A Dirac Delta Operator

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Abstract If $T$ is a (densely defined) self-adjoint operator acting on a complex Hilbert space $\mathcal{H}$ and $I$ stands for the identity operator, we introduce the delta function operator $\lambda \mapsto \delta (\lambda I - T)$ at $T$. When $T$ is a bounded operator, then $\delta (\lambda I - T)$ is an operator-valued distribution. If $T$ is unbounded, $\delta (\lambda I - T)$ is a more general object that still retains some properties of distributions. We provide an explicit representation of $\delta (\lambda I - T)$ in some particular cases, derive various operative formulas involving $\delta (\lambda I - T)$ and give several applications of its usage in Spectral Theory as well as in Quantum Mechanics.

Keywords Hilbert Space, Self-adjoint Operator, Vector-valued Distribution, Spectral Measure

1 The delta function $\delta (\lambda I - T)$

The scalar delta ‘function’ $\lambda \mapsto \delta (\lambda - \alpha)$ along with its derivatives were introduced by Paul Dirac in [1], and later in [2, Section 15], although its definition can be traced back to Heaviside. The rigorous treatment of this object in the context of distribution theory is due to Laurent Schwartz [6, 12]. In this paper we extend the definition of $\delta (\lambda - \alpha)$ from real numbers to self-adjoint operators on a Hilbert space $\mathcal{H}$. We denote by $\mathcal{D} (\mathbb{R}) = \lim D([-n, n])$ the linear space of infinitely differentiable complex-valued functions of compact support, equipped with the inductive limit topology. As usual in physics we shall assume that the scalar product in $\mathcal{H}$ is anti-linear for the first variable.

If $T$ is a densely defined self-adjoint operator\(^1\) on $\mathcal{H}$ and $I$ stands for the identity operator, we define the delta function operator $\lambda \mapsto \delta (\lambda I - T)$ at $T$ by

$$f (T) = \int_{-\infty}^{+\infty} f (\lambda) \delta (\lambda I - T) \, d\lambda \quad (1.1)$$

for each $f \in \mathcal{C} (\mathbb{R})$, i.e., for each real-valued continuous function $f (\lambda)$. Here $d\lambda$ is the Lebesgue measure of $\mathbb{R}$, but the right-hand side of (1.1) is not a true integral. If $T$ is a bounded operator, we shall see at once that $\delta (\lambda I - T)$ must be regarded as a vector-valued distribution, i.e., as a continuous linear map from the space $\mathcal{D} (\mathbb{R})$ into the locally convex space $\mathcal{L} (\mathcal{H})$ of the bounded linear operators on $\mathcal{H}$ equipped with the strong operator topology [10, 11], whose action on $f \in \mathcal{D} (\mathbb{R})$ we denote as an integral. If $T$ is unbounded we shall see that $\delta (\lambda I - T)$ still retains some useful distributional-like properties. The previous equation means

$$\langle y, f (T) x \rangle = \int_{-\infty}^{+\infty} f (\lambda) \langle y, \delta (\lambda I - T) x \rangle \, d\lambda \quad (1.2)$$

for each $(x, y) \in D (f (T)) \times \mathcal{H}$, where $D (f (T))$ stands for the domain of the self-adjoint operator $f (T)$.

Let us recall that if $T$ is a (densely defined) self-adjoint operator, there is a unique spectral family $\{ E_\lambda : \lambda \in \mathbb{R} \}$ of self-adjoint operators defined on the whole of $\mathcal{H}$ that satisfy (i) $E_\lambda \leq E_\mu$ and $E_\lambda E_\mu = E_\lambda$ for $\lambda \leq \mu$, (ii) $\lim_{\lambda \to 0^+} E_{\lambda + \epsilon} = E_\lambda x$, and (iii) $\lim_{\lambda \to -\infty} E_{\lambda} x = 0$ and $\lim_{\lambda \to +\infty} E_{\lambda} x = x$ in $\mathcal{H}$ for all $x \in \mathcal{H}$. The domain $D (T)$ of $T$ consists of those $x \in \mathcal{H}$ such that

$$\int_{-\infty}^{+\infty} |\lambda|^2 \, d \| E_\lambda x \|^2 < \infty.$$  

In this case, the spectral theorem (cf. [8, Section 107]) and the Borel-measurable functional calculus provide a self-adjoint operator $f (T)$ defined by

$$f (T) = \int_{-\infty}^{+\infty} f (\lambda) \, dE_\lambda \quad (1.3)$$

1In what follows $\sigma (T)$ will denote the spectrum of $T$. Recall that the residual spectrum of a self-adjoint operator $T$ is empty, so that $\sigma (T) = \sigma_p (T) \cup \sigma_c (T)$, where $\sigma_p (T)$ denotes the point spectrum (the eigenvalues) and $\sigma_c (T)$ the continuous spectrum of $T$. 


for each Borel-measurable function $f(\lambda)$, whose domain

$$D(f(T)) = \left\{ x \in \mathcal{H} : \int_{-\infty}^{+\infty} |f(\lambda)|^2 \, d\|E_\lambda x\|^2 < \infty \right\}$$

is dense in $\mathcal{H}$. Observe that if $T$ is bounded, $f(T)$ need not be bounded. Moreover, since $\lambda \mapsto E_\lambda$ is constant on the set $\mathbb{R} \setminus \sigma(T)$ of $T$, an open set in $\mathbb{R}$, equation (1.1) tells us that $f(\lambda)$ need not be defined on $\mathbb{R} \setminus \sigma(T)$.

Thanks to (1.3) the definition of $\delta(\lambda I - T)$ may be extended to Borel-measurable functions by declaring that the equation (1.1) holds for $(x, y) \in D(f(T)) \times \mathcal{H}$ and each Borel function $f$. But, by reasons that will become clear later, we shall restrict ourselves to those Borel functions which are continuous at each point of $\sigma_p(T)$. Moreover, working with the real and complex parts, no difficulty arises if the function $f$ involved in the equation (1.1) is complex-valued (except that $f(T)$ is no longer a self-adjoint operator whenever $Im f \neq 0$). Thus, unless otherwise stated, we shall assume that both in (1.1) and (1.3) the function is complex-valued. Note that the complex Stieltjes measure $d(E_\lambda x, y)$ need not be $d\lambda$-continuous. In what follows we shall denote by $B_p(\mathbb{R})$ the linear space over $\mathbb{C}$ consisting of all complex-valued Borel-measurable functions of one real variable which are continuous on $\sigma_p(T)$.

If $f_n \rightarrow f$ in $D(\mathbb{R})$, the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded and $f_n(x) \rightarrow f(x)$ at each $x \in \mathbb{R}$. So, if $T$ is bounded on $\mathcal{H}$ (equivalently, self-adjoint on the whole of $\mathcal{H}$) it turns out that $f_n(T) \rightarrow f(T)$ in the strong operator topology [3, 10.2.8 Corollary]. Therefore, in this case $\delta(\lambda I - T)$ is an $L(\mathcal{H})$-valued distribution.

