Quasi-particle Lifetime in a Mixture of Bose and Fermi Superfluids

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In this letter, to reveal the effect of quasi-particle interactions in a Bose-Fermi superfluid mixture, we consider the lifetime of quasi-particle of Bose superfluid due to its interaction with quasi-particles in Fermi superfluid. We find that this damping rate, i.e., inverse of the lifetime, has quite different threshold behavior at the BCS and the BEC side of the Fermi superfluid. The damping rate is constant nearby the threshold momentum in the BCS side, while it increases rapidly in the BEC side. This is because in the BCS side the decay process is restricted by constant density-of-state of fermion quasi-particle nearby Fermi surface, while such a restriction does not exist in the BEC side where the damping process is dominated by bosonic quasi-particles of Fermi superfluid. Our results are related to collective mode experiment in recently realized Bose-Fermi superfluid mixture.

Recently, for the first time, ENS group has realized a mixture of Bose and Fermi superfluids [1]. They prepare a mixture of bosonic $^7$Li atoms and two spin components of fermionic $^9$Li atoms nearby an s-wave Feshbach resonance between fermions. At low enough temperature, bosonic atoms condense and become a Bose superfluid, while fermionic atoms form pairs and become a Fermi superfluid. This experimental development generates many interesting questions on interaction effects between these two types of superfluid [2, 3].

Elementary excitations and their interactions play an important role in quantum many-body system. Here we can compare the low-energy elementary excitations of this superfluid mixture with other two widely studied mixtures, i.e., mixture of a BEC with normal Fermi gas [4] and mixture of two BECs [5]. A Bose-Fermi superfluid mixture exhibits two gapless bosonic modes (denoted by $B_b$ and $B_f$ in Fig. 1), corresponding to Goldstone modes of Bose superfluidity ($B_b$) and Fermi superfluidity ($B_f$), respectively, and a gapped fermionic excitation that describes the Cooper pair breaking (denoted by $F_f$ in Fig. 1). While in the mixture of a BEC with normal Fermi gas, there exists only one bosonic Goldstone mode and the fermionic excitation (particle or hole excitation) is always gapless at the Fermi surface. Mixture of two BECs also exhibits two bosonic Goldstone modes but there is no fermionic excitation in this system.

Moreover, in the cold atom system the Fermi superfluid can be continuously tuned from the BCS regime to the BEC regime by utilizing the Feshbach resonance. In the BCS limit, as schematically shown in Fig. 1(a), it is known that the $B_f$ mode has a quite large velocity proportional to $v_F/\sqrt{3}$ [6], while the gap of the $F_f$ mode is exponentially small. As approaching the BEC side, as shown in Fig. 1(b), the gap of the $F_f$ mode becomes larger and larger, and on the other hand, the velocity of the $B_f$ mode becomes smaller and smaller [7].

Therefore, the interplay between these three modes is quite unique in the Bose-Fermi superfluid mixture, and it will lead to different behaviors in the BCS and the BEC sides of Fermi superfluid. One manifestation of interaction between elementary excitations is the lifetime of quasi-particles. The most well-known effect is Landau-Delaiav damping in the Bose superfluid [8]. Interaction between bosonic mode itself gives rise to a finite lifetime of the bosonic quasi-particle. The damping rate, as the inverse of the lifetime $\gamma = 1/\tau$, is proportional to $k^3$ at zero-temperature and to $T^4$ at finite temperature [11, 12]. This effect has been experimentally studied in atomic BEC by measuring the damping rate of collective modes [13] and theoretically works have also been carried out in the content of cold atom systems [14-15]. Landau damping has also been studied for mixture of BEC with normal Fermi gas [9] and dipolar BEC [10].

In this letter we present an alternative damping channel for bosonic quasi-particle of Bose superfluid ($B_b$ mode) due to its interaction with quasi-particles in Fermi superfluid ($B_f$ and $F_f$ modes). We focus on the typical cold atom situation that Fermi superfluid is in the strongly interacting regime while Bose superfluid is in the weakly interacting regime. We show that this damping mechanism will be activated only when momentum of $B_b$ excitation exceeds a critical value $k_c$. We investigate the threshold behavior of damping rate $\gamma = C(k - k_c)^{\alpha}$, and the key result is that we find different $\alpha$ for the BCS...
side and the BEC side of Fermi superfluid.

