\textbf{\tau\,-INVIARNTS FOR KNOTS IN RATIONAL HOMOLOGY SPHERES}

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\textbf{Abstract.} Ozsváth and Szabó used the knot filtration on $\widehat{CF}(S^3)$ to define the $\tau$-invariant for knots in $S^3$. In this article we generalize their construction and define a collection of invariants $\{\tau_s(Y,K)\}_{s \in \text{Spin}^c(Y)}$ for rationally null-homologous knots in rational homology spheres. We also show that these invariants can be used to obtain a lower bound on the genus of a surface with boundary $K$ properly embedded in a negative definite 4-manifold with boundary $Y$.

1. Introduction

The $\tau$-invariant for knots in $S^3$ as defined by Ozsváth and Szabó has proven to be a useful and robust invariant for studying knot concordance. A key property of this invariant is that $\tau(K)$ is a lower bound for the 4-ball genus of $K$ \cite{OS03b}. In this paper, we introduce a generalization, a collection of rational numbers $\{\tau_s(Y,K)\}_{s \in \text{Spin}^c(Y)}$ associated to a knot $K$ in a rational homology sphere $Y$. Following the construction of $\tau(K)$ for $K \subset S^3$, we define $\tau_s(Y,K)$ in terms of the filtration on the Heegaard Floer chain complex coming from the knot. The identification of this filtration with rational numbers relies on Ni's definition of the Alexander grading \cite{Ni09}. We also show that these invariants satisfy properties similar to the original $\tau$-invariant. A different definition of $\tau_s(Y,K)$ has recently been given by Ni and Vafaee in \cite{NV16}, however our invariant satisfies the additional property that $\tau_s(-Y,K) = -\tau_s(Y,K)$ which will be important for Corollary 5.4. In addition, $\tau$-invariants for knots in lens spaces have also been investigated recently by Celoria \cite{Cel16}.

Our main theorem concerns a knot $K$ in the boundary of a negative definite 4-manifold $W$. We will not necessarily assume that our knot bounds a properly embedded surface in $W$, although the result simplifies in this case. In general, there exists an integer $p$ such that $p[K] = 0 \in H_1(W;\mathbb{Z})$, and we can consider a surface $\Sigma$ embedded in $W - \nu(K)$ with $[\partial \Sigma] = p[K] \in H_1(\nu(K);\mathbb{Z})$. We call $\Sigma$ a \textit{p-slicing surface} or \textit{rational slicing surface} for $K$. This definition is a 4-dimensional version of the notion of a rational Seifert surface for a knot in a 3-manifold.

A \textit{rational $q$-Seifert surface} for a rationally null-homologous knot $K$ in a 3-manifold $Y$ is a surface $F$ embedded in $Y - \nu(K)$ such that $[\partial F] = q[K] \in H_1(\nu(K);\mathbb{Z})$. In particular, $q$ is a multiple of the order of $K$. Calegari and Gordon define the rational genus of a knot to be

$$||K|| = \inf \frac{-\chi(F)}{2p}$$

where the infimum is taken over all $p$ and all $p$-Seifert surfaces without sphere components and, they obtain lower bounds on the rational genus for many knots in...
3-manifolds [CG13]. In addition, using Heegaard Floer type invariants, Ni [Ni09] and Ni and Wu [NW14] have found further bounds on the rational genus. In particular, Ni showed that the knot Floer homology of a knot in a rational homology sphere detects the rational genus [Ni09]. If \( A_{\text{max}} \) is the maximum Alexander grading of an element for which \( \widehat{HFK}(Y, K) \) is non-zero, then

\[
\|K\| = A_{\text{max}} - \frac{1}{2}.
\]

This raises the question, then, what happens when we consider \( Y \) as the boundary of a 4-manifold? Any \( q \)-Seifert surface can be pushed into the 4-manifold to obtain a \( q \)-slicing surface. However, it is also possible that \( K \) has a \( p \)-slicing surface for which \( p < q \). This makes the relationship between the genera of rational Seifert surfaces and rational slicing surfaces for \( K \) more subtle than the relationship between the 3-genus and 4-genus of a knot in \( S^3 \).

Let \( W \) be a negative definite 4-manifold with \( \partial W = Y \) a rational homology sphere. If \( K \subset Y \) is a knot of order \( q \in H_1(Y; \mathbb{Z}) \), we can form a new manifold \( W - n(K) \) by attaching a 2-handle to \( W \) along \( K \) with respect to a particular framing which we will define in Section 2.6. Our main theorem is the following:

**Theorem 1.1.** Let \( \Sigma \) be a rational \( p \)-slicing surface for \( K \). If \( n \) is sufficiently large, then for any embedded surface \( S \) such that \( [S] = [\Sigma \cup pC] \in H_2(W - n(K)) \) and any \( t \in \text{Spin}^c(W) \) such that \( t|_Y = s \) we have,

\[
\frac{1}{pq} \langle c_1(t), [q\Sigma \cup pF] \rangle + \frac{1}{pq^2} [q\Sigma \cup pF]^2 + 2\tau_s(Y, K) \leq -\frac{\chi(S)}{p} + 2 + (p - 1) \frac{(cr - nq)}{q}
\]

where \( F \) is a \( q \)-Seifert surface for \( K \) and \( \frac{cr}{q} \) is the unique representative of the rational self-linking number of \( K \) in \([0, 1)\).

When \( p = 1 \), and \( K \) is null-homologous in \( W \), we have the following corollary:

**Corollary 1.2.** Let \( F \) be a \( q \)-Seifert surface for \( K \). Then for any surface \( \Sigma \) such that \( \partial \Sigma = K \), and any \( s \) that extends over \( W \)

\[
\frac{1}{q} \langle c_1(t), [q\Sigma \cup F] \rangle + \frac{1}{q^2} [q\Sigma \cup F]^2 + 2\tau_s(Y, K) \leq 2g(\Sigma).
\]

We begin in Section 2 with the construction of the collection \( \{\tau_s(Y, K)\}_{s \in \text{Spin}^c(Y)} \) and conclude with a method for computing the Alexander grading from a Heegaard diagram. In Section 3 we turn to understanding how these invariants transform under change of orientation of \( Y \) and \( K \) as well as conjugation of \( s \) and connected sums. Section 4 is devoted to describing the relationship between the knot filtration and certain maps on Floer homology determined by cobordisms between 3-manifolds. In Section 5 we give the proof of Theorem 1.1. Finally in Section 6 we compute some explicit examples of \( \tau \)-invariants for knots in lens spaces.

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2. The definition of $\tau$

In order to define the $\tau$-invariant, we need to recall some facts about Heegaard Floer homology. Much of the set up is described in more detail in [OS11].

2.1. Heegaard diagrams for $(Y,K)$. A doubly pointed Heegaard diagram for an oriented knot $K$ in a 3-manifold $Y$ is determined by $(\Sigma, \alpha, \beta, w, z)$ where $\Sigma$ is a genus $g$-surface, $\alpha$ and $\beta$ are $g$-tuples of linearly independent curves on $\Sigma$ and $w$ and $z$ are base-points on $\Sigma$ which lie in the complement of the $\alpha$ and $\beta$-curves. The surface $\Sigma$ together with these sets of curves, determine a Heegaard splitting of $Y = U_\alpha \cup_\Sigma U_\beta$. As shown in [OS11] we can always construct such a doubly pointed Heegaard diagram so that the base-points $w$ and $z$ determine the knot $K \subset Y$ with its orientation as the union of two flow lines $\gamma_z - \gamma_w$.

It is also convenient to view $K$ as an oriented knot directly on the surface of $\Sigma$. According to the orientation convention described above, $K$ is the union of two oriented curves $\nu_\alpha$ and $\nu_\beta$ oriented so that $\nu_\alpha$ goes from $z$ to $w$ avoiding $\alpha$-curves and $\nu_\beta$ from $w$ to $z$ avoiding $\beta$.

In addition, we may assume that the final $\beta$-curve is the meridian of our knot and intersects only one of the $\alpha$-curves. In this case all of the generators of the Heegaard Floer chain complex, $\widehat{CF}(Y)$ of the form $x = (x_1, \ldots, x_g, p)$ where $p$ is the unique intersection point between $\alpha_g$ and $\beta_g$.

2.2. Relative spin$^c$-structures. The correspondence between spin$^c$-structures and homology classes of non-vanishing vector fields introduced by Turaev [Tur97], generalizes to 3-manifolds $M$ with torus boundary components as described in [OS11]. Specifically, when $M = Y - \nu(K)$, we define a relative spin$^c$-structure to be a homology class of non-vanishing vector field $v$ on $Y - \nu(K)$ subject to the condition that $v$ points outward on the boundary of $Y - \nu(K)$. There is also an affine correspondence between relative spin$^c$-structures and classes in $H^2(Y - \nu(K), \partial \nu(K)) = H^2(Y, K)$ which is analogous to the correspondence between Spin$^c(Y)$ and $H^2(Y)$. We denote the set of relative spin$^c$-structures on $Y - \nu(K)$ by Spin$^c(Y, K)$.

In addition, there is a natural “filling” map

$$G_{Y,K} : \text{Spin}^c(Y, K) \to \text{Spin}^c(Y).$$

Geometrically, if we let $v_\xi$ be a vector field on $Y - \nu(K)$ pointing outward on the boundary, representing $\xi$, we can visualize this map by extending $v_\xi$ over $\nu(K)$ in a specific way. Let $v_{\nu(K)}$ be the unique homotopy class of vector fields on $\nu(K) \cong S^1 \times D^2$ that points inward on the boundary and smoothly extends over each disk $p \times D^2$ so that it is everywhere transverse to $D^2$ and has $K$ as a closed orbit. Then $G_{Y,K}(\xi)$ is the homology class of the vector field on $Y$ that restricts to $v_\xi$ on $Y - \nu(K)$ and $v_{\nu(K)}$ on $\nu(K)$.

The filling map is equivariant with respect to the action of $H^2$, in the sense that given an element $\alpha \in H^2(Y, K)$ and the usual map $i^* : H^2(Y, K) \to H^2(Y)$, we have

$$G_{Y,K}(\xi + \alpha) = G_{Y,K}(\xi) + i^*(\alpha).$$

It is important to note that this map $G_{Y,K}$ depends on the orientation of $K$. In particular, if $K^\tau$ denotes $K$ with reverse orientation, we have

$$G_{Y,K^\tau}(\xi) = G_{Y,K}(\xi) - PD[K].$$
2.3. The Alexander grading and Alexander filtration. Let $K$ be a knot in $Y$ and $(\Sigma, \alpha, \beta, w, z)$ be a corresponding doubly pointed Heegaard diagram. Then the set of relative spin$^c$-structures determines a filtration of the chain complex $\widehat{CF}(Y)$ via a map

$$\sigma_{w,z} : T_\alpha \cap T_\beta \rightarrow \text{Spin}^c(Y,K).$$

The construction of this map is described in [OS11, Section 2.4] and is similar to the construction of the map $\sigma_{w} : T_\alpha \cap T_\beta \rightarrow \text{Spin}^c(Y)$.

When $K$ is a null-homologous knot in $Y$, the filtration levels can be associated with the integers via the Alexander grading [OS04a, Ras03]. Later, Ni introduced a generalized Alexander grading for knots in rational homology spheres which takes values in the rational numbers instead of the integers.

