A nonlinear adiabatic theorem for coherent states

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Abstract

We consider the propagation of wave packets for a one-dimensional nonlinear Schrödinger equation with a matrix-valued potential, in the semi-classical limit. For an initial coherent state polarized along an eigenvector, we prove that the nonlinear evolution preserves the separation of modes, in a scaling such that nonlinear effects are critical (the envelope equation is nonlinear). The proof relies on a fine geometric analysis of the role of spectral projectors, which is compatible with the treatment of nonlinearities. We also prove a nonlinear superposition principle for these adiabatic wave packets.

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1. Introduction

We consider the semi-classical limit $\varepsilon \to 0$ for the nonlinear Schrödinger equation

\begin{equation}
\begin{cases}
\text{i}\varepsilon \partial_t \psi + \frac{\varepsilon^2}{2} \partial_x^2 \psi = V(x) \psi + \Lambda |\psi|^2 \psi, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\
\psi|_{t=0} = \psi_0
\end{cases}
\end{equation}

where $\Lambda \in \mathbb{R}$. The data $\psi_0$ and the solution $\psi(t)$ are vectors of $C^N$, $N \geq 1$. The quantity $|\psi|^2_{C^N}$ denotes the square of the Hermitian norm in $C^N$ of the vector $\psi$. Finally, the potential $V$ is smooth and valued in the set of $N$ by $N$ Hermitian matrices.

Typically, when $N = 2$, such systems as (1.1) model a binary mixture of Bose–Einstein condensates (double condensate), under the effect of trap potentials [15]. Experimentally, the superfluidity of a double condensate is investigated by turning off the trap potentials in finite time, and letting the condensates expand freely, so even the study of the influence of $V$ in (1.1) on finite time intervals is relevant. Note that the analysis presented below becomes
We say that a function \( i \partial_t \psi_n + \frac{1}{2} \Delta \psi_n = V_n \psi_n + \sum_{k=1}^N \alpha_{n,k} \psi_k + \left( \sum_{k=1}^N g_{n,k} |\psi_k|^2 \right) \psi_n. \) (1.2)

We point out that we consider (1.1) in 1 space dimension: the analysis below is easily adapted to the case where \( |\psi|^2 \|_{L^2} \) is replaced by an anisotropic quadratic form such as \( \sum_{k=1}^N g_{n,k} |\psi_k|^2 \), \( g_{n,k} \in \mathbb{R} \). Finally, we want to stress the important literature about the numerical analysis of (1.2); see for example [1], where also the limit \( \varepsilon \to 0 \) in (1.1) is presented as an asymptotic regime for (1.2). Let us now define precisely the mathematical frame in which we are going to work.

**Definition 1.1.** We say that a function \( f \) is at most quadratic if \( f \in C^\infty(\mathbb{R}) \) and for all \( k \geq 2 \), \( f^{(k)} \in L^\infty(\mathbb{R}) \).

We make the following assumptions on the potential \( V \):

**Assumption 1.2.**

1. We have \( V(x) = D(x) + W(x) \) with \( D, W \in C^\infty(\mathbb{R}, \mathbb{R}^{N \times N}) \), \( D \) diagonal with at most quadratic coefficients, and \( W \) symmetric and bounded as well as its derivatives, \( W \in W^{\infty,\infty}(\mathbb{R}) \).
2. The matrix \( V \) has \( P \) distinct, at most quadratic, eigenvalues \( \lambda_1, \ldots, \lambda_P \) and
   \[ \exists c_0, n_0 \in \mathbb{R}^+, \forall j \neq k, \forall x \in \mathbb{R}, |\lambda_j(x) - \lambda_k(x)| \geq c_0 |x|^{-n_0}. \] (1.3)

Note that these assumptions are perfectly consistent with the models presented in [7], where \( D \) corresponds to a magnetic (or optical) trap, and \( W \) is constant. In addition, the gap condition (1.3) implies that the multiplicity of the eigenvalues is constant: this assumption allows us to avoid the problem of eigenvalue crossings, which is known to induce serious difficulties in the linear situation. In particular, if the potential has eigenvalue crossings, the adiabatic decoupling may fail (see for instance [10, 13, 14]). One does not expect the nonlinear case to be easier to handle.

Under these assumptions (the first point suffices), we can prove global existence of the solution \( \psi^\varepsilon \) for fixed \( \varepsilon > 0 \):

**Lemma 1.3.** If \( V \) satisfies assumption 1.2 and \( \psi_0^\varepsilon \in L^2(\mathbb{R}) \), there exists a unique, global solution to (1.1):

\[ \psi^\varepsilon \in C(\mathbb{R}; L^2(\mathbb{R})) \cap L^\infty_t L^2_{loc}(\mathbb{R}; L^4(\mathbb{R})). \]

The \( L^2 \)-norm of \( \psi^\varepsilon \) does not depend on time: \( \| \psi^\varepsilon(t) \|_{L^2(\mathbb{R})} = \| \psi_0^\varepsilon \|_{L^2(\mathbb{R})}, \forall t \in \mathbb{R}. \)

The proof of this lemma is sketched in appendix A.

In this nonlinear setting, the size of the initial data is crucial. As in [5], we choose to consider initial data of order 1 (in \( L^2 \)), and to introduce a dependence upon \( \varepsilon \) in the coupling constant (note that the nonlinearity is homogeneous). This leads to the equation

\[ i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \partial_x^2 \psi^\varepsilon = V(x) \psi^\varepsilon + A \varepsilon^{2\beta} |\psi^\varepsilon|^2 \|_{C^\infty} \psi^\varepsilon, \]

and we choose the exponent \( \beta = 3/4 \), which is critical for the type of initial data we want to consider (coherent state) when the potential \( V \) is scalar (see [5]). We are left with the nonlinear semi-classical Schrödinger equation

\[ i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \partial_x^2 \psi^\varepsilon = V(x) \psi^\varepsilon + A \varepsilon^{3/2} |\psi^\varepsilon|^2 \|_{C^\infty} \psi^\varepsilon, \quad \psi^\varepsilon_{\beta=0} = \psi_0^\varepsilon. \] (1.4)
We focus on the initial data which are perturbation of wave packets
\[
\psi_0^\varepsilon(x) = e^{-1/4} e^{i\varepsilon(x-x_0)/\varepsilon} a \left( \frac{x - x_0}{\sqrt{\varepsilon}} \right) \chi(x) + r_0^\varepsilon(x),
\]
where the initial error satisfies
\[
\|r_0^\varepsilon\|_{L^1(R)} + \|\varepsilon r_0^\varepsilon\|_{L^1(R)} + \|\varepsilon \partial_x r_0^\varepsilon\|_{L^1(R)} = O(\varepsilon^\kappa) \text{ for some } \kappa > \frac{1}{4}.
\]
The profile \(a\) belongs to the Schwartz class, \(a \in S(R; C)\), and the initial datum is polarized along an eigenvector \(\chi(x) \in C^\infty(R; C^N)\);
\[
V(x)\chi(x) = \lambda_1(x)\chi(x), \quad \text{with } |\chi(x)|_{C^\infty} = 1.
\]
Note that \(\lambda_1\) is simply a notation for some eigenvalues, up to a renumbering of eigenvalues. The \(L^2\)-norm of \(\psi_0^\varepsilon\) is independent of \(\varepsilon\), \(\|\psi_0^\varepsilon\|_{L^2(R)} = \|a\|_{L^2(R)}\). As pointed out above, this is equivalent to considering (1.1) with initial data of the same form (1.5), but of order \(\varepsilon^{3/4}\) in \(L^2(R)\). The evolution of such data when \(a\) is a Gaussian has been extensively studied by Hagedorn on the one hand, and by Hagedorn and Joyce on the other hand, in the linear context \(\Lambda = 0\) (see [13, 14]). These data are also particularly interesting for numerics (see [18] and references therein).

Because of the gap condition, the matrix \(V\) has smooth eigenprojectors and smooth eigenvalues of constant multiplicity (see [17]). Moreover, the gap condition (1.3) also implies that we control the growth of the eigenprojectors (see lemma B.2). Note however that in dimension 1 \((x \in R)\), one can have smooth eigenprojectors without any gap condition. We explain this fact below and give an example of projectors that we can consider; we also illustrate why things may be more complicated in higher dimensions \((d \geq 2)\).

**Example 1.4.** For \(N = 2\) and \(x \in R\), consider
\[
V(x) = (ax^2 + b)I_d + \begin{pmatrix} u(x) & v(x) \\ v(x) & -u(x) \end{pmatrix},
\]
for \(a, b \in R\), and \(u\) and \(v\) smooth and bounded with bounded derivatives. Such a potential satisfies assumption 1.2. Its eigenvalues are the two functions
\[
\lambda^\pm(x) = ax^2 + b \pm \sqrt{u(x)^2 + v(x)^2}.
\]
These functions are clearly smooth outside the set of points \(x_0\) such that \(u(x_0)^2 + v(x_0)^2 = 0\). In addition, for such points, one can renumber the modes in order to build smooth eigenvalues. More precisely, observe first that if \(u(x)^2 + v(x)^2 = O((x - x_0)^\infty)\) close to \(x_0\), the functions \(\lambda^\pm\) are smooth close to \(x_0\). Moreover, if \(u(x)^2 + v(x)^2 = (x - x_0)^k f(x)\) with \(f(x_0) \neq 0\), necessarily \(f(x_0) > 0\) and \(k = 2p\), so we have
\[
\lambda^\pm(x) = ax^2 + b \pm |x - x_0|^p \sqrt{f(x)}.
\]
For \(p\) even these functions are again smooth. However, when \(p\) is odd, they are no longer smooth and we perform a renumbering of the eigenfunctions, observing that
\[
x \mapsto ax^2 + b + (x - x_0)^p \sqrt{f(x)}
\]
are smooth eigenvalues of \(V\) close to \(x_0\).

**Example 1.5.** Resume the above example, now with \(x \in R^d, d \geq 2\). The smoothness of the eigenvalues is no longer guaranteed: suppose \(u(x) = x_1\) and \(v(x) = x_2\), then the functions \(\lambda_{\pm}\) are not smooth and one cannot find any renumbering which makes them smooth. Note that the gap condition (1.3) is not satisfied and that the multiplicity of \(\lambda_{\pm}\) is 2 in \(x = 0, 1\) elsewhere.

