On the use of Mellin transform to a class of
q-difference-differential equations

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Abstract

We explore the possibility of using the method of classical integral transforms to
solve a class of q-difference-differential equations. The Laplace and the Mellin transform
of q-derivatives are derived. The results show that the Mellin transform of the q-
derivative resembles most closely the corresponding expression in classical analysis,
and it could therefore be useful in solving certain q-difference equations.

Revised version
1. The study of q-analysis is an old subject, which dates back to the end of the 19th century ([1]-[4]). It has found many applications in such areas as the theory of partitions, combinatorics, exactly solvable models in statistical mechanics, computer algebra, etc [5]. Recent developments in the theory of quantum group has boosted further interests in this old subject [6, 7].

The subject of q-analysis concerns mainly the properties of the so-called q-special functions, which are the extensions of the classical special functions based on a parameter, or the base, q. The relations among these functions, and the difference equations satisfied by them are among the topics most studied so far. The q-difference equations involve a new kind of difference operator, the q-derivative, which can be viewed as a sort of deformation of the ordinary derivative. Solutions of the q-difference equations in one variable have been well studied in terms of the q-hypergeometric series (also called the basic hypergeometric series). Partial q-difference equations and q-difference-differential equations with more than one variables are generally studied by means of the method of separation of variables, or by the techniques of Lie symmetry in the literature ([3],[8]-[13]). The method of integral transforms, which is another powerful technique of solving differential equations in classical analysis, has not been, in our view, explored in q-analysis. The reason is not hard to understand. The main virtue of the classical integral transforms, particularly the Fourier and the Laplace transform, is to transform a differential equation into an algebraic equation, which can be solved easily. That this is possible is due to the fact that these transforms change the derivatives of a function to something proportional to the transform of the original function. As far as we know, integral transforms or q-integral transforms which could transform q-difference equations into algebraic equations have not been found. It should be mentioned that in fact q-analogues of Fourier transform, based on the Jackson q-integral, have been proposed recently [14, 15]. However, in order for the q-Fourier transform of the q-derivative
of a function $f(x)$ to be proportional to the $q$-Fourier transform of $f(x)$, the function $f(x)$ must satisfy very special conditions, such as $f(q^{-1}) = 0 = f(-q^{-1})$. Hence, while these $q$-Fourier transforms may be useful in proving certain identities among the $q$-special functions, their use in solving $q$-difference equations seems limited.

In this paper we shall explore the possibility of using the method of classical integral transform to solve a class of $q$-difference-differential equations. We derive the Laplace and the Mellin transform of $q$-derivative, and argue that the Mellin transform, which is not generally employed in solving differential equations in classical analysis, may still be useful in solving certain $q$-difference equations.

2. Suppose we want to solve the following $q$-diffusion equation

$$D_t^q y(x,t) = \frac{\partial^2}{\partial x^2} y(x,t) \quad (-\infty < x < \infty, \ t > 0) \quad (1)$$

subject to the initial condition

$$y(x,0) = f(x) \quad (2)$$

Here $D_t^q$ is the “forward” temporal $q$-derivative defined by [15, 16]

$$D_t^q h(t) := \frac{h(q^{-1}t) - h(t)}{(1-q)t} \quad (3)$$

for any function $h(x)$. We assume $0 < q < 1$ in this paper. The function $f(x)$ is assumed to vanish as $x \to \pm \infty$. One may as well use the more common definition of $q$-derivative [4]

$$D_t^q h(t) := \frac{h(t) - h(qt)}{(1-q)t} \quad (4)$$

We shall not employ this definition of the $q$-derivative here for reason to be explained later. We note here that $q$-difference and $q$-difference-differential equations of the diffusion type such as eq.(1) have been considered before ([3]-[13]), but mostly from the point of view of Lie symmetry, or by seperation of variables.
We can remove the partial differential operator in $x$ in (1) by a Fourier transform. The question now is to choose an appropriate integral transform to remove the $q$-derivative. In view of the positivity of the time variable, the two most natural choices are the Laplace and the Mellin transform.

