Density fluctuations and phase separation in a traffic flow model

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Abstract. Within the Nagel-Schreckenberg traffic flow model we consider the transition from the free flow regime to the jammed regime. We introduce a method of analyzing the data which is based on the local density distribution. This analyzes allows us to determine the phase diagram and to examine the separation of the system into a coexisting free flow phase and a jammed phase above the transition. The investigation of the steady state structure factor yields that the decomposition in this phase coexistence regime is driven by density fluctuations, provided they exceed a critical wavelength.

1 Introduction

Over the past few years much attention has been devoted to the study of traffic flow. Since the seminal work of Lighthill and Whitham in the middle of the 50’s many attempts have been made to construct more and more sophisticated models which incorporate various phenomena occurring in real traffic (for an overview see ). Recently, a new class of models, based on the idea of cellular automata, has been proven to describe traffic dynamics in a very efficient way ). Especially the transition from free flow to jammed traffic with increasing car density could be investigated very accurately. Nevertheless, besides various indications ), no unique description for a dynamical transition has been found (see for instance ) and references therein). In this article we consider a method of analysis which allows us to identify the different phases of the system and to describe the phase transition, i.e., considering the fluctuations which drive the transition, and determining the phase diagram.

We consider a one-dimensional cellular automaton of linear size $L$ and $N$ particles. Each particle is associated the integer values $v_i \in \{0, 1, 2, \ldots, v_{\text{max}}\}$ and $d_i \in \{0, 1, 2, 3, \ldots\}$, representing the velocity and the distance to the next forward particle . For each particle, the following update steps representing the acceleration, the slowing down, the noise, and the motion of the particles are done in parallel: (1) if $v_i < d_i$, then $v_i \rightarrow \text{Min}\{v_i + 1, v_{\text{max}}\}$, (2) if $v_i > d_i$, then $v_i \rightarrow d_i$, (3) with probability $P$, $v_i \rightarrow \text{Max}\{v_i - 1, 0\}$, and (4) $r_i \rightarrow r_i + v_i$, where $r_i$ denotes the position of the $i$-th particle.
2 Simulation and Results

Figure 1 shows a space-time plot of the system. Each dot corresponds to a particle at a given time step. The global density \( \rho_g = N/L \) exceeds the critical density and jams occur. Traffic jams are characterized by a high local density of the particles and by a backward movement of shock waves [1]. One can see from Fig. 1 that in the jammed regime the system is inhomogeneous, i.e., traffic jams with a high local density and free flow regions with a low local density coexist. In order to investigate this transition one has to take this inhomogeneity into account.

Traditionally one determines the so-called fundamental diagram, i.e., the diagram of the flow vs the density. The global flow is given by, \( \Phi = \rho_g \langle v \rangle \), where \( \langle v \rangle \) denotes the averaged velocity of the particles. This non-local measurements are not sensitive to the inhomogeneous character of the system, i.e., the information about the two different coexisting phases is lost. In the following we consider a method of analysis which is based on the measurement of the local density distribution \( p(\rho) \) [6]. The local density \( \rho \) is measured on a section of the system of size \( \delta \) according to

\[
\rho = \frac{1}{\rho_g \delta} \sum_{i=1}^{N} \theta(\delta - r_i).
\]

The local density distribution \( p(\rho) \) is plotted for various values of the global density \( \rho_g \) in Fig. 2. In the case of small values of \( \rho_g \), see Fig. 2a, the particles can be considered as independent (see below) and the local density distribution

\[\text{Fig. 1. Space-time plot for } v_{\text{max}} = 5, \ P = 0.5, \ \text{and } \rho_g > \rho_c. \ \text{Note the separation of the system in high and low density regions.}\]
Fig. 2. The local density distribution $p(\rho)$ for various values of the global density, $v_{\text{max}} = 5$, $P = 0.5$ and $\delta = 256$. The dashed line corresponds to the characteristic density of the free flow phase.

