Application of fuzzy random-based multi-objective linear fractional programming to inventory management problem

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ABSTRACT
This research article aims to study a multi-objective linear fractional programming (FMOLFP) problem having fuzzy random coefficients as well as fuzzy pseudorandom decision variables. Initially, the FMOLFP model is converted to a single objective fuzzy linear programming (FLP) model. Secondly, we show that a fuzzy random optimal solution of an FLP problem is resolved into a class of random optimal solution of relative pseudorandom linear programming (LP) model. As a result, some theorems show that a fuzzy random optimal solution of a fuzzy pseudorandom LP problem is combined with a series of random optimal solutions of relative pseudorandom LP problems. As an application, the developed approach is implemented to an inventory management problem by taking the parameters as trapezoidal fuzzy numbers, ultimately resulting in a new initiative for modelling real-world problems for optimization. In the last, some numerical examples are introduced to clarify the obtained results and their applicability.

1. Introduction
Fractional programming, i.e. the optimization of a fraction of two functions subject to some prescribed conditions, plays important role in modelling and optimization in the field of management, engineering, finance, economics and science. Recently, huge developments have taken place in this area. Charnes and Cooper (1962) proposed the programming with linear fractional functions, termed as fractional programming problem (FPP). Normally, FPP is a decision making model that aims to optimize the ratio subject to some constraints. In real-life situations, the decision maker (DM) sometimes may face to compute the ratio between stock of goods and sales, output and employee etc., with both denominator and numerator are linear. When one ratio is considered as an objective function under linear constraints, the problem is referred as linear FPP. As per applications scenario, FPP is used in the fields of traffic planning (Dantzig et al., 1966), and many more. In the meantime, some applications of FPP and the algorithms to solve this kind of problems were presented by Dinkelbach (1967). Luhandjula (1984) developed some fuzzy approaches to solve the multi-objective linear FPP. Sakawa and Yano (1988) proposed an approach for multi-objective linear FPPs. Guzel (2013) suggested a proposal for solving a multi-objective linear FPP.

Few decades ago, the multi-objective FPPs were proposed by many authors. Bitran and Novaes (1973) studied the LP problem with a fractional objective function. Ammar and Khalifa (2004, 2009) studied linear FPP with fuzzy parameters. Biswas and Dewan (2014) presented the priority based fuzzy goal programming (GP) methodology to fractional goals using dynamic programming. Many authors employed the fuzzy GP technique to solve the multi-level multi-objective LP problems, like for example Sadjadi et al. (2005), Dutta and Kumar (2015), Veeramani and Sumathi (2017), etc.

Many methodologies have been developed to counteract or curtail the uncertainty that is generated by decision making in fuzzy environment. In literature, Zadeh (1965) introduced a new idea, which is known as theory of fuzzy sets. The basic theory of uncertainty has been applied with immense success in numerous fields of day-to-day life. Several studies have been done on the topic of fuzzy sets. Zimmermann (1978) introduced the fuzzy programming and LP with multi-objective functions. Later, several authors presented their work on fuzzy set theory as introduced by Zadeh (1965).
LP problems with fuzzy random variable coefficient were presented and their applications in the area of distribution problems by Guangyuan and Zhong (1993a, 1993b). Tanaka and Asai (1984) presented the fuzzy LP problem with fuzzy numbers. Dutta et al. (1993) investigated the effect of tolerance in fuzzy linear FPP. Many authors have investigated the fuzzy random variables with various types of membership functions. Zhong et al. (1994) studied the fuzzy random LP problem with fuzzy as well as random nature, called fuzzy random LP problems. Liu and Liu (2002) derived a model for expected value of fuzzy variables. Pop and Stancu (2008) presented a method to solve fully fuzzified linear FPPs.

Recently, various applications of FPP were proposed by researchers. Atanassov (1986) developed the idea of intuitionistic fuzzy sets in many applications. Banerjee and Roy (2010) derived the solution methodology for single as well as multi-objective stochastic inventory models with fuzzy cost components. Das and Maiti (2013) studied the fuzzy stochastic inequality and equality possibility constraints and the applications in a production-inventory model using optimal control method. Chakraborty et al. (2013, 2016) presented an intuitionistic fuzzy method for pareto optimal solution of manufacturing problem. Singh and Yadav (2016) and Ali et al. (2018) derived a fuzzy programming method to solve an intuitionistic fuzzy linear FPP.

In literature, several authors considered fuzzy as well as random (probabilistic) nature of parameters of the problem. EI-Asharm and Girgis (1996) presented their research work on linear multi-objective programming in random and fuzzy environments. Chen (2005) developed an FPP method to solve the stochastic inventory control model. Nasseri and Bavandi (2019) proposed a study on fuzzy stochastic linear FPP based on fuzzy mathematical programming. Very recently, Yang et al. (2020) applied fractional programming to solve an agricultural planting problem. Valipour and Yaghoobi (2021) investigated some fuzzy linearization methods to solve FMOLFP problem. Table 1 illustrates the description of related work by different authors.

In this research work, a fuzzy random multi-objective linear FPP approach is proposed. The proposed approach is demonstrated with an application to inventory management problem. The cardinal contributions of the current work are demonstrated below:

- Introducing the concept of fuzziness and randomness in FPP.
- Applying FPP approach to solve the multi-item inventory management model.

| Author & References | Problem Type | Method used |
|---------------------|--------------|-------------|
| Luhandjula (1984)   | Multi-objective linear FPP | Fuzzy approach |
| Sakawa and Yano (1988) | Multi-objective linear FPP | Interactive fuzzy satisfying method |
| El-Asharm and Girgis (1996) | Multi-objective linear FPP | Randomness and fuzziness-based method |
| Ammar and Khalifa (2004) | Multi-criteria linear FPP | Parametric approach |
| Sadjadi et al. (2005) | Multi-objective FPP in inventory | Fuzzy approach |
| Guzel (2013) | Multi-objective linear FPP | Fuzzy approach |
| Dutta and Kumar (2015) | Multi-objective linear FPP in inventory | Intuitionistic fuzzy approach |
| Singh and Yadav, (2016) | Linear FPP with fuzzy cost of objective, resources and technological coefficients | Fuzzy approach using triangular fuzzy numbers. |
| Veeramani and Sumathi (2017) | Multi-objective linear FPP | Fuzzy approach using triangular fuzzy numbers. |
| Ali et al. (2018) | Multi-objective linear FPP in inventory | Intuitionistic fuzzy approach |
| Nasseri and Bavandi (2019) | Linear FPP | Fuzzy stochastic based programming method |
| Yang et al. (2020) | Linear FPP | Fuzzy approach |
| Valipour and Yaghoobi (2021) | Multi-objective linear FPP | Fuzzy linearization approach |
| Proposed paper | Multi-objective linear FPP | Fuzzy random coefficients and pseudorandom decision variables |

- Probabilistic constraints are converted to the corresponding crisp constraints.
- Demonstrating the problem and algorithm illustrating a numerical application of two-item inventory management problem.
- A comparative study is performed between the existing methods and the suggested approach.

The remainder of the paper is designed as follows: Section 2 demonstrates some preliminaries required in this paper. Section 3 introduces fuzzy random multi-objective linear FPP formulation. In Section 4, some of basic theorems with proof are presented. In Section 5, an inventory model is given as an application. In the end, some conclusions along with upcoming research ideas are reported in the last Section.

### 2. Preliminaries

This Section presents some of basic concepts.

