Finite good filtration dimension for modules over an algebra with good filtration.

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Dedicated to Eric Friedlander on his 60th birthday

Abstract

Let $G$ be a connected reductive linear algebraic group over a field $k$ of characteristic $p > 0$. Let $p$ be large enough with respect to the root system. We show that if a finitely generated commutative $k$-algebra $A$ with $G$-action has good filtration, then any noetherian $A$-module with compatible $G$-action has finite good filtration dimension.

1 Introduction

Consider a connected reductive linear algebraic group $G$ defined over a field $k$ of positive characteristic $p$. We say that $G$ has the cohomological finite generation property (CFG) if the following holds: Let $A$ be a finitely generated commutative $k$-algebra on which $G$ acts rationally by $k$-algebra automorphisms. (So $G$ acts on $\text{Spec}(A)$.) Then the cohomology ring $H^*(G, A)$ is finitely generated as a $k$-algebra. Here, as in [9, 1.4], we use the cohomology introduced by Hochschild, also known as ‘rational cohomology’.

In [13] we have shown that $SL_2$ over a field of positive characteristic has property (CFG), and in [14] we proved that $SL_3$ over a field of characteristic two has property (CFG). We conjecture that every reductive linear algebraic group has property (CFG). In this paper we show that this is at least a good heuristic principle: We derive one of the consequences of (CFG) for any simply connected semisimple linear algebraic group $G$ that satisfies the following

1
Hypothesis 1.1 Assume that for every fundamental weight $\varpi_i$ the symmetric algebra $S^*(\nabla_G(\varpi_i))$ on the fundamental representation $\nabla_G(\varpi_i)$ has a good filtration.

Recall that this hypothesis is satisfied if $p \geq \max_i(\dim(\nabla_G(\varpi_i)))$, by [1, 4.1(5) and 4.3(1)]. This inequality is not necessary. For instance, $SL_n$ satisfies the hypothesis for $n \leq 5$, by [13, Lemma 3.2]. When $p = 2$, the hypothesis does not hold for $SL_n$ with $n \geq 6$, by [13, 3.3].

In the sequel let $G$ be a connected reductive linear algebraic group over a field $k$ of characteristic $p > 0$ with simply connected commutator subgroup for which hypothesis 1.1 holds. Let $A$ be a finitely generated commutative $k$-algebra on which $G$ acts rationally by $k$-algebra automorphisms. Let $M$ be a noetherian $A$-module on which $G$ acts compatibly. This means that the structure map $A \otimes M \to M$ is a $G$-module map. Our main result is

Theorem 1.2 If $A$ has good filtration, then $M$ has finite good filtration dimension and each $H^i(G, M)$ is a noetherian $A^G$-module.

When $A = k$ the theorem goes back to [5] and does not need hypothesis [11]. Unlike the proofs in [13] and [14], the proof of our theorem does not involve any cohomology of finite group schemes and is thus independent of the work of Friedlander and Suslin [6]. But without their work we would not have guessed the theorem. For clarity we will pull some material of [13] free from finite group schemes.

2 Recollections

Some unexplained notations, terminology, properties, ... can be found in [9]. We choose a Borel group $B^+ = TU^+$ and the opposite Borel group $B^-$. The roots of $B^+$ are positive. If $\lambda \in X(T)$ is dominant, then $\text{ind}_{B^-}^G(\lambda)$ is the ‘dual Weyl module’ or ‘costandard module’ $\nabla_G(\lambda)$ with highest weight $\lambda$. The formula $\nabla_G(\lambda) = \text{ind}_{B^-}^G(\lambda)$ just means that $\nabla_G(\lambda)$ is obtained from the Borel-Weil construction: $\nabla_G(\lambda)$ equals $H^0(G/B^-, L)$ for a certain line bundle on the flag variety $G/B^-$. In a good filtration $0 = V_{-1} \subseteq V_0 \subseteq V_1 \ldots$ of a $G$-module $V = \bigcup_i V_i$ the nonzero layers $V_i/V_{i-1}$ are of the form $\nabla_G(\mu)$. As in [12] we will actually also allow a layer to be a direct sum of any number of copies of the same $\nabla_G(\mu)$, cf. [9, II.4.16 Remark 1]. This is much more convenient when working with infinite dimensional $G$-modules. It is shown
in [4] that a module of countable dimension that has a good filtration in our sense also has a filtration that is a good filtration in the old sense. Note that the module \( M \) in our theorem has countable dimension. It would do little harm to restrict to modules of countable dimension throughout.

