A symmetric Nörlund sum with application to inequalities

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Abstract Properties of an $\alpha, \beta$-symmetric Nörlund sum are studied. Inspired in the work by Agarwal et al., $\alpha, \beta$-symmetric quantum versions of Hölder, Cauchy–Schwarz and Minkowski inequalities are obtained.

1 Introduction

The symmetric derivative of function $f$ at point $x$ is defined as $\lim_{h \to 0} (f(x + h) - f(x - h))/(2h)$. The notion of symmetrically differentiable is interesting because if a function is differentiable at a point then it is also symmetrically differentiable, but the converse is not true. The best known example of this fact is the absolute value function: $f(x) = |x|$ is not differentiable at $x = 0$ but is symmetrically differentiable at $x = 0$ with symmetric derivative zero [6].

Quantum calculus is, roughly speaking, the equivalent to traditional infinitesimal calculus but without limits [4]. Therefore, one can introduce the symmetric quantum derivative of $f$ at $x$ by $(f(x + h) - f(x - h))/(2h)$. As in any calculus, it is then natural to develop a corresponding integration theory, looking to such integral as the inverse operator of the derivative.

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The main goal of this paper is to study the properties of a general symmetric quantum integral that we call, due to the so-called Nörlund sum, the \( \alpha, \beta \)-symmetric Nörlund sum.

The paper is organized as follows. In Section 2 we define the forward and backward Nörlund sums. Then, in Section 3 we introduce the \( \alpha, \beta \)-symmetric Nörlund sum and give some of its properties. We end with Section 4 proving \( \alpha, \beta \)-symmetric versions of Hölder’s, Cauchy–Schwarz’s and Minkowski’s inequalities.

2 Forward and backward Nörlund sums

This section is dedicated to the inverse operators of the \( \alpha \)-forward and \( \beta \)-backward differences, \( \alpha > 0, \beta > 0 \), defined respectively by

\[
\Delta_\alpha f(t) := \frac{f(t + \alpha) - f(t)}{\alpha}, \quad \nabla_\beta f(t) := \frac{f(t) - f(t - \beta)}{\beta}.
\]

**Definition 1.** Let \( I \subseteq \mathbb{R} \) be such that \( a, b \in I \) with \( a < b \) and \( \sup I = +\infty \). For \( f : I \to \mathbb{R} \) and \( \alpha > 0 \) we define the Nörlund sum (the \( \alpha \)-forward integral) of \( f \) from \( a \) to \( b \) by

\[
\int_a^b f(t) \Delta_\alpha t = \int_0^\infty f(t) \Delta_\alpha t - \int_0^\infty f(t) \Delta_\alpha t,
\]

where \( \int_0^\infty f(t) \Delta_\alpha t = \alpha \sum_{k=0}^{+\infty} f(x + k\alpha) \), provided the series converges at \( x = a \) and \( x = b \). In that case, \( f \) is said to be \( \alpha \)-forward integrable on \( [a, b] \). We say that \( f \) is \( \alpha \)-forward integrable over \( I \) if it is \( \alpha \)-forward integrable for all \( a, b \in I \).

Until Definition 2 (the backward/nabla case), we assume that \( I \) is an interval of \( \mathbb{R} \) such that \( \sup I = +\infty \). Note that if \( f : I \to \mathbb{R} \) is a function such that \( \sup I < +\infty \), then we can extend function \( f \) to \( f : \tilde{I} \to \mathbb{R} \), where \( \tilde{I} \) is an interval with \( \sup \tilde{I} = +\infty \), in the following way: \( f|_I = f \) and \( f|_{\tilde{I} \setminus I} = 0 \).

