THE UNIVERSAL GERBE, DIXMIER-DOUADY CLASS, AND GAUGE THEORY

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ABSTRACT. We clarify the relation between the Dixmier-Douady class on the space of self adjoint Fredholm operators (‘universal B-field’) and the curvature of determinant bundles over infinite-dimensional Grassmannians. In particular, in the case of Dirac type operators on a three dimensional compact manifold we obtain a simple and explicit expression for both forms.

0. INTRODUCTION.

Gerbes arise naturally in quantum field theory (QFT) in several different ways [Br] [CMM1]. The most recent manifestation is in string theory where it has been realised that the so-called B-field (a family of local 2-forms \{B_i\} each defined on an open set U_i in a cover of a space-time manifold M) actually defines the Dixmier-Douady (DD) class of a gerbe [FW], [K], [BM], [HM]. The DD class is an element of $H^3(M, \mathbb{Z})$ (Cech cohomology). When this class is not torsion it may also be given by a closed de Rham form $H_3$ defined locally on $U_i$ by $dB_i = H_3|_{U_i}$. In this letter we will be concerned only with the non-torsion case.

Now it is well known that classes in $H^3(M, \mathbb{Z})$ classify principal bundles over $M$ with fibre the projective unitary group of a complex Hilbert space $H$. Let $P$ be such a principal bundle over $M$ with DD class represented by a de Rham form $[H_3]$. In this situation $[H_3]$ has the interpretation of an obstruction to the existence of a prolongation of $P$ to a bundle with fibre the group of unitaries, $U(H)$, on $H$. This situation has recently been discussed in a string theory context, see e.g. [HM], where it was suggested that the B-field, can be obtained from a formula involving a connection on the bundle $P$ and its curvature form. However, it is difficult to...
make the proposed formula mathematically rigorous since it involves taking traces of manifestly non-traceclass operators.

Similar considerations have arisen in other contexts in standard quantum field theory constructions when chiral fermions are coupled to external fields [Se], [CMM], [CM] and these provide some of the ideas for the present paper.

In the present article we suggest how the construction in [HM] (which involves a path integral) of a connection on $P$ and its curvature should be replaced by a trace of a characteristic class constructed from a Bismut-Freed type superconnection. In addition, we point out an interesting relation between the 3-form DD class and the curvature form of the basic complex line bundle over Grassmannians modelled by $L_{p^+}$ Schatten ideals. (This latter line bundle is familiar from the study of anomalies in gauge QFT where the curvature gives the so-called commutator anomaly of the gauge symmetry group. It is discussed in [CMM], [MR])

We view the DD class on a given space-time manifold $M$ as being pulled back from a universal DD class initially defined on the space of self adjoint Fredholm operators. Our main result is to give an explicit formula for this universal class using a Bismut-Freed superconnection. However to obtain this formula we need to exploit an observation of Atiyah and Singer and replace the space of self adjoint Fredholm operators by the unitary operators which differ from the identity by a compact operator. In fact, we further replace compact operators by $L_{p^+}$ operators in order to be able to write explicit formulae for de Rham forms as traces of powers of a curvature form taking values in a Lie algebra of linear operators in a Hilbert space.

The basic problem with the proposal in [HM] is that the group $PU(H)$ is too large for a geometric analysis of this problem. Actually, QFT gives more structure. Elements of $PU(H)$, interpreted as Bogoliubov automorphisms of the fermion algebra, cannot in general be lifted to a fermionic Fock space. Instead, it is known that an operator $g$ on $H$ can be lifted to the fermionic Fock space over $H$ if and only if it belongs to the restricted unitary group on $H$, denoted $U_{res}$ (for details see [PS]). The restricted unitaries act as projective unitaries in the Fock space suggesting that $U_{res}$ may provide a substitute for $PU(H)$. Note that the connection with anomalies in gauge theories is via the fact that the group 2-cocycle on $U_{res}$ given by this projective action is infinitesimally just the commutator anomaly in the Lie algebra (Schwinger term).

The discussion starts in Section 1 with a suitable choice of the space carrying the universal Dixmier Douady class (this space is homotopy equivalent to the set of self adjoint Fredholm operators which are neither essentially positive nor essentially negative). Our choice is dictated by the existence for this space of a simple expression for the generator of degree three de Rham cohomology. Our earlier work [CM] provides a way to connect with quantum field theory in a geometric fashion bringing in the group $U_{res}$ and noting that the universal DD class is the DD class of the so-called ‘lifting bundle gerbe’ for $U_{res}$.

In Section 2 we define a Bismut-Freed type superconnection whose curvature gives a formula for this universal DD class. The Bismut-Freed formula gives the de
Rham form of the DD class on the base of a fiber bundle through an integration over fibers of a Chern class in the total space (which in turn can be written as a Dixmier trace of an appropriate operator form). This should be viewed as a replacement for the above mentioned ill-defined path integral formula. A derivation of the Dixmier-Douady class from the Bismut-Freed form of the families index theorem was discussed recently in [Lo]; in the case of Dirac operators coupled to vector potentials or metrics his results agree with the earlier computations in [CMM], [EM]. Since in the present paper we are studying a special family of Dirac type operators we obtain more explicit formulas.

