Stability of Yang Mills Vacuum State

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Abstract

We examine the phenomena of the chromomagnetic gluon condensation in the Yang-Mills theory and the problem of stability of the chromomagnetic vacuum fields. The apparent instability of the chromomagnetic vacuum fields is a result of quadratic approximation. The stability is restored when the nonlinear interaction of negative/unstable modes is taken into account in the case of chromomagnetic vacuum fields and the interaction of the zero modes in the case of (anti)self-dual covariantly-constant vacuum fields. All these vacuum fields are stable and indicate that the Yang-Mills vacuum is highly degenerate quantum state.

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1 Introduction

In the earlier investigations of the chromomagnetic gluon condensation [1, 2, 3, 4] it was realised that consideration of the vacuum polarisation in the quadratic approximation [3, 5] displays an apparent instability of the vacuum fields due to the negative/unstable modes [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Our aim is to demonstrate that the stability is restored when the nonlinear interaction of negative/unstable modes is taken into account in the case of chromomagnetic vacuum fields and the interaction of the zero modes in the case of (anti)self-dual covariantly-constant vacuum fields.

We will consider first the (anti)self-dual covariantly-constant vacuum fields (2.31) that have only positive/stable and infinitely many zero modes, so called chromons, as it was advocated by Leutwyller and Minkowski [12, 13, 14], and, importantly, there are no negative/unstable modes. To calculate the contribution of infinite number of zero modes we suggested a regularisation method that allows to sum the contribution of zero modes and get renormalised effective Lagrangian that has contribution of all positive/stable modes and zero modes. In the second approach we are taking into account a nonlinear interaction of zero modes that provides a necessary convergence of the path integral and leads to the same result for the effective Lagrangian that does not contain an imaginary part [1].

Next we are considering the stability of general covariantly-constant chromomagnetic vacuum fields (6.62). Instead of zero modes, here appear a plethora of negative/unstable modes [6, 7, 8, 9, 10, 11]. Generalising the calculation that was advocated earlier by Ambjorn, Nielsen, Olesen [8, 9, 10, 11], Flory [15] and other authors [12, 13, 14, 16, 17, 18, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40] we performed the integration over nonlinearly interacting negative/unstable modes and obtained the effective Lagrangian that includes contributions of positive/stable and negative/unstable modes and demonstrated the stability of covariantly-constant chromomagnetic vacuum fields. This consideration reflects a well known fact that a magnetic field does not produce work and cannot create particle pairs from the vacuum [1], opposite to what takes place in the case of an electric field [19, 20, 21, 22]. All these vacuum fields are stable and indicate that the Yang-Mills vacuum is a highly degenerate quantum state.

The rest of the article is devoted to the discussion of the chromomagnetic condensation and to the large \(N\) behaviour of the effective Lagrangian in the case of \(SU(N)\) group. A number of Appendixes are devoted to the technical details and the renormalisation group.

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\(^2\)Here, and afterwards, the phrase "vacuum fields" refers to the gauge fields that are the solutions of the sourceless Yang-Mills equation.



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2 Euclidean Path Integral

The quantum-mechanical amplitudes can be formulated as a sum over physical space-time trajectories [23, 24], as well as a sum over unphysical "trajectories" in Euclidean space [12, 13, 25, 26, 27, 28, 29]. Here we will analyse the amplitudes and the effective Lagrangian in Yang-Mills theory by using the Euclidean path integral representation [5, 29, 43].

For a single particle the Euclidean path integral determines the matrix elements of the operator \( \exp(-HT) \) [29], see Appendix D:

\[
\langle \vec{x}' | e^{-HT} | \vec{x} \rangle = \sum_n e^{-E_n T} \psi_n(\vec{x}') \psi_n^*(\vec{x}) = N \int_{\vec{x}'}^{\vec{x}} D\vec{x}(\tau) e^{-SE[\vec{x}(\tau)]},
\]

(2.1)

where the left-hand side of (2.1) is defined in terms of physical quantities, while the right-hand side in (2.1) is defined in unphysical Euclidean space [29]. A similar approach can be applied to the quantum gauge field theory when the physical states are described by gauge-invariant wave functionals \( \psi_n[\vec{A}] = \langle \vec{A}|E_n \rangle \) invariant with respect to infinitesimal gauge transformations of the three-dimensional vector gauge field \( \vec{A} \) [12, 13]:

\[
\langle \vec{A}' | e^{-HYMT} | \vec{A} \rangle = \sum_n e^{-E_n T} \psi_n[\vec{A}'] \psi_n^*[\vec{A}] = N \int_{\vec{A}'}^{\vec{A}} D\vec{A}(\tau) e^{-SE[\vec{A}(\tau)]} = N_f e^{-\int d^4x L_{eff}(\vec{A}', \vec{A})},
\]

(2.2)

E_a \equiv -i \frac{\delta}{\delta A^a_\mu} \psi, \quad H_a \equiv \epsilon_{ijk} \{ \partial_j A^a_k + \frac{1}{2} f^{abc} A^b_j A^c_k \} \psi(\vec{A}).

(2.4)

While the right-hand side of the equation (2.2) involves an integral over the Euclidean field \( A^a_i(\vec{x}, \tau) \) with the proper boundary values

\[
A^a_i(\vec{x}, 0) = A^a_i(\vec{x}), \quad A^a_i(\vec{x}, T) = A^a_i(\vec{x}).
\]

(2.5)

The Euclidean action is given by

\[
S_E = \frac{1}{4} \int d^4x G^{a}_{\mu\nu} G^a_{\mu\nu},
\]

(2.6)

where

\[
G^a_{\mu\nu}(\vec{A}) = \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu} + g f^{abc} A^b_{\mu} A^c_{\nu}.
\]
and \( x_0 = -i\tau_0 = -i\tau, A_0 = iA_0 \). The Euclidean fields cannot be interpreted directly in a physical space. In particular, a real electric field corresponds to an imaginary electric field in Euclidean space \( \vec{E} \to i\vec{E} \). Nevertheless, the integral over Euclidean fields interpolating between \( A^\alpha_i(x) \) and \( A_\mu^\alpha(x) \) does represent the physical quantities: the energy spectrum, the wave functions and the effective Lagrangian \( (2.2) \). The first term in \( (2.10) \) will be referred as unstable, while the negative eigenvalues produce a divergency of the path integral \( (2.11) \).

The path integral representation requires summation over Euclidean fields \( A_\mu^\alpha(x, \tau) \) that interpolate between the fixed boundary data \( A_\mu^\alpha(x) \) and \( A_\mu^\alpha(x) \). We will analyse the behaviour of vacuum fields in the vicinity of a given interpolating field \( B_\mu^\alpha(x) \) that starts at \( A_\mu^\alpha(x) \) and ends at \( A_\mu^\alpha(x) \):

\[
A_\mu^\alpha(x) = B_\mu^\alpha(x) + a_\mu^\alpha(x). \tag{2.7}
\]

Expanding the field strength \( G_{\mu\nu}^\alpha(A) \) in powers of quantum field \( a_\mu^\alpha(x) \) that has the zero boundary values

\[
G_{\mu\nu}^\alpha(A) = G_{\mu\nu}^\alpha(B) + \nabla_\mu^a b_\nu^b - \nabla_\nu^a b_\mu^b + g f^{abc} a_\mu^b a_\nu^c, \tag{2.8}
\]

where \( \nabla_\mu^a \) is a covariant derivative with respect to the interpolating field \( B_\mu^\alpha(x) \)

\[
\nabla_\mu^a b_\nu^b = \partial_\mu a_\nu^a + g f^{abc} B_\mu^b a_\nu^c, \tag{2.9}
\]

and \( [\nabla_\mu^a, \nabla_\nu^b]^\alpha \) = \( g f^{abc} G_{\mu\nu}^\alpha(B) \). For the Euclidean action \( (2.6) \) we will get:

\[
S_E = \frac{1}{4} \int d^4 x G_{\mu\nu}^\alpha(B) G_{\mu\nu}^\alpha(B) + \frac{1}{4} \int d^4 x G_{\mu\nu}^\alpha(B) \nabla_\mu^a b_\nu^a + \frac{1}{4} \int d^4 x \left( (\nabla_\mu^a b_\nu^b - \nabla_\nu^a b_\mu^b + g f^{abc} a_\mu^b a_\nu^c)^2 + 2 g G_{\mu\nu}^\alpha(B) f^{abc} a_\mu^b a_\nu^c \right). \tag{2.10}
\]

The first term in \( S_E \) doesn’t depend on the quantum field \( a_\mu^\alpha(x) \) and can be factorised in the path integral \( (2.2) \). The interpolating field \( B_\mu^\alpha(x) \) is supposed to satisfy the Euclidean equation of motion:

\[
\nabla_\mu G_{\mu\nu}(B) = 0, \tag{2.11}
\]

so that the second term in \( (2.10) \) that is linear in \( a_\mu^\alpha(x) \) vanishes and the effective Lagrangian is a gauge invariant functional \( (3) \). The quadratic part of the action has the following form:

\[
K_E = \frac{1}{4} \int d^4 x \left[ (\nabla_\mu^a b_\nu^b - \nabla_\nu^a b_\mu^b)^2 + 2 g G_{\mu\nu}^\alpha(B) f^{abc} a_\mu^b a_\nu^c \right]. \tag{2.12}
\]

The positive eigenvalues of the above quadratic form provide convergence of the path integral \( \int e^{-K_E} D a_\mu \) and will be referred as stable, while the negative eigenvalues produce a divergency of the path integral in the directions of the corresponding eigenfunctions and will be referred as unstable. The nonlinear terms in the Euclidean Lagrangian \( (2.10) \) are cubic and quartic in the quantum field \( a_\mu^\alpha(x) \):

\[
V_E = \int d^4 x \left[ - g f^{abc} a_\mu^b a_\nu^c \nabla_\mu^a d_\nu^d + \frac{g^2}{4} (f^{abc} a_\mu^b a_\nu^c)^2 \right]. \tag{2.13}
\]
The stability of the vacuum fields is directly connected with the stability of Euclidean field \( B^a_{\mu}(x) \) that interpolates between the boundary vacuum fields \( A^a_i(\vec{x}) \) and \( A'_i^a(\vec{x}) \). The interpolating field may or may not be stable. An interpolating field is stable if the action associated with it is smaller than the action of all neighbouring fields that have the same boundary values. The interpolating field is unstable if the action is exponentially diverging in some directions of the Hilbert space.

It follows that in order for the one-loop approximation to make sense, the amplitude \( e^{-S_E} \) when it is taken in a quadratic approximation \( e^{-K_E} \) (2.12), must decay in all directions of the quantum field \( a^a_\mu(x) \) over which we are integrating, that is, all eigenvalues of the quadratic form \( K_E \) must be positive. If this is not the case and some of the eigenvalues are negative, the background field \( B^a_{\mu}(x) \) displays an apparent instability in this quadratic approximation and stability should be reconsidered in the nonlinear regime by including the nonlinear interaction \( V_E \) of the quantum field \( a^a_\mu(x) \).

