Automorphic Equivalence in the Classical Varieties of Linear Algebras.

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Abstract

This research is a continuation of the [8]. In this paper we consider some classical varieties of linear algebras over the field $k$ such that $\text{char}(k) = 0$. We study the relation between the geometric equivalence and automorphic equivalence of the algebras of these varieties.

If we denote by $\Theta$ one of these varieties, then $\Theta^0$ is a category of the finite generated free algebras of the variety $\Theta$. In this paper we calculate for the considered varieties the quotient group $A/Y$, where $A$ is a group of the all automorphisms of the category $\Theta^0$ and $Y$ is a subgroup of the all inner automorphisms of this category. The quotient group $A/Y$ measures difference between the geometric equivalence and automorphic equivalence of algebras from the variety $\Theta$. The results of this paper and of the [8] are summarized in the table in the end of the Section 4.

We can see from this table that in the all considered varieties of the linear algebras the group $A/Y$ is generated by cosets which are presented by no more than two kinds of the strongly stable automorphisms of the category $\Theta^0$. One kind of automorphisms is connected to the changing of the multiplication by scalar and second one is connected to the changing of the multiplication of the elements of the algebras. In the Section 5 we present some examples of the pairs of linear algebras such that the considered automorphism provides the automorphic equivalence of these algebras but these algebras are not geometrically equivalent. These examples are presented for the all considered above varieties of algebras and for both these kinds of the strongly stable automorphisms, when they exist in the group $A/Y$. 

1
1 Introduction.

This is a paper from universal algebraic geometry. All definitions of the basic notions of the universal algebraic geometry can be found, for example, in [2, 3] and [4].

In universal algebraic geometry we consider some variety Θ of one-sorted algebras of the signature Ω. We denote by $X_0 = \{x_1, x_2, \ldots, x_n, \ldots\}$ a countable set of symbols, and by $\mathfrak{F}(X_0)$ the set of all finite subsets of $X_0$. We will consider the category $\Theta^0$, whose objects are all free algebras $F(X)$ of the variety $\Theta$ generated by finite subsets $X \in \mathfrak{F}(X_0)$. Morphisms of the category $\Theta^0$ are homomorphisms of free algebras.

We denote some time $F(X) = F(x_1, x_2, \ldots, x_n)$ if $X = \{x_1, x_2, \ldots, x_n\}$ and even $F(X) = F(x)$ if $X$ has only one element.

We consider a ”set of equations" $T \subset F \times F$, where $F \in \text{Ob}^{\Theta^0}$, and we ”resolve" these equations in $\text{Hom}(F,H)$, where $H \in \Theta$. The set $\text{Hom}(F,H)$ serves as an ”affine space over the algebra $H". We denote by $T_H'$ the set $\{\mu \in \text{Hom}(F,H) \mid T \subset \ker \mu\}$. This is the set of all solutions of the set of equations $T$. For every set of ”points" $R \subset \text{Hom}(F,H)$ we consider a congruence of equations defined by this set: $R_H' = \bigcap_{\mu \in R} \ker \mu$. For every set of equations $T$ we consider its algebraic closure $T''_H$ in respect to the algebra $H$. A set $T \subset F \times F$ is called $H$-closed if $T = T''_H$. An $H$-closed set is always a congruence.

Definition 1.1 Algebras $H_1, H_2 \in \Theta$ are geometrically equivalent if and only if for every $X \in \mathfrak{F}(X_0)$ and every $T \subset F(X) \times F(X)$ fulfills $T''_{H_1} = T''_{H_2}$.

We denote the family of all $H$-closed congruences in $F$ by $\text{Cl}_H(F)$. We can consider the category $\text{C}_\Theta(H)$ of the coordinate algebras connected with the algebra $H \in \Theta$. Objects of this category are quotient algebras $F(X)/T$, where $X \in \mathfrak{F}(X_0)$, $T \in \text{Cl}_H(F(X))$. Morphisms of this category are homomorphisms of algebras.

Definition 1.2 Let $\text{Id}(H,X) = \bigcap_{\varphi \in \text{Hom}(F(X),H)} \ker \varphi$ be the minimal $H$-closed congruence in $F(X)$. Algebras $H_1, H_2 \in \Theta$ are automorphically equivalent if and only if there exists a pair $(\Phi, \Psi)$, where $\Phi : \Theta^0 \rightarrow \Theta^0$ is an automorphism, $\Psi : \text{C}_\Theta(H_1) \rightarrow \text{C}_\Theta(H_2)$ is an isomorphism subject to conditions:

A. $\Psi(F(X)/\text{Id}(H_1,X)) = F(Y)/\text{Id}(H_2,Y)$, where $\Phi(F(X)) = F(Y)$,

B. $\Psi(F(X)/T) = F(Y)/\overline{T}$, where $T \in \text{Cl}_{H_1}(F(X))$, $\overline{T} \in \text{Cl}_{H_2}(F(Y))$,

C. $\Psi$ takes the natural epimorphism $\overline{\Psi} : F(X)/\text{Id}(H_1,X) \rightarrow F(X)/T$ to the natural epimorphism $\Psi(\overline{\Psi}) : F(Y)/\text{Id}(H_2,Y) \rightarrow F(Y)/T$,

D. for every $F(X_1), F(X_2) \in \text{Ob}^{\Theta^0}$ and every

$$\nu \in \text{Mor}_{\text{C}_\Theta(H_1)}(F(X_1)/\text{Id}(H_1,X_1), F(X_2)/\text{Id}(H_1,X_2))$$
If the diagram
\[
\begin{array}{ccc}
F(X_1) & \rightarrow & F(X_1)/Id(H_1,X_1) \\
\downarrow \mu & & \downarrow \nu \\
F(X_2) & \rightarrow & F(X_2)/Id(H_1,X_2)
\end{array}
\]
is commutative then the diagram
\[
\begin{array}{ccc}
F(Y_1) & \rightarrow & F(Y_1)/Id(H_2,Y_1) \\
\downarrow \Phi(\mu) & & \downarrow \Psi(\nu) \\
F(Y_2) & \rightarrow & F(Y_2)/Id(H_2,Y_2)
\end{array}
\]
is also commutative, where \(\mu \in \text{Mor}_{\Theta} (F(X_1), F(X_2))\), \(\delta_i\) and \(\tilde{\delta}_i\) are the natural epimorphisms, \(\Phi(F(X_i)) = F(Y_i)\) for \(i = 1, 2\).

