ON THE GROTHENDIECK RING OF THE SEQUENCE OF WALLED BRAUER ALGEBRAS

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Abstract. In this paper, we provide a diagrammatic approach to study the branching rules for cell modules on a sequence of walled Brauer algebras. This approach also allows us to calculate the structure constants of multiplication over the Grothendieck ring of the sequence.

1. Introduction

The walled Brauer algebra $B_{r,s}(\delta)$ is a subalgebra of the Brauer algebra $B_{r+s}(\delta)$, which was introduced independently by Koike [10] and Turaev [13]. As a generalization of the classical Schur-Weyl duality, the centralizer of the natural action of $GL_n(\mathbb{C})$ on a mixed tensor space $V^\otimes r \otimes W^\otimes s$, with $V = \mathbb{C}^n$ and $W = V^*$, was characterized as the walled Brauer algebra $B_{r,s}(n)$, for $n \geq r+s$. Walled Brauer algebras have been extensively studied, including the cellularity, semi-simplicity, decomposition numbers, Jucys-Murphy elements, block theory, Kazhdan-Lusztig theory and so on. We refer the reader to [1, 3, 4, 5, 9, 12] for details.

An important feature of a diagram algebra is that a low-dimensional algebra can be injected naturally as a subalgebra of a high-dimensional algebra. This enables one to consider diagram algebras in the tower framework. Wang [14] studied a tower of Temperley-Lieb algebras $TL_n$ and introduced the concept of walled modules to study the branching rules for cell modules of algebras $TL_m \otimes TL_n$ and $TL_m+n$. Using this concept, the structure constants of multiplication over the Grothendieck group can be calculated. The present study aimed to generalize Wang’s result [14] to walled Brauer algebras.

Note that the tensor product of two Temperley-Lieb diagrams $A$ and $B$ is defined as the juxtaposition, that is, diagram $A$ is to the left of diagram $B$. However, this definition cannot be generalized to walled Brauer diagrams directly since the juxtaposition is no longer a walled Brauer diagram. To overcome this obstacle, we introduce the so-called twisted tensor product of two walled Brauer diagrams. Interestingly, the twisted tensor product of walled Brauer algebras is actually isomorphic to their ordinary tensor product. As a result, we obtain the sequence of walled Brauer algebras with embedding

$$\rho_{r,s,n,m} : B_{r,s}(\delta) \otimes_k B_{n,m}(\delta) \rightarrow B_{r+n,s+m}(\delta).$$

Using the tensor product, we can investigate the restriction and induction functors between the module categories of algebras $B_{r,s}(\delta) \otimes_k B_{n,m}(\delta)$ and $B_{r+n,s+m}(\delta)$. In this process, the double-walled module, which can be viewed as a generalization of walled modules, is of primary importance. With these preparations, the structure constants of the Grothendieck ring for a sequence of walled Brauer algebras is calculated finally.

The paper is organized as follows. In Section 2, we provide a quick review of the cellular structure of a walled Brauer algebra. In Section 3, we introduce the so-called twisted tensor product of the walled Brauer algebra, which is proved to be isomorphic to the tensor product.
algebra. In order to characterize the restriction of cell modules, we study in Section 4
the double walled modules, and based on the main result of this section, we calculate the
structure constants of the Grothendieck ring for a sequence of walled Brauer algebras in
Section 5.

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2. Preliminaries

In this section, we shall provide a quick review of the definition and cellular structure of
walled Brauer algebras.

2.1. Definition of walled Brauer algebra. Let \( k \) be a field and \( \delta \in k \). For \( n \in \mathbb{N} \), recall
that \( \text{Brauer algebra} \ B_n(\delta) \) has a basis that includes all partitions of \( \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\} \) with
each part in a partition just being a pair. Such a partition can be represented by a so-called
\( n \)-diagram as follows. The diagram consists of two rows of \( n \) dots, with the top row labeled
by \( 1, \ldots, n \) in order from left to right, and similarly, with the bottom row labeled by \( \bar{1}, \ldots, \bar{n} \).
Then the dots belonging to the same part are joined with a smooth curve, i.e., an edge. An
edge is called a \textit{propagating edge} if it connects two dots in different rows; else, it is called an
\textit{arc}.

The multiplication \( A \cdot B \) of two \( n \)-diagrams \( A \) and \( B \) is given by concatenation, i.e., by
stacking \( A \) on the top of \( B \), identifying the bottom dots of \( A \) with the top dots of \( B \), and
following the lines from the top to bottom or within one row. Note that the result diagram
may contain some closed curses. The number of these closed curses is denoted by \( t \). Then
the multiplication \( A \cdot B \) is defined to be \( \delta^t C \), where \( C \) is the diagram obtained by removing
the aforementioned closed curses.

For two natural numbers \( r \) and \( s \), the \textit{walled Brauer algebra} \( B_{r,s}(\delta) \) (sometimes denoted
by \( B_{r,s} \) briefly) is a subalgebra of \( B_{r+s}(\delta) \). Its a basis consists of the so-called walled Brauer
diagrams satisfying certain conditions. Given an \( r+s \) Brauer diagram, a vertical line (wall)
is added to separate the first \( r \) top and bottom dots from the remainder. Then, an \( (r, s) \)-
walled Brauer diagram requires that no propagating edge crosses the wall and that every
arc does cross the wall. Let us illustrate the product of two walled Brauer diagrams when
\( n = 8, r = 3, \) and \( s = 5 \) (see Figure 1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Multiplication of two walled Brauer diagrams}
\end{figure}
The following lemma on the multiplication of diagram algebras is well-known. We write it here for the sake of description convenience later. It is helpful to point out that the lemma is valid for not only walled Brauer diagrams, but also Temperley-Lieb diagrams, partition diagrams, and so on.

**Lemma 2.1.** The number of propagating edges never increases and the number of arcs never decreases in a concatenation of two diagrams.

