A POSET METRIC FROM THE DIRECTED MAXIMUM COMMON EDGE SUBGRAPH

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Abstract. We study the directed maximum common edge subgraph problem (DMCES) for directed graphs. We use DMCES to define a metric on partially ordered sets. While most existing metrics assume that the underlying sets of the partial order are identical, and only the relationships between elements can differ, the metric defined here allows the partially ordered sets to be of different sizes. The proof that there is a metric based on DMCES involves the extension of the concept of line digraphs. Although this extension can be used directly to compute the metric, it is computationally feasible only for sparse graphs. We provide algorithms for computing the metric for dense graphs and transitivity-closed graphs.

Key words. Directed graphs, graph distance, partially ordered sets.

1. Introduction. In this paper we study the directed maximum common edge subgraph problem (DMCES) for directed graphs (digraphs). The maximum common edge subgraph problem (MCES) has been studied for undirected graphs [7]. DMCES has a natural application to defining a metric on partially ordered sets (posets), commonly denoted \((P, \leq)\). A (non-strict) partial order is a binary relation \(\leq\) over a set \(P\) that is reflexive, antisymmetric, and transitive. For two partially ordered sets \((P, \leq)\) and \((P', \leq')\), most metrics assume that the underlying sets of objects are identical, \(P = P'\), and only the relationships between elements can differ. Our metric measures the distance between posets where the underlying sets can be different, \(P \neq P'\).

In addition to comparing posets with different numbers of elements, we will compare partially ordered sets that are labeled, meaning there is a function \(\ell : P \rightarrow \mathcal{L}\) which maps elements of the poset to elements of a set of labels \(\mathcal{L}\). The addition of node labels is useful since labels can capture additional structure that a poset may have. For example, consider posets representing a dog pedigree where elements are names and \(\leq\) denotes ancestry. The additional structure of sex can be captured by a node labeling function which maps names to an element in the set \(\mathcal{L} = \{\sigma, \varphi\}\). The notation \((P, \leq, \ell)\) will refer to a labeled poset with labeling function \(\ell\).

A partially ordered set \((P, \leq)\) is often represented as a directed graph, where the \(\leq\) relation translates into edges between nodes corresponding to the elements of \(P\).

**Definition 1.1.** The digraph of a partial order \((P, \leq)\) is a directed graph \(D(P, \leq) = (P, E)\) with vertices \(P\) and edges \(E\) with \(v_1, v_2 \in E\) if and only if \(v_1 \leq v_2\) and \(v_1 \neq v_2\). The digraph of a labeled poset \(D(P, \leq, \ell)\) is a node-labeled graph which inherits the labeling function \(\ell\) from the poset.

In this paper we develop algorithms to compute for two directed, labeled graphs \(G\) and \(G'\), the size of the directed maximum common edge subgraph via a function that we denote

\[
\text{DMCES}(G, G').
\]

While we postpone the precise definition of \(\text{DMCES}(G, G')\) to Definition 3.1, we use DMCES to define a metric on partial orders, since every partial order can be viewed as a directed graph.

**Definition 1.2.** We define the following metric on partial orders. Let

\[
D(P, \leq, \ell) = G\quad\text{and}\quad D(P', \leq', \ell') = G'
\]

with edge sets \(E\) and \(E'\) respectively. The distance between \((P, \leq, \ell)\) and \((P', \leq', \ell')\) is

\[
d((P, \leq, \ell), (P', \leq', \ell')) := 1 - \frac{\text{DMCES}(G, G')}{\max(|E|, |E'|)}.
\]

This metric quantifies the maximal portion of relations which match under a label-preserving map between posets.

The main objectives of this paper are to

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1. prove that (1.1) satisfies the requirements of a metric,
2. provide algorithms for computing it, and
3. give complexity results for DMCES.

To prove the properties of a metric, an object called the extended line digraph is introduced, which is related to the well-known line (di)graph of a graph. It is used to demonstrate both that (1.1) is a metric, and that DMCES can be reduced to the maximum clique problem as has been done for undirected graphs in [7]. Algorithms based on the extended line digraph are inefficient except for sparse graphs, so special algorithms for dense graphs are introduced.

2. Preliminaries. The graphs discussed in this paper will have undirected edges, directed edges, or both.

**Definition 2.1.** Let $\mathcal{L}, \mathcal{L}'$ be sets of labels, let $V$ be a set of nodes, let $\ell : V \rightarrow \mathcal{L}$ be a labeling function between the set of vertices $V$ and the set of labels $\mathcal{L}$, and let $\ell_e : D \rightarrow \mathcal{L}_e$ be a labeling function between the set of edges $D$ and labels $\mathcal{L}_e$.

1. A node-labeled undirected graph is a triple $G = (V, E, \ell)$, where $E \subseteq V \times V$ is a set of undirected edges, i.e. unordered pairs of nodes. The notation $\{v_1, v_2\}$ will be used for an undirected edge.
2. A node-labeled directed graph (digraph) is a triple $G = (V, D, \ell)$, where $D \subseteq V \times V$ is a set of directed edges i.e. ordered pairs of nodes. We will consider only node-labeled digraphs that are finite, weakly connected, oriented (meaning no 2-cycles in the graph) and simple (meaning no self loops or multiple edges between the same ordered pair of nodes). We will denote a directed edge from $v_1$ to $v_2$ by $(v_1, v_2) \in D$. The class of all node-labeled digraphs satisfying these conditions will be denoted $\mathcal{G}$. Notice that $\mathcal{G} \subseteq \mathcal{Y}$.

3. A node-labeled mixed graph is a graph $G = (V, E, D, \ell)$, where $E \subseteq V \times V$ is a set of undirected edges and $D \subseteq V \times V$ is a set of directed edges.
4. A node- and edge-labeled digraph is a graph $G = (V, D, \ell, \ell_e)$ with directed edges, where the function $\ell$ labels nodes and $\ell_e$ labels edges. We will consider edge-labeled graphs in Section 5, and in that section, these graphs will be assumed to be finite, weakly connected, and simple, but not necessarily oriented. The class of all node- and edge-labeled digraphs satisfying these conditions will be denoted $\mathcal{G}$. Notice that $\mathcal{G} \subseteq \mathcal{Y}$.

The goal of the metric in Definition 1.2 is to measure the size of the largest subgraphs of two digraphs that are isomorphic to each other. In following sections, the machinery for doing this depends on the notion of isomorphism between node-labeled mixed graphs. We give the standard definition of a graph isomorphism and then generalize it to node-labeled mixed graphs.

**Definition 2.2.** Two unlabeled, undirected graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic, $G \cong G'$, if there is a bijection $\phi : V \rightarrow V'$ such that $\{v_1, v_2\} \in E$ if and only if $\{\phi(v_1), \phi(v_2)\} \in E'$.

For technical reasons in later proofs, it will be useful to impose additional structure on the mixed graphs by means of a partition of the edges, $E = \{E_1, E_2, \ldots, E_n\}$ and $D = \{D_1, D_2, \ldots, D_m\}$. The definitions below therefore incorporate the partition.

**Definition 2.3.** A map $\phi : U \rightarrow V'$ with $U \subseteq V$ between vertices of node-labeled (node- and edge-labeled) graphs respects labels if for all $u \in U$, $\ell(u) = \ell'(\phi(u))$ (for all $v \in U$, $\ell(v) = \ell'(\phi(v))$) and $(u, v) \in D$, $\ell_e((u, v)) = \ell'_e((\phi(u), \phi(v)))$.

In order to state a general definition of isomorphism, the node- and edge-labeled digraphs and node-labeled mixed graphs should be viewed as subsets of the larger class of node- and edge-labeled mixed graphs.

**Definition 2.4.** Let $G = (V, E, D, \ell, \ell_e)$ and $G' = (V', E', D', \ell', \ell'_e)$ be node- and edge- labeled mixed graphs with partitions $E = \{E_1, E_2, \ldots, E_n\}$, $D = \{D_1, D_2, \ldots, D_m\}$, and similarly for $E'$ and $D'$, with $n'$ and $m'$ the number of partitions. Let $\phi : V \rightarrow V'$ be a bijection. Then $\phi$ is an isomorphism, that is $G \cong G'$, if and only if
- $\phi$ respects labels
- $n = n'$ and $m = m'$
- $\{v_1, v_2\} \in D_i$ if and only if $\{\phi(v_1), \phi(v_2)\} \in D'_i$
- $\{v_1, v_2\} \in E_i$ if and only if $\{\phi(v_1), \phi(v_2)\} \in E'_i$
Note that node-labeled directed and undirected graphs inherit this definition of isomorphism by taking either $\mathcal{E} = \emptyset$ and $\mathcal{D} = \{D_1\}$ or $\mathcal{D} = \emptyset$ and $\mathcal{E} = \{E_1\}$ with no edge-labeling function $\ell_e$. Likewise, node-labeled mixed graphs lack $\ell_e$, and node- and edge-labeled digraphs have $\mathcal{E} = \emptyset$.

We now define a weaker notion of isomorphism between mixed graphs with partitions, where the direction of the edges is not required to be preserved between the two graphs.

**Definition 2.5.** Let $G = (V, \mathcal{E}, \mathcal{D}, \ell)$ be a node-labeled mixed graph. Let $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$ and $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$ be a partition of the edges. For each $D_i \in \mathcal{D}$ let $F_i$ be the set of undirected edges obtained by removing the direction of the edges in $\mathcal{D}$. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$. We refer to the undirected, unlabeled graph $S(G) = (V, \mathcal{F} \cup \mathcal{E})$ as the structure of $G$.

Notice that a node-labeled undirected graph $G = (V, \mathcal{E}, \ell)$ has structure $S(G) = (V, \mathcal{E})$ and a node-labeled digraph $G = (V, \mathcal{D}, \ell)$ has structure $S(G) = (V, \mathcal{D})$.

**Definition 2.6.** We say that two node-labeled mixed graphs $G = (V, \mathcal{E}, \mathcal{D}, \ell)$ and $G' = (V', \mathcal{E}', \mathcal{D}', \ell')$ are structurally isomorphic if $S(G) \cong S(G')$ as defined in Definition 2.2.

