A CRITICAL ACCRETION RATE FOR TRIGGERED STAR FORMATION

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ABSTRACT

We reexamine the similarity solution for a self-gravitating isothermal gas sphere and consider the implications for star formation in turbulent clouds. For adequately chosen parameters, the similarity solution describes an accreting isothermal gas sphere bounded by a spherical shock wave. The mass and radius of the sphere increase in proportion to time, while the central density decreases in proportion to the inverse square of time. The similarity solution is specified by the accretion rate and the infall velocity. The former has a critical value for a given infall velocity: when the accretion rate lies below this value, there exists a pair of similarity solutions determined by the given combination of accretion rate and infall velocity. One of these solutions is confirmed to be unstable against spherical perturbations. The implication is that the gas sphere will collapse to initiate star formation only when the accretion rate is higher than the critical value. We also examine the stability of the similarity solution against nonspherical perturbations, which are found to be damped.

Subject headings: accretion, accretion disks — hydrodynamics — shock waves — stars: formation

1. INTRODUCTION

Similarity solutions have contributed very much to our understanding of star formation processes. The classical solution by Larson (1969) and Penston (1969) elucidated the runaway nature of gravitational collapse. The density increases in proportion to the inverse square of time during the runaway-collapse phase. We learned from the similarity solutions of Shu (1977) and Hunter (1977) that the accretion rate of a protostar is of order \( c_s^2 / G \), where \( G \) and \( c_s \) denote the gravitational constant and the isothermal sound speed in the gas.

Similarity solutions have also been used to evaluate the effects of rotation and magnetic field. The solutions of Narita et al. (1984) and Saigo & Hanawa (1998) indicate that runaway collapse, once initiated, cannot be halted by rotation. The collapse of a rotating magnetized gas cloud is described by the similarity solution of Krasnopolsky & Königl (2002). Ambipolar diffusion is taken into account in the solution of Adams & Shu (2007). Tsai & Hsu (1995) extended the similarity solution to include a shock wave; Shu et al. (2002) further extended this solution for application to the “champagne flow” phase of \( \mathrm{H} \) II regions.

Tsai & Hsu (1995) found two classes of similarity solutions; the first describes accretion onto a protostar, while the second leads to a failure of star formation. The central density decreases in proportion to the inverse square of time \(( \rho_c \propto t^{-2} )\) in the second class. Although this second solution has not gained much attention, it can provide insight into the dynamical compression of a molecular cloud core. If a dense clump of gas is compressed by an external force, the increase in density over time may trigger gravitational collapse and star formation. One might surmise the existence of a threshold for gravitational collapse. If the dynamical compression is too weak or too brief, the clump will bounce back into expansion. A shock wave will be formed when the accreting gas is stopped by the expansion (see, e.g., Adams & Shu 2007). The similarity solution of Tsai & Hsu demonstrates that a spherical cloud can expand even when it is steadily compressed by a shock wave. On the other hand, a shock-compressed gas sphere will collapse owing to self-gravity if the shock is strong and lasts for a long enough period.

In this paper, we reexamine the similarity solution of Tsai & Hsu (1995) while keeping its competing aspects in mind: We find a condition for the existence of a similarity solution describing the expansion of a gas sphere. Conversely, this gives us the condition for a shock-compressed gas sphere to collapse under self-gravity. We also study the stability of the similarity solution; the solution denies collapse due to self-gravity only when it is stable.

We review the solution in § 2.1 and sketch the method of linear stability analysis in § 2.2. Technical details of the stability analysis are given in the Appendix. Properties of the similarity solutions, such as the accretion rate and infall velocity, are illustrated in § 3.1. The stability of the similarity solution is addressed in §§ 3.2 and 3.3. We discuss the implications of our analysis in § 4.

