**Abstract.** This is the first of a series of papers devoted to certain pairs of commuting nilpotent elements in a semisimple Lie algebra that enjoy quite remarkable properties and which are expected to play a major role in Representation theory. The properties of these pairs and their role is similar to those of the principal nilpotents. To any principal nilpotent pair we associate a two-parameter analogue of the Kostant partition function, and propose the corresponding two-parameter analogue of the weight multiplicity formula.

In a different direction, each principal nilpotent pair gives rise to a harmonic polynomial on the Cartesian square of the Cartan subalgebra, that transforms under an irreducible representation of the Weyl group. In the special case of $\mathfrak{sl}_n$, the conjugacy classes of principal nilpotent pairs and the irreducible representations of the Symmetric group, $S_n$, are both parametrised (in a compatible way) by Young diagrams. In general, our theory provides a natural generalization to arbitrary Weyl groups of the classical construction of simple $S_n$-modules in terms of Young’s symmetrisers.

First results towards a complete classification of all principal nilpotent pairs in a simple Lie algebra are presented at the end of this paper in an Appendix, written by A. Elashvili and D. Panyushev.

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**0. Introduction.**

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $G$ the corresponding adjoint group, i.e., the identity component of $\text{Aut}(\mathfrak{g})$.

Recall that an element $x \in \mathfrak{g}$ is called regular if $\mathfrak{z}_\mathfrak{g}(x)$, the centralizer of $x$ in $\mathfrak{g}$, has the minimal possible dimension, i.e., $\dim \mathfrak{z}_\mathfrak{g}(x) = \text{rk} \mathfrak{g}$. The most interesting, in a sense, among regular elements of $\mathfrak{g}$ are regular nilpotent elements, called "principal nilpotents". These elements form a single $\text{Ad}G$-orbit in $\mathfrak{g}$. We refer to the beautiful papers [K1], [K2] for a comprehensive study of principal nilpotents.

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We are going to "double" the above setup and replace the Lie algebra \( \mathfrak{g} \) by \( \mathcal{Z} \subset \mathfrak{g} \oplus \mathfrak{g} \), the set of all pairs \( (x_1, x_2) \in \mathfrak{g} \oplus \mathfrak{g} \), such that \([x_1, x_2] = 0\), called the commuting variety of \( \mathfrak{g} \), see [R]. It turns out that there is a remarkable "doubled" counterpart of the notion of a principal nilpotent, that we call a principal nilpotent pair. The underlying idea is best explained as follows. Let \( \mathcal{Z}^{\text{reg}} \) be the union of all \( \text{Ad}_{\mathcal{G}}\)-diagonal orbits of maximal dimension in \( \mathcal{Z} \). This is a smooth Zariski open, dense subset in \( \mathcal{Z} \). The natural \( \mathbb{C}^* \times \mathbb{C}^* \)-action on \( \mathfrak{g} \) by dilations gives rise to a \( \mathbb{C}^* \times \mathbb{C}^* \)-action on \( \mathfrak{g} \oplus \mathfrak{g} \), and \( \mathcal{Z}^{\text{reg}} \) is a \( \mathbb{C}^* \times \mathbb{C}^* \)-stable subvariety. A principal nilpotent pair is the one whose \( \text{Ad}_{\mathcal{G}} \)-diagonal orbit is a fixed point of the induced \( \mathbb{C}^* \times \mathbb{C}^* \)-action on \( \mathcal{Z}^{\text{reg}}/\text{Ad}_{\mathcal{G}} \). Here \( \mathcal{Z}^{\text{reg}}/\text{Ad}_{\mathcal{G}} \) denotes the naive set of orbits; it has no structure of an algebraic variety. This construction was motivated in part by our work on Hilbert schemes (joint work with R. Bezrukavnikov currently in progress).

The set of principal nilpotent pairs does not form a single orbit under \( \text{Ad}_{\mathcal{G}} \)-diagonal action on \( \mathcal{Z} \), but it consists of only finitely many such orbits. In the case \( \mathfrak{g} = \mathfrak{sl}_n \), for instance, conjugacy classes of principal nilpotent pairs are parametrized essentially (up to transposing matrices) by Young diagrams with \( n \)-boxes. This comes about as follows. Given a Young diagram:

\[
\lambda = \begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array}
\]

we enumerate its boxes in some order, and label the standard base vectors in \( \mathbb{C}^n \) by the box with the corresponding number. We define an endomorphism \( e_1 \in \mathfrak{sl}_n \) by letting it act "along the rows of the diagram", i.e., by sending the base vector labelled by a box to the base vector labelled by the next right box, if this box belongs to \( \lambda \), and to 0 otherwise. Similarly, we define \( e_2 \in \mathfrak{sl}_n \) by letting it act "along the columns", from bottom to top. It is easy to see that the operators \( e_1, e_2 \) thus defined commute, and form a principal nilpotent pair. Note that if \( \lambda \) consists of either a single row or a single column, then the corresponding principal nilpotent pair is either of the form \( (e, 0) \) or \( (0, e) \), where \( e \) is a principal nilpotent in \( \mathfrak{sl}_n \) in the ordinary sense. Moreover, any principal nilpotent pair for \( \mathfrak{g} = \mathfrak{sl}_n \) is associated to a Young diagram in the above way (up to conjugation and transposing matrices).

For Lie algebras of types other than \( \mathfrak{sl}_n \) the number of principal nilpotent pairs is typically less than one might have expected, because the set \( \mathcal{Z}^{\text{reg}}/\text{Ad}_{\mathcal{G}} \) is even farther away from being an algebraic variety (in the \( \mathfrak{g} = \mathfrak{sl}_n \)-case the
corresponding algebraic variety exists, in a sense; it is the Hilbert scheme of \( n \) points on \( \mathbb{C}^2 \).

In Section 2, to each principal nilpotent pair in an arbitrary semisimple Lie algebra \( \mathfrak{g} \), and each simple finite-dimensional \( \mathfrak{g} \)-module, we associate a certain two-variable analogue of Kostant’s partition function, and propose the corresponding \((s,t)\)-weight multiplicity formula. This formula may be related to double analogues of Kostka polynomials that arise in Macdonald theory [M2]. The geometry underlying our multiplicity formula will be discussed in the second paper of this series. Later on, we are also going to express the two-variable weight multiplicity function in terms of Intersection cohomology of an appropriate double-loop Grassmannian, similar to the way, the \( q \)-analogue of weight multiplicity introduced by Lusztig is related to perverse sheaves on the loop Grassmannian, see [L1], [Gi].

In section 4 we associate to a principal nilpotent pair in \( \mathfrak{g} \) a harmonic polynomial on the Cartesian square of a Cartan subalgebra of \( \mathfrak{g} \). This polynomial has very interesting properties, in particular transforms under an irreducible representation of the Weyl group. In the special case \( \mathfrak{g} = \mathfrak{sl}_n \) the polynomial in question has been known in Combinatorics as a double-analogue of the Vandermonde determinant. In the latter case, the irreducible representation of the Symmetric group generated by the polynomial turns out to be parametrized by the Young diagram labelling the principal nilpotent pair. Our construction is closely related to the theory of Springer representations.

In section 6 we compute the tangent space to \( \mathcal{Z}^{\text{reg}}/\text{Ad}G \) at a principal nilpotent conjugacy class and use this computation to derive some combinatorial identities involving a double-analogue of the notion of exponents of a semisimple Lie algebra. In the last section we introduce a distinguished “partial transverse slice” to each principal nilpotent conjugacy class, which is a sort of generalisation of Kostant-Slodowy slices to nilpotent orbits in \( \mathfrak{g} \). The parallelism with Kostant-Slodowy theory goes surprisingly far, see notably Theorem 7.4. This parallelism suggests the existence of a yet unknown double-analogue of Whittaker modules theory, similar to the one developed in [K3]. We hope to return to this issue elsewhere.

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1. Principal nilpotent pairs.

Given two elements \( x_1, x_2 \) in a Lie algebra \( \mathfrak{a} \), we write \( \mathbf{x} = (x_1, x_2) \), and let \( \mathfrak{z}_\mathfrak{a}(x_1) \), resp. \( \mathfrak{z}_\mathfrak{a}(\mathbf{x}) \), denote the centralizer, resp. the the simultaneous centralizer of \( x_1 \) and \( x_2 \), in \( \mathfrak{a} \).

Fix a complex semisimple Lie algebra \( \mathfrak{g} \) with adjoint group \( G \). We will write \( \mathfrak{z}(\mathbf{x}) \) instead of \( \mathfrak{z}_\mathfrak{g}(\mathbf{x}) \), for simplicity. Let \( \mathcal{Z} \subset \mathfrak{g} \oplus \mathfrak{g} \) be the commuting variety of \( \mathfrak{g} \), as defined in the Introduction. By a theorem of Richardson [R], the pairs \( \mathbf{x} = (x_1, x_2) \), where \( x_1, x_2 \) are in the same Cartan subalgebra of \( \mathfrak{g} \), form a dense subset in \( \mathcal{Z} \). In particular, for any \( \mathbf{x} \in \mathcal{Z} \) one has: \( \dim \mathfrak{z}(\mathbf{x}) \geq \text{rk} \mathfrak{g} \). We say that \( \mathbf{x} = (x_1, x_2) \in \mathcal{Z} \) is a regular pair if the equality \( \dim \mathfrak{z}(\mathbf{x}) = \text{rk} \mathfrak{g} \) holds.
Definition 1.1. A pair \( e = (e_1, e_2) \in g \oplus g \) is called a principal nilpotent pair if the following conditions (Reg) and (Nil) hold:

\[
\text{(Reg)} \quad e = (e_1, e_2) \in Z \text{ is a regular pair, i.e., } [e_1, e_2] = 0 \text{ and } \dim Z(e) = \text{rk } g;
\]

\[
\text{(Nil)} \quad \text{For any } (t_1, t_2) \in C^* \times C^*, \text{ there exists } g = g(t_1, t_2) \in G \text{ such that } (t_1 \cdot e_1, t_2 \cdot e_2) = (\text{Ad } g(e_1), \text{Ad } g(e_2)).
\]

Note that any elements \( e_1, e_2 \) satisfying the (Nil)-condition are necessarily nilpotent, since \( \text{Ad } G \)-conjugacy classes of \( e_1 \) and \( e_2 \) are both stable under dilation.

As a first example, take \( e \in g \) to be a regular (=principal) nilpotent in \( g \) (in the ordinary sense). Then \((e, 0)\) and \((0, e)\) are two principal nilpotent pairs in \( g \). In general, if \((e_1, e_2)\) is a principal nilpotent pair then so is \((e_2, e_1)\).

Write \( Z_G^{\text{unip}}(e) \) for the unipotent radical of the centralizer of the pair \( e \) in \( G \).

Theorem 1.2. Given a principal nilpotent pair \( e = (e_1, e_2) \), there exists a pair \( h = (h_1, h_2) \in Z \) formed by semisimple elements of \( g \) and such that one has:

\[
\begin{align*}
\text{(i)} & \quad [h_1, e_j] = \delta_{i,j} \cdot e_i, \quad i, j \in \{1, 2\}, \quad [h_1, h_2] = 0. \\
\text{(ii)} & \quad \text{The pair } h = (h_1, h_2) \text{ is regular, i.e., } \mathfrak{z}(h) \text{ is a Cartan subalgebra in } g. \\
\text{(iii)} & \quad \text{All eigenvalues of the operators } \text{ad } h_i : g \to g \text{ (} i = 1, 2 \text{) are integral.}
\end{align*}
\]

Furthermore, the semi-simple pair \( h \) is determined by the nilpotent pair \( e \) uniquely up to conjugacy, more precisely:

\[
\text{(iv)} \quad \text{For any two semisimple pairs } h = (h_1, h_2) \text{ and } h' = (h'_1, h'_2), \text{ each satisfying commutation relations } 1.2(\text{i}), \text{ there exists } u \in Z_G^{\text{unip}}(e) \text{ such that } h' = \text{Ad } u(h).
\]

A pair \( h = (h_1, h_2) \) of semisimple elements of \( g \) satisfying commutation relations 1.2(i) will be referred to as an associated semisimple pair.

Proof of the theorem will be done in stages and will occupy most of this section. We first derive a few consequences of Definition 1.1.

Lemma 1.3. For any commuting pair \( e = (e_1, e_2) \) satisfying the (Nil)-condition in (1.1), there exist integers \( m_1, m_2 > 0 \), and an algebraic group homomorphism \( \gamma : C^* \times C^* \to G \) such that

\[
(\text{Ad } \gamma(t_1, t_2)) e_i = t_i^{m_i} \cdot e_i, \quad \forall (t_1, t_2) \in C^* \times C^*, i = 1, 2.
\]

We will see later in Corollary 3.6 that in the case of a principal nilpotent pair \( e = (e_1, e_2) \) the integers \( m_1, m_2 \) can be in effect chosen to be equal to 1.

Proof of Lemma. Let \( M \subset C^* \times C^* \times G \) be the algebraic group formed by all triples:

\[
M = \{(t_1, t_2, g) \in C^* \times C^* \times G \mid \text{Ad } g(e_1) = t_1 \cdot e_1, \; \text{Ad } g(e_2) = t_2 \cdot e_2\}.
\]

The assumptions of the lemma imply that the projection \( p : M \to C^* \times C^* \) on the first two factors is surjective. This gives a short exact sequence

\[
1 \to Z_G(e_1, e_2) \to M \to C^* \times C^* \to 1
\] (1.4)
Let $M^o$ be the identity component of $M$, and $M^o = M_{\text{red}} \cdot M_{\text{unip}}$ the decomposition of $M^o$ as a semidirect product of a reductive subgroup, $M_{\text{red}}$, and the unipotent radical. The restriction of the homomorphism $p$ to $M^o$ remains surjective, and clearly vanishes both on $M_{\text{unip}}$ and on the semisimple part of $M_{\text{red}}$. Writing $C$ for the connected center of $M_{\text{red}}$, we see that the projection $p : M^o \rightarrow \mathbb{C}^* \times \mathbb{C}^*$ restricts to a surjective homomorphism $C \rightarrow \mathbb{C}^* \times \mathbb{C}^*$. Note that $C$ is a torus, and any surjective homomorphism of tori admits a quasi-splitting, i.e., there are integers $m_1, m_2 > 0$ and an algebraic homomorphism $s : \mathbb{C}^* \times \mathbb{C}^* \rightarrow C$ such that, $\forall (t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$, one has: $p \circ s(t_1, t_2) = (t_1^{m_1}, t_2^{m_2})$. The composition
\[
\gamma : \mathbb{C}^* \times \mathbb{C}^* \rightarrow C \hookrightarrow M \hookrightarrow \mathbb{C}^* \times \mathbb{C}^* \times G \overset{pr_3}{\rightarrow} G
\]
gives a homomorphism with the desired properties. □

Proof of parts (i), (ii) of Theorem 1.2. Using the homomorphism $\gamma$ of the Lemma, we define two elements, $h_1, h_2 \in \mathfrak{g}$, by
\[
h_t := \frac{1}{m_1} \cdot \frac{\partial \gamma}{\partial t}, \quad t = 1, 2.
\]
These elements are semisimple, since the image of $\gamma$ is contained in a torus in $G$, and by construction satisfy the commutation relations 1.2(i). This proves part (i) of Theorem 1.2.

Next observe that the commutation relations imply readily that, for the Ad-diagonal action, we have
\[
(\text{Ad} \exp t \cdot (e_1 + e_2)) \mathfrak{h} = \mathfrak{h} - t \cdot \mathfrak{e}, \quad \forall t \in \mathbb{C}. \quad (1.5)
\]

Define an algebraic family $\{\mathfrak{h}_t \in \mathcal{Z}\}$ parametrised by points $t \in \mathbb{C}^* \cup \{\infty\} = \mathbb{CP}^1 \setminus \{0\}$ as follows. If $t \in \mathbb{C}^*$ put $\mathfrak{h}_t := e - t^{-1} \cdot \mathfrak{h} = -t^{-1} \cdot (\text{Ad} \exp t \cdot (e_1 + e_2)) \mathfrak{h}$; and for $t = \infty$ put $\mathfrak{h}_\infty := \mathfrak{e}$. Formula (1.5) shows that $\mathfrak{h}_t \rightarrow \mathfrak{h}_\infty$, as $t \rightarrow \infty$.

To prove part (ii) of Theorem 1.2, we consider the family of vector spaces $\mathfrak{z}(\mathfrak{h}_t), t \in \mathbb{CP}^1 \setminus \{0\}$. We claim that the dimensions of all the spaces $\mathfrak{z}(\mathfrak{h}_t)$ in our family are independent of $t$. For $t \in \mathbb{C}^*$, formula (1.5) shows that $\mathfrak{z}(\mathfrak{h}_t) = (\text{Ad} \exp t \cdot (e_1 + e_2)) \mathfrak{z}(\mathfrak{h})$. Hence the centralizers $\mathfrak{z}(\mathfrak{h}_t), t \in \mathbb{C}^*$, all have the same dimension equal to $\dim \mathfrak{z}(\mathfrak{h})$. Now, by semicontinuity, the dimension of a special member of any algebraic family can not be less than the dimension of the general member. It follows that: $\dim \mathfrak{z}(\mathfrak{h}) \leq \dim \mathfrak{z}(\mathfrak{h}_\infty)$. On the other hand, we know that $\dim \mathfrak{z}(\mathfrak{h}_\infty) = \dim \mathfrak{z}(\mathfrak{e}) = \text{rk} \mathfrak{g}$, where the last equality is due to the (Reg)-condition in (1.1). The claim follows.

Thus, we have proved that $\dim \mathfrak{z}(\mathfrak{h}) = \text{rk} \mathfrak{g}$. It remains to observe that the centralizer of the pair $(h_1, h_2)$ of commuting semisimple elements is a Levi subalgebra in $\mathfrak{g}$. Being of dimension $\text{rk} \mathfrak{g}$ this Levi subalgebra must be a Cartan subalgebra. □

Observe that our proof implies also that in the Grassmannian of $\text{rk} \mathfrak{g}$-dimensional subspaces in $\mathfrak{g}$ we have
\[
\lim_{t \rightarrow \infty} (\text{Ad} \exp t \cdot (e_1 + e_2)) \mathfrak{z}(\mathfrak{h}) = \mathfrak{z}(\mathfrak{e}). \quad (1.6)
\]

Fix a principal nilpotent pair $\mathfrak{e} = (e_1, e_2)$ and an associated semisimple pair $\mathfrak{h} = (h_1, h_2)$. The adjoint action of commuting elements $(h_1, h_2)$ gives rise to a bigrading on $\mathfrak{g}$:
\[
\mathfrak{g} = \bigoplus_{(p, q) \in \mathbb{Q} \oplus \mathbb{Q}} \mathfrak{g}_{p,q}, \quad \mathfrak{g}_{p,q} = \{x \in \mathfrak{g} \mid [h_1, x] = p \cdot x, \ [h_2, x] = q \cdot x\}. \quad (1.7)
\]
We have: \( e_1 \in \mathfrak{g}_{1,0} \), and \( e_2 \in \mathfrak{g}_{0,1} \). Further, regularity of \( \mathfrak{h} \) implies that \( \mathfrak{g}_{0,0} = \mathfrak{z}(\mathfrak{h}) \) is a Cartan subalgebra in \( \mathfrak{g} \). A routine argument shows now that:

\[
[\mathfrak{g}_{p,q}, \mathfrak{g}_{p',q'}] = \mathfrak{g}_{p+p',q+q'}, \quad \mathfrak{g}_{p,q} \perp \mathfrak{g}_{p',q'} \quad \text{unless} \quad p = -p' \& q = -q',
\]

where ‘\( \perp \)’ is taken with respect to the Killing form on \( \mathfrak{g} \). This implies

**Corollary 1.8.** The Killing form induces a perfect pairing: \( \mathfrak{g}_{p,q} \times \mathfrak{g}_{-p,-q} \to \mathbb{C} \). \( \square \)

Observe that \([h_i, \mathfrak{z}(\mathfrak{e})] \subset \mathfrak{z}(\mathfrak{e}), \quad i = 1, 2\), hence the bigrading on \( \mathfrak{g} \) induces a bigrading \( \mathfrak{z}(\mathfrak{e}) = \bigoplus_{p,q} \mathfrak{z}_{p,q}(\mathfrak{e}) \).

**Proposition 1.9 (positive quadrant).** The algebra \( \mathfrak{z}(\mathfrak{e}) \) is graded by the ‘positive quadrant’, more precisely we have: \( \mathfrak{z}(\mathfrak{e}) = \bigoplus_{p,q \in \mathbb{Z}_{\geq 0}, (p,q) \neq (0,0)} \mathfrak{z}_{p,q}(\mathfrak{e}) \).

*Proof.* Observe that, for any \( h \in \mathfrak{g}_{0,0} \) and any \( t \in \mathbb{C}^* \), all non-vanishing bigraded components of the element \( (\text{Ad } \exp t \cdot (e_1 + e_2)) h \) are concentrated in bidegrees \((p,q) \in \mathbb{Z} \oplus \mathbb{Z}\) such that \( p,q \geq 0 \). Hence, the same holds for any element in \( \mathfrak{z}(\mathfrak{e}) \), due to (1.6).

It remains to prove that \( \mathfrak{z}_{0,0}(\mathfrak{e}) = 0 \), equivalently, that there is no non-zero element \( h \in \mathfrak{g}_{0,0} = \mathfrak{z}(\mathfrak{h}) \) that commutes with \((e_1, e_2)\). Assume that such an \( h \) exists.

We claim that \( \mathfrak{z}(\mathfrak{e}) \subset \mathfrak{z}(\mathfrak{h}) \). To prove this, consider the commuting pair \((h + e_1, h + e_2) \in \mathbb{Z}\). The Richardson theorem [R] mentioned at the beginning of this section yields the inequality: \( \dim \mathfrak{z}(h + e_1, h + e_2) \geq \text{rk } \mathfrak{g} \). On the other hand, since \( h \) and \( e_i \) are respectively the semisimple and nilpotent components of \( h + e_i, \; i = 1, 2 \), we have \( \mathfrak{z}(h + e_1, h + e_2) = \mathfrak{z}(h) \cap \mathfrak{z}(e_1, e_2) \). Therefore, if \( \mathfrak{z}(h) \) does not contain \( \mathfrak{z}(\mathfrak{e}) \), we get

\[
\dim \mathfrak{z}(h + e_1, h + e_2) = \dim (\mathfrak{z}(h) \cap \mathfrak{z}(e_1, e_2)) < \dim \mathfrak{z}(e_1, e_2) = \text{rk } \mathfrak{g}.
\]

This contradicts the Richardson’s inequality above, and the claim follows.

Next, we have a direct sum decomposition \( \mathfrak{g} = \bigoplus_{\nu \in \mathbb{C}} \mathfrak{g}(\nu) \) into \( \text{ad } h \)-eigenspaces. Clearly, for each \( \nu \in \mathbb{C} \), the eigen-space \( \mathfrak{g}(\nu) \) is stable under \( \text{ad } e_1 \) and \( \text{ad } e_2 \), since \( e_1, e_2 \) commute with \( h \) by assumption. Hence, the operators \( \text{ad } e_1 \) and \( \text{ad } e_2 \) form a commuting pair of nilpotent endomorphisms of \( \mathfrak{g}(\nu) \). It follows that there is a non-zero vector \( v \in \mathfrak{g}(\nu) \) annihilated by both endomorphisms. Choosing \( \nu \) here to be a non-zero eigenvalue, we conclude that \( \mathfrak{z}(\mathfrak{e}) \cap \mathfrak{g}(\nu) \neq 0 \). This contradicts the inclusion \( \mathfrak{z}(\mathfrak{e}) \subset \mathfrak{g}(0) = \mathfrak{z}(h) \) proved in the preceding paragraph, and the proposition follows. \( \square \)

From formula (1.6) and Proposition 1.9 we obtain:

**Corollary 1.10.** \( \mathfrak{z}(\mathfrak{e}) \) is an abelian Lie algebra consisting of nilpotent elements. \( \square \)

For example, let \( \mathfrak{g} = \mathfrak{sl}_3 \) and let the pair \( \mathfrak{e}_\lambda = (e_1, e_2) \) be associated to a Young diagram \( \lambda \) as explained in the Introduction. As will be shown in §5 below, the Lie algebra \( \mathfrak{z}(\mathfrak{e}) \) has a basis formed by the matrices: \( e_1^p \cdot e_2^q \), where the pairs \((p,q)\) run over the coordinates of all boxes of \( \lambda \), except for \((p,q) = (0,0)\), since
Thus, we observe that the set of points \((p, q)\) on the 2-plane such that \(\dim Z_p(q)(e_i) \neq 0\) forms a figure of shape \(\lambda\).

Motivated by this observation, and also by [K1, K2], given a principal nilpotent pair \(e\) in any semisimple Lie algebra \(g\), we define bieponents of \(g\) relative to \(e\) as the pairs of non-negative integers \((p, q)\), cf. Proposition 1.9, such that \(\dim Z_p(q)(e) \neq 0\). In more detail, fix a bi-homogeneous base \(z_1, \ldots, z_r\), \(r = \text{rk} g\), of the centralizer \(Z(e) = \bigoplus Z_p(q)(e)\). To each \(i = 1, \ldots, \text{rk} g\), assign the pair \((p_i, q_i) \in \mathbb{Z}_{\geq 0}^2\) such that \(z_i \in Z_{p_i, q_i}(e)\). We introduce the following

**Definition 1.11.** The subset \(\text{Exp}_e(g) = \{(p_1, q_1), \ldots, (p_r, q_r), r = \text{rk} g\} \subset \mathbb{Z}_{\geq 0}^2\) will be referred to as the collection of bieponents of \(g\) relative to \(e\).

A new feature of the "doubled" setup under investigation, making it quite different from the "classical" one, is that it is impossible, in general, to find \(\mathfrak{sl}_2\)-triples associated with \(e_1\) and \(e_2\) in such a way that they commute with each other. As a result, "hard Lefschetz" type equations, like \(\dim \mathfrak{g}_{p,q} = \dim \mathfrak{g}_{-p,-q}\), are typically false in the bigraded setup. The proposition below says that an analogue of the "weak Lefschetz theorem" still holds in our situation.

**Proposition 1.12 (weak Lefschetz).** For any principal pair \(e = (e_1, e_2)\), the map \(\text{ade}_1 : \mathfrak{g}_{p,q} \to \mathfrak{g}_{p+1,q}\) is injective whenever \(p < 0\), and is surjective whenever \(p \geq 0\). Similarly, the map \(\text{ade}_2 : \mathfrak{g}_{p,q} \to \mathfrak{g}_{p,q+1}\) is injective whenever \(q < 0\), and is surjective whenever \(q \geq 0\).

