Distributed Forward-Backward Methods without Central Coordination

Francisco J. Aragón-Artacho∗ Yura Malitsky† Matthew K. Tam‡
David Torregrosa-Belén§

Dedicated to the memory of Asen L. Dontchev

Abstract

In this work, we propose and analyse forward-backward-type algorithms for finding a zero in the sum of finitely many monotone operators, which are not based on reduction to a two operator inclusion in the product space. Each iteration of the studied algorithms requires one resolvent evaluation per set-valued operator, one forward evaluation per cocoercive operator, and two forward evaluations per monotone operator. Unlike existing methods, the structure of the proposed algorithms are suitable for distributed, decentralised implementation in ring networks without the need for a central coordinator to enforce consensus between nodes.

Keywords. monotone operator · monotone inclusion · splitting algorithm · forward-backward algorithm · distributed optimisation

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1 Introduction

In this work, we propose algorithms of forward-backward-type for solving structured monotone inclusions in a real Hilbert space \( \mathcal{H} \). Specifically, we consider the problem

\[
\text{find } x \in \mathcal{H} \text{ such that } 0 \in \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{m} B_i \right)(x),
\]

where \( A_1, \ldots, A_n : \mathcal{H} \rightrightarrows \mathcal{H} \) are maximally monotone operators, and \( B_1, \ldots, B_m : \mathcal{H} \to \mathcal{H} \) are either cocoercive, or monotone and Lipschitz continuous. Inclusions in the form (1) arise in a number of settings of fundamental importance in mathematical optimisation. In what follows, we describe three such examples.

Example 1.1 (Composite minimisation). Consider the minimisation problem given by

\[
\min_{x \in \mathcal{H}} \sum_{i=1}^{n} g_i(x) + \sum_{i=1}^{m} f_i(x),
\]

where \( g_1, \ldots, g_n : \mathcal{H} \to (-\infty, +\infty] \) are proper, lsc and convex, and \( f_1, \ldots, f_m : \mathcal{H} \to (-\infty, +\infty) \) are convex and differentiable with \( L \)-Lipschitz continuous gradients. Through its first order optimality condition, (2) can be posed as (1) with

\[
A_i = \partial g_i \quad \text{and} \quad B_i = \nabla f_i
\]

1Department of Mathematics, University of Alicante, Alicante, SPAIN. Email: francisco.aragon@ua.es
2Department of Mathematics, Linköping University, Linköping, SWEDEN. Email: yurii.malitskyi@liu.se
3School of Mathematics & Statistics, University of Melbourne, AUSTRALIA. Email: matthew.tam@unimelb.edu.au
4Department of Mathematics, University of Alicante, Alicante, SPAIN. Email: david.torregrosa@ua.es
where \( \partial g_i \) denotes the subdifferential of \( g_i \). Note that the operators \( B_1, \ldots, B_m \) are both L-Lipschitz and \( \frac{1}{m} \)-cocoercive, due to the Baillon–Haddad theorem [3, Corollaire 10].

**Example 1.2 (Structured saddle-point problems).** Consider the saddle-point problem given by

\[
\min_{x \in \mathcal{H}_1} \max_{y \in \mathcal{H}_2} \sum_{i=1}^n h_i(x) + \sum_{i=1}^m \Phi_i(x, y) - \sum_{i=1}^n g_i(y),
\]

(3)

where \( h_1, \ldots, h_n : \mathcal{H}_1 \to (-\infty, +\infty] \), \( g_1, \ldots, g_n : \mathcal{H}_2 \to (-\infty, +\infty] \) are proper, lsc and convex, and \( \Phi_1, \ldots, \Phi_m : \mathcal{H}_1 \times \mathcal{H}_2 \to (-\infty, +\infty] \) are differentiable convex-concave functions with Lipschitz continuous gradient. Assuming a saddle-point exists, (3) can be posed as (1) in the space \( \mathcal{H} := \mathcal{H}_1 \times \mathcal{H}_2 \) with

\[
A_i(x, y) = \left( \frac{\partial h_i(x)}{\partial g_i(y)} \right) \quad \text{and} \quad B_i(x, y) = \left( \nabla_x \Phi_i(x, y), -\nabla_y \Phi_i(x, y) \right),
\]

where we note that the operators \( B_1, \ldots, B_n : \mathcal{H} \to \mathcal{H} \) are monotone, due to [20, Theorem 2], and L-Lipschitz continuous, but generally not cocoercive.

**Example 1.3 (Structured variational inequalities).** Consider the variational inequality problem given by

\[
\text{find } x^* \in \mathcal{H} \text{ such that } \sum_{i=1}^n g_i(x) - \sum_{i=1}^n g_i(x^*) + \sum_{i=1}^m \langle B_i(x^*), x - x^* \rangle \geq 0 \quad \forall x \in \mathcal{H},
\]

(4)

where \( g_1, \ldots, g_n : \mathcal{H} \to (-\infty, +\infty] \) are proper, lsc and convex, and \( B_1, \ldots, B_m : \mathcal{H} \to \mathcal{H} \) are monotone and L-Lipschitz. Then (4) is of the form of (1) with \( A_1 = \partial g_1 \). An important special case of (4) is the constrained variational inequality problem given by

\[
\text{find } x^* \in \mathcal{H} \text{ such that } \sum_{i=1}^m \langle B_i(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C := \bigcap_{i=1}^n C_i,
\]

where \( C_1, \ldots, C_n \subseteq \mathcal{H} \) are nonempty, closed and convex sets. This formulation allows one to exploit a representation of the set \( C \) in terms of the simpler sets \( C_1, \ldots, C_n \).

**Splitting algorithms.** We focus on splitting algorithm for solving (1) of forward-backward-type, by which we mean those whose iteration can be expressed in terms of the resolvents of the set-valued operators \( A_1, \ldots, A_n \) and direct evaluations of the single-valued operators \( B_1, \ldots, B_m \). It is always possible to reduce this problem to the \( m = 1 \) case by combining the single-valued operators into a single operator \( F := \sum_{i=1}^m B_i \) whilst preserving the above features. However, since the resolvent of a sum is generally not related to the individual resolvents, the same cannot be said for the set-valued operators, and so it makes sense to distinguish algorithms for (1) based on the value of \( n \).

In the case \( n = 1 \), there are many methods satisfying the above criteria. Among them, the best known are arguably the forward-backward method given by

\[
x^{k+1} = J_{\lambda A_1}(x^k - \lambda F(x^k)),
\]

which can be used when \( F \) is cocoercive, and the forward-backward-forward method [23] given by

\[
\begin{align*}
\begin{cases}
    y^k & = J_{\lambda A_1}(x^k - \lambda F(x^k)) \\
    x^{k+1} & = y^k - \lambda F(y^k) + \lambda F(x^k),
\end{cases}
\end{align*}
\]

which can be used when \( F \) is monotone and Lipschitz. When \( n = 2 \), there are also many methods. For instance, if \( F \) is cocoercive, Davis–Yin splitting [12, 11, 1] which takes the form

\[
\begin{align*}
\begin{cases}
    x^k & = J_{\lambda A_1}(z^k) \\
    z^{k+1} & = z^k + J_{\lambda A_2}(2z^k - z^k - \lambda F(x^k)) - x^k
\end{cases}
\end{align*}
\]

can be applied, and if \( F \) is monotone and Lipschitz, then the backward-forward-reflected-backward methods [19] can be used.
However, for $n > 2$, the situation is drastically different. Most existing methods rely on a product space reformulation, either directly or implicitly. For instance, the iteration given by

$$
\begin{align*}
    x^k &= \frac{1}{n} \sum_{i=1}^{n} z_i^k \\
    z_i^{k+1} &= z_i^k + J_{A_i}(2x^k - z_i^k - \lambda B_i(x^k)) - x^k \quad \forall i \in [1,n]
\end{align*}
$$

(5)

for cocoercive $B_1, \ldots, B_n$, amounts to Davis–Yin splitting applied to the three operator inclusion

$$
\text{find } x = (x_1, \ldots, x_n) \in \mathcal{H}^n \text{ such that } 0 \in (N_D + A + B)(x),
$$

(6)

where $A := (A_1, \ldots, A_n)$, $B := (B_1, \ldots, B_n)$ and $N_D$ denotes the normal cone (see Example 2.7) to the diagonal subspace $D := \{(x_1, \ldots, x_n) \in \mathcal{H}^n : x_1 = \cdots = x_n\}$. Other methods for (1) with $n > 2$ include the generalised forward-backward method [18] and those from the projective splitting family [13, 14].

Indisputably, product space reformulations such as (6) provide a convenient tool that makes the derivation of algorithms for $n > 2$ operators an almost mechanical procedure. It is therefore natural to consider whether this tool is the only one at our disposal. In addition to academic importance in its own right, the discovery of new algorithms that do not fall within standard categories can provide new possibilities, both in terms of mathematical techniques and potential applications. Sometimes these applications can be quite unexpected, as we demonstrate next.

