MASS TRANSPORTATION AND CONTRACTIONS

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Abstract. According to a celebrated result of L. Caffarelli, every optimal mass transportation mapping pushing forward the standard Gaussian measure onto a log-concave measure \( e^{-W(x)} dx \) with \( D^2 W \geq Id \) is 1-Lipschitz. We present a short survey of related results and various applications.

Keywords: optimal transportation, Monge–Ampère equation, log-concave measures, Gaussian measures, isoperimetric inequalities, Sobolev inequalities

1. Introduction

Given a positive number \( \alpha \) we say that a mapping \( T: \mathbb{R}^d \rightarrow \mathbb{R}^d \) is \( \alpha \)-Lipschitz if

\[
|T(x) - T(y)| \leq \alpha |x - y|
\]

For a smooth \( T \) this is equivalent to the following:

\[
\sup_{x \in \mathbb{R}^d} \|DT(x)\| \leq \alpha,
\]

where \( \| \cdot \| \) is the operator norm. For the case \( \alpha = 1 \) we say that \( T \) is a contraction.

Similarly, a mapping \( T: X \rightarrow Y \) between metric spaces is called contraction, if \( \rho_Y(T(x_1), T(x_2)) \leq \rho_X(x_1, x_2) \).

Let \( \mu \) be a Borel measure on a metric space \((M, \rho)\). Given a Borel set \( A \subset M \) we define the corresponding boundary measure \( \mu^+ \) of \( \partial A \)

\[
\mu^+(\partial A) = \lim_{h \rightarrow 0} \frac{\mu(A_h) - \mu(A)}{h},
\]

where \( A_h = \{ x : \rho(x, A) \leq h \} \).

A set \( A \) is called isoperimetric if it has the minimal surface measure among of all the sets with the same measure \( \mu(A) \). The isoperimetric profile \( I_\mu \) of \( \mu \) is defined as the following function

\[
I_\mu(t) = \inf \{ \mu^+(\partial A) : \mu(A) = t \}.
\]

Generally, isoperimetric sets are not possible to find. Nevertheless, bounds for isoperimetric functions (the so-called isoperimetric inequalities) have many applications in analysis, geometry and probability theory. It is well-known, for instance, that isoperimetric inequalities imply Sobolev-type inequalities. See more in [9], [22], [18], [24], [27].

Numerous applications of contractions in analysis, probability and geometry rely on the following fact:

Let \( X, Y \) be two metric spaces and \( X \) is equipped with a measure \( \mu \). Assume that there exists a contraction \( T: X \rightarrow Y \) between metric spaces \( X \) and \( Y \). Then the image measure \( \nu = \mu \circ T^{-1} \) has a better isoperimetric profile

\[
I_\nu \geq I_\mu.
\]

In this paper we study mainly a special case of optimal transportations of measures. Given two Borel probability measures \( \mu \) and \( \nu \) we consider the optimal transportation map \( T \) minimizing the cost

\[
W_2^2(\mu, \nu) = \int |x - T(x)|^2 \ d\mu
\]

among of all the maps pushing forward \( \mu \) to \( \nu \). The latter means that \( \mu \circ T^{-1}(A) = \nu(A) \) for every Borel \( A \).

If \( \mu = \rho_0 \ dx \) and \( \nu = \rho_1 \ dx \) are absolutely continuous, then \( T \) does exist and can be obtained from the solution to the corresponding Monge-Kantorovich transportation problem. Moreover, this map is \( \mu \)-unique and has the form \( T = \nabla \Phi \), where \( \Phi \) is convex (see [27]). Assuming smoothness of \( \Phi \), one can easily verify that \( \Phi \) solves the following nonlinear PDE (the Monge–Ampère equation):

\[
\rho_1(\nabla \Phi) \det D^2 \Phi = \rho_0.
\]
This paper contains an overview of the results related to the contractivity of optimal transportation mappings. The first result in this direction has been established by L. Caffarelli (see [6]). Let $\mu$ be the standard Gaussian measure $\mu = \frac{1}{(2\pi)^{d/2}} e^{-\frac{x^2}{2}} dx$ and $\nu = e^{-W} dx$ with $D^2 W \geq \text{Id}$, then the corresponding $T$ is a contraction. This observation implies immediately the Bakry-Ledoux comparison theorem and various functional inequalities, including the log-Sobolev inequality for uniformly log-concave measures. Among other applications let us mention the Gaussian correlation conjecture and the Brascamp-Lieb inequality. We discuss several extensions of this result and some open problems.

2. Caffarelli’s contraction theorem

Remark 2.1. The Theorem 2.2 and Theorem 2.5 below will be both referred to as “Caffarelli’s contraction theorem”. Note, however, that the original formulation is given in Theorem 2.5.

**Theorem 2.2. (L. Caffarelli)** Let $T = \nabla \Phi$ be the optimal transportation mapping pushing forward a probability measure $\mu = e^{-V} dx$ onto a probability measure $\nu = e^{-W} dx$. Assume that $V$ and $W$ are twice continuously differentiable and $D^2 W \geq K$. Then for every unit vector $e$

$$
\sup_{x \in \mathbb{R}^d} \Phi^2_{ee} \leq \frac{1}{K} \sup_{x \in \mathbb{R}^d} V_{ee}.
$$

In particular, if $\mu$ is the standard Gaussian measure and $K \geq 1$, then $T$ is a contraction.

**Sketch of the proof:**

1) **Maximum principle proof.**

The proof based on the maximum principle is formal but elegant. Functions $V, W$ and $\Phi$ are assumed to be sufficiently regular. Note that smoothness of $\Phi$ can be justified in some favorable situations ($V, W$ are smooth and satisfy certain growth assumptions, see Theorem 4.14 of [27]). By the change of variables formula

$$
e^{-V} = e^{-W(\nabla \Phi)} \det D^2 \Phi.
$$

Taking the logarithm of both sides we get

$$
V = W(\nabla \Phi) - \log \det D^2 \Phi.
$$

We fix some unit vector $e$ and differentiate this formula twice along $e$. To this end we apply the following fundamental relation

$$
\partial_e \ln \det D^2 \Phi = \frac{\partial_e \det D^2 \Phi}{\det D^2 \Phi} = \text{Tr}(D^2 \Phi)^{-1} D^2 \Phi_e.
$$

Differentiating this formula along another direction $v$ and using that

$$
D^2 \Phi_v(D^2 \Phi)^{-1} + D^2 \Phi \left[(D^2 \Phi)^{-1}\right]_v = 0
$$

we obtain

$$
\partial_{ee} \ln \det D^2 \Phi = \text{Tr}(D^2 \Phi)^{-1} D^2 \Phi_{ee} - \text{Tr}\left[(D^2 \Phi)^{-1} D^2 \Phi_e(D^2 \Phi)^{-1} D^2 \Phi_e\right].
$$

