QUASI-COHERENT HECKE CATEGORY AND DEMAZURE DESCENT

SERGEY ARKHIPOV AND TINA KANSTRUP

To Boris Feigin on the occasion of his 60-th birthday
with gratitude and admiration

Abstract. Let $G$ be a reductive algebraic group with a Borel subgroup $B$. We define the quasi-coherent Hecke category for the pair $(G, B)$. For any regular Noetherian $G$-scheme $X$ we construct a monoidal action of the Hecke category on the derived category of $B$-equivariant quasi-coherent sheaves on $X$. Using the action we define the Demazure Descent Data on the latter category and prove that the descent category is equivalent to the derived category of $G$-equivariant sheaves on $X$.

1. Introduction

The present paper is the second one in the series devoted to the study of Demazure Descent. In [AK] we introduced the notion of Demazure Descent Data on a triangulated category. A category with Demazure Descent Data is a higher analog for a representation of the degenerate Hecke algebra (see [HLS]).

Such representations arise naturally from geometry. Let $X$ be a compact real manifold acted on by a compact simple Lie group $G$ with a fixed maximal torus $T$. Harada et al constructed a natural action of the degenerate Hecke algebra of the corresponding type on the $T$-equivariant $K$-groups of $X$. They showed that the $G$-equivariant $K$-groups are identified naturally with the invariants for the action. In a way we categorify the construction from [HLS] in our paper.

Let $G$ be a reductive algebraic group with a fixed Borel subgroup $B$. Recall that the (finite) Hecke algebra is defined classically as the algebra of $B(F_q)$-biequivariant functions on the group $G(F_q)$ with values in $\mathbb{C}$. The multiplication is provided by convolution. It turns out that the stack $B\backslash G/B$ is a universal geometric tool to produce "algebras", both in the usual and in the categorical sense. In particular, the categories of constructible sheaves and D-modules on $B\backslash G/B$ were studied in [Tan]. They were used as a natural source of finite Hecke algebra actions on categories of geometric origin.

In the present paper, we consider the quasi-coherent Hecke category $QCHecke(G, B)$, the derived category of $B$-biequivariant quasi-coherent sheaves on $G$. For technical reasons we prefer to work with the equivalent category - the derived category of $G$ equivariant quasi-coherent sheaves on $G/B \times G/B$.

Let us outline the structure of the paper. We recall the standard definitions and facts related to equivariant quasi-coherent sheaves on a scheme in Sections 2 and 3. We introduce the monoidal category $QCHecke(G, B)$ in Section 4. In Section 5 we recall the definitions...
2 SERGEY ARKHIPOV AND TINA KANSTRUP

of Demazure Descent Data on a triangulated category and of the corresponding Descent
category.

Let $X$ be a regular Noetherian scheme acted on by a reductive algebraic group $G$. Section 6 is devoted to the construction of Demazure Descent Data on the derived category of $B$-equivariant quasicoherent sheaves on $X$ in terms of the natural monoidal action of $\text{QCHecke}(G, B)$ on the category by convolution.

**Theorem:** For $w \in W$ let $\overline{O}_w$ denote the closure of the corresponding $G$ orbit in $G/B \times G/B$. Then the functors of convolution with the structure sheaf $O_{\overline{O}_w}$ define Demazure Descent Data on $D^+(\text{QCoh}^B(X))$.

In the last Section we study the corresponding Descent category. We prove the central result of the paper:

**Theorem:** $\text{Desc}(D^+(\text{QCoh}^B(X)), D_w, w \in W)$ is equivalent to $D^+(\text{QCoh}^G(X))$.

1.1. **Acknowledgements.** The authors are grateful to H.H. Andersen, C. Dodd, V. Ginzburg, M. Harada and R. Rouquier for many stimulating discussions. The project started in the summer of 2012 when the first named author visited IHES. S.A. is grateful to IHES for perfect working conditions. Part of the work was done while the second author visited Roman Bezrukavnikov at MIT in the Fall 2013. T.K. would like to express her deepest gratitude to Roman Bezrukavnikov for all that he taught her during her stay and for useful comments on previous versions of the text. T.K would also like to thank MIT for perfect working conditions.

Both authors' research was supported in part by center of excellence grants "Centre for Quantum Geometry of Moduli Spaces" and by FNU grant "Algebraic Groups and Applications".

2. **Equivariant quasi-coherent sheaves on a scheme**

2.1. **Equivariant quasi-coherent sheaves on a scheme.** In this Section $K$ denotes a not necessarily reductive linear algebraic group over an algebraically closed field $k$ of characteristic 0. Let $X$ be a Noetherian, regular $K$ scheme. Consider the action and the projection maps $ac, p : K \times X \to X$ and the coordinate embedding $s : X \to K \times X, x \mapsto (1, x)$. Consider further the multiplication map and two projections $m, p_0, p_1 : K \times K \times X \to K \times X$.

**Definition 2.1** (Equivariant quasi-coherent sheaf). A $K$-equivariant quasi-coherent sheaf on a $K$ scheme $X$ is a pair $(M, \theta)$, where $M \in \text{QCoh}(X)$ and $\theta$ is an isomorphism $ac^*M \xrightarrow{\sim} p^*M$ satisfying

\[ m^*\theta = p_0^*\theta \circ p_1^*\theta \]

\[ s^*\theta = \text{Id}_M. \]

The category of $K$-equivariant quasi-coherent sheaves on $X$ is denoted by $\text{QCoh}^K(X)$. 
Let $f : X \to Y$ be a $K$-equivariant map of $K$ schemes. The functors of push-forward and pull-back are extended naturally to the categories of equivariant sheaves:

\[ f^* : \text{QCoh}^K(Y) \to \text{QCoh}^K(X), \quad (M, \theta) \mapsto (f^*M, f^*\theta \circ \text{canon. isomorphisms}), \]
\[ f_* : \text{QCoh}^K(X) \to \text{QCoh}^K(Y), \quad (M, \theta) \mapsto (f_*M, (\text{Id} \times f)_*\theta \circ \text{canon. isomorphisms}). \]

