Approximation of fractals by manifolds and other graph-like spaces

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Abstract. We define a distance between energy forms on a graph-like metric measure space and on a discrete weighted graph using the concept of quasi-unitary equivalence. We apply this result to metric graphs and graph-like manifolds (e.g. a small neighbourhood of an embedded metric graph) as metric measure spaces with energy forms associated with canonical Laplacians, e.g., the Kirchhoff Laplacian on a metric graph resp. the (Neumann) Laplacian on a manifold (with boundary) and express the distance of the associated energy forms in terms of geometric quantities.

We showed in [PS17] that the approximating sequence of energy forms on weighted graphs used in the definition of an energy form on a pcf fractal converge in the sense that the distance in the quasi-unitary equivalence tends to 0. By transitivity of quasi-unitary equivalence, we conclude that we can approximate the energy form on a pcf fractal by a sequence of energy forms on metric graphs and graph-like manifolds. In particular, we show that there is a sequence of domains converging to a pcf fractal such that the corresponding (Neumann) energy forms converge to the fractal energy form.

Quasi-unitary equivalence of energy forms implies a norm estimate for the difference of the resolvents of the associated Laplace operators. As a consequence, suitable functions of the Laplacians are close resp. converge as well in operator norm, e.g. the corresponding heat operators and spectral projections. The same is true for the spectra and the eigenfunctions in all above examples.

1. Introduction

The aim of this article is to approximate energy forms on metric spaces by energy forms on discrete graphs. An energy form here is a closed non-negative and densely defined quadratic form in the corresponding $L_2$-space. We achieve the approximation by defining a sort of “distance” between two energy forms acting on different Hilbert spaces with the notion of quasi-unitary equivalence introduced in [Pos06] (see also [Pos12]). We consider in this article two main examples, namely we compare discrete weighted graphs with metric graphs and graph-like manifolds. We then apply these abstract results to pcf fractals where we prove that a suitable pcf fractal, and its corresponding sequence of metric graphs and graph-like manifolds together with their energy forms are close in the sense of quasi-unitary equivalence.

1.1. Main results

Let $G = (V, E)$ be a discrete graph with vertex and edge weight functions $\mu: V \rightarrow (0, \infty)$ and $\gamma: E \rightarrow (0, \infty)$ respectively. We consider the weighted Hilbert space $\ell_2(V, \mu)$
with norm given by \( \|f\|_{\ell^2(V,\mu)}^2 := \sum_{v \in V} |f(v)|^2 \mu(v) < \infty \) and the canonical non-negative quadratic form given by
\[
\mathcal{E}(f) := \sum_{e=\{v,v'\} \in E} \gamma_e |f(v) - f(v')|^2.
\]

We assume that this form is bounded (which is equivalent to the fact that the relative weight defined in (2.1) is bounded). Moreover, let \( X \) be a metric space with Borel measure \( \nu \), together with a non-negative and closed quadratic form \( \mathcal{E}_X \).

We embed the graph \( G \) into a metric measure space \( X \) with measure \( \nu \) and energy form \( \mathcal{E}_X \) via a partition of unity \( \psi_v : X \to [0,1] \) (for \( v \in V \)) such that \( X_v := \text{supp} \psi_v \) is compact and connected and such that \( X_v \cap X_{v'} \neq \emptyset \) if there is an edge between \( v \) and \( v' \in V \). The identification operators needed in order to express the notion of \emph{quasi-unitary equivalence} (see Definition A.1) are given by
\[
J : \ell^2(V,\mu) \to L^2(X,\nu), \quad Jf := c \sum_{v \in V} f(v) \psi_v
\]
and
\[
J' : L^2(X,\nu) \to \ell^2(V,\mu), \quad (J'u)(v) := \frac{1}{c} \cdot \frac{1}{\nu(v)} \int_X w \psi_v \, d\nu,
\]
where \( \nu(v) := \int_X \psi_v \, d\nu \) and where \( c > 0 \) is the so-called \emph{isometric rescaling factor}. We assume that \( \nu(v)/\mu(v) \) is equal or close to the constant \( c^{-2} \). Typically, we choose \( \psi_v \) to be \emph{harmonic} (in a suitable sense, specified later in the examples). We also need to relate the energy forms and therefore use also identification operators acting on the form domains, see the proof of Theorem 2.10 for details.

Our first main result is the following:

**1.1. Theorem (see Theorem 2.10).** Let the discrete weighted graph \( (G,\mu,\gamma) \) be uniformly embedded into the metric measure space \( (X,\nu,\mathcal{E}_X) \) (see Definition 2.3). Then, \( \mathcal{E} \) and \( \tilde{\mathcal{E}} := \tau \mathcal{E}_X \) are \( \delta \)-quasi-unitarily equivalent, where \( \delta \) can be expressed entirely in quantities of the weighted graph and the metric measure space.

Here, \( \tau \) is a suitable energy rescaling factor. It will give us some freedom in choosing a suitable length or weight scaling in our examples. The precise definition of \( \delta \) is given in Theorem 2.10 and its meaning is explained in Remarks 2.9 and 2.11.

**Quasi-unitary equivalence.** The notion of \( \delta \)-quasi-unitary equivalence (reviewed briefly in Appendix A) is a generalisation of two concepts (see also Remark A.2): if \( \delta = 0 \) then \( \delta \)-quasi-unitary equivalence is just ordinary \emph{unitary equivalence} and if \( J = J' = \text{id} \) then it implies that the operator norm of the difference of resolvents is bounded by \( \delta \). One condition to check for \( \delta \)-quasi-unitary equivalence is that
\[
(f - J'Jf)(v) = \frac{1}{\nu(v)} \sum_{v' \sim v} (f(v) - f(v')) \langle \psi_v, \psi_{v'} \rangle \quad \text{and}
\]
\[
u - J'J'\nu = \sum_{v \in V} \left( \nu - \frac{1}{\nu(v)} \int_X w \psi_v \, d\nu \right) \psi_v
\]
are small in suitable norms. The first one can be estimated by
\[
\|f - J'Jf\|_{\ell^2(V,\mu)}^2 \leq \frac{2\mu_\infty}{\gamma_0} \mathcal{E}(f), \quad \text{where} \quad \mu_\infty := \sup_{v \in V} \mu(v) \quad \text{and} \quad \gamma_0 := \inf_{e \in E} \gamma_e,
\]
and \( 2\mu_\infty/\gamma_0 \) is one of the terms appearing in \( \delta \). Note that \( \mu_\infty/\gamma_0 \) is related to the maximal inverse relative weight, i.e., if we want to approximate an unbounded form on \( X \) (as in Case A below), the minimum of the relative weight should tend to \( \infty \). The expression for \( u - J'J'\nu \) contains the orthogonal projection onto the first (constant) eigenfunction of a weighted eigenvalue problem on \( X_v \), and hence can be estimated by the inverse of the second (weighted) eigenvalue \( \lambda_2(X_v,\psi_v) \) times the energy form restricted to \( X_v \) (see Lemma 2.5 for details). Since \( \lambda_2(X_v,\psi_v) \) is bounded from below by a suitable isoperimetric...
or Cheeger constant, we need this isoperimetric constant to be uniformly large; this means that the sets $X_v$ are “well connected”.

From the abstract theory of quasi-unitary equivalence of energy forms we deduce the following (for more consequences we also refer to [KPT17 Sec. 3]):

**1.2. Theorem ([Pos12 Ch. 4]).** Assume that $E$ and $\tilde{E}$ are $\delta$-quasi-unitarily equivalent, then the associated operators $\Delta \geq 0$ and $\tilde{\Delta} \geq 0$ fulfil the following:

$$\|\eta(\Delta) - J^*\eta(\tilde{\Delta})J\| \leq C_\eta \delta, \quad \|\eta(\tilde{\Delta}) - J\eta(\Delta)J^*\| \leq C'_\eta \delta, \quad \text{dist}\left(\frac{1}{1 + \sigma(\Delta)}, \frac{1}{1 + \sigma(\tilde{\Delta})}\right) \leq \psi(\delta), \quad |\lambda_k(\Delta) - \lambda_k(\tilde{\Delta})| \leq C_k \delta,$$

where $\eta$ is a suitable function continuous in a neighbourhood of $\sigma(\Delta)$, e.g., $\eta_k(\lambda) = (\lambda - z)^{-1}$ (resolvent in $z$), $\eta_k(\lambda) = e^{-t\Delta}$ (heat operator) or $\varphi = 1_I$ with $\partial I \cap \sigma(\Delta) = \emptyset$ (spectral projection). Moreover, $C_\eta$ and $C'_\eta$ depend only on $\eta$, $\psi(\delta) \to 0$ as $\delta \to 0$ and dist denotes the Hausdorff distance. The last statement refers to the $k$-th eigenvalue of $\Delta$ resp. $\tilde{\Delta}$ (counted with respect to multiplicity), and $C_k$ depends only on upper bounds of $\lambda_k(\Delta)$ and $\lambda_k(\tilde{\Delta})$.

We also have convergence of corresponding eigenfunctions in energy norm, see [PS17 Prop. 2.5] for details.

**Applications.** We have the following applications in mind:

**A. Given a pcf fractal, choose a sequence of graphs:** Let $K$ be a pcf fractal, and let $G = G_m$ be a sequence of weighted graphs approximating the fractal and its energy form; this example has been treated in [PS17]; here, $c = 1$ and $\tau = 1$.

**B. Given a discrete graph construct a metric graph:** given a weighted discrete graph $G$ we construct a metric graph $M$ with standard (also called Kirchhoff) Laplacian and with edge lengths reciprocally proportional with the discrete edge weights; the corresponding energy forms then are $\delta$-quasi-unitarily equivalent, see Theorem 3.5 here, $\delta$ is of the same order as the maximal inverse relative weight defined by the discrete weighted graph (see (2.1) and Remark 2.2 (i)).

**C. Given a metric graph, construct a sequence of discrete subdivision graphs:** given a metric graph $M$, then there is a sequence of discrete weighted subdivision graphs $SG_m$ such that the corresponding metric and discrete energy forms are $\delta_m$-quasi-unitarily equivalent, where $\delta_m$ is of order of the mesh width of the (metric) subdivision graph $SM_m$, see Corollary 3.7; note that the energy forms on $M$ and $SM_m$ are unitarily equivalent as vertices of degree 2 have no effect.

**D. Given a discrete graph construct a graph-like manifold:** given a weighted discrete graph $G$ we construct a graph-like manifold $X$ with longitudinal length scale (edge lengths) again reciprocally proportional with the discrete edge weights; then the corresponding energy forms are $\delta$-quasi-unitarily equivalent; here, $\delta$ is of similar type as in the metric graph case, but has an additional parameter: the ratio of the transversal and longitudinal length scale, see Theorem 4.8 and Corollary 4.9.

**E. Given a sequence of discrete graphs, construct a sequence of metric graphs and graph-like manifolds:** We apply Cases B and D to the sequence $(G_m)_m$ of weighted discrete graphs approximating a given pcf fractal $K$ from Case A and hence obtain a sequence of metric graphs resp. graph-like manifolds $M_m$ resp. $X_m$ with energy forms being $\delta_m$-quasi-unitarily equivalent with the one on $G_m$, where $\delta_m \to 0$ exponentially fast, see Theorems 5.4 and 5.7.

For more information on fractals we refer to [Kig01], [Str06], see also Subsection 5.1 and [PS17 Sec. 3]. For more information on metric graphs and graph-like manifolds, we refer to the corresponding Subsections 3.1 and 4.1 and the references cited there.
Approximation of fractals by metric graphs. Let us here be more precise on Case E. Consider a pcf fractal $K$ with energy form $\mathcal{E}_K$ which can be approximated by a sequence of discrete graphs $(G_m)_{m\in\mathbb{N}_0}$ and energy forms $\mathcal{E}_{G_m}$. In [PS17] we proved that $\mathcal{E}_K$ and $\mathcal{E}_{G_m}$ are $\delta_m$-quasi-unitarily equivalent with $\delta_m = O((r/N)^{m/2})$ where $N \geq 2$ is the number of fixed points of the IFS and $r \in (0,1)$ the energy renormalisation parameter of the self-similarity (see Subsection 5.1 for details). Now, the idea is to show that the energy forms on the discrete graph $G_m$ and the corresponding metric graph $M_m$ are $\delta'_m$-quasi-unitarily equivalent with $\delta'_m = O((r/N)^{m/2})$ and then use the transitivity of quasi-unitary equivalence, see Propositions A.3 and A.6.

Let $G_m = (V_m,E_m)$ be one of the approximating discrete graphs with vertex and edge weights $\mu_m : V_m \to (0,\infty)$ and $\gamma_m : E_m \to (0,\infty)$. The length function $\ell_m : E_m \to (0,\infty)$ of the metric graph $M_m$ is chosen to be proportional to $1/\gamma_m$, in particular, $\ell_m,e$ decays exponentially in $m$. On a metric graph, we identify an edge $e \in E$ with the interval $[0,\ell_m,e]$ with boundary points identified according to the graph structure. Moreover the canonical measure on $M_m$ is given by the sum of the Lebesgue measures on the intervals. Hence, the associated Hilbert space is $L^2(M_m,\nu_m)$ with norm given by

$$||u||^2_{L^2(M_m,\nu_m)} = \sum_{e \in E} \int_0^{\ell_e} |u_e(x)|^2 \, dx$$

where we consider $u$ as a family $(u_e)_{e \in E_m}$ with $u_e : [0,\ell_e] \to \mathbb{C}$. A canonical energy form on $M_m$ is

$$\mathcal{E}_{M_m}(u) = ||u'||^2_{L^2(M_m,\nu_m)} = \sum_{e \in E} \int_{a_e}^{b_e} |u'_e(x)|^2 \, dx, \quad \text{dom}(\mathcal{E}_{M_m}) = H^1(M_m),$$

where $H^1(M_m)$ consists of functions $u_e \in H^1([0,\ell_e])$ such that $u$ is continuous on $M_m$. The associated operator is the usual standard (also called Kirchhoff) Laplacian. The functions in the partition of unity $(\psi_{m,v})_{v \in V_m}$ used for the identification operators $J$ and $J'$ fulfil $\psi_{m,v}(v) = 1$ and $\psi_{m,v}(v') = 0$ for all $v' \in V_m \setminus \{v\}$. Moreover, we assume that the functions $\psi_{m,v}$ are harmonic functions, i.e., affine linear on the edges. An application of Theorem 1.1 is as follows (for a version where we quantify the error term $\tilde{\delta}_m$, see Corollary 5.6): 

**1.3. Theorem.** [Corollary 5.5] The fractal energy form $\mathcal{E}_K$ and the rescaled approximating metric graph energy forms $\mathcal{E}_m = \tau_m \mathcal{E}_{M_m}$ are $\tilde{\delta}_m$-quasi-unitarily equivalent, where $\tilde{\delta}_m \to 0$ as $m \to \infty$.

A natural choice is to let the metric graphs $M_m$ be generated by the IFS, i.e., we start with $M_0$ (embedded as $K$ in $\mathbb{R}^d$), and define $M_m$ as the $m$-th iterate under the IFS with similitude factor $\theta \in (0,1)$ (see Subsection 5.1 for details). In this case, the lengths at generation $m$ are of order $\theta^m$, and the energy rescaling factor is $\tau_m = O((N\theta^2/r)^m)$. For the Sierpiński triangle, we have $N = 3, r = 3/5$ and $\theta = 1/2$, hence $\tau_m = O((5/4)^m)$, confirming analytically the energy rescaling factor for the Sierpiński triangle used in the numerical calculations of the eigenvalues in [BHS09 Sec. 3].

**Approximation of fractals by manifolds.** Similarly, we can construct a sequence $(X_m)_m$ of graph-like manifolds (see Figure 1.1 and for more details Subsection 4.1) with transversal length scale parameter $\epsilon_m$ (the “thickness” of an edge). One example of a graph-like manifold $X_m$ is the $\epsilon_m$-neighbourhood of $M_m$ (if embedded in $\mathbb{R}^d$). In this case, the corresponding operator associated with the energy form given by

$$\mathcal{E}_{X_m}(u) = \int_{X_m} |\nabla u(x)|^2 \, dx, \quad u \in \text{dom} \mathcal{E}_{X_m} = H^1(X_m),$$

is the Neumann Laplacian. We can also construct boundaryless manifolds such as the surface of a tubular neighbourhood of a metric graph $M_m$ embedded in $\mathbb{R}^3$. Again, we need an energy rescaling factor $\hat{\tau}_m$.
Figure 1. The beginning of the sequence of graphs \((G_m)_m\) approximating the Sierpiński triangle and a corresponding sequence of graph-like manifolds \((X_m)_m\) close to the graph \(G_m\) for the generations \(m = 0, 1, 2\).

1.4. Theorem (Corollary 5.11). Assume that the transversal length scale \(\varepsilon_m = \varepsilon_0 E^m (E \in (0, 1))\) decays faster than the longitudinal length scale \(\ell_{m, e} = O(\theta^m)\) (i.e., \(E < \theta\)), then the fractal energy form \(E_K\) and the approximating graph-like manifold energy forms \(\hat{E}_m := \hat{\tau}_m E_{X_m}\) are \(\hat{\delta}_m\)-quasi-unitarily equivalent, where \(\hat{\delta}_m \to 0\) as \(m \to \infty\). For a version where we quantify the error term \(\tilde{\delta}_m\), see Corollary 5.12. In particular, we can approximate the energy form on a fractal by (rescaled) energy forms on a family of smooth manifolds. Unfortunately, we cannot treat the case \(E = \theta\), i.e., when the longitudinal and transversal length scale are of the same. This case occurs e.g. when starting with a suitable neighbourhood \(X_0\) of the first metric graph \(M_0\), and then apply the IFS to generate a sequence of graph-like manifolds \((X_m)_m\), see Remark 5.10 for details.

1.2. Previous and related works

Variational convergence (such as \(\Gamma\)- or Mosco convergence) of discrete energy forms to suitable energy forms on metric measure spaces is an often treated topic; we mention here only two of them and refer to the references therein: Kasue [Kas10] considers sequences of compact metric spaces with resistance metric and energy forms, e.g. finite resistance networks (i.e., weighted graphs with trivial vertex weights \(\mu = 1\) and variable edge weights \(\gamma\)) and \(\Gamma\)-convergence of such sequences, e.g., to infinite graphs. Hinz and Teplyaev [HT15, Thm. 1.2] consider approximations of a bounded Dirichlet form by a sequence of finite weighted graphs in the sense of Mosco. The finite weighted graphs appear as Dirichlet forms on a finitely generated measure space. The partition of unity is just the corresponding finite family of indicator functions; as the limit Dirichlet form is bounded, these indicator functions are in its domain. Hinz and Teplyaev then use the fact that any Dirichlet form can be approximated by a bounded Dirichlet. Note that the Mosco convergence is equivalent with some notion of \(\text{strong}\) resolvent convergence for varying Hilbert spaces, see [KS03, Thm. 2.4], hence our results are stronger as they provide convergence in \(\text{operator norm}\); nevertheless we believe that the conditions of quasi-unitary equivalence are often easier to check than the one for Mosco convergence.

Discretisations of metric measure spaces have also been used in order to check certain types of Poincaré inequalities on the metric measure space by a suitable version on a graph, e.g. in [GL15] (see also the references therein). As in our work, Gill and Lopez embed the
graph into the metric measure space via a partition of unity. On the other hand, Cheeger and Kerner [CK15] use the opposite approach and construct metric measure spaces fulfilling a Poincaré inequality from an increasing sequence of discrete graphs (called metric measure graphs there).