Using the techniques of Aldwoah in his Ph.D. thesis [2], it can be proved that the \( \alpha \)-forward integral has the following properties:

**Theorem 1.** If \( f, g : I \to \mathbb{R} \) are \( \alpha \)-forward integrable on \( [a, b] \), \( c \in [a, b], k \in \mathbb{R} \), then

1. \( \int_a^a f(t) \Delta_\alpha t = 0 \);
2. \( \int_a^b f(t) \Delta_\alpha t = \int_a^c f(t) \Delta_\alpha t + \int_c^b f(t) \Delta_\alpha t \), when the integrals exist;
3. \( \int_a^b f(t) \Delta_\alpha t = -\int_b^a f(t) \Delta_\alpha t \);
4. \( kf \) is \( \alpha \)-forward integrable on \( [a, b] \) and \( \int_a^b kf(t) \Delta_\alpha t = k \int_a^b f(t) \Delta_\alpha t \);
5. \( f + g \) is \( \alpha \)-forward integrable on \( [a, b] \) and
\[ \int_a^b (f + g)(t) \Delta at = \int_a^b f(t) \Delta at + \int_a^b g(t) \Delta at; \]

6. if \( f \equiv 0 \), then \( \int_a^b f(t) \Delta at = 0 \).

**Theorem 2.** Let \( f : I \to \mathbb{R} \) be \( \alpha \)-forward integrable on \([a, b]\). If \( g : I \to \mathbb{R} \) is a non-negative \( \alpha \)-forward integrable function on \([a, b]\), then \( fg \) is \( \alpha \)-forward integrable on \([a, b]\).

**Proof.** Since \( g \) is \( \alpha \)-forward integrable, then both series \( \alpha \sum_{k=0}^{\infty} g(a + k\alpha) \) and \( \alpha \sum_{k=0}^{\infty} f(a + k\alpha) \) converge. We want to study the nature of series \( \alpha \sum_{k=0}^{\infty} fg(a + k\alpha) \) and \( \alpha \sum_{k=0}^{\infty} f(a + k\alpha) \). Since there exists an order \( N \in \mathbb{N} \) such that \( |fg(a + k\alpha)| \leq g(a + k\alpha) \) and \( |fg(a + k\alpha)| \leq g(a + k\alpha) \) for all \( k > N \), then both \( \alpha \sum_{k=0}^{\infty} fg(a + k\alpha) \) and \( \alpha \sum_{k=0}^{\infty} f(a + k\alpha) \) converge absolutely. The intended conclusion follows.

**Theorem 3.** Let \( f : I \to \mathbb{R} \) and \( p > 1 \). If \( |f| \) is \( \alpha \)-forward integrable on \([a, b]\), then \( |f|^p \) is also \( \alpha \)-forward integrable on \([a, b]\).

**Proof.** There exists \( N \in \mathbb{N} \) such that \( |f(b + k\alpha)|^p \leq |f(b + k\alpha)| \) and \( |f(a + k\alpha)|^p \leq |f(a + k\alpha)| \) for all \( k > N \). Therefore, \( |f|^p \) is \( \alpha \)-forward integrable on \([a, b]\).

**Theorem 4.** Let \( f, g : I \to \mathbb{R} \) be \( \alpha \)-forward integrable on \([a, b]\). If \( |f(t)| \leq g(t) \) for all \( t \in \{a + k\alpha : k \in \mathbb{N}_0\} \), then for \( b \in \{a + k\alpha : k \in \mathbb{N}_0\} \) one has \( \left| \int_a^b f(t) \Delta at \right| \leq \int_a^b g(t) \Delta at \).

**Proof.** Since \( b \in \{a + k\alpha : k \in \mathbb{N}_0\} \), there exists \( k_1 \) such that \( b = a + k_1 \alpha \). Thus,

\[
\left| \int_a^b f(t) \Delta at \right| = \left| \alpha \sum_{k=0}^{k_1-1} f(a + k\alpha) - \alpha \sum_{k=0}^{k_1-1} f(a + k\alpha) \right|
\]

\[
= \left| \alpha \sum_{k=0}^{k_1-1} f(a + k\alpha) - \alpha \sum_{k=0}^{k_1-1} f(a + k\alpha) \right|
\]

\[
\leq \alpha \sum_{k=0}^{k_1-1} |f(a + k\alpha)| \leq \alpha \sum_{k=0}^{k_1-1} g(a + k\alpha)
\]

\[
= \alpha \sum_{k=0}^{k_1-1} g(a + k\alpha) - \alpha \sum_{k=k_1}^{\infty} g(a + k\alpha) = \int_a^b g(t) \Delta at.
\]

**Corollary 1.** Let \( f, g : I \to \mathbb{R} \) be \( \alpha \)-forward integrable on \([a, b]\) with \( b = a + k\alpha \) for some \( k \in \mathbb{N}_0 \).

