Detecting Topology in a Nearly Flat Hyperbolic Universe

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Cosmic microwave background data shows the observable universe to be nearly flat, but leaves open the question of whether it is simply or multiply connected. Several authors have investigated whether the topology of a multiply connect hyperbolic universe would be detectable when $0.9 < \Omega < 1$. However, the possibility of detecting a given topology varies depending on the location of the observer within the space. Recent studies have assumed the observer sits at a favorable location. The present paper extends that work to consider observers at all points in the space, and (for given values of $\Omega_m$ and $\Omega_\Lambda$ and a given topology) computes the probability that a randomly placed observer could detect the topology. The computations show that when $\Omega = 0.98$ a randomly placed observer has a reasonable chance ($\sim 50\%$) of detecting a hyperbolic topology, but when $\Omega = 0.99$ the chances are low ($< 10\%$) and decrease still further as $\Omega$ approaches one.

Keywords: hyperbolic; topology; injectivity radius; injectivity profile.

1. Introduction

Analysis of recent cosmic microwave background (CMB) data suggests an approximately flat universe with the total energy density parameter $\Omega$ almost surely lying in the range $0.9 < \Omega < 1$. The near flatness of the observable universe does not preclude a multiconnected spatial topology, but may push the topology to a scale larger than the horizon radius, making it difficult or impossible to detect. Recent studies have examined the extent to which a nontrivial topology may or may not be observable in a locally spherical universe or a locally hyperbolic one. In a locally flat universe there is no a priori relationship between the topology scale and the horizon scale, so a great deal of luck would be required for the two to coincide.

In their most recent study of multiconnected hyperbolic universes, Gomero, Rebouças, and Tavakol examine the seven smallest known hyperbolic topologies and find that for a set of cosmological parameters given by Bond et al. with $\Omega = 0.99$, five of the seven topologies would be potentially detectable using CMB methods, while for a set of parameters given by Jaffe et al. with $\Omega = 0.98$, all seven would be potentially detectable.

A topology is considered potentially detectable by an observer at a point $p$ in the
space if the observer’s horizon radius $r_{\text{hor}}$ exceeds the injectivity radius $r_{\text{inj}}(p)$ at $p$. In a 3-torus and in many spherical topologies, the injectivity radius $r_{\text{inj}}$ is constant throughout the whole space, so if the topology is potentially detectable by an observer at point $p$, it is detectable by any other observer at any other point $q$ in the same space. In hyperbolic topologies, by contrast, the injectivity radius varies from point to point, so a hyperbolic topology might be detectable by an observer at point $p$ (where the injectivity radius $r_{\text{inj}}(p)$ is small), but undetectable by a different observer at some other point $q$ (where the injectivity radius $r_{\text{inj}}(q)$ is large).

In their most recent work, Gomero, Rebouças, and Tavakol consider the question of detectability from the most favorable point in the space, that is, from a point of minimal injectivity radius. The present article extends their work by considering the detectability of the topology at arbitrary locations in the space. The variation of the injectivity radius across the space will be summarized in an injectivity profile showing what fraction of the space’s volume has a given injectivity radius. Combining the injectivity profile (Section 2) with an estimate for the horizon radius (Section 3) reveals the probability that a randomly placed observer could potentially detect the topology, that is, it tells the fraction of the manifold’s volume in which $r_{\text{hor}} > r_{\text{inj}}$ (Section 4).

2. Injectivity Profiles

2.1. Definition

An injectivity profile is a histogram showing how much of a manifold’s volume has a given injectivity radius. For example, in the simple histogram in Fig. 1 the first bar shows that the injectivity radius lies in the range $[0.2, 0.3]$ for 10% of the manifold’s volume; the second bar shows that the injectivity radius lies in the range $[0.3, 0.4]$ for 20% of the manifold’s volume; and so on. In the studies presented below, each histogram will have 240 bins of width 0.0025, spanning the range of injectivity radii from 0 to 0.6. In the limit, as the bin width goes to zero, the assignment of a finite volume percentage $\Delta V/V$ to each bin of finite width $\Delta r$ is replaced by a limiting density distribution $(dV/V)/dr$ as shown in Fig. 2. The density distribution’s discontinuities reflect a preferred set of short closed geodesics and may be the subject of a future paper, but nevertheless hold no importance for the present study.