**Example 1.6.** For an example of a potential which satisfies (1.2), we simply choose \(V\) as in (1.7) with
\[
c_u u(x) = c_v v(x) = (x)^{\gamma_0}, \quad c_u^2 + c_v^2 \neq 0.
\]
1.1. The ansatz

We consider the classical trajectories \((x(t), \xi(t))\) solutions to

\[
\dot{x}(t) = \xi(t), \quad \dot{\xi}(t) = -\nabla \lambda_1(x(t)), \quad x(0) = x_0, \quad \xi(0) = \xi_0.
\]  

(1.8)

Because \(\lambda_1\) is at most quadratic, the classical trajectories grow at most exponentially in time (see e.g. [5]):

\[
\exists C > 0, \quad |\xi(t)| + |x(t)| \lesssim e^{Ct}.
\]  

(1.9)

We denote by \(S\) the action associated with \((x(t), \xi(t))\):

\[
S(t) = \int_0^t \left( \frac{1}{2}|\xi(s)|^2 - \lambda_1(x(s)) \right) ds.
\]  

(1.10)

We consider the function \(u = u(t, y)\) solution to

\[
i\partial_t u + \frac{1}{2} \partial_y^2 u = \frac{1}{2} \lambda_1'(x(t)) y^2 u + \Lambda |u|^2 u, \quad u(0, y) = a(y),
\]  

(1.11)

and we denote by \(\phi_\varepsilon\) the function associated with \(u, x, \xi, S\) by

\[
\phi_\varepsilon(t, x) = \varepsilon^{-1/4} u \left( t, \frac{x - x(t)}{\sqrt{\varepsilon}} \right) e^{i(S(t) + \xi(t)(x - x(t)))/\varepsilon}.
\]  

(1.12)

Global existence of \(u\) and control of its derivatives and momenta are proved in [6]. More precisely, we have the following result.

**Theorem 1.7 (From [6]).** Suppose \(a \in \mathcal{S}(\mathbb{R})\). There exists a unique, global solution \(u \in C(\mathbb{R}; L^2(\mathbb{R})) \cap L^8_{\text{loc}}(\mathbb{R}; L^4(\mathbb{R}))\) to (1.11). In addition, for all \(k, p \in \mathbb{N}\),

\[
\langle y \rangle^k \partial^p_y u \in C(\mathbb{R}; L^2(\mathbb{R}))
\]  

and

\[
\forall k, p \in \mathbb{N}, \quad \exists C > 0, \quad \forall t \in \mathbb{R}^+, \quad \|\langle y \rangle^k \partial^p_y u(t, \cdot)\|_{L^2(\mathbb{R})} \lesssim e^{Ct}.
\]  

(1.13)

In particular, note that \(\partial^p_y u(t, \cdot)\) is in \(L^\infty\) for all \(p \in \mathbb{N}\). These results have consequences on \(\phi_\varepsilon\). As far as the \(L^\infty\) norm is concerned, we infer, using (1.9),

\[
\forall p \in \mathbb{N}, \quad \|\varepsilon^{1/4} \partial_x^p \phi_\varepsilon(t)\|_{L^\infty} \lesssim e^{-1/4} e^{Ct}.
\]  

(1.14)

We use the time-dependent eigenvectors constructed in [14] (see also [13, 22]). To make the notation precise, we denote by \(d_j\) the multiplicity of the eigenvalue \(\lambda_j, 1 \leq j \leq P\) (note that \(\sum_{1 \leq j \leq P} d_j = N\)).

**Remark 1.8.** As we have already noted, condition (1.3) implies that the dimension \(d_j\) of the eigenspace associated with \(\lambda_j(x)\) does not depend on \(x\). The issue of eigenvalue crossing in the nonlinear case is a challenging issue.

**Proposition 1.9.** There exists a smooth orthonormal family \((\chi_\ell(t, x))_{1 \leq \ell \leq d_1}\) such that for all \(t\), \((\chi_\ell(t, x))_{1 \leq \ell \leq d_1}\) spans the eigenspace associated with \(\lambda_1, \chi^1(0, x) = \chi(x)\) and for \(m \in \{1, \ldots, d_1\}\),

\[
(\chi^m(t, x), \partial_t \chi_\ell(t, x) + \varepsilon(x) \partial_x \chi_\ell(t, x))_{C^0} = 0.
\]  

(1.15)

Moreover, for \(\ell \in \{1, \ldots, d_1\}, k, p \in \mathbb{N}\), there exists a constant \(C = C(p, k)\) such that

\[
|\partial^p_x \partial^k_x \chi_\ell(t, x)|_{C^0} \lesssim C e^{Ct} \langle x \rangle^{(k+p)(1+n_0)},
\]

where \(n_0\) appears in (1.3).
Note that equation (1.15) for \( m = \ell \) is true as soon as the eigenvector \( \chi^{\ell} \) is normalized and real-valued.

Let us motivate the use of time-dependent eigenvectors. The gap condition (1.3) guarantees the existence of complex-valued smooth normalized eigenvectors. Using these vectors leads to estimating their derivatives. For real-valued vectors, the derivative of a smooth normalized vector is orthogonal to this vector; however, for complex-valued vectors, it is only the real part of the scalar product of a vector with its derivative which is 0. Then, either it is possible to manage so that the imaginary part also vanishes by a smooth gauge transform (multiplication by a phase term of the form \( e^{i\theta(x)} \)), or it is not possible to do so. In the latter situation, the Laplace operator in the scalar Schrödinger equation for the projection of \( \psi^\varepsilon \) on one mode has to be replaced by a magnetic Laplace operator \( \varepsilon^2 (i\nabla_x - A(x))^2 \), where \( A(x) \) is the gauge potential of the so-called Berry connection (see [22, paragraph 2.3]). The analysis of magnetic nonlinear Schrödinger equation is rather complicated and the use of time-dependent eigenvectors of proposition 1.9 allows us to avoid this difficulty. The parallel transport stated in equation (1.15) can be thought of as a time-dependent gauge transform which allows us to eliminate the nonvanishing imaginary part of some of the scalar products of the eigenvectors with their derivatives.

The construction of these time-dependent eigenvectors is recalled in appendix C, where the control of their growth is also established.

**Notation.** In the case of a single coherent state, we complete the family \( (\chi^{\ell}(t, x))_{1 \leq \ell \leq d_1} \) as an orthonormal basis \( (\chi^j)_{1 \leq j \leq P} \) of \( C^N \) as follows:

- \( \chi^1 = \chi^{\ell} \),
- for \( j \geq 2 \) and \( 1 \leq \ell \leq d_j \), \( \chi^j \) does not depend on time,
- for \( j \geq 2 \), \( (\chi^j)_{1 \leq j \leq d_j} \) spans the eigenspace associated with \( \lambda_j \).

### 1.2. The results

We prove that there is adiabatic decoupling for the solution of (1.4) with initial data which are coherent states of the form (1.5): the solution maintains the same form and remains in the same eigenspace.

**Theorem 1.10.** Let \( a \in S(\mathbb{R}) \) and \( r_0^\varepsilon \) satisfy (1.6). Under assumption 1.2, consider \( \psi^\varepsilon \) solution to the Cauchy problem (1.4)–(1.5), and the approximate solution \( \psi^\varepsilon \) given by (1.12). There exists a constant \( C > 0 \) such that the function

\[
\sup_{|t| \leq C \log \log \frac{1}{\varepsilon}} (\|w^\varepsilon(t)\|_{L^2} + \|x w^\varepsilon(t)\|_{L^2} + \|\varepsilon \partial_x w^\varepsilon(t)\|_{L^2}) \longrightarrow 0.
\]

This adiabatic decoupling between the modes is well known in the linear setting and is at the basis of numerous results on semi-classical Schrödinger operator with matrix-valued potential in the framework of Born–Oppenheimer approximation for molecular dynamics. On this subject, the reader can consult the paper of Martinez and Sordoni [19], the paper of Spohn and Teufel [21], or the book of Teufel [22] for a review on the topic (see also [2] for an adiabatic result in a nonlinear context and [16] for application of adiabatic theory to the derivation of resolvent estimates).

The proof of validity of this approximation is established on time intervals which are not considerably large, since they are of order \( \log \log \frac{1}{\varepsilon} \). Note however that, as noted in section 1, even the case of finite time intervals may be interesting for physical interpretations.
Remark 1.11. Suppose that $V$ depends on $\varepsilon$ with $V' = D + \varepsilon W$, where $D$ and $W$ are as in assumption 1.2; this is so in the model presented in [1]. Then the above result remains true for $|t| \leq C \log(1/\varepsilon)$; we gain one logarithm. See remark 3.1 for the key arguments. Also, the assumption on the initial error can be relaxed: to prove the analogue of theorem 1.10 with an approximation in $L^2$ up to $C \log(1/\varepsilon)$, (1.6) can be replaced with

$$\exists \delta > 0, \quad \|r_0^\varepsilon\|_{L^2(R)} \leq \varepsilon^\delta \quad \text{as } \varepsilon \to 0.$$ 

The rate in (1.6) is due to the fact that we cannot use Strichartz estimates here.

Remark 1.12. In the linear setting, most semi-classical approximations are known to hold until the Ehrenfest time $\log(1/\varepsilon)$ (see [3, 4, 9] or [20]). More precisely, Bouzouina and Robert have proved in [4] that semi-classical estimates hold, in general, up to time $C_0 \log(1/\varepsilon)$ and precise constants $C_0$ are calculated. Moreover, they have found in [4] that this time can be improved to $\varepsilon^{-b_0}$, for some precise values of $b_0 > 0$ in very specific situations (see theorem 1.13 and remark 1.15 there, for integrable systems). On the other hand, they give examples in [4, section 6], which illustrate the sharpness of Ehrenfest time.

