Scaling Limit of RSOS Lattice Models
and TBA Equations

Paul A. Pearce and Bernard Nienhuis

1Department of Mathematics and Statistics, University of Melbourne,
Parkville, Victoria 3052, Australia

and

3Instituut voor Theoretische Fysica, Universiteit van Amsterdam,
Valckenierstraat 65, 1018XE Amsterdam, The Netherlands.

Abstract

We study the scaling limits of the $L$-state Restricted Solid-on-Solid (RSOS) lattice models and their fusion hierarchies in the off-critical regimes. Starting with the elliptic functional equations of Klümper and Pearce, we derive the Thermodynamic Bethe Ansatz (TBA) equations of Zamolodchikov. Although this systematic approach, in principle, allows TBA equations to be derived for all the excited states we restrict our attention here to the largest eigenvalue or groundstate in Regimes III and IV. In Regime III the TBA equations are massive while in Regime IV there is massless scattering describing the renormalization group flow between distinct $A_1^{(1)}$ coset conformal field theories. Regimes I and II, pertaining to $Z_{L-1}$ parafermions, will be treated in a subsequent paper.

1 Introduction

Many integrable Quantum Field Theories (QFT) can be obtained by perturbing Conformal Field Theories (CFT) with respect to an appropriate scaling operator. Examples include both massive and massless QFTs. From a microscopic point of view, these theories can be obtained as the continuum scaling limit of two-dimensional integrable lattice models. A primary tool for the study of such theories is the Thermodynamic Bethe Ansatz (TBA). For the most part, the TBA analysis has been restricted to the groundstate but the method has been generalized to treat a few excited states. A recent review of these and related topics is given by Mussardo.
In this paper we are interested in a systematic derivation of TBA equations starting at the level of the microscopic integrable lattice models. The approach developed here should be applicable to all excitations. Although the methods are general, we work here in the framework of the Restricted Solid-on-Solid (RSOS) lattice models of Andrews, Baxter and Forrester [11] and their fusion hierarchies [12]. The key to the derivation of the TBA equations is in essence to “solve” the elliptic TBA functional equations of Klümper and Pearce [13] in the appropriate scaling regime for the finite-size corrections to the transfer matrix eigenvalues. The approach of the present paper provides an alternative approach to the QFT transfer matrix approach introduced recently by Bazhanov, Lukyanov and Zamolodchikov [14].

The layout of the paper is as follows. We begin, in this section, by defining the RSOS lattice models and describing their relation to perturbed CFTs in the four distinct integrable regimes. We define the commuting transfer matrices and state the functional equations of Pearce and Klümper [13]. We also introduce the notion of the scaling limit and give an overview of the general structure of the TBA equations. In Sections 2 and 3 we derive the TBA equations relevant to Regimes III and IV respectively. Regimes I and II will be treated in a subsequent paper. The massive TBA equations in Regimes II and III have been derived previously by Bazhanov and Reshetikhin [7]. Their approach uses the Bethe ansatz equations and proceeds subject to a conjecture on the string solutions. Although conjectured by Zamolodchikov [6], there appears to be no previous derivation of the massless TBA equations in Regime IV. In some sense this is the most interesting regime since it describes the massless renormalization group flow between distinct coset conformal field theories. We conclude the paper with some discussion of future work.

1.1 RSOS Models

The RSOS($p, q$) models are restricted solid-on-solid models in which heights $a_i$ on the sites $i$ of the square lattice take values in the set $\{1, 2, 3, \ldots, L\}$, with $L \geq 3$, subject to the additional condition that the values $a_i, a_j$ of heights on adjacent sites of the lattice satisfy the constraints

$$0 \leq a_i \leq L, \quad 0 \leq (a_i - a_j + m)/2 \leq m, \quad m < a_i + a_j < 2L - m + 2 \quad (1.1)$$

where $m = p$ for a horizontal pair and $m = q$ for a vertical pair. The Boltzmann weights $W^{p,q}(u)$ depend on a spectral parameter $u$, a nome or temperature variable $t = \tilde{p}^2$ and a crossing parameter $\lambda = \pi/(L + 1)$. These weights are assumed to vanish unless the adjacency constraints are satisfied. There are four distinct off-critical physical regimes:

- Regime I: $-\pi/2 + \lambda \leq u \leq 0$, $-1 < t < 0$
- Regime II: $-\pi/2 + \lambda \leq u \leq 0$, $0 < t < 1$
- Regime III: $0 \leq u \leq \lambda$, $0 < t < 1$
- Regime IV: $0 \leq u \leq \lambda$, $-1 < t < 0 \quad (1.2)$
There are two lines of critical points at $t = 0$, one separating Regimes I and II and another separating Regimes III and IV. In particular, the RSOS($p, p$) models are conformally invariant on these critical lines. On the Regime III/IV critical line the models are described by the coset conformal field theories\[15\]
\[
(A_1^{(1)})_{L-p-1} \oplus (A_1^{(1)})_p \supset (A_1^{(1)})_{L-1}
\] (1.3)
with central charges
\[
c = \frac{3p}{p+2} \left( 1 - \frac{2(p+2)}{(L+1)(L+1-p)} \right).
\] (1.4)
On the Regime I/II critical line the models are described by $Z_{L-1}$ parafermion theories\[16\] with central charges
\[
c = \frac{2(L-2)}{L+1}.
\] (1.5)

The face weights of the RSOS($p, q$) models, with $\max[p, q] \leq L - 1$, are constructed by fusing blocks of $p \times q$ elementary faces together. The elementary faces with $(p, q) = (1, 1)$ are given by the ABF models. In this case the adjacency condition reduces to
\[
1 \leq a_i, a_j \leq L, \quad a_i - a_j = \pm 1.
\] (1.6)