Coming back to the change of variables formula we get

$$
V_e = \langle \nabla W(\nabla \Phi), D^2 \Phi \cdot e\rangle - \text{Tr}(D^2 \Phi)^{-1} D^2 \Phi_e
$$

and

$$
V_{ee} = \langle D^2 W(\nabla \Phi) D^2 \Phi \cdot e, D^2 \Phi \cdot e\rangle + \langle \nabla W(\nabla \Phi), \nabla \Phi_{ee}\rangle
$$

$$
- \text{Tr}(D^2 \Phi)^{-1} D^2 \Phi_{ee} + \text{Tr}\left[(D^2 \Phi)^{-1} D^2 \Phi_e\right]^2.
$$

Now assume that $\Phi_{ee}$ attains its maximum at $x_0$. Then

$$
\nabla \Phi_{ee}(x_0) = 0, \ D^2 \Phi_{ee} \leq 0.
$$

Note that $\text{Tr}\left[(D^2 \Phi)^{-1} D^2 \Phi_e\right]^2 \geq 0$ because it equals to $\text{Tr} C^2$, where

$$
C = (D^2 \Phi)^{-1/2} D^2 \Phi_e(D^2 \Phi)^{-1/2}
$$

is a symmetric matrix.

Clearly, $\text{Tr}(D^2 \Phi(x_0))^{-1} D^2 \Phi_{ee}(x_0) \leq 0$ and one gets

$$
V_{ee}(x_0) \geq K \|D^2 \Phi(x_0) \cdot e\| \geq K \Phi^2_{ee}(x_0).
$$
Finally that \( \lim_{n \to \infty} a_n \leq 1 \). The proof is complete.

2) Incremental quotients proof

Instead of differentiating the Monge-Ampère equation we consider the incremental quotient

\[ \delta_2 \Phi(x) = \Phi(x + th) + \Phi(x - th) - 2\Phi(x) \geq 0 \]

for some fixed vector \( h \in \mathbb{R}^d \) with \( |h| = 1 \). By approximation, one can assume that \( \text{supp}(\nu) \) is a bounded convex domain and \( V, W \) are locally Hölder. Caffarelli’s regularity theory assures that \( \Phi \in C^{2,\alpha}_{\text{loc}}(\mathbb{R}^d) \).

In addition, again by approximation, one can assume that \( \mu \) has at most Gaussian decay, meaning that \( V(x) \leq C_1 + C_2|x|^2 \) for some \( C_1, C_2 \geq 0 \). Then the following lemma holds (see Lemma 4 in [6])

**Lemma 2.3.** \( \lim_{x \to \infty} \delta_2 \Phi(x) = 0 \).

Thus there exists a maximum point \( x_0 \) of \( \delta_2 \Phi(x) \). Differentiating at \( x_0 \) yields

\[ \nabla \Phi(x_0 + th) + \nabla \Phi(x_0 - th) = 2\nabla \Phi(x_0), \tag{1} \]

\[ D^2 \Phi(x_0 + th) + D^2 \Phi(x_0 - th) \leq 2D^2 \Phi(x_0). \]

It follows from the concavity of the determinant that

\[ \det D^2 \Phi(x_0) \geq \det \left( \frac{D^2 \Phi(x_0 + th) + D^2 \Phi(x_0 - th)}{2} \right) \]

\[ \geq \left( \det D^2 \Phi(x_0 + th) \det D^2 \Phi(x_0 - th) \right)^{\frac{1}{2}}. \]

Applying the change of variables formula \( \det D^2 \Phi = e^{W(\nabla \Phi) - V} \) one finally gets

\[ V(x_0 + th) + V(x_0 - th) - 2V(x_0) \geq W(\nabla \Phi(x_0 + th)) + W(\nabla \Phi(x_0 - th)) - 2W(\nabla \Phi(x_0)). \tag{2} \]

It follows from [1] that \( v := \nabla \Phi(x_0 + th) - \nabla \Phi(x_0) = \nabla \Phi(x_0) - \nabla \Phi(x_0 - th) \). Hence we get by [2] that

\[ \sup V_{th} \cdot t^2 \geq K|\nabla \Phi(x_0 + th) - \nabla \Phi(x_0)|^2 = K|\nabla \Phi(x_0 - th) - \nabla \Phi(x_0)|^2 = K|v|^2. \]

By convexity of \( \Phi \)

\[ \Phi(x_0 + th) + \Phi(x_0 - th) - 2\Phi(x_0) \leq t \langle \nabla \Phi(x_0 + th) - \nabla \Phi(x_0 - th), h \rangle \]

\[ = 2t(v, h) \leq 2t|v|. \]

Finally

\[ \sup_{x \in \mathbb{R}^d} \frac{V_{th}}{K} \geq \left( \frac{\delta_2 \Phi(x_0)}{2t^2} \right)^2. \]

This clearly implies

\[ \Phi_{hh} \leq 2C \]

with \( C = \sqrt{\sup_{x \in \mathbb{R}^d} \frac{V_{th}}{K}} \). But this estimate is worse than the desired one. To get the sharp estimate we repeat the arguments and use the additional information that \( \Phi_{hh} \leq a_0 C \), where \( a_0 = 2 \). Apply the identity

\[ \Phi(x_0 + th) + \Phi(x_0 - th) - 2\Phi(x_0) = \int_0^t \langle \nabla \Phi(x_0 + sh) - \nabla \Phi(x_0 - sh), h \rangle \ ds. \]

By convexity of \( \Phi \)

\[ \langle \nabla \Phi(x_0 + sh) - \nabla \Phi(x_0 - sh), h \rangle \leq \langle \nabla \Phi(x_0 + th) - \nabla \Phi(x_0 - th), h \rangle. \]

One has

\[ \Phi(x_0 + th) + \Phi(x_0 - th) - 2\Phi(x_0) \leq \int_0^t \min \{2a_0 C s, 2|v| \} \ ds. \]

Computing the right-hand side and taking into account that \( |v| \leq Ct \), we get that

\[ \Phi(x_0 + th) + \Phi(x_0 - th) - 2\Phi(x_0) \leq a_1 C t^2. \]

where \( a_1 = \frac{3}{2} \). Hence \( \Phi_{hh} \leq a_1 C. \) Repeating this arguments infinitely many times we get that \( \Phi_{hh} \leq a_n C \) and \( \lim_{n \to \infty} a_n = 1 \). The proof is complete.

