ON THE ATTRACTOR FOR A SEMILINEAR WAVE EQUATION WITH VARIABLE COEFFICIENTS AND NONLINEAR BOUNDARY DISSIPATION

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Abstract. Long time behavior of a semilinear wave equation with variable coefficients with nonlinear boundary dissipation is considered. It is shown that the existence of global and compact attractors depends on the curvature properties of a Riemannian metric given by the variable coefficients.

1. Introduction and main results. Let $\Omega \subset \mathbb{R}^3$ be a bounded, connected set with a smooth boundary $\Gamma$. The exterior normal on $\Gamma$ is denoted by $\nu$. We consider the following system

$$
\begin{cases}
  u_{tt} - \text{div} A(x) \nabla u + f(u) = 0 & (x,t) \in \Omega \times (0, +\infty), \\
  \nu_A \cdot u + h(u_t) = 0 & (x,t) \in \Gamma \times (0, +\infty), \\
  u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & x \in \Omega,
\end{cases}
$$

(1.1)

where $\nu_A = A(x)\nu$ and $A(x) = (a_{ij}(x))$ are symmetric and positively definite matrices for all $x \in \Omega$, and $a_{ij}(x)$ are smooth functions on $\Omega$.

The coefficient matrices $A(x)$ are given by the material of the system. If $A(x) = I$, the identity matrix in $\mathbb{R}^n$, system (1.1) is said to be of constant coefficients. In the case of constant coefficients, there is a considerable amount of papers dealing with the asymptotic behavior for semilinear evolution systems, see [1, 2, 8, 10, 11, 12, 13, 22, 3, 4, 5, 6, 7, 18, 19, 9] and references therein. However, the majority of results deal with linear interior dissipation and subcritical growth imposed on a nonlinear function $f$. Probably the most general treatment for this class of problems is in [8] and in [2], where global attractors for the wave equation with even supercritical exponent have been established.

For the system (1.1) with constant coefficients, the global attractors for the wave equation with nonlinear boundary dissipation are established in [2] under assumption 1 below. Here we shall apply the geometrical approach to problem (1.1) to extend [2] to the case of variable coefficients. This method was introduced

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in [24] for the controllability of the wave equation with variable coefficients and then extended to the plate with variable coefficients, to the quasilinear wave equation, and to shells, see references in [27]. The global compact attractors for the system (1.1) with variable coefficients are obtained where existence of an escape vector field plays a key role (assumption 2 below).

In (1.1) \( f \) and \( h \) are nonlinear functions subject to the following assumption.

**Assumption 1.**

(f-1) \( f \in C^2(\mathbb{R}) \) such that \( |f''(s)| \leq c(1 + |s|) \) for all \( s \).

(f-2) \[ \lim_{|s| \to +\infty} \inf \frac{f(s)}{s} > -\lambda, \] (1.2)

where \( \lambda \) is the largest constant in the Poincare type inequality

\[ \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} \, dx + \int_{\Gamma} u^2 \, d\Gamma \geq \lambda \int_{\Omega} u^2 \, dx. \] (1.3)

(h-1) \( h \in C^1(\mathbb{R}) \) is monotone increasing such that \( 0 < m_1 \leq h'(s) \leq m_2 < \infty \) for all \( |s| > R \) for some sufficiently large \( R \).

(h-2) \( h(0) = 0 \).

Problem (1.1), after a transformation to a first order system generates a strongly continuous semiflow \( T(t), t \geq 0 \), on the space \( H \equiv H^1(\Omega) \times L^2(\Omega) \), which is often referred as the finite energy space. The main aim of this paper is to analyze the asymptotic (when time \( t \to \infty \)) behavior of this flow.

We define

\[ g = A^{-1}(x) \quad \text{for} \quad x \in \Omega \] (1.4)

to be a Riemannian metric on \( \Omega \) and consider the couple \( \langle \Omega, g \rangle \) as a Riemannian manifold with the inner product

\[ \langle X, Y \rangle_g = \langle A^{-1}(x)X, Y \rangle, \quad |X|_g^2 = \langle X, X \rangle_g \quad X, Y \in \mathbb{R}^3_x, \]

where \( \langle \cdot, \cdot \rangle \) is the Euclidean product of \( \mathbb{R}^3 \).

Denote by \( D \) the Levi-Civita connection in the Riemannian metric \( g \) and let \( H \) be a vector field on \( \Omega \). Then for each \( x \in \Omega \), the covariant differential \( DH \) of \( H \) determines a bilinear form:

\[ DH(X,Y) = \langle D_Y H, X \rangle_g \quad \forall X, Y \in \mathbb{R}^3_x, \] (1.5)

where \( D_Y H \) stands for the covariant derivative of the vector field \( H \) with respect to \( Y \).

Denote by \( \nabla_g \) the gradient operator in the Riemannian metric \( g \). Then

\[ \nabla_g u = A(x)\nabla u, \] (1.6)

\[ |\nabla_g u|_g^2 = \langle \nabla u, A(x)\nabla u \rangle = \sum_{i,j=1}^{3} a_{ij}(x)u_{x_i}u_{x_j} \quad \text{for} \quad x \in \bar{\Omega}, \] (1.7)

where \( \nabla u \) is the gradient of \( u \) in the standard metric.

For existence of global and compact attractors, we need the following assumption.

**Assumption 2.** There exists an \( C^1 \) escape vector field \( H \) on \( \Omega \), that is,

\[ DH(X,X) \geq \delta |X|_g^2 \quad \text{for} \quad X \in \mathbb{R}^3_x, x \in \bar{\Omega}, \] (1.8)

where \( \delta > 0 \) is a constant.
Assumption 3. Problem
\[
\begin{aligned}
\frac{\partial w}{\partial t} - \nabla \cdot A(x) \nabla w + q(x,t)w &= 0 & \text{on } & (0,T) \times \Omega, \\
\nu \cdot A(x) n(x) - w &= 0 & \text{on } & (0,T) \times \Gamma.
\end{aligned}
\]  
(1.9)

has the unique zero solution for some \( T > 0 \) large.

