PUSH-FORWARDS FOR WITT GROUPS OF SCHEMES

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Abstract. Using suitable symmetric monoidal closed structures on derived categories of schemes, as well as adjunctions of the type \((Lf^*, Rf_*)\) and \((Rf_*, f^!\) (i.e. Grothendieck duality theory), we define push-forwards for coherent Witt groups along proper morphisms between separated noetherian schemes. We discuss explicitly the special case of regular schemes.

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Introduction

Push-forwards (also known as transfers or norm maps) exist for many cohomology theories over schemes, e.g. for \(K\)-theory, (higher) Chow groups and algebraic cobordism. They are undoubtedly a useful tool for understanding and computing those cohomology theories. This article is about the construction of such push-forward maps for coherent\(^1\) Witt groups of schemes. A reader familiar with cohomology theories might think that constructing a push-forward is probably straightforward. He (or she) should be warned: Witt groups are not an oriented cohomology theory. In particular, push-forwards are, in some sense, only conditionally defined. For example, when \(X\) and \(Y\) are connected noetherian schemes of

\(^1\)The modern definition of Witt groups using triangulated categories with dualities (see [3]) can be applied either to the derived category of complexes of locally free sheaves to obtain “locally free” Witt groups or to the derived category of complexes with coherent cohomology to obtain “coherent” Witt groups. As with \(K\)-theory, it is the latter that is naturally covariant along proper morphisms, as we prove in this article. All schemes considered in this article are over \(\mathbb{Z}[1/2]\) so that the derived categories involved are \(\mathbb{Z}[1/2]\)-linear and the machinery of triangular Witt groups applies.
finite Krull dimension, smooth over a field, the Witt groups depend on a line bundle (used to define the duality) and the push-forward takes the form (see Theorem 6.7)

$$W^{i + \dim X}(X, \omega_X \otimes f^* L) \to W^{i + \dim Y}(Y, \omega_Y \otimes L)$$

where $\omega$ is the canonical bundle (the highest nontrivial exterior power of the cotangent bundle) and $L$ is an arbitrary line bundle on $Y$. In particular, if we pick a line bundle $K$ over $X$, there is no push-forward starting from $W^i(X, K)$ if $K$ is not isomorphic to $\omega_X \otimes f^* L$ for some $L$ (up to a square $M \otimes 2$ as $W^i(Y, K) \cong W^i(Y, K \otimes M \otimes 2)$, so only the class of $K$ in $\text{Pic}(X)/2$ really matters). This fundamental difference with oriented cohomology theories, where the push-forward is always possible, changes significantly classical computations, as one sees e.g. in [17], [38], [5]. The groups $W^{\dim X - i}(X, \omega_X)$ can be considered as a homology theory analogous to a non oriented complex homology theory in topology, but the construction of the push-forward here relies on triangulated monoidal methods.

Besides this article (and its precursors [9], [10] which only studied the case of regular schemes) there are already several articles available on the subject. In [16], Gille defined push-forwards along finite morphisms in the affine case. His approach is quite elementary in the sense that he uses direct computations (involving explicit injective resolutions, etc.), which is useful to get a hand on concrete forms. In [30] and [31], Nenashev adapts the oriented cohomology techniques of Panin and Smirnov to the (non-oriented) case of locally free Witt groups. He thus obtains push-forwards along projective morphisms between smooth quasi-projective varieties over fields. Still another approach using stable $\mathbb{A}^1$-representability of Witt groups can be found in [23]. (We understand that there is also some unpublished work of C. Walter on this subject.) Our approach is different: it uses derived functors and Grothendieck duality, so the dualities that appear are canonical and do not depend on choices as the other constructions mentioned above. If necessary, choices can be made in order to compare our constructions with others in the special cases where the latter are defined. Our abstract approach is very useful to prove computational properties such as base change in a simple and conceptual way, and we furthermore obtain the greater generality of singular schemes. An example of how those properties can be used for very concrete computations can be found in the computation of Balmer and the first author of the Witt group of Grassmann varieties (see [5]).

Witt groups are defined for triangulated categories $\mathcal{C}$ equipped with a duality, i.e. with a contravariant endofunctor $D$ on $\mathcal{C}$ together with a bidual isomorphism of functors $\text{Id} \to D^2 = D \circ D$ satisfying $D(\varpi_A) \circ \varpi_{DA} = \text{Id}_{DA}$ for all objects $A$ of $\mathcal{C}$. A morphism between Witt groups is naturally induced by an exact functor $F: \mathcal{C}_1 \to \mathcal{C}_2$ (both triangulated categories with dualities resp. $(D_1, \varpi_1)$ and $(D_2, \varpi_2)$) equipped with a morphism of exact functors $\phi: FD_1 \to D_2 F$ which explains how $F$ "commutes" with the dualities. This morphism $\phi$ should make the diagram

$$\begin{array}{ccc}
D_2 D_2 F & \xrightarrow{D_2 \phi} & D_2 FD_1 \\
\downarrow \varpi_2 F & & \downarrow \phi D_1 \\
FD_1 D_1 & \xrightarrow{F \varpi_1} & FD_1 D_1
\end{array}$$

commutative. In [11], we discuss such morphisms of functors and diagrams in the setting of symmetric monoidal closed categories. More precisely, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be symmetric monoidal closed categories (with tensor product denoted by $\otimes$ and internal Hom denoted by $[-,-]$). Given an object $K$, the functor $D_K := [-, K]$ together with the canonical natural transformation $\varpi_K : \text{Id} \to D^2_K$ defines a (not
necessarily strong) duality functor. Starting with an exact functor $f_* : C_1 \to C_2$ which has a left adjoint $f^*$ (which is monoidal) and a right adjoint $f^!$, there is a natural transformation

$$\zeta : f_* D_{f^! K} \to D_K f_*$$

such that the diagram (1), which becomes

(2)

$$\begin{array}{ccc}
f_* & \xrightarrow{\zeta_D^K} & D_K D_K f_* \\
\downarrow f_* & & \downarrow \zeta_D^K \\
D_K D_K f_* & & D_K D_K f_*
\end{array}$$

commutes. Therefore, as shown in [11], $f_*$ induces a morphism of Witt groups

$$W(C_1, D_{f^! K}, \varpi_{f^! K}) \to W(C_2, D_K, \varpi_K)$$

The present article is a description of how to apply this abstract monoidal closed setting to well-chosen derived categories of schemes, with the derived functors $Lf^*$, $Rf^*$, and its right adjoint $f^!$ constructed by Grothendieck duality theory.

The main result of this article is the definition of a push-forward along a proper morphism $f : X \to Y$ of separated noetherian schemes. In its most general form (Theorem 4.4), this push-forward is a morphism

$$W^i(X, f^! K) \to W^i(Y, K)$$

where $K$ is a dualizing complex on $Y$. This push-forward is induced by the derived functor $Rf_*$ and a suitable morphism of functors $\zeta_K : Rf_* D_{f^! K} \to D_K Rf_*$. This means that a form on a complex $A$ for the duality $D_{f^! K}$ is sent to a form on the complex $Rf_* A$ for the duality $D_K$. We further prove that this push-forward respects composition (Theorem 5.3).

Similarly, for morphisms of finite tor-dimension $f$ we define a pull-back (Theorem 4.1), that is a morphism

$$W^i(Y, K) \to W^i(X, Lf^* K)$$

respecting composition (Theorem 5.2).

We also prove a flat base change theorem (5.5) relating the push-forward and the pull-back in cartesian diagrams and a projection formula in the case of regular schemes (Theorem 5.7). Some explicit computations of transfers are provided in the last section.

We assume that schemes are separated and noetherian for the following technical reasons: quasi-compact and separated are necessary to have an equivalence between the derived category of quasi-coherent sheaves $D(Qcoh(X))$ and the subcategory $D_{qc}(X)$ of complexes with quasi-coherent homology in the derived category of all sheaves. Noetherian is used to ensure that the injectives in $Qcoh(X)$ remain injective in the category of all $O_X$-modules. Working without those assumptions would probably require significant improvements in the theory of Grothendieck duality; this is beyond the scope of this article which only intends to apply this theory to Witt groups.

Two main cases are discussed. The easier case is when all schemes considered are regular. Then their derived category $D_{b,c}$ of complexes with coherent and bounded homology is preserved under the derived tensor product $\otimes^L$ and the derived internal Hom $R\text{Hom}$. This endows $D_{b,c}$ with a natural structure of symmetric monoidal category. The dualizing complexes (see Definition 2.1) are line bundles or shifted line bundles. The coherent Witt groups are thus defined using the duality $R\text{Hom}(-, L)$
for some line bundle $L$. Furthermore, the derived pull-back $L^*$ for any morphism, the derived push-forward $RF_*$ and its right adjoint $f^!$ for proper morphisms also preserve $D_{b,c}$. Hence the abstract formalism of [11] applies on the nose, and we therefore obtain push-forwards and their classical properties of composition, base change and projection.