3) Proof via \( L^p \)-estimates

See Section 6.

**Remark 2.4.** We note that the original result from [6] was slightly different from the result stated above. Here is the exact statement proved by Caffarelli.
**Theorem 2.5. (L. Caffarelli)** Let $\mu = e^{-Q} \, dz$ be any Gaussian measure. Then for any measure $\nu = e^{-Q-P} \, dx$, where $P$ is convex, the corresponding optimal transportation $T$ is a contraction.

**Sketch of the proof:** Let us apply the maximum principle arguments. We are looking for a maximum of $\Phi(x)$ among all unit $e$ and $x \in \mathbb{R}^d$. Apply the relation obtained above

\[
Q_{ee} = \langle D^2(Q + P)(\nabla \Phi)D^2 \Phi \cdot e, D^2 \Phi \cdot e \rangle + \langle \nabla(Q + P)(\nabla \Phi), \nabla \Phi e \rangle
- \text{Tr}(D^2 \Phi)^{-1}D^2 \Phi e + \text{Tr}\left[ \left(D^2 \Phi)^{-1}D^2 \Phi e \right) \right]^2.
\]

By the same reasons as above

\[
Q_{ee} \geq \langle D^2(Q + P)(\nabla \Phi)D^2 \Phi \cdot e, D^2 \Phi \cdot e \rangle.
\]

Now take into account that $P$ is convex and, in addition, $e$ must be an eigenvector of $D^2 \Phi$. Hence we obtain

\[
Q_{ee} \geq \Phi_{ee}^2 \cdot Q_{ee}(\nabla \Phi).
\]

Taking into account that $Q_{ee}$ is constant, we obtain the claim.

### 3. General (uniformly) log-concave measures

The incremental quotients proof can be easily extended to the case of measures which are uniformly log-concave in a generalized case. The latter means that the potential $W$ satisfies

\[
W(x + y) + W(x - y) - W(x) \geq \delta(|y|)
\]

for some increasing function $\delta$. The following result has been proved in [15].

**Theorem 3.1.** Assume that $V$ and $W$ satisfy

\[
V(x + y) + V(x - y) - 2V(x) \leq A_p |y|^{p+1},
\]

\[
W(x + y) + W(x - y) - 2W(x) \geq A_q |y|^{q+1},
\]

for some $0 \leq p \leq 1$, $1 \leq q$, $A_p > 0$, $A_q > 0$.

Then $\Phi$ satisfies

\[
\Phi(x + th) + \Phi(x - th) - 2\Phi(x) \leq 2 \left( \frac{A_p}{A_q} \right) \frac{1}{q+1} t^{1+\alpha}
\]

for every unit vector $h \in \mathbb{R}^d$ with $\alpha = \frac{p+1}{q+1}$.

**Remark 3.2.** The constant in (3) is not optimal in general.

It follows from (3) that $\nabla \Phi$ is globally Hölder. This fact is actually true without any convexity assumption on $\Phi$, but the convex case is more simple and the result follows from the following lemma communicated to the authors by Sasha Sodin.

**Lemma 3.3.** For every convex $f$ and unit vector $h$ one has

\[
|\nabla f(x + th) - f(x)| \leq \frac{2}{t} \sup_{v : |v| = 1} \left( f(x + 2tv) + f(x - 2tv) - 2f(x) \right).
\]

Using this lemma one can extend the Hölder regularity result.

**Theorem 3.4.** Assume that

\[
V(x + y) + V(x - y) - 2V(x) \leq |y|^2,
\]

and

\[
W(x + y) + W(x - y) - W(x) \geq \delta(|y|)
\]

with some non-negative increasing function $\delta$. Then

\[
|\nabla \Phi(x) - \nabla \Phi(y)| \leq 8\delta^{-1}(4|x - y|^2).
\]
Applying this estimate one can transfer the famous Gaussian Sudakov-Tsirelson isoperimetric inequality to any (generalized) uniform log-concave measure. Recall (see [5]), that the standard Gaussian measure $\gamma$ satisfies the Gaussian isoperimetric inequality
\[ \gamma(A^*) \geq \Phi(\Phi^{-1}(\gamma(A)) + r), \]
where $A^* = \{ x \in \mathbb{R}^d : \exists a \in A : |a - x| < r \}$, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$.

Consequently, applying Theorem 3.3 to $\mu = \gamma$ and $\nu = e^{-W} dx$ with $W$ satisfying
\[ W(x + y) + W(x - y) - W(x) \geq \delta(|y|), \]
we get
\[ \nu(A_r) \geq \Phi(\Phi^{-1}(\nu(A)) + \frac{1}{2} \sqrt{\delta(r/8)}). \]

In particular, $\nu$ admits the following dimension-free concentration property:
\[ \nu(A_r) \geq 1 - \frac{1}{2} \exp\left(\frac{1}{8} \delta(r/8)\right) \]
with $\nu(A) \geq 1/2$. A similar result has been established by S. Sodin and E. Milman in [23] by localization arguments. Note that according to results of E. Milman [21] concentration and isoperimetric inequalities are in a sense equivalent for log-concave measures.

4. Lebesgue measure on a convex set

In this section we discuss the following problem.

**Problem 4.1.** Given a nice (product) probability measure $\mu$ (e.g. Gaussian or exponential) estimate effectively the Lipschitz constant of the optimal mapping pushing forward $\mu$ onto the normalized Lebesgue measure on a convex set $K$.

This problem was motivated in particular by the famous Kannan-Lovász-Simonovits conjecture (KLS-conjecture). Recall that the Cheeger $C_{ch}(K)$ constant of a convex body $K$ is the smallest constant such that the inequality
\[ \int_K \left| f - \frac{1}{\lambda(K)} \int_K f dx \right| dx \leq C_{ch}(K) \int_K |\nabla f| dx \]
holds for every smooth $f$.

**KLS conjecture.** There exists an universal constant $c$ such that
\[ C_{ch}(K) \leq c \]
for every convex $K \subset \mathbb{R}^d$ satisfying
\[ \int_K x_i dx = 0, \quad \frac{1}{\lambda(K)} \int_K x_i x_j dx = \delta_{ij}. \]

More on the KLS conjecture see in [12], [4], [21].