**Proof.** For any \(x \in \mathfrak{g}\), the operator \(\text{ad}x : \mathfrak{g} \to \mathfrak{g}\) is skew-adjoint with respect to the Killing form. Taking \(x = e_1\) and using Corollary 1.8 we see that, for any \((p, q)\), the adjoint of the operator on the left (below) equals, up to sign, the one on the right:

\[
\text{ade}_1 : \mathfrak{g}_{p,q} \to \mathfrak{g}_{p+1,q} \quad , \quad \text{ade}_1 : \mathfrak{g}_{-p-1,-q} \to \mathfrak{g}_{-p,-q}.
\]

Since a linear map is surjective if and only if the dual map is injective, it suffices to prove the injectivity part of the proposition only.

Set \(\mathfrak{g}_{p,*} = \bigoplus_q \mathfrak{g}_{p,q}\). We must prove that the map \(\text{ade}_1 : \mathfrak{g}_{p,*} \to \mathfrak{g}_{p+1,*}\) is injective for any \(p < 0\). If this map has a non-trivial kernel, \(\text{Ker}_p(\text{ade}_1) \subset \mathfrak{g}_{p,*}\), then the adjoint action of \(e_2\) gives an operator \(\text{ade}_2 : \text{Ker}_p(\text{ade}_1) \to \text{Ker}_p(\text{ade}_1)\). This operator is nilpotent, hence, has a non-trivial kernel \(\text{Ker}_p(\text{ade}_1) \cap \text{Ker}(\text{ade}_2) \neq 0\). It follows that \(\mathfrak{z}(e_1, e_2)\) has a non-zero intersection with \(\mathfrak{g}_{p,*}\). But this contradicts Proposition 1.9 ("positive quadrant"), because we assumed that \(p < 0\).

**Proof of parts (iii), (iv) of Theorem 1.2.** Proving integrality of the eigenvalues of the operators \(\text{ad} h_i\) amounts to showing that, in the bigrading \(\mathfrak{g} = \bigoplus \mathfrak{g}_{p,q}\), we have \(\mathfrak{g}_{p,q} = 0\) unless \((p, q) \in \mathbb{Z} \times \mathbb{Z}\). Assume there exists a pair \((p_0, q_0) \notin \mathbb{Z} \times \mathbb{Z}\) such that \(\mathfrak{g}_{p_0,q_0} \neq 0\). Set \(A = \{(p, q) \mid p \in p_0 + \mathbb{Z} \& q \in q_0 + \mathbb{Z}\}\), and let \(\mathfrak{g}_A = \bigoplus_{(p,q) \in A} \mathfrak{g}_{p,q}\). Clearly, \(\mathfrak{g}_A\) is a subspace in \(\mathfrak{g}\) stable under the action of \(\{\text{ade}_i\}_{i=1,2}\). Hence, there is a non-zero vector \(x \in \mathfrak{g}_A\) annihilated by both operators (which are nilpotent and commute). Thus, \(\mathfrak{z}(e)(x) \cap \mathfrak{g}_A \neq 0\). This contradicts Proposition 1.9, saying that \(\mathfrak{z}(e)(x)\) may be non-zero only for integral \((p, q)\), and part (iii) of Theorem 1.2 is proved.

To prove part (iv) fix an associated semisimple pair \(h = (h_1, h_2)\), and let \(\langle h_1, h_2 \rangle\) be the \(\mathbb{C}\)-linear span of \(h_1, h_2\).
We consider the algebraic group \( M \) from the exact sequence (1.4). Let \( \overline{M} \subset G \) be the image of \( M \) under the 3-d projection: \( \mathbb{C}^* \times \mathbb{C}^* \times G \to G \), and \( \overline{\mathfrak{m}} := \text{Lie} \overline{M} \). Thus, we have \( \overline{\mathfrak{m}} = \{ x \in \mathfrak{g} \mid \text{ad}x(e_i) \in \mathbb{C} \cdot e_i, \ i = 1, 2 \} \). Exact sequence (1.4) yields a Lie algebra semi-direct product decomposition \( \overline{\mathfrak{m}} = \langle h_1, h_2 \rangle \ltimes \mathfrak{z} \). Proposition 1.9 implies that \( \mathfrak{z} \) is the nilradical (= Lie algebra of the unipotent radical) of the algebraic Lie algebra \( \overline{\mathfrak{m}} \). Thus, \( \overline{\mathfrak{m}} \) is a solvable Lie algebra, and \( \langle h_1, h_2 \rangle \) is a maximal diagonalizable subalgebra in \( \overline{\mathfrak{m}} \).

Now, given another associated semisimple pair \( h' = (h'_1, h'_2) \), we will get the same way another maximal diagonalizable subalgebra \( \langle h'_1, h'_2 \rangle \) in \( \overline{\mathfrak{m}} \). Recall that any two maximal diagonalizable subalgebras of an algebraic solvable Lie algebra are conjugate by a unipotent element. We deduce that there exists a unipotent element \( u \in Z^0_G(\mathfrak{e}) \) such that \( \text{Ad}u(h_1, h_2) = \langle h'_1, h'_2 \rangle \). Thus, we may assume without loss of generality that \( \langle h'_1, h'_2 \rangle = \langle h_1, h_2 \rangle \). But then the commutation relations 1.2(i) uniquely determine the positions of \( h_1, h_2 \) inside the 2-dimensional space \( \langle h_1, h_2 \rangle \), hence \( h'_i = h_i \), for \( i = 1, 2 \). This completes the proof of the Theorem. \( \square \)

The following result providing an alternative characterisation of principal nilpotent pairs will be proved later in Section 2.

**Theorem 1.13.** A commuting pair \( e = (e_1, e_2) \in \mathcal{Z} \) is a principal nilpotent pair if and only if the following two conditions hold:

(a) There exists a regular semisimple pair \( h = (h_1, h_2) \in \mathcal{Z} \) such that:

\[
[h_i, e_j] = \delta_{i,j} \cdot e_i, \ i, j \in \{1, 2\}, \ cf. \ 1.2(i);
\]

(b) The subalgebra \( \mathfrak{z}(e) \) is graded by the ‘positive quadrant’ with respect to the corresponding bigrading \( \mathfrak{g} = \bigoplus_{p,q} \mathfrak{g}_{p,q} \); that is: \( \mathfrak{z}(e) \subset \bigoplus_{p,q \in \mathbb{Z}_{\geq 0}, (p,q) \neq (0,0)} \mathfrak{g}_{p,q} \).

Note that regularity of the pair \( e \) is not assumed in conditions (a)-(b) above, and that the grading on \( \mathfrak{z}(e) \) is required in (b) to be integral.

2. Bi-filtration associated to a nilpotent pair.

Given a vector space \( V \) equipped with an increasing \( \mathbb{Z}_{\geq 0} \oplus \mathbb{Z}_{\geq 0} \)-filtration (referred to as a "bi-filtration"), \( F_{i,j} V \), such that \( F_{i',j'} V \subset F_{i,j} V \), whenever \( i' < i \) or \( j' < j \), we define the corresponding associated bigraded space as follows:

\[
\text{gr}^F V := \bigoplus_{i,j} \text{gr}_{i,j} V, \ \ \ \text{gr}_{i,j} V = \frac{F_{i,j} V}{F_{i-1,j} V + F_{i,j-1} V}. \quad (2.1)
\]

To any commuting pair \( (e_1, e_2) \) of nilpotent endomorphisms of the vector space \( V \) one associates the bi-filtration: \( F_{i,j} (V) := \{ v \in V \mid e_1^{i+1} e_2^{j+1}(v) = 0 = e_1^i e_2^{j+1}(v) \} \). This bi-filtration on \( V \) induces, by restriction, a bifiltration on any vector subspace \( E \subset V \), and we write \( \text{gr} E \) for the corresponding associated bigraded space.

Let \( \dim E = r \), so that \( E \) gives a point in \( \text{Gr}_r(V) \), the Grassmannian of \( r \)-planes in \( V \). The natural action on \( \text{Gr}_r(V) \) of the one-parameter subgroup \( t \mapsto \exp t \cdot (e_1 + e_2) \in \text{GL}(V) \) gives an algebraic path \( \gamma : \mathbb{C} \to \text{Gr}_r(V) \) through the point \( E \); explicitly, \( \gamma(t) = \exp t \cdot (e_1 + e_2)(E) \). The Grassmannian being a compact
variety, the path $\gamma$ has a well-defined limit $\gamma(\infty) \in \text{Gr}_r(V)$, as $t \to \infty$. The limit point $\gamma(\infty)$ corresponds to an $r$-dimensional vector subspace in $V$, to be denoted $\lim_\mathbf{e} E$, where $\mathbf{e}$ indicates the commuting pair $e_1, e_2$.

Exploiting an idea of R. Brylinski one can give a more concrete description of the space $\lim_\mathbf{e} E$ in terms of the bi-filtration $F_{p,q} E$, which reduces in the special case $e_2 = 0$ to [Br, Lemma 2.5]:

**Lemma 2.2.** Assume that the sum $\sum_{p,q \geq 0} e_1^p e_2^q (F_{p,q} E)$ is a direct sum of vector subspaces in $V$. Then

(i) We have: $\lim_\mathbf{e} E = \bigoplus_{p,q \geq 0} e_1^p e_2^q (F_{p,q} E)$; in particular the space $\lim_\mathbf{e} E$ is annihilated by the operators $e_1, e_2$.

(ii) The direct sum (over all $p, q \geq 0$) of maps $e_1^p e_2^q : F_{p,q} E \to \lim_\mathbf{e} E$ induces a vector space isomorphism

$$\lim_\mathbf{e} : \text{gr} E = \bigoplus_{p,q} \frac{F_{p,q} E}{F_{p-1,q} E + F_{p,q-1} E} \overset{\sim}{\longrightarrow} \lim_\mathbf{e} E.$$  

To prove the lemma, note first that, for any $x \in F_{p,q} E$, one has a finite expansion

$$(\exp t \cdot (e_1 + e_2)) x = \sum_{i=0}^p \sum_{j=0}^q \frac{t^{i+j}}{i! j!} e_1^i e_2^j(x).$$

Observe next that the action on $V$ of the operator $e_1^p e_2^q$ kills the subspace $F_{p-1,q} E + F_{p,q-1} E$. From this, one proves by induction on $(p, q)$ that $(e_1^p e_2^q (F_{p,q} E)) \subseteq \lim_\mathbf{e} E$. Furthermore, the induced map $e_1^p e_2^q : F_{p,q} E/(F_{p-1,q} E + F_{p,q-1} E) \to \lim_\mathbf{e} E$ is injective. Using that $\dim E = \dim(\lim_\mathbf{e} E)$, one completes the proof by a similar induction. □

Now let $\mathfrak{g}$ be a semisimple Lie algebra. Fix a principal nilpotent pair $\mathbf{e} = (e_1, e_2)$, and an associated semisimple pair $\mathfrak{h} = (h_1, h_2)$. The construction above gives a bi-filtration on $\mathfrak{g}$, associated to the commuting pair of nilpotent endomorphisms $\text{ad} e_1, \text{ad} e_2$ on $\mathfrak{g} = \mathfrak{g}$. We first consider the subspace $E \equiv \mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \subset \mathfrak{g}$. Our construction gives a subspace $\lim_\mathbf{e} \mathfrak{h} \subset \mathfrak{g}$. Note that the assumption of Lemma 2.2 holds since: $e_1^p e_2^q (\mathfrak{h}) \subseteq \mathfrak{g}_{p,q}$, and the sum $\bigoplus \mathfrak{g}_{p,q}$ is direct. Hence the subspace $\lim_\mathbf{e} \mathfrak{h}$ is annihilated by $\mathbf{e}$, by Lemma 2.2(i). We conclude that $\lim_\mathbf{e} \mathfrak{h} \subset \mathfrak{z}(\mathbf{e})$. But $\dim \mathfrak{h} = \dim \mathfrak{z}(\mathbf{e})$, since $\mathbf{e}$ is a regular pair. Thus, the inclusion implies an equality: $\lim_\mathbf{e} \mathfrak{h} = \mathfrak{z}(\mathbf{e})$, which is just another form of (1.6).

Next, we take $E \equiv \mathfrak{z}(h_1, e_2)$. Since the operator $\text{ad} e_2$ annihilates the space $\mathfrak{z}(h_1, e_2)$, the bifiltration $F_{i,j} \mathfrak{z}(h_1, e_2)$ reduces to an ordinary filtration so that, for any $i, j \geq 0$, we have: $F_{i,j} \mathfrak{z}(h_1, e_2) = \text{Ker} (\text{ad}^{i+1} e_1 : \mathfrak{z}(h_1, e_2) \to \mathfrak{z}(h_1, e_2))$. The latter space being independent of $j$, we will simply write $F_i \equiv F_{i,j} \mathfrak{z}(h_1, e_2)$.

As we know, the map $\text{ad}^i e_1$ gives an imbedding $\text{ad}^i e_1 : F_i / F_{i-1} \hookrightarrow \mathfrak{z}(\mathbf{e})$. Furthermore, the assumption of Lemma 2.2 trivially holds. Hence, in the Grassmannian of $\text{rk} \mathfrak{g}$-planes, there exists a limit of the family of spaces $(\text{Ad} \exp t \cdot e_1) \mathfrak{z}(h_1, e_2)$, $t \to \infty$, and by Lemma 2.2(i) one has $\lim_\mathbf{e} \mathfrak{z}(h_1, e_2) \subset \mathfrak{z}(\mathbf{e})$. We note further that $(h_1, e_2) \in \mathfrak{Z}$ is a commuting pair. Hence the Richardson inequality and the inclusion above yield:

$$\text{rk} \mathfrak{g} \leq \dim \mathfrak{z}(h_1, e_2) = \dim(\lim_\mathbf{e} \mathfrak{z}(h_1, e_2)) \leq \dim \mathfrak{z}(\mathbf{e}) = \text{rk} \mathfrak{g}.$$

Thus, the inclusion: \( \lim \mathfrak{z}(h_1, e_2) \subset \mathfrak{z}(\mathfrak{e}) \) must be an equality, and we obtain the following analogue of (1.6):

\[
\lim \mathfrak{z}(h_1, e_2) = \lim_{t \to \infty} (\text{Ad exp } t \cdot e_1) \mathfrak{z}(h_1, e_2) = \bigoplus_{j \geq 0} \text{ad}^j e_1(\mathfrak{z}(h_1, e_2)) = \mathfrak{z}(\mathfrak{e}). \tag{2.3}
\]

**Proof of Theorem 1.13.** We already know that if \( \mathfrak{e} \) is a principal nilpotent pair, then properties (a)-(b) of Theorem 1.13 hold (by Theorem 1.2(i) and Proposition 1.9).

Thus, we must only show that any pair \( \mathfrak{e} \in \mathcal{Z} \) satisfying the conditions of Theorem 1.13 is a principal nilpotent pair. Condition 1.1(Nil) follows readily from the very existence of an associated semisimple pair \( \mathfrak{h} \) with the commutation relations as in Theorem 1.13(a). Thus, it suffices to show that \( \mathfrak{e} \) is a regular pair.

To this end, we note first that the only ingredient used in the proof of Proposition 1.12 was the ”positive quadrant” property, which is just condition (b) of Theorem 1.13. Therefore, the weak Lefschetz property holds for our pair \( \mathfrak{e} \), even though we don’t know yet that it is a principal nilpotent pair.

Write \( e_1^p \) for the \( r \)-th power of the adjoint action of \( e_1 \) on \( \mathfrak{g} \). The surjectivity part of the weak Lefschetz yields: \( \bigoplus_{p,q \in \mathbb{Z}_{\geq 0}} \mathfrak{g}_{p,q} = \bigoplus_{p,q \in \mathbb{Z}_{\geq 0}} e_1^p e_2^q(\mathfrak{g}_{0,0}) \). Using the ”positive quadrant” condition of Theorem 1.13(b) once again we deduce that \( \mathfrak{z}_{p,q}(\mathfrak{e}) \subset e_1^p e_2^q(\mathfrak{g}_{0,0}) \), \( \forall p, q \in \mathbb{Z}_{\geq 0} \). Hence, given \( x \in \mathfrak{z}_{p,q}(\mathfrak{e}) \), one can find \( h \in \mathfrak{g}_{0,0} \) such that \( x = e_1^p e_2^q(h) \). The condition \( x \in \mathfrak{z}_{p,q}(\mathfrak{e}) \) thus reads: \( 0 = e_1(x) = e_1^{p+1} e_2^q(h) \), and \( 0 = e_2(x) = e_1^{p} e_2^{q+1}(h) \).

By the assumptions of Theorem 1.13, the pair \( \mathfrak{h} \) is regular, hence \( \mathfrak{h} = \mathfrak{g}_{0,0} \) is a Cartan subalgebra in \( \mathfrak{g} \). We have the bi-filtration \( F_{i,j} \mathfrak{h} \) on \( \mathfrak{h} \) arising from the adjoint action of the commuting pair \( (e_1, e_2) \) on \( \mathfrak{g} \). The equations at the end of the previous paragraph say that \( h \in F_{p,q} \mathfrak{h} \). Hence, we have proved that \( \mathfrak{z}(\mathfrak{e}) = \bigoplus_{p,q \geq 0} e_1^p e_2^q(F_{p,q} \mathfrak{h}) \). On the other hand, in the special case \( \mathfrak{V} = \mathfrak{g} \) and \( E = \mathfrak{h} \), Lemma 2.1(i) reads: \( \dim \mathfrak{z}(\mathfrak{h}) = \bigoplus_{p,q \geq 0} e_1^p e_2^q(F_{p,q} \mathfrak{h}) \). Thus, \( \mathfrak{z}(\mathfrak{e}) = \bigoplus e_1^p e_2^q(F_{p,q} \mathfrak{h}) = \lim \mathfrak{h} \), and we deduce that \( \dim \mathfrak{z}(\mathfrak{e}) = \dim(\lim \mathfrak{h}) = \dim \mathfrak{h} \). This proves regularity of \( \mathfrak{e} \), and the theorem follows. \( \square \)

Now let \( \mathfrak{V} \) be a finite dimensional \( \mathfrak{g} \)-module. We write \( \mathfrak{V}^m \) for the subspace of \( \mathfrak{V} \) annihilated by a Lie subalgebra \( \mathfrak{m} \subset \mathfrak{g} \). Let \( T \) be the maximal torus in \( \mathfrak{G} \) corresponding to the Cartan subalgebra \( \mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \). Given a weight \( \mu \in \mathfrak{X}^*(T) := \text{Hom}_{\text{alg group}}(T, \mathbb{C}^*) \), let \( \mathfrak{V}(\mu) \) denote the corresponding weight space. We claim, inspired by [Br, Proposition 2.6], that

\[
\lim \mathfrak{V}(\mu) \subset \mathfrak{V}^{\mathfrak{z}(\mathfrak{e})}, \quad \forall \mu \in \mathfrak{X}^*(T). \tag{2.4}
\]

To prove this inclusion observe that every non-zero vector in \( \mathfrak{V}(\mu) \) gives a point in \( \mathbb{P}(\mathfrak{V}(\mu)) \) (= projectivisation of \( \mathfrak{V}(\mu) \)) fixed by the \( T \)-action, hence, by the infinitesimal \( \mathfrak{h} \)-action. It follows, by continuity, that each point of \( \mathbb{P}(\lim \mathfrak{V}(\mu)) \) is fixed by the infinitesimal \( \lim \mathfrak{h} \)-action. Hence \( \lim \mathfrak{V}(\mu) \) is a weight space for the subalgebra \( \lim \mathfrak{h} \). But since the subalgebra \( \lim \mathfrak{h} = \mathfrak{z}(\mathfrak{e}) \) consists of nilpotent elements, the weight in question must vanish, and our claim follows.

As an important consequence of (2.4), assume \( \mathfrak{V} \) is a rational \( \mathfrak{G} \)-module, so that all the weights of \( \mathfrak{V} \) belong to the root lattice of \( \mathfrak{g} \), in particular, zero is a weight of \( \mathfrak{V} \) and \( \mathfrak{V}^{\mathfrak{z}(\mathfrak{h})} \neq 0 \). Then, the subspace \( \lim \mathfrak{V}^{\mathfrak{z}(\mathfrak{h})} \subset \mathfrak{V}^{\mathfrak{z}(\mathfrak{e})} \) is independent
of the choice of an associated semisimple pair \( h \). Indeed, by Theorem 1.2(iv), any two such semisimple pairs are conjugate by the unipotent group \( Z_G^0(e) \). Hence, the corresponding Cartan subalgebras \( \mathfrak{h} = \frac{1}{2}(h) \) are conjugate by \( Z_G^0(e) \). It follows that the corresponding zero-weight spaces \( V^{\frac{1}{2}(h)} \), whence the limit-spaces \( \lim_{e} V^{\frac{1}{2}(h)} \), are conjugate by the \( Z_G^0(e) \)-action. But the latter action is trivial by (2.4), thus, the two limit-spaces coincide. Note that if \( \dim V^{\frac{1}{2}(h)} = \dim V^{\frac{1}{2}(e)} \), then (2.4) becomes an equality. We emphasize that unlike the case of an ordinary principal nilpotent, the inclusion \( \lim_{e} V^{\frac{1}{2}(h)} \subset V^{\frac{1}{2}(e)} \) may be strict, in general. This happens, for instance, for the pair \( e \) in \( \mathfrak{g} = \mathfrak{sl}_3 \), corresponding to the "hook" Young diagram, and \( V = S^3(C^3) \), the third Symmetric power of the fundamental representation.

In spite of this counter-example we expect that one of the most important results of [K2] continues to hold true in the double setup. Specifically, let \( \overline{O}_e \) denote the closure of the \( \text{Ad}G \)-diagonal orbit of \( e \), an affine subvariety in \( \mathfrak{g} \oplus \mathfrak{g} \). Write \( \mathbb{C}[X] \) for the coordinate ring of an affine algebraic variety \( X \). Then we have

**Conjecture 2.5.** There is a \( G \)-module isomorphism: \( \mathbb{C}[\overline{O}_e] \simeq \mathbb{C}[G/T] \).

Recall next that, associated to \( e \), we have defined the bifiltration \( F_{\bullet}, V \) on a finite dimensional \( \mathfrak{g} \)-module \( V \). Given a weight \( \mu \in X^*(T) \), we endow the weight space \( V(\mu) \) with the induced filtration \( F_{i,j} V(\mu) = V(\mu) \cap F_{i,j}(V) \), and let \( \text{gr} V(\mu) = \bigoplus_{i,j} \text{gr}_{i,j} V(\mu) \) denote the corresponding associated bigraded space, see (2.1). By Theorem 1.2(iv) the integers \( \dim \text{gr}_{i,j} V(\mu) \), called bi-exponents of \( V \) relative to \( e \), cf. Definition 1.11, are canonically associated to \( e \) and \( V \), i.e., do not depend on the choice of an associated pair \( (h_1, h_2) \).

We define an \( (s, t) \)-weight multiplicity of \( V(\mu) \) as the Poincaré polynomial:

\[
P_{\mu}(V, e) = \sum_{i,j \geq 0} s^i t^j \cdot \dim \text{gr}_{i,j} V(\mu).
\]

(2.6)

Our goal is to produce an explicit formula for the polynomials \( P_{\mu}(V, e) \), analogous to Lusztig’s \( q \)-weight multiplicity [L1], see also [Br].

First, write \( R \) for the root system of \( \mathfrak{g} \) with respect to the Cartan subalgebra \( \mathfrak{h} = \frac{1}{2}(h_1, h_2) \). There is a distinguished subset \( R_{\text{ne}} \subset R \) (where \( \text{ne} = \text{northeast} \) formed by the roots that occur in the “positive quadrant”: \( R_{\text{ne}} = \{ \alpha \in R \mid \alpha(h_1) \geq 0 \land \alpha(h_2) \geq 0 \} \). The set \( R_{\text{ne}} \) is clearly contained in an open half-space of \( \mathfrak{h}^* \). Thus, we can (and will) make a choice of the set \( R_{+} \subset R \) of positive roots in such a way that \( R_{\text{ne}} \subset R_+ \). By construction, we have a decomposition \( R_+ = R_{\text{ne}} \sqcup (R_+ \setminus R_{\text{ne}}) \). Furthermore, the set \( R_{\text{ne}} \) has a natural decomposition into a disjoint union of three subsets:

\[
R_{\text{ne}} = R^1_+ \sqcup R^2_+ \sqcup R_\bullet \quad \text{where} \quad R_\bullet = \{ \alpha \in R \mid \alpha(h_1) > 0 \land \alpha(h_2) > 0 \},
\]

(2.7)

\[
R^1_+ = \{ \alpha \in R \mid \alpha(h_1) > 0 = \alpha(h_2) \}, \quad R^2_+ = \{ \alpha \in R \mid 0 = \alpha(h_1) < \alpha(h_2) \}.
\]

Recall that since the group \( G \) is of adjoint type, the weight lattice \( X^*(T) \) may be identified with the root lattice. Let \( Q_{\text{ne}} \) be the sub-semigroup of \( X^*(T) \) generated by the set \( R_{\text{ne}} \).

Write \( e^{\beta} : T \to \mathbb{C}^* \) for the group character corresponding to a root \( \beta \in R \). For each \( s, t \in \mathbb{C}^* \), we define a rational function \( \Psi_e \) on the torus \( T \) by the following
product:

\[ \mathcal{P}_e = \prod_{\beta \in R_+ \setminus R_{ne}} (1 - e^\beta) \cdot \prod_{\mu \in R^1_+} (1 - s e^\mu) \cdot \prod_{\nu \in R^2_+} (1 - t e^\nu) \cdot \prod_{\alpha \in R_\bullet} \frac{(1 - s e^\alpha)(1 - t e^\alpha)}{(1 - st e^\alpha)}. \] (2.8)

Thus, the assignment: \( s, t, z \mapsto \mathcal{P}_e(s, t, z) \) gives a rational function on \( \mathbb{C}^* \times \mathbb{C}^* \times T \).

It is clear that the function \( \frac{1}{\mathcal{P}_e} \) has the following expansion:

\[ \frac{1}{\mathcal{P}_e} = \sum_{\alpha \in Q_{ne}} \varphi_\lambda(\alpha) \cdot e^\alpha, \quad \text{where } \varphi_\lambda(\alpha) \in \mathbb{Z}_{\geq 0}[s, t]. \]

Note further that if \( s = t = 1 \) then the product (2.8) specializes to the classical Weyl denominator: \( \mathcal{P}_e|_{s=t=1} = \prod_{\alpha \in R_+} (1 - e^\alpha) \), so that the function \( \varphi_\lambda \) becomes the Kostant’s partition function. Motivated by the classical case, we propose

**Definition 2.9.** The function \( \varphi_\lambda : Q_{ne} \to \mathbb{Z}_{\geq 0}[s, t], \ \alpha \mapsto \varphi_\lambda(\alpha) \), is called double-analogue of the Kostant partition function associated to the principal pair \( e \).

Note that for a principal nilpotent pair of the form \( e = (e, 0) \) or \( e = (0, e) \) the function \( \varphi_\lambda \) reduces to the \( q \)-analogue of Kostant’s partition function, introduced in [L1].