**Distributed algorithms.** Advances in hardware (parallel computation) and increasing the size of datasets (decentralised storage) have made distributed algorithms one of the most prevalent trends in algorithm development. Such algorithms rely on a network of devices that perform subtasks and are able to communicate with each other. For details on the topic, the reader is referred to the book of Bertsekas & Tsitsiklis [6] as well as [9] for recent advances.

From the perspective of distributed computing, the product space formulation only allows centralised communication. As a concrete example, consider a distributed implementation of (5) in which node $i$ performs the $z_i$-updates by using its operators, $A_i$ and $B_i$. In order to implement the $x$-update, it is necessary to have an additional node whose responsibility is to aggregate $z_1, \ldots, z_n$.

As this node must be connected to all other nodes, central communication naturally leads to a star network topology. There may be many reasons why this is not desirable including default network setting, privacy or cost issues.

Another important aspect of distributed communication is parallelism and synchronisation. Return to our example involving (5) from the previous paragraph, the product space reformulation provides a fully parallel algorithm in the sense that all nodes performing $z$-update can compute their updates in parallel before sending to the central coordinator. This parallelisation comes at cost of requiring global synchronisation between nodes. Specially, the algorithm (5) cannot move from $k$-th to $(k+1)$-th iteration until all nodes $1, \ldots, n$ have completed their computation. This can be overcome with asynchronous algorithms, that is, those which only require little or no global synchronisation. However, their development and mathematical analysis are significantly more delicate.

**Our contribution.** We propose and analyse algorithms of forward-backward-type for solving (1) which exploit problem structure. Note that by using the zero operator in (1) if necessary, we can always assume that $m = n - 1$. Applied to this problem with cocoercive operators $B_1, \ldots, B_{n-1}$, our algorithm can be expressed as the fixed point iteration $z^{k+1} = T(z^k)$ based on the operator $T : \mathcal{H}^{n-1} \to \mathcal{H}^{n-1}$ given by

$$
T(z) := z + \gamma \begin{pmatrix}
    x_2 - x_1 \\
    x_3 - x_2 \\
    \vdots \\
    x_n - x_{n-1}
\end{pmatrix}
$$
where \( x = (x_1, \ldots, x_n) \in \mathcal{H}^n \) depends on \( z = (z_1, \ldots, z_{n-1}) \in \mathcal{H}^{n-1} \) and is given by

\[
\begin{align*}
x_1 &= J_{\lambda A_1}(z_1), \\
x_i &= J_{\lambda A_i}(z_i + x_{i-1} - z_{i-1} - \lambda B_{i-1}(x_{i-1})) \quad \forall i \in [2, n-1], \\
x_n &= J_{\lambda A_n}(z_1 + x_{n-1} - z_{n-1} - \lambda B_{n-1}(x_{n-1})).
\end{align*}
\]

For the case where \( B_i \) are monotone and Lipschitz, the underlying operator is slightly more complicated and relies on an update similar to the one proposed in the forward-reflected-backward method [15].

Overall, the notable characteristics of the algorithms we propose are:

- They do not rely on existing product space reformulation: Instead, we extend the framework for backward operators, proposed in [16], which in turn is a generalisation of [21] for \( n > 3 \).
- They are decentralised and can be naturally implemented on a ring network for communication.
- They are asynchronous: iteration \( z_i^{k+1} \) can be computed without finishing evaluation of \( z_j^k, z_j^{k-1} \), etc., for \( j > i \).

Importantly, we believe that our work is an important starting point towards a more general template that will allow for different network topologies.

The remainder of this work is structured as follows: In Section 2, we recall notation and preliminaries for later use. In Section 3, we introduce and analyse a forward-backward type algorithm for solving (1) with cocoercive operators. In Section 4, we introduce and analyse a modification of the algorithm from Section 3 which can be used when \( B_1, \ldots, B_m \) are not necessarily cocoercive.

## 2 Preliminaries

Throughout this paper, \( \mathcal{H} \) denotes a real Hilbert space equipped with inner product \( \langle \cdot, \cdot \rangle \) and induced norm \( \| \cdot \| \). A set-valued operator is a mapping \( A : \mathcal{H} \rightrightarrows \mathcal{H} \) that assigns to each point in \( \mathcal{H} \) a subset of \( \mathcal{H} \), i.e., \( A(x) \subseteq \mathcal{H} \) for all \( x \in \mathcal{H} \). In the case when \( A \) always maps to singletons, i.e., \( A(x) = \{u\} \) for all \( x \in \mathcal{H} \), \( A \) is said to be a single-valued mapping and is denoted by \( A : \mathcal{H} \to \mathcal{H} \). In an abuse of notation, we may write \( A(x) = u \) when \( A(x) = \{u\} \). The domain, the graph, the set of fixed points and the set of zeros of \( A \), are denoted, respectively, by \( \text{dom} A \), \( \text{gra} A \), \( \text{Fix} A \) and \( \text{zer} A \); i.e.,

\[
\begin{align*}
\text{dom} A &:= \{x \in \mathcal{H} : A(x) \neq \emptyset\}, & \text{gra} A &:= \{(x, u) \in \mathcal{H} \times \mathcal{H} : u \in A(x)\}, \\
\text{Fix} A &:= \{x \in \mathcal{H} : x \in A(x)\}, & \text{zer} A &:= \{x \in \mathcal{H} : 0 \in A(x)\}.
\end{align*}
\]

The inverse operator of \( A \), denoted by \( A^{-1} \), is defined through \( x \in A^{-1}(u) \iff u \in A(x) \). The identity operator is denoted by \( \text{Id} \).

**Definition 2.1.** An operator \( B : \mathcal{H} \to \mathcal{H} \) is said to be

(i) L-Lipschitz continuous for \( L > 0 \) if

\[
\|B(x) - B(y)\| \leq L\|x - y\| \quad \forall x, y \in \mathcal{H}.
\]

(ii) \( \frac{1}{L} \)-cocoercive for \( L > 0 \) if

\[
\langle B(x) - B(y), x - y \rangle \geq \frac{1}{L}\|B(x) - B(y)\|^2 \quad \forall x, y \in \mathcal{H}.
\]

Note that, by the Cauchy–Schwarz inequality, a \( \frac{1}{L} \)-cocoercive operator is always L-Lipschitz continuous.

**Definition 2.2.** An operator \( \Gamma : \mathcal{H} \to \mathcal{H} \) is said to be
(i) quasi-nonexpansive if
\[ \|T(x) - y\| \leq \|x - y\| \quad \forall x \in \mathcal{H}, \forall y \in \text{Fix } T; \]

(ii) nonexpansive if it is 1-Lipschitz continuous, i.e.,
\[ \|T(x) - T(y)\| \leq \|x - y\| \quad \forall x, y \in \mathcal{H}; \]

(iii) strongly quasi-nonexpansive if there exists \( \sigma > 0 \) such that
\[ \|T(x) - y\|^2 + \sigma \|(\text{Id} - T)(x)\|^2 \leq \|x - y\|^2 \quad \forall x \in \mathcal{H}, \forall y \in \text{Fix } T; \]

(iv) averaged nonexpansive if there exists \( \alpha \in (0, 1) \) such that
\[ \|T(x) - T(y)\|^2 + \frac{1 - \alpha}{\alpha} \|(\text{Id} - T)(x) - (\text{Id} - T)(y)\|^2 \leq \|x - y\|^2 \quad \forall x, y \in \mathcal{H}. \]

In particular, the following implications hold: (iv) \( \Rightarrow \) (ii) \( \Rightarrow \) (i) and (iv) \( \Rightarrow \) (iii) \( \Rightarrow \) (i).

When we wish to explicitly specify the constants involved, we refer to the operators in Definition 2.2(iii) and (iv), respectively, as \( \sigma \)-strongly expansive and \( \alpha \)-averaged nonexpansive. Since the mapping \( \alpha \mapsto \frac{1 - \alpha}{\alpha} \) is a bijection from \( (0, 1) \) to \( (0, +\infty) \), there is a one-to-one relationship between the values of \( \sigma \) in (iii) and \( \alpha \) in (iv).

**Definition 2.3.** A set-valued operator \( A : \mathcal{H} \rightrightarrows \mathcal{H} \) is monotone if
\[ \langle x - y, u - v \rangle \geq 0 \quad \forall (x, u), (y, v) \in \text{gra } A. \]

Furthermore, \( A \) is said to be maximally monotone if there exists no monotone operator \( B : \mathcal{H} \rightrightarrows \mathcal{H} \) such that \( \text{gra } B \) properly contains \( \text{gra } A \).

**Proposition 2.4** ([4, Example 20.28]). Every continuous monotone operator is maximally monotone. In particular, every cocoercive operator is maximally monotone.

The resolvent operator, whose definition is given next, is one of the main building blocks of splitting algorithms.

