Differential rotation of the neutron star polar cap

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Abstract. The flow in the polar cap region of the neutron star is caused by magnetospheric electric current flow in the pulsar tube due to current closure in the crust and partially in the liquid surface layer. In this paper we consider the flow velocity in case of aligned pulsar with homogeneous vertical magnetic field. Changing of the liquid density and the conductivity with depth is taken into account.

1. Introduction
At present current losses are considered to be the main mechanism of the pulsar braking [1, 2]. According to this mechanism there are electric currents that flow along open field lines in the magnetosphere. When these currents enter the neutron star they may start closing and hence flowing across magnetic field lines. The latter generates the Lorentz force applied to neutron star matter and transfers momentum flux from magnetosphere to stellar matter and thus brakes the star [3]. In this paper we assume that a liquid layer (ocean) is located on the neutron star surface [4] and consider the flow at the ocean surface caused by electric current closure. Exact solution of MHD equations in case of axisymmetric flow with constant liquid density and conductivity was found in [5, 6]. Here we consider approximate solution in case of aligned pulsar with homogeneous vertical magnetic field and with taking into account the changing of density and conductivity with depth.

2. Model
We assume that the depth $L$ of the ocean is small, $L \ll r_{ns}$, where $r_{ns}$ is neutron star radius. Hence we consider the problem in plane-parallel approximation. We assume that stellar ocean is infinite flat layer $-L < z < 0$, rigid crust lies at $z < -L$ and force free magnetosphere corresponds to values $z > 0$, see figure 1. We consider the flow in the frame of reference corotating with the neutron star and assume that the flow is stationary in this frame of reference. Hence in the ocean we can consider stationary MHD equations and write them in the form

\begin{equation}
2 \left[\vec{\Omega} \times \vec{v}\right] + \left(\vec{v} \cdot \nabla\right)\vec{v} = -\frac{\nabla p}{\rho} + \frac{1}{\rho c} \left[\vec{j} \times \vec{B}\right] + \frac{\vec{F}_{vis}}{\rho} + \vec{g}
\end{equation}

\begin{equation}
\text{div}(\rho\vec{v}) = 0, \quad \text{div}\vec{B} = 0 \quad \text{and} \quad \text{rot}\vec{B} = \frac{4\pi}{c} \vec{j}
\end{equation}

\begin{equation}
-\nabla \Phi + \frac{1}{c} \left[\vec{v} \times \vec{B}\right] = R_{||}\vec{j}_{||} + R_{\perp}\vec{j}_{\perp} - R_{H} \left[\vec{c}_{B} \times \vec{j}_{\perp}\right]
\end{equation}
where $\vec{v}$ is the fluid velocity, $\rho$ is its density, $p = p(\rho)$ is its pressure, $\vec{g} = -g\vec{e}_z$ is the gravity force; we assume that $g$ is constant and $g > 0$, $\Omega = \Omega_0\vec{e}_z$ is the angular velocity of star rotation, $\Omega = 2\pi/P$, $P$ is its period, $\vec{B}$ is the magnetic field, $\vec{e}_B = \vec{B}/B$, $\vec{j}$ is electric current density, $j_{||} = \vec{j} \cdot \vec{e}_B$, $j_{\perp} = \vec{j} - \vec{e}_B j_{||}$, $\Phi$ is the electrostatic potential, $R_{||}$, $R_{\perp}$, $R_H$ are the liquid resistivities, $\vec{F}_{vis}$ is the viscous force, $\vec{F}_{vis} = \sigma_{vis}^{\alpha\beta} \vec{e}_\alpha \vec{e}_\beta$, $\sigma_{vis}^{\alpha\beta}$ is anisotropic viscous stress tensor in the magnetic field, taken from [7]; it is determined by five shear visosity coefficients $\eta_0 - \eta_4$ and two bulk visosity coefficients $\zeta$ and $\zeta_1$.

