Randomly Branched Polymers and Conformal Invariance

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ABSTRACT: We argue that the field theory that describes randomly branched polymers is not generally conformally invariant in two dimensions at its critical point. In particular, we show (i) that the most natural formulation of conformal invariance for randomly branched polymers leads to inconsistencies; (ii) that the free field theory obtained by setting the potential equal to zero in the branched polymer field theory is not even classically conformally invariant; and (iii) that numerical enumerations of the exponent \( \theta(\alpha) \), defined by \( T_N(\alpha) \sim \lambda^N N^{-\theta(\alpha)+1} \), where \( T_N(\alpha) \) is number of distinct configurations of a branched polymer rooted near the apex of a cone with apex angle \( \alpha \), indicate that \( \theta(\alpha) \) is not linear in \( 1/\alpha \) contrary to what conformal invariance leads one to expect.
I. Introduction

It is widely believed that statistical mechanical systems at the critical point of a second order phase transition are conformally invariant on scales much larger than any microscopic distance^{polyakov}. In two dimensions, the conformal algebra is infinite dimensional and conformal invariance places strong constraints on the fixed point correlation functions. The analysis of these constraints, starting with the paper of Belavin, Polyakov, and Zamolodchikov^{belavin}, has lead to a rather complete understanding of two dimensional critical theories^{cardy}. The conformal properties of most random geometrical systems: linear polymers, branched polymers with fixed topology, theta polymers, percolation, dense polymers, etc. in two dimensions are well understood^{duplantier}. Randomly branched polymers^{lubensky} are a notable exception. Numerical enumerations of randomly branched polymers in the wedge DBL85, and transfer matrix calculations of randomly branched polymers in the strip^derrida, yield results at variance with the naive predictions of conformal invariance.

In this paper we argue that the critical point of the theory describing randomly branched polymers is not generally conformally invariant in two dimensions. The arguments are quite simple. We state them here in outline.

(i) We first show that the most natural formulation of conformal invariance for the field theory that describes randomly branched polymers^{lubensky,parisi} leads to inconsistencies. We proceed by first assuming the field theory in question to be conformally invariant, and then by studying the consequences of this assumption.

In order to study the consequences of conformal invariance, we have first to say how the fields in the randomly branched polymer field theory transform under conformal transformations. This leads us to “the most natural formulation” of conformal invariance for ran-

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\(^{a}\) More precisely, translational, rotational, and scale invariance, in systems with only short range interactions, is thought to imply conformal invariance.
randomly branched polymers. The most natural formulation of conformal invariance follows immediately from the supposition that the most relevant fields in the randomly branched polymer problem are those which appear in the effective Hamiltonian that defines the field theory (see equation 1 below). This is a very mild requirement; the field theory in question describes the large distance physics of randomly branched polymers, so we expect that the most relevant operators in the branched polymer problem to be those of the field theory, i.e. those which appear in the effective Hamiltonian, and which are integrated over in the functional integrals.

Given, as seems likely, that the most relevant operators in the randomly branched polymer problem are those appearing in the effective Hamiltonian, it follows that the most relevant field in the effective Hamiltonian, the field with lowest scaling dimension, is primary in the sense of conformal field theory (that is to say, if the theory is conformally invariant, then the lowest dimension field must be primary). For if this field were not primary, it would, by the usual structure of conformal field theories, have to be a secondary field, and therefore the descendent of a field with smaller scaling dimension, and by assumption there is no such field.

Given that the lowest dimension field in the effective Hamiltonian is primary, it is straightforward to show, by conformally mapping the plane to the cone, that one obtains inconsistent predictions for the exponent $\theta(\alpha)$ defined by $T_N(\alpha) \sim \lambda^N N^{-\theta(\alpha)+1}$, where $T_N(\alpha)$ is number of distinct configurations of a branched polymer rooted near the apex of a cone with apex angle $\alpha$. From this we can conclude that either the most relevant operators in the branched polymer theory do not appear in the effective Hamiltonian (which, we repeat, seems to us unlikely), or that the field theory is not conformally invariant.

