Algebras generated by reciprocals of linear forms

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Abstract
Let $\Delta$ be a finite set of nonzero linear forms in several variables with coefficients in a field $K$ of characteristic zero. Consider the $K$-algebra $C(\Delta)$ of rational functions generated by $\{1/\alpha \mid \alpha \in \Delta\}$. Then the ring $\partial(V)$ of differential operators with constant coefficients naturally acts on $C(\Delta)$. We study the graded $\partial(V)$-module structure of $C(\Delta)$. We especially find standard systems of minimal generators and a combinatorial formula for the Poincaré series of $C(\Delta)$. Our proofs are based on a theorem by Brion-Vergne [BrV] and results by Orlik-Terao [OrT2].

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1 Introduction and main results

Let $V$ be a vector space of dimension $\ell$ over a field $K$ of characteristic zero. Let $\Delta$ be a finite subset of the dual space $V^*$ of $V$. We assume that $\Delta$ does not contain the zero vector and that no two vectors are proportional throughout this paper. Let $S = S(V^*)$ be the symmetric algebra of $V^*$. It is regarded as the algebra of polynomial functions on $V$. Let $S(0)$ be the field of quotients of $S$, which is the field of rational functions on $V$.

**Definition 1.1.** Let $C(\Delta)$ be the $K$-subalgebra of $S(0)$ generated by the set

$$\left\{ \frac{1}{\alpha} \mid \alpha \in \Delta \right\}.$$

Regard $C(\Delta)$ as a graded $K$-algebra with $\deg(1/\alpha) = 1$ for $\alpha \in \Delta$.

**Definition 1.2.** Let $\partial(V)$ be the $K$-algebra of differential operators with constant coefficients. Agree that the constant multiplications are in $\partial(V)$: $K \subset \partial(V)$.

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If \( x_1, \ldots, x_\ell \) are a basis for \( V^* \), then \( \partial(V) \) is isomorphic to the polynomial algebra \( K[\partial/\partial x_1, \ldots, \partial/\partial x_\ell] \). Regard \( \partial(V) \) as a graded \( K \)-algebra with \( \deg(\partial/\partial x_i) = 1 \) (\( 1 \leq i \leq \ell \)). It naturally acts on \( S_{[0]} \). We regard \( C(\Delta) \) as a graded \( \partial(V) \)-module. In this paper we study the \( \partial(V) \)-module structure of \( C(\Delta) \) (Theorem 1.3). In order to present our results we need several definitions. Let \( E_p(\Delta) \) be the set of all \( p \)-tuples composed of elements of \( \Delta \). Let \( E(\Delta) := \bigcup_{p \geq 0} E_p(\Delta) \). The union is disjoint. Write \( \prod \mathcal{E} := \alpha_1 \ldots \alpha_p \in S \) when \( \mathcal{E} = (\alpha_1, \ldots, \alpha_p) \in E_p(\Delta) \). Then one can write

\[
C(\Delta) = \sum_{\mathcal{E} \in E(\Delta)} K \left( \prod \mathcal{E} \right)^{-1}.
\]

Let

\[
E^i(\Delta) = \{ \mathcal{E} \in E(\Delta) \mid \mathcal{E} \text{ is linearly independent} \},
\]

\[
E^d(\Delta) = \{ \mathcal{E} \in E(\Delta) \mid \mathcal{E} \text{ is linearly dependent} \}.
\]

Note that \( \mathcal{E} \in E^d(\Delta) \) if \( \mathcal{E} \) contains a repetition. In a special lecture at the Japan Mathematical Society in 1992, K. Aomoto suggested the study of the finite-dimensional graded \( K \)-vector space

\[
AO(\Delta) := \sum_{\mathcal{E} \in E^i(\Delta)} K \left( \prod \mathcal{E} \right)^{-1}.
\]

Let

\[
\mathcal{A}(\Delta) = \{ \ker(\alpha) \mid \alpha \in \Delta \}.
\]

