Coordinates Distributions in Finite Uniformly Random Networks

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ABSTRACT: This work introduces the concept of angular distance distributions of the nodes in a Binomial point process-based model for wireless networks. The need for the derivation of these distributions is motivated by the use of high-frequency bands in the mmWave or even in the sub-THz bands. These frequencies are closely related to the beamforming capabilities of the transceivers in wireless networks. Therefore, the directional characteristics of the beams utilized by the communicating nodes, along with the random location of the latter, lead to an increasing demand for the distributions derived in this paper. A direct consequence of the Binomial process utilized to model the random location of a fixed number of nodes in a ball, is that the extracted distributions are based on the generalized beta distribution. The presented results are applicable to two- and three-dimensional networks and can be used to study various network characteristics among which are the interference, and the beam management.

INDEX TERMS: Binomial point process, finite wireless networks, stochastic geometry, mmWave, THz networks, beamforming.

I. INTRODUCTION
The wireless communication networks continuously expand to higher frequency bands and greater bandwidth. This trend stems from the demand for increased network capacity, average throughput per user, and area spectral efficiency [1], [2]. The availability of multiple antenna elements either at the Base Stations (BSs) or even at the User Equipment (UE), apart from the benefits of spatial multiplexing and diversity that provide increased throughput and reliability, allows for the application of beamforming techniques. The main advantage of beamforming is the compensation of large propagation loss due to the resulting beamforming or array gain, and the boost provided to the signal-to-interference-plus-noise ratio (SINR) by the inherent ability to suppress interference. Although the beamforming techniques are well applied to sub-6 GHz bands offering directional beams in uplink and downlink of mobile communication systems, the most important application area seems to be the millimeter wave (mmWave) bands, i.e., up to a frequency of roughly 100 GHz, and the sub-THz bands, i.e., frequencies from 100 GHz to roughly 330 GHz. Some of these high-frequency bands have been already considered in many commercial wireless systems including the 5G New Radio (NR) [3], the IEEE Std. 802.15.3c originally for wireless transmission of high-definition video in wireless area personal networks (WPANs) [4], in IEEE Std. 802.15.3d for point-to-point links with data rates of 100Gbit/s and higher at distances ranging from tens of centimeters up to a few hundred meters [5], and in the IEEE Std. 802.11ay for wireless local area networks (WLANs) [6].

The extreme bandwidth that is available in mmWave and sub-THz bands comes with challenging propagation characteristics. The first is the increased path loss with tens of dB extra margin required with respect to the sub-6 GHz band. Next, is the penetration loss that makes the blockage effects severe, and finally, it is the negligible diffraction propagation mechanism that significantly differentiates the line-of-sight (LOS) and the non-line-of-sight (NLOS) cases in these bands [7]. Nevertheless, due to the very small wavelength, the number of antenna elements that can be packed in small transceiver profiles is very large, and thus, highly directional antenna beams can be formed to address the aforementioned propagation issues.

Stochastic geometry tools have been used for the performance evaluation of wireless networks since they capture the spatial randomness in the geometry of network elements and usually lead to tractable analytical results [8]–[11]. The latter is a characteristic of the homogeneous Poisson point process (PPP), a model that is a good approximation for networks with known node density [12]. However, the PPP cannot be used to model finite wireless networks with a fixed and
finite number of nodes in a specific area. In such cases a suitable model is the Binomial point process (BPP) [12], [13]. The authors in [12] have investigated the distribution of the Euclidean internode distances in a finite uniformly random network confined in a ball of arbitrary dimensions and modeled as a BPP. The probability density function (PDF) was found to follow the generalized beta distribution, and this result has been used extensively in the literature for the study of various wireless network characteristics, such as energy consumption, interference, outage, and in applications like routing and node localization.

Meanwhile, stochastic geometry has been extensively utilized for the performance analysis of mmWave and sub-THz networks [14]–[21]. In [14], the authors proposed a general framework to evaluate the coverage and rate performance in mmWave cellular networks. Although the framework incorporates directional beamforming, the analysis is mainly distance-dependent. In [15], the coverage probability and the average rate of mmWave communication systems was analyzed, and the role of beam scanning in the directional cell discovery was highlighted. In [16], the authors study the co-channel interference in the THz band, and approximate the sum of interference as a gamma distribution. Moreover, they investigate the single-interferer approximation as a special case. Nevertheless, the angular coordinates have not been considered in the presented analysis. In [17], the authors proposed two inter-cell interference coordination schemes for mmWave cellular networks. A flat-top antenna pattern was assumed, and the angle of arrival (AoA) and/or departure (AoD) that determines the antenna gain, were uniformly distributed. In [18], the performance of a large intelligent surface-assisted mmWave network was investigated under a two-step user association policy. The uniform distribution was also considered for all the AoD and the AoA.

Stochastic geometry concepts have also been utilized in aerial networks. In [19], the authors modeled the spatial distribution of LEO satellite communication systems, and derived analytical expressions for the CDFs of the nearest neighbor and the contact distance in binomial point processes spatially-distributed on concentric spherical surfaces. In [20], the authors consider a finite network of unmanned aerial vehicles (UAVs), and derive the downlink coverage probability of a reference receiver based on the distribution of distances from the reference receiver to the serving and interfering nodes. The authors introduce a coverage probability approximation using dominant interferer-based approach. In [21], performance analysis was conducted in a realistic binomial UAV 5G network, modeled as a truncated octahedron. The analysis is again distance-dependent only. In [22], performance analysis in terms of coverage probability in a hybrid aerial-terrestrial network with a finite number of backhaul-enabled UAVs, was conducted. The mmWave backhaul links were subjected to possible beamforming misalignment errors, whereas a flat-top antenna pattern was used. In [23], the performance of a three-dimensional, two-hop cellular network with both terrestrial BSs and UAVs serving ground users was investigated. Although the authors consider realistic antenna patterns for both BSs and UAVs, they assume either a static direction of the antennas’ mainlobes, or an angle chosen uniformly at random.