It is also interesting to analyse the evolution of a solution associated with data which are the superposition of two data of the studied form. We suppose

$$\psi_0^\varepsilon(x) = \psi_1^\varepsilon(0, x) \chi_1(x) + \psi_2^\varepsilon(0, x) \chi_2(x),$$

where both functions $\psi_1^\varepsilon$ and $\psi_2^\varepsilon$ have the form (1.12), for two eigenvectors of $V$, $\chi_1$ and $\chi_2$, and phase space points $(x_1, \xi_1)$ and $(x_2, \xi_2)$. We assume

$$(\chi_1, x_1, \xi_1) \neq (\chi_2, x_2, \xi_2).$$

We associate with the phase space points $(x_j, \xi_j)$, $j \in \{1, 2\}$ the classical trajectories $(x_j(t), \xi_j(t))$, and the action $S_j(t)$ associated with $\lambda_j$, such that

$$V(x) \chi_j(x) = \lambda_j(x) \chi_j(x).$$

Note that we may have $\tilde{\lambda}_1 = \tilde{\lambda}_2$. Let us denote by $\chi_j^1(t), j \leq p$ a time-dependent orthonormal basis of eigenvectors defined according to proposition C.1 (see also proposition 1.9 above) with $\chi_1^1(0, x) = \chi_1(x), \chi_2^1(x) = \chi_2^1(0, x)$ if $\tilde{\lambda}_1 = \tilde{\lambda}_2, \chi_2^1(x) = \chi_2^1(0, x)$ otherwise, and by $\phi_j^\varepsilon$ the ansatz defined by (1.12). To unify the presentation, we write

$$\chi^1 = \chi_1^1, \quad \chi^2 = \begin{cases} \chi_1^1 & \text{if } \tilde{\lambda}_1 = \tilde{\lambda}_2, \\ \chi_2^1 & \text{otherwise}. \end{cases}$$

Theorem 1.13. Set $E_j = \frac{\kappa_{1,2}}{2} + \tilde{\lambda}_j(x_j)$ for $j \in \{1, 2\}$ and suppose

$$\Gamma = \inf_{x \in R} |\tilde{\lambda}_1(x) - \tilde{\lambda}_2(x) - (E_1 - E_2)| > 0.$$

There exists $C > 0$ such that the function

$$w^\varepsilon(t) = \psi^\varepsilon(t) - \phi_1^\varepsilon \chi^1(t, x) - \phi_2^\varepsilon \chi^2(t, x)$$

satisfies

$$\sup_{t \leq \log \frac{1}{\varepsilon}} (\|w^\varepsilon(t)\|_{L^2} + \|x w^\varepsilon(t)\|_{L^2} + \|\varepsilon \partial_t w^\varepsilon(t)\|_{L^2}) \to 0, \quad \varepsilon \to 0.$$ 

Note that if $\tilde{\lambda}_1 = \tilde{\lambda}_2$, one recovers the condition $E_1 \neq E_2$ of [5]. The proof of theorem 1.13 follows the same lines as in [5, section 6]. The constant $\Gamma$ controls the frequencies of time interval where trajectories cross.
Remark 1.14. In finite time, the situation is different whether \( \tilde{\lambda}_1 = \tilde{\lambda}_2 \) or not. If \( \tilde{\lambda}_1 = \tilde{\lambda}_2 \), the superposition holds in finite time without any condition on \( \Gamma \); this comes from the fact that the trajectories \( x_1(t) \) and \( x_2(t) \) only cross on isolated points (see [5]). However, if \( \tilde{\lambda}_1 \neq \tilde{\lambda}_2 \) one may have \( x_1(t) = x_2(t) \) on intervals of nonempty interior: the condition \( \Gamma \neq 0 \) prevents this situation from happening. For example, if

\[
V(x) = \begin{pmatrix}
\cos x & \sin x \\
\sin x & -\cos x
\end{pmatrix} + v(x) \text{Id}
\]

with \( v \) smooth and at most quadratic, we have \( \lambda_1(x) = v(x) - 1 \) and \( \lambda_2(x) = v(x) + 1 \): classical trajectories for both modes, issued from the same point of the phase space, are equal.

1.3. Strategy of the proof of theorem 1.10

The proof is more complicated than in the scalar case [5], due to the fact that the spectral projectors do not commute with the Laplace operator. From this perspective, a much finer geometric understanding is needed and we revisit [12–14, 19, 21, 22] by adapting to our nonlinear context ideas contained therein.

Observe first that the function \( \phi_\epsilon \) satisfies

\[
i\epsilon \partial_t \phi_\epsilon + \frac{\epsilon^2}{2} \partial^2_x \phi_\epsilon = T_\epsilon(t, x) \phi_\epsilon + \Lambda \epsilon^{3/2} |\phi_\epsilon|_C^2 \phi_\epsilon,
\]

where

\[
T_\epsilon(t, x) = \lambda_1(x(t)) + \lambda_1'(x(t))(x - x(t)) + \frac{1}{2} \lambda''(x(t))(x - x(t))^2.
\]

This term corresponds to the beginning of the Taylor expansion of \( \lambda_1 \) about \( x(t) \). Therefore, the function \( u_\epsilon(t, x) = \phi_\epsilon(t, x) - \phi_\epsilon(t, x)^1(t, x) \) satisfies \( u_{\epsilon|t=0} = r_0 \) and

\[
i\epsilon \partial_t u_\epsilon + \frac{\epsilon^2}{2} \partial^2_x u_\epsilon - V(x) u_\epsilon = \epsilon N\tilde{L}_\epsilon(t, x) + \epsilon \tilde{L}_\epsilon(t, x)
\]

where

\[
N\tilde{L}_\epsilon = \Lambda \epsilon^{1/2} (|\phi_\epsilon^1|_C^2 + |u_\epsilon|_C^2 (|\phi_\epsilon^1 + u_\epsilon| - |\phi_\epsilon^1| |\phi_\epsilon^1| - |\phi_\epsilon|^2 |\phi_\epsilon|_C^2)),
\]

\[
\tilde{L}_\epsilon = i\partial_t \phi_\epsilon + \epsilon \partial_x \phi_\epsilon^1 \partial_x \phi_\epsilon + \epsilon \partial^2_x \phi_\epsilon^1 + \epsilon^{-1} (\lambda_1(x) - T_\epsilon) \phi_\epsilon^1.
\]

Since \( \phi_\epsilon \) is concentrated near \( x = x(t) \) at scale \( \sqrt{\epsilon} \), we have

\[
(\lambda_1(x) - T_\epsilon) \phi_\epsilon = \mathcal{O}(\epsilon \sqrt{\epsilon} e^{Ct}) \quad \text{in} \quad L^2(\mathbb{R}),
\]

where we have used theorem 1.7. The term \( \tilde{L}_\epsilon \) \textit{a priori} presents an \( \mathcal{O}(1) \) contribution, which is an obstruction to infer that \( u_\epsilon \) is small by applying Gronwall’s lemma. Observing that in view of the estimates on the classical flow (see (1.9))

\[
i\epsilon \partial_t \phi_\epsilon = i\xi(t) \phi_\epsilon + \mathcal{O}(\sqrt{\epsilon} e^{Ct}) \quad \text{in} \quad L^2(\mathbb{R}),
\]

we write,

\[
\tilde{L}_\epsilon = i(\partial_t \phi_\epsilon + \xi(t) \partial_x \phi_\epsilon) \phi_\epsilon + \mathcal{O}(\sqrt{\epsilon} e^{Ct}) \quad \text{in} \quad L^2(\mathbb{R}).
\]

The choice of the time-dependent eigenvectors ensures that, for all time, the \( \mathcal{O}(1) \) contribution of \( \tilde{L}_\epsilon \) is orthogonal to the first mode (the eigenspace associated with \( \lambda_1^1 \)). Then, to get rid of these terms, we introduce a correction term to \( u_\epsilon \). We set

\[
\theta_\epsilon(t, x) = u_\epsilon(t, x) + \epsilon g_\epsilon(t, x), \quad g_\epsilon(t, x) = \sum_{2 \leq j \leq P} \sum_{1 \leq l \leq d_j} g_{\epsilon, l,j}^j(t, x) \chi_j^j(x),
\]
where for $j \geq 2$ and for $1 \leq \ell \leq d_j$, the function $g_{j,\ell}^\varepsilon(t, x)$ solves the scalar Schrödinger equation
\begin{equation}
\label{eq:1.17}
\begin{aligned}
\frac{i\varepsilon}{2} \partial_t g_{j,\ell}^\varepsilon + \frac{\varepsilon^2}{2} \partial_x^2 g_{j,\ell}^\varepsilon - \lambda_j(x) g_{j,\ell}^\varepsilon &= \psi^\varepsilon r_{j,\ell}, \\
g_{j,\ell}^\varepsilon|_{t=0} &= 0,
\end{aligned}
\end{equation}
where
\begin{equation}
\label{eq:1.18}
r_{j,\ell}(t, x) = -i \left( \partial_t \chi^1_j(t, x) + \xi(t) \partial_x \chi^1_j(t, x), \chi^\ell_j(x) \right)_{C^0}.
\end{equation}

The function $\theta^\varepsilon(t)$ then solves
\begin{equation}
\label{eq:1.19}
\begin{aligned}
\frac{i\varepsilon}{2} \partial_t \theta^\varepsilon(t, x) + \frac{\varepsilon^2}{2} \partial_x^2 \theta^\varepsilon(t, x) &= V(x) \theta^\varepsilon(t, x) + \varepsilon NL^\varepsilon(t, x) + \varepsilon L^\varepsilon(t, x), \\
\theta^\varepsilon|_{t=0} &= \theta^0,
\end{aligned}
\end{equation}
with
\begin{equation}
\label{eq:1.20}
NL^\varepsilon = \Lambda e^{1/2} \left( |\psi^\varepsilon \chi^1 + \theta^\varepsilon - \varepsilon g^\varepsilon|_{C^0}^2 (\psi^\varepsilon \chi^1 + \theta^\varepsilon - \varepsilon g^\varepsilon) - |\psi^\varepsilon|^2 \psi^\varepsilon \chi^1 \right),
\end{equation}
\begin{equation}
\label{eq:1.21}
L^\varepsilon = \tilde{L}^\varepsilon + \left( \frac{i\varepsilon}{2} \partial_t + \frac{\varepsilon^2}{2} \partial_x^2 - V(x) \right) g^\varepsilon(t, x)
= \mathcal{O}(\sqrt{\varepsilon} e^{C't}) + \sum_{2 \leq j \leq P} \sum_{1 \leq \ell \leq d_j} \left[ \frac{\varepsilon^2}{2} \partial_x^2 \chi^\ell_j \right] g_{j,\ell}^\varepsilon,
\end{equation}
where the $\mathcal{O}(\sqrt{\varepsilon} e^{C't})$ holds in $L^2$. The proof of the theorem then follows from a precise control of the functions $\chi^\ell_j$ and $g_{j,\ell}^\varepsilon$, which is achieved in appendix C and section 2, respectively. Then, the analysis of $\theta^\varepsilon$ as $\varepsilon$ goes to zero by an energy method is presented in section 3.

1.4. Organization of the paper

The proof of theorem 1.10 relies, as a first step, on the geometric reduction presented in the previous paragraph. After that step, our goal is two-fold: first we show in section 2 that the correctors $g_{j,\ell}^\varepsilon$ are bounded uniformly in $\varepsilon \in [0, 1]$, and that their norms grow at most exponentially in time in the functional spaces naturally associated with our framework (proposition 2.1). This implies that proving that the error $w^\varepsilon$ is small is equivalent to proving that the modified error $\theta^\varepsilon$ is small.

This second task is achieved in section 3. Note that proposition 2.1 also implies that $L^\varepsilon$, defined by (1.21), is globally $\mathcal{O}(\sqrt{\varepsilon} e^{C't})$ (and not only the first term). Then (1.19) appears as a nonlinear equation, with a small source term $L^\varepsilon$, and an interaction term $NL^\varepsilon$, defined in (1.20). This term can in turn be thought of as linear in $\theta^\varepsilon$, so the error estimate in theorem 1.10 follows from Gronwall’s lemma. The main task in the nonlinear analysis presented in section 3 is to make the first part of the previous sentence rigorous, thanks to energy estimates and Gagliardo–Nirenberg inequalities.