As all integrals considered so far are over $\sigma(T)$, we have

$$\delta(\lambda I - T) = 0 \quad \forall \lambda \notin \sigma(T) \quad (1.4)$$

Also $\delta(-\lambda I + T) = \delta(\lambda I - T)$ for all $\lambda \in \mathbb{R}$. On the other hand, if $\mu \in \sigma_p(T)$ and $y$ is an eigenvector corresponding to the eigenvalue $\mu$, clearly

$$\delta(\lambda I - T)y = \delta(\lambda - \mu)y \quad (1.5)$$

for every $\lambda \in \mathbb{R}$. In the particular case when $T_a$ is the linear operator defined on $\mathcal{H}$ by $T_a x = ax$ for a fixed $a \in \mathbb{R}$, then $T_a$ is a self-adjoint linear operator with $\sigma(T_a) = \sigma_p(T_a) = \{a\}$. In this case $\delta(\lambda I - T_a)x = \delta(\lambda - a)x$ for every $x \in \mathcal{H}$, i.e., $\delta(\lambda I - T_a) = \delta(\lambda - a)I$.

Since equality $\langle f(T)\gamma, y \rangle = \langle y, f(T)x \rangle$ holds for all $x, y \in D(f(T))$ and each $f \in B_p(\mathbb{R})$, we may infer that

$$\langle y, \delta(\lambda I - T)x \rangle = \langle \delta(\lambda I - T)y, x \rangle$$

holds (in a ‘distributional’ sense) for all $x, y \in D(T)$. This suggests that in certain sense $\delta(\lambda I - T)$ may be regarded (possibly for almost all $\lambda \in \mathbb{R}$) as a Hermitian operator on $D(T)$.

Let us also point out that as equation (1.1) holds for all $f \in D(\mathbb{R})$, in a distributional sense we have

$$\frac{d}{d\lambda} \langle y, E_\lambda x \rangle = \langle y, \delta(\lambda I - T)x \rangle \quad (1.6)$$

If $\lambda \mapsto Y(\lambda - \mu)$ denotes the unit step function at $\mu \in \mathbb{R}$, given by $Y(\lambda - \mu) = 0$ if $\lambda < \mu$ and $Y(\lambda - \mu) = 1$ if $\lambda \geq \mu$, since $E_\lambda = Y(\lambda I - T)$ for each $\lambda \in \mathbb{R}$, formally

$$dE_\lambda/d\lambda = Y'(\lambda I - T) \quad (1.7)$$

So, from (1.6) and (1.7) we get $Y'(\lambda I - T) = \delta(\lambda I - T)$.

**Proposition 1.** If $T$ is a bounded self-adjoint operator on $\mathcal{H}$ and $f \in C^1(\mathbb{R})$, then

$$\int_{-\infty}^{+\infty} f(\lambda) \, \delta'(\lambda I - T) \, d\lambda = -f'(T).$$

The same equality holds if $T$ is unbounded but $f \in D(\mathbb{R})$.

If $T$ is a self-adjoint operator and $f \in B_p(\mathbb{R})$, then

$$\int_{-\infty}^{+\infty} |f(\lambda)|^2 \, \delta(\lambda I - T) \, d\lambda = \int_{-\infty}^{+\infty} |f(\lambda)|^2 \, dE_\lambda,$$

where the latter equality is the definition of $|f(T)|^2$. So, we have the following result.

**Proposition 2.** If $T$ is self-adjoint and $f \in B_p(\mathbb{R})$, then

$$\langle f(T) y, f(T)x \rangle = \int_{-\infty}^{+\infty} |f(\lambda)|^2 \langle y, \delta(\lambda I - T) x \rangle \, d\lambda$$

for every $x, y \in D(f(T))$.

**Proof.** We adapt a classic argument. Indeed, for every $x, y \in D(f(T))$ we have

$$\langle f(T) y, f(T)x \rangle = \int_{-\infty}^{+\infty} \langle f(\lambda) \, d\langle f(T)y, E_\lambda x \rangle \rangle.$$

Since $E_\mu E_\lambda = E_\mu$ whenever $\mu \leq \lambda$, and $\langle E_\lambda y, x \rangle$ does not depend on $\mu$, by splitting the integral we get

$$\int_{-\infty}^{+\infty} f(\mu) \, d\langle E_\mu y, E_\lambda x \rangle = \int_{-\infty}^{\lambda} f(\mu) \, d\langle y, E_\mu x \rangle,$$

where clearly the first integral is $\langle f(T)y, E_\lambda x \rangle$. Plugging $d\langle f(T)y, E_\lambda x \rangle$ into (1.8), we are done. □

**Corollary 3.** Under the same conditions of the previous theorem, the equality

$$\|f(T)x\|^2 = \int_{-\infty}^{+\infty} |f(\lambda)|^2 \langle x, \delta(\lambda I - T) x \rangle \, d\lambda \quad (1.9)$$

holds for every $x \in D(f(T))$.

**Proposition 4.** If $T$ is self-adjoint and $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence in $B_p(\mathbb{R})$ such that $f_n \rightarrow f$ pointwise on $\mathbb{R}$ with $f \in B_p(\mathbb{R})$, then $f_n(T)x \rightarrow f(T)x$ for every $x \in D(T)$.

**Proof.** This is a straightforward consequence of preceding corollary and the Lebesgue dominated convergence theorem. □
This proposition holds in particular if \( f_n \to f \) in \( D(\mathbb{R}) \). Hence, even in the unbounded case, \( \delta (\lambda I - T) \) behaves as a vector-valued distribution-like object.

**Proposition 5.** Let \((\lambda, \mu) \mapsto g(\lambda, \mu)\) be a function defined on \(\mathbb{R}^2\) such that \( g(\lambda, \cdot) \in L_1(\mathbb{R}) \) for every \( \lambda \in \mathbb{R} \) and \( g(\cdot, \mu) \in B_p(\mathbb{R}) \) for every \( \mu \in \mathbb{R} \). If the parametric integral

\[
f(\lambda) = \int_{-\infty}^{+\infty} g(\lambda, \mu) \, d\mu
\]

is continuous on \(\mathbb{R}\) and makes sense if we replace \( \lambda \) by a self-adjoint operator \( T \), the value of the integral

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(\lambda, \mu) \delta(\lambda I - T) \, d\mu \, d\lambda
\]

does not depend on the integration ordering.

