Stability and error analysis of a splitting method using Robin–Robin coupling applied to a fluid–structure interaction problem

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Abstract
We analyze a splitting method for a canonical fluid structure interaction problem. The splitting method uses a Robin–Robin boundary condition to define an explicit coupling between the fluid and the structure. We prove the method is stable and, furthermore, we provide an error estimate that shows the error at the final time $T$ is $O(\sqrt{T\Delta t})$ where $\Delta t$ is the time step.

KEYWORDS
fluid–structure interaction, Robin–Robin coupling

1 | INTRODUCTION

In this work, we are interested in the stability analysis of a loosely coupled scheme for the approximation of the interaction of a viscous fluid and an elastic solid. In a loosely coupled (or explicit) scheme the two systems are solved separately in a staggered manner, passing interface data from one system to the other between the solves. It is well known that loosely coupled schemes for fluid structure interaction have severe stability problems in situations where the density ratio between the two phases is close to one. This is due to what is known as the added mass effect [1]. There has been intense research on approaches that allow for a partial or even complete decoupling of the two systems without loss of stability, however very few fully decoupled approaches have been developed with a satisfactory theoretical foundation.

A first step in the direction of decoupling the two systems is the semi-implicit coupling schemes [2–5], where the implicit part of the coupling, typically the elasticity system and the pressure velocity coupling in the fluid, guarantees stability, and the explicit step (transport in the fluid) reduces the computational cost. Such splitting methods nevertheless retain an implicit part of the same size as the original problem, although the equations are simplified. Fully explicit coupling was first achieved by Burman and Fernàndez [6] using a formulation based on Nitsche’s method, drawing on
an earlier, fully implicit formulation by Hansbo et al. [7]. Stability was achieved by the addition of a pressure stabilization that relaxed incompressibility in the vicinity of the interface. Although the proposed scheme was proved to be stable, it suffered from a strong consistency error of order $O(\tau/h)$ where $\tau$ and $h$ are the time and space discretization parameters, respectively. The source of this error was the penalty term of the Nitsche formulation. This led to the need for very small time steps combined with iterative corrections, for the method to yield sufficiently accurate approximations. In a further development, Burman and Fernández compared the Nitsche based method with a closely related method using a Robin type splitting procedure [8]. Robin type domain decomposition had already been applied for the preconditioning of monolithic fluid structure interaction problems by Badia et al. [9]. The loosely coupled scheme based on Robin type coupling of Reference [8] was proved to be stable, but only with the addition of the stabilization term on the pressure at the interface and using a weight in the Robin condition scaling similarly as the penalty term in the Nitsche method. It was however observed numerically that the Robin–Robin coupling method was stable also without such a pressure stabilizing term.

It is the objective of the present paper to revisit the analysis of the Robin–Robin method without any additional stabilization (what the authors of Reference [8] referred to as the genuine Robin–Robin method) and prove stability and error estimates for this method. To make the results cleaner and more transparent we do not discretize in space. Instead, our splitting scheme solves a fluid and a solid PDE on each time step. Assuming enough regularity of the local time PDEs, we give a rigorous error analysis that shows the error in a certain energy norm decreases as $O(\sqrt{T}\Delta t)$ for sufficiently smooth solutions, with a parameter $\lambda$ of the Robin condition chosen $O(1)$. This leads to convergence of the time discrete approximation independently of the space discretization. This was not the case in Reference [8], whereas mentioned above the convergence was hampered by the $h^{-1}$-scaling of the Robin parameter, imposing a very small time step and iterative correction steps to achieve sufficient accuracy. Observe that it is likely that the accuracy of the approach suggested here can be improved using correction steps for moderate values of the time step, thanks to the absence of the $h^{-1}$ scaling in the estimate. We would like to highlight that our estimates grow like $\sqrt{T}$ instead of an exponential growth. We accomplish this by using a technique used by Baker [10].

