STABILITY STRATEGIES OF MANUFACTURING-INVENTORY SYSTEMS WITH UNKNOWN TIME-VARYING DEMAND

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ABSTRACT. For a manufacturing-inventory system, its stability and robustness are of particular important. In the literature, most manufacturing-inventory models are constructed based on deterministic demand assumption. However, demands for many real-world manufacturing-inventory systems are non-deterministic. To minimize the gap between theory and practice, we construct two models for the inventory control problem involving multi-machine and multi-product manufacturing-inventory systems with uncertain time-varying demand, where physical decay rate and shelf life are accounted for in the models. We then design state feedback control strategies to stabilize such systems. Based on the Lyapunov stability theory, sufficient conditions for robust stability, stabilization and control are derived in the form of linear matrix inequalities. Numerical examples are presented to show the potential applications of the proposed models.

1. Introduction. Mathematical modelling has been utilized as a tool in the study of production-inventory systems in manufacturing enterprises. Analyzing such production-inventory models is essential for the understanding of the mechanisms of such systems and the development of appropriate control strategies. System dynamics methodologies are used in [10] in developing and analyzing production-inventory models. Mathematical models in the form of differential/difference equations are constructed in [1, 2, 14, 16, 19, 20, 22, 32] to investigate the behaviors of these production-inventory systems. Control theory is used to study the control structure for production-inventory systems in [18]. Recently, control-oriented
techniques, such as Laplace transform, Z- transform, optimal control and model predictive control, are proposed and applied in [6, 12, 13, 17, 24, 25, 26, 31] to address inventory management problems in production-inventory systems. In practice, one of the most important objectives of a controller in a production-inventory control system is to maintain the stability of the system in the face of exterior disturbances. In the literature, different stability conditions and their implications are obtained for different production inventory systems under varied settings [3, 7, 27, 28, 29, 30].

The demand functions in the above models are assumed to be either constant functions or deterministic time-varying functions, i.e., the demand is deterministic and known a priori. However, in real-world production-inventory systems, their demands are uncertain. Thus, the decision-making process for production and control under deterministic demand assumption is unlikely to be acceptable. In the literature, uncertainties are often modeled in stochastic framework. For example, an optimal production-inventory control system with stochastic manufacturing is investigated in [9], where the demand, the production capacity, and the processing time per unit are taken as random variables. In [23], a model for a serial production line with failure-prone machines and random demand is constructed using optimal control theory. In the literature, most uncertain demands in production-inventory systems are described in terms of stochastic processes with given probability distributions. However, in a real-world production-inventory system, it may not be able to identify the underlying probability distribution. In fact, such a probability distribution may not even exist. Thus, robust control theory is applied in [8] to the study of production-inventory systems with uncertain demands. Boukas et al. [4] designed a memoryless linear state feedback controller for an inventory-production delay system with uncertain demand. However, the shelf life of each product is not considered in the model.

In real life situations, the type of product of a production-inventory system has a significant influence on the dynamical behavior of the system. In particular, perishable product has shelf life, which should be taken into consideration when the model is being constructed. However, to the best of our knowledge, the shelf life of a perishable product has not been considered in production-inventory models in the literature.

In this paper, we investigate the inventory control for multi-machine, multi-product manufacturing-inventory systems with uncertain time-varying demand, where physical decay rate and shelf life are taken into account in the models. Two models are being constructed. The first model considers the production and inventory of perishable foods, while the second model looks into durable goods. State feedback control strategies are designed to stabilize these systems.

The reminder of the paper is organized as follows. In Section 2, the models describing the production and inventory of perishable products and durable products with unknown time-varying demand are presented. In Section 3, the stabilization strategies and the control of production-inventory systems for perishable products and durable products are derived. In Section 4, numerical examples are provided to demonstrate the effectiveness of the proposed methods. Conclusions are drawn in Section 5.
2. Model formulation. We consider a manufacturing system with one machine and one type of product. Let $v(t)$ and $d(t)$ denote the production rate of the manufacturing system and its demand rate, respectively. We assume that the product has physical decay rate $\beta$.

2.1. Model of perishable products. Perishable products have their shelf lives. We let $x(t)$ represent the stock level of perishable products within their shelf lives at time $t$. The dynamics of the stock level can be described by the following differential equation:

$$x'(t) = -\beta x(t) + v(t) - d(t), \quad x(t) = 0, \quad t = t_0 = 0, \quad t \neq t_k,$$

where $t_k$, $k = 1, 2, \ldots$ denote the respective expiry times of different batches, and $x_0$ is the initial stock level at time $t = 0$.