The general case, when schemes are not assumed to be regular, is more complicated. Indeed, in this case $\otimes L$ or $R\operatorname{Hom}$ do not necessarily preserve bounded homology and so there is no nice symmetric monoidal closed category structure on the category $D_{b,c}$ as the following affine example illustrates. Choose a field $k$ and set $X = \operatorname{Spec}(k[\varepsilon]/\varepsilon^2)$. Then consider the complex with $k$ concentrated in degree zero. A projective resolution of $k$ is given by

$$\cdots \xrightarrow{\varepsilon} k[\varepsilon]/\varepsilon^2 \xrightarrow{\varepsilon} k[\varepsilon]/\varepsilon^2 \xrightarrow{\varepsilon} k[\varepsilon]/\varepsilon^2 \xrightarrow{\varepsilon} k \to 0$$

Thus $k \otimes L k$ is the complex

$$\cdots 0 \to k[\varepsilon]/\varepsilon^2 \to k[\varepsilon]/\varepsilon^2 \to k[\varepsilon]/\varepsilon^2 \to k \to 0$$

which has unbounded homology. On the other hand, the unbounded derived category $D_{qc}$ of complexes with quasi-coherent cohomology admits a symmetric monoidal closed structure (this is not completely obvious, see Theorem 1.2). But this category is not suitable to define Witt groups, because there is no obvious (strong) duality on it and, anyway, as Eilenberg swindle type of arguments show for $K$-theory, unbounded categories are not the good framework to define cohomology theories. Still, the symmetric monoidal closed structure on $D_{qc}$ is useful to prove systematically the commutativity of diagrams (such as (1)). That is, we can use the framework of [11] to prove this commutativity in the large symmetric monoidal closed category $D_{qc}$ and then notice that all functors used in the definition of the duality ($R\operatorname{Hom}(\cdot, K)$ for some suitable $K$) and the push-forward ($RF_*$) actually restrict to $D_{b,c}$ under mild additional assumptions. Thus, the commutativity of the diagrams involved is proved in large categories by general symmetric monoidal closed methods, but the diagrams actually often live in a smaller category whose Witt groups are interesting.

A technical point arising is the construction of the functors involved in the symmetric monoidal structure as well as $L^*$, $RF_*$ and $f^!$ on the unbounded derived category $D_{qc}$. This relies on the work of Spaltenstein [34] as well as on the very useful book in preparation of Lipman [26]. The latter is only available online for the moment, but it is already a reference in the subject of Grothendieck duality, has been cited in several publications and, more importantly, contains very detailed explanations of all constructions; we allowed ourselves to quote some of its results.

The article is organized as follows. In Section 1, we recall the symmetric monoidal closed structures of the different categories we use. In Section 2, we use these structures to define triangulated categories with dualities and related Witt groups. Section 3 contains results on the derived functors $L^*$ and $RF_*$ and on Grothendieck duality, i.e. the construction of the right adjoint $f^!$ of $RF_*$. Section 4 contains the main result of the paper, namely Theorem 4.4. It explains how to use [11] to obtain the definition of push-forwards for the coherent Witt groups of schemes. It also contains a definition of the finite tor-dimension (e.g. flat) pull-back (Theorem 4.1). Section 5 explains the behavior of the push-forward and the pull-back under composition, and proves a base change formula relating them. Section 6 explains possible reformulations of the push-forward in different contexts and Section 7 studies in detail the push-forward the case of field extensions, regular embeddings and projective bundles, which is useful for computations and also allows a comparison with the transfer maps of other authors in the special cases where the latter are
defined. Everything except some specific computations in the last section works both for Grothendieck-Witt groups $GW$ and Witt groups $W$. For simplicity, we stated all results for $W$ only.

The present article is a generalization of the main results of the unpublished preprints [9] and [10] on regular schemes. To keep this article short, some applications established in [10] (dévissage/localization, Witt motives and partial results about their decomposition for cellular varieties) are not included here. Most important, all the abstract theorems about triangulated symmetric monoidal functors and adjunctions between them which are crucial for proving the theorems of this article are proven in the long article [11].

1. Symmetric monoidal closed categories

Let $\mathcal{S}ch$ denote the category of separated noetherian schemes and $\mathcal{R}eg$ its full subcategory of regular schemes. For any scheme $X$, let $K(X)$ (resp. $D(X)$) denote the homotopy (resp. derived) category of homological complexes of $O_X$-modules (without any restriction). We then add subscripts $+$ (bounded above, i.e. bounded where the differentials go), $-$ (bounded below), $b$ (bounded, i.e. below and above), $qc$ (quasi-coherent) and $c$ (coherent) to characterise the derived categories of complexes whose homology is as the subscript. For example $D_{b,c}(X)$ is the derived category of complexes of $O_X$-modules with coherent and bounded homology, and $D_{qc}(X)$ is the derived category of complexes with quasi-coherent homology. Note that we work with homological notation to be compatible with the literature on Witt groups, but it is easy to switch to cohomological notation by moving bounding subscripts to superscripts and exchanging $+$ and $-$ (i.e. $D_+ = D^-$).

For any scheme $X$, the usual tensor product $\otimes$ and internal Hom of complexes together with the obvious structure morphisms coming from the corresponding ones for sheaves turn $K(X)$ into a suspended symmetric monoidal closed category in the sense of [11, Section 3]. This is completely classical and is detailed in [11, Appendix], where a discussion on sign choices can be found. In particular, we have a functor $T : K(X) \to K(X)$ given by $(TA)_n = T(A_{n-1})$.

**Theorem 1.1.** Let $X$ be a scheme.

1. The tensor product on $K(X)$ admits a left derived functor

   $\otimes^L : D(X) \times D(X) \to D(X)$

   together with unit, associativity and symmetry morphisms.

2. It restricts to

   $D_{qc}(X) \times D_{qc}(X) \to D_{qc}(X)$.

3. When $X \in \mathcal{R}eg$, it furthermore restricts to

   $D_{b,c}(X) \times D_{b,c}(X) \to D_{b,c}(X)$.

4. The internal Hom on $K(X)$ has a right derived functor $\text{RHom}$

   $D(X)^\circ \times D(X) \to D(X)$

   which is a right adjoint to the derived tensor product in the usual special sense (natural in the three variables).

5. When $X \in \mathcal{S}ch$, $\text{RHom}$ restricts to

   $D_{b,c}(X)^\circ \times D_{b,c}(X) \to D_c(X)$

   as the usual $\text{RHom}$ (computed by replacing the second variable by a quasi-isomorphic complex of injectives in $\text{Qcoh}(X)$).

6. When $X \in \mathcal{R}eg$, this last restricted $\text{RHom}$ arrives in $D_{b,c}(X)$. 
Proof. See [34, Theorem A] or [26, 2.5.7] for the existence of the derived tensor product. It is based on the existence of a q-flat (also called K-flat) resolution for any complex $C$, i.e. the existence of a quasi-isomorphism $Q_C \to C$ where $Q_C$ is a complex such that $(- \otimes Q_C)$ preserves quasi-isomorphisms. These resolutions can even be constructed functorially (see [26, 2.5.5]). The derived tensor product can then be constructed by replacing one or (up to a canonical isomorphism) both variables by q-flat resolutions. The former case is used to define the unit morphism and the latter case to define the associativity and symmetry morphisms directly from the ones of $K(X)$ (see [34, Theorem A] or [26, 2.5.9]). See [26, 2.5.8] for the fact that $\otimes^L$ restricts to $D_{qc}$. In the regular case, by Point (3) of Proposition A.4, we can replace any complex in $D_{b,c}(X)$ by a bounded complex of locally free sheaves, in which case the derived tensor product obviously maps to $D_{b,c}(X)$.

Similarly, the derived internal Hom is constructed using q-injective (also called K-injective) resolutions: see [34, Section 1] for the definition of a q-injective complex and [34, Theorem A] or [26, 2.4.5] for the existence of $\mathcal{H}om$. Adjointness is also stated in [34, Theorem A] (see also [26, 2.6.1] for more details).

We now consider $\mathcal{R}hom(A, B)$ with $A, B \in D_{b,c}(X)$ for $X \in \text{Sch}$. By Corollary A.6, the right derived functor $\mathcal{R}hom$ here is computed as the one in [22, Prop. II.3.3]. This proves point (5). In the regular case, we can compute $\mathcal{R}hom$ by a locally free resolution of the first variable and then, up to isomorphism, also replace the second variable by a complex of locally free sheaves. As explained above, both complexes can be chosen to be bounded, and since $\mathcal{H}om(A, B)$ is coherent when $A$ and $B$ are [18, 5.3.5], this proves Point (6).

Now the subtle point is that $\mathcal{R}hom(M, N)$ is not necessarily an object in $D_{qc}(X)$ when $M$ and $N$ are. To fix this, we use the quasi-coherentator $Q : \text{Mod}(X) \to \text{Qcoh}(X)$ as introduced in [1, Lemma 3.2 p.187], which is right adjoint to the inclusion $\text{Qcoh}(X) \subset \text{Mod}(X)$. On an affine space $X = \text{Spec}(A)$, it takes a sheaf of $\mathcal{O}_X$-modules to the quasi-coherent sheaf associated to the $A$-module of its global sections. Its right derived functor is denoted by $\mathcal{R}Q$, as considered in [2, Remark 0.4], [26, Exercises 4.2.3] or [35, B.16]. It is a right adjoint to the inclusion $D_{qc}(X) \subset D(X)$, and $A \cong \mathcal{R}Q(A)$ when $A \in D_{qc}(X)$ (in particular for $A \in D_{b,c}(X)$) by [1, Exp. II, Prop. 3.5.2]. An alternative construction of $\mathcal{R}Q$ can be obtained from [29, Theorem 4.1].

