The Similarity Problem for J–nonnegative Sturm–Liouville Operators

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Vienna, Preprint ESI 1987 (2007)  
December 7, 2007

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at
The similarity problem for $J$-nonnegative Sturm-Liouville operators

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Abstract

New sufficient conditions for the similarity of a Sturm-Liouville operator $A := \frac{1}{r(x)} \left( -\frac{d^2}{dx^2} + q(x) \right)$ with an indefinite weight $r(x) = (\text{sgn } x)|r(x)|$ to a self-adjoint operator in $L^2(\mathbb{R}, |r| dx)$ are obtained. These conditions are formulated in terms of Weyl-Titchmarsh $m$-coefficients. New sufficient conditions for the regularity of the critical points $0$ and $\infty$ of $J$-nonnegative Sturm-Liouville operators are obtained too. This result is exploit to prove the regularity of the critical point zero for various classes of $J$-nonnegative Sturm-Liouville operators. This implies the similarity of the considered operators to self-adjoint ones. In particular, in the case $r(x) = \text{sgn } x$ and $q \in L^1(\mathbb{R}, (1 + |x|)dx)$ we prove that $A$ is similar to a self-adjoint operator if and only if $A$ is $J$-nonnegative. Moreover, the latter condition on $q$ is sharp, that is we construct a potential $q_0 \in \cap_{\gamma<1} L^1(\mathbb{R}, (1 + |x|^\gamma)dx)$ such that the operator $A$ is $J$-nonnegative with the singular critical point zero and hence is not similar to a self-adjoint one. For periodic or infinite-zone potentials $q$, we show the sufficiency of $J$-positivity for the similarity of $A$ to a self-adjoint operator. In the case $q \equiv 0$, we prove the regularity of the critical point $0$ of $A$ for a wide class of weight functions $r$. This yields new results for "forward-backward" diffusion equations.

Keywords: $J$-self-adjoint operator, non-self-adjoint operator, Sturm-Liouville operator, Weyl-Titchmarsh $m$-function, similarity, spectral function of $J$-nonnegative operators, critical points

Subject classification: 47E05, 34B24, 34B09, 34L10, 47B50

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1 Introduction

Consider a Sturm-Liouville equation

$$-y''(x) + q(x)y(x) = \lambda r(x)y(x), \quad x \in \mathbb{R},$$  \hspace{1cm} (1.1)

with a real potential $q$ and an indefinite weight $r$ having only one turning point $x = 0$, that is $r(x) = (\text{sgn} \ x)|r(x)|$. Assume that $q$ and $r$ satisfy conditions $q, r \in L^1_{\text{loc}}(\mathbb{R})$, and $|r(x)| > 0$ a.e. on $\mathbb{R}$.

Consider the operator $L = \frac{1}{|r(x)|} \left( -\frac{d^2}{dx^2} + q(x) \right)$ defined on its maximal domain $\mathfrak{D}$ in the Hilbert space $L^2(\mathbb{R}, |r|dx)$. If $L = L^*$ ($L \geq 0$), then the operator

$$A := \frac{(\text{sgn} \ x)}{|r(x)|} \left( -\frac{d^2}{dx^2} + q(x) \right), \quad \text{dom}(A) = \mathfrak{D},$$  \hspace{1cm} (1.2)

associated with (1.1) is called $J$-self-adjoint (resp., $J$-nonnegative). This means that $A$ is self-adjoint (nonnegative) with respect to the indefinite inner product

$$[f,g] := (Jf,g) = \int_{\mathbb{R}} f\overline{g}r \, dx, \quad r(x) = (\text{sgn} \ x)|r(x)|,$$

where the operator $J$ is defined by $J : f(x) \mapsto (\text{sgn} \ x)f(x)$.

In this paper, we will always assume that

$$L = L^*, \quad \text{i.e., the differential expression (1.1) is limit point at } +\infty \text{ and } -\infty.$$  \hspace{1cm} (1.3)

So the operator $A$ is $J$-self-adjoint. However, $A$ is non-self-adjoint in $L^2(\mathbb{R}, |r|dx)$.

The main problem we are concerned with is the problem of similarity of a $J$-nonnegative operator $A$ of type (1.2) to a self-adjoint operator. Recall that two closed operators $T_1$ and $T_2$ in a Hilbert space $\mathfrak{H}$ are called similar if there exist a bounded operator $S$ with the bounded inverse $S^{-1}$ in $\mathfrak{H}$ such that $S \text{dom}(T_1) = \text{dom}(T_2)$ and $T_2 = ST_1S^{-1}$.

Both ordinary and partial differential operators with indefinite weights have intensively been investigated during two last decades (see [32, 6, 11, 56, 57, 12, 18, 20, 60, 22, 36, 16, 17, 55, 40, 62, 44, 38, 7, 41] and references). The similarity of the operator (1.2) to a self-adjoint one is essential for
the solution of forward-backward boundary value problems arising in certain physical models and in the theory of random processes (see [19, 6, 26, 25, 13, 23, 37] and references therein).

Let us explain the difficulties arising in this problem by considering a very simple example. Let \( A \in \mathbb{C}^{n \times n} \) be a matrix such that \( A = J B \), where \( B = B^* > 0 \) and \( J = J^* \). Then \( A \) is similar to a self-adjoint matrix since \( B^{1/2} AB^{-1/2} = B^{1/2} JB^{1/2} = (B^{1/2} JB^{1/2})^* \). Moreover, this result remains valid if \( B \geq 0 \) and \( \ker A = \ker A^2 \). That is, \( A = J B \) with \( B = B^* \geq 0 \) is similar to a self-adjoint matrix if zero eigenvalue is simple. One can easily extend this result for the case of a bounded \( J \)-nonnegative operator \( A = J B \) in a Hilbert space \( H \) if \( 0 \notin \sigma_{\text{ess}}(B) \), but it does not hold true for an arbitrary \( B = B^* \geq 0 \) in \( H \) even if \( J = J^* = J^{-1} \) (simple examples were constructed in [59]). Thus additional requirements on \( B \) are needed.

Spectral theory of \( J \)-nonnegative operators was constructed by M.G. Krein and H. Langer [27, 48] (necessary notations and facts are contained in Subsection 2.4). If the resolvent set of a \( J \)-nonnegative operator \( A \) is nonempty, \( \rho(A) \neq \emptyset \), then the spectrum \( \sigma(A) \) of \( A \) is real. Moreover, \( A \) admits a spectral function \( E_A(\cdot) \) with properties similar to that of a spectral function of a self-adjoint operator. The main difference is the occurrence of critical points. Significantly different behavior of the spectral function \( E_A(\cdot) \) occurs at singular critical point in any neighborhood of which it is unbounded. The critical points, which are not singular, are called regular. It should be stressed that only 0 and \( \infty \) may be critical points for \( J \)-nonnegative operators. Furthermore, the problem of similarity to a self-adjoint operator is equivalent to the problem of their nonsingularity (see Proposition 2.6).

If the spectrum \( \sigma(A) \) is discrete, the similarity of \( A \) to a self-adjoint operator is equivalent to the Riesz basis property of eigenvectors. For this case, Beals [6] showed that the eigenfunctions of Sturm-Liouville problems of the type (1.1) form a Riesz basis if \( r(x) \) behaves like \( (\text{sgn } x)|x|^{\beta} \), \( \beta > -1/2 \), at 0. Improved versions of Beals’ condition were provided in [11, 57, 60, 20, 55]. In [11, 20], differential operators with indefinite weights and nonempty essential spectrum were considered and the regularity of the critical point \( \infty \) was proved for a wide class of weight functions under certain positivity assumptions. Indeed, it follows from the results of Curgus and Langer [11, Section 3] that \( \infty \) is a regular critical point of \( J \)-nonnegative \( J \)-self-adjoint operator (1.2) if the weight function \( r(\cdot) \) is absolutely continuous on certain intervals \( I_{\delta}^+ = (0, \delta), \ I_{\delta}^- = (-\delta, 0), \) and there exist constants \( s_+ > 0 \) and \( s_- > 0, \ s_\pm \neq 1, \) such that

\[
\lim_{x \to \pm 0} \frac{r(x)}{r(s_\pm x)} = s_\pm, \quad \text{and} \quad \left( \frac{r(x)}{r(s_\pm x)} \right)' \in L^\infty(I_{\delta}^\pm).
\]

(1.4)

In particular, (1.4) is valid if

\[
r(x) = (\text{sgn } x)p_\pm(x)|x|^{\beta_\pm}, \quad x \in I_{\delta}^\pm, \quad \beta_\pm > -1,
\]

where \( p_+ \in C^1[0, \delta], \ p_- \in C^1[-\delta, 0], \) and \( p_{\pm}(\pm x) > 0, \ x \in [0, \delta] \).

The existence of Sturm-Liouville operators (1.2) with the singular critical point \( \infty \) was established by Volkmer [60] in 1996, corresponding examples were constructed later (see [20, 1, 21, 55] and references).

The question of nonsingularity of 0 occurs much harder. Several abstract criteria for similarity and regularity of critical points may be found in [59, 3, 10, 9, 53, 51, 33, 34], but it is not easy to apply them to the operators of type (1.2). First results were obtained for the operators \( (\text{sgn } x)|x|^{-\alpha} \frac{d^2}{dx^2} \), \( \alpha > -1 \), by Curgus, Najman, and Fleige (see [12] for the case \( \alpha = 0 \), and [22] for an arbitrary \( \alpha > -1 \)). Their approach was based on the abstract regularity criterion [10, Theorem 3.2]. Another approach based on the resolvent criterion of similarity (see Theorem 2.2) was used by the authors of the present paper [35, 36, 40, 44, 41] as well as by Faddeev and Shertenberg [16, 17]. Namely, in
[35, 36], the result of [12] was reproved (see also [34]). It was shown in [16] that if \( r(x) = \text{sgn} \, x, \int_{\mathbb{R}}(1 + x^2)|q(x)|dx < \infty \) and \( \sigma(A) \subset \mathbb{R} \), then \( A \) is similar to a self-adjoint operator. The case when \( q \equiv 0 \) and \( r(x) \asymp \pm x^{\alpha_{\pm}+} \), \( \alpha_{\pm} > -1 \), as \( x \to \pm \infty \), was considered in [17, 44]. The complete analysis for the case of a finite-zone potential was done in [41]. However, the proof of the main theorem [41, Theorem 7.2] is technically complicated and can not be extended to the classes of potentials considered in this paper.

Our main aim is to present a simple and effective condition of the regularity for the critical point 0 as well as to apply it to various classes of Sturm-Liouville operators. In particular, we will show that restrictions imposed on the coefficients in [16, 17] are superfluous (see Remarks 4.3, 7.1) and give simple proofs for the results of [22, 44] and [41, Corollary 7.4].

Our method is based on two different ideas of the papers [41, 38] and on results of [11, Section 3] (condition (1.4)). Namely, the resolvent criterion \([9, 53, 51]\) was used in [41] to reduce the similarity problem to the theory of two weighted estimates for the Hilbert transform. This allowed to obtain several necessary and sufficient similarity conditions in terms of Weyl-Titchmarsh \( m \)-coefficients. In [38], another necessary resolvent similarity condition was combined with the Krein space method to prove that a slightly weaker condition is necessary and also to give its local version.

Several examples of Sturm-Liouville operators with the singular critical point 0 were constructed in [38]. In particular, it was proved that there exists a continuous potential \( q \in L^2(\mathbb{R}) \) such that the operator \((\text{sgn} \, x)(-d^2/dx^2 + q)\) is \( J \)-nonnegative and 0 is its singular critical point. The second aim of this paper is to present an explicit potential \( q_0 \) with the same property (see Theorem 5.2).

Some results of the present paper were announced without proofs in brief communications [40, 45].

First (see Section 3), we obtain a local regularity condition in terms of the Weyl-Titchmarsh \( m \)-coefficients \( M_\pm(\lambda) \) associated with the operator (1.2) on \( \mathbb{R}_+ \) (explicit definitions are given in Section 2.4). Namely, if

\[
\sup_{\lambda \in \Omega_R^0} \left| \frac{M_+(\lambda) + M_-(\lambda) - c}{M_+(\lambda) - M_-(\lambda)} \right| < \infty, \quad \Omega_R^0 := \{ \lambda \in \mathbb{C}_+ : |\lambda| < R \},
\]

where \( R > 0 \) and \( c \in \mathbb{R} \) are some constants, and the operator (1.2) is \( J \)-nonnegative, then 0 is not a singular critical point. This result is an improvement of [41, Theorem 5.9], where \( c = 0 \) and supremum is taken over \( \lambda \in \mathbb{C}_+ \).

We apply condition (1.5) to operators \( A \) with \( r(x) = \text{sgn} \, x \) and potentials \( q \) such that

\[
\int_{\mathbb{R}}(1 + |x|)|q(x)|dx < \infty.
\]

For this class, the following theorem gives a complete characterization of operators that are similar to self-adjoint ones.

**Theorem 1.1.** Suppose \( r(x) = \text{sgn} \, x, \ x \in \mathbb{R} \); so the \((J\text{-self-adjoint})\) operator \( A \) has the form \((\text{sgn} \, x)(-d^2/dx^2+q(x))\). If the potential \( q \) satisfies (1.6), then the following statements are equivalent:

(i) \( A \) is similar to a self-adjoint operator,

(ii) \( A \) is \( J \)-nonnegative (i.e., \( L \geq 0 \)),

(iii) the spectrum of \( A \) is real.

The proof is given in Section 4. For the general case (when \( \sigma(L) \cap (-\infty, 0) \) may be nonempty), we provide a complete spectral analysis of the operator \( A \). Namely, it is shown that \( \sigma_{\text{ess}}(A) = \mathbb{R} \) and \( A_{\text{ess}} \) is similar to a self-adjoint operator. Furthermore, \( A \) has no real eigenvalues and the discrete
spectrum $\sigma_{\text{disc}}(A)$ consists of a finite number of nonreal eigenvalues; we use results of [11] and [41] to describe their algebraic and geometric multiplicities both in terms of definitizing polynomials and Weyl-Titchmarsh $m$-coefficients (see Theorem 4.6).

In Section 5, it is shown that Theorem 1.1 is sharp in the sense that the condition (1.6) cannot be changed to $q \in L^1(\mathbb{R}, (1 + |x|)^\gamma dx)$ with $\gamma < 1$. Actually, we construct a potential $q_0$ such that

(i) $q_0(x) \approx 2(1 + |x|)^{-2}$ as $|x| \to \infty$,

(ii) the corresponding operator $A = (\text{sgn } x)(-d^2/dx^2 + q_0(x))$ is $J$-nonnegative,

(iii) $0$ is a singular critical point of $A$.

Note that under conditions (1.4) and $|r(x)| = \text{sgn}(x)r(x)$, the regularity of the critical point $\infty$ does not depend of local behavior of the coefficients $r$ and $q$ as well as on their ultimate behavior (for $J$-nonnegative operators, this follows from [11], for the case when $A$ is $J$-semi-bounded from below see [42]). It occurs that the latter is not true for the critical point 0. We compare the above example with another one and show that the regularity of the critical point 0 depends on a local behavior of the potential $q$ (see also [39, Remark 2]). This gives an answer on a question posed by Čurgus (see Subsection 5.2).

In Section 6, condition (1.5) is applied to the operators $(\text{sgn } x)(-d^2/dx^2 + q(x))$ with periodic potentials.

**Theorem 1.2.** Assume that the potential $q \in L^1_{\text{loc}}(\mathbb{R})$ is $T$-periodic, $q(x + T) = q(x)$ a.e., $T > 0$, and that the operator $L = -d^2/dx^2 + q(x)$ is nonnegative. Then the operator $A = (\text{sgn } x)L$ is similar to a self-adjoint operator.

Also, a similar result is obtained for the class of infinite-zone potentials. This class includes smooth periodic potential, generally, infinite-zone potentials are almost-periodic. The corresponding theorem for finite-zone potentials was obtained in [41, Corollary 7.4]. Let us note that the condition $L \geq 0$ is essential for the similarity even for operators with one-zone potentials [41, Theorem 7.1]. However, in the $J$-non-negative case we present a simple proof of [41, Corollary 7.4] based on local condition (1.5) (see Subsection 6.2).

Section 7 is devoted to the case $q \equiv 0$. The following theorem shows that several conditions imposed by Faddeev and Shterenberg [17, Theorem 3] are obsolete (see Remark 7.1).

**Theorem 1.3.** Let $r(x)\chi_{\pm}(x) = \pm p(x)|x|^\alpha_{\pm}$, where $\alpha_{\pm} > -1$ are constants and the function $p$ is positive a.e. on $\mathbb{R}$. Assume also that

\[
\pm \int_{-\infty}^{\infty} |x|^\alpha_{\pm}/2 |p(x) - c_{\pm}|dx < \infty,
\]

with certain constants $c_{\pm} > 0$. Then:

(i) $0$ is a regular critical point of the operator $A$;

(ii) if the weight $r$ also satisfies condition (1.4), then the operator $A$ is similar to a self-adjoint one.