There are quite some articles about the relation between (mostly infinite or classes of finite) graphs and (non-compact or classes of compact) Riemannian manifolds under the name discretisation of a manifold; most authors are interested in questions whether certain properties are invariant under so-called rough isometries, using a related property on a discrete graph. We mention here only the works of [DP76, Kan86a, Kan86b, Cou92, Man05, CGR16] and references therein; and the monograph [Cha01] where Chavel defines similar maps as our $J$ and $J'$, called smoothing and discretisation there. The interest in all these works is to have uniform control of classes of manifolds and discrete graphs, e.g. that $f - J' J f$ is bounded by a constant times the energy norm of $f$, but the constant is not supposed to be small as in our case. We will deal with the approximation of manifolds and their energy forms by discrete graphs in a forthcoming publication.

Kigami considers in [Kig03] energy forms (called resistance forms there) and shows that they can be approximated by limits of finite weighted nested graphs; the corresponding energy forms converge monotonously. In [Tep08] Teplyaev considers energy forms on sets with finitely ramified cell structure and uses an approximation by metric graphs e.g. to characterise the operator domain of the original energy form. In the recent preprint [HIM17] Hinz and Meinert use the metric graph approximation of a Sierpiński triangle to approximate some non-linear differential equations on a fractal.

For the approximation of fractals by open subsets or manifolds not so much is known: Berry, Heilman and Strichartz in [BHS09] provide numerical results on the eigenvalues of some pcf fractals $K$ approximated by Neumann Laplacians on open set $X_m \supset K$ constructed according to the IFS. We confirm analytically that their energy rescaling factor for the Sierpiński triangle and an approximating sequence of open subsets in $\mathbb{R}^2$ is $\tau_m = 5/4$ (see Case 1 in Example 5.9). Better numerical results are obtained by Blasiak, Strichartz and Uğurcan [BSU08, Sec. 3] by a sequence of open sets $X_m \supset K$ not constructed from the IFS. We follow in our manifold example a similar strategy as our approximating sets $X_m$ (if $K \subset \mathbb{R}^2$) are neither generated by the IFS; moreover, they are neither subsets nor supsets of $K$. Some analytic work is done in [MV15] (see also the references therein): Mosco and Vivaldi construct a sequence of weighted energy forms on open domains that Mosco-converge to an energy form on a nested fractal such as the Koch curve or the Sierpiński triangle. The weights are chosen in such a way that the passage along a vertex becomes very narrow. We encounter a similar problem, see Remark 5.10.

1.3. Structure of the article

In Section 2 we define when a discrete graph is uniformly embedded into a metric measure space and prove Theorem 1.1. In Sections 3 and 4 we apply these results to metric graphs and graph-like manifold. Finally, in Section 5 we apply the results from [PS17] and the transitivity of quasi-unitary equivalence (Propositions A.3 and A.6) and deduce that a large class of pcf fractals and its energy form can be approximated by a suitable sequence of metric graphs resp. graph-like manifolds together with their (renormalised) energy forms. Appendix A contains a brief introduction to the concept of quasi-unitary equivalence. In Appendix B we collect some estimates on graph-like manifolds.

1.4. Acknowledgements

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2. Convergence of energy forms on discrete graphs and metric spaces

In this section, we provide a rather general setting: we measure how “close” a discrete graph is to a metric space, both with a suitable energy form. The “distance” is measured in terms of a parameter $\delta \geq 0$ appearing in the definition of quasi-unitary equivalence, see Definition [A.1]

We say that $\mathcal{E}$ is an energy form on a measure space $(X, \nu)$, if $\mathcal{E}$ is a non-negative, densely defined and closed quadratic form, i.e., if $\mathcal{E}(f) \geq 0$ for all $f \in \text{dom } \mathcal{E}$, dom $\mathcal{E}$ is dense in $L^2(X, \nu)$ and dom $\mathcal{E}$ with norm given by

$$\|f\|_2^2 := \|f\|_{L^2(X, \nu)}^2 + \mathcal{E}(f)$$

is complete (i.e., a Hilbert space).

2.1. Metric spaces, energy forms and embedded graphs

The graph and its energy form. Let $G = (V, E, \partial)$ be a discrete graph, i.e., $V$ and $E$ are finite or countable sets and $\partial : E \rightarrow V \times V$, $\partial e = (\partial_- e, \partial_+ e)$, associates to each edge $e \in E$ its initial and terminal vertex $\partial_- e$ and $\partial_+ e$, respectively. We assume here that $G$ is simple, i.e., that there are no loops (i.e., edges with $\partial_- e = \partial_+ e$) nor multiple edges (i.e., edges $e_1, e_2$ with $\partial_+ e_1 = \partial_+ e_2$ and $\partial_- e_1 = \partial_- e_2$). We write $v \sim v'$ for two vertices if there is an edge $e$ with $v = \partial_+ e$ and $v' = \partial_- e$. We also set

$$E^+_v := \{ e \in E | \partial_+ e = v \} \quad \text{and} \quad E_v := E_v^+ \cup E_v^-.$$

In order to have a Hilbert space structure and an energy form, we need two weight functions $\mu : V \rightarrow (0, \infty)$ and $\gamma : E \rightarrow (0, \infty)$, the vertex and edge weights.

The following quantitative control using the variation of the weights (and the maximal degree) will be useful:

2.1. Definition. Let $(G, \mu, \gamma)$ be a weighted graph.

(i) The relative weight (also called $\mu$-degree) is the vertex weight defined by

$$\varrho(v) := \frac{\sum_{e \in E_v} \gamma_e}{\mu(v)}.$$  \hspace{1cm} (2.1)

(ii) We call

$$\frac{\mu_\infty}{\gamma_0} := \frac{\sup_{v \in V} \mu(v)}{\inf_{e \in E} \gamma_e}$$

the maximal inverse relative weight of the weighted graph.

(iii) Let $d_\infty \in \mathbb{N}$, $\bar{\mu} > 0$ and $\bar{\gamma} > 0$. The weighted graph is called $(d_\infty; \bar{\mu}, \bar{\gamma})$-uniform or simply uniform if the degree $\deg v := |E_v|$ is uniformly bounded, i.e., $\deg v \leq d_\infty$, and if

$$\mu_0 := \inf_{v \in V} \mu(v) \leq \mu_\infty := \sup_{v \in V} \mu(v) \leq \bar{\mu} \mu_0 \quad \text{and} \quad (2.2a)$$

$$\gamma_0 := \inf_{e \in E} \gamma_e \leq \gamma_\infty := \sup_{e \in E} \gamma_e \leq \bar{\gamma} \gamma_0.$$  \hspace{1cm} (2.2b)

2.2. Remark.

(i) The relative weight is bounded by

$$\frac{\gamma_0}{\mu_\infty} \leq \varrho(v) \leq \frac{d_\infty \gamma_\infty}{\mu_0}$$ \hspace{1cm} (2.3)

if the corresponding numbers are in $(0, \infty)$. In particular, $1/\varrho(v)$ is bounded from above by $\mu_\infty/\gamma_0$, hence the name maximal inverse relative weight.

Moreover, the relative weight is bounded for a $(d_\infty; \bar{\mu}, \bar{\gamma})$-uniform weighted graph, as the latter implies $0 < \mu_0 \leq \mu_\infty < \infty$ and $0 < \gamma_0 \leq \gamma_\infty < \infty$. 
(ii) For a \((d_{\infty}, \bar{\rho}, \bar{\tau})\)-uniform weighted graph, we also have
\[
\frac{1}{\bar{\tau} d_{\infty} \bar{\rho}}, \mu_{\infty} \leq \frac{1}{\bar{\rho}(v)} \leq \frac{\mu_{\infty}}{\gamma_0},
\]
i.e., the inverse of the relative weight is controlled by the maximal inverse relative weight \(\mu_{\infty}/\gamma_0\). The latter quotient will be used later on as error, e.g. in Proposition 2.8.

The Hilbert space associated with a weighted graph is
\[
\mathcal{H} := \ell_2(V, \mu), \quad \|f\|^2_{\ell_2(V, \mu)} := \sum_{v \in V} |f(v)|^2 \mu(v)
\]
and the energy form is given by
\[
\mathcal{E}(f) := \sum_{e \in E} \gamma_e |(df)_e|^2, \quad (df)_e := f(\partial_+ e) - f(\partial_- e).
\]
and bounded by \(2 \varrho_\infty := 2 \sup_v \varrho(v)\). Throughout this article, we assume that the relative weight \(\varrho\) is bounded, i.e., \(\varrho_\infty < \infty\). In particular, the corresponding discrete Laplacian \(\Delta_{(G, \mu, \gamma)}\) is also bounded by \(2 \varrho_\infty\), and acts as
\[
(\Delta_{(G, \mu, \gamma)} \varphi)(v) = \frac{1}{\mu(v)} \sum_{e \in E_v} \gamma_e (\varphi(v) - \varphi(v_e)),
\]
where \(v_e\) is the vertex on \(e\) opposite to \(v\).

**The metric measure space and energy form.** As space approximated by the graph \(G\) we choose a metric measure space \(X\) with Borel measure \(\nu\). Then we have a canonical Hilbert space, namely
\[
\tilde{\mathcal{H}} := L_2(X, \nu), \quad \|u\|^2_{L_2(X, \nu)} := \int_X |u|^2 \, d\nu.
\]
We assume that \(X\) has a natural energy form\(^1\) \(\tilde{\mathcal{E}}_X\) with domain \(\tilde{\mathcal{H}} :\! = \text{dom } \mathcal{E}_X\). We specify further properties later on. We are mainly interested in three examples together with their natural energy forms:

(i) \(X\) is a pcf self-similar fractal (see e.g. [PS17 Sec. 3]);
(ii) \(X\) is a metric graph;
(iii) \(X\) is a graph-like manifold.

We will explain metric graphs and graph-like manifolds in Subsections 3.1 and 4.1. As we need to rescale the energy in some cases, we introduce a constant \(\tau > 0\), called energy rescaling factor and set
\[
\tilde{\mathcal{E}} := \tau \mathcal{E}_X
\]
with the same domain. We specify \(\tau\) later on.

We come now to our main definition, relating a discrete graph with a metric measure space and its energy form:

**2.3. Definition.** Let \(X\) be a metric measure space with Borel measure \(\nu\) and energy form \(\mathcal{E}_X\). A weighted graph \((G, \mu, \gamma)\) is uniformly embedded into \((X, \nu, \mathcal{E}_X)\), if the following conditions hold:

(i) **Partition of unity and relation to graph structure:** There is a family of functions
\[
\Psi := (\psi_v)_{v \in V} \text{ with } \psi_v \in \text{dom } \mathcal{E}_X \text{ and } \sum_{v \in V} \psi_v = 1_X \text{ such that}
\]
\[
X_v := \text{supp } \psi_v \text{ is connected and } X_v \cap X_{v'} \neq \emptyset \iff v \sim v'.
\]
(ii) **Decomposition of energy:** There are energy forms \(\mathcal{E}_{X_v}\) on \((X_v, \nu(\cdot|X_v))\) \((v \in V)\) with domains \(\text{dom } \mathcal{E}_{X_v} = \{u|_{X_v} \mid u \in \text{dom } \mathcal{E}_X\}\) such that
\[
\mathcal{E}_X(u) \leq \sum_{v \in V} \mathcal{E}_{X_v}(u|_{X_v}) \leq 2 \mathcal{E}_X(u)
\]
\(^1\)It is indeed in all our examples a Dirichlet form but we will not need this fact in our analysis.
for all \( u \in \text{dom} \mathcal{E}_X \).

(iii) **Local energy is uniformly spectrally small:** The form \( \mathcal{E}_X \) is closable in the weighted Hilbert space

\[
\mathcal{L}_2(X_v, \psi_v) := \left\{ u \left| \|u\|_2^2 := \int_X |u|^2 \psi_v \, d\nu < \infty \right. \right\},
\]

and the spectrum is purely discrete. Moreover, the first eigenvalue is \( \lambda_1(X_v, \psi_v) = 0 \) with constant eigenfunction \( 1_{X_v} \), and the second eigenvalue (as family of \( v \in V \)) fulfills

\[
0 < \lambda_2 := \inf_{v \in V} \lambda_2(X_v, \psi_v).
\]

(iv) **(Almost) compatibility of the weights:** The weighted graph has finite maximal inverse relative weight \( \mu_\infty/\gamma_0 \in (0, \infty) \), where

\[
\mu_\infty := \sup_{v \in V} \mu(v) < \infty \quad \text{and} \quad \gamma_0 := \inf_{e \in E} \gamma_e > 0.
\]

Moreover,

\[
\nu(v) := \int_{X_v} \psi_v \, d\nu = \nu_0(v) + \bar{\nu}(v) \quad \text{and} \quad \frac{1}{c^2} := \frac{\nu_0(v)}{\mu(v)}
\]

for all \( v \in V \), where the so-called isometric rescaling factor \( c > 0 \) is assumed to be independent of \( v \in V \), and where

\[
\alpha_\infty := \sup_{v \in V} |\alpha(v)| \leq \frac{1}{2} \quad \text{with} \quad \alpha(v) := \frac{\bar{\nu}(v)}{\nu_0(v)}.
\]

We call \( X_v \) the \((enlarged) vertex neighbourhood\) of \( v \in V \) in \( X \). Moreover,

\[
\tilde{X}_v := \psi_v^{-1}\{1\} = \{ x \in X_v \mid \psi_v(x) = 1 \} \quad \text{and} \quad X_e := X_{\partial_+ e} \cap X_{\partial_- e}
\]

are called the **core vertex neighbourhood** of \( v \) and the **edge neighbourhood** of \( e \) in \( X \), respectively. We call \( \nu = (\nu(v))_{v \in V} \) the vertex weight of \( \Psi \) in \((X, \nu)\). We say that \( \nu \) and \( \mu \) are **compatible weights** if \( \nu(v)/\mu(v) \) is independent of \( v \), i.e., if \( \nu = \nu_0 \) or \( \alpha = 0 \).

**2.4. Remark.**

(i) Note that we have the relation

\[
\nu(v) = \sum_{v' \in V} \int_X \psi_v \psi_{v'} \, d\nu = \sum_{v' \in V} \langle \psi_v, \psi_{v'} \rangle = \|\psi_v\|_{\mathcal{L}_2(X_v)}^2 + \sum_{v' \in V, v' \neq v} \langle \psi_v, \psi_{v'} \rangle
\]

as the partition of unity reflects the discrete graph structure.

(ii) For ease of notation, we write \( \mathcal{E}_X(u) \) instead of the more precise notation \( \mathcal{E}_{X_v}(u|_{X_v}) \). Condition (2.7b) allows us to get finer estimates in the proof of Proposition 2.8.
(iii) Note that $L_2(X_v) \subset L_2(X_v, \psi_v)$ (see (2.13)), hence $\mathcal{E}_{X_v}$ need not to be closed in $L_2(X_v, \psi_v)$. Moreover, the condition $\lambda_2 > 0$ in particular implies that $\lambda_2(X_v, \psi_v) > 0$ for all $v \in V$, i.e., $\lambda_1(X_v, \psi_v) = 0$ is a simple eigenvalue. In our examples, this means that $X_v$ is connected (either as graph or as topological space) as we already assumed in (i).

(iv) Condition (2.7d) is a condition already for finite graphs, as it assumes that the “bad” part of $\nu$, i.e., the deviation $\tilde{\nu}(v)$ from being compatible, is small. The reason for assuming that $\alpha_\infty \leq 1/2$ becomes clear at the end of the proof of Proposition 2.8.

(v) For a single finite graph $G$, the conditions on $\lambda_2$ and $\mu_\infty/\gamma_0$ are always fulfilled. We stress the constants $\lambda_2, \epsilon, \mu_\infty/\gamma_0 \in (0, \infty)$ and $\alpha_\infty \in [0,1/2]$, if we consider an infinite graph or an infinite family of graphs. In both situations, we need a uniform control of certain parameters, see Proposition 2.8 and Theorem 2.10.

We denote the weighted average of $u$ by

$$\int_X u \, d\nu_v := \frac{1}{\nu(v)} \int_X u \, d\nu_v, \quad \text{where} \quad d\nu_v := \psi_v \, d\nu. \quad (2.10)$$

Note that

$$\int_X u \, d\nu_v = \int_X \psi_v \, d\nu_v \int_X u \psi_v \, d\nu_v = \frac{1}{\nu(v)} \langle u, 1_{X_v} \rangle_{L_2(X_v, \psi_v)},$$

the latter representation will be useful in the next lemma:

2.5. Lemma. Assume that $\mathcal{E}_{X_v}$ fulfills Definition 2.3 (iii) then

$$\int_X \left| u - \int_X u \, d\nu_v \right|^2 \, d\nu_v \leq \frac{1}{\lambda_2(X_v, \psi_v)} \mathcal{E}_{X_v}(u), \quad (2.11)$$

for all $u \in \text{dom} \mathcal{E}_{X_v}$.

**Proof.** Note first that $\|1_{X_v}\|^2_{L_2(X_v, \psi_v)} = \int_X \psi_v \, d\nu = \nu(v)$, hence

$$u - \frac{1}{\nu(v)} \langle u, 1_{X_v} \rangle 1_{X_v} = u - \int_X u \, d\nu_v$$

is the projection onto the orthogonal complement of the first eigenspace $C1_{X_v}$. The result then follows from the min-max characterisation of eigenvalues. □

In some cases, it will be easier to have an estimate of the corresponding unweighted eigenvalue problem. As above, $\lambda_2(X_v, \psi_v)$ denotes the second (first non-zero) eigenvalue of the operator associated with $\mathcal{E}_{X_v}$ in the weighted Hilbert space $L_2(X_v, \psi_v)$.

2.6. Lemma. Let $\Phi_{2,v}$ be a normalised eigenfunction associated with $\lambda_2(X_v, \psi_v)$. Moreover, denote by $\lambda_2(X_v)$ the second (first non-zero) eigenvalue of $\mathcal{E}_{X_v}$ in the unweighted Hilbert space $L_2(X_v)$. If $\Phi_{2,v} \in L_2(X_v)$, then we have

$$\lambda_2(X_v, \psi_v) \geq \lambda_2(X_v). \quad (2.12)$$

**Proof.** Note first that

$$\|u\|^2_{L_2(X_v, \psi_v)} = \int_{X_v} |u|^2 \psi_v \, d\nu \leq \int_{X_v} |u|^2 \, d\nu = \|u\|^2_{L_2(X_v)} \quad (2.13)$$

as $0 \leq \psi_v \leq 1$. In particular, $L_2(X_v) \subset L_2(X_v, \psi_v)$, but the inclusion is in general strict. By the min-max characterisation of the second eigenvalue, we have

$$\lambda_2(X_v) = \inf_{D_2} \sup_{u \in D_2 \setminus \{0\}} \frac{\mathcal{E}_{X_v}(u)}{\|u\|^2_{L_2(X_v)}} = \sup_{u \in D_2 \setminus \{0\}} \frac{\mathcal{E}_{X_v}(u)}{\|u\|^2_{L_2(X_v, \psi_v)}} \leq \sup_{u \in D_2 \setminus \{0\}} \frac{\mathcal{E}_{X_v}(u)}{\|u\|^2_{L_2(X_v, \psi_v)}}$$

where $D_2$ runs through all two-dimensional subspaces of $L_2(X_v) \cap \text{dom} \mathcal{E}_{X_v}$. As the first eigenfunction for both problems is the constant $1_{X_v}$ and as the second eigenfunction $\Phi_{2,v}$ of the weighted problem is also in the unweighted Hilbert space, we can choose $D_2 = C1_{X_v} + C\Phi_{2,v}$ as two-dimensional space. For this choice, the latter Rayleigh quotient becomes $\lambda_2(X_v, \psi_v)$, and the result follows. □
Remark. In our applications on metric graphs and graph-like manifolds, we choose $\psi_v$ to be harmonic, and due to the geometric assumptions, $\psi_v$ will be (piecewise) affine linear. For this choice, the second eigenfunction of the weighted problem is an Airy function hence continuous, and therefore belongs also to the unweighted Hilbert space.