If we will compare the geometric equivalence and the automorphic equivalence of the one-sorted universal algebras from the some variety \(\Theta\), we must take a countable set of symbols \(X_0 = \{x_1, x_2, \ldots, x_n, \ldots\}\) and consider all free algebras \(F(X)\) of the variety \(\Theta\), generated by finitely subsets \(X \subset X_0\). These algebras: \(\{F(X) \mid X \subset X_0, |X| < \infty\}\) - will be objects of the category \(\Theta^0\). Morphisms of the category \(\Theta^0\) will be homomorphisms of these algebras.

If our variety \(\Theta\) is a variety of one-sorted algebras and possesses the IBN property: for free algebras \(F(X), F(Y) \in \Theta\) we have \(F(X) \cong F(Y)\) if and only if \(|X| = |Y|\) - then we have [5, Theorem 2] the decomposition
\[
\mathfrak{A} = \mathfrak{I}\mathfrak{S},
\]
of the group \(\mathfrak{A}\) of all automorphisms of the category \(\Theta^0\). Hear \(\mathfrak{I}\) is a group of all inner automorphisms of the category \(\Theta^0\) and \(\mathfrak{S}\) is a group of all strongly stable automorphisms of the category \(\Theta^0\). The definitions of the notions of inner automorphisms and strongly stable automorphisms can be found, for example, in [5], [8] and [9]. But we will give these definitions here.

**Definition 1.3** An automorphism \(\Upsilon\) of a category \(\mathcal{R}\) is **inner**, if it is isomorphic as a functor to the identity automorphism of the category \(\mathcal{R}\).

It means that for every \(A \in \text{Ob}\mathcal{R}\) there exists an isomorphism \(s_A^\Upsilon : A \rightarrow \Upsilon(A)\) such that for every \(\psi \in \text{Mor}_{\mathcal{R}}(A, B)\) the diagram
\[
\begin{array}{ccc}
A & \rightarrow & \Upsilon(A) \\
\downarrow \psi & & \downarrow \Upsilon(\psi) \\
B & \rightarrow & \Upsilon(B)
\end{array}
\]
commutes.

**Definition 1.4.** An automorphism \(\Phi\) of the category \(\Theta^0\) is called **strongly stable** if it satisfies the conditions:
StSt1) $\Phi$ preserves all objects of $\Theta^0$.

StSt2) there exists a system of bijections $\{ s^\Phi_F : F \to F \mid F \in \text{Ob}\Theta^0 \}$ such that $\Phi$ acts on the morphisms $\psi : D \to F$ of $\Theta^0$ by this way:

$$\Phi(\psi) = s^\Phi_D \psi \left( s^\Phi_D \right)^{-1},$$

(1.2)

StSt3) $s^\Phi_F |_X = \text{id}_X$, for every free algebra $F = F(X)$.

The subgroup $\mathcal{Y}$ is a normal in $\mathfrak{A}$.

By [4] only strongly stable automorphism $\Phi$ can provide us automorphic equivalence of algebras which not coincides with geometric equivalence of algebras. Therefore, in some sense, difference from the automorphic equivalence to the geometric equivalence is measured by the quotient group $\mathfrak{A}/\mathcal{Y} \cong \mathcal{G}/\mathcal{S} \cap \mathcal{Y}$.

2 Verbal operations and strongly stable automorphisms.

In this paper, as in the [8] we use the method of verbal operations for the finding of the strongly stable automorphisms of the category $\Theta^0$. The explanation of this method there is in [5], [7] and [9].

We denote the signature of our variety $\Theta$ by $\Omega$, by $m_\omega$ we denote the arity of $\omega$ for every $\omega \in \Omega$. If $w = w(x_1, \ldots, x_{m_\omega}) \in F(x_1, \ldots, x_{m_\omega}) \in \text{Ob}\Theta^0$, then we can define in every algebra $H \in \Theta$ by using of the this word $w$ the new operation $\omega^*$:

$$\omega^*(h_1, \ldots, h_{m_\omega}) = w(h_1, \ldots, h_{m_\omega}) \text{ for every } h_1, \ldots, h_{m_\omega} \in H.$$ This operation we call the **verbal operation** defined on the algebra $H$ by the word $w$. If we have a system of words $W = \{ w_\omega \mid \omega \in \Omega \}$ such that $w_\omega \in F(x_1, \ldots, x_{m_\omega})$ then we denote by $H^*_W$ the algebra which coincide with $H$ as a set but instead the original operations $\{ \omega \mid \omega \in \Omega \}$ it has the system of the verbal operations $\{ \omega^* \mid \omega \in \Omega \}$ defined by words from the system $W$.

We suppose that we have the system of words $W = \{ w_\omega \mid \omega \in \Omega \}$ satisfies the conditions:

**Op1** $w_\omega(x_1, \ldots, x_{m_\omega}) \in F(x_1, \ldots, x_{m_\omega}) \in \text{Ob}\Theta^0$,

**Op2** for every $F = F(X) \in \text{Ob}\Theta^0$ there exists an isomorphism $\sigma_F : F \to F^*_W$ such that $\sigma_F |_X = \text{id}_X$.

It is clear isomorphisms $\sigma_F$ are defined uniquely by the system of words $W$.

The set $S = \{ \sigma_F : F \to F \mid F \in \text{Ob}\Theta^0 \}$ is a system of bijections which satisfies the conditions:

**B1** for every homomorphism $\psi : A \to B \in \text{Mor}\Theta^0$ the mappings $\sigma_B \psi \sigma_A^{-1}$ and $\sigma_B^{-1} \psi \sigma_A$ are homomorphisms;

**B2** $\sigma_F |_X = \text{id}_X$ for every free algebra $F \in \text{Ob}\Theta^0$. 

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So we can define the strongly stable automorphism by this system of bijections. This automorphism preserves all objects of \( \Theta^0 \) and acts on morphism of \( \Theta^0 \) by formula (1.2), where \( s_F^\Phi = \sigma_F \).

Vice versa if we have a strongly stable automorphism \( \Phi \) of the category \( \Theta^0 \) then its system of bijections \( S = \{ s_F^\Phi : F \rightarrow F \mid F \in \text{Ob}\Theta^0 \} \) defined uniquely. Really, if \( F \in \text{Ob}\Theta^0 \) and \( f \in F \) then

\[
s_F^\Phi (f) = s_F^\Phi \psi (x) = \left( s_F^\Phi \psi \left( s_F^\Phi \right)^{-1} \right) (x) = (\Phi (\psi)) (x), \tag{2.1}
\]

where \( D = F(x) \) - 1-generated free linear algebra - and \( \psi : D \rightarrow F \) homomorphism such that \( \psi (x) = f \). Obviously that this system of bijections \( S = \{ s_F^\Phi : F \rightarrow F \mid F \in \text{Ob}\Theta^0 \} \) fulfills conditions B1) and B2) with \( \sigma_F = s_F^\Phi \).