Model. We consider a mixture of bosons and spin-1/2 fermions, whose Hamiltonian is given by

\[
\hat{H}_f = \int d^3r \left\{ c_+^\dagger(r) H_{0,f} c_\sigma(r) - g_f c_+^\dagger(r) c_+^\dagger(r) c_\sigma(r) c_\sigma(r) \right\}
\]

\[
\hat{H}_b = \int d^3r \left\{ b_+^\dagger(r) H_{0,b} b_\sigma(r) + \frac{g_b}{2} b_+^\dagger(r) b_\sigma^\dagger(r) b_\sigma(r) b_\sigma(r) \right\}
\]

\[
\hat{H}_{bf} = \int d^3r b_+^\dagger(r) b_\sigma^\dagger(r) c_\sigma(r) c_\sigma(r)
\]

(1)

where \(H_{0,b} = -\hbar^2 \nabla^2/(2m_b) - \mu_b\) and \(i = b, f\) denotes bosons or fermions. Since interaction between fermions is nearby a Feshbach resonance, we shall get \(q\) to scattering length \(a_q\) as \(1/g_b = m/(4\pi\hbar^2 a_q) + \sum_k 1/|\delta_k(k)|^2\) with \(\epsilon_k(k) = \hbar^2 k^2/(2m_b)\). The ground state of \(\hat{H}_f\) is a superfluid of fermion pairs. Applying the BCS-BEC crossover mean-field theory to \(\hat{H}_f\) one can obtain a gapped fermion \(\phi^\dagger\) mode with excitation energy \(E_{\phi^\dagger} = \sqrt{\epsilon^2_k(k) - \mu_b^2} + \Delta^2\). As \(-1/(k^2 a_q)\) decreases from the BCS side to the BEC side, \(\mu_b\) decreases and \(\Delta\) increases. \(\hat{H}_f\) also has a bosonic \(\phi^\dagger\) mode that describes center-of-mass motion of Cooper pairs, which has a phonon-like dispersion \(E_{\phi^\dagger} = \hbar c_k k\), and \(c_k\) evolves smoothly from \(\sqrt{\phi^\dagger} \sqrt{\phi}\) to \(\sqrt{\hbar^2 a_m m_b/m_f^2}\) [17].

When magnetic field locates nearby a Feshbach resonance between fermions, generically \(g_{bf}\) and \(g_{fb}\) terms are in the weakly interacting regime and can be treated by Bogoliubov approximation. In the leading order of \(n_f\) \((-n_f = N_f/V, N_f\) is condensate bosonic atoms\), we replace two of \(\phi^\dagger\) or \(\phi\) operator with \(\sqrt{N_f}\) in the interaction part. From \(\hat{H}_b\) we obtain a Bogoliubov spectrum for Bose superfluid \(E_{\phi^\dagger} = \sqrt{\epsilon_f(k)b_\sigma(k) + 2g_{bf}b_\sigma(n_f)}\), where \(\epsilon_f(k) = \hbar^2 k^2/(2m_f)\) and \(g_{bf} = 4\pi\hbar^2 a_m/m_b\). When \(k \ll 1/\xi = \sqrt{\hbar^2 a_m n_b}\), the excitation is in the phonon regime with a linear dispersion \(\hbar^2 k^2/(2m_f)\), and \(g_{bf} = 4\pi\hbar^2 a_m^2 m_b/m_f^2\). When \(k \gg 1/\xi = \sqrt{\hbar^2 a_m n_b}\), the excitation is in the free-particle regime with a quadratic dispersion \(\epsilon_f(k) + g_{bf} n_b\). Also in the leading order, \(\hat{H}_{bf}\) becomes \(a \phi^\dagger \phi^\dagger c_\sigma^\dagger(r) c_\sigma(r)\), which simply provides a constant shift of chemical potential and will not affect spectrum and wave function of quasi-particles. In the sub-leading order of \(n_f\), only one \(\phi^\dagger\) or \(\phi\) operator is replaced by \(\sqrt{N_f}\), and it describes interaction between quasi-particles. In this order, \(\hat{H}_f\) leads to Landau-Beliaev damping discussed before [11,12]. As we will show later, \(\hat{H}_{bf}\) gives rise to interaction between quasi-particles of Bose superfluid and those of Fermi superfluid.