Let us specify the boundary fields \( A^a_i(\vec{x}) \) and \( A'_i^a(\vec{x}) \) at Euclidean time \( \tau = 0 \) and \( \tau = T \) to be equal \( A^a_i(\vec{x}) = A'_i^a(\vec{x}) \). We will consider boundary vacuum field \( A^a_i(\vec{x}) \) to be a covariantly-constant field \([41, 42, 3, 4]\) and therefore the interpolating field should obey the Euclidean Euler-Lagrange equation (2.11). A suitable four-dimensional gauge field \( B^a_{\mu}(x) \) that provides a possible solution has the following form:

\[
B^a_{\mu}(x) = -\frac{1}{2} F^{abc}_{\mu\nu} x^a_{\nu} S_3. \tag{2.14}
\]

The Euclidean transformations rotate the vectors \( E_i \) and \( H_i \) by two independent rotations, and we may therefore transform these vectors into the direction of the z-axis \([12]\):

\[
F_{12} = H, \quad F_{30} = E, \tag{2.15}
\]

and the gauge invariants are: \( F_E = \frac{1}{4} G^a_{\mu\nu} G^{a}_{\mu\nu} = \frac{H^2 + E^2}{2} \) and the \( G_E = \frac{1}{4} G^a_{\mu\nu} \tilde{G}^{a}_{\mu\nu} = H E \). In order to analyse the stability of the interpolating field we should initially consider the corresponding eigenvalue problem that appears in the quadratic approximation (2.12) of the action (2.10) for the quantum field \( a^a_\mu(x) \):

\[
-\nabla^a_{\mu} (\nabla^b_{\mu} a^c_{\nu} - \nabla^b_{\nu} a^c_{\mu}) + gf^{abc} G^b_{\mu\nu} a^c_{\mu} = \lambda a^a_{\nu}. \tag{2.16}
\]

It is convenient to decompose the field \( a_\mu \) into the neutral and charged components:

\[
a_\mu = (a^3_\mu, a^+_{\mu}, a^-_{\mu}), \quad a_\mu = a^1_{\mu} + i a^2_{\mu}, \quad a^-_{\mu} = a^1_{\mu} - i a^2_{\mu}. \tag{2.17}
\]

For the neutral and charged components we will get:

\[
-\partial_\mu (\partial_\nu a^3_{\mu} - \partial_\nu a^3_{\nu}) = \lambda a^3_{\nu}, \quad -\nabla_\mu (\nabla_\nu a^1_{\mu} - \nabla_\nu a^1_{\mu}) + ig F^a_{\mu\nu} a_{\mu} = \lambda a_{\nu}. \tag{2.18}
\]

\(^3\)We will consider the \( SU(2) \) gauge group in the remaining part of the article.
where $\nabla_\mu = \partial_\mu - \frac{1}{2}igF_{\mu\nu}x_\nu$. Taking into account that

$$[\nabla_\mu, \nabla_\nu] = igF_{\mu\nu}$$

and imposing the background gauge fixing condition on the quantum field \[3, 4, 5\]

$$\nabla_\mu \nabla_\mu = 0 \quad (2.19)$$

we will get for the charged components the following equation:

$$- \nabla_\mu \nabla_\mu a_\nu + 2igF_{\mu\nu}a_\mu = \lambda a_\nu. \quad (2.20)$$

Taking the interpolating field (2.14) in the z-direction (2.15) we will get

$$H_0 = -\nabla_\mu \nabla_\mu = -\left(\partial_\mu - \frac{1}{2}igF_{\mu\nu}x_\nu\right)\left(\partial_\mu - \frac{1}{2}igF_{\mu\rho}x_\rho\right) =$$

$$= -\partial_1^2 - \partial_2^2 + igH(x_2\partial_1 - x_1\partial_2) + \frac{g^2}{4}H^2(x_2^2 + x_1^2)$$

$$-\partial_3^2 - \partial_0^2 + igE(x_0\partial_3 - x_3\partial_0) + \frac{g^2}{4}E^2(x_3^2 + x_0^2). \quad (2.21)$$

$H_0$ is a sum of isomorphic oscillators in the $(1, 2)$ and $(3, 0)$ planes. Introducing the operators \[12, 13\]

$$c_i = \partial_i + \frac{g}{2}Hx_i, \quad c_i^+ = -\partial_i + \frac{g}{2}Hx_i, \quad i = 1, 2$$

$$d_j = \partial_j + \frac{g}{2}Ex_j, \quad d_j^+ = -\partial_j + \frac{g}{2}Ej, \quad j = 3, 0 \quad (2.22)$$

one can find that (see Appendix A)

$$H_0 = (c_1^+ + ic_2^+)(c_1 - ic_2) + (d_3^+ + id_0^+)(d_3 - id_0) + gH + gE. \quad (2.23)$$

The eigenstates of the operator $H_0$ therefore are:

$$\psi_{nm} = (c_1^+ + ic_2^+)^n(d_3^+ + id_0^+)^m\psi_{00} = (gH)^n(x_1 + ix_2)^n(gE)^m(x_3 + ix_0)^m\psi_{00}, \quad (2.24)$$

where

$$\psi_{00} = e^{-\frac{gH}{4}(x_1^2 + x_2^2)}e^{-\frac{gE}{4}(x_3^2 + x_0^2)} \quad (2.25)$$

and the corresponding eigenvalues $H_0\psi_{nm} = \lambda_0\psi_{nm}$ are:

$$\lambda_0 = (2n + 1)gH + (2m + 1)gE. \quad (2.26)$$

All eigenstates have infinite degeneracy because the states

$$\psi_{nm}(n_0, m_0) = (c_1^+ - ic_2^+)^{n_0}(d_3^+ - id_0^+)^{m_0}\psi_{nm} \quad (2.27)$$
have identical eigenvalues (2.26) and are indexed by two integers \( n_0, m_0 = 0, 1, 2, ... \) (see Appendix A for details). Now we can turn to the investigation of the eigenstates of the operator for the charged field (2.17)

\[
H_{\mu\nu} = -g_{\mu\nu} \nabla_\lambda \nabla_\lambda - 2igF_{\mu\nu},
\]

that appears in the equation (2.20). They are:

\[
b = a_1 + ia_2, \quad b^- = a_1 - ia_2, \quad h = a_3 + ia_0, \quad h^- = a_3 - ia_0.
\]

(2.29)

The eigenvalues corresponding to the fields \( b, b^- \) and \( h, h^- \) of the operator \( H_{\mu\nu} \) take the following form:

\[
\begin{align*}
  b^- &: \quad \lambda_1 = (2n + 1)gH + (2m + 1)gE + 2gH \\
  b &: \quad \lambda_2 = (2n + 1)gH + (2m + 1)gE - 2gH \\
  h^- &: \quad \lambda_3 = (2n + 1)gH + (2m + 1)gE + 2gE \\
  h &: \quad \lambda_4 = (2n + 1)gH + (2m + 1)gE - 2gE,
\end{align*}
\]

(2.30)

where \( n, m = 0, 1, 2,... \). For the conjugate field \( a^-_\mu \) one can find the identical eigenvalues \( \lambda_i, i = 5, ..., 8 \). As one can see, the negative eigenvalues appear in \( \lambda_2, \lambda_4 \) if \( E \neq H \). It follows that only in the case of (anti)self-dual interpolating field \([12, 13]\)

\[
E = H
\]

(2.31)

there are no negative/unstable eigenvalues in the spectrum. It also follows that when \( n = m = 0 \), there is an infinite number of zero eigenvalues \( \lambda_2 = \lambda_4 = 0 \) that appear due to the high symmetry of the self-dual field \( E = H \) and the degeneracy (2.27). These are Leutwyller zero mode chromons and they are linear combinations of (2.27) \((b^- = a_1 - ia_2 = 0, \ h^- = a_3 - ia_0 = 0)\):

\[
\begin{align*}
  a_1(\xi) &= ia_2(\xi) = \sum_{n_0, m_0} \xi_{n_0m_0} (c_1^+ - ic_2^+)^{n_0}(d_3^+ - id_4^+)^{m_0}\psi_{00} \\
  a_3(\eta) &= ia_0(\eta) = \sum_{n_0, m_0} \eta_{n_0m_0} (c_1^+ - ic_2^+)^{n_0}(d_3^+ - id_4^+)^{m_0}\psi_{00},
\end{align*}
\]

(2.32)

where the zero-mode amplitudes (collective variables) \( \xi_{n_0m_0} \) and \( \eta_{n_0m_0} \) are arbitrary complex numbers.\(^4\) For the zero modes the quadratic form (2.12) vanishes \( K_E = 0 \) and the stability of the interpolating field \( B_\mu(x) \) is determined by the nonlinear interaction term \( e^{-VE} \) (2.13) of the action \( S_E \).

\(^4\)The zero mode fields modify the interpolating field \( B_\mu \to B_\mu + a_\mu(\xi, \eta) \) in such a fashion that it remains self-dual even for arbitrary large zero-mode amplitudes \([12, 13]\). The interpolating field \( B_\mu \) that is the solution of the YM equation of motion (2.11) and boundary values (2.31) was found by Minkowski \([14]\).
In his original article Leutwyler remarked: "It does not seem to be possible to evaluate the integral over all self-dual fields exactly." And then: "Our motivation for restricting ourselves to small chromon amplitudes is of a technical nature: we do not know how to do better." Our aim is to calculate the contribution of the zero modes exactly. In order to calculate the contribution of zero modes we will suggest two alternative methods in the forthcoming sections. First we will develop the method of the infrared regularisation of zero modes.

3 Effective Lagrangian. Contribution of Positive/stable Modes

The Euclidean path integral \(\mathcal{L}_{\text{positive modes}}^{(1)}\) for positive/stable eigenvalues of the quadratic form \(K_E\) is defined through the determinants of the operators \(H_{\mu\nu}\) and \(H_0\) [3, 4]:

\[
\mathcal{L}_{\text{positive modes}}^{(1)} V_3 T = \frac{1}{2} \ln \text{Det} H - \ln \text{Det} H_0 = \frac{1}{2} \text{Deg} \sum_{n,m,i} \ln \lambda_i(n,m) - \text{Deg} \sum_{n,m} \ln \lambda_0(n,m) =
\]

\[
= -\frac{1}{2} \text{Deg} \int \frac{ds}{s} e^{-\sum_{n,m,i} \lambda_i(n,m)s} + \text{Deg} \int \frac{ds}{s} e^{-\sum_{n,m} \lambda_0(n,m)s},
\]

(3.33)

where \(\text{Deg} = \left(\frac{gH}{2\pi}\right)^2 V_3 T\) is the degeneracy of the eigenstates (see Appendix B (11.130)). After substituting the eigenvalues (2.30) we will get (see Appendix C)

\[
\mathcal{L}_{\text{positive modes}}^{(1)} = -\frac{g^2 H^2}{4\pi^2} \int \frac{ds}{s} \sum_{n,m=0}^{\infty} \left( e^{-2gH(n+m+2)s} + e^{-2gH(n+m)s} - e^{-2gH(n+m+1)s} \right)
\]

\[
= -\frac{1}{8\pi^2} \int \frac{ds}{s} \frac{g^2 H^2}{\sinh^2 gHs}.
\]

(3.34)

By using the renormalisation condition [1, 2]

\[
\frac{\partial \mathcal{L}}{\partial \mathcal{F}_E} |_{g^2 \mathcal{F}_E = \mu^4} = 1,
\]

(3.35)

where \(\mathcal{F}_E = \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a = H^2\) for the contribution of the positive/stable modes, we will get:

\[
\mathcal{L}_{\text{positive modes}}^{(1)} = -\frac{1}{8\pi^2} \int \frac{ds}{s} \left[ \frac{g^2 H^2}{\sinh^2 gHs} - \frac{g^2 H^2}{\sinh^2 \mu^2s} + \frac{g^2 H^2 \mu^2 s \cosh \mu^2 s}{\sinh^3 \mu^2 s} \right] = -\frac{g^2 H^2}{48\pi^2} \left[ \ln \frac{2g^2 H^2}{\mu^4} - 1 \right].
\]