2. MODEL AND METHODS OF COMPUTATION

2.1. Similarity Solution

We consider an isothermal gas that is distributed in a spherically symmetric fashion. In this case, the hydrodynamic equations can be expressed as

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \rho v \right) = 0, 
\]
\[
\frac{\partial v}{\partial t} + \frac{\partial P}{\partial r} + \frac{1}{\rho} \frac{\partial \rho}{\partial r} + \frac{GM_r}{r^2} = 0, 
\]
\[
\frac{\partial M_r}{\partial r} = 4\pi r^2 \rho, 
\]

where

\[
P = c_s^2 \rho. 
\]

Here \( \rho, v, P, \) and \( M_r \) denote the density, velocity, pressure, and mass inside the radius \( r, \) and \( G \) and \( c_s \) are again the gravitational constant and the isothermal sound speed, respectively. As originally shown by Larson (1969) and Penston (1969), these equations have similarity solutions,

\[
\rho(r, t) = \frac{\rho(\xi)}{4\pi G t}, \quad v(r, t) = \frac{v(\xi)}{c_s}, \quad M_r(r, t) = \frac{c_s^2 I}{G} \mu(\xi),
\]

where
where
\[ \xi = r/(ct). \] (7)

We restrict ourselves to the case \( t > 0 \) in the following. Substituting equations (5) through (7) into equations (1)–(3), we obtain
\[ \frac{\partial \rho}{\partial \xi} = -\frac{\mu - 2(\xi - u)}{(u - \xi^2 - 1)\xi}, \frac{\partial u}{\partial \xi} = \frac{\mu - 2(\xi - u)}{(u - \xi^2 - 1)\xi}, \] (8)
\[ \mu = \xi(\xi - u)\varrho. \] (9)

We assume that the density is finite at the center, since we are interested in application to a starless core, that is, the case of no star formation. Thus, the density and velocity should be expressed as
\[ \rho = \rho_0 - \frac{\rho_0}{6} \left( \frac{2}{3} \right) \xi^2 + \frac{\rho_0}{45} \left( \frac{2}{3} \right) \left( \rho_0 - \frac{1}{2} \right) + O(\xi^6), \] (10)
\[ u = \frac{2\xi}{3} - \frac{\rho_0}{45} \left( \frac{2}{3} \right) \xi^3 + O(\xi^5) \] (11)

near the center. Following Tsai & Hsu (1995), we assume that the flow has a shock front at some \( \xi = \xi_{sh} \). Then the Rankine-Hugoniot relations give us the conditions
\[ \frac{\varrho_u}{\varrho_d} = \frac{u_u - \xi_{sh}}{u_u - \xi_{sh}}(u_u - \xi_{sh}) = 1, \] (12)
where \( \varrho_u \) and \( \varrho_d \) denote the densities of the pre- and postshock gas, respectively, and \( u_u \) and \( u_d \) are the corresponding velocities. The preshock flow occupies the region \( \xi > \xi_{sh} \).

Far from the origin, the solution of the density and velocity equations (eq. [8]) approaches the asymptotic solution
\[ \rho = \frac{\dot{M}}{v_{inf}\xi^2} + O(\xi^{-3}), \quad u = -v_{inf} + O(\xi^{-1}). \] (13)

Here \( v_{inf} \) denotes the infall velocity at infinity, while \( \dot{M} \) is the accretion rate. The similarity solution can be specified in terms of either \( (\dot{\varrho}, \xi_{sh}) \) or \( (v_{inf}, \dot{M}) \). Note that the Larson-Penston solution has the same asymptotic form. Only when \( v_{inf} \) and \( \dot{M} \) vanish does the solution of equation (8) have a different asymptotic form, the “plus solutions” of Shu (1977).