If \( V \) is a \( G \)-module, and \( m \geq -1 \) is an integer so that \( H^{m+1}(G, \nabla_G(\mu) \otimes V) = 0 \) for all dominant \( \mu \), then we say as in [5] that \( V \) has good filtration dimension at most \( m \). The case \( m = 0 \) corresponds with \( V \) having a good filtration. And for \( m \geq 0 \) it means that \( V \) has a resolution

\[
0 \to V \to N_0 \to \cdots \to N_m \to 0
\]

in which the \( N_i \) have good filtration, in our sense. We say that \( V \) has good filtration dimension precisely \( m \), notation \( \dim_{\nabla}(V) = m \), if \( m \) is minimal so that \( V \) has good filtration dimension at most \( m \). In that case \( H^{i+1}(G, \nabla_G(\mu) \otimes V) = 0 \) for all dominant \( \mu \) and all \( i \geq m \). In particular \( H^{i+1}(G, V) = 0 \) for \( i \geq m \). If there is no finite \( m \) so that \( \dim_{\nabla}(V) = m \), then we put \( \dim_{\nabla}(V) = \infty \).

### 2.1 Filtrations

For simplicity assume also that \( G \) is semisimple. (Until remark [3.1]) If \( V \) is a \( G \)-module, and \( \lambda \) is a dominant weight, then \( V_{\leq \lambda} \) denotes the largest \( G \)-submodule all whose weights \( \mu \) satisfy \( \mu \leq \lambda \) in the dominance partial order [9 II.1.5]. For instance, \( V_{\leq 0} \) is the module of invariants \( V^G \). Similarly \( V_{< \lambda} \) denotes the largest \( G \)-submodule all whose weights \( \mu \) satisfy \( \mu < \lambda \). As in [12], we form the \( X(T) \)-graded module

\[
gr_{X(T)} V = \bigoplus_{\lambda \in X(T)} V_{\leq \lambda}/V_{< \lambda}.
\]

Each \( V_{\leq \lambda}/V_{< \lambda} \), or \( V_{\leq \lambda/\lambda} \) for short, has a \( B^+ \)-socle \( (V_{\leq \lambda/\lambda})^U = V_{\lambda}^U \) of weight \( \lambda \). We always view \( V_{\leq \lambda}^U \) as a \( B^- \)-module through restriction (inflation) along the homomorphism \( B^- \to T \). Then \( V_{\leq \lambda/\lambda} \) embeds naturally in its ‘good filtration hull’ \( \nabla_{\nabla}(V_{\leq \lambda/\lambda}) = \text{ind}_{B^-}^G V_{\lambda}^U \). This good filtration hull has the same \( B^+ \)-socle and by Polo it is the injective hull in the category \( \mathcal{C}_\lambda \) of \( G \)-modules \( N \) that satisfy \( N = N_{\leq \lambda} \). Compare [12 3.1.10].

We convert the \( X(T) \)-graded module \( gr_{X(T)} V \) to a \( \mathbb{Z} \)-graded module through an additive height function \( ht : X(T) \to \mathbb{Z} \), defined by \( ht = 2 \sum_{\alpha > 0} \alpha^\vee \), the sum being over the positive roots. (Our \( ht \) is twice the one

used by Grosshans [7], because we prefer to get even degrees rather than just integer degrees.) The Grosshans graded module is now

\[ \text{gr } V = \bigoplus_{i \geq 0} \text{gr}_i V, \]

with

\[ \text{gr}_i V = \bigoplus_{\text{ht}(\lambda) = i} V_{\lambda/\lambda}. \]

In other words, if one puts

\[ V_{\leq i} := \sum_{\text{ht}(\lambda) \leq i} V_{\lambda}, \]

then \( \text{gr } V \) is is the associated graded of the filtration \( V_{\leq 0} \subseteq V_{\leq 1} \cdots \).

Let us apply the above to our finitely generated commutative \( k \)-algebra with \( G \)-action \( A \). The Grosshans graded algebra \( \text{gr } A \) embeds in a good filtration hull, which Grosshans calls \( R \), and which we call \( \text{hull}_V(\text{gr } A) \),

\[ \text{hull}_V(\text{gr } A) := \text{ind}_G^U A_U = \bigoplus_i \bigoplus_{\text{ht}(\lambda) = i} \text{hull}_V(A_{\lambda/A<\lambda}). \]

Grosshans [7] shows that \( A_U, \text{gr } A, \text{hull}_V(\text{gr } A) \) are finitely generated \( k \)-algebras with \( \text{hull}_V(\text{gr } A) \) finite over \( \text{gr } A \). Mathieu studied \( \text{gr } A \) and \( \text{hull}_V(\text{gr } A) \) earlier in [11].

