HIGHER ORDER RECURRENCES FOR ANALYTICAL FUNCTIONS OF TCHEBYCHEFF TYPE

A. K. Kwaśniewski

Higher School of Mathematics and Applied Informatics
PL - 15-021 Bialystok, ul.Kamienna 17, Poland
e-mail: kwandr@uwb.edu.pl

Abstract

Relation of hyperbolons of volume one to generalized Clifford algebras is described in [1b] and there some applications are listed. In this note which is an extension of [8] we use the one parameter subgroups of the group of hyperbolons of volume one in order to define and investigate generalization of Tchebysheff polynomial system. Parallely functions of roots of polynomials of any degree are studied as possible generalization of symmetric functions considered by Eduard Lucas. It is found how functions of roots of polynomial of any degree are related to this generalization of Tchebysheff polynomials. The relation is explicit.

In a primary sense the considered generalization is in passing from $\mathbb{Z}_2$ to $\mathbb{Z}_n$ group decomposition of the exponential.

We end up with an application of the discovered generalization to quite large class of dynamical systems with iteration.

1 Introduction

Part One: “Tchebycheff $m$-polynomials” recurrence equation.

In order to establish a notation we quote the well known definitions:

$$T_n(x) = \cosh(n\alpha); \quad \cosh \alpha = x;$$
$$U_n(x) = \frac{\sinh(n\alpha)}{\sinh \alpha}; \quad \cosh \alpha = x.$$ (1.1)

Of course

$$\left[T_n(x) + \sqrt{x^2 - 1}U_n(x)\right]^r = T_{nr}(x) + \sqrt{x^2 - 1}U_{nr}(x)$$ (1.2)

It is however more useful to use “a&b” notation:

$$a_x(n) = T_n(x) \quad \text{and} \quad b_x(n) = \sqrt{x^2 - 1}U_n(x)$$ (1.3)
convenient also for the introduction of de Moivre matrix group [1b]:

\[
M_x = \left\{ \begin{pmatrix} a_x(n) & b_x(n) \\ b_x(n) & a_x(n) \end{pmatrix} \right\} \equiv M_x(n) \quad ; \quad \det M_x(n) = 1 \quad (1.4)
\]

where (see [3]; formula (50))

\[
a_x(-n) = a_x(n) \\
b_x(-n) = -b_x(n) \quad (1.5)
\]

Naturally \( M_x(n) M_x(m) = M_x(n+m) \) hence

\[
a_x(n+m) = a_x(n) a_x(m) + b_x(n) b_x(m) \\
a_x(n-m) = a_x(n) a_x(m) - b_x(n) b_x(m) \quad (1.6)
\]

and therefore:

\[
a_x(n+1) = 2a_x(1) a_x(n) - a_x(n-1) \\
a_x(0) = 1, a_x(1) = \cosh \alpha \equiv x \quad (1.7)
\]

This is the recurrence relation for \( T_n(x) = a_x(n) \) Tchebycheff polynomials.

In this note we shall investigate a following generalization of the recurrence relation (1.7) for \( T_n(x) = a_x(n) \); namely “Tchebycheff \( m \)-polynomials” (polynomials in \( m \) constrained variables) are defined by recurrence equation:

\[
x T_n(x) = \frac{1}{m} \sum_{s \in \mathbb{Z}_m} T_{n+w^s}(x), \quad (1.8)
\]

where \( \omega = \exp\left(\frac{2\pi i}{m}\right) \), \( m \geq 2 \), \( T_n(x) \equiv a_x(n) \) and \( a_x(n) \) is to be defined soon with the help of hyperbolic functions of higher order.

In the case of \( m = 2 \), (1.8) coincides with (1.7), if appropriate initial conditions are added and in order to derive (1.8) one uses de Moivre group [1b] for the arbitrary \( m \geq 2 \) in the similar way as above. Hyperbolic cosh and sinh functions – and hence Tchebycheff polynomials of the first and second type – are known to be strictly related to symmetric functions of the roots of quadratic equations (see (5) in [3]). Lucas has studied in [3] the relationship between these symmetric functions and the theories of divisibility, continued fractions, combinatorial analysis, determinants, quadratic diophantine analysis, continued radicals, etc.