We integrate the density and velocity equations (eq. [8]) with the fourth-order Runge-Kutta method from \( \xi = 0 \) for a given set of \( \varrho_c \) and \( \xi_{sh} \) to obtain a similarity solution. The infall velocity and accretion rate are obtained numerically as a function of these variables. We also obtain the mass enclosed within the shock front,
\[ M_c = \int_0^{\xi_{sh}} \varrho \xi^2 d\xi. \] (14)

2.2. Spherical and Nonspherical Perturbations

We performed a normal-mode analysis to examine the stability of the similarity solution. In this analysis, the density is expressed as
\[ \rho(r, t) = \frac{\varrho(\xi) + \rho(\xi) Y_n^m(\theta, \varphi)}{4\pi G\varrho}, \] (15)
where \( Y_n^m(\theta, \varphi) \) denotes the spherical harmonic functions. This particular form was chosen because the similarity solution has no specific timescale. An eigenmode has a growth timescale and an unstable perturbation grows exponentially when the unperturbed state is stationary and has a specific timescale. When the physical quantities vary according to a power law of time in the unperturbed state, the eigenmode grows (or decays) in proportion to a power of time, \( |t|^m \). Hanawa & Matsumoto (1999) provides the justification for equation (15). There, the linear stability of the Larson-Penston solution against a nonspherical perturbation is analyzed, using the coordinates \( (\xi, \theta, \varphi, ln t) \) instead of the usual \( (r, \theta, \varphi, t) \). It is shown that the similarity solution can be expressed as a steady state and that unstable mode grows in proportion to \( \exp(\ln t) \) in these coordinates.

The power index, \( \sigma \), is obtained as an eigenvalue of the perturbation equations, as shown in the Appendix. When the real part of \( \sigma \) is positive, the mode grows with time and the similarity solution is unstable. We call the real part of \( \sigma \) the “growth index” in what follows. When \( \sigma \) is complex-valued, the mode grows (or decays) while also oscillating. The imaginary part of \( \sigma \) is related to the frequency of oscillation, although the physical oscillation period increases with time.

3. RESULTS

3.1. Similarity Solutions

First we obtained a series of similarity solutions for a given \( \varrho_c \) by increasing \( \xi_{sh} \). The infall velocity decreases monotonically with increasing \( \xi_{sh} \). It reaches \( v_{inf} = 0 \) at some \( \xi_{sh} \) and the solution terminates. By tabulating these similarity solutions, we obtain the accretion rate \( \dot{M} \) as a function of \( v_{inf} \) and \( \xi_{sh} \). In Figure 1, the curves show \( \dot{M} \) as a function of \( \xi_{sh} \) for different values of \( v_{inf} \). The accretion rate has a maximum at a certain \( \xi_{sh} \), for a given \( v_{inf} \). It increases with increasing \( \xi_{sh} \) at lower \( \xi_{sh} \) (thin curves), while it decreases with increasing \( \xi_{sh} \) at higher \( \xi_{sh} \) (thick curves). This means that there exist two similarity solutions for a given set of \( v_{inf} \) and \( \dot{M} \).

Figure 2 shows the two solutions for \( v_{inf} = 1.4 \) and \( M = 1.204 \). The solid lines denote the solution with \( \xi_{sh} = 0.986 \), while the dashed lines are that with \( \xi_{sh} = 0.574 \). Both have the same density and velocity distributions in the region \( \xi \geq 0.986 \). The main difference is the location of the shock front. The shock-compressed gas sphere is denser and expands more slowly in the solution denoted by the dashed lines. Since the expansion is slow, the solution is similar to the Bonnor-Ebert solution for a self-gravitating
isothermal gas sphere (Ebert 1955; Bonnor 1956). Like the Bonnor-Ebert sphere, the latter ($\xi_{sh} = 0.574$) solution is unstable (see § 3.2).

Figure 3 indicates that the similarity solution has a limiting value of $\dot{M}$ for a given $v_{inf}$ (solid curve). Here the upper limit is highest, $\dot{M}_{max} = 1.312$, at $v_{inf} = 1.64$. Similarity solutions do not exist for $\dot{M} > 1.312$. The implications of this nonexistence are discussed in § 4.