**Theorem 1.2.** For any scheme $X$,

1. the derived tensor product $\otimes^L$ (together with the obvious morphisms) turns $D(X)$ into a symmetric monoidal category, closed by the $\mathcal{R}hom$, and suspended in the sense of [11, Section 3].

2. If $X \in \text{Sch}$, the functor

$$\mathcal{R}Q \circ \mathcal{R}hom : D_{qc}(X)^\circ \times D_{qc}(X) \to D_{qc}(X)$$

is a right adjoint (in the usual special way, see [25, (v) p. 97]) to the restricted tensor product $\otimes^L$ on $D_{qc}$. This turns $D_{qc}(X)$ into a suspended symmetric monoidal closed category.

3. If $X \in \text{Reg}$, the usual $\mathcal{R}hom$ is a right adjoint (in the usual special way) to the restricted tensor product $\otimes^L$ on $D_{b,c}$. This turns $D_{b,c}(X)$ into a suspended symmetric monoidal closed category.

**Proof.** The symmetric monoidal closed structure on $D(X)$ easily follows from Theorem 1.1. The fact that it is suspended follows, as explained in [11, Section 3], from the suspended bifunctor structure of $\mathcal{R}hom$. The symmetric monoidal structure on $D_{qc}(X)$ simply follows from the fact that $\otimes^L$ restricts to it. The fact that it is closed is a formal consequence of the fact that $D(X)$ is closed and that $\mathcal{R}Q$ is a
right adjoint to the (monoidal) inclusion $\iota : D_{qc}(X) \subseteq D(X)$:

$$\text{Hom}_{qc}(A \otimes^L B, C) = \text{Hom}(\iota(A \otimes^L B), \iota C) \simeq \text{Hom}(\iota A \otimes^L \iota B, \iota C)$$

$\simeq \text{Hom}(\iota A, \text{RHom}(\iota B, \iota C)) \simeq \text{Hom}_{qc}(A, \text{RQ}\text{RHom}(\iota A, \iota B))$.

(The closedness - that is the existence of the right adjoint to the derived tensor product - can also be deduced from Brown representability, in the spirit of the examples following [29, Theorem 4.1].) Point (3) follows from the same considerations, using Theorem 1.1 (3) and (6). \qed

**Notation 1.3.** To shorten the notation, let $[-, -]$ denote the functor $\text{RHom}$, right adjoint to the tensor product on the derived category $D$ and let $[-, -]'$ denote the functor $\text{RQ}\circ \text{RHom}$, right adjoint to the tensor product on the derived category $D_{qc}$.

Since the derived quasi-coherator is the identity on $D_{-qc}$ (see above), if $[A, B] \in D_{-qc}$ then $[A, B] \cong [A, B]'$.

We finish this section by pointing out a comment of Neeman: exploiting the fact that for $X \in \text{Sch}$, there are enough flat objects in $D_{qc}$ and his representability result more extensively gives an alternative approach for constructing a symmetric monoidal closed structure on $D_{qc}(X)$ directly without passing through $D(X)$.

## 2. Witt groups

From now on, we assume that all schemes are defined over $\mathbb{Z}[1/2]$.

To define a Witt group, we need a strong duality on a triangulated category. Using the previous framework of (triangulated) symmetric monoidal closed categories, we recall how $[-, K]$ and $[-, K']$ define dualities. The purpose of this section is to compare the restrictions of these dualities to the subcategory $D_{b,c}$ and to discuss when these dualities are strong on it. For any object $K$, let $\sharp_K$ (resp. $\sharp'_K$) denote the contravariant exact functor $[-, K]$ (resp. $[-, K']$).

Following [11, Section 3.2], applied to the symmetric monoidal closed structure on $D(X)$ with $X$ an arbitrary scheme, we may define the *bidual morphism*

$$\varpi_K : \text{Id} \to \sharp_K \sharp_K$$

as a morphism of triangulated endofunctors of $D(X)$. From [11, Cor. 3.2], we obtain that $(D(X), \sharp_K, \varpi_K)$ is a triangulated category with duality (in the sense of [11, Definition 2.1.1], so $\varpi_K$ is not necessarily an isomorphism). Similarly, when $X \in \text{Sch}$, we obtain a triangulated category with duality $(D_{qc}(X), \sharp_K, \varpi_K)$.

**Definition 2.1.** Let $K$ be an object of $D_{qc}(X)$. It is a *dualizing complex* (or it is *dualizing*) if

- the functor $[-, K]$ preserves $D_{b,c}$ and
- the bidual morphism $\varpi_K$ is an isomorphism on $D_{b,c}$.

If furthermore it has finite injective dimension (i.e. it is quasi-isomorphic to a finite complex of injectives), we say it is a *injectively bounded dualizing complex*.

In the terminology of [11, Definition 2.1.1], the second condition says that $\sharp_K$ is a strong duality.

Note that for any $X \in \text{Sch}$, a dualizing complex $K$ is automatically in $D_{b,c}(X)$ since the natural morphism $K \to [\mathcal{O}_X, K]$ (coming from the monoidal structure) is an isomorphism and $\mathcal{O}_X$ is coherent. In particular, our definition is exactly the “modern” [28, Definition 3.1], by Lemma 3.5 of loc. cit. Also note that our injectively bounded dualizing complexes are the “old” dualizing complexes of [22, V. §2]
Proposition 2.2. Let $X \in \text{Sch}$ and $K \in D_{qc}(X)$ be a dualizing complex. Then the functors $\sharp_K$ and $\sharp'_{K'}$ coincide and the bidual morphisms $\varpi_K$ and $\varpi'_{K'}$ are equal on the subcategory $D_{b,c}(X)$.

Proof. Since $[A,K] \in D_{b,c}(X)$ for any $A \in D_{b,c}(X)$, we have $[A,K]' \cong [A,K]$ by the remark after Notation 1.3 which proves that $\sharp_K \cong \sharp_{K'}$. The bidual morphisms are then equal by the large commutative diagram considered in the proof of [11, Theorem 4.1.2], in which the $f^*$ should be replaced by the inclusion $D_{qc}(X) \subset D(X)$, which is monoidal by definition of the tensor product on $D_{qc}(X)$.

Example 2.3. (1) A dualizing complex tensored by a shifted line bundle is still a dualizing complex. In fact, this is the only freedom: by [28, Lemma 3.9] (see also [22, Theorem V.3.1] for the injectively bounded case), a dualizing complex is unique up to tensoring by shifted line bundles (the shift can be different on different connected component of $X$).

(2) On a Gorenstein scheme $X$ (e.g. regular), $\mathcal{O}_X$ itself is dualizing, so by the previous point, the only dualizing complexes are the shifted line bundles.

Note that on a regular scheme, the category $D_{b,c}(X)$ itself is monoidal closed. It follows that dualizing complexes are dualizing objects in the sense of [11, Definition 3.2.2] in the category $D_{b,c}(X)$, for $X \in \text{Sch}$.

Theorem 2.4. Let $X \in \text{Sch}$ and $K$ be dualizing. Then $(D_{b,c}(X), \sharp_K, \varpi_K)$ is a triangulated category with strong duality in the sense of [11, Def. 2.1.1]. We denote it by $\mathcal{C}_K$, and its Witt groups [11, Definition 2.1.5] by $W^i(X,K)$, $i \in \mathbb{Z}$.

Proof. The functor $\sharp_K = \sharp'_{K'}$ is a contravariant endofunctor of $D_{b,c}(X)$ and $\varpi_K = \varpi'_{K'}$ is an isomorphism on this category by definition of dualizing complexes by Proposition 2.2. The necessary commutative diagrams that $\sharp_K$ and $\varpi_K$ must satisfy simply follow from the fact that they are already satisfied in $D_{qc}(X)$ since $(D_{qc}(X), \sharp_K, \varpi_K)$ is a triangulated category with duality.

We may thus think of the triangulated category with duality $(D_{b,c}(X), \sharp_K, \varpi_K)$ as being restricted from $(D(X), \sharp_K, \varpi_K)$ or from $(D_{qc}(X), \sharp_K, \varpi_K)$, both structures coinciding on $D_{b,c}(X)$.

Remark 2.5. In [3], all dualities considered are strict, i.e. they strictly commute with the suspension, but this assumption is only there for simplicity. Instead, in [11, Def. 2.1.1], we only assume commutativity up to a natural isomorphism, and all theorems in [3] are still true in this more general situation.

Remark 2.6. Recall (see e.g. [39, Def. 10.5.1]) that for a left exact functor $f$ between exact categories, despite the standard notation $Rf$ the right derived functor really is a couple $(Rf, s)$ with $s : qf \to (Rf)q$ and $q$ the morphism form the homotopy category to the derived category. It is only the couple $(Rf, s)$ which is unique up to unique isomorphism and therefore deserves being called the right derived functor. Consequently, the various derived functors, for example $\mathcal{R}\text{Hom}(-, K)$ (used to define the duality) and $\mathcal{R}f_*$ (used below to define the push-forward) together with all the morphisms of functors defining the symmetric monoidal structure can be considered as abstract exact functors and morphisms of exact functors. With them, it is possible to define coherent Witt groups and push-forwards by the methods discussed in this article, since these methods only involve the abstract triangulated categories and functors (the framework of [11]). But as such, there is no uniqueness of all these constructions. It is only if we keep (as extra data) all the structural morphisms of the derived functors (the $s$ part of the couples), and thus the relationship between the symmetric monoidal closed structure on $K(X)$ and the one on $D(X)$,
that the whole derived construction becomes unique, up to unique isomorphism, and thus the induced dualities, pull-backs and push-forwards.