**Definition 2.1 (Ni09).** Let $K$ be a knot in a rational homology sphere $Y$ with corresponding doubly pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$. Fix a rational Seifert surface $F$ for $K$. For an intersection point $x \in T_\alpha \cap T_\beta$ the Alexander grading of $x$ is given by

$$A(x) = \frac{1}{2[\mu] \cdot [F]} (\langle c_1(\sigma_{w,z}(x)), [F] \rangle - [\mu] \cdot [F]).$$

Note that when $K$ is null-homologous, this coincides with the definition of the Alexander grading given by Rasmussen and Ozsváth and Szabó. More generally, a pairing of this sort will exist for any rationally null-homologous knot $K$ in a 3-manifold. However, when $Y$ has $b_1 > 0$, the pairing will depend on the homology class of the rational Seifert surface $[F]$. This has been studied for null-homologous knots by Hedden [Hed08]. In this paper, we will focus on knots in rational homology spheres.

The Alexander grading gives rise to a $\mathbb{Q}$-filtration of the Floer chain complex where $\mathcal{F}_q = \{x \in \widehat{CF}(Y) | A(x) \leq q\}$ for each $q \in \mathbb{Q}$. Additionally, since $\widehat{CF}(Y)$ splits over spin$^c$-structures, the filtration splits as well. To define $\tau_\mathfrak{s}(Y,K)$, we will be interested in the restriction of this filtration to a particular $\widehat{CF}(Y,\mathfrak{s})$ summand. Specifically, we will see that for each $\mathfrak{s} \in \text{Spin}^c(Y)$, the $\mathbb{Q}$-valued Alexander filtration of $\widehat{CF}(Y,\mathfrak{s})$ can, in fact, be thought of as a $\mathbb{Z}$-filtration.

To this end, we fix $\mathfrak{s} \in \text{Spin}^c(Y)$, and consider $G_{Y,K}^{-1}(\mathfrak{s})$. If $\xi_1$ and $\xi_2$ are two lifts of $\mathfrak{s}$, then $\xi_2 - \xi_1 \in H^2(Y,K)$ is well a well defined element of $\ker(H^2(Y,K) \rightarrow H^2(Y))$. Thus, $\xi_2 - \xi_1 = m \text{PD}[\mu]$ for some $m \in \mathbb{Z}$.

If we consider the pairing from the Alexander grading, we can see that there exists a unique rational number $k_\mathfrak{s} \in [-\frac{1}{2}, \frac{1}{2})$ depending only on $\mathfrak{s}$ such that for each $\xi \in G_{Y,K}^{-1}(\mathfrak{s})$ there is a unique $m$ satisfying

$$\frac{1}{2[\mu] \cdot [F]} (\langle c_1(\xi), [F] \rangle - [\mu] \cdot [F]) = k_\mathfrak{s} + m.$$ 

In particular, there is a unique such relative spin$^c$-structure with $m = 0$. We call this $\xi_0^\mathfrak{s}$.

Alternatively, and equivalently we can think of $\xi_0^\mathfrak{s}$ as the unique relative spin$^c$-structure such that

$$0 \leq \langle c_1(\xi_0^\mathfrak{s}), [F] \rangle < 2[\mu] \cdot [F].$$
Now, for any \( x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \) such that \( s = s_m(x) = G_{Y,K} (\xi^s_0) \) we know that 
\[ s_{x,i}(x) = \xi^s_0 + m \text{PD} [\mu] \] 
for some integer \( m \). So we can write \( A(x) = k_s + m \) where

\[ k_s = \frac{1}{2|\mu| \cdot [F]} \langle (c_1(\xi^s_0), [F]) - [\mu] \cdot [F] \rangle. \]

Notice that \( k_s \) depends on the orientation of \( K \). In particular, we emphasize that \( F \) is a rational Seifert surface that inherits its orientation from \( K \).

Now we can easily see that the filtration coming from the Alexander grading is really a \( \mathbb{Z} \)-filtration of \( \hat{CF}(Y, s) \) where

\[ \mathcal{F}_{s,m} = \{ x \in \hat{CF}(Y, s) \mid A(x) - k_s = m \} \]

and the restriction of \( \mathcal{F}_q \) to a particular spin\(^c\)-structure will be \( \mathcal{F}_{k_s+m} \) where \( k_s + m \leq q \).

We will first define \( \tau_s(Y, K) \) in the simpler case that \( Y \) is an \( L \)-space. Recall that for \( Y \) an \( L \)-space, each \( \hat{HF}(Y, s) \) is generated by a single element. In this case, we can define \( \tau_s(Y, K) \) to be the minimum filtration level \( k_s + m \) such that the map 
\[ i_{m, s} : \mathcal{F}_{s,m} \rightarrow \hat{CF}(Y, s) \]
induces an isomorphism on homology.

Note that when \( Y \) is an \( L \)-space, there is no gap between the \( m \) for which \( i_m \) induces a nontrivial map on homology and the \( m \) for which \( i_m \) induces a surjective map on homology. In the general case, however, this gap may exist because \( \hat{HF}(Y, s) \) may be a vector space of dimension greater than one. In this setting, we will need to use more of the structure of \( CF^\infty(Y, s) \) to define \( \tau_s(Y, K) \).

### 2.4. Some facts about \( CF^\infty(Y) \)

Fix a Heegaard diagram \((\Sigma, \alpha, \beta, w)\) for \( Y \). Recall that \( CF^\infty(Y, s) \) is freely generated by elements \([x, i]\) where \( x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \) and \( i \in \mathbb{Z} \) and the sub-complex \( CF^-(Y) \) is obtained by restricting to the pairs \([x, i]\) where \( i < 0 \). These fit into a short exact sequence

\[ 0 \rightarrow CF^-(Y, s) \xrightarrow{\partial} CF^\infty(Y, s) \xrightarrow{\pi} CF^+(Y, s) \rightarrow 0 \]

where \( CF^+(Y) \) is the quotient complex.

In addition, we have a chain map \( U : CF^\infty(Y, s) \rightarrow CF^\infty(Y, s) \) defined by \([x, i] \mapsto [x, i - 1]\) which induces an isomorphism on homology. The \( U \) map allows us to view \( CF^\infty(Y, s) \) as a finitely generated \( \mathbb{F}[U, U^{-1}] \)-module. If we consider \( U \) restricted to either \( CF^-(Y, s) \) or \( CF^+(Y, s) \), we can see that it does not induce an isomorphism on homology. In fact, the complex \( \hat{CF}(Y, s) \) is defined as the kernel of \( U \) applied to \( CF^+(Y, s) \). Thus we have another short exact sequence:

\[ 0 \rightarrow \hat{CF}(Y, s) \xrightarrow{\partial} CF^+(Y, s) \xrightarrow{U} CF^+(Y, s) \rightarrow 0. \]

We can also view \( \hat{CF}(Y, s) \) as a cokernel. For any \( n \in \mathbb{Z} \), let \( CF^{<n}(Y, s) \) be the sub-complex of \( CF^\infty(Y, s) \) generated by \([x, i]\) where \( i < n \). Then taking \( n = 1 \) we see that \( \hat{CF}(Y, s) \) also fits into the following short exact sequence:

\[ 0 \rightarrow CF^{<1}(Y, s) \xrightarrow{U} CF^{<1}(Y, s) \xrightarrow{\psi} \hat{CF}(Y, s) \rightarrow 0. \]
In addition, note that $U$ maps $CF^{<1}(Y, s)$ isomorphically onto $CF^{-}(Y, s)$. In particular, we have an isomorphism of short exact sequences:

$$
\begin{array}{c}
0 \rightarrow CF^{<1}(Y, s) \xrightarrow{i} CF^{\infty}(Y, s) \xrightarrow{\pi} CF^{\geq 1}(Y, s) \rightarrow 0 \\
0 \rightarrow CF^{-}(Y, s) \xrightarrow{\iota} CF^{\infty}(Y, s) \xrightarrow{\pi} CF^{+}(Y, s) \rightarrow 0.
\end{array}
$$

Let $HF^{<n}(Y, s)$ denote the homology of the complex $CF^{<n}(Y, s)$. Then, putting the above information together, we see that

$$
\psi_{*} \, \hat{\iota}_{*} \, \pi_{*} \, \partial_{*}
$$

commutes.

2.5. The definition of $\tau_{s}(Y, K)$. If we fix a spin$^c$-structure $s \in \text{Spin}^{c}(Y)$, then for each $m \in \mathbb{Z}$ we can form the short exact sequence:

$$
0 \rightarrow F_{s,m} \xrightarrow{i_{m}} \widehat{CF}(Y, s) \xrightarrow{p_{m}} Q_{s,m} \rightarrow 0
$$

where $Q_{s,m}$ is the quotient. We denote the maps induced on homology by $i_{m}$ and $p_{m}$ as $I_{m}$ and $P_{m}$ respectively.

**Definition 2.2.** Let $K \subset Y$ be a knot in a rational homology sphere. Then

$$
\tau_{s}(Y, K) := \min\{k_{s} + m \mid \text{Im}(\rho_{*} \circ I_{m}) \cap \text{Im}(\pi_{*}) \neq 0\}
$$

where $\rho_{*} : \widehat{HF}(Y, s) \rightarrow HF^{+}(Y, s)$ and $\pi_{*} : HF^{\infty}(Y, s) \rightarrow HF^{+}(Y, s)$ are the maps induced on homology by $\rho$ and $\pi$.

Putting together several exact triangles, we form the diagram

$$
\begin{array}{ccc}
HF^{<1}(Y, s) & \xrightarrow{\psi_{*}} & \widehat{HF}(Y, s) \\
& \searrow \swarrow \partial' & \\
& U & \\
& \scriptstyle{I_{*}} & \scriptstyle{\pi_{*}} \searrow \\
& \scriptstyle{HF^{<1}(Y, s) \cong HF^{-}(Y, s)} & \rightarrow \\
& \scriptstyle{HF^{\infty}(Y, s)} & \rightarrow
\end{array}
$$

which commutes.

**Lemma 2.3.** Fix $s \in \text{Spin}^{c}(Y)$ and $m \in \mathbb{Z}$. Then the following are equivalent:

1. There exists $\beta \in H_{*}(F_{s,m})$ such that $\rho_{*} \circ I_{m}(\beta) \neq 0 \in \text{Im}(\pi_{*})$.
2. There exists $\alpha \in HF^{<1}(Y, s)$ which is non-torsion and $\beta \in H_{*}(F_{s,m})$ such that $I_{m}(\beta) = \psi_{*}(\alpha) \neq 0$. 


There exists $\alpha \in HF^{<1}(Y,s)$ such that $\pi_* \circ i_*(\alpha) \neq 0$ and $P_m(\psi_*(\alpha)) = 0$.

**Proof.** Suppose there exists $\beta \in H_*(F_{s,m})$ such that $\rho_* \circ I_m(\beta) \neq 0 \in \text{Im}(\pi_*)$. Then by commutativity of the diagram and exactness, $\partial \circ \rho_* \circ I_m(\beta) = \partial' \circ I_m(\beta) = 0$. Thus, there exists some element $\alpha \in HF^{<1}(Y,s)$ so that $I_m(\beta) = \psi_*(\alpha)$. On the other hand, $\rho_* \circ I_m(\beta) = \rho_* \circ \psi_*(\alpha) = \pi_* \circ i_*(\alpha)$ so we must have that $\alpha$ is non-torsion.

