Gauge Symmetries of the $N=2$ String

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Abstract: We study the underlying gauge symmetry algebra of the $N = 2$ string, which is broken down to a subalgebra in any spacetime background. For given toroidal backgrounds, the unbroken gauge symmetries (corresponding to holomorphic and antiholomorphic worldsheet currents) generate area-preserving diffeomorphism algebras of null 2-tori. A minimal Lie algebraic closure containing all the gauge symmetries that arise in this way, is the background–independent volume–preserving diffeomorphism algebra of the target Narain torus $T^{4,4}$. The underlying symmetries act on the ground ring of functions on $T^{4,4}$ as derivations, much as in the case of the $d = 2$ string. A background–independent spacetime action valid for noncompact metrics is presented, whose symmetries are volume–preserving diffeomorphisms. Possible extensions to $N = 2$ and $N = 1$ heterotic strings are briefly discussed.

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1. Introduction: Gauge Symmetries of Strings

String theories possess an infinite number of gauge symmetries. In any given spacetime background, most of these gauge symmetries will be spontaneously broken. This fact is fortunate for phenomenology’s sake, but makes it difficult to untangle the underlying symmetry structure. Explicit knowledge of the full gauge algebra would be useful in formulating background–independent string field theories, in constructing effective actions, and perhaps in understanding how the string selects a particular vacuum. In addition, it is not unreasonable to hope that gauge symmetries of this sort could play just as vital a role in the second quantization of string theories as they did for Yang–Mills theories.

It has been suggested [1] that there might be an unbroken phase of strings, perhaps a topological field theory, in which the full gauge symmetry algebra would be represented on the string’s Hilbert space. Indeed, spacetime backgrounds break spacetime diffeomorphism invariance, and a background–independent formulation of string theory would presumably be diffeomorphism–invariant, i.e., topological. We thus expect spacetime diffeomorphisms to play a crucial role in the full gauge algebra.

In string theory, worldsheet symmetries are directly related to symmetries of the spacetime effective action. From the point of view of the worldsheet, conformal field theories (CFTs) are associated with classical string vacua [2]. An exact symmetry algebra of a particular string vacuum is generated by the operator product algebras of the holomorphic (1, 0) currents and anti-holomorphic (0, 1) currents of the CFT. In general, a CFT can be deformed by truly marginal (1, 1) operators. The action of the symmetry generators on such deformations can be translated into an action on the couplings to the marginal operators, and therefore gives rise to symmetries on the moduli space of couplings. (This procedure has been used in [3] to show that target space duality symmetries in the flat case are residual discrete symmetries of this type, while restricting (locally) to the physical moduli space.) The couplings to (1, 1) operators become massless fields in spacetime, and the symmetries of the CFT thus translate into symmetries of the spacetime effective action.

The (1, 0) (or (0, 1)) operators which are not holomorphic (anti-holomorphic) in a given vacuum, may become holomorphic (anti-holomorphic) at some other points of the moduli space. In order to reveal the underlying symmetries of string theory, one should, therefore, consider at least all such operators. Such a program was discussed for low-energy effective actions of the heterotic string [4], and an infinite–dimensional gauge algebra (called
the “duality–invariant string gauge algebra”) was introduced, in order to construct an effective action valid throughout the moduli space of toroidal compactifications. However, the understanding of stringy gauge symmetries thus gained was incomplete. The algebra did not include gauge symmetries involving higher–spin states. Also, implementing it as a symmetry algebra of the effective action required introducing unphysical ultramassive ghost fields.

The complete description of the gauge symmetries of the $N = 1$ heterotic string is a difficult problem, due to the complexity of the vertex operator algebra. However, there are simpler string theories, for which one might hope the problem would be more tractable.

In this paper, we will investigate gauge symmetries of the closed string with $N = 2$ local worldsheet supersymmetry, in the critical dimension $[5][6]$. As explained above, these may be regarded either as gauge symmetries of the CFT in a particular background, or as symmetries of the effective spacetime action. Both points of view — worldsheet and spacetime — will prove useful. On the worldsheet, we study the CFTs associated to various backgrounds, and find gauge symmetry generators within the algebra of on–shell vertex operators. In section 2 we discuss the $N = 2$ string in Minkowski space, and in section 3 we consider general toroidal backgrounds. The gauge symmetries that we find generate area–preserving diffeomorphisms of two–dimensional null subspaces, and act on the ground ring of dimension $(0,0)$ operators as derivations. We show that for any on–shell dimension $(1,0)$ current, there is a toroidal background where the current becomes holomorphic, and the gauge symmetry that it generates is unbroken.

The gauge algebras in different backgrounds are all subalgebras of the full underlying off–shell algebra we seek. A minimal Lie algebraic closure, obtained by continuing the representation of the gauge generators as derivations to off–shell momenta, is the volume–preserving diffeomorphism algebra of the target Narain torus $[7]$. (Other closures exist; one possibility is a lattice algebra over the Narain lattice.) The underlying symmetries of the $N = 2$ string have some remarkable similarities with those of the $d = 2$ string, as we describe at the end of section 3.

An independent approach, from the point of view of spacetime, is to look for a background–independent effective action that reproduces the correlation functions of the $N = 2$ string when expanded around a particular background. This is the subject of section 4. We will succeed in finding such an action, valid for noncompact backgrounds, whose off–shell symmetries are exactly those obtained from the worldsheet operator algebra. It is the minimal action reproducing all on–shell amplitudes in the background $\mathbb{R}^{2,2}$. 

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The underlying symmetry algebra of the \( N = 2 \) string may shed light on the underlying structure of more realistic string theories. In section 5 we discuss a possible extension of our ideas to the \( N = 2 \) and \( N = 1 \) heterotic strings.

2. The \( N = 2 \) String in Minkowski Space

The critical \( N = 2 \) string has perhaps the simplest vertex operator algebra of any string theory. This is because, unlike other string theories, it contains a finite number of physical degrees of freedom.

The worldsheet action of the \( N = 2 \) string is

\[
S_0 = \int d^2zd^2\theta d^2\bar{\theta} K_0(X, \bar{X})
\]  

(2.1)

in terms of the \( N = 2 \) chiral superfield

\[
X^i(Z, \bar{Z}; \theta^-, \bar{\theta}^-) = x^i(Z, \bar{Z}) + \psi^i_L(Z, \bar{Z})\theta^- + \psi^i_R(Z, \bar{Z})\bar{\theta}^- + F^i(Z, \bar{Z})\theta^-\bar{\theta}^-
\]

(2.2)

(\text{where } i = s, t \text{ denote complex spacelike and timelike components, and bars denote complex conjugation}). For now, we will take as target space \( \mathbb{R}^{2,2} \), with \( K_0(X, \bar{X}) = X_s\bar{X}_s - X_t\bar{X}_t \).

