On the Λ-Convex Hull for Convex Integration
Applied to the Isentropic Compressible Euler System

Simon Markfelder

January 14, 2020

Institute of Mathematics, Würzburg University
Emil-Fischer-Str. 40, 97074 Würzburg, Germany

Abstract

We consider the isentropic compressible Euler equations in multiple space dimensions. In the past it has been shown via convex integration that this system allows for infinitely many solutions. However all non-uniqueness results available in the literature were achieved by reducing the equations to some kind of incompressible system and applying convex integration to the latter. This ansatz seems to be quite restrictive concerning the solutions that are obtained and also concerning the initial data for which the method works. A direct application of convex integration to the compressible Euler equations could overcome these restrictions. This paper can be viewed as the first step towards such a direct application as we present a new setup for convex integration and compute the corresponding Λ-convex hull.

Contents

1 Introduction 2
2 Preliminaries 4
  2.1 Adjusting the Problem 4
  2.2 Tartar’s Framework 5
  2.3 Wave Cone 5
3 On Convex and Λ-Convex Hulls 6
4 Computation of the Λ-Convex Hull of $K$ 8
5 Concluding Remarks 14
A Minkowski’s Theorem 14
1 Introduction

In this paper we consider the isentropic compressible Euler equations

\begin{align}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla p(\rho) &= 0.
\end{align}

The unknowns are the density \( \rho = \rho(t, x) \in \mathbb{R}^+ \) and the velocity \( u = u(t, x) \in \mathbb{R}^n \), which are both functions of time \( t \in [0, \infty) \) and position \( x \in \mathbb{R}^n \). In this paper the space dimension \( n \) can be anything larger than 1. Furthermore we exclude vacuum in our consideration, i.e. we assume \( \rho > 0 \). The pressure \( p = p(\rho) \) is a convex function of \( \rho \).

It turns out that it is more convenient to write system (1) in terms of momentum \( m \) instead of velocity \( u \), i.e.

\begin{align}
\partial_t \rho + \text{div} m &= 0, \\
\partial_t m + \text{div} \left( \frac{m \otimes m}{\rho} + p(\rho) I \right) &= 0.
\end{align}

Note again that since we exclude vacuum, the \( \rho \) in the denominator cannot cause any problems.

The isentropic compressible Euler system is a basic example of a hyperbolic system of conservation laws, which describes the time evolution of a compressible fluid by conservation of mass and momentum. Typically one considers initial value problems to such systems, i.e. one imposes the initial configuration of the fluid

\[ \rho(0, \cdot) = \rho_0, \quad m(0, \cdot) = m_0 \]

and the aim is to find out how the fluid behaves at later times. It is well-known that even for scalar conservation laws, strong solutions of the corresponding initial value problem do not exist globally in time, no matter how smooth or small the initial data are. To overcome this problem one typically considers weak solutions. Because of many simple examples of initial data for which there are infinitely many weak solutions, one imposes admissibility criteria to single out the physically relevant weak solutions. For scalar conservation laws it turned out that the entropy criterion is a satisfying criterion, in the sense that weak solutions fulfilling the entropy condition exist, are unique and depend continuously on the initial data, see Kružkov [13].

A similar well-posedness theory for the initial value problem for systems of conservation laws is far from being reached. On the one hand little is known in one space dimension, as Glimm [10] showed existence of weak entropy solutions for sufficiently small initial data and Bressan et al. [1] proved uniqueness of these solutions and continuous dependence on the initial data. In multiple space dimension even negative results are known. It was shown originally by De Lellis and Székelyhidi [7] that there exist bounded initial data for the Euler system [1] for which there are infinitely many weak solutions, all of which fulfill the corresponding entropy criterion which is in the case of compressible Euler the energy inequality

\[ \partial_t \left( \frac{1}{2} \frac{|m|^2}{\rho} + P(\rho) \right) + \text{div} \left[ \left( \frac{1}{2} \frac{|m|^2}{\rho} + P(\rho) + p(\rho) \right) \frac{m}{\rho} \right] \leq 0. \]
Here $P$ is the *pressure potential*, a function of the density $\varrho$ which is related to $p$ via $p(\varrho) = \varrho P'(\varrho) - P(\varrho)$. This non-uniqueness result for compressible Euler is based on convex integration applied to the *incompressible* Euler system.

The convex integration technique was developed by Gromov \cite{Gromov} in the context of partial differential relations. Later De Lellis and Székelyhidi \cite{DeLellisSzekelyhid1, DeLellisSzekelyhid2} applied this method to the incompressible Euler equations

\begin{equation}
\text{div } v = 0,
\partial_t v + \text{div} (v \otimes v) + \nabla p = 0,
\end{equation}

with unknown velocity $v = v(t, x)$ and pressure $p = p(t, x)$. More precisely they showed existence of bounded initial data to which there are infinitely many weak solutions of the initial value problem to \cite{DeLellisSzekelyhid1} where one can prescribe the kinetic energy $\frac{1}{2}|v|^2$ for all times $t$ and almost all positions $x$. If one sets the kinetic energy to be constant then one can achieve infinitely many weak solution to \cite{DeLellisSzekelyhid1} with constant pressure (here *constant* means constant in time and space).

It is easy to verify that $(\varrho, u)$ defined by $\varrho \equiv 1$ and $u = v$, where $v$ is a solution to the incompressible Euler equations \cite{DeLellisSzekelyhid1} with constant pressure, solves the compressible Euler system \cite{DeLellisSzekelyhid1}. This is how De Lellis and Székelyhidi deduced the above mentioned non-uniqueness result for compressible Euler from their theory for incompressible Euler.

This first result on non-uniqueness for the compressible Euler equations has been further improved. Chiodaroli \cite{Chiodaroli} as well as Feireisl \cite{Feireisl} showed that for any continuously differentiable initial density $\varrho_0$ there exists a bounded initial momentum for which there are infinitely many weak entropy solutions to the initial value problem for the compressible Euler system \cite{DeLellisSzekelyhid1}. Chiodaroli’s ansatz is to look for solutions with constant-in-time density $\varrho(t, \cdot) = \varrho_0$, $\forall t$. This leads to some kind of “incompressible system” for the momentum $m$ which includes the prescribed initial density $\varrho_0$. A slight modification of De Lellis’ and Székelyhidi’s convex integration then yields the result.