Some estimates of the Lipschitz constant for optimal transportation of convex bodies have been obtained in [15]. The arguments below generalize the maximum principle proof of Caffarelli. Let $\nabla \Phi$ be the optimal transportation mapping pushing forward $e^{-V} dx$ to $\frac{1}{\lambda(K)} 1_K$. Let us fix a unit vector $h$. We are looking for a function $\psi$ such that
\[ \psi(\Phi_h) + \log \Phi_{hh} \]
is bounded from above. Assume that $x_0$ is the maximum point. One has at this point
\[ \psi'(\Phi_h) \nabla \Phi_h + \frac{1}{\Phi_{hh}} \nabla \Phi_{hh} = 0 \quad (4) \]
\[ \psi''(\Phi_h) \nabla \Phi_h \otimes \nabla \Phi_h + \psi'(\Phi_h) D^2 \Phi_h + \frac{1}{\Phi_{hh}} D^2 \Phi_{hh} - \frac{1}{\Phi_{hh}} \nabla \Phi_{hh} \otimes \nabla \Phi_{hh} \leq 0. \quad (5) \]

Differentiation the change of variables formula gives (see Section 1)
\[ V_h = -\text{Tr}(D^2 \Phi)^{-1} D^2 \Phi_h, \]
\[ V_{hh} = -\text{Tr}(D^2 \Phi)^{-1} D^2 \Phi_{hh} + \text{Tr} \left[ (D^2 \Phi)^{-1} D^2 \Phi_h \right]^2. \]
Multiply \( \frac{3}{4} \) by \((D^2 \Phi)^{-1}\), take the trace and plug in the expression for \(V_{hh}\) into the formula. One obtains
\[
V_{hh} \geq \frac{1}{\Phi_{hh}} \text{Tr}[(D^2 \Phi)^{-1} \cdot \nabla \Phi_{hh} \otimes \nabla \Phi_{hh}] + \Phi_{hh} \cdot \psi''(\Phi_h) \text{Tr}[(D^2 \Phi)^{-1} \cdot \nabla \Phi_h \otimes \nabla \Phi_h] \\
+ \Phi_{hh} \cdot \psi'(\Phi_h) \text{Tr}[(D^2 \Phi)^{-1} D^2 \Phi_h] + \text{Tr}[(D^2 \Phi)^{-1} D^2 \Phi_h]^2.
\]

Remark that \( \text{Tr}[(D^2 \Phi)^{-1} \cdot \nabla \Phi_h \otimes \nabla \Phi_h] = \Phi_{hh} \). One obtains from (1) that \( \nabla \Phi_{hh} = -\Phi_{hh} \cdot \psi'(\Phi_h) \nabla \Phi_h \).

Plugging this into the inequality for \(V_{hh}\) one gets
\[
V_{hh} \geq \Phi_{hh}^2 \left[\psi'' - (\psi')^2\right] \circ \Phi_h + \Phi_{hh} \cdot \psi'(\Phi_h) \text{Tr}[(D^2 \Phi)^{-1} D^2 \Phi_h] + \text{Tr}[(D^2 \Phi)^{-1} D^2 \Phi_h]^2.
\]

Note that
\[
\text{Tr}[(D^2 \Phi)^{-1} D^2 \Phi_h] = \text{Tr}C, \quad \text{Tr}[(D^2 \Phi)^{-1} D^2 \Phi_h]^2 = \text{Tr}C^2,
\]

where
\[
C = (D^2 \Phi)^{-1/2}(D^2 \Phi)(D^2 \Phi)^{-1/2}
\]
is a symmetric matrix. Hence, by the Cauchy inequality
\[
V_{hh} \geq \Phi_{hh}^2 \left[\psi'' - \left(1 + \frac{d}{4}\right)(\psi')^2\right] \circ \Phi_h.
\]

Now assume that \(V_{hh}\) is bounded from above by a constant \(C\). Let \(\psi\) be a function satisfying
\[
\psi'' - \left(1 + \frac{d}{4}\right)(\psi')^2 \geq e^{2\psi}.
\]

Then we get
\[
C \geq \Phi_{hh}^2(x_0)e^{2\psi}(\Phi_h(x_0)) = \sup_{x \in \mathbb{R}^d} \Phi_{hh}^2 e^{2\psi}(\Phi_h).
\]

In particular, choosing carefully \(\psi\) one can obtain the following statement (see [15] for details).

**Theorem 4.2.** 1) Optimal transportation \(T\) of the standard Gaussian measure \(\gamma\) onto \(\frac{1}{\lambda(K)}\lambda|K\), where \(K\) is convex, satisfies
\[
\|DT\| \leq c \sqrt{d} \text{diam}(K),
\]

where \(c\) is an universal constant and \(\text{diam}(K)\) is the diameter of \(K\).

2) Optimal transportation \(T\) between \(\mu = e^{-V} dx\) and \(\frac{1}{\lambda(K)}\lambda|K\), with \(V_{hh} \leq C\), \(|V_h| \leq C\) for some \(C\), satisfies
\[
\|DT\| \leq c \text{ diam}(K),
\]

where \(c\) depends only on \(C\).

Unfortunately, estimates of Theorem 4.2 are not strong enough to recover even known results on the Cheeger constant for convex bodies. This gives rise to the following problem.

**Problem 4.3.** Does there exist any dimension-free estimate for \(\|DT\|\), when \(\mu = \gamma\) and \(\nu = \frac{1}{\lambda(K)}\lambda|K\)? The same for the case when \(\mu\) is the product of exponential distributions.

Note, that it would be enough for our purpose to have a integral norm estimate \(\int \|DT\|^p d\gamma\), \(p \geq 1\). This follows form the result of E. Milman [21] about equivalence of norms for log-concave measures.

5. CONTRACTION FOR THE MASS TRANSPORT GENERATED BY SEMIGROUPS

A contraction result for another type of mass transport has been obtained recently in [13] by Y.-H Kim and E. Milman. The idea of the construction of this transportation mapping goes back to J. Moser.

Consider the diffusion semigroup \(P_t = e^{tL}\) denerated by
\[
L = \Delta - \langle \nabla V, \nabla \rangle = e^V \text{div}(e^{-V} \cdot \nabla)
\]

and the flow of probability measures
\[
\nu_t = P_t(e^{-W+V}) \cdot \mu.
\]

Clearly, \(\mu\) is the invariant measure for \(P_t\), \(\nu_0 = \nu\), and \(\nu_\infty = \mu\).