1. If \( f(t) \geq 0 \) for all \( t \in \{a + k\alpha : k \in \mathbb{N}_0\} \), then \( \int_a^b f(t) \Delta at \geq 0 \).
2. If \( g(t) \geq f(t) \) for all \( t \in \{a + k\alpha : k \in \mathbb{N}_0\} \), then \( \int_a^b g(t) \Delta at \geq \int_a^b f(t) \Delta at \).

We can now prove the following fundamental theorem of the \( \alpha \)-forward calculus.
Theorem 5 (Fundamental theorem of Nörlund calculus). Let $f : I \to \mathbb{R}$ be $\alpha$-forward integrable over $I$. Let $x \in I$ and define $F(x) := \int_s^x f(t) \, \Delta_{\alpha} t$. Then, $\Delta_{\alpha} [F](x) = f(x)$. Conversely, $\int_a^b \Delta_{\alpha} [f](t) \, \Delta_{\alpha} t = f(b) - f(a)$.

Proof. If $G(x) = -\int_x^{+\infty} f(t) \, \Delta_{\alpha} t$, then

$$\Delta_{\alpha} [G](x) = \frac{G(x + \alpha) - G(x)}{\alpha} = -\alpha \sum_{k=0}^{+\infty} f(x + k\alpha) + \alpha \sum_{k=0}^{+\infty} f(x + k\alpha) = \sum_{k=0}^{+\infty} f(x + k\alpha) - \sum_{k=0}^{+\infty} f(x + (k+1)\alpha) = f(x).$$

Therefore, $\Delta_{\alpha} [F](x) = \Delta_{\alpha} \left( \int_a^{+\infty} f(t) \, \Delta_{\alpha} t - \int_x^{+\infty} f(t) \, \Delta_{\alpha} t \right) = f(x)$. Using the definition of $\alpha$-forward difference operator, the second part of the theorem is also a consequence of the properties of Mengoli’s series. Since

$$\int_a^{+\infty} \Delta_{\alpha} [f](t) \, \Delta_{\alpha} t = \alpha \sum_{k=0}^{+\infty} \Delta_{\alpha} [f](a + k\alpha) = \alpha \sum_{k=0}^{+\infty} f(a + k\alpha + \alpha) - f(a + k\alpha) = \sum_{k=0}^{+\infty} \left( f(a + (k+1)\alpha) - f(a + k\alpha) \right) = -f(a)$$

and $\int_a^b \Delta_{\alpha} [f](t) \, \Delta_{\alpha} t = -f(b)$, it follows that

$$\int_a^b \Delta_{\alpha} [f](t) \, \Delta_{\alpha} t = \int_a^{+\infty} f(t) \, \Delta_{\alpha} t - \int_b^{+\infty} f(t) \, \Delta_{\alpha} t = f(b) - f(a).$$

Corollary 2 ($\alpha$-forward integration by parts). Let $f, g : I \to \mathbb{R}$. If $f,g$ and $f \Delta_{\alpha} [g]$ are $\alpha$-forward integrable on $[a,b]$, then

$$\int_a^b f(t) \, \Delta_{\alpha} [g](t) \, \Delta_{\alpha} t = f(t) \, \Delta_{\alpha} [g] \mid_{a}^{b} - \int_a^b \Delta_{\alpha} [f](t) \, g(t + \alpha) \, \Delta_{\alpha} t$$