Feireisl’s approach is even more general. The idea is to apply Helmholtz decomposition to the momentum, i.e. $m = v + \nabla \Phi$ where $v$ is div-free and $\Phi$ is a scalar field. He then prescribes $\varrho$ and $\Phi$ such that they are compatible with the conservation of mass \cite{Feireisl}, i.e. $\partial_t \varrho + \Delta \Phi = 0$. Finally one ends up with some kind of “incompressible system” again, to which one applies a modified version of De Lellis’ and Székelyhidi’s convex integration.

In addition to that the literature provides non-uniqueness results for the initial value problem to compressible Euler with a special type of initial data that are inspired by one-dimensional Riemann problems, see \cite{Chiodaroli, Feireisl, Feireisl2}. These solutions are constructed with piecewise constant densities and again a slight modification of De Lellis’ and Székelyhidi’s convex integration.

**Remark.** To the best of the author’s knowledge the reduction of the problem to some kind of “incompressible system” for which a slight modification of De Lellis’ and Székelyhidi’s convex integration can be applied is common to all non-uniqueness results for compressible Euler available in the literature.

This is the reason why the solutions constructed by convex integration so far only contain oscillations in the momentum and not in the density. As shown by Feireisl et al. \cite{Feireisl} the set of initial data, for which one can construct convex integration solutions by now, is rather small.
This seems to be surprising since it contrasts with a result by Székelyhidi and Wiedemann [14] who showed that the opposite is true for incompressible Euler: The set of initial data for which there exist infinitely many solutions is $L^2$-dense in the set of all initial data. On the other hand a new convex integration approach for compressible Euler could work for a larger set of initial data.

The author’s motivation is to apply convex integration directly to the isentropic compressible Euler system [2]. This paper can be viewed as the first step in this direction since we will fix a suitable setup for which we will compute the $\Lambda$-convex hull that is needed in order to implement convex integration. In particular we do not produce any solutions in this paper. Hence we will forget about initial data. Note furthermore that convex integration itself does not care about admissibility criteria. Our setup will not respect the energy inequality [3]. We will come back to this issue in Section [2.1].

The paper is organized as follows. In Section 2 we fix a convex-integration-setup for the isentropic compressible Euler equations [2]. In particular we will find a set $K$ and consider the wave cone $\Lambda$ which corresponds to our setup. In Section 3 we present the definition of “$\Lambda$-convex” and discuss some general facts about $\Lambda$-convex hulls. The main part of this paper is Section 4 where the $\Lambda$-convex hull is computed. We finish with some remarks in Section 5.

2 Preliminaries

2.1 Adjusting the Problem

Mimicking the procedure in [6] and [7], where the kinetic energy $|v|^2 = \text{tr} (v \otimes v)$ was prescribed for the incompressible Euler system, we want to prescribe

$$\text{tr} \left( \frac{m \otimes m}{\rho} + p(\rho) I \right) = \frac{|m|^2}{\rho} + \frac{np(\rho)}{n} \equiv \tau,$$

where $\tau = \tau(t, x)$. Since we want to work with traceless matrices, we reformulate [2] as

$$\partial_t m + \text{div} \left( \frac{m \otimes m}{\rho} + p(\rho) I - \frac{c}{n} I \right) + \nabla \tau = 0.$$

For simplicity we look for solutions with constant $\tau =: c$ in this paper. Therefore we can rewrite (2) as

$$\partial_t \rho + \text{div} m = 0,$$

$$\partial_t m + \text{div} \left( \frac{m \otimes m}{\rho} + p(\rho) I - \frac{c}{n} I \right) = 0. \quad (5)$$

Remark. Note that if a monoatomic gas is considered, i.e. $p(\rho) = \rho^{\frac{2}{\gamma}+1}$, then $P(\rho) = \frac{C}{\gamma} p(\rho)$. Hence in this case we have

$$\frac{\tau}{2} = \frac{1}{2} \frac{|m|^2}{\rho} + P(\rho).$$

In other words we prescribe the energy. This means that in the case of a monoatomic gas, an energy criterion is contained in our setup. To be precise this criterion is not the energy...
inequality (3) but a global version which states that the total energy
\[ \int \left( \frac{1}{2} \frac{|m|^2}{\rho} + P(\rho) \right) \, dx \]
must not increase in time, which is true if \( \tau = c = \text{const.} \).

### 2.2 Tartar’s Framework

First we replace \( m \otimes m + P(\rho)I - cI \) in (5) by a new unknown \( U = U(t, x) \), which takes values in \( S_0^{n \times n} := \{ A \in \mathbb{R}^{n \times n} \mid A \text{ symmetric and } \text{tr} A = 0 \} \). Then we obtain a linear system with the following property which simply follows from the arguments in Section 2.1.

**Proposition 2.1.** If \((\rho, m, U)\) solves
\[ \begin{align*}
\partial_t \rho + \text{div} m &= 0, \\
\partial_t m + \text{div} U &= 0,
\end{align*} \tag{6} \]
and takes values in
\[ K := \left\{ (\rho, m, U) \in \mathbb{R}^+ \times \mathbb{R}^n \times S_0^{n \times n} \mid U = \frac{m \otimes m}{\rho} + \left( \frac{P(\rho) - c}{n} \right) I \right\}, \tag{7} \]
then \((\rho, m)\) solves (5) and hence (2) together with
\[ \frac{|m|^2}{\rho} + np(\rho) = c. \]

### 2.3 Wave Cone

The idea of convex integration is now to relax the set \( K \) to a larger set \( \mathcal{U} \). Due to Proposition 2.1 we call a triple \((\rho, m, U)\) a *solution* if it solves (6) and takes values in \( K \). A triple \((\rho, m, U)\) solving (6) and taking values in \( \mathcal{U} \) is called a *subsolution*. Next, one wants to construct a sequence of subsolutions by successively adding oscillations. In [6, 7] localized planar waves have been used for such oscillations. The relaxed set \( \mathcal{U} \) has to be chosen such that it is compatible with the oscillations that are added in each step, which we choose to be planar waves as well. In order to specify \( \mathcal{U} \) we have to study planar waves and the wave cone.