Explicitly, the non-zero face weights $W^{1,1}(u) = W(u)$ are given by
\[
W\left(\begin{array}{c|c}
a \pm 1 & a \\ a & a - 1 \end{array}\right) u \right) = \frac{\vartheta_1(\lambda - u)}{\vartheta_1(\lambda)}
\] (1.7)
\[
W\left(\begin{array}{c|c}
a & a \pm 1 \\ a + 1 & a \end{array}\right) u \right) = \frac{g_{a \pm 1}}{g_{a+1}} \left( \frac{\vartheta_1((a-1)\lambda)\vartheta_1((a+1)\lambda)}{\vartheta_1^2(a\lambda)} \right)^{1/2} \frac{\vartheta_1(u)}{\vartheta_1(\lambda)}
\] (1.8)
\[
W\left(\begin{array}{c|c}
a & a \pm 1 \\ a \pm 1 & a \end{array}\right) u \right) = \frac{\vartheta_1(a\lambda \pm u)}{\vartheta_1(a\lambda)}
\] (1.9)
where $\vartheta_1(u) = \vartheta_1(u, \overline{p})$ and the $g_a$ are arbitrary $u$-independent gauge factors that cancel out on a periodic lattice and do not effect the eigenvalues of the transfer matrices. In this paper we use standard elliptic theta functions as given in Gradshteyn
\[ \vartheta_1(u, \tilde{\rho}) = 2\tilde{\rho}^{1/4} \sin u \prod_{n=1}^{\infty} (1 - 2\tilde{\rho}^{2n} \cos 2u + \tilde{\rho}^{4n})(1 - \tilde{\rho}^{2n}) \tag{1.10} \]

\[ \vartheta_2(u, \tilde{\rho}) = 2\tilde{\rho}^{1/4} \cos u \prod_{n=1}^{\infty} (1 + 2\tilde{\rho}^{2n} \cos 2u + \tilde{\rho}^{4n})(1 - \tilde{\rho}^{2n}) \tag{1.11} \]

\[ \vartheta_3(u, \tilde{\rho}) = \prod_{n=1}^{\infty} (1 + 2\tilde{\rho}^{2n-1} \cos 2u + \tilde{\rho}^{2(2n-1)})(1 - \tilde{\rho}^{2n}) \tag{1.12} \]

\[ \vartheta_4(u, \tilde{\rho}) = \prod_{n=1}^{\infty} (1 - 2\tilde{\rho}^{2n-1} \cos 2u + \tilde{\rho}^{2(2n-1)})(1 - \tilde{\rho}^{2n}). \tag{1.13} \]

We express the nome \( \tilde{\rho} \) in terms of a real parameter \( \varepsilon > 0 \) by

\[ \tilde{\rho} = \begin{cases} e^{-\pi \varepsilon} & \text{Regimes II and III,} \\ ie^{-\pi \varepsilon} & \text{Regimes I and IV,} \end{cases} \tag{1.14} \]

so that \( t = \tilde{\rho}^{2} = \pm \exp(-2\pi \varepsilon) \). In particular, the \( \vartheta_1 \) functions satisfy the quasiperiodic properties

\[ \vartheta_1(u + \pi, \tilde{\rho}) = -\vartheta_1(u, \tilde{\rho}) \tag{1.15} \]

\[ \vartheta_1(u - i \ln \tilde{\rho}, \tilde{\rho}) = -\tilde{\rho}^{-1}e^{-2iu} \vartheta_1(u, \tilde{\rho}). \tag{1.16} \]

To define the RSOS(\( p, q \)) face weights it is convenient to fix a particular gauge

\[ g_a = (-1)^{a/2} \sqrt{\vartheta_1(a \lambda)}. \tag{1.17} \]

With this choice of gauge the RSOS(\( p, q \)) face weights are given by

\[ W^{p,q}(a_{q+1} \atop a_1 \ b_{q+1} \atop b_1) \bigg| u \bigg) = \prod_{k=0}^{q-2} s_k^p(u)^{-1} \sum_{a_1, \ldots, a_q} \prod_{k=1}^{q} W^{p,1}(a_{k+1} \atop a_k \ b_{k+1} \atop b_k) \bigg| u + (k - 1)\lambda \bigg) \tag{1.18} \]

independent of the values of the edge spins \( b_2, \ldots, b_q \). Here

\[ s_q^p = \prod_{j=0}^{p-1} \frac{\vartheta_1(u + (q - j)\lambda)}{\vartheta_1(\lambda)} \tag{1.19} \]

and the \( p \times 1 \) face weights are given in turn by

\[ W^{p,1}(b_{p+1} \atop a_1 \ a_{p+1}) \bigg| u \bigg) = \sum_{a_2, \ldots, a_p} \prod_{k=1}^{p} W(b_{k+1} \atop a_{k+1} \ a_k) \bigg| u + (k - p)\lambda \bigg) \tag{1.20} \]

independent of the edge spins \( b_2, \ldots, b_p \).
1.2 Transfer Matrices and Functional Equations

Consider the RSOS($p, q$) models and suppose that $a$ and $b$ are allowed height configurations for two consecutive rows of an $N$ column lattice with periodic boundary conditions. Then the elements of the row transfer matrices of the RSOS($p, q$) models are given by

$$\langle a | T_{p,q}(u) | b \rangle = \prod_{j=1}^{N} W_{p,q}(b_j \ b_{j+1} | a_j \ a_{j+1} | u)$$

(1.21)

where $a_{N+1} = a_1$ and $b_{N+1} = b_1$. The Yang-Baxter equations

$$\sum_{g} W_{p,q}(f \ g \ u) W_{p,s}(e \ d \ u + v) W_{q,s}(d \ c \ v) = \sum_{g} W_{q,s}(e \ g \ v) W_{p,s}(g \ c \ u + v) W_{p,q}(e \ d \ u)$$

(1.22)

imply that, for fixed $p$, these transfer matrices belong to commuting families

$$T_{p,q}(u) T_{p,q'}(v) = T_{p,q'}(v) T_{p,q}(u).$$

(1.23)

Starting with the fusion hierarchy of Bazhanov and Reshetikhin [18], Klümpner and Pearce have shown that the commuting transfer matrices satisfy the inversion identity hierarchy

$$T_{p,q}(u) T_{p,q-1}(u) = f_{p-1} f_{q} I + T_{p,q+1}(u) T_{p,q-1}(u), \quad 1 \leq q \leq L - 1$$

(1.24)

where

$$T_{p,q}(u) = T_{p,q}(u + k \lambda), \quad f_{p} = s_{p}^{N}.\quad$$

(1.25)

These functional equations close with

$$T_{p,-1}^{p} = 0, \quad T_{0}^{p,0} = f_{-1}^{p} I, \quad T_{0}^{p,L} = 0.$$ \quad