(3.36)

The zero eigenvalues in (2.30) and the corresponding eigenfunctions (2.32) that appear when \(n = m = 0\) should be considered separately. In quadratic approximation the zero-mode fields are not suppressed by the exponential factor \(e^{-K_E}\) because \(K_E = 0\) and the zero modes amplitudes can grow unboundly. The quantum field \(a_{\mu}^a(x)\) can deviate considerably from the self-dual field \(B_{\mu}(x)\) in the neighbourhood of which we are investigating. It merely means that the integration in these directions
is not Gaussian [5]. In order to handle the zero modes contribution one should replace the measure $\mathcal{D}\vec{A}(\tau)$ in the Euclidean path integral (2.2) by introducing the collective variables in the zero-mode directions (2.32):

$$\prod_{a,\mu} \mathcal{D}a^a_{\mu} = J \prod_{\eta,\mu=0} \mathcal{D}\xi_{\eta,\mu} \mathcal{D}\eta_{\eta,\mu} \mathcal{D}\eta^\prime_{\eta,\mu},$$

(3.37)

where $\xi_{n,\mu} = \zeta_{n,\mu} + i\zeta''_{n,\mu}$, $\eta_{n,\mu} = \eta'_{n,\mu} + i\eta''_{n,\mu}$ and $J$ is the Jacobian factor [5, 43]. As far as the quadratic part of the action $S_E$ does not suppress the zero mode fluctuations we have to consider the contribution of zero modes by taking into account the cubic and quartic terms $e^{-V_E}$ in the Euclidean action $S_E$ (2.13). We will develop this approach in the forthcoming sections.

We found it useful to develop first an alternative approach that allows to calculate the zero mode contribution to the path integral by suggesting an appropriate infrared regularisation. As we will see at the end of this investigation, both methods, the infrared regularisation and integration over the nonlinear interaction of zero modes, lead to the same result (4.42).

4 Infrared Regularisation of Zero Modes

Instead of performing the integration over the zero modes collective variables $(\xi, \eta)$ in the path integral (2.2), in this section we will consider an appropriate infrared regularisation of the spectrum (2.30). For that we will infinitesimally elevate the zero modes by adding the terms $\alpha gH$ and $\alpha gE$ to the zero eigenvalues spectrum (2.30):

$$\lambda_2 = (2n + 1)gH + (2m + 1)gE - 2gH + \alpha gH$$
$$\lambda_4 = (2n + 1)gH + (2m + 1)gE - 2gE + \alpha gE,$$

(4.38)

where $\alpha$ is a dimensionless infinitesimal parameter. As we will see, the result does not depend on $\alpha$. For the self-dual field $E = H$ the contribution of these modes $\lambda_{2,4} = (2n + 2m + \alpha)gH$ at $n = m = 0$ to the effective Lagrangian will be

$$\mathcal{L}_{\text{zero modes}}^{(1)} = -\frac{g^2H^2}{4\pi^2} \int \frac{ds}{s} e^{-\alpha gHs}.$$  

(4.39)

Thus the contribution of positive (3.34) and zero modes (4.39) to the effective Lagrangian will take the following form:

$$\mathcal{L}_E^{(1)} = \mathcal{L}_{\text{positive modes}}^{(1)} + \mathcal{L}_{\text{zero modes}}^{(1)} = -\frac{1}{2\pi^2} \int \frac{ds}{s} \left[ \frac{g^2H^2}{4\sinh^2\frac{gHs}{2}} + g^2H^2e^{-\alpha gHs} \right].$$

(4.40)

The Jacobian $J$ here is a gauge field strength independent function.
By using the renormalisation condition \((3.35)\) for the effective Lagrangian we will get

\[
\mathcal{L}_E^{(1)} = -\frac{1}{2\pi^2} \int \frac{ds}{s} \left[ \frac{g^2 H^2}{4 \sinh^2 gHs} - \frac{g^2 H^2}{4 \sinh^2 \mu^2 s} + \frac{g^2 H^2 \mu^2 s \cosh \mu^2 s}{4 \sinh^3 \mu^2 s} + \frac{g^2 H^2 \alpha^2 s}{2} \left( e^{-\alpha^2 Hs} - e^{-\alpha^2 \mu^2 s} + \alpha^2 \mu^2 s \right) \right] =
\]

\[
= -\frac{g^2 H^2}{48\pi^2} \left[ \ln \frac{2g^2 H^2}{\mu^4} - 1 \right] + \frac{g^2 H^2}{4\pi^2} \left[ \ln \frac{2g^2 H^2}{\mu^4} - 1 \right].
\]  

(4.41)

As one can see from the above result, the Lagrangian does not depend on the infrared regularisation parameter \(\alpha\) that was introduced earlier to elevate the zero mode degeneracy. This is in a good agreement with the independence of the effective Lagrangian on the other infrared regularisation parameter \(\mu\) in \((3.35)\) \([1, 2]\). It follows that the effective Lagrangian will take the following form:

\[
\mathcal{L}_E = H^2 + \frac{11g^2 H^2}{48\pi^2} \left[ \ln \frac{2g^2 H^2}{\mu^4} - 1 \right].
\]  

(4.42)

The Euclidean (anti)self-dual interpolating field is stable and the corresponding boundary covariantly-constant vacuum field is also stable against quantum fluctuations.

There is a perfect consistency between the result obtained in Euclidean formulation and the one obtained in a physical space-time \([1, 18]\):

\[
\mathcal{L}_{YM} = -\mathcal{F} - \frac{11g^2 \mathcal{F}}{48\pi^2} \left[ \ln \frac{2g^2 \mathcal{F}}{\mu^4} - 1 \right], \quad \mathcal{F} = \frac{1}{4} G_\mu^a G^a_{\mu \nu} > 0.
\]  

(4.43)

As it was already mentioned, the electric field in a physical space-time \(\vec{E}\) corresponds to an imaginary electric field in Euclidian space \(\vec{E} \to i\vec{E}\), therefore one can map the gauge invariant operators as

\[
\mathcal{F} = \frac{1}{2} \left( \vec{H}^2 - \vec{E}^2 \right) \Rightarrow \frac{1}{2} (\vec{H}^2 + \vec{E}^2)_E = \mathcal{F}_E.
\]  

(4.44)

For the self-dual vacuum field \(E = H\) we have \(\mathcal{F}_E = H^2\), and the effective Lagrangian \((4.43)\) transforms into the Euclidean effective Lagrangian \((4.42)\). Because the invariant \(\mathcal{F}_E\) is a positive quantity, it follows that in a physical space-time the corresponding \(\mathcal{F}\) is also positive and corresponds to the chromomagnetic field \([1, 18]\). A similar result takes place for the anti-self-dual field.

5  Nonlinear Interaction of Zero Modes

Here we will turn to the second approach taking into account a nonlinear interaction of zero modes by replacing the measure \(\mathcal{D} \vec{A}(\tau)\) in the Euclidean path integral \((2.2)\) by using the collective variables in the zero-mode directions \((3.37)\):

\[
\prod_{a, \mu} D a^a_\mu = J_\text{Deg} \prod_{\nu_{a, m} = 0} d\xi_{\nu_{a, m}} d\xi''_{\nu_{a, m}} d\eta_\nu d\eta''_\nu.
\]  

(5.45)
As far as the quadratic part \((2.12)\) of the action \(S_E\) vanishes and does not suppress the zero-mode fluctuations, it follows that one should take into account the contribution of the zero-mode fields through the cubic and quartic interactions in the Euclidean action \(S_E\) \((2.13)\). The cubic self-interaction term \((2.13)\) vanishes in the zero mode directions \((2.32)\):

\[
V_E^{(3)} = -g e^{abc} \int d^4x \, a^b_\mu a^c_\mu \nabla^a_\mu a^d_\nu = 0. \tag{5.46}
\]

Only the quartic interaction term is not vanishing in the zero mode directions and has the following form:

\[
Z^{\text{zero modes}} = N \int \exp \left[ -\frac{g^2}{4} \int d^4x \left( (a^0_\mu a^0_\mu)^2 - (a^0_\mu a^0_\nu)^2 \right) \right] \prod_{a,\mu} D a^0_\mu, \tag{5.47}
\]

where the integration is over all zero modes. The above zero mode partition function \((5.47)\), if expanded in the coupling constant \(g^2\), will generate an infinite number of multi-loop diagrams, and it seems impossible to calculate them. Nevertheless the exact calculation of the zero mode partition function is possible because the field strength dependence within the path integral \((5.47)\) can be factorised.

Let us first consider the lowest state \((2.25)\) when \(n = m = n_0 = m_0 = 0\) in \((2.24)\) and \((2.27)\). The solution \((2.32)\) corresponding to the lowest state with \(n_0 = m_0 = 0\) is:

\[
\begin{align*}
    a^1_1 &= \xi \psi_{00}, \quad a^2_1 = \xi'' \psi_{00}, \quad a^3_1 = 0 \\
    a^1_2 &= \xi'' \psi_{00}, \quad a^2_2 = -\xi \psi_{00}, \quad a^3_2 = 0 \\
    a^1_3 &= \eta \psi_{00}, \quad a^2_3 = \eta'' \psi_{00}, \quad a^3_3 = 0 \\
    a^1_0 &= \eta'' \psi_{00}, \quad a^2_0 = -\eta \psi_{00}, \quad a^3_0 = 0,
\end{align*}
\]

where \(\xi_{00} = \xi' + i\xi''\) and \(\eta_{00} = \eta' + i\eta''\). For that lowest zero mode field \((2.32), (5.48)\) the quartic term will take the following form (see Appendix B for details):

\[
V_E^{(4)} = \frac{g^2}{4} \int d^4x (e^{abc} a^b_\mu a^c_\mu)^2 = \frac{g^2}{2} (\xi'^2 + \xi''^2 + \eta'^2 + \eta''^2) \int d^4x |\psi_{00}(x)|^4, \tag{5.49}
\]

and the corresponding part of the partition function can be represented in the following form:

\[
Z^{\text{zero mode}} = \mu^4 \int \exp \left[ -\frac{g^2}{2} (\xi'^2 + \xi''^2 + \eta'^2 + \eta''^2) \int d^4x |\psi_{00}|^4 \right] d\xi' d\xi'' d\eta' d\eta'',
\]

\[
= \mu^4 \int \exp \left[ -\frac{g^2}{2} \left( \frac{gH}{4\pi} \right)^2 (\xi'^2 + \xi''^2 + \eta'^2 + \eta''^2) \right] d\xi' d\xi'' d\eta' d\eta'', \tag{5.50}
\]

where we used the expression \((12.135)\). Introducing the dimensionless variables \((\xi, \eta) \rightarrow (\xi, \eta)/(gH)^{1/2}\) allows to factorise the field strength dependence in the path integral, and we will get

\[
Z^{\text{zero mode}} = \left( \frac{\mu^2}{gH} \right)^2 \int \exp \left[ -\frac{g^2}{32} (\xi'^2 + \xi''^2 + \eta'^2 + \eta''^2) \right] d\xi' d\xi'' d\eta' d\eta'' = N_{00} \left( \frac{\mu^2}{gH} \right)^2. \tag{5.51}
\]
For the general zero mode field \( \psi_{00}(n_0, m_0) \) we will have (see Appendix B [12.139])

\[
Z_{\text{zero mode}}^{n_0,m_0} = \left( \frac{\mu^2}{gH} \right)^2 \int \exp \left[ -\frac{g^2}{32} \frac{\Gamma(n_0 + 1/2)\Gamma(m_0 + 1/2)}{\pi \Gamma(n_0 + 1)\Gamma(m_0 + 1)} \left( |\xi_{n_0,m_0}|^2 + |\eta_{n_0,m_0}|^2 \right)^2 \right] \, d^2\xi_{n_0,m_0} d^2\eta_{n_0,m_0}
\]

\[
= N_{n_0,m_0} \left( \frac{\mu^2}{gH} \right)^2.
\]