2.2. The identification operators

Let us now consider a weighted discrete graph $(G, \mu, \gamma)$ with vertex weight $\mu : V \to (0, \infty)$ and edge weight $\gamma : E \to (0, \infty)$. We have then the associated Hilbert space $\ell_2(V, \mu)$ as in Subsection 2.1. We assume that $(G, \mu, \gamma)$ is uniformly embedded into $(X, \nu, \mathcal{E}_X)$, see Definition 2.3 hence the isoperimetric rescaling factor $c > 0$ is defined.

We now define the identification operators $J$ and $J'$ from $\mathcal{H} = \ell_2(V, \mu)$ into $\tilde{\mathcal{H}} = L_2(X, \nu)$ and vice versa: Let

$$Jf := c \sum_{v \in V} f(v)\psi_v$$

for $f : V \to \mathbb{C}$ with finite support and

$$(J'u)(v) = \frac{1}{c} \int_X u \, d\nu_v.$$ (for the notation $\int_X$ and $d\nu_v$ see (2.10)).

2.7. Remark. If $X$ is a Riemannian manifold, then $J$ is also called smoothing of functions on $G$ and $J'$ is called discretisation of functions on $X$, see the monograph [Cha01, Sec. VI.5] for details (and also Subsection 1.2). Moreover, the concept of having a partition of unity with respect to a suitable cover of the manifold labelled by the graph vertices is called discretisation of $X$, see [Cha01, Sec. V.3.2] for details.

The energy form spaces are $\mathcal{H}^1 = \mathcal{H} = \ell_2(V, \mu)$ (if the relative weight is bounded) and $\tilde{\mathcal{H}}^1 = \text{dom} \mathcal{E}_X$ with norms given by

$$\|f\|^2_{\mathcal{H}^1} := \|f\|^2_{\ell_2(V, \mu)} + \mathcal{E}(f) \quad \text{and} \quad \|u\|^2_{\tilde{\mathcal{H}}^1} := \|u\|^2_{L_2(X, \nu)} + \tilde{\mathcal{E}}(u)$$

respectively, where $\tilde{\mathcal{E}}(u) = \tau\mathcal{E}_X$ and where the energy rescaling parameter $\tau > 0$ is specified later on. Note that although the spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}^1$ are the same, the norms differ, and this fact matters when considering families of graphs where $\|f\|^2_{\tilde{\mathcal{H}}^1}/\|f\|^2_{\mathcal{H}} \leq 1 + \sup_v \rho(v)$ is not uniformly bounded.

The following result is of abstract nature:

2.8. Proposition. Assume that $(G, \mu, \gamma)$ is uniformly embedded into $(X, \nu, \mathcal{E}_X)$, then the following holds:

(i) $J$ and $J'$ are bounded;

(ii) $J$ and $J'$ fulfil (A.3b) with $\delta$ replaced by $\delta_b$, where

$$\delta_b^2 = \max\left\{\frac{2\mu_\infty}{\gamma_0}, \frac{2}{\tau\lambda_2}\right\}.$$

(iii) $J$ and $J'$ fulfil (A.3a) with $\delta$ replaced by

$$\delta_a = 2\alpha_\infty,$$ (2.14)

where $\alpha_\infty$ is defined in (2.7d).

In particular, $J$ is $\delta$-quasi-unitary with adjoint $J' = J^*$ and $\delta = \max\{\delta_a, \delta_b\}$. 
Proof. \(\ell\) The boundedness of \(J: \mathcal{H} = \ell_2(V, \mu ) \rightarrow \mathcal{H} = \ell_2(X, \nu )\) follows from

\[
\| Jf \|^2_{\ell_2(X, \nu )} = \sum_{v \in V} \sum_{v' \in V} c^2 f(v) \overline{f(v')} \langle \psi_v, \psi_{v'} \rangle \leq c^2 \sum_{v \in V} |f(v)|^2 \sum_{v' \in V} \langle \psi_v, \psi_{v'} \rangle
\]

\[
= c^2 \sum_{v \in V} |f(v)|^2 \nu(v) \leq \sup_{v \in V} \frac{c^2 \nu(v)}{1 + \alpha(v)} \| f \|^2_{\ell_2(V, \mu )} \leq (1 + \alpha_\infty) \| f \|^2_{\ell_2(V, \mu )}, \tag{2.15a}
\]

where we used the partition of unity property (2.9) for the second equality. The boundedness of \(J'\) can be seen from

\[
\| J' u \|^2_{\ell_2(V, \mu )} = \frac{1}{c^2} \sum_{v \in V} \frac{\mu(v)}{\nu(v)^2} \left| \int_X u \, d\nu_v \right|^2 \leq \frac{1}{c^2} \sum_{v \in V} \frac{\mu(v)}{\nu(v)^2} \int_X |u|^2 \, d\nu_v \cdot \nu(v)
\]

\[
\leq \sup_{v \in V} \frac{1}{1 + \alpha(v)} \int_X |u|^2 \, d\nu \leq \frac{1}{1 - \alpha_\infty} \| u \|^2_{\ell_2(X, \nu )} \tag{2.15b}
\]

using the partition of unity property \(\sum_v d\nu_v = d\nu\) in the second last inequality.

(iii) We are now checking the conditions of (A.3b): We have

\[
f(v) - (J' J f)(v) = \frac{1}{\nu(v)} \sum_{v' \sim v} (f(v) - f(v')) \langle \psi_{v'}, \psi_v \rangle
\]

using (2.9), hence

\[
\| f - J' J f \|^2_{\mathcal{H}} = \sum_{v \in V} \frac{\mu(v)}{\nu(v)^2} \left( \sum_{v' \sim v} (f(v) - f(v')) \langle \psi_{v'}, \psi_v \rangle \right)^2
\]

\[
\leq \sum_{v \in V} \frac{\mu(v)}{\nu(v)^2} \sum_{v' \sim v} \gamma_e^{-1} \langle \psi_{v_e}, \psi_v \rangle^2 \sum_{v \in E_v} \gamma_e |f(v) - f(v_e)|^2.
\]

Here, \(v_e\) denotes the vertex opposite to \(v\) on \(e\).

We now continue with the second sum of the last formula and estimate

\[
\sum_{v \in E_v} \gamma_e^{-1} \langle \psi_{v_e}, \psi_v \rangle^2 \leq \frac{1}{\gamma_0} \sum_{v \in E_v} \left( \int_X \psi_{v_e} \overline{\psi_v} \, d\nu \right)^2
\]

\[
\leq \frac{1}{\gamma_0} \sum_{v \in E_v} \left( \int_X \psi_{v_e} \, d\nu \right) \left( \int_X \psi_v \, d\nu \right) = \frac{1}{\gamma_0} \left( \int_X \psi_v \, d\nu \right)^2 = \frac{\nu(v)^2}{\gamma_0}
\]

using (2.7d) for the first inequality, \(\psi_{v_e} \leq 1\) for the second inequality and the partition of unity property for the first equality in the last line. From this estimate we conclude

\[
\| f - J' J f \|^2_{\mathcal{H}} \leq \frac{\mu_\infty}{\gamma_0} \sum_{v \in V} \sum_{v \in E_v} \gamma_e |f(v) - f(v_e)|^2 \leq \frac{2 \mu_\infty}{\gamma_0} \sum_{e \in E} \gamma_e |(df)_e|^2 = \frac{2 \mu_\infty}{\gamma_0} \mathcal{E}(f)
\]

(the factor 2 appears because \(\sum_{v \in V} \sum_{v \in E_v} a_e = \sum_{v \in E} \sum_{v \in \partial_v} a_e = 2 \sum_{v \in E} a_e\).

For the second condition in (A.3b), we note that

\[
J' J u = \sum_{v \in V} \int_X u \, d\nu_v \psi_v
\]
Moreover, using the partition of unity property we have \( u = \sum_{v \in V} u \psi_v \) hence
\[
\| u - J' u \|_{\mathcal{D}}^2 = \int_X \left\| \sum_{v \in V} \left( u - \int_X u \, d\nu_v \right) \psi_v \right\|^2 \, d\nu
\]
\[
\leq \frac{CS}{\lambda_2} \sum_{v \in V} \left\| u - \int_X u \, d\nu_v \right\|^2 \psi_v \sum_{v \in V} \psi_v \, d\nu
\]
\[
= \sum_{v \in V} \int_X \left( u - \int_X u \, d\nu \right)^2 \psi_v \, d\nu
\]
\[
\leq \frac{1}{\lambda_2} \sum_{v \in V} \mathcal{E}_v (u) \lesssim \frac{2}{\lambda_2} \mathcal{E}_X (u) = \frac{2}{\tau \lambda_2} \tilde{\mathcal{E}} (u)
\]
using (2.11), (2.7c'), (2.7b) and the energy rescaling factor \( \tau \) for the last line.

(iii) For the second condition in (A.3a), we first define the function \( \Xi \) by
\[
\Xi (\xi) = \left| \sqrt{\xi} - \frac{1}{\sqrt{\xi}} \right| = 2 \sinh \left( \frac{1}{2} \log \xi \right)
\]
note that we have \( \Xi (1) = 0 \), \( \Xi (1/\xi) = \Xi (\xi) \) and \( 0 < \Xi (\xi) \leq \xi - 1 \) for \( \xi > 1 \). Then we have
\[
\left( \left\langle J f, u \right\rangle - \left\langle f, J' u \right\rangle \right) = \left| \sum_{v \in V} \left( c - \frac{\mu (v)}{c \nu (v)} \right) f (v) \psi_v, u \right|
\]
\[
\leq \sup_{v \in V} \Xi \left( \frac{c^2 \mu (v)}{\mu (v)} \right) \left( \sum_{v \in V} \sqrt{\mu (v)} f (v) \frac{1}{\sqrt{\nu (v)}} \psi_v, u \right)
\]
\[
\leq \sup_{v \in V} \Xi \left( 1 + \alpha (v) \right) \left( \sum_{v \in V} \mu (v) |f (v)|^2 \sum_{v \in V} \frac{1}{\nu (v)} \int_X |u|^2 \psi_v \, d\nu \int_X \psi_v \, d\nu \right)^{1/2}
\]
\[
\leq \max \left\{ \Xi (1 + \alpha_\infty), \Xi (1 + \inf_{v \in V} \alpha (v)) \right\} \| f \|_{\mathcal{L}_2 (\nu)} \| u \|_{\mathcal{L}_2 (X, \nu)}
\]
The first term in the maximum appears when \( \alpha (v) \geq 0 \), the second when \( \alpha (v) < 0 \). The latter one can further be estimated by
\[
\Xi (1 - \alpha_\infty) = \Xi (1/(1 - \alpha_\infty)) \leq 1/(1 - \alpha_\infty) - 1 = \alpha_\infty/(1 - \alpha_\infty) \leq 2 \alpha_\infty
\]
provided \( \alpha_\infty \leq 1/2 \). From (2.15a)–(2.15b) and the last estimate we see that
\[
\delta_a = \max \left\{ \sqrt{1 + \alpha_\infty} - 1, \frac{1}{\sqrt{1 - \alpha_\infty}} - 1, \Xi (1 + \alpha_\infty), 2 \alpha_\infty \right\}
\]
where the last term wins, hence we chose \( \delta_a = 2 \alpha_\infty \), see also Remark 2.4 (iv).

**2.9. Remark.** Let us comment on the error terms in \( \delta_b \) and \( \delta_a \):

(i) The maximal inverse relative weight \( \mu_\infty / \gamma_0 \) is an upper bound on the inverse of the relative weight \( \varrho (v) = \sum_{v \in E_v} \gamma_0 / \mu (v) \geq \gamma_0 / \mu_\infty \), hence a necessary condition for the maximal inverse relative weight \( \mu_\infty / \gamma_0 \) to be small is that the relative weight is large. For a uniform weighted graph, this condition is also sufficient (see Remark 2.2).

Note that if we plug in \( f = \delta_v \) into \( f - J'.Jf \), we see that
\[
\| f - J'.Jf \|^2 \geq \frac{1}{\varrho (v)}
\]
In particular, if the weighted graph is uniform, then \( 1/(\varrho (d_\infty \bar{\mu}) \cdot \mu_\infty / \gamma_0) \) is a lower bound, hence the error estimate is optimal.

(ii) The quotient \( 1/\lambda_2 \) in the second error term means that the cells \( X_v \) are “spectrally small”, i.e., they are small and sufficiently connected (we hence expect that the energy rescaling factor \( \tau \) is not small). Note that a lower bound on the second eigenvalue is given by a Cheeger-like isoperimetric constant; and a large Cheeger constant means a
"well-connected" space \( X_v \). In particular, if this constant is uniformly bounded and large, we will get a small error \( 1/\lambda_2 \).

This error is also optimal as one can see by plugging in the eigenfunction associated with \( \lambda_2(X_v, \psi_v) \) into the estimate.

(iii) The error term \( \alpha_\infty \) measures how far the weights \( (\nu(v))_{v \in V} \) given by \( \nu(v) = \int_X \psi_v \, d\nu \) are away from being a constant multiple of the vertex weights \( \mu \) on the graph.

### 2.3. Quasi-unitary equivalence of energy forms

Let us now show under some additional assumptions how to obtain the quasi-unitary equivalence of the energy forms on the metric space and the discrete graph:

#### 2.10. Theorem. Assume that \((G, \mu, \gamma)\) is uniformly embedded into \((X, \nu, \mathcal{E}_X)\) (see Definition 2.3). Moreover, we assume that

(i) for each \( v \in V \) there exists \( \Gamma_v : \text{dom} \mathcal{E}_{X_v} \rightarrow \mathbb{C} \) and \( \delta_c(v) \geq 0 \) such that

\[
\nu_0(v) \left| \Gamma_v u - \int_X u \, d\nu \right|^2 \leq \delta_c(v)^2 \mathcal{E}_{X_v}(u) \tag{2.16a}
\]

holds for all \( u \in \text{dom} \mathcal{E}_{X_v} \);

(ii) for each \( v \in V \) and \( e \in E_v \) there exists \( \Gamma_{v,e} : \text{dom} \mathcal{E}_{X_v} \rightarrow \mathbb{C} \) and \( \delta_d(v) \geq 0 \) such that

\[
c^2 \tau \mathcal{E}_X(u, \psi_v) = \sum_{e \in E_v} \gamma_e (\Gamma_{v,e} u - \Gamma_{v,e} u) \tag{2.16b}
\]

and

\[
\frac{1}{c^2} \sum_{e \in E_v} \gamma_e |\Gamma_{v,e} u - \Gamma_{v,e} u|^2 \leq \delta_d(v)^2 \mathcal{E}_{X_v}(u) \tag{2.16b'}
\]

hold for all \( u \in \text{dom} \mathcal{E}_{X_v} \).

Then \( \mathcal{E} \) and \( \widetilde{\mathcal{E}} = \tau \mathcal{E}_X \) are \( \delta \)-quasi-unitarily equivalent with isometric and energy rescaling factors \( c > 0 \) and \( \tau > 0 \), respectively, where

\[
\delta^2 := \max \left\{ 2 \alpha_\infty, \frac{2\mu_\infty}{\gamma_0}, \frac{2}{\tau \lambda_2}, \frac{2}{\tau} \sup_{v \in V} \delta_c(v)^2, \frac{4}{\tau} \sup_{v \in V} \delta_d(v)^2 \right\}. \tag{2.17}
\]

#### 2.11. Remarks.

(i) If points have positive capacity in \( X \), e.g. if \( X \) is a pcf fractal (see [PS17]) or a metric graph (see Section 3), one can choose

\[
\Gamma_v u = \Gamma_{v,e} u = u(v),
\]

and hence \( \delta_d(v) = 0 \). In this case, (2.16a) follows from a Hölder estimate of \( u \) and \( \delta_c(v)^2 \) is of order as the diameter of \( X_v \) (for suitable spaces \( X \)).

(ii) The condition in (2.16a) means that the "evaluation" \( \Gamma_v u \) is close to the (weighted) average on \( X_v \). In particular, we conclude from (2.16a) that \( \Gamma_v \mathbb{1}_{X_v} = 1 \), as \( \int_X \mathbb{1}_{X_v} \, d\nu = 1 \) and \( \mathcal{E}_{X_v}(\mathbb{1}_{X_v}) = 0 \) (see Definition 2.3 (iii)).

(iii) The choice \( \Gamma_v u = \int_X u \, d\nu \) is possible, but in our applications bad: Although then \( \delta_c(v) = 0 \), one obtains \( \Gamma_v u = u(v) \) from (2.16b) in the metric graph case. But then the estimate on \( u(v) - \int_X u \, d\nu \) appears in (2.16b) with the edge weight \( \gamma_e \) as factor: this weight is generally large, and the resulting error term \( \delta_d(v) \) will not be small (compared to the choice \( \Gamma_v u = \Gamma_{v,e} u = u(v) \) as in (i)), where we have to estimate \( u(v) - \int_X u \, d\nu \), together with the small factor \( \nu_0(v) \)). A similar remark holds for graph-like manifolds and fractals.

(iv) The conditions (2.16b)–(2.16b') can be understood as follows: It is not hard to see that

\[
\Gamma : \text{dom} \mathcal{E}_X \rightarrow \ell_2(V, \mu) \quad \text{with} \quad \Gamma u = (\Gamma_v u)_{v \in V}
\]

is bounded using (2.16a) and (2.7b). In particular, \( (\Gamma, \ell_2(V, \mu)) \) is a (generalised) boundary pair in the sense of [Pos16].
By (2.16b), \( \Gamma_v u \) and \( \Gamma_v \cdot u \) are close to each other. Assume here for simplicity that \( \Gamma_v u = \Gamma_v \cdot u \) as in [1]. If \( \psi_v \) is harmonic, i.e., if \( \psi_v \) minimises \( \mathcal{E}_X(\psi_v) \) among all functions \( u \in \text{dom} \mathcal{E}_X \) with \( \Gamma_v u = 1 \), then \( \mathcal{E}_X(u, \psi_v) = \langle \Lambda_0 \Gamma_v u, \delta_v \rangle_{\ell_2(V, \mu)} \), where \( \Lambda_0 \) is the Dirichlet-to-Neumann operator of the boundary pair \((\Gamma, \ell_2(V, \mu))\) (at the spectral value 0) and where \( \delta_v \) is the Kronecker delta, see [Pos16]. In particular, conditions (2.16b)–(2.16d) with \( \delta_0(v) = 0 \) mean that \( c^2 \tau \Lambda_0 = \Delta_{(G, \mu, \gamma)} \), i.e., that \( c^2 \tau \Lambda_0 \) equals the discrete Laplacian \( \Delta_{(G, \mu, \gamma)} \) given by (2.6). Note that this is the operator associated with the form \( \mathcal{E} \) on \((G, \mu, \gamma)\). In particular, the discrete energy form \( \mathcal{E} \) equals the (rescaled) Dirichlet-to-Neumann form of the boundary pair. The general case \( \delta_0(v) > 0 \) is just a small deviation from this situation.

