New Explicitly Diagonalizable Hankel Matrices Related to the Stieltjes–Carlitz Polynomials

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Abstract. Four new examples of explicitly diagonalizable Hankel matrices depending on a parameter $k \in (0, 1)$ are presented. The Hankel matrices are regarded as matrix operators on the Hilbert space $\ell^2(N_0)$ and the solution of the spectral problem is based on an application of the commutator method. Each of the Hankel matrices commutes with a Jacobi matrix which is related to a particular family of the Stieltjes–Carlitz polynomials. More examples of explicitly diagonalizable structured matrix operators are obtained when taking into account also weighted Hankel matrices.

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1. Introduction

To the authors’ best knowledge, the generalized Hilbert matrix

$$H(\theta)_{m,n} := 1/(m + n + \theta),$$

with $m, n \in N_0$ and $\theta \in \mathbb{R} \setminus (-N_0)$ being a parameter, is the only known example of a Hankel matrix which is explicitly diagonalizable if regarded as a self-adjoint matrix operator on the Hilbert space $\ell^2(N_0)$. A solution of the spectral problem for the Hilbert matrix $H(0)$ was known already to Magnus [13]. Later on Rosenblum described in [16] an explicit diagonalization of the generalized Hilbert matrix $H(\theta)$. Let us recall that $H(\theta)$ always represents a bounded operator on $\ell^2(N_0)$, its singular continuous spectrum is always empty, the absolutely continuous spectrum always fills the interval $[0, \pi]$, and for $\theta < 1/2$ the point spectrum of $H(\theta)$ is non-empty with the only possible eigenvalues $\pm \pi / \sin(\pi \theta)$ whose multiplicities are finite and depend on $\theta$. 
Rosenblum’s approach to the solution of the spectral problem is quite universal and is based on the powerful commutator method. In [16], he showed the matrix operator $H(\theta)$ to be unitarily equivalent to an integral operator on $L^2(\mathbb{R}_+, dx)$, and found a Sturm–Liouville operator on $\mathbb{R}_+$ commuting with that integral operator. Moreover, the Sturm–Liouville operator turned out to be explicitly diagonalizable with a simple spectrum. The desired result then followed rather straightforwardly.

The commutator method can be effectively used in various similar situations. For instance, a systematic application of the method to Hankel integral operators can be found in [20]. For our purposes it is substantial to note that it is possible to avoid an intermediate step in Rosenblum’s solution when $H(\theta)$ is transformed to an integral operator. As discussed in detail in [10], there exists a Jacobi matrix $J(\theta)$ commuting with $H(\theta)$. Moreover, $J(\theta)$ has quite nice properties since the associated orthogonal polynomial sequence is formed by the dual continuous Hahn polynomials and as such it is included in the Askey classification scheme. The corresponding normalized measure of orthogonality is unique (the determinate case) and is known explicitly [19]. For $J(\theta)$ this means that it is explicitly diagonalizable if regarded as a matrix operator on $\ell^2(\mathbb{N}_0)$. Moreover, this is a general feature that the spectrum of a Jacobi matrix operator is simple. Diagonalization of $H(\theta)$ is then a direct corollary provided one is able to evaluate the eigenvalues of $H(\theta)$. It turns out that, to this end, an additional piece of information is needed, namely a generating function for the orthogonal polynomial sequence in question written in an appropriate form.

The family of explicitly diagonalizable structured matrix operators can be substantially extended if one considers not only Hankel matrices but also weighted Hankel matrices. This possibility was systematically explored in [10] with the restriction that the commuting Jacobi matrix $J(a, b, c)$ is still related to the dual continuous Hahn polynomials which depend on three parameters $a, b, c$. In a recent paper [17] other orthogonal polynomial sequences from the Askey scheme are taken into account and several new examples of explicitly diagonalizable weighted Hankel matrices are presented.

In the current paper we still stick to this approach. In a recent study [18], it is shown that the generalized Hankel matrix is the only infinite-rank Hankel matrix which, if regarded as an operator on $\ell^2(\mathbb{N}_0)$, is diagonalizable by application of the commutator method to Jacobi matrices associated with polynomial families from the Askey scheme. Therefore, when attempting to find new diagonalizable Hankel matrices, we have to go beyond the Askey scheme. In this article, we focus on the Stieltjes–Carlitz polynomials whose basic properties were also well studied. A brief summary is given below in Sect. 2.2. This means in particular that the considered commuting Jacobi matrices have rather special form depending on five parameters. Of course, this implies, too, a restriction on the class of Hankel or weighted Hankel matrices we wish to explore. Notably, four explicitly diagonalizable Hankel matrices have been discovered within this class, and this is the main result presented in the paper. This list is then completed by additional examples of weighted Hankel matrices with the same property. In contradiction to the
Hilbert matrix, all studied matrix operators belong to the trace class and therefore they have a pure point spectrum.

Let us now make the settings of the present paper more precise. We seek Hankel matrices or, more generally, weighted Hankel matrices commuting with the Jacobi matrix

\[
J = \begin{pmatrix}
\beta_0 & \alpha_0 &  & \\
\alpha_0 & \beta_1 & \alpha_1 & \\
& \alpha_1 & \beta_2 & \alpha_2 \\
& & \ddots & \ddots & \ddots
\end{pmatrix}
\]  

(1.1)

whose entries are of the form

\[
\alpha_n := -\sqrt{(n + 1)(n + a + 1)(n + b + 1)(n + c + 1)}, \\
\beta_n := (k + k^{-1})n(n + \sigma).
\]  

(1.2)

Our choice of the parameters guarantees that \( J \) is a non-decomposable Hermitian matrix, namely \( k \in (0, 1) \), \( \sigma \in \mathbb{R} \), and \( a, b, c > -1 \). Later on, however, these parameters will be further specialized in order to obtain Jacobi operators with an explicitly solvable spectral problem.

Let us also note that in all cases studied in the sequel the Jacobi matrix (1.1), (1.2) represents a unique self-adjoint operator on \( \ell^2(\mathbb{N}_0) \). This is why we can afford to be less scrupulous in the notation when we are using the same symbol for a Jacobi matrix and the corresponding operator.

The paper is organized as follows. Section 2 summarizes some preliminary information which is then needed in the remainder of the paper. An important role in the entire paper is played by elliptic functions and integrals and this is the subject of Sect. 2.1. Section 2.2 is devoted to the Stieltjes–Carlitz polynomials. Section 3 contains some technical auxiliary results which are then used in the proofs of the presented theorems. In Sect. 4, a three-term recurrence equation is studied with coefficients depending linearly on the index. The purpose of this study is the fact that the commutation equation between a Hankel and a Jacobi matrix in our case finally leads to such a three-term recurrence. This type of equation is rather general, however, and, as we suppose, it may be encountered also in other problems. The main goal of Sect. 5 is to determine which Jacobi matrices of the form (1.1), (1.2) admit a nontrivial commuting Hankel matrix. Section 6 contains the main result of the paper, i.e. some examples of explicitly diagonalizable Hankel matrices. In addition, the list of explicitly diagonalizable structured matrix operators is extended in Sects. 7 and 8 by considering also weighted Hankel matrices.
2. Preliminaries

2.1. Jacobian Elliptic Functions

We start from recalling the definition of the complete elliptic integrals of the first kind,

\[ K = K(k) := \frac{\pi}{2} \, _2F_1 \left( \frac{1}{2}, \frac{1}{2} \bigg| k^2 \right) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \]

\[ K' = K'(k) := K(\sqrt{1 - k^2}), \]

and the elliptic nome

\[ q = q(k) := \exp(-\pi K'(k)/K(k)), \quad (2.1) \]

where \( k \in (0, 1) \); see, for example, [14, Chp. 19]. Then \( q \in (0, 1) \).

We shall also need the familiar integral representation of the Gauss hypergeometric function [14, Eq. 15.6.1]

\[ _2F_1 \left( a, b \bigg| c \bigg| z \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt, \quad (2.2) \]

where \( \text{Re}c > \text{Re}b > 0 \).

A remark to the notation. In displayed formulas we shall denote the hypergeometric functions as in Eq. (2.2). In in-line formulas, however, we prefer the equivalent expression \( _2F_1(a, b; c; z) \).

Further we recall several selected properties of the Jacobian elliptic functions \( \text{sn}(z) = \text{sn}(z,k) \), \( \text{cn}(z) = \text{cn}(z,k) \), and \( \text{dn}(z) = \text{dn}(z,k) \) that will be needed in the sequel. The reader is referred, for example, to [12] for the theory of the elliptic functions and to [14, Chp. 22] for an easily accessible review of their fundamental properties. As usual, the dependence of the elliptic functions on the parameter \( k \) will not be indicated explicitly in the notation in most cases.

First, the squares of the Jacobian elliptic functions are mutually related as follows [14, Eq. 22.6.1]

\[ \text{sn}^2(z) + \text{cn}^2(z) = k^2\text{sn}^2(z) + \text{dn}^2(z) = 1. \quad (2.3) \]

Second, we will need the formulas for the first derivatives [14, Table 22.13.1]

\[ \frac{d\text{sn}(z)}{dz} = \text{cn}(z)\text{dn}(z), \quad \frac{d\text{cn}(z)}{dz} = -\text{sn}(z)\text{dn}(z), \quad \frac{d\text{dn}(z)}{dz} = -k^2\text{sn}(z)\text{cn}(z). \quad (2.4) \]

Third, we have the special values [14, Table 22.5.1]

\[ \text{sn}(0) = 0, \quad \text{cn}(0) = 1, \quad \text{dn}(0) = 1. \quad (2.5) \]

and

\[ \text{sn}(K) = 1, \quad \text{cn}(K) = 0, \quad \text{dn}(K) = \sqrt{1 - k^2}. \quad (2.6) \]
Finally, recall the Fourier series [14, Eqs. 22.11.1-3]

\[
\text{sn}\left(\frac{2Kv}{\pi}\right) = \frac{2\pi}{kK} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 - q^{2n+1}} \sin\left((2n + 1)v\right), \quad (2.7)
\]

\[
\text{cn}\left(\frac{2Kv}{\pi}\right) = \frac{2\pi}{kK} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 + q^{2n+1}} \cos\left((2n + 1)v\right), \quad (2.8)
\]

\[
\text{dn}\left(\frac{Kv}{\pi}\right) = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} \cos(nv), \quad (2.9)
\]

and also [11]

\[
\text{sn}^2\left(\frac{Kv}{\pi}\right) = \frac{K - E(k)}{k^2K} - \frac{2\pi^2}{k^2K^2} \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} \cos(nv), \quad (2.10)
\]

\[
\text{sn}^3\left(\frac{2Kv}{\pi}\right) = \frac{\pi}{k^3K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 - q^{2n+1}} \left(1 + k^2 - \frac{(2n + 1)^2\pi^2}{4K^2}\right) \sin\left((2n + 1)v\right), \quad (2.11)
\]

where \(E\) is the complete elliptic integral of the second kind, see [14, Chp. 19]. The above Fourier expansions hold true for all \(v \in \mathbb{R}\) and \(k \in (0, 1)\).

### 2.2. The Stieltjes–Carlitz Polynomials

In [4], Carlitz investigated four families of orthogonal polynomials obtained from certain formulas for the Laplace transform of the Jacobian elliptic functions studied earlier by Stieltjes and Rogers. Two of the families are symmetric orthogonal polynomials. Consequently, each of these two families gives rise to other two families of orthogonal polynomials, see [5, Chp. I, Sec. 9 and Chp. VI, Sec. 9]. In total, there are six families of orthogonal polynomials intimately related to the Jacobian elliptic functions. For the sake of definiteness, we call them Family #1 - 6 because it seems that there are no commonly used names for these families in the literature.

Below we list their basic properties that will be needed further. All these properties are either contained directly in [4] or can be straightforwardly derived from the results in this reference. Namely, we recall the three-term recurrences, orthogonality relations, and generating functions. Let us note that, following the original paper, these polynomials are defined in their monic form, i.e., they fulfill a three-term recurrence of the form

\[
P_{n+1}(x) = (x - \beta_n)P_n(x) - \alpha_{n-1}^2 P_{n-1}(x), \quad n \in \mathbb{N}_0,
\]

\((\alpha_{-1} \text{ is arbitrary})\) with the standard initial conditions \(P_{-1}(x) = 0\) and \(P_0(x) = 1\). Moreover, all families depend on a parameter \(k\). It is always assumed that \(k \in (0, 1)\) in which case every family has a unique measure of orthogonality. In other words, the respective Hamburger moment problems are all determinate, see [5, Chp. VI, Sec. 9] or [7, Sec. 21.9] and references therein. Consequently, the Jacobi matrices corresponding to the families of orthogonal polynomials listed below give rise to unique self-adjoint Jacobi operators; see [2] for the general theory.
Family #1. The three-term recurrence:
\[ f_{n+1}(x) = (x+(k^2+1)(2n+1)^2) f_n(x) - k^2(2n-1)(2n)^2(2n+1) f_{n-1}(x), \quad n \geq 0. \] (2.12)

The orthogonality relation:
\[ \int_0^\infty f_n(-x) f_m(-x) \, d\mu(x) = k^{2n}(2n)! (2n+1)! \delta_{m,n}, \quad m, n \geq 0, \] (2.13)

where
\[ \mu = \frac{\pi^2}{K^2 k} \sum_{m=0}^{\infty} \frac{(2m+1) q^{m+1/2}}{1 - q^{2m+1}} \delta_{\lambda_m} \quad \text{and} \quad \lambda_m = \frac{\pi^2(2m+1)^2}{4K^2}. \] (2.14)

Here and below, \( \delta_x \) denotes the unit-mass Dirac delta measure supported on the one-point set \( \{x\} \).

