q-analogs of the generalized Stirling and Bell numbers

Miguel Méndez (1) and Adolfo Rodríguez (2)

(1) Departamento de Matemática, Instituto Venezolano de Investigaciones Científicas, A.P. 21827, Caracas 1020–A, Venezuela.
(2) Laboratoire de Combinatoire et d'Informatique Mathématique (LaCIM) Université du Québec à Montréal, CP 8888, Succ. Centre-ville Montréal (Québec) H3C 3P8, Canada.
E-mail: (1) mmendezenator@gmail.com
E-mail: (2) rodriguez.adolfo@courrier.uqam.ca

Abstract. Generalized Stirling numbers appear in a natural way as the coefficients of the normal ordering of a word in the Heisenberg-Weyl algebra of bosonic creation and annihilation operators. We introduce a new combinatorial model for the study of the q-analogs of the generalized Stirling numbers. Using this combinatorial model we obtain explicit formulas, recursive formulas and generating functions for those q-generalized Stirling numbers.

1. Introduction
The problem of the normal ordering of expressions in boson creation $a^\dagger$ and annihilation $a$ operators, satisfying the commutation rule $[a, a^\dagger] = 1$, has been the motive of a considerable amount of research in recent years (see for example [1, 2, 3, 4, 5, 9, 12, 14, 18, 19]).

The interpretation of the normal order coefficients of a word of ordinary creation and annihilation operators as rook numbers was given by Navon [15]. An alternative combinatorial model was introduced recently in [14] in terms of bugs, colonies and settlements, giving recursive formulas, closed formulas and generating functions for the so called generalized Stirling and Bell numbers. In this paper we extend these results to the normal order problem of the q-bosonic operators of type $M$ (see [10, 11, 13, 16, 17]) satisfying the commutation rule

$$[a_q, a_q^\dagger]_q = a_q a_q^\dagger - q a_q^\dagger a_q = 1.$$ (1)

Using algebraic tools we prove in section 2 the Dobinsky relation for the generalized $q$-exponential polynomials and closed formulas for the generalized $q$-Stirling and Bell numbers. In section 3 we describe our combinatorial model. As an application of this model we give a completely combinatorial proof of a recursive formula for the generalized $q$-Stirling numbers.

2. q-analog of the generalized Stirling numbers
Given two sequences of positive integers $b = (b_1, b_2, \ldots, b_n)$ and $f = (f_2, f_3, \ldots, f_n)$, define $d_0 = 0$ and $d_j = \sum_{i=1}^{j} (b_i - f_i)$, for $j = 1, 2, \ldots, n$. Define the generalized numbers $S_{b,f}^q(k)$ as the coefficients (polynomials in $q$) that appear in the expansion
Note that expression (2) is normally ordered, and that the general problem of normal ordering of a $q$-boson string reduces to compute the coefficients $S_{b,f}^q(k)$. This coefficients $S_{b,f}^q(k)$ are called the generalized $q$-Stirling numbers. The $q$–analog of the generalized exponential polynomials, sometimes also called Bell polynomials, are defined as

$$\Phi_{b,f}^q(x) = \sum_k S_{b,f}^q(k)x^k. \quad (3)$$

The generalized $q$-Bell numbers are defined as

$$B_{b,f}^q = \sum_{k=f_1}^{f_1+f_2+\ldots+f_n} S_{b,f}^q(k) = \Phi_{b,f}^q(1). \quad (4)$$

Consider the $q$-analog of the derivative $D_q$,

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \quad (6)$$

Denoting by $X$ the operator that multiplies a function by the variable $x$, $X(f(x)) = xf(x)$, $X$ and $D_q$ satisfy the relation

$$D_q X = qX D_q + I. \quad (7)$$

Since $X$ and $D_q$ satisfy the same commuting relations than $a_q^\dagger$ and $a_q$, we have that

$$X^{b_n} D_q^{f_n} \ldots X^{b_2} D_q^{f_2} X^{b_1} D_q^{f_1} = X^{d_n} \sum_k S_{b,f}^q(k) X^k D_q^k. \quad (8)$$

For $|q| < 1$, define the $q$-analog $[m]$ of a natural number $m$ as

$$[m] = 1 + q + \ldots + q^{m-1} = \frac{1 - q^m}{1 - q}. \quad (9)$$

The $q$-analog of the falling factorial $[m]_k$, $n \geq k$, is defined as follows

$$[m]_k = [m][m-1][m-2]\ldots[m-k+1]. \quad (10)$$

In particular, when $k = m$, $[m]_m = [m]! = [m][m-1]\ldots1.$

For integers $n \geq k \geq 0$, define the $q$-binomial coefficient

$$\left[\begin{array}{c} m \\ k \end{array}\right] = \frac{[m]!}{[k]![m-k]!} = \frac{[m]_k}{[k]!} \quad (11)$$