A plane wave solution to (6) is a solution \((\rho, m, U)\) of the form
\[ (\rho, m, U)(t, x) = (\overline{\rho}, \overline{m}, \overline{U}) h((t, x) \cdot \xi) \tag{8} \]
with a constant \((\overline{\rho}, \overline{m}, \overline{U}) \in \mathbb{R} \times \mathbb{R}^n \times S_0^{n \times n} \), a function \( h : \mathbb{R} \to \mathbb{R} \) and a direction in space-time \( \xi \in \mathbb{R}^{n+1} \setminus \{0\} \). Here \( \cdot \) denotes the scalar product in space-time \( \mathbb{R}^{n+1} \).

**Definition 2.2.** Define the *wave cone* as
\[ \Lambda := \left\{ (\overline{\rho}, \overline{m}, \overline{U}) \in \mathbb{R} \times \mathbb{R}^n \times S_0^{n \times n} \mid \exists \xi \in \mathbb{R}^{n+1} \setminus \{0\} \text{ such that } \overline{\rho} \xi_t + \overline{m} \cdot \xi_x = 0 \text{ and } \overline{m} \xi_t + \overline{U} \xi_x = 0 \right\}, \]
where we denote the components of \( \xi \in \mathbb{R}^{n+1} \) as \( \xi = (\xi_t, \xi_1, \ldots, \xi_n)^T \) and \( \xi_x := (\xi_1, \ldots, \xi_n)^T \).
The following proposition shows the meaning of Definition 2.2.

**Proposition 2.3.** Let \((\overline{\varphi}, \overline{m}, \overline{U}) \in \Lambda\) and \(h \in C^1(\mathbb{R}, \mathbb{R})\) arbitrary. Then \((\varphi, m, U)\) defined as in (8) with the corresponding \(\xi\), that exists according to the Definition 2.2, is a plane wave solution to (6).

**Proof.** The proof is a simple calculation. Indeed we have

\[
\partial_t \varphi + \text{div} m = \overline{\varphi} \partial_t \left( h((t, x) \cdot \xi) \right) + \overline{m} \cdot \nabla \left( h((t, x) \cdot \xi) \right) = h'((t, x) \cdot \xi) \left( \overline{\varphi} \xi_t + \overline{m} \cdot \xi_x \right) = 0
\]

and

\[
\partial_t m + \text{div} U = \overline{m} \partial_t \left( h((t, x) \cdot \xi) \right) + \overline{U} \cdot \nabla \left( h((t, x) \cdot \xi) \right) = h'((t, x) \cdot \xi) \left( \overline{m} \xi_t + \overline{U} \xi_x \right) = 0,
\]

where we have used the fact that \(\overline{U}\) is symmetric. \(\square\)

Note that \((\overline{\varphi}, \overline{m}, \overline{U}) \in \Lambda\) if and only if the kernel of the matrix

\[
\begin{pmatrix}
\overline{\varphi} & m^T \\
m & U
\end{pmatrix}
\]

contains \(\xi \neq 0\). This holds if and only if its determinant is zero. Hence we can write

\[
\Lambda = \left\{ (\overline{\varphi}, m, U) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_0^{n \times n} \mid \det \begin{pmatrix} \overline{\varphi} & m^T \\ m & U \end{pmatrix} = 0 \right\}.
\] (9)

As pointed out in the beginning of section 2.3 we have to choose \(\mathcal{U}\) in such a way that it is compatible with planar waves. This means that \(\mathcal{U}\) must be equal to the \(\Lambda\)-convex hull of \(K\), which can be seen from Lemma 3.5 below. For the definition of the \(\Lambda\)-convex hull we refer to Section 3.

### 3 On Convex and \(\Lambda\)-Convex Hulls

We will discuss some general facts about \(\Lambda\)-convex hulls in this section. These facts will help us the compute the \(\Lambda\)-convex hull of \(K\) later.

**Remark.** The statements presented here hold for every cone \(\Lambda \subseteq \mathbb{R}^N\) and every subset \(K \subseteq \mathbb{R}^N\) with \(N > 1\).

We first recall some definitions.

**Definition 3.1.** A set \(\Lambda \subseteq \mathbb{R}^N\) is called \textit{cone} if \(\Lambda \neq \emptyset\) and \(\forall p \in \Lambda, \forall \alpha \in \mathbb{R} : \alpha p \in \Lambda\).

**Remark.** Note that every cone contains 0. This will be used in the proof of Lemma 3.5.

**Definition 3.2.** A set \(S \subseteq \mathbb{R}^N\) is called
• convex if $\forall p, q \in S : [p, q] \subseteq S$,

• $\Lambda$-convex if $\forall p, q \in S$ with $p - q \in \Lambda : [p, q] \subseteq S$.

**Definition 3.3.**

- The convex hull $K^{\text{co}}$ is the smallest convex set which contains $K$.

- The $\Lambda$-convex hull $K^{\Lambda}$ is the smallest $\Lambda$-convex set which contains $K$.

Let us continue with some basic facts on the notions defined above.

**Proposition 3.4.**

(a) Every convex set is $\Lambda$-convex.

(b) $K^{\Lambda} \subseteq K^{\text{co}}$.

(c) If $p - q \in \Lambda$ for all $p, q \in K$, then $K^{\Lambda} = K^{\text{co}}$.

**Proof.** Part (a) is an immediate consequence of Definition 3.2 and (b) simply follows from (a). If $p - q \in \Lambda$ for all $p, q \in K$ then the notions convex and $\Lambda$-convex are equivalent which shows (c). \qed

**Remark.** Note that the wave cone $\Lambda$ for incompressible Euler is so large that part (c) of Proposition 3.4 holds, which is hidden in the proof of [6, Lemma 4.3] and [7, Proposition 4]. Then one has immediately $K^{\Lambda} = K^{\text{co}}$ which is an advantage since it is easier to compute convex hulls instead of $\Lambda$-convex hulls. The reason for this is in our context that for convex hulls one can apply Minkowski’s theorem (see Theorem A.1 in the appendix) which was also used by De Lellis and Székelyhidi in the proof of [7, Lemma 3].

