Asymptotic Expansions in Momenta and Masses
and Calculation of Feynman Diagrams

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Abstract

General results on asymptotic expansions of Feynman diagrams in momenta and/or masses are reviewed. It is shown how they are applied for calculation of massive diagrams.

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1. Earlier results.

The problem of asymptotic expansions of Feynman amplitudes in momenta and/or masses is rather old. A limit of large momenta and masses is characterized by a subdivision of the set of all external momenta and internal masses of a diagram $\Gamma$ into large $Q \equiv \{Q_1, \ldots, Q_i, \ldots\}$, $M \equiv \{M_1, \ldots, M_i, \ldots\}$, and small $q \equiv \{q_1, \ldots, q_i, \ldots\}$, $m \equiv \{m_1, \ldots, m_i, \ldots\}$ ones. The problem is to analyze the behavior of the corresponding Feynman integral $F_{\Gamma}(Q, q, M, m)$ in the limit $F_{\Gamma}(Q/\rho, q, M/\rho, m)$ as $\rho \to 0$. For brevity, let us denote this limit as $Q, M \to \infty$. Let us imply that the external momenta are not fixed at a mass shell.

In 1960 Weinberg [1] described the leading large momentum behavior. Logarithmic corrections were characterized in [2]. Later it was proved that the large momentum asymptotic expansions are always performed in powers and logarithms of the expansion parameter [3]. In [4, 5, 6, 7, 8, 9, 10, 11] asymptotic expansions in various limit of large momenta and masses were obtained. A typical result is the expansion of the form

$$F_{\Gamma}(Q/\rho, q, m) \overset{\rho \to 0}{\sim} \sum_{k,l} C_{k,l} \rho^k \log^l \rho.$$ (1)

However the coefficient functions in these expansions are cumbersome. They are expressed in terms of numerous parametric integrals or in terms of Mellin integrals. Thus, the first of the following two natural properties of asymptotic expansions does not hold:

(i) The coefficient functions $C_{k,l}$ are expressed in a simple way through renormalized and/or regularized Feynman amplitudes;

(ii) The expansion is in powers and logarithms.

2. Results in the simplest form for the large momentum limit.

To write down the asymptotic expansion with these two properties it is worthwhile to introduce dimensional (with $d = 4 - 2\varepsilon$) regularization even in case the original diagram is ultravioletly finite. The following proposition is valid.

In the large momentum limit,

$$F_{\Gamma}(Q/\rho, q, m; \varepsilon) \overset{Q \to \infty}{\sim} \sum_{\gamma} F_{\Gamma/\gamma}(q, m; \varepsilon) \circ T_{q, m, \varepsilon} F_\gamma(Q, q, m; \varepsilon),$$ (2)

where the sum is over subgraphs $\gamma$ of $\Gamma$ such that each $\gamma$ (a) contains all the vertices with the large external momenta and (b) is 1PI after contraction of these vertices.

3 The term ‘asymptotic’ implies that the corresponding remainder satisfies necessary estimates and one knows nothing about the radius of convergence. In contrast to expansions in coupling constants which typically have zero radii of convergence, the large mass/momentum expansions of Feynman diagrams seem to have always non-zero radii of convergence.

4 The results that are presented here hold both for Minkowski and Euclidean spaces. For Minkowski space, it is in fact sufficient to imply that the large external momenta are space-like. However, for the limit of large masses when all the momenta are small, there are no restriction on momenta. Another possible variant is to consider Feynman diagrams as distributions in momenta.

Furthermore, the operator $\mathcal{T}$ performs Taylor expansion in the corresponding set of variables; $q^\gamma$ are the light external momenta of the subgraph $\gamma$ (i.e., all its external momenta apart from the large external momenta of $\Gamma$); $m^\gamma$ is a set of masses of $\gamma$. Finally, if $F_{\Gamma/\gamma}$ is a Feynman integral corresponding to the reduced graph $\Gamma/\gamma$ and $P_\gamma$ is a polynomial in $q^\gamma$, then the expression $F_{\Gamma/\gamma} \circ P_\gamma$ denotes the Feynman integral that differs from $F_{\Gamma/\gamma}$ by insertion of $P$ into the vertex $v_\gamma$ (this is the vertex to which the subgraph $\gamma$ was collapsed). It is implied that the operators $\mathcal{T}$ act directly on the integrands of Feynman integrals over loop momenta.

