Abstract
The problem of efficiently sampling from a set of (undirected) graphs with a given degree sequence has many applications. One approach to this problem uses a simple Markov chain, which we call the switch chain, to perform the sampling. The switch chain is known to be rapidly mixing for regular degree sequences. We prove that the switch chain is rapidly mixing for any degree sequence with minimum degree at least 1 and with maximum degree \(d_{\text{max}}\) which satisfies \(3 \leq d_{\text{max}} \leq \frac{1}{2} \sqrt{M}\), where \(M\) is the sum of the degrees. The mixing time bound obtained is only an order of \(n\) larger than that established in the regular case, where \(n\) is the number of vertices.

1 Introduction
The switch chain is a natural Markov chain for sampling from a set of graphs with a given degree sequence. Each move of the chain selects two distinct, non-incident edges edges uniformly at random and attempts to replace these edges by a perfect matching of the four endvertices, chosen uniformly at random. The proposed move is rejected if a multiple edge would be formed.

We call each such move a switch. Ryser [21] used switches to study the structure of 0-1 matrices. Markov chains based on switches have been introduced by Besag and Clifford [3] for 0-1 matrices (bipartite graphs), Diaconis and Sturmfels [7] for contingency tables and Rao, Jana and Bandyopadhyay [20] for directed graphs.

The switch chain is aperiodic and its transition matrix is symmetric. It is well-known that the switch chain is irreducible for any (undirected) degree sequence: see [19] [24].

In order for the switch chain to be useful for sampling, it must converge quickly to its stationary distribution. (For Markov chain definitions not given here, see [10].)

Cooper, Dyer and Greenhill [5] [6] showed that the switch chain is rapidly mixing for regular undirected graphs. Here the degree \(d = d(n)\) may depend on \(n\), the number of vertices. The mixing time bound is given as a polynomial in \(d\) and \(n\). Earlier, Kannan, Tetali and Vempala [13] investigated the mixing time of the switch chain for regular directed graphs (that is, \(d\)-in, \(d\)-out directed graphs) is rapidly mixing, again for any \(d = d(n)\). Miklós, Erdős and Soukup [18] proved that the switch chain is rapidly mixing on half-regular bipartite graphs; that is, bipartite degree sequences which are regular for vertices on one side of the bipartition, but need not be regular for the other.

The proofs of all these mixing results used a multi-commodity flow argument [22]. In each case, regularity (or half-regularity) was only required for one lemma, which we will call the critical lemma. This is a counting lemma which is used to bound the maximum load of the flow (see [5] Lemma 4), [9] Lemma 5.6] and [18] Lemma 6.15).

In Section 3 we give an alternative proof of the critical lemma which does not require regularity. This establishes the following theorem, extending the rapid mixing result from [5] to irregular degree sequences which are not too dense.

Given a degree sequence \(d = (d_1, \ldots, d_n)\), write \(\Omega(d)\) for the set of all (simple) graphs with vertex set \([n] = \{1, 2, \ldots, n\}\) and degree sequence \(d\). Recall that \(d\) is called graphical when \(\Omega(d)\) is nonempty. We restrict our attention to graphical sequences. Write \(d_{\text{min}}\) and \(d_{\text{max}}\) for the minimum and maximum degree in \(d\), respectively, and let \(M = \sum_{j=1}^{n} d_j\) be the sum of the degrees.

Theorem 1.1. Let \(d = (d_1, \ldots, d_n)\) be a graphical degree sequence such that \(d_{\text{min}} \geq 1\) and

\[3 \leq d_{\text{max}} \leq \frac{1}{4} \sqrt{M}.
\]

The mixing time \(\tau(\epsilon)\) of the switch Markov chain with state space \(\Omega(d)\) satisfies

\[
\tau(\epsilon) \leq \frac{1}{10} d_{\text{max}}^{14} M^6 \left( M \log(M) + \log(\epsilon^{-1}) \right).
\]
This result covers many different degree sequences, for example:

- sparse graphs with constant average degree and maximum degree a sufficiently small constant times $\sqrt{n}$,
- dense graphs with linear average degree and maximum degree a sufficiently small constant times $n$.

The mixing time bound given above is at most a factor of $n$ larger than that obtained in [4, 6] in the regular case. (To see this, substitute $M = d_{\text{max}}n$, which holds when $d$ is regular: note that $M \leq d_{\text{max}}n$ always holds, as $M$ is the sum of the degrees.)

We expect that our approach also applies to directed graphs, which should allow the rapid mixing proof from [9] to be extended to irregular directed degree sequences, under conditions analogous to those in Theorem 1.1.

