A RELAXATION OF THE BORDEAUX CONJECTURE

RUNRUN LIU† AND XIANGWEN LI† AND GEXIN YU† ‡

† Department of Mathematics, Huazhong Normal University, Wuhan, 430079, China
‡ Department of Mathematics, The College of William and Mary, Williamsburg, VA, 23185, USA.

Abstract. A \((c_1, c_2, ..., c_k)\)-coloring of \(G\) is a mapping \(\varphi : V(G) \mapsto \{1, 2, ..., k\}\) such that for every \(i, 1 \leq i \leq k\), \(G[V_i]\) has maximum degree at most \(c_i\), where \(G[V_i]\) denotes the subgraph induced by the vertices colored \(i\). Borodin and Raspaud conjecture that every planar graph without intersecting triangles and 5-cycles is 3-colorable. We prove in this paper that every planar graph without intersecting triangles and 5-cycles is \((2,0,0)\)-colorable.

1. Introduction

It is well-known that the problem of deciding whether a planar graph is properly 3-colorable is NP-complete. Grötzsch [8] showed the famous theorem that every triangle-free planar graph is 3-colorable. A lot of research was devoted to find sufficient conditions for a planar graph to be 3-colorable, by allowing a triangle together with some other conditions. One of such efforts is the following famous conjecture made by Steinberg [12].

Conjecture 1.1 (Steinberg, [12]). All planar graphs without 4-cycles and 5-cycles are 3-colorable.

Some progresses have been made towards this conjecture, along two directions. One direction was suggested by Erdős to find a constant \(c\) such that a planar graph without cycles of length from \(4\) to \(c\) is 3-colorable. Borodin, Glebov, Raspaud, and Salavatipour [4] showed that \(c \leq 7\). For more results, see the recent nice survey by Borodin [1].

Another direction of relaxation of the conjecture is to allow some defects in the color classes. A graph is \((c_1, c_2, ..., c_k)\)-colorable if the vertex set can be partitioned into \(k\) sets \(V_1, V_2, ..., V_k\), such that for every \(i : 1 \leq i \leq k\) the subgraph \(G[V_i]\) has maximum degree at most \(c_i\). Thus a \((0,0,0)\)-colorable graph is properly 3-colorable. Chang, Havet, Montassier, and Raspaud [6] proved that all planar graphs without 4-cycles or 5-cycles are \((2,1,0)\)-colorable and \((4,0,0)\)-colorable. In [10, 11, 15], it is shown that planar graphs without 4-cycles or 5-cycles are \((3,0,0)\)- and \((1,1,0)\)-colorable.

Havel [9] asked if each planar graph with large enough distances between triangles (denotes \(d^V\)) is 3-colorable. This was resolved in a recent preprint of Dvořák, Král and Thomas [7]. Borodin
and Raspaud in 2003 made the following Bordeaux Conjecture, which has common features with Havel’s (1969) and Steinberg’s (1976) 3-color problems.

**Conjecture 1.2** (Borodin and Raspaud, [5]). Every planar graph with \( d^\triangledown \geq 1 \) and without 5-cycles is 3-colorable.

A relaxation of the Bordeaux Conjecture with \( d^\triangledown \geq 4 \) was confirmed by Borodin and Raspaud [5], and the result was improved to \( d^\triangledown \geq 3 \) by Borodin and Glebov [2] and, independently, by Xu [13]. Borodin and Glebov [3] further improved the result to \( d^\triangledown \geq 2 \).

Using the relaxed coloring notation, Xu [14] proved that all planar graphs without adjacent triangles and 5-cycles are \((1, 1, 1)\)-colorable, where two triangles are adjacent if they share an edge.

In this paper, we consider a relaxation of the Bordeaux Conjecture. Let \( \mathcal{G} \) be the family of plane graphs with \( d^\triangledown \geq 1 \) and without 5-cycles. Yang and Yerger [17] showed that planar graphs in \( \mathcal{G} \) are \((4, 0, 0)\)- and \((2, 1, 0)\)-colorable, but there is a flaw in one of their key lemmas (Lemma 2.4). We show the following result.

**Theorem 1.3.** A planar graph in \( \mathcal{G} \) is \((2, 0, 0)\)-colorable.

In fact, we will prove a stronger result. Let \( G \) be a graph and \( H \) be a subgraph of \( G \). We call \( (G, H) \) to be superextendable if any \((2, 0, 0)\)-coloring of \( H \) can be extended to \( G \) so that vertices in \( G-H \) have different colors from their neighbors in \( H \); in this case, we call \( H \) to be a superextendable subgraph.

**Theorem 1.4.** Every triangle or 7-cycle of a planar graph in \( \mathcal{G} \) is superextendable.

To see the truth of Theorem 1.3 by way of Theorem 1.4 we may assume that the planar graph contains a triangle \( C \) since \( G \) is 3-colorable if \( G \) has no triangle. Then color the triangle, and by Theorem 1.4 the coloring of \( C \) can be superextended to \( G \). Thus, we get a coloring of \( G \).

We will use a discharging argument to prove Theorem 1.4. The idea is to consider a minimal counterexample and assign an initial charge to each vertex and face so that the sum is 0. We shall design some rules to redistribute the charges among vertices and faces so that some local sparse structures appear, or otherwise all vertices and faces would have non-negative or even positive final charges. We will then show the coloring outside the sparse structures can be extended to include all vertices in the graph (that is, the local structure is reducible), to reach a contradiction.

As pointed out in [17], as we may have 4-cycles in the considered graphs, the proof is quite different from those relaxation of the Steinberg’s Conjecture.

The paper is organized as follows. In Section 2 we introduce some notations used in the paper. In Section 3 we show the reducible structures useful in our proof. In Section 4 we show the discharging process to finish the proof.