Let us write the transport equation for \(\nu_t\):
\[
\frac{d}{dt} \nu_t = LP_t(e^{-W+V}) \cdot \mu = \text{div} \left[ \nabla P_t(e^{-W+V}) \cdot e^{-V} \right] = \text{div} \left[ \nabla \log P_t(e^{-W+V}) \cdot \nu_t \right].
\]
The corresponding flow of diffeomorphisms is governed by the equation
\[
\frac{d}{dt} S_t = -\nabla \log P_t(e^{-W+V}) \circ S_t, \quad S_0 = \text{Id},
\] (6)
where \( \nu_t \) and \( S_t \) are related by
\[
\nu_t = \nu \circ S_t^{-1}.
\]
In particular, the limiting map \( S_\infty = \lim_{t \to \infty} S_t \) pushes forward \( \nu \) to \( \mu \). We denote the inverse mappings by \( T_t \):
\[
T_t \circ S_t = \text{Id}, \quad T = \lim_{t \to \infty} T_t.
\]

The contraction property for \( T = S^{-1} \) is equivalent to the expansion property of \( S \). It is sufficient to show that \( (DS_t)^* DS_t \geq \text{Id} \). Using (6) one gets
\[
\frac{d}{dt} DS_t(x) = -DW_t(S_t) \cdot DS_t, \quad W_t = \nabla \log P_t(e^{-W+V}).
\]
Hence
\[
\frac{d}{dt} (DS_t)^* DS_t = 2(DS_t)^* \cdot DW_t(S_t) \cdot DS_t.
\]
Clearly, if
\[
DW_t(S_t) = -D^2 \log P_t(e^{-W+V}) \geq 0,
\]
then \( S_t \) has the desired expansion property.

Assume now that the function \( U \) defined by
\[
\nu = e^{-U} \cdot \mu, \quad U = W - V,
\]
is convex. Then the property \(-D^2 \log P_t(e^{-W+V}) = -D^2 \log P_t e^{-U} \geq 0 \) means that \( P_t \) preserves log-concave functions. Thus we obtain

**Theorem 5.1.** Assume that \( U \) is convex. If \( U_t = -\log P_t e^{-U} \) is a convex function for every \( t \geq 0 \), then every \( T_t \) is a 1-contraction.

It should be noted that by a result from [14] the property to preserve all log-concave functions do admit only diffusion semigroups with Gaussian kernels. Nevertheless, Kim and Milman were able to show under certain symmetry assumptions log-concavity is preserved. The proof is based on the application of the maximum principle.

They get, in particularly, the following result (see [13] for a more general statement).

**Theorem 5.2.** Assume that \( \mu \) is a product measure, \( V \) and \( U \) are convex functions, \( U \) is unconditional \( U(x_1, \ldots, x_n) = U(\pm x_1, \ldots, \pm x_n) \), and \( V(x) = \sum_{i=1}^d \rho_i(|x_i|) \) with \( \rho_i'' \leq 0 \).

Then \( T \) is a contraction. In addition, the optimal transportation mapping \( T_{\text{opt}} \) pushing forward \( \mu \) onto \( \nu \) is a contraction too.

Let us very briefly explain the idea of the proof. Let \( t_0 \) be the first moment when the convexity of \( U_t \) fails. Assume that the minimum of \( \partial_{ee} U_{t_0} \) is attained at some point \( x_0 \) for some direction \( e \). Then
\[
(d/dt - \Delta) \partial_{ee} U_t |_{t_0, x_0} \leq 0.
\]
In addition, \( \nabla \partial_{ee} U_t = 0 \) and \( \nabla \partial_{ee} U_t = 0 \). Using this one can show that
\[
(d/dt - \Delta) \partial_{ee} U_t |_{t_0, x_0} = -\langle \nabla U_t, \nabla V_{ee} \rangle |_{t_0, x_0}.
\]
At the time \( t_0 \) the function \( U_t \) is still convex and it is easy to see that the right-hand side should be non-negative. This leads to a contradiction.

6. \( L^p \)-contractions

In this section we discuss an \( L^p \)-generalization of the Caffarelli’s theorem (see [16]). The proof below is obtained with the help of the so-called above-tangent formalism (see [10]). The huge advantage of this approach is that no a priori regularity of the function \( \Phi \) is required. See [16] for details and relations to the transportation inequalities.

**Remark 6.1.** The estimates obtained in this section can be considered as global dimension-free Sobolev a priori estimates for the optimal transportation problem. In particular, they can be generalized for infinite-dimensional measures.
**Theorem 6.2.** Assume that $D^2 W \geq K \cdot \text{Id}$. Then for every unit $e$, $p \geq 1$, one has

\[
K\|\Phi^2_{ee}\|_{L^p(\mu)} \leq \|(\nabla e x) + \|_{L^p(\mu)},
\]

\[
K\|\Phi^2_{ee}\|_{L^p(\mu)} \leq \frac{p+1}{2}\|V_e^2\|_{L^p(\mu)}.
\]

**Proof.** Fix unit vector $e$. According to a result of McCann \[19\] the change of variables formula

\[
V(x) = W(\nabla \Phi(x)) - \log D^2 \Phi
\]

holds $\mu$-almost everywhere. Here $D^2 \Phi$ is the absolutely continuous part of the second distributional derivative $D^2 \Phi$ (Alexandrov derivative). One has

\[
V(x + te) - V(x) = W(\nabla \Phi(x + te)) - W(\nabla \Phi(x)) - \log \left[ (\det_a D^2 \Phi(x))^2 - \det_a D^2 \Phi(x + te) \right].
\]

By the uniform convexity of $W$

\[
V(x + te) - V(x) \geq \langle \nabla \Phi(x + te) - \nabla \Phi(x), \nabla W(\nabla \Phi(x)) \rangle + \frac{K}{2} |\nabla \Phi(x + te) - \nabla \Phi(x)|^2 - \log \left[ (\det_a D^2 \Phi(x))^2 - \det_a D^2 \Phi(x + te) \right].
\]

Multiply this identity by $(\delta_{te} \Phi)^p$, where $p \geq 0$ and

\[
(\delta_{te} \Phi)^p = \Phi(x + te) + \Phi(x - te) - 2\Phi(x)
\]

and integrate over $\mu$. We apply the following simple lemma.

**Lemma 6.3.** Let $\varphi : A \to \mathbb{R}$, $\psi : B \to \mathbb{R}$ be convex functions on convex sets $A$, $B$. Assume that $\nabla \psi(B) \subset A$. Then

\[
\text{div}(\nabla \varphi \circ \nabla \psi) \geq \text{Tr} [D_2 \varphi(\nabla \psi) \cdot D_2 \psi] \, dx \geq 0,
\]

where div is the distributional derivative.