The expired perishable products need to be disposed of. Here, we assume that the manufacturer collects expired products in the warehouse and these expired products are disposed of at time $t = t_k$, $k = 1, 2, \ldots$. Such activities can be catered for by incorporating impulses in the model \[11, 21\]. The impulse is given by

$$\Delta x(t) = -\gamma_k x(t), \quad t = t_k,$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = \lim_{t \to t_k^\pm} x(t)$. $\gamma_k$, $k = 1, 2, \ldots$, denote the disposal rates, representing the percentages of the stock level that are expired.

Let $v(t)$ denote the production rate. In fact, each producer has limited inventory capacity, which affects the production capacity of the corresponding producer. We thus assume that the following standard constraint is satisfied

$$0 \leq v(t) \leq \bar{v}x(t),$$

i.e., the manufacturing system has a limited capacity. Here, $\bar{v}$ is a given positive number. The demand rate $d(t)$ is given by

$$d(t) = m + \omega(t),$$

where $m$ is a given constant and $\omega(t)$ is an energy bounded disturbance from $L_2[0, +\infty]$.

For a production-inventory system with several machines, producing several different kinds of products, the corresponding model is formulated as follows:

$$\begin{cases}
x'(t) = Ax(t) + Bv(t) - D + E\omega(t), \quad t \neq t_k, \\
\Delta x(t) = D_k x(t), \quad t = t_k, \\
x(t) = 0, \quad t = t_0 = 0
\end{cases}$$

where $x(t)$, $v(t)$, $D$ and $\omega(t)$ are vectors, standing for the stock levels of different products at time $t$, the production rates, the demand rates, and the disturbances on demand rate. $A$, $B$, $E$ are $n \times n$ constant matrices, $D$ is a constant vector, and $A = \{a_{ij}\}$, $0 < a_{ij} < 2$, $B = \{b_{ij}\}$, $0 < b_{ij} < 2$, $i, j = 1, 2, \ldots, n$. $D_k = \{d_{ij}^k\}$, and $-1 < d_{ij}^k < 0$, $i, j = 1, 2, \ldots, n$. It thus follows from (2.3) that there exists a positive scalar $\lambda$ such that

$$\|v(t)\|_2 \leq \lambda\|x(t)\|_2, \quad \text{for every } t.$$

Define

$$B_0u(t) = Bv(t) - D.$$

$$\text{(2.6)}$$
Then system (2.4) becomes
\[
\begin{cases}
x'(t) = Ax(t) + B_0u(t) + E\omega(t), \quad t \neq t_k, \\
\Delta x(t) = D_k x(t), \quad t = t_k, \\
x(t) = 0, \quad t = t_0 = 0,
\end{cases}
\] (2.7)
and it follows from (2.5) that there exists a positive \(\mu\) such that
\[\|u(t)\|_2 \leq \mu\|x(t)\|_2, \text{ for every } t.\] (2.8)

2.2. Model of durable products. Since durable products have no shelf life, manufacturing-inventory model of such products can be described by the following differential equation
\[
\begin{cases}
x'(t) = Ax(t) + B_0u(t) + E\omega(t), \\
x(t) = 0, \quad t = t_0 = 0,
\end{cases}
\] (2.9)
where \(x(t)\) and \(\omega(t)\) are vectors, standing for the stock levels of different kinds of products at time \(t\), and the disturbances on demand rate. \(u(t)\) is defined by (2.6). \(A, B_0\) and \(E\) are \(n \times n\) constant matrices.

Next, we recall the following definitions and lemmas which will be used in the proof of our main results.

**Definition 2.1** ([11]). System (2.7) (or (2.9)) is said to be asymptotically stable if the solution of (2.7) (or (2.9)) with \(\omega(t) = 0\) is Lyapunov stable and there exists \(\delta > 0\) such that if \(\|x(t_0)\| < \delta\) then \(\lim_{t \to \infty} \|x(t)\| = 0\).

**Lemma 2.1** ([11]). Assume system (2.7) (or (2.9)) has a point of equilibrium at \(x = 0\) when \(\omega(t) = 0\). Consider a function \(V(x) : \mathbb{R}^n \to \mathbb{R}\) such that
\(1\) \(V(x)\) is positive definite. That means \(V(x) = 0\) if and only if \(x = 0\); \(V(x) > 0\) if and only if \(x \neq 0\).
\(2\) \(V'(x) = \frac{dV(x(t))}{dt} < 0\) for all values of \(x \neq 0\).
Then system (2.7) (or (2.9)) is asymptotically stable.

