Critical points of Laplace eigenfunctions on polygons

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ABSTRACT
We study the critical points of Laplace eigenfunctions on polygonal domains with a focus on the second Neumann eigenfunction. We show that if each convex quadrilaterals has no second Neumann eigenfunction with an interior critical point, then there exists a convex quadrilateral with an unstable critical point. We also show that each critical point of a second-Neumann eigenfunction on a Lip-1 polygon with no orthogonal sides is an acute vertex.

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1. Introduction

A second Neumann eigenfunction \( u \) of the Laplacian approximates the temperature distribution of an insulated domain for large times. The ‘hot spots’ conjecture [1, 2] is the assertion that \( u \) does not assume its maximum value in the interior of the domain. The conjecture is false for some non-contractible plane domains [3, 4] but is still believed to be true for convex domains. The conjecture is known to be true when the domains are somewhat elongated, for example, the Lip-1 planar domains of [5]. In [6, 7] we show that the hot spots conjecture holds true for acute triangles thus resolving Polymath 7 [8].

In the present paper, we extend our study of critical points of eigenfunctions to general polygons and we encounter new phenomena. Note that every planar domain may be approximated by polygonal domains, and hence the weak form of the hot spots conjecture—some second Neumann eigenfunction has no interior maximum—for all planar domains would follow from the verification of the strong hot spots conjecture—every second Neumann eigenfunction has no interior maximum—for all polygonal domains.

Our general approach to the hot spots conjecture is based on the fact that eigenfunctions and their critical points vary continuously as one varies the domain. Roughly speaking, to show that a second Neumann eigenfunction \( u_0 \) on a polygon \( U_0 \) has no interior critical points, one constructs a path of polygons \( P_t \) and associated path of eigenfunctions \( u_t \) so that the eigenfunction \( u_1 \) on \( P_1 \) has no interior critical points. If one can show that the putative critical points of each \( u_t \) are ‘stable’ under perturbation, then \( u_0 \) also has no interior critical points.

1With the exception of rectangles, the critical set of a second Neumann eigenfunction on a simply connected polygon is finite [7].
In the case of triangles, we took $P_1$ to be a right isosceles triangle, and we established enough stability to successfully implement this strategy [6, 7]. Here we show that the strategy is likely to be more difficult to implement if the polygon has more sides.

**Theorem 1.1.** If each convex quadrilateral has no interior critical point, then there exists a convex quadrilateral $Q$, a second Neumann eigenfunction $u$ on $Q$, and a nonvertex critical point $p$ of $u$ that is not stable under perturbation.

By ‘stable under perturbation’ we mean that if $Q_n$ is a sequence of quadrilaterals that converges to $Q$ and $u_n$ is a sequence of second Neumann eigenfunctions on $Q_n$ that converges to $u$, then each $u_n$ has a critical point $p_n$ so that $p_n$ converges to $p$. We conjecture that instability does not hold for triangles.

On the other hand, we are able to successfully apply our strategy for resolving the hot spots conjecture on a large class of polygons.

**Theorem 1.2.** Suppose that $P_t$ is a path of polygons such that each $P_t$ has exactly two acute vertices, no two sides of $P_t$ are orthogonal, and $P_1$ is an obtuse triangle. Then the second Neumann eigenvalue of $P_0$ is simple, and the set of critical points of each eigenfunction consists of the two acute vertices.

The class of polygons described in Theorem 1.2 is exactly (up to rigid motion) the class of polygons that have no orthogonal sides and satisfy the Lip-1 condition of [5] (see Proposition 7.7). Thus, Theorem 1.2 provides a non-probabilistic proof of the weak hot spots conjecture for Lip-1 domains. Moreover, in contrast to the result of [5], we find that not only are there no interior critical points but there are also no critical points on the boundary other than the two acute vertices. Recently, Jonathan Rohleder [9] announced a non-probabilistic proof of the main result of [5].

We now outline the contents of this paper. In Sec. 2, we use the Bessel expansion of an eigenfunction $u$ to understand the nodal set of $Xu$ near a vertex where $X$ is a constant (resp. rotational) vector field. In particular, we show that whether or not an arc subset of the nodal set of $Xu$ ends at the vertex is essentially determined by the first two Bessel coefficients, the angle at $v$, and the angle between the vector field and the sides adjacent to $v$ (resp. location of central point). These criteria will be used crucially in the proof of Theorem 1.2.

We will need to rule out the possibility that critical points of a sequence of eigenfunctions, associated to a convergent sequence of polygons, converge to a vertex of the limiting polygon. In §3 we show in various contexts that if critical points converge to a vertex $v$, then the first two Bessel coefficients of the limiting eigenfunction equal zero. If the limiting polygon is simply connected then this is impossible (Proposition 6.3).

To check the stability of a critical point under perturbation, we will use a variant of the Poincaré-Hopf index. In §4, we define this invariant to include vertices and we prove a variant of the Poincaré-Hopf index formula for Neumann eigenfunctions $u$ on polygons. We relate the index of a critical point of $u$ located at a vertex $v$ with the first two Bessel coefficients of $u$ at $v$. We also show that the ‘total local index’ is unchanged under perturbation (Theorem 4.14). As a consequence each non-zero index critical point is stable (Lemma 4.15).

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1Here we regard each vector field $X$ as a first order differential operator.
In §5 we provide a local normal form for an eigenfunction in a neighborhood of a critical point \( p \) of \( u \) whose Poincaré-Hopf index equals zero (Lemma 5.1). Using this local normal form, we find that an index zero critical point cannot be a degree 1 vertex of the nodal set of \( Xu \) where \( X \) is either a constant or rotational vector field.

In §6 we specialize to simply connected polygons. For such domains, the nodal set of a second Neumann eigenfunction \( u \) is a simple arc, and from this fact we deduce that at least one of the first two Bessel coefficients at each vertex is nonzero. This implies a tighter relationship between the index of a vertex critical point of \( u \) and the first two Bessel coefficients (Corollary 6.4).

In §7 we prove Theorem 1.2 (Theorem 7.3). We first show that if a polygon \( P \) has at least one acute vertex and a second Neumann eigenfunction \( u \) on \( P \) has an interior critical point, then either \( u \) has four non-zero index critical points or there exists a side \( e \) of \( P \) such that the nodal set of the derivative of \( u \) in the direction of \( e \) has an arc that ends at a vertex \( v \) of \( P \). This leads us to consider, for the path \( u_t \) in Theorem 1.2, the number, \( S(t) \), of nonzero index critical points and the number, \( V(t) \), of vertices that are endpoints of a nodal arc of the derivative of \( u_t \) in the direction of a side of \( P \). We show that the set \( A \) of \( t \in [0, 1] \) such that either \( S(t) \geq 3 \) or \( V(t) \geq 1 \) is open and closed. For the obtuse triangle \( P_1 \), we have \( S(1) = 2 \) and \( V(1) = 0 \), and hence \( A \) is empty. In particular, the initial polygon \( P_0 \) has at most two non-zero index critical points, and from this we deduce using the results of §5 that there are no zero index critical points. Using the fact that \( V(0) = 0 \), we find that the two critical points are located at the vertices of \( P_0 \). These two critical points are the unique global extrema, and this implies that the eigenspace is one-dimensional.

In §8 we provide a criterion for the instability of a critical point on a quadrilateral. This criterion is based on the fact that the index of a vertex with angle less than \( \pi \) cannot equal \(-1\) (Corollary 6.4). In particular, an index \(-1\) critical point cannot cross from one side adjacent to a vertex to the other side of the vertex if the angle at the vertex is in \((\pi/2, \pi)\). Hence one is led to find a path of quadrilaterals \( Q_t \) such that \( Q_0 \) has an index \(-1\) critical point that lies on one side of a vertex and \( Q_1 \) and has an index \(-1\) critical point on the other side of the vertex.

In §9 we construct such a path of quadrilaterals and thus prove Theorem 1.1 (Theorem 9.5). The path is constructed by taking a nearly isosceles triangle whose vertex \( v \) of smallest angle is less than \( \pi/3 \), and then ‘breaking’ the side opposite to \( v \).

In §10 we specialize to convex polygons and find that if a second Neumann eigenfunction has only three critical points then one is a minimum, one is a maximum, and the third has index zero.

2. Solutions to \( \Delta u = \lambda u \) on a sector

To understand the behavior of an eigenfunction in a neighborhood of a vertex \( v \) of angle \( \beta \) of a polygon, we will consider its Fourier-Bessel expansion. By performing a rigid motion, we may assume that the vertex \( v \) is the origin, one side adjacent to \( v \) lies in the ray \( \{ z = r : r \geq 0 \} \) the nonnegative real axis, and the other side lies in the ray \( \{ z = r \cdot e^{i\beta} : r \geq 0 \} \). If \( u \) is a (real) solution to \( \Delta u = \lambda u \) with eigenvalue \( \mu \) that satisfies Neumann conditions on the rays \( \theta = 0 \) and \( \theta = \beta \), then separation of variables leads
to the Fourier-Bessel expansion:

\[
ur(r \exp(i\theta)) = \sum_{n=0}^{\infty} c_n \cdot J_\nu(\sqrt{\mu} \cdot r) \cdot \cos\left(\frac{n\pi\theta}{\beta}\right).
\]  

(1)

Here \(c_n \in \mathbb{R}\) and \(J_\nu\) denotes the Bessel function of the first kind of order \(\nu\) [10]

\[
J_\nu(x) = x^\nu \cdot \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k}}{2^{2k} \cdot \Gamma(k + \nu) \cdot \Gamma(k + \nu + 1)}
\]

(2)

where \(\Gamma\) is the Gamma function.

If \(u\) satisfies Dirichlet conditions on the rays \(\theta = 0\) and \(\theta = \beta\), then the one replaces \(\cos\) with \(\sin\), and if \(u\) satisfies Dirichlet conditions on the ray \(\theta = 0\) and Neumann conditions on the ray \(\theta = \beta\), then one replaces \(\cos(n\pi\theta/\beta)\) with \(\sin(n\pi\theta/2\beta)\) and \(J_{n\pi/\beta}\) with \(J_{n\pi/2\beta}\).

From (2) we find that, for each \(\nu \geq 0\), there exists an entire function \(g_\nu\) so that \(J_\nu(\sqrt{\mu} \cdot r) = r^\nu \cdot g_\nu(r^2)\). Note that neither \(g_\nu\) nor \(g'_\nu\) vanishes in a neighborhood of 0 for each \(\nu \geq 0\). With this notation, (1) takes a more compact form

\[
ur(r \exp(i\theta)) = \sum_{n=0}^{\infty} c_n \cdot r^{n\nu} \cdot g_{n\nu}(r^2) \cdot \cos(n \cdot \nu \cdot \theta)
\]

(3)

where \(\nu = \pi/\beta\). Note that we are suppressing the dependence of \(g\) on the eigenvalue \(\mu\).

Given a function \(f\), let \(Z(f) = f^{-1}(0)\) denote the nodal set of \(f\). For each \(\psi \in \mathbb{R}\), let \(L_\psi\) denote the constant vector field defined by

\[
L_\psi u = \cos(\psi) \cdot \partial_x + \sin(\psi) \cdot \partial_y.
\]

(4)

**Lemma 2.1.** Let \(u\) be a solution to \(\Delta u = \lambda u\) that satisfies Neumann conditions on the rays \(\theta = 0\) and \(\theta = \beta\).

(a) If \(c_0 \neq 0\) and either \(0 < \beta < \pi/2\) or \(c_1 = 0\), then there exists an arc in \(Z(L_\psi u)\) with an endpoint at the vertex \(\nu\) if and only if \(\psi \in [\pi/2, \pi/2 + \beta] \mod \pi\).

(b) If \(c_1 \neq 0\) and \(\pi/2 < \beta < \pi\), then there exists an arc in \(Z(L_\psi u)\) with an endpoint at the vertex if and only if \(\psi \in [\beta - \pi/2, \pi/2] \mod \pi\).

(c) If \(c_1 \neq 0\) and \(\beta = \pi\), then there exists an arc in \(Z(L_\psi u)\) with an endpoint at the vertex if and only if \(\psi = \pi/2 \mod \pi\). Moreover, near the vertex \(\nu\), this arc lies on \(\partial P\).

(d) If \(c_1 \neq 0\) and \(\beta > \pi\), then there exists an arc in \(Z(L_\psi u)\) with an endpoint at the vertex if and only if \(\psi \in [\pi/2, \beta - \pi/2] \mod \pi\).

Moreover, in all of the above situations, \(Z(L_\psi u)\) has at most one arc with an endpoint at the origin. Figure 1 describes some of these situations.
Proof. If $w = p/2 \mod p = 2$ (resp. $w = b/C_0 = p/2 \mod p$) then because $u$ satisfies Neumann conditions, the arc corresponding to $h = 0$ (resp. $h = b$) lies in $Z(L_w u)$.

Because $\partial_x = \cos(\theta) \cdot \partial_r - \sin(\theta) r^{-1} \cdot \partial_\theta$ and $\partial_y = \sin(\theta) \cdot \partial_r + \cos(\theta) r^{-1} \cdot \partial_\theta$, we have

$$L_w = \cos(\psi - \theta) \cdot \partial_r + \sin(\psi - \theta) \cdot \frac{1}{r} \partial_\theta.$$  \hfill (5)

If $c_0 \neq 0$ and either $c_1 = 0$ or $0 < \beta < \pi/2$, inspection of (3) shows that

$$u(re^{i\theta}) = a + b \cdot r^2 + f(r, \theta)$$  \hfill (6)

where $a$ and $b$ are constants and $|L_\Psi f| = o(r)$ and $|\partial_\theta L_\Psi f| = o(r)$. In particular, we find that

$$L_w u(re^{i\theta}) = 2b \cdot r \cdot \cos(\psi - \theta) + o(r),$$  \hfill (7)

and

$$\partial_\theta L_w u(re^{i\theta}) = -2b \cdot r \cdot \sin(\psi - \theta) + o(r).$$  \hfill (8)

If $\psi \in (\pi/2, \pi/2 + \beta) \mod \pi$, then from (7) we find that $L_w u(r)$ and $L_w u(re^{i\beta})$ have opposite signs for sufficiently small $r$. Thus, by the intermediate value theorem, there
exists $\gamma(r) \in (0, \beta)$ so that $(L_\psi u)(re^{iy(r)}) = 0$. Moreover, from (8), we find that the point
$\gamma(r)$ is unique for sufficiently small $r$, and by the implicit function theorem, $\gamma$ is smooth. The map $r \rightarrow re^{iy(r)}$ is the desired arc in $Z(L_\psi u)$. The uniqueness of $\gamma(r)$ implies that there is at most one arc.

Thus we have proven (a). To prove (b), note that if $\beta > \pi/2$ or $c_0 = 0$, then inspection of (3) shows that
\[
u u(r \cdot e^{i\theta}) = c_0 + c_1 \cdot r^\nu \cdot \cos(\nu \cdot \theta) + o(r^\nu).
\]
and hence a straightforward computation gives
\[
u L_\psi u(r \cdot e^{i\theta}) = c_1 \cdot \nu \cdot r^{\nu - 1} \cdot \cos(\psi + (\nu - 1) \cdot \theta) + o(r^{\nu - 1}).
\]
Note that $\cos(\psi + (\nu - 1) \cdot \theta)$ vanishes if and only if
\[
u \psi = \frac{\pi}{2} + (\beta - \pi) \cdot \frac{\theta}{\beta} \mod \pi.
\]
One argues as in the proof of part (a) to obtain a unique arc of $Z(L_\psi u)$ that ends at $v$.