(1.26)

If we further define

$$t_{p,q}^{0} = \frac{T_{0}^{p,q-1} T_{1}^{p,q+1}}{f_{-1}^{p} f_{q}}, \quad 1 \leq q \leq L - 2$$

(1.27)

then Klümpner and Pearce [13] have also shown that these equations can be recast in the form of TBA functional equations

$$t_{0}^{p,q} t_{1}^{p,q} = (I + t_{1}^{p,q-1})(I + t_{0}^{p,q+1}), \quad 1 \leq q \leq L - 2$$

(1.28)
where

\[ t_0^{p,0} = t_0^{p,L-1} = 0. \] (1.29)

From quasiperiodicity, it follows that

\[ \frac{T_{0}^{p,q}}{f_{(q-2)/2}^{p}} = \text{doubly periodic}, \quad t_0^{p,q} = \text{doubly periodic} \] (1.30)

with period rectangle in the complex \( u \) plane given by

\[
\text{period rectangle} = \begin{cases} 
\pi \times \pi i \varepsilon, & \text{Regimes II and III} \\
\pi \times 2 \pi i \varepsilon, & \text{Regimes I and IV}. 
\end{cases}
\] (1.31)

In Regimes I and IV there is an additional symmetry within the period rectangle

\[ t_{p,q}^{u,\pm \pi /2 + \pi i \varepsilon} = t_{p,q}^{u} . \] (1.32)

These statements apply to the entries of the transfer matrices and to their eigenvalues.

At the critical point with \( t = 0 \) and in the braid limit

\[ \lim_{u \to \pm i \infty} t_{p,q}^{u} = t_{\infty}^{p,q} \] (1.33)

the TBA functional equations reduce to a simple recursion relations which can be solved immediately to give

\[ t_{\infty}^{p,q} = \frac{\sin(q\phi)\sin((q + 2)\phi)}{\sin^2 \phi}, \quad 1 + t_{\infty}^{p,q} = \frac{\sin^2((q + 1)\phi)}{\sin^2 \phi} \] (1.34)

with the quantization

\[ \phi = \frac{m_j \pi}{L + 1}, \quad m_j = 1, 2, \ldots, L. \] (1.35)

### 1.3 Scaling Limit and Perturbed CFTs

Let \( a \) denote the lattice spacing. Then the (continuum) scaling limit of the RSOS models is given by

\[ N \to \infty, \quad a \to 0, \quad t \to 0, \quad \mu = N|t|^\nu = \text{fixed} \] (1.36)

where

\[
\nu = \begin{cases} 
\frac{L + 1}{4}, & \text{Regimes III and IV} \\
\frac{L + 1}{2(L - 1)}, & \text{Regimes I and II}.
\end{cases}
\] (1.37)
Here \( \nu \) is the critical exponent associated with the correlation length of the RSOS\((p, p)\) model which is in fact independent of the fusion level \( p \). The exponent \( \nu \) is related to the usual specific heat exponent \( \alpha \) by the hyperscaling relation \( 2 - \alpha = d \nu \). Strictly speaking, this relation breaks down in Regime IV for which \( \nu' = 2 \nu \). It is this fact that leads to Regime IV being massless. Also, in some cases for \( p > 1 \), the free energy is analytic \([12]\) so \( \alpha \) is not defined. We ignore these problems and scale precisely as indicated in (1.36) and (1.37).

In the scaling limit we introduce variables

\[
R = \lim_{N \to \infty, a \to 0} N a, \quad m = \lim_{t \to 0, a \to 0} \frac{4 t^{\nu}}{a} \tag{1.38}
\]

where \( R \) is the radial distance and \( m \) is a mass. It follows immediately from these definitions that

\[
4 \mu = m R. \tag{1.39}
\]

For the RSOS\((p, p)\) models in Regimes III and IV, the off-critical elliptic solution corresponds to perturbation away from the conformal critical point by the operator \( \Phi \) with conformal weights

\[
\Delta = \overline{\Delta} = \frac{L - 1}{L + 1}. \tag{1.40}
\]

For the RSOS\((1, 1)\) models this is the thermal operator \( \Phi = \Phi_{1,3} \).

### 1.4 Zamolodchikov’s TBA Equations

The Thermodynamic Bethe Ansatz (TBA) equations describe the scattering of \( n \) types of relativistic particles on a circle of radius \( R \). Here we will be interested in the TBA equations introduced by Zamoldchikov \([3]\). All of these take the form of a system of coupled nonlinear integral equations

\[
\epsilon_i(\theta) = m_i R \nu_i(\theta) - \sum_{j=1}^{n} \frac{1}{2 \pi} \int_{-\infty}^{\infty} d\theta' \Phi_{ij}(\theta - \theta') \log(1 + e^{-\epsilon_j(\theta')}) \tag{1.41}
\]

with the Casimir energy

\[
E(R) = - \sum_{j=1}^{n} \frac{m_j}{2 \pi} \int_{-\infty}^{\infty} d\theta \nu_j(\theta) \log(1 + e^{-\epsilon_j(\theta)}). \tag{1.42}
\]

Here \( m_i \) are the particle masses, \( \theta, \theta' \) are rapidities, \( \epsilon_i(\theta) \) are quasiparticle pseudo-energies, \( \nu_i(\theta) \) are energy functions and the symmetric kernel satisfies \( \Phi_{ij}(\theta) = \).
\( \Phi_{ij}(\theta) \). For diagonal scattering, which ensues in Regimes I and II but not Regimes III and IV, the kernel is related to the \( S \) matrix by

\[
\Phi_{ij} = -i \frac{\partial}{\partial \theta} \log S_{ij}(\theta).
\]

(1.43)

Typically, the energy functions are given by \( \nu(\theta) = \frac{1}{2} e^{\pm \theta} \) or \( \nu(\theta) = \cosh \theta \).