### 2.2. Cellular structures

Cellular algebras were introduced by Graham and Lehrer \[7\] in 1996 in order to study the non-semisimple specializations of many important algebras, including Hecke algebras \[6\], Brauer algebras, Birman-Wenzl algebras \[15\] and so on. In \[5\], Cox et al. proved that the walled Brauer algebras are cellular. In this section, we review the cellular structure.

Denote by \(\Sigma_n\) the symmetric group on \(n\) letters. Evidently, each element in \(\Sigma_n\) can be represented by an \(n\)-diagram without arcs. Denote \(\Sigma_i \times \Sigma_j\) by \(\Sigma_{i,j}\). If we regard the diagrams in \(\Sigma_{i,j}\) as the juxtaposition of diagrams in \(\Sigma_i\) and \(\Sigma_j\), then \(\Sigma_{i,j}\) is a subset of \(\Sigma_{i+j}\).

Moreover, we need to define a \(k\)-linear space consisting of half diagrams. Given an \((r,s)\)-walled diagram with \(l\) arcs in each row, we can obtain an upper “half diagram” by cutting the diagram horizontally in half. Considering the example of \(D_1\) (subsection 2.1), we get an upper \((3,5,2)\)-half diagram as follows.

\[
\begin{array}{c}
\text{Figure 2. Half diagram}
\end{array}
\]

Denote by \(V_{r,s}^l\), the \(k\)-linear space spanned by all \((r,s,l)\)-half diagrams. Then \(V_{r,s}^l\) has a natural \(B_{r,s}(\delta)\)-module structure with \(B_{r,s}(\delta)\)-action defined by the multiplication of diagrams. Hereinafter, \(V_{r,s}^l\) is called an \((r,s,l)\)-partial diagram module. Note that the result of this action is zero when the number of propagating edges decreases.

In \[5\], the walled Brauer algebra \(B_{r,s}(\delta)\) was proved to be the iterated inflation of group algebras \(k\Sigma_{i,j}\) along \(V_{r,s}^l\), and consequently, it is cellular according to the inflation theory introduced by Koenig and Xi in \[11\]. More precisely, there is a chain of two-sided ideals \(0 = J_0 \subseteq J_1 \subseteq \ldots \subseteq J_n = B_{r,s}(\delta)\) such that each subquotient \(J_i/J_{i-1}\) (called a layer) is a non-unital algebra of the form \(V_{r,s}^l \otimes V_{r,s}^l \otimes k\Sigma_{r-l,s-l}\). For arbitrary \(x \in J_i/J_{i-1}\) and \(y \in J_k/J_{k-1}\), we have \(xy \in J_{\min(l,k)}\), that is, the product cannot move to a higher layer.

Now, we give more details about this cellular structure. Let \(\lambda^L = (\lambda^L_1, \lambda^L_2, \ldots, \lambda^L_{l'})\) be a partition of \(r - l\), and \(\lambda^R = (\lambda^R_1, \lambda^R_2, \ldots, \lambda^R_{l''})\) a partition of \(s - l\). It is well-known that the cell modules of \(k\Sigma_{r-l,s-l}\) are precisely those modules of the form \(S^{\lambda^L} \boxtimes S^{\lambda^R}\), where \(S^{\lambda^L}\) is a Specht module of \(k\Sigma_{r-l}\) and \(S^{\lambda^R}\) a Specht module of \(k\Sigma_{s-l}\), and thus the modules can be labeled by pairs \((\lambda^L, \lambda^R)\). For each integer \(0 \leq l \leq \min(r,s)\), we set

\[
\Lambda_{r,s}(l) := \{ (\lambda^L, \lambda^R) \mid \lambda^L \vdash r - l; \lambda^R \vdash s - l \} \quad \text{and} \quad \Lambda_{r,s} := \bigcup_{l=0}^{\min(r,s)} \Lambda_{r,s}(l).
\]

Then, the cell modules of algebra \(B_{r,s}(\delta)\) are indexed by set \(\Lambda_{r,s}\) (see \[5\] for details), and cell module \(\Delta_{r,s}(\lambda^L, \lambda^R)\) has the form \(V_{r,s}^l \otimes (S^{\lambda^L} \boxtimes S^{\lambda^R})\). The \(B_{r,s}(\delta)\)-action on \(V_{r,s}^l \otimes (S^{\lambda^L} \boxtimes S^{\lambda^R})\) is defined as follows.
Given an \((r, s)\)-walled diagram \(x\) and a pure tensor \(v \otimes s \in V_{r,s}^{l} \otimes (S^{\lambda_{k}} \boxtimes S^{\lambda_{n}})\), we define
\[
x(v \otimes s) = \begin{cases} 
(xv) \otimes \pi(x,v)s & \text{if } xv \text{ has } l \text{ crossed arcs}, \\
0 & \text{otherwise},
\end{cases}
\]
where \(xv\) is as given above, and \(\pi(x,v) \in \Sigma_{r-l,s-l}\) is the permutation on the labeled dots of \(xv\).

To conclude this section, we generalize [8 Proposition 3] from Brauer algebras to walled Brauer algebras.

