OPTIMAL DIVIDENDS AND CAPITAL INJECTIONS FOR A SPECTRALLY POSITIVE LÉVY PROCESS

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Abstract. This paper investigates an optimal dividend and capital injection problem for a spectrally positive Lévy process, where the dividend rate is restricted. Both the ruin penalty and the costs from the transactions of capital injection are considered. The objective is to maximize the total value of the expected discounted dividends, the penalized discounted capital injections before ruin, and the expected discounted ruin penalty. By the fluctuation theory of Lévy processes, the optimal dividend and capital injection strategy is obtained. We also find that the optimal return function can be expressed in terms of the scale functions of Lévy processes. Besides, a series of numerical examples are provided to illustrate our consults.

1. Introduction. In recent years, quite a few papers deal with the optimal dividend problems in the spectrally positive Lévy risk model, which is also called the dual risk model. The dual model is an appropriate model for a company driven by inventions or discoveries. Other examples are commission-based business, such as real estate agent offices or insurance annuity business. In [1] and [2], the authors studied how the expectation of the discounted dividends until ruin can be calculated in the dual compound Poisson risk model. Recently, in [4], [6] and [18], the optimal dividend problems were studied in a general spectrally positive Lévy risk model.

Besides dividend payment, capital injection is another important approach to control the surplus process. In [3], [16] and [17], the optimal dividend and capital injection problem was considered in the compound Poisson dual model. For the general spectrally positive Lévy process, the optimal dividend and capital injection problems were studied in [4], [5] and [19], where there was no constraint on dividend rate. In [18], the optimal dividend with restricted dividend rate was considered for a general spectrally positive Lévy process, however, the capital injection was not involved the control problem. In addition, transaction cost, which usually

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includes two part - proportional cost and fixed cost -, is also an important factor in business activities. In [3], proportional costs were involved into the optimal dividend problem. In [16], [19] and [14] both proportional and fixed costs on capital injection were considered. Fixed costs on dividend were studied in [5].

In this paper, we discuss the optimal dividend and capital injection problem with the restricted dividend rate for a spectrally positive Lévy process, which will generalize our precious work in [18] and [19]. Both proportional and fixed costs from the transactions of capital injection are considered. In addition, as in [12] and [15], the ruin penalty is also considered. We firstly discuss the properties of the optimal return function. To our surprise, the monotonicity of the optimal return function varies as the change of ruin penalty. Then, by the fluctuation theory of Lévy processes, we find the optimal strategy and the explicit optimal return function. If the ceiling of the dividend rate goes to infinity, we can derive the results in [4] and [19]. When the positive jumps of the Lévy process are hyper-exponential compound Poisson jumps, the results in this paper are simplified into those in [17]. Besides, the results in [18] are also found.

The paper is organized as follows. Section 2 provides the formulation of the problem and the properties of the optimal return function. Section 3 discusses the case without capital injection. The case with incorporated capital injection is considered in Section 4. The optimal dividend and capital injection strategy is derived in Section 5. Section 6 gives some numerical results.

2. Model and optimal control problem.

2.1. Spectrally positive Lévy processes. Let \( X = \{X_t\}_{t \geq 0} \) be a spectrally positive Lévy process with non-monotone paths on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \( \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0} \) is generated by the process X and satisfies the usual conditions. The Lévy triplet of X is given by \((c, \sigma, \nu)\), where \( c > 0 \), \( \sigma \geq 0 \), and \( \nu \) is a Lévy measure on \((0, \infty)\) satisfying the integrability condition \( \int_0^\infty (1 \wedge x^2) \nu(dx) < \infty \).

The process \( X \) has paths of bounded variation if and only if \( \sigma = 0 \) and \( \int_1^{\infty} x \nu(dx) < \infty \). Correspondingly, the Laplace exponent (1) can be written as

\[
\psi(s) = \frac{1}{t} \log E e^{-sX_t} = \frac{s^2}{2} + cs + \int_0^\infty (e^{-sx} - 1 + sx 1_{0<x<1}) \nu(dx),
\]

where \( 1_A \) is an indicator function of a set A. The process \( X \) has paths of bounded variation if and only if \( \sigma = 0 \) and \( \int_0^1 x \nu(dx) < \infty \). Correspondingly, the Laplace exponent (1) can be written as

\[
\psi(s) = c_0 s + \int_0^\infty (e^{-sx} - 1) \nu(dx),
\]

where \( c_0 = c + \int_0^1 x \nu(dx) \). We rule out the case that \( X \) has monotone paths, and so \( c_0 > 0 \) is necessary when \( X \) is of bounded variation. The drift of \( X \) is given by \( \mu = EX_1 = -\psi'(0+) \). It is well known that if \( \int_0^\infty y \nu(dy) < \infty \), then \( \mu = -c + \int_1^\infty y \nu(dy) < \infty \). In this paper, we assume that \(-\infty < \mu < \infty \) to ensure that the problem is nontrivial. For more details on Lévy processes, the reader is referred to [6], [11] and [10].

2.2. Formulation of control problem. We assume that the surplus process of a company is modeled by the Lévy process \( X \), whose Laplace exponent is given by (1) and \( X_0 = x \geq 0 \). We incorporate dividend payment and capital injection into the risk process \( X \). Let \( L_t \) denote the cumulative amount of dividends paid up to time
$t$ and $l_t \in [0, l_0]$ be associated dividend rate at time $t$. Here $l_0 \in (0, \infty)$ is a ceiling on the dividend rate. The capital injection process $\{G_t = \sum_{n=1}^{\infty} 1_{\{\tau_n \leq t\}} \xi_n \}_{t \geq 0}$ is described by a sequence of increasing stopping times $\{\tau_n, n = 1, 2, \cdots\}$ and a sequence of random variables $\{\xi_n, n = 1, 2, \cdots\}$, which represent the times and the sizes of capital injections, respectively. A control policy $\pi$ is described by $\pi = (L^\pi; G^\pi) = (L^\pi; \tau^\pi_1, \cdots, \tau^\pi_n, \cdots; \xi^\pi_1, \cdots, \xi^\pi_n, \cdots)$. The controlled asset process associated with $\pi$ is modeled as

$$X^\pi_t = X_0 - L^\pi_t + \sum_{n=1}^{\infty} \xi^\pi_n 1_{\{\tau^\pi_n \leq t\}}, \quad t \geq 0.$$ 

**Definition 2.1.** A strategy $\pi$ is said to be admissible if

(i) $\{L^\pi_t\}_{t \geq 0}$ is an increasing, $\mathcal{F}$-adapted càdlàg process, and satisfies $L^\pi_t = \int_0^t l^\pi_s ds$.

(ii) $\tau^\pi_n$ is a stopping time w.r.t. $\mathcal{F}$ and $0 \leq \tau^\pi_1 < \tau^\pi_2 < \cdots < \tau^\pi_n < \cdots$, a.s.

(iii) $\xi^\pi_n$ is nonnegative and measurable with respect to $\mathcal{F}_{\tau^\pi_n}$, $n = 1, 2, \cdots$.

(iv) $P(\lim_{n \to \infty} \tau^\pi_n \leq T) = 0$, $\forall T \geq 0$.

Denote the set of admissible control strategies by $\Pi$. Define the time of ruin by

$$T^\pi = \inf \{t \geq 0 : X^\pi_t \leq 0\}.$$ 

Considering the fixed and proportional transaction costs when capital injection occurs, we assume that $(\phi - 1)\xi$ and $K > 0$ are respectively proportional cost and fixed cost to meet the capital injection of amount $\xi$, where $\phi > 1$. The force of interest $\delta > 0$ reflects the time preference of investors. In addition, we consider a constant penalty $P \in \mathbb{R}$ that must be paid (collected, if $P$ is negative) if and when ruin occurs. When $P > 0$, we think of $P$ as a penalty for ruin, the monetary cost when ruin occurs. When $P = 0$, we are in the case of no penalty. When $P < 0$, we think of $P$ as the salvage value of the company; for example, a company’s brand name or agency network might be of value to a potential customer. Under these assumptions, the performance function with the strategy $\pi \in \Pi$ is defined by

$$V(x; \pi) = E^\pi \left[ \int_0^{T^\pi} e^{-\delta s} l^\pi_s ds - Pe^{-\delta T^\pi} - \sum_{n=1}^{\infty} e^{-\delta \tau^\pi_n} (K + \phi \xi^\pi_n) 1_{\{\tau^\pi_n \leq T^\pi\}} \right]. \quad (3)$$

Our objective is to find the optimal return function, or the value function, defined as

$$V(x) = \sup_{\pi \in \Pi} V(x; \pi), \quad (4)$$

and the optimal strategy $\pi^* \in \Pi$ such that $V(x) = V(x; \pi^*)$.

### 2.3. Property of the value function.

**Proposition 2.1.** If the penalty $P \geq -\frac{l_0}{\phi}$, the value function $V(x)$ in $\{4\}$ is increasing on $[0, \infty)$; otherwise, the value function $V(x)$ in $\{4\}$ is decreasing on $[0, \infty)$.