In Section 3 we specialise to Dirac operators on odd dimensional manifolds (and 3-manifolds in particular) where we can calculate more explicit formulæ for this DD class using suitable renormalised traces.
1. A UNIVERSAL GAUGE GROUP.

Let $\mathcal{F}_*$ be the set of all self-adjoint Fredholm operators in an infinite-dimensional complex Hilbert space $H$, with both positive and negative essential spectrum. According to [AS] the space $\mathcal{F}_*$ is homotopy equivalent to the set of unitaries $g$ of the type $-1+$ a compact operator. The homotopy equivalence is constructed in several steps. The effect of the first stage is that $\mathcal{F}_*$ is a retract of self-adjoint operators with essential spectrum at $\pm 1$. (Actually, Atiyah and Singer talk about skew self-adjoint operators with essential spectrum $\pm i$.) In the last step such an operator $F$ is mapped to the unitary operator $g = \exp(i\pi F)$ with essential spectrum at $-1$. We prefer to use the parametrization $g = -\exp(i\pi F)$ and thus $g$ belongs to the group $G_c$ consisting of unitaries such that $g - 1$ is compact.

In this paper we shall restrict to the subspace of Dirac type Fredholm operators $\mathcal{F}_D$ and we can define the homotopy equivalence in a slightly more direct way. An unbounded self-adjoint operator $D$ is Dirac type if $|D|^{-p}$ is trace-class for some $p \geq 1$. The infimum of such $p$ is called the dimension of the underlying (noncommutative) space, [Co]. Let $D_0 \in \mathcal{F}_D$ be fixed and $D_A = D_0 + A$ a bounded perturbation of $D_0$. A generalized gauge transformation is a unitary map $g : H \to H$ such that $[D_0, g]$ is bounded. Then $g^{-1}D_A g = D_0 + A'$ is another bounded perturbation with the gauge transformed potential $A' = g^{-1}Ag + g^{-1}[D_0, g]$.

We can modify the construction of [AS], replacing $G_c$ by the group $G$ of unitaries differing from the identity by a trace class operator. (Note that Palais showed that $G$ and $G_c$ are homotopy equivalent, see [Q].) In this modification we have now a map $A \mapsto g_A \in G$ given by

$$A \mapsto D_A \mapsto F_A = D_A/\langle |D_A| + \beta(|D_A|) \rangle \mapsto g_A = -\exp(i\pi F_A),$$

where $\beta$ is any positive smooth function with $\beta(0) = 1$ and $\beta(x) \to 0$ faster than $|x|^{-p}$ as $x \to \infty$. The map from $\mathcal{F}_*$ to $G$ is essentially the same as the restriction of the homotopy equivalence in [AS] to Dirac type operators.

The drawback with the composite map described above is that it is not gauge covariant; it does not define a map from the space of gauge orbits $\mathcal{A}/\mathcal{G}$ to $G$, not even in the case when the space of perturbations $A$ is the space of smooth vector potentials $\mathcal{A}$ and $\mathcal{G}$ is the group of (based) gauge transformations. Next we shall explain a modification which gives a gauge covariant map.

Let us specialize to the case of smooth vector potentials $A = A^i\gamma_i$ on a connected compact spin manifold $M$ and smooth based gauge transformation $g \in \mathcal{G}$. Here the $\gamma_i$s are the Dirac gamma matrices with respect to a fixed Riemannian metric on $M$, $1 \leq i \leq \dim M$. We have $g_{A'} = g^{-1}g_A g$ for any gauge transformation $g$ and therefore we cannot use $g_A$ to define a map $\mathcal{A}/\mathcal{G} \to G$. Let $X = \mathcal{A}/\mathcal{G}$, the moduli space of gauge connections; this has the structure of a (Frechet) manifold because $\mathcal{G}$ acts freely on $\mathcal{A}$. We can view $\mathcal{A}$ as the total space of a principal $\mathcal{G}$ bundle over $X$. We can define an associated principal $U(H)$ bundle $Z$ over $X$ with total space $\mathcal{A} \times_{\mathcal{G}} U(H)$, where $\mathcal{G}$ acts naturally from the right on gauge potentials and from
the left on $U(H)$ through the embedding $G \subset U(H)$ via the gauge group action on square-integrable fermion fields.

Since $U(H)$ is contractible we can define a global section $X \to Z$, as a map $r : \mathcal{A} \to U(H)$ such that $r(A^2) = g^{-1}r(A)$ and $r(0) = 1$. Set next $\tilde{g}_A = r(A)^{-1}g_A r(A)$. This map is gauge invariant, giving a map $\tilde{g} : X \to G$.