**Proof.** Since \( g(\cdot, \mu) \in B_p(\mathbb{R}) \) for every \( \mu \in \mathbb{R} \), one has

\[
g(T, \mu) = \int_{-\infty}^{+\infty} g(\lambda, \mu) \delta(\lambda I - T) \, d\lambda,
\]

which implies

\[
f(T) = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} g(\lambda, \mu) \delta(\lambda I - T) \, d\lambda \right\} \, d\mu.
\]

On the other hand, by the definition of \( \delta(\lambda I - T) \) we have

\[
f(T) = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} g(\lambda, \mu) \, d\mu \right\} \delta(\lambda I - T) \, d\lambda,
\]

for \((x, y) \in D(T) \times \mathcal{H}\). So, the proposition follows. \( \square \)

**Theorem 6.** If \( T \) is a self-adjoint operator on \( \mathcal{H} \), then

\[
\int_{0}^{+\infty} f(\lambda) \delta(\lambda I - T^2) \, d\lambda = \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{\lambda}} \left( \delta\left(\sqrt{\lambda} I - T\right) - \delta\left(\sqrt{\lambda} I + T\right) \right) f(\lambda) \, d\lambda
\]

if \( \lambda > 0 \) and \( f \in B_p(\mathbb{R}) \), both members acting on \( D(T^2) \).

**Proof.** First note that \( T^2 \geq 0 \). Hence \( \sigma(T^2) \subseteq [0, +\infty) \), which implies that \( \delta(\lambda I - T^2) = 0 \) if \( \lambda < 0 \). Since \( T^2 \) is a self-adjoint operator, for \( f \in B_p(\mathbb{R}) \) we have

\[
\int_{0}^{+\infty} f(\lambda) \delta(\lambda I - T^2) \, d\lambda = f(T^2)
\]

On the other hand, it is clear that

\[
\int_{0}^{+\infty} \frac{f(\lambda)}{2\sqrt{\lambda}} \delta\left(\sqrt{\lambda} I - T\right) \, d\lambda = \int_{0}^{+\infty} f(\mu^2) \delta(\mu I - T) \, d\mu
\]

whereas, using that \( \delta(-\mu I + T) = \delta(\mu I - T) \), we have

\[
\int_{0}^{+\infty} \frac{f(\lambda)}{2\sqrt{\lambda}} \delta\left(\sqrt{\lambda} I + T\right) \, d\lambda = \int_{-\infty}^{0} f(\mu^2) \delta(\mu I - T) \, d\mu
\]

So, the right-hand side of (1.10) coincides with

\[
\int_{0}^{+\infty} f(\mu^2) \delta(\mu I - T) \, d\mu = f(T^2)
\]

since \( \mu \mapsto f(\mu^2) \) is a Borel function.

If we denote by \( L(\mathcal{H}) \) the linear space of all linear endomorphisms on \( \mathcal{H} \), the next theorem summarize some previous results.

**Theorem 7.** If \( T \) is a densely defined self-adjoint operator on a Hilbert space \( \mathcal{H} \), there is an \( L(\mathcal{H}) \)-valued linear map \( \delta_T \) on \( B_p(\mathbb{R}) \), whose action on \( f \in B_p(\mathbb{R}) \) we denote by

\[
\langle \delta_T, f \rangle = \int_{-\infty}^{+\infty} f(\lambda) \delta(\lambda I - T) \, d\lambda,
\]

such that \( \langle \delta_T, f \rangle = f(T) \). If \( \{f_n\} \subseteq B_p(\mathbb{R}) \) is uniformly bounded and \( f_n(t) \to f(t) \), with \( f \in B_p(\mathbb{R}) \), for all \( t \in \mathbb{R} \) then \( \langle \delta_T, f_n \rangle \to \langle \delta_T, f \rangle \) for all \( x \in \mathcal{H} \). If \( T \) is bounded, \( \delta_T \) is an \( L(\mathcal{H}) \)-valued distribution, so \( \langle \delta_T, f \rangle \) is a bounded operator on \( \mathcal{H} \). In addition \( \delta(\lambda I - T) = 0 \) if \( \lambda \notin \sigma(T) \) and \( \langle y, \delta(\lambda I - T) x \rangle = \langle \delta(\lambda I - T) y, x \rangle \) for \( x, y \in D(T) \).

# 2 Explicit form of \( \delta(\lambda I - T) \)

If \( Q \) is a vector-valued distribution, the Fourier transform of \( Q \) is defined as the vector valued distribution \( \mathcal{F}Q \) on \( S(\mathbb{R}) \) such that \( \langle \mathcal{F}Q, f \rangle = \langle Q, \mathcal{F}f \rangle \). As usual, we denote by \( \mathcal{F}^{-1} \) the inverse Fourier transform.

**Theorem 8.** If \( T \) is a self-adjoint operator, the identity

\[
\delta(\lambda I - T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{it(\lambda I - T)} \, dt
\]

holds for every \( \lambda \in \mathbb{R} \), and the action \( f(T) \) of \( \delta(\lambda I - T) \) on \( f \in S(\mathbb{R}) \) is given by

\[
f(T) = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \frac{f(\lambda)}{2\pi} e^{it(\lambda I - T)} \, d\lambda \right\} \, dt.
\]

**Proof.** Setting \( \delta_T(\lambda) = \delta(\lambda I - T) \) observe that

\[
\langle \mathcal{F}\delta_T, f \rangle = \langle \delta_T, \mathcal{F}f \rangle = \int_{-\infty}^{+\infty} \langle f(\lambda) \delta(\lambda I - T) \rangle \, d\lambda = \langle \mathcal{F}f, T \rangle = \int_{-\infty}^{+\infty} f(t) e^{-itT} \, dt.
\]

Indeed, if \( f \in S(\mathbb{R}) \) we have

\[
\langle \mathcal{F}\delta_T, f \rangle = \langle \delta_T, \mathcal{F}f \rangle = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-itT} \, dt.
\]

Consequently

\[
\delta_T = \mathcal{F}^{-1} \left\{ \frac{1}{\sqrt{2\pi}} e^{-itT} \right\}.
\]

Functionally, the action of \( \delta_T \) on \( f \in S(\mathbb{R}) \) by means of equation (2.2) becomes

\[
\langle \delta_T, f \rangle = \left\langle \frac{1}{\sqrt{2\pi}} e^{-itT}, \mathcal{F}^{-1} \langle f \rangle \right\rangle
\]

Consequently, we have

\[
\langle \delta_T, f \rangle = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \frac{f(\mu)}{2\pi} e^{it(\mu I - T)} \, d\mu \right\} \, dt
\]

with the order of the integration as stated. \( \square \)
The limit is well-defined since

$$H$$

for small

and the right-hand integral makes no sense (see [4] for a useful

Remark 10. Consider the one-parameter unitary group

$$\{ U(t) : t \in \mathbb{R} \}$$
generated by the self-adjoint operator $$T$$, that is, $$U(t) = \exp(-itT)$$ for every $$t \in \mathbb{R}$$. If $$\mathcal{F}$$ denotes the Fourier transform, equation (2.2) can be written as

$$\delta(\lambda I - T) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} (U)(\lambda). \quad (2.4)$$

