Abstract

In today’s ML, data can be twisted (changed) in various ways, either for bad or good intent. Such twisted data challenges the founding theory of properness for supervised losses which form the basis for many popular losses for class probability estimation. Unfortunately, at its core, properness ensures that the optimal models also learn the twist. In this paper, we analyse such class probability-based losses when they are stripped off the mandatory properness; we define twist-proper losses as losses formally able to retrieve the optimum (untwisted) estimate off the twists, and show that a natural extension of a half-century old loss introduced by S. Arimoto is twist proper. We then turn to a theory that has provided some of the best off-the-shelf algorithms for proper losses, boosting. Boosting can require access to the derivative of the convex conjugate of a loss to compute examples weights. Such a function can be hard to get, for computational or mathematical reasons; this turns out to be the case for Arimoto’s loss. We bypass this difficulty by inverting the problem as follows: suppose a blueprint boosting algorithm is implemented with a general weight update function. What are the losses for which boosting-compliant minimisation happens? Our answer comes as a general boosting algorithm which meets the optimal boosting dependence on the number of calls to the weak learner; when applied to Arimoto’s loss, it leads to a simple optimisation algorithm whose performances are showcased on several domains and twists.
1 Introduction

Modern supervised machine learning (ML) was founded almost four decades ago \cite{54}, one of its core parts being subsumed by even earlier contributions in normative economics on class probability estimation (CPE) \cite{41, 43, 47}. This so-called core part is the function chosen beforehand allowing an algorithm to compute the utility of models to solve the task at hand: the loss function. The founding normative theory of supervised losses is properness, which states that a good loss function should be such that Bayes rule be optimal for the loss \cite{41}. Standard losses of supervised learning such as the logistic, square, Matusita, or the 0/1-loss, are all proper. Further, some approaches have started learning or tailoring the proper loss minimized to the task at hand \cite{26, 34, 58} (and references therein). Properness alone is not sufficient to guarantee convergence to Bayes rule as algorithmic-, data- and model-dependent considerations need to be taken into account, but one certainty prevails: choosing a loss that is not proper guarantees convergence to a model that is probably not Bayes optimal.

Our first contribution in this paper analyses losses for class probability estimation when stripped off mandatory properness. Today’s ML relies on available data that can be corrupted in many complex ways such that the optimal classifier on such data represents more twist or noise than the actual signal of interest. Going beyond the now classical \cite{28}, recent work has demonstrated a flurry of concrete data twisting environments from label noise \cite{39} to adversarial examples \cite{17}, generated data \cite{24}, out-of-sample data \cite{6}, quantizers \cite{61}, data poisoners \cite{25} or just privacy enablers \cite{19, 36}. If one wants to make sure that the model learned fits the signal, then the loss needs to be amended to discard twists. Just as much as twists investigation, loss correction has a longstanding history in ML \cite{2, 4, 6, 13, 31, 36, 39, 53, 56, 60, 61, 62} (and many others), but the key problem is in fact where to put the correction. The papers just cited all share a common point that many others would as well: the correction is directly carved in the loss as it is optimized, i.e. usually its corresponding surrogate (Section 2). For a better understanding of how corrections operate, a more principled standpoint on the problem would open the loss’ black box and investigate corrections directly from its core ingredients: the partial losses \cite{41}. Such is our approach and since to the best of our knowledge there has been so far no investigation of such losses outside the proper lens, we first analyse their properties when the partial losses just meet standard mathematical properties (monotonicity, differentiability, etc.). We define a broad set of twists called "Bayes blunting", relevant to most of the aforementioned twisters, that keep the optimal label but decrease the "confidence" in the optimal prediction. We define twist-proper losses, having the desirable property that their minimizer can "correct" the twists and thus contains the (twist-free) Bayes rule. Further, we show that $\alpha$-loss, a loss function naturally extended and developed in \cite{29, 50} from its original form in \cite{5}, is twist-proper and comes with desirable properties for local and global twist correction.

Our second focus is algorithmic: boosting algorithms are powerful optimisation algorithms with the ability to output arbitrarily accurate classifiers by having access to a weak learner that outputs classifiers slightly different from random guessing \cite{44, 46}. Implementing a boosting algorithm with a CPE-based loss requires inverting a link function \cite{23, 40}. While such an inversion is simple for popular choices such as the log- or square- losses, it can be computationally demanding \cite{26, 34} or mathematically tricky; this turns out to be the case for the $\alpha$-loss.
Our second contribution circumvents this difficulty by inverting the problem (a sideways approach) as follows: instead of picking a loss and coming up with a boosting algorithm, suppose we implement a blueprint boosting algorithm with a function having the properties of an inverse link. The question, that we address, is then: "for what loss(es) and under which conditions does it grant boosting-compliant convergence?".

We propose such a general boosting algorithm accompanied with two formal results: (i) conditions on any potential twice differentiable loss to be granted boosting-compliant convergence via the algorithm, (ii) extension of this result to margin distributions. Our results rely on the classical weak learning assumption [27] and meets the general optimal bound of oracle calls to the weak learner [1]. Our result also involves two additional first- and second-order assumptions on the loss(es) of interest, the former being data-dependent while the latter is weak learner dependent. As boosting iterations increase, the data-dependent condition can become sharper, which may limit the number of boosting iterations, but regardless of the set of losses of interest, the boosting algorithm is essentially oblivious to its content as it never explicitly operates on any of its elements. To apply our theory to the $\alpha$-loss, we design a general clipped, easy to invert approximation of the link function based on partial losses, which can be of interest for general losses. This approximation is precise enough for the $\alpha$-loss to virtually get rid of the first- and second-order assumptions — therefore ending up with a boosting algorithm in the original model of boosting [46] but with no explicit computation about the loss of interest. The algorithm is tested experimentally against twisters tampering labels, features and an "insider twister" informed with the feature importance of the ML algorithms.

Section 2 presents key definitions on losses for class-probability estimation, Section 3 introduces and shows properties on twist-proper losses and the $\alpha$-loss. Section 4 presents our general approach on boosting and Section 5 applies it to the $\alpha$-loss. Section 6 presents experiments and two last Sections respectively discuss our findings (7) and conclude (8). Formal proofs and additional experiments are provided in an appendix.

2 Losses for class-probability estimation

Our setting is that of losses for class probability estimation (CPE) and our notations follow [40, 41]. Given a domain of observations $\mathcal{X}$, we wish to learn a classifier $h : \text{dom}(h) = \mathcal{X}$ that predicts the label $Y \in \mathcal{Y} = \{-1, 1\}$ (without loss of generality, we assume two classes or labels) associated with every instance of data drawn from $\mathcal{X}$. Traditionally, there are two kinds of outputs sought: one requires $\text{Im}(h) = [0, 1]$, in which case $h$ provides an estimate of $P[Y = 1|X]$, usually called Bayes posterior. This is the framework of class probability estimation. The other kind of output requires $\text{Im}(h) = \mathbb{R}$, but is usually completed by a mapping to $[0, 1]$, e.g. via the softmax in deep learning. A loss for class probability estimation, $\ell : \mathcal{Y} \times [0, 1] \to \mathbb{R}$, has the general definition

$$\ell(y, u) \doteq [y = 1] \cdot \ell_1(u) + [y = -1] \cdot \ell_{-1}(u),$$

where $[\cdot]$ is Iverson’s bracket (the indicator function). Functions $\ell_1, \ell_{-1}$ are called partial losses, basically assumed to satisfy $\text{dom}(\ell_1) = \text{dom}(\ell_{-1}) = [0, 1]$ and $|\ell_1(u)| \ll \infty, |\ell_{-1}(u)| \ll \infty, \forall u \in (0, 1)$ to be useful for ML. The pointwise conditional risk of local guess $u \in [0, 1]$
with respect to a ground truth \( v \in [0, 1] \) is (where \( B(v) \) defines a Bernoulli distribution with parameter \( v \)):

\[
L(u, v) = \mathbb{E}_{Y \sim B(v)}[\ell(Y, u)] = v \cdot \ell_1(u) + (1 - v) \cdot \ell_{-1}(u).
\]

(2)

Bayes tilted estimates We define the set-valued (pointwise) Bayes tilted estimate \( t_\ell \) as:

\[
t_\ell(v) = \arg \inf_{u \in [0, 1]} L(u, v),
\]

(3)

and the (pointwise) Bayes risk as \( L(v) = L(u, v), u \in t_\ell(v) \). If the loss is proper, \( v \in t_\ell(v) \) and, if strictly proper, \( t_\ell(v) = \{v\} \). The outputs in \([0, 1]\) and \( \mathbb{R} \) are related via convex duality of the losses. Let \( g^*(z) = \sup_t \{zt - g(t)\} \) the convex conjugate of \( g \). The surrogate \( F(z) \) of \( L \) is:

\[
F(z) = (-L)^*(-z), \forall z \in \mathbb{R}.
\]

(4)

For example, picking the log-loss as \( \ell \) gives the binary entropy for \( L \) and the logistic loss for \( F \). Convex duality implies that predictions in \([0, 1]\) and \( \mathbb{R} \) are related via the link of the loss, \( (-L)' \) where we use the notation \( f' \) to denote the derivative of a function \( f \) with respect to its argument.

3 Twist-proper losses and the \( \alpha \)-loss

To our knowledge, losses for class probability estimation have not received much coverage without substantial basic assumptions like properness. We provide such basic results and first summarize several simple but fundamental invariants (monotonicity being of primary importance).

**Lemma 3.1.** \( \forall \ell \) CPE loss, \( L \) is concave and continuous; \( F \) is convex, continuous and non-decreasing.

We now investigate the additional impact of common functional assumptions on the partial losses:

(M) Monotonicity: \( \ell_1 \) is non-increasing, \( \ell_{-1} \) is non-decreasing;

(D) Differentiability: \( \ell_1 \) and \( \ell_{-1} \) are differentiable;

(S) Symmetry: \( \ell_1(u) = \ell_{-1}(1 - u), \forall u \in [0, 1] \).

Commonly used proper losses like the log-, square- and Matusita-losses all satisfy the above three assumptions. We note that standard properties for \( t_\ell \) do not trivially follow from properties of the partial losses. For example, strict monotonicity of partial losses does not guarantee that \( t_\ell \) is not set-valued (Proof included in the proof of Lemma 3.2). The set valued inequality \( A \leq B \) means \( \forall a \in A, \exists b \in B, a \leq b \) and the set-valued (Minkowski) difference \( A - B = \{a - b : a \in A, b \in B\} \).
Lemma 3.2. The following properties of $t_\ell$ follow from assumptions $M$, $D$ or $S$ on partial losses:

(M) implies set-valued monotonicity: $\forall u_1 < u_3 \in [0,1],
\begin{align*}
(t_\ell(u_1) \leq t_\ell(u_3)) \land (t_\ell(u_1) \cap t_\ell(u_3) \subseteq t_\ell(u_2), \forall u_2 \in (u_1, u_3)).
\end{align*}

(D) and $t_\ell$ differentiable imply monotonicity: $\forall u \in [0,1], \ell'(t_\ell(u)) \leq \ell'_1(t_\ell(u)) \iff t'_\ell(u) \geq 0$;

(S) implies set-valued symmetry: $t_\ell(1-u) = \{1\} - t_\ell(u), \forall u \in [0,1]$.

The monotonicity of $t_\ell$ given by (D) holds without making monotonicity assumption on partial losses. Should we add strict monotonicity for at least one partial loss, we would get the invertibility of $t_\ell$.

Twist-proper losses Using more conventional ML notions [41], we now use $\eta_c$ to denote the “clean” posterior probability and “twist” refers to a general mapping $\eta_c \mapsto \eta_t$, which could be consequence of random noise, data augmentation or poisoning, etc. We refer to hyperparameter(s) of a loss as free variable(s) not appearing in the arguments.

Definition 3.1. A loss $\ell$ is said twist-proper (resp. strictly twist proper) iff for any twist, there exists hyperparameter(s) such that $\eta_c \in t_\ell(\eta_t)$ (resp. $\{\eta_c\} = t_\ell(\eta_t)$).

Hence, minimizing the loss "gets rid of the twist" in the twisted posterior. Any proper loss would fail at this objective. We emphasize the need for hyperparameters as otherwise, twist-properness would trivially enforce $t_\ell(.) = [0,1]$. Ideally, twist-properness would involve just 1 hyperparameter.

Definition 3.2. Twist $\eta_c \mapsto \eta_t$ is Bayes blunting iff $(\eta_c \leq \eta_t \leq 1/2) \lor (\eta_c \geq \eta_t \geq 1/2)$.