**Lemma 2.2** ([15]). Let \(G\) be a \(p \times q\) matrix, and \(G^TG \leq I\). Then, for every \(x \in \mathbb{R}^p, y \in \mathbb{R}^q\),
\[
2x^TGy \leq x^T x + y^T y.
\] (2.10)
When \(G = I\), (2.10) turns into \(2x^Ty \leq x^T x + y^T y\).

**Lemma 2.3** ([15]). Let \(P\) be a \(n \times n\) positive definite matrix, \(Q\) be an \(n \times n\) symmetric matrix. Then, for every \(x \in \mathbb{R}^n\),
\[
\lambda_{\min}(P^{-1}Q)x^TPx \leq x^T Qx \leq \lambda_{\max}(P^{-1}Q)x^TPx,
\] (2.11)
where \(\lambda_{\min}(P^{-1}Q)\) and \(\lambda_{\max}(P^{-1}Q)\) represent the minimum and maximum eigenvalue of \(P^{-1}Q\), respectively.

**Lemma 2.4** ([5]). (Schur Complement Theorem) Let \(M \in \mathbb{R}^{m \times n}\) be a symmetric positive definite matrix, \(N \in \mathbb{R}^{n \times n}\) be a symmetric matrix, and \(S \in \mathbb{R}^{m \times n}\). Then
\[
\begin{pmatrix} M & S \\ ST & N \end{pmatrix} > 0 \iff N - S^TM^{-1}S > 0.
\] (2.12)
3. Stability strategies. Suppose that the demand $\omega(t)$ is unknown and fluctuating. Our aim is to design an appropriate controller $u(t)$ for system (2.7) (or system (2.9)) such that the closed-loop system under this controller is asymptotically stable for $\omega(t) = 0$ and the mapping from an exogenous input $\omega(t)$ to a controller output

$$z(t) = Gx(t) + Hu(t)$$

(3.1)
is minimized in term of a $L_2$ gain, that is, for a given prescribed $H_\infty$ disturbance attenuation level $\gamma > 0$, the following condition holds

$$
\left( \int_0^{+\infty} z^2(t) dt \right)^{\frac{1}{2}} \leq \gamma \left( \int_0^{+\infty} \omega^2(t) dt \right)^{\frac{1}{2}},
$$

(3.2)

for any nonzero $\omega(t) \in L_2[0, +\infty)$, where $G$ and $H$ are constant matrices. In this situation, system (2.7)-(3.1) (or system (2.9)-(3.1)) is said to be asymptotically stable and has an $H_\infty$ performance $\gamma$.

After obtaining the controller $u(t)$ for system (2.7)-(3.1) (or system (2.9)-(3.1)), we can derive $v(t)$ from (2.6), given by

$$v(t) = (B^TB)^{-1}B(T_0u(t) + D).$$

(3.3)

Hence, in the following, we consider the design of $u(t)$.

3.1. Stability strategies for perishable products system. We shall design the following controller

$$u(t) = -Lx.$$  

(3.4)

Then the corresponding closed loop system (2.7)-(3.1) can be rewritten as

$$
\begin{align*}
x'(t) &= A_Lx(t) + E\omega(t), \quad t \neq t_k, \\
\Delta x(t) &= D_kx(t), \quad t = t_k, \\
z(t) &= G_Lx(t), \\
x(t) &= 0, \quad t = t_0 = 0,
\end{align*}
$$

(3.5)

where $A_L = A - B_0L$, $G_L = G - HL$, and $L$ is a constant matrix to be determined.

**Theorem 3.1.** Consider the closed loop system (3.5). For a given positive constant $\gamma$, if there exists a symmetric positive definite matrix $P \in R^{n \times n}$ such that

$$
\begin{align*}
0 &\leq \beta_k = \lambda_{\max}\left[P^{-1}(I + D_k)^TP(I + D_k)\right] \leq 1, \\
A_L^TP + PA_L + \gamma^{-2}PEE^TP + G_L^TG_L &< 0, \\
\mu^2I - L^2 &\geq 0,
\end{align*}
$$

(3.6) (3.7) (3.8)

then the close-loop system (3.5) is asymptotically stable and has an $H_\infty$ performance $\gamma$.