If we have a system of bijections \( S = \{ \sigma_F : F \rightarrow F \mid F \in \text{Ob}\Theta^0 \} \) which fulfills conditions B1) and B2) then we can define the system of words \( W = \{ w_\omega \mid \omega \in \Omega \} \) satisfies the conditions Op1) and Op2) by formula

\[
w_\omega (x_1, \ldots, x_{m_\omega}) = \sigma_{F_\omega} (\omega ((x_1, \ldots, x_{m_\omega}))) \in F_\omega, \tag{2.2}
\]

where \( F_\omega = F(x_1, \ldots, x_{m_\omega}) \).

By formulas (2.1) and (2.2) we can check that there are

1. one to one and onto correspondence between strongly stable automorphisms of the category \( \Theta^0 \) and systems of bijections satisfied the conditions B1) and B2)
2. one to one and onto correspondence between systems of bijections satisfied the conditions B1) and B2) and systems of words satisfied the conditions Op1) and Op2).

Therefore we can calculate the group \( \mathcal{S} \) if we can find the all system of words which fulfill conditions Op1) and Op2). For calculation of the group \( \mathcal{S} \cap \mathcal{Y} \) we also have a

**Criterion 2.1** The strongly stable automorphism \( \Phi \) of the category \( \Theta^0 \) which corresponds to the system of words \( W \) is inner if and only if for every \( F \in \text{Ob}\Theta^0 \) there exists an isomorphism \( c_F : F \rightarrow F^*_W \) such that \( c_F \psi = \psi c_D \) fulfills for every \( (\psi : D \rightarrow F) \in \text{Mor}\Theta^0 \).

### 3 Verbal operations in linear algebras.

From now on, the variety \( \Theta \) will be some specific variety of the linear algebras over field \( k \) with characteristic 0. We nether consider the vanished varieties, i.e., variety defined by identity \( x = 0 \) or variety defined by identity \( x_1 x_2 = 0 \). For linear algebras we can rewrite the equation \( t_1 = t_2 \), where \( t_1, t_2 \in F, F \in \text{Ob}\Theta^0 \), as \( t_1 - t_2 = 0 \). So we can assume that every set of equations \( T \) is a subset of \( F \in \text{Ob}\Theta^0 \) and this set \( T \subset F \) we understand as \( \{ t = 0 \mid t \in T \} \). Congruences
we can consider as two sided ideals of the algebra $F$. From now on, the word "ideal" means two sided ideal of the linear algebra.

We consider linear algebras as one-sorted universal algebras, i.e., multiplication by scalar we consider as 1-ary operation for every $\lambda \in k$: $H \ni h \mapsto \lambda h \in H$ where $H \in \Theta$. Hence the signature $\Omega$ of algebras of our variety contains these operations: 0-ary operation 0; $|k|$ 1-ary operations of multiplications by scalars; 2-ary operation $\cdot$ and 2-ary operation $\ast$. We will finding the system of words $W = \{ w_\omega \mid \omega \in \Omega \}$ satisfies the conditions Op1) and Op2). We denote the words corresponding to these operations by $w_0, w_\lambda$ for all $\lambda \in k, w_+, w_-$. So

$$W = \{ w_\omega \mid \omega \in \Omega \} = \{ w_0, w_\lambda (\lambda \in k), w_+, w_- \} \quad (3.1)$$

in our case. From this on we consider only these systems of words.

Some time we denote by $\lambda \ast$ the operation defined by the word $w_\lambda (\lambda \in k)$, by $\perp$ the operation defined by the word $w_+$ and by $\times$ the operation defined by the word $w_-$. We denote the group of all automorphisms of the field $k$ by $\text{Aut} k$.

We use in our research the familiar fact that every variety of the linear algebras over field with characteristic 0 is multi-homogenous. So, for example, every $F(X) \in \text{Ob} \Theta^0$ can be decompose to the direct sum of the linear spaces of elements which are homogeneous according the sum of degrees of generators from the set $X$: $F(X) = \bigoplus_{i=1}^{\infty} F_i$. We also denote the ideals $\bigoplus_{i=j}^{\infty} F_i = F^j$. $F_i F_j \subset F_{i+j}$ and $F^i F^j \subset F^{i+j}$ fulfills for every $1 \leq i, j < \infty$.

All our varieties $\Theta$ possess the IBN property, because $|X| = \dim F/F^2$ fulfills for all free algebras $F = F(X) \in \text{Ob} \Theta^0$. So we have the decomposition (1.1) for group of all automorphisms of the category $\Theta^0$.

## 4 Classical varieties of linear algebras.

In this Section we consider as the variety $\Theta$ the varieties of the all commutative algebras, of the all power associative algebras, i.e., the variety of linear algebras defined by identities

$$x (x^2) = (x^2) x,$$

$$x (x (x^2)) = x ((x^2) x) = (x (x^2)) x = ((x^2) x) x = (x^2) (x^2) \quad (4.1)$$

and so on, of the all alternative algebras, of the all Jordan algebras and arbitrary subvariety defined by identities with coefficients from $\mathbb{Z}$ of the variety of the all anticommutative algebras.

For the calculating of the group $\mathfrak{S}$ we consider an arbitrary strongly stable automorphism $\Phi$ of the category $\Theta^0$ and we will find for the all possible forms of the system of words $W$ which corresponds to the automorphism $\Phi$.

For the all considered varieties $F(\emptyset) = \{0\}$, so $w_0 = 0$.

The crucial point is the finding of the words $w_\lambda (x) \in F(x)$, where $\lambda \in k$. The system of words $W$ must fulfills conditions Op1) and Op2). By condition
all axioms of the variety \( \Theta \) must hold in the \( F^*_w \) for every \( F \in \text{Ob}\Theta^0 \). For every \( \lambda \in k^* \) must holds

\[
 w_{\lambda^{-1}} (w_\lambda (x)) = w_\lambda (w_{\lambda^{-1}} (x)) = x. \tag{4.2}
\]

So the mapping \( F(x) \ni x \to w_\lambda (x) \in F(x) \) can be extended to the isomorphism.

By \([6]\) the variety of the all commutative algebras is a Shreier variety and by \([ \tilde{H} ]\) all automorphisms of the free algebras of these varieties are tame. So if \( \Theta \) is the variety of the all commutative algebras, then for \( \lambda \in k^* \) we have that

\[
 w_\lambda (x) = \varphi (\lambda) x, \quad \tag{4.3}
\]

where \( \varphi (\lambda) \in k \). If \( \lambda = 0 \), then must fulfills \( w_\lambda (x) = 0 \), so in this case we also can write \((4.3)\), where \( \varphi (\lambda) = 0 \).