A Bayes blunting twist keeps Fisher consistency in the twist [23] and acts very specifically: Bayes blunting alters confidence in the optimal classification without changing its optimal polarity – it makes guessing harder just because the twisted posterior is closer to random guessing (note: our terminology is based on those established in the boosting framework [22, 46]). The term “blunting” is inherited from adversarial training [17], but the twists in Def. 3.2 also cover “gentler” twists historically overwhelmingly popular: label noise. Consider the symmetric label flip with probability $p$. The twisted posterior is $\eta_t = \eta_c(1-p) + (1-\eta_c)p$ and we readily deduce the following.

Lemma 3.3. Symmetric label flip is Bayes blunting if $p \leq 1/2$.

We can also deduce from Lemma 3.3 that Massart noise, where $p$ depends on $x$, is also a particular case of Bayes blunting twist [18].

The $\alpha$-loss was first introduced in information theory in the early 70s [5] and recently got increased scrutiny in privacy and ML [29, 50] (note that the terminology $\alpha$-loss is introduced in [29]). Let $\alpha \in (-\infty, \infty]$, and define the conjugate $\alpha^c$ such that $1/\alpha^c + 1/\alpha = 1$, using by extension $\alpha^c(\infty) = 1, \alpha^c(1) = \infty$. For $\alpha \geq 1$, $\alpha^c$ is known as the Hölder conjugate.

Definition 3.3. The $\alpha$-loss has the following partial losses ($\alpha$ implicit in notations):
\begin{align*}
\ell_1(u) &= \frac{\alpha}{\alpha - 1} \cdot (1 - u^{\frac{1}{\alpha}}) = \alpha^c \cdot (1 - u^{\frac{1}{\alpha^c}}); \quad \ell_{-1}(u) = \ell_1(1-u), \quad \forall u \in [0,1].
\end{align*}

and by continuity we let $\ell_1(u) \equiv -\log u$ for $\alpha = 1$ and $\ell_1(u) \equiv 1 - u$ for $\alpha = \infty$. 


Hence, the $\alpha$-loss is (S)ymmetric by construction. Our definition extends the previous definitions that either restricted [20] to $\alpha \geq 1$ or $\alpha \in \mathbb{R} \setminus\{1\}$ [5]. In our context, generality is convenient and desirable, as now explained. For any $u \in [0, 1]$, let $\iota(u) = \log(u/(1-u))$ denote the logit of $u$.

**Lemma 3.4.** The following properties hold for $\alpha$-loss:

(a) $\alpha$-loss meets all (M), (D), (S) assumptions, $\forall \alpha$,

(b) the Bayes tilted estimate of the $\alpha$-loss is:

$$ t_\ell(\eta) = \begin{cases} [0, 1] & \text{if } (\alpha = 0) \lor (\alpha = \infty \land \eta_c = 1/2) \\ \frac{\eta_c^\alpha}{\eta_c^\alpha + (1-\eta_c)^\alpha} & \text{otherwise (taking the limit if } \alpha = \infty) \end{cases} , \quad (7) $$

hence,

(c) $\alpha$-loss is twist-proper for $\alpha = \alpha^*$ with

$$ \alpha^* = \frac{\iota(\eta_c)}{\iota(\eta_t)} , $$

(d) for any Bayes blunting twist, $\alpha^* \geq 1$.

(proof straightforward) We observe from (7) that the Bayes tilted estimate of the $\alpha$-loss is invariant upon permuting ($\eta_t, \alpha$) and $(1-\eta_t,-\alpha)$ so that $\alpha < 0$ “reverse” the polarity of the twisted posterior in the Bayes tilted estimate. Lemma 3.4 is important theoretically because it shows that the $\alpha$-loss can be used to correct any twist. However, just as classification calibration leads to a pointwise form of consistency [7], twist-properness is a pointwise form of correction.

Extending twist-properness to domain $X$ requires a mapping $\alpha : X \rightarrow (-\infty, \infty]$. Without knowing the twist $\eta_t \mapsto \eta_c$, it is quite impossible to be twist-proper, but it can still be relevant practically, where the twist is engineered invertible and done to protect data, then used by a remote learner to train a model, received back and with twisted posterior corrected using (7). Such scenarios have been investigated, e.g. in [36]. To switch to population quantities, we assume a marginal distribution $M$ over $X$ [11] from which the expected value of a loss $\ell$ provides a (true) risk of a classifier $h$. Lemma 3.4 begs for the following question: are there “good” scalar values for $\alpha$, leading to substantial domain guarantee? We answer in the affirmative, and for this objective, switch to domain formulations of $\eta_t, \eta_c : X \rightarrow [0,1]$. Define the cross-entropy of the Bayes tilted estimate of the $\alpha$-loss:

$$ \text{CE}(\eta_t, \eta_c; \alpha) = \mathbb{E}_{X \sim M} [\eta_c(X) \cdot -\log t_\ell(\eta_c(X))] , \quad (8) $$

where $\alpha$ is hidden in the notation $t_\ell$. The reason why we focus on the cross-entropy is simple: if we subtract Shannon’s entropy of the clean posterior (its cross-entropy as $\text{CE}(\eta_t, \eta_c; 1)$), then we get the Kullback-Leibler divergence between the distributions on $X \times Y$ induced by $\eta_t, M$ on one hand, and $\eta_c, M$ on the other hand. Suppose the following property holds on the twisted posterior:

$$ \exists B > 0 : (1 + \exp(B))^{-1} \leq \eta_c(\cdot) \leq (1 + \exp(-B))^{-1} \quad \text{almost surely} \quad (9) $$
Algorithm 1 PilBoost

| Input sample $S = \{(x_i, y_i), i = 1, 2, ..., m\}$, number of iterations $T$, $a_f > 0$, Pil $\tilde{f}$; |
|---|
| Step 1: let $\beta \leftarrow 0$; // first classifier, $H_0 = 0$ |
| Step 2: for $t = 1, 2, ..., T$ |
| Step 2.1: for $i = 1, 2, ..., m$, let $w_i \leftarrow \tilde{f}(-y_iH_\beta(x_i))$ // Pil weights |
| Step 2.2: let $j \leftarrow \text{wl}(S, w)$ |
| Step 2.3: let $\eta_j \leftarrow (1/m) \cdot \sum_i w_iy_ih_j(x_i)$ |
| Step 2.4: let $\beta_j \leftarrow \beta_j + a_f\eta_j$ |
| Return $H_\beta$. |

(note that this can be straightforwardly ensured by clipping $\eta_j$) Define the logit-edge over $M$:

$$\eta = (1/B) \cdot \mathbb{E}_{X \sim M}[\eta_c(X)u(\eta_c(X))] \quad (\in [-1, 1]),$$

and let $q = (1 + \eta)/2$ (we use the term "edge" rather than "margin"). Define the binary entropy $H(u) = -u \cdot \log(u) - (1 - u) \cdot \log(1 - u), u \in [0, 1]$, with the convention $0 \cdot \log 0 = 1 \cdot \log 1 = 0$.

**Theorem 1.** Suppose we fix $\alpha = \alpha^*$ with $\alpha^* \doteq u(q)/B$. Then the following bound holds on the cross-entropy of the Bayes tilted estimate of the $\alpha$-loss:

$$\text{CE}(\eta, \eta_c; \alpha^*) \leq H(q).$$

The proof, in Section A.3, encompasses the more general setting where (9) does not hold. It is followed by a simple example where $\text{CE}(\eta, \eta_c; \alpha^*)$ can vanish while $\text{CE}(\eta, \eta_c; 1)$ is always larger than a constant $\geq 0.3$. While formal, Th. 1 has practical incidence that goes beyond the scope of this paper: the proof of Theorem 1 is refined to include the case where $\eta$ is estimated (from sampling) by some $\hat{\eta}$. In this case, the bound in Th. 1 incurs an additional penalty of order $O(|\eta - \hat{\eta}| \cdot |\hat{\eta}|/(1 - |\hat{\eta}|))$.

4 Sideways boosting a loss

Our setting is as follows: we have a training sample $S = \{(x_i, y_i), i \in [m]\} \subset X \times Y$ of examples, where $[m] = \{1, 2, ..., m\}$. We write $i \sim D$ to indicate sampling according to the observed distribution. We are interested in boosting algorithms to train a classifier $H$ from $S$ to minimise an expected loss with respect to $D$. Typically, $H$ is real-valued as in e.g. [46, 23], and the algorithm has two key components: (i) access to an oracle $\text{wl}$ returning (weak) classifiers $h_i$ slightly beating random classification and (ii) a way to combine those weak classifiers that complies with the accuracy and PTine requirements of the boosting model [46]. We focus on linear combinations of classifiers: following notations from [16, 35], we index weak hypotheses in $\mathbb{N}$, and let

$$H_\beta \doteq \sum_j \beta_jh_j.$$
The oracle \( \text{WL} \) returns an index \( j \in \mathbb{N} \) and the task of boosting is to learn the coordinates of \( \beta \), initialised to the null vector. In our general framework, the losses we consider are the surrogates \( F \) in Lemma 3.1 essentially convex and non-increasing functions, adding the condition that they are twice differentiable. We compute weights using the blueprint of [23], which uses the full \( H_\beta \):

\[
    w_i = -F'(y_i H_\beta(x_i)), \forall i \in [m]. \tag{12}
\]

Sometimes, boosting uses the mirror update [16, 35]:

\[
    w_i \leftarrow -F'(z_i + F'^{-1}(w_i)), \text{ where } z_i = y_i h(x_i). \tag{12}
\]

This has the main advantage that it does not require to compute the inverse \( F'^{-1} \), which is more convenient if \( F \) is not strictly convex. Both update rules ensure a popular property of boosting: weights are non-negative and tend to decrease for an example given the right class by the current weak classifier \( h_j \) – weighting puts emphasis on “hard” examples.

The issue of boosting for general \( CPE \) losses follows directly from (12): assuming strict concavity of the pointwise Bayes risk and assumption (D) in Section 3, we get from the definition of \( F \) in (4) that

\[
    F'(z) = L'^{-1}(-z) = (\ell_{-1} \circ t_\ell - \ell_1 \circ t_\ell)^{-1}(-z) \tag{13}
\]

(see the proof of point (D) in Lemma 3.2). We thus need to invert the difference of the partial losses to get to \( F' \) (and eventually \( F \) \( \text{nwLO} \) (Section 3)). The inversion is easy for the log-loss because of properties of the log function, and for the square loss because partial losses are quadratic functions. One can easily conjecture that the task could be substantially harder in general. This turns out to be the case for the \( \alpha \)-loss. We circumvent this difficulty by taking a fork to boosting \( F \): we propose an algorithm, PILBOOST, with a general weight update (Step 2.1) using a function \( \tilde{f} \) non-negative and increasing. By analogy with \( -L' \) being the (canonical) link of the loss \( \ell \) [37], we call \( \tilde{f} \) a pseudo-inverse link (PIL). We shall see in Section 5 a general way to construct \( \tilde{f} \) from the partial losses of interest with compelling properties for the \( \alpha \)-loss, but for now, we focus on providing conditions on any \( \tilde{f} \) to boost a loss \( F \) of interest and therefore analyse the general boosting abilities of PILBOOST. For this objective, we make two classical boosting assumptions on \( \text{WL} \) [46, 37].

**Assumption 1.** (R) The weak classifiers have bounded range: \( \exists M > 0 \) such that \( |h_j(x_i)| \leq M, \forall j \).

Let \( \tilde{\eta}_j = m \cdot \eta_j / (1^T w_j) \in [-M, M] \) be the normalized edge of the current weak classifier, where \( \eta_j \) is the (unnormalized) edge (Step 2.3 of PILBOOST). "WLA" denotes the Weak Learning Assumption.

**Assumption 2.** (WLA) The weak classifiers are not random: \( \exists \gamma > 0 \) such that \( |\tilde{\eta}_j| \geq \gamma \cdot M, \forall j \).

Since we want to analyze the boosting ability of PILBOOST for losses not directly related to the PIL chosen, we need two more functional assumptions on the first- and second-order derivatives of the losses of interest. The edge discrepancy of a function \( F \) on weak classifier \( h_j \) at iteration \( t \) is:

\[
    \Delta_j(F) = |\mathbb{E}_{x \sim D} [y_i h_j(x_i) F'(y_i H_\beta(x_i))] - \eta_j|, \tag{14}
\]
which is the absolute difference of the edge using (the derivative of) $F$ vs. using PILBOOST’s $\tilde{f}$.

