LENTH MINIMIZING PROPERTY, CONLEY-ZEHNDER INDEX
AND $C^1$-PERTURBATIONS OF HAMILTONIAN FUNCTIONS

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Abstract. The main purpose of this paper is to study the length minimizing property of Hamiltonian paths on closed symplectic manifolds $(M, \omega)$ such that there are no spherical homology class $A \in H_2(M)$ with
\[ \omega(A) > 0 \quad \text{and} \quad -n \leq c_1(A) < 0, \]
which we call very strongly semi-positive. We introduce the notion of positively $\mu$-undertwisted Hamiltonian paths and prove that any positively undertwisted quasi-autonomous Hamiltonian path is length minimizing in its homotopy class as long as it has a fixed maximum and a fixed minimum point that are generically under-twisted. This class of Hamiltonian can have non-constant large periodic orbits. The proof uses the chain level Floer theory, spectral invariants of Hamiltonian diffeomorphisms and the argument involving the thick and thin decomposition of Floer’s moduli space of perturbed Cauchy-Riemann equation. And then based on this theorem and some closedness of length minimizing property, we relate the Minimality Conjecture on the very strongly semi-positive symplectic manifolds to a $C^1$-perturbation problem of Hamiltonian functions on general symplectic manifolds, which we also formulate here.

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§1. Introduction

The celebrated Hofer’s norm of Hamiltonian diffeomorphisms introduced in [H] is defined by
\[ \|\phi\| = \inf_{H \mapsto \phi} \|H\| \] (1.1)
where \( H \mapsto \phi \) means that \( \phi = \phi^1_H \) is the time-one map of Hamilton’s equation
\[ \dot{x} = X_H(x) \]
and the norm \( \|H\| \) is defined by
\[ \|H\| = \int_0^1 \text{osc} \ H_t \ dt = \int_0^1 (\max H_t - \min H_t) \ dt. \] (1.2)
This induces a distance function, the so called the Hofer distance, on \( \text{Ham}(M, \omega) \) which in turn defines a topology on \( \text{Ham}(M, \omega) \). We call this topology the Hofer topology on \( \text{Ham}(M, \omega) \).

Our convention of the definition of Hamiltonian vector field will be
\[ X_h | \omega = dh \] (1.3)
for a smooth function \( h \) on \( M \). From now on we will always assume \( M \) is closed and, that the Hamiltonian functions are normalized, unless otherwise said, so that
\[ \int_M H_t \ d\mu = 0 \]
where \( d\mu \) is the Liouville measure of \((M, \omega)\).

The main purpose of the present paper is to study the length minimizing property of Hamiltonian paths without the standard condition that the corresponding Hamiltonian has no non-constant periodic orbits, but instead with some topological condition on the Conley-Zehnder indices of the periodic orbits. Our motivation for this attempt is to study the following prominent question posed by Polterovich [Conjecture 12.6.D, Po] and by Lalonde-McDuff-Slimowitz [LM2], [MS] in this regard.

[Minimality Conjecture]. Any autonomous Hamiltonian path that has no contractible periodic orbits of period less than equal to one is Hofer-length minimizing in its homotopy class with fixed ends.

Our attempt has led us to the following definition.

Definition 1.1. We call a symplectic manifold \((M, \omega)\) very strongly semi-positive if it does not carry any spherical homology class \( A \in H_2(M, \mathbb{Z}) \) such that
\[ \omega(A) > 0, \quad -n \leq c_1(A) < 0. \] (1.4)
We denote by
\[ \Sigma := \min \{|c_1(A)| \mid c_1(A) \neq 0\} \]
and call the minimal Chern number of the symplectic manifold.
This class of symplectic manifolds include all weakly exact symplectic manifolds, all (positively) monotone symplectic manifolds, and negatively monotone symplectic manifolds with $\Sigma \geq n + 1 = \frac{1}{2} \dim M + 1$. In particular, it includes any Fano manifold, e.g., $\mathbb{C}P^n$.

We would like to emphasize that although the very strongly semi-positive condition looks very much alike and slightly stronger than the usual semi-positivity condition [HS] in which the condition on $c_1$ in (1.4) is replaced by $-n + 3 \leq c_1(A) < 0$

in relation to the transversality question of pseudo-holomorphic curves, the origin of our very strongly semi-positive condition here is different from that of the semi-positivity condition. In particular, the enhanced machinery of virtual moduli cycles is irrelevant to our requirement and will not help removing this condition from the statement of the main result below. However under the very strongly semipositivity assumption, the technical aspect of the Floer theory in this paper is `elementary' in that it does not require such enhanced machinery. Here we avoid using the more natural name like 'strongly semi-positive' because it has been already used in [En] for the case where the $c_1$ condition in (1.4) is replaced by $-n + 2 \leq c_1(A) < 0$.

We now recall some basic definitions in the study of Hamiltonian diffeomorphisms and their paths. Two Hamiltonians $G$ and $F$ are called equivalent if there exists a smooth family $\{F_s\}_{0 \leq s \leq 1}$ with $F_0 = G$, $F_1 = F$ such that

$$
\phi_t F_s = \phi_t G
$$

for all $s \in [0,1]$. We denote $G \sim F$ in that case and say that two Hamiltonian paths $\phi_t G$ and $\phi_t F$ are homotopic to each other with fixed ends, or just homotopic to each other when there is no danger of confusion.

**Definition 1.2.** A Hamiltonian $H$ is called *quasi-autonomous* if there exists two points $x_-, x_+ \in M$ such that

$$
H(x_-, t) = \min_x H(x, t), \quad H(x_+, t) = \max_x H(x, t)
$$

for all $t \in [0,1]$.

To state the result on the length minimizing property in this paper and for the future purpose, the following general definition seems to be useful. This is a topological undertwistedness while the ones used in [En, MS] is dynamical. We denote by $[z, w]$ an element of the standard $\Gamma$-covering space $\tilde{\Omega}_0(M, \omega)$ [HS] of the contractible loop free loop space $\Omega_0(M)$, and by $\mu_H([z, w])$ its Conley-Zehnder index.

**Definition 1.3.** Let $H$ be any Hamiltonian and $z$ be a contractible one-periodic orbit of its Hamiltonian vector field $X_H$.

1. We say that $z$ is positively $\mu_H$-undertwisted if it allows a bounding disc $w_z$ such that

$$
-n \leq \mu_H([z, w_z]) \leq n \quad \text{and} \quad \int w_z^* \omega \geq 0
$$

for $[z, w_z]$. If this holds for all contractible one-periodic orbits of $H$, then we call $H$ positively $\mu$-undertwisted.

2. If $\int w_z^* \omega \geq -\epsilon$ holds for $\epsilon > 0$ instead, then we say the periodic orbit $z$ is $\epsilon$-positively $\mu_H$-undertwisted, and $H$ is $\epsilon$-positively $\mu$-undertwisted respectively.
Examples of positively $\mu$-undertwisted Hamiltonians will be any slow autonomous Hamiltonian in the sense of [En, MS], and any $C^2$-small perturbations of nondegenerate slow autonomous Hamiltonians are $\epsilon$-positively $\mu$-undertwisted for some sufficiently small $\epsilon > 0$.

**Theorem A.** Let $(M, \omega)$ be very strongly semi-positive. Suppose $H$ is a quasi-autonomous Hamiltonian such that

1. it has a fixed global maximum and a fixed global minimum point that are generically under-twisted.
2. $H$ is positively $\mu$-undertwisted and nondegenerate in the Floer theoretic sense.

Then the Hamiltonian path $\phi^t_H$ is length minimizing in its homotopy class with fixed ends.

We would like to emphasize that the Hamiltonian $H$ can have ‘large’ periodic orbits as long as they are positively $\mu_H$-undertwisted.

In fact, the proof of this theorem shows that the condition (2) can be weakened to the following

**Theorem A’.** Let $(M, \omega)$ be very strongly semi-positive. Let $H$ satisfy just (1) in Theorem A and $\epsilon_H > 0$ be a constant depending only on ‘the local behavior’ of $H$ at the maximum and minimum points introduced in Lemma 5.4 later. Suppose that there exists a sequence of $\epsilon \to 0$ such that $H$ allows a $C^1$-small perturbation $H'$ that is quasi-autonomous and satisfies the condition that

1. $H'$ is $\epsilon$-positively $\mu$-undertwisted and nondegenerate in the Floer theoretic sense.

Then the Hamiltonian path $\phi^t_H$ is length minimizing in its homotopy class with fixed ends.

This perturbation result is the precise formulation of the remark mentioned right after [Theorem I, Oh3]. We refer to section 5 for the precise meaning of ‘the local behavior’ in the statement.

We would like to emphasize that the perturbation in Theorem A’ is assumed to be $C^1$-small, not necessarily $C^2$-small. In particular, there is no a priori relation between the Conley-Zehnder indices of the nearby periodic orbits of $H$ and $H'$ at all. However combined with the closedness of the length minimizing property (Theorem 3.1) of Hamiltonian paths under the Hofer topology, this will be an important point in application to the Minimality Conjecture which lead us to the $C^1$ Perturbation Conjecture later.

Our proof of Theorem A and A’ will be based on the chain level Floer theory from [Oh3,5] using the scheme developed in [Oh6] via the usage of the spectral invariant $\rho(H; 1)$. Many arguments are combination of those from [Oh3,6].

First to recall this criterion, we rewrite the Hofer norm into

$$\|H\| = E^-(H) + E^+(H)$$

where $E^\pm$ are the negative and positive parts of the Hofer norms defined by

$$E^-(H) = \int_0^1 - \min H \, dt$$

$$E^+(H) = \int_0^1 \max H \, dt.$$
These are called the **negative Hofer-length** and the **positive Hofer-length** of $H$ respectively. We note

$$E^+ (H) = E^- (\overline{H})$$

where $\overline{H}$ is the Hamiltonian generating $(\phi_H^t)^{-1}$ defined by

$$\overline{H}(t,x) = -H(t, \phi_H^t(x)).$$

Therefore we will focus only on the semi-norm $E^-$. According to the criterion [Theorem III, Oh6], Theorem A$'$ will be an immediate corollary of Theorem 3.1 [Lemma 5.1, Oh3] and the following theorem in whose proof the very strongly semi-positive condition of $(M, \omega)$ enters in an essential way. We refer to section 3 for a brief outline of the construction from [Oh5] of the spectral invariant $\rho(H; 1)$ on non-exact symplectic manifolds.

**Theorem B.** Let $(M, \omega)$ and $H'$ be as in Theorem A'. Then we have

$$\rho(H'; 1) = E^- (H').$$

The same holds for the inverse Hamiltonian $\overline{H'}$.