### 1.1 A universal gerbe.

In [CM] we constructed a universal $\mathcal{U}_{\text{res}}$ bundle over the base $G$. The group $\mathcal{U}_{\text{res}} = \mathcal{U}_{\text{res}}(H,\epsilon)$ is defined as the group of unitaries in a Hilbert space $H$ such that the off-diagonal blocks with respect to a polarization $\epsilon$ are Hilbert-Schmidt. The construction was motivated by quantum field theoretic considerations. In particular, a family $X$ of self-adjoint Dirac operators on a manifold $M$ can be considered as a subset $X \subset G$ through the homotopy equivalence described above. To each point $A \in X$ one wants to associate a fermionic Fock space $V_A$ such that the representation of the canonical anticommutation relations algebra (CAR) in $V_A$ is equivalent to a representation defined by the ‘Dirac sea’ construction, that is, by the polarization of the 1-particle space $H$ to positive and negative energies with respect to the Dirac hamiltonian $D_A$. Since the group $\mathcal{U}_{\text{res}}$ acts only projectively in a fixed Fock space, through a central extension

$$1 \to S^1 \to \hat{\mathcal{U}}_{\text{res}} \to \mathcal{U}_{\text{res}} \to 1,$$

the problem of constructing Fock spaces parametrized by elements of $X$ reduces to the problem of prolonging a principal $\mathcal{U}_{\text{res}}$ bundle over $X$ to a principal $\hat{\mathcal{U}}_{\text{res}}$ bundle. The potential obstruction to this is the cohomology class on $X$ obtained as the pull-back of a basic 3-form on $G$, with respect to the embedding $X \to G$.

The basic 3-form $c_3$ on $G$ is equal to the Wess-Zumino-Witten-Novikov (WZWN) 3-form

$$c_3 = \frac{1}{24\pi^2} \text{tr} (dgg^{-1})^3. \tag{1}$$

The form is normalized such that it gives the generator of $H^3(G,\mathbb{Z})$. The form $c_3$ is the obstruction to prolonging the universal $\mathcal{U}_{\text{res}}$ bundle over $G$ to a $\hat{\mathcal{U}}_{\text{res}}$ bundle. For this reason it is the Dixmier-Douady class of the universal gerbe.

Thus we may pull back the three form $c_3$ on $G$ to give a three form on $\mathcal{A}/G$. The cohomology class of the pull-back does not depend on the choice of $r$. To see this consider a three dimensional cycle $\Sigma \subset X$ as a 3-disk $D^3 \subset \mathcal{A}$ such that the points on the boundary are gauge related. Let $h_t(A)$ be the deformation of $\tilde{g} : D^3 \to G$ given by $h_t(A) = r(tA)^{-1}g_A(t)r(tA)^{-1}$, where $g_A(t)$ is defined by

$$F_A(t) = (D_A + (t - 1)\lambda)/(|D_A + (t - 1)\lambda| + t\beta(|D_A|)) \tag{2}$$

and $\lambda$ is any real number not in the spectrum of $D_A$ for $A$ in the boundary of $D^3$ (this is a well defined condition as the $D_A$ for $A$ in the boundary are all gauge equivalent). Then $h_1(A) = \tilde{g}_A$ and $h_0(A) = -\exp(i\pi(D_A - \lambda)/|D_A - \lambda|)$. The integral of $h_0^*c_3$ does not depend anymore on $r$. 
The explicit formula for \( \tilde{g}^c_3 \) does depend on \( r \), and is complicated to evaluate. However, it is possible to build a ‘universal model’ with concrete formulas for the basic differential forms. The rest of this note is devoted to calculating these formulae.

In gauge theory based on the group \( U(N) \) the group of gauge transformations \( \text{Map}_0(M, U(N)) \) on an odd dimensional sphere \( M = S^k \) has the homotopy type of \( \Omega G \) in the limit \( N \to \infty \). Here \( \Omega G \) is the group of based smooth loops \( f : S^1 \to G \), \( f(1) \) is the neutral element in \( G \). Let us replace the gauge group \( G \) by \( \Omega G \) and the contractible space \( A \) by another contractible space \( PG \), the space of smooth maps \( f : [0, 2\pi] \to G \) such that \( f(0) = 1 \) and \( f^{-1}df \) is periodic at the end points. Each such \( f \) defines a vector potential \( A = f^{-1}df \) on the circle \( S^1 \). Conversely, each periodic vector potential \( A \) defines \( f \) uniquely by the given initial condition and the ordinary differential equation \( df = f \cdot A \).

The projection \( \pi : PG \to G \) is the evaluation \( f \mapsto f(2\pi) \). The fiber is \( \Omega G \). The pull-back \( \pi^*c_3 \) can be written as \( \pi^*c_3 = d\theta \) with

\[
\theta_f(X, Y) = \int_0^{2\pi} c_3(f^{-1}df, X, Y) = \frac{1}{8\pi^2} \int_0^{2\pi} \text{tr} (f^{-1}df)[X(t), Y(t)].
\]

The group \( \Omega G \) can be embedded in the restricted unitary group \( U_{res} = U_{res}(\mathcal{H}, \epsilon) \) in the Hilbert space \( \mathcal{H} = L^2(S^1, H) \), \([PS][CM]\). Here the polarization \( \epsilon \) is the sign of the (discrete) momentum on the circle (see [PS] for this construction). The embedding is defined by a point-wise action on \( H \)-valued functions and is a homotopy equivalence (as shown in [CM]). The group \( U_{res} \) is also homotopy equivalent to the larger group \( U_{cpt} \) defined by the condition \([\epsilon, g]\) is compact. The fact that it can be considered as the large \( N \) limit of the group of based \( U(N) \) valued gauge transformations over an odd dimensional sphere is an important motivation in the ‘universal gauge theory’ of [Ra].