In our setup for the compressible Euler system one can show that there exist $p, q \in K$ with $p - q \notin K$. Hence part (c) of Proposition 3.4 is not applicable. However we will show that we still have $K^{\Lambda} = K^{\text{co}}$.

The following lemma will be used later. Furthermore it enlightens why one has to use $K^{\Lambda}$ for the relaxed set $U$.

**Lemma 3.5.** Define for $i \in \mathbb{N}_0$

$$K^i := \begin{cases} K & \text{if } i = 0 \\ \{ s \in \mathbb{R}^N | \exists p, q \in K^{i-1} \text{ with } p - q \in \Lambda \text{ such that } s \in [p, q] \} & \text{if } i > 0. \end{cases}$$

Then $K^{\Lambda} = \bigcup_{i \in \mathbb{N}_0} K^i$.

**Remark.** Crippa et al. [5] use the notation of an $H_n$-condition which is similar to the $K^i$ defined above.
Proof. Note first, that $K_i \subseteq K_{i+1}$ for all $i \in \mathbb{N}_0$. Indeed if $y \in K_i$ then set $p = q := y$. Then $p, q \in K_i$ with $p - q = 0 \in \Lambda$ and $y \in [p, q]$, hence $y \in K_{i+1}$.

Now we show $K^\Lambda \subseteq \bigcup_{i \in \mathbb{N}_0} K^i$. By the observation above we have $K \subseteq \bigcup_{i \in \mathbb{N}_0} K^i$. To show that $\bigcup_{i \in \mathbb{N}_0} K^i$ is $\Lambda$-convex, let $s_1, s_2 \in \bigcup_{i \in \mathbb{N}_0} K^i$ with $s_1 - s_2 \in \Lambda$. Again with the observation above we find $j \in \mathbb{N}_0$ with $s_1, s_2 \in K^j$. Then $[s_1, s_2] \subseteq K^{j+1}$ and hence $[s_1, s_2] \subseteq \bigcup_{i \in \mathbb{N}_0} K^i$.

It remains to prove that $\bigcup_{i \in \mathbb{N}_0} K^i \subseteq K^\Lambda$. We show by induction over $i \in \mathbb{N}_0$ that $K^i \subseteq K^\Lambda$ for all $i \in \mathbb{N}_0$. Let first $i = 0$, then trivially $K^0 = K \subseteq K^\Lambda$. For the induction step we assume that $K^i \subseteq K^\Lambda$ for some $i \in \mathbb{N}_0$. Let $s \in K^{i+1}$. Then there exist $p, q \in K^i$ with $p - q \in \Lambda$ and $s \in [p, q]$. The induction assumption yields $p, q \in K^\Lambda$. Since $K^\Lambda$ is $\Lambda$-convex, we deduce $[p, q] \subseteq K^\Lambda$ which gives $s \in K^\Lambda$.

The following theorem is a simple observation. However it will be an important ingredient of the proof of our main result Theorem 4.3.

Theorem 3.6. Define $K^* := \bigcup_{\overline{x} \in \mathbb{R}} (K \cap \{x_1 = \overline{x}\})^\Lambda$. Then $K^* \subseteq K^\Lambda$.

Proof. We prove by induction over $i \in \mathbb{N}_0$ that $(K \cap \{x_1 = \overline{x}_i\})^i \subseteq K^i$. Obviously this is true for $i = 0$. Let for the induction step $x \in (K \cap \{x_1 = \overline{x}_i\})^{i+1}$. Then by definition there exist $p, q \in (K \cap \{x_1 = \overline{x}_i\})^i$ with $p - q \in \Lambda$ and $x \in [p, q]$. By induction assumption these $p, q$ lie also in $K^i$, which shows $x \in K^{i+1}$.

Let now $y \in K^*$. Then $y \in (K \cap \{x_1 = \overline{y}_i\})^\Lambda$. Hence by Lemma 3.5 there exists a $j \in \mathbb{N}_0$ such that $y \in (K \cap \{x_1 = \overline{y}_j\})^j$. What we showed above yields $y \in K^j$ and hence $y \in K^\Lambda$.

4 Computation of the $\Lambda$-Convex Hull of $K$

Now we turn our attention back to the Euler equations, i.e. we consider $K$ as in (7) and $\Lambda$ as in (3). For a symmetric matrix $A$ we define $\lambda_{\text{max}}(A)$ as the largest eigenvalue of $A$.

As in Theorem 3.6 we define

$$K^* = \bigcup_{\overline{y} \in \mathbb{R}^+} (K \cap \{y = \overline{y}\})^\Lambda. \quad (10)$$

We will finally prove that $K^* = K^\Lambda = K^{co}$, see Theorem 4.3 below. To this end we need the following two lemmas, the first of which (Lemma 4.1) is a “compressible variant” of [7, Lemma 3 (i)]. The second lemma (Lemma 4.2) can be deduced form [7, Lemma 3 (iv)] since here the density $\overline{y}$ is constant. For completeness we redo this proof.

Lemma 4.1. The mapping

$$(p, m, \mathbb{U}) \mapsto \lambda_{\text{max}} \left( \frac{m \otimes m}{\overline{y}} + p(\overline{y})\mathbb{1} - \mathbb{U} \right)$$

is convex.
Proof. We mimick the proof of [7, Lemma 3]. The first steps are exactly the same as in [7]. For completeness we redo them. First we show that

\[ \lambda_{\text{max}} \left( \frac{m \otimes m}{q} + p(q)I - U \right) = \max_{y \in \mathbb{R}^n, |y| = 1} y \cdot \left( \left( \frac{m \otimes m}{q} + p(q)I - U \right)y \right). \]

Since \( \frac{m \otimes m}{q} + p(q)I - U \) is symmetric, it is diagonalizable with orthogonal eigenvectors. Let \( \lambda_1 \leq \ldots \leq \lambda_n \) the eigenvalues and \( b_1, \ldots, b_n \in \mathbb{R}^n \) the corresponding normed eigenvectors. Then

\[ \lambda_{\text{max}} \left( \frac{m \otimes m}{q} + p(q)I - U \right) = \lambda_n = \langle b_n, b_n \rangle = \lambda_n \left( b_n \right) \]

\[ = b_n \cdot \left( \left( \frac{m \otimes m}{q} + p(q)I - U \right)b_n \right) \]

\[ \leq \max_{y \in \mathbb{R}^n, |y| = 1} y \cdot \left( \left( \frac{m \otimes m}{q} + p(q)I - U \right)y \right). \]