The generating function:
\[ \sum_{n=0}^{\infty} \frac{f_n(x)}{(2n+1)!^2} \text{sn}^{2n+1}(u) = \frac{\sinh(\sqrt{x} u)}{\sqrt{x}}. \] (2.15)

Family #2. The three-term recurrence:
\[ g_{n+1}(x) = (x+(k^2+1)(2n+2)^2) g_n(x) - k^2(2n)(2n+1)^2(2n+2) g_{n-1}(x), \quad n \geq 0. \] (2.16)

The orthogonality relation:
\[ \int_0^\infty g_n(-x) g_m(-x) \, d\mu(x) = \frac{k^{2n}(2n+1)! (2n+2)!}{2} \delta_{m,n}, \quad m, n \geq 0, \] (2.17)

where
\[ \mu = \frac{\pi^4}{K^4 k} \sum_{m=1}^{\infty} \frac{m^3 q^m}{1 - q^{2m}} \delta_{\lambda_m} \quad \text{and} \quad \lambda_m = \frac{\pi^2 m^2}{K^2}. \] (2.18)

The generating function:
\[ \sum_{n=0}^{\infty} \frac{g_n(x)}{(2n+1)!^2} \text{sn}^{2n+1}(u) = \frac{\sinh(\sqrt{x} u)}{\sqrt{x} \text{cn}(u) \text{dn}(u)}. \] (2.19)

Families #3 and #4. The three-term recurrences:
\[ p_{n+1}(x) = (x-k^2(2n)^2 -(2n+1)^2) p_n(x) - k^2(2n)^2(2n-1)^2 p_{n-1}(x), \quad n \geq 0, \] (2.20)

and
\[ q_{n+1}(x) = (x-(2n+1)^2-k^2(2n+2)^2) q_n(x) - k^2(2n+1)^2(2n)^2 q_{n-1}(x), \quad n \geq 0. \] (2.21)

The orthogonality relations:
\[ \int_0^\infty p_n(x) p_m(x) \, d\mu(x) = k^{2n}((2n)!)^2 \delta_{m,n}, \quad m, n \geq 0, \] (2.22)

and
\[ \int_0^\infty q_n(x) q_m(x) x \, d\mu(x) = k^{2n}((2n+1)!)^2 \delta_{m,n}, \quad m, n \geq 0, \] (2.23)
where
\[
\mu = \frac{2\pi}{Kk} \sum_{m=0}^{\infty} \frac{q^{m+1/2}}{1 + q^{2m+1}} \delta \lambda_m \quad \text{and} \quad \lambda_m = \frac{\pi^2 (2m + 1)^2}{4K^2}.
\] (2.24)

The generating functions:
\[
\sum_{n=0}^{\infty} \frac{(-1)^n p_n(x)}{(2n)!} \text{sn}^{2n}(u) = \frac{\cos(\sqrt{x}u)}{\text{cn}(u)}
\] (2.25)
and
\[
\sum_{n=0}^{\infty} \frac{(-1)^n q_n(x)}{(2n + 1)!} \text{sn}^{2n+1}(u) = \frac{\sin(\sqrt{x}u)}{\sqrt{x} \text{dn}(u)}.
\] (2.26)

**Families #5 and #6.** The three-term recurrences:
\[
r_{n+1}(x) = (x - (2n)^2 - k^2(2n+1)^2) r_n(x) - k^2(2n)^2(2n-1)^2 r_{n-1}(x), \quad n \geq 0,
\] (2.27)
and
\[
s_{n+1}(x) = (x - k^2(2n+1)^2 - (2n+2)^2) s_n(x) - k^2(2n+1)^2(2n)^2 s_{n-1}(x), \quad n \geq 0.
\] (2.28)

The orthogonality relations:
\[
\int_0^{\infty} r_n(x) r_m(x) \, d\mu(x) = k^{2n}((2n)!)^2 \delta_{m,n}, \quad m, n \geq 0,
\] (2.29)
and
\[
\int_0^{\infty} s_n(x) s_m(x) x \, d\mu(x) = k^{2n+2}((2n+1)!)^2 \delta_{m,n}, \quad m, n \geq 0,
\] (2.30)
where
\[
\mu = \frac{2\pi}{K} \sum_{m=0}^{\infty} \frac{q^m}{1 + q^{2m}} \delta \lambda_m \quad \text{and} \quad \lambda_m = \frac{\pi^2 m^2}{K^2}.
\] (2.31)

The generating functions:
\[
\sum_{n=0}^{\infty} \frac{(-1)^n r_n(x)}{(2n)!} \text{sn}^{2n}(u) = \frac{\cos(\sqrt{x}u)}{\text{dn}(u)}
\] (2.32)
and
\[
\sum_{n=0}^{\infty} \frac{(-1)^n s_n(x)}{(2n + 1)!} \text{sn}^{2n+1}(u) = \frac{\sin(\sqrt{x}u)}{\sqrt{x} \text{cn}(u)}.
\] (2.33)

### 3. Several Auxiliary Results

The asymptotic expansion to the leading order of the Stieltjes–Carlitz polynomials can be derived in a comparatively straightforward way and this fact was already exploited by some authors [9]. For the sake of completeness and since the asymptotic formulas will be of some importance in the sequel, the result is presented here, too.
Proposition 3.1. The leading terms in the asymptotic expansion of the Stieltjes–Carlitz polynomials $p_n(x)$, $q_n(x)$, $r_n(x)$ and $s_n(x)$, defined in (2.20), (2.21), (2.27) and (2.28), respectively, are as follows

\[
\frac{p_n(x)}{(2n)!} = \frac{(-1)^n}{\sqrt{n\pi}} \cos(\sqrt{x}K) + o\left(\frac{1}{n}\right),
\]
\[
\frac{q_n(x)}{(2n+1)!} = \frac{(-1)^n}{2(1-k^2)\sqrt{n\pi n^{3/2}}} \cos(\sqrt{x}K) + o\left(\frac{1}{n^2}\right),
\]
\[
\frac{r_n(x)}{(2n)!} = \frac{(-1)^{n+1}}{2(1-k^2)\sqrt{n\pi n^{3/2}}} \sqrt{x} \sin(\sqrt{x}K) + o\left(\frac{1}{n^2}\right),
\]
\[
\frac{s_n(x)}{(2n+1)!} = \frac{(-1)^n}{\sqrt{n\pi n}} \frac{\sin(\sqrt{x}K)}{\sqrt{x}} + o\left(\frac{1}{n}\right),
\]
as $n \to \infty$. Here $x$ is an arbitrary fixed complex number.

Proof. The generating function (2.25) can be rewritten as

\[
\sum_{n=0}^{\infty} \frac{(-1)^np_n(x)}{(2n)!} \xi^n = \frac{\cos(\sqrt{x}\text{sn}^{-1}(\sqrt{\xi}))}{\sqrt{1-\xi}} := g(\xi).
\]

The singularity of $g(\xi)$ located most closely to the origin occurs at $\xi = 1$. One can apply the Darboux method with the comparison function

\[
g_c(\xi) := \frac{\cos(\sqrt{x}K)}{\sqrt{1-\xi}},
\]

see [15, Sec. 8.9] and especially the refinements in §8.9.3. In order to get the asymptotic formula for $p_n(x)$ it suffices to observe that the restriction of $g(\xi) - g_c(\xi)$ to the unite circle $\xi = e^{i\theta}$, $\theta \in [0, 2\pi]$, is continuous. Moreover, the restriction of the derivative $g'(\xi) - g'_c(\xi)$ to the unite circle is continuous, too, except singularities at $\theta = 0$ and $\theta = 2\pi$ which are integrable, however. More precisely, the singularity at $\theta = 0$ is of order $\theta^{-1/2}$, and similarly for $\theta = 2\pi$.

As is the main idea of the Darboux method, these feature make it possible to effectively compare the coefficients in the power series expansions of $g(\xi)$ and $g_c(\xi)$. We skip further details of this standard approach.

Very analogously one can proceed in the case of polynomials $q_n(x)$, $r_n(x)$ and $s_n(x)$. Omitting additional details we confine ourselves to pointing out equations to which the Darboux method can be applied. By differentiating (2.26) and using the rules (2.3) and (2.4) we obtain

\[
\sum_{n=0}^{\infty} \frac{(-1)^nq_n(x)}{(2n)!} \xi^n = \frac{\cos(\sqrt{x}\text{sn}^{-1}(\sqrt{\xi}))}{\sqrt{1-\xi}} + \frac{k^2 \sin(\sqrt{x}\text{sn}^{-1}(\sqrt{\xi}))}{\sqrt{x}(1-k^2\xi)^{3/2}} \sqrt{\xi}.
\]

Similarly one can treat Eq. (2.32) to get

\[
\sum_{n=1}^{\infty} \frac{(-1)^nr_n(x)}{(2n-1)!} \xi^n = \frac{-\sqrt{x}\xi \sin(\sqrt{x}\text{sn}^{-1}(\sqrt{\xi}))}{\sqrt{1-\xi}} + \frac{k^2 \cos(\sqrt{x}\text{sn}^{-1}(\sqrt{\xi}))}{(1-k^2\xi)^{3/2}} \xi.
\]
Finally, Eq. (2.33) can be quite straightforwardly rewritten as
\[
\sum_{n=0}^{\infty} \frac{(-1)^n s_n(x)}{(2n+1)!} \xi^n = \left. \frac{\sin(\sqrt{x} s_n^{-1}(\sqrt{\xi}))}{\sqrt{x} \xi^{1/2} \sqrt{1-\xi}} \right|.
\]
The desired asymptotic formulas follow. \(\square\)

**Lemma 3.2.** We have
\[
\int_{0}^{K} \sn^2(u) \cn^2(u) \, du = \sqrt{\frac{\pi}{1-k^2}} \frac{1}{4n^{3/2}} (1 + o(1)),
\]
\[
\int_{0}^{K} \sn^2(u) \dn^2(u) \, du = \sqrt{\frac{(1-k^2)\pi}{2n^{1/2}}} (1 + o(1)), \quad \text{as } n \to \infty.
\]

**Proof.** The former integral, if written in the form
\[
\int_{0}^{K} e^{-np(u)} q(u) \, du,
\]
with \(p(u) := -2 \ln(\sn(u))\) and \(q(u) := \cn^2(u)\), admits a direct application of the Laplace method, see see for instance [15, Sec. 3.7]. Note that \(p(u)\) is strictly decreasing for \(u \in (0, K]\) and
\[
p(u) = (1-k^2)(u-K)^2 + O((u-K)^4),
\]
\[
q(u) = (1-k^2)(u-K)^2 + O((u-K)^4), \quad \text{as } u \to K.
\]
In the case of the latter integral we keep the function \(p(u)\) but now we let
\[
q(u) := \dn^2(u) = 1-k^2 + O((u-K)^2), \quad \text{as } u \to K.
\]
Again, the Laplace method gives the result. \(\square\)

**Proposition 3.3.** For \(x \in \mathbb{C}\) and the Stieltjes–Carlitz polynomials \(p_n(x), q_n(x), r_n(x)\) and \(s_n(x)\), defined in (2.20), (2.21), (2.27) and (2.28), respectively, it holds true that
\[
\sum_{n=0}^{\infty} \frac{(-1)^n E_n(x)}{(2n)!} p_n(x) = \int_{0}^{K} \cos(\sqrt{x} u) \cn(u) \, du, \quad (3.1)
\]
\[
\sum_{n=0}^{\infty} \frac{(-1)^n F_{n+1}(x)}{(2n+1)!} q_n(x) = \int_{0}^{K} \cos(\sqrt{x} u) \cn(u) \, du, \quad (3.2)
\]
\[
\sum_{n=0}^{\infty} \frac{(-1)^n F_n(x)}{(2n)!} r_n(x) = \int_{0}^{K} \cos(\sqrt{x} u) \dn(u) \, du \quad (3.3)
\]
and
\[
\sum_{n=0}^{\infty} \frac{(-1)^n E_{n+1}(x)}{(2n+1)!} s_n(x)
\]
\[
= -\frac{\sqrt{1-k^2} \sin(\sqrt{x} K)}{k^2 \sqrt{x}} + \frac{1}{k^2} \int_{0}^{K} \cos(\sqrt{x} u) \dn(u) \, du. \quad (3.4)
\]
where

\[ E_n(k) := \int_0^1 t^{2n} \sqrt{\frac{1-t^2}{1-k^2t^2}} \, dt = \frac{\pi(2n)!}{2^{2n+2}n!(n+1)!} 2F_1\left( \frac{n + \frac{1}{2}, \frac{1}{2}}{n+2} \left| k^2 \right. \right), \tag{3.5} \]

\[ F_n(k) := \int_0^1 t^{2n} \sqrt{\frac{1-t^2}{1-k^2t^2}} \, dt = \frac{\pi(2n)!}{2^{2n+1}(n+1)!} 2F_1\left( \frac{n + \frac{1}{2}, -\frac{1}{2}}{n+1} \left| k^2 \right. \right), \tag{3.6} \]

for \( n \geq 0 \) (the latter Eqs. in (3.5), (3.6) follow from (2.2)).

Proof. Substitution \( t = \text{sn}(u) \) in the integral (3.5) brings the LHS of (3.1) to the form

\[ \sum_{n=0}^{\infty} \frac{(-1)^n p_n(x)}{(2n)!} \int_0^K \text{sn}^2(u) \text{cn}^2(u) \, du. \]

Then after interchanging the integral and the sum and using the generating function (2.25) one arrives at the RHS of (3.1). The interchanging of summation and integration is justified by the Fubini theorem and the respective asymptotic formulas in Proposition 3.1 and Lemma 3.2.

Analogously, substitution \( t = \text{sn}(u) \) in the integral (3.6) brings the LHS of (3.2) to the form

\[ \sum_{n=0}^{\infty} \frac{(-1)^n q_n(x)}{(2n+1)!} \int_0^K \text{sn}^{2n+2}(u) \, du. \]

By interchanging the integral and the sum and using the generating function (2.26) one obtains the expression

\[ \int_0^K \frac{\sin(\sqrt{x}u)}{\sqrt{x}} \text{sn}(u) \, du. \]

Integrating by parts, using (2.4) and the special values (2.5), (2.6) leads to the RHS of (3.2). The interchanging of summation and integration is again possible owing to the respective asymptotic formulas in Proposition 3.1 and Lemma 3.2.

With the aid of the same substitution as above the LHS of (3.3) is transformed to

\[ \sum_{n=0}^{\infty} \frac{(-1)^n r_n(x)}{(2n)!} \int_0^K \text{sn}^2(u) \, du. \]

Relying on Proposition 3.1 and Lemma 3.2 one can interchange the integral and the sum, and using the generating function (2.32) one arrives at the RHS of (3.3).

Very analogously as in the foregoing equations the LHS of (3.4) is shown to be equal to

\[ \sum_{n=0}^{\infty} \frac{(-1)^n s_n(x)}{(2n+1)!} \int_0^K \text{sn}^{2n+2}(u) \text{cn}^2(u) \, du. \]
Interchanging the integral and the sum is again justifiable, and using the generating function (2.33) one obtains the expression
\[ \int_0^K \frac{\sin(\sqrt{x}u)}{\sqrt{x}} \text{sn}(u)\text{cn}(u) \, du. \]

Now we can integrate by parts while taking into account (2.4), (2.5) and (2.6), and we get the desired identity. \qed

4. A Three-Term Recurrence Equation with Coefficients Depending Linearly on the Index

The problem of finding Hankel matrices commuting with a given Jacobi matrix of the form (1.1) finally leads to a certain three-term recurrence equation. More explicitly, we will discuss the three-term recurrence

\[ (k + k^{-1})(n + \sigma)h_n - (n + \xi)h_{n-1} - (n + \eta)h_{n+1} = 0, \quad n \geq 1, \quad (4.1) \]

where \( k \in (0, 1) \) and \( \sigma, \xi, \eta \in \mathbb{C} \) are parameters. We claim that, up to a constant multiplier, Eq. (4.1) has exactly one square summable solution \((h_n)_{n \geq 0}\). One can show that this is true even for a somewhat more general type of equation.