The main new ingredient in the proof of Theorem B besides the scheme used in [Oh3,6] is our usage of the argument in section 6 that is based on the “thick and thin” decomposition of the Floer moduli space. Similar argument was previously used by the author [Oh1] in the context of Lagrangian intersection Floer homology theory for its application to the Maslov class and construction of the corresponding spectral sequence.

We recall that the Minimality Conjecture was answered affirmatively for the weakly exact case or of the case of surfaces by Lalonde-McDuff much earlier [LM2] by a different method. Our proof also works for the surface case without any further requirement on $G$ other than those in the conjecture, but does require (1) even for the weakly exact case when $\dim M \geq 4$. We, however, refer to section 2 and 6 for some discussion on how one might be able to improve this point in the general context. It would be very interesting to see if the very strongly semi-positive condition is an essential condition or just a technical artifact of our Floer theoretic approach. Roughly speaking, the very strongly semi-positive condition rules out a possible ‘quantum contribution’ to the minimization process. It seems to be of fundamental importance to understand how the quantum contribution affects the length minimization process in general. We hope to come back to this issue in the future.

On the other hand, based on Theorem A$'$ and the following $C^1$ Perturbation Conjecture on the Hamiltonians and Theorem 3.1 [Lemma 5.1, Oh3], we believe that for the autonomous Hamiltonian $G$ appearing in the Minimality Conjecture the condition (2) can be replaced by the condition “$G$ has no non-constant periodic orbit” at least for the very strongly semi-positive case.

[C$^1$ Perturbation Conjecture]. Let $(M, \omega)$ be any symplectic manifold. Suppose that $G$ is an autonomous Hamiltonian with a global maximum and a minimum that are generically under-twisted, and has no non-constant periodic orbits of period less than equal to one. Let $\epsilon_G > 0$ be the positive constant in Theorem A$'$. Then there is a $C^1$-small perturbation $H'$ of $G$ such that $H'$ is $\epsilon$-positively $\mu$-under-twisted where $\epsilon$ can be made arbitrarily small depending only on $G$. 


We refer to section 6 for a more precise formulation of the conjecture and for some further discussion. If this conjecture is true, then the Minimality Conjecture will follow from Theorem A′ at least for the very strongly semi-positive case. Based on the way how the proof goes in section 5, we suspect that the conjecture is not true in general.

The organization of the paper is in order. Section 2 through 4 deal with the cases of general symplectic manifolds. The very strongly semi-positivity condition enters only in section 5. In section 2, we recall a theorem proven in [Oh7] which states that the length minimizing property is closed under the Hofer topology of Hamiltonian paths. This theorem is a stronger statement than the one stated in [Lemma 5.1, Oh3].

In section 3, we recall the basic chain level Floer theory, and briefly outline the construction from [Oh3,5] of spectral invariants, especially the one \( \rho(H; 1) \) on non-exact symplectic manifolds, and state their basic properties. See also [Oh2], [Sc] for the construction in the exact case.

In section 4, we recall the notion of the canonical fundamental Floer cycle that was introduced in [Oh3,6] and compute its level. Since the scheme of the proof in [Proposition 4.3, Oh6] will play an essential role in the proof of Theorem A, for the reader’s convenience, we duplicate the proof here.

In section 5, we restrict to the very strongly semi-positive case and give the proof of Theorem A′ and B by proving that the above canonical fundamental cycle is tight in the sense of Definition 5.2 [Definition 4.2, Oh6] under the condition of Theorem A′ and B which will in turn prove Theorem B and hence Theorem A′.

In section 6, we formulate a precise version of the above \( C^1 \) Perturbation Conjecture and explain how the conjecture together with Theorem A′ would imply the Minimality Conjecture for the very strongly semi-positive \((M, \omega)\).

In the appendix, we provide complete details of a proof of the index formula

\[
\mu_H([z, w]) = \mu_H([z, w']) + 2c_1(w \# w')
\]

where we emphasize the sign ‘+’ in front of the Chern number term in this formula. An incorrect formula with the negative sign was written in [section 6.1, En] and [section 7, Oh3]. We also clarify the correct formula under the other commonly used package of conventions as in [Po] and others. Having a correct sign did not play any significant role in the works in [En], [Oh3] or in other previous literature on the symplectic Floer homology theory. Since our definition of the positively \( \mu \)-undertwistedness and the proof of the main theorems in this paper crucially depend on having the correct sign in this formula and we cannot locate any reference that contains a complete proof of the formula in any convention, we give complete details of the proof essentially from the scratch based on the definition of the Conley-Zehnder index given in [CZ], [SZ] on \( \mathbb{R}^{2n} \) for the reader’s convenience. The reference [HS] contains the formula (1.5) but their conventions do not completely agree with ours and we do not feel safe just to quote the formula, especially when we are not completely sure of what conventions the authors of [HS] are using. There have already been more than one instances of an incorrect formula written in the literature as in [En] and [Oh3].

We thank the unknown referee of the originally submitted version of the paper [Oh6] for pointing out that the Minimality Conjecture does not follow from the main result in the original version of [Oh6]. This remark prompted us to carefully
look over the delicate points of appearance of “small” periodic orbits, which we had overlooked in [Oh3] (see Erratum to [Oh3] for clarification of these points). The final writing has been carried out while we are visiting Korea Institute for Advanced Study during the winter of 2003-2004. We thank KIAS for its financial support and excellent research atmosphere during our stay.

§2. The Hofer topology and the length minimizing property

In this section, we first recall the notion of Hofer topology on the space of Hamiltonian paths, and the closedness of the length minimizing property of Hamiltonian paths under the topology proven in [Oh3,7]. Our presentation of the Hofer topology closely resembles that of the Hamiltonian topology that we introduce in [Oh7], which is however equivalent to the usual description of the Hofer topology in the literature.

We first recall the definition of Hamiltonian diffeomorphisms and smooth Hamiltonian paths.

Definition 2.1. (i) A $C^\infty$ diffeomorphism $\phi$ of $(M, \omega)$ is a Hamiltonian diffeomorphism if $\phi = \phi^1_H$ for a $C^\infty$ function $F : \mathbb{R} \times M \to \mathbb{R}$ such that
\[
F(t + 1, x) = F(x)
\]
for all $(t, x) \in \mathbb{R} \times M$. Having this periodicity in mind, we will always consider $F$ as a function on $[0, 1] \times M$.

(ii) A Hamiltonian path $\lambda : [0, 1] \to Ham(M, \omega)$ is a smooth map $\Lambda : [0, 1] \times M \to M$ such that

1. its derivative $\dot{\lambda}(t) = \frac{\partial}{\partial t} \circ (\lambda(t))^{-1}$ is Hamiltonian, i.e., the one form $\dot{\lambda}(t) \mid_\omega$ is exact for all $t \in [0, 1]$. We call the normalized function $H : \mathbb{R} \times M \to \mathbb{R}$ the generating Hamiltonian of $\lambda$ if it satisfies $\lambda(t) = \phi^t_H(\lambda(0))$ or equivalently $dH_t = \dot{\lambda}(t) \mid_\omega$.
2. $\lambda(0) := \Lambda(0, \cdot) : M \to M$ is a Hamiltonian diffeomorphism, and so is for all $\lambda(t) = \Lambda(t, \cdot), t \in \mathbb{R}$.

We denote by $\mathcal{P}(Ham(M, \omega))$ the set of Hamiltonian paths $\lambda : [0, 1] \to Ham(M, \omega)$, and by $\mathcal{P}(Ham(M, \omega), id)$ the set of $\lambda$ with $\lambda(0) = id$.

Here $\phi^1_H$ is the time-one map of the Hamilton equation
\[
\dot{x} = X_H(x).
\]
We will always denote by $\phi_H$ the corresponding Hamiltonian path
\[
\phi_H : t \mapsto \phi^t_H
\]
starting from the identity, and by $H \mapsto \phi$ when $\phi = \phi^1_H$. In the latter case, we say that the diffeomorphism $\phi$ is generated by the Hamiltonian $H$.

The Hofer length of the Hamiltonian path $\lambda \in \mathcal{P}(Ham(M, \omega))$ with $\lambda(t) = \phi^t_H(\lambda(0))$ is given by the norm of its generating Hamiltonian $H$ defined by
\[
leng(\lambda) = \|H\| = \int_0^1 (\max_x H_t - \min_x H_t) \, dt.
\]
We also denote by
\[
ev_1 : \mathcal{P}(Ham(M, \omega), id) \to Ham(M, \omega)
\]
the evaluation map $ev_1(\lambda) = \lambda(1) = \phi^1_H$. 

\[
ev_1 : \mathcal{P}(Ham(M, \omega), id) \to Ham(M, \omega)
\]

**Definition 2.2.** Consider the metric $d_H$ on $\mathcal{P}(\text{Ham}(M, \omega), \text{id})$ defined by

$$d_H(\lambda, \mu) := \text{leng}(\lambda^{-1} \circ \mu), \quad \lambda, \mu \in \mathcal{P}(\text{Ham}(M, \omega), \text{id})$$

(2.4)

where $\lambda^{-1} \circ \mu$ is the Hamiltonian path $t \in [0,1] \mapsto \lambda(t)^{-1} \mu(t)$. We call the induced topology the Hofer topology on $\mathcal{P}(\text{Ham}(M, \omega), \text{id})$. The Hofer topology on $\text{Ham}(M, \omega)$ is the weakest topology for which the evaluation map (2.3) is continuous.

It is easy to see that this definition of the Hofer topology of $\text{Ham}(M, \omega)$ coincides with the usual one induced by (1.1) which also shows that the Hofer topology is meterizable. Of course, nontriviality of the Hofer topology on $\text{Ham}(M, \omega)$ is a difficult theorem which was proven by Hofer himself for $\mathbb{C}^n$ and by Lalonde and McDuff in its complete generality [LM1].

We say that two Hamiltonians $H$ and $K$ are equivalent if their corresponding Hamiltonian paths $\phi_H$ and $\phi_K$ are path-homotopic relative to the end points on $\text{Ham}(M, \omega)$. In other words, for a given $\phi \in \text{Ham}(M, \omega)$ and $H$ and $K$ with $H, K \mapsto \phi$

are equivalent if they are connected by one parameter family of Hamiltonians $\{F_s\}_{0 \leq s \leq 1}$ such that $F^0 = H$, $F^1 = K$ and

$$\phi \mapsto F^s$$

for all $s \in [0,1]$.

To emphasize the time-one map of the Hamiltonian path $\phi_H$, we sometimes denote the Hamiltonian path $\phi_H$ also by $(\phi_H, H)$ and by $h = [\phi, H] = [H]$ the equivalence class of $(\phi, H)$. We denote by $\widetilde{\text{Ham}}(M, \omega)$ the set of equivalence classes $[\phi, H]$ with the quotient topology on it. Although $\text{Ham}(M, \omega)$ is not known to be locally path-connected in general, $\widetilde{\text{Ham}}(M, \omega)$ can be considered as the ‘universal covering space’ in the étale sense (see [Oh7] for a precise definition of topological étale covering).