In the scaling region, the finite-size corrections to the largest eigenvalue \( T(u) \) are related to the Casimir energy by

\[
- \log T(u) = N f(u) + \frac{R \sin \vartheta}{N} E(R) + o\left(\frac{1}{N}\right)
\]

(1.44)

where \( f(u) \) is the bulk free energy and the anisotropy angle is given by

\[
\vartheta = \begin{cases} 
(L + 1)u, & \text{Regimes III and IV} \\
- \frac{2(L + 1)u}{L - 1}, & \text{Regimes I and II}.
\end{cases}
\]

(1.45)

Despite the appearance of \( 1/N \) corrections, the system is not in general conformally invariant. The system, however, will be conformal at critical points. Such critical points can occur in the infrared limit (\( R \to \infty \) or \( \mu \to \infty \)) in addition to the critical point in the ultraviolet limit (\( R \to 0 \) or \( \mu \to 0 \)) with

\[
RE(R) \to -\pi c/6
\]

(1.46)

where \( c \) is the central charge.

The scaling behaviour is manifest in the behaviour of the zeros of \( T(u) \) in the complex \( u \) plane. The imaginary period of \( T(u) \) is either \( 2\pi \varepsilon \) or \( 4\pi \varepsilon \) where \( \pi \varepsilon = 2(\log N - \log \mu)/(L + 1) \). For large \( N \) and finite nonzero \( \mu \), the scaling behaviour occurs in the two regions

\[
\Im(\vartheta) = \pm (\theta + \log N - \log \mu)
\]

(1.47)

where the rapidity \( \theta \) is independent of \( N \). This equation relates the rapidity \( \theta \) to the spectral parameter \( u \) in these two scaling regions.

### 2 Regime III

In this section we derive the TBA equations in Regime III. The strategy is to “solve” the TBA functional equations in the scaling limit for the finite-size corrections to the eigenvalues \( t^{p,q}(u) \) of the transfer matrices \( t^{p,q}(u) \). In these equations the dependence on the temperature \( t \) enters only in the combination \( t^r \) which we replace with \( \mu/N \). Regarding \( \mu \) as fixed there only remains a dependence on \( N \), but since these functional equations are exact for finite \( N \) they contain all the information required to calculate the finite size corrections. To do this we follow closely Klümper and Pearce [13].
2.1 Integral Equations and Finite-Size Corrections

As in Klümper and Pearce we consider the analyticity strips of $t^{p,q}(u)$:

$$\frac{p-q-2}{2} \lambda < \text{Re}(u) < \frac{p-q+2}{2} \lambda.$$  \hspace{1cm} (2.1)

In Regime III the leading contributions to the eigenvalues $t^{p,q}(u)$ in its analyticity strip are given for large $N$ by

$$t^{p,q}(u) = \begin{cases} 
\sum \left[ \phi_{21}((L+1)u/2, t^\nu) \right]^N, & q = p \\
\sum \nu \left[ \phi_{22}((L+1)u/2, t^\nu) \right] \nu, & p \neq q 
\end{cases}$$  \hspace{1cm} (2.2)

where the constants are given by

$$t^{p,q}_{\text{const}} = \begin{cases} 
\frac{\sin(q\sigma) \sin((q+2)\sigma)}{\sin^2 \sigma}, & 1 \leq q \leq p-1 \\
4 \cos \sigma \cos \tau, & q = p \\
\frac{\sin((q-p)\tau) \sin((q-p+2)\tau)}{\sin^2 \tau}, & p+1 \leq q \leq L-2 
\end{cases}$$  \hspace{1cm} (2.3)

and

$$\sigma = \frac{m'_j \pi}{p+2}, \quad m'_j = 1, 2, \ldots, p+1$$  \hspace{1cm} (2.4)

$$\tau = \frac{m''_j \pi}{L+1-p}, \quad m''_j = 1, 2, \ldots, L-p.$$  \hspace{1cm} (2.5)

For the largest eigenvalue, we have

$$\phi = \frac{\pi}{L+1}, \quad \sigma = \frac{\pi}{p+2}, \quad \tau = \frac{\pi}{L+1-p}.$$  \hspace{1cm} (2.6)

Other choices will be needed for the excited states. Note that the leading contributions to the eigenvalues $t^{p,q}(u)$ give the correct periodicity, have the required zero and pole structure and that the constants are such that the functional equations (1.28) are satisfied including in the scaling limit.

We introduce functions of a real variable by restricting the eigenvalue functions to certain lines in the complex $u$ plane

$$a^q(x) = t^{p,q} \left( \frac{i}{L+1} x + \frac{p-q}{2} \lambda \right), \quad A^q(x) = 1 + a^q(x)$$  \hspace{1cm} (2.7)
Here and the sequel we will usually suppress the dependence on \( p \). Let us also introduce finite-size corrections terms \( \ell^q(x) \) by writing

\[
a^q(x) = e^q(x)\ell^q(x)
\]

with

\[
e^q(x) = \begin{cases} 
  \vartheta_1(ix/2, t\nu), & q = p \\
  \vartheta_2(ix/2, t\nu), & q \neq p \\
  1, & \text{otherwise}
\end{cases}
\]

\[
(2.9)
\]

The TBA functional equations then take the form

\[
\ell^q(x - \pi i/2)\ell^q(x + \pi i/2) = A^{q-1}(x)A^{q+1}(x).
\]

\[
(2.10)
\]

The functions \( \ell^q(x) \) and \( A^q(x) \) are analytic and nonzero in the strip \(-\pi < \text{Im}(x) < \pi\) with period \((L+1)\pi\varepsilon\). We represent their logarithmic derivatives by Fourier series

\[
[\log \ell^q(x)]' = \sum_{k=-\infty}^{\infty} L_k^q e^{2ikx/(L+1)\varepsilon} \ \ (2.11)
\]

\[
L_k^q = \frac{1}{(L+1)\pi\varepsilon} \int_{-(L+1)\pi\varepsilon/2}^{(L+1)\pi\varepsilon/2} [\log \ell^q(x)]' e^{-2ikx/(L+1)\varepsilon} \ dx \ \ (2.12)
\]

with similar equations relating the Fourier coefficients \( A_k^q \) to \( A^q(x) \).

