UNIFORM CONVEXITY, REFLEXIVITY, SUPERREFLEXIVITY AND $B$
CONVEXITY OF GENERALIZED SOBOLEV SPACES $W^{1,\Phi}$

ANNA KAMIŃSKA AND MARIUSZ ŻYLUK

Abstract. We investigate Sobolev spaces $W^{1,\Phi}$ associated to Musielak-Orlicz spaces $L^{\Phi}$. We first present conditions for the boundedness of the Volterra operator in $L^{\Phi}$. Employing this, we provide necessary and sufficient conditions for $W^{1,\Phi}$ to contain isomorphic subspaces to $\ell^\infty$ or $\ell^1$. Further we give necessary and sufficient conditions in terms of the function $\Phi$ for reflexivity, uniform convexity, $B$-convexity and superreflexivity of $W^{1,\Phi}$. As corollaries we obtain the corresponding results for Orlicz-Sobolev spaces $W^{1,\varphi}$ where $\varphi$ is an Orlicz function, the variable exponent Sobolev spaces $W^{1,p(\cdot)}$ and the Sobolev spaces associated to double phase functionals.

1. Introduction

The main goal of this paper is to study geometric properties of Sobolev spaces $W^{1,\Phi}$ induced by Musielak-Orlicz spaces $L^{\Phi}$, where $\Phi$ is an Orlicz function with parameter, called also Musielak-Orlicz function.

Musielak-Orlicz spaces appeared first time in the literature in 1951 in H. Nakano paper [30], and later J. Musielak and W. Orlicz in 1959 in the paper [29] gave a more general definition often more suitable for applications. Musielak-Orlicz spaces engendered some interest and were extensively studied during the seventies, eighties and nineties of the last century by various groups of mathematicians across the world. In particular the structural and geometrical properties of those spaces were well understood.

On the other hand, the Musielak-Orlicz-Sobolev spaces (MOS spaces) came to the light late in seventies. The first results about MOS spaces were established by H. Hudzik in series of paper between 1976-1979 (see e.g. [15]). We remark here that the author needed to assume some rather strong assumptions about the function $\Phi$ to establish his results, which could be common in the field of MOS spaces.

Parallel to the research on MO and MOS spaces the variable exponent spaces, also called the Nakano spaces, were of significant interest. In terms of MO spaces a variable exponent space $L^{p(\cdot)}$ is a space generated by a function $\Phi$ of the form $t^{p(x)} / p(x)$, where $p(x)$ is a measurable function such that $p(x) \geq 1$. Surprisingly, a serious investigation into Sobolev spaces based on $L^{p(\cdot)}$ began only in the nineties of the last century with the paper of O. Kovářík and J. Rákosník [23], where the authors proved some basic properties of variable exponent Sobolev spaces.

After the initial interest in the MOS spaces, the research in the area remained dormant for almost a decade. Surprisingly, the research was reinvigorated by the interest of those spaces for their application in physics. One of the applications of MOS came from modeling electrorheological fluids - fluids whose viscosity changes in the presence of an electrical field. In 2000 M. Ruzicka [31] provided a model for mechanics of those fluids that...
employs the variable exponent Sobolev spaces. In 2002 L. Diening in his dissertation [9] expanded the theory of Ruzicka and provided, among other things, the sufficient condition on the regularity of the exponent $p(x)$ to guarantee the boundedness of Hardy-Littlewood maximal operator. Those results rekindled the interest in MOS spaces and motivated other authors to study partial differential equations in the context of MOS spaces [2].

The paper consists of six sections. In the introductory part we define several notions related to Banach function spaces, Musielak-Orlicz spaces $L^\Phi$, their norms and useful inequalities of Musielak-Orlicz functions (MO functions) $\Phi$. We also consider variable exponent and double phase MO functions showing necessary and sufficient conditions for those functions and their conjugates to satisfy the growth condition $\Delta_2$. At the end we define Musielak-Orlicz-Sobolev spaces (MOS spaces) $W^{1,\Phi}$ on a finite interval $(\alpha, \beta)$.

In the second section we study integral operators between Musielak-Orlicz spaces. In particular we obtain a characterization of the bounded Volterta operators on $L^\Phi$ under the assumption of so called (V) condition. It appears that (V) condition is always satisfied in Orlicz spaces, variable exponent Lebesgue spaces as well as in the spaces induced by double phase functionals.

The third section is devoted to characterization of the Sobolev spaces $W^{1,\Phi}$ containing an isomorphic subspace of $\ell^\infty$. Before that we complete the analogous results in MO spaces and in particular in variable exponent Lebesgue spaces. It appears that the lack of the growth condition $\Delta_2$ of $\Phi$ is a sufficient condition for $W^{1,\Phi}$ to contain of $\ell^\infty$ and is also necessary whenever condition (V) is satisfied.

In the fourth section we do analogous investigations concerning the existence of a subspace isomorphic to $\ell^1$ in $W^{1,\Phi}$. We obtain that if either $\Phi$ or $\Phi^*$ does not satisfy $\Delta_2$ then $W^{1,\Phi}$ contains such a subspace. The necessity of this occurs under condition (V). It follows complete characterizations of the containment of $\ell^1$ in Orlicz-Sobolev spaces $W^{1,\varphi}$, variable exponent Lebesgue spaces $L^{p(\cdot)}$ or variable exponent Sobolev spaces $W^{1,p(\cdot)}$.

The short fifth section states necessary and sufficient conditions on reflexivity of $W^{1,\Phi}$ followed by corresponding corollaries in $W^{1,p(\cdot)}$ and $W^{1,\varphi}$.

Section sixth is the main part of this paper. Here we characterize uniform convexity of $W^{1,\Phi}$. It is expressed in terms of $\Phi$ and its conjugate $\Phi^*$. Since $W^{1,\Phi}$ is an isometric subspace of the product $L^\Phi \times L^\Phi$, the conditions for uniform convexity for $L^\Phi$ are sufficient for uniform convexity of $W^{1,\Phi}$. The key result here is to show that those conditions, $\Delta_2$ of $\Phi$ and uniform convexity of $\Phi$, are also necessary. In this part we also give a complete characterization of uniform convexity of $L^{p(\cdot)}$ as well as $W^{1,p(\cdot)}$ and $W^{1,\varphi}$.

In the last section seven, on the basis of the previous results we show that under condition (V), reflexivity, superreflexivity and $B$-convexity in $W^{1,\Phi}$ are equivalent, and they hold whenever $\Phi$ and $\Phi^*$ satisfy $\Delta_2$. We obtain analogous results in Orlicz-Sobolev spaces $W^{1,\varphi}$ and variable exponent Sobolev spaces $W^{1,p(\cdot)}$.

All results contained in this paper are proved for MO space $L^\Phi$ or MOS space $W^{1,\Phi}$, where $\Phi$ is a general MO function occasionally with mild assumption. In the particular case where $\Phi$ is a variable exponent function, most results hold true without any additional assumptions, and have not been known before.

Let further $\mathbb{R}$ be the set of real numbers, $\mathbb{N}$ the set of natural numbers and $\mathbb{R}_+ = [0, \infty)$. Let $(\Omega, \Sigma, \mu)$ be a measurable space, where $\Sigma$ is a $\sigma$-algebra of subsets of $\Omega$ and $\mu$ is a $\sigma$-finite, complete measure on $\Sigma$. By $L^0 = L^0(\Omega)$ denote the set of all $\mu$-measurable complex valued functions on $\Omega$. Recall that $(X, \| \cdot \|_X)$ is a Banach function space if $X \subset L^0$, and if $f \in L^0$, $g \in X$ and $|f| \leq |g|$ a.e. then $f \in X$ and $\|f\|_X \leq \|g\|_X$. We say that a Banach function space $(X, \| \cdot \|_X)$ has the Fatou property, if for any $0 \leq f_n \uparrow f$ a.e., $f_n \in X$, $f \in L^0$ and $\sup_n \|f_n\|_X < \infty$ then $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$ as $n \to \infty$. An element $f \in X$ is called order continuous whenever for any $0 \leq f_n \leq f$ with $f_n \downarrow 0$ a.e., we have $\|f_n\|_X \downarrow 0$. [1]
Letting $X_a$ be the set of all order continuous elements from $X$, and $X_b$ be the closure of all simple functions from $X$, we have $X_a \subset X_b$. Let $X^*$ be the dual space to $X$. The Köthe dual space $X'$ of $X$ \cite{3, 32} is defined as follows

$$X' = \left\{ f \in L^0 : \|g\|_{X'} = \sup \left\{ \int_{\Omega} fg \, d\mu : \|f\|_X \leq 1 \right\} < \infty \right\}.$$  

The space $X'$ equipped with the norm $\| \cdot \|_{X'}$ is a Banach function space satisfying the Fatou property. If $X_a = X_b$ and $X$ has the Fatou property then $(X_a)^*$ is isometrically isomorphic to $X'$. In this case $X' \simeq X' \oplus X^*_s$, where the symbol $\simeq$ denotes linear isometry, and $X^*_s = X^*_a$ is the set of all singular functionals that is the set of $S \in X^*$ such that $S(f) = 0$ for every $f \in X_a$. For references on function spaces see \cite{3, 22, 26, 27, 32}.

A function $\varphi : [0, \infty) \to [0, \infty]$ is called an Orlicz function with extended values, if $\varphi$ is not identically 0, $\lim_{t \to 0^+} \varphi(t) = \varphi(0) = 0$, and $\varphi$ is left continuous and convex on $(0, b_\varphi]$, where $b_\varphi = \sup \{ t > 0 : \varphi(t) < \infty \}$. If for every $t \geq 0$, $\varphi(t) < \infty$, then $\varphi$ is called an Orlicz function \cite{21, 27}.

A function $\Phi : \Omega \times [0, \infty) \to [0, \infty]$ is called a Musielak-Orlicz function with extended values (eMO function for short) if for a.a. $x \in \Omega$, $\Phi(x, \cdot)$ is an Orlicz function with extended values and for all $t \geq 0$, $\Phi(\cdot, t)$ is measurable. If in addition $\Phi(x,t) < \infty$ for a.a. $x \in \Omega$, $t \geq 0$, then it is called a Musielak-Orlicz function (MO function for short). Given a Musielak-Orlicz function $\Phi$ with extended values, the Musielak-Orlicz space (MO space), called also generalized Orlicz space $L^\Phi = L^\Phi(\Omega)$, consists of all functions $f \in L^0$ such that

$$I_\Phi(\lambda f) = \int_{\Omega} \Phi(x, \lambda|f(x)|) \, d\mu(x) < \infty$$

for some $\lambda > 0$. The Luxemburg norm

$$\|f\|_\Phi = \inf \{ \lambda > 0 : I_\Phi(f/\lambda) \leq 1 \},$$

and the Orlicz norm

$$\|f\|_{\Phi}^0 = \sup_{I_{\Phi^*}(g) \leq 1} \int_{\Omega} f(x)g(x) \, d\mu(x) = \sup_{I_{\Phi^*}(g) \leq 1} \int_{\Omega} fg \, d\mu$$

are two equivalent standard norms considered in $L^\Phi$. In fact, $\|f\|_\Phi \leq \|f\|_{\Phi}^0 \leq 2\|f\|_\Phi$ for any $f \in L^\Phi$. In particular case when $\Phi$ does not depend on the parameter, that is $\Phi(x,t) = \varphi(t)$, for a.a. $x \in \Omega$, $t \geq 0$, where $\varphi$ is an Orlicz function, then $L^\Phi$ is an Orlicz space. The Musielak-Orlicz space $L^\Phi$ with either norm is a Banach function lattice satisfying the Fatou property. For any MO function $\Phi$, $(L^\Phi)_a = (L^\Phi)_b$. If we do not mention otherwise, we will always consider the MO space $L^\Phi$ equipped with the Luxemburg norm. Extensive information about Musielak-Orlicz spaces one can find in \cite{8, 10, 14, 16, 17, 21, 28}.

Given an eMO function $\Phi$, by $\Phi^*$ denote the complementary function to $\Phi$, that is

$$\Phi^*(x,t) = \sup_{s \geq 0} \{ st - \Phi(x,s) \}, \quad a.a. \ x \in \Omega, \ t \geq 0.$$  

The reason that we also consider eMO functions in this paper is that even if a MO function $\Phi$ has finite values, its complementary function $\Phi^*$ may achieve infinite values. The function $\Phi(x,t) = t$, $x \in \Omega$, $t \geq 0$, is the simplest example of such functions. It is well known and not difficult to show that $\Phi^*$ is eMO function that is a Musielak-Orlicz function with extended values and $\Phi^{**} = \Phi$.

In view of the simple observation that $I_\Phi(f) \leq 1$ if and only if $\|f\|_\Phi \leq 1$, the following Hölder inequalities are satisfied for any $f \in L^\Phi$, $g \in L^{\Phi^*}$,

$$\int_{\Omega} fg \, d\mu \leq \|f\|_\Phi^0 \|g\|_{\Phi^*}, \quad \int_{\Omega} fg \, d\mu \leq \|f\|_\Phi \|g\|_{\Phi^*}^0.$$
We say that MO function $\Phi$ satisfies condition $\Delta_2$ if there exist $K > 0$ and a non-negative integrable function $h$ on $\Omega$, that is $h \in L^1 = L^1(\Omega)$ such that

$$\Phi(x, 2t) \leq K\Phi(x, t) + h(x), \quad \text{a.a. } x \in \Omega, \quad t \geq 0.$$ 

Let $\Phi$ be an Orlicz function, that is $\Phi(x, t) = \varphi(t)$ for a.a. $x \in \Omega$. One can show that when $\mu(\Omega) < \infty$ and $\mu$ is non-atomic then $\Phi$ satisfies $\Delta_2$ if and only if for some $k > 0$ and $u_0 \geq 0$,

$$(\Delta_2^\infty) \quad \varphi(2u) \leq k\varphi(u) \quad \text{for all } u \geq u_0.$$ 

The growth condition $\Delta_2$ of $\Phi$ plays an important role in the theory of MO spaces. Recall now some results in MO spaces which we will need later.

**Theorem 1.1.** [28] Theorem 7.6, Theorem 8.14 | [3] Proposition 3.10, Theorem 3.13

Let $\Phi$ be a MO function. The following properties are equivalent.

(i) The space $L^\Phi$ is order continuous that is $L^\Phi = (L^\Phi)_o = (L^\Phi)_b$.

(ii) The modular convergence $I_\Phi(u) \to 0$ is equivalent to norm convergence $\|u\|_\Phi \to 0$.

(iii) $\Phi$ satisfies $\Delta_2$.

**Theorem 1.2.** [28] Theorem 7.10

The MO space $L^\Phi(\Omega)$ is separable if and only if the measure $\mu$ is separable and $\Phi$ satisfies $\Delta_2$.

**Theorem 1.3.** Let $\Phi$ be MO function.

(i) [21] Theorem A4 | [28] $(L^\Phi, \| \cdot \|_\Phi)' = (L^{\Phi^*}, \| \cdot \|_{\Phi^*})$ and $(L^\Phi, \| \cdot \|_0)^* = (L^{\Phi^*}, \| \cdot \|_{\Phi^*})$.

(ii) [32]

$$(L^\Phi, \| \cdot \|_\Phi)^* \simeq (L^{\Phi^*}, \| \cdot \|_{\Phi^*}) \oplus (L^\Phi)_s,$$

where $(L^\Phi)_s = ((L^\Phi)_o)^\perp$.