**Definition 2.5.** Given an operator \( A : \mathcal{H} \rightrightarrows \mathcal{H} \), the resolvent of \( A \) with parameter \( \gamma > 0 \) is the operator \( J_{\gamma A} : \mathcal{H} \rightrightarrows \mathcal{H} \) defined by \( J_{\gamma A} := (\text{Id} + \gamma A)^{-1} \).

**Proposition 2.6** ([17] or [4, Corollary 23.11]). Let \( A : \mathcal{H} \rightrightarrows \mathcal{H} \) be monotone and let \( \gamma > 0 \). Then

(i) \( J_{\gamma A} \) is single-valued,

(ii) \( \text{dom } J_{\gamma A} = \mathcal{H} \) if and only if \( A \) is maximally monotone.

**Example 2.7.** Let \( C \subseteq \mathcal{H} \) a nonempty, closed and convex set. The normal cone of \( C \), which is the operator \( N_C : \mathcal{H} \rightrightarrows \mathcal{H} \) defined as
\[ N_C(x) = \begin{cases} \{ u \in \mathcal{H} : \langle u, c - x \rangle \leq 0, \forall c \in C \}, & \text{if } x \in C, \\ \emptyset, & \text{otherwise}, \end{cases} \]
is maximally monotone (see, e.g., [4, Example 20.26]). Furthermore, \( J_{N_C} = P_C \), where \( P_C : \mathcal{H} \rightarrow \mathcal{H} \) denotes the projector onto \( C \) (see, e.g., [4, Example 23.4]), which is defined at \( x \in \mathcal{H} \) by
\[ P_C(x) := \arg\min_{c \in C} \|x - c\|. \]

Fejér monotonicity is a key property in fixed point theory (see, e.g., [4, Chapter 5] or [8]).

**Definition 2.8.** Let \( C \) be a nonempty subset of \( \mathcal{H} \) and let \( (x^k) \) be a sequence in \( \mathcal{H} \). Then \( (x^k) \) is Fejér monotone with respect to \( C \) if, for all \( x \in C \), we have
\[ \|x^{k+1} - x\| \leq \|x^k - x\| \quad \forall k \in \mathbb{N}. \]
3 A Distributed Forward-Backward Method

Let \( n \geq 2 \) and consider the problem

\[
\text{find } x \in \mathcal{H} \text{ such that } 0 \in \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-1} B_i \right) (x),
\]

where \( A_1, \ldots, A_n : \mathcal{H} \rightrightarrows \mathcal{H} \) are maximally monotone and \( B_1, \ldots, B_{n-1} : \mathcal{H} \rightrightarrows \mathcal{H} \) are \( \frac{1}{\lambda} \)-cocoercive.

For the case when \( B_1 = \cdots = B_{n-1} = 0 \), Malitsky & Tam [16] proposed a splitting algorithm with \((n-1)\)-fold lifting for finding a zero of the sum of \( n \geq 2 \) maximally monotone operators; see also [7] for recent extensions. In this section, we adapt the methodology developed in [1] to obtain a splitting method of forward-backward-type for the inclusion (7) by modifying the splitting method in [16] without increasing the dimension of the ambient space.

Given \( \lambda \in (0, \frac{1}{\lambda}] \) and \( \gamma \in (0, 1 - \frac{\lambda}{2}] \) and an initial point \( z^0 = (z_1^0, \ldots, z_{n-1}^0) \in \mathcal{H}^{n-1} \), our proposed algorithm for (7) generates two sequences, \((z^k) \subseteq \mathcal{H}^{n-1}\) and \((x^k) \subseteq \mathcal{H}^n\), according to

\[
z^{k+1} = z^k + \gamma \begin{pmatrix}
x_2^k - x_1^k \\
x_3^k - x_2^k \\
\vdots \\
x_n^k - x_{n-1}^k
\end{pmatrix}
\]

(8a)

and

\[
\begin{align*}
x_1^k &= J_{\lambda A_1}(z_1^k), \\
x_i^k &= J_{\lambda A_i}(z_i^k + x_{i-1}^k - z_{i-1}^k - \lambda B_{i-1}(x_{i-1}^k)) \quad \forall i \in [2, n-1], \\
x_n^k &= J_{\lambda A_n}(x_{n-1}^k + x_n^k - x_{n-1}^k - \lambda B_{n-1}(x_{n-1}^k)).
\end{align*}
\]

(8b)

The structure of (8) lends itself to a distributed decentralised implementation, similar to the one in [16, Algorithm 2], without the need for a “central coordinator” to enforce consensus. More precisely, consider a cycle graph with \( n \) nodes labeled 1 through \( n \). Each node in the graph represents an agent, and two agents can communicate only if their nodes are adjacent. In our setting, this means that Agent \( i \) can only communicate with Agents \( i-1 \) and \( i+1 \) mod \( n \), for \( i \in [1, n] \). We assume that each agent only knows its operators in (1). Specifically, we assume that only Agent 1 knows the operator \( A_1 \) and that, for each \( i \in [2, \ldots, n] \), only Agent \( i \) knows the operators \( A_i \) and \( B_{i-1} \). The responsibility of updating \( x_i \) is assigned to Agent \( i \) for all \( i \in \{1, \ldots, n\} \) and the responsibility of updating \( z_i \) is assigned to Agent \( i \) for \( i \in \{2, \ldots, n\} \). Altogether, this gives rise to the protocol for distributed decentralised implementation of (8) described in Algorithm 1. To make the protocol simpler, we allow ourselves some duplication of variables/agents. In particular, Agent \( n \) does not need to send anything to Agent \( n+1 \), since they are the same. Similarly, we do not need a new variable \( z_n^k \) and we used it only as a notation convenience.

In order to analyse convergence of (8), we introduce the underlying fixed point operator \( T : \mathcal{H}^{n-1} \rightarrow \mathcal{H}^{n-1} \) given by

\[
T(z) := z + \gamma \begin{pmatrix}
x_2 - x_1 \\
x_3 - x_2 \\
\vdots \\
x_n - x_{n-1}
\end{pmatrix}
\]

(9)

where \( z = (x_1, \ldots, x_n) \in \mathcal{H}^n \) depends on \( z = (z_1, \ldots, z_{n-1}) \in \mathcal{H}^{n-1} \) and is given by

\[
\begin{align*}
x_1 &= J_{\lambda A_1}(z_1), \\
x_i &= J_{\lambda A_i}(z_i + x_{i-1} - z_{i-1} - \lambda B_{i-1}(x_{i-1})) \quad \forall i \in [2, n-1], \\
x_n &= J_{\lambda A_n}(x_1 + x_{n-1} - z_{n-1} - \lambda B_{n-1}(x_{n-1})).
\end{align*}
\]

(10)

In this way, the sequence \((z^k)\) given by (8a) satisfies \( z^{k+1} = T(z^k) \) for all \( k \in \mathbb{N} \).
**Remark 3.1.** Note that, although the sum of cocoercive operators is cocoercive (see, e.g., [4, Proposition 4.12]), considering the sum of \(n - 1\) operators in (1) gives the freedom of either applying each operator as a forward step before the corresponding backward step, or to apply the sum of all of them before a particular backward step (by setting all the operators to be equal to zero except for one of them, which would be equal to the sum).

**Remark 3.2** (Special cases). If \(n = 2\), then \(x_1 = x_{n-1}\) and \(T\) in (9) recovers the operator corresponding to Davis–Yin splitting [12, 11, 1] for finding a zero of \(A_1 + A_2 + B_1\). In turn, this includes the forward-backward algorithm and Douglas–Rachford splitting as special cases by further taking \(A_1 = 0\) or \(B_1 = 0\), respectively.

If \(B_1 = \cdots = B_{n-1} = 0\), then \(T\) in (9) reduces to the resolvent splitting algorithms proposed by the authors in [16]. This has been further studied in [5] for the particular case in which the operators \(A_i\) are normal cones of closed linear subspaces.

Although the number of set-valued and single-valued monotone operators in (7) differ by one, it is straightforward to derive a scheme where this is not the case by setting \(A_1 = 0\). In this case, \(x_1 = J_{\lambda A_1}(z_1) = z_1\) can be used to eliminate \(x_1\) so that (9) and (10) respectively become

\[
\begin{align*}
T(z) &:= z + \gamma \left( \begin{array}{c}
x_2 - z_1 \\
x_3 - x_2 \\
\vdots \\
x_n - x_{n-1}
\end{array} \right) \quad \text{where} \\
\begin{align*}
x_2 &= J_{\lambda A_2}(z_2 - \lambda B_1(z_1)), \\
x_3 &= J_{\lambda A_3}(x_1 + x_{i-1} - z_{i-1} - \lambda B_{i-1}(x_{i-1})), \\
\vdots &\quad \forall i \in [3, n-1], \\
x_n &= J_{\lambda A_n}(x_1 + x_{n-1} - z_{n-1} - \lambda B_{n-1}(x_{n-1})).
\end{align*}
\end{align*}
\]

While at first it may seem unusual that the number of set-valued and single-valued monotone operators in (7) are not the same, we note that this same situation arises in Davis–Yin splitting as described above.