The $q$–analogs of the exponential function $e^x$ are given by the following infinite products

$$e_q^x = \prod_{k=0}^{\infty} \frac{1}{1 + (q-1)q^kx}. \quad (12)$$
\[ E_q^x = \prod_{k=0}^{\infty} (1 - (q - 1)q^k x). \]  

We have the identities

\[ e^x_q = \sum_{m=0}^{\infty} \frac{x^m}{[m]!} \]  
\[ E^x_q = \sum_{m=0}^{\infty} q^m \frac{x^m}{[m]!} \]

It is clear from Eq. (12) and Eq. (13) that

\[ (e^x_q)^{-1} = E_q^{-x}. \]  

From Eq. (14), we obtain

\[ D_q e^x_q = e^x_q. \]

**Proposition 1.** Let \( p^q_{b,f}(m) = \prod_{j=1}^{n} [m + d_j]_{f_j} \). We have

\[ X^{b_1} D_{q_1}^{f_1} \cdots X^{b_n} D_{q_n}^{f_n} x^m = p^q_{b,f}(m)x^{m+d_n} \]  
\[ p^q_{b,f}(m) = \sum_k S^q_{b,f}(k)[m]_k \]

**Proof.** Eq. (18) is easy to prove. By Eq. (18) and (2) we have

\[ p^q_{b,f}(m)x^{m+d_n} = X^{b_1} D_{q_1}^{f_1} \cdots X^{b_n} D_{q_n}^{f_n} x^m = X^{d_n} \sum_k S^q_{b,f}(k)[m]_k x^m \]

\[ = \sum_k S^q_{b,f}(k)[m]_k x^{m+d_n}. \]

From that we obtain(19).

**Corollary 1.** (Dobinsky relations) We have the identity

\[ \Phi^q_{b,f}(x)e^x_q = \sum_{k=f_1}^{\infty} p^q_{b,f}(m)x^m/[m]!. \]

**Proof.** The coefficient of \( \frac{x^m}{[m]!} \) of the left hand side of (22) is equal to

\[ \sum_k \left[ \begin{array}{c} m \\ k \end{array} \right] S^q_{b,f}(k)[k]! = \sum_k S^q_{b,f}(k)[m]_k = p^q_{b,f}(m). \]

We obtain now explicit formulas for the generalized \( q \)-Stirling and Bell numbers.
Corollary 2. We have the formulas

\[ S_{b,f}^q(k) = \frac{1}{[k]!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} q^{\frac{j(j-1)}{2}} p_{b,f}^q(j) \]  \hspace{1cm} (24)

\[ B_{b,f}^q = \prod_{k=1}^{\infty} \left( 1 + (q-1)q^k \right) \sum_m \frac{p_{b,f}^q(m)}{m!}. \]  \hspace{1cm} (25)

Proof. By equation (22)

\[ \Phi_{b,f}^q(x) = E^{-x}q \sum_{m \geq 0} p_{b,f}^q(m) \frac{x^m}{m!}. \]  \hspace{1cm} (26)

Taking the coefficient of \( \frac{x^k}{[k]!} \) in both sides of Eq. (26) and using expression (15) for \( E^{-x}q \), we obtain (24). Eq. (25) is obtained by evaluating both sides of Eq. (26) in 1 using expansion (13) for \( E^{-x}q \).

3. The combinatorial approach

Varvak pointed out in [19] that the general coefficient in the normal ordered form of a generic \( q \)-bosonic word can be interpreted as a \( q \)-rook number. A different approach was given in [13] in terms of \( q \)-weighted Feynman diagrams. In this section we introduce an alternative combinatorial model that gives a combinatorial interpretation of the generalized \( q \)-Stirling numbers in terms of colonies of bugs. Using this model we give a simple combinatorial proof of a recursive formula for the generalized \( q \)-Stirling numbers.

Definition 1. A bug of type \((b, f)\) consists of a body and \( f \) legs. The body is formed by \( b \) linearly ordered empty cells. Each foot of the \( f \) legs is labelled with one number of an integer segment \( (m, m+f) = \{m+1, m+2, \ldots, m+f\} \). A worm is a bug with no legs.

