STABLE STANDING WAVES FOR A CLASS OF NONLINEAR SCHRÖDINGER-POISSON EQUATIONS

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Abstract. We prove the existence of orbitally stable standing waves with prescribed $L^2$-norm for the following Schrödinger-Poisson type equation

$$i\psi_t + \Delta \psi - (|x|^{-1} * |\psi|^2)\psi + |\psi|^{p-2}\psi = 0 \quad \text{in } \mathbb{R}^3,$$

when $p \in \{\frac{8}{3}\} \cup (3, \frac{10}{3})$. In the case $3 < p < \frac{10}{3}$ we prove the existence and stability only for sufficiently large $L^2$-norm. In case $p = \frac{8}{3}$ our approach recovers the result of Sanchez and Soler [18] for sufficiently small charges. The main point is the analysis of the compactness of minimizing sequences for the related constrained minimization problem. In a final section a further application to the Schrödinger equation involving the biharmonic operator is given.

1. Introduction

In this paper we study the following Schrödinger-Poisson type equation

$$(1.1) \quad i\psi_t + \Delta \psi - (|x|^{-1} * |\psi|^2)\psi + |\psi|^{p-2}\psi = 0 \quad \text{in } \mathbb{R}^3,$$

where $\psi(x,t) : \mathbb{R}^3 \times [0,T) \rightarrow \mathbb{C}$ is the wave function, * denotes the convolution and $2 < p < 10/3$. It is known that in this case the Cauchy problem associated to (1.1) is globally well-posed in $H^1(\mathbb{R}^3; \mathbb{C})$ (see e.g. [9]).

We are interested in the search of standing wave solutions of (1.1), namely solutions of the form

$$\psi(x,t) = e^{-i\omega t}u(x), \quad \omega \in \mathbb{R}, \quad u(x) \in \mathbb{C},$$

so we are reduced to study the following semilinear elliptic equation with a non local nonlinearity

$$(1.2) \quad -\Delta u + \phi_u u - |u|^{p-2}u = \omega u \quad \text{in } \mathbb{R}^3,$$

where we have set

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} dy.$$

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Evidently, \( \phi_u \) satisfies \(-\Delta \phi_u = 4\pi |u|^2 \), is uniquely determined by \( u \) and is usually interpreted as the scalar potential of the electrostatic field generated by the charge density \(|u|^2\).

Because of its importance in many different physical framework, many authors have investigated the Schrödinger-Poisson system (sometimes called Schrödinger-Poisson-Slater system). Besides the paper of Benci and Fortuna [4] on a bounded domain, many papers on \( \mathbb{R}^3 \) have treated different aspects of this system, even with an additional external and fixed potential \( V(x) \). In particular ground states, radially and non-radially solutions are studied, see e.g. [1, 8, 9, 10, 13, 17, 19]. However in all these papers the frequency \( \omega \) is seen as a parameter so the authors deal with the functional

\[
\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\omega}{2} \int_{\mathbb{R}^3} u^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx
\]

and look for its critical points in \( H^1(\mathbb{R}^3; \mathbb{R}) \). In this approach nothing can be said a priori on the \( L^2 \)-norm of the solution. On the other hand in [16] the problem has been studied in a bounded domain \( \Omega \) with a nonhomogeneous Neumann boundary condition on the potential \( \phi_u \): here the compatibility condition for \( \phi_u \) imposes to study a constrained problem on \( \{ u \in H^1_0(\Omega) : \|u\|_2 = 1 \} \).

In spite of the above cited papers on \( \mathbb{R}^3 \), we look for solutions \( u \) with a priori prescribed \( L^2 \)-norm. The natural way to study the problem is to look for the constrained critical points of the functional

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx
\]

on the \( L^2 \)-spheres in \( H^1(\mathbb{R}^3; \mathbb{C}) \)

\[
B_\rho = \{ u \in H^1(\mathbb{R}^3; \mathbb{C}) : \|u\|_2 = \rho \}.
\]

So by a solution of (1.2) we mean a couple \((\omega_\rho, u_\rho) \in \mathbb{R} \times H^1(\mathbb{R}^3; \mathbb{C}) \) where \( \omega_\rho \) is the Lagrange multiplier associated to the critical point \( u_\rho \) on \( B_\rho \).

Actually we are interested in the existence of solutions of (1.2) with minimal energy (constrained to the sphere), i.e. to the minimization problem

(1.3) \[
I_\rho = \inf_{B_\rho} I(u)
\]

that makes sense for \( 2 < p < 10/3 \); indeed it is well known that in this case the \( C^1 \) functional \( I \) is bounded from below and coercive on \( B_\rho \) (see Lemma 3.1). As far as we know the only results on constrained minimization for nonlinear Schrödinger-Poisson are [18] in case \( p = \frac{8}{3} \) and [14] for \( p = 3 \). In [18] the authors prove that all the minimizing sequence for (1.3) are compact provided that \( \rho \) is sufficiently small. In [14] the author proves that if \( \Lambda \) is sufficiently large then the infimum of the minimization problem

(1.4) \[
I_\rho = \inf_{B_\rho} I_\Lambda(u)
\]
where
\[ I_\Lambda(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u |u|^2 \, dx - \frac{\Lambda}{3} \int_{\mathbb{R}^3} |u|^3 \, dx \]
is achieved for any \( \rho \).

It is known that, in this kind of problems, that main difficulty concerns with the lack of compactness of the (bounded) minimizing sequences \( \{u_n\} \subset B_\rho \); indeed two possible bad scenarios are possible:

- \( u_n \rightharpoonup 0 \);
- \( u_n \rightharpoonup \bar{u} \neq 0 \) and \( 0 < \|\bar{u}\|_2 < \rho \).