His work was also aimed to be “the starting point for a more complete study of the properties of the symmetric functions of the roots of an algebraic equation with rational coefficients of any degree” [3].

To do that one uses in this note the properties of hyperbolic functions of higher order [1a,b],[2].

**Part Two: hyperbolic functions of higher order.**

In the following we shall introduce the hyperbolic functions of arbitrary \( m \geq 2 \) order using projection operators [2] which are important objects on their own.

Let us define this family of projection operators \( \{\Delta_k\}_{k \in \mathbb{Z}_m} \) acting on the linear space of functions of complex variable accordingly to:

\[
\Delta_k := \frac{1}{m} \sum_{s \in \mathbb{Z}_m} \omega^{-ks} \Omega^s; \quad (1.9)
\]
where \( (\Omega f)(z) := f(\omega z) \).

We used \( \omega \) to the negative power because of historical reasons - positive power shall appear more accurate.

The set \( \{\Delta_k\}_{k \in \mathbb{Z}_m} \) constitutes the family of orthogonal projection operators \([2]\):

\[
\Delta_i \Delta_j = \delta_{ij} \Delta_i ; \quad k \in \mathbb{Z}_m .
\] (1.10)

With the help of these projection operators \( \{\Delta_k\}_{k \in \mathbb{Z}_m} \) we define eigenfunctions of the \( \Omega \) operator acting on various linear spaces of functions. Here there are two examples:

1. \( \{h_k(z)\}_{k \in \mathbb{Z}_m} \) constitute the set of eigenfunctions of the \( \Omega \) \( (h_k \leftrightarrow m\)-hyperbolic \( k \)-series); \[ h_k := \Delta_k \exp \Rightarrow h_k(z) = \sum_{s \geq 0} \frac{z^{ms+k}}{(ms+k)!} \Rightarrow \Omega h_k = \omega^k h_k ; \quad k \in \mathbb{Z}_m . \]

2. \( \{g_k(z)\}_{k \in \mathbb{Z}_m} \) constitute the set of eigenfunctions of the \( \Omega \) \( (g_k \leftrightarrow m\)-geometric \( k \)-series); \[ g_k := \Delta_k \frac{1}{1-id} \left( \text{where} \quad \frac{1}{1-id}(z) := \frac{1}{1-z} \right) \Rightarrow g_k(z) = \sum_{s \geq 0} z^{ms+k} \Rightarrow \Omega g_k = \omega^k g_k ; \quad k \in \mathbb{Z}_m . \]

The operators with the algebraic properties of \( \Omega \) had been already profitably used in \([4]\) and \([5]\).

The eigenfunctions \( \{h_s(z)\}_{s \in \mathbb{Z}_m} \) of the \( \Omega \) \( (h_s \leftrightarrow m\)-hyperbolic \( s \)-series) are known since late 40’s as hyperbolic functions of \( m \)-th order (see \([1a,b]\) for the references).

These generalizations of \( \cosh \) and \( \sinh \) hyperbolic functions are given explicitly by:

\[
h_i(x) = \frac{1}{m} \sum_{k \in \mathbb{Z}_m} \omega^{-ki} \exp \left\{ \omega^k x \right\}; \quad i \in \mathbb{Z}_m; \quad \omega = \exp \left\{ \frac{2\pi i}{m} \right\} \] (1.11)

We call (1.11) — Euler formulae for hyperbolic functions of \( m \)-th order. In the next section we construct an analogue of Tchebycheff polynomial system – with the help of

\[
h_0(x) = \frac{1}{m} \sum_{k \in \mathbb{Z}_m} \exp \left\{ \omega^k x \right\}, \quad m \geq 2 . \] (1.12)

2 Tchebycheff Polynomials’ Systems for Higher Order Equations

One may define Tchebycheff -like functions via recurrences in the same manner as in (1.7) due to the identity

\[
\sum_{k \in \mathbb{Z}_m} h_0 \left( \alpha + \omega^k \beta \right) \equiv mh_0(\alpha) h_0(\beta) . \] (2.1)