## 2. Preliminaries

In this section, we introduce some notations used in the paper.

Graphs mentioned in this paper are all simple. A \( k \)-vertex (\( k^+ \)-vertex, \( k^- \)-vertex) is a vertex of degree \( k \) (at least \( k \), at most \( k \)). The same notation will apply to faces and cycles. We use \( b(f) \) to denote the vertex sets on \( f \). We use \( F(G) \) to denote the set of faces in \( G \). An \((l_1, l_2, \ldots, l_k)\)-face is a \( k \)-face \( v_1v_2 \ldots v_k \) with \( d(v_i) = l_i \), respectively. A face \( f \) is a pendant 3-face of vertex \( v \) if \( v \) is

\[1\] According to private communication with Yerger, they may have a way to fix the gap in their proofs.
not on $f$ but is adjacent to some 3-vertex on $f$. The pendant neighbor of a 3-vertex $v$ on a 3-face
is the neighbor of $v$ not on the 3-face.

Let $C$ be a cycle of a plane graph $G$. We use $int(C)$ and $ext(C)$ to denote the sets of vertices
located inside and outside $C$, respectively. The cycle $C$ is called a separating cycle if $int(C) \neq \emptyset \neq
ext(C)$, and is called a nonseparating cycle otherwise. We still use $C$ to denote the set of vertices
of $C$.

Let $S_1, S_2, \ldots, S_l$ be pairwise disjoint subsets of $V(G)$. We use $G[S_1, S_2, \ldots, S_l]$ to denote the
graph obtained from $G$ by identifying all the vertices in $S_i$ to a single vertex for each $i \in \{1,2,\ldots,l\}$.
Let $x(y)$ be the resulting vertex by identifying $x$ and $y$ in $G$.

A vertex $v$ is properly colored if all neighbors of $v$ have different colors from $v$. A vertex $v$ is nicely
colored if it shares a color (say $i$) with at most $\max\{s_i-1,0\}$ neighbors, where $s_i$ is the deficiency
allowed for color $i$; Thus if a vertex $v$ is nicely colored by a color $i$ which allows deficiency $s_i > 0$,
then an uncolored neighbor of $v$ can be colored by $i$.

3. Reducible configurations

Let $(G,C_0)$ be a minimum counterexample to Theorem 1.4 with minimum $\sigma(G) = |V(G)| + |
E(G)|$, where $C_0$ is a triangle or a 7-cycle in $G$ that is precolored.

The following are some simple observations about $(G,C_0)$.

Proposition 3.1. (a) Every vertex not on $C_0$ has degree at least 3.
(b) A $k$-vertex in $G$ can have at most one incident 3-face.
(c) No 3-face and 4-face in $G$ can have a common edge.

Similar to the lemmas in [14], we show Lemmas 3.2 to 3.6, which hold for all superextendable
$(c_1,c_2,c_3)$-coloring of $G \in \mathcal{G}$. The proofs are similar to those of [14], and for completeness, we
include the proofs here. If $C_0$ is a separating cycle, then $C_0$ is superextendable in both $G - ext(C_0)$
and $G - int(C_0)$. Thus, $C_0$ is superextendable in $G$, contrary to the choice of $C_0$. Thus, we may
assume that $C_0$ is the boundary of the outer face of $G$ in the rest of this paper.

Lemma 3.2. The graph $G$ contains neither separating triangles nor separating 7-cycles.

Proof. Let $C$ be a separating triangle or 7-cycle in $G$. Then $C$ is inside of $C_0$. By the minimality
of $G$, $(G - int(C), C_0)$ is superextendable, and after that, $C$ is colored. By the minimality of $G$
again, $(C \cup int(C), C)$ is superextendable. Thus, we have shown $(G,C_0)$ is superextendable, a
contradiction. \qed

Lemma 3.3. If $G$ has a separating 4-cycles $C_1 = v_1v_2v_3v_4$, then $ext(C_1) = \{b,c\}$ such that $v_1bc$
is a 3-cycle. Furthermore, the 4-cycle is the unique separating 4-cycle.

Proof. Suppose that the lemma is not true. Let $G_1 = G - int(C_1)$ and $G_2$ be the graph obtained
from $G - ext(C_1)$ by substituting $v_1v_2$ with $v_1w_1w_2w_3v_4$. Let $C_2 = v_1w_1w_2v_3v_2v_4v_1$.

Since $\sigma(G_1) < \sigma(G)$, $(G_1, C_0)$ is superextendable by the minimality of $G$. This means that $C_1$ is
colored and hence $C_2$ is colored. If $(G_2, C_2)$ is superextendable, then $(G, C_0)$ is superextendable, a
contradiction. Since $G \in \mathcal{G}$, no edge of $C_1$ is in any triangles. Therefore, $G_2 \in \mathcal{G}$. We now show
that $(G_2, C_2)$ is superextendable. For this goal, we need only to check that $\sigma(G_2) < \sigma(G)$. Note
that $\sigma(G_2) = \sigma(G - ext(C_1)) + 6$.

If $|C_0| = 7$, then $\sigma(C_0) - \sigma(C_0 \cap C_1) \geq 7$ as $C_1 \neq C_0$, and thus $\sigma(G_2) = \sigma(G - ext(C_1)) + 6 \leq
[\sigma(G) - (\sigma(C_0) - \sigma(C_0 \cap C_1))] + 6 < \sigma(G)$. Thus, we may assume that $|C_0| = 3$. 

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Lemma 3.6. Let \( G \) be a pair of diagonal vertices on a \(-\)face. If at most one of \( \{u, w\} \) is incident to a triangle, \( G[\{u, w\}] \subseteq G \).

Proof. Suppose that \( f = uvwx \). By Lemma 3.5, we may assume that \( w, x \notin C_0 \).