Expansion (2) was first written in an equivalent form in [12] (see also [13, 14, 15]). In a particular case of one large external momentum the expansion was found and justified within the so-called method of glueing [14].

In [16] an expansion similar to (2) was presented. Its validity was taken as a postulate. Later the approach of [16] resulted in the theory of As-operation [18]. According to prescriptions of the As-operation to obtain asymptotic expansions in a given limit it is necessary (a) perform a formal (naive) Taylor expansion in small momenta and masses, (b) study IR divergences that are induced in this formal expansion, (c) characterize the nature of these singularities in graph-theoretical language, (d) find analytical structure of the singularities and (e) construct other terms of the expansion. It should be noted that there is no need to perform this job for each limit and follow these multiple prescriptions: it is sufficient to use explicit formula (2) and similar formulae for other limits (see below) by writing down the corresponding expansions from the very beginning.

Asymptotic expansion (2) as well as asymptotic expansion in other situations (see below) was justified in [19] (see also [20]). There also exist another version of the proof based on the method of glueing (respectively, combinatorial [13] and analytical [14, 21] parts).

Practically, to write down the quantity $F_{\Gamma/\gamma} \circ \mathcal{T}_{\gamma}, F_\gamma$ it is necessary

1. when choosing the loop momenta of $\Gamma$, first, to choose a set of the loop momenta of $\gamma \subset \Gamma$;

2. to let large external momenta flow through $\gamma$.

Thus, if

$$F_{\Gamma}(Q, q_0, m; \varepsilon) = \int dk_1 \ldots dk_h \Pi_\Gamma(Q, q_0, k, m),$$

where $k \equiv k_1, \ldots k_h$ is the set of the loop momenta of $\Gamma$, and $\Pi_\Gamma \equiv \Pi_{\Gamma\gamma, \gamma}$ is the product of propagators associated with the given graph, then

$$F_{\Gamma/\gamma} \circ \mathcal{T}_{Q^\gamma, m^\gamma} F_\gamma = \int dk_1 \ldots dk_h \Pi_{\Gamma\gamma} \mathcal{T}_{Q^\gamma, m^\gamma} \Pi_\gamma$$

which is characterized by its author as ‘the main stream quantum field theory’ and ‘the most important development since 1972’ [17].

In the case of dimensional regularization, mathematical proofs of results on the As-operation are really absent. Since the basic principle of the As-operation is to apply distribution theory, it would be necessary to explain what are test functions in $d$ dimensions and to provide at least one non-trivial (non-zero) example of such a test function.
so that the small external momenta for the subgraph $\gamma$ turn out to be the small momenta $q$ of the initial graph itself as well as the loop momenta $k_{\Gamma/\gamma}$ of the reduced graph. (Note that according to these two rules, the loop momenta of $\Gamma$ are subdivided into the loop momenta of $\gamma$ and that of $\Gamma/\gamma$.)

Let us now observe that both properties (i) and (ii) are satisfied for expansion (2). In fact, if $T^{(j)}$ is contribution of the terms of the $j$-th order of the corresponding Taylor series, and $\omega(\gamma)$ is the degree of divergence of $\gamma$, then

$$
T^{(j)}_{q,m} F_\gamma(Q, q, m; \varepsilon) = \rho^{\omega(\gamma)+j-\varepsilon h(\gamma)} T^{(j)}_{q,m} F_{\gamma}(Q, q, m; \varepsilon),
$$

where $h(\gamma)$ is the number of loops of $\gamma$.