(5.52)

In order to calculate the contributions of all individual self-interacting zero-modes one should evaluate the product \( J \prod_{n_0=0,m_0=0}^{\text{Deg}} Z_{\text{zero mode}}^{n_0,m_0} \) taking into account the degeneracy of the zero modes (11.128):

\[
Z_{\text{zero modes}} = J \prod_{n_0=0,m_0=0}^{\text{Deg}} N_{n_0,m_0} \left( \frac{\mu^4}{g^2H^2} \right) = N \left( \frac{\mu^4}{g^2H^2} \right)^{\text{Deg}} = Ne^{-\frac{g^2H^2}{4\pi^2} \ln \frac{g^2H^2}{\mu^4} V_3T},
\]

(5.53)

where \( N = J \prod_{n_0=0,m_0=0}^{\text{Deg}} N_{n_0,m_0} \). In the limit \( T \to \infty \) we have \( Z_{\text{zero modes}} \to Ne^{-\mathcal{L}_{\text{eff}} V_3T} \) and for the zero mode contribution to the effective Lagrangian we will get

\[
\mathcal{L}_{\text{self-interacting zero modes}} = \frac{g^2H^2}{4\pi^2} \ln \frac{g^2H^2}{\mu^4}.
\]

(5.54)

This expression coincides with our previous result (4.39), (4.41) obtained by using the infrared regularisation. Now, adding the contribution of positive/stable modes (3.36) to the (5.54) for the effective Lagrangian we will get

The above result (5.53) pointed out to the fact that exact integration over the zero modes (2.32) can be performed not only for self-interacting but also for fully interacting zero modes. The full interaction term has the following form (to be compared with (12.138)):

\[
V_E = \frac{g^2}{4} \int d^4 x (e^{abc} g_\mu^a g_\nu^b x_{\nu}^c)^2 = \frac{g^2}{2} \int d^4 x \left[ \sum_{n_0,m_0} \xi_{n_0,m_0} \psi_{00}(n_0, m_0; x) \right]^2 + \sum_{n_0,m_0} \eta_{n_0,m_0} \psi_{00}(n_0, m_0; x) \right]^2 \right]^2 \, d^4 y.
\]

(5.55)

The corresponding partition function will take the following form:

\[
Z_{\text{zero modes}} = \int \exp \left\{ -\frac{g^2(gH)^2}{2} \int \left( \sum_{n_0,m_0} \xi_{n_0,m_0} \psi_{00}(n_0, m_0; y) \right)^2 + \sum_{n_0,m_0} \eta_{n_0,m_0} \psi_{00}(n_0, m_0; y) \right)^2 \right\} \, d^4 y
\]

\[
\times \prod_{n_0,m_0}^{\text{Deg}} \mu^4 d\xi_{n_0,m_0}' d\xi_{n_0,m_0}'' d\eta_{n_0,m_0}' d\eta_{n_0,m_0}'',
\]

(5.56)

where we introduced the dimensionless variables \( y_i = x_i (gH)^{1/2} \) in the integral \( V_E \), which allows to factorise the field strength dependence in the exponent of the path integral (5.56). The wave functions \( \psi_{00}(n_0, m_0; x) \) were normalised as in (12.139), and in terms of the dimensionless variables they are:

\[
\psi_{00}(n_0, m_0; y) = \frac{1}{\sqrt{\pi^2 n_0^2 + m_0^2 + 1}} \exp \left\{ -\frac{y_1^2 + y_2^2 + y_3^2 + y_4^2}{4} \right\}.
\]

6I would like to thank Konstantin Savvidy for pointing out to me the possibility of such generalisation.
By using the dimensionless collective variables \((\xi, \eta) \rightarrow (\xi, \eta)/(gH)^{1/2}\) introduced above (5.52) we get

\[
Z_{\text{zero modes}} = \int \exp \left\{ -\frac{g^2}{2} \int \left( \left| \sum_{n_0, m_0} \xi_{n_0m_0} \psi_{n_0m_0}(y) \right|^2 + \left| \sum_{n_0, m_0} \eta_{n_0m_0} \psi_{n_0m_0}(y) \right|^2 \right)^2 d^4 y \right\}
\]

\[
J \prod_{n_0, m_0} \left( \frac{\mu^4}{g^2 H^2} \right)^2 d\xi'_{n_0m_0} d\xi''_{n_0m_0} d\eta'_{n_0m_0} d\eta''_{n_0m_0}
\]

(5.57)

and the field strength dependence completely factorises as well and we have:

\[
Z_{\text{zero modes}} = N \prod_{n_0, m_0} \left( \frac{\mu^4}{g^2 H^2} \right) = N \left( \frac{\mu^4}{g^2 H^2} \right)^{\text{Deg}} = Ne^{-\frac{g^2 \mu^2}{4\pi^2} \ln \frac{g^2 \mu^2}{\mu^4} V_3 T},
\]

(5.58)

where

\[
N = \int \exp \left\{ -\frac{g^2}{2} \int \left( \left| \sum_{n_0, m_0} \xi_{n_0m_0} \psi_{n_0m_0}(y) \right|^2 + \left| \sum_{n_0, m_0} \eta_{n_0m_0} \psi_{n_0m_0}(y) \right|^2 \right)^2 d^4 y \right\}
\]

\[
J \prod_{n_0, m_0} d\xi'_{n_0m_0} d\xi''_{n_0m_0} d\eta'_{n_0m_0} d\eta''_{n_0m_0}.
\]

(5.59)

In the limit \(T \to \infty\) from (5.58) for the zero mode effective Lagrangian we will get

\[
\mathcal{L}_{\text{interacting zero modes}} = \frac{g^2 H^2}{4\pi^2} \ln \frac{g^2 H^2}{\mu^4}.
\]

(5.60)

The effective Lagrangian is a sum of positive/stable (3.36) and fully interacting zero modes (5.60):

\[
\mathcal{L}^{\text{eff}}_E = H^2 + \mathcal{L}^{(1)}_{\text{positive modes}} + \mathcal{L}_{\text{interacting zero modes}} = H^2 + \frac{11g^2 H^2}{48\pi^2} \left[ \ln \frac{2g^2 H^2}{\mu^4} - 1 \right],
\]

(5.61)

and it coincides with our previous results (4.42) and (5.54).

It is remarkable that the zero mode contribution that was calculated in terms of infrared regularisation of the spectrum (4.38), (4.39), (4.41), by integration over the self-interacting zero modes in (5.53) and then by integration over fully interacting zero modes in (5.58), all lead to the same result indicating the robustness of the logarithmic structure of the effective Lagrangian and that it is without an imaginary part [1]. It is interesting to investigate to what extent this behavioural robustness is rooted in the entropy factor, through the degeneracy of the positive and zero mode states and through the scaling invariance of the Yang-Mills action.

6 Deformation of (Anti)Self-Dual Field

Breaking the (anti)self-duality condition (2.31) will create negative/unstable modes (2.30) if we are considering the stability problem of the vacuum fields in the quadratic approximation [12, 13].

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The question is if the interaction of negative/unstable modes will improve the stability of general covariantly-constant chromomagnetic vacuum fields \( H > E \). As we will show below, even in that case, when there is a plethora of negative/unstable modes in the quadratic approximation, the vacuum fields turn out to be stable due to the nonlinear interaction of negative/unstable modes. Here we will generalise the calculation that were advocated earlier by Ambjorn, Nielsen, Olesen \[8, 9, 10, 11\], Flory \[15\] and other authors \[12, 13, 15, 16, 17, 18, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39\] by taking into account the interaction of negative/unstable modes.

Let us perform a deformation of the self-duality condition (2.31) of the interpolating field \( B^a_\mu(x) \) \( (2.15) \) in the following manner \[15\]:

\[
F_{12} = H, \quad F_{30} = E = \gamma H, \quad \gamma \leq 1,
\]

where \( \gamma \) is a real parameter that breaks the self-duality condition (2.31). At \( \gamma = 1 \) we will have the self-dual interpolating field (2.31). The consequences of this deformation are two-fold: First of all, there will be positive/stable modes in the spectrum when \( \gamma \neq 1 \) and, secondly, instead of zero modes the negative/unstable modes will appear in the spectrum. The problem of integration over a nonlinearly interacting zero modes now turns into the problem of integration over a nonlinearly interacting negative/unstable modes. After the substitution \( E \to \gamma H \) in (2.30) the spectrum will take the following form:

\[
\begin{align*}
b^- : & \quad \lambda_1 = (2n + 1)gH + (2m + 1)\gamma gH + 2gH \\
b : & \quad \lambda_2 = (2n + 1)gH + (2m + 1)\gamma gH - 2gH \\
h^- : & \quad \lambda_3 = (2n + 1)gH + (2m + 1)\gamma gH + 2\gamma gH \\
h : & \quad \lambda_4 = (2n + 1)gH + (2m + 1)\gamma gH - 2\gamma gH,
\end{align*}
\]

(6.63)

where the eigenvalues \( \lambda_2 \) are negative when \( n = 0 \) and \( 0 \leq m \leq \frac{1-\gamma}{2\gamma} \):

\[
\lambda_2 \mid_{n=0} = (2m\gamma - 1 + \gamma)gH.
\]

(6.64)

When \( \gamma \) is a small number close to zero, the amount of negative/unstable modes is large, and when \( \gamma \) tends to one, the number of negative/unstable modes decreases, and they completely disappear at \( \gamma = 1 \). At \( \gamma = 1 \) instead of negative/unstable modes we will have the zero modes \( \lambda_2 \) and \( \lambda_4 \) (2.30), the case of the (ani)self-dual interpolating field that we already considered in the previous sections.