(iii) [28] $L^\Phi$ is reflexive if and only if both $\Phi$ and $\Phi^*$ satisfy condition $\Delta_2$.

Given MO functions $\Phi_i, i = 1, 2$, the symbol $\Phi_2 \prec \Phi_1$ denotes that there exist a constant $K > 0$ and a non-negative function $h \in L^1$ such that

$$\Phi_2(x, Kt) \leq \Phi_1(x, t) + h(t), \quad \text{a.a. } x \in \Omega, \quad t \geq 0.$$ 

We say that $\Phi_1$ and $\Phi_2$ are equivalent if $\Phi_2 \prec \Phi_1$ and $\Phi_1 \prec \Phi_2$. The equivalence of two MO functions preserves condition $\Delta_2$.

**Theorem 1.4.** [28] Given MO functions $\Phi_i, i = 1, 2$, $L^{\Phi_1} \subset L^{\Phi_2}$ if and only if $\Phi_2 \prec \Phi_1$. The embedding of the spaces $L^{\Phi_1} \subset L^{\Phi_2}$ is automatically bounded. Consequently $L^{\Phi_1} = L^{\Phi_2}$ as sets with equivalent norms if and only if $\Phi_1$ is equivalent to $\Phi_2$.

**Proof.** The proof of the first part can be found in [28]. The automatic boundedness of the embedding among MO spaces follows from [3] Proposition 2.10, p. 13], in view of the Fatou property of MO spaces.

For a $eMO$ function $\Phi : \Omega \times \mathbb{R}_+ \to [0, \infty]$ define the generalized inverse of $\Phi$ by the formula

$$\Phi^{-1}(x, t) = \inf\{u : \Phi(x, u) \geq t\}, \quad \text{a.a. } x \in \Omega, \quad t \geq 0.$$ 

Clearly we have $\Phi(x, \Phi^{-1}(x,t)) \leq t$ for a.a. $x \in \Omega$ and $t \geq 0$.

**Proposition 1.5.** Let $\Phi$ be $eMO$ function. Then

(i) The Young inequality

$$st \leq \Phi(x, s) + \Phi^*(x, t), \quad \text{a.a. } x \in \Omega \quad s, t \geq 0,$$
(ii) For a.a. $x \in \Omega$, $t \geq 0$,

$$t \leq (\Phi^*)^{-1}(x,t)\Phi^{-1}(x,t) \leq 2t.$$  

Proof. (i) It is a direct consequence of the definition of the complementary function.

(ii) This is well known for Orlicz–N-functions (e.g. [24]). For the sake of completeness we provide a short proof here. For the right inequality, take $x \in \Omega$ and $t \geq 0$, and let $u = \Phi^{-1}(x,t)$ and $v = (\Phi^*)^{-1}(x,t)$. Then

$$(\Phi^*)^{-1}(x,t)\Phi^{-1}(x,t) = uv \leq \Phi(x,u) + \Phi^*(x,v) \leq 2t$$

by the Young inequality. For the left inequality, recall that

$$\Phi(x,t) = \int_0^t \Phi'(x,s)ds, \quad \Phi^*(x,t) = \int_0^t (\Phi')^{-1}(x,s)ds, \quad \text{a.a. } x \in \Omega, \ t \geq 0,$$

where $\Phi'(x,t)$ is the right derivative of $\Phi(x,t)$ with respect to $t \geq 0$ for a.a. $x \in \Omega$, and $(\Phi')^{-1}$ is a generalized inverse as defined above for $\Phi$. For any $x \in \Omega$ and $u > 0$ such that $\Phi(x,u) = \int_0^u \Phi'(x,s)ds \geq t$ we have

$$\frac{t}{u} \leq \frac{\Phi(x,u)}{u} = \frac{1}{u} \int_0^u \Phi'(x,s)ds \leq \frac{\Phi'(x,u)u}{u} = \Phi'(x,u).$$

It follows in view of $(\Phi')^{-1}(x,\Phi'(x,u)) \leq u$,

$$\Phi^*\left(x, -\frac{t}{u}\right) = \int_0^t (\Phi')^{-1}(x,s)ds \leq (\Phi')^{-1}\left(x, -\frac{t}{u}\right) \frac{t}{u} \leq (\Phi')^{-1}(x,\Phi'(x,u))\frac{t}{u} \leq t.$$

Applying now $(\Phi^*)^{-1}$ to both sides of the above inequality, for any $u, t \geq 0$ with $\Phi(x,u) \geq t$, we get

$$u(\Phi^*)^{-1}(x,t) \geq t.$$

Hence for a.a. $x \in \Omega$, $t \geq 0$,

$$\Phi^{-1}(x,t)(\Phi^*)^{-1}(x,t) = \inf\{u : \Phi(x,u) \geq t\}(\Phi^*)^{-1}(x,t) \geq t,$$

which is the left inequality of (1.1).

Lemma 1.6. Let $\Phi'(x,t)$ be the right derivative of a MO function $\Phi(x,t)$ for a.a. $x \in \Omega$ with respect to $t \geq 0$. If exists a constant $k > 1$ such that

$$\Phi'(x,2t) \geq k\Phi'(x,t), \quad \text{a.a. } x \in \Omega, \ t \geq 0,$$

then $\Phi^*$ satisfies condition $\Delta_2$.

Proof. Integrating the inequality (1.2), we get for $s \geq 0$,

$$\int_0^s \Phi'(x,2t)dt = \frac{1}{2} \int_0^{2s} \Phi'(x,u)du = \frac{1}{2}\Phi(x,2s) \geq k\int_0^s \Phi'(x,t)dt = k\Phi(x,s).$$

Hence $\Phi(x,2s) \geq 2k\Phi(x,s)$. It follows $\Phi(x,\frac{s}{2}) \leq \frac{1}{2k}\Phi(x,s)$, and thus

$$\Phi^*(x,2s) = \sup_{u \geq 0} \{us - \Phi\left(x, \frac{u}{2}\right)\} \geq \sup_{u \geq 0} \{us - \frac{1}{2k}\Phi(x,u)\} \geq \frac{1}{2k}\Phi^*(x,2ks).$$
Hence for a.a. \( x \in \Omega, s \geq 0, \)
\[
\Phi^*(x, ks) \leq 2k\Phi^*(x, s),
\]
which implies \( \Delta_2 \) condition of \( \Phi^* \) by the assumption \( k > 1. \)

The important class of Musielak-Orlicz spaces are Nakano spaces, recently also called variable exponent Lebesgue spaces. Let \( 1 \leq p(x) < \infty \) for a.a. \( x \in \Omega, \) be a measurable function and let

\[
(1.3) \quad \Phi(x,t) = \frac{t^{p(x)}}{p(x)}, \quad \text{a.a. } x \in \Omega, \ t \geq 0.
\]

Then \( L^{p(\cdot)} = L^\Phi \) is a variable exponent Lebesgue space. For \( 1 < p(x) < \infty \) a.e. in \( \Omega, \ t \geq 0, \)
\[
\Phi^*(x,t) = \frac{t^{q(x)}}{q(x)},
\]
where \( \frac{1}{p(x)} + \frac{1}{q(x)} = 1. \) Let further
\[
p^+ = \text{ess sup}_{x \in \Omega} p(x) \quad \text{and} \quad p^- = \text{ess inf}_{x \in \Omega} p(x).
\]

In the literature there is another version of the Nakano spaces given by the MO function \( t^{p(x)} \) for \( 1 \leq p(x) < \infty \) a.a. \( x \in \Omega, \ t \geq 0. \) The spaces defined in either way are equal as sets with equivalent norms. In order to avoid confusion we will consider here only the spaces given by formula (1.3).

**Theorem 1.7.** The variable exponent MO function \( \Phi(x,t) = \frac{t^{p(x)}}{p(x)}, \ 1 \leq p(x) < \infty, \) for a.a. \( x \in \Omega, \ t \geq 0, \) satisfies \( \Delta_2 \) if and only if \( p^+ < \infty. \) Its complement function \( \Phi^* \) satisfies \( \Delta_2 \) if and only if \( p^- > 1. \)

**Proof.** Assume that \( \Phi(x,t) = \frac{t^{p(x)}}{p(x)} \) satisfies \( \Delta_2. \) There exists \( C \geq 2 \) and an positive, integrable function \( h \) such that for a.a. \( x \in \Omega \) and \( t \geq 0 \) we have
\[
2^{p(x)} \frac{t^{p(x)}}{p(x)} = \Phi(x,2t) \leq C\Phi(x,t) + h(x) = C \frac{t^{p(x)}}{p(x)} + h(x).
\]
Hence,
\[
\left(2^{p(x)} - C\right) \frac{t^{p(x)}}{p(x)} \leq h(x).
\]
If \( p(x) = 1 \) for a.e. \( x \in \Omega, \) then clearly \( \Phi \) satisfies \( \Delta_2 \) condition. Now if \( p(x) \neq 1 \) for a.a. \( x \in \Omega, \) then for a.a. \( x \in \Omega \) it follows that \( \sup_{t>0} \frac{t^{p(x)}}{p(x)} = \infty. \) Therefore we conclude that
\[
2^{p(x)} \leq C, \quad \text{a.a. } x \in \Omega.
\]
It follows that for a.a. \( x \in \Omega \) we have \( p(x) \leq \log_2 C \) and so \( p^+ \leq \log_2 C < \infty. \) If on the other hand \( p^+ < \infty, \) then for a.a. \( x \in \Omega \) and any \( t \geq 0 \) we have
\[
\Phi(x,2t) = 2^{p(x)} \frac{t^{p(x)}}{p(x)} \leq 2^{p^+} \frac{t^{p(x)}}{p(x)} = 2^{p^+} \Phi(x,t),
\]
that is \( \Phi \) satisfies \( \Delta_2. \)

As for the second assertion, notice that \( \Phi^*(x,t) = \frac{t^{q(x)}}{q(x)} \), where \( q(x) = p(x)/(p(x) - 1). \) By the first part of the proof \( \Phi^*(x,t) \) satisfies \( \Delta_2 \) if and only if \( q^+ < \infty. \) By the fact that \( u/(u-1) \) is a decreasing function on \( (1, \infty) \) and \( \lim_{u \to 1^+} u/(u-1) = \infty, \) we have
\[
q(x) = \frac{p(x)}{p(x) - 1} \leq \frac{p^-}{p^- - 1}, \quad \text{a.a. } x \in \Omega.
\]
Therefore we conclude that $q^+ < \infty$ if and only if $p^- > 1$.

Another important class of MO functions is a class of double phase functionals consisting of $\Phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\Phi(x,t) = t^{p(x)} + a(x)t^{r(x)},$$

where $a,p,r$ are measurable functions on $\Omega$, $a(x) \geq 0$ a.e. and $1 \leq p(x) \leq r(x) < \infty$ a.e..

Denote by $r^+$ and $r^-$ analogously as $p^+$ and $p^-$ respectively.

**Theorem 1.8.** Let

$$\Phi(x,t) = t^{p(x)} + a(x)t^{r(x)},$$

where $a,p,r$ are measurable functions on $\Omega$, $a(x) \geq 0$ a.e. and $1 \leq p(x) \leq r(x) < \infty$ a.e.,

(i) If $r^+ < \infty$ then $\Phi$ satisfies $\Delta_2$.

(ii) If $p^- > 1$ then $\Phi^*$ satisfies condition $\Delta_2$.

(iii) If in addition we assume that $p(x) < r(x)$ for a.a. $x \in \Omega$, where $\Omega_1 = \text{supp} a$, then $\Phi$ satisfies condition $\Delta_2$ if and only if $p^+ < \infty$ and $r^+|_{\Omega_1} = \text{ess sup}_{x \in \Omega_1} r(x) < \infty$.

**Proof.** If $\mu(\Omega_1) = 0$ that is $a(x) = 0$ for a.a. $x \in \Omega$, then the space $L^\Phi$ is reduced to the Lebesgue variable space $L^{p(x)}$, and the conclusion follows from Theorem 1.7. Thus assume $\mu(\Omega_1) > 0$.

(i) If $p^+ < \infty$ and $r^+|_{\Omega_1} < \infty$ then

$$\Phi(x,2t) = 2^{p(x)}t^{p(x)} + 2^{r(x)}a(x)t^{r(x)} \leq 2^r(p^{p(x)} + a(x)t^{r(x)}) = 2^r\Phi(x,t),$$

where $r = \max(p^+,r^+|_{\Omega_1})$, and so $\Phi$ satisfies condition $\Delta_2$.

(ii) Let $p^- > 1$. Then for a.a. $x \in \Omega$, $t \geq 0$,

$$\Phi'(x,2t) = p(x)2^{p(x)-1}t^{p(x)-1} + a(x)r(x)2^{r(x)-1}t^{r(x)-1} \geq 2^{p^-+1}p(x)t^{p(x)-1} + a(x)r(x)2^{r^-+1}t^{r(x)-1} = 2^{p^-} \Phi'(x,t),$$

where $2^{p^-} > 1$. We conclude by Lemma 1.6.

(iii) Now let $\Phi$ satisfy $\Delta_2$ and $p^+ = \infty$ or $r^+|_{\Omega_1} = \infty$. Then by definition of $\Delta_2$, there exists $c > 0$ such that the function

$$h_c(x) = \sup_{t \geq 0} (\Phi(x,2t) - c\Phi(x,t))$$

belongs to $L^1$. We shall consider three cases.

Case 1. Let $p^+|_{\Omega \setminus \Omega_1} = \text{ess sup}_{x \in \Omega \setminus \Omega_1} p(x) = \infty$ if $\mu(\Omega \setminus \Omega_1) > 0$. Then $\Phi(x,t) = t^{p(x)}$ for a.a. $x \in \Omega \setminus \Omega_1$, and by Theorem 1.7 it is a contradiction because $\Phi$ cannot satisfy condition $\Delta_2$.

Case 2. Let $p^+|_{\Omega_1} = \infty$. Define

$$A_n = \{x \in \Omega_1 : p(x) > n\}, \quad n \in \mathbb{N}.$$ 

By assumptions for every $n \in \mathbb{N}$, $\mu(A_n) > 0$, and for a.a. $x \in A_n$,

$$r(x) > p(x) > n.$$ 

Choose $n \in \mathbb{N}$ such that $2^n > c$. Then for $x \in A_n$, $t \geq 0$,

$$\Phi(x,2t) - c\Phi(x,t) \geq 2^n t^{p(x)} + a(x)2^n t^{r(x)} - ct^{p(x)} - ca(x)t^{r(x)}$$

$$= (2^n - c)t^{p(x)} + a(x)t^{r(x)} = (2^n - c)\Phi(x,t).$$

Hence $h_c(x) = \infty$ for every $x \in A_n$ and $c < 2^n$. Therefore $h_c$ is not integrable over $\Omega$, and by monotonicity of $h_c$ with respect to $c$, $h_c \notin L^1$ for every $c > 0$. Consequently, $\Phi$ does
not satisfy $\Delta_2$.

