HOMOGENIZATION OF SYMMETRIC STABLE-LIKE PROCESSES IN STATIONARY ERGODIC MEDIUM

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Abstract. This paper studies homogenization of symmetric non-local Dirichlet forms with $\alpha$-stable-like jumping kernels in one-parameter stationary ergodic environment. Under suitable conditions, we establish homogenization results and identify the limiting effective Dirichlet forms explicitly. The coefficients of the jumping kernels of Dirichlet forms and symmetrizing measures are allowed to be degenerate and unbounded; and the coefficients in the effective Dirichlet forms can be degenerate.

Keywords: homogenization; symmetric non-local Dirichlet form; ergodic random medium; $\alpha$-stable-like operator

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1. INTRODUCTION AND RESULTS

1.1. Background. The aim of homogenization theory is to provide the macroscopic rigorous characterizations of microscopically heterogeneous media, which usually involve rapidly oscillating functions of the form $a = a(x/\varepsilon)$ with $\varepsilon$ being a small positive parameter that characterizes the microscopic length scale of the media. Homogenization has been a very active research area for a long time, and there is now a vast literature on this topic, see e.g. [1, 8, 31, 46].

Homogenization problems for random structures are widely studied. The first rigorous result for second order elliptic operators in divergence forms with stochastically homogeneous random coefficients was independently obtained by Kozlov [36] and by Papanicolaou and Varadhan [38]. The crucial point of their approaches is to construct the so-called corrector (i.e., the solution of certain associated elliptic equations) and prove that it grows sub-linearly. Later on a lot of homogenization problems were investigated for various elliptic and parabolic differential equations as well as system of equations in random stationary media. In particular, Bourgeat et al. in [11] introduced the stochastic version of the two-scale convergence method. Caffarelli, Souganidis and Wang in [13] studied the stochastic homogenization in the context of fully non-linear uniformly elliptic equations in stationary ergodic environment.

The goal of this paper is to address homogenization problem for non-local operators with random coefficients and to give a characterization of the homogenized limiting operators. We start with a brief review of some recent work on homogenization problems for non-local operators. Piatnitski and Zhizhina in [40] studied homogenization problem for integral operators of convolution type with dispersal kernels (or jumping kernels) that have random stationary ergodic coefficients and finite second moment. For discrete operators with ergodic weights associated with unbounded-range of jumps (but still having finite second moment), we refer to [27]. In these two cases the scaling order is naturally a Brownian scaling, and the limiting operator is a Laplacian. One key element in their approaches is that the corrector method or the two-scale convergence approach works when the jumping kernel has finite second moments. However, when the jumping kernel has infinite second moment, the scaling order and limiting process are completely different. Chen, Kim and Kumagai in [18] proved the Mosco convergence of non-local Dirichlet forms associated with symmetric stable-like random walks in independent long-range conductance model. They showed that the limiting process is a symmetric $\alpha$-stable Lévy process. Kassmann, Piatnitski and Zhizhina in [33] investigated homogenization of a class of symmetric stable-like processes in ergodic environment whose jumping kernels are of product form. Homogenization problem of symmetric stable-like processes in two-parameter ergodic environment was also studied in [33]. We shall mention that in [33] random coefficients of the jumping kernel are assumed to be uniformly elliptic and bounded. Recently, Flegel and Heida [28] considered the corresponding problem in the discrete setting under some moment conditions on coefficients in two-parameter ergodic environment. The stochastic homogenization of a class of fully non-linear integral-differential equations in ergodic environment
was studied by Schwab [45]. We refer the reader to [7, 12, 39, 44] and references therein for homogenization of integral equations (or jump processes) with periodic coefficients.

As mentioned above, known results concerning stochastic homogenization of stable-like Dirichlet forms in one-parameter ergodic environment require the coefficients enjoying very special forms (for example, the product form). The contribution of this paper is to systematically study homogenization problem for symmetric non-local operators in one-parameter ergodic environment under more general settings, where the corresponding random coefficients can be degenerate and unbounded.

1.2. Setting. Let $d \geq 1$, and $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space that describes the random environment on which a measurable group of transformations $\{\tau_x\}_{x \in \mathbb{R}^d}$ is defined with $\tau_0 = \text{id}$, the identity map on $\Omega$, and $\tau_x \circ \tau_y = \tau_{x+y}$ for every $x, y \in \mathbb{R}^d$. One may think of $\tau_x \omega := \tau_x(\omega)$ as a translation of the environment $\omega \in \Omega$ in the direction $x \in \mathbb{R}^d$. We assume that $\{\tau_x\}_{x \in \mathbb{R}^d}$ is stationary and ergodic; that is,

(i) $\mathbb{P}(\tau_x A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$ and $x \in \mathbb{R}^d$;

(ii) if $A \in \mathcal{F}$ and $\tau_x A = A$ for all $x \in \mathbb{R}^d$, then $\mathbb{P}(A) \in \{0, 1\}$;

(iii) the function $(x,\omega) \mapsto \tau_x \omega$ is $\mathcal{F}(\mathbb{R}^d) \times \mathcal{F}$-measurable.

Consider a random variable $\mu : \Omega \to [0, \infty)$ so that for every $\omega \in \Omega$, $\mu(\tau_x \omega) > 0$ for a.e. $x \in \mathbb{R}^d$ and $\mathbb{E}[\mu] = 1$, and a random function $\kappa : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to [0, \infty)$ so that for every $\omega \in \Omega$,

$$
\kappa(x,y;\omega) = \kappa(y,x;\omega), \quad \kappa(x+z,y+z;\omega) = \kappa(x,y;\tau_z \omega) \quad \text{for any } x,y,z \in \mathbb{R}^d,
$$

and

$$
x \mapsto \int (1 + |z|^2)^\frac{\kappa(x,x+z;\omega)}{|z|^{d+\alpha}} \, dz \in L^1_{\text{loc}}(\mathbb{R}^d;dx), \quad \mathbb{P}\text{-a.e.}
$$

For each $\omega \in \Omega$, these two functions determine a regular symmetric Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ on $L^2(\mathbb{R}^d;\mu(\tau_x \omega) \, dx)$ as follows.

Denote by $\Delta := \{(x,x) \in \mathbb{R}^d\}$ the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$ and $\mu^\omega(dx) := \mu(\tau_x \omega) \, dx$, which has full support on $\mathbb{R}^d$. Let $\Gamma$ be an infinite cone in $\mathbb{R}^d$ having non-empty interior that is symmetric with respect to the origin; that is, $\Gamma$ is a non-empty open subset of $\mathbb{R}^d$ so that $rx \in \Gamma$ for every $x \in \Gamma$ and $r \in \mathbb{R}$. For $\alpha \in (0,2)$, define

$$
\mathcal{E}^\omega(f,g) := \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} (f(x) - f(y))(g(x) - g(y)) \frac{\kappa(x,y;\omega)}{|x-y|^{d+\alpha}} \mathbb{I}_{\{y \in \Gamma\}} \, dx \, dy, \quad f, g \in \mathcal{F}^\omega,
$$

where $\mathcal{F}^\omega$ the closure of $C^1_c(\mathbb{R}^d)$ with respect to the norm $(\mathcal{E}^\omega(\cdot, \cdot) + \|\cdot\|_{L^2(\mathbb{R}^d;\mu^\omega(dx))}^2)^{1/2}$. Note that under (1.2), $\mathcal{E}^\omega(f,f) < \infty$ for all $f \in C^1_c(\mathbb{R}^d)$. Here and in what follows, $C^1_c(\mathbb{R}^d)$ (respectively, $C_0(\mathbb{R}^d)$) denotes the space of $C^1$-smooth (respectively, continuously) functions on $\mathbb{R}^d$ with compact support. Clearly, $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ is a regular symmetric Dirichlet form on $L^2(\mathbb{R}^d;\mu^\omega(dx))$. So there exist a Borel subset $\mathcal{N}^\omega \subset \mathbb{R}^d$ having zero $\mathcal{E}^\omega$-capacity, and a symmetric Hunt process $X^\omega := \{X^\omega_t, t \geq 0\} \subset \mathbb{R}^d \setminus \mathcal{N}^\omega$ on the state space $\mathbb{R}^d \setminus \mathcal{N}^\omega$; see [29, Chapter 7]. Note that $X^\omega_t$ is a time change of the process corresponding to the Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ on $L^2(\mathbb{R}^d;dx)$. When $\Gamma = \mathbb{R}^d$ and $\kappa(x,y;\omega)$ is bounded between two positive constants, this Hunt process is a symmetric $\alpha$-stable-like process studied in [19].

For any $\varepsilon > 0$, set $X^\varepsilon \omega = (X^\varepsilon \omega_t)_{t \geq 0} := (\varepsilon X^\omega_{t/\varepsilon^\alpha})_{t \geq 0}$. The following simple lemma characterizes the scaled processes $\{X^\varepsilon \omega : \varepsilon > 0\}$. Its proof is postponed to the appendix of this paper.

**Lemma 1.1.** For any $\varepsilon > 0$, the scaled process $X^\varepsilon \omega$ has a symmetrizing measure $\mu^\varepsilon \omega(dx) = \mu(\tau_{x/\varepsilon} \omega) \, dx$, and the associated regular Dirichlet form $(\mathcal{E}^\varepsilon \omega, \mathcal{F}^\varepsilon \omega)$ on $L^2(\mathbb{R}^d;\mu^\varepsilon \omega(dx))$ is given by

$$
\mathcal{E}^\varepsilon \omega(f,g) := \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} (f(x) - f(y))(g(x) - g(y)) \frac{\kappa(x,y;\varepsilon,x;\varepsilon;\omega)}{|x-y|^{d+\alpha}} \mathbb{I}_{\{y \in \Gamma\}} \, dx \, dy, \quad f, g \in \mathcal{F}^\varepsilon \omega,
$$

where $\mathcal{F}^\varepsilon \omega$ is the closure of $C^1_c(\mathbb{R}^d)$ with respect to the norm $(\mathcal{E}^\varepsilon \omega(\cdot, \cdot) + \|\cdot\|_{L^2(\mathbb{R}^d;\mu^\varepsilon \omega(dx))}^2)^{1/2}$.

Let $(\mathcal{L}^\omega, \text{Dom}(\mathcal{L}^\omega))$ and $(\mathcal{L}^\varepsilon \omega, \text{Dom}(\mathcal{L}^\varepsilon \omega))$ be the $L^2$-generator of the Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ on $L^2(\mathbb{R}^d;\mu^\omega)$ and the Dirichlet form $(\mathcal{E}^\varepsilon \omega, \mathcal{F}^\varepsilon \omega)$ on $L^2(\mathbb{R}^d;\mu^\varepsilon \omega)$, respectively. That is,

$$
\mathcal{L}^\omega f(x) = \lim_{\delta \to 0} \frac{1}{\mu(\tau_x \omega)} \int_{\{ y \in \mathbb{R}^d : |y-x| > \delta \}} (f(y) - f(x)) \frac{\kappa(x,y;\omega)}{|y-x|^{d+\alpha}} \mathbb{I}_{\{y-x \in \Gamma\}} \, dy \quad \text{for } f \in \text{Dom}(\mathcal{L}^\omega)
$$

(1.5)
and
\[ L^{\varepsilon,\omega}f(x) = \lim_{\delta \to 0} \frac{1}{\mu(\tau_{x/\delta}/\omega)} \int_{\{y \in \mathbb{R}^d : |y-x| > \delta\}} (f(y) - f(x)) \frac{\kappa(x/\varepsilon, y/\varepsilon, \omega)}{|y-x|^{d+\alpha}}1_{\{y-x \in \Gamma\}} \, dy \quad \text{for } f \in \text{Dom}(L^{\varepsilon,\omega}). \]  

(1.6)

It is easy to see that for each \( \varepsilon > 0 \), \( g \in \text{Dom}(L^{\varepsilon,\omega}) \) if and only of \( g^{(\varepsilon)} \in \text{Dom}(L^{\omega}) \), where \( g^{(\varepsilon)}(x) = g(\varepsilon x) \), and
\[ L^{\varepsilon,\omega}g(x) = \varepsilon^{-\alpha} L^{\omega}g^{(\varepsilon)}(x/\varepsilon). \]

For any \( \lambda > 0 \) and \( f \in C_c(\mathbb{R}^d) \), let \( u^{\varepsilon,\omega}_f \) be the solution to
\[ (\lambda - L^{\varepsilon,\omega})u^{\varepsilon,\omega}_f = f \]
in \( L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega}(dx)) \). The main goal of homogenization problem in our paper is to show, under suitable conditions, that almost surely, \( u^{\varepsilon,\omega}_f \) converges to a deterministic function \( u_f \) as \( \varepsilon \to 0 \) for every \( f \in C_c(\mathbb{R}^d) \), and that \( u_f \) is the solution of
\[ (\lambda - L)u_f = f, \]
where \( L \) is the \( L^2 \)-generator of certain regular symmetric Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \( L^2(\mathbb{R}^d; dx) \) whose jumping kernel can be degenerate. The convergence of \( u^{\varepsilon,\omega}_f \to u_f \) as \( \varepsilon \to 0 \) is in the resolvent topology; that is, for a.s. \( \omega \in \Omega \),
\[ \lim_{\varepsilon \to 0} \|u^{\varepsilon,\omega}_f - u_f\|_{L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega}(dx))} = 0. \]

See [42, 46] for background and [7, 12, 33] for recent study on homogenization problems related to non-local operators.

Let \( K(z) \) be a non-negative bounded measurable even function on \( \mathbb{R}^d \) (that is, \( K(z) = K(-z) \) for all \( z \in \mathbb{R}^d \)). Define a regular Dirichlet form \((\mathcal{E}^K, \mathcal{F}^K)\) on \( L^2(\mathbb{R}^d; dx) \) by
\[ \mathcal{E}^K(f, g) = \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} (f(x) - f(y))(g(x) - g(y)) \frac{K(x-y)}{|x-y|^{d+\alpha}}1_{\{x-y \in \Gamma\}} \, dx \, dy, \quad f, g \in \mathcal{F}^K, \]
where \( \mathcal{F}^K \) is the closure of \( C_c(\mathbb{R}^d) \) in with respect to the norm \( (\mathcal{E}^K(\cdot, \cdot) + \|\cdot\|_{L^2(\mathbb{R}^d; dx)})^{1/2} \). The limiting Dirichlet form \((\mathcal{E}, \mathcal{F})\) for homogenization problem considered in this paper is of this type. We emphasis that the symmetric cone \( \Gamma \) in (1.4) can be a proper subset of \( \mathbb{R}^d \) in our paper.

1.3. Main results. Our main results are divided into two cases, according to the explicit form of the coefficient \( \kappa(x, y; \omega) \) in (1.3). The first one is concerned on the case that \( \kappa(x, y; \omega) \) is of the summation form, and the second one on the case that is of a product form.

1.3.1. \( \kappa(x, y; \omega) \) of summation form. To state the statement of this part, we need the following assumption \((A)\) for \( \kappa(x, y; \omega) \).

(A1) For a.s. \( \omega \in \Omega \),
\[ \kappa(x, y; \omega) = \nu(y - x; \tau_x \omega) + \nu(x - y; \tau_y \omega) \quad \text{for every } x, y \in \mathbb{R}^d, \]
where \( \nu : \mathbb{R}^d \times \Omega \to [0, \infty) \) is a measurable random function such that

(i) There is a constant \( l > 0 \) such that for any \( n > 0 \) and \( x, z_1, z_2 \in \mathbb{R}^d \),
\[ \left| \text{Cov} \left( \nu_n(z_1; \cdot), \nu_n(z_2; \tau_x(\cdot)) \right) \right| = \left| \mathbb{E} \left[ \nu_n(z_1; \cdot) \cdot \nu_n(z_2; \tau_x(\cdot)) \right] - \mathbb{E} \left[ \nu_n(z_1; \cdot) \right] \mathbb{E} \left[ \nu_n(z_2; \cdot) \right] \right| \leq C_1(n) \left( 1 + |x|^{-l} \right), \]
where \( \nu_n = \nu \wedge n \) and \( C_1(n) \) is a positive constant depending on \( n \).

(ii) There is a non-negative measurable function \( \bar{\nu} \) on \( \mathbb{R}^d \) such that \( \mathbb{E}[(\nu(z; \varepsilon; \cdot)) \right] \) converges weakly to \( \bar{\nu}(z) \) in \( L^1_{\text{loc}}(\mathbb{R}^d; dx) \) as \( \omega \to 0 \); that is,
\[ \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} h(z) \mathbb{E}[\nu(z/\varepsilon; \cdot)] \, dz = \int_{\mathbb{R}^d} h(z) \bar{\nu}(z) \, dz \quad \text{for every } h \in L^\infty_{\text{loc}}(\mathbb{R}^d; dx). \]  

(1.10)
(A2) There are non-negative random variables $\Lambda_1 \leq \Lambda_2$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with
\[
\mathbb{E}[\Lambda_1^{-1} + \Lambda_2] < \infty,
\] (1.11)
for some constant $p > 1$ so that for a.s. $\omega \in \Omega$,
\[
\Lambda_1(\tau_x \omega) + \Lambda_1(\tau_y \omega) \leq \kappa(x, y; \omega) \leq \Lambda_2(\tau_x \omega) + \Lambda_2(\tau_y \omega) \quad \text{for every } x, y \in \mathbb{R}^d.
\] (1.12)

(A2') There are non-negative random variables $\Lambda_1 \leq \Lambda_2$ on $(\Omega, \mathcal{F}, \mathbb{P})$ so that (1.12) holds for a.s. $\omega \in \Omega$, and that
\[
\mathbb{E}[\Lambda_1^{-1}] < \infty.
\]

It is obvious that condition (A2') is weaker than condition (A2). Here are some comments on assumption (A).

Remark 1.2. (i) It is easy to see that any $\kappa(x, y; \omega)$ of form (1.8) enjoys the property (1.1). On the contrary, any $\kappa(x, y; \omega)$ satisfying (1.1) admits a representation of the form (1.8). This is because $\kappa(x, y; \omega) = \kappa(0, y - x; \tau_x \omega)$ and so by the symmetry of $\kappa(x, y; \omega)$ in $(x, y)$ we have
\[
\kappa(x, y; \omega) = \frac{1}{2}(\kappa(x, y; \omega) + \kappa(y, x; \omega)) = \frac{1}{2}(\kappa(0, y - x; \tau_x \omega) + \kappa(0, y - x; \tau_y \omega)).
\]
Hence we can write $\kappa(x, y; \omega)$ as
\[
\kappa(x, y; \omega) = \nu(y - x; \tau_x \omega) + \nu(x - y; \tau_y \omega),
\]
where
\[
\nu(x; \omega) := \kappa(0, x; \omega)/2.
\] (1.13)
Thus (1.1) is a symmetrized and long-range analogy of nearest neighborhood random walk models with balanced random conductance; see e.g. [9, 24, 25, 30]. Representing $\kappa(x, y; \omega)$ via (1.8) by a general $\nu(z; \omega)$ rather than that of (1.13) allows more flexibility in satisfying the mixing condition (1.9) on $\nu_n = \nu \wedge n$.

(ii) Unlike elliptic differential operators, we have a variable $(y - x)/\varepsilon$ by shifting operators $\tau_{x/\varepsilon}$ and $\tau_{y/\varepsilon}$ in the coefficient
\[
\kappa(x/\varepsilon, y/\varepsilon; \omega) = \kappa(0, (y - x)/\varepsilon; \tau_{x/\varepsilon} \omega) = \kappa(0, (x - y)/\varepsilon; \tau_{y/\varepsilon} \omega)
\]
of the scaled process $X_\varepsilon$ which corresponds to the long range property of the jumping kernel (see (1.4)). This prevents us to directly applying the ergodic theorem to deduce the almost sure convergence as indicated below. We thus assume some kind of mixing condition (1.9) on $\nu_n$, uniformly in $z_1, z_2 \in \mathbb{R}^d$, to guarantee this convergence. Similar assumption (without the variable $z_i \in \mathbb{R}^d$ and on $\nu$ itself) has been used in [32, Assumption A3] to establish the quenched functional central limit theorem for random walks on $\mathbb{R}^d$ where the random environment is i.i.d. in time and polynomially mixing in space, and in [2, Assumption A5] in the study of invariance principle for diffusions in time-space ergodic random environment. We mention that (1.9) includes the so-called “unit range of dependence” condition used in [3, p. xii, (0.6)].

(iii) Suppose that (1.9) holds with $C^*_1 := \limsup_{n \to \infty} C_1(n) < \infty$. Then, for any $x, z_1, z_2 \in \mathbb{R}^d$,
\[
|\text{Cov}(\nu(z_1; \cdot), \nu(z_2, \tau_x \cdot))| \leq C^*_1 \left(1 \wedge |x|^{-l}\right).
\]
This, along with the symmetry of of $\kappa(x, y; \omega)$ in $(x, y)$, yields that for every $x_1, y_1, x_2, y_2 \in \mathbb{R}^d$,
\[
\text{Cov}(\kappa(x_1, y_1; \cdot), \kappa(x_2, y_2; \cdot)) \leq 4 C^*_1 \left(1 \wedge \left(|x_1 - x_2| \wedge |y_1 - y_2| \wedge |y_1 - y_2| \wedge |y_1 - y_2|\right)^{-l}\right).
\] (1.14)
Thus, the mixing condition (1.14) is weaker than the mutually independent stable-like random conductance models investigated in [18, 16]. In details, (1.14) only requires the mixing condition on the position variable $x$, not on the jumping size variable $z$; while in [18, 16] the mutual independence is imposed on both variables $x$ and $z$, which was crucial to verify (A4*) (ii) in [18] (see also [16, Section 4]). In some sense the mutually independent assumption adopted in [18, 16] corresponds to the following mixing condition: there are constants $l, C_2 > 0$ so that for every $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$,
\[
\text{Cov}(\kappa(x_1, y_1; \cdot), \kappa(x_2, y_2; \cdot)) \leq C_2 \left(1 \wedge \left(|x_1 - x_2| + |y_1 - y_2| \wedge |x_1 - y_2| + |x_2 - y_1|\right)^{-l}\right)
\]
which is stronger than (1.14). Since in the present paper the mixing condition is of the weaker form (1.9) and acting on $\nu$, instead of $\nu$, the arguments in [18, 16] does not work. We use a different approach to deal with the homogenization problem.