(ii) The field theory that describes randomly branched polymers has a Parisi-Sourlas supersymmetry parisi. To further investigate the question of the conformal invariance of randomly branched polymers, we consider the simplest theory that has the same kind of
Parisi-Sourlas supersymmetry. This simple theory is obtained by setting the potential equal to zero in the branched polymer effective Hamiltonian. We show that this free field theory is not even classically conformally invariant. The reason for this is that the free Parisi-Sourlas theory is essentially a higher derivative Gaussian theory, and it is known that, in higher derivative theories, scale invariance does not imply conformal invariance \( zinn \). The lack of conformal invariance in the free theory does not of itself imply that the interacting theory of interest is not conformally invariant; it does however make it seem less likely.

(iii) Finally, we present numerical enumerations of of randomly branched polymers in the cone. If randomly branched polymers were conformally invariant, the exponent \( \theta(\alpha) \) in the cone would be simply related to the same exponent in the plane. More precisely, conformal invariance leads one to expect that \( \theta(\alpha) \) should be linear in \( \frac{1}{\alpha} \). Our enumerations indicate that this expected linear relation does not hold.

The plan of this paper is as follows. In section 2 we present the natural formulation of conformal invariance for randomly branched polymers, in section 3 we show that this formulation leads to inconsistencies, in section 4 we show that the free Parisi-Sourlas theory is not conformally invariant, in section 5 we present our results for the exponent \( \theta(\alpha) \) obtained by exact enumerations of branched polymers in the cone, and in section 6 we restate our conclusions.

II. Natural Formulation of Conformal Invariance

In this section we present the most natural formulation of conformal invariance for randomly branched polymers. The effective Hamiltonian of the field theory that describes the universal properties of randomly branched polymers\( lubensky \) can be written\( parisi \)

\[
H_{B.P.} = \int d^d r [\omega(-\nabla^2 \phi + V'(\phi)) + \frac{1}{2} \omega^2 + \bar{\psi}(-\nabla^2 + V''(\phi))\psi]. \tag{1}
\]

where \( V(\phi) = \frac{1}{2} \phi^2 + i^3_3 \phi^3 \). \( \phi \) and \( \omega \) are commuting scalars; \( \psi \) and \( \bar{\psi} \) anti-commute. \( H_{B.P.} \).
is invariant under the supersymmetry transformations \( \delta \phi = -a \epsilon_\mu x^\mu \psi \), \( \delta \omega = 2a \epsilon_\mu \partial^\mu \psi \), \( \delta \psi = 0 \), and \( \delta \bar{\psi} = a(\epsilon_\mu x^\mu \omega - 2 \epsilon_\mu \partial^\mu \phi) \) where \( a \) is an anti-commuting number and \( \epsilon_\mu \) an arbitrary vector.

The supersymmetry imposes relations between the correlation functions of the theory. For our purposes, the following identities are useful

\[
\langle \phi(r) \omega(0) \rangle = \langle \psi(r) \bar{\psi}(0) \rangle = 4 \frac{\partial}{\partial r^2} \langle \phi(r) \phi(0) \rangle.
\]

From these equations it follows that the scaling dimensions \( x_\phi \), \( x_\omega \), \( x_\psi \), and \( x_{\bar{\psi}} \) of the fields \( \phi \), \( \omega \), \( \psi \), and \( \bar{\psi} \) are related by

\[
x_\phi = x_\omega - 2 = x_\psi - 1 = x_{\bar{\psi}} - 1.
\]

\( \phi \) is therefore the lowest dimension field in \( H_{B.P.} \).

As stated in the introduction, the most natural formulation of conformal invariance for randomly branched polymers follows from the supposition that the most relevant fields in the randomly branched polymer problem appear in the effective Hamiltonian eq. 1. If this is the case, and it would be quite unusual if it were not, then it follows, by the argument given in the introduction, that the lowest dimension field, \( \phi \), is primary.