**Proof.** (I) First, we shall prove that $u(t)$ satisfies the standard constraint condition (2.8).

By (3.4), we have

$$
\|u(t)\|^2_2 = \| -Lx(t)\|^2_2 \leq \lambda_{\max}(L^2)\|x(t)\|^2_2.
$$

(3.9)

By (3.8), we get

$$
\mu^2 - \lambda_{\max}(L^2) \geq 0.
$$

Thus, it follows from (3.9) that

$$
\|u(t)\|^2_2 \leq \mu^2\|x(t)\|^2_2,
$$

which implies that controller (3.4) satisfies (2.8).
(II) Next, we shall prove that system (3.5) is asymptotically stable when \( \omega(t) = 0 \). Let

\[
V(x(t)) = x(t)^TPx(t),
\]

(3.10)

where \( P \) is a symmetric positive definite matrix in (3.6) and (3.7). Since \( P \) is positive definite, we know that \( V(x(t)) \) is positive definite as well. Then by the definition of Lyapunov function, \( V(x(t)) \) is a Lyapunov function. When \( \omega(t) = 0 \), it follows from (3.7) that

\[
A_L^TP + PA_L < -G_L^TG_L.
\]

Therefore,

\[
V'(t) = x'(t)^TPx(t) + x(t)^TPx'(t)
= [A_Lx(t)]^TPx(t) + x(t)^TPA_Lx(t)
= x(t)^T(A_L^TP + PA_L)x(t)
< -x(t)^TG_L^TG_Lx(t).
\]

(3.11)

Setting \( G_L^TG_L = Q \), then by (3.11) we obtain

\[
V'(t) < -x(t)^TQx(t).
\]

(3.12)

By Lemma 2.3, it follows that

\[
x(t)^TQx(t) \geq \lambda_{\min}(P^{-1}Q)x(t)^TPx(t).
\]

Thus, from (3.12), we obtain

\[
V'(t) + \eta V(t) < 0, \ t \in (t_k, t_{k+1}),
\]

where \( \eta = \lambda_{\min}(P^{-1}Q) > 0 \). From (3.5) and (3.10), it follows that

\[
V(t_k^+) = x(t_k^+)^TPx(t_k^+) = [(I + D_k)x(t_k^+)]^TP[(I + D_k)x(t_k^+)]
= x(t_k)^T[(I + D_k)^TP(I + D_k)]x(t_k)
= x(t_k)^TPP^{-1}(I + D_k)^TP(I + D_k)x(t_k)
\leq \beta_kx(t_k)^TPx(t_k)
= \beta_kV(t_k).
\]

Hence,

\[
V(t) < V(t_k^+) \exp[-\eta(t-t_k)]
= \beta_kV(t_k) \exp[-\eta(t-t_k)], \ t \in (t_k, t_{k+1}).
\]

When \( t \in (t_0, t_1) \), we have

\[
V(t) < V(t_0) \exp[-\eta(t-t_0)], \quad V(t_1) < V(t_0) \exp[-\eta(t_1-t_0)].
\]

When \( t \in (t_1, t_2) \), we get

\[
V(t_2) < V(t_1^+) \exp[-\eta(t-t_1)]
< \beta_k^2V(t_1) \exp[-\eta(t-t_1)]
< \beta_k^2V(t_0) \exp[-\eta(t-t_0)].
\]
Similarly, when \( t \in (t_k, t_{k+1}] \), we obtain
\[
V(t) < V(t_k^+) \exp[-\eta(t - t_k)] \\
< \beta_k^2 V(t_k) \exp[-\eta(t - t_k)] \\
< \prod_{i=1}^k \beta_i^2 V(t_0) \exp[-\eta(t - t_0)] \\
< V(t_0) \exp[-\eta(t - t_0)],
\]
which means that the closed-loop system (3.5) is asymptotically stable when \( \omega(t) = 0 \).

(III) Finally, we need to show that the system (3.5) is L2-gain.