(iv) It follows from (1.10) that $\bar{\nu}(z)$ is a radial process; that is, $\bar{\nu}(\lambda z) = \bar{\nu}(z)$ for any $\lambda > 0$. Moreover, under condition (A2), there are positive constants $C_3 \leq C_4$ so that

$$C_3 \leq \bar{\nu}(z) + \bar{\nu}(-z) \leq C_4 \quad \text{for all } z \in \mathbb{R}^d.$$ 

Note also that in our setting we always assume that (1.2) holds true. In fact, (1.2) is a consequence of assumption (A2). Indeed, suppose (A2) holds, and that

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^d} \kappa(x, x + z; \omega) dz dx < \infty.$$

In particular, we have $\mathbb{P}$-a.s.,

$$\int_{\mathbb{R}^d} \kappa(x, x + z; \omega) dz dx < \infty \quad \text{for every } R > 0.$$

**Theorem 1.3.** Suppose that (A1) and (A2) hold, and that $\mathbb{E}[\mu^p] < \infty$ for some $p > 1$. For $\varepsilon > 0$, let $U_{\varepsilon}^{x, \omega}$ be the $\lambda$-order resolvent of the Dirichlet form $(E, \mathcal{F})$ given by (1.4). There is $\Omega_0 \subset \Omega$ of full probability measure so that for every $\omega \in \Omega_0$, every $\lambda > 0$ and $f \in C_c(\mathbb{R}^d)$,

$$U_{\varepsilon}^{x, \omega} f \text{ converges to } U_{\lambda}^K f \text{ in } L^1(B(0,r); dx) \quad \text{as } \varepsilon \to 0$$

for every $r > 1$, and

$$\lim_{\varepsilon \to 0} \|U_{\varepsilon}^{x, \omega} f - U_{\lambda}^K f\|_{L^2(\mathbb{R}^d; \mu^{\varepsilon, \omega})} = 0,$$

where $U_{\lambda}^K$ is the $\lambda$-order resolvent of the symmetric Dirichlet form $(E^K, \mathcal{F}^K)$ on $L^2(\mathbb{R}^d; dx)$ given by (1.7) with

$$K(z) := \bar{\nu}(z) + \bar{\nu}(-z).$$

Clearly, by taking the smaller one, we can assume $p > 1$ in the condition $\mathbb{E}[\mu^p] < \infty$ is the same as the $p > 1$ in (A2). Note that by Remark 1.2(iv), $K(z)$ is a radial even function on $\mathbb{R}^d$ that is bounded between two positive constants. So the limiting Dirichlet form $(E^K, \mathcal{F}^K)$ is that of a symmetric, but not necessary rotationally symmetric $\alpha$-stable processes on $\mathbb{R}^d$. Since for any $g \in C^1_c(\mathbb{R}^d),$

$$E_{\varepsilon}^{x, \omega}(U_{\varepsilon}^{x, \omega} f, g) + \lambda(U_{\varepsilon}^{x, \omega} f, g)_{L^2(\mathbb{R}^d; \mu^{\varepsilon, \omega}(dz))} = \langle f, g \rangle_{L^2(\mathbb{R}^d, \mu^{\varepsilon, \omega}(dx))},$$

$$E^K(U_{\lambda}^K f, g) + \lambda(U_{\lambda}^K f, g)_{L^2(\mathbb{R}^d; dx)} = \langle f, g \rangle_{L^2(\mathbb{R}^d, dx)},$$

and by the Birkhoff ergodic theorem (see Proposition 2.1 below),

$$\lim_{\varepsilon \to 0} (U_{\varepsilon}^{x, \omega} f, g)_{L^2(\mathbb{R}^d; \mu^{\varepsilon, \omega}(dx))} = \langle U_{\lambda}^K f, g \rangle_{L^2(\mathbb{R}^d; dx)} \quad \text{and} \quad \lim_{\varepsilon \to 0} (f, g)_{L^2(\mathbb{R}^d, \mu^{\varepsilon, \omega}(dx))} = \langle f, g \rangle_{L^2(\mathbb{R}^d; dx)},$$

we conclude from (1.15) that

$$\lim_{\varepsilon \to 0} E_{\varepsilon}^{x, \omega}(U_{\varepsilon}^{x, \omega} f, g) = E^K(U_{\lambda}^K f, g) \quad \text{for every } g \in C^1_c(\mathbb{R}^d).$$

1.3.2. $\kappa(x, y; \omega)$ of product form. Motivated by [33, (Q1)], we next consider the case where the coefficient $\kappa(x, y; \omega)$ of (1.1) is of a product form. (We note that [33, (Q2)] is essentially an approximation of [33, (Q1)], under some additional assumptions on the random environment $(\Omega, \mathcal{F}, \mathbb{P})$ and on the coefficient $\kappa(x, y; \omega)$ of the jumping kernel.) We consider the following Assumption (B).

(B1) For a.s. $\omega \in \Omega,

$$\kappa(x, y; \omega) = \nu_1(\tau_x \omega) \nu_2(\tau_y \omega) + \nu_1(\tau_y \omega) \nu_2(\tau_x \omega) \quad \text{for every } x, y \in \mathbb{R}^d,$$

where $\nu_1$ and $\nu_2$ are non-negative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. 

(B2) There are non-negative random variables $\Lambda_1 \leq \Lambda_2$ on $(\Omega, \mathcal{F}, \mathbb{P})$ so that for a.s. $\omega \in \Omega$,
\[
\Lambda_1(\tau_x \omega) \Lambda_1(\tau_y \omega) \leq \kappa(x, y; \omega) \leq \Lambda_2(\tau_x \omega) \Lambda_2(\tau_y \omega) \quad \text{for every } x, y \in \mathbb{R}^d,
\]
and that
\[
\mathbb{E} [\Lambda_1^{-1} + \Lambda_2^2] < \infty. \tag{1.18}
\]

(B2') There are non-negative random variables $\Lambda_1 \leq \Lambda_2$ on $(\Omega, \mathcal{F}, \mathbb{P})$ so that (1.17) holds for a.s. $\omega \in \Omega$, and that
\[
\mathbb{E} [\Lambda_1^{-1}] < \infty.
\]

Clearly condition (B2) is stronger than condition (B2'). Conditions (B2) and (B2') are equivalent to the following, whose proof is postponed into the appendix of this paper.

Proposition 1.4. Suppose that $\kappa(x, y; \omega)$ is given by (1.16) for some non-negative random variables $\nu_1$ and $\nu_2$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Then condition (B2') holds if and only if $\mathbb{E} [(\nu_1 \nu_2)^{-1/2}] < \infty$, and condition (B2) holds if and only if
\[
\mathbb{E} \left[ (\nu_1 \nu_2)^{-1/2} + (\nu_1 + \nu_2)^2 \right] < \infty. \tag{1.19}
\]

Remark 1.5. Following the argument in Remark 1.2(iv) and using the elementary inequality $ab \leq (a^2 + b^2)/2$ for $a, b \geq 0$, we can easily verify that (1.2) is a consequence of assumption (B2). Moreover, if (1.17) holds and the function $x \mapsto \Lambda_2(\tau_x \omega)$ is locally bounded for a.s. $\omega \in \Omega$, we can establish (1.2) just under the first moment condition of $\Lambda_2$ (i.e. under the condition that $\mathbb{E}[\Lambda_2] < \infty$). Indeed, for any $R \geq 1$ and a.s. $\omega \in \Omega$,
\[
\int_{B(0,R)} \int_{\mathbb{R}^d} (1 \wedge |z|^2) \frac{\kappa(x, x + z; \omega)}{|z|^{d+\alpha}} \, dz \, dx \\
\leq \left[ \sup_{x \in B(0,2R)} \Lambda_2(\tau_x \omega)^2 \right] \int_{B(0,R)} \int_{B(0,R)} \frac{|z|^2}{|z|^{d+\alpha}} \, dz \, dx \\
+ \sum_{k = [\log R/\log 2]}^{\infty} 2^{-k(d+\alpha)} \int_{B(0,R)} \int_{\{2^k \leq |z| \leq 2^{k+1}\}} \Lambda_2(\tau_x \omega) \Lambda_2(\tau_{x+z} \omega) \, dz \, dx
\]
\[
\leq C_1(R; \omega) + \sum_{k = [\log R/\log 2]}^{\infty} 2^{-k(d+\alpha)} \left( \int_{B(0,R)} \Lambda_2(\tau_x \omega) \, dx \right) \cdot \left( \int_{B(0,2^{k+2})} \Lambda_2(\tau_y \omega) \, dy \right)
\]
\[
\leq C_1(R; \omega) + C_2(R; \omega) \sum_{k = [\log R/\log 2]}^{\infty} 2^{-k(d+\alpha)} 2^{(k+2)d} < \infty,
\]
where in the third inequality we used the Birkhoff ergodic theorem (see Proposition 2.1 below) and the last inequality follows from the local boundedness of $\Lambda_2(\tau_x \omega)$. The assumption such as the local boundedness of $x \mapsto \Lambda_2(\tau_x \omega)$ was assumed in [15] (see (a.3) on p.1536) to study the invariance principle for symmetric diffusions in a degenerate and unbounded stationary and ergodic random medium.

Note that any $\kappa(x, y; \omega)$ of form (1.16) enjoys the property (1.1). When the coefficient $\kappa(x, y; \omega)$ of the jumping kernel is of the product form (1.16), the corresponding symmetric Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ has the expression
\[
\mathcal{E}^\omega(f, f) := \frac{1}{2} \int \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} (f(x) - f(y))^2 \frac{\nu_1(\tau_x \omega) \nu_2(\tau_y \omega)}{|x - y|^{d+\alpha}} \mathbb{I}_{\{y-x \in \Gamma\}} \, dx \, dy \quad \text{for } f \in \mathcal{F}^\omega.
\]
In this case, we are able to drop the mixing condition (1.9) from Theorem 1.3.

Theorem 1.6. Suppose that assumptions (B1) and (B2) hold, and that $\mathbb{E}[\mu^p] < \infty$ for some $p > 1$. Then the conclusion of Theorem 1.3 holds with constant
\[
K(z) := \mathbb{E}[\nu_1] \mathbb{E}[\nu_2].
\]
1.4. Comments on main results. To the best of our knowledge, only two cases have been studied in literature concerning homogenization of $\alpha$-stable-like processes (or $\alpha$-stable-like operators) in ergodic random environment. The first one is [33], where the infinitesimal generator is given by (1.5) with $\Gamma = \mathbb{R}^d$ and $\kappa(x, y, \omega) = \lambda_1(\tau_x \omega) \lambda_2(\tau_y \omega)$ in one-parameter ergodic random environment. The second one is [28], which is under two-parameters ergodic environment. The setting of our paper is more general, and it also includes the symmetrization of those in [24, 30, 45]; see (A1) introduced above. Besides, as we will explain in Example 1.7 below, assumption (B) is also related to random conductance models associated with mutually independent site percolations.

As far as we know, for all the results in literature, the limiting process is always non-degenerate with $\Gamma = \mathbb{R}^d$, even if the coefficients of scaled processes are degenerate—see for instance [9, 15, 16, 33, 28]. Our paper provides examples for symmetric jump processes that both coefficients of scaled processes and the limiting process are degenerate. In details, in our paper not only the cone $\Gamma$ can be a proper subset of $\mathbb{R}^d$ but also the coefficient $\kappa(x, y, \omega)$ of jumping kernel can be degenerate and unbounded. Moreover, the coefficient $K(z)$ of jumping kernel for the limiting process can be a non-constant function, which in particular implies that the limiting process does not need to be a rotationally symmetric $\alpha$-stable process, but a more general symmetric $\alpha$-stable Lévy process on $\mathbb{R}^d$ that enjoying the scaling property.

Under assumption (A), we only assume the finiteness of negative 1-moment and positive $p$-moment with $p > 1$ for bounds of the coefficient $\kappa(x, y; \omega)$ to study the homogenization problem. We believe that the negative moment integrability condition is optimal and the positive moment integrability condition is almost optimal, since they are necessary to apply the ergodic theorem. We emphasis that under assumption (B), we also only require the finiteness of negative 1-moment. We also note that, under both negative 1-moment and positive 1-moment conditions, the annealed invariance principle for nearest neighbor random walks on random conductances was established in [22], and the quenched invariance principle was proven in [10] when $d = 1, 2$. For random divergence forms, one may follow the two-scale convergence method adopted in [47] to prove the $L^2$-convergence of associated resolvents under similar conditions.

It is natural to consider further the weak convergence of the scaled processes on the path space. Strong convergence of the resolvents that we have established so far corresponds to the convergence of the finite dimensional distributions of the scaled processes when the initial measure is absolutely continuous with respect to an invariant measure. In order to obtain the weak convergence of the scaled processes, we need to establish the tightness (with respect to the Skorohod topology) of the scaled processes. In fact, if the initial distribution is an invariant measure (or more generally it is absolutely continuous with respect to an invariant measure), then the tightness can be obtained by using the so-called forward-backward martingale decomposition (see [18, Proposition 3.4] for the corresponding statement in the discrete setting). Hence one can obtain the convergence of the processes on the path space under such initial condition (or under some weaker topology), see [18, Theorems 2.2 and 2.3] for more discussions in the discrete case. When $(x, y) \mapsto \kappa(x, y; \cdot)$ is bounded between two positive constants, we can use heat kernel estimates from [19] when $\Gamma = \mathbb{R}^d$ or parabolic Harnack inequality from [21] when $\Gamma \subset \mathbb{R}^d$ to establish the tightness and therefore the weak convergence of the scaled processes starting from any point. However, it is highly non-trivial to prove such convergence if the process starts at any fixed point (in other word, if the initial distribution is a Dirac measure) when $(x, y) \mapsto \kappa(x, y; \cdot)$ is not bounded between two positive constants. We will address this problem in a separate paper.

1.5. Example. A typical example of infinite symmetric cone for degenerate non-local Dirichlet forms given by (1.3) is

$$\Gamma = \left\{ z \in \mathbb{R}^d : \langle z, z_0 \rangle \geq \eta |z| \right\}$$

for some $z_0 \in S^{d-1}$ and $\eta \in [0, 1)$. In the deterministic case, the regularity estimates for non-local operators associated with such kind of degenerate Dirichlet forms have been studied in [26, 21]; see [26, Example 3] or [21, Example 1.2] for more details.

As an application of Theorem 1.6, we take the following example that improves [33, Theorem 3, Case (Q1)], where the coefficients $\lambda_i(\tau_x \omega) \ (i = 1, 2)$ are assumed to be uniformly bounded from above and below and $\Gamma = \mathbb{R}^d$. 
Example 1.7. Suppose that $\Gamma$ is an infinite symmetric cone in $\mathbb{R}^d$ that has non-empty interior. For any $\varepsilon > 0$, let $\mathcal{L}_{\varepsilon, \omega}$ be the Lévy-type operator given by

$$
\mathcal{L}_{\varepsilon, \omega} f(x) = \text{p.v.} \int_{\mathbb{R}^d} (f(y) - f(x)) \frac{\lambda_1(\tau_{x/\varepsilon} \omega) \lambda_2(\tau_{y/\varepsilon} \omega)}{|y - x|^{d+\alpha}} 1_{\Gamma}(y - x) \, dy,
$$

where $\lambda_1$ and $\lambda_2$ are non-negative measurable functions on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$
\lambda_2^{-1} \in L^1(\Omega; \mathbb{P}), \quad \lambda_2 \in L^2(\Omega; \mathbb{P}) \quad \text{and} \quad \lambda_2/\lambda_1 \in L^p(\Omega; \mathbb{P}),
$$

for some $p > 1$. Then, as $\varepsilon \to 0$, $\mathcal{L}_{\varepsilon, \omega}$ converges in the resolvent topology to

$$
\mathcal{L} f(x) = \text{p.v.} \int_{\mathbb{R}^d} (f(y) - f(x)) \frac{C_0}{|y - x|^{d+\alpha}} \, dy,
$$

where

$$
C_0 = \frac{(\mathbb{E}[\lambda_2])^2}{\mathbb{E}[\lambda_2/\lambda_1]}
$$

in the following sense. There is $\Omega_0 \subset \Omega$ of full probability measure so that for every $\omega \in \Omega_0$, $\lambda > 0$ and $f \in C_c(\mathbb{R}^d)$,

$$
\lim_{\varepsilon \to 0} \|U_{\lambda}^{\varepsilon, \omega} f - U_{\lambda} f\|_{L^2(\mathbb{R}^d; \mu^{\varepsilon, \omega}(dx))} = 0,
$$

where $\mu^{\varepsilon, \omega}(dx) := (\lambda_2/\lambda_1)(\tau_{x/\varepsilon} \omega) \, dx$, and $U_{\lambda}^{\varepsilon, \omega} f$ and $U_{\lambda} f$ are the $\lambda$-order resolvent function of $\mathcal{L}_{\varepsilon, \omega}$ and $\mathcal{L}$, respectively. In addition, $U_{\lambda}^{\varepsilon, \omega} f$ converges to $U_{\lambda} f$ in $L^1(B(0, r); dx)$, as $\varepsilon \to 0$, for every $r > 1$.

The operator (1.20) can be seen as a random long range randomly weighted site model. Indeed, if $\lambda_1(\tau_{x} \omega)$ and $\lambda_2(\tau_{y} \omega)$ are regarded as random weight at the site $x$ for initiating a jump and receiving a jump, respectively, then the long range effect of the media for the coefficient of the jump intensity from $x$ to $y$ is given by the product $\lambda_1(\tau_{x} \omega) \lambda_2(\tau_{y} \omega)$. At the first sight, the constant coefficient $C_0$ for the limiting operator $\mathcal{L}$ should be $\mathbb{E}[\lambda_1 \lambda_2]$, but with the idea of the time change as used in the proof of the assertion of Example 1.7 below, it turns out the correct one should be the one given by formula (1.21). We emphasize again that in this site model the mixing condition (1.9) of the media given in Assumption (A1) is not needed.

1.6. Organization of the paper. The rest of the paper is organized as follows. In Section 2, we will prove homogenization of stable-like Dirichlet forms under general sufficient conditions. The main results are Theorems 2.2 and 2.3. In Section 3, we study the weak convergence of non-local symmetric bilinear forms, and the $L^1$-precompactness of bounded functions with bounded Dirichlet energies. Both of them are of interest in their own. With those two at hand, we give proofs of Theorems 1.3 and 1.6, and the assertion of Example 1.7 in Subsection 3.3. In the appendix of this paper, in addition to presenting the proofs for Lemma 1.1 and Proposition 1.4, we study the Mosco convergence for $(\mathcal{E}_{\varepsilon, \omega}, \mathcal{F}_{\varepsilon, \omega})$.

1.7. Notations. We use := as a way of definition. Let $\mathbb{R}_+ := [0, \infty)$, $\mathbb{Z}_+ := \{0, 1, 2, \cdots\}$, $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$. For all $x \in \mathbb{R}^d$ and $r > 0$, set $B(x, r) = \{z \in \mathbb{R}^d : |z - x| < r\}$. For $p \in [1, \infty]$ and Lebesgue measurable $A \subset \mathbb{R}^d$, we use $|A|$ to denote the $d$-dimensional Lebesgue measure of $A$, $C_b(A)$ the space of bounded and continuous functions on $A$, $L^p(A; dx)$ the space of $L^p$-integrable functions on $A$ with respect to the Lebesgue measure, and $L^p_{loc}(\mathbb{R}^d; dx)$ the space of locally $L^p$-integrable functions on $\mathbb{R}^d$ with respect to the Lebesgue measure. Denote $(\cdot, \cdot)_{L^2(\mathbb{R}^d; \mu(dx))}$ the inner product in $L^2(\mathbb{R}^d; \mu(dx))$. Denote by $B(\mathbb{R}^d)$ the set of locally bounded measurable functions on $\mathbb{R}^d$, by $B_b(\mathbb{R}^d)$ the set of bounded measurable functions on $\mathbb{R}^d$, and by $B_c(\mathbb{R}^d)$ the set of bounded measurable functions on $\mathbb{R}^d$ with compact support. $C^1_0(\mathbb{R}^d)$ (respectively, $C^1(\mathbb{R}^d)$ or $C^\infty_c(\mathbb{R}^d)$) denotes the space of $C^1$-smooth (respectively, continuous or $C^\infty$-smooth) functions on $\mathbb{R}^d$ with compact support.

2. Homogenization of stable-like Dirichlet forms: general results

For any $\varepsilon > 0$, let $\mathcal{L}_{\varepsilon, \omega}$ be the generator of the Dirichlet form $(\mathcal{E}_{\varepsilon, \omega}, \mathcal{F}_{\varepsilon, \omega})$ on $L^2(\mathbb{R}^d; \mu^{\varepsilon, \omega}(dx))$ given by (1.6). Let $\mathcal{L}^{\mathbb{K}}$ be the generator of the Dirichlet form $(\mathcal{E}^{\mathbb{K}}, \mathcal{F}^{\mathbb{K}})$ of (1.7) on $L^2(\mathbb{R}^d; dx)$. The goal of homogenization theory is to construct homogenized characteristics and clarify whether the solutions for the operators $\mathcal{L}_{\varepsilon, \omega}$
are close to the solution for the operator \( L^K \). In this paper, we are concerned with the following question: how to prove that the solution to the equation
\[
(\lambda - L^\varepsilon,\omega) u^{\varepsilon,\omega} = f
\]
on \( L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega}(dx)) \) for any \( \lambda > 0 \) and \( f \in C_c(\mathbb{R}^d) \) converges in the resolvent topology, as \( \varepsilon \to 0 \), to the solution to the equation
\[
(\lambda - L^K) u = f
\]
on \( L^2(\mathbb{R}^d; dx) \).