1.1 Related work There are several approaches to the problem of sampling graphs with a given degree sequence, though none is known to be efficient for all degree sequences. The configuration model of Bollobás [1] gives expected polynomial time uniform sampling if $d_{\text{max}} = O(\sqrt{\log n})$. McKay and Wormald [10] adapted the configuration model to give an algorithm which performs uniform sampling from $\Omega(d)$ in expected polynomial time when $d_{\text{max}} = O(M^{1/4})$.

Jerrum and Sinclair [11] used a construction of Tutte's to reduce the problem of sampling from $\Omega(d)$ to the problem of sampling perfect matchings from an auxiliary graph. The resulting Markov chain algorithm is rapidly mixing if the degree sequence $d$ is stable: see [12]. Stable sequences are those in which small local changes to the degree sequences do not greatly affect the size of $|\Omega(d)|$. Many degree sequences which satisfy the conditions of Theorem 1.1 are stable; however, not all stable sequences satisfy the conditions of Theorem 1.1. (For example, if $d_{\text{min}} = n/9$ and $d_{\text{max}} = 4n/9$ then $d$ is stable [12] but then $\sqrt{M} \leq 2n/3$, which is not large enough for Theorem 1.1.)

Steger and Wormald [23] gave an easily-implementable algorithm for sampling regular graphs, and proved that their algorithm performs asymptotically uniform sampling in polynomial time when $d = o(n^{1/28})$ (where $d$ denotes the degree). Kim and Vu [14] gave a sharper analysis and established that $d = o(n^{1/3})$ suffices for efficient asymptotically uniform sampling. Bayati, Kim and Saberi [2] extended Steger and Wormald's algorithm to irregular degree sequences, giving polynomial-time asymptotically uniform sampling when $d_{\text{max}} = o(M^{1/4})$. From this they constructed a sequential importance sampling algorithm for $\Omega(d)$. Recently, Zhao [25] described and analysed a similar approach to that of [16], in a general combinatorial setting. Zhao shows that for sampling from $\Omega(d)$, when $d_{\text{max}} = o(M^{1/4})$, his algorithm performs asymptotically uniform sampling in time $O(M)$.

Finally we note that Barvinok and Hartigan [1] showed that the adjacency matrix of a random element of $\Omega(d)$ is “close” to a certain “maximum entropy matrix”, when the degree sequence is tame. The definition of tame depends on the maximum entropy matrix, but a sufficient condition is that $d_{\text{min}} \geq \alpha(n-1)$ and $d_{\text{max}} \leq \beta(n-1)$ for some constants $\alpha, \beta > 0$. Some degree sequences satisfying this latter condition are stable sequences, and many of these degree sequences also satisfy the condition of Theorem 1.1. It would be interesting to explore further the connections between stable degree sequences, tame degree sequences and the mixing rate of the switch Markov chain.

It is not known whether the corresponding counting problem (exact evaluation of $|\Omega(d)|$) is #P-complete. There are several results giving asymptotic enumeration formulae for $|\Omega(d)|$ under various conditions on $d$: see for example [1, 15, 17] and references therein.

2 The switch chain and multicommodity flow A transition of the switch chain on $\Omega(d)$ is performed as follows: from the current state $G \in \Omega(d)$, choose an unordered pair of two distinct non-adjacent edges uniformly at random, say $F = \{\{x, y\}, \{z, w\}\}$, and choose a perfect matching $F'$ from the set of three perfect matchings of (the complete graph on) $\{x, y, z, w\}$, chosen uniformly at random. If $F' \cap (E(G) \setminus F) = \emptyset$ then the next state is the graph $G'$ with edge set $(E(G) \setminus F) \cup F'$, otherwise the next state is $G'' = G$.

Define $M_2 = \sum_{j=1}^{n} d_j(d_j - 1)$. If $P(G, G') \neq 0$ and $G \neq G'$ then $P(G, G') = \frac{1}{\pi_{\text{st}}(d)}$, where

$$a(d) = \left(\frac{M/2}{2}\right) - \frac{1}{2} M_2$$

is the number of unordered pairs of distinct nonadjacent edges in $G$. This shows that the Markov chain is symmetric. The chain is aperiodic since by definition $P(G, G) \geq 1/3$ for all $G \in \Omega(d)$.

2.1 Multicommodity flow To bound the mixing time of the switch chain, we apply a multicommodity flow argument. Suppose that $G$ is the graph underlying a Markov chain $\mathcal{M}$, so that $xy$ is an edge of $G$ if and only if $P(x, y) > 0$. A flow in $G$ is a function $f: \mathcal{P} \rightarrow [0, \infty)$
such that
\[ \sum_{p \in \mathcal{P}_{xy}} f(p) = \pi(x)\pi(y) \quad \text{for all } x, y \in \Omega, \ x \neq y. \]