Case 3. Let $p^+ < \infty$ and $r^+|_{\Omega_1} = \infty$. We can assume that $c > 2p^+$. Thus

$$2p^+(x) - c < 0, \quad \text{a.a. } x \in \Omega.$$ 

Let $x \in \Omega_1$. Then $a(x) > 0$ and $p(x) - r(x) < 0$ by assumptions. Therefore there exists $T_x > 2$ such that for all $t > T_x$,

$$t^{p(x) - r(x)} < a(x).$$

Hence for a fixed $t > T_x$,

$$\Phi(x, 2t) - c\Phi(x, t) = t^{r(x)}[(2p(x) - c)t^{p(x) - r(x)} + (2r(x) - c)a(x)]$$

$$\geq t^{r(x)}[(2p(x) - c)a(x) + (2r(x) - c)a(x)]$$

$$= t^{r(x)}a(x)(2p(x) + 2r(x) - 2c) > t^{r(x)}a(x)(2r(x) - 2c).$$

Let for $n \in \mathbb{N}$,

$$B_n = \{x \in \Omega_1 : r(x) > n\}.$$ 

Since $r^+|_{\Omega_1} = \infty$, there is $N$ such that for all $n > N$, $\mu(B_n) > 0$ and $2^n > 2c$. Therefore for $t \geq 0$, $x \in B_n$ and $n > N$,

$$t^{r(x)}a(x)(2r(x) - 2c) \geq t^{r(x)}a(x)(2^n - 2c).$$

Thus for any $x \in B_n$ and $t_x > T_x$,

$$\sup_{t \geq 0}\{\Phi(x, 2t) - c\Phi(x, t)\} \geq \sup_{t \geq 0}[t^{r(x)}a(x)(2r(x) - 2c)] \geq t_x^{r(x)}a(x)(2^n - 2c).$$

Since $t_x$ can be taken arbitrary big it follows, for $n > N$ and for a.a. $x \in B_n$,

$$h_c(t) \geq \sup_{t \geq 0}[t^{r(x)}a(x)(2r(x) - 2c)] = \infty.$$ 

Hence $h_c(x) = \infty$ for every $x \in B_n$. Therefore, if $c < 2^n$, then $h_c$ is not integrable, but $n$ can be chosen arbitrary large, so for every $c > 0$, $h_c \notin L^1$.

\[\square\]

In applications there are often used double phase functionals where the functions $p(x)$ and $r(x)$ are constants. Consequently in view of Theorem 1.8 we get the corollary.

**Corollary 1.9.** Let $\Phi$ be a double phase functional of the following form

$$\Phi(x, t) = t^{p(x)} + a(x)t^r, \quad \text{a.a. } x \in \Omega, \quad t \geq 0,$$

where $1 \leq p \leq r < \infty$ and $a(x) \geq 0$ for a.a. $x \in \Omega$. Then $\Phi$ satisfies $\Delta_2$, and if $p > 1$ then $\Phi^{*}$ fulfills $\Delta_2$.

Let $\Omega = (\alpha, \beta)$, $-\infty < \alpha \leq \beta < \infty$ with the Lebesgue measure $\mu$. By $L^1_{\text{loc}} = L^1_{\text{loc}}(\Omega)$ denote the set of all locally integrable functions on $\Omega$, that is all functions $f \in L^0$ such that $\int_K |f| \, d\mu < \infty$ for every compact set $K \subset \Omega$. By $C^\infty_\text{loc}(\Omega)$ we denote the space of all smooth complex valued functions on $\Omega$ with compact supports. By a smooth function we mean a function which has all derivatives. Recall, a function $f \in L^1_{\text{loc}}$ is weakly differentiable if there exists a function $f' \in L^1_{\text{loc}}$ such that for every $u \in C^\infty_\text{c}(\Omega)$ we have

$$\int_a^\beta f(x)u'(x)dx = -\int_a^\beta f'(x)u(x)dx.$$
Then the function $f'$ is called the \textit{weak derivative} of $f$. If $f : \Omega \to \mathbb{R}$ is absolutely continuous on every compact subinterval of $\Omega$, then $f$ is weakly differentiable and $f'$ coincides with the classical derivative of $f$ a.e. \cite[Theorem 7.16]{13}. 

Given a $MO$ function $\Phi$, the Musielak-Orlicz-Sobolev space ($MOS$ space) $W^{1,\Phi}(\Omega)$ consists of all $f \in L^{1}_{loc}(\Omega)$ such that their weak derivative $f'$ exists and

\begin{equation}
\|f\|_{1,\Phi} = \|f\|_{\Phi} + \|f'\|_{\Phi} < \infty.
\end{equation}

The space $W^{1,\Phi}$ is often called a generalized Sobolev space, or if we know the context just a Sobolev space. If $\Phi$ is an Orlicz function $\varphi$ then $W^{1,\varphi}$ is an Orlicz-Sobolev space. In the case of variable exponent $MO$ function, the variable exponent Sobolev space is denoted by $W^{1,\nu}(\cdot)$. The space $W^{1,\Phi}$ equipped with the norm $\| \cdot \|_{1,\Phi}$ is a complete space \cite[Theorem 6.1.4]{13}. In the case of $W^{1,\nu}(\cdot)$ see also \cite[Theorem 6.6]{8}, \cite[Theorem 8.1.6]{10}).

Anytime further we will use Sobolev spaces $W^{1,\Phi}$, they will always be defined on the finite interval $(\alpha, \beta)$ equipped with the Lebesgue measure.

\section{Integral Operators in $L^\Phi$}

The integral operators in Orlicz spaces have been studied among others in \cite{24}. Here we will study the integral operators in Musielak-Orlicz spaces. Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space. For $MO$ functions $\Phi_1, \Phi_2$ define the function

\begin{equation}
\psi((x,y),t) = \Phi_2(x, \Phi_1^*(y,t)), \quad \text{a.a. } (x,y) \in \Omega \times \Omega, \ t \geq 0.
\end{equation}

Note first that $\Phi_1^*$ can achieve infinite values and so it is $eMO$ function. We will adopt the convention that if $\Phi_1^*(y,t) = \infty$ then $\psi((x,y),t) = \Phi_2(x,\infty) = \infty$. Therefore $\phi : (\Omega \times \Omega) \times \mathbb{R}_+ \to [0, \infty]$ is a $eMO$. Denote by $L^\Phi$ the Musielak-Orlicz space as a subspace of $L^0(\Omega \times \Omega)$. By $\| \cdot \|_\Phi$ and $\| \cdot \|_\Phi^0$ we mean the Luxemburg and Orlicz norm on $L^\Phi$, respectively.

\begin{lemma}
Let $\Phi_i, i = 1, 2$, be $MO$ functions on $\Omega$, where $\mu(\Omega) < \infty$. Assume $\int_{\Omega} \Phi_2(x,b) d\mu(x) < \infty$ for some $b > 0$. Let $\psi : (\Omega \times \Omega) \times \mathbb{R}_+ \to [0, \infty]$ be such that

\begin{equation}
\psi((x,y),t) = \Phi_2(x, \Phi_1^*(y,t)), \quad \text{a.a. } (x,y) \in \Omega \times \Omega, \ t \geq 0,
\end{equation}

and let $\phi = \psi^*$. Then there exists $l > 0$ such that whenever $u \in L^{\Phi_1}, v \in L^{\Phi_2}$ and

\begin{equation}
w(x,y) = u(y)v(x), \quad \text{a.a. } x,y \in \Omega,
\end{equation}

then

\begin{equation}
\|w\|_\phi^0 \leq l \|u\|_{\Phi_1} \|v\|_{\Phi_2^*}.
\end{equation}

\end{lemma}

\begin{proof}
By Young’s inequality applied to $\Phi_1$ and $\Phi_1^*$ and the measure $\frac{|v(x)|}{\|v\|_{\Phi_2^*}} d\mu(x)$, for $b > 0$ from the assumptions we get

\begin{align*}
&\left| \int_{\Omega} \int_{\Omega} bw(x,y)g(x,y) d\mu(x) d\mu(y) \right| \leq \|u\|_{\Phi_1} \|v\|_{\Phi_2^*} b \int_{\Omega} \int_{\Omega} g(x,y) \frac{|u(y)||v(x)|}{\|u\|_{\Phi_1} \|v\|_{\Phi_2^*}} d\mu(x) d\mu(y) \\
&\leq \|u\|_{\Phi_1} \|v\|_{\Phi_2^*} b \int_{\Omega} \left[ \int_{\Omega} \Phi_1^*(y,|g(x,y)|) \frac{|v(x)|}{\|v\|_{\Phi_2^*}} d\mu(x) + \int_{\Omega} \Phi_1(y,\frac{|u(y)|}{\|u\|_{\Phi_1}}) \frac{|v(x)|}{\|v\|_{\Phi_2^*}} d\mu(x) \right] d\mu(y) \\
&= \|u\|_{\Phi_1} \|v\|_{\Phi_2^*} (A + B),
\end{align*}

where

\begin{align*}
A &= \int_{\Omega} \int_{\Omega} g(x,y) \frac{|u(y)||v(x)|}{\|u\|_{\Phi_1} \|v\|_{\Phi_2^*}} d\mu(x) d\mu(y) \\
B &= \int_{\Omega} \int_{\Omega} \Phi_1^*(y,|g(x,y)|) \frac{|v(x)|}{\|v\|_{\Phi_2^*}} d\mu(x) d\mu(y).
\end{align*}
Applying now to term $A$, Young’s inequality to $\Phi_2$ and $\Phi_2^*$ and the obvious fact $I_{\Phi_2^*}(v/\|v\|_{\Phi_2^*}) \leq 1$, we get

$$A = b \int_\Omega \left[ \int_\Omega \Phi_1(y, |g(x,y)|) \frac{|v(x)|}{\|v\|_{\Phi_2^*}} \, d\mu(y) \right] \, d\mu(x) \leq b \int_\Omega \left[ \int_\Omega \Phi_2(x, \Phi_1^*(y, |g(x,y)|)) \, d\mu(x) + \int_\Omega \Phi_2^* \left( x, \frac{|v(x)|}{\|v\|_{\Phi_2^*}} \right) \, d\mu(x) \right] \, d\mu(y) \leq b \int_\Omega \int_\Omega \psi((x,y), |g(x,y)|) \, d\mu(x) \, d\mu(y) + b\mu(\Omega).$$

By Fubini’s theorem and $I_{\Phi_1}(u/\|u\|_{\Phi_1}) \leq 1$, and Young’s inequality applied to $\Phi_2$,

$$B = \int_\Omega \left[ \int_\Omega \Phi_1 \left( y, \frac{|u(y)|}{\|u\|_{\Phi_1}} \right) \, d\mu(y) \right] \, \frac{|v(x)|}{\|v\|_{\Phi_2}} \, d\mu(x) \leq \int_\Omega b \int_\Omega \frac{|v(x)|}{\|v\|_{\Phi_2}} \, d\mu(x) \leq \int_\Omega b \left( \frac{|v(x)|}{\|v\|_{\Phi_2}} \right) \, d\mu(x) \leq 1 + \int_\Omega \Phi_2(x, b) \, d\mu(x).$$

By the above,

$$\left| \int_\Omega \int_\Omega bw(x,y)g(x,y) \, d\mu(x) \, d\mu(y) \right| \leq \|u\|_{\Phi_1} \|v\|_{\Phi_2^*} \left( A + B \right) \leq \|u\|_{\Phi_1} \|v\|_{\Phi_2^*} \left( b \int_\Omega \int_\Omega \psi((x,y), |g(x,y)|) \, d\mu(x) \, d\mu(y) + b\mu(\Omega) + 1 + \int_\Omega \Phi_2(x, b) \, d\mu(x) \right) = \|u\|_{\Phi_1} \|v\|_{\Phi_2^*} \left( bI_\phi(g) + b\mu(\Omega) + 1 + \int_\Omega \Phi_2(x, b) \, d\mu(x) \right).$$

Finally,

$$\|w\|_\phi^0 = \sup_{l_\phi(y) \leq 1} \left| \int_\Omega \int_\Omega w(x,y)g(x,y) \, d\mu(x) \, d\mu(y) \right| \leq l \|u\|_{\Phi_1} \|v\|_{\Phi_2^*},$$

where $l = b + b\mu(\Omega) + 1 + \int_\Omega \Phi_2(x, b) \, d\mu(x) < \infty$ by assumption.

\[\square\]

**Theorem 2.2.** Let $\Phi_i$, $i = 1,2$, be MO functions on $\Omega$. Assume $\phi : (\Omega \times \Omega) \times \mathbb{R}_+ \to [0, \infty]$ is a cMO function and there exists $l > 0$ such that if $u \in L^{\Phi_1}$, $v \in L^{\Phi_2}$ and $w(x,y) = u(y)v(x) \in L^0$ then

$$\|w\|_\phi^0 \leq l \|u\|_{\Phi_1} \|v\|_{\Phi_2^*}.$$ 

Let $A$ be an integral operator for $u \in L^0$,

$$Au(x) = \int_\Omega k(x, y)u(y) \, d\mu(y), \quad a.a. \ x \in \Omega,$$

where the kernel $k(x,y) \in L^{\phi^*}$. Then $A : L^{\Phi_1} \to L^{\Phi_2}$ is bounded.

**Proof.** By Hölder’s inequality, for $u \in L^{\Phi_1}$, $v \in L^{\Phi_2}$,

$$\left| \int_\Omega Au(x)v(x) \, d\mu(x) \right| = \left| \int_\Omega \int_\Omega k(x,y)u(y)v(x) \, d\mu(x) \, d\mu(y) \right| \leq \|k\|_{\phi^*} \|w\|_\phi^0 \leq l \|k\|_{\phi^*} \|u\|_{\Phi_1} \|v\|_{\Phi_2^*}.$$
Recall that $I_{\Phi_2}(v) \leq 1$ if and only if $\|v\|_{\Phi_2} \leq 1$. Consequently,

$$\|Au\|_{\Phi_2}^0 = \sup_{I_{\Phi_2}(v) \leq 1} \left| \int_\Omega Au(x)v(x) \, d\mu(x) \right| \leq l\|k\|_{\phi^*}\|u\|_{\Phi_1},$$

and thus $A$ is bounded from $L^{\Phi_1}$ to $L^{\Phi_2}$.

\[\square\]

**Corollary 2.3.** Let $\Phi_i$, $i = 1, 2$, be MO functions on $\Omega$ and $\mu(\Omega) < \infty$. Assume there exists $b > 0$ such that $\int_\Omega \Phi_2(x, b) \, d\mu(x) < \infty$. Let $\psi : (\Omega \times \Omega) \times \mathbb{R}_+ \to [0, \infty]$ be such that

$$\psi((x, y), t) = \Phi_2(x, t), \quad a.a. \quad x, y \in \Omega, \quad t \geq 0.$$ 

If $k(x, y) \in L^\psi$, then the operator $Au(x) = \int_\Omega k(x, y)u(y) \, d\mu(y)$ is bounded from $L^{\Phi_1}$ to $L^{\Phi_2}$.

**Proof.** By Lemma 2.1 there is $l > 0$ such that for any $u \in L^{\Phi_1}$, $v \in L^{\Phi_2}$, and $w(x, y) = u(y)v(x)$ we have $\|w\|_{\phi^*}^0 \leq l\|u\|_{\Phi_1}\|v\|_{\Phi_2}$, where $\phi = \psi^*$. Consequently, the assumptions of Theorem 2.2 are satisfied and $A$ is bounded.

\[\square\]

The next result follows immediately from Corollary 2.3.