3.2. Spherical Perturbations

Figure 4 shows the growth index of a spherical perturbation, $\sigma_r$, as a function of $\xi_{sh}$ for a series of similarity solutions having a given $v_{inf}$. The solid curves denote the modes with real-valued growth indices. The dotted lines show the real part for complex indices. The imaginary part is shown in Figure 5. The $\sigma_r$ become positive, and the similarity solution is unstable, only when the shock radius $\xi_{sh}$ is smaller than a critical value. The condition for neutral stability coincides with that which yields the maximum accretion rate for a given $v_{inf}$, as expected. When $\xi_{sh}$ is a little larger than the critical value, the solution is stable, and the most slowly damped mode has a real index. For large enough $\xi_{sh}$, the solution is stable and the most slowly damped mode has a complex index.

3.3. $l = 2$ Mode

We have studied the $l = 2$ mode as a typical nonspherical perturbation. This is in part because the dipole ($l = 1$) mode is unlikely to be excited. If the dipole mode grows, the inner and outer parts of the gas sphere should move in opposite directions to maintain the center of gravity.

Figure 6 illustrates the eigenfrequencies of the $l = 2$ mode for similarity solutions with $v_{inf} = 1.0$. The abscissa denotes $\xi_{sh}$ while the ordinate denotes the index $\sigma$. All the modes are damped ($\sigma_r < -1$). The solid lines show the mode with the smallest negative growth index (or “damping index,” for brevity). This mode has a small imaginary part ($\sigma_r \approx \pm 0.1$). The mode having the second-smallest damping index is shown by the dashed line and has a real eigenfrequency (pure damping). The mode with the third-smallest damping index (dotted lines) has an imaginary part in the eigenfrequency. The imaginary part is similar to that of the mode with the smallest damping index. These three modes

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**Fig. 2.** Density and velocity distributions for two similarity solutions having the same $v_{inf}$ and $\dot{M}$.

**Fig. 3.** Illustration of the critical accretion rate $\dot{M}$, as a function of $v_{inf}$ (solid curve). The dashed curve denotes the asymptotic curve, $3.6/v_{inf}$.

**Fig. 4.** Real part of the eigenfrequency, $\sigma_r$, of a spherical perturbation as a function of the shock radius, $\xi_{sh}$ for a different values of $v_{inf}$.

**Fig. 5.** Same as Fig. 4, but for the imaginary part, $\sigma_r$. Our survey is limited to modes having low indices, that is, in the range $|\sigma_r| \leq 1.0$. It is however unlikely that we have missed unstable spherical perturbations. A spherical perturbation induces only sound waves, and these are confined within the gas sphere because it is bounded by the shock wave. A high-frequency sound wave has a shorter wavelength and is unlikely to be Jeans unstable.
have similar damping indices, $-1.1 \leq \sigma_r \leq -1.0$. The remaining mode (dash-dotted line) has a much larger damping index.

Figure 7 is the same as Figure 6, but for $v_{\text{inf}} = 4.0$. Again, all the modes are damped. The absolute value of the damping index is larger than unity ($\sigma_r < -1$). The oscillation frequency of the mode with the smallest damping index is larger than those for $v_{\text{inf}} = 1.0$.

One might ask why the similarity solution is stable against a bar mode ($l = 2$). We think that this mode is damped by the expansion of the gas sphere. Since the radius of the shock front increases with time, the asphericity of the shock-compressed gas sphere will decrease unless the displacement grows faster than the radius. When a gas sphere collapses, the bar mode can be excited, as shown by Lin et al. (1965) for a pressureless gas and Hanawa & Matsumoto (1999) for an isothermal gas. We have not yet studied modes with $l \geq 3$. However, they are also unlikely to be unstable, since self-gravity does not excite short-wavelength perturbations.