Parts (c) and (d) follow from arguments similar those that verified (a) and (b). \(\square\)

In order to succinctly formulate some corollaries of Lemma 2.1 we make the following definition.

**Definition 2.2.** Suppose $\beta \neq \pi/2$ and let $u$ be a solution to $\Delta u = \lambda u$ on the sector of angle $\beta$ that satisfies Neumann conditions. We define the **leading Bessel coefficient** of $u$ at $v$ to be

- $c_0$ if $\beta < \pi/2$, and
- $c_1$ if $\beta > \pi/2$.

**Corollary 2.3.** If $L$ is a nonzero constant vector field parallel to one of the boundary rays of the sector of angle $\beta \neq \pi/2$, then the leading coefficient vanishes if and only if an arc of $Z(Lu)$ ends at $v$.

**Proof.** Since $L$ is parallel to one of the boundary rays, the vector field $L$ is a multiple of $L_\psi$ with $\psi = 0$ or $\beta$. In particular, $Z(Lu) = Z(L_\psi u)$. By using the reflection symmetry about $\theta = \beta/2$, we may assume without loss of generality that $\psi = 0$.

Suppose that the leading coefficient does not vanish. If $\beta < \pi/2$, then part (a) of Lemma 2.1 implies that no arc in $Z(L_\psi u)$ ends at the vertex. Similarly, parts (b), (c), and (d) imply that no arc in $Z(L_\psi u)$ ends at the vertex in the other cases.

Conversely, suppose that the leading coefficient does equal zero. Let $k$ be the smallest positive integer such that $c_k \neq 0$. If $\beta < \pi/2$, then $c_0 = 0$ and from (3) we find that
\[
u u(r \cdot e^{i\theta}) = r^{k \cdot \nu} \cdot \cos(k \cdot \nu \cdot \theta) + o(r^{k \cdot \nu}).
\]
Hence using (4) we find that
\[
u L_\psi u(r \cdot e^{i\theta}) = k \cdot \nu \cdot r^{k \cdot \nu - 1} \cdot \cos((k \cdot \nu - 1) \cdot \theta) + o(r^{k \cdot \nu - 1}).
\]
Since $\beta < \pi/2$ and $k \geq 1$, the function $\cos((k \cdot \nu - 1) \cdot \theta)$ vanishes for some $\theta \in (0, \beta)$. An implicit function theorem argument establishes the existence of a smooth arc.

If $\beta > \pi/2$, then a similar argument applies to give the claim. \(\square\)
Corollary 2.4. Let $u$ be a solution to $\Delta u = \lambda u$ on a sector of angle $\beta$ that is not equal to an integral multiple of $\pi/2$ satisfying Neumann conditions. If one of the first two Bessel coefficients of $u$ is non-zero and if $L$ is a constant vector field such that

- some arc in $\mathcal{Z}(Lu)$ ends at the vertex, and
- $L$ is not orthogonal to a boundary ray of the sector,

then for each constant vector field $L'$ that is sufficiently close to $L$, some arc of $\mathcal{Z}(L'u)$ ends at the vertex.

Proof. Since $\beta$ is not equal to an integral multiple of $\pi/2$, by Proposition 4.4 [6], there exists a neighborhood $N$ of the vertex $v$ of the sector that contains no critical points of $u$.

Since at least one of the zeroth and the first Bessel coefficients of $u$ at $v$ is non-zero, by Lemma 2.1, the set $\mathcal{Z}(Lu)$ contains exactly one arc that ends at $v$. In particular, $u$ has opposite signs on the two rays of the sector. By continuity, for each constant vector field $L'$ that is sufficiently close to $L$, some arc of $\mathcal{Z}(L'u)$ must end at some point $p_{L'}$ near $v$ (depending on $L'$) that lies on the boundary of the sector.

To finish the proof it suffices to show that $p_{L'} = v$ for $L'$ sufficiently close to $L$. If $p_{L'}$ is not $v$, then since $L$ is not orthogonal to the sides of the sector, so is $L'$, and hence, $p_{L'}$ is a critical point of $u$. If $L'$ is sufficiently close then, by continuity, $p_{L'}$ lies in $N$. This contradicts the first paragraph of the proof. \hfill $\square$

Let $S_{\beta}$ denote the sector $\{ z = r \cdot e^{i\theta} : \theta \in [0, \beta] \mod \pi \}$. Recall that $R_w$ denotes the vector field that corresponds to rotation about $w \in \mathbb{C}$.

Corollary 2.5. Let $u$ be a solution to $\Delta u = \lambda u$ that satisfies Neumann conditions on the sides $\theta = 0$ and $\theta = \beta$. Suppose that $c_0$ and $c_1$ are not both equal to zero.

(1) If $\beta < \pi/2$

(a) and $c_0 \neq 0$, then an arc of $\mathcal{Z}(R_w)$ ends at the vertex if and only if $w$ lies in $S_{\beta}$.

(b) and $c_0 = 0$, then an arc of $\mathcal{Z}(R_w)$ ends at the vertex if and only if $w$ does not lie in $S_{\beta}$.

(2) If $\pi/2 < \beta < \pi$

(a) and $c_1 \neq 0$, then an arc of $\mathcal{Z}(R_w)$ ends at the vertex if and only if $w$ does not lie in $S_{\beta}$.

(b) and $c_1 = 0$, then an arc of $\mathcal{Z}(R_w)$ ends at the vertex if and only if $w$ lies in $S_{\beta}$.

Proof. If $w = \rho \cdot e^{i\phi}$, then a computation shows that the rotational vector field about $w$ takes the form

$$R_w = \partial_{\theta} + L_{\phi+\frac{\pi}{2}}.$$

Because $|\partial_{\theta}u| = o(r^\nu)$ and $|\partial_{\theta}^2u| = o(r^\nu)$, we find that (7) and (8) still hold with $L_\psi$ replaced by $R_w$ and $\psi$ replaced by $\phi + \pi/2$. Thus the argument given in the proof of Lemma 2.1 applies. \hfill $\square$
Remark 2.6. Similar statements hold for Dirichlet and mixed boundary conditions. We leave the formulation of the statements to the reader.

3. Critical points on a sector converging to a vertex

Let $S_n$ be a sequence of sectors that converges to a sector $S$. Let $u_n : S_n \rightarrow \mathbb{R}$ be a sequence of solutions to $\Delta u = \lambda u$ each satisfying Neumann conditions that converges to a Neumann eigenfunction $u : S \rightarrow \mathbb{R}$. In this section we show that if certain types of critical points of $u_n$ converge to the vertex of $S$, then the first two Bessel coefficients of $u$ must vanish. Some of these results are straightforward extensions of results in [6], but several are new.

Let $b_-$ and $b_+$ denote the distinct boundary rays of the sector $S$. Let $c_0$ and $c_1$ denote the respective Bessel coefficients of $u$ at the vertex of $S$. Let $\beta$ denote the vertex angle of $S$, and let $\nu = \pi/\beta$.

Lemma 3.1 (Compare Proposition 9.1 [6]). For each $n$, let $p_n$ be a critical point of $u_n$ that lies in the interior of $S_n$. If $p_n$ converges to the vertex of $S$, then $c_1 = 0$. If, in addition, $\beta < \pi$, then $c_0 = 0$.

Proof. Let $\beta_n$ be the angle of the sector $S_n$ and let $\nu_n = \pi/\beta_n$. By performing rigid motions if necessary, we may assume without loss of generality that the vertex $S$ and each of $S_n$ is 0 and that the boundary rays of $S_n$ are $\theta = 0, \beta_n$. Using (3) and the fact that $\sin(\chi)$ divides $\sin(k\chi)$ for each $k$, we find that

$$\partial_\theta u_n(r \cdot e^{i\theta}) = -\nu_n \cdot r^{\nu_n} \cdot \sin(\nu_n \cdot \theta) \cdot (c_1(n) \cdot g_{\nu_n}(r) + O(r^{\nu_n})).$$

Thus, since $p_n = r_n \exp(i\theta_n)$ is a critical point, $0 < \theta_n < \beta_n$, and $g_{\nu_n}(0) \neq 0$, we find that $c_1(n) = O(r_n^{\nu_n})$. In particular, since $u_n$ converges to $u$, we have $c_1 = \lim_{n \rightarrow \infty} c_1(n) = 0$.

From (3), we find

$$\partial_r u_n(r \cdot e^{i\theta}) = c_0(n) \cdot 2r \cdot g_0'(r^2) + c_1(n) \cdot \nu_n \cdot r^{\nu_n+1} \cdot g_{\nu_n}(r^2) \cdot \cos(\nu_n \theta) + O(r^{\nu_n}).$$

Thus, since $p_n = r_n \exp(i\theta_n)$ is a critical point, $g_0'(0) \neq 0$, and $c_1(n) = O(r_n^{\nu_n})$, we find that $c_0(n) = O(r_2^{2(\nu_n-1)}) + O(r_n^{\nu_n-1})$. If $\beta < \pi$, then there exists $\epsilon > 0$ so that for sufficiently large $n$, we have $\nu_n > 1 + \epsilon$. Hence $c_0 = \lim_{n \rightarrow \infty} c_0(n) = 0$. \hfill $\Box$

Lemma 3.2 (Compare Lemma 9.2 [6]). For each $n$, let $p_n$ be a critical point of $u_n$ that lies in the boundary ray of $S_n$ that converges to $b_-$, and let $q_n$ be a critical point of $u_n$ that lies in the boundary ray of $S_n$ that converges to $b_+$. If the sequences $p_n$ and $q_n$ both converge to the vertex of $S$, then $c_1 = 0$. If $\beta < \pi$, then we also have $c_0 = 0$.

Proof. Let $\beta_n$ be the angle of the sector $S_n$. By performing rigid motions if necessary, we may assume without loss of generality that the vertex $S$ and each of $S_n$ is 0 and that the boundary rays of $S_n$ are $\theta = 0, \beta_n$. Thus, there exist sequences $r_n$ and $s_n$ so that $p_n = r_n$ and $q_n = s_n e^{i\beta_n}$.
From (3) we find that
\[ \partial_r u_n(r) = c_0(n) \cdot 2r \cdot g'_0(r^2) + c_1(n) \cdot \nu_n \cdot r^{\nu_n-1} \cdot g_{\nu_n}(r^2) + O(r^{\nu_n+1} + r^{2\nu_n-1}) \]
\[ \partial_r u_n(s \cdot e^{\partial_x}) = c_0(n) \cdot 2s \cdot g'_0(s^2) - c_1(n) \cdot \nu_n \cdot s^{\nu_n-1} \cdot g_{\nu_n}(s^2) + O(s^{\nu_n+1} + s^{2\nu_n-1}) \]

Since \( p_n = r_n \) and \( q_n = s_n e^{\partial_x} \) are critical points, the radial derivative of \( u_n \) vanishes at these points, and hence
\[ 0 = c_0(n) \cdot 2r_n \cdot g'_0(r_n^2) + c_1(n) \cdot \nu_n \cdot r_n^{\nu_n-1} \cdot g_{\nu_n}(r_n^2) + O(r_n^{\nu_n+1} + r_n^{2\nu_n-1}) \]  \( \tag{9} \)
\[ 0 = c_0(n) \cdot 2s_n \cdot g'_0(s_n^2) - c_1(n) \cdot \nu_n \cdot s_n^{\nu_n-1} \cdot g_{\nu_n}(s_n^2) + O(s_n^{\nu_n+1} + s_n^{2\nu_n-1}). \]  \( \tag{10} \)

Let \( a_\nu(r) = 2g'_0(r^2)/g_\nu(r^2) \). Because, the functions \( g'_0 \) and \( g_\nu \) are continuous and positive near zero, so is \( a_\nu \). From (9) and (10) we find that
\[ c_0(n) \cdot (a_\nu(n) \cdot r_n^{2-\nu_n} + a_\nu(n) \cdot s_n^{2-\nu_n}) = O(r_n^2 + s_n^2 + r_n^{\nu_n} + s_n^{\nu_n}), \]  \( \tag{11} \)
and
\[ c_1(n) \cdot \left( \frac{r_n^{\nu_n-2}}{a_\nu(n)} + \frac{s_n^{\nu_n-2}}{a_\nu(n)} \right) = O(r_n^{\nu_n} + s_n^{\nu_n} + r_n^{2\nu_n-2} + s_n^{2\nu_n-2}). \]  \( \tag{12} \)

It follows from (12) that \( c_1(n) = O(r_n^2 + s_n^2 + r_n^{\nu_n} + s_n^{\nu_n}) \). Since \( \nu_n \) tends to \( \nu > 0 \), we have \( c_1 = \lim_{n \to \infty} c_1(n) = 0 \).

It follows from (11) that \( c_0(n) = O(r_n^{1+\epsilon} + s_n^{1+\epsilon} + r_n^{2\nu_n-2} + s_n^{2\nu_n-2}) \). If \( \beta < \pi \), then there exists \( \epsilon > 0 \) so that \( \nu_n > 1 + \epsilon \) and for sufficiently large \( n \). Thus, for \( n \) sufficiently large, we have \( c_0(n) = O(r_n^{1+\epsilon} + s_n^{1+\epsilon} + r_n^{2\nu_n-2} + s_n^{2\nu_n-2}) \). Therefore, \( c_0 = \lim_{n \to \infty} c_0(n) = 0 \).

**Lemma 3.3** (Compare Lemma 9.3 [6]). Let \( p_n \) be a critical point of \( u_n \) and suppose that \( p_n \) converges to the vertex of \( S \). If \( \beta < \pi/2 \), then \( c_0 = 0 \). If \( \beta > \pi/2 \), then \( c_1 = 0 \).

**Proof.** By performing rigid motions if necessary, we may assume without loss of generality the boundary rays of \( S_n \) are \( \theta = 0 \) and \( \theta = \beta_n \). By Lemma 3.1 passing to a subsequence, and applying a reflection across \( \theta = \beta_n/2 \) if necessary, we may assume, without loss of generality, that \( p_n = r_n \) lies in the positive real axis. As in the proof of Lemma 3.2 we have
\[ 0 = c_0(n) \cdot 2r_n \cdot g'_0(r_n^2) + c_1(n) \cdot \nu_n \cdot r_n^{\nu_n-1} \cdot g_{\nu_n}(r_n^2) + O(r_n^{\nu_n+1} + r_n^{2\nu_n-1}). \]  \( \tag{13} \)

If \( \beta < \pi/2 \), then there exists \( \epsilon > 0 \) so that \( \nu_n > 2 + \epsilon \) for sufficiently large. Hence, since \( g'_0(0) \neq 0 \), it follows from (13) that \( c_0(n) = O(r_n^2) \). It follows that \( c_0 = 0 \).