If \( \Theta \) is the variety of the all power associative algebras, then \( F(x) \) is the algebra of the polynomials of degrees no less than 1. Hence from \((4.2)\) we can conclude that \( \deg w_\lambda (x) = 1 \) and \((4.3)\) holds. Similar result we have for the variety of the all anticommutative algebras and for the variety of the all Jordan algebras, because these varieties are subvarieties of the variety of the all power associative algebras (see \([10\text{, Chapter 2, Theorem 2}]\) and \([10\text{, Chapter 3, Corollary from Theorem 8}]) \), so in these varieties \( F(x) \) is also the algebra of the polynomials of degrees no less than 1.

We conclude \((4.3)\) for the arbitrary subvariety defined by identities with coefficients from \( Z \) of the variety of the all anticommutative algebras from the fact that in this variety \( \dim F(x) = 1 \). Therefore in all our varieties we have \((4.3)\) for all \( \lambda \in k \).

\[
 \lambda * (\mu * x) = (\lambda \mu) * x \quad \text{must fulfills in } F(x) \text{ for every } \lambda, \mu \in k. \quad \text{We can conclude from this axiom as in } [8] \text{ that } \varphi (\lambda \mu) = \varphi (\lambda) \varphi (\mu). \text{ Also by using } [7\text{, Proposition 4.2}] \text{ we can prove that } \varphi : k \to k \text{ is a surjection.}
\]

After this we can conclude from axioms \( x_1 \perp 0 = x_1, 0 \perp x_2 = x_2, x_1 \perp x_2 = x_2 \perp x_1 \) and \( \lambda * (x_1 \perp x_2) = (\lambda * x_1) \perp (\lambda * x_2) \) as in \([8]\) that in all our varieties the

\[
 w_+ (x_1, x_2) = x_1 + x_2 \tag{4.4}
\]

holds. Here we must use the decomposition of \( F(x_1, x_2) \) to the direct sum of the linear spaces of elements which are homogeneous according the sum of degrees of generators, which was used in \([8]\).

From axiom \((\lambda + \mu) * x = \lambda * x + \mu * x \) for every \( \lambda, \mu \in k \) we conclude that \( \varphi (\lambda + \mu) = \varphi (\lambda) + \varphi (\mu) \). So \( \varphi \in \text{Aut} k \).

Now we must to find the all possible forms of the word \( w_\in F(x_1, x_2) \). Hear as in \([8]\) we use the decomposition of \( F(x_1, x_2) \) to the direct sum of the linear spaces of elements which are homogeneous according the degree of \( x_1 \), and after this according the degree of \( x_2 \). From axioms \( 0 \times x_2 = x_1 \times 0 = 0 \) and \( \lambda * (x_1 \times x_2) = (\lambda * x_1) \times x_2 = x_1 \times (\lambda * x_2) \) for every \( \lambda \in k \) we can conclude that \( w_\in F_2(x_1, x_2), w_\in F_0(x_1, x_2) = 0, (0, x_2) = 0 \). It means that for the variety of the all power associative algebras

\[
 w_\in (x_1, x_2) = \alpha_{1,2} x_1 x_2 + \alpha_{2,1} x_2 x_1, \tag{4.5}
\]
where $\alpha_{1,2}, \alpha_{2,1} \in k$. By condition Op2) in this variety the multiplication in $F_W^*$ can not by commutative or anticommutative, so $\alpha_{1,2} \neq \pm \alpha_{2,1}$.

For the varieties of the all commutative algebras, of the all Jordan algebras and for the arbitrary subvariety defined by identities with coefficients from $\mathbb{Z}$ of the variety of the all anticommutative algebras we have that

$$w. (x_1, x_2) = \alpha_{1,2} x_1 x_2,$$  \hspace{1cm} (4.6)

where $\alpha_{1,2} \neq 0$.

For the variety of the all alternative algebras we conclude from axiom $(x_1 \times x_1) \times x_2 = x_1 \times (x_1 \times x_2)$ that

$$w. (x_1, x_2) = \alpha_{1,2} x_1 x_2$$  \hspace{1cm} (4.7)

or

$$w. (x_1, x_2) = \alpha_{2,1} x_2 x_1,$$  \hspace{1cm} (4.8)

where $\alpha_{1,2}, \alpha_{2,1} \neq 0$.

Now we will prove for all our varieties that the systems of words $W$ defined above fulfill condition Op2). First of all we will prove that if $H \in \Theta$ then $H^*_W \in \Theta$. It means that we will check that in the $H^*_W$ the all axioms of the variety $\Theta$ hold. All these checking can be made by direct calculations. For all our varieties we must check only these axioms of linear algebra: $(x_1 + x_2) \times x_3 = (x_1 \times x_3) + (x_2 \times x_3)$ and $x_1 \times (x_2 + x_3) = (x_1 \times x_2) + (x_1 \times x_3)$, because other axioms are immediately concluded from the forms of the words of the system $W$.

For the varieties of the all commutative algebras and of the all Jordan algebras we must check the axiom $x_1 \times x_2 = x_2 \times x_1$.

Also for the variety of the all Jordan algebras we also must check the axiom

$$(x_1 \times x_1) \times x_1 = (x_1 \times x_1) \times (x_2 \times x_1).$$

For the variety of the all alternative algebras we must check the axioms $x_1 \times x_1 \times x_2 = x_1 \times (x_1 \times x_2)$ and $x_2 \times (x_1 \times x_1) = (x_2 \times x_1) \times x_1$.

For the variety of the all power associative algebras we must check the axioms

$$w. (x_1, x_2, \ldots, x_r) = \sum_{i=1}^{m} \lambda_i u_i = 0,$$

where $\lambda_i \in \mathbb{Z}$, $u_i \in F(x_1, \ldots, x_r)$, $F(x_1, \ldots, x_r) \in \text{Ob}\Theta^0$, $u_i$ are monomials with coefficients 1, deg $u_i = n$ for every $i$. $\lambda_i \in \mathbb{Z}$ because
all operations of the process of the homogenization of the identities can be made with coefficients from \( \mathbb{Q} \). We must check that for every \( H \in \Theta \) and every \( h_1, \ldots, h_r \in H \) the \( \sum_{i=1}^{m} \lambda_i * u_i^x (h_1, \ldots, h_r) = 0 \) holds. As above by induction we can prove that \( u_i^x (h_1, \ldots, h_r) = \alpha_{i,2}^{-1} \sum_{i=1}^{m} \lambda_i u_i (h_1, \ldots, h_r) \), so \( \sum_{i=1}^{m} \lambda_i * u_i^x (h_1, \ldots, h_r) = 0 \).