**Assumption 3.** $(O1, O2) \exists \xi, \pi \in [0, 1)$ such that:

1. $(O1)$ the edge discrepancy is bounded $\forall t$: $\Delta_j(F) \leq \xi \cdot \eta_j$, where $j$ is returned by $wl$ at iteration $t$;

2. $(O2)$ the curvature of $F$ is bounded: $F^* \equiv \sup_z F''(z) \leq (1 - \xi)(1 + \pi)/(aF M^2)$.

$(O2)$ is quite mild for specific sets of functions: for example proper canonical losses are Lipschitz [40], so $(O2)$ can in general be ensured by a simple renormalization of the loss, which does not change the ordering in models that the loss provides. In other cases, like for AdaBoost’s popular exponential loss, meeting $(O2)$ may require to limit the number of boosting iterations. The one assumption becoming progressively harder to ensure in general is $(O1)$, in particular if $wl$ runs out of options to keep $\eta_j$ not too small. Let $\tilde{w}_t = 1^T w_t$, the total weight at iteration $t$ in PILBOOST.

**Theorem 2.** Suppose $(R, WLA)$ hold on $wl$ and $(O1, O2)$ hold on function $F$, for each iteration of PILBOOST. Denote $Q(F) = 2F^*/(\gamma^2 (1 - \xi)^2 (1 - \pi^2))$. The following results hold:

- on the risk defined by $F$: $\forall z^* \in \mathbb{R}, \forall T > 0,$

$$\left( \sum_{t=0}^{T} \tilde{w}_t^2 \geq Q(F) \cdot (F(0) - F(z^*)) \right) \Rightarrow \mathbb{E}_{i \sim D} [F(y_i H_{\beta}(x_i))] \leq F(z^*). \quad (15)$$

- on edge distribution: $\forall \theta \geq 0, \forall \varepsilon \in [0, 1], \forall T > 0$, letting $F_{\varepsilon, \theta} = (1 - \varepsilon) \inf F + \varepsilon F(\theta)$,

$$\left( T \geq \frac{1}{\varepsilon^2} \cdot \frac{Q(F) \cdot (F(0) - F_{\varepsilon, \theta})}{f^2(-\theta)} \right) \Rightarrow \mathbb{P}_{i \sim D} [y_i H_{\beta}(x_i) \leq \theta] \leq \varepsilon. \quad (16)$$

Thus with Theorem 2, we give boosting compliant convergence on training. When classical assumptions about the loss of interest are satisfied, such as it being Lipschitz (ensured for proper canonical losses [40]), there is a natural extension to generalisation following standard approaches [8, 45].

Two remarks hold regarding convergence rate: first, the $1/\gamma^2$ dependence meets the general optimum for boosting [41]; second, the $1/\varepsilon^2$ dependence parallels classical training convergence of convex optimization [52] (and references therein). There is however a major difference with such work: PILBOOST requires no function oracles for $F$ (function values, (sub)gradients, etc.). This “sideways” fork to minimizing $F$ pays (only) a $1/(1 - \xi)^2$ factor in convergence.

We now apply it to the $\alpha$-loss.
5 Boosting for the \( \alpha \)-loss

We now connect Sections 3 and 4. If we were to exactly implement a boosting algorithm for the \( \alpha \)-loss, we would have to find the exact inverse of (13), which would require inverting 
\[ -L'(v) = \alpha^c \cdot t_\ell(v) - \alpha^c \cdot t_\ell(1-v). \]
Owing to the difficulty to carry this step, we choose a sidestep that makes inversion straightforward and can fall in the conditions to apply Theorem 2, thus making PILBOOST a boosting algorithm for the \( \alpha \)-loss of interest. The trick does not just hold for the \( \alpha \)-loss, so we describe it for a general loss \( \ell \) assuming for simplicity that \( \ell_1(1) = \ell_{-1}(0) = 0 \) and \( t_\ell, \ell_1, \ell_{-1} \) are invertible with \( \ell_1, \ell_{-1} \) non-negative, conditions that would hold for many popular losses (log, square, Matusita, etc.), and the \( \alpha \)-loss. We then approximate the link 
\[ -L'(v) \]
by using just one of \( \ell_1 \) or \( \ell_{-1} \) depending on their argument, while ensuring functions match in \( 0, 1/2, 1 \). We name \( \tilde{f}_\ell \) the clipped inverse link, CIL. Letting 
\[ a_{-\ell} = \ell_1(0)/(\ell_{-1}(0) = \ell_{-1}(1/2)) \]
and
\[ a_{+\ell} = \ell_{-1}(1)/(\ell_{-1}(1) - \ell_{-1}(1/2)), \]
our link approximation is
\[ f_\ell(u) = f_{+\ell}(u) \]
if \( u \leq 1/2 \) and \( f_{+\ell}(u) \) otherwise, with:
\[ f_{-\ell}(u) = a_{-\ell} \cdot (\ell_{1}(1/2) - \ell_1(t_\ell(u))) \]
\[ f_{+\ell}(u) = a_{+\ell} \cdot (\ell_{-1}(t_\ell(u)) - \ell_{-1}(1/2)). \]

Lemma 5.1. \( f_\ell(u) = -L'(u), \forall u \in \{0, 1/2, 1\} \); furthermore, the clipped inverse link \( \tilde{f}_\ell = f_\ell^{-1} \) is:
\[
\tilde{f}_\ell(z) = \begin{cases} 
0 & \text{if } z < -\ell_1(0), \\
-1 \cdot \ell_{1}(1/2) - \ell_1(0) & \text{if } -\ell_1(0) \leq z < 0, \\
-1 \cdot \ell_{-1}(1)/(\ell_{-1}(1) - \ell_{-1}(1/2)) & \text{if } 0 \leq z < \ell_{-1}(1), \\
1 & \text{if } z \geq \ell_{-1}(1). 
\end{cases}
\]

Furthermore, \( \tilde{f}_\ell \) is continuous and if \((S)\) and \((D)\) hold, then \( \tilde{f}_\ell \) is derivable on \( \mathbb{R} \) (with the only possible exceptions of \( \{-\ell_1(0), \ell_{-1}(1)\} \)).
The proof is immediate once we remark that \( \ell_1(1) = \ell_{-1}(0) = 0 \) bring "properness for the extremes", i.e. \( 0 \in t_\ell(0), 1 \in t_\ell(1) \). For space reasons, the appendix (Section A.6) gives expressions of \( f_\ell \) and \( \tilde{f}_\ell \) for the \( \alpha \)-loss. Fig. I shows the quality of approximation of the clipped inverse link for the \( \alpha \)-loss.

**Remark 5.1.** It could be tempting to think that the clipped inverse link trivially comes from clipping the partial losses themselves such as replacing \( \ell_1(u) \) by 0 if \( u \geq 1/2 \) and symmetrically for \( \ell_{-1}(u) \). This is not the case as it would lead to \( \mathbb{L} \) piecewise constant and therefore \( -\mathbb{L}' = 0 \) when defined.

We turn to a result that authorizes us to use Thm 2 while virtually not needing (O1) and (O2) for \( \alpha \)-loss. Denote \( \mathbb{I}_\alpha = \pm \alpha^\varepsilon \cdot [1 - (1/\alpha^4), 1] \) (See Fig. I).

**Lemma 5.2.** Suppose \( \alpha \geq 1.2 \). For \( \tilde{f}_\ell \) defined as in (18), \( \exists K \geq 0.133 \) such that \( \alpha \)-loss satisfies:

\[
\forall z \not\in \mathbb{I}_\alpha, |(\tilde{f}_\ell - (-\mathbb{L}')^{-1})(z)| \lesssim K/\alpha.
\] (19)

Remark the necessity of a trick as we do not compute \( (-\mathbb{L}')^{-1} \) in (19). The proof, in Section A.5, bypasses the difficulty by bounding the horizontal distance between the inverses. The Lemma can be read as: with the exception of an interval vanishing rapidly with \( \alpha \), the difference between \( \tilde{f}_\ell \) (that we can easily compute for the \( \alpha \)-loss) and \( (-\mathbb{L}')^{-1} \) (that we do not compute for the \( \alpha \)-loss), in order or just pointwise (typically for \( \alpha < 10 \)) is at most \( 0.14/\alpha \). We now show how we can virtually "get rid of" (O1) and (O2) in such a context to apply Theorem 2. Consider the following assumptions: (i) no edge falls in \( \mathbb{I}_\alpha \), (ii) the weak learner guarantees \( \gamma = 0.14 \), (iii) the average weights, \( \bar{w}_j = 1^\top w_j/m \), satisfies \( \bar{w}_j \geq 0.4 \). Looking at Figure I we see that (i) is virtually not limiting at all; (ii) is a reasonable assumption on WLA; remembering that a weight has the form \( w = \tilde{f}_\ell(-yH(x)) \), we see that (iii) requires \( H \) to be not "too good", see for example Figure I in which case \( w = 0.4 \) implies an edge \( yH \leq 0.8 \). We now observe that given (i), it is trivial to find \( a_f \) to satisfy (O2) since we focus only on one \( \alpha \)-loss. Suppose \( \alpha \geq 2.7 \), which approaches the average value of the \( \alpha \) in our experiments, and finally let \( \zeta = 2.5/2.7 \approx 0.926 \). Then we get the inequalities:

\[
\Delta(F) \lesssim \frac{M \cdot 0.14}{\alpha} \cdot \frac{\gamma M \cdot 1}{\alpha} \cdot \frac{1}{\alpha \bar{w}_j} \cdot \eta_j \leq \frac{2.5}{\alpha} \cdot \eta_j \leq \frac{2.5}{2.7} \cdot \eta_j \doteq \zeta \cdot \eta_j, \quad (20)
\]

and so (O1) is implied by the weak learning assumption. To summarise, PilBoost boosts the convex surrogate of the \( \alpha \)-loss without either computing it or its derivative, and achieves boosting compliant convergence using only the classical assumptions of boosting, (R, WLA).

The proof of Lemma 5.2 being very conservative, we can expect that the smallest value of \( K \) of interest is smaller than the one we use, indicating that (20) should hold for substantially smaller limit values in (ii, iii).

6 Experiments

We provide experimental results on PilBoost (for \( \alpha \in \{1.1, 2, 4\} \)) and compare with AdaBoost [21] and XGBoost [15] on four canonical binary classification datasets, namely, cancer [59], x6 [12], diabetes [49], and online shoppers intention [12]. For every result, we performed 10 runs per algorithm with randomization over the train/test split and the twisters.
Figure 2: Box and whisker plots reporting the accuracy of AdaBoost, PilBoost (for $\alpha \in \{1.1, 2, 4\}$), and XGBoost on the cancer dataset affected by the class noise twister with 0%, 15%, and 30% twist. Note that the orange line is the median, the green triangle is the mean, the box is the interquartile range, and the circles outside of the whiskers are outliers. All three algorithms were trained with decision stumps (depth 1 regression trees). For $\alpha = 1.1, 2,$ and $4$, we set $a_f = 7, 2,$ and $4$, respectively. Numeric values corresponding to the box and whisker plots are provided in Table 2 in Section B.3. We find that PilBoost has gains over AdaBoost and XGBoost when there is twist present, and $\alpha^*$ (of our set) increases as the amount of twist increases, which follows theoretical intuition (Lemma 3.4).

All experiments use regression decision trees (of varying depths 1-3) in order to align with XGBoost. All parameters of XGBoost were kept default in order to maintain the fairest comparison between the three algorithms; for more of these experimental details please refer to Section B.5 where we detail XGBoost parameters. In order to demonstrate the twist-properness of $\alpha$-loss as implemented in PilBoost, we augment the training examples of these datasets with three different (malicious) twisters.

Class Noise Twister (all datasets): This twister is equivalent to symmetric label noise in the training sample. Label noise has been very well studied in the literature [20] and has been shown to be difficult for many boosting algorithms [30]. Results on this twister for the cancer dataset are presented in Figure 2 and for the other three datasets in Section B.3. In general, we find that PilBoost is more robust to the Class Noise Twister than AdaBoost and XGBoost, and we find that $\alpha^*$ increases as the amount of twist increases, which complies with our theory (Lemmata 3.3, 3.4).

Feature Noise Twister (xd6 dataset): This twister perturbs the training sample by randomly flipping features. More precisely, for each training example, the example is selected if $\text{Ber}(p_1)$ returns 1. Then, for each selected training example, and for each feature independently, the feature is flipped (the features of xd6 are Booleans) to the other symbol if $\text{Ber}(p_2)$ also returns 1. Results on this twister are presented in Table 1 where $p_1 = p_2 = p$. In general, we find that PilBoost is more robust to the Feature Noise Twister than AdaBoost and XGBoost, and we find that $\alpha^*$ increases as the amount of twist increases.