Integrating by parts and applying this lemma we get

\[
\int \langle \nabla \Phi(x + te) - \nabla \Phi(x), \nabla W(\nabla \Phi(x)) \rangle (\delta_{te} \Phi)^p \, d\mu
\]

\[
= \int \langle \nabla \Phi(x + te) \circ (\nabla \psi) - x, \nabla W(x) \rangle (\delta_{te} \Phi)^p \circ (\nabla \psi) \, d\mu
\]

\[
\geq \int \left( \text{Tr} \left[ D_2 \Phi(x + te) \cdot (D_2 \Phi)^{-1} \right] \circ (\nabla \psi) - d \right) (\delta_{te} \Phi)^p \circ (\nabla \psi) \, d\nu
\]

\[
+ p \int \langle \nabla \Phi(x + te) \circ (\nabla \psi) - x, (D^2 \Phi) \circ \nabla \Phi(x) \circ (\nabla \psi) \rangle (\delta_{te} \Phi)^p \circ (\nabla \psi) \, d\nu.
\]

We note that

\[
\text{Tr} A - d - \log \det A \geq 0
\]

for any $A$ of the type $A = BC$, where $B$ and $C$ are symmetric and positive. Indeed,

\[
\text{Tr} A - d - \log \det A = \text{Tr} C^1 A^{1/2} B C^{1/2} - d - \log \det A^{1/2} B C^{1/2} = \sum \lambda_i - 1 - \log \lambda_i,
\]

where $\lambda_i$ are eigenvalues of $C^{1/2} B C^{1/2}$.

Consequently

\[
\int (V(x + te) - V(x)) (\delta_{te} \Phi)^p \, d\mu \geq \frac{K}{2} \int |\nabla \Phi(x + te) - \nabla \Phi(x)|^2 (\delta_{te} \Phi)^p \, d\mu
\]

\[
+ p \int \langle \nabla \Phi(x + te) - \nabla \Phi(x), (D^2 \Phi) \circ \nabla \Phi(x) \circ (\nabla \psi) \rangle (\delta_{te} \Phi)^p \circ (\nabla \psi) \, d\mu.
\]

Applying the same inequality to $-te$ and taking the sum we get

\[
\int (V(x + te) + V(x - te) - 2V(x)) (\delta_{te} \Phi)^p \, d\mu
\]

\[
\geq \frac{K}{2} \int |\nabla \Phi(x + te) - \nabla \Phi(x)|^2 (\delta_{te} \Phi)^p \, d\mu + \frac{K}{2} \int |\nabla \Phi(x - te) - \nabla \Phi(x)|^2 (\delta_{te} \Phi)^p \, d\mu
\]

\[
+ p \int \langle \nabla \delta_{te} \Phi, (D^2 \Phi)^{-1} \nabla \delta_{te} \Phi \rangle (\delta_{te} \Phi)^p \circ (\nabla \psi) \, d\mu.
\]
Note that the last term is non-negative. Dividing by $t^{2p}$ and passing to the limit we obtain
\[
\int V_e \Phi_e^p \, d\mu \geq K \int \|D^2 \Phi - e\|^2 \Phi_e \, d\mu + p \int (\langle D^2 \Phi \rangle^{-1} \nabla \Phi_e, \nabla \Phi_e) \Phi_e^p \, d\mu.
\] (7)

For the proof of the first part we note that
\[
\int V_e \Phi_e^p \, d\mu \geq K \int \Phi_e^{p+2} \, d\mu.
\]

Applying the Hölder inequality one gets
\[
\|V_e \Phi_e^p \|_{L^{(p+2)/2}(\mu)} \|\Phi_e^2 \|_{L^{(p+2)/p}(\mu)} \geq \int V_e \Phi_e^p \, d\mu.
\]

This readily implies the result.

To prove the second part we integrate by parts the left-hand side
\[
\int V_e \Phi_e^p \, d\mu = -p \int V_e \Phi_e \Phi_e^{p-1} \, d\mu + \int V_e^2 \Phi_e^p \, d\mu
\]

By the Cauchy inequality the latter does not exceed
\[
p \int (\langle D^2 \Phi \rangle^{-1} \nabla \Phi_e, \nabla \Phi_e) \Phi_e^{p-1} \, d\mu + \frac{B}{4} \int V_e (\langle D^2 \Phi e, e \rangle) \Phi_e^{p-1} \, d\mu + \int V_e^2 \Phi_e^p \, d\mu.
\]

Inequality (7) implies
\[
\frac{p + 4}{4} \int V_e^2 \Phi_e^p \, d\mu \geq K \int \|\nabla \Phi_e\|^2 \Phi_e \, d\mu \geq K \int \Phi_e^{p+2} \, d\mu.
\]

The rest of the proof is the same as in the first part.

\[\square\]

**Corollary 6.4.** In the limit $p \to \infty$ we obtain the contraction theorem of Caffarelli

\[K \|\Phi_e\|^2_{L^{\infty}(\mu)} \leq \|(V_e)\|_{L^{\infty}(\mu)} \cdot \|\cdot\|_{L^{\infty}(\mu)}.\]

A more difficult estimate for the operator norm $\|\cdot\|$ has been also obtained in [16].

**Theorem 6.5.** Assume that $D^2 W \geq K \cdot \text{Id}$. Then for every $r \geq 1$ one has

\[K \left( \int \|D^2 \Phi\|^{2r} \, d\mu \right)^{\frac{1}{r}} \leq \left( \int (\|D^2 V\|_{+} \|\cdot\|_{+}) \, d\mu \right)^{\frac{1}{r}}.
\]

7. **Contractions for infinite measures**

In this section we investigate contractions of infinite measures. Let us stress that unlike the probability case we don’t have a natural probabilistic normalization of the total volume.

We start with the following 1-dimensional example

**Example 7.1.** Let $d = 1$ and $\mu = \lambda_{\mathbb{R}^+}$, $\nu = I_{[0, +\infty)} \rho \, dx$ and $\rho \geq 1$. Then the standard monotone transportation $T$ is a contraction.

**Proof.** Indeed, this follows immediately from the explicit representation of $T$

\[\int_0^T \rho \, dx = x.\]

\[\square\]

Let us investigate what happens for $d = 2$ if the image measure is rotationally symmetric.