From (13), we have \( c_1(n) = O(r_n^{2-\nu_n}) + O(r_n^2 + r_n^{\nu_n}) \). If \( \beta > \pi/2 \), then there exists \( \epsilon > 0 \) so that \( \epsilon < \nu_n < 2 - \epsilon \) for sufficiently large. Hence, since \( g_\nu(0) \neq 0 \), it follows from (13) that \( c_1(n) = O(r_n^2) \). Thus \( c_1 = \lim_{n \to \infty} c_1(n) = 0 \).

**Lemma 3.4.** Suppose that \( \beta \neq \pi/2 \) and \( \beta < \pi \). Suppose that for each \( n \) the sector \( S_n \) is bounded by the rays \( \theta = 0 \) and \( \theta = \beta_n \), and there exist \( 0 < r_n \leq s_n \) such that \( \partial_r u(n) = 0 \) and \( \partial_r^2 u(s_n) = 0 \). If \( s_n \) converges to zero as \( n \) tends to infinity, then \( c_0 = 0 = c_1 \).

**Proof.** Because \( 0 < \beta < \pi \) and \( \beta_n \to \beta \), there exists \( \delta > 0 \) such that \( \pi \cdot \delta < \beta_n < \pi \cdot (1 + \delta)^{-1} \) and hence \( \delta^{-1} > \nu_n > \nu_n - 1 > \delta \). From (3) we have
\[ \partial_r u_n(r) = c_0(n) \cdot 2r \cdot g_0(r^2) + c_1(n) \cdot \nu_n \cdot r^{\nu_n - 1} \cdot g_\nu_n(r^2) + O(r^{\nu_n + 1} + r^{2\nu_n - 1}) \]
\[ \partial^2_r u_n(s) = c_0(n) \cdot (2 \cdot g_0(s^2) + 4s^2 \cdot g_0''(s^2)) + c_1(n) \nu_n (\nu_n - 1) s^{\nu_n - 1} g_\nu_n(s^2) + O(s^{\nu_n + 2\nu_n - 2}). \]

(14)

Let
\[ a_\nu(r) = \frac{2g'_0(r^2)}{\nu \cdot g_\nu(r^2)} \]
and
\[ b_\nu(s) = \frac{2g'_0(s^2) + 4s^2 \cdot g''_0(s^2)}{\nu \cdot g_\nu(s^2)}. \]

Because \( g_\nu \) and its derivatives are positive and continuous for \( r \) near zero, the functions \( a_\nu \) and \( b_\nu \) are also positive and continuous for small \( r \). Note that \( a_\nu(0)/b_\nu(0) = 1 \).

Since \( \partial_r u(r_n) = 0 \) and \( \partial^2_r u(s_n) = 0 \) we find from (14) that
\[ 0 = c_0(n) \cdot a_\nu_n(r_n) \cdot r_n^{2-\nu_n} + c_1(n) + O(r_n^{\nu_n + 1} + r_n^{2\nu_n - 1}) \]
\[ 0 = c_0(n) \cdot b_\nu_n(s_n) \cdot s_n^{2-\nu_n} + c_1(n) + O(s_n^{\nu_n} + s_n^{2\nu_n}). \]

(15)

By subtracting we have
\[ \frac{c_0(n) \cdot (a_\nu_n(r_n) \cdot r_n^{2-\nu_n} - b_\nu_n(s_n) \cdot s_n^{2-\nu_n})}{\nu_n - 1} = O(r_n^{\nu_n + 1} + r_n^{2\nu_n} + s_n^{\nu_n} + s_n^{2\nu_n}) \]

(16)

Suppose \( \beta > \pi/2 \), then \( \nu < 2 \) and so since \( \nu_n \to \nu \) there exists \( \epsilon > 0 \) so that for sufficiently large \( n \)
\[ \frac{1}{\nu_n - 1} \cdot \frac{b_\nu_n(s_n)}{a_\nu_n(r_n)} \geq 1 + \epsilon. \]

(17)

Since \( r_n \leq s_n \), we have \( r_n^{2-\nu_n} \leq s_n^{2-\nu_n} \). Therefore, from (16) we find that
\[ c_0(n) \cdot (-\epsilon) \cdot a(r_n) \cdot s_n^{2-\nu_n} = O(r_n^{\nu_n} + s_n^{\nu_n} + r_n^{2\nu_n - 1} + s_n^{2\nu_n - 2}) \]

Thus, since \( r_n \leq s_n \) we find that \( c_0(n) = O(r_n^{\nu_n} + s_n^{\nu_n} + r_n^{2\nu_n - 1} + s_n^{2\nu_n - 2}) \), and hence
\[ c_0(n) = O(r_n^{1+\delta} + s_n^{1+\delta} + r_n^{2\delta} + s_n^{2\delta}). \]

(18)

Therefore, \( c_0 = \lim_{n \to \infty} c_0(n) = 0 \).

Suppose \( \beta < \pi/2 \). Then since \( \nu_n \to \nu > 2 \), there exists \( \epsilon > 0 \) so that for sufficiently large \( n \)
\[ (\nu - 1) \cdot \frac{a_\nu_n(r_n)}{b_\nu_n(s_n)} \geq 1 + \epsilon. \]

(19)

Since \( r_n^{2-\nu_n} \geq s_n^{2-\nu_n} \), from (16) one deduces that \( c_0 = 0 \) in this case by arguing in a similar manner.
To show that $c_1 = 0$, we argue similarly. From (15) we find that

$$0 = c_0(n) + c_1(n) \cdot \frac{r_n^{\nu_n-2}}{a_{\nu_n}(r_n)} + O(r_n^2 + r_n^{\nu_n})$$

$$0 = c_0(n) + c_1(n) \cdot \frac{(\nu_n - 1) \cdot s_n^{\nu_n-2}}{b_{\nu_n}(s_n)} + O(s_n^2 + s_n^{\nu_n}).$$

and hence by subtracting

$$c_1(n) \cdot \left( \frac{r_n^{\nu_n-2}}{a_{\nu_n}(r_n)} - \frac{(\nu_n - 1) \cdot s_n^{\nu_n-2}}{b_{\nu_n}(s_n)} \right) = O(r_n^2 + r_n^{\nu_n} + s_n^2 + s_n^{\nu_n}).$$

Now argue as was done to show that $c_0 = 0$. In particular, in the case $\beta < \pi/2$ use (19), and in the case $\beta > \pi/2$ use (17).

**Corollary 3.5.** Suppose $\beta < \pi$ and $\beta \neq \pi/2$. Suppose that for each $n$, the points $p_n$ and $q_n$ are distinct critical points. If $p_n$ and $q_n$ both converge to the vertex of $S$, then $c_0 = 0 = c_1$.

**Proof.** By applying rigid motions we may assume that $S_n$ is bounded by the rays $\theta = 0$ and $\theta = \beta_n$. By Lemma 3.1 and Lemma 3.2, it suffices to assume that $p_n$ and $q_n$ lie in the same boundary ray, and by reflecting if necessary about $\theta = \beta_n/2$, we may assume that both $p_n$ and $q_n$ are real. By relabeling we may assume that $p_n < q_n$. By assumption $\partial_r(p_n) = 0 = \partial_r(q_n)$, and so Rolle’s theorem implies that there exist $s_n$ such that $p_n \leq s_n \leq q_n$ and $\partial_r^2(s_n) = 0$. The claim now follows from Lemma 3.4.

**Corollary 3.6.** Let $S$ be a sector with angle $\beta < \pi$ and not equal to $\pi/2$, and let $u : S \to \mathbb{R}$ be a Neumann eigenfunction. If the vertex $v$ is an accumulation point of the critical points of $u$, then $c_0 = 0 = c_1$.

**Proof.** Apply Corollary 3.5 with $S_n = S$ and $u_n = u$.

**Lemma 3.7.** Suppose $\beta < \pi$ and $\beta \neq \pi/2$. If $p_n$ is a degenerate critical point of $u_n$ that converges to the vertex of $S$, then $c_0 = 0 = c_1$.

**Proof.** By applying rigid motions we may assume that $S_n$ is bounded by $\theta = 0$ and $\theta = \beta_n$. By Lemma 3.1, by passing to a subsequence, and by applying a reflection across $\theta = \beta_n/2$ if necessary, we may assume that $p_n$ lies in the boundary ray $\theta = 0$. That is, $p_n = r_n > 0$ and $\partial_r u_n(r_n) = 0$.

Since $u_n$ satisfies Neumann conditions along the real axis, and $p_n$ is a degenerate critical point we have either $\partial_r^2 u_n(r_n) = 0$ or $\partial_y^2 u_n(p_n) = 0$. If $\partial_r^2 u_n(r_n) = 0$, then Lemma 3.4 with $s_n = r_n$ implies the claim.

Suppose then that $\partial_y^2 u_n(p_n) = 0$. Along the ray $\theta = 0$ we have $\partial_y^2 = r^{-1} \cdot \partial_r + r^{-2} \cdot \partial_0^2$. Since $\partial_y u_n(r_n) = 0$, we have $\partial_y^2 u_n(r_n) = \partial_0^2 u_n(r_n)$, and so

$$0 = (\partial_y^2 u_n)(r_n) = -c_1(n) \cdot r_n^2 \cdot r_n^{\nu_n} \cdot g_0(r_n^2) + O(r_n^2 \nu_n).$$
Since \( u_n \) satisfies the first equation in (13) we find that
\[
0 = \left( \frac{\partial^2}{\partial r^2} u_n \right)(r_n) = 2c_0(n) \cdot g_0'(r_n^2) + O(r_n^{\nu_n} + r_n^{2\nu_n-2}).
\]
Since \( \beta < \pi \), there exists \( \epsilon > 0 \) so that \( \nu_n > 1 + \epsilon \) for sufficiently large \( n \). Since \( g_0 \) and its derivative do not vanish at zero, it follows that \( c_0 = \lim_{n \to \infty} c_0(n) = 0 \) and \( c_1 = \lim_{n \to \infty} c_1(n) = 0 \).

Remark 3.8. Note that in the proof of Lemma 3.7 we used the condition \( \beta \neq \pi/2 \) only in the case that \( \partial^2_{rr} u_n(p_n) = 0 \). Indeed, the proof shows that if \( \partial^2_{rr} u_n(p_n) = 0 \), then \( c_0 = 0 = c_1 \) even if \( \pi = \beta/2 \).

4. A Poincaré-Hopf formula for critical points of eigenfunctions on a polygon

In this section, we provide a variant of the classical-Poincaré Hopf index theorem for the gradient of Laplace eigenfunctions on a planar polygonal domain \( P \). The discussion will focus on eigenfunctions satisfying Neumann boundary conditions, but the methods apply to give variants in the contexts of Dirichlet and mixed boundary conditions.

Each Neumann eigenfunction \( u : P \to \mathbb{R} \) extends continuously to the boundary \( \partial P \), and this extension is smooth at each nonvertex point in \( \partial P \). Let \( p \) lie in the closure \( \overline{P} \) of \( P \). Suppose that there exists a deleted disk neighborhood \( \hat{D} \) of \( p \) that contains no zeros of \( \nabla u \). Then the closure of each component of \( \hat{D} \cap \{ z : u(z) = u(p) \} \) is an arc.\(^3\) If such an arc contains \( p \), then we will say that the arc emanates from \( p \). Let \( n \) be the number of arcs in \( \{ z : u(z) = u(p) \} \) that emanate from \( p \), and define
\[
\text{ind}(u, p) = \begin{cases} 
1 - \frac{1}{2} \cdot n & \text{if } p \in P \\
1 - n & \text{if } p \in \partial P.
\end{cases}
\]
Note that if \( \text{ind}(u, p) \neq 0 \) and \( p \) is not a vertex of \( P \), then \( \nabla u(p) = 0 \).\(^4\) If \( p \) is a vertex and \( \text{ind}(u, p) \neq 0 \), then we will regard \( p \) as a critical point of \( u \).

Definition 4.1. A point \( p \in \overline{P} \) will be called a critical point of \( u \) if either
\begin{itemize}
  \item \( p \) is not a vertex and \( \nabla u(p) = 0 \), or
  \item \( p \) is a vertex and \( \text{ind}(u, p) \neq 0 \).
\end{itemize}

Assumption 4.2. In what follows we will assume that each critical point \( p \) is isolated. In particular, index \( \text{ind}(u, p) \) is well-defined for each \( p \).

In [11], we show that rectangles are the only simply-connected polygons whose second Neumann eigenfunctions have infinitely many critical points. Hence the assumption reduces to the assumption that the polygon is not a rectangle in the simply-connected case.

\(^3\)If the closure of some component were a loop, then the loop would bound a disk that contained a critical point. In this paper we use ‘arc’ to mean an embedded interval.

\(^4\)The converse is not true, namely there may be critical points with index equal to zero. See §5.
Let $\chi(S)$ denote the Euler characteristic of a surface $S$.\textsuperscript{5} For example, if $S$ is a polygonal domain obtained by removing $k$ disjoint simply connected polygons from the interior of a simply connected polygon, then $\chi(S) = 1 - k$. Let $\text{crit}(u)$ denote the set of critical points of $u$ including the vertices $v$ such that $\text{ind}(u, v) \neq 0$. The following is a variant of the classical Poincaré-Hopf formula [12].

**Proposition 4.3** (Index formula). Let $u : P \to \mathbb{R}$ be a Neumann eigenfunction such that the set $\text{crit}(u)$ is finite.

$$2 \cdot \chi(P) = \sum_{p \in \text{crit}(u) \cap P} 2 \cdot \text{ind}(u, p) + \sum_{p \in \text{crit}(u) \cap \partial P} \text{ind}(u, p).$$

**Proof.** Let $DP$ be the ‘double of $P$’, the closed surface without boundary obtained by gluing two disjoint copies of $P$ along their respective boundaries. The surface $DP$ has a natural real-analytic structure on the complement of the set $C$ of ‘cone points’ corresponding to the vertices of $P$. Because $u$ is a Neumann eigenfunction, $u$ extends to a real-analytic function $\tilde{u} : DP \setminus C \to \mathbb{R}$ that is invariant under the isometric involution that exchanges the two copies of $P$. For each $p \in DP$, we define $\text{ind}(\tilde{u}, p) = 1 - \frac{\alpha}{3}$. Because $u$ is a Neumann eigenfunction, we find that $\text{ind}(\tilde{u}, p) = \text{ind}(u, p)$ for $p \in P$ (including vertices).

Let $A$ be the union of the level sets of $\tilde{u}$ that contain critical points of $\tilde{u}$. The complement of $A$ consists of topological annuli, and hence, by the Euler-Poincaré formula, $\chi(DP) = \chi(A)$. On the other hand, the number of edges in $A$ equals $\frac{1}{2} \sum_{p \in C} n_p$ where $n_p$ is the valence of the graph $A$ at $p$. It follows that $\chi(DP) = \sum_{\text{crit}(\tilde{u})} \text{ind}(u, p)$ where $\text{crit}(\tilde{u})$ includes $p \in C$ such that $\text{ind}(\tilde{u}, p) \neq 0$. We have $\chi(DP) = 2 \cdot \chi(P)$ and for every interior critical point of $u$ we have two critical points of $\tilde{u}$ with the same index. The claimed formula follows. \hfill $\Box$

**Remark 4.4.** There are also variants of Proposition 4.3 in the contexts of Dirichlet and mixed boundary conditions. For example, if $u$ satisfies Dirichlet conditions, then formula (4.3) holds true if one redefines $\text{ind}(u, p) = 2 - k$ for each $p \in \partial P$.