**Proof.** We first prove the increase of the value function $V$ in the case of $P \geq -\frac{l_0}{\phi}$. For the initial surplus $x \geq 0$ and the admissible strategy $\pi$, we denote the ruin time by $T^\pi_x$. If $T^\pi_x < \infty$, for $y > x$, we construct a strategy $\tilde{\pi}$ as follows:

- $L^\tilde{\pi}_t = L^\pi_t$ and $G^\tilde{\pi}_t = G^\pi_t$ for $0 \leq t \leq T^\pi_x$.
- $L^\tilde{\pi}_t = l_0 \int_{T^\pi_x}^t e^{-\delta s} ds$ and $G^\tilde{\pi}_t = 0$ for $T^\pi_x < t \leq T^\pi_y$.
where \( T^\pi_y \) is the ruin time for the initial capital \( y \) and the strategy \( \pi \). Then \( \hat{\pi} \in \Pi \), and \( X^\pi_t + (y - x) = X^\pi_t \) a.s. for \( 0 \leq t \leq T^\pi_x \). By (3), we have
\[
V(y; \hat{\pi}) - V(x; \pi) = E^x \left[ l_0 \int_{T^\pi_x}^{\pi} e^{-\delta s} ds - P \left( e^{-\delta T^\pi_y} - e^{-\delta T^\pi_x} \right) \right] = - \left( \frac{l_0}{\delta} + P \right) E^y \left[ e^{-\delta T^\pi_y} - e^{-\delta T^\pi_x} \right].
\]

Due to \( P \geq -\frac{l_0}{\delta} \) and \( T^\pi_x \geq T^\pi_x \) a.s., we obtain
\[
V(x; \pi) \leq V(y; \hat{\pi}) \leq V(y).
\] (5)

If \( T^\pi_x = \infty \), for \( y > x \), we define a strategy \( \hat{\pi} = \pi \). Then \( T^\pi_y = T^\pi_x = \infty \), furthermore, \( V(x; \pi) = V(y; \hat{\pi}) \leq V(y) \). Combining with (5), we obtain \( V(x; \pi) \leq V(y) \) for all \( \pi \in \Pi \), and hence \( V(x) \leq V(y) \).

Now, we assume the penalty \( P < -\frac{l_0}{\delta} \). For the initial surplus \( y > 0 \) and the admissible strategy \( \pi_1 \), we denote the ruin time \( T^{\pi_1} \) by \( T^{\pi_1}_y \). For \( 0 \leq x < y \), let
\[
\tilde{T}^{\pi_1}_{y-x} = \sup \{ t \leq T^{\pi_1}_y : X^{\pi_1}_t \leq y - x \}.
\]

We construct the strategy \( \pi_2 = \pi_1 \) before \( \tilde{T}^{\pi_1}_{y-x} \). For the initial surplus \( x \) and the strategy \( \pi_2 \), the ruin time is denoted by \( T^{\pi_2}_x \). If \( \tilde{T}^{\pi_1}_{y-x} = \infty \), then \( T^{\pi_2}_x = \tilde{T}^{\pi_1}_{y-x} \) a.s.

Hence, we have
\[
V(y; \pi_1) - V(x; \pi_2) \leq E^x \left[ l_0 \int_{T^{\pi_1}_x}^{\pi_2} e^{-\delta s} ds - P \left( e^{-\delta T^{\pi_1}_y} - e^{-\delta T^{\pi_2}_x} \right) \right] \leq 0.
\]
Furthermore, we have
\[
V(y; \pi_1) \leq V(x; \pi_2) \leq V(x).
\] (6)

If \( \tilde{T}^{\pi_1}_{y-x} = \infty \), then \( T^{\pi_2}_x = T^{\pi_1}_y = \infty \), and so \( V(y; \pi_1) = V(x; \pi_2) \leq V(x) \). Combining with (6), we have \( V(y; \pi_1) \leq V(x) \) for any \( \pi_1 \in \Pi \), and hence \( V(y) \leq V(x) \). □

**Remark 2.1.** By the above proposition, if the penalty \( P < -\frac{l_0}{\delta} \), the value function \( V(x) \) is decreasing, that is, larger initial surplus results in smaller profit. Then it is not necessary to operate the company as \( P < -\frac{l_0}{\delta} \). Hence, we only discuss the case of \( P < -\frac{l_0}{\delta} \) in the following section.

**Proposition 2.2.** If the penalty \( P \geq -\frac{l_0}{\delta} \), the value function \( V(x) \) satisfies
\[
- P \leq V(x) \leq \frac{l_0}{\delta} + |P|_{P \leq 0}.
\] (7)

*Proof.* For the initial surplus \( x = 0 \) and the admissible strategy \( \pi_1 \): the company immediately declare ruin, the associated performance function \( V(0; \pi_1) = -P \), and so \( V(0) \geq -P \). By Proposition 2.1, the value function is increasing, and so the first inequality in (7) is obtained. The second inequality in (7) is followed by \( 0 \leq l^\pi_t \leq l_0 \) for all \( \pi \in \Pi \). □

To derive the optimal strategy and the value function, the Quasi-Variational-Inequality (QVI) (see, e.g., [9]) approach is adopted. Before that, we introduce some operators as following. Throughout the paper, a function \( f : D \to \mathbb{R} \) is called sufficiently smooth meaning that it belongs to \( C^1(D) \) if \( X \) is of bounded variation, otherwise it belongs to \( C^2(D) \). Suppose that a sufficiently smooth function \( v \) is a
candidate function for the value function. Let $\mathcal{M}$ denote the injection operator, defined by
\[
\mathcal{M}v(x) = \sup_{y \geq 0} \{v(x + y) - K - \phi y\}.
\]
Then $\mathcal{M}V(x)$ represents the value of the strategy that consists in choosing the best immediate capital injection. In addition, the operators used in this paper $\Gamma$ and $\mathcal{L}^l$ are defined, respectively, by
\[
\Gamma v(x) = \frac{\sigma^2}{2} v''(x) - cv'(x) + \int_0^\infty [v(x + y) - v(x) - v'(x)y1_{[0 < y < 1]}] \nu(dy),
\]
\[
\mathcal{L}^l v(x) = (\Gamma - \delta)v(x) + l(1 - v'(x)), \quad l \in [0, l_0].
\]
By Itô’s formula for semimartingales, we obtain the following lemma, whose proof is presented in Appendix A.

**Lemma 2.1.** Let $v(x)$ be an increasing, concave and sufficiently smooth function on $(0, \infty)$ satisfying
\[
\max\{\max_{0 \leq l \leq l_0} \mathcal{L}^l v(x), \mathcal{M}v(x) - v(x)\} \leq 0, \quad x > 0,
\]
\[
\max\{-v(0) - P, \mathcal{M}v(0) - v(0)\} \leq 0.
\]
Then we have $v(x) \geq V(x)$ as $P \geq -\frac{l_0}{\delta}$.

3. Optimal control problem without capital injection.

3.1. Formulation for the optimal problem without capital injection. We now study the optimal problem without capital injection. For this suboptimal problem, let $\Pi_p = \{\pi_p : \pi_p = (L^\pi_p; 0) \in \Pi\} \subset \Pi$ denote the set of all admissible strategies. The value function associated with $\pi_p$ is defined by
\[
V_p(x) = \sup_{\pi_p \in \Pi_p} V(x; \pi_p) = \sup_{\pi_p \in \Pi_p} E_x^x \left[ \int_0^{T^\pi_p} e^{-\delta s} l_{s}^\pi_p ds - P e^{-\delta T^\pi_p} \right].
\]
The objective is to find the value function $V_p(x)$ and the corresponding optimal strategy $\pi_p^\ast \in \Pi_p$ such that $V_p(x) = V(x; \pi_p^\ast)$. Similar Proposition 2.2, $V_p(x)$ is bounded and satisfies the inequalities (7).

If the value function $V_p(x)$ is sufficiently smooth, by the stochastic control theory, it satisfies the following QVI
\[
\max_{0 \leq l \leq l_0} \{\mathcal{L}^l v(x)\} = 0, \quad x > 0, \quad \text{and} \quad v(0) = -P.
\]
The following Theorem 3.1 and Corollary 3.1 are proved in Appendix B.

**Theorem 3.1.** If an increasing, concave and bounded function $g(x)$ is sufficiently smooth on $(0, \infty)$ and continuous from the right at 0, and satisfies the QVI (11), we have the following statements:
(i) For any strategy $\pi_p \in \Pi_p$, we have $g(x) \geq V(x; \pi_p)$, and so $g(x) \geq V_p(x)$;
(ii) Moreover, if there exists a point $x_p^\ast > 0$ such that
\[
g'(x) \geq 1, x \in (0, x_p^\ast]; \quad g'(x) \leq 1, x \in (x_p^\ast, \infty),
\]
then $g(x) = V_p(x) = V(x; \pi_p^\ast)$, where $\pi_p^\ast = \{L^\pi_p, 0\} \in \Pi_p$ is the optimal strategy such that
\[
X_{t}^\pi_p = X_{t} - L_{t}^\pi_p; \quad L_{t}^\pi_p = \int_0^t l_{s}^\pi_p ds = \int_0^t l_0 1_{\{X_{s}^\pi_p > x_p^\ast\}} ds.
\]
i.e. the optimal dividend strategy \( L^\pi_0 \) is a threshold dividend with parameters \( l_0 \) and \( x_p^* \).