Let \( \rho \) be the half-sum of positive roots in \( R_+ \). Write \( W \) for the Weyl group of \((g, h)\), and let \( \epsilon : w \mapsto \epsilon(w) = (-1)^{\text{length}(w)} \) denote the determinant of the \( w \)-action on \( h \). In the sequel to this paper we will prove the following double-analogue of Kostant’s weight multiplicity formula, expressing the \((s, t)\)-weight multiplicities in terms of the partition function \( \varphi_\lambda \):

**Theorem 2.10.** Let \( \lambda \) be a dominant weight such that \( \lambda \in Q_{ne} \), and \( V_\lambda \) the simple \( G \)-module with highest weight \( \lambda \). Then, one has:

\[ P_\mu(V_\lambda, e) = \sum_{w \in W} \epsilon(w) \cdot \varphi_\lambda(w(\lambda + \rho) - \mu - \rho), \quad \text{for any weight } \mu \in X^*(T). \]

In the case of \( e = (e, 0) \) where \( e \) is an ordinary principal nilpotent, the theorem above reduces to a result of R. Brylinski [Br]. Our proof consists of several steps which are analogous to the corresponding steps in [Br], but with considerable complications due to the difference between the classical geometry of conjugacy classes in \( g \) and that of double-orbital varieties to be introduced and studied in the subsequent paper of this series. Later on, we are also going to give an alternative geometric interpretation of the polynomials \( P_\mu(V_\lambda, e) \) in terms of intersection cohomology of certain varieties in a "double-loop" Grassmannian, cf. [L1], [Gi] for the affine case.

### 3. Two parabolics.

Fix a principal nilpotent pair \( e = (e_1, e_2) \) and an associated semisimple pair \( h = (h_1, h_2) \). We introduce two Levi subalgebras

\[ \mathfrak{g}^1 := \mathfrak{z}_g(h_2) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_{p,0}, \quad \text{and} \quad \mathfrak{g}^2 := \mathfrak{z}_g(h_1) = \bigoplus_{q \in \mathbb{Z}} \mathfrak{g}_{0,q}. \] (3.1)
We will analyze the first one, $g^1$, the structure of the second being entirely similar.

Let $g^1 = c(g^1) \oplus s^1$ be the decomposition of $g^1$ into its center and the semisimple derived Lie algebra. It is clear that $e_1 \in s^1$ and $h_1 \in g^1$. Write $h_1 = c_1 + s_1$, where $c_1 \in c(g^1)$ and $s_1 \in s^1$.

Let $h = \frac{1}{2}(h_1, h_2) = g_{00}$ be the Cartan subalgebra in $g$ associated to $h$. It is clear that $c(g^1) \subset h$. Furthermore, the space $h^1 := s^1 \cap h = z_{g^1}(s_1)$ is a Cartan subalgebra in $s^1$ since we have: $c(g^1) + h^1 = c(g^1) + z_{g^1}(s_1) = z_{g^1}(s_1) = z_{g^1}(h_1) = h$, is a Cartan subalgebra in $g^1$. In particular, $s_1$ is a regular element of $s^1$.

Write $R \subset h^*$ for the root system of $(g, h)$, and $R^1, R^2 \subset R$ for the root systems of $(g^1, h)$ and $(g^2, h)$, respectively. For each $\alpha \in R$, let $\alpha^\vee \in h$ denote the corresponding coroot, and $x_\alpha \in g$ the corresponding root vector. Let $R^1_+$ be the set of roots $\alpha \in R^1$ such that $\alpha(h_1) > 0$, see (2.7). Then $R^1_+$ is a system of positive roots for $R^1$, and we have the corresponding triangular decomposition $g^1 = g^1_+ \oplus h \oplus g^1_-$. Let $\Delta^1 \subset R^1_+$ be the set of simple roots.

**Proposition 3.2.** (i) $g^1_\pm$ are the nilradicals of two opposite Borel subalgebras in $s^1$:

- (ii) The element $s_1$ equals the half-sum of positive coroots of $h^1$, that is: $s_1 = \frac{1}{2} \sum_{\alpha \in R^1_+} \alpha^\vee$;
- (iii) $e_1 \in s^1_+$ is a principal nilpotent in $s^1$; furthermore, $e_1 = \sum_{\alpha \in \Delta^1} t_\alpha \cdot x_\alpha$, for certain $t_\alpha \in \mathbb{C}^*$;
- (iv) The elements $s_1, e_1$ are members of a principal $\mathfrak{sl}_2$-triple $(s_1, e_1, f_1)$ in $s^1$.

**Proof.** Part (i) has been already proved. To prove that $e_1$ is a regular element in $s^1$ we note that $rk g = rk g^1$ and $dim z_{g^1}(e_1) = dim z_{g^1}(h_2)$. But now isomorphism (2.3) yields $dim z_{g^1}(h_2) = rk g$. Thus, $e_1$ is regular in $g^1$, hence, in $s^1$.

To get the expression for $e_1$ given in (iii), we write $e_1$ as a linear combination of root vectors: $e_1 = \sum_{\alpha \in R^1_+} t_\alpha \cdot x_\alpha$, for some $t_\alpha \in \mathbb{C}$. It is known that, since $e_1$ is a regular nilpotent, the coefficient $t_\alpha$ corresponding to every simple root $\alpha \in \Delta^1$ must be non-zero. Furthermore, $e_1$ is an eigenvector for ad$s_1$. But since $s_1 \in h^1$ is regular, any ad$s_1$-eigenvector having a non-zero component for each simple root vector must be a linear combination of simple root vectors (with non-vanishing coefficients). This proves (iii).

The commutation relation $[s_1, e_1] = e_1$, where $e_1 = \sum_{\alpha \in R^1_+} t_\alpha \cdot x_\alpha$, with all $t_\alpha \neq 0$, forces $s_1 \in h^1$ to be equal to the half-sum of positive coroots: $s_1 = \frac{1}{2} \sum_{\alpha \in R^1_+} \alpha^\vee$. This proves (ii). Finally, it is known (and straightforward to check directly) that the elements $s_1, e_1$ given by the expressions above are members of a principal $\mathfrak{sl}_2$-triple, and part (iv) follows. \(\square\)

Write $g_{p,*} := \oplus q g_{p,q}$, and $g_{*,q} := \oplus p g_{p,q}$. Thus, $g^1 = g_{*,0}$, and $g^2 = g_{0,*}$. We introduce two parabolic subalgebras in $g$ with Levi factors $g^2$ and $g^1$, respectively:

$$p^{east} = g^2 \oplus \left( \bigoplus_{p \geq 0} g_{p,*} \right) = \bigoplus_{p \geq 0} g_{p,*}, \quad p^{north} = g^1 \oplus \left( \bigoplus_{q \geq 0} g_{*,q} \right) = \bigoplus_{q \geq 0} g_{*,q}. \quad (3.3)$$

Thus, the parabolic $p^{east}$ is the sum of $g_{p,q}$ over all $(p, q)$ in the right half-plane of the $(p, q)$-plane, resp. $p^{north}$ is the sum of $g_{p,q}$ over all $(p, q)$ in the upper half-plane.
Let \( G^2 \subset \mathcal{P}^{\text{anat}} \subset G \) denote the subgroups corresponding to the subalgebras \( g^2 \subset \mathfrak{p}^{\text{anat}} \subset \mathfrak{g} \), respectively. Note that \( \mathfrak{z}(e) \subset \mathfrak{p}^{\text{anat}} \cap \mathfrak{p}^{\text{north}} \). Therefore, the parabolics \( p^{\text{anat}} \), \( p^{\text{north}} \) are completely determined by \( e \) and do not depend on the choice of an associated semisimple pair \( h \), because the latter is determined up to conjugacy by an element of \( Z_G(e) = \exp \mathfrak{z}(e) \subset \mathcal{P}^{\text{anat}} \cap \mathcal{P}^{\text{north}} \).

Recall that an element \( e \) in the Lie algebra \( \mathfrak{p} \) of a parabolic subgroup \( P \subset G \) is called Richardson for \( \mathfrak{p} \) if the orbit \( \text{Ad} P(e) \) is open dense in the nilradical of \( \mathfrak{p} \).

It is clear that the space \( \bigoplus_{p \geq 1} \mathfrak{g}_{p,*} \) is the nilradical of \( \mathfrak{p}^{\text{anat}} \), hence is stable under the \( \text{Ad} \mathcal{P}^{\text{anat}} \)-action. Further, the space \( \mathfrak{g}_{1,*} \) contains the element \( e_1 \) and is stable under the \( \text{Ad} G^2 \)-action.

**Proposition 3.4.** (i) The \( \text{Ad} G^2 \)-orbit of \( e_1 \in \mathfrak{g}_{1,*} \) is Zariski open, dense in \( \mathfrak{g}_{1,*} \);

(ii) The \( \text{Ad} \mathcal{P}^{\text{anat}} \)-orbit of \( e_1 \) is Zariski open, dense in \( \bigoplus_{p \geq 1} \mathfrak{g}_{p,*} \), in particular, 

\( e_1 \) is a Richardson element for \( \mathfrak{p}^{\text{anat}} \). Similar results hold for \( e_2 \).

**Proof.** To prove (i), it suffices to show that the tangent space at \( e_1 \) to the \( \text{Ad} G^2 \)-orbit of \( e_1 \) equals \( \mathfrak{g}_{1,*} \), that is to show that \( [e_1, \mathfrak{g}^2] = \mathfrak{g}_{1,*} \). But this is immediate from the surjectivity claim of the weak Lefschetz. Part (ii) is proved in exactly the same way. \( \square \)

**Lemma 3.5.** (i) \( c(\mathfrak{g}^1) \cap c(\mathfrak{g}^2) = 0 \). (ii) The subspaces \( \mathfrak{g}^1 \) and \( \mathfrak{g}^2 \) generate \( \mathfrak{g} \) as a Lie algebra.

**Proof.** Recall that the centralizer of a principal \( \mathfrak{sl}_2 \)-triple in a semisimple Lie algebra is trivial. Hence, Proposition 3.2(iv) yields \( \mathfrak{z}(e_1, h_1) = c(\mathfrak{g}^1) \), and \( \mathfrak{z}(e_2, h_2) = c(\mathfrak{g}^2) \). Part (i) now follows from Proposition 1.9 (positive quadrant) and the equation

\[
c(\mathfrak{g}^1) \cap c(\mathfrak{g}^2) = \mathfrak{z}(e_1, h_1) \cap \mathfrak{z}(e_2, h_2) = \mathfrak{z}(e_1, e_2) \cap \mathfrak{z}(h_1, h_2) = \mathfrak{z}(e) \cap h = 0.
\]

To prove (ii), let \( \mathfrak{g} \) denote the Lie subalgebra in \( \mathfrak{g} \) generated by \( \mathfrak{g}^1 \) and \( \mathfrak{g}^2 \). The weak Lefschetz surjectivity result implies that the parabolic subalgebra \( \mathfrak{p}^{\text{anat}} \) is generated, as a Lie algebra, by the subalgebra \( \mathfrak{g}^2 \) and the element \( e_1 \). It follows that \( \mathfrak{g} \supset \mathfrak{p}^{\text{anat}} \). Further, if \( \sigma \) is the Cartan involution of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \), then clearly \( \sigma(\mathfrak{g}^1) = \mathfrak{g}^1 \), \( \sigma(\mathfrak{g}^2) = \mathfrak{g}^2 \), and \( \sigma(\mathfrak{p}^{\text{anat}}) \) is the parabolic opposite to \( \mathfrak{p}^{\text{anat}} \). Therefore, the algebra \( \mathfrak{g} = \sigma(\mathfrak{g}) \) contains both \( \mathfrak{p}^{\text{anat}} \) and \( \mathfrak{p}^{\text{anat}} \), and we are done. \( \square \)

**Corollary 3.6.** (i) For any semisimple pair \( (h_1, h_2) \) associated to a principal nilpotent pair \( e \), the integers \( m_1, m_2 \) in Lemma 1.3 can be chosen to be equal to \( 1 \);

(ii) The group \( Z_G(e) \), the simultaneous centralizer of \( e_1 \) and \( e_2 \) in \( G \), is connected.

**Proof.** Write \( T \) for the maximal torus in \( G \) corresponding to \( \mathfrak{h} \), and let \( R \subset \mathfrak{h}^* \) be the root system of \( (\mathfrak{g}, \mathfrak{h}) \). Recall that the lattice \( X_s(T) = \text{Hom}_{alg}(\mathbb{C}^*, T) \) is dual to the root lattice of \( \mathfrak{g} \), since the group \( G \) is of adjoint type. Theorem 1.2(iii) implies that, for any root \( \alpha \in R \), we have \( \alpha(h_i) \in \mathbb{Z} \). Thus, if we view \( X_s(T) \) as a lattice in \( \mathfrak{h} \) via taking the differentials of homomorphisms: \( \mathbb{C}^* \to T \) at the identity, then: \( h_i \in X_s(T), i = 1, 2 \). It follows that there exists \( \gamma : \mathbb{C}^* \times \mathbb{C}^* \to T \) such that \( \frac{\partial \gamma}{\partial t_1 |_{t_1=t_2=1}} = h_i \), for \( i = 1, 2 \). This proves part (i).
To prove (ii) note that the connected component of the group $Z_G(e)$ is a unipotent group, due to Proposition 1.9 ("positive quadrant"). Hence, any maximal reductive subgroup in $Z_G(e)$ is a finite group.

Recall now the short exact sequence (1.4). This sequence is split, by part (i) of the corollary. It follows, that $M$, the image of $M$ under the projection to $G$ considered in the proof of Theorem 1.2(iv), has the form of a semidirect product $M = (\mathbb{C}^* \times \mathbb{C}^*) \ltimes Z_G(e)$. Hence, if $\overline{M}_r \subset M$ is a maximal reductive subgroup containing $C\ltimes C^*$, then $Z_r := Z_G(e) \cap \overline{M}_r$ is a maximal reductive subgroup in $Z_G(e)$. In particular, $Z_r$ is a normal subgroup of $\overline{M}_r$, and there is a semi-direct product decomposition $\overline{M}_r = (\mathbb{C}^* \times \mathbb{C}^*) \ltimes Z_r$. The adjoint $\mathbb{C}^* \times \mathbb{C}^*$-action on $Z_r$ must be trivial, since the group $Z_r$ is finite. Hence, $Z_r$ commutes with the image of the homomorphism $\gamma : \mathbb{C}^* \times \mathbb{C}^* \to T$ constructed in the proof of part (i). It follows that $Z_r \subset T$, since $T = Z_G(h_1, h_2)$.

Let $z \in Z_r$. Since $z \in T$ commutes with $e_1$ and $e_2$, Proposition 3.2(ii) shows that, for any root $\alpha \in \Delta_1 \cup \Delta_2$ (sets of simple roots for $g^i$ and $g^2$), we have $\alpha(z) = 1$. By Lemma 3.5(ii) the set $\Delta_1 \cup \Delta_2$ generates the root lattice of $g$. It follows that $z$ commutes with $g$. Thus, $z$ belongs to the center of $G$, which is trivial because $G$ is of adjoint type. □

**Theorem 3.7.** Principal nilpotent pair $e = (e_1, e_2)$ is uniquely determined, up to conjugacy, by the associated semisimple pair $h = (h_1, h_2)$, that is, any two principal nilpotent pairs $e$ and $\tilde{e}$ that have the same associated semisimple pair $h$ are conjugate to each other by the maximal torus $T = Z_G(h)$.

We will use the following lemma; its proof is postponed until §6 (Corollary 6.10).

**Lemma 3.8.** Set $\mathfrak{z}_{p,q}(e_1) := g_{p,q} \cap \mathfrak{z}_g(e_1)$. Then, $3_{0,1}(e_1) = [e_2, \mathfrak{z}_{0,0}(e_1)]$.

**Proof of Theorem 3.7.** Let $e = (e_1, e_2)$ and $\tilde{e} = (\tilde{e}_1, \tilde{e}_2)$ be two principal nilpotent pairs with the same associated semisimple pair $h = (h_1, h_2)$. Put $\mathfrak{h} = \mathfrak{z}(h)$ and define the Levi subalgebras $g^i$ and $g^2$ as at the beginning of this section. We have $\mathfrak{h} = \mathfrak{c}(g^i) \oplus \mathfrak{h}^i$. Let $T^1 = \exp(h^i) \subset T$ be the torus corresponding to the Lie subalgebra $h^i$. Using Proposition 3.2(iii) we may, conjugating by $T^1$ if necessary, achieve that $e_1 = \tilde{e}_1$. We assume this, from now on.

Let $C = \exp(\mathfrak{c}(g^i)) \subset T$ be the torus corresponding to the center of $g^i$. Conjugating by $C$ acts trivially on $\mathfrak{g}^i$, hence does not affect $h$ and the equality $e_1 = \tilde{e}_1$. Therefore, it suffices to prove that $\tilde{e}_2$ is Ad$C$-conjugate to $e_2$. Note that, by construction, $e_2, \tilde{e}_2 \in 3_{0,1}(e_1)$, and that the space $3_{0,1}(e_1)$ is Ad$C$-stable. Thus, the theorem will follow provided we show that Ad$C$-orbits of $e_2$ and $\tilde{e}_2$ are both Zariski open in $3_{0,1}(e_1)$. To this end, we observe that the Ad$C$-orbit of $e_2$ is Zariski open in $3_{0,1}(e_1)$ if and only if its tangent space, $(\text{ad} \gamma(g^i))e_2$, equals the whole space, i.e.: $[e_2, \mathfrak{c}(g^i)] = 3_{0,1}(e_1)$. Since $\mathfrak{c}(g^i) = 3_{0,0}(e_1)$, the latter equation is insured by Lemma 3.8. Similar argument applies to $\tilde{e}_2$. □

**Theorem 3.9.** The number of Ad$G$-orbits of principal nilpotent pairs in $g$ is finite.

**First proof.** By Theorem 3.7 we only need to show that there are finitely many Ad$G$-conjugacy classes of all possible associated semisimple pairs $h = (h_1, h_2)$. To
this end, we fix a Cartan subalgebra $h \subset g$ and prove that the number of associated semisimple pairs $h = (h_1, h_2)$ such that $h = z(h)$ is finite.

Given $h$, let $g^1, g^2 \supset h$ be two Levi subalgebras that generate $g$ as a Lie algebra and such that $g^1 \cap g^2 = h$. Choose $\Delta^1$ and $\Delta^2$, bases of simple roots for $(g^1, h)$ and $(g^2, h)$, respectively. Note that there are finitely many choices of quadruples $(g^1, g^2, \Delta^1, \Delta^2)$ as above. We now prove that there is at most one associated semisimple pair $h = (h_1, h_2)$ compatible with such a quadruple.

To this end, assume that the quadruple $(g^1, g^2, \Delta^1, \Delta^2)$ and the semisimple pair $h = (h_1, h_2)$ come from a principal nilpotent pair $e$, as at the beginning of this section. Then $g^1 = c(g^1) \oplus s^1$ and $h_1 = c_1 + s_1$, where $s_1 = \frac{1}{2} \sum_{\alpha \in R^1_+} \alpha^\vee$. Further, let $h_2$ be any two parabolics with Levi factors $g^1$ and $g^2$, respectively. Assume that $g^1 \cap g^2 = h$ is a Cartan subalgebra of $g$, and for $i = 1, 2$, choose $\Delta^i$, the set of simple roots for $(g^i, h)$. Further, let $b^1$ be the Borel subalgebra in $g^1$ corresponding to the set $\Delta^1$. Write $\Delta \supset \Delta^1$ for the set of simple roots for $(g, h)$ corresponding to the unique Borel subalgebra $b$ in $g$ such that $b^1 \subset b \subset p''$. Let $h_2$ be the sum of the fundamental coweights that correspond to the simple roots in $\Delta \setminus \Delta^1$. Then $h_2$ takes the value 1 on every simple root for the Levi factor $g^2$ of $p'$. At the same time, $h_2$, defined similarly, takes the value 1 on every simple root for $g^1$.

We now give an alternative more geometric proof of Theorem 3.9, based on the following construction.

Let $p = l \oplus u$ be a parabolic subalgebra in $g$ with Levi factor $l$ and the nilradical $u$. Let $P = L \cdot U \subset G$ denote the corresponding (connected) parabolic subgroup. Fix a principal nilpotent $e_1 \in l$. Write $z_u(e_1)$ and $Z_p(e_1)$ for the centralizers of $e_1$ in $u$ and $P$, respectively.

**Proposition 3.12.** (i) Assume that $e_2 \in z_u(e_1) \subset u$ is a Richardson element for $p$. Set $e = (e_1, e_2)$. Then the following conditions (a) and (b) are equivalent:

(a) The orbit $\text{Ad}Z_p(e_1)e_2$ is Zariski open and dense in $z_u(e_1)$.

(b) $e = (e_1, e_2)$ is a principal nilpotent pair in $g$.

(ii) Any two pairs $(e_1, e_2)$ and $(e_1, e'_2)$ that satisfy the equivalent conditions (a)-(b) above are conjugate by the group $Z_p(e_1)$.

(iii) If $e = (e_1, e_2)$ is a principal nilpotent pair in $g$ and $p^\text{north} = \bigoplus_{q \geq 0} g_{\ast, q}$, the associated parabolic, see (3.3), then part (i) holds for $l := g^1 = g_{\ast, 0}$ and $p := p^\text{north}$.
**Proof.** Fix \( e_1 \) as in (i). Observe that we have \( \mathfrak{z}_p(e_1) = \mathfrak{z}_l(e_1) \oplus \mathfrak{z}_u(e_1) \). Furthermore, \( \dim \mathfrak{z}_l(e_1) = \dim \mathfrak{l} \), since \( e_1 \) is a principal nilpotent in \( \mathfrak{l} \). Hence, \( \dim \mathfrak{z}_p(e_1) = \dim \mathfrak{l} + \dim \mathfrak{z}_u(e_1) \).

If \( e_2 \in \mathfrak{u} \) is Richardson, then one has \( \mathfrak{z}_q(e_2) = \mathfrak{z}_p(e_2) \). Therefore, if \( e_2 \) is a Richardson element in \( \mathfrak{z}_u(e_1) \) then: \( \mathfrak{z}_q(e_1, e_2) = \mathfrak{z}_p(e_1, e_2) \), and the Richardson inequality for the pair \( e = (e_1, e_2) \) yields: \( \dim \mathfrak{z}_p(e) = \dim \mathfrak{z}_q(e) \geq \dim \mathfrak{g} = \dim \mathfrak{k} \). Thus we find

\[
\dim \mathfrak{z}_u(e_1) = \dim \mathfrak{z}_l(e_1) - \dim \mathfrak{l} \geq \dim \mathfrak{z}_l(e_1) - \dim \mathfrak{z}_l(e_1) = \dim \mathfrak{z}_l(e_1)
\]

We see that the equality in (3.13) holds if and only if \( \dim \mathfrak{z}_q(e) = \dim \mathfrak{g} \). Thus, the \( \text{Ad} \mathfrak{z}_p(e_1) \)-orbit of \( e_2 \) is Zariski open in \( \mathfrak{z}_u(e_1) \) if and only if the pair \( e = (e_1, e_2) \) is regular. Since \( \mathfrak{z}_u(e_1) \) is a vector space, hence an irreducible variety, it may contain at most one Zariski open orbit, which is then dense in \( \mathfrak{z}_u(e_1) \). This proves that \( i(b) \Rightarrow i(a) \), and also part (iii) of the Proposition.

To prove \( i(a) \Rightarrow i(b) \) assume that the \( \text{Ad} \mathfrak{z}_p(e_1) \)-orbit of \( e_2 \) is Zariski open dense in \( \mathfrak{z}_u(e_1) \). By (3.13) and the discussion following it, we must only show that the pair \( e = (e_1, e_2) \) satisfies the Nil-condition 1.2(ii). To that end, consider the natural \( \mathbb{C}^* \)-action on \( \mathfrak{g} \) by homotheties, and the induced \( \mathbb{C}^* \)-action on \( \mathfrak{z}_u(e_1) \). The \( \mathbb{C}^* \)-action clearly commutes with the \( \text{Ad} \mathfrak{z}_p(e_1) \)-action, hence, preserves the open \( \text{Ad} \mathfrak{z}_p(e_1) \)-orbit in \( \mathfrak{z}_u(e_1) \). It follows that, for any \( t_2 \in \mathbb{C}^* \), the pairs \( (e_1, t_2 \cdot e_2) \) and \( (e_1, 2 \cdot e_2) \) are \( \text{Ad} \mathfrak{z}_p(e_1) \)-conjugate. Now, the orbit \( \text{Ad} \mathfrak{l}(e_1) \) is nilpotent, hence, is a \( \mathbb{C}^* \)-stable subvariety in \( \mathfrak{l} \). We deduce that the \( \text{AdP} \)-diagonal orbit of \( e = (e_1, e_2) \) is a \( \mathbb{C}^* \times \mathbb{C}^* \)-stable subvariety in \( \mathfrak{p} \oplus \mathfrak{u} \). This implies condition 1.2(ii), and part (ii) follows.

To prove part (ii) assume \( e'_2 \) is another element satisfying conditions of part (i). Then, by \( i(a) \), the \( \text{Ad} \mathfrak{z}_p(e_1) \)-orbits of \( e_2 \) and \( e'_2 \) are both Zariski open dense in \( \mathfrak{z}_u(e_1) \). Hence, these orbits coincide. Therefore, \( e'_2 \in \text{Ad} \mathfrak{z}_p(e_1) \cdot e_2 \), and part (ii) is proved. \( \square \)

Let \( e_1 \) be a principal nilpotent in a Levi subalgebra \( \mathfrak{l} \), and let \( \mathfrak{p}^{(i)} \), \( i = 1, 2 \), be two parabolics with nilradicals \( u^{(i)} \) and the same Levi factor \( \mathfrak{l} \). Assume that \( e_2^{(i)} \in \mathfrak{z}_{u^{(i)}}(e_1) \), \( i = 1, 2 \), is a Richardson element for \( \mathfrak{p}^{(i)} \) satisfying conditions (a)-(b) of Proposition 3.12.

**Question 3.14.** Is it true that the principal nilpotent pairs \((e_1, e_2^{(1)})\) and \((e_1, e_2^{(2)})\) are always AdG-conjugate?

**Second proof of Theorem 3.9.** Part (iii) of Proposition 3.12 says that every principal nilpotent pair in \( \mathfrak{g} \) arises from a certain parabolic subalgebra \( \mathfrak{p} \subset \mathfrak{g} \) via the construction of Proposition 3.12(i). Note that the AdG-conjugacy class of the pair \( e \) arising from the construction does not depend on the choice of a Levi factor \( \mathfrak{l} \subset \mathfrak{p} \), since all such factors are AdP-conjugate. Moreover, part (ii) of Proposition 3.12 insures that any two principal nilpotent pairs arising from the same parabolic \( \mathfrak{p} \) are AdG-conjugate. Hence, the result follows from the finiteness of the number of AdG-conjugacy classes of parabolic subalgebras in \( \mathfrak{g} \). \( \square \)

In the next section (see Corollary 4.13) we will prove the following
Theorem 3.15. The conjugacy class \( \text{Ad}G(e_2) \subset \mathfrak{g} \) of the second member of a principal nilpotent pair \((e_1, e_2)\) is totally determined by the conjugacy class of \( e_1 \).