Definition 2. Consider two totally ordered sets \( A \) and \( B \), where \( |A| \leq |B| \). A placement \( p \) of the elements of \( A \) into the elements of \( B \) is a injective function \( p : A \to B \). A can though of as a set of linearly ordered balls and \( B \) as a set of linearly ordered boxes (or empty cells), and a placement as a distribution of balls into boxes where no more than one ball can be placed in each box.

An internal crossing of \( p \) is a pair of elements \( a_1 \) and \( a_2 \) in \( A \), such that \( a_1 <_A a_2 \) and \( p(a_1) >_B p(a_2) \). An external crossing of \( p \) is a pair \((a, b)\), \( a \in A \) and \( b \in B \), such that \( p^{-1}(b) = \emptyset \) (\( b \) is an empty box) and \( p(a) >_B b \). A placement could also be represented as a distribution of the set \( A \) of feet of a bug into a set \( B \) of cells of a worm. The internal crossings are pictorially represented as the crossings of the legs of the bug. The external crossings as the crossings of the legs of the bug with the legs of a ghost bug whose unlabelled feet are placed without crossings in the empty cells (see Figure 2). The weight of a placement \( p \) is defined to be

\[ w(p) = q^{I(p)+E(p)}, \]  \hspace{1cm} (27)
Consider a set of $P$ bugs. Once the $(j-1)$th bug is placed, the $j$th bug is placed by putting some of its feet on the floor in one of the empty cells of the bodies of the preceding bugs. The pair of sequences $(b_1, f_1)$ and $(b_2, f_2)$ with labels in $(f_1, f_1 + f_2]$ and so on. A colony is one of the possible ways of organizing the bugs using the following procedure. The first bug stand on the ground. Once the $(j-1)$th bug is placed, the $j$th bug is placed by putting some of its feet on the floor in one of the empty cells of the bodies of the preceding bugs. The pair of sequences $(b, f)$, $b = (b_1, b_2, \ldots, b_n)$, $f = (f_1, f_2, \ldots, f_k)$, is called the type of the colony. The legs of the colony standing on the ground are called free (see Figure 3).

A settlement is a colony with no free legs. As a consequence of the definition of settlement, its first bug has to be a worm (placed in the floor). The subjacent colony of a settlement is that which is obtained by deleting the worm and placing the corresponding feet on the floor in increasing order of the labels. A settlement whose first bug has exactly $m$ cells will be referred to as an $m$-settlement. The type of a settlement is defined to be the type of its subjacent colony (see Figure 4).

In order to $q$-enumerate colonies (and settlements) in a convenient way we shall follow the next conditions when drawing them.

(i) The body of each new bug has to be put above all the bugs already placed.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{Placement $p$ with $E(p) = 3$, $I(p) = 4$, and $w(p) = q^7$.}
\end{figure}

where $I(p)$ and $E(p)$ respectively denotes the number of internal and external crossings. For a set $P$ of placements, define the $q$-cardinality of $P$ as $|P|_q = \sum_{p \in P} w(p)$.

**Proposition 2.** Denote by $P_{m, k}$ the set of placements of a $k$-element set $A$ into an $m$-element set $B$, and by $\tilde{P}_{m, k}$ the subset of $P_{m, k}$ containing only placements with not internal crossings. We have

\begin{align}
|P_{m, k}|_q &= [m]_k \quad (28) \\
|\tilde{P}_{m, k}|_q &= \left[ \frac{m}{k} \right] \quad (29)
\end{align}

**Proof.** To prove Eq. (28) observe that $|P_{m, 1}|_q = 1 + q + q^2 + \cdots + q^{m-1} = [m]$. Then, all we have to prove is that $|P_{m, k}|_q = |m| |P_{m-1, k-1}|_q$ for $k > 1$. Denote by $P_{m, k}^{(r)}$ the subset of $P_{m, k}$ were the first element of $A$ (i.e. the element numbered by 1) is placed into the $r$th element of $B$. It is easy to see that $|P_{m, k}^{(r)}|_q = q^{r-1} |P_{m-1, k-1}|_q$. The result follows because $P_{m, k} = \bigcup_{r=1}^m P_{m, k}^{(r)}$.