**Corollary 2.4.** Let $\Phi$ be a MO function on $\Omega$ with $\mu(\Omega) < \infty$ and such that $\int_\Omega \Phi(x, b) \, d\mu(x) < \infty$ for some $b > 0$. Let

$$\psi((x, y), t) = \Phi(x, t), \quad x, y \in \Omega, \quad t \geq 0.$$ 

If $k(x, y) \in L^\psi$ then $Au(x) = \int_\Omega k(x, y)u(y) \, d\mu(y)$ is bounded from $L^\Phi$ to $L^\Phi$.

Now we wish to formulate a condition for the Volterra operator to be bounded from $L^\Phi(\alpha, \beta)$ to itself for $-\infty < \alpha < \beta < \infty$. It requires some preparations.

**Lemma 2.5.** Given a MO function $\Phi$ on $\Omega$, we have that $\text{ess inf}_{x \in \Omega} \Phi(x, a) > 0$ for some $a > 0$ if and only if there exists $c > 0$ with $\text{ess sup}_{x \in \Omega} \Phi^*(x, c) < \infty$.

**Proof.** If $\text{ess inf}_{x \in \Omega} \Phi(x, a) > 0$ for some $a > 0$, then $\Phi(x, a) \geq M > 0$ for a.a. $x \in \Omega$ and some $M$. Hence $a \geq \Phi^{-1}(x, M)$. Thus in view of (1.11), $(\Phi^*)^{-1}(x, M) \geq \Phi^{-1}(x, M) \geq \frac{M}{\alpha}$. Therefore $\alpha > M \geq \text{ess sup}_{x \in \Omega} \Phi^*(x, c)$ with $c = \frac{M}{\alpha}$. In the opposite direction the proof goes in a similar way.

\[\square\]

**Definition 2.6.** Let $\Omega = (\alpha, \beta)$, $-\infty < \alpha < \beta < \infty$, equipped with the Lebesgue measure. We say that MO function $\Phi$ satisfies condition (V) if for some $a, b > 0$,

(V) \quad \int_\alpha^\beta \Phi(x, b) \, dx < \infty \quad \text{and} \quad \text{ess inf}_{x \in \Omega} \Phi(x, a) > 0.

**Theorem 2.7.** Let $\Omega = (\alpha, \beta)$, $-\infty < \alpha < \beta < \infty$, equipped with the Lebesgue measure. Assume $\Phi$ is a MO function on $\Omega$ satisfying the condition (V). Then the Volterra operator

$$Au(x) = \int_\alpha^x u(y) \, dy, \quad x \in (\alpha, \beta),$$

is bounded on $L^\Phi$. 

Proof. By Lemma 2.5 we get \( \text{ess sup}_{x \in \Omega} \Phi^*(x, c) < \infty \) for some \( c > 0 \). Now by convexity of \( \Phi^* \), for all \( n \in \mathbb{N} \), \( \text{ess sup}_{x \in \Omega} \Phi^*(x, \frac{c}{n}) \leq \frac{1}{n} \text{ess sup}_{x \in \Omega} \Phi^*(x, c) \). Thus for for \( b \) from the assumptions and some \( c_1 > 0 \),

\[
\text{ess sup}_{x \in \Omega} \Phi^*(x, c_1) \leq b.
\]

Setting \( k(x, y) = \chi_{(\alpha,x)}(y) \), \( x, y \in (\alpha, \beta) \), we have \( Au(x) = \int_{\Omega} k(x, y)u(y) dy, u \in L^0 \). Let \( \psi \) be as in Corollary 2.11. Then

\[
I_\psi(c_1 k) = \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \psi((x, y), c_1 \chi_{(\alpha,x)}(y)) \, dx \, dy = \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \Phi(x, \Phi^*(y, c_1 \chi_{(\alpha,x)}(y))) \, dx \, dy
\]

\[
\leq \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \Phi(x, b) \, dx \, dy = (\beta - \alpha) \int_{\alpha}^{\beta} \Phi(x, b) \, dx < \infty.
\]

Hence the kernel \( k \in L^b \). Finally by Corollary 2.14 the Volterra operator is bounded on \( L^b \).

Since an Orlicz function satisfies condition (V), the next result is instant.

**Corollary 2.8.** Let \( \varphi \) be an Orlicz function and \( \Omega = (\alpha, \beta) \), where \( -\infty < \alpha < \beta < \infty \). Then the Volterra operator is bounded on Orlicz space \( L^\varphi \).

**Corollary 2.9.** Let \( 1 \leq p(x) < \infty \) a.e. on \( (\alpha, \beta) \), where \( -\infty < \alpha < \beta < \infty \). Then the Volterra operator is bounded on \( L^{p(x)} \).

**Proof.** Since for variable exponent MO function, \( \Phi(x, 1) = 1/p(x) \leq 1 \) and \( \Phi^*(x, 1) = 1/q(x) \leq 1 \) for a.a. \( x \in \Omega \), so in view of Lemma 2.5 the conditions in Theorem 2.7 are satisfied and the conclusion holds.

**Corollary 2.10.** Let \( \Phi \) be a double face functional, that is

\[
\Phi(x, t) = l^p(x) + a(x)r^p(x),
\]

where \( a, p, r \) are real measurable functions on \( (\alpha, \beta) \), \( -\infty < \alpha < \beta < \infty \), such that \( a(x) \geq 0, 1 \leq p(x) < \infty, 1 \leq r(x) < \infty \) a.e. on \( (\alpha, \beta) \). If \( a \in L^1 \) then the Volterra operator is bounded on \( L^\Phi \).

**Proof.** We have

\[
\int_{\alpha}^{\beta} \Phi(x, 1) \, dx = (\beta - \alpha) + \int_{\alpha}^{\beta} a(x) \, dx < \infty,
\]

and clearly \( \text{ess inf}_{x \in \Omega} \Phi(x, 1) \geq 1 \). In view of Theorem 2.7 we conclude the proof.

3. Copy of \( \ell^\infty \)

In this part we give conditions on \( \Phi \) in order to \( W^{1, \Phi} \) contain a subspace isomorphic to \( \ell^\infty \). We start with Musielak-Orlicz space \( L^\Phi \).

**Proposition 3.1.** [19] [20] Let \( (\Omega, \Sigma, \mu) \) be a non-atomic measure space. Then a eMO function \( \Phi \) does not satisfy condition \( \Delta_2 \) if and only if there exists a sequence of bounded and non-negative functions \( f_n \in L^\Phi \) such that \( f_n \wedge f_m = 0 \) for \( n \neq m \), \( I_\Phi(f_n) \leq 1/2^n \) and \( \| f_n \|_\Phi = 1 \) for all \( n \in \mathbb{N} \). Consequently,

\[
\left\| \sum_{n=1}^{\infty} f_n \right\|_\Phi = \| f_n \|_\Phi = 1
\]
for all \(n \in \mathbb{N}\).

The next theorem results directly from Proposition 3.1.

**Theorem 3.2.** Let the measure space \((\Omega, \Sigma, \mu)\) be separable and non-atomic. A Musielak-Orlicz space \(L^\Phi\) contains an isomorphic copy of \(\ell^\infty\) if and only if \(\Phi\) does not satisfy condition \(\Delta_2\).

**Proof.** If \(\Phi \notin \Delta_2\), then taking any element \(a = \{a_n\}_{n=1}^\infty \in \ell^\infty\) and the sequence \(\{f_n\}_{n=1}^\infty\) from Proposition 3.1, we get for every \(n \in \mathbb{N}\),

\[
|a_n| = \|a_n f_n\|_\Phi \leq \left\| \sum_{n=1}^\infty a_n f_n \right\|_\Phi \leq \|a\|_\infty.
\]

It follows

\[
\left\| \sum_{n=1}^\infty a_n f_n \right\|_\Phi = \|a\|_\infty,
\]

and in fact \(\ell^\infty\) is an isometric isomorphic subspace of \(L^\Phi\).

Now assume opposite that \(L^\Phi\) has a subspace isomorphic to \(\ell^\infty\). Then \(L^\Phi\) can not be separable, and so \(\Phi \notin \Delta_2\) by Theorem 1.2.

\[\square\]

**Corollary 3.3.** Let the measure space \((\Omega, \Sigma, \mu)\) be separable and non-atomic. The variable exponent Lebesgue space \(L^p(\cdot)\) contains an isomorphic subspace to \(\ell^\infty\) if and only if \(p^+ = \infty\).

**Proof.** By Theorem 1.7 the variable exponential function satisfies \(\Delta_2\) if and only if \(p^+ < \infty\). Therefore the proof is completed by application of Theorem 3.2.

\[\square\]

**Theorem 3.4.** Let \(\Phi\) be a MO function on \((\alpha, \beta)\), \(\infty < \alpha < \beta < \infty\) with the Lebesgue measure. If \(W^{1,\Phi}\) contains a subspace isomorphic to \(\ell^\infty\) then \(\Phi\) does not satisfy condition \(\Delta_2\).

**Proof.** Let \(A = \{(f, f') : f \in W^{1,\Phi}\}\) be a subspace of the product \(L^\Phi \times L^\Phi\). If \(L^\Phi \times L^\Phi\) is equipped with the norm \(\| (f, f') \|_{L^\Phi \times L^\Phi} = \| f \|_\Phi + \| f' \|_\Phi\), then the space \(W^{1,\Phi}\) is isometrically isomorphic to \(A\). Notice that \(L^\Phi \times L^\Phi\) is isomorphic to the MO space \(\overline{L^\Phi}(\Omega \times \{1, 2\})\) where \(\overline{\Phi} : \Omega \times \{1, 2\} \times [0, \infty) \to [0, \infty)\) is defined as

\[
\overline{\Phi}(x, y, t) = \Phi(x, t)\chi_{\{1\}}(y) + \Phi(x, t)\chi_{\{2\}}(y),
\]

for a.a. \(x \in \Omega, y \in \{1, 2\}\) and \(t \geq 0\). Indeed, the operator \(T : L^\Phi \times L^\Phi \to \overline{L^\Phi}(\Omega \times \{1, 2\})\) defined by

\[
(T(f_1, f_2))(x, y) = f_1(x)\chi_{\{1\}}(y) + f_2(x)\chi_{\{2\}}(y), \quad f_1, f_2 \in L^\Phi,
\]

for a.a. \(x \in \Omega, y \in \{1, 2\}\), is clearly a linear bijection and its inverse is given by

\[
(T^{-1}f)(x) = (f(x, 1), f(x, 2)), \quad f \in \overline{L^\Phi}(\Omega \times \{1, 2\})\).
\]

Moreover, taking any \((f_1, f_2) \neq (0, 0)\) where \((f_1, f_2) \in L^\Phi \times L^\Phi\),

\[
\int_{\Omega} \overline{\Phi} \left( \frac{T(f_1, f_2)}{2\| (f_1, f_2) \|_{L^\Phi \times L^\Phi}} \right) dx + \int_{\Omega} \overline{\Phi} \left( x, \frac{|f_2(x)|}{2\|f_1\|_\Phi + |f_2|_\Phi} \right) dx \leq 1.
\]
Therefore, for every $(f_1, f_2) \in L^\Phi \times L^\Phi$ we have
\[ \|T(f_1, f_2)\|_{L^\Phi} \leq 2\|f_1, f_2\|_{L^\Phi \times L^\Phi}. \]

On the other hand, if $0 \neq f \in L^\Phi(\Omega \times \{1, 2\})$, then for $j = 1, 2$,
\[ I_\Phi \left( \frac{f(\cdot, j)}{\|f\|_{\bar{L}^\Phi}} \right) = \int_\Omega \Phi \left( x, \frac{|f(x, j)|}{\|f\|_{\bar{L}^\Phi}} \right) \, dx \]
\[ \leq \int_\Omega \Phi \left( x, \frac{|f(x, 1)|}{\|f\|_{\bar{L}^\Phi}} \right) \, dx + \int_\Omega \Phi \left( x, \frac{|f(x, 2)|}{\|f\|_{\bar{L}^\Phi}} \right) \, dx = I_{\bar{L}^\Phi} \left( \frac{f}{\|f\|_{\bar{L}^\Phi}} \right) \leq 1. \]

Hence $\|f(\cdot, j)\|_\Phi \leq \|f\|_{\bar{L}^\Phi}$ for $j = 1, 2$. Consequently, for every $f \in L^\Phi(\Omega \times \{1, 2\})$ we have
\[ \|T^{-1}f\|_{L^\Phi \times L^\Phi} = \|f(\cdot, 1)\|_\Phi + \|f(\cdot, 2)\|_\Phi \leq 2\|f\|_{\bar{L}^\Phi}. \]

From this we conclude that indeed $L^\Phi \times L^\Phi$ is isomorphic to $L^\Phi(\Omega \times \{1, 2\})$.

If $W^{1, \Phi}$ contains $\ell^\infty$ isomorphically, then $L^\Phi \times L^\Phi$ does it too and so $L^\Phi$ must contain $\ell^\infty$, which implies that $\bar{L}^\Phi$ does not satisfy condition $\Delta_2$ by Theorem 3.2. Now we argue by contradiction. If $\Phi$ satisfies $\Delta_2$, then there exist a constant $C > 0$ and a non-negative function $h \in L^1(\Omega)$ such that for any $t \geq 0$ and a.a. $x \in \Omega,
\[ \Phi(x, 2t) \leq C\Phi(x, t) + h(x). \]

Hence for any $t \geq 0$ and a.a. $(x, y) \in \Omega \times \{1, 2\}$ we have
\[ \Phi(x, y, 2t) = \Phi(x, 2t)\chi_{\{1\}}(y) + \Phi(x, 2t)\chi_{\{2\}}(y) \leq (C\Phi(x, t) + h(x))\chi_{\{1\}}(y) + (C\Phi(x, t) + h(x))\chi_{\{2\}}(y) = C\Phi(x, y, t) + h(x)(\chi_{\{1\}}(y) + \chi_{\{2\}}(y)). \]

The function $H$ given by the formula $H(x, y) = h(x)(\chi_{\{1\}}(y) + \chi_{\{2\}}(y))$ is a non-negative element of $L^1(\Omega \times \{1, 2\})$. Therefore $\bar{L}^\Phi$ satisfies the $\Delta_2$ condition, a contradiction.

\[ \square \]

**Theorem 3.5.** Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$ and $\Phi$ be a MO function on $\Omega$ satisfying condition (V). If $\Phi$ does not satisfy condition $\Delta_2$ then the Sobolev space $W^{1, \Phi}$ contains a subspace isomorphic to $\ell^\infty$.

