The Spatial Scaling Laws of Compressible Turbulence

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This article proposed spatial scaling laws of the kinetic energy spectrum of compressible turbulence and its mass-weighted counterpart in terms of wavenumber. The study shown that the compressible turbulence kinetic and the mass-weighted energy spectrum does not behaves the complete similarity, but incomplete similarity as in equation (15) and (21), and both energy spectrum in $k^{-5/3}$ may still be preserved at small scales ($k\eta \to 0$) for compressible turbulence.

Keywords: compressible turbulence, energy spectrum, spatial scaling laws

INTRODUCTION

Compressible flow is the area of fluid mechanics that deals with fluids in which the fluid density varies significantly in response to a change in pressure. Compressibility effects are typically considered significant if the Mach number (the ratio of the flow velocity to the local speed of sound) of the flow exceeds 0.3, or if the fluid undergoes very large pressure changes. Compressible turbulence is of fundamental importance in a wide range of engineering flows and natural phenomena including high-temperature reactive flows, hypersonic aircrafts, solar plasma, interstellar clouds and star formation in galaxies. However, the basic processes occurring in compressible turbulence are less understood as compared to those occurring in incompressible turbulence. It is important to understand the physics behind of interscale transfer of kinetic energy in compressible turbulence [1–6, 12–30]. In this article, we will use Barenblatt incomplete similarity theory [7–10] to investigate the spatial scaling law of kinetic energy spectrum.

Compressible turbulence has drawn a great deal of attention for decades. Fully developed three-dimensional incompressible turbulence exhibits Kolmogorov’s $-5/3$ energy spectrum in the inertial range that is generated and maintained by a non-equilibrium process, viz., the conservative cascade of kinetic energy from large scales to small scales [8–11]. In compressible turbulence, there are nonlinear interactions between solenoidal and compressive modes of velocity fluctuations. Moreover, quasi-2D shock waves add a new type of flow structures in addition to quasi-1D intense vortices as in incompressible turbulence, which further complicates the kinetic energy transfer process in compressible turbulence. Interestingly, the Kolmogorov’s $-5/3$ energy spectrum of velocity has been reported in supersonic motions of interstellar media [12–14]. High resolution numerical simulations for supersonic isothermal turbulence showed the $-5/3$ spectrum of the density-weighted velocity $v = \rho^{1/3}u$, where $\rho$ is the density and $u$ is the velocity [4 and 15]. Aluie [16, 17] proved that the kinetic energy cascades conservatively in compressible turbulence, provided that the pressure-dilatation cosspectrum decays at a sufficiently rapid rate. This analysis was further supported by numerical simulations of both forced and decaying compressible turbulence [18]

Kovasznay [19] pointed out that the problem is characterized by the existence of acoustic, vortical, and entropy modes in interaction with each other. Systematic 2nd-approximation of these nonlinear interactions was given by Chu and Kovasznay [20]. In compressible turbulence, Lighthill [21] pointed out that its energy is continually radiated away in the form of sound waves which is ultimately converted into heat by the various processes of acoustic attenuation. Therefore, one may visualize the compressibility effects as acting like a source of energy dissipation in addition to that provided by viscosity and thermal conductivity [22]. As the nonlinear effects become prominent, the sound waves in a compressible fluid sharpen to form shock waves, vortex formation behind the shock waves then produces anisotropic shear turbulence. The investigation showed that passage of a shock wave also results in smaller parallel to the shock and compressed in the direction perpendicular to the shock. The shock formation and the shock interaction process lead to another source of energy dissipation in compressible turbulence.

Despite the anisotropy caused by the individual shock, Kadomtsev and Petviashvili [23] argued that the random orientation of the various shocks leads to the overall isotropy of the turbulent field. Using a Burgers equation type model, Kadomtsev and Petviashvili [23] gave a spectrum for kinetic energy

$$E(k) \sim c^{-1}k^{-2},$$

where $c$ is the speed of sound in the fluid, and $\varepsilon$ is the dissipation rate.

