On the boundedness of the denominators in the Zariski decomposition on surfaces

Thomas Bauer, Piotr Pokora, David Schmitz

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Abstract

Zariski decompositions play an important role in the theory of algebraic surfaces. For making geometric use of the decomposition of a given divisor, one needs to pass to a multiple of the divisor in order to clear denominators. It is therefore an intriguing question whether the surface has a “universal denominator” that can be used to simultaneously clear denominators in all Zariski decompositions on the surface. We prove in this paper that, somewhat surprisingly, this condition of bounded Zariski denominators is equivalent to the bounded negativity of curves that is addressed in the Bounded Negativity Conjecture. Furthermore, we provide explicit bounds for Zariski denominators and negativity of curves in terms of each other.

Introduction

The theorem on Zariski decomposition is a fundamental tool in the theory of algebraic surfaces. It was established by Zariski [8] for effective divisors and extended by Fujita [4] to the pseudoeffective case. The geometric significance of Zariski decompositions lies in the fact that, given a pseudoeffective integral divisor $D$ on $X$ with Zariski decomposition $D = P + N$, one has for every sufficiently divisible integer $m \geq 1$ the equality

$$H^0(X, O_X(mD)) = H^0(X, O_X(mP)).$$

In other words, all sections of $O_X(mD)$ come from the nef line bundle $O_X(mP)$. The term “sufficiently divisible” here means that one needs to pass to a multiple $mD$ that clears denominators in $P$ for the statement to hold. Of course, it would be most pleasant if one knew – beforehand and independently of $D$ – which multiple to take. This amounts to asking the following

**Question.** Let $X$ be a smooth projective surface. Does there exist an integer $d(X) \geq 1$ such that for every pseudoeffective integral divisor $D$ the denominators in the Zariski decomposition of $D$ are bounded from above by $d(X)$?

If such a bound $d(X)$ exists, then we say that $X$ has bounded Zariski denominators. Taking then the factorial $d(X)!$, one has in fact a uniform number that clears denominators in all Zariski denominators. Further details can be found in the paper.

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decompositions on $X$. It is an intriguing question as to whether a given smooth surface satisfies this boundedness condition.

We show in the present paper that, somewhat surprisingly, boundedness of Zariski denominators is equivalent to bounded negativity:

**Theorem.** For a smooth projective surface $X$ over an algebraically closed field the following two statements are equivalent:

(i) $X$ has bounded Zariski denominators.

(ii) $X$ has bounded negativity, i.e., there is a bound $b(X)$ such that for every irreducible curve $C$ on $X$ one has

$$C^2 \geq -b(X).$$

The Bounded Negativity Conjecture (BNC), explored in [1], is the conjecture that (ii) is true for every smooth projective surface over the field of complex numbers. This is open in general, even for the case where $X$ is the blow up of $\mathbb{P}^2$ in $s$ general points with $s \geq 10$ (see [5] for a nice introduction to this subject). By contrast, it is known that bounded negativity does not hold in general in positive characteristics – this is in accordance with Example 3.1, where we exhibit unbounded Zariski denominators on such surfaces.

Our result sheds new light on BNC: It says in particular that BNC is equivalent to boundedness of Zariski denominators on all smooth complex projective surfaces. We do not dare to suggest whether this makes it more likely or less likely that BNC holds – it definitely makes it more desirable to hold.

On the practical side, we provide explicit (while presumably not optimal) bounds for $d(X)$ and $b(X)$ in terms of each other (see Theorems 2.2 and 2.3). If, for instance, all negative curves on $X$ are known, then one has an effective bound on the Zariski denominators.

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## 1 Denominators in Zariski decompositions

Let $X$ be a smooth projective surface and $D$ a pseudo-effective integral divisor on $X$. Fujita’s extension [4] of Zariski’s result [8] states that $D$ can be written uniquely as a sum

$$D = P + N$$

of $\mathbb{Q}$-divisors such that

(i) $P$ is nef,

(ii) $N$ is effective and has negative definite intersection matrix if $N \neq 0$,

(iii) $P \cdot C = 0$ for every component of $N$.

For the question of bounded denominators in $P$ and $N$ it is of course enough to consider the denominators of $N = \sum_{i=1}^{k} a_i N_i$, i.e., the denominators of the coefficients $a_i$. In order to approach the problem, we use the following description of the coefficients $a_i$. They are given as the (unique) solution of the system of equations

$$D \cdot N_j = (P + \sum_{i=1}^{k} a_i N_i) \cdot N_j = \sum_{i=1}^{k} a_i N_i \cdot N_j$$

for all $j \in \{1, \ldots, k\}$. 