**Proposition 2.2.** Keep the notations as above. Let \(M\) and \(N\) be \(k\Sigma_{r-l,s-l}\)-modules, and let \(V_{r,s}^{l}\) be a partial diagram module of \(B_{r,s}\). Then
\[
\text{Hom}_{B_{r,s}}(V_{r,s}^{l} \otimes M, V_{r,s}^{l} \otimes N) \cong \text{Hom}_{k\Sigma_{r-l,s-l}}(M, N).
\]

**Proof.** Denote by \(v_{0} \in V_{r,s}^{l}\) the half diagram
\[
\begin{array}{c}
\cdots \cdots \cdots \\
\vdots \quad \quad \vdots \\
\delta \quad \quad \delta \\
\vdots \quad \quad \vdots \\
\cdots \cdots \cdots \\
\end{array}
\]
and define an idempotent \(e_{r,s,l}\) to be
\[
e_{r,s,l} = \frac{1}{d!} \begin{array}{c}
\cdots \cdots \cdots \\
\vdots \quad \quad \vdots \\
\delta \cdot \delta \\
\vdots \quad \quad \vdots \\
\cdots \cdots \cdots \\
\end{array}
\]
Let \(\phi \in \text{Hom}_{B_{r,s}}(V_{r,s}^{l} \otimes M, V_{r,s}^{l} \otimes N)\). For arbitrary \(m \in M\), assume that
\[
\phi(v_{0} \otimes m) = \sum_{i \in I} v_{i} \otimes n_{i}.
\]
Apply the idempotent \(e_{r,s,l}\) on both sides of the equality above. Note that
\[
e_{r,s,l}(v_{0} \otimes m) = v_{0} \otimes m.
\]
Moreover, each \(e_{r,s,l} \cdot v_{i}\) is either zero or a multiple of \(v_{0}\) and thus there exists certain \(n \in N\) such that
\[
e_{r,s,l} \cdot \sum v_{i} \otimes n_{i} = v_{0} \otimes n.
\]
Consequently, we obtain a map \(\hat{\phi}(m) = n\) from \(M\) to \(N\) induced by \(\phi\) and it is easy to check that \(\hat{\phi} \in \text{Hom}_{k\Sigma_{r-l,s-l}}(M, N)\).

On the other hand, for each \(v \in V_{r,s}^{l}\) there exists an element \(a \in B_{r,s}\) such that \(v = av_{0}\). Therefore, for each \(m \in M\) and \(v \in V_{r,s}^{l}\) we have
\[
\phi(v \otimes m) = \phi((av_{0}) \otimes m) = \phi(a(v_{0}) \otimes \pi(a,v_{0})\pi^{-1}(a,v_{0})m)
= a\phi(v_{0} \otimes \pi^{-1}(a,v_{0})m) = a(v_{0} \otimes \hat{\phi}(\pi^{-1}(a,v_{0})m))
= v \otimes \pi^{-1}(a,v_{0})\hat{\phi}(\pi^{-1}(a,v_{0})m) = v \otimes \hat{\phi}(m).
\]
That is, given a homomorphism \(\hat{\phi} \in \text{Hom}_{k\Sigma_{r-l,s-l}}(M, N)\), we can define a map \(\phi\) from previous equalities. Furthermore, we claim that \(\phi\) is a \(B_{r,s}\)-homomorphism. In fact, for arbitrary \(d \in B_{r,s}\),
\[
d\hat{\phi}(v \otimes m) = d(v \otimes \hat{\phi}(m)) = dv \otimes (\pi(d,v)\hat{\phi}(m))
= dv \otimes \hat{\phi}(\pi(d,v)m) = \phi(dv \otimes \pi(d,v)m)
= \phi(d(v \otimes m))
\]
and this completes the proof.
3. Tensor Product and Twisted Tensor Product

Let $B_{r,s}(\delta)$ and $B_{n,m}(\delta)$ be two walled Brauer algebras. The aim of this section is to show that the tensor product algebra $B_{r,s}(\delta) \otimes_k B_{n,m}(\delta)$ is actually a subalgebra of $B_{r+n,s+m}(\delta)$. The key is the so-called twisted tensor product of walled Brauer algebras. We define this product in this section. First, we need to define two kinds of mappings $\iota$ and $\zeta$, which are used to extend a walled Brauer diagram to a bigger one.

Denote by $I_n$ the diagram of the identity of a Brauer algebra $B_{n}(\delta)$. For $r, s, n,$ and $m \in \mathbb{N}$, let $D$ be an $(r, s)$-walled diagram in $B_{r,s}(\delta)$. Define $\iota_{n,m}(D)$ to be an $(r+n, s+m)$-walled diagram in $B_{r+n,s+m}(\delta)$ obtained by inserting diagrams $I_n$ and $I_m$ closing to the left and right sides of the wall, respectively, as shown in Fig. 3(a). Similarly, $\zeta_{n,m}(D)$ is an $(r+n, s+m)$-diagram, which is shown in Fig. 3(b) obtained by juxtaposing diagrams $I_n$ and $I_m$ to the left and right sides of diagram $D$, respectively.

According to the definition of $\iota_{n,m}$ and $\zeta_{n,m}$, the following two lemmas are obvious.

Lemma 3.1. Let $D_1, D_2$ be $(r, s)$-walled diagrams. Then

1. $\iota_{n,m}(D_1 \cdot D_2) = \iota_{n,m}(D_1) \cdot \iota_{n,m}(D_2)$
2. $\zeta_{n,m}(D_1 \cdot D_2) = \zeta_{n,m}(D_1) \cdot \zeta_{n,m}(D_2)$

Lemma 3.2. Let $r, s, n, m \in \mathbb{N}$. Let $D$ be an $(r, s)$-diagram and let $D'$ be an $(n, m)$-diagram. Then

$$\iota_{n,m}(D) \cdot \zeta_{r,s}(D') = \zeta_{r,s}(D') \cdot \iota_{n,m}(D)$$

Note that Lemma 3.2 can be clarified from the following figure.

Now we are in a position to give the definition of twisted tensor product of walled Brauer diagrams.
**Definition 3.3.** Let $D$ be an $(r, s)$-diagram and let $D'$ be an $(n, m)$-diagram. The *twisted tensor product* $\boxtimes$ of diagrams $D$ and $D'$ is an $(r + n, s + m)$-walled Brauer diagram defined by

$$D \boxtimes D' := \tau_{n,m}(D) \cdot \zeta_{r,s}(D').$$

Figure 5 shows this definition more intuitively.

![Twisted tensor product of walled Brauer diagrams](image)

**Figure 5.** Twisted tensor product of walled Brauer diagrams

The twisted tensor product is clearly associative. This fact is expressed in the form of the following lemma.