Since in the large momentum limit the graph itself contributes to the sum in (2) there is always the term

$$
T_{q,m} F_{\gamma}(Q, q, m; \varepsilon).
$$

This is nothing but the ‘naive’ part of the expansion, in the sense that this contribution appears as a result of Taylor expansion of the integrand of the initial Feynman integral with respect to the small parameters and that this part of expansion does not give a proper result. To see this it is sufficient to observe that this naive term (as well as other terms that involve Taylor operators) happen to involve infrared divergences starting from some minimal order of the expansion. On the other hand, the first factors of the form $F_{\gamma/\gamma} \circ P$ involve ultraviolet divergences after the degree of the insertion polynomial is enough great. A non-trivial point is that these induced divergences are mutually cancelled — see below. Appearance of these spurious divergences can be considered as the price to be paid to have explicit and simplest formulae of the expansion.

Note than expansion (2) is in powers of the expansion parameter once we keep the regularization. If the initial diagram is UV and IR finite it is nevertheless worthwhile to introduce regularization to have simple and explicit formulæ for the expansions. In the limit $\varepsilon \to 0$ one obtains an expansion in powers and logs, the maximal power of the logarithm being no greater than the number of loops. This is essentially a property of asymptotic expansions without external legs on a mass shell.

3. One-loop example.

Expansion (2) holds as well for ultravioletly divergent diagrams, provided one does not switch off the regularization.

Let us consider the simplest example: a one-loop propagator-type diagram with a mass $m_1 = m \neq 0$ and $m_2 = 0$. The corresponding Feynman integral is written as

$$
F_{\Gamma}(Q, m; \varepsilon) = \frac{1}{(2\pi)^d} \int \frac{dk}{(k^2 + m^2)(Q - k)^2}.
$$

\footnote{In this example we consider Euclidean space for simplicity.}
In the limit $k^2 \to \infty$ relevant subgraphs are $\Gamma$, $\{1\}$, and $\{2\}$. The subgraph $\{1\}$ does not contribute because it generates a massless vacuum diagram which is zero in dimensional regularization.

In accordance with (8), the contribution from $\gamma = \Gamma$ is given by (6) and looks like

$$T_m F_\Gamma(Q, m; \varepsilon) = \frac{1}{(2\pi)^d} \int \frac{dk}{(Q-k)^2} T_m \frac{1}{k^2 + m^2}.$$  \hspace{1cm} (8)

The contribution from $\gamma = \{2\}$ is

$$F_{\Gamma/\gamma} \circ T_q \cdot F_\gamma = \frac{1}{(2\pi)^d} \int \frac{dk}{k^2 + m^2} T_q \frac{1}{(Q-k)^2}.$$  \hspace{1cm} (9)

The (only) small external momentum $q^\gamma$ for $\gamma = \{2\}$ is just $k$ — the loop momentum of the whole graph. Applying once again the formula for geometrical series

$$T_k \frac{1}{(Q-k)^2} = T_k \frac{1}{Q^2 - (2Qk - k^2)} = \sum_{n=0}^{\infty} (Q^2)^{-1-n}(2Qk - k^2)^n,$$

using one loop formula, resp., for massless propagator type and vacuum massive Feynman integrals, and summing up two contributions from the subgraphs involved one gets

$$F_\Gamma(Q, m; \varepsilon) \overset{Q \to \infty}{\sim} = \sum_{\gamma} F_{\Gamma/\gamma} \circ T_{\gamma} \cdot F_{\gamma}$$

$$= \frac{1}{(4\pi)^{d/2}} (\mu^2/Q^2)^{\varepsilon} \left\{ \frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)} 1F_0(2\varepsilon-1; -m^2/Q^2) + (m^2/Q^2)^{1-\varepsilon}\Gamma(\varepsilon-1) 2F_1(1, \varepsilon; 2-\varepsilon; -m^2/Q^2)^2 \right\}.$$  \hspace{1cm} (11)

For brevity, hypergeometrical functions are here used to represent the series involved.

This diagram was used just an example of the general procedure of writing down explicit results for asymptotic expansions. Of course, result (11) can be obtained in a more efficient way. Note, however, that at 2-loop level we shall use the general result just for evaluation of diagrams — see below.