We define then the Tchebycheff polynomials’ systems of \( m \)-th order to be given by

\[
T_{\vec{n}}(x) = h_0(\vec{n}\alpha), \quad h_0(\alpha) \equiv x ; \] (2.2)

where \( \vec{n} \in \{n, n + \omega, ..., n + \omega^{m-1} \}; \quad n \in \mathbb{N} \).
The recurrence relation (1.8) for the system \( \{ T_{n} (x) \}_{\vec{n} \in \mathbb{N}} \) of Tchebycheff-like functions follows from (2.1) and (2.2):
\[
xT_{n} (x) = \frac{1}{m} \sum_{s \in \mathbb{Z}_{m}} T_{n+\omega^{s}} (x) .
\]

As a sufficient example we write this recurrence for the case of \( m = 3 \):
\[
\begin{cases}
T_{n+1} (x) = 3xT_{n} (x) - T_{n+\omega} (x) - T_{n+\omega^{2}} (x) \\
T_{0} (x) = 1 \quad T_{1} (x) = x
\end{cases}
\]

It is easily seen that \( e^{\alpha}, e^{\omega \alpha}, e^{\omega^{2} \alpha} \) are solutions of the characteristic equation of the recurrence (1.8)
\[
\lambda = 3\lambda - \lambda \omega - \lambda \omega^{2}.
\]
under the identification \( x = h_{0} (\alpha) \); hence the Binet form of the solution of the recurrence (1.8) for \( m = 3 \) and under the identification \( x = h_{0} (\alpha) \) reads as follows
\[
\frac{(e^{\alpha})^{n} + (e^{\omega \alpha})^{n} + (e^{\omega^{2} \alpha})^{n}}{3} \equiv h_{0} (n\alpha)
\]

In order to solve the recurrence (1.8) it is necessary to introduce for \( m = 3 \):
\[
x = h_{0} (\alpha) ; \quad x^{*} = h_{0} (-\alpha) ; \quad x^{**} = h_{0} ((2 + \omega) \alpha) .
\]

Then the following identities and identifications for \( m = 3 \) are useful:
0. \( T_{\beta} (x) := h_{0} (\beta \alpha) ; \beta \in \mathbb{C} ; \)
1. \( x^{*} = x = x^{**} \quad \text{iff} \quad m = 2 ; \quad 2. \ h_{0} (\omega^{l} \alpha) = h_{0} (\alpha) ; \quad l \in \mathbb{Z}_{3} ; \)
3. \( T_{\omega^{2}} (x) = T_{\omega} (x) = T_{1} (x) = x ; \quad 4. \ T_{2+\omega} (x) = x^{**} ; \)
5. \( T_{1+\omega^{2}} (x) = T_{1+\omega} (x) = x^{*} ; \quad 6. \ T_{2+\omega} (x) + T_{2+\omega^{2}} (x) = 3xx^{*} - 1 .
\]

In order to generate these formulae use the identity and the convolution formula [1b]:
\[
\sum_{k \in \mathbb{Z}_{3}} h_{k} (\alpha) h_{i-k} (\beta) \equiv h_{i} (\alpha + \beta) .
\]

Let us now define \( (m = 3) \) ordinary generating functions for the sequence \( \{ T_{\vec{n}} (x) \}_{\vec{n} \in \mathbb{N}} \) of Tchebycheff “3-polynomials” of the 0-th kind (\( Z_{m} \) - group enumeration terminology) as follows:
\[
T_{(0)} (x, x^{*}; z) = \sum_{n \geq 0} T_{n} (x, x^{*}) z^{n} , \quad T_{(s)} (x, x^{*}, x^{**}; z) = \sum_{n \geq 0} T_{n+\omega^{s}} (x, x^{*}, x^{**}) z^{n} , \quad s = 1, 2 .
\]

Using the above (2,..,6) identities and identifications one proves in a standard way that for the sequence of Tchebycheff “3-polynomials” from the “main stream” \( \{ T_{n} (x, x^{*}) \}_{n=0}^{\infty} \) we have
\[
T_{(0)} (x, x^{*}; z) = \frac{1 - 2xz + x^{*}z^{2}}{1 - 3xz + 3x^{*}z^{2} - z^{3}} ;
\]
while ordinary generating functions for “aside streams” are given by
\[
T_{(1)} (x, x^{*}, x^{**}; z) = \frac{x - (3x^{2} - x^{*}) z + x^{**}z^{2}}{1 - 3xz + 3x^{*}z^{2} - z^{3}} ;
\]
\[
T_{(2)} (x, x^{*}, x^{**}; z) = \frac{x - (3x^{2} - x^{*}) z + (3xx^{*} - x^{**} - 1) z^{2}}{1 - 3xz + 3x^{*}z^{2} - z^{3}} ;
\]