**Lemma 3.4.** The twisted tensor product $\boxtimes$ is associative.

The following lemma implies that the twisted tensor product ‘$\boxtimes$’ and multiplication ‘$\cdot$’ of diagrams are compatible. This enables us to accomplish the task of this section.

**Lemma 3.5.** Let $D_1, D_2$ be $(r, s)$-walled Brauer diagrams and let $D'_1, D'_2$ be $(n, m)$-walled Brauer diagrams. Then

$$(D_1 \boxtimes D_2) \cdot (D'_1 \boxtimes D'_2) = (D_1 \cdot D_2) \boxtimes (D'_1 \cdot D'_2)$$

**Proof.** According to Definition 3.3 and Lemma 3.2, we have

$$(D_1 \boxtimes D'_1) \cdot (D_2 \boxtimes D'_2) = \left(\tau_{n,m}(D_1) \cdot \zeta_{r,s}(D'_1)\right) \cdot \left(\tau_{n,m}(D_2) \cdot \zeta_{r,s}(D'_2)\right)$$

$$= \left(\tau_{n,m}(D_1) \cdot \tau_{n,m}(D_2)\right) \cdot \left(\zeta_{r,s}(D'_1) \cdot \zeta_{r,s}(D'_2)\right)$$

$$= \tau_{n,m}(D_1 \cdot D_2) \cdot \zeta_{r,s}(D'_1 \cdot D'_2)$$

$$= (D_1 \cdot D_2) \boxtimes (D'_1 \cdot D'_2).$$

The proof is completed. $\square$

By extending the twisted tensor product of diagrams linearly to algebras $B_{r,s}(\delta)$ and $B_{n,m}(\delta)$, we obtain the twisted tensor product algebra $B_{r,s}(\delta) \boxtimes B_{n,m}(\delta)$. Denote by $B_{r,s}(\delta) \otimes_k B_{n,m}(\delta)$ the ordinary tensor product algebra. Then we can give the main result of this section, which is derived directly from Lemma 3.5.

**Theorem 3.6.** The twisted tensor product $B_{r,s}(\delta) \boxtimes B_{n,m}(\delta)$ is a subalgebra of $B_{r+n,s+m}(\delta)$, and it is isomorphic to $B_{r,s}(\delta) \otimes_k B_{n,m}(\delta)$ as $k$-algebras.

**Proof.** Twisted tensor product algebra $B_{r,s}(\delta) \boxtimes B_{n,m}(\delta)$ is a clear subalgebra of $B_{r+n,s+m}(\delta)$. So we only need to prove that it is isomorphic to $B_{r,s}(\delta) \otimes_k B_{n,m}(\delta)$.

Define a bilinear map

$$g : B_{r,s}(\delta) \times B_{n,m}(\delta) \to B_{r,s}(\delta) \boxtimes B_{n,m}(\delta)$$

by

$$(D, D') \mapsto D \boxtimes D'.$$

It is easy to check that the map $g$ is surjective and injective. Denote by

$$i : B_{r,s}(\delta) \times B_{n,m}(\delta) \to B_{r,s}(\delta) \otimes B_{n,m}(\delta)$$

the canonical map. Then there exists a unique $k$-linear mapping

$$\tilde{g} : B_{r,s}(\delta) \otimes B_{n,m}(\delta) \to B_{r,s}(\delta) \boxtimes B_{n,m}(\delta)$$
satisfying \( \tilde{g}i = g \), which is bijective too. Moreover, it is seen from Lemma 5.5 that map \( \tilde{g} \) is a \( k \)-algebra homomorphism, and we have proven the lemma. \( \square \)

Clearly, a walled Brauer algebra \( B_{r,s}(\delta) \) can be embedded into \( B_{r+1,s}(\delta) \) or \( B_{r,s+1}(\delta) \) naturally. These embeddings make the walled Brauer algebras form something looking like a bicomplex.

\[
\begin{array}{cclclcl}
B_{0,0}(\delta) & \hookrightarrow & B_{0,1}(\delta) & \hookrightarrow & B_{0,2}(\delta) & \hookrightarrow & B_{0,3}(\delta) & \hookrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_{1,0}(\delta) & \hookrightarrow & B_{1,1}(\delta) & \hookrightarrow & B_{1,2}(\delta) & \hookrightarrow & B_{1,3}(\delta) & \hookrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_{2,0}(\delta) & \hookrightarrow & B_{2,1}(\delta) & \hookrightarrow & B_{2,2}(\delta) & \hookrightarrow & B_{2,3}(\delta) & \hookrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\end{array}
\]

Note that \( B_{0,n}(\delta) = B_{n,0}(\delta) = k\Sigma_n \), the group algebra of the symmetric group \( \Sigma_n \) on \( n \) letters.

If we denote by \( G_0(B_{i,j}(\delta)) \) the Grothendieck group of \( B_{i,j}(\delta) \), then we can obtain a bigraded \( \mathbb{Z} \)-module

\[
G_0(B(\delta)) := \bigoplus_{i,j \geq 0} G_0(B_{i,j}(\delta)).
\]