4. **The large momentum expansion of renormalized diagrams.**

In case one deals with renormalized diagrams the corresponding expansion (proved in [12, 13, 14]) looks like

$$RF_\Gamma(Q/\rho, q, m; \varepsilon) \overset{Q \to \infty}{\sim} = \sum_{\gamma} RF_{\Gamma/\gamma}(q, m; \varepsilon) \circ RT_{q/\gamma, m/\gamma} \cdot F_{\gamma}(Q, q^{\gamma}, m^{\gamma}; \varepsilon).$$  \hspace{1cm} (12)
In addition to (2) we use the following notation: $R$ is dimensional renormalization (e.g. the MS-scheme), and $\overline{R}$ is an incomplete $R$-operation — it does not include counterterms for subgraphs of $\Gamma/\gamma$ that contain the vertex $v_\gamma$ to which the subgraph $\gamma$ was collapsed.

The renormalized expansion possesses all the above mentioned properties of the unrenormalized expansion. For instance, it has both UV and IR divergences that are mutually cancelled after making summation over all the relevant subgraphs.

5. **Explicitly finite expansion of renormalized diagrams.**

To see that spurious ultraviolet and infrared divergences are indeed cancelled in (2) and (12) it is sufficient to write down the expansions in a manifestly finite form. In [15] it was proved that (12) can be written as (see also [13, 14, 19, 20])

$$RF_{\Gamma}(Q/\rho, q, m; \varepsilon) \xrightarrow{Q \rightarrow \infty} \sum_{\gamma} F_{\Gamma/\gamma}(q, m; \varepsilon) \circ R^* T q_\gamma, m_\gamma F_{\gamma}(Q, q_\gamma, m_\gamma; \varepsilon). \quad (13)$$

The changes are minimal: $\overline{R}$ is replaced by $R$ and $R$ by $R^*$. Here $R^*$ is the so-called $R^*$-operation introduced in [22] and developed in [23, 24]. It is a generalization of the renormalization procedure for the case of IR divergences. It is constructed using similarity of UV and IR divergences. The basic property of the $R^*$-operation is that it removes both UV and IR divergences from arbitrary Feynman integral (without restriction to a mass shell). This theorem was proved in [24, 15] (see also [20]).

The $R^*$-operation originated as a powerful technique in renormalization group calculations: for example it was successfully applied in the world record 5-loop calculations of the beta function and anomalous dimensions in the $\phi^4$-theory [25].

The manifestly finite expansion (13) was first found [20] for a particular example associated with the operator product expansion and then justified for the general case of expansion with one large momentum in [14].

6. **The large mass expansion.**

Explicit asymptotic expansions in other limits are written almost in the same form as above expansions in the large momentum limit. The only distinction is that, for each limit, it is necessary to write down the sum over specific class of subgraphs. For example, in the large mass limit when some masses are much greater than the other masses and all the momenta, one has the following analog of (13):

$$F_{\Gamma}(q, M, m; \varepsilon) \xrightarrow{M \rightarrow \infty} \sum_{\gamma} F_{\Gamma/\gamma}(q, m; \varepsilon) \circ T q_\gamma, m_\gamma F_{\gamma}(q_\gamma, M, m_\gamma; \varepsilon), \quad (14)$$

where the sum is now over subgraphs $\gamma$ of $\Gamma$ such that each $\gamma$ (it may be disconnected) (a) contains all the lines with the large mass momenta and (b) consists of connectivity components that are 1PI with respect to lines with small masses.

For example, if all the masses of the diagram are large, then the combinatorics of the expansion is trivial because only the graph itself contributes to the sum so that
the expansion consists only of the naive part and is just Taylor expansion in small momenta and masses:

\[ F_\Gamma(q, M; \varepsilon) \xrightarrow{M \to \infty} T_q F_\Gamma(q, M; \varepsilon). \]  

(15)

Here of course one can put \( \varepsilon = 0 \) if the initial diagram is convergent.

7. Evaluation of master 2-loop diagram: pure large mass expansion.

We shall see later how general theorems on asymptotic expansions in momenta and masses can be successfully applied for calculation of diagrams. But let us first consider an example of combinatorially trivial expansion (15) when the knowledge of general results is unnecessary. Let us consider the master two-loop diagram with non-zero masses (see Fig. 1a) in the large mass limit.