As shown in [CMM] the form \( \theta \) defines an extension of the Lie algebra of infinitesimal gauge transformations (by the so called Schwinger terms). The modified commutator is defined as

\[
[X, Y]' = [X, Y] + 2\pi i\theta_f(X, Y),
\]

that is, as a sum of the pointwise commutator of the loop algebra elements \( X, Y \) and a scalar function of \( f \). Thus the extension is defined by the abelian ideal of functions of \( f \). The cocycle \( \theta \) is equal, modulo a coboundary, to the cocycle

\[
c(X, Y) = \frac{i}{2\pi} \int \text{tr} X dY.
\]

\( c \) is the cocycle defining an affine Lie algebra. The coboundary relating the two cocycles is \( \delta w \) with \( w_f(X) = \frac{i}{4\pi} \int \text{tr} AX \).

There is a family of local closed 2-forms on \( \mathcal{A} \) which gives an alternative description of the gerbe arising from the DD class \( c_3 \). First, near the unit element in \( G \) we can write \( c_3 = d\psi \) with the parametrization \( g = -e^{i\pi F} \),

\[
\psi = \frac{1}{8} \text{tr} dF h(\text{ad}_{i\pi F})dF;
\]
where \( h(x) = \frac{\sinh(x) - x}{\pi^2} \) and \( ad_X(Y) = [X, Y] \). The local form \( \eta = \theta - \pi^*\psi \) is closed since \( d \) and the pull-back operation commute. Along gauge orbits the forms \( \eta \) and \( \theta \) are the same. However, one must remember that \( \psi \) is defined only in the open set where the exponential function is bijective whereas \( \theta \) is globally well-defined on \( \mathcal{A} \). We can cover \( \mathcal{A} \) with open sets \( V_\alpha = \pi^{-1}(U_\alpha) \), where each \( U_\alpha \) is of the type \( U_\alpha = g_\alpha U_1 \) and \( U_1 \) is an open neighborhood of the identity in \( G \). Each \( U_\alpha \) can be parametrized as \( g_\alpha e^{i\pi F} \) for \( F \in \text{Lie}(G) \). In this parametrization the local form \( \psi_\alpha \) on \( G \) has the same expression as before, near the identity. The curvature of the local line bundles \( L_{\alpha\beta} \) on the overlaps \( V_{\alpha\beta} \) is given as \( \frac{1}{2\pi} \omega_{\alpha\beta} = \pi^*(\psi_\alpha) - \pi^*(\psi_\beta) \) and is closed, again because \( d \) and the pull-back operation commute. Note however that although \( \omega_{\alpha\beta} \) can be pushed forward to \( G \), it is not a difference of closed forms on \( G \). On \( \mathcal{A} \) we have \( \omega_{\alpha\beta} = \omega_\alpha - \omega_\beta \) as a difference of closed forms with \( \frac{1}{2\pi} \omega_\alpha = \theta - \pi^*(\psi_\alpha) \). This reflects the fact that on \( \mathcal{A} \) the gerbe is necessarily trivial (but not down on \( G \)) which means that \( L_{\alpha\beta} = L_\alpha \otimes L_\beta^* \) for a family of local line bundles \( L_\alpha \) over the open sets \( V_\alpha \).

2. CONNECTION IN A UNIVERSAL BUNDLE AND THE DD CLASS.

In this section we use the superconnection formalism of [BF] to give a formula for the universal DD class. Let \( G \subset U(H) \) as before and let \( \pi : P \to G \) be a principal bundle over \( G \) with structure group \( \Omega G \) and total space \( P \) the space of smooth paths in \( G \) starting from the unit element and such that \( f^{-1}df \) is periodic at the end points \( x = 0, 2\pi \) of the paths \( f(x) \). The projection map \( \pi \) is \( f \mapsto f(2\pi) \).

The action of \( \Omega G \) on \( P \) is the pointwise right multiplication, \( (f \cdot g)(x) = f(x)g(x) \).

As before, each element \( f \in P \) can be viewed as a smooth vector potential on the circle \( S^1 \) (which is parametrized as \( e^{ix} \)) of the form \( A = f^{-1}df \).

We define a connection on \( P \). For this purpose choose any smooth real valued function \( \alpha \) on the interval \( [0, 2\pi] \) such that \( \alpha(0) = 0, \alpha(2\pi) = 1 \), and all the derivatives of \( \alpha \) vanish at the end points. The connection is defined as a \( \text{Lie}(\Omega G) \) valued 1-form \( \omega \) on \( P \),

\[
(7) \quad \omega = f^{-1}\delta f - \alpha(x)f(x)^{-1} (\delta f(2\pi)f(2\pi)^{-1}) f(x),
\]

where the exterior differentiation on \( P \) is denoted by \( \delta \) in contrast to the differentiation \( d \) on the circle.