Let us first derive the expression of the Laplace transform of the $q$-derivative. The Laplace transform of a function $h(t)$ is defined as $\mathcal{L}\{h(t), s\} = \int_0^\infty h(t) \exp(-st)dt$. For the $q$-derivative of $h(x)$, the Laplace transform is

$$
\mathcal{L}\{D^q_t h(t), s\} = \frac{1}{1-q} \left[ \int_0^\infty \frac{h(q^{-1}t)}{t} e^{-st}dt - \int_0^\infty \frac{h(t)}{t} e^{-st}dt \right]. \tag{5}
$$

To proceed we have to use the following relation of the Laplace transform

$$
\int_s^\infty \bar{h}(s')ds' = \int_0^\infty \frac{h(t)}{t} e^{-st}dt, \tag{6}
$$

provided the integral on the r.h.s. of (6) is well-defined. We may apply (6) to (5) directly if $h(0) = 0$. However, if $h(0) \neq 0$, the r.h.s. of (6) is not well-defined, and direct application of (6) to (5) leads to incorrect result which does not reduce to the usual expression of the Laplace transform of derivative in the classical limit $q \to 1^-$. In order to recover the classical limit correctly, we find it necessary to regularise (5) in the form

$$
\frac{1}{1-q} \left[ \int_0^\infty \frac{h(q^{-1}t) - h(0)}{t} e^{-st}dt - \int_0^\infty \frac{h(t) - h(0)}{t} e^{-st}dt \right]. \tag{7}
$$

We may now apply (5) to (7). Making use of

$$
\mathcal{L}\{h(t) - h(0), s\} = \bar{h}(s) - s^{-1}h(0) \tag{8}
$$

we finally obtained

$$
\mathcal{L}\{D^q_t h(t), s\} = \frac{1}{1-q} \int_s^{\ln q^{-1}} \bar{h}(s') ds' - \frac{\ln q^{-1}}{1-q} h(0). \tag{9}
$$

Eq.(9) reduces to the expression $s\bar{h}(s) - h(0)$ for the Laplace transform of ordinary derivative as $q \to 1^-$. 

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If one uses instead the definition (4) for the $q$-derivative, the Laplace transform would be

\[ \mathcal{L}\{D_q^t h(t), s\} = \frac{1}{1 - q} \int_s^\infty \tilde{h}(s') \, ds' - \frac{\ln q^{-1}}{1 - q} h(0). \tag{10} \]

It is now obvious that the Laplace transform is not useful in solving equations involving $q$-derivatives: it transforms such equations into integral equations!

3. We now consider the Mellin transform of a $q$-derivative. The Mellin transform is seldom being used in solving differential equations, because it generally transforms differential equations into difference equations instead of the much simpler algebraic equations. Now that the Fourier and the Laplace transform lose their virtues whenever $q$-derivatives are present, the Mellin transform is naturally the next one to be looked at. As we shall see below, the Mellin transform still transforms an equation containing $q$-derivatives into a difference equation of the transformed function, which is the best thing next to an algebraic equation one could get. Previously, the use of the Mellin transform in $q$-analysis is limited to proving various identities among the $q$-special functions \([2, 17]\).

The Mellin transform of a function $h(t)$ is defined as $h^*(s) := \mathcal{M}\{h(t), s\} = \int_0^\infty h(t) t^{s-1} \, dt$. For $q$-derivative defined in (3), we have

\[ \mathcal{M}\{D_q^t h(t), s\} = -[s - 1]_q h^*(s - 1). \tag{11} \]

Here $[x]_q$ is the $q$-number defined by

\[ [x]_q := \frac{1 - q^x}{1 - q}. \tag{12} \]