For the ordinary $f^*$ and $f_*$ of $\mathcal{O}$-modules $f^*$ is right adjoint to $f_*$. In the equivariant case, both the unit and the counit maps are morphisms in the equivariant category, so we have the same adjunction for the equivariant inverse and direct image functors:

\[ \text{Hom}(f^*F, \mathcal{K}) \simeq \text{Hom}(F, f_*\mathcal{K}). \]

**Proposition 2.2.** Let $X$ be a locally trivial (in the Zariski topology) principal $K$-bundle on $Y$ and let $\pi : X \to Y$ be the quotient map. Then the functors

\[ \pi_* : \text{QCoh}^K(X) \to \text{QCoh}(Y), \]
\[ \pi^* : \text{QCoh}(Y) \to \text{QCoh}^K(X), \]

are inverse to each other.

**Proof.** See [AG, Section 3].

In particular, there is an equivalence of categories $p^* : \text{QCoh}(X) \xrightarrow{\sim} \text{QCoh}^K(K \times X)$. This also works in the presence of additional equivariance.

**Proposition 2.3.** $p^* : \text{QCoh}^H(X) \to \text{QCoh}^{K \times H}(K \times X)$ is an equivalence of categories with the inverse functor being $p_* (\ )^K$.

### 3. Derived category

Below we show that the category $\text{QCoh}^K(X)$ has enough injective objects. It is known (see [Har], Exercise III.3.6(a)) that the category $\text{QCoh}(X)$ has enough injectives. Notice that the averaging functor $\text{Av} := ac_p^* : \text{QCoh}(X) \to \text{QCoh}^K(X)$ is exact since $K \times X$ is flat and affine over $X$ and that the functor $\text{Av}$ is right adjoint for the forgetful functor. By the standard argument is follows that the functor $\text{Av}$ takes injectives to injectives.

**Proposition 3.1.** $\text{QCoh}^K(X)$ has enough injective objects.

**Proof.** Let $M \in \text{QCoh}^K(X)$. We need to embed $M$ into an injective object in $\text{QCoh}^K(X)$. Since $\text{QCoh}(X)$ has enough injectives the exists an injective $I \in \text{QCoh}(X)$ with

\[ \text{Forget}(M) \hookrightarrow I. \]

The functor $\text{Av}$ is left exact, so we have

\[ \text{Av}(\text{Forget}(M)) \hookrightarrow \text{Av}(I) \]
By adjunction there is a canonical map $M \to \text{Av}(\text{Forget}(M))$. Below we show that this map is an inclusion. Denote the kernel by $\ker \gamma$ and the inclusion $\ker \hookrightarrow M$ by $\alpha$. Composing with the canonical map we get a map

$$\text{Hom}(\ker, M) \xrightarrow{\text{can}} \text{Hom}(\ker, \text{Av}(\text{Forget}(M))) \simeq \text{Hom}(\text{Forget}(\ker), \text{Forget}(M))$$

The composition is $\gamma \mapsto \text{Id} \circ \text{Forget}(\gamma)$. Now can $\circ \alpha = 0$ so $\text{Forget}(\alpha) = 0$ but $\text{Forget}$ is conservative so $\alpha = 0$. Thus, $\ker = 0$. \hfill \Box

In particular, any object in $\text{QCoh}^K(X)$ has an injective resolution in $\text{QCoh}^K(X)$. We obtain a model for the derived category $D^+(\text{QCoh}^K(X))$ so any left exact functor has a right derived functor.

**Lemma 3.2.** Let $K$ be a reductive group. Then for every $M_1, M_2 \in D^+(\text{QCoh}^K(X))$

$$\text{Ext}_{\text{QCoh}^K(X)}(M_1, M_2) \simeq (\text{Ext}_{\text{QCoh}(X)}(M_1, M_2))^K.$$

**Proof.** Notice that

$$\text{Hom}_{\text{QCoh}^K(X)}(M_1, M_2) = (\text{Hom}_{\text{QCoh}(X)}(M_1, M_2))^K.$$  

When $K$ is reductive the category $\text{Rep}(K)$ is semisimple and $(\cdot)^K$ is exact. Thus,

$$\text{Ext}_{\text{QCoh}^K(X)}(M_1, M_2) \simeq R(\text{Hom}_{\text{QCoh}(X)}(M_1, M_2))^K \simeq (\text{Ext}_{\text{QCoh}(X)}(M_1, M_2))^K.$$  

It follows that the RHS is a direct summand in $\text{Ext}_{\text{QCoh}(X)}(M_1, M_2)$. \hfill \Box

**Remark 3.3.** By the Lemma for a reductive group $K$ the forgetful functor

$$\text{Forget} : D^+(\text{QCoh}^K(X)) \to D^+(\text{QCoh}(X))$$

is conservative. In particular, to prove relations between functors $F$ and $G$ commuting with the forgetful functor it suffices to show the relation between $F \circ \text{Forget}$ and $G \circ \text{Forget}$.

In this paper we mostly consider equivariance with respect to a reductive group. Moreover all our functors will be push forwards or pull backs with the same equivariance on both sides so they all commute with forgetting equivariance. Thus, to prove relations between the equivariant functors it suffices to prove the relations for the non-equivariant functors.

4. **Convolution and the quasi-coherent Hecke category**

Let $Z, Y$ and $X$ be $K$ schemes. Consider the maps

$$i : Z \times Y \times X \to Z \times Y \times Y \times X, \quad (z, y, x) \mapsto (z, y, y, x),$$

$$\pi_{13} : Z \times Y \times X \to Z \times X, \quad (z, y, x) \mapsto (z, x).$$

**Remark 4.1.** One can define the equivariant pull-back for the diagonal embedding $i : Y \hookrightarrow Y \times Y$ under mild restrictions of existence of finite flat resolutions for quasi-coherent sheaves on $Y \times Y$. In the present paper we are only interested in the case where $Y = K/H$ for an algebraic group $K$ and a closed subgroup $H$. In this case the equivariant inverse image functor $i^*$ is exact. Indeed, consider the map

$$\rho : Z \times K/H \times K/H \times X \to Z \times K/H \times K/H \times X, \quad (z, k_1, k_2, x) \mapsto (z, k_1, k_1, x)$$
Let \( p \) be the projection \( Z \times K \times K \times X \to Z \times X \) and \( \pi \) the quotient map \( K \to K/H \).