From (3.7), we have
\[
A_k^T P + PA_L + \gamma^{-2} PEE^T P < -G_k^T G_L.
\]

Therefore,
\[
V'(t) = x'(t)^T Px(t) + x(t)^T Px'(t) \\
= [A_L x(t) + E\omega(t)]^T Px(t) + x(t)^T P[A_L x(t) + E\omega(t)] \\
= x(t)^T A_L^T Px(t) + x(t)^T PA_L x(t) + \omega(t)^T ET P x(t) + x(t)^T PE\omega(t) \\
= x(t)^T (A_k^T P + PA_L)x(t) + 2x(t)^T PE\omega(t).
\]

By Lemma 2.2, we have
\[
2x(t)^T PE\omega(t) \leq \gamma^{-2} x(t)^T PEE^T P x(t) + \gamma^2 \omega(t)^T \omega(t).
\]

Hence,
\[
V'(t) \leq x(t)^T (A_k^T P + PA_L + \gamma^{-2} PEE^T P)x(t) + \gamma^2 \omega(t)^T \omega(t) \\
< -x(t)^T G_k^T G_L x(t) + \gamma^2 \omega(t)^T \omega(t) \\
= -\|G_L x(t)\|^2 + \gamma^2 \|\omega(t)\|^2 \\
= -\|z(t)\|^2 + \gamma^2 \|\omega(t)\|^2.
\]

That is to say,
\[
\|z(t)\|^2 < -V'(t) + \gamma^2 \|\omega(t)\|^2, \quad t \in (t_k, t_{k+1}].
\]

Integrating both sides from 0 to \( \tau \) yields
\[
\int_0^\tau \|z(t)\|^2 dt \leq -\int_0^\tau V'(t) dt + \gamma^2 \int_0^\tau \|\omega(t)\|^2 dt, \quad \tau \in (t_k, t_{k+1}]. \quad (3.13)
\]

From (3.10), it follows that \( V(0) = 0, V(t_k) > 0 \). By \( 0 < \beta_k < 1 \), we obtain
\[
\int_0^\tau V'(t) dt \\
= \int_0^{t_1} V'(t) dt + \int_{t_1}^{t_2} V'(t) dt + \cdots + \int_{t_k}^{t_{k-1}} V'(t) dt + \int_{t_k}^{\tau} V'(t) dt \\
= V(t_1) - V(0) + V(t_2) - V(t_1^+) + \cdots + V(t_{k-1}) - V(t_k^+) + V(\tau) - V(t_k^+) \\
= \sum_{j=1}^k (1 - \beta_j) V(t_j) + V(\tau) \geq 0.
\]
Then, from (3.13), we have
\[ \int_0^\tau \|z(t)\|^2 dt < 2 \int_0^\tau \|w(t)\|^2 dt. \]
This implies the validity of inequality (3.2). The proof is completed. \( \square \)

**Theorem 3.2.** Let \( \beta_k \) be the maximal eigenvalue of \( P^{-1}(I + D_k)P(I + D_k), \gamma > 0, 0 \leq \beta_k \leq 1, k \in N \). Suppose that there exist a positive constant \( \varepsilon > 0 \), a symmetric positive definite matrix \( P \in \mathbb{R}^{n \times n} \), and a feedback control law designed as follow
\[ u(t) = -Lx(t), \quad L = \frac{1}{\varepsilon} B_0^T P \]
such that
\begin{align*}
& (A^T - \varepsilon^{-1} G^T HB_0^T)P + P(A - \varepsilon^{-1} B_0 H^T G) + P[\gamma^{-2} EE^T - (2\varepsilon)^{-1} B_0 B_0^T]P \\& + \frac{1}{\varepsilon^2} PB_0 H^T B_0^T P + G^T G < 0, \tag{3.14} \\
& \mu^2 I - \frac{1}{\varepsilon^2} PB_0 B_0^T P \geq 0. \tag{3.15}
\end{align*}
Then, the close-loop system (2.7)-(3.1) is asymptotically stable and has an \( H_\infty \) performance \( \gamma \).

**Proof.** Let \( u(t) = -Lx(t), \quad L = \frac{1}{\varepsilon} B_0^T P \). Then, by (3.7) and (3.8), we have
\begin{align*}
& (A^T - \varepsilon^{-1} G^T HB_0^T)P + P(A - \varepsilon^{-1} B_0 H^T G) + P[\gamma^{-2} EE^T - (2\varepsilon)^{-1} B_0 B_0^T]P \\& + \frac{1}{\varepsilon^2} PB_0 H^T B_0^T P + G^T G < 0, \\
& \mu^2 I - \frac{1}{\varepsilon^2} PB_0 B_0^T P \geq 0.
\end{align*}
This completes the proof. \( \square \)