The section is devoted to addressing this question under the following assumption.

**Assumption (H):** There is \( \Omega_0 \subset \Omega \) of full probability measure so that the following hold for every \( \omega \in \Omega_0 \).

(i) If \( \{ f_\varepsilon : \varepsilon \in (0, 1] \} \) is a sequence of functions on \( \mathbb{R}^d \) such that \( f_\varepsilon \in \mathcal{F}^{\varepsilon,\omega} \) for any \( \varepsilon \in (0, 1] \) and
\[
\lim_{\varepsilon \to 0} \sup_{\varepsilon} (\| f_\varepsilon \|_{\infty} + \mathcal{E}^{\varepsilon,\omega}(f_\varepsilon, f_\varepsilon)) < \infty,
\]
then \( \{ f_\varepsilon : \varepsilon \in (0, 1] \} \) is pre-compact as \( \varepsilon \to 0 \) in \( L^1(B(0, r); dx) \) for every \( r > 1 \) in the sense that for any sequence \( \{ \varepsilon_n : n \geq 1 \} \subset (0, 1] \) with \( \lim_{n \to 0} \varepsilon_n = 0 \), there are a subsequence \( \{ \varepsilon_{n_k} : k \geq 1 \} \) and a function \( f \in L^1_{\text{loc}}(\mathbb{R}^d; dx) \) so that \( f_{n_k} \) converges to \( f \) in \( L^1(B(0, r); dx) \) for every \( r > 1 \).

(ii) For any \( g \in C^\infty_c(\mathbb{R}^d) \),
\[
\lim_{\eta \to 0} \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} (g(x) - g(y))^2 \frac{K(x/\varepsilon, y/\varepsilon; \omega)}{|x - y|^{d+\alpha}} \mathbf{1}_{\{|x - y| \leq \eta\}} \ dx \ dy = 0
\]
and
\[
\lim_{\eta \to 0} \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} (g(x) - g(y))^2 \frac{K(x/\varepsilon, y/\varepsilon; \omega)}{|x - y|^{d+\alpha}} \mathbf{1}_{\{|x - y| \geq 1/\eta\}} \ dx \ dy = 0.
\]

(iii) There is a constant \( p > 1 \) such that
\[
\limsup_{\varepsilon \to 0} \int_{B(0,R)} \left( \int_{B(0,R)} \frac{\kappa(x/\varepsilon, y/\varepsilon; \omega) \ dy}{\| \mathcal{E}^{\varepsilon,\omega}(f_\varepsilon, f_\varepsilon) \|_{\infty} + \| f_\varepsilon \|_{\infty}} \right)^p \ dx < \infty \quad \text{for every} \ R \geq 1.
\]

(iv) For every \( \eta > 0 \), \( f \in B_b(\mathbb{R}^d) \) and \( g \in C^\infty_c(\mathbb{R}^d) \),
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} (f(x) - f(y))(g(x) - g(y)) \frac{\kappa(x/\varepsilon, y/\varepsilon; \omega)}{|x - y|^{d+\alpha}} \mathbf{1}_{\{\eta < |x - y| < 1/\eta, x-y \in \Gamma\}} \ dx \ dy
\]
\[
= \int_{\mathbb{R}^d} (f(x) - f(y))(g(x) - g(y)) \frac{K(x-y)}{|x - y|^{d+\alpha}} \mathbf{1}_{\{\eta < |x - y| < 1/\eta, x-y \in \Gamma\}} \ dx \ dy,
\]
where \( K(z) \) is a measurable even function on \( \mathbb{R}^d \) such that \( C_1 \leq K(z) \leq C_2 \) for some constants \( C_1, C_2 > 0 \).

The section is divided into two parts. We first consider the weak convergence of resolvents in the Dirichlet norm, and then study the strong convergence of resolvents in the \( L^2 \)-norm.

We will use the following Birkhoff ergodic theorem (see, for example, [31, Theorem 7.2]) several times in this paper.

**Proposition 2.1.** Suppose that \( \nu \geq 0 \) is a random variable on \( (\Omega, \mathcal{F}, P) \) with \( E[\nu] < \infty \). There is a subset \( \Omega_1 \subset \Omega \) of full probability measure so that for every \( \omega \in \Omega_1 \), the function \( x \mapsto \nu(\tau_{x/\varepsilon}\omega) \) converges weakly to \( E[\nu] \) in \( L^1_{\text{loc}}(\mathbb{R}^d, dx) \) as \( \varepsilon \to 0 \); that is, for every \( \omega \in \Omega_1 \), every bounded Lebesgue measurable set \( K \subset \mathbb{R}^d \) and every \( \varphi \in L^\infty(K; dx) \),
\[
\lim_{\varepsilon \to 0} \int_K \varphi(x)\nu(\tau_{x/\varepsilon}\omega) \ dx = E[\nu] \int_K \varphi(x) \ dx.
\]
Furthermore, if \( E[\nu^p] < \infty \) for some \( p > 1 \), then for every \( \omega \in \Omega_1 \), the function \( x \mapsto \nu(\tau_{x/\varepsilon}\omega) \) converges weakly to \( E[\nu] \) in \( L^p_{\text{loc}}(\mathbb{R}^d; dx) \) as \( \varepsilon \to 0 \).
2.1. Weak convergence of resolvents. Recall that $\mu$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ so that $\mathbb{E}[\mu] = 1$ and for any $\omega \in \Omega$, $\mu(\tau_x \omega) > 0$ for a.e. $x \in \mathbb{R}^d$. Denote by $U^\varepsilon_\lambda f$ the $\lambda$-order resolvent of the regular Dirichlet form $(\mathcal{E}^\varepsilon, \mathcal{F}^\varepsilon)$ on $L^2(\mathbb{R}^d; \mu^\varepsilon(dx))$, and $U^K_\lambda$ the $\lambda$-order resolvent of the regular Dirichlet form $(\mathcal{E}^K, \mathcal{F}^K)$ on $L^2(\mathbb{R}^d; dx)$. It is well known that $U^\varepsilon_\lambda f$ and $U^K_\lambda f$ are the unique solution to (2.1) and (2.2), respectively.

**Theorem 2.2.** Suppose that assumption (H) holds and that $\mathbb{E}[\mu^p] < \infty$ for some $p > 1$. Then there is a subset $\Omega_2 \subset \Omega$ of full probability measure so that for every $\omega \in \Omega_2$ and any $f \in C_c(\mathbb{R}^d)$,

$$U^\varepsilon_\lambda f \text{ converges to } U^K_\lambda f \text{ in } L^1(B(0,r); dx) \quad \text{as } \varepsilon \to 0$$

(2.5)

for every $r > 1$,

$$\lim_{\varepsilon \to 0} \langle U^\varepsilon_\lambda f, g \rangle_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} = \langle U^K_\lambda f, g \rangle_{L^2(\mathbb{R}^d; dx)} \text{ for every } g \in C_c(\mathbb{R}^d),$$

(2.6)

and

$$\lim_{\varepsilon \to 0} \mathcal{E}^\varepsilon(\lambda U^\varepsilon_\lambda f, g) = \mathcal{E}^K(\lambda U^K_\lambda f, g) \text{ for every } g \in C_c^\infty(\mathbb{R}^d),$$

(2.7)

where $K(z)$ is the function in assumption (H)(iv).

**Proof.** Let $\Omega_2 = \Omega_0 \cap \Omega_1$, where $\Omega_0$ and $\Omega_1$ are the subset of $\Omega$ in assumption (H) and in Proposition 2.1, respectively, both of them having full probability measure. Let $\omega \in \Omega_2$. For simplicity, throughout the proof we omit the parameter $\omega$ from $U^\varepsilon_\lambda f$ and $\mu^\varepsilon$.

Fix $\lambda > 0$. For any $\varepsilon > 0$ and $f \in C_c(\mathbb{R}^d)$, $U^\varepsilon_\lambda f$ is the unique element in $\mathcal{F}^\varepsilon$ so that

$$\lambda \langle U^\varepsilon_\lambda f, g \rangle_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} + \mathcal{E}^\varepsilon(U^\varepsilon_\lambda f, g) = \langle f, g \rangle_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} \quad \text{for } g \in \mathcal{F}^\varepsilon.$$  

(2.8)

We first treat the limits for the right hand side and the first term in the left hand side in (2.8). Note that

$$\|U^\varepsilon_\lambda f\|_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} \leq \lambda^{-1} \|f\|_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} \quad \text{for } \varepsilon \in (0, 1].$$

Thus by (2.8),

$$\lambda \|U^\varepsilon_\lambda f\|^2_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} + \mathcal{E}^\varepsilon(U^\varepsilon_\lambda f, U^\varepsilon_\lambda f) = \langle f, U^\varepsilon_\lambda f \rangle_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} \leq \lambda^{-1} \|f\|^2_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))}.$$  

(2.9)

Hence, according to $\mathbb{E}[\mu] = 1$ and Proposition 2.1,

$$\limsup_{\varepsilon \to 0} \left( \|U^\varepsilon_\lambda f\|^2_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} + \|U^\varepsilon_\lambda f\|^2_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} \right) < \infty.$$  

Note also that $\|U^\varepsilon_\lambda f\| \leq \lambda^{-1} \|f\|_{\infty}$ for all $\varepsilon \in (0, 1]$. By assumption (H)(i), $\{U^\varepsilon_\lambda f : \varepsilon \in (0, 1]\}$ is pre-compact as $\varepsilon \to 0$ in $L^1(B(0,r); dx)$ for every $r > 1$. In particular, for any sequence $\{\varepsilon_n : n \geq 1\} \subset (0, 1]$ with $\lim_{n \to \infty} \varepsilon_n = 0$, we can find a subsequence $\{\varepsilon_{n_k} : k \geq 1\}$ (for simplicity we will still denote it by $\{\varepsilon_n : n \geq 1\}$) and a function $U^\varepsilon_\lambda f \in L^1_{loc}(\mathbb{R}^d; dx)$ (which indeed may depend on $\omega$) so that

$$\lim_{n \to \infty} \int_{B(0,r)} |U^\varepsilon_\lambda f(x) - U^\varepsilon_\lambda f(x)| \, dx = 0 \quad \text{for every } r > 1.$$  

(2.10)

Since $\sup_{\varepsilon \in (0,1]} \|U^\varepsilon_\lambda f\|_{\infty} \leq \lambda^{-1} \|f\|_{\infty}$, we in fact have $\|U^\varepsilon_\lambda f\|_{\infty} \leq \lambda^{-1} \|f\|_{\infty}$ and so for every $q \in [1, \infty)$,

$$\lim_{n \to \infty} \int_{B(0,r)} |U^\varepsilon_\lambda f(x) - U^\varepsilon_\lambda f(x)|^q \, dx = 0 \quad \text{for every } r > 1.$$  

(2.11)

We next show that $U^\varepsilon_\lambda f = U^K_\lambda f$, where $U^K_\lambda$ is the $\lambda$-order resolvent associated with the operator $\mathcal{L}^K$.

Fix $g \in C_c(\mathbb{R}^d)$. We choose $R > 0$ such that $\text{supp}[g] \subset B(0, R_0)$. Then, by the Hölder inequality with $p > 1$ from assumption (H)(iii),

$$\int_{\mathbb{R}^d} |U^\varepsilon_\lambda f(x) - U^\varepsilon_\lambda f(x)| g(x) \mu(\tau_x/\varepsilon_n \omega) \, dx$$

$$\leq \|g\|_{\infty} \left( \int_{B(0,R_0)} |U^\varepsilon_\lambda f(x) - U^\varepsilon_\lambda f(x)|^{p/(p-1)} \, dx \right)^{(p-1)/p} \left( \int_{B(0,R_0)} \mu(\tau_x/\varepsilon_n \omega) p \, dx \right)^{1/p}.$$  

Since $\mathbb{E}[\mu^p] < \infty$, we get by Proposition 2.1 that for $\omega \in \Omega_2 \subset \Omega_1$,

$$\limsup_{n \to \infty} \int_{B(0,R_0)} \mu^p(\tau_x/\varepsilon_n \omega) \, dx < \infty.$$
This along with (2.11) yields
\[ \lim_{n \to \infty} \int_{\mathbb{R}^d} |U^\varepsilon_n f(x) - U^\varepsilon_n f(x)| g(x) \mu(\tau_{x/\varepsilon_n} \omega) \, dx = 0. \]

On the other hand, since \( E[\mu] = 1 \), we have again by Proposition 2.1,
\[ \lim_{n \to \infty} \int_{\mathbb{R}^d} U^\varepsilon_n f(x) g(x) \mu(\tau_{x/\varepsilon_n} \omega) \, dx = \lim_{n \to \infty} \int_{B(0,R_0)} U^\varepsilon_n f(x) g(x) \mu(\tau_{x/\varepsilon_n} \omega) \, dx \]
\[ = \int_{B(0,R_0)} U^\varepsilon_n f(x) g(x) \mathbb{E}[\mu] \, dx = \int_{\mathbb{R}^d} U^\varepsilon_n f(x) g(x) \, dx. \]

Putting both estimates above together yields that for any \( f, g \in C_c(\mathbb{R}^d) \)
\[ \lim_{n \to \infty} \lambda(U^\varepsilon_n f, g)_{L^2(\mathbb{R}^d, \mu^\varepsilon_n(dx))} = \lambda(U^\varepsilon_n f, g)_{L^2(\mathbb{R}^d, dx)} . \tag{2.12} \]

Similarly, according to \( E[\mu] = 1 \) and Proposition 2.1, we have for every \( f, g \in C_c(\mathbb{R}^d) \),
\[ \lim_{n \to \infty} \langle f, g \rangle_{L^2(\mathbb{R}^d, \mu^\varepsilon_n(dx))} = \langle f, g \rangle_{L^2(\mathbb{R}^d, dx)} . \tag{2.13} \]

In particular, by (2.12) and (2.13), we obtain that for every \( g \in C_c(\mathbb{R}^d) \),
\[ (U^\varepsilon_n f, g)_{L^2(\mathbb{R}^d, dx)} = \lim_{n \to \infty} (U^\varepsilon_n f, g)_{L^2(\mathbb{R}^d, \mu^\varepsilon_n(dx))} \]
\[ \leq \lim_{n \to \infty} \left( \|U^\varepsilon_n f\|_{L^2(\mathbb{R}^d, \mu^\varepsilon_n(dx))} \cdot \|g\|_{L^2(\mathbb{R}^d, \mu^\varepsilon_n(dx))} \right) \]
\[ \leq \lambda^{-1} \lim_{n \to \infty} \left( \|f\|_{L^2(\mathbb{R}^d, \mu^\varepsilon_n(dx))} \cdot \|g\|_{L^2(\mathbb{R}^d, \mu^\varepsilon_n(dx))} \right) \]
\[ = \lambda^{-1} \|f\|_{L^2(\mathbb{R}^d, dx)} \cdot \|g\|_{L^2(\mathbb{R}^d, dx)}, \]
which immediately that \( U^\varepsilon_n f \in L^2(\mathbb{R}^d; dx) \) and \( \|U^\varepsilon_n f\|_{L^2(\mathbb{R}^d, dx)} \leq \lambda^{-1} \|f\|_{L^2(\mathbb{R}^d, dx)} \).

We now treat the second term in the left hand side of (2.8) with \( g \in C_0^\infty(\mathbb{R}^d) \). According to Lemma 1.1, it holds that for any \( 0 < \eta < 1 \),
\[ 2\varepsilon_n^\eta(U^\varepsilon_n f, g) = \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \left( U^\varepsilon_n f(x) - U^\varepsilon_n f \right)(g(x) + \varepsilon_n f(x)) \frac{\kappa(0, z/\varepsilon_n; \tau_{x/\varepsilon_n}, \omega)}{|z|^{d+1}} 1_{\Gamma \cap \{|z| \leq \eta \}} \, dz \, dx \]
\[ + \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( U^\varepsilon_n f(x) + U^\varepsilon_n f \right)(g(x) + \varepsilon_n f(x)) \frac{\kappa(0, z/\varepsilon_n; \tau_{x/\varepsilon_n}, \omega)}{|z|^{d+1}} 1_{\Gamma \cap \{|z| > 1/\eta \}} \, dz \, dx \]
\[ =: I_1^{n, \eta} + I_2^{n, \eta} - 2I_3^{n, \eta}, \]
where
\[ \mathcal{L}_n f(x) = \int_{\mathbb{R}^d} (f(x) + \varepsilon_n f) \frac{\kappa(0, z/\varepsilon_n; \tau_{x/\varepsilon_n}, \omega)}{|z|^{d+1}} 1_{\Gamma \cap \{|z| \leq 1/\eta \}} \, dz. \]

Note that
\[ \mathcal{L}_0 f(x) := \text{p.v.} \int_{\mathbb{R}^d} (f(x) + \varepsilon_n f) \frac{\kappa(0, z/\varepsilon_n; \tau_{x/\varepsilon_n}, \omega)}{|z|^{d+1}} 1_{\Gamma(z)} \, dz \]
typically is the generator of the Dirichlet form \( (\mathcal{E}\varepsilon_n, \mathcal{F}\varepsilon_n, \omega) \) on \( L^2(\mathbb{R}^d; dx) \).

By the Cauchy-Schwarz inequality,
\[ I_1^{n, \eta} \leq \sqrt{2\varepsilon_n(U^\varepsilon_n f, U^\varepsilon_n f) \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} (g(x) + \varepsilon_n f(x) - g(x)) \frac{\kappa(0, z/\varepsilon_n; \tau_{x/\varepsilon_n}, \omega)}{|z|^{d+1}} 1_{\{|z| \leq \eta \}} \, dz \, dx}. \]

This along with (2.9) and assumption (H)(ii) yield that
\[ \lim_{\eta \to 0} \lim_{n \to \infty} I_1^{n, \eta} = 0. \]

Similarly, we can verify
\[ \lim_{\eta \to 0} \lim_{n \to \infty} I_2^{n, \eta} = 0. \]
Set
\[ \mathcal{L}^n_\eta g(x) := \int_{\mathbb{R}^d} (g(x + z) - g(x)) \frac{K(z)}{|z|^{d+\alpha}} \mathbb{1}_{\Gamma \cap \{ \eta < |z| < 1/\eta \}} \, dz. \]

It is obvious that
\[ \lim_{\eta \to 0} \mathcal{L}^n_\eta g(x) = \mathcal{L}^K g(x), \quad g \in C_c^\infty(\mathbb{R}^d), \]
where
\[ \mathcal{L}^K g(x) := \text{p.v.} \int_{\mathbb{R}^d} (g(x + z) - g(x)) \frac{K(z)}{|z|^{d+\alpha}} \mathbb{1}_\Gamma(z) \, dz \]
is the infinitesimal generator with respect to \((E^K, \mathcal{F}^K)\) on \(L^2(\mathbb{R}^d, dx)\). We find that
\[ |I^n_{3,\eta} - (U^\eta f, \mathcal{L}^n_\eta g)_{L^2(\mathbb{R}^d, dx)}| \leq |\langle U^{\varepsilon_n}_\lambda f - U^\eta f, \mathcal{L}^n_\eta g \rangle_{L^2(\mathbb{R}^d, dx)}| + \|U^\eta f, (\mathcal{L}^n_\eta g - \mathcal{L}^K g) \|_{L^2(\mathbb{R}^d, dx)} | =: I^n_{3,1} + I^n_{3,2}. \]

Observe that, for every \(x \in \mathbb{R}^d\),
\[ |\mathcal{L}^n_\eta g(x)| \leq 2\|g\|_{\infty} \eta^{-d-\alpha} \left( \int_{B(0, R_0 + 1/\eta)} \kappa(x/\varepsilon_n, y/\varepsilon_n; \omega) \, dy \right) \mathbb{1}_{\{ B(0, R_0 + 1/\eta) \}}(x). \]
This along with the Hölder inequality with \(p > 1\) in assumption (H)(iii) yields that
\[ I^n_{3,1} \leq c_{11}(\eta, \omega) \left( \int_{B(0, R_0 + 1/\eta)} |U^\varepsilon_n f(x) - U^\lambda f(x)|^{p/(p-1)} \, dx \right)^{1-1/p} \times \left( \int_{B(0, R_0 + 1/\eta)} \kappa(x/\varepsilon_n, y/\varepsilon_n; \omega) \, dy \right)^{p} \]
\[ \times \left( \int_{B(0, R_0 + 1/\eta)} \kappa(x/\varepsilon_n, y/\varepsilon_n; \omega) \, dy \right)^{1/p}. \]

Hence, by (2.11) and assumption (H)(iii), for each fixed \(\eta > 0\), \(\lim_{n \to \infty} I^n_{3,1} = 0\). On the other hand,
\[ \int_{\mathbb{R}^d} U^\lambda f(x) \mathcal{L}^n_\eta g(x) \, dx \]
\[ = -\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (U^\lambda f(x + z) - U^\lambda f(x))(g(x + z) - g(x)) \frac{\kappa(x/\varepsilon_n, (x + z)/\varepsilon_n; \omega)}{|z|^{d+\alpha}} \mathbb{1}_{\Gamma \cap \{ \eta < |z| < 1/\eta \}} \, dz \, dx. \]

Then, by applying assumption (H)(iv), we have
\[ \lim_{n \to \infty} \int_{\mathbb{R}^d} U^\lambda f(x) \mathcal{L}^n_\eta g(x) \, dx \]
\[ = -\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (U^\lambda f(x + z) - U^\lambda f(x))(g(x + z) - g(x)) \frac{K(z)}{|z|^{d+\alpha}} \mathbb{1}_{\Gamma \cap \{ \eta < |z| < 1/\eta \}} \, dz \, dx \]
\[ = \int_{\mathbb{R}^d} U^\lambda f(x) \mathcal{L}^K g(x) \, dx, \]
which implies \(\lim_{n \to \infty} I^n_{3,2} = 0\).

