Interaction of solitons with a qubit in an anisotropic Heisenberg spin chain

S K Varbev, R S Kamburova and M T Primatarowa
Georgi Nadjakov Institute of Solid State Physics, Bulgarian Academy of Sciences, 1784 Sofia, Bulgaria
E-mail: stanislavvarbev@issp.bas.bg

Abstract. The interaction between solitons, propagating in a magnetic chain, with a qubit is studied. The spin chain is described by the Heisenberg model in a nearest-neighbour approximation with anisotropy. The quantum bit (qubit) is modeled by a two level quantum system. The equation for the evolution of the qubit is obtained. The role of the parameters of the soliton for the quantum bit dynamics is investigated. The results are analyzed using a geometrical representation for the qubit as a Bloch sphere.

1. Introduction

Quantum bit in a quantum computer can be realised in many ways. In general, we could think of it as a two-level quantum system. A possible realization for a qubit is its implementation as an electron spin in Ref. [1]. Another proposed realization for a quantum bit is as a composed particle or a molecular cluster with magnetic moment and spin in Ref. [2].

The propagation and interaction of solitons in one-dimensional magnetic systems were studied in Refs. [3,4]. Propagation of soliton-like excitations along spin chains has been proposed as a possible way for transmitting both classical and quantum information between two distant parts of the system with negligible dispersion and dissipation in Refs. [5-7]. This gives the possibility to control the qubit levels, varying the parameters of the soliton.

In the present paper we study the interaction between a soliton, propagating in a magnetic chain, and a single-spin $\frac{\hbar}{2}$ qubit. The soliton is a solution of semiclassical equations of motion for the spin chain described by the Heisenberg model. The evolution of the qubit is described with the use of density operator formalism. The density operator for a qubit can be written in terms of the Bloch vector. The evolutionary equations for the components of the Bloch vector are determined. The system of equations is solved numerically for different values of the parameters. The dependence of the quantum bit dynamics on the parameters of the soliton is investigated. The results are analysed geometrically onto the Bloch sphere.

2. The model of the system

We consider the interaction of a ferromagnetic Heisenberg chain of $N$ spins with magnitude $S$ described in the nearest-neighbor approximation and a quantum bit by the following Hamiltonian

$$
\hat{H} = -J \sum_{n=1}^{N} \hat{S}_{n} \cdot \hat{S}_{n+1} - A \sum_{n=1}^{N} (\hat{S}_{n}^{z})^{2} - \mu H_{0} \sum_{n=1}^{N} \hat{S}_{n}^{z} - \nu H_{0} \hat{\sigma}_{j}^{z} + d_{xy}(\hat{S}_{j}^{x} \hat{\sigma}_{j}^{x} + \hat{S}_{j}^{y} \hat{\sigma}_{j}^{y}) + d_{z} \hat{S}_{j}^{z} \hat{\sigma}_{j}^{z}. (1)
$$
The identification of the quantities is as follows. \( \hat{S}_n \) are the spin operators for the spin chain, \( \hat{\sigma}_j \) is the spin operator for the qubit placed at an arbitrary site \( j \). \( J > 0 \) is the exchange integral, and \( A \) is the on-site anisotropy constant for the chain which can be positive (easy axis) or negative (easy plane). \( H_0 \) is the external magnetic field applied along the \( z \)-axis, so that in the ground state of the system all spins are aligned in the \( z \)-direction, \( \mu \) is the magnetic moment per spin in the chain, \( \nu \) is the magnetic moment per spin for the qubit and \( d_{xy} \) and \( d_z \) characterize the coupling interaction between the qubit and the spin from the chain labeled by the index \( j \).

We use for the scalar products of any spin vector operators in (1) the rule \( \langle \hat{\sigma}^+ \hat{\sigma}^- \rangle = 1 \) (quantum case) and the same rule (classical case). The envelopes \( \phi \) equal unity and the envelopes of \( \hat{\sigma}_j \) give zero.