Here \(\mathcal{P}_{xy}\) is the set of all simple directed paths from \(x\) to \(y\) in \(G\) and \(\mathcal{P} = \bigcup_{x \neq y} \mathcal{P}_{xy}\). Extend \(f\) to a function on oriented edges by setting \(f(e) = \sum_{p \ni e} f(p)\), so that \(f(e)\) is the total flow routed through \(e\). Write \(Q(e) = \pi(x)P(x, y)\) for the edge \(e = xy\). Let \(\ell(f)\) be the length of the longest path with \(f(p) > 0\), and let \(\rho(e) = f(e)/Q(e)\) be the load of the edge \(e\). The maximum load of the flow is \(\rho(f) = \max_e \rho(e)\). Using Sinclair [22 Proposition 1 and Corollary 6'], the mixing time of \(M\) can be bounded above by
\[
\tau(\epsilon) \leq \rho(f) \ell(f) \left( \log(1/\pi^*) + \log(\epsilon^{-1}) \right)
\]
where \(\pi^* = \min\{\pi(x) | x \in \Omega\}\).

2.2 Defining the flow

The definition of the multicommodity flow given in [5] Section 2.1 carries across to irregular degree sequences without change. This is because the flow from \(G\) to \(G'\) depends only on the symmetric difference \(G \triangle G'\) of \(G\) and \(G'\), treated as a 2-edge-coloured graph (with edges from \(G \setminus G'\) coloured blue and edges from \(G' \setminus G\) coloured red, say). The blue degree at a given vertex equals the red degree at that vertex, but in general the blue degree sequence will not be regular. Hence the multicommodity flow definition given in [5] is already general enough to handle irregular degree sequences.

The multicommodity flow is defined using a process which we now sketch. Given \(G, G' \in \Omega(d)\):

- Define a bijection from the set of blue edges incident at \(v\) to the set of red edges incident at \(v\), for each vertex \(v \in [n]\). The vector of these bijections is called a pairing \(\psi\), and the set of all possible pairings is denoted \(\Psi(G, G')\).

- The pairing gives a canonical way to decompose the symmetric difference \(G \triangle G'\) into a sequence of circuits, where each circuit is a blue/red-alternating closed walk.

- Each circuit is decomposed in a canonical way into a sequence of simpler circuits of two types: 1-circuits and 2-circuits. A 1-circuit is an alternating cycle in \(G \triangle G'\), while a 2-circuit is an alternating walk with one vertex of degree 4, the rest of degree 2, consisting of two odd cycles which share a common vertex. Each 1-circuit or 2-circuit has a designated start vertex. (The start vertex of a 2-circuit is the unique vertex of degree 4.) An important fact is that the 1-circuits and 2-circuits are pairwise edge-disjoint.

- Each 1-circuit or 2-circuit is processed in a canonical way to give a segment of the canonical path \(\gamma_\psi(G, G')\).

For full details see [5] Section 2.1.

3 Analysing the flow

Now we show how to bound the load of the flow by adapting the analysis from [6]. Note that some proofs in [5] used the assumption \(d = d(n) \leq n/2\), since (for regular sequences) the general result follows by complementation. This trick does not work for irregular degree sequences, so we cannot make a similar assumption here.

Given matrices \(G, G'\), \(Z \in \Omega(d)\), define the encoding \(L\) of \(Z\) (with respect to \(G, G'\)) by
\[
L + Z = G + G'
\]
by identifying each of \(Z, G, G'\) with their symmetric 0-1 adjacency matrices. Then \(L\) is a symmetric \(n \times n\) matrix with entries in \(\{-1, 0, 1, 2\}\) and with zero diagonal. Entries which equal \(-1\) or \(2\) are called defect entries. Treating \(L\) as an edge-labelled graph with edges labelled \(-1, 1, 2\) (and omitting edges corresponding to zero entries), a defect edge is an edge labelled \(-1\) or \(2\).

(In [5] these were called “bad edges”.) Specifically, we will refer to \((-1)\)-defect edges and to 2-defect edges. A 2-defect edge is present in both \(G\) and \(G'\) but is absent in \(Z\), while a \((-1)\)-defect edge is absent in both \(G\) and \(G'\) but is present in \(Z\).

We say that the degree of vertex \(v\) in \(L\) is the sum of the labels of the edges incident with \(v\) (equivalently, the sum of the entries in the row of \(L\) corresponding to \(v\)). By definition, the degree sequence of \(L\) equals \(d\).

Some proofs from [5, 6] also apply in the irregular case without any substantial change (after replacing \(d\) by \(d_{\max}\)). These proofs refer only to the symmetric difference and the process used to construct the multicommodity flow (and none of them use the assumption \(d \leq n/2\)). We state two of these results now.

**Lemma 3.1.** Suppose that \(G, G', Z, Z' \in \Omega(d)\) are such that \((Z, Z')\) is a transition of the switch chain which lies on the canonical path \(\gamma_\psi(G, G')\) for some \(\psi \in \Psi(G, G')\). Let \(L\) be the encoding of \(Z\) with respect to \((G, G')\). Then the following statements hold:

(i) ([5 Lemma 1]) From \((Z, Z')\), \(L\) and \(\psi\) it is possible to uniquely recover \(G\) and \(G'\).