Assumption (1.8) was introduced by [24] for the controllability of the wave equation with variable coefficients, which is also a useful condition for the controllability and the stabilization of the quasilinear wave equation (see [25, 26, 28, 27]). Existence of such a vector field depends on the sectional curvature of the Riemannian manifold \((\mathbb{R}^n, g)\). There are a number of methods and examples in [27] to find a vector field \( H \) that satisfies assumption 2.

The uniqueness assumption 3 is needed to eliminate the lower order terms by the compact-uniqueness argument when we apply the multiplier method to system (1.1). If there is a convex function on \( \Omega \) in the metric \( g \), then assumption 3 has been established in [23]. This is Theorem 1.4 later.

Our main results are stated below.

The first assertion is that of the existence of a global and compact attractor on the finite energy space \( \mathcal{H} = H^1(\Omega) \times L^2(\Omega) \).

**Theorem 1.1.** Under assumptions 1, 2, and 3, there exists a global, compact attractor \( B \subset \mathcal{H} \) to system (1.1).

Our subsequent results deal with the structure of the attractor \( B \). In order to state these results, we introduce the set of stationary points of \( T(t) \) denoted by \( \mathcal{N} \), i.e.,

\[
\mathcal{N} = \{ V \in \mathcal{H} : T(t)V = V, \ \forall t \geq 0 \}.
\]  
(1.10)

Every stationary point \( W \) has the form \( W = (\psi, 0) \), where \( \psi = \psi(x) \) solves the problem
\[
\begin{aligned}
- \nabla \cdot A(x) \nabla \psi + f(\psi) &= 0 & x \in \Omega, \\
\nu \cdot A(x) n(x) + \psi &= 0 & x \in \Gamma.
\end{aligned}
\]  
(1.11)

Let us define the unstable manifold \( M^u(\mathcal{N}) \) emanating from the set \( \mathcal{N} \) as a set of all \( Y \in \mathcal{H} \) such that there exists a full trajectory \( \gamma = \{ W(t) : t \in \mathbb{R} \} \) with the properties

\[
W(0) = Y \quad \text{and} \quad \lim_{t \to -\infty} dist_{\mathcal{H}}(W(t), \mathcal{N}) = 0,
\]  
\[
\]  
(1.12)

that is

\[
M^u(\mathcal{N}) = \{ Y \in \mathcal{H} : \exists W(t), \ W(0) = Y, \ \lim_{t \to -\infty} dist_{\mathcal{H}}(W(t), \mathcal{N}) = 0 \}.
\]  
(1.13)

Our next result asserts that the attractor \( B \) coincides with this unstable manifold.

**Theorem 1.2.** Under assumptions 1, 2, and 3, we have

1. \( B = M^u(\mathcal{N}) \);
2. \( \lim_{t \to +\infty} dist_{\mathcal{H}}(T(t)W, \mathcal{N}) = 0 \), for any \( W \in \mathcal{H} \).

The result stated below asserts the additional regularity of the attractor provided that the growth of nonlinear function \( f \) is subcritical. We recall that the same restriction is imposed in [2], where the regularity of attractors is established for semilinear wave equation with constant coefficients.

**Theorem 1.3.** Let assumptions 1, 2, and 3 hold. Let us assume that the nonlinear function \( f \) satisfies, in addition to assumption 1, an additional growth condition

\[
|f''(s)| \leq C(1 + |s|^{1-\varepsilon}) \quad \text{for some} \quad \varepsilon > 0.
\]  
(1.13)
Then $\mathcal{B}$ is a closed bounded set of $H^2(\Omega) \times H^1(\Omega)$.

A function $v$ on $\overline{\Omega}$ is said to be convex in the metric $g = A^{-1}(x)$, if
\[
\nabla_g^2 v(X, X) \geq \delta |X|^2_g \quad \text{for all} \quad X \in \mathbb{R}^3_x, \quad x \in \overline{\Omega},
\]
where $\delta > 0$ is a constant. In general extension of a convex function in the metric $g$ is also subject to the curvature of $g$. For some examples, see [27].

It follows from [23, Theorem 9.1] that under the assumption (1.14) the uniqueness problem (1.9) holds true. Thus we have the following.

**Theorem 1.4.** Suppose that there is a convex function on $\overline{\Omega}$ in the metric $g$. Let the corresponding assumptions on $f$ and $h$ hold true. Then the results in Theorems 1.1−1.3 hold.

2. **Preliminary estimates.** In what follows we shall use the classical (nonlinear semigroup) definitions of strong and generalized (weak) solutions to (1.1). Strong solutions are defined via the nonlinear semigroup theory for the initial data in the domain of the generator and satisfy differential inclusion almost everywhere in $t$. Generalized or weak solutions are defined as (strong) limits of strong solutions [20].

**Theorem 2.1. Part I-Generalized (weak) solutions.** Assume that the initial conditions satisfy $(u_0, u_1) \in \mathcal{H}$. Then there exists a unique generalized solution $(u, u_t) \in C([0, \infty); \mathcal{H})$ to (1.1). In addition, the following properties are valid for generalized solutions:
\[
u \in L^2_{\text{loc}}(\Sigma), \quad u_t \in L^2_{\text{loc}}(\Sigma); \quad u_{\nu} + u + h(u_t) = 0 \quad \text{in} \quad \Sigma, \quad (2.1)
\]
where $\Sigma = [0, +\infty) \times \Gamma$.

**Part II-Strong solutions.** Assuming, in addition, that $u_0 \in H^2(\Omega), u_1 \in H^1(\Omega)$ and $u_0, u_1$ satisfy the compatibility conditions on the boundary
\[
u u_{\nu} + u_0 + h(u_1) = 0 \quad \text{on} \quad \Gamma.
\]
Then the weak solution is strong and satisfies the regularity properties
\[
u \in L^\infty(0, \infty; H^2(\Omega)), \quad u_t \in L^\infty(0, \infty; H^1(\Omega)), \quad u_{tt} \in L^\infty(0, \infty; L^2(\Omega)).
\]

The above well-posedness of problem (1.1) can be obtained by the same arguments as in the case of constant coefficients in [2]. We omit the details.