One immediately checks that \( \omega \) is indeed a connection form: along vertical directions defined by the right \( \Omega G \) action it is tautological, \( \omega(X) = X \) for \( X \in \Omega g \), and it is equivariant, \( r_g \omega = g^{-1}\omega g \) for any \( g \in \Omega G \).

Let \( Q \) be the \( G \) bundle over \( S^1 \times G \) defined as follows. Start from \( \hat{Q} = S^1 \times P \times G \) viewed as a trivial \( G \) bundle over \( S^1 \times P \). The group \( \Omega G \) acts freely from the right as

\[
(8) \quad (x, f, a) \cdot g = (x, fg, g(x)^{-1}a).
\]
Thus we may pass to the quotient $Q = \tilde{Q}/\Omega G$ and this is a (nontrivial) $G$ bundle over $S^1 \times G$ since the right actions of $G$ and $\Omega G$ commute.

We can extend the connection $\omega$ to a connection in $Q$. It is a sum of two terms,

$$\omega = \omega^{(1,0)} + \omega^{(0,1)}$$

where the first term is a 1-form along $S^1$ given as $f^{-1}df$ in terms of a local section $G \to P$ which sends $a \in G$ to $f = f_a \in P$ with $f_a(2\pi) = a$. The second term is given by the formula (7) above, with respect to the same local section.

The curvature is easily computed to be $F = F^{(2,0)} + F^{(1,1)} + F^{(0,2)}$ with $F^{(2,0)} = 0$ (since dim$S^1 = 1$), and

$$F^{(1,1)} = -d\alpha f^{-1} (\delta f(2\pi)f(2\pi)^{-1}) f$$

$$F^{(0,2)} = (\alpha^2 - \alpha)f^{-1}(\delta f(2\pi)f(2\pi)^{-1})^2 f.$$  (9)

Following [BF] the odd Chern classes on the base $G$ of the bundle $P$ are then obtained by integrating along the fiber $S^1$ the ordinary even Chern classes in the base $S^1 \times G$ of the bundle $Q$. For example, the 3-form part is given as

$$c_3 = \frac{1}{8\pi^2} \int_{S^1} \text{tr} (F^2)^{(1,3)} = \frac{1}{4\pi^2} \int_{S^1} \text{tr} (d\alpha)(\alpha^2 - \alpha)f(x)^{-1}(\delta f(2\pi)f(2\pi)^{-1})^3 f(x)$$

$$= \frac{1}{24\pi^2} \text{tr} (\delta f(2\pi)f(2\pi)^{-1})^3$$  (10)

which is the canonical 3-form $\frac{1}{24\pi^2} \text{tr} (d\alpha a a^{-1})^3$ on the group $G$. The higher forms

$$\text{tr} (d\alpha a a^{-1})^{2k+1}$$

are obtained in exactly the same way, by starting from the higher Chern classes $\text{tr} F^{k+1}$.

The (1,1) form part of the curvature can be also written as

$$F^{(1,1)} = [D_f, \delta + \omega]$$

where $D_f$ is the (antihermitean) Dirac operator on $S^1$, $D_f = \frac{d}{dx} + A_f$, with $A_f = f(x)^{-1}f'(x)$. The three form can be written as

$$c_3 = \frac{1}{8\pi^2} \cdot 2\pi \text{tr}_+ \frac{1}{|D|} (F^2)^{(1,3)}.$$  (11)

Here $\text{tr}_+$ is the Dixmier trace. Note that all components of the curvature are bounded operators in the space $L^2(S^1,H)$.

This way of writing the 3-form leads immediately to a generalization. Let $p$ be an odd positive integer. Assume that $|D|^{-p}$ is in the ‘Dixmier ideal’ $L_{1+}$, with $D$ a linear operator in a dense domain in a Hilbert space $H$. (In the previous example
We assume that we have a bigraded differential algebra \( \Omega^{(*,*)} \) with a differential \( \Delta = d + \delta \), where \( \delta \) is the de Rham differential in the base space as before, increasing the second index by one unit, and \( d \) is a generalized differentiation in the sense of noncommutative geometry (NCG), anticommuting with \( \delta \). The \( k \)-forms in the \( d \) complex are constructed as linear combinations of operators of the form \( \phi = a_0 da_1 \ldots da_k \), where \( da = [D, a] \) and the \( a_i \)'s are elements in an associative algebra of bounded operators such that the commutators \([D, a_i]\) are bounded, [Co]. Naively, the exterior derivative of \( \phi \) is equal to \( da_0 da_1 \ldots da_k \).

However, there is an ambiguity in the expression for \( \phi \), as a differential polynomial in the algebra elements \( a_i \), and therefore one has to pass to an appropriate quotient algebra; see [Co], Section VI.1, for details.

Next we assume that \( \omega = \omega^{(1,0)} + \omega^{(0,1)} \) is a graded 1-form on the base of the type described above and we denote by \( \mathcal{F} \) its graded curvature form. For any positive odd \( k \) we have a closed form on the base defined by

\[
(12) \quad c_k = \frac{1}{n!(2\pi)^n b_p \text{tr}_+} \frac{1}{|D|^p} (\mathcal{F}^m)^{(p,k)},
\]

where \( 2n = p + k \) and \( b_p = (2\pi)^p/\text{vol}(S^{p-1}) \) is a normalization constant; the normalization is chosen such that \( b_p \text{tr}_+ \frac{h}{|D|^p} = \int_M h(x) dx \) when \( h \) is a multiplication operator by a smooth function in the spin bundle over the compact manifold \( M \) of dimension \( p = 2m + 1 \).