Proof. Since $\Delta_{\alpha} [f g](t) = \Delta_{\alpha} [f](t) \, g(t + \alpha) + f(t) \, \Delta_{\alpha} [g](t)$, then

$$\int_a^b f(t) \, \Delta_{\alpha} [g](t) \, \Delta_{\alpha} t = \int_a^b \left( \Delta_{\alpha} [f g](t) - \Delta_{\alpha} [f](t) \, g(t + \alpha) \right) \, \Delta_{\alpha} t$$

$$= \int_a^b \Delta_{\alpha} [f g](t) \, \Delta_{\alpha} t - \int_a^b \Delta_{\alpha} [f](t) \, g(t + \alpha) \, \Delta_{\alpha} t$$

$$= f(t) \, g(t) \mid_{a}^{b} - \int_a^b \Delta_{\alpha} [f](t) \, g(t + \alpha) \, \Delta_{\alpha} t.$$

Remark 1. Our study of the Nörlund sum is in agreement with the Hahn quantum calculus $[2,5,5]$. In $[4]$, $\int_a^b f(t) \, \Delta_{\alpha} t = \alpha \left[ f(a) + f(a + \alpha) + \cdots + f(b - \alpha) \right]$ for $a < b$ such that $b - a \in \alpha \mathbb{Z}$, $\alpha \in \mathbb{R}^+$. In contrast with $[4]$, our definition is valid for any two real points $a,b$ and not only for those points belonging to the time scale $\alpha \mathbb{Z}$. The definitions (only) coincide if function $f$ is $\alpha$-forward integrable on $[a,b]$. 


Similarly, we introduce the \( \beta \)-backward integral.

**Definition 2.** Let \( I \) be an interval of \( \mathbb{R} \) such that \( a, b \in I \) with \( a < b \) and \( \inf I = -\infty \). For \( f : I \to \mathbb{R} \) and \( \beta > 0 \) we define the \( \beta \)-backward integral of \( f \) from \( a \) to \( b \) by

\[
\int_a^b f(t) \, \nabla_\beta t = \int_{-\infty}^b f(t) \, \nabla_\beta t - \int_{-\infty}^a f(t) \, \nabla_\beta t,
\]

where \( \int_{-\infty}^x f(t) \, \nabla_\beta t = \sum_{k=0}^{+\infty} f(x - k\beta) \), provided the series converges at \( x = a \) and \( x = b \). In that case, \( f \) is called \( \beta \)-backward integrable on \([a, b]\). We say that \( f \) is \( \beta \)-backward integrable over \( I \) if it is \( \beta \)-backward integrable for all \( a, b \in I \).

The \( \beta \)-backward Nörlund sum has similar results and properties as the \( \alpha \)-forward Nörlund sum. In particular, the \( \beta \)-backward integral is the inverse operator of \( \nabla_\beta \).

### 3 The \( \alpha, \beta \)-symmetric Nörlund sum

We define the \( \alpha, \beta \)-symmetric integral as a linear combination of the \( \alpha \)-forward and the \( \beta \)-backward integrals.

**Definition 3.** Let \( f : \mathbb{R} \to \mathbb{R} \) and \( a, b \in \mathbb{R} \), \( a < b \). If \( f \) is \( \alpha \)-forward and \( \beta \)-backward integrable on \([a, b]\), \( \alpha, \beta \geq 0 \) with \( \alpha + \beta > 0 \), then we define the \( \alpha, \beta \)-symmetric integral of \( f \) from \( a \) to \( b \) by

\[
\int_a^b f(t) \, d_{\alpha, \beta} t = \frac{\alpha}{\alpha + \beta} \int_a^b f(t) \, \Delta_\alpha t + \frac{\beta}{\alpha + \beta} \int_a^b f(t) \, \nabla_\beta t.
\]

Function \( f \) is \( \alpha, \beta \)-symmetric integrable if it is \( \alpha, \beta \)-symmetric integrable for all \( a, b \in \mathbb{R} \).

**Remark 2.** Note that if \( \alpha \in \mathbb{R}^+ \) and \( \beta = 0 \), then \( \int_a^b f(t) \, d_{\alpha, \beta} t = \int_a^b f(t) \, \Delta_\alpha t \) and we do not need to assume in Definition 3 that \( f \) is \( \beta \)-backward integrable; if \( \alpha = 0 \) and \( \beta \in \mathbb{R}^+ \), then \( \int_a^b f(t) \, d_{\alpha, \beta} t = \int_a^b f(t) \, \nabla_\beta t \) and we do not need to assume that \( f \) is \( \alpha \)-forward integrable.