The \( N = 2 \) string around flat space \( \mathbb{R}^{2,2} \) has a single massless degree of freedom \( \phi \) describing Kähler deformations of the background geometry. The vertex operator to create a mode of \( \phi \) with complex 2-momentum \( p \) is

\[
V_p(X, \bar{X}) = \exp i(p \cdot \bar{X} + \bar{p} \cdot X)
\]

(2.3)

One calculates tree–level correlation functions by inserting \( V_p \) at \( n \) points on the sphere, and integrating over their positions modulo global superconformal transformations. Then one finds that the on–shell three–point function is

\[
\langle V_p V_q V_r \rangle = (q \cdot \bar{r} - r \cdot \bar{q})^2
\]

(2.4)

with \( p \cdot \bar{p} = q \cdot \bar{q} = r \cdot \bar{r} = 0 \) and \( p + q + r = 0 \). It turns out that four–point and probably also higher–point amplitudes of such operators vanish.
The superfield vertex operator has the following expansion in terms of component fields:

\[ V_p(X, \bar{X}) = e^{i(p \cdot \bar{x} + \bar{p} \cdot x)} \]

\[
\theta + \bar{\theta} - + (ip \cdot \partial \bar{x} - \bar{p} \cdot \partial x - (p \cdot \bar{\psi}_L)(\bar{p} \cdot \psi_L)) e^{i(p \cdot \bar{x} + \bar{p} \cdot x)} \theta + \bar{\theta} - \\
+ (ip \cdot \partial \bar{x} - i\bar{p} \cdot \partial x - (p \cdot \bar{\psi}_R)(\bar{p} \cdot \psi_R)) e^{i(p \cdot \bar{x} + \bar{p} \cdot x)} \theta + \bar{\theta} - \\
+ (ip \cdot \partial \bar{x} - i\bar{p} \cdot \partial x - (p \cdot \bar{\psi}_L)(\bar{p} \cdot \psi_L)) \\
\cdot (ip \cdot \partial \bar{x} - i\bar{p} \cdot \partial x - (p \cdot \bar{\psi}_R)(\bar{p} \cdot \psi_R)) e^{i(p \cdot \bar{x} + \bar{p} \cdot x)} \theta + \bar{\theta} - + \ldots \\
\equiv O_p(z, \bar{z}) + J_p(z, \bar{z})\theta + \bar{\theta} - + J_{\bar{p}}(z, \bar{z})\bar{\theta} + \theta - + V_p(z, \bar{z})\theta + \bar{\theta} - + \ldots \\
\]

where the dots refer to terms with an odd number of \theta’s and/or \bar{\theta}’s (namely, terms whose holomorphic and/or anti-holomorphic part is fermionic). It will be useful in what follows to factorize the various terms in eq. (2.5) into left- and right-moving (holomorphic and anti-holomorphic) parts. Corresponding respectively to the terms in (2.5), we denote the holomorphic and anti-holomorphic vertex operators as follows:

\[ V_p(X, \bar{X}) \equiv O_p(z)O_p(\bar{z}) + W_p(z)O_p(\bar{z})\theta + \bar{\theta} - + O_p(z)W_p(\bar{z})\bar{\theta} + \theta - \\
+ W_p(z)\bar{W}_p(\bar{z})\theta + \bar{\theta} - + \bar{\theta} - + \ldots \\
\]

where the exponential factors have been split using \( x(z, \bar{z}) = x_L(z) + x_R(\bar{z}). \) Thus, for example,

\[ O_p(z) = e^{i(p \cdot \bar{x}_L(z) + \bar{p} \cdot x_L(z))} \]

\[ W_p(z) = (ip \cdot \partial \bar{x} - i\bar{p} \cdot \partial x - (p \cdot \bar{\psi}_L)(\bar{p} \cdot \psi_L)) e^{i(p \cdot \bar{x}_L(z) + \bar{p} \cdot x_L(z))} \]

On the mass shell \( p \cdot \bar{p} = 0 \) these are holomorphic vertex operators of dimension 0 and 1, respectively, and the operators \( O_p, J_p, \bar{J}_p, \) and \( V_p \) appearing in eq.(2.5) have dimensions \((0, 0), (1, 0), (0, 1), \) and \((1, 1).\) Each of these four types of operators plays a distinct role in the vertex operator algebra.

The operators \( O_p(z, \bar{z}) \) form a ring (the “ground ring”) of dimension \((0, 0)\) operators. They obey the multiplication law

\[ O_p(z, \bar{z})O_q(w, \bar{w}) = O_{p+q}(w, \bar{w}) + O(|z - w|) \]

when \( p, q, \) and \( p + q \) are all on shell.
The ground ring is acted upon by the algebra of currents \( J_p(z, \bar{z}) \) of dimension \((1, 0)\) \textit{via} contour integration:

\[
J_p(O_q) \equiv \frac{1}{2\pi i} \oint_{C_w} dz \ J_p(z, \bar{z}) \ O_q(w, \bar{w})
= \frac{1}{2\pi i} \left( \oint_{C_w} dz \ W_p(z) O_q(w) \right) \bar{O}_p(z) \bar{O}_q(\bar{w}) + \mathcal{O}(C_w)
= \frac{1}{2}(p \cdot \bar{q} - \bar{p} \cdot q)O_{p+q}(w, \bar{w}) + \mathcal{O}(C_w)
\]

where the contour \( C_w \) is taken to enclose \( w \) once, and \( \mathcal{O}(C_w) \) refers to elements that depend on the contour \( C_w \). A convenient choice is to take \( C_w \) to be an infinitesimal contour around \( w \). With that choice, \( \mathcal{O}(C_w) \) can be shown to vanish for on-shell momenta, and the currents \( J \) act exactly as derivations on the ground ring.

The \( J \)'s amongst themselves form a Lie algebra under the bracket

\[
[J_p, J_q] \equiv J_p(J_q) - J_q(J_p) = (p \cdot \bar{q} - \bar{p} \cdot q) J_{p+q} + \mathcal{O}(C_w)
\]

Completely analogous statements hold for \( \bar{J} \) commutators as well. The commutator of \( J \) with \( \bar{J} \) is in general non-zero and will be discussed later.

As mentioned in the introduction, the \((1, 1)\) operators \( V_p(z, \bar{z}) \) generate marginal deformations

\[
S = S_0 + \delta S
\]

\[
\delta S = \int d^2z d^2\theta d^2\bar{\theta} \int d^4p \phi(p) V_p(Z, \bar{Z}) = \int d^2z \int d^4p \phi(p) V_p(z, \bar{z})
\]

The operator product expansion (OPE) of the marginal operators reproduces the 3-point function of eq.(2.4). The currents \( J \) act on the marginal operators as

\[
J_p(V_q) = \frac{1}{2}(p \cdot \bar{q} - q \cdot \bar{p})V_{p+q} + \ldots
\]

and likewise for \( \bar{J} \) (the dots in (2.12) refer to \((1, 1)\) operators which are not upper components of \( \mathcal{O}_{p+q} \)). A transformation of \( V_p \) is equivalent, by inspection of eq.(2.11), to a transformation of the couplings \( \phi(p) \). Thus, the transformations (2.12) relate different deformations of the action \( S_0 \).

The algebras (2.9), (2.10), and (2.12) are all dependent on the choice of a contour for the nonholomorphic current \( J_p \). Because it is not holomorphic, \( J_p \) is not conserved, so it
does not generate a true symmetry. However, as we will show in the next section, for any on–shell $J_p$, there is always a particular choice of a compactified background in which $J_p$ becomes holomorphic, and thus generates an unbroken gauge symmetry.