**Definition 2.1:** (Kaufmann & Gupta, 1988). A fuzzy set $\tilde{P}$ defined on the set of real numbers $\mathbb{R}$ is said to be fuzzy numbers if its membership function
\[ \mu_\beta(x) : \mathbb{R} \rightarrow [0, 1], \] satisfies the below mentioned conditions:

1. \( \mu_\beta(x) \) is an upper semi-continuous membership function;
2. \( \hat{P} \) is convex fuzzy set, i.e. \( \mu_\beta(\delta x + (1 - \delta)y) \geq \min\{\mu_\beta(x), \mu_\beta(y)\} \) for all \( x, y \in \mathbb{R}; 0 \leq \delta \leq 1; \)
3. \( \hat{P} \) is normal, i.e. \( \exists x_0 \in \mathbb{R} \) for which \( \mu_\beta(x_0) = 1; \)
4. \( \text{Supp}(\hat{P}) = \{x \in \mathbb{R} : \mu_\beta(x) > 0\} \) is the support of \( \hat{P} \), and the closure \( \text{cl} (\text{Supp}(\hat{P})) \) is compact set.

**Definition 2.2:** (Kaufmann & Gupta, 1988). A fuzzy number \( \tilde{A}(a_1, a_2, a_3, a_4) \) is a trapezoidal fuzzy number if its membership function

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\
\frac{a_2-x}{a_3-a_2}, & a_2 \leq x \leq a_3, \\
\frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4, \\
0, & \text{otherwise}
\end{cases}
\]

The interval confidence of \( \tilde{A}(a_1, a_2, a_3, a_4) \) is defined as follows:

\[
\tilde{A}_u = [(a_2 - a_1)\alpha + a_1, -(a_4 - a_3)\alpha + a_4]; \forall \alpha \in [0, 1].
\]

Let \( F_0(\mathbb{R}) \) be the set of all compact trapezoidal fuzzy numbers on \( \mathbb{R}. \) For any \( \tilde{A} \in F_0(\mathbb{R}), \tilde{A} \) satisfies the following conditions:

1. \( \exists x \in \mathbb{R} \sum \tilde{A}(x) = 1; \)
2. \( \tilde{A}_u = [a_{u}, d_{u}] \) in \( \mathbb{R}, \forall \alpha \in [0, 1], \) and \( a_u \leq d_u.\)

More definitions related to the intervals are all discussed in the Appendix.

### 3. Problem definition

A fuzzy random multi-objective linear fractional programming (FRMOLFP) problem is formulated as

\[
\begin{align*}
\text{Max} & \quad \tilde{Z}_k(\tilde{C}, \tilde{D}, \tilde{C}_0, \tilde{D}_0, \tilde{X}) \\
& = \frac{N_k(\tilde{C}, \tilde{C}_0, \tilde{X})}{D_k(\tilde{D}, \tilde{D}_0, \tilde{X})} = \frac{\tilde{C}_k^T \odot \tilde{X} \odot \tilde{C}_0^T}{D_k^T \odot \tilde{X} \odot D_k^T}, \quad k = 1, 2, \ldots, K
\end{align*}
\]

Subject to

\[
\tilde{S}(\tilde{A}, \tilde{B}, \tilde{X}) = \{ \tilde{X} \in \mathbb{R}^n : \tilde{A} \odot \tilde{X} \leq \tilde{B} \odot \tilde{X} \geq 0, \}
\]

where, \( \tilde{A} = (\tilde{a}_{ij})_{m \times n}, \tilde{B} = (\tilde{b}_i)_{m \times 1} \in \mathbb{R}^m, \)

\[
\tilde{C}_k = (\tilde{c}_{jk})_{1 \times n} \in \mathbb{R}^n, \tilde{D}_k = (\tilde{d}_{jk})_{1 \times n} \in \mathbb{R}^n,
\]

\[
\tilde{X} = (\tilde{x}_j) \in \mathbb{R}^n, \tilde{C}_0, \tilde{D}_0 \in \mathbb{R}, \tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_{jk}, \tilde{d}_{jk}, \tilde{c}_k, \tilde{d}_k \in F_0(\mathbb{R}),
\]

\[
i = 1, 2, \ldots, m; j = 1, 2, \ldots, n,
\]

\[
k = 1, 2, \ldots, K, \text{ and } D_k(\tilde{D}, \tilde{D}_0, \tilde{X}) = \tilde{D}_k^T \odot \tilde{X} \odot \tilde{D}_k^T > 0;
\]

\[
\forall k = 1, 2, \ldots, K; \forall \tilde{X} \in \tilde{S}.
\]

**Remark 3.1:** \( \tilde{X} \geq 0 \) means that \( (\tilde{x}_j)_\alpha \geq 0, j = 1, 2, \ldots, n, \) and for any \( \alpha \in (0, 1]. \)

Problems (1) and (2) can be rewritten as in the following form:

Find \( \tilde{X} \) such that

\[
\frac{\tilde{C}_k^T \odot \tilde{X} \odot \tilde{C}_0^T}{D_k^T \odot \tilde{X} \odot D_k^T} = \max_{\tilde{x} \in \tilde{S}} \left( \frac{\tilde{C}_k^T \odot \tilde{X} \odot \tilde{C}_0^T}{D_k^T \odot \tilde{X} \odot D_k^T} \right)
\]

**Definition 3.1:** (Nasseri & Bavandi, 2019). The \( \alpha \)-level set of \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{C}_0, \tilde{D}_0, \tilde{X} \) is defined as the ordinary set of \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{C}_0, \tilde{D}_0, \tilde{X} \) for which the degree of their membership functions exceeds the level \( \alpha. \)

Now, based on the \( \alpha \)-level set concept, the problem (3) can be rewritten as in the following non-fuzzy form:

\[
\frac{N_k(\tilde{C}, \tilde{C}_0, X)}{D_k(\tilde{D}, \tilde{D}_0, X)} = \frac{\tilde{C}_k^T \odot X \odot \tilde{C}_0^T}{D_k^T \odot X \odot D_k^T} = \max_{X \in S} \left( \frac{\tilde{C}_k^T \odot X \odot \tilde{C}_0^T}{D_k^T \odot X \odot D_k^T} \right)
\]

\[
(1, 2, \ldots, K, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{C}_0, \tilde{D}_0, \tilde{X}) \in (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{C}_0, \tilde{D}_0, \tilde{X})_\alpha
\]

where

\[
S(\tilde{A}, \tilde{B}, \tilde{X}) = \{ X \in \mathbb{R}_+^n : \tilde{A} \odot X \leq \tilde{B}, X \geq 0 \}.
\]

**Definition 3.2:** (Nasseri & Bavandi, 2019). \( X^* \in S(\tilde{A}, \tilde{B}, \tilde{X}) \) is called an \( \alpha \)-efficient solution to the problem (4) if and only if there does not exist another \( X \in S(\tilde{A}, \tilde{B}, \tilde{X}) \) for which the degree of their membership functions exceeds the level \( \alpha. \)

\[
Z_k(\tilde{C}, \tilde{D}, \tilde{C}_0, \tilde{D}_0, X^*) \leq Z_k(C, D, C_0, D_0, X), k = 1, 2, \ldots, K
\]

with \( Z_k(\tilde{C}, \tilde{D}, \tilde{C}_0, \tilde{D}_0, X^*) < Z_k(C, D, C_0, D_0, X) \) holds for at least one \( k. \)

Problem (3) is reduced to the following fuzzy random LP problem (Dinkelbach, 1967; Guzel, 2013):

\[
\max \left\{ \sum_{k=1}^{K} \frac{N_k(\tilde{C}, \tilde{C}_0, \tilde{X}) - \tilde{Z}_k^* D_k(\tilde{D}, \tilde{D}_0, \tilde{X})}{\tilde{X} \in S} \right\}
\]

where, \( \tilde{Z}_k^* = \frac{N_k(\tilde{C}, \tilde{C}_0, \tilde{X})}{D_k(\tilde{D}, \tilde{D}_0, \tilde{X})} = \max_{\tilde{x} \in \tilde{S}} \left( \frac{\tilde{C}_k^T \odot \tilde{X} \odot \tilde{C}_0^T}{D_k^T \odot \tilde{X} \odot D_k^T} \right);
\]

\[\forall k = 1, 2, \ldots, K.\]

**Definition 3.3:** (Nasseri & Bavandi, 2019). \( \tilde{X}^* \) which satisfies the constraints of problem (6), is called a fuzzy pseudodifferentiable optimality solution. While, if \( \tilde{X}^* \) is a fuzzy random
vector on $\Psi$, then it is a fuzzy random optimal solution of problem (6).