Insider Twister (online shoppers intention dataset): This twister assumes more knowledge about the model than the previous two twisters. In essence, the insider twister adds noise to a few of the most informative features for predicting the class. Results on this twister are presented in Figure 3 and further discussion in Section B.4. The case of the insider twister in interesting: post-twister, the feature importance profile of XGBoost is almost uniform,
| Dataset | Algorithm | Feature Noise Twister |
|---------|-----------|-----------------------|
| xd6     | AdaBoost  | \( p = \{0, 0.15, 0.25, 0.5\} \) |
|         | \( \alpha = 1.1 \) | 1.000 ± 0.000 | 0.988 ± 0.013 | 0.966 ± 0.013 | 0.884 ± 0.019 |
|         | \( \alpha = 2.0 \) | 1.000 ± 0.000 | **1.000 ± 0.000** | **1.000 ± 0.000** | **0.910 ± 0.026** |
|         | \( \alpha = 4.0 \) | 1.000 ± 0.000 | 0.997 ± 0.006 | 0.999 ± 0.002 | **0.958 ± 0.017** |
|         | XGBoost   | 1.000 ± 0.000 | 0.970 ± 0.016 | 0.962 ± 0.009 | 0.833 ± 0.027 |

Table 1: Accuracies on AdaBoost, PilBoost (for \( \alpha \in \{1.1, 2, 4\} \)), and XGBoost on the xd6 dataset affected by the feature noise twister with the flipping probability \( p = \{0, 0.15, 0.25, 0.5\} \). All three algorithms were trained with depth 3 regression trees. For each value of \( \alpha \), we set \( a_f = 8 \). Note that the xd6 dataset is perfectly classified (when there is no twist) by a Boolean formula on the features, given in [12], which explains the performance when \( p = 0 \). For \( p = 0.15 \), under Welch’s t-test, the difference between \( \alpha = 2 \) and XGBoost has \( p \)-value \( 3 \times 10^{-4} \); for \( p = 0.25 \), the \( p \)-value is \( 5 \times 10^{-7} \); for \( p = 0.5 \), the \( p \)-value is \( 5 \times 10^{-9} \).

displaying damages to the algorithm’s discriminative abilities (Figure 3, right), while this clearly does not happen for PilBoost.

7 Discussion

Studying data corruption in ML dates back to the eighties [55, Section 4]. Remarkably, the first twist models were assuming very strong corruption, possibly coming from an adversary with unbounded computational resources, but the data at hand was supposed to be binary. Hence, the feature space was as "complex" as the class space and twist models were lacking the unparalelled data complexity that we now face. Getting such twist models at scale with real world data has been a major problem in ML over the past decade for a number of reasons, not all of which are borne out of bad intent. Robustness inevitably comes to mind [51, 31, 4]. Data augmentation techniques also come to mind, with Vicinal Risk Minimization standing as a pioneer [13, 60]. Data poisoning techniques can be much more sophisticated [53]. Privacy techniques like differential privacy can also alter data with the objective to obfuscate specific information [19, 36]. Invariant risk minimisation aims at finding data representations yielding good classifiers but also invariant to "environment changes" [6]. Quantization can reduce the coding size of data to lower the computational cost of ML [61].

All these papers [6, 13, 31, 36, 53, 60, 61] study arguably much different problems, but they all have a commonpoint that goes substantially deeper than the superficial observation that they assume twisted data in some way: the core loss in all of them is a proper canonical loss (4’s is proper composite [40], a more sophisticated way to build a proper loss [58]). Therefore, they all start from the premise of a loss that inevitably fits the (unwanted) twist, and correct it mostly with a regularizer informed with the twist, on a "twist-per-twist" basis. More recently, some approaches have started to directly change the loss to tackle the twist at
Figure 3: Normalized feature importance bar charts for PilBoost with $\alpha = 1.1$ and $af = 7$ (left) and for XGBoost (right) on the online shoppers intention dataset (both for depth 3 trees) with and without the insider twister. The insider twister adds noise to three important features for classification, namely, feature 8 (page values - numeric type with range in $[-250, 435]$), feature 10 (month), and feature 15 (visitor type - ternary alphabet). For page values, the insider twister adds i.i.d. $N(0, 60)$ to the entries; for both month and visitor type, the insider twister independently increments (with probability $1/2$) the symbol according to their respective alphabets such that about 50% of each of these features are perturbed. We find that the insider twister significantly perturbs the feature importance of XGBoost as evidenced in the plot (far right). Under no twister, $\alpha = 1.1$, has accuracy $0.901 \pm 0.003$, and XGBoost has accuracy $0.892 \pm 0.003$. Under the insider twister, $\alpha = 1.1$, has accuracy $0.850 \pm 0.002$, and XGBoost has accuracy $0.829 \pm 0.016$; under the Welch t-test, the results have a $p$-value of 0.004. More details can be found in Section B.4.

Successful recent approaches include correcting the loss for class or label noise, such as [32, 39, 56] (and references therein). This latter approach is among the first to discuss the abstract problem of correcting label corruption, using reversible Markov transitions. Generalizing a previous approach held for symmetric proper canonical losses [38], it shows that sets of loss functions with a specific structure admit efficient analytic corrections – though not necessarily accessible experimentally. A key technical difference with us appears with the terminology of [39]: [56, Theorem 5] performs backward corrections via the Markov transitions, while the tilted estimate (Section 2) does in fact perform forward corrections. Recently, a correction for strict properness for a non-strictly proper loss (the focal loss) was designed, which can be assimilated to a tilted estimate [14]. All these approaches show the strength of directly coping with a loss to correct for a twist and the numerous examples above show there is undoubtedly traction to get there. The $\alpha$-loss has potential merits to get there — in fact, such merits have already been exemplified for specific twists: the $\alpha$-loss has been used as building block to correct the logistic loss (i) for light tailed predictions, with $\alpha \in \mathbb{R}_+$ [2, 3] (see e.g. (8) in [2]), and (ii) for class noise, with $\alpha \in (1, \infty]$ [62] (remark that the range is a perfect fit for Bayes blunting class noise, Lemma 3.4). [2, 3] use two different $\alpha$-losses as composite link for a Tsallis’ entropy [33]¹ and the latter heuristically changes and clamps the loss.

The approaches in [2, 4, 6, 13, 31, 32, 36, 39, 53, 60, 61, 62] share a higher-level technical commonality: they alter the loss via its surrogate. In the theory of CPE losses, the surrogate [4] is the end of the design chain; it follows from the Bayes risk, itself a product of the partial

¹Tsallis’ deformed logarithm is proportional to a reparameterisation of $\alpha$-loss [33].
losses (1). Derivations involves variational formulations of functions and so, for many of those approaches, it would be tricky to define the partial losses ending up with the surrogates of interest (see for example Remark 5.1). In this paper, we chose to study the direct alteration of the partial losses to correct a twist, with the concrete case of using $\alpha$-loss to carry the task. This, we believe, can bring a more general understanding of those twist corrections, which is much needed for "ML in the real world", the loss being part of the core engine of ML. We hope our theory can provide such a leverage, as we show that the $\alpha$-loss in fact fits to all twisters in the theory of losses for class-probability estimation (Lemma 3.4), constant $\alpha$s can at least partially correct domain-wide twists (Lemma 1); algorithmically, the difficulty of computing its surrogate and associated gradients does not even impede efficient formal boosting (Sections 4, 5), and finally experiments certainly demonstrate the applicability of the idea for diverse and potentially sophisticated twisters (Section 6). However, gaining the necessary altitude to cope with the problem at scale requires additional results barely mentioned. Two of them are: (i) extend Theorem 1 to confidence intervals when $\alpha$ is chosen constant, so that it can be reliably estimated, and (ii) get an overarching learning algorithm with appropriate theoretical guarantees for a functional $\alpha : X \rightarrow (-\infty, \infty]$ that would locally tune the loss to the data and twist at hand [35, 40].

All the approaches cited before, inclusive of ours, in fact beg for a unified understanding and/or theorisation of data twists in a field where all are much silo’ed subdomains, with some that should be treated with extreme caution: one could consider that data biases considered in fairness or ethics are relevant to our theory [57], or even intimate biases [48]. In such cases, very specific guarantees would be mandatory. How complex can the overall task be? It takes a single page decision tree to segment nineteen (19) families of the multifaceted dycotyledons [9, pp 312]. There is no reason not to believe that organising major families of twists can be achieved with a similar level of aggregation.

8 Conclusion

In this paper, we first study loss functions for class probability estimation when we strip off (until recently much-desired) mandatory properness. We show that an extension of the original $\alpha$-loss can correct for any twist and thereby recover Bayes clean posterior in its minimizers. Such twist properness does not exist for classical proper losses. We propose a general boosting algorithm with the desirable property that it can boost the convex surrogate of a loss without having access to its derivative to compute boosting weights. This is particularly interesting when this function is hard to compute, which holds for the $\alpha$-loss. Experiments showcase our algorithm vs different twists.

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Appendices

A  Proofs

A.1 Proof of Lemma 3.1

We study $U \doteq (-L)^*$, which is convex by definition, and show that it is non-decreasing. Monotonicity follows from the non-negativity of the argument of the partial losses and the definition of the convex conjugate: suppose $z' \geq z$ and let $u^* \in \arg \sup_u z u + L(u)$. We have

$$U(z') = \sup_{u \in [0,1]} z' u + L(u)$$

$$= \sup_{u \in [0,1]} (z' - z) u + z u + L(u)$$

$$\geq (z' - z) u^* + z u^* + L(u^*)$$

$$= (z' - z) u^* + U(z)$$

$$\geq U(z),$$

which completes the proof that $U$ is non-decreasing and therefore $F(z) = U(-z)$ non-increasing.

Concavity of $L$ follows from definition. We show continuity of $L$, the continuity of $F$ then following from the definition of the convex conjugate $F^{[10]}$. Let $a, u \in (0,1)$, let $u^* \in t_1(u), a^* \in t_1(a)$. We get:

$$L(u) = u \ell_1(u^*) + (1 - u) \ell_{-1}(u^*)$$

$$\leq u \ell_1(a^*) + (1 - u) \ell_{-1}(a^*)$$

$$= L(a) + (u - a) (\ell_1(a^*) - \ell_{-1}(a^*)),$$

(the inequality holds since otherwise $u^* \not\in t_1(u)$) Permuting the roles of $u$ and $a$, we also get

$$L(a) \leq L(u) + (a - u) (\ell_1(u^*) - \ell_{-1}(u^*)),$$

from which we get

$$|L(a) - L(u)| \leq Z \cdot |a - u|,$$

with $Z = \max_{u \in (a,u)} |\ell_1(t_1(u)) - \ell_{-1}(t_1(u))|$ (where we use set differences if $t_1$s are not singletons). Since $Z \ll \infty$, (30) is enough to show the continuity of $L$ (we have by assumption dom$\{L\} = [0,1]$).

A.2 Proof of Lemma 3.2

The result we show is slightly more general than the statement of the Lemma as we include a result on an additional assumption not in the main body:

**(E)** Extreme values: $\ell_1(1) = \ell_{-1}(0) = 0$, $\ell_1([0,1]) \subseteq \mathbb{R}_+$, $\ell_{-1}([0,1]) \subseteq \mathbb{R}_+$;
and the additional result we prove is: (E) implies properness on extreme values, as \(0 \in t_\ell(0), 1 \in t_\ell(1)\);

**Case (M)** – Suppose \(t_\ell(a) \cap t_\ell(a') \neq \emptyset\) for some \(a \neq a'\) and let \(v^* \in t_\ell(a) \cap t_\ell(a')\). It means
\[
\forall v \in [0, 1],
\]
\[
 a\ell_1(v^*) + (1 - a)\ell_1(v) \leq a\ell_1(v) + (1 - a)\ell_1(v), \quad (31)
\]
\[
a'\ell_1(v^*) + (1 - a')\ell_1(v) \leq a'\ell_1(v) + (1 - a')\ell_1(v), \quad (32)
\]
and so \(\forall \delta \in [0, 1]\), if we let \(a_\delta = a + \delta(a' - a)\), a \(1 - \delta, \delta\) convex combination of both inequalities yields \(\forall v \in [0, 1],\)
\[
a_\delta \ell_1(v^*) + (1 - a_\delta)\ell_1(v) \leq a_\delta \ell_1(v) + (1 - a_\delta)\ell_1(v), \forall v \in [0, 1], \quad (33)
\]
which implies \(v^* \in t_\ell(a_\delta)\) and shows the right part of (5).