One can easily check that $\overline{H} \# H'$ is given by the formula

$$(\overline{H} \# H')(t, x) = (H' - H)(t, (\phi_H'(x)))$$

and generates the flow $\phi_H^{-1} \circ \phi_{H'}$. Therefore we can also write

$$d_H(\phi_H, \phi_{H'}) = \text{leng}(\phi_H^{-1} \phi_{H'}) = \|\overline{H} \# H'\|.$$  (2.5)

The following theorem from [Oh7] is an improvement of [Lemma 5.1, Oh3]. For reader’s convenience, we reproduce the proof from [Oh3,7] here.

**Theorem 2.3 [Theorem 5.1, Oh7].** The length minimizing property of Hamiltonian path is closed in $\mathcal{P}(\text{Ham}(M, \omega))$ with respect to the Hofer topology.

**Proof.** Suppose that

1. the path $\phi_{G_i}$ converges to $\phi_{G_0}$
2. all $\phi_{G_i}$ are length minimizing in its homotopy class relative to end points.
Under these conditions, we need to prove that $\phi_{G_0}$ is length minimizing in its homotopy class relative to the end points again.

By definition of the Hofer topology, $\{G_i\}$ will satisfy $\|G_i \# G_0\| \to 0$ as $i \to \infty$. Suppose the contrary that there exists $F$ such that $F \sim G_0$, but $\|F\| < \|G_0\|$. Then there exists for some $\delta > 0$ such that

$$\|F\| < \|G_0\| - \delta$$

Therefore we have

$$\|F\| < \|G_i\| - \frac{\delta}{2}$$

for all sufficiently large $i$. We consider the Hamiltonian $F_i$ defined by

$$F_i := (G_i \# G_0) \# F \quad \text{i.e.,}$$

This generates the flow $\phi_{G_i} \circ (\phi_{G_0})^{-1} \circ \phi_F$ and so $F_i \sim G_i$. This implies, by the hypothesis that $G_i$ are length minimizing over $[0,1]$, we have

$$\|G_i\| \leq \|F_i\|.$$  \tag{2.8}$$

On the other hand, we have

$$\lim_{i \to \infty} \|F_i\| = \lim_{i \to \infty} \|(G_i \# G_0) \# F)\| \leq \lim_{i \to \infty} \|G_i \# G_0\| + \|F\| = \|F\|.$$  \tag{2.9}$$

Combining (2.7), (2.8) and (2.9), we get a contradiction. This finishes the proof. \qed

We would like to note that in terms of the generating Hamiltonians the Hofer topology is essentially $L^{(1,\infty)}$-topology which is much weaker than e.g., $C^1$-topology. This point will play an essential role when we will formulate our $C^1$ Perturbation Conjecture on the Hamiltonian functions and reduce the Minimality Conjecture to the $C^1$ Perturbation Conjecture in the very strongly semi-positive case based on the main theorem of this paper.

§3. Chain level Floer theory and spectral invariants

We first recall the construction of the spectral invariants from [Oh5] briefly. Let $\Omega_0(M)$ be the set of contractible loops and $\tilde{\Omega}_0(M)$ be its standard covering space in the Floer theory. Note that the universal covering space of $\Omega_0(M)$ can be described as the set of equivalence classes of the pair $(\gamma, w)$ where $\gamma \in \Omega_0(M)$ and $w$ is a map from the unit disc $D = D^2$ to $M$ such that $w|_{\partial D} = \gamma$: the equivalence relation to be used is that $[w \# w']$ is zero in $\pi_2(M)$. We say that $(\gamma, w)$ is $\Gamma$-equivalent to $(\gamma, w')$ if and only if

$$\omega([w' \# \bar{w}]) = 0 \quad \text{and} \quad c_1([w \# \bar{w}]) = 0$$

where $\bar{w}$ is the map with opposite orientation on the domain and $w' \# \bar{w}$ is the obvious glued sphere. And $c_1$ denotes the first Chern class of $(M, \omega)$. We denote by $[\gamma, w]$ the $\Gamma$-equivalence class of $(\gamma, w)$, by $\tilde{\Omega}_0(M)$ the set of $\Gamma$-equivalence classes.
and by $\pi: \tilde{\Omega}_0(M) \to \Omega_0(M)$ the canonical projection. We also call $\tilde{\Omega}_0(M)$ the $\Gamma$-covering space of $\Omega_0(M)$. The unperturbed action functional $A_0: \tilde{\Omega}_0(M) \to \mathbb{R}$ is defined by
\[
A_0(\gamma, w) = -\int w^*\omega.
\] (3.2)

Two $\Gamma$-equivalent pairs $(\gamma, w)$ and $(\gamma, w')$ have the same action and so the action is well-defined on $\tilde{\Omega}_0(M)$. When a periodic Hamiltonian $H : M \times (\mathbb{R}/\mathbb{Z}) \to \mathbb{R}$ is given, we consider the functional $A_H : \tilde{\Omega}(M) \to \mathbb{R}$ defined by
\[
A_H(\gamma, w) = -\int w^*\omega - \int H(\gamma(t), t)dt.
\]

We would like to note that under this convention the maximum and minimum are reversed when we compare the action functional $A_G$ and the (quasi-autonomous) Hamiltonian $G$.

We denote by $\text{Per}(H)$ the set of periodic orbits of $X_H$.

**Definition 3.1.** We define the action spectrum of $H$, denoted as $\text{Spec}(H) \subset \mathbb{R}$, by
\[
\text{Spec}(H) := \{A_H(z, w) \in \mathbb{R} \mid [z, w] \in \tilde{\Omega}_0(M), z \in \text{Per}(H)\},
\]
i.e., the set of critical values of $A_H : \Omega(M) \to \mathbb{R}$. For each given $z \in \text{Per}(H)$, we denote
\[
\text{Spec}(H; z) = \{A_H(z, w) \in \mathbb{R} \mid (z, w) \in \pi^{-1}(z)\}.
\]

Note that $\text{Spec}(H; z)$ is a principal homogeneous space modelled by the period group of $(M, \omega)$
\[
\Gamma_\omega = \Gamma(M, \omega) := \{\omega(A) \mid A \in \pi_2(M)\}
\]
and
\[
\text{Spec}(H) = \bigcup_{z \in \text{Per}(H)} \text{Spec}(H; z).
\]

Recall that $\Gamma_\omega$ is either a discrete or a countable dense subset of $\mathbb{R}$. It is trivial, i.e., $\Gamma_\omega = \{0\}$ in the weakly exact case. The followings were proved in [Oh3,4].

**Lemma 3.2.** $\text{Spec}(H)$ is a measure zero subset of $\mathbb{R}$.

For given $\phi \in \text{Ham}(M, \omega)$, we denote
\[
\mathcal{H}_m(\phi) = \{H \mid H \mapsto \phi, H \text{ normalized}\}.
\]

**Lemma 3.3.** Let $F, G \in \mathcal{H}_m(\phi)$ and $F \sim G$. Then we have
\[
\text{Spec}(G) = \text{Spec}(F)
\]
as a subset of $\mathbb{R}$.

This enables us to define the action spectrum over the universal covering space $\tilde{\text{Ham}}(M, \omega)$. 
**Definition 3.4.** Let \( h \in \widehat{Ham}(M, \omega) \) and let \( h = [\phi, H] \) for some Hamiltonian \( H \) with \( \phi = \phi^1_H \). Then we define the action spectrum of \( h \) by

\[
\text{Spec}(h) = \text{Spec}(H)
\]

for a (and so any) representative \([\phi, H]\).

Next we briefly recall the basic chain level operators in the Floer theory, and the definition and basic properties of spectral invariants \( \rho(H; a) \) from [Oh5].

For each given generic time-periodic \( H : M \times S^1 \to \mathbb{R} \), we consider the free \( \mathbb{Q} \) vector space over

\[
\text{Crit}_A H = \{ [z, w] \in \widehat{\Omega}_0(M) \mid z \in \text{Per}(H) \}.
\]

To be able to define the Floer boundary operator correctly, we need to complete this vector space downward with respect to the real filtration provided by the action \( A_H([z, w]) \) of the critical point \([z, w]\). More precisely, following [HS], [Oh3], we introduce

**Definition 3.5.** (1) We call the formal sum

\[
\beta = \sum_{[z, w] \in \text{Crit}_A H} a_{[z, w]}[z, w], \ a_{[z, w]} \in \mathbb{Q}
\]

(3.3)

a *Floer Novikov chain* if there are only finitely many non-zero terms in the expression (3.2) above any given level of the action. We denote by \( CF(H) \) the set of Novikov chains. We often simply call them *Floer chains*, especially when we do not need to work on the covering space \( \widehat{\Omega}_0(M) \) as in the weakly exact case.

(2) Two Floer chains \( \alpha \) and \( \alpha' \) are said to be *homologous* to each other if they satisfy

\[
\alpha' = \alpha + \partial H(\gamma)
\]

for some Floer chain \( \gamma \). We call \( \beta \) a *Floer cycle* if \( \partial \beta = 0 \).

(3) Let \( \beta \) be a Floer chain in \( CF(H) \). We define and denote the *level* of the chain \( \beta \) by

\[
\lambda_H(\beta) = \max_{[z, w]} \{ A_H([z, w]) \mid a_{[z, w]} \neq 0 \ \text{in} \ (3.3) \}
\]

if \( \beta \neq 0 \), and just put \( \lambda_H(0) = +\infty \) as usual.

(4) We say that \([z, w]\) is a *generator of or contributes to* \( \beta \) and denote

\[
[z, w] \in \beta
\]

if \( a_{[z, w]} \neq 0 \).

Let \( J = \{ J_t \}_{0 \leq t \leq 1} \) be a periodic family of compatible almost complex structures on \( (M, \omega) \).

For each given such periodic pair \((J, H)\), we define the boundary operator

\[
\partial : CF(H) \to CF(H)
\]
considering the perturbed Cauchy-Riemann equation
\[
\begin{cases}
\frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial \tau} - X_H(u) \right) = 0 \\
\lim_{\tau \to -\infty} u(\tau) = z^-, \lim_{\tau \to \infty} u(\tau) = z^+ 
\end{cases}
\] (3.4)

This equation, when lifted to \( \tilde{\Omega}_0(M) \), defines nothing but the negative gradient flow of \( A_H \) with respect to the \( L^2 \)-metric on \( \tilde{\Omega}_0(M) \) induced by the metrics \( g_j := \omega(\cdot, J \cdot) \). For each given \([z^-, w^-]\) and \([z^+, w^+]\), we define the moduli space
\[
\mathcal{M}_{(j, H)}([z^-, w^-], [z^+, w^+])
\]
of solutions \( u \) of (3.4) satisfying
\[
w^- \# u \sim w^+.
\] (3.5)

\( \partial \) has degree \(-1\) and satisfies \( \partial \circ \partial = 0 \).

