THE SET OF ALTERNATING SIGN MATRICES WHICH ARE DETERMINED BY THEIR X-RAY IS A MEMBER OF THE CATALAN FAMILY

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Abstract. We exhibit a bijection between Dyck paths and alternating sign matrices which are determined by their antidiagonal sums.

1. Introduction

A fundamental question in discrete tomography is whether a binary image can be reconstructed from a small number of projections. As a special case, one might restrict attention to permutation matrices, and try to determine which vectors of antidiagonal sums appear only once. This problem, considered by Bebeacua, Mansour, Postnikov and Severini [1], is apparently still open.

In this note, we consider the analogous problem for alternating sign matrices. An alternating sign matrix is a square matrix of 0s, 1s and −1s such that the sum of each row and each column is 1, and the nonzero entries in each row and in each column alternate in sign. For an $n \times n$-alternating sign matrix $A$, the $k$-th (antidiagonal) sum is $x_k = \sum_{i+j=k+1} A_{i,j}$ and the (antidiagonal) X-ray is the vector $x_1, \ldots, x_{2n-1}$. For example, the alternating sign matrices of size three together with their X-rays are as follows:

\begin{align*}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
&
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
&
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
&
\begin{pmatrix}
0 & 1 & 0 \\
1 & -1 & 1 \\
0 & 1 & 0
\end{pmatrix}
&
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
&
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\end{align*}

Note that all X-rays except $0/1/1/0$ occur precisely once. Thus, there are five alternating sign matrices determined by their X-rays. We can now state our main result:

Theorem 1.1. There is an explicit bijection between Dyck paths of semilength $n$ and $n \times n$-alternating sign matrices which are determined by their X-rays.

The coincidence described by the theorem was observed when submitting the statistic counting the number of alternating sign matrices with the same X-rays to the online database of combinatorial statistics FindStat [2] and looking at the first few generating functions automatically produced there. We currently have no explanation for any of the other terms in the distribution.

[1] http://www.findstat.org/St000889
2. The bijection

The map $A$ from Dyck paths to alternating sign matrices is defined as follows, see Figure 1 for an example. For more visual clarity in the pictures, we use (+)s and (−)s instead of 1s and −1s and omit 0s.

- Draw the Dyck path in an $n \times n$ square, beginning in the top left corner, taking east and south steps and terminating in the bottom right corner, never going below the main diagonal of the matrix.
- Add the reflection through the main diagonal of the Dyck path to the picture.
- For each peak of the Dyck path, fill the cells lying between the peak and its mirror image on the antidiagonal with 1s.
- For each valley of the Dyck path, fill the cells lying between the valley and its mirror image on the antidiagonal with −1s.
- Fill the remaining cells with 0s.

![Figure 1. The image of A Dyck path](image)

3. A map on diagonally symmetric alternating sign matrices

Because transposing a matrix preserves the X-ray, only diagonally symmetric alternating sign matrices may be reconstructible from their X-ray. We now present a map $M$ on diagonally symmetric alternating sign matrices that preserves the X-ray and is the identity precisely on the matrices in the image of the map $A$ from the previous section. Let $A$ be a diagonally symmetric alternating sign matrix, then $M(A)$ is obtained as follows, see Figure 2 for an example.

- Imagine a sun in the north-east, such that the 1s in $A$ cast shadows, and trace out a Dyck path by following the shadow line.
- Reflect the entries of $A$ which are strictly south-west of the entries just below the Dyck path through the subdiagonal.
- Into each cell just south-west of a valley of the Dyck path which is not on the subdiagonal and which contains a 0, place a −1, and place a 1 in the cell reflected through the subdiagonal.

Note that the Dyck path constructed in the first step returns to the main diagonal exactly once for each direct summand of $A$, regarding $A$ as a block diagonal matrix. Thus, the map $M$ is such that it can be applied to each direct summand of $A$ individually.
Lemma 3.1. The map $\mathcal{M}$, applied to a diagonally symmetric alternating sign matrix, produces an alternating sign matrix.

Proof. Let $A$ be a diagonally symmetric alternating sign matrix. Let us call the region in $A$ symmetric with respect to the subdiagonal, whose south-west border is the reflected Dyck path, the shade of $A$. This is the shaded region in Figure 2.

Consider a column $c$ of $A$, and its reflection $r$ through the subdiagonal. Thus, when $c$ is the first column, $r$ is the second row of $A$.