Given that \( \phi \) is primary, it follows from equation (2) that

\[
\omega = k \nabla^2 \phi + \bar{\omega}
\]

where \( \bar{\omega} \) is a sum of primary operators, and the constant \( k = -\frac{1}{x_\phi} \). This is because in a conformal field theory the two point function of operators belonging to different conformal families is always zero; since \( \langle \phi \omega \rangle \) is not zero, \( \omega \) must have a term that belongs to the conformal family of \( \phi \). Since the scaling dimension of \( \omega \) is \( x_\phi + 2 \), this term must be a descendent of \( \phi \) at level two, and the only descendent of \( \phi \) at level two which is also a scalar is \( \nabla^2 \phi \). Likewise, \( \bar{\omega} \) must be the sum of primary operators, since if \( \bar{\omega} \) has a term
which is not primary, that term must be the descendent of another operator with scaling
dimension $x_\omega - n$ where $n$ is a positive integer. But again there is no such operator in
the effective Hamiltonian and therefore, in accordance with our assumptions, none in the
theory (other than $\phi$; $\bar{\omega}$ is a commuting variable and therefore cannot be a first generation
descendent of $\psi$ or $\bar{\psi}$). It is important to note that conformal invariance together with eq.
2 imply that $\omega = k\nabla^2\phi + \bar{\omega}$ holds as an operator identity, and not just as an equality in
the plane.

The statement that $\phi$ is primary, together with eq. 4, constitute our formulation of
conformal invariance for branched polymers.

III. Consequences of Conformal Invariance

We now show that this natural formulation of conformal invariance for randomly
branched polymers leads to inconsistencies.

The correlation function $\langle \phi(r)\omega(0) \rangle$ is the generating function for the number $w_N(r)$
of randomly branched polymer configurations with $N$ bonds containing the sites 0 and

\[ \langle \phi(r)\omega(0) \rangle \sim \int_0^\infty dN e^{-\epsilon N} K_c^N w_N(r). \] (5)

Here $K$ is a fugacity for the number of bonds in the polymer, $K_c$ is the (non-universal)
value of the fugacity at which the average polymer size in the fixed fugacity ensemble
diverges, and $\epsilon$ measures the deviations of $K$ from $K_c$: $\epsilon = \frac{(K_c-K)}{K_c}$. Equation (5) holds
for $\epsilon$ small, and $\sim$ means that both sides of the equation have the same leading singular
behavior. In the scaling region we expect $\langle \phi(r)\phi(0) \rangle = \frac{f(r/\xi)}{r^{2x_\phi}}$, where the correlation length
$\xi \sim \epsilon^{-\nu}$. From equation (2), $\langle \phi(r)\omega(0) \rangle = \frac{g(r/\xi)}{r^{2x_\phi+2}}$. Using this scaling form and integrating
both sides of equation (5) with respect to $r$ and then taking the inverse Laplace transform
with respect to $\epsilon$ we have

\[ T_N^{unrooted} \sim \lambda^N N^{-\theta} \sim \left(\frac{1}{K_c}\right)^N N^{\nu(d-2x_\phi-2)-3} \] (6)
where $T_{unrooted}^N$ is the number of unrooted branched polymers of size $N$ in the plane, and 
$\sum_r w_N(r) = N^2 T_{unrooted}^N$. Setting $d = 2$ and using the approximate value of $\nu = 0.64$ derri da and the exact value of $\theta = 1$ parisi, we find $x_\phi = -1/\nu \simeq -1.5$. The field $\phi$ therefore has negative scaling dimension.

Using our proposal for conformal invariance, we can also calculate the exponent $\theta(\alpha)$, defined by $T_N(\alpha) \sim \lambda^N N^{-\theta(\alpha)+1}$, where $T_N(\alpha)$ is the the total number of randomly branched polymer configurations rooted near the apex of a cone with apex angle $\alpha$.\footnote{A cone can be thought of as a wedge in the plane, with opposite sides identified. By apex angle we mean the angle in the plane between the two sides of the wedge.} To calculate $\theta(\alpha)$ we need to know $\langle \phi \omega \rangle$ in the cone. The transformation $z \rightarrow \zeta = z^{\frac{\alpha}{2\pi}}$ maps the plane onto the cone. Since, by assumption, $\phi$ is primary, we have at criticality

$$
\langle \phi(z_1)\phi(z_2) \rangle_{\text{plane}} = \left| \frac{d\zeta}{dz}(z_1) \right|^{x_\phi} \left| \frac{d\zeta}{dz}(z_2) \right|^{x_\phi} \langle \phi(\zeta_1)\phi(\zeta_2) \rangle_{\text{cone}}
$$