Moiseev et al [22] applied group-invariance principles to Hopf-type functional formulation of the compressible case and gave the spectrum of the kinetic energy

$$E(k) \sim \rho c^2 \varepsilon^{2/3} k^{-5/3 - 1/3},$$

where $\rho$ is the density of the fluid and $\gamma$ is the ratio of specific heats of the fluid.
However, Shivamoggi [24] argued equation (2) is not completely correct and proposed a revised spectrum

\[ E(k) \sim \rho^{2/5} c^{2/5} \varepsilon^{4/5} k^{-5/3}. \]  

Fortunately, it is easy to verify the equation [11] and [3] have a common problem, i.e., the dimension of the right hand is not equal to the left due to the present of the speed of sound c.

Recently, for Mach number Ma near 1 the numerical simulation in [6] confirmed that fully developed three-dimensional compressible turbulence mass-weighted energy spectrum exhibited Kolmogorov’s −5/3 power law in the inertial range. Galtier and Banerjee [3] derived a relation for the scaling of compressible isothermal turbulence and shows only around the sonic scale, when the local Mach number drops to unity, the density-weighted energy spectrum approach the -5/3 power law.

Instead of remarkable progress in the compressible turbulence study, there are some fundamental are still remained. First question is that for arbitrary Mach number Ma, what kind scaling laws will be for the mass-weighted energy spectrum; second question will be can we use the kinetic energy to characterise the cascade of compressible turbulence. Due to the complicate nature of the compressible turbulence, the above question may not been fully answered by numerical simulations, other alternative ways must be purposed and tried. Although the nature of the compressible turbulence is still not fully understood, but we believe its physics must satisfy the dimensional laws and of course can be studied by dimensional analysis as its incompressible counterpart case, which is the central motivation of this study. The results in this article shown that the dimensional analysis can definitely capture the overall picture of the compressible turbulence cascade process as in the case of incompressible turbulence and leads to a fair rich information on the phenomena.

For incompressible turbulence, Kolmogorov (K41) [8, 9] defined the length, time and velocity scales of the smallest eddies of turbulence. The largest spatial scales are given by the energy-injection length scales L, whereas the smallest scales are given by the Kolmogorov length \( \eta = (\varepsilon^3/\nu)^{1/4} \), where the energy-dissipation rate \( \varepsilon = 2\nu \sum_{ij} s_{ij} \) and \( s_{ij} = \frac{1}{2} (u_i' u_j + u_j' u_i) \). The scale separation in a turbulent flow is thus \( l/\eta = R_e^{3/4} \), where viscosity \( \nu \), and \( R_e \) is the Taylor microscale Reynolds number, \( u_i' \) is fluctuation of flow velocity. Between these scales is the so-called, inertial range of turbulence. Kolmogorov (K41) stated the large-scale turbulence motion is roughly independent of viscosity. The small scale, however, is controlled by viscosity. In the inertial range, turbulence is controlled solely by the dissipation rate \( \varepsilon \) and the size of eddy. It is found that the spatial energy spectrum \( E(k) \) (its unit is \( L^3 t^{-2} \), \( L \) is length, \( t \) is time) can be formulated in terms of wave number \( k \) and dissipation rate \( \varepsilon \) as \( E(k) = \varepsilon^{2/3} k^{-5/3} F[(kn)^{4/3}] = C_k \varepsilon^{2/3} k^{-5/3} \), which is the famous Kolmogorov −5/3 law of incompressible turbulence, where \( F(\cdot) \) is a dimensionless function.

**KINETIC ENERGY SPECTRUM OF COMPRESSIBLE TURBULENCE FOR VELOCITY \( u \)**

For incompressible fluid there is no needs to consider the changes of mass density. However, the mass density change will be the central feature for compressible flow which must be taken into account in the formulation.