Then \( \mathcal{A}(\Delta) \) is a (central) arrangement of hyperplanes in \( V \). K. Aomoto conjectured, when \( K = \mathbb{R} \), that the dimension of \( AO(\Delta) \) is equal to the number of connected components of

\[
M(\mathcal{A}(\Delta)) := V \setminus \bigcup_{H \in \mathcal{A}(\Delta)} H.
\]

This conjecture was verified in [OrT2], where explicit \( K \)-bases for \( AO(\Delta) \) were constructed. This paper can be considered as a sequel to [OrT2]. (It should be remarked that constructions in [OrT2] were generalized for oriented matroids by R. Cordovil [Co].) We will prove the following

**Theorem 1.3.** Let \( \mathcal{B} \) be a \( K \)-basis for \( AO(\Delta) \). Let \( \partial(V)_+ \) denote the maximal ideal of \( \partial(V) \) generated by the homogeneous elements of degree one. Then

1. the set \( \mathcal{B} \) is a system of minimal generators for the \( \partial(V) \)-module \( C(\Delta) \),
2. \( C(\Delta) = \partial(V)_+ C(\Delta) \oplus AO(\Delta) \), and
3. \( \partial(V)_+ C(\Delta) = \sum_{\mathcal{E} \in E^i(\Delta)} K \left( \prod \mathcal{E} \right)^{-1} \). In particular, \( \partial(V)_+ C(\Delta) \) is an ideal of \( C(\Delta) \).
Let \( \text{Poin}(A(\Delta), t) \) be the Poincaré polynomial \([\text{OrT}1, \text{Definition 2.48}]\) of \( A(\Delta) \). (It is defined combinatorially and is known to be equal to the the Poincaré polynomial of \( M(A(\Delta)) \) when \( K = \mathbb{C} \) \([\text{OrS}, \text{OrT}1, \text{Theorem 5.93}]\).) Then we have

**Theorem 1.4.** The Poincaré series \( \text{Poin}(C(\Delta), t) \) of the graded module \( C(\Delta) \) is equal to \( \text{Poin}(A(\Delta), (1 - t)^{-1}) \).

In order to prove these theorems we essentially use a theorem by M. Brion and M. Vergne \([\text{BrV}, \text{Theorem 1}]\) and results from \([\text{OrT}2]\). By Theorem 1.4 and the factorization theorem (Theorem 2.4) in \([\text{Ter}1]\), we may easily show the following two corollaries:

**Corollary 1.5.** If \( A(\Delta) \) is a free arrangement with exponents \((d_1, \ldots, d_\ell)\) \([\text{Ter}1, \text{Definitions 4.15, 4.25}]\), then

\[
\text{Poin}(C(\Delta), t) = (1 - t)^{-(\ell + 1)} \prod_{i=1}^\ell (1 + (d_i - 1)t).
\]

**Example 1.6.** Let \( x_1, \ldots, x_\ell \) be a basis for \( V^* \). Let \( \Delta = \{x_i - x_j \mid 1 \leq i < j \leq \ell\} \). Then \( A(\Delta) \) is known to be a free arrangement with exponents \((0, 1, \ldots, \ell - 1)\) \([\text{OrT}1, \text{Example 4.32}]\). So, by Corollary 1.5, we have

\[
\text{Poin}(C(\Delta), t) = (1 - t)^{-(\ell + 1)} (1 + t)(1 + 2t)\ldots(1 + (\ell - 2)t).
\]

For example, when \( \ell = 3 \), we have

\[
\text{Poin}\left( K \left[ \frac{1}{x_1 - x_2}, \frac{1}{x_2 - x_3}, \frac{1}{x_1 - x_3} \right], t \right) = (1 + t)/(1 - t)^2 = 1 + 3t + 5t^2 + 7t^3 + 9t^4 + \ldots,
\]

which can be easily checked by direct computation.

When \( A(\Delta) \) is the set of reflecting hyperplanes of any (real or complex) reflection group, Corollary 1.5 can be applied because \( A(\Delta) \) is known to be a free arrangement \([\text{Sai1}, \text{Ter}2]\).

