Linear Quadratic Non-Zero Sum Differential Games of Backward Stochastic Differential Equations with Asymmetric Information

Guangchen Wang†  Hua Xiao‡

Abstract

This paper studies backward linear quadratic non-zero sum differential game problem with asymmetric information. Compared with the existing literature, there are two distinct features. One is that the information available to players is asymmetric. The other one is that the system dynamics is described by a backward stochastic differential equation. Nash equilibrium points are obtained for several cases of asymmetric information by stochastic maximum principle and technique of completion square. The systems of some Riccati equations and forward-backward stochastic filtering equations are introduced and the existence and uniqueness of the solutions are proved. Finally, the unique Nash equilibrium point for each case of asymmetric information is represented in a feedback form of the optimal filtering of the state, through the solutions of the Riccati equations.

Key words: Linear quadratic; Filtering; Backward Stochastic Differential Equation; Nash equilibrium; Asymmetric information.

1 Introduction

Throughout this article, we denote by $\mathbb{R}^k$ the $k$-dimensional Euclidean space, $\mathbb{R}^{k \times l}$ the collection of $k \times l$ matrices. The superscript $*$ denotes the transpose of vectors or matrices. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a complete filtered probability space in which $\mathcal{F}_t$ denotes a natural filtration generated by a three dimensional standard Brownian motion $(W_1(t), W_2(t), W_3(t))$, $\mathcal{F} = \mathcal{F}_T$, and $T > 0$ be a fixed time horizon. For a given Euclidean space, we denote by $\langle \cdot, \cdot \rangle$ (resp. $| \cdot |$) the scalar product (resp. norm). We also denote by $\mathcal{L}_{\mathcal{F}_t}^2(0, T; S)$ the space of all $S$-valued, $\mathcal{F}_t$-adapted and square integrable processes, by $\mathcal{L}_{\mathcal{F}_T}^2(\Omega; S)$ the space of all $S$-valued, $\mathcal{F}_T$-measurable and square integrable random variables, by $\mathcal{L}^2(0, T; S)$ the space of all $S$-valued functions sat-
isfying $\int_0^T |f(t)|^2 dt < \infty$, and by $f(t)^2$ the square of $f(t)$. For the sake of simplicity, we set

$$
\begin{align*}
\mathcal{F}^j_t & \triangleq \sigma\{W_j(s), 0 \leq s \leq t\} (j = 1, 2, 3), \\
\mathcal{F}^{1,2}_t & \triangleq \sigma\{W_1(s), W_2(s), 0 \leq s \leq t\}, \\
\mathcal{F}^{2,3}_t & \triangleq \sigma\{W_2(s), W_3(s), 0 \leq s \leq t\}, \\
\hat{h}(t) & \triangleq \mathbb{E}\left(h(t) | \mathcal{F}^{1,2}_t\right), \\
\hat{h}(t) & \triangleq \mathbb{E}\left(h(t) | \mathcal{F}^{2,3}_t\right), \\
\hat{h}(t) & \triangleq \mathbb{E}\left(h(t) | \mathcal{F}^3_t\right), \\
\hat{h}(t) & \triangleq \frac{dh(t)}{dt}.
\end{align*}
$$

(1)

This work is interested in backward linear quadratic (LQ, for short) non-zero sum stochastic differential game with asymmetric information. For simplicity, we only study the case of two players. Let us now begin to specify the problem. Consider the following one-dimensional differential game with asymmetric information. For simplicity, we only study the case of two players. We always use the subscript 1 (resp. 2) to characterize the control variable corresponding to Player 1 (resp. Player 2) the control processes of Player 1 and Player 2, respectively. We always use the subscript 1 (resp. 2) to characterize the control variable corresponding to Player 1 (resp. Player 2) and use the notation $(y^{v_1,v_2}, z_1^{v_1,v_2}, z_2^{v_1,v_2}, z_3^{v_1,v_2})$ to denote the dependence of the state on the control variable $(v_1, v_2)$.

Let $\mathcal{F}_t$ denote the full information up to time $t$ and $\mathcal{G}^i_t \subseteq \mathcal{F}_t$ be a given sub-filtration which represents the information available to Player $i$ ($i = 1, 2$) up to time $t \in [0, T]$. If $\mathcal{G}^1_t \subseteq \mathcal{F}_t$ and $\mathcal{G}^1_t \neq \mathcal{F}_t$, we call the available information partial or incomplete for Player $i$. If $\mathcal{G}^1_t \neq \mathcal{G}^2_t$, we call the available information asymmetric for Player 1 and Player 2. Now we introduce the admissible control set

$$
\mathcal{U}_i = \left\{v_i(\cdot) \in \mathcal{L}^2_{\mathcal{G}^i_t}(0,T; \mathbb{R}) \mid v_i(t) \in U_i, t \in [0, T]\right\},
$$

with $\mathcal{G}^i = \mathcal{G}^i_t (i = 1, 2)$. Each element of $\mathcal{U}_i$ is called an open-loop admissible control for Player $i$ ($i = 1, 2$). And $\mathcal{U}_1 \times \mathcal{U}_2$ is said to be the set of open-loop admissible controls for the players.

Suppose each player hopes to minimize her/his cost functional $J_i(v_1(\cdot), v_2(\cdot))$ by selecting a suitable admissible control $v_i(\cdot) (i = 1, 2)$. In this study, the problem is, under the setting of
asymmetric information, to look for \((u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2\) which is called the Nash equilibrium point of the game, such that

\[
\begin{aligned}
J_1(u_1(\cdot), u_2(\cdot)) &= \min_{v_1(\cdot) \in \mathcal{V}_1} J_1(v_1(\cdot), u_2(\cdot)), \\
J_2(u_1(\cdot), u_2(\cdot)) &= \min_{v_2(\cdot) \in \mathcal{V}_2} J_1(u_1(\cdot), v_2(\cdot)).
\end{aligned}
\]

We call the problem above an asymmetric information backward LQ non-zero sum stochastic differential game. For simplicity, we denote it by Problem \((AI \ BLQDE)\), and denote the state \((y_{u_1, u_2}, z_{1, u_1, u_2}, z_{2, u_1, u_2}, z_{3, u_1, u_2})\) corresponding to the Nash equilibrium point control \((u_1(\cdot), u_2(\cdot))\) by \((y, z_1, z_2, z_3)\).

The LQ problems constitute an extremely important class of optimal control or differential game problems, since they can model many problems in applications, and also reasonably approximate nonlinear control or game problems. On the other hand, there also exist so called partial and asymmetric information problems in real world. For example, investors only partially know the information from security market (see \([3, 24]\)); the principal faces information asymmetric and risk with regards to whether the agent has effectively completed a contract, when a principal hires an agent to perform specific duties (see, e.g. \([10, 11]\)). For more information about LQ control or game problems, the interested readers may refer the following partial list of the works including \([4, 8, 9, 14, 19, 25, 26]\) with complete information, and \([6]\) with partial information, and the references therein.

A BSDE is an Itô's stochastic differential equation (SDE) for which a random terminal condition on the state has been specified. General nonlinear BSDEs, introduced independently by Pardoux and Peng \([12]\) and Duffie and Epstein \([2]\), have received considerable research attention in recent years and wide applicability in number of different areas, such as stochastic control, differential games, recursive utility, partial differential equations, risk measure, mathematical finance. When we say backward stochastic control or backward differential games problems, we means that systems states are governed by BSDEs. For more information about backward LQ control or game problems, refer to \([16, 28]\) with complete information, and \([3, 17]\) with partial information.

It is very important and meaningful to find explicit Nash equilibrium points for differential game problems. When the available information is partial or asymmetric, we need to derive the corresponding optimal filtering of the states and adjoint variables which will be used to represent the Nash equilibrium points. It is very difficult to obtain the equations satisfied by the optimal filtering when the available information is asymmetric for Player 1 and Player 2. Up till now, it seems that there has been no literature about backward LQ differential games with asymmetric information \(\mathcal{G}_1^i\) and \(\mathcal{G}_2^i\). However, in case where \(\mathcal{G}_i^i (i = 1, 2)\) are chosen as certain special forms, we can still derive the filtering equations and then obtain the explicit form of the Nash equilibrium point. In the sequel, we shall study Problem \((AI \ BLQDE)\) under the following four classes of asymmetric information:

(i) \(\mathcal{G}_1^1 = \mathcal{F}_t^{1, 2}\) and \(\mathcal{G}_2^2 = \mathcal{F}_t^{2, 3}\), i.e., the two players possess the common partial information \(\mathcal{F}_t^2\);

(ii) \(\mathcal{G}_1^1 = \mathcal{F}_t^{1, 2}\) and \(\mathcal{G}_2^2 = \mathcal{F}_t^2\), i.e., Player 1 possesses more information than Player 2;

(iii) \(\mathcal{G}_1^1 = \mathcal{F}_t\) and \(\mathcal{G}_2^2 = \mathcal{F}_t\), i.e., Player 1 possesses the full information and Player 2 possesses the partial information;
(iv) \( G_t^1 = F_t^{1,2} \) and \( G_t^2 = F_t^3 \), i.e., the two players possess the mutually independent information.

In Section 3, we shall point out that some other cases similar to (i)-(iv) can be also solved by the same idea and method. To our knowledge, this paper is the first try to study backward LQ non-zero sum differential games in the setting of the asymmetric information.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries which will be used to derive the forward-backward filtering equations and prove the corresponding existence and uniqueness of the solutions. In Section 3.1 we study a special case of the two players possess the same available information \( F_t^i \), and solve the unique Nash equilibrium point which is represented by a feedback of the optimal filter of the state with respect to \( F_t^i \). In Section 3.2 we obtain the unique explicit Nash equilibrium point for each class of asymmetric information above. We also introduce some Riccati equations and represent the unique Nash equilibrium point in a feedback form of the optimal filtering of the state with respect to the corresponding asymmetric information, through the solutions of the Riccati equations. Some conclusions are given in Section 4.

## 2 Preliminary results

In this section, we are going to introduce two lemmas, which will be often used later. First, we present existence and uniqueness for the solutions of the forward-backward stochastic differential equation (FBSDE, for short), whose dynamics is described by

\[
\begin{cases}
-dy = f(t, x, y, z)dt - zdW, \\
\quad dx = b(t, x, y, z)dt + \sigma(t, x, y, z)dW; \\
\quad y(T) = \xi, \quad x(0) = \varphi(y(0)).
\end{cases}
\]

(5)

Here \( x(\cdot) \) satisfies an (forward) SDE, \( (y(\cdot), z(\cdot)) \) satisfies a backward stochastic differential equation, \( W(\cdot) \) is a \( d \)-dimensional standard Brownian motion, \( (x, (y, z)) \) takes value in \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \), and \( b, \sigma, f, \varphi \) are the mappings with suitable sizes. We can see that \( y \) is specified a terminal condition at \( T \) and \( x \) is coupled with \( y \) at initial time 0.