So taking the logarithmic derivative of \( (2.10) \) and equating Fourier coefficients gives

\[
\left[ e^{2ikx/(L+1)\varepsilon} + e^{-2ikx/(L+1)\varepsilon} \right] L_k^q = A_k^{q-1} + A_k^{q+1} \ \ (2.13)
\]

and hence

\[
[\log \ell^q(x)]' = \sum_{k=-\infty}^{\infty} \frac{e^{2ikx/(L+1)\varepsilon}}{e^{2ikx/(L+1)\varepsilon} + e^{-2ikx/(L+1)\varepsilon}} (A_k^{q-1} + A_k^{q+1}). \ \ (2.14)
\]

Substituting the integral expression for \( A_k^q(x) \) and evaluating the sum on \( k \) gives the nonlinear integral equation

\[
[\log \ell^q(x)]' = \int_{-(L+1)\pi\varepsilon/2}^{(L+1)\pi\varepsilon/2} dy \ k(x - y) \ \left\{ [\log A^{q-1}(y)]' + [\log A^{q+1}(y)]' \right\}. \ \ (2.15)
\]

After integration this yields

\[
\log a^q = \log e^q + k \log A^{q-1} + k \log A^{q+1} + D^q \ \ (2.16)
\]
where \( \ast \) denotes convolution (over the given interval) and \( D^q \) are integration constants.

The kernel is given in terms of standard Jacobian elliptic functions and theta functions by

\[
k(x) = \frac{1}{(L + 1)\pi \varepsilon} \sum_{k=-\infty}^{\infty} \frac{e^{2ikx}}{e^{(L+1)\varepsilon} + e^{-k\pi}}
\]

\[
= \frac{I'}{\pi^2} \text{dn} \left( \frac{2I'x}{\pi}, \tilde{q} \right) = \frac{1}{2\pi} \frac{\vartheta_4(0, \tilde{q}')}{\vartheta_2(i, \tilde{q}')} \vartheta_3(i, \tilde{q}')
\]

(2.17)

where \( I, I' \) respectively denote the complete elliptic integrals of nomes

\[
\tilde{q} = e^{-\pi(L+1)\varepsilon}, \quad \tilde{q}' = e^{-\pi(L+1)\varepsilon} = \frac{\mu^2}{N^2}.
\]

(2.18)

In the critical limit, when \( \varepsilon \to \infty \) and \( \tilde{q}' \to 0 \), the kernel simplifies to

\[
\lim_{\varepsilon \to \infty} k(x) = \hat{k}(x) = \frac{1}{2\pi \cosh x}.
\]

(2.19)

To evaluate the constants \( D^q \) we set

\[
\langle a \rangle = \frac{1}{(L + 1)\pi \varepsilon} \int_0^{(L+1)\pi \varepsilon} a(x) \, dx
\]

(2.20)

Then integrating (2.16) and using the result

\[
\int_0^{(L+1)\pi \varepsilon} k(x) \, dx = \frac{1}{2}
\]

(2.21)

we find that

\[
\langle \log \ell^q \rangle = \frac{1}{2} \left( \langle A^{q-1} \rangle + \langle A^{q-1} \rangle \right) + D^q.
\]

(2.22)

Comparing with (2.10) we see that, for each \( q \), \( D^q \) is a multiple of \( \pi i \) related to the branches of the logarithms and winding. In particular, for the largest eigenvalue, there is no winding and \( D^q = 0 \). Alternatively, we could work directly with the Fourier series of the logarithms rather than the logarithmic derivatives. In this case we would need to allow for multiples of \( 2\pi i \) on the right side of (2.13) with the same end result.

Let us factor the eigenvalues \( T^{p,q}(u) \) into bulk and finite-size correction terms

\[
T^{p,q}(u) = T^{p,q}_{\text{bulk}}(u) T^{p,q}_{\text{finite}}(u)
\]

(2.23)
then as in Klümper and Pearce

\[ T^{p,q}_{\text{bulk}}(u)T^{p,q}_{\text{bulk}}(u + \lambda) = f(u - \lambda)f(u + q\lambda) \]  

(2.24)

and

\[ T^{p,q}_{\text{finite}}(u)T^{p,q}_{\text{finite}}(u + \lambda) = 1 + t^{p,q}(u) \]  

(2.25)

where now \( T^{p,q}_{\text{finite}}(u) \) is a doubly periodic function of \( u \). It is again useful to introduce functions of a real variable by restricting the eigenvalue function to certain lines

\[ b^q(x) = T^{p,q}_{\text{finite}} \left( \frac{i}{L + 1} x + \frac{p - q + 1}{2} \lambda \right) \]  

(2.26)

so this relation becomes

\[ b^q(x - \pi i/2)b^q(x + \pi i/2) = A^q(x). \]  

(2.27)

Since \( b^q \) and \( A^q \) are non-zero and periodic we can solve as before by introducing the Fourier series of the logarithmic derivatives to obtain

\[ \log b^q = k \log A^q + C^q \]  

(2.28)

where \( C^q \) are constants and the convolution is over the finite interval. Explicitly, the finite-size \( 1/N \) corrections (2.28) for \( q = p \) are

\[ \log b^p(x) = \int_{-(L+1)\pi \varepsilon/2}^{(L+1)\pi \varepsilon/2} dy k(x - y) \log(1 + a^p(y)) + C^p \]  

(2.29)

We evaluate the constant as before by integrating (2.29) to obtain

\[ \langle b^p \rangle = \frac{1}{2}(A^p) + C^p. \]  

(2.30)

Comparing with (2.27) we see that, for each \( p \), \( C^p \) is a multiple of \( \pi i \) so it does not contribute to the \( 1/N \) corrections.

### 2.2 TBA Equations and Casimir Energy

So far our integral equations are completely general and are exact for finite \( N \). They could be used, for example, to study the exponentially small corrections to the eigenvalues off-criticality. In this subsection, however, we specialize the form of these equations appropriate to the scaling limit.