Damping Threshold. There are two different decay channels for bosonic quasi-particle \(B_0\) of Bose superfluid. The first is decay into two fermionic quasi-particles \(F_1\) of Fermi superfluid, i.e. \(B_0(k) \rightarrow F_1(k - q) + F_1(q)\), as shown in Fig. 1(a). In this case, the energy-momentum conservation requires \(E_{B_0}(k) = E_{F_1}(k - q) + E_{F_1}(q)\). Since \(E_{F_1}\) is gapped and the minimum of \(E_{F_1}(k)\) is \(\Delta\) occurring at \(k_0 = \sqrt{2m_f \mu_f}/\hbar^2\) for \(\mu_f > 0\) and \(k_0 = 0\) for \(\mu_f < 0\), a typical two-particle continuum for two \(F_1\) modes in the BCS side is shown in Fig. 2(a), which has a minimum of \(2\Delta\) for \(k < 2k_0\). For this channel, \(k_c\) is determined by \(E_{B_0}\), meeting this two-particle threshold. In the BCS side of resonance, \(k_c\) can be determined by equation \(E_{B_0}(k_c) = 2\Delta\) as long as the solution of \(k_c\) is smaller than \(2k_0\). Therefore, as \(-1/(k^2 a_q)\) decreases from the BCS side to unitary regime, \(k_c\) increases as shown in Fig. 2(c). Moreover, when \(\Delta \ll \hbar^2/(m_b \xi^2)\), \(k_c\) is in the phonon regime of \(B_0\) mode, while on contrary, when \(\Delta \gg \hbar^2/(m_b \xi^2)\), \(k_c\) is in the free-particle regime of \(B_0\) mode.

The second channel is decay into two bosonic quasi-particles \(B_1\), i.e. \(B_0(k) \rightarrow B_1(k - q) + B_1(q)\), or one \(B_1\) and one \(B_0\), i.e. \(B_0(k) \rightarrow B_0(k - q) + B_1(q)\), as shown in Fig. 1(b). Since in the strongly interacting regime of Fermi superfluid, \(c_\sigma\) is usually much larger than \(c_b\) because \(a_m = 0.6a_b \gg a_b\), it is easy to show that the two-particle threshold of \(B_0 + B_1\) is always lower than that of \(B_1 + B_1\). It is also straightforward to show that \(E_{B_0}(k)\) coincides with two-particle threshold of \(B_0 + B_1\) up to \(k_c\), as shown in Fig. 2(b). That means for \(k < k_c\), only the process with \(q = 0\) can happen which in fact does not lead to decay of quasi-particle. Thus, damp-
ing will be activated only when $\mathcal{E}_{B_b}(k)$ is above two-particle threshold when $k > k_c$, and $k_c$ is determined by $\partial \mathcal{E}_{B_b}(k)/\partial (h k) \big|_{k=k_c} = c_t$. Also due to $c_t \gg c_b$, $k_c$ is always located in the free-particle regime of $B_b$ mode.

Hence our following discussion can be divided into three representative cases, as shown in Fig. 2(c): Case A and B are both at the BCS side of Fermi superfluid, where damping is determined by the first process. For Case A, $\Delta \ll h^2/(m_b \xi^2)$ and therefore $k_c$ is in the phonon regime of $B_b$ mode. For Case B, $\Delta \gg h^2/(m_b \xi^2)$, and thus $k_c$ is in the free-particle regime of $B_b$ mode. Case C is at the BEC side of Fermi superfluid, where damping is determined by the second process, and $k_c$ is in the free-particle regime of $B_b$ mode.

Case A. In this regime we start with BCS mean-field Hamiltonian for $\hat{H}_t$ and Bogoliubov Hamiltonian for $\hat{H}_b$ given by

$$\hat{H}_t = \sum_k \mathcal{E}_{F_t}(k) (\hat{\beta}^\dagger_k \hat{\beta}_k + \hat{\gamma}^\dagger_k \hat{\gamma}_k),$$

$$\hat{H}_b = \sum_k \mathcal{E}_{B_b}(k) (\hat{\alpha}^\dagger_k \hat{\alpha}_k)$$

where quasi-particle $\hat{\alpha}_k$, $\hat{\beta}_k$ and $\hat{\gamma}_k$ are related to $\hat{b}_k$ and $\hat{c}_k$ via $\hat{b}_k = u^b_k \hat{\alpha}_k - v^b_k \hat{\alpha}^\dagger_k$, $\hat{c}_k = u^c_k \hat{\beta}_k + v^c_k \hat{\beta}^\dagger_k$, and $\hat{c}^\dagger_k = u^c_k \hat{\alpha}^\dagger_k - v^c_k \hat{\alpha}_k$. Here $u^b_k(v^b_k) = \sqrt{1/2 \left( 1 \pm \frac{\epsilon(k) - \mu}{\xi\eta(k)} \right)}$ and $u^c_k(v^c_k) = \sqrt{1/2 \left( 1 \pm \frac{\epsilon(k) + \mu}{\xi\eta(k)} \right)}$.