**Corollary 1.7.** If \( A(\Delta) \) is generic (i.e., \( |\Delta| \geq \ell \) and any \( \ell \) vectors in \( \Delta \) are linearly independent), then

\[
\text{Poin}(C(\Delta), t) = (1 - t)^{-(\ell + 1)} \sum_{i=0}^{\ell-1} \binom{|\Delta| - \ell + i - 1}{i} t^i.
\]

### 2 Proofs

In this section we prove Theorems 1.3 and 1.4. For \( \varepsilon \in E(\Delta) \), let \( V(\varepsilon) \) denote the set of common zeros of \( \varepsilon : \ V(\varepsilon) = \bigcap_{i=1}^p \ker(\alpha_i) \) when \( \varepsilon = (\alpha_1, \ldots, \alpha_p) \). Define

\[
L = L(\Delta) = \{ V(\varepsilon) \mid \varepsilon \in E(\Delta) \}.
\]
Agree that $V(\varepsilon) = V$ if $\varepsilon$ is the empty tuple. Introduce a partial order $\leq$ into $L$ by reverse inclusion: $X \leq Y \iff X \supseteq Y$. Then $L$ is equal to the intersection lattice of the arrangement $A(\Delta)$ [OrT1, Definition 2.1]. For $X \in L$, define

$$E_X(\Delta) := \{ \varepsilon \in E(\Delta) \mid V(\varepsilon) = X \}.$$ 

Then

$$E(\Delta) = \bigcup_{X \in L} E_X(\Delta) \quad \text{(disjoint)}.$$ 

Define

$$C_X(\Delta) := \sum_{\varepsilon \in E_X(\Delta)} K(\prod \varepsilon)^{-1}.$$ 

Then $C_X(\Delta)$ is a $\partial(V)$-submodule of $C(\Delta)$. The following theorem is equivalent to Lemma 3.2 in [OrT2]. Our proof is a rephrasing of the proof there.

**Proposition 2.1.**

$$C(\Delta) = \bigoplus_{X \in L} C_X(\Delta).$$

**Proof.** It is obvious that $C(\Delta) = \sum_{X \in L} C_X(\Delta)$. Suppose that $\sum_{X \in L} \phi_X = 0$ with $\phi_X \in C_X(\Delta)$. We will show that $\phi_X = 0$ for all $X \in L$. By taking out the degree $p$ part, we may assume that $\deg \phi_X = p$ for all $X \in L$. Let $S = \{X \in L \mid \phi_X \neq 0\}$. Suppose $S$ is not empty. Then there exists a minimal element $X_0$ in $S$ (with respect to the partial order by reverse inclusion). Let $X \in S \setminus \{X_0\}$ and write

$$\phi_X = \sum_{\varepsilon \in E_X(\Delta)} c_\varepsilon (\prod \varepsilon)^{-1}$$

with $c_\varepsilon \in K$. Let $\varepsilon \in E_X(\Delta)$. Because of the minimality of $X_0$, one has $X_0 \not\subseteq X$. Thus there exists $\alpha_0 \in \varepsilon$ such that $X_0 \not\subseteq \ker(\alpha_0)$. Let $I(X_0)$ be the prime ideal of $S$ generated by the polynomial functions vanishing on $X_0$. Then $\alpha_0 \notin I(X_0)$. Thus

$$(\prod \Delta)^p (\prod \varepsilon)^{-1} \in I(X_0)^p |\Delta_{X_0}| - p + 1,$$

where $\prod \Delta := \prod_{\alpha \in \Delta} \alpha$ and $\Delta_{X_0} = \Delta \cap I(X_0)$. Multiply $(\prod \Delta)^p$ to the both sides of

$$\phi_{X_0} = - \sum_{X \in S} \phi_X$$

to get

$$(\prod \Delta)^p \phi_{X_0} = - \sum_{\varepsilon \in E_X(\Delta) \setminus X_0} (\prod \Delta)^p \phi_X$$