4. DISCUSSION

We have obtained a critical accretion rate above which there exists no similarity solution for a self-gravitating isothermal gas sphere. This can also be interpreted as the minimum accretion rate for a high-density clump to initiate self-gravitational collapse. The critical rate can be rewritten as

$$\frac{dM}{dt} \bigg|_{\sigma_c} = \frac{3.6c_s^4}{Gv}$$

(16)

for $v \gtrsim 3c_s$, in dimensional form.

Equation (16) gives us an estimate for a converging flow to initiate gravitational collapse resulting in star formation. Consider a spherical region, the surface of which is surrounded by a converging flow. The radius, inflow velocity, and density are taken to be $r$, $v$, and $\rho$, respectively. Then gravitational collapse will be initiated when the mass accretion rate exceeds the critical value,

$$4\pi r^2 \rho v > 3.6c_s^4/(Gv).$$

(17)

Equation (17) can then be rewritten as

$$2r/\lambda_J > 0.3c_s/v,$$

(18)

where

$$\lambda_J = 2\pi c_s/\sqrt{4\pi G\rho}.$$  

(19)

Since $\lambda_J$ represents the Jeans length, equation (18) means that the effective Jeans length decreases in proportion to the reciprocal of the Mach number.

The Jeans mass is proportional to the cube of the Jeans length for a given density. Thus, the effective Jeans mass should reduce to

$$M_{J,\text{eff}} = M_J \left(\frac{\lambda_{J,\text{eff}}}{\lambda_J}\right)^3 = M_J \left(\frac{v}{0.3c_s}\right)^3.$$  

(20)

This implies that compression of a sub–Jeans-mass clump may result in gravitational collapse in a region of flow convergence. Note that the effective Jeans mass is several orders of magnitude smaller than the classical value when $v \gtrsim 3c_s$.

The compression should have to continue over a certain period of time in order for a dynamically compressed clump to collapse by virtue of self-gravity. If we evaluate the minimum such time as the effective Jeans length divided by the flow velocity, we find that it is shorter than the free-fall timescale by a factor of the square of the Mach number,

$$\tau_{\text{comp}} \approx \lambda_{J,\text{eff}}/v \approx \tau_{\text{ff}}(v/c_s)^{-2}.$$  

(21)

This timescale can be translated into the perturbation wavelength. A compressed clump can collapse under self-gravity if the wave-length of velocity perturbation is longer than the effective Jeans length. If turbulence contains long-wavelength velocity perturbations, gravitational collapse due to dynamical compression will take place somewhere in the cloud. In such a case, we would expect a number of clumps with masses that are much smaller than the classical Jeans mass.

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LINEAR STABILITY ANALYSIS

Since the density is given by equation (15) in our linear stability analysis, the gravitational potential should be expressed as

\[ \Phi(r, t) = c_s^2 \left[ \psi(\xi) + f^l \delta \psi(\xi)J_l^m(\theta, \varphi) \right]. \]  

(A1)

The velocity, \( v = (v_r, v_\theta, v_\varphi) \), is assumed to be given by

\[ v_r(r, t) = c_s [u(\xi) + f^l \delta u_r(\theta, \varphi)], \quad v_\theta(r, t) = -c_s f^l \frac{\delta u_\theta(\xi)}{l + 1} \frac{\partial}{\partial \theta} J_l^m(\theta, \varphi), \quad v_\varphi(r, t) = -c_s f^l \frac{\delta u_\varphi(\xi)}{(l + 1)\sin \theta} \frac{\partial}{\partial \varphi} J_l^m(\theta, \varphi), \]  

(A2)

in the usual spherical coordinates. We assume that the perturbation is discontinuous between the pre- and postshock flows and contains a change due to the shifting of the shock front.

\[ (l + 1)\delta u_r + \frac{d}{d\xi} (\xi \delta u_\theta) = 0. \]  

(A4)

This assumption is based on Kelvin’s circulation theorem, which states that the circulation,

\[ \Gamma \equiv \oint_C u \cdot d\mathbf{u}, \]  

(A5)

is conserved for an isothermal (barotropic) gas (see, e.g., Shu 1992). Since the Lagrangian loop \( C \) is expanding, the circulation velocity should decrease with time in order to conserve \( \Gamma \), although it can grow with time if measured at a given \( \xi \). (See Hanawa & Matsumoto 2000 for more details on the mathematical treatment of perturbations having vorticity.)