After all these checking we can conclude that for every \( F = F (X) \in \text{Ob} \Theta^0 \) there exists a homomorphism \( \sigma_F : F \rightarrow F_W^* \) such that \( \sigma_F |_X = \text{id}_X \). As in the [8] we can prove that \( \sigma_F \) is an isomorphism, so the systems of words defined above fulfill condition Op2). It completes the calculation of the group \( \mathcal{S} \).

For calculation of the group \( \mathfrak{Y} \cap \mathcal{S} \) we can prove as in the [8] that for the all considered varieties the strongly stable automorphism \( \Phi \) which corresponds to the defined above system of words \( W \) is inner if and only if \( \alpha_{2,1} = 0 \) and \( \varphi = \text{id}_k \).

So, as in the [8] we can prove that for the variety of the all power associative algebras \( \mathfrak{A} / \mathfrak{Y} \cong (U (kS_2) / U (k \{ e \}) \times \text{Aut} k \), where \( S_2 \) is the symmetric group of the set which has 2 elements, \( U (kS_2) \) is the group of all invertible elements of the group algebra \( kS_2 \), \( U (k \{ e \}) \) is a group of all invertible elements of the subalgebra \( k \{ e \} \), every \( \varphi \in \text{Aut} k \) acts on the algebra \( kS_2 \) by natural way: \( \varphi (ae + b (12)) = \varphi (a) e + \varphi (b) (12) \). But there is an isomorphism of groups \( U (kS_2) \ni ae + b (12) \rightarrow (a + b, a - b) \in k^* \times k^* \) so there is isomorphism

\[
U (kS_2) / U (k \{ e \}) = U (kS_2) / k^* e \ni (ae + b (12)) k^* e \rightarrow \frac{a + b}{a - b} \in k^*.
\]

Hence we prove the

**Theorem 4.1** For variety of the all power associative algebras

\( \mathfrak{A} / \mathfrak{Y} \cong k^* \times \text{Aut} k \)

holds.

By similar way we prove

**Theorem 4.2** For the variety of the all alternative algebras

\( \mathfrak{A} / \mathfrak{Y} \cong S_2 \times \text{Aut} k \)

holds.

And for other considered varieties we achieve
**Theorem 4.3** For the variety of the all commutative algebras

\[ \mathfrak{A}/\mathfrak{Y} \cong \text{Aut}_k \]

holds.

**Theorem 4.4** For the variety of the all Jordan algebras

\[ \mathfrak{A}/\mathfrak{Y} \cong \text{Aut}_k \]

holds.

**Theorem 4.5** For the arbitrary subvariety defined by identities with coefficients from \( \mathbb{Z} \) of the variety of the all anticommutative algebras

\[ \mathfrak{A}/\mathfrak{Y} \cong \text{Aut}_k \]

holds.

The results of this Section and of the [8] can be summarized in this table:

| Variety of the all | \[ \mathfrak{A}/\mathfrak{Y} \] |
|-------------------|---------------------------------|
| 1 linear algebras | \( k^* \times \text{Aut}_k \) |
| 2 commutative algebras | \( \text{Aut}_k \) |
| 3 power associative algebras | \( k^* \times \text{Aut}_k \) |
| 4 alternative algebras | \( \mathfrak{S}_2 \times \text{Aut}_k \) |
| 5 Jordan algebras | \( \text{Aut}_k \) |
| 6 arbitrary subvariety of anticommutative algebras defined by identities with coefficients from \( \mathbb{Z} \) | \( \text{Aut}_k \) |

### 5 Examples.

We can see from the previous Section that in the all considered varieties of the linear algebras the group \( \mathfrak{A}/\mathfrak{Y} \) is generated by cosets which are presented by no more than two kinds of the strongly stable automorphisms of the category \( \Theta^0 \). One kind of automorphisms is connected to the changing of the multiplication by scalar and second one is connected to the changing of the multiplication of the elements of the algebras. In this Section we will present some examples of the pairs of linear algebras such that the considered automorphism provides the automorphic equivalence of these algebras but these algebras are not geometrically equivalent. These examples will be presented for the all considered above varieties of algebras and for both these kinds of the strongly stable automorphisms, when they exist in the group \( \mathfrak{A}/\mathfrak{Y} \).

Same time we will use this simple

**Proposition 5.1** If \( F(X) \) is a free universal algebra of the variety \( \Theta \) and \( T \) is an arbitrary congruence in \( F(X) \times F(X) \), then \( T \) is an \( H \)-closed congruence, where \( H = F(X)/T \).
Proof. $(T)_H'' = \bigcap_{\varphi \in \text{Hom}(F(X), H), \ker \varphi \supseteq T} \ker \varphi$. There is the natural epimorphism $\tau : F(X) \to F(X)/T$, such that $\ker \tau = T$. So $(T)_H'' = T$. □

Example 1.

In this example we at first will denote by $\Theta$ the variety of the all linear algebras. We will consider a strongly stable automorphism $\Phi$ of the category $\Theta^0$ which corresponds to the system of words

$$W = \{w_0 = 0, w_\lambda (x) = \varphi (\lambda) x (\lambda \in k), w_+ (x_1, x_2) = x_1 + x_2, w. (x_1, x_2) = x_1 x_2\}$$

(5.1)

where $\varphi \in \text{Aut} k$. We assume about automorphism $\varphi$ that there is $\lambda \in k$ such $\varphi (\lambda) \neq \lambda$ and $\varphi (\lambda) \neq \lambda^{-1}$. These conditions fulfill, for example, if $k = k_0 (\theta_1, \theta_2)$ is a transcendental extension of degree 2 of the same subfield $k_0$ and $\varphi (\theta_1) = \theta_2$.

We will consider the free algebra of $\Theta$ with two free generators: $F = F(x_1, x_2)$. We consider the ideal $T = \langle t, F^3 \rangle$, where $t = \lambda x_1 x_2 + x_2 x_1$. We will denote $H = F/T$. By [2, Theorem 5.1] $H$ and $H_W^*$ are automorphically equivalent.

Proposition 5.2 $H$ and $H_W^*$ are not geometrically equivalent.

Proof. By Proposition [5.1] $T$ is an $H$-closed ideal. If $H$ and $H_W^*$ are geometrically equivalent then $T$ is an $H_W^*$-closed ideal. The system of words (5.1) is a transcendental extension of degree 2 of the same subfield $k$ and $\varphi (\theta_1) = \theta_2$.