We conclude that for \( m > 2 \) one gets the system of polynomials in \( m \) interdependent variables.
3 Cyclic-Symmetric Functions of Roots of Polynomials and Hyperbolic Functions

One may show (see formulae (3.6)) that the appropriate analogues of (1.6) formulae exist (of course \( m \) of them instead of two); also the relation of Tchebysheff \( m \)-polynomials in \( m \) dependent variables to functions of roots of polynomials generalizes to the case of \( m > 2 \).

We are now going to elaborate more on that hyperbolic-trigonometric character of the introduced Tchebysheff-like systems. To do that, let us recall de Moivre formulae in their matrix form [1]:

\[
H(\alpha) H(\beta) = H(\alpha + \beta),
\]

\[
H(\alpha) = \exp \{\gamma \alpha\},
\]

where \( \gamma = (\delta_{i,k-1}) \); \( k, i \in \mathbb{Z}_m \),

equivalent to their convolution form [1b]

\[
h_k(\alpha + \beta) = \sum_{i \in \mathbb{Z}_m} h_i(\alpha) h_{k-i}(\beta); \quad k \in \mathbb{Z}_m.
\] (3.2)

Then for \( m = 3 \) we have

\[
H(n\alpha) = \begin{pmatrix}
h_0(n\alpha) & h_1(n\alpha) & h_2(n\alpha) \\
h_2(n\alpha) & h_0(n\alpha) & h_1(n\alpha) \\
h_1(n\alpha) & h_2(n\alpha) & h_0(n\alpha)
\end{pmatrix} \equiv H^n(\alpha).
\] (3.3)

In order to present the main idea we restrict our attention from now on to the case of \( m = 3 \) (the generalization to the arbitrary \( m \) is straightforward) and we introduce the “\( a, b, c \)” notation:

\[
H(n\alpha) \equiv M_\alpha(n) = \begin{pmatrix}
a_\alpha(n) & b_\alpha(n) & c_\alpha(n) \\
c_\alpha(n) & a_\alpha(n) & b_\alpha(n) \\
b_\alpha(n) & c_\alpha(n) & a_\alpha(n)
\end{pmatrix}
\] (3.4)

thus obtaining de Moivre group [1b]:

\[
M_\alpha = \{M_\alpha(n)\}_{n \in \mathbb{Z}}.
\] (3.5)

Due to (3.2) or equivalently due to the group property of (3.5) one obtains \( (k, n \in \mathbb{Z}) \):

\[
a_\alpha(n+k) = 3a_\alpha(n) a_\alpha(n) - a_\alpha(n+k) - a_\alpha(n+k\omega^2) \quad (3.6a)
\]

\[
b_\alpha(n+k) = 3a_\alpha(n) b_\alpha(n) - b_\alpha(n) - b_\alpha(n+k\omega^2) \quad (3.6b)
\]

\[
c_\alpha(n+k) = 3a_\alpha(n) c_\alpha(n) - c_\alpha(n) - c_\alpha(n+k\omega^2) \quad (3.6c)
\]

i.e. the recurrent sequences \( \{a_\alpha(n)\}_{n \in \mathbb{Z}}, \{b_\alpha(n)\}_{n \in \mathbb{Z}}, \{c_\alpha(n)\}_{n \in \mathbb{Z}} \) obey the same rule of formation (put \( k = 1 \)) but differ in initial conditions. Hence we have Tchebycheff 3-polynomials of the zero-th, the first and the second kind (\( \mathbb{Z}_3 \)-group enumeration terminology) where, of course, the recurrence (3.6a) coincides with (2.3) for \( k = 1 \).

Parallely we shall propose a generalization of Lucas symmetric functions [3].