so that,

$$
\langle \phi(\zeta_1)\phi(\zeta_2) \rangle_{\text{cone}} = \left( \frac{2\pi}{\alpha} \right)^{2x_\phi} |\zeta_1|^{\left( \frac{2\pi}{\alpha} - 1 \right)x_\phi} |\zeta_2|^{\left( \frac{2\pi}{\alpha} - 1 \right)x_\phi} \left( \frac{1}{|\zeta_1^{\alpha/2\pi} - \zeta_2^{\alpha/2\pi}|^{2x_\phi}} \right)
$$

where we have normalized $\phi$ so that $\langle \phi(r)\phi(0) \rangle_{\text{plane}} = \frac{1}{|r|^{2x_\phi}}$. From eq. 4 it then follows that

$$
\langle \phi(\zeta_1)\omega(\zeta_2) \rangle_{\text{cone}} = \langle \phi(\zeta_1)(k\nabla^2 \phi + \bar{\omega}) \rangle_{\text{cone}} = k\nabla^2_{\zeta_2} \langle \phi(\zeta_1)\phi(\zeta_2) \rangle_{\text{cone}}
$$

where we used the orthogonality of $\phi$ and $\bar{\omega}$ to obtain the second equality. Combining eqs. 8 and 9 then gives

$$
\langle \phi(\zeta_1)\omega(\zeta_2) \rangle_{\text{cone}} = k\nabla^2_{\zeta_2} \left( \left( \frac{2\pi}{\alpha} \right)^{2x_\phi} |\zeta_1|^{\left( \frac{2\pi}{\alpha} - 1 \right)x_\phi} |\zeta_2|^{\left( \frac{2\pi}{\alpha} - 1 \right)x_\phi} \left( \frac{1}{|\zeta_1^{\alpha/2\pi} - \zeta_2^{\alpha/2\pi}|^{2x_\phi}} \right) \right).
$$

The correlation function $\langle \phi(\zeta_1)\omega(\zeta_2) \rangle_{\text{cone}}$ is the the generating function for the total number of branched polymers in the cone. This statement is true regardless of whether we put $\phi$ in the bulk and $\omega$ near the apex or $\omega$ near the apex and $\phi$ in the bulk.
other hand, we see from equation (10) that the exponent \( \theta(\alpha) \) depends on whether \( \phi \) or \( \omega \) is near the apex. For if we put \( \omega \) in the bulk, that is, if we integrate over \( \zeta_2 \), holding \( \zeta_1 \) fixed and near the apex of the cone, we find

\[
\theta(\alpha) = 2 - \frac{2\pi}{\alpha}
\]  

while if integrate over \( \zeta_1 \) holding \( \zeta_2 \) fixed we find something different, namely

\[
\theta(\alpha) = 2 - \frac{2\pi}{\alpha} - 2\nu
\]  

for \( \alpha < 2\pi \). Since we obtain different answers for \( \theta(\alpha) \) depending on whether we put \( \phi \) or \( \omega \) near the apex, equation (10) cannot be correct: the correlation function \( \langle \phi \omega \rangle \) in the cone is not equal to the conformal transformation of \( \langle \phi \omega \rangle \) in the plane, under the assumption that \( \phi \) transforms as a primary operator. There are in fact two additional problems with equations (11) and (12): (1) the the coefficient of \( \frac{2\pi}{\alpha} \) is negative, implying that the smaller the apex angle, the greater the number of rooted polymer configurations, which is impossible, and (2) the exponent \( \theta(\alpha) \) in equation (12) jumps discontinuously from one at \( \alpha = 2\pi \) to about \( \frac{1}{3} \) for \( \alpha \) just smaller than \( 2\pi \). Such a jump seems unlikely, and in any case is in the wrong direction (\( \theta(\alpha) \) should increase as \( \alpha \) decreases).