**Remark 3.3.** The algorithm given by (8) appears new even in the special case with \(A_1 = 0\) and \(B_i = \nabla f_i\) for convex smooth functions \(f_i\). In this case, one of the most popular algorithms for solving \(\min_{x} \sum_i f_i(x)\) in a decentralised way is EXTRA, proposed in [22]. They are similar in spirit, but also have quite different properties. In particular, the main update of EXTRA is

\[
x^{k+1} = (\text{Id} + W)x^k - \bar{W}x^{k-1} - \lambda[\nabla f(x^k) - \nabla f(x^{k-1})],
\]

where \(W\) and \(\bar{W}\) are certain mixing matrices and \(x^0 = Wx^0 - \lambda \nabla f(x^0)\). Undoubtedly, an advantage of EXTRA is the ability to use a wider range of mixing matrices which, in terms of communication, generalises better for network topology. On the other hand, our proposed algorithm can be implemented asynchronously, that is, we can update \(x^{k+1}\) without finishing the evaluation of \(x_k\) or even \(x^{k-1}\), etc.

In what follows, we first describe the relationship between the solutions of the monotone inclusion (7) and the fixed point set of the operator \(T\) in (9).
Lemma 3.4. Let \( n \geq 2 \) and \( \gamma, \lambda > 0 \). The following assertions hold.

(a) If \( \bar{x} \in \text{zer} \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-1} B_i \right) \), then there exists \( \bar{z} \in \text{Fix} T \).

(b) If \( (\bar{z}_1, \ldots, \bar{z}_{n-1}) \in \text{Fix} T \), then \( \bar{x} := J_{\lambda A_1}(\bar{z}_1) \in \text{zer} \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-1} B_i \right) \). Moreover,

\[
\bar{x} = J_{\lambda A_1}(\bar{z}_i - \bar{z}_{i-1} + \bar{x} - \lambda B_{i-1}(\bar{x})) = J_{\lambda A_n}(2\bar{x} - \bar{z}_{n-1} - \lambda B_{n-1}(\bar{x})) \quad \forall i \in [2, n-1].
\] (11)

Consequently,

\[
\text{Fix} T \neq \emptyset \iff \text{zer} \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-1} B_i \right) \neq \emptyset.
\]

Proof. (a): Let \( \bar{x} \in \text{zer} \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-1} B_i \right) \). Then there exists \( v = (v_1, \ldots, v_n) \in \mathcal{H}^n \) such that \( v_i \in A_i(\bar{x}) \) and \( \sum_{i=1}^{n} v_i + \sum_{i=1}^{n-1} B_i(\bar{x}) = 0 \). Define the vector \( \bar{z} = (\bar{z}_1, \ldots, \bar{z}_{n-1}) \in \mathcal{H}^{n-1} \) according to

\[
\begin{cases}
\bar{z}_1 := \bar{x} + \lambda v_1 \in (\text{Id} + \lambda A_1)\bar{x}, \\
\bar{z}_i := \lambda v_i + \bar{z}_{i-1} + \lambda B_{i-1}(\bar{x}) \in (\text{Id} + \lambda A_1)(\bar{x}) - \bar{x} + \bar{z}_{i-1} + \lambda B_{i-1}(\bar{x}) \quad \forall i \in [2, n-1].
\end{cases}
\]

Then \( \bar{x} = J_{\lambda A_1}(\bar{z}_1) \) and \( \bar{x} = J_{\lambda A_1}(\bar{z}_i - \bar{z}_{i-1} + \bar{x} - \lambda B_{i-1}(\bar{x})) \) for \( i \in [2, n-1] \). Furthermore, we have

\[
(\text{Id} + \lambda A_n)(\bar{x}) \ni \bar{x} + \lambda v_n = \bar{x} - \lambda v_1 - \lambda \sum_{i=2}^{n-1} (v_i + B_{i-1}(\bar{x})) - \lambda B_{n-1}(\bar{x})
\]

\[
= \bar{x} - (\bar{z}_1 - \bar{x}) - \sum_{i=2}^{n-1} (\bar{z}_i - \bar{z}_{i-1}) - \lambda B_{n-1}(\bar{x})
\]

\[
= 2\bar{x} - \bar{z}_{n-1} - \lambda B_{n-1}(\bar{x}),
\]

which implies that \( \bar{x} = J_{\lambda A_n}(2\bar{x} - \bar{z}_{n-1} - \lambda B_{n-1}(\bar{x})) \). Altogether, it follows that \( \bar{z} \in \text{Fix} T \).

(b): Let \( \bar{z} \in \text{Fix} T \) and set \( \bar{x} := J_{\lambda A_1}(\bar{z}_1) \). Then (11) holds thanks to the definition of \( T \). The definition of the resolvent therefore implies

\[
\begin{cases}
\lambda A_1(\bar{x}) \ni \bar{z}_1 - \bar{x}, \\
\lambda A_1(\bar{x}) \ni \bar{z}_i - \bar{z}_{i-1} - \lambda B_{i-1}(\bar{x}) \quad \forall i \in [2, n-1], \\
\lambda A_n(\bar{x}) \ni \bar{x} - \bar{z}_{n-1} - \lambda B_{n-1}(\bar{x}).
\end{cases}
\]

Summing together the above inclusions gives \( \bar{x} \in \text{zer} \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-1} B_i \right) \), as claimed. \( \square \)

Next, we study the nonexpansivity properties of the operator \( T \) in (9).

Lemma 3.5. For all \( \bar{z} = (\bar{z}_1, \ldots, \bar{z}_n) \in \mathcal{H}^{n-1} \) and \( \bar{z} = (\bar{z}_1, \ldots, \bar{z}_n) \in \mathcal{H}^{n-1} \), we have

\[
\| T(\bar{z}) - T(\bar{z}) \|_2^2 + \left( 1 - \frac{\gamma}{\lambda} - \frac{\lambda L}{2\gamma} \right) \| (\text{Id} - T)(\bar{z}) - (\text{Id} - T)(\bar{z}) \|_2^2
\]

\[
+ \frac{1}{\gamma} \| \sum_{i=1}^{n-1} (\text{Id} - T)(\bar{z})_i - \sum_{i=1}^{n-1} (\text{Id} - T)(\bar{z})_i \|_2^2 \leq \| \bar{z} - \bar{z} \|_2^2. \quad (12)
\]

In particular, if \( \lambda \in (0, \frac{2}{L}) \) and \( \gamma \in (0, 1 - \frac{\lambda L}{2\gamma}) \), then \( T \) is \( \alpha \)-averaged for \( \alpha = \frac{2\gamma}{\gamma - \lambda L} \in (0, 1) \).

Proof. For convenience, denote \( \bar{z}^+ := T(\bar{z}) \) and \( \bar{z}^- := T(\bar{z}) \). Further, let \( \bar{x} = (x_1, \ldots, x_n) \in \mathcal{H}^n \) be given by (10) and let \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in \mathcal{H}^n \) be given analogously. Since \( z_1 - x_1 \in \lambda A_1(x_1) \) and \( \bar{z}_1 - \bar{x}_1 \in \lambda A_1(\bar{x}_1) \), the monotonicity of \( \lambda A_1 \) implies

\[
0 \leq \langle x_1 - \bar{x}_1, (z_1 - x_1) - (\bar{z}_1 - \bar{x}_1) \rangle
\]

\[
\quad = \langle x_2 - \bar{x}_2, (z_1 - x_1) - (\bar{z}_1 - \bar{x}_1) \rangle + \langle x_1 - x_2, (z_1 - x_1) - (\bar{z}_1 - \bar{x}_1) \rangle. \quad (13)
\]
For $i \in [2, n - 1]$, $z_i - z_{i-1} + x_{i-1} - x_i - \lambda B_{i-1}(x_{i-1}) \in \lambda A_i(x_i)$ and $\tilde{z}_i - \tilde{z}_{i-1} + \tilde{x}_{i-1} - \tilde{x}_i - \lambda B_{i-1}(\tilde{x}_{i-1}) \in \lambda A_i(\tilde{x}_i)$. Thus monotonicity of $\lambda A_i$ yields

\[
0 \leq \langle x_n - \tilde{x}_n, (z_n - x_n) - (\tilde{z}_n - \tilde{x}_n) \rangle - \langle x_2 - \tilde{x}_2, (z_2 - x_2) - (\tilde{z}_2 - \tilde{x}_2) \rangle + \sum_{i=2}^{n} \langle x_i - x_{i+1}, (z_i - x_i) - (\tilde{z}_i - \tilde{x}_i) \rangle - \sum_{i=1}^{n-2} \langle \tilde{x}_i - \tilde{x}_{i+1}, (z_i - x_i) - (\tilde{z}_i - \tilde{x}_i) \rangle
\]

\[
- \lambda \sum_{i=2}^{n} \langle x_i - \tilde{x}_i, B_{i-1}(x_{i-1}) - B_{i-1}(\tilde{x}_{i-1}) \rangle.
\]

(14)