**Example 7.2. (F. Morgan)** Let $d = 2$ and $\mu = \lambda$, $\nu = \Psi(r) \, dx$. A natural transport mapping has the form

\[T(x) = \varphi(r) \cdot n, \quad n = \frac{x}{r} \]

Clearly

\[\nu(T(B_r)) = 2\pi \int_0^{\varphi(r)} s \Psi(s) \, dr = \pi r^2 = \mu(B_r).\]
Let us compute $\text{DT}$ in the frame $(n, v)$, where $v = \frac{(-x_2, x_1)}{r}$. One has
\[ \partial_n T = \phi' \cdot n \quad \partial_v T = \frac{\psi'}{r} \cdot v. \]

Clearly, a necessary and sufficient condition for $T$ to be a contraction is the following:
\[ \phi' \leq 1 \]
or $\phi' \geq 1$ for $\psi = \phi^{-1}$. From the change of variables formula we obtain
\[ \psi(r) = \sqrt{2 \int_0^r s \Psi(s) \, ds}. \]

Condition $\phi' \geq 1$ is equivalent to $\int_0^r s \Psi(s) \, ds \leq \frac{(r \Psi(r))^2}{2}$. The latter holds, for instance, if
\[ (s \Psi(s))' \geq 1. \]

Indeed, in this case
\[ \int_0^r s \Psi(s) \, ds \leq \int_0^r s \Psi(s)(s \Psi(s))' \, ds = \frac{(r \Psi(r))^2}{2}. \]

**Example 7.3.** Similarly in dimension $d$, a sufficient condition for the transportation mapping $T = \varphi(r) \frac{\partial}{\partial r}$ between $\lambda$ and $\Psi(r) \, dx$ to be a contraction is that
\[ (r \Psi^{\frac{1}{d-1}}(r))' \geq 1. \]

**Corollary 7.4.** In $d$-dimensional Euclidean space with density $\Psi(r)$ satisfying $(r \Psi^{\frac{1}{d-1}}(r))' \geq 1$, the Euclidean isoperimetric inequality holds.

Some example of contraction mappings arise naturally in differential geometry (see [20], Propositions 1.1 and 2.1).

**Proposition 7.5.** Let $M$ be the plane equipped with the metric
\[ dr^2 + g^2(r)r^2 d\theta^2 \]
(surface of revolution), $g \geq 1$. Then the identity mapping form $M$ to the Euclidean plane with measure $g \, dx$ is a volume preserving contraction.

In particular, $\cosh^2(r) \, dx$ is a Lipschitz image of $H^2$ (with metric $dr^2 + \cosh^2(r) d\theta^2$).

The following comparison result has been proved in [17]. It turns out that a natural model measure for the one-dimensional log-convex distributions has the following form:
\[ \nu_A = \frac{dx}{\cos A x}, \quad -\frac{\pi}{2 A} < x < \frac{\pi}{2 A}. \]

Its potential $V$ satisfies $V'' e^{-2V} = A^2$. Using a result [25] on symmetricity of the isoperimetric sets one can compute the isoperimetric profile of $\nu_A$:
\[ I_{\nu_A}(t) = e^{At/2} + e^{-At/2}. \]

**Proposition 7.6.** Let $\mu = e^W dx$ be a measure on $\mathbb{R}^1$ with even convex potential $W$. Assume that
\[ W'' e^{-2W} \geq A^2, \]
and $W(0) = 0$. Then $\mu$ is the image of $\nu_A$ under a $1$-Lipschitz increasing mapping.

**Proof.** Without loss of generality one can assume that $W$ is smooth and $W'' e^{-2W} > A^2$. Let $\varphi$ be a convex potential such that $T = \varphi'$ sends $\mu$ to $\nu_A$. In addition, we require that $T$ is antisymmetric. Clearly, $\varphi'$ satisfies
\[ e^W = \frac{\varphi''}{\cos A \varphi'}. \]

Assume that $x_0$ is a local maximum point for $\varphi''$. Then at this point
\[ \varphi'''(x_0) = 0, \quad \varphi''(x_0) \geq 0. \]

Differentiating the change of variables formula at $x_0$ twice we get
\[ W'' = \frac{\varphi''(4)}{\varphi''} + (\frac{\varphi''(3)}{\varphi''})^2 + \frac{A^2}{\cos^2 A \varphi'} (\varphi'')^2 + A \frac{\sin A \varphi'}{\cos A \varphi} \varphi''. \]
Consequently one has at $x_0$
\[ W'' \leq \frac{A^2}{\cos^2 A\varphi'}(\varphi'')^2 = A^2 e^{2W}. \]
But this contradicts to the main assumption.
Hence $\varphi''$ has no local maximum. Note that $\varphi$ is even. This implies that that $0$ is the global minimum of $\varphi''$. Hence $\varphi'' \geq \varphi''(0) = 1$. Clearly, $T^{-1}$ is the desired mapping. \qed

8. OTHER RESULTS AND APPLICATIONS

An immediate consequence of the contraction theorem is the Bakry-Ledoux comparison theorem, which is a probabilistic analog of the Lévy-Gromov comparison theorem for Ricci positive manifolds.

**Theorem 8.1.** Assume that $\mu = e^{-V}dx$, where $D^2V \geq 1d$, is a probability measure on $\mathbb{R}^d$. Then
\[ I_\mu \geq I_\gamma, \]
where $\gamma$ is the standard Gaussian measure.

In the same way the contraction theorem implies different functional and concentration inequalities for uniformly log-concave measures (log-Sobolev, Poincaré etc.).

The following unsolved problem is known as the "Gaussian correlation conjecture".

**Gaussian correlation conjecture.** Let $A$ and $B$ be symmetric convex sets and $\gamma$ be the standard Gaussian measure. Then
\[ \gamma(A \cap B) \geq \gamma(A) \gamma(B). \]  
(8)

The Gaussian correlation conjecture has quite a long history. This problem arose in 70th. The positive solution is known for two-dimensional sets and for the case when one of the sets is an ellipsoid. The ellipsoid case was proved by G. Hargé [10] by semigroup arguments.

**Theorem 8.2.** Let $B$ be an ellipsoid. Then (8) holds.

**Proof.** Applying a linear transformation of measures one can reduce the proof to the case when $B$ is a ball and $\gamma$ is a (non-standard) Gaussian measure. Consider the optimal transportation $T$ between $\gamma$ and $\gamma_A = \chi_{\{A\}}$. By Theorem 2.5 $T$ is a contraction and by the symmetry reasons $T(0) = 0$. Hence $T(B) \subset B$ and
\[ \frac{\gamma(A \cap B)}{\gamma(A)} = \gamma_A(B) = \gamma(T^{-1}(B)) \geq \gamma(B). \]
This completes the proof. \qed

The following beautiful observation [11] follows from the contraction theorem and properties of the Ornstein-Uhlenbeck semigroup
\[ P_t f(x) = \int f(x e^{-t} + \sqrt{1 - e^{-2t}} y) d\gamma(y). \]

**Theorem 8.3.** If $\gamma$ is a standard Gaussian measure, $g$ is symmetric convex and $f$ is symmetric log-concave, then
\[ \int f g d\gamma \leq \int f d\gamma \cdot \int g d\gamma. \]