Clearly, the result of Fleige and Najman [22, Theorem 2.7] on the similarity of $(\text{sgn } x)|x|^{-\alpha} d^2/dx^2$, $\alpha > -1$, is a particular case of Theorem 1.3.

**Notation:** $\mathcal{H}$, $\mathcal{H}$ denote separable Hilbert spaces. The scalar product and the norm in the Hilbert space $\mathcal{H}$ are denoted by $(\cdot, \cdot)_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$, respectively. The set of all bounded linear operators from
\[ \mathbf{\mathcal{H}} \text{ to } \mathcal{H} \text{ is denoted by } [\mathbf{\mathcal{H}}, \mathcal{H}] \text{ or } [\mathbf{\mathcal{H}}] \text{ if } \mathbf{\mathcal{H}} = \mathcal{H}. \text{ span}\{f_1, f_2, \ldots, f_N\} \text{ denotes the closed linear hull of vectors } f_1, f_2, \ldots, f_N. \]

Let \( T \) be a linear operator in a Hilbert space \( \mathbf{\mathcal{H}} \). In what follows \( \text{dom}(T), \ker(T), \text{ran}(T) \) are the domain, kernel, range of \( T \), respectively. \( R_T(\lambda) := (T - \lambda I)^{-1}, \lambda \in \rho(T) \), is the resolvent of \( T \). We denote by \( \sigma(T), \rho(T) \) the spectrum and the resolvent set of \( T \); \( \sigma_\rho(T) \) stands for the set of eigenvalues of \( T \). Recall that the discrete spectrum \( \sigma_{\text{disc}}(T) \) is the set of isolated eigenvalues of finite algebraic multiplicity; \( \sigma_{\text{ess}}(T) = \sigma(T) \setminus \sigma_{\text{disc}}(T) \) is the essential spectrum of \( T \).

We set \( \mathbb{C}_\pm := \{ \lambda \in \mathbb{C} : \pm \text{Im}\lambda > 0 \} \), \( \mathbb{Z}_\pm := \mathbb{N} \cup \{0\}, \mathbb{R}_+ := [0, +\infty), \mathbb{R}_- := (-\infty, 0] \). We denote by \( \chi_S(\cdot) \) the indicator function of a set \( S \subset \mathbb{R} \), and \( \chi_{R_{\pm}}(t) := \chi_{R_\pm}(t) \). We write \( f \in L^1_{\text{loc}}(\mathbb{R}) \) if \( f \) is Lebesgue integrable (absolutely continuous) on every bounded interval in \( \mathbb{R} \); \( f(x) \approx g(x) \ (x \to x_0) \) if both \( f/g \) and \( g/f \) are bounded functions in a certain neighborhood of \( x_0 \); \( f(x) \approx g(x) \ (x \to x_0) \) means that \( \lim_{x \to x_0} f(x)/g(x) = 1 \). We write \( f(x) = O(g(x)) (f(x) = O(g(x))) \) as \( x \to x_0 \) if \( f(x) = h(x)g(x) \) and \( h(x) \) is bounded in a certain neighborhood of \( x = x_0 \) (resp., \( \lim_{x \to x_0} h(x) = 0 \)).

\[ \sigma_{\text{ess}}(T) = \sigma(T) \setminus \sigma_{\text{disc}}(T) \]

\[ \sigma_{\text{ess}}(T) = \sigma(T) \setminus \sigma_{\text{disc}}(T) \]

\[ \text{disc} \]

\[ \text{ess} \]

\[ \text{discrete spectrum} \]

\[ \text{essential spectrum} \]

\[ \text{dom}(L) = \text{dom}(A) = \mathcal{D}, \]

\[ Lf = \ell[f], \quad Af = a[f] \quad \text{ for } f \in \mathcal{D}. \]

The operators \( A \) and \( L \) are closed in \( L^2(\mathbb{R}, |r(x)|dx) \), and the adjoint \( L^* \) of \( L \) is a closed symmetric operator with deficiency indices \((n, n)\), \( 0 \leq n \leq 2 \).

In the sequel, \((1.3)\) is supposed. This means that \( n = 0 \) and the operator \( L \) is self-adjoint in the Hilbert space \( L^2(\mathbb{R}, |r(x)|dx) \).

It is clear that \( A = JL \), where \( J \) is defined by

\[ (Jf)(x) = (\text{sgn}x)f(x), \quad f \in L^2(\mathbb{R}, |r(x)|dx). \]

Obviously, \( J^* = J^{-1} = J \) in \( L^2(\mathbb{R}, |r(x)|dx) \). Moreover, \( A^* = LJ \) and

\[ \text{dom}(A^*) = J\mathcal{D} = \{ f \in L^2(\mathbb{R}, |r(x)|dx) : Jf \in \mathcal{D} \} \neq \mathcal{D} = \text{dom}(A). \]

Thus the operator \( A \) is non-self-adjoint in \( L^2(\mathbb{R}, |r(x)|dx) \).

It is obvious that the following restrictions of the operators \( L \) and \( A \)

\[ L_{\text{min}} := L \upharpoonright \mathcal{D}_{\text{min}}, \quad A_{\text{min}} := A \upharpoonright \mathcal{D}_{\text{min}}; \quad \mathcal{D}_{\text{min}} := \mathcal{D} \cap J\mathcal{D} = \{ f \in \mathcal{D} : f(0) = f'(0) = 0 \}, \]

\[ L_{\text{min}} := L \upharpoonright \mathcal{D}_{\text{min}}, \quad A_{\text{min}} := A \upharpoonright \mathcal{D}_{\text{min}}; \quad \mathcal{D}_{\text{min}} := \mathcal{D} \cap J\mathcal{D} = \{ f \in \mathcal{D} : f(0) = f'(0) = 0 \}, \]

\[ L_{\text{min}} := L \upharpoonright \mathcal{D}_{\text{min}}, \quad A_{\text{min}} := A \upharpoonright \mathcal{D}_{\text{min}}; \quad \mathcal{D}_{\text{min}} := \mathcal{D} \cap J\mathcal{D} = \{ f \in \mathcal{D} : f(0) = f'(0) = 0 \}, \]

\[ L_{\text{min}} := L \upharpoonright \mathcal{D}_{\text{min}}, \quad A_{\text{min}} := A \upharpoonright \mathcal{D}_{\text{min}}; \quad \mathcal{D}_{\text{min}} := \mathcal{D} \cap J\mathcal{D} = \{ f \in \mathcal{D} : f(0) = f'(0) = 0 \}, \]
are closed densely defined symmetric operators in $L^2(\mathbb{R}, |r(x)|dx)$ with equal deficiency indices $n_\pm(L_{\text{min}}) = n_\pm(A_{\text{min}}) = 2$. The operator $A$ is a non-self-adjoint extension of $A_{\text{min}}$ and

$$D = \text{dom}(A) = \{ f \in D^*_{\text{min}} : f(+0) = f(-0), \quad f'(+0) = f'(-0) \}.$$  

(2.5)

Here $D^*_{\text{min}}$ stands for the domain of the adjoint operator $L^*_{\text{min}}$ of $L_{\text{min}}$, $D^*_{\text{min}} = \text{dom}(L^*_{\text{min}}) = \text{dom}(A^*_{\text{min}})$. By $A_0$ and $L_0$, we denote the self-adjoint extensions of the operators $A_{\text{min}}$ and $L_{\text{min}}$:

$$A_0 := A^*_{\text{min}} \upharpoonright \text{dom}(A_0), \quad L_0 := L^*_{\text{min}} \upharpoonright \text{dom}(L_0),$$

(2.6)

$$\text{dom}(A_0) = \text{dom}(L_0) = \{ f \in D^*_{\text{min}} : f'(+0) = f'(-0) = 0 \}.$$

It is obvious that $L_0 = JA_0$.

### 2.2 Weyl-Titchmarsh m-coefficients.

Let $c(x, \lambda)$ and $s(x, \lambda)$ denote the linearly independent solutions of equation

$$-y''(x) + q(x)y(x) = \lambda |r(x)|y(x), \quad x \in \mathbb{R},$$

(2.7)

satisfying the following initial conditions at zero

$$c(0, \lambda) = s'(0, \lambda) = 1; \quad c'(0, \lambda) = s(0, \lambda) = 0.$$  

(2.8)

Since equation (2.7) is limit-point at $+\infty$, then (see, for example, [50]) there exists a unique holomorphic function $m_+(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$, such that the solution $s(x, \lambda) - m_+(\lambda)c(x, \lambda)$ belongs to $L^2(\mathbb{R}, |r(x)|dx)$. Similarly, the limit point case at $-\infty$ yields the fact that there exists a unique holomorphic function $m_-(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$, such that $s(x, \lambda) + m_-(\lambda)c(x, \lambda) \in L^2(\mathbb{R}, |r(x)|dx)$. Note that if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $f_{\pm}(\cdot, \lambda)$ are $L^2(\mathbb{R} \pm, |r|dx)$-solution of equation (2.7) (which are unique up to a multiplicative constant), then

$$m_+(\lambda) = -\frac{f_+(0, \lambda)}{f'_+(0, \lambda)}, \quad m_-(\lambda) = \frac{f_-(0, \lambda)}{f'_-(0, \lambda)}, \quad \lambda \notin \mathbb{R}.$$  

(2.9)

The functions $f_{\pm}(\cdot, \lambda)$ and $m_{\pm}(\cdot)$ are called the Weyl solutions and the Weyl-Titchmarsh m-coefficients (or Weyl-Titchmarsh functions) for (2.7) on $\mathbb{R}_+$ and on $\mathbb{R}_-$, respectively. We put

$$M_\pm(\lambda) := \pm m_\pm(\pm \lambda); \quad \psi_\pm(x, \lambda) = (s(x, \pm \lambda) - M_\pm(\lambda)c(x, \pm \lambda))\chi_\pm(x).$$  

(2.10)

It is easily seen that $a[\psi_\pm(x, \lambda)] = \lambda \psi_\pm(x, \lambda)$, where $a[\cdot]$ is defined by (2.1), and, by definition of m-coefficients, $\psi_\pm(\cdot, \lambda) \in L^2(\mathbb{R}, |r(x)|dx)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The function $M_+(\cdot)$ ($M_-(\cdot)$) is said to be the Weyl-Titchmarsh m-coefficient for equation (1.1) on $\mathbb{R}_+$ (on $\mathbb{R}_-$).

It is known that for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the functions $\psi_\pm$ and $M_\pm$ are connected by

$$\int_0^{+\infty} |\psi_+(x, \lambda)|^2 r(x)dx = \frac{\text{Im} M_+(\lambda)}{\text{Im} \lambda}, \quad \int_{-\infty}^{0} |\psi_-(x, \lambda)|^2 r(x)dx = \frac{\text{Im} M_-(\lambda)}{\text{Im} \lambda}.$$  

(2.11)

These formulae imply that the functions $M_+$ and $M_-$ (as well as $m_+$ and $m_-$) belong to the class $(R)$, i.e., they are holomorphic in $\mathbb{C} \setminus \mathbb{R}$, $M_\pm(\bar{\lambda}) = \overline{M_\pm(\lambda)}$, and $\text{Im} \lambda \cdot \text{Im} M_\pm(\lambda) \geq 0$ for $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$. Moreover (see [46] and also [50, §II.5, Theorem 5.2] for the case $|r| \equiv 1$), the functions $M_+$ and $M_-$ admit the following integral representation

$$M_\pm(\lambda) = \int_{-\infty}^{+\infty} \frac{d\tau_\pm(s)}{s - \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$  

(2.12)
Proposition 4]). Therefore, the operator 
\[ \tilde{\tau}(s) = \frac{d\tau(s)}{1 + |s|}, \]
(2.13). Note that the function \( \tilde{\tau} \) at zero is a Friedrichs extension, that is a maximal nonnegative self-adjoint extension of the differential expression (1.2) and the boundary condition \( y'(\pm 0) = 0 \) is unitary equivalent to the multiplication by the independent variable in the Hilbert spaces \( L^2(\mathbb{R}, d\tau(x)) \). This fact implies
\[ \sigma(A_{0\pm}) = \text{supp}(d\tau_\pm). \] (2.14)

Notice that the functions \( \tau_+ \) and \( \tau_- \) are uniquely determined by the Stieltjes inversion formula [29]. The function \( \tau_\pm \) is called the spectral function of the boundary value problem
\[ -y''(x) + q(x)y(x) = \lambda r(x)y(x), \quad x \in \mathbb{R}_\pm; \quad y'(\pm 0) = 0. \] (2.13)

In other words, the self-adjoint operator \( A_{0\pm} \) determined in the Hilbert space \( L^2(\mathbb{R}_\pm, |r|dx) \) by the differential expression (1.2) and the boundary condition \( y'(\pm 0) = 0 \) is unitary equivalent to the multiplication by the independent variable in the Hilbert spaces \( L^2(\mathbb{R}, d\tau_\pm(x)) \). This fact implies
\[ \sigma(A_{0\pm}) = \text{supp}(d\tau_\pm). \] (2.14)

Here \( \text{supp} \) denotes the topological support of a Borel measure \( d\tau \) on \( \mathbb{R} \), i.e., \( \text{supp} \) \( d\tau \) is the smallest closed set \( \Omega \subset \mathbb{R} \) such that \( d\tau(\mathbb{R} \setminus \Omega) = 0 \).

We will say sometimes that \( M_\pm(\cdot) \) is the Weyl-Titchmarsh m-coefficient for the boundary value problem (2.13). Note that \( M_\pm(\cdot) \) and \( \psi_\pm(x, \cdot) \) can be extended naturally on \( \mathbb{C} \setminus \text{supp}(d\tau_\pm) \) using e.g. (2.12).

**Definition 2.1** ([29]). A function \( M(\cdot) \in (R) \) belongs to:
(i) the Krein–Stieltjes class \((S)\) if it is holomorphic on \( \mathbb{C} \setminus \mathbb{R}_+ \) and \( M(\lambda) \geq 0 \) for \( \lambda < 0 \);
(ii) the Krein–Stieltjes class \((S^{-1})\) if it is holomorphic on \( \mathbb{C} \setminus \mathbb{R}_+ \) and \( M(\lambda) \leq 0 \) for \( \lambda < 0 \).

If \( M(\cdot) \in (S) \) then it admits an integral representation (see [29, §5])
\[ M(\lambda) = c + \int_{-\infty}^{+\infty} \frac{d\tau(s)}{s - \lambda}, \quad \text{where} \quad c \geq 0, \quad \int_{-\infty}^{+\infty} (1 + s)^{-1}d\tau(s) < +\infty, \]
and \( \tau : \mathbb{R}_+ \to \mathbb{R}_+ \) is a nondecreasing function. The integral representation yields that the function \( M(\cdot) \in (S) \) is increasing on \( (-\infty, 0) \), and \( M(\lambda_0) = 0 \) for some negative \( \lambda \) exactly when \( M \equiv 0 \). Note also that \( M(\cdot) \in (S^{-1}) \) if and only if \( (M(\cdot))^{-1} \in (S) \).

The nonnegativity of the self-adjoint operator \( L \) determined by (2.3) can be described in terms of the Weyl-Titchmarsh m-coefficients \( m_+ \) and \( m_- \).

**Proposition 2.1** ([40]). The operator \( L \) is nonnegative if and only if \( (m_+^{-1}(\cdot) - m_-^{-1}(\cdot)) \in (S^{-1}) \). If in addition the weight \( |r(\cdot)| \) and the potential \( q(\cdot) \) are even, then \( L \geq 0 \) exactly when \( m_+(\cdot) \in (S) \).

**Proof.** Let us recall that the functions \( \tilde{m}_\pm(\cdot) = -1/m_\pm(\cdot) \in (R) \) are the Weyl-Titchmarsh m-coefficients of equation (2.7) on \( \mathbb{R}_\pm \) associated with the Dirichlet boundary problems
\[ -y''(x) + q(x)y(x) = \lambda |r(x)||y(x), \quad \pm x \geq 0; \quad y(\pm 0) = 0. \] (2.15)

If the operator \( L = L^* \) is nonnegative, then the symmetric operator \( L_{\text{min}} \) of the form (2.4) is nonnegative too. Moreover, the extension of \( L_{\text{min}} \) corresponding to the Dirichlet boundary condition at zero is a Friedrichs extension, that is a maximal nonnegative self-adjoint extension of \( L_{\text{min}} \) (cf. [14, Proposition 4]). Therefore, the operator \( L_{\text{min}} \) is nonnegative if and only if the self-adjoint operators \( L_{0\pm}^D \) associated with the problems (2.15) are nonnegative. This fact implies that \( m \)-functions \( \tilde{m}_+(\cdot) \) and \( \tilde{m}_-(\cdot) \) are analytic on \( \mathbb{C} \setminus \mathbb{R}_+ \) and are real on \( (-\infty, 0) \). The definitions of \( m_\pm \) also imply
\[ \left\{ \lambda < 0 : \tilde{m}_+(\lambda) + \tilde{m}_-(\lambda) = 0 \right\} = \sigma_p(L) \cap (-\infty, 0) = \sigma(L) \cap (-\infty, 0). \]
Hence $\tilde{m}_+(\lambda) + \tilde{m}_-(\lambda) \neq 0$ if $\lambda < 0$ since $L \geq 0$. Moreover, $\tilde{m}_+(\lambda) + \tilde{m}_-(\lambda) < 0$, ($\lambda < 0$), since $\tilde{m}_-(\infty) = -\infty$ (cf. [29, §1], [47], see also [14]). Thus, $\tilde{m}_+(\cdot) + \tilde{m}_-(\cdot) \in (S^{-1})$.