The definition of **DMCES** relies on the concept of a subgraph of a graph $G$.

**Definition 2.7.** Let $G = (V, \mathcal{D}, \ell)$ be a node-labeled directed graph.

1. Let $U \subseteq V$ and let $W \subseteq \mathcal{D}$ be a subset of edges such that $(u, v) \in W$ implies $u, v \in U$. Then $H = (U, W, \ell_{|U})$ is a subgraph of $G$.

2. Let $W \subseteq \mathcal{D}$. The $W$ edge-induced subgraph of $G$ is a graph $H = (U, W, \ell_{|U})$ with $U \subseteq V$ such that $U = \{v_1 \in V \mid (v_1, v_2) \in W \text{ or } (v_2, v_1) \in W\}$.

3. Let $U \subseteq V$. The $U$ node-induced subgraph is a graph $H = (U, W, \ell_{|U})$ with $W \subseteq \mathcal{D}$ such that $W = \{(v_1, v_2) \in \mathcal{D} \mid v_1, v_2 \in U\}$.

**Definition 2.8** ([2]). Let $\mathcal{G}$ be a set of graphs. A function $d : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}_{\geq 0}$ is called a graph distance metric (on $\mathcal{G}$) if, for $G_1, G_2, G_3 \in \mathcal{G}$, the following properties hold:

- **reflexivity:** $d(G_1, G_2) = 0 \Leftrightarrow G_1 \cong G_2$
- **symmetry:** $d(G_1, G_2) = d(G_2, G_1)$
- **triangle inequality:** $d(G_1, G_2) + d(G_2, G_3) \geq d(G_1, G_3)$

3. **DMCES.** The directed maximum common edge subgraph (DMCES) optimization problem given below is modified from the definition given in [7] for the maximum common edge subgraph (MCES) problem for undirected graphs.

**Definition 3.1.** Let $G = (V, \mathcal{D}, \ell)$ and $G' = (V', \mathcal{D}', \ell')$ be node-labeled digraphs. Define $\epsilon : V \times V \rightarrow \{0, 1\}$ and $\epsilon' : V' \times V' \rightarrow \{0, 1\}$ by

$$
\epsilon(v_1, v_2) := \begin{cases} 1 & \text{if } (v_1, v_2) \in \mathcal{D} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \epsilon'(v_1, v_2) := \begin{cases} 1 & \text{if } (v_1, v_2) \in \mathcal{D}' \\ 0 & \text{otherwise} \end{cases}.
$$

Let $U \subseteq V$ and $\phi : U \rightarrow V'$ be an injection which respects labels. We refer to the ordered pair $(U, \phi)$ as a feasible solution, and the set of all feasible solutions (to **DMCES**) as

$$
\text{DMCES}(G, G') := \{(U, \phi) \mid (U, \phi) \text{ is a feasible solution}\}.
$$

For any $(U, \phi) \in \text{DMCES}(G, G')$, we define the score of the feasible solution $(U, \phi)$ to be the function

$$
\mathcal{P}(U, \phi) := \sum_{(v_1, v_2) \in U \times U} \epsilon(v_1, v_2)\epsilon'(\phi(v_1), \phi(v_2)).
$$

Let $\mathcal{G}$ be the set of all node-labeled digraphs. We define the function

$$
\text{DMCES} : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{N}
$$

$$
\text{DMCES}(G, G') := \max\{\mathcal{P}(U, \phi) \mid (U, \phi) \in \text{DMCES}(G, G')\}.
$$

The Directed Maximal Common Edge Subgraph problem (**DMCES**) is to calculate, for inputs $G$ and $G'$ the value of $\text{DMCES}(G, G')$. We call a $(U, \phi)$ such that $\mathcal{P}(U, \phi) = \text{DMCES}(G, G')$ a solution to **DMCES**.
There is an alternative way of formulating DMCES that involves isomorphic subgraphs and is more amenable to computation. We now define the alternative Directed Maximal Common Edge Subgraph problem (aDMCES).

**Definition 3.2.** Let $G = (V, D, \ell)$ and $G' = (V', D', \ell')$ be node-labeled directed graphs.

A feasible solution to aDMCES is an ordered pair $(W, W')$ where the $W \subset D$ and $W' \subset D'$, and the edge-induced subgraphs of $W$ and $W'$ are isomorphic. We denote the set of all such feasible solutions as

$$\text{aDMCES}(G, G') := \{(W, W') \mid (W, W') \text{ is a feasible solution}\}$$

and define the function

$$\text{aDMCES} : \mathcal{G} \times \mathcal{G} \to \mathbb{N}$$

$$\text{aDMCES}(G, G') := \max\{|W| \mid (W, W') \in \text{aDMCES}(G, G')\}.$$  

The alternative DMCES problem (aDMCES) is to calculate, for inputs $G$ and $G'$, aDMCES$(G, G')$. We call a $(W, W')$ such that $|W| = \text{aDMCES}(G, G')$ a solution to aDMCES.

**Theorem 3.3.** DMCES is equivalent to aDMCES.

*Proof.* Given $G = (V, D, \ell)$ and $G' = (V', D', \ell')$, suppose $(U, \phi) \in \text{DMCES}(G, G')$. Let

$$(3.2)\quad W := \{(v_1, v_2) \in D \mid \epsilon(v_1, v_2)\epsilon'(\phi(v_1), \phi(v_2)) = 1\}.$$  

and let

$$W' := \{(\phi(v_1), \phi(v_2)) \in D' \mid \epsilon(v_1, v_2)\epsilon'(\phi(v_1), \phi(v_2)) = 1\}.$$  

Let $H$ and $H'$ be edge-induced subgraphs associated to $W$ and $W'$ respectively. Then $\phi$ is an isomorphism between $H$ and $H'$. To see this, we first observe that

$$(v_1, v_2) \in W \iff \epsilon(v_1, v_2)\epsilon'(\phi(v_1), \phi(v_2)) = 1$$  

$$\Rightarrow \epsilon'(\phi(v_1), \phi(v_2)) = 1$$  

$$\Leftrightarrow (\phi(v_1), \phi(v_2)) \in D'.$$

Since $\epsilon(v_1, v_2)\epsilon'(\phi(v_1), \phi(v_2)) = 1$, then $(\phi(v_1), \phi(v_2)) \in W'$ as well. Setting $w_1 := \phi(v_1), w_2 := \phi(v_2)$ we have

$$(w_1, w_2) \in W' \iff \epsilon(\phi^{-1}(w_1), \phi^{-1}(w_2))\epsilon'(w_1, w_2) = 1$$  

$$\Rightarrow \epsilon(\phi^{-1}(w_1), \phi^{-1}(w_2)) = 1$$  

$$\Leftrightarrow (\phi^{-1}(v_1), \phi^{-1}(v_2)) \in D$$  

$$\Leftrightarrow (v_1, v_2) \in D.$$

The first and last lines above imply $(v_1, v_2) \in W$. Putting the two arguments together, $(v_1, v_2) \in W \iff (\phi(v_1), \phi(v_2)) \in W'$. It now follows that $(W, W') \in \text{aDMCES}(G, G')$. Furthermore, by construction of the sets $W, W'$ we have

$$(3.3)\quad P(U, \phi) = |W| = |W'|.$$  

Now let $W \subset D, W' \subset D'$ such that $(W, W') \in \text{aDMCES}(G, G')$, i.e. $W$ and $W'$ are two sets of edges that form a feasible solution to aDMCES. Let $H = (U, W, \ell|_U)$ and $H' = (U', W', \ell'|_{U'})$ be the edge-induced subgraphs associated with $W$ and $W'$ respectively and let $\psi : U \to U'$ be the isomorphism between $H$ and $H'$. Then the pair $(U, \psi) \in \text{DMCES}(G, G')$. Since $G$ and $G'$ are simple, there is at most one edge from $v_i$ to $v_j$. Therefore

$$(3.4)\quad P(U, \psi) = |W| = |W'|.$$  

The equations (3.3)-(3.4) show that there is a solution of DMCES with score $P$ if and only if there is a solution to aDMCES with score $P$. 

\[\Box\]
4. The extended line digraph. The standard line (di)graph of \(G\) forms a dual to \(G\) in the sense that edges in \(G\) are converted to nodes in the line (di)graph of \(G\). In the line graph of an undirected graph \(G\), two nodes form an edge if the corresponding edges of \(G\) share a node in \(G\). In the line digraph of a directed graph \(G\), the head-to-tail relationships between edges of \(G\) become edges in the line digraph. In this section, we extend the standard idea of the line digraph to capture more information about the arrangement of edges in \(G\) and to account for node labels. We begin with the standard definition.

**Definition 4.1.** Given an undirected graph \(G = (V, \mathcal{E})\), the line graph of \(G\) is an undirected graph \(L(G) = (E, \mathcal{E}_L)\), with nodes that correspond to edges of \(G\). The edges \(E_L\) connect nodes in \(L(G)\) whenever there is a shared node between two edges \(e_1, e_2 \in \mathcal{E}\):

\[
E_L := \{\{e_1, e_2\} \in \mathcal{E} \times \mathcal{E} \mid e_1 \neq e_2 \text{ and } e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\} \text{ for some } v_1, v_2, v_3 \in V\}.
\]

The line digraph of a directed graph \(G = (V, \mathcal{D})\) is the directed graph \(L(G) = (\mathcal{D}, \mathcal{D}_L)\), where the set of edges \(\mathcal{D}_L\) corresponds to pair of edges that are aligned head-to-tail:

\[
\mathcal{D}_L := \{\{e_1, e_2\} \in \mathcal{D} \times \mathcal{D} \mid e_1 \neq e_2 \text{ and } e_1 = (v_1, v_2), e_2 = (v_2, v_3) \text{ for some } v_1, v_2, v_3 \in V\}.
\]

**Definition 4.2.** Given a node-labeled digraph \(G = (V, \mathcal{D}, \ell)\), its extended line digraph is a node-labeled mixed graph \(L(G) = (D, E_L, \mathcal{D}_L, \ell)\) with node set \(D\). The undirected partitioned edge set \(E_L = (E_1, E_2)\) and the directed edge set \(\mathcal{D}_L = (D)\) are defined in the following way:

- \(\{(v_1, v_2), (v'_1, v'_2)\} \in D \subset \mathcal{D} \times \mathcal{D}\) if and only if \(v_2 = v'_1\), implying the head of the edge \((v_1, v_2)\) meets the tail of \((v'_1, v'_2)\);
- \(\{(v_1, v_2), (v'_1, v'_2)\} \in E_1\) if and only if \(v_1 = v'_1\), implying the tail of the edge \((v_1, v_2)\) meets the tail of \((v'_1, v'_2)\);
- \(\{(v_1, v_2), (v'_1, v'_2)\} \in E_2\) if and only if \(v_2 = v'_2\), implying the head of the edge \((v_1, v_2)\) meets the head of \((v'_1, v'_2)\).