**Corollary 3.1.** If an increasing, concave and bounded function \( g(x) \) is sufficiently smooth on \((0, \infty)\) and continuous from the right at 0, and also satisfies, for some \( x_p^* > 0 \),
\[
\mathcal{L}^0 g(x) = 0, \quad x \in (0, x_p^*], \quad \mathcal{L}^x g(x) = 0, \quad x \in (x_p^*, \infty),
\]
with \( g(0) = -P \), we have \( g(x) = V_p(x) = V(x; \pi_p^*) \), where \( \pi_p^* \) is given by (13).

### 3.2. Threshold dividend strategies.

From the subsection 3.1, the optimal dividend strategy is a threshold dividend strategy under some hypotheses. Then we study the dividend problem when dividends are paid according to a threshold strategy with parameters \( l_0 \) and \( x_p^* > 0 \), i.e. \( \pi_p = \{L^\pi_0; l_0, x_p^*\} \) in (13), where \( x_p^* \) is replaced by \( x_p \). Then the performance function associated with the strategy \( \pi_p \) is
\[
V(x; \pi_p) = E^{x} \left[ \int_0^{T^{\pi}_x} l_0 e^{-\delta s} \mathbf{1}_{\{X^\pi_s > x_p\}} ds - Pe^{-\delta T^{\pi}_x} \right].
\]

Similar to Theorem 3.1 in [18], if \( V(x; \pi_p) \) is sufficiently smooth on \((0, \infty)\) \( \backslash \{x_p\} \), it satisfies the following integro-differential equation
\[
\mathcal{L}^0 V(x; \pi_p) = 0, \quad x \in (0, x_p); \quad \mathcal{L}^x V(x; \pi_p) = 0, \quad x \in (x_p, \infty),
\]
with the initial condition \( V(0; \pi_p) = -P \) and the continuity condition
\[
V(x_p^+; \pi_p) = V(x_p^-; \pi_p) = V(x_p; \pi_p),
\]
and furthermore, if \( X \) is of unbounded variation, \( V(x; \pi_p) \) also satisfies
\[
V'(x_p^+; \pi_p) = V'(x_p^-; \pi_p).
\]

For the uncontrolled surplus process \( X \), let
\[
T^{+}_{x_p} = \inf \{t \geq 0 : X_t > x_p\}; \quad T^{-}_{x_p} = \inf \{t \geq 0 : X_t < 0\}; \\
\Phi_1(\delta) = \sup \{s \geq 0 : \psi(s) + l_0 s = \delta\}.
\]
The following lemma is Theorem 3.2 in [18].

**Lemma 3.1.** For \( x > x_p \), we have
\[
V(x; \pi_p) = \frac{l_0}{\delta} + \left( V(x_p; \pi_p) - \frac{l_0}{\delta} \right) e^{\Phi_1(\delta)(x_p - x)} - l_0 e^{\Phi_1(\delta)(x - x_p)}.
\]

The definitions of scale functions \( W^{(\delta)}(x) \), \( Z^{(\delta)}(x) \) and \( (\delta)(x) \) are presented in Appendix C. Using the method of Theorem 3.3 in [18], we give the following theorem.

**Theorem 3.2.** Let \( \pi_p \) be the threshold dividend strategy with the parameters \( l_0 \) and \( x_p > 0 \), then the performance function \( V(x; \pi_p) \) in (13) is, for \( 0 \leq x \leq x_p \),
\[
V(x; \pi_p) = \frac{l_0}{\delta} Z^{(\delta)}(x_p - x) + \left( V(x_p; \pi_p) - \frac{l_0}{\delta} \right) Q(x_p - x),
\]
where
\[
Q(x) = e^{\Phi_1(\delta)x} + l_0 \Phi_1(\delta) \int_0^x W^{(\delta)}(y) e^{\Phi_1(\delta)(x - y)} dy,
\]
\[
V(x_p; \pi_p) = \frac{l_0}{\delta} - \frac{P + \frac{l_0}{\delta} Z^{(\delta)}(x_p)}{Q(x_p)}.
\]
Remark 3.1. From (21) and (22), we have
\[ V'(x_p^-; \pi_p) = -l_0W(\delta)(0+) - \Phi_1(\delta) \left( V(x_p^-; \pi_p) - \frac{l_0}{\delta} \right) \left( 1 + l_0W(\delta)(0+) \right); \]
\[ V'(x_p^+; \pi_p) = -\Phi_1(\delta) \left( V(x_p^+; \pi_p) - \frac{l_0}{\delta} \right). \]

When \( X \) is of bounded variation, and by (C.1), we have
\[ V(x; \pi_p) \in C^1((0, \infty)) \Leftrightarrow \Phi_1(\delta) \left( V(x_p^-; \pi_p) - \frac{l_0}{\delta} \right) + 1 = 0. \tag{25} \]

When \( X \) is of unbounded variation, we have
\[ V''(x_p^-; \pi_p) = l_0W(\delta)'(0+) + \Phi_1(\delta) \left( V(x_p^-; \pi_p) - \frac{l_0}{\delta} \right) \left( \Phi_1(\delta) + l_0W(\delta)'(0+) \right); \]
\[ V''(x_p^+; \pi_p) = \Phi_1^2(\delta) \left( V(x_p^+; \pi_p) - \frac{l_0}{\delta} \right). \]

Hence, if \( X \) is of unbounded variation, and by (C.2), we obtain \( V(x; \pi_p) \in C^1((0, \infty)) \) and
\[ V(x; \pi_p) \in C^2((0, \infty)) \Leftrightarrow \Phi_1(\delta) \left( V(x_p^-; \pi_p) - \frac{l_0}{\delta} \right) + 1 = 0. \tag{26} \]

3.3. Optimal dividend threshold. By Corollary 3.1, we will search for a threshold strategy \( \pi_p^* \) such that \( V(x_p^*; \pi_p) \) is increasing, concave and sufficiently smooth on \((0, \infty)\). Due to (25) and (26), the optimal dividend threshold \( x_p^* \) satisfies
\[ V(x_p^*; \pi_p) = \frac{l_0}{\delta} - \frac{1}{\Phi_1(\delta)}. \tag{27} \]

By (24), the above equation is equivalent to
\[ \frac{1}{\Phi_1(\delta)} Q(x_p^*) - \frac{l_0}{\delta} Z(\delta)(x_p^*) = P. \tag{28} \]

Lemma 3.2. The equation (28) has a solution \( x_p^* > 0 \) if and only if \( P > \frac{1}{\Phi_1(\delta)} - \frac{l_0}{\delta} \).

Proof. Let \( \theta(z) = \frac{1}{\Phi_1(\delta)} Q(z) - \frac{l_0}{\delta} Z(\delta)(z), \quad z \geq 0. \)

We have
\[ \theta(0) = \frac{1}{\Phi_1(\delta)} - \frac{l_0}{\delta}, \]
\[ \theta'(z) = l_0\Phi_1(\delta) \int_0^z W(\delta)(y)e^{\Phi_1(\delta)(z-y)}dy + e^{\Phi_1(\delta)z} > 0. \]

Furthermore, we obtain \( \theta'(z) > 0 \) and \( \lim_{z \to \infty} \theta(z) = \infty. \) Thus the equation (28) has a unique solution \( x_p^* > 0 \) if and only if \( P > \theta(0) = \frac{1}{\Phi_1(\delta)} - \frac{l_0}{\delta}. \)

Theorem 3.3. (i) If \(-\frac{l_0}{\delta} \leq P \leq \frac{1}{\Phi_1(\delta)} - \frac{l_0}{\delta}, \) we have \( V_p(x) = V(x; \pi_p^*) \) and
\[ V(x; \pi_p^*) = \frac{l_0}{\delta} - \left( P + \frac{l_0}{\delta} \right) e^{-\Phi_1(\delta)x}, \quad x \geq 0, \tag{30} \]
where $\pi^*_p$ is given by (13) and $x^*_p = 0$.

(ii) If $P > \frac{1}{\Phi_1(\delta)} - \frac{L_0}{\delta}$, we have $V_p(x) = V(x; \pi^*_p)$ and

$$V(x; \pi^*_p) = \begin{cases} \frac{l_0}{\delta} - \frac{1}{\Phi_1(\delta)}e^{\Phi_1(\delta)(x^*_p - x)}, & x > x^*_p, \\ \frac{l_0}{\delta}Z(\delta)(x^*_p - x) - \frac{1}{\Phi_1(\delta)}Q(x^*_p - x), & 0 \leq x \leq x^*_p, \end{cases}$$ (31)

where $Q(x)$ is defined by (23), $\pi^*_p$ is given by (13) and $x^*_p$ is determined by (28).

Proof. (i) If $-\frac{L_0}{\delta} \leq P \leq \frac{1}{\Phi_1(\delta)} - \frac{L_0}{\delta}$, we consider the threshold dividend strategy $\pi^*_p$ with parameters $l_0$ and $x^*_p = 0$, and then (30) is obtained from (21). Furthermore, by $V(0; \pi^*_p) = -P$, we have

$$0 \leq V'(x; \pi^*_p) = \Phi_1(\delta) \left(P + \frac{l_0}{\delta}\right) e^{-\Phi_1(\delta)x} \leq 1;$$

$$V''(x; \pi^*_p) = -\Phi^2_1(\delta) \left(P + \frac{l_0}{\delta}\right) e^{-\Phi_1(\delta)x} \leq 0.$$

Then $V(x; \pi^*_p)$ is increasing, concave, and sufficiently smooth on $(0, \infty)$. Recalling (8) and (16), we have $\max_{0 \leq \lambda \leq \delta} \mathcal{L} V(x; \pi^*_p) = \mathcal{L} V(x; \pi^*_p) = 0$. By Theorem 3.1, the results in (i) are obtained.