Accordingly, we have
\[ \bar{a}_i, \bar{b}_i, \bar{c}_{jk}, \bar{d}_{jk}, \bar{d}_k^L, \bar{d}_k^U \in \mathbb{R} \cup (\Psi). \]

For any $\alpha \in (0, 1)$, the $\alpha$-levels of $\bar{a}_i, \bar{b}_i, \bar{c}_{jk}, \bar{d}_k^L, \bar{d}_k^U$ are the random intervals:
\[ (\bar{a}_i)_\alpha = [(\bar{a}_i)_\alpha^L, (\bar{a}_i)_\alpha^U], \quad (\bar{b}_i)_\alpha = [(\bar{b}_i)_\alpha^L, (\bar{b}_i)_\alpha^U], \quad (\bar{c}_{jk})_\alpha = [(\bar{c}_{jk})_\alpha^L, (\bar{c}_{jk})_\alpha^U], \quad (\bar{d}_k^L)_\alpha = [(\bar{d}_k^L)_\alpha^L, (\bar{d}_k^L)_\alpha^U], \quad (\bar{d}_k^U)_\alpha = [(\bar{d}_k^U)_\alpha^L, (\bar{d}_k^U)_\alpha^U]. \]

Also, $(\hat{x})_\alpha = [(\hat{x})_\alpha^L, (\hat{x})_\alpha^U], j = 1, 2, \ldots, n$.

Denote
\[ \hat{A}_a^L = ((\hat{a}_i)_\alpha^L)_{m \times n}, \quad \hat{A}_a^U = ((\hat{a}_i)_\alpha^U)_{m \times n}, \]
\[ \hat{B}_a^L = ((\bar{b}_i)_\alpha^L)_{m \times 1}, \quad \hat{B}_a^U = ((\bar{b}_i)_\alpha^U)_{m \times 1}, \]
\[ \hat{C}_k^L = ((\bar{c}_{jk})_\alpha^L)_{1 \times n}, \quad \hat{C}_k^U = ((\bar{c}_{jk})_\alpha^U)_{1 \times n}, \]
\[ \hat{X}_a^L = ((\hat{x})_\alpha^L)_{1 \times n}, \quad \hat{X}_a^U = ((\hat{x})_\alpha^U)_{1 \times n}, \]
\[ \hat{S}_a(L, U) = \{X_a : \hat{A}_a^L X_a \leq \hat{B}_a^L X_a \leq \hat{B}_a^U X_a \leq \hat{C}_k^L X_a \leq \hat{C}_k^U X_a \leq \hat{D}_k^L X_a \leq \hat{D}_k^U X_a \geq 0\}, \]
\[ (\hat{x})_\alpha \in \{X : \Psi \rightarrow \mathbb{R}, j = 1, 2, \ldots, n\}. \]

For problem (6), the following programming problems are structured as follows:

Max \[ \sum_{k=1}^{K} \left( N_k(C_u^U, C_o^L, X_a) - \hat{Z}_k D_k(D_o^U, D_o^L, X_a) \right) : X_a \in \hat{S}_a \]
for any $\alpha \in (0, 1)$,  
(7)

Max \[ \sum_{k=1}^{K} \left( N_k(C_u^U, C_o^L, X_a) - \hat{Z}_k D_k(D_o^U, D_o^L, X_a) \right) : X_a \in \hat{S}_a \]
for any $\alpha \in (0, 1)$,  
(8)

Max \[ \sum_{k=1}^{K} \left( N_k(C_u^U, C_o^L, X_a) - \hat{Z}_k D_k(D_o^U, D_o^L, X_a) \right) : X_a \in \hat{S}_a \]
for any $\alpha \in (0, 1)$,  
(9)

Max \[ \sum_{k=1}^{K} \left( N_k(C_u^U, C_o^L, X_a) - \hat{Z}_k D_k(D_o^U, D_o^L, X_a) \right) : X_a \in \hat{S}_a \]
for any $\alpha \in (0, 1)$,  
(10)

Max \[ \sum_{k=1}^{K} \left( N_k(C_u^U, C_o^L, X_a) - \hat{Z}_k D_k(D_o^U, D_o^L, X_a) \right) : X_a \in \hat{S}_a \]
for any $\alpha \in (0, 1)$,  
(11)

Max \[ \sum_{k=1}^{K} \left( N_k(C_u^U, C_o^L, X_a) - \hat{Z}_k D_k(D_o^U, D_o^L, X_a) \right) : X_a \in \hat{S}_a \]
for any $\alpha \in (0, 1)$,  
(12)

It is observed that for any $\alpha \in (0, 1)$, problems (7)–(12) are LP problems with coefficients characterized by random variables. So, we may use the simplex method (Guangyuan & Zhong, 1993a). Now, we introduced Lemmas proved by Zhong et al. (1994).

Lemma 3.1: Let $\hat{A} \geq 0$, i.e. $(\hat{a}_i)_\alpha \geq 0$, for any $\alpha \in (0, 1)$. If $\hat{X} \in \hat{S}$, then $X_a^L \in \hat{S}_a^L$ and $X_a^U \in \hat{S}_a^U$, for any $\alpha \in (0, 1)$.

Lemma 3.2: (Zhong et al., 1994). Let $\hat{A} \leq 0$, i.e. $(\hat{a}_i)_\alpha \leq 0$, for any $\alpha \in (0, 1)$. If $\hat{X} \in \hat{S}$, then $X_a^L \in \hat{S}_a^L$ and $X_a^U \in \hat{S}_a^U$, for any $\alpha \in (0, 1)$.

Lemma 3.3: (Zhong et al., 1994). Let $\hat{A} \leq 0$ and $\hat{A}^U \geq 0$, i.e. $(\hat{a}_i)_\alpha \leq 0$ and $(\hat{a}_i)_\alpha^U \geq 0$ for any $\alpha \in (0, 1)$. If $\hat{X} \in \hat{S}$, then $X_a^L \in \hat{S}_a(L, U)$, for any $\alpha \in (0, 1)$.

4. Basics theorems

In this Section, some theorems with proofs point out the relations of optimal solution of problem (6) and the corresponding relative random programming problems (7)–(12).

Theorem 4.1: Assume that $\hat{A} \geq 0$, $\hat{C} \leq 0$, $\hat{D} \geq 0$, $\hat{S}_a^L \subset \{X_a^L : \hat{X} \in \hat{S}\}$ and $\hat{S}_a^U \subset \{X_a^U : \hat{X} \in \hat{S}\}$, where $\alpha \in (0, 1)$. If $\hat{X}$ is a fuzzy pseudorandom optimal solution of problem (6), then

1. $X_a^L$ is a pseudorandom optimal solution of problem (10).
2. $X_a^U$ is a pseudorandom optimal solution of problem (9), and
3. $Z^L_α = \max \left\{ \sum_{k=1}^{K} (N_k(C^L_{α}, C^0_a, X_a) - Z^L_k D_k(D^L_k, D^0_k, X_a)) : X \in S^U_a \right\}$,

$Z^U_α = \max \left\{ \sum_{k=1}^{K} (N_k(C^U_{α}, C^0_a, X_a) - Z^U_k D_k(D^U_k, D^0_k, X_a)) : X \in S^L_a \right\}$,

where, $\tilde{Z} = \max \left\{ \sum_{k=1}^{K} (N_k(\tilde{C}, \tilde{C}^0, \tilde{X}) - \tilde{Z}^k_k D_k(\tilde{D}, \tilde{D}^0, \tilde{X})) : \tilde{X} \in \tilde{S} \right\}$.