4. Harmonic polynomial attached to a principal nilpotent pair.

The main result of this section, Theorem 4.4 below, holds for all principal nilpotent pairs with one exception. The exception occurs for a particular principal nilpotent pair \( e_{\text{except}} = (e_1, e_2) \) in the simple Lie algebra \( \mathfrak{g} \) of type \( E_7 \). The elements \( e_1 \) and \( e_2 \) in the pair both belong to the same nilpotent orbit \( O_{\text{except}} \subset \mathfrak{g} \), the regular nilpotent conjugacy class in the Levi subalgebra of type \( A_4 + A_1 \) (according to the classification of §38, there is only one, up to conjugacy, principal nilpotent pair in \( \mathfrak{g} = E_7 \) with this property). The nilpotent orbit \( O_{\text{except}} \) is known to have an "exceptional" behavior in other respects as well, see Remark 4.12 below.

Definition 4.1. A commuting nilpotent pair \( e = (e_1, e_2) \in \mathcal{Z} \), resp. a nilpotent orbit \( O \), in a semisimple Lie algebra \( \mathfrak{g} \) is said to be non-exceptional if none of the components of \( e \), resp. \( O \), corresponding to the simple factors of \( \mathfrak{g} \) of type \( E_7 \) are conjugate to the pair \( e_{\text{except}} \), resp. are equal to \( O_{\text{except}} \).

Given a vector space \( V \), we write \( SV = \bigoplus_i S^i V \) for the Symmetric algebra of \( V \), identified with \( \mathbb{C}[V^*] \), the polynomial algebra on the dual space. Abusing the language we will often refer to elements of \( SV \) as "polynomials". We also have the completed algebra \( \hat{SV} = \prod_{i \geq 0} S^i V \simeq \mathbb{C}[V^*] \) of formal power series.

We keep the notations of the previous sections, in particular, we have fixed a principal nilpotent pair \( e \), and an associated semisimple pair \( h = (h_1, h_2) \). Let \( \mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \) be the corresponding Cartan subalgebra, and \( W \) the Weyl group. We write \( 1 \) for the trivial 1-dimensional representation of \( W \), and \( \varepsilon \) for the sign-representation. For any simple \( W \)-module \( E \), the space \( E \otimes (E^* \otimes \varepsilon) \) contains a unique 1-dimensional subspace, \( \varepsilon \cdot \text{Id}_E \subset \varepsilon \otimes \text{Hom}(E, E) = E \otimes (E^* \otimes \varepsilon) \), that transforms as the sign-representation under the diagonal \( W \)-action. Recall further that any simple \( W \)-module is defined over \( \mathbb{Q} \), hence is self-dual, \( E^* \simeq E \). Thus, there is also a distinguished line in \( E \otimes (E \otimes \varepsilon) \) that will be referred to as the sign-subspace.

To each element \( x = (x_1, x_2) \in \mathfrak{h} \oplus \mathfrak{h} \) we associate the following "doubled" analogue of a familiar Weyl character type alternating expression:

\[
\Delta_x := \sum_{w \in W} \varepsilon(w) \cdot e^{w(x)} = \sum_{w \in W} \varepsilon(w) \cdot w(e^{x_1} \otimes e^{x_2})
\]

\[
= \sum_{i, j \geq 0} \frac{1}{i! \cdot j!} \left( \sum_{w \in W} \varepsilon(w) \cdot w(x_1^i \otimes x_2^j) \right) \in \prod_{i, j \geq 0} S^{i+j}(\mathfrak{h} \oplus \mathfrak{h}),
\]  

where the group \( W \) acts diagonally on \( S_{\mathfrak{h}} \otimes S_{\mathfrak{h}} = S(\mathfrak{h} \oplus \mathfrak{h}) \). Note that the element \( \Delta_x \) is non-zero whenever \( x \) is regular, i.e., when all the points \( w(x) \), \( w \in W \) are distinct. For each non-negative integers \( (d_1, d_2) \), we consider the polynomial: \( \Delta_x(d_1, d_2) := (d_1! \cdot d_2!)^{-1} \cdot \sum_{w \in W} \varepsilon(w) \cdot w(x_1^{d_1} \otimes x_2^{d_2}) \), the \((d_1, d_2)\)-bihomogeneous component of the Taylor expansion of (4.2).
Recall the Levi subalgebras $g^i = z_p(h_2)$ and $g^2 = z_p(h_1)$. Write $c(g^i) \subset h$, $(i = 1, 2)$, for the center of $g^i$, and set $c(g^i)^{\text{reg}} := \{ h \in c(g^i) \mid z_p(h) = g^i \}$. We define:

$$c(g^1)^0 := \{ h \in h \mid \alpha(h) = 0, \forall \alpha \in R^1, \beta(h) \neq 0, \forall \beta \in R^2 \}$$

$$c(g^2)^0 := \{ h \in h \mid \beta(h) = 0, \forall \beta \in R^2, \alpha(h) \neq 0, \forall \alpha \in R^1 \}.$$

We have: $c(g^1)^{\text{reg}} \subset c(g^1)^0 \subset c(g^1)$. Put $c^o := \{ x = (x_1, x_2) \mid x_1 \in c(g^1)^0, x_2 \in c(g^1)^o \}$, (note the flip involved in the definition). It is clear from the definitions that: $h_1 \in c(g^1)^{\text{reg}}$, and $h_2 \in c(g^1)^{\text{reg}}$, hence $h \in c^o$.

We introduce two integers: $d_i := \# R^i_+ = \dim g^i_+$, $(i = 1, 2)$, see 2.2(i). By Corollary 6.10 (of section 6 below) one can rewrite these numbers in terms of bi-exponents:

$$d_1 = \sum_{i,j} i \cdot \dim j_{i,j}(e) = \sum_{(p,q) \in \text{Exp}_e(g)} p, \quad d_2 = \sum_{i,j} j \cdot \dim j_{i,j}(e) = \sum_{(p,q) \in \text{Exp}_e(g)} q.$$

The following result will be proved shortly.

**Lemma 4.3.** For any $x = (x_1, x_2) \in c^o$, we have:

(i) $\Delta_x(d_1, d_2) = 0$ whenever $d_1 < d_1$ or $d_2 < d_2$.

(ii) The polynomial $\Delta_x(d_1, d_2)$ is proportional to $\Delta_h(d_1, d_2)$, hence is independent, up to a constant factor, of the choice of $x \in c^o$.

We set

$$\Delta_e := \frac{1}{d_1! \cdot d_2!} \cdot \sum_{w \in W} \varepsilon(w) \cdot w(h_1^{d_1} \otimes h_2^{d_2}) \in S^{d_1} h \otimes S^{d_2} h.$$

According to the lemma, this polynomial of bi-degree $(d_1, d_2)$ is (if non-zero) the first non-vanishing term in the Taylor expansion of $\Delta_h$ (the the notation $\Delta_e$ is legitimate since $e$ is not in $h \oplus \hat{h}$, so that definition (4.2) doesn’t apply).

We recall a few standard results and notation concerning Weyl groups. First, associated to the Levi subalgebras $g^1$ and $g^2$, respectively, there are polynomials:

$$\pi_1 := \prod_{\alpha \in R^1_+} \alpha^\vee \in S^{d_1} h, \quad \text{and} \quad \pi_2 := \prod_{\alpha \in R^2_+} \alpha^\vee \in S^{d_2} h.$$

Let $\mathbb{C}[W]$ denote the group algebra of $W$. Write $E_i := \mathbb{C}[W] \cdot \pi_i \subset S^{d_i} h$, for the $W$-submodule generated by $\pi_i \{i = 1, 2\}$. It was shown in [M1] that $E_i$ is a simple $W$-module that occurs in $S^{d_i} h$ with multiplicity 1, and does not occur in $S^d h$, for any $d < d_i$. Furthermore, $\pi_i$ is a $W$-harmonic polynomial, cf. e.g., [CG, §6.3].

One of the main results of this paper is the following

**Theorem 4.4.** If $e$ is a non-exceptional (Def. 4.1) principal nilpotent pair, then

(i) We have: $E_2 \simeq E_1 \otimes e$.

(ii) $\Delta_e$ is a non-zero $W$-harmonic polynomial with respect to the diagonal $W$-action on $h \oplus \hat{h}$;

(iii) The $W \times W$-submodule in $S^{d_1} h \otimes S^{d_2} h$ generated by $\Delta_e$ equals $E_1 \otimes E_2$. 

and $\Delta_\alpha$ is a generator of the one-dimensional sign-subspace in $E_1 \otimes E_2$.

Proof of Lemma 4.3. Recall the decomposition $\mathfrak{g}^i = \mathfrak{c}(\mathfrak{g}^i) \oplus \mathfrak{s}^i$. Fix $s \in \mathfrak{h}^i = \mathfrak{h} \cap \mathfrak{s}^i$ such that $\alpha(s) \neq 0$, for all $\alpha \in R^i$. It is well-known that there is a non-zero constant $\text{const}_s$ such that the following identity holds:

$$
\sum_{w \in W^i} \varepsilon(w) \cdot w(s^d) = \begin{cases} 
\kappa_s \cdot \prod_{\alpha \in R^i_+} \alpha^\vee = \kappa_s \cdot \pi_1 & \text{if } d = d_1 \\
0 & \text{if } d < d_1 
\end{cases} 
$$

(4.5)

Moreover, $\kappa_s = d_1!$, if $s$ equals the half-sum of positive coroots of $\mathfrak{s}^i$. We claim first that the expression on the left of (4.5) remains unaffected if $s$ is replaced there by $x \in s + \mathfrak{c}(\mathfrak{g}^i)$. Indeed, for any $x = s + c$ where $c \in \mathfrak{c}(\mathfrak{g}^i)$, we find:

$$
\sum_{w \in W^i} \varepsilon(w) \cdot w(x)^d = \sum_{w \in W^i} \varepsilon(w) \cdot w(s + c)^d = \sum_{w \in W^i} \varepsilon(w) \cdot w(s)^d + \ldots 
$$

where dot-terms belong to the components $S^k(\mathfrak{c}(\mathfrak{g}^i)) \otimes S^{d-k}(\mathfrak{s}^i)$ with $k > 0$. But the sign-representation of $W^i$ does not occur in $S^d(\mathfrak{s}^i)$ for any $d < d_1$. Hence, all the dot-terms vanish, and the claim follows. Observe, that we have proved in particular that, for any $s \in \mathfrak{c}(\mathfrak{g}^i)^0 \subset \mathfrak{h}$, the identity (4.5) holds and, moreover, $\kappa_s \neq 0$.

Now fix $x = (x_1, x_2) \in \mathfrak{c}^i$ and some $d_1, d_2 \geq 0$, where $d_1 \leq d_1$. We have:

$$
x_2 \in \mathfrak{c}(\mathfrak{g}^i), \text{ hence the element } x_2^{d_2} \in S^{d_2} \mathfrak{h} \text{ is fixed by the } W^1\text{-action. Therefore, using (4.5) and the claim proved in the preceding paragraph for } s = x_1 \in \mathfrak{c}(\mathfrak{g}^i)^0 \text{ we get}
$$

$$
\sum_{w \in W^1} \varepsilon(w) \cdot w(x_1^{d_1} \otimes x_2^{d_2}) = \begin{cases} 
\kappa_{x_1} \cdot \pi_1 \otimes x_2^{d_2} \in S^{d_1}(\mathfrak{s}^i) \otimes S^{d_2} \mathfrak{h} & \text{if } d_1 = d_1 \\
0 & \text{if } d_1 < d_1 
\end{cases} 
$$

(4.6)

where $\kappa_{x_1} \neq 0$. The element $\Delta_x(d_1, d_2)$ is clearly obtained by alternating the expression on the LHS of (4.6) with respect to the diagonal $W$-action on $S^{d_1} \mathfrak{h} \otimes S^{d_2} \mathfrak{h}$. By (4.6), this expression vanishes whenever $d_1 < d_1$, and the lemma follows. □

Lemma 4.7. If the polynomial $\Delta_\alpha$ is non-zero, then all other claims of Theorem 4.4 hold.

Proof. If $\Delta_\alpha$ is nonzero, it is really the first non-vanishing term of $\Delta_{\mathfrak{h}}$, by Lemma 4.3. But the first non-vanishing term of any linear combination of exponents like in (4.2), i.e., a combination of $W$-conjugate exponents, is known to be $W$-harmonic, see e.g. [CG, Prop.6.4.4], hence part (ii) of the theorem.

To prove part (iii), we observe, as in the proof of Lemma 4.3, that $\Delta_\alpha$ is obtained by alternating the expression on the LHS of (4.6) with respect to the diagonal $W$-action. The RHS of (4.6) shows that the result of such an alternating procedure clearly belongs (for $d_i = d_i, i = 1, 2$) to the subspace $E_1 \otimes S^{d_2} \mathfrak{h} \subset S^{d_1} \mathfrak{h} \otimes S^{d_2} \mathfrak{h}$. Similar arguments, with the roles of $h_1$ and $h_2$ reversed, imply that $\Delta_\alpha \in S^{d_1} \mathfrak{h} \otimes E_2$. Now, part (iii) follows from the equality:

$$
\left( E_1 \otimes S^{d_2} \mathfrak{h} \right) \cap \left( S^{d_1} \mathfrak{h} \otimes E_2 \right) = E_1 \otimes E_2 
$$

Finally, using selfduality of all $W$-modules we get:

$$
\text{Hom}_W(E_2, E_1 \otimes \mathfrak{c}) = \left( E_2 \otimes (E_1 \otimes \mathfrak{c}) \right)^W = \text{Hom}_W(\mathfrak{c}, E_2 \otimes E_1),
$$

and $\Delta_\alpha$ is a non-zero element in the RHS. Hence, the LHS is non-zero, and part (i) of the theorem follows from Schur lemma. □

Thus, most of the remaining part of this section is devoted to the proof of
Proposition 4.8. If $e$ is non-exceptional (Definition 4.1) then, for any $x \in c^0$, we have: $\Delta_x(d_1, d_2) \neq 0$, in particular, $\Delta_x \neq 0$. On the contrary, $\Delta_{\text{except}} = 0$.

Let $\mathfrak{m} \mathfrak{B}$ denote the Flag variety of all Borel subalgebras in a Lie algebra $\mathfrak{m}$. We begin the proof by reinterpreting the representations $E_i$ geometrically, by means of Springer theory, cf. e.g., [CG, ch.3].

Given a nilpotent $x \in \mathfrak{g}$, set $C(x) := Z_G(x)/Z_G^0(x)$, and let $\mathfrak{B}_x \subset \mathfrak{B}$ denote the subvariety of the Borel subalgebras in $\mathfrak{g}$ containing $x$. By Springer theory, there is a natural $W$-action on each homology group $H_i(\mathfrak{B}_x) = H_i(\mathfrak{B}_x, \mathbb{C})$. Moreover, the $W$-action commutes with the natural $C(x)$-action, and one has a $W$-module isomorphism:

$$E_i = H_{\text{top}}(\mathfrak{B}_{e_i})^{C(e_i)}, \quad \text{where } "\text{top}" := \dim_{\mathbb{C}} B_{e_i} = 2d_i, \quad i = 1, 2.$$  \hspace{1cm} (4.9)

In general, given any nilpotent $x \in \mathfrak{g}$, we will write $H(\mathfrak{B}_x)$ for the representation of $W$ in the subspace of $C(x)$-invariants of the top homology of $\mathfrak{B}_x$.

Next, recall the concept of induction of nilpotent orbits. Given a Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$, we write $\text{Ind}_{\mathfrak{g}_i}^{\mathfrak{g}_j}(0)$ for the nilpotent orbit in $\mathfrak{g}$ that intersects the nilradical of some (hence, every) parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ by an open dense subset. We refer to [LS] for results, and [Spa] and [Ke] (resp. [El2]) for more concrete information concerning induced orbits in classical (resp. exceptional) Lie algebras. The results of [El2] were announced without proofs in [Spa].

It will be convenient for us to introduce the following

Definition 4.10. Two nilpotent orbits $O_1$ and $O_2$ in $\mathfrak{g}$ are said to be reciprocal, if there exist Levi subalgebras $\mathfrak{g}_i$ and $\mathfrak{g}_j$ such that the following conditions hold:

(i) $O_1 = \text{Ind}_{\mathfrak{g}_j}^{\mathfrak{g}_i}(0)$ and $O_2 = \text{Ind}_{\mathfrak{g}_i}^{\mathfrak{g}_j}(0)$.

(ii) $\mathfrak{g}_i \cap O_i$ is the regular nilpotent orbit in $\mathfrak{g}_i$, for $i = 1, 2$.

This notion is relevant because of Proposition 3.4 saying that, for a principal nilpotent pair $e = (e_1, e_2)$, the corresponding nilpotent orbits $O_1 = \text{Ad} G(e_1)$ and $O_2 = \text{Ad} G(e_2)$ are reciprocal orbits in $\mathfrak{g}$. We note that, for $\mathfrak{g}$ simple of type $E_7$, the exceptional orbit, $O_{\text{except}}$, is reciprocal to itself, see beginning of this section.

A crucial ingredient in our proof of the non-vanishing of $\Delta_x$ is the following

Proposition 4.11. If $O_1$ and $O_2$ are reciprocal non-exceptional (Definition 4.1) nilpotent orbits in $\mathfrak{g}$, then for any $x_1 \in O_1, x_2 \in O_2$, there is a $W$-module isomorphism: $H(\mathfrak{B}_{x_2}) \cong \varepsilon \otimes H(\mathfrak{B}_{x_1})$.

This result is false if $O_1 = O_2 = O_{\text{except}}$.

Remark 4.12. Lusztig has introduced (see [L2] or [Ca]) the notion of a special nilpotent orbit in $\mathfrak{g}$. He introduced also the notion of a special irreducible representation of the Weyl group [L3], and proved that the nilpotent orbit $\text{Ad} G(x)$ is special if and only if the Springer representation $H(\mathfrak{B}_x)$ is special. The orbit $O_{\text{except}}$ is special. The corresponding representation $H(\mathfrak{B}_x), x \in O_{\text{except}}$, is among very few of the special representations $E$, such that the representation $\varepsilon \otimes E$ is not special, see [L3]. Furthermore, Spaltenstein has defined (on a case-by-case basis) an involution $\sigma$ on the set of all special nilpotent orbits in $\mathfrak{g}$, see [Spa]. It is known, see e.g., [Ca, pp.373-374, 389] that in all but two exceptional cases in types $E_7, E_8$ the following holds. If $O_1$ and $O_2$ are special nilpotent orbits such that $O_2 = \sigma(O_1)$, then $H(\mathfrak{B}_{x_2}) \cong \varepsilon \otimes H(\mathfrak{B}_{x_1})$, for any $x_i \in O_i, i = 1, 2$. This property
does not hold for $O_1 = O_{\text{except}}$, in which case we have: $\sigma(O_{\text{except}}) = O_{\text{except}}$, but $H(B_{l_1}) \neq \mathfrak{e} \otimes H(B_{x_1})$, see [Ca].

**Sketch of first proof of Proposition 4.11.** For each simple Lie algebra $g$, there are available tables of all induced orbits of the form $\text{Ind}_{g}^g(0)$, see [El1] for exceptional Lie algebras and [Spa] (or [CM]) for classical Lie algebras. From this, it is a straightforward matter to derive a list all reciprocal pairs. We do not know at the moment which of the reciprocal pairs arise from principal nilpotent pairs. However, using explicit tables for the Springer correspondence, see [AL], [Sh] and references in [L2], one verifies case-by-case that the result of the proposition holds for every reciprocal pair in any simple Lie algebra, but the exceptional pair for $g$ of type $E_7$ (the special nilpotent orbit in $g$ of type $E_8$ with an "exceptional" behavior in the sense of Remark 4.12 does not give rise to any reciprocal pair). □

Recall that every induced orbit of the form $\text{Ind}_{g}^g(0)$ is known to be special. In particular, each member of any reciprocal pair $(O_1, O_2)$ is a special orbit. We have

**Lemma 4.13.** If $(O_1, O_2)$ is a reciprocal pair, then $O_2 = \sigma(O_1)$, where $\sigma$ is the Spaltenstein involution on the set of special orbits, see Remark 4.12.

**Proof.** Let $I \subset g$ be a proper Levi subalgebra. It was shown by Barbasch-Vogan [BV], see also [CM, Theorem 8.3.1], that if $O \subset g$ is a special nilpotent orbit such that $O := O \cap I$ is a special orbit in $I$, then $\sigma(O) = \text{Ind}_{I}^g(\sigma(O))$. It follows from Definition 4.10(ii) that, for a reciprocal pair $(O_1, O_2)$, the set $O_{g} := O_2 \cap g^1$ is a principal nilpotent orbit in $g^1$. Hence, $\sigma(O_{g}) = \{0\}$ is the zero-orbit in $g^1$. Therefore, by the above we get $\sigma(O_1) = \text{Ind}_{g}^g(0) = O_2$. □

**Second proof of Proposition 4.11.** If $(O_1, O_2)$ is a reciprocal pair, then $O_2 = \sigma(O_1)$ by the Lemma. The claim now follows from the result mentioned at the end of Remark 4.12, saying that $O_2 = \sigma(O_1)$ implies $H(B_{x_2}) \simeq \mathfrak{e} \otimes H(B_{x_1})$, except for the two cases in types $E_7, E_8$, which are handled separately. □

**Proof of Proposition 4.8.** By Proposition 4.11, we know that the vector space $E_1 \otimes E_2 \subset S^{d_1} \mathfrak{h} \otimes S^{d_2} \mathfrak{h}$ contains a distinguished 1-dimensional sign-subspace. Let $\Delta = \sum P_k \otimes P_k''$, $P_k \in S^{d_1} \mathfrak{h}$, $P_k'' \in S^{d_2} \mathfrak{h}$, be a non-zero element of this sign-subspace. We will show that, up to a non-zero constant factor, we must have: $\Delta = \Delta_\pi(d_1, d_2)$, hence $\Delta_\pi(d_1, d_2) \neq 0$.

To this end, observe first that $\Delta$ is clearly obtained by alternating the expression $\sum P_k \otimes P_k''$ with respect to the $W$-diagonal action. Recall further, that the module $E_2$ is $\mathbb{C}[W]$-generated by $\pi_2$. Hence, $P_k'' = u_k(\pi_2)$, $u_k \in \mathbb{C}[W]$, and it is easy to see that alternating $\sum P_k \otimes P_k''$ gives the same result as alternating an expression of the form $Q \otimes \pi_2$, where $Q \in S^{d_1} \mathfrak{h}$. Using the identity $\pi_2 = \kappa_\pi \cdot \sum_{w_1 \in W_1} \varepsilon(w_1) \cdot w(x_1^{d_1})$, see (4.5), we can further rewrite the latter alternating expression as follows:

$$\sum_{w \in W} \varepsilon(w) \cdot w(Q \otimes \pi_2) = \kappa_\pi \cdot \sum_{w \in W} \varepsilon(w) \cdot w(R \otimes x_1^{d_2})$$

$$= \kappa_\pi \cdot \frac{1}{|W|} \sum_{w \in W} \varepsilon(w) \cdot w\left(\sum_{w_1 \in W_1} \varepsilon(w_1) \cdot w_1(R) \otimes x_2^{d_2}\right).$$

But since $\pi_1$ is the only $W^1$-skew-invariant in $S^{d_1} \mathfrak{h}$, for any $R \in S^{d_1} \mathfrak{h}$, the polynomial $\sum_{w_1 \in W_1} \varepsilon(w_1) \cdot w_1(R)$ must either vanish, or else be proportional
to $\pi_1$. It cannot vanish in our case, since $\Delta \neq 0$. We conclude that $\Delta = \sum_{w \in W} \epsilon(w) \cdot w(R \otimes x_2^{d_2})$ equals (up to a non-zero constant factor, cf. (4.6)) to $\sum_{w \in W} \epsilon(w) \cdot w(\pi_1 \otimes x_2^{d_2}) = \Delta_x(d_1, d_2)$. □

It is instructive to give a conceptual proof of Proposition 4.11, at least in a special case. To this end, fix two Levi subalgebras $g^1, g^2$ in $g$ with (abstract) Weyl groups $W^1, W^2 \subset W$, respectively. Let $(O_1, O_2)$ be a reciprocal pair of nilpotent orbits in $g$, as in Definition 4.10. Choose $e_i \in O_i \cap g^i$, a principal nilpotent in $g^i$, $i = 1, 2$. By definition, there are parabolic subgroups $p^{\text{north}}, p^{\text{east}} \subset g$ with Levi factors $g^1$ and $g^2$, respectively, and such that the element $e_2$ is Richardson for $p^{\text{north}}$, while the element $e_1$ is Richardson for $p^{\text{east}}$. In general, given a parabolic $p$ and any nilpotent $x \in g$, let $P_x$ denote the variety of all parabolics of type $p$ that contain $x$ in their nilradical. In particular, we consider the sets $P^{\text{north}}_{e_2}$ and $P^{\text{east}}_{e_1}$. These sets are finite, since the elements in the corresponding subscript are Richardson.

**Proposition 4.14.** (i) There are natural $W$-module isomorphisms:

$$\text{Ind}_{w_1}^W 1 \simeq H_s(\mathfrak{g}B_{e_1}), \quad i = 1, 2.$$  

(ii) There are natural vector space isomorphisms:

$$H_0(P^{\text{north}}_{e_2}) \simeq \text{Hom}_W(\text{Ind}_{w_1}^W e, \text{Ind}_{w_2}^W 1) \simeq H^0(P^{\text{east}}_{e_1}).$$

(iii) The following conditions are equivalent:

$$Z_G(e_2) \subset P^{\text{north}} \iff Z_G(e_1) \subset P^{\text{east}} \iff \sharp P^{\text{north}}_{e_2} = 1 \iff \sharp P^{\text{east}}_{e_1} = 1.$$  

Proof. By Springer theory, for any nilpotent $x \in g^2$, there is a $W_x$-action on $H_s(\mathfrak{g}B_x)$, and a result of Borho-MacPherson [BM, 3.4] says that there is a natural $W$-module isomorphism:

$$\text{Ind}_{w}^W H_s(\mathfrak{g}B_x) \simeq H_s(\mathfrak{g}B_{e_2}).$$

In the special case of the principal nilpotent $x = e_2$ in $g^2$, the set $\mathfrak{g}B_{e_2}$ consists of a single point, hence, $H_s(\mathfrak{g}B_{e_2}) = 1$. Applying the Borho-MacPherson isomorphism in this case yields the isomorphism of part (i) of the Proposition.