Note that $[x]_q \to x$ as $q \to 1^-$. Hence (11) reduces to the expression $-(s - 1) h^*(s - 1)$ for the Mellin transform of the ordinary derivative as $q \to 1^-$. Repeated use of (11) leads to

\[ \mathcal{M}\{(D_q^t)^n h(t), s\} = (-1)^n [s - 1]_q [s - 2]_q \cdots [s - n]_q h^*(s - n), \quad n \geq 1. \tag{13} \]

This is the $q$-analogue of the corresponding formula in the classical case \([18]\).
For the definition (4), one has

\[ M\{D^q_t h(t), s\} = \left[1 - s\right]_q h^*(s - 1) \quad (14) \]

\[ = -q^{1-s}[s - 1]_q h^*(s - 1) . \quad (15) \]

Here an extra factor of \( q \) appears compared with (11). In order to simplify our presentation, we therefore adopt the definition (3) in this paper. We must, however, mention that all the arguments given below apply equally well to the corresponding cases with \( q \)-derivatives replaced by the definition (4).

4. Let \( Y^*(\xi, s) \) be the transformed function of \( y(x, t) \) obtained by taking the Mellin transform in \( t \) and a Fourier transform \( G(\xi) := \int_{-\infty}^{\infty} g(x) \exp(i\xi x) dx \) in \( x \). Making these transforms to (1), one obtains

\[ [s - 1]_q Y^*(\xi, s - 1) = \xi^2 Y^*(\xi, s) . \quad (16) \]

Fortunately solution to this equation can be readily found to be

\[ Y^*(\xi, s) = A(\xi)\xi^{-2s}\Gamma_q(s) , \quad (17) \]

where \( A(\xi) \) is some function of \( \xi \) only, and \( \Gamma_q(s) \) is the \( q \)-gamma function defined by

\[ \Gamma_q(s) := \frac{(q; q)_\infty}{(q^s; q)_\infty} (1 - q)^{1-s} , \quad 0 < q < 1 . \quad (18) \]

\[ (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k) . \quad (19) \]

\( \Gamma_q(s) \) satisfies

\[ \lim_{q \to 1^-} \Gamma_q(s) = \Gamma(s) , \quad (20) \]

\[ \Gamma_q(s + 1) = [s]_q \Gamma_q(s) , \quad \Gamma_q(1) = 1 . \quad (21) \]
Inverse-Mellin transform of $\xi^{-2s}\Gamma_q(s)$ in (17) is
\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \xi^{-2s}\Gamma_q(s)t^{-s}ds .
\] (22)

The poles of $\Gamma_q(s)$ are $s = 0, -1, -2, \ldots$. The residual of $\Gamma_q(s)$ at pole $s = -n$ ($n \geq 0$) is [4]:
\[
\frac{(1 - q)^{n+1}}{(q^{-n}; q)_n \ln q^{-1}} .
\] (23)

The symbol $(a; q)_n$ is the $q$-shifted factorial:
\[
(a; q)_0 := 1 , \quad n = 0 ,
(a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) , \quad n = 1, 2\ldots
\] (24)
(25)

Hence (22) becomes
\[
\frac{1 - q}{\ln q^{-1}} \sum_{n=0}^{\infty} \frac{[(1 - q)\xi^2t]^n}{(q^{-n}; q)_n} .
\] (26)

In view of the identity [4]
\[
(q^{-n}; q)_n = \left(-\frac{1}{q}\right)^n q^{-n(n-1)/2} (q;q)_n ,
\] (27)

(26) can be expressed as
\[
\frac{1 - q}{\ln q^{-1}} \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} [-q(1 - q)\xi^2t]^n}{(q; q)_n}
= \frac{1 - q}{\ln q^{-1}} E_q \left(-q(1 - q)\xi^2t\right) .
\] (28)

The function $E_q(z)$ (for complex $z$) is the $q$-exponential function defined by [4]
\[
E_q(z) := \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} z^n}{(q; q)_n} = (-z; q)_\infty .
\] (29)