The utilization of beamforming techniques and the corresponding directional beams in 5G and beyond networks, has introduced the need for statistical characterization of all the node coordinates in two (2D) or even in three dimensions (3D). The distributions of the angular coordinates in BPP wireless networks now play an important role in many network characteristics and operations, such as the beam alignment and beam management, the interference within the mainlobe or the sidelobe of the antenna array pattern, the node localization, and the corresponding network performance metrics affected by the directional characteristics of the links. Therefore, to the author’s best knowledge, this is the first time that the distributions of the distance from a reference direction in space to the nth nearest point in angles φ and θ are investigated. Along with the distance distributions in a 2D or 3D ball of specific radius R, given in [12], the angular distance distributions provide a full directional characterization of the spatial location of the nodes.

The main contribution of this paper is to propose a stochastic geometry framework to study the PDF, the cumulative distribution function (CDF), and the moments of angular internode distances in BPP-based networks. Initially, the 2D case is examined where the nodes are uniformly randomly distributed in a disk of radius R, and the distribution for the polar coordinate azimuth angle φ is derived using the concept of the circular or disk sector. Next, the analysis is extended to 3D networks, where the nodes are uniformly randomly distributed in a 3D ball of radius R, and the distribution for the spherical coordinate φ is derived using the concept of spherical wedge, whereas the spherical sector is utilized for the coordinate θ. Then, the conditional PDFs for the angular distances are derived, conditioned on the distance of the kth node from the reference direction. Finally, a discussion on the application of the derived results to the design and characteristics of wireless networks is provided, along with a comparison of diverse scenarios for the calculation of the received power from the dominant interferer in a 2D cellular network.

This paper is organized as follows. Section II introduces the framework for the derivation of the internode angular distance distributions in two and three dimensions. Moreover, the joint distance distributions in all coordinates are discussed. Section III provides the moments of the distances, whereas in Section IV the conditional distributions are derived. Finally, in Section V an extensive discussion of indicative application of the presented results in wireless networks is given.

II. ANGULAR DISTANCE INTERNODE DISTRIBUTIONS
The locations of the network nodes are modeled as a uniform BPP Φ, where a fixed number of nodes N are independently
uniformly distributed in a compact set $W \subset \mathbb{R}^d$, where $d$ is the dimension of the space. In the following, it is assumed that $W = b_d(\alpha, R)$, i.e., the BPP is isotropic, and the set $W$ is a $d$-dimensional ball of radius $R$ centered at the origin. The Lebesgue measure of this set is $|W| = R^d \pi^{d/2}/\Gamma(d/2 + 1)$, where $\Gamma(\cdot)$ is the gamma function. For any set $V \subset \mathbb{R}^d$, the number of points in $V$, i.e., $\Phi(V)$, is binomial $(n, p)$ with parameters $n = N$ and $p = |V \cap W|/|W|$. Moreover, assume that the set $V \subseteq W$.

The investigation of the distributions of the angular internode distances is based on the definition of the $n$-th nearest point in angle $\phi$ and/or in angle $\theta$ from a reference line, and in coordinate $r$ from a reference point that is assumed to be the origin. Depending on the variable for which we wish to evaluate the distribution of the distance to the $n$th nearest neighbor, we define a different set $V$. For the distance $r$, the set $V$ is either a disk, if $d = 2$, or a sphere if $d = 3$. Therefore, the probability $p$ is given by $p = (r/R)^d$. For the angle $\phi$ the set $V$ is either a circular sector of a disk of radius $R$, if $d = 2$, as shown in Fig. 1, or a spherical wedge of a sphere of radius $R$ with a dihedral angle of wedge $\phi$, as shown in Fig. 2. The area of the disk sector is $|V| = \phi R^2/2$ and $p = |V|/|W| = \phi/2\pi$. The volume of the spherical wedge is $|V| = \frac{\phi}{2} \phi R^3$ and $p = |V|/|W| = \phi/2\pi$, i.e., the probability is the same for $d = 2$ and $d = 3$. For the angle $\theta$ the set $V$ exists only for $d = 3$ and this is the spherical sector of a sphere of radius $R$, as shown in Fig. 3. The volume of the sector is $|V| = \frac{4\pi}{3} R^3 \left(1 - \cos \frac{\alpha}{2}\right) = \frac{4\pi}{3} R^3 \sin^2 \left(\frac{\alpha}{2}\right)$, and $p = \frac{1 - \cos \frac{\alpha}{2}}{2} = \sin^2 \left(\frac{\alpha}{2}\right)$. If one limits the radius of the disk or the sphere for the circular sector and the spherical wedge, respectively, to $r \in [0, R]$, the azimuth angle to $\phi$, and the elevation angle to $\theta$, then the probability $p$ for $d = 2$ is $p = \frac{\phi}{2\pi} R^2$, and for $d = 3$ is $p = \frac{\phi}{2\pi} R^3 \frac{1 - \cos \alpha}{2}$.

In the following, we are interested in the absolute angular distance, $|\phi|$, for the coordinate $\phi$ from a reference line and since $\phi \in [-\pi, \pi]$, we define $|\phi| \in [0, \pi]$, as shown in Fig. 1 and Fig. 2. Therefore, $p = \frac{|\phi|}{\pi}$. This assumption is quite useful when dealing with the angular distance from the direction of the maximum directivity in an antenna beampattern.

