Bias in IV with Unordered Treatments

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Abstract

This note revisits the identification argument of Kirkeboen et al. (2016) who showed how one may combine instruments for each type of education with information about individuals' ranking of treatment types to achieve identification while allowing for both observed and unobserved heterogeneity in treatment effects. First we show that the key assumptions underlying the identification argument of Kirkeboen et al. (2016) has testable implications. Second, we provide a new characterization of the bias based on principal strata, that may arise if these assumptions are violated. The strata are "next-best defiers", individuals who comply with the assigned treatment, but who otherwise choose a treatment other than the stated next-best alternative, and "irrelevance-defiers" who are shifted into other treatments than the assigned one. The bias due to each defier-type has a product structure: It depends on the number of defiers compared to compliers, multiplied by the difference between compliers and defiers in the average effect of one treatment compared to another. The bias becomes large only if there are both many defiers relative to compliers and there are large differences in the payoff between compliers and defiers. Lastly, we show that the shares of next-best or irrelevance defiers can be bounded, but not point identified. We derive sharp bounds – which are nontrivial – and, thus, provides testable implications of the additional assumptions of Kirkeboen et al. (2016). These results have also implications for the recent work of Nibbering et al. (2022), who propose an algorithm which aggregate fields into clusters based on estimated first-stage coefficients. The motivation for their approach is to avoid bias from irrelevance and next-best defiers. We show that this approach requires point identification of the shares of next-best and irrelevance defiers, and that it may produce biased estimates even if effects are constant across individuals.

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1 Introduction

Instrumental variables (IV) estimation of treatment effects is challenging if there are multiple unordered treatments. Not only does identification require (at least) one instrument per alternative, but it is also necessary to deal with the issue that units who choose the same treatment may have different next-best treatments. One way to resolve these challenges is to assume homogenous treatment effects. If effects are not constant then standard 2SLS does not identify the payoff to any individual or group of the population from choosing one treatment instead of another.

We revisits the identification argument of Kirkeboen et al. (2016) who showed how one may combine instruments for each type of education with information about individuals’ ranking of treatment types to achieve identification while allowing for both observed and unobserved heterogeneity in treatment effects. Applying this approach to data from Norway, they found that different fields have widely different payoffs, even after accounting for selection on unobservables.

This note shows that the key assumptions underlying the identification argument of Kirkeboen et al. (2016) has testable implications. Second, we provide a new characterization of the bias based on principal strata, that may arise if these assumptions are violated.

In Section 2, we begin by briefly reviewing IV in settings with multiple unordered treatments, laying the groundwork for our analysis. As in the analysis of binary treatments in Imbens and Angrist (1994), we allow for heterogeneous effects and assume that each instrument is exogenous and satisfies a monotonicity condition. Our point of departure is the key result in Kirkeboen et al. (2016): IV can then be used to identify local average treatment effects (LATEs) of unordered treatments under the additional assumptions that the analyst observes individuals’ next-best alternatives and an irrelevance condition on preferences.

In Section 3, we examine whether the additional assumptions of Kirkeboen et al. (2016) have testable implications and the bias that may arise if they are violated. To do so, it is necessary to stratify the population into a set of instrument-dependent groups sometimes referred to as principal strata. These groups are defined by the manner in which members of the population react to the instruments. In addition to the usual compliers, always takers, and never takers of Imbens and Angrist (1994), there are are two so-called defier groups (both of which are distinct from the usual defier group that exists if the monotonicity assumption fails). The first is the next-best defiers. In the context of the application of Kirkeboen et al. (2016) this group consists of individuals who would choose their preferred field if above the admission cutoff, but otherwise choose fields other than the stated next-best alternative. The others are the irrelevance-defiers. In our context, the irrelevance assumption means that if crossing the admission cutoff to a given field does not make an individual choose that field, it should not affect her choice of other fields either.

We next use this stratification of the population to characterize the bias in the IV estimands
that may arise in the presence of next-best defiers, or irrelevance defiers, or both. It is useful to observe that the bias due to each type of defier has a product structure: It depends on the number of defiers compared to compliers, multiplied by the difference between compliers and defiers in the average payoff to choosing one type of education compared to another. Thus, there will be zero bias if there either are no defiers or if the average payoff to choosing one type of education compared to another is the same for defiers and compliers. Furthermore, the bias becomes large only if there are both many defiers relative to compliers and there are large differences in the payoff between compliers and defiers.

Lastly, we show that the shares of next-best or irrelevance defiers can be bounded, but not point identified. We derive sharp bounds – which are nontrivial – and, thus, provides testable implications of the additional assumptions of Kirkeboen et al. (2016). These results have also implications for the recent work of Nibbering et al. (2022), who propose an algorithm which aggregate fields into clusters based on estimated first-stage coefficients. The motivation for their approach is to avoid bias from irrelevance and next-best defiers. We show that this approach requires point identification of the shares of next-best and irrelevance defiers, and that it may produce biased estimates even if effects are constant across individuals.

2 Assumptions and Notation

We assume individuals choose between three mutually exclusive and collectively exhaustive alternatives \(d \in \{0, 1, 2\}\). To fix ideas we envision these as enrolling in three different fields of study. We suppress the individual index and abstract from control variables. We want to interpret IV estimates of the equation

\[
y = \beta_0 + \beta_1 d_1 + \beta_2 d_2 + \epsilon
\]

where \(y\) is an observed outcome such as earnings, and \(d_j \equiv 1[d = j]\) is a treatment indicator. Without loss of generality we choose field 0 as reference field, so that \(\beta_{IV}^1\) \((\beta_{IV}^2)\) is the payoff from choosing field 1 \((2)\) over field 0.

We suppose individuals are randomly assigned to one of three mutually exclusive and collectively exhaustive groups \(Z \in \{0, 1, 2\}\) and let \(z_j = 1[Z = j]\) be an indicator variable that equals 1 if an individual is assigned to group \(j\) and 0 otherwise. The indicator \(z_j\) can be thought of as an instrument shifting the costs or benefits of choosing field \(j\). For each individual, this gives three potential field choices \(d z\) and nine potential outcomes \(y^{d, z}\).

We let \(d\) denote the column vector of treatment indicators and \(z\) the column vector of instruments and make the standard IV assumptions, and \(d_j^x \equiv 1[d^x = j]\) is an indicator variable that tells us whether an individual would choose field \(j\) for a given value of \(Z\).

Assumption 1. IV Assumptions

(a) **Exclusion:** \(y^{d, z} = y^d\) for all \(d, z\)
Table 1. Taxonomy of complier and defier groups with field 0 as the control.

| Group                  | Field Choice | Characteristics |
|------------------------|--------------|------------------|
|                        | $d^0$ | $d^1$ | $d^2$ |
| **Instrument 1**       |           |               |       |
| - Compliers            | $C_1$      | 0   | 1   | $d_1^1 - d_0^0 = 1$ $\land$ $d_2^1 = d_0^0 = 0$ |
| - Irrelevance Defiers  | $ID_1$     | 0   | 2   | $d_1^1 = d_0^0 = 0$ $\land$ $d_2^1 - d_0^2 = 1$ |
| - Next-best Defiers    | $ND_1$     | 2   | 1   | $d_1^1 - d_0^0 = 1$ $\land$ $d_2^1 - d_2^0 = -1$ |
| **Instrument 2**       |           |               |       |
| - Compliers            | $C_2$      | 0   | 2   | $d_2^2 - d_2^0 = 1$ $\land$ $d_2^1 = d_2^0 = 0$ |
| - Irrelevance Defiers  | $ID_2$     | 0   | 1   | $d_2^2 = d_2^0 = 0$ $\land$ $d_2^1 - d_0^1 = 1$ |
| - Next-best Defiers    | $ND_2$     | 1   | 2   | $d_2^2 - d_2^0 = 1$ $\land$ $d_2^1 - d_2^1 = -1$ |

**Note:** The table characterizes compliers, irrelevance defiers and next-best defiers based on their potential treatments.

(b) **Independence:** $y^d, d^z \perp Z$ for all $d, z$

(c) **Rank:** $E[zd^\top]$ has full rank

(d) **Monotonicity:** $d^k_k \geq d^k_{k'}$ for each assignment pair $k, k'$

We link observed and potential outcomes and choices as follows,

$$y = y^0d_0 + y^1d_1 + y^2d_2$$  \hspace{1cm} (2)

$$d_j = d^0_jz_0 + d^1_jz_1 + d^2_jz_2$$  \hspace{1cm} for $j = 0, 1, 2$  \hspace{1cm} (3)

In Table 1 we invoke assumptions 1(a)–1(d) and characterize the groups of individuals whose potential field choices depend on the instrument. The table does not include always takers of field 1 (2) (those who chose field 1 (2) irrespective of instrument value) and never takers of field 1 (2) (those who choose field 2 and 0 (1 and 0) irrespective of instrument value).

As shown in the table, there are two types of compliers, $C_1$ and $C_2$. The $C_1$ ($C_2$) compliers are individuals who choose field 1 (2) when the instrument takes value 1 (2), and the reference field 0 when the instrument takes the value 0. In addition, there are four types of defiers, irrelevance and next-best defiers of instruments 1 and 2. Irrelevance defiers $ID_1$ ($ID_2$) are individuals who choose field 2 (1) when the instrument takes value 1 (2) while choosing field 0 if the instrument takes value 0. Next-best defiers $ND_1$ ($ND_2$) are individuals who choose field 2 (1) when the instrument takes value 0 while choosing field 1 (2) if the instrument takes value 1 (2).

The table shows how the potential field choices transform into potential choice indicator variables, which are later used to derive the IV estimand.

Kirkeboen et al. (2016) suggest the following assumptions on the groups in 1 to obtain
identification.\footnote{Kirkeboen et al. (2016) are imprecise about whether assumption 2(b) is imposed on everyone or only those individuals whose treatment status depends on the instrument. However, this is immaterial for their results, as well as ours. The reason is that always takers and never takers drop out of the IV estimand because their treatment status does not change with the instrument.}

**Assumption 2. Auxiliary Assumptions**

(a) **Irrelevance:** \(d^k_k - d^0_k = 0 \implies d^k_k = d^0_k \) for all pairs \(k, k'\)

(b) **Next-best:** We are able to condition on \(d^0_0 = d^0_1 = 0\) i.e. \(d^0_0 = 1\).

The following lemma is immediate from these two assumptions.