Now assume (2). Since $\alpha$ is non-torsion, and $I_m(\beta) = \psi_*(\alpha) \neq 0$, we must have that $\pi_* \circ i_*(\alpha) \neq 0$. Thus, $\rho_* \circ \psi_*(\alpha) = \rho_* \circ I_m(\beta) \neq 0 \in \text{Im}(\pi_*)$. Thus, $\psi_*(\alpha) \in \ker(P_m)$.

Finally, (3) implies (1) since $\psi_*(\alpha) \neq 0$ and $\psi_*(\alpha) \in \ker(P_m) = \text{Im}(I_m)$. \qed

**Definition 2.4.** Let $K \subset Y$ be a knot in a rational homology sphere. Then

$$\tau_-(Y,K) := \max_{\alpha \in HF^{<1}(Y,s) \atop \pi_* \circ i_*(\alpha) \neq 0} \{k_s + m \mid P_m \circ \psi_*(\alpha) \neq 0\}$$

The two definitions are related in the following way:

**Proposition 2.5.** Let $Y$ be a rational homology 3-sphere, and $K \subset Y$. Then for each $s \in \text{Spin}^c(Y)$,

$$\tau_-(Y,K) = \tau_s(Y,K) - 1.$$

**Proof.** Suppose $k_s + m = \tau_s(Y,K)$. Then by Lemma 2.3, $k_s + m' = k_s + m - 1$ must be the maximum for which every $\alpha \in HF^{<1}(Y,s)$ such that $\pi_* \circ i_*(\alpha) \neq 0$ satisfies $P_m \circ \psi_*(\alpha) \neq 0$. \qed

2.6. **Framings for rationally null-homologous knots.** For any oriented knot $K$ in any 3-manifold there is a well defined meridian $\mu_K$, namely the homology class of the curve that generates the kernel of the map $H_1(\partial \nu(K)) \to H_1(\nu(K))$.

Geometrically, the meridian is the curve in $\partial \nu(K)$ that bounds a disk in $\nu(K)$. A longitude for $K$ is any choice of curve $\lambda$ such that $(\mu_K, \lambda)$ forms a basis for $H_1(\partial \nu(K))$. When $K$ is null-homologous, there is a canonical choice of longitude called the Seifert-framing or the 0-framing given by the curve $\lambda_0 = F \cap \partial \nu(K)$, where $F$ is a Seifert surface.

When $K$ is rationally null-homologous, we can use a rational Seifert surface to determine a canonical longitude, following Mark and Tosun [MT15]. First consider the map $i_* : H_1(\partial \nu(K)) \to H_1(Y - \nu(K))$ induced by inclusion. It is not hard to check that $\ker(i_*) \cong \mathbb{Z}$.

In particular, if $F$ is a rational Seifert surface for $K$, then $F \cap \partial \nu(K) = \partial F$ is a set of curves on $\partial \nu(K)$. While $[\partial F]$ may not be a primitive element in $H_1(\partial \nu(K))$, there exists an integer $c$ and a primitive element $\gamma$ so that $[\partial F] = c \cdot \gamma$. We call $c$ the complexity of $F$. Geometrically, $c$ represents the number of boundary components of $F$. If $F$ is a $q$-Seifert surface, and $\mu$ is the meridian of $K$ we have $[\partial F] : [\mu] = q$. The curve $\gamma$ will only be a longitude for $K$ if $c = q$. More generally, note that if we choose any longitude $\lambda$, then we can write $[\partial F] = c(d \lambda + r \mu)$.
where \( cd = q \) and \( r \) is an integer. Note that the number \( \frac{r}{d} \mod \mathbb{Z} \) is the rational self-linking number of \( K \). Any other longitude for \( K \) can be written as \( \lambda_m = \lambda + n\mu \). Thus, we have that \( [\partial F] = c(d\lambda_m - n\mu) + r\mu = c(d\lambda_m + (r-md)\mu) \). In particular, there is a unique choice, which we call the canonical longitude \( \lambda_{\text{can}} \) such that

\[
[\partial F] = c(d\lambda_{\text{can}} + r\mu)
\]

where \( 0 \leq r < d \). In other words, \( \lambda_{\text{can}} \) is the choice of longitude for which \( \frac{r}{d} \) is the unique representative of the rational self-linking number of \( K \) in \([0, 1)\).

2.7. Relative periodic domains and the \( c_1 \)-evaluation formula. The following gives a way of computing the Alexander grading of a generator directly from a doubly pointed Heegaard diagram for \((Y, K)\).

Recall that for knots in \( S^3 \) we can compute the Alexander grading using periodic domains on the Heegaard diagram for 0-surgery along \( K \) [OS04a]. We will describe an analogous construction for knots in rational homology spheres.

Let \((Y, K)\) be a rationally null-homologous knot with doubly pointed Heegaard diagram \((\Sigma, \alpha, \beta, w, z)\) and let \( F \) be a rational \( q \)-Seifert surface for \( K \). Let \( \lambda \) be a curve on \( \Sigma \) which is a longitude for \( K \). The following definition is given by Hedden and Plamenevskaya [HP13].

**Definition 2.6.** Let \((Y, K)\) be a rational knot with Heegaard diagram and longitude \( \lambda \) as described above. Let \( D_1, D_2, \ldots, D_r \) be the closures of the components of \( \Sigma - (\alpha \cup \beta \cup \gamma) \). A relative periodic domain is a relative 2-chain \( \mathcal{P} = \sum_i a_i D_i \) with

\[
\partial \mathcal{P} = q\lambda + \sum_{i=1}^{g} n_{\alpha_i}\alpha_i + \sum_{i=1}^{g} n_{\beta_i}\beta_i
\]

where the coefficients \( a_i \) are the local multiplicities of \( \mathcal{P} \).

Note that when \( m = 0 \) and \( n_w(P) = 0 \) this definition coincides with the definition given by Ozsváth and Szabó in [OS04a]. Moreover, a relative periodic domain gives rise to a map \( \Phi : F \to \Sigma \) where \( F \) is an oriented surface with boundary such that \( \partial F \) maps into \( \alpha \cup \beta \cup \gamma \).

**Lemma 2.7.** Given a rational \( q \)-Seifert surface, \( F \) for \( K \), satisfying \( \partial F = q\lambda_{\text{can}} + cr\mu \) we can always find a periodic domain \( \mathcal{P}_F \) for \( F \) which satisfies

\[
\partial \mathcal{P}_F = q\lambda_{\text{can}} + cr\mu + q\alpha_g + \sum_{i=1}^{g-1} n_{\alpha_i}\alpha_i + \sum_{i=1}^{g-1} n_{\beta_i}\beta_i
\]

where \( \lambda_{\text{can}} \) is the canonical longitude for \( K \).

**Proof.** Fix a Heegaard diagram for \((Y, K)\) when \( \beta_g = \mu \) and \( \beta_g \) and \( \alpha_g \) intersect in a single point \( p \). For any choice of longitude, there is a small neighborhood of \( \mu \) where \( \lambda \cap \alpha_g = \emptyset \).

Let \( \mathcal{P}_F \) be any periodic domain for \( F \). Then by definition,

\[
\partial \mathcal{P}_F = q\lambda + \sum_{i=1}^{g} n_{\alpha_i}\alpha_i + \sum_{i=1}^{g} n_{\beta_i}\beta_i
\]

\[
= q\lambda + n_{\alpha_g}\alpha_g + n_{\beta_g}\mu + \sum_{i=1}^{g-1} n_{\alpha_i}\alpha_i + \sum_{i=1}^{g-1} n_{\beta_i}\beta_i.
\]
Figure 1. A small neighborhood of $\mu$.

Since $\lambda = \lambda_{\text{can}} + n\mu$ for some integer $n$ we can change coordinates by making this substitution. Moreover, since $\mathcal{P}_F$ represents $F$, we should be able to cap off $\mathcal{P}_F$ with sums of $\alpha$ and $\beta$ curves to obtain a representative of $F$ with $[\partial F] = q\lambda_{\text{can}} + cr\mu$. Thus, we must have that $n\beta_b = cr - n$.

Moreover, from the diagram in Figure 1 we can compute that $n\alpha_g = q$. □

In addition, any relative periodic domain $\mathcal{P}$ gives rise to a relative homology class in $H_2(Y - \nu(K), \partial(Y - \nu(K)))$. Recall that when $K$ is null-homologous, there is a $c_1$ evaluation formula for periodic domains [OS04c]. Hedden and Levine have found a similar formula that holds in the relative case. First, recall the definition of the Euler measure:

**Definition 2.8 (OS04b).** Let $\mathcal{P} = \sum a_i D_i$ be a (relative) periodic domain. The Euler measure of $\mathcal{P}$ is given by,

$$\hat{\chi}(\mathcal{P}) = \sum_i n_i (\chi(D_i)) - \frac{1}{4} \#(\text{corners in } D_i)).$$

Then, we have a $c_1$-evaluation formula for relative periodic domains:

**Proposition 2.9 (HL).** Let $K$ be a rationally null-homologous knot in $Y$ with rational $q$-Seifert surface $F$. Let $(\Sigma, \alpha, \beta, w, z)$ be a doubly pointed Heegaard diagram for $(Y,K)$. For any relative periodic domain $\mathcal{P}$ representing $F$ and any $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$,

$$\langle c_1(s_{w,z}(x)), [F]\rangle - q = \hat{\chi}(\mathcal{P}) + 2n_x(\mathcal{P}) - \tilde{n}_{w,z}(\mathcal{P}).$$

### 3. Knot Floer homology for rationally null-homologous knots

To understand how $\tau_q(Y,K)$ transforms under certain operations, such as reversing the orientation of $K$ and $Y$, we must first understand how the knot Floer chain complex transforms under these operations. We begin by recalling the construction of $\text{CFK}^\infty(Y,K,\xi)$. 
3.1. **Knot Floer chain complexes.** Following [OS11 Section 3], we fix a doubly pointed Heegaard diagram \((\Sigma, \alpha, \beta, w, z)\) for \((Y, K)\) and a relative spin\(^c\)-structure \(\xi\) on \((Y, K)\). Then the complex \(\text{CFK}^\infty(Y, K, \xi)\) is generated by triples \([x, i, j]\) where \(x \in T_\alpha \cap T_\beta\) and \(i, j\) are integers such that
\[
\mathcal{S}_{w,z}(x) + (i - j) \text{PD}[\mu] = \xi
\]
and is equipped with the usual differential,
\[
\partial^\infty[x, i, j] = \sum_{y' \in T_\alpha \cap T_\beta} \sum_{\{\phi \in \pi_2(x, y') | \text{null}(\phi) = 1\}} \hat{M}(\phi)[y', i - n_w(\phi), j - n_z(\phi)].
\]

**Proposition 3.1** ([OS11 Proposition 3.2]). For any \(\xi \in \text{Spin}^c(Y, K)\) there is a natural identification of certain sub-complexes of \(\text{CFK}^\infty(Y, K, \xi) = C_\xi\) with the Floer chain complexes associated to \(Y\). Specifically,
\[
0 \to C_\xi\{i < 0\} \to C_\xi \to C_\xi\{i \geq 0\} \to 0
\]
and
\[
0 \to C_\xi\{i = 0\} \to C_\xi\{i \geq 0\} \overset{U}\to C_\xi\{i \geq 0\} \to 0
\]
can be naturally identified with
\[
0 \to \text{CF}^-(Y, s) \to \text{CF}^\infty(Y, s) \to \text{CF}^+(Y, s) \to 0
\]
where \(G_{Y,K}(\xi) = s\). In addition,
\[
0 \to C_\xi\{j < 0\} \to C_\xi \to C_\xi\{j \geq 0\} \to 0
\]
and
\[
0 \to C_\xi\{j = 0\} \to C_\xi\{j \geq 0\} \to C_\xi\{j \geq 0\} \to 0
\]
can be naturally identified with
\[
0 \to \text{CF}^-(Y, s') \to \text{CF}^\infty(Y, s') \to \text{CF}^+(Y, s') \to 0
\]
where \(G_{Y,-K}(\xi) = s'\).