**A. DISTRIBUTIONS IN 2D NETWORKS**

Let $|\phi_n|$ denote the random variable (r.v.) representing the angular distance from an arbitrary reference line to the $n$th nearest node of the BPP in the coordinate $|\phi|$. Without loss of generality one may assume that the reference line is the axis $x$, as shown in Fig. 1. The complementary CDF of $|\phi_n|$ is the probability that there are less than $n$ points in $V(x, |\phi_n|)$, i.e., in the corresponding set, and is given by

$$
\hat{F}_{|\phi_n|}(|\phi|) = \sum_{k=0}^{n-1} \binom{N}{k} p^k (1-p)^{N-k} = 1 - p^{N-n+1}, \quad 0 \leq |\phi| \leq \pi,
$$

where $p = |V \cap W|/|W|$, and $I_z(a, b)$ is the regularized incomplete beta function for which it holds that

$$
I_z(a, b) = \int_0^z t^{a-1} (1-t)^{b-1} dt \frac{B(a, b)}{B(a, b)} = 1 - I_{1-z}(b, a),
$$
where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function. The PDF of the angular distance is given by

$$f_{\phi_n}(\phi) = -\frac{dF_{\phi_n}(\phi)}{d\phi} = \frac{dp^{n-1}(1-p)^{N-n}}{d\phi} B(n, N-n+1)$$

$$= \frac{1}{\pi} p^{n-1}(1-p)^{N-n}$$

$$= \frac{1}{\pi} \frac{\phi}{n} B(n, N-n+1)$$

where in (a) the Leibniz integral rule has been used, and $\beta(\cdot, \cdot, \cdot)$ is the beta density function. According to [12] the Euclidean distance $R_n$ from the origin to its $n$th nearest node follows a generalized beta distribution

$$f_{R_n}(r) = \frac{2}{\pi} \frac{\Gamma(n+\frac{1}{2})\Gamma(N+1)}{\Gamma(n)\Gamma(N+\frac{1}{2})}$$

$$\times \beta\left(\frac{r^2}{R^2}; n + \frac{1}{2}, N-n+1\right). \quad (4)$$

**B. DISTRIBUTIONS IN 3D NETWORKS**

In a 3D BPP network the distributions that should be calculated refer to all spherical coordinates $(r, \theta, \phi)$. The PDF of the angular distance in $|\phi|$ is again given by (3).

Let $\theta_n$ denote the r.v. representing the angular distance from an arbitrary reference axis $z$ from the center of the sphere of radius $r$, to the nearest node of the BPP in the coordinate $\theta$, as shown in Fig. 3. Following the same procedure for $p = \frac{1}{1-\cos \theta}$, the corresponding PDF is given by

$$f_{\theta_n}(\theta) = -\frac{dF_{\theta_n}(\theta)}{d\theta} = \frac{dp^{n-1}(1-p)^{N-n}}{d\theta} B(n, N-n+1)$$

$$= \frac{\sin \theta}{2} \beta\left(1 - \cos \theta; n, N-n+1\right). \quad (5)$$

Fig. 4 depicts the PDF’s for both angles $|\phi|$ and $\theta$ for the ten nearest nodes in angle, using eq. (3) and (5). Since the nodes are sorted according to angle $|\phi|$ and $\theta$, it is observed that the probability of finding the nearest in angle node close to the reference line, is concentrated to small angular values, and as $n$ increases, the corresponding PDFs are shifted to greater values. Moreover, since the PDFs are overlapping, the probability that there are more than two nodes in a specific sector or wedge, is considerable. According to [12] the Euclidean distance $R_n$ from the origin to its $n$th nearest node follows a generalized beta distribution

$$f_{R_n}(r) = \frac{3}{R} \frac{\Gamma\left(n + \frac{2}{3}\right)\Gamma(N+1)}{\Gamma(n)\Gamma\left(N + \frac{5}{3}\right)}$$

$$\times \beta\left(\frac{r^3}{R^3}; n + \frac{2}{3}, N-n+1\right). \quad (6)$$

**C. JOINT DISTRIBUTIONS**

Of particular interest are the joint distributions of the distances in all coordinates. Since the polar and/or the spherical coordinates of the nodes are independent, the joint PDF is the product of the marginal PDFs. Let $n^p$, $n^\theta$, and $n^\phi$, denote the $n$th nearest neighbor to the reference line or point, in $|\phi|$, $\theta$, and $r$ coordinate, respectively. The joint PDF may be calculated as

$$f_{|\phi_n|, \theta_n, R_n}(|\phi|, \theta, r) = f_{|\phi_n|}(|\phi|) f_{\theta_n}(\theta) f_{R_n}(r), \quad (7)$$

where the marginal PDFs are given by equations (3), (5), and (6), respectively. In case of polar coordinates, i.e., $d = 2$, the marginal PDFs are given by (3), and (4). Therefore, one may easily calculate the joint PDF of e.g., the 1st node in $|\phi|$, the 2nd in $\theta$, and the 3rd in $r$, by substituting for $n^\phi = 1$, $n^\theta = 2$, and $n^r = 3$ respectively in (3), (5), and (6). Fig. 5 depicts the joint PDF for the angular distances, when $r = R$ and $N = 10$, for $n = 1, 5, 10$ nearest nodes. It is clearly depicted that the PDF of the first nearest node is concentrated close to $0^\phi$, the corresponding for the $5$th nearest node close to $90^\phi$, and the farthest one close to $180^\phi$ for both angles.