This system can be rewritten in matrix form as

$$S[a_1, \ldots, a_k]^t = [D \cdot N_1, \ldots, D \cdot N_k]^t,$$

where $S$ denotes the intersection matrix of the curves $N_1, \ldots, N_k$, i.e. $S = [N_i \cdot N_j]_{i,j} \in M_{k \times k}(\mathbb{Z})$. Since the matrix $S$ is negative definite, it has non-zero determinant, and using Cramer’s rule one has

$$a_i = \frac{\det(s_1, \ldots, s_{i-1}, b, s_{i+1}, \ldots, s_k)}{\det(S)},$$

where $s_i$ denotes the $i$-th column of the matrix $S$ and $b = [D \cdot N_1, \ldots, D \cdot N_k]^t$. Thus, for divisors with negative part $N$ supported on $N_1, \ldots, N_k$, the denominators of the Zariski decomposition are bounded by $|\det(S)|$.

**Remark 1.1.** Note that the above reasoning yields an upper bound for the denominators of the coefficients in the Zariski decomposition for any surface whose pseudoeffective cone is rational polyhedral, since in this case there are only finitely many possible sets $\{N_1, \ldots, N_k\}$ of components of negative parts, so we obtain the bound

$$d(X) = \max\{|\det(S_i)| \mid S_i \text{ principal negative definite submatrix of } S\},$$

where $S$ denotes the intersection matrix of all irreducible curves with negative self-intersection.

It is not clear a priori whether the corresponding supremum will be finite in the presence of infinitely many extremal rays. This more general situation is the topic of the following section.

## 2 Bounded denominators and bounded negativity

We start by reminding the reader of the following conjecture.

**Conjecture 2.1 (Bounded Negativity Conjecture, [1], [5]).** Let $X$ be a smooth complex projective surface. Then there exists an integer $b(X) \geq 0$ such that for every irreducible and reduced curve $C$ one has $C^2 \geq -b(X)$.

The aim of this section is to prove the theorem stated in the introduction. In particular, we thus show that Conjecture 2.1 is equivalent to the assertion that all smooth complex projective surfaces have bounded Zariski denominators.

**Theorem 2.2.** Let $X$ be a smooth projective surface on which the self-intersection of irreducible curves is bounded by $-b(X)$. Then $X$ has bounded Zariski denominators. More concretely, denoting by $\rho(X)$ the Picard number, we have

$$d(X) \leq b(X)^{\rho(X)-1}.$$

**Proof.** Let $D$ be any integral pseudoeffective divisor on $X$, with negative part $N = \sum a_i N_i$, $a_i > 0$, and let $S$ be the intersection matrix of the curves $N_1, \ldots, N_k$. According to the consideration in the first section, we know that the denominators of the $a_i$ can be at most $|\det S|$.

Since $S$ is negative definite, there exists an invertible matrix $U \in GL(k, \mathbb{R})$ and real numbers $\lambda_1, \ldots, \lambda_k$ such that $U^{-1}SU = \text{diag}(\lambda_1, \ldots, \lambda_k)$ and $\lambda_i < 0$ for each $i \in \{1, \ldots, k\}$. As the trace of a matrix is invariant under conjugation, we have

$$N_1^2 + \ldots + N_k^2 = \text{tr}(S) = \text{tr}(U^{-1}SU) = \lambda_1 + \ldots + \lambda_k.$$
The same holds for the determinant, thus
\[ |\det S| = |\det(U^{-1}SU)| = |\lambda_1 \cdot \ldots \cdot \lambda_k| = |\lambda_1| \cdot \ldots \cdot |\lambda_k|. \]

Using the inequality of arithmetic and geometric means we obtain
\[ |\det S| = |\lambda_1| \cdot \ldots \cdot |\lambda_k| \leq \left( \frac{\sum_i |\lambda_i|}{k} \right)^k = \left( \frac{-\text{tr}(S)}{k} \right)^k = \left( \frac{-N_1^2 - \ldots - N_k^2}{k} \right)^k. \]

By assumption, the self-intersection $N_i^2$ is at least $-b(X)$ for each $i \in \{1, \ldots, k\}$, hence
\[ |\det S| \leq \left( \frac{-N_1^2 - \ldots - N_k^2}{k} \right)^k \leq \left( \frac{k \cdot b(X)}{k} \right)^k. \]

Finally, by the Hodge Index Theorem, $k$ can be at most $\rho(X) - 1$, thus $d(X) = b(X)^\rho(X) - 1$ is a bound for the Zariski denominators of integral pseudo-effective divisors on $X$, which is independent of the particular support of the negative part.