To prove the second statement, let us note that $m_-(\cdot) = m_+(\cdot)$ if $q$ and $r$ are even. Hence, $\tilde{m}_+(\cdot) + \tilde{m}_-(\cdot) = 2\tilde{m}_+(\cdot) = -2m^+_1(\cdot) \in (S^{-1})$ or, equivalently, $m_+ \in (S)$.

**Remark 2.1.** In the recent paper [7], numbers of negative eigenvalues of self-adjoint operators in Krein spaces was investigated in terms of the corresponding abstract Weyl functions (cf. [40]). In particular, Proposition 2.1 was proved under additional assumptions on the operator $A_0$ (see Proposition 4.4 and Theorem 4.7 in [7]).

### 2.3 Similarity criterion.

Our approach to the similarity problem is based on the resolvent similarity criterion obtained in [53, 51] (under an additional assumption this criterion was obtained also in [9]).

**Theorem 2.2 ([53, 51]).** A closed operator $T$ on a Hilbert space $\mathfrak{H}$ is similar to a self-adjoint operator if and only if $\sigma(T) \subset \mathbb{R}$ and for all $f \in \mathfrak{H}$ the inequalities

$$
\sup_{\varepsilon > 0} \varepsilon \int_{\mathbb{R}} \| R_T(\eta + i\varepsilon) f \|^2 \, d\eta \leq K_1 \| f \|^2, \quad \sup_{\varepsilon > 0} \varepsilon \int_{\mathbb{R}} \| R_{T^*}(\eta + i\varepsilon) f \|^2 \, d\eta \leq K_{1*} \| f \|^2,
$$

hold with constants $K_1$ and $K_{1*}$ independent of $f$.

**Remark 2.2.** If $T$ is a $J$-self-adjoint operator, then $T^* = JTJ$ and the second inequality in (2.16) is equivalent to the first one since in this case $\| R_{T^*}(\lambda) f \| = \| R_T(\lambda) f \|, \; f \in \mathfrak{H}, \; \lambda \in \rho(T)$.

Recall that an operator $T$ in a Hilbert space $\mathfrak{H}$ is called completely non-self-adjoint if it has no nontrivial reducing subspace such that its restriction on that subspace is self-adjoint (sometimes completely non-self-adjoint symmetric operators are called simple).

Note that the symmetric operator $A_{\text{min}}$ defined by (2.4) is completely non-self-adjoint since its restrictions on half-lines $A_{\text{min}}^+ := A_{\text{min}} \upharpoonright \mathcal{D}_{\text{min}} \cap L^2(R_+, |r| \, dx)$ are completely non-self-adjoint (see e.g. [24]) and $A_{\text{min}} = A_{\text{min}}^+ \oplus A_{\text{min}}^-$. This implies the following statement.

**Proposition 2.3.** The operator $A$ defined by (1.2) is completely non-self-adjoint.

**Proof.** Assume the converse. Then $A$ admits a representation $A = A_0 \oplus A_1$, where $A_0 = A_0^*$ and $A_1$ is a completely non-self-adjoint operator. Hence, $A^* = A_0 \oplus A_1^*$ and therefore $A_0 \oplus (A_1 \cap A_1^*) = A \cap A^* = A_{\text{min}}$. Thus, $\mathfrak{H}_0 := \overline{\text{dom}(A_0)}$ reduces $A_{\text{min}}$ and $A_0 = A_0^* = A_{\text{min}} \upharpoonright \mathfrak{H}_0$. But $A_{\text{min}} = A_{\text{min}}^+ \oplus A_{\text{min}}^-$ is completely non-self-adjoint (see e.g. [24]), a contradiction.

The next proposition will be used to obtain more delicate information about spectra. It is based on the concept of characteristic functions of non-self-adjoint extensions of a symmetric operator (see e.g. [8]). We will use the definition given in [14] (see [41, §3] for details).

**Proposition 2.4 ([41]).** Let $A$ be the operator defined by (1.2) and let $M_\pm$ be the $m$-coefficients defined by (2.10).

(i) One of the characteristic functions $\theta_A(\cdot)$ of the operator $A$ admits the following representation

$$
\theta_A(\lambda) = \frac{1}{M_-(\lambda) - M_+(\lambda)} \begin{pmatrix} M_+(\lambda) + M_-(\lambda) & 2M_+(\lambda)M_-(\lambda) \\ 2 & M_+(\lambda) + M_-(\lambda) \end{pmatrix}.
$$

(2.17)
(ii) Suppose that $A$ is similar to a self-adjoint operator $T = T^*$ and that there exists a closed at most countable set $\{a_j\}_1^N \subset \mathbb{R}$, $N \leq \infty$, such that

$$
\sup_{\lambda \in \mathbb{C}_+ \setminus \Omega} \| \mathcal{J} - \theta_A(\lambda) \mathcal{J} \theta_A^*(\lambda) \| < \infty, \quad (2.18)
$$

for any domain $\Omega := \bigcup_1^N \mathbb{D}_{\varepsilon_j}(a_j) \cup \mathbb{D}_{\varepsilon_\infty}(\infty)$ with sufficiently small $\varepsilon_\infty, \varepsilon_1, \varepsilon_2, \ldots$. If, additionally, $\sigma_p(A) \cap \{a_j\}_1^N = \emptyset$, then the spectrum of $T$ is purely absolutely continuous, that is $A$ is similar to the self-adjoint operator with absolutely continuous spectrum.

Here $\mathbb{D}_\varepsilon(a) = \{ \lambda \in \mathbb{C}_+ : |\lambda - a| < \varepsilon \}$ for $a \in \mathbb{R}$ and $\mathbb{D}_\varepsilon(\infty) = \{ \lambda \in \mathbb{C}_+ : |\lambda| > 1/\varepsilon \}$.

Statements (i) and (ii) follow from Propositions 3.9 (i) and 3.8 of [41], respectively. Note only that the proof of [41, Proposition 3.8] obviously implies $\sigma_p(A) \subset \{a_j\}_1^N$.

### 2.4 Spectral functions of $J$-nonnegative operators.

Consider a Hilbert space $\mathcal{H}$ with a scalar product $(\cdot, \cdot)_\mathcal{H}$. Suppose that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+$ and $\mathcal{H}_-$ are closed subspaces of $\mathcal{H}$. Denote by $P_\pm$ the orthogonal projections from $\mathcal{H}$ onto $\mathcal{H}_\pm$. Let $\mathcal{J} = P_+ - P_-$ and $[\cdot, \cdot]_\mathcal{K} := (\mathcal{J}[\cdot, \cdot])_\mathcal{H}$. Then the pair $\mathcal{K} = (\mathcal{H}, [\cdot, \cdot]_\mathcal{K})$ is called a Krein space (see [48, 5] for the original definition). The form $[\cdot, \cdot]$ is called an inner product in the Krein space $\mathcal{K}$ and the operator $\mathcal{J}$ is called a fundamental symmetry.

Let $T$ be a closed densely defined operator in $\mathcal{H}$. The $\mathcal{J}$-adjoint operator of $T$ is defined by the relation

$$
[Tf, g] = [f, T^*[g]], \quad f \in \text{dom}(T),
$$

on the set of all $g \in \mathcal{H}$ such that the mapping $f \mapsto [Tf, g]$ is a continuous linear functional on $\text{dom}(T)$. The operator $T$ is called $\mathcal{J}$-self-adjoint (J-nonnegative) if $T = T^*$ ([$Tf, f] \geq 0$ for $f \in \text{dom}(T)$). It is easy to see that $T^*[g] := \mathcal{J}T^*g$ and the operator $T$ is $\mathcal{J}$-self-adjoint (J-nonnegative) if and only if $JT$ is self-adjoint (nonnegative). Note that $\mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1} = \mathcal{J}^*[\cdot, \cdot]$.

Let $\mathcal{S}$ be the semiring consisting of all bounded intervals with endpoints different from 0 and $\pm \infty$ and their complements in $\mathbb{R} := \mathbb{R} \cup \infty$.

**Theorem 2.5 ([48]).** Let $T$ be a $\mathcal{J}$-nonnegative $\mathcal{J}$-self-adjoint operator in $\mathcal{H}$ with a nonempty resolvent set $\rho(T) \neq \emptyset$. Then:

(i) The spectrum of $T$ is real, $\sigma(T) \subset \mathbb{R}$.

(ii) There exist a mapping $\Delta \rightarrow E(\Delta)$ from $\mathcal{S}$ into the set of bounded linear operators in $\mathcal{H}$ with the following properties ($\Delta, \Delta' \in \mathcal{S}$):

(E1) $E(\Delta \cap \Delta') = E(\Delta)E(\Delta')$, $E(\emptyset) = 0$, $E(\mathbb{R}) = I$, $E(\Delta) = E(\Delta)^*;$

(E2) $E(\Delta \cup \Delta') = E(\Delta) + E(\Delta')$ if $\Delta \cap \Delta' = \emptyset;$

(E3) the form $\pm[\cdot, \cdot]$ is positive definite on $E(\Delta)\mathcal{H}$, if $\Delta \subset \mathbb{R}_\pm$;

(E4) $E(\Delta)$ is in the double commutant of the resolvent $R_T(\lambda) = (T - \lambda)^{-1}$ and $\sigma(T \upharpoonright E(\Delta)\mathcal{H}) \subset \overline{\Delta};$

(E5) if $\Delta$ is bounded, then $E(\Delta)\mathcal{H} \subset \text{dom}(T)$ and $T \upharpoonright E(\Delta)\mathcal{H}$ is a bounded operator.

According to [48, Proposition II.4.2], a number $s \in \{0, \infty\}$ is called a critical point of $T$, if for each $\Delta \in \mathcal{S}$ such that $s \in \Delta$, the form $[\cdot, \cdot]$ is indefinite on $E(\Delta)\mathcal{H}$ (i.e., there exist $h_\pm \in E(\Delta)\mathcal{H}$ such that $[h_\pm, h_\pm] = 0$).
such that $[h_+, h_-] < 0$ and $[h_-, h_-] > 0$. The set of critical points is denoted by $c(T)$. If $\alpha \not\in c(T)$, then for arbitrary $\lambda_0, \lambda_1 \in \mathbb{R} \setminus c(T)$, $\lambda_0 < \alpha$, $\lambda_1 > \alpha$, the limits
\[
\lim_{\lambda \not\in \alpha} E([\lambda_0, \lambda]), \quad \lim_{\lambda \not\in \alpha} E([\lambda, \lambda_1])
\] (2.19)
exist in the strong operator topology. If $\alpha \in c(T)$ and the limits (2.19) do still exist, then $\alpha$ is called \textit{regular critical point} of $T$, otherwise $\alpha$ is called \textit{singular}. Here we agree that, if $\alpha = \infty$, then $\lambda_1 > \alpha (\lambda \uparrow \alpha)$ means $\lambda_1 > -\infty (\lambda \uparrow -\infty$, respectively).

The following proposition is well known (cf. [48, §6]).

\textbf{Proposition 2.6.} Let $T$ be a $J$-nonnegative $J$-self-adjoint operator in the Hilbert space $\mathfrak{H}$. Assume that $\rho(T) \neq \emptyset$ and $\ker T = \ker T^2$ (i.e., 0 is either a semisimple eigenvalue or a regular point of $T$). Then two following assertions are equivalent:

(i) $T$ is similar to a self-adjoint operator.

(ii) 0 and $\infty$ are not singular critical points of $T$.

The next proposition show that any $J$-nonnegative operator of type (1.2) has the spectral function $E_A(\cdot)$ with the properties (E1)-(E5) and only 0 and $\infty$ may be its critical points.

\textbf{Proposition 2.7} ([11], see also [38]). If the ($J$-self-adjoint) operator $A$ defined by (1.2) is $J$-nonnegative, then its spectrum $\sigma(A)$ is real.

Note also that $\infty$ is always a critical point of $A$. Combining Proposition 2.7 with [11, Theorem 3.6], one can easily obtain the following sufficient condition for the regularity of $\infty$.

\textbf{Proposition 2.8} ([11]). If the operator $A$ defined by (1.2) is $J$-nonnegative and assumption (1.4) is fulfilled, then $\infty$ is a regular critical point of $A$. If, in addition, $L \geq \varepsilon > 0$, then $A$ is similar to a self-adjoint operator.

\section{Sufficient conditions for similarity and the regularity of critical points}

Let $A$, $L$, and $J$ be the operators defined in Subsection 2.1, and $M_+$, $M_-$ the Weyl-Titchmarsh $m$-coefficients for (1.1) (see Subsection 2.2).

Consider the operator $A_{b,c} := A_{\min} \uparrow \ker(A_{b,c})$, where $A_{\min}^*$ is the adjoint of $A_{\min}$ (2.4),
\[
\ker(A_{b,c}) = \{ f \in \ker(A_{\min}): f(-0) + cf'(-0) = f(+0), \quad bf'(-0) = f'(+0) \},
\]
and $b, c \in \mathbb{R}$ are constants.

\textbf{Proposition 3.1.} (i) The operator $A_{b,c}$ is self-adjoint if and only if $b = -1$ and $c \in \mathbb{R}$;

(ii) $\sigma(A_{b,c}) \setminus \mathbb{R} = \{ \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- : M_- (\lambda) - b M_+ (\lambda) - c = 0 \}$;

(iii) If $\lambda \not\in \mathbb{R}$ and $\omega \in \rho(A_{b,c})$, then for all $f \in L^2(\mathbb{R}, |r|dx)$,
\[
(A_{b,c} - \lambda)^{-1} f = (A_0 - \lambda)^{-1} f + \frac{\mathcal{F}_-(f, \lambda) - \mathcal{F}_+(f, \lambda)}{M_- (\lambda) - b M_+ (\lambda) - c} (b \psi_+ (\cdot, \lambda) + \psi_- (\cdot, \lambda)),
\] (3.1)
where $\mathcal{F}_\pm(f, \lambda)$ are the generalized Fourier transforms of $f$, $\mathcal{F}_\pm(f, \lambda) := \int_{\mathbb{R}^\pm} f(x) \psi_\pm(x, \lambda)|r(x)|dx$. 

Proof. (i) can be obtained by simple calculations. Note also that (ii) follows from the proof of [41, Proposition 5.8]. Indeed, for the operator $A_{b,c}$, the matrix $B$ defined by [41, formula (5.24)] equals 
\[
\begin{pmatrix}
0 & b \\
-1 & c
\end{pmatrix}.
\]
So $A_{b,c} = A_{b,c}^*$ exactly when $B = B^*$ (see [14]). The proofs of (ii)-(iii) are similar to that of [38, Lemma 4.1] (see also [41, Proposition 4.4]).

\begin{remark}
The operator $A_{1,c}$ belongs to the class of operators with the so-called $\delta'$-interaction at zero ([4, 44]). The formal differential expression associated with $A_{1,c}$ is $r(x)^{-1}(-d^2/dx^2 + q(x) + c\delta'(x))$, where $\delta$ is the Dirac function.
\end{remark}

Note that $A_{1,0}$ coincides with the operator $A$ defined by (1.2).

**Theorem 3.2.** Suppose that there exists a constant $c \in \mathbb{R}$ such that

\[
\sup_{\lambda \in \mathbb{C}_+} \frac{|M_+(\lambda) + M_-(\lambda) - c|}{|M_+(\lambda) - M_-(\lambda)|} < \infty.
\]

Then the operator $A$ is similar to a self-adjoint operator.