Summing this inequality for $i \in [2, n - 1]$ and simplifying gives

\[
0 \leq \langle x_n - \tilde{x}_n, (z_n - x_n) - (\tilde{z}_n - \tilde{x}_n) \rangle - \langle x_2 - \tilde{x}_2, (z_2 - x_2) - (\tilde{z}_2 - \tilde{x}_2) \rangle + \sum_{i=2}^{n} \langle x_i - x_{i+1}, (z_i - x_i) - (\tilde{z}_i - \tilde{x}_i) \rangle - \sum_{i=1}^{n-2} \langle \tilde{x}_i - \tilde{x}_{i+1}, (z_i - x_i) - (\tilde{z}_i - \tilde{x}_i) \rangle
\]

Since $x_1 + x_{n-1} - x_n - z_{n-1} - \lambda B_{n-1}(x_{n-1}) \in \lambda A_n(x_n)$ and $\tilde{x}_1 + \tilde{x}_{n-1} - \tilde{x}_n - \tilde{z}_{n-1} - \lambda B_{n-1}(\tilde{x}_{n-1}) \in \lambda A_n(\tilde{x}_n)$, monotonicity of $\lambda A_n$ gives

\[
0 \leq \langle x_n - \tilde{x}_n, (x_1 + x_{n-1} - x_n - z_{n-1} - \lambda B_{n-1}(x_{n-1})) - (\tilde{x}_1 + \tilde{x}_{n-1} - \tilde{x}_n - \tilde{z}_{n-1} - \lambda B_{n-1}(\tilde{x}_{n-1})) \rangle
\]

\[
= \langle x_n - \tilde{x}_n, (x_1 - \tilde{x}_1) + (x_2 - \tilde{x}_2) + \sum_{i=2}^{n-1} (x_i - x_{i+1}) - \sum_{i=1}^{n-2} \tilde{x}_i - \tilde{x}_{i+1} \rangle
\]

\[
- \lambda \langle x_n - \tilde{x}_n, B_{n-1}(x_n) - B_{n-1}(\tilde{x}_n) \rangle
\]

(15)

Adding (13), (14) and (15) and rearranging gives

\[
0 \leq \sum_{i=1}^{n-1} \langle (x_i - \tilde{x}_i) - (x_{i+1} - \tilde{x}_{i+1}), \tilde{x}_i - x_i \rangle + \sum_{i=1}^{n-1} \langle (x_i - \tilde{x}_i) - (x_{i+1} - \tilde{x}_{i+1}), z_i - \tilde{z}_i \rangle
\]

\[
+ \frac{1}{2} (||x_1 - \tilde{x}_1||^2 - ||x_n - \tilde{x}_n||^2 - ||x_1 - x_n||^2 - ||\tilde{x}_1 - \tilde{x}_n||^2)
\]

\[
- \lambda \sum_{i=1}^{n-1} \langle x_{i+1} - \tilde{x}_{i+1}, B_{i}(x_i) - B_{i}(\tilde{x}_i) \rangle.
\]

(16)

The first term in (16) can be expressed as

\[
\sum_{i=1}^{n-1} \langle (x_i - \tilde{x}_i) - (x_{i+1} - \tilde{x}_{i+1}), \tilde{x}_i - x_i \rangle
\]

\[
= \frac{1}{2} \sum_{i=1}^{n-1} (||x_{i+1} - \tilde{x}_{i+1}||^2 - ||x_i - \tilde{x}_i||^2 - ||x_i - x_{i+1}||^2 - ||\tilde{x}_i - \tilde{x}_{i+1}||^2)
\]

\[
= \frac{1}{2} \left(||x_n - \tilde{x}_n||^2 - ||x_1 - \tilde{x}_1||^2 - \frac{1}{\gamma^2} (||z - z^+||^2 - ||\tilde{z} - \tilde{z}^+||^2) \right).
\]

(17)
and the second term in (16) can be written as
\[
\sum_{i=1}^{n-1} \langle (x_i - x_{i+1}) - (\bar{x}_i - \bar{x}_{i+1}), z_i - \bar{z}_i \rangle \\
= \frac{1}{\gamma} \sum_{i=1}^{n-1} \langle (z_i - z_i^+) - (\bar{z}_i - \bar{z}_i^+), z_i - \bar{z}_i \rangle \\
= \frac{1}{\gamma} \langle (z - z^+) - (\bar{z} - \bar{z}^+), z - \bar{z} \rangle \\
= \frac{1}{2\gamma} \left( ||z - z^+||^2 - ||\bar{z} - \bar{z}^+||^2 \right).
\] (18)

To estimate the last term, Young’s inequality and \(\frac{1}{\lambda}\)-cocoercivity of \(B_1, \ldots, B_{n-1}\) gives
\[
- \sum_{i=1}^{n-1} \langle x_{i+1} - x_i, B_i(x_i) - B_i(\bar{x}_i) \rangle \\
= \sum_{i=1}^{n-1} \langle (\bar{x}_{i+1} - \bar{x}_i) - (x_{i+1} - x_i), B_i(x_i) - B_i(\bar{x}_i) \rangle + \sum_{i=1}^{n-1} \langle \bar{x}_i - x_i, B_i(x_i) - B_i(\bar{x}_i) \rangle \\
\leq \frac{1}{4} \sum_{i=1}^{n-1} ||(\bar{x}_{i+1} - \bar{x}_i) - (x_{i+1} - x_i)||^2 + \frac{1}{\lambda} \sum_{i=1}^{n-1} ||B_i(x_i) - B_i(\bar{x}_i)||^2 \\
- \frac{1}{\lambda} \sum_{i=1}^{n-1} ||B_i(x_i) - B_i(\bar{x}_i)||^2 \\
= \frac{1}{4} \sum_{i=1}^{n-1} ||(\bar{x}_{i+1} - \bar{x}_i) - (x_{i+1} - x_i)||^2 \\
= \frac{L}{4\gamma^2} ||(z - z^+) - (\bar{z} - \bar{z}^+)||^2.
\] (19)

Thus, substituting (17) and (18) into (16), using (19) and simplifying gives the claimed inequality (12). Finally, to show that (12) implies \(T\) is \(\bar{\alpha}\)-averaged with \(\bar{\alpha} := \frac{2\beta}{2-\beta}\), note that \(\bar{\alpha} \in (0, 1)\) and satisfies \(\frac{1-\bar{\alpha}}{\bar{\alpha}} = \frac{1}{\gamma} - \frac{M_l}{\beta}\). This completes the proof. \(\square\)

The following theorem is our main convergence result regarding the algorithm given by (8).

**Theorem 3.6.** Let \(n \geq 2\), let \(A_1, \ldots, A_n : \mathcal{H} \rightrightarrows \mathcal{H}\) be maximally monotone and let \(B_1, \ldots, B_{n-1} : \mathcal{H} \rightrightarrows \mathcal{H}\) be \(\frac{1}{\lambda}\)-cocoercive with \(\text{zer} \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-1} B_i \right) \neq \emptyset\). Further, let \(\lambda \in (0, \frac{2}{3})\) and \(\gamma \in (0, 1 - \frac{M_l}{\beta})\).

Given \(z^0 \in \mathcal{H}^{n-1}\), let \((z^k) \subseteq \mathcal{H}^{n-1}\) and \((x^k) \subseteq \mathcal{H}^n\) be the sequences given by (8). Then the following assertions hold.

(a) The sequence \((z^k)\) converges weakly to a point \(z \in \text{Fix} T\).

(b) The sequence \((x^k)\) converges weakly to a point \((x, \ldots, x) \in \mathcal{H}^n\) with \(x \in \text{zer} \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-1} B_i \right)\).

(c) The sequence \((B_1(x^k))\) converges strongly to \(B_1(x)\) for all \(i \in [1, n-1]\).

**Proof.** (a): Since \(\text{zer} \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-1} B_i \right) \neq \emptyset\), Lemma 3.4(a) implies \(\text{Fix} T \neq \emptyset\). Since \(\lambda \in (0, \frac{2}{3})\) and \(\gamma \in (0, 1 - \frac{M_l}{\beta})\), Lemma 3.5 implies \(T\) is averaged nonexpansive. By applying [4, Theorem 5.15], we deduce that \((z^k)\) converges weakly to a point \(z \in \text{Fix} T\) and that \(\lim_{k \to \infty} ||z^{k+1} - z^k|| = 0\).