Next, given a $W$-module $M$, write $M^{(W)} : = \{ m \in M \mid w_1 \cdot m = \epsilon(w_1) \cdot m, \forall w_1 \in W^1 \}$, the $\mathfrak{e}|_{W^1}$-isotypic component of $M|_{W^1}$. By Frobenius reciprocity one has: $M^{(W)} \simeq \text{Hom}_W(\text{Ind}_{w_1}^W e, M)$. Borho-MacPherson have proved that, for any $i \geq 0$ and any nilpotent $x \in g$, the $\mathfrak{e}^{(W)}$-isotypic component of $M = H_s(\mathfrak{g}B_x)$ is given by the formula: $H_s(\mathfrak{g}B_x)^{(W)} \simeq H_{i-2d_1}(P^{\text{north}}_{x})$, $d_1 = \dim \mathfrak{g}B_x$, see [BM, 3.4]. From this formula and part (i) we find:

$$\text{Hom}_W(\text{Ind}_{w_1}^W e, \text{Ind}_{w_2}^W 1) \simeq (\text{Ind}_{w_2}^W 1)^{(W)} \simeq (H_s(\mathfrak{g}B_{e_2}))^{(W)} \simeq H_s(P^{\text{north}}_{e_2}).$$

Since the set $P^{\text{north}}_{e_2}$ is finite, its homology reduces to $H_0(P^{\text{north}}_{e_2})$, and the first isomorphism of part (ii) follows.

Further, using that $\text{Ind}_{w_1}^W e \simeq e \otimes (\text{Ind}_{w_1}^W 1)$ we obtain a chain of canonical isomorphisms:

$$\text{Hom}_W(\text{Ind}_{w_1}^W e, \text{Ind}_{w_2}^W 1) \simeq \text{Hom}_W(e \otimes (\text{Ind}_{w_1}^W e), e \otimes (\text{Ind}_{w_2}^W 1))$$

$$\simeq \text{Hom}_W(\text{Ind}_{w_1}^W 1, \text{Ind}_{w_2}^W e) \quad (4.15)$$

$$\simeq \left( \text{Hom}_W(\text{Ind}_{w_2}^W e, \text{Ind}_{w_1}^W 1) \right)^*.$$
The Hom-space in the last line is isomorphic, by an analogue for $e_1$ of the first isomorphism in (ii), to $H_0(T_{e_1}^{\text{gext}})$. Therefore we get $\left(\text{Hom}_W(\text{Ind}_{W_2}^W e, \text{Ind}_{W_1}^W 1)\right)^* \simeq (H_0(T_{e_1}^{\text{gext}}))^* \simeq H^0(P_{e_1}^{\text{north}})$, and part (ii) follows.

To prove (iii) we recall that if $x$ is Richardson for a parabolic $p$, then the group $Z_G(x)$ acts transitively on the set of parabolics of type $p$ that contain $x$ in their nilradical, see [Ca]. This fact, combined with the isomorphism $H_0(P_{e_2}^{\text{north}}) \simeq H^0(P_{e_1}^{\text{north}})$ of part (ii), yields the equivalences in (iii).

The following result gives a conceptual proof of Proposition 4.11 in a special case.

**Corollary 4.16.** Assume the equivalent conditions of Proposition 4.14(iii) hold, e.g., $g = sl_2$. Then we have: $H(B_{e_2}) \simeq H(B_{e_1}) \otimes e$. This $W$-module is the only common irreducible constituent of $\text{Ind}_{W_2}^W 1$ and $\text{Ind}_{W_1}^W e$; it occurs with multiplicity one in either of these modules.

**Proof.** By Proposition 4.14(ii) we have: $\dim \text{Hom}_W(\text{Ind}_{W_1}^W e, \text{Ind}_{W_2}^W 1) = \sharp P_{e_2} = 1$. The above formula implies that $\text{Ind}_{W_2}^W 1$ and $\text{Ind}_{W_1}^W e$ have only one irreducible constituent in common. Moreover, since $d_1 = \dim_{\mathbb{C}}(\theta^* B) = \dim_{\mathbb{C}} B_{e_2}$, the Borho-MacPherson formula yields: $H(B_{e_2})^e(W) = H_{2d_1}(B_{e_2})^e(W) \simeq H_0(T_{e_2}^{\text{north}}) \neq 0$. Thus, $H(B_{e_2})$ is this common irreducible constituent.

To complete the proof we note that (4.15) implies that $\text{Hom}_W(\text{Ind}_{W_2}^W e, \text{Ind}_{W_1}^W 1)$ is also 1-dimensional. Hence, the two induced representations involved in the Hom also have only one irreducible constituent in common. By the symmetry, this irreducible constituent is $H(B_{e_1})$. Thus, we must have $H(B_{e_1}) \otimes e \simeq H(B_{e_2})$, and the proof is complete.

In the special case considered above one has the following strengthening of Theorem 4.4 with the element $\Delta_e$ being replaced by $\Delta_h$. Moreover, during the proof below, we will effectively identify the element $\Delta_h$ with an intertwiner analogous to the classical "Young symmetriser" in the case of Symmetric groups.

**Proposition 4.17.** Let $e = (e_1, e_2)$ be a principal nilpotent pair such that the equivalent conditions of Proposition 4.14(iii) hold. Then, the $W \times W$-module generated by $\Delta_h$ is isomorphic to $E_1 \otimes E_2$.

**Proof.** Choose an element $\theta \in (\mathfrak{h}^*)^{W_1}$ generic enough so that the linear functions $\{w(\theta), w \in W/W_1\}$ separate points of the orbit $W \cdot h_1 \subset \mathfrak{h}$. We fix such a $\theta$, and set $\mathcal{F}_1 = \{f \in \mathbb{C}[\mathfrak{h}] \mid f = \sum_{w \in W} a_w \cdot e^{w(\theta)} \mid a_w \in \mathbb{C}\}$. The space $\mathcal{F}_1$ has an obvious $W$-module structure and, since $\theta$ is fixed by $W_1 \subset W$, there is a natural $W$-module isomorphism: $\mathcal{F}_1 \simeq \text{Ind}_{W_1}^W 1$, that sends $e^\theta$ to 1. We will view elements of $\mathcal{F}_1$ as holomorphic functions on $\mathfrak{h}$ (as opposed to the formal power series). Similarly, we define $\mathcal{F}_2 = \{\Psi \in S\mathfrak{h} \mid \Psi = \sum_{w \in W} a_w \cdot e^{w(h_2)}\}$, so that one has: $\mathcal{F}_2 \simeq \text{Ind}_{W_2}^W 1$.

Observe next that we may (and will) identify the element $\Delta_h$ with a map $\delta_h$:
\[ \mathcal{F}_1 \rightarrow \mathcal{F}_2 \text{ given by the formula:} \]

\[ f \mapsto \delta_h(f) = \sum_{w \in W} \epsilon(w) \cdot f(w(h_1)) \cdot e^{w(h_2)} \in \mathcal{F}_2. \]

The map \( \delta_h \) is non-zero, due to our choice of \( \theta \). Furthermore, it is clear from the definition that this map gives a \( W \)-module morphism \( \delta_h : \mathcal{F}_1 \rightarrow \mathcal{F}_2 \otimes \mathfrak{e} \).

Recall that \( \mathcal{F}_i \cong \text{Ind}_{\mathcal{W}_i}^W \mathbf{1}, \) and that \( E_1 \) is the only common irreducible constituent of \( \text{Ind}_{\mathcal{W}_1}^W \mathbf{1} \) and \( \text{Ind}_{\mathcal{W}_2}^W \mathbf{e} \), by Corollary 4.16(ii). It follows that \( \delta_h \), viewed as an element of \( \text{Hom}(\mathcal{F}_1, \mathcal{F}_2) = \mathcal{F}_1^* \otimes \mathcal{F}_2 \), belongs to \( E_1 \otimes E_2 \). \( \square \)

We conclude this section by indicating a link between our polynomial \( \Delta_e \) and canonical bases. Recall that Kazhdan and Lusztig have constructed a canonical basis \( \{ c_w, w \in W \} \) in the group algebra \( \mathbb{C}[W] \), cf. [KL], with remarkable properties. In the same paper, Kazhdan and Lusztig have partitioned the set \( W \) into two-sided cells \( C_{L,R} \), and have further partitioned each two-sided cell into left cells \( C_L \). To any left cell \( C_L \), Kazhdan and Lusztig have attached a \( \mathbb{C}[W] \)-module \( E(C_L) \) with a canonical basis formed by the elements \( \{ c_w, w \in C_L \} \). Furthermore, Lusztig has constructed a bijection: \( O \leftrightarrow C_{L,R}(O) \), between the set of special nilpotent orbits in \( \mathfrak{g} \) and the set of two-sided cells in \( W \). This bijection has the following property: if \( E \) is the special irreducible representation of \( W \) associated to a special nilpotent orbit \( O \) then, for every left cell \( C_L \subset C_{L,R}(O) \), the module \( E \) occurs with multiplicity one in \( E(C_L) \), and does not occur in any left-cell representations arising from any other two-sided cell.

**Conjecture 4.20.** Let \( e = (e_1, e_2) \) be a non-exceptional principal nilpotent pair, and \( O = \text{AdG}(e_1) \), a (special) nilpotent orbit in \( \mathfrak{g} \). Then, the corresponding two-sided cell representation of \( W \times W \) is irreducible, equivalently, for every left cell \( C_L \subset C_{L,R}(O) \), the \( W \)-module \( E(C_L) \) is irreducible.

The conjecture trivially holds for \( \mathfrak{g} = \mathfrak{sl}_n \), since all left cell representations for \( W = S_n \) are known to be irreducible, [KL, Theorem 1.3]. A. Elashvili has verified, using the tables of left cells given in [L3] and [Ca], that the conjecture holds for all non-exceptional, in the sense of Definition 4.1, principal nilpotent pairs known at the moment (the Conjecture is false for the exceptional pair).

Let \( e = (e_1, e_2) \) be a non-exceptional principal nilpotent pair. Set \( O = \text{AdG}(e_1) \), so that \( E_1 \simeq H(B_{e_1}) \), see (4.9), is the corresponding special representation. Then \( O = \text{AdG}(e_1) \), a (special) orbit in \( \mathfrak{g} \). If Conjecture 4.20 holds for \( e \) then, for every left cell \( C_L \subset C_{L,R}(O) \), we have: \( E_1 \simeq E(C_L) \). This isomorphism transfers the canonical basis \( \{ c_w, w \in C_L \} \) in \( E(C_L) \) to a basis in \( E_1 \). We remark that, for \( \mathfrak{g} = \mathfrak{sl}_n \), the basis in \( E_1 \) arising in this way is independent of the choice of the left cell, by [KL, Theorem 1.4]. It is likely that this is true for a non-exceptional principal nilpotent pair in an arbitrary semisimple Lie algebra.

We fix (non-exceptional) \( e \) and a left cell \( C_L \subset C_{L,R}(O) \) as above. Let \( w_0 \in W \) denote the element of maximal length. According to [KL, Corollary 3.2], the assignment \( x \mapsto x \cdot w_0 \) gives an involution \( C \leftrightarrow C \cdot w_0 \) on the sets of two-sided cells, left cells, etc.

**Proposition 4.21.** Assume Conjecture 4.20 holds for \( e \). Then

(i) There is a \( W \times W \)-module isomorphism: \( E_1 \otimes E_2 \simeq E(C_L) \otimes E(C_L \cdot w_0) \).
(ii) The polynomial \( \Delta_{c} \in E_{1} \otimes E_{2} \) goes under the isomorphism in (i) to the element: \( \Delta_{c} = \sum_{x \in \mathcal{C}_{L}} \varepsilon(x) \cdot c_{x} \otimes c_{x-w_{0}} \).

Proof. It is known that in general, for any left cell \( \mathcal{C}_{L} \), there is a natural isomorphism: \( E(\mathcal{C}_{L} \cdot w_{0}) \simeq \mathfrak{e} \otimes E(\mathcal{C}_{L})^{*} \), see e.g. [J2, 3.7]. Part (i) now follows from Theorem 4.4.

To prove (ii) we will use a more precise information about the isomorphism in (i). Specifically, for each \( w \in W \), we let \( \|m_{x,y}(w)\|_{x,y \in W} \) be the matrix in the canonical basis of the operator \( m(w) : \mathbb{C}[W] \to \mathbb{C}[W] \) given by left multiplication by \( w \), that is:

\[
w \cdot c_{x} = \sum_{y \in W} m_{x,y}(w) \cdot c_{y}.
\]

The inversion formula for the Kazhdan-Lusztig polynomials, see [KL, Theorem 3.1], implies the following identity, see [J1, 4.7]:

\[
m_{x,y}(w^{-1}) = \varepsilon(x \cdot y \cdot w) \cdot m_{x-w_{0},y-w_{0}}, \quad \forall x, y, w \in W.
\] (4.22)

Alternatively, one may deduce (4.22) from [KL, Corollary 3.2] as follows. Recall that by loc.cit., the \( W \)-graph (see [KL, pp.165, 167]) attached to the left cell \( \mathcal{C}_{L} \cdot w_{0} \), coincides with \( W \)-graph attached to \( \mathcal{C}_{L} \), except that the function: \( x \mapsto I_{x} \) on its set of vertices gets replaced by the function: \( x \mapsto S \setminus I_{x} \), where \( S \) denotes the set of simple reflections in \( W \). In the special case where \( w = s = w^{-1} \), is a simple reflection the identity (4.22) follows readily from the previous discussion and formula [KL, (1.0.a)]. The latter reads:

\[
m(s) : c_{x} \mapsto \begin{cases} -c_{x} & \text{if } s \in I_{x} \\ c_{x} + \sum_{\{y \in X \mid s \in I_{y}, \{y,x\} \in Y\}} \mu(y,x) \cdot c_{y} & \text{if } s \not\in I_{x} \end{cases}.
\]

Next, in \( \mathbb{C}[W] \) define a new basis: \( \{ c_{x}^{0} := \varepsilon(x) \cdot c_{x-w_{0}}, x \in W \} \). The identity in (4.22) means that the matrix of the operator \( m(w^{-1}) : \mathbb{C}[W] \to \mathbb{C}[W] \) in the basis \( \{ c_{x}^{0} \} \) is equal to \( \|m_{x,y}(w)\|^{-T} \), the transpose of the matrix of the operator \( m(w) : \mathbb{C}[W] \to \mathbb{C}[W] \) in the basis \( \{ c_{x} \} \). This implies that the element \( \sum_{x \in \mathcal{C}_{L}} \varepsilon(x) \cdot c_{x} \otimes c_{x-w_{0}} \) transforms as the sign-representation under the \( W \)-diagonal action, and part (ii) of the proposition follows from Theorem 4.4(iii). \( \square \)

5. Distinguished nilpotent pairs; \( \mathfrak{sl}_{n} \)-case.

Given a semisimple Lie algebra \( \mathfrak{g} \), we call an arbitrary pair \( \mathfrak{e} = (e_{1}, e_{2}) \in \mathcal{Z} \) satisfying condition 1.1(\text{Nil}), but not necessarily the regularity condition 1.1(\text{Reg}), a \textit{nil-pair}.

Given a nil-pair \( \mathfrak{e} \), choose a Cartan (= maximal diagonalizable) subalgebra in \( \mathfrak{z}(\mathfrak{e}) \), and let \( \mathfrak{i} \) be the centralizer of this subalgebra in \( \mathfrak{g} \). Thus, \( \mathfrak{i} \) is a Levi subalgebra of \( \mathfrak{g} \) containing the pair \( \mathfrak{e} = (e_{1}, e_{2}) \). Let \( \mathfrak{z}_{i}(\mathfrak{e}) = \mathfrak{i} \cap \mathfrak{z}(\mathfrak{e}) \) be the centralizer of \( \mathfrak{e} \) in \( \mathfrak{i} \). By construction, every semisimple element of \( \mathfrak{z}_{i}(\mathfrak{e}) \) belongs to the center of \( \mathfrak{i} \).

We claim that \( \mathfrak{e} \) is a nil-pair in \( \mathfrak{i} \). To prove this, we note that Lemma 1.3 still applies to any nil-pair in \( \mathfrak{g} \), hence to \( \mathfrak{e} \). Therefore, we can define a semisimple pair \( \mathfrak{h} = (h_{1}, h_{2}) \), and the Lie subalgebra \( \mathfrak{m} := \{ x \in \mathfrak{g} \mid \text{ad} x(e_{i}) \in \mathbb{C} \cdot e_{i}, \ i = 1, 2 \} \),
in the same way as we have done in the proof of Theorem 1.2(iv). Let \( \m = \m_{\text{red}} \) be a maximal reductive subalgebra in \( \m \). The argument in the proof of Lemma 1.3 shows that one can choose the pair \( \mathfrak{h} = (h_1, h_2) \) inside the center of \( \m_{\text{red}} \) so that one has a Lie algebra direct sum decomposition: \( \m_{\text{red}} = \mathfrak{z}(\mathfrak{e})_{\text{red}} \oplus \langle h_1, h_2 \rangle \). With this choice of \( \mathfrak{h} \), the elements \( h_1, h_2 \) clearly commute with any Cartan subalgebra in \( \mathfrak{z}(\mathfrak{e}) \), hence belong to the Levi subalgebra \( \mathfrak{l} \) defined earlier. It follows that \( \mathfrak{e} \) is a nil-pair in \( \mathfrak{l} \), and the claim is proved. Note further that since all Cartan subalgebras in \( \mathfrak{z}(\mathfrak{e}) \) are conjugate to each other, all Levi subalgebras arising from our construction are conjugate.

In general, we call a nil-pair \( \mathfrak{e} \) in a reductive Lie algebra \( \mathfrak{l} \) pre-distinguished if every semisimple element of \( \mathfrak{z}(\mathfrak{e}) \) belongs to the center of \( \mathfrak{l} \). Proposition 1.9 implies that any principal nilpotent pair is pre-distinguished. Thus, to each nil-pair \( \mathfrak{e} \) in \( \mathfrak{g} \) we have associated a Levi subalgebra \( \mathfrak{l} \subset \mathfrak{g} \) such that \( \mathfrak{e} \) is a pre-distinguished pair in \( \mathfrak{l} \). Observe further that the argument used in the proof of Theorem 1.2(iv) applies to any pre-distinguished pair \( \mathfrak{e} \), and not only to a principal nilpotent pair. Thus, we have proved the following result

**Proposition 5.1.** (a) There is a bijection between the following sets:

\[
\left\{ \text{nil-pairs in } \mathfrak{g} \right\}/\text{Ad} \mathbf{G} \leftrightarrow \left\{ \text{Levi subalgebras } \mathfrak{l} \subset \mathfrak{g}, \text{ and conjugacy classes of pre-distinguished pairs in } \mathfrak{l} \right\}/\text{Ad} \mathbf{G}
\]

(b) The semisimple pair \( \mathfrak{h} \) associated to a pre-distinguished pair \( \mathfrak{e} \) is unique, up to conjugacy by \( Z_{\mathbf{G}}^{\text{unip}}(\mathfrak{e}) \). □

**Definition 5.2.** A commuting pair \( \mathfrak{e} = (\mathfrak{e}_1, \mathfrak{e}_2) \in \mathcal{Z} \) will be called distinguished if the following holds:

(i) Lie algebra \( \mathfrak{z}(\mathfrak{e}) \subset \mathfrak{g} \) consists of nilpotent elements, and

(ii) There exists a regular semisimple pair \( \mathfrak{h} = (h_1, h_2) \in \mathcal{Z} \) such that:

\[
[h_i, \mathfrak{e}_j] = \delta_{i,j} \cdot \mathfrak{e}_i, \ i, j \in \{1, 2\}.
\]

Thus the pair \( \mathfrak{e} = (\mathfrak{e}_1, \mathfrak{e}_2) \) is distinguished if and only if it is pre-distinguished and the associated semi-simple pair \( \mathfrak{h} \) (which is unique up to conjugacy, due to Proposition 5.1(b)) is regular. We believe that a good deal of results that we prove here for principal nilpotent pairs have natural generalisations to distinguished nilpotent pairs. For example, we make the following

**Conjecture 5.3.** If \( V \) is a rational \( \mathbf{G} \)-module and \( \mathfrak{e} \) a distinguished pair, then the space \( \lim_{\to} \mathfrak{z}(\mathfrak{h}) V^{\mathfrak{z}(\mathfrak{h})} \) is annihilated by \( \mathfrak{z}(\mathfrak{e}) \), and hence is independent of the choice of an associated semisimple pair \( \mathfrak{h} \).

**Conjecture 5.4.** The number of \( \text{Ad} \mathbf{G} \)-conjugacy classes of all distinguished pairs in \( \mathfrak{g} \) is finite.

**Remark 5.5.** The following example shows that, even for \( \mathfrak{g} = \mathfrak{sl}_n \), not every pre-distinguished pair is distinguished, and moreover, the classification of \( \text{Ad} \mathbf{G} \)-conjugacy classes of nil-pairs (hence by Proposition 5.1, of all pre-distinguished pairs) is a wild problem; in particular there exist continuous families of such \( \text{Ad} \mathbf{G} \)-conjugacy classes.

Given a finite dimensional vector space \( E \) and any finite collection \( E_i, i = 1, \ldots, r \) of its vector subspaces, form the following commutative ladder-shaped
Let $V$ be the direct sum of all the spaces in the diagram, let $e_1 : V \to V$ be the map induced by all horizontal arrows in the diagram, and let $e_2 : V \to V$ be the map induced by all vertical arrows in the diagram. This way, to any collection $(E, E_1, E_2, \ldots, E_r)$ as above one associates the nil-pair $(e_1, e_2)$ in $\mathfrak{sl}(V)$. We see that the classification problem of Ad$_G$-conjugacy classes of nil-pairs contains as a subproblem the classification of $r$-tuples of vector subspaces of a vector space. This is known to be a wild problem, in general (4-tuples of lines in a 2-plane have a well-defined cross-ratio, a continuous invariant). □

From now on, we assume that $g = \mathfrak{sl}_n$. We are going to classify all nil-pairs $e = (e_1, e_2)$ in $\mathfrak{sl}_n$ that have a regular associated semisimple pair $h$.

Let $\lambda \subset \mathbb{Z} \oplus \mathbb{Z}$ be a finite subset, thought of as a collection of boxes on the 2-plane. We say that $\lambda$ is connected if any two boxes in $\lambda$ can be joined by a sequence of boxes of $\lambda$, such that every pair of adjacent boxes in the sequence has a common edge. Let $\lambda_1$ and $\lambda_2$ be two Young diagrams with the same southwest corner, and such that $\lambda_1 \subset \lambda_2$ and $\lambda_2 \setminus \lambda_1$ is a connected $n$-element set. Connected sets of the form $\lambda := \lambda_2 \setminus \lambda_1$ will be referred to as skew-diagrams. We write $(-\lambda)$ for the set whose boxes have coordinates opposite to those of $\lambda$.

Given any connected set $\lambda$ with $n$ boxes, we label the standard base vectors in $\mathbb{C}^n$ by the boxes of the set (in some way), and associate to $\lambda$ a pair $e_\lambda = (e_1, e_2)$ of nilpotent linear transformations of $\mathbb{C}^n$ in the same way as it has been done in the Introduction. In more detail, we let $e_1$ act along the rows of the set, moving one step to the right (if this is possible and act by zero otherwise), and let $e_2$ act along the columns of the set, moving one step up (if this is possible and act by zero otherwise).

**Theorem 5.6.** (a) A commuting pair $e$ in $\mathfrak{sl}_n$ is distinguished if and only if it is conjugate to a pair of the form $e = e_{\pm \lambda}$, where $\lambda$ is a skew-diagram with $n$ boxes.

(b) The pair $e_{\pm \lambda}$ corresponding to a skew-diagram $\lambda$ is a principal nilpotent pair if and only if the skew-diagram $\lambda$ is a Young diagram.

We need some notation. Given a collection of the form $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$, where each $\lambda_i$ is either a skew-diagram or minus skew-diagram, write $|\lambda_i|$ for the number of boxes in $\lambda_i$, and put $|\lambda| = |\lambda_1| + \ldots + |\lambda_r|$. We let $e_\lambda := \bigoplus_i e_{\lambda_i}$ denote the commuting pair of endomorphisms of $\mathbb{C}^{|\lambda|} = \bigoplus_i \mathbb{C}^{|\lambda_i|}$, given by the direct sum.

**Lemma 5.7.** A nil-pair in $\mathfrak{sl}_n$ has a regular associated semisimple pair if and only if it is conjugate to a pair of the form $e_\lambda$, for a certain collection of skew-diagrams $\lambda = \{\pm \lambda_1, \pm \lambda_2, \ldots, \pm \lambda_r\}$ with $|\lambda| = n$. 

---

**Diagram:**

\[
\begin{array}{c}
E_1 \rightarrow E \xrightarrow{id} E \xrightarrow{id} E \\
\uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow \\
E_2 \rightarrow E \xrightarrow{id} E \\
\uparrow \quad \quad \quad \uparrow \\
E_3 \rightarrow E \\
\ldots
\end{array}
\]
Proof of Lemma. We observe that any commuting pair of diagonal matrices \( h = (h_1, h_2) \) gives a bigrading \( \mathbb{C}^n = \bigoplus_{p, q} V_{p, q} \), where \( V_{p, q} = \{ v \in \mathbb{C}^n \mid h_1(v) = p \cdot v, h_2(v) = q \cdot v \} \) is a joint weight space of \((h_1, h_2)\). It is clear that the pair \( h = (h_1, h_2) \) is regular if and only if all bigraded components \( V_{p, q} \) are at most 1-dimensional. In this case, one breaks up the set, \( \text{Spec}(h) \), of all pairs \((p, q)\) such that \( \dim V_{p, q} = 1 \) into subsets: \( \text{Spec}(h) = S_1 \sqcup \ldots \sqcup S_m \) in such a way that \((p, q)\) and \((p', q')\) belong to the same subset if and only if \( p' - p \in \mathbb{Z} \) \& \( q' - q \in \mathbb{Z} \). Now, choosing an arbitrary pair \((p_0, q_0) \in S_i\) one may identify each subset \( S_i \) with the subset in the 2-plane formed by the points with coordinates \( \{(p - p_0, q - q_0)\}_{(p, q) \in S_i} \). Thus we see that if the nil-pair \( e = (e_1, e_2) \) in \( \mathfrak{sl}_n \) has a regular associated semisimple pair, then \( e = e_{\lambda} \), for a certain collection of connected subsets \( \lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_r\} \).

Next, we restrict our attention to one such subset \( \lambda = \lambda_i \) in the collection. Observe that, for the maps \( e_1 \) and \( e_2 \) in the pair \( e_\lambda = (e_1, e_2) \) to commute, every \( 2 \times 2 \)-square of boxes must commute. This automatically holds for any \( 2 \times 2 \)-square contained inside \( \lambda \). Therefore one only has to check commutativity of the \( 2 \times 2 \)-squares that intersect both \( \lambda \) and its complement. It is straightforward to see that commutativity of all such squares amounts to the requirement that either \( \lambda \) or \( (-\lambda) \) has the shape of a skew-diagram. This proves the lemma. \( \Box \)

Proof of Theorem 5.6. It is clear that the centralizer of a pair \( e = e_\lambda \) consisting of more than one skew-diagram contains a diagonal matrix (that restricts to an arbitrary scalar on each skew-diagram of the collection \( \lambda \)), hence the pair \( e \) can not be distinguished in this case. Thus, to prove part (a) it suffices to show that, for any skew-diagram \( \lambda \), the Lie algebra \( \mathfrak{z}(e_\lambda) \) consists of nilpotent endomorphisms of \( \mathbb{C}^n \) only.