![Figure 1: Two-loop self-energy diagrams.](image)

The corresponding analytically and dimensionally regularized Feynman integral up to a trivial factor is written as

\[ J(\nu; m; q) = \int \frac{d^d k \, d^d l}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}; \]  

(16)

where \( \nu_i \) are the powers of the denominators \( D_i \equiv p_i^2 - m_i^2 + i0, \) \( p_i \) being the momentum of the corresponding line (\( p_i \) are constructed from the loop momenta \( k \) and \( l \) and the external momentum \( q \)). This is a very complicated function of five dimensionless variables. The most interesting case is when all \( \nu \)'s are integer. The cases when some of the \( \nu \)'s are zero correspond to reducing lines in Fig. 1a to points. In such a way, self-energy diagrams with four or three internal lines can also be described. Moreover, by trivial decomposition (partial fractioning) of the first and the fourth denominators (provided that \( \nu_1 \) and \( \nu_4 \) are integer) one can reduce the integral corresponding to Fig. 1b to the integrals (16) with \( \nu_1 \) or \( \nu_4 \) equal to zero (such a decomposition is required only if \( m_1 \neq m_4 \)). So, in the general case of self-energy diagrams (with integer \( \nu \)'s) it is sufficient to consider only the integrals (16).

In [27] a large number of first terms of the large mass expansion was analytically calculated. The problem was reduced to calculation of vacuum two-loop Feynman integrals with three different masses:

\[ I(\nu; m) = \int \frac{d^d k \, d^d l}{[p^2 - m_1^2]^{\nu_1} [q^2 - m_2^2]^{\nu_2} [(p - q)^2 - m_3^2]^{\nu_3}}; \]  

(17)
For example, in the case of three different masses and unique indices $\nu_i = 1$ one has

$$I(1, 1, 1; m) = \pi^4 - 2\varepsilon (m_3^2) \frac{A(\varepsilon)}{2} \left\{ -\frac{1}{\varepsilon^2} (1 + x + y) + \frac{2}{\varepsilon} (x \ln x + y \ln y) - x \ln^2 x - y \ln^2 y + (1 - x - y) \ln x \ln y - \lambda^2 \Phi(x, y) \right\}, \quad (18)$$

with

$$x = \frac{m_1^2}{m_3^2}, \quad y = \frac{m_2^2}{m_3^2}, \quad \lambda = \sqrt{(1 - x - y)^2 - 4xy}, \quad A(\varepsilon) = \Gamma^2 (1 + \varepsilon)/(1 - \varepsilon)(1 - 2\varepsilon),$$

$$\Phi(x, y) = \frac{1}{\lambda} \left\{ 2 \ln((1 + x - y - \lambda)/2) \ln((1 - x + y - \lambda)/2) - \ln x \ln y - 2\text{Li}_2((1 + x - y - \lambda)/2) - 2\text{Li}_2((1 - x + y - \lambda)/2) + \pi^2/3 \right\}, \quad (19)$$

and $\text{Li}_2(z)$ di-logarithm. (Similar results were obtained in \cite{28}.)

By comparing the obtained results with numerical calculations based on a two-fold parametric representation \cite{29}, in \cite{27} it was shown that several first terms of the expansion provide a very good approximation for the diagram if the value of the momentum is not too close to a threshold.

**8. Evaluation of master 2-loop diagram: the large momentum expansion.**

For the large momentum limit, the naive Taylor expansion alone does not produce a correct result so that we need to apply general formula (2). The corresponding set of relevant subgraphs is shown in Fig. 2.

In case $\gamma = \Gamma$ (Type 1), all denominators of (16) should be expanded in masses:

$$\mathcal{T}_m \ J(\nu; m; Q) = \sum_{j_1, \ldots, j_5 = 0}^{\infty} \frac{(\nu_1)_{j_1} \cdots (\nu_5)_{j_5}}{j_1! \cdots j_5!} (m_1)^{j_1} \cdots (m_5)^{j_5} \ J(\nu + j; Q; Q), \quad (20)$$

where

$$(\nu)_j \equiv \frac{\Gamma(\nu + j)}{\Gamma(\nu)} \quad (21)$$

is the Pochhammer symbol.