We now turn to the converse implication.

**Theorem 2.3.** Let $X$ be a smooth projective surface. If Zariski denominators on $X$ are bounded by $d(X)$, then $X$ has bounded negativity. More concretely, denoting by $\Delta$ the discriminant of the Néron-Severi lattice $N^1(X)$, we have
\[ b(X) \leq d(X) \cdot d(X)! \cdot \Delta. \]

The proof rests on the following lemma.

**Lemma 2.4.** Let $X$ be a smooth projective surface with bound $d(X)$ of Zariski denominators. Then there exists for every negative curve $C$ on $X$ an (integral) ample line bundle $A$ on $X$ such that the gcd of the numbers $C^2$ and $A \cdot C$ is a divisor of $d(X)! \cdot \Delta$.

Granting the lemma, we give the proof of the theorem.

**Proof of the theorem.** Let $C$ be any irreducible negative curve on $X$. By Lemma 2.4 there exists an ample divisor $A$ such that
\[ \gcd(C^2, AC) \mid d(X)! \cdot \Delta. \]

For a sufficiently large integer $k$, the line bundle $A + kC$ has Zariski decomposition
\[ A + kC = (A + \alpha C) + (k - \alpha)C \]
where the divisor in parentheses is the positive part and
\[ \alpha = -\frac{A \cdot C}{C^2}. \]

Now, the denominator of $\alpha$ is exactly $\frac{-C^2}{\gcd(C^2, AC)}$. In particular, by the assumed boundedness of the denominators,
\[ d(X) \geq \frac{-C^2}{\gcd(C^2, AC)} \geq \frac{-C^2}{d(X)! \cdot \Delta}. \]

Therefore, the self-intersection of any curve $C$ is bounded from below by $-d(X) \cdot d(X)! \cdot \Delta$. \(\square\)
Now we turn to the proof of Lemma 2.4. The first step is:

**Lemma 2.5.** Let $X$ be a smooth projective surface with Zariski denominators bounded by $d(X)$. If $C$ is a negative curve and $t$ an integer that divides $A \cdot C$ for all ample line bundles $A$ on $X$, then $t$ is a divisor of $d(X)! \cdot \Delta$.

**Proof.** Let $F$ be the minimal integer divisor class in the ray in $N^1(X)_{\mathbb{R}}$ spanned by $C$. Then $C = kF$ for some integer $k \geq 1$, and $F$ is pseudo-effective with Zariski decomposition

$$F = \frac{1}{k} \cdot C.$$ 

By the boundedness assumption of Zariski denominators we have $k \leq d(X)$.

Note that if the hypothesis of the lemma is satisfied, then

$$t \text{ divides } D \cdot C \text{ for all (integral) divisors } D \text{ on } X.$$

In fact, if $D$ is any (integral) divisor, then $mA + D$ is ample for sufficiently large integers $m$, and hence $t$ divides both $mA \cdot C$ and $(mA + D) \cdot C$, so that it also divides $D \cdot C$.

Choosing now a lattice basis of $N^1(X)$, we may think of the intersection form as given—after modding out torsion—on $\mathbb{Z}^{\rho(X)}$ by an integral matrix $S$ of determinant $\Delta$, and of classes $C, D$ in $N^1(X)$ as represented by vectors $c, d$ in $\mathbb{Z}^{\rho(X)}$, with

$$C \cdot D = c^t S d.$$

In these terms, condition (*) implies that every entry in the vector $c^t S$ is divisible by $t$. Using now the adjugate matrix $S_{\text{adj}}$ of $S$, we infer that the vector

$$c^t S_{\text{adj}} = c^t \det(S) = c^t \Delta$$

is divisible by $t$. Representing $F$ as an integral vector $f$, we obtain that $t$ is a divisor of

$$f^t k \Delta.$$

Now this implies that in fact $k \Delta$ is divisible by $t$, since otherwise every entry in the vector $f$ would be divisible by $t$, which in turn would mean that the class of $F$ is not primitive. Since $k \leq d(X)$, in particular $t$ is a divisor of $d(X)! \cdot \Delta$. \qed

We now prove Lemma 2.4 using Lemma 2.5.