Combining all the estimates for \(I^n_i, i = 1, 2, 3\) with the fact that \(U^\lambda f \in L^2(\mathbb{R}^d; dx)\), first letting \(n \to \infty\) and then letting \(\eta \to 0\), we obtain
\[ \lim_{n \to \infty} E^{\varepsilon_n}(U^\varepsilon_n f, g) = -\int_{\mathbb{R}^d} U^\lambda f(x) \mathcal{L}^K g(x) \, dx. \] (2.14)

Putting this with (2.8), (2.12) and (2.13) together, we see that for any \(f \in C_c(\mathbb{R}^d)\) and \(g \in C_c^\infty(\mathbb{R}^d)\),
\[ \langle (\lambda - \mathcal{L}^K)g, U^\lambda f \rangle_{L^2(\mathbb{R}^d, dx)} = \langle f, g \rangle_{L^2(\mathbb{R}^d, dx)}. \]

In particular, since the above holds for any \(g \in C_c^\infty(\mathbb{R}^d)\), we have \(U^\lambda f = U^\varepsilon f\); that is, \(U^\varepsilon\) is the \(\lambda\)-order resolvent corresponding to the operator \(\mathcal{L}^K\). For every \(f \in C_c(\mathbb{R}^d)\) and \(g \in C_c^\infty(\mathbb{R}^d)\),
\[ \lim_{n \to \infty} E^{\varepsilon_n}(U^{\varepsilon_n} f, g) = E^K(U^K f, g), \]
and by (2.10), \(U^{\varepsilon_n} f\) converges to \(U^K f\) in \(L^1(B(0, r); dx)\), as \(n \to \infty\), for every \(r > 1\). Since these hold for any sequence \(\{\varepsilon_n\}_{n \geq 1}\) that converges to 0, we get (2.7) and (2.5). The property (2.6) follows from (2.12).
2.2. Strong convergence of resolvents.

**Theorem 2.3.** Suppose that assumption (H) holds and $\mathbb{E} [\mu^p] < \infty$ for some $p > 1$. Let $\Omega_2$ be the subset of $\Omega$ in Theorem 2.2 that is of full probability measure. Then for every $\omega \in \Omega_2$ and every $f \in C_c(\mathbb{R}^d)$,

$$
\lim_{\varepsilon \to 0} \| U^\varepsilon f \|_{L^2(\mathbb{R}^d; \mu^\varepsilon(\omega))} = \| U^0 f \|_{L^2(\mathbb{R}^d; dx)}
$$

(2.15)

and

$$
\lim_{\varepsilon \to 0} \| U^\varepsilon f - U^0 f \|_{L^2(\mathbb{R}^d; \mu^\varepsilon(\omega))} = 0.
$$

(2.16)

In particular,

$$
\lim_{\varepsilon \to 0} \| U^\varepsilon f \|_{L^2(\mathbb{R}^d; \mu^\varepsilon(\omega))} = \| U^0 f \|_{L^2(\mathbb{R}^d; dx)}.
$$

(2.17)

To prove Theorem 2.3, we need the following lemma. Recall that $\Gamma$ is an infinite symmetric cone in $\mathbb{R}^d$ that has non-empty interior. We define the Dirichlet form $(\mathcal{E}_0, \mathcal{F}_0)$ on $L^2(\mathbb{R}^d; dx)$ as follows

$$
\mathcal{E}_0(f, f) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} \mathbf{1}_\Gamma(x - y) \, dx \, dy, \quad f \in \mathcal{F}_0,
$$

(2.18)

where $\mathcal{F}_0$ is the closure of $C_c^\infty(\mathbb{R}^d)$ under the norm $(\mathcal{E}_0(\cdot, \cdot) + \| \cdot \|^2_{L^2(\mathbb{R}^d; dx)})^{1/2}$.

**Lemma 2.4.** There exists a constant $c_0 > 0$ such that for all $f \in C_c(\mathbb{R}^d)$,

$$
\| f \|^2_{L^2(\mathbb{R}^d, dx)} \leq c_0 \mathcal{E}_0(f, f)^{d/(d+\alpha)} \| f \|_{L^1(\mathbb{R}^d, dx)}^{2\alpha/(d+\alpha)}.
$$

Proof. This inequality is well-known when $\Gamma = \mathbb{R}^d$ (see, for instance, [19, Proposition 3.1]). By [6, Theorem 1.1], there is a constant $c_1 > 0$ which may depend on $\Gamma$ so that for every $f \in L^2(\mathbb{R}^d; dx)$,

$$
\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} \, dx \, dy \leq c_1 \int_{\mathbb{R}^d \setminus \Delta} \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} \mathbf{1}_\Gamma(x - y) \, dx \, dy.
$$

(2.19)

This immediately gives the desired result. In fact, the result of [6, Theorem 1.1] is more general which shows that (2.19) holds with $\mathbb{R}^d$ being replaced by any ball $B$ in $\mathbb{R}^d$ on both sides of (2.19). In the $\mathbb{R}^d$ case, one can establish (2.19) directly by using the Fourier transform. For reader’s convenience, we give a such proof below.

For $f \in C_c(\mathbb{R}^d)$, let

$$
\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\langle \xi, y \rangle} f(y) \, dy
$$

be the Fourier transform of $f$. Then,

$$
\int_{\mathbb{R}^d} f(x)^2 \, dx = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \, d\xi, \quad \mathcal{E}_0(f, f) = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \phi(\xi) \, d\xi,
$$

where

$$
\phi(\xi) = \int_{\Gamma} \frac{1 - e^{i\langle \xi, z \rangle}}{|z|^{d+\alpha}} \, dz = \int_{\Gamma} \frac{1 - \cos \langle \xi, z \rangle}{|z|^{d+\alpha}} \, dz.
$$

(2.20)

Let $S^{d-1}$ be the unit sphere in $\mathbb{R}^d$ and $\Theta = \Gamma \cap S^{d-1}$. Denote by $z = (r, \theta)$ the spherical coordinates for $0 \neq z \in \mathbb{R}^d$, and $\sigma$ the Lebesgue surface measure on $S^{d-1}$. Since $\Theta$ has non-empty interior, there are positive constants $\delta_1$ and $\delta_2$ such that for any $\eta \in S^{d-1}$, there exists $A(\eta) \subset \Theta$ (which may depend on $\eta$) so that $\sigma(A(\eta)) \geq \delta_1$ and

$$
1 - \cos(r\langle \eta, \theta_0 \rangle) \geq \delta_2 \quad \text{for all } 1/2 \leq r \leq 1 \text{ and } \theta_0 \in A(\eta).
$$

Thus for any $\xi \in \mathbb{R}^d$,

$$
\phi(\xi) = c_2 |\xi|^\alpha \int_{0}^{\infty} r^{-1-\alpha} \int_{\Theta} (1 - \cos(r\langle \xi/|\xi|, \theta \rangle)) \, \sigma(d\theta) \, dr \\
\geq c_3 |\xi|^\alpha \int_{1/2}^1 \int_{A(\xi/|\xi|)} (1 - \cos(r\langle \xi/|\xi|, \theta \rangle)) \, d\theta \, dr \\
\geq \frac{c_3 \delta_1 \delta_2}{2} |\xi|^\alpha.
$$

(2.21)
Therefore, for every $f \in C^1_c(\mathbb{R}^d)$,
\[
\mathcal{E}_0(f, f) \geq c_4 \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |\xi|^\alpha d\xi = c_4 \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} \, dx \, dy.
\]
This establishes (2.19). \hfill \Box

**Proof of Theorem 2.3.** Throughout this proof, we again suppress the parameter $\omega$ for simplicity. We claim that (2.15) holds for any $f \in C_c(\mathbb{R}^d)$ and $\omega \in \Omega_2$. Indeed, let $X := \{X_t, t \geq 0\}$ be the symmetric Lévy process associated with the Dirichlet form $(\mathcal{E}_K, \mathcal{F}_K)$ on $L^2(\mathbb{R}^d; dx)$. The Lévy process $X$ has Lévy exponent
\[
\psi(\xi) = \int \left(1 - \cos(\xi, z)\right) \frac{K(z)}{|z|^{d+\alpha}} \, dz, \quad \xi \in \mathbb{R}^d.
\]
Since $0 < C_1 \leq K(z) \leq C_2$, we have by (2.20) and (2.21) that $\psi(\xi) \geq c_0 |\xi|^\alpha$ on $\mathbb{R}^d$. Hence the Lévy process $X$ has a jointly continuous transition density function $p(t, x, y) = p(t, x - y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$, where
\[
p(t, x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle \xi, x \rangle} e^{-t\psi(\xi)} \, d\xi, \quad x \in \mathbb{R}^d.
\]
Since
\[
\mathcal{E}_0(g, g) \leq C_1^{-1} \mathcal{E}_K(g, g)
\]
for any $g \in C^1_c(\mathbb{R}^d)$, we have by Lemma 2.4 that
\[
\|g\|^2_{L^2(\mathbb{R}^d; dx)} \leq c_1 \left(\mathcal{E}_K(g, g)\right)^{d/(d+\alpha)} \|g\|^{2\alpha/(d+\alpha)}_{L^1(\mathbb{R}^d; dx)}, \quad g \in C^1_c(\mathbb{R}^d).
\]
This along with [14, Theorems (2.1) and (2.9)] yields
\[
p(t, x - y) \leq c_2 t^{-d/\alpha} \quad \text{for every } t > 0 \text{ and } x, y \in \mathbb{R}^d.
\]
Since $K(z) \leq C_2$ on $\mathbb{R}^d$, by the proof of [5, Theorem 1.4], we in fact have
\[
p(t, x - y) \leq c_3 \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right) \quad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^d.
\]
In the following, we fix $f \in C_c(\mathbb{R}^d)$. Choose $R_0 \geq 1$ so that $\text{supp}[f] \subset B(0, R_0)$. For any $x \in \mathbb{R}^d$ with $|x| > 2R_0$,
\[
U^K_x f(x) = \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p(t, x - y) f(y) \, dy \, dt
\]
\[
\leq c_4 \|f\|_\infty \int_0^\infty \int_{B(0,R_0)} e^{-\lambda t} \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right) \, dy \, dt
\]
\[
\leq c_5 |x|^{-d-\alpha}.
\]
where $c_5 > 0$ is a constant that depends on $\|f\|_\infty$, $\lambda$ and $R_0$.

As mentioned in the beginning of Theorem 2.2, $\Omega_2 = \Omega_0 \cap \Omega_1$, where $\Omega_0$ and $\Omega_1$ are the subset of $\Omega$ in assumption (H) and in Proposition 2.1, respectively. According to Proposition 2.1, for every $\omega \in \Omega_2$, $k \geq 0$ and every $\varphi \in L^\infty(B(0, 2^k); dx)$
\[
\lim_{\varepsilon \to 0} \int_{B(0, 2^k)} \varphi(x) \mu(\tau_{x/\varepsilon}\omega) \, dx = \mathbb{E}[\mu] \int_{B(0, 2^k)} \varphi(x) \, dx = \int_{B(0, 2^k)} \varphi(x) \, dx.
\]
(2.23)
In particular, taking $\varphi \equiv 1$ and $k = 0$ yields that
\[
\lim_{\varepsilon \to 0} \varepsilon^d \int_{B(0, 1/\varepsilon)} \mu(\tau_{x/\varepsilon}\omega) \, dx = \lim_{\varepsilon \to 0} \int_{B(0, 1)} \mu(\tau_{x/\varepsilon}\omega) \, dx = |B(0, 1)|.
\]
Hence, for every $\omega \in \Omega_2$, there exists $\varepsilon_0(\omega) > 0$ such that
\[
\int_{B(0, 1/\varepsilon)} \mu(\tau_{x/\varepsilon}\omega) \, dx \leq 2\varepsilon^{-d}|B(0, 1)| \quad \text{for all } \varepsilon \in (0, \varepsilon_0(\omega)).
\]
(2.24)
Let $Q_k := \{ x \in \mathbb{R}^d : 2^{k-1} \leq |x| < 2^k \}$ for $k \geq 1$. Therefore, by (2.22) and (2.24), we obtain for every $\omega \in \Omega_2$ and $N \geq 1$ with $2^N > \max\{ 2R_0, \varepsilon_0(\omega)^{-1} \}$,

$$\limsup_{\varepsilon \to 0} \int_{\{ |x| \geq 2^N \}} U^K_\lambda f(x)^2 \mu(\tau_{x/\varepsilon} \omega) \, dx = \limsup_{\varepsilon \to 0} \sum_{k=N+1}^\infty \int_{Q_k} U^K_\lambda f(x)^2 \mu(\tau_{x/\varepsilon} \omega) \, dx$$

$$\leq \varepsilon_0^2 \limsup_{\varepsilon \to 0} \sum_{k=N+1}^\infty 2^{-2(k-1)(d+\alpha)} \int_{B(0,2^{k+1})} \mu(\tau_{x/\varepsilon} \omega) \, dx$$

$$= \varepsilon_0^2 \limsup_{\varepsilon \to 0} \sum_{k=N+1}^\infty 2^{-2(k-1)(d+\alpha)} \int_{B(0,2^{k+1})} \mu(\tau_{x/\varepsilon} \omega) \, dx$$

$$\leq c_6 \sum_{k=N}^\infty 2^{-2(k-1)(d+\alpha)} \varepsilon^d \leq c_7 2^{-(d+2\alpha)N}.$$

On the other hand, according to (2.23), for every $\omega \in \Omega_2$,

$$\lim_{\varepsilon \to 0} \int U^K_\lambda f(x)^2 \mathbb{1}_{\{ |x| < 2^N \}} \mu(\tau_{x/\varepsilon} \omega) \, dx = \int U^K_\lambda f(x)^2 \mathbb{1}_{\{ |x| < 2^N \}} \, dx$$

for every $N \geq 1$.

This together with (2.25) and by taking $N \to \infty$ gives us (2.15).

For $\lambda > 0$, define

$$\mathcal{E}^\varepsilon_\lambda(u,v) = \mathcal{E}(u,v) + \lambda \langle u, v \rangle_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))}$$

for $u, v \in \mathcal{F}^\varepsilon$, and

$$\mathcal{E}^K_\lambda(u,v) = \mathcal{E}^K(u,v) + \lambda \langle u, v \rangle_{L^2(\mathbb{R}^d; dx)}$$

for $u, v \in \mathcal{F}^K$.

For $f \in C_c(\mathbb{R}^d)$ and $g \in C^\infty_c(\mathbb{R}^d)$,

$$\langle U^K_\lambda f, f \rangle_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} = \mathcal{E}^\varepsilon_\lambda(U^K_\lambda f, U^K_\lambda f)$$

$$= \mathcal{E}^\varepsilon_\lambda(U^K_\lambda f - g, U^K_\lambda f - g) + \mathcal{E}^\varepsilon_\lambda(g, g) + 2 \mathcal{E}^\varepsilon_\lambda(U^K_\lambda f - g, g)$$

$$= \mathcal{E}^\varepsilon_\lambda(U^K_\lambda f - g, U^K_\lambda f - g) + 2 \langle f, g \rangle_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} - \mathcal{E}^\varepsilon_\lambda(g, g) .$$

By (ii) and (iv) of assumption (H) and Proposition 2.1,

$$\lim_{\varepsilon \to 0} \mathcal{E}^\varepsilon_\lambda(g, g) = \mathcal{E}^K_\lambda(g, g) \quad \text{and} \quad \lim_{\varepsilon \to 0} \langle f, g \rangle_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} = \langle f, g \rangle_{L^2(\mathbb{R}^d; dx)} .$$

Hence after taking $\varepsilon \to 0$ in (2.26), we get together with (2.6) that

$$\lim_{\varepsilon \to 0} \mathcal{E}^\varepsilon_\lambda(U^K_\lambda f - g, U^K_\lambda f - g) = -2 \langle f, g \rangle_{L^2(\mathbb{R}^d; dx)} + \mathcal{E}^K_\lambda(g, g) + (U^K_\lambda f, f)_{L^2(\mathbb{R}^d; dx)} .$$

Take $\{ g_k ; k \geq 1 \} \subset C^\infty_c(\mathbb{R}^d)$ so that $g_k \to U^K_\lambda f$ in $\sqrt{\mathcal{E}^K_\lambda}$-norm. Then

$$\lim_{k \to \infty} \left( 2 \langle f, g_k \rangle_{L^2(\mathbb{R}^d; dx)} - \mathcal{E}^K_\lambda(g_k, g_k) - (U^K_\lambda f, f)_{L^2(\mathbb{R}^d; dx)} \right)$$

$$= 2 \langle f, U^K_\lambda f \rangle_{L^2(\mathbb{R}^d; dx)} - \mathcal{E}^K_\lambda(U^K_\lambda f, U^K_\lambda f) - (U^K_\lambda f, f)_{L^2(\mathbb{R}^d; dx)} = 0 .$$

In particular, we have by (2.27) that

$$\lim_{k \to \infty} \lim_{\varepsilon \to 0} ||U^K_\lambda f - g_k||_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} = 0 .$$

On the other hand, by Proposition 2.1, (2.6) and (2.15),

$$\lim_{\varepsilon \to 0} ||g_k - U^K_\lambda f||^2_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} = \lim_{\varepsilon \to 0} \left( ||g_k||^2_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} - 2 \langle U^K_\lambda f, g_k \rangle_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} + ||U^K_\lambda f||^2_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} \right)$$

$$= ||g_k||^2_{L^2(\mathbb{R}^d; dx)} - 2 \langle U^K_\lambda f, g_k \rangle_{L^2(\mathbb{R}^d; dx)} + ||U^K_\lambda f||^2_{L^2(\mathbb{R}^d; dx)} .$$

Hence we have

$$\lim_{k \to \infty} \lim_{\varepsilon \to 0} ||g_k - U^K_\lambda f||^2_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} = 0 .$$

This together with (2.28) gives (2.16).
Consequently, we have by (2.15) that
\[
\lim_{\varepsilon \to 0} \|U_{\varepsilon}^f\|_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} = \lim_{\varepsilon \to 0} \|U_{\varepsilon}^K f\|_{L^2(\mathbb{R}^d; \mu^\varepsilon(dx))} = \|U_{\varepsilon}^K f\|_{L^2(\mathbb{R}^d; dx)}.
\]
This completes the proof of the theorem. \(\square\)

The proof for the property (2.16) (in particular see (2.28) and (4.3) below) immediately implies the following.

**Corollary 2.5.** Assume that assumption (H) holds and \(E[\mu^p] < \infty\) for some \(p > 1\). Let \(\Omega_2 \subset \Omega\) be as in Theorem 2.2 of full probability measure. Then for every \(\omega \in \Omega_2\) and every \(f \in C_c(\mathbb{R}^d)\), \(U_{\varepsilon} f\) strongly converges in \(L^2\)-spaces to \(U_{\varepsilon}^K f\) as \(\varepsilon \to 0\).

The definition of strong convergence in \(L^2\)-spaces with changing reference measures can be found in the appendix of this paper. Corollary 2.5 can also be proved by using the Mosco convergence of Dirichlet forms – See Subsection 4.2 in the appendix for this alternative approach.

3. **Either Assumption (A) or (B) Implies Assumption (H)**

In the section, we will prove that either assumption (A) or (B) implies assumption (H). For this, we first consider the weak convergence of non-local bilinear forms, and then study the compactness of functions with uniformly bounded Dirichlet forms. We need the following lemma, which is an extension of Proposition 2.1.

**Lemma 3.1.** Let \((\Omega, \mathcal{F}, P)\) be a probability space on which there is a stationary and ergodic measurable group of transformations \(\{\tau_x\}_{x \in \mathbb{R}^d}\) with \(\tau_0 = \text{id}\).

(i) Suppose that \(\nu(z; \omega)\) is a non-negative measurable function on \(\mathbb{R}^d \times \Omega\) such that the function \(z \mapsto E[\nu(z;\cdot)^p]\) is locally integrable for some \(p > 1\). Then there is a subset \(\Omega_0 \subset \Omega\) of full probability measure so that for every \(\omega \in \Omega_0\) and every compactly supported \(f \in L^q(\mathbb{R}^d \times \mathbb{R}^d; dx dy)\) with \(q = p/(p-1)\),
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, z) \nu(z; \tau_{x/\varepsilon} \omega) \, dx \, dz = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, z) E[\nu(z;\cdot)] \, dx \, dz.
\]

(ii) Suppose that \(\nu_1\) and \(\nu_2\) are two non-negative random variables on \((\Omega, \mathcal{F}, P)\) so that \(E[\nu_1^p + \nu_2^p] < \infty\) for some \(p > 1\). Then there is a subset \(\Omega_0 \subset \Omega\) of full probability measure so that for every \(\omega \in \Omega_0\) and every compactly supported \(f \in L^q(\mathbb{R}^d \times \mathbb{R}^d; dx dy)\) with \(q = p/(p-1)\),
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) \nu(\tau_{x/\varepsilon} \omega) \nu_1(\tau_{x/\varepsilon} \omega) \, dx \, dy = E[\nu_1] \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) \, dx \, dy.
\]

**Proof.** (i) By the Fubini theorem and Proposition 2.1, for any bounded \(A, B \in \mathcal{B}(\mathbb{R}^d)\) and a.s. \(\omega \in \Omega\),
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} 1_{A \times B}(x, z) \nu(z; \tau_{x/\varepsilon} \omega) \, dz \, dx = \lim_{\varepsilon \to 0} \int_A \left( \int_B \nu(z; \tau_{x/\varepsilon} \omega) \, dz \right) \, dx = \int_A E[\int_B \nu(z;\cdot) \, dz] \, dx
\]
\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} 1_{A \times B}(x, z) E[\nu(z;\cdot)] \, dz \, dx.
\]

The above also holds with \(\nu(z; \omega)^p\) in place of \(\nu(z; \omega)\).