$\hat{Z}_α = [Z^L_α, Z^U_α]$.

**Proof:** Suppose that $\tilde{X}^o$ is a fuzzy random optimal solution of problem (6), then

$\hat{X}^o \in \tilde{S}$ and $\sum_{k=1}^{K} (N_k(\tilde{C}, \tilde{C}^0, \tilde{X}^o) - Z^U_k D_k(\tilde{D}, \tilde{D}^0, \tilde{X}^o))$

$= \max X \in S \left( \sum_{k=1}^{K} (N_k(\tilde{C}, \tilde{C}^0, X^o) - Z^U_k D_k(\tilde{D}, \tilde{D}^0, X)) \right)$.

Using Lemma 3.3 and since $\tilde{A} \geq 0$, we have

$X^o \in S^L_a, X^U_a \in S^U_a, \tilde{X}^o = (X^o_α : \tilde{X} \in \tilde{S})$, and

$\tilde{X}^U_a = \{X^U_α : \tilde{X} \in \tilde{S}\}$.

Based on the condition $\tilde{C} \geq 0, \tilde{D} \leq 0$, and from Definition A3 and Lemma A1, we have

$$\max \left\{ \sum_{k=1}^{K} (N_k(C^L_{α}, C^0_a, X^U_a) - Z^L_k D_k(D^U_k, D^0_k, X^U_a)) \right\}$$

$$\times \sum_{k=1}^{K} (C^L_{α} C^0_a, X^L_a) - Z^L_k D_k(D^L_k, D^0_k, X^L_a) \right\}$$

$$= \left[ \sum_{k=1}^{K} \sum_{j=1}^{n} (N_k(c_j^L(c^0_k a, (x_j^o)_α))^U)

- Z^L_k D_k(c_j^L(c^0_k a, (x_j^o)_α))^L \right] \times \sum_{k=1}^{K} \sum_{j=1}^{n} (N_k(c_j^L(c^0_k a, (x_j^o)_α))^L

- (\tilde{Z}^L_k)_α D_k(c_j^L(c^0_k a, (x_j^o)_α))^L)$

$$= \sum_{k=1}^{K} \sum_{j=1}^{n} \left[ (N_k(c_j^L(c^0_k a, (x_j^o)_α)) - (\tilde{Z}^L_k)_α D_k(c_j^L(c^0_k a, (x_j^o)_α))^L) \right] D_k(c_j^L(c^0_k a, (x_j^o)_α))^L)

+ (\tilde{Z}^L_k)_α D_k(c_j^L(c^0_k a, (x_j^o)_α))^L)\right]$$

$$\times (N_k((c_j^L(c^0_k a, (x_j^o)_α)) - Z^L_k D_k((d_j^L(c^0_k a, (x_j^o)_α)))\left[ (x_j^L(c_j^L(c^0_k a, (x_j^o)_α))

+ (\tilde{Z}^L_k)_α D_k(c_j^L(c^0_k a, (x_j^o)_α))^L)\right]$$

Thus for any $\alpha \in (0, 1)$, and from problems (13) and (14) we have

$$\sum_{k=1}^{K} (N_k(C^U_{α}, C^0_a, X^U_a) - Z^U_k D_k(D^U_k, D^0_k, X^U_a))$$
The solution is 

\[ \sum_{k=1}^{K} (N_k(C_a^L, C_\alpha^L, X_a) - \langle \tilde{F}_k^* \rangle_a D_k(D_a^L, D_\alpha^L, X_a)) \]

\[ \max_{ SL } \sum_{k=1}^{K} (N_k(C_a^L, C_\alpha^L, X_a) - \langle \tilde{F}_k^* \rangle_a D_k(D_a^L, D_\alpha^L, X_a)) \]

\[
\begin{align*}
\text{Remark 4.1: } & \text{A similar result can be discussed under the conditions } \hat{A} & \geq 0, \hat{C} \geq 0 \text{ and } \hat{D} \geq 0 \\
\text{Example 4.1: } & \text{Consider the following problem} \\
& \text{Max} \tilde{Z}_1 = \\
& \text{Max} \tilde{Z}_2 = \\
& \text{Subject to} \\
& \text{Subject to} \\
& \text{At first, let us determine the individual maximum for each objective function with respect to the given constraints as:} \\
& \text{At } \alpha = 1, \text{ the individual maximum for each objective are as follows } \max \tilde{Z}_1 = \\
& \text{Subject to} \\
& \text{The solution is } Z_1^* = 0.45 \text{ at } (x_1^*, x_2^*, x_3^*) = (0, 1.818182, 0) \\
& \text{Also,} \\
& \text{Max} \tilde{Z}_2 = \frac{24x_1 + 25x_2 + 19x_3 + 6}{20x_1 + 27x_2 + 25x_3 + 18} \\
& \text{Subject to} \\
& 19x_1 + 22x_2 + 27x_3 \leq 40, \\
& 0.07x_1 + 0.08x_2 + 0.06x_3 \leq 0.9, \\
& 10x_1 + 10x_2 + 14x_3 \leq 30, \\
& x_1, x_2, x_3 \geq 0. \\
& \text{The solution is } Z_2^* = 0.9404553 \text{ at } (x_1^* , x_2^* , x_3^*) = (2.105263, 0.0) \\
& \text{At } \alpha = 0.7, \text{ and according to problem } (9), \text{ we have } \\
& \max Z_2^\alpha(x) = -2.7672x_1 - 9.69566x_2 - 19.2771x_3 - 23.4898 \\
& \text{Subject to} \\
& 20.8x_1 + 22.6x_2 + 27.9x_3 \leq 58.5, \\
& 0.076x_1 + 0.086x_2 + 0.063x_3 \leq 0.93, \\
& 10.9x_1 + 11.5x_2 + 15.8x_3 \leq 33, \\
& x_1, x_2, x_3 \geq 0. \\
& \text{The solution is } Z_2^\alpha(x) = -23.48980 \text{ at } ((\tilde{x}_1)_\alpha, (\tilde{x}_2)_\alpha, (\tilde{x}_3)_\alpha) = (0, 0, 0) \\
& \text{The problem } (15) \text{ according to problem } (10) \text{ becomes } \\
& \max Z_2^\alpha(x) = 15.560x_1 + 11.8695x_2 + 3.37158x_3 - 4.72115 \\
& \text{Subject to} \\
& 14.9x_1 + 15.4x_2 + 23.5x_3 \leq 25.7, \\
& 0.024x_1 + 0.044x_2 + 0.023x_3 \leq 0.59, \\
& 5.4x_1 + 3.5x_2 + 10.1x_3 \leq 17, \\
& x_1, x_2, x_3 \geq 0. \\
& \text{The solution is } Z_2^\alpha(x) = 22.11724 \text{ at } ((\tilde{x}_1)_\alpha, (\tilde{x}_2)_\alpha, (\tilde{x}_3)_\alpha) = (1.724842, 0, 0) \\
& \text{Thus, the solution of problem } (15) \text{ at } \alpha = 0.7 \text{ is given by } \\
& \tilde{Z}_\alpha = [-23.48980, 22.11724] \text{ at } (\tilde{x}_1)_\alpha \in [0, 1.724832], \\
& (\tilde{x}_2)_\alpha = (\tilde{x}_3)_\alpha = 0. \\
& \text{The simulation environment is MATLAB 2020a, the oper-}
Theorem 4.2: Assume that $\tilde{A} \leq 0$, $\tilde{C} \geq 0$, $D \geq 0$, $\tilde{S}_a \subset \{x^U: \tilde{X} \in \tilde{S}\}$ and $\tilde{S}_a^U \subset \{x^U: \tilde{X} \in \tilde{S}\}$, where $a \in (0, 1]$. Let $\tilde{X}$ be a fuzzy pseudorandom optimal solution of problem (6). Then, we have