**Remark 4.5.** The classical Poincaré-Hopf theorem applies to a smooth vector field $X$ on an oriented closed surface $S$ that has finitely many critical points. If $\gamma$ is a simple oriented loop that encloses at most one zero $p$ of $X$, then the restriction of $X/|X|$ to $\gamma$ defines a map from the unit circle to itself. The index of $X$ at $p$ is the degree of this self-map of the circle. (See, for example, [12] §1.10.) If $X = \nabla f$, then this index equals $1 - \frac{k}{2}$ where $k$ is the number of components of $f^{-1}(f(p)) \setminus \{p\}$. In the context of a vector field $X$, the Poincaré-Hopf index formula gives that the sum of the indices of the zeros of $X$ equals the Euler characteristic of $S$.

**Proposition 4.6.** The point $p \in P$ is a local extremum if and only if $\text{ind}(u, p) = 1$.

**Proof.** By Assumption 4.2, each critical point is isolated. We have $\text{ind}(u, p) = 1$ if and only if there exists a punctured disk neighborhood $\tilde{D}$ of $p$ so that $u(z) \neq u(p)$ for each.

\textsuperscript{5}See for example [12] or [17].
Proof. Without loss of generality, we have either $u(z) > u(p)$ for all $z \in \tilde{D}$ or $u(z) < u(p)$ for all $z \in \tilde{D}$. This occurs if and only if $p$ is a local extremum of $u$. \hfill \square

Suppose that $v$ is a vertex of $P$ that is not a limit point of the zeros of $\nabla u$. The index $\text{ind}(u, v)$ is determined by the Bessel expansion (3) of $u$ near $v$.

**Lemma 4.7.** Let $P$ be a polygon, let $v$ be a vertex of $P$ with angle $\beta$, and let $u$ be a Neumann eigenfunction on $P$. Let $k \geq 1$ be the smallest positive integer so that $c_k \neq 0$ and suppose that $v$ is a critical point of $u$.

(i) If $u(v) = 0$ or $\beta > k \cdot \frac{\pi}{2}$, then $\text{ind}(u, v) = 1 - k$.

(ii) If $u(v) \neq 0$ and $\beta < k \cdot \frac{\pi}{2}$, then $\text{ind}(u, v) = 1$.

(iii) If $u(v) \neq 0$ and $\beta = k \cdot \frac{\pi}{2}$, then $1 - k \leq \text{ind}(u, v) \leq 1$.

In particular, if $\beta \neq \pi/2$ or $3\pi/2$, then $\text{ind}(u, v)$ equals either 1 or $1 - k$.

A similar statement can be derived in the cases of Dirichlet or mixed boundary conditions.

**Proof.** Without loss of generality, $v = 0$ and the sides adjacent to $v$ bound the sector $0 < \theta < \beta$.

If $u(0) = 0$ or $\beta > k \cdot \frac{\pi}{2}$, then from (3) there exist $b \neq 0$ and $a$ so that

$$u(r \cdot e^{i\theta}) = a + b \cdot r^{k\nu} \cdot \cos(k\nu \theta) + o(r^{k\nu}).$$

Using, for example, the implicit function theorem, one finds that there exists a disk neighborhood $D$ of 0 such that $D \cap u^{-1}(u(v)) \setminus \{v\}$ consists of $k$ arcs each with an endpoint at $v$. It follows that $\text{ind}(u, 0) = 1 - k$.

Suppose $u(v) \neq 0$ and $\beta < k \cdot \frac{\pi}{2}$. Then $k \cdot \nu > 2$ and hence from (3) we find that

$$u(z) = a + b \cdot r^2 + o(r^2)$$

where $a \neq 0 \neq b$. Hence $v$ is a local extremum of $u$, and so by Proposition 4.6, $\text{ind}(u, v) = 1$.

If $u(v) \neq 0$ and $\beta = k\pi/2$, then from (3) we have

$$u(r \cdot e^{i\theta}) = a + r^2(b + c \cdot \cos(k\nu \theta)) + o(r^2)$$

where $a$, $b$ and $c$ are nonzero constants. If $b = -c$, then $\text{ind}(u, v)$ will depend on the $o(r^2)$ error term. In this case $1 - k \leq \text{ind}(u, v) \leq 1$. \hfill \square

**Corollary 4.8.** Suppose $\beta$ is not a multiple of $\pi/2$.

(1) If $c_1 = 0$, then $\text{ind}(u, v) \neq 0$.

(2) If $\beta > \pi/2$, then $c_1 = 0$ if and only if $\text{ind}(u, v) \neq 0$.

**Proof.** Let $k$ be as in the statement of Lemma 4.7. If $c_1 = 0$, then $k > 1$, and hence Lemma 4.7 implies that $\text{ind}(u, v) \neq 0$. If $\beta > \pi/2$ and $c_1 \neq 0$, then $k = 1$ and we are in case (i) of Lemma 4.7. Thus, $\text{ind}(u, v) = 0$. \hfill \square
Remark 4.9. In the case that $\beta$ is a multiple of $\pi/2$ and $u(\nu) \neq 0$, part (iii) of Lemma 4.7 provides only an inequality for $\text{ind}(u, \nu)$. Yet, one can determine the index in finitely many steps. In particular if $k \cdot \nu = 2$, then

$$u(z) - u(\nu) = r^2 \cdot (a + \cos(2\theta)) + o(r^2)$$

where $a = (c_0 \cdot g_0(0))/(c_k \cdot g_2(0))$. If $|a| > 1$, then $\text{ind}(u, \nu) = 1$ and if $|a| < 1$, then $\text{ind}(u, \nu) = 1 - k$. If $|a| = 1$, then by considering more terms of the Bessel expansion, one can identify $\text{ind}(u, \nu)$.

If $p$ is an isolated critical point of an eigenfunction $u$ that lies in the interior of a polygon $P$, then $\text{ind}(u, p)$ equals the degree of the mapping $\nabla u / |\nabla u| \circ \gamma$ as described in Remark 4.5. If $p$ lies in the interior of a side of $P$, then one may reflect a Neumann eigenfunction across the side to $\tilde{u}$, and then find that $\text{ind}(u, p)$ equals degree of the map $\nabla \tilde{u} / |\nabla \tilde{u}| \circ \gamma$.

If $p$ is a vertex, we may also interpret $\text{ind}(u, v)$ in terms of the degree of the self-map of the circle induced by a vector field. Indeed, let $D$ be a disk centered at $p$ that intersects no sides of $P$ other than the side(s) adjacent to $p$ and so that $\partial D \setminus \{p\}$ contains no critical points of $u$ other than possibly $p$. By applying a rigid motion we may assume that $p = 0$ and that $D \cap P$ lies in the sector $S$ bounded by the rays $\theta = 0$ and $\theta = \beta$. Moreover, by rescaling if necessary, we may assume that $D$ is the unit disk. The map $z \mapsto z^\beta$ maps $H = \{z \in \mathbb{C} : |z| < 1 \text{ and } y > 0\}$ to the sector $D \cap P$. In particular, the function $w(z) = u(z^\beta)$ is defined on $H$. If $u$ is given by (3), then

$$w(r \cdot e^{i\theta}) = \sum_{j=0} c_j \cdot r^j \cdot g_{j,\nu}(r^2) \cdot \cos(j \cdot \theta). \quad (20)$$

We may extend $w$ smoothly to $D \setminus \{0\}$ by setting $w(z) = w(\bar{z})$.

Lemma 4.10. The degree of the restriction of $\frac{\nabla w}{|\nabla w|}$ to the unit circle equals $2 \cdot \sum \text{ind}(u, q)$ where the sum is over critical points $q$ of $u$ that lie in $D$.

Proof. Suppose $q \neq 0$ is a critical point of $u$. If $q$ lies in the interior of $P$, then $q$ corresponds to two critical points $q_+$ and $q_-$ of $w$ which have the same indices as $q$. By Remark 4.5, since $w$ is smooth at $q_\pm$, the index $\text{ind}(w, q_\pm)$ equals the degree of the restriction of $\nabla w / |\nabla w|$ to a small circle centered at $q_\pm$. If $q \neq 0$ lies on the boundary of $P$, then $q$ corresponds to a single critical point $q'$ of $w$, and $\text{ind}(w, q')$ equals the degree of $\nabla w / |\nabla w|$ on a small circle centered at $q'$. By choosing disjoint circles, and applying a standard argument\(^6\), we find that it suffices to assume that $u$ has no critical points in $\partial D \setminus \{0\}$.

Since $u$ has no critical points in $\partial D \setminus \{0\}$, the function $w$ has no critical points in $\partial D \setminus \{0\}$. In particular, since $\partial w$ vanishes on the real line, it follows that $w(z) \neq w(0)$ for each $z \neq 0$ on the real axis. Hence the number of arcs in $\{z : w(z) = w(0)\}$ that emanate from 0 equals twice the number of arcs in $\{z : u(z) = u(0)\}$ that emanate from 0. Thus, to complete the proof, it suffices to show that the degree of $\frac{\nabla w}{|\nabla w|} \circ \gamma$ where $\gamma$ is the unit circle equals $1 - n/2$ where $n$ is the number of arcs of $\{z : w(z) = w(0)\}$ that emanate from 0.

\(^6\)See for example, the proof of Proposition 20.2 in [12]
Let \( h(z) := w(z) - w(0) \) and let \( k \) be the smallest positive integer such that \( c_k \neq 0 \). Then
\[
\partial_\theta h(r \cdot e^{i\theta}) = -c_k \cdot k \cdot r^{k-1} \cdot g_{k,v}(\dot{r}) \cdot \sin(k \cdot \theta) + O(r^k),
\]
and so there exists \( r_0 > 0 \) so that if \( 0 < r \leq r_0 \), then the set \( \{ \theta : \partial_\theta h(r \cdot e^{i\theta}) = 0 \} \) consists of exactly \( 2k \) elements, \( \theta_0(r), \ldots, \theta_{2k-1}(r) \). Using the implicit function theorem, we find that, for each \( j \), the map \( r \mapsto \theta_j(r) \) is smooth. By relabeling if necessary, we may assume that \( \lim_{r \to 0} \theta_j(r) = j \cdot \pi/k \). The function \( h \) has no critical points in \( \vec{D} \setminus \{ 0 \} \), and so the degree of \( \nabla h/|\nabla h| \circ \gamma \) equals the degree of the map \( \nabla h/|\nabla h| \circ \gamma_0 \) where \( \gamma_0 \) is the standard counterclockwise parameterization of \( r = r_0 \).

Choose a homeomorphism \( \psi : \vec{D} \to \vec{D} \) that is isotopic to the identity map, that is smooth away from 0, and that maps each ray \( \theta = j \cdot \pi/k \) to the arc \( \theta_j \). Then if we define \( \vec{h}(z) = h \circ \psi \), then the degree of \( \nabla \vec{h}/|\nabla \vec{h}| \circ \gamma \) equals the degree of \( \nabla h/|\nabla h| \circ \gamma_0 \) and \( \text{ind}(\vec{h},0) = \text{ind}(h,0) \).

Let \( j \in \{ 1, \ldots, 2k \} \) and let \( \theta_j = j \pi/k \). Since \( \vec{h} \) has no critical points in \( \vec{D} \setminus \{ 0 \} \), the mean value theorem implies that \( r \mapsto |\vec{h}(re^{i\theta})| \) is strictly increasing and thus \( \vec{h}(re^{i\theta}) \neq 0 \) for each \( r \in (0,r_0) \). Let \( \epsilon_j \in \{ +1, -1 \} \) denote the sign of the function \( r \mapsto \vec{h}(re^{i\theta}) \). Note that \( \epsilon_j \) is also the sign of \( \partial_\theta \vec{h}(re^{i\theta}) \).

The number arcs in \( \{ z : \vec{h}(z) = 0 \} \) emanating from the number of \( j \in \{ 1, \ldots, 2k \} \) such that \( \epsilon_j \neq \epsilon_{j+1} \). Indeed, for each fixed \( r \), the restriction of \( \theta \mapsto \vec{h}(re^{i\theta}) \) to the interval \( I_j := [\theta_j, \theta_{j+1}] \) is monotone, and hence \( \theta \mapsto \vec{h}(re^{i\theta}) \) assumes the value 0 at most once, and it assumes the value 0 if and only if \( \epsilon_j \neq \epsilon_{j+1} \). In other words, \( \text{ind}(\vec{h},0) \) equals the number of \( j \) such that \( \epsilon_j \neq \epsilon_{j+1} \).

To compute the degree of \( \nabla h/|\nabla h| \circ \gamma_0 \), we first regard this map as a map \( X : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z} \). In particular, for each \( \theta \) there exists a unique \( X(\theta) \) so that \( \nabla \vec{h}/|\nabla \vec{h}|(e^{i\theta}) \) corresponds to the point \( e^{iX(\theta)} \) in the unit circle. In other words, \( X(\theta) \) is the angle between the vector \( \partial_x \vec{h} \) and \( \nabla \vec{h} \) measured counterclockwise.

We have \( \nabla h = \partial_\theta \vec{h} \cdot \partial_\theta + r^{-2} \cdot \partial_\theta \vec{h} \cdot \partial_r \). Since \( \partial_\theta \vec{h}(re^{i\theta}) = 0 \) we have \( \nabla \vec{h} = \partial_\theta \vec{h} \cdot \partial_r \), and so
\[
X(\theta_j) = \begin{cases} 
\theta_j \mod 2\pi & \text{if } \epsilon_j = +1, \\
\theta_j + \pi \mod 2\pi & \text{if } \epsilon_j = -1.
\end{cases}
\]
We also have \( \partial_\theta \vec{h}(r_0e^{i\theta}) > 0 \) if and only if \( X(\theta) \in (\theta, \theta + \pi) \mod 2\pi \), and \( \partial_\theta \vec{h}(r_0e^{i\theta}) < 0 \) if and only if \( X(\theta) \in (\theta - \pi, \theta) \). In particular, we have either \( X(\theta) \in [\theta, \theta + \pi] \) for each \( \theta \in I_j \) or \( X(\theta) \in [\theta - \pi, \theta] \) for each \( \theta \in I_j \).

If \( \epsilon_j = +1 = \epsilon_{j+1} \), then \( X(\theta_j) = \theta_j \) and \( X(\theta_{j+1}) = \theta_{j+1} \) and either \( \theta \leq X(\theta) \leq \theta + \pi \) for each \( \theta \in I_j \) or \( \theta - \pi \leq X(\theta) \leq \theta \) for each \( \theta \in I_j \). It follows that the restriction of \( X \) to \( I_j \) is homotopic to the identity map rel endpoints. Similarly, if \( \epsilon_j = -1 = \epsilon_{j+1} \), then the restriction of \( X \) to \( I_j \) is homotopic to the identity map rel endpoints.