The system size \( N \) only enters the integral equations through the function \( e^p(x) \). This function is periodic in \( x \) with period \( 2(\log N - \log \mu) \). For \( N \) large the function
is exponentially small in $N$ except in the two scaling regions when $x$ is of the order of $\log N$ or $-\log N$. Let us set

$$z^2 = e^{-(x+\log N)} = \frac{e^{-x}}{N}, \quad t^\nu = \frac{\mu}{N}. \tag{2.31}$$

Then in these scaling regions we find

$$\log \hat{e}_\pm^p(x) = \lim_{N \to \infty} \log e^p(\pm(x + \log N))$$

$$= \lim_{N \to \infty} \log \left[ \left( \frac{1 - z^2}{1 + z^2} \right)^N \left( \frac{1 - t^{2\nu} z^2}{1 + t^{2\nu} z^2} \right)^N \left( \frac{1 - t^{2\nu} z^{-2}}{1 + t^{2\nu} z^{-2}} \right)^N \ldots \right]$$

$$= -2(e^{-x} + \mu^2 e^x). \tag{2.32}$$

We assume that the functions $a^q$ and $A^q$ scale similarly and set

$$\hat{a}_\pm^q(x) = \lim_{N \to \infty} a^p(\pm(x + \log N)) \tag{2.33}$$

$$\hat{A}_\pm^q(x) = \lim_{N \to \infty} A^p(\pm(x + \log N)) = 1 + \hat{a}_\pm^q(x). \tag{2.34}$$

Scaling $x$ and $y$ in the same way, the integral equations (2.16) take the following simplified form in the scaling limit

$$\log \hat{a}^q = \log \hat{e}^q + \hat{k}^* \log \hat{A}^{q-1} + \hat{k}^* \log \hat{A}^{q+1} + D^q \tag{2.35}$$

where we suppress the subscripts $\pm$ and

$$\log \hat{e}^q(x) = \begin{cases} 0, & q \neq p \\ -2(e^{-x} + \mu^2 e^x), & q = p. \end{cases} \tag{2.36}$$

Clearly, this reduces back to the case of Klümper and Pearce when $\mu \to 0$. Indeed, the modified equation is obtained just by replacing $-2e^{-x}$ by $-2(e^{-x} + \mu^2 e^x)$ in $\hat{e}^p(x)$.

If we now set

$$\theta = x + \log \mu \tag{2.37}$$

and

$$\hat{a}^q(x) = \hat{a}^q(\theta - \log \mu) = e^{-\epsilon_q(\theta)} \tag{2.38}$$

then $-2(e^{-x} + \mu^2 e^x) = -4\mu \cosh \theta$ and we obtain the TBA equations (1.41)

$$\epsilon_q(\theta) + \frac{1}{2\pi} \int_{-\infty}^\infty d\theta' \frac{\log(1 + e^{-\epsilon_q(\theta')})}{\cosh(\theta - \theta')} + \frac{1}{2\pi} \int_{-\infty}^\infty d\theta' \frac{\log(1 + e^{-\epsilon_{q+1}(\theta')})}{\cosh(\theta - \theta')}$$

$$+ D^q = mR \cosh \theta \delta_{pq} \tag{2.39}$$
where \( q = 1, 2, \ldots, L - 2 \) and we identify

\[
4\mu = mR.
\]

The Casimir energy is given by the scaling limit of the finite size correction. The integral in (2.28) is over one period \((L + 1)\pi \epsilon = 2(\log N - \log \mu)\). For \( N \) large, the integrand is of the order \( o(1/N) \) unless \( y \) is in one of the scaling regions where \( y \) is of the order of \( \log N \) or \( -\log N \). Hence

\[
\log b^p(x) = \frac{1}{2} \int_{-\log N - \log \mu}^{\log N - \log \mu} dy \, k(x - y - \log N) \log(1 + a^p(y + \log N))
\]

\[
+ \frac{1}{2} \int_{-\log N - \log \mu}^{\log N - \log \mu} dy \, k(x + y + \log N) \log(1 + a^p(-y - \log N)) + C^p
\]

\[
= \frac{1}{2\pi N} \int_{-\infty}^{\infty} dy \left( e^{-y} + \mu^2 e^{y} \right) \log(1 + \hat{a}^p_+(y))
\]

\[
+ \frac{1}{2\pi N} \int_{-\infty}^{\infty} dy \left( e^{-y} + \mu^2 e^{y} \right) \log(1 + \hat{a}^p_-(y)) + C^p + o\left(\frac{1}{N}\right)
\]

Here we have used the fact that

\[
k(\pm(x + \log N)) = \frac{e^{-x} + \mu^2 e^{x}}{\pi N} + o\left(\frac{1}{N}\right).
\]

For the largest eigenvalue we have \( \hat{a}_+(y) = \hat{a}_-(y) \) and

\[
\log T_{\text{finite}}^{p,p}(u) = \frac{\cosh x}{\pi N} \int_{-\infty}^{\infty} dy \left( e^{-y} + \mu^2 e^{y} \right) \log(1 + \hat{a}^p(y))
\]

\[
= \frac{2\mu \cosh x}{\pi N} \int_{-\infty}^{\infty} d\theta \cosh \theta \log(1 + e^{-\epsilon_p(\theta)})
\]

\[
= -\frac{R \sin(L + 1)u}{N} E_p(R)
\]

where the Casimir energy is

\[
E_p(R) = -\frac{m}{2\pi} \int_{-\infty}^{\infty} d\theta \cosh \theta \log(1 + e^{-\epsilon_p(\theta)})
\]

and we have used \( 4\mu = mR \). In the isotropic case \( u = \pi/2(L+1) \) and \( \sin(L+1)u = 1 \).

3 Regime IV

In this section we derive the TBA equations for Regime IV. We use precisely the same scaling as for Regime III with \( t < 0 \). It turns out that in this regime, with one exception when \( p = (L - 1)/2 \), the resulting TBA equations are massless.
### 3.1 Integral Equations and Finite-Size Corrections

In Regime IV the analyticity strips are the same as for Regime III. The leading bulk contributions to the eigenvalues \( t^{p,q}(u) \) in its analyticity strip are given for large \( N \) by

\[
\begin{cases}
    t^{p,q}_{\text{const}} \left[ \frac{\partial_1((L+1)u/2, |t|^{2\nu})}{\partial_2((L+1)u/2, |t|^{2\nu})} \right]^N, & q = p \\
    t^{p,q}_{\text{const}} \left[ \frac{\partial_4((L+1)u/2 + (L-2p-1)\pi/4, |t|^{2\nu})}{\partial_3((L+1)u/2 + (L-2p-1)\pi/4, |t|^{2\nu})} \right]^N, & q = L - p - 1 \\
    t^{p,q}_{\text{const}}, & \text{otherwise}
\end{cases}
\]

where the constants are the same as in Regime III. Again this has the required periodicity and zero and pole structure. This applies for \( p < (L - 1)/2 \). For the case \( p > (L - 1)/2 \), the zeros and poles of order \( N \) can be obtained from the previous case using the relation

\[
t^{p,q}_0 = t^{p',q}_1, \quad p' = L - 1 - p. \quad (3.2)
\]