Throughout the paper we shall use the linear energy function
\[
E(t) = \frac{1}{2} \int_\Omega (|\nabla_g u|^2 + |u_t|^2) \, dx + \frac{1}{2} \int_\Gamma |u|^2 \, d\Gamma, \quad (2.2)
\]
and the (nonlinear) energy function
\[
\varepsilon(t) = E(t) + \int_\Omega F(u) \, dx, \quad (2.3)
\]
where $F(z) = \int_0^z f(s) \, ds$.

**Theorem 2.2.** Let $(u_0, u_1) \in \mathcal{H}$. Then the solution to the boundary value problem (1.1) satisfies the estimate
\[
\varepsilon(0) = \int_0^t \int_\Omega h(u_t) u \, dx \, dt + \varepsilon(t), \quad (2.4)
\]
which implies $\varepsilon(t)$ is decreasing.
Proof. Multiplying the equation in (1.1) by \( u_t \) and integrating over \( \Omega \times (0,t) \). Equality (2.4) follows from Green’s formula.

**Lemma 2.3.** There exists constant \( C > 0 \), independent of the initial data, such that
\[
\varepsilon(t) \leq CE(t), \quad (2.5)
\]
\[
E(t) \leq C[\varepsilon(t) + 1]. \quad (2.6)
\]

Proof. At first we show that there exists a constant such that
\[
\varepsilon(t) \leq CE(t), \quad (2.7)
\]
For that we need to estimate only the nonlinear term in \( \varepsilon(t) \). The continuous embedding \( H^1(\Omega) \subset L^4(\Omega) \) gives
\[
\int_\Omega F(u)dx = \int_\Omega \int_0^u f(z)dzdx \leq \int_\Omega \int_0^{|u|} (|f(z)| + |f(-z)|)dzdx
\]
\[
\leq C \int_\Omega \int_0^{|u|} (1 + |z|^3)dzdx \leq C \int_\Omega (|u| + |u|^4)dx
\]
\[
\leq C(|u|_{L^4(\Omega)} + |u|_4^4) \leq C||u||_{H^1(\Omega)}, \quad (2.8)
\]
which shows that (2.5) holds.

Next we will show an estimate in the opposite direction. Assumption (f-2) implies that there exist \( \delta > 0 \) and \( N = N(\delta) > 0 \) such that \( F(s) \geq -\lambda - \delta s^2 |s| \geq N \).

This can be seen from the following calculation. Let \( M \) be a positive constant such that \( f(z)/z + \lambda \geq 2\delta \) for \( |z| \geq M \) and some \( \delta > 0 \). Then we have
\[
F(s) + \lambda \frac{s^2}{2} = \int_0^M \left( \frac{f(z)}{z} + \lambda \right)zdz + \int_M^s \left( \frac{f(z)}{z} + \lambda \right)zdz
\]
\[
\geq -C + 2\delta \left( \frac{s^2}{2} - \frac{M^2}{2} \right) \geq \delta \frac{s^2}{2} + (-C + \delta \frac{s^2}{2} - \delta M^2)
\]
\[
\geq \delta \frac{s^2}{2} \quad \text{for} \quad s^2 \geq M^2 + 2C/\delta, \quad (2.9)
\]
where
\[
C = |\Omega| \max_{|z| \leq M} |F(z)| + \frac{\lambda M^2}{2}.
\]
For negative \( s \) we have to repeat the same computation with \( M \) replaced by \(-M\). Now
\[
\int_\Omega F(u)dx = \int_{|u| \leq N} F(u)dx + \int_{|u| > N} F(u)dx \geq -\frac{\lambda - \delta}{2} \int_\Omega |u|^2 - C,
\]
where \( C = |\Omega| \max_{|u| \leq N} |F(s)|. \)

This leads to
\[
\varepsilon(t) = \epsilon E(t) + (1 - \epsilon)E(t) + \int_\Omega F(u)dx
\]
\[
\geq \epsilon E(t) + (1 - \epsilon) \left( \frac{\lambda}{2} \int_\Omega |u|^2dx - \frac{\lambda - \delta}{2} \int_\Omega |u|^2dx - \frac{\lambda - \delta}{2} \int_\Omega |u|^2dx - C = \epsilon E(t) - C, \quad (2.10)
\]
where \( \epsilon = \delta / \lambda \). Thus (2.6) follows. \( \square \)
3. Existence of an absorbing set. The main results of this section is the following.

**Theorem 3.1.** Let assumptions 1, 2, and 3 hold. Let \( H \) be an escape vector field on \( \bar{\Omega} \) for the metric \( g = A^{-1}(x) \). Then there exists an absorbing set for problem (1.1), i.e., there exists a \( M > 0 \) for all \( R_0 > 0 \) and initial data \((u_0, u_1) \in \mathcal{H}\) with the property \(\|(u_0, u_1)\|_{\mathcal{H}} \leq R_0\), there exist a \( t_0 = t(R_0) \) such that
\[
\varepsilon(t) \leq M \quad \text{for} \quad t > t_0.
\]

The proof of Theorem 3.1 is based on the following two lemmas.

**Lemma 3.2.** Let assumptions 1 and 2 hold. Let \((u_0, u_1) \in \mathcal{H}\). Then the solution to the boundary value problem (1.1) satisfies the estimate
\[
\varepsilon(T) \leq C_1(T, \|u\|_{L^\infty(0, T; L^2(\Gamma))}) \int_0^T \int_\Gamma b(u_t) u_t d\Gamma dt + C_2(T)L(u) + K(T),
\]
provided \( T \) is sufficiently large. Here
\[
L(u) = (1 + \|u\|_{L^\infty(0, T; L^2(\Gamma))}^2) \|u\|_{H^{1/2}(\Gamma)}^2 + \int_0^T \|u\|_{H^{1/2}(\Omega)}^4 dt + (1 + \|u\|_{L^\infty(0, T; L^2(\Gamma))}^2) \int_0^T \|u\|_{H^{3/2}(\Omega)}^6 dt + \|u\|_{L^\infty(0, T; L^2(\Gamma))}^2
\]
and \( K \) is a constant that depends only on \( \Omega, f \) and \( T \) and not on \( E(0) \).