The Heisenberg equations of motion for \( \hat{S}_n \) are

\[
i \hat{S}_n = [\hat{S}_n, \hat{H}],
\]

which in this case yield

\[
\pm i \frac{\partial \hat{S}_n}{\partial t} &= -J[\hat{S}_n^{\pm}(\hat{S}_{n-1}^+ + \hat{S}_{n+1}^-) - \hat{S}_n^\pm(\hat{S}_{n-1}^- + \hat{S}_{n+1}^+)] \\
&+ A(\hat{S}_n^x \hat{S}_n^z + \hat{S}_n^z \hat{S}_n^x) + d_{xy} \delta_{jn} \hat{S}_j^x \hat{S}_j^z - d_z \delta_{jn} \hat{S}_j^z \hat{S}_j^z + \mu H_0 \hat{S}_n^z.
\]  

For now on, we assume that \( S \) is large enough for the semiclassical approximation to be valid, and hence, using \( \alpha_n = \hat{S}_n^z / S, \alpha_n^* = \hat{S}_n^- / S, \hat{S}_n^- / S = \sqrt{1 - |\alpha_n|^2} \), we obtain

\[
i \frac{\partial \alpha_n}{\partial t} = - JS \left[ (\alpha_{n+1} + \alpha_{n-1}) \sqrt{1 - |\alpha_n|^2} - \alpha_n \left( \sqrt{1 - |\alpha_{n-1}|^2} + \sqrt{1 - |\alpha_{n-1}|^2} \right) \right] + 2 AS \delta_{jn} \sqrt{1 - |\alpha_n|^2} + \mu H_0 \alpha_n.
\]

3. Soliton solutions

We shall look for solutions in the form of amplitude-modulated waves

\[
\alpha_n(t) = \varphi_n(t) e^{i(kn - \omega t)},
\]

where \( k \) and \( \omega \) are the wave number and the frequency of the carrier waves (the lattice constant equals unity) and the envelopes \( \varphi_n(t) \) are slowly varying functions of the position and time. In the continuum limit equations (6) transform into the following modified Nonlinear Schrödinger equation for the envelopes in Ref. [8]

\[
i \left( \frac{1}{S} \frac{\partial \varphi}{\partial t} + 2 J \sin k \frac{\partial \varphi}{\partial x} \right) = (\varepsilon - \omega S^{-1}) \varphi - J \cos k \frac{\partial^2 \varphi}{\partial x^2} + g|\varphi|^2 \varphi,
\]

where

\[
\varepsilon = \mu H_0 S^{-1} - 2g, \quad g = J(\cos k - 1 - A/J).
\]
For a bright soliton $g \cos k < 0$, equation (8) has the following solution

$$\varphi(x, t) = \varphi_0 \text{sech} \frac{x - vt}{L},$$

$$\varphi_0^2 = -\frac{2J \cos k}{gL^2}, \quad \omega = \varepsilon S - \frac{JS \cos k}{L^2}, \quad v = 2JS \sin k,$$

where the parameters $L$ and $v$ are the width and the velocity of the soliton. The region for the $k$-values where this soliton solution exists in the case of easy axis anisotropy ($A > 0$) is $0 \leq k < \pi/2$.

4. Qubit dynamics

The equation of the evolution for the qubit is

$$\frac{i}{\hbar} \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}],$$

where the reduced density operator for the qubit is given via

$$\hat{\rho} = \frac{1}{2}(\hat{1} + \hat{a} \cdot \hat{\sigma}_j),$$

$\hat{1}$ is the identity operator, and

$$\hat{a} = \text{Tr}(\rho \hat{\sigma}_j)$$

is the expectation value of the spin operator for the qubit (the Bloch vector). Evidently, $\hat{H}$ is the Hamiltonian of the system, after averaging over states of the spin chain.

From the Liouville equation (11) we find the following system of ordinary differential equations for the components of the Bloch vector

$$i \frac{d}{dt} (a^-) = (d_z S \sqrt{1 - \varphi^2} - \nu H_0) a^- - d_{xy} S \varphi e^{-i(k_j - \omega t)} a^z,$$

$$i \frac{d}{dt} (a^+) = -(d_z S \sqrt{1 - \varphi^2} - \nu H_0) a^+ + d_{xy} S \varphi e^{i(k_j - \omega t)} a^z,$$

$$i \frac{d}{dt} (a^z) = \frac{d_{xy}}{2}(S \varphi e^{-i(k_j - \omega t)} a^+ + S \varphi e^{i(k_j - \omega t)} a^-).$$

Using $a^+ = (a^-)^*$ and $a^2 = 1$ we obtain for $a^+$ and $a^-$ the following solutions

$$a^+ = R(t)e^{ig(t)}, \quad a^- = R(t)e^{-ig(t)},$$

where $R(t)$ and $g(t)$ are real functions. Finally, for $a^z(t)$ we find

$$a^z(t) = \pm \sqrt{1 - R^2(t)}.$$

The dynamics of the qubit is governed by two real functions. One of these functions $R(t)$ is explicitly related to $a^z(t)$ component of the Bloch vector, which is the expectation value for the spin $z$ component. The other function $g(t)$ represents the azimuth angle of the Bloch sphere.
5. Numerical results