In order to formulate the compressible turbulence scaling law, [30] extended the Kolmogorov assumption of incompressible turbulence into compressible case. The ideas of this extension is from the dimensional analysis of lift of wing, in which the lift force \( F_L \) is the function of velocity \( V \), air density \( \rho \), characteristic length \( L \), angle of attack \( \alpha \), viscosity \( \mu \), and speed of sound \( c \), that is \( F_L = f(V, L, \rho, \mu, c, \alpha) \), by dimensional analysis which can be reduced to \( C_F = F_L/(1/2\rho AV^2) = f(Re, Ma, \alpha) \).

**Extended Kolmogorov assumption for compressible turbulence:** In the inertial range, the compressible turbulence is controlled not only by the dissipation rate \( \varepsilon \) but also by the size of eddy \( k \) and fluid density \( \rho \) or Mach number.

In this extended Kolmogorov assumption, there are two set of variables in the formulation, one set has mass density and another has fluid velocity. For the first set, there are six variables and constants must be considered, they are \( E \) the energy spectrum (its unit is \( L^3 t^{-2} \)), \( \varepsilon \) dissipation rate(its unit is \( L^2 t^{-3} \)), \( k \) wave number(its dimension is \( L^{-1} \)), \( \nu \) kinematic viscosity(its unit is \( L^2 t^{-1} \)), \( \rho \) the current mass density (its unit is \( ML^{-3} \)), and \( \rho_0 \) the local ”reservoir values” of mass density or stagnation density [31]. The mass density and its reservoir values can be replaced by fluid velocity \( V \) and the speed of sound \( c \) because their relationship of heat capacities \[ \gamma = c_p/c_v^{-1} \] is ratio of heat capacities [31]. The energy spectrum \( E \) can be expressed as the function of \( (\nu, \varepsilon, k, \rho_0, \rho) \)

\[ E = f(\nu, \varepsilon, k, \rho_0, \rho). \]  

Within the six variables, there are three basic unit such as time \( t \), mass \( M \) and length \( L \), from the Buckingham Pi theorem, we can choose three variables as independent ones, such as wavenumber \( k \), dissipation rate \( \varepsilon \), and density \( \rho \); the independent variables are the energy spectrum \( E(k) \), the kinematic viscosity \( \nu \) and \( \rho_0 \), all primary dimensions are listed in the Table [11].

From the Buckingham II theorem of dimensional analysis, we have three dimensional variables as \( \Pi_1 = E \varepsilon^{-3/2} k^{5/3}, \Pi_2 = \nu \varepsilon^{-1/3} k^{4/3} \), and \( \Pi_3 = \rho \rho_0^{-1} \), then we have the scaling law of the energy spectrum \( \Pi_1 = \)
Existence of the limit of \( F \) as \( \alpha, \beta \) variables there is complete similarity with respect to the variable \( C \) simply takes the constant value 1, so for finite value of the Mach number, the function \( F((k\eta)^{4/3}, Ma) \rightarrow F(0, Ma) \). Kolmogorov assumed that in the limits \( k\eta \rightarrow 0 \), the function \( F(x,0) \) simply takes the constant value \( C_K \). In other words, there is complete similarity with respect to the variables \( k\eta \rightarrow 0 \), then we have the Kolmogorov -5/3 law as \( E(k) = C_K \varepsilon^{2/3} k^{-5/3} \).

Unfortunately, for the compressible turbulence, the existence of the limit of \( F(x, Ma) \) as \( x \rightarrow 0 \) is a question due to intermittency - the fluctuations of the energy dissipation rate about its mean value \( \varepsilon \). According to Barenblatt, the incomplete similarity in the variable \( k\eta \) would require the nonexistence of a finite and nonzero limit of \( F(x, Ma) \) as \( x \rightarrow 0 \).

In equation (7), we have arrived at the energy spectrum relation

\[
E(k, Ma) = \varepsilon^{2/3} k^{-5/3} A((k\eta)^{4/3}, Ma). \tag{7}
\]

Then we have the scaling law of the energy spectrum in terms of Mach number as \( E(k, Ma) \), that is

\[
E(k, Ma) = \varepsilon^{2/3} k^{-5/3} A((k\eta)^{4/3}, Ma). \tag{8}
\]

where \( A(Ma) \) is a function of the Mach number, and \( B(Ma) \) is the so-called intermittency exponent, believed to be small.