**Proof.** Let $\{f_k\} \subset L^\Phi$ satisfy the hypothesis of Proposition 3.1. Since they are bounded on $\Omega$, so $f_k \in L^1$. Define
\[ g_k(x) = \int_{\alpha}^{x} f_k(y) \, dy, \quad x \in (\alpha, \beta), \ k \in \mathbb{N}. \]
We have that $g_k \in W^{1, \Phi}$. Indeed $g'_k = f_k \in L^\Phi$, and by the assumption (V), the Volterra operator is bounded on $L^\Phi$, and so $\|g_k\|_\Phi \leq l\|f_k\|_\Phi < \infty$ for some constant $l$. Moreover for every $k \in \mathbb{N},$
\[ 1 = \|f_k\|_\Phi \leq \|g_k\|_{L^1} = \|f_k\|_\Phi + \|g_k\|_\Phi \leq (1 + l)\|f_k\|_\Phi = 1 + l. \]

Analogously, for every $m \in \mathbb{N},$
\[ 1 \leq \left\| \sum_{k=1}^{m} g_k \right\|_{1, \Phi} \leq \left\| \left( \sum_{k=1}^{m} g_k \right)' \right\|_{\Phi} + \left\| \sum_{k=1}^{m} g_k \right\|_{\Phi} = \left\| \sum_{k=1}^{m} f_k \right\|_{\Phi} + \left\| \sum_{k=1}^{m} g_k \right\|_{\Phi} \leq (1 + l)\left\| \sum_{k=1}^{m} f_k \right\|_{\Phi} \leq 1 + l. \]
Hence $\sum_{k=1}^{\infty} g_k \in W^{1,\Phi}$. Notice that $g_k \geq 0$ and $f_k \geq 0$. Therefore in view of (3.1), for every element $a = (a_k) \in \ell^\infty$, $m \in \mathbb{N}$,

\begin{equation}
\left\| \sum_{k=1}^{m} a_k g_k \right\|_{1,\Phi} = \left\| \sum_{k=1}^{m} a_k g_k' \right\|_{\Phi} + \left\| \sum_{k=1}^{m} a_k g_k \right\|_{\Phi} \leq \left\| \sum_{k=1}^{m} |a_k| g_k' \right\|_{\Phi} + \left\| \sum_{k=1}^{m} |a_k| g_k \right\|_{\Phi} \\
= \left\| \sum_{k=1}^{m} |a_k| f_k \right\|_{\Phi} + \left\| \sum_{k=1}^{m} g_k |a_k| \right\|_{\Phi} \leq \|a\|_\infty \left( \left\| \sum_{k=1}^{m} f_k \right\|_{\Phi} + \left\| \sum_{k=1}^{m} g_k \right\|_{\Phi} \right) \leq (1 + l)\|a\|_\infty \left\| \sum_{k=1}^{m} f_k \right\|_{\Phi} \leq (1 + l)\|a\|_\infty.
\end{equation}

On the other hand for every $m, k \in \mathbb{N}$,

\begin{equation}
\left\| \sum_{k=1}^{m} a_k g_k \right\|_{1,\Phi} \geq \left\| \sum_{k=1}^{m} a_k f_k \right\|_{\Phi} = \left\| \sum_{k=1}^{m} |a_k| f_k \right\|_{\Phi} \geq \|a\|_\infty \left\| \sum_{k=1}^{m} f_k \right\|_{\Phi} = \|a\|_\infty |a_k|.
\end{equation}

Hence for every $m \in \mathbb{N}$,

\begin{equation}
\left\| \sum_{k=1}^{m} a_k g_k \right\|_{1,\Phi} \geq \|a\|_\infty.
\end{equation}

Combining (3.2) and (3.3) we have that $\ell^\infty$ is an isomorphic copy in $W^{1,\Phi}$. 

Since (V) is always satisfied in Orlicz space over $(\alpha, \beta)$, $-\infty < \alpha < \beta < \infty$, we get instantly the following result.

**Corollary 3.6.** Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. The Orlicz-Sobolev space $W^{1,\Phi}$ contains an isomorphic subspace to $\ell^\infty$ if and only if $\Phi$ does not satisfy condition $\Delta_2^\infty$.

**Corollary 3.7.** Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. The variable exponent Sobolev space $W^{1,p(\cdot)}$ contains an isomorphic subspace to $\ell^\infty$ if and only if $p^+ = \infty$.

**Proof.** By Corollary 2.9 the variable exponent function satisfies condition (V). Thus the proof is an immediate consequence of Theorems 3.4 and 3.5. 

The next result follows from Proposition 1.8.

**Corollary 3.8.** Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. Let the Sobolev space $W^{1,\Phi}$ be induced by the double phase functional

$$\Phi(x, t) = t^{p(x)} + a(x) t^{r(x)}$$

with $a, p, r$ measurable functions on $\Omega$, such that $a(x) > 0$ and $p(x) < r(x)$ for a.a. $x \in \Omega$. Then $W^{1,\Phi}$ contains an isomorphic subspace to $\ell^\infty$ if and only if $p^+ = \infty$ or $r^+ = \infty$.

4. Copy of $\ell^1$

In this section we will characterize the spaces $W^{1,\Phi}$ that contain an isomorphic copy of $\ell^1$. First we need to recall an analogous result for MO spaces $L^\Phi$.

**Theorem 4.1.** Let $(\Omega, \Sigma, \mu)$ be a non-atomic and separable measure space. A MO space $L^\Phi$ contains an isomorphic copy of $\ell^1$ if and only if $\Phi$ or $\Phi^*$ do not satisfy condition $\Delta_2$. 

Proof. If $\Phi$ does not satisfy condition $\Delta_2$ then $L^\Phi$ contains isomorphically $\ell^\infty$ by Theorem 3.2. and so $\ell^1$ is an isomorphic subspace of $L^\Phi$ [4, Corollary 6.8]. If $\Phi^*$ does not satisfy $\Delta_2$ then $L^{\Phi^*}$ contains an isomorphic subspace of $\ell^\infty$ again by Theorem 3.2. Thus applying Theorem 3.3, the dual space $(L^\Phi)^* \simeq L^{\Phi^*} \oplus (L^{\Phi^*})_*$ must also contain an isomorphic copy of $\ell^\infty$. Finally by general result in Banach spaces [25, Proposition 2.e.8], the space $L^\Phi$ must contain a subspace isomorphic to $\ell^1$.

If $L^\Phi$ contains a subspace isomorphic to $\ell^1$, then it can not be reflexive, and so by Theorem 1.3, $\Phi$ or $\Phi^*$ does not satisfy $\Delta_2$.

\[ \square \]

**Corollary 4.2.** Let $(\Omega, \Sigma, \mu)$ be a non-atomic and separable measure space. The variable exponent Lebesgue space $L^{p(\cdot)}$ contains an isomorphic subspace to $\ell^1$ if and only if $p^+ = \infty$ or $p^- = 1$.

**Proof.** The proof follows from Theorems 1.7 and 1.1

\[ \square \]

**Theorem 4.3.** Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. If a MO function $\Phi$ and its conjugate $\Phi^*$ satisfy condition $\Delta_2$ then $W^{1,\Phi}$ does not contain an isomorphic subspace of $\ell^1$.

**Proof.** If both $\Phi$ and $\Phi^*$ satisfy condition $\Delta_2$ then the space $L^\Phi$ is reflexive by Theorem 1.3. Hence $L^\Phi \times L^\Phi$ equipped with norm $\| \cdot \|_{L^\Phi \times L^\Phi} = \| \cdot \|_\Phi + \| \cdot \|_\Phi$ is reflexive and so $W^{1,\Phi}$ as its closed subspace is reflexive too. Therefore it can not contain a subspace $\ell^1$.

\[ \square \]

**Theorem 4.4.** Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. Let $\Phi$ be MO function satisfying condition (V). If $W^{1,\Phi}$ does not contain a subspace isomorphic to $\ell^1$ then both $\Phi$ and $\Phi^*$ satisfy $\Delta_2$.

**Proof.** Let first $\Phi$ do not satisfy $\Delta_2$. By Theorem 3.5 $\ell^\infty$ is an isomorphic copy in $W^{1,\Phi}$. Thus $\ell^1$ is contained isomorphically in $\ell^\infty$ by [1, Corollary 6.8] and so in $W^{1,\Phi}$.

Assume now that $\Phi^*$ does not satisfy $\Delta_2$. Then in view of Proposition 3.1 there exists a sequence of bounded and non-negative functions $\{f_k\} \subset L^{\Phi^*}$ such that $f_k \land f_i = 0$, $k \neq i$, and

$$
(4.1) \quad \left\| \sum_{k=1}^{\infty} f_k \right\|_{\Phi^*} = \|f_k\|_{\Phi^*} = 1, \quad k \in \mathbb{N}.
$$

Hence

$$
(4.2) \quad 1 = \|f_k\|_{\Phi^*} \leq \|f_k\|_{\Phi^*}^\epsilon \leq 2\|f_k\|_{\Phi^*} = 2, \quad k \in \mathbb{N}.
$$

By the definition of the Orlicz norm $\| \cdot \|_{\Phi^*}$ for $\epsilon > 0$, each $k \in \mathbb{N}$, there exists a non-negative, bounded function $g_k \in L^\Phi$ such that ess supp $g_k \subset \text{ess supp } f_k$, $I_\Phi(g_k) \leq 1$ and

$$
(4.3) \quad \|f_k\|_{\Phi^*}^\epsilon \leq \frac{\epsilon}{2k} + \int_{\Omega} f_k(x) g_k(x) \, dx.
$$

Define

$$
<h_k(x) = \int_{\alpha}^{x} g_k(y) \, dy, \quad x \in (\alpha, \beta), \quad k \in \mathbb{N}.
$$

Thus its derivative $h'_k = g_k \in L^\Phi$. Moreover, by the boundedness of the Voltera operator $\|h_k\|_{\Phi} \leq l \|g_k\|_{\Phi} \leq l$, which implies that $h_k$ also belongs to $L^\Phi$.

Clearly for every $k \in \mathbb{N}$,

$$
\|h_k\|_{1,\Phi} = \|g_k\|_{\Phi} + \|h_k\|_{\Phi} \leq 1 + l.
$$
Hence for all $a = (a_k) \in \ell^1$, $m \in \mathbb{N}$,
\[
\left\| \sum_{k=1}^m a_k h_k \right\|_{1,\Phi} \leq \sum_{k=1}^m |a_k| \|h_k\|_{1,\Phi} \leq (1+l)\|a\|_1.
\]

On the other hand,
\[
\left\| \sum_{k=1}^m a_k h_k \right\|_{1,\Phi} = \left\| \sum_{k=1}^m a_k h_k \right\|_{\Phi} + \left\| \sum_{k=1}^m a_k h'_k \right\|_{\Phi} \geq \left\| \sum_{k=1}^m a_k g_k \right\|_{\Phi},
\]
and by H"older's inequality and by (4.1), (4.2),
\[
\frac{1}{2} \int_{\Omega} \left( \sum_{k=1}^m a_k g_k(x) \right) \left( \sum_{k=1}^m (\text{sign } a_k) f_k(x) \right) dx \leq \frac{1}{2} \sum_{k=1}^m |a_k| g_k(x) f_k(x) dx = \frac{1}{2} \int_{\Omega} \sum_{k=1}^m |a_k| f_k(x) g_k(x) dx = \frac{1}{2} \sum_{k=1}^m |a_k| f_k(x) g_k(x) dx
\]
\[
\frac{1}{2} \sum_{k=1}^m |a_k| f_k(x) g_k(x) dx \geq \frac{1}{2} \sum_{k=1}^m |a_k| \left( \|f_k\|_{\Phi^*} - \frac{\epsilon}{2k} \right)
\]
\[
\geq \frac{1}{2} \left( \sum_{k=1}^m |a_k| - \sum_{k=1}^m \frac{\epsilon}{2k} \right) \geq \frac{1}{2} \left( \sum_{k=1}^m |a_k| - \epsilon \right).
\]
Since $\epsilon > 0$ and $m \in \mathbb{N}$ were arbitrary, combining the above inequalities we get
\[
\frac{1}{2} \|a\|_1 \leq \left\| \sum_{k=1}^\infty a_k h_k \right\|_{1,\Phi} \leq (1+l)\|a\|_1,
\]
which shows that $W^{1,\Phi}$ contains an isomorphic subspace of $\ell^1$ and completes the proof.

In view of (V) satisfied in any Orlicz space the next result is an instant corollary of Theorems 4.3 and 4.4.

**Corollary 4.5.** Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$, and $\varphi$ be an Orlicz function. Then the space $W^{1,\varphi}$ does not contain isomorphic copy of $\ell^1$ if and only if $\varphi$ and $\varphi^*$ satisfy condition $\Delta_2^\infty$.

**Corollary 4.6.** Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. The space $W^{1,p(\cdot)}$ does not contain isomorphic copy of $\ell^1$ if and only if $1 < p^- \leq p^+ < \infty$.

**Proof.** We observe first that the Voltera operator is bounded on $L^{p(\cdot)}$ by Corollary 2.9. Therefore the conclusion follows by Theorem 4.1 and Theorems 4.3 and 4.4.

**Corollary 4.7.** Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. Let the Sobolev space $W^{1,\Phi}$ be induced by the double phase function
\[
\Phi(x,t) = t^{p(x)} + a(x)t^{r(x)}
\]
with $a, p, r$ measurable functions on $\Omega$, such that $a(x) \geq 0$ and $p(x) \leq r(x)$ for a.a. $x \in \Omega$. If $r^+ > \infty$ and $p^- > 1$ then $W^{1,\Phi}$ does not contain an isomorphic subspace to $\ell^1$. 

Proof. The Volterra operator is bounded on \( L^\Phi \) by Corollary 2.10. Therefore by Theorems 4.3 and 4.4, \( W^{1,\Phi} \) does not contain an isomorphic subspace to \( \ell^1 \) if and only if \( \Phi \) and \( \Phi^* \) satisfy \( \Delta_2 \). On the other hand in view of Proposition 1.8 (i), by the assumption that \( r^+ < \infty \), \( \Phi \) satisfies condition \( \Delta_2 \), and by the assumption \( p^- > 1 \), \( \Phi^* \) satisfies \( \Delta_2 \), and the conclusion follows.

\[ \square \]

5. Reflexivity of \( W^{1,\Phi} \)

Theorem 5.1. Let \( \Omega = (\alpha, \beta) \), where \( -\infty < \alpha < \beta < \infty \), and \( \Phi \) be a MO function.

If both \( \Phi \) and \( \Phi^* \) satisfy condition \( \Delta_2 \) then \( W^{1,\Phi} \) is reflexive.

Let \( \Phi \) be MO function satisfying condition (V). If the space \( W^{1,\Phi} \) is reflexive then both \( \Phi \) and \( \Phi^* \) satisfy condition \( \Delta_2 \).

Proof. If both \( \Phi \) and \( \Phi^* \) satisfy \( \Delta_2 \) then the MO space \( L^\Phi \) is reflexive by Theorem 4.3 (iii), and so is \( W^{1,\Phi} \).

Let assume now condition (V) that the Volterra operator is bounded on \( L^\Phi \). If \( W^{1,\Phi} \) is reflexive then it can not contain isomorphic copy of \( \ell^\infty \). Therefore by Theorem 3.5 \( \Phi \) satisfies \( \Delta_2 \). Similarly \( W^{1,\Phi} \) can not contain an isomorphic copy of \( \ell^1 \), and thus by Theorem 4.4 \( \Phi^* \) also satisfies \( \Delta_2 \).

\[ \square \]

Since (V) is fulfilled in any Orlicz space, so we have the following corollary.

Corollary 5.2. Let \( \Omega = (\alpha, \beta) \) where \( -\infty < \alpha < \beta < \infty \), and \( \varphi \) be an Orlicz function. Then the space \( W^{1,\varphi} \) is reflexive if and only if \( \varphi \) and \( \varphi^* \) satisfy condition \( \Delta_2^\infty \).

Corollary 5.3. Let \( \Omega = (\alpha, \beta) \) where \( -\infty < \alpha < \beta < \infty \). The space \( W^{1,\varphi(\cdot)} \) is reflexive if and only if \( 1 < p^- \leq p^+ < \infty \).