We will consider the free algebra of $\Theta$ with two free generators: $F = F(x_1, x_2)$. We consider the ideal $T = \langle t, F^3 \rangle$, where $t = \lambda x_1 x_2 + x_2 x_1$. We will denote $H = F/T$. By [2, Theorem 5.1] $H$ and $H_W^*$ are automorphically equivalent.
0, \alpha_1\alpha_{21} = 0. If \alpha_1 = 0 then \alpha (x_2) \in F^2 and ker \psi \supset sp_k \{x_1x_2, x_2x_1\}. If \alpha_{21} = 0 then \alpha (x_1) \in F^2 and also ker \psi \supset sp_k \{x_1x_2, x_2x_1\}.

Case 2. \alpha_2 = \alpha_{21} = 0. In this case \varphi (\lambda) \alpha_{11} \alpha_{22} = \rho \lambda, \alpha_{11} \alpha_{22} = \rho, so \varphi (\lambda) \rho = \rho \lambda. Therefore, by our assumption about \varphi, \rho = 0 and as above ker \psi \supset sp_k \{x_1x_2, x_2x_1\}.

Case 3. \alpha_{11} = \alpha_{21} = 0. In this case \alpha (x_1) \in F^2 and ker \psi \supset sp_k \{x_1x_2, x_2x_1\}.

Case 4. \alpha_2 = \alpha_{22} = 0. In this case \alpha (x_2) \in F^2 and ker \psi \supset sp_k \{x_1x_2, x_2x_1\}.

So in all these cases (s_F (T))''_H \supset sp_k \{x_1x_2, x_2x_1\} and s_F (T) \cap F_2 \neq (s_F (T))''_H \cap F_2. Therefore s_F (T) \neq (s_F (T))''_H and s_F (T) is is not an H-closed ideal. This contradiction finishes the proof. ■

It is clear that this example is valid for the variety of the all power associative algebras and for the variety of the all alternative algebras. Indeed, if we consider the free algebra \( F = F(x_1, x_2) \) in one of these varieties, ideal \( T = \langle t, F^3 \rangle \subset F \), the quotient algebra \( H = F/T \) and the system of words \( W \) as in \([5,1]\), then, as in the previous calculations, we can prove that algebras \( H \) and \( H^*_W \) are automorphically equivalent but are not geometrically equivalent.

Example 2

In this example we at first will denote by \( \Theta \) the variety of the all commutative algebras. We will consider a strongly stable automorphism \( \Phi \) of the category \( \Theta^k \) which corresponds to the system of words \([5,1]\). We assume about automorphism \( \varphi \) that \( \varphi \neq id_k \). It means there is \( \lambda \in k \) such that \( \varphi (\lambda) \neq \lambda \).

We will consider the free algebra of \( \Theta \) with two free generators: \( F = F(x_1, x_2) \). We consider the ideal \( T = \langle t, F^4 \rangle \), where \( t = \lambda x_1 (x_1 x_2) + x_2 (x_1^2) \). We will denote \( H = F/T \). Algebras \( H \) and \( H^*_W \) are automorphically equivalent.

Proposition 5.3 \( H \) and \( H^*_W \) are not geometrically equivalent.

Proof. As in the proof of the Proposition \([5,2]\) \( s_F (t) = \langle s_F (t), F^4 \rangle \). But now \( s_F (t) = \varphi (\lambda) x_1 (x_1 x_2) + x_2 (x_1^2) \). As in that proof we will consider all endomorphisms \( \alpha \in \text{End} F \) such that \( \alpha (s_F (t)) \in T \) and will calculate the ker \psi where \( \psi = \tau \alpha, \tau \) is the natural epimorphism \( F \to F/T = H \). As above \( \alpha (x_1) = \alpha_{11} x_1 + \alpha_{21} x_2 + f_i \) where \( i = 1, 2, f_i \in F^2 \).

\[
\alpha (s_F (t)) = (\varphi (\lambda) + 1) \alpha_{11} \alpha_{12} x_1^3 + (\varphi (\lambda) \alpha_{12} \alpha_{22} + (\varphi (\lambda) + 2) \alpha_{11} \alpha_{12} \alpha_{21}) x_1 (x_1 x_2) + (\varphi (\lambda) \alpha_{11} \alpha_{21} \alpha_{22} + \alpha_{12} \alpha_{22}) x_1 (x_1 x_2) + \]

\[
(\varphi (\lambda) \alpha_{11} \alpha_{21} \alpha_{22} + \alpha_{12} \alpha_{22}) x_1 (x_1 x_2) + (\varphi (\lambda) \alpha_{11} \alpha_{12} \alpha_{21} + \alpha_{12} \alpha_{22}) x_1 (x_1 x_2) + (\varphi (\lambda) + 2) \alpha_{11} \alpha_{21} \alpha_{22}) x_2 (x_1 x_2) + (\varphi (\lambda) + 1) \alpha_{21} \alpha_{22} x_1^3 (\text{mod} F^4) .
\]

So if \( \alpha (s_F (t)) \in T \) then \( \alpha_{11} \alpha_{22} = 0 \) and \( \alpha_{21} \alpha_{22} = 0 \). As above we must consider the following four cases:

Case 1. \( \alpha_{11} = \alpha_{22} = 0 \). In this case we conclude from \( \alpha (s_F (t)) \in T \) that \( \alpha_{12} \alpha_{21} = 0 \) holds. If \( \alpha_{12} = 0 \) then \( \alpha (x_2) \in F^2 \) and

\[
\text{ker } \psi \supset sp_k \{x_1 (x_1 x_2), x_1 (x_1^2), x_2 (x_1^2), x_2 (x_1 x_2)\} .
\] (5.2)

If \( \alpha_{21} = 0 \) then \( \alpha (x_1) \in F^2 \) and also \([5,2]\) holds.
Case 2. $\alpha_{12} = \alpha_{21} = 0$. In this case we conclude from $\alpha (s_F (t)) \in T$ that $\varphi (\lambda) \alpha_1^2 \alpha_{22} = \rho \lambda$, $\alpha_1^2 \alpha_{22} = \rho$, where $\rho \in k$. $\varphi (\lambda) \neq \lambda$, so $\rho = 0$ and $\alpha_{11} \alpha_{22} = 0$. So, as above $\alpha (x_1) \in F^2$ or $\alpha (x_2) \in F^2$ and \((5.2)\) holds.

Case 3. $\alpha_{11} = \alpha_{21} = 0$. In this case $\alpha (x_1) \in F^2$ and \((5.2)\) holds.

Case 4. $\alpha_{12} = \alpha_{22} = 0$. In this case $\alpha (x_2) \in F^2$ and \((5.2)\) holds.