**Lemma 1.** Suppose Assumptions 1–2 hold. Then \(\beta _{1}^{IV}, \beta _{2}^{IV} \) have a causal interpretation as positively weighted averages of treatment effects for compliers, and

\[
\beta _{1}^{IV} = \mathbb{E}[y^1 - y^0 \mid C_1] \\
\beta _{2}^{IV} = \mathbb{E}[y^2 - y^0 \mid C_2]
\]

**Proof.** For a proof, see Kirkeboen et al. (2016). \[\square\]

The core of Lemma 1 is that the IV estimand of \(\beta _{1} \) (\(\beta _{2} \)) can be given an interpretation as a local average treatment effect (LATE) of an instrument-induced shift from field 0 to field 1 (2) for compliers when irrelevance and next-best defiers are assumed away.

3 Interpretation of IV Estimand if Auxiliary Assumptions Fail

If Assumptions 2(a)–2(b) do not hold, the IV estimand of \(\beta _{1} \) (\(\beta _{2} \)) does not have a causal interpretation as a positively weighted average of treatment effects of choosing field 1 (2) over field 0. In the following, we characterize the bias that will occur in this case, and discuss in which situations the bias will be large and small.\footnote{Throughout the paper, we use the word *bias* to describe the difference between two population quantities, namely the IV estimand and the parameter of interest, that is the positively weighted average of treatment effects for some complier group.}

3.1 Assuming Only Next-best

The IV estimands of \(\beta _{1} \) and \(\beta _{2} \) can be decomposed into a LATE for compliers and a bias term using IV moment conditions. In particular, if only next-best holds, but not irrelevance, we get the following decomposition, as shown in Appendix A.
Proposition 1. Suppose Assumptions 1(a)–1(d) and 2(b) hold. Then $\beta_1^{IV}$, $\beta_2^{IV}$ do not have a causal interpretation as positively weighted averages of treatment effects for compliers,

$$
\beta_1^{IV} = \frac{E[y^1 - y^0 | C_1]}{A} + \left( \frac{P(ID_1)P(ID_2)}{\omega_1} \right) \times \left( \frac{E[y^1 - y^0 | C_1] - E[y^1 - y^0 | ID_2]}{\Delta_1} \right) - \left( \frac{P(ID_1)P(C_2)}{\omega_2} \right) \times \left( \frac{E[y^2 - y^0 | C_2] - E[y^2 - y^0 | ID_1]}{\Delta_2} \right)
$$

where $W' = P(C_1)P(C_2) - P(ID_1)P(ID_2)$ and the expression for $\beta_2^{IV}$ follows by symmetry. $A$ is the complier LATE, $\omega_1$ and $\omega_2$ are defier group weights, and $\Delta_1$ and $\Delta_2$ are differences in the causal effects between compliers and irrelevance defiers.

Proof. See appendix A.

Imposing the constant effects assumption implies that the differences in the causal effects between defier groups ($\Delta_1$, $\Delta_2$) go to zero. In this case, $\beta_1^{IV}$ ($\beta_2^{IV}$) would recover the causal effect, $E[y^1 - y^0]$ ($E[y^2 - y^0]$). Imposing irrelevance implies that the defier weights ($\omega_1$, $\omega_2$) go to zero. In this case, $\beta_2^{IV}$ ($\beta_2^{IV}$) would recover the complier LATE, $E[y^1 - y^0 | C_1]$ ($E[y^2 - y^0 | C_2]$).

A central question is when the bias in Proposition 1 is large. To answer this question, it is useful to observe that the two bias terms in equation 4 are the products of a difference in causal effects and a defier weight consisting of the product of the propensities of irrelevance defiers divided by the difference between complier and defier propensity products.

Note that as long as $P(C_1)P(C_2) > 2 \times P(ID_1)P(ID_2)$ the weight $\omega_1$ is below 1. This will occur when there are many compliers relative to defiers. When the weight is below 1, the corresponding bias term will always be smaller than the difference in causal effects. Due to the product structure ($\omega_j \times \Delta_j$) the bias due to violations of the irrelevance assumption will be very small when both $\omega_j$ and $\Delta_j$ are small. Conversely, in order for a large bias to occur, there needs to be both many defiers relative to compliers and a large difference in causal effects between the different groups.

We illustrate this with two examples. In both examples, we fix the LATE for compliers at $1000$. We focus on the first instrument, fixing the propensities of compliers and irrelevance defiers of instrument 2 to a medium alternative, $P(ID_2) = 0.2$ and $P(C_2) = 0.8$, and, for simplicity, assume no always takers or never takers for any of the instruments, such that $P(C_1) = 1 - P(ID_1)$.

In Figure 1a we show how the bias varies with the propensity of irrelevance defiers. We let the difference in causal effects between compliers and instrument 2-defiers be fixed at three different levels: 10%, 20% and 50% of the complier LATE. In Figure 1b we show the bias from the first term when varying the difference in causal effects between compliers and defiers. We let the propensity of irrelevance defiers be fixed at three different levels: low (0.1), medium
Figure 1. Bias from defiers under different defier weights and levels of heterogeneity.

(0.2) and high (0.5). The key take away is that the bias will be small even when there is a sizable number of defiers and a nontrivial difference in causal effects between the compliers and the defiers.

3.2 Assuming Only Irrelevance

If irrelevance holds, but next-best is not observed, we may decompose the IV estimand into a complier LATE and a bias term.

**Proposition 2.** Suppose Assumptions 1(a)–1(d) and 2(a) hold. Then $\beta_1^{IV}, \beta_2^{IV}$ do not have a causal interpretation as positively weighted averages of treatment effects for compliers,

$$\beta_1^{IV} = \frac{E[y_1^1 - y_0^0 | C_1]}{A} + \frac{P(ND_1)P(C_2)}{\hat{W}} \omega_3 \times \left( E[y_1^1 - y_0^0 | ND_1] - E[y_1^1 - y_0^0 | C_1] \right) \Delta_3$$

$$- \frac{P(ND_1)P(C_2)}{\hat{W}} \omega_4 \times \left( E[y_2^2 - y_0^0 | ND_1] - E[y_2^2 - y_0^0 | C_2] \right) \Delta_4$$

$$+ \frac{P(ND_1)P(ND_2)}{\hat{W}} \omega_5 \times \left( E[y_1^1 - y_0^0 | ND_1] - E[y_1^1 - y_0^0 | ND_2] \right) \Delta_5$$
\[ \hat{W} = P(C_1)P(C_2) + P(C_1)P(ND_2) + P(ND_1)P(C_2) \]

where \( \hat{W} = P(C_1)P(C_2) + P(C_1)P(ND_2) + P(ND_1)P(C_2) \) and the expression for \( \beta_{IV}^2 \) follows by symmetry. \( A \) is the complier LATE, \( \omega_3 \) through \( \omega_6 \) are defier group weights and \( \Delta_3 \) through \( \Delta_6 \) are differences in the causal effects between complier and defier groups.

**Proof.** See appendix A.

Imposing the constant effects assumption implies that the differences in causal effects between defier groups (\( \Delta_3 \) through \( \Delta_6 \)) go to zero. In this case, \( \beta_{IV}^1 \) (\( \beta_{IV}^2 \)) would recover the causal effect, \( E[y^1 - y^0 | C_1] \) (\( E[y^2 - y^0 | C_2] \)). Observing the next-best alternative implies that the defier weights (\( \omega_3 \) through \( \omega_6 \)) go to zero. In this case, \( \beta_{IV}^1 \) (\( \beta_{IV}^2 \)) would recover the complier LATE, \( E[y^1 - y^0 | C_1] \) (\( E[y^2 - y^0 | C_2] \)).

As in equation (4), the bias terms in equation (5) are the products of a difference in causal effects and a weight consisting of the product of the propensities of next-best defiers divided by the sum of complier and defier propensity products.

Note that the weight in the first and second terms of equation (5) (\( \omega_3, \omega_4 \)) are below 1, but that the weights for the two latter terms (\( \omega_5, \omega_6 \)) can be above 1 if \( P(ND_1)P(ND_2) > P(C_1)P(C_2) + P(C_1)P(ND_2) + P(ND_1)P(C_2) \). When the weight is below 1, the bias from the term will always be smaller than the difference in causal effects. Due to the product structure (\( \omega_j \times \Delta_j \)) the bias due to violations of the next-best assumption will be very small when both \( \omega_j \) and \( \Delta_j \) are small. Conversely, in order for a large bias to occur, we need both many defiers relative to compliers and a large difference in causal effects between the different groups.

We keep the same numerical example as in Section 3.1 and focus on the term \( \omega_3 \times \Delta_3 \). In Figure 2a we show how the bias from this term varies with the propensity of next-best defiers. We let the difference in causal effects between compliers and defiers be fixed at three different levels: at 10%, 20% and 50% of the complier LATE. In Figure 2b we show the bias when varying the difference in causal effects between compliers and defiers. We let the propensity of next-best defiers be fixed at three different levels: low (0.1), medium (0.2) and high (0.5). The key take away is as above that the bias will be small even when there is a sizable number of defiers and a nontrivial difference in causal effects between the compliers and the defiers.

### 3.3 Assuming Neither Irrelevance Nor Next-best

If one neither makes the irrelevance assumption nor the next-best assumption, the IV estimand becomes the sum of the complier LATE, all bias terms from Propositions 2 and 1, as well as a third set of interacted bias terms.
**Note:** Panel (a) shows one term of the bias from next-best defiers for different defier propensities. The red line assumes a difference in causal effects between compliers and defiers at 10% of the complier LATE, the green at 20% and the blue at 50%. Panel (b) shows the bias from irrelevance defiers for different levels of treatment heterogeneity. The red line assumes 10%, the green 20% and the blue 50% irrelevance defiers. The number of defiers and compliers for instrument 2 is fixed at 20% and 80%.