To show the left part of (5), we add to (31) and (32) we now add the inequality:
\[
a\ell_1(v^*) + (1 - a)\ell_1(v) \leq a\ell_1(v) + (1 - a)\ell_1(v), \quad (34)
\]
with therefore \(v^* \in t_\ell(a)\), implying \(a\ell_1(v^*) + (1 - a)\ell_1(v^*) = a\ell_1(v^*) + (1 - a)\ell_1(v^*)\) as otherwise one of \(v^*, v^*\) would not be in \(t_\ell(a)\). We then get
\[
a'\ell_1(v^*) + (1 - a')\ell_1(v) = a\ell_1(v^*) + (1 - a)\ell_1(v^*) + (a' - a) \cdot (\ell_1(v^*) - \ell_1(v))
\]
\[
= a\ell_1(v^*) + (1 - a)\ell_1(v^*) + (a' - a) \cdot (\ell_1(v^*) - \ell_1(v))
\]
\[
= a'\ell_1(v^*) + (1 - a')\ell_1(v^*) + (a' - a) \cdot \Delta, \quad (35)
\]
with \(\Delta = \ell_1(v^*) - \ell_1(v) = (\ell_1(v^*) - \ell_1(v))\). Considering (35), we deduce from (32) that to have \(v^* \in t_\ell(a)\), we equivalently need \((a' - a) \cdot \Delta \leq 0\). We also know by assumption that \(\ell_1\) is non-increasing and \(\ell_1\) is non-decreasing, so \(g(u) = \ell_1(u) - \ell_1(u)\) is non-increasing. We thus have \((a' - a) \cdot \Delta \leq 0\) iff one of the two possibilities hold:

- \(a' \geq a\) and \(v^* \geq v^*\), or
- \(a' \leq a\) and \(v^* \leq v^*\),

which shows the right part of (5).

**Case (E)** – we have \(L(0) = \inf_{v \in [0, 1]} \ell_1(v) = 0\) for \(v = 0\), hence \(0 \in t_\ell(0)\). Similarly, \(L(1) = \inf_{v \in [0, 1]} \ell_1(v) = 0\) for \(v = 1\), hence \(1 \in t_\ell(1)\).

**Case (D)** – we have
\[
\frac{d}{du}L(u) = \ell_1(t_\ell(u)) + u\ell'_1(t_\ell(u))t'_\ell(u) - \ell_1(t_\ell(u)) + (1 - u)\ell'_1(t_\ell(u))t'_\ell(u)
\]
\[
= \ell_1(t_\ell(u)) - \ell_1(t_\ell(u)) + t'_\ell(u) \cdot (u\ell'_1(t_\ell(u)) + (1 - u)\ell'_1(t_\ell(u))), \quad (36)
\]
but since \(v = t_\ell(u)\) is the solution to (31) it satisfies \(u\ell'_1(t_\ell(u)) + (1 - u)\ell'_1(t_\ell(u)) = 0\), so that (37) simplifies to
\[
\frac{d}{du}L(u) = \ell_1(t_\ell(u)) - \ell_1(t_\ell(u)), \quad (37)
\]
Figure 4: Comparison between the cross-entropy of the logistic loss ($\alpha = 1$) and that of the $\alpha$-loss for the scalar correction in (70) in Theorem 3.

and since $L$ is concave and the partial losses are differentiable,

$$\frac{d^2}{du^2}L(u) = t'_\ell(u) \cdot (t'_\ell(t_\ell(u)) - t'_{-1}(t_\ell(u))) \leq 0, \forall u,$$  (38)

which proves the statement of the Lemma.

**Case (S)** – Suppose $v^* \in t_\ell(a)$, which implies

$$a\ell_1(v^*) + (1-a)\ell_{-1}(v^*) \leq a\ell_1(v) + (1-a)\ell_{-1}(v), \forall v \in [0,1].$$  (39)

We also note that since symmetry holds, $a\ell_1(v^*) + (1-a)\ell_{-1}(v^*) = (1-a)\ell_1(1-v^*) + a\ell_{-1}(1-v^*)$, which implies because of (39) $1 - v^* \in t_\ell(1-a)$.

**Remark:** even if we assume the partial losses to be strictly monotonic, the tilted estimate can still be set valued. To see this, craft the partial losses such that $v \in t_\ell(u)$ and then for some $w > v$, replace the partial losses in the interval $[v, w]$ by affine parts w/ slope $-a < 0$ for $\ell_1$, $b > 0$ for $\ell_{-1}$ and such that $b/a = u/(1-u)$ which guarantees $L(u, v) = L(u, w)$ and thus $w \in t_\ell(u)$;

**A.3 Proof of Theorem 1**

As explained in the main body, we prove a result more general than Theorem 1. Let $B > 0$ be fixed, and denote $M(B)$ the distribution restricted to the support for which we have a.s.
the following bound on the twisted posterior:

$$\frac{1}{1 + \exp B} \leq \eta_c(x) \leq \frac{\exp B}{1 + \exp B}, \quad (40)$$

and let $p(B)$ be the weight of this support in $M$. Let $M(B)$ denote the restriction of $M$ to the complement of this support. Define the logit-edge over $M(B)$:

$$\eta(B) \doteq \frac{\mathbb{E}_{X \sim M(B)} [\eta_c(X) t(\eta_c(X))]}{B} \in [-1, 1], \quad (41)$$

and

$$\eta(\alpha) \doteq \frac{\mathbb{E}_{X \sim M(B)} [\eta_c(X) \cdot \|\alpha t(\eta_c(x)) < 0\| \cdot |t(\eta_c(x))|]}{B} \geq 0, \quad (42)$$

which discards, in $M(B)$, the logits whose sign agree with that of the parameter chosen for the $\alpha$-loss. Let $q(B) \doteq (1 + \eta(B))/2$.

**Theorem 3.** Suppose we fix $\alpha = \alpha^*$ with

$$\alpha^* \doteq \frac{t(q(B))}{B}. \quad (43)$$

then the following bound holds on the cross-entropy of the Bayes tilted estimate of the $\alpha$-loss:

$$\text{CE}(\eta_c, \eta_c; \alpha) \leq p(B) \cdot H(q(B)) + (1 - p(B)) \cdot (\eta(\alpha^*) \cdot |\alpha^*| + \exp |t(q(B))|) \quad (44)$$

To prove Theorem 3 we remark that the cross-entropy can be split as:

$$\text{CE}(\eta_c, \eta_c; \alpha) = \mathbb{E}_{X \sim M} \left[ \eta_c(X) \log \left( 1 + \left( \frac{1 - \eta_c(X)}{\eta_c(X)} \right)^\alpha \right) \right] = p(B) \cdot K(\alpha) + (1 - p(B)) \cdot L(B)$$

with

$$K(\alpha) \doteq \mathbb{E}_{X \sim M(B)} \left[ \eta_c(X) \log \left( 1 + \left( \frac{1 - \eta_c(X)}{\eta_c(X)} \right)^\alpha \right) \right], \quad (46)$$

$$J(B) \doteq \mathbb{E}_{X \sim M(B)} \left[ \eta_c(X) \log \left( 1 + \left( \frac{1 - \eta_c(X)}{\eta_c(X)} \right)^\alpha \right) \right], \quad (47)$$

where the dependencies in variables indicate that we are going to choose $\alpha$ to minimise $L$ and upperbound $J$ as a function of $B$. Denote for short $z(x) \doteq \log((1 - \eta_c(x))/\eta_c(x)) = -t(\eta_c(x)) \in [-B, B]$ over $M(B)$ negative the twisted logit. The optimal value $\alpha$ minimizing $K(\alpha)$, is such that:

$$K'(\alpha) = \mathbb{E}_{X \sim M(B)} \left[ \eta_c(X) \cdot \exp(\alpha \cdot z(X)) \cdot z(X) \right] = 0. \quad (48)$$

We remark that $K''(\alpha) \geq 0$ so to have the optimal $\alpha$ strictly positive, we need

$$-K'(0) = \mathbb{E}_{X \sim M(B)} [\eta_c(X) t(\eta_c(X))] = \eta(B) > 0. \quad (49)$$

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Notice that this makes sense as \( \eta(B) \) measures a correlation between \( \eta_c \) and \( \eta_t \). Having \( \alpha < 0 \) implies a very "damaging" twist. It is also clear that there is a single solution to (48) (at least assuming wlog that we do not have \( \eta_c(\cdot)z(\cdot) = 0 \) a.s.), but it is hard to get the \( \alpha \) in closed form, so we are going to find an approximate expression with guarantees, using the simple fact that for all \( |z| \leq 1, \forall \alpha \in \mathbb{R} \),

\[
\log(1 + \exp(\alpha z)) \leq \frac{1 + z}{2} \cdot \log(1 + \exp(\alpha)) + \frac{1 - z}{2} \cdot \log(1 + \exp(-\alpha)) \tag{50}
\]

\[
= \log(1 + \exp(\alpha)) - \frac{1 - z}{2} \cdot \alpha, \tag{51}
\]

which indeed holds as the LHS of (50) is convex and the RHS is the equation of a line passing through the points \((-1, \log(1 + \exp(-\alpha)))\) and \((1, \log(1 + \exp(\alpha)))\). Hence if instead \(|z| \leq B\), then

\[
\log(1 + \exp(\alpha z)) = \log\left(1 + \exp\left(\alpha B \cdot \frac{z}{B}\right)\right) \leq \log(1 + \exp(\alpha B)) - \frac{B - z}{2} \cdot \alpha, \tag{53}
\]

so we get:

\[
K(\alpha) \leq \log(1 + \exp(B\alpha)) - \frac{B - \mathbb{E}_{X \sim M(B)}[\eta_c(X)z(X)]}{2} \cdot \alpha
\]

\[
= \log(1 + \exp(B\alpha)) - \frac{B + B\eta(B)}{2} \cdot \alpha. \tag{54}
\]

We have

\[
L'(\alpha) = B \cdot \left(\frac{\exp(B\alpha)}{1 + \exp(B\alpha)} - \frac{1 + \eta(B)}{2}\right), \tag{55}
\]

which zeroes for

\[
\alpha^* = \frac{1}{B} \cdot \log \left(\frac{1 + \eta(B)}{1 - \eta(B)}\right) = \frac{\log(q(B))}{B}, \tag{56}
\]

and yields

\[
K(\alpha^*) \leq \log(1 + \exp(B\alpha^*)) - B \cdot \frac{1 + \eta(B)}{2} \cdot \alpha^*
\]

\[
= -\log \left(\frac{1 - \eta(B)}{2}\right) - \frac{1 + \eta(B)}{2} \cdot \log \left(\frac{1 + \eta(B)}{1 - \eta(B)}\right) \tag{58}
\]

\[
= -\frac{1 + \eta(B)}{2} \log \left(\frac{1 + \eta(B)}{2}\right) - \frac{1 - \eta(B)}{2} \log \left(\frac{1 - \eta(B)}{2}\right) \tag{59}
\]

\[
= H\left(\frac{1 + \eta(B)}{2}\right). \tag{60}
\]
We now focus on $J(B)$. Since $\log(1+\exp(-z)) \leq \exp(-z)$, $\forall z$ via an order-1 Taylor expansion, it follows that if $z \geq C$ for some $C > 0$, then $\log(1 + \exp(-z)) \leq \exp(-C)$. Equivalently, we get

$$z \geq C \implies \log(1 + \exp(z)) \leq z + \exp(-C). \quad (61)$$

By symmetry, we have

$$z \leq -C \implies \log(1 + \exp(z)) \leq \exp(-C), \quad (62)$$

so we get

$$|z| \geq C \implies \log(1 + \exp(z)) \leq \max\{0, z\} + \exp(-C). \quad (63)$$

It follows that over the support of $M(B)$, we have

$$\log(1 + \exp(\alpha^* z(x))) \leq \max\{0, \alpha^* z(x)\} + \exp\left(\frac{1 - \eta(B)}{1 + \eta(B)}\right) \quad (64)$$

$$= \max\{0, -\alpha^* u_\eta(x)\} + \exp\left(\frac{1 - \eta(B)}{1 + \eta(B)}\right) \quad (65)$$

$$= |\alpha^*| \cdot \max\{0, -\text{sign}(\alpha^*) u_\eta(x)\} + \frac{1 + \eta(B)}{1 - |\eta(B)|}, \quad (66)$$

so using quantity $\eta(\alpha)$, we get for our choice of $\alpha$,

$$J(B) \leq \frac{\eta(\alpha^*)}{B} \cdot \log\left(\frac{1 + |\eta(B)|}{1 - |\eta(B)|}\right) + \frac{1 + |\eta(B)|}{1 - |\eta(B)|} \quad (67)$$

$$= \eta(\alpha^*) |\alpha^*| + \exp(B|\alpha^*|), \quad (68)$$

which completes the proof.