**Lemma 4.1.** In the three-term recurrence equation

\[ (k + k^{-1})(1 + s_n)h_n - (1 + x_n)h_{n-1} - (1 + y_n)h_{n+1} = 0, \quad n \geq 1, \quad (4.2) \]

where \( k \in (0, 1) \) and \((s_n)_{n \geq 1}\), \((x_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) are given complex sequences, assume that \( 1 + x_n \neq 0 \) for all \( n \geq 1 \). If

\[ \lim_{n \to \infty} s_n = \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0 \]

then, up to a constant multiplier, there exists exactly one solution to (4.2) which is square summable.

**Proof.** The proof is based on routine manipulations and thus we only give a sketch of it. It clearly suffices to consider Eq. (4.2) on a neighborhood of \( \infty \) determined by a lower bound \( n \geq N \), with \( N \in \mathbb{N} \) sufficiently large. This will be specified more precisely later on. In view of the assumption \( 1 + x_n \neq 0 \) for \( n \geq 1 \), the beginning of the sequence \((h_n)_{n \geq 0}\) can be obtained by a descending three-term recurrence.

Denote by \( e_n, \ n = 0, 1, 2, \ldots \), the semi-infinite column vectors with all zero entries except a unit on the \( n \)th position (counting from 0 upwards). Furthermore, \( k \) is another semi-infinite column vector,

\[ k := (1, k, k^2, \ldots)^T. \]

Next we introduce two semi-infinite matrices, \( L \) and \( G \), defined as follows,

\[ L_{m,n} := (k + k^{-1})\delta_{m,n} - \delta_{m,n+1} - \delta_{m+1,n}, \quad G_{m,n} := \frac{k^{m-n+1}}{1 - k^2}, \quad m, n \in \mathbb{N}_0. \quad (4.3) \]
The matrices satisfy the equations

\[ \begin{align*}
    L G &= I + \frac{k^2}{1 - k^2} e_0 k^T, \\
    G L &= I + \frac{k^2}{1 - k^2} k e_0^T,
\end{align*} \tag{4.4} \]

where \( I \) is the unit matrix. The matrix product of \( L \) and \( G \) makes good sense since \( L \) is a band matrix.

Let us write

\[ \tilde{h}_n := h_{N+n}, \quad n \in \mathbb{N}_0. \]

Then \((\tilde{h}_n)_{n \geq 0}\) is supposed to solve the equation

\[ (k+k^{-1})\tilde{h}_n - \tilde{h}_{n-1} - \tilde{h}_{n+1} = (- (k+k^{-1})s_{N+n}\tilde{h}_n + x_{N+n}\tilde{h}_{n-1} + y_{N+n}\tilde{h}_{n+1}), \tag{4.5} \]

for \( n \geq 1 \). Let us introduce yet another semi-infinite matrix \( R \),

\[ R_{m,n} := (- (k+k^{-1})s_{N+m}\delta_{m,n} - x_{N+m}\delta_{m,n+1} - y_{N+m}\delta_{m+1,n}), \quad m, n \in \mathbb{N}_0. \]

Then (4.5) means exactly that

\[ (L\tilde{h})_m = (R\tilde{h})_m \quad \text{for} \quad m \geq 1, \tag{4.6} \]

where \( \tilde{h} \) is the column vector with entries \( \tilde{h}_n, \quad n \geq 0. \)

\( G \) and \( R \) can be regarded as matrix operators on \( \ell^2(\mathbb{N}_0) \). As such, it is clear that both of them are bounded. We can even estimate

\[ \|R\| \leq (k + k^{-1} + 2) \sup_{k \geq N} \max\{|s_k|, |x_k|, |y_k|\}. \]

We choose \( N \) sufficiently large so that \( \|GR\| < 1 \).

For a square summable solution to (4.6) we can choose the vector

\[ \tilde{h} := (I - GR)^{-1}k = k + GR\tilde{h}. \]

Note that \((Lk)_n = 0 \) for \( n \geq 1 \). Referring to (4.4) we actually get, for \( n \geq 1 \),

\[ (L\tilde{h})_n = \left( I + \frac{k^2}{1 - k^2} e_0 k^T \right) (R\tilde{h})_n. \]

Conversely, bearing in mind (4.4) one finds that (4.6) implies, for \( n \geq 0 \),

\[ \tilde{h}_n = (GL\tilde{h})_n - \frac{k^2}{1 - k^2} \tilde{h}_0 k^n \]

\[ = (GR\tilde{h})_n - G_{n,0}(R\tilde{h})_0 + G_{n,0}(L\tilde{h})_0 - \frac{k^2}{1 - k^2} \tilde{h}_0 k^n. \]

In view of the form of \( G \) in (4.3) one concludes that there exists \( c \in \mathbb{C} \) such that

\[ \tilde{h} = GR\tilde{h} + ck. \]

It means that, for \( N \) sufficiently large so that \( \|GR\| < 1 \), \( \tilde{h} = c(I - GR)^{-1}k \)
which implies the uniqueness. \( \square \)
Remark 4.2. The uniqueness part of Lemma 4.1 can be also shown by using a generalization of Carleman’s criterion to complex Jacobi matrices. See for instance the discussion in Subsec. 2.2 in [3], and particularly Example 2.7 therein. The criterion has been shown for symmetric complex Jacobi matrices which is not our case, however. Fortunately, the symmetrization of Eq. (4.2) is a simple transformation which, owing to our assumptions, does not influence the square summability.

Further we wish to present a quadratic identity for the hypergeometric functions which will be helpful in the sequel. The identity may be new. At least we were not able to trace it out in the most common literature dedicated to the hypergeometric functions.

We first recall several identities for contiguous relations [1, Eq. 15.2.10]
\[
(c - a)\, _2F_1\left(\begin{array}{c}
a - 1, b \\
c \end{array} \right| z) + (2a - c + (b - a)z)\, _2F_1\left(\begin{array}{c}
a, b \\
c \end{array} \right| z)
\]
\[+ a(z - 1)\, _2F_1\left(\begin{array}{c}
a + 1, b \\
c \end{array} \right| z) = 0
\] (4.7)
and [1, Eqs. 15.2.18, 15.2.20]
\[
2\, _2F_1\left(\begin{array}{c}
a, b + 1 \\
c \end{array} \right| z) = \frac{a - c}{b(z - 1)}\, _2F_1\left(\begin{array}{c}
a - 1, b \\
c \end{array} \right| z) + \frac{c - a - b}{b(z - 1)}\, _2F_1\left(\begin{array}{c}
a, b \\
c \end{array} \right| z),
\] (4.8)
\[
2\, _2F_1\left(\begin{array}{c}
a, b \\
c + 1 \end{array} \right| z) = \frac{c}{(c - b)z}\, _2F_1\left(\begin{array}{c}
a - 1, b \\
c \end{array} \right| z) + \frac{c(z - 1)}{(c - b)z}\, _2F_1\left(\begin{array}{c}
a, b \\
c \end{array} \right| z).
\] (4.9)

Proposition 4.3. For \(a, b, c, d, z \in \mathbb{C}, |z| < 1\), it holds true that
\[
(a - c + 1)\, _2F_1\left(\begin{array}{c}
a, b \\
c \end{array} \right| z)\, _2F_1\left(\begin{array}{c}
a - c + 2, b - c + 1 \\\n2 - c \end{array} \right| z) - a\, _2F_1\left(\begin{array}{c}
a + 1, b \\
c \end{array} \right| z)
\times _2F_1\left(\begin{array}{c}
a - c + 1, b - c + 1 \\\n2 - c \end{array} \right| z) = (1-c)(1-z)^{-a-b+c-1}
\] (4.10)
provided all hypergeometric functions occurring in the expression are well defined.

Proof. The verification is very straightforward though rather tedious. We omit some computational details. Let
\[
F(z) := (1 - z)^{a+b-c+1}\left(\begin{array}{c}
a - c + 1 \\
c \end{array} \right)\, _2F_1\left(\begin{array}{c}
a, b \\
c \end{array} \right| z)\, _2F_1\left(\begin{array}{c}
a - c + 2, b - c + 1 \\\n2 - c \end{array} \right| z)
\]
\[- a\, _2F_1\left(\begin{array}{c}
a + 1, b \\
c \end{array} \right| z)\, _2F_1\left(\begin{array}{c}
a - c + 1, b - c + 1 \\\n2 - c \end{array} \right| z)
\].

We are going to show that \(F'(z) = 0\). The constant value of \(F(z)\) is then determined by putting \(z = 0\).

To evaluate the derivative one can use the well-known rule
\[
\frac{d}{dz} \, _2F_1\left(\begin{array}{c}
a, b \\
c \end{array} \right| z) = \frac{ab}{c}\, _2F_1\left(\begin{array}{c}
a + 1, b + 1 \\
c + 1 \end{array} \right| z).
\]
Afterwards we apply (4.8), (4.9) so that all hypergeometric functions occurring in the resulting expression have for the second parameter either \( b \) or \( b - c + 1 \) and, similarly, for the third parameter either \( c \) or \( 2 - c \). This way we get an equation of the form

\[
(1 - z)^{-a-b+c} F'(z) = \sum_{j=0}^{3} A_j \binom{a-c+j}{b-c+1} \binom{a-c+j}{b-c+1}
\]

where \( A_j \)'s are linear combinations of \( \binom{a-c+j}{b-c+1} \) for the second parameter either \( b \) or \( b - c + 1 \) and, similarly, for the third parameter either \( c \) or \( 2 - c \). Then one can use (4.7) to express the \( A_j \)'s as linear combinations of \( \binom{a-c+j}{b-c+1} \) and \( \binom{a-c+j}{b-c+1} \) only. Explicitly,

\[
A_0 = -\frac{(a - 1)a(a - c + 1)}{b z} F_1 \left( \frac{a + 1}{c} \right)
\]

\[
A_1 = \frac{a(a - c + 1)(a + b - c + 2)}{b z} F_1 \left( \frac{a}{c} \right) - \frac{a(a - c + 1)(c - 2a + (a - b)z)}{b z} F_1 \left( \frac{a + 1}{c} \right)
\]

\[
A_2 = \frac{(a - c + 1)(a + b - c + 2)(c - 2a - 2 + (a - b + 1)z)}{b z} F_1 \left( \frac{a}{c} \right) + \frac{a(a - c + 1)^2(z - 1)}{b z} F_1 \left( \frac{a + 1}{c} \right)
\]

\[
A_3 = -\frac{(a - c + 1)(a - c + 2)(a + b - c + 2)(z - 1)}{b z} F_1 \left( \frac{a}{c} \right)
\]

The equation can be then rewritten as

\[
(1 - z)^{-a-b+c} F'(z) = B_0 F_1 \left( \frac{a}{c} \right) + B_1 F_1 \left( \frac{a + 1}{c} \right)
\]

But with the aid of (4.7) it can be seen quite straightforwardly that \( B_0 = B_1 = 0 \).

**Lemma 4.4.** Assume that \( k \in (0, 1) \), \( \xi, \eta, \sigma \in \mathbb{C} \) and \( \xi - \eta \notin \mathbb{Z} \). Then the sequences \( (h_n^{(I)})_{n \geq N}, (h_n^{(II)})_{n \geq N} \), with

\[
h_n^{(I)} := k^n F_1 \left( \frac{n + \eta, \omega(\xi, \eta, \sigma)}{\eta - \xi} \left| 1 - k^2 \right| \right)
\]

\[
h_n^{(II)} := k^n \Gamma(n + \frac{\xi + 1}{n + \eta}) \Gamma(n + \eta) F_1 \left( \frac{n + \xi + 1, \omega(\xi, \eta, \sigma) + \xi - \eta + 1}{\xi - \eta + 2} \left| 1 - k^2 \right| \right)
\]

\[
\omega(\xi, \eta, \sigma) := \frac{-\xi - k^2 \eta + (1 + k^2)\sigma}{1 - k^2}
\]

are well defined for \( N \in \mathbb{N} \) sufficiently large and solve Eq. (4.1) for \( n > N \). Moreover,

\[
h_{n+1}^{(II)} - h_n^{(II)} = \frac{\Gamma(n + \xi + 1)}{\Gamma(n + \eta + 1)} (\xi - \eta + 1)k^{-2\xi - 2\omega(\xi, \eta, \sigma) - 1}, n \geq N.
\]
Hence these solutions are linearly independent.

Proof. With our assumptions, $h_n^{(I)}$ is well defined for all $n \in \mathbb{Z}$ and $h_n^{(II)}$ is well defined for all $n \in \mathbb{Z}, -n - \xi \notin \mathbb{N}$. Furthermore, it is straightforward to verify with the aid of (4.7) that both $(h_n^{(I)})$ and $(h_n^{(II)})$ satisfy (4.1) for $n > N$. Finally, (4.12) is a direct consequence of (4.10). □

Let us recall that

$$
2F_1\left( \begin{array}{c} a, b \\ a + b - c + 1 \end{array} \middle| 1 - z \right) = \frac{\Gamma(1 + a + b - c)\Gamma(1 - c)}{\Gamma(1 + a - c)\Gamma(1 + b - c)} 2F_1\left( \begin{array}{c} a, b \\ c \end{array} \middle| z \right)
+ \frac{\Gamma(1 + a + b - c)\Gamma(c - 1)}{\Gamma(a)\Gamma(b)} z^{1-c} 2F_1\left( \begin{array}{c} a - c + 1, b - c + 1 \\ 2 - c \end{array} \middle| z \right),
$$

(4.13)

see [1, Eq.15.3.6].

Proposition 4.5. Assume that $k \in (0,1)$, $\xi, \eta, \sigma \in \mathbb{C}$ and $-\xi \notin \mathbb{N}$. Let

$$
h_n^{(+)} := \frac{(1 - k^2)^{-\xi+n-1}k^n \Gamma(n + \xi + 1)}{\Gamma(n + \omega(\xi, \eta, \sigma) + \xi + 1)} 2F_1\left( \begin{array}{c} n + \eta, \omega(\xi, \eta, \sigma) \\ n + \xi + \omega(\xi, \eta, \sigma) + 1 \end{array} \middle| k^2 \right) = \frac{k^n \Gamma(n + \xi + 1)}{\Gamma(n + \omega(\xi, \eta, \sigma) + \xi + 1)} 2F_1\left( \begin{array}{c} n + \xi + 1, \omega(\xi, \eta, \sigma) + \xi - \eta + 1 \\ n + \omega(\xi, \eta, \sigma) + \xi + 1 \end{array} \middle| k^2 \right),
$$

(4.14)

with $\omega(\xi, \eta, \sigma)$ being defined in (4.11). Then, up to a constant multiplier, $(h_n^{(+)})_{n \geq 0}$ is the unique square summable solution of Eq. (4.1). Moreover,

$$
h_n^{(+)} = (1 - k^2)^{-\omega(\xi, \eta, \sigma) - \xi + n - \omega(\xi, \eta, \sigma)} n^\omega(\xi, \eta, \sigma) \left( 1 + O\left( \frac{1}{n} \right) \right)
$$

(4.15)

as $n \to \infty$.

