On the Signature Calculus for finite fields of order square of prime numbers

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Abstract

In [Huang-Raskind 2009], the authors proved that the discrete logarithm problem in a prime finite field is random polynomial time equivalent to computing the ramification signature of a real quadratic field. In this paper, we do this for a quadratic extension of a prime field.

Keywords: discrete logarithm problem, signature calculus, real quadratic field, class field theory, etale fundamental group

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1 Introduction

Let $p$ be a big prime number and $E$ be an elliptic curve over $\mathbb{F}_p$. The computational complexity of the discrete logarithm problem in $\mathbb{F}_p^\times$ or in $E(\mathbb{F}_p)$ is used in public-key encryption schemes. Estimating the greatest lower bound in the complexity of solving the discrete logarithm problem is an important and difficult problem.

In the case of $\mathbb{F}_p^\times$, the best algorithm known so far is the number field sieve. For example, see [Gordon 1993] and [Schirokauer 1993]. It solves the discrete logarithm problem in $\mathbb{F}_p^\times$ in a conjectural running time $L_p\left(\frac{1}{3}, c\right) = \exp((c + o(1))(\log p)^{1/3}(\log \log p)^{2/3}))$, with $c = (64/9)^{1/3}$. In the case of $E(\mathbb{F}_p)$, the problem can be reduced to the discrete logarithm problem in $\mathbb{F}_q^\times$ using the MOV attack ([MOV 1993]), where $q = p^k$ for some integer $k$. The discrete logarithm problem in $\mathbb{F}_q^\times$ can be solved using the function field sieve ( [Adleman 1994] ; [Adleman and Huang 1999]) or a modified number field sieve ([Schirokauer 2000]). Let $e$ be the real number such that $k = (\log q/\log \log q)^e$. Then, the running time of the function field sieve is conjecturally equal to $L_q(\max\{\frac{1}{3}, 1 - e\}, O(1))$, and that of the modified number field sieve is conjecturally equal to $L_q(\max\{\frac{1}{3}, \frac{1}{1+e}\}, O(1))$.

Alternatively, we can estimate the greatest lower bound by studying an equivalent problem of a discrete logarithm problem. In [Huang-Raskind 2009], the authors lifted the discrete logarithm problem in $\mathbb{F}_p^\times$ to a real quadratic field. They defined the “ramification signature” for the real quadratic field and proved that the discrete logarithm problem in $\mathbb{F}_p^\times$ is random polynomial time equivalent to computing the ramification signature of the real quadratic

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field under two heuristic assumptions, namely, an assumption on the class number and an assumption on a global unit of the real quadratic field.

In this paper, we lift the discrete logarithm problem in $\mathbb{F}_p^\times$ to a real quadratic field. We then define the "ramification signature" for the real quadratic field and prove that the discrete logarithm problem in $\mathbb{F}_q^\times$ is random polynomial time equivalent to computing the ramification signature of the real quadratic field, with one heuristic assumptions on the class number. We also show that in the proof of the equivalence in [Huang-Raskind 2009] one can remove the assumption on the global unit. More precisely, we give an improvement (Step 4 in section 3.2.b in the text) on the construction of real quadratic field and global unit that makes the condition in proposition 2 in section 4.1 in [Huang-Raskind 2009] be satisfied automatically.

In section 2, we define the ramification signature for a real quadratic field. In section 3, we prove the equivalence of the discrete logarithm problem in $\mathbb{F}_q^\times$ and the computation of a ramification signature of a real quadratic field. Consequently, we also prove the equivalence in [Huang-Raskind 2009] without the assumption on the global unit.

2 Signature

To define the ramification signature for a real quadratic field, we need a proposition.

Proposition 2.1. Let $l$ and $p$ be two distinct odd prime numbers, and $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field that splits over $l$ and inerts over $p$. We denote the ring of integers in $K$ by $A_K$, the points over $l$ by $u$ and $\tilde{u}$, and the point over $p$ in $\text{Spec}A_K$ by $v$. Let $I_u, I_\tilde{u}$, and $I_v$ be the prime ideals of $A_K$ corresponding to $u$, $\tilde{u}$, and $v$, respectively. Let $Z := \{u, v\}$, and $U := \text{Spec}A_K \setminus Z$. Let $A_u$ and $A_v$ be the completions of $A_K$ at $u$ and $v$ respectively. Denote $A_U := \Gamma(U, O_U)$.