(ii) If $P > \frac{1}{\Phi_1(\delta)} - \frac{L_0}{\delta}$, the equation (28) has a unique solution $x^*_p > 0$. From (21), (22) and (27), $V(x; \pi^*_p)$ in (31) is obtained. Note that

$$V'(x; \pi^*_p) = \begin{cases} e^{\Phi_1(\delta)(x^*_p - x)}, & x > x^*_p, \\ Q(x^*_p - x), & 0 < x < x^*_p, \end{cases}$$

then $V'(x^*_p - ; \pi^*_p) = V'(x^*_p + ; \pi^*_p) = 1$ and $V'(x; \pi^*_p) > 0$. Furthermore

$$V''(x; \pi^*_p) = \begin{cases} -\Phi_1(\delta)e^{\Phi_1(\delta)(x^*_p - x)}, & x > x^*_p, \\ k(x) - \Phi_1(\delta)e^{\Phi_1(\delta)(x^*_p - x)}, & 0 < x < x^*_p, \end{cases}$$

where $k(x) = -l_0\Phi^2_1(\delta) \int_{x^*_p - x}^{\infty} W(\delta)(y)e^{\Phi_1(\delta)(x^*_p - x - y)}dy - l_0\Phi_1(\delta)W(\delta)(x^*_p - x)$. Then $V''(x^*_p - ; \pi^*_p) = -\Phi_1(\delta) - l_0\Phi_1(\delta)W(\delta)(0)$, $V''(x^*_p + ; \pi^*_p) = -\Phi_1(\delta)$ and $V''(x; \pi^*_p) < 0$ on $(0, x^*_p)$ and $(x^*_p, \infty)$. By (C.1), $V''(x^*_p - ; \pi^*_p) = V''(x^*_p + ; \pi^*_p)$ if and only if the Lévy process $X$ is of unbounded variation. Hence, $V(x; \pi^*_p)$ in (31) is increasing, concave and sufficiently smooth on $(0, \infty)$. Finally, by (16), we know $\mathcal{L} V(x; \pi^*_p) = 0$ for $x \in (0, x^*_p)$ and $\mathcal{L} V(x; \pi^*_p) = 0$ for $x \in (x^*_p, \infty)$. By Corollary 3.1, we obtain the results in (ii).

**Remark 3.2.** From (28), the optimal threshold $x^*_p$ increases as the ruin penalty $P$ increases. The value function $V_p$ in (31) is a decreasing function with respect to $x^*_p$. Combining with (30), the value function $V_p$ defined by (10) decreases with the increase of $P$. There is a reasonable economic explanation for this phenomenon. As the ruin penalty $P$ increases, the company should raise the dividend threshold $x^*_p$ in order to lower the risk of ruin. Certainly, the higher penalty results in the smaller the profits (value function).
4. Optimal problem with forced capital injections to prevent ruin.

4.1. Formulation for the optimal problem without ruin. We assume that the company survives forever by forced capital injections. Let \( \Pi_q \) denote the set of admissible strategies of this suboptimal problem, i.e.,

\[
\Pi_q = \{ \pi_q = (L^{\pi_q}; G^{\pi_q}) : \pi_q \in \Pi \text{ such that } X_t^{\pi_q} > 0 \text{ for all } t \geq 0 \}.
\]

The value function \( V_q(x) \) is defined by

\[
V_q(x) = \sup_{\pi_q \in \Pi_q} V(x; \pi_q) = \sup_{\pi_q \in \Pi_q} E^x \left[ \int_0^\infty e^{-\delta s} t^{\pi_q} ds - \sum_{n=1}^\infty e^{-\delta \tau_n^{\pi_q}} (K + \phi \xi_n^{\pi_q}) \right].
\]

(32)

The objective is to find the value function \( V_q(x) \) and the corresponding optimal strategy \( \pi_q^* \in \Pi_q \) such that \( V_q(x) = V(x; \pi_q^*) \). Similar to Proposition 2.2, \( V_q(x) \) satisfies 0 < \( \pi_q^* \) and satisfies the QVI (33), we have the following statements:

\[
\max_{0 \leq t \leq t_0} \mathcal{L} V_q(x), \mathcal{M} V_q(x) - V_q(x) = 0, \quad \mathcal{M} V_q(0) \leq V_q(0).
\]

(33)

We give the following theorem and corollary, which are proved in Appendix B.

**Theorem 4.1.** If an increase and concave function \( h(x) \) is sufficiently smooth on \((0, \infty)\) and satisfies the QVI (35), we have the following statements:

(i) For any strategy \( \pi_q \in \Pi_q \), \( h(x) \geq V(x; \pi_q) \), and so \( h(x) \geq V_q(x) \) for \( x \geq 0 \);

(ii) Moreover, if there exists two points \( x_q^* > x_p^* \) such that

\[
h'(x) \geq 1, \quad x \in (0, x_p^*) \quad \text{and} \quad h'(x) \leq 1, \quad x \in (x_p^*, \infty),
\]

and

\[
h(0) = h(\eta) - \phi \eta - K; \quad h'(\eta) = \phi,
\]

then \( h(x) = V_q(x) = V(x; \pi_q^*) \), where \( \pi_q^* = (L^{\pi_q^*}, G^{\pi_q^*}) \in \Pi_q \) is the associated optimal strategy such that

\[
0 < X_t^{\pi_q^*} = X_t - L_t^{\pi_q^*} + G_t^{\pi_q^*};
\]

\[
L_t^{\pi_q^*} = \int_0^t I^{\pi_q^*} (X_s^{\pi_q^*} > x_q^*) ds,
\]

\[
G_t^{\pi_q^*} = \sum_{n=1}^\infty \xi_n^{\pi_q^*} 1_{\{\tau_n^{\pi_q^*} \leq t\}}, \quad \xi_n^{\pi_q^*} = \eta, n = 1, 2, \ldots,
\]

\[
\tau_1^{\pi_q^*} = \inf\{ t \geq 0 : X_t^{\pi_q^*} = 0 \}, \tau_n^{\pi_q^*} = \inf\{ t > \tau_{n-1}^{\pi_q^*} : X_t^{\pi_q^*} = 0 \}, n = 2, 3, \ldots.
\]

**Corollary 4.1.** If an increase and concave function \( h(x) \) is sufficiently smooth on \((0, \infty)\), and satisfies

\[
\mathcal{L}^0 h(x) = 0, \quad x \in (0, x_q^*) \quad \text{and} \quad \mathcal{L}^0 h(x) = 0, \quad x \in (x_q^*, \infty),
\]

(37)

and \( [\eta, x_q^*] \) for some pair \( (\eta, x_q^*) \) with \( 0 < \eta < x_q^* \), we have \( h(x) = V_q(x) = V(x; \pi_q^*) \), where \( \pi_q^* \) is given in \( [36] \).
4.2. Optimal strategy for the optimal problem without ruin. From the above arguments, we consider the dividend strategies \( \pi_q \) in [30] with parameters \( x_q > 0 \) and \( \eta \), where \( x_q^* \) is replaced by \( x_q \). The strategy \( \pi_q \) is a band strategy with the upper threshold \( x_q \) and the lower reflecting barrier \( 0 \). Following this strategy, the company’s management takes no action as long as the asset process takes value in \((0, x_q)]\). Whenever the asset value exceeds \( x_q \), dividends are paid at the maximal rate \( l_0 \); The asset immediately jumps to \( \eta \) by injecting capitals once it reaches 0. The performance function associated with the strategy \( \pi_q \) is

\[
V(x; \pi_q) = E^x \left[ \int_0^\infty l_0 e^{-\delta s} \mathbf{1}_{\{X^\pi_q > x_q\}} ds - \sum_{n=1}^\infty e^{-\delta \tau^\pi_n} (K + \phi \xi^\pi_n) \right]. \tag{38}
\]

Similar to the case of the threshold dividend without injecting capital, and by Itô’s formula, if \( V(x; \pi_p) \) is sufficiently smooth on \((0, \infty)\/\{x_p\}\), it satisfies the following integro-differential equation

\[
\mathcal{L}^0 V(x; \pi_q) = 0, \quad x \in (0, x_q); \quad \mathcal{L}^0 V(x; \pi_q) = 0, \quad x \in (x_q, \infty), \tag{39}
\]

with the continuity condition

\[
V(x_q^-; \pi_q) = V(x_q^+; \pi_q) = V(x_q; \pi_q), \tag{40}
\]

and furthermore, if \( X \) is of unbounded variation, \( V(x; \pi_q) \) also satisfies

\[
V'(x_q^-; \pi_q) = V'(x_q^+; \pi_q). \tag{41}
\]

We use the hitting times \( T^- \) in [20] and \( T^+ \) in \( \inf\{t \geq 0 \colon X_t > x_q\} \). Similar to the proof of Theorem 3.2 in [18], we can prove, for \( x > x_q \),

\[
V(x; \pi_q) = \frac{l_0}{\delta} + \left( V(x_q; \pi_q) - \frac{l_0}{\delta} \right) e^{\Phi(\delta)(x_q-x)}. \tag{42}
\]