Besides $O$, $J$, $\bar{J}$, and $V$, there are additional states at discrete values of the momenta $\mathbb{R}$. For the background $\mathbb{R}^{2,2}$, the only discrete states are at $p = 0$ (although we shall find many more when we consider toroidal compactifications below). Vertex operators for dimension–one discrete states are formed from off–shell $J_p$’s by choosing an appropriate normalization as $p$ goes to zero:

$$D_\epsilon(z) = i\epsilon \cdot \partial \bar{x} - i\bar{\epsilon} \cdot \partial x = \lim_{\substack{p \to 0 \\bar{p}=\epsilon}} \frac{1}{|p|} W_p(z) \bar{O}_p(\bar{z})$$  \hspace{1cm} (2.13)

where $\epsilon_i$ is a constant complex two-vector of unit norm. The idea here is to send $p$ to zero with a particular polarization, scaling out the factor of $p$ appearing in $W_p$ that makes $W_p$ go to zero. In order for the normalization $1/|p|$ to make sense, we must continue to off–shell momenta (with $|p| \neq 0$) before taking the on–shell limit $p \to 0$. Although the polarization $\epsilon_i$ must be non–null in taking the limit in eq. (2.13), arbitrary polarizations can be obtained by taking linear combinations of the $D_\epsilon$. We shall refer to the vector components of $D_\epsilon$ and $\bar{D}_\epsilon$ as

$$D_i \equiv \partial x^i \hspace{1cm} \bar{D}_i \equiv \bar{\partial} x^i$$  \hspace{1cm} (2.14)

(where $i$ may be either a holomorphic or an antiholomorphic index).

The discrete operator $D_\epsilon$ is related by supersymmetry transformations to the dimension zero operator $O_\epsilon = \epsilon \cdot \bar{x} - \bar{\epsilon} \cdot x$. However, $O_\epsilon$ is not conformal, so $D_\epsilon$ is not an upper component of a lower–dimensional conformal field. The holomorphic current $D_\epsilon$ (and likewise $\bar{D}_\epsilon$) generates global isometries in the direction $\epsilon$. As we shall see in the next section, these discrete operators are naturally incorporated into the operator algebra.

In addition to $D_i$, we can also form the discrete (1,1) operators

$$V_{ij}(z, \bar{z}) = \partial x^i \bar{\partial} x^j$$  \hspace{1cm} (2.15)

which generate deformations of the metric moduli.

It should be stressed that the operator algebras (2.8) and (2.10) apply properly to on–shell momenta only, i.e., $p \cdot \bar{p} = q \cdot \bar{q} = (p + q) \cdot (\bar{p} + \bar{q}) = 0$.\footnote{If $(p + q) \cdot (\bar{p} + \bar{q}) < 0$, the $\sigma(C_w)$ in (2.9), (2.10) can not be set to 0 by a choice of a contour. Declaring that $\sigma(C_w) = 0$ anyway will lead to an algebra that violates the Jacobi identity.} Strictly speaking, then,
in defining these algebras $O_p$, $J_p$, and $V_p$ should be indexed by momenta restricted to a subspace $V_{\text{null}}$ of null vectors, such that the sum of any two momenta $p, q \in V_{\text{null}}$ is also null: $p \cdot \bar{q} + \bar{p} \cdot q = 0$. A maximal such subspace of $\mathbb{R}^{2,2}$ is spanned by any two perpendicular null vectors. We will refer to algebras of operators with momenta lying in a null plane of this sort, as on–shell algebras $[10]$.

A nondegenerate triple of on–shell momenta $(p, q, -(p + q))$ determines a unique null plane. The spacelike and timelike components of $(p, q, -(p + q))$ may each be thought of as forming the edges of two congruent triangles [6], which may have equal or opposite orientation. If the triangles’ orientations are equal, then the corresponding 3–point function vanishes, while for opposite orientations it is nonvanishing. Thus to specify a pair of momenta in this two–dimensional null space it is sufficient to specify only their (complex) spacelike components $p_s, q_s$, if we are interested in the nontrivial part of the operator algebra. Let us restrict attention to this nontrivial subalgebra and accordingly let us label its generators by the unrestricted spacelike momenta $p_s$. Then the on–shell algebra (2.10) can be written in terms of the $p_s$ as

$$[J_{p_s}, J_{q_s}] = 2(p_s \bar{q}_s - q_s \bar{p}_s)J_{p_s + q_s} \quad (2.16)$$

Up to a rescaling, this is nothing but the Lie bracket in the algebra of area–preserving diffeomorphisms of the plane as generated by the basis

$$L_k \equiv ie^{ik\bar{x} + \bar{k}x}(\bar{k}\partial_{\bar{x}} - k\partial_x) \quad (2.17)$$

In fact,

$$[L_k, L_{k'}] = (kk' - k'k)L_{k+k'} \quad (2.18)$$

so the identification of $J_k$ with $2L_k$ gives a Lie algebra isomorphism between (2.18) and (2.16). Geometrically, the $J$’s generate area–preserving diffeomorphisms of the null plane $V_{\text{null}}$, and act on the ring of normalizable functions on $V_{\text{null}}$, which is the ground ring generated by the $O_p(z, \bar{z})$.

We conclude that area–preserving diffeomorphisms of the null plane are a subalgebra of the off–shell algebra of vertex operators. Indeed, area–preserving diffeomorphisms of any null plane are contained in the full algebra. As we will argue in the next section, a natural candidate for this off–shell algebra is the algebra of volume–preserving diffeomorphisms of the target space, $\text{vdiff}(M)$.
3. Toroidal backgrounds

So far, we have just considered the background $\mathbb{R}^{2,2}$. A straightforward generalization is to compactify some or all of the coordinates — including timelike dimensions — on a torus. Compactification of timelike dimensions is quite unphysical, but will prove to be a useful trick in determining an underlying gauge algebra. Namely, for each on–shell subalgebra, we will be able to find a background in which that symmetry is unbroken. More precisely, what we will find is that for every null plane algebra of the form (2.10), there is a point in the moduli space of toroidal compactifications at which the currents become holomorphic. At this point, the currents $J_p$ and $J_q$ are conserved, their definitions as contour integrals (2.9) become independent of the contours chosen, and the gauge symmetry is realized exactly on physical states.

The most general toroidal compactification is constrained by the level–matching requirement, which must be satisfied by both on– and off–shell states:

$$\left( |p_Ls|^2 - |p_Lt|^2 \right) - \left( |p_Rs|^2 - |p_Rt|^2 \right) \in 2\mathbb{Z} \quad (3.1)$$

This condition is satisfied by the vectors of an even Lorentzian lattice of signature $(4,4)$. Modular invariance of the formal 1-loop partition function also constrains the lattice to be self–dual, so we just have a Narain compactification [7] on $\Gamma^{4,4}$.