**Figure 2.** Bias from defiers under different defier weights and levels of heterogeneity.

**Proposition 3.** Suppose Assumptions 1(a)–1(d) holds. Then \( \beta_1^{IV}, \beta_2^{IV} \) do not have a causal interpretation as positively weighted averages of treatment effects for compliers,

\[
\beta_1^{IV} = \mathbb{E}[y^1 - y^0 | C_1] \times \frac{P(ID_1)P(ID_2)}{W} + \frac{P(ID_1)P(C_2)}{W} \times \frac{\mathbb{E}[y^1 - y^0 | C_1] - \mathbb{E}[y^1 - y^0 | ID_2]}{\Delta_1} \\
- \frac{P(ID_1)P(C_2)}{W} \times \frac{\mathbb{E}[y^2 - y^0 | C_2] - \mathbb{E}[y^2 - y^0 | ID_1]}{\Delta_2} \\
+ \frac{P(ND_1)P(C_2)}{W} \times \frac{\mathbb{E}[y^1 - y^0 | ND_1] - \mathbb{E}[y^1 - y^0 | C_1]}{\Delta_3} \\
- \frac{P(ND_1)P(C_2)}{W} \times \frac{\mathbb{E}[y^2 - y^0 | ND_1] - \mathbb{E}[y^2 - y^0 | C_2]}{\Delta_4} \\
+ \frac{P(ND_1)P(ND_2)}{W} \times \frac{\mathbb{E}[y^1 - y^0 | ND_1] - \mathbb{E}[y^1 - y^0 | ND_2]}{\Delta_5} \\
- \frac{P(ND_1)P(ND_2)}{W} \times \frac{\mathbb{E}[y^2 - y^0 | ND_1] - \mathbb{E}[y^2 - y^0 | ND_2]}{\Delta_6}
\]
where

\[
\bar{W} = P(C_1)P(C_2) + P(C_1)P(ND_2) + P(ND_1)P(C_2) \\
+ P(ND_1)P(ID_2) + P(ID_1)P(ND_2) - P(ID_1)P(ID_2)
\]

and the expression for \( \beta_{IV}^2 \) follows by symmetry.

**Proof.** See appendix A.

A is the complier LATE, \( \omega_1 \) and \( \omega_2 \) are defier weights which also occur when observing the next-best alternative, \( \omega_3 \) through \( \omega_8 \) are defier weights which also occur under irrelevance and \( \omega_7 \) through \( \omega_9 \) are defier weights which occur only when neither assumption holds. \( \Delta_1, \Delta_2 \) and \( \Delta_7 \) are differences in the causal effects between irrelevance defiers and compliers, \( \Delta_3, \Delta_4 \) and \( \Delta_8 \) are differences in the causal effects between next-best defiers and compliers, while \( \Delta_5, \Delta_6 \) and \( \Delta_9 \) are differences in the causal effects between next-best defiers and irrelevance defiers as well as between next-best defiers for the two different instruments.

Imposing the constant effects assumption implies that the differences in causal effects between defier groups (\( \Delta_1 \) through \( \Delta_9 \)) go to zero. In this case, \( \beta_{IV}^1 \) (\( \beta_{IV}^2 \)) would recover the causal effect, \( E[y^1 - y^0 | C_1] \ (E[y^2 - y^0 | C_2]) \). Imposing the next-best assumption yields the result from Proposition 1, as weights \( \omega_1 \) through \( \omega_9 \) go to zero. Imposing the irrelevance assumption yields Proposition 2, as weights \( \omega_1, \omega_2 \) and \( \omega_7 \) through \( \omega_9 \) go to zero. Imposing both irrelevance and observing the next-best alternative make all defier weights (\( \omega_1 \) through \( \omega_9 \)) go to zero. Then \( \beta_{IV}^1 \) (\( \beta_{IV}^2 \)) would recover the complier LATE, \( E[y^1 - y^0 | C_1] \ (E[y^2 - y^0 | C_2]) \).

Note that the bias in Proposition 3 is the sum of all bias terms from Propositions 2 and 1, in addition to three new bias terms (except for a different denominator of the weights). These are terms following from interactions between irrelevance and next-best defiers, and rely on both types of defiers being present and having differences in causal effects between each other and with the complier group. As a result, the bias will be small unless there are relatively many of both types of defiers and the causal effects are materially different between these groups and the compliers.
4 Testable Implications

We have the first stage equations for the IV estimates of $d$ as

\[ d_1 = \alpha^0_1 + \alpha^1_1 z_1 + \alpha^2_1 z_2 + v_1 \]  \hspace{1cm} (7) \\
\[ d_2 = \alpha^0_2 + \alpha^1_2 z_1 + \alpha^2_2 z_2 + v_2 \]  \hspace{1cm} (8)

A natural next step would be to ask if it is possible to devise a test of whether the auxiliary assumptions hold empirically. To answer this question, it is useful to characterize the quantities that the first stage coefficients recover:

**Lemma 2.** Suppose Assumptions 1(a)–1(d) hold. Then

\[ \alpha^0_1 = P(AT_1) \]
\[ \alpha^0_2 = P(AT_2) \]
\[ \alpha^1_1 = P(C_1) + P(ND_1) \]
\[ \alpha^2_1 = P(ID_1) - P(ND_1) \]
\[ \alpha^0_2 = P(C_2) + P(ND_2) \]
\[ \alpha^2_2 = P(ID_2) - P(ND_2) \]

\[ P(NT_1) = 1 - \alpha^0_1 - \alpha^0_2 - \alpha^1_1 - \alpha^2_1 \]
\[ P(NT_2) = 1 - \alpha^0_1 - \alpha^0_2 - \alpha^2_1 - \alpha^2_2 \]

and

\[ P(C_1) + P(AT_1) + P(NT_1) + P(OT_1) + P(ID_1) + P(ND_1) = 1 \]
\[ P(C_2) + P(AT_2) + P(NT_2) + P(OT_2) + P(ID_2) + P(ND_2) = 1 \]

where $NT_1$ ($NT_2$) are never takers of field 1 (2) choosing field 0 when the instrument takes values 0 and 1 and $OT_1$ ($OT_2$) are always takers of the other field, choosing 2 (1) irrespective of which value the instrument takes.

**Proof.** See appendix B.

This result paves the way for the main result on the testability of the irrelevance and next best assumptions:

**Proposition 4.** Suppose Assumptions 1(a)–1(d) hold. Then $P(ID_1)$ and $P(ND_1)$ are partially identified.

\[ P(ND_1) \in [\max \{0, -\alpha^1_2\}, \min \{\alpha^1_1, \alpha^0_2\}] \]
\[ P(ID_1) \in [\max \{0, -\alpha^1_2\}, \max \{0, \alpha^1_2 + \min \{\alpha^1_1, \alpha^0_2\}\}] \]

where results for $P(ID_2)$ and $P(ND_2)$ follow by symmetry.
Proof. See appendix B.

If either assumption 2(a) or 2(b) is known to hold, the other assumption can be tested separately and \( P(ID_1) \) or \( P(ND_1) \) is point identified.

**Corollary 1.** Suppose Assumptions 1(a)–1(d) and 2(b) hold. Then \( P(ND_1) = P(ND_2) = 0 \) and we can test whether assumption 2(a) (irrelevance) holds, as \( \alpha_1^2 = P(ID_1) \) and \( \alpha_2^2 = P(ID_2) \).

**Corollary 2.** Suppose Assumptions 1(a)–1(d) and 2(a) hold. Then \( P(ID_1) = P(ID_2) = 0 \) and we can test whether assumption 2(b) (next-best) holds, as \( \alpha_1^1 = -P(ND_1) \) and \( \alpha_2^1 = -P(ND_2) \).

The practical implications of Proposition 4 is that we cannot point identify the defier propensities without further assumptions, but the assumptions are in principle testable as the bounds will generally be nontrivial.

## 5 How Aggregation May Cause Violations of the Exclusion Restriction

Nibbering et al. (2022) propose an algorithm which aggregate fields into clusters based on estimated first-stage coefficients. The motivation for their approach is to avoid bias from irrelevance and next-best defiers. Before discussing their approach, it is important to observe that the resulting IV estimates between such clusters will, at best, identify a positively weighted average of the causal effects of choosing one field versus a linear combination of the other fields, for example, the effects of choosing field 1 versus field 0 or 2. Hence, this approach involves moving the goalpost from clearly defined field contrasts that govern individuals’ educational investments to clusters of different fields.

### 5.1 Bias From Exclusion Violation

We continue to consider the situation with three fields, discussed above. The algorithm takes as a starting point all individuals with a certain reported next-best alternative (in our case taken to be 0), and test the hypothesis that the off-diagonal coefficients, \( \alpha_1^2 \) and \( \alpha_2^1 \), are zero. If this hypothesis is rejected, the sign of the coefficient is evaluated and the treatments are clustered according to the rules laid out in Table 2. For example, if \( \alpha_1^2 \) is negative and \( \alpha_2^1 \) is either zero or positive, fields 0 and 2 become the control cluster and field 1 the treatment cluster. Conversely, if \( \alpha_1^2 \) is either zero or positive and \( \alpha_2^1 \) is negative, fields 0 and 1 become the control cluster and field 2 the treatment cluster.

After performing the clustering based algorithm, Nibbering et al. (2022) estimate cluster treatment effects.

We let \( \tilde{d}(d) = 1_{d \in S_1} \) be the binary cluster treatment indicator and \( \tilde{z}(Z) = 1[Z = d \in S_1] \) the cluster instrument indicator. The no clustering-scenario is equivalent to the field level. In the
two other scenarios (control clustering or treatment clustering) we consider IV estimates of the equation
\[
y = \tilde{\beta}_0 + \tilde{\beta}_1 \tilde{d} + \epsilon
\]
where the first stage is
\[
\tilde{d} = \pi_0 + \pi_{1,0} \tilde{z} + \nu
\]
and \(\pi_{1,0}\) is the first stage coefficient. Observed and potential outcomes and choices are linked as
\[
y = \hat{y}_0 (1 - \tilde{d}) + \hat{y}_1 \tilde{d} \quad (9)
\]
\[
\tilde{d} = d^0 + (d^1 - d^0) \tilde{z} \quad (10)
\]
where \(\tilde{d}^j \equiv 1_{[d^j=1]}\) denotes the cluster-level potential treatment and \(\hat{y}^j\) is the potential outcome in cluster \(j\). In section C we show that this IV estimand does not, under Assumptions 1(a)–1(d), have a causal interpretation as a positive weighted average of treatment effects for the cluster complier groups. This result is summarized in Proposition 5.