For the convenience of presentation put \( m = 3 \) and consider \( V, U, W \) functions of roots of the equation: (the case \( a = b = c \) is excluded)

\[
(x - a)(x - b)(x - c) = 0 \iff x^3 = Px^2 + Qx + R \quad (3.7)
\]
as follows

\[ V_n(a, b, c) = a^n + b^n + c^n \]  \hspace{1cm} (3.8)

where

\[
\begin{cases}
V_{n+3} = PV_{n+2} + QV_{n+1} + RV_n \\
V_0 = 3, \quad V_1 = -P, \quad V_2 = a^2 + b^2 + c^2 ;
\end{cases}
\]

\[ U_n(a, b, c) = \frac{a^n + \omega b^n + \omega^2 c^n}{a + \omega b + \omega^2 c} \]  \hspace{1cm} (3.9)

where

\[
\begin{cases}
U_{n+3} = PU_{n+2} + QU_{n+1} + RU_n \\
U_0 = 0, \quad U_1 = 1, \quad U_2 = \frac{a^2 + \omega b^2 + \omega^2 c^2}{a + \omega b + \omega^2 c} ;
\end{cases}
\]

\[ W_n(a, b, c) = \frac{a^n + \omega^2 b^n + \omega c^n}{a + \omega^2 b + \omega c} \]  \hspace{1cm} (3.10)

where

\[
\begin{cases}
W_{n+3} = PW_{n+2} + QW_{n+1} + RW_n \\
W_0 = 0, \quad W_1 = 1, \quad W_2 = \frac{a^2 + \omega^2 b^2 + \omega c^2}{a + \omega^2 b + \omega c} ;
\end{cases}
\]

One identifies the values of \( P, Q, R \) from (3.7) to be:

\[-P = a + b + c; \quad Q = ab + ac + bc; \quad -R = abc. \]  \hspace{1cm} (3.11)

Note that \( V_n(a, b, c) \) – functions are symmetric, while \( U_n(a, b, c) = U_n(b, c, a) = U_n(c, a, b) \), i.e. \( U_n(a, b, c) \) – functions are cyclic-symmetric. Similarly \( W_n(a, b, c) = W_n(b, c, a) = W_n(c, a, b) \).

Recall now that Lucas considered the following symmetric functions of the roots \( a \) and \( b \) of the quadratic equation \( z^2 = Pz - Q \) with \( P, Q \in \mathbb{Z} \) and \( P, Q \) relatively prime.

\[ V_n(a, b) = V_n(b, a) ; \]

\[ U_n(a, b) = U_n(b, a) ; \]

\[ V_n(a, b) = a^n + b^n \]

\[ U_n(a, b) = a^n - b^n \]  \hspace{1cm} (3.12)

Lucas proved [3] the complete analogy of the \( V_n \) and \( U_n \) symmetric functions of roots with the circular and hyperbolic functions of \( m = 2 \) order due to the following \textit{“Lucas formulae”}:

\[ V_n(a, b) = 2Q^{\frac{n}{2}} \cosh \left[ \frac{n}{2} \ln \frac{a}{b} \right] ; \]  \hspace{1cm} (3.13)

\[ U_n(a, b) = \frac{2Q^{\frac{n}{2}}}{\sqrt{\Delta}} \sinh \left[ \frac{n}{2} \ln \frac{a}{b} \right] ; \]  \hspace{1cm} (3.14)

where \( \Delta = P^2 - 4Q \). The equations (3.11) and (3.12) define a one to one correspondence between formulae in plane (hyperbolic) trigonometry and analogous formulae for symmetric functions \( V_n(a, b) \) and \( U_n(a, b) \). (Below we shall propose an available generalization of these formulae to the case of arbitrary \( m \)).