**Proof of Theorem 2.10.** We have already shown in Proposition 2.8 that \( J \) is max\{\( \delta_a, \delta_b \}\)-quasi-unitary with adjoint \( J^* \), explaining the first three members in the definition of \( \delta \).

For the remaining conditions of quasi-unitary equivalence, we have to define the identification operators on the level of the energy forms. Namely, we set

\[
J_1 : \mathcal{H} = \ell_2(V, \mu) \longrightarrow \tilde{\mathcal{H}}^1 = \text{dom} \mathcal{E}_X, \quad J_1 f = \tilde{f},
\]

and this is well-defined as \( \psi_v \in \text{dom} \tilde{\mathcal{E}} \) and \( \tilde{J} f \in \text{dom} \tilde{\mathcal{E}} \) for any \( f \in \mathcal{H} = \ell_2(V, \mu) \). For the opposite direction, we define

\[
J_1^* : \tilde{\mathcal{H}}^1 \longrightarrow \mathcal{H} = \ell_2(V, \mu) \quad \text{by} \quad (J_1^* u) (v) := \frac{1}{c} \langle \Gamma_v u, v \rangle, \quad v \in V.
\]

The first condition of (A.3c) is trivially fulfilled, and for the second, we estimate

\[
\|J_1 - J_1^* u\|^2_{\mathcal{E}_X(V, \mu)} = \sum_{v \in V} \nu_0 (v) \left\| \int_X u \, d\nu_v - \Gamma_v u \right\|^2 \leq \sup_{v \in V} \delta_0 (v)^2 \sum_{v \in V} \mathcal{E}_{X_v} (u) \leq \frac{2}{\tau} \sup_{v \in V} \delta_0 (v)^2 \tilde{\mathcal{E}} (u).
\]

using (2.7d) for the first equality, using (2.16a) and (2.7b). We now check estimate (A.3d): we have

\[
\mathcal{E}(f, J_1^* u) - \tilde{\mathcal{E}} (J f, u) = \frac{1}{c} \sum_{v \in E} (df)_e (d\Gamma \nu)_e \gamma_e - c \sum_{v \in V} f (v) \tilde{\mathcal{E}} (\psi_v, u)
\]

\[
= \frac{1}{c} \sum_{v \in E} \gamma_e (df)_e \left( (\Gamma_\partial_\pm \nu - \Gamma_{\partial_\pm e, e} \nu) - (\Gamma_{\partial_\pm e} \nu - \Gamma_{\partial_\pm e, e} \nu) \right)
\]

using (2.16b), where \( \Gamma_v u = (\Gamma_v u)_e \nu_v \) and \( (dh)_e = h (\partial_\pm e) - h (\partial_\pm e) \), and where we use the reordering \( \sum_{v \in V} \sum_{e \in E_v} = \sum_{e \in E} \sum_{v \in \partial_\pm e} \). In particular,

\[
\left| \mathcal{E}(f, J_1^* u) - \tilde{\mathcal{E}} (J f, u) \right|^2 \leq \frac{2}{c^2} \mathcal{E}(f) \sum_{e \in E} \gamma_e \sum_{v \in \partial_\pm e} | \Gamma_v u - \Gamma_{v, e} u |^2
\]

\[
= \frac{2}{c^2} \mathcal{E}(f) \sum_{v \in V} \sum_{e \in E_v} \gamma_e \left| \Gamma_v u - \Gamma_{v, e} u \right|^2
\]

\[
\leq 2 \mathcal{E}(f) \sum_{v \in V} \delta_0 (v)^2 \mathcal{E}_{X_v} (u) \leq \frac{4}{\tau} \sup_{v \in V} \delta_0 (v)^2 \mathcal{E}(f) \tilde{\mathcal{E}} (u)
\]

using (2.16b) for the second last estimate and again (2.7b) for the last one. \( \square \)

### 3. Convergence of energy forms on metric and discrete graphs

In this section, the metric measure space \( X \) is a metric graph, called \( M \) here.
3.1. Metric graphs

We briefly introduce the notion of a metric graph here. More details on metric graphs can be found in [Pos12] or [BK13]. For a metric graph, we need a discrete graph \((V, E, \partial)\) together with a function \(\ell: \overline{E} \to (0, \infty)\). We will interpret \(\ell_e > 0\) as the length of an edge \(e\). A metric graph \(M\) is now given by \((G, \ell)\) and can be defined as the topological space
\[
M := \bigcup_{e \in E} M_e/\omega,
\]
where \(M_e := [0, \ell_e]\), and where \(\omega: \bigcup_e \{0, \ell_e\} \to V\) identifies \(0 \in M_e\) with the initial vertex \(\partial_- e \in V\) and \(\ell_e \in M_e\) with the terminal vertex \(\partial_+ e \in V\). The topological space \(M\) is a metric space by choosing as distance of two points \(x, y \in M\) the length of the (not necessarily unique) shortest path \(\gamma_{x,y}: [0, a] \to M\) in \(M\) realising the distance \(d(x, y) = a\). Moreover, we have a canonical measure \(\nu\) on \(M\), given by the sum of the Lebesgue measures \(dx_e\) on each interval \(M_e\).

The Hilbert space is here
\[
\mathcal{H} = L_2(M, \nu), \quad \|u\|^2_{L_2(M, \nu)} = \sum_{e \in E} \int_0^{\ell_e} |u_e(x)|^2 \, dx,
\]
where we consider \(u = (u_e)_{e \in E}\) with \(u_e: [0, \ell_e] \to \mathbb{C}\). The energy form on \(M\) is
\[
\mathcal{E}_M(u) = \|u\|^2_{L_2(M, \nu)} = \sum_{e \in E} \int_0^{\ell_e} |u'_e(x)|^2 \, dx
\]
with
\[
\mathcal{H}^1 = H^1(M) = C(M) \cap \bigoplus_{e \in E} H^1([0, \ell_e]),
\]
i.e., \(u \in H^1(M)\) if and only if \(u_e \in H^1([0, \ell_e])\); \(\sum_e \|u_e\|^2_{H^1([0, \ell_e])} < \infty\) (the latter condition is only necessary if \(E\) is not finite) and
\[
u_e(v) := \begin{cases} u_e(0), & v = \partial_- e, \\ u_e(\ell_e), & v = \partial_+ e \end{cases}
\]
for all \(v \in V\). Note that the corresponding operator \(\Delta_M\) acts as \((\Delta_M f)_e = -f''_e\) on each edge with \(f \in \bigoplus_e H^2(M_e)\), where \(f\) is continuous at the vertices and where \(\sum_{e \in E_v} f'_e(v) = 0\). Here, \(f'_e(v)\) denotes the inwards derivative of \(f\) at \(v\) along \(e \in E_v\). This operator is called the standard (or by many authors also) Kirchhoff Laplacian.

3.2. Quasi-unitary equivalence of metric and discrete graphs

Let us first specify the partition of unity related to the graph structure (see Definition 2.3). Let \(\psi_v: M \to [0, 1]\) be the function, affine linear on each edge \(M_e\), such that \(\psi_v(v) = 1\) and \(\psi_v(v') = 0\) if \(v' \in V \setminus \{v\}\). Then the vertex neighbourhood (i.e., the support of \(\psi_v\)) is
\[
M_v := \text{supp } \psi_v = \bigcup_{e \in E_v} M_e/\omega,
\]
i.e., the star graph around \(v\) consisting of all edges adjacent with \(v\). Note that with this definition, \(M_v \cap M_{v'} \neq \emptyset\) if and only if \(v \sim v'\) and that the edge neighbourhood \(M_e = M_{\partial_- e} \cap M_{\partial_+ e}\) is the edge (as interval) \(M_e\) as already defined above. Moreover, the vertex core is \(M_v = \psi_v^{-1}\{1\} = \{v\}\), i.e., a point in \(M\).

The vertex weights here are given by
\[
\nu(v) = \int_M \psi_v \, dx = \sum_{e \in E_v} \frac{1}{2} \ell_e \int_0^{\ell_e} x \, dx = \frac{1}{2} \sum_{e \in E_v} \ell_e = \frac{1}{2} \nu(M_v).
\]

The following definition assures that the weights \(\mu\) and \(\nu\) are compatible:
3.1. Definition. We say that a metric graph \( M \) and a weighted discrete graph \((G, \mu, \gamma)\) are compatible, if the underlying discrete graphs are the same and if there exists \( c > 0 \) and \( \tau > 0 \) such that the length function \( \ell \) and the weights \( \mu \) and \( \gamma \) fulfil
\[
\frac{1}{2\mu(v)} \sum_{e \in E_v} \ell_e = \frac{1}{c^2} \quad \text{(3.3a)}
\]
for all \( v \in V \) and
\[
\ell_e \gamma_e = c^2 \tau \quad \text{(3.3b)}
\]
for all \( e \in E \).

3.2. Remark. Let \( M \) and \((G, \mu, \gamma)\) be compatible, then the following holds:

(i) The measures \( \mu \) and \( \nu \) are compatible in the sense of Definition 2.3. Moreover, \( \nu_0(v) = \nu(v) \) and \( \alpha_\infty = 0 \), see (2.7d').

(ii) Condition (3.3b) is dictated by (3.7). Moreover, a lower bound \( \gamma_0 := \inf_{e \in E} \gamma_e > 0 \) implies an upper bound
\[
\ell_\infty := \sup_{e \in E} \ell_e = \frac{c^2 \tau}{\gamma_0} < \infty. \quad \text{(3.4)}
\]

(iii) There is still some freedom in the choice of the parameters \( \ell_e, \tau \) and \( c \) (and the parameters \( \mu(v) \) and \( \gamma_e \) from the weighed graphs), as they have to fulfil only two equations (3.3a)–(3.3b). Given \( \ell_e, \mu(v) \) and \( \gamma_e \), we conclude
\[
c = \left( \frac{1}{2\mu(v)} \sum_{e \in E_v} \ell_e \right)^{-1/2} \quad \text{and} \quad \tau = \frac{1}{2\mu(v)} \sum_{e \in E_v} \ell_e^2 \gamma_e. \quad \text{(3.5)}
\]

(iv) Note that (3.3a) and (3.3b) give a restriction on the vertex and edge weights of the weighted graph, namely that
\[
\frac{1}{2\mu(v)} \sum_{e \in E_v} \frac{1}{\gamma_e} = \frac{1}{c^4 \tau} \quad \text{(3.6)}
\]
is independent of \( v \in V \). This condition looks a bit surprising — a natural condition for a weighted graph would be that the relative weight is independent of \( v \in V \) (see (2.1)); in this case, the discrete Laplacian and a corresponding weighted adjacency operator are related by an affine linear transformation.

For Sobolev spaces of order 1 on one-dimensional spaces, the evaluation in a point is well-defined, hence \( u \mapsto u(v) \) makes sense for \( u \in H^1(M) \) (see e.g. [Pos12, Ch. 2]). We therefore define
\[
\Gamma_v u := u(v).
\]

Let us now check (2.16a), (2.16b)–(2.16b'):

3.3. Lemma. Let \( u \in H^1(M) \), then (2.16a) holds, i.e.,
\[
\nu(v) \left| \int_{M_e} u \, d\nu_v - \Gamma_v u \right|^2 \leq \delta_v(v)^2 \mathcal{E}_{M_e}(u) \quad \text{with} \quad \delta_v(v)^2 := \frac{\ell_v^2}{2}.
\]

If we also have (3.3b) then (2.16b) and (2.16b') hold with \( \Gamma_v u = \Gamma_v u \) and \( \delta_u(v) = 0 \).

Proof. Let \( v \in V \) and \( e \in E_v \). Assume that \( v \) is the terminal vertex for all \( e \in E_v \), i.e., \( v \) corresponds to \( \ell_e \in M_e \) for all \( e \in E_v \). Then \( d\nu_v(x) = (x/\ell_e) \, dx \) on \( M_e \). The proof of the first assertion is an application of the fundamental theorem of calculus, namely we have
\[
u_v(x) - u(v) = \int_{\ell_e}^{x} u'_e'(t) \, dt.
\]
After integration with respect to \( x \) and the probability measure
\[ \int_{M_v} u \, d\nu_v - u(v) = \int_{M_v} (u - u(v)) \, d\nu_v = \frac{1}{\nu(v)} \sum_{e \in E_v} \int_0^{\ell_e} (u_e(x) - u(v)) \, d\nu_v(x) = \frac{1}{\nu(v)} \sum_{e \in E_v} \int_0^{\ell_e} u_e'(t) \, dt \, d\nu_v(x), \]

hence
\[
\left| \int_{M_v} u \, d\nu_v - u(v) \right|^2 \leq \frac{1}{\nu(v)^2} \left( \sum_{e \in E_v} \int_0^{\ell_e} |u_e'(t)| \, dt \, d\nu_v(x) \right)^2 = \frac{1}{\nu(v)^2} \left( \sum_{e \in E_v} \frac{\ell_e}{2} |u_e'(t)| \, dt \right)^2 \leq \frac{\text{CS}}{4\nu(v)^2} \left( \sum_{e \in E_v} \ell_e^3 \right) \int_{M_v} |u|^2 \, d\nu \leq \frac{\max_{e \in E_v} \ell_e^2}{2\nu(v)} \mathcal{E}_{M_v}(u),
\]

using \( d\nu_v(x) = (x/\ell_e) \, dx \) for the equality and (3.2) for the last inequality.

For the validity of the last assertion we calculate
\[
e^2 \tau \mathcal{E}_M(u, \psi_v) = \sum_{e \in E_v} \frac{\ell_e^2}{\ell_e} \int_0^{\ell_e} u_e'(x) \, dx = \sum_{e \in E_v} \gamma_e (\Gamma_v u - \Gamma_v \psi_v)
\]

using \( (\psi_e')_e = 1/\ell_e \) for the first and (3.3b) for the second equality. Note that \( v \) corresponds to \( \ell_e \) and \( v_e \) to 0. In particular, we can choose \( \Gamma_v u = \Gamma_v \psi_v \), hence \( \delta_3(v) = 0. \)

We need a lower bound on the second eigenvalue:

**3.4. Lemma.** We have
\[
\frac{2}{\ell_\infty^2} \leq \lambda_2(M_v, \psi_v).
\]

**Proof.** Eigenfunctions of the weighted problem on an edge are solutions of the ODE
\[-u''(x) = \lambda x u(x), \quad x \in [0, \ell_e], \quad 0 \text{ corresponds to the vertex of degree 1 on the star graph. Such eigenfunctions are linear combinations of (rescaled) Airy functions, and hence continuous also at } x = 0. \]

In particular, the solutions are also in the unweighted Hilbert space \( L^2(V) \), and we obtain the estimate \( \lambda_2(V, \psi_v) \geq \lambda_2(X_v) \) from Lemma 2.6.

Consider now the Rayleigh quotient for the unweighted problem, it is given by
\[
\frac{\mathcal{E}_{M_v}(u)}{\|u\|_{L^2(X_v)}^2} = \frac{\sum_{e \in E_v} \|u_e'(x)\|^2 \, dx}{\sum_{e \in E_v} \int_0^{\ell_e} |u_e(x)|^2 \, dx} = \frac{\sum_{e \in E_v} \frac{1}{\ell_e} \int_0^{\ell_e} \tilde{u}_e(t)^2 \, dt}{\sum_{e \in E_v} \ell_e \int_0^{\ell_e} |\tilde{u}_e(t)|^2 \, dt},
\]

where \( x = t \ell_e \) and \( \tilde{u}(t) = u(t \ell_e) \); and the latter expression is monotonously decreasing in \( \ell_e \). In particular, as \( \ell_e \) is bounded from above by \( \ell_\infty := \max_{e \in E_v} \ell_e \), we have \( \lambda_2 \geq \lambda_{2,0}/\ell_\infty^2 \), where \( \lambda_{2,0} = \pi^2/4 \geq 2 \) is the second eigenvalue of a star graph with all edges having length 1.

We are now prepared to prove the main result of this subsection.

**3.5. Theorem.** Assume that \((G, \mu, \gamma)\) is a weighted graph with
\[
\mu_\infty := \sup_{e \in V} \mu(v) < \infty, \quad \text{and} \quad 0 < \gamma_0 := \inf_{e \in E} \gamma_e \leq \gamma_\infty := \sup_{e \in E} \gamma_e < \infty.
\]

Assume in addition that \( M \) is a metric graph compatible with \((G, \mu, \gamma)\), i.e., its edge lengths \( \ell_e \) fulfil (3.3a)–(3.3b), namely
\[
\frac{1}{2\mu(v)} \sum_{e \in E_v} \ell_e = \frac{1}{c^2} \quad \text{and} \quad \ell_e = \frac{c^2 \tau}{\gamma_e}
\]
for some $c > 0$ and $\tau > 0$, independently of $v \in V$ and $e \in E$. Then the graph energy form $E$ associated with the weighted discrete graph $(G, \mu, \gamma)$ and the rescaled metric graph energy form $\tilde{E} = \tau E_M$ are $\delta$-quasi-unitarily equivalent with

$$\delta^2 := 2 \frac{\gamma_\infty}{\gamma_0} \cdot \frac{\mu_\infty}{\gamma_0}.$$

Proof. Note first that $(G, \mu, \gamma)$ is uniformly embedded into $(M, \nu, E_M)$ (see Definition 2.3) by the assumptions of the theorem: in particular, $\lambda_2(M, \nu) \geq 2/\ell^2_\infty$ by Lemma 3.4. The remaining assumptions of Theorem 2.10 are fulfilled by Lemma 3.3. Let us now check the individual terms in the error $\delta$ in (2.17). As the weights are compatible, we have $\alpha_\infty = 0$. Moreover, the error term $2\mu_\infty/\gamma_0$ is already covered as $\gamma_\infty/\gamma_0 \geq 1$. For the third term in (2.17) we have

$$\frac{2}{\tau \lambda_2} \leq \frac{\ell^2_\infty}{\tau} = \frac{c^2_0}{\gamma_0^2} \leq \frac{2\mu_\infty}{\gamma_0^2} = \delta^2$$

using again Lemma 3.4 for the first, (3.4) for the second and (3.6) for the third step. The fourth error term in (2.17) (the one with $\delta_\zeta(v)$) is treated in the same way as the third one as $2\delta_\zeta(v)^2/\tau \leq \ell^2_\infty/\tau$ by Lemma 3.3. The last error term is $0$ by Lemma 3.3. □

From Definition 2.1 (iii) and (2.4) we immediately conclude:

3.6. Corollary. Assume that $(G, \mu, \gamma)$ is a $(d_\infty, \overline{\mu}, \overline{\gamma})$-uniform weighted graph with corresponding compatible metric graph lengths then $E$ and $\tilde{E}$ are $\delta$-quasi-unitarily equivalent with

$$\delta^2 = 2 \overline{\gamma}^2 d_\infty \overline{\mu} \cdot \frac{1}{\varrho_0},$$

where $\varrho_0 = \inf_{v \in V} \varrho(v) = \inf_{v \in V} \frac{1}{\mu(v)} \sum_{e \in E_v} \gamma_e$ is a lower bound on the relative weight.

**Metric graphs approximated by discrete weighted subdivision graphs.** We have another application of our result: A subdivision graph $SG$ of a discrete graph $G = (V, E, \partial)$ is a discrete graph with additional vertices on the edges. We denote the graph objects associated with $SG$ by $V(SG)$, $E(SG)$ etc.