(ii) ([5 Lemma 2]) There are at most four defect edges in \(L\). The labelled graph consisting of the defect edges in \(L\) must form a subgraph of one of the five labelled graphs shown in Figure 7, where “?” represents a label which may be either \(-1\) or \(2\).
Next, if $y$ is incident with two edges of defect 2 then it must be that one is an odd chord for a 1-circuit $C_1$ and one is a shortcut edge for a 2-circuit $C_2$. Then $y$ is incident in $G$ with an edge of $C_1$, an edge of $C_2$ and the two edges \{x, y\}, \{y, z\} which are 2-defect edges in $L$. Since $C_1$ and $C_2$ are edge-disjoint and no defect edge belongs to $G \Delta G'$, it follows that $d_y \geq 4$, proving (ii).

We may adapt this argument to prove (iii), noting that a $(-1)$-defect may only arise from a shortcut edge or an odd chord which is absent in $G$ and $G'$ and present in $Z$.

Now we extend the term “encoding” to refer to any symmetric $n \times n$ matrix with entries in $\{-1, 0, 1, 2\}$ which has zero diagonal and row sums given by $d$. We say that an encoding $L$ is consistent with $Z$ if $L+Z$ only takes entries in $\{0, 1, 2\}$. Say that an encoding is valid if it satisfies the conclusions of Lemma 3.1(ii), and that a valid encoding is good if it also satisfies the conclusion of Lemma 3.2. Let $L(Z)$ be the set of valid encodings which are consistent with $Z$, and let $L^*(Z)$ be the set of good encodings which are consistent with $Z$. In $[5]$ the set $L(Z)$ was studied, but we require a bit more information about our encodings, so we will focus on the smaller set $L^*(Z)$.

**Lemma 3.3.** ([5, Lemma 5] and [6, Lemma 1]) The load $f(e)$ on the transition $e = (Z, Z')$ satisfies

$$f(e) \leq d_{\text{max}}^{14} \frac{|L^*(Z)|}{|\Omega(d)|}.$$  

**Proof.** In [5, Lemma 5] and [6, Lemma 1] it was shown that $f(e) \leq d^{14} |L(Z)|/|\Omega(d)|^2$ when $d = (d, d, \ldots, d)$ is a regular degree sequence. (The assumption $d \leq n/2$ is not used in this proof.) The proof relied on the fact that $L(Z)$ contains all encodings which may arise along a canonical path. But the same is true for $L^*(Z)$, by Lemma 3.1(ii) and Lemma 3.2 so the proof goes through without change in the irregular setting (after replacing $d$ by $d_{\text{max}}$).

The switch operation can be extended to encodings in the natural way: each switch reduces two edge labels by one and increases two edge labels by one, without changing the degrees. It was shown in [5, Lemma 3] that from any valid encoding, one could obtain a graph (with no defect edges) by applying a sequence of at most three switches. In [5, Lemma 4] we used this fact to bound the ratio $|L(Z)|/|\Omega(d)|$ for regular degree sequences. This provided an upper bound for the flow $f(e)$ through a transition $e = (Z, Z')$ (as in Lemma 3.3 above).

The proof of [5, Lemma 3] uses regularity to prove the existence of certain edges which are needed in order
to find switches to remove the defect edges. This argument fails for irregular degree sequences. Instead, we consider a slightly more complicated operation than a switch, which we call a 3-switch. (This operation is called a “circular C₆-swap” in [8].)

A 3-switch is described by a 6-tuple \((a_1, b_1, a_2, b_2, a_3, b_3)\) of distinct vertices such that \(a_1 b_1, a_2 b_2, a_3 b_3\) are all edges and \(a_2 b_1, a_3 b_2, a_1 b_3\) are all non-edges. The 3-switch deletes the three edges \(a_1 b_1, a_2 b_2, a_3 b_3\) from the edge set and replaces them with \(a_2 b_1, a_3 b_2, a_1 b_3\), as shown in Figure 2.

![Figure 2: A 3-switch](image)

Let \(\mathcal{L}(p, q)\) be the set of encodings in \(\mathcal{L}^c(Z)\) with precisely \(p\) defect edges labelled 2 and precisely \(q\) defect edges labelled -1, for \(p \in \{0, 1, 2\}\) and \(q \in \{0, 1, 2, 3\}\). Then \(\Omega(d) = \mathcal{C}(0, 0)\) and

\[
\mathcal{L}^c(Z) = \bigcup_{p=0}^{2} \bigcup_{q=0}^{3} \mathcal{C}(p, q),
\]

where this union is disjoint. (Note that \(\mathcal{C}(2, 3) = \emptyset\), by Lemma 3.1(ii).)