1. $X^{UU}_a$ is a pseudorandom optimal solution of problem (9),
2. $X^U_a$ is a pseudorandom optimal solution of problem (10), and

$$Z^L_a = \max \left\{ \sum_{k=1}^{K} (N_k(C^L_a, C^U_a, X^{L}_a) - Z^L_k D_k(D^L_a, D^U_a, X^{L}_a)) : X^L_a \right\} \left\{ \sum_{k=1}^{K} (N_k(C^L_a, C^U_a, X^{U}_a) - Z^U_k D_k(D^L_a, D^U_a, X^{U}_a)) \right\}$$

$$Z_a = \max \left\{ \sum_{k=1}^{K} (N_k(C, C^0_a, X) - Z^L_k D_k(D, D^0, X)) \right\}$$

Proof: Let $\tilde{X}^\circ$ be a fuzzy random optimal solution of problem (6). Then

$$\tilde{X}^\circ \in \tilde{S} \sum_{k=1}^{K} (N_k(\tilde{C}, \tilde{C}^0_a, \tilde{X}^\circ) - Z^L_k D_k(\tilde{D}, \tilde{D}^0, \tilde{X}^\circ))$$

$$= \max_{X \in \tilde{S}} \left( \sum_{k=1}^{K} (N_k(\tilde{C}, \tilde{C}^0_a, \tilde{X}) - Z^L_k D_k(\tilde{D}, \tilde{D}^0, \tilde{X})) \right)$$

Using Lemma 3.3 and since $\tilde{A} \leq 0$, we have

$$X^L_a \in \tilde{S}^L_a, X^U_a \in \tilde{S}_a^U = \{x^U: \tilde{X} \in \tilde{S}\}, \text{ and}$$

$$\tilde{S}_a = \{x^L_a: \tilde{X} \in \tilde{S}\}.$$
\[
\max_{\tilde{X} \in \tilde{S}} \sum_{k=1}^{K} n_k(c_{\alpha}^L, c_{\alpha}^0, x_{\alpha}^U) - (\tilde{z}_k^+ D_k(d_{\alpha}^L, d_{\alpha}^0, x_{\alpha}))
\]

Thus for any \( \alpha \in (0, 1) \), and from problems (15) and (16) we have the following expression

\[
\sum_{k=1}^{K} (n_k(c_{\alpha}^U, c_{\alpha}^0, x_{\alpha}) - (\tilde{z}_k^+ D_k(d_{\alpha}^U, d_{\alpha}^0, x_{\alpha}))
\]

Based on the condition \( \tilde{C} \geq 0, \tilde{D} \leq 0 \), and from Definition A3 and Lemma A1, we have

\[
\sum_{k=1}^{K} (n_k(c_{\alpha}^L, c_{\alpha}^0, x_{\alpha}^L) - (\tilde{z}_k^+ D_k(d_{\alpha}^L, d_{\alpha}^0, x_{\alpha}^L))
\]

**Theorem 4.3:** Assume that \( \tilde{A} \leq 0, \tilde{C} \geq 0, \tilde{D} \leq 0 \), and let \( \tilde{X}^\circ \) be a fuzzy pseudorandom optimal solution of problem (6). Then, we have

1. \( X^L_{\alpha} \) is a pseudorandom optimal solution of problem (8).
2. \( X^U_{\alpha} \) is a pseudorandom optimal solution of problem (7), and

\[
Z^L_{\alpha} = \max \left\{ \sum_{k=1}^{K} n_k(c_{\alpha}^L, c_{\alpha}^0, x_{\alpha}) - (\tilde{z}_k^+ D_k(d_{\alpha}^L, d_{\alpha}^0, x_{\alpha})) : \tilde{X} \in \tilde{S}^U \right\},
\]

\[
Z^U_{\alpha} = \max \left\{ \sum_{k=1}^{K} n_k(c_{\alpha}^U, c_{\alpha}^0, x_{\alpha}) - (\tilde{z}_k^+ D_k(d_{\alpha}^U, d_{\alpha}^0, x_{\alpha})) : \tilde{X} \in \tilde{S}^U \right\},
\]

where \( \tilde{Z} = \max \left\{ \sum_{k=1}^{K} n_k(c_{\alpha}^L, c_{\alpha}^0, x_{\alpha}) - (\tilde{z}_k^+ D_k(d_{\alpha}^L, d_{\alpha}^0, x_{\alpha})) : \tilde{X} \in \tilde{S} \right\} \)

\[
\tilde{Z} = [\tilde{Z}^L_{\alpha}, \tilde{Z}^U_{\alpha}]
\]

**Proof:** Let \( \tilde{X}^\circ \) is a fuzzy random optimal solution of problem (6). Then

\[
\tilde{X}^\circ \in \tilde{S} \text{ and } \sum_{k=1}^{K} n_k(c_{\alpha}^L, c_{\alpha}^0, x_{\alpha}) - (\tilde{z}_k^+ D_k(D_{\alpha}^L, D_{\alpha}^0, \tilde{X}^\circ))
\]

\[
= \max_{\tilde{X} \in \tilde{S}} \left\{ \sum_{k=1}^{K} n_k(c_{\alpha}^L, c_{\alpha}^0, x_{\alpha}) - (\tilde{z}_k^+ D_k(D_{\alpha}^L, D_{\alpha}^0, \tilde{X})) \right\}
\]

Using Lemma 3.2 and since \( \tilde{A} \leq 0 \), we have

\[
X^L_{\alpha} \in \tilde{S}^L_{\alpha}, X^U_{\alpha} \in \tilde{S}^U_{\alpha}, x_{\alpha}^\circ = (x_{\alpha}^L : \tilde{X} \in \tilde{S}), \text{ and } \tilde{S}^U_{\alpha} = (x_{\alpha}^U : \tilde{X} \in \tilde{S}).
\]

\[
\tilde{X}^\circ \in \tilde{S} \text{ and } \sum_{k=1}^{K} n_k(c_{\alpha}^L, c_{\alpha}^0, x_{\alpha}) - (\tilde{z}_k^+ D_k(D_{\alpha}^L, D_{\alpha}^0, \tilde{X}^\circ))
\]

\[
= \max_{\tilde{X} \in \tilde{S}} \left\{ \sum_{k=1}^{K} n_k(c_{\alpha}^L, c_{\alpha}^0, x_{\alpha}) - (\tilde{z}_k^+ D_k(D_{\alpha}^L, D_{\alpha}^0, \tilde{X})) \right\}
\]

On the other hand, by using Definitions A3, A4 and A5, and Lemma A1, we see that

\[
\left( \max_{\tilde{X} \in \tilde{S}} \left\{ \sum_{k=1}^{K} n_k(c_{\alpha}^L, c_{\alpha}^0, \tilde{X}) - (\tilde{z}_k^+ D_k(D_{\alpha}^L, D_{\alpha}^0, \tilde{X})) \right\} \right)\alpha
\]

\[
= \max_{\tilde{X} \in \tilde{S}} \left\{ \sum_{k=1}^{K} n_k(c_{\alpha}^L, c_{\alpha}^0, \tilde{X}) \tilde{z}_k^+ D_k(D_{\alpha}^L, D_{\alpha}^0, \tilde{X}) \right\} \alpha
\]

\[
= \max_{\tilde{X} \in \tilde{S}} \left\{ \sum_{k=1}^{K} n_k(c_{\alpha}^L, c_{\alpha}^0, \tilde{X}) \tilde{z}_k^+ D_k(D_{\alpha}^L, D_{\alpha}^0, \tilde{X}) \right\} \alpha
\]

\[
= \max_{\tilde{X} \in \tilde{S}} \left\{ \sum_{k=1}^{K} n_k(c_{\alpha}^L, c_{\alpha}^0, \tilde{X}) \tilde{z}_k^+ D_k(D_{\alpha}^L, D_{\alpha}^0, \tilde{X}) \right\} \alpha
\]
Thus, for any \( \alpha \in (0, 1) \), and from problems (17) and (18) we have

\[
\sum_{k=1}^{K} (N_k(c^U, c^L, x^U, x^L)) = \text{Max} \sum_{k=1}^{K} (N_k(c^U, c^L, x^U, x^L) - (\bar{Z}_k^+)_a D_k(D^L, D^U, X_a))
\]

\[
= \text{Max} \sum_{k=1}^{K} (N_k(c^U, c^L, x^U, x^L) - (\bar{Z}_k^+)_a D_k(D^L, D^U, X_a))
\]

Subject to

\[-x_1 + 3x_2 \leq 0, x_2 \leq 6, x_1, x_2 \geq 0.\]

The optimum value is \( Z_1^* = 0 \).

