Detecting Incapacity

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(Dated: December 9, 2012)

Using unreliable or noisy components for reliable communication requires error correction. But which noise processes can support information transmission, and which are too destructive? For classical systems any channel whose output depends on its input has the capacity for communication, but the situation is substantially more complicated in the quantum setting. We find a generic test for incapacity based on any suitable forbidden transformation—a protocol for communication with a channel passing our test would also allow us to implement the associated forbidden transformation.

Our approach includes both known quantum incapacity tests—positive partial transposition (PPT) and antidegradability (no cloning)—as special cases, putting them both on the same footing. We also find a physical principle explaining the nondistillability of PPT states: Any protocol for distilling entanglement from such a state would also give a protocol for implementing the forbidden time-reversal operation.
istence of a suitable forbidden (or unphysical) map on the states. One motivation of this work is to understand incapacity of quantum channels, but our approach is sufficiently general to include such things as generalized probabilistic theories [14] as well as the discrete quantum mechanics of [15]. Our findings will also apply to classical systems with prescribed operations.

Preliminaries—The theories we will consider have minimal structure. We assume a physical state space $B$ and a set of allowed physical operations $\mathbf{P}$ from $B \to B$ that is closed under composition. For quantum mechanics, $B$ will be the set of density matrices and $\mathbf{P}$ will be the set of trace-preserving completely positive maps. We will also require a nonphysical operation $R : B \to B$ with $R \notin \mathbf{P}$. This $R$ will need the following crucial property:

**Definition 1:** ($\mathbf{P}$-commutation) An unphysical map $R$ is $\mathbf{P}$-commutative if for every $D \in \mathbf{P}$ there is a $D^* \in \mathbf{P}$ such that $R \circ D = D^* \circ R$. See Fig. 2.

Note that any nonphysical $R$ acting on $N$ of a channel, $\mathcal{N}$ indicating this dimension whenever possible. To streamline notation we will suppress labels in nonphysical operations. To every dimension $d$ nonphysical maps acting on $N$ is dimension $d$ implicitly acts only on some particular part of $N$. To completely characterize the zero-capacity channels detected by our method. This is achieved through the following theorem originally communicated to us by Choi [16]. We present here an alternative proof that makes an explicit connection to the theory of group representations.

**Theorem 1:** Take states and channels to be the usual quantum mechanics with $d$-level systems. If $R$ is linear, invertible, preserves system dimension and trace, and is $\mathbf{P}$-commutative it is either of the form $R(\rho) = (1 - p)\rho^T + pI/d$ or $R(\rho) = (1 - p)\rho + pI/d$.

**Proof:** Let $R$ be such a map, and consider the requirements of $\mathbf{P}$-commutation. Defining $N_U(\rho) = U\rho U^\dagger$, we have $N_U^* = R \circ N_U \circ R^{-1}$ and see immediately that since $N_U$ is invertible, so is $N_U^*$. Since it preserves dimension and is a physical channel, $N_U^*$ must be simply conjugation by a unitary $V_U$. Furthermore, $N_U^* V_U = R \circ N_U^* V_U \circ R^{-1} = R \circ N_U^* \circ R^{-1} \circ R \circ N_U^* \circ R^{-1} = N_U^* \circ N_U^*$, so that $V_U$ must be a $d$-dimensional represen-
of U(d). Since \( N_U = R^{-1} \circ N_U^* \circ R \), this representation must be faithful (i.e., invertible) up to an overall phase which will cancel under conjugation. There are only two such representations: the fundamental and complex conjugate representations.

Suppose we have \( N_U^* = N_V \). Then, for all \( \rho \), \( U^\dagger R(U \rho U^\dagger)U = R(\rho) \), so that letting \( |\phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle |i\rangle \) and \( |\phi_d^\dagger\phi_d^\Gamma\rangle \) denote its partial transpose, we have

\[
U^\dagger \otimes U^\dagger (I \otimes R)(|\phi_d\rangle|\phi_d\rangle^\Gamma) U \otimes U = (I \otimes R)(|\phi_d\rangle|\phi_d\rangle^\Gamma).
\]

As a result, we have

\[
(I \otimes R)(|\phi_d\rangle|\phi_d\rangle^\Gamma) = aI + bF \quad \text{where} \quad F = \sum_{i,j} |i\rangle |j\rangle |j\rangle |i\rangle
\]

so that \( (I \otimes R)(|\phi_d\rangle|\phi_d\rangle) = aI + dB|\phi_d\rangle|\phi_d\rangle \). Requiring \( R \) to preserve trace, we thus find \( R(\rho) = pI/d + (1 - p)\rho \).