So in all these cases $(s_F (T))''_H \supset s_F \{x_1 (x_1 x_2), x_1 (x_2^2), x_2 (x_1^2), x_2 (x_1 x_2)\}$ and $s_F (T) \cap F_3 \neq (s_F (T))''_H \cap F_3$. Therefore $s_F (T) \neq (s_F (T))''_H$. As above the proof is finished. \(
\)

It is clear that this example is also valid for the variety of the all Jordan algebras.

Example 3

The Uroboros 1 program designed for symbolic computation in Lie algebras was used for the finding of this example. We denote by $\Theta$ the variety of the all Lie algebras. As in the both previous examples we will consider a strongly stable automorphism $\Phi$ of the category $\Theta^0$ which corresponds to the system of words \((5.1)\). We assume about automorphism $\varphi$ that $\varphi \neq id_k$. It means there is $\lambda \in k$ such $\varphi (\lambda) \neq \lambda$.

$L = L (x_1, x_2)$ is a free Lie algebra with two free generators. The algebra $L/L^6$ has a basis

\[
\begin{align*}
\{x_1 = e_1, x_2 = e_2, [x_1, x_2] = e_3, [x_1, [x_1, x_2]] = e_4, [[x_1, x_2], x_2] = e_5, \\
[x_1, [x_1, [x_1, x_2]]] = e_6, [x_1, [[x_1, x_2], x_2]] = e_7, [[[x_1, x_2], x_2], x_2] = e_8, \\
[[x_1, [x_1, [x_1, x_2]]]] = e_9, [x_1, [x_1, [[x_1, x_2], x_2]]] = e_{10}, [x_1, [[[x_1, x_2], x_2], x_2]] = e_{11}, \\
[[x_1, [x_1, x_2]], [x_1, x_2]] = e_{12}, [[[x_1, x_2], [x_1, x_2]], x_2] = e_{13}, [[[x_1, x_2], x_2], x_2] = e_{14}, \\
\end{align*}
\]

where multiplication in $L$ we denote by Lie brackets. It will be more punctual to write $e_1 = x_1 + L^6$ and so on, but we chose the shorter form of the notation. This basis was found by program Uroboros 1. We consider the ideal $T = \langle t, L^6 \rangle$, where $t = \lambda e_{10} + e_{12}$. As above we denote $H = L/T$. Algebras $H$ and $H'_W$ are automorphically equivalent.

Proposition 5.4 $H$ and $H'_W$ are not geometrically equivalent.

Proof. As in the proof of the Proposition \((5.2)\) $s_F (T) = \langle s_F (t), L^6 \rangle$. But now $s_F (t) = \varphi (\lambda) e_{10} + e_{12}$. As above we will consider all endomorphisms $\alpha \in End F$ such that $\alpha (s_F (t)) \in T$ and will calculate the $\ker \psi$ where $\psi = \tau \alpha$, $\tau$ is the natural epimorphism $L \to L/T = H$. As above $\alpha (x_i) = \alpha_{1,i} x_1 + \alpha_{2,i} x_2 + f_i$, where $i = 1, 2, f_i \in L^2$. $\alpha (s_F (t)) \equiv \sum_{i=9}^{14} \alpha_i e_i \pmod{L^6}$, where

\[
\begin{align*}
\alpha_9 &= -\varphi (\lambda) \alpha_{1,1}^2 \alpha_{1,2}^2 (\alpha_{1,1} \alpha_{2,2} - \alpha_{1,2} \alpha_{2,1}), \\
\alpha_{10} &= \varphi (\lambda) \alpha_{1,1} \alpha_{1,2} (\alpha_{1,1} \alpha_{2,2} - \alpha_{1,2} \alpha_{2,1}) (\alpha_{1,1} \alpha_{2,2} + 2 \alpha_{1,2} \alpha_{2,1}),
\end{align*}
\]

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Therefore ker $\alpha$ of the coefficient $\alpha$ above from this equation we conclude that $\alpha$

This variety of the all linear algebras was denoted by $\Theta$. We consid-

$\alpha \subseteq \alpha \cap \alpha \cap \alpha$.

These coefficients also were found by program Uroboros 1. If $\alpha(s_F(t)) \in T$ then $\alpha(s_F(t)) \equiv \lambda \rho e_{10} + \rho e_{12} \pmod{L^6}$, where $\rho \in k$. So if $\alpha(s_F(t)) \in T$ then $-\varphi(\lambda) \alpha_{1,1}^{-1} \alpha_{1,2} (\alpha_{1,1} - \alpha_{1,2}) = 0$ and $\varphi(\lambda) \alpha_{1,2} (\alpha_{1,2} - \alpha_{1,2})$.

If $\alpha_{1,1} - \alpha_{1,2} = 0$ then $\alpha(x_1) \equiv \beta \alpha(x_2) \pmod{L^2}$, where $\beta \in k$. Hence $\alpha(x_1, x_2) \in L^3$ and $\alpha(L^3) \subseteq L^6$. Therefore ker $\psi \geq \sp_k \{e_9, \ldots, e_{14}\}$.

If $\alpha_{1,1} - \alpha_{1,2} \neq 0$ then $\alpha(s_F(t)) \in T$ we conclude $\alpha_{1,1} = 0, \alpha_{2,1} = 0$. So as above we must consider the following four cases:

Case 1. $\alpha_{1,1} = \alpha_{2,2} = 0$. In this case we conclude from $\alpha(s_F(t)) \in T$

by consideration of the coefficient $\alpha_{11}$ that $\alpha_{1,2} = 0$. As above from this equation we conclude that $\alpha(x_1) \in L^2$ or $\alpha(x_2) \in L^2$. So $\alpha(x_1, x_2) \in L^3$ and as above ker $\psi \geq \sp_k \{e_9, \ldots, e_{14}\}$.

Case 2. $\alpha_{1,2} = \alpha_{1,1} = 0$. In this case we conclude from $\alpha(s_F(t)) \in T$

by consideration of the coefficient $\alpha_{10}$ that $\varphi(\lambda) \alpha_{1,2}^2 = \lambda \rho$ and by consideration

of the coefficient $\alpha_{12}$ that $\alpha_{1,2}^2 = \rho$. $\varphi(\lambda) \neq \lambda$, so $\rho = 0$ and $\alpha_{1,2} = 0$.

As above from this equation we conclude that $\alpha(x_1) \in L^2$ or $\alpha(x_2) \in L^2$.

Therefore ker $\psi \geq \sp_k \{e_9, \ldots, e_{14}\}$.

In the Case 3: $\alpha_{1,1} = \alpha_{2,1} = 0$ and in the Case 4: $\alpha_{1,2} = \alpha_{2,2} = 0$ - also $\alpha(x_1) \in L^2$ or $\alpha(x_2) \in L^2$. So ker $\psi \geq \sp_k \{e_9, \ldots, e_{14}\}$.