Fig. 3. The local density distribution $p(\rho_g, \rho)$ as a function of the global density (horizontal axis) and local density (vertical axis), respectively. The colors correspond to the values of the probability $p(\rho_g, \rho)$, increasing from black to red.
is simply Gaussian with the mean values $\rho_g$ and a width which scales with $\sqrt{\delta}$. Increasing the global density, jams occur and the distribution displays two different peaks (Fig. 2c). The first peak corresponds to the density of free particles and in the phase coexistence regime the position of this peak does not depend on the global density (see the dashed lines in Fig. 2). The second peak is located at larger densities and characterizes the jammed phase. With increasing density the second peak occurs in the vicinity of the critical density $\rho_c$ (Fig. 2b) and grows further (Fig. 2c) until it dominates the distribution in the sense that the first peak disappears (Fig. 2d). The two peak structure of the local density distribution clearly reflects the coexistence of the free flow and jammed phase above the critical value $\rho_c$. In Fig. 3 we present the probability distribution as function of the global and local density. Above a certain value of the global density $\rho_g$ the two peak structure occurs. The behavior of the first peak yields a criterion to determine the transition point [6] and one gets $\rho_c = 0.0695 \pm 0.0007$ for $P = 0.5$ and $v_{\text{max}}$, respectively.

In order to describe the spatial decomposition of the coexisting phases we measured the steady state structure factor [7]

$$S(k) = \frac{1}{L} \left\langle \left| \sum_{r=1}^{L} \eta(r) e^{ikr} \right|^2 \right\rangle,$$  

(2)

where $\eta(r) = 1$ if the lattice site $r$ is occupied and $\eta(r) = 0$ otherwise. In Fig. 4 we plot the structure factor $S(k)$ for the same values of the global density as in
Fig. 5. The phase diagram of the Nagel-Schreckenberg model. Note that in the non-deterministic region $0 < P < 1$ the density of the maximum flow exceeds the density of the transition point.

Fig. 2, i.e., below, in the vicinity, above and far away of the transition point. It is remarkable that $S(k)$ exhibits a maximum for all considered values of the global density at $k_0 \approx 0.72$ (dashed lines in Fig. 3). This value correspondence to the characteristic wave length $\lambda_0 = \frac{2\pi}{k_0}$ of the density fluctuations in the free flow phase. The steady state structure factor is related to the Fourier transform of the real space density-density correlation function. The wave length $\lambda_0$ corresponds to a maximum of the correlation function, i.e., $\lambda_0$ describes the most likely distance of two particles in the free flow phase. For low densities the structure factor is almost independent of the density and displays a minimum for small $k$ values indicating the lack of long-range correlations. Crossing the transition point the smallest mode $S(k = \frac{2\pi}{L})$ increases quickly. This suggests that the jammed phase is characterized by long-range correlations which decay in the limit $\rho_0 \gg \rho_c$ algebraically as one can see from the log-log plot in Fig. 4d.

Up to now we only considered the case $P = 0.5$. The phase diagram in Fig. 5 shows the $P$ dependence of the transition density $\rho_c$. $f$ denotes the free flow phase and $f+j$ corresponds to the coexistence region where the system separates in the free flow and jammed phase. The dashed line displays the $P$ dependence of the maximum flow obtained from an analysis of the fundamental diagram [8]. The critical densities $\rho_c$, where the phase transition takes place, are lower than the density values of the maximum flow. Measurements of the relaxation time, which is expected to diverge at a transition point [4], confirm this result [9] (see Fig. 3). But one has to mention that the determination of the critical density via relaxation times leads in the coexistence regime $f+j$ to unphysical results, in the sense that the relaxation time becomes negative [8,9].
3 Conclusions

In conclusion we have studied numerically the Nagel-Schreckenberg traffic flow model using a local density analysis. Crossing the critical line of the system a phase transition takes place from a homogeneous regime (free flow phase) to an inhomogeneous regime which is characterized by a coexistence of two phases (free flow traffic and jammed traffic). The decomposition in the phase coexistence regime is driven by density fluctuations, provided they exceed a critical wavelength $\lambda_c$.

References

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