**Proposition 5.** Suppose Assumptions 1(a)–1(d) hold.

(a) Under control clustering, \(\tilde{\beta}_{1IV}\) does not have a causal interpretation as a positively weighted average of treatment effects for the cluster complier group. If the clustering

| Scenario          | Conditions | Clusters | Implied Restrictions on Defiers |
|-------------------|------------|----------|-------------------------------|
| Control           | \(< 0 = 0\) | \(0,2\) | \(1\) | \(P(ND_1) > P(ID_1) \geq 0 \land P(ID_2) = P(ND_2) \geq 0\) |
| Clustering        | \(< 0 > 0\) | \(0,1\) | \(2\) | \(P(ID_1) = P(ND_1) \geq 0 \land P(ID_2) = P(ND_2) \geq 0\) |
| Treatment         | \(> 0 = 0\) | \(0\)   | \(\{1, 2\}\) | \(P(ID_1) = P(ND_1) \geq 0 \land P(ID_2) = P(ND_2) \geq 0\) |
| Clustering        | \(= 0 \geq 0\) | \(0\)   | \(\{1\}\) | \(P(ID_1) = P(ND_1) \geq 0 \land P(ID_2) = P(ND_2) \geq 0\) |
| No Clustering     | \(= 0 \geq 0\) | \(0\)   | \(\{2\}\) | \(P(ID_1) = P(ND_1) \geq 0 \land P(ID_2) = P(ND_2) \geq 0\) |
| Undefined\(^1\)  | \(< 0 < 0\) | \(0\)   | \(\{2\}\) | \(P(ND_1) > P(ID_1) \geq 0 \land P(ID_2) = P(ND_2) \geq 0\) |

**Note:** The table shows different clusterings ensuing from the algorithm proposed by Nibbering et al. (2022) and their implied restrictions on defiers. The algorithm tests the null hypothesis of coefficients being zero. The conditions in columns two and three specify which estimates must be observed for the clustering to be chosen, where \(> 0 (< 0)\) indicate rejecting the null and observing a positive (negative) coefficient, while \(\sim = 0\) indicates not being able to reject.

It is unclear what Nibbering et al. (2022) do when both coefficients are negative. In that case, the ordering of the coefficients will matter.
Proof. See Appendix C. \qed

Imposing the irrelevance assumption under control clustering implies that the defier weights \((\tilde{\omega}_1, \tilde{\omega}_2)\) go to zero. In this case, \(\tilde{\beta}_{1,0}^{IV}\) recovers a positively weighted average of the causal effect of choosing field 1 over 0 for compliers of instrument 1 and next-best defiers of instrument 2, and of choosing field 1 over 2 for compliers of instrument 2 and next-best defiers of instrument 1, weighted by the number of compliers and defiers. Under control clustering, this is the new parameter of interest.

Imposing the next-best assumption under treatment clustering implies that the defier weights \((\tilde{\omega}_3, \tilde{\omega}_4)\) go to zero. In this case, \(\tilde{\beta}_{1,0}^{IV}\) recovers a positively weighted average of the causal effect of choosing field 1 over 0 for compliers of instrument 1 and irrelevance defiers of instrument 2, and of choosing field 2 over 0 for compliers of instrument 2 and irrelevance defiers of instrument 1, weighted by the number of compliers and defiers. Under treatment clustering, this is the new parameter of interest.
If neither irrelevance nor next-best assumptions hold, the IV estimand does not have a causal interpretation as a positively weighted average of treatment effects for the cluster complier group. The bias terms reflect that individuals may in response to changes in the cluster instrument be switching across fields in the treatment cluster and/or across fields in the control cluster. Such switches will generally involve changes in potential outcomes, yet no change in the cluster treatment status. Thus, the exclusion restriction at the cluster level will be violated. The reason for this bias is that the algorithm equates the sign of the off-diagonal coefficients with the presence and absence of irrelevance and next best defiers. As shown in Proposition 2, this is wrong. The off-diagonal coefficients tell us only if there are more or less next best defiers than irrelevance defiers. One cannot in general use the sign of $\alpha_1^2$ ($\alpha_2^2$) to show that there are no irrelevance defiers of instrument 1 (2) if $\alpha_2^1 < 0$ ($\alpha_1^2 < 0$) and no next-best defiers of instrument 1 (2) if $\alpha_1^2 > 0$ ($\alpha_2^1 > 0$).

It is also important to observe that the constant effects assumption is not sufficient for $\tilde{\beta}_{IV1,0}$ to recover a positively weighted average of treatment effects between clusters 0 and 1 and obtain a causal interpretation. This result is summarized in Proposition 6.

**Proposition 6.** Suppose Assumptions 1(a)–1(d) hold and we further assume constant treatment effects.

(a) Under control clustering, $\tilde{\beta}_{IV1,0}$ does not recover the causal effect. If the clustering is $S_1 = \{1\}$ and $S_0 = \{2, 0\}$, we have

$$
\tilde{\beta}_{IV1,0} = \frac{P(C_1 \cup ND_2)}{\pi_{1,0}} E[y^1 - y^0] + \frac{P(C_2 \cup ND_1)}{\pi_{1,0}} E[y^1 - y^2] + \frac{P(ID_1) - P(ID_2)}{\pi_{1,0}} E[y^2 - y^0] - \frac{\Delta_1}{\hat{\omega}_1}
$$

where $\pi_{1,0} = P(C_1 \cup C_2 \cup ND_1 \cup ND_2)$. $A$ is a positively weighted average of the causal effects of choosing field 1 over 0 and of choosing field 1 over 2, $\hat{\omega}_1$ is a difference between defier group weights, and $\Delta_1$ is the difference in potential outcomes for irrelevance defiers in cluster $S_0$, i.e. never takers of the clustered treatment. The result for the clustering $S_1 = \{2\}$ and $S_0 = \{1, 0\}$ is symmetric.

(b) Under treatment clustering, $\tilde{\beta}_{IV1,0}$ does not recover the causal effect. We have

$$
\tilde{\beta}_{IV1,0} = \frac{P(C_1 \cup ID_2)}{\pi_{1,0}} E[y^1 - y^0] + \frac{P(C_2 \cup ID_1)}{\pi_{1,0}} E[y^2 - y^0]
$$
\[
+ \frac{P(ND_1) - P(ND_2)}{\pi_{1,0}} \underbrace{\mathbb{E}[y^1 - y^2]}_{\Delta_2}
\]

where \( \pi_{1,0} = P(C_1 \cup C_2 \cup ID_1 \cup ID_2) \). \( A \) is a positively weighted average of the causal effects of choosing field 1 over 0 and of choosing field 2 over 0, \( \omega_2 \) is a difference between defier group weights, and \( \Delta_2 \) is the difference in potential outcomes for irrelevance defiers in cluster \( S_1 \), i.e. always takers of the clustered treatment.

**Proof.** The constant effects assumption reduces all conditional expectations to unconditional expectations, i.e. \( \mathbb{E}[y^j - y^k \mid G] = \mathbb{E}[y^j - y^k] \) for any group \( G \) and any combination of fields \( j, k \). The result is immediate. \( \square \)

One exception to this result is the particular case when the number of defiers for each instrument happen to be equal, i.e. that \( P(ID_1) = P(ID_2) \) under control clustering or \( P(ND_1) = P(ND_2) \) under treatment clustering.

In contrast, the approach of Kirkeboen et al. (2016) recovers the causal effect under the constant effects assumption. This shows that the clustering method relies on different, not weaker assumptions than Kirkeboen et al. (2016). The clustering approach achieves identification under different, not weaker assumptions.

The following auxiliary exclusion restriction can be made to obtain identification under the clustering approach.

**Assumption 3. Cluster Exclusion Assumptions**

(a) **Control Cluster Exclusion:** \( \bar{d}^1 = \bar{d}^0 = 0 \implies \bar{y}^{0,1} = \bar{y}^{0,0} \)

(b) **Treatment Cluster Exclusion:** \( \bar{d}^1 = \bar{d}^0 = 1 \implies \bar{y}^{1,1} = \bar{y}^{1,0} \)

Assumptions 3(a) and 3(b) ensure that the bias from switchers within clusters (irrelevance defiers under control clustering and next-best defiers under treatment clustering) disappear, irrespective of the number of switchers. These assumptions are homogeneity restrictions on potential outcomes across different fields, and, thus, difficult to justify. Nevertheless, if one is willing to invoke Assumptions 3(a) and 3(b), one may obtain the following identification result:

**Proposition 7.** Under control clustering, suppose assumptions 1(a)–1(d) and 3(a) hold. \( \bar{\beta}^{IV}_I \) has a causal interpretation as the positively weighted average of treatment effects for cluster compliers. If the clustering is \( S_1 = \{1\} \) and \( S_0 = \{2, 0\} \), we have

\[
\bar{\beta}^{IV}_{I,0} = \frac{P(C_1 \cup ND_2)}{\pi_{1,0}} \mathbb{E}[y^1 - y^0 \mid C_1 \cup ND_2] + \frac{P(C_2 \cup ND_1)}{\pi_{1,0}} \mathbb{E}[y^1 - y^2 \mid C_2 \cup ND_1]
\]

where \( \pi_{1,0} = P(C_1 \cup C_2 \cup ND_1 \cup ND_2) \). The result for clustering \( S_1 = \{2\} \) and \( S_0 = \{1, 0\} \) is symmetric.
Under treatment clustering, suppose Assumptions 1(a)–1(d) and 3(b) hold. $\tilde{\beta}_1^{IV}$ has a causal interpretation as a positively weighted average of treatment effects for cluster compliers, and

$$\tilde{\beta}_{1,0}^{IV} = \frac{P(C_1 \cup ID_1)}{\pi_{1,0}} E[y^1 - y^0 | C_1 \cup ID_1] + \frac{P(C_2 \cup ID_1)}{\pi_{1,0}} E[y^2 - y^0 | C_2 \cup ID_1]$$

where $\pi_{1,0} = P(C_1 \cup C_2 \cup ID_1 \cup ID_2)$.

**Proof.** Assumption 3(a) (3(b)) eliminates the bias terms in the results from Proposition 5 by letting $\tilde{\Delta}_1, \tilde{\Delta}_2$ ($\tilde{\Delta}_3, \tilde{\Delta}_4$) go to zero. The result is immediate. 