$$= - \sum_{X \in S} \sum_{\varepsilon \in E_X(\Delta) \setminus X_0} c_\varepsilon (\prod \Delta)^p (\prod \varepsilon)^{-1} \in I(X_0)^p |\Delta_{X_0}| - p + 1.$$
Since $\prod \Delta / (\prod \Delta X_0) \in S \setminus I(X_0)$ and $I(X_0)^{|\Delta X_0| - p + 1}$ is a primary ideal, one has

$$(\prod \Delta X_0)^p \phi X_0 \in I(X_0)^{|\Delta X_0| - p + 1}.$$  

This is a contradiction because

$$\deg(\prod \Delta X_0)^p \phi X_0 = p|\Delta X_0| - p.$$  

Therefore $S = \phi$. Q.E.D.

Next we will study the structure of $C_X(\Delta)$ for each $X \in L$. Let $A_O X(\Delta)$ be the $K$-subspace of $A_O(\Delta)$ generated over $K$ by

$$\{(\prod \varepsilon)^{-1} \mid \varepsilon \in E(\Delta) \cap E_X(\Delta)\}.$$  

Then

$$A_O(\Delta) = \bigoplus_{X \in L} A_O X(\Delta)$$  

by Proposition 2.1. Let $B_X$ be a $K$-basis for $A_O X(\Delta)$. Then we have

**Proposition 2.2.** The $\partial(V)$-module $C_X(\Delta)$ can also be regarded as a free $\partial(V/X)$-module with a basis $B_X$. In other words, there exists a natural graded isomorphism

$$\partial(V/X) \otimes_K A_O X(\Delta) \simeq C_X(\Delta).$$  

**Proof.** First assume that $\Delta$ spans $V^*$ and $X = \{0\}$. Then $E(\Delta)^i \cap E_X(\Delta)$ is equal to the set of $K$-bases for $V^*$ which are contained in $\Delta$. Thus $A_O X(\Delta)$ is generated over $K$ by

$$\{(\prod \varepsilon)^{-1} \mid \varepsilon \in E(\Delta) \text{ is a basis for } V\}.$$  

Similarly $C_X$ is spanned over $K$ by

$$\{(\prod \varepsilon)^{-1} \mid \varepsilon \in E(\Delta) \text{ spans } V\}.$$  

Then Theorem 1 of [BrV] is exactly the desired result. Next let $X \in L$ and $\overline{V} = V/X$. Regard the dual vector space $\overline{V}$ as a subspace of $V^*$ and the symmetric algebra $S := S(\overline{V})$ of $\overline{V}$ as a subring of $S$. Then $\Delta_X := I(X) \cap \Delta$ is a subset of $\overline{V}$ and $\Delta_X$ spans $\overline{V}$. Consider $A_O(\Delta_X)$ and $C(\Delta_X)$ which are both contained in $S(0)$. Note that $C_X(\Delta_X)$ can be regarded as a $\partial(V/X)$-module because $\partial(X)$ annihilates $C_X(\Delta)$. Denote the zero vector of $\overline{V}$ by $\overline{X}$. Then it is not difficult to see that

$$C_X(\Delta_X) \simeq C_X(\Delta) \quad \text{(as } \partial(\overline{V})\text{-modules),}$$  

$$A_O X(\Delta_X) \simeq A_O X(\Delta) \quad \text{(as } K\text{-vector spaces).}$$  

Since there exists a natural graded isomorphism

$$C_X(\Delta_X) \simeq \partial(\overline{V}) \otimes_K A_O X(\Delta_X),$$  

\text{(5)}
Proof of Theorem 1.3. By Proposition 2.2, $C_X(\Delta)$ is generated over $\partial(V)$ by $AO_X(\Delta)$. Since

$$C(\Delta) = \bigoplus_{X \in L} C_X(\Delta) \quad \text{(Proposition 2.1)},$$

and

$$AO(\Delta) = \bigoplus_{X \in L} AO_X(\Delta),$$

the $\partial(V)$-module $C(\Delta)$ is generated by $AO(\Delta)$. So $B$ generates $C(\Delta)$ over $\partial(V)$. Define