Now, we consider the performance function \( V(x; \pi_q) \) in (38) for \( 0 \leq x \leq x_q \). By the law of total probability and the strong Markov property, for \( 0 < x < x_q \), we have

\[
V(x; \pi_q) = h_1(x) + h_2(x) V(0; \pi_q). \tag{43}
\]

where \( h_1(x) = E^x \left[ e^{-\delta T^+_\pi} V(X_{T^+_\pi}; \pi_q) \mathbf{1}_{\{T^+_\pi < T^-\}} \right] \) and \( h_2(x) = E^x \left[ e^{-\delta T^-_\pi} \mathbf{1}_{\{T^-_\pi < T^+_\pi\}} \right] \). By the definition of \( \pi_q \), we know

\[
V(0; \pi_q) = V(\eta; \pi_q) - (\phi \eta + K). \tag{44}
\]

By the proof of Theorem 3.3 in [18], we obtain

\[
h_1(x) = \frac{l_0}{\delta} \left( Z^{(\delta)}(x_q - x) - W^{(\delta)}(x_q - x) \frac{Z^{(\delta)}(x_q)}{W^{(\delta)}(x_q)} \right) + \left( V(x_q; \pi_q) - \frac{l_0}{\delta} \right) \left( Q(x_q - x) - \frac{W^{(\delta)}(x_q - x)}{W^{(\delta)}(x_q)} Q(x_q) \right),
\]

where \( Q(x) \) is defined by (23). By (8.8) in [11], we have \( h_2(x) = \frac{W^{(\delta)}(x_q - x)}{W^{(\delta)}(x_q)} \). Substituting \( h_1(x), \ h_2(x) \) into (43), and using the conditions in (40) and (41), we get

\[
\frac{l_0}{\delta} Z^{(\delta)}(x_q) + \left( V(x_q; \pi_q) - \frac{l_0}{\delta} \right) Q(x_q) - V(0; \pi_q) = 0.
\]
Then, for $0 \leq x \leq x_q$, we have
\[ V(x; \pi_q) = \frac{l_0}{\delta} Z^{(\delta)}(x_q - x) + \left( V(x_q; \pi_q) - \frac{l_0}{\delta} \right) Q(x_q - x). \]

By \[(42)\] and Remark 3.1, in order that the above $V(x; \pi_q)$ is sufficiently smooth on $(0, \infty)$, we set $V(x_q; \pi_q) = \frac{l_0}{\delta} - \frac{1}{\Phi_1(\delta)}$. Therefore, we obtain
\[ V(x_q; \pi_q) = \frac{l_0}{\delta} Z^{(\delta)}(x_q - x) - \frac{1}{\Phi_1(\delta)} Q(x_q - x). \]

**Theorem 4.2.** The value function in \[(32)\] satisfies $V_q(x) = V(x; \pi_q^*)$, where $V(x; \pi_q^*)$ is given by
\begin{equation}
V(x; \pi_q^*) = \begin{cases}
\frac{l_0}{\delta} Z^{(\delta)}(x_q^* - x), & x > x_q^*, \\
\frac{l_0}{\delta} Z^{(\delta)}(x_q^*-x) - \frac{1}{\Phi_1(\delta)} Q(x_q^*-x), & 0 \leq x \leq x_q^*,
\end{cases}
\end{equation}

and $Q$ is defined by \[(23)\], and the optimal strategy $\pi_q^*$ is given in \[(36)\] with parameters $x_q^*$ and $\eta$ determined by
\begin{align}
Q(x_q^*-\eta) &= \phi, \tag{46} \\
\frac{l_0}{\delta} \left( Z^{(\delta)}(x_q^*-\eta) - Z^{(\delta)}(x_q^*) \right) + \frac{1}{\Phi_1(\delta)} (Q(x_q^*) - Q(x_q^*-\eta)) &= \phi \eta + K. \tag{47}
\end{align}

**Proof.** By the discussions at the beginning of this subsection, if there exist $0 < \eta < x_q^*$ satisfying the equations \[(46)\] and \[(47)\], the performance function $V(x; \pi_q^*)$ associated with $\pi_q^*$ \[(36)\] is given by \[(45)\]. Similar to $V(x; \pi_p^*)$ in \[(31)\], we can show that $V(x; \pi_q^*)$ is increasing, concave and sufficiently smooth on $(0, \infty)$. Moreover, by \[(39)\], \[(46)\] and \[(47)\], we know that $V(x; \pi_q^*)$ satisfies \[(35)\] and \[(37)\]. The results are obtained by Corollary 4.1. Therefore, we only need to prove that there exists a pair of $(\eta, x_q^*)$ that \[(46)\] and \[(47)\] with $0 < \eta < x_q^*$.

Making the change of variable $z = x_q - \eta$, we can rewrite \[(46)\] as $Q(z) = \phi$. Recalling the definition in \[(23)\], we have
\[ Q(0) = 1 < \phi; \quad Q'(z) > 0 \quad (z > 0); \quad \lim_{y \to \infty} Q(y) = \infty. \]

Then there exists a unique $x_1 > 0$ such that $Q(x_1) = \phi$. We define a function in $x_q$ as follows, for $x_q \geq x_1$,
\[ \alpha(x_q) = \frac{l_0}{\delta} \left( Z^{(\delta)}(x_1) - Z^{(\delta)}(x_q) \right) + \frac{1}{\Phi_1(\delta)} (Q(x_q) - Q(x_1)) - \phi(x_q - x_1) - K. \tag{48} \]

Then the equation \[(47)\] can be rewritten as $\alpha(x_q) = 0$. Due to
\[ \alpha(x_1) = -K < 0; \quad \lim_{x_q \to \infty} \alpha(x_q) = \infty; \]
\[ \alpha'(x_q) = \alpha(x_q) - \phi > \alpha(x_1) - \phi = 0, \quad x_q > x_1, \]
we know that there exists an $x_q^* > x_1$ such that $\alpha(x_q^*) = 0$, so $\eta = x_q^* - x_1$ is also determined. \(\square\)
5. Optimal joint dividend and capital injection strategy. From the definitions of $V_p$, $V_q$ and $V$, we can easily get the relationship

$$V(x) \geq \max(V_p(x), V_q(x)), \quad x \geq 0.$$  

Lemma 5.1. For initial capital $x \geq 0$, if the functions $g(x)$ and $h(x)$ satisfy the conditions of Theorem 3.1 and Theorem 4.1, respectively, then

(i) If $\mathcal{M} g(0) \leq g(0)$, we have $g(x) = V(x)$ and the optimal strategy $\pi^* = \pi^*_p$ given by (13);

(ii) If $h(0) \geq -P$, we have $h(x) = V(x)$ and the optimal strategy $\pi^* = \pi^*_q$ given by (36);

(iii) In particular, if the conditions in (i) and (ii) hold, we have $g(x) = h(x) = V(x)$ and the optimal strategy $\pi^* = \pi^*_p$ or $\pi^* = \pi^*_q$.

Proof. (i) If $g'(0) \leq \phi$, $\mathcal{M} g(x) = g(x) - K < g(x)$. If $g'(0) > \phi$, we let

$$\bar{\eta} = \sup\{y > 0 : g'(y) \geq \phi\},$$

then $0 < \bar{\eta} < x^*_p$ and $g'(\bar{\eta}) = \phi$. Similar to (B.3), we have $\mathcal{M} g(x) \leq g(x)$ for all $x \geq 0$. Then we have $g(x) \geq V(x)$ by Lemma 2.1. Due to $g(x) = V_p(x) \leq V(x)$ by Theorem 3.1, the results are obtained.

(ii) The results are derived by Lemma 2.1 and Theorem 4.1.

(iii) In this case, there is no difference between injecting capital to rescue the company and declaring ruin on the edge of ruin. In fact, it is not necessary to rescue the company by capital injections when the surplus hits 0, because it can not profit from capital injections. Then the optimal strategy $\pi^* = \pi^*_p$.

Lemma 5.2. Let $g(x) = V(x; \pi^*_p)$ and $h(x) = V(x; \pi^*_q)$ given by (31) and (33), respectively, we have

(i) $\mathcal{M} g(0) \leq g(0)$ if and only if $x^*_p \leq x^*_q$;

(ii) $h(0) \geq -P$ if and only if $x^*_p \geq x^*_q$.

Proof. (i) Since $\mathcal{M} g(0) - g(0) = \max_{y \geq 0} \{g(y) - \phi y - K - g(0)\}$ and $g'(y) - \phi < 0$ for $x \geq x^*_p$, we have

$$0 \leq \bar{\eta} = \arg \max_{y \geq 0} [g(y) - \phi y - K - g(0)] < x^*_p.$$  

By the concavity of $g(x)$, we obtain

$$\bar{\eta} = 0 \text{ if and only if } g'(0) \leq \phi;$$

$$0 < \bar{\eta} < x^*_p \text{ if and only if } g'(\bar{\eta}) = \phi.$$  

We first consider the case of $\bar{\eta} = 0$. Recalling the function $Q$ in (29), we have

$$g'(0) = l_0 \Phi_1(\delta) \int_0^{x^*_p} W^\delta(y) e^{\Phi_1(\delta)(x^*_p - y)} \delta y + e^{\Phi_1(\delta)x^*_p} = Q(x^*_p).$$

By the increase of $Q$ and $Q(x_1) = \phi$, we conclude $g'(0) \leq \phi$ if and only if $x^*_p \leq x_1$, and so $\mathcal{M} g(0) - g(0) = g(0) - K - g(0) < 0$ if and only if $x^*_p \leq x_1 \leq x^*_q$.