(b): By nonexpansivity of resolvents, L-Lipschitz continuity of \(B_1, \ldots, B_{n-1}\), and boundedness of \((z^k)\), it follows that \((x^k)\) is also bounded. Further, (9) and the fact that \(\lim_{k \to \infty} ||z^{k+1} - z^k|| = 0\) implies that
\[
\lim_{k \to \infty} ||x^k_i - x^k_{i-1}|| = 0 \quad \forall i = 2, \ldots, n.
\] (20)
Next, using the definition of the resolvent together with (8b), we have

\[
S \left( \begin{array}{c}
(z_n^k - x_n^k) - (z_{n-1}^k - x_{n-1}^k) + \lambda(B_n(x_n^k) - B_{n-1}(x_{n-1}^k)) \\
(z_{n-1}^k - x_{n-1}^k) - (z_{n-2}^k - x_{n-2}^k) + \lambda(B_{n-2}(x_{n-2}^k) - B_{n-2}(x_{n-2}^k)) \\
\vdots \\
(z_1^k - x_1^k) - (z_0^k - x_0^k) + \lambda(B_1(x_0^k) - B_1(x_1^k))
\end{array} \right) \equiv \left( \begin{array}{c}
x_n^k - x_n^k \\
x_{n-1}^k - x_{n-1}^k \\
\vdots \\
x_1^k - x_1^k + \lambda \sum_{i=1}^{n-1} (B_i(x_{i+1}^k) - B_i(x_i^k))
\end{array} \right), \tag{21}
\]

where the operator \( S: \mathcal{H}^n \rightarrow \mathcal{H}^n \) is given by

\[
S := \left( \begin{array}{c}
(\lambda A_1)^{-1} - 1 \\
(\lambda (A_2 + B_1))^{-1} - 1 \\
\vdots \\
(\lambda (A_{n-1} + B_{n-2}))^{-1} - 1 \\
(\lambda (A_n + B_{n-1}))^{-1} - 1
\end{array} \right) + \left( \begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
1 \text{Id} \ldots \text{Id} 0
\end{array} \right). \tag{22}
\]

As the sum of two maximally monotone operators is again maximally monotone provided that one of the operators has full domain [4, Corollary 24.4(i)], it follows that \( S \) is maximally monotone. Consequently, it is demiclosed [4, Proposition 20.32]. That is, its graph is sequentially closed in the weak-strong topology.

Let \( w \in \mathcal{H}^n \) be an arbitrary weak cluster point of the sequence \( (x^k) \). As a consequence of (20), \( w = (x, \ldots, x) \) for some \( x \in \mathcal{H} \). Taking the limit along a subsequence of \( (x^k) \) which converges weakly to \( w \) in (21), using demiclosedness of \( S \) together with L-Lipschitz continuity of \( B_1, \ldots, B_{n-1} \), and unravelling the resulting expression gives

\[
\begin{aligned}
\lambda A_1(x) &\ni z_1 - x, \\
\lambda(A_i + B_{i-1})(x) &\ni z_i - z_{i-1} \quad \forall i \in [2, n-1], \\
\lambda(A_n + B_{n-1})(x) &\ni x - z_{n-1},
\end{aligned}
\]

which implies \( z \in \text{Fix } T \) and \( x = J_{A_1}(z_1) \in \text{zer } \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-1} B_i \right) \).

In other words, \( w = (x, \ldots, x) \in \mathcal{H}^n \) with \( x := J_{A_1}(z_1) \) is the unique weak sequential cluster point of the bounded sequence \( (x^k) \). We therefore deduce that \( (x^k) \) converges weakly to \( w \), which completes this part of the proof.

(c): For convenience, denote \( y^k = (y_1^k, \ldots, y_n^k) \) where

\[
\begin{cases}
y_1^k := z_1^k, \\
y_i^k := z_i^k + x_{i-1}^k - z_{i-1}^k - \lambda B_{i-1}(z_{i-1}) \quad \forall i \in [2, n-1], \\
y_n^k := x_n^k + x_{n-1}^k - z_{n-1}^k - \lambda B_{n-1}(x_{n-1}^k),
\end{cases}
\]

so that \( x_i^k = J_{A_i}(y_i^k) \) for all \( i \in [1, n] \). Define \( y = (y_1, \ldots, y_n) \) in an analogous way with \( z \) in place of \( z^k \) and \( (x, \ldots, x) \) in place of \( x^k \), so that \( x = J_{A_i}(y_i) \) for all \( i \in [1, n] \). Using firm nonexpansivity
of resolvents yields

\[ 0 \leq \sum_{i=1}^{n} \langle J_{\lambda A_i}(y_i^k) - J_{\lambda A_i}(y_1), (\text{Id} - J_{\lambda A_i})(y_i^k) - (\text{Id} - J_{\lambda A_i})(y_1) \rangle \]

\[ = \langle x_i^k - x_i, (z_i^k - x_i^k) - (z_i - x) \rangle \]

\[ + \sum_{i=2}^{n-1} \langle x_i^k - x_i, ((z_i^k - x_i^k) - (z_{i-1}^k - x_{i-1}^k)) - \lambda B_{i-1}(x_{i-1}^k) \rangle - (z_i - z_{i-1} - \lambda B_{i-1}(x)) \]

\[ + \langle x_n^k - x_m, (x_{n-1}^k - x_{n-1}^k) - \lambda B_{n-1}(x_{n-1}^k) \rangle - (x - z_{n-1} - \lambda B_{n-1}(x)) \]

\[ = \langle x_n^k - x_n, (z_n^k - x_n^k) - (z_1 - x) \rangle + \langle x_n^k - x, (z_n^k - x_n^k) - (z_1 - x) \rangle \] (23)

\[ + \sum_{i=2}^{n-1} \langle x_i^k - x_i, (z_i^k - x_i^k) - (z_{i-1}^k - x_{i-1}^k) - (z_i - z_{i-1}) \rangle \]

\[ + \langle x_n^k - x, (z_n^k - x_n^k) - (z_n^k - x_n^k) - (z_n - z_1) \rangle \]

\[ - \lambda \sum_{i=1}^{n-1} \langle x_{i+1}^k - x_i^k, B_i(x_i^k) - B_i(x) \rangle - \lambda \sum_{i=1}^{n-1} \langle x_i^k - x, B_i(x_i^k) - B_i(x) \rangle \]

\[ + \langle x_n^k - x, x_n^k - x_n^k \rangle - \langle x_n^k - x, (z_{n-1}^k - x_{n-1}^k) + (x - z_{n-1}) \rangle. \]

Rearranging (23) followed by applying \( \frac{1}{\lambda} \)-cocoercivity of \( B_1, \ldots, B_{n-1} \) gives

\[ \langle x_n^k - x, x_n^k - x_n^k \rangle + \langle x_i^k - x_n^k, (z_i^k - x_i^k) - (z_i - x) \rangle - \lambda \sum_{i=1}^{n-1} \langle x_{i+1}^k - x_i^k, B_i(x_i^k) - B_i(x) \rangle \]

\[ + \sum_{i=2}^{n-1} \langle x_i^k - x_i^k, ((z_i^k - x_i^k) - (z_{i-1}^k - x_{i-1}^k)) - (z_i - z_{i-1}) \rangle \]

\[ \geq \lambda \sum_{i=1}^{n-1} \langle x_i^k - x, B_i(x_i^k) - B_i(x) \rangle \geq \lambda \sum_{i=1}^{n-1} \|B_i(x_i^k) - B_i(x)\|^2. \] (24)

Note that the left-hand side of (24) converges to zero due to (20) and the boundedness of sequences \( (z^k), (x^k) \) and \( \langle B_i(x_i^k) \rangle \) for \( i \in [1, n-1] \). It then follows that \( B_i(x_i^k) \to B_i(x) \) for all \( i \in [1, n-1] \), as claimed.

**Remark 3.7 (Attouch–Théra duality).** Let \( I \subseteq \{1, \ldots, n-1\} \) be a non-empty index set with cardinality denoted by \( |I| \). Express the monotone inclusion (1) as

\[ \text{find } x \in \mathcal{H} \text{ such that } 0 \in \sum_{i \in I} B_i(x) + \left( \sum_{i=1}^{n} A_i + \sum_{i \notin I} B_i \right)(x), \] (25)

and note that the first operator \( \sum_{i \in I} B_i \) is \( \frac{1}{|I|} \)-cocoercive (see, e.g., [4, Proposition 4.12]). The **Attouch–Théra dual** [2] associated with (25) takes the form

\[ \text{find } u \in \mathcal{H} \text{ such that } 0 \in \left( \sum_{i \in I} B_i \right)^{-1}(u) - \left( \sum_{i=1}^{n} A_i + \sum_{i \notin I} B_i \right)^{-1}(-u), \] (26)

where we note that the first operator \( \left( \sum_{i \in I} B_i \right)^{-1} \) is \( \frac{1}{|I|} \)-strongly monotone. Hence, as a strongly monotone inclusion, (26) has a unique solution \( \bar{u} \in \mathcal{H} \). Moreover, for any solution \( \bar{x} \in \mathcal{H} \) of (25), [2, Theorem 3.1] implies \( \bar{u} = (\sum_{i \in I} B_i)\left(\bar{x}\right) \). In the context of the previous result, Theorem 3.6(c) implies \( \sum_{i \in I} B_i(x_i^k) \to u \) as \( k \to \infty \). In other words, the algorithm in (8) also produces a sequence which converges strongly to the unique solution of the dual inclusion (26).
Remark 3.8. (i) When $B_1 = \cdots = B_{n-1} = 0$, Theorem 3.6 recovers [16, Theorem 4.5].