Let
\[
\mathcal{I} := \left\{ f(x, z) = \sum_{i=1}^m a_i 1_{A_i \times B_i}(x, z) : m \in \{1, 2, \cdots\}, a_i \in \mathbb{Q}, \ A_i, B_i \in \mathcal{B}_Q(\mathbb{R}^d) \right\}.
\]
Here \(\mathbb{Q}\) denotes the set of all rational numbers, and \(\mathcal{B}_Q(\mathbb{R}^d)\) denotes the collection of all bounded cubes in \(\mathbb{R}^d\) whose end points are rational numbers. Then there is a set \(\Omega_0 \subset \Omega\) of full probability measure so that for every \(\omega \in \Omega_0\), (3.1) holds for every \(f \in \mathcal{I}\), and
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} 1_{A \times B}(x, z) \nu(z; \tau_{x/\varepsilon} \omega)^p \, dz \, dx = \int_{\mathbb{R}^d \times \mathbb{R}^d} 1_{A \times B}(x, z) E[\nu(z;\cdot)^p] \, dz \, dx.
\]
For general compactly supported \( f \in L^q(\mathbb{R}^d \times \mathbb{R};dx \, dy) \), take \( A, B \in \mathcal{B}_0(\mathbb{R}^d) \) so that \( \text{supp}[f] \subset A \times B \). Since \( C_0(A \times B) \) is dense in \( L^q(A \times B;dx \, dy) \) and \( \mathcal{S} \) is dense in \( C_0(A \times B) \) under the uniform norm, we can find a sequence \( \{\varphi_n\}_{n \geq 1} \subset \mathcal{S} \) such that
\[
\lim_{n \to \infty} \|\varphi_n - f\|_{L^q(A \times B;dx \, dy)} = 0.
\]
Note that \( q = p/(p-1) > 1 \) is the conjugate of \( p > 1 \). It follows (3.3) and (3.4), as well as the fact (3.1) holds for every \( f \in \mathcal{S} \) and \( \omega \in \Omega_0 \), that for every \( \omega \in \Omega_0 \),
\[
\begin{align*}
\limsup_{\varepsilon \to 0} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \begin{array}{c}
f(x, z)\nu(z; \tau_{x/\varepsilon} \omega) \, dz \, dx - \int_{\mathbb{R}^d} f(x, z)\nu(z; \cdot) \, dz \, dx \end{array} + \int_{A \times B} & \int_{A \times B} \varphi_n(x, z)\nu(z; \tau_{x/\varepsilon} \omega) \, dz \, dx - \int_{A \times B} \varphi_n(x, z) \nu(z; \cdot) \, dz \, dx \\
& + \int_{A \times B} \int_{A \times B} |f(x, z) - \varphi_n(x, z)| \nu(z; \tau_{x/\varepsilon} \omega) \, dz \, dx + \int_{A \times B} |f(x, z) - \varphi_n(x, z)| \nu(z; \cdot) \, dz \, dx \\
& \leq \|f - \varphi_n\|_{L^q(A \times B; dx \, dy)} \left( \limsup_{\varepsilon \to 0} \left( \int_{A \times B} \nu(z; \tau_{x/\varepsilon} \omega)^p \, dz \, dx \right)^{1/p} + \left( \int_{A \times B} (\nu(z; \cdot))^p \, dz \, dx \right)^{1/p} \right) \leq 2\|f - \varphi_n\|_{L^q(A \times B; dx \, dy)} \left( \int_{A \times B} \nu(z; \cdot)^p \, dz \, dx \right)^{1/p}.
\end{align*}
\]
By taking \( n \to \infty \), we get
\[
\limsup_{\varepsilon \to 0} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x, z)\nu(z; \tau_{x/\varepsilon} \omega) \, dz \, dx - \int_{\mathbb{R}^d} f(x, z)\nu(z; \cdot) \, dz \, dx \right| = 0.
\]
This establishes (3.1).

(ii) Note that by Proposition 2.1 that there is a subset \( \Omega_0 \subset \Omega \) of full probability measure so that for every \( \omega \in \Omega_0 \), any bounded \( A, B \in \mathcal{B}(\mathbb{R}^d) \) and \( \gamma = 1 \) or \( p \),
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{A \times B}(x, z) \left( \nu_1(\tau_{x/\varepsilon} \omega)\nu_2(\tau_{y/\varepsilon} \omega) \right)^\gamma \, dx \, dy = \lim_{\varepsilon \to 0} \left( \int_A \nu_1(\tau_{x/\varepsilon} \omega)^\gamma \, dx \right) \left( \int_B \nu_2(\tau_{y/\varepsilon} \omega)^\gamma \, dy \right) = \mathbb{E}[\nu_1^\gamma] \mathbb{E}[\nu_2^\gamma] \, |A \times B|.
\]
With this at hand, the assertion can be proved in exactly the same way as that for (i).

In the proof of Proposition 3.3(i) below, we need the following maximal ergodic theorem for multiplicative additive processes with continuous parameter.

**Proposition 3.2.** Suppose that \( 0 \leq F \in L^1(\Omega; \mathbb{P}) \). Then, there is a constant \( C > 0 \) such that for all \( \lambda, R_0 > 0 \),
\[
\mathbb{P}\left\{ \omega \in \Omega : \sup_{\varepsilon \in (01]} \int_{[0, R_0]^d} F(\tau_{x/\varepsilon} \omega) \, dx > \lambda \right\} \leq CR_0^d \mathbb{E}[F]/\lambda. \quad (3.5)
\]
Although the maximal ergodic theorem for multiplicative additive processes has been used in some literature, we can not find a suitable reference for its proof in the continuous parameter setting. For safe of the completeness, here we will provide the

**Proof of Proposition 3.2.** Without loss of generality, we assume \( R_0 = 1 \) in (3.5).

Let \( \mathcal{I} = \{[x, x + k2^m]^d : x \in \mathbb{Z}^d, m, k \in \mathbb{Z}_+\} \), and define
\[
F_I(\omega) := \int_I F(\tau_{x} \omega) \, dx, \quad I \in \mathcal{I}.
\]
It is easy to verify \( \{F_I(\omega)\}_{I \in \mathcal{I}} \) satisfies (2.1)–(2.3) in [37, Page 201] (indeed, (2.2) in [37, Page 201] holds with equality), and so \( \{F_I(\omega)\}_{I \in \mathcal{I}} \) is a (discrete) additive process on the integer lattice \( \mathbb{Z}^d \). Let \( Q_m := [0, 2^m]^d \)
for $m \geq 0$, and
\[ F_\varepsilon(\omega) := \sup_{m > 0} \frac{1}{2^{md}} \int_{Q_m} F(\tau_x \omega) \, dx. \]
According to [37, Page 205, Corollary 2.7], there is a constant $c_0 > 0$ so that for all $\lambda > 0$,
\[ \mathbb{P}(\tilde{F}_\varepsilon(\omega) > \lambda) \leq c_0 \lambda^{-1} \mathbb{E}[F]. \] (3.6)
Now, we define
\[ F_\varepsilon(\omega) := \sup_{\varepsilon \in (0,1]} \left[ \varepsilon^d \int_{[0,\varepsilon^{-1}]^d} F(\tau_x \omega) \, dx \right] = \sup_{\varepsilon \in (0,1]} \int_{[0,1]^d} F(\tau_x / \varepsilon) \, dx. \]
Then, for any $m \geq 0$ and $\varepsilon \in (2^{-(m+1)}, 2^{-m}]$,
\[ 2^{-md} \int_{Q_m} F(\tau_x \omega) \, dx - \varepsilon^d \int_{[0,\varepsilon^{-1}]^d} F(\tau_x \omega) \, dx \]
\[ \leq 2^{-md} \int_{Q_{m+1} \setminus Q_m} F(\tau_x \omega) \, dx + (2^{-md} - 2^{-(m+1)d}) \int_{Q_{m+1}} F(\tau_x \omega) \, dx \]
\[ \leq (2^{d+1} - 1) \frac{\int_{Q_{m+1}} F(\tau_x \omega) \, dx}{|Q_{m+1}|} \leq (2^{d+1} - 1) \tilde{F}_\varepsilon(\omega). \]
Hence, for any $m \geq 0$,
\[ \sup_{\varepsilon \in (2^{-(m+1)}, 2^{-m}]} \left[ \varepsilon^d \int_{[0,\varepsilon^{-1}]^d} F(\tau_x \omega) \, dx \right] \leq 2^{-md} \int_{Q_m} F(\tau_x \omega) \, dx + (2^{d+1} - 1) \tilde{F}_\varepsilon(\omega) \leq 2^{d+1} \tilde{F}_\varepsilon(\omega), \]
which implies that
\[ F_\varepsilon(\omega) = \sup_{m > 0} \sup_{\varepsilon \in (2^{-(m+1)}, 2^{-m}]} \left[ \varepsilon^d \int_{[0,\varepsilon^{-1}]^d} F(\tau_x \omega) \, dx \right] \leq 2^{d+1} \tilde{F}_\varepsilon(\omega). \]
Therefore, by (3.6), we obtain
\[ \mathbb{P}(\{\omega \in \Omega : F_\varepsilon(\omega) > \lambda\}) \leq \mathbb{P}(\{\omega \in \Omega : \tilde{F}_\varepsilon(\omega) > 2^{-(d+1)\lambda}\}) \leq c_0 2^{d+1} \mathbb{E}[F] / \lambda. \]
This proves (3.5).

### 3.1. Weak convergence of bilinear forms
Recall that $\Gamma \subset \mathbb{R}^d$ is an infinite symmetric cone that has non-empty interior. When $d = 1$, $\Gamma$ is just $\mathbb{R}$.

**Proposition 3.3.**
(i) Suppose that (A1) holds and that there is a non-negative random variables $\Lambda$ on $(\Omega, \mathcal{F}, \mathbb{P})$ so that $\mathbb{E}[\Lambda^p] < \infty$ for some $p > 1$ and
\[ \kappa(x,y; \omega) \leq \Lambda(\tau_x \omega) + \Lambda(\tau_y \omega) \quad \text{for every } \omega \in \Omega, \; x,y \in \mathbb{R}^d. \] (3.7)
Then there is a subset $\Omega_1 \subset \Omega$ of full probability measure so that for every $\omega \in \Omega_1$, any $\eta > 0$, $f \in B(\mathbb{R}^d)$ and $g \in B_c(\mathbb{R}^d)$,
\[ \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \int_{\{\eta < |z| < 1/\eta, z \in \Gamma\}} \frac{(f(x+z) - f(x))(g(x+z) - g(x))}{|z|^{d+\alpha}} \kappa(x, (x+z)/\varepsilon; \omega) \, dz \, dx = \int_{\mathbb{R}^d} \int_{\{\eta < |z| < 1/\eta, z \in \Gamma\}} \frac{(f(x+z) - f(x))(g(x+z) - g(x))}{|z|^{d+\alpha}} (\tilde{\nu}(z) + \tilde{\nu}(-z)) \, dz \, dx, \]
where $\tilde{\nu}$ is a non-negative measurable function given in assumption (A1).
(ii) Let $\nu_1$ and $\nu_2$ be non-negative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[
u_1^p + \nu_2^p] < \infty$ for some $p > 1$. Then there is a subset $\Omega_2 \subset \Omega$ of full probability measure so that for every $\omega \in \Omega_2$, any $\eta > 0$, $f \in B(\mathbb{R}^d)$ and $g \in B_c(\mathbb{R}^d)$,
\[ \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \int_{\{\eta < |x-y| < 1/\eta, x-y \in \Gamma\}} \frac{(f(y) - f(x))(g(y) - g(x))}{|x-y|^{d+\alpha}} \nu_1(\tau_x / \varepsilon; \omega) \nu_2(\tau_y / \varepsilon; \omega) \, dx \, dy \]
Then, for notational convenience, we will prove the above for probability measure so that for every $\omega$ and so, by (1.10), the function

\[ F(x,y) := \mathbb{1}_{\{\eta<|x-y|<1/\eta, x,y \in \Gamma\}} \frac{(f(x) - f(y))(g(x) - g(y))}{|x-y|^{d+\alpha}} \]

is a bounded and compactly supported function on $\mathbb{R}^d \times \mathbb{R}^d$. Since

\[ \int_{\mathbb{R}^d} \int_{\eta<|x-y|<1/\eta, x,y \in \Gamma} \frac{(f(x) - f(y))(g(x) - g(y))}{|x-y|^{d+\alpha}} \nu_1(x/\varepsilon, \omega) \nu_2(y/\varepsilon, \omega) \, dx \, dy = 2 \int_{\mathbb{R}^d \times \mathbb{R}^d} F(x,y) \nu_1(\tau_{x/\varepsilon} \omega) \nu_2(\tau_{y/\varepsilon} \omega) \, dx \, dy. \]

Proposition 3.3(ii) follows directly from Lemma 3.1(ii).

(i) Let $\eta>\lim \sup_{B \to \infty} (R_{x,y}) = \{ \varepsilon = 2 \varepsilon \}$ be the bounded and compactly supported function defined above. Note that

\[ \int_{\mathbb{R}^d} \int_{\{\eta<|x-y|<1/\eta, x,y \in \Gamma\}} \nu_1(x/\varepsilon, \omega) \nu_2(y/\varepsilon, \omega) \, dx \, dy \]

is bounded. Proposition 3.3(i) can be proved in exactly the same way as that of Lemma 3.1(i) once one can verify that there is a subset $\Omega_2 \subset \Omega$ of full probability measure so that for every $\omega \in \Omega_2$ and any $A, B \in \mathcal{B}_Q(\mathbb{R}^d)$,

\[ \limsup_{\varepsilon \to 0} \int_{A \times B} \nu_1(z/\varepsilon, \tau_{x/\varepsilon} \omega) \, dz \, dx \leq 2^p \limsup_{\varepsilon \to 0} \int_{A \times B} (\Lambda(\tau_{x/\varepsilon} \omega)^p + \Lambda(\tau_{(x+z)/\varepsilon} \omega)^p) \, dz \, dx \]

where $A + B = \{x+y \in \mathbb{R}^d : x \in A, y \in B\}$. Note also that by (A1) and (3.7),

\[ \sup_{x \in \mathbb{R}^d} \mathbb{E}[\nu(x; \cdot)] \leq \sup_{x \in \mathbb{R}^d} \mathbb{E}[\kappa(x, 0; \cdot)] \leq \mathbb{E}[\Lambda] + \sup_{x \in \mathbb{R}^d} \mathbb{E}[\Lambda(\tau_x(\cdot))] = 2\mathbb{E}[\Lambda] \leq 2(\mathbb{E}[\Lambda^p])^{1/p} < \infty, \]

and so, by (1.10), the function $\bar{\nu}$ in (A1) is bounded. Proposition 3.3(i) can be proved in exactly the same way as that of Lemma 3.1(i) once one can verify that there is a subset $\Omega_2 \subset \Omega$ of full probability measure so that for every $\omega \in \Omega_2$ and any $A, B \in \mathcal{B}_Q(\mathbb{R}^d)$,

\[ \int_{A \times B} \nu_1(z/\varepsilon, \tau_{x/\varepsilon} \omega) \, dz \, dx = \int_{A \times B} \bar{\nu}(z) \, dz \, dx. \]

Furthermore, by (1.10), in order to verify (3.8) it suffices to prove that there is a subset $\Omega_3 \subset \Omega_2$ of full probability measure so that for every $\omega \in \Omega_3$ and any $A, B \in \mathcal{B}_Q(\mathbb{R}^d)$,

\[ \lim_{\varepsilon \to 0} \left| \int_{A \times B} (\nu_1(z/\varepsilon, \tau_{x/\varepsilon} \omega) - \mathbb{E}[\nu_1(z/\varepsilon; \cdot)]) \, dz \, dx \right| = 0. \]

For notational convenience, we will prove the above for $A = B = [0,1]^d$. The proof for the general case is similar.

Recall that $\nu_k(z, \omega) := \nu(z, \omega) \wedge k$. Fix $\delta > 1$. For any $\varepsilon > 0$, choose $m \geq 1$ such that $\delta^{m-1} \leq 1/\varepsilon < \delta^m$. Then,

\[ \left| \int_{[0,1]^d \times [0,1]^d} (\nu_1(z/\varepsilon, \tau_{x/\varepsilon} \omega) - \mathbb{E}[\nu_1(z/\varepsilon; \cdot)]) \, dz \, dx \right| \]

\[ = \left| \varepsilon^{2d} \int_{[0,1]^d \times [0,1]^d} (\nu_1(z, \tau_{x} \omega) - \mathbb{E}[\nu_1(z; \cdot)]) \, dz \, dx \right| \]

\[ \leq \delta^{-2(m-1)d} \left| \int_{[0,\delta^{-m-1}]^d \times [0,\delta^{-m-1}]^d} (\nu_1(z, \tau_{x} \omega) - \mathbb{E}[\nu_1(z; \cdot)]) \, dz \, dx \right| \]

\[ + \delta^{-2(m-1)d} \left| \int_{[0,\delta^{-m-1}]^d \times [0,\delta^{-m-1}]^d \setminus [0,\delta^{-m-1}]^d \times [0,\delta^{-m-1}]^d} (\nu_1(z, \tau_{x} \omega) - \mathbb{E}[\nu_1(z; \cdot)]) \, dz \, dx \right| \]
\[ \left\| \delta^{-2(m-1)d} \int_{[0, \delta^{-m-1}d] \times [0, \delta^{-m-1}d]} \left( \nu_k(z; \tau_x \omega) - E[\nu_k(z; \cdot)] \right) dz \right\| \]

\[ + \delta^{-2(m-1)d} \int_{[0, \delta^{-m}]d \times [0, \delta^{-m}]d \times [0, \delta^{-m-1}d]} \nu_k(z; \tau_x \omega) dz dx \]

\[ + \delta^{-2(m-1)d} \int_{[0, \delta^{-m}]d \times [0, \delta^{-m}]d \times [0, \delta^{-m-1}d]} E[\nu_k(z; \cdot)] dz dx \]

\[ =: \sum_{i=1}^{3} I_i^m. \]

We consider \( I_1^m \) first. For any \( m \in \mathbb{Z}_+ \) and \( x = (x^{(1)}, x^{(2)}, \ldots, x^{(d)}) \in \delta^{-m} \mathbb{Z}^d \), set

\[ Q_x^m = \prod_{1 \leq i \leq d} [x^{(i)} + \delta^{-m}]. \]

Define

\[ \mathcal{I}_m = \{ x \in \delta^{-m} \mathbb{Z}^d : Q_x^m \subset [0, 1]^d \}, \]

\[ \mathcal{F}_m = \{ x \in \delta^{-m} \mathbb{Z}^d : Q_x^m \cap [0, 1]^d \neq \emptyset, Q_x^m \cap ((0, 1)^d)^c \neq \emptyset \}, \]

\[ F_m(\omega) = \int_{[0, 1]^d} \nu_k(\delta^m z; \omega) dz. \]

We have

\[ \delta^{-2md} \int_{[0, \delta^{-m}]d \times [0, \delta^{-m}]d} \nu_k(z; \tau_x \omega) dz dx \]

\[ = \int_{[0, 1]^d \times [0, 1]^d} \nu_k(\delta^m z; \tau_{\delta^m x} \omega) dz dx = \int_{[0, 1]^d} F_m(\tau_{\delta^m x} \omega) dx \]

\[ = \sum_{i : x_i \in \mathcal{I}_m} \int_{Q_{x_i}^m} F_m(\tau_{\delta^m z} \omega) dz + \sum_{j : x_j \in \mathcal{F}_m} \int_{Q_{x_j}^m \cap [0, 1]^d} F_m(\tau_{\delta^m z} \omega) dz \]

\[ =: L_1^m + L_2^m. \]

For \( x \in \delta^{-m} \mathbb{Z}^d \), let

\[ F_{x,m}(\omega) = \int_{Q_x^m} F_m(\tau_{\delta^m z} \omega) dz. \]

Since \( 0 \leq \nu_k(z; \cdot) \leq k \) for all \( z \in \mathbb{R}^d \), there is a constant \( c_1(k) > 0 \) (which may depend on \( k \)) such that

\[ \sup_{x \in \delta^{-m} \mathbb{Z}^d} \text{Var}(F_{x,m}) \leq \sup_{x \in \delta^{-m} \mathbb{Z}^d} E \left[ F_{x,m}^2 \right] \leq c_1(k) \delta^{-2md}. \]

Hence,

\[ \text{Var}(L_1^m) = \text{Var} \left( \sum_{i : x_i \in \mathcal{I}_m} F_{x_i,m} \right) = \sum_{n=0}^{[\delta^m]} \sum_{i,j : x_i, x_j \in \mathcal{I}_m, d \delta^{-m} \leq |x_i - x_j| \leq d(n+1)\delta^{-m}} \text{Cov}(F_{x_i,m}, F_{x_j,m}) \]

\[ + \sum_{n=1}^{[\delta^m]} \sum_{i,j : x_i, x_j \in \mathcal{I}_m, d \delta^{-m} \leq |x_i - x_j| \leq d(n+1)\delta^{-m}} \text{Cov}(F_{x_i,m}, F_{x_j,m}) \]

\[ =: J_1^m + J_2^m. \]