**Remarks:** Theorem 3 calls for several remarks:

**Gains with respect to the "proper" choice $\alpha = 1$:** the case we develop is simplistic but allows a graphical comparison of the gains that Theorem allow to get compared to the choice $\alpha = 1$, which we recall corresponds to the (proper) logistic loss. Suppose $p(B) = 1$ so the cross-entropy $CE(\eta_t, \eta_c; \alpha)$ in (45) reduces to $K(.)$. Suppose to simplify $B = 1$ and $z \in \{\pm 1\}$, with $p$ the proportion in $M(1)$ for which $z = -1 = -1$. Denote $q = \mathbb{E}_{X \sim M(B)}[\eta_c(X)[z = -1]] = 1$ and suppose to (overly)simplify that $q = \mathbb{E}_{X \sim M(B)}[\eta_c(X)[z = 1]]$. Remark that in this extreme case, Shannon’s entropy of the clean posterior is zero:

$$CE(\eta_t, \eta_c; 1) = 0, \quad (69)$$

so in theory $CE(\eta_t, \eta_c; \alpha^*)$ can be as small as possible. We show that this indeed can happen while it never happens for $CE(\eta_t, \eta_c; 1)$. We have $\eta(1) = pq - (1 - p)q = 2p - 1$. In this case, we obtain:

$$\alpha^* = \log\left(\frac{p}{1 - p}\right).$$
so

\[ K(\alpha^*) = \mathbb{E}_{X \sim M(1)} [\eta_c(X) \log (1 + \exp(\alpha^* z(X)))] \]

\[ = p \log \left( 1 + \exp \left( -\log \left( \frac{p}{1-p} \right) \right) \right) + (1-p) \log \left( 1 + \exp \left( \log \left( \frac{p}{1-p} \right) \right) \right) \]

\[ = \log \left( 1 + \exp \left( \log \left( \frac{p}{1-p} \right) \right) \right) - p \log \left( \exp \left( \log \left( \frac{p}{1-p} \right) \right) \right) \]

\[ = -p \log(p) - (1-p) \log(1-p) = H(p), \]

while for the properness choice \( \alpha^* = 1 \), we get

\[ K(1) = \mathbb{E}_{X \sim M(1)} [\eta_c(X) \log (1 + \exp(z(X)))] \]

\[ = p \log(1 + \exp(-1)) + (1-p) \log(1 + \exp(1)). \]

\[ = \log(1+e) - p. \]

Figure 4 plots \( \text{CE}(\eta_1, \eta_c; \alpha^*) \) \( \text{CE}(\eta_1, \eta_c; 1) \). We remark that \( \text{CE}(\eta_1, \eta_c; \alpha^*) \leq \text{CE}(\eta_1, \eta_c; 1) \), and the difference is especially large as \( p \to 0, \) for which \( \text{CE}(\eta_1, \eta_c; \alpha^*) \to 0 \) while we always have \( \text{CE}(\eta_1, \eta_c; 1) > 0.3, \forall p. \)

**Incidence of computing \( \alpha^* \) on an estimate of \( \eta(B) \):** Theorem 3 can be refined if, instead of the true value \( \eta(B) \) we have access to an estimate \( \hat{\eta}(B) \). In this case, we can refine the proof of the Theorem from the series of eqs in (60) and instead get, using \( \log(1+z) \leq z, \)

\[ K(\alpha^*) \leq H \left( \frac{1 + \hat{\eta}(B)}{2} \right) + \frac{\eta(B) - \hat{\eta}(B)}{2} \cdot \log \left( 1 + \frac{\hat{\eta}(B)}{1 - \hat{\eta}(B)} \right) \]

\[ = H \left( \frac{1 + \hat{\eta}(B)}{2} \right) + \frac{\eta(B) - \hat{\eta}(B)}{2} \cdot \log \left( 1 + \text{sign}(\hat{\eta}(B)) \cdot \frac{2\hat{\eta}(B)}{1 - \text{sign}(\hat{\eta}(B)) \cdot \hat{\eta}(B)} \right) \]

\[ \leq H \left( \frac{1 + \hat{\eta}(B)}{2} \right) + \frac{\eta(B) - \hat{\eta}(B)}{1 - |\hat{\eta}(B)|}. \]

**Polarity of \( \alpha^* \):** as presented in the main body, the state of the art defines the \( \alpha \)-loss only for \( \alpha \geq 0 \). The proof of Theorem 3 and more specifically its proof, hints at why alleviating this constraint is important and corresponds to especially difficult cases. We have the general rule \( \alpha^* \leq 0 \) iff \( \hat{\eta}(B) \leq 0 \), which indicates that the twisted posterior tends to be small when the clean posterior tends to be large. Since the Bayes tilted estimate is invariant if we switch the couple \( (\alpha, \eta_c) \) for \( (-\alpha, 1 - \eta_c) \), \( \alpha^* \leq 0 \) provokes a change of polarity in the Bayes tilted estimate compared to the twisted posterior. It thus corrects the twisted posterior. We emphasize that such a situation happens for especially damaging twists (in particular, not Bayes blunting).

**A general method to choose \( \alpha^* \):** our choice of \( \alpha^* \) is an approximation of the optimum sought for the cross-entropy; in the general case, one can directly solve (48). This equation is interesting because while it is not exactly computable in the general case – we do not know \( \eta_c \) –, it could be used to guess, for some particular twists, confidence intervals for \( \alpha^* \), or even just its polarity.
A.4 Proof of Theorem \[2\]

We proceed in two steps, assuming (WLA) holds for WL and (R) holds for the weak classifiers.

In step 1, we show that for any loss defined by \( F \) twice differentiable, convex and non-increasing, for any \( z^* \in \mathbb{R} \), as long as \( F \) satisfies assumptions (1O) and (2O) for \( T \) iterations such that

\[
T \geq \frac{2F^*M^2(F(0) - F(z^*))}{\gamma^2(1 - \zeta)^2(1 - \pi^2)},
\]

we have the guarantee on the risk defined by \( F \):

\[
\mathbb{E}_{i \sim D}[F(y, H_T(x_i))] \leq F(z^*).
\]

Let \( F \) be any twice differentiable, convex and non-increasing function. We wish to find a lower bound \( \Delta \) on the decrease of the expected loss computed using \( F \):

\[
\mathbb{E}_{i \sim D}[F(y, H_t(x_i))] - \mathbb{E}_{i \sim D}[F(y, H_{t+1}(x_i))] \geq \Delta,
\]

where \( D \) denotes the empirical distribution. We make use of the same proof technique as in [37](Theorem 7). Suppose

\[
H_{t+1} = H_t + \delta_j \cdot h_j,
\]

index \( j \) being returned by WL at iteration \( t \). For any such index \( j \), any \( g : \mathbb{R} \to \mathbb{R}_+ \) and any \( H \in \mathbb{R}^X \), let

\[
\eta(j, g, H) = \mathbb{E}_{i \sim D}[y_i h_j(x_i) \cdot g(y_i H(x_i))]
\]

denote the expected edge of \( h_j \) on weights defined by the couple \((g, H)\). There are two quantities we define. First,

\[
X = \mathbb{E}_{i \sim D}[(y_i H_t(x_i) - y_i H_{t+1}(x_i)) F'(y_i H_t(x_i))]
\]

\[
= \delta_j \cdot \mathbb{E}_{i \sim D}[y_i h_j(x_i) \cdot -F'(y_i H_t(x_i))]
\]

\[
\geq \delta_j \cdot \mathbb{E}_{i \sim D} [y_i h_j(x_i) \cdot \tilde{f}(-y_i H_t(x_i))] - \delta \cdot \Delta(-F', \tilde{f}_s)
\]

\[
= a \eta^2(j, \tilde{f}_s, H_t) - a \eta(j, \tilde{f}_s, H_t) \cdot \Delta(-F', \tilde{f}_s)
\]

\[
\geq a(1 - \zeta)\eta^2(j, \tilde{f}_s, H_t),
\]

where \( \tilde{f}_s(z) = \tilde{f}_s(-z) \) and

\[
\Delta(g_1, g_2) = |\eta(j, g_1, H_t) - \eta(j, g_2, H_t)|,
\]

and finally \([88]\) makes use of assumption (1O). The second quantity we define is:

\[
Y(Z) = \mathbb{E}_{i \sim D}[(y_i H_t(x_i) - y_i H_{t+1}(x_i))^2 F''(z_i)]
\]

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where $Z = \{z_1, z_2, ..., z_m\} \subset \mathbb{R}^m$. Using assumption (R) and letting $F^*$ being any real such that $F^* \geq \sup F''(z)$, we obtain:

\[
Y(Z) \leq F^* \cdot \mathbb{E}_{i \sim D} \left[ (y_i H_t(x_i) - y_i H_{t+1}(x_i))^2 \right] = F^* \cdot \delta_j^2 \cdot \mathbb{E}_{i \sim D} \left[ (y_i h_j(x_i))^2 \right] \\
\leq F^* \cdot \delta_j^2 \cdot M^2 = F^* \cdot a^2 M^2 \cdot \eta^2(j, \tilde{f}_s, H_t). \tag{91}
\]

A second order Taylor expansion on $F$ brings that there exists $Z = \{z_1, z_2, ..., z_m\} \subset \mathbb{R}^m$ such that:

\[
\mathbb{E}_{i \sim D} [F(y_i H_t(x_i))] = \mathbb{E}_{i \sim D} [F(y_i H_{t+1}(x_i))] + \mathbb{E}_{i \sim D} [(y_i H_t(x_i) - y_i H_{t+1}(x_i))F'(y_i H_t(x_i))] \\
+ \mathbb{E}_{i \sim D} \left[ (y_i H_t(x_i) - y_i H_{t+1}(x_i))^2 \cdot \frac{F''(z_i)}{2} \right], \tag{92}
\]

So,

\[
\mathbb{E}_{i \sim D} [F(y_i H_t(x_i))] - \mathbb{E}_{i \sim D} [F(y_i H_{t+1}(x_i))] = X + \frac{Y(Z)}{2} \geq \left(1 - \zeta - \frac{F^* a M^2}{2}\right) a \cdot \eta^2(j, \tilde{f}_s, H_t). \tag{93}
\]

Suppose we fix $\pi \in [0, 1]$ and choose any

\[
a \in \frac{1 - \zeta}{F^* M^2} \cdot [1 - \pi, 1 + \pi]. \tag{94}
\]

We can check that

\[
Z(a) \geq \frac{(1 - \zeta)^2(1 - \pi^2)}{2F^* M^2}, \tag{95}
\]

and so

\[
\mathbb{E}_{i \sim D} [F(y_i H_t(x_i))] - \mathbb{E}_{i \sim D} [F(y_i H_{t+1}(x_i))] \geq \frac{(1 - \zeta)^2(1 - \pi^2)}{2F^* M^2} \cdot \eta^2(j, \tilde{f}_s, H_t). \tag{96}
\]

So, taking into account that for the first classifier, we have $\mathbb{E}_{i \sim D} [F(y_i H_0(x_i))] = F(0)$, if we take any $z^* \in \mathbb{R}$ and we boost for a number of iterations $T$ satisfying (we use notation $\eta_t$ as a summary for $\eta^2(j, \tilde{f}_s, H_t)$ with respect to PilBoost):

\[
\sum_{t=1}^T \eta_t^2 \geq \frac{2F^* M^2 (F(0) - F(z^*))}{(1 - \zeta)^2 (1 - \pi^2)}, \tag{97}
\]

then $\mathbb{E}_{i \sim D} [F(y_i H_T(x_i))] \leq F(z^*)$. We now assume (WLA) holds, the LHS of (97) is $\geq T \gamma^2$. Given that we choose $a = a_f$ in PilBoost, we need to make sure (94) is satisfied for the loss defined by $F$, which translates to

\[
F^* \in \frac{1 - \zeta}{a_f M^2} \cdot [1 - \pi, 1 + \pi], \tag{98}
\]

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and defines assumption (2O). To complete Step 1, we normalise the edge. Letting \( \tilde{w}_t = w_t / \sum_k w_k \), \( \tilde{w}_t = \frac{\eta_t}{\tilde{w}_t} \in [-M, M] \),

\[
\tilde{w}_t = \frac{\eta_t}{\tilde{w}_t} \in [-M, M], \tag{99}
\]

which is then properly normalized and such that (97) becomes equivalently:

\[
\sum_{t=0}^{T} \tilde{w}_t^2 \tilde{\eta}_t^2 \geq \frac{2F^*M^2(F(0) - F(z^*))}{(1 - \zeta)^2(1 - \pi^2)}, \tag{100}
\]

and so under the (weak learning) assumption on \( \tilde{\eta}_t \) that \( |\tilde{\eta}_t| \geq \gamma \cdot M \), a sufficient condition for (100) is then

\[
\sum_{t=0}^{T} \tilde{w}_t^2 \geq \frac{2F^*(F(0) - F(z^*))}{\gamma^2(1 - \zeta)^2(1 - \pi^2)}, \tag{101}
\]

completing step 1 of the proof.