Now we discuss $\hat{H}_{bf}$ in the order of $\sqrt{\mathbb{N}_b}$, by replacing one of $\hat{b}$ or $\hat{b}^\dagger$ operator as $\sqrt{\mathbb{N}_b}$, which leads to

$$\hat{H}_{bf} = g_{bf} \sqrt{\frac{m_b}{V}} \sum_{k_q} (\hat{c}^\dagger_{k+q} \hat{c}_{q} \hat{b}_k + \text{h.c.}).$$

We can further rewrite $\hat{H}_{bf}$ in term of quasi-particle operators $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$. Here we focus on zero-temperature damping rate (or lifetime) of bosonic $\alpha$ mode, thus, only one term retains as [19]

$$g_{bf} \sqrt{\frac{m_b}{V}} \sum_{k_q} M_{kq} \hat{\beta}^\dagger_{k+q} \hat{\beta}_{k} \hat{\gamma}^\dagger_{k} \hat{\gamma}_{k},$$

$$M_{kq} = (u^b_k - v^b_k) (u^c_k - v^c_k)$$

This term describes the process that one $B_b$ mode decays into two $F_t$ modes, as schematically drawn in Fig. 1(a). With Fermi-Golden rule, the damping rate is given by

$$\gamma(k) = \frac{2\pi}{\hbar} n_b g_{bf}^2 \sum_{k_q} \left| M_{kq} \right|^2 \delta \left[ \mathcal{E}_{F_t}(k - q) + \mathcal{E}_{F_t}(q) - \mathcal{E}_{B_b}(k) \right].$$

When $k_c$ is in the phonon regime, we can approximate $u^b_k(v^b_k) = \sqrt{\frac{m_b}{2\xi\eta(k)}} \pm \frac{\hbar c}{2\sqrt{2}m_b}$. And since the decay products of $F_t$ mode locate nearby its minimum of dispersion $\mathcal{E}_{F_t}(k)$ at $k_0$, to the leading order we can approximate $u^b_k(v^b_k) = 1/\sqrt{2}$. Therefore $M_{kq} \approx \sqrt{\frac{\hbar c k}{2\xi\eta(k)}}$. Moreover, in this regime we can approximate $\mathcal{E}_{B_b}(k) = \hbar c |k|$ and $\mathcal{E}_{F_t}(k) = \Delta + \frac{\hbar^2 k^2}{2m^*} - \hbar c_0 (k - k_c)$. With these approximations, the damping rate $\gamma(k)$ can be simplified as

$$\gamma(k) = \frac{g_{bf}^2 c_b k}{8\pi^2 g_b} \int d^3\mathbf{q} \delta \left[ \frac{\hbar^2 (\eta_{k-q}^2 + \eta_{q}^2)}{2m^*} - \hbar c_0 (k - k_c) \right].$$

Basicly this integration is to count for the density-of-state that satisfies energy conservation. With quite straightforward calculation [19] we find that

$$\gamma(k) = \frac{g_{bf}^2 m_b}{4\pi^2 h} \int d^3\mathbf{q} \delta \left[ \frac{\hbar^2 (u_{k-q}^2 + u_{q}^2)}{2m^*} - \hbar^2 (k^2 - k_c^2) \right],$$

which gives rise to a damping rate

$$\gamma(k) = \frac{g_{bf}^2 m_b}{h^2 k} \frac{\Delta m^2}{k_c} \Theta(k - k_c) \approx \frac{g_{bf}^2 m_b}{h^2 k_c} \frac{\Delta m^2}{k_c} \left( 1 - \frac{k - k_c}{k_c} \right) \Theta(k - k_c).$$

The leading order is still a constant and the sub-leading order gives a slow decreasing of $\gamma(k)$ as $k$ increases. However, we shall also note that because the approximations implemented, our results are only valid nearby $k_c$ and cannot be extended to very large momentum.
Case C. In this regime the damping is due to coupling between $B_0$ mode and $B_1$ mode. A comprehensive description of $B_1$ mode and its coupling to $B_0$ mode can be obtained from fluctuation theory of Fermi superfluid [20]. Here to highlight the essential physics we take a simpler approach by treating the Fermi superfluid at the BEC side as molecular condensate, and we consider a Hamiltonian of molecular BEC as

$$H_m = \int d^3 r \left\{ \hat{d}^\dagger(r) H_{0,m} \hat{d}(r) + \frac{g_m}{2} \hat{d}^\dagger(r) \hat{d}(r) \hat{d}(r) \right\}$$  \hspace{1cm} (12)

where $H_{0,m} = -\frac{\hbar^2 \nabla^2}{2m_m} - \mu_m$, and $g_m = 4\pi\hbar^2 a_m/m_m$. $\hat{d}$ represents a creation operator for a bosonic molecule. The coupling between the molecular BEC and Bose superfluid is due to scattering between bosonic atoms and molecules, which can be effectively described by

$$H_{bm} = g_{bm} \int d^3 r \hat{b}^\dagger(r) \hat{b}(r) \hat{d}^\dagger(r) \hat{d}(r)$$  \hspace{1cm} (13)