We label each node in the extended line digraph \(L(G)\) by the pair of node labels associated to the corresponding edge of \(G\):

\[
\ell : (v_1, v_2) \mapsto (\ell(v_1), \ell(v_2)).
\]

Notice that the directed edges in \(\mathcal{D}_L\) are the edges in the traditional line digraph associated to the directed graph \(G\).

Tracking of the head-to-head and tail-to-tail adjacencies in the extended line digraph allows us to prove an isomorphism theorem, Theorem 4.9, that extends the Whitney isomorphism theorem \([8]\), stated for undirected graphs (Figure 1) are defined as

\[
Y := \{\{a, b, c, d\}, \{a, d\}, \{b, d\}, \{c, d\}\}
\]

\[
\Delta := \{\{a, b, c\}, \{a, b\}, \{b, c\}, \{c, a\}\}.
\]

Note that these two graphs have isomorphic line graphs. The Whitney isomorphism theorem states that these are the only non-isomorphic graphs that have isomorphic line graphs.

**Theorem 4.4** (Whitney \([8]\)). Two finite, connected, undirected graphs \(G = (V, \mathcal{E})\) and \(G' = (V', \mathcal{E}')\) are isomorphic if and only if their line graphs are isomorphic, with the single exception when \(G \cong Y\) (respectively \(G' \cong Y\)) and \(G' \cong \Delta\) (respectively \(G \cong \Delta\)).

**Figure 1. The \(Y\) and \(\Delta\) graphs.**
The main theoretical result of this paper is Theorem 4.9, which states that node-labeled, directed graphs $G$ and $G'$ are isomorphic if and only if $L(G) \cong L(G')$, without exceptions. We first prove the following result relating the structure of a graph (Definition 2.5) with the construction of a line graph.

**Lemma 4.5.** Let $G = (V, D, \ell)$ be a digraph. Then the structure of the extended line graph is isomorphic to the line graph of the structure of $G$

$$S(L(G)) \cong L(S(G)).$$

**Proof.** Recall that $S(G) = (V, F)$ where $F = \{\{v_1, v_2\} | (v_1, v_2) \in D\}$, so that $L(S(G)) = (F, \mathcal{E}_L)$, where $\{e_1, e_2\} \in \mathcal{E}_L$ if and only if $e_1$ and $e_2$ share a node. On the other hand,

$$L(G) = (D, \mathcal{E}_L, D_L, \ell),$$

where $\mathcal{E}_L$ is the set of undirected edges associated to head-to-head and tail-to-tail connections between edges in $D$, and $D_L$ is the set of directed edges for head-to-tail connections in $D$. The structure

$$S(L(G)) = (D, \mathcal{E}_L)$$

has edges

$$\mathcal{E}_L = \mathcal{E}_L \cup \{\{e_1, e_2\} | (e_1, e_2) \in D_L\}.$$ 

The fact that $G$ is a simple and oriented digraph means that there is a bijection $\phi : D \rightarrow F$ defined by

$$\phi : (v_1, v_2) \mapsto \{v_1, v_2\}.$$ 

Suppose $e_1 = (u, v) \in D$ and $e_2 = (w, z) \in D$. Then $\{\phi(e_1), \phi(e_2)\} \in \mathcal{E}_L$ if and only if $\{u, v\} \cap \{w, z\} \neq \emptyset$. Therefore $e_1$ and $e_2$ share a head-to-tail, head-to-head, or tail-to-tail connection. This is true if and only if

$$\{e_1, e_2\} \in \mathcal{E}_L.$$ 

Thus $\phi$ is an isomorphism between $S(L(G))$ and $L(S(G))$.

**Lemma 4.6.** Let $G$ and $G'$ be two node-labeled digraphs satisfying the two conditions

1. $G \neq G'$ and
2. $S(G)$ and $S(G')$ are isomorphic to $\Delta$ and $Y$; that is, $S(G) \cong A, S(G') \cong B$, with $\{A, B\} = \{\Delta, Y\}$. Then $L(G) \not\cong L(G')$.

**Proof.** In Figure 2 we calculate the extended line digraphs of all graphs structurally isomorphic to $\Delta$ or $Y$. Since no two graphs in the right column of Figure 2 are isomorphic this proves the Lemma.

The next two results establish that isomorphism between extended line digraphs implies structural isomorphism between digraphs.

**Lemma 4.7.** Given two digraphs $G$ and $G'$, if $L(G) \cong L(G')$, then $S(G) \cong S(G')$.

**Proof.** It is easy to see that $L(G) \cong L(G')$ implies $S(L(G)) \cong S(L(G'))$, since only information about labels and direction of edges is lost in the structure. By Lemma 4.5, it follows that $L(S(G)) \cong L(S(G'))$. Next, by the contrapositive of Lemma 4.6, $L(G) \cong L(G')$ implies either that $G = G'$ in which case the Lemma trivially holds, or at least one of $S(G), S(G')$ is not isomorphic to either $\Delta$ or $Y$. Since $G$ and $G'$ are weakly connected by assumption, then $S(G)$ and $S(G')$ are connected. We may then directly apply the Whitney Graph Isomorphism Theorem [8] to show that $S(G) \cong S(G')$.

**Corollary 4.8.** Let $G = (V, D)$ be a digraph with a subgraph $H = (U, W)$ such that $S(H)$ is isomorphic to the $Y$ graph. Let $G' = (V', D')$ be a digraph such that $\phi : D \rightarrow D'$ is an isomorphism between the extended line digraphs $L(G)$ and $L(G')$. Then the edge-induced subgraph $H' \subset G'$, induced by the set of edges $\phi(W)$, has a structure isomorphic to the $Y$ graph, $S(H') \cong Y$.

**Proof.** We apply Lemma 4.7 to $\phi|_W$, which is an isomorphism between $L(H)$ and $L(H')$. Then $S(H) \cong S(H')$ follows.

Corollary 4.8 is used in the proof of the main result, Theorem 4.9, the proof of which has been removed to Appendix A due to length.
Figure 2. The extended line digraphs of graphs structurally isomorphic to $\Delta$ or $Y$. For edges of the extended line digraphs $\mathcal{L}(G) = (D, \mathcal{E}, D_L, \bar{\ell})$, undirected edges in $\mathcal{E}_2$ are shown as solid lines (those that belong to $\mathcal{E}_1$) and dotted lines (those that belong to $\mathcal{E}_2$). Directed lines indicate edges in $D_L$. Note that the letters appearing in nodes are not labels and are used to distinguish which nodes in the extended line digraphs correspond to which edges in the original graph.
Theorem 4.9. Let $G = (V, D)$ and $G' = (V', D')$. Then

$$\mathcal{L}(G) \cong \mathcal{L}(G') \quad \text{if and only if} \quad G \cong G'.$$

The importance of Theorem 4.9 is that node-labeled digraphs are uniquely associated to an extended line digraph. As will be shown in the following section, a standard metric on the extended line digraph induces the metric in (1.1).

5. Reduction to a maximum clique problem.

Definition 5.1. An ordered pair of subsets of nodes, $(U, U')$, is a feasible solution (to MCIS) if the node-induced subgraphs $H$ and $H'$ are isomorphic. The set of feasible solutions (to MCIS) is

$$\text{MCIS}(G, G') := \{(U, U') \mid H = (U, W, e_{[U]}, e_{[W]}) \text{ and } H' = (U', W', e'_{[U']}, e'_{[W']}) \text{ are isomorphic}\},$$

where the edge set $W$ may have undirected and/or directed edges, and the labeling functions may or may not be present. We define the function

$$\text{MCIS} : \mathcal{G} \times \mathcal{G} \to \mathbb{N}$$

$$(G, G') \mapsto \text{MCIS}(G, G').$$

We define the maximum common node-induced subgraph problem (MCIS) to be the task of finding $\text{MCIS}(G, G')$ for inputs $G$ and $G'$. We call a feasible solution $(U, U')$ such that $|U| = \text{MCIS}(G, G')$ a solution (of MCIS).

Lemma 5.2. Let $G = (V, D, \ell)$ and $G' = (V', D', \ell')$ and let $R \subset D, R' \subset D'$. Let $H = (U, R, \ell_{[U]})$ and $H' = (U', R', \ell'_{[U']})$ be edge-induced subgraphs of $G$ and $G'$ respectively. Let

$$\mathcal{L}(G) = (D, (E_1, E_2), D_L, \bar{\ell}), \quad \mathcal{L}(G') = (D', (E'_1, E'_2), D'_L, \bar{\ell}')$$

$$\mathcal{L}(H) = (R, (\bar{E}_1, \bar{E}_2), D_L, \bar{\ell}_R), \quad \mathcal{L}(H') = (R', (\bar{E}'_1, \bar{E}'_2), D'_L, \bar{\ell}'_{R'})$$

$$J = (R, (\bar{E}_1, \bar{E}_2), D_L, \bar{\ell}_R), \quad J' = (R', (\bar{E}'_1, \bar{E}'_2), D'_L, \bar{\ell}'_{R'})$$

where $J$ and $J'$ are the node-induced subgraphs of the extended line graphs $\mathcal{L}(G), \mathcal{L}(G')$, using the nodes $R \subset D, R' \subset D'$, respectively. Then

1. $\mathcal{L}(H) \cong J$, $\mathcal{L}(H') \cong J'$ and
2. $H \cong H'$ if and only if $J \cong J'$.