**Proof.** We shall prove this lemma by modifying the method in [2]. Assume the \( u \) is a solution to (1.1) and that \( T \) is a positive real number. Moreover, we set \( Q_T = (0, T) \times \Omega \) and \( \Sigma_T = (0, T) \times \Gamma \). Let \( p \) be a function on \( \Omega \). Using the multipliers \( H(u) \) and \( pu \) we obtain the following identities. For their proofs, see [27].
\[
\int_0^T \int_\Gamma 2\nu_A H(u) d\Gamma dt + \int_0^T \int_\Gamma (u_t^2 - |\nabla_g u|^2_g) \langle H, \nu \rangle d\Gamma dt = 2(u_t, H(u))_0^T + \int_0^T \int_\Omega 2f(u) H(u) dx dt + \int_0^T \int_\Omega 2DH(\nabla_g u, \nabla_g u) dx dt + \int_0^T \int_\Omega (u_t^2 - |\nabla_g u|^2_g) \text{div} H dx dt,
\]
where
\[
(u_t, H(u)) = \int_\Omega u_t H(u) dx.
\]
\[
\int_0^T \int_\Omega 2(u_t^2 - |\nabla_g u|^2_g) p dx dt = \int_0^T \int_\Gamma u_t^2 p_{\nu_A} d\Gamma dt - \int_0^T \int_\Gamma 2pu_{\nu_A} d\Gamma dt + 2(u_t, up)_0^T + \int_0^T \int_\Omega 2pf(u) ud\Gamma dt - \int_0^T \int_\Omega u^2 (\text{div} A(x) \nabla p) dx dt.
\]
Set
\[
P = \frac{1}{2} (\text{div} H - \delta).
\]
After (3.4) minus (3.6) and using assumption 2, we obtain

\[ \delta \int_0^T \int_{\Omega} \left( u_t^2 + |\nabla u|^2 \right) dx \, dt \]
\[ = \int_0^T \int_{\Omega} \left( u_t^2 - |\nabla u|^2 \right) \delta \, dx \, dt + 2\delta \int_0^T \int_{\Omega} |\nabla u|^2 dx \, dt \]
\[ \leq \int_0^T \int_{\Omega} \left( u_t^2 - |\nabla u|^2 \right) (\div H - 2\rho) \, dx \, dt + 2 \int_0^T \int_{\Omega} D H(\nabla u, \nabla u) \, dx \, dt \]
\[ = \int_0^T \int_{\Gamma} \left( 2(pu + H(u)) u_{\nu,\lambda} + \left( u_t^2 - |\nabla u|^2 \right) (H, \nu) - u^2 p_{\nu,\lambda} \right) \, d\Gamma \, dt \]
\[ + 2 \left( u_t \cdot p + H(u) \right) \bigg|_0^T - \int_0^T \int_{\Omega} 2(pu + H(u)) f(u) \, dx \, dt \]
\[ + \int_0^T \int_{\Omega} u^2 (\div A(x) \nabla p) \, dx \, dt. \]  

(3.7)

Then we have a brief look at the nonlinear function \( h \). For \( s \geq 2R \) we have because of \( h(0) = 0 \)

\[ h(s) = \int_0^s h'(\tau) \, d\tau = s \int_0^1 h'(s\tau) \, d\tau \geq s \inf_{s \geq 2R} \int_0^1 h'(s\tau) \, d\tau \]
\[ \geq s \inf_{s \geq 2R} \int_0^1 h'(s\tau) \, d\tau \geq \frac{s}{2} \inf_{2 < s \leq 1} \frac{1}{s} h'(s) = \frac{1}{2} \inf_{2 < s \leq 1} h'(s) = \frac{s}{2} m_1. \]  

(3.8)

Similarly, for \( s \leq -2R \) we obtain \( h(s) \leq sm_1/2 \). Furthermore, with a given \( \epsilon > 0 \) we have for \( s \in (\epsilon, 2R) \)

\[ h(s) = \int_0^s h'(\tau) \, d\tau \geq \int_0^\epsilon h'(\tau) \, d\tau = m_\epsilon \geq \frac{1}{2} m_\epsilon \frac{s}{2} \]  

with an analogous inequality for \( s \in (-2R, -\epsilon) \).

These two formulas combined result in

\[ h(s) \geq \frac{s^2}{2} m_1 \text{ for } |s| > 2R \text{ and } h(s) \geq \frac{s^2}{2} C_\epsilon \text{ for } |s| \geq \epsilon. \]  

(3.10)

In a similar way we obtain

\[ h(s) \leq s \max \left\{ \sup_{|\xi| \leq R} h'(\xi), m_2 \right\} \text{ for } s > 0. \]  

(3.11)

From (3.10) and the boundary condition in (1.1) we obtain

\[ \int_{|u_t| \geq \varepsilon} u_t^2 d\Gamma \, dt \leq C \int_{\Sigma_T} |u_t|(|u| + |u_{\nu,\lambda}|) \, d\Gamma \, dt, \]

which yields

\[ \int_{|u_t| \geq \varepsilon} u_t^2 d\Gamma \, dt \leq C(||u||^2_{L^2(\Sigma_T)} + ||\nabla u||^2_{L^2(\Sigma_T)}) + C \int_{|u_t| \leq \varepsilon} u_t^2 d\Gamma \, dt. \]  

(3.12)

The first term in the boundary integral on the right hand side of (3.7) can be estimated by using Cauchy’s inequality and (3.12) as follows.

\[ \int_0^T \int_{\Gamma} \left[ 2(pu + H(u)) u_{\nu,\lambda} + \left( u_t^2 - |\nabla u|^2 \right) (H, \nu) - u^2 p_{\nu,\lambda} \right] \, d\Gamma \, dt \]
Similarly, we can estimate the second term on the right hand side of (3.7) as follows.