where $g_{bm}$ is determined by atom-molecule scattering length calculated in Ref. [3]. Bogoliubov approximation can be applied to $H_m$ which gives

$$\dot{H}_m = \sum_\mathbf{k} \mathcal{E}_{B_1}(\mathbf{k}) \chi_\mathbf{k}^\dagger \chi_\mathbf{k},$$  \hspace{1cm} (14)

where $\mathcal{E}_{B_1}(\mathbf{k}) = \sqrt{\epsilon_m(\mathbf{k})}+2g_{bm}n_m$ with $\epsilon_m(\mathbf{k}) = \hbar^2 k^2/(2m_m)$. $\chi_\mathbf{k}$ relates to $\hat{d}$ as $\chi_\mathbf{k} = u^m_\mathbf{k} \hat{d}_\mathbf{k} - v^m_\mathbf{k} \hat{d}_\mathbf{k}^\dagger$,

$$\text{where } u^m_\mathbf{k} (v^m_\mathbf{k}) = \frac{1}{2} \left( \frac{\epsilon_m(\mathbf{k}) + g_{bm}n_m}{\mathcal{E}_{B_1}(\mathbf{k})} \pm 1 \right).$$

Similarly, in the order proportional to $n_b$ or $n_m$, $\dot{H}_{bm}$ is simply a constant chemical potential shift for both Bose superfluid and molecular condensate.

Similar as analysis in case A, by replacing one of $\hat{d}$ (or $\hat{d}$) operator as $\sqrt{N_m}$ or one of $\hat{b}^\dagger$ (or $\hat{b}$) operator as $\sqrt{N_b}$, it can be expanded into quite a few terms that describe quasi-particle interactions, among which only one term contributes to decay of $B_0$ mode with a lower critical velocity, as discussed above [19]. This term is given by

$$g_{bm} \sqrt{\frac{n_m}{V}} \sum_{\mathbf{k}\mathbf{q}} Q_{\mathbf{k}\mathbf{q}} \chi_{\mathbf{k}+\mathbf{q}}^\dagger \hat{d}_{\mathbf{q}} \chi_\mathbf{k},$$  \hspace{1cm} (15)

$$Q_{\mathbf{k}\mathbf{q}} = (v^m_{\mathbf{q}} - v^b_{\mathbf{q}})(u^m_{\mathbf{k}+\mathbf{q}} - u^b_{\mathbf{k}+\mathbf{q}}) + v^b_{\mathbf{k}} v^b_{\mathbf{q}}.$$  \hspace{1cm} (16)

In this regime we can approximate $u^m_{\mathbf{k}} (v^m_{\mathbf{k}}) = \sqrt{\frac{g_{bm}n_b}{2\hbar c k}} \pm \frac{1}{2} \sqrt{\frac{\hbar c k}{g_{bm}n_m}}$, $u^b_{\mathbf{k}} \approx 1$ and $v^b_{\mathbf{k}} \approx 0$, therefore $Q_{\mathbf{k}\mathbf{q}}$ becomes $\sqrt{\frac{\hbar c k}{g_{bm}n_m}}$. Furthermore, we can approximate $\mathcal{E}_{B_0}(\mathbf{k})$ by $\epsilon_b(\mathbf{k}) + g_m n_b$, $\mathcal{E}_{B_1}$ as $\mathcal{E}_{cl}(\mathbf{k})$, and the damping rate is

$$\gamma(\mathbf{k}) = \frac{g^2_{bm}c_f}{8\pi^2 g_m} \int d^3 q |q| \delta \left\{ \mathcal{E}_{cl}(q) + \frac{\hbar^2 (|k-q|^2 - k^2)}{2m_b} \right\}.$$  \hspace{1cm} (17)

Straightforward evaluation of this integral gives [19]

$$\gamma(\mathbf{k}) = \frac{2g^2_{bm}c_e}{3\pi\hbar^2 g_m} (k - k_e)^3 \Theta(k - k_e).$$  \hspace{1cm} (18)

At leading order $\gamma(\mathbf{k})$ fast increases as $(k - k_e)^3$ once $k$ is above threshold.

Conclusion. The results of damping rate for three cases are presented in Fig. [3] We choose $n_b/k^3_b = 0.1$, $1/(k_F a_b) = 100$ and $1/(k_F a_d) = 100$. For three different cases, we choose $1/(k_F a_b) = -2.5$, $1/(k_F a_d) = -0.5$, and $1/(k_F a_d) = 0.5$, respectively. We find a different threshold behavior $\gamma(\mathbf{k}) \propto (k - k_e)^\alpha$ with $\alpha = 0$ in the BCS regime and $\alpha = 3$ in the BEC regime. This finding, on one hand, is a unique manifestation of quasi-particle interaction effect in the Bose-Fermi superfluid mixture; on the other hand, reveals fundamental different between Fermi superfluid in the BCS side and in the BEC side. In the BCS side, the low-energy physics is dominated by fermionic quasi-particles nearby the Fermi surface, and the damping processes are also restricted by the constant density-of-state nearby Fermi surface, which is basically the origin of constant damping rate. While such restriction does not exist in the BEC side where the low-energy physics is dominated by bosonic mode.