Proof. Notice that because $J, J'$ are subgraphs of $\mathcal{L}(G), \mathcal{L}(G')$, then $\bar{E}_1 \subset E_1, \bar{E}_2 \subset E_2, \bar{D}_L \subset D_L$, and $\bar{\ell}_R = \bar{\ell}_{[R]}$, where $\bar{\ell}_{[R]}$ is the labeling function $\bar{\ell}$ restricted to the node set $R \subset D$. Further notice that the labeling function of $\mathcal{L}(H), \mathcal{L}(H')$, is also $\bar{\ell}_R$, since if $(u, v) \in R$, then $u, v \in U$. So the labeling functions of $J$ and $\mathcal{L}(H)$ are the same. Now consider the partitions of the edges.

Choose $\{e_1, e_2\} \in \bar{E}_1$. Then as observed above, $\{e_1, e_2\} \in E_1$. This is true if and only if the edges $e_1, e_2 \in R$ share a tail-to-tail relationship. Moreover, the edges $e_1, e_2 \in R$ in the graph $H$ share a tail-to-tail relationship if and only if $\{e_1, e_2\} \in E_1$ of $\mathcal{L}(H)$, the extended line digraph of $H$. Therefore, $\bar{E}_1 = \bar{E}_1$. Similar arguments show $\bar{E}_2 = E_2$ and $\bar{D}_L = D_L$, so that $J = \mathcal{L}(H)$. A similar argument shows that $J' = \mathcal{L}(H')$. This shows the first statement of the Lemma.

Theorem 4.9 applied to digraphs $H$ and $H'$ gives that $\mathcal{L}(H) \cong \mathcal{L}(H')$ if and only if $H \cong H'$. This concludes the proof.

Lemma 5.3. Consider digraphs $G = (V, D, \ell)$ and $G' = (V', D', \ell')$ and their extended line digraphs $\mathcal{L}(G)$ and $\mathcal{L}(G')$. Then $(W, W') \in aDMCES(G, G')$ if and only if $(W, W') \in \text{MCIS}(\mathcal{L}(G), \mathcal{L}(G'))$.

Furthermore

$$aDMCES(G, G') = \text{MCIS}(\mathcal{L}(G), \mathcal{L}(G')).$$

Proof. Let $R \subset D$ and $R' \subset D'$ and let $(R, R') \in \text{MCIS}(\mathcal{L}(G), \mathcal{L}(G'))$, which means that $J = (R, W, \bar{\ell}_R)$ and $J' = (R', W', \bar{\ell}'_{R'})$ are isomorphic node-induced subgraphs of $\mathcal{L}(G)$ and $\mathcal{L}(G')$ of $G, G'$, respectively. By Lemma 5.2, the isomorphism between $J$ and $J'$ exists if and only if there is an isomorphism between edge-induced subgraphs $H = (U, R, \ell_{[U]})$ and $H' = (U', R', \ell'_{[U']})$ of $G, G'$, respectively. Therefore $(R, R') \in aDMCES(G, G')$ if and only if $(R, R') \in \text{MCIS}(\mathcal{L}(G), \mathcal{L}(G'))$. For both the MCIS and aDMCES problems a feasible solution $(R, R')$ is maximal if there are no other feasible solutions $(T, T')$ for which $|T| > |R|$. Thus a feasible solution to MCIS is a solution if and only if it is a solution to aDMCES satisfying $aDMCES(G, G') = \text{MCIS}(\mathcal{L}(G), \mathcal{L}(G'))$.
**Corollary 5.4.** aDMCES can be reduced in polynomial time to MCIS.

*Proof.* Given a digraph \( G \), the construction of \( \mathcal{L}(G) \) can be done by iterating over all pairs of edges in \( G \) and determining for each pair its adjacency type. Since the number of pairs of edges is polynomial in number of vertices, the construction of \( \mathcal{L}(G) \) can be done in polynomial time. The corollary now follows from Lemma 5.3. \( \square \)

We now summarize our results up to this point. By Theorem 3.3, DMCES and aDMCES are equivalent and Lemma 5.3 says in part that

\[
aDMCES(G, G') = \text{MCIS}(\mathcal{L}(G), \mathcal{L}(G')).
\]

Furthermore, by Corollary 5.4, the aDMCES problem can be reduced to the MCIS problem in polynomial time. We will use the equality between feasible solution sets \( aDMCES(G, G') \) and \( \text{MCIS}(\mathcal{L}(G), \mathcal{L}(G')) \) to construct a graph distance on \( \mathcal{G} \) using the function DMCES\((G, G')\).

**Theorem 5.5 ( [4]).** Let \( \mathcal{G} \) be the space of node- and edge-labeled digraphs which are not necessarily oriented, as introduced in Definition 2.1. Then,

\[
d_n : \mathcal{G} \times \mathcal{G} \to \mathbb{R}
\]

is a metric an \( \mathcal{G} \).

**Definition 5.6.** Let \( \mathcal{G}_L \) be the space of all extended line graphs. Define a map

\[
A : \mathcal{G}_L \to \mathcal{G},
\]

\[
(D, E_L, D_L, \bar{\ell}) \mapsto (D, E, \bar{\ell}, \ell_e)
\]

where

\[
E = D_L \cup \{(v, u) \mid (u, v) \in E \} \cup \{(u, v) \mid (u, v) \in E \}.
\]

Note that undirected edges between vertices \( u, v \in E \) correspond to two directed edges \( (v, u), (u, v) \in \mathcal{E} \). We define the edge-labeling function by

\[
\ell_e : (u, v) \mapsto \begin{cases}
E_1 & \text{if } \{u, v\} \in E_1 \\
E_2 & \text{if } \{u, v\} \in E_2 \\
D & \text{if } (u, v) \in D_L
\end{cases}
\]

The map \( A \) takes a node-labeled mixed graph with a partition of edges to a directed, node- and edge-labeled graph, where the edge labeling function retains information about the original edge partitions in the extended line graph. We formalize this observation in the following Lemma.

**Lemma 5.7.** Let \( \mathcal{L}(G) = (D, E_L, D_L, \bar{\ell}) \) and \( \mathcal{L}(G') = (D', E'_L, D'_L, \bar{\ell}') \) be extended line digraphs. Then

\[
\mathcal{L}(G) \cong \mathcal{L}(G') \quad \text{if and only if} \quad A \circ \mathcal{L}(G) \cong A \circ \mathcal{L}(G').
\]

*Proof.* Let \( A \circ \mathcal{L}(G) = (D, E', \bar{\ell}, \ell_e) \) and \( A \circ \mathcal{L}(G') = (D', E''_L, \bar{\ell}', \ell_{e'}) \). Let \( \psi : D \to D' \) be a bijection. Note that the nodes of \( \mathcal{L}(G) \) and \( \mathcal{L}(G') \) are the same set \( D \) and likewise those of \( \mathcal{L}(G') \) and \( \mathcal{A} \circ \mathcal{L}(G') \) are the same set \( D' \).

We wish to show \( \psi \) is an isomorphism between node- and edge-labeled digraphs \( A \circ \mathcal{L}(G) \) and \( A \circ \mathcal{L}(G') \) if and only if it is an isomorphism between node-labeled mixed graphs \( \mathcal{L}(G) \) and \( \mathcal{L}(G') \). We first observe that \( \psi \) respects node labels in \( \mathcal{L}(G) \) and \( \mathcal{L}(G') \) if and only if it respects the node labels in \( A \circ \mathcal{L}(G) \) and \( A \circ \mathcal{L}(G') \) since they share the node labeling functions \( \bar{\ell} \) and \( \bar{\ell}' \), respectively.

If \( \psi \) is an isomorphism between \( \mathcal{L}(G) \) and \( \mathcal{L}(G') \), then it must preserve the partitions \( E_1, E_2, \) and \( D_L \). In other words, \( \{e_1, e_2\} \in E_1 \) if and only if \( \{\psi(e_1), \psi(e_2)\} \in E'_1 \), and similarly for the other two partitions.
If $\psi$ is an isomorphism between $A \circ L(G)$ and $A \circ L(G')$, then instead $\psi$ must preserve the edge labels $E_1$, $E_2$, and $D$.

To establish the lemma, it is sufficient to show that $\psi$ preserves partitions between $L(G)$ and $L(G')$ if and only if it preserves edge labels between $A \circ L(G)$ and $A \circ L(G')$.

First consider $E_2$ and $E_1$. Notice that $(e_1, e_2), (e_2, e_1) \in \mathcal{E}$ and $\ell_e((e_1, e_2)) = E_1$ in $A \circ L(G)$ if and only if $(e_1, e_2) \in E_1$ by definition.

If $\psi : L(G) \to L(G')$ is an isomorphism this implies $(\psi(e_1), \psi(e_2)) \in E_1$, which by definition of the map $A$ is equivalent to

\[(\psi(e_1), \psi(e_2)), (\psi(e_2), \psi(e_1)) \in \mathcal{E}' \text{ and } \ell'_e((\psi(e_1), \psi(e_2))) = E_1.\]

Thus $\psi : L(G) \to L(G')$ preserving the partition $E_1$ implies that $\psi : A \circ L(G) \to A \circ L(G')$ preserves the edge label $E_1$. Now if $\psi : A \circ L(G) \to A \circ L(G')$ is an isomorphism, then we have $(\psi(e_1), \psi(e_2)), (\psi(e_2), \psi(e_1)) \in \mathcal{E}'$ and $\ell'_e((\psi(e_1), \psi(e_2))) = E_1$, which is equivalent to $(\psi(e_1), \psi(e_2)) \in E_1$. Therefore the partition $E_1$ is preserved under $\psi$ if and only if the edge label $E_1$ is preserved under $\psi$.

A similar argument works for the partition $E_2$ and the label $E_2$, and the partition $D_L$ and the label $D$.