Suppose that $p^2 - 1$ is divisible by $l$, the class number of $K$ is not divisible by $l$, and there is a unit $\alpha \in A_K^\times$, such that

$$\alpha^{l-1} \not\equiv 1 \mod I_u^2, \quad \alpha^{p^2 - 1} \not\equiv 1 \mod I_v.$$

Then, we have the following:

a. There is an exact sequence

$$1 \longrightarrow A_K^\times / A_K^{\times l} \longrightarrow \frac{A_u^\times / A_u^{\times l}}{A_\tilde{u}^\times / A_\tilde{u}^{\times l}} \longrightarrow A_v^\times / A_v^{\times l} \longrightarrow \pi_1(U)^{ab} / \pi_1(U)^{abl} \longrightarrow 1.$$  \hspace{1cm} (2.2.1)

b. $\text{dim}_{\mathbb{Z}/l\mathbb{Z}} \pi_1(U)^{ab} / \pi_1(U)^{abl} = 1$, where the $\pi_1(U)$ is the etale fundamental group of $U$ (see, for example, [Milne 1980]);

c. For any nontrivial character $\chi : \pi_1(U) \longrightarrow \mathbb{Z}/l\mathbb{Z}$, $\chi$ is ramified at both $u$ and $v$.

Proof.

a. Let us consider the following commutative diagram:

$$\begin{array}{ccccccc}
1 & \longrightarrow & A_K^\times / A_K^{\times l} & \longrightarrow & \frac{A_u^\times / A_u^{\times l}}{A_\tilde{u}^\times / A_\tilde{u}^{\times l}} & \longrightarrow & \frac{A_v^\times / A_v^{\times l}}{\{\pm 1\}^\oplus 2} \oplus \frac{K_\times / A_u^{\times l}}{K_\times / A_v^{\times l}} \oplus \{\pm 1\}^\oplus 2 \oplus \frac{\text{Div}(K)}{\text{Div}(K)} \longrightarrow 0.
\end{array}$$
Through the snake lemma and class field theory, we have the following exact sequence:

\[ A_K^\times \rightarrow \{ \pm 1 \} \oplus A_u^\times /A_u^\times l \oplus A_v^\times /A_v^\times l \rightarrow \pi_1(U)^{ab}/Im(A_u^\times l \oplus A_v^\times l) \rightarrow Cl(K) \rightarrow 1 \]

where the term \( Im(A_u^\times l \oplus A_v^\times l) \) is the image of \( A_u^\times l \oplus A_v^\times l \) under the reciprocity map

\[ K_u^\times \oplus K_v^\times \rightarrow \text{Gal}(K_u^{ab}/K_u) \oplus \text{Gal}(K_v^{ab}/K_v) \rightarrow \pi_1(U)^{ab}. \]

As the class number of \( K \) is assumed non-divisible by \( l \), a diagram chasing shows an exact sequence

\[ A_K^\times /A_K^{\times l} \rightarrow A_u^\times /A_u^{\times l} \oplus A_v^\times /A_v^{\times l} \rightarrow \pi_1(U)^{ab}/\pi_1(U)^{abl} \rightarrow 1. \]

The hypothesis on the existence of the global unit shows that the left morphism is nonzero. Thus it is injective, since \( A_u^\times l \oplus A_v^\times l \) is an \( \mathbb{Z}/l\mathbb{Z} \)-linear space of dimension 1. Therefore, we obtain the exact sequence (2.2.1).

b. The complete discrete valuation rings \( A_u \) and \( A_v \) are isomorphic to \( \mathbb{Z}_d \) and \( W(\mathbb{F}_{p^2}) \) (the Witt ring over \( \mathbb{F}_{p^2} \)) respectively. Therefore, the middle term in the sequence (2.2.1) is isomorphic to \( (\mathbb{Z}/l^2\mathbb{Z})^\times /((\mathbb{Z}/l^2\mathbb{Z})^\times \oplus \mathbb{F}_l^\times /\mathbb{F}_p^\times l \) and is of \( \mathbb{Z}/l\mathbb{Z} \)-dimension 2. Since we know that \( A_K^\times /A_K^{\times l} \) is a \( \mathbb{Z}/l\mathbb{Z} \)-linear space of dimension 1, the right term in (2.2.1) has \( \mathbb{Z}/l\mathbb{Z} \)-dimension 1.