After some manipulation, the perturbation equations can be rewritten as

\[ (\sigma + 1)\delta \varrho + \frac{1}{\xi^2} \frac{d}{d\xi} (\xi^2 \delta f) + \frac{1}{\xi} \frac{\partial}{\partial \theta} \delta \psi = 0, \]  

(A6)

\[ (\sigma + 2)\delta f + \frac{1}{\xi^2} [2w \delta f + (1 - w^2)] - \left( \varrho_0 w + \frac{2}{\xi} \right) \delta \varrho + \varrho_0 \delta \gamma + \frac{1}{\xi} \frac{\partial}{\partial \theta} \delta u_\theta = 0, \]  

(A7)

\[ \frac{d}{d\xi} \delta \gamma = \frac{2}{\xi} \delta \gamma - \frac{l(l + 1)}{\xi^2} \delta \psi, \]  

(A8)

where

\[ w = u - \xi, \quad \delta u_\theta = \frac{1 + 1}{(\sigma + 1)\xi} \left( w \delta u_r + \frac{\delta \varrho}{\varrho_0} + \delta \psi \right), \quad \delta f = \varrho_0 \delta u_r + w \delta \varrho, \quad \delta \gamma = \frac{\delta}{\delta \xi} \delta \psi. \]  

(A9)

The perturbation equations do not contain the perturbation in the potential, \( \delta \psi \), is continuous even at \( \xi = \xi_{sh} \). These perturbation equations enable us to connect the perturbation in the postshock flow (\( \xi < \xi_{sh} \)) with that in the preshock flow (\( \xi > \xi_{sh} \)).

After some manipulations we obtain the jump conditions,

\[ (w^+ - w^-)[\delta f] - (\varrho^+ - \varrho^-)[w \delta u_r + \delta \varrho/\varrho_0] = 0, \quad (\sigma + 1)[\delta \gamma] + [\delta f] = 0, \]  

(A10)

where brackets denote the difference between quantities evaluated at \( \xi = \xi_{sh} \pm \epsilon \). Variables with plus and minus subscripts similarly denote values at \( \xi = \xi_{sh} + \epsilon \) and \( \xi = \xi_{sh} - \epsilon \), respectively. The perturbation in the potential, \( \delta \psi \), is continuous even at \( \xi = \xi_{sh} \). These jump conditions enable us to connect the perturbation in the postshock flow (\( \xi < \xi_{sh} \)) with that in the preshock flow (\( \xi > \xi_{sh} \)).

When the perturbation is spherically symmetric (\( l = 0 \)), equations (A6) and (A8) are linearly dependent and we obtain for \( l = 0 \)

\[ \delta \gamma = -\delta f / (\sigma + 1). \]  

(A11)
Thus, we need two independent boundary conditions for \( l = 0 \) and three for \( l \neq 0 \). The spherical perturbation should vanish in the preshock flow, since the unperturbed flow is supersonic. Otherwise, the perturbation will diverge when the mode is unstable. Thus, the effective outer boundary is set at \( \xi = \xi_{sh} \), and the jump condition is applied there. The other boundary condition is set so that the velocity perturbation vanishes at the origin, \( \xi = 0 \), and is proportional to \( \xi \) near the origin. We integrated the perturbation equation numerically with the Runge-Kutta method from the origin and searched for the eigenvalue, \( \sigma \), by trial and error.