Note that if we consider the case $\nu_1 = \ldots = \nu_5 = 1$, the first term of the expansion (20) (with $j_1 = \ldots = j_5 = 0$) gives the well-known result [30, 31]: $-6\zeta(3)\pi^4/Q^2$. Two-loop massless integrals with higher integer powers of denominators occurring on the right-hand side of (20) can be evaluated by use of the integration-by-parts technique [32] or by Gegenbauer polynomial technique [31].

A typical contribution of the second type (see Fig. 2), when $\gamma$ is obtained from $\Gamma$ by removing line 1, looks like

$$\int \frac{d^d k}{[k^2 - m_1^{2\nu_1}]^{\nu_1}} \mathcal{T}_{m_2, \ldots, m_5; k} \int \frac{d^d l}{[(l^2 - m_2^{2\nu_2})^2 - (Q - k)^2 - m_3^{2\nu_3}]^{\nu_3} [(Q - l)^2 - m_4^{2\nu_4}]^{\nu_4}}. \quad (22)$$

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After expanding the integrand of the $l$-integral in masses and $k$, one obtain products of massless one-loop integrals and massive tadpoles with numerators [34].

For Type 3 one obtains:

$$
\int \frac{d^d k}{[k^2 - m_3^2]^{\nu_3}} \times T_{m_1,m_2,m_4,m_5;k} \int \frac{d^d l}{[(Q-k-l)^2 - m_1^2]^{\nu_1} [Q-l)^2 - m_2^2]^{\nu_2} [(k-l)^2 - m_3^2]^{\nu_3} [l^2 - m_5^2]^{\nu_5}}.
$$

The resulting integrals are of the same type as in the previous case.

For Type 4 there are no loop integrations in the subgraph $\gamma$, and we get for the first contribution of the fourth type:

$$
\int \int \frac{d^d k}{[k^2 - m_1^2]^{\nu_1} [l^2 - m_5^2]^{\nu_5}} \frac{d^d l}{[(Q-l)^2 - m_2^2]^{\nu_2} [(Q-k-l)^2 - m_3^2]^{\nu_3} [(Q-k)^2 - m_4^2]^{\nu_4}} \times T_{m_2,m_3,m_4;k,l} \left( \frac{1}{[(Q-l)^2 - m_2^2]^{\nu_2} [(Q-k-l)^2 - m_3^2]^{\nu_3} [(Q-k)^2 - m_4^2]^{\nu_4}} \right).
$$

As a result, one obtains products of two one-loop tadpoles with numerators (also for the second contribution).
For Type 5 one obtains non-trivial two-loop vacuum integrals. For example, the first contribution of the fifth type gives:

\[
\int \int \frac{d^d k \, d^d l}{[k^2 - m_1^2][l^2 - m_3^2][(k - l)^2 - m_5^2]} \nu_1 \nu_2 \nu_3 \nu_4 \nu_5 \times T_{m_4, m_5, k, l} \left( \frac{1}{[(Q - k)^2 - m_4^2][(Q - l)^2 - m_5^2]} \right).
\] (25)

Expanding the denominators, one obtains two-loop vacuum integrals with numerators, which are then reduced [34] to the same integrals (17) as in the opposite limit.

For example, in the case \( \nu_i = 1, i = 1, \ldots, 5 \) and \( \epsilon = 0 \) one has [34]

\[
J(1, \ldots, 1; m; Q) I_{d=4} = J(m; Q) = -\frac{\pi^4}{Q^2} M(m; Q) = -\frac{\pi^4}{Q^2} \sum_{j=0}^{\infty} M_j (Q^2)^j,
\] (26)

where the coefficient functions \( M_j \) include powers of masses and logarithms of masses and momentum squared. It is easy to see that the only integral contributing to \( M_0 \) is \( J(0)(1,1,1,1,1) \) in (20) and that the expansion starts from

\[ M_0 = 6\zeta(3). \] (27)