As we argued above, it is crucial to consider the nonlinear interaction of negative/unstable modes. Let us first calculate the contribution of the positive/stable modes. Substituting the eigenvalues
where $\zeta$ This expression represents a converging integral in the infrared region $(m \to 0)$ and the corresponding field components (2.29) are: 

\[
\gamma \text{ and in the case of (anti)self-dual field } \gamma = 1 \text{ the } L^{(1)}_{\text{positive modes}} \text{ reduces to the expression } (3.36). \]

Our main concern now is the contribution of the negative/unstable modes. The corresponding wave functions (2.24) are:

\[
\psi_{0m} = \zeta_m (d^+_3 + id^+_0)^m \psi_{00}, \quad n = 0, \quad 0 \leq m \leq \frac{1}{2\gamma} - \frac{1}{2},
\]

and the corresponding field components (2.24) are:

\[
b = a_1 + ia_2 = \zeta_m (d^+_3 + id^+_0)^m \psi_{00}, \quad b^- = a_1 - ia_2 = 0,
\]

where $\zeta_m$ are the amplitudes of the negative/unstable modes and $b = b^- = 0$. For the lowest state $m = 0$ we have $\zeta_0 \equiv \zeta' + i\zeta''$:

\[
a^1_0 = 0, \quad a^2_0 = 0, \quad a^3_0 = 0.
\]
The contribution of the negative/unstable modes will take the following form:

\[ Z_{\text{negative modes}} = \int \exp \left[ - \int d^4x \left( \frac{1}{2} a^a_{\mu}(-\delta_{\mu\nu} \nabla_r \nabla^r - 2gf^{abc}G^{b\mu}_{\nu}a^c_{\nu} + \frac{g^2}{4}((a^a_{\mu}a^a_{\mu})^2 - (a^a_{\mu}a^a_{\nu})^2)) \right) \right] \prod_{a\mu} \mathcal{D}a^a_{\mu}, \]  

(6.71)

where the integration is over all negative/unstable modes:

\[ \prod_{a\mu} \mathcal{D}a^a_{\mu} = \mathcal{J} \prod_{n_0,m_0} \prod_{m=0} d\zeta'_m d\zeta''_m. \]  

(6.72)

The computation of the above negative mode partition function seems impossible if one ignores the scaling invariance of the Yang-Mills action and the degeneracy of the quantum states. At first we will consider the integration only over self-interacting negative/unstable modes.

The cubic self-interaction term (5.46) for the negative/unstable modes (6.69) vanishes. For the negative/unstable modes the quartic term is

\[ V_E = \frac{g^2}{2} |\zeta_m|^4 \int d^4x |\psi_{0m}(x)|^4 \]  

(6.73)

and the four-dimensional Euclidean integral is

\[ \frac{1}{C_0^2} \frac{1}{C_m^2} \int dx_1 dx_2 dx_3 dx_0 |\psi_{0m}(x)|^4 = \frac{\gamma(\frac{gH}{4\pi})^{2}}{\pi^\frac{1}{4} \Gamma(m+1) \Gamma(m+2)}} \]  

(6.74)

By substituting the negative/unstable modes (6.64), (6.69) into the (6.71) one can get:

\[ Z_{\text{self-interacting negative modes}} = \mu^2 \prod_{m=0} \left( \frac{\mu^2}{gH} \right) \int \exp \left[ gH(1 - \gamma - 2m\gamma)|\zeta_m|^2 - \frac{\gamma g^4 H^2}{32\pi^2} \frac{\Gamma(m+1/2)}{\Gamma(m+1)} |\zeta_m|^4 \right] d\zeta'_m d\zeta''_m, \]  

(6.74)

where the product is over all negative/unstable modes (6.68). Introducing dimensionless collective variables \( \zeta_m \rightarrow \zeta_m/(gH)^{1/2} \) one can factorise the field dependence from each integral in the product, and we will get

\[ Z_{\text{self-interacting negative modes}} = \prod_{m=0} \left( \frac{\mu^2}{gH} \right) \int \exp \left[ (1 - \gamma - 2m\gamma)|\zeta_m|^2 - \frac{\gamma g^4 H^2}{32\pi^2} \frac{\Gamma(m+1/2)}{\Gamma(m+1)} |\zeta_m|^4 \right] d\zeta'_m d\zeta''_m, \]  

(6.75)

Evaluating the integrals and the product one can get

\[ Z_{\text{self-interacting negative modes}} = \prod_{m=0} N_m \left( \frac{\mu^2}{gH} \right) = Ne^{-\frac{(1+\gamma) \gamma}{2}} \ln \frac{gH}{\mu^2}. \]  

(6.76)
Taking into account the degeneracy of each negative/unstable mode one can get

$$Z_{\text{negative modes}} = \prod_{\text{self-interacting}} Z_{\text{negative modes}} = N e^{-\frac{2a^2\mu^2(1+\gamma)}{4\pi^2} \ln \frac{gH}{\nu} V_0 T},$$  \hspace{1cm} (6.77)

where in the given case \(6.62\) the degeneracy \(11.130\) is \(D_{\text{eg}} = \frac{g^2H^2}{4\pi^2} V_3 T\), and we will get

$$\mathcal{L}_E^{\text{negative modes}} = \frac{g^2H^2}{8\pi^2} (1+\gamma) \ln \frac{gH}{\mu^2}.$$  \hspace{1cm} (6.78)

The total unrenormalised effective Lagrangian is a sum of the positive modes contribution \(6.66\) and of the self-interacting negative/unstable modes \(6.78\).

The above consideration of self-interacting negative/unstable modes shows that the exact integration over these modes can be performed for fully interacting modes. The general negative/unstable mode wave function that includes all degenerate states \(\psi_{0m}(n_0, m_0)\) is given in \(2.24\) and \(2.27\), and the gauge field components \(2.29\) have the following form:

$$a_1(\zeta) = ia_2(\zeta) = \sum_{n_0, m_0} \sum_{m=0}^{1/2} \zeta_m(n_0, m_0) (c_1^{+} - i c_2^{+})^{n_0} (d_1^{+} - i d_0^{+})^{m_0} (d_2^{+} + i d_1^{+})^{m_0} \psi_{00}(x)$$

$$a_3(\eta) = ia_0(\eta) = 0,$$  \hspace{1cm} (6.79)

where \(\zeta_m(n_0, m_0)\) are the amplitudes (collective variables) of the negative/unstable modes that should be normalised similarly to \(12.139\). The cubic interaction term \(5.46\) vanishes and the quartic term gives

$$V_E = \frac{g^2}{2} \int \left| \sum_{n_0, m_0} \sum_{m=0}^{1/2} \zeta_m(n_0, m_0) \psi_{0m}(n_0, m_0; x) \right|^4 d^4 x,$$  \hspace{1cm} (6.80)

and it can be transformed into the following expression:

$$V_E = \gamma \frac{g^2(gH)^2}{2} \int \sum_{n_0, m_0} \sum_{m=0}^{1/2} \zeta_m(n_0, m_0) \psi_{0m}(n_0, m_0; y) \right|^4 d^4 y,$$  \hspace{1cm} (6.81)

where, as above in \(5.30\), we introduced the dimensionless variables \(y_i = x_i(gH)^{1/2}\), and the field strength dependence completely factorises from the wave function \(\psi_{0m}(n_0, m_0; y)\). Now the integration measure for the negative/unstable modes in the partition function \(6.71\) is

$$\prod_{a_{\mu}} D a_{\mu}^a = J \prod_{n_0, m_0} \prod_{m=0}^{1/2} d\zeta_m(n_0, m_0) d\zeta_{m'}(n_0, m_0).$$  \hspace{1cm} (6.82)

The contribution of the negative/unstable modes to the partition function will take the following
Thus the contribution of negative/unstable modes that follows from the expression (6.85) is

\[
\mathcal{L}_{\text{negative modes}} = \int \exp \left\{ \sum_{n_0, m_0, m} gH(1 - \gamma - 2m\gamma)|\zeta_m(n_0, m_0)|^2 - \gamma \frac{g^2}{2} \frac{H^2}{2} \right\} \left| \sum_{n_0, m_0, m} \zeta_m(n_0, m_0) \psi_0 m(n_0, m_0; y) \right|^4 d^4 y \right\}
\]

\[
J \prod_{n_0, m_0 = 0}^{1-\gamma \frac{1}{2\gamma}} \prod_{m=0}^{1-\gamma \frac{1}{2\gamma}} d\zeta_m(n_0, m_0)d\zeta_m^\prime(n_0, m_0).
\]

Introducing dimensionless collective variables \(\zeta_m(n_0, m_0) \rightarrow \zeta_m(n_0, m_0)/(gH)^{1/2}\) we will get:

\[
\mathcal{L}_{\text{negative modes}} = \int \exp \left\{ \sum_{n_0, m_0, m} (1 - \gamma - 2m\gamma)|\zeta_m(n_0, m_0)|^2 - \gamma \frac{g^2}{2} \frac{H^2}{2} \right\} \left| \sum_{n_0, m_0, m} \zeta_m(n_0, m_0) \psi_0 m(n_0, m_0; y) \right|^4 d^4 y \right\}
\]

\[
J \prod_{n_0, m_0 = 0}^{1-\gamma \frac{1}{2\gamma}} \prod_{m=0}^{1-\gamma \frac{1}{2\gamma}} \left( \frac{\mu^2}{gH} \right) d\zeta_m(n_0, m_0)d\zeta_m^\prime(n_0, m_0).
\]

The dependence on the field strength completely factorises now from the integral over collective variables \(\zeta_m(n_0, m_0)\) and we get

\[
\mathcal{L}_{\text{negative modes}} = \int \exp \left\{ \sum_{n_0, m_0, m} (1 - \gamma - 2m\gamma)|\zeta_m(n_0, m_0)|^2 - \gamma \frac{g^2}{2} \frac{H^2}{2} \right\} \left| \sum_{n_0, m_0, m} \zeta_m(n_0, m_0) \psi_0 m(n_0, m_0; y) \right|^4 d^4 y \right\}
\]

\[
J \prod_{n_0, m_0 = 0}^{1-\gamma \frac{1}{2\gamma}} \prod_{m=0}^{1-\gamma \frac{1}{2\gamma}} d\zeta_m(n_0, m_0)d\zeta_m^\prime(n_0, m_0).
\]

Thus the contribution of negative/unstable modes that follows from the expression (6.85) is

\[
\mathcal{L}_{\text{negative modes}} = \frac{g^2 H^2 (1 + \gamma) \ln \frac{gH}{\mu^2}}{8\pi^2},
\]

and the contribution of all modes to the effective Lagrangian is a sum of (6.66) and (6.85):

\[
\mathcal{L}(\gamma) = \epsilon^{(1)}_{\text{positive modes}}(\gamma) + \epsilon_{\text{negative modes}}(\gamma).
\]

The conclusion is that the general chromomagnetic vacuum fields (6.62) are also stable because the effective Lagrangian is a real function and is without imaginary terms. In the case of a pure
chromomagnetic field $\gamma = 0$ the contribution of positive modes (6.66) can be exactly integrated (6.67) and together with the contribution of negative/unstable modes (6.86) will take the following form:

$$L^{\text{eff}}_E = \frac{H^2}{2} + \frac{11g^2H^2}{48\pi^2} \left( \ln \frac{gH}{\mu^2} - \frac{1}{2} \right).$$  \hspace{1cm} (6.88)

This demonstrates the robustness of the logarithmic structure of the effective Lagrangian functional \[1\]. In the case of (anti)self-dual field $\gamma = 1$ the sum (6.87) reduces to the expression (4.42). The conclusion is that the stability of the vacuum fields takes place not only for (anti)self-dual fields but also for the more general covariantly-constant vacuum fields defined in (6.62).