Note that \(|\{(i, j) : x_i, x_j \in \mathcal{I}_m, |x_i - x_j| \leq d \delta^{-m}\}| \leq c_2 \delta^{md} \) for some constant \( c_2 > 0 \). Then, by (3.11) and the Cauchy-Schwarz inequality,

\[ |J_1^m| \leq c_3(k) \delta^{-md}. \]
On the other hand, it follows from (1.9) that when \( dn \delta^{-m} < |x_i - x_j| \leq d(n + 1)\delta^{-m} \) for some \( n \geq 1 \),

\[
\text{Cov}(F_{x_i,m}, F_{x_j,m}) = \int_{Q_{x_i}^m} \int_{Q_{x_j}^m} \text{Cov} \left( F_m(\tau_{\delta^{-m} \cdot}), F_m(\tau_{\delta^{-m} y} \cdot) \right) \, dy \, dx \\
= \int_{Q_{x_i}^m} \int_{Q_{x_j}^m} \text{Cov} \left( F_m(\tau_{\delta^{-m} \cdot}), F_m(\tau_{\delta^{-m} (y-x)} \cdot) \right) \, dy \, dx \\
\leq c_4(k) \int_{Q_{x_i}^m} \int_{Q_{x_j}^m} |\delta^m (y-x) - 1| \, dy \, dx \leq c_5(k) n^{-1} \delta^{-2md},
\]

where the first inequality follows from (1.9), and the second one is due to (3.11) and the fact that \( |x - y| \geq c_6 n \delta^{-m} \) for all \( x \in Q_{x_i}^m \) and \( y \in Q_{x_j}^m \) with \( |x_i - x_j| > dn \delta^{-m} \). Since

\[
|\{(i,j) : x_i, x_j \in \mathcal{G}_m, dn \delta^{-m} < |x_i - x_j| \leq d(n + 1)\delta^{-m}\}| \leq c_7 \delta^{-md} n^{d-1}
\]

for some constant \( c_7 > 0 \) independent of \( m \) and \( n \), we get

\[
|J_2^m| \leq c_8(k) \delta^{-md} \sum_{n=1}^{[\delta^m]} n^{-(l+1-d)} \leq c_9(k) m \delta^{-m(d+1)} \log \delta.
\]

Therefore, according to these two estimates for \( J_1^m \) and \( J_2^m \), we obtain

\[
\text{Var}(F_{i,m}^m) \leq c_{10}(k) m \delta^{-m(d+1)} \log \delta.
\]

In particular, by the Markov inequality, for any \( \gamma > 0 \),

\[
\sum_{m=1}^{\infty} P \left( \sup_{i : x_i \in \mathcal{G}_m} \| F_{x_i,m} - \mathbb{E}[F_{x_i,m}] \|^2 > \gamma \right) < \infty.
\]

Hence, by the Borel-Cantelli lemma, for a.a. \( \omega \in \Omega \),

\[
\lim_{m \to \infty} I_1^m = \lim_{m \to \infty} \sum_{i : x_i \in \mathcal{G}_m} F_{x_i,m} = \lim_{m \to \infty} \sum_{i : x_i \in \mathcal{G}_m} \mathbb{E}[F_{x_i,m}] = \lim_{m \to \infty} \int_{B_{x_i}^m} \int_{[0,1]^d} \mathbb{E}[\nu_k(\delta^m \cdot \cdot \cdot)] \, dz \, dx \\
= \lim_{m \to \infty} \int_{[0,1]^d \times [0,1]^d} \mathbb{E}[\nu_k(\delta^m \cdot \cdot \cdot)] \, dz \, dx = \lim_{m \to \infty} \delta^{-2md} \int_{[0,\delta^m]^d \times [0,\delta^m]^d} \mathbb{E}[\nu_k(\cdot \cdot \cdot)] \, dz \, dx,
\]

where the fourth equality we have used the fact \( \lim_{m \to \infty} \bigcup_{i : x_i \in \mathcal{G}_m} Q_{x_i}^m = [0,1]^d \). Note that

\[
\{|x_i \in \delta^{-m} \mathbb{Z}^d : Q_{x_i}^m \cap [0,1]^d \neq \emptyset, Q_{x_i}^m \cap ([0,1]^d)^c \neq \emptyset\} \leq c_{11} \delta^{-m(d-1)}.
\]

This along with (3.11) and the Cauchy-Schwarz inequality gives us

\[
\mathbb{E}[|L_2^m|^2] \leq |\{x_i \in \delta^{-m} \mathbb{Z}^d : Q_{x_i}^m \cap [0,1]^d \neq \emptyset, Q_{x_i}^m \cap ([0,1]^d)^c \neq \emptyset\}|^2 \sup_{x_i \in \delta^{-m} \mathbb{Z}^d} \mathbb{E}[F_{x_i,m}^2] \leq c_{12}(k) \delta^{-2m},
\]

which in turn implies that for any \( \gamma > 0 \),

\[
\sum_{m=1}^{\infty} P \left( |L_2^m| > \gamma \right) < \infty.
\]

Hence by the Borel-Cantelli lemma, \( \lim_{m \to \infty} L_2^m = 0 \) a.s.

Combining both estimates for \( L_1^m \) and \( L_2^m \) with (3.10) yields that

\[
\lim_{m \to \infty} \delta^{-2md} \int_{[0,\delta^m]^d \times [0,\delta^m]^d} \nu_k(z; \tau_{x \omega}) \, dz \, dx = \lim_{m \to \infty} \delta^{-2md} \int_{[0,\delta^m]^d \times [0,\delta^m]^d} \mathbb{E}[\nu_k(z; \cdot)] \, dz \, dx;
\]

that is,

\[
\lim_{m \to \infty} I_1^m = \lim_{m \to \infty} \delta^{-2(m-1)d} \int_{[0,\delta^{m-1}]^d \times [0,\delta^{m-1}]^d} (\nu_k(z; \tau_{x \omega}) - \mathbb{E}[\nu_k(z; \cdot)]) \, dz \, dx = 0.
\]
By the argument for $I^n_2$ (in particular, by applying the Borel-Cantelli lemma), we have that for a.s. $\omega \in \Omega$ and every $k \geq 1$, there exists a constant $m_0(k, \omega) > 0$ such that for every $m \geq m_0(k, \omega)$,
\[
\int_{[0, \delta^n] \times [0, \delta^n] \times [0, \delta^{n-1}] \times [0, \delta^{n-1}]} \nu_k(z; x, \omega) \, dz \, dx \leq \int_{[0, \delta^n] \times [0, \delta^n] \times [0, \delta^{n-1}] \times [0, \delta^{n-1}]} (E[\nu_k(z; \cdot)] + 1) \, dz \, dx
\]
\[
\leq c_{13}(k)(\delta^{2m} - \delta^{2(m-1)}d) = c_{14}(k)\delta^{2md}(1 - \delta^{-2d}).
\]
In particular, for a.s. $\omega \in \Omega$,
\[
\limsup_{m \to \infty} I^n_2 \leq c_{14}(k)(\delta^{2d} - 1).
\]
Since $0 \leq \nu_k(z; \omega) \leq k$, it is obvious that
\[
\limsup_{m \to \infty} I^n_3 \leq c_{15}(k)(\delta^{2d} - 1).
\]
Putting all the above estimates together, we conclude that for every fixed $\delta > 1$ and $k \geq 1$, there exists a $P$-null set $N_{\delta, k}$ such that for every $\omega \in \Omega \setminus N_{\delta, k}$,
\[
\limsup_{\varepsilon \to 0} \left| \int_{[0, 1]^d \times [0, 1]^d} (\nu_k(z/\varepsilon; x, \omega) - E[\nu_k(z/\varepsilon; \cdot)]) \, dz \, dx \right| \leq \limsup_{m \to \infty} \sum_{i=1}^3 I^n_i \leq c_{16}(k)(\delta^{2d} - 1). \tag{3.12}
\]
Let $N := \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty N_{\delta, k}$ and $\Omega_2 := \Omega \setminus N$. Clearly $P(\Omega_2) = 1$, and (3.12) holds for every $\delta = 1 + 1/n$, $k \geq 1$ and $\omega \in \Omega_2$. Letting $n \to \infty$, we have for every $\omega \in \Omega_2$,
\[
\lim_{\varepsilon \to 0} \left| \int_{[0, 1]^d \times [0, 1]^d} (\nu_k(z/\varepsilon; x, \omega) - E[\nu_k(z/\varepsilon; \cdot)]) \, dz \, dx \right| = 0 \quad \text{for every } k \in \mathbb{Z}_+ \tag{3.13}
\]
For $k \geq 1$, let
\[
A_k := \left\{ \omega \in \Omega : \sup_{\varepsilon \in (0, 1)} \left| \int_{[0, 1]^d \times [0, 1]^d} \nu_k(z/\varepsilon; x, \omega) \, dz \, dx - \int_{[0, 1]^d \times [0, 1]^d} \nu(z/\varepsilon; x, \omega) \, dz \, dx \right| > 1/k \right\}.
\]
According to (1.8) and (3.7),
\[
P(A_k) \leq P\left( \left\{ \omega \in \Omega : \sup_{\varepsilon \in (0, 1)} \int_{[0, 1]^d \times [0, 1]^d} \nu(z/\varepsilon; x, \omega) 1_{\{\nu(z/\varepsilon; x, \omega) > 2k\}} \, dz \, dx > 1/k \right\} \right)
\]
\[
\leq P\left( \left\{ \omega \in \Omega : \sup_{\varepsilon \in (0, 1)} \int_{[0, 1]^d \times [0, 1]^d} \nu(z/\varepsilon; x, \omega)^p 2^{k(p-1)} \, dz \, dx > 1/k \right\} \right)
\]
\[
\leq P\left( \left\{ \omega \in \Omega : \sup_{\varepsilon \in (0, 1)} \int_{[0, 1]^d \times [0, 1]^d} (\Lambda(x/\varepsilon, \omega) + \Lambda(x+z/\varepsilon, \omega))^p \, dz \, dx > k^{-1}2^{k(p-1)} \right\} \right)
\]
\[
\leq P\left( \left\{ \omega \in \Omega : \sup_{\varepsilon \in (0, 1)} \int_{[0, 2]^d} \Lambda(x/\varepsilon, \omega)^p \, dx > 2^{-p}k^{-1}2^{k(p-1)} \right\} \right).
\]
Here $p > 1$ is given in the assumption of Proposition 3.3(i), and in the last inequality we used the facts that $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ for $a, b \geq 0$ and that
\[
\int_{[0, 1]^d \times [0, 1]^d} \Lambda(x+z/\varepsilon, \omega)^p \, dz \, dx \leq \int_{[0, 2]^d} \Lambda(x/\varepsilon, \omega)^p \, dx.
\]
Hence by Proposition 3.2,
\[
P\left( \left\{ \omega \in \Omega : \sup_{\varepsilon \in (0, 1)} \int_{[0, 2]^d} \Lambda(x/\varepsilon, \omega)^p \, dx > 2^{-p}k^{-1}2^{k(p-1)} \right\} \right) \leq c_{17}k2^{-k(p-1)}E[\Lambda^p],
\]
where $c_{17} > 0$ is independent of $k$. Putting all the estimates above together, we get $\sum_{k=1}^\infty P(A_k) < \infty$. By the Borel-Cantelli lemma again, we can find a subset $\Omega_3 \subset \Omega_2$ with full probability measure such that for
every \( \omega \in \Omega_3 \), there exists \( k_0(\omega) > 0 \) such that
\[
\left| \int_{[0,1]^d \times [0,1]^d} \nu_{2k}(z/\varepsilon; \tau_x/\varepsilon \omega) \, dz \, dx - \int_{[0,1]^d \times [0,1]^d} \nu(z/\varepsilon; \tau_x/\varepsilon \omega) \, dz \, dx \right| \leq 1/k
\]
for every \( k \geq k_0(\omega) \) and \( \varepsilon \in (0,1) \).

Combining this with (3.13) yields (first letting \( \varepsilon \to 0 \) and then \( k \to \infty \)) that for every \( \omega \in \Omega_3 \),
\[
\lim_{\varepsilon \to 0} \left| \int_{[0,1]^d \times [0,1]^d} \nu(z/\varepsilon; \tau_x/\varepsilon \omega) \, dz \, dx - \int_{[0,1]^d \times [0,1]^d} \mathbb{E}[\nu(z/\varepsilon; \cdot)] \, dz \, dx \right| = 0.
\]
Thus (3.9) holds with \( A = B = [0,1] \) for every \( \omega \in \Omega_3 \). This completes the proof of assertion (i) in Proposition 3.3.

\[\square\]

3.2. Pre-compactness in \( L^1 \)-spaces. In this part, we give the compactness for a sequence of uniformly bounded functions whose associated scaled Dirichlet forms are also uniformly bounded. The following is the main result of this subsection.

**Proposition 3.4.** Suppose that either (A2') or (B2') holds. Then there is a subset \( \Omega_0 \subset \Omega \) of full probability measure so that, for every \( \omega \in \Omega_0 \) and any collection of functions \( \{f_\varepsilon : \varepsilon \in (0,1)\} \) with \( f_\varepsilon \in \mathcal{F}^{\varepsilon, \omega} \) for any \( \varepsilon \in (0,1) \) having
\[
\limsup_{\varepsilon \to 0} (\|f_\varepsilon\|_\infty + \mathcal{E}^{\varepsilon, \omega}(f_\varepsilon, f_\varepsilon)) < \infty,
\]
\( \{f_\varepsilon : \varepsilon \in (0,1)\} \) is pre-compact as \( \varepsilon \to 0 \) in \( L^1(B(0,r); dx) \) for all \( r > 1 \).

**Lemma 3.5.** Suppose that either (A2') or (B2') holds. Then there is subset \( \Omega_0 \subset \Omega \) of full probability measure so that the following holds for every \( \omega \in \Omega_0 \). Suppose that \( \{f_\varepsilon : \varepsilon \in (0,1)\} \) is a collection of functions with \( f_\varepsilon \in \mathcal{F}^{\varepsilon, \omega} \) for \( \varepsilon \in (0,1) \) and
\[
\limsup_{\varepsilon \to 0} \mathcal{E}^{\varepsilon, \omega}(f_\varepsilon, f_\varepsilon) < \infty.
\]
Then, for every \( r > 1 \) and \( 0 < |h| \leq r/3 \),
\[
\limsup_{\varepsilon \to 0} \sup_{x_0 \in B(0,r)} \int_{B_{2h}(x_0, r)} |f_\varepsilon(x+h) - f_\varepsilon(x)| \, dx \leq c_0(r)h^{\alpha/2} \limsup_{\varepsilon \to 0} \mathcal{E}^{\varepsilon, \omega}(f_\varepsilon, f_\varepsilon)^{1/2}, \quad 1 \leq i \leq d, \tag{3.14}
\]
where \( \{e_i : 1 \leq i \leq d\} \) is the orthonormal basis of \( \mathbb{R}^d \), \( B_{2h}(x_0, r) := \{y \in B(x_0, r) : |y - \partial B(x_0, r)| > 2h\} \) and \( c_0(r) \) is a positive constant depending on \( r \) but independent of \( h, f, \varepsilon \) and \( \omega \).

**Proof.** It suffices to prove (3.14) for every fixed \( e_i \) and the case that \( h > 0 \). The argument below is partially motivated by the proof of the compact embeddings in fractional Sobolev spaces; see [23, Theorem 4.54, p. 216].

For any \( z \in \mathbb{R}^d \), denote by \( (r(z), \theta(z)) \in \mathbb{R}_+ \times S^{d-1} \) its spherical coordinate. Let \( \Theta := \Gamma \cap S^{d-1} \), which has non-empty interior. Hence there are a non-empty open set \( \overline{\Theta} \subset \Theta \) and a constant \( N \geq 1 \) large enough so that
\[
\left\{ \theta \left( z - \frac{e_i}{N} \right) : 1 \leq r(z) \leq 2, \theta(z) \in \overline{\Theta} \right\} \subset \Theta.
\]
Consequently, \( \theta(z - he_i) = \theta(z/(hN) - e_i/N) \in \Theta \) for any \( z = (r(z), \theta(z)) \) with \( Nh \leq r(z) \leq 2Nh \) and \( \theta(z) \in \Theta \). Let
\[
G_{h, \overline{\Theta}} := \left\{ z \in \mathbb{R}^d : Nh \leq r(z) \leq 2Nh, \theta(z) \in \overline{\Theta} \right\} \quad \text{and} \quad G_{h, \Theta} := \left\{ z \in \mathbb{R}^d : Nh \leq r(z) \leq 2Nh, \theta(z) \in \Theta \right\}.
\]
Clearly \( G_{h, \overline{\Theta}} \subset G_{h, \Theta} \) and \( |G_{h, \overline{\Theta}}| \asymp |G_{h, \Theta}| \asymp h^d \).
Similarly, we have

\[
\int_{B_{2h}(x_0, r)} |f_\varepsilon(x + he_i) - f_\varepsilon(x)| \, dx \\
\leq c_1 h^{-d} \int_{B_{2h}(x_0, r)} \int_{x + G_{h, \Theta}} |f_\varepsilon(x + he_i) - f_\varepsilon(x)| \, dz \, dx
\]

\[
\leq c_1 h^{-d} \left( \int_{B_{2h}(x_0, r)} \int_{x + G_{h, \Theta}} |f_\varepsilon(x + he_i) - f_\varepsilon(z)| \, dz \, dx + \int_{B_{2h}(x_0, r)} \int_{x + G_{h, \Theta}} |f_\varepsilon(x) - f_\varepsilon(z)| \, dz \, dx \right)
\]

\[
= c_1 h^{-d} \left( I_{x_0, h, 1}(f_\varepsilon) + I_{x_0, h, 2}(f_\varepsilon) \right).
\]

By a change of variables,

\[
I_{x_0, h, 1}(f_\varepsilon) \leq \int_{B(0, 2r)} \int_{G_{h, \Theta} - he_i} |f_\varepsilon(x + z + he_i) - f_\varepsilon(x + he_i)| \, dz \, dx
\]

\[
\leq \int_{B(0, 2r)} \int_{G_{h, \Theta}} |f_\varepsilon(x + z + he_i) - f_\varepsilon(x + he_i)| \, dz \, dx
\]

\[
\leq \int_{B(0, 3r)} \int_{G_{h, \Theta}} |f_\varepsilon(x + z) - f_\varepsilon(x)| \, dz \, dx
\]

(3.15)

Similarly, we have

\[
I_{x_0, h, 2}(f_\varepsilon) \leq \int_{B(0, 2r)} \int_{G_{h, \Theta}} |f_\varepsilon(x + z) - f_\varepsilon(x)| \, dz \, dx.
\]

(3.16)

(i) Assume that \( (A2') \) holds. By (3.16), the Hölder inequality and Proposition 2.1,

\[
\limsup_{\varepsilon \to 0} \sup_{x_0 \in B(0, r)} I_{x_0, h, 1}(f_\varepsilon) \leq \limsup_{\varepsilon \to 0} \left( \int_{B(0, 3r)} \int_{G_{h, \Theta}} \frac{(f_\varepsilon(x + z) - f_\varepsilon(x))^2}{|z|^{d+\alpha}} \Lambda_1(\tau_{x/\varepsilon} \omega) \, dz \, dx \right)^{1/2}
\]

\[
\times \limsup_{\varepsilon \to 0} \left( \int_{B(0, 3r)} \int_{G_{h, \Theta}} |z|^{d+\alpha} \Lambda_1(\tau_{x/\varepsilon} \omega)^{-1} \, dz \, dx \right)^{1/2}
\]

\[
\leq c_2 \limsup_{\varepsilon \to 0} \left[ \left( \int_{B(0, 3r)} \Lambda_1(\tau_{x/\varepsilon} \omega)^{-1} \, dx \right)^{1/2} \Xi_{\varepsilon, \omega}(f_\varepsilon, f_\varepsilon) \right] h^{d+\alpha/2}
\]

\[
= c_2 h^{d+\alpha/2} \{ |B(0, 3r)| \Xi_{\varepsilon, \omega}(f_\varepsilon, f_\varepsilon) \}^{1/2}
\]

\[
= c_3 h^{d+\alpha/2} \limsup_{\varepsilon \to 0} \Xi_{\varepsilon, \omega}(f_\varepsilon, f_\varepsilon)^{1/2},
\]

where \( c_2, c_3 \) are positive constants independent of \( x_0, h \) and \( \varepsilon \), but may depend on \( N \) and \( r \).

Similarly, we have

\[
\limsup_{\varepsilon \to 0} \sup_{x_0 \in B(0, r)} I_{x_0, h, 2}(f_\varepsilon) \leq c_4 h^{d+\alpha/2} \limsup_{\varepsilon \to 0} \Xi_{\varepsilon, \omega}(f_\varepsilon, f_\varepsilon)^{1/2},
\]

where \( c_4 \) is a positive constant independent of \( x_0, h \) and \( \varepsilon \). Thus we have by (3.15),

\[
\limsup_{\varepsilon \to 0} \sup_{x_0 \in B(0, r)} \int_{B_{2h}(x_0, r)} |f_\varepsilon(x + he_i) - f_\varepsilon(x)| \, dx \leq c_5 h^{\alpha/2} \limsup_{\varepsilon \to 0} \Xi_{\varepsilon, \omega}(f_\varepsilon, f_\varepsilon)^{1/2},
\]

which establishes (3.14).
(ii) Next we assume that (B2') holds. According to (3.16) and the Hölder inequality, for any $0 < h < r/3,$
\[
\limsup_{\varepsilon \to 0} \sup_{x_0 \in B(0,r)} I_{x_0,h,1}(f_\varepsilon) \\
\leq \limsup_{\varepsilon \to 0} \left( \int_{B(0,3r)} \int_{G_{h,\Theta}} \frac{(f_\varepsilon(x + z) - f_\varepsilon(x))^2}{|z|^{d+\alpha}} \Lambda_1(\tau_{x/\varepsilon \omega}) \Lambda_1(\tau_{(x+z)/\varepsilon \omega}) \, dz \, dx \right)^{1/2} \\
\times \limsup_{\varepsilon \to 0} \left( \int_{B(0,3r)} \int_{G_{h,\Theta}} |z|^{d+\alpha} \Lambda_1(\tau_{x/\varepsilon \omega})^{-1} \Lambda_1(\tau_{(x+z)/\varepsilon \omega})^{-1} \, dx \, dz \right)^{1/2} \\
\leq c_7 h^{(d+\alpha)/2} \limsup_{\varepsilon \to 0} \left( \mathbb{E}^\varepsilon(\tau_{x/\varepsilon \omega}, f_\varepsilon) \int_{B(0,3r)} \int_{x+G_{h,\Theta}} \Lambda_1(\tau_{x/\varepsilon \omega})^{-1} \Lambda_1(\tau_{y/\varepsilon \omega})^{-1} \, dy \, dx \right)^{1/2}.
\]

Here $c_7$ is a positive constant independent of $h$ and $\varepsilon.