When magnetospheric current is absent we assume that star is in hydrostatic equilibrium $\vec{v} = 0$. Hence, equations (1-3) can be written as

$$\nabla p^{(0)} = -\rho^{(0)} g\vec{e}_z, \quad \nabla \Phi^{(0)} = 0, \quad \text{div}\vec{B}^{(0)} = 0 \quad \text{and} \quad \text{rot}\vec{B}^{(0)} = 0$$

(4)

where values corresponding to hydrostatic equilibrium are marked by index (0). To simplify equations we assume that $\vec{B}^{(0)} = B^{(0)}\vec{e}_z$, $B^{(0)} = \text{const}$. Also we assume that $p^{(0)}$, $\rho^{(0)}$, $R_{||}$, $R_{\perp}$, $R_H$ and viscosity coefficients $\eta_0 - \eta_4$, $\zeta^{(0)}$, $\zeta_1^{(0)}$ depend only on $z$ coordinate. Now let us consider small perturbations caused by the current from the magnetosphere: $p = p^{(0)} + \delta p$, $\rho = \rho^{(0)} + \delta \rho$, $\Phi = \Phi^{(0)} + \delta \Phi$ and $\vec{B} = \vec{B}^{(0)} + \delta \vec{B}$. Values $\vec{v}$ and $\vec{j}$ are also considered as small perturbations. Equations (1-3) linearized over perturbations can be written as

$$2 \left[ \Omega \times \vec{v} \right] = -\nabla \delta p - g\vec{e}_z \delta \rho + \frac{B^{(0)}}{\rho^{(0)}} \left[ \vec{j} \times \vec{e}_z \right] + \frac{\vec{F}_{vis}}{\rho^{(0)}}$$

(5)

$$\text{div} \delta \vec{B} = 0 \quad \text{and} \quad \text{rot} \delta \vec{B} = \frac{4\pi}{c} \vec{j}$$

(6)

$$-\nabla \delta \Phi + \frac{B^{(0)}}{c} \left[ \vec{v} \times \vec{e}_z \right] = R_{||}^{(0)} \vec{j}_z + R_{\perp}^{(0)} \vec{j}_{\perp} - R_H^{(0)} \left[ \vec{e}_z \times \vec{j}_{\perp} \right]$$

(7)

where $j_{\perp} = \vec{j} - j_z \vec{e}_z$ and $\sigma_{vis}^{\alpha\beta}$ contain $\eta_0 - \eta_4$, $\zeta^{(0)}$, $\zeta_1^{(0)}$. Let us consider a star with surface magnetic field $B^{(0)} = 10^{12}$ G and period $P = 1$ s. We assume that $L \sim 10^2$ m, $\rho \sim 10^6$ g / cm$^3$ [4], $\eta_0 \sim 10^4$ g / cm s [8, 9], $R_{||} \sim 10^{-13}$ CGS and $R_{\perp} \sim 10^{-15}$ CGS [10]. Hence, Ekman number is $E^{-1} = \Omega L^2 \rho/\eta_0 \sim 10^{14}$ and Hartman number can be estimated as...
$H a = (B^2 L^2 / (\eta R L \rho_c^2))^{1/2} \sim 10^{11}$. Because of $H a^2 \gg \eta^{-1}$, 1 outside thin $L / H a \sim 10^{-7}$ cm boundary layers equation (5) may be replaced by [11]

$$\frac{B^{(0)}}{c} [\vec{j} \times \vec{e}_z] = \nabla \delta p + \vec{e}_z \delta \rho$$

Within the boundary layers we assume that all perturbations depend only on coordinate $z$ and unperturbed values are constants.

At $z > 0$ the force free magnetosphere with extremely rarefied matter is assumed. We assume that $\vec{j} = J_{ext} \vec{e}_z$ is here. Hence using equations (6) we obtain $\partial j_{ext} / \partial z = 0$ and $j_{ext} = J_{ext}(r, \phi)$ here [5], where $(r, \phi, z)$ is cylindrical coordinate system. Because of $\rho \approx 0$ at $z > 0$ we demand $v_z = 0$ and $\partial v_r / \partial z = 0$, $\partial v_\phi / \partial z = 0$ at boundary $z = 0$ [5]. At $z < -L$ we have rigid crust, there we assume $\vec{v}_0 = 0$, thus $-\nabla \delta \Phi = R_{||} (0) \vec{j}_z + R_{||} (0) \vec{j}_{\perp} - R_{H} (0) [\vec{e}_z \times \vec{j}_{\perp}]$ is here. In order to obtain analytical solution we assume that crust resistivities depend on $z$ as $R_{A}^{(0)} = R_{A} e^{\beta (z + L)}$, $A = ||, \perp, H$. At $z = -L$ we demand $\vec{v}_0 = 0$. Also it is assumed that $J_z$, $\delta \Phi$ and $\delta \vec{B}$ are continuous on both boundaries $z = 0$ and $z = -L$ and all perturbation are zero at $r \to +\infty$ and at $z \to -\infty$. We also demand $\delta B_0 \to 0$, $\alpha = r, z$, at $z \to +\infty$.