7 Consideration in Physical Space-time

A similar integration of negative/unstable modes can be performed in the Minkowski space-time \[18\], where instead of the Euclidean spectrum (2.30) the spectrum of the charge vector bosons in pure chromomagnetic vacuum field $H$ will take the following form \[6, 7\]:

$$k_0^2 = k_1^2 + gH(2n + 1) \pm 2gH,$$  \hspace{1cm} (7.89)

and it has the negative/unstable mode $k_0^2 = k_1^2 - gH$ when $k_1^2 < gH$. If one ignores for a while the nonlinear interaction of negative/unstable modes, one can conclude that there is an imaginary term in the vacuum energy density \[6, 7\] :

$$\text{Im} \epsilon^{(1)} = \text{Im} \frac{gH}{4\pi^2} \int_{-\infty}^{\infty} dk_1\sqrt{k_1^2 - gH - i\epsilon} = -\frac{g^2H^2}{8\pi}.$$  

The conclusion is that one should take into account the nonlinear interaction of negative/unstable modes. First let us consider the contribution of the positive modes into the energy density \[6, 21\]

$$\epsilon^{\text{positive modes}} = \int_{-\infty}^{\infty} dk_1\left( \sum_{n=0}^{\infty} \sqrt{k_1^2 + gH(2n + 3)} + \sum_{n=1}^{\infty} \sqrt{k_1^2 + gH(2n - 1)} \right),$$  \hspace{1cm} (7.90)

which after the renormalisation will take the following form:

$$\epsilon^{\text{positive modes}} = -\frac{g^2H^2}{48\pi^2} \left( \ln \frac{gH}{\mu^2} - \frac{1}{2} \right) + \frac{g^2H^2}{8\pi^2} \left( \ln \frac{gH}{\mu^2} - \frac{1}{2} \right).$$  \hspace{1cm} (7.91)

Now let us consider the contribution of the negative/unstable mode by taking into account their nonlinear interaction. The negative/unstable mode wave function has the following form \[8, 11\]:

$$a_1(x) = ia_2(x) = \int \frac{dk}{2\pi} e^{-\frac{1}{2}gH(x_1-k_2/gH)^2+ik_2x_2} \phi k_2(x_3, x_0) \frac{1}{2^{1/4}},$$  \hspace{1cm} (7.92)

Here the gauge fields are defined as $a_0^\mu$ and $a_\mu = (a_\mu + ia_0^\mu)/\sqrt{2}$.  

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where \( \phi_k(x_3, x_0) \) is the dimensionless amplitude (collective variable) of negative/instable modes, and it is analogous to the amplitudes \( \zeta_m(n_0, m_0) \) in the Euclidean path integral formulation (6.79). The energy spectrum (7.89) does not depend on the continuous momentum variable \( k_2 \) and exposes the true infinite degeneracy (11.130) of the spectrum (7.89). If one introduces the dimensionless variables \( k_\mu \rightarrow k_\mu/\sqrt{gH} \), \( x_\mu \rightarrow x_\mu/\sqrt{gH} \) in the wave function, then it takes the following form [18]:

\[
\alpha_1(x) = i\alpha_2(x) = (gH)^{1/2} \int \frac{dk_2}{2\pi} e^{-\frac{1}{g} (k_1 - k_2)^2} \phi_k(x_3, x_0) \frac{2^{1/4}}{\pi^{1/2}}.
\]  

(7.93)

In terms of the dimensionless variables the part of the action that corresponds to the negative/instable mode [8] will take the following form [18]:

\[
S_{\text{negative mode}} = \int \frac{dk_2}{\sqrt{2\pi}} dx_0 dx_3 \left( |\partial_\mu \phi_{k_2}|^2 - \frac{1}{2} g^2 \int \frac{dp dq}{(2\pi)^2} e^{-\frac{1}{g} \phi_{k_2+p} \phi_{k_2+q} \phi_{k_2+p+q}} \right),
\]  

(7.94)

and it contains the tachyonic mass term \( |\phi_{k_2}|^2 \) and the positive definite quartic interaction term that provides a convergence of the path integral over the negative/instable modes. In terms of the dimensionless variable the dependence on the chromomagnetic vacuum field completely factorises from the action (7.94) while it appears only in front of the wave function (7.93) and in the path integral measure \( \prod_{k_2}^{\text{negative modes}} (\mu^2/gH)^{1/2} D\phi_{k_2} \). Thus the contribution of the negative/instable mode into the partition function appears only through the integration measure and the degeneracy of the negative/instable modes:

\[
Z_{\text{negative mode}} = N \left( \frac{\mu^2}{gH} \right)^{1/2} \langle \phi \rangle^2 V_5 T = Ne^{-\frac{2\mu^2 g}{3\pi gH} \log \frac{2H}{gH}} V_5 T.
\]  

(7.95)

The vacuum energy density is a real function of the chromomagnetic vacuum field and is a sum of positive (7.94) and negative/instable mode (7.95) contributions:

\[
e = \frac{H^2}{2} + e^{(1)}_{\text{negative mode}} + e_{\text{negative mode}} = \frac{H^2}{2} + \frac{g^2 H^2}{48\pi^2} \left( \ln \frac{gH}{\mu^2} - \frac{1}{2} \right).
\]  

(7.96)

This confirms that the energy density does not have an imaginary part [1] [18].

One can also include the interaction of the negative/instable mode with the neutral mode \( \alpha_\mu^3 \) of the Yang-Mills field. The interaction term was derived in [8] [11], and in terms of dimensionless variables introduced above it will take the following form:

\[
S_{\text{neutral mode}} = \int dx_3 dx_0 \left( \frac{dk_1 dk_2}{(2\pi)^3/2} e^{-(k_1^2 + k_2^2)/4} \int \frac{dk_2' dk_2''}{2\pi} e^{-k_1^2 - k_2^2} \phi_{k_2} \phi_{k_2'} \delta(k_2 + k_2' - k_2'') e^{ik_1(k_2' + k_2'')/2} \right.
\]

\[
\left. \left[ - i g a_\mu^3 \phi_{k_2} \phi_{k_2'} \phi_{k_2} - g^2 (a_\mu^3)^2 \phi_{k_2} \phi_{k_2'} \phi_{k_2} \right] \right).
\]  

(7.97)

This part of the Yang-Mills action also does not depend on the field strength. The contribution of the instable and neutral modes into the partition function appears only through the integration measure and the degeneracy of these states, and therefore does not alter the previous result (7.96).
Chromomagnetic Gluon Condensate

We showed that the effective Lagrangian of the SU(N) Yang-Mills theory does not have an imaginary term and has the following form [1]:

\[ \mathcal{L} = -\mathcal{F} - \frac{11N}{96\pi^2} g^2 \mathcal{F} \left( \ln \frac{2g^2\mathcal{F}}{\mu^4} - 1 \right), \quad \mathcal{F} = \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} > 0. \quad (8.98) \]

It follows from this expression that the chromomagnetic magnetic induction \( \vec{B}_a \) of the YM vacuum is

\[ \vec{B}_a = -\frac{\partial \mathcal{L}}{\partial \vec{H}_a} = \vec{H}_a \left[ 1 + \frac{11g^2N}{96\pi^2} \log \frac{g^2\vec{H}_a^2}{\mu^4} \right] = \mu_{\text{vac}} \vec{H}_a \quad (8.99) \]

and that the YM vacuum behaves as a paramagnet with a magnetic permeability of the following form [1]:

\[ \mu_{\text{vac}} = -\frac{\partial \mathcal{L}}{\partial \mathcal{F}} = 1 + \frac{11g^2N}{96\pi^2} \log \frac{g^2\vec{H}_a^2}{\mu^4} = \frac{11g^2N}{96\pi^2} \log \frac{g^2\vec{H}_a^2}{\Lambda^4_S}. \quad (8.100) \]

The paramagnetism of the YM vacuum at \( g^2\vec{H}_a^2 \geq \Lambda^4_S \) means that there is an amplification of the chromomagnetic vacuum fields very similar to the Pauli paramagnetism, an effect associated with the polarisation of the electron spins. In YM theory the polarisation of the virtual vector boson spins is responsible for the vacuum fields amplification. This also can be seen from the vacuum energy density with its new minimum outside of the perturbative vacuum \( \langle \mathcal{F} \rangle = 0 \) at the renormalisation group invariant field strength [1]

\[ \langle 2g^2\mathcal{F} \rangle_{\text{vac}} = \mu^4 \exp \left( -\frac{32\pi^2}{11g^2(\mu)} \right) = \Lambda^4_S \quad (8.101) \]

or, in terms of the strong coupling constant,

\[ \langle \frac{\alpha_s}{\pi} G^2_{\mu\nu} \rangle_{\text{vac}} = \langle \frac{g^2}{4\pi^2} G^2_{\mu\nu} \rangle_{\text{vac}} = \frac{\Lambda^4_S}{2\pi^2}. \quad (8.102) \]

Using the relation \( \Lambda_P/\Lambda_S \approx 2.11 \) derived in [9] and that \( \Lambda_P \approx 200MeV \) one can get [18]

\[ \langle \frac{\alpha_s}{\pi} G^2_{\mu\nu} \rangle_{\text{vac}} \approx 0.0000041 \text{ GeV}^4. \]

As we have seen, a large class of chromomagnetic vacuum fields is stable and indicates that the Yang-Mills vacuum is a highly degenerate quantum state. It is also appealing that even a larger class of alternative vacuum fields have also been considered in the recent publications [44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57], and some of them expose a natural chaotic behaviour. In this respect one should ask whether there exist physical systems that have high degeneracy of the vacuum state. Turning to the statistical spin systems, one can observe that the classical 3D Ising system has a double degeneracy of all its excited states and of the vacuum state. It is this
symmetry that allows to construct a dual gauge invariant representation of the 3D Ising model [58].

The extensions of the 3D Ising model that have a direct ferromagnetic and one quarter of the next to nearest neighbour antiferromagnetic interaction constructed in [59], as well as a model with a zero intersection coupling constant \((k = 0)\) [60, 61, 63, 64], have exponential degeneracy of the vacuum state. In recent publications this symmetry was referred as the subsystem symmetry [67]. This higher symmetry allows to construct the dual representations of the same systems and in various dimensions [60, 61, 62]. As a consequence of the high degeneracy of the vacuum state, these systems have rich physical properties, including a glass behaviour [65, 66] and exotic fracton excitations [67].

9 Large N Behaviour

Let us consider the behaviour of the effective Lagrangian from the renormalisation group point of view and in the limit of large \(N\) [70]. When \(\mathcal{G} = \vec{E}_a \vec{H}_a = 0\), we have

\[
\mathcal{M}(t, g) = \frac{\partial \mathcal{L}}{\partial F} = -\frac{g^2}{g^2(t)}, \quad \frac{d\bar{g}}{dt} = \bar{\beta}(g). \tag{9.103}
\]

The vacuum magnetic permeability (8.99) will take the following form [4]:

\[
\mu_{\text{vac}} = \frac{g^2}{g^2(t)}, \quad \mathcal{G} = 0. \tag{9.104}
\]

The Callan-Symanzik beta function can be calculated by using (8.98):

\[
\bar{\beta} = \frac{1}{2} g \frac{\partial \mathcal{M}}{\partial t} \bigg|_{t=0} = -\frac{11N}{96\pi^2} g^3, \tag{9.105}
\]

and the effective coupling constant as a function of the vacuum field strength has the following form:

\[
\bar{g}^2(F) = \frac{g^2}{1 + \frac{11g^2N}{96\pi^2} \ln \frac{2g^2F}{\mu^4}}. \tag{9.106}
\]

Let us consider the field strength \(F_0\) at which the vacuum energy density vanishes \(\epsilon(F_0) = 0\):

\[
2g^2F_0 = \mu^4 \exp \left(1 - \frac{96\pi^2}{11g^2N}\right) = \epsilon(2g^2F)_{\text{vac}}. \tag{9.107}
\]

The effective coupling constant (9.106) at this field strength has the value

\[
\bar{g}^2(F_0) = \frac{96\pi^2}{11N}. \tag{9.108}
\]

It follows that the effective coupling constant at the intersection point \(F_0\) is small:

\[
\bar{g}^2(F_0) = \frac{96\pi^2}{11N} \ll 1 \quad \text{if} \quad N \gg \frac{96\pi^2}{11}. \tag{9.109}
\]
The energy density curve $\epsilon(F)$ intersects the zero energy density level at a nonzero angle $\theta$:

$$\tan \theta = \frac{11g^2N}{96\pi^2} > 0.$$  (9.110)

This means that i) the vacuum state is below the perturbative vacuum and that ii) there is a nonzero vacuum field condensate. Now, the question is, how far into the infrared region one can continue the energy density curve by using the perturbative results? Let us consider the vacuum fields that are close to the infrared pole. This can be achieved by using the following parametrisation:

$$F_\alpha = e^{1-\alpha \langle F \rangle_{vac}},$$  (9.111)

where the parameter $\alpha \leq 1$. When $\alpha$ is close to one, we will have $F_\alpha \rightarrow \langle F \rangle_{vac}$. At this field the value of the effective coupling constant (9.106) tends to zero,

$$\bar{g}^2(F_\alpha) = \frac{96\pi^2}{11N(1-\alpha)} \rightarrow 0,$$  (9.112)

if the product $N(1-\alpha)$ is large and the t’Hooft coupling constant $g^2N = \lambda$ is fixed and is small [70]. The energy density curve can be continuously extended infinitesimally close to the value of the vacuum field $\langle F \rangle_{vac}$ in this limit.