3. The functors $L^f_*$, $R^f_*$ and $f^!$

We now introduce the functors $L^f_*$, $R^f_*$ and $f^!$ associated to a morphism of schemes $f$ and explain how they behave with respect to the monoidal structure. The first two functors are derived functors, whereas the third one is right adjoint to $R^f_*$ at the level of derived categories, but is not the derived functor of some underlying functor on the category of $O_X$-modules. The construction of $f^!$ is the heart of Grothendieck duality theory, for which we refer the reader to [22], [36], [29], [12] or [26].

**Proposition 3.1.** Let $f : X \to Y$ be a morphism of schemes.

1. The functor $f^*$ admits a left derived functor $L^f_* : D(Y) \to D(X)$ which restricts to $D_{qc}(Y) \to D_{qc}(X)$.

2. If $f$ is of finite tor-dimension (see e.g. [26, Example (2.7.6)]) or if $X, Y \in \text{Reg}$, then $L^f_*$ restricts to $D_{b,c}(Y) \to D_{b,c}(X)$.

**Proof.** For existence, see [34, Theorem A (iii) or Prop. 6.7] or [26, Example 2.7.3]. For the fact that it restricts to $D_{qc}$, see [26, 3.9.1]. It restricts to $D_{b,c}$ in the finite tor-dimension case because $L^f_*$ is then bounded and it respects the coherence of the cohomology by [22, Proposition II.4.4] (bearing in mind Proposition A.7). The case $X, Y \in \text{Reg}$ follows from Point (3) of Proposition A.4 and Proposition A.7. □

**Proposition 3.2.** The usual isomorphism $f^*(A \otimes B) \to f^*A \otimes f^*B$ induces an isomorphism of triangulated bifunctors (in the sense of [11, Def. 1.4.14])

$$\alpha : L^f_*(- \otimes -) \to L^f_*(-) \otimes L^f_*(-)$$

which turns $L^f_*$ into a suspended symmetric monoidal functor (in the sense of [11, Section 4]).

**Proof.** See [34, Prop. 6.8]. The morphism $\alpha$ is defined as the corresponding one on $K(X)$ after having replaced both variables by q-flat resolutions. It is already an isomorphism on $K(X)$. The commutative diagrams required (compatibility with the associativity, unit and symmetry of the monoidal structures) then easily follow from the corresponding ones on $K(X)$, using Proposition A.3, Points (1) and (3). □

By [11, Proposition 4.1.1] applied to the symmetric monoidal structure and $L^f_*$ on $D(X)$, there is a natural morphism

$$\beta : L^f_*[-,-] \to [L^f_*(-), L^f_*(-)].$$

We also obtain a morphism

$$\beta' : L^f_*[-,-]' \to [L^f_*(-), L^f_*(-)]'.$$

using $D_{qc}(X)$ instead of $D(X)$.

**Proposition 3.3.** Let $X, Y \in \text{Sch}$ and $A, B \in D_{qc}(Y)$. Assuming $[A, B] \in D_{qc}(Y)$ and $[L^f_*A, L^f_*B] \in D_{qc}(X)$, the morphisms $\beta_{A,B}$ and $\beta'_{A,B}$ coincide. In particular, when $K$ and $L^f_*$ are dualizing, $\beta_K$ and $\beta'_K$ coincide.

**Proof.** This follows from the commutative diagram

$$\begin{array}{ccc}
L^f_*[A, B]' & \xrightarrow{\beta'} & [L^f_*A, L^f_*B]'\\
\downarrow & & \downarrow \\
L^f_*[A, B] & \xrightarrow{\beta} & [L^f_*A, L^f_*B]
\end{array}$$
This follows from \[ \text{Proof.} \] that the \( \rho \) \( D \)-isomorphism on objects in \( \text{first variable coherent and the second in} \) \( D \) on \( X \), \[ \text{Proof.} \[ \text{of} \] \] \( X, Y \) or when \[ \text{Proof.} \[ \text{functors restrict).} \] \[ \text{Proof.} \[ \text{adjoint to} \) \( Lf \) \] \( \text{Proposition 3.4.} \) \[ \text{For existence, see} \) \( \text{Proposition 3.5.} \) \[ \text{For any morphism} \) \( f \) \[ \text{of schemes, the functor} \) \( \text{Proposition 3.6.} \) \[ \text{By} \) \( \text{Proposition 3.9.} \) \[ \text{Theorem 3.7.} \] \[ \text{Let} \) \( f : X \to Y \) \[ \text{be quasi-compact and quasi-separated (e.g. proper).} \] \[ \text{Theorem 3.8.} \] \[ \text{For any separated morphism} \) \( f : X \to Y \) \[ \text{Proposition 3.9.} \] \[ \text{Proposition 3.10.} \]
and restrict to an affine open set $U$ of $X$. As $f$ is of finite type, we have a factorization $U \to \mathbb{A}^n \times Y \to Y$ for some $n$ where the left arrow is a closed embedding. Taking the closure of $U$ in $\mathbb{P}^n \times Y$, we see that $U$ can be embedded as an open subset of a closed subset of some $Y \times \mathbb{P}^n$. Hence we only have to show that closed immersions, open immersions and projections $Y \times \mathbb{P}^n \to Y$ respect dualizing complexes. The case of closed immersions is done in [28, Theorem 3.14, Remark 3.17 and Lemma 3.18] (closed immersions are finite). The case of open immersions is [28, Theorem 3.12]. For projective morphisms $f: \mathbb{P}_Y^n \to Y$, one uses that $\text{RHom}(A, f^!K) \cong \text{RHom}(A, f^!\mathcal{O} \otimes f^* K) \cong \text{RHom}(A, f^* K) \otimes f^!\mathcal{O}$, using Lemma 5.6 below and that $f^!\mathcal{O}$ is a shifted line bundle by [22, Section VII.4]. Then one checks the conditions of Definition 2.1 for $A$ objects of the form $f^*B$ and $\mathcal{O}(i)$ which by a theorem of Beilinson generate $D_{b,c}(\mathbb{P}_Y^n)$ as a thick triangulated category.

4. Pull-back and push-forward for Witt groups

We can now state the main result of this article: the definition of the push-forward for coherent Witt groups along proper morphisms (Theorem 4.4). This section also contains a definition of the pull-backs for morphisms of finite tor-dimension (Theorem 4.1).

Let $f: X \to Y$ be a morphism of schemes. By [11, Theorem 4.1.2] (applied to the monoidal categories $D(-)$), $\beta_K : Lf^*\mathcal{O}_K \to \mathcal{O}_X \otimes Lf^* \mathcal{O}_X$ defines a duality preserving functor $\{Lf^*, \beta_K\}$ between triangulated category with (weak) duality, from $(D(Y), \mathcal{O}_Y, \omega_Y)$ to $(D(X), \mathcal{O}_X, \omega_X)$. Let $f: X \to Y$ be a morphism of schemes such that

- the objects $K$ and $Lf^*K$ are dualizing.
- $Lf^*$ preserves $D_{b,c}$,
- $\beta_K$ is an isomorphism on $D_{b,c}(Y)$.

Then $\{Lf^*, \beta_K\}$ induces a morphism on Witt groups

$$f^* : W^i(Y, K) \to W^i(X, Lf^*K)$$

that we call pull-back. In particular, this pull-back exists if $K$ and $Lf^*K$ are dualizing and

- $f$ is of finite tor-dimension and $X, Y \in \text{Sch}$ or
- for any $f$ and $X, Y \in \text{Reg}$ (in which case $K$ dualizing imples $Lf^*K$ dualizing by Example 2.3).

Proof. This follows from [11, Theorem 4.1.2 and Lemma 2.2.6 (1)]. The Theorem of loc. cit. ensures the existence of the appropriate commutative diagrams in $D(X)$. The requirements in the Lemma of loc. cit. that the dualities given by $K$ and $f^*K$ restrict as strong dualities to $D_{b,c}$ are satisfied by assumption, and the requirement that $\beta_K$ is an isomorphism when restricted to $D_{b,c}$ follows from Proposition 3.4.

Remark 4.2. Note that we obtain the very same push-forward when starting with the monoidal structure on $D_{qc}$ instead of $D$. This follows from Proposition 3.3.

Remark 4.3. In [28, Theorem 3.12], it is proved that if $f$ is an open immersion, then $Lf^*K$ is automatically dualizing if $K$ is dualizing.

Let $X \in \text{Sch}$, $K \in D_{qc}(X)$ and $f: X \to Y$ be a separated morphism. From [11, Theorem 4.2.9] applied to the closed monoidal category $D_{qc}(X)$, we obtain a morphism of functors $\zeta_K : Rf_* f^! K \to \mathcal{O}_X \otimes Rf_* f^! \mathcal{O}_X$. By loc. cit., the pair $\{Rf_*, \zeta_K\}$ is duality preserving (i.e. Diagram (1) commutes).
Theorem 4.4. Let $X, Y \in \text{Sch}$ and $f : X \to Y$ be a separated morphism such that $Rf_*$ preserves $D_{b,c}$. Let $K$ and $f^!K$ be dualizing. Then $\{Rf_*, \zeta_K\}$ induces a morphism of Witt groups

$$f_* : W^i(X, f^!K) \to W^i(Y, K)$$

that we call push-forward. In particular, this push-forward is defined if $f$ is proper and both $K$ and $f^!K$ are dualizing (e.g. $K$ is an injectively bounded dualizing complex, see Proposition 3.9).