c. We consider the dual sequence

\[
0 \rightarrow \text{Hom}(\pi_1(U)^{ab}, \mathbb{Z}/l\mathbb{Z}) \rightarrow j^* \text{Hom}(A_u^\times /A_u^{\times l} \oplus A_v^\times /A_v^{\times l}, \mathbb{Z}/l\mathbb{Z}) \rightarrow \text{Hom}(A_K^\times /A_K^{\times l}, \mathbb{Z}/l\mathbb{Z}) \rightarrow 0
\]

(2.2.2)

of (2.2.1). Denote the image of \( \alpha \) under the morphism \( i \) by \( (\alpha_u, \alpha_v) \). For any \( \chi \neq 0 \in \text{Hom}(\pi_1(U)^{ab}, \mathbb{Z}/l\mathbb{Z}) \), denote the image of \( \chi \) under the morphism \( j^* \) by \( (\chi_u, \chi_v) \), we then have

\[
\langle \alpha_u, \chi_u \rangle + \langle \alpha_v, \chi_v \rangle = 0
\]

(2.2.3)

by (2.2.2). Therefore, the following four conditions are equivalent:

(i) \( \chi \) is ramified at \( u \),
(ii) \( \langle \alpha_u, \chi_u \rangle \neq 0 \),
(iii) \( \langle \alpha_v, \chi_v \rangle \neq 0 \),
(iv) \( \chi \) is ramified at \( v \).

The map \( j^* \) is injective, indicating that there is not non-trivial character \( \chi : \pi_1(U) \rightarrow \mathbb{Z}/l\mathbb{Z} \) such that it is unramified at both points \( u \) and \( v \). Therefore, for any non-trivial \( \chi \in \text{Hom}(\pi_1(U), \mathbb{Z}/l\mathbb{Z}) \), \( \chi \) must be ramified at both \( u \) and \( v \).

The following corollary is proved in the proof of Proposition 2.1c.

**Corollary 2.2.** Under the conditions in proposition 2.1, for any non-trivial character \( \chi : \pi_1(U) \rightarrow \mathbb{Z}/l\mathbb{Z} \), we have the following:

(i) \( \langle \alpha_u, \chi_u \rangle \neq 0 \),
(ii) \( \langle \alpha_v, \chi_v \rangle \neq 0 \),
(iii) \( \langle \alpha_u, \chi_u \rangle + \langle \alpha_v, \chi_v \rangle = 0 \).
Through the natural isomorphism $A^\times_K/A^\times_u \cong (\mathbb{Z}/l^2\mathbb{Z})^\times/(\mathbb{Z}/l^2\mathbb{Z})^\times_l$, $A^\times_K/A^\times_u$ is generated by $1 + l$. For any generator $g$ of $\mathbb{F}^\times_{p^l}/\mathbb{F}^\times_{p^2}$, we regard it as a generator of $A^\times_K/A^\times_u$ through the natural isomorphism $A^\times_v/A^\times_u \cong \mathbb{F}^\times_{p^l}/\mathbb{F}^\times_{p^2}$. Clearly, $(1 + l, \chi_u)^{-1}(g, \chi_v)$ is independent of the choice of $\chi \neq 0 \in \text{Hom}(\pi_1(U), \mathbb{Z}/l\mathbb{Z})$. We call this term the ramification signature of $U$ with respect to $g$.

3 Signature computation problem and discrete logarithm problem in $\mathbb{F}^\times_{p^2}$

In this section, we show that the discrete logarithm problem in $\mathbb{F}^\times_{p^2}$ is random polynomial time equivalent to computing the ramification signature of some real quadratic field.

3.1 Reduction from signature computation problem to discrete logarithm problem

Suppose given $p, l, K = \mathbb{Q}(\sqrt{D}), U, u, \tilde{u}, v, \alpha, g$, as in Proposition 2.1. Then the computation of the ramification signature of $U$ with respect to $g$ can be reduced to a discrete logarithm problem in $\mathbb{F}_{p^2}$ as follows by using Corollary 2.2.