When we are given a family \((j, \mathcal{H})\) with \( \mathcal{H} = \{H^s\}_{0 \leq s \leq 1} \) and \( j = \{J^s\}_{0 \leq s \leq 1} \), the chain homomorphism
\[
h_{(j, \mathcal{H})} : CF(H^0) \to CF(H^1)
\]
is defined by the non-autonomous equation
\[
\begin{cases}
\frac{\partial u}{\partial \tau} + J \rho_1(\tau) \left( \frac{\partial u}{\partial \tau} - X_{H^2(\tau)}(u) \right) = 0 \\
\lim_{\tau \to -\infty} u(\tau) = z^-, \lim_{\tau \to \infty} u(\tau) = z^+
\end{cases}
\] (3.6)

where \( \rho_i, i = 1, 2 \) is functions of the type \( \rho : \mathbb{R} \to [0, 1] \),
\[
\rho(\tau) = \begin{cases}
0 & \text{for } \tau \leq -R \\
1 & \text{for } \tau \geq R
\end{cases}
\]

for some \( R > 0 \). We denote by
\[
\mathcal{M}^{(j, \mathcal{H})}([z^-, w^-], [z^+, w^+])
\]
or sometimes with \( j \) suppressed the set of solutions of (3.6) that satisfy (3.5). The chain map \( h_{(j, \mathcal{H})} \) is defined similarly as \( \partial \) using this moduli space instead. \( h_{(j, \mathcal{H})} \) has degree 0 and satisfies
\[
\partial_{(j^1, H^1)} \circ h_{(j, \mathcal{H})} = h_{(j, \mathcal{H})} \circ \partial_{(j^0, H^0)}.
\]

The following identity can be proven by a straightforward calculation, but has played a fundamental role in [Oh2-7] and will also play an essential role in the proof of Theorem A’ and B.

**Lemma 3.6.** Let \( H, K \) be any Hamiltonian not necessarily non-degenerate and \( j = \{J^s\}_{s \in [0, 1]} \) be any given homotopy and \( \mathcal{H}^{lin} = \{H^s\}_{0 \leq s \leq 1} \) be the linear homotopy \( H^s = (1 - s)H + sK \). Suppose that (3.5) has a solution satisfying (3.6). Then we have the identity
\[
\mathcal{A}_K([z^+, w^+]) - \mathcal{A}_H([z^-, w^-])
= -\int \left| \frac{\partial u}{\partial \tau} \right|^2_{J \rho_1(\tau)} - \int_{-\infty}^{\infty} \rho_2(\tau) \left( K(t, u(\tau, t)) - H(t, u(\tau, t)) \right) dt d\tau.
\] (3.7)

Now we recall the definition and some basic properties of spectral invariant \( \rho(H; a) \) from [Oh5]. We refer readers to [Oh5] for the complete discussion on general properties of \( \rho(H; a) \).
Definition & Theorem 3.7 [Oh5]. Let $a \neq 0$ be a given quantum cohomology class in $QH^\ast(M)$, and denote by $a^\flat \in FH_\ast$ the Floer homology class dual to $a$ in the sense of [Oh5]. For any given Hamiltonian path $\lambda = \phi_H \in \mathcal{P}(Ham(M,\omega),id)$, we define
\[ \rho(\lambda; a) := \rho(H; a) = \inf_{\alpha \in \ker \partial H} \{\lambda_H(\alpha) \mid [\alpha] = a^\flat\} \]
where $a^\flat$ is the dual to the quantum cohomology class $a$ in the sense of [Oh5]. We call any of these spectral invariants. The map $\rho_a : \lambda = \phi_H \mapsto \rho(H; a)$ defines a continuous function
\[ \rho_a = \rho(\cdot; a) : C^\infty([0,1] \times M,\mathbb{R}) \to \mathbb{R} \]
with respect to Hofer topology, and for two smooth functions $H \sim K$ it satisfies
\[ \rho(H; a) = \rho(K; a) \]
for all $a \in QH^\ast(M)$. In particular, for each given $h \in \widehat{Ham}(M,\omega)$, the following definition is well-defined:
\[ \rho(h; a) = \rho(H; a) \]
any representative $[\phi, H] = h$.

Now we focus on the invariant $\rho(h; 1)$ for $1 \in QH^\ast(M)$. We first recall the following quantities
\[ E^-(h) = \inf_{[\phi, H] = h} E^-(H) \]
\[ E^+(h) = \inf_{[\phi, H] = h} E^+(H) \]
defined for smooth $h$. The following is an immediate consequence of the proofs of similar inequalities from [Theorem II, Oh5].

Proposition 3.9. Let $(M,\omega)$ be arbitrary, especially non-exact, closed symplectic manifold. For any $h \in \widehat{Ham}(M,\omega)$, we have
\[ \rho(h; 1) \leq E^-(h) \quad \rho(h^{-1}; 1) \leq E^+(h). \]

The following theorem is an immediate consequence of Theorem 3.7 and Proposition 3.9 applied to the smooth case.

Theorem 3.10. Let $G : [0,1] \times M \to \mathbb{R}$ be a quasi-autonomous Hamiltonian. Suppose that $G$ satisfies
\[ \rho(G; 1) = E^-(G) \]
Then $G$ is negative Hofer-length minimizing in its homotopy class with fixed ends.

So far in this section, we have presumed that the Hamiltonians are time one-periodic. Now we explain how to dispose the periodicity and extend the definition of $\rho(H; a)$ for arbitrary time dependent Hamiltonians $H : [0,1] \times M \to \mathbb{R}$. Note that it is obvious that the semi-norms $E^\pm(H)$ and $\|H\|$ are defined without assuming the periodicity. For this purpose, the following lemma from [Oh3] is important. We leave its proof to readers or to [Oh3].
Lemma 3.11. Let $H$ be a given Hamiltonian $H : [0, 1] \times M \to \mathbb{R}$ and $\phi = \phi_H^t$ be its time-one map. Then we can re-parameterize $\phi_H^t$ in time so that the re-parameterized Hamiltonian $H'$ satisfies the following properties:

1. $\phi_H^t = \phi_{H'}^t$
2. $H' \equiv 0$ near $t = 0, 1$ and in particular $H'$ is time periodic. We call such Hamiltonians boundary flat.
3. Both $E^\pm (H' - H)$ can be made as small as we want
4. If $H$ is quasi-autonomous, then so is $H'$
5. For the Hamiltonians $H', H''$ generating any two such re-parameterizations of $\phi_H^t$, there is canonical one-one correspondences between $\text{Per}(H')$ and $\text{Per}(H'')$, and $\text{Crit}_{A_H}$ and $\text{Crit}_{A_{H''}}$ with their actions fixed.

Furthermore this re-parameterization is canonical with the “smallness” in (3) can be chosen uniformly over $H$ depending only on the $C^0$-norm of $H$. In particular, this approximation can be done with respect to the Hofer topology.

In fact, the above approximation can be done under the stronger topology, the strong Hamiltonian topology introduced by the author [Oh7]. Since this stronger statement will not be needed in the present paper, we will be content with stating this lemma under the Hofer topology here.

§4. Canonical fundamental Floer cycles

In this section, we start with our study of length minimizing property of Hamiltonian paths following the scheme used in [Oh6]. This section is largely a duplication of section 4 of [Oh6]. Partly because the details of the proofs, not the theorems therein, will be needed in the proof of Theorem B in the next section and also to make this paper self-contained, we repeat the whole section 4 of [Oh6] here with minor simplification for the readers’ convenience.

We first recall the basic definitions in relation to the dynamics of Hamiltonian flows.

Definition 4.1. Let $H : M \times [0, 1] \to \mathbb{R}$ be a Hamiltonian which is not necessarily time-periodic and $\phi_H^t$ be its Hamiltonian flow.

1. We call a point $p \in M$ a time $T$ periodic point if $\phi_H^t(p) = p$. We call $t \in [0, T] \mapsto \phi_H^t(p)$ a contractible time $T$-periodic orbit if it is contractible.
2. When $H$ has a fixed critical point $p$ over $t \in [0, T]$, we call $p$ over-twisted as a time $T$-periodic orbit if its linearized flow $d \phi_H^t(p); t \in [0, T]$ on $T_pM$ has a closed trajectory, other than the fixed origin, of period less than or equal to $T$. Otherwise we call it under-twisted. If in addition the linearized flow has only the origin as the fixed point, then we call generically under-twisted.

For the proof of Theorem B, we need to unravel the definition of the particular spectral invariant $\rho(G; 1)$ from [Oh5], and its realization as the level of some optimal Floer cycles for nondegenerate (one periodic) Hamiltonians $G$. According to the definition from [Oh5] for nondegenerate Hamiltonians, we consider the Floer homology class dual to the quantum cohomology class $1 \in H^*(M) \subset QH^*(M)$, which we denote by $1^\flat$ following the notation of [Oh5] and call the semi-infinite fundamental class of $M$. Then according to [Definition 4.2 & Theorem 4.5, Oh5], we have

$$\rho(G; 1) = \inf_{\gamma} \{ \lambda_G(\gamma) \mid \gamma \in \ker \partial_G \subset CF(G) \text{ with } [\gamma] = 1^\flat \}.$$  

(4.1)
\( \rho \) is then extended to arbitrary Hamiltonians by continuity in \( C^0 \)-topology. Therefore to prove Theorem B for the nondegenerate Hamiltonians, we need to first construct cycles \( \gamma \) with \( [\gamma] = 1 \) whose level \( \lambda_G(\gamma) \) become arbitrarily close to \( E^-(G) \), and then to prove that the cycle cannot be pushed down by the Cauchy-Riemann flow.

We recall the following concept of homological essentialness in the chain level theory from [Oh3,6].

**Definition 4.2.** We call a Floer cycle \( \alpha \in CF(H) \) tight if it satisfies the following non-pushing down property under the Cauchy-Riemann flow (3.4): for any Floer cycle \( \alpha' \in CF(H) \) homologous to \( \alpha \) (in the sense of Definition 3.1 (2)), it satisfies

\[
\lambda_H(\alpha') \geq \lambda_H(\alpha). \tag{4.2}
\]

Now we will need to construct a tight fundamental Floer cycle of nondegenerate quasi-autonomous \( H \) whose level is precisely \( E^-(H) \). As a first step, we first construct a fundamental cycle of \( H \) whose level is \( E^-(H) \) which may not be tight in general.