Our results can be experimentally verified by studying damping rate of collective mode, as done in previous BEC experiments [13]. In the recent experiment, ENS group has find damping of collective oscillation when the relative velocity between Bose and Fermi superfluid exceeds a critical velocity. At the unitary regime and in the BEC side the damping rate increases rapidly when velocity is above the critical velocity [11]. They also find a nearly constant damping rate at the BCS side [21]. The underlying mechanism of this experimental finding may be connected to the physics discussed in this work.

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Supplemental Material

The damping rate in the BCS side

From Eq. (1) in the text, the Hamiltonian describing the interaction between the Bosons and Fermions in momentum space is given by

\[ H_{bf} = \frac{g_{bf}}{V} \sum_{q,p,k,\sigma} \hat{c}^\dagger_{q-k,\sigma} \hat{c}_{q,\sigma} \hat{b}^\dagger_{p+k} \hat{b}_{p}. \]  

(19)

By replacing one of the \( \hat{b} \) or \( \hat{b}^\dagger \) operators as \( \sqrt{N_b} \), this Hamiltonian becomes

\[ H_{bf} = g_{bf} \sqrt{\frac{N_b}{V}} \sum_{k,q,\sigma} \left( \hat{c}^\dagger_{k+q,\sigma} \hat{c}_{q,\sigma} \hat{b}_k + \text{h.c.} \right). \]  

(20)
Then we rewrite the Hamiltonian in terms of the quasi-particle operators as

$$
H_{bf} = g_{bf} \sqrt{\frac{n_{bf}}{V}} \sum_{k,q} \left( u_{k}^{\dagger} - v_{k}^{\dagger} \right) \left( u_{k-q}^{\dagger} v_{k}^{\dagger} + v_{k-q}^{\dagger} u_{k}^{\dagger} \right) \beta_{k-q}^{\dagger} \alpha_{k} + \text{h.c.}
$$

$$
+ g_{bf} \sqrt{\frac{n_{bf}}{V}} \sum_{k,q} \left( u_{k}^{\dagger} - v_{k}^{\dagger} \right) \left( u_{k+q}^{\dagger} v_{k}^{\dagger} - v_{k+q}^{\dagger} u_{k}^{\dagger} \right) \beta_{k+q}^{\dagger} \alpha_{k} + \text{h.c.}
$$

$$
+ g_{bf} \sqrt{\frac{n_{bf}}{V}} \sum_{k,q} \left( u_{k}^{\dagger} - v_{k}^{\dagger} \right) \left( u_{k+q}^{\dagger} v_{k}^{\dagger} - v_{k+q}^{\dagger} u_{k}^{\dagger} \right) \gamma_{k+q}^{\dagger} \alpha_{k} + \text{h.c.}
$$

(21)

Here we have ignored the terms such as $\hat{\beta}^{\dagger} \hat{\alpha}$ or $\hat{\beta}^{\dagger} \hat{\alpha}^{\dagger}$, since they do not conserve the energy, and will not contribute to the decay process. At the zero temperature, only the first term contribute to the damping of the $B_b$ mode.

Employing the approximations discussed in the text, we obtain the integral as Eq. (8) in the text. To calculate this integral, we first make the substitution: $k - q \rightarrow \frac{k}{2} - q$ and $q \rightarrow \frac{k}{2} + q$, so that the integral becomes

$$
\gamma \left( k \right) = \frac{g_{bf}^{2} c_{b} k}{8 \pi^{2} g_{b}} \int d^{3} q \delta \left[ \frac{h^{2}}{2m^{*}} \left( \eta_{q}^{2} - q \right) - \hbar c_{b} \left( k - k_{c} \right) \right].
$$

(22)