**III. MOMENTS OF THE DISTANCES**

The $y$th moment of the distance $|\phi_n|$ is calculated in Appendix A and is given by

$$\mathbb{E}[|\phi_n|^y] = \frac{\pi^y \Gamma(N+1)\Gamma(n+y)}{\Gamma(N+y+1)\Gamma(n)} = \frac{\pi^y \eta^y}{(N+1)\gamma}, \quad (8)$$
where \( n^{[\nu]} = \Gamma(n + \gamma)/\Gamma(n) \) is the rising Pochhammer symbol. In this case, the expected distance in angle \(|\phi|\) to the \( n \)th nearest node is

\[
E[|\phi_n|] = \pi \frac{n}{N + 1},
\]
and the variance is

\[
\text{Var}[|\phi_n|] = \pi^2 \frac{\Gamma(n + 2)}{(N + 1)^2} - \pi^2 \left( \frac{n}{N + 1} \right)^2 = \frac{2\pi^2 n}{(N + 1)^2} \frac{N + 1 - n}{N + 2}.
\]

The \( \gamma \)th moment of \( \theta_n \) is calculated in Appendix B as follows

\[
E[\theta_n^\gamma] = \frac{1}{B(n, N - n + 1)} \int_0^1 2^\gamma (\text{ArcSin} (\sqrt{t}))^\gamma n^{-1}(1 - t)^N - n dt.
\]

A simple expression is obtained for \( \gamma = 1 \), i.e., for the expected distance

\[
E[\theta_n] = \pi - \sqrt{\pi} \frac{\Gamma(N + 1) \Gamma(n + \frac{1}{2})}{\Gamma(N - n + 1) \Gamma(n + 1)^2} \times 3F_2 \left[ n, \frac{1}{2}, n - N; n + 1, n + 1, 1 \right].
\]

The variance may be numerically calculated by

\[
\text{Var}[\theta_n] = E[\theta_n^2] - (E[\theta_n])^2.
\]

The \( \gamma \)th moment of \( R_n \) is given in [12] as

\[
E[R_n] = \frac{\Gamma(N + 1) \Gamma(n + \frac{1}{2})}{\Gamma(n) \Gamma(n + 1)^2} = \frac{R_n^{[\frac{1}{2}]} \Gamma(N + 1)}{(N + 1)^{\frac{1}{2}}},
\]
and the variance is given by

\[
\text{Var}[R_n] = \frac{R_n^{[\frac{1}{2}]} \Gamma(N + 1)}{(N + 1)^{\frac{1}{2}}} - \left( \frac{R_n^{[\frac{1}{2}]} \Gamma(n + 1)}{(n + 1)^{\frac{1}{2}}} \right)^2.
\]

IV. CONDITIONAL DISTANCE DISTRIBUTIONS

The conditional distance distributions are of great interest in localization and interference calculations. Assume that we know that the \( k \)th nearest neighbor is at an angle \(|\phi_k|\) from the reference line. This knowledge may become available by using a beam sweeping procedure from the base station in a cellular network, or an access point (AP) in any wireless network. The received signal strength may be used then to identify the order of the nodes in the angular domain. The first \((k - 1)\) nodes are uniformly randomly distributed in \( b_d(\alpha, R) \setminus V(x, |\phi_i|) \). The distance distributions of the first \((k - 1)\) nearest nodes from the reference line can be written as in (3), by setting \( N = k - 1 \), and replacing \( \pi \) with \(|\phi_k|\).

\[
f_{|\phi|/|\phi_k|}(\phi | |\phi_k| = |\phi|) = \frac{1}{|\phi_k|} \beta \left( \frac{|\phi|}{|\phi_k|}, n, k - n \right), \quad n < k.
\]

For the remaining nodes, i.e., for \( n > k \), and for \(|\phi| \in [|\phi_k|, \pi]\)

\[
f_{|\phi|/|\phi_k|}(\phi | |\phi_k| = |\phi|) = \frac{-d}{d|\phi|} I_{1-q}(N - n + 1, n - k), \quad n > k,
\]

where \( q = \frac{|\phi| - |\phi_k|}{\pi - |\phi_k|} \). Therefore,

\[
f_{|\phi|/|\phi_k|}(\phi | |\phi_k| = |\phi_k|) = \frac{d}{d|\phi|} B(\pi - |\phi_k|, N - n + 1, n - k) = \frac{1}{\pi - \phi_k} \beta \left( \frac{|\phi| - |\phi_k|}{\pi - |\phi_k|}, N - n + 1, n - k \right), \quad n > k.
\]

In the following the conditional PDFs for the angle \( \theta \) are calculated. In the set \( V \) with volume \(|V| = \frac{4\pi^2}{3} R^3 \) there are \((k - 1)\) nodes. Therefore, \( p = \frac{1 - \cos \theta_k}{2} \) and for \( n < k \)

\[
f_{\theta_k}(\theta | \theta_k = \theta) = \frac{d}{d\theta} B(n - k, n) = \frac{\sin \theta}{1 - \cos \theta_k} \beta \left( \frac{1 - \cos \theta_k}{1 - \cos \theta}, n, k - n \right).
\]

For the remaining nodes, i.e., for \( n > k \), \( \theta \in [\theta_k, \pi] \)

\[
f_{\theta_k}(\theta | \theta_k = \theta_k) = \frac{d}{d\theta} B(n - 1 - \theta, n - k) = \frac{\cos \theta_k - \cos \theta}{1 + \cos \theta_k}.
\]

Therefore,

\[
f_{\theta_k}(\theta | \theta_k = \theta_k) = \frac{d}{d\theta} B(n - 1 - \theta, n - k) = \frac{\cos \theta_k - \cos \theta}{1 + \cos \theta_k}.
\]

for \( n > k \). For the sake of completeness the conditional PDFs in distance \( r \) evaluated in [12] are given in the following

\[
f_{R_k}(r | R_k = s) = \frac{d}{ds} B(n - 1 - \theta, n - k) \times \beta \left( \frac{1 - \cos \theta_k}{1 - \cos \theta}, n - 1 - \theta, n - k \right), \quad n < k,
\]

\[
f_{R_k}(r | R_k = s) = \frac{d}{ds} B(n - 1 - \theta, n - k) \times \beta \left( \frac{1 - \cos \theta_k}{1 - \cos \theta}, n - 1 - \theta, n - k \right), \quad n > k,
\]

for \( r \in [s, R] \) and \( q = (r - s)/(|R^d| - s^d) \).