Due to (3.13) and (3.14) we get the following identification (via rescaling of \( V \)’s & \( U \)’s):

\[ a_\alpha(n) := Q^{\frac{n}{2}} \frac{1}{2} V_n; \quad b_\alpha(n) := Q^{\frac{n}{2}} \frac{\sqrt{\Delta}}{2} U_n \]  \hspace{1cm} (3.15)
According to this (see also: formulae (49), (50), (51) in [3]) one has de Moivre group for \( m = 2 \)
\[
\left\{ \begin{array}{l}
(a_\alpha(n), b_\alpha(n)) \\
b_\alpha(n), a_\alpha(n)
\end{array} \right\} ; n \in \mathbb{Z} = \left\{ \begin{array}{l}
(a_\alpha(1), b_\alpha(1)) \\
b_\alpha(1), a_\alpha(1)
\end{array} \right\} ; n \in \mathbb{Z} ; \quad (3.16)
\]
where
\[
\alpha = \frac{1}{2} [\ln a - \ln b] . \quad (3.17)
\]
One may show that it is possible to extend Lucas formulae to the case of the cyclic-symmetric functions of the roots of the polynomial with rational, real or complex coefficients of any degree. For that purpose let us consider cyclic-symmetric functions defined by (3.8), (3.9), (3.10). We state that the following identifications hold (generalization to arbitrary \( m \) is trivial):
\[
a_\alpha(n) = R^{-\frac{n}{3}} \frac{V_n(A, A^\omega, A^{\omega^2})}{3} ; \quad (3.18a)
\]
\[
c_\alpha(n) = R^{-\frac{n}{3}} \frac{U_n(A, A^\omega, A^{\omega^2})}{3} h_1(\ln A) ; \quad (3.18b)
\]
\[
b_\alpha(n) = R^{-\frac{n}{3}} \frac{W_n(A, A^\omega, A^{\omega^2})}{3} h_2(\ln A) ; \quad (3.18c)
\]
Here
\[
\alpha = \frac{1}{3} [\ln a + \omega \ln b + \omega^2 \ln c] ; \quad (3.19)
\]
and
\[
A^3 = ab^\omega c^{\omega^2} . \quad (3.20)
\]
Accordingly to this one has de Moivre group for \( m = 3 \):
\[
H(n\alpha) \equiv \left( \begin{array}{ccc}
a_\alpha(n) & b_\alpha(n) & c_\alpha(n) \\
c_\alpha(n) & a_\alpha(n) & b_\alpha(n) \\
b_\alpha(n) & c_\alpha(n) & a_\alpha(n)
\end{array} \right) ; \quad n \in \mathbb{Z} \quad (3.21)
\]
The corresponding formulae for the case of \( m = 2(\omega = -1, h_1 \equiv \sinh) \) are:
\[
a_\alpha(n) = Q^{-\frac{n}{2}} \frac{V_n(A, A^{\omega})}{2} ; \quad (3.18a*)
\]
\[
b_\alpha(n) = Q^{-\frac{n}{2}} \frac{U_n(A, A^{\omega})}{2} h_1(\ln A) ; \quad (3.18b*)
\]
with
\[
\alpha = \frac{1}{2} [\ln a + \omega \ln b] ; \quad (3.19*)
\]
and
\[
A^2 = ab^\omega . \quad (3.20*)
\]
Of course for the \( Q = 1 \) case (via obvious rescaling of \( V \)’s & \( U \)’s one may always get to this case):
\[
V_n(a, b) = V_n(A, A^{\omega}) ; \quad (3.22)
\]
\[
U_n(a, b) = U_n(A, A^{\omega}) . \quad (3.23)
\]
and both \( \{a_n(n)\}_{n \in \mathbb{Z}} \), \( \{b_n(n)\}_{n \in \mathbb{Z}} \) sequences and \( \{V_n(a,b)\}_{n \in \mathbb{Z}} \), \( \{U_n(a,b)\}_{n \in \mathbb{Z}} \) recurrent sequences obey the same rule of formation but differ in initial conditions. For \( m = 3 \) we have

\begin{align}
V_n \left( A, A^\omega, A^{\omega^2} \right) &\neq V_n \left( a, b, c \right), \quad (3.24a) \\
U_n \left( A, A^\omega, A^{\omega^2} \right) &\neq U_n \left( a, b, c \right), \quad (3.24b) \\
W_n \left( A, A^\omega, A^{\omega^2} \right) &\neq W_n \left( a, b, c \right). \quad (3.24c)
\end{align}

Hence for \( m = 3 \) the “proper” generalization of Lucas formulae (3.13) and (3.14) or equivalently (3.15) is given by formulae (3.18).

