NORMAL CURVATURE BOUNDS VIA MEAN CURVATURE SMOOTHING

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Abstract. Let $F_0 : M^n \to \mathbb{M}^{n+1}$ be a complete, immersed hypersurface with bounded second fundamental form in a complete ambient manifold with bounded geometry. Let $F_t : M^n \to \mathbb{M}^{n+1}$ be a smooth solution to the mean curvature flow with initial data $F_0$. We show that the supremum and infimum of the normal curvature of the immersion $F_t$ vary at a bounded rate. This is an analog of a result of Kapovitch on Ricci flow.

1. Introduction

In a recent paper [K] Kapovitch proved that the supremum and infimum of the sectional curvature of a complete manifold vary at a bounded rate under the Ricci flow, which generalized a result in Rong [R] to the noncompact case. In this short note, we will prove a mean curvature flow analog of this result. More precisely, we have the following

**Theorem** Let $F_0 : M^n \to \mathbb{M}^{n+1}$ be a complete, immersed hypersurface with bounded second fundamental form in a complete ambient manifold with bounded geometry. Let $F_t : M^n \to \mathbb{M}^{n+1}$ be a smooth solution to the mean curvature flow on $M^n \times [0,T]$ with initial data $F_0$. Then there exists a constant $C$ depending only on $n, T$, the initial bound of the second fundamental form and the ambient manifold, such that $\inf \kappa_0 - Ct \leq \kappa_t \leq \sup \kappa_0 + Ct$, where $\kappa_t$ is the normal curvature function of the immersion $F_t$.

(Here, as usual, by bounded geometry we mean that $\mathbb{M}^{n+1}$ has bounded injectivity radius and (norms of) covariant derivatives of the curvature tensor.)

Recall that the short time existence of the mean curvature flow in our situation is established by Ecker-Huisken [EH,Theorem 4.2]. (Actually Ecker-Huisken [EH] only consider the case $\mathbb{M}^{n+1} = \mathbb{R}^{n+1}$, but their proof of Theorem 4.2 in [EH] can be easily adapted here, since in the general case one need only add some lower order terms in the evolution equation of the second fundamental form, which do not affect the original proof much.) Moreover from [EH] (cf. also [CY]) we know that the following estimates hold on $M^n \times [0,T]$:

\[
\begin{align*}
|\nabla^m A| &\leq \frac{C}{(m+1)^2}, \\
|H| &\leq C,
\end{align*}
\]

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\[ |R_{\eta_1}\eta_1| \leq C, \text{ and} \]
\[ |\frac{\partial \eta_1}{\partial t}| \leq C. \]
(Here and below, we use \( C \) to denote various constants depending only on \( n, T \), the initial bound of the second fundamental form and the ambient manifold.)

As in [K], the proof of our theorem is an application of the maximum principle combining with a technique of cut-off function, which is given in the second section.

2. Proof of Theorem

The proof is a modification of that of Kapovitch [K]. For simplicity we only consider the case \( M^{n+1} = R^{n+1} \), the general case can be treated similarly as remarked in the first section.

As in [K] we choose a nonnegative, nonincreasing, smooth function \( \chi : R \rightarrow R \) with

1. \( \chi(s) = 1 \) for \( s \leq 1 \), \( \chi(s) = 0 \) for \( s \geq 2 \),
2. \( \frac{\chi'(s)}{\chi(s)} \leq 16 \), and
3. \( \chi''(s) \leq 8 \).

Given a point \( z \in M \), let \( d_z(x,t) = d_{g_t}(x,z) \) be the distance w.r.t. \( g_t \), where \( g_t \) is the induced metric on \( M \) from the immersion \( F_t \). Set \( \xi_z(x,t) = \chi(d_z(x,t)) \). Then as in [K] we have

1. \( 0 \leq \xi_z \leq 1 \), \( \frac{\partial \xi_z}{\partial t} \leq C \),
2. \( \sum_{i \leq 1} \xi_z^2 \leq C \), and
3. \( \Delta \xi_z \geq C \) in the barrier sense.