Suppose first that the top most nonzero entry of $c$ and the right most nonzero entry of $r$ within the shade of $A$ are both 1. This is the case when $c$ (or $r$) does not contain a peak of the Dyck path, or the valley below (or to the left of) the peak contains a $-1$.

In this case, column $c$ and row $r$ of $\mathcal{M}(A)$ satisfy the alternating sign matrix conditions, because after reflecting through the subdiagonal the top most nonzero entry of all columns, and the right most nonzero entry of all rows within the shade is 1.

Let us now consider the second scenario, where the top most nonzero entry of $c$ within the shade of the original matrix $A$ is $-1$. In the example of Figure 2 this happens in the sixth column.

In this case, the Dyck path must have a peak in this column. Let $v$ be the cell just south-west of the valley below the peak. Note that $v$ must contain a 0.

The cell $v$ must be strictly above the diagonal, because otherwise the reflection of the $-1$ below it through the main diagonal would lie on or above the Dyck path. Thus, by definition of $\mathcal{M}$, we place a $-1$ into the cell $v$. The effect of this is that column $c$ of $\mathcal{M}(A)$ is alternating.

Furthermore, we place a 1 in the cell $v'$ corresponding to $v$ reflected through the subdiagonal. This satisfies the alternating sign matrix conditions, because after reflecting through the subdiagonal, the row containing $v'$ begins with a $-1$.

Lemma 3.2. The map $\mathcal{M}$, applied to a diagonally symmetric alternating sign matrix $A$, is the identity if and only if $A$ is in the image of $A$.

Proof. If $A$ is in the image of $A$, the shade of $A$ is symmetric. Moreover, the cells just south-west of the valleys which are above the subdiagonal all contain $-1$s. Thus, $\mathcal{M}(A) = A$.

Otherwise, since $A$ is symmetric, and the shade of $A$ is reflected through the subdiagonal, $\mathcal{M}(A)$ cannot be symmetric.
4. Reconstructing the alternating sign matrix

To complete the proof of the main theorem, we have to show the following:

**Lemma 4.1.** The X-ray corresponding to an alternating sign matrix in the image of \( A \) determines the matrix unambiguously.

**Proof.** Consider the antidiagonal sums beginning at the north-west corner. Suppose that the entries of the first \( k \) antidiagonals are uniquely determined by their X-rays \( x_1/x_2/\ldots/x_k \), and suppose that \( x_k \neq 0 \), \( x_{k+1} = \cdots = x_{\ell-1} = 0 \) and \( x_\ell \neq 0 \).

For simplicity, assume that \( x_k > 0 \). By hypothesis, the alternating sign matrix then has the following form:

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

Let us first note that there cannot be any nonzero entries on the antidiagonals \( k+1, \ldots, \ell-1 \), since all these have sum zero. More precisely, suppose for the sake of contradiction that there is such an antidiagonal and consider the first of these. Because every row and every column of an alternating sign matrix must begin with a 1, this antidiagonal can have \(-1\)s only in the triangular region south-east of the sequence of 1s in the \( k\)-th antidiagonal - shaded grey in the example above. However, there cannot be any 1s on the same antidiagonal, necessarily outside of this triangular region: any such 1 below the main diagonal would be followed by another 1 in the same column above it.

We now distinguish two cases: if \( x_\ell < 0 \), by hypothesis \( x_\ell \) is so large that all cells of the antidiagonal within the triangular region defined above are filled with \(-1\)s. Thus, in this case the entries on the \( \ell\)-th antidiagonal are also uniquely determined by the antidiagonal sum.

On the other hand, if \( x_\ell > 0 \), by hypothesis \( x_\ell \) is so large that all cells of the antidiagonal that lie within the triangular region shaded red in the example above are filled with 1s. Since there cannot be any 1s on the same antidiagonal outside of the red triangular region, also in this case the entries of the \( \ell\)-th antidiagonal are uniquely determined by their sum.

\[\square\]

**References**

[1] Cecilia Bebeacua et al. “On the X-rays of permutations”. In: Proceedings of the Workshop on Discrete Tomography and its Applications. Vol. 20. Electron. Notes Discrete Math. Elsevier Sci. B. V., Amsterdam, 2005, pp. 193–203. doi: [10.1016/j.endm.2005.05.063](https://doi.org/10.1016/j.endm.2005.05.063) arXiv: [math.CO/0506334](https://arxiv.org/abs/math.CO/0506334)

[2] Martin Rubey, Christian Stump, et al. FindStat - the combinatorial statistics database. 2017. URL: [http://findstat.org](http://findstat.org)

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