This establishes that $\psi$ is an isomorphism between both $L(G), L(G')$ and $A \circ L(G), A \circ L(G')$, and finishes the proof. $\square$

**Lemma 5.8.** $MCIS(L(G), L(G')) = MCIS(A \circ L(G), A \circ L(G'))$.

**Proof.** Consider extended line digraphs $L(G) = (D, E_L, D_L, \bar{e})$ and $L(G') = (D', E'_L, D'_L, \bar{e}')$. Further let $U \subseteq D$ and $U' \subseteq D'$ be two sets of nodes in the extended line digraphs. Let $H$ and $H'$ be the node-induced subgraphs of $L(G)$ and $L(G')$ by $U, U'$, respectively. Let $J$ and $J'$ be the $U$ and $U'$ node induced subgraphs of $A \circ L(G)$ and $A \circ L(G')$. Then

\[J = A(H), \quad J' = A(H')\]

and by Lemma 5.7 applied to pair of extended line digraphs $(H, H')$ instead of the pair $(L(G), L(G'))$, it follows that $J \cong J'$ if and only if $H \cong H'$.

It follows now that if $(U, U') \in MCIS(L(G), L(G'))$, then $(U, U') \in MCIS(A \circ L(G), A \circ L(G'))$ as the $U$ and $U'$ node induced subgraphs $H \cong H'$ of $L(G)$ and $L(G')$ are isomorphic if and only if the $U$ and $U'$ node induced subgraphs $J \subseteq A \circ L(G)$ and $J' \subseteq A \circ L(G')$ are isomorphic.

Similarly if $(U, U') \in MCIS(A \circ L(G), A' \circ L(G))$ then $(U, U') \in MCIS(L(G), L(G'))$. **Definition 5.1** completes the proof. $\square$

**Theorem 5.9.** Let $\mathcal{G}$ be the set of all node-labeled digraphs. Let

\[d_e : \mathcal{G} \times \mathcal{G} \to [0, 1]\]

\[d_e(G, G') = 1 - \frac{DMCES(G, G')}{\max(|D|, |D'|)}\]

Then $d_e$ is a graph distance metric.

**Proof.** From Lemma 5.7 $A$ is injective i.e. if $A(G) \not\cong A(G')$ then $G \not\cong G'$. By Theorem 4.9 $L$ is also injective in the same sense, i.e up to isomorphism and therefore the composition $A \circ L$ is injective up to isomorphism. Since by Lemma 5.3

\[aDMCES(G, G') = MCIS(L(G), L(G'))\]

and by Lemma 5.8

\[MCIS(L(G), L(G')) = MCIS(A \circ L(G), A \circ L(G'))\]

it follows that $d_e(G, G') = d_e(A \circ L(G), A \circ L(G'))$, referring to (5.1). The injectivity up to an isomorphism implies that $d_e$ inherits all properties of a graph distance metric from $d_n$. $\square$

The reference [1] shows that even in the case of a mixed graph the MCIS problem can be reduced to the maximum clique problem.

**Theorem 5.10.** [1] Let $G$ and $G'$ be node-labeled mixed graphs. Then the MCIS can be reduced to the maximum clique problem.

We will apply this theorem to graphs $L(G)$ and $L(G')$. We outline the main ideas of this reduction.
Definition 5.11. For extended line digraphs \( \mathcal{L}(G) = (\mathcal{D}, \mathcal{E}_L, \mathcal{D}_L, \ell) \) and \( \mathcal{L}(G') = (\mathcal{D}', \mathcal{E}_L', \mathcal{D}_L', \ell') \). Let
\[
M = \{((n, n') \in \mathcal{D} \times \mathcal{D}' | \ell(e) = \ell'(e'))\}
\]
be a collection of pairs of nodes in \( \mathcal{L}(G) \) and \( \mathcal{L}(G') \) with matching labels. Define the compatibility graph of \( \mathcal{L}(G) \) and \( \mathcal{L}(G') \) as an undirected graph \( \mathcal{C}(\mathcal{L}(G), \mathcal{L}(G')) = (M, W) \), where the edge set \( W \) is
\[
\{(n, n'), (m, m') \in W \text{ if and only if} \}
\[
(n, m) \in \mathcal{D}_L \iff (n', m') \in \mathcal{D}_L',
\{n, m\} \in \mathcal{E}_1 \iff \{n', m'\} \in \mathcal{E}_1',
\{n, m\} \in \mathcal{E}_2 \iff \{n', m'\} \in \mathcal{E}_2'.
\]

Cliques in the compatibility graph \( \mathcal{C}(\mathcal{L}(G), \mathcal{L}(G')) \) give a subgraph isomorphism between node-induced subgraphs of \( \mathcal{L}(G) \) and \( \mathcal{L}(G') \). Let \( Q \subset M \) be a clique in \( \mathcal{C}(\mathcal{L}(G), \mathcal{L}(G')) \), i.e., a collection of nodes that induce a complete subgraph. Define \( R = \{n | (n, n') \in Q \} \subset \mathcal{D} \) and \( R' = \{n' | (n, n') \in Q \} \subset \mathcal{D}' \). Then
\[
\phi : R \to R'
\]
\[
n \mapsto n'
\]
is an isomorphism between the node-induced subgraphs of \( \mathcal{L}(G) \) and \( \mathcal{L}(G') \) corresponding to \( R, R' \), respectively. Therefore, maximum cliques in \( \mathcal{C}(\mathcal{L}(G), \mathcal{L}(G')) \) correspond to the maximal node-induced subgraphs of \( \mathcal{L}(G) \) and \( \mathcal{L}(G') \) (see [1]).

We briefly discuss the computational complexity of the compatibility graph. First note that nodes in \( \mathcal{C}(\mathcal{L}(G), \mathcal{L}(G')) \) can be computed by iterating over all edges of \( \mathcal{L}(G) \) and \( \mathcal{L}(G') \). Edges of \( \mathcal{C}(\mathcal{L}(G), \mathcal{L}(G')) \) can be computed by iterating over all pairs of nodes in \( \mathcal{C}(\mathcal{L}(G), \mathcal{L}(G')) \). Therefore \( \mathcal{C}(\mathcal{L}(G), \mathcal{L}(G')) \) can be calculated in polynomial time.

We have shown that \( \text{DMCES} \) with input \( (G, G') \) can be reduced to \( \text{MCIS} \) with input \( (\mathcal{L}(G), \mathcal{L}(G')) \) and that can be, in turn, reduced to the maximum clique problem. This result has applications outside of complexity. Many methods for solving the \( \text{MCES} \) problem [7], which we briefly mentioned in the introduction of \( \text{DMCES} \), and the \( \text{MCIS} \) problem first formulate it as a maximum clique problem and then compute the solution using well known maximum clique algorithms. Our results show that efficient algorithms for computing the maximum clique problem could be leveraged to compute the \( \text{DMCES} \) problem. However, since the size of the compatibility graph and the size of the corresponding maximal clique problem is proportional to the product \( |\mathcal{D}| \cdot |\mathcal{D}| \) these methods are inefficient for dense graphs. We address this issue in the next section.

6. Techniques for Transitive Closures. This section establishes some technical properties that can be leveraged in algorithms for calculating the graph distance metric established in Section 5 for graphs that are transitively closed. The first subsection introduces the existence of a solution to \( \text{DMCES} \) that has the maximum number of nodes. This is true for any node-labeled digraph with the properties in Definition 2.1. The second subsection establishes the “order-respecting” property that holds for graphs that are in addition transitively closed. The last subsection discusses the algorithm.

Throughout this section we will use the definition of \( \text{DMCES} \) given by Definition 3.1. Recall that a feasible solution to \( \text{DMCES} \) for two graphs \( G = (V, E, \ell) \) and \( G' = (V', E', \ell') \) is an ordered pair \( (U, \phi) \), where \( U \subset V \) and \( \phi : U \to V' \) is injective and respects labels, and the set of such feasible solutions is \( \text{DMCES}(G, G') \). A solution to \( \text{DMCES} \) is some \( (U, \phi) \in \text{DMCES}(G, G') \) such that \( \mathcal{P}(U, \phi) \) is maximized, where \( \mathcal{P}(U, \phi) \) is the score (see Equation (3.1)), i.e. \( \mathcal{P}(U, \phi) = \text{DMCES}(G, G') \).

6.1. Maximal cardinality solutions.

Definition 6.1. We say a feasible solution \( (U, \phi) \in \text{DMCES}(G, G') \) is a maximal cardinality solution (to \( \text{DMCES} \)) if \( \mathcal{P}(U, \phi) = \text{DMCES}(G, G') \), and for all \( (U', \phi') \in \text{DMCES}(G, G') \), \( |U'| \leq |U| \).

Theorem 6.2. Let \( G = (V, E, \ell) \) and \( G' = (V', E', \ell') \) be digraphs. Then there exists a maximal cardinality solution to \( \text{DMCES} \).
Proof. First we determine the maximal value of $|U|$ for any $(U, \phi) \in \text{DMCES}(G, G')$. Let $a \in \mathcal{L}$ be a label, and let $U \subset V$. Define $U_a := \{v \in U \mid \ell(v) = a\}$, i.e. all nodes in $U$ which have label $a$. Now define

$$N_a(G, G') := \min \{ \ell^{-1}(a), |\ell^{-1}(a)| \}$$

We claim for all $(U, \phi) \in \text{DMCES}(G, G')$, $|U_a| \leq N_a(G, G')$. To see this, note $U_a = \ell^{-1}(a) \cap U$, so clearly $|U_a| \leq |\ell^{-1}(a)|$. Also, $\phi$ is an injection which respects labels, so

$$|U_a| = |\phi(U_a)| = |\phi(U) \cap \ell^{-1}(a)| \leq |\ell^{-1}(a)|$$

implying

$$|U_a| \leq \min \{ |\ell^{-1}(a)|, |\ell^{-1}(a)| \} = N_a(G, G').$$

To continue the main argument we observe that, as $U$ is a disjoint union of $U_a$,

$$U = \bigcup_{a \in \ell(U)} U_a \Rightarrow |U| = \sum_{a \in \ell(U)} |U_a|,$$

We use this to obtain a bound on $|U|$ given by

$$|U| = \sum_{a \in \ell(U)} |U_a| \leq \sum_{a \in \ell(U)} N_a(G, G') \leq \sum_{a \in \mathcal{L}} N_a(G, G') =: N(G, G').$$

We observe that this bound holds for any feasible solution. Then $\forall(U, \phi) \in \text{DMCES}(G, G')$, $|U| \leq N(G, G')$. Next we prove the following claim

$$\exists (U, \phi) \in \text{DMCES}(G, G') \text{ such that } P(U, \phi) = \text{DMCES}(G, G') \text{ and } |U| = N(G, G').$$

Let $(\bar{U}, \bar{\phi}) \in \text{DMCES}(G, G')$ be any feasible solution such that $P(\bar{U}, \bar{\phi}) = \text{DMCES}(G, G')$. Suppose $|\bar{U}| < N(G, G')$. We construct a feasible solution with the desired properties as follows. Define a $U \supset \bar{U}$ such that for each label $a \in \mathcal{L}$, $U$ contains $N_a(G, G')$ vertices with label $a$. Such a $U$ exists from the definition of $N_a(G, G')$. We first observe that $|U| = N(G, G')$. We extend $\bar{\phi}$ to $\phi$ in such a way that the restriction $\phi|_{\bar{U}} = \bar{\phi}$ and $\phi$ is an injection which respects labels. This extension is possible, because the definition of $N(G, G')$ and our construction ensures that for each $a \in \mathcal{L}$, $|U_a| \leq |\ell^{-1}(a)|$, so there is an injection $U_a \to V'$ that respects labels. We can then assemble these injections piecewise.