Functions \( V_n \left( A, A^\omega, A^{\omega^2} \right) \) are of course symmetric in \( A, A^\omega, A^{\omega^2} \) arguments but are no more symmetric functions of roots \( a, b, c \); they are however cyclic symmetric functions of roots.

Functions \( U_n \left( A, A^\omega, A^{\omega^2} \right), W_n \left( A, A^\omega, A^{\omega^2} \right) \) are of course cyclic-symmetric in \( A, A^\omega, A^{\omega^2} \) arguments, but are no more cyclic-symmetric functions of roots \( a, b, c \).

### 4 Geometrical Representation of Tchebycheff \( m \)-Polynomials of \( k \)-th Kind.

Consider the case \( m = 2 \) i.e. the ordinary Tchebycheff polynomials in one variable \( x = \cosh \alpha \) or in two dependent variables: \( x = \cosh \alpha \) and \( y = \sinh \alpha \) then one has

\[
T_n \left( x, y \right) = \sum_{k=0}^{\left[ \frac{n}{2} \right]} \binom{n}{2k} x^{n-2k} y^{2k},
\]

where \( x, y \) are coordinates of a point from a hyperbola given by the group of hyperbolons of volume one [1b]

\[
\det \left( \begin{array}{ccc} x & y & z \\ y & x & z \\ z & y & x \end{array} \right) = 1.
\]

Consider now the case \( m = 3 \). Then one may show that

\[
h_0 (nx) = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{n-k} \binom{n-k}{i} \delta (i-k) h_0^{n-k-i} (x) h_1^i (x) h_2^k (x),
\]

i.e. for Tchebycheff 3-polynomials of zero-th kind we get

\[
T_n \left( x, y, z \right) = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{n-k} \binom{n-k}{i} \delta (i-k) x^{n-k-i} y^i z^k,
\]

where \( x, y, z \) are coordinates of a point from the surface given by the group of hyperbolons of volume one [1b]

\[
\det \left( \begin{array}{ccc} h_0(\alpha) & h_1(\alpha) & h_2(\alpha) \\ h_2(\alpha) & h_0(\alpha) & h_1(\alpha) \\ h_1(\alpha) & h_2(\alpha) & h_0(\alpha) \end{array} \right) \equiv \det \left( \begin{array}{ccc} x & y & z \\ z & x & y \\ y & z & x \end{array} \right) = 1.
\]

This surface defined by the equation \( x^3 + y^3 + z^3 - 3xyz = 1 \).
Tchebycheff 3-polynomials of the first and second kind are also functions on the one-parameter subgroup of hyperbolons of volume one [1] as seen from the formulae

\[ U_n(x, y, z) \equiv \sum_{k=0}^{n} \binom{n}{k} \delta \left( \frac{i}{i} - \frac{k}{k} \right) x^{n-k-i} y^i z^k, \]  

\[ (4.6) \]

\[ W_n(x, y, z) \equiv \sum_{k=0}^{n} \binom{n}{k} \delta \left( \frac{i}{i} - \frac{k}{k} + 1 \right) x^{n-k-i} y^i z^k. \]

\[ (4.7) \]

The formulae (4.4), (4.6) and (4.7) may be obtained with the help of the following identities:

\[ (a + b + c)^n \equiv \sum_{k=0}^{n} \binom{n}{k} a^{n-k} c^k \equiv \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{n-k} \binom{n-k}{i} a^{n-k-i} b^i c^k \]

\[ (4.8) \]

\[ \frac{1}{3} \left\{ \omega^0 + \omega^{k+2i} + \omega^{i+2k} \right\} = \delta \left( \frac{i}{i} - \frac{k}{k} \right) \quad \omega = \exp \left\{ \frac{2\pi i}{3} \right\} \]

\[ (4.9) \]

where \(-\equiv subtracting \) mod 3.