Proof. We observe first that the Volterra operator is bounded on \( L^{\varphi(\cdot)} \) by Corollary 2.9. Therefore the conclusion follows by Theorem 5.1.

\[ \square \]

6. Uniform convexity of \( W^{1,\Phi} \)

The concept of uniform convexity of a Banach space \( X \) was first introduced by James A. Clarkson in 1936 where the author showed the uniform convexity of \( \ell^p \) and \( L^p \) spaces for \( 1 < p < \infty \). Since then many authors had studied this property in other instances of Banach spaces as Orlicz or Musielak-Orlicz spaces [6, 18, 16].

Let \( (X, \| \cdot \|) \) be a Banach space equipped with the norm \( \| \cdot \| \). We denote by \( S(X) \) and \( B(X) \) the unit sphere and unit ball of \( X \) respectively. We say the \( X \) is strictly convex whenever \( \frac{\|x+y\|}{2} < 1 \) for any \( x, y \in S(X), x \neq y \). Recall that \( X \) is uniformly convex (for short UC) if

\[ \forall \epsilon \in (0, 1) \ \exists \delta > 0 \ \forall x, y \in B(X) \ |x - y| > \epsilon \Rightarrow \left\| \frac{x + y}{2} \right\| < 1 - \delta. \]

It is not difficult to show that equivalently \( X \) is UC whenever for any sequences \( \{x_n\}, \{y_n\} \subset B(X), \|x_n + y_n\| \to 2 \) implies \( \|x_n - y_n\| \to 0 \) as \( n \to \infty \).

A MO function \( \Phi \) is called strictly convex if for a.a. \( x \in \Omega \), the function \( t \mapsto \Phi(x, t) \) is strictly convex on \( \mathbb{R}_+ \), that is

\[ \exists A \subset \Omega, \ \mu(A) = 0 \ \forall x \in \Omega \setminus A, \forall u \neq v \in \mathbb{R}_+ \ \forall \lambda \in (0, 1) \]

\[ \Phi(x, \lambda u + (1 - \lambda)v) < \lambda \Phi(x, u) + (1 - \lambda)\Phi(x, v). \]
Following [6, 16] we say that a MO function $\Phi$ is **uniformly convex** if

$$\forall \epsilon > 0 \exists \delta > 0 \exists 0 \leq f \in L^0, \int_{\Omega} \Phi(x, f(x)) \, dx \leq \epsilon,$$

and whenever $\forall u, v \geq 0 \forall a.a. \, x \in \Omega \, |u - v| \geq \epsilon \max\{u, v\}$ and $|u - v| \geq f(x)$, then $\Phi\left(x, \frac{u + v}{2}\right) \leq \frac{1 - \delta}{2}(\Phi(x, u) + \Phi(x, v))$.

It is not difficult to show that $\Phi$ is uniformly convex if and only if for every $\epsilon > 0$,

$$\lim_{c \to 0} \int_{\Omega} \Phi(x, P_{c,x}(x)) \, dx = 0,$$

where

$$P_{c,x}(x) = \sup\{u - v : (u, v) \in E_{c,\epsilon,x}\}, \text{ with } E_{c,\epsilon,x} = \{(u, v) : u, v \geq 0, |u - v| > \epsilon \max\{u, v\}, \text{ and } \Phi\left(x, \frac{u + v}{2}\right) > \frac{1}{2}(1 - c)(\Phi(x, u) + \Phi(x, v))\}.$$

First version of uniform convexity for an Orlicz function $\varphi$ was defined in the doctoral thesis of W.A.J. Luxemburg [27]. He showed that $L^\varphi$ is uniformly convex if $\varphi$ satisfies condition $\Delta_2$ and is uniformly convex. Later in [18], three versions of uniform convexity of $\varphi$ have been introduced, one for the case of non-atomic infinite measure, another one for non-atomic finite measure and one for counting measure. It was also proved that appropriate $\Delta_2$ condition and uniform convexity of $\varphi$ are necessary and sufficient conditions of uniform convexity of $L^\varphi$. Combining all three conditions in the case of MO function results in the present form (6.1) where we have a function $f$ with small modular $I_\Phi(f) \leq \epsilon$.

If $f = 0$ a.e. on $\Omega$ then in fact the inequality defining uniform convexity of $\Phi$ is the same condition as in the original paper [27], uniform for every parameter $x \in \Omega$.

The definition of uniform convexity of $\Phi$ seems to be quite complicated, but in the next remark is explained that this condition not only implies strict convexity of $\Phi$, but something more, a sort of uniform strict convexity. In fact it can be expressed by the uniform inequalities of ratios of its derivatives. This was first observed in [1] for Orlicz functions, and later in [17] for MO functions.

**Remark 6.1.** (1) It is standard to show that a uniformly convex function $\Phi$ is strictly convex on $\mathbb{R}_+$ for a.e. $x \in \Omega$.

(2) [17] The condition for $\Phi$ being uniformly convex can be expressed in terms of its derivatives. Let $\Phi'(x, t)$ denote the right derivative of $\Phi(x, t)$ with respect to $t > 0$ for a.a. $x \in \Omega$. Then $\Phi$ is uniformly convex if for every $\epsilon > 0$ there exists a constant $k_\epsilon > 1$ such that

$$\Phi'(x, (1 + \epsilon)t) \geq k_\epsilon \Phi'(x, t)$$

for a.a. $x \in \Omega$, every $t \geq 0$.

Characterization of $UC$ of $L^\Phi$ was considered in [16] and later in [6]. Partial results on uniform convexity of $L^\Phi$ or $L^{p(\cdot)}$ have been given in Section 2.4 in [10]. Recall the complete characterization of $UC$ in $L^\Phi$.

**Theorem 6.2.** [6, Theorem 5.15] Let $(\Omega, \Sigma, \mu)$ be a non-atomic separable measure space. The MO space $(L^\Phi, \| \cdot \|_\Phi)$ is uniformly convex if and only if $\Phi$ satisfies $\Delta_2$ and $\Phi$ is uniformly convex.
Lemma 6.3. Let $\Phi(x,t) = \frac{p(x)}{p(x)}$, $t \geq 0$, $1 \leq p(x) < \infty$ a.e. $x \in \Omega$. If $p^- > 1$ then $\Phi$ is uniformly convex.

Proof. Let $p^- > 1$. Then for a.a. $x \in \Omega$,

$$
\Phi'(x, (1 + \epsilon)t) = (1 + \epsilon)^{p(x)} - 1 \geq (1 + \epsilon)^{p(x)} - 1 = k_\epsilon \Phi'(x, t),
$$

where $k_\epsilon = (1 + \epsilon)^{p^-} > 1$. Thus we showed (6.4), and $\Phi$ is uniformly convex.

Lemma 6.4. Let $\Phi : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a double phase functional, that is

$$(6.5) \Phi(x,t) = a(x)r(x),$$

where $a,p,r$ are measurable functions on $\Omega$, $a(x) \geq 0$ a.e. and $1 \leq p(x) < r(x) < \infty$ a.e.. If $p^- > 1$ then $\Phi$ is uniformly convex.

Proof. Let $p^- > 1$. Then for a.a. $x \in \Omega$, $t \geq 0$,

$$
\Phi'(x, (1 + \epsilon)t) = p(x)(1 + \epsilon)^{p(x)} - 1 + a(x)r(x)(1 + \epsilon)^{r(x)} - 1
$$

$$
\geq (1 + \epsilon)^{p^-} - 1 p(x)^{p(x)} - 1 + a(x)r(x)(1 + \epsilon)^{p^-} - 1 r(x) - 1 = (1 + \epsilon)^{p^-} - 1 \Phi'(x, t),
$$

where $(1 + \epsilon)^{p^-} - 1 > 1$. It follows uniform convexity of $\Phi$ by (6.4)

Partial results of the next corollary are known [10], but here we present a complete criterion of uniform convexity of $L^{\hat{p}(\cdot)}$.

Corollary 6.5. Let $(\Omega, \Sigma, \mu)$ be a non-atomic separable measure space. The variable exponent Lebesgue space $L^{\hat{p}(\cdot)}$ is uniformly convex if and only if $1 < p^- \leq p^+ < \infty$.

Proof. If $1 < p^- \leq p^+ < \infty$ then $\Phi(x,t) = \frac{p(x)}{p(x)}$ satisfies $\Delta_2$ by Theorem [17] and is uniformly convex by Lemma 6.3. Thus the space is uniformly convex in view of Theorem 6.2. On the other hand uniform convexity of $L^{\hat{p}(\cdot)}$ implies reflexivity of the space, and so it cannot have an isomorphic subspace of $l^p$, and thus $1 < p^- \leq p^+ < \infty$ by Corollary 4.2

Corollary 6.6. Let $(\Omega, \Sigma, \mu)$ be a non-atomic separable measure space. If $\Phi$ is a double phase functional of the form (6.5), then the space $L^\Phi$ is uniformly convex if $1 < p^-$ and $r^+ < \infty$.

Proof. If $p^- > 1$ then by Lemma 6.4 $\Phi$ is uniformly convex. If $r^+ < \infty$ then $\Phi$ satisfies $\Delta_2$ by Theorem [18]. We conclude in view of Theorem 6.2.

Now we proceed to consider uniform convexity of MOS space $W^{1,\Phi}$.

Theorem 6.7. Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. Let $\Phi$ be a MO function and $W^{1,\Phi}$ be the Sobolev space equipped with the norm $\|f\|_{1,\Phi} = \|f\|_\Phi + \|f\|_\Phi$, $f \in W^{1,\Phi}$.

If $\Phi$ satisfies condition $\Delta_2$ and $\Phi$ is uniformly convex then the space $W^{1,\Phi}$ is uniformly convex.

If in addition $\Phi$ satisfies (V), then $\Delta_2$ and uniform convexity of $\Phi$ are also necessary conditions for the space $W^{1,\Phi}$ to be uniformly convex.

Proof. By Theorem 6.2 if $\Phi$ satisfies condition $\Delta_2$ and $\Phi$ is uniformly convex then the space $L^\Phi$ is uniformly convex. Therefore $L^\Phi \times L^\Phi$ equipped with norm (14) is also uniformly convex, and so is $W^{1,\Phi}$.
Let now \( \Phi \) satisfy (V) and \( W^{1,\Phi} \) be uniformly convex. Then the space \( W^{1,\Phi} \) is reflexive, and so it cannot have a subspace isomorphic to \( \ell^\infty \). By Theorem 3.3 \( \Phi \) needs to satisfy condition \( \Delta_2 \).

Thus assume that \( \Phi \) satisfies \( \Delta_2 \) and \( \Phi \) is not uniformly convex. It follows by (6.2) that there exists \( \epsilon > 0 \) and a sequence \( \{c_k\} \subset (0,1) \) with \( \lim_{k \to \infty} c_k = 0 \) and

\[
\lim_{k \to \infty} \int_\Omega \Phi(x, P_{c_k}(x)) \, dx > 0.
\]

Hence there are \( \delta > 0 \), \( N > 0 \) such that for all \( k > N \),

\[
(6.6) \quad \int_\Omega \Phi(x, P_{c_k}(x)) \, dx > 2\delta.
\]

In view of (6.3) for every \( c \in (0,1) \) there exists a sequence \( \{u^c_k, v^c_k\} \) of non-negative measurable functions satisfying the following conditions,

\[
(6.7) \quad |u^c_k(x) - v^c_k(x)| \uparrow P_{c}(x) \text{ if } k \to \infty, \text{ for a.a. } x \in \Omega.
\]

If \( u^c_k(x) \neq v^c_k(x) \) then \( \Phi(x, |u^c_k(x) - v^c_k(x)|) \geq \max\{\Phi(x, cu^c_k(x)), \Phi(x, cv^c_k(x))\} \) and

\[
\Phi\left(x, \frac{u^c_k(x) + v^c_k(x)}{2}\right) > \frac{1 - c}{2} (\Phi(x, u^c_k(x)) + \Phi(x, v^c_k(x))).
\]

By (6.6) and (6.7), for every \( k \in \mathbb{N} \) there exists \( j_k \in \mathbb{N} \) such that for all \( k \in \mathbb{N} \),

\[
\int_\Omega \Phi(x, |u^c_{j_k}(x) - v^c_{j_k}(x)|) \, dx > \frac{1}{2} \int_\Omega \Phi(x, P_{c_k}(x)) \, dx > \delta.
\]

Letting \( u_k = u^c_{j_k}, v_k = v^c_{j_k} \) and applying the above inequalities we obtain for every \( k \in \mathbb{N} \),

\[
(6.8) \quad I_\Phi(u_k - v_k) = \int_\Omega \Phi(x, |u_k(x) - v_k(x)|) \, dx > \delta,
\]

and if \( u_k(x) \neq v_k(x) \) then

\[
(6.9) \quad \Phi(x, |u_k(x) - v_k(x)|) > \max\{\Phi(x, cu_k(x)), \Phi(x, cv_k(x))\},
\]

\[
(6.10) \quad \Phi\left(x, \frac{1}{2}(u_k(x) + v_k(x))\right) > \frac{1 - c_k}{2} (\Phi(x, u_k(x)) + \Phi(x, v_k(x))).
\]

In view of \( \Delta_2 \), by Theorem 1.1 there exists \( \gamma \in (0,\delta) \) such that for all \( u \in L^\Phi \),

\[
(6.11) \quad I_\Phi(u) < \gamma \Rightarrow \|u\|_\Phi < \epsilon.
\]

By (6.8) for every \( k \in \mathbb{N} \) we find the sets \( E_k \) satisfying

\[
E_k \subset \{x \in \Omega : u_k(x) \neq v_k(x)\} \quad \text{and} \quad \int_{E_k} \Phi(x, |u_k(x) - v_k(x)|) \, dx = \gamma.
\]

If \( x \in E_k \) then \( u_k(x) \neq v_k(x) \) and by (6.9), \( \Phi(x, |u_k(x) - v_k(x)|) > \Phi(x, cu_k(x)) \). Hence

\[
(6.12) \quad \int_{E_k} \Phi(x, cu_k(x)) \, dx \leq \int_{E_k} \Phi(x, |u_k(x) - v_k(x)|) \, dx = \gamma.
\]

It follows in view of (6.11) that \( \|cu_k\chi_{E_k}\|_\Phi < \epsilon \). Consequently and by symmetry, for all \( k \in \mathbb{N} \),

\[
\|u_k\chi_{E_k}\|_\Phi \leq 1 \quad \text{and} \quad \|v_k\chi_{E_k}\|_\Phi \leq 1.
\]

Let for each \( k \in \mathbb{N} \),

\[
\Omega_k = \{x \in E_k : \Phi(x, u_k(x)) \geq \Phi(x, v_k(x))\} \quad \text{and} \quad \overline{\Omega_k} = E_k \setminus \Omega_k.
\]
Set also
\[ \alpha_k = \int_{\Omega_k} (\Phi(x, u_k(x)) - \Phi(x, v_k(x))) \, dx \quad \text{and} \quad \bar{\alpha}_k = \int_{\Omega_k} (\Phi(x, v_k(x)) - \Phi(x, u_k(x))) \, dx. \]