Also, we have

\[
\max Z_2 = \frac{-x_1 + 5}{x_2 + 1}
\]

Subject to

\[-x_1 + 3x_2 \leq 0, x_2 \leq 6, x_1, x_2 \geq 0.\]

The optimum value is \( Z_2^* = 5 \).

At \( \alpha = 0.6 \), the problem according to problem (8) can be rewritten as follows:

\[
\text{Max} \bar{Z}_a^U = -x_1 - 7x_2 - 9.2
\]

Subject to

\[-0.6x_1 + 3.4x_2 \leq 0,\]

\[1.4x_2 \leq 6.8, x_1, x_2 \geq 0.\]

The solution is \( \bar{Z}_a^U = -9.2 \) at \( (0, 0) \).

The problem according to problem can be rewritten as follows:

\[
\text{Max} \bar{Z}_a^L = -3x_2 + 4.6
\]

Subject to

\[-2.4x_1 + 1.6x_2 \leq 0,\]

\[0.6x_2 \leq 4.2, x_1, x_2 \geq 0.\]

The solution is \( \bar{Z}_a^L = -4.600000 \) at \( (0, 0) \).

**Theorem 4.4:** Assume that \( A^L_\alpha \leq 0, A^U_\alpha \geq 0, C^L_\alpha \leq 0, C^U_\alpha \geq 0, D^L_\alpha \geq 0, \) and \( D^U_\alpha \geq 0, S^U_\alpha \subset \{x^U : \bar{x} \in \tilde{S}\} \), where \( \alpha \in (0, 1) \). Also, let \( \bar{x}^\alpha \) be a fuzzy pseudorandom optimal solution of problem (6). Then, for any \( \alpha \in (0, 1) \), \( x^U_\alpha \) is a pseudorandom optimal solution of problem (11).

**Proof:** Let \( \bar{x}^\alpha \in \tilde{S} \) and

\[
\max \sum_{k=1}^{K} (N_k(c^U, c^L, x^U_\alpha) - \bar{Z}_k^+ D_k(D^L, D^U, \bar{x}^\alpha))
\]

Subject to

\[-(3, -2, -1, 0)x_1 + (1, 2, 3, 4)x_2 \leq 0,\]

\[(-1, 0, 1, 2)x_2 \leq (3, 5, 6, 8), x_1, x_2 \geq 0\]

At \( \alpha = 1 \), we have

\[
\max Z_1 = \frac{x_1 - 4}{-2x_2 + 3}
\]
\[
\begin{align*}
&\times \sum_{k=1}^{K} \sum_{j=1}^{n} (N_k ((c_j)^{0L}_{\alpha}(c_k^{0L}_{\alpha}, (x_j^{0L}_{\alpha}) - (\tilde{Z}_{k})_{\alpha} D_k ((c_j)^{0L}_{\alpha}(c_k^{0L}_{\alpha}, (x_j^{0L}_{\alpha})))) \\
&= \sum_{k=1}^{K} \sum_{j=1}^{n} ((N_k ((c_j)^{0L}_{\alpha}(c_k^{0L}_{\alpha}, (x_j^{0L}_{\alpha}) - Z_k D_k ((c_j)^{0L}_{\alpha}(c_k^{0L}_{\alpha}, (x_j^{0L}_{\alpha})))) \\
&\times (N_k ((c_j)^{0L}_{\alpha}(c_k^{0L}_{\alpha}, (x_j^{0L}_{\alpha}) - (\tilde{Z}_{k})_{\alpha} D_k ((c_j)^{0L}_{\alpha}(c_k^{0L}_{\alpha}, (x_j^{0L}_{\alpha})))) \\
&= \sum_{k=1}^{K} \sum_{j=1}^{n} N_k ((c_j)^{0L}_{\alpha}(c_k^{0L}_{\alpha}, (x_j^{0L}_{\alpha}) \\
&- (\tilde{Z}_{k})_{\alpha} D_k ((d_j)_{\alpha}(d_k^{0L}_{\alpha}, (x_j^{0L}_{\alpha}))) \\
&= \sum_{k=1}^{K} (N_k ((\tilde{C}, \tilde{C}, \tilde{X}) - Z_k D_k (\tilde{D}, \tilde{D}, \tilde{X}))_a)
\end{align*}
\]

On the other hand, we obtain the following expression

\[
\left( \max_{\tilde{x} \in \tilde{S}} \left\{ \sum_{k=1}^{K} (N_k ((\tilde{C}, \tilde{C}, \tilde{X}) - Z_k D_k (\tilde{D}, \tilde{D}, \tilde{X})) \right\} \right)_{\alpha}
\]

\[
= \max_{\tilde{x} \in \tilde{S}} \left\{ \sum_{k=1}^{K} (N_k ((\tilde{C}, \tilde{C}, \tilde{X}) - Z_k D_k (\tilde{D}, \tilde{D}, \tilde{X})) \right\}_{\alpha}
\]

\[
= \max_{\tilde{x} \in \tilde{S}} \left\{ \sum_{k=1}^{K} \sum_{j=1}^{n} N_k ((\tilde{C}, \tilde{C}, \tilde{X}) \right\}_{\alpha}
\]

\[
= \max_{\tilde{x} \in \tilde{S}} \left\{ \sum_{k=1}^{K} \sum_{j=1}^{n} [(N_k ((\tilde{C}, \tilde{C}, \tilde{X}) \right\}_{\alpha}
\]

\[
= \max_{\tilde{x} \in \tilde{S}} \left\{ \sum_{k=1}^{K} \sum_{j=1}^{n} [(N_k ((\tilde{C}, \tilde{C}, \tilde{X}) \right\}_{\alpha}
\]

Thus, for any \( \alpha \in (0, 1) \), and from problems (19), (20) and (21), we observe that \( \tilde{X}_{\alpha} \) is a pseudorandom optimal solution of problem (11).

\[\textbf{Theorem 4.5:} \text{Assume that } A_{\alpha}^L \leq 0, A_{\alpha}^U \geq 0, \tilde{C} \geq 0, \tilde{D} \geq 0, \text{ and } S_{\alpha} \subset \{X_{\alpha} : X \in \tilde{S}\}, \text{ where } \alpha \in (0, 1]. \text{ Also, let } \tilde{X}^{\alpha} \text{ be a fuzzy pseudorandom optimal solution of problem (6). Then, for any } \alpha \in (0, 1), \tilde{X}^{\alpha}_{\alpha} \text{ is a pseudorandom optimal solution of problem (12).} \]

\[\textbf{Proof:} \text{The proof is similarly to Theorem 4.} \]

5. Case study (Multi-item inventory problem)

We consider the multi-objective linear fractional inventory model as Kumar and Dutta (2015):

\[\textbf{Nomenclature} \]

The following nomenclature is adopted to deal with the suggested approach.