**Remark 1.** Note that by Lemma 2.4, inequalities (3.14) and (3.15) are equivalent to
\[ \begin{pmatrix} (A^T - \varepsilon^{-1} G^T HB_0^T)P + P(A - \varepsilon^{-1} B_0 H^T G) + G^T G & PE & PB_0 & PB_0 \\ E^T P & -\gamma^{-2} I & 0 & 0 \\ B_0^T P & 0 & (2\varepsilon)^{-1} I & 0 \\ B_0^T P & 0 & 0 & -\varepsilon^{-2} H^T H \end{pmatrix} < 0, \]
and
\[ \begin{pmatrix} -\mu^2 I & B_0^T P \\ PB_0 & -\varepsilon^2 I \end{pmatrix} \leq 0. \]
They are in the form of LMIs and hence can be solved by the technique proposed in [5].

Suppose that each machine produces one particular kind of product, and that the productivity and the due date of each batch of different products are the same. Then, \( d_{11} = d_{22} = \cdots = d_{nn} = d_k, d_{ij} = 0, \ i \neq j \), and system (2.7)-(3.1) under this scenario reduces to
\[ \begin{align*}
& x'(t) = Ax(t) + B_0 u(t) + E \omega(t), \quad t \neq t_k, \\
& \Delta x(t) = d_k x(t), \quad t = t_k, \\
& z(t) = Gx(t) + H u(t), \\
& x(t) = 0, \quad t = t_0 = 0.
\end{align*} \tag{3.16} \]
We obtain the following corollary.
Corollary 3.1. Let \( \gamma = \varepsilon = 1, -1 \leq d_k \leq 0, k = 1, 2, \ldots, n \). Suppose that there exist a symmetric positive definite matrix \( P \in \mathbb{R}^{n \times n} \), and a feedback control law given by

\[
    u(t) = -B_0^T P x(t)
\]

such that

\[
    (A^T - G^T H B_0^T)P + P(A - B_0 H^T G) + P(EE^T - \frac{1}{2} B_0 B_0^T)P
    + P B_0 H^T H B_0^T P + G^T G < 0,
\]

\[
    \mu^2 I - PB_0 B_0^T P \geq 0.
\]

Then, the close-loop system (3.16) is asymptotically stable and has an \( H_{\infty} \) performance \( \gamma \).

Proof. Since \( d_{11} = d_{22} = \cdots = d_{nn} = d_k \), we have

\[
    P^{-1}(I + D_k)^T P(I + D_k) = (1 + d_k)^2.
\]

By \(-1 \leq d_k \leq 0\), it is clear that \( 0 < (1 + d_k)^2 < 1 \). Thus,

\[
    0 < \beta_k = \lambda_{\max}[P^{-1}(I + D_k)^T P(I + D_k)] < 1.
\]

Let \( \gamma = \varepsilon = 1 \). By Theorem 3.2, the conditions of the theorem follow readily. \( \square \)

3.2. Stability strategies for durable products system. If \( D_k = 0 \), the manufacturing-inventory model of perishable products (2.7) reduces to the model of durable products (2.9).

Theorem 3.3. Let \( \gamma > 0 \). Suppose that there exist a positive constant \( \varepsilon > 0 \) and a symmetric positive definite matrix \( P \in \mathbb{R}^{n \times n} \), and a feedback control law designed as

\[
    u(t) = -\varepsilon B_0^T P x(t),
\]

such that

\[
    (A^T - \varepsilon^{-1} G^T H B_0^T)P + P(A - \varepsilon^{-1} B_0 H^T G) + P(\gamma^{-2} EE^T - (2\varepsilon)^{-1} B_0 B_0^T)P
    + \frac{1}{\varepsilon^2} P B_0 H^T H B_0^T P + G^T G < 0,
\]

\[
    \mu^2 I - \frac{1}{\varepsilon^2} PB_0 B_0^T P \geq 0.
\]

Then, the close-loop system (2.9)-(3.1) is asymptotically stable and has an \( H_{\infty} \) performance \( \gamma \).