6 Summary

This note revisits the identification argument of Kirkeboen et al. (2016) who showed how one may combine instruments for each type of education with information about individuals' ranking of treatment types to achieve identification while allowing for both observed and unobserved heterogeneity in treatment effects. First we show that the key assumptions underlying the identification argument of Kirkeboen et al. (2016) has testable implications. Second, we provide a new characterization of the bias based on principal strata, that may arise if these assumptions are violated. The strata are "next-best defiers", individuals who comply with the assigned treatment, but who otherwise choose a treatment other than the stated next-best alternative, and "irrelevance-defiers" who are shifted into other treatments than the assigned one. The bias due to each defier-type has a product structure: It depends on the number of defiers compared to compliers, multiplied by the difference between compliers and defiers in the average effect of one treatment compared to another. The bias becomes large only if there are both many defiers relative to compliers and there are large differences in the payoff between compliers and defiers. Lastly, we show that the shares of next-best or irrelevance defiers can be bounded, but not point identified. We derive sharp bounds – which are nontrivial – and, thus, provides testable implications of the additional assumptions of Kirkeboen et al. (2016). These results have also implications for the recent work of Nibbering et al. (2022), who propose an algorithm which aggregate fields into clusters based on estimated first-stage coefficients. The motivation for their approach is to avoid bias from irrelevance and next-best defiers. We show that this approach requires point identification of the shares of next-best and irrelevance defiers, and that it may produce biased estimates even if effects are constant across individuals.

References

Imbens, G. W. and Angrist, J. D. (1994). Identification and estimation of local average treatment effects. *Econometrica*, 62(2):467–475.

Kirkeboen, L. J., Leuven, E., and Mogstad, M. (2016). Field of study, earnings, and self-selection. *The Quarterly Journal of Economics*, 131(3):1057–1111.

Nibbering, D., Oosterveen, M., and Silva, P. L. (2022). Clustered local average treatment effects: fields of study and academic student progress. Discussion Paper No. 15159, IZA Institute for Labor Economics.
A Proof Of Bias When Auxiliary Assumptions Fail

Proof. We build on the notation from Section 2. IV uses the three moment conditions:

\[ \mathbb{E}[\varepsilon] = 0, \quad \mathbb{E}[\varepsilon z_1] = 0 \quad \text{and} \quad \mathbb{E}[\varepsilon z_2] = 0 \]

Expressing \( \varepsilon \) in terms of potential outcomes, we get:

\[
\varepsilon = (y^0 - \beta_0) + (y^1 - y^0 - \beta_1)d_1 + (y^2 - y^0 - \beta_2)d_2
\]

(11)

\[
= (y^0 - \beta_0) + (y^1 - y^0 - \beta_1)(d_1^0 + (d_1^1 - d_1^0)z_1 + (d_1^2 - d_1^0)z_2)
\]

\[
+ (y^2 - y^0 - \beta_2)(d_2^0 + (d_2^1 - d_2^0)z_1 + (d_2^2 - d_2^0)z_2)
\]

We substitute into the moment conditions, and solve. Under independence, we get:

\[
\mathbb{E}[(y^1 - y^0 - \beta_1)(d_1^0 - d_1^0) + (y^2 - y^0 - \beta_2)(d_1^0 - d_2^0)] = 0
\]

\[
\mathbb{E}[(y^1 - y^0 - \beta_1)(d_1^0 - d_1^0) + (y^2 - y^0 - \beta_2)(d_1^0 - d_2^0)] = 0
\]

As shown by Kirkeboen et al. (2016), this implies, for \( k = 1,2, k' = 2,1 \), that:

\[
\mathbb{E}[y^k - y^0 - \beta_k \mid d_k^0 - d_0^0 = 1, d_k^1 - d_0^1 = 0] \times P[d_k^0 - d_0^0 = 1, d_k^1 - d_0^1 = 0] = 0
\]

(12)

\[
+ \mathbb{E}[(y^k - y^0 - y^{k'} - y^0) - (\beta_k - \beta_{k'}) \mid d_k^0 - d_0^0 = 1, d_k^1 - d_0^1 = -1] \times P[d_k^0 - d_0^0 = 1, d_k^1 - d_0^1 = -1] = 0
\]

(13)

\[
+ \mathbb{E}[y^k - y^0 - \beta_{k'} \mid d_k^0 - d_0^0 = 0, d_k^1 - d_0^1 = 1] \times P[d_k^0 - d_0^0 = 0, d_k^1 - d_0^1 = 1] = 0
\]

where we have assumed

\[
P[d_k^0 - d_0^0 = -1, d_k^1 - d_0^1 = 0] = P[d_k^0 - d_0^0 = 0, d_k^1 - d_0^1 = -1] = 0
\]

under monotonicity. To simplify notation, we rewrite equation 12 in terms of the notation from Table 1:

\[
\mathbb{E}[y^k - y^0 - \beta_k \mid C_k] \times P(C_k)
\]

\[
+ \mathbb{E}[(y^k - y^0 - y^{k'} - y^0) - (\beta_k - \beta_{k'}) \mid ND_k] \times P(ND_k)
\]

\[
+ \mathbb{E}[y^{k'} - y^0 - \beta_{k'} \mid ID_k] \times P(ID_k) = 0
\]

We isolate \( \beta_k \) for \( k = 1,2 \):

\[
\beta_k = \beta_{k'} \frac{P(ND_k) - P(ID_k)}{P(C_k) + P(ND_k)} + \frac{\mathbb{E}[y^k - y^0 \mid C_k]P(C_k)}{P(C_k) + P(ND_k)}
\]

(14)

\[
+ \frac{\mathbb{E}[y^k - y^0 - y^{k'} - y^0 \mid ND_k]P(ND_k)}{P(C_k) + P(ND_k)} + \frac{\mathbb{E}[y^{k'} - y^0 \mid ID_k]P(ID_k)}{P(C_k) + P(ND_k)}
\]
A.1 No Auxiliary Assumptions

We substitute equation (14) with \( k = 2 \) into (14) with \( k = 1 \) and get:

\[
\beta_1 = \frac{\mathbb{E}[y^1 - y^0 \mid C_1]P(C_1)}{P(C_1) + P(ND_1)}
\]

\[
+ \mathbb{E}[y^2 - y^0 \mid ID_1] \frac{P(ID_1)}{P(C_1) + P(ND_1)}
+ \frac{\mathbb{E}[y^1 - y^0 - y^2 - y^0 \mid ND_1]P(ND_1)}{P(C_1) + P(ND_1)}
\]

\[
+ \frac{P(ND_1) - P(ID_1)}{P(C_1) + P(ND_1)} \times \left[ \frac{\mathbb{E}[y^2 - y^0 \mid C_2]P(C_2)}{P(C_2) + P(ND_2)} \right.
+ \mathbb{E}[y^1 - y^0 \mid ID_2] \frac{P(ID_2)}{P(C_2) + P(ND_2)}
\]

\[
+ \frac{\mathbb{E}[y^2 - y^0 - y^1 - y^0 \mid ND_2]P(ND_2)}{P(C_2) + P(ND_2)}
+ \beta_1 \frac{P(ND_2) - P(ID_2)}{P(C_2) + P(ND_2)}
\]

Letting

\[
\bar{W} = 1 - \frac{(P(ND_1) - P(ID_1))(P(ND_2) - P(ID_2))}{(P(C_1) + P(ND_1))(P(C_2) + P(ND_2))}
\]

and gathering \( \beta_1 \)-terms on the LHS gives:

\[
\beta_1 \bar{W} = \mathbb{E}[y^1 - y^0 \mid C_1] \times \frac{P(C_1)}{P(C_1) + P(ND_1)}
\]

\[
+ \mathbb{E}[y^2 - y^0 \mid ID_1] \times \frac{P(ID_1)}{P(C_1) + P(ND_1)}
\]

\[
+ \mathbb{E}[y^1 - y^0 - y^2 - y^0 \mid ND_1] \times \frac{P(ND_1)}{P(C_1) + P(ND_1)}
\]

\[
+ \mathbb{E}[y^1 - y^0 \mid ID_2] \times \frac{(P(ND_1) - P(ID_1))P(ID_2)}{(P(C_1) + P(ND_1))(P(C_2) + P(ND_2))}
\]

\[
+ \mathbb{E}[y^2 - y^0 \mid C_2] \times \frac{(P(ND_1) - P(ID_1))P(C_2)}{(P(C_1) + P(ND_1))(P(C_2) + P(ND_2))}
\]

\[
+ \mathbb{E}[y^2 - y^0 - y^1 - y^0 \mid ND_2] \times \frac{(P(ND_1) - P(ID_1))P(ND_2)}{(P(C_1) + P(ND_1))(P(C_2) + P(ND_2))}
\]
Adding and subtracting
\[
\mathbb{E}[y^1 - y^0 \mid C_1] - \frac{P(ND_1)}{P(C_1) + P(ND_1)} + \mathbb{E}[y^1 - y^0 \mid C_1] \frac{(P(ND_1) - P(ID_1))(P(ND_2) - P(ID_2))}{(P(C_1) + P(ND_1))(P(C_2) + P(ND_2))}
\]
on the RHS and gathering terms gives:
\[
\beta_1 \bar{W} = \mathbb{E}[y^1 - y^0 \mid C_1] \bar{W} - \mathbb{E}[y^1 - y^0 \mid C_1] \times \frac{P(ND_1)(P(C_2) + P(ND_2))}{(P(C_1) + P(ND_1))(P(C_2) + P(ND_2))}
\]
\[
+ \mathbb{E}[y^1 - y^0 \mid C_1] \times \frac{(P(ND_1) - P(ID_1))(P(ND_2) - P(ID_2))}{(P(C_1) + P(ND_1))(P(C_2) + P(ND_2))}
\]
\[
+ \mathbb{E}[y^2 - y^0 \mid ID_1] \times \frac{P(ID_1)(P(C_2) + P(ND_2))}{(P(C_1) + P(ND_1))(P(C_2) + P(ND_2))}
\]
\[
+ \mathbb{E}[y^2 - y^0 \mid ID_1] \times \frac{(P(ND_1) - P(ID_1))(P(ND_2) - P(ID_2))}{(P(C_1) + P(ND_1))(P(C_2) + P(ND_2))}
\]
\[
+ \mathbb{E}[y^1 - y^0 - y^2 - y^0 \mid ND_1] \times \frac{P(ND_1)(P(C_2) + P(ND_2))}{(P(C_1) + P(ND_1))(P(C_2) + P(ND_2))}
\]
\[
+ \mathbb{E}[y^1 - y^0 \mid ID_2] \times \frac{(P(ND_1) - P(ID_1))(P(ND_2) - P(ID_2))}{(P(C_1) + P(ND_1))(P(C_2) + P(ND_2))}
\]
\[
+ \mathbb{E}[y^2 - y^0 \mid ID_1] \times \frac{(P(ND_1) - P(ID_1))(P(C_2) + P(ND_2))}{(P(C_1) + P(ND_1))(P(C_2) + P(ND_2))}
\]
\[
+ \mathbb{E}[y^2 - y^0 - y^1 - y^0 \mid ND_2] \times \frac{(P(ND_1) - P(ID_1))(P(ND_2))}{(P(C_1) + P(ND_1))(P(C_2) + P(ND_2))}
\]
Dividing by \( W \) on both sides, and letting
\[
\bar{W} = (P(C_1) + P(ND_1))(P(C_2) + P(ND_2)) - (P(ND_1) - P(ID_1))(P(ND_2) - P(ID_2))
\]
gives
\[
\beta_1 = \mathbb{E}[y^1 - y^0 \mid C_1] - \mathbb{E}[y^1 - y^0 \mid C_1] \times \frac{P(ND_1)(P(C_2) + P(ND_2))}{W}
\]
\[
+ \mathbb{E}[y^1 - y^0 \mid C_1] \times \frac{(P(ND_1) - P(ID_1))(P(ND_2) - P(ID_2))}{W}
\]
\[
+ \mathbb{E}[y^2 - y^0 \mid ID_1] \times \frac{P(ID_1)(P(C_2) + P(ND_2))}{W}
\]
\[
+ \mathbb{E}[y^1 - y^0 - y^2 - y^0 \mid ND_1] \times \frac{P(ND_1)(P(C_2) + P(ND_2))}{W}
\]
\[
+ \mathbb{E}[y^1 - y^0 \mid ID_2] \times \frac{(P(ND_1) - P(ID_1))(P(ID_2))}{W}
\]
\[
+ \mathbb{E}[y^2 - y^0 \mid C_2] \times \frac{(P(ND_1) - P(ID_1))(P(ND_2))}{W}
\]
\[ + \mathbb{E}[y^2 - y^0 - y^1 - y^0 \mid ND_2] \times \frac{(P(ND_1) - P(ID_1))P(ND_2)}{\bar{W}} \]