So, equation (2.3) reads as

$$f(T) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left( \mathcal{F}^{-1} f \right)(t) U(t) \, dt. \quad (2.5)$$

In what follows we shall compute the spectral family $$\{E_\lambda : \lambda \in \mathbb{R} \}$$ for some useful self-adjoint operators of Quantum Mechanics by means of the delta $$\delta(\lambda I - T)$$. Nonetheless, although $$E_\lambda = Y(\lambda I - T)$$, the identification

$$Y(\lambda I - T) = \int_{-\infty}^{+\infty} Y(\lambda - \mu) \delta(\mu I - T) \, d\mu$$

might be not well-defined because $$\mu \mapsto Y(\lambda - \mu)$$ has a jump discontinuity at $$\mu = \lambda$$. Indeed, if $$\lambda \in \sigma_p(T)$$ and $$x$$ is an eigenvector corresponding to $$\lambda$$, then

$$\int_{-\infty}^{+\infty} Y(\lambda - \mu) \delta(\mu I - T) \, x \, d\mu = \left\{ \int_{-\infty}^{\lambda} \delta(\mu - \lambda) \, d\mu \right\} x$$

and the right-hand integral makes no sense (see [4] for a useful discussion). If $$\lambda \notin \sigma_p(T)$$ we define

$$E_\lambda = \int_{-\infty}^{+\infty} Y(\lambda - \mu) \delta(\mu I - T) \, d\mu \quad (2.6)$$

If $$\lambda$$ belongs to $$\sigma_p(T)$$, then $$(\mu \mapsto Y(\lambda - \mu)) \notin B_p(\mathbb{R})$$. In order to define $$E_\lambda$$ we enlarge a little the interval of integration by considering the integral

$$\int_{-\infty}^{\lambda + \epsilon} \delta(\mu - \lambda) \, d\mu$$

for small $$\epsilon > 0$$. So, if $$\lambda \in \sigma_p(T)$$ we define

$$E_\lambda = \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} Y(\lambda + \epsilon - \mu) \delta(\mu I - T) \, d\mu. \quad (2.7)$$

The limit is well-defined since $$\lim_{\epsilon \to 0^+} E_{\lambda + \epsilon} = E_\lambda$$ pointwise on $$\mathcal{H}$$. In the particular case when $$\lambda$$ belongs to $$\sigma_d(T)$$, the discrete part of $$\sigma_p(T)$$, $$\lambda$$ is isolated in $$\sigma_p(T)$$.

Example 11. The spectral family of the (up to a sign) one-dimensional Quantum Mechanics momentum operator of the free particle $$P = iD$$, where $$D \varphi = \varphi'$$, acting on the Hilbert space $$\mathcal{H} = L_2(\mathbb{R})$$ is given by

$$(E_\lambda \varphi)(x) = \frac{1}{2} \varphi(x) + \frac{1}{2\pi i} \text{p.v.} \int_{-\infty}^{+\infty} \frac{e^{i\lambda(s-x)}}{s-x} \varphi(s) \, ds$$

for every regularly compactly supported $$\varphi \in D(P)$$.

Proof. As is well-known $$P$$ is a self-adjoint operator with $$D(P) = H^{2,1}(\mathbb{R})$$ and $$\sigma_c(P) = \mathbb{R}$$. Since

$$(e^{-itP}\varphi)(x) = (e^{itP}\varphi)(x) = \varphi(x + t)$$

for a regular enough $$\varphi \in D(P)$$, by Corollary 9 we have

$$\{ \delta(\mu I - P) \varphi \}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\mu t} \varphi(x + t) \, dt.$$

Note that the integral of the right-hand side does exist because $$\varphi$$ has compact support.

According to the definition of $$E_\lambda$$ for the continuous spectrum and keeping in mind the order of integration as indicated in Corollary 9, one has

$$\{E_\lambda \varphi \}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} Y(\lambda - \mu) e^{i\mu t} \varphi(x + t) \, d\mu \, dt.$$ 

So, since

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} Y(\lambda - \mu) e^{i\mu t} \, d\mu = \mathcal{F}(Y)(t) \cdot e^{i\lambda t},$$

bearing in mind the distributional relation

$$\mathcal{F}(Y)(t) = \sqrt{\frac{\pi}{2}} \left( \delta(t) + \frac{1}{i\pi} \text{p.v.} \frac{1}{t} \right),$$

we get

$$\{E_\lambda \varphi \}(x) = \frac{1}{2} \varphi(x) + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i\lambda(s-x)}}{s-x} \varphi(s) \, ds$$

where the last integral must be understood in Cauchy’s principal value sense.

Example 12. The spectral family of the one-dimensional Quantum Mechanics kinetic energy term of the free particle, corresponding to the Laplace operator $$T = -D^2$$ on $$\mathcal{H} = L_2(\mathbb{R})$$, where $$D^2 \varphi = \varphi''$$, is given by

$$(E_\lambda \varphi)(x) = \frac{1}{i\pi} \text{p.v.} \int_{-\infty}^{+\infty} \cos(\lambda(s-x)) \frac{1}{s-x} \varphi(s) \, ds$$

for $$\lambda > 0$$ and $$E_\lambda = 0$$ whenever $$\lambda < 0$$, where $$\varphi$$ is a regular function with compact support belonging to $$D(T)$$.

Proof. In this case $$T$$ is a self-adjoint operator with $$\sigma(T) = [0, +\infty)$$. Since $$T = (iD)^2$$, according to (1.10) we have

$$\delta(\lambda I - T) = \frac{1}{2\lambda} \left\{ \delta \left( \sqrt{\lambda} I - iD \right) - \delta \left( \sqrt{\lambda} I + iD \right) \right\}$$
regarded as a functional on $\mathcal{S}(\mathbb{R})$ through $d\lambda$-integration over $[0, +\infty)$. Plugging

$$
(\delta (\mu I \mp iD) \varphi) (x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\mu t} \varphi (x \pm t) \, dt
$$

into the previous expression and keeping in mind the correct order of integration, we see that

$$
\int_{0}^{\infty} f (\lambda) \left( \delta (\mu I - T) \varphi \right) (x) \, d\lambda =
\frac{1}{4\pi} \int_{0}^{\infty} \int_{-\infty}^{+\infty} f (\lambda) \frac{e^{i\sqrt{\mu} t}}{\sqrt{\lambda}} \left[ \varphi (x + t) - \varphi (x - t) \right] \, d\lambda \, dt
$$

for every $f \in \mathcal{S}(\mathbb{R})$. By the definition of $E_\lambda$ if $\lambda > 0$ and the fact that $\delta (\mu I - T) = 0$ whenever $\mu < 0$, we have

$$
(E_\lambda \varphi) (x) = \int_{0}^{+\infty} Y (\lambda - \mu) \left( \delta (\mu I - T) \varphi \right) (x) \, d\mu.
$$