\[ \int_{\Omega} |u|^4 \, dx \leq C \int_{\Omega} |u_{\tau}|^4 \, dx + C \left( \|u\|_{L^4(\Omega)}^4 + \|u\|_{H^{\frac{3}{2}}(\Omega)}^4 \right) \]

\[ \int_{\Gamma} |u|^4 \, d\tau \leq C \int_{\Gamma} |u_{\tau}|^4 \, d\tau + C \left( \|u\|_{L^4(\Gamma)}^4 + \|u\|_{H^{\frac{3}{2}}(\Gamma)}^4 \right) \]

Then we estimate the term involving the non-linear function \( f(u) \).

\[
\int_{\Omega} H(u)f(u)\, dx = \int_{\Omega} \langle H, \nabla(F(u)) \rangle \, dx = \int_{\Omega} \text{div} (F(u)H) \, dx - \int_{\Omega} F(u) \text{div} H \, dx \\
- \int_{\Gamma} F(u)H \cdot \nu \, d\tau - \int_{\Omega} F(u) \, H \, dx.
\]

Hence, we have

\[ -2 \int_{\Omega} H(u)f(u)\, dx \leq C \int_{\Gamma} |F(u)| \, d\tau + C \int_{\Omega} |F(u)| \, dx. \]  

(3.16)

The continuous embedding \( H^{\frac{3}{2}}(\Omega) \subset L^4(\Omega) \) give

\[
- \int_{\Omega} f(u)\, dx \leq \int_{\Omega} (|u| + |u|^4) \, dx \leq C \|u\|_{L^4(\Omega)} + C \|u\|_{H^{\frac{3}{2}}(\Omega)}^4
\]

(3.17)

and

\[
\int_{0}^{T} \int_{\Omega} |F(u)| \, dx \, dt \leq C \int_{0}^{T} \int_{\Omega} (|u| + |u|^4) \, dx \, dt \leq C \int_{0}^{T} \left( \|u\|_{L^4(\Omega)} + C \|u\|_{H^{\frac{3}{2}}(\Omega)}^4 \right) \, dt
\]

\[ \leq C_1 \|u\|_{L^2(\Sigma_T)} + C_1 \int_{0}^{T} \|u\|_{H^{\frac{3}{2}}(\Omega)}^4 \, dt. \]  

(3.18)

We use similar ideas when we estimate the integral of \( F(u) \) on the boundary \( \Sigma \). Thus,

\[
\int_{0}^{T} \int_{\Gamma} |F(u)| \, d\tau \, dt
\]

\[ \leq C \int_{0}^{T} \int_{\Gamma} (|u| + |u|^4) \, d\tau \, dt \leq C \int_{0}^{T} \left( \|u\|_{L^4(\Gamma)} + C \|u\|_{H^{\frac{3}{2}}(\Gamma)}^4 \right) \, dt
\]

\[ \leq C \|u\|_{L^2(\Sigma_T)}^2 + C_1 \int_{0}^{T} \|u\|_{H^{\frac{3}{2}}(\Gamma)}^4 \, dt \leq C \|u\|_{L^2(\Sigma_T)} + C \int_{0}^{T} \|u\|_{H^{1}(\Gamma)}^2 \|u\|_{L^2(\Gamma)}^2 \, dt
\]

\[ \leq C \|u\|_{L^2(\Sigma_T)}^2 + C \int_{0}^{T} \|u\|_{L^2(\Sigma_T)}^2 \, dt \leq C \|u\|_{L^2(\Sigma_T)}^2 + C \int_{0}^{T} \|\nabla u\|_{L^2(\Sigma_T)}^2 \, dt
\]

\[ \leq C \|u\|_{L^2(\Sigma_T)}^2 + C \|u\|_{L^2(\Sigma_T)}^2 \left( \|u\|_{L^2(\Sigma_T)}^2 + \|\nabla u\|_{L^2(\Sigma_T)}^2 \right), \]

(3.19)

where we used the continuity of the embedding \( H^{\frac{3}{2}}(\Gamma) \subset L^4(\Gamma) \) and the interpolation inequality in Sobolev spaces

\[ \|u\|_{H^{\frac{3}{2}}(\Gamma)}^2 \leq C \|u\|_{H^1(\Gamma)} \|u\|_{L^2(\Gamma)}. \]  

(3.20)

For the tangential derivatives \( \nabla_{\tau} u \), we rely on a result in [17] Lemma 7.2. For small \( \alpha > 0 \) and \( \delta > 0 \) there exists a constant \( C = C(\alpha, \delta, T) \), such that

\[
\int_{0}^{T-\alpha} \int_{\Gamma} |\nabla_{\tau} u|^2 \, d\tau \, dt
\]
Hence, we have
\[\|f(u)\|_{H^{-\frac{1}{2}+\delta}(\Omega)} \leq \left(\int_{\Omega} (1 + |u|^3)^{\frac{6}{5\sigma}} \, dx \right)^{\frac{5\sigma}{6}} \leq C\|u\|_{L^{\frac{18}{5}}(\Omega)}^{\frac{18}{5\sigma}} + K \leq C\|u\|_{H^{\frac{1}{6}+\frac{\delta}{2}}(\Omega)}^3 + K. \quad (3.22)\]

Hence,
\[\|f(u)\|_{H^{-\frac{1}{2}+\delta}(Q_T)} \leq \|f(u)\|_{L^2(0,T;H^{-\frac{1}{2}+\delta}(\Omega))} = \int_0^T \|f(u)\|_{H^{-\frac{1}{2}+\delta}(\Omega)}^2 \, dt \leq C \int_0^T \|u\|_{H^{\frac{1}{6}+\frac{\delta}{2}(\Omega)}}^6 \, dt + K_1. \quad (3.23)\]

We like to point out that the estimate (3.7) is not only valid in the interval (0,T); we can also apply it on the interval (a, T - a). With this in mind we combine now the previous estimates (3.13),(3.14),(3.19),(3.21) and (3.23) into
\[\int_0^T E(t) \, dt \leq C(T)\left(1 + \|u\|_{L^\infty(0,T;L^2(\Omega))}^2\right) \left(\|u_t\|_{L^2(\Sigma_T)}^2 + \|u_{tt}\|_{L^2(\Sigma_T)}^2\right) + C(T)L(u) + C(E(0) + E(T)) + K_1. \quad (3.24)\]