Let \( \lambda \) be a skew diagram. A subset \( \nu \subset \lambda \) will be called an \textit{in-subset} (resp. \textit{out-subset}) if \( \nu \) is connected, and for any box \((p, q) \in \nu\), all boxes \((i, j) \in \lambda\) such that \( i \leq p \) and \( j \leq q \) (resp. \( i \geq p \) and \( j \geq q \)) belong to \( \nu \) (note that any in-subset has the shape of a skew-diagram, and any out-subset has the shape of a ”minus” skew-diagram). For any pair of integers \( p, q \), let \( S_{p, q}(\lambda) \) be the set of pairs \((\nu_{\text{in}}, \nu_{\text{out}})\), where \( \nu_{\text{in}} \) is an in-subset of \( \lambda \), \( \nu_{\text{out}} \) is an out-subset of \( \lambda \) and, moreover, \( \nu_{\text{out}} \) is obtained from \( \nu_{\text{in}} \) via translation \( T: \nu_{\text{in}} \sim \nu_{\text{out}} \) by the vector \((p, q)\) (translation of the plane \( p \) steps horizontally and \( q \) steps vertically). Thus, \( \nu_{\text{in}} \) and \( \nu_{\text{out}} \) have the same shape.

Put \( n = |\lambda| \), and label base vectors of \( \mathbb{C}^n \) by the boxes of \( \lambda \). Given a pair \( \nu = (\nu_{\text{in}}, \nu_{\text{out}}) \in S_{p, q}(\lambda) \), we define a linear map \( f_\nu: \mathbb{C}^n \rightarrow \mathbb{C}^n \) as follows. The map \( f_\nu \) takes every base vector labelled by a box \((i, j) \in \nu_{\text{in}}\) to the base vector labelled by the translated box, \((i + p, j + q) \in \nu_{\text{out}}\), and takes all other base vectors (i.e., those labelled by the boxes of \( \lambda \setminus \nu_{\text{in}}\)) to 0. Now write \( \mathfrak{z}(e_\lambda) = \bigoplus_{p, q} \mathfrak{z}_{p, q}(e_\lambda) \) for the bigrading on the centraliser of the pair \( e_\lambda \). Verification of the following crucial observation is straightforward and is left to the reader.

**Claim 5.8.** For any skew-diagram (or minus skew-diagram) \( \lambda \), and any \( p, q \geq 0 \), the endomorphisms \( \{f_\nu \mid \nu = (\nu_{\text{in}}, \nu_{\text{out}}) \in S_{p, q}(\lambda)\} \) form a basis of \( \mathfrak{z}_{p, q}(e_\lambda) \) \( \Box \)

Next, given \( \nu = (\nu_{\text{in}}, \nu_{\text{out}}) \in S_{i, j}(\lambda) \), such that \( \nu_{\text{out}} \) is obtained from \( \nu_{\text{in}} \) via translation \( (i, j) \rightarrow (i + p, j + q) \), define an operator \( T_\nu : \nu_{\text{in}} \rightarrow \nu_{\text{out}} \) to be the restriction of that translation to the set \( \nu_{\text{in}} \). Given a second pair \( \nu' = (\nu'_{\text{in}}, \nu'_{\text{out}}) \in \)
Thus, \( \text{dim}(\mathfrak{p}, \mathfrak{q})(\lambda) \), one finds by a direct calculation

\[
f_{\nu} \cdot f_{\nu'} = f_{\nu''}, \quad \text{where} \quad \nu'' = (\nu''_{\text{in}}, \nu''_{\text{out}}) \quad \text{is defined by:}
\]

\[
\nu''_{\text{in}} = T_{\nu}^{-1}(\nu_{\text{in}} \cap \nu'_{\text{out}}) \subset \nu'_{\text{in}} \quad \text{and} \quad \nu''_{\text{out}} = T_{\nu}^{-1}(\nu_{\text{in}} \cap \nu'_{\text{out}}) \subset \nu_{\text{out}}.
\]

Observe next that since the skew-diagram \( \lambda \) is connected, it can not contain a subset which is both an in-subset and an out-subset at the same time. It follows that \( \nu_{\text{in}} \neq \nu'_{\text{out}} \). Hence, we deduce: \( |\nu''_{\text{in}}| < \max(|\nu_{\text{in}}|, |\nu'_{\text{in}}|) \). This implies that any product of the maps of the form \( f_{\nu} \) that contains more than \( n \) factors vanishes, and part (a) of Theorem 5.6 follows.

Let the pair \( e_\lambda = (e_1, e_2) \) be associated to a Young diagram \( \lambda \). We claim that \( (e_1, e_2) \) is a principal nilpotent pair. The only non-obvious part of (1.1) that needs to be verified is the equation \( \text{dim}(\mathfrak{z}(e)) = \text{rk}\mathfrak{g} \). To prove this, note that the base vector \( v_0 \in \mathbb{C}^n \) labelled by the box with coordinates \((0,0)\), the south-west “corner” of the diagram, is a cyclic vector for the operators \( e_1, e_2 \), in the sense that \( \mathbb{C}^n = \mathbb{C}[e_1, e_2] \cdot v_0 \), where \( \mathbb{C}[e_1, e_2] \) denotes an abstract polynomial ring in the variables \( e_1, e_2 \). Therefore, any operator \( x \in \mathfrak{z}(e) \) is completely determined by the vector \( x(v_0) \in \mathbb{C}^n \). It follows that any element of \( \mathfrak{z}(e) \) can be expressed as a polynomial without constant term in the operators \( e_1 \) and \( e_2 \). Hence, \( \mathfrak{z}(e) + \mathbb{C}\text{Id} = \mathbb{C}[e_1, e_2]/I \), where \( I \) is the ideal annihilating \( v_0 \). It is easy to see that this ideal is spanned by all monomials \( e_1^r e_2^s \), such that \( (r, s) \) are not coordinates of a box of \( \lambda \). Thus, \( \text{dim}(\mathbb{C}[e_1, e_2]/I) = n \), hence: \( \text{dim}(\mathfrak{z}(e)) = n - 1 \), so that \( e \) is a regular pair.

Finally, it is easy to verify using Claim 5.8 that, for any skew-diagram (resp. minus skew-diagram) \( \lambda \) which is not a Young diagram (resp. minus Young diagram), one has \( \text{dim}(\mathfrak{z}(e_\lambda)) > n - 1 \). Part (b) of Theorem 5.6 follows. \( \square \)

Theorem 5.6 shows that the pairs corresponding to skew-diagrams may be thought of as ”double-analogues” of nilpotent matrices that have a single Jordan block in their Jordan form.

**Example 5.9.** We give an example of distinguished pairs in the orthogonal Lie algebra \( \mathfrak{so}_n \). Let \( \lambda \) be a skew diagram with \( n \) boxes, and \( e_\lambda = (e_1, e_2) \) the pair of endomorphisms of \( \mathbb{C}^n \) constructed as above. Assume, in addition, that the diagram \( \lambda \) is centrally symmetric with respect to the origin \((0,0)\) \( \in \mathbb{Z}^2 \), i.e., \( (p,q) \in \lambda \implies (-p,-q) \in \lambda \). We define a symmetric bilinear form on \( \mathbb{C}^n \) as follows:

\[
\omega(v_{p,q}, v_{p',q'}) = \begin{cases} 
(-1)^{p+q} & \text{if} \quad p + p' = 0 = q + q' \\
0 & \text{otherwise}
\end{cases}
\]

where \( v_{p,q} \) denotes the base vector in \( \mathbb{C}^n \) corresponding to a box \( (p,q) \in \lambda \). It is clear that the form \( \omega \) is non-degenerate, and that both \( e_1 \) and \( e_2 \) are skew-symmetric relative to \( \omega \), hence form a commuting pair in the Lie algebra \( \mathfrak{so}(\mathbb{C}^n, \omega) \).

The proof of Theorem 5.6 implies that the centralizer of the pair \( e_\lambda \) in \( \mathfrak{sl}_n \), hence in \( \mathfrak{so}_n \), consists of nilpotent matrices. Thus, to any centrally symmetric skew diagram \( \lambda \) we have attached a distinguished pair in \( \mathfrak{so}_n \). We do not know if every distinguished pair in \( \mathfrak{so}_n \) is obtained in this way. Neither do we know for which \( \lambda \), apart from rectangular ones, the pair \( e_\lambda \) is a principal nilpotent pair in \( \mathfrak{so}_n \).

From now until the end of this section, we will assume that \( \mathfrak{g} \) is the reductive Lie algebra \( \mathfrak{gl}_n \), rather than \( \mathfrak{sl}_n \). This slight modification will become more convenient.
shortly. Write \( \mathfrak{h} \) for the Cartan subalgebra of diagonal \( n \times n \)-matrices, and identify it with \( \mathbb{C}^n \). Set \( \mathbf{u} = (u_1, \ldots, u_n) \) and \( \mathbf{v} = (v_1, \ldots, v_n) \), and view \( S(\mathfrak{h} \oplus \mathfrak{h}) \) as the polynomial ring, \( \mathbb{C}[\mathbf{u}, \mathbf{v}] \), in \( 2n \) variables. We let the Symmetric group \( W = S_n \) act on the \( n \)-tuples \( \mathbf{u} \) and \( \mathbf{v} \) by permutations.

Fix a Young diagram \( \lambda \) with \( n \) boxes, let \( \mathbf{e} = e_\lambda \in \mathfrak{gl}_n \) be the corresponding principal nilpotent pair. To write an associated pair \((h_1, h_2)\) of diagonal matrices explicitly, enumerate \( n \) boxes of the diagram \( \lambda \) in some order, and write \((a_i, b_i)\) for the coordinates of the \( i \)-th box, starting counting the coordinates with \((0,0)\), see fig. (0.1). This way we get a collection of non-negative integers \( a_1, b_1, b_2, \ldots, a_n, b_n \). One may then choose an associated semisimple pair \( \mathbf{h} = (h_1, h_2) \) to be: \( h_1 = \text{diag}(a_1, \ldots, a_n) \) and \( h_2 = \text{diag}(b_1, \ldots, b_n) \). Note that these diagonal matrices do not have zero trace, hence are not in \( \mathfrak{sl}_n \). The corresponding semisimple pair in \( \mathfrak{sl}_n \) is: \((h_1 - x_1, h_2 - x_2)\), where \( x_i = 1/n \cdot \text{Tr}(h_i) \cdot \text{Id.}\)

All the considerations of the previous sections extend to reductive Lie algebras without troubles. In particular, for \( \mathfrak{g} = \mathfrak{gl}_n \), we have the element \( \Delta_{\mathbf{h}} \in \mathbb{C}[\mathbf{u}, \mathbf{v}] \), which is explicitly given by the formula:

\[
\Delta_{\mathbf{h}}(\mathbf{u}, \mathbf{v}) = \sum_{w \in S_n} \varepsilon(w) \cdot e^{w(h)} = \sum_{w \in S_n} \varepsilon(w) \cdot e^{\sum_i (a_{w(i)}u_{w(i)} + b_{w(i)}v_{w(i)})} \quad (5.10)
\]

A special feature of the \( \mathfrak{gl}_n \)-case is that one can make a change of variables by setting: \( x_i := e^{u_i} \) and \( y_i := e^{v_i} \), \( i = 1, \ldots, n \). Then, \( \Delta_{\mathbf{h}} \) becomes a polynomial (as opposed to the exponential expression above) in the new variables: \( \mathbf{x} = (x_1, \ldots, x_n) \) and \( \mathbf{y} = (y_1, \ldots, y_n) \). In view of (5.10), this polynomial may be written in the form of the following determinant:

\[
\Delta_{\mathbf{h}}(\mathbf{x}, \mathbf{y}) = \det \begin{vmatrix} x_1^{a_1} y_1^{b_1} & x_2^{a_1} y_2^{b_1} & \cdots & x_n^{a_1} y_n^{b_1} \\ x_1^{a_2} y_1^{b_2} & x_2^{a_2} y_2^{b_2} & \cdots & x_n^{a_2} y_n^{b_2} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{a_n} y_1^{b_n} & x_2^{a_n} y_2^{b_n} & \cdots & x_n^{a_n} y_n^{b_n} \end{vmatrix} \quad (5.11)
\]

where \( d_1 = \sum a_i = \text{Tr}(h_1) \) and \( d_2 = \sum b_i = \text{Tr}(h_2) \). The determinant on the RHS has been first introduced by P.A. MacMahon "Combinatory Analysis", Cambridge University Press (1916), as a "double analogue" of the Vandermonde determinant; see also the paper by A.M. Garsia and M. Haiman "A graded representation model for Macdonald’s polynomials", Proc. Nat. Acad. Sci. USA, 90 (1993), 3607–3610.

We claim next that the \( W \)-harmonic polynomial \( \Delta_{\mathbf{e}} \in \mathbb{C}[\mathbf{u}] \otimes \mathbb{C}[\mathbf{v}] \) is given by the same determinant (5.11), where now \( \mathbf{x} \) is replaced by \( \mathbf{u} \), and \( \mathbf{y} \) by \( \mathbf{v} \), that is we have

**Proposition 5.12.** In the \( \mathfrak{gl}_n \)-case the elements \( \Delta_{\mathbf{h}} \) and \( \Delta_{\mathbf{e}} \) can be obtained from each other by a change of variables, i.e.: \( \Delta_{\mathbf{e}}(e^{\mathbf{u}}, e^{\mathbf{v}}) = \frac{1}{d_1! \cdot d_2!} \cdot \Delta_{\mathbf{h}}(\mathbf{u}, \mathbf{v}) \).

This explains Corollary 4.16. We see also that the "difficult" non-vanishing part of Theorem 4.4(ii) follows, in the \( \mathfrak{gl}_n \)-case, directly from the non-vanishing of \( \Delta_{\mathbf{h}} \), which is trivial.
Proof of Proposition 5.12. Our argument will be based on the identity:

$$\sum_{y \in W^i} \varepsilon(y) \cdot e^{y(h_i)} \bigg|_{e^u = x} = \prod_{\alpha \in R_+^i} \langle \alpha^\vee, x \rangle = \pi_i(x) = \frac{1}{d_i!} \sum_{y \in W^i} \varepsilon(y) \cdot \langle y(h_i), x \rangle^{d_i}. \quad (5.13)$$

In this identity, $i = 1, 2$, the LHS is the standard Weyl denominator for the Levi subalgebra $g^i$ which, for $g = g_{\mathfrak{h}_i}$, is equal to a product of Vandermonde determinants corresponding to the columns (if $i = 1$), resp. rows (if $i = 2$), of the Young diagram. It is clear that, upon substitution $e^u = x$, each Vandermonde determinant becomes the corresponding polynomial $\pi$, the product of positive coroots that stands at the middle of formula (5.13). The equation on the right of (5.13) is nothing but identity (4.5).

In the formulas below we abuse the notation slightly so that $w(\langle h, u \rangle)$ should be understood as $w$ acting on either of the two variables $h$ or $u$, that is: $\langle w(h), u \rangle = \langle h, w(u) \rangle$, and $w(e^{\langle h, u \rangle})$ has a similar meaning. With this understood, we calculate

$$\Delta_h = \sum_{w \in W} \varepsilon(w) \cdot w(e^{h_1, u}) \otimes e^{h_2, v})$$

$$= \sum_{w \in W} \varepsilon(w) \cdot \frac{1}{d_1!} \cdot w \left( \sum_{w_1 \in W^1} \varepsilon(w_1) \cdot e^{w(h_1, h)} \otimes e^{h_2, v} \right) \bigg|_{e^u = x} = \text{by (5.13)}$$

$$= \frac{1}{d_1!} \cdot \sum_{w \in W} \varepsilon(w) \cdot \langle w(h_1), x \rangle^{d_1} \otimes e^{w(h_2, v)}$$

$$= \frac{1}{d_1!} \sum_{w \in W} \varepsilon(w) \cdot \langle h_1, x \rangle^{d_1} \otimes \left( \frac{1}{d_2!} \sum_{w_2 \in W^2} \varepsilon(w_2) \cdot e^{w_2(h_2, v)} \right) \bigg|_{e^v = y} = \text{by (5.13)}$$

$$= \frac{1}{d_1! \cdot d_2!} \sum_{w \in W} \varepsilon(w) \cdot \langle w(h_1), x \rangle^{d_1} \otimes \langle w(h_2), y \rangle^{d_2} = \frac{1}{d_1! \cdot d_2!} \cdot \Delta_e(x, y). \quad \Box$$

6. Some cohomology and generating functions.

Fix a principal nilpotent pair $e = (e_1, e_2)$, an associated semisimple pair $h = (h_1, h_2)$, and let $g = \bigoplus_{p, q} g_{p, q}$ be the corresponding bigrading.

We will be concerned with the following 3-term complex:

$$g \xrightarrow{\partial'} g \otimes g \xrightarrow{\partial''} g, \quad g_{p-1, q-1} \xrightarrow{\partial'_{p, q}} g_{p, q-1} \oplus g_{p-1, q} \xrightarrow{\partial''_{p, q}} g_{p, q}, \quad (6.1)$$

where the differentials are given by the formulas

$$\partial' := ade_1 \oplus ade_2, \quad \partial'' := (ade_2) \oplus (-ade_1).$$
It is clear that $\partial' \circ \partial'' = 0$, and we put $H_{p,q} := \text{Ker}(\partial''_{p,q})/\text{Im}(\partial''_{p,q})$.

Observe that the two maps below are adjoint to each other with respect to the Killing form (see Corollary 1.8):

$$\partial'_{p+1,q+1} : g_{p+1,q} \rightarrow g_{p,q} + g_{p+1,q+1}, \quad \partial''_{p,q} : g_{p-1,q} \oplus g_{p,q} \rightarrow g_{p,q}.$$ 

Since $\text{Ker}(\partial''_{p,q}) = \mathfrak{z}_{p-1,q-1}(e)$, we get by duality $\text{Coker}(\partial''_{p,q}) \simeq \mathfrak{z}^*(p,q)(e)$, and $\text{Im}(\partial''_{p,q}) = \mathfrak{z}_{p-1,q}(e) \simeq \mathfrak{z}^*(p,q)/(\mathfrak{z}^*(p,q)(e))$. From these formulas we deduce:

$$H_{p,q} = \frac{\text{Ker}(g_{p,q-1} \oplus g_{p-1,q} \rightarrow g_{p,q-1}^* / \mathfrak{z}^*(p,q)(e))}{\text{Im} (g_{p,q-1} / \mathfrak{z}_{p,q-1} \hookrightarrow g_{p,q-1} \oplus g_{p-1,q})}.$$ (6.2)

To put the complex (6.1) in an adequate context, we regard $g$ as a module over an abstract two-dimensional abelian Lie algebra $e$ whose base vectors act on $g$ via the operators $\{\text{ad}e_i\}_{i=1,2}$. Then (6.1) becomes the standard Koszul complex: $g \rightarrow e^* \otimes g \rightarrow (\wedge^2 e^*) \otimes g$, computing the Lie algebra cohomology $H^1(e,g)$.

The group $H^1(e,g)$ has the following geometric interpretation in terms of the commuting variety $Z$. First, it is easy to verify that condition 1.1(Reg) insures that $e$ is a smooth point of $Z$. Assume further (without justification) that the space of diagonal $AdG$-orbits on $Z$ has the structure of a smooth algebraic variety, $Z/AdG$, at least locally near $e$. By general principles, the tangent space to $Z/AdG$ at the point $O = AdG(e)$ is then given by: $T_O(Z/AdG) = H^1(e,g)$. In other words, $H^1(e,g)$ is the normal space at a point $x \in O$ to the submanifold $O$ inside $Z$.

Next, we introduce the following generating functions in two variables: $H_a(s,t) :=$

$$\sum_{p,q} s^pt^q \dim H_{p,q}, \quad H_a(s,t) := \sum_{p,q} s^pt^q \dim g_{p,q}, \quad H_a(s,t) := \sum_{p,q} s^pt^q \dim \mathfrak{z}_{p,q}(e).$$

**Lemma 6.3.** $H_a(s,t) = st \cdot H_a(s,t) + H_a(s^{-1}, t^{-1}) - (s-1)(t-1) \cdot H_a(s,t)$.

**Proof.** From (6.2) we find: $\dim H_{p,q} = \dim g_{p,q-1} + \dim g_{p-1,q} -$

$$- \dim g_{p-1,q-1} - \dim g_{p,q-1} + \dim \mathfrak{z}_{p,q-1}(e) + \dim \mathfrak{z}_{p-1,q}(e).$$

Multiply each side by $s^pt^q$, and use that $\dim g_{p,q-1} = \dim g_{p,q}$, by Corollary 1.8. Summing up over all $p,q \in \mathbb{Z}$, we find

$$H_a(s,t) = t \cdot H_a(s,t) + s \cdot H_a(s,t) - st \cdot H_a(s,t) - H_a(s,t) + st \cdot H_a(s,t) + H_a(s^{-1}, t^{-1}).$$

Recall that the centralizer of $e_i$, $(i = 1,2)$, is a bigraded subalgebra $\mathfrak{z}(e_i) = \bigoplus_{p,q} \mathfrak{z}_{p,q}(e_i) \subset g$. The structure of the cohomology $H^1(e,g)$ is described by

**Theorem 6.4.** (i) $\dim H^1(e,g) = 2 \text{rk} g$, and $H_{p,q} = 0$ whenever $p \cdot q \geq 0$.

(ii) Furthermore, we have

$$H_{p,q} \simeq \begin{cases} \text{Coker} (\text{ade}_1 : \mathfrak{z}_{p,q-1}(e_2) \rightarrow \mathfrak{z}_{p,q-1}(e_2)) & \text{if } p < 0 \& q \geq 0 \\
\text{Coker} (\text{ade}_2 : \mathfrak{z}_{p,q-1}(e_1) \rightarrow \mathfrak{z}_{p,q-1}(e_1)) & \text{if } p \geq 0 \& q < 0 \end{cases}$$
Proof. Let \( z \in \mathfrak{j}_{p,q-1}(e_2) \). Then, for the element \( h_{p,q} := (z,0) \in \mathfrak{g}_{p,q-1} \oplus \mathfrak{g}_{p-1,q} \)
we clearly have: \( \partial'_{p,q}(h_{p,q}) = \text{ade}_2(z) - \text{ade}_1(0) = 0 \). Hence, \( h_{p,q} \) represents a
class in \( H_{p,q} \), see (6.1). The element \( h_{p,q} \) is a coboundary if and only if there
exists \( y \in \mathfrak{g}_{p-1,q-1} \) such that: \( \text{ade}_2(y) = 0 \) and \( \text{ade}_1(y) = z \). The first equation
above implies that \( y \in \mathfrak{j}_{p-1,q-1}(e_2) \). Therefore, the second equation implies that
the assignment \( z \mapsto (z,0) \) gives an injection

\[
\text{Coker} (\text{ade}_1 : \mathfrak{j}_{p-1,q-1}(e_2) \to \mathfrak{j}_{p,q-1}(e_2)) \hookrightarrow H_{p,q}.
\tag{6.7}
\]

We now study in more detail the case when \( p < 0 \) & \( q \geq 0 \). We observe that the
operator \( \text{ade}_1 \) in (6.7) indeed maps \( \mathfrak{j}_{p-1,q-1}(e_2) \) into \( \mathfrak{j}_{p,q-1}(e_2) \), since \( e_1 \)
commutes with \( e_2 \). Moreover, for all \( p < 0 \), this map is injective, due to the weak Lefschetz.
This way we see, applying \( \text{ade}_1 \) several times that, for any \( i = -p > 0 \), we have
an injective map \( \text{ad}^i e_1 : \mathfrak{j}_{p-1,q-1}(e_2) \to \mathfrak{j}_{p,q-1}(e_2) \subset \mathfrak{j}(h_1,e_2) \). Thus, the sequence of
spaces \( F^i := \bigoplus_{q \geq 0} \text{ad}^i e_1 (\mathfrak{j}_{p,q-1}(e_2)) , \ i = 0,1, \ldots \), gives a decreasing filtration
of the subalgebra \( \mathfrak{j}(h_1,e_2) \), which is, in a sense, dual to the increasing filtration \( F_* \)
considered in \( \S 2 \). Observe further that the weak Lefschetz insures that \( \mathfrak{j}_{p,q}(e_2) = 0 \),
for all \( q < 0 \). Hence, summation over all \( q \) yields, for any \( i > 0 \), a vector space
isomorphism

\[
\text{ad}^i e_1 : \text{Coker} (\text{ade}_1 : \mathfrak{j}_{p-1,q-1}(e_2) \to \mathfrak{j}_{p,q-1}(e_2)) \xrightarrow{\sim} F^i / F^{i-1}.
\]

Thus, writing \( \text{gr} \mathfrak{j}(h_1,e_2) := \bigoplus_i F^i / F^{i-1} \), we obtain a graded space isomorphism

\[
\bigoplus_i \left( \text{Coker} (\text{ade}_1 : \mathfrak{j}_{p,q-1}(e_2) \to \mathfrak{j}_{p,q-1}(e_2)) \right) \xrightarrow{\sim} \text{gr} \mathfrak{j}(h_1,e_2).
\tag{6.8}
\]

Recall now that \( \dim \mathfrak{j}(h_1,e_2) = \dim \mathfrak{j}(e) = \text{rk} \mathfrak{g} \), by (2.3). Hence, we get
\( \text{dim}(\text{gr} \mathfrak{j}(h_1,e_2)) = \dim \mathfrak{j}(h_1,e_2) = \text{rk} \mathfrak{g} \). It follows now from (6.7) and (6.8)
that we have: \( \text{rk} \mathfrak{g} = \text{dim}(\text{gr} \mathfrak{j}(h_1,e_2)) \leq \text{dim}(\bigoplus_{p<0,q\geq 0} H_{p,q}) \). In the same way one
obtains a similar inequality for the southeast quadrant: \( p \geq 0 \) & \( q < 0 \), i.e.,
\( \text{rk} \mathfrak{g} \leq \text{dim}(\bigoplus_{p\geq 0,q<0} H_{p,q}) \). The two inequalities combined together imply that

\[
2 \cdot \text{rk} \mathfrak{g} \leq \text{dim} \left( \bigoplus_{p<0,q} H_{p,q} \right) .
\tag{6.9}
\]

On the other hand, setting \( s = t = 1 \) in the formula of Lemma 6.3, we find
\( \dim H^1(\mathfrak{e}, \mathfrak{g}) = 2 \cdot \text{dim} \mathfrak{j}(e) = 2 \cdot \text{rk} \mathfrak{g} \). It follows that the inequality in (6.9) has to
be an equality. This implies that the inclusion in (6.7), and a similar inclusion for the quadrant \( p \geq 0 \) & \( q < 0 \), are in effect both isomorphisms. Furthermore, the
groups \( H_{p,q} \) must vanish for the two other quadrants, that is for all \( p \cdot q \geq 0 \). This
completes the proof of the theorem. \( \square \)

Part (i) of the theorem implies, in view of formula (6.7) that, for any \( p,q \geq 0 \),
the LHS of (6.7) vanishes. This gives

**Corollary 6.10.** The maps \( \text{ade}_1 : \mathfrak{j}_{p,q}(e_2) \to \mathfrak{j}_{p+1,q}(e_2) \) and \( \text{ade}_2 : \mathfrak{j}_{p,q}(e_1) \to \mathfrak{j}_{p,q+1}(e_1) \) are both surjective, for any \( p,q \geq 0 \). \( \square \)

Write \( \mathfrak{j}_{p,q}(e_1 \cdot e_2) := \text{Ker}(\text{ade}_1 \cdot \text{ade}_2 : \mathfrak{g}_{p,q} \to \mathfrak{g}_{p+1,q+1}) \). Corollary 6.10 may be
reformulated in the following way
Corollary 6.12. \( \mathfrak{z} \subseteq \mathfrak{y} \).

