Chiral multicritical points driven by isospin density in the Ginzburg–Landau approach

Yuhei Iwata, Hiroaki Abuki and Katsuhiko Suzuki

Department of Physics, Tokyo University of Science, Tokyo 162-8601, Japan

Abstract. We study how a chiral tricritical point (TCP) on QCD phase diagram is affected by the imbalance of up and down quark densities (isospin density), using the generalized Ginzburg–Landau (GL) approach. The resulting phase diagram near TCP shows a rich fine structure which includes inhomogeneities of both the chiral and the charged pion condensations. It turns out that the TCP splits into multicritical points.

Keywords: QCD phase diagram, quark matter, critical point, inhomogeneous pion condensate

PACS: 12.38.Mh, 21.65.Qr

INTRODUCTION

Study of the phase structures of QCD at finite temperature and quark density is intriguing subject. In particular, it is suggested there exists a critical end point (CEP) in the 2-flavor QCD phase diagram [1], which leads to a tricritical point (TCP) in the chiral limit, although the location of the CEP is not well determined. According to recent work based on the Ginzburg–Landau (GL) approach [2, 3] and effective models [4] for the isospin symmetric 2-flavor quark matter, the TCP is shown to be replaced with the Lifshitz point, at which three different phases meet: the Wigner phase, the chiral symmetry broken phase with the spatially homogeneous chiral condensate (HCC), and the broken phase with the inhomogeneous chiral condensate (ICC).

How the phase structures near the TCP will change under realistic situations for the hadronic matter? We focus on the effect of the isospin asymmetry, which may be realized in the quark matter under the charge neutrality condition or in the realistic heavy-ion collisions. There are only a few model-based works on the TCP with the isospin asymmetry [5]. On the other hand, the charged pion condensed phase is known to show up as the true ground state of QCD at large isospin density [6]. We therefore try to present a systematic model-independent approach based on the generalized GL method to study effects of the finite isospin density on the TCP and its neighborhood.

In this report we focus on the phase structure in the vicinity of TCP for the massless $u$ and $d$ quark matter based on our recent work [7]. Taking into account isospin dependent contributions to the GL functional, we obtain new fine structures near TCP. As a result, the charged pion condensates extend over sizable domains in the GL parameter space. The finite isospin density also brings about a splitting of TCP into four independent critical points, which will be shown below.
SYMMETRY BREAKING DUE TO THE ISOSPIN DENSITY

In the vicinity of the chiral TCP, the local free energy for the chiral phase transition can be expressed in terms of the chiral 4-vector \( \phi = (\sigma, \pi^\alpha) \) as order parameters, where \( \sigma \) is proportional to the chiral condensate \( \langle \bar{q}q \rangle \) and \( \pi^\alpha \) the pion condensates \( \langle \bar{q}i\gamma^\delta \tau^i q \rangle \) \( (i = 1, 2, 3) \) with doublet quark fields \( q = (u, d) \). In order to provide a minimal description of TCP, we expand the GL potential up to the sixth order of the order parameters, and their spatial derivatives as follows,

\[
\Omega_{\text{GL}}[\sigma(x), \pi^\alpha(x)] = \frac{\alpha_2}{2} \phi^2 + \frac{\alpha_4}{4} (\phi^2)^2 + \frac{\alpha_{4b}}{4} (\nabla \phi)^2 + \frac{\alpha_6}{6} (\phi^2)^3 + \frac{\alpha_{6b}}{6} (\phi, \nabla \phi)^2
+ \frac{\alpha_{6c}}{6} [\phi^2 (\nabla \phi)^2 - (\phi, \nabla \phi)^2] + \frac{\alpha_{6d}}{6} (\Delta \phi)^2 + \delta \omega_c + \delta \omega_I, \tag{1}
\]

where \( \alpha_i, \beta_i \) are the GL parameters, \( \delta \omega_c \) stands for the contribution from current quark mass \( m_c \), and \( \delta \omega_I \) expresses the contribution from the finite isospin density. Eq. (1) except \( \delta \omega_c \) and \( \delta \omega_I \) is invariant under the chiral \( \text{SU}(2)_L \times \text{SU}(2)_R \) transformation. The explicit chiral symmetry breaking term, \( \delta \omega_c \), may be given by \( -h \sigma \) with a small constant \( h \). We set \( h = 0 \) throughout this report.

For the isospin symmetry breaking term, \( \delta \omega_I \), we consider the GL potential written with the charged pion condensates, \( \pi_c^\alpha = (\pi^1, \pi^2) \), which is given by

\[
\delta \omega_I = \frac{\beta_2}{2} (\pi_c^\alpha)^2 + \frac{\beta_4}{4} (\pi_c^\alpha)^4 + \frac{\beta_{4b}}{4} [\phi^2 - (\pi_c^\alpha)^2] (\pi_c^\alpha)^2 + \frac{\beta_{4e}}{4} (\nabla \pi_c^\alpha)^2. \tag{2}
\]

Here, we retain the \( \pi_c^4 \) contributions in addition to the leading \( \pi_c^2 \) terms, because we apply our GL analysis to the region where the order parameters become comparable with isospin density. Without loss of generality, we set \( \pi^1 = \pi, \pi^2 = \pi^3 = 0 \) from symmetry.