Since \( G \in G \), \( G \) has no 3-path joining \( u \) and \( w \), thus no new triangle can be obtained from the identification of \( u \) and \( w \). Since at most one vertex in \( \{u, w\} \) is incident to a triangle, the identification
of $u$ and $w$ produces no intersecting triangles. If $G[\{u, w\}]$ has a 5-cycle, then $G$ has a 5-path $P'$ joining $u$ and $w$. If one of $v$ and $x$ is in $P'$, then $b(f) \cup P'$ has a 5-cycle, a contradiction. So, $v, x \notin V(P')$, and hence either $P' \cup uwv$ or $P' \cup uxw$ is a separating 7-cycle; both contradict Lemma 3.2. Therefore, $G[\{u, w\}] \in \mathcal{G}$.

For convenience, let $f = v_1v_2...v_k$ have corresponding degrees $(d_1, d_2,...d_k)$.

**Lemma 3.7.** Let $f = uvwx$ be a 4-face in $F_4 \cup F'_4$. Then (1) if $b(f) \cap C_0 = \{u\}$, then each of $u$ and $w$ is incident to a triangle. (2) if $f \in F_4$ is a $(4^-,3^+,4^-,3^+)$-face, then each of $v$ and $x$ is incident to a triangle.

*Proof.* (1) Suppose otherwise that at most one of $u$ and $w$ is incident to a triangle. By Lemma 3.6, $G[\{u, w\}] \in \mathcal{G}$. By Lemma 3.5, $|N(w) \cap C_0| = 0$. Since $\sigma(G[\{u, w\}]) = \sigma(G) - 3$, $(G[\{u, w\}], C_0)$ is superextendable such that the color of $u(w)$ is different from $v$ and $x$. But then $(G, C_0)$ is superextendable, by coloring $u$ and $w$ with the color of $u(w)$ and keeping the colors of the other vertices, a contradiction.

(2) Suppose to the contrary that at most one of $v$ and $x$ is incident to a triangle. By Lemma 3.6, $G[\{v, x\}] \in \mathcal{G}$. Then $(G[\{v, x\}], C_0)$ is superextendable. Color $v, x$ with the color of $v(x)$ and keep the colors of the other vertices, we obtain a coloring of $(G, C_0)$, unless $v(x)$ is colored with 1 and $u$ (or $w$) is colored with 1 as well and one of the other two neighbors of $u$ (or $w$) is colored with 1. Note that $v$ and $x$ have no other common neighbors than $u$ and $w$ by Lemma 3.3. In this case, we recolor $u$ (or $w$) properly and get a coloring of $(G, C_0)$.

**Lemma 3.8.** Every 3-vertex in $\text{int}(C_0)$ has either a neighbor on $C_0$ or a $5^+$-neighbor.

*Proof.* Let $v \in \text{int}(C_0)$ be a 3-vertex with no neighbor on $C_0$. If all neighbors of $v$ have degree at most 4, by minimality of $G$, $(G - v, C_0)$ is superextendable. We may assume that all neighbors of $v$ are colored differently and $u$ be the neighbor of $v$ that is colored with 1. Then either two neighbors of $u$ are colored with 1, or $u$ is nicely colored. In the former case, we recolor $u$ with the color not in its neighbors and color $v$ with 1, and in the latter case, we color $v$ with 1, a contradiction.

We could say more on the degrees of the neighbors of a 3-vertex on a triangle. For a 3-vertex $u$, let $u'$ be the pendant neighbor of $u$ on a 3-face $f = uwv$.

**Lemma 3.9.** Let $f = uwv$ be a $(3,3,5^-)$-face in $G$ with $d(u) = d(v) = 3$. If $(b(f) \cup \{u'\}) \cap C_0 = \emptyset$, then $d(u') \geq 5$.

*Proof.* The result is true for all $(3,3,4^-)$-faces in $F_3$ by Lemma 3.8. So we may assume $d(w) = 5$ and $d(u') \leq 4$. By the minimality of $G$, $(G - \{u, v\}, C_0)$ is superextendable. Properly color $v$, and $u$ cannot be properly colored only if $u', v, w$ are colored differently. Note that $d(u') \leq 4$. If $u'$ is colored with 1, then either $u'$ is nicely colored or two neighbors of $u'$ are colored with 1. In the former case, we color $u$ with 1; in the latter case, we color $u'$ with the color not in its neighbors and color $u$ with 1. If $v$ is colored with 1, then we color $u$ with 1 as well. So we may assume that $w$ is colored with 1, then either $w$ is nicely colored or two of the three other neighbors of $w$ other than $u, v$ are colored with 1. In the former case, we color $u$ with 1; in the latter case, we recolor $w$ properly and color $u$ and $v$ with 1.

We define some special faces from $F_3$. First of all, $(3,4,4)$-faces and $(3,3,5^-)$-faces in $F_3$ are special. Then we use a recursive method to define special $(3,5,5)$-faces. The initial special $(3,5,5)$-faces are those $(3,5,5)$-faces whose two 5-vertices have 6 pendant $(3,3,5^-)$-faces or $(3,4,4)$-faces.
altogether; then a \((3, 5, 5)\)-face is \textit{special} if the two 5-vertices have 6 pendant \((3, 3, 5^-)\)-faces, or \((3, 4, 4)\)-faces, or initial or subsequent special \((3, 5, 5)\)-faces altogether. Clearly, special \((3, 5, 5)\)-faces are well-defined. We call a 3-face \textit{special} if it is a \((3, 3, 5^-)\)-face, or a \((3, 4, 4)\)-face, or a special \((3, 5, 5)\)-face.

The following is a technical lemma which we will use many times in the proofs of later lemmas.

**Lemma 3.10.** Let \(f = uvw\) be a special 3-face with \(d(u) = 3\) and \(u' \not\in C_0\). Then a desired coloring of \((G - \{u, u'\}, C_0)\) can be extended to the desired coloring of \(G - u'\) such that \(u\) is colored with 1.