Consider the commutative diagram

\[
\begin{array}{c}
\text{QCoh}^{H \times H}(Z \times X) \xrightarrow{\text{Forget}} \text{QCoh}^{H \Delta}(Z \times X) \\
\uparrow \quad \uparrow
\end{array}
\]

\[
\begin{array}{c}
\text{QCoh}^{K \times K \times H \times H}(Z \times K \times K \times X) \xrightarrow{\rho^*} \text{QCoh}^{K \times K \times H \Delta}(Z \times K \times K \times X) \\
\uparrow \quad \uparrow
\end{array}
\]

\[
\begin{array}{c}
\text{QCoh}^{K \times K}(Z \times K/H \times K/H \times X) \xrightarrow{i^*} \text{QCoh}^{K}(Z \times K/H \times X) \\
\end{array}
\]

The forgetful functor is exact so \( i^* \) is exact.

The convolution product \( * \) is defined as

\[
* : D^+(\text{QCoh}^K(Z \times Y)) \times D^+(\text{QCoh}^K(Y \times X)) \to D^+(\text{QCoh}^K(Z \times X)),
\]

\[
M_1 * M_2 := R\pi_{13*}i^*(M_1 \boxtimes M_2).
\]

Next we check that the convolution is associative (up to a non-specified isomorphism).

**Proposition 4.2.** The convolution \( * \) is associative.

**Proof.** Consider the diagram

\[
\begin{array}{c}
Z \times Y \times Y \times T \\
\downarrow^{1 \times \pi_X \times 1} \\
Z \times Y \times Y \times X \times T \\
\downarrow^{1 \times i_X \times 1} \\
Z \times Y \times X \times X \times T \\
\downarrow^{i_Y \times 1} \\
Z \times Y \times X \times X \times T \\
\downarrow^{1 \times i_X} \\
Z \times Y \times X \times X \times T \\
\end{array}
\]

\[
\begin{array}{c}
Z \times Y \times T \\
\downarrow^{\pi_X \times 1} \\
Z \times T \\
\end{array}
\]

\[
\begin{array}{c}
Z \times Y \times X \times T \\
\downarrow^{1 \times \pi_Y \times 1} \\
Z \times X \times X \times T \\
\downarrow^{1 \times i_X} \\
Z \times X \times X \times T \\
\downarrow^{\pi_Y \times 1} \\
Z \times X \times X \times T \\
\downarrow^{i_X} \\
Z \times X \times X \times T \\
\end{array}
\]

The squares are cartesian so using flat base change we get

\[
i_X^* R(\pi_Y \times 1 \times 1)_* \simeq R(\pi_Y \times 1 \times 1)_*(1 \times i_X)^*
\]

\[
i_Y^* R(1 \times \pi_X \times 1)_* \simeq R(\pi_X \times 1 \times 1)(i_Y \times 1)^*
\]
Inserting this into the formula for convolution we obtain:
\[
R_{\pi_{13}}i_X^* R(\pi_Y \times 1 \times 1)_*(i_Y \times 1 \times 1)^*
\]
\[
\simeq R_{\pi_{13}}R(\pi_Y \times 1)_*(1 \times i_X)^*(i_Y \times 1 \times 1)^*
\]
\[
\simeq R(\pi_{13} \circ (\pi_Y \times 1))_*(i_Y \times 1 \circ (1 \times i_X))^*
\]
\[
\simeq R_{\pi_{13}}R(\pi_X \times 1)_*(i_Y \times 1 \times 1)^*(1 \times i_X \times 1)^*
\]
\[
\simeq R_{\pi_{13}}i_Y^* R(1 \times \pi_X \times 1)_*(1 \times i_X \times 1)^*.
\]
This proves associativity. □

Remark 4.3. We have proved and will be using associativity of convolution up to a non-specified isomorphism.

In particular, \(D^+(\text{QCoh}^K(Y \times X))\) is a monoidal category. Consider the \(K\) equivariant diagonal embedding \(\Delta : Y \to Y \times Y\) and denote \(\Delta_* \mathcal{O}_Y\) by \(\mathcal{O}_Y\).

**Lemma 4.4.** \(\mathcal{O}_Y\) is the unit in \(D^+(\text{QCoh}^K(Y \times X))\) and acts as identity on \(D^+(\text{QCoh}^K(Y \times X))\).