If $M$ is a metric graph with underlying discrete graph $G$ and length function $\ell: E \rightarrow (0, \infty)$, then we call $SM$ a metric subdivision graph if the lengths $\ell_{e_1}, \ldots, \ell_{e_r}$ of the additional edges $e_1, \ldots, e_r$ on the original edge $e$ add up to the original length $\ell_e$, i.e.,

$$\sum_{j=1}^r \ell_{e_j} = \ell_e.$$

Note that additional vertices of degree 2 on an edge lead to unitarily equivalent metric graph energy forms and Laplacians with natural unitary map. In particular, the energy form and the Laplacian on a metric subdivision graph are unitarily equivalent with the energy form and the Laplacian on the original metric graph. We define

$$\ell_0(SM) := \inf_{e \in E(SG)} \ell_e \quad \text{and} \quad \ell_\infty(SM) := \sup_{e \in E(SG)} \ell_e,$$

the minimal and the maximal mesh width of the subdivision graph $SM$, respectively. We have now the following result:

3.7. Corollary. Assume that $M$ is a metric graph with edge length fulfilling $0 < \ell_0 \leq \ell_e \leq \ell_\infty < \infty$ for all $e \in E$ and uniformly bounded degree, i.e., $\deg v \leq d_\infty$ for all $v \in V$. Then there is a sequence of metric subdivision graphs $SM_m$ and compatible weighted discrete subdivision graphs $(SG_m, \mu_m, \gamma_m)$ such that the associated discrete energy form $E_m$ is $\delta_m$-unitarily equivalent with the energy form $E_M$ of the original metric graph, where

$$\delta_m^2 \leq d_\infty \ell_\infty(SM_m)^3/\ell_0(SM_m).$$
The decomposition into vertex and edge neighbourhoods is, of course, not unique. Let \( SM_m \) be a sequence of metric subdivision graphs with edge length function denoted by \( \ell_m: E(SM_m) \to (0, \infty) \) with \( \ell_0(SM_m) \to 0 \). Define the weights \( \mu_m \) and \( \gamma_m \) of the discrete underlying subdivision graph \( SG_m \) by

\[
\mu_m(v) := \frac{1}{2} \sum_{e \in E_v(SG_m)} \ell_{m,e} \quad \text{and} \quad \gamma_{m,e} := \frac{1}{\ell_{m,e}}.
\]

In particular, we then have \( c = 1 \) and \( \tau = 1 \) and \( SM_m \) and \( (SG_m, \mu_m, \gamma_m) \) are compatible. From Theorem 3.5 we conclude that the energy forms associated with the metric subdivision graph \( E_{SM_m} \) and the weighted discrete subdivision graph \( E_{SG_m} \) are \( \delta_m \)-quasi-unitarily equivalent with

\[
\delta^2_m = \frac{2\ell_\infty(SM_m)}{\ell_0(SM_m)} \sup_{v \in V} \sum_{e \in E_v(SG_m)} \frac{\ell_{m,e}/2}{\ell_\infty(SM_m)^{-1}} \leq d_\infty \ell_\infty(SM_m)^3 / \ell_0(SM_m).
\]

Note that the maximal degree of a subdivision graph is the same as for the original graph and that \( E_{SM_m} \) and \( E_M \) are unitarily equivalent. \( \square \)

4. Convergence of energy forms on graph-like manifolds and discrete graphs

4.1. Graph-like manifolds

We introduce here the notion of a graph-like manifold. More details can be found in \cite{Pos12}. Let \( X \) be a Riemannian manifold of dimension \( d \geq 2 \). The standard example of a graph-like manifold with boundary is the thickened metric graph as in Example 4.2 or Figure 4.1.

4.1. Definition. We say that \( X \) is a graph-like manifold with associated discrete graph \((V, E, \partial)\) and edge length function \( \ell: E \to (0, \infty) \) if there are compact subsets \( \bar{X}_v \) and \( X_e \) of \( X \) with the following properties:

(i) \( X = \bigcup_{v \in V} \bar{X}_v \cup \bigcup_{e \in E} X_e \) and \( \bar{X}_v \cap X_e \neq \emptyset \) if and only if \( e \in E_v \); all other sets \( \bar{X}_v \) and \( X_e \) are pairwise disjoint;

(ii) \( X_v \) is isometric with \( M_v \times Y_v \), where \( M_v = [0, \ell_v] \) for some \( \ell_v > 0 \) and some \( (d - 1) \)-dimensional Riemannian manifold \( Y_v \);

(iii) there exists \( \kappa \in (0, 1] \) such that \( \partial_e \bar{X}_v := \bar{X}_v \cap X_e \) (isometric with \( Y_v \)) has a \( \kappa \ell_v \)-collar neighbourhood \( X_{v,e} \) inside \( \bar{X}_v \), i.e., \( X_{v,e} \) is isometric with \([0, \kappa \ell_v] \times Y_v \); we assume that \( (X_{v,e})_{e \in E_v} \) are pairwise disjoint.

We call \( \bar{X}_v \) the core vertex neighbourhood of \( v \in V \) and \( X_e \) the edge neighbourhood of \( e \in E \). We call \( Y_v \) the transversal manifold of \( e \). Moreover, we call \( X_v := \bar{X}_v \cup \bigcup_{e \in E_v} X_e \) the (enlarged) vertex neighbourhood of \( v \).

Remark.

(i) A graph-like manifold may have boundary or not; the boundary may even be Lipschitz (see \cite[App. A]{MT99}) for a precise definition).

(ii) A graph-like manifold \( X \) can be constructed from a metric graph \( M \) with the same edge length function \( \ell \). In this case, \( X \) is defined as an abstract space. For the case when the metric graph and the graph-like manifold are embedded in \( \mathbb{R}^d \), see Example 4.2.

(iii) The decomposition into vertex and edge neighbourhoods is, of course, not unique.

(iv) For a compact graph-like manifold, condition Definition 4.1 (iii) follows from (iii):

\[
\text{Take away a little piece of } X_e \text{ and add it to } X_v, \text{ this means that } \ell_e \text{ becomes a bit smaller.}
\]

An important example is a neighbourhood of a metric graph embedded in \( \mathbb{R}^d \):

4.2. Example (Thickened metric graph as graph-like manifold). Let \( M \) be a compact metric graph embedded in \( \mathbb{R}^d \) in such a way that the edges \( M_e \) are line segments. Then the edge lengths of an edge \( e \) is \( |\partial_+ e - \partial_- e| \) (where \( V \) is considered as a subset of \( \mathbb{R}^d \)). Let \( X \) be
the closed $\varepsilon$-neighbourhood of $M$ in $\mathbb{R}^d$. If $\varepsilon > 0$ is small enough, it can be seen that $M$ is a graph-like manifold. In particular, one can choose $\ell_e = (1 - 2\varepsilon)|\partial_+ e - \partial_- e|$ as edge length function.

Metric graphs with edges embedded as curved segments can also be treated as a perturbation of abstract metric graphs (not necessarily embedded) with straight edges, see [Pos12, Sec. 5.4 and 6.7]).

Let $\nu$ denote the Riemannian measure on $X$. The associated Hilbert space is

$$\mathcal{H} = L^2(X, \nu), \quad \|u\|_{L^2(X, \nu)}^2 = \int_X |u(x)|^2 \, d\nu(x).$$

The energy form on $X$ is

$$\mathcal{E}_X(u) = \int_X |\nabla u(x)|^2_x \, d\nu(x)$$

(4.1)

where $\nabla$ is the gradient and $|\cdot|_x$ is the norm induced by the Riemannian metric tensor at $x \in X$ and where $\mathcal{H}^1 = H^1(X)$ is the closure of Lipschitz continuous functions with compact support in $X$ with respect to the norm given by $\|u\|_{H^1(X)}^2 = \|u\|_{L^2(X)}^2 + \mathcal{E}_X(u)$.

4.2. Quasi-unitary equivalence of discrete graphs and graph-like manifolds

The choice of the partition of unity $\Psi = (\psi_v)_{v \in V}$ is almost obvious now: Let

$$\psi_v(x) = 1 \quad \text{if } x \in \bar{X}_v \quad \text{and} \quad \psi_v(x) = \frac{1}{\ell_e} t, \quad \text{where } x = (t, y), \; t \in M_e, \; y \in Y_e$$

are coordinates on $X_e$. We assume here that $v = \partial_+ e$ is the terminal vertex, i.e., $v$ corresponds to $\ell_e \in M_e = [0, \ell_e]$. Let $\psi_v(x) = 0$ for any other point $x$ not in $\bar{X}_v$ and $X_e$, $e \in E_v$.

In particular, $\psi_v$ is Lipschitz continuous on $X$. As $X_e$ is a product, $\psi_v$ is harmonic on $X_e$ (affine linear in the longitudinal direction times a constant function in the transversal direction). Now, it is obvious, that $X_e$ is the edge neighbourhood of $e \in E$ also in the sense of Definition 2.3, and $\bar{X}_v$ is the (core) vertex neighbourhood of $v \in V$ in $X$. Moreover, the (enlarged) vertex neighbourhood $X_v$ is

$$X_v = \bar{X}_v \cup \bigcup_{e \in E_v} X_e.$$
The vertex measure here is
\[ \nu(v) = \int_X \psi_v \, dx = \sum_{e \in E_v} \text{vol} \, Y_e \int_0^{\ell_e} \frac{1}{\ell_e} \, dt + \text{vol} \, \bar{X}_v = \frac{1}{2} \sum_{e \in E_v} \text{vol} \, X_e + \text{vol} \, \bar{X}_v \] (4.2)
and hence \( \text{vol} \, X_v/2 \leq \nu(v) \leq \text{vol} \, X_v \). Here, \( \text{vol} = \nu \) is the \( d \)-dimensional volume on \( X \); with the exception that \( \text{vol} \, Y_e \) denotes the \((d-1)\)-dimensional volume of \( Y_e \). For the decomposition of \( \nu(v) \) in (2.7d') we choose
\[ \nu_0(v) := \frac{1}{2} \sum_{e \in E_v} \text{vol} \, X_e = \frac{1}{2} \sum_{e \in E_v} \ell_e (\text{vol} \, Y_e) \quad \text{and} \quad \bar{\nu}(v) := \text{vol} \, \bar{X}_v \] (4.3)
for all \( v \in V \). The following definition assures that a graph-like manifold is well-adopted to a given weighted graph:

4.3. Definition.
(i) We say that a graph-like manifold \( X \) and a weighted discrete graph \((G, \mu, \gamma)\) are compatible, if the underlying discrete graphs are the same and if there exists \( c > 0 \) and \( \tau > 0 \) such that the edge length function \( \ell \), the weights \( \mu \) and \( \gamma \) and the transversal volumes \( \text{vol} \, Y_e \) fulfil
\[ \frac{1}{2} \mu(v) \sum_{e \in E_v} \ell_e (\text{vol} \, Y_e) = \frac{1}{c^2} \] (4.4a)
for all \( v \in V \) and
\[ \frac{\gamma_e \ell_e}{\text{vol} \, Y_e} = c^2 \tau \] (4.4b)
for all \( e \in E \).
(ii) We say that \( X \) has uniformly small (core) vertex neighbourhoods, if
\[ \alpha_\infty = \sup_{v \in V} \alpha(v) \leq \frac{1}{2} \quad \text{and} \quad \alpha_0 = \inf_{v \in V} \alpha(v) > 0, \] (4.4c)
where
\[ \alpha(v) = \frac{2 \text{vol} \, \bar{X}_v}{\sum_{e \in E_v} \text{vol} \, X_e}, \] (4.4d)
and if
\[ \lambda_2 := \inf_{v \in V} \lambda_2(X_v) > 0. \] (4.4e)
(iii) We say that \( X \) has uniform transversal volume if
\[ 0 < \text{vol}_0 := \inf_{e \in E} \text{vol} \, Y_e \leq \text{vol}_\infty := \sup_{e \in E} \text{vol} \, Y_e < \infty. \] (4.4f)
(iv) We say that a graph-like manifold \( X \) and a weighted discrete graph \((G, \mu, \gamma)\) are uniformly compatible, if they are compatible and if \( X \) has uniformly small vertex neighbourhoods and uniform transversal volume.

4.4. Remark.
(i) Condition (4.4a) is the compatibility of the vertex weights \( \nu_0 \) and \( \mu \).
(ii) Note that if we consider an embedded metric graph with length function \( \ell \) together with a small \( \varepsilon \)-neighbourhood as graph-like manifold \( X \) as in Example 4.2, then the edge length function of \( X \) is \((1 - 2\varepsilon)\ell \); the common factor \((1 - 2\varepsilon)\) does not destroy the compatibility in the sense of Definition 4.3 (i), it just changes the factors \( c \) and \( \tau \) slightly.
(iii) Equation (4.8) forces that \( \gamma_e \) is given by (4.4b).
(iv) As for metric graphs, there is still some freedom in the choice of the parameters \( \ell_e, \text{vol} \, Y_e, \tau \) and \( c \), as they have to fulfil only two equations in (4.4a) and (4.4b). Nevertheless, from (4.4a) and (4.4b) we conclude
\[ c = \left( \frac{1}{2} \mu(v) \sum_{e \in E_v} \ell_e \text{vol} \, Y_e \right)^{-1/2} \quad \text{and} \quad \tau = \frac{1}{2} \mu(v) \sum_{e \in E_v} \ell_e^2 \gamma_e. \] (4.5)
(v) As for metric graphs, the compatibility conditions \((4.4a)\) and \((4.4b)\) give a restriction on the vertex and edge weights of the weighted graph and the transversal volume \(\text{vol} Y_e\), namely that

\[
\frac{1}{2\mu(v)} \sum_{e \in E_v} (\text{vol} Y_e)^2 = \frac{1}{c^4 \tau}
\]

is independent of \(v \in V\).

(vi) Note that \(\alpha(v) = \mathcal{P}(v)/\nu_0(v) > 0\) and \(\alpha_\infty = \sup_v \alpha(v)\) as in \([2.7d]\). The upper bound \(\alpha_\infty\) is needed for \(J'\) being close to the adjoint of \(J\), see Proposition \(2.8\) while the lower bound \(\alpha_0\) is needed in Lemma \(4.5\).

As “evaluation” \(\Gamma_v u\), we set

\[
\Gamma_v u := \int_{X_v} u \, d\nu.
\]

Note that the pointwise evaluation \(u \mapsto u(v)\) does not make sense here, as \(X\) is at least 2-dimensional and hence evaluation on points is not defined on \(H^1(X)\).

Let us first check the condition in \((2.16a)\):

4.5. **Lemma.** We have

\[
\nu_0(v) \left| \Gamma_v u - \int_X u \, d\nu \right|^2 \leq \delta_c(v)^2 \mathcal{E}_{X_v}(u)
\]

with

\[
\delta_c(v)^2 = \frac{1}{\alpha(v)} \max_{e \in E_v} \left\{ \frac{9}{2\lambda_2(X_v, \psi_e)}, 4\kappa \ell_e^2 \right\}
\]

(see \([4.3]\) for the definition of \(\nu_0(v)\) and \([4.4d]\) for the definition of \(\alpha(v)\)).

**Proof.** Summing \((3.1)\) over \(e \in E_v\), we obtain

\[
\|u\|_{L_2(X_v)}^2 \leq \max \left\{ \frac{9\kappa}{2}, 1 \right\} \|u\|_{L_2(X_v, d\nu_0)}^2 + 4\kappa \max_{e \in E_v} \ell_e^2 \mathcal{E}_{X_v}(u)
\]

(note that the contribution of \(u\) on \(X_v \setminus \bigcup_{e \in E_v} X_e, e\) also adds on the right hand side, hence the 1 on the right hand side); the maximum can be estimated by \(9/2\) as \(0 < \kappa \leq 1\). Now we plug in \(u - \int_{X_v} u \, d\nu\) instead of \(u\) in the last inequality and obtain

\[
\left| \Gamma_v u - \int_{X_v} u \, d\nu \right|^2 \leq \max_{e \in E_v} \left\{ \frac{9}{2\lambda_2(X_v, \psi_e)}, 4\kappa \ell_e^2 \right\} \mathcal{E}_{X_v}(u)
\]

using Lemma \(2.5\) Now we have

\[
\left| \Gamma_v u - \int_{X_v} u \, d\nu \right|^2 = \int_{X_v} (u - \int_{X_v} u \, d\nu) \, d\nu \leq \frac{1}{\text{vol} X_v} \|u - \int_{X_v} u \, d\nu\|_{L_2(X_v)}^2
\]

hence the result follows with \(\delta_c(v)^2\) as above using \(\alpha(v) = \text{vol} \bar{X}_v / \nu_0(v)\).

For \((2.16b)\)–\((2.16b')\) we set

\[
\Gamma_{v,e} u := \int_{\partial_e X_v} u,
\]

where \(\partial_e X_v \cong Y_e\) is the boundary component of \(X_v\) at the edge neighbourhood \(X_e\).

4.6. **Lemma.** Assume that \((4.4b)\) holds then \((2.16b)\)–\((2.16b')\) are fulfilled with

\[
\delta_d(v)^2 = \tau \max_{e \in E_v} \left\{ \kappa + \frac{2}{\kappa \ell_e^2 \lambda_2(X_v)} \right\}.
\]

**Proof.** The proof of \((2.16b)\) is again an application of the fundamental theorem of calculus: Assume that the vertex \(v\) corresponds to the endpoint \(t = \ell_e\) of each adjacent \(X_e, e \in E_v\). Then \(\psi_{v,e}(t, y) = t / \ell_e\) for \(x = (t, y) \in X_v = [0, \ell_e] \times Y_e\) with derivative \(1/\ell_e\) and we have

\[
c^2 \tau \mathcal{E}_X(u, \psi_v) = \sum_{e \in E_v} \frac{c^2 \tau}{\ell_e} \int_0^{\ell_e} \int_{Y_e} u'_e(t, y) \, dt \, dy = \sum_{e \in E_v} \gamma_e (\Gamma_{v,e} u - \Gamma_{v,e} u)
\]

(4.8)
using (4.4b) for the last equality. For (2.16b) we apply (B.2) and obtain
\[
\sum_{e \in E_v} \gamma_e |\Gamma_{v,e} u - \Gamma_v u|^2 \leq \max_{e \in E_v} \frac{\gamma_e}{\vol Y_e} \left( \kappa \ell_e + \frac{2}{\kappa \ell_e \lambda_2(X_v)} \right) \|u\|_{L^2(\hat{X}_v)}^2
\]
using (4.4b) for the last equality. For (2.16b) we apply (B.2) and obtain
\[
\sum_{e \in E_v} \gamma_e |\Gamma_{v,e} u - \Gamma_v u|^2 \leq \max_{e \in E_v} \frac{\gamma_e}{\vol Y_e} \left( \kappa \ell_e + \frac{2}{\kappa \ell_e \lambda_2(X_v)} \right) \|u\|_{L^2(\hat{X}_v)}^2
\]
using again (4.4b) for the last equality.

4.7. Lemma. We have
\[
\lambda_2(X_v) \leq \lambda_2(X_v, \psi_v)
\]
Proof. The proof is almost the same as the proof of Lemma 3.4. Note that the function \(\psi_v\) is harmonic on \(X_v\), namely affine linear and constant in transversal direction \(Y_e\), hence the second eigenfunction is again an Airy function on \(X_v\) (in longitudinal direction), hence continuous and also in the unweighted Hilbert space \(L^2(X_v)\); the result then follows from Lemma 2.6.