In order to avoid the above two cases and to show that the infimum is achieved, we prove a lemma (Lemma 2.1) in an abstract framework that guarantees the compactness of the minimizing sequences in the right norm. We recall that the abstract lemma is essentially contained in [2] and here it has been modified for the application to a wider class of functionals. Roughly speaking, this lemma is a version of the Concentration Compactness principle of [15] having in mind the application to a constrained minimization problem for functionals of the form
\[ I(u) = \frac{1}{2}\|u\|_{H^m,2}^2 + T(u). \]

The lemma we prove says that if \( \bar{u} \neq 0 \) and \( T(u) \) has a splitting property, i.e
\[ T(u_n - \bar{u}) + T(\bar{u}) = T(u_n) + o(1) \]
and the infima are subadditive in the following sense
\[ I_{\rho} < I_\mu + I_{\sqrt{\rho^2 - \mu^2}} \] for any \( 0 < \mu < \rho \),
then \( \|u_n - \bar{u}\|_{H^m} = o(1) \) and, as a consequence, \( \|\bar{u}\|_2 = \rho \).

As a consequence of the abstract minimization lemma we prove the following

**Theorem 1.1.** Let \( p \in \left\{ \frac{8}{3} \right\} \cup (3, \frac{10}{3}) \). Then there exist \( \rho_1 > 0 \) and \( \rho_2 > 0 \) (depending on \( p \)) such that all the minimizing sequences for (1.3) are precompact in \( H^1(\mathbb{R}^3; \mathbb{C}) \) up to translations provided that
\[ 0 < \rho < \rho_1 \quad \text{if} \quad p = \frac{8}{3} \]
\[ \rho_2 < \rho < +\infty \quad \text{if} \quad 3 < p < \frac{10}{3}. \]

In particular there exists a couple \( (\omega_\mu, u_\mu) \in \mathbb{R} \times H^1(\mathbb{R}^3; \mathbb{R}) \) solution of (1.2).

**Remark 1.1.** We underline that the result for \( p = \frac{8}{3} \) it has been proved first by [18] with a different approach to that developed in this paper. However it is interesting that our result for \( p = \frac{8}{3} \) is proved within the same general framework that is applied for \( 3 < p < \frac{10}{3} \).
As a matter of fact there are few results concerning the orbital stability of standing waves for Schrödinger-Poisson equation. We mention [12] and [14] where the orbital stability is achieved by following the original approach of [11]. On the other hand, following [7] and [18], the compactness of minimizers on $H^1(\mathbb{R}^3; \mathbb{C})$ and the conservation laws give rise to the orbital stability of the standing waves $\psi = e^{-i\omega t}u_\rho$ without further efforts; so we get the following

**Theorem 1.2.** Let $p \in \{\frac{8}{3}\} \cup (3, \frac{10}{3}).$ Then the set

$$S_\rho = \{e^{i\theta}u(x) : \theta \in [0, 2\pi), \|u\|_2 = \rho, I(u) = I_\rho\}$$

is orbitally stable.

The definition of *orbital stability* is recalled in the Section 4.

We underline that Lemma 2.1 can be applied to a wider class of minimization problems involving, for instance the biharmonic operator. For this reason, in the final Section 5 we study the following minimization problem

$$J_\rho = \inf_{B_\rho} \left( \frac{1}{2}\|\Delta u\|_2^2 + \int_{\mathbb{R}^N} F(u)dx \right)$$

where $B_\rho = \{u \in H^2(\mathbb{R}^N) : \|u\|_2 = \rho\}$ and the nonlinear local term $F : H^2(\mathbb{R}^N) \to \mathbb{R}$ fulfills some suitable assumptions that will be specified later. As a byproduct we obtain the orbital stability for the standing waves of the following Schrödinger equation involving the bilaplace operator

$$i\psi_t - \Delta^2 \psi - F'(|\psi|)\frac{\psi}{|\psi|} = 0, \ (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$  

1.1. **Notation.** In all the paper it is understood the all the functions, unless otherwise stated, are complex-valued, but we will write simply $L^s(\mathbb{R}^3), H^1(\mathbb{R}^3)$. ... where, for any $1 \leq s < +\infty$, $L^s(\mathbb{R}^3)$ is the usual Lebesgue space endowed with the norm

$$\|u\|_s^s := \int_{\mathbb{R}^3} |u|^sdx,$$

and $H^1(\mathbb{R}^3)$ the usual Sobolev space endowed with the norm

$$\|u\|_{H^1}^2 := \int_{\mathbb{R}^3} |\nabla u|^2dx + \int_{\mathbb{R}^3} |u|^2dx.$$  

In order to state the abstract lemma let us the space $D^{m,2}(\mathbb{R}^N)$. It is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{m,2}}^2 := \sum_{\alpha_1 + \ldots + \alpha_N = m} \int_{\mathbb{R}^N} |D^{\alpha}u|^2dx \text{ where } \alpha \in \mathbb{N}^N, D^{\alpha} = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N}.$$  

We need also $H^m(\mathbb{R}^N)$, the usual Sobolev space with norm

$$\|u\|_{H^m}^2 := \|u\|_{D^{m,2}}^2 + \|u\|_2^2.$$
We will use $C$ to denote a suitable positive constant whose value may change also in the same line and the symbol $o(1)$ to denote a quantity which goes to zero. We also use $O(1)$ to denote a bounded sequence.