The \( M_1 \) term already includes contributions of all terms (20)–(25) (with the exception of (24) that begins to contribute starting from \( M_2 \)); this yields

\[
M_1 = \frac{m_1^2}{2} \left\{ \ln^2 \left( -\frac{Q^2}{m_1^2} \right) + 4 \ln \left( -\frac{Q^2}{m_1^2} \right) - \ln \frac{m_2^2}{m_1^2} \ln \frac{m_3^2}{m_1^2} + 6 \right\}
\]

\[ + \left\{ \text{analogous terms with } m_2^2, m_4^2, m_5^2 \right\} \]

\[ + \frac{m_3^2}{2} \left\{ 2 \ln^2 \left( -\frac{Q^2}{m_3^2} \right) + 4 \ln \left( -\frac{Q^2}{m_3^2} \right) - \ln \frac{m_2^2}{m_3^2} \ln \frac{m_4^2}{m_3^2} - \ln \frac{m_3^2}{m_4^2} \ln \frac{m_5^2}{m_3^2} \right\} \]

\[ + \frac{1}{2} \left\{ F(m_1^2, m_2^2, m_3^2) + F(m_4^2, m_5^2, m_3^2) \right\}, \] (28)

where the symmetric function \( F \) is defined by

\[ F(m_1^2, m_2^2, m_3^2) \equiv m_3^2 \lambda^2 (x, y) \Phi (x, y) \] (29)

and (19). By use of the REDUCE system [35], analytical results for the general massive case of the integral (26) (when all five masses are arbitrary) were obtained for the coefficient functions up to \( M_6 \) [34].

These results are in complete agreement with known explicit expressions in cases where non-zero masses are equal [33, 36, 37]. Furthermore, as in the large mass limit, a comparison with numerical calculations based on the same two-fold representation
shows that only first few terms provide a very good approximation provided we are not close to the highest threshold.

Thus, in situations when one is below the lowest threshold or above highest threshold one can substitute the general master diagram by sufficient number of terms of small (resp., large) momentum expansion.

9. Evaluation of master 2-loop diagram: the large mass expansion with zero thresholds.

Let us now consider the small momentum (large mass) expansion when some masses are large while the other masses are zero. If there no zero thresholds one can apply the large mass expansion of Section 7. There exist four independent zero-threshold configurations:

Case 1: one zero-2PT (e.g. $m_2 = m_5 = 0$);
Case 2: two zero-2PT’s ($m_1 = m_2 = m_4 = m_5 = 0$);
Case 3: one zero-3PT (e.g. $m_2 = m_3 = m_4 = 0$);
Case 4: one zero-2PT and one zero-3PT (e.g. $m_2 = m_3 = m_4 = m_5 = 0$).

Note that, in case 1, one more vanishing mass (e.g. $m_3$ or $m_4$) does not produce a new zero-threshold configuration, and the corresponding cases can be considered together with case 1, e.g.:

Case 1a = Case 1 with $m_3 = 0$;
Case 1b = Case 1 with one more mass (not $m_3$) equal to zero.

The set of relevant subgraphs that should be involved in general formula (14) is shown in Fig. 3.

In the resulting expansion one has

(a) two-loop vacuum diagrams with two (or one) massive and one (or two) massless lines;
(b) products of a one-loop massless diagram with external momentum $k$ and a one-loop massive tadpole;
(c) two-loop massless diagrams with one (or two) powers of the denominators being non-positive.

The results of calculation are described in [38]. In particular, for unique indices one has

$$J(m; q) \equiv J(1, 1, 1, 1; m; q) = -\pi^4 \sum_{j=0}^{\infty} C_j (q^2)^j,$$

where, for zero-threshold configurations, the coefficients $C_j$ can depend on logarithms of $q^2$:

$$C_j = C_j^{(2)} \ln^2(-q^2) + C_j^{(1)} \ln(-q^2) + C_j^{(0)}.$$

For cases when one has only one massive parameter $m$ (all non-zero masses are equal), the results obtained are in a full agreement with [33, 36, 37].