For \(v \in [n]\), given an encoding \(L\), write \(N_L(v)\) to denote the set of \(w \in [n] \setminus \{v\}\) such that \(L(v, w) = 1\). This is the set of neighbours of \(v\) in \(L\), where neighbours along defect edges are not included. If \(L \in \mathcal{C}(p, q)\) there are precisely \(M/2 - 2p + q\) non-defect edges in \(L\). (To see this, note that the sum of all entries in the matrix \(L\) must equal \(M\), and \(L\) has zero diagonal.)

**Lemma 3.4.** Suppose that \(d\) satisfies \(d_{\min} \geq 1\) and \(3 \leq d_{\max} \leq \frac{1}{4} \sqrt{M}\). Let \(Z \in \Omega(d)\). Then

\[
|\mathcal{L}^c(Z)| \leq \frac{1}{5} M^6 |\Omega(d)|.
\]

**Proof.** We prove that any \(L \in \mathcal{L}^c(Z)\) can be transformed into an element of \(\Omega(d)\) (with no defect edges) using a sequence of at most three 3-switches. The strategy is as follows: in Phase 1 we aim to remove two defects per 3-switch (one 2-defect and one \((-1)\)-defect), then in Phase 2 we remove one 2-defect per 3-switch, and finally in Phase 3 we remove one \((-1)\)-defect per 3-switch. There is at most one step in Phase 1, though the other phases may have more than one step: any phase may be empty. Each 3-switch we perform gives rise to an upper bound on certain ratios of the sizes of the sets \(\mathcal{C}(p, q)\), by double counting. The proof is completed by combining these bounds. (Such an argument is often called a “switching argument” in the asymptotic enumeration literature: see [17] for example.)

**Phase 1.** If \(p + q \leq 3\) then Phase 1 is empty: proceed to Phase 2. Otherwise, suppose that \(L \in \mathcal{C}(p, q)\) where \(p + q = 4\). Then \((p, q) \in \{(2, 2), (1, 3)\}\), and it follows from Figure 3 that there must be a vertex \(b_1\) which is incident with a 2-defect \(L(a_1, b_1) = 2\) and a \((-1)\)-defect \(L(a_2, b_1) = -1\). We count the number of 3-switches \((a_1, a_2, b_2, a_3, b_3)\) which may be applied to \(L\) to produce an encoding \(L' \in \mathcal{C}_{p-1, q-1}\). This operation is shown in Figure 3 where defect edges are shown using thicker lines: a thick solid line is a 2-defect edge while a thick dashed line is a \((-1)\)-defect edge.

![Figure 3: A 3-switch with \(L(a_1, b_1) = 2\), \(L(a_2, b_1) = -1\).](image)

Given \((a_1, b_1, a_2)\), there is at least one vertex \(b_2 \in N_L(a_2) \setminus \{a_1\}\). To see this, first suppose that \(a_2\) is not incident with a 2-defect. Then \(N_L(a_2)\) has at least \(d_{a_2} + 1 \geq 2\) elements, leaving at least one which is distinct from \(a_1\). Otherwise, if \(a_2\) is incident with a 2-defect then it can be incident with at most one 2-defect, since \(p \leq 2\). Then there are at least \(d_{a_2} - 2\) choices for \(b_2\) in \(N_L(a_2) \setminus \{a_1\}\), and this number is positive by Lemma 3.2(iii).

Next, we choose \((a_3, b_3)\) such that all six vertices are distinct, \(L(a_3, b_3) = 1\) and \(L(a_3, b_2) = L(a_1, b_1) = 0\). There are \(M - 4p + 2q\) possibilities for \((a_3, b_3)\) with \(L(a_3, b_2) = 1\), but we must reject those which are incident with the four vertices already chosen, or which are incident to a neighbour of \(a_1\) or \(b_2\). We need to be careful with \((-1)\)-defect edges. Hence, for all \(x \in [n]\), let \(\eta_x\) be the number of \((-1)\)-defect edges other than \(\{a_2, b_1\}\) which are incident with \(x\) in \(L\). Then \(\sum_{x \in [n]} \eta_x \leq 4\) since there are at most two more \((-1)\)-defect edges in \(L\). Furthermore, \(\eta_{a_1} + \eta_{b_2} \leq 3\). The
number of bad choices for \((a_3, b_3)\) is at most
\[
2 \left( |N_L(b_1)| + \sum_{x \in N_L(a_1)} |N_L(x)| + \sum_{y \in N_L(b_2)} |N_L(y)| \right).
\]
To see this, note that \(a_2 \in N_L(b_2)\) so all edges incident with \(a_2\) are counted in the final sum (with \(y = a_2\)). Furthermore, for each \(x \in N_L(a_1)\), the edge from \(a_1\) to \(x\) is among those counted by \(|N_L(x)|\), so the first sum covers all edges incident with \(a_1\) or a neighbour of \(a_1\) (and similarly for the second sum). Hence the number of bad choices for \((a_3, b_3)\) is at most
\[
2 \left( d_{b_1} - 1 + \eta_{a_1} + \sum_{x \in N_L(a_1)} (d_x + \eta_x) \right)
+ \sum_{y \in N_L(b_2)} (d_y + \eta_y)
\]
\[
\leq 2 \left( d_{\text{max}} - 1 + \eta_{a_2} + \eta_{b_1} \right)
+ d_{\text{max}} (d_{a_1} + d_{b_2} - 2 + \eta_{a_1} + \eta_{b_2})
+ \sum_{x \notin \{a_1, b_1, a_2, b_2\}} 2 \eta_x
\]
\[
\leq 2 \left( 2d_{\text{max}}^2 + 2d_{\text{max}} + 1 \right).
\]
The final inequality follows from setting \(\eta_{a_1} + \eta_{b_2} = 3\), the maximum possible, and letting \(\eta_x = 1\) for some \(x \notin \{a_1, b_1, a_2, b_2\}\) (as well as bounding \(d_{a_1}\) and \(d_{b_2}\) by \(d_{\text{max}}\)).