We begin by deriving the on–shell algebra. The on-shell condition requires that the two expressions in parentheses in eq.(3.1) be separately zero — namely, that the on–shell vectors generate a subset of the null vectors of $\Gamma^{4,4}$. Sublattices of $\Gamma^{4,4}$ of mutually perpendicular null vectors are at most four–dimensional. It is thus advantageous to study compactifications for which all the vectors in such a null sublattice are on–shell, in order to describe a maximal on–shell algebra. In fact, there always exists a four–dimensional subspace of mutually perpendicular on–shell null momenta if the $(4,4)$–dimensional lattice is a direct sum of two $(2,2)$–dimensional ones

$$\Gamma^{4,4} = \Gamma_L^{2,2} \oplus \Gamma_R^{2,2} \quad (3.2)$$

On–shell momenta $p_L$ and $p_R$ may then be taken to lie in two 2–dimensional null sublattices and vertex operators carry indices $(p_L, p_R)$. If we wanted to, we could as before label on–shell momenta (giving rise to non-trivial 3-point functions) by their spacelike components.

† The signature $(4, 4)$ should be regarded as a notational shorthand for $(2, 2; 2, 2)$, where the semicolon separates the signatures of the left and right movers.
only; however, in order to allow generalization to off–shell momenta, we will take \((p_L, p_R)\) to represent the full \((4, 4)\)-dimensional lattice vector.

Recall that we are interested in any \((1, 0)\) (or \((0, 1)\)) operator that may become holomorphic (anti-holomorphic) somewhere in the moduli space of toroidal backgrounds. All operators with momenta which are null with respect to the Lorentzian norm in eq.\((3.1)\) have this property. To see this, let \(p = (p_L, p_R)\) be any null lattice vector in \(\Gamma^{4,4}\). By rotating \(\Gamma^{4,4}\) by the set of \(SO(4, 4)\) transformations we can cover the full moduli space of \((4,4)\) lattices. Now \(SO(4, 4)\) transformations act on individual 8–real–component momenta \((p_{Ls}, p_{Rs}, p_{Lt}, p_{Rs})\) in the fundamental representation, and an \(SO(4, 4)\) transformation can always be found, which rotates \(p\) into a null vector of the form \((p'_{Ls}, 0, p'_{Lt}, 0)\). That is, there always exist points in moduli space where \(J_p = J_{(p_L, 0)}\) is holomorphic. At such a point, \(J\) is exactly conserved, and the charge given by the contour integral of \(J\) will be independent of the choice of contour. Indeed, more is true: given two operators \(J_p\) and \(J_q\) with \(p, q,\) and \(p + q\) all on–shell, there is always an \(SO(4, 4)\) rotation taking \(p\) and \(q\) simultaneously to \((p_L, 0)\) and \((q_L, 0)\). This is seen by first rotating \(p\) by a particular \(SO(4, 4)\) transformation. The subgroup of \(SO(4, 4)\) preserving \(p\) may then be used to rotate the orthogonal vector \(q\) into the desired form. Hence there is always a point in the moduli space of toroidal backgrounds where the Lie bracket \((2.10)\) is exact, independent of the choices of contours for \(J_p\) and \(J_q\). Once we move away from the special point where the currents are holomorphic, the corresponding gauge symmetries will be spontaneously broken. Taken together, the on–shell currents generate an enormous symmetry algebra of the full theory, which is broken down to a subalgebra by any given toroidal background.

Even the unbroken subalgebra in a particular background may be infinite–dimensional. For example, in a toroidal background of the type \((3.2)\) there are an infinite number of holomorphic (and anti-holomorphic) currents, corresponding to momenta of the form \((p_L, 0)\) (and \((0, p_R)\)). These currents give rise to exact infinite symmetry algebras of the CFT: the area–preserving diffeomorphisms of null 2-tori in the \((2, 2)_L\) (and \((2, 2)_R\)) torus.

For a general toroidal compactification, the on–shell algebra generated by the \(J\) and \(\bar{J}\) is

\[
\begin{align*}
[J_{(p_L, p_R)}, J_{(q_L, q_R)}] &= (p_L \cdot q_L - q_L \cdot \bar{p}_L)J_{(p_L + q_L, p_R + q_R)} \\
[\bar{J}_{(p_L, p_R)}, \bar{J}_{(q_L, q_R)}] &= (p_R \cdot q_R - q_R \cdot \bar{p}_R)\bar{J}_{(p_L + q_L, p_R + q_R)} \\
[J_{(p_L, p_R)}, \bar{J}_{(q_L, q_R)}] &= \frac{1}{2} (p_L \cdot q_L - q_L \cdot \bar{p}_L)\bar{J}_{(p_L + q_L, p_R + q_R)} \\
&\quad + \frac{1}{2} (p_R \cdot q_R - q_R \cdot \bar{p}_R)J_{(p_L + q_L, p_R + q_R)} + \ldots \tag{3.3}
\end{align*}
\]
where the dots refer to \((1,0)\) and \((0,1)\) operators which are not upper components of \(O(p_L, q_L, p_R, q_R)\). The OPE of the \(V\)’s is

\[
V_{(p_L, p_R)}(z, \bar{z}) \cdot V_{(q_L, q_R)}(w, \bar{w}) = \frac{1}{4} \frac{(p_L \cdot \bar{q}_L - q_L \cdot \bar{p}_L)(p_R \cdot \bar{q}_R - q_R \cdot \bar{p}_R)}{|z - w|^2} V_{(p_L + q_L, p_R + q_R)}(w, \bar{w}) + \cdots
\]  

(3.4)

By a simple extension of eq.\((2.17)\), we can obtain a representation of the algebra of the \(J\)’s and \(\bar{J}\)’s as follows. Let

\[
L_{(p_L, p_R)}^\pm = J_{(p_L, p_R)} \pm \bar{J}_{(p_L, p_R)}
\equiv ie^{i(p_L \cdot \bar{x}_L + p_R \cdot \bar{x}_R + \bar{p}_L \cdot x_L + \bar{p}_R \cdot x_R)}[(\bar{p}_R \cdot \partial_{\bar{x}_R} - p_R \cdot \partial_{x_R}) \pm (p_L \cdot \partial_{x_L} - \bar{p}_L \cdot \partial_{\bar{x}_L})]
\]  

(3.5)

The \(L^+\) and \(L^-\) act on the ground ring of functions on the Narain \((4,4)\)-torus as derivations, \textit{i.e.}, diffeomorphisms. In fact, the \(L^+\) and \(L^-\) generate algebras of symplectic diffeomorphisms. Before describing these algebras, we should briefly recall some basic facts about symplectic geometry.