The GL potential Eq. (1), (2) contains parameters \( \alpha_i, \beta_i \), which are, in principle, the function of temperature and quark/isospin chemical potentials. In order to reduce the number of the independent parameters in Eq. (1), (2), we make use of the feedback of the quark loops to the potential energy, assuming that they are the dominant contributions. We systematically relate the GL parameters with the \( n \)-th quark loop integrals, which certainly depend on the \( u \) and \( d \) quark chemical potential, \( \mu_u \) and \( \mu_d \). Expanding the resulting integrals up to the 2nd order of the perturbative isospin chemical potential \( \mu_I \equiv (\mu_u - \mu_d)/2 \), we can show the GL parameters, \( \alpha(\mu_I) \), \( \beta(\mu_I) \) are proportional to \( \alpha(\mu_I = 0) \) or \( \mu_I^{-1} \alpha(\mu_I = 0) \) [7]. Hence, we finally obtain the potential:

\[
\Omega_{\text{GL}}(\mu_I) = \alpha_2^{(0)} \sigma^2/2 + (\alpha_2^{(0)} - \mu_I^2 \alpha_4^{(0)}/2) \pi^2/2
+ (\alpha_4^{(0)} + \mu_I^2 \alpha_6^{(0)}) [\sigma^4 + \sigma^2 \pi^2 + (\sigma')^2] /4 + \alpha_4^{(0)} [\sigma^4 + \sigma^2 \pi^2 + (\pi')^2] /4
+ \alpha_6^{(0)} [(\sigma^2 + \pi^2)^3 + 5(\sigma \sigma' + \pi \pi')^2 + 3(\sigma \pi' - \pi \sigma')^2 + [(\sigma'')^2 + (\pi'')^2] /2] /6, \tag{3}
\]

where \( \alpha^{(0)} \equiv \alpha(\mu_I = 0) \). We restrict the analysis to one-dimensional structures and accordingly a primed quantity in the above formula means its derivative with respect to \( z \)-direction, for instance, \( \sigma' = \partial_z \sigma \). We simply choose \( \alpha_6^{(0)} = 1 \) to set the energy
FIGURE 1. Phase structure near the TCP for \( \mu_I \neq 0, h = 0 \). The solid and dashed curves indicate boundaries of the first and second order phase transitions, respectively. (a) Phase diagram with the HCC and HPC: Asymptotic behaviors of the first order critical line separating the HCC phase from the HPC phase is analytically derived:\( \alpha_2 \to \frac{3}{2} \mu_I^2 \alpha_4 / 4 + \mu_I^4 / 24 + O(\mu_I^6) \) as \( \alpha_4 \to -\infty \), while \( \alpha_4 \to 0 \). (b) Phase diagram with the ICC and IPC in addition to homogeneous condensates: The second order critical line connecting TCP' to Q is characterized by \( \alpha_2 = \frac{3}{8}(\alpha_4 + \mu_I^2)^2 / \alpha_4 \). The second order critical line linking TCP' to R is given by \( \alpha_2 = \frac{5}{36}(\alpha_4 + \mu_I^2)^2 / \alpha_4 \). The first order critical line between the ICC and IPC phases (a curve QR) is obtained by a numerical computation.

TABLE 1. The location of chiral multicritical points; TCP' and P in Fig. 1(a), Q, R in Fig. 1(b).

| \( \alpha_2 \) | \( \alpha_4 \) | type |
|---|---|---|
| TCP' | 0 | \( -\mu_I^2 \) | Lifshitz tricritical point |
| P | 0 | 0 | bicritical point |
| Q | \( \frac{3}{32} \mu_I^4 \) | \( -\frac{3}{2} \mu_I^2 \) | Lifshitz bicritical point |
| R | 0.21 \( \mu_I^4 \) | \( -2.22 \mu_I^2 \) | critical (end) point |

scale. By virtue of the appropriate scaling of the GL parameters \[3\], the potential Eq. (3) is characterized by two independent parameters, \( \{ \alpha_2 / \alpha_4, \mu_I^2 / |\alpha_4| \} \). Hence, we shall describe the phase diagram in the \( (\alpha_2^{(0)}, \alpha_4^{(0)}) \) parameter space with the given \( \mu_I \). Hereafter, we suppress the subscript \( (0) \) for \( \alpha_s \) to avoid notational confusion.