Proof. This follows from [11, Theorem 4.2.9 and Lemma 2.2.6. (1)]. For the Theorem of loc. cit., consider the triangulated monoidal closed category $D_{qc}$. The fact that $\zeta_K$ is an isomorphism follows from [11, Prop. 4.3.3] using Theorem 3.7. Then, apply the Lemma of loc. cit. to the subcategories $D_{b,c}$, to which the dualities restrict by definition of a dualizing object. Note that when $X$ and $Y$ are regular, the complete proof works using directly $D_{b,c}$ as the triangulated monoidal closed category in [11, Theorem 4.2.9]. □

5. Properties

We now show that both push-forwards and pull-backs respect composition and that they commute in an appropriate way ("base change") provided certain standard conditions hold. We also prove a projection formula for regular schemes.

Theorem 5.1. For any $f : X \to Y$ and $g : Y \to Z$,

1. there is an isomorphism $Lf^* \circ Lg^* \to L(g \circ f)^*$ between functors on $D(-)$ which is associative in the usual sense.

2. There is an isomorphism $R(g \circ f)_* \to Rg_* \circ Rf_*$ between functors on $D(-)$ which is associative in the usual sense, and respects the adjoint couple $(L(-)^*, R(-)_*)$ in the sense of [11, Def. 5.1.5].

3. When the schemes are separated and quasi-compact, and both $f$ and $g$ are separated, there is an isomorphism $f^! \circ g^! \to (g \circ f)^!$ between functors on $D_{qc}(-)$ which is associative in the usual sense, and which respects the adjoint couple $(R(-)_*, (-)^!)$ in the sense of [11, Def. 5.1.5].

Proof. For the functors $Lf^*$ on $D$, the isomorphism is in [34, Theorem A (iii)] or [26, 3.6.4]. For a proof that it is associative, see [26, Scholium 3.6.10]. The other points follow from the first one by [11, Lemma 5.1.6]. □

Theorem 5.2. The pull-back respects composition: the diagram

$$
\begin{array}{ccc}
W^i(Z, K) & \xrightarrow{g^*} & W^i(Y, Lg^*K) \\
\downarrow & & \downarrow \\
W^i(X, f^*Lg^*K) & \xrightarrow{(gf)^*} & W^i(X, L(gf)^*K)
\end{array}
$$

commutes, under the conditions for the existence of the pull-backs $f^*$ and $g^*$ of Theorem 4.1.

Proof. This follows from [11, Theorem 5.1.3 and Cor. 5.1.4] applied to the structures on $D(-)$. □

Theorem 5.3. The push-forward respects composition: the diagram

$$
\begin{array}{ccc}
W^i(X, f^!g^!K) & \xrightarrow{f_*} & W^i(Y, g^!K) \\
\downarrow & & \downarrow \\
W^i(X, (gf)^!K) & \xrightarrow{(gf)_*} & W^i(Z, K)
\end{array}
$$

commutes, under the conditions for the existence of the push-forwards $f_*$ and $g^!$ of Theorem 4.1.
commutes, under the conditions for the existence of the push-forward of Theorem 4.4.

Proof. This follows from [11, Theorem 5.1.9 and Cor. 5.1.10] applied to the structures on $D_{qc}(-)$.

We now prove a base change formula. Let us consider a pull-back diagram

\[
\begin{array}{ccc}
V & \xrightarrow{g} & Y \\
\downarrow{f} & & \downarrow{f} \\
X & \xrightarrow{g} & Z
\end{array}
\]

By [11, Section 5.2], we obtain a morphism of functors

\[
\varepsilon : Lf^* Rg_* \to R\tilde{g}_* L\tilde{f}^*
\]

between functors on $D(X)$.

**Proposition 5.4.** If all schemes are in $Sch$ and the diagram is tor-independent (e.g. $f$ flat), the morphism $\varepsilon$ is an isomorphism on $D_{qc}(X)$.

Proof. The case where $f$ is flat is [26, Proposition 3.9.5] (all maps between schemes in $Sch$ are “concentrated” in the sense of loc. cit.). The more general case is [26, Theorem 3.10.3].

Then, when the schemes are in $Sch$, still by [11, Section 5.2], $\varepsilon$ induces a morphism

\[
\gamma : L\tilde{f}_* g^! \to \tilde{g}_! Lf^*
\]

between functors on $D_{qc}(X)$. It is an isomorphism on the subcategory $D_{qc}$ by [26, Corollary 4.4.3]. In particular, $\gamma_K$ is an isomorphism when $K$ is dualizing (and thus in $D_{b,c}(Z)$).

**Theorem 5.5.** (base change) Under the assumptions of Proposition 5.4 and the ones for the pull-backs along $f$ and $\tilde{f}$ and the push-forwards along $g$ and $\tilde{g}$ to exist (Theorems 4.1 and 4.4), the pull-back and push-forward satisfy a base change formula: the diagram

\[
\begin{array}{ccc}
W^i(V, g^! Lf^* K) & \xrightarrow{\tilde{g}_!} & W^i(Y, Lf^* K) \\
\downarrow{f^*} & & \downarrow{f^*} \\
W^i(X, g^! K) & \xrightarrow{\gamma} & W^i(Z, K)
\end{array}
\]

commutes.

Proof. This follows from [11, Theorem 5.2.1 and Corollary 5.2.2] applied to the structures on $D_{qc}(-)$, keeping in mind Remark 4.2.

We conclude this section with a projection formula for Witt groups, in the case of regular schemes. For this, we first need to introduce another natural morphism that will anyway be of some use even in the case of non regular schemes.

When $f : X \to Y$ is a separated morphism in $Sch$, using the functors $Lf^*$ (monoidal), $Rf_*$ and $f_!$ between the categories $D_{qc}(X)$ and $D_{qc}(Y)$, and the fact
that the projection morphism $\pi$ is an isomorphism by Theorem 3.7, we obtain a morphism of functors
\[
\theta : f^!(\cdot) \otimes^L Lf^*(\cdot) \to f^!(\cdot \otimes^L \cdot)
\]
by [11, Proposition 4.3.1].

**Lemma 5.6.** The morphism $\theta : f^!(A) \otimes^L Lf^*(B) \to f^!(A \otimes^L B)$ is an isomorphism when $B$ is a perfect complex.

**Proof.** The morphism $\theta$ is compatible with open immersions (by Diagram 41 of [11, Proposition 5.2.5]), and so we can restrict to the case of bounded complexes of vector bundles, then to vector bundles, then again using open immersions to the trivial bundle $O_Y$. In that case, easy commutative diagrams using the monoidal structures on $D_{qc}$ show that $\theta_{A,O_Y}$ coincides with the unit morphism of the monoidal structure $f^!(A) \otimes^b O_X \to f^!(A)$, and is therefore an isomorphism. □

For any scheme $X$ in $\text{Reg}$, the derived tensor product preserves $D_{b,c}(X)$ (Theorem 1.1 (3)). This gives two different products on Witt groups by the formalism of [17] using [11, Proposition 4.4.6 and Corollary 4.4.7] applied to the closed monoidal structure of $D_{b,c}(X)$. We fix one of these products (say, the left one) for the following results. When $K$ and $L$ are shifted line bundles (thus dualizing complexes), the product is a pairing
\[
W^i(X, K) \times W^j(X, L) \to W^{i+j}(X, K \otimes L).
\]

**Theorem 5.7.** (projection formula) For any proper morphism $f : X \to Y$ with $X, Y \in \text{Reg}$, the pull-back and push-forward satisfy a projection formula: If $K, L$ are shifted line bundles on $Y$, $x \in W^i(X, f^!K)$ and $y \in W^j(Y, L)$, then $f_*(I(x, f^*y)) = f_*(x) \cdot y$ in $W^{i+j}(y, K \otimes L)$ with $I$ the isomorphism from $W^i(X, f^!K \otimes f^!L)$ to $W^i(X, f^!(K \otimes L))$ induced by $\theta_{K,L}$.

**Proof.** First note that $L$ being a shifted line bundle explains the absence of derivations in the pull-back and tensor products above. Then, the morphism $\theta_{K,L}$ is an isomorphism by Lemma 5.6, thus the result follows from [11, Theorem 5.5.1 and Corollary 5.5.2] applied to the monoidal closed structure on $D_{b,c}$. □

6. Reformulations in special cases

In this section, we give other canonical ways of writing the push-forward, under additional assumptions.

**Notation 6.1.** For an equidimensional morphism $f : X \to Y$ of relative dimension $n$, let $\omega_f$ denote the object $f^!(\mathcal{O}_Y)[-n] := T^{-n} f^!(\mathcal{O}_Y)$.

This notation is motivated by the fact that in several cases, this object can be identified with a geometric object called a relative dualizing sheaf and usually denoted similarly: see sections 7.2 and 7.3 for the examples of regular embeddings and projective spaces.