We choose a Morse function \( f \) such that \( f \) has the unique global minimum point \( x^- \) and

\[
f(x^-) = 0, \quad f(x^-) < f(x_j) \tag{4.3}
\]

for all other critical points \( x_j \). Then we choose a fundamental Morse cycle

\[
\alpha = \alpha_{ef} = [x^-, w_{x^-}] + \sum_j a_j [x_j, w_{x_j}]
\]

as in [Oh3] where \( x_j \in \text{Crit}_2n(-f) \). Recall that the positive Morse gradient flow of \( \epsilon f \) corresponds to the negative gradient flow of \( A_{ef} \) in our convention.

Considering Floer’s homotopy map \( h_L \) over the linear path

\[
L : s \mapsto (1 - s)\epsilon f + sH
\]

for sufficiently small \( \epsilon > 0 \), we transfer the above fundamental Morse cycle \( \alpha \) and define a fundamental Floer cycle of \( H \) by

\[
\alpha_H := h_L(\alpha) \in CF(H). \tag{4.4}
\]

We call this particular cycle the canonical fundamental Floer cycle of \( H \).

The following important property of this fundamental cycle was proved by Kerman-Lalonde [KL] for the aspherical case and by the author [Oh6] in general.

**Proposition 4.3.** Suppose that \( H \) is a generic one-periodic Hamiltonian such that \( H_t \) has the unique non-degenerate global minimum \( x^- \) which is fixed and generically under-twisted for any \( t \in [0, 1] \). Suppose that \( f : M \to \mathbb{R} \) is a Morse function such that \( f \) has the unique global minimum point \( x^- \) and \( f(x^-) = 0 \). Then the canonical fundamental cycle has the expression

\[
\alpha_H = [x^-, w_{x^-}] + \beta \in CF(H) \tag{4.5}
\]
for some Floer Novikov chain $\beta \in CF(H)$ with the inequality
\[ \lambda_H(\beta) < \lambda_H([x^-, w^-]) = \int_0^1 -H(t, x^-)\,dt. \] (4.6)

In particular its level satisfies
\[ \lambda_H(\alpha_H) = \lambda_H([x^-, w^-]) \]
\[ = \int_0^1 -H(t, x^-)\,dt = \int_0^1 -\min H\,dt. \] (4.7)

The proof is based on the following simple fact used by Kerman and Lalonde (see the proof of [Proposition 4.2, KL]). We refer to [Lemma 3.5, Oh6] for the details of its proof.

**Lemma 4.4.** Let $H$ and $f$ as in Proposition 3.3. Then for all sufficiently small $\epsilon > 0$, the function $G^H$ defined by
\[ G^H(t, x) = H(t, x^-) + \epsilon f \]
satisfies
\[ G^H(t, x^-) = H(t, x^-) \]
\[ G^H(t, x) \leq H(t, x) \] (4.8)
for all $(t, x)$ and equality holds only at $x^-$.  

**Proof of Proposition 4.3.** Since $x^-$ is a under-twisted fixed minimum of both $H$ and $f$, we have the Conley-Zehnder index
\[ \mu_H([x^-, w^-]) = \mu_f([x^-, w^-])(= n) \]
and so the moduli space $\mathcal{M}_\mathcal{L}([x^-, w^-], [x^-, w^-])$ has dimension zero. Let $u \in \mathcal{M}_\mathcal{L}([x^-, w^-], [x^-, w^-])$.

We note that the Floer continuity equation (3.6) for the linear homotopy
\[ \mathcal{L} : s \to (1 - s)\epsilon f + sH \]
is unchanged even if we replace the homotopy by the homotopy
\[ \mathcal{L}' : s \to (1 - s)G^H + sH. \]
This is because the added term $H(t, x^-)$ in $G^H$ to $\epsilon f$ does not depend on $x \in M$ and so
\[ X_{\epsilon f} \equiv X_{G^H}. \]
Therefore $u$ is also a solution for the continuity equation (3.6) under the linear homotopy $\mathcal{L}'$. Using this, we derive the identity
\[ \int \left| \frac{\partial u}{\partial \tau} \right|_{J^{p_1}(\tau)}^2 \,dt\,d\tau = A_{G^H}([x^-, w^-]) - A_H([x^-, w^-]) \]
\[ - \int_{-\infty}^{\infty} \rho'(\tau) \left( H(t, u(\tau, t))\,dt\,d\tau - G^H(t, u(\tau, t)) \right) \,dt\,d\tau \] (4.9)
from (3.7). Since we have
\[ A_H([x^-, w_{x^-}]) = A_{GH}([x^-, w_{x^-}]) = \int_0^1 -\min H \, dt \] (4.10)
and \( G^H \leq H \), the right hand side of (4.9) is non-positive. Therefore we derive that
\( M^C([x^-, w_{x^-}], [x^-, w_{x^-}]) \) consists only of the constant solution \( u \equiv x^- \). This in particular gives rise to the matrix coefficient of \( h_L \) satisfying
\[ \langle [x^-, w_{x^-}], h_L([x^-, w_{x^-}]) \rangle = \#(M^C([x^-, w_{x^-}], [x^-, w_{x^-}])) = 1. \]

Now consider any other generator of \( \alpha_H \)
\[ [z, w] \in \alpha_H \quad \text{with} \quad [z, w] \neq [x^-, w_{x^-}]. \]
By the definitions of \( h_L \) and \( \alpha_H \), there is a generator \([x, w_x] \in \alpha \) such that
\[ M_L([x, w_x], [z, w_x]) \neq \emptyset. \] (4.11)
Then for any \( u \in M^C([x, w_x], [z, w]) \), we have the identity from (3.7)
\[ A_H([z, w]) - A_{GH}([x, w_x]) = -\int \left| \frac{\partial u}{\partial \tau} \right|_{J_{p_1}(\tau)}^2 \, dt \, d\tau \]
\[ \quad - \int_{-\infty}^{\infty} \rho'(\tau) \left( H(t, u(\tau, t)) - G^H(t, u(\tau, t)) \right) \, dt \, d\tau. \]
Since \(-\int \left| \frac{\partial u}{\partial \tau} \right|_{J_{p_1}(\tau)}^2 \leq 0 \), and \( G^H \leq H \), we have
\[ A_H([z, w]) \leq A_{GH}([x, w_x]) \] (4.12)
with equality holding only when \( u \) is stationary. There are two cases to consider, one for the case of \( x = x^- \) and the other for \( x = x_j \) for \( x_j \neq x^- \) for \([x_j, w_{x_j}] \in \alpha \).

For the first case, since we assume \([z, w] \neq [x^-, w_{x^-}]\), \( u \) cannot be constant and so the strict inequality holds in (4.12), i.e,
\[ A_H([z, w]) < A_{GH}([x^-, w_{x^-}]). \] (4.13)

For the second case, we have the inequality
\[ A_H([z, w]) \leq A_{GH}([x_j, w_{x_j}]) \] (4.14)
for some \( x_j \neq x^- \) with \([x_j, w_{x_j}] \in \alpha \). We note that (4.3) is equivalent to
\[ A_{GH}([x_j, w_{x_j}]) < A_{GH}([x^-, w_{x^-}]). \]
This together with (4.14) again give rise to (4.13). On the other hand we also have
\[ A_{GH}([x^-, w_{x^-}]) = A_H([x^-, w_{x^-}]) \]
because \( G^H(t, x^-) = H(t, x^-) \) from (4.5). Altogether, we have proved
\[ A_H([z, w]) < A_H([x^-, w_{x^-}]) = \int_0^1 -H(t, x^-) \, dt \]
for any \([z, w] \in \alpha_H \) with \([z, w] \neq [x^-, w_{x^-}]. \) This finishes the proof of (4.3). \( \Box \)
§5. Proof of Theorem A’ and Theorem B

In this section, we will give the proof of Theorem B and hence Theorem A’ via
the scheme of the previous section. We now rephrase Theorem B here.

**Theorem 5.1.** Suppose that \((M, \omega)\) is very strongly semi-positive. Let \(H, H'\) be as in Theorem A’.
Then we have

\[
\rho(H'; 1) = E^-(H') = \int_0^1 - \min H' \, dt.
\]

(5.1)

The theorem is an immediate consequence of the following tightness result of
the canonical fundamental Floer cycle constructed in section 4. Here the very strongly
semi-positive condition enters in an essential way.

**Proposition 5.2.** Let \((M, \omega)\) and \(H, H'\) be as in Theorem 5.1 and let \(\alpha_H'\) be the
canonical fundamental Floer cycle of \(H'\). Then \(\alpha_H'\) is tight: i.e., for any Floer
Novikov cycle \(\alpha \in CF(H')\) homologous to \(\alpha_H'\), we have

\[
\lambda_H'(\alpha) \geq \lambda_H'(\alpha_H').
\]

(5.2)

In particular, we have

\[
\rho(H'; 1) = \lambda_H'(\alpha_H')(= E^-(H')).
\]

**Proof.** Suppose that \(\alpha\) is homologous to \(\alpha_H'\), i.e.,

\[
\alpha = \alpha_H' + \partial_H'(\gamma)
\]

(5.3)

for some Floer Novikov chain \(\gamma \in CF(H')\). We need to check whether \([x^-, w_x^-]\) in
the cycle \(\alpha_H'\) can be cancelled by \(\partial_H'(\gamma)\) for a suitable choice of a Floer chain \(\gamma\).
For this purpose, we study the matrix element

\[
([x^-, w_x^-], \partial_H'(z, w])
\]

for each \([z, w] \in \text{Crit}_A\), which in turn need to study the integers

\[
\#(M_{J, H'}([z, w], [x^-, w_x^-])).
\]

There are two types of \([z, w]\) to consider, one \(z = x^-\) and the other \(z \neq x^-\).

We first state the following perturbation lemma whose proof we omit and refer to
[Appendix, KL] for the details, which however considers only the \(C^2\)-perturbations,
but immediately generalizes to the \(C^1\)-perturbations as long as we do not perturb
\(H\) near \(x^-\). See the proof of Proposition 6.4 for some relevant adjustment needed.

**Lemma 5.3.** Suppose that \(H\) has a generically under-twisted local minimum point
\(x^-\). Then there exists a fixed neighborhood \(U_{x^-}\) depending only on \(H\) such that

1. for any sufficiently \(C^1\)-small perturbation \(H'\) of \(H\) with \(H' \equiv H\) on \(U_{x^-}\),
\(x^-\) is the only critical point of \(H'\) in \(U_{x^-}\) which is generically under-twisted and
2. for any non-constant contractible periodic orbit \(z\) of \(H'\), \(\text{Im} \ z \subset M \setminus U_{x^-}\).