The paper is organized as follows: Section 2 is devoted to the minimization problem and to the proof of the abstract lemma. Section 3 concerns the proof of the main theorem while in Section 4 the orbital stability of the standing waves is proved. In the final Section 5 the abstract lemma is applied to the biharmonic Schrödinger equation.

2. The minimization problem

As we have anticipated, we first prove an abstract result on a constrained minimization problem on Sobolev spaces $H^m(\mathbb{R}^N), N \geq 3$. Let we consider the following problem

$$I_\rho = \inf_{B_\rho} I(u)$$

where $B_\rho := \{ u \in H^m(\mathbb{R}^N) \text{ such that } \| u \|_2 = \rho \}$ and

$$I(u) := \frac{1}{2} \| u \|_{B^m,2}^2 + T(u)$$

(2.1)

Under suitable assumption on $T$ we have the strong convergence of the weakly convergent minimizing sequence.

**Lemma 2.1.** Let $T$ be a $C^1$ functional on $H^m(\mathbb{R}^N)$ and $\{ u_n \} \subset B_\rho$ be a minimizing sequence for $I_\rho$ such that $u_n \rightharpoonup \bar{u} \not= 0$; let us set $\mu = \| \bar{u} \|_2 \in (0, \rho]$.

Assume also that

$$T(u_n - \bar{u}) + T(\bar{u}) = T(u_n) + o(1);$$

(2.2)

$$T(\alpha_n(u_n - \bar{u})) - T(u_n - \bar{u}) = o(1)$$

(2.3)

where $\alpha_n = \sqrt{\rho^2 - \mu^2}/\| u_n - \bar{u} \|_2$ and finally that

$$I_\rho < I_\mu + I(\sqrt{\rho^2 - \mu^2}) \text{ for any } 0 < \mu < \rho.$$  

(2.4)

Then $\bar{u} \in B_\rho$.

Moreover if, as $n, m \to +\infty$

$$< T'(u_n) - T'(u_m), u_n - u_m > = o(1)$$

(2.5)

$$< T'(u_n), u_n > = O(1)$$

(2.6)

then $\| u_n - \bar{u} \|_{H^m(\mathbb{R}^N)} \to 0$. 


Proof. We argue by contradiction and assume that \( \mu_0 < \rho \). Since \( u_n - \bar{u} \to 0 \),
\[
\|u_n - \bar{u}\|^2 + \|\bar{u}\|^2 = \|u_n\|^2 + o(1)
\]
hence
\[
(2.7) \quad \alpha_n = \frac{\sqrt{\rho^2 - \mu^2}}{\|u_n - \bar{u}\|} \to 1.
\]
Since \( \{u_n\} \) is a minimizing sequence, we get
\[
\frac{1}{2}\|u_n\|^2_{D^m} + T(u_n) = I_\rho + o(1)
\]
and by (2.2), we deduce also
\[
\frac{1}{2}\|u_n - \bar{u}\|^2_{D^m} + \frac{1}{2}\|\bar{u}\|^2_{D^m} + T(u_n - \bar{u}) + T(\bar{u}) = I_\rho + o(1).
\]
Hence using (2.7) and (2.3) we infer
\[
\frac{1}{2}\|\alpha_n(u_n - \bar{u})\|^2_{D^m} + \frac{1}{2}\|\bar{u}\|^2_{D^m} + T(\alpha_n(u_n - \bar{u})) + T(\bar{u}) = I_\rho + o(1).
\]
Finally, notice that \( \|\alpha_n(u_n - \bar{u})\|_2 = \sqrt{\rho^2 - \mu^2} \), therefore
\[
I_{\sqrt{\rho^2 - \mu^2}} + I_\mu \leq I_\rho + o(1)
\]
which is in contradiction with (2.4). This implies that \( \|\bar{u}\|_2 = \rho \).

To prove the second assertion, we may assume, by the Ekeland variational principle, that \( \{u_n\} \) is a Palais-Smale sequence for the functional \( I \). From \( \bar{u} \in B_\rho \) it follows that \( \|u_n - \bar{u}\|_2 = o(1) \), hence it remains to show that \( \|u_n - \bar{u}\|_{D^m} = o(1) \) up to a sub-sequence. By assumptions there exists a sequence \( \{\lambda_n\} \subset \mathbb{R} \) such that for the functional \( I \) defined in (2.1)
\[
< I'(u_n) - \lambda_n u_n, v > = o(1), \quad \forall v \in H^m(\mathbb{R}^N)
\]
where \( < , , > \) denotes the duality pairing. It follows that
\[
< I'(u_n) - \lambda_n u_n, u_n > = o(1)
\]
since \( \|u_n\|_{H^m} \) is bounded. From this and assumption (2.6) it follows that the sequence \( \{\lambda_n\} \) is bounded, hence up to a sub-sequence there exists \( \lambda \in \mathbb{R} \) with \( \lambda_n \to \lambda \).

We now have
\[
< I'(u_n) - I'(u_m) - \lambda_n u_n + \lambda_m u_m , u_n - u_m > = o(1) \quad \text{as } n, m \to \infty
\]
hence, using that \( (\lambda_n - \lambda_m) < u_m, u_n - u_m > = o(1) \),
\[
\|u_n - u_m\|^2_{D^m} + < T'(u_n) - T'(u_m), u_n - u_m > - \lambda_n\|u_n - u_m\|^2_2 = o(1)
\]
Since \( \|u_n - u_m\|_2 = o(1), \lambda_n \to \lambda \) and (2.5) holds, we obtain that \( \{u_n\} \) is a Cauchy sequence in \( H^m(\mathbb{R}^N) \). Hence \( \|u_n - \bar{u}\|_{H^m} \to 0. \) \( \Box \)
3. Proof of the Main Theorem

We want to apply the previous theorem to the functional $I : H^1 \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx,$$

with

$$T(u) := N(u) + M(u)$$

where

$$N(u) = \frac{1}{4} \int_{\mathbb{R}^3} \phi_u |u|^2 dx, \quad M(u) = -\frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx.$$

Before to prove the main theorem some preliminaries are in order: the next lemma shows that the functional is bounded from below on $B_\rho$.