**Proof.** The proof is similar to that of [41, Theorem 5.9]. We present a sketch.

Let $c \in \mathbb{R}$. Note that $A_0 = A_0^*$ (see (2.6)) and $A_{-1,c} = A_{-1,c}^*$. Hence inequalities (2.16) hold for the resolvents of both the operators $A_0$ and $A_{-1,c}$. Therefore (3.1) implies that

\[
\sup_{\varepsilon > 0} \int_{-\infty}^{+\infty} \varepsilon \left\| \frac{\psi_+(x, \mu + i\varepsilon) F_+(f, \mu + i\varepsilon)}{M_+(\mu + i\varepsilon) + M_-(\mu + i\varepsilon) - c} \right\|^2 d\mu \leq C_1 \|f\|^2, \quad f \in L^2(\mathbb{R}, |r| dx).
\]

The same arguments show that the operator $A = A_{1,0}$ is similar to a self-adjoint one exactly when

\[
\sup_{\varepsilon > 0} \int_{-\infty}^{+\infty} \varepsilon \left\| \frac{\psi_+(x, \mu + i\varepsilon) F_+(f, \mu + i\varepsilon)}{M_+(\mu + i\varepsilon) - M_-(\mu + i\varepsilon)} \right\|^2 d\mu \leq C_2 \|f\|^2, \quad f \in L^2(\mathbb{R}, |r| dx).
\]

Combining (3.3) with (3.2), we get (3.4). \hfill \Box

Theorem 3.2 is valid for $J$-self-adjoint (not necessary $J$-nonnegative) Sturm-Liouville operators. If $c = 0$, this result coincides with [41, Theorem 5.9]. Note that condition (3.2) with arbitrary $c \in \mathbb{R}$ has a wider field of applicability than its special case $c = 0$ (see the proofs of Theorems 1.1 and 1.2).

Proposition 2.7 and Theorem 2.5 allows us to obtain a local version of Theorem 3.2.

**Theorem 3.3.** Assume that the operator $A$ is $J$-nonnegative.

(i) If ratio

\[
\frac{|M_+(\lambda) + M_-(\lambda) - c|}{|M_+(\lambda) - M_-(\lambda)|},
\]

is bounded on $\Omega^0_R := \{\lambda \in \mathbb{C}_+ : |\lambda| < R\}$ for certain constants $R > 0$ and $c \in \mathbb{R}$, then $0$ is not a singular critical point of $A$.

(ii) If the function (3.5) is bounded on the set $\Omega^\infty_R := \{\lambda \in \mathbb{C}_+ : |\lambda| > R\}$ for certain constants $R > 0$ and $c \in \mathbb{R}$, then $\infty$ is not a singular critical point of $A$. 

Proof. (i) Under the assumptions of the theorem, the operator $A$ has the following properties:

(A1) $A$ is a $J$-self-adjoint $J$-nonnegative operator;

(A2) for some $R > 0$ and $c \in \mathbb{R}$, the function (3.5) is bounded on $\Omega_R^0$.

By (A1) and Proposition 2.7, the operator $A$ has a spectral function $E_A(\Delta)$ (see Theorem 2.5). Therefore $P_R := E_A([-R/2, R/2])$ is a bounded $J$-orthogonal projection. Using properties (E4)–(E5) of $E_A(\Delta)$, we obtain the decomposition

$$A = A^0 + A^\infty,$$

$$A^0 := A \upharpoonright \mathfrak{H}_0, \quad A^\infty := A \upharpoonright \mathfrak{H}_\infty, \quad L^2(\mathbb{R}, |r(x)| dx) = \mathfrak{H}_0 + \mathfrak{H}_\infty,$$

where $\mathfrak{H}_0 := \text{ran}(P_R)$ and $\mathfrak{H}_\infty := \text{ran}(I - P_R)$. Moreover,

$$\sigma(A^0) \subset [-R/2, R/2], \quad \sigma(A^\infty) \subset (-\infty, -R/2] \cup [R/2, +\infty).$$

Obviously, $A^0$ is $J$-self-adjoint $J$-nonnegative operator. Let us remark that $A^0$ has the singular critical point 0 if and only if so does $A$.

Let us prove that (2.16) holds for the resolvent of $A^0$ and, therefore, the operator $A^0$ is similar to a self-adjoint one. Put $\mathcal{I}_\varepsilon := [-\sqrt{R^2 - \varepsilon^2}, \sqrt{R^2 - \varepsilon^2}]$ if $\varepsilon < R$, and $\mathcal{I}_\varepsilon = \emptyset$ if $\varepsilon \geq R$. Using (A2), (3.1), and arguing as in proof of Theorem 3.2, one gets

$$\varepsilon \int_{\mathcal{I}_\varepsilon} \|(A^0 - (\mu + i\varepsilon))^{-1} f\|^2 d\mu = \varepsilon \int_{\mathcal{I}_\varepsilon} \|(A - (\mu + i\varepsilon))^{-1} f\|^2 d\mu \leq C_1 \|f\|^2, \quad f \in \mathfrak{H}_0,$$

(3.6)

with a constant $C_1 > 0$ independent of $f \in \mathfrak{H}_0$. Further, note that $A^0$ is bounded and $\sigma(A^0) \subset [-R/2, R/2]$. Thus $\|(A^0 - \lambda)^{-1}\| \leq C_2 |\lambda|^{-1}$ for $|\lambda| > R$, and hence

$$\varepsilon \int_{R \setminus \mathcal{I}_\varepsilon} \|(A^0 - (\mu + i\varepsilon))^{-1} f\|^2 d\mu \leq C_2 \|f\|^2 \varepsilon \int_{R \setminus \mathcal{I}_\varepsilon} |\mu + i\varepsilon|^{-2} d\mu \leq C_2 \pi \|f\|^2.$$

(3.7)

Combining (3.6) and (3.7) with Remark 2.2, we see that $A^0$ is similar to a self-adjoint operator. Hence 0 is not a singular critical point of $A^0$. This completes the proof (i). The proof of (ii) is similar. \qed

In Section 5, we will use the following necessary condition for regularity.

**Theorem 3.4** ([38]). Assume that the operator $A$ defined by (1.2) is $J$-nonnegative. Then:

(i) If 0 is not a singular critical point of $A$ and $\ker A = \ker A^2$, then for all $R > 0$

$$\sup_{\lambda \in \Omega_R^0} \left| \frac{\text{Im}(M_+(\lambda) + M_-(\lambda))}{M_+(\lambda) - M_-(\lambda)} \right| = C_R < \infty.$$

(3.8)

(ii) If $\infty$ is not a singular critical point of $A$, then for all $R > 0$ function (3.8) is bounded on $\Omega_R^\infty$.

If $\text{Re}(M_+(\lambda) + M_-(\lambda)) - c = O(\text{Im}(M_+(\lambda) - M_-(\lambda)))$ as $\lambda \to 0$, $\lambda \in \mathbb{C}_+$, the necessary conditions of Theorem 3.4 yields the sufficient conditions of Theorem 3.3. The results of the following sections yield that this is the case for several classes of coefficients.
4 Sturm-Liouville operators with decaying potentials and regular critical point 0

In this section, we consider the operator
\[ A = (\text{sgn } x) \left( -\frac{d^2}{dx^2} + q(x) \right), \quad \text{dom}(A) = \mathcal{D}. \] (4.1)

with the potential \( q \in L^1_{\text{loc}}(\mathbb{R}) \) having a finite first moment. That is we consider the case when \( r(x) = \text{sgn } x \) and \( q \) satisfies
\[ \int_{\mathbb{R}} (1 + |x|)|q(x)|dx < \infty. \] (4.2)

4.1 The asymptotic behavior of the Weyl-function.

Denote by \( z^{1/2}, z \in \mathbb{C} \), the branch of the multifunction \( \sqrt{z} \) with cut along the negative semi-axis \( \mathbb{R}^- \) such that \((-1 + i0)^{1/2} = i\). As in Subsection 2.2, \( s(\cdot, \lambda) \) and \( c(\cdot, \lambda) \) stand for the solutions of
\[ -y''(x) + q(x)y(x) = \lambda y(x), \quad x \in \mathbb{R}, \] (4.3)
satisfying initial conditions (2.8).

**Lemma 4.1.** Let \( q(\cdot) \) satisfy (4.2) with \( \mathbb{R}^+ \) in place of \( \mathbb{R} \) and let \( m(\cdot) \) be the Weyl-Titchmarsh \( m \)-coefficient of equation (4.3) on \( \mathbb{R}^+ \) subject to the Neumann condition at zero, \( y'(0) = 0 \).

(i) If the solution \( s(\cdot, 0) \) of (4.3) with \( \lambda = 0 \) is unbounded on \( \mathbb{R}^+ \), then
\[ m(\lambda) = \frac{a_0}{b_0 + \sqrt{-\lambda}[1 + o(1)]}, \quad |\lambda| \to 0, \] (4.4)
with certain \( a_0 > 0 \) and \( b_0 \in \mathbb{R} \); 

(ii) If \( s(\cdot, 0) \) is bounded on \( \mathbb{R}^+ \), then
\[ m(\lambda) = -k\sqrt{-\lambda}[1 + o(1)], \quad |\lambda| \to 0, \] (4.5)
with certain \( k > 0 \).

**Proof.** (i) Recall that if \( q \in L^1(\mathbb{R}^+) \), then \( m(\cdot) \) admits the representation (see [58, Chapter V])
\[ m(\lambda) = \frac{a(\lambda)}{b(\lambda)}, \quad \lambda \in \mathbb{C}^+, \] (4.6)
where the functions
\[ a(\lambda) = \frac{1}{2\sqrt{-\lambda}} + \frac{1}{2\sqrt{-\lambda}} \int_0^{+\infty} q(t) e^{-\sqrt{-\lambda}t} s(t, \lambda)dt, \]
\[ b(\lambda) = \frac{1}{2} + \frac{1}{2\sqrt{-\lambda}} \int_0^{+\infty} q(t) e^{-\sqrt{-\lambda}t} c(t, \lambda)dt, \] (4.7)
are analytic in \( \mathbb{C}^+ \). Note that it suffices to prove (4.4) and (4.5) only for \( \lambda \in \mathbb{C}^+ \), since \( m \) is an \( R \)-function and hence \( \overline{m(\lambda)} = m(\overline{\lambda}), \ \lambda \in \mathbb{C}^+ \cup \mathbb{C}^- \).
In order to estimate \( c(x, \lambda) \) and \( s(x, \lambda) \), we use transformation operators preserving initial conditions at zero. Indeed, it is well known (see [49] and [52, §1.2]) that the solutions \( c(x, \lambda) \) and \( s(x, \lambda) \) of (4.3) admit the following representation by means of transformation operators

\[
c(x, \lambda) = \cos x \sqrt{\lambda} + \int_{-x}^{x} K(x, t) \cos t \sqrt{\lambda} dt, \tag{4.8}
\]

\[
s(x, \lambda) = \frac{\sin x \sqrt{\lambda}}{\sqrt{\lambda}} + \int_{-x}^{x} K(x, t) \sin t \sqrt{\lambda} dt, \tag{4.9}
\]

where the kernel \( K(x, t) \) is a unique solution of the integral equation

\[
K(x, t) = \frac{1}{2} \int_{0}^{(x+t)/2} q(y) dy + \int_{0}^{(x-t)/2} q(\alpha + \beta) K(\alpha + \beta, \alpha - \beta) d\beta. \tag{4.10}
\]

Moreover, \( K(x, t) \) satisfies the estimate (see [49], [52, §1.2])

\[
|K(x, t)| \leq \frac{1}{2} w_0(x + t)e^{\bar{w}_1(x) - w_1(x)} \leq \frac{1}{2} w_0(x + t)e^{\bar{w}_1(x)} \quad \text{for all} \quad x, t \in \mathbb{R}_+.
\]

Under assumption (4.2), one can simplify (4.11) as follows

\[
|K(x, t)| \leq \frac{1}{2} w_0(x + t)e^{\bar{w}_1(x)}, \quad \bar{w}_1(x) := \int_{0}^{x} |q(y)| dy, \quad w_1(x) := \int_{0}^{x} w_0(y) dy.
\]

Combining this fact with (4.8) and (4.9), one obtains

\[
|c(x, \lambda)| \leq (1 + C_1 x) e^{x|\text{Im} \sqrt{\lambda}|}, \quad |\sqrt{\lambda} s(x, \lambda)| \leq (1 + C_1 x) e^{x|\text{Im} \sqrt{\lambda}|}, \tag{4.13}
\]

for all \( x \in \mathbb{R}_+ \) and \( \lambda \in \mathbb{C}_+ \cup \mathbb{R} \). We also need the following inequality for \( s(x, \lambda) \) (see [52, §3.1])

\[
|s(x, \lambda)| \leq x e^{x|\text{Im} \sqrt{\lambda}|} e^{\bar{w}_1(x)} \leq C_2 x e^{x|\text{Im} \sqrt{\lambda}|}, \quad C_2 := e^{C_0}, \quad x \in \mathbb{R}_+, \quad \lambda \in \mathbb{C}_+ \cup \mathbb{R}, \tag{4.14}
\]

which is better than (4.13) as \( \lambda \to 0 \).

Further, we put

\[
\tilde{a}(\lambda) = 1 + \int_{0}^{+\infty} q(t) e^{-\sqrt{\lambda} t} s(t, \lambda) dt, \quad \tilde{b}(\lambda) = \int_{0}^{+\infty} q(t) e^{-\sqrt{\lambda} t} c(t, \lambda) dt, \tag{4.15}
\]

and

\[
a_0 := \tilde{a}(0) = 1 + \int_{0}^{+\infty} q(t) s(t, 0) dt, \quad b_0 := \tilde{b}(0) = \int_{0}^{+\infty} q(t) c(t, 0) dt. \tag{4.16}
\]

By (4.2), (4.13), and (4.14), the integrals (4.15) and (4.16) exist and are finite for all \( \lambda \in \mathbb{C}_+ \cup \mathbb{R} \). Note also that \( a_0, b_0 \in \mathbb{R} \), since \( q(\cdot), c(\cdot, 0), \) and \( s(\cdot, 0) \) are real functions.

Let us show that \( a_0 = 0 \) if and only if the solution \( s(x, 0) \) is bounded. Indeed, integrating the equation

\[
-y''(x) + q(x) y(x) = 0, \quad x > 0, \tag{4.17}
\]

we get

\[
s'(x, 0) = 1 + \int_{0}^{x} q(t) s(t, 0) dt, \quad x \geq 0,
\]
since \( s(0,0) = 1 \). By (4.16), \( a_0 = 0 \) exactly when \( s'(x,0) = o(1) \) as \( x \to +\infty \). On the other hand, equation (4.17) with \( q(\cdot) \) satisfying (4.2) has two linearly independent solutions \( y_1(x) \) and \( y_2(x) \) such that

\[
y_1(x) = 1 + o(1), \quad y_1'(x) = o(1), \quad \text{and} \quad y_2(x) = x + o(1), \quad y_2'(x) = 1 + o(1), \quad x \to +\infty.
\]

Hence, \( s(x,0) = c_1 y_1(x) + c_2 y_2(x) \). So we conclude that \( s'(0) = o(1) \) as \( x \to +\infty \) if and only if \( s(\cdot,0) = c_1 y_1(\cdot) \in L^\infty(\mathbb{R}_+) \).

Note that \( c(x,\lambda) \) and \( s(x,\lambda) \) are entire functions of \( \lambda \) for every \( x \in \mathbb{R}_+ \). Combining this fact with (4.13), (4.14), and first Helly’s theorem, we obtain that functions (4.15) are continuous on \( \mathbb{C}_+ \cup \mathbb{R} \). Due to the assumption \( s(\cdot,0) \notin L^\infty(\mathbb{R}_+) \), we have \( a_0 \neq 0 \). Therefore,

\[
a(\lambda) = \frac{a_0}{2\sqrt{-\lambda}}(1 + o(1)), \quad b(\lambda) = \frac{1}{2} + \frac{b_0}{2\sqrt{-\lambda}}(1 + o(1)), \quad |\lambda| \to 0, \quad (4.18)
\]

and (4.4) easily follows from (4.6).

To complete the proof of (i), it remains to note that \( a_0 > 0 \) since \( m(\cdot) \) is an \( R \)-function.