We also need one more simple lemma about the lower bound for the energy of
the Floer trajectory connecting \(x^-\) and any other periodic orbits \(z\). Again we omit
its proof which is a simple consequence of compactness arguments (see e.g., [Oh3]
for such an argument).
Lemma 5.4. Let $H$ be any smooth Hamiltonian which is not necessarily regular. Suppose $H$ has the unique critical point $x$ in a neighborhood $U_x$ and the image of no periodic orbits intersect $U_x$. Then there exists a constant $e_H > 0$ such that for any finite energy solution $u : \mathbb{R} \times S^1 \to M$ of (3.4) such that

$u(-\infty) = z, u(\infty) = x$

with $z \neq x$, we have

$$\int \left| \frac{\partial u}{\partial \tau} \right|^2 J \geq e_H > 0$$

(5.4)

where $e_H$ does not depend on $u$.

Let $U_x$ be as in Lemma 5.3. We recall that only the critical points $[z, w]$ with

$$\mu_H([z, w]) = \mu_H([x^-, w_x^-]) + 1 = n + 1$$

(5.5)

can have non-zero matrix element $\langle [x^-, w_x^-], \partial H[z, w] \rangle$. Here the latter equality follows from the generically-undertwistedness of $[x^-, w_x^-]$ and the definition of the Conley-Zehnder index. To provide some intuition on why this is so, we quote the following general result concerning the Conley-Zehnder index and the eigenvalues of the linearization matrix of the autonomous Hamiltonians. We would like to note that [SZ] uses a different convention of the Hamiltonian vector field from ours. According to their convention, our $X_H$ is the negative of theirs. For example, if we apply $S = -\partial^2 G(x^-)$, we have $\mu_G([x^-, w_x^-]) = 2n - n = n$ since $x^-$ is a maximum point of $-G$. One may regard the condition of generically undertwistedness in Definition 4.1 [KL] as the non-autonomous analog to this case.

Lemma 5.5 [Theorem 3.3 (iv), SZ]. Consider the matrix $\Psi(t) = \exp(J_0 St)$ where $J_0$ is the standard almost complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and $S = ST \in M^{2n \times 2n}(\mathbb{R})$ is a non-singular symmetric matrix such all the eigenvalues $\lambda$ satisfies

$$|\lambda| < 2\pi.$$  

(5.6)

Then the Conley-Zehnder index $\mu(\Psi)$ satisfies

$$\mu(\Psi) = \mu^-(S) - n$$

(5.7)

where $\mu^-(S)$ denotes the number of negative eigenvalues of $S$ counted with multiplicity.

We also have the general index formula for the Conley-Zehnder index and the first Chern number

$$\mu_H([z, w]) = \mu_H([z, w']) + 2c_1([w#\overline{w'}]).$$

(5.8)

(In the literature there are various other different conventions used in relation to the definition of $X_H$, that of action functional, and also both homological and cohomological notations have been used. This makes the question about the correct sign for the corresponding formula very confusing. All of our conventions are the same with those in [En] and [Oh3], except that there is an error in the corresponding formula: in [En] and [Oh3], this formula is written as

$$\mu_H([z, w]) = \mu_H([z, w']) - 2c_1([w#\overline{w'}]).$$
At this stage, we first consider the case where $z = x^-$, i.e., $[z, w] = [x^-, w]$ for an arbitrary $w$ bounding disc of $x^-$ which is nothing but a sphere passing through the point $x^-$. In this case (5.8) becomes

$$
\mu_{H'}([x^-, w]) = \mu_{H'}([x^-, w_{x^-}]) + 2c_1([w\#x^-])
$$

when applied to $[z, w'] = [x^-, w_{x^-}]$. We also recall that, only when

$$
\mu_{H'}([x^-, w]) - \mu_{H'}([x^-, w_{x^-}]) = 1,
$$

there can be a Floer trajectory that is issued at $[x^-, w]$ and landing at $[x^-, w_{x^-}]$, and provides a non-trivial matrix element. However this formula is impossible because $2c_1([w\#x^-])$ is an even number. Therefore the matrix elements $([x^-, w_{x^-}], \partial_{H'}[z, w])$ are all zero when $z = x^-$. For this we do not need the condition of $\epsilon$-positively $\mu$-undertwistedness of $H'$ but only the generically undertwistedness of $x^-$. 

Next we consider the case $z \neq x^-$. We first recall that $\mu_{H'}([x^-, w_{x^-}]) = n$ and so for $[z, w]$ to give a nontrivial contribution to the Floer matrix element $([x^-, w_{x^-}], \partial_{H'}[z, w])$ it must be the case that

$$
\mu_{H'}([z, w]) = n + 1.
$$

(5.9)

By the hypothesis of the $\mu$-undertwistedness on $z$ in Theorem 5.1, we have a bounding disc $w_z$ such that the Conley-Zehnder index of $[z, w_z]$ satisfies

$$
-n \leq \mu_{H'}([z, w_z]) \leq n
$$

(5.10)

for any one-periodic orbit $z$ of $H'$.

Now suppose that there exists $[z, w] \in \text{Crit}_{A_{H'}}$ that gives a non-trivial matrix coefficient

$$
([x^-, w_{x^-}], \partial_{H'}[z, w]) \neq 0
$$

and let $u$ be a trajectory from $[z, w]$ to $[x^-, w_{x^-}]$. We consider the glued sphere $w_z\#w\#x^- = w_z\#u$ and $w\#w_z$.

Combining (5.8)-(5.10) applied to $w' = w_z$, we derive

$$
1 \leq 2c_1([w\#w_z]) \leq 2n + 1
$$

and hence

$$
\frac{1}{2} \leq c_1([w\#w_z]) \leq n + \frac{1}{2}.
$$

Because $c_1$ is an integer, we indeed must have

$$
0 < c_1([w\#w_z]) \leq n
$$

or equivalently

$$
-n \leq c_1([w\#w_z]) < 0.
$$

(5.11)

(This inequality is the origin of our imposing the very strongly semi-positive condition in Theorem A.)
Next we consider the actions of \([z, w]\). From (3.5), \(w\#u\#w_x = w\#u\) is contractible, and so we have

\[0 = \int w\#u\omega = \int w\#w_x\omega + \int w_z\omega\]  \hfill (5.12)

We rewrite the second term into

\[\int w_z\omega = \int w_z^*\omega + \int u^*\omega.\]  \hfill (5.13)

For the second term here, we have

\[\int u^*\omega = \int_{-\infty}^{\infty} \int_0^1 \omega \left( \frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial t} \right) dt d\tau\]

\[= \int_{-\infty}^{\infty} \int_0^1 \omega \left( \frac{\partial u}{\partial \tau}, \int \frac{\partial u}{\partial \tau} + X_{H'}(u) \right) dt d\tau\]

\[= \int \left| \frac{\partial u}{\partial \tau} \right|^2_j + \int_{-\infty}^{\infty} \int_0^1 \omega \left( \frac{\partial u}{\partial \tau}, X_{H'}(u) \right) dt d\tau\]

\[= \int \left| \frac{\partial u}{\partial \tau} \right|^2_j + \int_{0}^{1} \left( -H'(t, x^-) + H'(t, z(t)) \right) dt \geq \int \left| \frac{\partial u}{\partial \tau} \right|^2_j. \]  \hfill (5.14)

For the last inequality, we have used the fact that \(x^-\) is the (global) minimum point of \(H'\).

On the other hand, since \(z\) does not intersect \(U_{x^-}\), it follows from Lemma 5.4 that we have

\[\int \left| \frac{\partial u}{\partial \tau} \right|^2_j \geq \epsilon_H > 0. \]  \hfill (5.15)

Therefore, combining (5.12)-(5.15), we obtain

\[\int w_z\omega = \int w_z^*\omega + \int u^*\omega \geq \int w_z^*\omega + \epsilon_H.\]

And by the condition on \(w_z\) in the definition of \(\epsilon\)-positively \(\mu\)-untwistedness of \(H'\), we must have

\[\int w_z^*\omega > -\epsilon, \quad 0 < \epsilon < \epsilon_H\]

and so

\[\int w_z\omega > 0\]  \hfill (5.16)

Then (5.12) and (5.16) imply that we have \(\int w\#w_z < 0\), i.e.,

\[\int w\#w_z > 0\]  \hfill (5.17)

for all critical points \([z, w]\) such that \(z \neq x^-\) and

\([x^-, w_x^-], \partial_{H'}[z, w]) \neq 0.\]
However it follows from the very strongly semi-positive hypothesis that (5.11) and (5.17) together are prohibited and so there is no critical point \([z, w], z \neq x^-\) such that \([\langle x^-, w_{x^-} \rangle, \partial H'[z, w]\rangle \neq 0\).

Altogether, addition of \(\partial H(\gamma)\) to \(\alpha_{H'}\) cannot kill the term \([x^-, w_{x^-}]\) away from the cycle

\[
\alpha_{H'} = [x^-, w_{x^-}] + \beta
\]

in (5.3), and hence we have

\[
\lambda_{H'}(\alpha) = \lambda_{H'}(\alpha_{H'} + \partial H'(\gamma)) \geq \lambda_{H'}([x^-, w_{x^-}])
\]

by the definition of the level \(\lambda_{H'}\). Combining (4.10) and (5.18), we have finished the proof (5.2).

\[\square\]

**Remark 5.7.** Note that (5.17) is not possible for the weakly exact case and so already proves Theorem B in that case without assuming the \(\epsilon\)-positively \(\mu\)-unperturbedness hypotheses for \(H'\).

Finish-up of the proof of Theorem A'. Similar consideration simultaneously applied to \(x^+\), then proves \(\rho(\overline{H'}; 1) = E^-(\overline{H'})\).

Now let \(F \sim H'\). Then Theorem 3.7 implies

\[
\rho(F; 1) = \rho(H'; 1).
\]

Combining this with (3.11) and Theorem 5.1 for \(H'\) and \(\overline{H'}\), we derive

\[
E^-(H') \leq E^-(F) \quad (5.19)
\]

\[
E^-(\overline{H'}) \leq E^-(\overline{F}) \quad (5.20)
\]

Recall that (5.20) is equivalent to

\[
E^+(H') \leq E^+(F) \quad (5.21)
\]

Adding (5.19) and (5.21) then gives rise to the inequality \(||H'|| \leq ||F||\). This proves \(H'\) is length minimizing. Now let \(\epsilon \to 0\). Then we have \(H' \to H\) in the \(C^1\)-topology and so \(\phi_{H'} \to \phi_H\) in the Hofer topology. Since Proposition 5.2 proves that all \(\phi_{H'} : t \mapsto \phi_{H'}^t\) is length minimizing, by letting \(\epsilon \to 0\), we derive from Theorem 2.3 that the Hamiltonian path \(\phi_H : t \mapsto \phi_H^t\) itself is length minimizing.

\[\square\]

\section*{§6. Minimality and the \(C^1\) Perturbation Conjecture}

In this section, we restrict to the case of autonomous Hamiltonians \(G\) and discuss how one might try to prove the Minimality Conjecture, at least for the very strongly semi-positive \((M, \omega)\).