**Lemma 3.1.** If $2 < p < \frac{10}{3}$, then for every $\rho > 0$ the functional $I$ is bounded from below and coercive on $B_\rho$.

**Proof.** We apply the following Sobolev inequality

$$||u||_q \leq b_q ||u||_2^{1-\frac{2}{q}} ||\nabla u||_2^{\frac{2}{q}}$$

that holds for $2 \leq q \leq 2^*$ when $N \geq 3$. Therefore if $||u||_2 = \rho$ it follows $||u||_p^p \leq b_{p,\rho} ||\nabla u||_2^{\frac{3p}{2}-3}$ and

$$I(u) \geq \int \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p} |u|^p \right) dx \geq \frac{1}{2} ||\nabla u||_2^2 - b_{p,\rho} ||\nabla u||_2^{\frac{3p}{2}-3}$$

Since $p < \frac{10}{3}$, it results $\frac{3p}{2} - 3 < 2$ and

$$I(u) \geq \frac{1}{2} ||\nabla u||_2^2 + O(||\nabla u||_2^2).$$

which concludes the proof.

Notice that if we set $u_\lambda(\cdot) = \lambda^\alpha u(\lambda^\beta(\cdot)), \alpha, \beta \in \mathbb{R}, \lambda > 0$, then

$$\phi_{u_\lambda}(x) = \int_{\mathbb{R}^3} \frac{\lambda^{2\alpha+\beta} |u(\lambda^\beta y)|^2}{|\lambda^\beta x - \lambda^\beta y|^2} dy = \lambda^{2(\alpha-\beta)} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|\lambda^\beta y|^2} dy = \lambda^{2(\alpha-\beta)} \phi_u(\lambda^\beta x).$$

Now we prove some subadditivity properties that are crucial for the proof of Theorem 1.1.

**Lemma 3.2.** Let $p = \frac{8}{3}$, then there exists $\rho_1 > 0$ such that $I_\mu < 0$ for all $\mu \in (0, \rho_1)$ and

$$I_\mu < I_\mu + I_{\sqrt{\rho_1^2 - \mu^2}}.$$
for all $0 < \mu < \rho < \rho_1$.
If $3 < p < \frac{10}{3}$, then there exists $\rho_2 > 0$ such that $I_\mu < 0$ for all $\mu \in (\rho_2, +\infty)$ and
\[
I_\rho < I_\mu + I_{\sqrt{\rho^2 - \mu^2}}
\]
for all $\rho > \rho_2$ and $0 < \mu < \rho$.

**Proof.** We define $u_\theta(x) = \theta^{1-2\beta} u(\frac{x}{\theta})$ (so that $\|u_\theta\|_2 = \theta\|u\|_2$), then we have the following scaling laws:

\[
\begin{align*}
A(u_\theta) &:= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_\theta|^2 \, dx = \theta^{2-2\beta} A(u) \\
N(u_\theta) &:= \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_\theta} |u_\theta|^2 \, dx = \theta^{4-\beta} N(u) \\
M(u_\theta) &:= -\frac{1}{p} \int_{\mathbb{R}^3} |u_\theta|^p \, dx = \theta^{(1-\frac{4}{3}\beta)p+3\beta} M(u).
\end{align*}
\]

We get
\[
I(u_\theta) = \theta^2 \left( I(u) + (\theta^{-2\beta} - 1) A(u) + (\theta^{2-\beta} - 1) N(u) + (\theta^{(1-\frac{4}{3}\beta)p+3\beta-2} - 1) M(u) \right),
\]

(3.4) \quad $I(u_\theta) = \theta^2 (I(u) + f(\theta, u))$

where
\[
(3.5) \quad f(\theta, u) := (\theta^{-2\beta} - 1) A(u) + (\theta^{2-\beta} - 1) N(u) + (\theta^{(1-\frac{4}{3}\beta)p+3\beta-2} - 1) M(u).
\]

We distinguish between the case $p = \frac{8}{3}$ and $3 < p < \frac{10}{3}$.

**Case $p = \frac{8}{3}$:**

We notice that for $\beta = -2$ we get
\[
I(u_\theta) = \theta^6 A(u) + \theta^6 N(u) + \theta^{4p-6} M(u)
\]
and that $4p - 6 < 6$ for $2 < p < 3$. Hence for $\theta \to 0$ we have $I(u_\theta) \to 0^-$ which proves the first claim.