Let us consider the following commutative diagram:

$$
\begin{array}{cccc}
A^\times_K & \longrightarrow & A^\times_u & \sim \longrightarrow & \mathbb{Z}^\times_l & \longrightarrow & \mathbb{Z}^\times_l/\mathbb{Z}^\times_l \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\mathbb{Z}/l^2\mathbb{Z})^\times & \longrightarrow & (\mathbb{Z}/l^2\mathbb{Z})^\times/(\mathbb{Z}/l^2\mathbb{Z})^\times_l.
\end{array}
$$

If the image in $(\mathbb{Z}/l^2\mathbb{Z})^\times$ of $\alpha$ equals $\xi(1 + l)^y$, where $\xi$ is an $(l - 1)$-st root of unity, then its image in $(\mathbb{Z}/l^2\mathbb{Z})^\times/(\mathbb{Z}/l^2\mathbb{Z})^\times_l$ will be equal to $(1 + l)^y$. We can easily compute $\xi, y$ and consequently the first term in (2.2.3) $\langle \alpha_u, \chi_u \rangle = y(1 + l, \chi_v)$.

For the second term in (2.2.3), if the image of $\alpha$ under the morphism $A^\times_K \longrightarrow A^\times_v/A^\times_u \cong \mathbb{F}^\times_{p^l}/\mathbb{F}^\times_{p^2}$ is $a = g^m$, then $\langle \alpha_v, \chi_v \rangle = m(\langle g, \chi_v \rangle)$.

By Corollary 2.2, if we can compute $m$ from $a = g^m$, then we can compute $$(1 + l, \chi_u)^{-1}(g, \chi_v) = -m^{-1}y \in \mathbb{Z}/l\mathbb{Z}.$$

3.2 Reduction from discrete logarithm problem to signature computation problem

Let $g$ be a generator of $\mathbb{F}^\times_{p^2}$, $a \in \mathbb{F}^\times_{p^2}$ and $l$ be a prime dividing $p^2 - 1$. The computation of discrete logarithm $\log_g a \mod l$ can then be reduced to computing the ramification signature of a real quadratic field as follows, by using Corollary 2.2.

Let $m \equiv \log_g a \mod l$. If $a \in \mathbb{F}^\times_{p^2}$, then we have $m \equiv 0 \mod l$. Thus we can suppose $a \notin \mathbb{F}^\times_{p^l}$.

a. If $l \nmid p - 1$, we must have $l|p + 1$. Let $\tilde{a} := a^{p-1}, \tilde{g} := g^{p-1}$. We then have

$$
\tilde{a} = \tilde{g}^m, \quad Nm(\tilde{a}) = Nm(\tilde{g}) = 1, \quad \tilde{a} \notin \mathbb{F}^\times_{p^2}.
$$
We take \( t \in \mathbb{F}_p \) such that \( \left( \frac{t}{p} \right) = -1 \). We have \( \mathbb{F}_{p^2} = \mathbb{F}(\sqrt{t}) \). We put \( \bar{a} = a_0 + b_0 \sqrt{t} \), where \( a_0, b_0 \in \mathbb{F}_p \). We can assume \( b_0 \neq 0 \); otherwise, \( \bar{a}^2 = 1 \) and \( m = \frac{p+1}{2} \) or \( p+1 \).

We have \( a_0^2 - b_0^2t = Nm(a) \equiv 1 \mod p \). Hence, for any \( k \in \mathbb{Z} \), the following holds:

\[
\left( \frac{(a_0 + kp)^2 - 1}{p} \right) = \left( \frac{a_0^2 - 1}{p} \right) = \left( \frac{b_0^2t}{p} \right) = \left( \frac{t}{p} \right) = -1.
\]

We choose \( k \in \{0, 1, \ldots, l-1\} \) randomly, until \( \left( \frac{(a_0 + kp)^2 - 1}{p} \right) = 1 \). Lemma 3.1 below for \( c = 1 \) shows that we can obtain such \( k \) with probability about 50% each time.