Finally, we should mention recent papers for the simpler case of interaction between a fluid and a thin structure that also have rigorous convergence analysis [11–13]. For the case of thick solids, the paper of Reference [14] seems to be the first paper with a rigorous error analysis of a thick wall structure. The method considered in Reference [14] is a Robin–Neumann coupling that first appeared in Reference [15]. There stability is achieved by handling the inertial effects of the solid in an implicit coupling with the fluid. This is then combined with extrapolation to reduce the splitting error. The leading error in that method for this approach is $O(\Delta t/\sqrt{h})$ which scales like our error estimates if $\Delta t = O(h)$. It should be mentioned that the constant of their estimates grows exponentially with $T$.

The outline of the present paper is as follows. In Section 2, we introduce the linear model problem. The proposed Robin–Robin loosely coupled scheme is introduced in Section 3 and the stability is analyzed in Section 4. Finally, in Section 5, we derive the truncation error of the splitting and use this result together with the stability estimate to prove the error estimate.

2 | THE MODEL PROBLEM

Let $\Omega_s$ and $\Omega_f$ be two domains with a matching interface $\Sigma = \partial \Omega_s \cap \Omega_f$. We also set $\Sigma_i = \partial \Omega_i \setminus \Sigma$ for $i = s, f$ (see Figure 1 for example). Our fluid is modeled by the Stokes equation on the
Our structure is modeled by the classical linear elasticity equations on the structure domain $\Omega_s$. 

\[
\begin{cases}
\rho_s \partial_t \mathbf{e} - \text{div} \sigma_s = 0, & \text{in } \Omega_s \times (0, T), \\
\text{div} \mathbf{e} = 0, & \text{in } \Omega_s \times (0, T), \\
\mathbf{e} = 0, & \text{on } \Sigma_s \times (0, T).
\end{cases}
\] (2)

Here, $\mathbf{u}$ is the velocity of the fluid, $\mathbf{p}$ is the pressure of the fluid, and $\mathbf{e}$ is the displacement of the structure. The constants $\rho_f, \rho_s$ are the fluid and solid densities, and $\mathbf{n}$ and $\mathbf{n}_s$ represent the outward-facing normal of the fluid and solid domains, respectively. Furthermore, $\sigma_F, \sigma_S$ denote the fluid and solid stress tensors, respectively, and are given by

\[
\sigma_F = 2\mu \varepsilon(\mathbf{u}) - \mathbf{pI},
\]

\[
\sigma_S = 2L_1 \varepsilon(\mathbf{e}) + L_2 (\text{div } \mathbf{e}) \mathbf{I},
\]

where $\varepsilon$ denotes the symmetric gradient, $\mu$ the viscosity coefficient and $L_1 > 0, L_2 \geq 0$ the Lamé constants. Then the two problems are coupled via the following interface conditions:

\[
\mathbf{u} = \partial_t \mathbf{e} \quad \text{on } \Sigma, \tag{3.1}
\]

\[
\sigma_S \mathbf{n}_s + \sigma_F \mathbf{n} = 0 \quad \text{on } \Sigma. \tag{3.2}
\]

### 3 SPLITTING METHOD

In Reference [8], several splitting methods were given for the following FSI problem. We will consider one such method. In order to describe the splitting method, we consider a uniform grid for the interval $[0, T]$ with step size $\Delta t$. We assume that there is an integer $N$ so that $N\Delta t = T$ and we let $t_n = \Delta tn$.

For what follows, we may consider the variables $\mathbf{e}, \mathbf{u}, \mathbf{p}$ to be the exact solutions to our FSI system defined in (1)–(3.2). Consequently, the variables $\mathbf{n}^n, \mathbf{i}^n, \mathbf{u}^n, \mathbf{p}^n$ defined below may be considered approximate solutions to (1)–(3.2), as they are solutions to our splitting method that will be shown to converge to $\mathbf{e}, \mathbf{u}, \mathbf{p}$ in an appropriate norm as $\Delta t \to 0$ (see Section 5).
The splitting method sequentially solves the following two sub-problems. The first is the solid problem:

Find $\eta^{n+1}$ and $\dot{\eta}^{n+1}$ such that

$$\rho_s \partial_t \eta^{n+1} - \text{div} \sigma_s^{n+1} = 0 \quad \text{in} \quad \Omega_s \times [t_n, t_{n+1}],$$

$$\dot{\eta}^{n+1} = \partial_t \eta^{n+1} \quad \text{in} \quad \Omega_s \times [t_n, t_{n+1}]$$

$$\sigma_s^{n+1} = 2L_1 \varepsilon(\eta^{n+1}) + L_2 (\text{div} \eta^{n+1}) I \quad \text{in} \quad \Omega_s \times [t_n, t_{n+1}]$$

We require $\lambda > 0$. This strict positivity determines the balancing of the two interface coupling conditions (3.1) and (3.2). We may think of these coupling conditions as two distinct linear combinations of (3.1) and (3.2), where $\lambda$ allows us to control the contribution of each condition. For example, a large value of $\lambda$ will emphasize the continuity of velocities and a small value that of stresses.

Before proceeding, we define the space–time norm on $X$, a Hilbert space

$$\|v\|_{L^2(t_1, t_2; X)}^2 := \int_{t_1}^{t_2} \|v(s)\|_X^2 ds.$$
Finally, the following quantities will allow us to state our stability estimates.

\[
\text{It induces the norm:} \\
\|v - w\|_{L^2(\Sigma)}^2 = \frac{1}{2} \left( \|v\|_{L^2(\Sigma)}^2 - \|w\|_{L^2(\Sigma)}^2 + \|v - w\|_{L^2(\Sigma)}^2 - \|v - w\|_{L^2(\Sigma)}^2 \right). 
\]  

(6)

Additionally, if we set \( \tilde{w}(x) = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} w(x, s) ds \) then we see that

\[
\int_{t_n}^{t_{n+1}} \|\tilde{w}\|_{L^2(\Sigma)}^2 = \Delta t \int_{\Sigma} (\tilde{w}(x))^2 = \frac{1}{\Delta t} \int_{\Sigma} \left( \int_{t_n}^{t_{n+1}} w(x, s) ds \right)^2 \leq \int_{t_n}^{t_{n+1}} \int_{\Sigma} (w(x, s))^2 ds = \int_{t_n}^{t_{n+1}} \|w(s)\|_{L^2(\Sigma)}^2 ds. 
\]  

(7)

For the analysis that follows, we define the bilinear form \( a_s(w, v) \) to be

\[
a_s(w, v) := 2L_1(\epsilon(w), \epsilon(v))_S + L_2(\div w, \div v)_S. 
\]

It induces the norm:

\[
\|w\|_{S}^2 := 2L_1|\epsilon(w)|_{L^2(\Omega)}^2 + L_2|\div w|_{L^2(\Omega)}^2. 
\]

Finally, the following quantities will allow us to state our stability estimates

\[
E^n := \frac{\rho_f}{2} \|u^n(t_n)\|_{L^2(\Omega)}^2 + \frac{\rho_s}{2} \|\tilde{u}^n(t_n)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\eta^n(t_n)\|_{S}^2, 
\]

\[
T^n := 2\mu \int_{t_{n-1}}^{t_n} \|\sigma(u^n)(s)\|_{L^2(\Omega)}^2 ds + \lambda \int_{t_{n-1}}^{t_n} \|\tilde{u} - \tilde{u}^{n-1}(s)\|_{L^2(\Sigma)}^2 ds, 
\]

\[
S^n := \frac{1}{2\lambda} \int_{t_{n-1}}^{t_n} \|\sigma_f^n(s)u^n\|_{L^2(\Sigma)}^2 ds + \frac{\lambda}{2} \int_{t_{n-1}}^{t_n} \|u^n(s)\|_{L^2(\Sigma)}^2 ds, 
\]

for \( n \geq 1 \),

\[
S^0 := \frac{\Delta t}{2\lambda} \|\sigma_f(t_0)u_0\|_{L^2(\Sigma)}^2 + \frac{\lambda\Delta t}{2} \|U(t_0)\|_{L^2(\Sigma)}^2. 
\]

The stability result is given in the following theorem.