Clearly, \( \alpha_k, \bar{\alpha}_k > 0 \). Therefore for each \( k \in \mathbb{N} \) there exists \( F_k \subset \Omega_k \) satisfying
\[ \int_{F_k} (\Phi(x, u_k(x)) - \Phi(x, v_k(x))) \, dx = \frac{\alpha_k}{2}. \]

Hence
\[ \int_{F_k} (\Phi(x, u_k(x)) - \Phi(x, v_k(x))) \, dx = \int_{\Omega_k \setminus F_k} (\Phi(x, u_k(x)) - \Phi(x, v_k(x))) \, dx, \]

which implies that
\[ \int_{F_k} \Phi(x, v_k(x)) \, dx + \int_{\Omega_k \setminus F_k} \Phi(x, u_k(x)) \, dx = \int_{\Omega_k \setminus F_k} \Phi(x, v_k(x)) \, dx + \int_{F_k} \Phi(x, u_k(x)) \, dx. \]

Analogously for every \( k \in \mathbb{N} \) there exists \( \tilde{F}_k \subset \tilde{\Omega}_k \) such that
\[ \int_{\tilde{F}_k} \Phi(x, v_k(x)) \, dx + \int_{\tilde{\Omega}_k \setminus \tilde{F}_k} \Phi(x, u_k(x)) \, dx = \int_{\tilde{\Omega}_k \setminus \tilde{F}_k} \Phi(x, v_k(x)) \, dx + \int_{\tilde{F}_k} \Phi(x, u_k(x)) \, dx. \]

Setting now for \( k \in \mathbb{N} \),
\[ \hat{x}_k = u_k \chi_{F_k \cup (\tilde{\Omega}_k \setminus \tilde{F}_k)} + v_k \chi_{\tilde{F}_k \cup (\Omega_k \setminus F_k)}, \]
\[ \hat{y}_k = v_k \chi_{F_k \cup (\tilde{\Omega}_k \setminus \tilde{F}_k)} + u_k \chi_{\tilde{F}_k \cup (\Omega_k \setminus F_k)}, \]

we get
\[ [F_k \cup (\tilde{\Omega}_k \setminus \tilde{F}_k)] \setminus [F_k \cup (\tilde{\Omega}_k \setminus \tilde{F}_k)] = E_k, \quad [F_k \cup (\tilde{\Omega}_k \setminus \tilde{F}_k)] \cap [F_k \cup (\Omega_k \setminus F_k)] = \emptyset \quad \text{and} \quad I_\Phi(\hat{x}_k) = I_\Phi(\hat{y}_k) \]

for all \( k \in \mathbb{N} \). In view of \( (6.9) \) and \( (6.12) \),
\[ \delta > \gamma = I_\Phi((u_k - v_k)\chi_{E_k}) \geq I_\Phi(\epsilon \max\{u_k, v_k\}\chi_{E_k}), \]

and so by \( (6.11) \), \( \|\epsilon \max\{u_k, v_k\}\chi_{E_k}\|_\Phi \leq \epsilon \). Hence \( \|\max\{u_k, v_k\}\chi_{E_k}\|_\Phi \leq 1 \) and consequently
\[ I_\Phi(\max\{u_k, v_k\}\chi_{E_k}) \leq 1. \]

Since \( \hat{x}_k \leq \max\{u_k, v_k\}\chi_{E_k} \) and \( \hat{y}_k \leq \max\{u_k, v_k\}\chi_{E_k} \), we get
\[ \beta_k := I_\Phi(\hat{x}_k) = I_\Phi(\hat{y}_k) \leq I_\Phi(\max\{u_k, v_k\}\chi_{E_k}) \leq 1. \]

Now for every \( k \in \mathbb{N} \), in view of \( \mu(\Omega \setminus E_k) > 0 \), there exist \( G_k \subset \Omega \setminus E_k \) and \( \sigma_k > 0 \) such that
\[ \int_{G_k} \Phi(x, \sigma_k) \, dx = 1 - \beta_k. \]

Finally let
\[ x_k = \hat{x}_k \chi_{E_k} + \sigma_k \chi_{E_k}, \quad y_k = \hat{y}_k \chi_{E_k} + \sigma_k \chi_{E_k}. \]

Then for all \( k \in \mathbb{N} \),
\[ I_\Phi(x_k) = I_\Phi(y_k) = 1. \]

Moreover by \( (6.12) \),
\[ 0 < \gamma = I_\Phi((u_k - v_k)\chi_{E_k}) = I_\Phi(x_k - y_k) = I_\Phi(\hat{x}_k - \hat{y}_k) \leq I_\Phi(\max\{\hat{x}_k, \hat{y}_k\}) = I_\Phi(\max\{u_k, v_k\}\chi_{E_k}) \leq 1. \]

Consequently for all \( k \in \mathbb{N} \),
\[ 0 < \gamma \leq I_\Phi(x_k - y_k) \leq \|x_k - y_k\|_\Phi. \]
Since \( \|x_k\|_\Phi = \|y_k\|_\Phi = 1, \|x_k + y_k\|_\Phi \leq 1 \). Hence

\[
(6.15) \quad \left\| \frac{x_k + y_k}{2} \right\|_\Phi \geq I_\Phi \left( \frac{x_k + y_k}{2} \right).
\]

Moreover,

\[
I_\Phi \left( \frac{x_k + y_k}{2} \right) = I_\Phi \left( \frac{x_k + y_k}{2} \chi_{E_k} \right) + I_\Phi (\sigma_k \chi_{E_k}) = \frac{1}{2} I_\Phi (x_k) + \frac{1}{2} I_\Phi (y_k) - \frac{c_k}{2} \int_{E_k} (\Phi(x, u_k(x)) + \Phi(x, v_k(x))) \, dx + \int_{G_k} \Phi(x, \sigma_k) \, dx
\]

\[
\geq 1 - c_k \to 1 \quad \text{by (6.13),}
\]

when \( k \to \infty \).

Combining the above and \( (6.14), (6.15) \), it follows that \( L^\Phi \) is not uniformly convex.

Now we will proceed to show that \( W^{1,\Phi} \) is not uniformly convex either.

Recall that a measurable function is called simple if it assumes finite number of values. A function \( f : (\alpha, \beta) \to \mathbb{C} \) is called a step function if there exists a finite partition \( \alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_m = \beta \) and the numbers \( \{a_i\}_{i=1}^m \subset \mathbb{C} \) such that \( f(x) = \sum_{i=1}^m a_i \chi_{(\alpha_{i-1}, \alpha_i)}(x), \quad x \in (\alpha, \beta) \).

First observe that the functions \( u_k^c, v_k^c \) satisfying \( (6.7) \) can be chosen to be simple functions. Therefore the functions \( x_k \) and \( y_k \) can be also chosen as simple functions. The next observation is that these functions can be replaced by step functions. In fact it follows from the regularity of the Lebesgue measure on \( \Omega = (\alpha, \beta) \) and the assumption of \( \Delta_2 \) condition. By Theorem 1.1 \( L^\Phi \) is order continuous under the assumption of \( \Delta_2 \). It implies in particular that for any \( f \in L^\Phi \) and every \( \epsilon > 0 \) there is \( \delta > 0 \) such that \( \forall x \in (\alpha, \beta) \) whenever \( \mu(A) < \delta \) then \( \|f \chi_A\|_\Phi < \epsilon \). By regularity of the Lebesgue measure, for any measurable \( A \subset \Omega \) with \( \chi_A \in L^\Phi \) and any \( \delta > 0 \), there exist disjoint open intervals \( G_1, \ldots, G_m \) such that \( \mu((A \setminus \cup_{i=1}^m G_i) \cup (\cup_{i=1}^m G_i \setminus A)) < \delta \). Hence \( \|\chi_A - \chi_{\cup_{i=1}^m G_i}\|_\Phi < \epsilon \). Therefore we can approximate any measurable subset of \( \Omega \) by a finite union of disjoint open intervals. So any simple function can be replaced by a step function. It follows that the functions \( x_k \) and \( y_k \) can be taken as step functions. Recall that \( x_k \) and \( y_k \) are non-negative.

By the above discussion, without loss of generality, assume for every \( k \in \mathbb{N} \),

\[
x_k = a_{k1} \chi_{(\alpha_0, \alpha_1)} + a_{k2} \chi_{(\alpha_1, \alpha_2)} + \cdots + a_{kM_k} \chi_{(\alpha_{M_k-1}, \alpha_{M_k})},
\]

where \( \alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_{M_k} = \beta \) and \( \{a_{ki}\}_{i=1}^{M_k} \subset [0, \infty) \). Similarly let

\[
y_k = b_{k1} \chi_{(\beta_0, \beta_1)} + b_{k2} \chi_{(\beta_1, \beta_2)} + \cdots + b_{JK_k} \chi_{(\beta_{J_k-1}, \beta_{J_k})},
\]

with \( \alpha = \beta_0 < \beta_1 < \cdots < \beta_{J_k} = \beta \) and \( \{b_{kj}\}_{j=1}^{J_k} \subset [0, \infty) \). Now let \( \{(\gamma_{i-1}, \gamma_i)\}_{i=1}^m \) be the family of all intersections of the intervals \( (\alpha_{p-1}, \alpha_p) \cap (\beta_{j-1}, \beta_j), \quad p = 1, \ldots, M_k, \quad j = 1, \ldots, J_k \), which are neither empty nor reduced to one point. Let \( \gamma_i \) be ordered as...
\( \alpha = \gamma_0 \prec \gamma_1 \prec \cdots \prec \gamma_m = \beta. \) The numbers \( \gamma_i \) and \( m \in \mathbb{N} \) depend on \( k. \) Both functions \( x_k \) and \( y_k \) are constant on every interval \((\gamma_i-1, \gamma_i).\)

Let \( l \in \{1, \ldots, m\} \) be fixed. Divide \((\gamma_{l-1}, \gamma_l)\) into \( 2n_l \) equal subintervals for \( n_l \in \mathbb{N},\)

\[
(\gamma_1^l, \gamma_2^l), (\gamma_2^l, \gamma_3^l), \ldots, (\gamma_{2n_l}^l, \gamma_{2n_l+1}^l),
\]
such that

\[
\int_{\gamma_{2i-1}}^{\gamma_{2i}^l} \max\{x_k, y_k\} < \frac{1}{2^k}, \quad i = 1, \ldots, n_l,
\]

\[
\int_{\gamma_{2i}}^{\gamma_{2i+1}^l} \max\{x_k, y_k\} < \frac{1}{2^k}, \quad i = 1, \ldots, n_l.
\]

Since each \( x_k \) or \( y_k \) is constant on the interval \((\gamma_{l-1}, \gamma_l),\) so

\[
\int_{\gamma_{2i-1}}^{\gamma_{2i}^l} x_k = \int_{\gamma_{2i}}^{\gamma_{2i+1}^l} x_k, \quad \int_{\gamma_{2i-1}}^{\gamma_{2i}^l} y_k = \int_{\gamma_{2i}}^{\gamma_{2i+1}^l} y_k, \quad i = 1, \ldots, n_l.
\]

Define on each interval \((\gamma_{l-1}, \gamma_l), \) \( l = 1, \ldots, m,\)

\[
\tilde{x}_k = x_k \chi_{(\gamma_1^l, \gamma_2^l)} - x_k \chi_{(\gamma_2^l, \gamma_3^l)} + \cdots + x_k \chi_{(\gamma_{2n_l}^l, \gamma_{2n_l+1}^l)} - x_k \chi_{(\gamma_{2n_l}^l, \gamma_{2n_l+1}^l)} - x_k \chi_{(\gamma_{2n_l}^l, \gamma_{2n_l+1}^l)},
\]

\[
\tilde{y}_k = y_k \chi_{(\gamma_1^l, \gamma_2^l)} - y_k \chi_{(\gamma_2^l, \gamma_3^l)} + \cdots + y_k \chi_{(\gamma_{2n_l}^l, \gamma_{2n_l+1}^l)} - y_k \chi_{(\gamma_{2n_l}^l, \gamma_{2n_l+1}^l)}.
\]

We defined \( \tilde{x}_k, \) \( \tilde{y}_k \) on every \((\gamma_{l-1}, \gamma_l),\) so they are well defined on \((\alpha, \beta).\) If \( x \in (\gamma_{l-1}, \gamma_l)\) then either \( x \in (\gamma_{2i-1}^l, \gamma_{2i}^l) \) or \( x \in (\gamma_{2i}^l, \gamma_{2i+1}^l) \) for some \( i = 1, \ldots, n_l. \) For \( x \in (\gamma_{2i-1}^l, \gamma_{2i}^l),\)

\[
\left| \int_{\gamma_{l-1}}^{x} \tilde{x}_k \right| = \left| \left( \int_{\gamma_{l-1}}^{\gamma_1^l} x_k - \int_{\gamma_{2i-1}}^{\gamma_{2i}^l} x_k \right) + \cdots + \int_{\gamma_{2i-1}^l}^{\gamma_{2i+1}^l} x_k \right| \leq \int_{\gamma_{2i-1}}^{\gamma_{2i}^l} x_k < \frac{1}{2^k}.
\]

For \( x \in (\gamma_{2i}^l, \gamma_{2i+1}^l),\)

\[
\left| \int_{\gamma_{l-1}}^{x} \tilde{x}_k \right| = \left| \left( \int_{\gamma_{l-1}}^{\gamma_1^l} x_k - \int_{\gamma_{2i-1}}^{\gamma_{2i}^l} x_k \right) + \cdots + \int_{\gamma_{2i}^l}^{\gamma_{2i+1}^l} x_k \right| = \int_{\gamma_{2i-1}}^{\gamma_{2i}^l} x_k - \int_{\gamma_{2i}^l}^{\gamma_{2i+1}^l} x_k < \frac{1}{2^k}.
\]

Combining the above we get for every \( k \in \mathbb{N} \) and \( x \in (\gamma_{l-1}, \gamma_l),\)

\[
\int_{\gamma_{l-1}}^{\gamma_l} \tilde{x}_k = 0, \quad \left| \int_{\gamma_{l-1}}^{x} \tilde{x}_k \right| < \frac{1}{2^k}.
\]

Similarly we get for every \( k \in \mathbb{N} \) and \( x \in (\gamma_{l-1}, \gamma_l),\)

\[
\int_{\gamma_{l-1}}^{\gamma_l} \tilde{y}_k = 0, \quad \left| \int_{\gamma_{l-1}}^{x} \tilde{y}_k \right| < \frac{1}{2^k}.
\]

Since the above inequalities are satisfied for every \( l = 1, \ldots, m \) we obtain for every \( x \in (\alpha, \beta),\)

\[
(6.16) \quad \left| \int_{\alpha}^{x} \tilde{x}_k \right| < \frac{1}{2^k}, \quad \left| \int_{\alpha}^{x} \tilde{y}_k \right| < \frac{1}{2^k}.
\]