Working out the penultimate integral with $\mu$ instead of $\lambda$ and $f (\mu) = Y (\lambda - \mu)$, we obtain

$$
\int_{0}^{+\infty} \int_{-\infty}^{+\infty} Y (\lambda - \mu) \frac{e^{i\sqrt{\mu} t}}{\sqrt{\lambda}} \left[ \varphi (x + t) - \varphi (x - t) \right] \, d\mu \, dt
= \int_{-\infty}^{\infty} \int_{0}^{+\infty} \left\{ \int_{-\infty}^{\lambda} e^{i\sqrt{\mu} t} \, dt \right\} \left[ \varphi (x + t) - \varphi (x - t) \right] \, dt
$$

for $\lambda > 0$. So, by setting $u = \sqrt{\mu}$ we get

$$
(E_\lambda \varphi) (x) = \int_{0}^{+\infty} \frac{dt}{2\pi} \left[ \varphi (x + t) - \varphi (x - t) \right] \int_{0}^{\lambda} e^{iut} \, du.
$$

Now we have

$$
\frac{1}{\sqrt{2\pi}} \int_{0}^{\lambda} e^{iut} \, du = \left( 1 - e^{i\lambda t} \right) F^{-1} (Y) (t),
$$

so, using that $F^{-1} (Y (v)) = F (1 - Y (v)) (t)$ as well as equation (2.8), we get

$$
\frac{1}{\sqrt{2\pi}} \int_{0}^{\lambda} e^{iut} \, du = \left( 1 - e^{i\lambda t} \right) \sqrt{\frac{\pi}{2}} \left( \delta (t) - \frac{1}{i\pi} \mathrm{p.v.} \frac{1}{t} \right)
$$

which implies

$$
(E_\lambda \varphi) (x) =
- \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{\varphi (s) \, ds}{s - x} + \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{\cos (\lambda (s - x))}{s - x} \varphi (s) \, ds
$$

where the integrals are understood in Cauchy’s principal value sense. 

Example 13. Spectral family of the (up to a sign) one-dimensional Quantum Mechanics momentum operator $S$ for a bounded particle on $\mathcal{H} = L_2 [-\pi, \pi]$ with domain

$$
\{ \varphi \in L_2 [-\pi, \pi] : \varphi' \in L_2 [-\pi, \pi], \varphi (-\pi) = \varphi (\pi) \}
$$

As is well-known this is a self-adjoint operator with discrete spectrum $\sigma (S) = \mathbb{Z}$ whose eigenfunction system $\{ \varphi_n : n \in \mathbb{Z} \}$, with $\varphi_n (x) = (2\pi)^{-1/2} e^{-inx}$, are the solutions of the eigenvalue problem $i \varphi'' = \lambda \varphi$ with $\varphi (-\pi) = \varphi (\pi)$. So, for $\varphi \in D (S)$ we have $E_\lambda \varphi \equiv \sum_{n \in \mathbb{Z}} c_n \varphi_n$ with

$$
c_n = \langle \varphi, \varphi_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi (x) e^{inx} \, dx
$$

for every $n \in \mathbb{Z}$. Since $\sigma (S) = \sigma_d (S)$, recalling the definition of the operator $E_\lambda$ for $\lambda \in \sigma_d (S)$, clearly we have

$$
(E_\lambda \varphi) (x) = \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} Y (\lambda + \epsilon - \mu) \left( \delta (\mu I - S) \varphi \right) (x) \, d\mu
$$

for every $\lambda \in \mathbb{R}$. So, the fact that $E_\lambda$ is a bounded operator yields

$$
E_\lambda \varphi = \sum_{n \in \mathbb{Z}} c_n E_\lambda \varphi_n.
$$

Using that $\delta (\mu I - S) e^{-inx} = \delta (\mu - n) e^{-inx}$ and that $Y (\lambda + 0 - n) = Y (\lambda - n)$, we get

$$
(E_\lambda \varphi) (x) = \sum_{n \in \mathbb{Z}} \frac{c_n}{\sqrt{2\pi}} Y (\lambda - n) e^{-inx} = \sum_{n \in \mathbb{Z}, n \leq |\lambda|} \frac{e^{-inx}}{\sqrt{2\pi}}.
$$

Remark 14. Since in the previous example $S$ is bounded on $\mathcal{H} = L_2 [-\pi, \pi]$, the delta operator $\delta (\lambda I - S)$ should be regarded as a continuous endomorphism as well. In this case

$$
\delta (\lambda I - S) \varphi = \sum_{n \in \mathbb{Z}} c_n \delta (\lambda - n) \varphi_n.
$$

Example 15. The one-dimensional Quantum Mechanics position operator on $L_2 (\mathbb{R})$. This operator is defined on $\mathcal{H} = L_2 (\mathbb{R})$ by $(Q \varphi) (x) = x \varphi (x)$ for every $x \in \mathbb{R}$. Clearly $\sigma_c (Q) = \mathbb{R}$ and $\varphi \in D (Q)$ if $(x \mapsto x \varphi (x)) \in L_2 (\mathbb{R})$. Moreover, it is clear that

$$
\{ \exp (it (\lambda - Q)) \varphi \} (x) = e^{i(\lambda - x)t} \varphi (x).
$$

So we have

$$
(\delta (\lambda I - Q) \varphi) (x) = \delta (\lambda - x) \varphi (x).
$$

Hence, in this case we can write

$$
\{ E_\lambda \varphi \} (x) = \int_{-\infty}^{+\infty} Y (\lambda - \mu) \delta (\mu - x) \varphi (x) \, d\mu
$$

Therefore, if $\lambda \neq x$ we get

$$
\{ E_\lambda \varphi \} (x) = Y (\lambda - x) \varphi (x).
$$

Example 16. Explicit form of $\delta (\lambda I - M)$ for the Hermitian matrix of $\mathcal{H} = \mathbb{C}^3$.

$$
M = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}.
$$
Proof. In this case \( M = PJ_MP^{-1} \) with \( \sigma(M) = \{-1, 2\} \) and
\[
J_M = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 2
\end{bmatrix}, \quad P = \begin{bmatrix}
1 & 1 & 1 \\
-1 & 0 & 1 \\
0 & -1 & 1
\end{bmatrix}
\]
Using (2.1) we get
\[
\delta(\lambda I - M) = P \begin{bmatrix}
\delta(\lambda + 1) & 0 & 0 \\
0 & \delta(\lambda + 1) & 0 \\
0 & 0 & \delta(\lambda - 2)
\end{bmatrix} P^{-1}
\]
Let us compute the spectral family and the projection operator onto the eigenspace \( \ker(M + I) \). Clearly
\[
E_\lambda = P \begin{bmatrix}
Y(\lambda + 1) & 0 & 0 \\
0 & Y(\lambda + 1) & 0 \\
0 & 0 & Y(\lambda - 2)
\end{bmatrix} P^{-1}
\]
for every \( \lambda \in \mathbb{R} \). If \( \lambda_1 = -1 \), the orthogonal projection \( P_{\lambda_1} \) onto \( \ker(I + M) \) is
\[
P_{\lambda_1} = \frac{1}{3} P \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} P^{-1} = \frac{1}{3} \begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{bmatrix}
\]
since \( P_{\lambda_1} = E_{\lambda_1} - E_{\lambda_1 - 0} = E_{\lambda_1} \).