6.1. Relative dualizing sheaf.

**Theorem 6.2.** Let $f : X \to Y$ be a proper morphism in $\text{Sch}$, $K$ a dualizing complex on $Y$ such that $f^!K$ is a dualizing complex and assume that $\theta_{\mathcal{O}_Y,K} : f^!(\mathcal{O}_Y) \otimes^L f^!K \to f^!(K)$ is an isomorphism. Then we can rewrite the push-forward of Theorem 4.4 as
\[
f_* : W^{i+d}(X, \omega_f \otimes^L f^!K) \to W^i(Y, K).
\]
In particular, this holds if $K$ is dualizing and either of the two following conditions hold.
- $f$ is quasi-perfect (see below, e.g. of finite tor-dimension) and $f^!K$ is dualizing.
- $Y$ is a Gorenstein scheme (e.g. regular).

Proof. The reformulation of the push-forward is [11, Definitions 6.1.3 and 6.1.4]. When $f$ is quasi-perfect, [27, Proposition 2.1] shows that $\theta$ is an isomorphism on all $D_{gr}(Y)$. Example 2.2 in loc. cit. shows that if $f$ is of finite tor-dimension, it is quasi-perfect. When $Y$ is Gorenstein, the only dualizing complexes are shifted line bundles, for which $\theta$ is an isomorphism by Lemma 5.6. \hfill $\square$

Let $g : Y \to Z$ be another proper morphisms in $\text{Sch}$ and $M$ a dualizing complex on $Z$ and let

$$\iota_{f,g} : \omega_f \otimes^L Lf^*(\omega_g \otimes^L Lg^*M) \to \omega_{gf} \otimes^L L(gf)^*M$$

be the morphism defined in [11, Theorem 6.1.5].

Theorem 6.3. The push-forward of Theorem 6.2 respects composition: let $I$ denote the isomorphism of Witt groups induced by $\iota_{f,g}$, then the push-forward on Witt groups defined above satisfies $g_*f_* = (gf)_*I$.

Proof. This follows from [11, Theorem 6.1.5, Lemma 2.2.6 (2)]. Note that $\iota_{f,g}$ is an isomorphism because it is a composition of isomorphisms (under the assumptions for the reformulated push-forward to exist). \hfill $\square$

Theorem 6.4. In the situation of Theorem 5.5 and under the assumptions of the reformulation of the push-forward above for the morphisms $g$ and $\bar{g}$, the base-change theorem 5.5 becomes

$$f^*g_* = \bar{g}_*I\bar{f}_*$$

where $I$ is the isomorphism of Witt groups induced by the isomorphism

$$\iota : L\bar{f}^*(\omega_{\bar{g}} \otimes^L L\bar{g}^*K) \to \omega_{\bar{g}} \otimes^L L\bar{g}^*\bar{f}^*K.$$ 

Proof. This follows from [11, Theorem 6.1.7]. Note that $\gamma_{O_S}$ is an isomorphism (see before Theorem 5.5). \hfill $\square$

In the regular case, the projection formula 5.7 becomes the following.

Theorem 6.5. Let $f : X \to Y$ be a proper equidimensional morphism of relative dimension $d$ with $X, Y \in \mathcal{R}_{\text{reg}}$. Then the push-forward of Theorem 6.2 and the pull-back of Theorem 4.1 satisfy $f_I(x, f^*(y)) = f_*(x, y)$ in $W^{i+1}(Y, L \otimes K)$ for any $x \in W^{i+d}(X, \omega_f \otimes f^*L)$ and $y \in W^i(Y, K)$.

Proof. See [11, Theorem 6.1.9 and Corollary 6.1.10]. \hfill $\square$

6.2. Smooth schemes over a base. We now fix a base scheme $S \in \text{Sch}$ with a dualizing complex $K_S$ and consider the category $\text{SmPr}/S$ of schemes in $\text{Sch}$ that are smooth, equidimensional and proper over $S$. For such a scheme $X$, let the structural morphism be denoted by $p_X : X \to S$ and its relative dimension over $S$ by $d_X$. Note that any separated morphism between schemes in $\text{SmPr}/S$ is proper, being the composition of a closed embedding (its graph) and a proper projection.

Notation 6.6. Let $X \in \text{SmPr}/S$. We set $\omega_X = f^!(K_S)[-d_X]$. Observe that $\omega_X = \omega_{p_X}$ if $K_S = O_S$.

Theorem 6.7. Let $f : X \to Y$ be a separated morphism, $X, Y \in \text{SmPr}/S$ and let $L$ be a line bundle on $Y$. The push-forward can be written

$$f_* : W^{i+d_X}(X, \omega_X \otimes f^*L) \to W^{i+d_Y}(Y, \omega_Y \otimes L)$$
Proof. First, let us note that when pulling back or tensoring by a line bundle, there is nothing to derive. This is why no \(L\) appear in front of \(f^*\) and \(\otimes\). We then use Definitions [11, Definitions 6.3.3 and 6.3.4]. We need to check that the morphism

\[ \omega_X \otimes L \cong f^! \omega_Y \otimes f^* L \to f^! (\omega_Y \otimes L) \]

is an isomorphism. This is the case by lemma 5.6. \(\square\)

**Theorem 6.8.** The push-forward of Theorem 6.7 respects composition.

**Proof.** See [11, Theorem 6.3.5 and Corollary 6.3.6]. \(\square\)

**Theorem 6.9.** The push-forward of Theorem 6.7 satisfies flat base change.

**Proof.** See [11, Theorem 6.3.7 and Corollary 6.3.8]. \(\square\)

7. Examples

Note that \(f^!\) is unique up to unique isomorphism whenever it is defined, because it is always defined as a right adjoint to \(Rf_*\). This allows us to use computations of \(f^!\) from [22] and other sources in the examples below.

7.1. Field extensions. The simplest example of proper morphism is the case of a field extension \(E/F\) giving rise to a finite morphism

\[ f : X = \text{Spec}(E) \to \text{Spec}(F) = Y. \]

The tilde construction gives equivalences of categories \(\text{Mod}(F) \cong \text{Qcoh}(Y)\) and \(\text{Mod}(E) \cong \text{Qcoh}(X)\), and the subcategories of finite dimensional vector spaces correspond to coherent sheaves of modules. We thus describe all objects and functors through these equivalences of categories. The only dualizing complex (up to shifts and isomorphisms) on \(Y\) is \(F\) itself. The functors \(f^* = (- \otimes_F E)\) and \(f_* = (-)|_F\) are exact, there is nothing to derive. The functor \(f^!\) is given by [22, III §6] as \(\mathcal{H}om_F(E, -)\) (mapping to \(E\)-vector spaces) and the unit and counit of the adjunction \((f_*, f^!\)) are respectively given by

\[
\begin{align*}
\left\{
\begin{array}{c}
V \to \mathcal{H}om_F(E, V|_F) \\
a \mapsto (e \mapsto e.a)
\end{array}
\right\} & \quad \left\{
\begin{array}{c}
\mathcal{H}om_F(E, V')|_F \to V' \\
\phi \mapsto \phi(1)
\end{array}
\right\}
\end{align*}
\]

for an \(E\)-vector space \(V\) and an \(F\)-vector space \(V'\). For fields, the only nonzero Witt group modulo 4 is \(W_0\) which is the classical Witt group of the field. So we are reduced to study push-forward for forms on vector spaces (i.e. complexes concentrated in degree zero). Following the construction, it is easy to check that for any \(E\)-vector space \(V\), the morphism \(\zeta : f_*[V, f^! F] \to [f_* V, F]\) coincides with the Cartan isomorphism of \(F\)-vector spaces

\[ \mathcal{H}om_F(V, \mathcal{H}om_F(E, F)|_F) \cong \mathcal{H}om_F(V|_F, F) \]

which sends a morphism \(\phi : V \to \mathcal{H}om_F(E, F)\) to the morphism \((a \mapsto \phi(a)(1))\). Thus, the push-forward \(f_* : W^0(E, \mathcal{H}om_F(E, F)) \to W^0(F)\) is a Scharlau transfer (see [33, p. 48]) with respect to the usual trace \(Tr : E \to F\). To see this, note that \(Tr\) factors as \(E \cong_r \mathcal{H}om_F(E, F) \to F\) where the isomorphism is given by \(e \mapsto (x \mapsto Tr(e.x))\) and \(\mathcal{H}om_F(E, F) \to F\) is the evaluation at 1.
7.2. Regular embeddings. Let $F$ be a vector bundle of rank $d > 0$ over $X$ with a regular section $s : O_X \to F$, i.e. such that the corresponding embedding $f : Z \subset X$ of the zero locus is a closed regular embedding of codimension $d$ and the augmented Koszul resolution is exact \cite{14, IV, §2, Proposition 3.1} and thus yields a quasi-isomorphism

\[
(3) \quad K_F = \begin{array}{c}
\Lambda^d F^\vee \\
\Lambda^{d-1} F^\vee \\
\vdots \\
F^\vee \\
O_X \\
0
\end{array} \quad \xrightarrow{\text{qis}} \quad f_* O_Z = \begin{array}{c}
0 \\
f_* O_Z \\
0
\end{array}
\]