**Proof.** Let \(M\) be an object in \(D^+(\text{QCoh}^K(Y \times X))\). Consider the inclusion
\[
j : Y_\Delta \times Y \times X \hookrightarrow Y \times Y \times Y \times X.
\]
Since \(\mathcal{O}_Y \boxtimes M\) is supported on \(Y_\Delta \times Y \times X\) it can be written as \(Rj_*(\mathcal{O}_Y \boxtimes M)\). There is a cartesian diagram
\[
\begin{array}{ccc}
Y \times Y \times X & \to & Y \times Y \times Y \\
\pi_{13} & \downarrow & \downarrow \pi_{134} \\
Y \times X & \to & Y \times Y \times X
\end{array}
\]
Using flat base change we obtain:
\[
\mathcal{O}_Y \boxtimes M \simeq i'^* R\pi_{134*} Rj_*(\mathcal{O}_Y \boxtimes M)
\]
\[
\simeq i'^* R(\pi_{134} \circ j)_*(\mathcal{O}_Y \boxtimes M)
\]
\[
\simeq i'^* R\text{Id}_*(\mathcal{O}_Y \boxtimes M)
\]
\[
\simeq i'^* (\mathcal{O}_Y \boxtimes M).
\]
Let \(pr_1\) be the projection of the first factor \(Y \times Y \times X \to Y\) and \(pr_2\) the projection of the last two factors \(Y \times Y \times X \to Y \times X\).
\[
i'^*(\mathcal{O}_Y \boxtimes M) \simeq i'^*(pr_1^* \mathcal{O}_Y \boxtimes \mathcal{O}_{Y \times Y \times X} \otimes \mathcal{O}_{Y \times X} \otimes \mathcal{O}_{Y \times Y \times X})
\]
\[
\simeq i'^* pr_1^* \mathcal{O}_Y \boxtimes \mathcal{O}_{Y \times X} \otimes i'^* pr_2^* M
\]
\[
\simeq \mathcal{O}_{Y \times X} \boxtimes \mathcal{O}_{Y \times X} \otimes i'^* (pr_2 \circ i')^* M
\]
\[
\simeq \text{Id}^* M \simeq M.
\]
Similarly, one can prove that \( M \ast \mathcal{O}_Y \simeq M \) in the monoidal category.

**Definition 4.5.** Fix a reductive algebraic group \( K \) with an algebraic subgroup \( H \). Consider the \( K \)-scheme \( Y = K/H \). The category

\[
(D^+(\text{QCoh}^K(K/H \times K/H)), 
\ast)
\]

is called the quasi-coherent Hecke category and it's denoted by \( \text{QCHecke}(K, H) \).

Notice that for a \( K \)-scheme \( X \) we have

\[
D^+(\text{QCoh}^H(X)) \simeq D^+(\text{QCoh}^K(K/H \times X)).
\]

Taking \( Z = Y = K/H \) we get the monoidal action (again, associative up to a non-specified isomorphism).

\[
\text{QCHecke}(K, H) \times D^+(\text{QCoh}^H(X)) \to D^+(\text{QCoh}^H(X)).
\]

5. **Demazure Descent**

5.1. **Notations.** From now on \( G \) is a reductive algebraic group. Let \( T \) be a Cartan subgroup of \( G \) and let \((I, X, Y)\) be the corresponding root data, where \( I \) is the set of vertices of the Dynkin diagram, \( X \) is the weight lattice of \( G \) and \( Y \) is the coroot lattice of \( G \). Choose a Borel subgroup \( T \subset B \subset G \). Denote the set of roots for \( G \) by \( \Phi = \Phi^+ \sqcup \Phi^- \). Let \( \{\alpha_1, \ldots, \alpha_n\} \) be the set of simple roots. The Weyl group \( W = \text{Norm}(T)/T \) of the fixed maximal torus acts naturally on the lattices \( X \) and \( Y \) and on the \( \mathbb{R} \)-vector spaces spanned by them, by reflections in root hyperplanes. The simple reflection corresponding to an \( \alpha_i \) is denoted by \( s_i \). The elements \( s_1, \ldots, s_n \) form a set of generators for \( W \). For \( w \in W \) denote the length of a minimal expression of \( w \) via the generators by \( \ell(w) \). The unique longest element in \( W \) is denoted by \( w_0 \). We have a partial ordering on \( W \) called the Bruhat ordering. \( w' \leq w \) if there exists a reduced expression for \( w' \) that can be obtained from a reduced expression for \( w \) by deleting a number of simple reflections.

The monoid \( \text{Br}^+ \) with generators \( \{T_w, \ w \in W\} \) and relations

\[
T_{w_1}T_{w_2} = T_{w_1w_2} \quad \text{if} \quad \ell(w_1) + \ell(w_2) = \ell(w_1w_2) \quad \text{in} \ W
\]

is called the braid monoid of \( G \).

5.2. **Demazure Descent.** Fix a root data \((I, X, Y)\) of the finite type, with the Weyl group \( W \) and the braid monoid \( \text{Br}^+ \). Consider a triangulated category \( \mathcal{C} \).

**Definition 5.1.** A weak braid monoid action on the category \( \mathcal{C} \) is a collection of triangulated functors

\[ D_w : \mathcal{C} \to \mathcal{C}, \quad w \in W \]

satisfying braid monoid relations, i.e. for all \( w_1, w_2 \in W \) there exist isomorphisms of functors

\[ D_{w_1} \circ D_{w_2} \simeq D_{w_1w_2}, \quad \text{if} \quad \ell(w_1w_2) = \ell(w_1) + \ell(w_2). \]

Notice that we neither fix the braid relations isomorphisms nor impose any additional relations on them.
Definition 5.2. Demazure Descent Data on the category $\mathcal{C}$ is a weak braid monoid action $\{D_w\}$ such that for each simple root $s_k$ the corresponding functor $D_{s_k}$ is a comonad for which the comonad map $D_{s_k} \to D_{s_k}^2$ is an isomorphism.

Here is the central construction of the paper. Consider a triangulated category $\mathcal{C}$ with a fixed Demazure Descent Data $\{D_w, w \in W\}$ of the type $(I, X, Y)$.

Definition 5.3. The Descent category $\text{Desc}(\mathcal{C}, D_w, w \in W)$ is the full subcategory in $\mathcal{C}$ with objects $M$ such that for all $k$ the cones of the counit maps $D_{s_k}(M) \xrightarrow{\epsilon} M$ are isomorphic to $0$.

Remark 5.4. Suppose that $\mathcal{C}$ has functorial cones. Then $\text{Desc}(\mathcal{C}, D_w, w \in W)$ a full triangulated subcategory in $\mathcal{C}$ being the intersection of kernels of $\text{Cone}(D_{s_k} \to \text{Id})$. However, one can prove this statement not using functoriality of cones.

Lemma 5.5. An object $M \in \text{Desc}(\mathcal{C}, D_w, w \in W)$ is naturally a comodule over each $D_{s_k}$.