Finally, theorem 1.13 is established in section 4. The strategy consists in resuming the same approach as in the case of a single coherent state. By doing so, we do not only double the number of terms, since new terms appear, due to the interactions of the approximating wave packets. These new terms are proved to be small on $[0, C \log \log \frac{1}{\varepsilon}]$ for some $C > 0$, provided that $\Gamma > 0$. 

2. Analysis of the correction terms

In this section, our aim is to prove that the correction terms $g_{j,\ell}(t)$ are uniformly bounded in $\varepsilon > 0$ for all $t > 0$ and to obtain an exponential control of their norm as $t$ grows. We will make use of the following norms defined for $p \in \mathbb{N}$,

$$\|f\|_{\Sigma^p} = \sup_{a^\beta \leq p} \|x^a \varepsilon^\beta f(b(x))\|_{L^2}.$$ 

We associate with this norm the functional space $\Sigma^p$ defined by

$$\Sigma^p = \{ f \in L^2(\mathbb{R}^d), \|f\|_{\Sigma^p} < \infty \}.$$

In view of (1.9) and (1.14), for all $p \in \mathbb{N}$, there exists $c(p)$ such that

$$\|\psi_{\varepsilon}(t)\|_{\Sigma^p} \lesssim e^{c(p)t}, \quad \forall t \geq 0.$$ 

(2.1)

We can obviously take $c(0) = 0$ by conservation of the $L^2$-norm, but in general, the norm of $\psi_{\varepsilon}$ in $\Sigma^p$ potentially grows exponentially in time (see [6]). We denote by $U_{\varepsilon}^k(t)$ the semi-group associated with the operator $-\frac{\varepsilon^2}{2} \partial^2_x + \lambda_k(x)$ and we observe that for $p \in \mathbb{N}$, there exists a constant $C(p)$ such that

$$\|U_{\varepsilon}^k(t)\|_{L(\Sigma^p)} \leq C(p)e^{C(p)|t|}.$$ 

(2.2)

The main result of this section is the following proposition 2.1. It comes from an asymptotic orthogonality property of the evolution operators $U_{\varepsilon}^j(t)$ and $U_{\varepsilon}^k(t)$ for $k \neq j$ which are involved in the definition of $g_{j,\ell}(t)$ (see lemma 2.2 below).

**Proposition 2.1.** For $p \in \mathbb{N}$, there exists $C(p)$ such that for all $j \geq 2$ and all $\ell \in \{1, \ldots, d_j\}$,

$$\|g_{j,\ell}(t)\|_{\Sigma^p} \lesssim e^{C(p)t}, \quad \forall t \geq 0,$$

where $g_{j,\ell}$ is defined in (1.17).

**Proof.** We use Duhamel’s formula and write

$$g_{j,\ell}(t) = \frac{1}{i\varepsilon} \int_0^t U_{\varepsilon}^j(t-s)(\psi^\varepsilon(s)r_{j,\ell}(s))\, ds.$$

In addition, if $\tilde{\psi}_{j,\ell}(t, x) = \psi^\varepsilon(t, x)r_{j,\ell}(t, x)$, then we have,

$$\left(\frac{i\varepsilon^2}{2} \partial^2_x - \lambda_1(x)\right)\tilde{\psi}_{j,\ell} = \frac{i\varepsilon^2}{2}(\partial_{x^j}r_{j,\ell}(t, x))^2[\psi^\varepsilon + \frac{\varepsilon^2}{2}(\partial_{x^j}r_{j,\ell}(t, x))]\psi^\varepsilon.$$

Therefore, we can write

$$\tilde{\psi}_{j,\ell}(t) = U_{\varepsilon}^j(t)\tilde{\psi}_{j,\ell}(0) - i \int_0^t U_{\varepsilon}^j(t-s)\tilde{\psi}_{j,\ell}(s)\, ds,$$

whence

$$g_{j,\ell}(t) = \frac{1}{i\varepsilon} \int_0^t U_{\varepsilon}^j(t-s)U_{\varepsilon}^j(s)\, ds\tilde{\psi}_{j,\ell}(0) - \frac{1}{\varepsilon} \int_0^t \int_0^s U_{\varepsilon}^j(t-s)U_{\varepsilon}^j(s-\tau)\tilde{\psi}_{j,\ell}(\tau)\, d\tau\, ds$$

$$= \frac{1}{i\varepsilon} \int_0^t U_{\varepsilon}^j(t-s)U_{\varepsilon}^j(s)\, ds\tilde{\psi}_{j,\ell}(0) - \int_0^t \left[ \frac{1}{\varepsilon} \int_0^\tau U_{\varepsilon}^j(t-s)U_{\varepsilon}^j(s-\tau)\, ds \right] \tilde{\psi}_{j,\ell}(\tau)\, d\tau.$$

In order to estimate these terms, we use lemma 2.2 which is proved at the end of this section.
Lemma 2.2. For \( T > 0 \) and \( k \neq j \), there exists a constant \( C \) such that
\[
\forall t \in [0, T], \quad \forall p \in \mathbb{N}, \quad \left\| \frac{1}{i \varepsilon} \int_0^t U_k^\varepsilon (-s) U_j^\varepsilon (s) \, ds \right\|_{L^\infty(S_{\varepsilon^{(p+3)n_0+n_1} \Sigma_1})} \leq C e^{Ct}.
\]

Lemma 2.2 yields
\[
\| g_{j,\ell}^\varepsilon (t) \|_{\Sigma_1 p \varepsilon} \lesssim e^{Ct} + \int_0^t e^{C(t-\tau)} \| \tilde{r}^\varepsilon (\tau) \|_{\Sigma_1 q \varepsilon} \, d\tau,
\]
with \( q = p + 2 + (p + 3)(1 + n_0) \). Let us now study \( \tilde{r}^\varepsilon \). We write \( \tilde{r}^\varepsilon = \tilde{r}_1^\varepsilon + \tilde{r}_2^\varepsilon \) with
\[
\tilde{r}_1^\varepsilon (t, x) = i \varepsilon r_{j,\ell} \varphi^\varepsilon + \frac{\varepsilon}{2} (\partial^2_x + r_{j,\ell}(t, x)) \varphi^\varepsilon.
\]
In view of corollary C.2 and of (2.1), we have for all \( q \in \mathbb{N} \),
\[
\| \tilde{r}_1^\varepsilon (t) \|_{\Sigma_1 q \varepsilon (R)} \lesssim e^{C(q)t}.
\]
A very rough estimate yields
\[
\| \tilde{r}_2^\varepsilon (t) \|_{\Sigma_1 q \varepsilon} \lesssim e^{Ct}.
\]
This completes the proof of proposition 2.1. \( \Box \)

To conclude this section, it remains to prove lemma 2.2.

Proof of lemma 2.2. We first observe that
\[
i \varepsilon \partial_t (U_k^\varepsilon (-t) U_j^\varepsilon (t)) = U_k^\varepsilon (-t) \varphi^\varepsilon (x) - \lambda_k (x) U_j^\varepsilon (t).
\]
Indeed, if \( f \in L^2(R) \) and \( f^\varepsilon (t) = U_k^\varepsilon (-t) U_j^\varepsilon (t) f \). We have
\[
i \varepsilon \partial_t f^\varepsilon (t, x) = -\left( -\frac{\varepsilon^2}{2} \partial^2_x + \lambda_k (x) \right) f^\varepsilon (t) + U_k^\varepsilon (-t) \left( -\frac{\varepsilon^2}{2} \partial^2_x + \lambda_j (x) \right) U_j^\varepsilon (t) f
\]
because \( U_k^\varepsilon (-t) \) commutes with \( -\frac{\varepsilon^2}{2} \partial^2_x + \lambda_k (x) \). We use equation (2.3) to perform an integration by parts:
\[
U_k^\varepsilon (-t) U_j^\varepsilon (t) = U_k^\varepsilon \left( -t \right) (\lambda_j - \lambda_k)^{-1} U_j^\varepsilon \left( -t \right) (\lambda_j - \lambda_k) U_j^\varepsilon (t)
\]
\[
= i \varepsilon U_k^\varepsilon \left( -t \right) (\lambda_j - \lambda_k)^{-1} U_j^\varepsilon \left( -t \right) \partial_t U_k^\varepsilon \left( -t \right) U_j^\varepsilon (t).
\]
Therefore,
\[
\frac{1}{i \varepsilon} \int_0^t U_k^\varepsilon (-s) U_j^\varepsilon (s) \, ds = \left[ U_k^\varepsilon (-s) (\lambda_j - \lambda_k)^{-1} U_j^\varepsilon (s) \right]_0^t
\]
\[
+ \int_0^t \partial_s (U_k^\varepsilon (-s) (\lambda_j - \lambda_k)^{-1} U_j^\varepsilon (s)) U_k^\varepsilon (-s) U_j^\varepsilon (s) \, ds.
\]
Set
\[
\gamma_{j,k} = (\lambda_k - \lambda_j)^{-1}.
\]
The behaviour of these functions as \( x \) goes to infinity is studied in appendix B (see lemma B.1). It is proven there that for all \( \beta \in \mathbb{N} \),
\[
|\partial^\beta x \gamma_{j,k}(x)| \lesssim (x)^{n_0 + |\beta| (1 + n_0)}.
\]
Since the propagators $U_k^\varepsilon(t)$ and $U_j^\varepsilon(t)$ map continuously $\Sigma^p$ into itself uniformly with respect to $\varepsilon$, we have
\[
\|U_k^\varepsilon(-s)Y_{j,k}U_k^\varepsilon(s)\|_{L^2(\Sigma^{p+1+n}\Sigma')} \lesssim e^{C(p)t},
\]
where in all these paragraphs, $C(p)$ denotes a generic constant depending only on the parameter $p \in \mathbb{N}$. Moreover, we observe that
\[
\partial_s(U_k^\varepsilon(-s)Y_{j,k}U_k^\varepsilon(s)) = \frac{1}{i\varepsilon} \left[ -\frac{\varepsilon^2}{2} \partial_x^2 + \lambda_{j,k} \right] U_k^\varepsilon(s).
\]
In view of
\[
\frac{1}{i\varepsilon} \left[ -\frac{\varepsilon^2}{2} \partial_x^2 + \lambda_{j,k} \right] = \frac{1}{i\varepsilon} \left[ -\frac{\varepsilon^2}{2} \partial_x^2 + \gamma_{j,k} \right] = i \gamma'_{j,k}(x)\varepsilon \partial_x + i \varepsilon \gamma''_{j,k}(x),
\]
and of
\[
\|\partial_s(U_k^\varepsilon(-s)\gamma'_{j,k}(x)\varepsilon \partial_x U_k^\varepsilon(s))\|_{L^2(\Sigma^{p+3+n}\Sigma')} + \|\partial_s(U_k^\varepsilon(-s)\gamma''_{j,k} U_k^\varepsilon(s))\|_{L^2(\Sigma^{p+3+n}\Sigma')} \lesssim e^{Ct},
\]
which comes from (2.2) and lemma B.1, we get
\[
\|\partial_s(U_k^\varepsilon(-s)Y_{j,k}U_k^\varepsilon(s))\|_{L^2(\Sigma^{p+2+n}\Sigma')} \lesssim e^{Ct},
\]
which concludes the proof. □