One can analyse the effective coupling constant (9.108) and the intersection point (9.107) at the two-loop level. The two-loop effective Lagrangian has the following form [4]:

$$\mathcal{L} = -F - \left(\frac{11}{6(4\pi)^2}g^2N + \frac{34}{6(4\pi)^4}(g^2N)^2\right)\mathcal{F}\left(\ln \frac{2g^2\mathcal{F}}{\mu^4} - 1\right).$$  (9.113)

The field at the intersection point (9.107) is shifted by an exponentially small correction

$$2g^2F_0 = \mu^4 \exp \left(1 - \frac{96\pi^2}{11\lambda} \cdot \frac{1}{1 + \frac{1}{17\pi^2}\lambda}\right),$$  (9.114)

At this field the effective coupling constant is smaller by the factor $1/1 + \frac{17}{88\pi^2}\lambda$:

$$\bar{g}^2(F_0) = \frac{96\pi^2}{11N} \cdot \frac{1}{1 + \frac{17}{88\pi^2}\lambda} \ll 1,$$  (9.115)

and the inequality (9.115) is fulfilled at smaller values of $N$ than in the first approximation (9.109).

The chromomagnetic condensate in the two-loop approximation will take the following form:

$$\langle 2g^2F \rangle_{vac} = \mu^4 \exp \left( - \frac{1}{\beta_1 g^4} \left[ 1 - \frac{\beta_2 g^2}{\beta_1} \ln(1 + \frac{\beta_1}{\beta_2 g^2}) \right] \right).$$  (9.116)

The high-loop corrections can be obtained by using the renormalisation group results (14.147), (9.103). Then the value of the chromomagnetic condensate is [1]:

$$\langle 2g^2F \rangle_{vac} = \mu^4 \exp \left( 2 \int g \frac{dg}{\beta(g)} \right).$$  (9.117)

The expression (9.116) is recovered at the two-loop level.

---

8The beta function (9.103) coefficients $\beta = -\beta_1 g^2 - \beta_2 g^4 + ..$ are given by $\beta_1 = \frac{11N}{6\pi^2}$ and $\beta_2 = \frac{34N^2}{72\pi^2}$ [68, 69].
Conclusion

It is remarkable that the zero mode contribution that was calculated in terms of infrared regularisation of the spectrum (4.39), by integration over the self-interacting zero modes in (5.53) and then by integration over fully interacting zero modes in (5.58), all lead to the same result indicating the robustness of the logarithmic structure of the effective Lagrangian and that it is without an imaginary part [1]. A similar phenomenon took place when we were considering the contribution of the negative/unstable modes by integrating over self-interacting (6.77) and fully interacting modes (6.85). It is interesting to investigate to what extent this behavioural robustness is rooted in the entropy factor, through the degeneracy of the quantum states and through the scaling invariance of the Yang-Mills action.

One can consider the above approach of calculating the effective action as an alternative loop expansion of the effective action. The expansion is organised by rearranging the perturbative expansion in a vacuum field so that the interaction of all negative/unstable modes is included into the propagator of the gauge field $G(x, y; A)$ and the loop expansion is performed in terms of the cubic and quartic cross-mode vertices between positive/stable and negative/unstable modes of the YM action [18]. Some technical details are presented in the Appendixes.

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11 Appendix A. Degeneracy of Eigenstates

A component representation of the operator $H_0 = -\nabla_\mu \nabla_\mu$ is given in (2.21). By using the operators (2.22) one can find that

$$c_1^+ + ic_2^+ = -\partial_1 - i\partial_2 + \frac{g}{2}H(x_1 + ix_2), \quad c_1 - ic_2 = \partial_1 - i\partial_2 + \frac{g}{2}H(x_1 - ix_2),$$

and that the magnetic part of the $H_0$ is

$$(c_1^+ + ic_2^+)(c_1 - ic_2) = -\partial_1^2 - \partial_2^2 + igH(x_2\partial_1 - x_1\partial_2) + \frac{g^2}{4}H^2(x_2^2 + x_1^2) - gH.$$
The vacuum wave function is defined as
\[
(c_1 - ic_2)\psi_0 = 0, \quad \psi_0 = e^{-gH(x_1^2 + x_2^2)} \quad (11.120)
\]
and the wave functions of the excited states (2.24) are
\[
\psi_n = (c_1^+ + ic_2^+)n\psi_0 = (gH)^n(x_1 + ix_2)^n\psi_0. \quad (11.121)
\]
The set of operators (corresponding to the centre of the cyclotron motion of charge particle in a magnetic field) that defines the degeneracy of the eigenstates is
\[
c_1^+ - ic_2^+ = -\partial_1 + i\partial_2 + \frac{g}{2}H(x_1 - ix_2), \quad c_1 + ic_2 = \partial_1 + i\partial_2 + \frac{g}{2}H(x_1 + ix_2),
\]
\[
[c_1^+ - ic_2^+ ; c_1 + ic_2] = -2gH. \quad (11.122)
\]
The wave functions of the degenerate vacuum state are
\[
\psi(n_0) = (c_1^+ - ic_2^+)n_0\psi_0 = (gH)^{n_0}(x_1 - ix_2)^{n_0}\psi_0. \quad (11.123)
\]
They are the eigenstates of the operator \( L = i(x_1\partial_2 - x_2\partial_1) \):
\[
L \psi(n_0) = n_0 \psi(n_0). \quad (11.124)
\]
All of them have the ground state eigenvalue in (2.26):
\[
\lambda_0 = gH. \quad (11.125)
\]
The normalisation of the wave functions \( \psi_{n_0} \) is
\[
C_{n_0} = \int dx_1 dx_2|\psi(n_0)|^2 = 2^{n_0 + 1}\pi n_0!(gH)^{n_0 - 1}, \quad (11.126)
\]
and these wave functions (11.123) are orthogonal:
\[
\int dx_1 dx_2\psi^*(n_0)\psi(\bar{n}_0) = C_{n_0}\delta_{n_0,\bar{n}_0}. \quad (11.127)
\]
The average size of the orbit of the degenerate state \( \psi(n_0) \) is
\[
\langle r^2 \rangle = \langle x_1^2 + x_2^2 \rangle = \frac{1}{C_{n_0}} \int dx_1 dx_2(x_1^2 + x_2^2)|\psi(n_0)|^2 = \frac{2(n_0 + 1)}{gH}. \quad (11.124)
\]
It follows then that the degeneracy, the number of the charge particle orbits of the size less than \( r^2 \leq R^2 \), is
\[
n_0 + 1 \leq \frac{gH}{2}\frac{R^2}{\pi} = \frac{gH}{2\pi}R^2 = \frac{gH}{2\pi}A_{12}, \quad (11.128)
\]
where $A_{12}$ is two-dimensional area on the plane $(x_1, x_2)$. The same is true for the chomoelectric part of the $H_0$ operator (2.21) on the plane $(x_3, x_0)$:

$$m_0 + 1 \leq \frac{gE}{2} R^2 = \frac{gE}{2\pi} R^2 = \frac{gE}{2\pi} A_{03},$$

(11.129)

thus the total degeneracy is

$$\text{Deg} = \left( \frac{gH}{2\pi} \right) \left( \frac{gE}{2\pi} \right) V_3 T,$$

(11.130)

where $V_3$ is a three-volume and $T$ play the role of a time-like parameter in (2.1).

12 Appendix B. Self-Interaction of Zero Modes

The zero modes are the solutions of the equation (2.21)

$$H_0 a_\nu + 2igF_{\mu\nu}a_\mu = 0$$

(12.131)

with $\lambda = 0$, and the solution (2.32) corresponding to the lowest state with $n_0 = m_0 = 0$ is:

$$a_1^1 = \xi' \psi_{00}, \quad a_1^2 = \xi'' \psi_{00}, \quad a_1^3 = 0$$

$$a_2^1 = \xi'' \psi_{00}, \quad a_2^2 = -\xi' \psi_{00}, \quad a_2^3 = 0$$

$$a_3^1 = \eta' \psi_{00}, \quad a_3^2 = \eta'' \psi_{00}, \quad a_3^3 = 0$$

$$a_0^1 = \eta'' \psi_{00}, \quad a_0^2 = -\eta' \psi_{00}, \quad a_0^3 = 0,$$

(12.132)

where $\xi_{00} = \xi' + i\xi''$ and $\eta_{00} = \eta' + i\eta''$. The interaction cubic term (2.13) vanishes on this solution:

$$V_E^{(3)} = -g\epsilon^{abc} a_\nu^b a_\mu^c \nabla_\mu a_\nu^d = 0,$$

(12.133)

and the quartic term will take the following form:

$$V_E^4 = \frac{g^2}{4} \int d^4x (\epsilon^{abc} a_\mu^b a_\nu^c)^2 = \frac{g^2}{4} \int d^4x ((a_\mu^a a_\mu^a)^2 - (a_\mu^a a_\nu^a)^2) = \frac{g^2}{2} (\xi'^2 + \xi''^2 + \eta'^2 + \eta''^2) \int d^4x |\psi_{00}|^4,$$

(12.134)

where $(a_\mu^a a_\mu^a)^2 = 4(\xi'^2 + \xi''^2 + \eta'^2 + \eta''^2)$, $(a_\mu^a a_\nu^a)^2 = 2(\xi'^2 + \xi''^2 + \eta'^2 + \eta''^2)$. We have to calculate the quartic integral for the lowest state $\psi_{00}$:

$$\frac{1}{C_4^4} \int dx_1 dx_2 dx_3 dx_0 |\psi_{00}|^4 = \left( \frac{gH}{2\pi} \right)^4 \left( \frac{\pi}{gH} \right)^2 = \left( \frac{gH}{4\pi} \right)^2.$$

(12.135)

The quartic term for the general zero mode wave function (2.27), (2.32) will take the following form:

$$V_E^4 = \frac{g^2}{2} (|\xi_{n0a}|^2 + |\eta_{n0a}|^2)^2 \int d^4x |\psi_{n0a}|^4,$$