Notice that every element $p \in P_{m, k}$ can be identified with a pair $(\hat{p}, p')$, where $\hat{p} \in P_{m, k}$ and $p'$ is a placement in $P_{k, k}$ obtained by permuting the balls in the boxes occupied by $\hat{p}$. Since $w(p) = w(\hat{p})w(p')$, we have that $|\tilde{P}_{m, k}|_q |P_{k, k}|_q = |P_{m, k}|_q$. By Eq. (28) we obtain Eq. (29). \qed

**Definition 3.** Consider a set of $n$ bugs, the first one of type $(b_1, f_1)$ and feet labelled with labels in $(0, f_1)$, the second with type $(b_2, f_2)$ with labels in $(f_1, f_1 + f_2]$ and so on. A colony is one of the possible ways of organizing the bugs using the following procedure. The first bug stand on the ground. Once the $(j-1)$th bug is placed, the $j$th bug is placed by putting some of its $f_j$ feet on the floor and the rest on each of the rest in one of the empty cells of the bodies of the preceding bugs. The pair of sequences $(b, f)$, $b = (b_1, b_2, \ldots, b_n)$, $f = (f_1, f_2, \ldots, f_n)$, is called the type of the colony. The legs of the colony standing on the ground are called free (see Figure 3).

A settlement is a colony with no free legs. As a consequence of the definition of settlement, its first bug has to be a worm (placed in the floor). The subjacent colony of a settlement is that which is obtained by deleting the worm and placing the corresponding feet on the floor in increasing order of the labels. A settlement whose first bug has exactly $m$ cells will be referred to as an $m$-settlement. The type of a settlement is defined to be the type of its subjacent colony (see Figure 4).

In order to $q$-enumerate colonies (and settlements) in a convenient way we shall follow the next conditions when drawing them.

(i) The body of each new bug has to be put above all the bugs already placed.
(ii) Each new free leg has to be placed at the right hand side of the free legs already placed.

(iii) If one leg of the \( j \)th bug is placed in the cell of the \( i \)th bug, the arc of this leg has to pass at the right hand side of the bodies of the \( j - 1, j - 2, \ldots, i + 1 \) bugs.

(iv) The upper part of the legs of each bug has to be drawn from left to right following the order of the feet labels.

(v) After placing all the bugs, a ghost bug is placed. Its legs do not cross among them and occupy all the empty cells of the colony (settlement). The cells occupied by the ghost legs are called free.

**Definition 4.** The weight of a colony \( C \), \( w(C) \), is defined to be \( q^\#C \), where \( \#C \) is the number of crossings of the legs in the colony. Observe that the weight of an \( m \)-settlement \( S \), \( w(S) \), is the weight of the subjacent colony times the weight of the placement of the free legs into the \( m \) floor cells. Denote respectively by \( C_{b,f}(k) \) and by \( T_{b,f}(m) \) the sets of colonies of type \( (b,f) \) with exactly \( k \) free legs and the set of \( m \)-settlements of the same type. The \( q \)-cardinal of each of them is defined as the sum of the weights of their respective elements,

\[
|C_{b,f}(k)|_q = \sum_{C \in C_{b,f}(k)} w(C) \quad (30)
\]

\[
|T_{b,f}(m)|_q = \sum_{S \in T_{b,f}(m)} w(S) \quad (31)
\]

**Theorem 1.**

We have the following identities

\[
|T_{b,f}(m)|_q = p_{b,f}(m), \quad (32)
\]

\[
|C_{b,f}(k)|_q = S_{b,f}(k). \quad (33)
\]

**Proof.** To prove Eq. (32) let us assume that we have an \( m \)-settlement \( S_1 \) of type \((b_1, b_2, \ldots, b_{j-1}), (f_1, f_2, \ldots, f_{j-1})\), \( j \) an integer between 1 and \( n - 1 \). Adding the \( j \)th bug,
of type $(b_j, f_j)$, will produce a settlement $S_2$ of type $((b_1, b_2, \ldots, b_j), (f_1, f_2, \ldots, f_j))$. The weight of $S_2$ is equal to the weight of $S_1$ times $q^l$, where $l$ is the number of crossing legs produced when positioning the feet of the $j$th bug. This positioning is exactly a placement of the $f_j$ feet of the bug into the free cells of $S_1$ linearly ordered from left to right and from top to bottom. The reader can check that conditions (i)-(v) of above assure that the counting of crossing legs and the counting of crossings according with definition 2 coincide. The number of free cells in $S_1$ is equal to $m + d_{j-1}$. Then

$$|T_{b,f}(m)|_q = |P_{m,f_1}|_q |P_{m+d_1,f_2}|_q \cdots |P_{m+d_{n-1},f_n}|_q = p^q_{b,f}. \quad (34)$$