We set \( \mathcal{F}_t^W \triangleq \sigma\{W(s), 0 \leq s \leq t\} \) and introduce the notations

\[
\quad u = (x, y, z)^*, \quad A(t, u) = (-f, b, \sigma)^*(t, u)
\]

and make the following assumption.

**(H1)** \( A(t, u) \) and \( \varphi \) are uniformly Lipschitz continuous with respect to their variables; for each \( y, \varphi(y) \) is in \( \mathcal{L}^2_{\mathcal{F}_T^W}(\Omega; \mathbb{R}^n) \); for every \( (\omega, t) \in \Omega \times [0, T] \), \( b(\omega, t, 0, 0, 0) \in \mathcal{L}^2_{\mathcal{F}_t^W}(0, T; \mathbb{R}^n) \), \( \sigma(\omega, t, 0, 0, 0) \in \mathcal{L}^2_{\mathcal{F}_t^W}(0, T; \mathbb{R}^{n \times d}) \) and \( f(\omega, t, 0, 0, 0) \in \mathcal{L}^2_{\mathcal{F}_t^W}(0, T; \mathbb{R}^n) \).

We also make the following assumption.

**(H2)** The functions \( A(t, u) \) and \( \varphi \) satisfy the monotonic conditions:

\[
\begin{cases}
\langle A(t, u_1) - A(t, u_2), u_1 - u_2 \rangle \leq -\kappa_1|x_1 - x_2|^2 - \kappa_2(|y_1 - y_2|^2 + |z_1 - z_2|^2), \\
\langle \varphi(y_1) - \varphi(y_2), y_1 - y_2 \rangle \leq -\kappa_3|y_1 - y_2|^2, \\
\forall u_1 - u_2 = (x_1 - x_2, y_1 - y_2, z_1 - z_2),
\end{cases}
\]

(6)
Lemma 2.1 (Yu and Ji [27]) If the assumptions (H1) and (H2) hold, then (5) has a unique triple \((x(\cdot), y(\cdot), z(\cdot)) \in \mathcal{L}^2_{\mathcal{F}_t}(0,T; \mathbb{R}^{n+n+n\times d})\).

Remark 2.1 If we assume \(\sigma \equiv 0\), \(\xi\) and all functions are deterministic, then (5) is reduced to a forward-backward ordinary differential equation (FBODE for short)

\[
\begin{align*}
-dy &= f(t, x, y)dt, \\
\quad dx &= b(t, x, y)dt, \\
\quad y(T) &= \xi, \quad x(0) = \phi(y(0)).
\end{align*}
\]

We define the notation \(u = (x, y)^\ast\), \(G(t, u) = (-f, b)^\ast(t, u)\). If \(b, f, \phi\) and \(G\) satisfy the assumptions (H1) and (H2) with \(\mathcal{L}^2_{\mathcal{F}_t}(0,T; S)\) replaced by \(\mathcal{L}^2(0,T; S)\) and \(\phi\) is uniformly bounded, then (7) has a unique solution \((x, y) \in \mathcal{L}^2(0,T; \mathbb{R}^{n+n})\).

The following lemma is from the monograph by Chung [1] (see the example, Section 9.2).

Lemma 2.2 If \(\mathcal{F}_1, \mathcal{F}_2\) and \(\mathcal{F}_3\) are three \(\sigma\)-algebras, and \(\mathcal{F}_1 \vee \mathcal{F}_2\) is independent of \(\mathcal{F}_3\), then, for any integrable random variable \(X \in \mathcal{F}_1\), we have \(E[X|\mathcal{F}_2 \vee \mathcal{F}_3] = E[X|\mathcal{F}_2]\).

3 Nash equilibrium point

In this section, we shall derive the explicit form of the Nash equilibrium point for Problem (AI BLQDE), applying stochastic maximum principle for partial information optimal control problem and the technique of complete square. Further, we also introduce the Riccati equations and represent the Nash equilibrium point as a feedback of the optimal filters \(\hat{y}, \tilde{y}\) and \(\bar{y}\), through the solutions to the Riccati equations.

Appealing to the stochastic maximum principle for partial information backward stochastic control problems (see [5]), or partial information nonzero-sum backward stochastic differential games problems (see [24]), we can derive the following necessary conditions for Problem (AI BLQDE).

Lemma 3.1 If \((u_1, u_2)\) is a Nash equilibrium point for Problem (AI BLQDE), then we have

\[
\begin{align*}
\left\{ \begin{array}{l}
u_1(t) = m_1^{-1}(t)b_1(t)\mathbb{E}\left[x_1(t)|\mathcal{G}_t^1\right], \\
u_2(t) = m_2^{-1}(t)b_2(t)\mathbb{E}\left[x_2(t)|\mathcal{G}_t^2\right],
\end{array} \right.
\end{align*}
\]

where \(((y, z_1, z_2, z_3), x_1, x_2)\) is a solution of the following FBSDE

\[
\begin{align*}
-dy &= \left[a(y) + b_1^2m_1^{-1}\mathbb{E}\left(x_1(t)|\mathcal{G}_t^1\right) + b_2^2m_2^{-1}\mathbb{E}\left(x_2(t)|\mathcal{G}_t^2\right) + \sum_{j=1}^{3} f_jz_j + c\right]dt \\
-z_1dW_1 - z_2dW_2 - z_3dW_3 \\
\quad dx_1 &= [ax_1 - l_1y]dt + f_1x_1dW_1 + f_2x_1dW_2 + f_3x_1dW_3, \\
\quad dx_2 &= [ax_2 - l_2y]dt + f_1x_2dW_1 + f_2x_2dW_2 + f_3x_2dW_3, \\
y(T) &= \xi, \quad x_1(0) = -r_1y(0), \quad x_2(0) = -r_2y(0).
\end{align*}
\]
Lemma 3.2 \( (u_1, u_2) \) in \( \mathcal{S} \) is indeed a Nash equilibrium point for Problem (AI BLQDE).

Proof. For any \( v_1(\cdot) \in \mathcal{U}_1 \), we have

\[
J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) = \frac{1}{2} \mathbb{E} \int_0^T \left[ l_1(y^{v_1,u_2} - y)^2 + m_1(v_1 - u_1)^2 \right] dt + \frac{1}{2} \mathbb{E} \left[ r_1(y^{v_1,u_2}(0) - y(0))^2 \right] + \Theta,
\]

where

\[
\Theta = \mathbb{E} \int_0^T \left[ l_1y(y^{v_1,u_2} - y) + m_1u_1(v_1 - u_1) \right] dt + \mathbb{E} \left[ r_1y(0)(y^{v_1,u_2}(0) - y(0)) \right].
\]

We apply Itô’s formula to \( x_1(y^{v_1,u_2} - y) \) and get

\[
\Theta = \mathbb{E} \int_0^T \left( m_1(t)u_1(t) - b_1(t)x_1(t) \right) (v_1(t) - u_1(t)) dt + \mathbb{E} \left[ r_1y(0)(y^{v_1,u_2}(0) - y(0)) \right].
\]

Then, because \( l_1 \) and \( r_1 \) are nonnegative, and \( m_1 \) is positive, we have

\[
J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \geq 0.
\]

Similarly, for any \( v_2(\cdot) \in \mathcal{U}_2 \), we also have

\[
J_2(u_1(\cdot), v_2(\cdot)) - J_2(u_1(\cdot), u_2(\cdot)) \geq 0.
\]

Therefore, we can conclude that \( (u_1, u_2) \) in \( \mathcal{S} \) is a Nash equilibrium point for Problem (AI BLQDE) indeed. \( \square \)

Combining Lemma 3.1 and Lemma 3.2, we obtain the following theorem.

Theorem 3.1 \((u_1, u_2)\) is a Nash equilibrium point for Problem (AI BLQDE) if and only if \((u_1, u_2)\) has the form denoted by \(\mathcal{S}\) and \(((y, z_1, z_2, z_3), x_1, x_2)\) satisfies FBSDE \((\mathcal{H})\).

Remark 3.1 If \(\mathcal{H}\) has a unique solution, then Problem (AI BLQDE) has a unique Nash equilibrium point. If \(\mathcal{H}\) has many solutions, then Problem (AI BLQDE) may have many Nash equilibrium points. If \(\mathcal{H}\) has no solution, Problem (AI BLQDE) has no Nash equilibrium point. The existence and uniqueness of the Nash equilibrium point for Problem (AI BLQDE) is equivalent to the existence and uniqueness of \(\mathcal{H}\).

Note that under the two general asymmetric information \(\mathcal{G}_1^1\) and \(\mathcal{G}_2^2\), the optimal filters \(\mathbb{E}(y_i(t)|\mathcal{G}_i^t)\) \((i = 1, 2)\) are very abstract which leads to the difficulty in finding the filtering
equations satisfied by \( \mathbb{E} (y_i(t) | G^1_t) \) \( (i = 1, 2) \). In the following, we begin to study Problem (AI BLQDE) under several classes of particular asymmetric information. Though the chosen observable information is a bit special, the mathematical deductions are still highly complicated, and the derived results are interesting and meaningful.

In Section 3.1 we consider the particular case where the two players possess the same available information \( \mathcal{F}_t^1 \), i.e., \( \mathcal{G}_t^1 = \mathcal{G}_t^2 = \mathcal{F}_t^2 \), whose results will be used in the asymmetric information cases in Section 3.2.