(ii) Let the solution \( s(x,0) \) be bounded, i.e., \( s(\cdot,0) \in L^\infty(\mathbb{R}_+) \). It is well known (see [52, \S3.1]) that under condition (4.2), one of the the Weyl solutions \( f(x,\lambda) \) of (4.3) admits a representation

\[
f(x,\lambda) = e(x,\lambda) := e^{i\sqrt{\lambda}x} + \int_x^{+\infty} \tilde{K}(x,t)e^{i\sqrt{\lambda}t} dt = (I + \tilde{K})e^{i\sqrt{\lambda}x}, \quad x > 0, \quad \lambda \in \mathbb{C}_+ \quad (4.19)
\]

where the kernel \( \tilde{K}(x,t) \) satisfies the following estimates for \( x, t \geq 0 \)

\[
|\tilde{K}(x,t)| \leq \frac{1}{2} \tilde{\omega}_0 \left( \frac{x + t}{2} \right) e^{\tilde{\omega}(x)}, \quad \tilde{\omega}_0(x) := \int_x^{\infty} |q(t)| dt, \quad \tilde{\omega}(x) = \int_x^{\infty} \tilde{\omega}_0(t) dt. \quad (4.20)
\]

Due to the estimate (4.20), the representation (4.19) remains valid for \( \lambda \in \mathbb{R} \). In particular,

\[
e(x,0) = 1 + \int_x^{\infty} \tilde{K}(x,t) dt
\]

is a bounded solution of (4.17). Hence,

\[
e(x,0) = c_0 s(x,0) \quad \text{with} \quad c_0 = -\tilde{K}(0,0) + \int_0^{\infty} K'_x(0,t) dt \neq 0, \quad \text{and} \quad e(0,0) = c_0 s(0,0) = 0.
\]

In this case, \( e(0,\lambda) \) has the form (see [52, \S3.2])

\[
e(0,\lambda) = i\sqrt{\lambda}\tilde{K}_1(-\sqrt{\lambda}), \quad \tilde{K}_1(x) = \int_x^{\infty} \tilde{K}(0,t) dt, \quad (4.21)
\]

where \( \tilde{K}_1(\lambda) := \int_0^{\infty} K_1(t)e^{-i\lambda t} dt \). Moreover, \( \tilde{K}_1 \) is continuous at zero since \( K_1 \in L^1(\mathbb{R}_+) \), and \( \tilde{c}_0 := \tilde{K}_1(0) \neq 0 \) (see [52, \S3.2]). Noting that \( e'(0,0) = c'_0 s'(0,0) = c'_0 \neq 0 \), and taking into account (2.9), we arrive at the desired relation

\[
m(\lambda) = -\frac{e(0,\lambda)}{e'(0,\lambda)} = -\frac{i\sqrt{\lambda}\tilde{K}_1(0)}{c'_0}(1 + o(1)), \quad (\mathbb{C}_+ \ni) \lambda \to 0, \quad (4.22)
\]

which proves (ii) with \( k = -\frac{\tilde{c}_0}{c'_0} \). The inequality \( k > 0 \) follows from the inclusion \( m(\cdot) \in (R) \). \( \square \)
Proposition 4.2. Let (4.2) be fulfilled. Then the operator $A$ defined by (4.1) has no real eigenvalues, $\sigma_p(A) \cap \mathbb{R} = \emptyset$.

Proof. First, let $\lambda = 0$. Since $L = JA$, one gets $\ker A = \ker L$. But $\ker L = \{0\}$ (see e.g. [52, §3.2]).

Let $\lambda > 0$ and $f(x) \in \ker(A - \lambda)$ (the case $\lambda < 0$ is analogous). Then $f \in L^2(\mathbb{R})$ solves (4.3) with $\lambda > 0$. Under assumption (4.2), equation (4.3) has two linearly independent solutions of the form (see [52, Lemma 3.1.3])

$$e_+(x, \lambda) = e^{ix\sqrt{x}} + \int_x^\infty K(t, x)e^{it\sqrt{t}}dt, \quad e_-(x, \lambda) = e^{-ix\sqrt{x}} + \int_x^\infty \tilde{K}(x, t)e^{-it\sqrt{t}}dt; \quad x \geq 0,$$

with $\tilde{K}$ satisfying (4.20). So $f(x) = c_+e_+(x, \lambda) + c_-e_-(x, \lambda)$ for $x > 0$ with certain $c_+ \in \mathbb{C}$. Hence $f(x) = c_+e^{ix\sqrt{x}} + c_-e^{-ix\sqrt{x}} + o(1)$ as $x \to +\infty$. The latter yields $c_+ = c_- = 0$ since $f \in L^2(\mathbb{R})$. Therefore, $f(x) = 0$, $x > 0$. Since $f$ is a solution of (4.3), we get $f \equiv 0$.

Remark 4.1. Assume that $q$ satisfies (4.2) on $\mathbb{R}_+$ and that the minimal symmetric operator

$$L_{\min}^+ = -\frac{d^2}{dx^2} + q(x), \quad \text{dom}(L_{\min}^+) = \mathcal{D}_{\min}^+,$$

is nonnegative in $L^2(\mathbb{R}_+)$. The Friedrichs (maximal nonnegative) extension $L_F^+ = (L_F^+)^*$ of $L_{\min}^+$ is determined by the Dirichlet boundary condition at zero

$$\text{dom}(L_F^+) = \{f \in \text{dom}((L_{\min}^+)^*) : f(+0) = 0\}.$$ Recall that the corresponding $m$-coefficient is $\tilde{m}_+(\cdot) = -1/m_+(\cdot)$. If $s(\cdot, 0) \in L^\infty(\mathbb{R}_+)$, then (4.5) yields $\tilde{m}_F^+(0) := \tilde{m}_+(0) = +\infty$. It is known (see for instance [47] and [14, Proposition 4]) that the latter condition holds if and only if $L_F^+$ is a unique nonnegative self-adjoint extension of $L_{\min}^+$. Thus, Lemma 4.1 leads to the following criterion for the nonnegative operator $L_{\min}^+$:

$L_F^+$ is a unique nonnegative self-adjoint extension of $L_{\min}^+$ if and only if $s(\cdot, 0) \in L^\infty(\mathbb{R}_+)$. 

4.2 The case of the nonnegative operator $L$.

The proof of Theorem 1.1 is contained in this and the next subsections. The most essential part, the implication $(ii) \Rightarrow (i)$, is given by the following theorem.

Theorem 4.3. Let $A = (\text{sgn} x)(-d^2/dx^2 + q(x))$ and let $q(\cdot)$ satisfy (4.2). If the operator $A$ is $J$-nonnegative, then it is similar to a self-adjoint operator with absolutely continuous spectrum.

Proof. 1) Assume that the operator $A$ is $J$-nonnegative. By Proposition 2.7, $\sigma(A) \subset \mathbb{R}$. Since $r(x) = \text{sgn} x$ obviously satisfies (1.4), Proposition 2.8 implies that $\infty$ is a regular critical point of $A$. Moreover, (4.2) implies $\ker A = \{0\}$ (see Proposition 4.2). Hence the similarity of $A$ is equivalent to the nonsingularity of the critical point zero of the operator $A$ (see Proposition 2.6).

By Lemma 4.1 and (2.10), one of the asymptotic formulas (4.4), (4.5) holds for the function $m_+(\lambda) = M_+(\lambda)$. And the same is true for $m_-(-\lambda) = M_-(-\lambda)$. Consider the following four cases.

(a) Let the solution $s(\cdot, 0)$ of (4.3) be bounded on $\mathbb{R}$, $s(\cdot, 0) \in L^\infty(\mathbb{R})$. By Lemma 4.1 (iii),

$$M_\pm(\lambda) = \mp k_+ \sqrt{\pm \lambda(1 + o(1))}, \quad \lambda \to 0; \quad k_+ > 0.$$
Therefore, using the identity \( \sqrt{-\lambda} = -i\sqrt{\lambda}, \lambda \in \mathbb{C}_+ \), we obtain
\[
\frac{M_+(\lambda) + M_-(-\lambda)}{M_+(\lambda) - M_-(-\lambda)} = \frac{-k_+\sqrt{-\lambda} + k_-\sqrt{\lambda}}{-k_+\sqrt{\lambda} - k_-\sqrt{-\lambda}}(1 + o(1)) = \frac{ik_+ + k_-}{ik_+ - k_-}(1 + o(1)), \quad \lambda \to 0.
\]

(b) Let \( s(\cdot, 0) \notin L^\infty(\mathbb{R}_+) \), but \( s(\cdot, 0) \in L^\infty(\mathbb{R}_-) \). Then, by Lemma 4.1,
\[
M_+(\lambda) = \frac{a_+}{b_+ + \sqrt{-\lambda}}(1 + o(1)), \quad M_-(-\lambda) = k_-\sqrt{\lambda}(1 + o(1)); \quad \lambda \to 0,
\]
where \( a_+ > 0, b_+ \in \mathbb{R} \), and \( k_- > 0 \). Hence
\[
\frac{M_+(\lambda) + M_-(-\lambda)}{M_+(\lambda) - M_-(-\lambda)} = \frac{a_+ + k_-\sqrt{\lambda}(b_+ + \sqrt{-\lambda})}{a_+ - k_-\sqrt{\lambda}(b_+ + \sqrt{-\lambda})}(1 + o(1)) = 1 + O(\sqrt{|\lambda|}), \quad \lambda \to 0.
\]

(c) The case when \( s(\cdot, 0) \in L^\infty(\mathbb{R}_+) \) and \( s(\cdot, 0) \notin L^\infty(\mathbb{R}_-) \) is similar to (b).

(d) Let \( s(\cdot, 0) \notin L^\infty(\mathbb{R}_+) \) and \( s(\cdot, 0) \notin L^\infty(\mathbb{R}_-) \). Then, by Lemma 4.1 (ii), one gets
\[
M_\pm(\lambda) = \pm a_\pm(b_\pm + \sqrt{\pm\lambda})^{-1}(1 + o(1)), \quad \lambda \to 0,
\]
where \( a_\pm > 0 \) and \( b_\pm \in \mathbb{R} \). Hence,
\[
\frac{M_+(\lambda) + M_-(-\lambda) - c}{M_+(\lambda) - M_-(-\lambda)} = \frac{a_+(b_+ + \sqrt{\lambda}) - a_-(b_- + \sqrt{-\lambda}) - c(b_+ + \sqrt{-\lambda})(b_- + \sqrt{\lambda})}{a_+(b_+ + \sqrt{\lambda}) + a_-(b_- + \sqrt{-\lambda})}(1 + o(1)),
\]
as \( \lambda \to 0 \). If \( b_+ \cdot b_- = 0 \), then the left part of (4.24) with \( c = 0 \) has the asymptotic behaviour similar to one of the cases (a), (b), or (c). Otherwise, we put \( c := a_+b_+^{-1} - a_-b_-^{-1} \) and get
\[
\frac{M_+(\lambda) + M_-(-\lambda) - c}{M_+(\lambda) - M_-(-\lambda)} = \frac{(a_+ - cb_+)\sqrt{\lambda} - (a_- + cb_-)\sqrt{-\lambda}}{a_+(b_+ + \sqrt{\lambda}) + a_-(b_- + \sqrt{-\lambda})}(1 + o(1)) = O(1), \quad \lambda \to 0.
\]

From the above considerations, we conclude that there exists \( c \in \mathbb{R} \) such that ratio (3.5) is bounded in a neighborhood of zero. By Theorem 3.3 (i), zero is not a singular critical point of \( A \). Combining this fact with Propositions 2.6 and 2.7, we complete the proof of similarity of \( A \) to a self-adjoint operator.

2) Let us show that the operator \( A \) is similar to a self-adjoint one with absolutely continuous spectrum. For this purpose, we investigate the characteristic function \( \theta_A(\cdot) \) determined by (2.17).

Since \( q \) satisfies (4.2), the m-coefficients \( M_+ \) and \( M_- \) may be extended by continuity from \( \mathbb{C}_+ \) on \( \mathbb{C}_+ \setminus \{0, \lambda_1^+, \ldots, \lambda_k^+, \lambda_1^-, \ldots, \lambda_n^-\} \), where \( n_+ \in \mathbb{Z}_+ \) and \( \{\lambda_k^+\}_{k=1}^{n_+} \{\lambda_k^-\}_{k=1}^{n_-} \) is the set of negative (resp., positive) poles of \( M_+ \) (resp., \( M_- \)) [49, 52].

If \( \lambda_0 < 0 \) is a pole of \( M_+ \), then \( \theta_A(\cdot) \) is bounded at \( \lambda_0 \) since \( M_- \) is continuous on \( \mathbb{R}_- \setminus \{0\} \). Therefore, we can assume that both \( M_+ \) and \( M_- \) have no poles on \( \mathbb{R} \setminus \{0\} \) and hence are continuous on \( \mathbb{C}_+ \setminus \{0\} \). Thus \( \theta_A(\cdot) \) is unbounded in a neighborhood of \( \lambda_0 \in \mathbb{C}_+ \cup \mathbb{R} \setminus \{0\} \) if and only if \( M_+(\lambda_0) - M_-(\lambda_0) = 0 \).

Note that \( \sigma(A) \subset \mathbb{R} \) and hence, by Proposition 3.1 (i), \( M_+(\lambda) - M_-(-\lambda) \neq 0 \) for all \( \lambda \in \mathbb{C}_+ \). Moreover, one can easily show that \( M_+(\lambda) - M_-(-\lambda) \neq 0 \) if \( \lambda \in \mathbb{R} \setminus \{0\} \). Indeed, let \( \lambda = \lambda_0 > 0 \) (the case \( \lambda < 0 \) may be considered similarly). Then \( M_-(\lambda_0) = -M_-(\lambda_0) \in \mathbb{R} \) since, by Proposition
2.1, \(-(m_+^{-1} + m_-^{-1})\in(S^{-1})\). Further, it follows easily from (2.11), (4.19), and the invertibility of transformation operator \((I + \tilde{K})\) that

\[
\text{Im} \, M_+(\lambda_0) = \text{Im} \, m_+(\lambda_0) = \lim_{y \to +0} \text{Im} \, m_+(\lambda_0 + iy) = \lim_{y \to +0} y \frac{\|e(\cdot, \lambda_0 + iy)\|^2_{L^2(\mathbb{R}_+)}}{\|e_x'(0, \lambda_0 + iy)\|^2} \geq C_2 \lim_{y \to +0} y \frac{\|e(\cdot, \lambda_0 + iy)\|^2_{L^2(\mathbb{R}_+)}}{\|e_x'(0, \lambda_0 + iy)\|^2} = C_2 \frac{\sqrt{\lambda_0}}{2} \|e_x'(0, \lambda_0)\|^2, \quad C_2 := \|(I + \tilde{K})^{-1}\|^{-1} > 0.
\]

Here \(e(\cdot, \lambda)\) is the Weyl solution of (4.3) determined by (4.19). It is known that the function \(e_x'(0, \lambda)\) is continuous and has no zeros on \((0, +\infty)\) (see [49, 52]). Hence \(\text{Im} \, M_+(\lambda_0) > 0\) for \(\lambda_0 > 0\) and \(\text{Im} \,(M_+(\lambda_0) - M_-(\lambda_0)) = \text{Im} \, M_+(\lambda_0) > 0\).

Therefore, \(\theta_A(\cdot)\) may be unbounded only near zero and infinity. That is for each \(\varepsilon > 0\),

\[
\sup_{\lambda \in \mathbb{C}_+ \setminus \{\mathbb{D}_z(0) \cup \mathbb{D}_z(\infty)\}} \|\theta_A(\lambda)\| < \infty.
\]

Since \(\ker L = \ker A = \{0\}\), Proposition 2.4 (ii) implies that the operator \(A\) is similar to a self-adjoint operator with absolutely continuous spectrum.

\[
\square
\]

In passing, we have proved the following facts for any (not necessarily \(J\)-nonnegative) operators (4.1)-(4.2).

**Proposition 4.4.** Let \(A = (\text{sgn} \, x)(-d^2/dx^2 + q(x))\) with \(q\) satisfying (4.2). Then:

(i) there exist \(c \in \mathbb{R}\) such that the ratio (3.5) is bounded in some neighborhood of zero,

(ii) if the characteristic function \(\theta_A(\cdot)\) is unbounded near \(\lambda_0 \in \mathbb{R}\), then \(\lambda_0 = 0\).

Note that Proposition 4.2 is a corollary of statement (b) since inclusion \(\lambda_0 \in \sigma_p(A)\) yields that \(\theta_A(\cdot)\) is unbounded in a neighborhood of \(\lambda_0\) (see [8]). Note also that for the operators (1.2) with \(r = \text{sgn} \, x\) the latter also follows from [41, Theorem 4.2].

**Remark 4.2.** It should be pointed out that for the nonnegative operator \(L\) only the case (d) in the proof of Theorem 4.3 is realized. Actually, if \(s(\cdot, 0) \in L^\infty(\mathbb{R}_+)\) then (4.5) yields \((-m_+(x))^{-1} \uparrow +\infty\) as \(x \uparrow -0\). Therefore \((-m_+)^{-1} + (-m_-)^{-1}\) takes positive values on \(\mathbb{R}_-\). But \(L \geq 0\) and Proposition 2.1 implies \((-m_+)^{-1} + (-m_-)^{-1} \in(S^{-1})\). This contradiction shows that \(s(\cdot, 0) \notin L^\infty(\mathbb{R}_+)\).

### 4.3 The operator \(L\) with negative eigenvalues.

It is known that, under the condition (4.2), the negative spectrum of the operator \(L = JA = -d^2/dx^2 + q(x)\) either consists of a finite number \(\kappa_-(A)\) of simple eigenvalues or else is empty (see e.g. [49, 52]). So Propositions 1.1 and 2.5 of [11] imply that \(A\) is a definitizable operator (for the definitions and basic facts see [28, 48, 11] and [20, Appendix B]). The latter means that \(\rho(A) \neq \emptyset\) and there exists a real polynomial \(p\) such that \([p(A)f, f] \geq 0\) for all \(f \in \text{dom}(A^k)\), where \(k = \deg p\); the polynomial \(p\) is called definitizing. Since \(\kappa_-(A)\) is finite, there is a definitizing polynomial \(p\) of minimal degree and of the form

\[
p(z) = zq(z)\overline{q(z)}, \quad \deg q \leq \kappa_-(A) \quad \text{(see [48, 11])}.
\]  (4.25)
The polynomial \( q(z) \) is uniquely determined under the assumption that its coefficient of the highest power of \( z \) is one and all its zeros belongs to \( \mathbb{C}_+ \cup \mathbb{R} \). A definitizable operator admits a spectral function \( E(\Delta) \) with, possibly, some critical points (which belong to the set \( \infty \cup \{ \lambda \in \mathbb{R} : p(\lambda) = 0 \} \)). The properties of \( E(\Delta) \) similar to that of Theorem 2.5.