We choose the direction of $k$ as the $q_z$ axis, and transform into the cylindrical polar coordinates. The coordinate transformation is given by

$$
q_{z} = p_{z},
$$

$$
q_{x} = (k_{0} \sin \theta_{0} + p_{p}) \cos \phi,
$$

$$
q_{y} = (k_{0} \sin \theta_{0} + p_{p}) \sin \phi.
$$

where $\theta_{0}$ is defined as $\cos \theta_{0} = \frac{k_{0}}{k_{c}}$. The Jacobi determinant of this coordinate transformation is $d q_{z} d q_{x} d q_{y} = (k_{0} \sin \theta_{0} + p_{p}) d p_{z} d p_{\rho} d \phi$. Then the $\eta_{q}^{2} - q$ can be expanded in the new coordinates as

$$
\eta_{q}^{2} - q = \left| \frac{k}{2} \pm q \right| - k_{0} \approx \sin \theta_{0} p_{\rho} \pm \cos \theta_{0} p_{z},
$$

where the high order terms of $p_{\rho}$ and $p_{z}$ are ignored. Then the integral becomes

$$
\gamma \left( k \right) = \frac{g_{bf}^{2} c_{b} k}{8 \pi^{2} g_{b}} \int_{-\infty}^{\infty} dp_{z} \int_{-k_{0} \sin \theta_{0}}^{\infty} dp_{\rho} \int_{0}^{2\pi} d \phi
$$

$$
\times \left( k_{0} \sin \theta_{0} + p_{p} \right) \delta \left[ \frac{h^{2}}{2m^{*}} \left( p_{\rho}^{2} \sin^{2} \theta_{0} + \frac{p_{z}^{2}}{\cos^{2} \theta_{0}} \right) - \hbar c_{b} \left( k - k_{c} \right) \right].
$$

(23)

We apply a coordinate transformation again as

$$
-\frac{h}{\sqrt{m^{*}}} p_{\rho} \sin \theta_{0} = r \cos \zeta,
$$

$$
\frac{h}{\sqrt{m^{*}}} p_{z} \cos \theta_{0} = r \sin \zeta.
$$

where the corresponding Jacobi determinant is $d p_{z} d p_{\rho} = \frac{m^{*}}{\sqrt{m^{*} \sin \theta_{0} \cos \theta_{0}}} r d r d \zeta$. So Eq. 23 becomes

$$
\gamma \left( k \right) = \frac{g_{bf}^{2} c_{b} k}{4 \pi g_{b}} \int_{0}^{\infty} dr \int_{0}^{2\pi - \zeta_{0}} d \zeta \frac{m^{*}}{\sqrt{m^{*} \sin \theta_{0} \cos \theta_{0}}} r
$$

$$
\times \left( k_{0} \sin \theta_{0} - \frac{\sqrt{m^{*}}}{\hbar \sin \theta_{0}} r \cos \zeta \right) \delta \left[ r^{2} - \hbar c_{b} \left( k - k_{c} \right) \right].
$$

(24)

where $\zeta_{0}$ is given by $| \cos \zeta_{0} | = \frac{hk_{0} \sin \theta_{0}}{\sqrt{2m^{*} \hbar c_{b} (k - k_{c})}}$. Since we are Considering the threshold behavior, we have $\zeta_{0} = 0$. The damping rate is obtained as

$$
\gamma \left( k \right) = \frac{g_{bf}^{2} c_{b} k_{0}^{2} m^{*}}{2 \hbar^{2} g_{b}} \Theta \left( k - k_{c} \right)
$$

(25)
Substituting the expression of the effective mass, \( m^* = \frac{\Delta m^2}{\hbar^2 k_0} \), to the upper formula, one obtains the Eq. (9) in the text.

In the free boson regime, \( \Delta \gg \hbar^2 / (m_b\xi^2) \), using the same integral skill, one obtains the damping rate as:

\[
\gamma (k) = \frac{\gamma_{\text{d}} n_b k_0^2 m^*}{\hbar^3 k} \Theta (k - k_c)
\]  

(26)

Substituting the expression of the effective mass to the upper formula, we have the Eq. (11) in the text.

**Damping rate in the BEC side**

From Eq. (12), in the BEC side the Boson-molecular interaction Hamiltonian in the momentum space is given by

\[
H_{bm} = \frac{g_{bm}}{V} \sum_{q;p,k} \hat{d}^\dagger_{q-k} \hat{d} \hat{b}_{q+p+k} \hat{b}_p.
\]

(27)

By replacing one of the \( \hat{d} \) or \( \hat{d}^\dagger \) operators by the \( \sqrt{N_m} \), we obtain

\[
H_{bm}^{(1)} = g_{bm} \sqrt{N_m} \sum_{k,q} \left( \hat{d} - \hat{d}^\dagger \right) \hat{b}_{k-q}^\dagger \hat{b}_k.
\]

(28)