Finally, note that

$$P(U, \phi) \geq P(\bar{U}, \bar{\phi})$$

because $\bar{U} \subset U$, $\phi|_{\bar{U}} = \bar{\phi}$, and from the definition of $P$ in Equation (3.1). Since $P(\bar{U}, \bar{\phi}) = \text{DMCES}(G, G')$ it follows that $P(U, \phi) = P(\bar{U}, \bar{\phi})$. Therefore $(U, \phi)$ is the solution advertised in the Theorem. □

6.2. The order-respecting property.

**Definition 6.3.** We say a feasible solution $(U, \phi)$ respects order on labels if there is no $v, u \in U$ with $\ell(v) = \ell(u)$ such that $(v, u) \in E$ and $(\phi(u), \phi(v)) \in E'$.

**Lemma 6.4.** Let $G = (V, E, \ell)$ and $G' = (V', E', \ell')$ be digraphs such that $G$ and $G'$ are transitive closures. For all $(U, \phi) \in \text{DMCES}(G, G')$, define

$$\mathcal{X}(U, \phi) := \{ u, v \subset U \mid \ell(v), \ell(u) \in E \text{ and } (\phi(v), \phi(u)) \in E' \}$$

Let $(U, \phi) \in \text{DMCES}(G, G')$ such that $\mathcal{X}(U, \phi) \neq \emptyset$. Fix $\{u, v\} \in \mathcal{X}(U, \phi)$. Let $\psi : U \to V'$ be an injection which is identical to $\phi$, with the exception that $\psi(u) = \phi(v)$ and $\psi(v) = \phi(u)$. Then

$$P(U, \phi) + 1 \leq P(U, \psi).$$
Proof. Let \((U, \phi) \in \text{DMCES}(G, G')\) such that \(X(U, \phi) \neq \emptyset\). Fix \(\{u, v\} \in X(U, \phi)\). Let \(\psi : U \to V'\) be an injection which is identical to \(\phi\), with the exception that \(\psi(u) = \phi(v)\) and \(\psi(v) = \phi(u)\). Recall the definition of \(P\), given in Equation (3.1),

\[
P(U, \phi) := \sum_{(v_1, v_2) \in U \times U} \epsilon(v_1, v_2) \epsilon'(\phi(v_1), \phi(v_2))
\]

Let

\[
C := \sum_{(v_1, v_2) \in U \times U \atop v_1, v_2 \notin \{u, v\}} \epsilon(v_1, v_2) \epsilon'(\phi(v_1), \phi(v_2))
\]

For each \(x \in U \setminus \{u, v\}\), let \(U(x) := \{u, v, x\}\). The proof of this Lemma relies on the observation that

\[
P(U, \phi) = \sum_{x \in U \setminus \{u, v\}} (P(U(x), \phi|_{U(x)}) ) + C
\]

To see this, we first recall that

\[
P(U(x), \phi|_{U(x)}) := \epsilon(x, u) \epsilon'(\phi(x), \phi(u)) + \epsilon(x, v) \epsilon'(\phi(x), \phi(v)) + \epsilon(u, x) \epsilon'(\phi(u), \phi(x))
\]

\[+ \epsilon(u, v) \epsilon'(\phi(u), \phi(v)) + \epsilon(v, x) \epsilon'(\phi(v), \phi(u)) + \epsilon(v, u) \epsilon'(\phi(v), \phi(u)).\]

Note that since, for all \(x \in U \setminus \{u, v\}\), nodes \(u, v \in U(x)\) it appears that the terms \(\epsilon(u, v) \epsilon'(\phi(u), \phi(v))\) or \(\epsilon(v, u) \epsilon'(\phi(v), \phi(u))\) may be counted multiple times as we sum over \(x \in U \setminus \{u, v\}\). However, recall that \(G\) and \(G'\) are oriented and we assume that \(\{u, v\} \in X(U, \phi)\). This means that the orientation of the edge \((u, v) \in E\) (resp. \((v, u) \in E\)) and \((\phi(u), \phi(v)) \in E'\) (resp. \((\phi(v), \phi(u)) \in E'\)) do not agree. Therefore both terms \(\epsilon(u, v) \epsilon'(\phi(u), \phi(v)) = 0\) and \(\epsilon(v, u) \epsilon'(\phi(v), \phi(u)) = 0\). This verifies the formula 6.1.

We now observe that for the new function \(\psi\)

\[
P(U, \psi) = \sum_{x \in U \setminus \{u, v\}} (P(U(x), \psi|_{U(x)}) - 1) + C + 1
\]

To explain the term \(-1\) inside the summand, observe that exactly one of summands

\[
\epsilon(v, u) \epsilon'(\phi(v), \phi(u)) = \epsilon(u, v) \epsilon'(\phi(u), \phi(v))
\]

will be equal \(+1\), by the assumption that \(G\) and \(G'\) are oriented, while the other will be zero. To avoid counting this term multiple times we subtract it from the sum over all subgraphs indexed by \(x\) and add \(+1\) at the end of the equation to account for this term exactly once.

We will now show that

\[
P(U(x), \psi|_{U(x)}) + 1 \leq P(U(x), \psi|_{U(x)})
\]

which will be sufficient to complete the proof. Shown in Appendix B are all possible arrangements of the subgraphs induced by \(U(x) = \{u, v, x\}\), under the assumptions that \(G\) and \(G'\) are oriented and are transitive closures. We use the notation \(x' = \phi(x) = \psi(x), v' = \psi(v), u' = \psi(u), v' = \phi(u), u' = \phi(v)\). In each case, the Equation (6.3) is valid.

We remark that without the assumption of transitive closure, we cannot guarantee that \(P(U, \phi) + 1 \leq P(U, \psi)\). For example, Figure 3 shows an instance where \(P(U, \phi) = P(U, \psi)\), when \(G\) and \(G'\) are not transitive closures.

**Theorem 6.5.** Let \(G = (V, E, \ell)\) and \(G' = (V', E', \ell')\) be node-labeled digraphs such that \(G\) and \(G'\) are transitive closures. If \((U, \phi) \in \text{DMCES}(G, G')\) such that \(\mathcal{P}(U, \phi) = \text{DMCES}(G, G')\), then \((U, \phi)\) respects order on labels.
for all possible pairings of branch becomes inviable, that is corresponds to the largest maximal solution will be returned all the way to the top of the tree. If a are compared and the largest is returned. In this way, only the value from the branch of the recursive tree can make with
\[ L \]
for elements with label count. This is checked by the line: if
\[ v' = \psi(v), \quad u' = \phi(u), \quad v'' = \phi(u), \quad u'' = \phi(v). \]
The edges shown in blue are matched in \((U(x), \psi|_{U(x)})\), and the dashed edges are matched in \((U(x), \phi|_{U(x)})\).

**Proof.** Suppose not. Then there exists \((U, \phi) \in \text{DMCES}(G, G')\) with \(\mathcal{P}(U, \phi) = \text{DMCES}(G, G')\) and \(X(U, \phi) \neq \emptyset\). Let \(\{u, v\} \in X(U, \phi)\). Let \(\psi : U \rightarrow V'\) be an injection which is identical to \(\phi\), with the exception that \(\psi(u) = \phi(v)\) and \(\psi(v) = \phi(u)\). Then by Lemma 6.4, \(\mathcal{P}(U, \phi) + 1 \leq \mathcal{P}(U, \psi)\), which is a contradiction, as we assumed \(\mathcal{P}(U, \phi) = \text{DMCES}(G, G')\), i.e. \(\mathcal{P}(U, \phi)\) was maximal.

**Corollary 6.6.** Let \(G = (V, E, \ell)\) and \(G' = (V', E', \ell')\) be node-labeled digraphs. If \(G\) and \(G'\) are transitive closures, then there exists \((U, \phi) \in \text{DMCES}(G, G')\) such that both
- \((U, \phi)\) is a maximal cardinality solution to \(\text{DMCES}\) and
- \((U, \phi)\) respects order on labels.