We will assume in this section that we have a lower bound on the unweighted eigenvalue of the form
\[
\frac{\lambda_{2,0}}{\ell_\infty^2} \leq \lambda_2(X_v)
\]
for some constant \(\lambda_{2,0} \in (0, 2]\) independent of \(v \in V\), where \(\ell_\infty := \sup_e \ell_e\). Later, in our application with shrinking graph-like manifolds in Corollary 4.9 or with graph-like manifolds approximating fractals in Subsection 5.3, we will check that we can even choose \(\lambda_{2,0} = 1\) (see Proposition B.3).

We are now prepared to prove the main result of this subsection:

4.8. Theorem. Assume that \((G, \mu, \gamma)\) is a weighted graph with
\[
\mu_\infty := \sup_{v \in V} \mu(v) < \infty, \quad \text{and} \quad 0 < \gamma_0 := \inf_{e \in E} \gamma_e \leq \gamma_\infty := \sup_{e \in E} \gamma_e < \infty.
\]
Assume in addition that \(X\) is a graph-like manifold uniformly compatible with \((G, \mu, \gamma)\), i.e., the weights \(\mu(v)\), \(\gamma_v\), the edge lengths \(\ell_v\), the transversal volumes \(\vol Y_e\) and the core vertex neighbourhoods \(\hat{X}_v\) fulfil (4.4a)–(4.4c) and (4.4e)–(4.4f). Moreover, we assume that (4.9) holds. Then \(\ell_0 := \inf_{e \in E} \ell_e > 0\), and the graph energy form \(\mathcal{E}\) associated with the weighted discrete graph \((G, \mu, \gamma)\) and the rescaled graph-like manifold energy form \(\mathcal{E} = \tau \mathcal{E}_X\) are \(\delta\)-quasi-unitarily equivalent with
\[
\delta^2 := \max \left\{ 2\alpha_\infty, \frac{18}{\lambda_{2,0} \alpha_0 \gamma_0} \left( \frac{\vol_\infty}{\vol_0} \right)^2, \frac{\mu_\infty}{\gamma_0}, \frac{\kappa + \frac{2}{\kappa \ell_0^2 \lambda_2}}{\lambda_{2,0}} \right\}.
\]
Remark. The error terms have the following meaning:
(i) The first containing \(\alpha_\infty\) ensures that the core vertex manifold volume is small compared with the edge neighbourhood manifold volume.
(ii) The second term is similarly as for metric graphs, having again the factor \(\mu_\infty/\gamma_0\) inside which becomes small if the relative weight is large (similarly as in Corollary 3.6). Here we have an additional term \(1/\alpha_0\) which in general is large, making the error a bit worse than in the metric graph case.
(iii) The last term can best be understood in the setting of the next corollary or Subsection 5.3; namely, if we introduce \(\epsilon\) as length scale parameter, i.e., if \(\epsilon \hat{X}_v\) denotes the manifold \(\hat{X}_v\) with metric \(\epsilon^2 g_{\hat{X}_v}\), then \(\lambda_2(\epsilon \hat{X}_v) = \epsilon^{-2} \lambda_2(X_v)\), and \(\kappa \ell_0\) is of order \(\epsilon\), hence the entire last term is of order \(\epsilon^2/\ell_0\).

Proof of Theorem 4.8. Note first that \((G, \mu, \gamma)\) is uniformly embedded into \((X, \nu, \mathcal{E}_X)\) (see Definition 2.3) by the assumptions of the theorem: in particular, \(\lambda_2(X, \psi) \geq \lambda_2(X) \geq \lambda_{2,0}/\ell_\infty^2\) by (4.9) and Lemma 4.7. The remaining assumptions of Theorem 2.10 are fulfilled by Lemmas 4.5 and 4.6. We now estimate the terms in the definition of \(\delta\) in (2.17) in our
model here: The first term is $2\alpha_\infty$ as above; the second one is $2\mu_\infty/\gamma_0$ contained already in the second term above; for the third one we estimate
\[
\frac{2}{\tau \lambda_2} \leq \frac{2\ell^2_\infty}{\lambda_2 \gamma_0^2} \leq c^2 \tau \cdot \frac{2 \gamma_\infty}{\lambda_2} \cdot \frac{\gamma_\infty}{\lambda_2 \gamma_0^2} = \frac{4}{\lambda_2} \frac{\gamma_\infty}{\gamma_0} \left( \frac{\gamma_\infty}{\gamma_0} \right)^2 \cdot \frac{\mu_\infty}{\gamma_0},
\]
where we have shown the first inequality already above, and where we used (4.4b), (4.4f) and $\gamma_\epsilon \geq \gamma_0$ for the second estimate and (4.6) and (4.4f) for the last. Since $\alpha_0 \leq 1/2$, this term is already contained in the second term above.

The fourth term in (2.17) (the one with $2\sup_v \delta_\epsilon(v^2)/\tau$) contains one term of the form $2/(\tau \lambda_2) \cdot 9/(2\alpha_0)$ by Lemma 4.3 and (2.7c). The other term from Lemma 4.3 is of the form $8\kappa/\alpha_0 \cdot \ell^2_\infty/\tau$ and can be treated as above, hence we have
\[
\frac{8\kappa}{\alpha_0} \frac{\ell^2_\infty}{\tau} \leq \frac{16\kappa}{\alpha_0} \frac{\gamma_\infty}{\gamma_0} \left( \frac{\gamma_\infty}{\gamma_0} \right)^2 \cdot \frac{\mu_\infty}{\gamma_0}.
\]
As $16\kappa < 18/\lambda_2$, this term is already in the above list for $\delta$. Finally, the last term in Theorem 2.10 (the one with $4\sup_v \delta_\epsilon(v^2)/\tau$) gives the last term in the list for $\delta$ by Lemma 4.6.

We assume now that $Y_{\epsilon,\alpha} = \varepsilon Y_\epsilon$ with $\vol Y_\epsilon = 1$ and that $X_{\epsilon,\alpha} = \varepsilon X_\epsilon$. Here, $rX$ is the Riemannian manifold $(X, r^2 g)$ if $X$ is the Riemannian manifold $(X, g)$; the factor $r$ is hence a change of length scale. In particular, the $\varepsilon$-scaling of $X_\epsilon$ implies that the length of the collar neighbourhood of Definition 4.1 is $\kappa \ell_\epsilon$ is of order $\varepsilon$. We denote the resulting $\varepsilon$-depending graph-like manifold by $X_{\epsilon,\alpha}$.

4.9. Corollary. Assume that $(G, \mu, \gamma)$ is a $(d_\infty, \mu, \gamma)$-uniform weighted graph. Assume in addition that $X$ is a corresponding (unscaled) graph-like manifold with (unscaled) transversal manifold $Y_\epsilon$ being isometric with a fixed one $Y_0$ with volume $\vol Y_0 = 1$. Moreover, we assume that there exist $c_1 \geq 0$ and $\tau > 0$ with
\[
\frac{1}{2\mu(v)} \sum_{e \in E_v} \ell_\epsilon = \frac{1}{c_1} \quad \text{and} \quad \gamma_\epsilon \ell_\epsilon = c_2^2 \tau
\]
for all $v \in V$ and $e \in E$. Finally, we assume that there are constants $\tilde{\mu}_0, \tilde{\vol}_0$ and $\tilde{\lambda}_2$ such that
\[
0 < \tilde{\vol}_0 \leq \vol X_\epsilon \leq \vol_\infty < \infty \quad \text{and} \quad \lambda_2(X_\epsilon) \geq \tilde{\lambda}_2 > 0.
\]
Then $(G, \mu, \gamma)$ and $X_\epsilon$ are uniformly compatible (see Definition 4.3) and $\ell_0 := \inf \ell_\epsilon > 0$ and $\ell_\infty := \sup \ell_\epsilon < \infty$. Moreover, there exists a constant $C_0 > 0$ such that the graph energy form $\mathcal{E}$ associated with the weighted discrete graph $(G, \mu, \gamma)$ and the energy form on the scaled graph-like manifold $\tilde{X} = \tau \mathcal{E}X_\epsilon$ are $\delta$-quasi-unitarily equivalent with
\[
\delta^2 := \max \left\{ O\left( \frac{\varepsilon}{\ell_0} \right), \frac{\ell_0}{\varepsilon}, \frac{\mu_\infty}{\gamma_0} \right\}
\]
for all $0 < \varepsilon \leq C_0^{-2} \ell_0$, where the errors depend only on the above-mentioned constants $d_\infty$, $\mu$, $\gamma$, $\tau$, $\bar{\ell} := \ell_\infty/\ell_0$, $\bar{\vol}_0$, $\vol_\infty$ and $\tilde{\lambda}_2$.

Proof. Note first that $\ell_\epsilon = c_2^2 \tau/\gamma_\epsilon$, hence $\ell_0 = c_2^2 \tau/\gamma_\infty > 0$ and $\ell_\infty = c_2^2 \tau/\gamma_0 < \infty$. Denote objects associated with the $\varepsilon$-depending manifold $X_\epsilon$ also with a subscript $(\cdot)_\epsilon$. We apply simple scaling arguments such as $\vol(\varepsilon X_\epsilon) = \varepsilon^d \vol X_\epsilon$ etc.

Let us now check that $(G, \mu, \gamma)$ and $X_\epsilon$ are uniformly compatible: From (4.10) we conclude (4.4a)–(4.4b) with $c_\epsilon = \varepsilon^{-d-3/2} c_1$ and $\tau_\epsilon = \tau$. Moreover, $\alpha_\epsilon(v) = \varepsilon \alpha_\epsilon(v)$, hence $\alpha_\epsilon := \sup \alpha_\epsilon(v) = \varepsilon \alpha_\infty \leq 1/2$ once $\varepsilon \leq 1/(2\alpha_\infty)$; moreover, we have $\alpha_\infty \geq \tilde{\vol}_0/\ell_0 < \infty$ by (4.11); in particular, we need $\varepsilon/\ell_0 \leq 1/(2\tilde{\vol}_0)$. Similarly, $\alpha_{\epsilon,0} = \varepsilon \alpha_0$ with $\alpha_0 := \inf \alpha_\epsilon(v) \geq \vol_0/(d_\infty \ell_\infty) > 0$, hence (4.4c) is fulfilled. Conditions (4.4c)–(4.4f) follow directly from the assumptions.

We now check the individual terms in the definition of $\delta = \delta_\epsilon$ in Theorem 4.8: we have $\alpha_{\epsilon,0} = \varepsilon \alpha_\infty \leq \tilde{\vol}_0(\varepsilon/\ell_0)$, hence the first term is of order $O(\varepsilon/\ell_0)$. For the second term
in Theorem 4.8 we need the estimate \(\alpha_{\varepsilon,0} = \varepsilon \alpha_0 \geq \tilde{v} \lambda_0 / (d_{\infty} \ell) \cdot (\varepsilon / \ell_0)\); moreover, for the validity of (4.9) with \(\lambda_2 \varepsilon = 1\), namely the existence of \(C_0 > 0\) such that \(\lambda_2 (X_{\varepsilon,0}) \geq 1 / \ell_0\) for all \(0 < \varepsilon \leq \varepsilon_0 = C_0^{-2} \ell_0\), we refer to Proposition B.3. The constant \(C_0\) depends on lower estimates on \(\lambda_2 (Y_{\varepsilon}) = \lambda_2 (Y_0)\) and \(\lambda_2 (X_{\varepsilon,0}) \geq \lambda_2 \varepsilon > 0\), and on an upper estimate of \(\text{vol} \ X_{\varepsilon,0} / \text{vol} \ Y_0 = \text{vol} \ X_{\varepsilon,0} / \deg \ v \leq \text{vol} \ X_0\), hence \(C_0 := \sup_\varepsilon C_\varepsilon < \infty\).

The estimate on the last term in the definition of \(\delta\) in Theorem 4.8 follows now from the scaling \(\lambda_2 (\varepsilon X_{\varepsilon}) = \varepsilon^{-2} \lambda_2 (X_{\varepsilon})\); moreover, the length of the collar neighbourhood \(\kappa \ell_\varepsilon\) is of order \(\varepsilon\) in the sense that

\[
0 < \varepsilon = \kappa \ell_0 \leq \kappa \ell_\varepsilon \leq \kappa \ell_\infty = \kappa \ell_0 = 7 \varepsilon
\]

for all \(e \in E\). In particular we have the estimate

\[
\kappa + \frac{2}{\kappa \inf_\varepsilon \ell_\varepsilon^2 \inf_\varepsilon \lambda_2 (X_{\varepsilon})} \leq \frac{\varepsilon}{\ell_0} + \frac{2}{\varepsilon \ell_0 \inf_\varepsilon \lambda_2 (\varepsilon X_{\varepsilon})} \leq \left(1 + \frac{2}{\lambda_2}\right) \frac{\varepsilon}{\ell_0}.
\]

\[
\square
\]

5. Convergence of energy forms on fractals and graph-like spaces

5.1. Symmetric post-critically finite fractals

For details on post-critically finite fractals we refer to our first article [PS17], and, of course, to the monographs [Str06, Kig01]. We consider here only, what we call symmetric fractals, which basically means that the corresponding quantity is independent of the index \(j\) of the iterated function system. The symmetry might appear from an underlying dihedral symmetry of \(K\), but what matters for us is mostly the fact that we have explicit formulas for various quantities. We need the symmetry assumption mainly for the compatibility assumptions (3.3a) and (4.4a).

A symmetric fractal (in our setting here) is a compact subset of \(\mathbb{R}^d\) which is invariant under an iterated function system \(F = \{F_j\}_{j=1}^N\), i.e., for which we have \(K = F(K) := \bigcup_{j=1}^N F_j (K)\). Each member \(F_j : \mathbb{R}^d \rightarrow \mathbb{R}^d\) is supposed to be a \(\theta\)-similitude, i.e., if there is \(\theta \in (0, 1)\) such that

\[
|F_j (x) - F_j (y)| = \theta |x - y| \quad \text{for all } x, y \in \mathbb{R}^d.
\]

Let \(V_0\) be a non-empty subset of the fixed points of the \(F_j\)'s, called boundary of the fractal, and set \(N_0 := |V_0|\) (then \(N_0 \leq N\)). We assume that the fractal is post-critically finite (pcf), i.e., that \(F_j (K) \cap F_{j'} (K) \subset F_j (V_0) \cap F_{j'} (V_0)\) for all \(1 \leq j < j' \leq N\). For such fractals, there is a recursively defined sequence of simple graphs \(G_m = (V_m, E_m)\), starting with the complete graph \(G_0\) over the set of boundary points \(V_0\), and such that \(V_{m+1} := F(V_m)\), and \(e = \{x, y\} \in E_{m+1}\) if there exists an edge \(e' = \{x', y'\} \in E_m\) and \(j \in \{1, \ldots, N\}\) such that \(F_j (x') = x\) and \(F_j (y') = y\) (for short, we write \(F_j (e') = e\)).

On \(G_m\), we assume that there is a (discrete) energy form

\[
\mathcal{E}_m (f) = \sum_{\{x,y\} \in E_m} \gamma_m (x, y) |f (x) - f (y)|^2.
\]

We call the sequence \((\mathcal{E}_m)_m\) self-similar and symmetric if there exists \(r \in (0, 1)\) with \(\mathcal{E}_{m+1} (f) = \sum_{j=1}^N \frac{1}{r} \mathcal{E}_m (f \circ F_j)\) for all \(f : V_m \rightarrow \mathbb{C}\). We call \(r\) the energy renormalisation parameter (of the self-similarity). We call the sequence \((\mathcal{E}_m)_m\) compatible if the vertex sets of \(G_m\) are nested (i.e., \(V_m \subset V_{m+1}\)) and if

\[
\mathcal{E}_G (\varphi) = \min \{ \mathcal{E}_G (f) \mid f : V_{m+1} \rightarrow \mathbb{C}, f \big|_{V_m} = \varphi\}
\]

for all \(\varphi : V_m \rightarrow \mathbb{C}\). We assume here that \((\mathcal{E}_m)_m\) is self-similar, symmetric and compatible. In this case, the discrete edge weights \(\gamma_m : E_m \rightarrow (0, \infty)\) are given by \(\gamma_{m,e} = r^m \gamma_{0, e}\)

if there is a word \(w \in W_m := \{1, \ldots, N\}^m\) of length \(m\) such that \(F_w (e_0) = e\), where \(F_w := F_{w_1} \circ \cdots \circ F_{w_m}\) if \(w = (w_1, \ldots, w_m)\).
For a symmetric, self-similar and compatible sequence \((\mathcal{E}_m)_m\), there exists a self-similar and symmetric energy form \(\mathcal{E}_K\) on \(K\), i.e., a closed non-negative quadratic form with domain \(\text{dom} \, \mathcal{E}_K \subset C(K)\) such that

\[
\mathcal{E}_K(u) = \sum_{j=1}^{N} \frac{1}{r} \mathcal{E}_K(u \circ F_j)
\]

for all continuous \(u: V_* := \bigcup_m V_m \to \mathbb{C}\). For this energy form, a given \(m \in \mathbb{N}_0\) and \(v \in V_m\), there is a unique function \(\psi_{m,v}: K \to [0,1]\) with \(\psi_{m,v}(v') = 1\) if \(v = v'\) and 0 if \(v \neq v'\). Moreover, \(\psi_{m,v} \in \text{dom} \, \mathcal{E}_K\) and \(\mathcal{E}_K(\psi_{m,v})\) is the minimal value among all \(\mathcal{E}_K(u)\) with \(u|_{V_m} = \psi_{m,v}|_{V_m}\). The function \(\psi_{m,v}\) is called \(m\)-harmonic. Note that \((\psi_{m,v})_{v \in V_m}\) is a partition of unity on \(K\).

Moreover, we assume that we have a self-similar symmetric measure, i.e., a finite measure \(\mu\) on \(K\) such that

\[
\mu(F_m(K)) = N^{-m} \mu(K) \quad \text{for any word } \quad w \in W_m := \{1, \ldots, N\}^m.
\]

For simplicity, we assume that \(\mu(K) = 1\). We define

\[
\mu_m(v) := \int_K \psi_{m,v} \, d\mu.
\]

We call \((G_m, \mu_m, \gamma_m)_m\) a sequence of approximating weighted graphs for the pcf fractal \(K\).

Let us summarise the above discussion and introduce the symmetry of the boundary:

**5.1. Definition.** Let \(K\) be a pcf fractal given by some iterated function system \(F = (F_j)_{j=1, \ldots, N}\).

(i) We say that \(K\) is symmetric if the similitude factor \(\theta\) is the same for all functions in the iterated function system, i.e., if \((5.1)\) holds.

(ii) We say that a self-similar energy form \(\mathcal{E}_K\) on \(K\) is symmetric or homogeneous if the energy renormalisation parameter \(r \in (0,1)\) is the same for all \(j \in \{1, \ldots, N\}\), i.e., if \((5.3)\) holds.

(iii) We say that a self-similar measure \(\mu\) on \(K\) is symmetric if the measure self-similar factor is the same for all \(j \in \{1, \ldots, N\}\), i.e., if \((5.4)\) holds.

(iv) We say that the boundary \(V_0\) of a fractal \(K\) is symmetric if \(\mu_0\) gives the same mass to all points, i.e., if \(\mu_0(v_0) = 1/\mathcal{N}_0\) for all \(v_0 \in V_0\) and if there is \(\mathcal{N}_0 > 0\) such that

\[
\sum_{e_0 \in E_{e_0}(G_0)} \frac{1}{\gamma_{0,e_0}} = \mathcal{N}_0
\]

holds for all \(v_0 \in V_0\).

If all four conditions hold, we also say that \((K, \mathcal{E}_K, \mu, V_0)\) is symmetric. We also call \((G_m, \mu_m, \gamma_m)\) the \(m\)-th approximation of \((K, \mathcal{E}_K, \mu, V_0)\) by a finite weighted graph.