### 3.1 The same available information: \( \mathcal{G}_t^1 = \mathcal{G}_t^2 = \mathcal{F}_t^2 \)

In this case, from the notations defined by (1), we have \( \mathbb{E} (x_1(t) | G^1_t) = \tilde{x}_1(t) \) and \( \mathbb{E} (x_2(t) | G^2_t) = \tilde{x}_2(t) \). Then Theorem 3.1 can be rewritten as follows:

**Theorem 3.2** \((u_1, u_2)\) is a Nash equilibrium point if and only if \((u_1, u_2)\) has the following form:

\[
\begin{align*}
    u_1(t) &= m_1^{-1}(t)b_1(t)\tilde{x}_1(t), \\
    u_2(t) &= m_2^{-1}(t)b_2(t)\tilde{x}_2(t),
\end{align*}
\]

(11)

where \(((y, z_1, z_2, z_3), x_1, x_2)\) is a solution of the following FBSDE

\[
\begin{align*}
    -dy &= \left[ ay + b_1^2m_1^{-1}\tilde{x}_1 + b_2^2m_2^{-1}\tilde{x}_2 + \sum_{j=1}^3 f_j z_j + c \right] dt - \sum_{j=1}^3 z_j dW_j, \quad (12a) \\
    dx_1 &= \left[ ax_1 - l_1 y \right] dt + f_1 x_1 dW_1 + f_2 x_1 dW_2 + f_3 x_1 dW_3, \quad (12b) \\
    dx_2 &= \left[ ax_2 - l_2 y \right] dt + f_2 x_2 dW_1 + f_2 x_2 dW_2 + f_3 x_2 dW_3, \quad (12c) \\
    y(T) &= \xi, \quad x_1(0) = -r_1 y(0), \quad x_2(0) = -r_2 y(0). \quad (12d)
\end{align*}
\]

Using Lemma 5.4 in Xiong [23] to (12), we get the optimal filtering \((\tilde{y}, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3), \tilde{x}_1, \tilde{x}_2)\) of \(((y, z_1, z_2, z_3), x_1, x_2)\) with respect to \( \mathcal{F}_t^2 \), which is governed by

\[
\begin{align*}
    -d\tilde{y} &= \left[ a\tilde{y} + b_1^2m_1^{-1}\tilde{x}_1 + b_2^2m_2^{-1}\tilde{x}_2 + \sum_{j=1}^3 f_j \tilde{z}_j + c \right] dt - \tilde{z}_2 dW_2, \quad (13a) \\
    d\tilde{x}_1 &= \left[ a\tilde{x}_1 - l_1 \tilde{y} \right] dt + f_2 \tilde{x}_1 dW_2, \quad (13b) \\
    d\tilde{x}_2 &= \left[ a\tilde{x}_2 - l_2 \tilde{y} \right] dt + f_2 \tilde{x}_2 dW_2, \quad (13c) \\
    \tilde{y}(T) &= \tilde{\xi}, \quad \tilde{x}_1(0) = -r_1 \tilde{y}(0), \quad \tilde{x}_2(0) = -r_2 \tilde{y}(0). \quad (13d)
\end{align*}
\]

Note that (13) is not a standard FBSDE because the additional filtering estimates \( \tilde{z}_1 \) and \( \tilde{z}_3 \) are involved, then generally its existence and uniqueness are not an immediate result of Lemma 2.1 whose detail proof is postponed and put in Theorem 3.3 in order to make the paper easier to read.

We first introduce the systems of ordinary differential equations (ODEs for short)

\[
\begin{align*}
    \dot{\alpha}_1 - b_1^2m_1^{-1}\alpha_1^2 - (2a + f_1^2 + f_3^2 + f_3^2)\alpha_1 - b_2^2m_2^{-1}\alpha_2 + l_1 &= 0, \quad (14a) \\
    \dot{\beta}_1 - (a + b_1^2m_1^{-1}\alpha_1 + f_1^2 + f_2^2 + f_3^2)\beta_1 - b_2^2m_2^{-1}\alpha_2 - ca_1 &= 0, \quad (14b) \\
    \alpha_1(0) &= -r_1, \quad \beta_1(0) = 0, \quad (14c)
\end{align*}
\]
Lemma 3.3 Under the assumption (H3), there exists a unique solution \((\alpha_1, \beta_1, \alpha_2, \beta_2)\) to (14) and (15).

Proof. Let \(\alpha = \alpha_1 + \alpha_2\). From (H3), we have

\[
\dot{\alpha} - b_2^2 m_{-1}^{-1} \alpha^2 + (2a + f_1^2 + f_2^2 + f_3^2) \alpha + l_1 + l_2 = 0 \quad \text{on } (0, T], \quad \alpha(0) = -(r_1 + r_2).
\]  

(16)

Since (16) is a standard Riccati equation, it has a unique solution \(\alpha(\cdot)\). Introduce two auxiliary equations

\[
\begin{align*}
\dot{\alpha}_1 + [(2a + f_1^2 + f_2^2 + f_3^2) - b_1^2 m_{-1}^{-1} \alpha] \alpha_1 + l_1 = 0 & \quad \text{on } (0, T], \quad \dot{\alpha}_1(0) = -r_1, \quad (17) \\
\dot{\alpha}_2 + [(2a + f_1^2 + f_2^2 + f_3^2) - b_2^2 m_{-2}^{-1} \alpha] \alpha_2 + l_2 = 0 & \quad \text{on } (0, T], \quad \dot{\alpha}_2(0) = -r_2, \quad (18)
\end{align*}
\]

where \(\alpha\) is the solution to (16). Obviously, ODEs (17) and (18) have unique solutions \(\bar{\alpha}_1\) and \(\bar{\alpha}_2\), respectively. In addition, we can check that \(\alpha_1\) and \(\alpha_2\) in (14a) and (15a) are also the solutions to equations (17) and (18), respectively. From the uniqueness of solutions of equations (17) and (18), it follows that

\[
\bar{\alpha}_1 = \alpha_1, \quad \bar{\alpha}_2 = \alpha_2,
\]

which implies in turn that (14a) and (15a) have the unique solutions \(\alpha_1\) and \(\alpha_2\).

Let \(\beta = \beta_1 + \beta_2\), then we have

\[
\dot{\beta} - (a + b_1^2 m_{-1}^{-1} \alpha + f_1^2 + f_2^2 + f_3^2) \beta - c \alpha = 0 \quad \text{on } (0, T], \quad \beta(0) = 0,
\]  

(19)

where \(\alpha\) is the solution to equation (16). Note that ODE (19) has a unique solution \(\beta\). Introduce two another auxiliary equations

\[
\begin{align*}
\dot{\beta}_1 - (a + f_1^2 + f_2^2 + f_3^2) \beta_1 - b_1^2 m_{-1}^{-1} \alpha_1 \beta - c \alpha_1 = 0 & \quad \text{on } (0, T], \quad \dot{\beta}_1(0) = 0, \quad (20) \\
\dot{\beta}_2 - (a + f_1^2 + f_2^2 + f_3^2) \beta_2 - b_2^2 m_{-2}^{-1} \alpha_2 \beta - c \alpha_2 = 0 & \quad \text{on } (0, T], \quad \dot{\beta}_2(0) = 0, \quad (21)
\end{align*}
\]

where \(\alpha_1, \alpha_2\) and \(\beta\) are the solutions to (17), (18) and (19), respectively. Similarly, we can prove that (14b) and (15b) also have unique solutions \(\beta_1\) and \(\beta_2\) satisfying

\[
\bar{\beta}_1 = \beta_1, \quad \bar{\beta}_2 = \beta_2.
\]

Based on the arguments above, we can derive the unique analytical expressions for \(\alpha_1, \alpha_2, \beta_1, \beta_2, \alpha\) and \(\beta\). The proof is completed. \(\square\)
We introduce a standard FBSDE

\[
\begin{align*}
-d\tilde{y} &= \left[ a\tilde{y} + (b^2_1m_1^{-1} + f^2_3\alpha_1^{-1})\tilde{x}_1 + (b^2_2m_2^{-1} + f^2_1\alpha_2^{-1})\tilde{x}_2 + f_2\tilde{z}_2 + c \right] dt - \tilde{z}_2 dW_2, \quad (22a) \\
\tilde{x}_1 &= \left[ a\tilde{x}_1 - l_1\tilde{y} \right] dt + f_2\tilde{x}_1 dW_2, \quad (22b) \\
\tilde{x}_2 &= \left[ a\tilde{x}_2 - l_2\tilde{y} \right] dt + f_2\tilde{x}_2 dW_2, \quad (22c) \\
\tilde{y}(T) &= \xi, \quad \tilde{x}_1(0) = -r_1\tilde{y}(0), \quad \tilde{x}_2(0) = -r_2\tilde{y}(0) \quad (22d)
\end{align*}
\]

and an additional assumption as follows.

\((H_4) \ b^2_1(t)m_1^{-1}(t) + f^2_3(t)\alpha_1^{-1}(t) > 0 \) and \( f^2_2(t)\alpha_1^{-1}(t) = f^2_1(t)\alpha_2^{-1}(t), \) \( t \in [0, T]. \)

In the sequel, we set about proving the following lemma.

**Lemma 3.4** Under the assumptions \((H_3)\) and \((H_4)\), equation \( (\tilde{y}, \tilde{z}_2, \tilde{x}_1, \tilde{x}_2) \in L^2_{F_T}(0, T; \mathbb{R}^4) \).

**Proof.** We first introduce another FBSDE

\[
\begin{align*}
-dp &= \left[ ap + (b^2_1m_1^{-1} + f^2_3\alpha_1^{-1})n + f_2q + c \right] dt - q dW_2, \\
-dn &= \left[ an - (l_1 + l_2)p \right] dt + f_2ndW_2, \\
p(T) &= \tilde{\xi}, \quad n(0) = -(r_1 + r_2)p(0).
\end{align*}
\]

If \( (\tilde{y}, \tilde{z}_2, \tilde{x}_1, \tilde{x}_2) \) is a solution to \( (22) \), then \( (n, p, q) \) is a solution to \( (23) \), where

\[
p = \tilde{y}, \quad q = \tilde{z}_2, \quad n = \tilde{x}_1 + \tilde{x}_2.
\]

On the other hand, if \( (p, q, n) \) is a solution to \( (23) \), we introduce the following stochastic differential equation (SDE for short)

\[
\begin{align*}
dn_1 &= \left[ an_1 - l_1p \right] dt + f_2n_1 dW_2, \\
dn_2 &= \left[ an_2 - l_2p \right] dt + f_2n_2 dW_2, \\
n_1(0) &= -r_1p(0), \quad n_2(0) = -r_2p(0). \quad (24)
\end{align*}
\]

From the existence and uniqueness of SDE, \( (24) \) has a unique solution \( (n_1, n_2) \) with \( n_1 + n_2 = n \). Further, we can check that \( (p, q, n_1, n_2) \) is a solution to \( (22) \). In other words, the existence and uniqueness of \( (22) \) is equivalent to that of \( (23) \). It is easy to check that \( (23) \) satisfies the assumptions \((H_1)\) and \((H_2)\). From Lemma \( 2.1 \) it has a unique solution \( (p, q, n) \). So does \( (22) \). \( \square \)

**Theorem 3.3** Under the assumptions \((H_3)\) and \((H_4)\), equation \( (y, \tilde{z}_2, \tilde{x}_1, \tilde{x}_2) \in L^2_{F_T}(0, T; \mathbb{R}^6) \).

**Proof.** We will prove the equivalence between \( (13) \) and \( (22) \) by two steps, which together with Lemma \( 3.4 \) implies the existence and uniqueness of \( (13) \).