If we find such \( k \), let \( a_1 := a_0 + kp \in \mathbb{Z}_x \). We have \( \sqrt{a_1^2 - 1} \in \mathbb{Z}_d \) because \( \left( \frac{a_1^2 - 1}{p} \right) = 1 \). If \( (a_1 + \sqrt{a_1^2 - 1})^{l-1} \equiv 1 \mod l^2 \), we know also \( (a_1 - \sqrt{a_1^2 - 1})^{l-1} \equiv 1 \mod l^2 \), let \( x = a_1 \). Else, let \( x = a_1 + pl \). Lemma 3.2 below for \( c = 1 \) shows that \( (x + \sqrt{x^2 - 1})^{l-1} \not\equiv 1 \mod l^2 \) and \( (x - \sqrt{x^2 - 1})^{l-1} \not\equiv 1 \mod l^2 \).

Let \( K := \mathbb{Q}(\sqrt{x^2 - 1}) \). Then, \( K \) inverts over \( p \) and splits over \( l \) because \( \left( \frac{x^2 - 1}{p} \right) = -1 \), and \( \left( \frac{x^2 - 1}{l} \right) = 1 \). Let \( v \in \text{Spec}A_K \) be the point over \( p \) and \( u \in \text{Spec}A_K \) be a point over \( l \). We have \( \sqrt{x^2 - 1} \equiv \pm b_0 \sqrt{t} \mod v \), because \( x^2 - 1 \equiv a^2 - 1 \equiv b^2t \mod v \). Let \( \alpha := x + \sqrt{x^2 - 1} \in \mathcal{A}_K \), if \( \sqrt{x^2 - 1} \equiv \pm b_0 \sqrt{t} \mod v \); or \( \alpha := x - \sqrt{x^2 - 1} \in \mathcal{A}_K \), if else. We then have \( \alpha^{l-1} \not\equiv 1 \mod I_u \) and

\[
\alpha \equiv a_0 + \sqrt{a_0^2 - 1} \equiv a_0 + b_0 \sqrt{t} \equiv \bar{a} \equiv \bar{g}^m \mod v
\]

implying that \( \alpha^{l-1} \not\equiv 1 \mod I_v \), as \( \bar{a} \not\in \mathbb{F}_p^\times l \). As \( \alpha := x + \sqrt{x^2 - 1} \in \mathcal{A}_K \) and \( Nm(\alpha) = x^2 - (x^2 - 1) = 1 \), we have \( \alpha \in \mathcal{A}_K^\times \).

Let \( U := \text{Spec}A_K \setminus \{u, v\} \). We assume that \( l \nmid h_K \), which is likely to be satisfied. Proposition 2.1 then shows \( \langle \alpha, \chi \rangle + \langle \alpha, \chi \rangle = 0 \), for any \( \chi \neq 0 \in \text{Hom}(\pi_1(U), \mathbb{Z}/l\mathbb{Z}) \). Let \((1 + l)^y \) be the image of \( \alpha \) under the morphism

\[
A_K^\times \to A_u^\times \cong \mathbb{Z}_l^\times \to (\mathbb{Z}/l^2\mathbb{Z})^\times / (\mathbb{Z}/l^2\mathbb{Z})^\times.
\]

For the first term in (2.2.3), we have \( \langle \alpha, \chi \rangle = y(1 + l, \chi) \). For the second term in (2.2.3), we have \( \langle \alpha, \chi \rangle = m(\bar{g}, \chi) \). By Corollary 2.2.(iii), we obtain

\[
y(1 + l, \chi) + m(\bar{g}, \chi) = 0.
\]

Therefore, if we can compute the ramification signature \( \langle \chi, 1 + l \rangle^{-1} \langle \chi, g \rangle \) of \( U \) with respect to \( g \), then we can compute \( m = -y(\chi, 1 + l)^{-1} \langle \chi, g \rangle \).

\[\textbf{b.}\] If \( l \mid p - 1 \), we have \( Nm(a) = Nm(g)^m \) as elements in \( \mathbb{F}_p \). The construction in [Huang-Raskind 2009] gives us a real quadratic field and a global unit in the field that enable us to reduce the computation of \( m \) satisfying \( Nm(a) = Nm(g)^m \) to the signature computation problem of the real quadratic field using the algorithm in [Huang-Raskind 2009]. However, the construction requires some conditions on the class number of the field and the unit to be satisfied ([Huang-Raskind 2009], Section 4.2). We give an improvement in Step 4 below on the construction recalled below. With the improvement, one of the condition (the condition 2 in Proposition 2 of section 4.1 in [Huang-Raskind 2009]) is satisfied automatically.