Proof. The latter Eq. in (4.14) follows from the familiar identity [1, Eq.15.3.3]

$$
2F_1\left( \begin{array}{c} a, b \\ c \end{array} \middle| z \right) = (1 - z)^{c-a-b} 2F_1\left( \begin{array}{c} c - a, c - b \\ c \end{array} \middle| z \right).
$$

To show that $(h_n^{(+)})_{n \geq 0}$ is the sought solution we can make use of solutions $(h_n^{(I)})_{n \geq 0}$ and $(h_n^{(II)})_{n \geq 0}$ from Lemma 4.4 which are well defined for all $n \in \mathbb{Z}$ provided $\xi, -\eta \notin \mathbb{Z}$. Then a direct application of (4.13) yields

$$(1 - k^2)^{\xi-\eta+1}h_n^{(+)} = \frac{\Gamma(1 + \xi - \eta)}{\Gamma(1 + \omega(\eta, \xi, \sigma))} h_n^{(I)} + \frac{(1 - k^2)^{1+\xi-\eta}\Gamma(\eta - \xi - 1)}{\Gamma(\omega(\xi, \eta, \sigma))} h_n^{(II)}.
$$

Hence, under these restrictions, $(h_n^{(+)})_{n \geq 0}$ is also a solution to (4.1). But from (4.14) it is seen that if $-\xi \notin \mathbb{N}$ then $h_n^{(+)}$ is defined for all $n \in \mathbb{N}_0$ and depends continuously on $\xi$ and $\eta$. Hence the restriction on $\xi$ and $\eta$ can be relaxed while keeping only the assumption $-\xi \notin \mathbb{N}$. If so, $h_n^{(+)}$ satisfies (4.1).
To get the asymptotic expansion we can use Eq. 15.3.5 in [1],
\[ 2F_1\left(\frac{a, b}{cz} \bigg| z\right) = (1 - z)^{-b} 2F_1\left(b, c - a \bigg| \frac{z}{z - 1}\right), \]
and Eq. 15.7.1 ibidem,
\[ 2F_1\left(\frac{a, b}{c} \bigg| z\right) = 1 + O\left(\frac{1}{|c|}\right) \text{ as } |c| \to \infty, \]
with \(a, b, z\) fixed, to find that
\[ 2F_1\left(\frac{n + a, b}{n + c} \bigg| z\right) = (1 - z)^{-b} 2F_1\left(b, c - a \bigg| \frac{z}{z - 1}\right) = (1 - z)^{-b} \left(1 + O\left(\frac{1}{n}\right)\right) \]
as \(n \to \infty\). Strictly speaking, this reasoning is applicable only for \(|z| < 1/2\) but the asymptotic expansion is known to be valid also for \(|z| < 1\), see [6]. Furthermore, by Stirling’s formula,
\[ \frac{\Gamma(n + a)}{\Gamma(n + b)} = n^{a-b} \left(1 + O\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty. \]
Equation (4.15) follows. \(\square\)

5. General Commuting Hankel Matrix

Not all Jacobi matrices (1.1) with coefficients of the form (1.2) admit a non-trivial commuting Hankel matrix. The goal of the current section is to explore all possible cases within this class of Jacobi matrices when such a Hankel matrix exists. The starting point is the following lemma which is proven in [18, Lemma 3].

**Lemma 5.1.** Let \(p\) and \(q\) be complex functions which are meromorphic in a neighborhood of \(\infty\) and assume that the order of the pole at \(\infty\) equals 2 for both of them. Further let \(\epsilon \in \mathbb{C}, \epsilon \neq 0\), and put, for \(z, w \in \mathbb{C}\) sufficiently large,
\[ M(z, w) := \begin{pmatrix} p(z + \epsilon) - p(w - \epsilon) & q(z + \epsilon) - q(w - \epsilon) \\ p(z - \epsilon) - p(w + \epsilon) & q(z - \epsilon) - q(w + \epsilon) \end{pmatrix}. \]
Let us write the determinant of \(M(z, w)\) in the form
\[ \det M(z, w) = ((z - w)^2 - 4\epsilon^2) \delta(z, w). \]
If at least one of the functions \(p(z)\) and \(q(z)\) is not a polynomial in \(z\) of degree 2 and the set of functions \(\{1, p, q\}\) is linearly independent, then one of the following two cases happens:
(i) for every \(w \in \mathbb{C}\) sufficiently large there exists \( \lim_{z \to \infty} \delta(z, w) \in \mathbb{C}\\setminus\{0\}\),
(ii) for every \(w \in \mathbb{C}\) sufficiently large there exists \( \lim_{z \to \infty} z\delta(z, w) \in \mathbb{C}\\setminus\{0\}\).
Consequently, for every \(w \in \mathbb{C}\) sufficiently large there exists \(R(w) > 0\) such that for all \(z \in \mathbb{C}, |z| > R(w)\), the matrix \(M(z, w)\) is regular.
Consider a semi-infinite Jacobi (tridiagonal) matrix $J$, indexed by $m, n \in \mathbb{N}_0$, which is of the form (1.1) and is determined by the sequences $(\alpha_n)$ and $(\beta_n)$ given in (1.2), with $k \in (0,1)$, $a, b, c > -1$ and $\sigma \in \mathbb{R}$. Asymptotically we have

$$\alpha_n = -n^2 - (\xi + 2)n + A + O\left(\frac{1}{n}\right) \quad \text{as} \quad n \to \infty,$$

(5.1)

where

$$\xi = \frac{a + b + c}{2}.$$  

(5.2)

and

$$A = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc - 4a - 4b - 4c - 8.$$  

(5.3)

Note that $\alpha_{-1} = 0$.

Suppose $H$ is a Hankel matrix, $H_{m,n} = h_{m+n}$. Then $H$ and $J$ commute if and only if it holds true

$$(\alpha_n - \alpha_m)h_{n+m+1} + (\beta_n - \beta_m)h_{n+m} + (\alpha_{n-1} - \alpha_{m-1})h_{n+m-1} = 0,$$

(5.4)

for all $m, n \geq 0$. In particular, letting $m = 0$ we have

$$(\alpha_n - \alpha_0)h_{n+1} + (\beta_n - \beta_0)h_n + \alpha_{n-1}h_{n-1} = 0 \quad \text{for all} \quad n \geq 1.$$

Taking into account the descending recurrence it is clear that, for any $n \in \mathbb{N}_0$,

if $h_n = h_{n+1} = 0$ then $h_0 = h_1 = \ldots = h_n = h_{n+1} = 0$.

**Proposition 5.2.** Let $\alpha_n, \beta_n$ be the coefficients given in (1.2) and let $J$ be the associated Jacobi matrix. If there exists a nonzero Hankel matrix commuting with $J$ then $\alpha_n$ depends polynomially on $n$.

**Proof.** We will proceed by contradiction. Let us assume that $\alpha_n$ is not a polynomial in $n$ and a nontrivial solution $(h_n)_{n \geq 0}$ to (5.4) does exist. Without loss of generality we can assume that $h_n$ is real for all $n$. We will make use of the fact that $\alpha_n$ may be regarded as an analytic functions in $n$ for $n$ sufficiently large. Of course, we can make use as well of the fact that $\beta_n$ is a polynomial in $n$.

Along with (5.4) we will consider the equation

$$(\alpha_{n-1} - \alpha_{m+1})h_{n+m+1} + (\beta_{n-1} - \beta_{m+1})h_{n+m} + (\alpha_{n-2} - \alpha_{m})h_{n+m-1} = 0.$$  

(5.5)

From the asymptotic behavior, as $n \to \infty$, 

$$\alpha_n = -n^2 - (\xi + 2)n + O(1), \quad \alpha_{n-1} = -n^2 - \xi n + O(1),$$

it is obvious that $\{\alpha_n, \alpha_{n-1}, 1\}$ is linearly independent as a set of functions in $n$. Clearly, the same is true for both $\{\alpha_n, \beta_n, 1\}$ and $\{\beta_n, \alpha_{n-1}, 1\}$ since $\beta_n$ is a polynomial in $n$. 

Let

\[
\delta_1(n, m) := \det \begin{pmatrix}
\beta_n - \beta_m & \alpha_{n-1} - \alpha_{m-1} \\
\beta_{n-1} - \beta_{m+1} & \alpha_{n-2} - \alpha_m
\end{pmatrix},
\]

\[
\delta_2(n, m) := -\det \begin{pmatrix}
\alpha_n - \alpha_m & \alpha_{n-1} - \alpha_{m-1} \\
\alpha_{n-1} - \alpha_{m+1} & \alpha_{n-2} - \alpha_m
\end{pmatrix},
\]

\[
\delta_3(n, m) := \det \begin{pmatrix}
\alpha_n - \alpha_m & \beta_n - \beta_m \\
\alpha_{n-1} - \alpha_{m+1} & \beta_{n-1} - \beta_{m+1}
\end{pmatrix}.
\]

According to Lemma 5.1, for all \( m \) sufficiently large there exists \( R_m \in \mathbb{N} \) such that for all \( n \geq R_m \), \( \delta_j(n, m) \neq 0 \) for \( j = 1, 2, 3 \). Then, by Eqs. (5.4) and (5.5), the vectors

\((h_{n+1}, h_n, h_{n+1}^{-1}), (\delta_1(n, m), \delta_2(n, m), \delta_3(n, m))\)

are linearly dependent.

Fix sufficiently large \( m \in \mathbb{N}_0 \). Then for all \( n \in \mathbb{N}_0, n \geq m_0 := R_m + m \), we have

\[ h_{n+1} = \psi(n)h_n, \quad \text{with} \quad \psi(n) := \frac{\delta_1(n-m, m)}{\delta_2(n-m, m)}. \]

It is of importance that \( \psi(n) \) can be regarded as a meromorphic function of \( n \) in a neighborhood of \( \infty \). Particularly, \( \psi(n) \) has an asymptotic expansion to all orders as \( n \to \infty \).

In view of Lemma 5.1 there are only three possible types of asymptotic behavior of \( \psi(n) \) as \( n \to 0 \):

(I) \( \psi(n) = \lambda_1 \left( 1 + O\left( \frac{1}{n} \right) \right) \),

(II) \( \psi(n) = \lambda_2 n \left( 1 + O\left( \frac{1}{n} \right) \right) \),

(III) \( \psi(n) = \frac{\lambda_3}{n} \left( 1 + O\left( \frac{1}{n} \right) \right) \).

Note that in any case \( \lambda_j \neq 0 \). From here one can deduce the asymptotic behavior of

\[ h_n = h_{m_0} \prod_{k=m_0}^{n-1} \psi(k). \]

In case (I) we have

\[ h_n = c_1 \lambda_1^n n^{s_1} \left( 1 + O\left( \frac{1}{n} \right) \right) \quad \text{as} \quad n \to \infty, \]

for some \( c_1, s_1 \in \mathbb{R}, c_1 \neq 0 \). In case (II) we have

\[ h_n = c_2 \lambda_2^n n! n^{s_2} \left( 1 + O\left( \frac{1}{n} \right) \right) \quad \text{as} \quad n \to \infty, \]

for some \( c_2, s_2 \in \mathbb{R}, c_2 \neq 0 \). In case (III) we have

\[ h_n = \frac{c_3 \lambda_3^n n^{s_3}}{n!} \left( 1 + O\left( \frac{1}{n} \right) \right) \quad \text{as} \quad n \to \infty, \]
for some $c_3, s_3 \in \mathbb{R}, c_3 \neq 0$.

Rewriting (5.4) and taking into the account the asymptotic behavior of $\alpha_n$ we obtain

$$h_{n+1} + \frac{\beta_{n-m} - \beta_m}{\alpha_{n-m} - \alpha_m} h_n + \frac{\alpha_{n-m-1} - \alpha_{m-1}}{\alpha_{n-m} - \alpha_m} h_{n-1} = h_{n+1} - (k + k^{-1}) \left(1 + O\left(\frac{1}{n}\right)\right) h_n + \left(1 + O\left(\frac{1}{n}\right)\right) h_{n-1} = 0.$$

(5.6)

It is readily seen that the asymptotic behavior of $h_n$ of type (II) and (III) is incompatible with (5.6). Hence the only admissible asymptotic behavior of $h_n$ is that of type (I). Without loss of generality we can suppose that $c_1 = 1$. Moreover, from (5.6) it is also seen that $\lambda_1$ should solve the equation

$$\lambda_1^2 - (k + k^{-1})\lambda_1 + 1 = 0$$

whence $\lambda_1 = k$ or $\lambda_1 = k^{-1}$. For definiteness let us assume that $\lambda_1 = k$. The case $\lambda_1 = k^{-1}$ can be treated analogously.

Furthermore, for the sake of simplicity we will drop the index in $s_1$. Thus we obtain

$$h_n = k^n (n+1)^s \varphi(n)$$

(5.7)

where

$$\varphi(n) = 1 + \frac{\varphi_1}{n} + O\left(\frac{1}{n^2}\right) \text{ as } n \to \infty,$$

(5.8)

with some (undetermined) coefficient $\varphi_1$. But in fact, $\varphi(n)$ has an asymptotic expansion to all orders as $n \to \infty$.

Plugging (5.7) into (5.4) we obtain

$$(\alpha_n - \alpha_m)k \left(1 + \frac{1}{n + m + 1}\right)^s \varphi(n + m + 1) + (\beta_n - \beta_m) \varphi(n + m)$$

$$+ (\alpha_{n-1} - \alpha_{m-1})k^{-1} \left(1 - \frac{1}{n + m + 1}\right)^s \varphi(n + m - 1) = 0.$$  

(5.9)

The asymptotic expansion of the LHS of (5.9) as $n \to \infty$, with $m$ being fixed but otherwise arbitrary, while taking into account (5.8)) and (5.2), yields the expression

$$\frac{(1 - k^2)s - (1 + k^2)(\xi - \sigma) - 2k^2}{k} n + V(k, a, b, c, \sigma, s, \varphi_1) + (k - k^{-1}) sm$$

$$- \beta_m - k \alpha_m - k^{-1} \alpha_{m-1} + O\left(\frac{1}{n}\right)$$

where $V(k, a, b, c, \sigma, s)$ is a function of the indicated variables but independent of $m$ and $n$. Necessarily,

$$s = \frac{(1 + k^2)(\xi - \sigma) + 2k^2}{1 - k^2}.$$

Furthermore,

$$\beta_m + k \alpha_m + k^{-1} \alpha_{m-1} = (k - k^{-1}) sm + V(k, a, b, c, \sigma, s, \varphi_1), \forall m \geq 0.$$  

(5.10)

The asymptotic expansion (5.1) can be made more precise. For $n$ large we have

$$\alpha_n = -n^2 - (\xi + 2)n + A + B(n)$$
where $A$ is a constant given in (5.3) and
\begin{equation}
B(z) = \sum_{j=1}^{\infty} \frac{b_j}{z^j}
\end{equation}
is an analytic function in a neighborhood of $\infty$.