**Theorem 1** Let \( \lambda > 0 \) and suppose that \( \eta^{n+1} \) solves (4.1)–(4.6) and \( u^{n+1}, p^{n+1} \) solve (5.1)–(5.6) for \( 0 \leq n \leq N - 1 \). Then we have

\[
E^N + \sum_{n=1}^{N} T^n + S^N \leq E^0 + S^0. 
\]
Proof. We multiply (4.1) and (5.1) by $\dot{\eta}^{n+1}$ and $u^{n+1}$, respectively, integrate, and add the results to get
\[
\frac{\rho_f}{2} \partial_t \|u^{n+1}\|_{L^2(\Omega)}^2 + 2\mu \|\varepsilon(u^{n+1})\|_{L^2(\Omega)}^2 + \frac{\rho_e}{2} \partial_t \|\eta^{n+1}\|_{L^2(\Omega)}^2 + \frac{1}{2} \partial_t \|\dot{\eta}^{n+1}\|_2^2 = J,
\] (8)
where
\[
J := \int_\Sigma \sigma_f^{n+1} n \cdot \dot{u}^{n+1} + \int_\Sigma \sigma_s^{n+1} n_s \cdot \dot{\eta}^{n+1}.
\]
We can then write
\[
J = \int_\Sigma \sigma_f^{n+1} n \cdot (u^{n+1} - \dot{\eta}^{n+1}) + \int_\Sigma (\sigma_s^{n+1} n_s + \sigma_f^{n+1} n) \cdot \dot{\eta}^{n+1}.
\]
From (4.5) and (5.5) we get
\[
1 \sigma_s^{n+1} n_s + \sigma_f^{n+1} n = \lambda (\tilde{u}^n - u^{n+1}),
\]
\[
u^{n+1} - \dot{\eta}^{n+1} = \frac{1}{\lambda} (\tilde{\sigma}_f n - \sigma_f^{n+1} n).
\]
Thus,
\[
J = \frac{1}{\lambda} \int_\Sigma \sigma_f^{n+1} n \cdot (\tilde{\sigma}_f n - \sigma_f^{n+1} n) + \lambda \int_\Sigma (\tilde{u}^n - u^{n+1}) \cdot \dot{\eta}^{n+1}.
\] (9)
By the relation (6) and the fact that $\frac{1}{\lambda} \|\tilde{\sigma}_f n - \sigma_f^{n+1} n\|_{L^2(\Sigma)}^2 = \lambda \|\dot{\eta}^{n+1} - u^{n+1}\|_{L^2(\Sigma)}^2$, we obtain
\[
J = \frac{\lambda}{2} (||\tilde{u}^n||_{L^2(\Sigma)}^2 - ||u^{n+1}||_{L^2(\Sigma)}^2) + \frac{1}{2\lambda} (||\tilde{\sigma}_f n||_{L^2(\Sigma)}^2 - ||\sigma_f^{n+1} n||_{L^2(\Sigma)}^2) - \frac{\lambda}{2} ||\dot{\eta}^{n+1} - u^n||_{L^2(\Sigma)}^2.
\]
If we plug this into (8) we arrive at
\[
\frac{\rho_f}{2} \partial_t \|u^{n+1}\|_{L^2(\Omega)}^2 + 2\mu \|\varepsilon(u^{n+1})\|_{L^2(\Omega)}^2 + \frac{\rho_e}{2} \partial_t \|\eta^{n+1}\|_{L^2(\Omega)}^2 + \frac{1}{2} \partial_t \|\dot{\eta}^{n+1}\|_2^2
\]
\[
+ \frac{1}{2\lambda} \|\sigma_f^{n+1} n\|_{L^2(\Sigma)}^2 + \frac{\lambda}{2} \|u^{n+1}\|_{L^2(\Sigma)}^2 = \frac{1}{2\lambda} \|\tilde{\sigma}_f n\|_{L^2(\Sigma)}^2 + \frac{\lambda}{2} ||\tilde{u}^n||_{L^2(\Sigma)}^2 - \frac{\lambda}{2} ||\dot{\eta}^{n+1} - u^n||_{L^2(\Sigma)}^2.
\]
After integrating on $[t_n, t_{n+1}]$ and using (7), we obtain
\[
E^{n+1} + T^{n+1} + S^{n+1} \leq E^n + S^n.
\]
The result now follows after summing the above inequalities over all $n$ from 0 to $N - 1$. 