This term describes the \( B_0 \) mode scattered by the phonon mode \( B_f \) in the Fermi superfluid. We rewrite this Hamiltonian in terms of the quasi-particle operators

\[
H_{bm}^{(1)} = g_{bm} \sqrt{N_m} \sum_{k,q} \left( u^m - i v^m \right) (u_{k-q}^b v_k^b + v_{k-q}^b u_k^b) \chi_{q-k} \hat{\alpha}_{k-q} \hat{\alpha}_k + \text{h.c.}
\]  

\[
- g_{bm} \sqrt{N_m} \sum_{k,q} \left( u^m - i v^m \right) v_{k-q}^b u_k^b \chi_{q-k} \hat{\alpha}_{q-k} \hat{\alpha}_k + \text{h.c.,}
\]

(29)

where such terms as \( \hat{\alpha} \chi \hat{\alpha} \) or \( \hat{\alpha}^\dagger \chi \hat{\alpha}^\dagger \) are ignored, since they do not conserve the energy. At the zero temperature, only the first term contribute to the damping, which describes the process \( B_0 (k) \rightarrow B_0 (k - q) + B_f (q) \). The critical momentum of this process can be obtained by energy-momentum conservation as \( \partial \mathcal{E}_{B_0}(k) / \partial (\hbar k) \big|_{k = k_c} = c_f \). In the free boson regime, we have \( \hbar k_c = m_c c_f \).

By replacing one of the \( \hat{b} \) or \( \hat{b}^\dagger \) operators by the \( \sqrt{N_b} \) in Eq. (27) we obtain

\[
H_{bm}^{(2)} = - g_{bm} \sqrt{N_b} \sum_{k,q} \left( u_k^b - i v_k^b \right) u_{k-q}^m v_q^m \chi_{k-q} \hat{\chi}_{q-k} \hat{\chi}_k + \text{h.c.}
\]  

\[
+ g_{bm} \sqrt{N_b} \sum_{k,q} \left( u_k^b - i v_k^b \right) \left( v_{k+q}^m v_q^m + u_{k+q}^m u_q^m \right) \chi_{k+q} \hat{\chi}_{q-k} \hat{\chi}_k + \text{h.c.,}
\]

(30)

At zero temperature, only the first term contribute to the damping, which describes the process \( B_0 (k) \rightarrow B_f (k - q) + B_f (q) \). Using the energy-momentum conservation, one can determine the critical momentum for this process by \( \mathcal{E}_{B_0}(k_c) = 2\mathcal{E}_{B_0}(k_c/2) \). In the free boson regime, we have \( \hbar k_c \approx 2m_c c_f \), which is larger than the critical momentum of the process discussed above. So we will focus on the threshold behavior of the process \( B_0 \rightarrow B_0 + B_f \).

Employing the approximations discussed in the text, we obtain damping rate as Eq. (17) in the text:

\[
\gamma (k) = \frac{g_{bm} c_f}{8\pi^2 g_b} \int \frac{d^3 q}{|q|} \left[ \hbar c_f |q| + \frac{\hbar^2 (k - q)^2}{2m_b} - \frac{\hbar^2 k^2}{2m_b} \right] |q|.
\]

(31)

The Dirac function in this integral gives

\[
\delta \left[ \hbar c_f |q| + \frac{\hbar^2 (k - q)^2}{2m_b} - \frac{\hbar^2 k^2}{2m_b} \right] = \frac{m_b}{\hbar^2 k q} \left( \frac{\hbar q + 2m_b c_f}{2\hbar k} - \cos \theta \right),
\]

(32)
where \( \theta \) is the angle between \( q \) and \( k \). So the integral becomes

\[
\gamma (k) = \frac{g^2 \alpha_m m_b c_l}{4 \pi h^2 g_m k} \int_0^\infty q^2 dq \int_0^\pi d \cos \theta \delta \left( \frac{h q + 2 m_b c_l}{2 h k} - \cos \theta \right).
\] (33)

Since we have \(-1 < \cos \theta < 1\), the integral regime of \( q \) is determined as

\[-2 \left( k + \frac{m_b c_l}{\hbar} \right) < q < 2 \left( k - \frac{m_b c_l}{\hbar} \right).\] (34)

The integral can be simplified into

\[
\gamma (k) = \frac{g^2 \alpha_m m_b c_l}{4 \pi h^2 g_m k} \int_0^{2 \left( k - m_b c_l / \hbar \right)} q^2 dq.
\]

\[
= \frac{2 g^2 \alpha_m m_b c_l}{3 \pi h^2 g_m k} (k - m_b c_l / \hbar)^3 \Theta (k - m_b c_l / \hbar).
\]

\[
= \frac{2 g^2 \alpha_m m_b c_l}{3 \pi h^2 g_m k} (k - k_c)^3 \Theta (k - k_c).
\] (35)

Here we can see \( k_c = m_b c_l / \hbar \), which reproduces the critical momentum discussed above.