Since $\beta_m$ is a polynomial in $m$, from (5.10) it is seen that the coefficients in the asymptotic expansion of
\[ kB(m) + k^{-1}B(m-1), \text{ as } m \to \infty, \]
vanish to all orders. Referring to (5.11), from here one straightforwardly deduces by mathematical induction in $\ell$ that $b_j = 0$ for $1 \leq j \leq \ell - 1$ and all $\ell \geq 1$. Whence $B(z) = 0$. In fact, if $b_j = 0$ for $1 \leq j \leq \ell - 1$ and some $\ell \in \mathbb{N}$ then
\[ b_{\ell} = \lim_{m \to \infty} m^{\ell}B(m) = \lim_{m \to \infty} (m-1)^{\ell}B(m-1) = \lim_{m \to \infty} m^{\ell}B(m-1) \]
whence $(k + k^{-1})b_{\ell} = 0$.

Thus we conclude that $\alpha_n$ is a polynomial in $n$, a contradiction. \qed

From now on we shall focus on the case when $\alpha_n$ is a polynomial in $n$ while $\beta_n$ is the same as in (1.2). Hence
\begin{equation}
\alpha_n = -(n+1)(n+a+1), \quad \beta_n = (k+k^{-1})n(n+\sigma)
\end{equation}
and $\xi$ in (5.2) simplifies to $\xi = a$. Furthermore, Eq. (5.4) reduces to
\begin{equation}
(k+k^{-1})(n+\sigma)h_n - (n+a)h_{n-1} - (n+a+2)h_{n+1} = 0, \quad n \geq 1.
\end{equation}

Referring to Lemma 4.1 and Proposition 4.5 we have the following result.

**Theorem 5.3.** Let $\alpha_n$ and $\beta_n$ be given by (5.12). Denote by $J$ the respective Jacobi matrix (1.1). Then, up to a constant multiplier, the only square summable solution to (5.13) reads
\begin{equation}
h_n = \frac{k^n \Gamma(n+a+1)}{\Gamma(n+a+\omega(a,\sigma)+1)} \frac{2F_1}{k^2} \left( \begin{array}{c} n+a+1, \omega(a,\sigma)-1 \\ n+a+\omega(a,\sigma)+1 \end{array} \right) \bigg| k^2 \bigg), \quad n \geq 0,
\end{equation}
where
\[ \omega(a,\sigma) := \frac{-2k^2 + (1+k^2)(\sigma-a)}{1-k^2}. \]

The solution also admits an integral representation,
\begin{equation}
h_n = \frac{k^n}{\Gamma(\omega(a,\sigma))} \int_0^1 t^{n+a} \left( \frac{1-t}{1-k^2t} \right)^{\omega(a,\sigma)-1} dt.
\end{equation}

The asymptotic behavior of the solution is as follows,
\begin{equation}
h_n = (1-k^2)^{-\omega(a,\sigma)+1} k^n n^{-\omega(a,\sigma)} \left( 1 + O\left( \frac{1}{n} \right) \right) \text{ as } n \to \infty.
\end{equation}

Then, again up to a constant multiplier, $H_{m,n} = h_{m+n}$, $m, n \in \mathbb{N}_0$, is the only Hankel matrix commuting with $J$ with square summable columns.
Theorem 6.1. Each of the Hankel matrices $H_{m+n}$ represents a trace class operator on $\ell^2(\mathbb{N}_0)$. Actually, a sufficient condition is that
\[
\sum_{m,n=0}^{\infty} |H_{m,n}| < \infty
\]
which is apparently the case. This fact will be further confirmed in concrete cases by explicit formulas for the eigenvalues of $H$.

6. The Main Theorem

Let us introduce four Hankel matrices $H^{(p)}$, $H^{(q)}$, $H^{(r)}$, $H^{(s)}$, depending on a parameter $k \in (0,1)$,

\[
H^{(j)}_{m,n} := h^{(j)}_{m+n}, \quad \text{for } m, n \in \mathbb{N}_0, \quad j = p, q, r, s, \quad (6.1)
\]

where
\[
h^{(p)}_n := \frac{k^n \Gamma(n+1/2)}{(n+1)!} {}_2F_1\left(\frac{n+1/2, 1/2}{n+2} \right | k^2 \right) = \frac{4k^n}{\sqrt{\pi}} \int_0^1 t^{2n} \sqrt{\frac{1 - t^2}{1 - k^2 t^2}} \, dt, \quad (6.2)
\]
\[
h^{(q)}_n := \frac{k^n \Gamma(n+3/2)}{(n+1)!} {}_2F_1\left(\frac{n+3/2, -1/2}{n+2} \right | k^2 \right) = \frac{2k^n}{\sqrt{\pi}} \int_0^1 t^{2n+2} \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} \, dt, \quad (6.3)
\]
\[
h^{(r)}_n := \frac{k^n \Gamma(n+1/2)}{n!} {}_2F_1\left(\frac{n+1/2, -1/2}{n+1} \right | k^2 \right) = \frac{2k^n}{\sqrt{\pi}} \int_0^1 t^{2n+1} \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} \, dt, \quad (6.4)
\]
\[
h^{(s)}_n := \frac{k^n \Gamma(n+3/2)}{(n+2)!} {}_2F_1\left(\frac{n+3/2, 1/2}{n+3} \right | k^2 \right) = \frac{4k^n}{\sqrt{\pi}} \int_0^1 t^{2n+2} \sqrt{\frac{1 - t^2}{1 - k^2 t^2}} \, dt. \quad (6.5)
\]

Note that the latter equality in each row of this array of equations follows from the integral representation (2.2).

Recall definitions of the Stieltjes–Carlitz polynomials $p_n(x)$, $q_n(x)$, $r_n(x)$ and $s_n(x)$ in (2.20), (2.21), (2.27) and (2.28), respectively.

**Theorem 6.1.** Each of the Hankel matrices $H^{(j)}$, $j = p, q, r, s$, represents a positive trace class operator on $\ell^2(\mathbb{N}_0)$ with simple eigenvalues. The eigenvalues of $H^{(j)}$, $j = p, q, r, s$, if enumerated in descending order, are respectively
\[
\nu^{(p)}_m = \frac{4\sqrt{\pi}}{k} \frac{q^{m+1/2}}{1 + q^{2m+1}}, \quad m \geq 0,
\]
\[
\nu^{(q)}_m = \frac{2\sqrt{\pi}}{k} \frac{q^{m+1/2}}{1 + q^{2m+1}}, \quad m \geq 0,
\]
\[
\nu^{(r)}_m = 2\sqrt{\pi} \frac{q^m}{1 + q^{2m}}, \quad m \geq 0,
\]
\[
\nu^{(s)}_m = \frac{4\sqrt{\pi}}{k^2} \frac{q^m}{1 + q^{2m}}, \quad m \geq 1,
\]

where the parameter $q = q(k)$ occurring on the RHS in each of these equations is the elliptic nome, see (2.1). Eigenvector $\Psi^{(j)}_m$ corresponding to $\nu^{(j)}_m$, $j =$
\( p, q, r, s, \) can be chosen with the entries

\[
\begin{align*}
\Psi(p)_m^n &= \frac{(-1)^n}{k^n(2n)!} p_n \left( \frac{\pi^2(2m+1)^2}{4K^2} \right), \\
\Psi(q)_m^n &= \frac{(-1)^n}{k^n(2n+1)!} q_n \left( \frac{\pi^2(2m+1)^2}{4K^2} \right), \\
\Psi(r)_m^n &= \frac{(-1)^n}{k^n(2n)!} r_n \left( \frac{\pi^2m^2}{K^2} \right), \\
\Psi(s)_m^n &= \frac{(-1)^n}{k^n(2n+1)!} s_n \left( \frac{\pi^2m^2}{K^2} \right),
\end{align*}
\]

with \( n \in \mathbb{N}_0 \). The \( \ell^2 \)-norms of the eigenvectors equal

\[
\begin{align*}
\| \Psi(p)_m \|^2 &= \frac{kK}{2\pi} \frac{1 + q^{2m+1}}{q^{m+1/2}}, \\
\| \Psi(q)_m \|^2 &= \frac{2kK^3}{\pi^3} \frac{1 + q^{2m+1}}{(2m+1)^2 q^{m+1/2}}, \\
\| \Psi(r)_m \|^2 &= \frac{K}{2\pi} \frac{1 + q^{2m}}{q^m}, \\
\| \Psi(s)_m \|^2 &= \frac{k^2K^3}{2\pi^3} \frac{1 + q^{2m}}{m^2 q^m}.
\end{align*}
\]

**Proof.** Let us first summarize some general features which are applicable in each of the four cases. As is explicitly indicated below, each sequence \( (h_j)_n \), \( j = p, q, r, s, \) coincides with a solution \( h_n \) given in (5.14) for a particular choice of parameters \( a > -1 \) and \( \sigma > 0 \), and in each case we have \( \omega(a, \sigma) > 0 \). As already observed in Remark 5.4, each Hankel matrix \( H^{(j)} \) represents a trace class operator on \( \ell^2(\mathbb{N}_0) \).

Furthermore, as can be deduced from Theorem 5.3 and Eq. (5.12), each Hankel matrix \( H^{(j)} \) commutes with a certain Jacobi matrix \( J = J^{(j)} \) of the form (1.1) where

\[
\alpha_n = -k(n+1)(n+a+1), \quad \beta_n = k(k+k^{-1})n(n+\sigma) + d,
\]

and \( J^{(j)} \) is therefore determined by a proper choice of the parameters \( a \) and \( \sigma \) (note that the multiplicative constant \( k \) and the additive constant \( d \) are inessential for the commutation relation). Jacobi matrix \( J^{(j)} \) turns out to have a pure point spectrum which is necessarily simple. In addition, the Jacobi matrix in each case is known to determine a unique self-adjoint operator on \( \ell^2(\mathbb{N}_0) \) (determinate case). Then from the formal commutation relation between \( H^{(j)} \) and \( J^{(j)} \) on the level of semi-infinite matrices as well as from the fact that the matrix \( H^{(j)} \) corresponds to a bounded operator it readily follows that \( H^{(j)} \) preserves the domain of \( J^{(j)} \) (provided \( H^{(j)} \) and \( J^{(j)} \) are both regarded as operators). From the simplicity of the spectrum of \( J^{(j)} \) one deduces that every eigenvector of \( J^{(j)} \) is at the same time an eigenvector of \( H^{(j)} \). Hence a diagonalization of \( J^{(j)} \) provides, too, a diagonalization of \( H^{(j)} \).
Let us briefly mention some familiar facts concerned with eigenvectors of Jacobi matrix operators and needed in the sequel. The related formulas are most conveniently expressed in terms of normalized orthogonal polynomials. The reader is warned that this is unlike in Sect. 2.2 where all formulas for the Stieltjes–Carlitz polynomials are given in terms of monic polynomials.

If \( \lambda \) is an eigenvalue of \( J^{(j)} \) then for a corresponding eigenvector one can choose the vector
\[
\Psi(\lambda) := \left( \hat{P}_0(\lambda), \hat{P}_1(\lambda), \hat{P}_2(\lambda), \ldots \right)^T
\]
where \( (\hat{P}_n(x))_{n \geq 0} \) is the orthonormal polynomial sequence associated with \( J^{(j)} \) which is unambiguously determined by letting \( \hat{P}_0(x) = 1 \). If 
\[
\text{spec}_p J^{(j)} = \{ \lambda_m; \ m \geq 0 \}
\]
is the point spectrum of \( J^{(j)} \) and \( \mu \) is the respective orthogonality measure (normalized as a probability measure and supported on \( \text{spec}_p J^{(j)} \)) then the orthogonality relation reads
\[
\sum_{\ell=0}^{\infty} \mu_\ell \hat{P}_m(\lambda_\ell) \hat{P}_n(\lambda_\ell) = \delta_{m,n},
\]
where \( \mu_\ell := \mu(\{\lambda_\ell\}) \), and dually,
\[
\sum_{m=0}^{\infty} \hat{P}_m(\lambda_\ell) \hat{P}_m(\lambda_r) = \frac{1}{\mu_\ell} \delta_{\ell,r}.
\]
Hence
\[
\|\Psi(\lambda)\|^2 = \frac{1}{\mu_\ell}.
\]

Denoting by \( \{e_n; n \geq 0\} \) the canonical basis in \( \ell^2(\mathbb{N}_0) \) we obtain a unitary mapping
\[
U: \ell^2(\mathbb{N}_0) \rightarrow L^2 \left( (0, \infty), d\mu \right): e_n \mapsto \hat{P}_n,
\]
which diagonalizes both \( J^{(j)} \) and \( H^{(j)} \) (as defined in (6.6) and (6.1), respectively). The operator \( UH^{(j)}U^{-1} \) is a multiplication operator by a function \( h^{(j)}(x) \) which obeys
\[
h^{(j)}(x) = h^{(j)}(x) \hat{P}_0(x) = \sum_{n=0}^{\infty} H^{(j)}_{n,0} \hat{P}_n(x).
\]
Then the values \( h^{(j)}(\lambda_m), m \geq 0, \) are exactly the eigenvalues of \( H^{(j)} \).

(i) The sequence \( h_n^{(p)} \), \( n \geq 0 \), in (6.2) coincides with the solution \( h_n \) given in (5.14) for the values of parameters
\[
a = -\frac{1}{2}, \quad \sigma = \frac{1}{1 + k^2}, \quad \text{whence} \quad \omega(a, \sigma) = \frac{3}{2}.
\]
Theorem 5.3 then implies that the Hankel matrix \( H^{(p)} \) commutes with a Jacobi matrix \( J^{(p)} \) of the form (1.1) where \( \alpha_n \) and \( \beta_n \) are replaced by
\[
\alpha_n^{(p)} := -2k(n + 1)(2n + 1) \quad \text{and} \quad \beta_n^{(p)} := k^2(2n)^2 + (2n + 1)^2.
\]
The recurrence (2.20) means that
\[ p_{n+1}(x) = (x - \beta_n^{(p)}) p_n(x) - \left( \alpha_{n-1}^{(p)} \right)^2 p_{n-1}(x), \quad n \geq 0. \]

Hence the Jacobi matrix \( J^{(p)} \) corresponds to the Family #3 of the Stieltjes–Carlitz polynomials.