(ii) In the special case when $n = 2$, (12) from Lemma 3.5 simplifies to give the stronger inequality

$$\|T(z) - T(\bar{z})\|^2 + \left(\frac{2-\gamma}{\gamma} - \frac{\lambda L}{2\gamma}\right) \|\left(\text{Id} - T\right)(z) - \left(\text{Id} - T\right)(\bar{z})\|^2 \leq \|z - \bar{z}\|^2.$$ (27)

This assures averagedness of $T$ provided that $\gamma \in (0, 2 - \frac{\lambda L}{4})$, which is larger than the range of permissible values for $\gamma$ in the statement of Theorem 3.6. However, by using (27), a proof similar to that of Theorem 3.6 guarantees the convergence for a larger range of parameter values, namely, when $\lambda \in (0, \frac{4}{L})$ and $\gamma \in (0, 2 - \frac{\lambda L}{4})$. For details, see [11, 1].

4 A Distributed Forward-Reflected-Backward Method

Let $n \geq 3$ and consider the problem

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in \left(\sum_{i=1}^{n} A_i + \sum_{i=1}^{n-2} B_i\right)(x),$$ (28)

where $A_1, \ldots, A_n: \mathcal{H} \rightrightarrows \mathcal{H}$ are maximally monotone and $B_1, \ldots, B_{n-2}: \mathcal{H} \rightrightarrows \mathcal{H}$ are monotone and $L$-Lipschitz continuous.

Developing splitting algorithms which use forward evaluations of Lipschitz continuous monotone operators is generally more intricate than those exploiting cocoercivity, such as the one in the previous section. For concreteness, consider the special case of (28) with two operators given by

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in (A_1 + B_1)(x).$$ (29)

It is well known that the forward-backward method for (29) given by

$$x^{k+1} = J_{\lambda A_1}(x^k - \lambda B_1(x^k))$$ (30)

fails to converge for any $\lambda > 0$. However, a small modification of (30) gives rise to

$$x^{k+1} = J_{\lambda A_1}(x^k - 2\lambda B_1(x^k) + \lambda B_1(x^{k-1})),$$ (31)

which is known as the forward-reflected-backward method [15]. Unlike (30), it converges for any $\lambda < \frac{4}{L}$. While (31) is not the only constant stepsize scheme for solving (29), as there are a few which are fundamentally different [23, 10], it is arguably one of the simplest. In this section, we develop a modification of the method from the previous section which converges for Lipschitz continuous operators by drawing inspiration from the differences between (31) and (30).

Given $\lambda \in (0, \frac{1}{4L})$ and $\gamma \in (0, 1 - 2L\lambda)$ and an initial point $z^0 = (z_1^0, \ldots, z_{n-1}^0) \in \mathcal{H}^{n-1}$, our proposed algorithm for (28) generates two sequences, $(z^k) \subseteq \mathcal{H}^{n-1}$ and $(x^k) \subseteq \mathcal{H}^n$, according to

$$z^{k+1} = z^k + \gamma \begin{pmatrix} x_2^k - x_1^k \\ x_3^k - x_2^k \\ \vdots \\ x_n^k - x_{n-1}^k \end{pmatrix}$$ (32a)

and

$$\begin{align*}
x_1^k &= J_{\lambda A_1}(z_1^k), \\
x_2^k &= J_{\lambda A_2}(x_2^k + x_1^k - z_1^k - \lambda B_1(x_1^k)), \\
x_3^k &= J_{\lambda A_3}(x_3^k + x_2^k - z_2^k - \lambda B_1(x_2^k) - \lambda B_{i-1}(x_{i-1}^k) - \lambda(B_{i-2}(x_{i-2}^k) - B_{i-2}(x_{i-2}^k))) \quad \forall i \in [3, n-1], \\
x_n^k &= J_{\lambda A_n}(x_n^k + x_{n-1}^k - z_{n-1}^k - \lambda(B_{n-2}(x_{n-2}^k) - B_{n-2}(x_{n-2}^k))) \quad \forall i \in [3, n-1].
\end{align*}$$ (32b)
Lemma 4.2. Let \( \lambda \). In this way, the sequence \( z^k \) can be eliminated from (32) to give

\[
\begin{align*}
x_2^k &= J_{\lambda A} \left( z_2^k - \lambda B_1(z_1^k) \right) \\
x_1^k &= z_1^k + \gamma (x_2^k - z_1^k) \\
x_3^k &= z_2^k + \gamma (z_1^k - z_3^k) - \lambda (B_1(x_2^k) - B_1(z_3^k)).
\end{align*}
\]

To better understand the relationship between this and (31) clearer, it is instructive to consider the limiting case with \( \gamma = 1 \). Indeed, when \( \gamma = 1 \), \( x_2^k \) and \( z_2^k \) can be eliminated to give

\[
z_1^{k+1} = J_{\lambda A} \left( z_1^{k-1} - 2\lambda B_1(z_1^k) + \lambda B_1(z_1^{k-1}) \right).
\]

Although this closely resembles (31) for finding zero of \( A_2 + B_1 \), it is not exactly the same due to the index of the first term inside the resolvent.

In order to analyse (32), we introduce the underlying fixed point operator \( \tilde{\bar{T}}: \mathcal{H}^{n-1} \to \mathcal{H}^{n-1} \) given by

\[
\tilde{\bar{T}}(z) := z + \gamma \begin{pmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{pmatrix},
\]

where \( x = (x_1, \ldots, x_n) \in \mathcal{H}^n \) depends on \( z = (z_1, \ldots, z_n) \in \mathcal{H} \) and is given by

\[
\begin{align*}
x_1 &= J_{\lambda A_1}(z_1), \\
x_2 &= J_{\lambda A_2}(z_2 + x_1 - z_1 - \lambda B_1(x_1)), \\
x_i &= J_{\lambda A_1}(z_i + x_{i-1} - z_{i-1} - \lambda B_{i-1}(x_{i-1}) - \lambda (B_{i-2}(x_{i-1}) - B_{i-2}(x_{i-2}))) \quad \forall i \in [3, n-1], \\
x_n &= J_{\lambda A_n}(x_1 + x_{n-1} - z_{n-1} - \lambda (B_{n-2}(x_{n-1}) - B_{n-3}(x_{n-3}))).
\end{align*}
\]

In this way, the sequence \( (z^k) \) given by (32) satisfies \( z^{k+1} = \tilde{\bar{T}}(z^k) \) for all \( k \in \mathbb{N} \).

The following lemma analyses the nonexpansivity properties of the operator \( \tilde{T} \).

Lemma 4.2. Let \( \hat{z} = (\hat{z}_1, \ldots, \hat{z}_{n-1}) \in \text{Fix} \bar{T} \). Then, for all \( z = (z_1, \ldots, z_{n-1}) \in \mathcal{H}^{n-1} \), we have

\[
\| \tilde{T}(z) - \hat{z} \|^2 + \left( \frac{1 - \gamma}{\gamma} - \frac{2\lambda}{\gamma} \right) \| (\text{Id} - \tilde{T})(z) \|^2 + \frac{1}{\gamma} \sum_{i=1}^{n-1} \| (\text{Id} - \tilde{ar{T}})(z)_i \|^2 \\
+ \gamma \lambda \| (\text{Id} - \tilde{T})(z) \|_1^2 + \gamma \lambda \| (\text{Id} - \tilde{T})(z) \|_{n-1}^2 \leq \| z - \hat{z} \|^2.
\]

In particular, if \( \lambda \in (0, \frac{1}{2\lambda}) \) and \( \gamma \in (0, 1 - 2\lambda) \), then \( \bar{T} \) is \( \sigma \)-strongly quasi-nonexpansive for \( \sigma = \frac{1 - \gamma - 2\lambda}{\gamma} > 0 \).