Therefore in all these cases $(s_F(T)) \geq L^3 \geq \sp_k \{e_9, \ldots, e_{14}\}$.

Example 4

In [8] Section 5] was given an example of the algebras in the variety of the

all linear algebras which are automorphically equivalent but not geometrically

equivalent. The variety of the all linear algebras was denoted by $\Theta$. We consid-

ered the strongly stable automorphism $\Phi$ of the category $\Theta^0$ corresponding to

the system of words

$W = \{w_0 = 0, w_\lambda(x) = \lambda x (\lambda \in k), w^+ (x_1, x_2) = x_1 + x_2, w^- (x_1, x_2) = ax_1x_2 + bx_2x_1\}$,

where $b \neq 0$. In the free algebra $F(x_1, x_2) \in \Ob\Theta^0$ we considered the verbal

ideal $I$ generated by identity $(x_1x_1) x_2 = 0$. By [7] Theorem 5.1] the algebras

$H = F(x_1, x_2) / I$ and $H_W$ are automorphically equivalent. We proved in [8] Proposition 5.1] that the algebras $H$ and $H_W$ are not geometrically equivalent.

From this proof it is clear that this example is valid for the variety of the all

power associative algebras.

Example 5
In this example we denote by $\Theta$ the variety of the all alternative algebras. We will consider a strongly stable automorphism $\Phi$ of the category $\Theta^0$ which corresponds to the system of words

\[ W = \{ w_0 = 0, w_\lambda (x) = \lambda x (\lambda \in k), w_+ (x_1, x_2) = x_1 + x_2, w_0 (x_1, x_2) = x_2 x_1 \} . \]

We will consider the free algebra of $\Theta$ with two free generators: $F = F (x_1, x_2)$. This is an associative algebra. In this algebra we will consider the verbal ideal $I$ generated by identity $x_1 x_2^2 = 0$. We will denote $H = F/I$. As above $H$ and $H^*_W$ are automorphically equivalent. We will prove the

Proposition 5.5 $H$ and $H^*_W$ are not geometrically equivalent.

Proof. In this proof we use the method of [8, Proposition 5.1]. The ideal $I = \langle \alpha (x_1 x_2^2) \mid \alpha \in \text{End} F \rangle$ will be the smallest $H$-closed set in $F$, because $I = (0)^H_H$, where $0 \in F$. If algebras $H$ and $H^*_W$ are geometrically equivalent then the structures of the $H$-closed sets and of the $H^*_W$-closed sets in $F$ coincide. Hence $I$ must be the smallest $H^*_W$-closed set in $F$.

By [7, Remark 5.1]

\[ T \rightarrow s_F T \quad (5.3) \]

is a bijection from the structure of the $H^*_W$-closed sets in $F$ to the structure of the $H$-closed sets in $F$. It is clear that the bijection (5.3) preserves inclusions of sets. So it transforms the smallest $H^*_W$-closed set to the smallest $H$-closed set, hence $I = s_F I$ must fulfills.

We will get more information about the subspace $(I + F^4)/F^4$. If as above $\alpha \in \text{End} F$ such that $\alpha (x_i) = \alpha_1 x_1 + \alpha_2 x_2 + f_i$, where $i = 1, 2, f_i \in F^2, \alpha_j \in k$, then

\[ \alpha (x_1 x_2^2) = \alpha_1 x_1^3 + \alpha_2 x_2^3 + \alpha_{12} x_1 x_2 + x_1 x_2 x_1 + \alpha_{11} x_1^2 x_2 + \alpha_{12} x_1 x_2^2 + \alpha_{22} x_2^2 + x_2 x_1 x_2 + x_1 x_2^2 \]

\[ \alpha_{12} \alpha_1 x_1 x_2^2 + \alpha_2 \alpha_1 x_1 x_2 + x_1 x_2 x_1 + \alpha_{12} \alpha_1 x_2^2 + \alpha_{22} x_2^2 + x_2 x_1 x_2 + x_1 x_2 x_2 + \alpha_{12} \alpha_1 x_1 x_2 + x_2 x_1 + x_2 x_1 x_2 + x_1 x_2 + x_2 x_1 x_2 + x_1 x_2 x_2 \]

So $(I + F^4)/F^4 \subseteq \text{sp}_k \left\{ \overline{x_1 x_2 x_2 + x_1 x_2 x_1}, \overline{x_1 x_2 x_2 + x_2 x_1 x_2 + x_1 x_2 x_2} \right\} = V$, where $x_1 = x_1^3 + F^4$ and so on. $s_F (x_1 x_2^2) = x_1^2 x_1 \in s_F I$, but $x_2^2 x_1 \notin V$ and $(s_F I + F^4)/F^4 \neq (I + F^4)/F^4$, so $I \neq s_F I$. This contradiction finishes the proof.

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References

[1] J. Lewin, On Schreier varieties of linear algebras. Trans. Amer. Math. Soc. 132 (1968), pp. 553–562.

[2] B. Plotkin, Varieties of algebras and algebraic varieties. Categories of algebraic varieties. Siberian Advanced Mathematics, Allerton Press, 7:2, (1997), pp. 64 – 97.

[3] B. Plotkin, Some notions of algebraic geometry in universal algebra, Algebra and Analysis, 9:4 (1997), pp. 224 – 248, St. Petersburg Math. J., 9:4, (1998), pp. 859 – 879.

[4] B. Plotkin, Algebras with the same (algebraic) geometry, Proceedings of the International Conference on Mathematical Logic, Algebra and Set Theory, dedicated to 100 anniversary of P.S. Novikov, Proceedings of the Steklov Institute of Mathematics, MIAN, 242, (2003), pp. 17 – 207.

[5] B. Plotkin, G. Zhitomirski, On automorphisms of categories of free algebras of some varieties, Journal of Algebra, 306:2, (2006), pp. 344 – 367.

[6] A. I. Shirshov, Subalgebras of the free commutative and free anticommutative algebras. Matematicheskij Sbornik, 34(76):1, (1954), pp. 81-88. (In Russian.)

[7] A. Tsurkov, Automorphic equivalence of algebras. International Journal of Algebra and Computation. 17:5/6, (2007), pp. 1263–1271.

[8] A. Tsurkov, Automorphic equivalence of linear algebras, http://arxiv.org/abs/1106.4853 Accepted in the Journal of Algebra and Its Applications.

[9] A. Tsurkov, Automorphic equivalence of many-sorted algebras. http://arxiv.org/abs/1304.0021

[10] K. A. Zhevlakov, A. M. Slin’ko, I. P. Shestakov, A. I. Shirshov, Almost associative rings. Moscow, ”Nauka”, 1978. (In Russian.)