Example 17. Consider a compact self-adjoint operator \( K \) acting on a separable Hilbert space \( \mathcal{H} \) which does not admit the eigenvalue zero. Let \( \{u_i : i \in \mathbb{N}\} \) be a Hilbert basis of \( \mathcal{H} \) with its corresponding sequence of real eigenvalues \( \{\lambda_i : i \in \mathbb{N}\} \), where \( |\lambda_{i+1}| \leq |\lambda_i| \) for every \( i \in \mathbb{N} \). Let us compute the action of the operator \( (\lambda I - K)^{-1} \) on any \( x \in \mathcal{H} \) and the operator \( \delta(\lambda I - T) \).

Proof. If \( x \in \mathcal{H} \), we can write \( x = \sum_{i=1}^{\infty} \langle x, u_i \rangle u_i \). Since \( (\lambda I - K)^{-1} \) is a bounded operator whenever \( \lambda \not\in \sigma(K) \), we have
\[
(\lambda I - K)^{-1} x = \sum_{i=1}^{\infty} \frac{1}{\lambda - \mu_i} \langle x, u_i \rangle u_i
\]
so we obtain the classic series
\[
(\lambda I - K)^{-1} x = \sum_{i=1}^{\infty} \frac{1}{\lambda - \mu_i} (x, u_i) u_i.
\]
For the solution of the equation \( (I - zK) x = y \) with \( z \in \mathbb{C} \), we get the Schmidt series
\[
x = (I - zK)^{-1} y = \sum_{i=1}^{\infty} \frac{1}{1 - z \mu_i} \langle y, u_i \rangle u_i
\]
whenever \( z^{-1} \not\in \sigma(T) \). On the other hand, since \( \delta(\lambda I - K) \) acts on \( \mathcal{H} \) as a continuous endomorphism, equation
\[
\delta(\lambda I - K) x = \sum_{i=1}^{\infty} (x, u_i) \delta(\lambda - \mu_i) u_i.
\]
holds for every \( x \in \mathcal{H} \).

If \( T \) is an unbounded self-adjoint operator then \( D(T) \neq \mathcal{H} \) and \( D(T^n) \) becomes smaller as \( n \) grows. So, the following result, makes sense only if the operator \( T \) is bounded.

Theorem 18. In general, if \( T \) is a bounded self-adjoint operator, one has
\[
\delta(\lambda I - T) = \sum_{n=0}^{\infty} (\mu I - \lambda) n! T^n
\]
which is the Taylor series of \( \delta(\lambda I - T) \) at \( \lambda I \).

Proof. Developing the operator function \( \exp(itT) \), which is well-defined by the spectral theorem, we get
\[
\delta(\lambda I - T) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-i\lambda t} \sum_{n=0}^{\infty} \frac{(it)^n}{n!} T^n dt,
\]
so that, formally interchanging the sum and the integral, we may write
\[
\delta(\lambda I - T) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \mathcal{F}\{(it)^n\}(\lambda) T^n.
\]
Using the fact that
\[
\mathcal{F}\{(it)^n\}(\lambda) = (-1)^n \sqrt{2\pi} \delta^{(n)}(\lambda)
\]
for every \( n \in \mathbb{N} \), we obtain (2.9).

3 The resolvent operator and \( \delta(\lambda I - T) \)

Recall that the spectrum \( \sigma(T) \) of a (densely defined) self-adjoint operator on a complex Hilbert space \( \mathcal{H} \) is a closed subset of \( \mathbb{C} \) contained in \( \mathbb{R} \) (see for instance [9, 3.2]). If \( z \in \mathbb{C} \setminus \sigma(T) \), i.e., if \( z \) is a regular point of \( T \), and
\[
\mathcal{R}(z, T) = (z - T)^{-1}
\]
denotes the resolvent operator of \( T \) at \( z \) (see [7, Definition 8.2]), the function \( \lambda \mapsto (z - \lambda)^{-1} \) is continuous on \( \sigma(T) \). The resolvent is well-defined over \( \mathcal{H} \), so it is a bounded normal operator. If \( z \in \mathbb{C} \setminus \sigma(T) \) then \( \mathcal{R}(z, T) \) is even self-adjoint. From (1.1) it follows that
\[
\mathcal{R}(z, T) = \int_{-\infty}^{+\infty} \frac{1}{z - \lambda} \delta(\lambda I - T) d\lambda
\]
which is the integral form of the resolvent of \( T \). So, by considering the complex-valued function \( f(\lambda) = (z - \lambda)^{-1} \) with \( z \in \mathbb{C} \setminus \sigma(T) \) and using the fact that
\[
\mathcal{F}^{-1}\left(\frac{1}{\lambda - z}\right)(t) = \sqrt{2\pi} i e^{izt} Y(t)
\]
then, according to (2.5), for \( Imz > 0 \) we have
\[
(z - I)^{-1} = -i \int_{0}^{\infty} e^{izt} U(t) dt.
\]
From here, it follows that
\[ \mathcal{R} (z, iT) = i \mathcal{R} (iz, -T) = (LU^{-1})(z) \]
if \( Imz > 0 \), where \( \mathcal{L} \) is the Laplace transform. This is the Hille-Yosida theorem which relates the resolvent with the one-parameter group of unitary transformations \( \{ U(t) : t \in \mathbb{R} \} \) generated by the self-adjoint operator \( T \).