2.1 A Basic Example. Replace \( G \) above by \( \mathcal{G} = \text{Map}_0(S^{2m}, G) \), the group of based maps. For a topologist, \( \mathcal{G} \) is the same as \( G \) but for the following geometric analysis there is a difference. Set \( P = \text{Map}_0(B^{2m+1}, G) \), where \( B^{2m+1} \) is a disk with boundary \( S^{2m} \). The maps are based on a radius connecting the origin of the disk to the base point on \( S^{2m} \). We can view \( P \) as a principal bundle over \( \mathcal{G} \), the projection is the restriction onto the boundary and the fiber is equal to the group of based maps from \( S^{2m+1} \) to \( G \); we identify functions on the disk \( B^{2n+1} \) which are constant on the boundary as functions on \( S^{2n+1} \). This bundle has a connection defined as above,

\[
(13) \quad \omega = f^{-1} \delta f - \alpha(r) f^{-1} \left( \delta f(r = 1) f(1)^{-1} \right) f,
\]

where \( \alpha \) is a smooth function of the radius \( r \) alone and otherwise it satisfies the same conditions as in the 1-dimensional case. There is a superconnection defined as \( A = A^{(1,0)} + A^{(0,1)} \) where \( A^{(0,1)} = \omega \) and \( A^{(1,0)} \) is a \( \text{Lie}(G) \) valued vector potential on \( S^{2m+1} \) defined by

\[
(14) \quad A^{(1,0)} = f^{-1} df - \alpha(r) f^{-1} \left( df(1) f(1)^{-1} \right) f.
\]

Here \( d \) is just the classical de Rham differentiation. It is a special case of the NCG differential operator \( d \); for example, a vector potential \( A^i dx_i \) corresponds to the operator \( A^i \gamma_i \), which can be written as a linear combination of operators \( a_0[D, a_1] \).
where $a_0, a_1$ are multiplication operators by smooth functions and $D$ is the Dirac operator determined by the standard metric on the disk $B^{2m+1}$.

Thus the superconnection can be written as

\begin{equation}
A = f^{-1} \Delta f - \alpha(r)f^{-1}(\Delta f(1)f(1)^{-1})f,
\end{equation}

where we denote $\Delta = d + \delta$. The curvature of this superconnection is

\begin{equation}
F = -d\alpha f^{-1}(\Delta f(1)f(1)^{-1})f + (\alpha^2 - \alpha)f^{-1}(\Delta f(1)f(1)^{-1})^2f.
\end{equation}

It can be decomposed as $F = F^{(2,0)} + F^{(1,1)} + F^{(0,2)}$. The Chern character is generated by the powers $\text{tr} F^k$ and the forms on the base $G$ of the bundle $P$ are given as

\begin{equation}
c_k = \frac{1}{n!(2\pi)^n} \int_{B^{2m+1}} \text{tr} (F^n)^{(2m+1,k)},
\end{equation}

with $2n = 2m+1+k$. In fact, by Stokes’ theorem the integral reduces to a boundary integral over $S^{2m}$, since the integrand is of the form

\[
d\alpha(\alpha^2 - \alpha)^{n-1}\text{tr}(\Delta f(1)f(1)^{-1})^{2n-1}
\]

and the radial integration gives just the numerical factor

\[
\int_0^1 \alpha'(r)(\alpha^2 - \alpha)^{n-1}dr.
\]

Thus

\begin{equation}
c_k = N_{k,m} \int_{S^{2m}} \text{tr} ((\Delta f(1)f(1)^{-1})^{2n-1})^{(2m,k)},
\end{equation}

where $N_{k,m}$ is a normalization constant. In the case $m = 0$ the integration reduces to a calculation of $f(1)$ in a single point and we recover the formula in the case of the universal bundle over $G$.

We observe that the integration in eq. (17) can be replaced, up to a normalization constant, by the Dixmier trace $\text{tr}_+ \frac{1}{|D|^p}(F^n)^{(2m+1,k)}$ with $p = 2m+1$. Thus eq. (17) is indeed a special case of (12).
3. AN APPLICATION TO GAUGE GROUPS AND GERBES IN ODD DIMENSIONS.