**Proof.** Let \(f\) be a \((3, 3, 5^-)\)-face. Note that \(G - \{u, u'\}\) is \((2, 0, 0)\)-colorable. If \(w\) is not colored with 1, we can color \(u\) with 1. Thus, we may assume that \(w\) is colored with 1. If \(v\) is colored with 1, then \(w\) has at most one neighbor (other than \(u\) and \(v\)) which was colored with 1. In this case, we recolor \(v\) properly and then color \(u\) with 1. Thus, assume further that \(v\) is not colored with 1. The vertex \(u\) cannot be colored with 1 if and only if \(w\) has two neighbors (other than \(u\) and \(v\)) which are colored with 1. In this case, since \(d(w) \leq 5\), \(w\) can be nicely colored with 2 or 3 and then we recolor \(v\) properly and color \(u\) with 1.

Let \(f\) be a \((3, 4, 4)\)-face. If \(u\) cannot be colored with 1, then \(w\) or \(v\) is colored with 1. If both \(w\) and \(v\) are colored with 1, then either \(v\) or \(w\), say \(v\), has a neighbor (other than \(u\) and \(w\)) colored with 1. In this case, we recolor \(v\) properly and color \(u\) with 1. If \(v\) is colored with color 1 and \(w\) is not colored with 1, then \(v\) has two neighbors (other than \(u\) and \(w\)) colored with 1. Then we recolor \(v\) properly and color \(u\) with 1.

Let \(f = uww\) be a special \((3, 5, 5)\)-face. Assume first that \(f\) is an initial special \((3, 5, 5)\)-face, that is, its two 5-vertices have 6 pendant \((3, 3, 5^-)\) or \((3, 4, 4)\)-faces. We recolor \(u\) and \(w\), by the argument above, each of the six 3-vertices on pendant 3-faces that adjacent to \(v\) and \(w\) can be recolored with 1, then we can recolor \(v\) and \(w\) with 2 and 3, respectively, and color \(u\) with 1. Next, assume that \(f = uww\) is a subsequent special \((3, 5, 5)\)-face. Then by induction, the six neighbors of \(v\) and \(w\) on either previous pendant special \((3, 5, 5)\)-faces or other pendant special 3-faces can be recolored with 1. Thus, we can recolor \(v\) and \(w\) with 2 and 3, respectively, and then color \(u\) with 1.  

**Lemma 3.11.** Let \(v\) be a 4-vertex with neighbors \(v_1, v_2, v_3\) and \(v_4\). Then \(v\) cannot be incident to a \((3, 4, 5^-)\)-face \(f_1 = v_3v_4v_v\) and \((3, 4, 3, 5^+)\)-face \(f_2 = v_1vv_2w\) with \((b(f_1) \cup b(f_2)) \cap C_0 = \emptyset\).

**Proof.** Suppose otherwise. By the minimality of \((G, C_0)\), \((G - \{v, v_1, v_2, v_3\}, C_0)\) is superextendable. Properly color \(v_1, v_2\) and \(v_3\). Let both \(v_1\) and \(v_2\) be colored with 1. If one of \(v_3\) and \(v_4\) is colored with 1, then color \(v\) properly; if neither \(v_3\) and \(v_4\) is colored with 1, then color \(v\) with 1. Thus, we may assume that at most one of \(v_1\) and \(v_2\) is colored with 1. We can color \(v\) with 1, unless \(v_4\) is colored with 1 and two neighbors (other than \(v\) and \(v_3\)) of \(v_4\) are colored with 1, in which case, we recolor \(v_4\) properly and then \(v_3\) properly and color \(v\) with 1. So in either case, we have a contradiction.

**Lemma 3.12.** Let \(v \not\in C_0\) be a 5-vertex with neighbors \(v_i\), \(0 \leq i \leq 4\). Then each of the following holds.

1. \(v\) cannot be incident to a \((3, 5, 3^+)\)-face \(f = v_4v_0v_0v_1\) with \(d(v_0') \leq 4\) and \((b(f) \cup \{v_4'\}) \cap C_0 = \emptyset\) and adjacent to 3 pendant special 3-faces;
2. \(v\) cannot be adjacent to 4 pendant special 3-faces;
3. \(v\) cannot be incident to five 4-faces from \(F_1\) with at least three \((4^-, 3^+, 5, 5^+)\)-faces \(u_i\) such that at most one of \(v_i\) and \(v_{i+1}\) is incident with a triangle.
Proof. (1) Suppose otherwise that \( v \) is incident to a \((3,5,3^+)\)-face \( f = vuwv_0 \) with \( d(v') \leq 4 \) and \( (b(f) \cup \{v'_i\}) \cap C_0 = \emptyset \) and adjacent to 3 pendant special 3-faces from \( F_3 \). By the minimality of \( G \), \((G - \{v, v_1, v_2, v_3\}, C_0)\) is superextendable. By Lemma 3.10 we recolor \( v_1, v_2, v_3 \) with 1. If \( v \) cannot be colored, then, without loss of generality, \( v_4 \) and \( v_0 \) are colored with 2 and 3, respectively. Since \( d(v'_j) \leq 4 \), \( v'_4 \) can always be nicely colored with 1 or properly colored with 2 or 3. So we recolor \( v_4 \) by 1, then color \( v \) by 2, a contradiction.

(2) Suppose otherwise that \( v \) is adjacent to 4 pendant special 3-faces, and the pendant neighbors are \( v_1, v_2, v_3, v_4 \). By minimality of \( G \), \((G - \{v, v_1, v_2, v_3, v_4\}, C_0)\) is superextendable. By Lemma 3.10 \( v_i \) with \( i \in \{1, 2, 3, 4\} \) can be colored with 1. Then we can properly color \( v \), a contradiction.