The next proposition follows from [11, Subsection 1.3].

**Proposition 4.5** ([11]). Let \( A = (\sgn x)(-d^2/dx^2 + q(x)) \) and \( q \in L^1(\mathbb{R}, (1 + |x|)dx) \). Let \( q \) be defined by (4.25). Then:

(i) \( \lambda \in \mathbb{R} \setminus \{ 0 \} \) is a zero of \( q(\cdot) \) if and only if it is a critical point of \( A \);

in this case, \( \lambda \) is also an eigenvalue of \( A \).

(ii) \( \lambda \in \mathbb{C}_+ \setminus \{ \mathbb{R} \} \) is a zero of \( q(\cdot) \) (\( q(\overline{\cdot}) \)) if and only if it is a nonreal eigenvalue of \( A \);

in this case, the algebraic multiplicity of \( \lambda \) is finite.

Taking Propositions 4.2 and 4.4 into account, we obtain the following description for essential and discrete parts of the operator \( A \).

**Theorem 4.6.** Let \( A = (\sgn x)(-d^2/dx^2 + q(x)) \) and \( q(\cdot) \) satisfy (4.2). Then:

(i) The nonreal spectrum \( \sigma(A) \setminus \mathbb{R} \) is finite and consists of eigenvalues of finite algebraic multiplicity. If \( \lambda_0 \in \mathbb{C} \setminus \mathbb{R} \) is an eigenvalue of \( A \), then its algebraic multiplicity is equal to the multiplicity of \( \lambda_0 \) as a zero of the holomorphic function \( M_+(\lambda) - M_-(\lambda) \). Its geometric multiplicity equals 1.

(ii) \( \sigma_p(A) = \sigma_{\text{disc}}(A) = \sigma(A) \setminus \mathbb{R}, \quad \sigma_{\text{ess}}(A) = \mathbb{R}, \) and there exist a skew direct decomposition

\[
L^2(\mathbb{R}) = \mathcal{H}_{\text{ess}} + \mathcal{H}_{\text{disc}}
\]

such that

\[
A = A_{\text{ess}} + A_{\text{disc}}, \quad A_{\text{ess}} = A \upharpoonright (\text{dom}(A) \cap \mathcal{H}_{\text{ess}}), \quad A_{\text{disc}} = A \upharpoonright (\text{dom}(A) \cap \mathcal{H}_{\text{disc}}),
\]

and

\[
\sigma_{\text{disc}}(A) = \sigma(A_{\text{disc}}) = \sigma(A) \setminus \{ 0 \}, \quad \sigma_{\text{ess}}(A) = \sigma(A_{\text{ess}}) = \mathbb{R};
\]

the subspace \( \mathcal{H}_{\text{disc}} \) is finite-dimensional.

(iii) \( A_{\text{ess}} \) is similar to a self-adjoint operator.

**Proof.** (i) follows from Proposition 4.5 (i) and [41, Proposition 4.3 (5)].

(ii) follows from (i) and Proposition 4.2 (see e.g. [41, Section 6]). Note only that \( \sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A_0) = \mathbb{R} \) (see e.g. [41, Proposition 4.3 (1)]).

(iii) The operator \( A \) is a definitizable and admits a spectral function \( E_A(\Delta) \). Moreover, \( \sigma_p(A) \cap \mathbb{R} = \emptyset \). So Proposition 4.5 (i) implies that \( q \) has no real zeros and that the only possible critical points of the operator \( A \) are zero and infinity. Further, \( \infty \) is a regular critical point due to [11, Theorem 3.6]. Using Proposition 4.4 (i) and arguing as in the proof of Theorem 3.3 (ii), one can prove that zero is not a singular critical point of \( A \). Hence \( A_{\text{ess}} \), the part of \( A \) corresponding to the real spectrum, is similar to a certain self-adjoint operator \( T \).

**Corollary 4.7.** Let \( A = (\sgn x)(-d^2/dx^2 + q(x)) \) and \( q(\cdot) \) satisfy (4.2). Then:

(i) \( \lambda_0 \) is an eigenvalue of \( A \) if and only if it is a zero of \( \overline{q(z)q(z)} \); moreover, its algebraic multiplicity coincides with the multiplicity as a zero of \( q(z)q(\overline{z}) \).

(ii) \( \sigma(A) \subset \mathbb{R} \) if and only if \( A \) is \( J \)-nonnegative.
Proof. (i) Since $\sigma_p(A) \cap \mathbb{R} = \emptyset$, $q(0) \neq 0$ due to Proposition 4.5 (i). It follows from these facts that equality holds in [11, formula (1.3)]. Combining this and [48, Proposition II.2.1], we see that the degree $\deg p$ of polynomial $p(z) = zq(z)q'(-z)$ is greater or equal than $2\kappa_-(A)$. From this and (4.25), we obtain $\deg p = 2\kappa_-(A) + 1$ and $\deg q = \kappa_-(A)$. Applying the equality in [11, formula (1.3)] and [48, Proposition II.2.1] again, one gets statement (i).

(ii) If $A$ is not $J$-nonnegative, then $\kappa_-(A) \geq 1$ and therefore $q(\cdot) \neq 1$. So it has at least one zero $\lambda_1$, which is an eigenvalue of $A$ due to statement (i). By Proposition 4.2, the eigenvalue $\lambda_1$ is nonreal. Proposition 2.7 completes the proof. 

Now we are ready to prove Theorem 1.1.

The proof of Theorem 1.1. Note that the implication (ii) $\Rightarrow$ (i) follows from Theorem 4.3. The implication (i) $\Rightarrow$ (iii) is obvious. To prove Theorem 1.1, it suffices to mention that the equivalence (ii) $\Leftrightarrow$ (iii) was established in Corollary 4.7 (ii).

Recall that the function $M_+(-) - M_-(-)$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}$. The next result follows easily from Theorem 4.6 (i) and Corollary 4.7 (i).

Corollary 4.8. Let $A = (\text{sgn} x)(-d^2/dx^2 + q(x))$ and $q(\cdot)$ satisfy (4.2). Assume also that $A$ is not $J$-nonnegative.

(i) Let $\{z_j\}_1^n$ be the set of nonreal zeros of the function $M_+(-) - M_-(-)$, and let $\{k_j\}_1^n$ be their multiplicities. Then $p = z \prod_{1}^{n}(z - z_j)^{k_j}$ is a definitizing polynomial of minimal degree for $A$.

(ii) The operator $A$ is similar to a normal operator if and only if $k_j = 1$ for all $1 \leq j \leq n$.

Let us mention that assumption (4.2) cannot be weaken. That is, in Section 5, we construct a potential $q$ such that $q \in \cap_{\gamma < 1}L^1(\mathbb{R}, (1 + |x|^\gamma)dx)$ and the corresponding operator $A$ is $J$-nonnegative, but $A$ is not similar to a self-adjoint one.

Remark 4.3. Under the additional assumption $q \in L^1(\mathbb{R}, (1 + |x|^2)dx)$, the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 1.1 was proved in [16] by using another approach. Note also that inclusion $\sigma(A) \subset \mathbb{R}$ was established in [16, Corollary 4] under the assumption $m_{\pm} \in (S)$ (cf. Proposition 2.1 of the present paper).

5 Sturm-Liouville operators with decaying potentials and singular critical point 0

If $r(x) = \text{sgn} x$, then a $J$-self-adjoint operator $A$ of the form (1.2) is similar to a self-adjoint one whenever $L(= JA)$ is uniformly positive (see Proposition 2.8). If $0 \in \sigma_{\text{ess}}(L)$, then it may occur that 0 is a critical point of $A$. Sturm-Liouville operators with the singular critical point 0 were constructed in [38]. A $J$-nonnegative operator of type $\text{sgn} x(-d^2/dx^2 + q(x))$ with the singular critical point 0 have not been constructed, but existence of such an operator was proved in [38, Section 6.2]. The goal of this section is to construct explicitly an example of such type. Our example also shows that the condition (1.6) in Theorem 1.1 cannot be changed to $q \in L^1(\mathbb{R}, (1 + |x|^\gamma)dx)$ with $\gamma < 1$.  

5.1 Example.

**Lemma 5.1.** Let
\[ q_0(x) = -\chi_{[0, \pi/4]}(x) + 2\frac{\chi_{(\pi/4, +\infty)}(x)}{(1 + x - \pi/4)^2}, \quad x \in \mathbb{R}_+. \] (5.1)

Then the function
\[ m_0(\lambda) = \frac{\sin(\pi \sqrt{\lambda + 1/4})/\sqrt{\lambda + 1} + m_1(\lambda) \cos(\pi \sqrt{\lambda + 1/4})}{\cos(\pi \sqrt{\lambda + 1/4}) - m_1(\lambda) \sqrt{\lambda + 1} \sin(\pi \sqrt{\lambda + 1/4})}, \quad \lambda \in \mathbb{C}_+, \] (5.2)

where
\[ m_1(\lambda) = \frac{1 + \sqrt{-\lambda}}{1 + \sqrt{-\lambda} - \lambda}, \quad \lambda \in \mathbb{C}_+, \] (5.3)

is the Weyl-Titchmarsh \(m\)-coefficient of the boundary value problem
\[-y''(x) + q_0(x)y(x) = \lambda y(x), \quad x \geq 0; \quad y'(0) = 0. \] (5.4)

**Proof.** Consider the Sturm-Liouville equation
\[-y''(x) + 2 \left(1 + x\right)^2 y(x) = \lambda y(x), \quad x \geq 0. \] (5.5)

It is easy to check that \(f_1(x, \lambda) = e^{i\sqrt{\lambda + 1}}(\sqrt{\lambda + 1} + i/(x + 1))\) solves (5.5) and \(f_1(\cdot, \lambda) \in L^2(\mathbb{R})\) for \(\lambda \in \mathbb{C}_+.\) Further, we obtain \(f_1(0, \lambda) = e^{i\sqrt{\lambda}}(\sqrt{\lambda} + i)\) and \(f_1'(0, \lambda) = e^{i\sqrt{\lambda}}(-\sqrt{\lambda} + i\lambda - i).\) By (2.9), we get that (5.3) is the Weyl-Titchmarsh \(m\)-coefficient of (5.5) subject to the Neumann boundary condition at zero.

Using (5.1), we obtain that the function
\[ f_0(x, \lambda) = \left(f_1(0, \lambda) \cos((x - \pi/4)\sqrt{\lambda + 1}) + f_1'(0, \lambda) \frac{\sin((x - \pi/4)\sqrt{\lambda + 1})}{\sqrt{\lambda + 1}}\right) \chi_{[0, \pi/4]}(x) + f_1(x - \pi/4, \lambda) \chi_{(\pi/4, +\infty)}(x), \quad x \geq 0, \] (5.6)

is the Weyl solution of (5.4) for \(\lambda \in \mathbb{C}_+.\) To complete the proof it remains to substitute (5.6) in (2.9).

Let us consider the indefinite Sturm-Liouville operator
\[ A = (\text{sgn} x) \left(-\frac{d^2}{dx^2} + q_0(|x|)\right), \quad \text{dom}(A) = W^2_2(\mathbb{R}), \] (5.7)

with \(q_0\) determined by (5.1).

**Theorem 5.2.** Let \(A\) be the operator determined by (5.7) and (5.1). Then:

(i) \(A\) is \(J\)-self-adjoint, \(J\)-nonnegative, and \(\sigma(A) \subset \mathbb{R}\).

(ii) 0 is a simple eigenvalue of \(A\), i.e., its algebraic multiplicity is 1.

(iii) 0 is a singular critical point of \(A\).

(iv) \(A\) is not similar to a self-adjoint operator.
Proof. (i) Note that $q_0$ is bounded on $\mathbb{R}$. Hence $A$ is $J$-self-adjoint. Next, we show that the operator $L = JA = -d^2/dx^2 + q_0(|x|)$ is nonnegative. The potential is even, hence, by Lemma 5.1, $m_+(\lambda) = m_-(\lambda) = m_0(\lambda)$ ($m_0$ is defined by (5.2)). It is easy to see that $m_1$ is a Krein-Stieltjes function, $m_1 \in (S)$, since it is analytic and positive on $(-\infty, 0)$. It is not difficult to see that, the latter implies $m_0 \in (S)$. Proposition 2.1 yields $L \geq 0$. Hence $A = JL$ is $J$-nonnegative and, by Proposition 2.7, $\sigma(A) \subseteq \mathbb{R}$.

(ii) It is easily seen that $\lim_{t \to 0} \lambda m_0(\lambda) = k = d\tau(\{0\}) \neq 0$, where $d\tau$ is the spectral measure of the problem (5.4). So $\lambda = 0$ is the eigenvalue of (5.4) and, therefore, $(c(x, 0)\chi_+)(x) \in L^2(\mathbb{R}_+)$. Furthermore, the potential $q_0(|x|)$ is even, hence $c(x, 0)\chi_+(x) \in L^2(\mathbb{R}_+)$ and $c(x, 0) \in \ker L$. Since $s(x, 0) \notin L^2(\mathbb{R})$, we obtain $\ker L = \text{span}\{c(\cdot, 0)\}$. The equality $\ker A = \ker L$ implies $0 \in \sigma_p(A)$.

Further, by (5.2) and (5.3), we get

$$m_0(\lambda) = \frac{1 + m_1(\lambda)}{1 - m_1(\lambda)}(1 + O(\lambda)) = \left(1 + \frac{2}{\sqrt{-\lambda}} - \frac{2}{\lambda}\right)(1 + O(|\lambda|)), \quad |\lambda| \to 0.$$  

Note that $M_+(\cdot) = -M_-(\cdot) = m_0(\cdot)$ since $m_+(\cdot) = m_-(\cdot) = m_0(\cdot)$. Hence,

$$\frac{\text{Im}(M_+(iy) + M_-(iy))}{M_+(iy) - M_-(iy)} = \frac{\text{Im} m_0(iy)}{\text{Re} m_0(iy)} = \frac{1/\sqrt{2y} + 1/y}{1 + 1/\sqrt{2y}} (1 + O(y)) = \sqrt{\frac{2}{y} (1 + O(\sqrt{y}))}, \quad (5.8)$$

as $y \to +0$. Combining (3.1) with (2.12), (5.8), and the inequality $\| (A_0 - iy)^{-1/2} \| \leq y^{-1}$, after simple calculations we arrive at

$$\| (A - iy)^{-1/2} \| \leq O(y^{-3/2}), \quad y \to +0.$$  

Therefore, $\ker A = \ker A^2$. This completes the proof of (ii).

(iii) Combining (5.8) with Theorem 3.4 (i), we conclude that 0 is a singular critical point of $A$.

(iv) follows from Proposition 2.6 and (iii). \qed

5.2 On a question of B. Ćurgus.

It is known that infinity is a critical point of the operator (1.2). Moreover, the results of [11, 60, 20, 55] shows that the regularity of the critical point $\infty$ of a definitizable operator of type (1.2) depends only on behavior of the weight function $r$ in a neighborhood of its turning point (in our case, in a neighborhood of $x = 0$). At 6th Workshop on Operator Theory in Krein Spaces (TU Berlin, 2006), B. Ćurgus posed the following problem: does the regularity of the critical point zero of a $J$-nonnegative operator of type (1.2) depend only on behavior of the coefficients $q$ and $r$ at infinity?

Below we give the negative answer to this question.

Consider the operator

$$A_1 = (\text{sgn} \ x) \left(-\frac{d^2}{dx^2} + 2 \frac{\chi(\pi/4, +\infty)(|x|)}{(1 + |x| - \pi/4)^2}\right), \quad \text{dom}(A) = W^2_2(\mathbb{R}).$$

It is easy to see that $A_1$ is $J$-self-adjoint and $J$-nonnegative, since the potential is bounded and positive on $\mathbb{R}$. Arguing as in the proof of Lemma 5.1, we obtain that the corresponding Weyl-Titchmarsh m-coefficients are

$$M_+(\lambda) = -M_-(\lambda) = m_2(\lambda) := \frac{\sin(\pi \sqrt{\lambda}/4)/\sqrt{\lambda} + m_1(\lambda) \cos(\pi \sqrt{\lambda}/4)}{\cos(\pi \sqrt{\lambda}/4) - m_1(\lambda) \sqrt{\lambda} \sin(\pi \sqrt{\lambda}/4)}, \quad \lambda \in \mathbb{C}_+,$$
where \( m_1(\cdot) \) is given by (5.3). It is easily seen that \( m_2(\lambda) = (1 + \pi/4 + O(\sqrt{\lambda})) \), as \( \lambda \to 0 \), since \( m_1(\lambda) = 1 + O(\sqrt{\lambda}) \), as \( \lambda \to 0 \). Hence we obtain

\[
\lim_{c+\to 0} \left| \frac{M_+(\lambda) + M_-(\lambda)}{M_+(\lambda) - M_-(\lambda)} \right| = \left| \frac{(1 + \pi/4) - (1 + \pi/4)}{(1 + \pi/4) + (1 + \pi/4)} \right| = 0 < \infty,
\]

and, by Theorem 3.3 (i), 0 is not a singular critical point of \( A_1 \).