Proof. By definition the comonad maps
$$
\eta : D_{s_k} \to D_{s_k}^2, \quad \epsilon : D_{s_k} \to \text{Id}
$$
makes the following diagram commutative

![Diagram](image)

For Demazure Descent Data we require that $\eta$ is an isomorphism, so $\epsilon D_{s_k}$ is also an isomorphism. Let $M \in \text{Desc}(\mathcal{C}, D_w, w \in W)$. That $\text{Cone}(D_{s_k}(M) \xrightarrow{\epsilon} M)$ is isomorphic to $0$ is equivalent to saying that $D_{s_k}(M) \xrightarrow{\epsilon^{-1}} M$ is an isomorphism. This gives the commutative diagram.

![Diagram](image)

Thus, $\epsilon^{-1}$ satisfies the axiom for the coaction. \qed

Remark 5.6. Recall that in the usual Descent setting either in Algebraic Geometry or in abstract Category Theory (Barr-Beck Theorem) Descent Data includes a pair of adjoint functors and their composition which is a comonad. By definition, the Descent category for such data is the category of comodules over this comonad. Our definition of $\text{Desc}(\mathcal{C}, D_w, w \in W)$ for Demazure Descent Data formally is not about comodules, yet the previous Lemma...
demonstrates that every object of $\mathcal{D}_{\text{Res}}(C, D_w, w \in W)$ is naturally equipped with structures of a comodule over each $D_k$ and any morphism in $\mathcal{D}_{\text{Res}}(C, D_w, w \in W)$ is a morphism of $D_k$-comodules.

6. Demazure Descent for $D^+(\text{QCoh}^G(G/B \times X))$

In this Section we introduce a Demazure Descent Data on the category $D^+(\text{QCoh}^B(X))$ for a regular Noetherian $G$ scheme $X$. For each $w \in W$ we construct a functor $D_w$ acting on $D^+(\text{QCoh}^G(G/B \times X))$. The functor is defined in terms of the monoidal action of $\text{QCHecke}(G,B)$.

We consider the orbits for the diagonal action of $G$ on $G/B \times G/B$. For a simple root $\alpha_k$ take the standard parabolic subgroup $P_k$ of $G$ containing $B$ whose Levi subgroup has the root system $\{\alpha_k, -\alpha_k\}$. Recall that $G \Delta$ orbits in $G/B \times G/B$ are in bijection with elements of the Weyl group $[CG$, Theorem 3.1.9]. For $w \in W$ let $O_w$ be the closure of the corresponding orbit. E.g. for $s_k$ a simple reflection $O_{s_k}$ is a union of two orbits.

$O_k := O_{s_k} = G/B \cup G \cdot \{(B,pkB) \mid pk \in P_k, pk \notin B\}$

Consider the $G$-equivariant sheaf $\mathcal{O}_{\overline{O}_k} \in \text{QCHecke}(G,B)$. We get an endofunctor $D_w : D^+(\text{QCoh}^G(G/B \times X)) \to D^+(\text{QCoh}^G(G/B \times X))$, $D_w(M) := \mathcal{O}_{\overline{O}_w} \ast M$.

For simple reflections we write $D_k := D_{s_k}$. Our aim is to prove the following Theorem

**Theorem 6.1.** The $\{D_w, w \in W\}$ is Demazure Descent Data on $D^+(\text{QCoh}^G(G/B \times X))$.

Consider the quotient map $p_k : G/B \times X \to G/P_k \times X$. Notice that the map $p_k$ is flat, so the functor $p_k^*$ is exact. We obtain a comonad structure on $D_k$ by realizing it as a composition of the adjoint functors $p_k^*$ and $R_{p_{k*}}$.

**Lemma 6.2.** There is an isomorphism of functors $D_k \simeq p_k^* R_{p_{k*}}$.

**Proof.** Let $M \in D^+(\text{QCoh}^G(G/B \times X))$. Consider the following inclusion and projection

$j_k : \overline{O}_k \times G/B \times X \hookrightarrow G/B \times G/B \times G/B \times X,$

$p_{34} : \mathcal{O}_{\overline{O}_k} \times G/B \times X \rightarrow G/B \times X.$

Notice that $\mathcal{O}_{\overline{O}_k} \boxtimes M$ is supported on $\overline{O}_k \times G/B \times X$ so $\mathcal{O}_{\overline{O}_k} \boxtimes M \simeq R_{j_k} p_{34}^*(M)$. Consider the cartesian diagram

$$
\begin{array}{ccc}
G/B \times G/B \times X \ar[r]^i \ar[d]_{\pi_{13}} & G/B \times G/B \times G/B \times X \ar[d]_{\pi_{134}} \\
G/B \times X \ar[r]^j & G/B \times G/B \times X
\end{array}
$$


Applying flat base change we obtain:
\[
\mathcal{O}_{\overline{\mathcal{U}}_k} * M \simeq (i')^* R(\pi_{134} \circ j_k)_* p_{34}^*(M) \\
\simeq i'^* R\pi_{134} p_{34}^*(M)
\]
Notice that \(\mathcal{O}_{\overline{\mathcal{U}}_k} \simeq G/B \times_{G/P_k} G/B\). There’s a cartesian diagram
\[
\begin{array}{ccc}
G/B \times_{G/P_k} G/B \times X \xrightarrow{i_k} & G/B \times_{G/P_k} G/B \times G/B \times X \simeq G/B \times P_k/B \times G/B \times X \\
pr_1 \downarrow & & \downarrow \pi_{134} \circ j_k \\
G/B \times X \xrightarrow{i'} & G/B \times G/B \times X
\end{array}
\]
Using flat base change we get
\[
i'^* R\pi_{134} p_{34}^*(M) \simeq Rpr_1 \circ i_k^* p_{34}^*(M) \\
\simeq Rpr_1 \circ pr_2^*(M),
\]
where \(pr_2\) is the projection on the second factor in the fiber product. These projections fits into another cartesian diagram
\[
\begin{array}{ccc}
G/B \times_{G/P_k} G/B \times X \xrightarrow{pr_2} & G/B \times X \xrightarrow{p_k} G/P_k \times X \\
pr_1 \downarrow & & \downarrow p_k \\
G/B \times X \xrightarrow{p_k} & G/P_k \times X
\end{array}
\]
The \(p_k\) are locally trivial fibrations. In particular they are flat so \(Rpr_1 \circ pr_2^*(M) \simeq p_k^* \mathcal{O}_{P_k} (M)\).
This finishes the proof.