POSSIBLE PHASE STRUCTURE AT FINITE ISOSPIN DENSITY

Let us first assume the spatially constant distribution for the chiral and charged pion condensates. We show the phase diagram near the TCP in Fig. 1 (a), and locations of multicritical points on the GL parameter space in TABLE 1. In this case, the ground state favors either HCC phase or homogeneous pion condensed (HPC) phase. There is no room for the coexistence phase of \( \pi \) and \( \sigma \). Compared with the result for the \( \mu_I = 0 \) matter, large domains of the phase diagram is occupied by the HPC phase. This drastic change is due to the presence of \( (\alpha_2 - \mu_I^2 \alpha_4 / 2) \pi^2 / 2 \) term in the GL potential Eq. (3),
which favors the HPC phase for arbitrary small $\mu_I$ with $\alpha_4 > 0$. In addition, while the TCP is located at the origin of the phase diagram for the $\mu_I = 0$ matter, the finite $\mu_I$ brings about a shift of the location of TCP to $(0, -\mu_2^2)$, newly labeled by TCP'. Origin of this shift is easily understood by looking at the coefficient of the $\sigma^4$ term in Eq.(3), $\alpha_4^{(0)} + \mu_2^2 \alpha_6$, ($\alpha_6 = 1$). Our result is consistent with the model-based calculations [5].

Next, we consider possible inhomogeneous structures for the chiral and charged pion condensates with the finite isospin chemical potential. Following the previous work, we consider the one-dimensional solitonic structure expressed by the Jacobi’s elliptic function “sn”, for the inhomogeneous phases of $\sigma$ and $\pi$ condensates [3]. In fact, the Jacobi’s elliptic function is shown to be a solution of the Euler-Lagrange equation for the GL potential in the chiral limit. We assume $\sigma(x) = \sqrt{\nu} k \text{sn}(kz; \nu)$, $\pi = 0$ for the ICC phase, and $\sigma = 0$, $\pi(x) = \sqrt{\nu} k \text{sn}(kz; \nu)$ for inhomogeneous charged pion condensed (IPC) phase, where the positive parameters $\nu$ and $k$ are determined by the variational principle in each phase. The resulting phase diagram is depicted in Fig. 1 (b).

The TCP is now replaced with the Lifshitz tricritical point. Compared with isospin symmetric matter, a large part of the ICC phase is taken over by the IPC. This is because the spatial derivative term of $(\pi')^2$ gets lower energy compared with the $(\sigma')^2$ contribution as clearly seen in Eq. (3). The boundary between ICC and IPC also provides two new critical points, Q and R, listed in Table 1.

For completeness, we also consider possible $\sigma$-$\pi$ coexistence phases. We take the ansatz of chiral kink spiral: $\sigma(x) = m \text{cn}(kz; \nu)$, $\pi(x) = m \text{sn}(kz; \nu)$ with $m$ a new variational parameter. This includes the standard chiral spiral $\sigma(x) = m \cos(kz)$, $\pi(x) = m \sin(kz)$ [8] in a particular limit $\nu \to 0$. We always find either IPC or ICC being favored, indicating that the $\sigma$-$\pi$ coexistence state may be ruled out near TCP.

To summarize, we studied the phase structures in the proximity of TCP at finite $\mu_I$, using the model-independent GL approach. The sizable domain of the phase diagram was found to be occupied by the charged pion condensation. We found not only the shift of the chiral TCP, but also the appearance of the new critical points. The new fine structure obtained in the GL parameter space may be mapped onto the low temperature and intermediate density of the $(T, \mu)$ QCD phase diagram, as suggested in the previous work [4]. The inhomogeneous charged pion condensate of the quark matter may smoothly continue to the classical pion condensate discussed in the nuclear matter [9].

REFERENCES

1. M. Asakawa and K. Yazaki, Nucl. Phys. A 504, 668 (1989).
2. D. Nickel, Phys. Rev. Lett. 103, 072301 (2009) [arXiv:0902.1778 [hep-ph]].
3. H. Abuki, D. Ishibashi and K. Suzuki, Phys. Rev. D 85, 074002 (2012) [arXiV:1109.1615 [hep-ph]].
4. D. Nickel, Phys. Rev. D 80, 074025 (2009) [arXiv:0906.5295 [hep-ph]].
5. D. Toublan and J. B. Kogut, Phys. Lett. B 564, 212 (2003) [hep-ph/0301183]; A. Barducci, R. Casalbuoni, G. Pettini and L. Ravagli, Phys. Rev. D 69, 096004 (2004) [hep-ph/0402104].
6. D. T. Son and M. A. Stephanov, Phys. Rev. Lett. 86, 592 (2001) [hep-ph/0005225].
7. Y. Iwata, H. Abuki and K. Suzuki, arXiv:1206.2870 [hep-ph].
8. E. Nakano and T. Tatsumi, Phys. Rev. D 71, 114006 (2005) [hep-ph/0411350].
9. A.B. Migdal, Sov. Phys. JETP 36, 1052 (1973); R.F. Sawyer, Phys. Rev. Lett. 29, 382 (1972); D.J. Scalapino, Phys. Rev. Lett. 29, 386 (1972); T. Takatsuka, K. Tamiya, T. Tatsumi and R. Tamba-gaki, Prog. Theor. Phys. 59, 1933 (1978).