* A priori the complex \(\text{CFK}^\infty(Y, K, \xi)\) depends on a relative spin\(^c\)-structure. However, Ozsváth and Szabó show that if \(\xi_1\) and \(\xi_2\) in \(\text{Spin}^c(Y, K)\) map to the same \(s \in \text{Spin}^c(Y)\) under \(G_{Y,K}\), then \(\text{CFK}^\infty(Y, K, \xi_1)\) and \(\text{CFK}^\infty(Y, K, \xi_2)\) differ by a shift in their \(j\)-filtration.

Since we have picked a distinguished element of \(\xi^0 \in \hat{G}_{Y,K}(s)\) for each \(s\), we define the complex
\[
\text{CFK}^\infty(Y, K, s) := \text{CFK}^\infty(Y, K, \xi^0).
\]
3.2. Symmetries of \( \text{CFK}^\infty(Y,K,\mathfrak{s}) \). We are interested in how the complexes \( \text{CFK}^\infty(Y,K,\mathfrak{s}) \) change under orientation reversal of \( Y \) and \( K \) as well as under conjugation of \( \mathfrak{s} \). Following convention, we will use the notation \( \text{CFK}^*_\infty(Y,K,\mathfrak{s}) \) to denote the dual complex, \( \text{Hom}_{\mathbb{F}_2}(\text{CFK}^*_\infty(Y,K,\mathfrak{s}),\mathbb{F}_2) \).

**Lemma 3.2.** Let \( Y \) be a rational homology 3-sphere and \( K \subset Y \) a knot. Let \( -Y \) denote \( Y \) with its reverse orientation. Then for any \( \mathfrak{s} \in \text{Spin}^c(Y) \),

\[
\text{CFK}^\infty(-Y,K,\mathfrak{s}) \cong \text{CFK}^*_\infty(Y,K,\mathfrak{s}).
\]

**Proof.** Let \( (\Sigma,\alpha,\beta,w,z) \) be a Heegaard diagram for \((Y,K)\). Then \((-\Sigma,\alpha,\beta,w,z)\) is a Heegaard diagram for \((-Y,K)\). Note that for \( x \in T_\alpha \cap T_\beta \) the map \( s_w(x) \) will be independent of the orientation of \( \Sigma \), and thus the orientation of \( Y \). Suppose \( x,y \in T_\alpha \cap T_\beta \) and \( \phi \in \pi_2(x,y) \). Then we can construct \( \phi' = J \circ \phi \in \pi_2(y,x) \) where \( J \) denotes complex conjugation. In particular, the disk \( \phi \) will have a holomorphic representative if and only if \( \phi' \) does.

Moreover, since \( n_z(\phi) = n_z(\phi') \) and \( n_w(\phi) = n_w(\phi') \), we have,

\[
s_{w,z}(y) - s_{w,z}(x) = (n_z(\phi') - n_w(\phi')) \text{PD}[\mu] = (n_z(\phi) - n_w(\phi)) \text{PD}[\mu].
\]

Thus, the duality map

\[
D : \text{CFK}^\infty(Y,K,\mathfrak{s}) \to \text{CFK}^*_\infty(-Y,K,\mathfrak{s})
\]

taking elements \([x,i,j] \mapsto [x,-i,-j]\) is an isomorphism. \( \square \)

To understand the dependence on the orientation of the knot \( K \subset Y \) we recall the involution on relative spin\(^c\)-structures

\[
\tilde{J} : \text{Spin}^c(Y,K) \to \text{Spin}^c(Y,K)
\]

and the related involution,

\[
J : \text{Spin}^c(Y) \to \text{Spin}^c(Y)
\]

of spin\(^c\)-structures. Geometrically, if \( v \) is a vector field, representing \( \mathfrak{s} \in \text{Spin}^c(Y) \), then \(-v\) represents \( J\mathfrak{s} \). This involution behaves appropriately with respect to the filling maps in the following sense,

**Lemma 3.3.**

\[
G_{Y,K^r}(\tilde{J}\xi) = JG_{Y,K}(\xi).
\]

**Proof.** As in [OS08, Section 3.7] we can think of a relative spin\(^c\)-structure \( \xi \in \text{Spin}^c(Y,K) \) as represented by a nowhere zero vector field \( v \) which has \( K \) as a closed orbit. Thus, \(-v\) is a vector field representing \( J\xi \) that has \( K^r \) as a closed orbit. \( \square \)

To show how the complex \( \text{CFK}^\infty(Y,K,\mathfrak{s}) \) transforms under reversal of orientation of \( K \), we first investigate how \( \text{CFK}^\infty(Y,K,\xi) \) where \( \xi \in \text{Spin}^c(Y,K) \) transforms under this orientation reversal.

**Lemma 3.4 ([OS08, Lemma 3.12]).** Let \( (\Sigma,\alpha,\beta,w,z) \) be a Heegaard diagram for \((Y,K)\), then \((-\Sigma,\alpha,\beta,w,z)\) is a diagram for \((Y,K^r)\). Let

\[
s_{w,z} : T_\alpha \cap T_\beta \to \text{Spin}^c(Y,K)
\]

be the map determined by \((\Sigma,\alpha,\beta,w,z)\), and

\[
s'_{w,z} : T_\alpha \cap T_\beta \to \text{Spin}^c(Y,K)
\]
be the map determined by \((-\Sigma, \beta, \alpha, w, z)\). Then,
\[
\mathfrak{s}_{w,z}(x) = \tilde{J}_s'_{w,z}(x).
\]

Proof. If \(f\) is a Morse function compatible with \((\Sigma, \alpha, \beta, w, z)\) then \(-f\) is compatible with \((-\Sigma, \beta, \alpha, w, z)\). Now, if \(v\) is a vector field representing \(\mathfrak{s}_{w,z}(x)\), then \(-v\) will represent \(\mathfrak{s}'_{w,z}(x)\).

Applying the lemma, we obtain the following, which is the analogue of [OS04a, Proposition 3.9]:

**Proposition 3.5.** For each \(\xi \in \text{Spin}^c(Y, K)\), we have
\[
\text{CFK}^\infty(Y, K, \xi) \cong \text{CFK}^\infty(Y, K^r, \tilde{J}_\xi).
\]

**Proof.** Suppose \([x, i, j]\) is a generator of \(\text{CFK}^\infty(Y, K, \xi)\), then
\[
\mathfrak{s}_{w,z}(x) + (i - j) \text{PD}[\mu_K] = \xi
\]
where \(\mu_K\) is the meridian of \(K\). On the other hand since \(\mathfrak{s}_{w,z}(x) = \tilde{J}_s'_{w,z}(x)\), we must also have
\[
\tilde{J}_s'_{w,z}(x) + (i - j) \text{PD}[\mu_K] = \xi
\]
which is equivalent to
\[
\mathfrak{s}'_{w,z}(x) + (i - j) \text{PD}[\mu_{K^r}] = \tilde{J}_\xi
\]
where \(\mu_{K^r}\) is the meridian of \(K^r\). Moreover, if \(\phi \in \pi_2(x, y)\) is a holomorphic disk contributing to the differential in \(\text{CFK}^\infty(Y, K, \xi)\) then the disk \(-\phi \in \pi_2(x, y)\) is holomorphic and contributes to the differential in \(\text{CFK}^\infty(Y, K^r, \tilde{J}_\xi)\).

Thus we may conclude that in fact,

**Proposition 3.6.**
\[
\text{CFK}^\infty(Y, K, \mathfrak{s}) \cong \text{CFK}^\infty(Y, K^r, \mathfrak{s}).
\]

**Proof.** Using the previous proposition, we know that
\[
\text{CFK}^\infty(Y, K, \mathfrak{s}) = \text{CFK}^\infty(Y, K, \xi_0^\mathfrak{s}) \cong \text{CFK}^\infty(Y, K^r, \tilde{J}_\xi_0^\mathfrak{s}).
\]
Thus, we only need to show that \(\xi_0^\mathfrak{s} = \tilde{J}_\xi_0^\mathfrak{s}\).

Note that \(\langle c_1(\xi_0^\mathfrak{s}), [F] \rangle \in [0, 2[\mu_K] \cdot [F]]\), where \(F\) is a rational Seifert surface for \(K\). Also, note that \(c_1(\tilde{J}_\xi_0^\mathfrak{s}) = -c_1(\xi_0^\mathfrak{s})\). Thus, we have
\[
\langle c_1(\tilde{J}_\xi_0^\mathfrak{s}), -[F] \rangle = \langle c_1(\xi_0^\mathfrak{s}), [F] \rangle \in [0, 2[\mu_{K^r}] \cdot [-F]] = [0, 2[\mu_K] \cdot [F]]
\]
where \(-F\) is a rational Seifert surface associated to \(K^r\).

### 3.3. Connected sums

We will also be interested in how \(\tau_\mathfrak{s}(Y, K)\) behaves under connected sum. Recall that for a pair of knots \((Y_1, K_1)\) and \((Y_2, K_2)\) we can form their connected sum, \((Y_1, K_1) \# (Y_2, K_2) := (Y_1 \# Y_2, K_1 \# K_2)\). Moreover, given Heegaard diagrams \((\Sigma_1, \alpha_1, \beta_1, w_1, z_1)\) and \((\Sigma_2, \alpha_2, \beta_2, w_2, z_2)\) we can form a Heegaard diagram for the connected sum given by \((\Sigma_1 \# \Sigma_2, \alpha_1 \cup \alpha_2, \beta_1 \cup \beta_2, w_1, z_2)\) where the connected sum of \(\Sigma_1\) and \(\Sigma_2\) is performed by identifying neighborhoods of \(w_2\) and \(z_1\).

In addition, we can “glue” \(\text{spin}^c\)-structures over the connected sum to obtain a map:
\[
\text{Spin}^c(Y_1, K_1) \times \text{Spin}^c(Y_2, K_2) \to \text{Spin}^c(Y_1 \# Y_2, K_1 \# K_2)
\]
which sends \((\xi_1, \xi_2) \mapsto \xi_1 \# \xi_2\) and is equivariant with respect to the action of

\[H^2(Y_1, K_1) \oplus H^2(Y_2, K_2) \to H^2(Y_1 \# Y_2, K_1 \# K_2).\]

Moreover, given an intersection point \(x_1 \otimes x_2 \in \mathbb{T}_{\alpha_1 \cup \alpha_2} \cap \mathbb{T}_{\beta_1 \cup \beta_2}\), we have

\[s_{u_1, z_2}(x_1 \otimes x_2) = s_{u_1, z_1}(x_1) \# s_{u_2, z_2}(x_2).\]

The following theorem comes from [OS11, Theorem 5.1]:

**Theorem 3.7.** Fix \(\xi_i \in \text{Spin}^c(Y_i, K_i)\) for \(i = 1, 2\). There is a filtered chain homotopy equivalence

\[
\bigoplus_{\xi_1 \# \xi_2 = \xi_3} \text{CFK}^\infty(Y_1, K_1, \xi_1) \otimes_{\text{CFK}^\infty(Y_2, K_2, \xi_2)} \text{CFK}^\infty(Y_2, K_2, \xi_2) \xrightarrow{\sim} \text{CFK}^\infty(Y_1 \# Y_2, K_1 \# K_2, \xi_3).
\]

**Proof.** This is [OS11, Theorem 5.1] however, there seems to be a typographical error in the statement there. The statement above amends this and the map sending \([x_1, i_1, j_1] \otimes [x_2, i_2, j_2]\) to

\[
\sum_{y \in \mathbb{T}_{\alpha_1 \cup \alpha_2} \cap \mathbb{T}_{\beta_1 \cup \beta_2}} \sum_{(\psi \in \pi_2(x_1 \otimes \theta_2, \theta_1 \otimes x_2, y))} \# \tilde{M}(\psi) \cdot [y, i_1 + i_2 - n_{u_1}(\psi), j_1 + j_2 - n_{z_2}(\psi)]
\]

defines the isomorphism. \(\square\)

For knots in rational homology spheres, we can improve this to a statement about absolute spin\(^c\)-structures. In particular, it remains to show how this map respects the filtration defined by the Alexander grading.