Denote by $T_{b,f}(m,k)$ the set of $m$–settlements of type $(b,f)$ with exactly $k$ feet placed in the floor cells. It is clear that

$$|T_{b,f}(m)|_q = \sum_{k=f_1}^m \sum_{S \in T_{b,f}(m,k)} w(S) \quad (35)$$

Since every settlement $S$ in $T_{b,f}(m,k)$ consists of a colony $C \in C_{b,f}(k)$ and a placement of the $k$ feet of $C$ in the $m$ floor cells, we have

$$\sum_{S \in T_{b,f}(m,k)} w(S) = \sum_{C \in C_{b,f}(k)} w(C) \sum_{p \in P_{m,k}} w(p) = |C_{b,f}(k)|_q |m| \quad (36)$$

Then, by equation (32)

$$p^q_{b,f}(m) = \sum_{k=f_1}^m |C_{b,f}(k)|_q |m|. \quad (38)$$

By Eq. (19), Identity (33) follows.

We introduce the notation $b \uplus b_{n+1} = (b_1, b_2, \ldots, b_n, b_{n+1})$ and $f \uplus f_{n+1} = (f_1, f_2, \ldots, f_n, f_{n+1})$. 

Figure 4. A 6-settlement of type $(b, f) = ((2, 3, 1), (3, 2, 3))$ and weight $q^{11}$. 

\[ \text{Symmetry and Structural Properties of Condensed Matter IOP Publishing} \]

\text{Journal of Physics: Conference Series 104 (2008) 012019 doi:10.1088/1742-6596/104/1/012019} \]
where the internal sum ranges over the set of all possible placements into the free cells of $C$. This is because $d_n = \sum_{i=1}^n b_i - f_i$ and $k - j = d_n - k - f_{n+1}$. The number of free cells in each colony $C_{n+1}$ is obtained by placing a bug $B_{n+1}$ on a colony $C_n$ of type $(b,f)$. If the bug has exactly $j$ legs in the floor, $C_n$ has to be in $C_{b,f}(k-j)$. The number of crossings in $C_{n+1}$ is equal to $w(C_n)q^{nc(p(B_{n+1}))}$, where $nc(p(B_n))$ is the number of new crossing produced by the placement $p$ of the feet of the bug $B_{n+1}$. Then

$$S^q_{b\oplus b_{n+1}, f_{n+1}}(k) = |C_{b\oplus b_{n+1}, f_{n+1}}(k)|_q = \sum_{j=0}^{f_{n+1}} |C_{b,f}(k-j)|_q \sum_{p} w(C_n)q^{nc(p(B_{n+1}))}$$

where the internal sum ranges over the set of all possible placements $p$ of $f_{n+1}-j$ feet of $B_{n+1}$ into the free cells of $C_n$. The number of free cells in each colony $C_n \in C_{b,f}(k-j)$ is equal to $(\sum_{i=1}^n b_i - f_i) + k - j = d_n + k - j$ (Number of cells - Number of legs + Number of free legs). The number of free cells in each colony $C_n$ after placing the feet of $B_{n+1}$ is equal to $d_n + k - j - (f_{n+1} - j) = d_n + k - f_{n+1}$. We claim that

$$\sum_{p} q^{nc(p(B_{n+1}))} = \left| P_{f_{n+1},j} \right|_q |P_{f_{n+1}-j,d_n+k-j}|_q q^{j(d_n+k-f_{n+1})}$$

$$= \left[ f_{n+1} \atop j \right] [d_n + k - j]_{f_{n+1}-j} q^{j(d_n+k-f_{n+1})}. \quad (43$$

This is because $|P_{f_{n+1},j}|_q q$-counts the crossings when choosing the set of $j$ legs to be placed in the floor (a placement without internal crossings). $|P_{f_{n+1}-j,d_n+k-j}|_q q$-counts the crossings
when placing the $f_{n+1} - j$ feet into the $d_n + k - j$ free cells of $C_n$. Finally, $q^{i(d_n+k-f_{n+1})}$ is the number of crossings of the $j$ floor legs with the ghost legs that occupy the $d_n + k - f_{n+1}$ free cells of $C_n$ after placing the legs of $B_{n+1}$ (see Figure 5).