In the limit $q \to 1^-$, eq.(28) tends to the usual exponential function $\exp(-\xi^2t)$. Finally, performing an inverse Fourier transform we obtain the solution of the $q$-diffusion equation
\[
y(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\xi) \left\{ \frac{1 - q}{\ln q^{-1}} E_q \left(-q(1 - q)\xi^2x\right) \right\} e^{-ix\xi} d\xi .
\] (30)
Setting $t = 0$ in \((30)\) shows that

\[
\frac{1 - q}{\ln q^{-1}} A(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(x, 0) e^{i\xi x} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\xi x} \, dx = F(\xi)
\]

is the Fourier transform of $y(x, 0) = f(x)$. So the final solution of the initial problem is

\[
y(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) E_q \left(-q(1 - q)\xi^2 t\right) e^{-i\xi x} \, d\xi .
\] (32)

This is the $q$-analogue of the solution given in [19] for the corresponding classical case. One can easily check that (32) indeed satisfies (1) by using the following identity

\[
D_q^t E_q(\lambda t) = \frac{\lambda}{q(1 - q)} E_q(\lambda t).
\] (33)

Let us consider an example. Suppose the initial profile is $f(x) = \exp(-x^2/4b)/\sqrt{2b}$, $(b > 0)$. Its Fourier transform is $F(\xi) = \exp(-b\xi^2)$. Then from (32) and (29), we get

\[
y(x, t) = E_q \left(q(1 - q)t \frac{d}{db}\right) f(x) .
\] (34)

In the limit $q \to 1^-$, eq.(34) gives the classical solution

\[
y(x, t) = e^{t\frac{d}{db}} \left(\frac{1}{\sqrt{2b}} e^{-\frac{x^2}{4b}}\right) = \frac{1}{\sqrt{2(t+b)}} e^{-\frac{x^2}{4(t+b)}} .
\] (35)

5. As another example, let us consider the following wave equation

\[
(D_q^t)^2 y(x, t) = \frac{\partial^2}{\partial x^2} y(x, t) \quad (-\infty < x < \infty, \ t > 0)
\] (36)

with initial conditions

\[
y(x, 0) = f(x), \quad D_q^t y(x, 0) = g(x) .
\] (37)
We assume that both $f(x)$ and $g(x)$ vanish as $x \to \pm\infty$. In this case the Fourier-Mellin transformed function $Y^*(\xi, s)$ obeys

\[ [s - 1]_q[s - 2]_q Y^*(\xi, s - 2) = -\xi^2 Y^*(\xi, s) . \tag{38} \]

The general solution is

\[ Y^*(\xi, s) = \left[ A(\xi) (-i\xi)^{-s} + B(\xi) (i\xi)^{-s} \right] \Gamma_q(s) , \tag{39} \]

where $A(\xi)$ and $B(\xi)$ are some functions of $\xi$. Performing the inverse-Mellin transform, we get

\[ Y(\xi, t) = \frac{1 - q}{\ln q^{-1}} \left\{ A(\xi) E_q (iq(1 - q)\xi t) + B(\xi) E_q (-iq(1 - q)\xi t) \right\} . \tag{40} \]

Here $Y(\xi, t)$ is the Fourier transform of $y(x, t)$ with respect to $x$. Now we rewrite (40) in terms of the $q$-Sine and the $q$-Cosine function which are defined by

\[ \sin_q x = \frac{E_q(ix) - E_q(-ix)}{2i} , \quad \cos_q x = \frac{E_q(ix) + E_q(-ix)}{2} . \tag{41} \tag{42} \]