V. APPLICATIONS FOR WIRELESS NETWORKS

A. USER ASSOCIATION POLICY AND INTERFERENCE

The performance evaluation of wireless networks is heavily based on the achieved SINR. The received power of the desired signal as well as the interference power from many
interfering nodes depend not only on the distance between the involved nodes, but also on directional characteristics that bring into play the transmitting and receiving antenna beams. In 5G and beyond networks the beamforming capabilities of the antenna systems have a central role in the performance of the systems. The antenna characteristics have been modeled by 3GPP and used extensively in the corresponding technical reports and specifications [24]–[26]. Moreover, the interference characteristics, especially in mmWave networks, are largely dependent on the directional beams, since the corresponding beamwidths contribute the most to the interfering power that is brought to the receiver. Thus, a nearby interferer may contribute less than a distant one due to the loss imposed by a beam misalignment of the transmitting and receiving nodes. This is due to the fact that the AoA at the receiver falls outside the half power beamwidth of the antenna pattern, although, the AoD from the transmitting node may coincide with the direction of maximum directivity at the transmitter. This brings the necessity for the stochastic modeling of the directional characteristics of the resulting links. A first application of the analysis provided in the previous sections is the nearest and the farthest node in a coordinate, or jointly in all coordinates. Therefore, for either 2D or 3D wireless networks the PDF for the nearest node is given by

\[
f_{\theta}(|\phi|) = \frac{N}{\pi} \left(1 - \frac{|\phi|}{\pi}\right)^{N-1} = \frac{N}{\pi^N} (\pi - |\phi|)^{N-1},
\]

(24)

\[
f_{\theta}(\theta) = \frac{N \sin \theta}{2} \left(1 + \cos \theta \right)^{N-1},
\]

(25)

\[
f_{R}(r) = \frac{dN}{r} \left(1 - \left(\frac{r}{R}\right)^d\right)^{N-1} \left(\frac{r}{R}\right)^d.
\]

(26)

It is noted that the reference line for the determination of the angular distance is, without loss of generality, arbitrarily selected to be the direction of the antenna maximum directivity. Up to now, the association policy, i.e., the user association to a BS in sub-6 GHz, mmWave or THz networks is usually based on the strongest fading average received signal, which is translated to the nearest point in the corresponding point process that models the BSs in the network [27]. Now, the nearest node in angle |\phi| may be, e.g., used for the determination of the serving node, whereas the second nearest one may be the dominant interferer. Nevertheless, the decision should be based also on the distance r, i.e., the joint PDF for the first and correspondingly the second nearest in angle |\phi| and the distance r, could be utilized.

Based on the previous discussion, an indicative comparison is presented in the following, regarding the received power from the dominant interferer, in the absence of fading, for three different scenarios in a 2D cellular network. The model assumes the downlink transmissions from BSs that form a BPP in a geographical area of radius R. The serving BS is selected to be the nearest one, i.e., at distance \(R_1\) from the typical user, which is placed at the origin. One simplifying assumption is that the maximum gain of the transmit antennas is always directed towards the typical user. The first examined scenario is based on the presented results and considers the dominant interferer as the node that is the nearest in angle |\phi| from the direction of the serving BS, i.e., at \(|\phi_1|\), and the second nearest in distance r from the typical user, i.e., at \(R_2\).

The beampattern of the receiving antenna is a realistic one as that given in [24], i.e., the antenna gain (in dB) is given by

\[
G(\phi) = G_m - \min\left\{12\left(\frac{\phi}{\phi_{3dB}}\right)^2, SLA\right\},
\]

(27)

where \(SLA = 30\) dB is the sidelobe level attenuation [29], \(\phi_{3dB}\) is the half power beamwidth, and \(G_m = 8\) dBi is the maximum directional antenna gain. In the second scenario, the same antenna pattern is used, the dominant interferer is the second nearest in distance r from the typical user, and the angle is assumed to be uniformly distributed in \([-\pi, \pi]\). In the third scenario, a flat-top pattern is assumed for the typical user within the half power beamwidth, and the dominant interferer is the second nearest in distance r and always within the half power beamwidth of the receiving typical user. Thus, for the third scenario the gain of the receiving antenna towards the dominant interferer is fixed and equal to the maximum value \(G_m\). Fig. 6 presents the CDF of the interference power from the dominant interferer for all scenarios. This power is calculated as \(P_I = K g(|\phi|) r^{-\alpha}\), where \(g(|\phi|)\) is the multiplicative antenna gain factor, \(\alpha = 2\) is the path loss factor, \(K = \left(\frac{c}{2\pi f}\right)^2\), \(f = 26.5\) GHz is the carrier frequency, and \(c = 3 \times 10^8\) meters/sec is the speed of light. The values of the simulation parameters are \(N = 10\) BSs in an area of radius \(R = 100m\), and the half power beamwidth is \(\phi_{3dB} = \pi/3\). A first observation is that the third scenario provides a CDF close to the first one but more pessimistic. The reason for this behavior is the assumption for a constant maximum receiver gain for all values of the angle within the half power beamwidth. The second scenario provides a far more optimistic curve.