Let \( \bar{y} \in \mathbb{R}^n, |\bar{y}| = 1 \) such that

\[ \max_{y \in \mathbb{R}^n, |y| = 1} y \cdot \left( \left( \frac{m \otimes m}{q} + p(q)I - U \right)y \right) = \bar{y} \cdot \left( \left( \frac{m \otimes m}{q} + p(q)I - U \right)\bar{y} \right). \]

Because \( b_1, \ldots, b_n \) form a basis of \( \mathbb{R}^n \), there exist unique coefficients \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) such that \( \bar{y} = \sum_{i=1}^{n} \alpha_i b_i \), and from \( |\bar{y}| = 1 \) and the fact that \( b_1, \ldots, b_n \) form an orthonormal basis we deduce \( \sum_{i=1}^{n} \alpha_i^2 = 1 \). So we obtain

\[ \max_{y \in \mathbb{R}^n, |y| = 1} y \cdot \left( \left( \frac{m \otimes m}{q} + p(q)I - U \right)y \right) = \bar{y} \cdot \left( \left( \frac{m \otimes m}{q} + p(q)I - U \right)\bar{y} \right) = \sum_{i,j=1}^{n} \alpha_i \alpha_j b_i \cdot \left( \left( \frac{m \otimes m}{q} + p(q)I - U \right)b_j \right) = \sum_{i=1}^{n} \alpha_i^2 \lambda_i \leq \lambda_n. \]

An easy calculation yields

\[ y \cdot \left( \left( \frac{m \otimes m}{q} + p(q)I - U \right)y \right) = \frac{1}{q} y \cdot (mm^\top y) + p(q) - y \cdot (Uy) = \frac{1}{q} (y \cdot m)^2 + p(q) - y \cdot (Uy) \]

for all \( y \in \mathbb{R}^n \) with \( |y| = 1 \).

Hence we have to show that

\[ (q, m, U) \mapsto \max_{y \in \mathbb{R}^n, |y| = 1} \left( \frac{1}{q} (y \cdot m)^2 + p(q) - y \cdot (Uy) \right) \]
is convex. From here on the proof slightly differs from the one of [7, Lemma 3].

Let \((q_1, m_1, U_1), (q_2, m_2, U_2) \in \mathbb{R}^+ \times \mathbb{R}^n \times S_0^{n \times n}\) and \(\tau \in [0, 1]\). Furthermore let \(\mathbf{y} \in \mathbb{R}^n\), \(|\mathbf{y}| = 1\) such that

\[
\max_{y \in \mathbb{R}^n, |y| = 1} \left( \frac{1}{\tau q_1 + (1 - \tau) q_2} \left( y \cdot (\tau m_1 + (1 - \tau) m_2) \right)^2 + p(\tau q_1 + (1 - \tau) q_2) - y \cdot ((\tau U_1 + (1 - \tau) U_2) y) \right)
\]

\[
= \frac{1}{\tau q_1 + (1 - \tau) q_2} \left( \mathbf{y} \cdot (\tau m_1 + (1 - \tau) m_2) \right)^2 + p(\tau q_1 + (1 - \tau) q_2) - \mathbf{y} \cdot ((\tau U_1 + (1 - \tau) U_2) \mathbf{y})
\]

(11)

We consider each summand separately in (11) and obtain

\[
p(\tau q_1 + (1 - \tau) q_2) \leq \tau p(q_1) + (1 - \tau) p(q_2)
\]

since \(p(\cdot)\) is convex. Furthermore

\[
-\mathbf{y} \cdot ((\tau U_1 + (1 - \tau) U_2) \mathbf{y}) = -\tau \mathbf{y} \cdot (U_1 \mathbf{y}) - (1 - \tau) \mathbf{y} \cdot (U_2 \mathbf{y}).
\]

What remains is to look at the first summand in (11). To handle this summand, we show that for all \(a, b > 0, c, d \in \mathbb{R}, \tau \in [0, 1]\) it holds that

\[
\frac{1}{\tau a + (1 - \tau)b} \left( \tau c + (1 - \tau)d \right)^2 \leq \frac{\tau c^2}{a} + (1 - \tau)\frac{d^2}{b}.
\]

(12)

Obviously we have

\[
0 \leq \tau (1 - \tau) (ad - bc)^2 = \tau^2 ab^2 + (1 - \tau)^2 abd^2 + \tau(1 - \tau)(a^2 d^2 + b^2 c^2) - \tau^2 abc^2 - (1 - \tau)^2 abd^2 - 2\tau(1 - \tau)abcd,
\]

what is equivalent to

\[
\tau^2 ab^2 + (1 - \tau)^2 abd^2 + 2\tau(1 - \tau)abcd \leq \tau^2 abc^2 + (1 - \tau)^2 abd^2 + \tau(1 - \tau)(a^2 d^2 + b^2 c^2).
\]

This yields

\[
ab(\tau c + (1 - \tau)d)^2 \leq (\tau a + (1 - \tau)b)(\tau c^2 b + (1 - \tau)d^2 a).
\]

Deviding by the positive expression \(ab(\tau a + (1 - \tau)b)\) leads to (12).

