Counting Dirac braid relators and hyperelliptic Lefschetz fibrations

Hisaaki Endo and Seiichi Kamada

Abstract

We define an invariant $w$ for hyperelliptic Lefschetz fibrations over closed oriented surfaces, which counts the number of Dirac braids included intrinsically in the monodromy, by using chart description introduced by the second author. As an application, we prove that two hyperelliptic Lefschetz fibrations of genus $g$ over a given base space are stably isomorphic if and only if they have the same numbers of singular fibers of each type and they have the same value of $w$ if $g$ is odd. We also give examples of pair of hyperelliptic Lefschetz fibrations with the same numbers of singular fibers of each type which are not stably isomorphic.

1. Introduction

Since the seminal works of Donaldson [7] and Gompf [18] in 1998, Lefschetz fibrations have been investigated from various viewpoints. Several kinds of equivalence classes of a Lefschetz fibration such as isomorphism class, diffeomorphism type, homeomorphism type, and homotopy type, and their relations have been studied by many authors. The above work of Donaldson together with a theorem of Gompf [17] implies that there exists a Lefschetz fibration with prescribed fundamental group, which means the classification of all homotopy types of Lefschetz fibrations is not possible in principle. Baykur [4] proved that any symplectic 4-manifold which is not a rational or ruled surface, after sufficiently many blow-ups, admits an arbitrary number of nonsingular Lefschetz fibrations of the same genus (see also Park and Yun [32, 33]). It clarified a significant difference between isomorphism classes and diffeomorphism types. Two Lefschetz fibrations of the same genus over a given base space are called stably isomorphic if they become isomorphic after fiber-summed with the same number of copies of a ‘universal’ Lefschetz fibration. Auroux [1] obtained a sufficient condition for two Lefschetz fibrations over the 2-sphere each of which admits a section to be stably isomorphic. Hasegawa, Tanaka, and the authors [10] relaxed Auroux’s condition to obtain a necessary and sufficient condition for two Lefschetz fibrations over a closed surface to be stably isomorphic. Thus the stable isomorphism problem, in contrast to the isomorphism one, turned out to be within reach.

Hyperelliptic Lefschetz fibrations are Lefschetz fibrations for which the image of the monodromy is included in the hyperelliptic mapping class group. They are considered to be a natural generalization of elliptic surfaces because several properties are common to these two kinds of fibrations. For instance, many of fibrations can be obtained by branched covering construction, the signature of a fibration localizes on the singular fibers, typical fibrations are used as building blocks for constructions of more complicated fibrations and 4-manifolds.
etc. (see Siebert and Tian [35], Fuller [15], Endo [8], and Endo and Nagami [12]). Although the classification of isomorphism classes of irreducible hyperelliptic Lefschetz fibrations was partially established in genus two case by Siebert and Tian [36], it seems there is little prospect for a complete classification of isomorphism classes of hyperelliptic Lefschetz fibrations. In fact, there are infinitely many distinct Lefschetz fibrations of genus two with the same numbers of singular fibers of each type (see Baykur and Korkmaz [5], cf. Ozbagci and Stipsicz [30] and Korkmaz [27]).