Hence, the number of possible 3-switches \((a_1, b_1, a_2, b_2, a_3, b_3)\) such that \(L(a_1, b_1) = 2\) and \(L(a_1, b_3) = -1\) is at least
\[
(3.3) \quad M - 4p + 2q - 2 \left( 2d_{\text{max}}^2 + 2d_{\text{max}} + 1 \right)
\geq M - 2 \left( 2d_{\text{max}}^2 + 2d_{\text{max}} + 3 \right)
\geq M - 6d_{\text{max}}^2
\geq M/2
\]

since \(3 \leq d_{\text{max}} \leq \sqrt{M} / 4\). Each such 3-switch produces an encoding \(L' \in \mathcal{C}(p-1, q-1)\).

Now we consider the reverse of this operation, which is given by reversing the arrow in Figure 3. Given \(L' \in \mathcal{C}(p-1, q-1)\), we need an upper bound on the number of 6-tuples \((a_1, b_1, a_2, b_2, a_3, b_3)\) such that \(L'(a_1, b_1) = L'(a_3, b_3) = 0\) and \(L'(a_2, b_2) = L'(a_3, b_1) = 1\). Since the encoding \(L \in \mathcal{C}(p, q)\) produced by this reverse operation must be consistent with \(Z\), it follows that \(\{a_2, b_1\}\) must be an edge of \(Z\). Hence there are precisely \(M\) choices for \((a_2, b_1)\). There are at most \(d_{a_1} + \eta_{a_1}\) ways to choose \(a_1 \in N_L(b_1)\) and at most \(d_{b_1} - 1 + \eta_{b_1}\) ways to choose \(b_3 \in N_L(a_1) \setminus \{b_1\}\).

From Figure 4 if \(\eta_{a_1} = 2\) then \(\eta_{b_1} = 0\), and if \(\eta_{b_1} = 1\) then \(\eta_{a_1} \leq 1\). Furthermore, \(\eta_{b_1} \leq 1\). (Otherwise, the reverse switching would produce an encoding which is not valid.) Therefore,
\[
(d_{b_1} + \eta_{b_1})(d_{a_1} - 1 + \eta_{a_1}) \leq d_{\text{max}} (d_{\text{max}} + 1)
\leq 4 / 3 d_{\text{max}}^2.
\]

Finally we must choose \((a_3, b_2)\) such that \(L(a_3, b_2) = 1\), the vertices \(a_3, b_2\) are distinct from the four vertices chosen so far and \(L'(a_2, b_2) = L'(a_3, b_1) = 0\). When \((p, q) = (2, 2)\) we ignore all conditions except \(L(a_3, b_2) = 1\), and take
\[
M - 4(p-1) + 2(q-1) = M - 2 \leq M
\]
as an upper bound for the number of good choices of \((a_3, b_2)\). When \((p, q) = (1, 3)\) there are no 2-defects in \(L'\), as \(L' \in \mathcal{C}(0, 2)\), so there are at most
\[
M - 4(p-1) + 2(q-1) - (d_{a_1} + d_{b_1} + d_{a_2} + d_{b_3})
\leq M - 4p + 2q - 2
\leq M
\]
good choices for \((a_3, b_2)\). (The existence of any additional \((-1)\)-defect edges incident with \(a_1, b_1, a_2\) or \(b_3\) can only help here.) Hence the number of ways to apply the reverse operation to \(L' \in \mathcal{C}(p-1, q-1)\) to produce a consistent encoding \(L \in \mathcal{C}(p, q)\) is at most \(4 / 3 d_{\text{max}}^2 M^2\).