Note that if \(K_1\) has order \(q_1\) and \(K_2\) has order \(q_2\), their connected sum will have order \(\text{lcm}(q_1, q_2)\) in \(Y_1 \# Y_2\). Following Calegari and Gordon [CG13], given connected rational Seifert surfaces \(F_1\) and \(F_2\), we can construct a rational Seifert surface \(F\) for \(K_1 \# K_2\) by taking \(q_1\) copies of \(F_2\) and \(q_2\) copies of \(F_1\) and taking their boundary connected sum along \(q_1 q_2\) arcs. Note that \(F\) has Euler characteristic

\[\chi(F) = q_2 \chi(F_1) + q_1 \chi(F_2) - q_1 q_2.\]

**Lemma 3.8.** Let \(K_1 \subset Y_1\) and \(K_2 \subset Y_2\) be knots in rational homology spheres. Then there is a Heegaard diagram for the connected sum and if \(x_1 \otimes x_2\) an intersection point in the Heegaard diagram for \((Y_1 \# Y_2, K_1 \# K_2)\),

\[A(x_1 \otimes x_2) = A(x_1) + A(x_2).
\]

**Proof.** Let \(q_1\) and \(q_2\) denote the order of \(K_1\) and \(K_2\), and fix a connected rational Seifert surface for each of \(K_1\) and \(K_2\). Let \(P_1\) and \(P_2\) be periodic domains representing \(F_1\) and \(F_2\) respectively. Following the construction above, we can construct a periodic domain \(P\) for a rational Seifert surface \(F\) for \(K_1 \# K_2\) by taking \(q_2\) copies of \(P_1\) and \(q_1\) copies of \(P_2\) and connecting them along \(q_1 q_2\) arcs. Now we can compute \(A(x_1 \otimes x_2)\) using Proposition 2.9 and \(P\).

In particular, from the diagram in Figure 2 we have:

- \(\chi(P) = q_2 \chi(P_1) + q_1 \chi(P_2) - q_1 q_2.\)
- \(n_{x_1 \otimes x_2}(P) = q_2 n_{x_1}(P_1) + q_1 n_{x_2}(P_2).\)
- \(n_{u_1, z_2}(P) = q_1 q_2.\)
Putting this together, we have

\[ HF_a = \text{rational homology sphere}. \]

In particular, we will use the fact that the generator which we call \( c \) is referred to as the "tower." Moreover, we have the following lemma.

Before we give the proof, we also stress that Proposition 3.10 requires that

\[ \tau(x_1, x_2) \]

is the reverse mirror of \( \tau(2) \). Property (1) and (3) together imply that

\[ \tau \] satisfies properties similar to the \( \tau \)-invariant for knots in \( S^3 \). Property (1) and (2) together imply that \( \tau(-K) = -\tau(K) \) where \(-K\) is the inverse of \( K \) in the knot concordance group.

**Proposition 3.10.** Let \( K \subset Y \) be a knot in a rational homology sphere. Then for any \( s \in \text{Spin}^c(Y) \), we have:

1. \( \tau_s(-Y, K) = -\tau_s(Y, K) \).
2. \( \tau_s(Y, K^r) = \tau_{Js}(Y, K) \).
3. If \( K_1 \) and \( K_2 \) are knots in rational homology spheres \( Y_1 \) and \( Y_2 \). Then for \( \text{spin}^c \)-structures \( s_1 \in \text{Spin}^c(Y_1) \) and \( s_2 \in \text{Spin}^c(Y_2) \), we have

\[ \tau_{s_1 \# s_2}(Y_1 \# Y_2, K_1 \# K_2) = \tau_{s_1}(Y_1, K_1) + \tau_{s_2}(Y_2, K_2). \]

Before we give the proof, we also stress that Proposition 3.10 requires that \( Y \) is a rational homology sphere. In particular, we will use the fact that \( HF^+(Y, s) \cong \mathcal{T}^+ \oplus H\text{F}_{\text{red}}(Y, s) \) when \( Y \) is a rational homology sphere where \( \mathcal{T}^+ = F_2[U, U^{-1}]/UF_2[U] \) and is referred to as the "tower." Specifically, there is a distinguished generator which we call \( c_Y \in HF^+(Y, s) \) which is the lowest graded element of the "tower." Moreover, we have the following lemma.

**Theorem 3.9.** We have a filtered chain homotopy equivalence

\[ CFK^\infty(Y_1 \# Y_2, K_1 \# K_2, s) \cong \bigoplus_{s_1 \# s_2 = s} CFK^\infty(Y_1, K_1, s_1) \otimes_{F_2[U, U^{-1}]} CFK^\infty(Y_2, K_2, s_2) \]

where \( F_{s_1 \# s_2}(x_1 \otimes x_2) = F_{s_1}(x_1) + F_{s_2}(x_2) \).

**Diagram:**

![Figure 2. Neighborhoods of the meridian of \( K_1 \) and \( K_2 \) respectively.](image-url)
Lemma 3.11. Let $Y$ be a rational homology sphere, then
\[ \text{Im}(\rho_\ast) \cap \text{Im}(\pi_\ast) = \text{Im}(\pi_\ast \circ \iota_\ast) = c_Y. \]

Proof of Proposition 3.11. First we note that property (2) follows directly from Proposition 3.6.

Now consider property (1). The proof is similar in spirit to the proof of Lemma 3.3]. Let $r = \tau_\ast(Y, K)$ and consider $r' \geq r$. Note that we could instead have written $r = k_0 + m$ for some $m$ and $r' = k_0 + m'$ for some $m' > m$. Most importantly, we emphasize that without loss of generality in what follows, we may assume that $r$ and $r'$ differ by an integer. For clarity, we will use the notation $\mathcal{F}_{r'}(Y, K, s)$ to denote the Alexander filtration at level $r'$ of $\widehat{C}\mathcal{F}(Y, s)$ coming from the knot $K$ and $Q_{r'}(Y, K, s)$ for the corresponding quotient complex.

As in the case where $K$ is null-homologous, reversing the orientation of $Y$ changes the sign of the Alexander grading of each generator in $\widehat{C}\mathcal{F}(Y, s)$. In particular if we consider the short exact sequence,
\[ 0 \to \mathcal{F}_{r'}(Y, K, s) \to \widehat{C}\mathcal{F}(Y, s) \to Q_{r'}(Y, K, s) \to 0 \]
we can naturally identify $Q_{r'}(Y, K, s)$ with $\mathcal{F}_{r'-1}(-Y, K, s)$. In fact, we have an isomorphism of short exact sequences:

\[
\begin{array}{ccccccc}
0 & \to & \mathcal{F}_{r'}(Y, K, s) & \xrightarrow{i_{r'}} & \widehat{C}\mathcal{F}_{r}(Y, s) & \xrightarrow{p_{r'}} & Q_{r'}(Y, K, s) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathcal{F}_{r'-1}(-Y, K, s) & \xrightarrow{i_{r'-1}} & \widehat{C}\mathcal{F}_{r'-1}(-Y, s) & \xrightarrow{p_{r'-1}} & Q_{r'-1}(-Y, K, s) & \to & 0 \\
\end{array}
\]

by Lemma 3.2.

Since $\tau_\ast(Y, K)$ is defined in terms of the map $\rho_\ast$ to $HF^+(Y, s)$ we must also understand how this map transforms under reversal of orientation of $Y$. Recall from [OS03b] that $HF^+(Y, s) \cong HF^{<1}(-Y, s)$. Thus, we have the following diagram

\[
\begin{array}{ccccccc}
HF_{r'}^<(-Y, s) & \xrightarrow{\psi} & \widehat{HF}_{r'}^<(-Y, s) & \xrightarrow{P_{r'}} & H_\ast(Q_{r'}(Y, K, s)) \\
\downarrow & = & \downarrow & = & \downarrow \\
HF_{r'-1}^<(-Y, s) & \xrightarrow{\rho_\ast} & \widehat{HF}_{r'-1}^<(-Y, s) & \xrightarrow{I_{r'-1}} & H_\ast(\mathcal{F}_{r'-1}(-Y, K, s)) \\
\end{array}
\]

which commutes. Since $r' \geq r$, by Lemma 2.3 there exists $\alpha \in HF^<1(Y, s)$ such that $\pi_\ast \circ \iota_\ast(\alpha) \neq 0$, but $P_{r'} \circ \psi_\ast(\alpha) = 0$. By the commutative diagram (2), if $P_{r'} \circ \psi_\ast(\alpha) = 0$, we also have $I_{r'-1} \circ \rho_\ast(\alpha) = 0$. Moreover, since we are using $\mathbb{F}_2$ coefficients, we can view $I_{r'-1} \circ \rho_\ast(\alpha)$ as a map
\[ \alpha \circ \rho_\ast \circ I_{r'-1} : H_\ast(\mathcal{F}_{r'-1}(-Y, K, s)) \to \mathbb{F}_2 \]
which must be the trivial map.

On the other hand, we also know that $\pi_\ast \circ \iota_\ast(\alpha) = \iota_\ast \circ \pi_\ast(\alpha) \neq 0$,
\[
\begin{array}{ccccccc}
HF_{r'}^<(-Y, s) & \xrightarrow{\iota_\ast} & HF_{r'}^<(-Y, s) & \xrightarrow{\pi_\ast} & HF^+_r(Y, s) \\
\downarrow & = & \downarrow & = & \downarrow \\
HF_{r'-1}^<(-Y, s) & \xrightarrow{\pi_\ast} & HF_{r'-1}^<(-Y, s) & \xrightarrow{\iota_\ast} & HF_{r'-1}^<(-Y, s) \\
\end{array}
\]
so by duality, the map \( \alpha \circ \pi_\ast \circ \iota_\ast : HF_{\ast < 1}^c(-Y, s) \to \mathbb{F}_2 \) must be nontrivial. In addition, by Lemma 3.11 we have \( \text{Im}(\pi_\ast \circ \iota_\ast) = c_{-Y}. \) In particular, this means that \( \alpha \) must be nontrivial on the element \( c_{-Y}. \) Thus, \( c_{-Y} \notin \text{Im}(\rho_\ast \circ I_{-r'-1}). \) Therefore, 
\[ -r' - 1 < \tau_s(-Y, K) \quad \text{or} \quad -\tau_s(Y, K) \leq \tau_s(-Y, K). \]

Now suppose this inequality is strict. Rearranging, we have
\[ -\tau_s(-Y, K) < \tau_s(Y, K). \]

Let \( \tilde{r} = -\tau_s(-Y, K). \) Since \( \tilde{r} < \tau_s(Y, K), \) then for all \( \alpha \in HF_{\ast < 1}^c(Y, K) \) such that \( \pi_\ast \circ \iota_\ast(\alpha) \neq 0, \) we have \( P_{\tilde{r}} \circ \psi(\alpha) \neq 0. \) So, \( I_{-\tilde{r}-1} \circ \rho^*(\alpha) \neq 0 \) which means that the map
\[ \alpha \circ \rho_\ast \circ I_{-\tilde{r}-1} : H_s(F_{\tilde{r}-1}(Y, K, s)) \to \mathbb{F}_2 \]
is nontrivial.