We like to extend the integral on the right hand side over the interval (0,T). For that sake we can make use of formula (2.6) and the fact that the energy is non-increasing, see (2.4):
\[\int_0^T E(t) \, dt + \int_{T-\alpha}^T E(t) \, dt \leq C\left(\int_0^\alpha \varepsilon(t) \, dt + \int_{T-\alpha}^T \varepsilon(t) \, dt + 2\alpha\right) \leq C(2\alpha\varepsilon(0) + 2\alpha). \quad (3.25)\]

With a similar argument we can replace the linear energy terms on the right hand side in (3.24) by the nonlinear energy function, i.e.,
\[E(0) + E(T) \leq C\left[\varepsilon(0) + \varepsilon(T) + 2\right]. \quad (3.26)\]

Applying the last two formulas (3.25) and (3.26) to the estimate (3.24) we obtain
\[\int_0^T E(t) \, dt \leq C(T)\left(1 + \|u\|_{L^\infty(0,T;L^2(\Omega))}^2\right) \left(\|u_t\|_{L^2(\Sigma_T)}^2 + \|u_{tt}\|_{L^2(\Sigma_T)}^2\right) + C(T)L(u) + C(\varepsilon(0) + \varepsilon(T)) + 2(1 + \alpha) + K_1. \quad (3.27)\]

Now we use the fact that
\[\int_0^T \varepsilon(t) \, dt \geq T\varepsilon(T), \quad (3.28)\]
which follows from the energy identity (2.4). Thus,
\[T\varepsilon(T) \leq C(T)\left(1 + \|u\|_{L^\infty(0,T;L^2(\Omega))}^2\right) \left(\|u_t\|_{L^2(\Sigma_T)}^2 + \|u_{tt}\|_{L^2(\Sigma_T)}^2\right) + C(T)L(u) + C\varepsilon(T) + C \int_0^T h(u_t)u_t \, dt + K, \quad (3.29)\]

where \(K\) is a positive constant.

If we choose now \(T > 2C\), we can absorb the energy term on the right hand side of (3.29) into the left hand side. Hence,
\[\varepsilon(T) \leq C(T)\left(1 + \|u\|_{L^\infty(0,T;L^2(\Omega))}^2\right) \left(\|u_t\|_{L^2(\Sigma_T)}^2 + \|u_{tt}\|_{L^2(\Sigma_T)}^2\right)\]
\[ + C(T)L(u) + C \int_0^T \int_{\Gamma} h(u_t)u_t d\Gamma dt + K. \] (3.30)

We introduce the following notation to estimate of the boundary traces on the right hand side of (3.30).

\[ \Sigma_A = \{(x, t) \in \Sigma_T, |u_t| \geq 2R, a.e. \} \quad \text{and} \quad \Sigma_B = \Sigma_T \setminus \Sigma_A. \] (3.31)

Taking the boundary condition and the growth conditions (3.10), (3.11) into consideration we obtain

\[
\|u_t\|^2_{L^2(\Sigma_T)} + \|u_\nu\|^2_{L^2(\Sigma_T)} \\
\leq \|u_t\|^2_{L^2(\Sigma_T)} + \|h(u_t)\|^2_{L^2(\Sigma_T)} + \|u\|^2_{L^2(\Sigma_T)} \\
\leq 4R^2|\Sigma_B| + \frac{2}{m_1} \int_{\Sigma_A} h(u_t)u_t d\Gamma dt \\
+ \max\{ \sup_{0 < |\xi| < R} h'(\xi), m_2 \} \int_0^T \int_{\Gamma} h(u_t)u_t d\Gamma dt + \|u\|^2_{L^2(\Sigma_T)}. \] (3.32)

We obtain (3.2) by inserting (3.32) into (3.30).

The next lemma shows that the lower order terms in Lemma 3.2 can be absorbed.

**Lemma 3.3.** Let assumptions 1, 2, and 3 hold. Assume that \((u, u_t) \in C([0, T); \mathcal{H}) \) is a solution to (1.1) with \(\|(u_0, u_1)\|_{\mathcal{H}} \leq R_0\). Then there exist two constants \(C = C(T, E(0))\) and \(K = K(T)\) (independent of \(E(0)\)) such that

\[ \|u_t\|_{L^2(\Sigma_T)} + \|u_\nu\|_{L^2(\Sigma_T)} \leq C \int_0^T \int_{\Gamma} h(u_t)u_t d\Gamma dt + K(T) \] (3.33)

provided \(T\) and \(K(T)\) are sufficiently large.

**Proof.** The proof relies on an indirect compactness/uniqueness argument. Suppose that the desired estimate is not true. Then there exists a sequence \((u_n, u_{nt}) \subset C([0, T); \mathcal{H})\) of solutions to (1.1) such that

\[ L(u^n) > K \quad \text{for all} \quad n, \] (3.34)

\[ \lim_{n \to +\infty} \int_0^T \int_{\Gamma} h(u^n_t)u^n_t d\Gamma dt = 0, \] (3.35)

\[ E(u^n(0)) = M_0 \quad \text{for all} \quad n, \] (3.36)

where \(M_0\) is some positive constant. Of course, we have to choose

\[ K > L(u) \] (3.37)

for all stationary solutions \(u\), i.e., solutions of (1.1) that satisfy \(u_t \equiv 0\).