In this section we employ numerical study of (14). We choose the initial condition for the Bloch vector to be $\vec{a} = (0, 0, 1)$ - spin up. We define the energy scale with $J = 1$, the spin of the chain in terms of $\hbar$ is $S = 1$, and the width of soliton ($L = 6$) is taken to be larger than the lattice constant, which is unity. The lattice constant defines the scale of the distance. The terms with the external field are taken to be of the order of the exchange interaction in the spin chain $\mu H_0 = 0.5$, $\nu H_0 = 1.5$, where we supposed that the magnetic moment per spin for the qubit is larger than the magnetic moment per spin in the chain. The site $j$ where the qubit is situated and is coupled with the spin $S_j$ from the chain is arbitrary. $x$ stands for the distance between the peak of the soliton and the site $j$ at the initial moment. The result does not depend from $j$ up to translation, which is equivalent to renumeration of the sites in the chain. For this reason we can take $j = 10$.

First, let us assume that the soliton is far away from the qubit in the initial moment, after approaching it the interaction takes place, and finally, the soliton is going away from the qubit. In this case $x = 50$ means that in the initial moment the soliton is far away from the qubit. The soliton as a function of time is shown in [figure 1(a)]. When the soliton is in the vicinity of the qubit, the interaction takes place. This interaction does not change the initial state much and one cannot switch the qubit spin from up to down $\vec{a} = (0, 0, -1)$. This can be seen in the time evolution of the $z$-component of the Bloch vector which is shown in [figure 1(b)]. We see that when the soliton retrieves, the qubit restores its initial state. The path of the Bloch vector onto the Bloch sphere is shown [figure 1(c)].

Second, we consider the case where the soliton resides onto the $j$-th spin at the initial moment [figure 2(a)], i.e. the soliton peaks onto the $j$-th site. It is seen that the interaction changes the $z$-component of the Bloch vector, which is a measure how close we are to spin up or spin down state. This is shown for different values of the final time [figures 2(b) - 2(d)]. We also plot the respective Bloch vector path onto the Bloch sphere [figures 2(e) - 2(g)]. For long enough time $a^z(t)$ tends to asymptotic value between 0.8 and 1, which implies non-negligible probability to find the qubit in spin down state.

With an increase of the interaction between the soliton and the qubit in the $x - y$ plane, the expectation value $a^z(t)$ oscillates reaching smaller values, see [figures 3(a) - 3(c)]. For long enough time $a^z(t)$ tends to an asymptotic value with higher probability to find the qubit in spin down state.
Figure 2. Soliton-qubit evolution for parameters $A = 1, d_{xy} = 20, d_z = 5, x = 0, k = \frac{\pi}{90}$. (a) soliton solution $\varphi(t)$ for fixed $x = 0$. Pairs of frames representing the $z$-component of the Bloch vector and the Bloch sphere are plotted for different final time $t$: (b) and (e) $t = 1$, (c) and (f) $t = 10$, (d) and (g) $t = 100$.

In the case of small interaction [figures 3(d) and 3(e)], the Bloch vector covers only a small part of the sphere around the "north pole". For larger interaction it covers almost all the states onto the sphere [figure 3(f)], which means that almost arbitrary superpositions are possible.

6. Conclusion
We have studied the interaction between a quantum bit and a soliton in a magnetic chain. The magnetic chain was described by a Heisenberg model in a nearest-neighbour approximation with anisotropy. The evolutionary differential equation for the density operator for the qubit is used to describe its dynamics. It is shown that for large enough values of the interaction constant between the soliton and the qubit in the $x - y$ plane $d_{xy}$ and long enough time the Bloch vector covers almost all states onto the sphere. We showed that the spin switching and creation of superpositions are in principle possible.
Figure 3. Soliton-qubit system for parameters $A = 1$, $d_z = 5$, $x = 0$, $k = e^{-\pi/30}$. Pairs of frames representing the $z$-component of the Bloch vector and the Bloch sphere are plotted for different values of the interaction $d_{xy}$: (a) and (d) 1, (b) and (e) 10, (c) and (f) 100.

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