**Proof.** From Theorem 6.2 we know that there exists \((U, \phi) \in \text{DMCES}(G, G')\) such that \((U, \phi)\) is a maximal cardinality solution. Then by definition of maximal cardinality solution, \(\mathcal{P}(U, \phi) = \text{DMCES}(G, G')\), so applying Theorem 6.5 we see \((U, \phi)\) must respect direction of labels. □

### 6.3. Description of the algorithm.

The following pseudo-code, written in a python style, gives an algorithm for calculating \(\text{DMCES}(G, G')\) for node-labeled digraphs \(G = (V, E, \ell)\) and \(G' = (V', E', \ell')\) that leverages Theorem 6.2. The idea of the algorithm is to, at every recursive call, create a separate branch for each way we can grow the current feasible solution \((U, \phi)\) into a feasible solution \((U', \phi')\) such that \(U \subseteq U'\) and \(\phi'|_U = \phi\). We do not keep track of \(U\) explicitly, rather it is the domain of the map \(\phi\). The map \(\phi\) is represented as a set of ordered pairs \(\Phi \subseteq V \times V'\) with the property that if \((v_1, v'_1), (v_2, v'_2) \in V \times V'\), then \(v_1 \neq v_2\) and \(v'_1 \neq v'_2\). Moreover, \(\ell(v_1) = \ell'(v'_1)\) whenever \((v_1, v'_1) \in \Phi\). The pairs \(\Phi\) define a label-preserving, injective function \(\phi : U \rightarrow V'\). As a shortcut, we sometimes refer to the collection of first elements of \(\Phi\) as the domain of \(\Phi\), and similarly refer to the second elements as the range. These are in fact the domain and range of \(\phi\). final_num_nodes is a dictionary keyed by label that gives the number of elements with a label \(L\) that will appear in \(U\) for any feasible solution \((U, \phi)\) with maximal \(|U|\) so final_num_nodes = \(N^*_L(G, G')\).

The calculation of this is given by the proof of Theorem 6.2. The set \(X \subseteq V\) is an ordered list of nodes.

At each recursive call of \(\text{pick_nodes}\) the function parameters are the list of nodes \(X \subseteq V\) and a set of ordered pairs of nodes \(\Phi \subseteq V \times V'\), see Figure 4. At the initial call of \(\text{pick_nodes}\), \(X = V\) and \(\Phi = \emptyset\).

The first element of the list \(X, X[0]\), is stored as \(m\). The function then determines all possible nodes \(n \in V'\) that both share a label with \(m\) and do not appear in any element of \(\Phi\). For each such \(n\) a new recursive call \(\text{pick_nodes}\) is made in which \(X' = X \setminus \{m\}\) and \(\Phi' = \Phi \cup \{(m, n)\}\). A new recursive call may also be made with \(X' = X \setminus \{m\}\) and \(\Phi' = \emptyset\), if adding the edge \(\{(m, n)\}\) to \(\Phi\) would exceed the maximum node count. This is checked by the line: if \(|\Phi_L| + |X_L| > \text{final_num_nodes}[L]\) (the subscript \(L\) denotes a subset of elements with label \(L\)). This recursion continues until an instance occurs with \(X = \emptyset\) at which point the score of \(\Phi\), given by Equation (3.1), is calculated and returned (the base case of the algorithm).

During each instance of \(\text{pick_nodes}\) the return values of all recursive calls made within the instance are compared and the largest is returned. In this way, only the value from the branch of the recursive tree that corresponds to the largest maximal solution will be returned all the way to the top of the tree. If a branch becomes inviable, that is \(X \neq \emptyset\) and no more recursive calls can be made, then 0 is returned.

Since each node \(v \in V\) corresponds to a different level of recursion, and since a recursive call is made for all possible pairings of \(v\) to a node in \(V'\), there will be a branch for every maximal cardinality solution.
Therefore, since the graph size resulting from every branch is compared, the maximum common subgraph size \( \text{DMCES}(G, G') \) will be returned.

```python
def DMCES(G, G'):
    global final_num_nodes = find_final_num_nodes(G, G')
    return pick_nodes(nodes(G), φ)

def pick_nodes(X, Φ)
    if X == φ:
        return P(Φ)
    score = 0
    m = X[0]
    L = ℓ(m)
    for n ∈ V' such that ℓ(n) == L and n not in the image of Φ
        score = max(score, pick_nodes(X \ {m}, Φ ∪ {(m, n)})
    if |Φ_L| + |X_L| > final_num_nodes[L]
        score = max(score, pick_nodes(X \ {m}, Φ))
```

Suppose now that \( G \) and \( G' \) are transitive closures. Using Corollary 6.6 we can improve our algorithm to only consider solutions that are order-respecting.

```python
def DMCES(G, G'):
    global final_num_nodes = find_final_num_nodes(G, G')
    return pick_nodes(topologically_sort(nodes(G)), φ)

def pick_nodes(X, Φ)
    if X == φ:
        return P(Φ)
    score = 0
    m = X[0]
    L = ℓ(m)
    cross = predecessors(φ({k ∈ predecessors(n) | k ∈ domain of Φ}))
    for n ∈ V' such that ℓ(n) == L, n not in image of Φ, and n not in cross
        score = max(score, pick_nodes(X \ {m}, Φ ∪ {(m, n)})
    if |Φ_L| + |X_L| > final_num_nodes[L]
        score = max(score, pick_nodes(X \ {m}, Φ))
```

Here, \( cross \) is the set of nodes \( n \) for which adding \( Φ \) to include \( (n, m) \) would cause \( Φ \) not to respect order on labels. Note that since \( X \) is always topologically sorted, only predecessors of \( m \) could be involved in a \( Φ \) that does not respect order on labels since \( m \) has no successors in the domain of \( Φ \).

We can further improve the algorithm in the case when subgraphs induced by all nodes of given label are directed path graphs. That is, graphs that are isomorphic to graphs of form

\[
V = \{v_1, v_2, ..., v_n\}
\]
\[
E = \{(v_1, v_2), (v_2, v_3), ..., (v_{n-1}, v_n)\}.
\]

This type of graph helps structure the partial orders in [3], which is work to appear. With this added assumption we can further improve the algorithm by, for a feasible solution \((U, φ)\), keeping track of what nodes in \( V' \) will not be in the image of \( φ' \) for any extension \((U', φ')\). This is stored as the set \( Y ⊆ V' \) in the following algorithm. If adding \((m, n)\) to \( Φ \) will cause \(|V'_L| - |Y_L| < \text{final_num_nodes}[L]\) then the branch is not continued as it can not lead to a maximal cardinality solution.
def DMCES($G, G'$)
    global final_num_nodes = find_final_num_nodes($G, G'$)
    return pick_nodes(topologically_sort(nodes($G$)), 0, 0)

def pick_nodes($X, Y, \Phi$)
    if $X$ is empty
        return $P(\Phi)$
    score = 0
    $m = X[0]$
    $L = \ell(m)$
    for $n \in V'$ such that $\ell(n) == L, n \notin Y$
        $\hat{Y} = \{v \in predecessors(n) | v$ not in range of $\Phi$ and $\ell'(v) = L\}$
        if final_num_nodes[$L$] $\leq |V'_\hat{L}|$ $-$ $|Y_L \cup \hat{Y}_L|$
            score = max(score, pick_nodes($X \setminus \{m\}, Y \cup \hat{Y}, \Phi \cup \{(m, n)\}$))
    if $|\Phi_L| + |X_L| > final_num_nodes[\Phi[3]]$
        score = max(score, pick_nodes($X \setminus \{m\}, Y, \Phi)$)

    \begin{figure}
    \centering
    \includegraphics[width=\textwidth]{recursion_tree.png}
    \caption{The head of the pick_nodes() recursion tree. Each box is an instance of the function, where $X$ and $\Phi$ are the input parameters. $m$ is found by simply taking the first element of $X$. Lines indicate which function makes each recursive call. In this example $\ell(v_1) = \ell'(v'_1)$ and $\ell(v_1) = \ell'(v'_2) = \ell'(v'_3)$.}
    \end{figure}

7. Discussion. We have shown that there exists a graph distance metric based on an edge-induced maximum common subgraph. This graph distance metric is particularly applicable to assessing similarity between partially ordered sets, as we will do in work to appear [3]. To prove that we have a defined metric, we build an extended version of the line digraph of a directed graph that captures both label information and oriented edge information. This changes the process of finding an edge-induced subgraph of a digraph $G$ to a node-induced subgraph of the extended line digraph $L(G)$. A metric using node-induced subgraphs on $L(G)$ then transfers to a metric using edge-induced subgraphs on $G$.

We further show that finding a maximum common node-induced subgraph of $L(G)$ can be reduced in polynomial time to the maximum clique problem. Although this algorithm could in principle be directly implemented, in practice the construction of the extended line digraph is prohibitively expensive for dense
graphs. Since our interest is in transitively closed graphs induced by partial orders, a different algorithm is necessary. We prove that there are two properties, maximum cardinality and order-preserving, that lead to algorithms directly on the graph \( G \), rather than on \( \mathcal{L}(G) \), which provide substantial savings in computational time for transitively closed graphs.

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Appendix A. Proof of Theorem 4.9.

Definition A.1. The degree of a node \( v \) in a digraph \( G \), denoted \( \text{deg} v \), is the number of incoming and outgoing edges incident to the node \( v \).

Proof of Theorem 4.9. The reverse direction is immediate, so consider the forward direction, \( \mathcal{L}(G) \cong \mathcal{L}(G') \) implies \( G \cong G' \).

Let \( G = (V, D, \ell) \) and \( G' = (V', D', \ell') \) be node-labeled digraphs. Let \( \mathcal{L}(G) = (D, E_L, D_L, \bar{\ell}) \) and \( \mathcal{L}(G') = (D', E'_L, D'_L, \bar{\ell}') \) be the extended line digraphs of \( G \) and \( G' \), where \( E_L = (E_1, E_2) \) and \( E'_L = (E'_1, E'_2) \). Recall that \( E_1 \) records tail-to-tail incident pairs of edges, \( E_2 \) records the head-to-head incidence, and \( D_L \) records head-to-tail incidence. The theorem is easy to verify for \( |V|, |V'| \leq 2 \) so assume \( |V|, |V'| > 2 \).

Let \( \phi : D \rightarrow D' \) be an isomorphism between \( \mathcal{L}(G) \) and \( \mathcal{L}(G') \) in the sense of Definition 2.4 for mixed labeled graphs. We will show that there is a map \( \gamma : V \rightarrow V' \) satisfying \( \phi((v_1, v_2)) = (\gamma(v_1), \gamma(v_2)) \) that is an isomorphism between \( G \) and \( G' \). We remark that our construction of \( \gamma \) follows the outline of a proof due to [6], given in [5].