![Figure 6. CDF of the interference power from the dominant interferer for three scenarios, along with the aggregate interference form \(N = 10\) BSs in an area of radius \(R = 100m\), and a \(\phi_{3dB} = \pi/3\).](image-url)
since the assumption of uniform angle in \([-\pi, \pi]\) allows for angles out of the half power beamwidth and consequently, quite smaller values for the receiving antenna gain. An extra curve denoted as aggregate in the figure depicts the total interference from all BSs in the considered area. This curve is very close to that of the first scenario, and therefore, the identification of the dominant interferer, and the way it is calculated, based on angle \(|\phi_1|\) and distance \(R_2\), provides a realistic estimation of the expected interference. In addition, one may observe that the spread of the total interference curve is larger than that of the first scenario, as expected, since the latter considers only the nearest in angle, i.e., \(|\phi_1|\), and the second nearest in distance \(r\), i.e., \(R_2\). Finally, the theoretical curve depicted in the figure has been extracted using the following PDF

\[
 f_{P_1}(z) = \int_{0}^{\min\left\{ \frac{\pi}{2\pi+r}, \frac{\text{area}}{10} \right\}} \frac{\phi^{3dB}}{Kx^2 \ln(10)} a(\pi R^2)^N \times \left( \pi - \phi^{3dB} \frac{G_m - 10 \log_{10}(10)}{12} Kx^2 \frac{z}{\phi^{3dB}} \right) \times \left( R^2 - \left( \frac{Kx}{z} \right)^2 \right)^{N-2} \frac{1}{1.2 \left( G_m - 10 \log_{10}(10) \right)^{2N-1}} \right) dx, 
\]

where

\[
 z \in \left[ \frac{G_m - 12(N-1)}{10} \frac{\sqrt{\pi}}{2\pi} \right] K R^{-\alpha}, \infty \right). \]

The proof of (28) is given in Appendix C.

Another useful application of the presented results is the calculation of the void probability of the point process, i.e., the probability of there being no point/node of the process in an arbitrary test set \(B\). For a BPP with \(N\) points distributed over a set \(W\), and assuming that \(B \subseteq W\), the void probability is given by \(p^0_B = (1 - p)^N\). This is of particular interest when we are dealing with a spherical sector defined by \(|\phi|^{3dB}\) and \(\theta^{3dB}\), i.e., the half power beamwidth of an antenna. Therefore, the void probability for the antenna half power beamwidth is given by

\[
 p^0 = \left( 1 - \frac{|\phi|^{3dB}}{2\pi} (1 - \cos \theta^{3dB}) \right)^N. \tag{29} 
\]

If, in addition, the distance \(r\) is also considered as a limiting coordinate, the void probability is given by

\[
 p^0 = \left( 1 - \frac{|\phi|^{3dB}}{2\pi} (1 - \cos \theta^{3dB}) \frac{r^{2\gamma}}{R^2} \right)^N. \tag{30} 
\]

The distance \(r\) is related to power, through the path loss exponent, and if it is used at the receiver as a limiting factor, it denotes either the maximum distance as imposed by the sensitivity of the receiver, at which the serving node may be located, or the minimum distance, at which an interferer should be found. Moreover, one may use (1) to compute the probability that there are less than \(N\) nodes in a region \(V\) by replacing the probability \(p\), as in (30).

\[B. \ BEAM \ MANAGEMENT\]

Beam management in 5G and beyond networks is based on the measurement and reporting of beamformed reference signals (RSs) transmitted either by the BS or by the UE. Beam sweeping refers to the process where the BS or the UE uses sequentially different analog beams when transmitting or receiving RSs. The BS and the UE have pre-determined analog beam codebooks, and try to find an acceptable beam pair for the link [28]. The limited number of analog beams in the codebook leads to a misalignment during the association of a user to a BS. Therefore, not only realistic antenna patterns for the BS and the UE are needed, [29], to calculate the loss due to misalignment, but also the stochastic characterization of the angular distance of the maximum directivity directions should be considered for the performance evaluation of the followed beam management procedure. The situation is getting worse when considering the UE mobility [30]. The mobility-induced beam misalignment means that the transmitting and receiving beams are not aligned with each other, due to the time duration between the synchronization signals utilized to perform the beam reselection. The misalignment may be thought of as a translation of the reference line from which the angular distances investigated in previous sections are measured. Therefore, the nearest node and the corresponding PDF is of interest.

\[C. \ LOCALIZATION\]

The coordinate distance statistics of the nodes in a wireless network provide valuable information for the expected location of the nodes. Based on the assumption that some of the nodes can accurately estimate their angular distance from a reference line that connects the node to a BS or an AP, the conditional moments may be calculated from (17) and (18) for the angle \(|\phi|\), and form (19) and (21) for the angle \(\theta\).

Assume that the \(k\)th nearest neighbor in angle \(|\phi|\) is at angle \(|\phi_k|\), then the moments of \(|\phi|\) for \(n < k\), are given by

\[
 \mathbb{E}[|\phi_n|^{\alpha} | |\phi_k| = |\phi_1|] = |\phi_1|^{\gamma} \frac{Kx^2}{|\phi|^{\alpha}} \frac{1}{Kx^2}. \tag{31} 
\]

The proof of (31) is given in Appendix D. For \(n > k\)

\[
 \mathbb{E}[|\phi_n|^{\alpha} | |\phi_k| = |\phi_1|] = |\phi_1| \times 2F1 \left[ -\alpha, n-k; N-n+1; 1 - \frac{\pi}{|\phi_1|} \right]. \tag{32} 
\]

The proof of (32) is given in Appendix E.

Fig. 7 depicts the conditional expected value of the angular distance of the higher order nodes \(|\phi_n|\), versus the angular distance \(|\phi_1|\) of the nearest node. The total number of nodes is \(N = 10\). The higher the value of \(|\phi_1|\), the more concentrated in space are the remaining nodes. This result is quite useful also for the imposed interference from the nearest nodes in angle \(|\phi|\). Therefore, a significant observation is that if \(|\phi_1|\) is close to zero, the expected value for the second or even the third nearest in \(|\phi|\) node, takes values less than the half of the half power beamwidth, that is assumed in antenna arrays utilized in 5G and beyond networks [25].