The following perturbation conjecture seems to be the crux towards the proof of the Minimality Conjecture at least for the very strongly semi-positive cases. Applying Lemma 5.3 and Lemma 5.4 to the autonomous Hamiltonian \(G\) in the Minimality Conjecture, the conjecture for the very strongly semi-positive case will immediately follow from the following general perturbation conjecture, via Theorem A' and B.
[C^1 Perturbation Conjecture]. Let (M, ω) be any symplectic manifold and let G be an autonomous Hamiltonian as in the Minimality Conjecture and let ε > 0 be given. Then we can choose a quasi-autonomous smooth Hamiltonian H that satisfies the following properties in addition:

1. H is non-degenerate in the Floer theoretic sense and ε-positively μ-undertwisted.
2. H has the unique fixed minimum point x^- and the unique fixed maximum point x^+ that are generically under-twisted.
3. The C^1-norm ∥G - H∥_{C^1} can be made arbitrarily small as we want.

A more conservative version of the conjecture will be the one restricted to the case when the autonomous Hamiltonian has only isolated critical points. The only non-trivial point of this conjecture lies in the condition 'ε-positively μ-undertwistedness'. Furthermore we will now show that ε-positivity can be automatically ensured by any sufficiently C^1-small perturbation of G. For the remaining section, we will explain this claim and how the C^1 Perturbation Conjecture implies the Minimality Conjecture on the very strongly semi-positive symplectic manifold.

We first introduce the notion of center of mass of the closed curve.

Definition & Lemma 6.1. Fix any Riemannian metric e.g., the already used compatible metric g = ω(·, J·) on M. We denote by inj(g) > 0 the injectivity radius and by exp the exponential map of g. Let z : S^1 → M be a closed continuous curve whose diameter is less than the injectivity radius of M. Then there exists a unique point x_z and a unique closed continuous curve ξ_z : S^1 → T_{x_z}M such that

\[ z(t) = \exp_{x_z} \xi_z(t). \]  (6.1)

The point x_z is called the center of mass of z. If z is C^k, then so is ξ_z.

We refer readers to [K], for example, for the proof of this lemma.

Next we introduce the notion of canonical small bounding disc of any closed curve z : S^1 → M that is contained in a convex ball in the sense of Riemannian geometry.

Definition 6.2. Let B ⊂ M be a closed strongly convex ball. Suppose that the image of a continuous closed curve z : S^1 → M is contained in IntB. Consider the center of mass representation (6.1) of the curve z. We call the disc w_z : D = D^2 → M defined by

\[ w_z(r, t) = \exp_{x_z} (r\xi_z(t)) \]  (6.2)

in the polar coordinates the canonical small bounding disc of z. Furthermore this representation is independent of reparameterization of the curve.

It follows that w_z is always continuous, and is unique up to homotopy in that any two bounding discs of z whose images are contained in the convex ball B are homotopic to each other relative to the boundary.

It is also C^k away from 0 if z is C^k. In particular if z is C^1, the integral

\[ \int_{w_z} \omega \]

is well-defined by defining it to be the integral of any C^1-approximation of w_z inside IntB and fixing its boundary to be z. With this definition, we have the following easy lemma.
Lemma 6.3. Consider the compatible metric \( g = \omega(\cdot, J \cdot) \). If \( z \) is as in Definition 6.2 and is \( C^1 \), then the Riemannian area \( \text{Area}_g(w_z) \) is well-defined and we have the inequality
\[
\left| \int_{w_z} \omega \right| \leq \text{Area}_g(w_z). \tag{6.3}
\]

With these preparations, we prove the following proposition.

Proposition 6.4. Let \( G \) be the autonomous Hamiltonian as in the Minimality Conjecture. Then for any given small \( 0 < \epsilon < \frac{1}{2} \text{inj}(g) \), there exists sufficiently small \( \delta_1, \delta_2 > 0 \) such that for any smooth Hamiltonian \( H : [0, 1] \times M \to \mathbb{R} \) with \( \| H - G \|_{C^1} < \delta_1 \) the following holds: for any periodic orbit \( z \) of period \( T \) with \( 1 - \delta_2 \leq T \leq 1 + \delta_2 \), we have

1. \[
\text{diam}(z) \leq \frac{1}{2} \text{inj}(g) \tag{6.4}
\]

and

2. \[
d_{C^1}(z, x_z) \leq \epsilon \tag{6.5}
\]

and the canonical small bounding disc \( w_z : D \to M \) with \( \partial w_z = z \) satisfies
\[
\int_D w_z^* \omega \geq -\epsilon. \tag{6.6}
\]

Proof. We will prove this by contradiction. Suppose the contrary that there exists some \( \epsilon_0 > 0 \) for which we can choose sequences of one-periodic Hamiltonians \( H_i \) with \( \| H_i - G \|_{C^1} \to 0 \) and a sequence \( z_i \) of periodic orbit of \( X_{H_i} \) of period \( |T_i - 1| \to 0 \) such that one of the following occurs

(i) either \( \text{diam}(z_i) > \frac{1}{2} \text{inj}(g) \) or

(ii) \( \text{diam}(z_i) \leq \frac{1}{2} \text{inj}(g) \) but \( d_{C^1}(z_i, x_{z_i}) > \epsilon_0 \) or

(iii) \( \text{diam}(z_i) \leq \frac{1}{2} \text{inj}(g) \) and \( d_{C^1}(z_i, x_{z_i}) \leq \epsilon_0 \) but \( \int_D w_z^* \omega < -\epsilon_0 \).

Using the fact that the period \( T_i \) satisfies \( |T_i - 1| \to 0 \), we consider the reparamaterized Hamiltonians \( H_i' \) defined by
\[
H_i'(t, x) := T_i H_i(T_i t, x).
\]

We note that \( H_i' \) is not one-periodic but \((1/T_i)\)-periodic. But this is irrelevant as long as it is a smooth function on \( \mathbb{R} \times M \) which certainly is. The reparamaterized orbit
\[
\tilde{z}_i(t) = z_i(T_i t)
\]

now defines a smooth one-periodic map \( \tilde{z}_i : S^1 = \mathbb{R}/\mathbb{Z} \to M \) that satisfies
\[
\dot{\tilde{z}}_i(t) = X_{H_i'}(\tilde{z}_i(t)). \tag{6.7}
\]

From the hypothesis \( \| H_i - G \|_{C^1} \to 0 \) and \( T_i \to 1 \), we have
\[
\max_{(t, x) \in [0, 1] \times M} |X_G(t, x) - X_{H_i'}(t, x)| \to 0. \tag{6.8}
\]
It follows from (6.7) and (6.8) that $|\dot{z}_i|_{C^0} < C$, $C$ independent of $i$. Therefore the sequence $z_i : S^1 \to M$ is equi-continuous and so uniformly converges to a continuous map $z_\infty : S^1 \to M$ which can be easily shown to be a weak solution of the equation $\dot{z} = X_G(z)$. Using the smoothness of $G$, $z_\infty$ is a genuine one-periodic solution which is smooth. Therefore by the hypothesis in the Minimality Conjecture, $z_\infty$ must be a constant solution. This in turn implies that the sequence $\tilde{z}_i$ and so $z_i$ uniformly converges to a constant map. This already rules out the possibility (i) and (ii).

For the case of (iii), since $\tilde{z}_i$ is smooth (and so $C^1$) and

$$\text{diam}(\tilde{z}_i) = \text{diam}(z_i) \leq \frac{1}{2}\text{inj}(g)$$

we can represent $\tilde{z}_i$ in its center of mass representation

$$\tilde{z}_i = \exp_x(\xi'_i), \quad x = x_{\tilde{z}_i} = x_{z_i}.$$  

It follows from the general properties of the exponential map that we have

$$\text{Area}(w'_i) \leq 2\pi C \max_{t \in [0,1]} |\xi'_i|$$

with a constant $C$ depending only on $M$. This converges to zero since $\tilde{z}_i$ uniformly converges to a constant map and so $\xi'_i \to 0$. On the other hand, we have

$$\left| \int_{w'_i} \omega \right| \leq \text{Area}(w'_i) \to 0 \quad (6.9)$$

from (6.3). However (6.9) then contradicts to the third possibility (iii). This finishes the proof. □

Here we would like to emphasize that in this proof we have used only $C^1$-small condition for the perturbation, which only gives rise to the $C^0$-closeness of the corresponding Hamiltonian vector fields. Recall that a $C^0$-small perturbation of a vector field does not imply the uniform convergence of the whole flow. The upshot is that we were only interested in periodic orbits with periods $1 - \delta < T < 1 + \delta$ which enters in the above proof in a crucial way.

By this proposition, we have only to achieve the $\mu$-undertwistedness only for those critical points

$$[z, w_z] \in \text{Crit}_{A_H}$$

that bifurcate from the critical points of the autonomous $G$. The main point of the $C^1$ Perturbation Conjecture is that we hope to be able to change the Conley-Zehnder index arbitrarily using a $C^1$ small but $C^2$ big perturbation so that the Conley-Zehnder index for these critical points lie in the region $-n \leq \mu_H([z, w_z]) \leq n$. As we mentioned before, the requirement about $\mu$-undertwistedness would not be possible if a $C^2$-small perturbation were asked for. This is because both nondegeneracy of periodic orbits and the Conley-Zehnder index are stable under $C^2$-small perturbations.

Since all other requirements will be satisfied by any $C^1$ small perturbation, once we achieve this index condition, we can make another $C^2$-small perturbation afterwards to achieve the Floer theoretic nondegeneracy keeping all other properties
Theorem A will then imply that $H$ is length minimizing. We can apply Theorem 2.3 to prove that $G$ is length minimizing, by letting the $C^1$ perturbation go to zero. This will finish proof of the Minimality Conjecture for the very strongly semi-positive case, once the above $C^1$ Perturbation Conjecture is proven.

One final additional remark is that our conjecture does not contradict to the Arnold conjecture

$$HF^*(H) \cong H^*(M; \Lambda_\omega).$$

It suggests that near the autonomous Hamiltonian as in the Minimality Conjecture, one can find a $C^1$-close nondegenerate Hamiltonian whose Floer complex is very special in that its homologically essential part of the Floer complex resembles that of the Morse complex of nearby Morse functions.

Appendix: Proof of the index formula (1.5)

As we mentioned before, different sign conventions have been used in the definitions of various objects in the literature of symplectic geometry. The only thing that enters in the definition of the Maslov index is, however, a periodic solution of the Hamilton’s equation

$$\dot{x} = X_H(x)$$
on a symplectic manifold $(M, \omega)$ for a one-periodic Hamiltonian function $H : S^1 \times M \to \mathbb{R}$. Our convention is that $X_H$ is defined by

$$X_H | \omega = dH$$
or equivalently $dH(x)(\xi) = \omega(X_H, \xi)$. 