**Proof.** Let $T(x) = x + \nabla \varphi(x)$ be the optimal transportation of $\gamma$ onto $\frac{\gamma_A}{\int f d\gamma}$. Thus we need to prove that
\[ \int g(x + \nabla \varphi(x)) d\gamma \leq \int g d\gamma. \]
Set:
\[ \psi(t) = \int g(x + P_t(\nabla \varphi(x))) d\gamma, \]
where $P_t = e^{tL}$ is the Ornstein-Uhlenbeck semigroup generated by $L = \Delta - \langle x, \nabla \rangle$. Note that
\[ \frac{\partial}{\partial t} \psi(t) = \frac{\partial}{\partial t} \int g(x + P_t(\nabla \varphi(x))) d\gamma = \int (\nabla g(x + P_t(\nabla \varphi(x))), L P_t(\nabla \varphi(x))) d\gamma. \]
Integrating by parts we get
\[ \frac{\partial}{\partial t} \psi(t) = -\int \text{Tr} \left[ D^2 g(x + P_t(\nabla \varphi(x))) \cdot (I + M) M \right] d\gamma, \]
\[ M = DP_t(\nabla \varphi(x)) = e^{-t/2}P_t(D^2 \varphi). \]

Clearly, by the contraction theorem \( I + M \geq 0 \) and \( M \leq 0 \). Hence \( \text{Tr} \left[ D^2 g \cdot (I + M)M \right] \leq 0 \) and \( \psi(t) \) is increasing. Note that \( P_{+\infty}(\nabla \varphi) = \int \nabla \varphi \, d\gamma = \left( \int \frac{x f \, d\gamma}{f} \right) = 0 \). Hence \( \int g(x + \nabla \varphi(x)) \, d\gamma \leq \psi(+\infty) = \int g \, d\gamma \). The proof is complete. \[ \square \]

Some other applications to correlation inequalities have been obtained in [8, 13]. A generalization of Theorem 8.2 to non-Gaussian measures have been obtained in [13] (see Corollary 4.1).

Other applications obtained in [9, 8, 10, 13] concern inequalities of the type
\[ \int \Gamma(x) \, d\mu \leq \int \Gamma(x) \, d\nu, \]
where \( \Gamma(x) \) is convex (moment inequalities etc.).

The following theorem was obtained in [7] with the help of the contraction theorem. In particular, it solves the so-called (B)-conjecture from the theory of Gaussian measures.

**Theorem 8.4.** Let \( K \) be a symmetric convex set and \( \gamma \) is a standard Gaussian measure. Then the function
\[ t \to \gamma(e^t K) \]
is log-concave.

In particular, \( \gamma(\sqrt{ab} K)^2 \geq \gamma(aK)\gamma(bK) \) for every \( a > 0, b > 0 \).

**Sketch of the proof.** Since \( \gamma(e^{t_1+t_2} K) = \gamma(e^{t_1}(e^{t_2} K)) \), it is sufficient to show that \( g(t) = \gamma(e^t K) \) is log-concave at zero. This is equivalent to the inequality \( g''(0)g(0) \leq (g'(0))^2 \). Computing the derivatives of \( g \) we get that this is equivalent to
\[ \int |x|^4 \, d\gamma_K - \left( \int |x|^2 \, d\gamma_K \right)^2 \leq 2 \int x^2 \, d\gamma_K, \]
where \( \gamma_K = \frac{1}{\gamma(K)} f_K \cdot \gamma \). Let us prove a more general relation: if \( \mu = e^{-W} \, dx \) is a log-concave measure with \( D^2 W \geq \text{Id} \) and \( f \) is a function, satisfying \( \int f \, d\mu = 0 \), \( \int \nabla f \, d\mu = 0 \), then the following Poincaré-type inequality holds:
\[ \int f^2 \, d\mu \leq \frac{1}{2} \int \| \nabla f \|^2 \, d\mu. \]

Applying (9) to \( f = ||x||^2 - \int |x|^2 \, d\mu \), we get the desired inequality for \( \mu \). Then it remains to approximate \( \gamma_K \) by measures of this type.

Note that by the Caffarelli’s theorem it is sufficient to prove inequality (9) only for the standard Gaussian measure. But in this case (9) is well-known and can be obtained from the expansion of \( f \) on the basis formed by the Hermite polynomials. The proof is complete.

Note that apart from the observations of the previous section nothing is known about contractions of manifolds.

The following result was obtained by S. I. Valdimarsson (see [26]). For every nonnegative symmetric \( M \) let us denote by \( \gamma_M \) the Gaussian measure with density
\[ \sqrt{\det M} e^{-\gamma(M, x)}. \]

**Theorem 8.5.** Let \( A, G \) and \( B \) be positive definite symmetric linear transformations, \( A < G \), \( GB = BG \), \( H \) is a convex function, and \( \mu_0 \) is a probability measure. The optimal transportation \( T = \nabla \Phi \) between probability measures
\[ \mu = \gamma_B^{-1/2} GB^{-1/2} \ast \mu_0 \text{ and } \nu = Ce^{-H} \cdot \gamma_B^{-1/2} A^{-1} B^{-1/2} \]
satisfies
\[ D^2 \Phi \leq G. \]

A particular form of the measure \( \mu \) allows Valdimarsson (after F. Barthe [2]) to obtain by the transportation arguments a new form of the well-known Brascamp-Lieb inequality. See [26] for details.

We finish with the following observation from [3].
Proposition 8.6. Let \( \mu = I_{[0, +\infty)}e^{-x} \) dx be the one-sided exponential measure and \( \nu = e^{x} \cdot \mu \) with \( |g'| \leq c \) for some \( c < 1 \). Then the monotone map \( T \) which transports \( \nu \) to \( \mu \) satisfies

\[
T'(x) \in [1 - c, 1 + c]
\]

for all \( x \in [0, \infty) \). The inverse map \( S = T^{-1} \) is a \( \frac{1}{1-c} \)-contraction.

The result follows from the explicit representation of \( T \) but can heuristically proved by the maximum principle arguments applied to \( S \). Indeed

\[
g(S) - S + \log S' = -x.
\]

If \( x_0 \) is the maximum point for \( S' \), one has \( S''(x_0) = 0 \). In addition,

\[
g'(S(x_0))S'(x_0) - S'(x_0) + \frac{S''(x_0)}{S'(x_0)} = -1.
\]

Clearly \( S'(x_0) = \frac{1}{1 - g'(S(x_0))} \leq \frac{1}{1-c} \).

Using this property one can give a transportation proof of the 1-dimensional Talagrand inequality for the exponential law (see [3], Proposition 6.6).

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