Let \( \mathcal{A} \) be the space of \( g \) valued vector potentials on a compact spin manifold \( M \) of dimension \( p = 2k + 1 \). Let \( D_0 \) be a fixed Dirac operator (defined, for example, with the vector potential \( A = 0 \)). Denote \( D_A = D_0 + A \) with \( A = i \sum \gamma^k A_k \) for \( A \in \mathcal{A} \). Denote \( F_A = D_A(D_A^2 + m^2)^{-1/2} \) where \( m \) is a positive constant. Then \( F_A = \epsilon \in L_{p+} \), where \( \epsilon = D_0/|D_0| \); if zero is an eigenvalue of \( D_0 \) we put it on the positive side in the spectrum. To each \( D_A \) we can associate the unitary operator \( g_A = -\exp i\pi F_A \). We claim that this operator differs from the unit operator by a perturbation which is in the ideal \( L_{p+} \) consisting of operators \( T \) with \( |T|^p \in L_{1+} \). To see this we introduce \( \epsilon_A = D_A/|D_A| \), with the same convention for zero eigenvalues as for \( \epsilon \), and note that \( g_A = \exp i\pi \epsilon_A(|F_A| - 1) \) so that it suffices to prove that \( |F_A| - 1 \in L_{p+} \). Using \( |F_A| - 1 = (1 + |F_A|)^{-1}(F_A^2 - 1) \) and \( F_A^2 - 1 = -m^2 \cdot (D_A^2 + m^2)^{-1} \) we have

\[
(D_A^2 + m^2)^{-1} = D_A^{-2} - m^2(D_A^2 + m^2)^{-1}D_A^{-2}
\]

and the RHS clearly lies in \( L_{p+} \) so that \( |F_A| - 1 \) lies in \( L_{p+} \).

Thus in this setting we need to extend the forms \( c_j \) in the previous section to the the group \( U^{(p+)} \) of unitaries \( g \) such that \( g - 1 \in L_{p+} \). (Note that if \( U^{(p+)} \) is equipped with the Banach Lie group structure arising from the norm on \( L_{p+} \) it is not homotopy equivalent to \( G \) as the former is not separable. This will not be an issue in our discussion.) For that purpose we use the method explained in [LMR]. Replace \( g \) by a \( 2 \times 2 \) operator matrix and define the operators \( \epsilon, \Gamma \),

\[
g \mapsto \tilde{g} = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Replace the variation \( \delta \) by \( \pm \Gamma \delta \) and denote \( d\tilde{g} = [\sigma, \tilde{g}] \). We have once again a NCG bigraded differential algebra and a total differential \( \Delta = d + \delta \), where now the first differential \( d \) is defined as \( d\tilde{g} = [\sigma, \tilde{g}] \) for 0-forms, and in general \( d\phi = d\alpha_0 d\alpha_1 \ldots d\alpha_j \) for \( \phi = a_0 a_1 \ldots a_j \) for j-forms \( \phi \). Note that in the present setting \( d\phi = [\sigma, \phi] \) for even degree forms and \( d\phi = \sigma \phi + \phi \sigma \) for odd forms. One must select the sign convention in the definition of \( \delta \) on odd/even forms such that \( d, \delta \) anticommute, giving \( \Delta^2 = 0 \).

For any odd positive integers \( j \) and \( q = 2l + 1 > p \) we can now set

\[
(19) \quad c_{j,q} = -\left(\frac{1}{2\pi i}\right)^{l+1} \frac{l!}{(2l+1)!} \text{tr} \Gamma ((\tilde{g}^{-1} \Delta \tilde{g})^q)^{(q-j,l)}
\]

and this is a closed degree \( j \)-form on \( U^{(p+)} \). For example, for \( j = 1 \) this expression is, up to a constant, equal to

\[
\text{tr}(g + g^{-1} - 2)^{(q-1)/2} g^{-1} \delta g.
\]
In the case of even $q$ the trace vanishes identically. In the case of $g - 1 \in L_1$ we can set $q = 1$ and we get the standard expression $\frac{1}{2\pi i} \text{tr}(g^{-1}\delta g)$ for the 1-form on $G$. When $p = j$ we have $c_j \sim \text{tr}(g^{-1}\delta g)^j$, which is logarithmically diverging but can be replaced by a renormalized trace, see below.

The case of $q > j > 1$ gives a slightly more complicated expression. The reason is that the operators $\tilde{g}^{-1}\delta \tilde{g}$ and $\tilde{g}^{-1}d\tilde{g}$ do not commute and the trace has to be computed separately for all possible combinations of products of these operators. However, for pseudodifferential operators (PSDO’s) on a compact manifold the borderline case $q = j = 2k + 1$ still gives a nice formula,

\begin{equation}
(20) \quad c_{p,p} = -\left(\frac{1}{2\pi i}\right)^k \frac{k!}{(2k + 1)!} \text{TR}(g^{-1}\delta g)^p,
\end{equation}

where the renormalized trace TR is defined as follows. Fix a positive PSDO $Q$ of order one. Then

\begin{equation}
(21) \quad \zeta(z; T) = \text{tr}(Q^{-z}T)
\end{equation}

is defined and analytic when Re$(z)$ is large and positive. For a classical PSDO $T$ the trace at $z = 0$ has at most a simple pole; subtracting the pole one gets a finite expression and this is defined as TR$T$. (In physics literature this is called the dimensional regularization.) The following lemma is proven in [CDMP]:

**Lemma.** If $A, B$ is a pair of PSDO’s on a compact manifold $M$ such that the order of $AB$ is less or equal to $-\dim M$ then $\text{TR}[A, B] = 0$.