We now consider \( N_U^* = N_U^* \), which works similarly. Specifically, we now have \( (I \otimes R)(|\phi_d\rangle|\phi_d\rangle) = aI + bF \) so that, requiring \( R \) to preserve trace, we find \( R(\rho) = pI/d + (1 - p)\rho^T \). □

**Decoder-dependent R and no cloning** — The above theorem shows that the only nonphysical \( \mathcal{P} \)-commutative linear maps are of the form \( R(\rho) = (1 - p)\rho^T + pI/d \). In order to make statements about capacities, \( R^{\otimes n} \) must also be \( \mathcal{P} \)-commutative. The only such nonphysical map is \( R(\rho) = \rho^T \). As a result detecting incapacity in non-PPT channels requires \( R \) to be nonlinear.

In order to accommodate nonlinear maps, we must generalize our ideas about \( \mathcal{P} \)-commutation. Specifically, we introduce the following definition of \( \mathcal{P} \)-commutation for a class of unphysical operations.

**Definition 2** (\( \mathcal{P} \)-commutation) A set of unphysical maps \( \{ R_D \}_{D \in \mathcal{D}} \) with \( R_D \notin \mathcal{P} \) is \( \mathcal{P} \)-commutative if for all maps \( D \in \mathcal{P} \) there is a \( D^* \in \mathcal{P} \) and an unphysical \( R \) with \( R_D \circ D = D^* \circ R \).

In Eq. (2), we were able to commute \( R \) and \( D \) by replacing \( D \) with \( D^* \). The motivation behind our new definition is to allow more freedom in finding decoders that remain physical after commutation with \( R \), at the cost of having channel-dependent nonphysical maps \( R_D \). Note that if \( R \) is invertible, \( D^* = R_D \circ D \circ R^{-1} \) will define \( * \).

Now, given a \( \mathcal{P} \)-commuting family of maps \( \{ R_D \} \) that are unphysical on a qubit with associated \( * \) and \( R \), any channel \( \mathcal{N} \) for which \( R \circ \mathcal{N} \) is physical cannot transmit all states in some set \( S \) reliably. Within quantum mechanics, given \( R_D^{(n)} \), \( R^{(n)} \), and \( * \) then if \( R^{(n)} \circ \mathcal{N}^{\otimes n} \) is physical, \( \mathcal{N} \) has no quantum capacity.

In quantum mechanics, any decoder can be written in terms of a unitary \( U \) as \( D(\psi) = \text{Tr}_E U(\psi \otimes |0\rangle\langle 0|) U^\dagger \approx \psi \) which implies \( U(\psi \otimes |0\rangle\langle 0|) U^\dagger \approx \psi \otimes \sigma \) for some \( \sigma \) independent of \( \psi \). We therefore only need consider the simpler set of nonphysical maps \( R_U \) with \( U^\dagger (U \psi \otimes |0\rangle\langle 0|) U^\dagger \) defining \( \* \) to detect the incapacity of any \( \mathcal{N} \) with \( R \circ (\mathcal{N} \otimes |0\rangle\langle 0|) \) physical.

Every quantum channel has an isometric extension to an environment. An antidegradable channel is a channel for which the environment, by further processing, can simulate the original channel [17, 18]. As a result of the no cloning theorem, such channels can be shown have zero quantum capacity [10, 11]. We now use the nonphysical cloning operation to give a simple proof that antidegradable channels have zero quantum capacity.

If \( \mathcal{M} \) is antidegradable with input \( A \) and output \( B \), then there is an extension of \( \mathcal{M} \), \( \mathcal{M}_{12} \), from \( A \) to \( B_1B_2 \), such that for all \( \rho \), \( \text{Tr}_{B_2} \mathcal{M}_{12}(\rho) = \text{Tr}_{B_1} \mathcal{M}_{12}(\rho) = \mathcal{M}(\rho) \). Any \( \psi \) on \( B \) has a unique decomposition \( \psi = \psi + \sigma \), with \( \psi \) in the range of \( \mathcal{M} \) and \( \sigma \) in its orthogonal complement.