Below, denote
\[
K_h := \{(x,y) : x \in B(0,3r), y \in x + G_{h,\Theta}\} \subset \mathbb{R}^d \times \mathbb{R}^d.
\]

If we directly apply Lemma 3.1(ii) to estimate $\int_{K_h} \Lambda_1(\tau_{x/\varepsilon \omega})^{-1} \Lambda_1(\tau_{y/\varepsilon \omega})^{-1} \, dx \, dy,$ then it requires that $\Lambda_1^{-1} \in L^p(\Omega; \mathbb{P})$ for some $p > 1.$ Instead we will adopt a different approach under the weak condition $\Lambda_1^{-1} \in L^1(\Omega; \mathbb{P})$ as in (1.17) of assumption (B2'). For every $0 < h \leq r/3,$ we can find $\{x_i\}_{i=1}^m \subset \mathbb{R}^d$ such that $B(0,3r) \subset \bigcup_{i=1}^m B(x_i,h)$ and $\sum_{i=1}^m |B(x_i,h)| \leq c_0 |B(0,3r)|,$ where $c_0$ is independent of $h, r, m$ and $\{x_i\}_{i=1}^m$, but the integer $m$ and $\{x_i\}_{i=1}^m$ may depend on $h$. It is easy to verify that $K_h \subset \bigcup_{i=1}^m B(x_i,h) \times (B(x_i, (2N+1)h) \setminus B(x_i, (N-1)h)).$

Hence,
\[
\lim_{\varepsilon \to 0} \int_{K_h} \Lambda_1(\tau_{x/\varepsilon \omega})^{-1} \Lambda_1(\tau_{y/\varepsilon \omega})^{-1} \, dx \, dy \\
\leq \lim_{\varepsilon \to 0} \sum_{i=1}^m \int_{B(x_i,h)} \int_{B(x_i,(2N+1)h) \setminus B(x_i,(N-1)h)} \Lambda_1(\tau_{x/\varepsilon \omega})^{-1} \Lambda_1(\tau_{y/\varepsilon \omega})^{-1} \, dx \, dy \\
= \sum_{i=1}^m \left( \lim_{\varepsilon \to 0} \int_{B(x_i,h)} \Lambda_1(\tau_{x/\varepsilon \omega})^{-1} \, dx \right) \cdot \left( \lim_{\varepsilon \to 0} \int_{B(x_i,(2N+1)h) \setminus B(x_i,(N-1)h)} \Lambda_1(\tau_{y/\varepsilon \omega})^{-1} \, dy \right) \\
= \sum_{i=1}^m \mathbb{E}[\Lambda_1^{-1}] \cdot |B(x_i,h)| \cdot |B(x_i,(2N+1)h) \setminus B(x_i,(N-1)h)| \leq c_8 h^d \sum_{i=1}^m |B(x_i,h)| \leq c_9 h^d,
\]

where $c_8$ and $c_9$ are positive constants independent of $h$ (but may depend on $N$ and $r$). Here, in the second equality above we have used Proposition 2.1 (so only $\Lambda_1^{-1} \in L^1(\Omega; \mathbb{P})$ is required), and the last inequality is due to the fact $\sum_{i=1}^m |B(x_i,h)| \leq c_0 |B(0,3r)|.$ Therefore, combining this estimate with (3.17), we find that
\[
\limsup_{\varepsilon \to 0} \sup_{x_0 \in B(0,r)} I_{x_0,h,1}(f_\varepsilon) \leq c_{10} \left( \limsup_{\varepsilon \to 0} \mathbb{E}^\varepsilon(\tau_{x/\varepsilon \omega}, f_\varepsilon; f_\varepsilon)^{1/2} \right) h^{d+\alpha/2}.
\]

Similarly, we can obtain
\[
\limsup_{\varepsilon \to 0} \sup_{x_0 \in B(0,r)} I_{x_0,h,2}(f_\varepsilon) \leq c_{11} \left( \limsup_{\varepsilon \to 0} \mathbb{E}^\varepsilon(\tau_{x/\varepsilon \omega}, f_\varepsilon; f_\varepsilon)^{1/2} \right) h^{d+\alpha/2}.
\]

Thus we have by (3.15),
\[
\lim_{\varepsilon \to 0} \sup_{x_0 \in B(0,r)} \int_{B_{2h}(x_0,r)} |f_\varepsilon(x + he_i) - f_\varepsilon(x)| \, dx \leq c_{12} h^{\alpha/2} \limsup_{\varepsilon \to 0} \mathbb{E}^\varepsilon(\tau_{x/\varepsilon \omega}, f_\varepsilon; f_\varepsilon)^{1/2},
\]

which establishes (3.14).
Proof of Proposition 3.4. We first claim that for every \( r > 1 \) and \( h \in \mathbb{R}^d \) with \(|h| < r/3\)

\[
\limsup_{\varepsilon \to 0} \int_{B_2|h|/(0,r)} |f_\varepsilon(x + h) - f_\varepsilon(x)| \, dx \leq c_0(r; \omega)h^{\alpha/2}.
\]  

(3.18)

Indeed, writing \( h = (h^{(1)}, \ldots, h^{(i)}, \ldots, h^{(d)}) \in \mathbb{R}^d \), we have for any \( 0 \leq i \leq d \),

\[
\int_{B_2|h|/(0,r)} |f_\varepsilon(x + (h^{(1)}, \ldots, h^{(i)}, h^{(i+1)}, 0, \ldots, 0)) - f_\varepsilon(x + (h^{(1)}, \ldots, h^{(i)}, 0, \ldots, 0))| \, dx
\]

\[
= \int_{B_2|h|/(h^{(1)}, \ldots, h^{(i)}, 0, \ldots, 0, r)} |f_\varepsilon(x + (0, \ldots, 0, h^{(i+1)}, 0, \ldots, 0)) - f_\varepsilon(x)| \, dx
\]

\[
\leq \int_{B_2|h|/(h^{(1)}, \ldots, h^{(i)}, 0, \ldots, 0, r)} |f_\varepsilon(x + (0, \ldots, 0, h^{(i+1)}, 0, \ldots, 0)) - f_\varepsilon(x)| \, dx.
\]

Here, we set \((h^{(1)}, \ldots, h^{(i)}, 0, \ldots, 0) = 0\) when \( i = 0 \). This along with (3.14) gives (3.18).

On the other hand, for any \( \delta > 0 \),

\[
\limsup_{\varepsilon \to 0} \int_{B(0,r) \setminus B_{2\delta}(0,r)} |f_\varepsilon(x)| \, dx \leq \left( \limsup_{\varepsilon \to 0} \|f_\varepsilon\|_\infty \right) |B(0, r) \setminus B_{2\delta}(0, r)| \leq c_1(r)\delta,
\]

where \( c_1(r) \) is a positive constant independent of \( \delta \) and \( \varepsilon \).

Therefore, for every \( r > 1 \) and \( \zeta > 0 \), there exists a constant \( \delta := \delta(r, \zeta; \omega) \) such that for every \( h \in \mathbb{R}^d \) with \(|h| < \delta\),

\[
\limsup_{\varepsilon \to 0} \int_{B_\delta(0,r)} |f_\varepsilon(x + h) - f_\varepsilon(x)| \, dx \leq \zeta
\]

(3.19)

and

\[
\limsup_{\varepsilon \to 0} \int_{B(0,r) \setminus B_{\delta}(0,r)} |f_\varepsilon(x)| \, dx \leq \zeta.
\]

(3.20)

It then follows from (3.19), (3.20) and [23, Theorem 1.95, p. 37] that \( \{f_\varepsilon : \varepsilon \in (0,1]\} \) is pre-compact as \( \varepsilon \to 0 \) in \( L^1(B(0,r); dx) \) for all \( r > 1 \). The proof is complete. \( \square \)

3.3. Proofs of Theorems 1.3 and 1.6, and the assertion of Example 1.7.

Proof of Theorem 1.3. According to Theorems 2.2 and 2.3, we only need to verify that assumption (A) implies assumption (H). By Propositions 3.3 and 3.4, assumptions (A) implies properties (i) and (iv) in assumption (H). So it remains to show that properties (ii) and (iii) in assumption (H) hold as well.

Suppose assumption (A2) holds. There is a subset \( \Omega_1 \subset \Omega \) so that Proposition 2.1 holds with \( \Lambda_2 \) in place of \( \nu \). For \( g \in C^1_c(\mathbb{R}^d) \), let \( R_0 > 1 \) be such that \( \text{supp}[g] \subset B(0, R_0) \). For every \( \omega \in \Omega_1 \) and \( \eta \in (0, 1/(2R_0)) \),

\[
\limsup_{\varepsilon \to 0} \int_{\{|x-y| \leq \eta\}} (g(x) - g(y))^2 \frac{\kappa(x/\varepsilon, y/\varepsilon; \omega)}{|x-y|^{d+\alpha}} \, dx \, dy \leq 2\limsup_{\varepsilon \to 0} \int_{\{|x-y| \leq \eta\}} (g(y) - g(x))^2 \frac{\Lambda_2(\tau_x/\varepsilon \omega)}{|x-y|^{d+\alpha}} \, dx \, dy
\]

\[
\leq 2\|\nabla g\|_\infty \limsup_{\varepsilon \to 0} \int_{B(0,R_0) \setminus \eta} \left( \int_{\{|z| \leq \eta \}} \frac{1}{|z|^{d+\alpha-2}} \, dz \right) \Lambda_2(\tau_x/\varepsilon \omega) \, dz \leq c_1 \eta^{2-\alpha} |B(0, R_0 + \eta)| \mathbb{E} [\Lambda_2].
\]

(3.21)

In particular, we have for every \( \omega \in \Omega_1 \),

\[
\limsup_{\varepsilon \to 0} \int_{\{|x-y| \leq \eta\}} (g(x) - g(y))^2 \frac{\kappa(x/\varepsilon, y/\varepsilon; \omega)}{|x-y|^{d+\alpha}} \, dx \, dy = 0.
\]
On the other hand, noting that $1/\eta > 2R_0$, we have

$$\limsup_{\varepsilon \to 0} \int_{\{|x-y| > 1/\eta\}} (g(y) - g(x))^2 \frac{\kappa(\tau_x/\varepsilon, \tau_y/\varepsilon; \omega)}{|x-y|^{d+\alpha}} \, dx \, dy \leq 2 \limsup_{\varepsilon \to 0} \int_{\{|x-y| > 1/\eta\}} (g(y) - g(x))^2 \frac{\Lambda_2(\tau_x/\varepsilon)}{|x-y|^{d+\alpha}} \, dx \, dy$$

$$\leq 2 \limsup_{\varepsilon \to 0} \int_{B(0,R_0)} \left( \int_{\{|y-x| > 1/\eta\}} \frac{1}{|y-x|^{d+\alpha}} \, dy \right) g(x)^2 \Lambda_2(\tau_x/\varepsilon) \, dx$$

$$+ 2 \limsup_{\varepsilon \to 0} \int_{B(0,R_0)^c} \left( \int_{\{y \in B(0,R_0) : |y-x| > 1/\eta\}} \frac{g(y)^2}{|y-x|^{d+\alpha}} \, dy \right) \Lambda_2(\tau_x/\varepsilon) \, dx$$

$$\leq c_2 \|g\|_\infty^2 |B(0,R_0)| \eta^{\alpha/2} \left( \eta^{\alpha/2} \mathbb{E}[\Lambda_2] + \limsup_{\varepsilon \to 0} \int_{B(0,R_0)^c} \Lambda_2(\tau_x/\varepsilon) \, dx \right).$$

Recall that, by Proposition 2.1, for every $\omega \in \Omega_1$,

$$\lim_{\varepsilon \to 0} \varepsilon^d \int_{B(0,\varepsilon)} \Lambda_2(\tau_x \omega) \, dx = \lim_{\varepsilon \to 0} \int_{B(0,1)} \Lambda_2(\tau_x \omega) \, dx = |B(0,1)| \mathbb{E}[\Lambda_2].$$

Hence, for every $\omega \in \Omega_1$, there exists $\varepsilon_0(\omega) > 0$ such that

$$\int_{B(0,1/\varepsilon)} \Lambda_2(\tau_x \omega) \, dx \leq 2 \mathbb{E}[\Lambda_2] |B(0,1)| \varepsilon^{-d}, \quad \varepsilon \in (0, \varepsilon_0(\omega)).$$

Next, we choose $k_0 := k_0(\omega) \geq 1$ such that $2^{k_0} \geq \varepsilon_0(\omega)^{-1}$. Thus, for every $\omega \in \Omega_1$,

$$\limsup_{\varepsilon \to 0} \int_{B(0,R_0)^c} \frac{\Lambda_2(\tau_x \omega)}{|x|^{d+\alpha/2}} \, dx \leq \limsup_{\varepsilon \to 0} \int_{B(0,2^{k_0}) \setminus B(0,R_0)} \frac{\Lambda_2(\tau_x \omega)}{|x|^{d+\alpha/2}} \, dx + \limsup_{\varepsilon \to 0} \sum_{k=k_0+1}^{\infty} \int_{Q_k} \frac{\Lambda_2(\tau_x \omega)}{|x|^{d+\alpha/2}} \, dx$$

$$= \limsup_{\varepsilon \to 0} I_1^\varepsilon + \limsup_{\varepsilon \to 0} I_2^\varepsilon,$$

where $Q_k = \{ x \in \mathbb{R}^d : 2^{k-1} < |x| \leq 2^k \}$. According to Proposition 2.1 again, we obtain

$$\limsup_{\varepsilon \to 0} I_1^\varepsilon = \int_{B(0,2^{k_0}) \setminus B(0,R_0)} \frac{\mathbb{E}[\Lambda_2]}{|x|^{d+\alpha/2}} \, dx \leq \int_{B(0,R_0)^c} \frac{\mathbb{E}[\Lambda_2]}{|x|^{d+\alpha/2}} \, dx < \infty.$$

Meanwhile, by (3.23), it holds that

$$\limsup_{\varepsilon \to 0} I_2^\varepsilon \leq \limsup_{\varepsilon \to 0} \sum_{k=k_0+1}^{\infty} 2^{-(k-1)(d+\alpha/2)} \int_{B(0,2^k)} \Lambda_2(\tau_x \omega) \, dx$$

$$= \limsup_{\varepsilon \to 0} \sum_{k=k_0+1}^{\infty} 2^{-(k-1)(d+\alpha/2)} \varepsilon^d \int_{B(0,2^k/\varepsilon)} \Lambda_2(\tau_x \omega) \, dx$$

$$\leq 2 |B(0,1)| \mathbb{E}[\Lambda_2] \limsup_{\varepsilon \to 0} \sum_{k=k_0+1}^{\infty} 2^{-(k-1)(d+\alpha/2)} \varepsilon^d (2^k/\varepsilon)^d$$

$$= c_4 \sum_{k=k_0+1}^{\infty} 2^{-k\alpha/2} \leq c_4 \sum_{k=1}^{\infty} 2^{-k\alpha/2} < \infty.$$

Combining all the estimates above with (3.22) shows that for every $\omega \in \Omega_1$,

$$\lim \limsup_{\eta \to 0} \int_{\{|x-y| > 1/\eta\}} (g(y) - g(x))^2 \frac{\kappa(x/\varepsilon, y/\varepsilon; \omega)}{|x-y|^{d+\alpha}} \, dx \, dy = 0,$$

that is, property (ii) of assumption (H) holds.
Evidently, there is $\Omega_2 \subset \Omega_1$ of full probability measure so that the conclusion of Proposition 2.1 holds with $\Lambda^2_2$ in place of $\nu$.  In view of (1.12) of assumption (A2), for every $\omega \in \Omega_2$, by Hölder’s inequality,

$$
\limsup_{\varepsilon \to 0} \int_{B(0,R)} \left( \int_{B(0,R)} \kappa(x/\varepsilon, y/\varepsilon; \omega) \, dy \right)^p \, dx \leq c_5 R^{(p-1)d/p} \limsup_{\varepsilon \to 0} \int_{B(0,R) \times B(0,R)} \kappa(x/\varepsilon, y/\varepsilon; \omega)^p \, dx \, dy
$$

$$
\leq 2^{p+1} c_5 R^{(p-1)d/p} \limsup_{\varepsilon \to 0} \int_{B(0,R)} \int_{B(0,R)} \Lambda_2(\tau_x/\varepsilon \omega)^p \, dx \, dy
$$

$$
= 2^{p+1} c_5 R^{(p-1)d/p} |B(0,R)|^2 \mathbb{E} [\Lambda^2_2] < \infty.
$$

Hence property (iii) of assumption (H) holds as well.  This shows that assumption (A2) yields properties (ii) and (iii) of assumption (H), and so proves Theorem 1.3.

Proof of Theorem 1.6.  By Propositions 3.3 and 3.4, assumption (B) implies properties (i) and (iv) in assumption (H).  So it remains to show that properties (ii) and (iii) of assumption (H) hold as well.  Note that under assumption (B2), for any $\omega \in \Omega$ and $x, y \in \mathbb{R}^d$,

$$
\kappa(x, y; \omega) \leq \Lambda_2(\tau_x \omega) \Lambda_2(\tau_y \omega) \leq \frac{1}{2}(\Lambda_2(\tau_x \omega)^2 + \Lambda_2(\tau_y \omega)^2).
$$

With this at hand, we can follow the argument in the proof of Theorem 1.3 to show that assumption (B2) implies property (ii) of assumption (H).  Furthermore, an analogous argument also shows that assumption (B2) implies property (iii) of assumption (H).

We point out that the second moment condition on $\Lambda_2$ in assumption (B2) is only used in establishing (2.3) and (1.2).  The remaining properties in (ii) and (iii) of assumption (H) hold under a weaker assumption that $\mathbb{E} [\Lambda_2^p] < \infty$ for some $p > 1$.  In particular, (2.4) follows from the corresponding argument in the proof of Theorem 1.3 by using Lemma 3.1(ii) instead of Proposition 2.1.

Proof of the assertion in Example 1.7.  Define

$$
\mu := \frac{1}{Z_\mu} \frac{\lambda_2}{\lambda_1}, \quad \text{where } Z_\mu := \mathbb{E} [\lambda_2 / \lambda_1].
$$

Clearly, $\mathbb{E}[\mu] = 1$, and we can rewrite (1.20) as

$$
\mathcal{L}^{\varepsilon, \omega} f(x) = \frac{1}{Z_\mu \mu(\tau_x/\varepsilon \omega)} \text{p.v.} \int_{\mathbb{R}^d} (f(y) - f(x)) \frac{\lambda_2(\tau_x/\varepsilon \omega) \lambda_2(\tau_y/\varepsilon \omega)}{|y - x|^{d+\alpha}} \mathbb{1}_\Gamma(y - x) \, dy.
$$

Thus, the operator $\mathcal{L}^{\varepsilon, \omega}$ is symmetric in $L^2(\mathbb{R}^d; \mu(\tau_x/\varepsilon \omega) \, dx)$, and is associated with the symmetric Dirichlet form $(\mathcal{E}^{\varepsilon, \omega}, \mathbb{F}^{\varepsilon, \omega})$ on $L^2(\mathbb{R}^d; \mu(\tau_x/\varepsilon \omega) \, dx)$ given by

$$
\mathcal{E}^{\varepsilon, \omega}(f, g) = \frac{1}{2Z_\mu} \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(x) - f(y))(g(x) - g(y)) \frac{\lambda_2(\tau_x/\varepsilon \omega) \lambda_2(\tau_y/\varepsilon \omega)}{|x - y|^{d+\alpha}} \mathbb{1}_\Gamma(y - x) \, dx \, dy.
$$

Under assumptions of the example, we see that (1.16) and (1.19) are satisfied with

$$
\nu_1 = \nu_2 = \frac{1}{\sqrt{Z_\mu}} \lambda_2,
$$

and $\mathbb{E}[\mu^p] < \infty$.  Hence, by Proposition 1.4, assumption (B) is fulfilled.  Therefore, the conclusion of the example follows readily from Theorem 1.6.
4. Appendix

4.1. Proofs of Lemma 1.1 and Proposition 1.4.

Proof of Lemma 1.1. Let \( \mathcal{L}^\omega \) and \( (P_t^\omega)_{t \geq 0} \) be the generator and the semigroup of the process \( X^\omega \), respectively. Let \( \mathcal{L}_{\varepsilon,\omega} \) and \( (P_t^{\varepsilon,\omega})_{t \geq 0} \) be the generator and the semigroup of the process \( X_{\varepsilon,\omega} \), respectively. Then, for any \( f \in C_c^\infty(\mathbb{R}^d) \),

\[
P_t^{\varepsilon,\omega} f(x) = E_x^\omega \left[ f(x_{t/\varepsilon}) \right] = P_t^{\varepsilon,\omega} f^{(\varepsilon)}(x/\varepsilon),
\]

where \( f^{(\varepsilon)}(x) = f(\varepsilon x) \). Thus,

\[
\mathcal{L}_{\varepsilon,\omega} f(x) = \frac{dP_t^{\varepsilon,\omega} f(x)}{dt} \bigg|_{t=0} = \varepsilon^{-\alpha} \mathcal{L}^\omega f^{(\varepsilon)}(x/\varepsilon).
\]

For any \( f, g \in C_c^\infty(\mathbb{R}^d) \), by the facts that the operator \( \mathcal{L}^\omega \) is symmetric on \( L^2(\mathbb{R}^d; \mu^\omega(dx)) \) and

\[
- \int_{\mathbb{R}^d} \mathcal{L}^\omega f(x) g(x) \mu^\omega(dx) = \mathcal{E}^\omega(f, g),
\]

we find that

\[
- \int_{\mathbb{R}^d} \mathcal{L}_{\varepsilon,\omega} f(x) g(x) \mu^{\varepsilon,\omega}(dx) = - \int_{\mathbb{R}^d} \mathcal{L}^\omega f^{(\varepsilon)}(x) g^{(\varepsilon)}(x) \mu^{\varepsilon}(\tau_{x/\varepsilon} \omega) dx.
\]

Hence, the desired assertion follows.