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82PT and 3PT mean two- and three-particle thresholds, respectively.
For Case 1 with three non-equal masses one has, in particular,

\[ C_0 = \frac{1}{4(m_1^2 - m_3^2)(m_1^2 - m_3^2)(m_1^2 - m_4^2)} \left\{ 2 \left( \ln \left( -\frac{q^2}{m_1^2} \right) + \ln \left( -\frac{q^2}{m_1^2} \right) - 4 \right) \right. \\
\times \left( m_1^2(m_4^2 - m_3^2) \ln \frac{m_1^2}{m_3^2} - m_4^2(m_1^2 - m_3^2) \ln \frac{m_4^2}{m_3^2} \right) \\
+ (m_1^2 - m_3^2)(m_4^2 - m_3^2) \ln \frac{m_1^2}{m_4^2} \left( \ln \frac{m_1^2}{m_3^2} + \ln \frac{m_2^2}{m_3^2} \right) \\
- 2m_3^2(m_1^2 - m_4^2) \ln \frac{m_3^2}{m_1^2} \ln \frac{m_4^2}{m_1^2} + 2(m_1^2 + m_2^2)(m_4^2 - m_3^2) \mathcal{H}(m_1^2, m_3^2) \\
- 2(m_3^2 + m_4^2)(m_1^2 - m_3^2) \mathcal{H}(m_4^2, m_3^2) \right\}, \tag{32} \]

where \( \mathcal{H} \) is a dimensionless function defined as

\[ \mathcal{H}(m_1^2, m_3^2) = 2 \text{Li}_2 \left( 1 - \frac{m_1^2}{m_3^2} \right) + \frac{1}{2} \ln^2 \left( \frac{m_1^2}{m_3^2} \right). \tag{33} \]

In [38] the coefficients were analytically calculated up to \( C_3 \). In Case 3 with two different non-zero masses, one has, for instance,

\[ C_0 = \frac{1}{4m_1^2m_3^2} \left\{ -(m_1^2 + m_2^2) \left( \ln^2 \frac{m_1^2}{m_3^2} + \frac{2\pi^2}{3} \right) - 2(m_1^2 - m_3^2) \mathcal{H}(m_1^2, m_3^2) \right\}, \tag{34} \]
\[ C_1 = \frac{1}{8m_1^4m_5^4(m_1^2 - m_5^2)} \left\{ 2m_1^2m_5^2(m_1^2 - m_5^2) \left( \ln \left( -\frac{q^2}{m_1^2} \right) + \ln \left( -\frac{q^2}{m_5^2} \right) - 3 \right) 
\right. \\
\left. - (m_1^2 - m_5^2)(m_1^4 + m_5^4) \left( \ln^2 \frac{m_1^2}{m_5^2} + \frac{2\pi^2}{3} \right) + 2m_1^2m_5^2(m_1^2 + m_5^2) \ln \frac{m_1^2}{m_5^2} \right. \\
\left. - 2(m_1^2 - m_5^2)^2 (m_1^2 + m_5^2) \mathcal{H}(m_1^2, m_5^2) \right\}, \quad (35) \]

In [38] these coefficients were analytically calculated up to \( C_5 \).

By comparison with numerical integration [29, 39, 40] it was found that substitution of the initial function by several terms of its asymptotic expansion provides a satisfactory agreement unless we are very close to the first non-zero threshold.

10. Other approaches and possibilities.

The general formulæ of asymptotic expansions were applied for calculation of Feynman diagrams at 3- and 4-loop level in other various situations — see e.g. [41].

Results for other two-loop massive diagrams, e.g. the ‘setting sun’ (≡ London transport) diagram, as well as presentation of other methods can be found in [39, 42, 43, 44, 45, 46, 47]. It looks natural to apply general results for asymptotic expansions in momenta and masses for massive 3-point 2-loop diagrams. In ‘combinatorially trivial’ case of the large mass expansion calculations were performed in [46]. For 3-point diagrams one can also use analogous two-fold parametric representation [40] for comparison.

Note that there are no general results for limits with some legs at a mass shell in specifically Minkowskian situations. See, e.g., [48] for discussion of asymptotic behavior in the Sudakov limit.

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