**Example 1.** Consider the case all the bugs are of type $(1,1)$, $(b,f) = ((1,1, \ldots, 1), (1,1, \ldots, 1))$. The Stirling numbers $S_{b,f}^q(k) = S_{(1,1, \ldots, 1), (1,1, \ldots, 1)}^q(k) = S^q(n,k)$ are one of the two types of the classical $q$-Stirling numbers of the second kind studied in the literature (see [8]). We recover the recursive formula

$$S^q(n+1,k) = [k]S^q(n,k) + q^{k-1}S^q(n,k-1).$$

The combinatorial interpretation of $S^q(n,k)$ in terms of colonies of $n$ bugs of type $(1,1)$ with $k$ free legs is equivalent to the intertwining combinatorial interpretation in [6] (see Figure 6), and also to the definition by Garsia and Remmel in [7].

**Example 2.** In the case where all the bugs are of type $(r,1)$, the colonies can be represented as forests of $r$-ary increasing trees (see [14], section v). Using the notation $S_{(r,1)}^q(k) = S_{(r,r, \ldots, r), (1,1, \ldots, 1)}^q(k)$, the general recursive formula gives us

$$S_{(r,1)}^{q(n+1)}(k) = [n(r-1) + k]S_{(r,1)}^q(k) + q^{n(r-1)+k-1}S_{(r,1)}^q(k-1).$$

Hence, the general approach for $q$-counting colonies given here provides new statistics for the $q$-counting of forests of increasing trees and other families of combinatorial structures. In a subsequent publication we shall study this kind of applications of the present theory.

**Acknowledgments**

M. Méndez wish to thank James Louck, a source of encouragement and intellectual stimulus while finishing this research.
References

[1] Blasiak P, Horzela A, Penson K, Duchamp G and Solomon A 2005 Boson Normal Ordering via Substitutions and Sheffer-type Polynomials Phys. Lett. A 338 108
[2] Blasiak P, Penson K and Solomon A 2003 The Boson Normal Ordering Problem and Generalized Bell Numbers, Ann. Comb. 7 127
[3] Blasiak P, Penson K and Solomon A 2003 The general boson normal ordering problem Phys. Lett. A 309 198-205
[4] Blasiak P, Penson K and Solomon A 2004 Combinatorial coherent states via normal ordering of bosons Lett. Math. Phys. 67 13-23
[5] Blasiak P, Penson K, Solomon A, Horzela A and Duchamp G 2005 Some useful combinatorial formulas for bosonic operators, J. Math. Phys. 46
[6] Ehrenborg R and Readdy M 1996 Juggling and applications to q-analogues Discrete Math. 157 107-125
[7] Garsia A and Remmel J 1986 q-Counting rook configuration and a formula of Frobenius J. Combin. Theory Ser. A 41 246-275
[8] Gould H 1961 The q-Stirling numbers of the first and second kinds Duke Math. J. 32 281-289
[9] Katriel J 2000 Bell numbers and coherent states Phys. Lett. A 237 159-161
[10] Katriel J and Kibler M 1992 Normal ordering for deformed boson operators and operator-valued deformed Stirling numbers, J. Phys. A: Math. Gen. 25 2683-2691
[11] Katriel J and Duchamp G 1995 Ordering relations for q-boson operators, continued fractions techniques, and the q-CBH enigma, J. of Phys. A 28 7209-7225
[12] Mansour T, Schork M and Severini S A 2006 generalization of the boson normal ordering Preprint arXiv:quant-ph/0608081
[13] Mansour T, Shork M and Severini S 2007 Wicks's theorem for q-deformed boson operators Preprint arXiv:quant-ph/0703086v1
[14] Méndez M, Blasiak P, Penson K 2005 Combinatorial approach to generalized Bell and Stirling numbers and boson normal ordering problem J. Math. Phys. 46 1
[15] Navon A M 1973 Combinatorics and fermion algebra Nuovo Cimento 16 324-330
[16] Schork M 2006 Normal ordering q-bosons and combinatorics Phys. Lett. A 355 293-297
[17] Schork M 2003 On the combinatorics of normal ordering bosonic operators and deformations of it J. Phys. A: Math. Gen. 36 4651-4665.
[18] Solomon A, Duchamp G, Blasiak P, Horzela A and Penson K 2004 Normal Order: Combinatorial Graphs, in 3rd International Symposium on Quantum Theory and Symmetries World Scientific Publishing arXiv:quant-ph/0402082
[19] Varvak A 2004 Rook numbers and the normal ordering problem, in 16th Annual International Conference on Formal Power Series and Algebraic Combinatorics Vancouver B.C., Canada arXiv:math.CO/0402376