Let \(\omega\) be a closed 2–form on a 2\(n\)–dimensional manifold \(M\) whose \(n\)th power is proportional to the volume form on \(M\), and let \(f\) denote any differentiable function on \(M\). Then there is a vector field \(v_f\) associated with \(f\), whose interior product with the symplectic form \(\omega\) is \(df\):

\[
i(v_f)\omega = df
\]  

(3.6)

Such a vector field is said to be symplectic with respect to \(\omega\), and the Lie derivative of \(\omega\) in the direction \(v_f\) is automatically 0; that is, the flow generated by \(v_f\) preserves \(\omega\). The symplectic vector fields form the Lie algebra of symplectic diffeomorphisms \(\text{sdiff}_\omega(M)\), whose Lie bracket may be shown to satisfy

\[
[v_f, v_g] = v_{\{f, g\}}
\]  

(3.7)

where \(\{f, g\} = \omega(v_f, v_g)\) is the Poisson bracket with respect to \(\omega\). Since the \(n\)th power of \(\omega\) is proportional to the volume form on \(M\), a symplectic diffeomorphism is automatically volume–preserving. (There may be additional diffeomorphisms preserving \(\omega\); these correspond to closed 1-forms on \(M\) which are not derived from any \(f\). Such 1-forms are precisely the elements of \(H^1(M)\).)
Any Kähler manifold comes equipped with a natural symplectic structure, given simply by the Kähler form. The Narain (4,4) torus is a Kähler manifold with Kähler form (in an appropriate basis)

\[ k = dx^1_L \wedge d\bar{x}^1_L - dx^2_L \wedge d\bar{x}^2_L - dx^1_R \wedge d\bar{x}^1_R + dx^2_R \wedge d\bar{x}^2_R \]  

(3.8)
corresponding to the Lorentzian metric on \( T^{4,4} \) implicit in (3.1). It is with respect to this \( k \) that the \( L^+ \) are symplectic. Indeed, the symplectic vector field

\[ L^+_f \equiv k^{ij} \partial_i f \partial_j \]  

(3.9)
where \( k^{ij} \) is the inverse of the matrix \( k_{ij} \) representing \( k \) readily reduces to \( L^+_p \) when \( e^{ip} : x \) is substituted for \( f \). Thus, for on-shell momenta the \( L^+_p \) generate a null subalgebra of \( \text{sdiff}_k(T^{4,4}) \).

The \( L^- \) also close on themselves as an algebra of symplectic diffeomorphisms relative to a different symplectic form

\[ \tilde{k} = dx^1_L \wedge d\bar{x}^1_L - dx^2_L \wedge d\bar{x}^2_L + dx^1_R \wedge d\bar{x}^1_R - dx^2_R \wedge d\bar{x}^2_R \]  

(3.10)
In addition, the Lie bracket of an \( L^+_p \) with an \( L^- \) will generate volume-preserving diffeomorphisms symplectic with respect to still other forms:

\[ [L^+_f, L^-_g] = (k^{ij} \tilde{k}^{kl} - k^{il} \tilde{k}^{kj}) \partial_j (\partial_i f \partial_k g) \partial_l \]  

(3.11)
The right side can always be decomposed in terms of symplectic forms on \( T^{4,4} \)

\[ k^{ij} \tilde{k}^{kl} - k^{il} \tilde{k}^{kj} = \sum_I A^I_{ik} \omega^j_I \]  

(3.12)
where \( I \) indexes a basis of symplectic matrices on \( T^{4,4} \). (This decomposition follows from the fact that any antisymmetric matrix can be written as a sum of symplectic matrices.)

In terms of the \( \omega_I \), we may rewrite eq. (3.11) as

\[ [L^+_f, L^-_g] = \sum_I \omega^j_I \partial_j (A^I_{ik} \partial_i f \partial_k g) \partial_l \]  

(3.13)
which makes it clear that the Lie bracket is always a sum of symplectic diffeomorphisms. The additional symplectic diffeomorphisms thus generated correspond to the dots in eq. (3.3).
Besides the above operators, we can again form discrete states $D_\epsilon(z)$ and $\bar{D}_\epsilon(\bar{z})$ as in (2.13), corresponding to the $U(1)_L^4 \times U(1)_R^4$ isometries of the toroidal background. These states combine naturally with the symplectic diffeomorphisms. In fact, they are precisely the extra volume–preserving diffeomorphisms which are not symplectic with respect to any $\omega$, and are generated by vector fields $v$ for which the left side of eq.(3.6) is closed but not exact.

The geometric interpretation of the zero–momentum discrete states merits a brief digression. The dimension $(1,1)$ operators $V_{ij}$ appearing in eq.(2.15) generate modular deformations of $(4,4)$–dimensional Narain compactifications \[11\]. As such they are naturally associated with elements of $H^{(1,1)}(M)$. In a more general curved background, the zero–momentum condition on the dimension $(1,1)$ discrete operators becomes the condition that the $(1,1)$–form corresponding to the associated compactification modulus be harmonic. Thus there is a correspondence between discrete operators of worldsheet dimension $(1,1)$ and closed $(1,1)$ forms on $M$. Likewise, the dimension–one operators $D_\epsilon$ and $\bar{D}_\epsilon$ generating isometries of the Lorentzian torus are also related to the cohomology of $M$. In general, the discrete states of dimension one will be associated to nonsymplectic volume–preserving diffeomorphisms of $M$, which are in one-to-one correspondence with the elements of $H^1(M,\mathbb{R})$.

Actually, now that we are working in a compact background there are many more discrete $(1,0)$ and $(0,1)$ states, at momenta $(0,p_R)$ and $(p_L,0)$ with $p_L^2 = p_R^2 = 0$ (here $(p_L,p_R)$ includes the time-like components). The corresponding $(1,0)$ operators $D_{(\epsilon_L,p_R)}(z,\bar{z}) \equiv D_{\epsilon_L}(z)\bar{O}_{p_R}(\bar{z})$ act on the $J$’s as follows:

$$[D_{(\epsilon_L,p_R)},J_{(q_L,q_R)}] = (\epsilon_L \cdot \bar{q}_L - \bar{\epsilon}_L \cdot q_L)J_{(q_L,p_R+q_R)}$$

The complete underlying algebra generated by $J, \bar{J}, D$ and $\bar{D}$ is given by their representations as derivations:

$$D_{(\epsilon_L,p_R)} \equiv i\epsilon(p_{\bar{\epsilon}R}+\bar{p}_{\epsilon R})(\bar{\epsilon}_L \partial_{\bar{x}_L} - \epsilon_L \partial x_L)$$

and similarly for $\bar{D}_{(p_L,\epsilon_R)}$. We immediately recognize $D_{(\epsilon_L,0)} = D_{\epsilon_L}$ and $\bar{D}_{(0,\epsilon_R)} = \bar{D}_{\epsilon_R}$ as generators of isometries of the torus.

The other $D_{(\epsilon_L,p_R)}$, with $p_R \neq 0$, are actually symplectic diffeomorphisms, with respect to symplectic forms that mix left and right movers, of the form

$$\omega_{ij} \, dx^i_L \wedge dx^j_R$$
If coordinates are chosen so that the vector $\epsilon_L$ is dual to the one-form $dx^1_L$ and $p_R$ is dual to $dx^1_R$, then a symplectic structure generating $D_{(\epsilon_L,p_R)}$ is

$$\omega_{(\epsilon_L,p_R)} = dx^1_L \wedge dx^1_R + \text{(terms \perp to $\epsilon_L$ and $p_R$)}$$

Explicitly, we obtain

$$D_{(\epsilon_L,p_R)} = \omega_{ij}^{(\epsilon_L,p_R)} \partial_i f(x_R) \partial_j$$

with the function $f(x_R) = e^{ip_R \cdot x_R}$. The discrete operators $D_{(\epsilon_L,p_R)}$ are already included in the algebra generated by the $L^+$ and $L^-$, as is seen by taking, for example, $f = f(x_R)$ and $g = g(x_R)$ to be independent of $x_L$ in eq.(3.11).