**Lemma 6.3.** The comonad map \(D_k \to D_k^2\) is an isomorphism.

**Proof.** Since \(D_k \simeq p_k^* \mathcal{O}_{P_k}\) it suffices to show that for all \(M \in \mathcal{D}^+ (\text{QCoh}^G (G/B \times X))\) the adjunction map \(M \to R\mathcal{O}_{P_k} \mathcal{O}(M)\) is an isomorphism. The Lemma follows from the following statement.

**Claim 6.4.** If \(p : X \to Y\) is a locally trivial fibration with fiber \(Z\) s.t.
\[
H^{>0}(Z, \mathcal{O}_Z) = 0 \quad \text{and} \quad H^0(Z, \mathcal{O}_Z) = k,
\]
then the adjunction map \(M \to R\mathcal{O}_{P_k} \mathcal{O}(M)\) is an isomorphism. \(\Box\)

**Remark 6.5.** Notice that \(\overline{\mathcal{U}}_{w_0} = G/B \times G/B\). Replacing \(\overline{\mathcal{U}}_k\) by \(G/B \times G/B\) and \(p_k\) by the projection \(p : G/B \times X \to G/G \times X \simeq X\) the argument from Lemma 6.2 shows that \(D_{w_0} \simeq p^* \mathcal{O}_{P_k}\). Thus, we get a comonad structure on \(D_{w_0}\). By the Borel-Weil-Bott Theorem \(p\) satisfies the condition in Claim 6.4 so the adjunction \(D_{w_0} \to D_{w_0}^2\) is an isomorphism.

The key part in proving the remaining part of Theorem 6.1 is the following Proposition.

**Proposition 6.6.** Let \(w = s_{k_1} \cdots s_{k_n}\) be a reduced expression. Then we have
\[
\mathcal{O}_{\overline{\mathcal{U}}_{k_1}} * \cdots * \mathcal{O}_{\overline{\mathcal{U}}_{k_n}} \simeq \mathcal{O}_{\overline{\mathcal{U}}_w}
\]
Proof. Looking at the proof of associativity of $*$ we see that
\[ \mathcal{O}_{\overline{G}_{k_1}} \ast \cdots \ast \mathcal{O}_{\overline{G}_{k_n}} \simeq R\pi_{1n}^*(i \times \cdots \times i)^*(\mathcal{O}_{\overline{G}_{k_1} \times \cdots \times \overline{G}_{k_n}}). \]
Let $j$ be the inclusion $\overline{O}_{k_1} \times \cdots \times \overline{O}_{k_n} \hookrightarrow G/B \times \cdots \times G/B$. Then
\[ \mathcal{O}_{\overline{O}_{k_1} \times \cdots \times \overline{O}_{k_n}} \simeq Rj_*(\mathcal{O}_{\overline{O}_{k_1} \times \cdots \times \overline{O}_{k_n}}). \]
We now make the same kind of flat base change trick as in the proof of Lemma 6.2

\[ \begin{array}{ccc}
G/B \times \cdots \times G/B & \xrightarrow{(i \times \cdots \times i)} & G/B \times \cdots \times G/B \\
\downarrow \pi_{1n} & & \downarrow \pi_{135 \cdots 2n} \\
G/B \times G/B' & \xrightarrow{i'} & G/B \times \cdots \times G/B
\end{array} \]

Thus, we get
\[ R\pi_{1n}^*(i \times \cdots \times i)^*(\mathcal{O}_{\overline{O}_{k_1} \times \cdots \times \overline{O}_{k_n}}) \simeq i'^* R(\pi_{135 \cdots 2n} \circ j)_*(\mathcal{O}_{\overline{O}_{k_1} \times \cdots \overline{O}_{k_n}}) \]

Set
\[ \overline{O}_{k_1 \cdots k_n} := \overline{O}_{k_1} \times G/B \cdots \times G/B \overline{O}_{k_n}. \]
Consider the cartesian diagram

\[ \begin{array}{ccc}
\overline{O}_{k_1 \cdots k_n} & \xrightarrow{i'} & \overline{O}_{k_1} \times \cdots \times \overline{O}_{k_n} \\
\downarrow \pi'_{1n} & & \downarrow \pi_{135 \cdots 2n} \circ j \\
G/B \times G/B' & \xrightarrow{i''} & G/B \times \cdots \times G/B
\end{array} \]

Notice that $\pi_{135 \cdots 2n} \circ j$ is a locally trivial fibration. Applying flat base change we get
\[ i''^* R(\pi_{135 \cdots 2n} \circ j)_*(\mathcal{O}_{\overline{O}_{k_1} \times \cdots \overline{O}_{k_n}}) \simeq R\pi'_{1n}^* i'^* (\mathcal{O}_{\overline{O}_{k_1} \times \cdots \overline{O}_{k_n}}) \simeq R\pi_{1n}^*(\overline{O}_{k_1 \cdots k_n}). \]