In the marginal case \( p = p' = (L - 1)/2 \) we find

\[
\begin{cases}
    t^{p,q}_{\text{const}} \left[ \frac{\partial_1((L+1)u/2, |t|^{\nu})}{\partial_2((L+1)u/2, |t|^{\nu})} \right]^N, & q = p \\
    t^{p,q}_{\text{const}}, & \text{otherwise}.
\end{cases}
\]

As in Regime III we introduce finite-size correction terms \( \ell^q(x) \) by writing

\[
a^q(x) = t^{p,q} \left( \frac{i}{L+1} x + \frac{p - q}{2\lambda} \right) = e^q(x)\ell^q(x) \quad (3.4)
\]

Here, for \( p < (L - 1)/2 \),

\[
e^q(x) = \begin{cases} 
    \left[ \frac{\partial_1(ix/2, |t|^{2\nu})}{\partial_2(ix/2, |t|^{2\nu})} \right]^N, & q = p \\
    \left[ \frac{\partial_4(ix/2, |t|^{2\nu})}{\partial_3(ix/2, |t|^{2\nu})} \right]^N, & q = L - p - 1 \\
    1, & \text{otherwise}
\end{cases} \quad (3.5)
\]
and, for \( p = (L - 1)/2 \),

\[
e_{q}(x) = \begin{cases} 
\left[ i \vartheta_{1}(ix/2, |t|^{\nu}) \right]^{N} & , \quad q = p \\
1, & \text{otherwise}
\end{cases}
\] (3.6)

Following the same Fourier series analysis as in Regime III and allowing for the change in periodicity leads to the integral equations

\[
\log a_{q} = \log e_{q} + k \ast \log A_{q-1} + k \ast \log A_{q+1} + D_{q}
\] (3.7)

In this regime the kernel is given by

\[
k(x) = \frac{1}{2(L + 1)\pi \varepsilon} \sum_{k = -\infty}^{\infty} \frac{e^{\frac{2i k x}{2(L + 1)\varepsilon}}}{e^{\frac{k}{2(L + 1)\varepsilon}} + e^{-\frac{k}{2(L + 1)\varepsilon}}} = \frac{\vartheta_{2}(0, \tilde{q}^{'} \vartheta_{3}(0, \tilde{q}^{'} \vartheta_{3}(ix, \tilde{q}^{'}))}{2\pi \vartheta_{2}(ix, \tilde{q}^{')}}}
\] (3.8)

with the elliptic nomes

\[
\tilde{q} = e^{-\frac{\pi}{2(L + 1)\varepsilon}}, \quad \tilde{q}^{'} = e^{-2\pi(L + 1)\varepsilon} = \frac{\mu^{4}}{N^{4}}.
\] (3.9)

In the critical limit, the kernel again simplifies to

\[
\lim_{\varepsilon \to \infty} k(x) = \hat{k}(x) = \frac{1}{2\pi \cosh x}.
\] (3.10)

The constants \( D_{q} \) can be evaluated as in Regime III, allowing for the different period, by setting

\[
\langle a \rangle = \frac{1}{2(L + 1)\pi \varepsilon} \int_{0}^{2(L + 1)\pi \varepsilon} a(x) \, dx
\] (3.11)

Again \( D_{q} \) is a multiple of \( \pi i \) with \( D_{q} = 0 \) for the largest eigenvalue.

Repeating the analysis of the finite-size corrections for Regime IV leads to the result

\[
\log b^{p}(x) = \int_{-(L + 1)\pi \varepsilon}^{(L + 1)\pi \varepsilon} dy \, k(x - y) \log(1 + a^{p}(y)) + C^{p}
\] (3.12)

where \( C^{p} \) is a multiple of \( \pi i \) that does not contribute to the \( 1/N \) corrections. The only difference with Regime III is in the period.
3.2 TBA Equations and Casimir Energy

In the scaling regions we find

\[
\log \hat{e}_\pm^p = \lim_{N \to \infty} \log e^p(\pm(x + \log N))
\]

\[
= \begin{cases} 
-2e^{-x}, & p < (L - 1)/2, \quad q = p \\
-2\mu^2 e^x, & p < (L - 1)/2, \quad q = L - 1 - p \\
-2(e^{-x} + \mu^2 e^x), & q = p = (L - 1)/2.
\end{cases} \tag{3.13}
\]

Passing to the scaling limit of the integral equations (3.7) thus yields

\[
\log \hat{a}_q = \log \hat{e}_q + \hat{k} \log \hat{A}_q^{-1} + \hat{k} \log \hat{A}_q^{q+1} + D^q \tag{3.14}
\]

where again we suppress the subscripts \(\pm\). Next, introducing rapidity variables as before we obtain the TBA equations. For \(p = (L - 1)/2\) we obtain the same massive TBA equation as in Regime III. For \(p < (L - 1)/2\), however, we obtain the massless TBA equations (1.41)

\[
\epsilon_q(\theta) + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta' \frac{\log(1 + e^{-\epsilon_{q+1}(\theta')})}{\cosh(\theta - \theta')} + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta' \frac{\log(1 + e^{-\epsilon_{q+1}(\theta')})}{\cosh(\theta - \theta')}
\]

\[
+ D^q = \frac{1}{2} m Re^{-\theta} \delta_{pq} + \frac{1}{2} m Re^\theta \delta_{L-1-p,q} \tag{3.15}
\]

where \(q = 1, 2, \ldots, L - 2\) and we again identify \(4\mu = mR\).