The representation of the on–shell algebra as a derivation algebra turns out to be more than a convenient encoding; it also suggests a natural underlying off–shell algebraic extension.

The generalization of the algebras in (3.3) and (3.14) to off–shell momenta begins with the observation that there is an on–shell algebra associated with any null sublattice $\Gamma_{null}$. There are many possible choices of $\Gamma_{null}$, and the full algebra should contain all the resulting on–shell algebras. A natural candidate for this off–shell algebra is the derivation algebra generated from (3.5) and (3.15) over the set of all null momenta of the $(4,4)$ lattice. Since in fact there is a basis for $\Gamma_{4,4}^\perp$ consisting entirely of null momenta, we immediately obtain generators for each lattice momentum $(p_L, p_R)$ as Lie brackets of on–shell generators $L^\pm$. The $L^+$ and $L^-$ each generate algebras of symplectic diffeomorphisms $\text{sdiff}_k(T^{4,4})$ and $\text{sdiff}_k(\tilde{T}^{4,4})$. Together with the $D$’s they generate the full volume–preserving diffeomorphism algebra $\text{vdiff}(T^{4,4})$. (This is a nontrivial statement since there are 105 independent symplectic structures on $T^{4,4}$; it can be checked by taking linear combinations of the derivations appearing on the right–hand side of (3.11).) The off–shell algebraic extension to $\text{vdiff}(T^{4,4})$ is background–independent, because all the Narain tori are isomorphic.

In the standard decompactification limit, where $p_L = p_R$, the full algebra reduces to the algebra of the $L^+$’s. The $L^-$ act trivially on themselves and on the ground ring of functions on the target space $\mathbb{R}^{2,2}$, and the $L^+$ generate $\text{sdiff}(\mathbb{R}^{2,2})$.

It is also interesting to consider the case where only the spacelike dimensions are compactified. Now the on-shell condition requires that $(p_{Ls}, p_{Rs})$ be a null vector in $\Gamma_{s,2}^2$: $(p_{Ls})^2 - (p_{Rs})^2 = 0$ (because $p_{Lt} = p_{Rt}$ for non-compact time). Since a maximal sublattice $\Gamma_{null}$ of the Narain lattice $\Gamma_{s,2}^2$ is two–dimensional, the on–shell algebra in this case reduces
to the area-preserving diffeomorphism algebra of a two-dimensional null torus. The underlying off–shell algebra generated by the derivations is isomorphic to \( \text{vdiff}(T^{2,2}_s) \), namely, to the volume–preserving diffeomorphisms of the Narain torus of this compactification.

We conclude this section by pointing out some similarities between the algebraic structures of the \( N = 2 \) and \( d = 2 \) strings.

The on–shell algebra of the \( N = 2 \) string contains the on–shell algebra of the \( d = 2 \) string in a trivial way. To see this, let \( p^{(1)}_L \) and \( p^{(2)}_L \) be two basis vectors for the left–moving two–dimensional null sublattice of \( \Gamma_L \) in (3.2), and similarly choose a basis for the right–movers. Following [12], define the holomorphic (and anti-holomorphic) dimension 0 operators

\[
x = \mathcal{O}_{p^{(1)}_L}(z) \\
y = \mathcal{O}_{p^{(2)}_L}(z) \\
x' = \bar{\mathcal{O}}_{p^{(1)}_R}(\bar{z}) \\
y' = \bar{\mathcal{O}}_{p^{(2)}_R}(\bar{z})
\]

(3.19)

With this choice, a subring of the ground ring is generated by polynomials in the four variables

\[
a_1 = xx', \quad a_2 = yy', \quad a_3 = xy', \quad a_4 = yx'
\]

(3.20)

and their inverses, with the relation

\[
a_1a_2 - a_3a_4 = 0.
\]

(3.21)

The algebra (3.3) acts on the 3-dimensional cone of such \( a_i \). This is quite similar to the situation studied in ref.[12], where the symmetry algebra of the \( d = 2 \) string (at the self–dual point) acts as the algebra of volume–preserving diffeomorphisms of the cone (3.21).

In case \( p_L \) and \( p_R \) both live in the same Narain lattice \( \Gamma^{2,2}_L = \Gamma^{2,2}_R \) (namely, the symmetric toroidal compactification \( \Gamma^{2,2}_L = \Gamma^{2,2}_R \) extended symmetry), the choice of basis (3.19) is more restricted. At the self-dual point (with \( SU(2)_L^2 \times SU(2)_R^2 \) extended symmetry), a good choice is

\[
p^{(1)}_{Ls} = p^{(1)}_{Rs} = \frac{1}{\sqrt{2}}(1, 1); \quad p^{(2)}_{Ls} = p^{(2)}_{Rs} = \frac{1}{\sqrt{2}}(1, -1).
\]

(3.22)

With this choice, the OPE (3.4) coincides with the algebra found in [13] for the \( d = 2 \) closed string at the self-dual point. This can be shown as follows: amongst the vectors \((p^{(i)}_{Ls}, p^{(j)}_{Rs})\) only three (denoted \( v^i \)) are independent

\[
v^1 = \frac{1}{\sqrt{2}}(1, 1, 1); \quad v^2 = \frac{1}{\sqrt{2}}(1, -1, 1, -1); \quad v^3 = \frac{1}{\sqrt{2}}(1, 1, 1, -1).
\]

(3.23)
On–shell momenta are now labeled by the space-like components

\[ p_s = \sum_{i=1}^{3} p_i v_i; \quad q_s = \sum_{i=1}^{3} q_i v_i \]  

(3.24)

where \( p_i, q_i \) are integers. Denoting \( V_p = V_{(p_1,p_2,p_3)} \equiv V_{j_p,m_p,m'_p} \) with \( j_p = \frac{1}{2}(p_1 + p_2 + p_3), m_p = \frac{1}{2}(p_1 - p_2 + p_3), m'_p = \frac{1}{2}(p_1 - p_2 - p_3) \), one finds for the OPE (3.4):

\[
V_{j_p,m_p,m'_p}(z,\bar{z}) \cdot V_{j_q,m_q,m'_q}(w,\bar{w}) = \frac{(j_p m_q - j_q m_p)(j_p m'_q - j_q m'_p)}{|z - w|^2} V_{j_p+j_q,m_p+m_q,m'_p+m'_q}(w,\bar{w}) + \cdots
\]  

(3.25)

reproducing the algebra of [13].

The remarkable similarity between the \( d = 2 \) and \( N = 2 \) string algebras is suggestive of a deep connection between the two theories, which should be investigated further.

The full off–shell operator algebra is of course harder to obtain directly. We have already discussed a good candidate; yet it is far from unique. In fact, another straightforward off–shell extension of the on–shell algebras is a Lorentzian lattice algebra, of the type discussed in [4]. Further arguments will be required to justify our choice of an off–shell algebra. The justification will come in the following section when we consider gauge symmetries from the point of view of the spacetime action.