(3) Suppose otherwise that such five 4-faces from \( F_4 \) exist. By the hypothesis, we assume, without loss of generality, that \( f_0 = u_0v_0v_1v_2 \) and \( f_2 = u_2v_2v_3v_4 \) are two 4-faces such that \( d(u_j) \leq 4 \) and at most one of \( v_i \) and \( v_{i+1} \) is incident with a triangle for \( j \in \{0, 2\} \). Let \( H = G\{v_0, v_1, v_2, v_3\} \). By Lemma 3.6 \( H \in \mathcal{G} \). By the minimality of \((G, C_0), (H, C_0)\) is superextendable. We now go back to color the vertices of \( G \). We color \( v_0 \) and \( v_1 \) with the color of \( v_0(v_1) \), and color \( v_2 \) and \( v_3 \) with the color of \( v_2(v_3) \), and keep the colors of the other vertices. The coloring is valid, unless the following two cases (by symmetry): (a) both \( v_0(v_1) \) and one neighbor of \( u_0 \) other than \( v_0(v_1) \) are colored with 1 in \( H \), or (b) all of \( v_0(v_1) \), \( v_2(v_3) \) and \( v \) are colored with 1 in \( H \) (there may be one neighbor of \( u_0 \) other than \( v_0(v_1) \) is colored with 1 in \( H \) or one neighbor of \( u_2 \) other than \( v_2(v_3) \) are colored with color 1 in \( H \)). In the former case, since \( d(u_0) \leq 4 \), we can recolor \( u_0 \) properly. In the latter case, if one neighbor of \( u_0 \) other than \( v_0(v_1) \) is colored with 1 in \( H \) or one neighbor of \( u_2 \) other than \( v_2(v_3) \) are colored with 1 in \( H \), we recolor \( u_0 \) or \( u_2 \) as in the former case, and then color \( v \) properly. □

Lemma 3.12 (3) tells us that if \( v \notin C_0 \) is a 5-vertex, then it cannot be incident to five 4-faces from \( F_4 \) with at least three \((3,3,5,5^+)\)-faces. Moreover, if \( v \) is incident to five 4-faces from \( F_4 \) with two \((3,3,5,5^+)\)-faces, then it cannot be incident to a \((3,4,5,5)\)-face from \( F_4 \).

Lemma 3.13. Let \( w \) be a 6-vertex with \( h \) pendant special 3-faces. Then \( w \) cannot be incident with a \((3,3,6)\)-face \( f = uvw \) such that \( \min\{d(u'), d(v')\} \leq 4 \) and \( (b(f) \cup \{u', v'\}) \cap C_0 = \emptyset \) and \( h = 4 \); In addition, if \( \max\{d(u'), d(v')\} \leq 4 \), then \( h \leq 2 \).

Proof. Suppose to the contrary that \( w \) is incident to a \((3,3,6)\)-face such that \( d(u') \leq 4 \) and \( (b(f) \cup \{u', v'\}) \cap C_0 = \emptyset \) and adjacent to four pendant special 3-faces. Let \( N(w) = \{u, v, w_1, w_2, w_3, w_4\} \). By the minimality of \( G \), \((G - (N(w) \cup \{w\}), C_0)\) is superextendable. If \( h = 4 \), then by Lemma 3.10 \( w_i \) can be colored by 1 for \( i \in \{1, 2, 3, 4\} \). Since \( d(u') \leq 4 \), \( u' \) can be nicely colored. Then we color \( u \) by 1 and color \( v \) and \( w \) properly to get a desired coloring of \( G \), a contradiction.

Assume that \( \max\{d(u'), d(v')\} \leq 4 \) and \( h \geq 3 \). As \( d(v') \leq 4 \), \( v' \) is nicely colored. Thus we color \( v \) with 1. Let \( w_4 \) be the vertex that may not be on a special 3-face. As in the proof above, \( w_i \) can be colored with 1 for \( i \in \{1, 2, 3\} \). Since \( d(u') \leq 4 \), \( u \) can be colored with 1. Then all neighbors of \( w \) except \( w_4 \) are colored with 1. So we properly color \( w \) to get a coloring of \( G \), a contradiction again. □

4. Discharging Procedure

In this section, we will finish the proof of the main theorem by a discharging argument. Let the initial charge of vertex \( u \in G \) be \( \mu(u) = 2d(u) - 6 \), and the initial charge of face \( f \neq C_0 \) be
\[ \mu(f) = d(f) - 6 \] and \( \mu(C_0) = d(C_0) + 6. \) Then
\[
\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = 0.
\]

Let \( h \) be the number of pendant special 3-faces of a vertex \( u \).

The discharging rules are as follows.

(R1) Let \( u \notin C_0 \). Then in (R1.1)-(R1.5) \( u \) gives charges only to incident or pendant faces that are disjoint from \( C_0 \), in the following ways:

(R1.1) \( d(u) = 4. \)

(R1.1.1) \( u \) gives \( \frac{3}{2} \) to each incident \((3, 4, 5^+)\)-face, and 1 to other incident 3-faces.

(R1.1.2) \( u \) gives 1 to the incident \((3, 4, 3, 5^+)\)-face if \( u \) is incident to a 3-face, otherwise gives \( \frac{1}{2} \) to each incident 4-face.

(R1.2) \( d(u) = 5 \).

(R1.2.1) \( u \) gives 1 to the incident 3-face if \( h = 3 \), \( \frac{3}{2} \) if \( h = 2 \), and 2 if \( h \leq 1 \).

(R1.2.2) \( u \) gives 1 to each incident \((3, 3, 5, 5^+)\)-face, and if \( u \) is incident with a 3-face then 1 to each incident 4-face; if \( u \) is not incident with a 3-face, then \( u \) gives \( \frac{3}{4} \) to each incident \((3, 4, 5, 5^+)\)-face, and \( \frac{1}{2} \) to each other incident 4-faces.