5 | ERROR ESTIMATES

We now show that the splitting method with Robin–Robin type boundary conditions described above is, in fact, weakly consistent. In fact, we will prove that the error is $\sqrt{T\Delta t}$. Consider the solutions $U, \mathcal{V}, \sigma_T, E, \sigma_S$ of (1)–(3.2). We use the notation $U^{n+1}(t, x) = U(t, x)$ for $t_n \leq t \leq t_{n+1}$ and $x \in \Omega$; this similarly holds for the other variables. We then set the errors:
\[
\epsilon_u^n = U^n - u^n, \quad \epsilon_f^n = \sigma_T^n - \sigma_f^n
\]
\[
\epsilon_s^n = \sigma_S^n - \sigma_s^n, \quad \epsilon_H^n = E^n - \eta^n, \quad \epsilon_\eta^n = \dot{\mathcal{E}}^n - \dot{\eta}^n
\]
We also define the following quantities which will be useful to describe our error estimates:

\[ E^n := \frac{\rho_1}{2} \| e^n_u(t_n) \|^2_{L^2(\Omega)} + \frac{\rho_2}{2} \| \dot{e}^n_u(t_n) \|^2_{L^2(\Omega)} + \frac{1}{2} \| e^n_i(t_n) \|^2_S, \]

\[ T^n := 2\mu \int_{t_{n-1}}^{t_n} \| \varepsilon(e^n_u(s)) \|^2_{L^2(\Omega)} ds + \frac{\lambda}{4} \int_{t_{n-1}}^{t_n} \| \ddot{e}^n_u - \ddot{e}^{n-1}_u \|^2_{L^2(\Sigma)} ds, \]

\[ S^n := \frac{1}{2} \int_{t_{n-1}}^{t_n} \| \dot{e}^n_j(s) n \|^2_{L^2(\Sigma)} ds + \frac{\lambda}{2} \int_{t_{n-1}}^{t_n} \| e^n_u(s) \|^2_{L^2(\Sigma)} ds, \quad \text{for } n \geq 1, \]

\[ S^0 := \frac{\Delta t}{2\alpha} \| e^0_j(t_0) n \|^2_{L^2(\Sigma)} + \frac{\lambda \Delta t}{2} \| e^0_u(t_0) \|^2_{L^2(\Sigma)}. \]

We note that \( E^0 = 0 \) and \( S^0 = 0 \).

For the proof of the error estimates, we will make use of the following lemma. We define

\[ g^{n+1}_3 := \lambda (\mathcal{U}^{n+1} - \mathcal{U}^n), \]

\[ g^{n+1}_2 := (\sigma^{n+1}_r n - \widetilde{\sigma}^n_r n). \]

**Lemma 1** For \( \mathcal{U}^n \) and \( \sigma^n_r \) defined above, we have for \( n \geq 1 \)

\[ \int_{t_n}^{t_{n+1}} \| g^{n+1}_3(s) \|^2_{L^2(\Sigma)} ds \leq C \lambda^2 (\Delta t)^2 \int_{t_{n-1}}^{t_{n+1}} \| \partial_t \mathcal{U}(s) \|^2_{L^2(\Sigma)} ds, \tag{12} \]

\[ \int_{t_n}^{t_{n+1}} \| g^{n+1}_2(s) \|^2_{L^2(\Sigma)} ds \leq C (\Delta t)^2 \int_{t_{n-1}}^{t_{n+1}} \| \partial_t \sigma_r(s) n \|^2_{L^2(\Sigma)} ds. \tag{13} \]