Using the similar approach in [2], the \( B(Ma) \) can be further simplified as a linear form \( B(Ma) = \alpha + \beta \varepsilon \), where \( \alpha, \beta \) are to be determined constants, and the function of the Mach number \( Ma \),

\[
E(k, Ma) = \varepsilon^{2/3} k^{-5/3} A((k\eta)^{4/3})^{\alpha + \beta \varepsilon}. \tag{9}
\]

The small perturbation parameter \( \varepsilon \) is a function of the Mach number vanishing when \( Ma \rightarrow \infty \) as \( Re \rightarrow \infty \).

Applying the second hypothesis, when the viscosity tends to very small, the Reynolds number tends to very large, so does to the Mach number \( Ma \) because of relationship with the Reynolds number \( Re = Ma^{2/3} \), for a well-defined limit of \( E(k, Ma) \) exists only if \( \alpha = 0 \). Then equation (10) can be rewritten as

\[
E(k, Ma) = \varepsilon^{2/3} k^{-5/3} A e^{(\beta \varepsilon \ln k\eta)/3}. \tag{11}
\]

In equation (11), when viscosity vanishes \( k\eta \rightarrow -\infty \), if \( \varepsilon \) tended to be zero as \( Ma \rightarrow \infty \) faster than \( (1/\ln Ma) \) then the exponent would tend to zero and we would return to the case of complete similarity, which not supposed to be the case in terms of compressible turbulence. However, if \( \varepsilon \) tends to zero slower than \( 1/\ln Ma \), a well-defined limit of the energy spectrum does not exist which will violate the second hypothesis. Therefore, the only choice compatible with the hypotheses must be

\[
\varepsilon = \frac{1}{\ln Ma} \tag{12}
\]
which was firstly obtained by Barenblatt in\cite{7} for the boundary turbulence flow. Then the equation\cite{11} would be in the form

\[ E(k, Ma) = \varepsilon^{2/3} k^{-5/3} A[(k \eta)^{1/3}]^\beta \ln Ma. \]  

(13)

Since the Kolmogorov length \( \eta = (\varepsilon^3/\nu)^{1/4} \), so equation\cite{13} becomes

\[ E(k, Ma) = \varepsilon^{2/3} k^{-5/3} A[(k(\varepsilon/\nu)^{1/4})^{4/3}]^{\frac{\beta}{\ln Ma}}, \]  

(14)

or rewritten as in a compact form as

\[ E(k, Ma) = C_{\varepsilon}^{\frac{2}{3} + \frac{\beta}{\ln Ma^3}} k^{\left(-\frac{5}{3} + \frac{4\beta}{3\ln Ma^3}\right)}, \]  

(15)

where the constant

\[ C = A\nu^{-\frac{\beta}{\ln Ma^3}}, \]  

(16)

in which \( C \) appears to be the inversely proportional to the \( \ln Ma \).

For a very large but not infinite Mach number, the equation\cite{15} together with equation\cite{16} is the Barenblatt-type incomplete scaling law for compressible turbulence. The modified exponents represent the intermittency of the process. The constants \( C \) and \( \beta \) must be determined by either numerical or experimental data.

Despite the unknown constants of \( A \) and \( \beta \), the equations\cite{15} and\cite{16} can still give us fair rich information on how the kinetic energy varies with the wavenumber and dissipation rate. Although the equation\cite{15} was formulated for a large Mach number, fortunately we found it is still valid for quite a wide range of Mach number \( Ma \). Having the modified scaling law in equation\cite{15}, we can also obtain scaling laws for following four limit cases: i). \( Ma = 0 \); ii). \( Ma = 1 \); iii). supersonic turbulence.

i. For incompressible turbulence, the Mach number is zero, then \( \ln Ma \rightarrow -\infty \), so for any \( \beta \) the exponents \( \frac{2}{3} + \frac{\beta}{\ln Ma^3} \rightarrow 2/3 \) and \( \frac{4\beta}{3\ln Ma^3} \rightarrow -5/3 \), the equation\cite{11} is reduced to the Kolmogorov\(-5/3\) scaling law

\[ E(k, Ma)\big|_{Ma=0} = E(k, 0) = A\varepsilon^{2/3} k^{-5/3}. \]  