Let \( \bar{a} = \bar{g}^m \) in \( \mathbb{F}_p^\times \) where \( m \) is to be computed. If \( \bar{a}^\frac{p-1}{m} = 1 \), then \( m \equiv 0 \mod l \). Thus suppose \( \bar{a}^\frac{p-1}{m} \neq 1 \). We will lift \( \bar{a} \) to some unit \( \alpha \) of a real quadratic field \( K \) such that
\[ \alpha \equiv \tilde{a} \mod v \] for some place \( v \) of \( K \) over \( p \), \( \alpha^{l-1} \not\equiv 1 \mod I_u^2 \), and \( \alpha^{l-1} \not\equiv 1 \mod I_u^2 \) for the two places \( u \) and \( u' \) of \( K \) over \( l \). We do it as follows:

1. Compute \( \tilde{b} \in \mathbb{F}_p^\times \) such that \( \tilde{a} \tilde{b} = 1 \) in \( \mathbb{F}_p^\times \).

2. Put \( c := \frac{\tilde{a}+\tilde{b}}{2} \), \( d := \frac{\tilde{a}+\tilde{b}}{2} \). Note that \( c^2 - d^2 = 1 \) and \( \tilde{a} = c + d \). We can assume \( d \neq 0 \); otherwise, \( \tilde{a}^2 = 1 \) and \( m = \frac{a^2}{2} \) or \( p - 1 \).

3. Lift \( d \) to an integer. We have \( \left( \frac{a^2+d^2}{p} \right) = 1 \). We choose \( k \in \{0,1,...,l-1\} \) randomly until \( \left( \frac{(a+d)k^2+1}{p} \right) = 1 \). Lemma 3.1 below for \( c = -1 \) shows that we can obtain such \( k \) with probability of about 50% each time.

4. If we find such \( k \), let \( d_1 := d + kp \in \mathbb{Z}_l^\times \). We may take \( \sqrt{d_1^2+1} \in \mathbb{Z}_l^\times \) since \( \left( \frac{d_1^2+1}{l} \right) = 1 \). If \( (d_1 + \sqrt{d_1^2 + 1})^{l-1} \equiv 1 \mod l^2 \), let \( x = d_1 \); otherwise let \( x = d_1 + pl \). Lemma 3.2 below for \( c = -1 \) shows that \( (x + \sqrt{x^2+1})^{l-1} \not\equiv 1 \mod l^2 \), \( (x - \sqrt{x^2+1})^{l-1} \not\equiv 1 \mod l^2 \).

5. Let \( K := \mathbb{Q}(\sqrt{x^2+1}) \), \( \alpha := x + \sqrt{x^2+1} \in \mathcal{O}_K \). Note that \( \text{Nm}(\alpha) = 1 \), so \( \alpha \) is a unit of \( K \).

6. Let \( v \) be the point in \( \text{Spec} \mathcal{O}_K \) corresponding to the prime ideal \((p, \sqrt{x^2+1} - c)\), \( v' \) be the point in \( \text{Spec} \mathcal{O}_K \) responding to the prime ideal \((p, \sqrt{x^2+1} + c)\), \( u \) and \( u' \) be the points in \( \text{Spec} \mathcal{O}_K \) over \( l \). Thus, \( \alpha \equiv d + c \equiv \tilde{a} \mod v \), \( \alpha \equiv d - c \equiv -\tilde{b} \mod v' \), \( \alpha^{l-1} \not\equiv 1 \mod I_u^2 \) and \( \alpha^{l-1} \not\equiv 1 \mod I_u^2 \).

In [Huang-Raskind 2009], they proved the reduction from a signature computation to a discrete logarithm problem in \( \mathbb{F}_p \) without any heuristic assumption. Therefore, we conclude that the discrete logarithm problems in \( \mathbb{F}_{p^2} \) and \( \mathbb{F}_{p} \) are random polynomial time equivalent to some signature computation problem with only one assumption, namely, that on the class number.

The following are the statements and proofs of lemma 3.1 and lemma 3.2.