Combining this with (3.3) shows that whenever \(p + q = 4\), by double counting,
\[
(3.4) \quad \frac{|\mathcal{C}(p, q)|}{|\mathcal{C}(p-1, q-1)|} \leq \frac{8}{3} d_{\text{max}} M.
\]

**Phase 2.** Once Phase 1 is complete, we have reached an encoding \(L \in \mathcal{C}(p, q)\) with \(p + q \leq 3\). If \(p = 0\) then Phase 2 is empty: proceed to Phase 3. Otherwise, we have \((p, q) \in \{(2, 1), (2, 0), (1, 2), (1, 1), (1, 0)\}\). We count the number of ways to perform a 3-switch to reduce the number of 2-defect edges by one, as shown in Figure 4.

Choose an ordered pair \((a_1, b_1)\) such that \(L(a_1, b_1) = 2\), in \(2p\) ways. The number of ways to
choose the ordered pair \((a_2, b_2)\) such that \(a_1, b_1, a_2, b_2\) are all distinct, \(L(a_2, b_2) = 1\) and \(L(a_2, b_1) = 0\), is at least

\[
M - 4p + 2q - 2\left(\left|N_L(a_1)\right| + \sum_{x \in N_L(b_1)} |N_L(x)|\right)
\]

\[
\geq M - 4p + 2q - 2\left(d_{a_1} - 2 + \eta_{a_2} + \sum_{x \in N_L(b_1)} (d_x + \eta_x)\right)
\]

\[
\geq M - 2\left(d_{a_1} + 2 + d_{a_2} + \eta_{a_1} + \sum_{x \in N_L(b_1)} (d_x + \eta_x)\right)
\]

\[
\geq M - 2\left(d_{a_1} + 2 + \eta_{a_1} + \eta_{b_1}
\right.
\]

\[
+ d_{a_2} (d_{a_1} + d_{a_2} - 2 + \eta_{a_1} + \eta_{b_1})
\]

\[
+ \sum_{x \notin \{a_1, b_1, a_2, b_2\}} 2\eta_x
\]

\[
\geq M - 2\left(2d_{\text{max}} + 2d_{\text{max}} + 4\right)
\]

\[
\geq M - 8d_{\text{max}}^2,
\]

ways, arguing as above. (Again, the worst case is when \(\eta_{a_1} + \eta_{b_2} = 3\) and \(\eta_{b_2} = 1\) for some \(x \notin \{a_1, b_1, a_2, b_2\}\).) Hence there are at least

\[
2(8 - 4d_{\text{max}}^2)(M - 8d_{\text{max}}^2) \geq \frac{1}{2} M^2
\]

such choices for \((a_1, b_1, a_2, b_2, a_3, b_3)\), using the stated upper bound on \(d_{\text{max}}\).

For the reverse operation, let \(L' \in C(p - 1, q)\) where \((p, q) \in \{(2, 1), (2, 0), (1, 2), (1, 1), (1, 0)\}\). We need an upper bound on the number of 6-tuples \((a_1, b_1, a_2, b_2, a_3, b_3)\) with \(L(a_1, b_1) = L(a_2, b_1) = L(a_3, b_1) = 1\) and \(L(a_2, b_2) = L(a_3, b_2) = 0\).

There are at most \(M - 4p + 2q \leq M\) choices for \((a_1, b_1)\) with \(L(a_1, b_1) = 1\), and then there are at most

\[
(d_{a_1} - 1 + \eta_{a_1})(d_{b_1} - 1 + \eta_{b_1}) \leq d_{\text{max}}^2
\]

choices for \((a_2, b_3)\). This uses the fact that there are at most two defect edges in \(L'\), and hence \(\eta_{a_1} + \eta_{b_1} \leq 2\), by choice of \((a_1, b_1)\). Finally there are at most \(M - 4p + 2q \leq M\) choices for \((a_3, b_2)\), so the number of 6-tuples where the reverse operation can be performed is at most \(d_{\text{max}}^2 M^2\).

Combining this with (3.5), it follows that

\[
|C(p, q)| \leq 2d_{\text{max}}^2
\]

holds for \((p, q) \in \{(2, 1), (2, 0), (1, 2), (1, 1), (1, 0)\}\).

**Phase 3.** After Phase 2, we may suppose that \(p = 0\). Let \(L \in C(0, q)\) where \(q \in \{1, 2, 3\}\). We count the number of 6-tuples \((a_1, b_1, a_2, b_2, a_3, b_3)\) where a 3-switch can be performed with \(L(a_2, b_1) = -1\). Performing this 3-switch will produce \(L' \in C(0, q - 1)\), as illustrated in Figure 5.