A final point must be considered, namely that the cone is not conformally related to the plane, but to the punctured plane. In most theories, the distinction between the plane and the punctured plane is irrelevant. A puncture in the continuum theory corresponds to a finite size hole in the lattice theory. The presence of the hole changes the interactions between sites variables, and so corresponds to the presence of an energy like operator. The energy operator on the lattice can be written as a sum of operators in the continuum theory. In general, one expects every operator in the continuum theory to appear in this sum unless it is forbidden to do so by some symmetry. The lowest dimension operator in the sum is the most relevant, and dominates the large distance physics. In most theories the
lowest dimension operator with the same symmetry as the energy operator is the identity operator, so a defect in the lattice does not change the large distance behavior of correlation functions. But in the theory we are considering (and in the Yang-Lee theory) the operator $\phi$ has the same symmetry as the energy operator and has a smaller scaling dimension than the identity operator. If, therefore, $\phi$ appears in the sum (and there is no symmetry which forbids it from doing so) the large distance behavior of correlation functions on a lattice with a single defect would be different from the behavior of correlation functions on the perfect lattice. In the continuum this means that correlation functions in the plane would not be the same as those in the punctured plane. The exponent $\theta$ in the punctured plane, or on a lattice with a finite size hole, would therefore be different from $\theta$ in the plane. Since correlation functions in the cone are conformally related to those in the punctured plane, they would also be modified, as would predictions for $\theta(\alpha)$.

This proposal, while raising interesting questions, has a number of problems. First, it is difficult to see how the presence of a finite size hole on the lattice could alter the exponent $\theta$. To check this, we have enumerated all branched polymer configurations (rooted near the origin) with twelve and fewer bonds on a square lattice with the site at the origin removed. Branched polymer configurations containing the origin were disallowed. The ratio of the number of polymer configurations in the presence of the puncture to the number in the plane appears to converge to a constant near .4 as the number of bonds in the polymer increases. This indicates that that the defect does not change $\theta$. Second, we expect that the presence of the puncture should not affect the boundary conditions at infinity. Thus the state at infinity should still be the $SL(2, C)$ invariant vacuum. In the Hilbert space formulation, the state corresponding to the operator $\phi$ is an energy eigenstate, and is orthogonal to the $SL(2, C)$ invariant vacuum. Hence the partition function $Z$ should not be affected by the puncture. Correlation functions, however, will now have an extra $\phi(0)$ inserted in them. Since we are interested in two point functions, the correlation functions
with the extra insertion of $\phi(0)$ are three point functions and are computable. Performing these computations, one can show explicitly that the correlation function $\langle \phi \omega \rangle$ in the punctured plane still depends on whether $\phi$ or $\omega$ is put at the origin.

IV. Free Parisi-Sourlas Theory Not Conformally Invariant

In order to gain insight into what might go wrong with conformal invariance in the branched polymer theory, we consider here the simplest theory with a Parisi-Sourlas supersymmetry. This is obtained by setting the potential $V(\phi)$ in equation (1) equal to zero. The commuting fields $\phi$ and $\omega$ then decouple from the anti-commuting fields $\psi$ and $\bar{\psi}$. Moreover, the fermion part of the free Hamiltonian $H_F$ has the form of an ordinary Gaussian model, and is conformally invariant with $\psi$ and $\bar{\psi}$ transforming as primary fields with conformal dimension zero. The remaining part of the free theory is

$$H_F = \int d^d x [\omega(-\nabla^2 \phi) + \frac{1}{2} \omega^2].$$  \hfill (13)

or, integrating out the field $\omega$,

$$H_F = \int d^2 r (\nabla^2 \phi)^2$$  \hfill (14)

a higher derivative Gaussian theory. For $H_F$ to be scale invariant, $\phi$ must have scaling dimension $-1$, or conformal dimension $-1/2$. Let us assume, as we did for the branched polymers, that $\phi$ transforms under conformal transformations as a primary operator. Then, under the transformation $z \rightarrow z + \epsilon(z)$,

$$\phi(z, \bar{z}) \rightarrow (1 - \frac{d\epsilon}{dz})^{-\frac{1}{2}} (1 - \frac{d\bar{\epsilon}}{d\bar{z}})^{-\frac{1}{2}} \phi(z, \bar{z}).$$  \hfill (15)

Taking into account the transformation of the measure $d^2 r$ and the derivatives $\partial_\mu$, we find the variation in $H_F$

$$\delta H_F = \int d^2 r \left[ \frac{d^2 \epsilon(z)}{dz^2} (\partial \bar{\phi})(\partial \bar{\phi}) + \text{c.c.} \right].$$  \hfill (16)