(R1.3) \( d(u) = 6 \), then \( u \) gives 3 if \( h \leq 2 \), \( \frac{5}{2} \) if \( h = 3 \), and 2 if \( h = 4 \), to each incident 3-face.

(R1.4) \( 7^+\)-vertex gives 1 to each pendant 3-face; 5- or 6-vertex gives 1 to each special pendant 3-face and \( \frac{1}{2} \) to each of the other pendant 3-faces.

(R1.5) \( 6^+\)-vertex gives 1 to each incident 4-face, and \( 7^+\)-vertex gives 3 to each incident 3-face.

(R1.6) \( 4^+\)-vertex which is incident to a triangle gives \( \frac{1}{2} \) to each incident 4-face from \( F'_4 \).

(R2) If \( u \in C_0 \), then \( u \) gives 1 to each incident 4-face from \( F''_4 \) or each pendant face from \( F'_3 \), \( \frac{3}{2} \) to each incident face from \( F''_3 \), and 3 to each incident face from \( F'_3 \).

(R3) \( C_0 \) gives 2 to each 2-vertex on \( F(G) \cup V(G) \) other than \( C_0 \) which has final charge \( \mu^*(x) \geq 0 \) and \( \mu^*(C_0) > 0 \).

First we consider faces. As \( G \) contains no 5-faces and \( 6^+\)-faces other than \( C_0 \) are not involved in the discharging procedure, we will first consider 3- and 4-faces other than \( C_0 \).

Let \( f \) be a 3-face. Note that \( f \) has initial charge \( 3 - 6 = -3 \). By Lemma 3.4, \( |b(f) \cap C_0| \leq 2. \) If \( |b(f) \cap C_0| = 1 \), then by (R2), \( \mu^*(f) \geq -3 + 3 = 0; \) if \( |b(f) \cap C_0| = 2 \), then by (R2), \( \mu^*(f) = -3 + \frac{3}{2} \times 2 = 0. \) So we may assume that \( b(f) \cap C_0 = \emptyset \). Let \( f = uvw \) with corresponding degrees \( (d_1, d_2, d_3) \).

(1) \( f \) is a \((3, 3, 5^-)\)-face. By Lemmas 3.8 and 3.9, the neighbors of \( f \) to the 3-vertices are either on \( C_0 \) or have degree at least 5. In latter case, \( f \) is a special pendant 3-face to them. Thus by (R2) or (R1.4), each of these neighbors gives 1 to \( f \), plus the 4- or 5-vertex on \( f \), if exists, gives at least 1 to \( f \) by (R1.1.1) or (R1.2.1), thus \( \mu^*(f) \geq -3 + 1 \times 3 = 0. \)

(2) \( f \) is a \((3, 3, 6)\)-face. Let \( w \) be the 6-vertex of \( f \). If \( f \) has a pendant neighbor on \( C_0 \), then by (R2), it gets 1 from \( C_0 \), and by (R1.3), gets at least 2 from \( w \). Thus, \( \mu^*(f) \geq -3 + 1 + 2 = 0. \) We assume that \( f \) has no pendant neighbors on \( C_0 \). If each of the 3-vertices’ pendant neighbors is of degree at most 4, then by Lemma 3.13, \( w \) is adjacent to at most 2 pendant 3-faces. Thus
Thus, we now assume that \( \mu^*(f) \leq -3 + 1 = -2 \). By (R1) and (R2), \( u \in C_0 \) and \( w \in C_0 \) and \( v \in C_0 \) and \( w \in C_0 \) and \( v \in C_0 \). Note that \( f \) has initial charge \( 4 - 6 = -2 \). By Lemma 3.3, \( |b(f) \cap C_0| \leq 2 \). If \( |b(f) \cap C_0| = 1 \), say \( u \in b(f) \cap C_0 \), then by (R2), \( u \) gives \( \frac{1}{2} \) to \( f \). By Lemma 3.7 each of \( u \) and \( w \) is incident to a triangle. So by (R1) \( w \) gives \( \frac{1}{2} \) to \( f \). So \( \mu^*(f) \geq -2 + \frac{1}{2} + \frac{1}{2} = 0 \). If \( |b(f) \cap C_0| = 2 \), then by (R2), \( \mu^*(f) = -2 + 1 \times 2 = 0 \). Thus, we now assume that \( b(f) \cap C_0 = \emptyset \).

Note that by Lemma 3.7 we only need to consider the following situations.

1. \( f \) is a \((3, 3, 5^+, 5^+)\)-face. By (R1.2) and (R1.5), \( f \) gets at least 1 from both \( w \) and \( x \). Thus, \( \mu^*(f) \geq -2 + 1 \times 2 = 0 \).
2. \( f \) is a \((3, 4^+, 3, 5^+)\)-face. By Lemma 3.7 both \( v \) and \( x \) are incident to triangles. Thus by (R1.1) and (R1.2) and (R1.5), \( f \) gets at least 1 from both \( v \) and \( x \). It follows that \( \mu^*(f) \geq -2 + 1 \times 2 = 0 \).
3. \( f \) is a \((3, 4^+, 4, 5^+)\)-face. By Lemma 3.7 both \( v \) and \( x \) are incident to triangles. Thus by (R1.1) and (R1.2) and (R1.5), \( f \) gets at least 1 from \( x \) and \( \frac{1}{2} \) from each of \( v \) and \( w \). It follows that \( \mu^*(f) \geq -2 + 1 + \frac{1}{2} \times 2 = 0 \).
4. \( f \) is a \((3, 4^+, 5^+, 5^+)\)-face. If \( d(x) = d(w) = 5 \), then by (R1.1) \( f \) gets \( \frac{1}{2} \) from \( v \) and by (R1.2) \( f \) gets at least \( \frac{3}{2} \) from both \( w \) and \( x \). This implies that \( \mu^*(f) \geq -2 + \frac{1}{2} + \frac{3}{2} = 0 \). Otherwise, by (R1.1), (R1.2) and (R1.5), \( \mu^*(f) \geq -2 + \frac{1}{2} + \frac{3}{2} + 1 > 0 \).
(5) $f$ is a $(3, 5^+, 5^+, 5^+)$-face. By (R1.2.2) and (R1.5), $f$ gets at least $\frac{2}{3}$ from each of the $5^+$-vertices, thus $\mu^*(f) \geq -2 + \frac{2}{3} \times 3 = 0$.