$$J(\Delta) := \sum_{\varepsilon \in E(\Delta)} K(\prod \varepsilon)^{-1},$$

which is an ideal of $C(\Delta)$. Then it is known by [OrT2, Theorem 4.2] that

$$C(\Delta) = J(\Delta) \oplus AO(\Delta) \quad \text{(as $K$-vector spaces)}.$$

It is obvious to see that

$$\partial(V)C(\Delta) \subseteq J(\Delta).$$

On the other hand, we have

$$C(\Delta) = \partial(V)AO(\Delta) = \partial(V)AO(\Delta) + AO(\Delta) = \partial(V)C(\Delta) + AO(\Delta).$$

Combining these, we have (2) and (3) at the same time. By (2), we know that $B$ minimally generates $C(\Delta)$ over $\partial(V)$, which is (1).

If $M = \bigoplus_{p \geq 0} M_p$ is a graded vector space with $\dim M_p < +\infty \quad (p \geq 0)$, we let

$$\text{Poin}(M, t) = \sum_{p=0}^{\infty} (\dim M_p) t^p$$

be its Poincaré (or Hilbert) series. Recall [OrT1, 2.42] the (one variable) Möbius function $\mu : L(\Delta) \to \mathbb{Z}$ defined by $\mu(V) = 1$ and for $X > V$ by $\sum_{Y \subseteq X} \mu(Y) = 0$. Then the Poincaré polynomial $\text{Poin}(A(\Delta), t)$ of the arrangement $A(\Delta)$ is defined by

$$\text{Poin}(A(\Delta), t) = \sum_{X \in L} \mu(X)(-t)^{\text{codim } X}.$$
Recall $C(\Delta)$ is a graded $\partial(V)$-module. Since $C(\Delta)$ is infinite dimensional, $\text{Poin}(C(\Delta), t)$ is a formal power series. We now prove Theorem 1.4 which gives a combinatorial formula for $\text{Poin}(C(\Delta), t)$.

**Proof of Theorem 1.4.** We have

$$\text{Poin}(C(\Delta), t) = \sum_{X \in L} \text{Poin}(C_X(\Delta), t) = \sum_{X \in L} \text{Poin}(\partial(V/X), t)\text{Poin}(AO_X(\Delta), t)$$

by Propositions 2.1 and 2.2. Since the $K$-algebra $\partial(V/X)$ is isomorphic to the polynomial algebra with $\text{codim} X$ variables, we have

$$\text{Poin}(C(\Delta), t) = \sum_{X \in L} (1 - t)^{-\text{codim} X} \text{Poin}(AO_X(\Delta), t).$$

By Proposition 2.3, we have

$$\text{Poin}(AO_X(\Delta), t) = (-1)^{\text{codim} X} \mu(X)t^{\text{codim} X}.$$

Thus

$$\text{Poin}(C(\Delta), t) = \sum_{X \in L} (-1)^{\text{codim} X} \mu(X) \left( \frac{t}{1 - t} \right)^{\text{codim} X} = \text{Poin}(A(\Delta), (1 - t)^{-1}t). \qed$$

Let $\text{Der}$ be the $S$-module of derivations:

$$\text{Der} = \{ \theta : \theta : S \to S \text{ is a } K\text{-linear derivations} \}. $$

Then $\text{Der}$ is naturally isomorphic to $S \otimes_K V$. Define

$$D(\Delta) = \{ \theta \in \text{Der} | \theta(\alpha) \in \alpha S \text{ for any } \alpha \in \Delta \},$$

which is naturally an $S$-submodule of $\text{Der}$. We say that the arrangement $A(\Delta)$ is **free** if $D(\Delta)$ is a free $S$-module [OrT1, Definition 4.15]. An element $\theta \in D(\Delta)$ is said to be **homogeneous of degree** $p$ if

$$\theta(x) \in S_p \text{ for all } x \in V^*.$$

When $A(\Delta)$ is a free arrangement, let $\theta_1, \cdots, \theta_\ell$ be a homogeneous basis for $D(\Delta)$. The $\ell$ nonnegative integers $\text{deg} \theta_1, \cdots, \text{deg} \theta_\ell$ are called the **exponents** of $A(\Delta)$. Then one has