4. Spacetime action

In the previous section, we constructed on–shell generators of exact gauge algebras in different backgrounds, and found a natural off–shell Lie algebraic closure in the algebra of volume–preserving diffeomorphisms of the target Narain torus. Here we present independent evidence that vdiff(\( T^{4,4} \)) is an underlying gauge symmetry algebra of the \( N = 2 \) string.

A spacetime action leading to the correct on–shell amplitudes in uncompactified Minkowski space is [6]

\[
S_\phi = \int d^2 x_1 d^2 x_2 \left[ \frac{1}{2} \eta^{ij} \partial_i \phi \partial_j \phi + \frac{1}{3} \epsilon^{ijkl} \phi \partial_i \phi \partial_j \phi \partial_k \phi \right].
\]  

(4.1)

This action implies the three–point function

\[
\langle V_p V_q V_r \rangle = -4(q \cdot \bar{r})(\bar{q} \cdot r) + 4(q \cdot \bar{q})(r \cdot \bar{r})
\]  

(4.2)
which reduces to eq. (2.4) when all momenta are on–shell.

We can generalize the action (4.1) to an arbitrary background, in a way that makes the off–shell symmetry manifest, as follows. From the Kähler potential and the metric we can construct

$$S = \frac{1}{3} \int K \wedge k \wedge k = \frac{1}{3} \int d^4x \sqrt{g}K$$

(4.3)

where $k$ is the Kähler (1,1)-form derived from the (0,0)-form $K$ and $g$ is to be treated as a function of $K$. This is essentially the unique action (up to total derivatives and, as we shall explain, terms arising from the existence of discrete states), which involves only the Kähler potential and its derivatives, is invariant under Kähler transformations, and reproduces all 3- and higher–point correlation functions when expanded around flat Minkowski space. Upon substitution of $K_0 + \phi$ for $K$ (where for example $K_0$ may be taken to be the Kähler potential for $\mathbb{R}^{2,2}$; $K_0 = x_1 x_\bar{1} - x_2 x_\bar{2}$) this action reverts to the cubic term in Ooguri–Vafa action (1.1), plus a (somewhat arbitrary) quadratic term, a tadpole and a $\phi$–independent term:

$$S = S_\phi + \int \left( \frac{1}{2} \phi \partial \bar{\partial} \phi \wedge \partial \bar{\partial} K_0 + \phi \partial \bar{\partial} K_0 \wedge \partial \bar{\partial} K_0 + \frac{1}{3} K_0 \partial \bar{\partial} K_0 \wedge \partial \bar{\partial} K_0 \right)$$

(4.4)

Of course, if $K_0$ is a solution of the equations of motion, the tadpole will vanish. Varying within the class of Kähler metrics – i.e., with respect to $K$ – we obtain the equation of motion

$$\det g = 0$$

(4.5)

The solutions to (4.5) lie on the boundary of the moduli space of Kähler geometries, and the functional measure for the action (4.3) is therefore peaked about singular metrics, just as in topological theories of gravity. We will shortly see how to modify the action in order to obtain nonsingular backgrounds as solutions to the equations of motion as well.

The symmetries of the action (4.3) are manifest; it is invariant under diffeomorphisms that preserve the Kähler form $k$. Such diffeomorphisms clearly leave the volume element $k \wedge k$ invariant, and a diffeomorphism that preserves $k$ can only change $K$ by a Kähler gauge transformation. Under a Kähler transformation $K(x, \bar{x}) \to K(x, \bar{x}) + \text{Re} f(x)$, the Lagrangian changes by a total derivative, as seen by integrating by parts twice. (The same sort of gauge invariance up to a total derivative is also a property of the Chern–Simons action in 2+1 dimensions.) Assuming that the total derivative can be ignored, this shows that $k$–symplectic diffeomorphisms are indeed symmetries.

These are precisely the symmetries generated by the $L^+$ in eq. (3.3), which formed the surviving part of the off–shell algebra in the limit of decompactification to $\mathbb{R}^{2,2}$. 16
Because of the presence of a tadpole, the action (4.3) is not equivalent to the action (4.1) derived by Ooguri and Vafa. However, there is a term we can add to it, which cancels the tadpole without violating any symmetries. This is the quadratic term

$$\frac{1}{2} \int d^4x G^{ij} K \partial_i \partial_j K = -\frac{1}{2} \int d^4x G_{\bar{i} \bar{j}} \epsilon^{\bar{i} \bar{j}} \partial_{\bar{i}} \partial_{\bar{j}} K$$

(4.6)

For constant $G_{ij}$, this term is invariant under symplectic diffeomorphisms preserving $k$. Adding it to the background–independent action $S$, we obtain

$$S_G = \int \left( \frac{1}{3} K \wedge k \wedge k - \frac{1}{2} K \wedge k \wedge k_G \right)$$

(4.7)

where $K_G$ is the Kähler potential for $G$ and $k_G$ is the $(1,1)$–form derived from $K_G$.

With this modified action, the equation of motion for $K$ is

$$\partial \bar{\partial} K \wedge \partial \bar{\partial} K = \partial \bar{\partial} K \wedge \partial \bar{\partial} K_G$$

(4.8)

which is solved (for example) by $K = K_G$. In other words, $K_G$ should be regarded as the classical background $K_0$, in order for the tadpole to vanish. Expanding around $K_0 = K_G = \eta_{i\bar{j}} x^i x^{\bar{j}}$, we recover exactly the original action (4.1) (plus a \(\phi\)-independent term).* The equation of motion then becomes

$$\partial \bar{\partial} \phi \wedge \partial \bar{\partial} \phi + \partial \bar{\partial} K_0 \wedge \partial \bar{\partial} \phi = 0$$

(4.9)

as found in \cite{6}. This is equivalent to the Plebanski equation \cite{14}.

There is another reason for adding the term (4.6), besides cancelling the tadpole, involving the existence of discrete states. In a flat Minkowski background, for example, there are discrete states corresponding to the 0–momentum graviton vertex operator

$$V^{i\bar{j}} = \partial x^i \partial x^{\bar{j}} + \partial x^{\bar{i}} \partial x^j$$

(4.10)

We associate this operator with a constant background metric $G_{i\bar{j}}$ derived from the Kähler potential $\tilde{K}_0 = G_{i\bar{j}} x^i x^{\bar{j}}$. For convenience, we may choose coordinates so that $\tilde{K}_0 = x^1 x^{\bar{1}} - x^2 x^{\bar{2}}$.

* Another solution to the equation of motion (4.8) is $K = 0$. Expanding $K$ around this classical solution, we also recover the action (4.1) if $K_G = -\eta_{i\bar{j}} x^i x^{\bar{j}}$. (Note that the metric $-\eta_{i\bar{j}}$ is equivalent to $\eta_{i\bar{j}}$ in signature (2,2).)
Now there is a nonvanishing 3–point function involving the graviton discrete state
\[ \langle V_i V_p V_{-p} \rangle = p_i p_j + p_i p_j \] (4.11)
The tadpole–cancelling term (4.6) can therefore be identified as a contribution of the
discrete states to the background–independent effective action in (4.3).