Proof of Proposition 1.4. (i) Suppose that (1.19) holds. Clearly (1.18) and (1.17) hold by taking \( \Lambda_1(\omega) := \sqrt{2} \nu_1(\omega) \nu_2(\omega) \) and \( \Lambda_2(\omega) := \nu_1(\omega) + \nu_2(\omega) \).

(ii) Conversely, suppose that there are non-negative random variables \( \Lambda_1 \leq \Lambda_2 \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) so that (1.18) and (1.17) hold. Taking \( x = y = 0 \) in (1.17) yields from (1.16) that

\[
\Lambda_1^2(\omega) \leq 2 \nu_1(\omega) \nu_2(\omega) \leq \Lambda_2(\omega)^2.
\]

This in particular implies that

\[
\mathbb{E}(\nu_1(\omega) \nu_2(\omega)^{-1/2}) \leq 2^{1/2} \mathbb{E}(\Lambda_1^{-1}) < \infty.
\]

We claim that there is a constant \( C > 0 \) so that

\[
\nu_i(\omega) \leq C \Lambda_2(\omega) \quad \mathbb{P} \text{-a.s. for } i = 1, 2.
\]

This together with (4.1) will imply that (1.19) holds. Since \( \mathbb{P}(0 < \Lambda_1 \leq \Lambda_2 < \infty) = 1 \), there are constants \( 0 < a < b \) so that \( \mathbb{P}(\Omega_1) > 3/4 \), where

\[
\Omega_1 := \{ \omega \in \Omega : a \leq \Lambda_1(\omega) \leq \Lambda_2(\omega) \leq b \text{ and } a \leq \min\{\nu_1(\omega), \nu_2(\omega)\} \leq \max\{\nu_1(\omega), \nu_2(\omega)\} \leq b \}.
\]

Define

\[
A = \{ \omega \in \Omega : \tau_x \omega \in \Omega_1 \text{ for some } x \in \mathbb{R}^d \}.
\]
Clearly, $\Omega_1 \subset A$ and $\tau_x A \subset A$ for every $x \in \mathbb{R}^d$. Since $\tau_x \circ \tau_y = \tau_{x+y}$ for every $x, y \in \mathbb{R}^d$ and $\tau_0 = \text{id}$, we have $\tau_x A = A$ for every $x \in \mathbb{R}^d$. Hence $\mathcal{P}(A) = 1$ as the family of shift operators $\{\tau_x; x \in \mathbb{R}^d\}$ is ergodic. By (1.16) and (1.17),

$$\nu_1(\tau_x \omega)\nu_2(\tau_y \omega) + \nu_1(\tau_y \omega)\nu_2(\tau_x \omega) \leq \Lambda_2(\tau_x \omega)\Lambda_2(\tau_y \omega).$$

For every $\omega \in A$, take $x = 0$ and $y \in \mathbb{R}^d$ so that $\tau_y \omega \in \Omega_1$ in the above display. Then this implies that

$$\nu_1(\omega) + \nu_2(\omega) \leq (b/a)\Lambda_2(\omega).$$

This proves the claim (4.2), and hence completes the proof of the Proposition.

4.2. Mosco convergence of Dirichlet forms. In this part, we will study the Mosco convergence for the Dirichlet forms $(\mathcal{E}^{\varepsilon_n,\omega},\mathcal{F}^{\varepsilon_n,\omega})$, which yields the strong convergence of the associated semigroups and resolvents in $L^2$-senses along any fixed sequence $\{\varepsilon_n\}_{n \geq 1}$ with $\varepsilon_n \to 0$ as $n \to \infty$.

For this, we first recall some known results from [41, 34, 35] on Mosco convergence with changing reference measures, and adapt them to our setting. Consider the Hilbert spaces $L^2(\mathbb{R}^d;\mu^{\varepsilon_n,d}(dx))$ and $L^2(\mathbb{R}^d;dx)$ as these in the present paper. For simplicity, we drop the parameter $\omega$, and write $L^2(\mathbb{R}^d;\mu^{\varepsilon_n,d}(dx))$ as $L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))$. Then, by Proposition 2.1 and $E[\mu] = 1$, there is $\Omega_0 \subset \Omega$ of full probability measure so that for all $\omega \in \Omega_0$ and $f \in C_c^\infty(\mathbb{R}^d)$,

$$\lim_{n \to \infty} \|f\|_{L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))} = \|f\|_{L^2(\mathbb{R}^d;dx)};$$

that is, $L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))$ converges to $L^2(\mathbb{R}^d;dx)$ in the sense of [41], see [41, p. 611] or [35, Definition 2.1].

Following [41, Definitions 2.4 and 2.5] or [35, Definitions 2.2 and 2.2], we say that a sequence of functions $\{f_n\}_{n \geq 1}$ with $f_n \in L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))$ for all $n \geq 1$ strongly converges in $L^2$-spaces to $f \in L^2(\mathbb{R}^d;dx)$, if there exists a sequence of functions $\{g_m\}_{m \geq 1} \subset C_c^\infty(\mathbb{R}^d)$ such that

$$\lim_{m \to \infty} \|g_m - f\|_{L^2(\mathbb{R}^d;dx)} = 0, \quad \lim_{m \to \infty} \limsup_{n \to \infty} \|g_m - f_n\|_{L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))} = 0; \tag{4.3}$$

we say that a sequence of functions $\{f_n\}_{n \geq 1}$ with $f_n \in L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))$ for all $n \geq 1$ weakly converges in $L^2$-spaces to $f \in L^2(\mathbb{R}^d;dx)$, if for every sequence $\{g_n\}_{n \geq 1}$ with $g_n \in L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))$ for all $n \geq 1$ and strongly convergent to $g \in L^2(\mathbb{R}^d;dx)$,

$$\lim_{n \to \infty} \langle f_n, g_n \rangle_{L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))} = \langle f, g \rangle_{L^2(\mathbb{R}^d;dx)}.$$ 

It is obvious that strong convergence in $L^2$-spaces is stronger than weak convergence in $L^2$-spaces; see [41, Lemma 2.1(4)].

For any $\varepsilon > 0$, let $(\mathcal{E}^{\varepsilon,\omega},\mathcal{F}^{\varepsilon,\omega})$ be a regular Dirichlet form on $L^2(\mathbb{R}^d;\mu^{\varepsilon,\omega}(dx))$ given by (1.4), and $(\mathcal{E}^K,\mathcal{F}^K)$ be a regular Dirichlet form on $L^2(\mathbb{R}^d;dx)$ given by (1.7). Following [41, Definition 2.11], a sequence of Dirichlet forms $\{(\mathcal{E}^{\varepsilon_n},\mathcal{F}^{\varepsilon_n})\}_{n \geq 1}$ on $L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))$ is said to be Mosco convergent to a Dirichlet form $(\mathcal{E}^K,\mathcal{F}^K)$ on $L^2(\mathbb{R}^d;dx)$, if

(i) for every sequence $\{f_n\}_{n \geq 1}$ with $f_n \in L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))$ for all $n \geq 1$ and converging weakly to $f \in L^2(\mathbb{R}^d;dx)$,

$$\liminf_{n \to \infty} \mathcal{E}^{\varepsilon_n}(f_n, f_n) \geq \mathcal{E}^K(f, f).$$

(ii) for any $f \in L^2(\mathbb{R}^d;dx)$, there is a sequence $\{f_n\}_{n \geq 1}$ with $f_n \in L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))$ for all $n \geq 1$ and converging strongly to $f$ such that

$$\limsup_{n \to \infty} \mathcal{E}^{\varepsilon_n}(f_n, f_n) \leq \mathcal{E}^K(f, f).$$

In the above definition, we have extended the definition of $\mathcal{E}^{\varepsilon_n}(f, f)$ and $\mathcal{E}^K(f, f)$ by $\mathcal{E}^{\varepsilon_n}(f, f) = \infty$ for $f \in L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx)) \setminus \mathcal{F}^{\varepsilon_n}$ and $\mathcal{E}^K(f, f) = \infty$ for $f \in L^2(\mathbb{R}^d;dx) \setminus \mathcal{F}^K$, respectively.

**Remark 4.1.** (i) The condition (i) in the definition of Mosco convergence holds true, if for every sequence $\{f_n\}_{n \geq 1}$ with $f_n \in L^2(\mathbb{R}^d;\mu_{\varepsilon_n}(dx))$ for all $n \geq 1$ and converging weakly to $f \in L^2(\mathbb{R}^d;dx)$, then $\liminf_{n \to \infty} \mathcal{E}^{\varepsilon_n}(f_n, f) \geq \mathcal{E}^K(f, f)$. See [18, p. 726, the proof Theorem 4.7] for the proof.
(ii) Note that $C^\infty_c(\mathbb{R}^d) \subset \mathcal{F}$ is a core of both Dirichlet forms $(\mathcal{E}^{\varepsilon_n}, \mathcal{F}^{\varepsilon_n})$ and $(\mathcal{E}^K, \mathcal{F}^K)$. According to [35, Lemma 2.8], the condition (ii) in the definition of Mosco convergence holds true, if and only if for any $f \in C^\infty_c(\mathbb{R}^d)$, $\lim_{n \to \infty} \mathcal{E}^{\varepsilon_n}(f, f) = \mathcal{E}^K(f, f)$. See [4, Theorem 2.3] or [18, Lemma 8.2] for the case of the Mosco convergence without changing reference measures.

**Theorem 4.2.** Under assumption (H), there is $\Omega_0 \subset \Omega$ of full probability measure so that for any $\omega \in \Omega_0$ and any $\{\varepsilon_n\}_{n \geq 1}$ with $\varepsilon_n \to 0$ as $n \to \infty$, Dirichlet forms $(\mathcal{E}^{\varepsilon_n, \omega}, \mathcal{F}^{\varepsilon_n, \omega})$ convergence to $(\mathcal{E}^K, \mathcal{F}^K)$ in the sense of Mosco as $\lim_{n \to \infty} \varepsilon_n = 0$.

**Proof.** Throughout the proof, we drop the parameter $\omega$. According to (ii) and (iv) of assumption (H) and Proposition 2.1, we know that there is $\Omega_0 \subset \Omega$ of full probability measure so that for any $\omega \in \Omega_0$ and $f \in C^\infty_c(\mathbb{R}^d)$,

$$
\lim_{n \to \infty} \mathcal{E}^{\varepsilon_n}(f, f) = \mathcal{E}^K(f, f).
$$

By Remark 4.1(ii), property (ii) in the definition of Mosco convergence holds true. So it remains to verify property (i).

Let $\{f_n\}_{n \geq 1}$ be such that $f_n \in L^2(\mathbb{R}^d; \mu_{\varepsilon_n}(dx))$ for any $n \geq 1$ and converging weakly to $f$ in $L^2(\mathbb{R}^d; dx)$.

Without loss of generality, we assume that $\mathcal{E}^{\varepsilon_n}(f_n, f_n)$ converges and

$$
\sup_{n \geq 1} \left( \mathcal{E}^{\varepsilon_n}(f_n, f_n) + \|f_n\|_{L^2(\mathbb{R}^d; \mu_{\varepsilon_n}(dx))}^2 \right) < \infty,
$$

thanks to [41, Lemma 2.3]. For any $N > 0$, let $f^N := (-N) \vee (f(x) \wedge N)$. Note that $\mathcal{E}^{\varepsilon_n}(f^N_n, f^N_n) \leq \mathcal{E}^{\varepsilon_n}(f_n, f_n)$. According to (4.4), the fact $\|f^N_n\|_\infty \leq N$ and (i) in assumption (H), $\{f^N_n\}_{n \geq 1}$ is a pre-compact set in $L^1_{\text{loc}}(\mathbb{R}^d; dx)$. So there exist a measurable function $f^{*, N} \in L^1_{\text{loc}}(\mathbb{R}^d; dx)$ and a subsequence $\{f^N_{n_k}\}_{k \geq 1}$ such that for all $r \geq 1$,

$$
\lim_{k \to \infty} \|f^N_{n_k} - f^{*, N}\|_{L^1(B(0,r); dx)} = 0.
$$

Since $\|f^N_n\|_\infty \leq N$ for all $n \geq 1$, $\|f^{*, N}\|_\infty \leq N$ and so for any $r \geq 1$, $N > 0$ and $1 < q < \infty$,

$$
\lim_{k \to \infty} \|f^N_{n_k} - f^{*, N}\|_{L^q(B(0,r); dx)} = 0.
$$

Moreover, it is easy to see that $f^{*, N}(x) = f^{*, M}(x)$ for all $N \leq M$ and a.e. $x \in \mathbb{R}^d$ with $|f^{*, N}(x)| \leq N$. Therefore, we can find a measurable function $f_0 \in L^1_{\text{loc}}(\mathbb{R}^d; dx)$ such that $f^N_0(x) = f^{*, N}(x)$ for a.e. $x \in \mathbb{R}^d$ and $N > 0$. Combining this with (4.5) and the fact that $f_n$ converges weakly to $f$ yields that $f_0 = f$ a.e. and (4.5) holds with $f^{*, N}$ replaced by $f^N$. That is, for any $r \geq 1$, $N > 0$ and $1 < q < \infty$,

$$
\lim_{k \to \infty} \|f^N_{n_k} - f^N\|_{L^q(B(0,r); dx)} = 0.
$$

For notational simplicity, in the following we denote the subsequence $\{f^N_{n_k}\}_{k \geq 1}$ by $\{f_n\}_{n \geq 1}$.

We first suppose that $f \in C^\infty_c(\mathbb{R}^d)$. Take $N > \|f\|_\infty$; that is, $f(x) = f^N(x)$. Then, following the argument for (2.14) in the proof of Theorem 2.2 line by line,

$$
\lim_{n \to \infty} \mathcal{E}^{\varepsilon_n}(f^N_n, f) = \mathcal{E}^K(f, f),
$$

which along with Remark 4.1(i) gives us that

$$
\lim_{n \to \infty} \mathcal{E}^{\varepsilon_n}(f_n, f_n) \geq \lim_{n \to \infty} \mathcal{E}^{\varepsilon_n}(f^N_n, f^N_n) \geq \mathcal{E}^K(f, f).
$$

For the general case, we partly follow the proof of [18, Theorem 4.7]. Without loss of generality, we assume that $f \in \mathcal{F}^K$. For fixed $\gamma > 0$, we choose $f_\ast \in C^\infty_c(\mathbb{R}^d)$ such that

$$
\mathcal{E}^K(f - f_\ast, f - f_\ast) + \|f - f_\ast\|_{L^2(\mathbb{R}^d; dx)}^2 \leq \gamma^2.
$$

For any $\eta \in (0, 1)$, denote by

$$
\mathcal{E}^{\varepsilon_n}_\eta(f, f) = \frac{1}{2} \int_{\{\eta < |z| < 1/\eta, x \in \Gamma\}} (f(x + z) - f(x))^2 \frac{\kappa(0, z/\varepsilon_n; \tau_x/\varepsilon_n \omega)}{|z|^{d-\alpha}} \, dz \, dx
$$

and

$$
\mathcal{L}^{\varepsilon_n}_\eta f(x) = \int_{\{\eta < |z| < 1/\eta, x \in \Gamma\}} (f(x + z) - f(x)) \frac{\kappa(0, z/\varepsilon_n; \tau_x/\varepsilon_n \omega)}{|z|^{d+\alpha}} \, dz.
$$
Similarly, we can define $\mathcal{E}_n^K(f, f)$ and $\mathcal{L}_n^K f$. Then,

$$
|\mathcal{E}_n^K(f_n, f_\omega) - \mathcal{E}_n^K(f_\omega, f_\omega)| \leq \left| \langle f_n - f_\omega, (\mathcal{L}_n^K - \mathcal{L}_\omega^K) f_\omega \rangle \right|_{L^2(\mathbb{R}^d; dx)} + \left| \langle f_n - f_\omega, \mathcal{L}_\omega^K f_\omega \rangle \right|_{L^2(\mathbb{R}^d; dx)} + \left| \langle f_\omega - f_n, \mathcal{L}_n^K f_\omega \rangle \right|_{L^2(\mathbb{R}^d; dx)} + \left| \langle f - f_\omega, \mathcal{L}_n^K f_\omega \rangle \right|_{L^2(\mathbb{R}^d; dx)}
$$

$$
=: I_1 + I_2 + I_3 + I_4.
$$

By the Cauchy-Schwarz inequality,

$$
I_4 \leq \sqrt{\mathcal{E}_n^K(f_\omega, f_\omega) \mathcal{E}_n^K(f_\omega, f_\omega)} \leq \gamma \sqrt{\mathcal{E}_n^K(f_\omega, f_\omega) + \gamma^2}.
$$

Since $f_n \in C_0^\infty(\mathbb{R}^d)$, $\mathcal{L}_n^K f$ has a compact support. Due to (4.6), the argument for $I_{3, \gamma}^{\mathcal{L}}$ in the proof of Theorem 2.2 yields that $\lim_{n \to \infty} I_2 = 0$, thanks to (iii) in assumption (H). Similarly, using (iv) in assumption (H) and following the argument for $I_{3, \gamma}^{\mathcal{L}}$ in the proof of Theorem 2.2, we can also verify that $\lim_{n \to \infty} I_1 = 0$. Obviously, by the Cauchy-Schwarz inequality again,

$$
I_3 \leq \|f_n - f\|_{L^2(\mathbb{R}^d; dx)} \mathcal{L}_n^K f_\omega \|_{L^2(\mathbb{R}^d; dx)} = \|f I\{\{f_\omega > N\} \|_{L^2(\mathbb{R}^d; dx)} \| \mathcal{L}_n^K f_\omega \|_{L^2(\mathbb{R}^d; dx)}
$$

Therefore, putting all the estimates together, we find that

$$
\mathcal{E}_n^K(f_\omega, f_\omega) \leq \limsup_{n \to \infty} \mathcal{E}_n^K(f_n, f_\omega) + \gamma \sqrt{\mathcal{E}_n^K(f_\omega, f_\omega) + \gamma^2} + \|f I\{\{f_\omega > N\} \|_{L^2(\mathbb{R}^d; dx)} \| \mathcal{L}_n^K f_\omega \|_{L^2(\mathbb{R}^d; dx)}
$$

$$
\leq \limsup_{n \to \infty} \mathcal{E}_n^K(f_n, f_\omega) \sqrt{\mathcal{E}_n^K(f_\omega, f_\omega) + \gamma \sqrt{\mathcal{E}_n^K(f_\omega, f_\omega) + \gamma^2} + \|f I\{\{f_\omega > N\} \|_{L^2(\mathbb{R}^d; dx)} \| \mathcal{L}_n^K f_\omega \|_{L^2(\mathbb{R}^d; dx)}
$$

where in the second inequality we used the Cauchy-Schwarz inequality and (iv) in assumption (H). Letting $N \to \infty$, we obtain

$$
\mathcal{E}_n^K(f_\omega, f_\omega) \leq \sqrt{\limsup_{n \to \infty} \mathcal{E}_n^K(f_n, f_\omega) \sqrt{\mathcal{E}_n^K(f_\omega, f_\omega) + \gamma \sqrt{\mathcal{E}_n^K(f_\omega, f_\omega) + \gamma^2} + \|f I\{\{f_\omega > N\} \|_{L^2(\mathbb{R}^d; dx)} \| \mathcal{L}_n^K f_\omega \|_{L^2(\mathbb{R}^d; dx)}
$$

Note that

$$
\mathcal{E}_n^K(f_\omega, f_\omega) - \gamma \leq \mathcal{E}_n^K(f_\omega, f_\omega) \leq \mathcal{E}_n^K(f_\omega, f_\omega) + \gamma.
$$

Then, combining two inequalities above together, and letting $\gamma \to 0$ and $\eta \to 0$,

$$
\lim_{n \to \infty} \mathcal{E}_n(f_n, f_\omega) \geq \mathcal{E}(f_\omega, f_\omega).
$$

The proof is complete. \qed

We say that a sequence of bounded operators $\{T_n\}$ on $L^2(\mathbb{R}^d; \mu_{x_n}(dx))$ strongly converges to an operator $T$ on $L^2(\mathbb{R}^d; dx)$, if for every sequence $\{u_n\}_{n \geq 1}$ with $u_n \in L^2(\mathbb{R}^d; \mu_{x_n}(dx))$ for all $n \geq 1$ and strongly converging in $L^2$-spaces to $u \in L^2(\mathbb{R}^d; dx)$, the sequence $\{T_n u_n\}_{n \geq 1}$ strongly converges in $L^2$-spaces to $Tu$; see [41, Definition 2.6] or [35, Definition 2.4].

Let $(P_t^{\varepsilon, \omega})_{t \geq 0}$ and $(U_{\lambda}^{\varepsilon, \omega})_{\lambda > 0}$ be the semigroup and the resolvent associated with the Dirichlet form $(\mathcal{E}^{\varepsilon, \omega}, \mathcal{F}^{\varepsilon, \omega})$, respectively. Let $(P_t^K)_{t \geq 0}$ and $(U_{\lambda}^{K})_{\lambda > 0}$ be the semigroup and the resolvent associated with the Dirichlet form $(\mathcal{E}^K, \mathcal{F}^K)$, respectively. Then, by Theorem 4.2 and [41, Theorem 2.4], (also thanks to the fact that Theorem 4.2 holds for any $\{\varepsilon_n\}_{n \geq 1}$ that converges to 0 which is independent of $\omega \in \Omega_0$), we have

**Corollary 4.3.** Under assumption (H), there is $\Omega_0 \subset \Omega$ of full probability measure so that for every $\omega \in \Omega_0$, $P_t^{\varepsilon, \omega}$ strongly converges in $L^2$-spaces to $P_t^K$ for every $t > 0$ as $\varepsilon \to 0$; equivalently, $U_{\lambda}^{\varepsilon, \omega}$ strongly converges in $L^2$-spaces to $U_{\lambda}^K$ for every $\lambda > 0$ as $\varepsilon \to 0$.

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