3. Consistency

We now prove theorem 1.10. We go back to equation (1.19), that we recall:
\[
\begin{cases}
\varepsilon i \partial_t \theta^\varepsilon(t, x) + \frac{\varepsilon^2}{2} \partial_x^2 \theta^\varepsilon(t, x) = V(x)\theta^\varepsilon(t, x) + \varepsilon NL^\varepsilon(t, x) + \varepsilon L^\varepsilon(t, x), \\
\theta^\varepsilon|_{t=0} = r^\varepsilon_0,
\end{cases}
\]
where $NL^\varepsilon$ and $L^\varepsilon$ are defined in (1.20) and (1.21), respectively. The standard $L^2$-estimate yields:
\[
\|\theta^\varepsilon(t)\|_{L^2} \leq \|r^\varepsilon_0\|_{L^2} + \int_0^t (\|NL^\varepsilon(s)\|_{L^2} + \|L^\varepsilon(s)\|_{L^2}) \, ds.
\]
In view of (1.21), proposition C.1 and proposition 2.1, we have
\[
\|L^\varepsilon(t)\|_{L^2} \lesssim \sqrt{\varepsilon e^{Ct}}.
\]
Moreover, we observe
\[
\|NL^\varepsilon(t)\|_{L^2} \lesssim \sqrt{\varepsilon} \|(\varphi^\varepsilon(t)\|^2 + |\theta^\varepsilon(t)\|^2_{L^2} + \varepsilon^2 |g^\varepsilon(t)|^2_{L^2})(\theta^\varepsilon(t) - \varepsilon g^\varepsilon(t))\|_{L^2} 
\lesssim \sqrt{\varepsilon} \|\varphi^\varepsilon(t)\|^2_{L^2} + \|\theta^\varepsilon(t)\|^2_{L^2} + \varepsilon^2 \|g^\varepsilon(t)\|^2_{L^2} + \varepsilon g^\varepsilon(t)\|_{L^2}.
\]
In view of (1.14), we have $\|\varphi^\varepsilon(t)\|_{L^\infty} \lesssim \varepsilon^{-1/2} e^{Ct}$. On the other hand, proposition 2.1 implies, in view of the Gagliardo–Nirenberg inequality
\[
\|f\|_{L^\infty} \lesssim \varepsilon^{-1/2} \|f\|_{L^2}^{1/2} \|\varepsilon \partial_x f\|_{L^2}^{1/2},
\]
the estimate
\[
\varepsilon^2 \|g^\varepsilon(t)\|^2_{L^2} \lesssim \varepsilon e^{Ct}.
\]
Therefore, it is natural to perform a bootstrap argument assuming, say
\[
\|\theta^\varepsilon(t)\|_{L^\infty} \lesssim \varepsilon^{-1/4} e^{Ct}.
\]
Note that we fixed the value of the constant in factor of the right-hand side equal to one. We did so because $\theta^\varepsilon$, as an error term, is expected to be smaller than $\phi^\varepsilon$ (the approximate solution) in the limit $\varepsilon \to 0$. As long as (3.2) holds, the $L^2$-estimate implies, in view of (1.6)
\[ \|\theta^\varepsilon(t)\|_{L^2} \lesssim \varepsilon^k + \int_0^t (\sqrt{\varepsilon} e^{Cs} + e^{Cs} \|\theta^\varepsilon(s)\|_{L^2}) \, ds. \]
By Gronwall’s lemma, we obtain
\[ \|\theta^\varepsilon(t)\|_{L^2} \leq C(\varepsilon^k + \sqrt{\varepsilon}) e^{Ct}. \] (3.3)
It remains to check how long the bootstrap assumption (3.2) holds. For this, we use the Gagliardo–Nirenberg inequality (3.1), and we look for a control of the norm of $\theta^\varepsilon(t)$ in $\Sigma^1$.

Differentiating the system (1.19) with respect to $x$, we find
\[ i\varepsilon \partial_t (\varepsilon \partial_x \theta^\varepsilon) + \frac{\varepsilon^2}{2} \partial_x^2 (\varepsilon \partial_x \theta^\varepsilon) = V(x) \varepsilon \partial_x \theta^\varepsilon + \varepsilon V'(x) \theta^\varepsilon + \varepsilon^2 \partial_x NL^\varepsilon + \varepsilon^2 \partial_x L^\varepsilon. \]
We observe that since $V$ is at most quadratic,
\[ |V'(x) \theta^\varepsilon| \lesssim (|\phi^\varepsilon(t, x)|^2 + |\theta^\varepsilon(t, x)|^2 + \varepsilon^2 |g^\varepsilon(t, x)|^2_{C^0}) |\partial_x \theta^\varepsilon(t, x)|_{C^0}. \]

By proposition 2.1, we have
\[ \|x L^\varepsilon(t)\|_{L^2} + \|\partial_x L^\varepsilon(t)\|_{L^2} \lesssim \sqrt{\varepsilon} e^{Ct}. \]
In addition,
\[ |x NL^\varepsilon(t, x)|_{C^0} \lesssim (|\phi^\varepsilon(t, x)|^2 + |\theta^\varepsilon(t, x)|^2_{C^0} + \varepsilon^2 |g^\varepsilon(t, x)|^2_{C^0}) |\partial_x \phi^\varepsilon(t, x)|_{C^0} \]
\[ + \varepsilon \|\phi^\varepsilon(t, x)\|_{C^0} \times \|\theta^\varepsilon(t, x)\|_{C^0} \times \|\theta^\varepsilon(t, x)\|_{C^0}. \]

Arguing as before and using again (1.14), we obtain that under (1.2) we have
\[ \|\partial_x \theta^\varepsilon(t)\|_{L^2} + \|x \theta^\varepsilon(t)\|_{L^2} \lesssim (\varepsilon^k + \sqrt{\varepsilon}) e^{Ct}. \]

The Gagliardo–Nirenberg inequality then implies
\[ \|\theta^\varepsilon(t)\|_{L^\infty} \lesssim \varepsilon^{-1/2} (\varepsilon^k + \sqrt{\varepsilon}) e^{Ct}. \]

We infer that (3.2) holds (at least) as long as
\[ (\varepsilon^k + 1) e^{Ct} \ll \varepsilon^{-1/4} e^{Ct}, \]
which is ensured provided that $t \leq C \log \log(\frac{1}{\varepsilon})$, for some suitable constant $C$, since $\kappa > 1/4$.

This concludes the bootstrap argument: we infer
\[ \sup_{|t| \leq C \log \log(\frac{1}{\varepsilon})} (\|\theta^\varepsilon(t)\|_{L^2} + \|x \theta^\varepsilon(t)\|_{L^2} + \|\partial_x \theta^\varepsilon(t)\|_{L^2}) \to 0. \]

Theorem 1.10 then follows from the above asymptotics, together with the relation $\theta^\varepsilon = w^\varepsilon + \varepsilon g^\varepsilon$, and proposition 2.1.
Remark 3.1. In the case where $V^\varepsilon = D + \varepsilon W$, as in remark 1.11, the proof can be adapted, in order to reproduce the argument given in [5]. The main point to note is that (local in time) Strichartz estimates are available for the propagator associated with $-\frac{C}{2}\partial_t^2 + D(x)$, thanks to [11]. Then in the presence of the power $\varepsilon$ in front of $W$, the potential $\varepsilon W$ can be considered as a source term in the error estimates: the factor $\varepsilon$ is crucial to avoid a singular power of $\varepsilon$ due to the presence of $\varepsilon$ in front of the time derivative in (1.19). The proof in [5, section 6] for the cubic, one-dimensional Schrödinger equation can be reproduced: another bootstrap argument can be invoked, which does not involve Gagliardo–Nirenberg inequalities, since a useful a priori estimate for the envelope $u$ is available.

4. Superposition

As explained in section 1, the only difficulty in the proof of theorem 1.13 is to treat a nonlinear interaction term. Indeed, we set

$$w^\varepsilon = \psi^\varepsilon - \psi_1^\varepsilon \chi^1 - \psi_2^\varepsilon \chi^2 + \varepsilon g^\varepsilon,$$

where $g^\varepsilon$ is the sum of two correction terms, similar to the one introduced in section 1.3. More precisely, set $p(1) = 1$, and $p(2) = 1$ if $\lambda_1 = \lambda_2$, $p(2) = 2$ otherwise. Define $g^\varepsilon = g_1^\varepsilon + g_2^\varepsilon$, with

$$g_1^\varepsilon = \sum_{1 \leq j \leq P, j \neq p(1)} \sum_{1 \leq \ell \leq d_j} g_{j,1,\ell}^\varepsilon(t, x) \chi_j^1(t, x),$$

$$g_2^\varepsilon = \sum_{1 \leq j \leq P, j \neq p(2)} \sum_{1 \leq \ell \leq d_j} g_{j,2,\ell}^\varepsilon(t, x) \chi_j^2(t, x),$$

where for $k = \{1, 2\}$, $j \neq p(k)$ and $1 \leq \ell \leq d_j$, the function $g_{j,k,\ell}^\varepsilon(t, x)$ solves the scalar Schrödinger equation

$$\text{i}\varepsilon \partial_t g_{j,k,\ell}^\varepsilon + \frac{\varepsilon^2}{2} \partial_x^2 g_{j,k,\ell}^\varepsilon - \lambda_j(x) g_{j,k,\ell}^\varepsilon = \psi^\varepsilon r_{j,k,\ell}, \quad g_{j,k,\ell}^\varepsilon(t=0) = 0, \quad (4.1)$$

where

$$r_{j,k,\ell}(t, x) = -i(\partial_x \chi_j(t, x) + \varepsilon \rho_{j,k}(t) \partial_x \chi_k(t, x), \chi_j(t, x)) e^{\varepsilon t}.$$

The function $w^\varepsilon(t)$ then solves

$$\text{i}\varepsilon \partial_t w^\varepsilon + \frac{\varepsilon^2}{2} \partial_x^2 w^\varepsilon = V(x) w^\varepsilon + \varepsilon NL^\varepsilon + \varepsilon L^\varepsilon, \quad w^\varepsilon(t=0) = 0,$$

with

$$L^\varepsilon = O(\sqrt{\varepsilon} C^\varepsilon) + \sum_{k=1,2} \sum_{1 \leq j \leq P} \sum_{1 \leq \ell \leq d_j} \left[ \frac{\varepsilon^2}{2} \partial_x^2 \chi_j^k \right] g_{j,k,\ell}^\varepsilon = O(\sqrt{\varepsilon} C^\varepsilon).$$