Now define \( \check{R}(\psi) = \mathcal{M}_{12} \circ \mathcal{M}^{-1}(\psi) + \sigma \otimes \sigma \) where the pseudo-inverse \( M^{-1} \) maps \( \psi \) to its unique preimage in the orthogonal complement of the kernel of \( \mathcal{M} \). This \( \check{R} \) is continuous and \( \text{Tr}_1 \check{R}(\psi) = \text{Tr}_2 \check{R}(\psi) = \psi \). Furthermore, \( \check{R} \circ \mathcal{M} = \mathcal{M}_{12} \) is physical. We can similarly extend \( \check{R} \) to a continuous \( \check{R} \) from \( BE \) to \( B_1E_1E_2 \) with \( R \circ (\mathcal{M} \otimes |0\rangle\langle 0|) = \mathcal{M}_{12} \otimes |0\rangle\langle 0| \), \( \mathcal{M}_{12} \otimes |0\rangle\langle 0| \), and \( \text{Tr}_1 \check{R}(\psi) = \text{Tr}_2 \check{R}(\psi) = \psi \).

Now, let \( U^* = U \otimes U \). Choosing \( R_U(\rho_{BE}) = U \otimes U R(U^\dagger \rho_{BE} U)(U^\dagger \otimes U^\dagger) \) and \( N_U(\rho) = U \rho U^\dagger \), we have \( R_U \circ N_U = N_U \circ R \) and \( R \circ (\mathcal{M} \otimes |0\rangle\langle 0|) = \mathcal{M}_{12} \otimes |0\rangle\langle 0| \). \( \text{Tr}_1 R_U(\psi \otimes |0\rangle\langle 0|) = \text{Tr}_2 R_U(\psi \otimes |0\rangle\langle 0|) = \psi \otimes |0\rangle\langle 0| \), so \( R_U \) can clone an arbitrary state and is therefore...
unphysical.

Having demonstrated a nonphysical $R_U$, a $\ast$ and physical $R \circ (M \otimes |0\rangle\langle 0|_E)$, we have so far shown that a single use of an anti-degradable channel can’t be used to transmit a quantum state with high fidelity. However, since the tensor product of two anti-degradable channels is again anti-degradable, this also shows that many copies cannot transmit quantum information either. As a result, the capacity must be zero.

Discussion— We have presented a general approach for detecting the incapacity for quantum communication using unphysical transformations, and shown that both known incapacity tests fall into this framework. Furthermore we have discovered a connection to the theory of representations of the unitary group, and both positive partial transposition and antidegradability correspond to simple representations. This paves the way for the discovery of new incapacity tests, and Theorem 1 suggests a fruitful direction, namely forbidden transformations that are not linear.

We have focused primarily on the standard (one-way) quantum capacity, but these ideas can also be extended to the two-way capacity and questions of entanglement distillation. For example, in the appendix we demonstrate non-distillability of PPT states by showing that any successful distillation protocol could be used to implement the unphysical time-reversal operation. Our argument there makes crucial use of the linearity of time reversal, which in light of Theorem 1 severely restricts the detection of two-way capacity and distillability. Finding an argument relating two-way capacities and nonlinear forbidden transformations is an important challenge.

We end on a speculative note. We have shown that both known incapacity tests are derived from fundamentally unphysical transformations on state space: time reversal and cloning. Could it be that any zero quantum capacity channel has such a “reason” for its incapacity? Formally, of course, the answer is “yes”—the forbidden transformation could just be a successful encoding or decoding operation for the channel. However, we would be much more satisfied with something less tautological. A good starting point might be to identify a minimal set of primitive forbidden operations that includes cloning and time-reversal as examples.

Acknowledgments: We are grateful to Man-Duen Choi for telling us about Theorem 1 and for very informative discussions. Thanks also to Toby Cubitt and Robert Koenig for helpful suggestions and Charlie Bennett for advice on the manuscript. This work was supported by the DARPA QUEST program under contract no. HR0011-09-C-0047.