For $w \in W$ the Schubert variety $X_w$ is the closure of $BwB/B$ in $G/B$. Consider the $B \times B \times \cdots \times B$ action on $P_{k_1} \times P_{k_2} \times \cdots \times P_{k_n}$ given by $(b_1, b_2, \ldots, b_n) \cdot (p_1, p_2, \ldots, p_n) = (p_1 b_1 b_1^{-1} p_2 b_2, \ldots, b_n^{-1} p_n b_n)$. The quotient by the action is called the Bott-Samelson scheme and is denoted by
\[ Z_w := P_{k_1} \times^B P_{k_2} \times^B \cdots \times^B P_{k_n}/B. \]
The multiplication $P_{k_1} \times P_{k_2} \times \cdots \times P_{k_n} \to P_{k_1} P_{k_2} \cdots P_{k_n} \subset G$ factors through the quotient by the action so it induces a map
\[ \phi_w : Z_w \to X_w. \]
Observe that
\[
\overline{O}_{k_1...k_n} \simeq (G/B \times_{G/P_1} G/B) \times_{G/B} \cdots \times_{G/B} (G/B \times_{G/P_n} G/B)
\]
\[
\simeq G/B \times_{G/P_1} G/B \times_{G/P_2} \cdots \times_{G/P_n} G/B
\]
\[
\simeq G \times B P_{k_1} \times B P_{k_2} \times \cdots \times B P_{k_n}/B
\]
\[
\simeq \frac{G \times Z_w}{B}.
\]
Thus, we have the following equivalences of categories
\[
\text{QCoh}^G(\overline{O}_{k_1...k_n}) \simeq \text{QCoh}^G\left(\frac{G \times Z_w}{B}\right) \simeq \text{QCoh}^B(Z_w).
\]
\(\overline{O}_{k_1...k_n}\) can also be considered as tuples of Borel subgroups (see [CG, Proposition 3.1.29])
\[
\overline{O}_{k_1...k_n} = \{(B_0 = B, B_1, \ldots, B_n) \in \mathcal{B}^n \mid (B_{j-1}, B_j) \text{ in relative position } s_k\}
\]
Hence, the image of the projection is pairs of Borels in relative position \(w\)
\[
\pi'_1(\overline{O}_{k_1...k_n}) = \overline{O}_w.
\]
With these identifications the following diagram is commutative
\[
\begin{array}{ccc}
G \times Z_w & \xrightarrow{\phi_w} & G \times X_w \\
\downarrow \wr & & \downarrow \wr \\
\overline{O}_{k_1...k_n} & \xrightarrow{\pi'_1} & \overline{O}_w
\end{array}
\]
Thus, on the level of categories we have
\[
\xymatrix{ D^+(\text{QCoh}^B(Z_w)) \ar[r]^{R\phi_w} \ar[d]_{\psi_1} & D^+(\text{QCoh}^B(X_w)) \ar[d]_{\psi_2} \\
D^+(\text{QCoh}^G(\overline{O}_{k_1...k_n})) \ar[r]^{R\pi'_1} & D^+(\text{QCoh}^G(\overline{O}_w))}
\]
By Propositions 5.2 and 5.3 in [Dem]
\[
R^j \phi_w^* \mathcal{O}_{Z_w} = 0 \quad \text{for } j > 0
\]
Since \(Z_w\) is complete, \(\phi_w\) is proper. It’s surjective and by [Dem] Proposition 3.2 it’s also birational. It’s known from [And] that \(X_w\) is normal. By Zariski’s main Theorem (see [EGA, III, 4.3.12]) this implies that
\[
\phi_w^* \mathcal{O}_{Z_w} \simeq \mathcal{O}_{X_w}.
\]
Inserting this we get
\[
\pi'_1(\mathcal{O}_{\overline{O}_{k_1...k_n}}) \simeq \pi'_1 \psi_1(\mathcal{O}_{Z_w}) \simeq \psi_2 R\phi_w^* \mathcal{O}_{Z_w}
\]
\[
\simeq \psi_2 \mathcal{O}_{X_w} \simeq \mathcal{O}_{\overline{O}_w}.
\]
This finishes the proof. □

Proof of Theorem 7.1 The only remaining part is to show that for all $w_1, w_2 \in W$ with $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$

$$D_{w_1} \circ D_{w_2} \simeq D_{w_1w_2}.$$ 

Pick reduced expressions $w_1 = s_{k_1} \cdots s_{k_n}$ and $w_2 = s_{j_1} \cdots s_{j_m}$. Since $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ the expression $w_1w_2 = s_{k_1} \cdots s_{k_n} s_{j_1} \cdots s_{j_m}$ is reduced so by the Proposition

$$D_{w_1} \circ D_{w_2} \simeq \mathcal{O}_{\mathcal{U}_{w_1}} * \mathcal{O}_{\mathcal{U}_{w_2}} * -$$

$$\simeq \mathcal{O}_{\mathcal{U}_{k_1}} * \cdots * \mathcal{O}_{\mathcal{U}_{k_n}} * \mathcal{O}_{\mathcal{U}_{j_1}} * \cdots * \mathcal{O}_{\mathcal{U}_{j_m}} * -$$

$$\simeq \mathcal{O}_{\mathcal{U}_{w_1w_2}} * -$$

$$\simeq D_{w_1w_2}.$$ 

This finishes the proof that $\{D_w, w \in W\}$ is Demazure Descent Data. □

7. DESCENT CATEGORY

Now that we have proved that the $D_k$'s form Demazure Descent Data we may consider the Descent category. We aim to prove the following Theorem

**Theorem 7.1.** $\text{Desc}(D^+(\text{QCoh}^B(X), D_w, w \in W))$ is equivalent to $D^+(\text{QCoh}^G(X))$.

We consider an object in $D^+(\text{QCoh}^G(G/B \times X))$ to be in $D^+(\text{QCoh}^G(G/P_k \times X))$ if it can be written as pull-back along the quotient map $p_k : G/B \times X \to G/P_k \times X$.

**Lemma 7.2.** An object $M$ in $D^+(\text{QCoh}^G(G/B \times X))$ is also in $D^+(\text{QCoh}^G(G/P_k \times X))$ if and only if $D_k(M) \simeq M$.

**Proof.** In lemma 6.2 we proved that $D_k \simeq p_k^* Rp_k$. Thus, $M \simeq D_k(M)$ implies that $M$ is in $D^+(\text{QCoh}^G(G/P_k \times X))$. Assume that $M = p_k^*(N)$ for some $N \in D^+(\text{QCoh}^G(G/P_k \times X))$.