Here, the $O(\sqrt{\varepsilon} C^\varepsilon)$ holds in $\Sigma_\varepsilon$, from proposition 2.1. Moreover,

$$NL^\varepsilon = \sqrt{\varepsilon} (|w^\varepsilon|^2 + |\psi_1^\varepsilon\chi^1|^2 + |\psi_2^\varepsilon\chi^2|^2 + \varepsilon |g^\varepsilon|^2)(w^\varepsilon + |\psi_1^\varepsilon\chi^1| + |\psi_2^\varepsilon\chi^2| + \varepsilon g^\varepsilon)
- |\psi_1^\varepsilon|^2 |\psi_1^\varepsilon\chi^1| - |\psi_2^\varepsilon|^2 |\psi_2^\varepsilon\chi^2|.$$

Adding and subtracting the term $\sqrt{\varepsilon}(|\psi_1^\varepsilon\chi^1| + |\psi_2^\varepsilon\chi^2|)^2$, we have $|NL^\varepsilon| \leq N_S^\varepsilon + N_T^\varepsilon$,

where we have the pointwise estimates

$$N_T^\varepsilon \lesssim \sqrt{\varepsilon}(|\psi_1^\varepsilon|^2 + |\psi_2^\varepsilon|^2),
N_S^\varepsilon \lesssim \sqrt{\varepsilon}(|\psi_1^\varepsilon|^2 + |\psi_2^\varepsilon|^2 + |w^\varepsilon|^2 + \varepsilon |g^\varepsilon|^2 + |w^\varepsilon|^2 + \varepsilon |g^\varepsilon|^2)).$$
The semilinear term $N^{l}_{s}$ can be treated exactly in the same manner as in section 3. It remains to analyse $\int_{0}^{T} \|N L^{l}_{s}(s)\|_{\Sigma_{1}} ds$. We observe

\[
\sqrt{\varepsilon} \int_{0}^{t} \|\varphi^{l}_{s}(s)^{2}\varphi^{2}_{s}(s)\|_{L^{2}} ds = \int_{0}^{t} \left\| u_{1}(s, y - \frac{x_{1}(s) - x_{2}(s)}{\sqrt{\varepsilon}}) \right\|_{L^{2}}^{2} u_{2}(s, y) ds,
\]

can be treated exactly in the same manner as in section 3. It remains

\[
\text{Lemma 4.2. Set } \text{and the next lemma yields the conclusion.}
\]

whence

\[
\text{Lemma 4.1. Let } T \in \mathbb{R}, 0 < \gamma < 1/2 \text{ and }
\]

\[
I^{\varepsilon}(T) = \{ t \in [0, T], |x_{1}(t) - x_{2}(t)| \leq \varepsilon^{\gamma} \}.
\]

Then, for all $\varepsilon \in \mathbb{N}$, there exists a constant $C_{k}$ such that

\[
\int_{0}^{T} \|N L^{l}_{s}(t)\|_{\Sigma_{1}} dt \lesssim (M_{k}^{(2)}(T))^{3}(T \varepsilon^{(1/2 - \gamma)} + |I^{\varepsilon}(T))| e^{C_{k}T},
\]

with

\[
M_{k}(T) = \sup \| (x)^{a} \partial_{\beta} u_{j} \|_{L^{\infty}([0, T], L^{2}(\mathbb{R}))}; \ j \in \{1, 2\}, \ a + \beta \leq k\].

In view of this lemma and of equation (1.13), we obtain

\[
\int_{0}^{T} \|N L^{l}_{s}(t)\|_{\Sigma_{1}} dt \lesssim e^{C_{k}T}(T \varepsilon^{(1/2 - \gamma)} + |I^{\varepsilon}(T))|),
\]

and the next lemma yields the conclusion.

\[
\text{Lemma 4.2. Set }
\]

\[
\Gamma = \inf_{x \in \mathbb{R}} [\lambda_{1}(x) - \lambda_{2}(x) - (E_{1} - E_{2})],
\]

and suppose $\Gamma > 0$. Then for $0 < \gamma < 1/2$, there exists $C_{0}, C_{1} > 0$ such that

\[
|I^{\varepsilon}(t)| \lesssim \varepsilon^{\gamma} \Gamma^{-2} e^{C_{1}t}, \quad 0 \leq t \leq C_{1}\log \left( \frac{1}{\varepsilon} \right).
\]

\textbf{Proof.} Consider $I^{\varepsilon}(t)$ an interval of maximal length included in $I^{\varepsilon}(t)$, and $N^{\varepsilon}(t)$ the number of such intervals. The result comes from the estimate

\[
|I^{\varepsilon}(t)| \leq N^{\varepsilon}(t) \times \max |J^{\varepsilon}(t)|,
\]

with

\[
|J^{\varepsilon}(t)| \lesssim \varepsilon^{\gamma} e^{C_{1} \Gamma^{-1}} \text{ and } N^{\varepsilon}(t) \lesssim t e^{C_{1} \Gamma^{-1}},
\]

provided that $\varepsilon^{\gamma} e^{C_{1} \Gamma^{-1}} \ll 1$. Let us prove the first property: consider $t, \sigma \in I^{\varepsilon}(t)$. There exists $t^{*} \in [\tau, \sigma]$ such that

\[
|(x_{1}(\tau) - x_{2}(\tau)) - (x_{1}(\sigma) - x_{2}(\sigma))| = |\tau - \sigma||\xi_{1}(t^{*}) - \xi_{2}(t^{*})|,
\]

whence

\[
|\tau - \sigma| \leq |\xi_{1}(t^{*}) - \xi_{2}(t^{*})|^{-1} \times 2\varepsilon^{\gamma}.
\]

On the other hand,

\[
|\xi_{1}(t^{*}) - \xi_{2}(t^{*})| \geq ||\xi_{1}(t^{*})| - |\xi_{2}(t^{*})|| \geq \frac{\|\xi_{1}(t^{*})\|^{2} - |\xi_{2}(t^{*})|^{2}}{|\xi_{1}(t^{*})| + |\xi_{2}(t^{*})|}.
\]
We use
\[ |\xi_1(t^*)| + |\xi_2(t^*)| \lesssim e^{C t}, \]
\[ |\xi_1(t^*)|^2 - |\xi_2(t^*)|^2 = 2(E_1 - E_2 - \tilde{\lambda}_1(x_1(t^*)) + \tilde{\lambda}_2(x_2(t^*))), \]
and infer
\[ |E_1 - E_2 - \tilde{\lambda}_1(x_1(t^*)) + \tilde{\lambda}_2(x_2(t^*))| \geq |E_1 - E_2 - \tilde{\lambda}_1(x_1(t^*)) + \tilde{\lambda}_2(x_2(t^*))| \]
\[ \geq \Gamma - C \varepsilon \gamma e^{C t}, \]
where we have used the fact that \( \tilde{\lambda}_2 \) is at most quadratic. Therefore, if \( \varepsilon \gamma e^{C t} \) is sufficiently small,
\[ |E_1 - E_2 - \tilde{\lambda}_1(x_1(t^*)) + \tilde{\lambda}_2(x_2(t^*))| \geq \frac{\Gamma}{2}. \]
We infer
\[ |\tau - \sigma| \lesssim \varepsilon \gamma e^{C t} \Gamma^{-1}, \]
provided \( \varepsilon \gamma e^{C t} \ll 1. \)

Consider now \( N^*(t) \). We note that as \( t \) is large, \( N^*(t) \) is comparable to the number of distinct intervals of maximal size where \( |x_1(t) - x_2(t)| \geq \varepsilon \gamma \). More precisely, \( N^*(t) \) is smaller than \( t \) divided by the minimal size of these intervals. Therefore, we consider one interval \( [\tau, \sigma] \) of this type and we look for lower bound of \( \sigma - \tau \). We have
\[ |x_1(\tau) - x_2(\tau)| = |x_1(\sigma) - x_2(\sigma)| = \varepsilon \gamma \quad \text{and} \quad \forall \tau \in [\tau, \sigma], \quad |x_1(t) - x_2(t)| \geq \varepsilon \gamma. \]
Moreover, inside \( [\tau, \sigma] \), \( x_1(t) - x_2(t) \) has a constant sign that we can suppose to be + (one argues similarly if it is -). Under this assumption, we have
\[ \xi_1(\tau) - \xi_2(\tau) > 0 \quad \text{and} \quad \xi_1(\sigma) - \xi_2(\sigma) < 0. \]
Using the exponential control of \( \lambda_j'(x_j(t)) \) for \( j \in \{1, 2\} \), we obtain
\[ (\xi_1(\tau) - \xi_2(\tau)) - (\xi_1(\sigma) - \xi_2(\sigma)) \lesssim e^{C t}(\sigma - \tau). \] (4.4)
We write
\[ \xi_1(\tau) - \xi_2(\tau) = |\xi_1(\tau) - \xi_2(\tau)| \geq \frac{|\xi_1(\tau)|^2 - |\xi_2(\tau)|^2}{|\xi_1(\tau)| + |\xi_2(\tau)|} \]
(4.5)
and
\[ -\xi_1(\sigma) + \xi_2(\sigma) = |\xi_1(\tau) - \xi_2(\tau)| \gtrsim e^{-C t}|\xi_1(\sigma)|^2 - |\xi_2(\sigma)|^2|. \] (4.6)
As before, we prove
\[ ||\xi_1(\tau)||^2 - |\xi_2(\tau)||^2 + ||\xi_1(\sigma)||^2 - |\xi_2(\sigma)||^2| \gtrsim \Gamma, \]
provided that \( \varepsilon \gamma e^{C t} \ll 1. \) Therefore, plugging the latter equation, (4.5) and (4.6) into (4.4), we obtain
\[ \sigma - \tau \gtrsim e^{-2C t} \Gamma \quad \text{thus} \quad N^*(t) \lesssim te^{C t} \Gamma^{-1} \lesssim e^{C t} \Gamma^{-1}, \]
which completes the proof of theorem 1.13. \qed

Remark 4.3. The proof shows that if the approximation of theorem 1.10 is proven to be valid on some time interval \([0, C \log(1/\varepsilon)]\), then theorem 1.13 will also be valid on a time interval of the same form.
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Appendix A. Global existence of the exact solution