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[1] C. E. Shannon, Bell Syst. Tech. J. 27, 379 (1948).
[2] T. M. Cover and J. A. Thomas, Elements of Information Theory (Wiley & Sons, 1991).
[3] C. H. Bennett and P. W. Shor, Science 303, 1784 (2004).
[4] S. Lloyd, Phys. Rev. A 55, 1613 (1997).
[5] P. W. Shor, lecture notes, MSRI Workshop on Quantum Computation, 2002. Available online at http://www.msri.org/publications/ln/msri/2002/quantumcrypto/shor/1/.
[6] I. Devetak, IEEE Trans. Inf. Theory 51, 44 (2005), arXiv:quant-ph/0304127.
[7] M. Hastings, Nat. Phys. 5, 255 (2009), arXiv:0809.3972.
[8] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996).
[9] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[10] C. H. Bennett, D. P. DiVincenzo, and J. A. Smolin, Phys. Rev. Lett. 78, 3217 (1997).
[11] D. Bruss, D. P. DiVincenzo, A. Ekert, C. A. Fuchs, C. Macchiavello, and J. A. Smolin, Phys. Rev. A 57, 2368 (1998), arXiv:quant-ph/9705038.
[12] W. Wootters and W. Zurek, Nature 299, 802 (1982).
[13] G. Smith and J. Yard, Science 321, 1812 (2008).
[14] J. Barrett, Phys. Rev. A 75, 032304 (2007).
[15] B. Schumacher and M. D. Westmoreland (2010), arXiv:1010.2929.
[16] M. Choi, private communication.
[17] I. Devetak and P. W. Shor, Communications in Mathematical Physics 256, 287 (2005), ISSN 0010-3616, arXiv:quant-ph/0304127. URL http://dx.doi.org/10.1007/s00220-005-1317-6
[18] V. Giovannetti and R. Fazio, Phys. Rev. A. 71, 032314 (2005), arXiv:quant-ph/0405110.
[19] C. H. Bennett, D. P. D. and J. A. Smolin, and W. K. Wootters, Phys. Rev. A. 54, 3824 (1996), arXiv:quant-ph/9604024.
[20] M. Choi, Linear Algebra and Its Applications pp. 285–290 (1975).

[21] Recall that a positive map $\gamma$ has the property that $\gamma(\rho) \geq 0$, for all $\rho \geq 0$ but not necessarily $(\sum_i \gamma(A_i)) \geq 0$ for some $\rho \geq 0$ on a larger space. A completely positive map $\eta$ has $(\sum_i \eta(A_i)) \geq 0$ for all $\rho \geq 0$.
[22] The Choi matrix of a channel $N$ completely characterizes the channel and is defined as $(I \otimes T)((I \otimes N)(\otimes \phi_d)\langle \phi_d))$, where $|\phi_d\rangle = \sqrt{d} \sum_{i=1}^d |i\rangle |i\rangle$. 

Appendix: Two-way capacities and distillable entanglement— We now argue that if the Choi matrix of a channel is PPT, it cannot be distillable via local operation and classical communication (LOCC). Since the Choi matrix of $N$ is PPT if $T \circ N$ is a physical channel, showing this will also prove that such a channel has no $Q_2$, or quantum capacity assisted by LOCC.

By teleporting through the Choi matrix of a channel $N$, an LOCC protocol with Kraus operators $A_i \otimes B_i$ can be used to prepare the state

$$d_{\text{out}} - d_{\text{in}} \sum_i \frac{1}{d_{\text{in}}^2} \sum_u B_i \left( N(A_i^\dagger \sigma_u \psi \sigma_d^\dagger A_i^\dagger) \right) B_i^\dagger \otimes |u\rangle \langle u|.$$  \hspace{1cm} (3)

where the $\sigma_u$s are generalized Pauli matrices.

Furthermore, if the Choi matrix can be distilled to a maximally entangled state, there is an LOCC protocol
such that
\[ \psi = \frac{2}{d} \sum B_i \mathcal{N}(A_i^T \psi A_i^*) B_i^\dagger \] (4)

for an arbitrary qubit state \( \psi \). Letting \( T(\rho) = \rho^T \), this implies that
\[ T(\psi) = \frac{2}{d} \sum B_i^* T \circ \mathcal{N}(A_i^T \psi A_i^*) B_i^T, \] (5)

and
\[ T(\sigma_u \psi \sigma_u^\dagger) = \frac{2}{d} \sum B_i^* T \circ \mathcal{N}(A_i^T \sigma_u \psi \sigma_u^\dagger A_i^*) B_i^T. \] (6)

This latter can be used to show that
\[ T(\psi) = \frac{2}{d} \sum \sigma_u^* B_i^* T \circ \mathcal{N}(A_i^T \sigma_u \psi \sigma_u^\dagger A_i^*) B_i^T \sigma_u^T \] (7)

the right hand of which, using the fact that \( T \circ \mathcal{N} \) is physical combined with Eq. (3), gives a recipe for physically implementing \( T \) using the LOCC operation with Kraus operators \( A_i \otimes B_i^* \). As a result, the Choi matrix of \( \mathcal{N} \) must not be distillable.