(17)

This is the Kolmogorov incompressible turbulence scaling law, the constant \( A \) equal to the Kolmogorov universal constant \( CK \), i.e., \( A = CK = 1.5 \).

ii. For sonic turbulence, \( Ma = 1 \), then \( \ln Ma = \ln 1 = 0 \), so to have a well-defined limit energy spectrum, the parameter \( \beta \) must be zero, that is \( \beta = 0 \). The equation\cite{15} is reduced to the \(-5/3\) power law

\[ E(k, Ma)\big|_{Ma=1} = E(k, 1) = A\varepsilon^{2/3} k^{-5/3}, \]  

(18)

in which \( A \) must be determined by using other approaches.

iii. In the case of supersonic turbulence \( \left(\frac{5}{3} + \frac{\beta}{\ln Ma^3}\right) = \frac{1}{3} \), then \( \beta = -1/3 \ln Ma \), \( B(Ma) = -19/9 \) and the kinetic energy spectrum becomes

\[ E(k) = A\nu^{1/3}\varepsilon^{1/3} k^{-19/9}, \]  

(19)

in which the exponent \(-19/9\) is bigger than \(-5/3\) (the K41 scaling) indicates that supersonic turbulence is significantly steeper than K41 scaling. This is shown in numerical simulations by Kritsk et al.\cite{4}. Galtier and Bauer-jec\cite{3} obtained the exponent of \(-19/9\) for mass-weighted energy spectrum, however our result in equation\cite{19} is formulated for kinetic energy spectrum.

**KINETIC ENERGY SPECTRUM OF COMPRESSIBLE TURBULENCE FOR THE DENSITY-WEIGHTED VELOCITY** \( \mathbf{v} = \rho^{1/3} \mathbf{u} \)**

Due to the density change of compressible turbulence, the numerical simulations usually use the density-weighted velocity \( \mathbf{v} = \rho^{1/3} \mathbf{u} \), where \( \rho \) is the density and \( \mathbf{u} \) is the velocity, then the density-weighted energy spectrum as denoted as \( E_\rho \) can be expressed as

\[ E_\rho(k, \rho) = f(\nu, \varepsilon, k, \rho, \gamma), \]  

(20)

where the \( \gamma \) is the dimensionless specific heat of fluid, all primary dimensions are listed in the Table\cite{111}.

| \( \) | \( E_\rho \) | \( \nu \) | \( \varepsilon \) | \( k \) | \( \rho \) | \( \gamma \) |
|---|---|---|---|---|---|---|
| \( M^{1/3} L^2 t^{-2} L^2 t^{-3} L^{-1} \) | \( E \) | \( k \) | \( \rho \) | \( \gamma \) |

The table has six parameters and three primary dimensions \( (M,L,t) \), from the Buckingham \( \Pi \) theorem of dimensional analysis, we have two \( \Pi \) as \( \Pi_1 = E_\rho^{-1/3} \varepsilon^{-2/3} k^{5/3} \) and \( \Pi_2 = (k \eta)^{4/3} \), then we have \( \Pi_1 = f(\Pi_2, \gamma) \), ie.

\[ E(k, \rho) = \rho^{1/3} \varepsilon^{2/3} k^{-5/3} F((k \eta)^{4/3}, \gamma), \]  

(21)

which has the same power exponents as given by\cite{21} without the F\((((k \eta)^{4/3}, \gamma)\).

The compressibility depends on \( \gamma \) from the relation \( \rho_0 \rho^{-1} = (1 + \frac{\gamma - 1}{2} (\frac{\gamma}{\gamma - 1})^2) \), where \( \gamma = c_p/c_v-1 \) is ratio of heat capacities\cite{31}. If \( \gamma \rightarrow \infty \), then \( \rho_0 \rho^{-1} \rightarrow 1 \) for incompressible turbulence; if \( \gamma \rightarrow 1 \), then \( \rho_0 \rho^{-1} \rightarrow \infty \) which is an extreme complete compressible case.