In the proof of lemma 6.3 we proved that the comonad map $\text{Id} \to Rp_k p_k^*$ is an isomorphism. Thus,

$$D_k(M) \simeq p_k^* Rp_k p_k^*(N) \simeq p_k^*(N) = M.$$

**Remark 7.3.** The same proof shows that an object $M$ in $D^+(\text{QCoh}^G(G/B \times X))$ is also in $D^+(\text{QCoh}^G(X))$ if and only if $D_{w_0}(M) \simeq M$.

**Proof of Theorem 7.4.** Let $M \in D^+(\text{QCoh}^G(G/B \times X))$. By the Lemma $M \in D^+(\text{QCoh}^G(G/P_k \times X))$ if and only if $D_k(M) \simeq M$. Set $C_k := \text{Cone}(D_k \to \text{Id})$. By definition of a comonad we have the commutative diagram

$$\begin{array}{ccc}
D_k & \xrightarrow{\eta} & D_k^2 \\
\downarrow \text{Id} \circ D_k & \nearrow & \downarrow \epsilon_{D_k} \\
& D_k & \end{array}$$
Since \( \eta \) is an isomorphism so is \( \epsilon D_k \) and thus \( \operatorname{Cone}(D_k(D_k(M)) \to D_k(M)) = 0 \). Hence, \( D_k(M) \approx M \) is equivalent to \( M \in \ker C_k \). The same argument applies to \( D_{w_0} \) so \( M \in D^+(\text{QCoh}^G(G/G \times X)) \) if and only if \( M \in \ker(C) \) where \( C := \operatorname{Cone}(D_{w_0} \to \text{Id}) \). Choose a reduced expression \( s_{k_1} \cdots s_{k_n} \) for \( w_0 \). Then

\[
D_{k_1} \circ \cdots \circ D_{k_n}(M) \approx (\mathcal{O}_{\mathfrak{u}_{k_1}} \ast \cdots \ast \mathcal{O}_{\mathfrak{u}_{k_n}}) \ast M \\
\approx \mathcal{O}_{G/B \times G/B} \ast M = D_{w_0}(M).
\]

Thus, \( D_k(M) \approx M \) for all \( k \) implies that \( D_{w_0}(M) \approx M \). In other words we have shown that

\[
D^+(\text{QCoh}^G(X)) \approx \ker(C) \supseteq \bigcap_k \ker(C_k) = \mathcal{O}_{\text{Desc}}(D^+(\text{QCoh}^B(X)), D_w, w \in W)
\]

The quotient map factors as \( G/B \times X \to G/P_k \times X \to G/G \times X \) so \( D^+(\text{QCoh}^G(G/G \times X)) \subseteq D^+(\text{QCoh}^G(G/P_k \times X)) \) for all \( k \). This finishes the proof. \( \square \)

References

[AG] S. Arkhipov, D. Gaitsgory, *Another realization of the category of modules over the small quantum group*. Advances in Mathematics 173 (2003) 114-143.

[AK] S. Arkhipov, T. Kanstrup, *Demazure descent and representations of reductive groups*, arXiv:1303.3780

[BB] A. Beilinson, J. Bernstein, *Localisation de \( g \)-modules*. C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 1, 15-18.

[BN] D. Ben-Zvi, D. Nadler, *Beilinson-Bernstein localization over the Harish-Chandra center*, arXiv:1209.0188v1.

[CPS] E. Cline, B. Parshall and L. Scott. *Induced Modules and Extensions of Representations*. Inventiones mathematicae 47, 41–51 (1978).

[CG] N. Chriss and V. Ginzburg, *Representation Theory and Complex Geometry*. Birkhäuser (1997)

[Dem] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Annales scientifiques de l’É.N.S. 4e série, tome 7, n° 1 (1974), p. 53-88.

[EAG] J. Dieudonné, A. Grothendieck, *Éléments de géométrie algébrique*, Publ. Math. I.H.E.S.

[GMS] S. Gelfand, Y. Manin, *Methods of homological algebra* second edition. Springer (2002).

[Har] R. Hartshorne, *Algebraic geometry*. Springer (1977).

[And] H. H. Andersen, *Schubert varieties and Demazure’s character formula*. Invent. math. 79, 611-618 (1985).

[HLS] M. Harada, G. Landweber, R. Sjamaar, *Divided differences and the Weyl character formula in equivariant K-theory*. Math. Res. Lett. 17 (2010), no. 3, 507-527.

[Hum] J. E. Humphreys, *Linear Algebraic Groups* corrected fourth printing. Springer (1995).

[Hum2] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press (1997).

[Jant] J. C. Jantzen, *Representations of Algebraic Groups*. Academic Press, Inc (1987).

[Mac] S. MacLane. *Categories for the Working Mathematician* second edition. Springer (1997).

[Mum] D. Mumford, *The Red Book of Varieties and Schemes* second expanded edition. Springer (1999).

[Ser] J. P. Serre, *Espaces fibrés algébriques*. Sém. C. Chevalley, t. 3, Anneaux de Chow et applications, E.N.S. Lecture Notes (1958).

[Spr] T. A. Springer, *Linear Algebraic Groups* second edition. Birkhäuser (1998).

[Stacks] Stacks Project, Version 9885e24, available at http://stacks.math.columbia.edu

[Tan] T. Tanisaki, *Hodge modules, equivariant K-theory and Hecke algebras*. Publ. Res. Inst. Math. Sci. 23 (1987), no. 5, 841-879.
S.A. Matematisk Institut, Aarhus Universitet, Ny Munkegade, DK-8000, Århus C, Denmark, email: hippie@qgm.au.dk

T.K. Centre for Quantum Geometry of Moduli Spaces, Aarhus Universitet, Ny Munkegade, DK-8000, Århus C, Denmark, email: tina@qgm.au.dk