From (2.22) we know that the set of functions
\[ \hat{P}_n(x) := (-1)^n \frac{p_n(x)}{k^n (2n)!}, \quad n \in \mathbb{N}_0, \]
is an orthonormal basis of the Hilbert space \( L^2 ((0, \infty), d\mu) \) where \( \mu \) is given in (2.24). Referring to (6.7) and (6.8), the diagonalized operator \( U H^{(p)} U^{-1} \)
is a multiplication operator by a function \( h^{(p)}(x) \) which can be computed as follows (see (6.1) and (6.2))
\[ h^{(p)}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{h_n^{(p)} p_n(x)}{k^n (2n)!} = \frac{4}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{p_n(x)}{(2n)!} \int_0^1 t^{2n} \sqrt{1 - t^2} \frac{1}{1 - k^2 t^2} \, dt. \]

From (3.1) we find that
\[ h^{(p)}(x) = \frac{4}{\sqrt{\pi}} \int_0^K \cos(\sqrt{x} u) \csc(u) \, du. \]

Finally, the last equation combined with the Fourier series (2.8) allows us to evaluate the function \( h^{(p)}(x) \) at the spectral points \( \lambda_m \) of the Jacobi matrix \( J^{(p)} \), as given in (2.24). The obtained values read
\[ h^{(p)}(\lambda_m) = \frac{4\sqrt{\pi}}{k} \frac{q^{m+1/2}}{1 + q^{2m+1}}, \quad m \geq 0, \]
and these are in fact the eigenvalues of \( H^{(p)}. \)

(ii) The sequence \( h_n^{(q)} \), \( n \geq 0 \), in (6.3) coincides with the solution \( h_n \)
given in (5.14) for the values of parameters
\[ a = \frac{1}{2}, \quad \sigma = \frac{1 + 2k^2}{1 + k^2}, \quad \text{whence} \quad \omega(a, \sigma) = \frac{1}{2}. \]

Theorem 5.3 then implies that the Hankel matrix \( H^{(q)} \) commutes with a Jacobi matrix \( J^{(q)} \) of the form (1.1) where \( \alpha_n \) and \( \beta_n \) are replaced by
\[ \alpha_n^{(q)} := -2k(n+1)(2n+3) \quad \text{and} \quad \beta_n^{(q)} := (2n+1)^2 + k^2(2n+2)^2. \]
The recurrence (2.21) means that
\[ q_{n+1}(x) = (x - \beta_n^{(q)}) q_n(x) - \left( \alpha_{n-1}^{(q)} \right)^2 q_{n-1}(x), \quad n \geq 0. \]

Hence the Jacobi matrix \( J^{(q)} \) corresponds to the Family #4 of the Stieltjes–Carlitz polynomials.

From (2.23) we know that the set of functions
\[ \hat{Q}_n(x) := \frac{(-1)^n q_n(x)}{k^n (2n+1)!}, \quad n \geq 0, \]
is an orthonormal basis of the Hilbert space \( L^2 ((0, \infty), x d\mu(x)) \) where \( \mu \) is given by (2.24). Referring to (6.7) and (6.8), with \( \hat{Q}_n \) instead of \( \hat{P}_n \), the
diagonalized operator $UH^{(q)}U^{-1}$ is a multiplication operator by a function $h^{(q)}(x)$ which can be computed as follows (see (6.1) and (6.3))

$$h^{(q)}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{h^{(q)}_n(x)}{k^n(2n+1)!} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{q_n(x)}{(2n+1)!} \int_0^1 t^{2n+2} \sqrt{\frac{1-k^2t^2}{1-t^2}} \, dt.$$  

From (3.2) we find that

$$h^{(q)}(x) = \frac{2}{\sqrt{\pi}} \int_0^K \cos(\sqrt{x}u)\mathrm{cn}(u) \, du.$$  

Finally, the last equation combined with the Fourier series (2.8) allows us to evaluate the function $h^{(q)}(x)$ at the spectral points $\lambda_m$ of the Jacobi matrix $J^{(q)}$, as given in (2.24). The obtained values read

$$h^{(q)}(\lambda_m) = \frac{2\sqrt{\pi}}{k} \frac{q^{m+1/2}}{1+q^{2m+1}} , \quad m \geq 0,$$

and these are in fact the eigenvalues of $H^{(q)}$.

(iii) The sequence $h^{(r)}_n, n \geq 0$, in (6.4) coincides with the solution $h_n$ given in (5.14) for the values of parameters

$$a = -\frac{1}{2} , \quad \sigma = \frac{k^2}{1+k^2} , \quad \text{whence } \omega(a, \sigma) = \frac{1}{2}.$$  

Theorem 5.3 then implies that the Hankel matrix $H^{(r)}$ commutes with a Jacobi matrix $J^{(r)}$ of the form (1.1) where $\alpha_n$ and $\beta_n$ are replaced by

$$\alpha_n^{(r)} := -2k(n+1)(2n+1) \quad \text{and} \quad \beta_n^{(r)} := (2n)^2 + k^2(2n+1)^2.$$  

The recurrence (2.27) means that

$$r_{n+1}(x) = (x - \beta_n^{(r)}) r_n(x) - \left(\alpha_n^{(r)}\right)^2 r_{n-1}(x), \quad n \geq 0.$$  

Hence the Jacobi matrix $J^{(r)}$ corresponds to the Family #5 of the Stieltjes–Carlitz polynomials.

From (2.29) we know that the set of functions

$$\hat{R}_n(x) := \frac{(-1)^n r_n(x)}{k^n(2n)!} , \quad n \geq 0,$$

is an orthonormal basis of the Hilbert space $L^2((0, \infty), d\mu(x))$ where $\mu$ is given by (2.31). Referring to (6.7) and (6.8), with $\hat{R}_n$ instead of $\hat{P}_n$, the diagonalized operator $UH^{(r)}U^{-1}$ is a multiplication operator by a function $h^{(r)}(x)$ which can be computed as follows (see (6.1) and (6.4))

$$h^{(r)}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{h^{(r)}_n(x)}{k^n(2n)!} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{r_n(x)}{(2n)!} \int_0^1 t^{2n} \sqrt{\frac{1-k^2t^2}{1-t^2}} \, dt.$$
From (3.3) we find that
\[ h^{(r)}(x) = \frac{2}{\sqrt{\pi}} \int_0^K \cos(\sqrt{x}u)dn(u) \, du. \]

Finally, the last equation combined with the Fourier series (2.9) allows us to evaluate the function \( h^{(r)}(x) \) at the spectral points \( \lambda_m \) of the Jacobi matrix \( J^{(r)} \), as given in (2.31). The obtained values read
\[ h^{(r)}(\lambda_m) = \frac{2\sqrt{\pi}q^m}{1 + q^{2m}}, \quad m \geq 0, \]
and these are in fact the eigenvalues of \( H^{(r)} \).

(iv) The sequence \( h^{(s)}_n, n \geq 0 \), in (6.2) coincides with the solution \( h_n \) given in (5.14) for the values of parameters
\[ a = \frac{1}{2}, \quad \sigma = \frac{2 + k^2}{1 + k^2}, \quad \text{whence} \quad \omega(a, \sigma) = \frac{3}{2}. \]

Theorem 5.3 then implies that the Hankel matrix \( H^{(s)} \) commutes with a Jacobi matrix \( J^{(s)} \) of the form (1.1) where \( \alpha_n \) and \( \beta_n \) are replaced by
\[ \alpha_n^{(s)} := -2k(n+1)(2n+3) \quad \text{and} \quad \beta_n^{(s)} := k^2(2n+1)^2 + (2n+2)^2. \]

The recurrence (2.28) means that
\[ s_{n+1}(x) = (x - \beta_n^{(s)})s_n(x) - \left( \alpha_n^{(s)} \right)^2 s_{n-1}(x), \quad n \geq 0. \]

Hence the Jacobi matrix \( J^{(s)} \) corresponds to the Family \#6 of the Stieltjes–Carlitz polynomials.

From (2.30) we know that the set of functions
\[ \hat{S}_n(x) := \frac{(-1)^n s_n(x)}{k^n(2n+1)!}, \quad n \geq 0, \]
is an orthonormal basis of the Hilbert space \( L^2((0, \infty), k^{-2}xd\mu(x)) \) where \( \mu \) is given in (2.31). Referring to (6.7) and (6.8), with \( \hat{S}_n \) instead of \( \hat{P}_n \), the diagonalized operator \( UH^{(s)}U^{-1} \) is a multiplication operator by a function \( h^{(s)}(x) \) which can be computed as follows (see (6.1) and (6.2))
\[ h^{(s)}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n h_n^{(s)} s_n(x)}{k^n(2n+1)!} = \frac{4}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{s_n(x)}{(2n+1)!} \int_0^1 t^{2n+2} \sqrt{1 - t^2} \frac{1 - t^2}{1 - k^2 t^2} \, dt. \]

From (3.4) we find that
\[ h^{(s)}(x) = -4 \sqrt{1 - k^2 \sin(\sqrt{x}K)} - \frac{4}{k^2 \sqrt{\pi x}} \int_0^K \cos(\sqrt{x}u)dn(u) \, du. \]

Finally, the last equation combined with the Fourier series (2.9) allows us to evaluate the function \( h^{(s)}(x) \) at the spectral points \( \lambda_m \) of the Jacobi
matrix \( J^{(s)} \), as given in (2.31). The obtained values read

\[
h^{(s)}(\lambda_m) = \frac{4\sqrt{\pi}}{k^2} \frac{q^m}{1+q^{2m}}, \quad m \geq 1,
\]

and these are in fact the eigenvalues of \( H^{(s)} \). Note that \( \lambda_0 = 0 \) is not an eigenvalue of \( J^{(s)} \) since it does not belong to the support of the measure of orthogonality in (2.30). Consequently, \( h^{(s)}(0) \) is not an eigenvalue of \( H^{(s)} \). \( \square \)

7. Families #1 and #2, a Generalization to Weighted Hankel Matrices

The main focus of the paper so far was on Hankel matrices which admit an explicit solution of the spectral problem owing to their close relationship to the Stieltjes–Carlitz polynomials. This concerns Families #3, #4, #5 and #6 only. Our approach does not lead to explicitly diagonalizable Hankel matrices in case of Families #1 and #2. In this section we propose an extension of the foregoing results by considering also weighted Hankel matrices of the form

\[
H_{m,n} = w_m w_n h_{m+n}, \quad m,n \in \mathbb{N}_0. \tag{7.1}
\]

The weights \( w_n \) are supposed to be positive. Admitting nontrivial (non-constant) weights we are able to enrich the list of explicitly diagonalizable matrix operators by several additional items and, in particular, the generalized approach can be applied to Families #1 and #2 as well. Apart of this generalization the basic scheme remains practically the same as in Sect. 6. This is why we try to be rather brief in the current section and we omit some details for routine steps in the derivations to follow.

First of all we have to modify Eq. (5.4). The formal commutation relation \( HJ = JH \) between a weighted Hankel matrix \( H \) and a Jacobi matrix of the form (1.1), where the multiplication is understood on the level of semi-infinite matrices, is satisfied if and only if

\[
(\beta_m - \beta_n)w_m w_n h_{m+n} + (\alpha_{m-1} w_{m-1} w_n - \alpha_{n-1} w_m w_{n-1})h_{m+n-1} + (\alpha_m w_{m+1} w_n - \alpha_n w_m w_{n+1})h_{m+n+1} = 0
\]

holds for all \( m,n \in \mathbb{N}_0, \ m < n \). Here and everywhere in what follows we assume that \( \alpha_{-1} = 0 \). An easy computation leads to the following lemma.

**Lemma 7.1.** Let \( J \) be a Jacobi matrix (1.1) with entries given by (1.2). Then a matrix \( H \) with entries (7.1) commutes formally, on the level of semi-infinite matrices, with \( J \) provided the sequence \((h_n)_{n \geq 0}\) satisfies the difference equation

\[
(k+k^{-1})(n+\sigma)h_n - (n+a)h_{n-1} - (n+b+c+2)h_{n+1} = 0, \quad n \geq 1, \tag{7.2}
\]

and

\[
w_n = \sqrt{\frac{(b+1)n(c+1)n}{n!(a+1)n}}, \quad n \geq 0. \tag{7.3}
\]
Note that Eq. (7.2) is again of type (4.1) which has been studied in Sect. 4.

Similarly as in Sect. 3 we shall need some auxiliary results. All of them can be derived in a routine way by using standard methods. The following proposition was shown in [8, Thm. 3.1] and in the course of the proof of Theorem 3.3.1 in [9] though the notation therein was different from ours.

**Proposition 7.2.** The leading terms in the asymptotic expansion of the Stieltjes–Carlitz polynomials \( f_n(x) \) and \( g_n(x) \) defined in (2.12) and (2.16), respectively, are as follows

\[
\frac{f_n(-x)}{(2n)!} = \frac{1}{\sqrt{\pi n(1-k^2)}} \cos(\sqrt{x}K) + o\left(\frac{1}{n}\right),
\]

\[
\frac{g_n(-x)}{(2n+1)!} = \frac{1}{\sqrt{\pi n(1-k^2)}} \frac{\sin(\sqrt{x}K)}{\sqrt{x}} + o\left(\frac{1}{n}\right),
\]

as \( n \to \infty \). Here \( x \) is an arbitrary fixed complex number.

**Proposition 7.3.** For \( x \in \mathbb{C} \) and the Stieltjes–Carlitz polynomials \( f_n(x) \) and \( g_n(x) \), defined in (2.12), and (2.16), respectively, it holds true that

\[
\frac{\pi}{4} \sum_{n=0}^{\infty} \frac{f_n(-x)}{4^n n!(n+1)!} = \int_0^K \cos(\sqrt{x}u)\text{cn}(u) \, du,
\]

(7.4)

\[
\frac{\pi}{8} \sum_{n=0}^{\infty} \frac{g_n(-x)}{4^n n!(n+2)!} = -\frac{\sqrt{1-k^2}}{k^2} \frac{\sin(\sqrt{x}K)}{\sqrt{x}} + \frac{1}{k^2} \int_0^K \cos(\sqrt{x}u)\text{dn}(u) \, du.
\]

(7.5)

**Proof.** Equation (7.4) can be derived from the formula for the generating function (2.15). One has to write \(-x\) instead of \(x\) and differentiate the formula term-wise with respect to \(u\) and use the integral identity

\[
\int_0^K \text{sn}^2(u)\text{cn}(u) \, \frac{d\text{sn}(u)}{du} \, du = \int_0^1 y^{2n} \sqrt{1-y^2} \, dy = \frac{\pi}{2^{2n+2} \, n!(n+1)!},
\]

for \( n \geq 0 \).