If \((K, \mathcal{E}_K, \mu, V_0)\) is symmetric, then we have

\[
\mu_m(v) = \int_K \psi_{m,v} \, d\mu = \sum_{w \in W_m} \int_{F_w(K)} \psi_{m,w} \, d\mu = \sum_{w \in W_m} \frac{1}{N^m} \int_K \psi_{m,F_w^{-1}v} \, d\mu = \frac{1}{\mathcal{N}_0 N^m} \sum_{w \in W_m} |W_{m,v}| \quad \text{(5.6)}
\]

as \(\text{supp} \, \psi_{m,v} \subset \bigcup_{w \in W_m,v} F_w(K)\) (second equality) and as \(\mu\) is self-similar and symmetric, as well as \(\mathcal{E}_K\) is symmetric (third equality). The last equality uses the symmetry of the boundary.

**5.2. Lemma.** Let \((G_m, \mu_m, \gamma_m)\) be the sequence of weighted graphs associated with a symmetric \((K, \mathcal{E}_K, \mu, V_0)\) then \((G_m, \mu_m, \gamma_m)\) is \((N_1(N_0 - 1), N_1, C_2/C_1)\)-uniform, where

\[
N_1 := \sup_{m \in \mathbb{N}_0, v \in V_m} |W_{m,v}|, \quad C_1 := \min_{e_0 \in E_0} \gamma_{0,e_0} \quad \text{and} \quad C_2 := \max_{e_0 \in E_0} \gamma_{0,e_0}.
\]
In particular, all constants are independent of $m \in \mathbb{N}_0$.

**Proof.** Each cell has degree maximal to $N_0 - 1$ (as we start with the complete graph in generation 0 on $N_0$ vertices), and the maximal number of cells intersecting in one vertex is given by $N_1$. Moreover, $\mu_m(v) = |W_{m,v}|/(N_0 N^m)$ hence $\mu_m,\infty/\mu_m,0 = N_1$. Finally, $\gamma_{m,e} = r^{-m}\gamma_{0,e}$ if $e = F_w(0)$ for some word $w \in W_m$, hence $\gamma_{m,\infty}/\gamma_{m,0} = C_2/C_1$. \(\square\)

### 5.2. Quasi-unitary equivalence of fractals and metric graphs

We will now apply Theorem 3.5 to the discrete graphs $G = G_m$ with vertex weights $\mu = \mu_m$ and edge weights $\gamma = \gamma_m$ from the fractal approximation, and a corresponding metric graph $M = M_0$. We fix the edge lengths in a way that $M_m$ and $(G_m, \mu_m, \gamma_m)$ are compatible in the sense of Definition 3.1.

**5.3. Lemma.** Assume that $(K, \mathcal{E}_K, \mu, V_0)$ is symmetric, that $(G_m, \mu_m, \gamma_m)$ is a corresponding member of an approximating sequence of weighted graphs and that $M_m$ is a metric graph according to $G_m$ with edge length given by

$$\ell_{m,e} = \frac{\ell_{0,0}}{\gamma_{0,e}} \cdot \Lambda^m$$

for some $\Lambda \in (0,1)$ and $\ell_{0,0} > 0$, where $e = F_w(0)$ with $w \in W_m$. Then the metric graph $M_m$ and the weighted discrete graph $(G_m, \mu_m, \gamma_m)$ are compatible. Moreover, the isometric rescaling factor $c = c_m$ and the energy rescaling factor $\tau = \tau_m$ are given by

$$c_m^2 = \frac{2}{C_0 N_0} \frac{1}{(N\Lambda)^m} \quad \text{and} \quad \tau_m = \frac{\ell_{m,e} \gamma_{m,e}}{c_m^2} = \frac{C_0 N_0}{\ell_{0,0}} \cdot \left( \frac{N\Lambda^2}{r} \right)^m.$$  

**Proof.** We have

$$\nu_m(v) = \frac{1}{2} \sum_{e \in E_v(G_m)} \ell_{m,e} = \frac{\ell_{0,0} \Lambda^m}{2} \sum_{w \in W_{m,v}} \sum_{e \in E_{F_w^{-1}(G_0)}} \frac{1}{\gamma_{0,e}} = \frac{\ell_{0,0} \Lambda^m}{2} |W_{m,v}| C_0$$

using $\gamma_{m,e} = r^{-m}\gamma_{0,F_w^{-1}e}$, hence

$$c_m^2 = \frac{\mu_m(v)}{\nu_m(v)} = \frac{\ell_{0,0} \Lambda^m |W_{m,v}| C_0}{2 |W_{m,v}|} = \frac{\ell_{0,0} C_0 N_0}{2} \cdot (N\Lambda)^m$$

using (5.6). In particular, the vertex weights are compatible, i.e., (3.3a) holds, and $c_m$ is given as in (5.9). Moreover, we have

$$c_m^2 \tau_m = \ell_{m,e} \gamma_{m,e} \frac{\Lambda^m}{\gamma_{0,e}} \frac{\gamma_{0,e}}{r^m} = \frac{\Lambda^m}{r^m},$$

i.e., (3.3b) holds, too, and $\tau_m$ is given as in (5.9). \(\square\)

We can play a bit with the choice of parameters:

**Case 1 (geometric case):** If we set $\Lambda = \theta$, then the length scale shrinks as the IFS with similitude factor $\theta$. In this case, $\ell_m$, $c_m$ and $\tau_m$ are given as in (5.8) and (5.9) with $\Lambda$ replaced by $\theta$.

**Case 2 (edge weight is inverse of edge length):** Set $\Lambda = r$ and $\ell_{0,0} = 1$, then we have $\ell_{m,e} = 1/\gamma_{m,e}$ and

$$\ell_{m,e} = \frac{r^m}{\gamma_{0,e}}, \quad c_m^2 = \frac{2}{C_0 N_0} \cdot \frac{1}{(N\Lambda)^m} \quad \text{and} \quad \tau_m = \frac{C_0 N_0}{2} \cdot (N\Lambda)^m$$

if $e = F_w(0)$ for some word $w \in W_m$.

**Case 3 (no energy rescaling factor):** We can also fix $\tau_m = 1$, then $\Lambda = \sqrt{r/N}$. Moreover,

$$\ell_{m,e} = \sqrt{\frac{2}{C_0 N_0} \cdot \frac{1}{\gamma_{0,e}}} \cdot \left( \frac{r}{N} \right)^{m/2} \quad \text{and} \quad c_m^2 = \sqrt{\frac{2}{C_0 N_0} \cdot \frac{1}{(N\Lambda)^m}}$$

if $e = F_w(0)$ for some word $w \in W_m$. 
We have the following orders of the length scale, the isometric rescaling factor and the energy renormalisation factor:

| Case | $\ell_{m,e} =$ | $c_m =$ | $\tau_m =$ |
|------|----------------|----------|-----------|
| 1    | $O(\theta^m)$  | $O((N\theta)^{-m/2})$ | $O((N\theta^2/r)^m)$ |
| 2    | $O(r^m)$       | $O((Nr)^{-m/2})$   | $O((N)^m)$        |
| 3    | $O((r/N)^{m/2})$ | $O((r)^{-m/4})$    | 1                    |

For the Sierpiński triangle, we have $N = 3$, $r = 3/5$ and $\theta = 1/2$, hence the parameters have the following order:

| Case | $\ell_{m,e} =$ | $c_m =$ | $\tau_m =$ |
|------|----------------|----------|-----------|
| 1    | $O(1/2^m)$    | $O((2/3)^{m/2})$ | $O((5/4)^m)$ |
| 2    | $O(3/5)^m$    | $O((5/9)^{m/2})$ | $O((9/5)^m)$ |
| 3    | $O(1/5^{m/2})$ | $O((5/9)^{m/4})$ | 1                |

5.4. Theorem. Let $(G_m, \mu_m, \gamma_m)$ be the $m$-th generation of a symmetric pcf fractal given by $(K, E_K, \mu, V_0)$ (see Definition 3.1). Moreover, let $\mathcal{E}_m$ be the discrete energy functional of $(G_m, \mu_m, \gamma_m)$ (see (5.2)), and let $\tilde{\mathcal{E}}_m = \tau_m \mathcal{E}_{M_m}$ be the rescaled metric graph energy (see (3.1)) for the metric graph $M_m$ constructed according to $G_m$ with edge lengths $(\ell_{m,e})_e$ as in (5.8). Then $\mathcal{E}_m$ and $\tilde{\mathcal{E}}_m$ are $\delta_m$-unitarily equivalent with $\delta_m = O((r/N)^{m/2})$.

Proof. Note first that $M_m$ and $(G_m, \mu_m, \gamma_m)$ are compatible by Lemma 5.3. Then we apply Theorem 3.5 and calculate the error term $\delta = \delta_m$ defined there. Note first that $\overline{\gamma} = \gamma_\infty/\gamma_0 = \max_{e_0 \in E_0} \gamma_{e_0}/\min_{e_0 \in E_0} \gamma_{e_0}$. Moreover,

$$\frac{\mu_{m,\infty}}{\gamma_{m,0}} = \max_{v \in V_m} \min_{e \in E_m} \gamma_{m,e} = \frac{N_1}{N_0^m N_1} \cdot \frac{r^m}{C_1} = \frac{N_1}{N_0 C_1} \cdot \left(\frac{r}{N}\right)^m = O\left(\left(\frac{r}{N}\right)^m\right)$$

where $N_1$, $C_1$ and $C_2$ are defined in (5.7).

Using the $O((r/N)^{m/2})$-quasi-unitary equivalence of the fractal energy $\mathcal{E}_K$ and the discrete graph energy $\mathcal{E}_{G_m}$ proven in [PS17] and the transitivity of quasi-unitary equivalence (see Proposition A.3), we obtain:

5.5. Corollary. Let $K$ be a pcf fractal such that $(K, E_K, \mu, V_0)$ is symmetric. Moreover, let $\tilde{\mathcal{E}}_m = \tau_m \mathcal{E}_{M_m}$ be the rescaled metric graph energy for the metric graph $M_m$ constructed as above. Then $\tilde{\mathcal{E}}_m$ and the fractal energy form $\mathcal{E}_K$ are $\tilde{\delta}_m$-unitarily equivalent with $\tilde{\delta}_m \to 0$ as $m \to \infty$.

If we want to quantify the error estimate, we need the weaker notion of operator quasi-unitary equivalence, see Definition A.4 and Proposition A.5, and the transitivity for this notion in Proposition A.6.

5.6. Corollary. Let $(K, E_K, \mu, V_0)$ be a symmetric pcf fractal. Moreover, let $\tilde{\Delta}_m$ be the metric graph Laplacian associated with the metric graph $M_m$ as above. Then $\tilde{\Delta}_m$ and the fractal Laplacian $\Delta_K$ (associated with $E_K$) are $\tilde{\delta}_m$-unitarily equivalent, where $\tilde{\delta}_m$ is of order $O((r/N)^{m/2})$.

For the Sierpiński triangle, we have $N = 3$, $r = 3/5$ and $\theta = 1/2$, hence the error estimate of $\tilde{\delta}_m$ is $O(1/5^{m/2})$.

5.3. Quasi-unitary equivalence of fractals and graph-like manifolds

Lastly, we will apply Corollary 4.9 to the case of a family of discrete weighted graphs $G = G_m$ with vertex weights $\mu = \mu_m$ and edge weights $\gamma = \gamma_m$ and a corresponding family of graph-like manifolds $X_m$. The scaling of the transversal and vertex manifold are

$$Y_{m,e} = \varepsilon_m Y_e \quad \text{and} \quad \hat{X}_{m,v} = \varepsilon_m \hat{X}_v$$

(5.10)

(5.10)

(as before, $rX$ is the Riemannian manifold $(X, r^2 g)$ if $X$ is the Riemannian manifold $(X, g)$; the factor $r$ is hence a change of length scale).
We now determine the parameters of the graph-like manifold having exponential dependency on \(m\), namely

\[
\ell_{m,e} = \frac{\ell_{e,0}}{\gamma_{e,0}} \Lambda^{m}, \quad \text{and} \quad \varepsilon_{m} = \varepsilon_{0}E^{m}, \quad \text{where} \quad 0 < E < \Lambda < 1 \tag{5.11}
\]

and where \(e = F_{v}(e_{0})\) for some word \(w \in W_{m}\). Here, \(\ell_{e,0} > 0\) and \(\varepsilon_{0} > 0\) are some constants. Moreover, we assume that \(Y_{e}\) is isometric with a fixed manifold \(Y_{0}\) with \(\text{vol} Y_{0} = 1\) (for simplicity only). Finally, we assume that there are constants \(\text{vol}_{0}, \text{vol}_{\infty}\) and \(\lambda_{2}\) such that the unscaled manifold \(X_{v}\) fulfil

\[
0 < \text{vol}_{0} \leq \text{vol} X_{v} \leq \text{vol}_{\infty} < \infty \quad \text{and} \quad \lambda_{2}(X_{v}) \geq \lambda_{2} > 0 \tag{5.12}
\]

for all \(e \in E\) and \(v \in V\).

Our main result is now the following:

**5.7. Theorem.** Let \((G_{m}, \mu_{m}, \gamma_{m})\) be the \(m\)-th generation of a symmetric pcf fractal given by \((K, \mathcal{E}_{K}, \mu, V_{0})\) (see Definition 5.1). Moreover, let \(\mathcal{E}_{m}\) be the discrete energy functional of \((G_{m}, \mu_{m}, \gamma_{m})\) (see (5.2)), and let \(\mathcal{E}_{m} = \tau_{m}\mathcal{E}_{X_{m}}\) be the rescaled graph-like manifold energy (see (4.11)) for the graph-like manifold \(X_{m}\) constructed according to \(G_{m}\) with edge lengths \((\ell_{m,e})_{e}\), transversal manifolds \((Y_{m,e})_{e}\) and core vertex neighbourhoods \((\text{vol} X_{m,v})_{v}\), (5.10)–(5.12). Finally, we assume that

\[
\frac{r}{N}\Lambda < E < \Lambda.
\]

Then \(\mathcal{E}_{m}\) and \(\widetilde{\mathcal{E}}_{m}\) are \(\delta_{m}\)-unitarily equivalent with

\[
\delta_{m} = \max\left\{ O\left(\left(\frac{E}{\Lambda}\right)^{m/2}\right), O\left(\left(\frac{\Lambda}{E}\frac{r}{N}\right)^{m/2}\right)\right\}. \tag{5.13}
\]

In particular, if \(E\) is the geometric mean of \(\Lambda\) and \((r/N)\Lambda\), i.e., \(E = (r/N)^{1/2}\Lambda\), then the error estimate is \(\delta_{m} = O((r/N)^{m/4})\), the best possible choice.

**Proof.** The result follows from the assumptions and Corollary 4.9. Note that (4.10) follows as in Lemma 5.3 since the situation is the same as for metric graphs. Note also that \(\varepsilon_{m}/\ell_{m} \to 0\) as \(m \to \infty\), hence the condition \(\varepsilon_{m} \leq \ell_{m}/C_{0}^{2}\) is eventually fulfilled.

**5.8. Remark.** Note that since \(\text{vol} Y_{e} = 1\), the compatibility conditions for the graph-like manifold in (4.4a)–(4.4b) are formally the same as for a metric graph in (3.3a)–(3.3b), hence the different cases for choices of the parameters in Subsection 5.2 also apply here, we just have to take into account, that the isometric rescaling factor \(c_{m}\) also depends on \(\varepsilon_{m}\), namely, \(c_{m}\) contains an extra factor \(\varepsilon_{m}^{(d-1)/2} = \varepsilon_{0}^{(d-1)/2}(E^{(d-1)/2})^{-m}\), while the energy renormalisation factor \(\tau_{m}\) remains the same. In particular, we have for \(\Lambda = \theta, r\) and \((r/N)^{1/2}\) the following cases:

| Case | \(\ell_{m,e}\) | \(\varepsilon_{m}\) | \(E \in\) | \(c_{m}\) | \(\tau_{m}\) |
|------|----------------|----------------|----------|----------------|----------------|
| 1    | \(O(\theta^{m})\) | \(O(E^{m})\) | \(\theta^{m}/N, \theta\) | \(O((E^{d-1}N^{2})^{m/2})\) | \(O((N\theta^{2}/r)^{m})\) |
| 2    | \(O(r^{m})\) | \(O(E^{m})\) | \((r^{m}/N, r)\) | \(O((E^{d-1}N^{2}r)^{m/2})\) | \(O((N\theta^{2}/r)^{m})\) |
| 3    | \(O((r/N)^{m/2})\) | \(O(E^{m})\) | \((r/N)^{m/2}, (r/N)^{1/2}\) | \(O((E^{d-1}N^{2}r)^{m/4})\) | 1 |

**5.9. Example.** For a Sierpiński triangle we have \(N = 3\), \(r = 3/5\) and we can choose \(\Lambda = 1/2\), the length scale factor. If we choose \(E = 1/(2\sqrt{5})\), then the error estimate has the optimal rate \(\delta_{m} = O((1/5)^{m/4})\). In particular, the approximating manifold has longitudinal edge lengths scaling as \(\Lambda^{m} = 1/2^{m}\) in generation \(m\), while the transversal manifold has radius of order \(E^{m} = 1/(2\sqrt{5})^{m}\), which shrinks faster than the longitudinal scale \(\Lambda^{m}\). This is Case 1 in the tabular below. Other choices for \(\Lambda\) are \(1/2, 3/5\) and \(1/5^{1/2}\) (the \(E\) with optimal error rate then is \(1/(2\sqrt{5}), 3/(5\sqrt{5})\) and \(1/5\):
| Case | $\ell_{m,c} =$ | $\varepsilon_m =$ | $E \in$ | $\varepsilon_m =$ | $\tau_m =$ |
|------|-----------------|-----------------|----------|-----------------|----------------|
| 1    | $O(1/2^m)$     | $O(E^m)$       | $(1/10, 1/2)$ | $O((E^{-d-1}2/3)^{m/2})$ | $O((5/4)^m)$ |
| 2    | $O((3/5)^m)$   | $O(E^m)$       | $(3/25, 3/5)$ | $O((E^{-d-1}5/9)^{m/2})$ | $O((9/5)^m)$ |
| 3    | $O((1/5)^{m/2})$ | $O(E^m)$       | $(1/5^{3/2}, 1/5^{1/2})$ | $O((E^{-d-1}5/9)^{m/2})$ | 1 |

We can also choose $E$ as close to $\Lambda$ (i.e., as large) as we want, the price is a worse error estimate.

5.10. Remark. Note that we cannot assume directly $E = \Lambda$. This case is interesting since it would allow to apply directly the IFS to a suitable starting compact neighbourhood $X_0$ of the (metric) graph associated with $G_0$. In this case $E = \Lambda = \theta$, where $\theta$ is the the similitude factor, but here, the graph-like manifold is not shrinking fast enough in transversal direction (the transversal scale is $\varepsilon_m = \varepsilon_0 E^m$, while the longitudinal scale $\ell_{m,c}$ is of order $\Lambda^m$).

Nevertheless we conjecture that if $E = \Lambda$ and if the starting transversal parameter $\varepsilon_0$ is sufficiently small, one can still conclude convergence results such as convergence of eigenvalues of graph-like manifolds (i.e., a sequence of graph-like manifolds $X_m$ generated by the IFS) with corresponding (Neumann) eigenvalues converging to the eigenvalues of the fractal. Note that $E < \Lambda$ is only used in the parameter $\varepsilon/\ell_0$ resp. $\alpha_\infty$ in Corollary 4.9 resp. Corollary 4.9 and $\alpha_\infty$ is only needed for $\|J' - J^*\| \leq 2\alpha_\infty$ in Proposition 2.8 (iii). We will treat such questions in a subsequent publication.