If $\epsilon(z)$ is constant, corresponding to a translation, or linear in $z$, corresponding to a dilatation or rotation, $\delta H_F$ vanishes identically, while if $\epsilon(z)$ is quadratic in $z$, the integrand is
a total derivative, and so $\delta H_F$ vanishes in this case as well. These three transformations generate the special conformal group, $SL(2, C)$. On the other hand, the integrand is not a total derivative for more general conformal transformations. Thus, assuming that $\phi$ is primary, the action in equation (14) is invariant under the group $SL(2, C)$, transformations mapping the plane to itself, but not invariant under the full conformal group in two dimensions. Furthermore, there is no way to transform $\phi$ so that $\delta H_F = 0$. To see this, suppose that under $z \to z + \epsilon(z)$
\[
\phi(z, \bar{z}) \to (1 - \frac{d\epsilon}{dz})^{-\frac{1}{2}}(1 - \frac{d\bar{\epsilon}}{d\bar{z}})^{-\frac{1}{2}}\phi(z, \bar{z}) + \delta\phi(z, \bar{z}).
\]
with $\delta\phi$ linear in $\phi$ and $\epsilon$. Then
\[
\delta H_F = \int d^2 r [(\frac{d^2 \epsilon(z)}{dz^2}) (\partial\bar{\partial}\phi) - 2(\partial\bar{\partial}\phi)(\partial\bar{\partial}\delta\phi)] + c.c. \tag{18}
\]
In order for $\delta H_F$ to vanish, we need to choose $\delta\phi$ in such a way that the integrand is a total derivative. Considering the first term in equation 18 we see that this is only possible if $\delta\phi$ is of the form $a\epsilon(z)\partial\phi + b\partial\epsilon(z)\phi$. But we cannot add a term of this kind to the transformation law for $\phi$, because this has already been fixed by translational and scale invariance (that is to say, if we add term of this form to transformation law for $\phi$, $H_F$ will not be invariant under the special conformal group). Thus there is no way to transform $\phi$ so that $\delta H_F = 0$ under general conformal transformations. Hence the simplest theory with a Parisi-Sourlas supersymmetry is not conformally invariant even at the classical level. This may account for the problems encountered in the more complicated theory that describes branched polymers.

V. Numerical Results

If randomly branched polymers were conformally invariant, the form of the transformation law eq.7 for primary operators leads one to expect a linear relationship between the exponent $\theta(\alpha)$ and the reciprocal of the cone angle $\alpha$ for polymers confined to a cone.
A similar relationship would be expected for randomly branched polymers confined to a wedge but was not found in a previous analysis of exact enumeration data.\textsuperscript{DBL85}

We have analysed exact enumeration data for lattice trees confined to a cone. The cone is formed by applying cyclic boundary conditions to a wedge, cut from a square lattice, in such a way that corresponding lattice sites on the two boundaries of the wedge become identified as a single site on the surface of the cone. The number of (weak) embeddings of trees rooted at the apex of the cone were enumerated for trees with up to 16 vertices.

We assume that the number of trees on the surface of the cone diverges as

\[ T_N(\alpha) \sim \lambda^N N^{-\theta(\alpha)+1} \]  \hspace{1cm} (19)

\( \lambda \) is the growth constant for lattice trees on the square lattice and is the same for trees confined to a wedge or cone as it is for trees in the bulk lattice. The \( y \)th moment of the generating function for \( t_N(\alpha) \) will therefore have critical behaviour described by

\[ G^{(y)}(x) = \sum_N T_N x^N N^y \sim (x_c - x)^{2+y-\theta} \]  \hspace{1cm} (20)

From previous work\textsuperscript{DLZtree}, the value of \( x_c \) is known to be

\[ x_c = 1/\lambda = 0.19445^{+0.00001}_{-0.00002} \]  \hspace{1cm} (21)

To obtain estimates of \( \theta(\alpha) \) a Baker-Hunter confluent singularity analysis\textsuperscript{BH} was applied to the exact enumeration data at each of the cone angles considered. This type of analysis was used since we expect the presence of confluences in the generating function for lattice trees\textsuperscript{MADRAS}. The initial analysis used the central estimate of \( x_c \) given above and, since \( \theta(\alpha) \) is bigger than unity for the smaller angles, the first, second and third moments of the generating function were analysed\textsuperscript{DBL85}. The results are shown in table 1. As this type of analysis may be sensitive to the value of \( x_c \) assumed, we also performed an analysis which eliminates the need to specify \( x_c \). To do this the ratio of the number of embeddings in the
bulk lattice $T_N$ and the corresponding number of embeddings in a cone with wedge angle $\alpha$, $T_N(\alpha)$, was formed

$$r_N = T_N/T_N(\alpha) \sim N^{\theta(\alpha) - \theta}$$

(22)