Let $u_n$ be a minimizing sequence in $B_\mu$ with $I_\mu < 0$, then
\[
a_1 < A(u_n) < a_2 \\
m_1 < M(u_n) < m_2.
\]

We get for $\beta = 0$
\[
f(\theta, u_n) = (\theta^2 - 1) N(u_n) + (\theta^{p-2} - 2) M(u_n).
\]

We have $\frac{d}{d\theta} f(\theta, u_n)|_{\theta=1} < 0$ provided that
\[
(3.6) \quad 2N(u_n) + (2 - \frac{8}{3}) ||u_n||_{L^\frac{8}{3}} < 0.
\]
Relation (3.6) holds for \( \mu \) sufficiently small recalling the following inequality

\[
N(u_n) \leq C ||u_n||_{\frac{4}{3}}^\frac{4}{3} ||u_n||_{L^\infty}^\frac{8}{3}.
\]

Indeed we have

\[
2N(u_n) + (2 - \frac{8}{3}) ||u_n||_{L^\infty}^\frac{8}{3} \leq ||u_n||_{L^\infty}^\frac{8}{3} (C \mu^\frac{4}{3} + (2 - \frac{8}{3})) < 0.
\]

We notice that

\[
d^2 \frac{d}{d\theta^2} f(\theta, u_n) = 2N(u_n) - \frac{1}{p} \left( \frac{8}{3} - 2 \right) \left( \frac{8}{3} - 3 \right) \theta^{\frac{8}{3} - 4} ||u_n||_{L^\infty}^\frac{8}{3} > 0,
\]

and that

\[
d \frac{d}{d\theta} f(\theta, u_n) = 0
\]

for \( \tilde{\theta} \) fulfilling

\[
\tilde{\theta}^{-\frac{4}{3}} = 2 \frac{N(u_n)}{||u_n||_{L^\infty}^\frac{8}{3}} \leq C \mu^\frac{4}{3}
\]

Therefore we find that for \( \mu \) sufficiently small \( f(\theta, u_n) < 0 \) for \( \theta \in \left( 1, C^\frac{8}{3} \right) \). We get

\[
I_{\theta \mu} < \theta^2 I(u_n) = \theta^2 I_\mu,
\]

for \( \theta \in \left( 1, C^\frac{8}{3} \right) \).

Now we argue as in [15] observing that for \( \mu \) sufficiently small

\[
I_p = I_{p \mu} < \frac{\rho^2}{\mu^2} I_\mu = \frac{\rho^2 - \mu^2 + \mu^2}{\mu^2} I_\mu = \frac{\rho^2 - \mu^2}{\mu^2} I_{\sqrt{\rho^2 - \mu^2} - \mu^2} + I_\mu < I_\mu + I_{\sqrt{\rho^2 - \mu^2}}.
\]

**Case 3** \( p < \frac{10}{3} \):

We notice that for \( \beta = -2 \) we get

\[
I(u_\theta) = \theta^6 A(u) + \theta^6 N(u) + \theta^{4p-6} M(u)
\]

and that \( 4p - 6 > 6 \) for \( 3 < p < \frac{10}{3} \). Hence for \( \theta \) sufficiently large we have \( I(u_\theta) < 0 \) which proves the first claim.

Let \( u_n \) be a minimizing sequence in \( B_\mu \) with \( I_\mu < 0 \), then

\[
0 < k_1 < A(u_n) < k_2
\]

\[
0 < \eta_1 < |M(u_n)| < \eta_2.
\]

Indeed if \( A(u_n) = o(1) \) we have \( |M(u_n)| = o(1) \) and \( I_\mu = 0 \). For \( \beta = -2 \) we have

\[
f(\theta, u_n) := (\theta^4 - 1) A(u) + (\theta^4 - 1) N(u) + (\theta^{4p-8} - 1) M(u),
\]
with \(4p - 8 > 4\) and
\[
(3.8) \quad \frac{d}{d\theta} f(\theta, u_n)|_{\theta=1} < 0, \quad \frac{d^2}{d\theta^2} f(\theta, u_n) < 0 \text{ for all } \theta > 1.
\]
In order to show (3.8) we have first
\[
(3.9) \quad \frac{d}{d\theta} f(1, u_n)|_{\theta=1} = 4(A(u_n) + N(u_n)) + (4p - 8)M(u_n) < k < 0.
\]
Moreover we have
\[
(3.10) \quad \frac{d^2}{d\theta^2} f(\theta, u_n) = 12\theta^2(A(u_n) + B(u_n)) + (4p - 8)(4p - 9)\theta^{4p - 10}M(u_n) < k < 0.
\]
Thanks to (3.9) and (3.10) we get
\[
f(\theta, u_n) < k(\theta) < 0 \text{ for all } \theta > 1, \; n \in \mathbb{N},
\]
and hence
\[
I_{\theta\mu} < \theta^2 I(u_n) = \theta^2 I_\mu.
\]
Let us suppose that \(\mu < \sqrt{\rho^2 - \mu^2}\). We distinguish three cases
\begin{itemize}
  \item \(\mu < \sqrt{\rho^2 - \mu^2} < \rho_2\)
  \item \(\mu < \rho_2 < \sqrt{\rho^2 - \mu^2}\)
  \item \(\rho_2 < \mu < \sqrt{\rho^2 - \mu^2}\)
\end{itemize}
The first case is trivial. For the second one we have \(I_{\sqrt{\rho^2 - \mu^2}} > I_\rho\) and we conclude.
For the third case we argue as for \(p = \frac{8}{3}\). \(\square\)

**Proposition 3.1.** If \(2 < p < \frac{10}{3}\), then the functionals \(N\) and \(M\) fulfill (2.2), (2.3), (2.5), (2.6).

**Proof.** By Lemma 3.1 any minimizing sequence is bounded in the \(H^1\)-norm. Hence \(\{u_n\}\) is bounded in all \(L^s\) norms for \(s \in [2, 2^*]\) and there exists \(\bar{u} \in H^1(\mathbb{R}^3)\) such that \(u_n \rightharpoonup \bar{u}\) in \(H^1(\mathbb{R}^3)\).