From (3.35) we infer that

\[ \lim_{n \to +\infty} h(u^n_t) = 0 \quad \text{in} \quad L^2(\Sigma_T). \] (3.38)

Moreover, we also claim

\[ \lim_{n \to +\infty} u^n_t = 0 \quad \text{in} \quad L^2(\Sigma_T). \] (3.39)

This can be seen as follows. As before we set

\[ \Sigma_{A} = \{(x, t) \in \Sigma_T, |u_t| \geq 2\epsilon, a.e. \} \quad \text{and} \quad \Sigma_{B} = \Sigma_T \setminus \Sigma_{A}. \] (3.40)
By formula (3.10) for any $\epsilon > 0$ we have
\[
\int_{\Sigma_T} |u^n_t|^2 = \int_{\Sigma^0_T} |u^n|^2 + \int_{\Sigma^0_T} |u^n|^2 \leq \frac{2}{C_\epsilon} \int^T_0 \int_{\Gamma} h(u^n_\nu)u^n_\nu d\Gamma dt + \epsilon^2 |\Sigma_T|, \tag{3.41}
\]
which implies that
\[
\lim_{n \to +\infty} \int_{\Sigma_T} |u^n_t|^2 \leq \epsilon^2 |\Sigma_T| \quad \text{for every } \epsilon > 0.
\tag{3.42}
\]
Thus we obtain (3.39). By Theorem 2.1 there exists a constant $C$ such that
\[
\|(u^n, u^n_\nu)\|_{\mathcal{H}} \leq C \quad \text{for all } n \text{ and } t \in [0, \infty).
\tag{3.43}
\]
Consequently, a subsequence denoted again by the superscript $n$ converges weakly to some limit function $u$, i.e.,
\[
\omega^* = \lim_{n \to +\infty} u^n = u \quad \text{in } L^\infty(0, T; H^1(\Omega)); \tag{3.44}
\]
\[
\omega^* = \lim_{n \to +\infty} u^n_\nu = u_\nu \quad \text{in } L^\infty(0, T; L^2(\Omega)). \tag{3.45}
\]
Hence,
\[
\omega = \lim_{n \to +\infty} u^n = u \quad \text{in } H^1(Q_T). \tag{3.46}
\]

The compactness of the embedding $H^1(\Omega) \subset H^{1-\epsilon}(\Omega)$ for $\epsilon > 0$ and Aubin’s lemma [21, Corollaries 4 and 5], yield
\[
\lim_{n \to +\infty} L(u^n) = L(u). \tag{3.47}
\]

The convergence (3.44), (3.45) allows to show that the limit function $u$ satisfies weak form of differential (1.1). Regarding the nonlinear term, we have by means of Holder’s inequality and the continuous embedding $H^1(\Omega) \subset L^6(\Omega)$
\[
(f(u^n) - f(u), \phi)
\leq \|u^n - u\|_{L^2(\Omega)} \left\| \int^T_0 f'(su^n + (1 + s)u)ds\phi\right\|_{L^2(\Omega)}
\leq \|u^n - u\|_{L^2(\Omega)}(1 + \|u\|^2_{L^6(\Omega)} + \|u^n\|^2_{L^6(\Omega)})\|\phi\|_{L^6(\Omega)}
\leq \|u^n - u\|_{L^2(\Omega)}(1 + \|u\|^2_{H^1(\Omega)} + \|u^n\|^2_{H^1(\Omega)})\|\phi\|_{H^1(\Omega)}, \tag{3.48}
\]
which proves
\[
\lim_{n \to +\infty} f(u^n) - f(u) = 0 \quad \text{in } [H^1(\Omega)]' \quad \text{a.e. } t.
\tag{3.49}
\]

Thus, the limit function $u$ is a weak solution to the differential (1.1). Moreover $u$ satisfies the boundary condition
\[
u_\nu + u = 0 \quad \text{on } \Sigma_T, \tag{3.50}
\]

Setting $w = u_\nu$ and differentiating (1.1) with respect to time we obtain
\[
\begin{cases}
w_{tt} - \text{div} A(x)\nabla w + f'(u)w = 0 & (x, t) \in \Omega \times (0, T),
w_{\nu, \nu} = u = 0 & (x, t) \in \Gamma \times (0, T). \tag{3.51}
\end{cases}
\]

Let $\Delta_g$ be the Laplacian in the metric of $g$, then
\[
\Delta_g w = \text{div} A(x)\nabla w - \langle Dw, Dv_g \rangle, \tag{3.52}
\]
where $2v_g = \log \det A(x)$ for $x \in \bar{\Omega}$
By assumption 3, we conclude that $u_t \equiv 0$ in $\mathbb{R} \times \Omega$. Hence, the limit function $u$ is a solution to the stationary boundary problem

$$\begin{align*}
\text{div} A(x) \nabla u + f(u) &= 0 & x &\in \Omega, \\
u_{\nu A} + u &= 0 & x &\in \Gamma.
\end{align*}$$

We multiply the first equation in (3.53) by $u$ and integrate by parts to get that the solutions to (3.53) are bounded, i.e., $\|u\|_{H^1(\Omega)} \leq C$ for some constant $C$ by using (f-2).

On the other hand we have by (3.34) and (3.47) $L(u) \geq K$ which contradicts (3.37).

Now we are in a position to complete the proof of Theorem 3.1.

**Proof of Theorem 3.1.** By the same arguments as in the case of constant coefficients [2, p.1924], there exists a constant $T_0(E(0))$ such that $\varepsilon(t) \leq K(T) + 1$ for all $t > T_0(E(0))$. (3.54)

The final statement of Theorem 3.1 follows now from (3.54) and (2.6).

4. **Existence of global and compact attractor.** The main result of this section is the following.

**Theorem 4.1.** With reference to system (1.1) under assumptions 1, 2, and 3, there exists a global, compact attractor $\mathcal{B} \subset \mathcal{H}$.

The proof of this theorem follows [2]. Here we just present a sketch of the proof. For continuous dynamical systems, we have the following lemmas [12].

**Lemma 4.2.** If $T(t) : X \to X, t \geq 0$, is asymptotically smooth, point dissipative, and orbits of bounded sets are bounded, then there exists a global attractor $\mathcal{B}$.

**Lemma 4.3.** Suppose $T(t) = S(t) + U(t)$ where $U(t)$ is completely continuous for $t \geq 0$ and there is a continuous function $k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $k(t, r) \to 0$ as $t \to \infty$ and $|S(t)x| \leq k(t, r)$ for $t \geq 0, |x| \leq r$. Then $T(t)$ is asymptotically smooth.