from the Koszul complex $K_F$ to $f_* O_Z$ concentrated in degree 0. Since $f$ is a closed embedding (thus finite), $f_*$ is exact and coincides with $Rf_*$. Let $\Delta_F = \Lambda^d F$ be the determinant of $F$. In this situation, we have $\Delta^d A = f^* \Delta_F [-d] \otimes L_i^* A$ for all $A \in D_{b,c}(X)$ (this may be extracted from \cite{22, III §7}, see also \cite{36, Proposition 1}, after applying Lemma 5.6 and using that $F$ is dual to the cotangent sheaf). By tensoring the augmented Koszul resolution with $\Delta_F$ and using the canonical isomorphisms $\Lambda^d F^\vee \otimes \Delta_F \cong \Lambda^{d-1} F$ and $f_* O_Z \otimes \Delta_F \cong f_* f^* \Delta_F$, we obtain the trace map $f_* f^* O_X \to O_X$ (counit of the adjunction $(Rf_*, f^*)$) in the derived category as the composition of a usual map of complexes followed by the inverse of a quasi-isomorphism ($O_X$ is in degree 0 and $f_* f^* \Delta_F$ in degree $-d$):

\[
f_* f^* O_X = \begin{array}{c}
0 \\
0 \\
\vdots \\
f_* f^* \Delta_F \\
0
\end{array} \quad \xrightarrow{\text{id}} \quad \begin{array}{c}
0 \\
F \\
\Delta_F \\
f_* f^* \Delta_F \\
0
\end{array} \quad \xrightarrow{\text{id}} \quad \begin{array}{c}
0 \\
O_X \\
0 \\
\vdots \\
0
\end{array}
\]

Now assume $Z$ is Gorenstein. Then $O_Z$ is dualizing, and the isomorphism $O_Z \to [O_Z, O_Z]$ (adjoint to $O_Z \otimes L_i^* O_Z \simeq O_Z$ defines a form on $O_Z$, denoted by $1_Z$. On the other hand, there is a well-known form $\theta_F : K_F \to \mathcal{H}om(K_F, \Delta_F^\vee [d])$ (see \cite{7, §4}) given by the canonical isomorphism $\Lambda^i F^\vee \simeq (\Lambda^{d-i} F^\vee)^\vee \otimes \Lambda^d F^\vee$ in degree $i$, with a sign chosen so that when $F = \oplus L_i$ is a direct sum of line bundles, this form is the tensor product of forms $\theta_{L_i}$ on Koszul complexes of length one

\[
(4) \quad \mathcal{K}_{L_i} = \begin{array}{c}
0 \\
L_i^\vee \\
\vdots \\
O_X \\
0
\end{array} \quad \xrightarrow{\theta_{L_i}} \quad \begin{array}{c}
0 \\
L_i^\vee \\
\vdots \\
O_X \\
0
\end{array}
\]

representing as a concrete description of the push-forward of $1_Z$ along $f$.

**Proposition 7.1.** Let $Z$ and $X$ be Gorenstein schemes and $f : Z \to X$ be a closed regular embedding of codimension $d$ defined as the zero locus of a regular section of a vector bundle $F$ of rank $d$ whose determinant $\Lambda^d F$ is denoted by $\Delta_F$. Then the image of the form $1_Z : O_Z \xrightarrow{\sim} [O_Z, O_Z]$ (adjoint to $O_Z \otimes^L O_Z \simeq O_Z$) under the composition

\[
W^0(Z, O_Z) \simeq W^d(Z, f^! \Delta_F^\vee) \xrightarrow{f_*} W^d(X, \Delta_F^\vee)
\]
is a form \( \phi \) such that the following diagram in \( D(X) \) commutes.

\[
\begin{array}{ccc}
K_F \xrightarrow{\sim} f_*O_Z \\
\phi \downarrow & \sim & \downarrow \theta \\
\text{Hom}(K_F, \Delta^\vee_F[d]) \xrightarrow{\sim} [f_*O_Z, \Delta^\vee_F[d]]
\end{array}
\]

**Proof.** Let \( \delta_F : K_F \otimes K_F \rightarrow K_F \) in \( D(X) \) be the composition

\[
K_F \otimes K_F \xrightarrow{qis \otimes qis} f_*O_Z \otimes f_*O_Z \lambda \xrightarrow{\sim} f_*O_Z \otimes K \xrightarrow{\sim} f_*O_Z \xrightarrow{\sim} K_F
\]

where \( \lambda \) is the morphism from [11, Proposition 4.2.1]. Note that \( \delta_F \) is in fact represented by a morphism of complexes (not just a fraction): one can check that the map from \( K_F \otimes K_F \) to \( K_F \) in degree \( i \) is a sum of the canonical morphisms \( \Lambda^i F \otimes \Lambda^{i-k} F \rightarrow \Lambda^i F \) with appropriate signs. We also consider the map \( \sigma_F : K_F \rightarrow \Delta^\vee_F[d] \) given by

\[
\begin{array}{c}
\Delta^\vee_F[d] \\
\downarrow \sigma_F \\
K_F
\end{array} = \begin{pmatrix}
0 & \Lambda^1 F^\vee & \cdots & F^\vee & 0
\end{pmatrix}
\]

The following three lemmas together clearly imply the theorem.

**Lemma 7.2.** The morphism of complexes \( \pi_F : K_F \rightarrow \text{Hom}(K_F, \Delta^\vee_F[d]) \) defined as the composition

\[
K_F \xrightarrow{\text{Hom}(K_F, K_F \otimes K_F)} (\delta_F)_* \xrightarrow{\text{Hom}(K_F, K_F)} (\sigma_F)_* \xrightarrow{\text{Hom}(K_F, \Delta^\vee_F[d])}
\]

coincides with the form \( \theta \) where the first map is the unit of the adjunction of the tensor product and the internal Hom in the homotopy category.

**Lemma 7.3.** The form \( \phi : f_*O_Z \rightarrow [f_*O_Z, \Delta^\vee_F[d]] \) coincides with the composition

\[
\begin{array}{c}
[f_*O_Z, f_*O_Z \otimes L f_*O_Z] \\
\downarrow \lambda \\
[f_*O_Z, f_*O_Z] \xrightarrow{\sim} [f_*O_Z, f_*O_Z]
\end{array}
\]

where the first map is the unit of the adjunction of the tensor product and the internal Hom in \( D(X) \), the penultimate one is the identification \( O_Z \simeq f^! \Delta^\vee_F[d] \) and the last one is induced by the counit of the adjunction \( (f_*, f^!) \), i.e. the trace map described above.

**Lemma 7.4.** The composition of Lemma 7.2 coincides with the one of Lemma 7.3 when \( f_*O_Z \) is identified with \( K_F \) using \( qis \).

**Proof of Lemma 7.2.** As we are dealing with honest morphisms of complexes we may first reduce to open subsets on which \( F \) is a sum of line bundles \( L_i \) (note that two morphisms in \( D(X) \) are not necessarily equal if they are equal when restricted to all open sets of an affine covering, see for example [6]). We then reduce to the case of codimension \( d = 1 \), by multiplicativity of Koszul complexes: let \( d = d_1 + d_2 \) and let \( F = F_1 \oplus F_2 \) where \( F_1 \) (resp. \( F_2 \)) is the sum of the first \( d_1 \) line bundles (resp. last \( d_2 \)). Let \( f_1 : O_{Z_1} \rightarrow X \) (resp. \( f_2 : O_{Z_2} \rightarrow X \)) be the corresponding
regular subschemes. Then \( \Delta_f \simeq \Delta_{f_1} \otimes \Delta_{f_2} \) and \( K_F \simeq K_{F_1} \otimes K_{F_2} \). We leave it to the reader to show that the diagram

\[
\begin{array}{c}
K_{F_1} \otimes K_{F_2} \xrightarrow{\simeq} K_F \\
\xrightarrow{x_{F_1} \otimes x_{F_2}} \quad \xrightarrow{x_F} \quad \text{Hom}(K_F, \Delta_F^\vee[d])
\end{array}
\]

commutes, where \( \tau \) is the morphism defined as in [11, Definition 4.4.1], using the monoidal structure on the homotopy category. Since by definition, \( \theta_F, \theta_{F_1}, \) and \( \theta_{F_2} \) make the same diagram commutative when replacing \( x_F, x_{F_1}, \) and \( x_{F_2} \). Hence it suffices to show the lemma for one line bundle \( L \) and its associated Koszul complex of length one, which can be checked by hand.

**Proof of Lemma 7.3.** By definition, the form \( \phi \) is given by the composition

\[
f_*O_Z \xrightarrow{f_1^L} f_*[O_Z, O_Z] \xrightarrow{\simeq} f_*[O_Z, f^!\Delta^\vee_F[d]] \xrightarrow{\zeta} [f_*O_Z, \Delta^\vee_F[d]].
\]

One proves using the monoidal closed structure that it coincides with the composition

\[
f_*O_Z \xrightarrow{\simeq} f_*f^!\Delta^\vee_F[d] \xrightarrow{[f_*O_Z, f^!\Delta^\vee_F[d]]} [f_*O_Z, \Delta^\vee_F[d]]
\]

where the second map is adjoint to the unit morphism of the monoidal structure. Then, looking back at the definition of \( \zeta \) and \( \mu \) in [11, Proposition 4.2.2 and Theorem 4.2.9], one sees that \( \phi \) is the composition around the lower left corner of the commutative diagram

\[
\begin{array}{c}
\xymatrix{f_*O_Z \ar[r] & [f_*O_Z, f_*O_Z] \ar[r] & [f_*O_Z, f_*O_Z] \\
\ar[r] & [f_*f^!\Delta^\vee_F[d], f_*O_Z] \ar[r] & [f_*f^!\Delta^\vee_F[d], f_*O_Z] \\
\ar[r] & [f_*f^!\Delta^\vee_F[d], f_*O_Z] \ar[r] & [f_*f^!\Delta^\vee_F[d], f_*O_Z] \\
\ar[r] & [f_*f^!\Delta^\vee_F[d], f_*O_Z] \ar[r] & [f_*O_Z, \Delta^\vee_F[d]]
}\end{array}
\]

which thus proves the lemma (all squares in this diagram are commutative by obvious functorial reasons, and the triangle by adjunction).