Moreover, thanks to Theorem 6.6 we have embeddings

\[
\rho_{r,s,n,m} : B_{r,s}(\delta) \otimes_k B_{n,m}(\delta) \rightarrow B_{r+s+n+m}(\delta),
\]

which enable us to define the multiplication for \( G_0(B(\delta)) \) and make \( G_0(B(\delta)) \) a ring. In fact, let \( M \) be a left \( B_{r,s}(\delta) \)-module, and \( N \) be a left \( B_{n,m}(\delta) \)-module. Recall that tensor product \( M \otimes_k N \) is a left \( B_{r,s}(\delta) \otimes_k B_{n,m}(\delta) \)-module with action \( (a \otimes b) \cdot (w \otimes u) = aw \otimes bu \) for \( a \in B_{r,s}(\delta), b \in B_{n,m}(\delta), w \in M \) and \( u \in N \). Then in \( G_0(B(\delta)) \), the multiplication of the isomorphism classes of a \( B_{r,s}(\delta) \)-module \( M \) and a \( B_{n,m}(\delta) \)-module \( N \) can be defined by the induction product:

\[
[M] \cdot [N] = \left[ \text{Ind}_{B_{r,s}(\delta) \otimes B_{n,m}(\delta)}^{B_{r+s+n+m}(\delta)}(M \otimes N) \right],
\]

where

\[
\text{Ind}_{B_{r,s}(\delta) \otimes B_{n,m}(\delta)}^{B_{r+s+n+m}(\delta)}(M \otimes N) = B_{r+s+n+m}(\delta) \otimes_{B_{r,s}(\delta) \otimes B_{n,m}(\delta)} (M \otimes N)
\]

\[
= \frac{B_{r+s+n+m}(\delta) \otimes_k (M \otimes N)}{\langle a \otimes \left[ (b \otimes c)(w \otimes u) \right] - [a \rho_{r,s,n,m}(b \otimes c)] \otimes w \otimes u \rangle}
\]

for \( a \in B_{r+s+n+m}(\delta), \ b \in B_{r,s}(\delta), \ c \in B_{n,m}(\delta), \ w \in M \) and \( u \in N \).

To ensure the multiplication is well-defined and admits associativity, we require that the walled Brauer algebras are semi-simple (see [2, Theorem 3.5]). We refer the reader to [5, Theorem 6.3] for relative results on semi-simplicity.

The structure constants of the Grothendieck ring are calculated in Section 5.

## 4. Double-walled modules

In this section, we characterize the restriction of cell modules. To this end, we introduce double-walled modules, which will be useful for describing the branching rules for cell modules over a sequence of the walled Brauer algebras. We also give a composition series of the restricted module in Theorem 4.3.
4.1. **Double-walled diagram and actions.** The notions given in the section are inspired by the following observations. Let \( r_1, r_2, s_1, s_2 \in \mathbb{N} \) with \( r_1 + r_2 = r \) and \( s_1 + s_2 = s \). For a partial one-row \((r, s, l)\) diagram \( v \in V_{r,s}^l \) we can add two walls between dots \( r_1, r_1 + 1 \), and dots \( s_1, s_1 + 1 \), respectively. Then the diagram is divided into four blocks, says \( A, B, C, D \) with \( r_1, r_2, s_2, \) and \( s_1 \) dots, respectively (see the left diagram of Fig. 6 for an example).

We next endow the diagram with a 4-tuple

\[
(T_{AC}, T_{BD}, T_{AD}, T_{BC}),
\]

wherein \( T_{XY} \) is the number of the arcs jointing from block \( X \) to \( Y \). It is clear that \( T_{AC} + T_{BD} + T_{AD} + T_{BC} = l \).

For a diagram \( v \), define a transformation \( f \) by moving its blocks \( B, C \), together with the joining arcs horizontally to the right side of the block \( D \). We illustrate \( f \) as follows.

We call the image of \( f \) an \((r_1|s_1, r_2|s_2)\)-double-walled diagram, and denote it by symbol \( \tilde{v} \). In fact, such a diagram can also be obtained by first juxtaposing two partial one-row diagrams and then possibly jointing single dots from block \( A \) to \( C \), or from \( D \) to \( B \).

We can define a \( B_{r_1, s_1}(\delta) \circ B_{r_2, s_2}(\delta) \)-action on an \((r_1|s_1, r_2|s_2)\)-double-walled diagram as in the case of a partial diagram module. Let us describe the action by using the right part of Fig. 6. Taking an \((r_1, s_1)\)-diagram \( D_1 \) (with parts I and IV) and an \((r_2, s_2)\)-diagram \( D_2 \) (with parts II and III), then \( D_1 \circ D_2 \) can be viewed as the juxtaposition of these two diagrams. Let \( \tilde{v} \) be an \((r_1|s_1, r_2|s_2)\)-double walled diagram with the total number of arcs \( l \). The action is given by concatenating diagrams \( D_1 \) with parts \( AD \) and \( D_2 \) with parts \( BC \). Then, by identifying the corresponding dots gives a new \((r_1|s_1, r_2|s_2)\)-double-walled diagram \( v' \) possibly with \( t \) closed curves. Then

\[
(D_1 \circ D_2) \tilde{v} = \begin{cases} 0, & \text{if } t > l; \\ \delta^t v', & \text{otherwise}. \end{cases}
\]

Therefore, if we denote by \( \tilde{V}_{r_1|s_1,r_2|s_2}^l \) the linear space spanned by all \((r_1|s_1, r_2|s_2)\)-double-walled diagrams with \( l \) arcs, then it is a \( B_{r_1,s_1}(\delta) \circ B_{r_2,s_2}(\delta) \)-module with the action defined above.

**Lemma 4.1.** Let \( r_1, r_2, s_1 \) and \( s_2 \in \mathbb{N} \) with \( r_1 + r_2 = r \) and \( s_1 + s_2 = s \). Let \( \tilde{v} \) be an \((r_1|s_1, r_2|s_2)\)-double-walled diagram with the tuple \((T_{AC}, T_{BD}, T_{AD}, T_{BC})\), and the total number of arcs being \( l \). For an \((r_1, s_1)\)-diagram \( D_1 \) and an \((r_2, s_2)\)-diagram \( D_2 \), suppose that \( (D_1 \circ D_2) \tilde{v} \) has \( l' \) arcs with the tuple \((T_{AC}, T_{BD}, T_{AD}, T_{BC})\). Then \( l \leq l' \). If \( l = l' \), then

1. the numbers \( T_{AD}, T_{BC} \) of arcs jointing blocks never increases;
2. the numbers \( T_{AC}, T_{BD} \) of arcs jointing blocks never decreases.