Using the approximations described above and neglecting corrections like $\sim 1 / H a$ we have solved equations (6-8) and found that the liquid velocity at the star surface $z = 0$ is equal to

$$\vec{v} = (v_r' + \bar{v}_r) \vec{e}_r + (v_\phi' + \bar{v}_\phi) \vec{e}_\phi$$

where $v_r'$ and $v_\phi'$ correspond to contribution of the stellar ocean to the surface velocity

$$v_r' = S' \cdot \frac{1}{r} \frac{\partial j_{ext}}{\partial \phi} , \quad v_\phi' = -S' \cdot \frac{\partial j_{ext}}{\partial r} , \quad \text{where} \quad S' = \frac{c}{B^{(0)}} \int_{-L}^{0} R_{||} (z) dz$$

and $\bar{v}_r$ and $\bar{v}_\phi$ correspond to contribution of the rigid crust to the surface velocity

$$\bar{v}_r = \sum_{m=-\infty}^{+\infty} i m e^{i m \phi} \int_{0}^{+\infty} k \gamma_R^{(0)} R_{||} \cdot \frac{c}{B^{(0)}} j_{ext}(k, m) \cdot \frac{J_m(kr)}{kr} \ kdk$$

$$\bar{v}_\phi = -\sum_{m=-\infty}^{+\infty} e^{i m \phi} \int_{0}^{+\infty} k \gamma_R^{(0)} R_{||} \cdot \frac{c}{B^{(0)}} j_{ext}(k, m) \cdot J_m'(kr) \ kdk$$

where

$$\gamma(k) = \frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + \frac{k^2 R_{||} R_{||} + R_{H}^2}{R_{||}^2 + R_{H}^2}}$$

$$\tilde{j}_{ext}(k, m) = \int_{0}^{+\infty} r dr \int_{0}^{2\pi} \frac{d\phi}{2\pi} \left( j_{ext}(r, \phi) J_m(kr) e^{-im\phi} \right)$$

In case of $\beta \ell \gg 1$, where $\ell \sim L$ is scale of current $j_{ext}$ changing, it may be rewritten as

$$\bar{v}_r = S \cdot \frac{1}{r} \frac{\partial j_{ext}}{\partial \phi} , \quad \bar{v}_\phi = -S \cdot \frac{\partial j_{ext}}{\partial r} , \quad \text{where} \quad S = \frac{c}{B^{(0)}} \cdot \frac{R_{||}}{\beta}$$

Contributions of boundary layers to the surface velocity give corrections $\sim 1 / H a$, so we neglect them.
3. Discussion
We consider approximate solution of MHD equations and find expression of the flow velocity on the ocean surface. In case of axisymmetric flow and constant density and resistivities it coincides with result [5] upto correction of order $\sim 1/H_a$. In case of magnetospheric current $\vec{j}_{ext}$ depending on $\phi$ it gives the same order of the flow velocity $v \sim 10^{-10} - 10^{-8}$ cm/s at ordinary radio pulsar parameters $B^{(0)} \sim 10^{12}$ G and $P \sim 1$ s. Also it is worth to note that as well as in [5] the flow is almost force free $[\vec{j} \times \vec{B}] \approx 0$ and almost all electric current closes in rigid crust. Thus, braking torque is transferred directly from the magnetosphere to the rigid crust without significant participation of viscous stresses.

In the paper, in order to obtain analytical solution, we assume plane geometry, neglect centrifugal force and consider only the case of homogeneous vertical magnetic field. This approximation is correct in case of slow rotating aligned pulsar $P \sim 1$ s with dipolar magnetic field if electric current closes not so far from the polar cap. The current closure in case of spherical geometry with dipolar magnetospheric magnetic field, entirely rigid star and more realistic resistivity profile was considered by [3]. In [3] it is shown that electric current closes at distance $\sim 100 - 300$ m from polar cap in case of pulsars with period $P \sim 1$ s.

Acknowledgments
We sincerely thank A.I. Chugunov for comments and useful discussions.

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