For \( n = 0 \) we have

\[ \int_{t_0}^{t_1} \| g^1_3(s) \|^2_{L^2(\Sigma)} ds \leq C \lambda^2 (\Delta t)^2 \int_{t_0}^{t_1} \| \partial_t \mathcal{U}(s) \|^2_{L^2(\Sigma)} ds, \tag{14} \]

\[ \int_{t_0}^{t_1} \| g^1_2(s) \|^2_{L^2(\Sigma)} ds \leq C (\Delta t)^2 \int_{t_0}^{t_1} \| \partial_t \sigma_r(s) n \|^2_{L^2(\Sigma)} ds. \tag{15} \]

**Proof.** We only prove (12) as the proof of the other estimates are similar. We have, for \( s \in [t_n, t_{n+1}] \),

\[ \mathcal{U}^{n+1}(s) - \mathcal{U}^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (\mathcal{U}(s) - \mathcal{U}(r)) dr = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_r^s \partial_t \mathcal{U}(\theta) d\theta dr. \]

Hence,

\[ \int_{t_n}^{t_{n+1}} \| \lambda (\mathcal{U}^{n+1}(s) - \mathcal{U}^n) \|^2_{L^2(\Sigma)} ds = \lambda^2 \int_{t_n}^{t_{n+1}} \int_\Sigma \left( \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_r^s \partial_t \mathcal{U}(\theta) d\theta dr \right)^2 ds \]

\[ \leq 2\lambda^2 \int_{t_n}^{t_{n+1}} \int_\Sigma \int_{t_n}^{t_{n+1}} \int_r^s (\partial_t \mathcal{U}(\theta))^2 d\theta dr ds \]

\[ \leq C \lambda^2 (\Delta t)^2 \int_{t_{n-1}}^{t_{n+1}} \| \partial_t \mathcal{U}(\theta) \|^2_{L^2(\Sigma)}. \]
The error estimates are given in the following theorem. Note that we will implicitly assume the regularity mentioned in Remark (1). In addition, we assume that $\partial_t u \in L^2(0,T;L^2(\Sigma))$ and $\partial_t \sigma_F n \in L^2(0,T;L^2(\Sigma))$.

**Theorem 2** Let $\mathcal{U}, \mathcal{P}, \sigma_F, \mathcal{E}, \sigma_\Theta$ solve (1)–(3.2). Furthermore, let $u^{n+1}, \sigma_F^{n+1}, p^{n+1}$ solve (5.1)–(5.6) and $\eta^{n+1}, \Phi^{n+1}$ solve (4.1)–(4.6). If $T = N\Delta t$ with $N \geq 1$, the following estimate holds:

$$
\mathbb{E}^N + \sum_{n=1}^N \mathbb{T}^n + \mathbb{S}^N \leq C\Delta t T \left( \lambda \|\partial_t \mathcal{U}\|_{L^2(0,T;L^2(\Sigma))}^2 + \frac{1}{\lambda} \|\partial_t \sigma_F n\|_{L^2(0,T;L^2(\Sigma))}^2 \right).
$$

**Proof.** Using (4.5), (5.5) and (3.1)–(3.2) we see that

$$
e^{n+1}_s n_s + \lambda \epsilon^{n+1}_\eta = \lambda \epsilon^{n}_u - \epsilon^{n}_f n + g_1^{n+1},
$$

$$e^{n+1}_f n + \lambda \epsilon^{n+1}_u = \epsilon^{n+1}_\eta + \epsilon^{n}_f n + g_2^{n+1},
$$

where $g_2^{n+1}$ is given in (11) and

$$g_1^{n+1} := \lambda (\mathcal{U}^{n+1} - \mathcal{U}^n) + (\sigma_F^n - \sigma_F^{n+1} n).
$$