Let us set \(a := q_1, b := q_2, c := \mathbf{y} \cdot m_1, d := \mathbf{y} \cdot m_2\) in (12) to estimate the first summand in (11). We obtain

\[
\frac{1}{\tau q_1 + (1 - \tau) q_2} \left( \mathbf{y} \cdot m_1 + (1 - \tau) \mathbf{y} \cdot m_2 \right)^2 \leq \tau \left( \frac{\mathbf{y} \cdot m_1}{q_1} \right)^2 + (1 - \tau) \left( \frac{\mathbf{y} \cdot m_2}{q_2} \right)^2.
\]

All in all we have

\[
\max_{y \in \mathbb{R}^n, |y| = 1} \left( \frac{1}{\tau q_1 + (1 - \tau) q_2} \left( y \cdot (\tau m_1 + (1 - \tau) m_2) \right)^2 + p(\tau q_1 + (1 - \tau) q_2) - y \cdot ((\tau U_1 + (1 - \tau) U_2) y) \right)
\]

\[
\leq \tau \left( \frac{1}{q_1} (\mathbf{y} \cdot m_1)^2 + p(q_1) - \mathbf{y} \cdot (U_1 \mathbf{y}) \right) + (1 - \tau) \left( \frac{1}{q_2} (\mathbf{y} \cdot m_2)^2 + p(q_2) - \mathbf{y} \cdot (U_2 \mathbf{y}) \right)
\]

\[
\leq \tau \max_{y \in \mathbb{R}^n, |y| = 1} \left( \frac{1}{q_1} (\mathbf{y} \cdot m_1)^2 + p(q_1) - \mathbf{y} \cdot (U_1 \mathbf{y}) \right) + (1 - \tau) \max_{y \in \mathbb{R}^n, |y| = 1} \left( \frac{1}{q_2} (\mathbf{y} \cdot m_2)^2 + p(q_2) - \mathbf{y} \cdot (U_2 \mathbf{y}) \right).
\]

\(\square\)

10
Lemma 4.2. It holds that
\[ K^* = \left\{ (\varrho, \mathbf{m}, \mathbf{U}) \in \mathbb{R}^+ \times \mathbb{R}^n \times S_0^{n \times n} \mid \lambda_{\max} \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho) \mathbb{I} - \mathbf{U} \right) \leq \frac{c}{n} \right\}, \]
where \( K^* \) was defined in (10).

Proof. Let us fix \( \overline{\varrho} \in \mathbb{R}^+ \). We want to show that
\[ (K \cap \{ \varrho = \overline{\varrho} \})^\Lambda = \left\{ (\varrho, \mathbf{m}, \mathbf{U}) \in \mathbb{R}^+ \times \mathbb{R}^n \times S_0^{n \times n} \mid \lambda_{\max} \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho) \mathbb{I} - \mathbf{U} \right) \leq \frac{c}{n} \text{ and } \varrho = \overline{\varrho} \right\}, \tag{13} \]
which proves the lemma.

The proof of (13) is the same as the proof of [7, Lemma 3]. For completeness we recall that proof and furthermore we present more details.

Let \((\varrho_1, \mathbf{m}_1, \mathbf{U}_1), (\varrho_2, \mathbf{m}_2, \mathbf{U}_2) \in K \cap \{ \varrho = \overline{\varrho} \}\). Then we claim that \((\varrho_1, \mathbf{m}_1, \mathbf{U}_1) - (\varrho_2, \mathbf{m}_2, \mathbf{U}_2) \in \Lambda\). In order to show this we must look at
\[ \begin{pmatrix} \varrho_1 - \varrho_2 & \mathbf{m}_1^T - \mathbf{m}_2^T \\ \mathbf{m}_1 - \mathbf{m}_2 & \mathbf{U}_1 - \mathbf{U}_2 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{m}_1^T - \mathbf{m}_2^T \\ \mathbf{m}_1 - \mathbf{m}_2 & \frac{1}{\varrho}(\mathbf{m}_1 \otimes \mathbf{m}_1 - \mathbf{m}_2 \otimes \mathbf{m}_2) \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{m}_1^T - \mathbf{m}_2^T \\ \mathbf{m}_1 - \mathbf{m}_2 & \frac{1}{\varrho}\mathbf{m}_1 \mathbf{m}_1^T - \frac{1}{\varrho} \mathbf{m}_2 \mathbf{m}_2^T \end{pmatrix}, \]
since \( \varrho_1 = \overline{\varrho} = \varrho_2 \) and \((\varrho_1, \mathbf{m}_1, \mathbf{U}_1), (\varrho_2, \mathbf{m}_2, \mathbf{U}_2) \in K\). To compute the determinant, observe that
\[ \begin{pmatrix} 1 & 0^T \\ -\frac{1}{\varrho}\mathbf{m}_1 & \mathbb{I} \end{pmatrix} \cdot \begin{pmatrix} 0 & \mathbf{m}_1^T - \mathbf{m}_2^T \\ \mathbf{m}_1 - \mathbf{m}_2 & \frac{1}{\varrho}\mathbf{m}_1 \mathbf{m}_1^T - \frac{1}{\varrho} \mathbf{m}_2 \mathbf{m}_2^T \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{1}{\varrho} \mathbf{m}_2^T \\ 0 & \mathbb{I} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{m}_1^T - \mathbf{m}_2^T \\ \mathbf{m}_1 - \mathbf{m}_2 & \mathbb{O} \end{pmatrix}, \]
where \( \mathbb{O} \in \mathbb{R}^{n \times n} \) denotes the zero matrix. Obviously
\[ \det \begin{pmatrix} 1 & 0 \end{pmatrix} = 1, \]
\[ \det \begin{pmatrix} 1 \end{pmatrix} = 1, \]
\[ \det \begin{pmatrix} 0 & \mathbf{m}_1^T - \mathbf{m}_2^T \\ \mathbf{m}_1 - \mathbf{m}_2 & \mathbb{O} \end{pmatrix} = 0, \]
and hence
\[ \det \begin{pmatrix} 0 & \mathbf{m}_1^T - \mathbf{m}_2^T \\ \mathbf{m}_1 - \mathbf{m}_2 & \frac{1}{\varrho}\mathbf{m}_1 \mathbf{m}_1^T - \frac{1}{\varrho} \mathbf{m}_2 \mathbf{m}_2^T \end{pmatrix} = 0. \]

Now Proposition [3.4] (c) yields that \((K \cap \{ \varrho = \overline{\varrho} \})^\Lambda = (K \cap \{ \varrho = \overline{\varrho} \})^\mathbb{C}^\Lambda\), which means that we can use Minkowski’s theorem [A.1] in order to find \((K \cap \{ \varrho = \overline{\varrho} \})^\Lambda\).