Very similarly, Eq. (7.5) can be derived from the formula for the generating function (2.19). This time the integral identity

\[
\int_0^K \text{sn}^2(u)\text{cn}^3(u) \, \frac{d\text{sn}(u)}{du} \, du = \int_0^1 y^{2n} (1-y^2)^{3/2} \, dy = \frac{3\pi}{2^{2n+3} \, n!(n+2)!},
\]

for \( n \geq 0 \), turns out to be useful. This case is slightly more complicated since finally one has to integrate by parts on the RHS to get the desired expression.

Manipulations used during the derivation in both cases can be justified with the aid of Proposition 7.3. □

Below we present two weighted Hankel matrices which have comparatively simple form and which are related to Families #1 and #2 of the
Stieltjes–Carlitz polynomials,
\[ H_{m,n}^{(f)} := \binom{2n}{m} \binom{2m}{n} \left(\frac{k}{4}\right)^{m+n} \frac{\sqrt{(2n+1)(2m+1)}}{m+n+1}, \quad (7.6) \]
\[ H_{m,n}^{(g)} := \binom{2n+1}{m} \binom{2m+1}{n} \left(\frac{k}{4}\right)^{m+n} \frac{\sqrt{(m+1)(n+1)}}{m+n+2}, \quad (7.7) \]
for \( m, n \in \mathbb{N}_0 \).

**Theorem 7.4.** The weighted Hankel matrices \( H^{(f)} \) and \( H^{(g)} \) represent both positive trace class operators on \( \ell^2(\mathbb{N}_0) \) with simple eigenvalues. We have:

(i) Eigenvalues of \( H^{(f)} \) enumerated in descending order are
\[ \nu_m^{(f)} = \frac{4}{k} \frac{q^{m+1/2}}{1 + q^{2m+1}}, \quad m \geq 0. \]

An eigenvector \( \Psi_m^{(f)} \) corresponding to \( \nu_m^{(f)} \) can be chosen with the entries
\[ \left( \Psi_m^{(f)} \right)_n = \frac{1}{k^n(2n)!\sqrt{2n+1}} f_n \left( -\frac{\pi^2(2m+1)^2}{4K^2} \right), \quad n \geq 0, \]
and its \( \ell^2 \)-norm equals
\[ \| \Psi_m^{(f)} \|^2 = \frac{kK^2}{\pi^2} \frac{1 - q^{2m+1}}{(2m+1)q^{m+1/2}}. \]

(ii) Eigenvalues of \( H^{(g)} \) enumerated in descending order are
\[ \nu_m^{(g)} = \frac{8}{k^2} \frac{q^m}{1 + q^{2m}}, \quad m \geq 1. \]

An eigenvector \( \Psi_m^{(g)} \) corresponding to \( \nu_m^{(g)} \) can be chosen with the entries
\[ \left( \Psi_m^{(g)} \right)_n = \frac{1}{k^n(2n+1)!\sqrt{n+1}} g_n \left( -\frac{\pi^2m^2}{K^2} \right), \quad n \geq 0, \]
and its \( \ell^2 \)-norm equals
\[ \| \Psi_m^{(g)} \|^2 = \frac{k^2K^4}{\pi^4} \frac{1 - q^{2m}}{m^3q^m}. \]

**Proof.** The basic scheme remains literally the same as explained in the introductory part of the proof of Theorem 6.1. Comparing (7.2) to (4.1) we let, in the latter equation,
\[ \xi = a, \quad \eta = b + c + 2. \]
Recall also definition (4.11) of \( \omega(\xi, \eta, \sigma) \).

(i) Consider the entries \( \alpha_n, \beta_n \), as given in (1.2), for the values of parameters
\[ a = 0, \quad b = 1/2, \quad c = -1/2, \quad \sigma = 1, \] whence \( \xi = 0, \quad \eta = 2, \quad \omega(\xi, \eta, \sigma) = 1. \)
Then the weight given in (7.3) equals
\[ w_n^{(f)} = \sqrt{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}\right)_n} n!(1)_n = \frac{\sqrt{2n+1}}{2^{2n}} \binom{2n}{n}. \]
Referring to Proposition 4.5, we can choose for \( h_n^{(f)} \) the square summable solution (4.14) of Eq. (4.1),
\[
h_n^{(f)} = \frac{k^n \Gamma(n+1)}{\Gamma(n+2)} \bar{F}_1\left(\frac{n+1,0}{n+2}\right) k^2 = \frac{k^n}{n+1}.
\]

One can check that \( H_{m,n}^{(f)} = w_m^{(f)} w_n^{(f)} h_{m+n}^{(f)} \) coincides with (7.6). By Lemma 7.1, the weighted Hankel matrix \( H^{(f)} \) commutes with \( J = J^{(f)} \) introduced in (1.1) where we put \( \alpha_n = \alpha_n^{(f)} \), \( \beta_n = \beta_n^{(f)} \),
\[
\alpha_n^{(f)} := -2k(n+1)\sqrt{(2n+1)(2n+3)}, \quad \beta_n^{(f)} := (k^2 + 1)(2n + 1)^2.
\]

From (2.12) it is seen that the Jacobi matrix \( J^{(f)} \) corresponds to the Family \#1 of the Stieltjes–Carlitz polynomials.

From the asymptotic expansion
\[
\binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}} \left(1 + O\left(\frac{1}{n}\right)\right) \quad \text{as } n \to \infty
\]
it seen that \( \sum_{m,n=0}^{\infty} H_{m,n}^{(f)} < \infty \) and therefore the matrix \( (H_{m,n}^{(f)}) \) determines a trace class operator on \( l^2(\mathbb{N}_0) \).

By the orthogonality relation (2.13), the functions
\[
\hat{F}_n(x) := \frac{f_n(-x)}{k^n (2n)! \sqrt{2n + 1}}, \quad n \in \mathbb{N}_0,
\]
form an orthonormal basis of the Hilbert space \( L^2((0,\infty), d\mu) \) where \( \mu \) is defined in (2.14). With this orthonormal basis at hand and with the canonical basis in \( l^2(\mathbb{N}_0) \) we can construct a unitary mapping \( U \) which diagonalizes \( J^{(f)} \) and, at the same time, \( H^{(f)} \). \( UH^{(f)}U^{-1} \) becomes a multiplication operator by a function \( h^{(f)}(x) \) on \( L^2((0,\infty), d\mu) \). In view of (7.4), we have the formula
\[
h^{(f)}(x) = \sum_{n=0}^{\infty} H_{0,n}^{(f)} \hat{F}_n(x) = \sum_{n=0}^{\infty} \frac{f_n(-x)}{4^n n! (n+1)!} = \frac{4}{\pi} \int_0^K \cos(\sqrt{x}u) \text{cn}(u) \, du.
\]
This formula in combination with the Fourier series (2.8) allows us to evaluate \( h^{(f)}(x) \) at the spectral points \( \lambda_m \) of the Jacobi matrix \( J^{(f)} \), as given in (2.14).

The obtained values read
\[
h^{(f)}(\lambda_m) = \frac{4}{k} \frac{q^{m+1/2}}{1 + q^{2m+1}}, \quad m \geq 0,
\]
and these are in fact the eigenvalues of \( H^{(f)} \). Respective eigenvectors and their norms can be derived exactly in the same way as described in the beginning of the proof of Theorem 6.1.

(ii) Now we consider the entries \( \alpha_n, \beta_n \) in (1.2) for the values of parameters
\[
a = 1, \quad b = 1/2, \quad c = 1/2, \quad \sigma = 2, \quad \text{whence } \xi = 1, \eta = 3, \quad \omega(\xi, \eta, \sigma) = 1.
\]

Then the weight given in (7.3) equals
\[
w_n^{(g)} = \left(\frac{3}{2}\right)_n \sqrt{\frac{n+1}{2^n}} \binom{2n+1}{n},
\]
and the square summable solution (4.14) of Eq. (4.1), as described in Proposition 4.5, equals

\[
h_n^{(g)} = \frac{k^n \Gamma(n + 2)}{\Gamma(n + 3)} 2F_1\left(\begin{array}{c} n + 2, 0 \\ n + 3 \end{array} \left| k^2 \right. \right) = \frac{k^n}{n + 2}.
\]

Thus \( H_{m,n}^{(g)} = w_m^{(g)} w_n^{(g)} h_{m+n}^{(g)} \) coincides with (7.7). By Lemma 7.1, the weighted Hankel matrix \( H^{(g)} \) commutes with \( J = J^{(g)} \) introduced in (1.1) where we let

\[
\alpha_n^{(g)} := -2k(2n + 3)\sqrt{(n + 1)(n + 2)}, \quad \beta_n^{(g)} := (k^2 + 1)(2n + 2)^2.
\]

From (2.16) it is seen that the Jacobi matrix \( J^{(g)} \) corresponds to the Family #2 of the Stieltjes–Carlitz polynomials.

Similarly as in the forgoing case one can argue that \( \sum_{m,n=0}^{\infty} H_{m,n}^{(g)} < \infty \) and therefore \( H^{(g)} \) is a trace class operator.

In view of (2.17), the functions

\[
\hat{G}_n(x) := \frac{g_n(-x)}{k^n(2n + 1)!\sqrt{n + 1}} , \quad n \in \mathbb{N}_0,
\]

form an orthonormal basis in \( L^2 ((0, \infty), d\mu) \) where \( \mu \) is defined in (2.18). Using \( \{\hat{G}_n\} \) we can again construct a common eigenbasis for both \( J^{(g)} \) and \( H^{(g)} \) and, consequently, a unitary mapping \( U \) such that \( UJ^{(g)}U^{-1} \) is a multiplication operator by \( x \) and \( UH^{(g)}U^{-1} \) is a multiplication operator by a function \( h^{(g)}(x) \), both acting on \( L^2 ((0, \infty), d\mu) \). According to (7.5), we have

\[
h^{(g)}(x) = \sum_{n=0}^{\infty} H_{n,0}^{(g)} \hat{G}_n(x) = \sum_{n=0}^{\infty} \frac{g_n(-x)}{4^n n!(n + 2)!}
\]

\[
= -\frac{8\sqrt{1-k^2}}{\pi k^2} \frac{\sin(\sqrt{x}K)}{\sqrt{x}} + \frac{8}{\pi k^2} \int_0^K \cos(\sqrt{x}u)dn(u) \, du.
\]

With the aid of this formula and (2.9) we can evaluate \( h^{(g)}(x) \) at the spectral points \( \lambda_m \) of the Jacobi matrix \( J^{(g)} \), as given in (2.18). The obtained values read

\[
h^{(g)}(\lambda_m) = \frac{8}{k^2} \frac{q^m}{1 + q^{2m}} , \quad m \geq 1,
\]

and these are in fact the eigenvalues of \( H^{(g)} \). Note that in this case, too, \( \lambda_0 = 0 \) is not an eigenvalue of \( J^{(g)} \) and \( h^{(g)}(0) \) is not an eigenvalue of \( H^{(g)} \).

Respective eigenvectors and their norms can be derived as in the forgoing cases.

\[ \square \]

8. Some More Weighted Hankel Matrices

Another set of explicitly diagonalizable weighted Hankel matrices can be obtained by permuting the parameters \( a, b, \) and \( c \). In fact, note that permuting \( a, b, \) and \( c \) does not change the Jacobi matrix defined in (1.1), (1.2), but this need not be the case for the weight \( w_n \) defined in (7.3). Hence while keeping
First we apply this observation to the Jacobi matrices $J^{(q)}$ and $J^{(s)}$ corresponding to Families #4 and #6 of the Stieltjes–Carlitz polynomials, respectively. In both cases we can put $a = b = 1/2$, $c = 0$ (compare (1.2) to (6.6)). Then the weight in (7.3) is trivial, $w_n = 1$. Another possible choice of the parameters, $a = 0$, $b = c = 1/2$, leads to a nontrivial weight but with the Jacobi matrix remaining untouched. The result is described in detail in Theorem 8.1 below.

Naturally, one can attempt to apply the same procedure to the Jacobi matrices $J^{(p)}$ and $J^{(r)}$ corresponding to the Families #3 and #5 of the Stieltjes–Carlitz polynomials, respectively. Now we put, in both cases, $a = b = -1/2$, $c = 0$, and consequently we have $w_n = 1$. Unfortunately, the permutation $a = 0$, $b = c = -1/2$, does not yield results of notable interest.

As far as $J^{(p)}$ is concerned, we were not able, for the moment, to evaluate eigenvalues of the newly obtained weighted Hankel matrix explicitly. As for $J^{(r)}$, the new weighted Hankel matrix turns out to be of rank 1 and therefore of little interest.

Let us introduce another couple of weighted Hankel matrices,

$$H^{(q')}_{m,n} := (2m+1)(2n+1) \binom{2m}{m} \binom{2n}{n} \left(\frac{k}{4}\right)^{m+n} \frac{1 + (1-k^2)(m+n+1)}{(m+n+1)(m+n+2)},$$

$$H^{(s')}_{m,n} := \binom{2m}{m} \binom{2n}{n} \left(\frac{k}{4}\right)^{m+n} \frac{(2m+1)(2n+1)}{(m+n+1)(m+n+2)},$$

for $m, n \in \mathbb{N}_0$.

**Theorem 8.1.** The weighted Hankel matrices $H^{(q')}$ and $H^{(s')}$ represent both positive trace class operators on $\ell^2(\mathbb{N}_0)$ with simple eigenvalues. We have:

(i) Eigenvalues of $H^{(q')}_{m,n}$ enumerated in descending order are

$$\nu^{(q')}_{m} = \frac{2\pi}{kK} \frac{(2m+1)q^{m+1/2}}{1 - q^{2m+1}}, \ m \geq 0.$$

For an eigenvector corresponding to $\nu^{(q')}_{m}$ one can choose the vector $\Psi^{(q')}_{m}$, as introduced in Theorem 6.1.

(ii) Eigenvalues of $H^{(s')}_{m,n}$ enumerated in descending order are

$$\nu^{(s')}_{m} = \frac{4\pi}{k^2K} \frac{mq^m}{1 - q^{2m}}, \ m \geq 1.$$

For an eigenvector corresponding to $\nu^{(s')}_{m}$ one can choose the vector $\Psi^{(s')}_{m}$, as introduced in Theorem 6.1.

**Proof.** The proof again follows the same scheme as in the proof of Theorems 6.1 or 7.4. Most steps are quite routine and thus we confine ourselves to pointing out only some features which are particular for the matrices in question.