In the case of $0 < \bar{\eta} < x^*_p$, we have $g'(\bar{\eta}) = \phi$, i.e., $Q(x^*_p - \bar{\eta}) = \phi$, and then $x^*_p - \bar{\eta} = x_1$. Note

$$\mathcal{M} g(0) - g(0) = g(\bar{\eta}) - \phi \bar{\eta} - K - g(0) = \alpha(x^*_p),$$

where $\alpha$ is defined by (48). By the increase of $\alpha$ and $\alpha(x^*_p) = 0$, we obtain $\mathcal{M} g(0) - g(0) \leq 0$ if and only if $x^*_p \leq x^*_q$.

(ii) By the definition of $h$, we have $h(0) = -\theta(x^*_q)$, where $\theta$ is defined in (29). Due to the increase of $\theta$, we have $h(0) \geq -P = g(0) = -\theta(x^*_q)$ if and only if $x^*_p \geq x^*_q$.  \[\square\]
Theorem 5.1. For the general optimal control problem in Section 2,
(i) If \(-\frac{\nu}{\lambda} \leq P \leq \frac{1}{\Phi_{1}(\delta)} - \frac{\nu}{\lambda}\), we have the value function \(V(x) = V_{p}(x) = V(x; \pi_{p}^{*})\) given by (30) and the optimal control strategy \(\pi^{*} = \pi_{p}^{*}\) given by (13) and the optimal threshold \(x' = x_{p}^{*} = 0\); 
(ii) If \(P > \frac{1}{\Phi_{1}(\delta)} - \frac{\nu}{\lambda}\) and \(x_{p}^{*} \leq x_{q}^{*}\), we have the value function \(V(x) = V_{p}(x) = V(x; \pi_{p}^{*})\) given by (30) and the optimal control strategy \(\pi^{*} = \pi_{p}^{*}\) given by (13) and the optimal threshold \(x' = x_{p}^{*} > 0\);
(iii) If \(P > \frac{1}{\Phi_{1}(\delta)} - \frac{\nu}{\lambda}\) and \(x_{p}^{*} > x_{q}^{*}\), we have the value function \(V(x) = V_{q}(x) = V(x; \pi_{q}^{*})\) given by (41) and the optimal control strategy \(\pi^{*} = \pi_{q}^{*}\) given by (50) the optimal threshold \(x' = x_{q}^{*} > 0\);
In other words, the value function \(V(x) = \max\{V_{p}(x), V_{q}(x)\}\) and the optimal threshold \(x' = \min\{x_{p}^{*}, x_{q}^{*}\}\).

Proof. If \(-\frac{\nu}{\lambda} \leq P \leq \frac{1}{\Phi_{1}(\delta)} - \frac{\nu}{\lambda}\), \(V_{p}(x) = V(x; \pi_{p}^{*})\) is given by (30). Due to \(V_{p}(x) \leq 1\), it follows that \(\mathbb{M} V_{p}(x) = V_{p}(0) - K < V_{p}(x)\) for all \(x \geq 0\). Then \(V_{p}(x)\) satisfies (9). We obtain \(V_{p}(x) \geq V(x)\) from Lemma 2.1. The results in (i) are proved. By Lemma 5.1 and Lemma 5.2, the results in (ii) and (iii) are obtained.

6. Numerical examples. Recently, in [8], the authors considered the spectrally negative phase-type Lévy process, whose scale function admits an analytical expression; they proposed an approach to approximate the scale function for a general spectrally negative Lévy process. The numerical results of this paper are based on their approximation method. For simplicity, we discuss the cases of the Lévy process \(X\) with hyper-exponential and Gamma distributed compound Poisson positive jumps, respectively.

We start from the case where \(X\) has hyper-exponential compound Poisson positive jumps, i.e.,
\[
\nu(dy) = \lambda \sum_{i=1}^{n} w_{i} \alpha_{i} e^{-\alpha_{i}y} dy \quad (y \geq 0),
\]
where \(\lambda > 0\), \(0 < \alpha_{1} < \alpha_{2} < \cdots < \alpha_{n}\), and \(w_{i} > 0\) for \(1 \leq i \leq n\) such that \(\sum_{i=1}^{n} w_{i} = 1\). The Laplace exponent \(\psi(s)\) is then
\[
\psi(s) = \frac{\sigma^{2}}{2} s^{2} + c_{0} s + \lambda \sum_{i=1}^{n} w_{i} \frac{\alpha_{i}}{\alpha_{i} + s} - \lambda.
\]
If \(\sigma > 0\), the equation \(\psi(s) = \delta\) has \(n + 2\) roots, denoted by \(r_{0} > 0, r_{1}, r_{2}, \cdots, r_{n+1}\), and
\[
r_{n+1} < -\alpha_{n} < r_{n} < -\alpha_{n-1} < \cdots < r_{1} < 0 < r_{0}.
\]
Letting \(\Phi_{1}(\delta) = s_{0}\), and by partial fraction decomposition and some simplifications, we have
\[
\frac{1}{s_{0}} = \frac{t_{0} \prod_{i=1}^{n} \frac{\alpha_{i} + s_{0}}{\alpha_{i}}}{\delta} \prod_{j=0}^{n} \frac{r_{j}}{r_{j} - s_{0}}; \tag{49}
\]
\[
W(\delta)(x) = \frac{1}{\delta} \sum_{k=0}^{n+1} r_{k} \prod_{i=1}^{n} \frac{\alpha_{i} + r_{k}}{\alpha_{i}} \prod_{j=0,j \neq k}^{n+1} \frac{r_{j}}{r_{j} - r_{k}} e^{r_{k}x}, \quad x > 0;
\]
\[
Z(\delta)(x) = \sum_{k=0}^{n+1} \prod_{i=1}^{n} \frac{\alpha_{i} + r_{k}}{\alpha_{i}} \prod_{j=0,j \neq k}^{n+1} \frac{r_{j}}{r_{j} - r_{k}} e^{r_{k}x}, \quad x > 0.
\]
Furthermore, letting
\[ p(x) = \frac{l_0}{\delta} Z(x) - \frac{1}{s_0} Q(x), \quad x > 0, \]
and by some complicated simplifications, we obtain
\[ p(x) = \frac{l_0}{\delta} \sum_{k=0}^{n+1} \frac{s_0}{s_0 - r_k} \prod_{i=1}^{n} \frac{\alpha_i + r_k}{\alpha_i} \prod_{j=0, j \neq k}^{n+1} \frac{r_j}{r_j - r_k} e^{r_k x}, \quad x > 0. \quad (50) \]
Substituting (49) and (50) into (30), (31) and (45), and letting \( P = 0 \), we obtain the results in [17].

In the following, we assume \( \nu(dy) = 3ye^{-y}dy, \quad y \geq 0 \), to illustrate our results. That is, the size of jumps follows the Gamma(2,1) distribution, and the number of jumps follows the Poisson process with parameter 3. In the following section, we will discuss the influences of \( P, \phi, K, l_0, \delta \), and \( \sigma \) on the optimal strategy and the value function.

- **The influence of \( P \)**

We first consider the impact of penalty \( P \) on the optimal threshold. Let \( c_0 = 5, \sigma = 0, \delta = 0.1, l_0=1, \phi = 1.1 \) and \( K = 0.1 \). Then \( \Phi_1(\delta) = 0.113146 \) and \( \frac{1}{\Phi_1(\delta)} - \frac{l_0}{\delta} = 0.1619 \). Due to (i) in Theorem 5.1, it follows that \( x^* = x_p^* = 0 \) if \( P \in \mathcal{I} = [-10, -1.1619] \). From (46) and (49), it yields that \( x_p^* = 1.7590 \). We give the values of \( x_p^* \) in Table 1 as the penalty \( P \) varies. It shows that \( x_p^* \) increases with the increase of \( P \) from Table 1. The optimal strategy \( \pi^* \) switches from \( \pi_p^* \) to \( \pi_q^* \) with the increase of the penalty. The management would choose to avoid ruin by injecting capital for large penalty. Here, the maximum penalty \( P \) that the company can afford is 0.8380. In other word, when \(-\infty < P \leq 0.8380\), we have \( x^* = x_p^* \leq 1.7590 \) and \( V(x) = V_p(x) \); and when \( P > 0.8380 \), we have \( x^* = x_q^* = 1.7590 \) and \( V(x) = V_q(x) \). Certainly, the value function decreases with the increase of the penalty because the value function is decreasing with respect to the level of threshold.

| \( P \uparrow \) | \( \mathcal{I} \) | -1 | 0 | 0.5 | 0.8380 | 1 | 1.4 | 1.5 |
|---|---|---|---|---|---|---|---|---|
| \( x_p^* \uparrow \) | 0 | 0.1601 | 1.0765 | 1.4922 | 1.7590 | 1.8830 | 2.1794 | 2.2509 |
| \( x_q^* \equiv \) | 1.7590 | 1.7590 | 1.7590 | 1.7590 | 1.7590 | 1.7590 | 1.7590 | 1.7590 |
| \( x^* \uparrow \) | \( x_p^* \) | \( x_p^* \) | \( x_p^* \) | \( x_p^* \) | \( x_q^* = x_q^* \) | \( x_q^* \) | \( x_q^* \) | \( x_q^* \) |