**Example 1.** Let \( f(t) = 1/t^2 \). Then \( \int_1^3 \frac{1}{t^2} \, d_{2, 2} t = \frac{10}{9} \).

The \( \alpha, \beta \)-symmetric integral has the following properties:

**Theorem 6.** Let \( f, g : \mathbb{R} \to \mathbb{R} \) be \( \alpha, \beta \)-symmetric integrable on \([a, b]\). Let \( c \in [a, b] \) and \( k \in \mathbb{R} \). Then,

1. \( \int_a^c f(t) \, d_{\alpha, \beta} t = 0 \);
2. \( \int_a^b f(t) \, da \, \beta t = \int_a^c f(t) \, da \, \beta t + \int_c^b f(t) \, da \, \beta t \), when the integrals exist;

3. \( \int_a^b f(t) \, da \, \beta t = - \int_b^a f(t) \, da \, \beta t \);

4. \( kf \) is \( \alpha, \beta \)-symmetric integrable on \([a, b]\) and \( \int_a^b kf(t) \, da \, \beta t = k \int_a^b f(t) \, da \, \beta t \);

5. \( f + g \) is \( \alpha, \beta \)-symmetric integrable on \([a, b]\) and \( \int_a^b (f + g)(t) \, da \, \beta t = \int_a^b f(t) \, da \, \beta t + \int_a^b g(t) \, da \, \beta t \);

6. \( fg \) is \( \alpha, \beta \)-symmetric integrable on \([a, b]\) provided \( g \) is a nonnegative function.

**Proof.** These results are easy consequences of the \( \alpha \)-forward and \( \beta \)-backward integral properties.

The next result follows immediately from Theorem 3 and the corresponding \( \beta \)-backward version.

**Theorem 7.** Let \( f : \mathbb{R} \to \mathbb{R} \) and \( p > 1 \). If \( |f| \) is symmetric \( \alpha, \beta \)-integrable on \([a, b]\), then \( |f|^p \) is also \( \alpha, \beta \)-symmetric integrable on \([a, b]\).

**Theorem 8.** Let \( f, g : \mathbb{R} \to \mathbb{R} \) be \( \alpha, \beta \)-symmetric integrable functions on \([a, b]\), \( \mathcal{A} := \{a + ka : k \in \mathbb{N}_0\} \) and \( \mathcal{B} := \{b - k\beta : k \in \mathbb{N}_0\} \). For \( b \in \mathcal{A} \) and \( a \in \mathcal{B} \) one has:

1. if \( |f(t)| \leq g(t) \) for all \( t \in \mathcal{A} \cup \mathcal{B} \), then \( \int_a^b f(t) \, da \, \beta t \leq \int_a^b g(t) \, da \, \beta t \);

2. if \( f(t) \geq 0 \) for all \( t \in \mathcal{A} \cup \mathcal{B} \), then \( \int_a^b f(t) \, da \, \beta t \geq 0 \);

3. if \( g(t) \geq f(t) \) for all \( t \in \mathcal{A} \cup \mathcal{B} \), then \( \int_a^b g(t) \, da \, \beta t \geq \int_a^b f(t) \, da \, \beta t \).

**Proof.** It follows from Theorem 4 and Corollary 1 and the corresponding \( \beta \)-backward versions.

In Theorem 9 we assume that \( a, b \in \mathbb{R} \) with \( b \in \mathcal{A} := \{a + ka : k \in \mathbb{N}_0\} \) and \( a \in \mathcal{B} := \{b - k\beta : k \in \mathbb{N}_0\} \), where \( \alpha, \beta \in \mathbb{R}_0^+, \alpha + \beta \neq 0 \).