(6) $f$ is a $(4^+, 4^+, 4^+, 4^+)$-face. by (R1.1.2), (R1.2.2) and (R1.5), $f$ gets at least $\frac{1}{2}$ from each of the four vertices. Thus, $\mu^*(f) \geq -2 + \frac{1}{2} \times 4 = 0$.

Now we consider vertices. Note that $int(C_0)$ contains no $2^-$-vertices. For a vertex $u$, let $p$ be the number of 4-faces incident with $u$, $q$ be the number of pendant 3-faces adjacent to $u$ and $r$ be the number of 3-faces incident with $u$.

First let $u \notin C_0$. Note that if $d(u) = 3$ then $u$ is not involved in the discharging process thus $\mu^*(u) = \mu(u) = 0$.

1. $d(u) = 4$. If $u$ is not incident with any 3-face, then $u$ is incident with at most four 4-faces. By (R1.1.2), $\mu^*(u) \geq 2 - \frac{1}{2} \times 4 = 0$. Thus, we may assume that $u$ is incident with a 3-face. In this case, by our assumption, $u$ is incident with a 3-face and at most one 4-face. If the 3-face is not a $(3, 4, 5^+)$-face or the 4-face is not a $(3, 4, 3, 5^+)$-face, then by (R1.1.1), (R1.2.1) and (R1.6), $u$ gives at most $\max\{1 + 1, \frac{2}{3} + \frac{1}{3}\} = 2$ to the 3-face and the 4-face, thus $\mu^*(u) \geq 2 - 2 = 0$. If the 3-face $f_1$ is a $(3, 4, 5^-)$-face and the 4-face $f_2$ is a $(3, 4, 3, 5^+)$-face, then by Lemma 3.11 one of the vertices on the faces must be on $C_0$. By (R1.1.1) and (R1.6), either $u$ only gives at most $\frac{1}{2}$ to the $f_2$ (in this case $|b(f_1) \cap C_0| \geq 1$) or $u$ gives $\frac{2}{3}$ to $f_1$ and at most $\frac{1}{2}$ to $f_2$ (in this case $|b(f_1) \cap C_0| = 0$), thus $\mu^*(u) \geq 2 - \frac{2}{3} - \frac{1}{2} = 0$.

2. $d(u) = 5$. Assume first that $u$ is incident with a 3-face. In this case, $u$ is adjacent at most three pendant 3-faces or at most two incident 4-faces but not both. If $u$ is incident with two 4-faces, then by (R1.2.1), (R1.2.2) and (R1.6), $\mu^*(u) \geq 4 - 2 - 1 \times 2 = 0$. If $u$ is incident with one 4-face, then $u$ is adjacent at most one pendant 3-face. In this case, by (R1.2.1) and (R1.2.2), $\mu^*(u) \geq 4 - 2 - 1 - 1 = 0$. If $u$ is not incident with 4-face, let $u$ be adjacent to $h$ pendant special 3-faces. Then $u$ is adjacent to at most $3 - h$ pendant 3-faces (not special). If $h = 3$, then by (R1.2.1) and (R1.2.2), $\mu^*(u) \geq 4 - 1 - 3 \times 1 = 0$; if $h = 2$, then by (R1.2.1) and (R1.2.2), $\mu^*(u) \geq 4 - \frac{3}{2} - 2 \times 1 - \frac{1}{2} = 0$; if $h = 1$, then by (R1.2.1) and (R1.2.2), $\mu^*(u) \geq 4 - 2 - 2 \times 1 - \frac{1}{2} = 0$; if $h = 0$, then by (R1.2.1) and (R1.2.2), $\mu^*(u) \geq 4 - 2 - 3 \times \frac{1}{2} = \frac{5}{2} > 0$. Thus, we may assume that $u$ is not incident with any 3-face. If $u$ is incident with five 4-faces, then by Lemma 3.12 (3), $u$ is incident with at most two $(3, 3, 5, 5^+)$-faces. Moreover, if $u$ is incident with two $(3, 3, 5, 5^+)$-faces, it cannot be incident with $(3, 4, 5)$-faces. Thus, if $u$ is incident with at most one $(3, 3, 5, 5^+)$-face, then by (R1.2.2), $\mu^*(u) \geq 4 - 1 - \frac{3}{2} \times 4 = 0$; if $u$ is incident with two $(3, 3, 5, 5^+)$-faces, then by (R1.2.2), $\mu^*(u) \geq 4 - 1 \times 2 - \frac{3}{2} \times 3 = 0$. If $u$ is incident with $k(1 \leq k \leq 4)$ 4-faces, then $u$ is adjacent to at most $4 - k$ pendant 3-faces. By (R1.2.2) and (R1.4), $\mu^*(u) \geq 4 - k - (4 - k) = 0$. If $u$ is not incident with any 4-face, then by Lemma 3.12 (2), it adjacent to at most three pendant special 3-faces. So by (R1.4), $\mu^*(u) \geq 4 - 3 - \frac{4}{3} \times 2 = 0$.