Rearranging, we get:

\[
\beta_1^{IV} = \mathbb{E}[y^1 - y^0 \mid C_1] + \frac{P(ND_1)P(C_2)}{\bar{W}} \times (\mathbb{E}[y^1 - y^0 \mid ND_1] - \mathbb{E}[y^1 - y^0 \mid C_1]) \tag{15}
\]

\[
+ \frac{P(ND_1)P(C_2)}{\bar{W}} \times (\mathbb{E}[y^2 - y^0 \mid C_2] - \mathbb{E}[y^2 - y^0 \mid ND_1])
\]

\[
+ \frac{P(ND_1)P(ND_2)}{\bar{W}} \times (\mathbb{E}[y^1 - y^0 \mid ND_1] - \mathbb{E}[y^1 - y^0 \mid ND_2])
\]

\[
+ \frac{P(ID_1)P(ID_2)}{\bar{W}} \times (\mathbb{E}[y^1 - y^0 \mid ID_1] - \mathbb{E}[y^1 - y^0 \mid ID_2])
\]

\[
+ \frac{P(ID_1)P(C_2)}{\bar{W}} \times (\mathbb{E}[y^2 - y^0 \mid ID_1] - \mathbb{E}[y^2 - y^0 \mid C_2])
\]

\[
+ \frac{P(ID_1)P(ND_2)}{\bar{W}} \times (\mathbb{E}[y^1 - y^0 \mid ND_2] - \mathbb{E}[y^1 - y^0 \mid C_1])
\]

\[
+ \frac{P(ID_1)P(ID_2)}{\bar{W}} \times (\mathbb{E}[y^2 - y^0 \mid ID_1] - \mathbb{E}[y^2 - y^0 \mid ND_2])
\]

\[
+ \frac{P(ND_1)P(ID_2)}{\bar{W}} \times (\mathbb{E}[y^1 - y^0 \mid ID_2] - \mathbb{E}[y^1 - y^0 \mid ID_1])
\]

where we can rearrange the denominator such that

\[
\bar{W} = P(C_1)P(C_2) + P(C_1)P(ND_2) + P(ND_1)P(C_2)
\]

\[
+ P(ND_1)P(ID_1) + P(ID_1)P(ND_2) - P(ID_1)P(ID_2)
\]

and the expression for \( \beta_2^{IV} \) follows by symmetry.

\[ A.2 \quad \text{Assuming Only Next-best} \]

We now want to find an expression of the bias assuming only next-best.

\[ \textbf{Proof.} \quad \text{Next-best ensures } P(ND_1) = P(ND_2) = 0. \text{ Equation 15 then reduces to} \]

\[
\beta_1^{IV} = \mathbb{E}[y^1 - y^0 \mid C_1] + \frac{P(ID_1)P(ID_2)}{W'} \times (\mathbb{E}[y^1 - y^0 \mid C_1] - \mathbb{E}[y^1 - y^0 \mid ID_2]) \tag{16}
\]

\[
+ \frac{P(ID_1)P(C_2)}{W'} \times (\mathbb{E}[y^2 - y^0 \mid ID_1] - \mathbb{E}[y^2 - y^0 \mid C_2])
\]

where

\[
W' = P(C_1)P(C_2) - P(ID_1)P(ID_2)
\]
A.3 Assuming Only Irrelevance

We now want to find an expression of the bias assuming only irrelevance.

**Proof.** Irrelevance ensures $P(ID_1) = P(ID_2) = 0$. Equation 15 then reduces to

\[
\beta_{IV} = \mathbb{E}[y^1 - y^0 | C_1] + \frac{P(ND_1)P(C_2)}{\hat{W}} \times (\mathbb{E}[y^1 - y^0 | ND_1] - \mathbb{E}[y^1 - y^0 | C_1]) \\
+ \frac{P(ND_1)P(C_2)}{\hat{W}} \times (\mathbb{E}[y^2 - y^0 | C_2] - \mathbb{E}[y^2 - y^0 | ND_1]) \\
+ \frac{P(ND_1)P(ND_2)}{\hat{W}} \times (\mathbb{E}[y^1 - y^0 | ND_1] - \mathbb{E}[y^1 - y^0 | ND_2]) \\
+ \frac{P(ND_1)P(ND_2)}{\hat{W}} \times (\mathbb{E}[y^2 - y^0 | ND_2] - \mathbb{E}[y^2 - y^0 | ND_1])
\]  

(17)

where

\[
\hat{W} = (P(C_1) + P(ND_1))(P(C_2) + P(ND_2)) - P(ND_1)P(ND_2) \\
= P(C_1)P(C_2) + P(C_1)P(ND_2) + P(ND_1)P(C_2)
\]
Table 3. Detailed taxonomy of behavioral groups.

| Behavioral Type | Potential Field Choice | Group       | Behavioral Type | Potential Field Choice | Group       |
|-----------------|------------------------|-------------|-----------------|------------------------|-------------|
|                 | d₀  d₁  d₂             |             |                 | d₀  d₁  d₂             |             |
| Compliers       |                        | C₁ ∩ C₂     | Always takers   |                        |             |
| C₁              | 0  1  2                |             | AT₁            | 1  1  1                |             |
|                 | 0  1  0                | C₁ ∩ NT₂    | Next-best Defiers | ND₁       | 2  1  2                |             |
| NT₁             | 0  0  2                | NT₁ ∩ C₂    | Irrelevance Defiers | ID₁       | 0  2  2                |             |
| OT₁             | 2  2  2                | OT₁ ∩ AT₂   |                 |                        |             |

Note: The table decomposes the behavioral groups from Table 1 into subgroups. The table shows how each individual has a behavioral response to all states of the instrument. Note that other takers OT₁ (OT₂) refers to always takers of field 2 (1).

B Proof of Testable Implications

B.1 First Stage Quantities

We start by proving Proposition 2

Proof. We start by introducing a richer decomposition of behavioral groups, building on Table 1. This is presented in Table 3.

Focusing on $k = 1$, we take expectations on both sides in equation (7). As $\mathbb{E}[v_1] = 0$, we get:

$$\mathbb{E}[d_1] = \alpha_1^0 + \alpha_1^1 \times \mathbb{E}[z_1] + \alpha_1^2 \times \mathbb{E}[z_2]$$

We decompose the LHS into potential outcomes, using that $z_0 = 1 - z_1 - z_2$. Under independence we have:

$$\mathbb{E}[d_1] = \mathbb{E}[d_1^0] + \mathbb{E}[d_1^1 - d_1^0] \times \mathbb{E}[z_1] + \mathbb{E}[d_1^2 - d_1^0] \times \mathbb{E}[z_2]$$

Using Table 1, as groups are disjoint, we have

$$\mathbb{E}[d_1^0] = P(d_1^0 = 1) = P(\text{AT}_1)$$

$$\mathbb{E}[d_1^1 - d_1^0] = P(d_1^1 - d_1^0 = 1) = P(C_1) + P(\text{ND}_1)$$

$$\mathbb{E}[d_1^2 - d_1^0] = P(d_1^2 - d_1^0 = 1) - P(d_1^2 - d_1^0 = -1) = P(\text{ID}_2) - P(\text{ND}_2)$$

where we in both instances have assumed monotonicity and AT₁ denotes always takers. This
turns equation (18) into:

\[
\begin{align*}
\alpha_1^0 - P(AT_1) \\
+ [\alpha_1^1 - (P(C_1) + P(ND_1))] \times E[z_1] \\
+ [\alpha_2^2 - (P(ID_2) - P(ND_2))] \times E[z_2] &= 0
\end{align*}
\]

By the rank condition (and symmetry for \( k = 2 \)), this implies:

\[
\begin{align*}
P(AT_1) &= \alpha_1^0 \\
P(C_1) + P(ND_1) &= \alpha_1^1 \\
P(ID_1) - P(ND_1) &= \alpha_2^1
\end{align*}
\]

(20)

\[
\begin{align*}
P(AT_2) &= \alpha_2^0 \\
P(C_2) + P(ND_2) &= \alpha_2^2 \\
P(ID_2) - P(ND_2) &= \alpha_1^2
\end{align*}
\]

(21)

\[
\begin{align*}
P(C_1) + P(AT_1) + P(NT_1) + P(OT_1) + P(ID_1) + P(ND_1) &= 1 \\
P(C_2) + P(AT_2) + P(NT_2) + P(OT_2) + P(ID_2) + P(ND_2) &= 1
\end{align*}
\]