**Proof of Lemma 2.4.** Let $C$ be any negative curve on $X$. Assume by way of contradiction that the conclusion of the lemma is false. By the factorization theorem, then the following holds:

$$+ \text{ For every ample divisor } A \text{ on } X \text{ there exists a prime power } p^r \text{ such that }$$

$$p^r \mid C^2, \quad p^r \mid A \cdot C, \quad p^r \nmid d(X)! \cdot \Delta.$$ 

Note that there are only finitely many possibilities for prime powers satisfying (+), namely those $p^r$ that divide $C^2$ and do not divide $d(X)! \cdot \Delta$. Let $p_1, \ldots, p_s$ be the prime factors of $C^2$ such that there exists a power which divides $C^2$ but not $d(X)! \cdot \Delta$, and denote for each $i$ by $n_i$ the smallest number such that $p_i^{n_i}$ divides $C^2$ but not $d(X)! \cdot \Delta$.

We claim:
There exists an $i \in \{1, \ldots, s\}$ such that for all ample line bundles $A$, the intersection number $A \cdot C$ is divisible by $p^n_i$.

If $(++)$ does not hold, then there is for every $i$ an ample divisor $A_i$ such that $p^n_i$ does not divide $A_i \cdot C$. Consider now the ample line bundle

$$A := p^{n_2}_2 \cdots p^{n_r}_r A_1 + p^{n_1}_1 p^{n_3}_3 \cdots p^{n_r}_r A_2 + \cdots + p^{n_1}_1 \cdots p^{n_{r-1}}_{r-1} A_r = \sum_{i=1}^r \frac{p^n_i}{p^{n_i}} A_i .$$

By assumption $(+)$, both $C^2$ and $A \cdot C$ are divisible by some $p^n_i$ that does not divide $d(X)! \cdot \Delta$. Therefore, by the minimality of the $n_i$ we have $r \geq n_i$, and $A \cdot C$ is also divisible by $p^n_i$. We can assume $i$ to be 1. Now, $p^{n_1}_1$ divides all terms in the sum

$$p^{n_2}_2 \cdots p^{n_r}_r A_1 \cdot C + p^{n_1}_1 p^{n_3}_3 \cdots p^{n_r}_r A_2 \cdot C + \cdots + p^{n_1}_1 \cdots p^{n_{r-1}}_{r-1} A_r \cdot C$$

except for possibly the first one, and it divides the sum (which is $A \cdot C$). It must therefore also divide the first one, and hence it divides $A_1 \cdot C$, which is a contradiction with the choice of $A_1$.

We conclude that $(++)$ holds. So the number $p^{n_1}_1$ that we have found divides $A \cdot C$ for all ample line bundles $A$, thus by Lemma 2.3, it divides $d(X)! \cdot \Delta$. Now, this a contradiction with the choice of $n_1$, thus $(+)$ is false and the lemma follows.

\[ \square \]

### 3 Examples

**Example 3.1** (Surfaces with unbounded Zariski denominators in positive characteristic). Let $C$ be a curve of genus $g \geq 2$ defined over a finite field of characteristic $p > 0$. The surface $X = C \times C$ is then known to have unbounded negativity (see [11, Sect. 2]). Indeed, taking for $n \in \mathbb{N}$ the graph $\Gamma_n$ of the Frobenius morphism obtained by taking $p^n$-th powers, we have $\Gamma^2_n = p^n (2 - 2g) \to -\infty$ (see [6, Ex. V.1.10]). By Theorem 2.3, $X$ must have unbounded Zariski denominators. In the particular case at hand, these are in fact quickly detected: Denote by $F_2$ a fiber of the second projection $X \to C$, and consider the divisor $D_n = F_2 + \Gamma_n$. The negative part of its Zariski decomposition has support $\Gamma_n$ with coefficient

$$\frac{D_n \cdot \Gamma}{\Gamma^2_n} = \frac{1 + \Gamma^2_n}{2 \Gamma^2_n} = \frac{1}{2} + \frac{1}{2} \Gamma^2_n .$$

Since numerator and denominator are coprime for all $n$, we see that the Zariski denominator is $-1/\Gamma^2_n = p^n (2g - 2)$ and hence tends to infinity.

Next, we determine concrete bounds on the Zariski denominators for classes of surfaces $X$ for which bounded negativity holds and explicit bounds $b(X)$ are known.