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4One may substitute the last scattering surface at $z \simeq 1100$ for the absolute horizon at $z = \infty$ with little effect on the horizon radius.
5Given a point $p$ in a multiconnected space, the injectivity radius at $p$ is defined to be the radius of the largest ball centered at $p$ whose interior does not overlap itself. Equivalently, twice the injectivity radius is the length of the shortest topologically nontrivial path from $p$ to itself.
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Fig. 1. An injectivity profile shows how much of a manifold’s volume has a given injectivity radius.

Fig. 2. As the bin width goes to zero, the histogram becomes a density distribution.

2.2. Computational Overview

To construct an injectivity profile, begin with a fundamental domain for the manifold. The present study used a Dirichlet domain computed by the computer program SnapPea. SnapPea presents the Dirichlet domain in the Klein model of hyperbolic space, where it looks just like an ordinary Euclidean polyhedron. Cover the Klein model with a grid of points. The grid is rectangular relative to the Euclidean geometry of the model, although of course it’s nonrectangular relative to the intrinsic hyperbolic geometry. The present study used a $200 \times 200 \times 200$ grid. The program scanned the grid, first rejecting grid points lying on or beyond the sphere-at-infinity in the Klein model, and then rejecting points lying outside the Dirichlet domain. For each grid point lying within the Dirichlet domain, the program computed the injectivity radius at that point (Subsection 2.3) to determine the correct bin in the histogram, then computed the true hyperbolic volume of the surrounding grid cell (Subsection 2.4) and added that volume to the correct bin. After all grid points were processed, the program printed the volume in each bin to a file.

2.3. Computing the Injectivity Radius

At first glance, computing the injectivity radius at a point $p$ is trivially easy: just apply elements $\gamma$ of the holonomy group $\Gamma$ to $p$ and see what the minimum trans-
lation distance $dist(p, \gamma(p))$ is. The injectivity radius will be half that minimum: $r_{\text{inj}}(p) = \min_{\gamma \in \Gamma} \frac{dist(p, \gamma(p))}{2}$. The only problem is that $\Gamma$ is an infinite group, so we must decide ahead of time how many elements—and which elements—of $\Gamma$ to apply.

Our selection criterion for choosing a finite subset of $\Gamma$ depends on the concept of an isometry $\gamma$’s basepoint translation distance, defined to be the distance $dist(O, \gamma(O))$ that $\gamma$ translates the origin $O$. Say we want to find all $\gamma \in \Gamma$ that translate a given point $p$ a distance at most $\ell$. Any such isometry $\gamma$ has basepoint translation distance $dist(O, \gamma(O)) \leq dist(O, p) + dist(p, \gamma(p)) + dist(\gamma(p), \gamma(O)) \leq D_{out} + \ell + D_{out}$, where $D_{out}$ is the outradius of the Dirichlet domain $D$. In other words, if we consider all isometries $\gamma$ with basepoint translation distance at most $\ell + 2D_{out}$, we are sure to find all translates of $p$ closer than the distance $\ell$.

How large must $\ell$ be? A simple argument shows that every point $p$ has a translate $\gamma(p)$ lying at a distance less than $2D_{in} + 2D_{out}$. So if we were to choose $\ell$ in the preceding paragraph to be $2D_{in} + 2D_{out}$, we would be guaranteed to find the nearest translate of every point $p$, and would therefore know the injectivity radius $r_{\text{inj}}(p)$. This method, while rigorous, is inefficient: the number of isometries $\gamma$ with basepoint translation distance at most $\ell$ grows exponentially in $\ell$, and the majority of those isometries are unneeded. A more efficient algorithm is simply to guess a plausible value of $\ell$, find all isometries $\gamma$ with basepoint translation distance at most $\ell + 2D_{out}$, and verify afterwards that for each point $p$ we found a translate $\gamma(p)$ at a distance $dist(p, \gamma(p))$ less than $\ell$. There is no guarantee that such a translate will always be found, but if one is found then we rigorously know the minimum value of $dist(p, \gamma(p))$ and hence the injectivity radius $r_{\text{inj}}(p) = (dist(p, \gamma(p))/2$. In the present study a value of $\ell = 1.3$ worked in all cases.

Given a Dirichlet domain $D$ and its face pairings, how do we find all isometries $\gamma \in \Gamma$ with basepoint translation distance $dist(O, \gamma(O))$ at most $\ell$? A simple recursion does the job: start with the the original Dirichlet domain $D$; add its immediate neighbors in the universal covering space, which we think of as the translates $\gamma(D)$ for each of $D$’s face pairing isometries $\gamma$; then add the neighbors’ neighbors, which correspond to products $\gamma_2 \gamma_1(D)$ of pairs of face pairing isometries; and so on. Continue recursively, keeping those images $\gamma(D)$ with $dist(O, \gamma(O)) \leq \ell$ and discarding those with $dist(O, \gamma(O)) > \ell$. Proposition 3.1 of shows that this algorithm finds all isometries with $dist(O, \gamma(O)) \leq \ell$; that is, we needn’t worry that the recursion will terminate when it encounters only unwanted translates with $dist(O, \gamma(O)) > \ell$.