Proof. For convenience, denote \( z^+ = \tilde{T}(z) \). Further, let \( x = (x_1, \ldots, x_n) \in \mathcal{H}^n \) be given by (34) and let \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \in \mathcal{H}^{n-1} \) be given analogously. Note that the expression of \( \hat{x} \) is justified as \( \hat{z} = \tilde{T}(z) \). Using the same argument as in Lemma 3.5 with (34) instead of (10) gives

\[
\| \tilde{T}(z) - \hat{z} \|^2 + \left( \frac{1 - \gamma}{\gamma} - \frac{2\lambda}{\gamma} \right) \| (\text{Id} - \tilde{T})(z) \|^2 + \frac{1}{\gamma} \sum_{i=1}^{n-1} \| (\text{Id} - \tilde{T})(z)_i \|^2 \leq \| z - \hat{z} \|^2
\]

\[
- 2\gamma \lambda \sum_{i=2}^{n-1} (x_i - \hat{x}, B_{i-1}(x_{i-1}) - B_{i-1}(\hat{x})) - 2\gamma \lambda \sum_{i=3}^{n} (x_i - \hat{x}, B_{i-2}(x_{i-1}) - B_{i-2}(x_i)) \leq \| z - \hat{z} \|^2.
\]
Using monotonicity of $B_1, \ldots, B_{n-2}$, the second last term can be estimated as

\[- \sum_{i=2}^{n-1} \langle x_i - \bar{x}, B_{i-1}(x_i) - B_{i-1}(\bar{x}) \rangle \leq \sum_{i=2}^{n-1} \langle x_i - \bar{x}, B_{i-1}(x_i) - B_{i-1}(x_i) \rangle \]  

and, using L-Lipschitz continuity of $B_1, \ldots, B_{n-2}$, the last term can be estimated as

\[- \sum_{i=3}^{n} \langle x_i - \bar{x}, B_{i-2}(x_i) - B_{i-2}(x_i) \rangle \]

\[
= - \sum_{i=3}^{n} \langle x_i - \bar{x}, B_{i-2}(x_i) - B_{i-2}(x_i) \rangle + \sum_{i=3}^{n} \langle x_i - \bar{x}, B_{i-2}(x_i) - B_{i-2}(x_i) \rangle \\
\leq - \sum_{i=3}^{n} \langle x_i - \bar{x}, B_{i-2}(x_i) - B_{i-2}(x_i) \rangle + \frac{L}{2} \sum_{i=3}^{n} (\|x_{i-1} - x_i\|^2 + \|x_{i-1} - x_i\|^2) \\
= - \sum_{i=2}^{n} \langle x_i - \bar{x}, B_{i-1}(x_i) - B_{i-1}(x_i) \rangle + \frac{L}{2} \sum_{i=2}^{n} \|x_i - x_{i-1}\|^2 \\
- \frac{L}{2} \|x_2 - x_1\|^2 - \frac{L}{2} \|x_n - x_{n-1}\|^2 \\
= - \sum_{i=2}^{n-1} \langle x_i - \bar{x}, B_{i-1}(x_i) - B_{i-1}(x_i) \rangle + \frac{L}{2} \|\|Id - \tilde{T}\|z_i\|^2 \\
- \frac{L}{2} \|\|Id - \tilde{T}\|z_i\|^2 - \frac{L}{2} \|\|Id - \tilde{T}\|z_{n-1}\|^2. 
\]

Thus, substituting (37) and (38) into (36) gives (35), which completes the proof. \( \Box \)

**Remark 4.3.** Compared to Lemma 3.5 from the previous section, the conclusions of Lemma 4.2 are weaker in two ways. Firstly, the permissible stepsize range of $\lambda \in (0, \frac{1}{2L})$ is smaller than in Lemma 3.5, which allowed $\lambda \in (0, \frac{1}{4})$. And, secondly, the operator $\tilde{T}$ is only shown to be strongly quasi-nonexpansive in Lemma 4.2 whereas $T$ is known to be averaged nonexpansive.

The following theorem is our main result regarding convergence of (32).

**Theorem 4.4.** Let $n \geq 3$, let $A_1, \ldots, A_n : \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone and let $B_1, \ldots, B_{n-2} : \mathcal{H} \rightrightarrows \mathcal{H}$ be monotone and L-Lipschitz continuous with $\text{zer} \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-2} B_i \right) \neq \emptyset$. Further, let $\lambda \in (0, \frac{1}{2L})$ and $\gamma \in (0, 1 - 2\lambda L)$. Given $z^0 \in \mathcal{H}^{n-1}$, let $(z^k) \subseteq \mathcal{H}^{n-1}$ and $(x^k) \subseteq \mathcal{H}^n$ be the sequences given by (32). Then the following assertions hold.

(a) The sequence $(z^k)$ converges weakly to a point $z \in \text{Fix} \tilde{T}$.

(b) The sequence $(x^k)$ converges weakly to a point $(x, \ldots, x) \in \mathcal{H}^n$ with $x \in \text{zer} \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-2} B_i \right)$.

**Proof.** (a): Since $\text{zer} \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-2} B_i \right) \neq \emptyset$, Lemma 3.4(a) implies that the set of fixed points of operator $T$ in (9)-(10) (with $B_{n-1} = 0$) is nonempty. The latter set coincides with the set of fixed points of operator $\tilde{T}$ in (33)-(34), so $\text{Fix} \tilde{T} \neq \emptyset$. Since $\lambda \in (0, \frac{1}{2L})$ and $\gamma \in (0, 1 - 2\lambda L)$, Lemma 4.2 implies that $(z^k)$ is Fejér monotone with respect to $\text{Fix} \tilde{T}$ and that $\lim_{k \to +\infty} \|z^{k+1} - z^k\| = 0$. By nonexpansivity of resolvents, L-Lipschitz continuity of $B_2, \ldots, B_{n-1}$, and boundedness of $(z^k)$, it follows that $(x^k)$ is also bounded. Further, (33) and the fact that $\lim_{k \to +\infty} \|x^{k+1} - x^k\| = 0$ implies that

\[ \lim_{k \to +\infty} \|x^k - x_{k-1}\| = 0 \quad \forall i = 2, \ldots, n. \]  

(39)

Let $u = (u_1, \ldots, u_{n-1}) \in \mathcal{H}^{n-1}$ be an arbitrary weak cluster point of $(z^k)$. Then, due to (39), there exists a point $x \in \mathcal{H}$ such that $(u, w)$ is a weak cluster point of $(z^k, x^k)$, where $w = (x, \ldots, x) \in \mathcal{H}^n$. 

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Let $S$ denote the maximally monotone operator defined by (22) when $B_{n-1} = 0$. Then (32b) implies

$$
S \left( \begin{array}{c}
\frac{z_1^k - x_1^k}{\lambda B_1(x_1^k) - B_1(x_1^k)} \\
(\frac{z_2^k - x_2^k}{\lambda B_2(x_2^k) - B_2(x_2^k)}) \\
\vdots \\
(\frac{z_n^k - x_n^k}{\lambda B_n(x_n^k) - B_n(x_n^k)})
\end{array} \right) = \left( \begin{array}{c}
x_1^k - x_1^k \\
x_2^k - x_2^k \\
\vdots \\
x_n^k - x_n^k
\end{array} \right).
$$

Taking the limit along a subsequence of $(z^k, x^k)$ which converges weakly to $(u, w)$ in (40), using demiclosedness of $S$ together with L-Lipschitz continuity of $B_2, \ldots, B_{n-1}$, and unravelling the resulting expression gives that $u \in \text{Fix } \tilde{T}$ and $x = J_{\lambda A_1}(u) \in \text{zer} \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-2} B_i \right)$. Thus, by [4, Theorem 5.5], it follows that $(z^k)$ converges weakly to a point $z \in \text{Fix } \tilde{T}$.

(b) Follows by using an argument analogous to the one in Theorem 3.6(c). □

**Remark 4.5 (Exploiting cocoercivity).** If a Lipschitz continuous operator $B_1$ in (28) is actually cocoercive, then it is possible to reduce the number evaluations of $B_1$ per iteration by combining the ideas in Sections 3 and 4. In fact, we can consider the problem

$$
\text{find } x \in \mathcal{H} \text{ such that } 0 \in \left( \sum_{i=1}^{n} A_i + \sum_{i=1}^{n-2} B_i \right) (x),
$$

where $B_1, \ldots, B_{n-2}$ are either monotone and Lipschitz continuous or cocoercive, and $B_{n-1}$ is cocoercive. For this problem, we can replace (34) in the definition of $\tilde{T}$ with

$$
\begin{align*}
x_1 &= J_{\lambda A_1}(z_1), \\
x_2 &= J_{\lambda A_1}(z_2 + x_1 - z_1 - \lambda B_1(x_1)), \\
x_i &= J_{\lambda A_1}(z_i + x_{i-1} - z_{i-1} - \lambda B_{i-1}(x_{i-1}) - \lambda y_{i-1}) \quad \forall i \in [3, n-1], \\
x_n &= J_{\lambda A_1}(x_1 + x_{n-1} - z_{n-1} - \lambda B_{n-1}(x_{n-1}) - \lambda y_{n-1}),
\end{align*}
$$

where $y_2, \ldots, y_{n-1} \in \mathcal{H}$ are given by

$$
y_i = \begin{cases} 
0 & \text{if } B_i \text{ is cocoercive} \\
B_{i-1}(x_i) - B_{i-1}(x_{i-1}) & \text{if } B_i \text{ is monotone and Lipschitz}.
\end{cases}
$$

This modification can be shown to converge using a proof similar to Theorem 4.4 for $\lambda \in (0, \frac{1}{\|A_1\|})$. However, it is not straightforward to recover Theorem 3.6 as a special case of such a result because the stepsizes range of $\lambda \in (0, \frac{1}{\|A_1\|})$ is larger than the range in the mixed case. Moreover, Theorem 3.6(c) (strong convergence to dual solutions) does not have an analogue in the statement of Theorem 4.4. In addition, keeping the two cases separate allows the analysis to be as transparent as possible.

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