**Proof.** Formula \( l \leq l' \) is a direct corollary of Lemma 2.1. For (1-2), we first change the structure of a double-walled diagram from “left-right” to “up-down” being illustrated as in Figure 7. The process is as follows. Given a double walled diagram \( \tilde{v} \), rotate the right side...
of the vertical dashed line (saying blocks $BC$) clockwise, until two walls are connected with each other. Then (1-2) follow from the module action being essentially a concatenation of diagrams.

\[\square\]

Remarks

(1) When $l = l'$, Lemma 4.1 implies that $T_{AC}$ or $T_{BD}$ will be certain to decrease by one once $T_{AD}$ or $T_{BC}$ increases by one, and vice versa.

(2) $V^t_{r,s}$ also admits a $B_{r_1,s_1}(\delta) \otimes B_{r_2,s_2}(\delta)$-action in terms of the mapping $\bar{g}$ in Theorem 3.6 For a partial diagram $v \in V^t_{r,s}$, an $(r_1,s_1)$-diagram $D_1$ and an $(r_2,s_2)$-diagram $D_2$, define

\[(D_1 \otimes D_2).v := \bar{g}(D_1 \otimes D_2).v = (D_1 \boxtimes D_2).v\]

As a $B_{r_1,s_1}(\delta) \otimes B_{r_2,s_2}(\delta)$-module, $V^t_{r,s}$ is isomorphic to $\hat{V}^t_{r_1|s_1,r_2|s_2}$. In fact, the map $f$ defined in the beginning of this section is clearly invertible, and

\[f((D_1 \otimes D_2).v) = f((D_1 \boxtimes D_2).v) = (D_1 \otimes D_2)\hat{v} = (D_1 \otimes D_2).f(v)\]

4.2. Filtration. First of all, we introduce a schematic manner to express partial diagrams:

Here, in the schematic presentation, ‘free dots’ (that are not vertices of arcs) are contained in a dashed frame with a single dot, and arcs crossing the wall become a single one with number $T_{AB} (= l)$ on it. For example, changing Fig. 6 to the schematic presentation, as shown in Fig. 8, the left diagram is acted by twisted product $B_{r_1,s_1}(\delta) \boxtimes B_{r_2,s_2}(\delta)$, and the right one is acted by algebra $B_{r_1,s_1}(\delta) \otimes B_{r_2,s_2}(\delta)$.
Lemma 4.1. double-walled modules in our study.

(1) Since each partial one-row diagram can be viewed as a double walled diagram, we define

\[ W_r \] of \( (r_1|s_1,r_2|s_2) \)-double-walled diagrams with \( l \) arcs (equally, the partial one-row \((r,s,l)\) diagrams), and endow \( I_{(r_1|s_1,r_2|s_2)}^l \) with a linear order by ordered lexicographically with \( T_{AC}, T_{BD} \) using the order of the natural numbers, while \( T_{AD}, T_{BC} \) using the inverse order of the natural numbers.

Let \( r_1 + r_2 = r \) and \( s_1 + s_2 = s \). Suppose that \( t = (T_{AC}, T_{BD}, T_{AD}, T_{BC}) \) is a possible tuple of an \((r_1|s_1,r_2|s_2)\)-double-walled diagram with

\[ T_{AC} + T_{BD} + T_{AD} + T_{BC} = l. \]

Define \( \tilde{V}_{(r_1|s_1,r_2|s_2)}^{\leq t} \) to be the space spanned by all \((r_1|s_1,r_2|s_2)\)-double-walled diagrams with \( l \) arcs such that the indices in \( I_{(r_1|s_1,r_2|s_2)}^l \) no more than \( t \). It is clearly a \( B_{r_1,s_1}(\delta) \otimes B_{r_2,s_2}(\delta) \)-module.

Definition 4.2. The module \( \tilde{V}_{(r_1|s_1,r_2|s_2)}^{\leq t} \) is called an \((r_1|s_1,r_2|s_2)\)-double-walled module with the tuple \( t \). Moreover, let \( \lambda^L = r - l, \lambda^R = s - l \) and

\[ W_{(r_1|s_1,r_2|s_2)}^{(\lambda^L, \lambda^R)}(t) := \tilde{V}_{(r_1|s_1,r_2|s_2)}^{\leq t} \otimes (S^{\lambda^L} \boxtimes S^{\lambda^R}), \]

which will be called a double-walled module with pair \((\lambda^L, \lambda^R)\).

Remarks

(1) Since each partial one-row diagram can be viewed as a double walled diagram, we replace partial one-row diagrams with double walled diagrams and consequently use double-walled modules in our study.

(2) The module \( W_{(r_1|s_1,r_2|s_2)}^{(\lambda^L, \lambda^R)}(t) \) is a submodule of the restriction module

\[ \text{Res}_{B_{r_1,s_1} \otimes B_{r_2,s_2}} (\Delta_{r,s}(\lambda^L, \lambda^R)). \]

The fact that \( W_{(r_1|s_1,r_2|s_2)}^{(\lambda^L, \lambda^R)}(t) \) is actually a \( (B_{r_1,s_1} \otimes B_{r_2,s_2}) \)-module is a direct corollary of Lemma 4.1.

According to Definition 4.2, we have a chain of modules indexed by \( I_{(r_1|s_1,r_2|s_2)}^l \):

\[ 0 \subset \cdots \subset W_{(r_1|s_1,r_2|s_2)}^{(\lambda^L, \lambda^R)}(t') \subset W_{(r_1|s_1,r_2|s_2)}^{(\lambda^L, \lambda^R)}(t) \subset \cdots \subset \text{Res}_{B_{r_1,s_1} \otimes B_{r_2,s_2}} (\Delta_{r,s}(\lambda^L, \lambda^R)). \] (*)

Now we are in a position to give the main result of this section, which play an important role in the calculation shown in the next section.