By adding the two equations we get

$$e^{n+1}_s n_s + e^{n+1}_f n = \lambda (\epsilon^{n}_u - \epsilon^{n+1}_u) + g_3^{n+1},
$$

where $g_3^{n+1}$ is given in (10). Also, we re-arrange the second equation and write

$$e^{n+1}_u - \epsilon^{n}_\eta = \frac{1}{\lambda} (\epsilon^{n}_f n - \epsilon^{n+1}_f n) + \frac{1}{\lambda} g_2^{n+1}.
$$

We may therefore proceed with the same initial steps from the stability analysis. This yields

$$\frac{\rho_f}{2} \partial_t \|e^{n+1}_u\|_{L^2(\Omega)}^2 + 2\mu \|\epsilon(e^{n+1}_u)\|_{L^2(\Omega)}^2 - \int_\Sigma e^{n+1}_f n \cdot e^{n+1}_u = 0,$$

$$\frac{\rho_s}{2} \partial_t \|\epsilon^{n+1}_\eta\|_{L^2(\Omega)}^2 + \frac{1}{\lambda} \partial_t \|\epsilon^{n+1}_\eta\|_{L^2(\Omega)}^2 - \int_\Sigma e^{n+1}_s n_s \cdot e^{n+1}_\eta = 0.$$

If we set

$$p^{n+1} := \frac{\rho_f}{2} \partial_t \|e^{n+1}_u\|_{L^2(\Omega)}^2 + 2\mu \|\epsilon(e^{n+1}_u)\|_{L^2(\Omega)}^2 + \frac{\rho_s}{2} \partial_t \|\epsilon^{n+1}_\eta\|_{L^2(\Omega)}^2 + \frac{1}{\lambda} \partial_t \|\epsilon^{n+1}_\eta\|_{L^2(\Omega)}^2,$$

we have that

$$p^{n+1} = \int_\Sigma e^{n+1}_f n \cdot e^{n+1}_u + \int_\Sigma e^{n+1}_s n_s \cdot e^{n+1}_\eta$$

$$= \int_\Sigma e^{n+1}_f n \cdot (e^{n+1}_u - e^{n+1}_\eta) + \int_\Sigma (e^{n+1}_s n_s + e^{n+1}_f n) \cdot e^{n+1}_\eta$$

$$= \frac{1}{\lambda} \int_\Sigma e^{n+1}_f n \cdot (e^{n+1}_\eta - e^{n+1}_u) + \lambda \int_\Sigma e^{n+1}_\eta - e^{n+1}_u \cdot e^{n+1}_\eta$$

$$+ \frac{1}{\lambda} \int_\Sigma e^{n+1}_f n \cdot g_2^{n+1} + \int_\Sigma g_3^{n+1} \cdot e^{n+1}_\eta.$$
In the last equality we used (16.1) and (16.2). Also, the following holds after using (16.2)
\[
\|\tilde{e}_j^m - e_j^{m+1}\|_{L^2(\Sigma)}^2 = \|\lambda (\tilde{e}_u^{m+1} - e_u^{m+1}) - g_2^{m+1}\|_{L^2(\Sigma)}^2, \\
\]
\[
= \lambda^2 \|\tilde{e}_u^{m+1} - e_u^{m+1}\|_{L^2(\Sigma)}^2 + \|g_2^{m+1}\|_{L^2(\Sigma)}^2 - 2\lambda \int_{\Sigma} g_2^{m+1} \cdot (\tilde{e}_u^{m+1} - e_u^{m+1}).
\]

If we use the above equations and (6) we obtain
\[
l^{m+1} = A + \frac{1}{\lambda} \int_{\Sigma} e_j^{m+1} \cdot g_2^{m+1} + \int_{\Sigma} g_3^{m+1} \cdot e_u^{m+1} - \frac{1}{2\lambda} \|g_2^{m+1}\|_{L^2(\Sigma)}^2 + \int_{\Sigma} g_2^{m+1} \cdot (\tilde{e}_u^{m+1} - e_u^{m+1}),
\]
where
\[
A := \frac{1}{2\lambda} (\|\tilde{e}_j^m\|_{L^2(\Sigma)}^2 - \|e_j^{m+1}\|_{L^2(\Sigma)}^2) + \frac{\lambda}{2} (\|\tilde{e}_u^m\|_{L^2(\Sigma)}^2 - \|e_u^{m+1}\|_{L^2(\Sigma)}^2) - \|\tilde{e}_u^m\|_{L^2(\Sigma)}^2.
\]