Furthermore the canonical symplectic form of on $T^*\mathbb{R}^n = \mathbb{R}^{2n} \cong \mathbb{C}^n$ in the coordinates $z_j = q_j + ip_j$ is given by

$$\omega_0 = \sum_{j=1}^{2n} dq^j \wedge dp_j. \quad (A.2)$$

This means that on $\mathbb{R}^{2n}$, $X_H = J_0 \nabla H$ where $J_0$ is the standard complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ obtained by multiplication by the complex number $i$. With these being mentioned, we give the proof of the index formula (1.5) in several steps.

0. There is another package of conventions that have been consistently used by Polterovich [Po] and others. In that convention, there are two things to watch out in relation to the index formula, when compared to our convention. The first thing is that their definition of the Hamiltonian vector field, also called as the symplectic gradient and denoted by $\text{sgrad} H$, is given by

$$\text{sgrad} H | \omega = -dH. \quad (A.3)$$

Therefore we have $X_H = -\text{sgrad} H$. The second thing is that their definition of the canonical symplectic form on $T^*\mathbb{R}^n = \mathbb{R}^{2n} \cong \mathbb{C}^n$ in the coordinates $z_j = q_j + ip_j$ is given by

$$\omega'_0 = \sum_{j=1}^{2n} dp_j \wedge dq_j = -\omega_0 \quad (A.4)$$

Cancelling out two negatives, the definition of the Hamiltonian vector field of a function $H$ on $\mathbb{R}^{2n}$ in this package becomes the same vector field as ours that is given by

$$J_0 \nabla H$$
where $\nabla H$ is the usual gradient vector field of $H$ with respect to the standard Euclidean inner product on $\mathbb{R}^{2n}$.

1. We follow the definition from [CZ], [SZ] of the Conley-Zehnder index for a path $\alpha$ lying in $SP^*(1)$ where we denote

$$SP^*(1) = \{ \alpha : [0,1] \to Sp(2n,\mathbb{R}) \mid \alpha(0) = id, \det(\alpha(1) - id) \neq 0 \}$$  \hspace{1cm} (A.5)

following the notation from [SZ]. We denote by $\mu_{CZ}(\alpha)$ the Conley-Zehnder index of $\alpha$ given in [SZ]. Note that the definition of $Sp(2n,\mathbb{R})$ are the same in both of the above two conventions and so the Conley-Zehnder index function $\mu_{CZ} : SP^*(1) \to \mathbb{Z}$ is the same under the above two conventions.

2. A given pair $[\gamma, w] \in \tilde{\Omega}_0(M)$ determines a preferred homotopy class of trivialization of the symplectic vector bundle $\gamma^*TM$ on $S^1 = \partial D^2$ that extends to a trivialization $\Phi_w : w^*TM \to D^2 \times (\mathbb{R}^{2n}, \omega_0)$ over $D^2$ of where $D^2 \subset \mathbb{C}$ is the unit disc with the standard orientation.

3. Let $z : \mathbb{R}/\mathbb{Z} \times M$ be a one-periodic solution of $\dot{x} = X_H(x)$. Any such one-periodic solution has the form $z(t) = \phi_H^t(p)$ for a fixed point $p = z(0) \in \text{Fix}(\phi_H^1)$. When we are given a one-periodic solution $z$ and its bounding disc $w : D^2 \to M$, we consider the one-parameter family of the symplectic maps $d\phi_H^t(z(0)) : T_{z(0)}M \to T_{z(t)}M$

and define a map $\alpha_{[z,w]} : [0,1] \to Sp(2n,\mathbb{R})$ by

$$\alpha_{[z,w]}(t) = \Phi_w(z(t)) \circ d\phi_H^t(z(0)) \circ \Phi_w(z(0))^{-1}.$$  \hspace{1cm} (A.6)

Obviously we have $\alpha_{[z,w]}(0) = id$, and nondegeneracy of $H$ implies that

$$\det(\alpha_{[z,w]}(1) - id) \neq 0$$

and hence

$$\alpha_{[z,w]} \in SP^*(1).$$  \hspace{1cm} (A.7)

Then the Conley-Zehnder index of $[z,w]$ is, by definition, given by

$$\mu_H([z,w]) := \mu_{CZ}(\alpha_{[z,w]}).$$  \hspace{1cm} (A.8)

4. When we are given two maps $w, w' : D^2 \to M$

with $w|_{\partial D^2} = w'|_{\partial D^2}$, we define the glued map $u = w \# w' : S^2 \to M$ in the following way:

$$u(z) = \begin{cases} 
  w(z) & z \in D^+ \\
  w'(1/\bar{z}) & z \in D^-.
\end{cases}$$

Here $D^+$ is $D^2$ with the same orientation, and $D^-$ with the opposite orientation. This is a priori only continuous but we can deform to a smooth one without changing
its homotopy class by ‘flattening’ the maps near the boundary: In other words, we may assume
\[ w(z) = w(z/|z|) \quad \text{for } |z| \geq 1 - \epsilon \]
for sufficiently small \( \epsilon > 0 \). We will always assume that the bounding disc will be assumed to be flat in this sense. With this adjustment, \( u \) defines a smooth map from \( S^2 \).

5. For the given \([z, w], [z, w']\) with a periodic solution \( z(t) = \phi^t_H(z(0)) \), we impose the additional marking condition
\[ \Phi_w(z(0)) = \Phi_{w'}(z(0)) \]  
\[ \text{(A.9)} \]
as a map from \( T_{z(0)}M \) to \( \mathbb{R}^{2n} \) for the trivializations
\[ \Phi_w, \Phi_{w'} : w^*TM \to D^2 \times (\mathbb{R}^{2n}, \omega_0) \]
which is always possible. With this additional condition, we can write
\[ \alpha_{[z, w]}(t) = S_{ww'}(t) \cdot \alpha_{[z, w']}(t) \]  
\[ \text{(A.10)} \]
where \( S_{ww'} : S^1 = \mathbb{R}/\mathbb{Z} \to Sp(2n, \mathbb{R}) \) is the loop defined by the relation (A.10).

Note that this really defines a loop because we have
\[ \alpha_{[z, w]}(0) = \alpha_{[z, w']}(0)(= id) \]  
\[ \alpha_{[z, w]}(1) = \alpha_{[z, w']}(1) \]  
\[ \text{(A.12)} \]
\[ \text{(A.13)} \]
where (A.13) follows from the marking condition (A.9). In fact, it follows from the definition of (A.5) and (A.9) that we have the identity
\[ S_{ww'}(t) = \left( \Phi_w(z(t)) \circ d\phi^t_H(z(0)) \circ \Phi_w(z(0))^{-1} \right) \]
\[ \circ \left( \Phi_{w'}(z(t)) \circ d\phi^t_H(z(0)) \circ \Phi_{w'}(z(0))^{-1} \right)^{-1} \]
\[ = \Phi_w(z(t)) \circ \left( d\phi^t_H(z(0)) \circ \Phi_w(z(0))^{-1} \circ \Phi_{w'}(z(0)) \circ (d\phi^t_H)^{-1}(z(0)) \right) \]
\[ \circ (\Phi_{w'}(z(t)))^{-1}. \]  
\[ \text{(A.14)} \]
Then the marking condition (A.9) implies the middle terms in (A.14) are cancelled away and hence we have proved
\[ S_{ww'}(t) = \Phi_w(z(t)) \circ \Phi_{w'}(z(t))^{-1} \]  
\[ \text{(A.15)} \]
Then we derive the following formula, from the definition \( \mu_{CZ} \) in [CZ] and from (A.10),
\[ \mu_{CZ}(\alpha_{[z, w]}) = 2 \ \text{wind} (\tilde{S}_{ww'}) + \mu_{CZ}(\alpha_{[z, w']}) \]  
\[ \text{(A.16)} \]
where \( \tilde{S}_{ww'} : S^1 \to U(n) \) is a loop in \( U(n) \) that is homotopic to \( S_{ww'} \) inside \( Sp(2n, \mathbb{R}) \). Such a homotopy always exists and is unique up to homotopy because
$U(n)$ is a deformation retract to $Sp(2n, \mathbb{R})$. And $\text{wind}(\hat{S}_{ww'})$ is the degree of the obvious determinant map

$$\det_C(\hat{S}_{ww'}) : S^1 \to S^1.$$ 

6. Finally, we recall the definition of the first Chern class $c_1$ of the symplectic vector bundle $E \to S^2$. We normalize the Chern class so that the tangent bundle of $S^2$ has the first Chern number 2, which also coincides with the standard convention in the literature. We decompose $S^2 = D^+ \cup D^-$, and consider the trivializations $\Phi_+ : E|_{D^+} \to D^2 \times (\mathbb{R}^{2n}, \omega_0)$ and $\Phi_- : E|_{D^-} \to D^2 \times (\mathbb{R}^{2n}, \omega_0)$. Denote by the transition matrix loop

$$\phi_{+-} : S^1 \to Sp(2n, \mathbb{R})$$

which is the loop determined by the equation

$$\Phi_+|_{S^1} \circ (\Phi_-|_{S^1})^{-1}(t, \xi) = (t, \phi_{+-}(t)\xi)$$

for $(t, \xi) \in E|_{S^1}$, where $S^1 = \partial D^+ = \partial D^-$. Then, by definition, we have

$$c_1(E) = \text{wind}(\hat{\phi}_{+-}) \quad (A.17)$$

Now we apply this to $u^*(TM)$ where $u = w\#w'$ and $\Phi_w$ and $\Phi_{w'}$ are the trivializations given in 4. It follows from (A.15) that $S_{ww'}$ is the transition matrix loop between $\Phi_w$ and $\Phi_{w'}$. Then by definition, the first Chern number $c_1(u^*TM)$ is provided by the number $\text{wind}(\hat{S}_{ww'})$ of the loop of unitary matrices

$$\hat{S}_{ww'} : t \mapsto \hat{S}_{ww'}(t) : S^1 \to U(n). \quad (A.18)$$

One can easily check that this winding number, not that of the inverse loop $\hat{S}_{ww'}^{-1}$, is indeed 2 when applied to the tangent bundle of $S^2$ and so consistent with the convention of the Chern class that we are adopting.

7. Combining these steps, we have finally proved

**Theorem C.** Let $(M, \omega)$ be a symplectic manifold and $X_H$ a Hamiltonian vector field defined by

$$X_H|_{\omega} = dH$$

of any one-periodic Hamiltonian function $H : [0, 1] \times M \to \mathbb{R}$. For a given one-periodic solution $z : S^1 = \mathbb{R}/\mathbb{Z} \to M$ of $\dot{x} = X_H(x)$ and two given bounding discs $w, w'$, we have the identity

$$\mu_H([z, w]) = \mu_H([z, w']) + 2c_1([w\#w']).$$

The same formula holds for the other package of conventions given in the paragraph 0 without change.
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