3. $d(u) = 6$. If $u$ is incident with a 3-face, then $u$ is incident with at most three 4-faces or adjacent to at most four pendant 3-faces but not both. Thus, we need to consider the three cases when $h \in \{4, 3\}$ or $h \leq 2$. By (R1.3),(R1.5) and (R1.6), $\mu^*(u) \geq 6 - \max\{2 + 4, \frac{2}{3} + 3 + \frac{1}{3}, 3+1\} = 0$ . If $u$ is not incident with any 3-face, then by (R1.3) and (R1.5), $\mu^*(u) \geq 6 - 1 \times 6 = 0$.

4. $d(u) \geq 7$. Since $u$ is incident with at most one 3-face, $r \leq 1$. By (R1.4),(R1.5) and (R1.6), $u$ gives at most 1 to each incident 4-face and each pendant 3-face, and 3 to each of the $r$ incident 3-faces. Thus $\mu^*(u) \geq 2d(u) - 6 - (p+q+3r) \geq 2d(u) - 6 - (p+q+2r+1) > 2d(u) - 6 - (d(u)+1) \geq 0.$
Now we consider the case that \( u \in C_0 \). For \( l = 3, 4 \), by Lemma 3.4 each \( l \)-face \( f \) in \( G \) satisfies that \( |b(f) \cap C_0| \leq 2 \) and furthermore, when \( |b(f) \cap C_0| = 2, f \) and \( C_0 \) share a common edge.

1. \( d(u) = 2 \). By (R3), \( \mu^*(u) = 2 \times 2 - 6 + 2 = 0 \).

2. \( d(u) = 3 \). \( u \) is not incident with a face from \( F_3' \) or \( F_4' \). By (R2), \( \mu^*(u) \geq -\frac{3}{2} + \frac{3}{2} = 0 \).

3. \( d(u) = 4 \). Assume first that \( u \) is incident with a 3-face \( f \). If \( f \in F_3' \), then by (R2) and (R3), \( \mu^*(u) = 2 - 3 + 1 = 0 \). If \( f \in F_3'' \), then it incident to a 4-face from \( F_3'' \) or adjacent to a pendant 3-face from \( F_3 \); by (R2) and (R3), \( \mu^*(u) \geq 2 - \frac{3}{2} - 1 + 1 = \frac{1}{2} > 0 \). Thus we may assume \( u \) is not incident with any 3-face. By Lemma 3.7 \( u \) is not incident face from \( F_4' \). Thus we consider that \( u \) is incident with \( k(\leq 2) \) 4-faces from \( F_4'' \). Then \( u \) is adjacent to at most \( 2 - k \) pendant 3-faces from \( F_3 \). By (R2) and (R3), \( \mu^*(u) \geq 2 - k - (2 - k) = 0 \).

4. \( d(u) = k \geq 5 \). If \( u \) is not incident with any 3-face, then by Lemma 3.7 \( u \) is not incident face from \( F_4' \), so by (R2), \( \mu^*(u) \geq 2k - 6 - 1 \cdot (k - 2) \geq 1 > 0 \). Thus, we first assume that \( u \) is incident with a face from \( F_3' \). Let \( s \) be the number of 4-faces in \( F_3' \) incident with \( u \). If \( s = 0 \), then by (R2), \( \mu^*(u) \geq 2k - 6 - (k - 4) - 3 \geq 0 \); and if \( s \geq 1 \), then \( s \leq k - 4 \). By (R2), \( \mu^*(u) \geq 2k - 6 - \frac{3}{2}s - (k - s - 4) = k - \frac{3}{2}s - 5 \geq \frac{1}{2} \). Next, we assume that \( u \) is incident with a face from \( F_3'' \). By (R2), \( \mu^*(u) \geq 2k - 6 - (k - 3) - \frac{3}{2} \geq \frac{1}{2} \); if \( s \geq 1 \), then \( s \leq k - 4 \). By (R2), \( \mu^*(u) \geq 2k - 6 - \frac{3}{2}s - \frac{3}{2} - (k - s - 3) = k - \frac{3}{2}s - \frac{9}{2} \geq \frac{1}{2}s - \frac{1}{2} \geq 0 \).

Now we consider the outer-face \( C_0 \). Let \( t_i \) be the number of \( i \)-vertices on \( C_0 \), then \( d(C_0) \geq t_2 + t_3 + t_4 \). Note that \( d(C_0) \in \{3, 7\} \). By (R3),

\[
\mu^*(C_0) = d(C_0) + 6 - 2t_2 - \frac{3}{2}t_3 - t_4 \geq d(C_0) + 6 - \frac{3}{2}(t_2 + t_3 + t_4) - \frac{t_2}{2} \\
\geq d(C_0) + 6 - \frac{3}{2}d(C_0) - \frac{t_2}{2} = 6 - d(C_0) - \frac{t_2}{2}
\]

If \( d(C_0) = 3 \) or \( t_2 \leq 5 \), then \( \mu^*(C_0) \geq 0 \). Thus, we may assume that \( d(C_0) = 7 \) and \( (t_2, t_3, t_4) \in \{(6, 1, 0), (7, 0, 0)\} \). If \( t_2 = 7 \), then \( G = C_0 \) and it is trivially superextendable. If \( t_2 = 6 \) and \( t_3 = 1 \), then by (R3), \( C_0 \) gets 1 from the adjacent face which has degree more than 7, so \( \mu^*(C_0) \geq \frac{1}{2} > 0 \).

We have shown that all vertices and faces have non-negative final charges. Furthermore, the outer-face has positive charges, except when \( d(C_0) = 7 \) and \( t_2 = 5 \), in which there must be a face other than \( C_0 \) having degree more than 7, thus the face has positive final charge. So \( \sum_{x \in V(G) \cup F(G)} \mu^*(x) > 0 \), a contradiction.

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