Tchebycheff \(m\)-polynomials of the \(k\)-th kind are of course also functions on the one-parameter subgroup of hyperbolons of volume one [1b] in the \(m\)-th dimensional space – the whole group being represented by points of the corresponding surface. For example in \(m = 4\) case the group of hyperbolons of volume one is represented by points of the surface defined by equation [6]

\[-x^4 + y^4 - z^4 + t^4 + 4x^2yt + 4xy^2z + 4z^2yt - 4t^2xz + 2x^2z^2 - 2y^2t^2 = 1.\]

\[ (4.10) \]

5 Cayley-Hamilton Theorem and Matrix Recurrences for Tchebycheff \(m\)-Polynomials

Let us consider the matrix

\[ A = \begin{pmatrix} P & -Q \\ 1 & 0 \end{pmatrix}. \]

\[ (5.1) \]

This matrix \(A\) is the representation of equivalence class of all matrices \((2 \times 2)\) with trace \(P\) and determinant \(Q\). According to Cayley-Hamilton Theorem it satisfies the equation

\[ A^2 = PA - QI, \]

\[ (5.2) \]

which is of the form \(z^2 = Pz - Q\) i.e. of the form of characteristic equation for the corresponding recurrences (compare with (3.12) and with (10) in [3]):

\[ V_{n+2} = PV_{n+1} - QV_n, \quad U_{n+2} = PU_{n+1} - QU_n. \]

\[ (5.3) \]

Matrix \(A\) generates solutions of these recurrences due to

\[ \begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} P & -Q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}. \]

\[ (5.4) \]
For example if we take \( F_0 = 0, \ F_1 = 1 \) as initial values then [7]

\[
A^n = \begin{pmatrix} F_{n+2} & -QF_{n+1} \\ QF_n & -F_n \end{pmatrix}; \quad n \in N. \tag{5.5}
\]

Let us consider now the matrix

\[
A = \begin{pmatrix} P & Q & R \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.6}
\]

This matrix \( A \) is the representation of equivalence class of all matrices \((3 \times 3)\) with trace \( P \), determinant \( R \) and sum of corresponding minors = \(-Q\). According to Cayley-Hamilton Theorem \( A \) satisfies

\[
A^3 = PA^2 + QA + RI, \tag{5.7}
\]

which is of the form \((x - a)(x - b)(x - c) = 0 \iff x^3 = Px^2 + Qx + R\), i.e. of the form of characteristic equation for the corresponding recurrences (compare with (3.8), (3.9) and (3.10)):

\[
\begin{cases}
F_{n+3} = PF_{n+2} + QF_{n+1} + RF_n \\
\text{initial...values}
\end{cases}
\]

Matrix \( A \) generates solutions of these recurrences due to

\[
\begin{pmatrix} F_{n+3} \\ F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} P & Q & R \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+2} \\ F_{n+1} \\ F_n \end{pmatrix}; \quad n = 0, 1, 2, \ldots. \tag{5.9}
\]

For example if we take \( F_0 = 0, \ F_1 = 1, \ F_2 = 1 \) as initial values then [7]

\[
A^n = \begin{pmatrix} F_{n+2} & QF_{n+1} + RF_n & RF_{n+1} \\ F_{n+1} & QF_n + RF_{n-1} & RF_n \\ F_n & RF_{n+1} + RF_{n-2} & RF_{n-1} \end{pmatrix}; \quad n = 2, 3, \ldots. \tag{5.10}
\]

In general case of arbitrary \( m > 1 \) one obtains the matrix \( A \) adjoint representation of an endomorphism \( A \) – given by its invariants from Cayley-Hamilton theorem – after choosing the basis

\[
\{e_k = A^{m-1-k}; \quad k \in Z_m\}. \tag{5.11}
\]

Of course

\[
A^m = \sum_{k \in Z_m} \alpha_k A^k; \tag{5.12}
\]

therefore in this general case

\[
A = \begin{pmatrix} \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \cdots & \alpha_1 & \alpha_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{5.13}
\]

Matrix \( A \) may be considered as a generator of a dynamical system \( \{A^n; \quad n \in Z\} \) in which dynamics is introduced via iteration [7] and thus is governed by recurrent sequences.
Naturally the appropriate choice of invariants \( \{\alpha_k; k \in \mathbb{Z}_m\} \) leads to Tchebycheff \( m \)-polynomials as exemplified by \( m = 2 \) and \( m = 3 \) cases.

This paper extends primary results from [8] – however it is self-contained. Further development of the investigation presented above is to be found in [9] soon.

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