Proof. Let \( x \in \mathfrak{g}_{p,q} \) be such that \( \text{ad} e_1 \cdot \text{ad} e_2(x) = 0 \). Put \( y := \text{ad} e_2(x) \). Then \( y \in \mathfrak{z}_{p+1,q}(e_1) \). Hence, by Corollary 6.10, there exists \( z \in \mathfrak{z}_{p,q}(e_1) \) such that \( y = \text{ad} e_2(z) \). Therefore we have: \( \text{ad} e_2(x - z) = y - y = 0 \). Thus, \( x = z + (x - z) \in \mathfrak{z}_{p,q}(e_1) + \mathfrak{z}_{p,q}(e_2) \), and the result is proved.

Conversely, one may show that the Lemma implies Corollary 6.10. □

By Theorem 6.4 we can write \( H^1(\mathfrak{e}, \mathfrak{g}) = H_{nw} \bigoplus H_{se} \), where \( \mathfrak{nw} = \text{northwest}, \mathfrak{se} = \text{southeast} \), and we put \( H_{nw} := \bigoplus_{p<0, q \geq 0} H_{p,q} \), and \( H_{se} := \bigoplus_{p \geq 0, q < 0} H_{p,q} \).

Corollary 6.12. The Killing form on \( \mathfrak{g} \) induces a perfect pairing: \( H_{nw} \times H_{se} \rightarrow \mathbb{C} \). In particular, the space \( H^1(\mathfrak{e}, \mathfrak{g}) \simeq H_{nw} \bigoplus H_{nw}^* \) has a natural symplectic structure.

Proof. Corollary 1.8 insures that the two squares below are obtained from each other by duality induced by the Killing form:

\[
\begin{array}{cccc}
\mathfrak{g}_{p-1,q} & \xrightarrow{e_1} & \mathfrak{g}_{p,q} & \xrightarrow{e_1} \\
\uparrow e_2 & & \uparrow e_2 & \\
\mathfrak{g}_{p-1,q-1} & \xrightarrow{e_1} & \mathfrak{g}_{p,q-1} & \\
\end{array}
\quad
\begin{array}{cccc}
\mathfrak{g}_{p,q} & \xrightarrow{e_1} & \mathfrak{g}_{p,q+1} & \xrightarrow{e_1} \\
\uparrow e_2 & & \uparrow e_2 & \\
\mathfrak{g}_{p,q} & \xrightarrow{e_1} & \mathfrak{g}_{p+1,q} & \\
\end{array}
\]

where "\( e_i \)" stands for the map given by the \( \text{ad} e_i \)-action.

The duality of the above squares induces a perfect duality between the following two spaces:

\[
\text{Coker}(\text{Ker}_{p-1,q-1}(e_2) \xrightarrow{e_1} \text{Ker}_{p,q-1}(e_2)) \quad \text{Coker}(\text{Ker}_{p,q}(e_1) \xrightarrow{e_2} \text{Ker}_{p,q+1}(e_1)).
\]

The result follows. □

We complete this section with a few numerical identities involving bi-exponents of \( \mathfrak{g} \) relative to \( \mathfrak{e} \).

Proposition 6.13. The following identities hold

\[
\dim \mathfrak{g}^i_+ = \sum_{p,q} p \cdot \dim \mathfrak{z}_{p,q}(\mathfrak{e}) \quad \text{and} \quad \dim \mathfrak{g}^2_+ = \sum_{p,q} q \cdot \dim \mathfrak{z}_{p,q}(\mathfrak{e}).
\]

Proof. Recall isomorphism (2.3). The spaces \( \mathfrak{z}(h_1, e_2), \lim_{p} \mathfrak{z}(h_1, e_2), \) and \( \mathfrak{z}(\mathfrak{e}) \) entering (2.3) are all stable under the \( \text{ad} h_2 \)-action. Further, since \( e_1 \) commutes with \( h_2 \), the limit construction in (2.3), hence the isomorphism itself, commutes with the \( \text{ad} h_2 \)-action. Separating different weight components of \( \text{ad} h_2 \), and using Lemma 2.2(ii) we get an isomorphism:

\[
\text{gr} \mathfrak{z}_{0,q}(h_1, e_2) \xrightarrow{\sim} \mathfrak{z}_{*,q}(\mathfrak{e}) \quad \forall q \in \mathbb{Z}.
\]

Hence, \( \dim \mathfrak{z}_{*,q}(\mathfrak{e}) = \dim(\text{gr} \mathfrak{z}_{0,q}(h_1, e_2)) = \dim \mathfrak{z}_{0,q}(e_2) \), \( \forall q \). Summing up over all \( q \) we obtain: \( \sum_{q \geq 0} q \cdot \dim \mathfrak{z}_{*,q}(\mathfrak{e}) = \sum_{q \geq 0} q \cdot \dim \mathfrak{z}_{0,q}(e_2) \). Clearly, we may concentrate on the terms with \( q > 0 \). Observe that \( \mathfrak{z}_{0,q}(e_2) \) is by definition the \( q \)-eigenspace of the operator \( \text{ad} h_2 \) acting on \( \mathfrak{z}_{0}(e_2) \). If \( q > 0 \) we may replace \( \mathfrak{g}^2 \) here by \( \mathfrak{s}^2 \), and also replace the \( \text{ad} h_2 \)-action by that of \( \text{ad} s_2 \), c.f. Proposition 3.2. By parts (iii)-(iv) of that Proposition, \( e_2 \) is a principal nilpotent in \( \mathfrak{s}^2 \), and the theory of principal \( \mathfrak{s}_2 \)-triples [K1] applied to the Lie algebra \( \mathfrak{s}^2 \) says that one has an identity: \( \sum_{q > 0} q \cdot \dim \mathfrak{z}_{0,q}(e_2) = \dim \mathfrak{g}^2_+ \). The first equation is proved similarly. □
Proposition 6.14. One has:
\[
\prod_{(p,q) \in \text{Exp}_e(g)} \frac{1 - s^{p+1}t^{q+1}}{1 - s \cdot t} = \prod_{\alpha \in R_{\text{nw}}} \frac{1 - s^{(\alpha,h_1)+1}t^{(\alpha,h_2)+1}}{1 - s^{(\alpha,h_1)+1}t^{(\alpha,h_2)} - 1} \cdot \frac{1 - s^{(\alpha,h_1)}t^{(\alpha,h_2)+1}}{1 - s^{(\alpha,h_1)}t^{(\alpha,h_2)} + 1}.
\]

Proof. The LHS of the formula above equals:
\[
\prod_{i,j \geq 0} (1 - s^it^j)^{\dim \mathfrak{z}_{i,j}(e)}.
\]

Put \(g_{i,j} := \dim \mathfrak{z}_{i,j}\). The RHS of the formula above can be rewritten as
\[
\prod_{i,j \geq 0} \left( \frac{(1 - s^{i+1}t^{j+1}) \cdot (1 - s^itj)}{(1 - s^{i+1}t^j) \cdot (1 - s^it^{j+1})} \right)^{g_{i,j}} = \prod_{i,j \geq 0} \frac{(1 - s^it^j)^{g_{i+1,j+1}} \cdot (1 - s^it^j)^{g_{i,j}}}{(1 - s^it^j)^{g_{i+1,j}} \cdot (1 - s^it^j)^{g_{i,j+1}}}.
\]

Theorem 6.4 yields an exact sequence:
\[
0 \to \mathfrak{z}_{i,j}(e) \to \mathfrak{g}_{i,j} \to \mathfrak{g}_{i+1,j} \oplus \mathfrak{g}_{i,j+1} \to \mathfrak{g}_{i+1,j+1} \to 0
\]

Hence, \(\dim \mathfrak{g}_{i,j} - \dim \mathfrak{g}_{i+1,j} - \dim \mathfrak{g}_{i,j+1} + \dim \mathfrak{g}_{i+1,j+1} = \dim \mathfrak{z}_{i,j}(e)\), and the RHS of (6.16) reduces to: \(\prod_{i,j \geq 0} (1 - s^it^j)^{\dim \mathfrak{z}_{i,j}(e)}\), which is (6.15). \(\square\)

The proposition above reduces in the special case \(e = (e,0)\), where \(e \in \mathfrak{g}\) is the principal nilpotent, to the following classical identity, see e.g. \([K1]\):
\[
\prod_{1 \leq i \leq r} \frac{1 - t^{m_i+1}}{1 - t} = \prod_{\alpha \in R_{+}} \frac{1 - t^{\text{height}(\alpha)+1}}{1 - t^{\text{height}(\alpha)}}
\]

where \(m_1, \ldots, m_r, r = \text{rk} \mathfrak{g}\) denote the exponents of \(\mathfrak{g}\).

Further, choose a bi-homogeneous base \(z_1, \ldots, z_r, r = \text{rk} \mathfrak{g}\), of the space \(H_{\text{nw}} := \bigoplus_{p<0,q \geq 0} H_{p,q}\). To each \(i = 1, \ldots, \text{rk} \mathfrak{g}\), we assign the pair \((p_i,q_i) \in \mathbb{Z}^2\) such that \(z_i \in H_{p_i,q_i}(e)\). The subset \(\text{Exp}_e^\text{nw}(g) = \{(p_1,q_1), \ldots, (p_r,q_r), r = \text{rk} \mathfrak{g}\}\) of the second quadrant will be referred to as the collection of higher bieponents of \(\mathfrak{g}\) relative to \(e\). Also, write \(R_{\text{nw}} \subset R\) for the set of those roots of \((\mathfrak{g},\mathfrak{h})\) that occur in the root-decomposition of \(\mathfrak{g}_{\text{nw}} := \bigoplus_{p<0,q \geq 0} \mathfrak{g}_{p,q}\). One can similarly prove the following identity:
\[
\prod_{(p,q) \in \text{Exp}_e^\text{nw}(g)} \frac{1 - s^{p+1}t^{q+1}}{1 - s \cdot t} = \prod_{\alpha \in R_{\text{nw}}} \frac{1 - s^{(\alpha,h_1)+1}t^{(\alpha,h_2)+1}}{1 - s^{(\alpha,h_1)+1}t^{(\alpha,h_2)}} \cdot \frac{1 - s^{(\alpha,h_1)}t^{(\alpha,h_2)+1}}{1 - s^{(\alpha,h_1)}t^{(\alpha,h_2)} + 1}.
\]
7. Partial slices.

Given a set $\Sigma \subset \mathbb{Z}^2$ and a $\mathbb{Z}^2$-graded vector space $a = \bigoplus_{p,q} a_{p,q}$, we will use the notation $a_{\Sigma}$ for $\bigoplus_{p,q \in \Sigma} a_{p,q}$; for example, $a_{(p-1)} = \bigoplus_{p \leq -1, q \in \mathbb{Z}} a_{p,q}$.

Fix a principal nilpotent pair $e = (e_1, e_2)$, an associated semisimple pair $h = (h_1, h_2)$, and let $g = \bigoplus g_{p,q}$ be the corresponding bi-grading. We put $g_- := g_{(p \leq 0)} \oplus g_{(q \leq 0)} \subset g \oplus g$, and consider the affine subspace $e + g_- \subset g \oplus g$.

The following result and its proof are double-analogues of [K2, Lemma 10].

**Lemma 7.1.** The set $\mathcal{Z} \cap (e + g_-)$ consists of regular pairs.

**Proof.** Let $x \in \mathcal{Z} \cap (e + g_-)$. The bi-grading on $g$ gives rise to an increasing bi-filtration: $g_{\leq p,q} := g_{(k \leq p, l \leq q)}$. The bi-filtration on $g$ induces a similar bi-filtration $j(x)_{\leq p,q} := j(x) \cap g_{\leq p,q}$ on the Lie subalgebra $j(x)$. Write $gr g$ and $gr j(x)$ for the corresponding associated bi-graded spaces. It is clear that the natural projection: $g_{\leq p,q} \rightarrow g_{p,q}$ yields an isomorphism $gr_{p,q} g \cong gr_{p,q} j(x)$, hence, induces an imbedding $\sigma : gr_{p,q} j(x) \hookrightarrow gr_{p,q} j(x)$.

We claim that $\sigma (gr_{p,q} j(x)) \subset j(x) (e)$. To see this, for $i = 1, 2$, write $x_i = e_i + a_i$, where $a_1 \in g_{(p \leq 0)}$ and $a_2 \in g_{(q \leq 0)}$. Choose $y \in j(x)_{\leq p,q}$, so that $[x_i, y] = 0$, $i = 1, 2$. Write $y = \sum k \leq p, l \leq q y_{k,l}$, and $[x_i, y] = \sum k, l [x_i, y]_{k,l}$, for the corresponding decompositions into graded components: $y_{k,l}, [x_i, y]_{k,l} \in g_{k,l}$. Then we find:

$$0 = [x_1, y]_{p+1,q} = [e_1 + a_1, y_{p,q}]_{p+1,q} = [e_1, y_{p,q}]_{p,q}.$$

Similarly, computing the $(p,q + 1)$-th component of $[x_2, y]$, we deduce: $0 = [x_2, y]_{p,q+1} = [e_2, y_{p,q}]$. Thus $\sigma (y) = y_{p,q} \in j(x)$, and our claim is proved.

The claim implies that the map $\sigma$ gives an imbedding: $gr j(x) \hookrightarrow j(x)$. It follows that $\dim j(x) = \dim (gr j(x)) \leq \dim j(x) = rk g$. On the other hand, since $x \in \mathcal{Z}$, the Richardson inequality yields $\dim j(x) \geq rk g$, and the lemma follows. \(\square\)

The argument above yields the following result

**Corollary 7.2.** For any $x \in \mathcal{Z} \cap (e + g_-)$, we have a natural isomorphism: $gr j(x) \cong j(x)$. In particular, $j(x) \cap g_{(p \leq 0 \text{ or } q \leq 0)} = 0$.

**Proof.** The first claim is clear. To prove the second, fix $y \in j(x) \cap g_{(p \leq 0 \text{ or } q \leq 0)}$. By our choice of $y$, there exist a pair of integers $p, q$ such that at least one of them is non-positive and such that $\sigma (y) \neq 0$, where $\sigma : gr_{p,q} j(x) \rightarrow g_{p,q}$ is the symbol-map. But we know that $\sigma (gr_{p,q} j(x)) \subset j(x)$. Thus, $\sigma (y) \in j(x)$, which contradicts the "positive quadrant" property: $j(x) \cap g_{(p \leq 0 \text{ or } q \leq 0)} = 0$. \(\square\)

We come to the main point of this section. Recall the notation of Theorem 6.4. For each $p, q$ such that $p < 0 \& q \geq 0$, choose an arbitrary subspace $S_{p,q} \subset j_{p,q-1}(e_2)$ complementary to $\text{Image} (\text{ad} e_1 : j_{p-1,q-1}(e_2) \rightarrow j_{p,q-1}(e_2))$, and form the subspace $S_{nw} := \bigoplus_{p < 0, q \geq 0} S_{p,q} \subset g$. Let $S_{nw} := S_{nw} \oplus \{0\}$ denote the corresponding subspace in $g \oplus g$, and define $S_{nw} := \{0\} \oplus S_{nw} \subset g \oplus g$ similarly. Observe that the affine linear spaces: $e + S_{nw}, e + S_{nw} \subset g \oplus g$ are both contained in $\mathcal{Z}$, the commuting variety. These affine spaces play the role of "partial slices"
to the orbit \( \text{Ad}G(e) \) in \( \mathcal{Z} \). For example, if \( (e, h, f) \) is an \( \mathfrak{s}\mathfrak{l}_2 \)-triple associated to the regular nilpotent \( e \), then for the principal nilpotent pair \( e = (e, 0) \) we have:

\[
e + S_{nw} = (e, \_3\_\_e) \quad \text{and} \quad e + S_{sa} = (e + \_3\_\_e, 0).
\]

Further, assume \( \mathfrak{g} = \mathfrak{s}\mathfrak{l}_n \), and let \( e = e_\lambda \) be the principal nilpotent pair associated to a Young diagram \( \lambda \). Then the spaces \( S_{nw} \) and \( S_{sa} \) can be described as follows. Given a box \((p, q)\) of the diagram \( \lambda \), let \((p, q_{\max})\) denote the top box in the same column as \((p, q)\), and \((q_{\max}, q)\) denote the rightmost box in the same row as \((p, q)\). In the notation of Claim 5.8 set \( \nu(p, q) := (\nu_{in}, \nu_{out}) \), where

\[
\nu_{in} = \{(0, q_{\max}), \ldots, (p, q_{\max})\}, \quad \nu_{out} = \{(p_{\max} - p, q), (p_{\max} - p + 1, q), \ldots, (p_{\max}, q)\}.
\]

Then, one verifies that \( n \) matrices \( \{f_{\nu(p, q)} \mid (p, q) \in \lambda, (p, q) \neq (0, q_{\cop})\} \) form a basis of \( S_{sa} \). Here \((0, q_{\cop})\) stand for the coordinates of the top box in the leftmost column of \( \lambda \); the operator \( f_{\nu(0, q_{\cop})} \) corresponding to this box is a rank one diagonal matrix with trace 1, which is therefore not in \( \mathfrak{s}\mathfrak{l}_n \). Flipping the roles of rows and columns of \( \lambda \) one similarly obtains a basis of \( S_{nw} \).

We will now show that, for an arbitrary semisimple Lie algebra \( \mathfrak{g} \), the "partial slices" have quite remarkable properties, similar to the properties of the standard transversal slice to the regular nilpotent orbit \( \text{Ad}G(e) \subset \mathfrak{g} \), established by Kostant [K1]-[K3]. We restrict our attention to the north-west quadrant and the slice \( S_{nw} \), the situation with \( S_{sa} \) being entirely similar.

We remark first that every pair \( x \in e + S_{nw} \) is regular, due to Lemma 7.1. Furthermore, there is a \( \mathbb{C}^* \)-action on \( e + S_{nw} \) with a single fixed point, \( e \), that contracts the space \( e + S_{nw} \) to this point. Specifically, let \( \gamma : \mathbb{C}^* \to G \) be the homomorphism such that \( \frac{d}{dt} |_{t=1} = h_1 \), see Corollary 3.6(i). Define a \( \mathbb{C}^* \)-action on \( \mathfrak{g} \oplus \mathfrak{g} \) by the formula: \( \mathbb{C}^* \ni t : (x_1, x_2) \mapsto (t \cdot \text{Ad} \gamma(t^{-1}) x_1, \text{Ad} \gamma(t^{-1}) x_2) \). It is clear that this \( \mathbb{C}^* \)-action preserves the commuting variety \( \mathcal{Z} \), and takes any \( \text{Ad}G \)-diagonal orbit in \( \mathfrak{g} \oplus \mathfrak{g} \) into another \( \text{Ad}G \)-diagonal orbit. Furthermore, it keeps the point \( e \) fixed, and contracts the space \( \mathfrak{g}_{(p \leq 0)} \oplus \{e_2\} \) to \((0, e_2)\), hence, contracts \( e + S_{nw} \) to \( e \).

We introduce the nilpotent Lie subalgebra \( \mathfrak{n}_{nw} := \mathfrak{g}_{(p \leq -1, q \geq 0)} \subset \mathfrak{g} \), and write \( N_{nw} \) for the corresponding unipotent group. Further, set

\[
\mathfrak{g}_{nw} = \mathfrak{g}_{nw}^1 \bigoplus \mathfrak{g}_{nw}^2 \subset \mathfrak{g} \bigoplus \mathfrak{g}, \quad \mathfrak{g}_{nw}^1 := \mathfrak{g}_{(p \leq 0, q \geq 0)}; \quad \mathfrak{g}_{nw}^2 := \mathfrak{g}_{(p \leq -1, q \geq 1)}.
\]  

We consider the affine subspace \( e + \mathfrak{g}_{nw} \subset \mathfrak{g} \oplus \mathfrak{g} \), and observe that it is stable under \( \text{Ad}N_{nw} \)-diagonal action. Let \( (\mathcal{Z} \cap (e + \mathfrak{g}_{nw}))_{\text{red}} \) denote the intersection \( \mathcal{Z} \cap (e + \mathfrak{g}_{nw}) \) as an algebraic variety with reduced scheme structure. The following result is a double analogue of [K3, Theorem 1.2]. Note that even in the classical case the proof given below is simpler than the argument in [K3].

**Theorem 7.4.** The action-map gives an isomorphism of algebraic varieties:

\[
N_{nw} \times (e + S_{nw}) \xrightarrow{\sim} (\mathcal{Z} \cap (e + \mathfrak{g}_{nw}))_{\text{red}}.
\]

**Proof.** We know, by Corollary 7.2, that for any point in \( \mathcal{Z} \cap (e + \mathfrak{g}_{nw}) \), the Lie algebra of isotropy group of the \( \text{Ad}N_{nw} \)-action is trivial. It follows, since the
group $N_{\text{nw}}$ is unipotent, that $\text{Ad} N_{\text{nw}}$-diagonal action on $\mathcal{Z} \cap (e + g_{\text{nw}})$ is free. We claim further that the action-map: $N_{\text{nw}} \times (e + S_{\text{nw}}) \rightarrow (\mathcal{Z} \cap (e + g_{\text{nw}}))_{\text{red}}$ is an injective open morphism. To prove injectivity it suffices to show that each $\text{Ad} N_{\text{nw}}$-orbit meets the space $e + S_{\text{nw}}$ in at most one point. This is clear locally near $e$ since the space $e + S_{\text{nw}}$ was chosen to be a transverse slice to the $\text{Ad} N_{\text{nw}}$-orbit of $e$. Hence, the same holds globally on $e + g_{\text{nw}}$, because the $C^*$-action on $g \oplus g$ defined two paragraphs before the theorem preserves the variety $\mathcal{Z} \cap (e + g_{\text{nw}})$, takes $\text{Ad} N_{\text{nw}}$-diagonal orbits into $\text{Ad} N_{\text{nw}}$-diagonal orbits and, moreover, contracts the partial slice $e + S_{\text{nw}}$ to the point $e$. Finally, by definition of a transverse slice (see the geometric meaning of the space $H^1(e, g)$ explained in §6) the morphism: $N_{\text{nw}} \times (e + S_{\text{nw}}) \rightarrow \mathcal{Z} \cap (e + g_{\text{nw}})$ has a surjective differential at the point $(1_{N_{\text{nw}}}, e)$, hence is an open morphism.

Next, consider the affine subspace $e_2 + g_{\text{nw}}^2 = e_2 + g_{(p \leq -1, q \geq 1)} \subset g$. It is clear from (7.3) that: $[n_{\text{nw}}, g_{\text{nw}}^2] \subset g_{\text{nw}}^2$, and that $[n_{\text{nw}}, e_2] \subset g_{\text{nw}}^2$. It follows that the set $e_2 + g_{\text{nw}}^2 \subset g$ is $\text{Ad} N_{\text{nw}}$-stable, see e.g. [CG, Lemma 1.4.12]. Furthermore, the last inclusion is actually an equality, due to the weak Lefschetz. Hence, the $\text{Ad} N_{\text{nw}}$-orbit of $e_2$ is Zariski open in $e_2 + g_{\text{nw}}^2$. The group $N_{\text{nw}}$ being unipotent, the orbit has to be closed, and we conclude that $e_2 + g_{\text{nw}}^2 = \text{Ad} N_{\text{nw}}(e_2)$. Thus, any point of $\mathcal{Z} \cap (e + g_{\text{nw}})$ is $\text{Ad} N_{\text{nw}}$-conjugate to a point of the form $(e_1 + x, e_2)$, where $x \in g_{(p \leq 0, q \geq 1)}$. Note that the condition $q \geq 0$ is superfluous, since $\mathfrak{z}(e_2)_{p, q} = 0$ for all $q < 0$ by the weak Lefschetz.

Consider the Lie algebra $\mathfrak{z}_{\text{nw}}(e_2) := \mathfrak{z}(e_2)_{(p \leq -1)} \subset n_{\text{nw}}$, and write $Z_{\text{nw}}(e_2)$ for the corresponding unipotent group. We have shown in the previous paragraph that the second projection: $(x_1, x_2) \mapsto x_2$ induces, set theoretically, an isomorphism of $N_{\text{nw}}$-equivariant fibrations:

\[
(\mathcal{Z} \cap (e + g_{\text{nw}}) \rightarrow \text{Ad} N_{\text{nw}}(e_2)) \simeq \left( N_{\text{nw}} \times Z_{\text{nw}}(e_2) \frac{(e_1 + \mathfrak{z}_{\text{nw}}(e_2)_{(p \leq 0})}{\mathfrak{n}_{\text{nw}}(e_2)} \rightarrow \frac{N_{\text{nw}}}{Z_{\text{nw}}(e_2)} \right).
\]

(7.6)

The space on the RHS here is an affine bundle over the base $N_{\text{nw}}/Z_{\text{nw}}(e_2)$, which is a smooth connected affine variety. It follows that $(\mathcal{Z} \cap (e + g_{\text{nw}}))_{\text{red}}$ is itself a smooth connected affine variety. Hence, proving the theorem amounts to showing that the action-map: $\varphi : N_{\text{nw}} \times (e + S_{\text{nw}}) \rightarrow \mathcal{Z} \cap (e + g_{\text{nw}})$ is bijective. We already know, by the first paragraph of the proof, that $\varphi$ is an injective morphism with Zariski open image. It is clear that the algebraic variety $N_{\text{nw}} \times (e + S_{\text{nw}})$ is isomorphic to a vector space $\mathbb{C}^k$. The group $N_{\text{nw}}$ being unipotent, the variety $N_{\text{nw}}/Z_{\text{nw}}(e_2)$ is also isomorphic to a vector space $\mathbb{C}^k$. Hence, $\mathcal{Z} \cap (e + g_{\text{nw}})$, the total space of the fibration (7.6), is isomorphic topologically (in fact algebraically) to $\mathbb{C}^k$. Thus, we are reduced to proving the following:

Let $\varphi : U \hookrightarrow X$ be a Zariski open imbedding of irreducible affine algebraic varieties such that both $U$ and $X$ are topologically isomorphic to $\mathbb{C}^k$. Then $U = X$.