Nonspherical perturbations should also be regular at the origin and vanishingly small at infinity. We calculated asymptotic solutions around the origin and around infinity to examine the conditions for a perturbation to be regular according to Hanawa & Matsumoto (1999). Some asymptotic solutions are divergent, since the perturbation equations are singular at the origin and infinity. When \( l \neq 0 \), the perturbation should be expressed by the asymptotic solution

\[
\delta \rho = B \rho_0 \xi^l, \quad \delta u_r = -A l \xi^{l-1} - C (l+2) \xi^{l+1}, \quad \delta \psi = [A (\sigma + 1 - \frac{1}{2} l) - B] \xi^l, \tag{A12}
\]

where

\[
C = \frac{1}{4l + 6} \left[ l \left( \frac{2}{3} \phi_0 - \frac{2}{3} \right) A + \left( \sigma - \frac{l-1}{3} \right) B \right] \tag{A13}
\]

and \( A \) and \( B \) are arbitrary constants. The expressions for \( \delta \rho \) and \( \delta \psi \) in equation (A12) indicate that the density and potential perturbations can be expanded in a series of polynomials starting from the \( l \)th power in Cartesian coordinates. Otherwise the perturbation diverges at the origin. The velocity perturbation is also expanded in another series of polynomials starting from the \( (l - 1) \)th order, since it should balance the gradient of the potential perturbation.

The other boundary condition is given by the asymptotic solution

\[
\delta \rho = \frac{2D(l + 1) \rho_0}{(\sigma + l + 2)(\sigma + l + 3) \xi^{l+3}}, \quad \delta u_r = \frac{D(l + 1)}{(\sigma + l + 2) \xi^{l+2}}, \quad \delta \psi = \frac{D}{\xi^{l+1}}, \tag{A14}
\]

for \( \xi \gg 1 \). Here \( D \) is an arbitrary constant. The expression for \( \delta \psi \) in equation (A14) means that the potential perturbation in the region \( \xi \gg 1 \) is dominated by the aspherical density distribution near the center—in other words, the density perturbation is too small to affect the gravitational potential. The expression for \( \delta u_r \) indicates that the velocity perturbation is induced by the gravitational perturbation. Finally, the expression for \( \delta \rho \) means that the density perturbation is induced by the velocity perturbation. Thus, our boundary conditions guarantee that the perturbation is induced by internal changes in the flow.

We integrated the perturbation equations from both the origin (\( \xi = 10^{-2} \)) and a very large \( \xi \) (\( \approx 100 \)) with the Runge-Kutta method. We surveyed the eigenvalue \( \sigma \) by examining whether the numerical solutions from both ends satisfied the jump condition at the shock front.

REFERENCES

Adams, F. C., & Shu, F. H. 2007, ApJ, 671, 497
Bonnor, W. B. 1956, MNRAS, 116, 351
Ebert, R. 1955, Z. Astrophys., 37, 217
Hanawa, T., & Matsumoto, T. 1999, ApJ, 521, 703
Hunter, C. 1977, ApJ, 218, 834
Krasnopolsky, R., & Königl, A. 2002, ApJ, 580, 987
Larson, R. B. 1969, MNRAS, 145, 271
Lin, C. C., Mestel, L., & Shu, F. H. 1965, ApJ, 142, 1431
Narita, S., Hayashi, C., & Miyama, S. M. 1984, Prog. Theor. Phys., 72, 1118
Penston, M. V. 1969, MNRAS, 144, 425
Saigo, K., & Hanawa, T. 1998, ApJ, 493, 342
Shu, F. H. 1977, ApJ, 214, 488
———. 1992, The Physics of Astrophysics, Vol. 2, Gas Dynamics (Mill Valley, CA: Univ. Sci.), chap. 6
Shu, F. H., Lizano, S., Galli, D., Cantó, J., & Laughlin, G. 2002, ApJ, 580, 969
Tsai, J. C., & Hsu, J. J.-L. 1995, ApJ, 448, 774