- \( i \): Number of items, \( i = 1, 2, 3, \ldots, n \).
- \( \lambda \): Fixed cost per order.
- \( B \): Maximum available budget for all items.
- \( F \): Maximum available space for all items.
- \( N_o \): Maximum number of orders placed.
- \( \tilde{Q}_i \): Ordering quantity of item \( i \), (decision variable).
- \( h_i \): Holding cost per item per unit time for item \( i \).
- \( \tilde{P}_i \): Purchasing price of item \( i \).
- \( \tilde{C}_i \): Selling price of item \( i \).
- \( \tilde{D}_i \): Demand per unit time of item \( i \).
- \( \tilde{S}_i \): Space required per unit for item \( i \).
- \( \tilde{O}C_i \): Ordering cost of item \( i \).

\[\textbf{Assumptions} \]

The following assumptions are considered to formulate the suggested model:

i. Multi-item inventory model without shortages is assumed.
ii. Infinite time horizon considered.
iii. Zero lead time considered.
iv. Holding cost is constant for each item.
v. Demand is inversely related to the selling price:

\[\tilde{D}_i = \tilde{D}_i(S_i) = \beta \tilde{S}_i^{-\gamma},\]

where \( \beta > 0 \) is a scaling constant, and \( \gamma > 1 \) is the coefficient of price-elasticity. For notational simplicity, \( \tilde{D}_i \) and \( \tilde{D}_i(S_i) \) may be used interchangeably in this research work.
vi. There is no discount, i.e. the purchase price is constant for each item.

Following the above assumptions, this inventory model is formulated as multi-objective linear fractional inventory model as follows (Kumar & Dutta, 2015):

\[
\begin{align*}
\text{Max } & \quad \tilde{Z}_1 = \frac{\sum_{i=1}^{n} (\beta S_i - \tilde{P}_i) \tilde{Q}_i}{\sum_{i=1}^{n} \tilde{h}_i \tilde{Q}_i} \quad (21) \\
\text{Min } & \quad \tilde{Z}_2 = \frac{\sum_{i=1}^{n} \tilde{h}_i \tilde{Q}_i}{\sum_{i=1}^{n} \tilde{Q}_i} \\
\text{Subject to: } & \quad \tilde{Q}_i \geq \frac{\beta}{\alpha} S_i, \quad \forall i = 1, 2, \ldots, n, \\
& \quad N_0 > 0, \\
& \quad \beta > 0, \\
& \quad \gamma > 1, \\
& \quad \tilde{Q}_n > 0, \forall i = 1, 2, \ldots, n, \\
& \quad OC_n > 0, \forall i = 1, 2, \ldots, n.
\end{align*}
\]

where, \( \sum_{i=1}^{n} (\tilde{S}_i - \tilde{P}_i) \tilde{Q}_i \) represents the profit,

\( \sum_{i=1}^{n} (\beta S_i - \tilde{Q}_i) \) represents the back ordered quantity,

\( \sum_{i=1}^{n} \tilde{h}_i \tilde{Q}_i \) represents the holding cost,

\( \sum_{i=1}^{n} \tilde{Q}_i \) represents the total ordered quantity,

Constraint I represents the restriction on total budget, Constraint II represents the restriction on storage space, Constraint III represents the upper limit on number of orders, Constraint IV represents the budgetary constraint on ordering cost, Constraint V represents the non-negative restrictions.

We illustrate the following numerical example for the proposed model.

**Example 5.1:** (Two-item inventory problem)

**(Model in Crisp Environment)** Consider inventory model of two items with the following input data (in proper units) as presented in the following Table 2:

At \( \alpha = 0.6 \), the input data are presented in the following Table 3, as follows:

Based on the data in as shown in Table 3, the multi-objective inventory problem can be expressed as:

\[
\begin{align*}
\text{Max } & \quad \tilde{Z}_1 = \frac{[3,122] \tilde{Q}_1 |_{\alpha=0.6} + [28,82] \tilde{Q}_2 |_{\alpha=0.6}}{[228,84] \tilde{Q}_1 |_{\alpha=0.6} + [378,05] \tilde{Q}_2 |_{\alpha=0.6}} \\
\text{Min } & \quad \tilde{Z}_2 = \frac{[4,6,6] \tilde{Q}_1 |_{\alpha=0.6} + [6,6,8,8] \tilde{Q}_2 |_{\alpha=0.6}}{[6,6,8,8] \tilde{Q}_1 |_{\alpha=0.6}} \\
\text{Subject to: } & \quad [118, 127] \tilde{Q}_1 |_{\alpha=0.6} + [146, 164] \tilde{Q}_2 |_{\alpha=0.6} \leq 90,000, \\
& \quad [1.6, 2.8] \tilde{Q}_1 |_{\alpha=0.6} + [2.6, 4.4] \tilde{Q}_2 |_{\alpha=0.6} \leq 300, \\
& \quad [344, 1375, 718, 213] \tilde{Q}_1 |_{\alpha=0.6} + [549, 492, 675, 339] \tilde{Q}_2 |_{\alpha=0.6} \geq 16,000, \\
& \quad \tilde{Q}_1 |_{\alpha=0.6} \geq [8.8604, 24.6554], \\
& \quad \tilde{Q}_2 |_{\alpha=0.6} \geq [8.8214, 13.4095].
\end{align*}
\]

Problem (24) according to problem (7) can be rewritten as follows:

\[
\begin{align*}
\text{Max } & \quad Z_1 = (0.4 Q_1 |_{\alpha=0.6} + 23.4 (Q_2 |_{\alpha=0.6} - 378.054)) \\
\text{Subject to } & \quad 118 Q_1 |_{\alpha=0.6} + 146 Q_2 |_{\alpha=0.6} \leq 90,000, \\
& \quad 1.6 Q_1 |_{\alpha=0.6} + 2.6 Q_2 |_{\alpha=0.6} \leq 300, \\
& \quad 718.213 Q_1 |_{\alpha=0.6} + 675.339 Q_2 |_{\alpha=0.6} \geq 16,000, \\
& \quad (Q_1 |_{\alpha=0.6} \geq 24.6554, Q_2 |_{\alpha=0.6} \geq 13.4095. (24)
\end{align*}
\]

The solution of problem (25) is given by

\[
Z_1 = 1976.7700, \quad (Q_1 |_{\alpha=0.6} = 24.6554, (Q_2 |_{\alpha=0.6} = 100.2121.
\]

Also, problem (24) according to problem (8) can be rewritten as follows

\[
\begin{align*}
\text{Max } & \quad Z_2 = (116.2 Q_1 |_{\alpha=0.6} + 75.2 (Q_2 |_{\alpha=0.6} - 228.8466)) \\
\text{Subject to } & \quad 127 Q_1 |_{\alpha=0.6} + 164 Q_2 |_{\alpha=0.6} \leq 90,000, \\
& \quad 2.8 Q_1 |_{\alpha=0.6} + 4.4 Q_2 |_{\alpha=0.6} \leq 300, \\
& \quad 344.1375 Q_1 |_{\alpha=0.6} + 549.492 Q_2 |_{\alpha=0.6} \geq 16,000, \\
& \quad (Q_1 |_{\alpha=0.6} \geq 8.8604, Q_2 |_{\alpha=0.6} \geq 13.4095. (25)
\end{align*}
\]

The solution is given by

\[
Z_2 = 10780.9700, \quad (Q_1 |_{\alpha=0.6} = 86.0708, (Q_2 |_{\alpha=0.6} = 13.4095.
\]

---

**Table 2.** Input data for Example 4.1.

| Item | \( \tilde{h}_i \) | \( \tilde{p}_i \) | \( \tilde{S}_i \) | \( \tilde{Q}_i \) | \( \tilde{h}_i \) | \beta | \gamma | N_0 | \lambda | F | B |
|------|-----------------|-----------------|----------------|----------------|--------------|-------|-------|-------|-------|---|---|
| i = 1 | (8, 10, 12, 16) | (115, 120, 125, 130) | (100, 150, 200, 300) | (0, 1, 2, 4) | 80,000 | 1.20 | 5 | 7 | 300 | 90,000 |
| i = 2 | (12, 14, 16, 20) | (140, 150, 160, 170) | (70, 80, 90, 100) | (2, 3, 4, 5) | 20,000 | 1.20 | 5 | 7 | 300 | 90,000 |
6. Conclusions

In this work, the solution method of fuzzy random linear multi-objective fractional programming (FRLMOFP) problem has investigated. With a-level set concept, the FRLMOFP can be classified into six subproblems according different criteria. The relations of these models as well as the FRLMOFP have mathematically formulated. An inventory problem with price-dependant has introduced as an application to support the suggested method. This approach comprises of the deterministic nature of demand. This research work assumes the demand function which depends exclusively on selling price. Further research directions can be explored for different type demand functions, variable deterioration and use of preservation technology to reduce the deterioration. One can also introduce the impact of lead time as random variable that may occur in supply chain during the present COVID 19 pandemic outbreak. Another direction for further research is to use the proposed approach to model different dynamics of decision sciences, such as multi-criterion decision making, networking, scheduling problem, etc.