If \( \epsilon_j = -1 \) and \( \epsilon_{j+1} = +1 \), then \( \partial_\theta \vec{h}(r_0e^{i\theta}) \geq 0 \) for each \( \theta \in I_j \), and so \( X(\theta) \in [\theta, \theta + \pi] \mod 2\pi \). We also have \( X(\theta_j) = \theta_j + \pi \mod 2\pi \) and \( X(\theta_{j+1}) = \theta_{j+1} \mod 2\pi \). It follows that \( X \) is homotopic rel endpoints to the linear map \( Y_j^+ : I_j \to \mathbb{R}/2\pi\mathbb{Z} \) defined by
\[
Y_j^+(\theta) = (1 - k) \cdot (\theta - \theta_j) + \theta_j + \pi \mod 2\pi.
\]
Similarly, if $\epsilon_j = +1$ and $\epsilon_{j+1} = -1$, then one finds that $X|_{I_j}$ is homotopic rel endpoints to the map $Y_j^- : I_j \to \mathbb{R}/2\pi\mathbb{Z}$ defined by

$$Y_j^-(\theta) = (1 - k) \cdot (\theta - \theta_j) + \theta_j \mod 2\pi.$$  

Using the identity map on $I_j$ when $\epsilon_j = \epsilon_{j+1}$ and the maps $Y_j^+$ and $Y_j^-$ when $\epsilon_j \neq \epsilon_{j+1}$, one constructs a piecewise linear map $Y : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ that is homotopic to $X$. An elementary argument shows that $Y$ is in turn homotopic to the map $Z$ defined by $Z(\theta) = (1 - \frac{2}{n}) \cdot \theta \mod 2\pi$ where $n$ is the number of $j$ such that $\epsilon_j \neq \epsilon_{j+1}$. The claim follows. \hfill \Box

**Remark 4.11.** If the angle $\beta$ at $\nu$ is not a multiple of $\pi/2$, then $k \cdot \nu \neq 2$, and the proof of Lemma 4.10 can be significantly shortened. Indeed, one can use expansion (20) as in the proof of Lemma 4.7. However, if $\beta = \pi/2$ or $3\pi/2$, then using expansion (20) is more cumbersome. See Remark 4.9.

**Definition 4.12.** Let $P_k$ be a sequence of $n$-gons and let $P$ be an $n$-gon. We will say that $P_k$ converges to $P$ if and only if there exists a sequence of homeomorphisms $\varphi_k : \bar{P} \to \bar{P}_k$ that are $C^2$ diffeomorphisms on the complement of the vertices such that $\varphi_k$ converges uniformly in $C^2$ to the identity map on each compact subset of $P$ that does not include the vertices. Given continuous functions $u_k : P_k \to \mathbb{C}$ and $u : P \to \mathbb{C}$, we will say that $u_k$ converges to $u$ if and only if $u_k \circ \varphi_k$ converges to $u$.

**Remark 4.13.** Our notion of convergence requires that the number, $n$, of vertices be constant. On the other hand, we will sometimes want to analyze a sequence $P_k$ of polygons with $n$ vertices converging to a polygon $P$ with $n - 1$ vertices. In this case, we add a vertex to some side of $P$ to obtain an $n$-gon $P'$. The ‘new’ polygon $P'$ then has $n$ vertices and the ‘new’ vertex has angle $\pi$. **Definition 4.12** can now be used to test the convergence of $P_k$ to $P'$.

It is important to note that our notion of convergence precludes the possibility that two distinct vertices of $P_k$ converge to a single vertex of $P'$. However, it would be interesting to analyze the convergence of eigenfunctions in this case.

**Proposition 4.14** (Stability of the total index). Suppose that $P_n$ is a sequence of polygons that converges to $P$, and suppose that $u_n : P_n \to \mathbb{R}$ is a sequence of Neumann eigenfunctions that converge to a Neumann eigenfunction $u : P \to \mathbb{R}$. Let $p \in P$ and suppose that $\bar{D} \subset \mathbb{C}$ is an open disk neighborhood of $p$ such that $\partial D$ contains no zeros of $\nabla u$. Let $A$ (resp. $A_n$) denotes the set of critical points of $u$ (resp. $u_n$) that lie in $D$. If $A$ and $A_n$ are finite, then for each $n$ sufficiently large

$$\sum_{q \in A} \text{ind}(u,q) = \sum_{q \in A_n} \text{ind}(u_n,q).$$

**Proof.** The gradient $\nabla u_n$ converges to $\nabla u$, and so the sets $A_n$ converges to $A$.

First we suppose that $p$ lies in the interior of $P$. Let $\gamma$ be a counterclockwise parameterization of $\partial D$. By Proposition 20.2 in [12], we have that $\sum_{q \in A} \text{ind}(u,q) = \text{deg}(\nabla u/|\nabla u| \circ \gamma)$ and $\sum_{q \in A_n} \text{ind}(u_n,q) = \text{deg}(\nabla u_n/|\nabla u_n| \circ \gamma)$. But the vector field
\[ \nabla u_n / |\nabla u_n| \circ \gamma \] converges to \[ \text{deg}(\nabla u / |\nabla u| \circ \gamma) \], and hence the degrees converge. Since the degree is an integer invariant, the degrees coincide for all sufficiently large \( n \).

If \( p \) lies on the boundary of \( P \), then we apply Lemma 4.10. We have \( \sum_{q \in A} \text{ind}(u, q) = \text{deg}(\nabla w / |\nabla w| \circ \gamma) \) and \( \sum_{q \in A_n} \text{ind}(u_n, q) = \text{deg}(\nabla w_n / |\nabla w_n| \circ \gamma) \) where \( w \) and \( w_n \) are constructed as in (20). Since \( w_n \) converges to \( w \), the degree of \( \nabla w_n / |\nabla w_n| \circ \gamma \) converges to the degree of \( \nabla w / |\nabla w| \circ \gamma \). The claim follows from Lemma 4.10.

Lemma 4.15. Let \( P_n \) be a sequence of polygons that converges to a polygon \( P \) and let \( u_n : P_n \to \mathbb{R} \) be a sequence of Neumann eigenfunctions that converge to a Neumann eigenfunction \( u : P \to \mathbb{R} \). If \( u \) has finitely many nonzero index critical points, then there exists \( N \) such that if \( n > N \), then the number of nonzero index critical points of \( u_n \) is greater than or equal to the number of critical points of \( u \).

**Proof.** Let \( p \) be a nonzero index critical point of \( P \). In particular, \( p \) is isolated, and so there exists a disk neighborhood \( D_p \) of \( p \) that contains no critical points of \( u \) other than \( p \). Since \( \text{ind}(u, p) \neq 0 \), Proposition 4.14 implies that, for sufficiently large \( n \), at least one nonzero index critical points of \( u_n \) lies in \( D_p \). Since the various disks \( D_p \) are disjoint, the claim follows.

Lemma 4.16. Let \( u \) be a nonconstant Neumann eigenfunction on a polygon \( P \). If the set of critical points of \( u \) is discrete, then each local extremum \( p \) of the restriction \( u|_{\partial P} \) is a critical point of \( u \).

**Proof.** Suppose that \( p \) lies in the interior of a side \( e \) of \( p \). Since \( p \) is a local extremum of \( u|_{\partial P} \), we have \( L_e u(p) = 0 \). Thus, since \( u \) satisfies Neumann conditions at \( p \), we have \( \nabla u = 0 \).

5. Index zero critical points on a side of a polygon

In this section \( u \) is a Neumann eigenfunction on a polygon \( P \), and \( p \) is an isolated critical point of \( u \) that lies in a side \( e \) of \( P \). We show in Lemma 5.1 that if \( \text{ind}(u, p) = 0 \), then the level set \( \{ z : u(z) = u(p) \} \) is a ‘cusp’ that is tangent to \( e \) (see Lemma 5.1). We then use this to show that if the nodal set of \( Xu \), where \( X \) is either a rotational or constant vector field, has a degree 1 vertex, then the vertex is a critical point with nonzero index (Proposition 5.2).

By applying a rigid motion to \( P \) we may assume that \( p = 0 \) and that the side that contains \( p \) lies in the real axis.

Lemma 5.1. Suppose that \( p \) is an index zero critical point of a Neumann eigenfunction \( u \) that belongs to the side \( e \). Then there exist real-analytic functions \( c : \mathbb{C} \to \mathbb{R} \) and \( \rho : \mathbb{R} \to \mathbb{R} \) and an odd integer \( k \geq 3 \) so that \( c(0) \neq 0 \), \( \rho(0) \neq 0 \), and

\[ u(z) = u(0) + c(z) \cdot (y^2 - x^k \cdot \rho(x)). \]  \hspace{1cm} \text{(21)}

**Proof.** Because the index of the critical point \( p \) of \( u \) equals zero, the Hessian of \( u \) at \( p \) has exactly one nonzero eigenvalue. Indeed, the Morse lemma implies that
nondegenerate critical points have Poincaré-Hopf index equal to 1 or −1. Hence p is
degenerate and zero is an eigenvalue of the Hessian of u at p. If the associated eigen-
space were two-dimensional, then ∆u(p) = 0, and so p would be a nodal critical point.
But the result of [13] shows that a nodal critical point cannot have index zero.
Therefore, zero is not the only eigenvalue of the Hessian.

The eigenspace E that corresponds to the nonzero eigenvalue is invariant under the
reflection z → ¯z. Thus E is either the real or the imaginary axis. We claim that E is not
the real axis. Indeed, suppose to the contrary that E is the real axis. Then ∂xu(0) = 0
but ∂2x u(0) ≠ 0. The Weierstrass preparation theorem applies to provide unique real-
analytic functions a, b1, and b2 defined near 0, so that a(0) ≠ 0, b1(0) = 0 = b2(0), and

\[ u(z) - u(0) = a(z) \cdot (x^2 + b_1(y) \cdot x + b_2(y)) \]

for z near p = 0. Since the factorization is unique and u(¯z) = u(z), we have b_j(y) =
b_j(−y) for j = 1, 2. In particular, the discriminant D(y) := b_1(y)^2 − 4 ⋅ b_2(y) is an even
function. If D were to vanish on a neighborhood of 0, then we would have u(z) −
u(0) = a(z) ⋅ (x + b_1(y)/2)^2 and hence

\[ \nabla u(z) = \left( x + \frac{b_1(y)}{2} \right)^2 \cdot \nabla a(z) + 2a(z) \cdot \left( x + \frac{b_1(y)}{2} \right) \cdot \nabla \left( x + \frac{b_1(y)}{2} \right). \]

Thus, \nabla u would vanish along the level set of u that contains p = 0, but by assumption
p = 0 is an isolated zero of \nabla u. Since D is even and \text{ind}(u, 0) ≠ 1, it follows that D(y) > 0
for y ≠ 0 sufficiently small, and hence there exists a neighborhood U of p = 0 so
that the intersection of \text{u}⁻¹(u(p)) − {p} and U consists of four arcs. This contradicts
the assumption that p is a zero index critical point of u.

Therefore E coincides with the imaginary axis. By use of the Weierstrass preparation
theorem we find that

\[ u(z) - u(0) = a(z) \cdot (y^2 + b_1(x) \cdot y + b_2(x)) \]

for unique real-analytic functions a, b1, and b2 defined near 0 where a(0) ≠ 0 and
b_1(0) = 0 = b_2(0). Since the factorization is unique and u(¯z) = u(z), we find that
b_1(x) = 0. Because p is an isolated critical point, there exists ε > 0 so that b_2(x) ≠ 0 if
0 < |x| < ε. We claim that, moreover, b_2(x) ⋅ b_2(−x) < 0 if 0 < |x| < ε. Indeed, other-
wise b_2(x) ⋅ b_2(−x) > 0, and thus from (22) we find that there exists a neighborhood U
of p = 0 so that the intersection of \text{u}⁻¹(u(p)) − {p} and U consists of four arcs. This
contradicts the assumption that p is a zero index critical point of u.

Since b_2(x) ⋅ b_2(−x) < 0 the first nonzero term in the Taylor series of b_2 about zero
has odd degree k, and since ∂xu(0) = 0 we also have k ≥ 3. The claim follows.

\[ \square \]

**Proposition 5.2.** Let X be either a constant vector field or a rotational vector field. If p is
a degree 1 vertex of \mathcal{Z}(Xu) that is not a vertex of P, then p is a critical point with non-
zero index.

**Proof.** Since p is not a vertex, p lies in the interior of a side \text{e}. Since p is a degree 1 ver-
tex of \mathcal{Z}(Xu), the vector \text{X}(p) is independent of the normal vector to \partial P at p, and in
particular p is a critical point of u. It remains to show that p has nonzero index.
As above, we may suppose without loss of generality that $e$ lies in the real-axis and that $p = 0$. We will consider the case in which $X$ is a constant vector field of the form $X = \cos(\psi)\partial_x + \sin(\psi)\partial_y$ where $\psi \neq \pi/2 \mod \pi$.

Suppose to the contrary that the index of $p$ were to equal zero. Then by Lemma 5.1, near $p$, the function $u$ would satisfy (21) where $c(0) \neq 0 \neq c(0)$ and $k \geq 3$ is odd. Direct computation shows that $\partial_y X u(0) = 2\cos(\psi) \cdot c(0) + \sin(\psi) \partial_y X u(0) \neq 0$. Thus, by the implicit function theorem, there exists a function $f : (-\epsilon, \epsilon) \to \mathbb{R}$ so that $f(0) = 0$ and

$$X u(x + i \cdot f(x)) = 0.$$  

From (21), we find that for each real $x$

$$(Xu)(x) = -\cos(\psi) \cdot c(0) \cdot k \cdot x^{k-1} + O(|x|^k).$$  

(24)

By repeatedly differentiating (23) with respect to $x$ and using (24) we find that $\partial_j^x f(0) = 0$ for each $j < k - 1$ and $\partial_j^{k-1} f(0) \neq 0$. Since $k - 1$ is even and greater than 0, the function $f$ is of one sign in a deleted neighborhood of 0. Thus, there exists a neighborhood $U$ of $p = 0$ such that $(U \cap \mathcal{Z}(Xu)) \{p\}$ intersects $p$ in either no arcs or two arcs. Hence $p$ is not a degree 1 vertex, a contradiction.

The following Lemma follows from the discussion in §7 of [6]. We provide a statement and proof for the convenience of the reader.

**Lemma 5.3.** Let $u$ be a second Neumann eigenfunction on a polygon $P$, and let $p$ be an index zero critical point that lies in the side $e$. If $L_e$ is a constant vector field that is parallel to $e$, then $\mathcal{Z}(L_e u)$ intersects the interior of $P$ and has at least two degree 1 vertices in $\partial P \setminus e$.

**Proof.** By Lemma 5.1, $u$ has the expression

$$u(x, y) = u_{00} + u_{02} \cdot y^2 + u_{30} \cdot (x^3 - 3xy^2) + O(|z|^3)$$

in a neighborhood of $p$, where $u_{02} \neq 0$. If $u_{30} = 0$ then Proposition 7.4 of [6] provides the claim. If $u_{30} \neq 0$ then the first paragraph of Proposition 7.6 of [6] provides the claim. \square

### 6. Critical points of second Neumann eigenfunctions on simply connected polygons

In this section, we restrict attention to a polygon $P$ that is simply connected and to an eigenfunction $u$ that is associated to the second Neumann eigenvalue.