Let us now check the assumptions of Theorem [A.1]. We first show that the set
\[ C := \left\{ (\varrho, \mathbf{m}, \mathbf{U}) \in \mathbb{R}^+ \times \mathbb{R}^n \times S_0^{n \times n} \mid \lambda_{\max} \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho) \mathbb{I} - \mathbf{U} \right) \leq \frac{c}{n} \text{ and } \varrho = \overline{\varrho} \right\} \]
is compact and convex. To prove compactness it is enough to show that $C$ is closed and bounded. The fact that $C$ is closed follows since the map $A \mapsto \lambda_{\max}(A)$ is continuous and $C$ is the pre-image of a closed set under $\lambda_{\max}$. For the boundedness let $(\varrho, m, U) \in C$ arbitrary. First notice that $\varrho = \varrho$ and hence $|\varrho| \leq \varrho$. Furthermore

$$\frac{|m|^2}{\varrho} = \text{tr} \left( \frac{m \otimes m}{\varrho} - U \right) \leq \text{tr} \left( \frac{m \otimes m}{\varrho} - U \right) + np(\varrho) = \text{tr} \left( \frac{m \otimes m}{\varrho} + p(\varrho)I - U \right) \leq n\lambda_{\max} \left( \frac{m \otimes m}{\varrho} + p(\varrho)I - U \right) \leq c.$$ 

Therefore $|m| \leq \sqrt{c}$. To show the bound of $U$ we use the matrix norm (note that all norms are equivalent) which is given by $\|U\| = \max_i |\lambda_i|$ for symmetric matrices $U$, where $\lambda_i$ are the eigenvalues of $U$. Hence $\|U\| = \max\{|\lambda_{\min}(U)|, |\lambda_{\max}(U)|\}$. First we show that $|\lambda_{\min}(U)|$ is bounded. Since $\varrho \in \mathcal{S}_0^{n \times n}$, we have $\lambda_{\min}(U) \leq 0$ and therefore

$$|\lambda_{\min}(U)| = -\min_{y \in \mathbb{R}^n, |y| = 1} y \cdot (Uy) \leq \max_{y \in \mathbb{R}^n, |y| = 1} \left( \frac{1}{\varrho} (y \cdot m)^2 + p(\varrho) - y \cdot (Uy) \right) = \lambda_{\max} \left( \frac{m \otimes m}{\varrho} + p(\varrho)I - U \right) \leq \frac{c}{n}.$$ 

The fact that $|\lambda_{\max}(U)|$ is bounded, too, follows because $U$ is traceless. Indeed

$$|\lambda_{\max}(U)| \leq \sum_{\text{pos. EV}} |\lambda_i| = \sum_{\text{neg. EV}} |\lambda_i| \leq (n - 1)|\lambda_{\min}(U)|.$$ 

The convexity of $C$ is a simple consequence of Lemma 111. Furthermore $K \cap \{\varrho = \varrho\} \subseteq C$. Indeed, let $(\varrho, m, U) \in K \cap \{\varrho = \varrho\}$. Then

$$\lambda_{\max} \left( \frac{m \otimes m}{\varrho} + p(\varrho)I - U \right) = \lambda_{\max} \left( \frac{c}{n}I \right) = \frac{c}{n}$$

holds, in addition to $\varrho = \varrho$.

Hence the assumptions of Minkowski’s theorem A.1 hold. What remains is to prove that the extreme points of $C$ lie in $K \cap \{\varrho = \varrho\}$. In order to do this, let $(\varrho, m, U) \in C$ but $(\varrho, m, U) \notin K \cap \{\varrho = \varrho\}$. It suffices to show that this implies that $(\varrho, m, U)$ is not an extreme point of $C$.

From $(\varrho, m, U) \in C$ we obtain $\varrho = \varrho$ and

$$\lambda_{\max} \left( \frac{m \otimes m}{\varrho} + p(\varrho)I - U \right) \leq \frac{c}{n}. \quad (14)$$

Since the matrix

$$\frac{m \otimes m}{\varrho} + p(\varrho)I - U$$

is symmetric, it is diagonizable. More precisely there exists an orthogonal matrix $T$ such that

$$\frac{m \otimes m}{\varrho} + p(\varrho)I - U = T \begin{pmatrix} \lambda_1 \\ & \ddots \\ & & \lambda_n \end{pmatrix} T^{-1}. \quad (15)$$

12
We may assume without loss of generality that the eigenvalues are ordered \( \lambda_1 \leq \ldots \leq \lambda_n \). We denote the normed eigenvector, which corresponds to the \( i \)-th eigenvalue \( \lambda_i \), with \( b_i \). Then the vectors \( b_1, \ldots, b_n \) form an orthonormal basis of \( \mathbb{R}^n \) and \( T = (b_1 \cdots b_n) \).

From (14) we deduce that \( \lambda_i \leq \frac{c}{n} \) for all \( i = 1, \ldots, n \). Assume that \( \lambda_1 = \frac{c}{n} \). Then we have \( \frac{c}{n} \leq \lambda_1 \leq \ldots \leq \lambda_n \), i.e. \( \lambda_i = \frac{c}{n} \) for all \( i = 1, \ldots, n \). Hence with (15) we get

\[
\frac{m \otimes m}{\varrho} + p(\varrho)I - U = T \frac{c}{n} I T^{-1} = \frac{c}{n} I,
\]

which means that \((\varrho, m, U) \in K \cap \{ \varrho = \varrho \} \), a contradiction. Therefore \( \lambda_1 < \frac{c}{n} \).

Because \( b_1, \ldots, b_n \) form a basis of \( \mathbb{R}^n \), there exist unique coefficients \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) such that \( m = \sum_{i=1}^{n} \alpha_i b_i \). Let us define

\[
(\hat{\varrho}, \hat{m}, \hat{U}) := \left( 0, b_1, \frac{m \otimes b_1 + b_1 \otimes m - 2\alpha_1 b_1 \otimes b_1}{\varrho} \right).
\]

Obviously \( \hat{U} \) is symmetric and furthermore

\[
\text{tr} \hat{U} = \frac{2}{\varrho} (m \cdot b_1 - \alpha_1 |b_1|^2) = 0
\]
due to the facts that \( m \cdot b_1 = \alpha_1 \) and \( |b_1| = 1 \). In other words \( \hat{U} \in S_0^{n \times n} \).