- **The influences of \( \phi \) and \( K \)**

Let \( c_0 = 5, \sigma = 0, \delta = 0.1 \) and \( l_0 = P = 1 \). It can verify that these parameters satisfy \( P > \frac{1}{\Phi_1(\delta)} - \frac{l_0}{\delta} \). By Table 1, we know the optimal threshold \( x_p^* = 1.8830 \). From Table 2, the thresholds \( x_q^* \) and \( x^* \) increase with the increase of \( \phi \) or \( K \), while the amount of capital injection \( \eta \) increases when \( \phi \) decreases or \( K \) increases. Then the value function decreases as \( \phi \) or \( K \) increases due to (31) and (45). Larger \( \phi \) or \( K \) means higher costs of capital injection, the company would reserve more money instead of paying more dividends in order to reduce or avoid capital injection, which calls for higher dividend threshold. As \( K \) increases or \( \phi \) decreases, it would increase the amount of capital injection \( \eta \) so as to cut down costs of capital
injection. The optimal strategy $\pi^*$ switches from $\pi_q^*$ to $\pi_p^*$ with the increase of $\phi$ or $K$. Furthermore, when $\phi = 1.1$ and $K \geq 0.1256$; or when $K = 0.1$ and $\phi \geq 1.1226$, the optimal strategy $\pi^* \equiv \pi_p^*$, i.e. the company prefer declaring ruin to injecting capital whenever it is on the edge of ruin.

**Table 2. The influences of $\phi$ and $K$ on $\eta$, $x_q^*$ and $x_p^*$**

| $K$ | $\phi = 1.1$ | $\phi = 1.256$ | $\phi = 1.4$ | $K = 0.1$ | $\phi = 1.12$ | $\phi = 1.1226$ | $\phi = 1.14$ |
|-----|--------------|----------------|--------------|-----------|--------------|----------------|--------------|
| $\eta$ | $1.1753$ | $1.2011$ | $1.2649$ | $\downarrow$ | $1.0623$ | $1.0604$ | $1.0481$ |
| $x_q^*$ | $1.8572$ | $1.8830$ | $1.9467$ | $\uparrow$ | $1.8687$ | $1.8830$ | $1.9755$ |
| $x_p^*$ | $x_q^* = x_p^*$ | $x_q^* = x_p^*$ | $x_p^*$ | $x_q^* = x_p^*$ | $x_q^* = x_p^*$ | $x_q^* = x_p^*$ | $x_q^* = x_p^*$ |

**The influence of $l_0$**

We now consider the effect of the dividend rate ceiling $l_0$ on the optimal control problem. Let $c_0 = 5$, $\sigma = 0$, $\phi = 1.2$, $K = 0.2$ and $P = 1.5$. We plot the levels of the optimal dividend thresholds and the value functions under the condition $P > \frac{1}{\Psi(\phi)} = \frac{1}{2}$. From the left figure in Figure 1, larger $l_0$ results in the higher levels of $\eta$, $x_q^*$ and $x_p^*$. When $l_0$ increases, the dividend may be paid at a relatively rapid speed. In contrast to early dividend payout, it is better to reserve more money to hedge against financial risks. The optimal strategy $\pi^*$ switches from $\pi_p^*$ to $\pi_q^*$ as $l_0$ increases. The management becomes optimistic over the company’s prospects with the increase of $l_0$, he would choose to inject capital if $l_0$ is large enough. The more benefits (larger value function) are obtained for the higher level of dividend rate, which is implied from the right figure of Figure 1.

**Figure 1.** LEFT: The influence of $l_0$ on $\eta$, $x_p^*$, $x_q^*$ and $x^*$. RIGHT: The influence of $l_0$ on the value function.

**The influence of $\delta$**

We assume that $c_0 = 5$, $\sigma = 0$, $P = l_0 = 1$, $\phi = 1.1$ and $K = 0.1$. From the left figure in Figure 2, it follows that the larger force of interest results in the lower levels of $\eta$, $x_q^*$ and $x_p^*$, but the change of $\eta$ is not sensible with the increase of $\delta$. When the time discounted value is larger, it should be earlier to dividend, which calls for

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lower dividend thresholds. Certainly, the increase of discounted value decreases the profits, which is shown in the right figure of Figure 2. As $\delta$ increases, the optimal strategy $\pi^*$ switches from $\pi^*_q$ to $\pi^*_p$. In another word, as $\delta$ is higher, it would be better not to inject capital on the edge of ruin since the present value of the costs by capital injections would far outweigh the present value of the dividends distributed.

**The influence of $\sigma$**

Finally, we discuss the impact of $\sigma$ on the optimal control problem. Let $c_0 = 5$, $\delta = 0.1$, $l_0 = 1$, $P = 0.5$, $\phi = 1.02$ and $K = 0.12$. As $\sigma$ increases, $x^*_p$ decreases while $x^*_q$ and $\eta$ increase. Larger volatility brings the company higher risk which calls for a lower threshold $x^*_p$ to distribute a greater proportion of the surplus as dividend. As $\sigma$ increases, the cash flow of the company becomes more and more wobbly, if the ruin is forbidden, the threshold level $x^*_q$ and $\eta$ have to increase to delay the coming of injecting. The optimal strategy $\pi^*$ switches from $\pi^*_q$ to $\pi^*_p$ as $\sigma$ increases. That is, if the volatility is large enough, it may prefer declaring ruin to rescuing it by capital injections whenever it is on the edge of ruin. Certainly, the value function decreases with the increase of volatility.
Appendix

A. Proof of Lemma 2.1. Similar to Lemma 3.2 in [13], we first give the following lemma.

**Lemma A.1.** It is optimal to postpone the capital injection as long as possible, i.e., if capital injection occurs, it happens only at the moment when the surplus process hits the barrier 0.

**Proof of Lemma 2.1** Due to Lemma A.1 and that there are only positive jumps for the reserve process, it only needs to consider the strategies such that the jumps of capital injection and the positive jumps of $X$ do not occur at the same time. For a policy $\pi \in \Pi$, define

$\Lambda_t = \{ s \leq t : X_{s-} \neq X_s \}$,

$\Lambda_i^\pi = \{ s \leq t : G^\pi_{s-} \neq G^\pi_{s} \} = \{ \tau^\pi_t : \tau^\pi_t \leq t, i = 1, 2, \ldots \}$.

Let $\{T_m\}_{m \geq 1}$ be a sequence of stopping times defined by $T_m = \inf\{ t \geq 0 : X_t^\pi > m \text{ or } X_t^\pi < \frac{1}{m} \}$. Since the controlled surplus process $X^\pi$ is a semi-martingale, and recalling the sufficient smoothness of $v$, we apply Itô’s formula on $e^{-\delta(t\wedge T_m)}v(X_{t\wedge T_m}^\pi)$ to have

$$
e^{-\delta(t\wedge T_m)}v(X_{t\wedge T_m}^\pi)
= v(x) + \int_0^{t\wedge T_m} e^{-\delta s} \left[\frac{\sigma^2}{2} v''(X^\pi_s) - \delta v(X^\pi_s) \right] ds + \int_0^{t\wedge T_m} e^{-\delta s} v'(X^\pi_s) dX^\pi_s
+ \sum_{s \in (\Lambda_{t\wedge T_m} \cup \Lambda_{t\wedge T_m}^\pi)} e^{-\delta s} [v(X^\pi_{s-}) - v(X^\pi_{s}) - v'(X^\pi_{s}) \Delta X^\pi_{s}]
+ \left\{ \int_0^{t\wedge T_m} e^{-\delta s} v'(X^\pi_s) dX_s + c_s - \sum_{u \in \Lambda_s} \Delta X_u 1_{\{|\Delta X_u| > 1\}} \right\}
+ \left\{ \sum_{s \in \Lambda_{t\wedge T_m}^\pi} e^{-\delta s} [v(X^\pi_{s-} + \xi^\pi_s) - v(X^\pi_{s-})] - \int_0^{t\wedge T_m} \int_0^{\infty} e^{-\delta s} [v(X^\pi_{s-} + y) - v(X^\pi_{s-}) - v'(X^\pi_{s}) y 1_{\{|y| < 1\}}] \nu(dy) ds \right\}.
$$