**Proposition 6.1.** The nodal set $\mathcal{Z}(u)$ is a simple arc whose intersection with $\partial P$ consists of its two endpoints. Moreover, the endpoints of this arc lie in distinct sides of $P$, and $\mathcal{Z}(u)$ does not contain any critical point of $u$.

**Proof.** The first statement is a well-known consequence of Courant’s nodal theorem and Polya’s inequality.\(^7\) The second statement follows from Lemma 3.3 [6] and Theorem 2.5 in [13]. \square

\(^7\)See, for example, Theorem 5.2 [5].
Proposition 6.2. Let \( p \) be a critical point of a second Neumann eigenfunction \( u \) that is not a vertex. Then the index \( \text{ind}(u, p) \) equals either 1, 0, or \(-1\).

Proof. Let \( \tilde{u} \) be the lift of \( u \) to the double \( DP \), and let \( \tilde{p} \in DP \) correspond to \( p \). If \( \text{ind}(u, p) < -1 \), then more than four arcs in \( \tilde{u}^{-1}(\tilde{u}(\tilde{p})) \) emanate from \( \tilde{p} \). It follows that, in the natural coordinates at \( \tilde{p} \), we have \( \tilde{u}(z) - \tilde{u}(\tilde{p}) = o(|z - \tilde{p}|^2) \). In particular, the degree two homogeneous polynomial \( h_2 \) consisting of second order terms in the Taylor expansion of \( \tilde{u} \) at \( p \) vanishes identically. But \( (Dh_2)(\tilde{p}) = \mu \cdot \tilde{u}(\tilde{p}) \) and so \( u(\tilde{p}) = 0 \). This contradicts Proposition 6.1. \( \square \)

Next we consider the possible indices of a critical point of \( u \) that lies at a vertex of \( P \). Let \( c_k \) be the coefficient in the Bessel expansion (3) at a point \( p \in \partial M \)

Proposition 6.3. Let \( p \in \partial P \). Either \( c_0 \neq 0 \) or \( c_1 \neq 0 \).

Proof. If \( c_0 = 0 = c_1 \), then by inspecting (3) one finds that at least two nodal arcs emanate from \( p \). This contradicts Proposition 6.1. \( \square \)

Corollary 6.4. Let \( v \) be a vertex whose angle \( \beta \) is not a multiple of \( \pi/2 \).

\begin{enumerate}
  \item If \( c_0 = 0 \), then \( \text{ind}(u, v) = 0 \).
  \item If \( \beta < \pi/2 \), then \( c_0 \neq 0 \) if and only if \( v \) is a local extremum.
  \item If \( \beta < \pi \) and \( c_1 = 0 \), then \( v \) is a local extremum.
  \item If \( \pi/2 < \beta < \pi \), then \( c_1 = 0 \) if and only if \( v \) is a local extremum.
  \item If \( \beta < \pi \), then \( \text{ind}(u, v) = 0 \) or \( \text{ind}(u, v) = 1 \).
\end{enumerate}

Proof. Let \( k \) be the smallest positive integer such that \( c_k \neq 0 \). If \( c_0 = 0 \), then by Proposition 6.3, we have \( c_1 \neq 0 \), and hence by Lemma 4.7 we have \( \text{ind}(u, v) = 0 \).

If \( \beta < \pi/2 \) and \( c_0 \neq 0 \), then we are in case (ii) of Lemma 4.7, and hence \( \text{ind}(u, v) = 1 \). By Proposition 4.6 we have \( \text{ind}(u, v) = 1 \) if and only if \( v \) is a local extremum.

Suppose that \( \pi/2 < \beta < \pi \). If \( c_1 = 0 \), then Proposition 6.3 implies that \( c_0 \neq 0 \), and so part (ii) of Lemma 4.7 implies that \( \text{ind}(u, v) = 1 \). If \( c_1 \neq 0 \), then part (i) of Lemma 4.7 implies that \( \text{ind}(u, v) = 0 \). \( \square \)

Corollary 6.5. Suppose that \( v \) is an acute vertex of \( P \) contained in the side \( e \). If \( v \) is not a local extremum, then \( Z(L_c u) \) has an arc that ends at \( v \).

Proof. This follows from Corollary 2.3 and Corollary 6.4. \( \square \)

7. No hot spots on certain polygons with two acute vertices

Bañuelos and Burdzy \cite{14} used probabilistic methods to show that the second Neumann eigenfunction \( u \) of an obtuse triangle has no interior critical points. In \cite{6, 7}, we used a variational approach to show that the two acute vertices are the only critical points of \( u \) and hence they are the global extrema of \( u \). In this section, we extend this latter result to a large class of \( n \)-gons that have two acute vertices. At this end of the
section we identify this class of polygons as those that satisfy the Lip-1 condition of [5] and which have no orthogonal sides.

**Lemma 7.1.** Let $u$ be a second Neumann eigenfunction on a simply connected polygon $P$ with at least one acute vertex. If $u$ has an interior critical point, then either $u$ has at least four nonzero index critical points or there is a side $e$ such that $Z(L_e u)$ has an arc that ends at a vertex of $P$.

**Proof.** Suppose that for each side $e$, the nodal set $Z(L_e u)$ does not have an arc that ends at a vertex. Thus, if $v$ is a vertex and $e$ is a side containing $v$, then Corollary 2.3 implies that the leading Bessel coefficient of $u$ at $v$ is nonzero. In particular, each acute vertex has index $+1$ by Corollary 6.4, and each obtuse vertex is not a critical point by Proposition 4.8.

Since $u$ has a critical point in the interior of $P$, for any side $e$, the nodal set $Z(L_e u)$ has at least two degree 1 vertices in $\partial P$. Since $Z(L_e u)$ does not have an arc that ends at a vertex of $P$, each of these degree 1 vertices is a non-vertex point on $\partial P$.

By Proposition 5.2, each of these degree 1 vertices is a nonzero index critical point of $u$.

Thus, since $P$ has at least one acute vertex, $u$ has at least three nonzero index critical points on $\partial P$.

In the following we consider paths $P_t$ of polygons with $n$ vertices where the topology on the space of $n$-gons is given by Definition 4.12. Recall from Remark 4.13 that, by adding a vertex to the side of a polygon, we can consider a polygon with $n-1$ vertices as a polygon with $n$ vertices. For example, a triangle $T$ may be regarded as a quadrilateral if we declare that some point $p$ on a side of $T$ is a vertex that has angle $\pi$.

Let $u_t$ be a path of second Neumann eigenfunctions associated to the path $P_t$. For each $t$, let $V(t)$ denote the number of vertices $v$ of $P_t$ with angle not equal to $\pi$ such that there exists a side $e$ of $P_t$ so that an arc in $Z(L_e u_t)$ ends at $v$. Let $S(t)$ denote the number of nonzero index critical points of $u_t$.

**Lemma 7.2.** Suppose that $P_t$ is a path of $n$-gons such that no two sides of $P_t$ are orthogonal and $P_t$ has exactly two acute vertices for each $t \in [0, 1]$. Let $u_t$ be an associated path of eigenfunctions. If $S(0) \geq 3$ or $V(0) \geq 1$, then either $S(1) \geq 3$ or $V(1) \geq 1$.

**Proof.** It suffices to show that the set, $A$, of $t \in [0, 1)$ such that either $S(t) \geq 3$ or $V(t) \geq 1$ is both open and closed in $[0, 1)$.

$(A$ is open) If $S(t) \geq 3$, then Lemma 4.15 implies that there exists $\epsilon > 0$ such that if $|s - t| < \epsilon$, then $S(s) \geq 3$. Hence, to prove openness, it suffices to assume that $V(t) \geq 1$, and show that there exists $\epsilon > 0$ so that if $|s - t| < \epsilon$ then either $V(s) \geq 1$ or $S(s) \geq 3$.

If $V(t) \geq 1$, then there exists a vertex $v$ of $P_t$, a side $e$ of $P_t$, and an arc in $Z(L_e u)$ that ends at $v$.

If the leading coefficient at $v$ is nonzero then for $s$ near $t$ the leading coefficient at the corresponding vertex is also nonzero. Since no two sides of $P_t$ are orthogonal and
since the corresponding edge and sector vary continuously in \( s \), we find from Lemma 2.1 that \( V(s) \geq 1 \) for each \( s \) near \( t \). Therefore, we may assume that there exists a vertex \( v \) such that the leading coefficient of \( u_t \) at \( v \) equals zero.

We may assume without loss of generality that \( v \) is an acute vertex. Indeed, if \( v \) were obtuse with \( c_1(t) = 0 \), then Lemma 4.7 would imply that \( v \) is a critical point with nonzero index. If \( c_0(t) \) were not to vanish at each of the two acute vertices, then Lemma 4.7 would imply that each of these vertices have index equal to one.

Hence, \( S(t) \geq 3 \), and so \( S(s) \geq 3 \) for \( s \) near \( t \) by Lemma 4.15. Thus, we may assume that \( v \) is acute.

Suppose that \( c_0(t) = 0 \) at an acute vertex \( v \). Corollary 6.4 implies that \( \text{ind}(u, v) = 0 \). By Proposition 4.16, the eigenfunction \( u_t \) has at least two nonzero index critical points. Thus, it follows from Lemma 4.15 that there exists \( \epsilon > 0 \) such that if \( |s - t| < \epsilon \), then there exist two nonzero index critical points of \( u_s \) that are distinct from \( v \). Suppose that \( 0 < |s - t| < \epsilon \). If \( c_0(s) \neq 0 \), then, since \( v \) is acute, Corollary 6.4 implies that \( \text{ind}(u_s, v) \neq 0 \), and hence \( S(s) \geq 3 \). On the other hand, if \( c_0(s) = 0 \), then Corollary 6.5 implies that \( Z(L_\circ u_s) \) has an arc that ends at \( v \) where \( e \) is a side adjacent to \( v \). In sum, if \( |s - t| < \epsilon \), then either \( S(s) \geq 3 \) or \( V(s) \geq 1 \).

(A is closed) By assumption, for each \( t \in [0, 1) \), no two sides of \( P_t \) are orthogonal, and so the set of \( t \) such that \( V(t) \geq 1 \) is closed by Lemma 2.1. Suppose that \( S(t_n) \geq 3 \) with \( t_n \to t \). To prove that \( A \) is closed it suffices to show that either \( S(t) \geq 3 \) or \( V(t) \geq 1 \).

If the eigenfunction \( u_t \) has an interior critical point, then Lemma 7.1 implies that \( S(t) \geq 4 \) or \( V(t) \geq 1 \). Thus, we may assume that \( u_t \) has no interior critical points. By Proposition 4.16, the two index 1 critical points, \( p^+ \) and \( p^- \) lie in \( \partial P \). Suppose that there exists a third critical point \( p \). If the index of \( p \) is nonzero, then \( S(t) \geq 3 \). Thus, in the following we assume that \( \text{ind}(u, p) = 0 \).

If some acute vertex \( v \) has is not a local extremum, then by Corollary 6.5 an arc of \( Z(L_\circ v) \) ends at \( v \), and so \( V(t) \geq 1 \). Thus, we may assume that each acute vertex is a local extremum. If there are three local extrema, then \( S(t) \geq 3 \). Hence we may assume that the acute vertices correspond to the index 1 critical points \( p^+ \) and \( p^- \).

Because \( S(t_n) \geq 3 \), for each \( n \) there exists a critical point \( p_n \) on a side that is distinct from \( p^+ \) and \( p^- \). Suppose that \( p_n \) converges to a vertex \( v \) of \( P_t \) whose angle does not equal \( \pi \). Then, by Lemma 3.3 the leading coefficient—\( c_0(t) \) if \( v \) is acute and \( c_1(t) \) if \( v \) is obtuse—equals zero. If \( v \) is obtuse then Corollary 4.8 implies that \( v \) is a nonzero index critical point and so \( S(t) \geq 3 \). If \( v \) is acute, then by Corollary 6.5 we have \( V(t) \geq 1 \).

Therefore, we may assume that \( p_n \) converges to a critical point \( p \) of \( u_t \) that lies in the interior of a side \( e \). Since \( p \neq p^\pm \), the critical point has index equal to zero. Thus, by Lemma 5.3, the graph \( Z(L_\circ u_t) \) intersects the interior of \( P_t \) and has at least two degree 1 vertices. If one of these degree 1 vertices equals a vertex of \( P_t \) then \( V(t) \geq 1 \). If a degree one vertex lies in the interior of a side then it is a nonzero index critical point by Proposition 5.2, and hence \( S(t) \geq 3 \) since the acute vertices \( p^\pm \) are also nonzero index critical points.

\begin{theorem}
Suppose that \( P_t \) is a path of \( n \)-gons such that no two sides of \( P_t \) are orthogonal. If \( P_1 \) is an obtuse triangle, then each second Neumann eigenfunction of \( P_0 \) has
\end{theorem}
exactly two critical points, a global maximum at one acute vertex and a global minimum at the other acute vertex. Moreover, the second Neumann eigenspace of $P_0$ is one-dimensional.

**Proof.** By the method of Lemma 12.2 of [6], one may modify the path $P_t$ so that there exists a continuous family of second Neumann eigenfunctions $u_t$ connecting any $u_0$ to any $u_1$. If $u_1$ is a second Neumann eigenfunction for an obtuse triangle $P_1$, then by [6, 7], the acute vertices are the only critical points of $u_1$, and in particular each is a global extremum and $S(1) = 2$. Thus Proposition 4.6 and Corollary 6.4 imply that the coefficient $c_0$ of $u_1$ at each acute vertex is nonzero. Given an acute vertex $v$, the angle between the opposite side and one of the sides adjacent to $v$ is greater than $\pi/2$. Hence it follows from Lemma 2.1 that for each side $e$ of $P_1$ there does not exist an arc in $Z(L_e u_1)$ that ends at an acute vertex. The obtuse vertex is not a local extremum and hence $c_1$ of $u_1$ at this vertex is nonzero. Thus, it follows from Lemma 2.1 that for each side $e$ of $P_1$, no arc of $Z(L_e u_1)$ ends at the obtuse vertex. In sum, $S(1) = 2$ and $V(1) = 0$.

Thus, Lemma 7.2 implies that $S(0) = 2$ and $V(0) = 0$. In particular, $u_0$ has exactly two nonzero index critical points and these are necessarily the global extrema of $u_0$. Each global extremum must be an acute vertex. Indeed if an acute vertex $v$ were not a local extremum, then by Corollary 6.5 we would have that $Z(L_e v)$ has an arc that ends at $v$ where $e$ is a side adjacent to $v$, contradicting $V(0) = 0$.

Suppose that there exists a critical point $p$ of $u_0$ that were distinct from the acute vertices. Then $p$ has index zero and lies in a side of $P_0$. Thus, $p$ lies in the interior of a side $e$ of $P$, and hence by Lemma 5.3, the graph $Z(L_e u_0)$ intersects the interior of $P_t$ and has at least two degree 1 vertices. If a degree 1 vertex $p$ lies in the interior of a side, then $\text{ind}(u, p) \neq 0$ by Proposition 5.2, a contradiction. Therefore, the acute vertices are the only critical points of $u_0$.