When $g = -\exp(i\pi F_A)$ we can define a 1-parameter family

\begin{equation}
(22) \quad g_t = -\exp(i\pi(1 - t)\epsilon + i\pi tF_A)
\end{equation}

in the group $U^{(p+)}$. As in the proof of Poincare’s lemma we can define a form $q_{j-1,p}$ such that $dq_{j-1,p} = c_{j,p}$ by the formula

\begin{equation}
(23) \quad q_{j-1,p} = \int_0^1 i_{\pi(F_A - \epsilon)}c_{j,p}(g_t)dt,
\end{equation}

where $i_X$ is the interior product of a vector field $X$ with a differential form.

In the case of Dirac operators coupled to vector potentials on a compact manifold $M$ of dimension 3 we can be a bit more explicit. In this case $p = 3$ and using the above lemma we can set $j = 3$. By (6) we have now $c_3 = c_{3,3} = d\psi$; note that in the present case $F_A - \epsilon \in L_{3+}$. In particular, if $F^2 = 1$ the form $\psi$ reduces to

\begin{equation}
(24) \quad \psi = \frac{1}{16\pi} \text{TR} F(\delta F)(\delta F), \text{ for } F^2 = 1.
\end{equation}

This is the curvature form of the determinant line bundle over the $L_{3+}$ Grassman-nian. It gives the Schwinger terms when evaluated along gauge directions, [MR],
[LM]. Here gauge transformations are of the form $F \mapsto aFa^{-1}$, with $a$ unitary. Denoting by $X, Y$ a pair of infinitesimal gauge transformations the Schwinger term becomes

\begin{equation}
(25) \quad s(F; X, Y) = \frac{1}{4} \text{TR} F[X, F][Y, F].
\end{equation}

Up to a coboundary, this is the same as in [MR],

\begin{equation}
(26) \quad s'(F; X, Y) = \frac{1}{8} \text{TR}(F - \epsilon)[[\epsilon, X], [\epsilon, Y]].
\end{equation}

In the case when $F - \epsilon$ is trace-class the difference $s - s'$ is equal to the coboundary of the Lie algebra 1-cochain

$$
\eta(X; F) = \frac{1}{2} \text{tr}\{(\epsilon - F)X + \frac{1}{4} F[\epsilon, X]\}.
$$

One can derive the cohomology classes $c_{j,p}$ on the base $U^{(p+)}$ as in the case of trace class perturbations of the unit operator, Section 2.

We can define a graded 1-form $A$ on the path space $\mathcal{P}$ of $U^{(p+)}$. As before, we consider paths $f(x)$ (with $0 \leq x \leq 2\pi$) starting from $f(0) = 1$ and with periodic boundary conditions on $f^{-1}df$. The form $A$ is defined as

$$
A^{(1,0)} = \tilde{f}^{-1} \tilde{d}\tilde{f} - \alpha(x)\tilde{f}^{-1} \left( [\sigma, \tilde{f}(2\pi)]\tilde{f}(2\pi)^{-1} \right) \tilde{f},
$$

$$
A^{(0,1)} = \tilde{f}^{-1} \delta \tilde{f} - \alpha(x)\tilde{f}^{-1} \left( \delta \tilde{f}(2\pi)\tilde{f}(2\pi)^{-1} \right) \tilde{f}.
$$

Here $\tilde{d}\tilde{f}(x) = \tilde{f}'(x)dx + [\sigma, \tilde{f}(x)]$. For higher order $(\sigma, \delta)$ forms one must associate a sign $s = \pm 1$ with the symbol $dx$ in order to guarantee that the various differentials anticommute. Note that $A^{(1,0)}$ can be further split into a form $\tilde{f}^{-1}\tilde{f}'dx$ along the circle parametrized by $x$, and the Fredholm module form

$$
\tilde{f}^{-1}[\sigma, \tilde{f}] - \tilde{f}^{-1} \left( [\sigma, \tilde{f}(2\pi)]\tilde{f}(2\pi)^{-1} \right) \tilde{f}.
$$

The forms $\text{tr} \mathcal{F}^n$, where $n/2 > p$ and $\mathcal{F}$ is the curvature of the connection $A$, can be 'integrated' to give odd forms on the base. Integration is to be understood in the NCG sense: It involves a true integration along the circle and an operator trace in the algebra of 2 by 2 operator matrices; the entries are operators in the Hilbert space $\hat{H}$ where the action of $U^{(p+)}$ is defined. The $x-$ integration is easily performed as in the section 2 to give the formula

\begin{equation}
(27) \quad c_{j,2n-1} = -\left(\frac{1}{2\pi i}\right)^n \frac{(n-1)!}{(2n-1)!} \text{tr} (\tilde{f}(2\pi)^{-1} \Delta \tilde{f}(2\pi))^{(2n-1-j,j)}
\end{equation}
which agrees with (19), with \( \tilde{g} \) replaced by \( \tilde{f}(2\pi) \) and \( \Delta g = \delta g + [\sigma, g] \) and \( q = 2n-1 \); in the case of PSDO’s on a compact manifold we can go down to the borderline case \( q = p \) by replacing the trace ‘tr’ by the renormalized trace ‘TR’.

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