The functionals \(M\) and \(N\) satisfy the condition (2.2) (see [5] and Lemma 2.2 in [20]).

We have, by the convolution and Sobolev inequalities
\[
N(u_n) = \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \leq C \|u_n\|_{12/5}^4 \leq C \|u_n\|_2^3 \|\nabla u_n\|_2
\]
and than the relation (2.3) follows from
\[
N(\alpha_n(u_n - \bar{u})) - N(u_n - u) = (\alpha_n^4 - 1)N(u_n - \bar{u}) = o(1)
\]
\[
M(\alpha_n(u_n - \bar{u})) - M(u_n - u) = (\alpha_n^p - 1)M(u_n - \bar{u}) = o(1)
\]
since \(\alpha_n \to 1\). Notice that thanks to the classical interpolation inequality we have
\[
\|u_n - u_m\|_p \leq \|u_n - u_m\|_2^{1-\alpha} \|\nabla u_n - \nabla u_m\|_2^{1-\alpha} \quad \text{where} \quad \frac{\alpha}{2} + \frac{(1 - \alpha)}{2^*} = \frac{1}{p}
\]
and then on the minimizing sequence we get

\[ ||u_n - u_m||_p = o(1). \]

We obtain, for \( q = p/(p-1) \)

\[ \int_{\mathbb{R}^3} |u_n|^{p-1}|u_n - u| \, dx \leq \left( \int_{\mathbb{R}^3} |u_n|^q \, dx \right)^{\frac{p}{q}} \left( \int_{\mathbb{R}^3} |u_n - u_m|^p \, dx \right)^{\frac{1}{p}} = o(1) \]

and then

\[ \left| \int_{\mathbb{R}^3} (|u_n|^{p-1} - |u_m|^{p-1})(u_n - u_m) \, dx \right| \leq C \|u_n - u_m\|_p = o(1). \]

This proves (2.5) for \( M \). The verification of (2.5) for \( N \) follows from

\[ \int_{\mathbb{R}^3} \phi_{u_n} u_n(u_n - u_m) \, dx \leq \|\phi_{u_n}\|_6 \|u_n\|_2 \|u_n - u_m\|_3 \]

\[ \leq C \|u_n\|_{H^1(\mathbb{R}^3)}^2 \|u_n\|_2 \|u_n - u_m\|_3 = o(1) \]

Then condition (2.6) is trivial. \[\Box\]

Now we can conclude the proof of Theorem 1.1. In case \( p = \frac{8}{3} \) we can fix \( \rho \in (0, \rho_1) \) due to the fact that \( I_\rho < 0 \) for all \( \rho \in (0, \rho_1) \). In case \( 3 < p < \frac{10}{3} \) we fix \( \rho \in (\rho_2, +\infty) \).

Let \( \{u_n\} \) be a minimizing sequence in \( B_\rho \). Notice also that for any sequence \( y_n \in \mathbb{R}^n \) we have that \( u_n(\cdot + y_n) \) is still a minimizing sequence for \( I_\rho \). This implies that the proof of the Theorem can be concluded provided that we show the existence of a sequence \( y_n \in \mathbb{R}^3 \) such that the weak limit of \( u_n(\cdot + y_n) \) belongs to \( B_\rho \) and that the convergence is strong in \( H^1(\mathbb{R}^3) \). Notice that if

\[ \lim_{n \to \infty} \left( \sup_{y \in \mathbb{R}^n} \int_{B(y,1)} |u_n|^2 \, dx \right) = 0 \]

then \( u_n \to 0 \) in \( L^q(\mathbb{R}^3) \) for any \( q \in (2, 2^*) \), where \( B(a, r) = \{ x \in \mathbb{R}^3 : |x - a| \leq r \} \).

Since \( I_\rho < 0 \) we have that

\[ \sup_{y \in \mathbb{R}^n} \int_{B(y,1)} |u_n|^2 \, dx \geq \mu > 0. \]

In this case we can choose \( y_n \in \mathbb{R}^3 \) such that

\[ \int_{B(0,1)} |u_n(\cdot + y_n)|^2 \, dx \geq \mu > 0 \]

and hence, due to the compactness of the embedding \( H^1(B(0,1)) \subset L^2(B(0,1)) \), we deduce that the weak limit of the sequence \( u_n(\cdot + y_n) \) is not the trivial function, so \( u_n \to \bar{u} \neq 0 \). Since the subadditivity condition holds, we can apply the abstract Lemma 2.1 and conclude the proof.
4. The orbital stability

In this section we prove Theorem 1.2 following the ideas of [7]. First of all we recall the definition of orbital stability.

We define

\[ S_\rho = \{ e^{i\theta} u(x) : \theta \in [0, 2\pi), \| u \|_2 = \rho, \ I(u) = I_\rho \}. \]

We say that \( S_\rho \) is orbitally stable if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( \psi_0 \in H^1(\mathbb{R}^3) \) with

\[ \inf_{v \in S_\rho} \| v - \psi_0 \|_{H^1(\mathbb{R}^3; \mathbb{C})} < \delta \]

we have

\[ \forall t > 0 \quad \inf_{v \in S_\rho} \| \psi(t, \cdot) - v \|_{H^1(\mathbb{R}^3; \mathbb{C})} < \varepsilon \]

where \( \psi(t, \cdot) \) is the solution of (1.1) with initial datum \( \psi_0 \). We notice explicitly that \( S_\rho \) is invariant by translation, i.e. if \( v \in S_\rho \) then also \( v(\cdot - y) \in S_\rho \) for any \( y \in \mathbb{R}^3 \).