![Figure 5: A 3-switch with \(L(a_2, b_1) = -1\).](image-url)
for \((a_3, b_3)\) is at least
\[
M + 2q - 2 \left( \sum_{x \in N_L(a_1)} |N_L(x)| + \sum_{y \in N_L(b_2)} |N_L(y)| \right)
\geq M + 2q - 2 \left( \sum_{x \in N_L(a_1)} (d_x + \eta_x) + \sum_{y \in N_L(b_2)} (d_y + \eta_y) \right)
\geq M - 2 \left( d_{\max}(2d_{\max} + \eta_a + \eta_b) - 1 + \sum_{x \in \{a_1, b_2\}} 2\eta_x \right)
\geq M - 2 \left( 2d_{\max}^2 + 3d_{\max} + 1 \right)
\geq M - 8d_{\max}^2.
\]

The penultimate line follows by substituting \(\eta_a + \eta_b = 3\) and letting \(\eta_x = 1\) for some \(x \notin \{a_1, b_1, a_2, b_2\}\). Hence the number of 3-switches which can be performed in \(L\) to reduce the number of 2-defects by exactly one is at least
\[
2q(d_{b_1} + 1) d_{a_2} (M - 8d_{\max}^2) \geq 4q(M - 8d_{\max}^2)
\geq 2M,
\]
using the given bounds on \(d_{\max}\).

For the reverse operation, let \(L' \in C(0, q - 1)\), where \(q \in \{1, 2, 3\}\). We need an upper bound on the number of 6-tuples such that \(L(a_1, b_3) = L(a_3, b_2) = 1\), \(L(a_1, b_1) = L(b_1, a_2) = L(a_2, b_2) = L(a_3, b_3) = 0\) and \(\{a_2, b_1\}\) is an edge of \(Z\). There are at most \(M\) choices for \((a_2, b_1)\) satisfying the latter condition, then at most \(M + 2(q - 1) - 2(d_{a_2} + d_{b_1}) \leq M\) ways to choose \((a_3, b_2)\) with \(L(a_3, b_2) = 1\) and \(a_1, a_3, b_2, b_3\) all distinct. Similarly, there at most \(M\) ways to choose \((a_1, b_3)\). Hence the number of reverse operations is at most \(M^3\).

Combining this with (3.7) shows that
\[
\frac{|C(0, q)|}{|C(0, q - 1)|} \leq \frac{1}{2} M^2
\]
holds for \(q \in \{1, 2, 3\}\), by double counting.

**Consolidation.** Define
\[
B_{(2,-1)} = \frac{8}{3} d_{\max}^2 M, \quad B_{(2)} = 2d_{\max}^2, \quad B_{(-1)} = \frac{1}{2} M^2.
\]
It follows from (3.4)–(3.8) that
\[
\frac{|L^*(Z)|}{|\Omega(d)|} = \sum_{p=0}^{2} \sum_{q=0}^{3} \frac{|C(p, q)|}{|C(0, 0)|} \leq 1 + B_{(2)} + B_{(2)}^2 + B_{(-1)} + B_{(2)} B_{(-1)} B_{(2)}^2 
+ B_{(-1)} B_{(2)} B_{(2)} + B_{(-1)} B_{(2)}^2 + B_{(-1)}^2 B_{(2)} + B_{(-1)}^2 B_{(2)}^2 
+ B_{(-1)} B_{(2)} + B_{(-1)}^2 
\leq \frac{1}{5} M^6,
\]
using the upper bound on \(d_{\max}\) and the fact that \(M \geq 144\). This completes the proof of Lemma 3.4 \[\blacksquare\]

Since \(M \leq d_{\max} n\), the bound \(\frac{1}{5} M^6\) is at most a factor \(n/10\) bigger than the analogous bound \(2d^6 n^5\) given in [3, Lemma 4] in the regular case.

Finally we can prove Theorem 1.1.

**Proof.** (Proof of Theorem 1.1) We wish to apply (2.2). It follows from the configuration model (see [17, Equation (1)]) that the set \(\Omega(d)\) has size
\[
|\Omega(d)| \leq \frac{M!}{2M/2!} \prod_{j=1}^{d_j} d_j! \leq \exp \left( \frac{1}{2} M \log(M) \right).
\]
Hence the smallest stationary probability \(\pi^*\) satisfies
\[
\log(1/\pi^*) = \log(|\Omega(d)|) \leq M \log(M).
\]
Next, \(\ell(f) \leq M/2\) since each transition along a canonical path replaces an edge of \(G\) by an edge of \(G'\).

Finally, if \(e = (Z, Z')\) is a transition of the switch chain then \(1/Q(e) = 6a(d) \leq M^2\), using (2.1). Combining this with Lemmas 3.3 and 3.4 gives \(\rho(f) \leq \frac{1}{5} d_{\max}^6 M^6\). Substituting these expressions into (2.2) gives the claimed bound on the mixing time. \[\blacksquare\]

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