Step 1: The solution \( (\tilde{y}, \tilde{z}_2, \tilde{x}_1, \tilde{x}_2) \) of \( (22) \) is a solution of \( (13) \).
Applying Itô’s formula to \( \tilde{y}, \tilde{z}_2, \tilde{x}_1, \tilde{x}_2 \) to (22), then for the known \( \tilde{x}_1 \) and \( \tilde{x}_2 \), similar to Lemma 3.4 equation (12) has a unique solution \( (\tilde{y}, z_1, z_2, z_3), x_1, x_2). \) Noting the terminal condition of (12), we set
\[
x_i = \alpha_i y + \beta_i \text{ with } \alpha_i(0) = -r_i \text{ and } \beta_i(0) = 0, \quad i = 1, 2.
\]

Applying Itô’s formula to \( x_1 \) in (25) subject to (12a), we obtain
\[
dx_1 = \left[ (\dot{\alpha}_1 - a\alpha_1)y - b_1^2 m_1^{-1}\alpha_1 \tilde{x}_1 - b_2^2 m_2^{-1}\alpha_1 \tilde{x}_2 - \sum_{j=1}^{3} f_j \alpha_1 z_j \right. \\
+ \left. (\dot{\beta}_1 - c\alpha_1) \right] dt + \sum_{j=1}^{3} \alpha_1 z_j dW_j.
\]

Substituting (25) into (12b) and comparing the coefficients between (12b) and (26), we have
\[
\alpha_1 z_j = f_j x_1 \equiv f_j (\alpha_1 y + \beta_1), \quad j = 1, 2, 3,
\]

\[
\left( \dot{\alpha}_1 - (2a + \sum_{j=1}^{3} f_j^2)\alpha_1 + l_1 \right) y - b_1^2 m_1^{-1}\alpha_1 \tilde{x}_1 - b_2^2 m_2^{-1}\alpha_1 \tilde{x}_2 + \dot{\beta}_1 - \left( a + \sum_{j=1}^{3} f_j^2 \right) \beta_1 - c\alpha_1 = 0.
\]

Taking \( \mathbb{E} [\cdot | F_t^2] \) on both sides of (25), (27) and (28), it yields
\[
\tilde{x}_i = \alpha_i \tilde{y} + \beta_i, \quad i = 1, 2,
\]

\[
\tilde{z}_1 = f_1 \alpha_1^{-1} \tilde{x}_1, \quad \tilde{z}_2 = f_2 \alpha_1^{-1} \tilde{x}_1, \quad \tilde{z}_3 = f_3 \alpha_1^{-1} \tilde{x}_1
\]
and
\[
\left( \dot{\alpha}_1 - (2a + \sum_{j=1}^{3} f_j^2)\alpha_1 + l_1 \right) \tilde{y} - b_1^2 m_1^{-1}\alpha_1 \tilde{x}_1 - b_2^2 m_2^{-1}\alpha_1 \tilde{x}_2 + \dot{\beta}_1 - \left( a + \sum_{j=1}^{3} f_j^2 \right) \beta_1 - c\alpha_1 = 0.
\]

Plugging (29) in (31), we derive (14).

Applying Itô’s formula to \( x_2 \) in (25), similarly we have
\[
\alpha_2 z_j = f_j x_2 \equiv f_j (\alpha_2 y + \beta_2), \quad j = 1, 2, 3,
\]

\[
\left( \dot{\alpha}_2 - (2a + \sum_{j=1}^{3} f_j^2)\alpha_2 + l_2 \right) y - b_1^2 m_1^{-1}\alpha_2 \tilde{x}_1 - b_2^2 m_2^{-1}\alpha_2 \tilde{x}_2 + \dot{\beta}_2 - \left( a + \sum_{j=1}^{3} f_j^2 \right) \beta_2 - c\alpha_2 = 0.
\]

Taking \( \mathbb{E} [\cdot | F_t^2] \) on both sides of (32) and (33), it yields
\[
\tilde{z}_1 = f_1 \alpha_2^{-1} \tilde{x}_2, \quad \tilde{z}_2 = f_2 \alpha_2^{-1} \tilde{x}_2, \quad \tilde{z}_3 = f_3 \alpha_2^{-1} \tilde{x}_2
\]
and
\[
\left( \dot{\alpha}_2 - (2a + \sum_{j=1}^{3} f_j^2) \alpha_2 + l_2 \right) \dot{y} - b_1^2 m_1^{-1} \alpha_2 \ddot{x}_1 - b_2^2 m_2^{-1} \alpha_2 \ddot{x}_2 + \dot{\beta}_2 = -\left( a + \sum_{j=1}^{3} f_j^2 \right) \beta_2 - c \alpha_2 = 0
\] (35)
subject to (29). Plugging (29) in (35), we derive (15).

In addition, we can affirm from (30) and (34) that
\[
\alpha_1^{-1} \dddot{x}_1 \equiv \alpha_2^{-1} \dddot{x}_2, \quad \alpha_1^{-1} \beta_1 \equiv \alpha_2^{-1} \beta_2,
\] (36)
\[
\dddot{z}_2 = f_2 \dddot{y} + f_2 \alpha_1^{-1} \beta_1 \equiv f_2 \dddot{y} + f_2 \alpha_2^{-1} \beta_2.
\] (37)

According to the third equality of (30) and the first equality of (34), it is easy to see that the solution of (13) is a solution of (12).

Step 2: The solution of (13) is a solution of (12).

If there exists a solution \((\tilde{y}, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{x}_1, \tilde{x}_2)\) to (13), then for the known \(\tilde{x}_1\) and \(\tilde{x}_2\), equation (12) still has a unique solution \(((y, z_1, z_2, z_3, x_1, x_2)\). As shown in Step 1, we can conclude that
\[
\tilde{z}_1 = f_1 \alpha_2^{-1} \tilde{x}_2, \quad \tilde{z}_3 = f_3 \alpha_1^{-1} \tilde{x}_1,
\] (38)
where \(\alpha_1\) and \(\alpha_2\) are the unique solutions of (14) and (15). Putting (38) into (13), we derive (22), which means that the solution of (13) is a solution of (22).

To the end, we are the position to solve the explicit representations of \(((\tilde{y}, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3), \tilde{x}_1, \tilde{x}_2)\). Due to (29), (22a) can be rewritten as
\[
\begin{cases} 
- \dot{d} \tilde{y} = [(a + b_1^2 m_1^{-1} \alpha + f_1^2 + f_3^2) \tilde{y} + f_2 \dddot{z}_2 + (b_1^2 m_1^{-1} + f_3^2 \alpha_1^{-1}) \beta + c] dt - \dddot{z}_2 dW_2, \\
\tilde{y}(T) = \tilde{\xi}.
\end{cases}
\] (39)

Then there exists a unique solution
\[
\tilde{y}(t) = \mathbb{E} \left[ \xi \Gamma^T_t + \int_t^T \Gamma^s_t ( (b_1^2 m_1^{-1}(s) + f_3^2(s) \alpha_1^{-1}(s)) \beta(s) + c(s) ) ds \right]|_{\mathcal{F}^2_t}
\] (40)
with \(\Gamma^s_t = \exp \left\{ \int_t^s [(a(r) + b_1^2 (r)m_1^{-1}(r) \alpha(r) + f_1^2 (r) + f_3^2 (r) - \frac{1}{2} f_3^2 (r)) dr + f_2 (r)dW_2 (r)] \right\} \) and \(\alpha_1\) and \(\beta = \beta_1 + \beta_2\) uniquely given by (14) and (15).

Set
\[
g_1 \triangleq a + b_1^2 m_1^{-1} \alpha + f_1^2 + f_2^2 + f_3^2, \quad g_2 \triangleq (b_1^2 m_1^{-1} + f_3^2 \alpha_1^{-1}) \beta + f_2 \alpha_2^{-1} \beta_2 + c.
\]
Due to (37), (39) can be rewritten as
\[
\begin{cases} 
- \dot{d} \tilde{y} = (g_1 \tilde{y} + g_2) dt - (f_2 \dddot{y} + f_2 \alpha_2^{-1} \beta_2) dW_2, \\
\tilde{y}(T) = \tilde{\xi}.
\end{cases}
\] (41)

Further, from (29) and (30) or (29) and (34), the explicit representations of \((\tilde{x}_1, \tilde{x}_2, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3)\) are derived. Then we have the Nash equilibrium point in the feedback form of the optimal filter \(\tilde{y}(\cdot)\) of the state \(y\) with respect to \(\mathcal{F}^2_t\) as follows.
Theorem 3.4 Under the assumptions \((H_3)\) and \((H_4)\), the unique Nash equilibrium point is denoted by

\[
\begin{align*}
  u_1(t) &= m_1^{-1}(t)b_1(t)(\alpha_1(t)y(t) + \beta_1(t)), \\
  u_2(t) &= m_2^{-1}(t)b_2(t)(\alpha_2(t)y(t) + \beta_2(t)),
\end{align*}
\]

where \(y\) is given by \((30)\), and \(\alpha_i\) and \(\beta_i\) \((i = 1, 2)\) are uniquely determined by \((14)\) and \((15)\).

3.2 The asymmetric information

3.2.1 \(G^1_t = F_t^{1,2}\) and \(G^2_t = F_t^{2,3}\).

In this case, we have \(\mathbb{E}(x_1(t)|G^1_t) = \hat{x}_1(t)\) and \(\mathbb{E}(x_2(t)|G^2_t) = \hat{x}_2(t)\). Hereinafter, we simply call \(\hat{x}_1\) and \(\hat{x}_2\) the optimal filters of \(x_1\) and \(x_2\), respectively, if there is no ambiguity from the notations and context. Then Theorem 3.1 can be rewritten as follows:

Theorem 3.5 \((u_1, u_2)\) is a Nash equilibrium point for Problem (AI BLQDE) if and only if \((u_1, u_2)\) has the form of

\[
\begin{align*}
  u_1(t) &= m_1^{-1}(t)b_1(t)\hat{x}_1(t), \\
  u_2(t) &= m_2^{-1}(t)b_2(t)\hat{x}_2(t),
\end{align*}
\]

where \(((y, z_1, z_2, z_3), x_1, x_2)\) is a solution of the following FBSDE

\[
\begin{align*}
  -dy &= \left[ay + b_1^1m_1^{-1}\hat{x}_1 + b_1^2m_2^{-1}\hat{x}_2 + \sum_{j=1}^{3}f_jz_j + c\right]dt - \sum_{j=1}^{3}z_jdW_j, \\
  dx_1 &= [ax_1 - l_1]\hat{y}dt + f_1x_1dW_1 + f_2x_1dW_2 + f_3x_1dW_3, \\
  dx_2 &= [ax_2 - l_2]\hat{y}dt + f_1x_2dW_1 + f_2x_2dW_2 + f_3x_2dW_3, \\
  y(T) &= \xi, \quad x_1(0) = -r_1\hat{y}(0), \quad x_2(0) = -r_2\hat{y}(0).
\end{align*}
\]

Now it is the position to seek the dynamics of \(\hat{x}_1(t)\) and \(\hat{x}_2(t)\) which will be used to construct the analytical representation of the Nash equilibrium point. Applying Lemma 5.4 in Xiong [23] and Lemma 2.2, we obtain the optimal filters \(\hat{y}\) and \(\hat{x}_1\) of \(y\) and \(x_1\) in \((43a)\) and \((43b)\) with respect to \(F_t^{1,2}\) for Player 1 which satisfy

\[
\begin{align*}
  -d\hat{y} &= \left[a\hat{y} + b_1^1m_1^{-1}\hat{x}_1 + \sum_{j=1}^{3}f_j\hat{z}_j + b_1^2m_2^{-1}\hat{x}_2 + c\right]dt - \hat{z}_1dW_1 - \hat{z}_2dW_2, \\
  d\hat{x}_1 &= [a\hat{x}_1 - l_1]\hat{y}dt + f_1\hat{x}_1dW_1 + f_2\hat{x}_1dW_2, \\
  \hat{y}(T) &= \hat{\xi}, \quad \hat{x}_1(0) = -r_1\hat{y}(0),
\end{align*}
\]

where \(\hat{x}_2(\cdot)\) in \((44a)\) is given by \((29)\) and \((40)\).