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4 The holonomy group $\Gamma$ is also known as the group of covering transformations. We follow the convention of saying holonomy group when thinking of $\Gamma$ geometrically as a group of isometries, but saying group of covering transformations when thinking of $\Gamma$ topologically as a group of homeomorphisms.

4 Let $\gamma$ be the face pairing isometry taking one of the Dirichlet domain’s faces closest to the origin $O$ to its mate. Because the chosen face is maximally close to the origin, $dist(O, \gamma(O)) = 2D_{in}$, where $D_{in}$ is the inradius of the Dirichlet domain $D$. Thus for any point $p$ in the Dirichlet domain, $dist(p, \gamma(p)) \leq dist(p, O) + dist(O, \gamma(O)) + dist(\gamma(O), \gamma(p)) \leq D_{out} + 2D_{in} + D_{out}$.
4. Computing the Volume of a Grid Cell

Inscribe the Klein model of hyperbolic space in a cube with corners at \((\pm 1, \pm 1, \pm 1)\), and cover the cube with a 200 \times 200 \times 200 grid. Each grid cell is then a small cube of side length \(10^{-2}\) and Euclidean volume \(10^{-6}\). The hyperbolic volume represented by each grid cell differs from the apparent Euclidean volume, and varies from cell to cell.

It’s easy to compute the ratio of the true hyperbolic volume to the apparent Euclidean volume. The grid cells are small enough that the volume ratio doesn’t vary much within a given cell, so pick any point \((x, y, z)\) within the cell, and let \(r = \sqrt{x^2 + y^2 + z^2}\) be its distance from the origin. By symmetry the volume ratio depends only on \(r\), not on the particular point \((x, y, z)\). So take a spherical shell of radius \(r\) and thickness \(dr\), and compare its true hyperbolic volume to its apparent Euclidean volume. The apparent Euclidean volume is

\[
V_E = 4\pi r^2 dr.
\]

(1)

To compute the shell’s true hyperbolic volume, project from the Klein model onto the hyperboloid model of hyperbolic space (Fig. 3). Similar triangles show that the hyperbolic radius \(\rho\) has \(\cosh \rho = 1/\sqrt{1 - r^2}\) and \(\sinh \rho = r/\sqrt{1 - r^2}\). Thus the area of the sphere is \(4\pi \sinh^2 \rho = 4\pi r^2 / (1 - r^2)\). To find its thickness, note that \(\tanh \rho = r\) so \(d\rho = \cosh^2 \rho \, dr = dr / (1 - r^2)\). Hence the hyperbolic volume of the spherical shell is

\[
V_H = \frac{4\pi r^2 dr}{(1 - r^2)^2}
\]

(2)

and the required ratio is

\[
\frac{V_H}{V_E} = \frac{4\pi r^2 dr / (1 - r^2)^2}{4\pi r^2 dr} = \frac{1}{(1 - r^2)^2}.
\]

(3)
3. Horizon Radius

The variable $r_{\text{hor}}$ represents the horizon radius in comoving coordinates, that is, in units of the curvature radius of the ambient hyperbolic space. In a standard Friedmann-Lemaître model with matter density today $\Omega_m$ and cosmological constant $\Omega_\Lambda$, one may compute $r_{\text{hor}}$ by integrating the path of a photon backwards from $z = 0$ to $z = \infty$, obtaining

$$r_{\text{hor}} = \sqrt{|\Omega_m + \Omega_\Lambda - 1|} \int_0^\infty \frac{dz}{\sqrt{\Omega_\Lambda + (1 - \Omega_m - \Omega_\Lambda)(z + 1)^2 + \Omega_m(z + 1)^3}}$$  \hspace{1cm} (4)

Replacing the upper limit of $z = \infty$ with the redshift $z = 1100$ of last scatter would have little effect on the result.

For comparison with Gomero et al., the remainder of the present article will consider two sets of plausible choices for the density parameters:

- $\Omega_m = 0.37$ and $\Omega_\Lambda = 0.61$, for which $\Omega_{\text{total}} = 0.98$ and $r_{\text{hor}} = 0.43$, and
- $\Omega_m = 0.37$ and $\Omega_\Lambda = 0.62$, for which $\Omega_{\text{total}} = 0.99$ and $r_{\text{hor}} = 0.30$.