We recall that the energy and the charge associated to \( \psi(x, t) \) evolving according to (1.1) are given by

\[
E(\psi(x, t)) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \psi|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} (|x|^{-1} * |\psi|^2)|\psi|^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} |\psi|^p dx
\]

and

\[
C(\psi(x, t)) := \frac{1}{2} \int_{\mathbb{R}^3} |\psi|^2 dx = C(\psi(x, 0)).
\]

So our action functional \( I \) is exactly the energy. In order to prove Theorem 1.2 we argue by contradiction assuming that there exists a \( \rho \) such that \( S_\rho \) is not orbitally stable. This means that there exists \( \varepsilon > 0 \) and a sequence of initial data \( \{ \psi_{n,0} \} \subset H^1(\mathbb{R}^3) \) and \( \{ t_n \} \subset \mathbb{R} \) such that the maximal solution \( \psi_n \), which is global and \( \psi_n(0, \cdot) = \psi_{n,0} \), satisfies

\[
\lim_{n \to \infty} \inf_{v \in S_\rho} \| \psi_{n,0} - v \|_{H^1(\mathbb{R}^3)} = 0 \quad \text{and} \quad \inf_{v \in S_\rho} \| \psi_n(t_n, \cdot) - v \|_{H^1(\mathbb{R}^3)} \geq \varepsilon
\]

Then there exists \( u_\rho \in H^1(\mathbb{R}^3) \) minimizer of \( I_\rho \) and \( \theta \in \mathbb{R} \) such that \( v = e^{i\theta} u_\rho \) and

\[ \| \psi_{n,0} \|_2 \to \| v \|_2 = \rho \quad \text{and} \quad I(\psi_{n,0}) \to I(v) = I_\rho \]

Actually we can assume that \( \psi_{n,0} \in B_\rho \) (there exist \( \alpha_n = \rho/\| \psi_{n,0} \|_2 \to 1 \) so that \( \alpha_n \psi_{n,0} \in B_\rho \) and \( I(\alpha_n \psi_{n,0}) \to I_\rho \), i.e. we can replace \( \psi_{n,0} \) with \( \alpha_n \psi_{n,0} \)).

So \( \{ \psi_{n,0} \} \) is a minimizing sequence for \( I_\rho \), and since

\[
I(\psi_n(\cdot, t_n)) = I(\psi_{n,0}),
\]

also \( \{ \psi_n(\cdot, t_n) \} \) is a minimizing sequence for \( I_\rho \). Since we have proved that every minimizing sequence has a subsequence converging (up to translation) in \( H^1 \)-norm to a minimum on the sphere \( B_\rho \), we readily have a contradiction.
Finally notice that, since in general, if \( \psi(x, t) = |\psi(x, t)|e^{iS(x, t)} \) then
\[
I(\psi(x, t)) = I(|\psi(x, t)|) + \int_{\mathbb{R}^3} |\psi(x, t)|^2|\nabla S(x, t)|^2 dx,
\]
we easily conclude that the minimizer \( u_\rho \) has to be real valued.

### 5. Application to a biharmonic Schrödinger equation

In this final section we apply the above abstract result to the following Schrödinger equation involving the biharmonic operator

\[
(i\psi_t - \Delta^2 \psi - F'(|\psi|)\frac{\psi}{|\psi|}) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N \quad N > 4.
\]

The search of standing wave solution \( \psi(x, t) = u(x)e^{-i\omega t} \) lead us to study the following semilinear equation

\[
\Delta^2 u + F(u) = -\omega u
\]

which will be studied by minimizing the functional \( J : H^2(\mathbb{R}^N) \to \mathbb{R} \) given by
\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\Delta u|^2 dx + \int_{\mathbb{R}^N} F(u)dx
\]
on \( B_\rho = \{ u \in H^2(\mathbb{R}^N) : \|u\|_2 = \rho \} \), namely studying the minimization problem
\[
J_\rho = \min_{B_\rho} J(u)
\]
where \( \omega \) is seen as the Lagrange multiplier.

We make the following hypothesis on the nonlinearity

\[(F_p) \quad |F'(s)| \leq c_1 |s|^q + c_2 |s|^p \quad \text{for some} \quad 2 < q \leq p < \frac{N+4}{N-4}
\]

\[(F_0) \quad F(s) \geq -c_1 s^2 - c_2 s^{\frac{2+4}{N}} \quad \text{with} \quad c_1, c_2 > 0
\]

\[(F_1) \quad \exists s_0 \in (0, +\infty) \quad \text{such that} \quad F(s_0) < 0.
\]

So we get the following result.

**Theorem 5.1.** Let \((F_p), (F_0)\) and \((F_1)\) hold. Then there exists \( \rho_0 \) such that for all \( \rho > \rho_0 \), \( J_\rho \) is achieved on \( u_\rho \) and \((\omega_\rho, u_\rho)\) is a solution of \((5.2)\).

In order to apply the abstract Lemma 2.1 to the functional
\[
J(u) = \frac{1}{2} \|u\|_{H^2}^2 + T(u) \quad \text{where} \quad T(u) = \int_{\mathbb{R}^N} F(u)dx
\]
we need to prove the boundedness of \( J \) on \( B_\rho \) and the subadditivity condition.