Finally, we show that the second Neumann eigenspace of $P_0$ is one-dimensional. Let $u_+$ and $u_-$ be second Neumann eigenfunctions of $P_0$ and let $v$ be an acute vertex. Then there exist $a_+, a_- \in \mathbb{R}$ so that $a_+ \cdot u_+(v) + a_- \cdot u_-(v) = 0$. We claim that $u^* := a_+ \cdot u_+ + a_- \cdot u_- \equiv 0$. Indeed, if not then $u^*$ would be a second Neumann eigenfunction and in particular would be orthogonal to the constant functions. Thus both the maximum value and the minimum value of $u$ would be nonzero. But by Theorem 7.3, the acute vertex $v$ is a global extremum of $u^*$ and hence we have a contradiction.

We now show that the set of polygons that satisfy the hypotheses of Theorem 7.3 is the interior of the set of polygons that satisfy the Lip-1 condition of [5]. First we recall the notion of Lip-K domain. Let $f_+ : [-b, b] \to \mathbb{R}$ and $f_- : [-b, b] \to \mathbb{R}$ be a pair of Lipschitz functions such that

- $f_+(\pm b) = f_-(\pm b)$,
- $f_-(x) < f_+(x)$ for $x \in (-b, b)$, and
- the Lipschitz constant of $f_\pm$ is at most $K$.

The domain $\{(x, y) : f_-(x) < y < f_+(x)\}$ is called a Lip-K domain.
Recall that if \( \Omega \) is a domain with Lipschitz boundary \( \partial \Omega \) then the outward unit normal vector \( \nu(p) \) is defined for almost every \( p \in \partial \Omega \).

**Proposition 7.4.** A simply connected Lipschitz domain \( \Omega \) is isometric to a Lip-1 domain if and only if there exists a partition of \( \partial \Omega \) into two connected sets \( \Gamma^+ \) and \( \Gamma^- \) so that if \( p, p' \in \Gamma^\pm \) then \( \nu(p) \cdot \nu(p') \geq 0 \) and if \( p \in \Gamma^+ \) and \( q \in \Gamma^- \) then \( \nu(p) \cdot \nu(q) \leq 0 \).

**Proof.** \((\Rightarrow)\) After applying an isometry, we may suppose that \( \Omega \) is bounded by the graphs of the Lip-1 functions \( f_+ \) and \( f_- \) as above. Let \( \Gamma^+ \) be the graph of \( f^+ \) and let \( \Gamma^- \) be the graph of \( f^- \). Suppose that \( \nu(p) = (x,y) \). Since \( f^+ \) is Lip-1 we have that \( p \in \Gamma^+ \) implies that \( y > |x| \), and since \( f^- \) is Lip-1 we have that \( p \in \Gamma^- \) implies that \( y < -|x| \). It follows that if \( p, p' \in \Gamma^\pm \) then \( \nu(p) \cdot \nu(p') \geq 0 \) and if \( p \in \Gamma^+ \) and \( q \in \Gamma^- \) then \( \nu(p) \cdot \nu(q) \leq 0 \).

\((\Leftarrow)\) Let \( p^+_n \in \Gamma^+ \) and \( p^-_n \in \Gamma^- \) be sequences such that \( \lim_{n \to \infty} \nu(p^+_n) \cdot \nu(p^-_n) \) equals the supremum of \( \{ \nu(p) \cdot \nu(q) : p \in \Gamma^+, \ q \in \Gamma^- \} \). Let \( w \) be a limit point of the sequence \( (\nu(p^+_n) - \nu(p^-_n))/|\nu(p^+_n) - \nu(p^-_n)| \). A computation shows that for each \( p \in \Gamma^+ \) we have \( \nu(p) \cdot w \geq 1/\sqrt{2} \) and for each \( p \in \Gamma^- \) we have \( \nu(p) \cdot w \leq -1/\sqrt{2} \). Choose coordinates in the plane so that the vector \( w \) is the vector \( (0,1) \). Then for each \( p \in \Gamma \) we have \( \nu(p) = (x,y) \) where \( y \geq |x| \). From this it follows that \( \Gamma^+ \) is the graph of a Lip-1 function \( f_+ : [a_+, b_+] \to \mathbb{R} \). Similarly, \( \Gamma^- \) is the graph of a Lip-1 function \( f_- : [a_-, b_-] \to \mathbb{R} \). Because \( \Gamma^+ \) and \( \Gamma^- \) form a partition of \( \partial \Omega \) we have \( f_+(a_+) = f_-(a_-) \) and \( f_+(b_+) = f_-(b_-) \). Because \( \nu(p) \) is the outward normal vector for a domain we have \( f_+ > f_- \).

**Corollary 7.5.** A triangle \( T \) is a Lip-1 domain if and only if \( T \) is not an acute triangle.

**Proof.** Let \( e_1, e_2, e_3 \) be the sides of the triangle and let \( \nu_1, \nu_2 \) and \( \nu_3 \) be the associated outward normal vectors. The angle between \( e_1 \) and \( e_2 \) is acute if and only if \( \nu_1 \cdot \nu_2 < 0 \). The claim follows from Proposition 7.4.

**Proposition 7.6.** Suppose that \( P_t \) is a path of polygons such that no two sides of \( P_t \) are orthogonal and \( P_0 \) is isometric to a Lip-1 domain. Then each \( P_t \) is also isometric to a Lip-1 domain.

Note that we are allowing for the possibility that some vertices have angle \( \pi \) for some \( t \).

**Proof.** Since \( P_0 \) is a Lip-1 domain, there exists a partition \( \{ \Gamma^+, \Gamma^- \} \) of \( \partial P_0 \) that satisfies the criteria of Proposition 7.4. In particular, \( \Gamma^+ \) is the union of sides with outward unit normal vectors \( \nu^+_j(0), ..., \nu^+_j(0) \), the set \( \Gamma^- \) is the union of sides with outward unit normal vectors \( \nu^-_j(0), ..., \nu^-_j(0) \), and these normal vectors satisfy \( \nu^+_j(0) \cdot \nu^-_j(0) \geq 0 \) and \( \nu^+_j(0) \cdot \nu^-_j(0) \leq 0 \). Since no two sides of \( P_0 \) are orthogonal, each inequality is strict. The quantities \( \nu^+_j(t) \cdot \nu^+_j(t) \) and \( \nu^+_j(t) \cdot \nu^-_j(t) \) depend continuously in \( t \) and cannot vanish since no two sides of \( P_t \) are orthogonal. Thus the inequalities persist for all \( t \), and thus each \( P_t \) is a Lip-1 domain by Proposition 7.4.
Proposition 7.7. If $P$ is a Lip-1 polygonal domain with no two sides orthogonal, then there exists a path $P_t$ of polygons with no two sides orthogonal such that $P_1 = P$ and $P_0$ is an obtuse triangle.

Proof. We will argue via induction on the number, $n$, of sides of $P$. If $n = 3$, then the claim follows from 7.5. Suppose that the claim is true if a Lip-1 polygon has $n$ sides no two of which are orthogonal. Let $P$ be a Lip-1 polygon with $n+1$ sides such that no two sides are orthogonal. Proposition 7.4 implies that the sides of $P$ can be partitioned into sides $e_1^+, ..., e_n^+$ and $e_1^-, ..., e_n^-$, so that the associated outward unit normal vectors $\nu_1^+, ..., \nu_n^+$ and $\nu_1^-, ..., \nu_n^-$ satisfy the inequalities $\nu_i^+ \cdot \nu_j^- > 0$ and $\nu_i^- \cdot \nu_j^+ < 0$. Because $P$ has nonempty interior, by relabeling if necessary, we may assume that $\nu_1^+ \neq \nu_2^+$ and the sides $e_1^+$ and $e_2^+$ are adjacent. Let $v$ be the vertex shared by $e_1^+$ and $e_2^+$, and let $v'$ be the midpoint of the segment that joins the other two vertices of $e_1^+$ and $e_2^+$. Define $P_t$ to be the polygon obtained from $P$ by replacing $v$ with $v_t = (1-t) \cdot v + t \cdot v'$. A straightforward computation shows that both $n_1^+(t)$ and $n_2^+(t)$ are convex combinations of $n_1^+$ and $n_2^+$, and so it follows that $P_t$ is a Lip-1-polygon with no orthogonal sides. The polygon $P_1$ may be regarded as a Lip-1 polygon with only $n$ sides no two of which are orthogonal. Thus, by the inductive hypothesis, we may concatenate the path $P_t$ with another path to obtain the desired path to an obtuse triangle. \qed

8. Instability via blocking

In this section we provide criteria—Proposition 8.1—that guarantee the existence of a quadrilateral with a second Neumann eigenfunction that has an unstable critical point. In §9, we will construct families of quadrilaterals that meet the criteria under the assumption that these quadrilaterals have no interior critical points.

The statement and proof of Proposition 8.1 are somewhat complicated, but the basic idea is simple: Suppose that we have a continuous family of quadrilaterals $Q_t$ with an obtuse vertex $w_t$ and sides $e_t^-$ and $e_t^+$ adjacent to $w_t$. Suppose further that for the associated family of eigenfunctions $u_t$, we know that $u_0$ (resp. $u_1$) has only one nonvertex critical point $p_0$ (resp. $p_1$), that this critical point lies on the side $e_0^-$ (resp. $e_1^+$), and that this critical point has index $-1$. One might naively expect that the index $-1$ critical point varies continuously in $t$, and therefore, for some time $t$, the critical point lies at the obtuse vertex $w_t$. However, Lemma 3.1 would then imply that $c_1 = 0$ at $w_t$, and then Corollary 6.4 would imply that $w_t$ is an index $+1$ critical point. Thus, the index of the critical point would abruptly change which is not possible by Proposition 4.14. Roughly speaking, the obtuse vertex ‘blocks’ the index $-1$ critical point.

Under additional assumptions, we show that this ‘blocking phenomenon’ implies the existence of an unstable critical point.

Proposition 8.1. Let $Q_t$ be a continuous family of quadrilaterals such that for each $t \in [0,1]$ the quadrilateral $Q_t$ has three acute vertices, and the angle of the fourth vertex, $w_t$, lies in $(\pi/2, \pi)$ for each $t \in (0,1)$. Let $e_t$ be a side of $Q_t$ that is adjacent to $w_t$ so that $t \mapsto e_t$ is continuous. Let $u_t : Q_t \to \mathbb{R}$ be a second Neumann eigenfunction, and suppose that $t \mapsto u_t$ is continuous. Suppose that
for each $t$, the eigenfunction $u_t$ has no interior critical points,
(2) for each $t$, each nonzero index critical point of $u_t$ either is a vertex or belongs to a side adjacent to $w_t$,
(3) for each $t$, each acute vertex of $Q_t$ is a local extremum of $u_t$,
(4) $u_0$ has exactly one nonvertex critical point and it belongs to the interior of $e_0$,
(5) $u_1$ has no critical points on $e_1$ except for the acute vertex.

Then there exists $t \in (0, 1)$ such that $u_t$ has an unstable critical point.

Proof. For each $t \in [0, 1]$, let $A_t$ be the set of critical points $p$ of $u_t$ such that either $p = w_t$ or $p$ lies in the interior of a side of $Q_t$ that is adjacent to $w_t$. We claim that there exists $\delta > 0$ so that for all $t$ no element of $A_t$ is within distance $\delta$ of an acute vertex. Indeed, if not, then there would exist $t \in [0, 1]$, a sequence $t_n \to t$, and a sequence of critical points $p_n$ of $u_n$ that converges to an acute vertex $v$. Lemma 3.1 would then imply that $c_0 = 0$ at $v$, but this would contradict part (3) of Corollary 6.4 and condition (3) above.

By condition (4), the set $A_0$ has exactly one element $p_0$, and it follows from Proposition 4.3 that the index of $p_0$ equals $-1$. Thus, Proposition 4.14 implies that the sum of the indices of the critical points in $A_t$ equals $-1$.

Let $t^*$ be the supremum of $t \in [0, 1]$ such that $A_s$ contains exactly one nonzero index point, $p_s$, for each $s \leq t$. It follows from Proposition 4.14 that $s \to p_s$ is continuous on $[0, t^*)$ and the index of each $p_s$ equals $-1$. Moreover, as $s \to t^*$ the point $p_s$ converges to a point $p_{t^*}$. Indeed, if $p_t$ were to have more than one limit point as $t \to t^*$, then, since $t \to p_t$ is continuous for $t < t^*$, we would have a nontrivial continuum of critical points. But since $Q_{t^*}$ is not a rectangle, the function $u_{t^*}$ has only finitely many critical points [11].

Proposition 4.14 implies that the index of $p_{t^*}$ equals $-1$. It follows that the critical point $p_{t^*}$ lies in the interior of $e_{t^*}$. Indeed, otherwise, condition (5) would imply that $p_s = w_s$ for some $s \leq t^*$. But this would contradict part (5) of Corollary 6.4.

If $A_{t^*}$ contains a critical point $q$ that is distinct from $p_{t^*}$ then $q$ is an unstable critical point since $p_s$ is the only critical point in $A_s$ for $s < t^*$. For the remainder of the proof we will suppose that $p_{t^*}$ is the only element of $A_{t^*}$.

By the definition of $t^*$, there exists a sequence $t_n \searrow t^*$ such that $A_{t_n}$ consists of more than one nonzero index critical point. Since $p_{t_n}$ is the only critical point in $A_{t_n}$ these points converge to $p_{t^*}$, and in particular for $n$ sufficiently large, the set $A_{t_n}$ lies in the interior of $e_{t_n}$. By Proposition 6.2, each nonzero index critical point has index $+1$ or $-1$. Hence since the sum of the indices equals $-1$, the set $A_{t_n}$ contains at least three critical points.

But this is impossible. Indeed, if three critical points of $u_{t_n}$ were to lie in the interior of $e_{t_n}$, then $\mathcal{Z}(L_{e_{t_n}} u_{t_n})$ would have three degree 1 vertices that lie in $\partial Q_{t_n} \setminus e_{t_n}$, and in particular some degree 1 vertex would lie in the interior of a side not adjacent to $w_{t_n}$. But this degree 1 vertex would be a nonzero index critical point by Proposition 5.2, thus contradicting (2).

9. Breaking acute triangles along a side

In this section, we will construct families of quadrilaterals that satisfy the hypotheses (2) through (5) of Proposition 8.1. The construction consists of:
(1) Producing a nonempty open set \( \mathcal{N} \) of acute triangles \( T \) such that the set of critical points of each second Neumann eigenfunction \( u \) consists only of the three vertices and an index \(-1\) critical point;

(2) ‘Breaking’ the side that contains the index \(-1\) critical point of \( T \in \mathcal{N} \) to create quadrilaterals for which each acute vertex is a critical point and for which the only sides that may contain critical points in their interior are the sides adjacent to the new obtuse vertex;

(3) Choosing a path \( w_t \) of break points so that the resulting path \( Q_t \) of quadrilaterals forces ‘blocking’ to occur.

We now provide the details of this construction. Define \( \mathcal{N} \) to be the set of acute triangles \( T \) such that if \( u \) is any second Neumann eigenfunction on \( T \), then

(1) each vertex of \( T \) is a local extremum of \( u \),

(2) \( u \) has exactly one non-vertex critical point \( p \).

(3) the critical point \( p \) is nondegenerate.