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Appendix. Definitions, Lemma and some theorems.

Definition A.1: (Kaufmann & Gupta, 1988): Assume that $\tilde{A}_\alpha = [a_L^\alpha, a_U^\alpha]$ and $\tilde{B}_\alpha = [b_L^\alpha, b_U^\alpha]$, we have

1. **Addition:**
   
   $\tilde{A}_\alpha \oplus \tilde{B}_\alpha = [a_L^\alpha + b_L^\alpha, a_U^\alpha + b_U^\alpha].$

2. **Subtraction:**
   
   $\tilde{A}_\alpha \ominus \tilde{B}_\alpha = [a_L^\alpha - b_L^\alpha, a_U^\alpha - b_U^\alpha].$

3. **Multiplication:**
   
   $\tilde{A}_\alpha \otimes \tilde{B}_\alpha = \left[ \min(a_L^\alpha, b_L^\alpha), \max(a_L^\alpha, b_L^\alpha), a_U^\alpha b_U^\alpha, a_U^\alpha b_U^\alpha \right].$

4. **The order relation** ($\leq$) **id** defined as
   
   $\tilde{A}_\alpha (\leq) \tilde{B}_\alpha \iff a_L^\alpha \leq b_L^\alpha \land a_U^\alpha \leq b_U^\alpha.$

5. **Minimum:**
   
   $\bigwedge_{i \in I} a_i^\alpha = \left[ \bigwedge_{i \in I} a_i^\alpha, \bigwedge_{i \in I} a_i^\alpha \right], \bigwedge_{i \in I} a_i^\alpha > -\infty,$

6. **Maximum:**
   
   $\bigvee_{i \in I} a_i^\alpha = \left[ \bigvee_{i \in I} a_i^\alpha, \bigvee_{i \in I} a_i^\alpha \right], \bigvee_{i \in I} a_i^\alpha < \infty,$

where $I$ is the index set.

Definition A.2: (Zhong et al., 1994): Let $[g_t : t \in I] \subseteq F_0(\mathbb{R}), \alpha \in (0, 1]$.

1. **Minimum:**
   
   $\bigwedge_{t \in I} g_t \text{ is defined by } h_\alpha = \bigwedge_{t \in I} (g_t)_\alpha.$

2. **Maximum:**
   
   $\bigvee_{t \in I} g_t \text{ is defined by } h_\alpha = \bigvee_{t \in I} (g_t)_\alpha.$

**Lemma A.1:** (Zhong et al., 1994): Let $g, h \in F_0(\mathbb{R})$. Then for any $\alpha \in (0, 1]$, we get

$$(g * h)_\alpha = g_\alpha * h_\alpha, \text{ where } * \text{ be an algebraic operation as } "+", "\cdot", "\wedge", "\vee", \ldots$$

**Definition A.3:** (Zhong et al., 1994): Let $(\Psi, A, P)$ be a probability measure space. A mapping $\tilde{A} : \Psi \rightarrow F_0(\mathbb{R})$ is said to be a fuzzy random variable on $(\Psi, A, P)$ if and only if

**Reference:**

Valipour, E., & Yaghoobi, M. A. (2020). A solving approach for fuzzy multi-objective linear fractional programming and application to an agricultural planting structure optimization problem. *Chaos Solitons & Fractals*, 141, 110352. [https://doi.org/10.1016/j.chaos.2020.110352](https://doi.org/10.1016/j.chaos.2020.110352)
\[ \hat{A}_\alpha(u) = \{ x : x \in \mathbb{R} : \hat{A}(u)(x) \geq \alpha \} = [a_L^\alpha(u), a_U^\alpha(u)], \quad \text{for any} \quad \alpha \in (0, 1) \text{is a random interval on } (\Psi, A), \text{i.e.} \ a_L^\alpha(u) \text{ and } a_U^\alpha(u) \text{ are two random variables on } (\Psi, A). \]

**Remark A.1:** Let \( \text{FR}(\Psi) \) denotes the set of all fuzzy random variable on \((\Psi, A), u \in \Psi. \)

**Definition A.4:** (Zhong et al., 1994): Assume that \(*\) be an algebraic operation on \( \mathbb{R}, \hat{A}, \hat{B} \in F_0(\mathbb{R}). \) The algebraic operation \(*\) on \( \text{FR}(\Psi) \) is defined as:

\[ (\hat{A} * \hat{B})(u) = \hat{A}(u) * \hat{B}, \text{for any } u \in \Psi, \]

and \( \hat{A} \leq \hat{B} \iff \hat{A}(u) \leq \hat{B}(u) \text{for any } u \in \Psi, \)

where \( \hat{A}, \hat{B} \in \text{FR}(\Psi). \)

If \( \{\hat{A}_i : i \in I\} \subset \text{FR}(\Psi), \) where \( I \) is the index set, then

\[ \big( \bigwedge_{i \in I} \hat{A}_i \big)(u) = \bigwedge_{i \in I} \hat{A}_i(u), \text{for any } u \in \Psi. \]

**Definition A.5:** (Zhong et al., 1994): Suppose that \( \hat{A} \in \text{FR}(\Psi) \) satisfies \( \hat{A}_\alpha(u) = [a_L^\alpha(u), a_U^\alpha(u)], \) where \( \alpha \in (0, 1], u \in \Psi. \)

(i) If \( H_L^\alpha(x) \) and \( H_U^\alpha(x) \) are the probability distribution functions of \( a_L^\alpha(u) \) and \( a_U^\alpha(u) \) respectively, then the function

\[ \tilde{H}(x) = \bigcup_{\alpha \in (0, 1]} \alpha [H_L^\alpha(x), H_U^\alpha(x)] \]

is said to be fuzzy probability distribution function of \( \hat{A}, x \in \mathbb{R}. \)

(ii) If \( E(a_L^\alpha) \) and \( E(a_U^\alpha) \) are the expectation of \( a_L^\alpha \) and \( a_U^\alpha, \) respectively, then the expectation of \( \hat{A} \) is defined as follows:

\[ E(\hat{A}) = \bigcup_{\alpha \in (0, 1]} \alpha [E(a_L^\alpha), E(a_U^\alpha)]. \]

**Definition A.6:** (Zhong et al., 1994): \( \bar{x} \in \tilde{S}(\hat{A}, \hat{B}, \hat{X}) \) is said to be fuzzy pseudorandom efficient solution of problem (3) if and only if

\[ \tilde{Z}_k(\hat{C}, \hat{D}, \hat{C}_0, \hat{D}_0, \hat{X}) \leq \tilde{Z}_k(\hat{C}, \hat{D}, \hat{C}_0, \hat{D}_0, \hat{X}) \]

with \( \tilde{Z}_k(\hat{C}, \hat{D}, \hat{C}_0, \hat{D}_0, \hat{X}) < \tilde{Z}_k(\hat{C}, \hat{D}, \hat{C}_0, \hat{D}_0, \hat{X}) \) holds for at least one \( k = 1, 2, \ldots, K. \)