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means that at least two nodes out of ten in the considered space will contribute with high directivity to the interference at the receiver. Nevertheless, the half power beamwidth is a function of the number of beams the antenna array can produce in a specified sector.

Now assume that the $k$th nearest neighbor in angle $\theta$ is at angle $\theta_n$, then the moments of $\theta_n$ for $n < k$, are given by

$$E[\theta_n^\alpha | \theta_k = \theta_n] = \int_0^{\theta_n} \theta_n^\alpha \frac{\sin \theta}{1 - \cos \theta_n} \frac{p^{n-1}(1 - p)^{k-n-1}}{B(n, k - n)} d\theta$$

$$= \int_0^1 \left[2\text{ArcSin} \left(\sqrt{\frac{p(1 - \cos \theta_n)}{2}}\right)\right]^\alpha \frac{p^{n-1}(1 - p)^{k-n-1}}{B(n, k - n)} dp.$$  \hspace{1cm} (33)

The conditional expected value, i.e., for $\alpha = 1$, is given by

$$E[\theta_n | \theta_k = \theta_n] = 2 \sin \left(\frac{\theta_n}{2}\right) \frac{n^{\frac{1}{2}}}{k^{\frac{1}{2}}}$$

$$\times \text{F} \left[\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, k + \frac{1}{2}; \sin^2 \left(\frac{\theta_n}{2}\right)\right].$$  \hspace{1cm} (34)

For $n > k$

$$E[\theta_n^\alpha | \theta_k = \theta_n] = \int_0^\pi \theta_n^\alpha \frac{\sin \theta}{1 + \cos \theta_n} \frac{q^{n-k-1}(1 - q)^{N-n}}{B(N - n + 1, n - k)} d\theta$$

$$= \int_0^1 \left[\text{ArcCos} \left(1 - q \cos \theta_n \right)\right]^\alpha \frac{q^{n-k-1}(1 - q)^{N-n}}{B(N - n + 1, n - k)} dq.$$  \hspace{1cm} (35)

Fig. 8 depicts the conditional expected value of the angular distance of the higher order nodes $\theta_n$, versus the angular distance $\theta_1$ of the nearest node. The total number of nodes is $N = 10$. As expected, the higher the $\theta_1$, the more concentrated in space are the remaining nodes. As compared to the previous comment for the angular distance in $|\phi|$, the expected value for the second nearest and the higher order nodes in $\theta$ are greater, which implies a lower probability of suffering from major interferers lying in the $3\text{dB}$.

VI. CONCLUSION

This paper presented the distributions of all coordinate distances in finite uniformly random networks. The angular distance distributions were analytically derived for 2D and 3D BPP wireless networks, where $N$ nodes are independently uniformly randomly distributed in a two or three dimensional ball. The results are applicable to 5G and beyond wireless networks where the beamforming capabilities of BSs and UEs significantly affect their performance. The introduction of new bands in mmWave or even in sub-THz frequency areas, imposes the need for the directional characterization of the nodes, the corresponding antennas beam patterns, and the relative position of the transmitting and receiving beams of the links. In this context, the distributions, the conditional distributions, the moments, and the conditional moments of the distances in all coordinates are derived, and a discussion on the application of the presented results in various applications is given.

The application of the derived concepts and equations in the calculation of performance analysis parameters and metrics, is left for future research. Indeed, the antenna gain and correspondingly the path loss should be considered as random variables that depend on the angular distances of the communicating nodes. A first example of such an application is given in Appendix C. This model affects the received power from the desired and the unwanted transmitters, and determines not only the selection of the serving BS in a cellular network, but also the calculation of interference, and thus the corresponding SINR. The latter is closely associated with the half power beamwidth of the receiving antenna array, since the number of interfering nodes that fall within the beamwidth as well as the received power should consider the angular distribution of the nodes.
APPENDIX A

PROOF OF EQUATION (11)
The $\gamma$th moment of the distance $|\phi_n|$ is calculated using (3) and $p = \frac{\sin^2 \theta}{\sin^2 \frac{\theta}{2}}$ as follows:
\[
\mathbb{E}[|\phi_n|^\gamma] = \int_0^{2\pi} |\phi|^\gamma f_{\phi_n}(|\phi|) d|\phi|
\]
\[
= \frac{1}{\pi} \frac{1}{B(n, N - n + 1)} \times \int_0^{\pi} |\phi|^\gamma \left(\sin^2 \frac{\theta}{2}\right)^{n-1} \left(1 - |\phi| \frac{\pi}{\pi}ight)^{N-n} d|\phi|
\]
\[
= \frac{\pi^\gamma}{B(n, N - n + 1)} \frac{1}{\pi} \frac{1}{\Gamma(n + 1) \Gamma(\alpha + \gamma)} \times \frac{1}{\Gamma(n + \gamma + 1) \Gamma(n)} = \frac{\pi^\gamma n^\gamma}{(N + 1)^\gamma}, \tag{36}
\]
where (a) is obtained by making the substitution $\phi = t \pi$, $B(x; a, b)$ is the incomplete beta function, which for $x = 1$ gives $B(1; a, b) = B(a, b)$, and $n^\gamma = (n + \gamma)/\Gamma(n)$, is the rising Pochhammer symbol.