In both cases we have $\sum_{m,n=0}^{\infty} H_{m,n} < \infty$ implying that the matrices represent trace class operators.
As already mentioned, in both cases we let $a = 0$, $b = c = 1/2$. Hence, comparing (7.2) to (4.1), we have to put $\xi = a = 0$, $\eta = b + c + 2 = 3$. Also the weight (7.3) is the same in both cases,

$$w_n = \frac{1}{n!} \left( \frac{3}{2} \right)_n = \frac{2n + 1}{2^n} \binom{2n}{n}.$$  

(i) The choice of $\sigma$ corresponds to the Jacobi matrix $J^{(q)}$, see the proof of Theorem 6.1 ad (ii). We have

$$\sigma = \frac{1 + 2k^2}{1 + k^2}, \text{ whence } \omega(\xi, \eta, \sigma) = 1,$$

see (4.11). The square summable solution (4.14) of Eq. (4.1) equals

$$h_n^{(q)} = \frac{k^n}{n + 1} \frac{\Gamma(n + 1)}{\Gamma(n + 2)} \left[ 1 - \frac{k^2}{n + 1} + \frac{k^2}{(n + 1)(n + 2)} \right] \left( -1 \right)^n q_n(x).$$

$H^{(q)}$ can be diagonalized by the same unitary transform as the matrix $H^{(q)}$ which is described in the proof of Theorem 6.1 ad (ii). Using some routine manipulations, quite similarly as in the proof of Proposition 3.3, one can transform $H^{(q)}$ to a multiplication operator by the function

$$h^{(q)}(x) = \sum_{n=0}^{\infty} H_{n,0} \hat{Q}_n(x)$$

$$= \sum_{n=0}^{\infty} \frac{(3/2)_n}{n!} \left[ 1 - \frac{k^2}{n + 1} + \frac{k^2}{(n + 1)(n + 2)} \right] \left( -1 \right)^n q_n(x)$$

$$= \frac{4}{\pi} \int_0^K \frac{\sin(\sqrt{x}u)}{\sqrt{x}} \left[ (1 + k^2)\sin(u) - 2k^2 \sin^3(u) \right] \mathrm{d}u.$$  

This formula in combination with the Fourier series (2.7) and (2.11) makes it possible to evaluate $h^{(q)}(x)$ at the spectral points $\lambda_m$ of the Jacobi matrix $J^{(q)}$, as given in (2.24). The obtained values read

$$h^{(q)}(\lambda_m) = \frac{2\pi}{kK} \frac{(2m + 1)q^{m+1/2}}{1 - q^{2m+1}}, \quad m \geq 0,$$

and these are in fact the eigenvalues of $H^{(q)}$.

(ii) The choice of $\sigma$ corresponds to the Jacobi matrix $J^{(s)}$, see the proof of Theorem 6.1 ad (iv). We have

$$\sigma = \frac{2 + k^2}{1 + k^2}, \text{ whence } \omega(\xi, \eta, \sigma) = 2,$$

see (4.11). The square summable solution (4.14) of Eq. (4.1) equals

$$h_n^{(s)} = \frac{k^n}{(n + 1)(n + 2)} \frac{\Gamma(n + 1)}{\Gamma(n + 3)} \left[ 1 - \frac{k^2}{n + 3} \right] \left( -1 \right)^n q_n(x).$$

$H^{(s)}$ can be diagonalized by the same unitary transform as the matrix $H^{(s)}$ which is described in the proof of Theorem 6.1 ad (iv). Using some
routine manipulations, similarly as in the proof of Proposition 3.3, one can transform $H^{(s')}(x)$ to a multiplication operator by the function
\[
    h^{(s')}(x) = \sum_{n=0}^{\infty} H^{(s')}_{n,0} \hat{s}_n(x) = \sum_{n=0}^{\infty} \frac{(3/2)_n}{(n+2)!} \frac{(-1)^n s_n(x)}{(2n+1)!}
\]
\[
= \frac{4}{\pi} \left( \sin(\sqrt{x}K) \sqrt{x} - \int_{0}^{K} \cos(\sqrt{x}u) \sin^2(u) \, du \right).
\]

This formula in combination with the Fourier series (2.10) makes it possible to evaluate $h^{(s')}(x)$ at the spectral points $\lambda_m$ of the Jacobi matrix $J^{(s)}$, as given in (2.31). The obtained values read
\[
    h^{(s')}(\lambda_m) = \frac{4\pi}{k^2 K} \frac{mq^m}{1-q^{2m}}, \quad m \geq 1,
\]
and these are in fact the eigenvalues of $H^{(s')}$. \hfill \Box

Finally we consider permutations of the parameters $a, b$ and $c$ in case of the matrices $H^{(f)}$ and $H^{(g)}$ from Theorem 7.4. As for $H^{(f)}$, the original values were $a = 0$, $b = 1/2$, $c = -1/2$. Permuting $a, b, c$ in (7.3) provides us with two new weights. In case of $H^{(g)}$ we have $a = 1$, $b = c = 1/2$, and permuting $a, b, c$ leads to just one new weight.

These considerations lead us to introducing three weighted Hankel matrices
\[
    H^{(f')}_{m,n} := \frac{k^{m+n} \Gamma(m+n+3/2)}{\sqrt{(2m+1)(2n+1)(m+n+1)!}} 2F_1 \left( \frac{m+n+3/2, 1/2}{m+n+2} \right)
\]
\[
= \frac{2k^{m+n}}{\sqrt{\pi}(2m+1)(2n+1)} \int_{0}^{1} \frac{x^{2m+2n+2}}{\sqrt{(1-x^2)(1-k^2x^2)}} \, dx,
\]
\[
    H^{(f'')}_{m,n} := \frac{k^{m+n} \sqrt{(2m+1)(2n+1)} \Gamma(m+n+1/2)}{(m+n+1)!} 2F_1 \left( \frac{m+n+1/2, -1/2}{m+n+2} \right)
\]
\[
= \frac{4k^{m+n} \sqrt{(2m+1)(2n+1)}}{\sqrt{\pi}} \int_{0}^{1} x^{2m+2n} \sqrt{(1-x^2)(1-k^2x^2)} \, dx,
\]
\[
    H^{(g')}_{m,n} := \frac{k^{m+n} \sqrt{(m+1)(n+1)} \Gamma(m+n+3/2)}{(m+n+2)!} 2F_1 \left( \frac{m+n+3/2, -1/2}{m+n+3} \right)
\]
\[
= \frac{4k^{m+n} \sqrt{(m+1)(n+1)}}{\sqrt{\pi}} \int_{0}^{1} x^{2m+2n+2} \sqrt{(1-x^2)(1-k^2x^2)} \, dx,
\]
m, n \in \mathbb{N}_0. \text{ Here we have again used the integral representation (2.2).}

**Theorem 8.2.** Each of the weighted Hankel matrices $H^{(f')}$, $H^{(f'')}$ and $H^{(g')}$ represents a positive trace class operator on $\ell^2(\mathbb{N}_0)$ with simple eigenvalues. We have:
(i) Eigenvalues of $H_{f'}$ enumerated in descending order are
$$\nu_m^{(f')} = \frac{4K}{\sqrt{\pi}k} \frac{q^{m+1/2}}{(2m+1)(1-q^{2m+1})}, \ m \geq 0.$$  
For an eigenvector corresponding to $\nu_m^{(f')}$ one can choose the vector $\Psi^{(f)}_m$, as introduced in Theorem 7.4 ad (i).

(ii) Eigenvalues of $H_{f''}$ enumerated in descending order are
$$\nu_m^{(f'')} = \frac{2\pi^{3/2}}{kK} \frac{(2m+1)q^{m+1/2}}{1-q^{2m+1}}, \ m \geq 0.$$  
For an eigenvector corresponding to $\nu_m^{(f'')}$ one can choose the vector $\Psi^{(f)}_m$, as introduced in Theorem 7.4 ad (i).

(iii) Eigenvalues of $H_{g'}$ enumerated in descending order are
$$\nu_m^{(g')} = \frac{2\pi^{3/2}}{k^2K} \frac{mq^m}{1-q^{2m}}, \ m \geq 1.$$  
For an eigenvector corresponding to $\nu_m^{(g')}$ one can choose the vector $\Psi^{(g)}_m$, as introduced in Theorem 7.4 ad (ii).

Proof. Similarly as in the proof of Theorem 8.1 we confine ourselves to pointing out only some features which are particular for the matrices treated in this theorem. Otherwise the basic scheme is still the same as in the proof of Theorems 6.1 or 7.4.

It is easy to verify in each of the three cases that $\sum_{m,n=0}^{\infty} H_{m,n} < \infty$ which implies that the matrices represent trace class operators.

(i) In this case $a = 1/2$, $b = 0$ and $c = -1/2$. The choice of $\sigma = 1$ corresponds to the Jacobi matrix $J^{(f)}$, see the proof of Theorem 7.4 ad (i). We put $\xi = a = 1/2$, $\eta = b + c + 2 = 3/2$, and then $\omega(\xi, \eta, \sigma) = 1/2$, see (4.11). The weight (7.3) equals
$$w_n^{(f')} = \sqrt{\frac{(1)n}{n!} \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n}} = \frac{1}{\sqrt{2n+1}}.$$  
Furthermore, the square summable solution (4.14) of Eq. (4.1) equals
$$h_n^{(f')} = \frac{k^n\Gamma(n+3/2)}{(n+1)!} 2F_1\left(n+3/2, 1/2 \middle| \frac{k^2}{n+2}\right)$$  
$$= \frac{2k^n}{\sqrt{\pi}} \int_0^1 \frac{x^{2n+2}}{\sqrt{(1-x^2)(1-k^2x^2)}} dx.$$  

$H^{(f')}$ can be diagonalized by the same unitary transform as the matrix $H^{(f)}$ which is described in the proof of Theorem 7.4 ad (i). Using some routine manipulations, quite similarly as in the proof of Proposition 3.3, one
can transform $H^{(f')}$ to a multiplication operator by the function
\[
h^{(f')}(x) = \sum_{n=0}^{\infty} H_{n,0}^{(f')} \hat{F}_n(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{f_n(-x)}{(2n+1)!} \int_0^K \sin^{2n+2}(u) \, du
\]
\[
= \frac{2}{\sqrt{\pi}} \int_0^K \frac{\sin(\sqrt{x}u)}{\sqrt{x}} \sin(u) \, du.
\]

This formula in combination with the Fourier series (2.7) makes it possible to evaluate $h^{(f')}(x)$ at the spectral points $\lambda_m$ of the Jacobi matrix $J^{(f)}$, as given in (2.14). The obtained values read
\[
h^{(f')}(\lambda_m) = \frac{4K}{\sqrt{\pi k}} \frac{q^{m+1/2}}{2(m+1)(1-q^{2m+1})}, \quad m \geq 0,
\]
and these are in fact the eigenvalues of $H^{(f')}$. (ii) In this case $a = -1/2$, $b = 0$ and $c = 1/2$. The choice of $\sigma = 1$ again corresponds to the Jacobi matrix $J^{(f)}$, see the proof of Theorem 7.4 ad (i). We put $\xi = a = -1/2$, $\eta = b + c + 2 = 5/2$, and then $\omega(\xi, \eta, \sigma) = 3/2$, see (4.11). The weight (7.3) equals
\[
w^{(f'')}_n = \sqrt{\frac{(1)n(\frac{3}{2})n}{n!(\frac{1}{2})n}} = \sqrt{2n + 1}.
\]
Furthermore, the square summable solution (4.14) of Eq. (4.1) equals
\[
h^{(f'')}_n = \frac{k^n \Gamma(n + 1/2)}{(n + 1)!} \frac{\Gamma(1 - n/2)}{2\Gamma(n + 1)} \left(\frac{1}{2}\right)_{n+2} = 4k^n \int_0^{1} x^{2n} \sqrt{(1 - x^2)(1 - k^2x^2)} \, dx.
\]

$H^{(f'')}$ can be diagonalized by the same unitary transform as the matrix $H^{(f)}$ which has been described in the proof of Theorem 7.4 ad (i). After some routine manipulations, similarly as in the proof of Proposition 3.3, one can transform $H^{(f'')}$ to a multiplication operator by the function
\[
h^{(f'')}_{(x)} = \sum_{n=0}^{\infty} H_{n,0}^{(f'')} \hat{F}_n(x) = \frac{4}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{f_n(-x)}{(2n)!} \int_0^K \sin^{2n}(u) \cos^2(u) \, du \]
\[
= \frac{4}{\sqrt{\pi}} \left(\cos(\sqrt{x}K) + \int_0^K \sin(\sqrt{x}u) \sin(u) \, du\right).
\]

This formula in combination with the Fourier series (2.7) makes it possible to evaluate $h^{(f'')}(x)$ at the spectral points $\lambda_m$ of the Jacobi matrix $J^{(f)}$, as given in (2.14). The obtained values read
\[
h^{(f'')}_{(x)} = \frac{2\pi^{3/2}}{kK} \frac{(2m + 1)q^{m+1/2}}{1 - q^{2m+1}}, \quad m \geq 0,
\]
and these are in fact the eigenvalues of $H^{(f''')}$. (iii) In this case $a = b = 1/2$, $c = 1$. The choice of $\sigma = 2$ corresponds to the Jacobi matrix $J^{(g)}$, see the proof of Theorem 7.4 ad (ii). We put $\xi = a = 1/2$, $\eta = b + c + 2 = 7/2$, and then $\omega(\xi, \eta, \sigma) = 3/2$, see (4.11).
The weight (7.3) equals $w_n(g') = \sqrt{n + 1}$. Furthermore, the square summable solution (4.14) of Eq. (4.1) equals

$$h_n(g') = \frac{k^n \Gamma(n + 3/2)}{(n + 2)!} _2F_1\left(\frac{n + 3/2, -1/2}{n + 3}, k^2\right)$$

$$= \frac{4k^n}{\sqrt{\pi}} \int_0^1 x^{2n+2} \sqrt{(1 - x^2)(1 - k^2x^2)} \, dx.$$  

$H(g')$ can be diagonalized by the same unitary transform as the matrix $H(g)$ which has been described in the proof of Theorem 7.4 ad (ii). After some routine manipulations, similarly as in the proof of Proposition 3.3, one can transform $H(g')$ to a multiplication operator by the function

$$h(g')(x) = \sum_{n=0}^{\infty} H_n^{(g')}(x) \hat{G}_n(x)$$

$$= \frac{4}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{g_n(-x)}{(2n + 1)!} \int_0^K \sin^{2n+2}(u)\cos^2(u)\sin^2(u) \, du$$

$$= \frac{1}{\sqrt{\pi}} \left( \frac{\sin(\sqrt{x}K)}{\sqrt{x}} - \int_0^K \cos(\sqrt{x}u)\sin^2(u) \, du \right).$$

This formula in combination with the Fourier series (2.10) makes it possible to evaluate $h(g')(x)$ at the spectral points $\lambda_m$ of the Jacobi matrix $J(g)$, as given in (2.18). The obtained values read

$$h(g') (\lambda_m) = \frac{2\pi^{3/2}}{k^2K} \frac{mq^m}{1 - q^{2m}} , \ m \geq 1,$$

and these are in fact the eigenvalues of $H(g')$. □

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