Consider a vertex \( v \in V \) and let \( P(v) \subseteq D \) be the set of edges incident on \( v \), i.e.

\[
P(v) = \{(u, w) \in D \mid u = v \text{ or } w = v\}.
\]

First suppose \( \text{deg} v > 1 \). Let \( e_1 \) and \( e_2 \) be two edges connected to \( v \) in \( G \). Then \( e_1, e_2 \in D \) is a pair of directed edges that have either a head-to-tail, head-to-head, or tail-to-tail relationship. Since \( \phi \) is an isomorphism it preserves the adjacency between \( e_1 \) and \( e_2 \). Thus, \( \phi(e_1), \phi(e_2) \in D' \) share some node \( v' \in G' \) which is to say \( \phi(e_1), \phi(e_2) \in P(v') \). Since \( G' \) is simple and oriented, \( \phi(e_1) \) and \( \phi(e_2) \) can share a maximum of one node so \( v' \) is uniquely determined by the isomorphism \( \phi \).

Now assume there is another edge \( e_3 \neq e_1, e_2 \) connected to \( v \). Then \( \phi(e_1) \) and \( \phi(e_3) \) is a pair of edges in \( D' \) that share a node \( v'' \). Similarly, \( \phi(e_2), \phi(e_3) \) is a pair of edges that share a node \( v''' \). Notice that the \( \{e_1, e_2, e_3\} \) edge induced subgraph of \( G \) has structure isomorphic to the \( Y \) graph shown in Figure 1. By Corollary 4.8, the \( \{\phi(e_1), \phi(e_2), \phi(e_3)\} \) edge induced subgraph of \( G' \) must also have structure isomorphic to \( Y \). This implies that \( v' = v'' = v''' \) in the \( Y \) subgraph of \( G' \). Therefore, \( \phi(P(v)) \subseteq P(v') \). For the same reason, for any edge \( e' \neq \phi(e_1), \phi(e_2) \) connected to \( v' \) the \( \{e', \phi(e_1), \phi(e_2)\} \) edge induced subgraph of \( G' \) has structure isomorphic to \( Y \) and so the \( \{\phi^{-1}(e'), e_1, e_2\} \) edge induced subgraph of \( G \) must have structure isomorphic to \( Y \), implying \( \phi^{-1}(e') \in P(v) \). Thus, \( \phi(P(v)) = P(v') \) and \( v' \) is uniquely determined by \( \phi \). We can then define the injection

\[
\gamma\mid_w : W \rightarrow V'
\]

\[
v \mapsto v'
\]
where \( W \subseteq V \) is the subset of nodes of \( V \) with degree greater than 1 and \( v' \) is the unique node in \( V' \) such that \( \phi(P(v)) = P(v') \).

Next suppose \( \deg v = 1 \). Let \( u \) be the neighbor of \( v \) and let \( e_1 \) be the directed edge connecting \( u \) and \( v \). Since the digraphs are weakly connected and we assume that the number of vertices of \( G \) is greater than 2, \( \deg u > 1 \). Then \( \gamma|_W \) is well defined on \( u \) and \( \phi(u') = \gamma|_W(u) \). Then \( \phi(e_1) \in P(u') \) and we let \( v' \) be the other node of the edge \( \phi(e_1) \). We now show that \( \deg v' = 1 \). Indeed, if \( \deg v' = 1 \) then \( w := \gamma|_W^{-1}(v') \) is a \( \deg \) \( 1 \) vertex in \( V \), with \( w \neq v \) since \( \deg v = 1 \). However, since there is an edge \( \phi(e_1) \in P(u') \cap P(v') \), then \( \phi^{-1}(P(u')) \cap \phi^{-1}(P(v')) \neq \emptyset \). In other words, there exists an edge \( e_2 \) connecting \( w \) and \( u \). Then since \( \phi \) is an isomorphism, \( \phi(e_2) \in P(u') \cap P(v') \). Since \( e_1 \neq e_2 \), there are two edges \( \phi(e_1) \neq \phi(e_2) \) connecting the same vertices \( u', v' \). This is a contradiction to the fact that \( G' \) is a simple and oriented graph. It follows that \( \deg v' = 1 \), with \( \phi(P(v)) = P(v') \), where \( v' \) is uniquely determined. We therefore extend \( \gamma|_W \) to the injection
\[
\gamma : V \to V'
\]
\[
v \mapsto v'.
\]

The map \( \gamma \) is in fact a bijection. To see this, assume by contradiction that \( \gamma \) is not surjective. Then there exists a \( v' \in V' \) such that \( \gamma^{-1}(v') \) does not exist. Since \( \phi^{-1} \) is an isomorphism, this means that \( v' \) participates in no edges in \( D' \); i.e. \( v' \) is an isolated node. But this contradicts the fact that \( G' \) is weakly connected, so \( \gamma \) must be surjective.

We have shown that \( \gamma \) is a bijection between nodes in \( G \) and \( G' \). However, the manner of the proof says nothing about the edges between the nodes.

Let \( e \) be an edge connecting two nodes \( u, v \in V \) which is to say \( e \in P(v) \cap P(u) \). Then \( \phi(e) \in \phi(P(v) \cap P(u)) = \phi(P(v)) \cap \phi(P(u)) \) since \( \phi \) is injective. Thus, \( \phi(e) \in P(\gamma(v)) \cap P(\gamma(u)) \) by the definition of \( \gamma \) so \( \phi(e) \) is an edge connecting \( \gamma(u) \) and \( \gamma(v) \). This means \( \phi \) maps a edge connecting \( u \) and \( v \) to a edge connecting \( \gamma(v) \) and \( \gamma(u) \). Now we need to consider the orientation of these edges.

Consider a directed edge \( e := (u,v) \in D \) connecting two nodes \( u, v \in G \). Since \( G \) is weakly connected with \( |V| > 2 \), either \( u \) or \( v \) has degree greater than one.

Assume first that there is an edge \( \tilde{e} \neq e \) incident on \( v \), connecting \( v \) and \( w \), so that \( \phi(\tilde{e}) = (\gamma(w), \gamma(v)) \) or \( \phi(\tilde{e}) = (\gamma(v), \gamma(w)) \). Let \( q \) be the edge in \( L(G) \) connecting \( e \) and \( \tilde{e} \). Either \( \tilde{e} = (v,w) \) and thus \( q \in D_L \), indicating \( q = (e,\tilde{e}) \) and a head-to-tail relationship between the edges, or \( \tilde{e} = (w,v) \) and thus \( q \in E_2 \), indicating a head-to-head relationship between \( e \) and \( \tilde{e} \). Let \( q' \) be the edge connecting \( \phi(e) \) and \( \tilde{\phi}(e) \). Since \( \phi \) is an isomorphism between \( L(G) \) and \( L(G') \) then either \( q' \in D'_L \) (and \( q \in D_L \)) or \( q' \in E'_2 \) (and \( q \in E_2 \)). If \( q' \in D'_L \), then
\[
q' = (\phi(e), \tilde{\phi}(e)) \text{ and } (\gamma(u), \gamma(v)), (\gamma(v), \gamma(u)) \in D.
\]
If \( q' \in E'_2 \), then
\[
q' = \{\phi(e), \tilde{\phi}(e)\} \text{ and } (\gamma(u), \gamma(v)), (\gamma(v), \gamma(u)) \in D.
\]
In both cases it follows that orientation of \( \phi(e) \) is given by \( \phi(e) = (\gamma(u), \gamma(v)) \).

Assume now that there is an edge \( \tilde{e} \neq e \) incident on \( u \), connecting \( u \) and \( w \). Let \( q \) be the edge in \( L(G) \) connecting \( e \) and \( \tilde{e} \). Either \( \tilde{e} = (w,u) \) and thus \( q \in D_L \), or \( \tilde{e} = (u,w) \) and thus \( q \in E_1 \). Let \( q' \) be the edge connecting \( \phi(e) \) and \( \phi(\tilde{e}) \). By a similar argument, we conclude that if \( q' \in D'_L \), then
\[
q' = (\phi(e), \phi(\tilde{e})) \text{ and } (\gamma(w), \gamma(u)), (\gamma(u), \gamma(w)) \in D.
\]
and if \( q' \in E'_1 \), then
\[
q' = \{\phi(e), \phi(\tilde{e})\} \text{ and } (\gamma(u), \gamma(v)), (\gamma(v), \gamma(u)) \in D.
\]
This shows that \( \gamma \) conserves orientation of the edges.

All that remains is to show \( \gamma \) respects labels. The isomorphism \( \phi \) ensures that labels between \( L(G) \) and \( L(G') \) are respected; in other words,
\[
(\ell(u), \ell(v)) = \ell((u,v)) = \ell'((u',v')) = (\ell'(u'), \ell'(v')) = (\ell'(\gamma(u)), \ell'(\gamma(v))).
\]
So \( \ell(u) = \ell'(\gamma(u)) \) for all \( u \in V \), thus \( \gamma : G \to G' \) is an isomorphism between labeled digraphs. This completes the proof.

**Appendix B. Figures for Lemma 6.4.** The following figure shows all possible arrangements of the subgraphs induced by \( U_{(\ell)} = \{u,v,x\} \), and the corresponding images \( \phi(U_{(\ell)}) \) and \( \psi(U_{(\ell)}) \) under the
assumptions that $G$ and $G'$ are oriented, transitive closures. We use the notation $x' = \phi(x) = \psi(x)$, $v' = \psi(v)$, $u' = \phi(u)$, $v' = \phi(u)$, and $u' = \phi(v)$. Under each graph we list the score of both feasible solutions $(U(x), \phi)$ and $(U(x), \psi)$, and the cardinality of the set $\mathcal{X}$ in both cases. For ease of notation, we write $\phi|_{U(x)}$ as $\phi$, and similarly $\psi|_{U(x)}$ as $\psi$. In each of the following cases, the Equation (6.3) is verified.
\[ \mathcal{P}(W, \phi) = 0 \quad |X(W, \phi)| = 1 \quad |X(W, \psi)| = 0 \]

\[ \mathcal{P}(W, \psi) = 1 \]

\[ \mathcal{P}(W, \phi) = 0 \quad |X(W, \phi)| = 1 \quad |X(W, \psi)| = 0 \]

\[ \mathcal{P}(W, \psi) = 1 \]

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