Again using (16.2) and applying Cauchy-Schwarz and Young's inequalities, we have
\[
l^{m+1} = A + \frac{\lambda}{4} \|\tilde{e}_u^{m+1} - e_u^{m+1}\|_{L^2(\Sigma)}^2 + \frac{1}{\lambda} \left(1 + \frac{1}{2\delta}\right) \|g_3^{m+1}\|_{L^2(\Sigma)}^2 + \frac{1}{\lambda} \left(1 + \frac{1}{2\delta}\right) \|g_2^{m+1}\|_{L^2(\Sigma)}^2 \\
+ \frac{\delta}{2\lambda} \|\tilde{e}_j^m\|_{L^2(\Sigma)}^2 + \frac{\lambda\delta}{2} \|\tilde{e}_u^m\|_{L^2(\Sigma)}^2,
\]
where \(\delta > 0\).

Taking the integral on \([t_n, t_{n+1}]\) and applying (7) we have:
\[
1 \|\tilde{E}^{n+1} + T^{n+1} + S^{n+1} \leq E^n + (1 + \delta) S^n + G^{n+1},
\]
where
\[
G^{n+1} := \frac{1}{\lambda} \left(1 + \frac{1}{2\delta}\right) \int_{t_n}^{t_{n+1}} \|g_3^{m+1}(s)\|_{L^2(\Sigma)}^2 ds + \frac{1}{\lambda} \left(1 + \frac{1}{2\delta}\right) \int_{t_n}^{t_{n+1}} \|g_2^{m+1}(s)\|_{L^2(\Sigma)}^2 ds.
\]

We then clearly have
\[
E^{n+1} + T^{n+1} + S^{n+1} \leq E^n + S^n + \delta \max_{1 \leq m \leq N} S^m + G^{n+1}.
\]

If we sum from 0 to \(M - 1\) with \(1 \leq M \leq N\) and set \(\delta = \frac{M}{2T}\) we obtain
\[
E^M + \sum_{n=0}^{M-1} T^{n+1} + S^M \leq \frac{1}{2} \max_{1 \leq m \leq N} S^m + \sum_{n=0}^{M-1} G^{n+1}.
\]

Here we used that \(E^0 = 0\) and \(S^0 = 0\). Since this holds for any \(1 \leq M \leq N\) we have
\[
\frac{1}{2} \max_{1 \leq m \leq N} S^m \leq \sum_{n=0}^{N-1} G^{n+1}.
\]

Thus, we have
\[
E^N + \sum_{n=1}^{N} T^{n} + S^N \leq 2 \sum_{n=0}^{N-1} G^{n+1}.
\]

(17)
Using Lemma 1 and that $\delta = \frac{\Delta t}{2T}$ we immediately have

$$\sum_{n=0}^{N-1} G^{n+1} \leq C \Delta t T \left( \lambda \| \partial_t \mathbf{u} \|_{L^2(0,T;L^2(\Sigma))}^2 + \frac{1}{\lambda} \| \partial_t \sigma F n \|_{L^2(0,T;L^2(\Sigma))}^2 \right).$$

Here we used that $T \geq \Delta t$. Combining this with (17) completes the proof. □

6 | CONCLUSION

In this paper we analyzed a Robin–Robin splitting scheme for an FSI problem. We showed that the error is bounded by $\sqrt{T \Delta t}$. Since the splitting does not discretize in space this gives several possibilities for spatial discretization. In a forthcoming paper, we will analyze a fully discrete scheme and present numerical experiments. Of particular interest to us in the fully discrete case is how to deal with the fluid stresses on the interface. More specifically, one particular challenge that we address in our follow-up paper is how to express the fluid stress at the interface in a variationally consistent manner.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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