**Example 3.2** (Surfaces with nef anticanonical bundle). Let $X$ be a smooth projective surface with $-K_X$ nef. As a consequence of the adjunction formula, we have the negativity bound $b(X) = 2$. Indeed, for every irreducible curve one has $2g(C) - 2 = K_X \cdot C + C^2 \leq C^2$, and hence $C^2 \geq -2$.

So, for every pseudo-effective integral divisor $D$ on $X$, the Zariski decomposition of $2g-1! \cdot D$ is integral.

**Example 3.3** (Surfaces with $d(X) = 1$). Let $X$ be a smooth projective surface, such that all negative curves on $X$ are $(-1)$-curves. Then the Zariski decomposition of pseudo-effective
integral divisors on $X$ is integral. Indeed, note that if $X$ contains only $(-1)$-curves, then every intersection matrix $S$ of negative curves, which are in the support of the negative part of the Zariski decomposition of a divisor $D$ has the form $-I_k = \text{diag}(-1, \ldots, -1)$. (This follows from negative definiteness.) Equation (1) shows then that the coefficient of a component $N_i$ of the negative part of $D$ is the integer $-D \cdot N_i$.

So we have $d(X) = 1$ for instance on del Pezzo surfaces (see also [2, Sect. 3]), and conjecturally (according to a weaker form of the SHGH conjecture, cf. [3]) for all blowups of $\mathbb{P}^2$ in several points in very general position.

**Example 3.4** (Surfaces with large $d(X)$). By contrast, note that on the blow-up of $\mathbb{P}^2$ in three collinear points, fractional Zariski decompositions occur (see [2, Example 2.3.20]). This is in accordance with the fact that a $(-2)$-curve exists on that surface. More generally, denoting on the blow-up of $k$ points on a line $L$ in $\mathbb{P}^2$, the strict transform of $L$ by $\tilde{L}$, the pull-back of a general line by $H$ and the exceptional divisors by $E_1, \ldots, E_k$, the coefficient of $\tilde{L}$ in the negative part of the divisor $\tilde{L} + H + \sum_{i=1}^{r} E_i$ for $0 \leq r \leq k - 3$ is

$$a = \frac{k - r - 2}{k - r - 1}.$$  

Thus the numbers $2, \ldots, k - 1$ occur as denominators, in particular,

$$d(X) \geq k - 1 = b(X).$$

In order to see that the occurring denominators can in fact be larger than the least self-intersection dictates, consider the blow-up of $\mathbb{P}^2$ in $r = k_1 + k_2$ points of which $k_1$ lie exclusively on a line $L_1$ and $k_2$ exclusively on a second line $L_2$. Assume further $k_1$ and $k_2$ to be coprime and both $\geq 4$. A computation shows that the divisor $H + \tilde{L_1} + \tilde{L_2}$ has negative part supported on $\tilde{L_1}$ and $\tilde{L_2}$ with coefficients

$$a_i = \frac{k_1 k_2 - k_1 - k_2 - k_i}{k_1 k_1 - k_1 - k_2},$$

respectively. In these expressions numerators and denominators are coprime, hence the Zariski denominator is $k_1 k_2 - k_1 - k_2$. So we found that

$$d(X) \geq k_1 k_2 - k_1 - k_2,$$

On the other hand, it is not hard to see that

$$b(X) = \max(k_1 - 1, k_2 - 1).$$

So we have constructed a series of examples where $d(X)$ grows at least quadratically in $r$, whereas $b(X)$ grows linearly in $r$.

Note that in the examples above we are still far from the theoretical upper bound $d(X) \leq b(X)^{\rho(X) - 1}$ given by Theorem [2.2] because $\rho(X)$ also grows linearly in $r$ in these cases. It would therefore be very interesting to know the answer to the following

**Question 3.5.** Is there a sequence of surfaces where $d(X)$ is not bounded by a polynomial in $b(X)$?
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Thomas Bauer, Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Straße, D-35032 Marburg, Germany.

E-mail address: tbauer@mathematik.uni-marburg.de

Piotr Pokora, Instytut Matematyki, Pedagogical University of Cracow, Podchorążych 2, PL-30-084 Kraków, Poland.

Current Address: Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Straße, D-35032 Marburg, Germany.

E-mail address: piotrpkr@gmail.com, piotrpokora@daad-alumni.de

David Schmitz, Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Straße, D-35032 Marburg, Germany.

E-mail address: schmitzd@mathematik.uni-marburg.de