Proof. By \( D_k = 0 \), we obtain \( \beta_k = 0 \) which satisfies \( 0 \leq \beta_k \leq 1 \). The conclusion of the theorem follows readily from Theorem 3.2. \( \square \)

Corollary 3.2. Let \( \gamma = \varepsilon = 1 \). Suppose that there exists a symmetric positive definite matrix \( P \in \mathbb{R}^{n \times n} \), and a feedback control law designed as

\[
    u(t) = -B_0^T P x(t),
\]

such that

\[
    (A^T - G^T H B_0^T)P + P(A - B_0 H^T G) + G^T G + P(EE^T - \frac{1}{2} B_0 B_0^T)P
    + PB_0 H^T H B_0^T P < 0,
\]

\[
    \mu^2 I - PB_0 B_0^T P \geq 0.
\]
Then, the close-loop system (2.9)-(3.1) is asymptotically stable and has an $H_{\infty}$ performance $\gamma$.

Proof. The conclusion follows as a direct consequence of Theorem 3.3 with $\gamma = \varepsilon = 1$. \hfill \square

Remark 2. Note that by Lemma 2.4, inequalities (3.17) and (3.18), (3.20) and (3.21), (3.23) and (3.24) can all be expressed in terms of LMIs. Thus, the problem can be solved by the technique proposed in [5].

For a manufacturing enterprise, a stable inventory is a necessity for maintaining a normal production operation. On one hand, overproduction results in overstocked products, which require extra warehousing costs. On the other hand, underproduction leads to understock, and as such the demands of customers cannot be met. Since the demand for the products changes over time, it is essential to control the production to meet and stabilize the demand. We then obtain a stable manufacturing-inventory system. Theorems 3.2 and 3.3 provide effective strategies to control the production to suppress unstable effects caused by uncertain customer demand. Under such control strategies, a manufacturing enterprise is able to avoid overproduction and underproduction and as such minimize its operation costs.

4. Example. In order to illustrate the applicability of our results, we consider a manufacturing system with two machines producing two different kinds of products.

4.1. Example of perishable products. Assume that machine I produces type 1 products, and machine II produces type 2 products. Let

\[
A = \begin{pmatrix} -0.4 & 0 \\ 0 & -0.3 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, \\
D_k = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.2 \end{pmatrix}, \quad G = \begin{pmatrix} -0.2 & 0 \\ 0 & -0.1 \end{pmatrix}, \quad H = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.4 \end{pmatrix}.
\]

Then, system (2.7) with (3.1) can be written as

\[
\begin{cases}
\dot{x}(t) = \begin{pmatrix} -0.4 & 0 \\ 0 & -0.3 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix} \omega(t), \quad t \neq t_k, \\
\Delta x(t) = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.2 \end{pmatrix} x(t), \quad t = t_k, \\
\dot{z}(t) = \begin{pmatrix} -0.2 & 0 \\ 0 & -0.1 \end{pmatrix} x(t) + \begin{pmatrix} 0.8 & 0 \\ 0 & 0.4 \end{pmatrix} u(t), \\
x(t) = 0, \quad t = t_0 = 0,
\end{cases}
\]

(4.1)

For $\gamma = \varepsilon = 1$, it follows from using LMI Toolbox of Matlab that

\[
P = \begin{pmatrix} 0.0180 & 0 \\ 0 & 0.0535 \end{pmatrix},
\]

which satisfies the conditions (3.14) and (3.15). Through simple calculation, we have $0 < \beta_k = 0.64 < 1$. Then, the feedback control law is obtained as

\[
u(t) = -Lx(t), \quad L = \begin{pmatrix} 0.0180 & 0 \\ 0 & 0.0374 \end{pmatrix}.
\]

Therefore, it follows that the close-loop system (4.1) is asymptotically stable and has an $H_{\infty}$ performance 1 by Theorem 3.2.
4.2. Example of durable products. Assume that machine I produces type 1 products, and machine II produces type 2 products. Let

\[ A = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.4 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.9 \end{pmatrix}, \quad E = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.8 \end{pmatrix}, \quad G = \begin{pmatrix} -0.6 & 0 \\ 0 & -0.3 \end{pmatrix}, \quad H = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.6 \end{pmatrix}. \]

Then, system (2.9) with (3.1) can be written as

\[
\begin{cases}
    x'(t) = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.4 \end{pmatrix} x(t) + \begin{pmatrix} 0.8 & 0 \\ 0 & 0.9 \end{pmatrix} u(t) + \begin{pmatrix} 0.3 & 0 \\ 0 & 0.8 \end{pmatrix} \omega(t), \\
    z(t) = \begin{pmatrix} -0.6 & 0 \\ 0 & -0.3 \end{pmatrix} x(t) + \begin{pmatrix} 0.9 & 0 \\ 0 & 0.6 \end{pmatrix} u(t), \\
    x(t) = 0, \quad t = t_0 = 0.
\end{cases}
\] (4.2)