First we consider the case that \( \sup \kappa_t > 0 \) for all \( t \in [0,T] \). Let \( \overline{A}(t) = \sup \kappa_t \) and \( \overline{A}_z(t) = \max\{0, \max_{(x,v)}\xi_z(x,t)\kappa_t(x,v)\} \), where \( x \) runs over \( M \), and \( v \) runs over unit vectors (w.r.t. \( g_t \)) in \( T_xM \). Of course \( \overline{A}(t) = \sup \{z | \overline{A}_z(t) \} \).

We wish to prove that the upper right-hand derivative of \( \overline{A}_z(t) \) (which will be denoted by \( \overline{A}_z(t) \)) satisfies \( \overline{A}_z(t) \leq C \) uniformly.

Let \( \phi_z(x,v,t) = \xi_z(x,t)\kappa_t(x,v) \), by Hamilton [Ha,Lemma 3.5] we need only show that given \( t_0 \in [0,T] \), \( \frac{\partial \phi_z}{\partial t}(x,v_0,t_0) \leq C \) for any maximum point \((x,v_0)\) of \( \phi_z(\cdot,v_0,t_0) \) such that \( \kappa_{t_0}(x,v_0) > 0 \).

Now we extend the vector \( v_0 \) by parallel translation along geodesics emanating radially out of \( x_0 \) w.r.t. \( g_{t_0} \). Denote this vector field by \( v_0 \).

Let \( \Phi_z(x,t) = \xi_z(x,t)\kappa_t(x,v_0) = \xi_z(x,t)v_0^iv_0^j \), where \( h_{ij} \) is the second fundamental form of the immersion \( F_t \), and \( v_0^i \) is the \( i \)-th component of \( v_0 \) in (say ) a normal coordinate system at \( x_0 \) w.r.t. \( g_{t_0} \).

Using the evolution equation for the second fundamental form

\[ \frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2H h_{il} h_{mj} + |A|^2 h_{ij} =: \Delta h_{ij} + P_{ij}, \]
( cf. Huisken [Hu]) we compute

\[ \frac{\partial \Phi_z}{\partial t}(x_0,t_0) \]

\[ = \Delta \Phi_z(x_0,t_0) - 2\nabla \xi_z \cdot \nabla h_{ij} v_0^iv_0^j(x_0,t_0) - h_{ij}v_0^iv_0^j \Delta \xi_z(x_0,t_0) \]
\[ - \xi_z h_{ij} \Delta (v_0^iv_0^j)(x_0,t_0) + \xi_z P_{ij} v_0^iv_0^j(x_0,t_0) + h_{ij} v_0^iv_0^j \frac{\partial \chi}{\partial t}(x_0,t_0) \]
\[ + \xi_z h_{ij} v_0^iv_0^j \frac{\partial}{\partial t}(v_0^iv_0^j)(x_0,t_0). \]

Note \( \Delta \Phi_z(x_0,t_0) \leq 0 \), since \( \Phi_z(\cdot,t_0) \) has a local maximum at \( x_0 \). Utilizing \( \nabla \Phi_z(x_0,t_0) = 0 \) and the property (ii) of \( \xi_z \), we see that the second term in RHS is also bounded above. Then the third term is bounded above is due to

\[ \nabla \Phi_z(x_0,t_0) = 0 \]
\( \kappa_{t_0}(x_0,v_0) > 0 \) and the property (iii) of \( \xi_z \). As in [R] we have \(|\nabla^2 v_0|((x_0,t_0) \leq C, |\nabla^1 v_0|_{g_0}((x_0,t_0) \leq C, so the fourth term and the seventh term are also bounded. Finally that the fifth term and the sixth term are bounded follows trivially from property (i) of \( \xi_z \) (and the smoothing property of MCF). Then we obtain the desired estimate

\[
\frac{\partial \phi_z}{\partial t}(x_0,v_0,t_0) = \frac{\partial \Phi_z}{\partial t}(x_0,t_0) \leq C.
\]

It follows that \( A(t) \leq C \).

As in [K], the general case can be easily reduced to this one by considering \( \kappa_t + C \) instead, and the argument for \( \inf \kappa_t \) is similar.

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