On the other hand, the operator \( A \) considered in the previous subsection is an additive perturbation of \( A_1 \) by a potential with a compact support. However, 0 is a singular critical point of \( A \) due to Theorem 5.2 (iii). Thus, the regularity of the critical point zero of the operator (1.2) depends not only on behavior of the weight function \( r \), but also on local behavior of the potential \( q \).

### 6 Periodic and almost-periodic potentials

Throughout this section we assume \( r(x) = \text{sgn} \, x \), so the operators \( L \) and \( A \) have the forms \( L = -d^2/dx^2 + q(x) \) and \( A = (\text{sgn} \, x)L \). All the asymptotic formulas in this section are considered in \( \mathbb{C}_+ \).

#### 6.1 The case of a periodic potential \( q \).

First, we consider the case of \( T \)-periodic potential \( q \in L^1_{\text{loc}}(\mathbb{R}) \), i.e., \( q(x+T) = q(x) \) a.e. on \( \mathbb{R} \), \( T > 0 \). It is known that in this case equation (2.7) is limit point at both \( +\infty \) and \( -\infty \). Hence, the maximal operator \( L \) corresponding to the differential expression \( -d^2/dx^2 + q(x) \) is self-adjoint in \( L^2(\mathbb{R}) \).

Let \( c(x,\lambda) \) and \( s(x,\lambda) \) be the functions defined by (2.7), (2.8). Recall that for any \( x \in \mathbb{R} \), \( c(x,\lambda) \), \( s(x,\lambda) \), \( c'(x,\lambda) \), and \( s'(x,\lambda) \) are entire functions of \( \lambda \), hence so are

\[
\Delta_+(\lambda) := (c(T,\lambda) + s'(T,\lambda))/2 \quad \text{and} \quad \Delta_-(\lambda) := (c(T,\lambda) - s'(T,\lambda))/2. \tag{6.1}
\]

The function \( 2\Delta_+(\cdot) \) is the trace of the monodromy matrix and it is called Hill’s discriminant (or the Lyapunov function).

As before, we denote by \( \tilde{m}_\pm(\lambda) \) (\( m_\pm(\lambda) \)) the Weyl-Titchmarsh \( m \)-coefficient for (2.7) on \( \mathbb{R}_\pm \) corresponding to the Dirichlet (Neumann, resp.) boundary condition at 0. Then,

\[
\tilde{m}_\pm(\lambda) = -\frac{1}{m_\pm(\lambda)} = \frac{\mp \Delta_-(\lambda) + \sqrt{\Delta_+^2(\lambda) - 1}}{s(T,\lambda)}, \tag{6.2}
\]

where the branch of the multifunction \( \sqrt{\Delta_+^2(\lambda) - 1} \) is chosen such that both \( \tilde{m}_\pm(\cdot) \) (and so \( m_\pm(\cdot) \)) belong to the class \( (R) \). For continuous \( q(\cdot) \), formula (6.2) may be found e.g. in [58, §21], the proof for \( q \in L^1[0,T] \) is the same.

Let us recall the following statement (see [58] for the case of continuous \( q \), and e.g. [43, 31] for \( q \in L^1_{\text{loc}}(\mathbb{R}) \) as well as for the case of a singular potential).

**Lemma 6.1.** Let \( L \) be a Sturm-Liouville operator with a \( \tau \)-periodic potential \( q \in L^1_{\text{loc}}(\mathbb{R}) \). Let also \( \lambda_0 := \inf \sigma(L) \). Then:

(i) \( (-\infty <) \lambda_0 \) is a first order zero of \( \Delta_+(\lambda) - 1 \) and \( \Delta'_+(\lambda_0) < 0 \);

(ii) \( s(T,\lambda_0) > 0 \).
Proof of Theorem 1.2. Consider the operator \( L = -d^2/dx^2 + q(x) \) with a \( T \)-periodic potential \( q \) and assume that \( \lambda_0 = \inf \sigma(L) \geq 0 \). It follows from (2.10) and (6.2) that Weyl-Titchmarsh m-coefficients for the operator \( A = (\text{sgn} x)L \) have the form

\[
M_\pm(\lambda) = \frac{s(T, \pm\lambda)}{\sqrt{\Delta_-(\pm\lambda) \mp \sqrt{\Delta_+(\pm\lambda)}} - 1}.
\]

(6.3)

By Proposition 2.8, \( \infty \) is a regular critical point of \( A \). At the same time, by Proposition 2.8, it suffices to consider only the case \( \lambda_0 = 0 \).

Assuming \( \lambda_0 = 0 \), consider two cases.

(a) Let \( \Delta_-(0) = 0 \). Lemma 6.1 (i) yields that \( \lambda_0 = 0 \) is a first order zero of the entire function \( \Delta_+(\lambda) - 1 \). By Lemma 6.1 (ii), \( s(T, 0) > 0 \) and, therefore, (6.3) implies

\[
M_\pm(\lambda) = \frac{s(T, 0)(1 + O(\lambda))}{\pm\lambda(\Delta_+(0) + O(\lambda)) \pm \sqrt{\pm\lambda(2\Delta_+(0) + O(\lambda))}} = \pm i C_1 \sqrt{\pm\lambda}[1 + O(\lambda)],
\]

(6.4)

as \( \lambda \to 0 \). Here \( C_1 = s(T, 0)/\sqrt{-2\Delta_+(0)} > 0 \) and. Substituting (6.4) for \( M_\pm(\lambda) \) in (3.5) with \( c = 0 \), we see that Theorem 3.3 (i) implies that \( 0 \) is not a singular critical point of \( A \).

(b) Suppose \( \Delta_-(0) \neq 0 \). Note that \( \Delta_-(\lambda) \) and \( \Delta_+(\lambda) \) are real if \( \lambda \in \mathbb{R} \). Combining (6.3) with Lemma 6.1 (ii), we get

\[
M_\pm(\lambda) = \frac{s(T, 0)}{\Delta_-(0) \pm iC_2 \sqrt{\pm\lambda}[1 + O(\lambda)]}, \quad \lambda \to 0,
\]

(6.5)

with \( C_2 = \sqrt{-2\Delta_+(0)} > 0 \). Using Theorem 3.3 (i) with \( c = 2s(T, 0)/\Delta_-(0) \in \mathbb{R} \setminus \{0\} \), we see that \( 0 \) is not a singular critical point of \( A \).

Thus the operator \( A \) is \( J \)-nonnegative and has no singular critical points. Moreover, it is well known that \( \sigma(L) = \sigma_{ac}(L) \) and therefore \( \ker A = \ker L = \{0\} \) (see e.g. [49], the last fact follows also from (6.3)). Applying Proposition 2.6, we complete the proof of Theorem 1.2.

\[\square\]

### 6.2 Infinite-zone and finite-zone potentials.

In this subsection we consider the cases of (real) infinite- and finite-zone potentials (see e.g. [49]).

Following [49], we briefly recall the definitions. First note that the spectrum of the operator \( L = -d^2/dx^2 + q(x) \) with an infinite-zone potential \( q \) is absolutely continuous and has the zone structure, i.e.,

\[
\sigma(L) = \sigma_{ac}(L) = [\mu^r_0, \mu^l_1] \cup [\mu^r_1, \mu^l_2] \cup \cdots,
\]

(6.6)

where \( \{\mu^r_j\}_0^\infty \) and \( \{\mu^l_j\}_{j=1}^\infty \) are sequences of real numbers such that

\[
\mu^r_0 < \mu^l_1 < \mu^r_1 < \cdots < \mu^r_{j-1} < \mu^l_j < \mu^r_j < \cdots,
\]

(6.7)

and

\[
\lim_{j \to \infty} \mu^r_j = \lim_{j \to \infty} \mu^l_j = +\infty.
\]

In the case of a finite-zone potential, the corresponding sequences \( \{\mu^r_j\}_0^N \), \( \{\mu^l_j\}_{j=1}^N \) are finite, \( N < \infty \), the spectrum of \( L \) is also absolutely continuous and is given by

\[
\sigma(L) = \sigma_{ac}(L) = [\mu^r_0, \mu^l_1] \cup [\mu^r_1, \mu^l_2] \cup \cdots \cup [\mu^r_N, +\infty).
\]

(6.8)
Let $N \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Consider also sets of real numbers $\{\xi_j\}_1^N$ and $\{\epsilon_j\}_1^N$ such that $\xi_j \in [\mu_j^-, \mu_j^+]$ and $\epsilon_j \in \{-1, +1\}$ for all $j \leq N$. Define polynomials $R(\lambda)$, $P(\lambda)$, and $Q(\lambda)$ by
\[
P(\lambda) = \prod_{j=1}^{N}(\lambda - \xi_j), \quad R(\lambda) = (\lambda - \mu_0^-) \prod_{j=1}^{N}(\lambda - \mu_j^-)(\lambda - \mu_j^+), \quad Q(\lambda) = P(\lambda) \sum_{j=1}^{N} \frac{\epsilon_j \sqrt{-R(\xi_j)}}{p(\xi_j)(\lambda - \xi_j)}. \tag{6.9}
\]
Then there exists (see [49, Lemma 8.1.1]) a real polynomial $S(\lambda)$ of degree $\deg S = N + 1$ such that
\[
S(\lambda) = \prod_{j=0}^{N}(\lambda - \tau_j), \quad \tau_0 \in (-\infty, \mu_0^[-1]), \quad \tau_j \in [\mu_j^-, \mu_j^+], \quad j \in \{1, \ldots, N\}, \tag{6.11}
\]
and the following identity holds
\[
P(\lambda)S(\lambda) - Q^2(\lambda) = R(\lambda). \tag{6.12}
\]

According to [49, formulas (8.1.9) and (8.1.10)] the functions
\[
m_{\pm}(\lambda) := \pm \frac{P(\lambda)}{Q(\lambda) \mp i \sqrt{R(\lambda)}} \tag{6.13}
\]
are the Weyl-Titchmarsh m-coefficients corresponding to the Neumann boundary value problems on $\mathbb{R}_+$ for some Sturm-Liouville operator $L = -d^2/dx^2 + q(x)$ with a quasi-periodic potential $q = \bar{q}$. Here the multifunction $\sqrt{R(\cdot)}$ is considered on $\mathbb{C}$ with cuts along the union of intervals (6.8). The branch $\sqrt{R(\cdot)}$ of the multifunction is chosen in such a way that $\sqrt{R(\lambda_0 + i0)} > 0$ for some $\lambda_0 \in (\mu_N^+, +\infty)$. So both $m_{\pm}(\cdot)$ belong to the class $(R)$. In this case the spectrum of $L$ is given by (6.8).

**Definition 6.1** ([49]). A real quasi-periodic potential $q$ is called a finite-zone potential if the Weyl-Titchmarsh m-coefficients $m_{\pm}$ admit the representations (6.13).

Note that if a potential $q$ is $T$-periodic and equation $\Delta_+(\lambda) = 1$ (see (6.1)) has only a finite number of simple roots, then $q$ is a finite-zone potential (see [49, 52]). Moreover, in this case $\mu_j^+$ and $\mu_j^-$ denote simple roots of $\Delta_+^2(\lambda) - 1 = 0$ listed in a natural order. Note also that every finite-zone potential $q$ is bounded and its $n$-th derivative $d^n/dx^n q$ is bounded on $\mathbb{R}$ for any $n \in \mathbb{N}$ (see [49]).

In [41], a criterion of similarity to a self-adjoint operator for (not necessary $J$-nonnegative) operator $A = (\text{sgn} x)(-d^2/dx^2 + q(x))$ with a finite-zone potential have been obtained (see [41, Theorems 7.1 and 7.2]). In the case of a $J$-nonnegative operator $A$, we present a new simple proof of [41, Corollary 7.4] based on Theorem 3.3.

**Theorem 6.2** ([41]). Let $q(x)$ be a finite-zone potential and $\mu_0^+ \geq 0$. Then $A = (\text{sgn} x)(-d^2/dx^2 + q(x))$ is similar to a self-adjoint operator.

**Proof.** Consider the operator $L = -d^2/dx^2 + q(x)$ with a finite-zone potential $q$ and assume that $L \geq 0$. This is equivalent to $\mu_0^+ \geq 0$ due to (6.8).

Combining (2.10) with (6.13) and (6.12), we get
\[
M_{\pm}(\lambda) = \frac{P(\pm \lambda)}{Q(\pm \lambda) \mp i \sqrt{R(\pm \lambda)}} = \frac{Q(\pm \lambda) \mp i \sqrt{R(\pm \lambda)}}{S(\pm \lambda)}. \tag{6.14}
\]
It is easy to see that
\[
M_\pm(\lambda) = \pm \frac{i}{\sqrt{\pm \lambda}}[1 + O(\lambda^{-1/2})], \quad \lambda \to \infty, \quad \lambda \in \mathbb{C}_+ \cup \mathbb{R}. \tag{6.15}
\]
This implies that the function \((M_+ + M_-)(M_+ - M_-)^{-1}\) is bounded in a certain neighborhood of \(\infty\). So \(\infty\) is a regular critical point due to Theorem 3.3 (ii).

Let us prove that 0 is not a singular critical point. As in the periodic case, we note that 0 is not a critical point if \(\mu_0^* > 0\). Further, assume that \(\mu_0^* = 0\) and consider the cases analogous to the proof of Theorem 1.2.

(a) Let \(\tau_0 = 0 = (\mu_0^*)\), where \(\tau_0\) is the number defined in (6.11). Then \(R(0) = S(0) = 0\), and it follows from (6.12) that \(Q(0) = 0\). By definition, \(P(0) = P(\mu_0^*) \neq 0\) and, therefore, (6.13) implies that (6.4) holds with \(C_1 = \frac{\prod_{j=1}^{N} \xi_j}{(\prod_{j=1}^{N} \mu_j^*)^{1/2}} > 0\).

(b) Let \(\tau_0 \neq 0\) (actually, this yields \(\tau_0 < 0\), see (6.11)). Then \(S(0) \neq 0\). Further, \(R(0) = 0\), \(P(0) \neq 0\) and (6.12) implies that \(Q(0) \neq 0\). Using the second representation of \(M_\pm(\lambda)\) from (6.14), one can check that
\[
M_\pm(\lambda) = C_2 \pm i C_3 \sqrt{\pm \lambda} + o(|\lambda|^{1/2}), \quad \lambda \to 0, \tag{6.16}
\]
where \(C_2 = Q(0)/S(0) \in \mathbb{R} \setminus \{0\}\) and \(C_3 = |C_1/S(0)| > 0\).

The arguments of subsection 6.1 conclude the proof. \(\square\)

In the proof of Theorem 6.2, we proved the regularity of the critical point \(\infty\) using the asymptotic formula (6.15) for \(M_\pm\) and sufficient regularity condition (Theorem 3.3 (ii)). On the other hand, this fact follows from Proposition 2.8.

Now consider infinite sequences \(\{\mu_j^*\}_0^\infty\), \(\{\mu_j^\prime\}_1^\infty\), \(\{\xi_j\}_j^\infty\), and \(\{\epsilon_j\}_j^\infty\) such that \(\xi_j \in [\mu_j^\prime, \mu_j^*]\) and \(\epsilon_j \in \{-1, +1\}\) for all \(j \geq 1\), assumptions (6.7) and inequalities
\[
\sum_{j=1}^{\infty} \mu_j^\prime (\mu_j^* - \mu_j^\prime) < \infty, \quad \sum_{j=1}^{\infty} \frac{1}{\mu_j^\prime} < \infty. \tag{6.17}
\]
are fulfilled. For every \(N \in \mathbb{N}\), put
\[
g_N = \prod_{j=1}^{N} \frac{\xi_j - \lambda}{\mu_j^\prime}, \quad f_N = (\lambda - \mu_0^*) \prod_{j=1}^{N} \frac{\lambda - \mu_j^\prime}{\mu_j^\prime}, \tag{6.18}
\]
\[
k_N(\lambda) = g_N(\lambda) \sum_{j=1}^{N} \frac{\xi_j \sqrt{-f_N(\xi_j)}}{g_N(\xi_j)(\lambda - \xi_j)}, \quad h_N(\lambda) = \frac{f_N(\lambda) + k_N^2(\lambda)}{g_N(\lambda)}. \tag{6.19}
\]

It is easy to see from (6.17) that \(g_N\) and \(f_N\) converge uniformly on every compact subset of \(\mathbb{C}\). Denote \(\lim_{N \to \infty} g_N(\lambda) =: g(\lambda), \lim_{N \to \infty} f_N(\lambda) =: f(\lambda)\). [49, Theorem 9.1.1] states that there exist limits \(\lim_{N \to \infty} h_N(\lambda) =: h(\lambda), \lim_{N \to \infty} k_N(\lambda) =: k(\lambda)\) for all \(\lambda \in \mathbb{C}\). Moreover, the functions \(g, f, h,\) and \(k\) are holomorphic in \(\mathbb{C}\).