Turning to the finite-size corrections we see that the integral in (3.12) is over one period given by \(2(L + 1)\pi \varepsilon = 4\log(N/\mu)\). Hence

\[
\log b^p(x) = \int_{-\log(N/\mu)}^{\log(N/\mu)} dy \int_{-\log(N/\mu)}^{\log(N/\mu)} dk(x + y - \log N/\mu) \log(1 + a^p(y + \log N/\mu))
\]

\[
+ \int_{-\log(N/\mu)}^{\log(N/\mu)} dy \int_{-\log(N/\mu)}^{\log(N/\mu)} dk(x + y - \log N/\mu) \log(1 + a^p(-y - \log N/\mu)) + C^p
\]

\[
= \frac{\mu}{\pi N} \int_{-\infty}^{\infty} dy \ e^{-y} \log(1 + \hat{a}_+(y - \log \mu))
\]

\[
+ \frac{\mu}{\pi N} \int_{-\infty}^{\infty} dy \ e^{-y} \log(1 + \hat{a}_-(y - \log \mu)) + C^p + o\left(\frac{1}{N}\right) \tag{3.16}
\]

Here we have used the fact that

\[
k(\pm(x + \log N/\mu)) = \frac{\mu e^{-x}}{\pi N} + o\left(\frac{1}{N}\right). \tag{3.17}
\]
For the largest eigenvalue we have \( \hat{a}_+(y) = \hat{a}_-(y) \) and

\[
\log T_{\text{finite}}^{p,p}(u) = \frac{2 \cosh x}{\pi N} \int_{-\infty}^{\infty} dy \, e^{-y} \log(1 + \hat{a}(y))
\]

\[
= \frac{2 \mu \cosh x}{\pi N} \int_{-\infty}^{\infty} d\theta e^{-\theta} \log(1 + e^{-\epsilon_p(\theta)}) \quad (3.18)
\]

Hence the Casimir energy is

\[
E_p(R) = -\frac{m}{2\pi} \int_{-\infty}^{\infty} d\theta e^{-\theta} \log(1 + e^{-\epsilon_p(\theta)})
\]

\[
= -\frac{m}{4\pi} \int_{-\infty}^{\infty} d\theta \left[ e^{-\theta} \log(1 + e^{-\epsilon_p(\theta)}) + e^{\theta} \log(1 + e^{-\epsilon_{L-1-p}(\theta)}) \right] \quad (3.19)
\]

where we have identified \( 4\mu = mR \) and used the symmetry between \( \epsilon_p(\theta) \) and \( \epsilon_{L-1-p}(-\theta) \). In the marginal case when \( p = (L-1)/2 \) we see that \( \epsilon_p(\theta) \) is even and

\[
E_p(R) = -\frac{m}{2\pi} \int_{-\infty}^{\infty} d\theta \cosh \theta \log(1 + e^{-\epsilon_p(\theta)}) \quad (3.20)
\]

in agreement with the massive case in Regime III.

4 Discussion

In this paper we have given a systematic derivation, at least for the largest eigenvalue, of the TBA equations for the RSOS lattice models and their fusion hierarchies in the off-critical Regimes III and IV related to \( A^{(1)}_1 \) coset models. Interestingly, in the case of Regime IV, this appears to give the first derivation of Zamolodchikov’s massless TBA equations describing the renormalization group flow between distinct coset theories. In a subsequent paper we will similarly, derive the massive TBA equations for Regimes I and II pertaining to \( Z_{L-1} \) parafermions. In this case the TBA equations for the largest eigenvalue in Regimes I and II turn out to be exactly the same due to a duality between the two regimes.

The systematic derivation of TBA equations introduced in this paper seems to afford many advantages over the approach based on Bethe ansatz equations. First, and most importantly, the analysis in this paper should in principle generalize to allow for the treatment of all the excited states. Second, the present methods can also be extended to treat systems with a boundary. We hope to explore these possibilities fully in future work. Already, for the case of \( A_4 \) in the massless Regime IV with fixed boundaries, it is possible \([19]\) to obtain the TBA equations for all excited states and to give a complete classification of these eigenvalues. This enables a complete mapping of the flow of eigenvalues from the tricritical Ising to the critical Ising conformal fixed points.
Acknowledgements

This work began while PAP was visiting Amsterdam and Bonn Universities. We thank Uwe Grimm for help in the early stages of this work. We also thank Vladimir Bazhanov and Leung Chim for useful discussions. This research is supported by the Australian Research Council.

References

[1] A. B. Zamolodchikov, JETP Letts. 46 (1987) 160.
[2] A. B. Zamolodchikov, Adv. Stud. in Pure Math 19 (1989) 641.
[3] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B241 (1984) 333.
[4] R. J. Baxter, Exactly Solved Models in Statistical Mechanics, Academic Press, London, 1982.
[5] C. N. Yang and C. P. Yang, J. Math. Phys. 10 (1969) 1115.
[6] A. B. Zamolodchikov, Nucl. Phys. B 342 (1990) 695; Phys. Lett. B 253 (1991) 391; Nucl. Phys. B 358 (1991) 497; Nucl. Phys B 358 (1991) 524; Nucl. Phys B 366 (1991) 122.
[7] V. V. Bazhanov and N. Yu. Reshetikhin, Progress of Theoretical Physics Supplement 102 (1990) 301.
[8] M. J. Martins, Phys. Rev. Lett. 67 (1991) 419.
[9] P. Fendley, Nucl. Phys. B 374 (1992) 667.
[10] G. Mussardo, Phys. Reports 218 (1992) 215.
[11] G. E. Andrews, R. J. Baxter and P. J. Forrester, J. Stat. Phys. 35 (1984) 193.
[12] E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Nucl. Phys. B290 (1987) 231.
[13] P. A. Pearce and A. Klümper, Phys. Rev. Lett. 66 (1991) 974; A. Klümper and P. A. Pearce, J. Stat. Phys. 64 (1991) 13; Physica A 183 (1992) 304.
[14] V. V. Bazhanov, S. L. Lukyanov and A.B. Zamolodchikov, Commun. Math. Phys. 177 (1996) 381; Nucl. Phys. B 489 (1997) 487.
[15] P. Goddard, A. Kent and D. Olive, Phys. Lett. B152 (1985) 88; Commun. Math. Phys. 103 (1986) 105.

[16] V. A. Fateev and A. B. Zamolodchikov, Sov. Phys. JETP 62 (1985) 215.

[17] I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series and Products, Academic Press, 1980.

[18] V. V. Bazhanov and N. Yu. Reshetikhin, Int. J. Mod. Phys. A4 (1989) 115.

[19] L. Chim and P. A. Pearce, in preparation.