4. Results

A computer program, written in the C programming language and freely available from [ftp://ftp.northnet.org/weeks/NearlyFlatHyperbolic](ftp://ftp.northnet.org/weeks/NearlyFlatHyperbolic), computes injectivity profiles using the algorithm outlined in Subsection 2.2 along with the details explained in Subsections 2.3 and 2.4. For the first ten low-volume hyperbolic manifolds from the Hodgson-Weeks census [11], which include the seven manifolds considered by Gomero et al., the program obtains the injectivity profiles shown in Fig. 4. Shaded backgrounds mark the cutoffs $r_{\text{hor}} = 0.43$ and $r_{\text{hor}} = 0.30$, corresponding to $\Omega_{\text{total}} = 0.98$ and $\Omega_{\text{total}} = 0.99$, respectively. Only in the fraction of the manifold to left of each cutoff would the topology be detectable. Table 1 summarizes the results.

| manifold | $\Omega_{\text{total}} = 0.98$ | $\Omega_{\text{total}} = 0.99$ |
|----------|-----------------|-----------------|
| 1  m003(-3, 1) | 78% | 4% |
| 2  m003(-2, 3) | 55% | 2% |
| 3  m007(3, 1) | 12% | 0% |
| 4  m003(-4, 3) | 69% | 4% |
| 5  m004(6, 1) | 70% | 16% |
| 6  m004(1, 2) | 32% | 5% |
| 7  m009(4, 1) | 7% | 0% |
| 8  m003(-3, 4) | 25% | 9% |
| 9  m003(-4, 1) | 41% | 8% |
| 10 m004(3, 2) | 39% | 5% |
Fig. 4. Injectivity profiles for the manifolds of Table 1.
The manifolds in Table 1 contain no closed geodesics shorter than 0.36. However, there exist families of hyperbolic 3-manifolds containing arbitrarily short closed geodesics. The manifolds within each family approach a limiting cusped manifold which is, in effect, a manifold with a geodesic of length zero. Furthermore, the injectivity profiles of the manifolds in each family approach the injectivity profile of the limiting cusped manifold. Fig. 5 shows the injectivity profiles for the two smallest limiting cusped manifolds, \( m_{003} \) and \( m_{004} \), while Table 2 shows the fraction of each in which an observer could potentially detect the topology. Note that for \( \Omega_{\text{total}} \) close to one, the fraction of the manifold in which the topology is detectable is roughly proportional to \( 1 - \Omega_{\text{total}} \). The Dirichlet domains for these cusped manifolds have infinite outradius, so in the algorithm of Subsection 2.3 we considered all isometries of basepoint translation distance less than the plausible but arbitrary value of 2.5; thus these results for cusped manifolds are not rigorous, but they are most likely correct nonetheless. The beginning of each profile is noisy because at most a few grid points fall into each bin; refining the grid diminishes this effect at the expense of a slower computation.

![Fig. 5. Injectivity profiles for the two smallest limiting cusped manifolds.](image)

Table 2. The fraction of each cusped manifold’s volume in which its topology is potentially detectable when \( \Omega_{\text{total}} \) is close to one.

| manifold | \( \Omega_{\text{total}} = 0.98 \) | \( \Omega_{\text{total}} = 0.99 \) | \( \Omega_{\text{total}} = 0.995 \) | \( \Omega_{\text{total}} = 0.9975 \) |
|----------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| \( m_{003} \) | 17% | 8% | 4% | 2% |
| \( m_{004} \) | 67% | 32% | 16% | 8% |

*The cusped manifolds \( m_{000}, m_{001}, \) and \( m_{002} \) are nonorientable and not the limit of any sequence of closed manifolds.*
5. Conclusions

In the case $\Omega_{\text{total}} = 0.98$, the horizon radius $r_{\text{hor}} = 0.43$ is large enough that an observer at a random location in a small hyperbolic universe would have a reasonable chance of detecting the topology. However, in the case $\Omega_{\text{total}} = 0.99$, the horizon radius $r_{\text{hor}}$ drops to 0.30 and a random observer would have little or no chance of detecting the topology.

There exist closed hyperbolic 3-manifolds with arbitrarily short closed geodesics, so no matter how close $\Omega_{\text{total}}$ is to one, there will always be infinitely many hyperbolic manifolds in which well-placed observers could detect the topology. However, the probability that an observer could detect the topology from a random point in such a manifold decreases in proportion to $1 - \Omega_{\text{total}}$.

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