(23)

(24)

By combining equation (23) with equations (20)-(22) we get

\[
\begin{align*}
P(NT_1) &= 1 - \alpha_1^0 - \alpha_2^0 - \alpha_2^1 - \alpha_1^2 \\
P(NT_2) &= 1 - \alpha_1^0 - \alpha_2^0 - \alpha_2^2 - \alpha_1^2
\end{align*}
\]

(25)

(26)

It follows that

\[
\begin{align*}
\alpha_1^0 &= P(AT_1) \\
\alpha_1^1 &= P(C_1) + P(ND_1) \\
\alpha_2^1 &= P(ID_1) - P(ND_1)
\end{align*}
\]

\[
\begin{align*}
\alpha_2^0 &= P(AT_2) \\
\alpha_2^2 &= P(C_2) + P(ND_2) \\
\alpha_1^2 &= P(ID_2) - P(ND_2)
\end{align*}
\]

\[
\begin{align*}
P(NT_1) &= 1 - \alpha_1^0 - \alpha_2^0 - \alpha_1^1 - \alpha_2^2 \\
P(NT_2) &= 1 - \alpha_1^0 - \alpha_2^0 - \alpha_1^2 - \alpha_2^2
\end{align*}
\]

and

\[
\begin{align*}
P(C_1) + P(AT_1) + P(NT_1) + P(OT_1) + P(ID_1) + P(ND_1) &= 1 \\
P(C_2) + P(AT_2) + P(NT_2) + P(OT_2) + P(ID_2) + P(ND_2) &= 1
\end{align*}
\]
B.2 Partial Identification Of Defiers

We continue by proving Proposition 4

Proof. From Proposition 2, we get the following information on $P(\text{ND}_1)$:

$$
P(\text{ND}_1) = \begin{cases} 
-\alpha_2^1 + P(\text{ID}_1) \\
\alpha_2^0 - P(\text{OT}_1) \\
\alpha_1^1 - P(\text{C}_1)
\end{cases}
$$

(27)

where the first line follows from equation (22), the second from (21) and the third from combining equation 23 with 25 and 20. From equation (22) we know that $P(\text{ID}_1) = \alpha_2^1 + P(\text{ND}_1)$. Combining this with the information in equation (27) we have:

$$
P(\text{ID}_1) = \begin{cases} 
\alpha_2^1 + P(\text{ND}_1) \\
\alpha_2^1 + \alpha_2^0 - P(\text{OT}_1) \\
\alpha_2^1 + \alpha_2^1 - P(\text{C}_1)
\end{cases}
$$

This gives the following bounds on $P(\text{ID}_1)$ and $P(\text{ND}_1)$

$$
\begin{align*}
P(\text{ND}_1) & \geq -\alpha_2^1 & P(\text{ID}_1) & \geq \alpha_2^1 \\
P(\text{ND}_1) & \leq \alpha_2^0 & P(\text{ID}_1) & \leq \alpha_2^1 + \alpha_2^0 \\
P(\text{ND}_1) & \leq \alpha_1^1 & P(\text{ID}_1) & \leq \alpha_2^1 + \alpha_1^1
\end{align*}
$$

where also, trivially, $P(\text{ID}_1), P(\text{ND}_1) \geq 0$. It follows that the bounds on $P(\text{ID}_1)$ are:

$$
\begin{align*}
\max\{0, -\alpha_2^1\} & \leq P(\text{ND}_1) \leq \min\{\alpha_1^1, \alpha_2^0\} \\
\max\{0, \alpha_2^1\} & \leq P(\text{ID}_1) \leq \max\{0, \alpha_2^1 + \min\{\alpha_1^1, \alpha_2^0\}\}
\end{align*}
$$

and results for instrument 2 are symmetric.

B.3 Assuming Next-best

We now prove Corollary 1.

Proof. Assuming next-best, we have $P(\text{ND}_1) = P(\text{ND}_2) = 0$. This turns equation (21) into:

$$
\begin{align*}
P(\text{AT}_1) &= \alpha_2^0 \\
P(\text{AT}_2) &= \alpha_2^0 \\
P(\text{C}_1) &= \alpha_2^1 \\
P(\text{C}_2) &= \alpha_2^2 \\
P(\text{ID}_1) &= \alpha_2^1 \\
P(\text{ID}_2) &= \alpha_2^1
\end{align*}
$$
B.4 Assuming Irrelevance

Lastly, we prove Corollary 2

Proof. Assuming irrelevance, we have \( P(ID_1) = P(ID_2) = 0 \). This turns equation (21) into:

\[
\begin{align*}
P(AT_1) &= \alpha_1^0 \\
P(C_1) &= \alpha_1^1 + \alpha_2^1 \\
P(ND_1) &= -\alpha_2^1 \\
P(AT_2) &= \alpha_2^0 \\
P(C_2) &= \alpha_2^2 + \alpha_1^2 \\
P(ND_2) &= -\alpha_1^2
\end{align*}
\]
C Proof Of Violation of Exclusion Under Clustering

In the following, we derive an expression for the IV estimand under binary clustering, as presented in Section 5.1.

C.1 Introduction

As mentioned in Section 5.1, we have the binary IV estimand in our set-up as:

\[
\tilde{\beta}^{IV}_{1} = \frac{\theta_{1}}{\pi_{1}}
\]

where \(\theta_{1}\) is the reduced form and \(\pi_{1}\) is the first stage between when clustering treatments in two clusters, \(S_{0}\) and \(S_{1}\), and seeking to estimate the effect of going from the former to the latter. In the following we will derive a general expression for this estimand.

C.1.1 First Stage

We have the first stage given by the relation

\[
\tilde{d} = \pi_{0} + \pi_{1} \tilde{z} + \nu
\]

Taking expectations on both sides with \(E[\nu] = 0\), we get

\[
E[\tilde{d}] = \pi_{0} + \pi_{1} \times E[\tilde{z}]
\]

Decomposing the LHS into potential outcomes using \(\tilde{d} = \tilde{d}^{0} + (\tilde{d}^{1} - \tilde{d}^{0}) \times \tilde{z}\) we get:

\[
E[\tilde{d}] = E[\tilde{d}^{0}] + E[\tilde{d}^{1} - \tilde{d}^{0}] \times E[\tilde{z}]
\]

i.e. we have

\[
\pi_{0} + \pi_{1} \times E[\tilde{z}] = E[\tilde{d}^{0}] + E[\tilde{d}^{1} - \tilde{d}^{0}] \times E[\tilde{z}]
\]

C.1.2 Reduced Form

With respect to the reduced form, we have:

\[
\theta_{1} = E[y \mid \tilde{z} = 1] - E[y \mid \tilde{z} = 0]
\]

We substitute for potential outcomes with \(y = \tilde{y}^{0} \times (1 - \tilde{d}) + \tilde{y}^{1} \times \tilde{d}\)

\[
\theta_{1} = E[\tilde{y}^{0}(1 - \tilde{d}) + \tilde{y}^{1}\tilde{d} \mid \tilde{z} = 1] - E[\tilde{y}^{0}(1 - \tilde{d}) + \tilde{y}^{1}\tilde{d} \mid \tilde{z} = 0]
\]

Since we do not assume cluster-level exclusion, we need to keep potential treatments and outcomes instrument-dependent. Rearranging we get:

\[
\theta_{1} = E[\tilde{y}^{0,1}\tilde{d}_{0} \mid \tilde{z} = 1] + E[\tilde{y}^{1,1}\tilde{d}_{1} \mid \tilde{z} = 1]
\]
Table 4. Taxonomy of response groups under control clustering

| Type          | Cluster Level | Field Level | Group |
|---------------|---------------|-------------|-------|
|               | $\bar{d}^0$  | $\bar{d}^1$ | $d^0$ | $d^2$ | $d^1$ | Field | Cluster |
| Compliers     | 0             | 1           | 0     | 1     |       | $C_1$ |         |
|               | 0             | 1           | 0     | 2     | 1     | $\bar{C}$ | $C_2$ |
|               | 0             | 1           | 2     | 1     |       | $ND_1$ |       |
|               | 0             | 1           | 0     | 1     |       | $ND_2$ |       |
| Never Takers  | 0             | 0           | 2     | 0     |       | $\bar{NT}$ | $ID_1$ |
|               | 0             | 0           | 2     | 0     |       | $ID_2$ |       |

Note: The table shows potential treatments for field and cluster instruments for groups impacted by the cluster instrument under control clustering. At the field level, $d^0$ indicates which treatment is taken given $Z = 0, d^2$ indicates which treatment is taken given $Z = 2$ and $d^1$ indicates which treatment is taken when $Z = 1$. The notation is equivalent at the cluster level. Relative to the clustered instrument, $\bar{C}$ are compliers and $\bar{NT}$ are never takers. Relative to the field instrument, $C$ are compliers, $ND$ are next-best defiers and $ID$ are irrelevance defiers, all relative to some field level instrument corresponding to a treatment in $S_1$.

\[
- \mathbb{E}[y^{0,0} | \bar{d}^0 = 0, \bar{z} = 0] - \mathbb{E}[y^{1,0} | \bar{d}^1 = 0, \bar{z} = 0]
\]

Rearranging, this becomes:

\[
\theta_1 = \mathbb{E}[y^{0,1} | \bar{d}^1 = 0]P(\bar{d}^1 = 0) + \mathbb{E}[y^{1,1} | \bar{d}^1 = 1]P(\bar{d}^1 = 1) - \mathbb{E}[y^{0,0} | \bar{d}^0 = 0]P(\bar{d}^0 = 0) - \mathbb{E}[y^{1,0} | \bar{d}^0 = 1]P(\bar{d}^0 = 1)
\]

(30)

Under control clustering, we will have

\[
S_1 = \{1\}, \ S_0 = \{0, 2\} \quad \text{or} \quad S_1 = \{2\}, \ S_0 = \{0, 1\}
\]

and under treatment clustering we will have

\[
S_1 = \{1, 2\}, \ S_0 = \{0\}
\]

We will treat these scenarios separately, but focussing on the former control clustering scenario as these are symmetric.