Similarly, we can obtain the optimal filters \(\hat{y}\) and \(\hat{x}_2\) of \(y\) and \(x_2\) in \((43a)\) and \((43c)\) with respect to \(F_t^{2,3}\) for Player 2 which satisfy

\[
\begin{align*}
  -d\hat{y} &= \left[a\hat{y} + b_2^1m_2^{-1}\hat{x}_2 + \sum_{j=1}^{3}f_j\hat{z}_j + b_2^2m_1^{-1}\hat{x}_1 + c\right]dt - \hat{z}_2dW_2 - \hat{z}_3dW_3, \\
  d\hat{x}_2 &= [a\hat{x}_2 - l_2]\hat{y}dt + f_1\hat{x}_2dW_2 + f_2\hat{x}_2dW_3, \\
  \hat{y}(T) &= \hat{\xi}, \quad \hat{x}_2(0) = -r_2\hat{y}(0),
\end{align*}
\]

12
where $\tilde{x}_1(\cdot)$ in (45a) is given by (29) and (40).

We can see that the additional term $\tilde{z}_3$ (resp. $\tilde{z}_1$) appears in (44) (resp. 45) which results in the difficulty of proving the existence and uniqueness of solutions. To overcome this, we introduce the ODEs

$$
\begin{align*}
\dot{\gamma}_1 &= -b_1^2m_1^{-1} \gamma_1^2 - (2a + f_1^2 + f_2^2 + f_3^2)\gamma_1 + l_1 = 0, \\
\dot{\gamma}_2 &= (a + f_1^2 + f_2^2 + b_2^2m_2^{-1}\gamma_1 + g_1)\gamma_2 - b_2^2m_2^{-1} \alpha_2 \gamma_1 = 0, \\
\dot{\gamma}_3 &= (a + f_1^2 + f_2^2 + f_3^2 + b_2^2m_2^{-1}\gamma_1)\gamma_3 + (f_2^2\alpha_2^{-1}\beta_2 - g_2)\gamma_2 - (c + b_2^2m_2^{-1}\beta_2)\gamma_1 = 0, (46c) \\
\gamma_1(0) &= -r_1, \gamma_2(0) = 0, \gamma_3(0) = 0
\end{align*}
$$

and

$$
\begin{align*}
\dot{\tau}_1 &= -b_2^2m_2^{-1} \tau_1^2 - (2a + f_1^2 + f_2^2 + f_3^2)\tau_1 + l_2 = 0, \\
\dot{\tau}_2 &= (a + f_1^2 + f_2^2 + b_3^2m_3^{-1}\tau_1 + g_1)\tau_2 - b_3^2m_3^{-1} \alpha_1 \tau_1 = 0, \\
\dot{\tau}_3 &= (a + f_1^2 + f_2^2 + f_3^2 + b_3^2m_3^{-1}\tau_1)\tau_3 + (f_2^2\alpha_2^{-1}\beta_2 - g_2)\tau_2 - (c + b_2^2m_2^{-1}\beta_1)\tau_1 = 0, (47c) \\
\tau_1(0) &= -r_2, \tau_2(0) = 0, \tau_3(0) = 0,
\end{align*}
$$

which are obtained similar to (44) and (45) and have unique solutions $(\gamma_1, \gamma_2, \gamma_3)$ and $(\tau_1, \tau_2, \tau_3)$. See also Lemma 3.5 for more detail. In addition, we introduce the standard FBSDEs

$$
\begin{align*}
-d\dot{y} &= \left[ a\dot{y} + (b_1^2m_1^{-1} + f_2^2\gamma_1^{-1})\dot{x}_1 + f_1\dot{z}_1 + f_2\dot{z}_2 + b_2^2m_2^{-1}\dot{x}_2 + c \right] dt \\
&\quad -\dot{z}_1 dW_1 - \dot{z}_2 dW_2, \\
d\dot{x}_1 &= \left[ a\dot{x}_1 - l_1\dot{y} \right] dt + f_1\dot{x}_1 dW_1 + f_2\dot{x}_1 dW_2, \\
\dot{y}(T) &= \zeta, \quad \dot{x}_1(0) = -r_1\dot{y}(0)
\end{align*}
$$

and

$$
\begin{align*}
-d\dot{y} &= \left[ a\dot{y} + (b_2^2m_2^{-1} + f_3^2\tau_1^{-1})\dot{x}_2 + f_2\dot{z}_2 + f_3\dot{z}_3 + b_2^2m_2^{-1}\dot{x}_1 + c \right] dt \\
&\quad -\dot{z}_2 dW_2 - \dot{z}_3 dW_3, \\
d\dot{x}_2 &= \left[ a\dot{x}_2 - l_2\dot{y} \right] dt + f_2\dot{x}_2 dW_2 + f_3\dot{x}_2 dW_3, \\
\dot{y}(T) &= \zeta, \quad \dot{x}_2(0) = -r_2\dot{y}(0)
\end{align*}
$$

which are subject to the additional assumption as follows.

$$(H_5) \quad b_1^2m_1^{-1} + f_2^2\gamma_1^{-1} > 0 \text{ and } b_2^2m_2^{-1} + f_3^2\tau_1^{-1} > 0.$$

Appealing to Lemma 2.1 under the assumptions $(H_3)$ and $(H_5)$, (48) and (49) have unique solutions. If we can prove the equivalence between (44)-(45) and (48)-(49), then (44) and (45) have unique solutions $((\tilde{y}, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3), \tilde{x}_1)$ and $((\tilde{y}, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3), \tilde{x}_2)$, respectively, which is stated as follows.

**Lemma 3.5** If the assumptions $(H_3)$-$(H_5)$ hold, there exist the unique solutions $((\tilde{y}, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3), \tilde{x}_1)$ and $((\tilde{y}, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3), \tilde{x}_2)$ to (44) and (45), respectively.

**Proof.** The equivalence between (44)-(45) and (48)-(49) is proved by two steps.

Step 1: A solution of (44)-(45) is a solution of (48)-(49).
If (44) and (45) have solutions \(((\hat{y}, \hat{z}_1, \hat{z}_2, \hat{z}_3), \hat{x}_1)\) and \(((\bar{y}, \bar{z}_1, \bar{z}_2, \bar{z}_3), \bar{x}_2)\), we can see that \(\hat{x}_1\) and \(\bar{x}_2\) are dependent on \(\hat{x}_2\) and \(\hat{x}_1\), respectively. Due to \(\hat{x}_i = \alpha_i \hat{y} + \beta_i (i = 1, 2)\) from Section 3.1 \(\hat{x}_1\) and \(\bar{x}_2\) are dependent on \(\bar{y}\). After we plug \(\hat{x}_1\) and \(\bar{x}_2\) into (43), according to Lemma 2.1 and the initial conditions, (43) has a unique solution \(((y, z_1, z_2, z_3), x_1, x_2)\) with \(x_1\) and \(x_2\) depending on both \(y\) and \(\bar{y}\). So we set

\[
x_1 = \gamma_1 y + \gamma_2 \bar{y} + \gamma_3, \tag{50}
\]
\[
x_2 = \tau_1 y + \tau_2 \bar{y} + \tau_3 \tag{51}
\]

with \(\gamma_1(0) = -r_1, \gamma_2(0) = 0, \gamma_3(0) = 0, \tau_1(0) = -r_2, \tau_2(0) = 0, \tau_3(0) = 0, y\) and \(\bar{y}\) satisfying (43a) and (41).

Applying Itô’s formula to \(x_1\) in (50), we have

\[
dx_1 = \left[ (\gamma_1 - a \gamma_1) y + (\gamma_2 - g_1 \gamma_2) \bar{y} - b_1^2 m_1^{-1} \gamma_1 \hat{x}_1 - b_2^2 m_2^{-1} \gamma_1 \bar{x}_2 - \gamma_1 (f_1 z_1 + f_2 z_2 + f_3 z_3) + \gamma_3 - g_2 \gamma_2 + c \right] dt + \gamma_1 z_1 dW_1 + \left[ \gamma_1 \bar{z}_2 + \gamma_2 (f_2 \bar{y} + f_2 a_1^{-1} \beta_2) \right] dW_2 + \gamma_1 z_3 dW_3. \tag{52}
\]

Comparing the drift and diffusion coefficients between (43b) and (52), we conclude that

\[
\begin{aligned}
z_1 &= f_1 \gamma_1^{-1} x_1, \\
z_2 &= \gamma_1^{-1} \left( f_2 \hat{x}_1 - f_2 \gamma_2 \bar{y} - f_2 a_2^{-1} \beta_2 \gamma_2 \right), \\
(\dot{\gamma}_1 - (a + f_1^2 + f_2^2 + f_3^2) \gamma_1) y + (\dot{\gamma}_2 - (g_1 + f_1^2 + f_3^2) \gamma_2) \bar{y} - b_1^2 m_1^{-1} \gamma_1 \hat{x}_1 - b_2^2 m_2^{-1} \gamma_1 \bar{x}_2 + \dot{\gamma}_3 - (f_1^2 + f_2^2 + f_3^2) \gamma_3 + (f_2 a_2^{-1} \beta_2 - g_2) \gamma_2 - c \gamma_1 &= (a \gamma_1 - l_1) y + a \gamma_2 \bar{y} + a \gamma_3. \tag{54}
\end{aligned}
\]

Taking \(\mathbb{E} \left[ \cdot | \mathcal{F}_t^{1, 2} \right] \) on (50), (53) and (54), we have

\[
\begin{aligned}
\hat{z}_1 &= f_1 \gamma^{-1} \hat{x}, \\
\hat{z}_2 &= \gamma^{-1} \left( f_2 \hat{x}_1 - f_2 \gamma \bar{y} - f_2 a^{-1} \beta_2 \gamma \right), \\
(\dot{\gamma}_1 - (a + f_1^2 + f_2^2 + f_3^2) \gamma_1 - b_1^2 m_1^{-1} \gamma_1) \hat{y} + (\dot{\gamma}_2 - (g_1 + f_1^2 + f_3^2 + b_1^2 m_1^{-1} \gamma_1) \gamma_2 - b_2^2 m_2^{-1} \gamma_2) \bar{y} + \dot{\gamma}_3 - (f_1^2 + f_2^2 + f_3^2 + b_1^2 m_1^{-1} \gamma_1) \gamma_3 + (f_2 a^{-1} \beta_2 - g_2) \gamma_2 - (b_2^2 m_2^{-1} \beta_2 + c) \gamma_1 &= (a \gamma_1 - l_1) \hat{y} + a \gamma_2 \bar{y} + a \gamma_3, \tag{57}
\end{aligned}
\]

Then (46) is obtained from (57). Substituting \(\hat{z}_3\) denoted by (56) into (44), we derive (48).