For \( \tau \in \mathbb{R} \) we compute

\[
T^{-1} \left( \frac{m + \tau \hat{m}}{\varrho + \tau \hat{\varrho}} \right) + p(\varrho + \tau \hat{\varrho})I - (U + \tau \hat{U}) T
\]

\[
= T^{-1} \left( \frac{m \otimes m}{\varrho} + p(\varrho)I - U \right) T + \tau T^{-1} \left( \frac{m \otimes b_1 + b_1 \otimes m}{\varrho} - \hat{U} \right) T + \tau^2 T^{-1} \frac{b_1 \otimes b_1}{\varrho} T
\]

\[
= \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_n
\end{pmatrix} + (2\alpha_1 \tau + \tau^2) T^{-1} \frac{b_1 \otimes b_1}{\varrho} T.
\]

Note that

\[
T^{-1} b_1 \otimes b_1 T = \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix},
\]

due to the facts that \( T \) is orthogonal, the columns of \( T \) are the vectors \( b_1, \ldots, b_n \) and these vectors form an orthonormal basis. Hence

\[
\lambda_{\max} \left( \frac{m + \tau \hat{m}}{\varrho + \tau \hat{\varrho}} \right) + p(\varrho + \tau \hat{\varrho})I - (U + \tau \hat{U}) = \max \left\{ \lambda_1 + \frac{2\alpha_1 \tau + \tau^2}{\varrho}, \lambda_n \right\} \leq \frac{c}{n},
\]

as long as \( |\tau| \) is sufficiently small, because \( \lambda_1 < \frac{c}{n} \) and \( \lambda_n \leq \frac{c}{n} \). Furthermore we have \( \varrho + \tau \hat{\varrho} = \varrho = \varrho \) for all \( \tau \in \mathbb{R} \) since \( \hat{\varrho} = 0 \). In other words we have \((\varrho + \tau \hat{\varrho}, m + \tau \hat{m}, U + \tau \hat{U}) \in C\) for sufficiently small \( |\tau| \). Since \((\varrho, m, U)\) is a convex combination of \((\varrho \pm \tau \hat{\varrho}, m \pm \tau \hat{m}, U \pm \tau \hat{U})\), \((\varrho, m, U)\) is not an extreme point of \( C \). This finishes the proof.
The following theorem is our main result.

**Theorem 4.3.** It holds that $K^* = K^\Lambda = K^\infty$.

**Proof.** By Theorem 3.6 and Proposition 3.4 (b) we have $K^* \subseteq K^\Lambda \subseteq K^\infty$. Lemmas 4.1 and 4.2 show that $K^*$ is convex. Since $K \subseteq K^*$, we have $K^\infty \subseteq K^*$. This yields the claim. \qed

5 Concluding Remarks

As already pointed out, the computation of the $\Lambda$-convex hull, see Theorem 4.3, can be viewed as the first step towards a genuinely compressible convex integration which yields solutions with oscillations in the density as well.

Let us compare our result with Feireisl’s convex integration for compressible Euler [8], where Helmholtz decomposition was used and which is the most general among the available convex integration ansatzes for compressible Euler. In Feireisl’s method a subsolution is more or less a triple of functions $(\rho, m, U)$ solving (6) and taking values in $K^*$. It is surprising that the notion of a subsolution to our setup, which seems to be more general, coincides with the notion of a subsolution to Feireisl’s setup (because $K^* = K^\Lambda$).

A Minkowski’s Theorem

**Theorem A.1.** Let $C \subseteq \mathbb{R}^d$ be a compact convex set and let $M \subseteq C$. Then

$$C = M^\infty \iff \text{ext}(C) \subseteq M,$$

where $\text{ext}(C)$ denotes the set of extreme points of $C$.

References

[1] A. Bressan, G. Crasta and B. Piccoli: *Well-posedness of the Cauchy problem for $n \times n$ systems of conservation laws*. Mem. Amer. Math. Soc. 146(694), 1–134 (2000)

[2] E. Chiodaroli: *A counterexample to well-posedness of entropy solutions to the compressible Euler system*. J. Hyperbolic Differ. Equ. 11(3), 493–519 (2014)

[3] E. Chiodaroli, C. De Lellis and O. Kreml: *Global ill-posedness of the isentropic system of gas dynamics*. Comm. Pure Appl. Math. 68(7), 1157–1190 (2015)

[4] E. Chiodaroli and O. Kreml: *On the energy dissipation rate of solutions to the compressible isentropic Euler system*. Arch. Ration. Mech. Anal. 214(3), 1019–1049 (2014)

[5] G. Crippa, N. Gusev, S. Spirito and E. Wiedemann: *Failure of the chain rule for the divergence of bounded vector fields*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 17(1), 1–18 (2017)

[6] C. De Lellis and L. Székelyhidi Jr.: *The Euler equations as a differential inclusion*. Ann. of Math. (2) 170(3), 1417–1436 (2009)
[7] C. De Lellis and L. Székelyhidi Jr.: *On admissibility criteria for weak solutions of the Euler equations.* Arch. Ration. Mech. Anal. 195(1), 225–260 (2010)

[8] E. Feireisl: *Maximal dissipation and well-posedness for the compressible Euler system.* J. Math. Fluid Mech. 16, 447–461 (2014)

[9] E. Feireisl, C. Klingenberg and S. Markfelder: *On the density of wild initial data for the compressible Euler system.* Submitted, arXiv: 1812.11802 (2018)

[10] J. Glimm: *Solutions in the large for nonlinear hyperbolic systems of equations.* Comm. Pure Appl. Math. 18, 697–715 (1965)

[11] M. Gromov: *Partial differential relations.* Ergebnisse der Mathematik und ihrer Grenzgebiete Vol. 9, Springer (1986)

[12] C. Klingenberg and S. Markfelder: *The Riemann problem for the multidimensional isentropic system of gas dynamics is ill-posed if it contains a shock.* Arch. Ration. Mech. Anal. 227(3), 967–994 (2018)

[13] S. N. Kružkov: *First order quasilinear equations with several independent variables.* Mat. Sb. 81(123), 228–255 (1970)

[14] L. Székelyhidi Jr. and E. Wiedemann: *Young measures generated by ideal incompressible fluid flows.* Arch. Ration. Mech. Anal. 206(1), 333–366 (2012)