Let for \( k \in \mathbb{N}, \) \( x \in (\alpha, \beta),\)

\[
f_k(x) = \int_{\alpha}^{x} \tilde{x}_k, \quad g_k(x) = \int_{\alpha}^{x} \tilde{y}_k.
\]
Then
\[ f_k(x) = \tilde{x}_k(x), \quad g_k(x) = \tilde{y}_k(x), \]
for a.a. \( x \in (\alpha, \beta) \). By (6.16) for every \( \lambda > 0 \),
\[(6.17) \quad I_\Phi(\lambda f_k) = \int_{\Omega} \Phi \left( x, \lambda \left| \frac{x}{\alpha} \right| \right) dx \leq \frac{1}{2k} \int_{\Omega} \Phi(x, \lambda) dx.\]
Now by assumption (V) (see Definition 2.6) and \( \Delta_2, \int_{\Omega} \Phi(x, \lambda) dx < \infty \). Hence the right side of (6.17) tends to zero when \( k \to \infty \). It follows that
\[ \| f_k \|_\Phi \to 0 \quad \text{if} \quad k \to \infty. \]
Similarly
\[ \| g_k \|_\Phi \to 0 \quad \text{as} \quad k \to \infty. \]
In view of (6.13), we have for every \( k \in \mathbb{N} \),
\[ \| \tilde{x}_k \|_\Phi = \| \tilde{x}_k \| = \| x_k \| = 1 \quad \text{and} \quad \| \tilde{y}_k \|_\Phi = \| \tilde{y}_k \| = \| y_k \| = 1. \]
Consequently,
\[ \| f_k \|_{1, \Phi} = \| f_k \|_\Phi + \| f'_k \|_\Phi = \| f_k \|_\Phi + \| \tilde{x}_k \|_\Phi \to 1, \]
\[ \| g_k \|_{1, \Phi} = \| g_k \|_\Phi + \| g'_k \|_\Phi = \| g_k \|_\Phi + \| \tilde{y}_k \|_\Phi \to 1, \]
as \( k \to \infty \).
Moreover \( \frac{\tilde{x}_k + \tilde{y}_k}{2} \) for every \( k \in \mathbb{N} \). Thus in view of (6.15),
\[ \left\| \frac{f_k + g_k}{2} \right\|_{1, \Phi} = \left\| \frac{f_k + g_k}{2} \right\|_\Phi + \left\| \frac{\tilde{x}_k + \tilde{y}_k}{2} \right\|_\Phi \geq \frac{\| x_k + y_k \|_\Phi}{2} \geq I_\Phi \left( \frac{x_k + y_k}{2} \right) \geq 1 - c_k \to 1, \]
as \( k \to \infty \). We also have by (6.14),
\[ \| f_k - g_k \|_{1, \Phi} = \| f_k - g_k \|_\Phi + \| \tilde{x}_k - \tilde{y}_k \|_\Phi \geq \| \tilde{x}_k - \tilde{y}_k \|_\Phi = \| x_k - y_k \|_\Phi \geq \gamma \]
for all \( k \in \mathbb{N} \). It shows that \( W^{1, \Phi} \) is not uniformly convex and the proof is finished.

The next result follows from Theorems 6.2 and 6.7.

**Corollary 6.8.** Let \( \Omega = (\alpha, \beta), \infty < \alpha < \beta < \infty \). Let \( \Phi \) satisfy (V). Then \( L^\Phi \) is uniformly convex if and only if \( W^{1, \Phi} \) is uniformly convex. This in turn is equivalent to \( \Phi \) satisfying \( \Delta_2 \) and being uniformly convex.

**Corollary 6.9.** Let \( \Omega = (\alpha, \beta), \infty < \alpha < \beta < \infty \). The variable exponent Sobolev space \( W^{1, p(\cdot)} \) is uniformly convex if and only if \( 1 < p^- \leq p^+ < \infty \).

**Proof.** It follows immediately from Corollaries 6.8 and 6.5.

**Corollary 6.10.** Let \( \Omega = (\alpha, \beta), \infty < \alpha < \beta < \infty \). If \( \Phi \) is a double phase functional of the form (6.5), then the space \( W^{1, \Phi} \) is uniformly convex if \( 1 < p^- \) and \( p^+ < \infty \).

**Proof.** It follows from Corollary 6.6.
In the same case of finite and non-atomic measure space, $\Phi$ is uniformly convex if and only if $\varphi$ satisfies the following condition\[18],
\begin{equation}
(6.18) \quad \forall \varepsilon > 0 \exists \delta > 0 \exists u_0 \geq 0 \forall u, v \geq u_0 \quad |u - v| \geq \max\{u, v\}
\end{equation}
$$\implies \varphi \left( \frac{u + v}{2} \right) \leq \frac{1 - \delta}{2} (\varphi(u) + \varphi(v)).$$

In paper\[7\] the authors gave a characterization of uniform convexity of $W^{1,\varphi}$ under additional assumption that $\varphi$ is a $N$-function. Moreover, their methods are very specific for Orlicz functions only, and they are not applicable in the case of MO functions (see Theorem 1.4). It follows that $\Omega = (\alpha, \beta)$, where $-\infty < \alpha < \beta < \infty$. For an Orlicz function $\varphi$, the condition (V) is always satisfied on $\Omega = (\alpha, \beta)$, that is the Volterra operator is bounded on $L^\infty$ (Theorem 2.7). By Theorem 6.7 and the above remarks we arrive at the following result.

**Corollary 6.11.** Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. For an Orlicz function $\varphi$, the Orlicz-Sobolev space $W^{1,\varphi}$ is uniformly convex if and only if $\varphi$ satisfies $\Delta_2^\infty$ and is uniformly convex in the sense of (6.18).

7. **Superreflexivity and $B$-convexity of $W^{1,\Phi}$**

A Banach space $(X, \| \cdot \|)$ is said to be $B$-convex if there exist a $\delta > 0$ and an integer $n \geq 2$ such that for any $x_1, \ldots, x_n \in X$ we can choose $\epsilon = \{\epsilon_k\}_{k=1}^n$, $\epsilon_k = \pm 1$, in such a way that
$$\left\| \frac{1}{n} \sum_{k=1}^{n} \epsilon_k x_k \right\| \leq (1 - \delta) \max_{1 \leq k \leq n} \|x_k\|.$$
A Banach space $X$ is said to be superreflexive if every Banach space $Y$ which is finitely representable in $X$ is reflexive\[2\]. A uniformly convex Banach space is superreflexive. Any superreflexive Banach space has a uniformly convex equivalent norm\[2\] Problem 11.6].

**Lemma 7.1.**\[17\] Lemma 1.1.5] Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite non-atomic measure space. Given a MO function $\Phi$ on $\Omega$ if $\Phi^*$ satisfies condition $\Delta_2$, then there exists a uniformly convex MO function $\Psi$ equivalent to $\Phi$.

**Theorem 7.2.** Let $\Omega = (\alpha, \beta)$, $-\infty < \alpha < \beta < \infty$ and let $\Phi$ satisfy condition (V). Then the following conditions are equivalent.

(i) $W^{1,\Phi}$ is reflexive.
(ii) $W^{1,\Phi}$ is superreflexive.
(iii) $W^{1,\Phi}$ is $B$-convex.
(iv) Both $\Phi$ and $\Phi^*$ satisfy condition $\Delta_2$.

**Proof.** (ii) $\implies$ (i) is clear.
(i) $\implies$ (iv) If (i) is satisfied, that is the space $W^{1,\Phi}$ is reflexive, then by Theorem 5.1 both $\Phi$ and $\Phi^*$ satisfy $\Delta_2$, so (iv) holds.
(iv) $\implies$ (ii) By the assumption that $\Phi^* \in \Delta_2$, in view of Lemma 7.1 there exists a uniformly convex function $\Psi$ equivalent to $\Phi$. Since $\Phi \in \Delta_2$, the function $\Psi \in \Delta_2$. Now by Theorem 6.2 the space $W^{1,\Psi}$ is uniformly convex and thus superreflexive\[2\] Problem 11.6]. Since $\Psi$ is equivalent to $\Phi$, the spaces $W^{1,\Phi}$ and $W^{1,\Psi}$ coincide as sets with equivalent norms (see Theorem 1.3). It follows that $W^{1,\Phi}$ as isomorphic to $W^{1,\Psi}$ is superreflexive.
(iv) $\implies$ (iii) By\[12\] Example 3 (ii), p. 118, any uniformly convex space is $B$-convex. By $\Phi^* \in \Delta_2$, in view of Lemma 7.1 there exists a uniformly convex function $\Psi$ equivalent to $\Phi$. Since $\Delta_2$ is preserved by equivalence, $\Psi \in \Delta_2$. In view of Theorem 6.2 the space $W^{1,\Psi}$ is uniformly convex and thus is $B$-convex. In the same paper\[12\], in Corollary 6 it was proved that if Banach spaces $X$ and $Y$ are isomorphic, then $X$ is $B$-convex if and
only if $Y$ is $B$-convex. It follows that $W^{1,\Phi}$ is $B$-convex as $W^{1,\Psi}$ and $W^{1,\Phi}$ coincide as sets with equivalent norms.

(iii) $\Rightarrow$ (iv) If $\Phi \notin \Delta_2$, then $W^{1,\Phi}$ contains a subspace isomorphic to $\ell^\infty$ by Theorem 3.5 and since $\ell^\infty$ is not $B$-convex [12 Example 3, (iv)], it contradicts the assumption of $B$-convexity of $W^{1,\Phi}$.

If $\Phi^* \notin \Delta_2$ then $W^{1,\Phi}$ contains a subspace isomorphic to $\ell^1$ by Theorem 4.4. However $\ell^1$ is not $B$-convex [12 Example 3, (iv)] or [11], and so $W^{1,\Phi}$ cannot be $B$-convex.

\[\square\]

In the next result let $\Phi(x,t) = \varphi(t)$ for all $x \in \Omega$, $t \geq 0$. Then $\varphi$ is an Orlicz function and $L^\varphi$ is an Orlicz space. For $\Omega = (\alpha, \beta)$, $-\infty < \alpha < \beta < \infty$, and an Orlicz function $\varphi$, the condition (V) is always satisfied by Theorem 2.7, and the condition $\Delta_2$ achieves a simpler form ($\Delta_\infty^\varphi$). By these remarks and Theorem 7.2 we get the following corollary in Orlicz-Sobolev spaces.

**Corollary 7.3.** Let $\Omega = (\alpha, \beta)$, $-\infty < \alpha < \beta < \infty$ and $\varphi$ be an Orlicz function. Then the following conditions are equivalent.

(i) $W^{1,\varphi}$ is reflexive.

(ii) $W^{1,\varphi}$ is superreflexive.

(iii) $W^{1,\varphi}$ is $B$-convex.

(iv) Both $\varphi$ and $\varphi^*$ satisfy condition $\Delta_\infty^\varphi$.

Since the Voltera operator is always bounded on $L^{p(\cdot)}$ by Corollary 2.9, the next corollary is an immediate result from Theorem 7.2.

**Corollary 7.4.** Let $\Omega = (\alpha, \beta)$, $\infty < \alpha < \beta < \infty$, and $\Phi(x,t) = \frac{p(x)}{p(x)}$, $1 \leq p(x) < \infty$ a.e. in $\Omega$. Then the following conditions are equivalent.

(i) $W^{1,p(\cdot)}$ is reflexive.

(ii) $W^{1,p(\cdot)}$ is superreflexive.

(iii) $W^{1,p(\cdot)}$ is $B$-convex.

(iv) $1 < p^- \leq p^+ < \infty$.

**References**

[1] V. Akimovic, *On uniformly convex and uniformly smooth Orlicz spaces*, Teoria Funkcii Funk. Anal, i Pril. 15 (1970), 114–120 (in-Russian).

[2] F. Albiak, N. J. Kalton, *Topics in Banach Space Theory*, Springer 2006.

[3] C. Bennet and R. Sharpley, *Interpolation of Operators*, Academic Press 1988.

[4] N. L. Carothers, *A Short Course in Banach Space Theory*, Cambridge University Press 2004.

[5] I. Chlebicka, P. Gwiazda, A. Świerczewska-Gwiazda, A. Wróblewska-Kamińska, *Partial Differential Equations in Anisotropic Musielak–Orlicz Spaces*, Springer Monographs in Mathematics, 2021.

[6] S. Chen, *Geometry of Orlicz Spaces*, Dissertationes Mathematicae, Warszawa, 1996.

[7] Sh. Chen, Chang. Hu and Charles X. Zhao, *Uniform rotundity of Orlicz-Sobolev spaces*, Soochow J. Math. 29, No. 3, (2003), 299–312.

[8] D. V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces*, Birkhäuser 2013.

[9] L. Diening, *Theoretical and numerical results for electrorheological fluids*, PhD thesis, Univ. Freiburg im Breisgau, Mathematische Fakultät, 2002.

[10] L. Diening, P. Harjulehto, P. Hästö and M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, Springer 2017.

[11] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge University Press 1995.

[12] D. P. Giesy, *On convexity condition in normed linear spaces*, Trans. Math. Amer. Soc. 125 (1966), 114–146.

[13] G. Leoni *A First Course in Sobolev Spaces*, Second Edition, Graduate Studies in Mathematics Vol. 181, 2017.

[14] P. Harjulehto and P. Hästö, *Orlicz Spaces and Generalized Orlicz Spaces*, Springer 2019.
[15] H. Hudzik, The problem of separability, duality, reflexivity and comparison for generalized Orlicz-Sobolev space $W^{k,M}(\Omega)$ Comment. Math. Parace Mat. 21 (1979), 315-324.

[16] H. Hudzik, Uniform convexity of Musielak-Orlicz spaces with Luxembury’s norm, Com. Math. 23 (1983), 21–32.

[17] H. Hudzik and A. Kamińska, On uniformly convexifiable and $B$-convex Musielak-Orlicz spaces, Commentationes Mathematicae 25 (1985), 59–75.

[18] A. Kamińska, On uniform convexity of Orlicz spaces, Indag. Math. 44(1) (1982), 27–36.

[19] A. Kamińska, Some convexity properties of Musielak-Orlicz spaces of Bochner type, Supplemento ai Rendiconti del Circolo Matemático di Palermo, Serie II 5 (1984), 63–73.

[20] A. Kamińska, Indices, convexity and concavity in Musielak-Orlicz spaces, Functiones Math. 26 (1998), 67–84. Special volume on the 70th birthday of J.Musielak.

[21] A. Kamińska and D. Kubiak, The Daugavet property in the Musielak-Orlicz spaces, J. Math. Analysis 427 (2015), 873–898.

[22] L. V. Kantorovich and G. P. Akilov, Functional Analysis, Second Edition, Pergamon Press 1982.

[23] O. Kováčik, J. Rákosník On spaces $L^{p(x)}$ and $W^{1,p(x)}$, Czechoslovak Math. J. 41 (116) (1991), 592–618.

[24] M. A. Krasnoselkii and Ya. B. Rutickii, Convex Functions and Orlicz Spaces, Groningen 1961.

[25] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces, Springer-Verlag 1977.

[26] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces II, Springer-Verlag 1979.

[27] W. A. J. Luxemburg, Banach Function Spaces, Thesis, Technische Hogeschool te Delft 1955.

[28] J. Musielak, Orlicz Spaces and Modular Spaces, Lecture Notes in Math., vol. 1034, Springer-Verlag, Berlin 1983.

[29] J. Musielak and W. Orlicz, On modular spaces, Studia Mathematica 18.1 (1959), 49–65.

[30] H. Nakano, Modulated sequence spaces Proc. Japan Acad. 27 (9) (1951), 508–512.

[31] M. Ruzička, Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Mathematics 1748, Springer 2000.

[32] A. C. Zaanen, Integration, North-Holland Publishing Co., Amsterdam 1967.

Email address: kaminska@memphis.edu

Email address: mzyluk@gmail.com