Applying Itô’s formula to \(x_2\) in (51), comparing the coefficients of equations and taking \(\mathbb{E} \left[ \cdot | \mathcal{F}_t^{2, 3} \right] \), we can similarly obtain

\[
\begin{aligned}
\bar{x}_1 &= \gamma_1 \bar{y} + \gamma_2 \hat{y} + \gamma_3, \\
\bar{x}_2 &= \tau_1 \bar{y} + \tau_2 \hat{y} + \tau_3, \tag{58}
\end{aligned}
\]
\[
\begin{aligned}
\bar{z}_1 &= f_1 \tau_1^{-1} \bar{x}_2, \\
\bar{z}_2 &= \tau_1^{-1} \left( f_2 \bar{x}_2 - f_2 \gamma \bar{y} - f_2 a^{-1} \beta_2 \gamma \right), \\
(\dot{\tau}_1 - (a + f_1^2 + f_2^2 + f_3^2) \tau_1 - b_1^2 m_1^{-1} \tau_1) \bar{y} + (\dot{\tau}_2 - (g_1 + f_1^2 + f_3^2 + b_2^2 m_2^{-1} \gamma_1) \tau_2 - b_1^2 m_1^{-1} \alpha_1 \gamma_1) \bar{y} + \dot{\tau}_3 - (f_1^2 + f_2^2 + f_3^2 + b_2^2 m_2^{-1} \gamma_1) \tau_3 + (f_2 a^{-1} \beta_2 - g_2) \tau_2 - (b_2^2 m_2^{-1} \beta_1 + c) \tau_1 &= (a \tau_1 - l_2) \bar{y} + a \tau_2 \hat{y} + a \tau_3. \tag{60}
\end{aligned}
\]

Then (57) is derived from (60). Substituting \(\hat{z}_1\) denoted by (59) into (45), we derive (49). So A solution of (44)–(45) is a solution of (48)–(49).
Step 2: A solution of (48)-(49) is a solution of (44)-(45).

We assume that (48) and (49) have solutions \((\hat{y}, \hat{z}_1, \hat{z}_2)\) and \((\tilde{y}, \tilde{z}_2, \tilde{z}_3, \tilde{x}_2)\). Then (43) has a unique solution \(((y, z_1, z_2, z_3, x_1, x_2))\) for the known \(\hat{x}_1\) and \(\tilde{x}_2\). Similar to Step 1, we can derive the relation:

\[
 f_3 \hat{z}_3 = f_3^2 \gamma_1^{-1} \hat{x}_1, \quad f_1 \hat{z}_1 = f_1^2 \tau_1^{-1} \tilde{x}_2,
\]

which implies (44) and (45) hold, i.e., a solution of (48)-(49) is a solution of (44)-(45).

We set

\[
 g_3 \triangleq a + f_3^2 + b_1^2 m_1^{-1} \gamma_1
\]

\[
 g_4 \triangleq (b_2^2 m_2^{-1} \alpha_2 + b_1^2 m_1^{-1} \gamma_2 + f_3^2 \gamma_1 \gamma_2) \hat{y} + (b_1^2 m_1^{-1} + f_3^2 \gamma_1^{-1}) \gamma_3 + b_2^2 m_2^{-1} \beta_2 + c.
\]

Due to (29), (55) and (56), (49a) can be rewritten as

\[
 \begin{cases}
 -d\hat{y} = [g_3 \hat{y} + f_1 \hat{z}_1 + f_2 \hat{z}_2 + g_4] \, dt - \hat{z}_1 dW_1 - \hat{z}_2 dW_2, \\
 \hat{y}(T) = \hat{\xi},
\end{cases}
\]

which has a unique solution

\[
 \hat{y}(t) = \mathbb{E} \left[ \hat{\xi} T T + \int_0^T \Psi_t \, ds \bigg| F_t^{1,2} \right]
\]

with \(\Psi_t = \exp \left\{ \int_t^s \left[ g_3(r) - \frac{1}{2} f_3^2(r) - \frac{1}{2} f_2^2(r) \right] \, dr + \int_t^s f_1(r) \, dW_1(r) + \int_t^s f_2(r) \, dW_2(r) \right\} \).

We set

\[
 g_5 \triangleq a + f_3^2 + b_2^2 m_2^{-1} \tau_1
\]

\[
 g_6 \triangleq (b_2^2 m_2^{-1} \tau_2 + b_1^2 m_1^{-1} \alpha_1 + f_3^2 \tau_1^{-1}) \tilde{y} + (b_1^2 m_1^{-1} + f_3^2 \tau_1^{-1}) \tau_3 + b_2^2 m_2^{-1} \beta_1 + c.
\]

Due to (29), (58) and (59), (49a) can be rewritten as

\[
 \begin{cases}
 -d\tilde{y} = [g_3 \tilde{y} + f_2 \tilde{z}_2 + f_3 \tilde{z}_3 + g_6] \, dt - \tilde{z}_2 dW_2 - \tilde{z}_3 dW_3, \\
 \tilde{y}(T) = \tilde{\xi},
\end{cases}
\]

which has a unique solution

\[
 \tilde{y}(t) = \mathbb{E} \left[ \tilde{\xi} T T + \int_0^T \Psi_t \, ds \bigg| F_t^{2,3} \right]
\]

with \(\Psi_t = \exp \left\{ \int_t^s \left[ g_5(r) - \frac{1}{2} f_3^2(r) - \frac{1}{2} f_2^2(r) \right] \, dr + \int_t^s f_2(r) \, dW_2(r) + \int_t^s f_3(r) \, dW_3(r) \right\} \).

Based on the arguments above, we derive the Nash equilibrium point which is represented in the feedback of the optimal filters \(\hat{y}, \tilde{y}\) and \(\hat{\gamma}\) of the state \(y\). Then Theorem 3.5 can be rewritten as follows.

**Theorem 3.6** Under the assumption (H3)-(H5), Problem (AI BLQDE) has a unique Nash equilibrium point denoted by

\[
 \begin{cases}
 u_1(t) = m_1^{-1}(t)b_1(t)(\gamma_1(t)\hat{y}(t) + \gamma_2(t)\tilde{y}(t) + \gamma_3(t)), \\
 u_2(t) = m_2^{-1}(t)b_2(t)(\tau_1(t)\hat{y}(t) + \tau_2(t)\tilde{y}(t) + \tau_3(t)),
\end{cases}
\]

where \(\hat{y}, \tilde{y}\) and \(\hat{\gamma}, \tilde{\gamma}\) are shown as (40), (65) and (69) respectively, and \(\gamma_i\) and \(\tau_i\) \((i = 1, 2, 3)\) are uniquely determined by the systems of equations (46) and (47), respectively.
3.2.2 \( \mathcal{G}_t^1 = \mathcal{F}_{t}^{1,2} \) and \( \mathcal{G}_t^2 = \mathcal{F}_{t}^2 \).

In this case, we have \( \mathbb{E} (y_1(t)|\mathcal{G}_t^1) = \tilde{y}_1(t) \) and \( \mathbb{E} (y_2(t)|\mathcal{G}_t^2) = \tilde{y}_2(t) \). Applying the similar methods shown in Section 3.2.1, we can obtain the following theorem.

**Theorem 3.7** \((u_1, u_2)\) is a Nash equilibrium point for Problem (AI BLQDE) if and only if

\[
\begin{align*}
  u_1(t) &= m_1^{-1}(t)b_1(t)\tilde{x}_1(t), \\
  u_2(t) &= m_2^{-1}(t)b_2(t)\tilde{x}_2(t),
\end{align*}
\]

where \(((y, z_1, z_2, z_3), x_1, x_2)\) is a solution of the following FBSDE

\[
-\frac{dy}{dt} = ay + b_1^2 m_1^{-1} \frac{dx_1}{dt} + b_2^2 m_2^{-1} \frac{dx_2}{dt} + \sum_{j=1}^{3} f_j z_j + c dt - \sum_{j=1}^{3} z_j dW_j,
\]

\[
\begin{align*}
  dx_1 &= [a x_1 - l_1 y] dt + f_1 x_1 dW_1 + f_2 x_1 dW_2 + f_3 x_1 dW_3, \\
  dx_2 &= [a x_2 - l_2 y] dt + f_1 x_2 dW_1 + f_2 x_2 dW_2 + f_3 x_2 dW_3, \\
  y(T) &= \xi, \quad x_1(0) = -r_1 y(0), \quad x_2(0) = -r_2 y(0).
\end{align*}
\]

Under the assumptions (H3)-(H5), we can check that the filtering equations (13), (14) and the linear relations (29) and (55) still hold, and the systems of equations (14), (15) and (16) are still uniquely solvable. Then we have the following theorem.

**Theorem 3.8** If (H3)-(H4) hold and \( b_1^2 m_1^{-1} + f_3^2 \gamma_1^{-1} > 0 \), then Problem (AI BLQDE) has a unique Nash equilibrium point denoted by

\[
\begin{align*}
  u_1(t) &= m_1^{-1}(t)b_1(t)\gamma_1(t)\tilde{y}(t) + \gamma_2(t)\tilde{y}(t) + \gamma_3(t), \\
  u_2(t) &= m_2^{-1}(t)b_2(t)\alpha_2(t)\tilde{y}(t) + \beta_2(t),
\end{align*}
\]

where \(\tilde{y}\) and \(\tilde{y}\) are shown in (40) and (55), respectively.

**Remark 3.2** In the similar cases, such as \( \mathcal{G}_t^1 = \mathcal{F}_{t}^{2,3} \) and \( \mathcal{G}_t^2 = \mathcal{F}_{t}^2 \), \( \mathcal{G}_t^1 = \mathcal{F}_{t}^{1,3} \) and \( \mathcal{G}_t^2 = \mathcal{F}_{t}^1 \), the corresponding results can be easily derived.