By the Lévy-Itô decomposition the expression between the first pair of curly brackets is a zero-mean martingale, and by the compensation formula the expression between the second pair of curly brackets is also a zero-mean martingale. Hence we derive that

$$
e^{-\delta(t\wedge T_m)}v(X_{t\wedge T_m}^\pi)
= v(x) + \int_0^{t\wedge T_m} e^{-\delta s} \left[\frac{\sigma^2}{2} v''(X^\pi_s) - \delta v(X^\pi_s) \right] ds + \int_0^{t\wedge T_m} e^{-\delta s} v'(X^\pi_s) dX^\pi_s
+ \sum_{\tau^\pi_t \leq t\wedge T_m} e^{-\delta \gamma \tau^\pi_t} [v(X^\pi_{\tau^\pi_t-} + \xi^\pi_t) - v(X^\pi_{\tau^\pi_t-})] + M_t \tag{A.1}.$$
where \( \{M_t\}_{t \geq 0} \) is a zero-mean martingale. By (6), we have \( \mathcal{L}^l v(x) \leq 0 \) for \( l \in [0, l_0] \) and \( v(x + y) - v(x) \leq cy + K \). Noting that \( v(X_{t \wedge T_m}^\pi) \geq v(0) \geq -P \), and taking expectations on both sides of (A.1), we obtain

\[
v(x) \geq E^x \left[ \int_0^{t \wedge T_m} e^{-\delta s} l_s^s \, ds - Pe^{-\delta (t \wedge T_m)} - \sum_{\tau \leq t \wedge T_m} e^{-\delta \tau} (\phi \xi_\tau + K) \right]
\]

Due to the inequalities (7), we can assume that the right-hand side of the above equation is bounded for large enough \( t \) and \( m \). Letting \( t \) and \( m \) go to infinity, using the dominated convergence theorem, and noting (3.5), we have

\[
\lim_{t \to \infty} \lim_{m \to \infty} E^x \left[ \int_0^{t \wedge T_m} e^{-\delta s} l_s^s \, ds - Pe^{-\delta (t \wedge T_m)} - \sum_{\tau \leq t \wedge T_m} e^{-\delta \tau} (\phi \xi_\tau + K) \right] = 0.
\]

Thus (B.1) can be written to

\[
e^{-\delta (t \wedge T_m)} g(X_{t \wedge T_m}^\pi)
\]

\[
= g(x) + \int_0^{t \wedge T_m} e^{-\delta s}[\Delta g(X_{s}^\pi) - l_s^s g'(X_{s}^\pi) + l_s^s] \, ds
\]

\[
= g(x) - \int_0^{t \wedge T_m} l_s^s \, ds + M_t.
\]

Since \( g(x) \) is bounded, \( T_m \to T_\pi^\pi \) \( (m \to \infty) \), \( X_{T_m}^\pi = 0 \) a.s., and \( g(0) = -P \), by the dominated convergence theorem, we have

\[
\lim_{t,m \to \infty} E^x \left[ e^{-\delta (t \wedge T_m)} g(X_{t \wedge T_m}^\pi) \right] = E^x \left[ g(0)e^{-\delta T_\pi^\pi} \right] = -E^x \left[ Pe^{-\delta T_\pi^\pi} \right].
\]

Taking expectations and letting \( m,t \to \infty \) on both sides of the above equation, we have

\[
g(x) = V(x; \pi^*_p) = V_p(x).
\]

**Proof of Corollary 3.1** Due to the sufficient smoothness of \( g(x) \) and \( \mathcal{L}^0 g(x) = \mathcal{L}^{l_0} g(x) = 0 \), it follows that \( g'(x^*_p) = 1 \). Furthermore, the concavity of \( g \) implies that

\[
g'(x) \geq 1, x \in [0, x^*_p]; \quad g'(x) \leq 1, x \in (x^*_p, \infty),
\]
which together with (14) yields
\[
\max_{0 \leq t \leq L_0} \{ L^t g(x) \} = \begin{cases} 
L^0 g(x) = 0, & 0 \leq x \leq x^*_p; \\
L^{L_0} g(x) = 0, & x > x^*_p. 
\end{cases}
\]
The results are followed from Theorem 3.1.

\textbf{Proof of Theorem 4.1}  The proof of (i) is similar to that of Lemma 2.1. Now we only give the proof of (ii). Since \( h \) is increasing and concave, and satisfies (35), the function \( F(y) = h(x + y) - \phi y - K \) is increasing in \([0, \eta - x]\) and decreasing in \([\eta - x, \infty)\) for \( 0 \leq x \leq \eta \). Then
\[
\mathcal{M} h(x) = \begin{cases} 
 h(\eta) - \phi(\eta - x) - K, & 0 \leq x \leq \eta, \\
 h(x) - K, & \eta < x < \infty, 
\end{cases}
\]
Furthermore, \( \mathcal{M} h(x) < h(x) \) for \( x > 0 \) and \( \mathcal{M} h(0) = h(0) \). Due to (33), it follows that \( \max_{0 \leq t \leq L_0} \mathcal{L}^t h(x) = 0 \) for \( x \in (0, \infty) \). Similar to (B.2),
\[
\int_0^t e^{-\delta s}[\Gamma - \delta]h(X^\pi_s^x) - t^\pi_s^x h'(X^\pi_s^x) + l^\pi_s^x]ds = 0.
\]
Under the strategy \( \pi^*_q \) given by (36),
\[
\sum_{\tau^*_n \leq t} e^{-\delta \tau^*_n} [h(X^\pi_s^x + \xi^\pi_s^x) - h(X^\pi_s^x)] = \sum_{\tau^*_n \leq t} e^{-\delta \tau^*_n} [h(\xi^\pi_s^x) - h(0)] \\
= \sum_{\tau^*_n \leq t} e^{-\delta \tau^*_n} (\phi \xi^\pi_s^x + K),
\]
Thus the equation (A.1) can be written as
\[
e^{-\delta t} h(X^\pi_t^x) = h(x) - \int_0^t e^{-\delta s} l^\pi_s^x ds + \sum_{\tau^*_n \leq t} e^{-\delta \tau^*_n} (\phi \xi^\pi_s^x + K) + M_t.
\]
Taking expectation and letting \( t \to \infty \) on both sides of the above equation, it yields \( h(x) = V(x; \pi^*_q) = V^*_q(x) \).

\textbf{Proof of Corollary 4.1}  By the proof of Corollary 3.1, we know that \( h(x) \) satisfies (34) and that \( \max_{0 \leq t \leq L_0} \mathcal{L}^t h(x) = 0 \) for \( x \in (0, \infty) \). From the proof of Theorem 4.1, we have \( \mathcal{M} h(x) \leq h(x) \) for \( x \in (0, \infty) \) and \( \mathcal{M} h(0) = h(0) \), then \( h(x) \) satisfies the QVI (33). The results are followed from Theorem 4.1.

\textbf{C. Introduction of scale function.}  We now recall the definition of the q-scale function for the spectrally positive Lévy process \( X \), whose Laplace exponent \( \psi \) is given by (1). For \( q > 0 \), there exists a continuous and strictly increasing function \( W(q) : \mathbb{R} \to [0, \infty) \), called the q-scale function defined in such a way that \( W(q)(x) = 0 \) for all \( x < 0 \) and on \([0, \infty)\) its Laplace transform is given by
\[
\int_0^\infty e^{-sx} W(q)(x)dx = \frac{1}{\psi(s) - q}, \quad s > \Phi(q),
\]
where \( \Phi(q) := \sup \{ s \geq 0 : \psi(s) = q \} \). Note that the Laplace exponent \( \psi \) in (1) is known to be zero at the origin, convex on \( \mathbb{R}_+ \). Then \( \Phi(q) \) is well-defined and is
strictly positive as \( q > 0 \). We give the function \( Z^{(q)}(x) \), closely related to \( W^{(q)}(x) \), by

\[
Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)dy, \quad x \in \mathbb{R},
\]
and its anti-derivative

\[
Z^{-1}(q)(x) = \int_0^x Z(q)(y)dy = x + q \int_0^x \int_0^y W^{(q)}(z)dz, \quad x \in \mathbb{R}.
\]

Noting that \( W^{(q)}(x) \) is uniformly zero on the negative half line, we have \( Z^{(q)}(x) = 1 \) and \( Z^{-1}(q)(x) = x \) for \( x \leq 0 \).

From [7], we know the following facts about the scale function. If \( X \) has paths of bounded variation, we have that \( W^{(q)} \in C^1((0, \infty)) \) if and only if the Lévy measure \( \nu \) has no atoms. Particularly, if \( \nu \) is absolutely continuous with respect to Lebesgue measure, \( W^{(q)} \in C^1((0, \infty)) \). In the case that \( X \) has paths of unbounded variation, it is known that \( W^{(q)} \in C^1((0, \infty)) \). Moreover, if \( \sigma > 0 \), \( C^1((0, \infty)) \) may be replaced by \( C^2((0, \infty)) \). Hence, \( Z^{(q)} \in C^1((0, \infty)) \), \( Z^{-1}(q) \in C^1(\mathbb{R}) \) and \( Z^{-1}(q) \in C^2((0, \infty)) \) for the bounded variation case, while \( Z^{(q)} \in C^1(\mathbb{R}) \), \( Z^{-1}(q) \in C^2((0, \infty)) \), \( Z^{-1}(q) \in C^3((0, \infty)) \) for the unbounded variation case. Considering the asymptotic behavior near zero, we have

\[
W^{(q)}(0+) = \begin{cases} 
0, & \text{if } X \text{ is of unbounded variation,} \\
\frac{1}{c_0}, & \text{if } X \text{ is of bounded variation.}
\end{cases}
\]

and

\[
W^{(q)'}(0+) = \begin{cases} 
\frac{2}{\sigma^2}, & \text{if } \sigma > 0, \\
\infty, & \text{if } \sigma = 0 \text{ and } \nu(0, \infty) = \infty, \\
\frac{q + \nu(0, \infty)}{c_0^2}, & \text{if } X \text{ is of bounded variation.}
\end{cases}
\]

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