**Theorem 9 (Mean value theorem).** Let \( f, g : \mathbb{R} \to \mathbb{R} \) be bounded and \( \alpha, \beta \)-symmetric integrable on \([a, b]\) with \( g \) nonnegative. Let \( m \) and \( M \) be the infimum and the supremum, respectively, of function \( f \). Then, there exists a real number \( K \) satisfying the inequalities \( m \leq K \leq M \) such that \( \int_a^b f(t) \, g(t) \, da \, \beta t = K \int_a^b g(t) \, da \, \beta t \).

**Proof.** Since \( m \leq f(t) \leq M \) for all \( t \in \mathbb{R} \) and \( g(t) \geq 0 \), then \( mg(t) \leq f(t) \, g(t) \leq Mg(t) \) for all \( t \in \mathcal{A} \cup \mathcal{B} \). All functions \( mg \), \( f \) and \( Mg \) are \( \alpha, \beta \)-symmetric integrable on \([a, b]\). By Theorems 5 and 8 \( m \int_a^b g(t) \, da \, \beta t \leq \int_a^b f(t) \, g(t) \, da \, \beta t \leq \int_a^b g(t) \, da \, \beta t \). If \( \int_a^b g(t) \, da \, \beta t = 0 \), then \( \int_a^b f(t) \, g(t) \, da \, \beta t = 0 \); if \( \int_a^b g(t) \, da \, \beta t > 0 \), then \( m \leq \frac{\int_a^b f(t) \, g(t) \, da \, \beta t}{\int_a^b g(t) \, da \, \beta t} \leq M \). Therefore, the middle term of these inequalities is equal to a number \( K \), which yields the intended result.
4 $\alpha, \beta$-Symmetric Integral Inequalities

Inspired in the work by Agarwal et al. [11], we now present $\alpha, \beta$-symmetric versions of Hölder, Cauchy–Schwarz and Minkowski inequalities. As before, we assume that $a, b \in \mathbb{R}$ with $b \in \mathbb{R}_+ := \{a + k\alpha : k \in \mathbb{N}_0\}$ and $a \in \mathbb{R}_- := \{b - k\beta : k \in \mathbb{N}_0\}$, where $\alpha, \beta \in \mathbb{R}_0^+$. $\alpha + \beta \neq 0$.

**Theorem 10 (Hölder’s inequality).** Let $f, g : \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$. If $|f|$ and $|g|$ are $\alpha, \beta$-symmetric integrable on $[a, b]$, then

$$\int_a^b |f(t)g(t)|d_{\alpha, \beta}t \leq \left(\int_a^b |f(t)|^p d_{\alpha, \beta}t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q d_{\alpha, \beta}t\right)^{\frac{1}{q}},$$

where $p > 1$ and $q = p/(p - 1)$.

**Proof.** For $\alpha, \beta \in \mathbb{R}_0^+$, $\alpha + \beta \neq 0$, the following inequality holds: $\alpha^p \beta^q \leq \frac{\alpha}{p} + \frac{\beta}{q}$.

Without loss of generality, suppose that $\int_a^b |f(t)|^p d_{\alpha, \beta}t \neq 0$ (note that both integrals exist by Theorem [11]). Set $\xi(t) = |f(t)|^p / \int_a^b |f(t)|^p d_{\alpha, \beta}t$ and $\gamma(t) = |g(t)|^q / \int_a^b |g(t)|^q d_{\alpha, \beta}t$. Since both functions $\alpha$ and $\beta$ are symmetric $\alpha, \beta$-integrable on $[a, b]$, then [11] holds:

$$\int_a^b \frac{|f(t)|}{\left(\int_a^b |f(t)|^p d_{\alpha, \beta}t\right)^{\frac{1}{p}}} \frac{|g(t)|}{\left(\int_a^b |g(t)|^q d_{\alpha, \beta}t\right)^{\frac{1}{q}}} d_{\alpha, \beta}t = \int_a^b \xi(t)^{\frac{1}{p}} \gamma(t)^{\frac{1}{q}} d_{\alpha, \beta}t$$

$$\leq \int_a^b \left(\frac{\xi(t)}{p} + \frac{\gamma(t)}{q}\right) d_{\alpha, \beta}t$$

$$= \frac{1}{p} \int_a^b \left(\frac{|f(t)|^p}{\int_a^b |f(t)|^p d_{\alpha, \beta}t}\right) d_{\alpha, \beta}t + \frac{1}{q} \int_a^b \left(\frac{|g(t)|^q}{\int_a^b |g(t)|^q d_{\alpha, \beta}t}\right) d_{\alpha, \beta}t = 1.$$ 

The particular case $p = q = 2$ of [11] gives the Cauchy–Schwarz inequality.