To prove this claim we use the standard long exact sequence of Borel-Moore homology, see e.g. [CG, ch.2]:

\[
\ldots \rightarrow H_i^{BM}(X) \xrightarrow{\varphi^*} H_i^{BM}(U) \rightarrow H_i^{BM}(X \setminus U) \rightarrow H_i^{BM}(X) \xrightarrow{\varphi^*} H_i^{BM}(U) \rightarrow \ldots
\]

By assumption, the restriction map $\varphi^*$ here is an isomorphism between two spaces of dimension 1 if $i = 2k$, and of dimension zero otherwise. It follows from the
exact sequence that all Borel-Moore homology groups of $X \setminus U$ vanish. But this contradicts the fact that the algebraic variety $X \setminus U$ has a non-zero fundamental class in $H^{BM}_{\top}(X \setminus U)$, unless $X \setminus U = \emptyset$. □

Next, we study the south-west quadrant, which turns out to be much simpler.

Introduce the Lie subalgebra $n_{sw} := \mathfrak{g}_{(p \leq -1, q \leq -1)}$, and let $N_{sw} \subset G$ denote the corresponding (connected) unipotent subgroup. We also put $\mathfrak{g}_{sw} := \mathfrak{g}_{(p \leq 0, q \leq 0)} \oplus \mathfrak{g}_{(p \leq 0, q \leq 0)} \subset \mathfrak{g} \oplus \mathfrak{g}$.

**Proposition 7.7.** The group $N_{sw}$ acts freely on $Z \cap (e + \mathfrak{g}_{sw})$, and $Z \cap (e + \mathfrak{g}_{sw}) = \text{Ad} N_{sw}(e)$ is a single $\text{Ad} N_{sw}$-diagonal orbit.

**Proof.** Since the group $N_{sw}$ is unipotent, to prove the first claim it suffices to show that, for any $x \in Z \cap (e + \mathfrak{g}_{sw})$, one has: $n_{sw} \cap j(x) = 0$. This follows from Corollary 7.2.

To prove the second claim, we introduce a bi-filtration on $\mathfrak{g}_{sw}$ as follows:

$$\mathfrak{g}_{\leq p,q} := \{(a_1, a_2) \mid a_1 \in \mathfrak{g}_{(i \leq p)}, a_2 \in \mathfrak{g}_{(j \leq q)}\}.$$ 

Let $\sigma : \mathfrak{g}_{\leq p,q} \to \mathfrak{g}_{p,*} \oplus \mathfrak{g}_{*,q}$ denote the corresponding "symbol-map". We will prove by induction on $(p,q)$, where $\mathbb{Z} \oplus \mathbb{Z}$ is viewed as a partially ordered set, that for any $a \in \mathfrak{g}_{\leq p,q}$, such that $x = e + a \in Z \cap (e + \mathfrak{g}_{sw})$, the element $x$ is $\text{Ad} N_{sw}$-conjugate to $e$.

Fix $x = e + a \in Z \cap (e + \mathfrak{g}_{sw})$, where $x_i = e_i + a_i$, and $a \in \mathfrak{g}_{\leq p,q}$. We may assume that $\sigma(a) \neq 0$. Then the leading term of the equation $[x_1, x_2] = 0$ reads:

$$0 = [e_1 + a_1, e_2 + a_2] = [e_1, a_2] + [e_2, a_1].$$

The equation shows that $(a_2, a_1) \in \mathfrak{g} \oplus \mathfrak{g}$ is a cocycle giving a class in $H^1(e, \mathfrak{g})$, see (6.1). Since, $a \in \mathfrak{g}_{sw}$, both $p$ and $q$ are non-positive. Hence, by Theorem 6.4 the corresponding cohomology group vanishes, so that there exists $y \in \mathfrak{g}$ such that $a_i = [e_i, y]$, $i = 1, 2$. Note that since $a \in \mathfrak{g}_{sw}$, the equations $a_i = [e_i, y]$ imply that $y \in \mathfrak{g}_{(p \leq -1, q \leq -1)} = n_{sw}$. Therefore, we get: $\text{Ad exp}(-y)(e) = e + a + e$, where $e$ belongs to lower terms of the bi-filtration on $\mathfrak{g}_{sw}$. Hence, $(\text{Ad exp} y)(e)$ belongs to lower terms of the bi-filtration again, and the induction hypothesis implies: $e - (\text{Ad exp} y)(e) \in \text{Ad} N_{sw}(e)$. Thus we find:

$$\text{(Ad exp } y)(x) = (\text{Ad exp } y)(e + a)$$

$$= (\text{Ad exp } y)(\text{Ad exp}(-y)(e) - e) = e - (\text{Ad exp } y)(e) \in \text{Ad} N_{sw}(e),$$

and the claim follows. □
8. Appendix: TOWARDS A CLASSIFICATION OF PRINCIPAL NILPOTENT PAIRS.

A. Elashvili* and D. Panyushev**

In connection with the preceding article, V. Ginzburg asked us whether there existed non-trivial examples of principal nilpotent pairs in exceptional Lie algebras. We present here a full description of principal nilpotent pairs (= pn-pairs) in the exceptional case and some results towards a complete classification in classical Lie algebras.

As has been observed in section 1 of the main body of the paper, equalities like \( \dim g_{p,q} = \dim g_{-p,-q} \) are typically false for the \( \mathbb{Z}^2 \)-grading associated with a pn-pair. We give first a classification of an interesting class of pn-pairs satisfying these equalities.

Definition. A pn-pair \( e = (e_1, e_2) \in g \times g \) is said to be rectangular if \( e_1 \) and \( e_2 \) can be included in commuting \( \mathfrak{sl}_2 \)-triples \( \{e_1, h_1, f_1\} \) and \( \{e_2, h_2, f_2\} \).

The name is explained by the following observation. For \( g = \mathfrak{sl}_N \), a pn-pair \( e \) is rectangular if and only if the Young diagram of \( e_1 \) (or \( e_2 \)) is a rectangle, see below. In particular, non-trivial rectangular pairs exist if and only if \( N \) is not a prime.

Remarks. 1. It is easy to see that a pn-pair \( e \) is rectangular if and only if there exists an \( \mathfrak{sl}_2 \)-triple \( \{e_1, h_1, f_1\} \) such that \( [e_2, f_1] = 0 \).
2. We assume that \( [h_i, e_i] = 2e_i \), hence the pair \( (h_1, h_2) \) in the rectangular case is twice the associated semisimple pair, in the sense of section 1.
3. Note that if \( \{e_1, h_1, f_1\} \) and \( \{e_2, h_2, f_2\} \) are commuting \( \mathfrak{sl}_2 \)-triples, then condition (ii) of the Definition above is automatically satisfied. That is, such a pair \( (e_1, e_2) \) is principal if and only if \( \dim \mathfrak{z}(e) = \text{rk } g \).

A general description of rectangular pn-pairs can quickly be obtained without using structure theory of pn-pairs developed in sections 1 and 2. To this end, we briefly recall the structure of the centralizer \( \mathfrak{z}(e) \) of a nilpotent element \( e \in g \) (the Dynkin-Kostant theory, see e.g. [CM, ch.4]). Let \( \{e, h, f\} \) be an \( \mathfrak{sl}_2 \)-triple and \( g = \bigoplus_{i \in \mathbb{Z}} g_i \) the corresponding \( \mathbb{Z} \)-grading. Then \( \mathfrak{z}(e) = \bigoplus_{i \geq 0} \mathfrak{z}(e)_i \) and \( \mathfrak{z}(e)_0 \) is a maximal reductive subalgebra in \( \mathfrak{z}(e) \). Moreover, \( \mathfrak{z}(e)_0 = \mathfrak{z}(e, f) = \mathfrak{z}(e, h, f) \). One has \( \mathfrak{z}(e) \simeq g_0 \oplus g_1 \) as \( \mathfrak{z}(e)_0 \)-module. The element \( e \) is called even whenever all the eigenvalues of \( \text{ad } h \) are even, i.e., \( g_i = 0 \) for \( i \) odd. Obviously, \( e \) is even if and only if \( g_1 = 0 \). In this case the weighted Dynkin diagram of \( e \) contains only numbers 0 and 2 [Dy].

Theorem 8.1. The following conditions are equivalent:
(i) \( e \) is a member of a rectangular pn-pair;
(ii) \( e \) is even, and any regular nilpotent element in \( \mathfrak{r} := \mathfrak{z}(e)_0 \) is regular in \( g_0 \) as well.

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Under condition (ii), if $e'$ is any regular nilpotent element in $\mathfrak{k}$, then $(e, e')$ is a pn-pair.

**Proof.** (i) $\Rightarrow$ (ii) For any nilpotent element $e' \in \mathfrak{k}$, we can choose an $\mathfrak{sl}_2$-triple \{e', h', f'\} lying inside $\mathfrak{k}$. Application of $\text{ad} f$ yields an isomorphism of $\mathfrak{k}$-modules $z(e, e') = \mathfrak{g}_0$ and $z(e, e') = \mathfrak{g}_1$. Therefore

$$
\dim z(e, e') = \dim \mathfrak{g}_0(e') + \dim \mathfrak{g}_1(e') \geq \text{rk} \mathfrak{g} + \dim \mathfrak{g}_0(e')
$$

Thus if $\dim z(e, e') = \text{rk} \mathfrak{g}$, then $\mathfrak{g}_0 = 0$ and $e'$ is regular in $\mathfrak{g}_0$. The latter implies that $e'$ is regular in $\mathfrak{k}$, too.

(ii) $\Rightarrow$ (i) Reverse the previous argument. \hfill \Box

As an immediate consequence we obtain the following uniqueness statement in the rectangular case:

**Corollary 8.2.** Given an $\mathfrak{sl}_2$-triple \{e, h, f\} containing the first member of a rectangular pn-pair, the second member is determined uniquely up to conjugacy by an element of the connected group $K = Z_G(e, h, f)$.  

It is likely that a kind of uniqueness holds for arbitrary principal nilpotent pairs. Using theorem 8.1, we find the rectangular pn-pairs in the exceptional simple Lie algebras. The tables in [El1] contain the information on $\mathfrak{g}_0$ and $\mathfrak{k}$ for all nilpotent orbits. To verify condition (ii) of theorem 8.1, we need the description of inclusion $\mathfrak{k} \subset \mathfrak{g}_0$. The latter is determined by the structure of $\mathfrak{g}_2$ as $\mathfrak{g}_0$-module, since $\mathfrak{k}$ is a generic stabilizer. An algorithm for describing $\mathfrak{g}_0$-representation in $\mathfrak{g}_2$ is given in [El1] as well. Usually, a nilpotent orbit in exceptional Lie algebra is denoted by a Cartan label. This label is said to be the type of nilpotent orbit. The idea of such notation goes back to Dynkin [Dy], who studied minimal regular* reductive subalgebras in $\mathfrak{g}$ containing a given simple 3-dimensional subalgebra or, what is the same, a given nilpotent element. The type of an orbit represents one of these subalgebras, namely, a unique, up to conjugation, minimal Levi subalgebra of $\mathfrak{g}$ containing an element of the nilpotent orbit under consideration. We refer to [CM, 8.4] for the tables of nilpotent orbits and the corresponding labels, where some more explanation concerning this notation is found.

**Theorem 8.3.** The following list contains all the rectangular pn-pairs in exceptional Lie algebras and their bi-exponents:

| $\mathfrak{g}$ | $(\mathfrak{b}_1, \tilde{\mathfrak{a}}_2)$ | $(\mathfrak{b}_3, \tilde{\mathfrak{a}}_2)$ |
|----------------|---------------------------------|---------------------------------|
| $\mathfrak{f}_4$ | $\{(1,0), (5,0), (0,1), (2,0)\}$ | $\{(1,0), (5,0), (0,1), (2,0), (3,0)\}$ |
| $\mathfrak{e}_6$ | $\{2\mathfrak{a}_2\}$ | $\{(1,0), (5,0), (0,1), (2,0), (3,0)\}$ |
| $\mathfrak{e}_7$ | $\{(1,0), (5,0), (0,1), (2,1), (3,2), (4,3)\}$ | $\{(1,0), (5,0), (0,1), (2,1), (3,2), (4,3), (5,0)\}$ |
| $\mathfrak{e}_8$ | $\{(1,0), (5,0), (0,1), (2,3), (3,1), (4,1)\}$ | $\{(1,0), (5,0), (0,1), (2,3), (3,1), (4,1)\}$ |

*Proof.** We give a sample of our computations. Consider the nilpotent orbit $O_1 = \text{Ad}G(e_1)$ of type $A_2 + 3A_1$ in $\mathfrak{g} = \mathfrak{e}_7$. The weighted Dynkin diagram of $O_1$ is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

*a subalgebra is called regular if its normalizer in $\mathfrak{g}$ contains a Cartan subalgebra.
Therefore \([\mathfrak{g}_0, \mathfrak{g}_0] \simeq \mathfrak{sl}_7 = \mathfrak{sl}(\mathcal{V})\) and \(\mathfrak{sl}_7\)-module \(\mathfrak{g}_2\) is isomorphic to \(\wedge^3 \mathcal{V}\). Then \(\mathfrak{k} \simeq \mathbf{G}_2\), and the embedding \(\mathbf{G}_2 \hookrightarrow \mathfrak{g}_2\) corresponds to the unique 7-dimensional representation of \(\mathbf{G}_2\). It is easy to show that the restriction of this representation to a principal \(\mathfrak{sl}_2\) in \(\mathbf{G}_2\) yields a 7-dimensional irreducible representation of this \(\mathfrak{sl}_2\) (the single Jordan block). This precisely means that a (any) regular nilpotent element in \(\mathfrak{g}_2\) is still regular in \(\mathfrak{sl}_7\). It remains to determine the type of \(O_2 = \text{Ad}G(e_2)\), where \(e_2\) is a regular nilpotent element in \(\mathbf{G}_2\). Since \(e_2\) is also regular in \(\mathfrak{g}_2\), we see that \(\mathfrak{g}_0\) is a minimal Levi subalgebra intersecting \(O_2\). Thus \(O_2\) is of type \(A_6\). Therefore the weighted Dynkin diagram of \(O_2\) is 
\[
\begin{array}{c}
0 \\
0 \quad 2 \quad 0 \quad 2 \quad 0 \quad 0
\end{array}
\]

In each of these cases, it is not hard to determine the structure of \(\mathfrak{g}\) as \(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2\)-module and then find the bi-exponents of the \(pn\)-pair. Indeed, let \(R(d)\) denote the irreducible \(\mathfrak{sl}_2\)-module of dimension \(d + 1\) and let \(\mathfrak{g}|_{\mathfrak{sl}_2 \oplus \mathfrak{sl}_2} = \bigoplus_{i=1}^{rk \mathfrak{g}} R(n_i) \otimes R(m_i)\). Since nilpotent elements in both \(\mathfrak{sl}_2\)-triples are even, the integers \(n_i,m_i\) are even. Then the bi-exponents are \(\{(n_i/2, m_i/2) \mid i = 1, \ldots, rk \mathfrak{g}\}\).

Theorem 8.1 applies to the classical Lie algebras as well, but in this case it is easier to obtain the classification of \(pn\)-pairs in another way. Namely, we exploit a simple relationship between the simplest and the adjoint representation of a classical Lie algebra. Let \(\mathfrak{g} = \mathfrak{g}(\mathcal{V})\) be a classical Lie algebra, \(\mathcal{V}\) being its tautological representation. Then
\[
\mathfrak{g} \simeq \mathcal{V} \otimes \mathcal{V}^* \oplus \{\text{triv. 1-dim repr.}\} \quad \text{for} \quad \mathfrak{sl}(\mathcal{V}),
\]
\[
\mathfrak{g} \simeq S^2(\mathcal{V}) \quad \text{for} \quad \mathfrak{sp}(\mathcal{V}),
\]
\[
\mathfrak{g} \simeq \wedge^2(\mathcal{V}) \quad \text{for} \quad \mathfrak{so}(\mathcal{V}).
\]

Consider the subalgebra \(\mathfrak{s} = '\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \subset \mathfrak{g}(\mathcal{V})\), where \('\mathfrak{sl}_2 = \langle e_1, h_1, f_1 \rangle\) and \("\mathfrak{sl}_2 = \langle e_2, h_2, f_2 \rangle\) and the corresponding \(\mathfrak{sl}_2\)-triples commute. We wish to determine those \(\mathfrak{s}\) that correspond to principal nilpotent pairs of rectangular type. By the very definition, it is equivalent to that \(\mathfrak{g}(\mathcal{V})\) as \(\mathfrak{s}\)-module is a sum of exactly \(rk \mathfrak{g}\) irreducible submodules. Therefore we assume that
\[
\mathcal{V} |_{\mathfrak{s}} = \bigoplus_{i=1}^{p} R(n_i - 1) \otimes R(m_i - 1) \tag{8.4}
\]

and then compute the decomposition \(\mathfrak{g} |_{\mathfrak{s}}\) making use of the above 3 relations and variations of the Clebsch-Gordan formula. Of course, appropriate constraints on parity of \(n_i,m_i\) should be satisfied in the orthogonal and symplectic cases. An advantage of this approach is that the decomposition \(\mathcal{V} |_{\mathfrak{s}}\) immediately yields the description of \(e_1,e_2\) in terms of partitions, i.e., if \(n_1 \geq n_2 \geq \ldots\), then \(e_1\) corresponds to the partition \((\underbrace{n_1,\ldots,n_1}_m,\underbrace{n_2,\ldots,n_2}_m,\ldots)\) and likewise for \(e_2\). Note that \(\dim \mathcal{V} = \sum n_i m_i\). The output of our computations is as follows:

**Theorem 8.5.** Suppose the embedding \(\mathfrak{s} \hookrightarrow \mathfrak{g}(\mathcal{V})\) is given by (8.4). Then \((e_1,e_2)\) is a (rectangular) \(pn\)-pair in \(\mathfrak{g}(\mathcal{V})\) if and only if the set \(\mathcal{I} = \{(n_i,m_i) \mid i = 1,\ldots,p\}\) is of the form

(i) for \(\mathfrak{sl}(\mathcal{V})\): \(p = 1\);

(ii) for \(\mathfrak{sp}(\mathcal{V})\): \(p = 1\) and \(n_1,m_1\) have different parity;

(iii) for \(\mathfrak{so}(\mathcal{V})\): either \(p = 1\) and \(n_1,m_1\) have the same parity or
$p = 2$ with $\mathcal{J} = \{(n,m), (1,1)\}$ or $\mathcal{J} = \{(n,1), (1,m)\}$, where $n,m$ are both odd.

This means in particular that we obtain a single rectangle for $\mathfrak{sl}(V)$ and $\mathfrak{sp}(V)$ and at most 2 rectangles for $\mathfrak{so}(V)$. Let us give an explicit presentation of the respective $pn$-pair for the most interesting (last) case. Let $e_1$ (resp. $e_2$) be a regular nilpotent element in $\mathfrak{so}_n$ (resp. $\mathfrak{so}_m$), where $n,m$ are odd. Consider the embeddings $\mathfrak{so}_n \times \mathfrak{so}_m \hookrightarrow \mathfrak{so}_{nm} \hookrightarrow \mathfrak{so}_{nm+1}$ (the first one corresponds to the tensor product and the second one is natural) and $\mathfrak{so}_n \times \mathfrak{so}_m \hookrightarrow \mathfrak{so}_{n+m}$ (the direct sum). Then the image of $(e_1,e_2)$ under these embeddings is a rectangular $pn$-pair.

As for the classification of arbitrary $pn$-pairs, one may use the following approach. Results in section 3 concerning ‘associated Levi subalgebras’ shows that if $e$ is a $pn$-pair, then the respective pair of nilpotent orbits $(O_1,O_2)$ is reciprocal, see Definition 4.10. So, the initial step is to describe all reciprocal pairs in the simple Lie algebras. Making use of the explicit description of the induced orbits given in [El2], one easily finds the pairs of reciprocal orbits in the exceptional Lie algebras. Of course, the pairs of orbits indicated in Theorem 8.3 are reciprocal. The total number of non-trivial pairs of reciprocal orbits in $G_2$, $F_4$, $E_6$, $E_7$, $E_8$ is 0, 2, 4, 7, 5 respectively. For instance, the five reciprocal pairs in $E_8$ correspond to the following 5 pairs of Levi subalgebras:

$$(E_6, D_4), (A_6, D_4+A_2), (D_5, D_5), (A_6+A_1, A_4+A_2+A_1), (D_5+A_2, A_4+A_2).$$

Below we give the complete description of $pn$-pairs in the exceptional simple Lie algebras.

**Theorem 8.6.**
1. There are no $pn$-pairs in $G_2$. The rectangular $pn$-pair of theorem 8.3 is the unique, up to conjugacy, $pn$-pair in $F_4$.
2. There are four $AdG$-orbits of $pn$-pairs in $E_6$. The three orbits of non-rectangular $pn$-pairs correspond to the following reciprocal pairs of orbits, where the last column gives the bi-exponents:

$$(D_5, 2A_1) \quad \{(1,0), (4,0), (5,0), (7,0), (0,1), (3,1)\}$$

$$(A_4+A_1, A_2+2A_1) \quad \{(1,0), (4,0), (0,1), (2,1), (3,1), (1,2)\}$$

$$(A_4, A_3) \quad \{(1,0), (3,0), (4,0), (0,1), (0,3), (2,2)\} .$$

3. There are five $AdG$-orbits of $pn$-pairs in $E_7$. The unique orbit of non-rectangular $pn$-pairs corresponds to the following reciprocal pairs of orbits:

$$(A_4+A_1, A_4+A_1) \quad \{(1,0), (0,1), (0,4), (1,3), (2,2), (3,1), (4,0)\} .$$

4. The rectangular $pn$-pair of theorem 8.3 is the unique, up to conjugation, $pn$-pair in $E_8$.

**Proof.** All the proofs are based on explicit calculations with centralizers. Having a suitable candidate $e_1$ for the first member of $pn$-pair, we try to select an $e_2 \in \mathfrak{j}(e_1)$ in order to meet all the requirements of Definition 1.1. To establish whether some commutators in $\mathfrak{j}(e_1)$ are equal or not equal to zero, we have used computer program GAP [GP].

1. For $F_4$, we need only to demonstrate that the reciprocal pair of orbits $(C_3, A_2)$ do not produce a $pn$-pair. If $e_1$ lies in the orbit of type $C_3$, then $\dim \mathfrak{j}(e_1) = 10$ and $\mathfrak{j}(e_1)_0$ is a 1-dimensional toral subalgebra. Therefore a nonzero $h \in \mathfrak{j}(e_1)_0$ is, up to conjugation, the only possible candidate for $h_2$, the second
member of an associated semisimple pair. However, explicit computations show that \( \dim \mathfrak{z}(e_1, e_2) \geq 6 \) for any \( \text{ad} h \)-weight vector \( e_2 \in \mathfrak{z}(e_1)_{\geq 0} \).

2. For \( \mathbf{E}_6 \), we show that each reciprocal pair of orbits yields a unique \( pn \)-pair. For instance, consider an element \( e_1 \) in the orbit of type \( D_5 \). Then \( \dim \mathfrak{z}(e_1) = 10 \) and \( \dim \mathfrak{z}(e_1)_{i} \) is equal to \( 1, 1, 2, 1, 1, 3, 0, 1 \) for \( i = 0, 2, 4, 6, 8, 10, 12, 14 \) respectively, \( \mathfrak{g}_{14} \) being the greatest nonzero subspace in the \( \mathbb{Z} \)-grading defined by \( \text{ad} h_1 \). These data are easily derived from the weighted Dynkin diagram of \( e_1 \). Therefore \( \mathfrak{z}(e_1)_{0} \) is a 1-dimensional toral algebra and \( \mathfrak{z}(e_1)_{2} = \langle e_1 \rangle \). The weights of \( \mathfrak{z}(e_1)_{0} \) in \( \mathfrak{z}(e_1)_{4} \) are nonzero (and opposite) and we take either of weight vectors as \( e_2 \). Then a straightforward computation shows that \( \dim \mathfrak{z}(e_1, e_2)_{i} \) is equal to \( 0, 1, 1, 0, 1, 2, 0, 1 \) for \( i = 0, 2, 4, 6, 8, 10, 12, 14 \) respectively. Let \( t \) be a nonzero element in \( \mathfrak{z}(e_1)_{0} \) normalized so that \( [t, e_2] = e_2 \). Then the elements \( \tilde{h}_1 = h_1/2 - 2t \) and \( \tilde{h}_2 = t \) satisfy the commutator relations \( [\tilde{h}_i, e_j] = \delta_{i,j}e_j \) and \( [\tilde{h}_1, \tilde{h}_2] = 0 \). Hence \( (\tilde{h}_1, \tilde{h}_2) \) is an associated semisimple pair in the sense of section 1 and \( e_1, e_2 \) is a \( pn \)-pair.

Since the eigenvalues of \( \text{ad} h_1 \) are known, it only suffices to compute the eigenvalues of \( \text{ad} t \) on \( \mathfrak{z}(e_1, e_2) \) in order to determine the bi-exponents in this case.

3.4. Because calculations for \( \mathbf{E}_7, \mathbf{E}_8 \) are not illuminating, too, we omit them.

\[ \square \]

**Remark.** It could have happened, a priori, that some reciprocal pair \( (O_1, O_2) \) would give rise to several \( \text{Ad} G \)-orbits of \( pn \)-pairs. We see, as a result of straightforward calculations, that this never happens.

Finally, we give an example of non-rectangular \( pn \)-pair in a classical Lie algebra. Let \( \mathfrak{g} = \mathfrak{so}(\mathbb{V}) = \mathfrak{D}_{2n+1} \). Take the reciprocal pair of orbits corresponding to the partitions \( (2n+1, 2n+1) \) and \( (2, \ldots, 2, 1, 1) \). This pair of orbits really produces a \( pn \)-pair. An explicit matrix presentation of \( (e_1, e_2) \) is the following. Let \( v_1, \ldots, v_{4n+2} \) be a basis of \( \mathbb{V} \) such that the \( \mathfrak{g} \)-invariant quadratic form is \( x_1x_{4n+2} + \cdots + x_{2n+1}x_{2n+2} \). Then \( e_1 \) acts on the basis vectors by \( e_1(v_j) = v_{j-1} \) \( (j \geq 2n + 3) \), \( e_1(v_j) = -v_{j-1} \) \( (2 \leq j \leq 2n + 1) \), \( e_1(v_1) = e_1(v_{2n+2}) = 0 \); \( e_2 \) acts by \( e_2(v_j) = (-1)^{j}v_{2n+2-j} \) \( (j \geq 2n + 3) \), \( e_2(v_j) = 0 \) \( (j \leq 2n + 2) \). The simultaneous centralizer \( \mathfrak{z}(e_1, e_2) \) has the basis consisting of the following matrices: \( e_1, e_1^2, \ldots, e_1^{2n-1}, e_2, e_1^2e_2, \ldots, e_1^{2n-2}e_2, x \). Here \( x \) is the operator sending \( v_{4n+2} \) to \( v_{2n+2} \) and \( v_{2n+1} \) to \( -v_1 \). We leave it to the reader as an exercise to write down an associated semisimple pair and then the bi-exponents in this case.

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