**In Step 2**, we show a result on the distribution of edges, *i.e.* margins. (101) contains all the intuition about how the rest of the proof unfolds, as we have two major steps: in step 2.1, we translate the guarantee of (101) on margins, and in step 2.2, we translate the "margin" based (101) in a readable guarantee in the boosting framework (we somehow "get rid" of the \( \tilde{w}_t^2 \) in the LHS of (101)).

**Step 2.1.** Let \( Z = \{z_1, z_2, ..., z_m\} \subset \mathbb{R} \) a set of reals. Since \( F \) is non-increasing, we have \( \forall u \in [0, 1], \forall \theta \geq 0 \),

\[
P[z_i \leq \theta] > u \Rightarrow \mathbb{E}_i[F(z_i)] > (1 - u) \inf_z F(z) + uF(\theta) \]

\[
\doteq (1 - u)F^\circ + uF(\theta), \tag{102}
\]

so if we pick \( z^* \) in (101) such that

\[
F(z^*) = (1 - u)F^\circ + uF(\theta), \tag{103}
\]

then (101) implies \( \mathbb{E}_{i \sim D}[F(y_i H_T(x_i))] \leq (1 - u)F^\circ + uF(\theta) \) and so by the contraposition of (102) yields:

\[
P_{i \sim D}[y_i H_T(x_i) \leq \theta] \leq u, \tag{104}
\]

which yields our margin based guarantee.

**Step 2.2.** At this point, the key (in)equalities are (101) (for boosting) and (104) (for margins). Fix \( \kappa > 0 \). We have two cases:

- **Case 1:** \( \tilde{w}_t \) never gets too small, say \( \tilde{w}_t \geq \kappa, \forall t \geq 0 \). In this case, granted the weak learning assumption holds on \( \tilde{\eta}_t \), (101) yields a direct lowerbound on iteration number \( T \) to get \( P_{i \sim D}[y_i H_\alpha(x_i) \leq \theta] \leq u \);
• Case 2: \( \tilde{w}_t \leq \kappa \) at some iteration \( t \). Since the smaller it is, the better classified are the examples, if we pick \( \kappa \) small enough, then we can get \( \mathbb{P}_{i \sim D}[y_i H_T(x_i) \leq \theta] \leq u "\text{straight}"\).

This suggest to use the notion of "denseness" for weights\(^{11}\).

**Definition A.1.** The weights at iteration \( t \) is called \( \kappa \)-dense iff \( \tilde{w}_t \geq \kappa \).

We now have the following Lemma.

**Lemma A.1.** For any \( t \geq 0, \theta \in \mathbb{R}, \kappa > 0 \), if weights produced in Step 2.1 of PilBoost fail to be \( \kappa \)-dense, then

\[
\mathbb{P}_{i \sim D}[y_i H_T(x_i) \leq \theta] \leq \frac{\kappa}{\tilde{f}(-\theta)}. \tag{105}
\]

**Proof.** Let \( Z = \{z_1, z_2, ..., z_m\} \subset \mathbb{R} \) a set of reals. Since \( \tilde{f} \) is non-decreasing, we have \( \forall \theta \in \mathbb{R}, \)

\[
\mathbb{E}_i[\tilde{f}(z_i)] \geq \mathbb{P}_i[z_i < -\theta] \cdot \inf_z \tilde{f}(z) + \mathbb{P}_i[z_i \geq -\theta] \cdot \tilde{f}(-\theta)
\]

\[
= \mathbb{P}_i[z_i \geq -\theta] \cdot \tilde{f}(-\theta) \tag{106}
\]

since by assumption \( \inf \tilde{f} = 0 \). Pick \( z_i = -y_i H_T(x_i) \). We get that if \( \mathbb{P}_{i \sim D}[y_i H_T(x_i) \geq -\theta] = \mathbb{P}_{i \sim D}[y_i H_T(x_i) \leq \theta] \geq \xi \), then \( w_i = \mathbb{E}_{i \sim D}[\tilde{f}(-y_i H_T(x_i))] \geq \xi \cdot \tilde{f}(-\theta) \). If we fix

\[
\xi = \frac{\kappa}{\tilde{f}(-\theta)},
\]

then \( \tilde{w}_t < \kappa \) implies (105), which ends the proof of Lemma A.1 \( \square \)

From Lemma A.1 we let \( \kappa = \xi_s \cdot \tilde{f}(-\theta) \) and \( u = \xi_s \) in (104). If at any iteration, \( H_T \) fails to be \( \kappa \)-dense, then \( \mathbb{P}_{i \sim D}[y_i H_\alpha(x_i) \leq \theta] \leq \xi_s \) and classifier \( H_\alpha \) satisfies the conditions of Theorem 2 (this is Case 2 above).

Otherwise, suppose it is always \( \kappa \)-dense (this is our Case 1 above). We then have at any iteration \( T \sum_{t < T} \tilde{w}_t^2 \geq T \xi_s^2 \cdot \tilde{f}^2(-\theta) \) and so a sufficient condition to get (104) is then

\[
T \geq \frac{2F^*(F(0) - F(z^*))}{\xi_s^2 \tilde{f}^2(-\theta) \gamma^2(1 - \zeta)^2(1 - \pi^2)}, \tag{108}
\]

where we recall \( z^* \) is chosen so that \( F(z^*) = (1 - \xi_s)F^\circ + \xi_sF(\theta) \). This ends the proof of Theorem 2 (with the change of notation \( \xi_s \leftrightarrow \epsilon \)).

### A.5 Proof of Lemma 5.2

Define for short

\[
F(u) \doteq \left( \frac{u^\alpha}{u^\alpha + (1 - u)^\alpha} \right)^{\alpha \epsilon} - \left( \frac{(1 - u)^\alpha}{u^\alpha + (1 - u)^\alpha} \right)^{\alpha \epsilon} \tag{109}
\]

\[
G(u) \doteq 1 - \left( \frac{2 \cdot (1 - u)^\alpha}{u^\alpha + (1 - u)^\alpha} \right)^{\frac{1}{\alpha}}, \tag{110}
\]
that we study for $u \geq 1/2$ (the bound also holds by construction for $u < 1/2$). Define the following functions:

\begin{align*}
g(u) &\equiv 1 - (2u)_{\frac{1}{\alpha}} , \\
h(z) &\equiv \frac{1}{1 + z} , \\
i_\alpha(u) &\equiv u^\alpha ,
\end{align*}

and $u_\alpha \equiv h \circ i_{-\alpha} \circ h^{-1}(u)$, $f(u) \equiv i_{\alpha'}(1 - u) - i_{\alpha'}(u)$. We remark that $g$ is convex if $\alpha \geq 1$ while $f$ is concave. Both derivatives match in $1/2$ if

\begin{equation}
(\alpha')^2 2^{1-\alpha} = 1 ,
\end{equation}

whose roots are $\alpha' < 6$. It means if $\alpha \geq 6/5 = 1.2$, then $(g - f)' \geq 0$, and so if we measure

\begin{equation}
k^* = \arg \sup_k \sup_{x,x': g(x) = f(x')} \left| x - x' \right| ,
\end{equation}

then $k^*$ is obtained for $x = 1$, for which $g(x) = 1 - 2^{\frac{1}{\alpha}} = k^*$. We then need to lowerbound $x'$ such that $f(x') = 1 - 2^{\frac{1}{\alpha}}$, which amounts to finding $x^*$ such that $f(x^*) \geq 1 - 2^{\frac{1}{\alpha}}$, since $f$ is strictly decreasing. Fix

\begin{equation}
x^* = 1 - \frac{K}{\alpha} ,
\end{equation}

A series expansion reveals that for $x = x^*$ and $K = \log 2$,

\begin{equation}
f(x^*) = g(x^*) + O \left( \frac{1}{\alpha^2} \right) ,
\end{equation}

and we thus get that there exists $K \geq \log 2$ such that

\begin{equation}
\sup_k \sup_{x,x': g(x) = f(x')} \left| x - x' \right| \leq \frac{K}{\alpha} ,
\end{equation}

or similarly for any ordinate value, the difference between the abscissae giving the value for $f$ and $g$ are distant by at most $K/\alpha$. The exact value of the constant is not so much important than the dependence in $1/\alpha$: we now plug this in the $u_\alpha$'s notation and ask the following question: suppose $f(u_\alpha) = g(v_\alpha) = k$. Since $|u_\alpha - v_\alpha| \leq K/\alpha$, what is the maximum difference $|u - v|$ as a function of $\alpha$? We observe

\begin{align*}
\frac{\partial}{\partial u} u_\alpha &= -\frac{\alpha(u(1 - u))^{\alpha-1}}{(u^\alpha + (1 - u)^\alpha)^2} , \\
\frac{\partial^2}{\partial u^2} u_\alpha &= \alpha \cdot \frac{(u(1 - u))^{\alpha-2}((\alpha - 2u + 1)u^\alpha - (\alpha + 2u - 1)(1 - u)^\alpha)}{(u^\alpha + (1 - u)^\alpha)^3} ,
\end{align*}

which shows the convexity of $u_\alpha$ as long as

\begin{equation}
\left( \frac{u}{1 - u} \right)^\alpha \geq \frac{\alpha + 2u - 1}{\alpha - 2u + 1} ,
\end{equation}

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a sufficient condition for which – given the RHS increases with \( u \) – is

\[
u \geq \frac{\left(\frac{4}{\alpha-1}\right)^{1 \over \alpha}}{1 + \left(\frac{4}{\alpha-1}\right)^{1 \over \alpha}}.\tag{121}\]

Since \( u \geq 1/2 \), we note the constraint quickly vanishes. In particular, if \( \alpha \geq 5 \), the RHS is \( \leq 1/2 \), so \( u_\alpha \) is strictly convex. Otherwise, scrutinising the maximal values of the derivative for \( \alpha \geq 1 \) reveals that if we suppose \( v \leq \delta \) for some \( \delta \), then \( |u - v| \) is maximal for \( v = \delta \). So, suppose \( v_\alpha = \varepsilon \) and we solve for \( u_\alpha = K/\alpha + \varepsilon \), which yields

\[
u = \frac{(1 - K/\alpha - \varepsilon)^{1 \over \alpha}}{(K/\alpha + \varepsilon)^{1 \over \alpha} + (1 - K/\alpha - \varepsilon)^{1 \over \alpha}}.\tag{122}\]

while the \( v \) producing the largest \( |u - v| \) is:

\[
v = \frac{(1 - \varepsilon)^{1 \over \alpha}}{\varepsilon^{1 \over \alpha} + (1 - \varepsilon)^{1 \over \alpha}}.\tag{124}\]

so

\[
|v - u| = (v - u)(\varepsilon) = \frac{(1 - \varepsilon)^{1 \over \alpha}}{\varepsilon^{1 \over \alpha} + (1 - \varepsilon)^{1 \over \alpha}} - \frac{(1 - \varepsilon)\alpha - K)^{1 \over \alpha}}{(K + \varepsilon \alpha)^{1 \over \alpha} + ((1 - \varepsilon)\alpha - K)^{1 \over \alpha}}.\tag{125}\]

If we fix

\[
\varepsilon = \frac{1}{\alpha^4},\tag{126}\]

then we get after separate series are computed in \( \alpha \to +\infty \),

\[
|v - u| = (v - u)(\varepsilon) = \frac{\log(1 + \log K)}{4\alpha} + O\left(\frac{1}{\alpha^2}\right)\tag{127}
\]

\[
\lesssim \frac{0.133}{\alpha} \tag{128}\]

The "forbidden interval" for \( v \) is then

\[
\left[\frac{(\alpha^4 - 1)^{1 \over \alpha}}{1 + (\alpha^4 - 1)^{1 \over \alpha}}, 1\right] \approx \left[\frac{1}{2} + \frac{\log \alpha}{\alpha}, 1\right]; \tag{129}\]

what is more interesting for us is the corresponding forbidden images for \( g(v_\alpha) \), which are thus

\[
g \not\in \alpha^\varepsilon \cdot \left[1 - \frac{1}{\alpha^4}, 1\right] = \mathbb{I}_\alpha, \tag{130}\]

where we use shorthand \( z \cdot [a, b] = [az, bz] \). This, we note, translates directly into observable edges since \( g \) is the function we invert. Summarizing, we have shown that if (i) \( \alpha \geq 1.2 \) then for any \( u, v \) such that \( F(u) = G(v) \not\in \mathbb{I}_\alpha \), then \( |u - v| \lesssim 0.133/\alpha \). It suffices to remark that \( \mathbb{I}_\alpha \) represents the set of forbidden weights to get the statement of the Lemma.
A.6 Additional results

We first give the expression of the formulas of interest regarding Lemma 5.1 for the $\alpha$-loss.