(12.136)
and then using the integral of the wave function \([11.123]\) one can get:

\[
\int dx_1 dx_2 |\psi(n_0)|^4 = \pi \Gamma(2n_0 + 1)(gH)^{2n_0 - 1}.
\]  \(12.137\)

The zero mode \([2.32]\) self-interaction term \([2.13]\) has the following form:

\[
V_{E \text{zero modes}} = \frac{g^2}{4} \int d^4x (\epsilon^{abc} a^b_\mu a^c_\mu)^2 = \frac{g^2}{2} (|\xi_{n_0 m_0}|^2 + |\eta_{n_0 m_0}|^2) \int d^4x |\psi_{00}(n_0, m_0; x)|^4,
\]  \(12.138\)

where \(\psi_{00}(n_0, m_0; x) = (gH)^{n_0} (x_1 - ix_2)^{n_0} (gE)^{m_0} (x_3 - ix_0)^{m_0} \psi_{00}\). For the normalised wave function one can get:

\[
\frac{1}{C_{n_0}^2} \frac{1}{C_{m_0}^2} \int d^4x |\psi_{00}(n_0, m_0; x)|^4 = \left(\frac{gH}{4\pi}\right)^2 \frac{\Gamma(n_0 + 1/2)\Gamma(m_0 + 1/2)}{\pi \Gamma(n_0 + 1)\Gamma(m_0 + 1)}.
\]  \(12.139\)

13 Appendix C. Sum of Eigenvalues

The spectral sums \([3.34]\) and \([6.66]\) can be evaluated by using the following formulas:

\[
\sum_{n, m = 0}^{\infty} e^{-gH(2n + 2m + 3 + \gamma)s} = e^{-2gH \gamma s} \frac{1}{4\sinh(gH s) \sinh(g\gamma H s)}
\]

\[
\sum_{n, m = 0}^{\infty} e^{-gH(2n + 2m + 1 + 3\gamma)s} = e^{-2g\gamma H s} \frac{1}{4\sinh(gH s) \sinh(g\gamma H s)}
\]

\[
\sum_{n, m = 0}^{\infty} e^{-gH(2n + 2m + 1 - \gamma)s} = e^{2g\gamma H s} \frac{1}{4\sinh(gH s) \sinh(g\gamma H s)}
\]

\[
\sum_{n, m = 0}^{\infty} e^{-gH(2n + 2m - 1 + \gamma)s} = e^{2g\gamma H s} \frac{1}{4\sinh(gH s) \sinh(g\gamma H s)}
\]

\[
\sum_{n, m = 0}^{\infty} e^{-gH(2n + 2m - 1 - \gamma)s} = \frac{1}{4\sinh(gH s) \sinh(g\gamma H s)}
\]

\[
\frac{1}{2^{n + \frac{1}{2}}} \sum_{m = 0}^{\infty} e^{-gH(2m - 1 + \gamma)s} = e^{gH s(1 - \gamma)} \frac{1 - e^{-gH s(1 + \gamma)}}{1 - e^{-2g\gamma H s}}
\]  \(13.140\)

The integrals appearing in the effective Lagrangian have the following form:

\[
\int_0^{\infty} \frac{ds}{s^{1-k} \sinh^2(as)} = \frac{4}{(2a)^k} \Gamma(k) \zeta(k - 1),
\]  \(13.141\)

\[
\int_0^{\infty} \frac{\cosh(bs)ds}{s^{1-k} \sinh(as)} = \frac{\Gamma(k)}{(2a)^k} \left[\zeta(k, \frac{1}{2}(1 - \frac{b}{a}) + \frac{1}{2} \left(1 + \frac{b}{a}\right))\right], \quad b \neq a,
\]

\[
\int_0^{\infty} \frac{ds}{s^{1-k} \sinh(as)} = \frac{2^k - 1}{2^{k-1} a^k} \Gamma(k) \zeta(k), \quad \int_0^{\infty} \frac{\cosh(as)ds}{s^{1-k} \sinh^2(as)} = \frac{2^{k-1} - 1}{2^{k-2} a^k} \Gamma(k) \zeta(k - 1)
\]

\[
\int_0^{\infty} \frac{\sin(as)ds}{s^{1-k}} = \frac{\Gamma(k)}{a^k} \sin \frac{k\pi}{2}, \quad \int_0^{\infty} \frac{\cos(as)ds}{s^{1-k}} = \frac{\Gamma(k)}{a^k} \cos \frac{k\pi}{2},
\]

where \(k\) can be considered as a dimensional regularisation parameter and the integrals should be calculated in the limit \(k \to 0\) \([4]\).
Appendix E. Renormalisation Group

The exact expression for the effective Lagrangian can be derived by using the renormalisation group equation \[1, 2\]. The effective action \( \Gamma \) is a renormalisation group invariant quantity:

\[
\Gamma = \sum_n \int dx_1...dx_n \Gamma^{(n)}_{\mu_1...\mu_n}(x_1,...,x_n) A^{a_1}_{\mu_1}(x_1) ... A^{a_n}_{\mu_n}(x_n),
\]

because the vertex functions and gauge fields transform as follows:

\[
\Gamma^{(n)}_{\mu_1...\mu_n}(x_1,...,x_n) = Z_n^{n/2} \Gamma^{(n)}_{\mu_1...\mu_n}(x_1,...,x_n), \quad A^a_{\mu}(x) = Z_3^{-1} A^a_{\mu}(x), \quad g_r = Z_3^{1/2} g_{un}.
\]

The renormalisation group equation takes the form

\[
\{ \mu^2 \frac{\partial}{\partial \mu^2} + \beta(g) \frac{\partial}{\partial g} + \gamma(g) \int d^4x A^a_{\mu}(x) \frac{\delta}{\delta A^a_{\mu}(x)} \} \Gamma = 0,
\]

where \( \beta(g) \) is the Callan-Symanzik beta function, \( \gamma(g) \) is the anomalous dimension. When \( G = \vec{E} \vec{H} = 0 \) it reduces to the form

\[
\{ \mu^2 \frac{\partial}{\partial \mu^2} + \beta(g) \frac{\partial}{\partial g} + 2 \gamma(g) F \frac{\partial}{\partial F} \} L = 0,
\]

where in the covariant background gauge \( \beta = -g \gamma \) \[4\]. By introducing a dimensionless quantity

\[
M(g, t) = \frac{\partial L}{\partial F}, \quad t = \frac{1}{2} \ln(2 g^2 F/\mu^4),
\]

one can get

\[
\{ -\frac{\partial}{\partial t} + \bar{\beta}(g) \frac{\partial}{\partial g} + 2 \bar{\gamma}(g) \} M(g, t) = 0,
\]

where

\[
\bar{\gamma} = \frac{\gamma}{1 - \gamma}, \quad \bar{\beta} = \frac{\beta}{1 - \gamma}
\]

and \( 3.35 \) plays the role of the boundary condition:

\[
M(g, 0) = -1.
\]

From equations \[14.143\] and \[14.145\] it follows that

\[
\bar{\gamma} = -\frac{1}{2} \frac{\partial M(g, t)}{\partial t} \bigg|_{t=0}, \quad \bar{\beta} = \frac{1}{2} g \frac{\partial M(g, t)}{\partial t} \bigg|_{t=0}.
\]

The solution of the renormalisation group equation \[14.143\] in terms of the effective coupling constant \( \bar{g}(g, t) \) with the boundary condition \( \bar{g}(g, 0) = g \) has the following form \[1, 2\]:

\[
\frac{\partial L}{\partial F} = -\frac{g^2}{\bar{g}^2(t)}, \quad \frac{dg}{dt} = \bar{\beta}(g).
\]
The behaviour of the effective Lagrangian at large fields is similar to the behaviour of the gauge theory at a large momentum. It follows that $M(g,t)$ is completely determined for all values of $t$ in terms of its first derivative $(14.146)$ at $t = 0$. The above results allow to obtain the renormalisation group expressions for the physical quantities considered in a one-loop approximation. With these expressions in hand one can calculate different observables of physical interest that will include the vacuum energy density and pressure, the magnetic permeability, the effective coupling constants, and their behaviour as a function of vacuum fields [71].

15 Appendix D. Euclidean Formulation of Quantum Mechanics

The matrix elements of the evolution operator $e^{iHt}$ are defined in terms of path integral over trajectories in the physical space-time as [23]:

$$
\langle \vec{x}' | e^{iHt} | \vec{x} \rangle = \sum_n e^{-iE_n T} \psi_n(\vec{x}) \psi_n^*(\vec{x}) = N \int_{\vec{x}} e^{iS[\vec{x}(t)]}.
$$

(15.148)

The integration runs over all trajectories $\vec{x}(t)$ that start at time $t = 0$ in the point $\vec{x}$ and end at time $t = T$ in the point $\vec{x}'$. The path integral is defined in terms of a sum over the physical space-time trajectories. The vectors $\psi(\vec{x}) = \langle x| n \rangle$ are the Schrödinger wave functions. The $H$ is the Hamiltonian of the system and $\psi_n(\vec{x})$ is the eigenfunction $H \psi_n = E_n \psi_n$.

With the rotation $t \rightarrow -i \tau$ to the Euclidean time the matrix elements transform to the matrix elements of the operator $e^{-HT}$ and are represented in terms of Euclidean path integral [29]:

$$
\langle \vec{x}' | e^{-HT} | \vec{x} \rangle = \sum_n e^{-E_n T} \psi_n(\vec{x}) \psi_n^*(\vec{x}) = N \int_{\vec{x}} e^{-S_E[\vec{x}(\tau)]}.
$$

(15.149)

The path integral is defined as an integral in unphysical Euclidean space. The integration runs over all Euclidean trajectories $\vec{x}(\tau)$ that start at Euclidean time $\tau = 0$ in the point $\vec{x}$ and end at Euclidean time $\tau = T$ in the point $\vec{x}'$. The $S_E$ is the Euclidean action associated with a given trajectory:

$$
S_E = \int_0^T d\tau \left( \frac{1}{2} m \vec{x}^2 + V(\vec{x}) \right).
$$

(15.150)

The boundary conditions in both of the formulations are identical, the difference is in the geometry of the trajectories. In the first case the trajectories are in a real physical space-time where a particle moves in a potential $V$ bounded from below. In the Euclidean formulation a particle ”moves” in a potential $-V$ which is unbounded from below, its ”trajectories” are in unphysical Euclidean space and do not directly correspond to the trajectories in a physical space. At large values of $T$ the leading
term in (15.149) defines the ground state energy \( E_0 \) and the corresponding eigenfunction \( \psi_0(\vec{x}) \):

\[
\sum_n e^{-E_n T} \psi_n(\vec{x}') \psi^*_n(\vec{x}) \rightarrow e^{-E_0 T} \psi_0(\vec{x}') \psi^*_0(\vec{x}).
\] (15.151)

The above relation leads to the following observation: In some cases the Euclidean path integral can readily be evaluated in the semiclassical limit when it is dominated by the stationary "trajectory" of \( S_E \). Considering a standard oscillator with boundary condition \( x' = x = 0 \) one can find that the stationary "trajectory" is simply \( \hat{x} = 0 \) and leads to the relation \[29\]

\[
e^{-E_0 T} |\psi_0(0)|^2 = \left( \frac{\omega}{\pi} \right)^{1/2} e^{-\omega T / 2}
\] (15.152)

allowing to extract the value of the ground state energy \( E_0 = \omega / 2 \) and of the \( |\psi_0(0)|^2 = \left( \frac{\omega}{\pi} \right)^{1/2} \).

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