3.2.3 \( \mathcal{G}_t^1 = \mathcal{F}_t \) and \( \mathcal{G}_t^2 = \mathcal{F}_t^2 \).

In this case, we have \( \mathbb{E} (x_1(t)|\mathcal{G}_t^1) = x_1(t) \) and \( \mathbb{E} (x_2(t)|\mathcal{G}_t^2) = \tilde{x}_2(t) \). Then we have the following theorem.

**Theorem 3.9** \((u_1, u_2)\) is a Nash equilibrium point for Problem (AI BLQDE) if and only if

\[
\begin{align*}
  u_1(t) &= m_1^{-1}(t)b_1(t)x_1(t), \\
  u_2(t) &= m_2^{-1}(t)b_2(t)\tilde{x}_2(t),
\end{align*}
\]

where \(((y, z_1, z_2, z_3), x_1, x_2)\) is a solution of the following FBSDE

\[
\begin{align*}
  -\frac{dy}{dt} &= ay + b_1^2 m_1^{-1} x_1 + b_2^2 m_2^{-1} \tilde{x}_2 + \sum_{j=1}^{3} f_j z_j + c dt - \sum_{j=1}^{3} z_j dW_j, \\
  dx_1 &= [a x_1 - l_1 y] dt + f_1 x_1 dW_1 + f_2 x_1 dW_2 + f_3 x_1 dW_3, \\
  dx_2 &= [a x_2 - l_2 y] dt + f_1 x_2 dW_1 + f_2 x_2 dW_2 + f_3 x_2 dW_3, \\
  y(T) &= \xi, \quad x_1(0) = -r_1 y(0), \quad x_2(0) = -r_2 y(0).
\end{align*}
\]

16
Under the assumptions \((H_3)-(H_4)\), we can check that \(\tilde{y}, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{x}_1, \tilde{x}_2\)) still satisfies (13). From Section 3.1, we know that \(\tilde{y}\) is shown as (40) and \(\tilde{x}_2\) is uniquely represented by (29). Then (73a) and (73d) can be rewritten as

\[
\begin{align*}
-dy &= \left[ay + \sum_{j=1}^{3} f_j y_j + b_1^1 m_1^{-1} x_1 + b_2^1 m_1^{-1} \alpha_2 \tilde{y} + b_2^2 m_2^{-1} \beta_2 + c\right] dt - \sum_{j=1}^{3} z_j dW_j, \\
 dx_1 &= \left[ax_1 - l_1 y\right] dt + f_1 x_1 dW_1 + f_2 x_1 dW_2 + f_3 x_1 dW_3,
\end{align*}
\]

(74a) and (74b)

\[
y(T) = \xi, \quad x_1(0) = -r_1 y(0).
\]

(74c)

From Lemma 2.1, we can say that (74) has a unique solution \((y, z_1, z_2, z_3, x_1)\). Further, the relation between \(x_1\) and \((y, \tilde{y})\) is as follows:

\[
x_1 = \gamma_1 y + \gamma_2 \tilde{y} + \gamma_3,
\]

(75)

where \((\gamma_1, \gamma_2, \gamma_3)\) is the solution to (40), and

\[
y(t) = \mathbb{E} \left[ \xi Y_t^y + \int_0^T Y_t^y g_8(s) ds \big| \mathcal{F}_t \right]
\]

(76)

with

\[
Y_t^y = \exp \left\{ \int_t^s [g_7(r) - \frac{1}{2} \sum_{j=1}^{3} f_j^2(r)] dr + \int_t^s \sum_{j=1}^{3} f_j(r) dW_j(r) \right\},
\]

\[
g_7 = a + b_1^2 m_1^{-1} \gamma_1,
\]

\[
g_8 = (b_1^2 m_1^{-1} \gamma_2 + b_2^2 m_2^{-1} \alpha_2) \tilde{y} + b_2^2 m_1^{-1} \gamma_3 + b_2^2 m_2^{-1} \beta_2 + c.
\]

Then we have the following theorem.

**Theorem 3.10** Under the assumptions \((H_3)\) and \((H_4)\), Problem (AI BLQDE) has a unique Nash equilibrium point denoted by

\[
\begin{align*}
\left\{\begin{array}{l}
u_1(t) = m_1^{-1}(t) b_1(t) (\gamma_1(t) y(t) + \gamma_2(t) \tilde{y}(t) + \gamma_3(t)), \\
u_2(t) = m_2^{-1}(t) b_2(t) (\alpha_2(t) \tilde{x}(t) + \beta_2(t)),
\end{array}\right.
\end{align*}
\]

where \(\tilde{y}\) and \(y\) are shown as (40) and (76), respectively.

**Remark 3.3** In the similar cases, such as \(G^1_t = \mathcal{F}_t^1\) and \(G^2_t = \mathcal{F}_t^2\), \(G^1_t = \mathcal{F}_t\) and \(G^2_t = \mathcal{F}_t^3\), the corresponding results can be easily derived.

**3.2.4** \(G^1_t = \mathcal{F}_t^{1,2}\) and \(G^2_t = \mathcal{F}_t^3\).

In this case, we have \(\mathbb{E} \left( x_1(t) | G^1_t \right) = \tilde{x}_1(t)\) and \(\mathbb{E} \left( x_2(t) | G^2_t \right) = \tilde{x}_2(t)\). Throughout this subsection, we make an additional assumption on equation (2):

\[
(H_6) \quad f_1(t) = f_2(t) = f_3(t) = 0, \quad t \in [0, T].
\]

Similar to Section 3.2.2 and 3.2.3, we directly present the following theorem.
Theorem 3.11 \((u_1, u_2)\) is a Nash equilibrium point for Problem (AI BLQDE) if and only if
\[
\begin{align*}
\dot{u}_1(t) &= m_1^{-1}(t)b_1(t)\hat{x}_1(t), \\
\dot{u}_2(t) &= m_2^{-1}(t)b_2(t)\hat{x}_2(t),
\end{align*}
\] (77)
where \((y, z_1, z_2, z_3, x_1, x_2)\) is a solution of the following FBSDE
\[
\begin{align*}
-d\eta &= \left[ a\eta + b_1^2 m_1^{-1} \dot{x}_1 + b_2^2 m_2^{-1} \dot{x}_2 + c \right] dt - \sum_{j=1}^{3} \eta_j dW_j, \\
dx_1 &= \left[ a\dot{x}_1 - l_1 y \right] dt, \\
dx_2 &= \left[ a\dot{x}_2 - l_2 y \right] dt, \\
y(T) &= \xi, \quad x_1(0) = -r_1 y(0), \quad x_2(0) = -r_2 y(0).
\end{align*}
\] (81a)

Using the similar method shown in Section 3.2.1, we obtain the optimal filters of \(y\) and \(x_1\) in (78a) and (78b) with respect to \(\mathcal{F}^{1,2}_t\) which satisfies
\[
\begin{align*}
-d\hat{y} &= \left[ a\hat{y} + b_1^2 m_1^{-1} \dot{x}_1 + b_2^2 m_2^{-1} \dot{x}_2 + c \right] dt - \sum_{j=1}^{2} \hat{\eta}_j dW_j, \\
\dot{x}_1 &= \left[ a\dot{x}_1 - l_1 \hat{y} \right] dt, \\
\hat{y}(T) &= \hat{\xi}, \quad \dot{x}_1(0) = -r_1 \hat{y}(0).
\end{align*}
\] (79a)

Here we denote by \(\mathbb{E}_T\) the mathematical expectation \(\mathbb{E}(\eta(t))\) of the variable \(\eta(t)\) and omit \(t\) for simplicity. Similarly, we can obtain the optimal filters of \(x\) and \(y_2\) in (78a) and (78c) with respect to \(\mathcal{F}^2_t\) which satisfy
\[
\begin{align*}
-d\hat{y} &= \left[ a\hat{y} + b_1^2 m_1^{-1} \dot{x}_1 + b_2^2 m_2^{-1} \dot{x}_2 + c \right] dt - \hat{\eta}_3 dW_3, \\
\dot{x}_2 &= \left[ a\dot{x}_2 - l_2 \hat{y} \right] dt, \\
\hat{y}(T) &= \hat{\xi}, \quad \dot{x}_2(0) = -r_2 \hat{y}(0).
\end{align*}
\] (80a)

In addition, \(\mathbb{E}x_1\) and \(\mathbb{E}x_2\) together with \(\hat{\mathbb{E}}y\) satisfy an FBODE
\[
\begin{align*}
-d\hat{\mathbb{E}}y &= a\hat{\mathbb{E}}y + b_1^2 m_1^{-1} \hat{\mathbb{E}}x_1 + b_2^2 m_2^{-1} \hat{\mathbb{E}}x_2 + c, \\
\hat{\mathbb{E}}x_1 &= a\hat{\mathbb{E}}x_1 - l_1 \hat{\mathbb{E}}y, \\
\hat{\mathbb{E}}x_2 &= a\hat{\mathbb{E}}x_2 - l_2 \hat{\mathbb{E}}y, \quad \hat{\mathbb{E}}y(T) = \hat{\mathbb{E}}\xi, \quad \hat{\mathbb{E}}x_1(0) = -r_1 \hat{\mathbb{E}}y(0), \quad \hat{\mathbb{E}}x_2(0) = -r_2 \hat{\mathbb{E}}y(0).
\end{align*}
\] (81a)

where \(\hat{\mathbb{E}}\eta\) denotes \(\frac{d\mathbb{E}(\eta(t))}{dt}\) for \(\eta = y, x_1, x_2\).

It is clear that (81) is an FBODE independent of (79) and (80). Using the similar method shown in Lemma 3.3 and Remark 2.1, we conclude that (81) has a unique solution \((\mathbb{E}y, \mathbb{E}x_1, \mathbb{E}x_2)\) under the assumption \((H_3)\). Plugging the solutions \(\mathbb{E}x_2\) and \(\mathbb{E}x_1\) into (79) and (80) respectively and applying Lemma 2.1 we conclude that (79) and (80) have the unique solutions \((\hat{y}, \hat{z}_1, \hat{z}_2, \hat{x}_1)\) and \((\hat{y}, \hat{z}_3, \hat{x}_2)\), respectively. Then we derive the more explicit representation of the Nash equilibrium point in (77) as follows.
Theorem 3.12 Under the assumption \((H_3)\), Problem \((AI\ BLQDE)\) has a unique Nash equilibrium point denoted by

\[
\begin{align*}
\begin{cases}
u_1(t) = \mathcal{m}_1^{-1}(t)b_1(t)\hat{x}_1(t), \\
u_2(t) = \mathcal{m}_2^{-1}(t)b_2(t)\hat{x}_2(t),
\end{cases}
\end{align*}
\]

where \(\hat{x}_1\) and \(\hat{x}_2\) are uniquely determined by the systems of equations \((79)-(81)\).