It follows from [49, Subsection 9.1.2] that the functions
\[
m_\pm(\lambda) := \pm \frac{g(\lambda)}{k(\lambda)} \mp i \sqrt{f(\lambda)} \tag{6.20}
\]
are the Weyl-Titchmarsh m-coefficients on \(\mathbb{R}_\pm\) (corresponding to the Neumann boundary conditions) for some Sturm-Liouville operator \(L = -d^2/dx^2 + q(x)\) with a real bounded potential \(q(\cdot)\). The branch \(\sqrt{f(\cdot)}\) of the multifunction is chosen such that both \(m_\pm(\cdot)\) belong to the class \((R)\).
Definition 6.2 ([49]). A real potential $q$ is called an infinite-zone potential if the Weyl-Titchmarsh m-coefficients $m_{\pm}$ admit representations (6.20).

Let $q$ be an infinite-zone potential defined as above. Since $q$ is bounded, the operator $L = -d^2/dx^2 + q(x)$ is self-adjoint. Its spectrum is given by (6.6). B. Levitan proved that under the additional condition $\inf(\mu_{j+1} - \mu_j) > 0$, the potential $q$ is almost-periodical (see [49, Chapter 11]). Note that for a $T$-periodic potential $q$ first inequality in (6.17) implies $q \in W^{2}_{2}[0,T]$, and second inequality in (6.17) obviously follows from the asymptotic formulas for the periodic (antiperiodic) eigenvalues (see [49, 52] for the details).

The following theorem is the main result of this subsection.

**Theorem 6.3.** Let $L = -d^2/dx^2 + q(x)$ be a Sturm-Liouville operator with an infinite-zone potential $q$. Assume also that the spectrum $\sigma(L)$ satisfy (6.17) and $L \geq 0$ (i.e., $\mu_{0}^r \geq 0$). Then the operator $A = (\text{sgn } x)L$ is similar to a self-adjoint operator.

The asymptotic formula (6.15) does not hold true in the infinite-zone case. Therefore, we use Proposition 2.8 to prove that $\infty$ is a regular critical point. The rest of the proof is also close to subsection 6.1.

**Proof of Theorem 6.3.** It is sufficient to consider the case $\mu_{0}^r = 0$. Recall that the functions $g$, $f$, $k$, and $h$ defined above are holomorphic in $\mathbb{C}$. Moreover, $g(\cdot)$ and $\varphi(\cdot)$ admit the following representations

$$g(\lambda) = \prod_{j=1}^{\infty} \frac{\xi_j - \lambda}{\mu_j}, \quad f(\lambda) = \lambda \prod_{j=1}^{N} \frac{\lambda - \mu_j^l \lambda - \mu_j^r}{\mu_j^l \mu_j^r},$$

(6.21)

where the infinite products converge uniformly on all compact subsets of $\mathbb{C}$ due to assumptions (6.17) (see [49, Section 9]). In particular, this and $\xi_j > \mu_{0}^r = 0$, $j \in \mathbb{N}$, imply

$$f(0) = 0, \quad g(0) \neq 0.$$ (6.22)

It follows from (6.19) that

$$h_N(\lambda)g_N(\lambda) - k^2_N(\lambda) = f_N(\lambda) \quad \text{and, therefore,} \quad h(\lambda)g(\lambda) - k^2(\lambda) = f(\lambda).$$ (6.23)

As above, the latter yields

$$M_{\pm}(\lambda) = \frac{g(\pm \lambda)}{k(\pm \lambda) \mp i \sqrt{f(\pm \lambda)}} = \frac{k(\pm \lambda) \pm i \sqrt{f(\pm \lambda)}}{h(\pm \lambda)}.$$ (6.24)

(a) Let $k(0) = 0$. Then (6.22) and the first equality in (6.24) yield that (6.4) holds with $C_{1} = \prod_{j=1}^{\infty} \xi_j (\prod_{j=1}^{\infty} \mu_j^l \mu_j^r)^{-1/2} > 0$ (as above, the product converges due to (6.17)).

(b) Let $k(0) \neq 0$. Then (6.23) and (6.22) yield $h(0) \neq 0$. Using the second representation of $M_{\pm}(\lambda)$ from (6.24), we get (6.16) with the constants $C_{2} = k(0)/h(0) \in \mathbb{R}$, $C_{3} = |C_{1}/h(0)| > 0$.

Theorem 3.3 (i) and Proposition 2.8 complete the proof.

If the potential $q$ is periodic or finite(infinite)-zone and $\inf \sigma(L) = 0$, it is easy to show that $0$ is a critical point of $A$. So we have proved that $0$ is a regular critical point in these cases. However, this is not important for the proofs given above.
7 The case \( q \equiv 0 \)

In this section, we consider the operator (1.2) assuming that \( q \equiv 0 \) and \( r \notin L^1(\mathbb{R}_+, dx) \). In the following \( \omega(\cdot) \) stands for \(|r(\cdot)|\). Let us denote the corresponding operator by

\[
A_\omega := -\frac{(\text{sgn} \ x)}{\omega(x)} \frac{d^2}{dx^2}, \quad \text{dom}(L_\omega) = \mathcal{D}.
\] (7.1)

Note that \( A_\omega \) is \( J \)-self-adjoint in \( L^2(\mathbb{R}, \omega dx) \) since \( \omega \notin L^1(\mathbb{R}_+, dx) \). Moreover, it is \( J \)-nonnegative and hence, by Proposition 2.7, the spectrum of \( A_\omega \) is real, \( \sigma(A_\omega) \subseteq \mathbb{R} \).

The main aim of this section is to prove Theorem 1.3. But first we need two preparatory lemmas.

Consider the following spectral problem

\[
y''(x) = \lambda x^\alpha y(x), \quad x \geq 0; \quad y'(0) = 0,
\] (7.2)

with \( \alpha > -1 \). Denote by \( z^{1/(2+\alpha)} \), \( z \in \mathbb{C} \), the branch of the complex root with cut along the negative semi-axis \( \mathbb{R}_- \) such that \((-1 + i0)^{1/(2+\alpha)} = e^{i\pi/(2+\alpha)}\).

**Lemma 7.1** ([15]). Let \( \alpha > -1 \). Then the function

\[
m_\alpha(\lambda) := \frac{C_\nu}{(-\lambda)^\nu}, \quad \lambda \notin \mathbb{R}_+, \quad \nu = \frac{1}{\alpha + 2}, \quad C_\nu := \frac{\Gamma(1 + \nu)}{\nu^2 \nu \Gamma(1 - \nu)},
\] (7.3)

is the Weyl-Titchmarsh \( m \)-coefficient of the problem (7.2) (here \( \Gamma(\cdot) \) is the classical \( \Gamma \)-function).

This result was obtained in [15], using an explicit form of the Weyl solution of equation (7.2) (see [30, 2.162 (1a)]). For the sake of completeness we present different and more simple proof of Lemma 7.1 (but without computing \( C_\nu \)).

**Proof.** Since \( \alpha > -1 \) and \( c(x,0) \equiv 1 \notin L^2(\mathbb{R}_+, x^\alpha) \), equation (7.2) is in the limit-point case at \( +\infty \). The spectral problem (7.2) has a positive spectrum and the corresponding Weyl-Titchmarsh \( m \)-coefficient belongs to the class \((S)\). Let \( f_\alpha(x, \lambda), \ \lambda \in \mathbb{C} \setminus \mathbb{R}_+ \), denote the Weyl solution of (7.2) and let \( \rho \) be a positive number, \( \rho > 0 \). Then \( f_\alpha(\rho x, \lambda) \) is the Weyl solution of (7.2) with \( \rho^{\alpha+2} \lambda \) in place of \( \lambda \), i.e., \( f_\alpha(x, \rho^{\alpha+2} \lambda) \) belongs to \( L^2(\mathbb{R}_+, x^\alpha) \) and solves (7.2) with \( \rho^{\alpha+2} \lambda \) in place of \( \lambda \). Further, by (2.9), we obtain

\[
\rho^{-1} m_\alpha(\lambda) = m_\alpha(\rho^{\alpha+2} \lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+, \quad \rho > 0.
\] (7.4)

Substituting \( \rho = (-\lambda)^{-1/(2+\alpha)} \) with \( \lambda < 0 \) in (7.4), we get

\[
m_\alpha(\lambda) = m_\alpha(-1) \cdot (-\lambda)^{1/(2+\alpha)}, \quad \lambda < 0.
\] (7.5)

To complete the proof of (7.3) it suffices to mention that \( m_\alpha(\cdot) \in (S) \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \). \( \square \)

It was proved in [22, Theorem 2.7] (see also [12]) that the operator \(-{\text{sgn} \ x \over |x|^{\alpha}} \frac{d^2}{dx^2} \), \( \alpha > -1 \), is similar to a self-adjoint one and has no eigenvalues. As a corollary of Lemma 7.1, we obtain the following improvement of [22] (as well as a simple proof).

**Theorem 7.2.** The operator \( A_{|x|^\alpha} = -{\text{sgn} \ x \over |x|^{\alpha}} \frac{d^2}{dx^2} \), \( \alpha > -1 \), is similar to a self-adjoint operator with absolutely continuous spectrum.
Proof. (i) Clearly, the operator $A_{|x|^\alpha}$ is $J$-self-adjoint (since $x \notin L^2(\mathbb{R}_+, |x|^\alpha)$) and $J$-nonnegative. By Lemma 7.1, we have $M_+(-\nu) = -M_-(-\nu) = m_\alpha(-\nu)$. Hence,

$$M_+(\lambda) + M_-(-\lambda) = \frac{1 + \exp\{i\pi\nu\}}{1 - \exp\{i\pi\nu\}}, \quad \lambda \in \mathbb{C}_+.$$ (7.6)

By Theorem 3.2, $A_{|x|^\alpha}$ is similar to a self-adjoint operator.

(ii) Using the explicit form of Weyl-Titchmarsh functions $M_+(-\nu)$ and $M_-(-\nu)$, we get (see (2.17))

$$\theta_{A_{|x|^\alpha}}(\lambda) = \left(\begin{array}{c}
\frac{1 + \exp\{i\pi\nu\}}{1 - \exp\{i\pi\nu\}} \\
\frac{1 + \exp\{i\pi\nu\}}{1 - \exp\{i\pi\nu\}}
\end{array}\right), \quad \lambda \in \mathbb{C}_+.$$ (7.7)

It is obvious that $\theta_{A_{|x|^\alpha}}(\nu)$ is unbounded only at zero and at infinity. Moreover, ker $A_{|x|^\alpha} = \{0\}$ since $1 \notin L^2(\mathbb{R}_+, |x|^\alpha)$. Hence $0 \notin \sigma_p(A_{|x|^\alpha})$ if $\alpha > -1$ and, by Proposition 2.4(ii), the operator $A_{|x|^\alpha}$ is similar to a self-adjoint operator with absolutely continuous spectrum.

Lemma 7.3. Let $\omega > -1$ and $\omega(x) = p(x)x^\alpha$, $x > 0$, with the positive function $p$ satisfying (1.7) on $\mathbb{R}_+$ with $\alpha_+ = \alpha$ and $c_+ > 0$. Let $m_+(-\nu)$ be the Weyl-Titchmarsh function of the spectral problem

$$-\frac{d^2y(x)}{dx^2} = \nu p(x)|x|^\alpha y(x), \quad x > 0; \quad y'(0) = 0.$$ (7.8)

Then

$$m_+(\lambda) = m_\alpha(\lambda)(c_+ + o(1)) = \frac{C_\nu}{(-\lambda)^\nu}(c_+ + o(1)), \quad \lambda \to 0; \quad \nu = 1/(2 + \alpha).$$ (7.9)

Proof. Assume that $c_+ = 1$ and denote by $f_p(x, \lambda)$ the Weyl solution of (7.7)\footnote{we write $f_1(x, \lambda)$ if $p = 1$}. Then $f_p(x, \lambda)$ solves the following integral equation

$$f_p(x; \lambda) = f_1(x; \lambda) + \lambda \int_x^{+\infty} |t|^\alpha(p(t) - 1)G(x, t; \lambda)f_p(t; \lambda)dt,$$ (7.10)

where the Green function is determined by

$$G(x, t; \lambda) = \varphi_1(t, \lambda)f_1(x, \lambda) - \varphi_1(x, \lambda)f_1(t, \lambda).$$ (7.11)

Here $\varphi_1(x, \lambda)$ is the solution of (7.7) such that $W(f_1, \varphi_1) := f_1(x, \lambda)\varphi_1(x, \lambda) - f_1(x, \lambda)\varphi_1(x, \lambda) \equiv 1$.

It is known (see [30, §2.162 (1a)]) that the general solution of (7.7) is

$$y_1(x, \lambda) = c_+\sqrt{x}H^{(1)}(2\nu \sqrt{x} \lambda^{1/2}) + c_-\sqrt{x}H^{(2)}(2\nu \sqrt{x} \lambda^{1/2}), \quad c_\pm \in \mathbb{C},$$ (7.12)

where $H^{(1)}(\cdot)$ and $H^{(2)}(\cdot)$ are the Hankel functions (see [61, §3.6]). Moreover, (see [61, §3.3])

$$W(H^{(1)}(\nu), H^{(2)}(\nu)) = \frac{-4i}{\pi z}.$$ (7.13)

The asymptotic of these functions is well known. Indeed, if $-\pi + \delta \leq \arg z \leq \pi - \delta$, $\delta > 0$, then [61, §7.2]

$$H^{(1)}(\nu) = \left(\frac{2}{\pi z}\right)^{1/2} e^{\pm i(\nu \pi/2 - \pi/4)}(1 + O(z^{-1})), \quad |z| \to \infty;$$ (7.14)

$$H^{(2)}(\nu) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-\pm i(\nu \pi/2 - \pi/4)}(1 + O(z^{-1})), \quad |z| \to \infty.$$ (7.15)
Remark 7.1. Faddeev and Shterenberg [17] proved the similarity of the operator $A_{\omega}$, $\omega(x) = p(x)x^{\alpha_{\pm}}$, $x \in \mathbb{R}_{\pm}$, under the following additional assumptions

(1) $\alpha_{+} = \alpha_{-} = \alpha > -1$;

(2) $0 < c \leq p(x) \leq C < \infty$, $x \in \mathbb{R}$, the function $p$ and its first derivative $p'$ are locally absolutely
continuous on $\mathbb{R}$;
\[
\int_{\mathbb{R}} \frac{|3(u''(x))^2/u'(x) - 2u'''(x)|}{(u'(x))^2(|u(x)|^{\alpha}/2 + 1)\text{sgn} \alpha} \, dx < \infty,
\]
where $u(x)$ is defined by
\[
u(x) := (\text{sgn} x) \left( \left| \frac{\alpha + 2}{2} \int_x^0 \sqrt{p(t)} |q(t)/2\, dt \right| \right)^{2/(2+\alpha)}.
\]

Note that in this case it is rather difficult to apply Theorem 3.2 because under assumption (1.4) the asymptotic behavior of $M_+$ and $M_-$ at infinity is more complicated. In [17], Faddeev and Shterenberg used condition (7.19) to obtain the following uniform asymptotic for $\lambda \in \mathbb{C}_+$
\[
m_\pm(\lambda) = m_v(\lambda) (1 + o(1)), \quad \lambda \to \infty.
\]
Indeed, assumption (2) allows to apply a Liouville type transform to (7.7),
\[
- \frac{1}{|x|^\alpha} \frac{d^2y(x)}{dx^2} = \lambda p(x)y(x) \quad \iff \quad - \frac{1}{|v|^\alpha} \frac{d^2\tilde{y}(v)}{dv^2} + q(v)\tilde{y}(v) = \lambda \tilde{y}(v).
\]
In this case, (7.19) is equivalent to the condition $\int_{\mathbb{R}} \frac{v^\alpha q(v)}{(|v|^{\alpha}/2 + 1)\text{sgn} \alpha} \, dv < \infty$, and asymptotic formula (7.20) has been obtained in [17] using a standard technique.

We avoid this difficulty using the spectral theory of $J$-nonnegative operators, that makes it possible to apply Theorem 3.3 with $c = 0$.

**Acknowledgments**

The authors thank Branko Ćurgus for useful discussions. AK gratefully acknowledges support from the Junior Research Fellowship Program of the Erwin Schrödinger Institute for Mathematical Physics. IK would like to thank to the University of Calgary for support from the Postdoctoral Fellowship Program.

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