**Proposition 5.1.** If \((F_p), (F_0)\) and \((F_1)\) hold, then there exists \( \rho_0 \) such that for all \( \rho > \rho_0 \)

- \(-\infty < J_\rho < 0;\)
Proof. By arguing as in [3] we have that $J_\rho > -\infty$ and that the functional is coercive. We build a sequence of radial functions $\{u_n\}$ in $H^2(\mathbb{R}^N)$ such that $J(u_n) < 0$ for large $n$. The sequence is defined as follows:

$$u_n(r) = \begin{cases} 
  s_0 & r < R_n; \\
  s_0 \cos^2\left(\frac{\pi}{2}(r - R_n)\right) & R_n \leq r \leq R_n + 1; \\
  0 & r > R_n + 1.
\end{cases}$$

We show that $J(u_n) < 0$ when $R_n \to +\infty$. Notice that for a radial function $u$ the laplacian is given by

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \left(\frac{N - 1}{r}\right).$$

After some computation we have

$$\frac{\partial u_n(r)}{\partial r} = \begin{cases} 
  0 & r < R_n; \\
  -\pi s_0 \cos\left(\frac{\pi}{2}(r - R_n)\sin\left(\frac{\pi}{2}(r - R_n)\right)\right) & R_n \leq r \leq R_n + 1; \\
  0 & r > R_n + 1
\end{cases}$$

and

$$\frac{\partial^2 u_n(r)}{\partial r^2} = \begin{cases} 
  0 & r < R_n; \\
  \frac{s_0}{2} \sin^2\left(\frac{\pi}{2}(r - R_n)\right) - \cos^2\left(\frac{\pi}{2}(r - R_n)\right) & R_n \leq r \leq R_n + 1; \\
  0 & r > R_n + 1
\end{cases}$$

Then we get

$$J(u_n) = \int_{\mathbb{R}^N} \frac{1}{2} |\Delta u_n|^2 + F(u_n) \, dx \leq$$

$$\leq C_1 \int_{R_n}^{R_n + 1} \left[ \left| \frac{\partial^2 u_n(r)}{\partial r^2} + \frac{\partial u_n(r)}{\partial r} \left(\frac{N - 1}{r}\right)\right|^2 + \sup_{|s| < s_0} F(s) \right] r^{N-1} dr +$$

$$+ C_2 \int_0^{R_n} F(s_0) r^{N-1} dr$$

where $C_1$ and $C_2$ are strictly positive constants. We have $F(s_0) < 0$ and, thus, an easy growth estimate gives $J(u_n) < 0$ for $R_n \to +\infty$. \(\square\)

**Proposition 5.2.** For any $\rho > \rho_0$ and $0 < \mu < \rho$ the following subadditivity condition holds

(5.3) \hspace{1cm} J_\rho < J_\mu + \sqrt{\rho^2 - \mu^2}.

Proof. Let us define $u_\lambda(x) = u(\frac{x}{\lambda})$ (so that $\|u_\lambda\|_2 = \lambda\|u\|_2$). We have

$$J(u_\lambda) = \frac{\lambda^2 - \frac{\mu^2}{2}}{2} \|u\|_{D^{2,2}}^2 + \lambda^2 T(u).$$
and then

\[ J_{\lambda \mu} \leq \lambda^2 \left( \frac{1}{2} \|u_n\|_{D^{2,2}}^2 + T(u_n) \right) + \frac{1}{2} \left( \lambda^2 - \frac{4}{N} - \lambda^2 \right) \|u_n\|_{D^{2,2}}^2 \]

for any minimizing sequence \( \{u_n\} \subset B_\mu \). Taken \( \mu \) such that \( J_\mu < 0 \) and \( \lambda > 1 \)
we obtain

\[ J_{\lambda \mu} < \lambda^2 J(\mu). \]

By arguing as in Lemma 3.2 we have (5.3). \( \square \)

**Proposition 5.3.** If \((F_p),(F_0)\) and \((F_1)\) hold then the functional \(T\) fulfills (2.2), (2.3), (2.5), (2.6)

**Proof.** It follows as in Proposition 3.1. Condition (2.2) follows from standard arguments. \( \square \)

**Proof of Theorem 5.1.** We argue as in the proof of Theorem 1.1. Recall that if \( \{u_n\} \) is a bounded sequence in \( H^2(\mathbb{R}^N) \) such that

\[ \lim_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B(0,1)} |u_n|^2 \right) = 0 \]

then \( u_n \to 0 \) in \( L^q(\mathbb{R}^n) \) for any \( q \in (2, \frac{2N}{N-4}) \). The proof of this fact is given in [3]. Finally we apply Lemma 2.1 to the functional \( J(u) \). \( \square \)

Finally the orbital stability of the standing waves is proved. As in the previous section, we define

\[ S_\rho = \{ e^{i\theta} u(x) : \theta \in [0, 2\pi), \|u\|_2 = \rho, J(u) = J_\rho \} \]

and we say that \( S_\rho \) is orbitally stable if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( \psi_0 \in H^2(\mathbb{R}^N) \) with \( \inf_{v \in S_\rho} \|v - \psi_0\|_{H^2(\mathbb{R}^N)} < \delta \) we have

\[ \forall \, t > 0 \quad \inf_{v \in S_\rho} \|\psi(\cdot, t) - v\|_{H^2(\mathbb{R}^N)} < \varepsilon \]

where \( \psi(\cdot, t) \) is the solution of (5.1) with initial datum \( \psi_0 \). Arguing as for the Schrodinger-Poisson equation we obtain the following

**Corollary 5.1.** Let \((F_p),(F_0)\) and \((F_1)\) hold. Then there exists \( \rho_0 > 0 \) such that for any \( \rho \in (\rho_0, +\infty) \) the standing waves \( \psi_\rho(t, x) = e^{-i\omega t} u_\rho(x) \) are orbitally stable solutions of (5.1).

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