**Proof of Lemma 7.4.** This follows from the computation of the resolution of \( f_*O_Z \) by \( K_F \) when computing the derived functors \( [f_*O_Z, -] \) and \( -\otimes^L f_*O_Z \).

This finishes the proof of Proposition 7.1.

**Remark 7.5.** If \( F = F' \oplus L_1 \), with \( L_1 \) a line bundle, \( s = (s', s_1) \), \( s' \) and \( s_1 \) transverse, the push-forward of \( 1_Z \) is zero: decompose the inclusion \( Z \subset X \) as \( Z \subset Z(s') \subset X \) where \( Z(s') \) is the zero locus of \( s' \). Push-forwards respect composition and the push-forward of \( 1_Z \) along \( Z \subset Z(s') \) is already zero since it is the form (4) which is the cone of a (degenerate) form \( s : L'_1 \to O_X \).

On the other hand, an example when this push-forward is nonzero can be extracted from [5]. Let \( k \) be a field and let \( Gr_k(2, 4) \) be the Grassmannian of 2-planes in \( A_k^4 \). A nonzero map from \( A_k^4 \) to \( A_k^4 \) induces a section of the dual \( W^\vee \) of the universal subbundle \( W \subset A_k^4 \) of rank 2. Its zero locus is a copy of \( P^2 \) regularly embedded in \( Gr(2, 4) \). The push-forward of the unit form of \( P^2 \) to \( Gr(2, 4) \) is nonzero by [5] where it is proved that it is an element of a basis of the total Witt group of \( Gr(2, 4) \) as a \( W(k) \)-module.
7.3. Projective space. Let $Y \in \text{Sch}$ be a Gorenstein scheme, let $E$ be a vector bundle of rank $r + 1$ on $Y$ and let us examine when the unit form on $X = \mathbb{P}_Y(E)$ can be pushed forward to $Y$ along $f : X \to Y$. Since $f$ is smooth (thus flat), we can use section 6.1. In the case of a smooth morphism $f$, the canonical sheaf $\omega_f$ is a line bundle, and it is the maximal exterior power of the relative cotangent bundle (see [22, Ch. VII §4]). Here, since $f$ is a projective bundle, it is given by $\omega_f = f^*(\det E)^r \otimes \mathcal{O}(-r - 1)$ (see e.g. [13, Appendix B.5.8]). If $r + 1$ is even, we can push-forward the unit form $1_X : \mathcal{O}_X \simeq [\mathcal{O}_X, \mathcal{O}_X]$ from $W^0(X, \mathcal{O}_X)$ by using the composition

$$W^0(X, \mathcal{O}_X) \simeq W^0(X, \mathcal{O}(-r - 1)) = W^0(X, \omega_f \otimes f^*(\det E)) \to W^{-r}(Y, \det E)$$

where the first isomorphism is given by tensoring with the canonical form $\phi_r = [\mathcal{O}(-r+1)/2] \simeq [\mathcal{O}((r+1)/2) \otimes \mathcal{O}(-r+1)] \cong \text{Hom}_\mathcal{O}([\mathcal{O}(-r+1)/2], \mathcal{O}(-r+1))$ and the last map in the push-forward in the form of Theorem 6.2. Computing the image of $1_X$ through this composition means therefore computing the image of $\phi_r$ by the push-forward. The complex on which $K$-injective) if the triangulated functor $(\cdot \otimes A) : K(X) \to K(X)$ preserves quasi-isomorphisms. We say that $A$ is q-injective (or K-injective) if the triangulated functor $\text{Hom}_A(\cdot, A) : K(X) \to K(X)$ preserves quasi-isomorphisms.

**Example A.2.** A bounded above complex of flat $\mathcal{O}_X$-modules is q-flat. A bounded below complex of injectives is q-injective.
A discussion of q-flat and q-injective complexes can be found in [34, Sections 1 and 5]. See in particular Propositions 1.5 and 5.3.

**Proposition A.3.** Let $A$ and $B$ be objects in $K(X)$ or $K(Y)$ and let $f : X \rightarrow Y$ be a morphism of schemes.

1. If $A$ and $B$ are q-flat, then so is $A \otimes B$.
2. If $A$ is q-flat and $B$ is q-injective, then $\text{Hom}(A, B)$ is q-injective.
3. If $A$ is q-flat, then $f^*A$ is q-flat.

**Proof.** See [34, Prop. 5.3 and 5.4].

The following two propositions summarize the equivalences of categories and the properties of injectives that we need. Let $\text{Qcoh}(X)$ denote the abelian category of quasi-coherent sheaves on $X$ and $\text{C}_b(\text{Vect}(X))$ (resp. $\text{C}_b(\text{Coh}(X))$) the category of bounded complexes of locally free (resp. coherent) sheaves on $X$.

**Proposition A.4.** Let $X \in \text{Sch}$.

1. The natural functor $D(\text{Qcoh}(X)) \rightarrow D_{\text{qc}}(X)$ is an equivalence of categories and thus the same is true for their homologically bounded, bounded below or bounded above subcategories and the subcategories with coherent homology.
2. The natural functor $D_b(\text{Coh}(X)) \rightarrow D_{b,c}(\text{Qcoh}(X))$ is an equivalence.
3. If $X \in \text{Reg}$, then the natural functor $D(\text{C}_b(\text{Vect}(X))) \rightarrow D_{b,c}(X)$ is an equivalence of categories.

**Proof.** Point (1) is [8, Cor. 5.5]. In point (2), fully faithful follows from [24, Theorem 12.1], second part: For affine schemes, use [24, Example 12.3]. In general, take a finite affine cover of the noetherian scheme $X$ and then take the direct sum over the coherent sheaves on $X$ obtained by extending the coherent sheaves on the affine subschemes using [21, I.6.9.7]. Essentially surjective can be found in [15, Example 2.5.2] which follows from [37, Lemme 2.1.2.c)] and an induction argument. Point (3) can then be proved as follows. Let $\text{C}_b(\text{Coh}(X))$ be the category of bounded complexes of coherent sheaves. Decompose the functor $D(\text{C}_b(\text{Vect}(X))) \rightarrow D_{b,c}(X)$ as

$$D(\text{C}_b(\text{Vect}(X))) \rightarrow \cdots \rightarrow D_b(\text{Coh}(X)) \rightarrow D_{b,c}(\text{Qcoh}(X)) \rightarrow D_{b,c}(X).$$

All these functors are equivalences of categories: the first one by the fact since $X$ is regular, every coherent sheaf has a finite resolution by locally free sheaves by [32, §7 Point 1], the second one by [24, Lemma 11.7], the third one by Point (2) and the fourth by Point (1).

**Proposition A.5.** Let $X \in \text{Sch}$.

1. The category $\text{Qcoh}(X)$ has enough injectives by [35, B.3].
2. The natural inclusion $\text{Qcoh}(X) \rightarrow \text{Mod}(X)$ preserves injectives by [35, B.4].
3. Every bounded below complex of quasi-coherent $\mathcal{O}_X$-modules admits a quasi-isomorphism into a complex of $\text{Qcoh}(X)$-injectives by (1) and [22, I.4.6].
4. Every bounded below complex of $\text{Qcoh}(X)$-injectives is q-injective in both $K(\text{Qcoh}(X))$ and $K(X)$ by (2).

**Corollary A.6.** Let $X \in \text{Sch}$. On objects in $D_{-\text{qc}}(X)$ or $D_-(X)$, the unbounded right derived functors computed by q-injective resolutions (as in [34]) coincide with the more classical bounded below right derived functors computed by replacing a complex by a bounded below complex of injectives (as in [22]).

**Proof.** By the proposition, any object $A \in D_{-\text{qc}}(X)$ is quasi-isomorphic to a bounded below complex of $\text{Qcoh}(X)$-injective sheaves which are also $\text{Mod}(X)$-injective (resp. a bounded below complex of $\text{Mod}(X)$-injective sheaves) and this complex is q-injective.
Similarly, for q-flat resolutions, we have a weaker statement (sufficient for our purposes).

**Proposition A.7.** Let \( X \in \text{Sch} \). On objects in \( D_{+,-q}(X) \) or \( D_{-}(X) \), the unbounded left derived functors computed using q-flat resolutions can be computed by replacing a complex by a bounded above complex of flat \( O_{X} \)-modules.

**Proof.** This follows from the fact that any \( O_{X} \)-module is a quotient of a flat one (see [26, 2.5.5]). \( \square \)

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