Proposition 4.3 implies that \( p \) has index equal to \(-1\). The main theorem of [7] implies that \( p \) lies on a side of \( T \). Note that equilateral triangles do not belong to \( \mathcal{N} \), and hence by a result of Siudeja [15], the second Neumann eigenspace of \( T \) is one dimensional for each \( T \in \mathcal{N} \).

**Lemma 9.1.** The set \( \mathcal{N} \) is open in the space of acute triangles.

**Proof.** By Corollary 6.4, a vertex \( v \) of an acute triangle is a local extremum if and only if \( u(v) \neq 0 \). Thus, condition (1) is open. A critical point \( p \) is nondegenerate if and only if the determinant of the Hessian at \( p \) is nonzero, and hence condition (3) is also an open condition.

Thus, if \( \mathcal{N} \) were not open, then there would exist \( T \in \mathcal{N} \) and a sequence \( T_n \) converging to \( T \) such that condition (2) is not satisfied for each \( n \). In particular, for each \( n \) there would exist a second Neumann eigenfunction \( u_n \) on \( T_n \) with distinct nonvertex critical points \( p_n \) and \( q_n \).

By passing to a subsequence if necessary, we may assume without loss of generality that \( u_n \) converges to an eigenfunction \( u \) on \( T \). Neither of the sequences \( p_n \) nor \( q_n \) can converge to a vertex of \( T \) because then, by Lemma 3.1, we would have \( c_1 = 0 \) contradicting (1). Thus, by (2), both sequences converge to the unique nonvertex critical point \( p \) of \( T \), and it would follow that \( p \) is a degenerate critical point, contradicting (3). \( \square \)

The set \( \mathcal{N} \) is also nonempty.

**Lemma 9.2.** Let \( T \) be an isosceles triangle with reflection symmetry \( \sigma \), and let \( u \) be a second Neumann eigenfunction of \( T \). If the angle of the apex vertex \( v \) fixed by \( \sigma \) is less than \( \pi/3 \), then

(1) each vertex is a local extremum of \( u \),

(2) \( u \) has exactly one non-vertex critical point \( p \), the midpoint of the side \( e \) opposite to \( v \),

(3) \( p \) is nondegenerate with index \(-1\),

(4) \( u(z) \neq 0 \) for each \( z \in e \).
**Proof.** By Lemma 3.1 in [16] the second Neumann eigenvalue of $T$ has multiplicity one, and $u$ is symmetric with respect to $\sigma$. It follows that the midpoint $p$ of the side $e$ preserved by $\sigma$ is a critical point. Let $T_+, T_- \subset T$ be the two right triangles such that $\sigma(T_+) = T_-$ and $T_+ \cup T_- = T$. Since $u$ is symmetric with respect to $\sigma$, the restriction of $u$ to $T_\pm$ is a second Neumann eigenfunction of $T_\pm$.

By Theorem 4.1 in [7], the restriction of $u$ to the right triangle $T_\pm$ has no nonvertex critical points and each acute vertex of $T_\pm$ is a local extremum. It follows that each vertex of $T$ is a local extremum of $u$ and the midpoint $p$ of $e$ is the only other critical point of $u$. Thus, Theorem 4.3 implies that $p$ has index $-1$.

Next we show that $u$ does not vanish on $e$. By Proposition 6.1, the nodal set $Z(u)$ does not contain a critical point and hence does not contain the midpoint $p$. Thus, if there did exist $z \in e$ with $u(z) = 0$, then $z \neq p$ and hence $\sigma(z) \neq z$. Since $u$ is symmetric, we would have $u(\sigma(z)) = 0$ but then $\sigma(z)$ would be a second endpoint of $Z(u)$ that lies in $e$, a contradiction. Therefore, $u$ does not vanish on $e$.

Finally, by examining the Taylor expansion of $u$ about $p$, we find that $p$ is non-degenerate. Indeed, without loss of generality, $p = 0$ and $e$ lies in the $x$-axis. Since $u \circ \sigma = u$, the restriction of $u$ to $e$ is an even function of $x$. In particular, the Taylor coefficient $a_{30} = 0$. Thus, if $p$ were degenerate, then Theorems 7.3 and 7.4 in [6] would imply that $u$ has an additional non-vertex critical point, a contradiction.

Next, we will 'break' each $T \in \mathcal{N}$ along the side that contains the index $-1$ critical point. We first give a precise definition of 'breaking': Let $T$ be a triangle with vertices $v_1, v_2, v_3$. Let $e$ be a side of $T$, let $w$ be a point that lies in the interior of $e$, and let $n_w$ be the outward pointing unit normal vector at $w$. For each $\epsilon \geq 0$, define $w(\epsilon) = w + \epsilon \cdot n_w$, and define $Q(T, w, \epsilon)$ to be the convex hull of $\{v_1, v_2, v_3, w(\epsilon)\}$. For $\epsilon > 0$, the polygon $Q(T, w, \epsilon)$ is a nondegenerate quadrilateral. We say that $Q(T, w, \epsilon)$ is the result of breaking $T$ along $e$ at the point $w$ at distance $\epsilon$.

**Lemma 9.3.** Let $T \in \mathcal{N}$ and let $e$ be the side of $T$ that contains the index $-1$ critical point. Let $K$ be a compact subset of the interior of $e$. There exists $\delta > 0$ such that if $0 \leq \epsilon < \delta$ and $w \in K$, then

a. the second Neumann eigenfunction $u$ of $Q(T, w, \epsilon)$ is unique up to scalar multiplication,

b. each acute vertex of $Q(T, w, \epsilon)$ is a local extremum of $u$,

c. if $e'$ is a side that does not contain the obtuse vertex $w$, then the interior of $e'$ does not contain a critical point of $u$.

**Proof.** The simplicity of the second Neumann eigenvalue is an open condition, and the second eigenvalue of each $T \in \mathcal{N}$ is simple by [15]. It follows that there exists $\delta' > 0$, so that (a) holds for each $Q(T, w, \epsilon)$ with $\epsilon < \delta'$ and $w \in K$. Corollary 6.4 implies that condition (b) is an open condition. In particular, $c_0 \neq 0$ at each acute vertex.

Thus, if the claim were false, then there would exist a sequence $\epsilon_n \to 0$ and $w_n \in K$ such that $Q_n := Q(T, w_n, \epsilon_n)$ has a second Neumann eigenfunction $u_n$ with a nonvertex critical point $p_n$ on a side $e'$ that does not contain $w_n(\epsilon_n)$. The sequence $Q_n$ converges...
to $T$, and thus by passing to a subsequence if necessary, we may assume that $u_n$ converges to an eigenfunction $u$ on $T$. If the sequence $p_n \in e'$ were to converge to a vertex $v$ of $T$, then Lemma 3.1 would imply $c_1 = 0$ at $v$, a contradiction. If the sequence $p_n$ converges to a point $p$ in the interior of $e'$, then $p$ is a critical point of $u$, contradicting the assumption that the ‘unbroken’ sides of $T$ contain no critical points.

Let $\delta_{T,K}$ denote the supremum of all possible $\delta$ for which the statement of Proposition 9.3 is true for the given compact set $K$.

**Lemma 9.4.** Let $T \in \mathcal{N}$ and let $e$ be the side of $T$ that contains the index $-1$ critical point $p$. Let $w_t$ be a path in the interior of $e$ so that $w_0$ and $w_1$ lie in distinct components of $e \setminus \{p\}$. If $K$ is the image of the path $w_t$, then for each $\epsilon \in (0, \delta_{T,K})$, the path $Q_t := Q(T, w_t, \epsilon \cdot \sin(t \cdot \pi))$ has an associated path $u_t$ of second Neumann eigenfunctions that satisfy the conditions (2) through (5) of Proposition 8.1.

**Proof.** By the definition of $\delta_{T,K}$, the quadrilateral $Q(T, w_t, \epsilon)$ satisfies (a), (b), and (c) of Lemma 9.3. Condition (a) implies that there exists a path $u_t$ of eigenfunctions of $Q_t$. Condition (b) implies that $u_t$ satisfies condition (3) in Proposition 8.1, and condition (c) implies that condition (2) is satisfied.

Let $e_0$ be the side so that $e_0$ is the component of $e \setminus \{p\}$ that contains $p$. It follows that conditions (4) and (5) of Proposition 8.1 are satisfied.

**Theorem 9.5.** Suppose that each convex quadrilateral has no interior critical points. Let $T \in \mathcal{N}$ and let $e$ be the side of $T$ that contains the index $-1$ critical point. Then for each $\eta > 0$ there exists $\epsilon \in (0, \eta)$ and $w$ in the interior of $e$ so that each second Neumann eigenfunction $u$ of $Q(T, w, \epsilon)$ has an unstable critical point.

**Proof.** Lemma 9.4 provides us with a family of quadrilaterals $Q_t$ and second Neumann eigenfunctions $u_t$ that satisfy conditions (2) through (5) of Proposition 8.1. If each second Neumann eigenfunction on a quadrilateral was to have no interior critical points, then each $u_t$ would also satisfy condition (1). Therefore, Proposition 8.1 would imply that for some $t$ the function $u_t$ has an unstable critical point.

10. Second Neumann eigenfunctions on convex polygons

**Proposition 10.1.** Suppose that $P$ is convex without right angles and suppose that $w$ lies in the interior of $P$. An arc of $\mathcal{Z}(R_w u)$ ends at a vertex $v$ of $P$ if and only if $v$ is a local extremum of $u$.

**Proof.** If $w$ lies in $P$, then it lies in the interior of the sector associated to $v$. By assumption the angle at $v$ lies in either $(0, \pi/2)$ or $(\pi/2, \pi)$. The claim then follows from combining Corollary 2.5, Corollary 4.8, and Corollary 6.4.

With additional hypotheses, we can expand the scope of Proposition 5.2 to include degree 1 vertices of $\mathcal{Z}(R_w u)$ that are vertices of $P$. 
Corollary 10.2. Let \( u \) be a second Neumann eigenfunction of a convex polygon \( P \) with no right angles. If \( w \) lies in the interior of \( P \) then each degree one vertex of \( \mathcal{Z}(R_wu) \) is a non-zero index critical point.

Proof. Each degree 1 vertex \( p \) of \( \mathcal{Z}(R_wu) \) lies in \( \partial P \). If \( p \) lies in the interior of an edge, then Proposition 5.2 applies. If \( p \) is a vertex, then Proposition 10.1 applies. \( \square \)

Proposition 10.3. Let \( u \) be a second Neumann eigenfunction \( u \) on a convex polygon \( P \). If \( u \) has a critical point \( p \) that lies in the interior of \( P \), then \( u \) has at least four nonzero index critical points on the boundary. In particular, \( u \) has at least five critical points.

Proof. Without loss of generality \( p = 0 \). Since \( p \) is a critical point of \( u \) we have
\[
    u(z) = u(0) + a \cdot x^2 + b \cdot xy + c \cdot y^2 + O(|z|^3)
\]
for some constants \( a, b \) and \( c \). We have \( R_p u = -y \partial_x + x \partial_y \) and hence
\[
    R_p u(z) = b \cdot (x^2 - y^2) + 2(c - a) \cdot xy - b \cdot y^2 + O(|z|^3).
\]
In particular, \( p = 0 \) is a nodal critical point of the Laplace eigenfunction \( R_p u \). Thus, by the result of [13], the valence of \( \mathcal{Z}(R_p u) \) at \( p \) is at least four. By Proposition 6.2 in [6], the nodal set \( \mathcal{Z}(R_p u) \) is a tree whose degree 1 vertices lie in the boundary of \( P \). Thus \( \mathcal{Z}(R_p u) \) has at least four degree 1 vertices, and each of these is a nonzero index critical point by Corollary 10.2. \( \square \)

Corollary 10.4. If \( u \) has only three critical points, then each critical point lies on the boundary. Moreover, one critical point is a global maximum, one critical point is a global minimum, and the third critical point has index zero.

Proof. By Proposition 10.3, each critical point lies on the boundary. Since \( u \) is nonconstant, at least two of these critical points are global extrema. The index of each global extremum is +1. Thus, if there are exactly three critical points, then it follows from Proposition 4.3 that two critical points have index 1 and the third has index zero. \( \square \)

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References

[1] Kawohl, B. (1985). *Rearrangements and Convexity of Level Sets in PDE*. Lecture Notes in Mathematics, Vol. 1150, MR 87a:35001. Berlin: Springer.
[2] Rauch, J. (1975). Five problems: An introduction to the qualitative theory of partial differential equations. In: Partial Differential Equations and Related Topics (Program, Tulane Univ., New Orleans, La., 1974). Lecture Notes in Mathematics, Vol. 446. Berlin: Springer, pp. 355–369.

[3] Burdzy, K. (2005). The hot spots problem in planar domains with one hole. Duke Math. J. 129(3):481–502. DOI: 10.1215/S0012-7094-05-12932-5.

[4] Burdzy, K., Werner, W. (1999). A counterexample to the “hot spots” conjecture. Ann. Math. (2). 149(1):309–317. DOI: 10.2307/121027.

[5] Atar, R., Burdzy, K. (2004). On Neumann eigenfunctions in lip domains. J. Am. Math. Soc. 17(2):243–265. DOI: 10.1090/S0894-0347-04-00453-9.

[6] Judge, C., Mondal, S. (2020). Euclidean triangles have no hot spots. Ann. Math. (2). 191(1):167–211. DOI: 10.4007/annals.2020.191.1.3.

[7] Judge, C., Mondal, S. (2022). Erratum: Euclidean triangles have no hot spots. Ann. Math. (2). 195(1):337–362.

[8] Polymath project 7, Thread 5 Hots spots conjecture. August 9, 2013. https://polymathprojects.org/2013/08/09/polymath7-research-thread-5-the-hot-spots-conjecture/.

[9] Rohleder, J. A new approach to the hot spots conjecture. https://arxiv.org/pdf/2106.05224.pdf.

[10] Lebedev, N. N. (1972). Special Functions and Their Applications. Translated from the Russian and edited by R. A. Silverman. New York: Dover.

[11] Judge, C., Mondal, S. Hypersurfaces of critical points of Laplace eigenfunctions (in preparation). https://arxiv.org/abs/2204.11968

[12] Taylor, M. E. (2011). Partial Differential Equations I. Basic Theory. Applied Mathematical Sciences. 2nd ed., Vol. 115. New York: Springer.

[13] Cheng, S. Y. (1976). Eigenfunctions and nodal sets. Comment. Math. Helv. 51(1):43–55. DOI: 10.1007/BF02568142.

[14] Bañuelos, R., Burdzy, K. (1999). On the “hot spots” conjecture of J. Rauch. J. Funct. Anal. 164(1):1–33. DOI: 10.1006/jfan.1999.3397.

[15] Siudeja, B. (2015). Hot spots conjecture for a class of acute triangles. Math. Z. 280(3–4):783–806. DOI: 10.1007/s00209-015-1448-1.

[16] Miyamoto, Y. (2013). A planar convex domain with many isolated “hot spots” on the boundary. Jpn. J. Ind. Appl. Math. 30(1):145–164. DOI: 10.1007/s13160-012-0091-z.

[17] Thurston, W. P. (1997). Three-Dimensional Geometry and Topology. Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton, NJ: Princeton University Press.