If \( T \) is a bounded self-adjoint operator, \( \gamma \) is a closed Jordan contour that encloses \( \sigma(T) \) and \( f(z) \) is holomorphic inside the connected region surrounded by the path \( \gamma \), the Dunford integral formula asserts that
\[ \frac{1}{2\pi i} \int_{\gamma} f(z) \mathcal{R}(z, T) \, dz = f(T). \]

In [13] is pointed out that \( (2\pi i)^{-1} \mathcal{R}(z, T) \) can be considered as the indicatrix of a vector-valued distribution with values in \( \mathcal{L}(\mathcal{H}) \). Dunford integral formula is easily obtained by using the \( \delta(\lambda I - T) \) operator since, if we apply the Proposition 5 with \( \varphi(\lambda, \mu) = f(z(\mu)) (z(\mu) - \lambda)^{-1} \), where \( z(\mu) = \gamma(\mu) \) and \( 0 \leq \mu \leq 1 \), then
\[ \int_{\gamma} f(z) \mathcal{R}(z, T) \, dz = \int_{-\infty}^{\infty} \left\{ \int_{\gamma} \frac{f(z)}{z - \lambda} \, dz \right\} \delta(\lambda I - T) \, d\lambda = 2\pi i \int_{-\infty}^{\infty} f(\lambda) \delta(\lambda I - T) \, d\lambda = 2\pi i f(T). \]

**Example 19.** Derivation of the orthogonal projection operator onto \( \ker(M + I) \) of the Hermitian matrix \( M \) of the Example 16 by the resolvent technique. We must compute
\[ P_{\lambda} = \frac{1}{2\pi i} \int_{[\lambda+1]} \mathcal{R}(z, M) \, dz. \]

Clearly, we have
\[ \mathcal{R}(z, M) = \frac{1}{z^2 - z - 2} \begin{bmatrix} z - 1 & 1 & 1 \\ 1 & z - 1 & 1 \\ 1 & 1 & z - 1 \end{bmatrix}. \]

Using that
\[ \int_{[\lambda+1]} \frac{1, z - 1}{(z + 1)(z - 2)} \, dz = \left\{ \frac{-2\pi i}{3}, \frac{4\pi i}{3} \right\} \]
we reproduce the result we got earlier.

## 4 The \( \delta(\lambda I - T) \) operator as a limit

As \( \mu \to (\lambda \pm i\epsilon - \mu)^{-1} \) is continuous, for self-adjoint \( T \)
\[ ((\lambda - i\epsilon) I - T)^{-1} - ((\lambda + i\epsilon) I - T)^{-1} \]
\[ = \int_{-\infty}^{\infty} \left( \frac{1}{\lambda - i\epsilon - \mu} - \frac{1}{\lambda + i\epsilon - \mu} \right) \delta(\mu I - T) \, d\mu. \]

If \( f \in \mathcal{D}(\mathbb{R}) \), Proposition 5 yields
\[ \int_{-\infty}^{\infty} \frac{f(\lambda)}{2\pi i} \left( ((\lambda - i\epsilon) I - T)^{-1} - ((\lambda + i\epsilon) I - T)^{-1} \right) \, d\lambda \]
\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{\epsilon}{(\lambda - \mu)^2 + \epsilon^2} \, d\lambda \right\} \delta(\mu I - T) \, d\mu. \]

Since in the sense of distributions
\[ \frac{1}{2\pi i} \left( \frac{1}{\lambda - i\epsilon - \mu} - \frac{1}{\lambda + i\epsilon - \mu} \right) \to \delta(\lambda - \mu) \]
as \( \epsilon \to 0^+ \), we have
\[ \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{(\lambda - \mu)^2 + \epsilon^2} \, d\lambda \to \int_{\mathbb{R}} \frac{1}{\lambda - \mu} \, d\lambda \]
as \( \epsilon \to 0^+ \). Hence, if \( g_n \) is defined by the left-hand side \( \mu \)-parametric integral with \( \epsilon = 1/n \), then \( g_n \to f \) pointwise on \( \mathbb{R} \).

Thus, by [3, 10.2.8 Corollary] one has \( g_n (T) \to f(T) \) in the strong operator topology, that is
\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu} \, d\lambda \right\} \delta(\mu I - T) \, d\mu \]
as \( \epsilon \to 0^+ \) in the strong operator topology of \( \mathcal{L}(\mathcal{H}) \). Therefore, if \( T \) is bounded and \( f \in \mathcal{D}(\mathbb{R}) \) then
\[ \int_{-\infty}^{+\infty} \frac{f(\lambda)}{2\pi i} \left( ((\lambda - i\epsilon) I - T)^{-1} - ((\lambda + i\epsilon) I - T)^{-1} \right) \, d\lambda \]
goes to \( f(T) \) as \( \epsilon \to 0^+ \). This proves that for bounded \( T \)
\[ \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \left( ((\lambda - i\epsilon) I - T)^{-1} - ((\lambda + i\epsilon) I - T)^{-1} \right) \]
coincides with \( \delta(\lambda I - T) \) as an \( \mathcal{L}(\mathcal{H}) \)-valued distribution.

## 5 Unitary equivalence of \( \delta(\lambda I - T) \)

**Theorem 20.** If \( T \) is a self-adjoint operator defined on the whole of \( \mathcal{H} \), there exist a finite measure \( \mu \) on the Borel sets of the compact space \( \sigma(T) \) and a linear isometry \( U \) from \( L_2(\sigma(T), \mu) \) onto \( \mathcal{H} \) such that
\[ U^{-1} \delta(\lambda I - T) U = \delta(\lambda I - Q) \]
where \( (Q\varphi)(x) = x\varphi(x) \) is the position operator.

**Proof.** According to [5] there exist a finite measure \( \mu \) on the Borel sets of the compact space \( \sigma(T) \) and a linear isometry \( U \) from \( L_2(\sigma(T), \mu) \) onto \( \mathcal{H} \) such that
\[ (U^{-1}TU) \varphi = Q\varphi \]
for every \( \varphi \in L_2(\sigma(T), \mu) \). So, since \( U^{-1}TU \) is a self-adjoint operator on \( L_2(\sigma(T), \mu) \), we have
\[ U^{-1}\delta(\lambda I - T) U = \delta(\lambda I - U^{-1}TU) = \delta(\lambda I - Q) \]
as stated.