Suppose \( \beta \in H_s(F_{\tilde{r}-1}(-Y, K, s)) \) is an element such that \( \alpha \circ \rho_\ast \circ I_{-\tilde{r}-1}(\beta) \neq 0. \) Since \( -\tilde{r} - 1 < -\tilde{q} = \tau_s(-Y, K), \) we must have that \( c_{-Y} \notin \text{Im}(\rho_\ast \circ I_{-\tilde{r}-1}). \)

Now let \( \alpha' = \alpha -(\rho_\ast \circ I_{-\tilde{r}-1})^\ast. \) Now the map \( \alpha' \circ \rho_\ast \circ I_{-\tilde{r}-1} \) must be trivial on \( \beta \) but we still have \( \pi_\ast \circ \beta(\alpha') \neq 0, \) so we must still have \( I_{-\tilde{r}-1} \circ \rho^*(\alpha') \neq 0. \) In particular, if we continue to adjust \( \alpha \) in this way, we will continue to create nontrivial maps, which is a contradiction since there will only be finitely many linearly independent torsion elements in \( HF^+(Y, s). \) We may conclude that \( -\tau_s(Y, K) = \tau_s(-Y, K). \)

Finally we prove (3). Let \( m = \tau_{a_1 # a_2}(Y_1 # Y_2, K_1 # K_2). \) Then there exists \( \alpha \in HF_{< 1}(Y_1 # Y_2, K_1 # K_2) \) which is non-torsion and \( \beta \in H_s(F_m) \) so that \( I_m(\beta) = \psi_\ast(\alpha). \) By the Künneth formula \( OS[41] \) \( \alpha \) decomposes as \( \alpha_1 \otimes \alpha_2 \) where \( \alpha_1 \in HF_{< 1}(Y_1, a_1) \) and \( \alpha_2 \in HF_{< 1}(Y_2, a_2). \) Moreover, \( \alpha \) is non-torsion if and only if both \( \alpha_1 \) and \( \alpha_2 \) are non-torsion. Moreover, since \( \psi_\ast(\alpha) \neq 0 \) we must have that \( \psi_\ast(\alpha_1) \neq 0 \) and \( \psi_\ast(\alpha_2) \neq 0. \) Finally, by Lemma 3.8 the Alexander grading of \( \psi_\ast(\alpha_1) \otimes \psi_\ast(\alpha_2) \) must decompose as \( r = r_1 + r_2 \) where \( r_1 \) is the Alexander grading of \( \psi_\ast(\alpha_1) \) and \( r_2 \) is the Alexander grading of \( \psi_\ast(\alpha_2). \)

\[ \square \]

4. LARGE SURGERY

Ozsváth and Szabó proved that the Heegaard Floer homology of surgery along a null-homologous knot could be computed from the chain complex \( CFK^\infty(Y, K, [m]) \) using their “large surgery” formula \( OS[44]. \) In \( OS[11], \) they showed that a similar construction works for rationally null-homologous knots. However, their theorem depends on a choice of longitude for \( K. \) We will give a refinement of this theorem using \( \lambda_{\text{can}} \) as the choice of longitude. Specifically, this choice of longitude will allow us to enumerate 2-handle cobordism maps on Floer homology using the Alexander grading of \( K. \)

Recall that if \( K \) is a knot in a rational homology sphere \( Y, \) we can construct a cobordism \( X_{-n}(K) \) from \( Y \) to \( Y_{-n}(K) \) by attaching a 4-dimensional 2-handle to \( K \times \{1\} \subset Y \times I \) with \((-n)\)-framing with respect to the canonical longitude. For each spin\(^c\)-structure \( t \in \text{Spin}^c(X_{-n}(K)), \) this cobordism induces a map on Floer homology:
\[ F_{X_{-n}(K), t} : \tilde{HF}(Y, t|_Y) \to \tilde{HF}(Y_{-n}(K), t|_{Y_{-n}(K)}). \]

As in the case when \( K \) is null-homologous, the large surgery theorem tells us when we can compute the above map in terms of information from the \( CFK^\infty(Y, K, \xi) \) complexes. Recall from \( OS[33] \) that when \( K \) is null-homologous, we can enumerate these maps in terms of the Alexander grading. We will describe the analogue for rationally null-homologous knots.
Let $X_{-n}(K)$ be the 4-manifold described above. Let $C$ be the core of the added 2-handle in $X_{-n}(K)$. Then $[C]$ represents the generator of $H_2(X_{-n}(K), Y) \cong \mathbb{Z}$. Now fix a rational $q$-Seifert surface $F$ for $K$. Attaching $q$ parallel copies of $C$ to $F$ along $K$ we obtain a 2-complex $F \cup qC$ which represents a homology class in $H_2(X_{-n}(K))$. Moreover, under the map $\iota: H_2(X_{-n}(K)) \to H_2(X_{-n}(K), Y)$ coming from the long exact sequence of the pair $(X_{-n}(K), Y)$ we have $\iota([F \cup qC]) = q[C]$. Thus, for $\alpha \in H^2(X_{-n}(K); \mathbb{Z})$ we can define a $\mathbb{Q}$-valued pairing

$$\langle \alpha, [C] \rangle = \frac{1}{q} \langle \alpha, [F \cup qC] \rangle.$$  
Similarly, we also have,

$$[C] \cdot [C] = \frac{1}{q^2} [F \cup qC] \cdot [F \cup qC].$$

Moreover, since we have $[\partial F] = c(d\lambda_{can} + r\mu)$, so we can compute $[F \cup qC]^2$ directly as the linking number of $[\partial F]$ and $q[C]$ inside $\partial \nu(K)$.

\begin{equation}
[\partial F] \cdot q[C] = (q\lambda_{can} + cr\mu) \cdot (q\lambda_{can} - q\mu) = q(cr - q\mu).
\end{equation}

4.1. The large surgery theorem for rationally null-homologous knots. Recall that we can always choose our Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ for $(Y, K)$ so that $\beta = \beta_0 \cup \mu$ and $\alpha_{g}$ and $\beta_{g}$ intersect in a single point $p$. Now consider an annular neighborhood of $\mu$ in the Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$. Let $\lambda_{-n} = \lambda_{can} - n\mu$. We can use $\lambda_{-n}$ to form a Heegaard diagram $(\Sigma, \alpha, \beta_0 \cup \lambda_{-n}, w, z)$ for $Y_{-n}(K)$. In particular, $\lambda_{-n}$ intersects $\alpha_{g}$ in exactly $n$ points in a small neighborhood of the meridian which we refer to as the winding region.

Note that every intersection point $x \in T_\alpha \cap T_\beta$ has $x_g = p$. Let $y \in T_\alpha \cap T_\gamma$. We say $y$ is supported in the winding region if $y_g$ is one of the $n$ points in the winding region. If $y \in T_\alpha \cap T_\gamma$ is supported in the winding region, then there is a “closest” point $x \in T_\alpha \cap T_\beta$. Let $\theta$ be the top generator in homology of $\overline{HF}(#_{g-1}(S^1 \times S^2))$. Moreover, there is a canonical “small triangle” $\psi \in \pi_2(x, \theta, y)$ supported there. Recall the function

$$f(x) = \langle c_1(s_w(\psi)), [C] \rangle + [C]^2 - 2(n_w(\psi) - n_z(\psi))$$

defined by Ozsváth and Szabó in [OS11] where $\psi \in \pi_2(x, \theta, y)$ is a small triangle. Ozváth and Szabó showed that this function depends only on $x$ this argument is revisited in [IT15]. In fact, by using Lemma 2.7 we will see that this function actually computes the Alexander grading of the generator $x$.

First, we recall a few more facts and definitions from [OS11]. Note that the tuple $(\Sigma, \alpha, \beta, \gamma)$ is called a Heegaard triple-diagram and specifies a 4-manifold one of whose boundary components is $#_{g-1}(S^1 \times S^2)$. Capping this boundary component off with $\mathbb{Z}_{g-1}S^1 \times B^3$ gives us a description of the cobordism $X_{-n}(K)$. In this 4-manifold, a triply periodic domain is a 2-chain $P$ whose boundary is a sum of $\alpha$, $\beta$, and $\gamma_g$ curves. Given a surface in $X_{-n}(K)$ we can represent it via a triply periodic domain.

Recall that for a triply periodic domain we also have a $c_1$-evaluation formula so that if $\psi$ is a Whitney triangle,

\begin{equation}
\langle c_1(s_w(\psi)), P_{\alpha\beta\gamma} \rangle = \chi(P_{\alpha\beta\gamma}) + \# \partial P_{\alpha\beta\gamma} + 2\sigma(\psi, P_{\alpha\beta\gamma}) - 2n_w(P_{\alpha\beta\gamma})
\end{equation}
Figure 3. The winding region for $-6$-surgery along $K$.

where $\sigma(\psi, \mathcal{P}_{\alpha\beta\gamma})$ is the dual spider number [OS11 Section 2.5].

**Lemma 4.1.** Let $(\Sigma, \alpha, \beta, w, z)$ be a doubly pointed Heegaard diagram for $K \subset Y$, and let $(\Sigma, \alpha, \beta, \gamma_n)$ be the related Heegaard diagram for the cobordism $X_{-n}(K)$ described above. Then for any $x \in T_\alpha \cap T_\beta$

$$f(x) = 2A(x)$$

where $A(x)$ is the Alexander grading of $x$.

**Proof.** By Lemma 2.7, we can find a relative periodic domain representing $[F], \mathcal{P}_F$ with

$$\partial \mathcal{P}_F = q\lambda_{can} + cr\mu + q\alpha_g + \sum_{i=1}^{g-1} n_\alpha \alpha_i + \sum_{i=1}^{g-1} n_\beta \beta_i.$$

We can also find a triply periodic domain $\mathcal{P}_{\alpha\beta\gamma}$ representing $[F \cup pC]$ with

$$\partial \mathcal{P}_{\alpha\beta\gamma} = q\lambda_n + (qn + cr)\mu + q\alpha_g + \sum_{i=1}^{g-1} n_\alpha \alpha_i + \sum_{i=1}^{g-1} n_\beta \beta_i.$$

We claim that

$$\langle c_1(s_{w,z}(x)), \mathcal{P}_F \rangle - q = \langle c_1(s_w(\psi)), \mathcal{P}_{\alpha\beta\gamma} \rangle - qn + cr - 2q(n_w(\psi) - n_z(\psi)).$$

Moreover, since the right hand side does not depend on $\psi$, it suffices to prove the statement for the Whitney triangle $\psi$ with $n_w(\psi) = n_z(\psi) = 0$. This follows by applying the first Chern class formulas. The diagram in Figure 3 shows the winding region for $-6$-surgery along $K$ and the multiplicities for the corresponding periodic domain $\mathcal{P}_{\alpha\beta\gamma}$.