**Corollary 3 (Cauchy–Schwarz’s inequality).** Let $f, g : \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$. If $f$ and $g$ are $\alpha, \beta$-symmetric integrable on $[a, b]$, then

$$\int_a^b |f(t)g(t)|d_{\alpha, \beta}t \leq \sqrt{\left(\int_a^b |f(t)|^2 d_{\alpha, \beta}t\right) \left(\int_a^b |g(t)|^2 d_{\alpha, \beta}t\right)}.$$ 

We prove the Minkowski inequality using Hölder’s inequality.

**Theorem 11 (Minkowski’s inequality).** Let $f, g : \mathbb{R} \to \mathbb{R}$ and $a, b, p \in \mathbb{R}$ with $a < b$ and $p > 1$. If $f$ and $g$ are $\alpha, \beta$-symmetric integrable on $[a, b]$, then

$$\left(\int_a^b |f(t) + g(t)|^p d_{\alpha, \beta}t\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(t)|^p d_{\alpha, \beta}t\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p d_{\alpha, \beta}t\right)^{\frac{1}{p}}.$$
Proof. One has

$$\int_a^b |f(t) + g(t)|^p d_{\alpha, \beta}t = \int_a^b |f(t) + g(t)|^{p-1} |f(t) + g(t)| d_{\alpha, \beta}t$$

$$\leq \int_a^b |f(t)| |f(t) + g(t)|^{p-1} d_{\alpha, \beta}t + \int_a^b |g(t)| |f(t) + g(t)|^{p-1} d_{\alpha, \beta}t.$$ 

Applying Hölder’s inequality (Theorem 10) with $q = p/(p - 1)$, we obtain

$$\int_a^b |f(t) + g(t)|^p d_{\alpha, \beta}t \leq \left( \int_a^b |f(t)|^p d_{\alpha, \beta}t \right)^{\frac{1}{p}} \left( \int_a^b |g(t)|^{(p-1)q} d_{\alpha, \beta}t \right)^{\frac{1}{q}}$$

$$+ \left( \int_a^b |g(t)|^p d_{\alpha, \beta}t \right)^{\frac{1}{p}} \left( \int_a^b |f(t) + g(t)|^{(p-1)q} d_{\alpha, \beta}t \right)^{\frac{1}{q}}$$

$$= \left[ \left( \int_a^b |f(t)|^p d_{\alpha, \beta}t \right)^{\frac{1}{p}} + \left( \int_a^b |g(t)|^p d_{\alpha, \beta}t \right)^{\frac{1}{p}} \right] \left( \int_a^b |f(t) + g(t)|^{(p-1)q} d_{\alpha, \beta}t \right)^{\frac{1}{q}}.$$ 

Therefore,

$$\frac{\int_a^b |f(t) + g(t)|^p d_{\alpha, \beta}t}{\left( \int_a^b |f(t) + g(t)|^{(p-1)q} d_{\alpha, \beta}t \right)^{\frac{1}{q}}} \leq \left( \int_a^b |f(t)|^p d_{\alpha, \beta}t \right)^{\frac{1}{p}} + \left( \int_a^b |g(t)|^p d_{\alpha, \beta}t \right)^{\frac{1}{p}}.$$ 

Our $\alpha, \beta$-symmetric calculus is more general than the standard $h$-calculus. In particular, all our results give, as corollaries, results in the classical quantum $h$-calculus by choosing $\alpha = h > 0$ and $\beta = 0$.

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