For \( \gamma = 1 \) and \( \varepsilon = 0.5 \), it follows from using LMI Toolbox of Matlab that

\[ P = \begin{pmatrix} 0.0969 & 0 \\ 0 & 0.0866 \end{pmatrix}, \]

which satisfies the conditions (3.20) and (3.21). Then, the feedback control law is obtained as

\[ u(t) = -L x(t), \quad L = \begin{pmatrix} 0.1550 & 0 \\ 0 & 0.1558 \end{pmatrix}, \]

Therefore, it follows that the close-loop system (4.2) is asymptotically stable and has an \( H_\infty \) performance 0.5 by Theorem 3.3.

4.3. Numerical simulation results. Now we use numerical simulations to illustrate the effects of the proposed feedback control on the behaviors of the model. The time history of the stock level of perishable products is shown in Figure 1. When \( t < 90 \), feedback control is not applied to the system and the stock level remains positive with impulses. When appropriate control is applied to the system for \( t \geq 90 \), we notice that the stock level is minimized and the impulsive effects disappear. The proposed control stabilizes the system since condition (3.2) is satisfied (see Figure 2).

For non-perishable products, there is no impulse in the model and the stock level is positive for \( t < 50 \) (see Figure 3). Assume that the feedback control is applied to the system when \( t \geq 50 \). The simulation results indicate that the stock levels of both products approach to 0. The stabilization of the system with the designed control is guaranteed since condition (3.2) is satisfied (see Figure 4).

5. Conclusion. In this paper, we construct two models for the manufacturing-inventory systems – one produces perishable products and the other one produces non-perishable products. Impulses were incorporated in the perishable product model so as to cater for the disposal of expired products. Based on the structures of these models, feedback control strategies are designed to stabilize the systems. Both mathematical proofs and numerical simulations are carried out to show the effectiveness of the proposed control strategies. The results presented in this article are clearly applicable to the stabilization of real manufacturing-inventory systems.
6. Figures.

Figure 1. Time history of system (4.1) with \( u(t) = [u_1(t), u_2(t)]^T = [1.6, 1.2]^T \) and \( \omega(t) = [\omega_1(t), \omega_2(t)]^T = [\omega_{01} \sin(a_1 t + b_1), \omega_{02} \sin(a_2 t + b_2)]^T \), where \( \omega_{01} = 0.15, \omega_{02} = 0.1, \ a_1 = 16, b_1 = 10, a_2 = 20 \) and \( b_2 = 2 \). Feedback control is applied to the system when \( t = 90 \).

Figure 2. Time history of \( z(t) \) and \( \omega(t) \) for system (4.1) with feedback control applied to the system. Here, \( u(t) = [u_1(t), u_2(t)]^T = [1.6, 1.2]^T \) and \( \omega(t) = [\omega_1(t), \omega_2(t)]^T = [\omega_{01} \sin(a_1 t + b_1), \omega_{02} \sin(a_2 t + b_2)]^T \), where \( \omega_{01} = 0.15, \omega_{02} = 0.1, \ a_1 = 16, b_1 = 10, a_2 = 20 \) and \( b_2 = 2 \).
Figure 3. Time history of system (4.2) with \( u(t) = [u_1(t), u_2(t)]^T = [1.5, 0.9]^T \) and \( \omega(t) = [\omega_1(t), \omega_2(t)]^T = [\omega_{01} \sin(a_1 t + b_1), \omega_{02} \sin(a_2 t + b_2)]^T \), where \( \omega_{01} = 0.055, \omega_{02} = 0.04, a_1 = 16, b_1 = 10, a_2 = 20 \) and \( b_2 = 2 \). Feedback control is applied to the system when \( t = 50 \).

Figure 4. Time history of \( z(t) \) and \( \omega(t) \) for system (4.2) with feedback control applied to the system. Here, \( u(t) = [u_1(t), u_2(t)]^T = [1.5, 0.9]^T \) and \( \omega(t) = [\omega_1(t), \omega_2(t)]^T = [\omega_{01} \sin(a_1 t + b_1), \omega_{02} \sin(a_2 t + b_2)]^T \), where \( \omega_{01} = 0.055, \omega_{02} = 0.04, a_1 = 16, b_1 = 10, a_2 = 20 \) and \( b_2 = 2 \).
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