First consider the domain $\mathcal{P}_F$. Since $x$ is of the form $x = (x_1, \ldots, x_{g-1}, p)$ we can compute from Figure 3 that

$$n_p(\mathcal{P}_F) = \frac{1}{4}(0 - cr + q + cr) = \frac{2q - 2cr}{2} = \frac{1}{2}(q - cr)$$
and that
\[ \tilde{n}_{w,z}(P_F) = \frac{1}{2}(0 + q) + \frac{1}{2}(-cr + q - cr) = q - cr. \]
Thus \(2n_y(P_F) - \tilde{n}_{w,z}(P_F) = 0\). In particular, by Proposition 2.9 for the relative periodic domain \(P_F\) we have that
\[
\langle c_1(s_{w,z}(x)), P_F \rangle - q = \hat{\chi}(P_F) + 2n_x(P_F) - \tilde{n}_{w,z}(P_F).
\]

On the other hand, if we fix a point \(y = (y_1, \ldots, y_q)\) in the interior of the Whitney triangle, we compute using the \(c_1\)-evaluation formula of equation (4) for the triply periodic domain \(P_{\alpha\beta\gamma}\) that \(2n_y(P_{\alpha\beta\gamma}) - 2n_w(P_{\alpha\beta\gamma}) = -2(n + 1)q\). Thus,
\[
\langle c_1(s_u(\psi)), P_{\alpha\beta\gamma} \rangle = \hat{\chi}(P_{\alpha\beta\gamma}) + 2q + qn + cr
+ \sum_{i=1}^{g-1} n_{\alpha_i} + \sum_{i=1}^{g-1} n_{\beta_i} + 2\sum_{i=1}^{g-1} n_{\gamma_i} - 2(n + 1)q
= \hat{\chi}(P_{\alpha\beta\gamma}) - qn + cr + \sum_{i=1}^{g-1} n_{\alpha_i} + \sum_{i=1}^{g-1} n_{\beta_i} + 2\sum_{i=1}^{g-1} n_{\gamma_i}.
\]

Outside of the winding region, we have that \(2n_{x_i}(P_F) = 2n_{y_i}(P_{\alpha\beta\gamma}) + n_{\alpha_i} + n_{\beta_i}\).

Putting this all together, along with the fact that \(\hat{\chi}(P_F) = \hat{\chi}(P_{\alpha\beta\gamma})\) we obtain the result.

Lemma 4.1 gives us the following refinement of [OS11, Theorem 4.1]:

**Theorem 4.2.** Let \(K \subset Y\) be a rationally null-homologous knot in a closed oriented 3-manifold of order \(q\). Then for all sufficiently large \(n\), there is a map:
\[
G_{-n} : \text{Spin}^c(Y, K) \to \text{Spin}^c(Y_{-n}(K))
\]
with the property that for all \(\xi \in \text{Spin}^c(Y, K)\) the complex \(CF^+(Y_{-n}(K), G_{-n}(\xi))\) can be represented by \(\mathcal{C}_a\{\min(i, j - m) \geq 0\}\) where \(\xi = \xi_0^n + mPD[\mu]\) and \(|m| \leq \frac{n}{2}\) in the sense that there are isomorphisms
\[
\Psi_{-n,m}^* : \mathcal{C}_a\{\min(i, j - m) \geq 0\} \xrightarrow{\sim} CF^+(Y_{-n}(K), G_{-n}(\xi)).
\]

Furthermore, if \(t_m = G_{-n}(\xi) = G_{-n}(\xi_0^n + mPD[\mu]) \in \text{Spin}^c(Y_{-n}(K))\). There is a unique extension, \(t \in \text{Spin}^c(Y_{-n}(K))\) satisfying
\[
\langle c_1(t), [F \cup qC]\rangle - nq + cr = 2q(k_2 + m)
\]
so that the following diagram
\[
\begin{array}{ccc}
CF^+(Y, \mathcal{S}) & \xrightarrow{F_{-n,m}^*} & CF^+(Y_{-n}(K), t_m) \\
\downarrow & & \downarrow \\
\mathcal{C}_a\{i \geq 0\} & \xrightarrow{\psi_{-n,m}^*} & \mathcal{C}_a\{\min(i, j - m) \geq 0\}
\end{array}
\]
commutes.
4.2. A four-dimensional interpretation of \( \tau_4(Y, K) \). The large surgery theorem now allows us to relate \( \tau_4(Y, K) \) to the map induced on Floer homology by the cobordism \( X_{-n}(K) \).

Let \( K \) be a knot of order \( q \) in a rational homology sphere \( Y \). Let

\[
\widetilde{F}_{-n,m}^*: \tilde{HF}(Y, \mathfrak{s}) \to \tilde{HF}(Y_{-n}(K), \tau_m)
\]

be the map induced by the cobordism \( X_{-n}(K) : Y \to Y_{-n}(K) \). Where \( \tau_m \) is the restriction to \( Y_{-n}(K) \) of the unique spin\(^c\)-structure \( t_m \) on \( X_{-n}(K) \) satisfying \( t_m|_Y = \mathfrak{s} \) and

\[
\langle c_1(t_m), [F \cup qC] \rangle = -nq + cr = 2q(k_s + m).
\]

**Proposition 4.3.** Let

\[
\mathcal{S} = \text{Im}(\rho_s) \cap \text{Im}(\pi_*) \subset HF^+(Y, \mathfrak{s}).
\]

Then for all \( |n| \) sufficiently large, we have the following:

- If \( k_s + m < \tau_s(Y, K) \), then for all \( \beta \in \rho_s^{-1}(\mathcal{S}) \), we have \( \widetilde{F}_{-n,m}^*(\beta) \neq 0 \).
- If \( k_s + m > \tau_s(Y, K) \) then there exists \( \beta \in \rho_s^{-1}(\mathcal{S}) \) such that \( \widetilde{F}_{-n,m}^*(\beta) = 0 \).

**Proof.** We adapt the argument given in [OS03a]. Let \( \mathcal{C} = CFK^\infty(Y, K, \mathfrak{s}) \). Note that the diagram

\[
\begin{array}{cccccc}
0 & \to & C_{(i=0, j \leq m)} & \cong & \mathcal{F}_{k,m} & \xrightarrow{i_m} & C_{(i=0)} & \cong & \tilde{\mathcal{C}F}(Y, \mathfrak{s}) & \xrightarrow{p_m} & C_{(i=0, j > m)} & \cong & Q_{k,m} & \to & 0 \\
0 & \to & C_{(i \geq 0, j = m)} & \to & C_{(\min(i, j - m) = 0)} & \xrightarrow{j} & C_{(i=0, j > m)} & \to & 0 \\
0 & \to & C_{(i=0, j > m)} & \to & C_{(i=0, j > m)} & \to & 0 \\
\end{array}
\]

commutes. By Theorem 4.2 when \( |n| \) is sufficiently large, we can also identify the sub-quotient complex \( C_{(\min(i, j - m) = 0)} \) with \( \tilde{\mathcal{C}F}(Y_{-n}(K), \tau_m) \) and the map induced by \( \tilde{f} \) on homology with the map \( \widetilde{F}_{-n,m}^* \). To prove the first part, assume that \( k_s + m < \tau_s(Y, K) \). Let \( \gamma \in \rho_s^{-1}(\mathcal{S}) \). By the assumption, \( \gamma \) is not in the image of \( I_m \). Thus, by exactness, \( P_m(\gamma) \neq 0 \) and in particular, since the diagram commutes, \( \widetilde{F}_{-n,m}^*(\gamma) \neq 0 \).

On the other hand, note that the map \( \tilde{f} \) factors through \( p_{m-1} \). Thus, if \( k_s + m > \tau_s(Y, K) \), then there exists an element \( \gamma \in \rho_s^{-1}(\mathcal{S}) \) such that \( \gamma \in \text{Im}(I_{m-1}) \). Thus, \( P_{m-1}(\gamma) = 0 \) and by commutativity, \( \widetilde{F}_{-n,m}^*(\gamma) = 0 \) as well. \( \square \)

5. Genus bounds

Consider a negative definite 4-manifold \( W \) with boundary a rational homology sphere \( Y \). By removing a small ball from \( W \), we obtain a 4-manifold which is a cobordism \( W - B^4 \) from \( S^3 \) to \( Y \). In this setting, the maps induced on Floer homology satisfy certain properties. The first of which is a consequence of [OS03a, Theorem 9.6] and will act as the analogue to [OS03b, Lemma 3.4].

**Lemma 5.1.** Let \( W \) be a negative definite 4-manifold with boundary \( Y \) a rational homology sphere. Then for any spin\(^c\)-structure \( \mathfrak{s} \) on \( Y \) that extends over \( W \), the map

\[
\widetilde{F}_{W-B^4,4}^*: \tilde{HF}(S^3) \to \tilde{HF}(Y, \mathfrak{s})
\]
is nontrivial where $t$ is the extension of $s$ to $\text{Spin}^c(W)$. In particular, $\widehat{HF}(S^3)$ is generated by a single element which maps to $c_Y \in HF^+(Y, s)$ under the composition

$$\widehat{HF}(S^3) \to \widehat{HF}(Y, s) \to HF^+(Y, s).$$

In addition, we will need the following lemma:

**Lemma 5.2 ([OS03], Lemma 3.5).** Let $N$ be the total space of a disk bundle with Euler number $n > 0$ over an oriented two manifold $S$ of genus $g > 0$. The map

$$\hat{F}_{N-B,s} : \widehat{HF}(S^3) \to \widehat{HF}(\partial N, \delta|\partial N)$$

is trivial whenever

$$(c_1(s), [S]) + [S] : [S] > 2g(S) - 2.$$

Now consider a knot $K$ in $Y = \partial W$. Adding a 2-handle to $W$ along $K$ with $-n$-framing, we form a new 4-manifold $W_n(K)$ which decomposes as $W \cup_Y X_n(K)$ where $X_n(K)$ is the 2-handle cobordism from $Y$ to $Y_n(K)$. Looking at the Mayer-Vietoris sequence,

$$0 \to H_2(W) \oplus H_2(X_{-n}) \to H_2(W_n(K)) \to H_1(Y) \to \ldots$$

we can see that $i$ is injective. Suppose $\Sigma$ is a rational p-slicing surface for $K$ and $F$ is a rational $q$-Seifert surface for $K$. If we consider the homology class of the 2-complex $\Sigma \cup pC$ in $H_2(W_n(K))$, we can see that $q[\Sigma \cup pC] \in \ker(j)$. Thus, it must split as a class in $H_2(W) \oplus H_2(X_{-n}(K))$ and we can think of this splitting geometrically,

$$q[\Sigma \cup pC] = i([q\Sigma \cup pF] \oplus [pF \cup pqC]).$$

**Theorem 5.3.** Let $W$ be a negative definite 4-manifold with $\partial W = Y$ a rational homology 3-sphere. Let $K \subset Y$ be a knot of order $q$ in $H_1(Y; \mathbb{Z})$. Let $\Sigma$ be a rational p-filling surface for $K$. If $n$ is sufficiently large, then for any embedded surface $S$ such that $[S] = [\Sigma \cup pC] \in H_2(W_n(K))$ and any $t \in \text{Spin}^c(W)$ such that $t|Y = s$ we have,

$$\frac{1}{pq} (c_1(t), [q\Sigma \cup pF]) + \frac{1}{pq^2} [q\Sigma \cup pF]^2 + 2\tau_s(Y, K) \leq \frac{\chi(S)}{p} + 2 + (p - 1)[C]^2$$

where $F$ is a $q$-Seifert surface for $K$.