The result is

\[ y(\xi, t) = \frac{1 - q}{\ln q^{-1}} \left\{ C(\xi) \cos_q (q(1 - q)\xi t) + D(\xi) \sin_q (q(1 - q)\xi t) \right\} , \tag{43} \]

where the functions $C(\xi)$ and $D(\xi)$ are linear combinations of $A(\xi)$ and $B(\xi)$. The inverse-Fourier transform of (43) is

\[ y(x, t) = \frac{1 - q}{\sqrt{2\pi} \ln q^{-1}} \int_{-\infty}^{\infty} \left\{ C(\xi) \cos_q (q(1 - q)\xi t) + D(\xi) \sin_q (q(1 - q)\xi t) \right\} e^{-i\xi x} d\xi . \tag{44} \]

Letting $t = 0$ in (44), one can check that the function $C(\xi)$ is related to the Fourier transform of $f(x)$ by

\[ F(\xi) = \frac{1 - q}{\ln q^{-1}} C(\xi) . \tag{45} \]
Making use of the following relations, which can be obtained by means of (33):

\[ D^q_t \sin(q \lambda t) = \frac{\lambda}{q(1-q)} \cos(q \lambda t), \]  
\[ D^q_t \cos(q \lambda t) = -\frac{\lambda}{q(1-q)} \sin(q \lambda t), \]  

we can relate \( D(\xi) \) to the Fourier transform \( G(\xi) \) of \( g(x) \) as follows:

\[ G(\xi) = \frac{1-\lambda}{\ln q^{-1}} D(\xi) \xi. \] 

With these results, we finally obtain the solution to the initial problem of eq.(36):

\[ y(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ F(\xi) \cos(q(1-q)\xi t) + \frac{G(\xi)}{\xi} \sin(q(1-q)\xi t) \right\} e^{-i\xi x} d\xi. \]

This solution is the \( q \)-analogue of the solution to the corresponding classical case given in [19].

6. We now see how the above steps are generalised to the equation:

\[ (D^q_t)^n y(x, t) = \frac{\partial^2}{\partial x^2} y(x, t) \quad (-\infty < x < \infty, \ t > 0, \ n \geq 2) \]

with initial conditions

\[ y(x, 0) = f(x), \quad (D^q_t)^k y(x, 0) = g_k(x), \quad k = 1, \ldots, n-1, \]

where the functions \( f(x) \) and \( g_k(x) \) are assumed to vanish as \( x \to \pm \infty \). The Fourier-Mellin transformed function \( Y^*(\xi, s) \) obeys

\[ (-1)^n[s - 1]_q[s - 2]_q \cdots [s - n]_q Y^*(\xi, s - n) = -\xi^2 Y^*(\xi, s). \]

The general solution is

\[ Y^*(\xi, s) = \Gamma_q(s)\xi^{-2n} \sum_{m=0}^{n-1} A_m(\xi) \left[ -e^{-\frac{(2m+1)\pi i}{n}} \right]^s. \]
where $A_m(\xi)$ are some functions of $\xi$. We can now perform the inverse Mellin and Fourier transforms to get the final solution, which is given formally as

$$y(x, t) = \frac{1 - q}{\sqrt{2\pi \ln q}^{-1}} \int_{-\infty}^{\infty} A_m(\xi) E_q \left( q(1 - q)e^{(m+1)\frac{\pi i}{n} \xi^2 t} \right) e^{-i\xi x} d\xi .$$

The functions $A_m(\xi)$ can then be related to the Fourier transforms of the functions $f(x)$ and $g_k(x)$ from the initial conditions.

7. To summarise, we show that the Mellin transform of the $q$-derivative resembles most closely the corresponding expression in classical analysis, whereas transforms such as the Fourier and the Laplace transform fail in this respect. As such the Mellin transform can be useful in solving certain $q$-difference equations. We illustrated this fact with a few examples. However, for the Mellin transform to be really useful, a more complete knowledge of the properties of the $q$-special functions under various integral transforms (Fourier, Laplace, Mellin, etc) and their inverses has yet to be attained. What is more desirable is to invent integral transforms or $q$-integral transforms that possess the virtue of the Fourier and the Laplace transform in the classical analysis mentioned in the introduction.

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