Theorem 4.3. Keep notations as above. Then

(1) Let \( \Delta_{(r_1|s_1,r_2|s_2)}^{(\lambda^L, \lambda^R)}(t) \) be a subquotient of the chain (*). Then

\[ \Delta_{(r_1|s_1,r_2|s_2)}^{(\lambda^L, \lambda^R)}(t) \simeq \tilde{V}_{(r_1|s_1,r_2|s_2)}^t \otimes (S^{\lambda^L} \boxtimes S^{\lambda^R}), \]

where the space \( \tilde{V}_{(r_1|s_1,r_2|s_2)}^t \) is spanned by all \((r_1|s_1,r_2|s_2)\)-double walled diagrams with index \( t \).

Figure 9. Schematic presentation of Fig. 9.
which are important for dealing with the general case. As a result, we only need to prove (2). Let us first consider two special cases.

**Proof.** The Statement (1) follows directly from the definition of walled modules, and (3) is isomorphic to \( B_{r,s} \). Clearly, the action of each element of \( B_{r,s} \) containing at least one arc on subquotients is zero. This implies that as a \( B_{r,s} \)-module, \( \Delta_{\{\sigma,\sigma'\}} \) is filtered by modules of the form:

\[
(V_{r_1,s_1}^0 \otimes (S^{\mu_1} \boxtimes S^{\mu_2})) \otimes (V_{r_2,s_2}^0 \otimes (S^{\mu_1} \boxtimes S^{\mu_2})).
\]

We draw a schematic diagram to help comprehend the case (see Fig. 11).

**Special case 2:** \( T_{AC} = 0, T_{BD} = 0 \)
Note that $S^\lambda L \otimes S^\lambda R$, in the double walled module $\tilde{V}^{t}_{(r_1|s_1,r_2|s_2)} \otimes (S^\lambda L \otimes S^\lambda R)$, corresponds to the free dots of diagrams. This is shown in Fig. 12 (a) with $S^\lambda L$ corresponding to the free dots of block $A$ jointed $B$ as $S^\lambda R$ corresponding to free dots of block $C$ jointed $D$. Then $\lambda_L \vdash r - T_{AD} - T_{BC}, \lambda_R \vdash s - T_{AD} - T_{BC}$. Therefore, from the characteristic-free version of the Littlewood-Richardson Rule [13], we have that the restricted module $S^\lambda L \otimes_{\Sigma_{r_1-T_{AD}\times T_{BC}}} S^\lambda R$ has a filtration with subquotients $S^\lambda L \otimes S^\lambda R$, where $S^\lambda L$ and $S^\lambda R$ are shown in Fig.12 (b). Let us give some explain here. Module $S^\lambda L$ corresponds to the free dots of block $A$ on the left side of the wall with $\lambda_L \vdash r - T_{AD}$, as $S^\lambda R$ corresponds to the right free dots of block $B$ with $\lambda_R \vdash s - T_{BC}$. Similarly, we can study the restriction of $S^\lambda R$. Note that any diagram of $B_{r_1,s_1}$ with strictly more than $T_{AD}$ arcs will kill the summand, as does any diagram of $B_{r_2,s_2}$ with strictly more than $T_{BC}$ arcs. Hence, as a $B_{r_1,s_1} \otimes B_{r_2,s_2}$-module, $\Delta_{(r_1|s_1,r_2|s_2)}(t)$ is filtered by modules of the form:

$$ (V^{T_{AD}}_{r_1,s_1} \otimes (S^\lambda L \otimes S^\lambda R)) \otimes (V^{T_{BC}}_{r_2,s_2} \otimes (S^\lambda L \otimes S^\lambda R)). $$

**General case.** Given a double-walled diagram $\tilde{v}$ with index $(T_{AC}, T_{BD}, T_{AD}, T_{BC})$, it can be divided into three different parts:

1) the arcs jointing blocks $AD$ or $BC$;
2) the arcs jointing blocks $AC$ or $BD$;
3) and free dots (jointing no arcs).

As our study of the special cases above, when we consider restricting to $B_{r_1,s_1} \otimes B_{r_2,s_2}$, part (1) determine the half diagram module $V^{T_{AD}}_{r_1,s_1}, V^{T_{BC}}_{r_2,s_2}$ respectively; part (2) determine new free dots (with dashed circle in Fig.13) corresponding Specht modules $S^{\mu_1}, S^{\mu_2}$ as special case 1; for part (3), as special case 2, old free dots (with dashed frame) correspond Specht modules $S^\lambda L, S^\lambda R, S^\lambda L$ and $S^\lambda R$. Note that if a block, say $A$, contains both $r_1 - T_{AB}$ old and $T_{AC}$ new free dots, then the Specht modules corresponding the free dots should be a
the double-walled module is filtered by modules of the form shown in the theorem, as shown


Throughout this section, all walled Brauer algebras are assumed to be semisimple.

Then we have

Theorem 5.1. Let $B(\delta)$ be a semi-simple sequence of the walled Brauer algebras with $\delta \in k$. Then we have

$$[\Delta_{r_1,s_1}(\nu^L_1, \nu^R_1)][\Delta_{r_2,s_2}(\nu^L_2, \nu^R_2)] = \sum_\chi C_{\tilde{\nu}_1|\tilde{\nu}_2}^{\chi}[\Delta_{r,s}(\lambda^L, \lambda^R)],$$

where $\tilde{\nu}_i = (\nu^L_i, \nu^R_i)$ for $i = 1, 2$, and the pair $\chi = (\lambda^L, \lambda^R)$ in sum runs over all indices of cell modules of $B_{r,s}(\delta)$ and the structure constant

$$C_{\tilde{\nu}_1|\tilde{\nu}_2}^{\chi} = \sum_{\bar{r}} \left( \prod_{i=1,2} C_{\lambda^L_i|\mu_i}^{\nu^L_i} \cdot C_{\lambda^R_i|\mu_i}^{\nu^R_i} \right),$$