The generating function for these ratios has a critical behaviour given by

$$G_r(x) = \sum_N r_N x^N \sim (1 - x)^{1+\theta(\alpha) - \theta}.$$  

(22)

This has the advantage that the critical point ($x_c = 1$) is known exactly and it is not necessary to use higher moments as the exponent is always greater than 1. The resulting estimates of $\theta(\alpha)$ are shown in column 4 of table 1.

In most cases the spread in the central estimates of $\theta(\alpha)$ in the columns of table 1, for a given value of $\alpha$, is comparable with the error estimate in a single column obtained by considering the variation in the estimate from different Padé approximants to the Baker-Hunter auxiliary function. In order to make a comparison with the expected linear relation we take the overall estimate of $\theta(\alpha)$ to have a central value given by an average, over the columns of table 1, of the central estimates and error bounds such that all of the central estimates fall within these bounds. These values are plotted against $1/\alpha$ in figure 1. Even allowing for some degree of subjectivity in the value of the error bounds the estimates of $\theta(\alpha)$ are clearly not consistent with a linear relationship. Consequently, although we can not altogether rule out short series effects as the reason for this deviation from a linear relation, the results reported here together with the earlier work on trees confined to a wedge lend support to the conclusion that randomly branched polymers are not conformally invariant.

VI. Conclusions

In this paper we have presented three arguments against the conformal invariance, at criticality, of the supersymmetric field theory that describes randomly branched polymers.
We showed first that the most natural formulation of conformal invariance leads to inconsistencies, then that the free Parisi-Sourlas Hamiltonian is not even classically conformally invariant, and finally, from numerical enumerations, that $\theta(\alpha)$ is not linear in the inverse cone angle $\alpha$. None of these arguments constitutes a proof; however taken together they strongly suggest that randomly branched polymers are not conformally invariant. This is interesting, because every other geometrical critical system with which we are familiar is described by a conformal field theory. The reason for problems with conformal invariance may be related to non-unitarity of the model. Since the theory is non-unitary, the Hamiltonian in the Hilbert space formulation of the problem is not Hermitian, and therefore need not be diagonalizable. If $H$ is not diagonalizable, the usual arrangement of operators in conformal field theories in highest weight representations of the Virasoro algebra $^{belavin}$, would be impossible.

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**TABLE 1** Estimates of $\theta(\alpha)$ for various cone angles $\alpha$. Columns labelled 1, 2 and 3 tabulate results obtained from the first, second and third moment of the generating function respectively. The column labelled 4 tabulates the results obtained by taking the ratio of the number of trees in the bulk lattice to the number on the cone.

| Angle | 1       | 2       | 3       | 4       |
|-------|---------|---------|---------|---------|
| 90°   | 2.091   | 2.134   | 2.126   | 2.114   |
|       | ±0.017  | ±0.025  | ±0.017  | ±0.014  |
| 127°  | 1.98    | 1.96    | 1.85    | 1.852   |
|       | ±0.11   | ±0.09   | ±0.03   | ±0.007  |
| 143°  | 1.85    | 1.83    | 1.78    | 1.73    |
|       | ±0.06   | ±0.05   | ±0.10   | ±0.09   |
| 180°  | 1.55    | 1.551   | 1.550   | 1.553   |
|       | ±0.02   | ±0.005  | +0.005  | ±0.010  |
| 233°  | 1.418   | 1.4080  | 1.39    | 1.39    |
|       | ±0.010  | ±0.007  | ±0.09   | ±0.04   |
| 270°  | 1.2539  | 1.250   | 1.252   | 1.236   |
|       | ±0.0012 | ±0.006  | ±0.004  | ±0.014  |
FIGURE 1 Overall estimates of $\theta(\alpha)$ plotted against $1/\alpha$. The bulk value of $\theta = 1$ has been included as the value for $\alpha = 360^\circ$. 