The proof of lemma 1.3 follows classical arguments; see [23] (or [8]) for more details. We suppose \( \epsilon = 1 \) without loss of generality. We use the decomposition \( V(x) = D(x) + W(x) \) of assumption 1.2 and we denote by \( U(t) \) the unitary propagator of \( -\frac{1}{2} \partial_x^2 + D(x) \). Let \( X_T \) be the set

\[
X_T = \{ \psi \in C(I_T, \Sigma_1^1), \psi, x\psi, \nabla \psi \in L^8(I_T, L^4(R, C^N)) \}, \quad I_T = [s - T, s + T[ \text{ for } s \in R \text{ and } T \in R \text{ to be fixed later.} \]

The proof consists of a fixed point argument for the function \( \Phi_1(s) : \psi \mapsto \Phi_1(s)(\psi) \) where for \( s \in R \), the function \( \Phi_1(s)(\psi) \) is defined by

\[
\Phi_1(s)(\psi)(t) = U(t - s)\psi(s) - i \Lambda \int_s^t U(t - \tau)(|\psi|^2 \psi)(\tau) \, d\tau - i \int_s^t U(t - \tau)(W \psi(\tau)) \, d\tau.
\]

By [11], local in time Strichartz estimates are available for \( U \). Strichartz estimates and Hölder inequality imply that there exists a constant \( C > 0 \) such that

\[
\| \Phi_1(s)(\psi) \|_{L^8(I_T, L^4) \cap L^\infty(I_T, L^2)} \leq C \| \psi(s) \|_{L^2} + C \| \psi \|^2_{L^8(I_T, L^4)} \| \psi \|_{L^8(I_T, L^4)} + C \| W \psi \|_{L^1(I_T, L^2)}.
\]

Using the boundedness of the coefficients of \( W \) and Hölder inequality in time, we obtain

\[
\| \Phi_1(s)(\psi) \|_{L^8(I_T, L^4) \cap L^\infty(I_T, L^2)} \leq C \| \psi(s) \|_{L^2} + C \sqrt{T} \| \psi \|^3_{L^8(I_T, L^4)} + CT \| \psi \|_{L^\infty(I_T, L^2)}.
\]

We can then infer that \( \Phi_1 \) is a contraction on a ball of \( X_T \) for some \( T \) which depends only on \( \| \psi(s) \|_{L^2} \). Then, the conservation of \( \| \psi(t) \|_{L^2} \) yields the lemma.

Appendix B. Control of the growth of the eigenvectors

B.1. Some formulae involving the projectors

In this section, we list and prove some formulae which will be used in the course of the computations in the next paragraph. We consider here the more general case \( x \in R^d \), with \( d \geq 1 \). Fix once and for all in this paragraph \( j \in \{1, \ldots, P\} \) and \( \ell \in \{1, \ldots, d\} \). First, as shown in appendix C, since \( \Pi_j^2 = \Pi_j \),

\[
\Pi_j (\partial_\ell \Pi_j) \Pi_j = 0. \tag{B.1}
\]

Differentiating the relation \( \Pi_j^2 = \Pi_j \), we find \( \forall j \in \{1, \ldots, P\}, \forall \ell \in \{1, \ldots, d\}, \)

\[
\partial_\ell \Pi_j = (\partial_\ell \Pi_j) \Pi_j + \Pi_j (\partial_\ell \Pi_j). \tag{B.2}
\]

We now show \( \forall j \in \{1, \ldots, P\}, \forall \ell \in \{1, \ldots, d\}, \)

\[
\partial_\ell \Pi_j = \sum_{k \neq j} (\Pi_k (\partial_\ell \Pi_j) \Pi_j + \Pi_j (\partial_\ell \Pi_j) \Pi_k) \]

\[= \sum_{1 \leq k \leq P} (\Pi_k (\partial_\ell \Pi_j) \Pi_j + \Pi_j (\partial_\ell \Pi_j) \Pi_k), \tag{B.3}
\]
where the last equality stems from (B.1). To prove (B.3), simply write
\[ \partial_t \Pi_j = \sum_{k,m} \Pi_k (\partial_t \Pi_j) \Pi_m, \]
where we have used \( \sum_k \Pi_k = \text{Id}. \) Then, observing that \( \Pi_k \Pi_j = \delta_{jk} \Pi_j \) yields
\[ \Pi_k (\partial_t \Pi_j) + (\partial_t \Pi_k) \Pi_j = -(\partial_t \Pi_k) \Pi_j. \]
The fact that \( \Pi_j \Pi_m = 0 \) for all \( m \neq j \) gives (B.3).

The last formulae we wish to establish involve the spectral gap. Since we have a basis of eigenfunctions, we have
\[ V \Pi_j = \lambda_j \Pi_j. \]
Differentiating with respect to \( x_\ell \), we infer
\[ (\partial_\ell \Pi_j) V + \Pi_j \partial_\ell V = \lambda_j \partial_\ell \Pi_j + (\partial_\ell \lambda_j) \Pi_j. \]
For \( k \in \{1, \ldots, P\} \), multiply this relation by \( \Pi_k \) on the right, and use the property
\[ V \Pi_k = \lambda_k \Pi_k: \]
\[ \lambda_k (\partial_\ell \Pi_j) \Pi_k + \Pi_j (\partial_\ell V - \partial_\ell \lambda_j) \Pi_k = \lambda_j (\partial_\ell \Pi_j) \Pi_k, \]
whence
\[ (\lambda_j - \lambda_k)(\partial_\ell \Pi_j) \Pi_k = \Pi_j (\partial_\ell V - \partial_\ell \lambda_j) \Pi_k. \]
Similarly, we have
\[ (\lambda_j - \lambda_k)(\partial_\ell \Pi_j) \Pi_k = \Pi_k (\partial_\ell V - \partial_\ell \lambda_j) \Pi_j. \]

B.2. About the growth of the eigenvectors at infinity

This section is devoted to the proof of estimates at infinity for the eigenprojectors associated with a potential \( V \) satisfying assumption 1.2. We will use a lemma on the derivatives of the inverse of the gap between two different eigenvalues. For \( j, k \in \{1, \ldots, P\} \), \( j \neq k \), we recall that we have set (see (2.4))
\[ \forall x \in \mathbb{R}, \quad \gamma_{j,k}(x) = (\lambda_j(x) - \lambda_k(x))^{-1}. \]
Since the results are not specific to the space dimension one, we prove them for potentials depending on \( x \in \mathbb{R}^d, \ d \geq 1. \)

**Lemma B.1.** Assume (1.3) is satisfied with \( n_0 \in \mathbb{N} \) and that the functions \( V \) and \( \lambda_j \) \( (j \in \{1, \ldots, P\}) \) are at most quadratic. Then, for \( \beta \in \mathbb{N}^d \) and for \( j, k \in \{1, \ldots, P\} \) with \( j \neq k \),
\[ |\partial_\Gamma^\beta \gamma_{j,k}(x)| \lesssim \langle x \rangle^{n_0+|\beta|}. \]

**Proof.** For \( \beta = 1 \), we immediately obtain
\[ |\partial_\ell \gamma_{j,k}(x)| \lesssim \langle x \rangle^{2n_0+1}. \]
from (1.3), and the fact that \( \lambda_j \) and \( \lambda_k \) are at most quadratic.

Set \( \Lambda_{j,k} = \lambda_j - \lambda_k \): it is at most quadratic. Moreover, for \( \beta \in \mathbb{N}^d \), we have
\[ \partial_\ell^\beta (\gamma_{j,k}) = \sum_{a_1, \ldots, a_p = 1}^\beta a_{i_1} \cdots a_{i_p} \Lambda_{j,k}^{-1-p} \partial_\ell^{a_{i_1}} \Lambda_{j,k} \cdots \partial_\ell^{a_{i_p}} \Lambda_{j,k} \]
\[ \lesssim \langle x \rangle^{2n_0+1}. \]
for some real numbers \(a_{\alpha_1,\ldots,\alpha_p}\). The result then follows by observing that 
\[
|\Lambda_{j,k}^{-\beta} \partial^{\alpha} \Lambda_{j,k} \cdots \partial^{\alpha_p} \Lambda_{j,k}| \lesssim \langle \xi \rangle^{n_0(1+\beta)} \langle \xi \rangle^p,
\]
from (1.3), and the property \(|\partial^{\alpha} \Lambda_{j,k}| \lesssim \langle \xi \rangle^{2-|\alpha|}\), which follows from the fact that \(\Lambda_{j,k}\) is at most quadratic, in the sense of definition 1.1. \(\square\)

We now consider the eigenprojectors \(\Pi_j\) associated with the eigenvalues \(\lambda_j\) of the matrix \(V\). Because of the gap condition, these functions are smooth in \(\mathbb{R}^d\). We prove the following

**Lemma B.2.** Let \(\Pi_j\) be an eigenprojector of \(V\) for \(j \in \{1, \ldots, P\}\), we have for \(\beta \in \mathbb{N}^d\)
\[
|\partial^{\beta} \Pi_j|_{C^{N,N}} \lesssim \langle \xi \rangle^{|\beta|(1+n_0)},
\]
where the norm \(|\cdot|_{C^{N,N}}\) denotes the matricial norm.

**Proof.** The case \(|\beta| = 0\) is immediate since \(\Pi_j\) is a projector. In view of (B.3), relations (B.5) and (B.4) imply (B.7) for \(|\beta| = 1\).

We now argue by induction. We suppose that (B.7) holds for any \(\gamma \in \mathbb{N}^d\) with \(|\gamma| = K\) for some \(K \in \mathbb{N}\) and we consider \(\beta\) with \(|\beta| = K + 1\) and \(\beta_1 \neq 0\). Differentiation of order \(\beta - 1_\ell\) of (B.2) and multiplication on both sides by \(\Pi_j\) yield
\[
\Pi_j (\partial^{\beta} \Pi_j) \Pi_j = \Pi_j \left( \sum_{0 < |\alpha| < |\beta|} a_\alpha \partial^{\alpha} \Pi_j \partial^{\beta-\alpha} \Pi_j \right) \Pi_j,
\]
where all along this proof, \(a_\alpha\) will denote real numbers whose exact value is unimportant. We obtain
\[
|\Pi_j (\partial^{\beta} \Pi_j) \Pi_j|_{C^{N,N}} \lesssim \langle \xi \rangle^{|\beta|(1+n_0)}. \tag{B.8}
\]
Then, for all \(k \neq j\), we estimate \((\partial^{\beta} \Pi_j) \Pi_k\). To do so, we differentiate (B.4) and get
\[
(\partial^{\beta} \Pi_j) \Pi_k = \sum_{0 < |\alpha| < |\beta|} a_\alpha \partial^{\alpha} \Pi_j \partial^{\beta-\alpha} \Pi_k + \sum_{\alpha_1 + \cdots + \alpha_p = \beta - 1_\ell} b_{\alpha_1,\ldots,\alpha_p} (\lambda_j - \lambda_k)^{-1} \partial^{\alpha_1} \Pi_j \partial^{\alpha_2} \partial_{\alpha_3} (V - \lambda_j) \partial^{\alpha_p} \Pi_k.
\]
In the first sum, the induction assumption yields
\[
\|\partial^{\alpha} \Pi_j \partial^{\beta-\alpha} \Pi_k\|_{C^{N,N}} \lesssim \langle \xi \rangle^{|\alpha|(1+n_0) + |\beta-\alpha|(1+n_0)} = \langle \xi \rangle^{|\beta|(1+n_0)}. \tag{B.9}
\]
In addition, for each term in the second sum, we write
\[
\|\partial^{\alpha_1} ((\lambda_j - \lambda_k)^{-1}) \partial^{\alpha_2} \Pi_j \partial^{\alpha_3} \partial_{\alpha_4} (V - \lambda_j) \partial^{\alpha_5} \Pi_k\|_{C^{N,N}} \lesssim \langle \xi \rangle^{n_0 + |\alpha_1|(1+n_0) + |\alpha_2|(1+n_0) + |\alpha_3| + |\alpha_4| + |\alpha_5|},
\]
where \(r_+ = \max(r, 0)\) and where we have used the fact that \(V\) and \(\lambda_j\) are at most quadratic, together with the induction assumption and lemma B.1. We have the two alternatives:

- If \(\alpha_3 = 0\), then
  \[
  n_0 + |\alpha_1|(1 + n_0) + (1 - |\alpha_3|) + (1 + n_0)(|\alpha_2| + |\alpha_4|) = n_0 + |\alpha_1|(1 + n_0) + 1 + (1 + n_0)(|\alpha_2| + |\alpha_4|) = (1 + n_0)(1 + |\alpha_1| + |\alpha_2| + |\alpha_4|) = (1 + n_0)|\beta|,
  \]
  since \(\alpha_1 + \alpha_2 + \alpha_4 = \beta - 1_\ell\).
We consider the Hamiltonian curves of $1$ proof of [14]. More generally, we prove the following result which implies proposition 1.9. 

satisfying (1.15), and analysing the behaviour of their derivatives for large time. We follow the 

In this section we prove proposition 1.9, recalling the construction of the eigenvectors 

Appendix C. Proof of proposition 1.9

In this section we prove proposition 1.9, recalling the construction of the eigenvectors satisfying (1.15), and analysing the behaviour of their derivatives for large time. We follow the proof of [14]. More generally, we prove the following result which implies proposition 1.9. 

We consider the Hamiltonian curves of $V^2\Lambda_1 + \lambda_j (x)$, that we denote by $(x_j (t), \xi_j (t))$.

**Proposition C.1.** There exists a smooth orthonormal basis of $\mathcal{C}^N (x_j (t), x)$ that spans the eigenspace associated with $\lambda_j$, with $\chi_j (0, x) = \chi (x)$ and for $m \in \{1, \ldots, d_j\}$,

$$(x_j^n (t, x), \partial_x \chi_j^m (t, x) + \xi_j (t) \partial_x \chi_j^m (t, x))_{C^\infty} = 0.$$ 

Moreover, for $\ell \in \{1, \ldots, d_j\}, k, p \in \mathbb{N}$, there exists a constant $C = C(p, k)$ such that

$$|\partial^{k}_{\beta} \partial^{p}_{x} \chi_{j}^{m} (t, x)|_{C^\ell} \leq C e^{C_{1} (x) (k + p)(4n_0)}\,,$$

where $n_0$ appears in (1.3).

**Proof of proposition C.1.** We consider a smooth basis of eigenvectors $(x_j^{\ell} (0))_{1 \leq \ell \leq d_j}$ such that $x_j^{1} (0) = \chi$. Then, we denote by $\Pi_{j} (x)$ the smooth eigenprojector associated with the eigenvalue $\lambda_j (x)$ and define

$$K_{j} (x) = -i[\Pi_{j} (x), \partial_{x} \Pi_{j} (x)].$$

We set $z = x - x_j (t)$ and we consider the Schrödinger-type equation

$$i\partial_{t} Y_{j}^{\ell} (t, z) = \xi_j (t) K_j (z + x_j (t)) Y_{j}^{\ell} (t, z), \quad Y_{j}^{\ell} (0, z) = \chi_{j}^{\ell} (x_j (0) + z). \quad (C.1)$$

Let us prove that the vector $Y_{j}^{\ell} (t, z)$ is in the eigenspace of $\lambda_j (x_j (t) + z)$. Indeed, the evolution of $Z_{j}^{\ell} (t, z)$ = $(\mathrm{Id} - \Pi_{j} (x_j (t) + z)) Y_{j}^{\ell} (t, z)$ obeys $Z_{j}^{\ell} (0, z) = 0$ and

$$\partial_{t} Z_{j}^{\ell} (t, z) = -\xi_j (t) \partial_{x} \Pi_{j} (x_j (t) + z) Y_{j}^{\ell} - \xi_j (t) (\mathrm{Id} - \Pi_{j} (x_j (t) + z)) [\Pi_{j} (x_j (t) + z), \partial_{x} \Pi_{j} (x_j (t) + z)] Y_{j}^{\ell}$$

$$= -\xi_j (t) \partial_{x} \Pi_{j} (x_j (t) + z) (\mathrm{Id} - \Pi_{j} (x_j (t) + z)) Y_{j}^{\ell}$$

$$= -\xi_j (t) \partial_{x} \Pi_{j} (x_j (t) + z) Z_{j}^{\ell},$$
where we have used \( \partial_t \Pi_j = \partial_x (\Pi_j^2) = \Pi_j \partial_t \Pi_j + (\partial_x, \Pi_j) \Pi_j \), whence
\[
\Pi_j (\partial_t, \Pi_j) \Pi_j = \Pi_j (\Pi_j \partial_t, \Pi_j) + (\partial_x, \Pi_j) \Pi_j = 2 \Pi_j (\partial_t, \Pi_j) \Pi_j = 0.
\]
Therefore, \( Z_j^j(t) \) satisfies an equation of the form \( \partial_t Z_j^j = A(t, z) Z_j^j \), which combined with \( Z_j^j(0) = 0 \) implies \( Z_j^j(t) = 0 \) for all \( t \in \mathbb{R} \): the vectors \( Y_j^j(t, z) \) are eigenvectors of \( V(x_j(t) + z) \) for the eigenvalue \( \lambda_j(x_j(t) + z) \). Moreover, since \( \xi_j(t) K_j (z + x_j(t)) \) is self-adjoint, \( Y_j^j(t, z) \) is normalized for all \( t \), and the family \( \{Y_j^j(t)\}_{t \leq t \leq \delta} \) is orthonormal. We define \( \chi_j^j(t, x) \) by
\[
\chi_j^j(t, x) = Y_j^j(t, x - x_j(t))
\]
and we obtain an orthonormal basis of eigenvectors of \( V(x) \).

This concludes the first part of proposition 1.9. It remains to study the behaviour at infinity of the vectors \( \chi_j^j(t, x) \) and of their derivatives.

By the definition of \( \chi_j^j(t, x) \) in (C.2), it is enough to prove
\[
|\partial_t^p \partial_x^k Y_j^j(t, x)|_{C^N} \lesssim c^{Ct} (x_j(t) + z)^{(p+k)(1+\eta)}.
\]

For this, we crucially use the estimates of lemma B.2 and we argue by induction. First, consider the case \( p = 1 \) and \( k = 0 \). By lemma B.2, \( |K_j(x)| \lesssim \langle x \rangle^{1+\eta} \), whence (C.1) gives
\[
|\partial_t Y_j^j(t, x)| \lesssim |\xi_j(t)| |K_j(x)| \lesssim c^{Ct} (x)^{1+\eta}.
\]

Now suppose \( k \geq 1 \) and \( p = 0 \). We observe that \( \partial_x^k Y_j^j(t, z) \) solves
\[
\begin{align*}
\partial_t^p \partial_x^k Y_j^j(t, z) &= -\partial_x (t) K_j(z + x_j(t)) \partial_x^k Y_j^j(t, z) + f(t, z), \\
\partial_x^k Y_j^j(0, z) &= \partial_x^k \chi_j^j(0, z + x_j(0)),
\end{align*}
\]

where
\[
f(t, z) = \sum_{0 \leq \gamma \leq k-1} c_{\gamma} \xi_j(t) \partial^{\gamma} K_j(z + x_j(t)) \partial_x^{k-\gamma} Y_j^j(t, z),
\]

for some complex numbers \( c_{\gamma} \) independent of \( t \) and \( z \). We obtain
\[
\partial_x^k Y_j^j(t, z) = U_j(t, 0) \partial_x^k \chi_j^j(z + x_j(0)) + \int_0^t U_j(t, s) f(s, z) \, ds,
\]

where \( U_j(t, s) \) denotes the unitary propagator associated with (C.1) (when the initial time is equal to \( s \)). We have by lemma B.2
\[
|\partial_x^k Y_j^j(0, z)|_{C^N} \lesssim (z + x_j(0))^{(1+\eta)},
\]

therefore the induction assumption
\[
\forall \gamma \in [0, \ldots, k - 1], \quad |\partial_x^\gamma Y_j^j(t, z)|_{C^N} \lesssim c^{Ct} (x_j(t) + z)^{\gamma(1+\eta)}
\]
implies, along with lemma B.2,
\[
|\partial_k x Y_{j}^{\ell}(t, z)|_{C^N} \lesssim e^{Ct} \langle x_j(t) + z \rangle^{(1+n_0)}.
\]
We have obtained the estimate for \( p = 0, k \in \mathbb{N} \), and for \( p = 1, k = 0 \). Note that equation (C.3) yields
\[
\forall k \in \mathbb{N}, \quad |\partial_t \partial_k x Y_{j}^{\ell}(t, z)|_{C^N} \lesssim e^{Ct} \langle x_j(t) + z \rangle^{(1+n_0)}(1+k)(1+n_0),
\]
and allows us to prove the general estimate for time derivatives by an induction argument which crucially uses the fact that we have an exponential control of the derivatives in time of \( \xi_j(t) \). This property follows by induction from (1.8), (1.9), and the fact that \( \lambda_j \) is at most quadratic.

Before concluding this section, note that in view of the definition of the function \( r_{j,\ell} \) in (1.18), proposition 1.9 gives the following corollary.

**Corollary C.2.** For all \( p \in \mathbb{N} \) and \( k \in \mathbb{N} \), there exists a constant \( C = C(p, k) \) such that, for \( x \in \mathbb{R}, \ j \in \{1, \ldots, P\} \) and \( \ell \in \{1, \ldots, d_j\} \),
\[
|\partial_t^p \partial_k x r_{j,\ell}(t, x)| \lesssim e^{Ct} \langle x \rangle^{(1+p+k)(1+n_0)}.
\]

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