**Lemma A.2.** We have for the $\alpha$-loss,

$$f_\ell(u) = \alpha^c \cdot \left\{ \begin{array}{ll} \left(\frac{2u^\alpha}{u^\alpha+(1-u)^\alpha}\right)^{\frac{1}{\alpha c}} - 1 & \text{if } u \leq 1/2, \\ 1 - \left(\frac{2(1-u)^{\alpha c}}{u^\alpha+(1-u)^\alpha}\right)^{\frac{1}{\alpha c}} & \text{if } u \geq 1/2 \end{array} \right. ,$$

(131)

$$\tilde{f}(z) = \left\{ \begin{array}{ll} 0 & \text{if } z \leq -\alpha^c, \\ \frac{(\alpha^c+z)^{\frac{\alpha^c}{\alpha}}}{(\alpha^c+z)^{\frac{\alpha^c}{\alpha}}+(2\alpha^c\alpha^c-(\alpha^c+z)^{\alpha^c})^{\frac{\alpha c}{\alpha}}} & \text{if } -\alpha^c \leq z \leq 0, \\ \frac{(2\alpha^c\alpha^c-(\alpha^c-z)^{\alpha^c})^{\frac{\alpha c}{\alpha}}}{(2\alpha^c\alpha^c-(\alpha^c-z)^{\alpha^c})^{\frac{\alpha c}{\alpha}}+1} & \text{if } 0 \leq z \leq \alpha^c, \\ 1 & \text{if } z \geq \alpha^c. \end{array} \right. ,$$

(132)

Rewritten, we have that

$$\tilde{f}(z) = \left\{ \begin{array}{ll} 0 & \text{if } z \leq -\frac{\alpha}{\alpha-1}, \\ \frac{(\frac{\alpha}{\alpha-1}+z)^{\frac{\alpha}{\alpha-1}}}{(\frac{\alpha}{\alpha-1}+z)^{\frac{\alpha}{\alpha-1}}+(2(\frac{\alpha}{\alpha-1})^{\frac{\alpha}{\alpha-1}}-(\frac{\alpha}{\alpha-1}+z)^{\frac{\alpha}{\alpha-1}})^{\frac{\alpha}{\alpha-1}}} & \text{if } -\frac{\alpha}{\alpha-1} \leq z \leq 0, \\ \frac{(2(\frac{\alpha}{\alpha-1})^{\frac{\alpha}{\alpha-1}}-(\frac{\alpha}{\alpha-1}-z)^{\frac{\alpha}{\alpha-1}})^{\frac{\alpha}{\alpha-1}}}{(\frac{\alpha}{\alpha-1}-z)^{\frac{\alpha}{\alpha-1}}+(2(\frac{\alpha}{\alpha-1})^{\frac{\alpha}{\alpha-1}}-(\frac{\alpha}{\alpha-1}-z)^{\frac{\alpha}{\alpha-1}})^{\frac{\alpha}{\alpha-1}}} & \text{if } 0 \leq z \leq \frac{\alpha}{\alpha-1}, \\ 1 & \text{if } z \geq \frac{\alpha}{\alpha-1}. \end{array} \right. ,$$

(133)

Figure 5 provides some examples.

**B Additional Experimental Results**

In this section, we provide additional experimental results and discussion to accompany Section 6 in the main text.

**B.1 General Details**

All experiments were performed over the course of a month on a 2015 MacBook Pro with a 2.2 GHz Quad-Core Intel Core i7 processor and 16GB of memory. Averaged experiments employed 10-fold cross validation, and when twisters were present, randomization occurred over the twisted samples as well. All algorithms across all experiments ran for 1000 iterations.

**B.2 Discussion of $a_f$ and $\alpha$**

In general, we found that for most experiments, $0.1 \leq a_f \leq 15$. From the theory, we know that if $a_f$ is too small, you have to boost forever, and if $a_f$ is too large, almost no loss
fits to O2 (equivalently, O2 fails for us). We also generally found that PILBOOST was not particularly sensitive to the choice of \(a_f\) as long as it was in the “right ballpark”, hence our use of integer or rational values of \(a_f\) for all experiments. When there is twist present, we found that \(\alpha > 1\) performed best, where \(\alpha^*\) increased as the amount of twist increased (both observations are consistent with our theory, see for example Lemma 3.4). Regarding the relationship between \(a_f\) and \(\alpha\), this appeared to depend on the dataset and depth of the decision trees.

### B.3 Random Class Noise Twister

![Figure 5: A plot of \(\tilde{f}(z)\) as a function of \(\alpha\) as given in (133).](image)

| Dataset | Algorithm | \(p = 0\)  | 0.15  | 0.3  |
|---------|-----------|------------|-------|------|
| cancer  | AdaBoost  | 0.966 \(\pm\) 0.015 | 0.905 \(\pm\) 0.027 | 0.856 \(\pm\) 0.033 |
|         | \((\alpha = 1.1)\) | 0.944 \(\pm\) 0.029 | 0.912 \(\pm\) 0.013 | 0.861 \(\pm\) 0.042 |
|         | \((\alpha = 2.0)\) | 0.956 \(\pm\) 0.018 | 0.938 \(\pm\) 0.017 | 0.905 \(\pm\) 0.039 |
|         | \((\alpha = 4.0)\) | 0.957 \(\pm\) 0.014 | 0.917 \(\pm\) 0.012 | 0.922 \(\pm\) 0.032 |
|         | XGBoost   | 0.971 \(\pm\) 0.012 | 0.861 \(\pm\) 0.033 | 0.733 \(\pm\) 0.031 |

Table 2: cancer feature random class noise. Accuracies reported for each algorithm and level of twister. Depth one trees. For \(\alpha = 1.1\), \(a_f = 7\), for \(\alpha = 2\), \(a_f = 2\), and for \(\alpha = 4\), \(a_f = 1\).
Figure 6: Random class noise twister on the diabetes dataset. Depth 3 trees. $a_f = 0.1$ for all $\alpha$.

| Dataset   | Algorithm   | Random Class Noise Twister |
|-----------|-------------|----------------------------|
|           |             | $p = 0$ | 0.15 | 0.3  |
|           | AdaBoost    | 1.000 ± 0.000 | 0.949 ± 0.016 | 0.830 ± 0.043 |
|           | xd6 us ($\alpha = 1.1$) | 1.000 ± 0.000 | 0.981 ± 0.013 | 0.886 ± 0.033 |
|           | xd6 us ($\alpha = 2.0$) | 1.000 ± 0.000 | 0.992 ± 0.009 | 0.900 ± 0.027 |
|           | xd6 us ($\alpha = 4.0$) | 1.000 ± 0.000 | 0.999 ± 0.003 | 0.927 ± 0.023 |
|           | XGBoost     | 1.000 ± 0.000 | 0.912 ± 0.016 | 0.776 ± 0.041 |

Table 3: xd6 random class noise. Accuracies reported for each algorithm and level of twister. Depth three trees. $a_f = 8$ for all $\alpha$. Note that for 0% noise $\alpha = 4$ used $a_f = 0.1$.

| Dataset          | Algorithm   | Random Class Noise Twister |
|------------------|-------------|----------------------------|
|                  |             | $p = 0$ | 0.10 | 0.20 | 0.30 |
| Online Shopping  | AdaBoost    | 0.902 ± 0.002 | 0.900 ± 0.004 | 0.898 ± 0.005 | 0.894 ± 0.004 |
|                  | xd6 us ($\alpha = 1.1$) | 0.901 ± 0.005 | 0.899 ± 0.003 | 0.897 ± 0.004 | 0.890 ± 0.004 |
|                  | xd6 us ($\alpha = 2.0$) | 0.901 ± 0.004 | 0.895 ± 0.004 | 0.895 ± 0.003 | 0.894 ± 0.004 |
|                  | xd6 us ($\alpha = 4.0$) | 0.898 ± 0.003 | 0.873 ± 0.009 | 0.892 ± 0.005 | 0.889 ± 0.005 |
|                  | XGBoost     | 0.893 ± 0.005 | 0.874 ± 0.002 | 0.842 ± 0.006 | 0.782 ± 0.008 |

Table 4: Accuracies reported for each algorithm and level of twister. Random training sample selected with probability $p$. Then, for selected training sample, boolean feature flipped with probability $p$ for each feature, independently. Depth three trees. For $\alpha = 1.1$, $a_f = 7$, for $\alpha = 2$, $a_f = 8$, and for $\alpha = 4$, $a_f = 15$. 

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Figure 7: Box and whisker visualization of scores associated with Figure 3. For all insider twister results, we fixed $a_f = 7$.

### B.4 Insider Twister

### B.5 Discussion of XGBoost

| Algorithm | Average Compute Times | cancer | xd6 | diabetes | shoppers |
|-----------|-----------------------|--------|-----|----------|----------|
| AdaBoost  |                       | 1.41   | 0.75| 1.11     | 13.68    |
| us ($\alpha = 1.1$) |                   | 2.19   | 2.01| 2.19     | 30.88    |
| us ($\alpha = 2.0$) |                   | 1.11   | 0.79| 2.09     | 21.85    |
| us ($\alpha = 4.0$) |                   | 0.96   | 1.35| 1.82     | 13.01    |
| XGBoost   |                       | 0.29   | 0.28| 0.46     | 3.16     |

Table 5: Average compute times per run (10 runs) in seconds across the datasets. Note that the values of $a_f$ are chosen identically to choices in Section B.3.

XGBoost is a very fast, very well engineered boosting algorithm. It employs many different hyperparameters and customizations. In order to report the fairest comparison between AdaBoost, PilBoost, and XGBoost, we opted to keep as many hyperparameters fixed (and similar, e.g., depth of decision trees) as possible. That being said, it appears that XGBoost inherently uses pruning, so the algorithm pruned while the other two did not. Further details regarding three other important points related to XGBoost:

1. Please refer to Table 5 for averaged compute times for the three different algorithms. In general, XGBoost had the far faster computation time among the three. However, note that PilBOOST was not particularly engineered for speed. Indeed, we estimate that the computation of $\tilde{f}$ accounts for $40 - 50\%$ of the total computation time, which we believe can be improved. Thus, we leave the further computational optimization of PilBOOST for future work.
2. For details regarding regularization, refer to Figure 8, where we report a comparison of regularized XGBoost and PilBoost such that the training data suffers from the insider twister. We find that regularization improves the ability of XGBoost to combat the twister, but it is not as effective as PilBoost.

3. For details regarding early stopping, refer to Figure 10, where we report a comparison of early-stopped XGBoost (on un-twisted validation data, i.e., cheating) and PilBoost such that the training data suffers from the insider twister. We find that even early-stopping does not improve XGBoost’s ability to combat the insider twister as effectively as PilBoost.

Early stopping - on an untwisted hold-out set contradicts our experiment. With early stopping enabled on a twisted hold-out set, XGBoost generally did not early stop.

![Figure 8](image)

**Figure 8**: With regularization (where $\lambda = 20$), we still observe that the feature importance of XGBoost is perturbed. Note that PilBoost is not regularized.
Figure 9: Scores associated with Figure 8.

Figure 10: With early stopping (where XGBoost has access to clean validation data - cheating scenario), we still observe that the feature importance of XGBoost is perturbed. Note that PILBOOST is not early stopped.
Figure 11: Scores associated with Figure 10.