**Lemma 3.1.** Let \( l \) be an odd prime, \( c \in \mathbb{F}_l^\times \). Define a map \( f : \mathbb{Z}/l\mathbb{Z} \rightarrow \{0,1,-1\} \) by \( a \mapsto \left( \frac{a^2-c}{l} \right) \). Then, we have

\[
|f^{-1}(0)| = 2, \quad |f^{-1}(1)| = (l-3)/2, \quad |f^{-1}(-1)| = (l-1)/2 \quad \text{if } \left( \frac{c}{l} \right) = 1,
\]

\[
|f^{-1}(0)| = 0, \quad |f^{-1}(1)| = (l-1)/2, \quad |f^{-1}(-1)| = (l+1)/2 \quad \text{if } \left( \frac{c}{l} \right) = -1.
\]

**Proof.** Let \( X \) be the curve defined by \( y^2 = x^2 - c \) over \( \mathbb{F}_l \). For any \( a \in \mathbb{F}_l \), the cardinality of the set \( \{ \mathbb{F}_l \text{-rational point of } X \text{ that has first coordinate } a \} \) is \( f(a) + 1 \). Therefore, the following holds:

\[
\sum_{a \in \mathbb{F}_l} (f(a) + 1) = |X(\mathbb{F}_l)|.
\]

The curve \( X \) is isomorphic to the affine scheme defined by \( z\omega = 1 \) over \( \mathbb{F}_l \), which implies \( |X(\mathbb{F}_l)| = l - 1 \), and \( \sum_{a \in \mathbb{F}_l} f(a) = |X(\mathbb{F}_l)| - l = -1 \). Clearly,

\[
f^{-1}(0) = \{\sqrt{c}, -\sqrt{c}\} \quad \text{if } \left( \frac{c}{l} \right) = 1,
\]

\[
f^{-1}(0) = \emptyset \quad \text{if } \left( \frac{c}{l} \right) = -1.
\]

and the lemma follows easily from \(|f^{-1}(1)|+|f^{-1}(-1)|+|f^{-1}(0)| = l\) and \(|f^{-1}(1)|-|f^{-1}(-1)| = \sum_{a \in \mathbb{F}_l} f(a) = -1\). □
Lemma 3.2. Let $p$ and $l$ be two distinct odd prime numbers. Let $c$ be an integer such that $c^{l-1} \equiv 1 \mod l^2$ and $a$ be an integer such that $(a^2 - c)^{l-1} \equiv 1 \mod l^2$. We denote a square root of $a^2 - c$ in $\mathbb{Z}_l^*$ by $\sqrt{a^2 - c}$. If $(a + \sqrt{a^2 - c})^{l-1} \not\equiv 1 + l^2 \mathbb{Z}_l$, then we have $((a + pl) + \sqrt{(a + pl)^2 - c})^{l-1} \not\equiv 1 + l^2 \mathbb{Z}_l$ and $((a + pl) - \sqrt{(a + pl)^2 - c})^{l-1} \not\equiv 1 + l^2 \mathbb{Z}_l$.

Proof. By Hensel's lemma, there is a unique square root $\sqrt{(a + x)^2 - c}$ of $(a + x)^2 - c$ in $\mathbb{Z}_l[[x]]$ such that its image under the morphism $x \mapsto 0 : \mathbb{Z}_l[[x]] \to \mathbb{Z}_l$ is $\sqrt{a^2 - c}$. Let $h(x) := (a + x) + \sqrt{(a + x)^2 - c}$, we then have

$$h(pl) \equiv h(0) + h'(0)pl \mod l^2,$$

where $h'(0) = 1 + \frac{a}{\sqrt{a^2 - c}} = \frac{h(0)}{\sqrt{a^2 - c}}$. Therefore, we have

$$h(pl) \equiv h(0)(1 + \frac{p}{\sqrt{a^2 - c}}) \mod l^2.$$

The term $\frac{p}{\sqrt{a^2 - c}}$ is not divided by $l$, which implies $h(pl)^{l-1} \not\equiv h(0)^{l-1} \mod l^2$. Hence, we have

$$((a + pl) + \sqrt{(a + pl)^2 - c})^{l-1} \not\equiv 1 \mod l^2.$$

The fact that $((a + pl) + \sqrt{(a + pl)^2 - c})^{l-1}((a + pl) - \sqrt{(a + pl)^2 - c})^{l-1} = c^{l-1} \equiv 1 \mod l^2$ shows

$$((a + pl) - \sqrt{(a + pl)^2 - c})^{l-1} \not\equiv 1 \mod l^2.$$  

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