APPENDIX B

PROOF OF EQUATION (11)
The $\gamma$th moment of $\theta_n$ is calculated as follows:
\[
\mathbb{E}[\theta_n^\gamma] = \int_0^{\pi} \theta^\gamma f_{\theta_n}(\theta) d\theta. \tag{37}
\]
Using (5) with $p = \frac{1 - \cos \theta}{\sin \theta} = \sin^2 \left(\frac{\theta}{2}\right)$
\[
\mathbb{E}[\theta_n^\gamma] = \frac{1}{2} \frac{1}{B(n, N - n + 1)} \times \frac{\pi}{\sin^2 \frac{\theta}{2}} \left(\sin^2 \frac{\theta}{2}\right)^{n-1} \left(1 - \sin^2 \left(\frac{\theta}{2}\right)\right)^{N-n} d\theta
\]
\[
= \frac{1}{B(n, N - n + 1)} \times \int_0^{\pi} \left(\sin^2 \frac{\theta}{2}\right)^{N-n} dt, \tag{38}
\]
where in (a) the substitution $t = \sin^2 \left(\frac{\theta}{2}\right)$ is used.

APPENDIX C

PROOF OF EQUATION (28)
The PDF of $R_2^\alpha$ is first calculated. The CDF of $R_2^\alpha$ is given by
\[
F_{R_2^\alpha}(x) = \mathbb{P}[R_2^\alpha \leq x] = 1 - F_{R_2} \left(\frac{1}{x}\right)^\alpha. \tag{39}
\]
By differentiation of (39) the corresponding PDF, $f_{R_2^\alpha}(x)$, is derived
\[
f_{R_2^\alpha}(x) = \frac{2N(N - 1)}{aR_2^{2\alpha}} \left(\frac{2}{1-x}\right)^\alpha \left(\frac{4\pi^2}{\alpha}\right), \tag{40}
\]
for $x \in [R^{-\alpha}, \infty)$. Moreover, building on $|\phi_1|$, the PDF of $g(|\phi_1|)$ is given in closed form by
\[
f_{g(|\phi_1|)}(x) = \frac{N}{\pi} \frac{\phi^{3dB}}{x \ln(10)} \frac{5}{\sqrt{G_{m-10\log_{10}(x)}}} \times \left(\pi - \phi^{3dB} \sqrt{G_{m-10\log_{10}(x)}}\right)^{N-1}, \tag{41}
\]
where $10^{\frac{G_{m-12\left(\frac{G_{m-10}}{10}\right)^2}{10} x \leq 10^{\frac{G_{m-12\left(\frac{G_{m-10}}{10}\right)^2}{10}}}}$. Then, the PDF of the product of the random variables $Z = g(|\phi_1|)R_2^\alpha$ is given directly by
\[
f_Z(z) = \int_0^{\min\left[\frac{G_{m-12\left(\frac{G_{m-10}}{10}\right)^2}{10}}{\pi}, 10^{\frac{G_{m-12\left(\frac{G_{m-10}}{10}\right)^2}{10}}\right]} \frac{1}{\pi} f_{g(|\phi_1|)}(x) f_{R_2^\alpha}(z/x) dx
\]
\[
= \int_0^{\min\left[\frac{G_{m-12\left(\frac{G_{m-10}}{10}\right)^2}{10}}{\pi}, 10^{\frac{G_{m-12\left(\frac{G_{m-10}}{10}\right)^2}{10}}\right]} \frac{\phi^{3dB}}{x^2 \ln(10)} N^2(N - 1) \times \left(\pi - \phi^{3dB} \sqrt{G_{m-10\log_{10}(x)}}\right)^{N-1} \left(\frac{x}{z}\right)^{\frac{4\pi^2}{\alpha}}
\]
\[
\times \left[R^2 - \left(\frac{x}{z}\right)^2\right]^{N-2} \frac{1}{1.2 \sqrt{G_{m-10\log_{10}(x)}}} dx, \tag{42}
\]
for
\[
z \in \left[\frac{G_{m-12\left(\frac{G_{m-10}}{10}\right)^2}{10}, \infty\right].
\]
Now, the PDF of $P_1$ can directly be obtained through the transformation $P_1 = KZ$ and by applying change of variable in (42), which results in (28) immediately, and this completes the proof.

APPENDIX D

PROOF OF EQUATION (31)
The moments of $|\phi_n|$ for $n < k$, are given by
\[
\mathbb{E}[|\phi_n|^\alpha | |\phi_k| = |\phi_k|] = \int_0^{\min\left[\frac{G_{m-12\left(\frac{G_{m-10}}{10}\right)^2}{10}}{\pi}, 10^{\frac{G_{m-12\left(\frac{G_{m-10}}{10}\right)^2}{10}}\right]} |\phi|^\alpha \left(1 - p\right)^{k-n-1} B(n, k - n) d\phi
\]
\[
= \int_0^{\min\left[\frac{G_{m-12\left(\frac{G_{m-10}}{10}\right)^2}{10}}{\pi}, 10^{\frac{G_{m-12\left(\frac{G_{m-10}}{10}\right)^2}{10}}\right]} \frac{\phi^{3dB}}{B(n, k - n)} B(n, k - n) \frac{\phi^{3dB}}{B(n, k - n)} d\phi
\]
\[
= |\phi_k|^\alpha \left(\frac{\Gamma(n + \alpha + \Gamma(k)}{\Gamma(k + \alpha)} \Gamma(n)\tag{43}
\]
where in (a) the substitution $|\phi| = p|\phi_k|$ is used.

APPENDIX E

PROOF OF EQUATION (32)
The moments of $|\phi_n|$ for $n > k$, are given by
\[
\mathbb{E}[|\phi_n|^\alpha | |\phi_k| = |\phi_k|] = \frac{1}{\pi - |\phi_k|} \times \int_0^{\pi} \frac{|\phi|^\alpha}{B(n - k, N - n + 1)} q^{n-k-1}(1 - q)^{N-n} d|\phi|
\]
\[
= \int_0^{\pi} \frac{q^{n-k-1}(1 - q)^{N-n} d|\phi|}{B(n - k, N - n + 1)} \tag{44}
\]
where in (a) the substitution $q = \frac{|\phi| - |\phi_k|}{|\phi_k|}$ is used.