with the indices

$$\bar{r} = \begin{cases} \lambda^L_1 + (r_1 - T_{AD} - T_{AC}), & \mu_1 + T_{AC}, \\ \lambda^R_1 + (s_1 - T_{AD} - T_{BD}), & \mu_2 + T_{BD}, \\ \lambda^L_2 + (r_2 - T_{BC} - T_{BD}), & \mu_2 + T_{BD}, \\ \lambda^R_2 + (s_2 - T_{BC} - T_{AC}). \end{cases}$$

running over all $(T_{AC}, T_{BD}, T_{AD}, T_{BC}) \in I_{(r_1, s_1, r_2, s_2)}$ such that

$$l_1 = T_{AD} \text{ and } l_2 = T_{BC}.$$
Proof. Firstly, we have
\[
[\Delta_{r_1,s_1}(\nu^L_1,\nu^R_1)] \cdot [\Delta_{r_2,s_2}(\nu^L_2,\nu^R_2)] = \text{Ind}(\Delta_{r_1,s_1}(\nu^L_1,\nu^R_1) \otimes \Delta_{r_2,s_2}(\nu^L_2,\nu^R_2))
\]
\[
= \sum_{\lambda} C_{\nu_1,\nu_2}^\lambda [\Delta_{r,s}(\lambda^L,\lambda^R)],
\]
where
\[
C_{\nu_1,\nu_2}^\lambda := \dim \text{Hom} \left( \text{Ind}(\Delta_{r_1,s_1}(\nu^L_1,\nu^R_1) \otimes \Delta_{r_2,s_2}(\nu^L_2,\nu^R_2)), \Delta_{r,s}(\lambda^L,\lambda^R) \right)
\]
\[
= \dim \text{Hom} \left( \Delta_{r_1,s_1}(\nu^L_1,\nu^R_1) \otimes \Delta_{r_2,s_2}(\nu^L_2,\nu^R_2), \text{Res}(\Delta_{r,s}(\lambda^L,\lambda^R)) \right).
\]

From Theorem 4.3 (2), the restricted module \(\text{Res}(\Delta_{r,s}(\lambda^L,\lambda^R))\) is filtered by cell modules of the following form
\[
V^{T_{r_1,s_1}}_{r_1,s_1} \otimes \left( \text{Ind}(S^\lambda L \otimes S^\mu_1) \otimes \text{Ind}(S^\lambda R \otimes S^\mu_2) \right) \quad (\text{left})
\]
\[
\otimes
\]
\[
V^{T_{r_2,s_2}}_{r_2,s_2} \otimes \left( \text{Ind}(S^\lambda L \otimes S^\mu_2) \otimes \text{Ind}(S^\lambda R \otimes S^\mu_1) \right) \quad (\text{right}),
\]
with \((T_{AC}, T_{BD}, T_{AD}, T_{BC}) \in \mathcal{I}_{(r_1,s_1,r_2,s_2)}\) and the indices
\[
\left\{
\begin{array}{l}
\lambda^L_1 \vdash (r_1 - T_{AD} - T_{AC}), \quad \mu_1 \vdash T_{AC}, \\
\lambda^R_2 \vdash (s_1 - T_{AD} - T_{BD}), \\
\lambda^L_2 \vdash (r_2 - T_{BC} - T_{BD}), \quad \mu_2 \vdash T_{BD}, \\
\lambda^R_2 \vdash (s_2 - T_{BC} - T_{AC}).
\end{array}
\right.
\]
Note that for \(i = 1, 2\), \(\Delta_{r_i,s_i}(\nu^L_i,\nu^R_i) = V^{T_{r_i,s_i}}_{r_i,s_i} \otimes (S^\nu_i \otimes S^\mu_i)\), and simply write \(T_1 = T_{AD}, T_2 = T_{BC}\). Thanks to Proposition 2.2, we get
\[
\text{Hom} \left( V^{T_{r_i,s_i}}_{r_i,s_i} \otimes (S^\nu_i \otimes S^\mu_i), V^{T_{r_i,s_i}}_{r_i,s_i} \otimes \left( \text{Ind}(S^\lambda L \otimes S^\mu_1) \otimes \text{Ind}(S^\lambda R \otimes S^\mu_2) \right) \right)
\]
\[
= \left\{
\begin{array}{l}
\text{Hom}(S^\nu_i \otimes S^\mu_i, \text{Ind}(S^\lambda L \otimes S^\mu_1) \otimes \text{Ind}(S^\lambda R \otimes S^\mu_2)) \quad \text{if } l_i = T_i, \text{ for } i = 1, 2; \\
0, \quad \text{otherwise}.
\end{array}
\right.
\]
By Frobenius reciprocity, \((\bullet)\) is equal to
\[
\text{Hom}(\text{Res}(S^\nu_i), S^\lambda L \otimes S^\mu_i) \otimes \text{Hom}(\text{Res}(S^\mu_i), S^\lambda R \otimes S^\mu_i)$
\]
and the dimension of the factors is the Littlewood-Richardson coefficient:
\[
C^R_{\lambda^L \vdash \mu_1} \cdot C^L_{\lambda^R \vdash \mu_2} \quad \text{for } i = 1, 2; \quad C_{\lambda^L \vdash \mu_1} \cdot C^R_{\lambda^R \vdash \mu_2} \quad \text{for } i = 2.
\]
Therefore, we get the dimension of previous Hom-space as follows:
\[
\left\{
\begin{array}{l}
\prod_{i=1,2} C^R_{\lambda^L \vdash \mu_1} \cdot C^L_{\lambda^R \vdash \mu_2}, \quad \text{if } l_1 = T_{AD} \text{ and } l_2 = T_{BC}, \\
0, \quad \text{otherwise}.
\end{array}
\right.
\]
Consequently, the conclusion follows when index \((T_{AC}, T_{BD}, T_{AD}, T_{BC})\) runs over all \(\mathcal{I}_{(r_1,s_1,r_2,s_2)}\) such that \(l_1 = T_{AD}\) and \(l_2 = T_{BC}\). \(\square\)
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