Since we already know that there exists an absorbing set $\mathcal{B}$, it suffices to show that the semigroup $T(t)$ is asymptotically smooth. To accomplish this, it is enough to establish a decomposition of the flow into an uniformly stable part and a compact part. To this end we propose the following decomposition (see [14, 15] where similar decomposition has been used in the context of plate equations), depending on a positive constant $\gamma$ to be selected later. With $u$ solution to (1.1) we consider the following system of equations in the variables $y$ and $z$.

$$\begin{align*}
y_{tt} - \text{div} A(x) \nabla y + f(y) + \gamma y &= 0 & (x,t) &\in Q_T, \\
y_{\nu A} + y + h(y_t + z_t) - h(z_t) &= 0 & (x,t) &\in \Sigma_T, \\
y(0) = u_0, & \quad y_t(0) = u_1
\end{align*}$$

$$\begin{align*}
z_{tt} - \text{div} A(x) \nabla z + f(u - z) - \gamma(u - z) &= 0 & (x,t) &\in Q_T, \\
z_{\nu A} + z + h(z_t) &= 0 & (x,t) &\in \Sigma_T, \\
z(0) = z_t(0) &= 0
\end{align*}$$

By applying the same arguments as in [2, sec.4] one can obtain the existence and uniqueness of a global generalized solution to (4.1) and (4.2).
Let us denote by $T(t)$ the semigroup generated on $\mathcal{H}$ by the original problems (1.1). In view of (4.1) and (4.2) we have the following decomposition
\begin{equation}
T(t) = S(t) + K(t),
\end{equation}
where
\begin{align*}
S(t)(u_0, u_1) &= (y(t), y_t(t)), \\
K(t)(u_0, u_1) &= (z(t), z_t(t)).
\end{align*}
[2, Theorem 4.4] shows that for a specific choice of $\gamma$ the semigroup $K(t)$ is compact for each $t \geq 0$.

In the following we will show that the semigroup $S(t)$ is uniformly stable on $\mathcal{H}$. We observe that the feedback function $h_0(s) = h(s + a) - h(a)$ is monotone, and $h_0(0) = 0$ for all $a \in \mathbb{R}$. Furthermore, the function $h_0(s)$ satisfies a growth condition for large $s$, in particular (see [2] Proposition 4.3)
\begin{align*}
h_0'(s) &\leq M \quad \text{for all } s, a \in \mathbb{R}, \\
sh_0(s) &\geq ms^2 \quad \text{for all } a \in \mathbb{R} \text{ and } |s| \geq 1,
\end{align*}
with a suitable positive constants $m, M$.

By applying the same arguments as in Sec. 3 and [16, Proposition 3.5] one can show that
\begin{equation}
E(t)dt \leq Ce^{-\alpha t}, \quad t > T_0
\end{equation}
for some $T_0 > 0$, where $C > 0$ and $\alpha > 0$ depend on $E(0)$.

(4.8) implies the semigroup $S(t)$ is uniformly stable on $\mathcal{H}$. As a consequence of Lemma 4.2, the proof of Theorem 4.1 is thus completed.

5. Structure of the global attractor. In this section we follows [2] to obtain the result on the structure of the attractor $B$.

As above we denote by $T(t)$ the evolution semigroup generated by problem (1.1) in the space $\mathcal{H}$. By Theorem 4.1 this evolution semigroup possesses a global compact attractor $B$ which belongs to $\mathcal{H}$. If $W_0 = (u_0, u_1) \in B$, then the invariance of $B$, implies that there exists a full trajectory $\gamma = \{W(t) : t \in \mathbb{R}\} \subset B$ such that $W(0) = (u_0, u_1)$.

Let $N$ be the set of stationary points of $T(t)$, i.e.,
\begin{equation}
N = \{V \in \mathcal{H} : T(t)V = V, \quad \forall t \geq 0\}.
\end{equation}
Since $N \subset B$ and $B$ is a compact set in $\mathcal{H}$, $N$ is a compact set in $\mathcal{H}$. It is clear that every stationary point $W$ has the form $W = (\psi, 0)$, where $\psi = \psi(x)$ solves the problem
\begin{equation}
\begin{cases}
-\text{div} \, A(x) \nabla \psi + f(\psi) = 0 & x \in \Omega, \\
\psi_{\nu_x} + \psi = 0 & x \in \Gamma.
\end{cases}
\end{equation}
Let us define the unstable manifold $M^u(N)$ emanating from the set $N$ as a set of all $Y \in \mathcal{H}$ such that there exists a full trajectory $\gamma = \{W(t) : t \in \mathbb{R}\}$ with the properties $W(0) = Y$, and \( \lim_{t \to -\infty} \text{dist}_\mathcal{H}(W(t), N) = 0 \).

That is
\begin{equation}
M^u(N) = \{Y \in \mathcal{H} : \exists W(t), \ W(0) = Y, \lim_{t \to -\infty} \text{dist}_\mathcal{H}(W(t), N) = 0\}.
\end{equation}
We have the following result on the structure of the global and long-time behavior of individual solutions.
Theorem 5.1. Under assumptions 1, 2, and 3, we have
\( \mathcal{B} = M(u)(N) \);
\( \lim_{t \to +\infty} \text{dist}_\mathcal{H}(T(t)W, N) = 0 \), for any \( W \in \mathcal{H} \), i.e., any trajectory stabilizes to the set \( N \) of the stationary points.

The proof of the Theorem 5.1 can be obtained by the same arguments as in the case of constant coefficients but with a few modifications, for example, see [2, Theorem 5.1 and Theorem 5.2]. We omit the details.

6. Regularity of the attractor. In this section we show that the attractor obtained in Theorem 4.1 is, in fact, more regular as in [2]. To accomplish this we need to impose additional assumptions restricting the growth of nonlinear function \( f(u) \).

Theorem 6.1. Let assumptions in Theorem 1.3 hold. Then \( \mathcal{B} \) is a closed bounded set of \( H^2(\Omega) \times H^1(\Omega) \).

The proof of the Theorem 6.1 can be obtained by the same arguments as in the case of constant coefficients but with a few modifications, for example, see [2, Theorem 6.1]. We omit the details.

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