**Example 2.2** Consider the multicone [10]

\[ k[G/U] := \text{ind}_U^G k = \text{ind}_B^G \text{ind}_{B^+} B^+ k = \text{ind}_B^G k[T] = \bigoplus_{\lambda \text{ dominant}} \nabla_G(\lambda). \]

It is its own Grosshans graded ring. Recall [10] that it is generated as a \( k \)-algebra by the finite dimensional sum of the \( \nabla_G(\varpi_i) \), where \( \varpi_i \) denotes the \( i \)th fundamental weight.

**Lemma 2.3** Let \( A \) have a good filtration, so that \( \text{gr } A = \text{hull}_V(\text{gr } A) \). Let \( R = \oplus_i R_i \) be a graded algebra with \( G \)-action such that \( R_i = (R_i)_{\leq i} \). Then every \( T \)-equivariant graded algebra homomorphism \( R^U \to (\text{gr } A)^U \) extends uniquely to a \( G \)-equivariant graded algebra homomorphism \( R \to (\text{gr } A) \).
Proof. Use that $\text{hull}_V(\text{gr } A)$ is an induced module.

2.4 A graded polynomial $G \times D$-algebra with good filtration

We now extract a construction from [13]. It is hidden in the study of a Hochschild-Serre spectral sequence which in the present situation would correspond with the case where as normal subgroup one takes the trivial subgroup!

As the algebra $(\text{gr } A)^D$ is finitely generated, it is also generated by finitely many weight vectors. Consider one such weight vector $v$, say of weight $\lambda$. Clearly $\lambda$ is dominant. If $\lambda = 0$, map a polynomial ring $P_v := k[x]$ with trivial $G$-action to $\text{gr } A$ by substituting $v$ for $x$. Also put $D_v := 1$. Next assume $\lambda \neq 0$. Let $\ell$ be the rank of $G$. Define a $T$-action on the $X(T)$-graded algebra

$$P = \bigotimes_{i=1}^{\ell} S^*(\nabla_G(\varpi_i))$$

by letting $T$ act on $\bigotimes_{i=1}^{\ell} S^{m_i}(\nabla_G(\varpi_i))$ through weight $\sum_i m_i \varpi_i$. So now we have a $G \times T$-action on $P$. Observe that by our key hypothesis 1.1 and the tensor product property [9, Ch. G] the polynomial algebra $P$ has a good filtration for the $G$-action. Let $D$ be the scheme theoretic kernel of $\lambda$. So $D$ has character group $X(D) = X(T)/\mathbb{Z}\lambda$ and $D = \text{Diag}(X(T)/\mathbb{Z}\lambda)$ in the notations of [9, I.2.5]. The subalgebra $P^{1 \times D}$ is a graded algebra with good filtration such that its subalgebra $P^{U \times D}$ contains a polynomial algebra on one generator $x$ of weight $\lambda \times \lambda$. In fact, this polynomial subalgebra contains all the weight vectors in $P^{U \times D}$ of weight $\mu \times \nu$ with $\text{ht}(\mu) \geq \text{ht}(\nu)$. The other weight vectors in $P^{U \times D}$ also have weight of the form $\mu \times \nu$ with $\nu$ a multiple of $\lambda$. These other weight vectors span an ideal in $P^{U \times D}$. Now assume $A$ has a good filtration. By lemma 2.3 one easily constructs a $G$-equivariant algebra homomorphism $P^{1 \times D} \rightarrow \text{gr } A$ that maps $x$ to $v$. Write it as $P^{1 \times D}_v \rightarrow \text{gr } A$, to stress the dependence on $v$.

As new $P$ we take the tensor product of the finitely many $P_v$ and as diagonalized group $D$ we take the direct product of the $D_v$. Then we have a graded algebra map $P^D \rightarrow \text{gr } A$. It is surjective because its image has good filtration ([9, Ch. A]) and contains $(\text{gr } A)^U$. The $G \times D$-algebra $P$ is an example of what we called in [13] a graded polynomial $G \times D$-algebra with good filtration. We have proved
Lemma 2.5 If $A$ has a good filtration, then there is a graded polynomial $G \times D$-algebra $P$ with good filtration and a graded $G$-equivariant surjection $P^D \to \text{gr} A$.