**Proof.** Consider our decomposition of the 4-manifold $W_n(K)$ as

$$W_n(K) = W \cup_Y X_n(K).$$

If we remove a small ball $B$ from the interior of $W_n(K)$ we can view $W_n(K) - B$ as a cobordism from $S^3$ to $Y_{-n}(K)$ which is the composition of the cobordisms $W - B$ from $S^3$ to $Y$ and $X_{-n}(K)$ from $Y$ to $Y_{-n}(K)$.

Let $s$ be a spin$^c$-structure on $Y$ which extends to a spin$^c$-structure $t$ on $W$. Then by Lemma 5.1 the map

$$F_{W-B,t} : \widehat{HF}(S^3) \to \widehat{HF}(Y, s)$$

maps the generator of $\widehat{HF}(S^3)$ to an element $\gamma \in \rho_{-1}(S)$.

Now for any $m < \tau_s(Y, K) - k_s$, we can choose $|n|$ large enough that Lemma 4.3 holds. In addition, we can also associate a spin$^c$-structure $t_m$ on $X_{-n}(K)$ satisfying $t|Y = s$ and

$$\langle c_1(t_m), [F \cup qC] \rangle + [F \cup qC]^2 = 2q(m + k_s) < 2q(\tau_s(Y, K))$$
Thus, so that the map
\[ F_{n,m}^s : \widehat{HF}(Y, s) \to \widehat{HF}(Y_{-n}(K), v_m) \]
must satisfy \( F_{n,m}^s(\gamma) \neq 0 \) and therefore, the composition
\[ F_{n,m}^s \circ F_{W-B,t} : \widehat{HF}(S^3) \to \widehat{HF}(Y, v_m) \]
is nontrivial.

On the other hand, since \( W_{-n}(K) - B \) is a 4-manifold, we may factor the cobordism through any intervening 3-manifolds. In particular, let \( S \) be an embedded surface representing the homology class of \( \Sigma \cup pC \). We can view \( W_{-n}(K) = W_1 \cup W_2 \) where \( W_1 \) is the tubular neighborhood of \( S \) and \( W_2 \) is the complement of this tubular neighborhood.

For now, we will assume that \( g(S) > 1 \). Lemma 5.2 implies that for \( \hat{t} = t#t_m \) we must have
\[ \langle c_1(\hat{t}), [S] \rangle + |S|^2 \leq 2g(S) - 2 \]
since the map \( F_{n,m}^s \circ F_{W-B,t} \) is nontrivial. Applying equation (6) to rewrite the left-hand side and using equation (4) to compute the self intersection number of \( [F \cup qC] \) shows that
\[
\langle c_1(\hat{t}), [S] \rangle + |S|^2 = \frac{1}{q} \langle c_1(t), [q\Sigma \cup pF] \rangle + \frac{1}{q^2} [q\Sigma \cup pF]^2 \\
+ \frac{p}{q} \left( \langle c_1(t_m), [F \cup qC] \rangle + \frac{1}{q} [F \cup qC]^2 + \frac{p}{q} \frac{1}{2} [F \cup qC]^2 \right) \\
= \frac{1}{q} \langle c_1(t), [q\Sigma \cup pF] \rangle + \frac{1}{q^2} [q\Sigma \cup pF]^2 \\
+ 2p(k_a + m) + \frac{p(p - 1)(cr - nq)}{q}.
\]

Thus,
\[
\frac{1}{q} \langle c_1(t), [q\Sigma \cup pF] \rangle + \frac{1}{q^2} [q\Sigma \cup pF]^2 + 2p(k_a + m) + \frac{p(p - 1)(cr - nq)}{q} \leq -\chi(S).
\]

Moreover, this holds for all \( k_a + m < \tau_a(Y, K) \) and thus for \( k_a + m \leq \tau_a(Y, K) - 1 \). Making this substitution and simplifying we obtain,
\[
\frac{1}{pq} \langle c_1(t), [q\Sigma \cup pF] \rangle + \frac{1}{pq^2} [q\Sigma \cup pF]^2 + 2\tau_a(Y, K) \leq \frac{-\chi(S)}{p} + 2 + \frac{(p - 1)(cr - nq)}{q}
\]

Now suppose that \( g(S) = 0 \). Consider the surface \( \Sigma_K \) obtained by taking the intersection of \( S \) with \( W \) inside of \( W_{-n}(K) \). This must be a surface with boundary that satisfies \( [\partial \Sigma_K] = p[K] \in H_1(\nu(K)) \). In particular, this surface is a \( p \)-slicing surface for \( K \).

Now let \( W' = W\sharp B^4 \) where \( \sharp \) denote the boundary connected sum and consider the knot \( K' = K\# J \subset W' \) where \( J \) is the right handed trefoil in \( S^3 \). We can form a \( p \)-slicing surface for \( K' \) by attaching \( p \) copies of the minimal genus slicing surface \( \Sigma_J \) for \( J \) to \( \Sigma_K \) along \( p \) arcs to form \( \Sigma_{K'} \). We can form an embedded surface representing the homology class of \( \Sigma_{K'} \cup pC \) by cutting out \( p \)-disks from \( S \) and gluing in \( p \)-copies of \( \Sigma_J \). Thus, if \( S' \) is an embedded surface representing the homology class \( [\Sigma_{K'} \cup pC] \) we must have \(-\chi(S') \leq -\chi(S) + 2p\).
Finally, applying the genus 1 case and using the fact that
\[ \tau_s(Y \# S^3, K') = \tau_s(Y, K) + 1 \]
we may conclude that
\[ \frac{1}{pq} (c_1(t), [q \Sigma \cup pF]) + \frac{1}{pq^2} [q \Sigma \cup pF]^2 + 2\tau_s(Y, K) + 2 \leq -\chi(S) + 2p + 2 + (p-1) \frac{(cr - nq)}{q} \]
so we obtain the same bound as in the first case. \( \square \)

The theorem will simplify significantly in the case when \( p = 1 \). In particular, when \( p = 1 \) the final term vanishes and \(-\chi(S) = 2g(S) - 2 = 2g(\Sigma) - 2\), so we obtain:

**Corollary 5.4.** Let \( W \) be a negative definite 4-manifold and \( K \) a knot in \( \partial W = Y \). Let \( F \) be a \( q \)-Seifert surface for \( K \). Then for any surface \( \Sigma \) such that \( \partial \Sigma = K \), and any \( \mathfrak{s} \) that extends over \( W \)
\[ \frac{1}{q} (c_1(t), [q \Sigma \cup F]) + \frac{1}{q^2} [q \Sigma \cup F]^2 + 2\tau_s(Y, K) \leq 2g(\Sigma). \]
In particular, when \( W \) is a rational homology 4-ball, we have
\[ |\tau_s(Y, K)| \leq g(\Sigma). \]

6. Examples

6.1. **A slice knot and a non-slice knot in \( L(4, 1) \).** If we consider the lens space \( L(4, 1) \), we can see via the diagram in Figure 4 that \( L(4, 1) \) bounds a rational homology 4-ball which we call \( W \). Moreover, we can compute that the map
\[ H_1(L(4, 1)) \to H_1(W) \cong \mathbb{Z}_2 \]
is reduction modulo two. Thus, any knot of order two in \( H_1(L(4, 1)) \) will be null-homologous in \( W \) and applying Corollary 5.4 to any such knot should give us a bound on the genus of a surface bounded by the knot.

Let \( K \) be the knot of order two in \( L(4, 1) \) shown on the left in Figure 5. We can construct a genus one Heegaard diagram for \( K \) as shown on the left in Figure 6 and compute the Alexander grading of each element by finding a periodic domain for
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Figure 5. The knots $K$ and $J$ in $L(4, 1)$.

Figure 6. Heegaard diagrams for $K$ and $J$ in $L(4, 1)$.

Table 1. Alexander gradings for generators of $\hat{CF}(Y, K)$.

| $x$ | $a$ | $b$ | $c$ | $d$ |
|-----|-----|-----|-----|-----|
| $A(x)$ | 0   | $\frac{1}{2}$ | 0   | $-\frac{1}{2}$ |

Since each intersection point in the diagram represents a distinct spin$^c$-structure, we can denote $s_w(a)$ by $s_a$ and denote the rest of the spin$^c$-structures accordingly. Since there are no holomorphic disks in this diagram, the Alexander grading of each element is equal to $\tau_{s_w(x)}(L(4, 1), K)$. It is not hard to check that $K$ is actually slice in the rational homology ball in Figure 4. Thus, the two spin$^c$-structures for which $\tau_s(Y, K)$ vanishes must be the spin$^c$-structures that extend.

Now consider the knot $J$ shown on the right in Figure 5. This knot is also order two in $H_1(L(4, 1))$ and we can construct a Heegaard diagram for $J$ as pictured on the right in Figure 6. From the Heegaard diagram we can compute the Alexander gradings of each element. If we consider the rational 3-genus of the knot $J$, using Ni’s formula from equation (1) we see that if $F$ is a rational Seifert surface for $J$, a rational Seifert surface $F$ and using the $c_1$-evaluation formula from Proposition 2.9. The Alexander gradings are listed in the table in Figure 1.
then $\chi(F) \geq 4$. Since a minimal genus Seifert surface will have either one or two boundary components, this means that we must have $g(F) \geq 2$.

We can also determine the structure of $\text{CFK}^\infty(L(4,1), J, s_c)$ from the Heegaard diagram. In particular, we have the following spin$^c$-equivalence classes for the generators:

- $s_a = s_w(a)$;
- $s_\alpha = s_w(b_1) = s_w(b_2) = s_w(b_3)$;
- $s_c = s_w(c_1) = s_w(c_2) = s_w(c_3)$;
- $s_d = s_w(d_1) = s_w(d_2) = s_w(d_3)$.

Putting this information together we can compute $\tau_s(Y, K)$ for each spin$^c$-structure as well as the corresponding $\tau$-invariants.

$$\tau_{s_a}(L(4,1), J) = 0; \quad \tau_{s_b}(L(4,1), J) = \frac{3}{2};$$
$$\tau_{s_c}(L(4,1), J) = 1; \quad \tau_{s_d}(L(4,1), J) = \frac{1}{2}.$$

Using our previous calculation for $\tau_s(L(4,1), K)$, we can compute that $s_a$ and $s_c$ both extend over the rational homology ball bounded by $L(4,1)$, thus we see that $J$ is not slice. In particular, if $\Sigma$ is a slicing surface for $J$, then $g(\Sigma) \geq 1$. Note that the bound we obtain is lower than the 3-dimensional genus bound coming from equation (1). On the other hand, by a remark of Celoria [Cel16, Remark 17], since $\tau_{s_a}(L(4,1), J) = 0$, we also know that $J$ cannot be rationally concordant to a connected sum of knots $K \# K'$ for $K'$ in $S^3$ with $\tau(K') = 1$. 

| x  | a  | b_1 | b_2 | b_3 | c_1 | c_2 | c_3 | d_1 | d_2 | d_3 |
|----|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| A(x) | 0  | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}$ | $-1$ | $0$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |

Table 2. Alexander gradings for generators of $\hat{CF}(Y, J)$. 

Figure 7. $\text{CFK}^\infty(L(4,1), J, s_c)$
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