Using again the $O((r/N)^{m/2})$-quasi-unitary equivalence of the fractal energy $E_K$ and the discrete graph energy $E_{G_m}$ proven in [PS17] and the transitivity of quasi-unitary equivalence (see Proposition A.3), we obtain:

5.11. Corollary. Let $K$ be a pcf fractal such that $(K, E_K, \mu, V_0)$ is symmetric. Moreover, let $\hat{E}_m = \tau_m E_{X_m}$ be the rescaled energy form of the graph-like manifold $X_m$ constructed according to $G_m$ as in Theorem 5.7. Then $\hat{E}_m$ and the fractal energy form $E_K$ are $\hat{d}_m$-unitarily equivalent with $\hat{d}_m \to 0$ as $m \to \infty$.

As for the metric graph approximation, we obtain a quantified error estimate using the weaker notion of operator quasi-unitary equivalence, see Definition A.4 and Proposition A.5. The transitivity for this notion in Proposition A.6 gives a precise error estimate:

5.12. Corollary. Let $K$ be a pcf fractal such that $(K, E_K, \mu, V_0)$ is symmetric. Moreover, let $\hat{\Delta}_m = \tau_m \Delta_{X_m}$ be the Laplacian (associated with $\hat{E}_m = \tau_m E_{X_m}$) on the graph-like manifold $X_m$ constructed as above. Then $\hat{\Delta}_m$ and the fractal Laplacian $\Delta_K$ (associated with $E_K$) are $\delta_m$-unitarily equivalent, where $\delta_m$ is of order as in (5.13).

Appendix A. An abstract norm resolvent convergence result

In this appendix, we briefly present a general framework which assures a generalised norm resolvent convergence for operators $\Delta_m$ converging to $\Delta_\infty$ as $\varepsilon \to 0$, see [Pos12] for details. Each operator $\Delta_m$ acts in a Hilbert space $\mathcal{H}_m$ for $m \in \mathbb{N}$; and the Hilbert spaces are allowed to depend on $m$.

In one of our applications, the Hilbert spaces $\mathcal{H}_m$ are of the form $L_2(X_m) = L_2(V_m, \mu_m)$ and a “limit” metric measure space $(X, \mu)$ with Hilbert space $\mathcal{H} = L_2(X, \mu)$.

In order to define the convergence, we define a sort of “distance” $\delta_m$ between $\Delta := \Delta_m$ and $\tilde{\Delta} := \Delta_\infty$, in the sense that if $\delta_m \to 0$ then $\Delta_m$ converges to $\Delta_\infty$ in the above-mentioned generalised norm resolvent convergence. We start now with the general concept:

Let $\mathcal{H}$ and $\mathcal{H}^1$ be two separable Hilbert spaces. We say that $(\mathcal{E}, \mathcal{H}^1)$ is an energy form in $\mathcal{H}$ if $\mathcal{E}$ is a closed, non-negative quadratic form in $\mathcal{H}$, i.e., if $\mathcal{E}(f) := \mathcal{E}(f, f)$ for some sesquilinear form $\mathcal{E}: \mathcal{H}^1 \times \mathcal{H}^1 \to \mathbb{C}$, denoted by the same symbol, if $\mathcal{E}(f) \geq 0$ and if $\mathcal{H}^1 := \text{dom} \mathcal{E}$, endowed with the norm defined by

$$\|f\|_1^2 := \|f\|_{\mathcal{H}^1}^2 := \|f\|_{\mathcal{H}^1}^2 + \mathcal{E}(f),$$

(A.1)
We say that the energy forms $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are $\delta$-quasi-unitarily equivalent, if and only if the following operator norm estimates hold:
\[ \|\mathcal{E} - \tilde{\mathcal{E}}\| \leq \delta \|\cdot\|_1 \|\cdot\|_1 \quad (f \in \mathcal{H}, u \in \tilde{\mathcal{H}}). \] (A.3d)

We say that $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are $\delta$-quasi-unitarily equivalent, if (A.3a)–(A.3d) are fulfilled, i.e., if the following operator norm estimates hold:
\[ \|\mathcal{E} - \tilde{\mathcal{E}}\| \leq \delta \quad (f \in \mathcal{H}, u \in \tilde{\mathcal{H}}). \] (A.3ad)

(iii) We say that $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are $\delta$-quasi-unitarily equivalent, if and only if the following operator norm estimates hold:
\[ \|\mathcal{E} - \tilde{\mathcal{E}}\| \leq \delta \|\cdot\|_1 \|\cdot\|_1 \quad (f \in \mathcal{H}, u \in \tilde{\mathcal{H}}). \] (A.3ad)

A.2. Remark. Let us explain the notation in two extreme cases assuring that $\delta$ is in some sense a “distance” between the two forms:

(i) “$\delta$-quasi-unitary equivalence” is a quantitative generalisation of “unitary equivalence”:
Note that if $\delta = 0$, $J$ is 0-quasi-unitary if and only if $J$ is unitary with $J^* = J$. Moreover, $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are 0-quasi-unitarily equivalent if and only if $\Delta = \tilde{\Delta}$.

(ii) “$\delta_m$-quasi-unitary equivalence” is a generalisation of “norm resolvent convergence”:
If $\mathcal{H} = \tilde{\mathcal{H}}$, $J = J^* = \text{id}_\mathcal{H}$, then the first two conditions (A.3a)–(A.3b) are trivially fulfilled with $\delta = 0$. Moreover, if $\Delta = \Delta_m$ and $\delta = \delta_m \to 0$ as $m \to \infty$, then $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are $\delta_m$-quasi-unitarily equivalent.
as \( m \to \infty \), hence this implies that \( \| R_m - R \| \to 0 \), i.e., \( \Delta_m \) converges to \( \Delta \) in norm resolvent sense.

Now from the concept of quasi-unitary equivalence of forms, many more results follow (see e.g. [Pos12]).

We now state the transitivity of \( \delta \)-quasi-unitary of energy forms. Assume that \( \mathcal{H}, \tilde{\mathcal{H}} \) and \( \tilde{\mathcal{H}} \) are three Hilbert spaces with non-negative operators \( E, \tilde{E} \) and \( \tilde{E} \), respectively. Moreover, assume that

\[
J : \mathcal{H} \to \mathcal{H}, \quad \tilde{J} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}, \quad J' : \tilde{\mathcal{H}} \to \mathcal{H} \quad \text{and} \quad J' : \tilde{\mathcal{H}} \to \mathcal{H},
\]

\[
J^1 : \mathcal{H}^1 \to \mathcal{H}^1, \quad \tilde{J}^1 : \tilde{\mathcal{H}}^1 \to \tilde{\mathcal{H}}^1, \quad J^1 : \tilde{\mathcal{H}}^1 \to \tilde{\mathcal{H}}^1 \quad \text{and} \quad J^1 : \tilde{\mathcal{H}}^1 \to \tilde{\mathcal{H}}^1.
\]

are bounded operators. We define

\[
\tilde{J} := \tilde{J}J : \mathcal{H} \to \tilde{\mathcal{H}}, \quad \tilde{J} := J'\tilde{J} : \mathcal{H} \to \mathcal{H},
\]

\[
\tilde{J}^1 := \tilde{J}^1J : \mathcal{H}^1 \to \tilde{\mathcal{H}}^1, \quad \tilde{J}^1 := J'\tilde{J} : \tilde{\mathcal{H}}^1 \to \tilde{\mathcal{H}}^1.
\]

In addition to Definition A.1 we assume that the identification operators \( J^1 \) and \( \tilde{J}^1 \) in Definition A.1 are \((1 + \delta)\)- resp. \((1 + \tilde{\delta})\)-bounded, i.e.

\[
\| J^1 \|_{1 \to 1} \leq 1 + \delta \quad \text{resp.} \quad \| \tilde{J}^1 \|_{1 \to 1} \leq 1 + \tilde{\delta}.
\]

A.3. Proposition ([Pos12] Prp. 4.4.16). Assume that \( 0 \leq \delta \) and \( \tilde{\delta} \leq 1 \). Assume in addition that \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \) are \( \delta \)-quasi-unitarily equivalent with identification operators \( J, J^1, J' \) and \( J^1 \), and that \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \) are \( \delta \)-quasi-unitarily equivalent with identification operators \( \tilde{J}, \tilde{J}^1, \tilde{J}' \) and \( \tilde{J}^1 \). Then \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \) are \( \delta \)-quasi-unitarily equivalent with identification operators \( \tilde{J}, \tilde{J}' \) and \( \tilde{J}^1 \), where \( \delta = \delta(\delta, \tilde{\delta}) \to 0 \) as \( \delta \to 0 \) and \( \tilde{\delta} \to 0 \).

If we want to quantify the error we need a slightly weaker notion of unitary equivalence for operators:

A.4. Definition. Let \( \delta \geq 0 \), and let \( J : \mathcal{H} \to \tilde{\mathcal{H}} \) and \( J' : \tilde{\mathcal{H}} \to \mathcal{H} \) be bounded linear operators.

(i) We say that \( J \) is \( \delta \)-quasi-unitary with \( \delta \)-quasi-adjoint \( J' \) (for the operators \( \Delta \) and \( \tilde{\Delta} \)) if and only if

\[
\| Jf \| \leq (1 + \delta)\| f \|, \quad \| Jf, u \| - \| Jf, J'u \| \leq \delta \| f \| \| u \| \quad (f \in \mathcal{H}, \ u \in \tilde{\mathcal{H}}), \tag{A.4a}
\]

\[
\| f - J'f \| \leq \delta \| f \|, \quad \| u - J'Ju \| \leq \delta \| u \|, \quad (f \in \mathcal{H}^2, \ u \in \tilde{\mathcal{H}}^2). \tag{A.4b}
\]

(ii) We say that the operators \( \Delta \) and \( \tilde{\Delta} \) are \( \delta \)-close if and only if

\[
\| Jf, \tilde{\Delta}u \| - \| J\Delta f, u \| \leq \delta \| f \| \| u \| \quad (f \in \mathcal{H}^2, \ u \in \tilde{\mathcal{H}}^2). \tag{A.4c}
\]

(iii) We say that \( \Delta \) and \( \tilde{\Delta} \) are \( \delta \)-quasi-unitarily equivalent, if (A.4a)–(A.4c) are fulfilled, i.e., we have the following operator norm estimates

\[
\| J \| \leq 1 + \delta, \quad \| J' - J' \| \leq \delta \tag{A.4a}
\]

\[
\| (\text{id}_{\mathcal{H}} - J'J)R \| \leq \delta, \quad \| (\text{id}_{\tilde{\mathcal{H}}^-} - J'J)\tilde{R} \| \leq \delta, \tag{A.4b}
\]

\[
\| R\tilde{J} - JR \| \leq \delta. \tag{A.4c}
\]

We have the following relation:

A.5. Proposition ([Pos12] Prp. 4.4.15). If the forms \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \) are \( \delta \)-quasi unitarily equivalent then the operators \( \Delta \) and \( \tilde{\Delta} \) are \( 4\delta \)-quasi-unitarily equivalent.

The transitivity for operator quasi-unitary equivalence gives a more explicit error estimate:

\[\text{The result in [Pos12] Prp. 4.4.16] is stated with a linear error term } \tilde{\delta} = O(\delta) + O(\tilde{\delta}), \text{ relying on a wrong estimate in [Pos12] Thm. 4.2.9]. \text{ We will correct this estimate in a forthcoming publication.}\]
A.6. Proposition ([Pos12] Prp. 4.2.5]). Assume that \(0 \leq \delta \) and \(\tilde{\delta} \leq 1\). Assume in addition that \(\Delta = (\Delta, \tilde{\Delta})\) are \(\delta\)-quasi-unitarily equivalent with identification operators \(J, J'\), and that \(\Delta = (\Delta, \tilde{\Delta})\) are \(\tilde{\delta}\)-quasi-unitarily equivalent with identification operators \(\tilde{J}, \tilde{J}'\). Then \(\Delta = (\Delta, \tilde{\Delta})\) are \(\tilde{\delta}\)-quasi-unitarily equivalent with identification operators \(\tilde{J} = \tilde{J}\tilde{J} \) and \(\tilde{J}' = \tilde{J}'\tilde{J}'\), where \(\tilde{\delta} = 22\delta + 43\tilde{\delta}\).

Appendix B. Some estimates on graph-like manifolds

We need some estimates on our graph-like manifold with respect to a norm weighted by a harmonic function and also a lower bound on the second eigenvalue of a graph-like manifold. We use the notation of Section 4.

B.1. Lemma. We have

\[
\|u(t, \cdot)\|^2_{L^2(Y_e)} \leq \frac{9}{2}\|u\|^2_{L^2(X_e, \nu, d\nu)} + 4\kappa_\ell\|du\|^2_{L^2(X_e, \nu, d\nu)},
\]

\[
\|u\|^2_{L^2(X_e, \nu, d\nu)} \leq \frac{9\kappa_\ell}{2}\|u\|^2_{L^2(X_e, \nu, d\nu)} + 4\kappa_\ell^2\|du\|^2_{L^2(X_e, \nu, d\nu)}.
\]

Proof. The proof of the first assertion is again an application of the fundamental theorem of calculus: Assume that the vertex \(v\) corresponds to the endpoint \(\ell_v\) of each adjacent \(X_e, e \in E_v\). Note that \(X_e \cong \{0, \ell_v\} \times Y_e\) and \(X_{v,e} \cong \{\ell_v, (1 + \kappa)\ell_v\} \times Y_v\); so we can use a common coordinate \(t \in [0, (1 + \kappa)\ell_v]\) for the first variable on both \(X_e\) and \(X_{v,e}\). Then \(\psi_{v,e}(x) = t/\ell_v\) for \(x = (t, y)\) and \(t \in M_v\), with derivative \(1/\ell_v\).

Let \(\chi_v(t) = (t/\ell_v)^{3/2}\) if \(t \in [0, \ell_v]\) and \(\chi_v(t) = 1\) if \(t \in [\ell_v, (1 + \kappa)\ell_v]\). Then \(\chi_v\) is Lipschitz continuous, \(\chi_v u \in H^1(X_e \cup X_{v,e})\) and \(\chi_v(0) = 0\). Moreover,

\[
\sum_{t \in E_v} \gamma_v \left| f_{X_e} u - \sum_{e \in E_v} \gamma_v \left| \frac{\partial}{\partial\ell_v} X_v u \right|^2 \leq \max_{e \in E_v} \frac{\gamma_v}{\text{vol} Y_v} \left( \kappa_\ell \ell_v + \frac{2}{\kappa_\ell \ell_v \lambda_2(\lambda_2)} \right) \|du\|^2_{L^2(X_v)}.
\]
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see e.g. [Pos12] Prp. 5.1.1 and Cor. A.2.12] (we omit the natural measures of the spaces). We then have

\[
\sum_{e \in E_e} \gamma_e |f_{\tilde{X}e} u - f_{\partial_e \tilde{X}e} u|^2 = \sum_{e \in E_e} \gamma_e |f_{\partial_e \tilde{X}e} (u - f_{\tilde{X}e} u)|^2
\]

\[
\leq \sum_{e \in E_e} \frac{\gamma_e}{\text{vol} Y_e} \int_{\partial_e \tilde{X}e} |u - f_{\tilde{X}e} u|^2
\]

\[
\leq \sum_{e \in E_e} \frac{\gamma_e}{\text{vol} Y_e} \left( \kappa \ell_e \|du\|_{L^2(X_{v,e})}^2 + \frac{2}{\kappa \ell_e} \|u - f_{\tilde{X}e} u\|_{L^2(X_{v,e})}^2 \right)
\]

\[
\leq \max_{e \in E_e} \frac{\gamma_e}{\text{vol} Y_e} \left( \kappa \ell_e + \frac{2}{\kappa \ell_e \lambda_2(X_v)} \right) \|du\|_{L^2(X_v)}^2
\]

using (B.3) for the last two estimates.

Let us now provide a lower bound on the second eigenvalue of a graph-like manifold \(X_{v,e}\), independently of the shrinking parameter \(\varepsilon\). Here, \(X_{v,e}\) is a graph-like manifold according to a star graph \(M_v\) with central vertex \(v\) and \(e \in E_e\) adjacent edges isometric with \(M_e = [0, \ell_e]\) with vertices \(v_e\) of degree 1: The core vertex neighbourhood is scaled as \(\varepsilon X_v\), the edge neighbourhood as \(X_{v,e} = [0, \ell_e] \times \varepsilon Y_e\) (the notation is explained in the paragraph before Corollary 4.9).

**B.3. Proposition.** Assume that

\[
0 < \ell_0 \leq \ell_e \leq \ell_\infty < \infty \quad \text{for all } e \in E.
\]

Then there exists a constant \(C_v\) depending only on \(\tilde{\ell} = \ell_\infty / \ell_0\) and upper estimates on the unscaled quantities \(\text{vol} X_v / (\sum_{e \in E_v} \text{vol} Y_e), 1/\lambda_2(Y_e)\) and \(1/\lambda_2(X_v)\) such that

\[
\lambda_2(X_{v,e}) \geq \frac{1}{\tilde{\ell}_\infty^2} \quad \text{for all } 0 < \varepsilon \leq \varepsilon_0 := \frac{\ell_0}{C_v^2}.
\]

**Proof.** We use a simple scaling argument to replace the parameters \(\ell_e\) and \(\varepsilon\) by \(\ell_e/\ell_0 \in [1, \tilde{\ell}]\) and \(\kappa = \varepsilon/\ell_0\). Denote by \(\ell_0^{-1} X_{v,e}\) the scaled graph-like manifold (with metric \(\ell_0^{-2} g_{v,e}\)) and similarly denote by \(\ell_0^{-1} M_v\) the metric graph with edge length \(\ell_e/\ell_0\). Now, the edge lengths are in \([1, \tilde{\ell}]\), and we can apply the convergence result for graph-like manifolds, proven e.g. in [EP05], giving us

\[
|\lambda_k(\ell_0^{-1} X_{v,e}) - \lambda_k(\ell_0^{-1} M_v)| \leq C_v \kappa^{1/2}
\]

(the error estimate \(\kappa^{1/2}\) is proven e.g. in [Pos12] Thm. 6.4.1 and Thm. 4.6.4]; note that \(\kappa\) is the transversal thickness, called \(\varepsilon\) in the cited works). Since

\[
\lambda_k(\ell_0^{-1} X_{v,e}) = \ell_0^2 \lambda_k(X_{v,e}) \quad \text{and} \quad \lambda_k(\ell_0^{-1} M_v) = \ell_0^2 \lambda_k(M_v)
\]

we obtain

\[
\lambda_2(X_{v,e}) \geq \lambda_2(M_v) - \frac{C_v}{\tilde{\ell}_0^2} \left( \frac{\varepsilon}{\ell_0} \right)^{1/2} \geq \frac{2}{\tilde{\ell}_0^2} - \frac{C_v}{\tilde{\ell}_0^2} \left( \frac{\varepsilon}{\ell_0} \right)^{1/2}
\]

using Lemma 3.4 for the last estimate. Choosing \(0 < \varepsilon \leq \varepsilon_0 := C_v^{-2} \ell_0\) and \(\ell_0 \leq \ell_\infty\) we obtain the desired result.

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