Now recall $M$ is a noetherian $A$-module on which $G$ acts compatibly, meaning that the structure map $A \otimes M \to M$ is a map of $G$-modules. Form the ‘semi-direct product ring’ $A \rtimes M$ whose underlying $G$-module is $A \oplus M$, with product given by $(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + a_2m_1)$. By Grosshans $\text{gr}(A \rtimes M)$ is a finitely generated algebra, so we get

Lemma 2.6 $\text{gr} M$ is a noetherian $\text{gr} A$-module.

This is of course very reminiscent of the proof of the lemma [8, Thm. 16.9] telling that $M^G$ is a noetherian module over the finitely generated $k$-algebra $A^G$. We will tacitly use its counterpart for diagonalized actions, cf. [2], I.2.11.

Taking things together we learn that if $A$ has a good filtration, then $P \otimes_{P^D} \text{gr} M$ is what we called in [13] a finite graded $P$-module. Lemma [13] Lemma 3.7 then tells us

Lemma 2.7 Let $A$ have good filtration. Then $P \otimes_{P^D} \text{gr} M$ has finite good filtration dimension and each $H^i(G, P \otimes_{P^D} \text{gr} M)$ is a noetherian $P^G$-module.

Extend the $D$-action on $P$ to $P \otimes_{P^D} \text{gr} M$ by using the trivial action on the second factor. Then we have a $G \times D$-module structure on $P \otimes_{P^D} \text{gr} M$. As $D$ is diagonalized, $P^D$ is a direct summand of $P$ as a $P^D$-module [2], I.2.11] and $(P \otimes_{P^D} \text{gr} M)^{1 \times D} = \text{gr} M$ is a direct summand of the $G$-module $P \otimes_{P^D} \text{gr} M$. It follows that $\text{gr} M$ also has finite good filtration dimension and it follows that each $H^i(G, P \otimes_{P^D} \text{gr} M)^{1 \times D} = H^i(G, \text{gr} M)$ is a noetherian $P^{G \times D}$-module. But the action of $P^{G \times D}$ on $\text{gr} M$ factors through $(\text{gr} A)^G$, so we see that each $H^i(G, \text{gr} M)$ is a noetherian $(\text{gr} A)^G$-module. And one always has $(\text{gr} A)^G = (\text{gr}_0 A)^G = A^G$. We conclude

Lemma 2.8 Let $A$ have good filtration. Then $\text{gr} M$ has finite good filtration dimension and each $H^i(G, \text{gr} M)$ is a noetherian $A^G$-module.
3 Degrading

We still have to get rid of the grading. The filtration $M_{\leq 0} \subseteq M_{\leq 1} \cdots$ induces a filtration of the Hochschild complex \cite[1.4.14]{Hoch} whence a spectral sequence

$$E(M) : E_{ij}^{ij} = H^{i+j}(G, \text{gr}_{-i} M) \Rightarrow H^{i+j}(G, M).$$

It lives in an unusual quadrant.

Assume that $A$ has good filtration. Then by Lemma 2.8 $E_1(M)$ is a finitely generated $A^G$-module. So the spectral sequence lives in only finitely many bidegrees $(i, j)$. Thus there is the same kind of convergence as one would have in a more common quadrant.

Choose $A^G$ as ring of operators to act on the spectral sequence $E(M)$. As $E_1(M)$ is a noetherian $A^G$-module, it easily follows (even without the spectral sequence) that $H^*(G, M)$ is a noetherian $A^G$-module. To finish the proof of the theorem, we note that $A \otimes k[G/U]$ is also a finitely generated algebra with a good filtration and that $M \otimes k[G/U]$ is a noetherian module over it. So what we have just seen tells that $H^*(G, M \otimes k[G/U])$ is a noetherian $(A \otimes k[G/U])^G$-module. In particular, there is an $m \geq -1$ so that $H^{m+1}(G, M \otimes k[G/U]) = 0$.

\begin{remark}
Somewhere along the way we made the simplifying assumption that $G$ is semisimple. So for the original $G$ we have now proved that $M$ has finite good filtration dimension with respect to the commutator subgroup $H$ of $G$. But that is the same as having finite good filtration dimension with respect to $G$. Also, the fact that $H^i(H, M)$ is a noetherian $A^H$-module implies that $H^i(G, M)$ is a noetherian $A^G$-module by taking invariants under the diagonalizable center $Z(G)$.
\end{remark}

\begin{remark}
We did not prove that $M$ has a finite resolution by noetherian $A$-modules with compatible $G$-action and good filtration. We do not know how to start. One may embed $M$ into the $A$-module $M \otimes k[G]$ with compatible $G$-action. It has good filtration, but it is not noetherian as an $A$-module.
\end{remark}

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