**Proposition 2.4. (Factorization Theorem [Ter1], [OrT1, Theorem 4.137])**

*If $A(\Delta)$ is a free arrangement with exponents $d_1, \cdots, d_\ell$, then*

$$\text{Poin}(A(\Delta), t) = \prod_{i=1}^\ell (1 + d_i t).$$
By Theorem 1.4 and Proposition 2.4, we immediately have Corollary 1.5.

The arrangement $A(\Delta)$ is generic if $|\Delta| \geq \ell$ and any $\ell$ vectors in $\Delta$ are linearly independent. In this case, it is easy to see that [OrT1, Lemma 5.122]

$$\text{Poin}(A(\Delta), t) = (1 + t)^{\ell - 1} \sum_{i=0}^{\ell - 1} \binom{|\Delta| - 1}{i} t^i.$$  

Proof of Corollary 1.7. By Theorem 1.4, one has

$$\text{Poin}(C(\Delta), t) = \frac{1}{1-t} \sum_{i=0}^{\ell - 1} \binom{|\Delta| - 1}{i} \left( \frac{t}{1-t} \right)^i.$$  

On the other hand, we have

$$\sum_{j=0}^{k} (-1)^j \binom{|\Delta| - 1}{k-j} \binom{\ell-k+j-1}{j} = \binom{|\Delta| - \ell + k - 1}{k}$$

by equating the coefficients of $x^k$ in $(1+x)^{|\Delta| - \ell + k - 1}$ and $(1+x)^{|\Delta| - 1}(1+x)^{-(\ell-k)}$. This proves the assertion.

We now consider the nbc (=no broken circuit) bases [Bjo1] [Bjo2] [BjZ] [InT] [OrT2] p.72. Suppose that $\Delta$ is linearly ordered: $\Delta = \{\alpha_1, \cdots, \alpha_n\}$. Let $X \in L$ with codim $X = p$. Define

$$\text{nbc}_X(\Delta) := \{\varepsilon \in E_X(\Delta) \mid \varepsilon = (\alpha_{i_1}, \cdots, \alpha_{i_p}), i_1 < \cdots < i_p, \text{ contains no broken circuits}\}.$$  

Let $B_X = \{(\prod \varepsilon)^{-1} \mid \varepsilon \in \text{nbc}_X(\Delta)\}$ for $X \in L$. Then we have

Proposition 2.5. (OrT2, Theorem 5.2) Let $X \in L$. The set $B_X$ is a $K$-basis for $AO_X(\Delta)$.

Thanks to Propositions 2.1, 2.2 and 2.3, we easily have

Proposition 2.6. Let $B = \bigcup_{X \in L} B_X = \{\phi_1, \cdots, \phi_m\}$. Write $\text{supp}(\phi_i) = X$ if $\phi_i \in B_X$. Then, for any $\phi \in C(\Delta)$ and $j \in \{1, \cdots, m\}$, there uniquely exists $\theta_j \in \partial(V/\text{supp}(\phi_j))$ such that

$$\phi = \sum_{j=1}^{m} \theta_j(\phi_j).$$  

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Remark 2.7. Suppose that $\Delta$ spans $V^*$ and that $AO_{\{0\}}(\Delta) = \sum_{j=1}^{q} K \phi_j$, where $q = |\mu(\{0\})|$. Then the mapping

$$
\phi \mapsto \sum_{j=1}^{q} \theta_j^{(0)}(\phi_j) \in AO_{\{0\}}(\Delta)
$$

is the restriction to $C(\Delta)$ of the Jeffrey-Kirwan residue $[BrV, Definition 6]$ $[Sze]$. Here $\theta_j^{(0)}$ is the degree zero part of $\theta_j$ ($j = 1, \ldots, q$).

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