Within the inertial subrange \((k \eta \rightarrow 0)\), in the case of highly compressible \((\gamma \rightarrow 1)\), the equation\cite{22} can be simplified to

\[ E(k, \rho) = F(0,1) \rho^{1/3} \varepsilon^{2/3} k^{-5/3}. \]  

(22)

It is clear that \( F(0,1) \) must be an universal constant. The formula\cite{23} was confirmed by numerical simulation as illustrated in the FIG.1 of\cite{6} and FIG.3.1 of\cite{32}, the \( F(0,1) \) was determined by curve fitting as \( F(0,1) = 1.5 \) which is very close to incompressible turbulence Kolmogorov universal constant \( CK = 1.5 \sim 2.1 \).

Equation\cite{22} indicate that a mass-weighted energy spectrum in \( k^{-5/3} \) may still be preserved at small scales \((k \eta \rightarrow 0)\).
if the density-weighted fluid velocity \( \mathbf{v} = \rho^{1/3} \mathbf{u} \), is used, this conclusion was also mentioned in [3].

It might be worth to note the relationship between kinetic energy spectrum \( E \) and its mass-weighted counterpart \( E_\rho \) can be established from equations (15), (7) and (22) as

\[
E(k, \rho)/E(k, Ma) = \rho^{1/3} F(k\eta, \gamma)/F(k\eta, Ma).
\]

(23)

It must be pointed out that the functions \( F(k\eta, \gamma) \) and \( F(k\eta, Ma) \) are totally different from each other, and general their ratio is not a constant.

At small scales \( (k\eta \rightarrow 0) \) and in a limit case of complete compressible \( Ma \rightarrow \infty \) and/or \( \gamma \rightarrow 1 \), if the ratio in equation (23) does exist, then we have

\[
E_\rho/E = \lambda \rho^{1/3}.
\]

(24)

From [32] numerical simulation at Mach number close to 1, the coefficient can be estimated as \( \lambda \approx 1 \). This relation answered our one question, which state the compressible turbulence can be studied by the mass-weighted kinetic energy spectrum as well as the kinetic energy spectrum.

**DISCUSSION AND CONCLUSION**

This article proposed spatial scaling laws of the kinetic energy spectrum of compressible turbulence flow and its mass-weighted counterpart in terms of wavenumber. The study shown that the compressible turbulence kinetic and the mass-weighted energy spectrum does not behaves the complete similarity, but incomplete similarity as in equation (15) and (21), and both energy spectrum in \( k^{-5/3} \) may still be preserved at small scales \( (k\eta \rightarrow 0) \) for compressible turbulence.

Why the energy spectrum in \( k^{-5/3} \) can still be preserved? As we known in compressible turbulence, the kinetic energy spectrum can be decomposed into compressible and solenoidal parts. There are nonlinear interactions between solenoidal and compressive modes of velocity fluctuations. The numerical simulations shown that the compressive kinetic energy is significantly larger than its solenoidal counterpart, and their cascade scaling are different from each other [3]. The compressible kinetic energy cascade follows \(-19/9 \approx -2.1 \) powers laws of wave number, the solenoidal follows the \( k^{-5/3} \) powers laws. It means that the solenoidal dominate the energy spectrum for small wavenumber \( k \), and overall will leads to the total kinetic energy spectrum exhibits a \(-5/3\) scaling.

It might be worth to mention we treated a largely ignored problem so far, namely, deriving the energy spectrum of compressible turbulence from a dimensional analysis, the reason was generally believed that the complexity of the matter (compressible turbulence with a mixture of shocks, vortices, boundary layers, etc.) renders the dimensional argument irrelevant. This study shown that the dimensional argument is still working for the compressible turbulence.

The similar strategy has been applied to the investigation of temporal scaling laws of compressible turbulence by the author [32].

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