In the sequel, we only present the results and omit the deduction procedures, because the method and technique are parallel to those in Section 3.2.1.

The relation between \(\mathcal{E} x_i\) and \(\mathcal{E} y\) is as follows:

\[
\mathcal{E} x_i = \alpha_i \mathcal{E} y + \beta_i \quad (i = 1, 2),
\]

where \(\alpha_i, \beta_i, \alpha\) and \(\beta\) are the unique solutions to the systems of equations \((14)-(16)\) and \((19)\) with \(f_i(\cdot) (i = 1, 2, 3)\) replaced by 0, and

\[
\mathcal{E} y(t) = \Gamma^T_t \mathcal{E} x + \int_t^T \Gamma_s \left[ (b_2^2(s)m_1^{-1}(s)\beta(s) + c(s)) \right] ds
\]

with \(\Gamma_t^s = \exp \left\{ \int_t^s [a(r) + b_1^2(r)m_1^{-1}(r)\alpha(r)] dr \right\} .

The relation between \(\hat{x}_1\) and \(\hat{y}, \mathcal{E} y\) is as follows:

\[
\hat{x}_1 = \gamma_1 \hat{y} + \gamma_2 \mathcal{E} y + \gamma_3
\]

where \(\gamma_i (i = 1, 2, 3)\) is the solution to \((46)\) with \(f_i(\cdot) (i = 1, 2, 3)\) replaced by 0, and

\[
\hat{y}(t) = \mathbb{E} \left[ \Gamma_t^T \hat{x} + \int_t^T \Gamma_s \mathcal{E} y] ds | \mathcal{F}_t^{1,2} \right]
\]

with

\[
\Gamma_t^s = \exp \left\{ \int_t^s [a(r) + b_1^2(r)m_1^{-1}(r)\gamma_1(r)] dr \right\}
\]

and

\[
g_0 = (b_2^2m_2^{-1}\alpha_2 + b_1^2m_1^{-1}\gamma_2)\mathcal{E} y + b_2^2m_2^{-1}\gamma_3 + b_2^2m_2^{-1}\beta_2 + c.
\]

The relation between \(\hat{x}_2\) and \(\hat{y}, \mathcal{E} y\) is as follows:

\[
\hat{x}_2 = \tau_1 \hat{y} + \tau_2 \mathcal{E} y + \tau_3
\]

where \(\tau_i (i = 1, 2, 3)\) is the unique solution to \((47)\) with \(f_i(\cdot) (i = 1, 2, 3)\) replaced by 0, and

\[
\hat{y}(t) = \mathbb{E} \left[ \Psi_t^T \hat{x} + \int_t^T \Psi_s \mathcal{E} y] ds | \mathcal{F}_t^{1,2} \right]
\]

with

\[
\Psi_t^s = \exp \left\{ \int_t^s [a(r) + b_2^2(r)m_2^{-1}(r)\tau_1(r)] dr \right\}
\]

and

\[
g_{10} = (b_2^2m_2^{-1}\tau_2 + b_1^2m_1^{-1}\alpha_1)\mathcal{E} y + b_2^2m_2^{-1}\beta_1 + b_2^2m_2^{-1}\tau_3 + c.
\]

Then Theorem 3.12 can be rewritten as follows.
Theorem 3.13 Under the assumption $(H_3)$, Problem (AI BLQDE) has a unique Nash equilibrium point denoted by

\[
\begin{align*}
   u_1(t) &= m_1^{-1}(t)b_1(t)(\gamma_1(t)\dot{y}(t) + \gamma_2(t)\mathbb{E}y(t) + \gamma_3(t)), \\
   u_2(t) &= m_2^{-1}(t)b_2(t)(\tau_1(t)\dot{y}(t) + \tau_2(t)\mathbb{E}y(t) + \tau_3(t)),
\end{align*}
\]

where $\mathbb{E}y, \dot{y}$ and $\dot{y}$ are shown in (83), (85) and (87) respectively, and $\gamma_i$ and $\tau_i$ ($i = 1, 2, 3$) are uniquely determined by the systems of equations (46) and (47) with $f_i(\cdot)$ ($i = 1, 2, 3$) replaced by 0, respectively.

Remark 3.4 In the similar cases 4, $G_1^1 = F_1^{1,3}$ and $G_2^1 = F_1^{2,3}$, and $G_1^2 = F_1^1$, the corresponding results can be easily derived.

4 Conclusion remark

In this paper, we investigate backward LQ non-zero sum differential game problem where the information available to players is asymmetric. We discuss the game problem under the four classes of cases: (i) $G_1^1 = F_1^{1,2}$ and $G_2^1 = F_1^{2,3}$; (ii) $G_1^1 = F_1^{1,2}$ and $G_2^2 = F_2^1$; (iii) $G_1^1 = F_1$ and $G_2^2 = F_2^1$; (iv) $G_1^1 = F_1^{1,2}$ and $G_2^2 = F_2^1$. Some forward-backward stochastic filtering equations with respect to the asymmetric information $G_1^1$ and $G_2^2$ are introduced and the existence and uniqueness of the solutions are proved. Finally, the corresponding unique Nash equilibrium point is represented in a feedback form of the optimal filtering of the state, through the solutions of some Riccati equations.

References

[1] K.L. Chung, A course in probability theory, Third edition. Academic Press, Inc., San Diego, CA, 2001.

[2] C. Duffie and L. Epstein, Stochastic differential utility, Econometrica, vol. 60, pp. 353-394, 1992.

[3] G. Gennotte, Optimal portfolio choice under incomplete information, Journal of Finance, vol. 41, pp. 733-746, 1986.

[4] S. Hamadène, Nonzero sum linear-quadratic stochastic differential games and backward-forward equations. Stochastic Analysis and Applications, vol. 17, no. 1, pp. 117-130, 1999.

[5] J. Huang, G. Wang and J. Xiong, A maximum principle for partial information backward stochastic control problems with applications. SIAM Journal on Control and Optimization, vol. 48, no. 4, pp. 2106-2117, 2009.

[6] Y. Hu, and B. Øksendal, Partial information linear quadratic control for jump diffusions, SIAM Journal on Control and Optimization, vol. 47, no. 4, pp. 1744-1761, 2008.

[7] E. Hui and H. Xiao, Differential games of partial information forwardbackward doubly stochastic differential equations and applications, ESAIM: Control, Optimisation and Calculus of Variations, vol. 20, no. 1, pp. 78-94, 2014.
[8] Q. Meng, General linear quadratic optimal stochastic control problem driven by a Brownian motion and a poisson random martingale measure with random coefficients, Stochastic Analysis and Applications, vol. 32, no. 1, pp. 88-109, 2014.

[9] L. Mou and J. Yong, Two-person zero-sum linear quadratic stochastic differential games by a Hilbert space method, Journal of Industrial and Management Optimization, vol. 2, no. 1, pp. 95-117, 2006.

[10] H. Muller, The first-best sharing rule in the continuous-time principal-agent problem with exponential utility, Journal of Economic Theory, vol. 79, pp. 276-280, 1998.

[11] H. Muller, Asymptotic efficiency in Dynamic Principal-Agent Problems? Journal of Economic Theory, vol. 91, pp. 292-301, 2000.

[12] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation, Systems and Control Letter, vol. 14, no. 1, pp. 55-61, 1990.

[13] S. Tang, The maximum principle for partially observed optimal control of stochastic differential equations, SIAM Journal on Control and Optimization, vol. 36 no. 5, pp. 1596-1617, 1998.

[14] S. Tang, General linear quadratic optimal stochastic control problems with random coefficients: Linear stochastic hamilton systems and backward stochastic riccati equations, SIAM Journal on Control and Optimization, vol. 42, no. 1, pp. 53-75, 2003.

[15] G. Wang, Z. Wu and J. Xiong, Maximum principles for forward-backward stochastic control systems with correlated state and observation noises, SIAM Journal on Control and Optimization, vol. 51, no. 1, pp. 491-524, 2013.

[16] G. Wang and Z. Yu, A pontryagins maximum principle for non-zero sum differential games of BSDEs with applications, IEEE Transactions on Automatic Control, vol. 55, pp. 1742-1747, 2010.

[17] G. Wang and Z. Yu, A partial information non-zero sum differential game of backward stochastic differential equations with applications, Automatica J. IFAC, vol. 48, no. 2, pp. 342-352, 2012.

[18] G. Wang, C. Zhang and W. Zhang, Stochastic maximum principle for mean-field type optimal control under partial information, IEEE Transactions on Automatic Control vol. 59, no. 2, pp. 522-528, 2014.

[19] Z. Wu and Z. Yu, Linear quadratic nonzero-sum differential games with random jumps. Applied Mathematics and Mechanics, vol. 26, no. 8, pp. 1034-1039, 2005.

[20] H. Xiao, The maximum principle for partially observed optimal control of forwardbackward stochastic systems with random jumps, Journal of Systems Science and Complexity, vol. 24, no. 6, pp. 1083-1099, 2011.

[21] H. Xiao and G. Wang, The filtering equations of forward-backward stochastic systems with random jumps and applications to partial information stochastic optimal control, Stochastic Analysis and Applications, vol. 28, no. 6, pp. 1003-1019, 2010.
[22] H. Xiao and G. Wang, A necessary condition for optimal control of initial coupled forward-backward stochastic differential equations with partial information, Journal of Applied Mathematics and Computing, vol. 37, pp. 347-359, 2011.

[23] J. Xiong, An introduction to stochastic filtering theory, London: Oxford University Press, 2008.

[24] J. Xiong and X. Zhou, Mean-variance portfolio selection under partial information, SIAM Journal on Control and Optimization, vol. 46, no. 1, pp. 156-175, 2007.

[25] J. Yong, A leader-follower stochastic linear quadratic differential game, SIAM Journal on Control and Optimization, vol. 41, no. 4, pp. 1015-1041, 2002.

[26] Z. Yu, Linear-quadratic optimal control and nonzero-sum differential game of forward-backward stochastic system, Asian Journal of Control, vol. 14, pp. 173-185, 2012.

[27] Z. Yu and S. Ji, Linear-quadratic nonzero-sum differential game of backward stochastic differential equations, Proceedings of the 27th Chinese Control Conference, July 16-18, Kunming, Yunnan, China, pp. 562-566, 2008.

[28] A. E. B. Lim and X. Y. Zhou, Linear-quadratic control of backward stochastic differential equations, SIAM Journal on Control and Optimization, vol. 40, no. 2, pp. 450-474, 2001.