**Remark 21.** For such linear isometry \( U \) the equation
\[ (U^{-1}\delta(\lambda I - T) U \varphi)(x) = \delta(\lambda - x) \varphi(x) \]
holds for every \( \varphi \in L_2(\sigma(T), \mu) \).
6 Commutation relations

Let $S$ and $T$ be two self-adjoint operators defined on the whole of $\mathcal{H}$ for which equations $[S, [S, T]] = [T, [S, T]] = 0$ hold. In this case

$$[-iS, [-iS, -isT]] = it^2s [S, [S, T]] = 0$$

and the Baker-Campbell-Hausdorff formula yields

$$\delta (\lambda I - S) \delta (\mu I - T) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{it(\lambda + s\mu)} e^{-isT} e^{-i(sT + sT)} ds dt = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{it(\lambda + s\mu)} e^{sT} e^{-isT} e^{-i(sT + sT)} ds dt.$$

Likewise, since $[T, [T, S]] = [S, [T, S]] = 0$ one has

$$\delta (\mu I - T) \delta (\lambda I - S) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{it(\lambda + s\mu)} e^{sT} e^{-i(sT + sT)} ds dt = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{it(\lambda + s\mu)} e^{sT} e^{-i(sT + sT)} ds dt.$$

So, using that

$$\exp \left( \frac{ist}{2} i [S, T] \right) - \exp \left( - \frac{ist}{2} i [S, T] \right) = 2i \sin \left( \frac{ist}{2} [S, T] \right)$$

we have

$$\int_{\mathbb{R}} \sin \left( \frac{ist}{2} [S, T] \right) e^{it(\lambda + s\mu)} e^{-i(sT + sT)} ds dt.$$

For position $Q$ and momentum $P$ of a one-dimensional particle, one has $\mathcal{H} = L_2(\mathbb{R})$ and $[Q, P] = i\hbar I$. Therefore $[Q, [Q, P]] = [P, [Q, P]] = 0$ and

$$\int_{\mathbb{R}} \sin \left( \frac{st}{2} [S, T] \right) e^{it(\lambda + s\mu)} e^{-i(sT + sT)} ds dt.$$

According to Theorem 8, if $\delta (\lambda I - S), \delta (\mu I - T)$ acts on $f (\lambda) = \lambda$, formally we have

$$\int_{-\infty}^{+\infty} \lambda \left[ \delta (\lambda I - S), \delta (\mu I - T) \right] d\lambda = \frac{i}{2\pi^2} \int_{\mathbb{R}} \sin \left( \frac{ist}{2} [S, T] \right) e^{it\lambda} d\lambda.$$

and integrating by parts, it follows that

$$\int_{-\infty}^{+\infty} \lambda \left[ \delta (\lambda I - S), \delta (\mu I - T) \right] d\lambda = -\frac{i}{2\pi} \left[ S, T \right] \int_{-\infty}^{+\infty} se^{is(\mu I - T)} ds.$$

Observe that a second application of equation (6.1) and a second integration by parts yield

$$\int_{-\infty}^{+\infty} \lambda \mu \left[ \delta (\lambda I - S), \delta (\mu I - T) \right] d\lambda d\mu = -\frac{i}{2\pi} \left[ S, T \right] \int_{-\infty}^{+\infty} se^{isT} \left\{ \int_{-\infty}^{+\infty} \mu e^{is\mu} d\mu \right\} ds = \left[ S, T \right] \int_{-\infty}^{+\infty} \delta (s) \left[ 1 - isT \right] e^{-isT} ds = [S, T]$$

as expected.

7 A remark on the Stone formula

Let $T$ be a self-adjoint operator densely defined on a Hilbert space $\mathcal{H}$. If $A$ is a Borel set in $\sigma (T)$, defining

$$E (A) := \int_{-\infty}^{+\infty} \chi_A (\lambda) \, dE (\lambda)$$

(7.1)

where $\chi_A$ stands for the characteristic function of $A$ (which is a bounded Borel function), then $E$ is an $\mathcal{L} (\mathcal{H})$-valued finitely additive and pointwise countably additive measure (i.e., countably additive under the strong operator topology of $\mathcal{L} (\mathcal{H})$) on the $\sigma$-algebra $\mathcal{A}$ of Borel subsets of $\sigma (T)$. So, if the characteristic function $\chi_A$ of $A$ with respect to $\mathbb{R}$ is continuous on $\sigma (T)$ then

$$E (A) = \int_{-\infty}^{+\infty} \chi_A (\lambda) \delta (\lambda I - T) \, d\lambda.$$

For $-\infty < a < b < \infty$ and $\epsilon > 0$, we have

$$\int_a^b \left\{ \int_{-\infty}^{+\infty} \left( \frac{1}{\lambda - i\epsilon - \mu} - \frac{1}{\lambda + i\epsilon - \mu} \right) \delta (\mu I - T) \, d\mu \right\} d\lambda$$

$$= \int_{-\infty}^{+\infty} \left\{ \int_a^b \frac{2i\epsilon d\lambda}{(\lambda - \mu)^2 + \epsilon^2} \right\} \delta (\mu I - T) \, d\mu = 2i \int_{\mathbb{R}} \left\{ \arg \tan \left( \frac{b - \mu}{\epsilon} \right) - \arg \tan \left( \frac{a - \mu}{\epsilon} \right) \right\} \delta (\mu I - T) \, d\mu.$$

If the limit as $\epsilon \to 0^+$ the bracketed function is equal to 0 if $\mu \in \mathbb{R} \setminus [a, b]$, equal to $\pi$ if $a < \mu < b$ and equal to $\pi/2$ if $\mu \in \{a, b\}$. So, if $a, b \notin \sigma_p (T)$ so that $\chi_{(a, b)}$ and $\chi_{[a, b]}$ both belong to $B_p (\mathbb{R})$, setting

$$g_n (\mu) := \frac{1}{\pi} \int_a^b \frac{2i n^{-1} d\lambda}{(\lambda - \mu)^2 + n^{-2}}$$

for each $n \in \mathbb{N}$ and

$$f (\mu) := \chi_{(a, b)} (\mu) + \chi_{[a, b]} (\mu),$$

and $f (\mu)$ is a simple function.

$$\int_{-\infty}^{+\infty} \lambda \left[ \delta (\lambda I - S), \delta (\mu I - T) \right] d\lambda = -\frac{i}{2\pi} \left[ S, T \right] \int_{-\infty}^{+\infty} se^{is(\mu I - T)} ds.$$
then $g_n(\mu) \to f(\mu)$ for every $\mu \in \mathbb{R}$ and $\sup_{n \in \mathbb{N}} \|g_n\|_\infty \leq 1$ which, according to Proposition 4, implies that $g_n(T)x \to f(T)x$ for every $x \in D(T)$. In other words

$$
\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{a}^{b} \left( (\lambda - i\epsilon - T)^{-1} - (\lambda + i\epsilon - T)^{-1} \right) d\lambda = \frac{1}{2} \int_{-\infty}^{+\infty} \left( \chi_{(a,b)} + \chi_{[a,b]} \right) \delta (\mu - T) \, d\mu,
$$

holds pointwise on the domain $D(T)$ of $T$. Hence, by virtue of (7.1) we get

$$
\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{a}^{b} \left( (\lambda - i\epsilon - T)^{-1} - (\lambda + i\epsilon - T)^{-1} \right) d\lambda = \frac{1}{2} E((a,b)) + \frac{1}{2} E([a,b]) = E(a,b) + \frac{1}{2} E(a) + \frac{1}{2} E(b)
$$

which is Stone’s formula.

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