The discrete states \( D^i \) of conformal dimension one also play a role in the spacetime action. Consider making the following Kähler transformation
\[ K(x, \bar{x}) \rightarrow K(x, \bar{x}) + a_i x^i + b_{\bar{i}} x^{\bar{i}} \] (4.12)
where the \( a_i \) and \( b_{\bar{i}} \) are complex constants. Such a transformation leaves the metric and the equations of motion unchanged, but shifts the one–form \( dK \) by \( a_i dx^i + b_{\bar{i}} d\bar{x}^{\bar{i}} \). Recall from section 2 that the discrete state \( D^i \) is an upper component of the non–normalizable field \( x^i \); the shift of \( K \) in (4.12) is therefore associated with \( D^i \) and \( D^{\bar{i}} \). The effect of the Kähler transformation (4.12) on the action (4.7) is to introduce a background gauge potential \( A = a_i dx^i + b_{\bar{i}} d\bar{x}^{\bar{i}} \)
\[ S_{G,A} = \int \left[ -\frac{1}{3} (\partial K + A) \wedge (\bar{\partial} K + \bar{A}) \wedge \partial \bar{\partial} K + \frac{1}{2} (\partial K + A) \wedge (\bar{\partial} K + \bar{A}) \wedge k_G \right] \] (4.13)
The action (4.7) (derived for a flat Minkowski background) may be extended to more general backgrounds. The simplest extension is to complexified Kähler structures \( \bar{13} \), with nonvanishing constant two–form field \( B_{i\bar{j}} \). In such backgrounds, it is natural to work with a complex Kähler potential, satisfying \( \partial_i \partial_{\bar{j}} K = G_{i\bar{j}} + iB_{i\bar{j}} \). The action (4.7) then becomes complex. The on–shell 3–point amplitudes are real in Minkowski space, and are unaffected if we take the real part of this complex action.

The generalization to compact backgrounds is more problematic. We derived the action (4.7) from on–shell amplitudes in a noncompact Minkowski background, and we might not expect it to apply in a toroidal background. On a torus, the off–shell three–point function (4.2) no longer admits an obvious decomposition into left-handed and right-handed parts; it is hard to see how a generalization of the intrinsically 4–dimensional action (1.3) could produce the compactified on–shell 3-point amplitude contained in (3.4). After all, the Narain torus is 8–dimensional, and momenta have separate left– and right–moving components, which are not reflected in the momentum spectrum of \( K \).

Alternatively, we may choose to work in the 4–dimensional picture of Narain, Sarmadi, and Witten \( [11] \), but then we must figure out how to include winding modes. From the
point of view of a spacetime action, these would be nonlocal excitations. We might hope to find them appearing as solitons, as cosmic strings wrapped around the spacetime torus. If the winding modes of the string are actually solitons of the field \( K \), then it may be that the spacetime action (4.13) is already complete for any toroidal background. Shifting the winding sector \( w \) of \( K \) would allow us to obtain a state of any \( p_L = p + w \) and \( p_R = p - w \). Then the additional symmetries not contained in the action’s manifest symmetry algebra \( \text{sdiff}_k(M) \) would act by mixing up local momentum excitations of \( K \) with solitonic winding states. An analogous situation arises in \( N = 2 \) super Yang–Mills theories, where the full theory may have an extra duality symmetry relating local electric and nonlocal solitonic magnetic monopoles, which however is not manifest in the local action.

To recapitulate, we have found an action that gives back the correct on–shell amplitudes and is invariant under the algebra of symplectic diffeomorphisms, which coincides with the derivation algebra derived in section 3 from the worldsheet point of view. The discrete states played a key role in the construction: they contributed a background–dependent term that allowed us to shift the tadpole away in a flat Minkowski background.

The effective action was derived in two stages. First, following ref. [6], we wrote down the minimal Lorentz–invariant action reproducing all on–shell correlation functions at tree level, and next we constructed the unique (up to total derivatives) spacetime action for the Kähler potential, which reduced to the Ooguri–Vafa action when expanded around Minkowski space.

Additional terms with vanishing on–shell contributions could also have been added; however, (4.7) is the unique minimal action reproducing all on–shell amplitudes. Its invariance under the same off–shell symmetry algebra found in section 3 in the decompactification limit, \( \text{sdiff}_k(M) \), provides strong evidence that this is the correct off–shell extension of the on–shell algebras of area–preserving diffeomorphisms of each of the null planes. This infinite–dimensional gauge symmetry will strongly constrain the form of higher–loop corrections.

5. Discussion: The Heterotic (\( N = 2 \)) String

We have studied the underlying gauge symmetry structure of the closed \( N = 2 \) superstring in toroidal backgrounds. Within the natural underlying symmetry algebra \( \text{vdiff}(T^{4,4}) \), the contribution of the left-moving and the right-moving sectors of the closed
string are manifest. Moreover, the underlying algebra is background-independent, as all (4, 4)–dimensional Narain tori are isomorphic.

The next step towards revealing the underlying structure of the $N = 1$ heterotic string is to determine the underlying symmetries of the $N = 2$ heterotic string. This is left for future work; here we will limit ourselves to a few speculations.

The left-handed sector of the critical $N = (2, 0)$ heterotic string is the $N = 2$ superstring in spacetime signature (2,2), and the right-handed sector is a bosonic string in (2, 26) dimensions [18]. To find a maximal on–shell gauge algebra, we may consider total compactification on a (4,28)–dimensional Narain torus of the form $T_L^{2,2} \times T_R^{2,26}$. As before, the left–moving on–shell algebra will generate area–preserving diffeomorphisms of a null 2-torus, while the right–moving on–shell algebra will include both gauge and diffeomorphism generators. As a maximal on–shell sublattice of $\Gamma^{2,26}$ we may take the lattice generated by the root vectors of $g = E_8^3$ (or any rank 24 Niemeier lattice) and 2 perpendicular null vectors. If there had just been one perpendicular null vector, the resulting algebra of currents would have been the affine Lie algebra $\hat{g}$ [4] [19]. But with two null vectors available, we have a doubly–indexed affine Lie algebra. Now as before, there are many choices for on–shell algebras, and there will always be points in the moduli space of (4,28) Narain lattices at which any given on–shell algebra is realized exactly. The off–shell algebra must include all possible on–shell algebras as subalgebras, and in particular, symplectic diffeomorphisms associated to all null momenta in $\Gamma^{4,28}$ must be included. Thus it seems reasonable to expect that the full off–shell algebra will contain $\text{vdiff}(T^{4,28})$. This, however, can not be the full algebra, which in addition must include generators corresponding to $e^{ipR \times R}$, where $p_R$ are length 2 vectors of the rank 24 internal lattice.

It is also plausible that the underlying symmetry structure of the $N = 1$ heterotic string is related to $\text{vdiff}(T^{10,26})$, $T^{10,26}$ being the Narain torus of the totally compactified target space. However, the proliferation of oscillator modes leads us to suspect that in this case the algebra will be much larger, perhaps as large as the loop algebra $\mathcal{L}(\text{vdiff}(T^{10,26}))$.

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