C.2 Control Clustering

We have $S_1 = \{1\}, \ S_0 = \{0, 2\}$ and seek to find an expression of the first stage, reduced form and IV estimand. For brevity of notation, we use the taxonomy in Table 4 to denote complier and defier groups.
C.2.1 First Stage  Applying the taxonomy to the expectation in equation (28), under field level monotonicity we get:

$$E[\tilde{d}^1 - \tilde{d}^0] = P[\tilde{d}^1 - \tilde{d}^0 = 1] - P[\tilde{d}^1 - \tilde{d}^0 = -1] = P(\overline{C})$$

From equation (29) we hence have by the rank condition

$$\pi_{1,0} = P(\overline{C})$$

C.2.2 Reduced Form  We use Table 4 to decompose the expectations in equation (30). Under independence and field level monotonicity, we get:

$$\theta_1 = E[y_1^1 \mid \overline{NT}] \times P(\overline{NT})$$

$$+ E[y_1^1 \mid C] \times P(C)$$

$$- E[y_0^0 \mid \overline{C} \cup \overline{NT}] \times P(\overline{C} \cup \overline{NT})$$

Since sets are disjoint, we can rearrange:

$$\theta_1 = E[y_1^1 - y_0^0 \mid \overline{C}] \times P(\overline{C})$$

$$- E[y_0^0 - y_0^0 \mid \overline{NT}] \times P(\overline{NT})$$

Using Table 4 to turn cluster level groups into field level groups, changing outcome indices to reflect instruments relevant to the group in question, we get:

$$\theta_1 = E[y_1^1 - y_0^0 \mid C_1] \times P(C_1)$$

$$+ E[y_1^1 - y_2^2 \mid C_2] \times P(C_2)$$

$$+ E[y_1^1 - y_0^2 \mid ND_1] \times P(ND_1)$$

$$+ E[y_1^1 - y_0^2 \mid ND_2] \times P(ND_2)$$

$$- E[y_0^0 - y_2^2 \mid ID_1] \times P(ID_1)$$

$$- E[y_2^1 - y_0^0 \mid ID_2] \times P(ID_2)$$

At the field level, we assume exclusion, hence:

$$\theta_1 = E[y_1^1 - y_0^0 \mid C_1] \times P(C_1)$$

$$+ E[y_1^1 - y_2^2 \mid C_2] \times P(C_2)$$

$$+ E[y_1^1 - y_0^2 \mid ND_1] \times P(ND_1)$$

$$+ E[y_1^1 - y_0^2 \mid ND_2] \times P(ND_2)$$

$$+ E[y_2^1 - y_0^0 \mid ID_1] \times P(ID_1)$$

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Table 5. Taxonomy of response groups under treatment clustering.

| Type          | Cluster Level | Field Level | Group |
|---------------|---------------|-------------|-------|
|               | $\bar{d}^0$ | $\bar{d}^1$ | $d^0$ | $d^1$ | $d^2$ | Field | Cluster |
| Compliers     | 0             | 1           | 0     | 1     | 2     | $C_1$ |
|               | 0             | 1           | 0     | 1     |       | $C_2$ |
|               | 0             | 1           | 0     | 2     |       | $C$   |
|               | 1             | 1           | 1     | 2     |       | $C_1$ |
| Always Takers | 1             | 1           | 1     | 2     |       | $C_2$ |
|               | 1             | 1           |       |       |       | $ID_1$|         |
|               | 1             | 2           |       |       |       | $ID_2$|         |

Note: The table shows potential treatments for field and cluster instruments for groups impacted by the cluster instrument under treatment clustering. At the field level, $d^0$ indicates which treatment is taken given $Z = 0$, $d^1$ indicates which treatment is taken when $Z = 1$ and $d^2$ indicates which treatment is taken given $Z = 2$. The notation is equivalent at the cluster level. Relative to the clustered instrument, $C$ are compliers and $AT$ are always takers. Relative to the field instrument, $C$ are compliers, $ID$ are irrelevance defiers and $ND$ are next-best defiers.

\[-\mathbb{E}[y^2 - y^0 \mid ID_2] \times P(ID_2)\]

We divide by the first stage and rearrange. This gives us:

\[
\tilde{\beta}_{IV}^1 = \frac{P(C_1)}{\pi_{1,0}} \mathbb{E}[y^1 - y^0 \mid C_1] + \frac{P(C_2)}{\pi_{1,0}} \mathbb{E}[y^1 - y^2 \mid C_2]
\]

\[
+ \frac{P(ND_1)}{\pi_{1,0}} \mathbb{E}[y^1 - y^0 \mid ND_1] + \frac{P(ND_2)}{\pi_{1,0}} \mathbb{E}[y^1 - y^2 \mid ND_2]
\]

\[
+ \frac{P(ID_1)}{\pi_{1,0}} \mathbb{E}[y^2 - y^0 \mid ID_1] - \frac{P(ID_2)}{\pi_{1,0}} \mathbb{E}[y^2 - y^0 \mid ID_2]
\]

where

\[
\pi_{1,0} = P(C_1 \cup C_2 \cup ND_1 \cup ND_2)
\]

This can be rewritten as:

\[
\tilde{\beta}_{IV}^1 = \frac{P(C_1 \cup ND_2)}{\pi_{1,0}} \mathbb{E}[y^1 - y^0 \mid C_1 \cup ND_2] + \frac{P(C_2 \cup ND_1)}{\pi_{1,0}} \mathbb{E}[y^1 - y^2 \mid C_2 \cup ND_1]
\]

\[
+ \frac{P(ID_1)}{\pi_{1,0}} \mathbb{E}[y^2 - y^0 \mid ID_1] - \frac{P(ID_2)}{\pi_{1,0}} \mathbb{E}[y^2 - y^0 \mid ID_2]
\]

C.3 Treatment Clustering

We have $S_1 = \{1, 2\}$, $S_0 = \{0\}$ and seek to find an expression of the first stage, reduced form and IV estimand. We use the taxonomy in Table 5 to denote complier and defier groups.
C.3.1 First Stage  Applying the taxonomy to the expectation in equation (28), under field level monotonicity we get:

\[ \mathbb{E}[\tilde{d}^1 - \tilde{d}^0] = P[\tilde{d}^1 - \tilde{d}^0 = 1] - P[\tilde{d}^1 - \tilde{d}^0 = -1] = P(\overline{C}) \]

From equation (29) we hence have by the rank condition

\[ \pi_{1,0} = P(\overline{C}) \]

C.3.2 Reduced Form  We use Table 5 to decompose the expectations in equation (30). Under independence and field level monotonicity, we get:

\[
\begin{align*}
\theta_1 &= \mathbb{E}[\tilde{y}^{1,1} | \overline{C}] \times P(\overline{C}) \\
&+ \mathbb{E}[\tilde{y}^{1,1} | \overline{AT}] \times P(\overline{AT}) \\
&- \mathbb{E}[\tilde{y}^{0,0} | \overline{C}] \times P(\overline{C}) \\
&- \mathbb{E}[\tilde{y}^{1,0} | \overline{AT}] \times P(\overline{AT})
\end{align*}
\]

This rearranges to:

\[
\theta_1 = \mathbb{E}[\tilde{y}^{1,1} - \tilde{y}^{0,0} | \overline{C}] \times P(\overline{C}) \\
- \mathbb{E}[\tilde{y}^{0,1} - \tilde{y}^{0,0} | \overline{AT}] \times P(\overline{AT})
\]

Using Table 4 to turn cluster level groups into field level groups, further using that groups are disjoint, and changing outcome indices to reflect instruments relevant to the group in question, we get:

\[
\begin{align*}
\theta_1 &= \mathbb{E}[y^{2,2} - y^{0,0} | C_1] \times P(C_1) \\
&+ \mathbb{E}[y^{1,1} - y^{0,0} | C_2] \times P(C_2) \\
&+ \mathbb{E}[y^{1,2} - y^{0,0} | ID_1] \times P(ID_1) \\
&+ \mathbb{E}[y^{2,1} - y^{0,0} | ID_2] \times P(ID_2) \\
&- \mathbb{E}[y^{2,2} - y^{1,0} | ND_1] \times P(ND_1) \\
&- \mathbb{E}[y^{1,1} - y^{2,0} | ND_2] \times P(ND_2)
\end{align*}
\]

At the field level, we assume exclusion, hence:

\[
\begin{align*}
\theta_1 &= \mathbb{E}[y^2 - y^0 | C_1] \times P(C_1) \\
&+ \mathbb{E}[y^1 - y^0 | C_2] \times P(C_2) \\
&+ \mathbb{E}[y^1 - y^0 | ID_1] \times P(ID_1)
\end{align*}
\]
\[ + \mathbb{E}[y^2 - y^0 \mid ID_2] \times P(ID_2) \]
\[ - \mathbb{E}[y^2 - y^1 \mid ND_1] \times P(ND_1) \]
\[ - \mathbb{E}[y^1 - y^2 \mid ND_2] \times P(ND_2) \]

We divide by the first stage and rearrange. This gives us:

\[
\tilde{\beta}^{IV} = \frac{P(C_1)}{\pi_{1,0}} \mathbb{E}[y^2 - y^0 \mid C_1] + \frac{P(C_2)}{\pi_{1,0}} \mathbb{E}[y^1 - y^0 \mid C_2] + \frac{P(ID_1)}{\pi_{1,0}} \mathbb{E}[y^1 - y^0 \mid ID_1] + \frac{P(ID_2)}{\pi_{1,0}} \mathbb{E}[y^2 - y^0 \mid ID_2] + \frac{P(ND_1)}{\pi_{1,0}} \mathbb{E}[y^1 - y^2 \mid ND_1] - \frac{P(ND_2)}{\pi_{1,0}} \mathbb{E}[y^1 - y^2 \mid ND_2]
\]

where

\[
\pi_{1,0} = P(C_1 \cup C_2 \cup ID_1 \cup ID_2)
\]

This may be rewritten to

\[
\tilde{\beta}_{1,0}^{IV} = \frac{P(C_1 \cup ID_1)}{\pi_{1,0}} \mathbb{E}[y^1 - y^0 \mid C_1 \cup ID_1] + \frac{P(C_2 \cup ID_1)}{\pi_{1,0}} \mathbb{E}[y^2 - y^0 \mid C_2 \cup ID_1] + \frac{P(ND_1)}{\pi_{1,0}} \mathbb{E}[y^1 - y^2 \mid ND_1] - \frac{P(ND_2)}{\pi_{1,0}} \mathbb{E}[y^1 - y^2 \mid ND_2]
\]