In the present paper, we define a $\mathbb{Z}_2$-valued invariant $w$ for hyperelliptic Lefschetz fibrations over closed oriented surfaces by using chart description introduced by the second author [21] (Definition 5.1 and Proposition 5.3). This invariant counts the number of ‘Dirac braids’ included intrinsically in the monodromy representation of a hyperelliptic Lefschetz fibration of genus $g$. The Dirac braid is a full twist on all strands in the $(2g+2)$-string braid group $B_{2g+2}(S^2)$ of a $2$-sphere, which corresponds to a Dehn twist around all marked points in the mapping class group $\mathcal{M}_{2g+2}$ of a $2$-sphere with $2g+2$ marked points, and to a maximal chain relator in the hyperelliptic mapping class group $\mathcal{H}_g$ of a connected closed oriented surface of genus $g$, under natural homomorphisms (see §3.1 for details). The relator $r_{4}$ in $\mathcal{M}_{2g+2}$ corresponding to the Dirac braid is called the Dirac braid relator (see §3.2). The proof of the invariance of $w$ under chart moves is the most technical part of this paper (Propositions 4.2 and 4.4). We expect the invariant $w$ to coincide with a $\mathbb{Z}_2$-valued invariant for hyperelliptic Lefschetz fibrations of odd genus mentioned by Auroux and Smith [3] (Remark 5.8). Employing the invariant $w$, we prove that two hyperelliptic Lefschetz fibrations of genus $g$ over a given base space are stably isomorphic if and only if they have the same numbers of singular fibers of each type and they have the same value of $w$ if $g$ is odd (Theorem 5.6). Two hyperelliptic Lefschetz fibrations of genus $g$ over a given base space are called stably isomorphic if they become isomorphic after fiber-summed with the same number of copies of a hyperelliptic Lefschetz fibration on $\mathbb{CP}^2\#(4g+5)\mathbb{CP}^2$, which is a natural generalization of the rational elliptic surface $E(1)$ (Definition 5.4). We also give examples of pair of hyperelliptic Lefschetz fibrations with the same numbers of singular fibers of each type which are not stably isomorphic (Examples 6.3 and 6.4).

2. Lefschetz fibrations and hyperelliptic structures

In this section, we review a precise definition and basic properties of Lefschetz fibrations and introduce a notion of hyperellipticity for Lefschetz fibrations. See also Matsumoto [29], Gompf and Stipsicz [18], and Endo and Kamada [11].

2.1. Lefschetz fibrations and their monodromies

We begin with a precise definition of Lefschetz fibration. Let $\Sigma_g$ be a connected closed oriented surface of genus $g$.

**Definition 2.1.** Let $M$ and $B$ be connected closed oriented smooth 4-manifold and 2-manifold, respectively. A smooth map $f: M \to B$ is called an (achiral) **Lefschetz fibration** of genus $g$ if it satisfies the following conditions:

(i) the set $\Delta \subset B$ of critical values of $f$ is finite and $f$ is a smooth fiber bundle over $B - \Delta$ with fiber $\Sigma_g$;

(ii) for each $b \in \Delta$, there exists a unique critical point $p$ in the singular fiber $F_b := f^{-1}(b)$ such that $f$ is locally written as $f(z_1, z_2) = z_1z_2$ or $\bar{z}_1z_2$ with respect to some local complex coordinates around $p$ and $b$ which are compatible with orientations of $M$ and $B$;

(iii) no fiber contains a $(\pm 1)$-sphere.
We call $M$ the total space, $B$ the base space, and $f$ the projection. We call $p$ a critical point of positive type (respectively of negative type) and $F_0$ a singular fiber of positive type (respectively of negative type) if $f$ is locally written as $f(z_1, z_2) = z_1 z_2$ (respectively $f(z_1, z_2) = \bar{z}_1 \bar{z}_2$) in (ii).

For a regular value $b \in B$ of $f$, $f^{-1}(b)$ is often called a general fiber.

**Definition 2.2.** Let $f: M \to B$ and $f': M' \to B$ be Lefschetz fibrations of genus $g$ over the same base space $B$. We say that $f$ is isomorphic to $f'$ if there exist orientation preserving diffeomorphisms $H: M \to M'$ and $h: B \to B$ which satisfy $f' \circ H = h \circ f$. If we can choose such an $h$ isotopic to the identity relative to a given base point $b_0 \in B$, we say that $f$ is strictly isomorphic to $f'$.

Let $\mathcal{M}_g$ be the mapping class group of $\Sigma_g$, namely the group of all isotopy classes of orientation preserving diffeomorphisms of $\Sigma_g$. We assume that $\mathcal{M}_g$ acts on the right: the symbol $\varphi \psi$ means that we apply $\psi$ first and then $\varphi$ for $\varphi, \psi \in \mathcal{M}_g$.

Let $f: M \to B$ be a Lefschetz fibration of genus $g$ as in Definition 2.1. Take a base point $b_0 \in B - \Delta$ and an orientation preserving diffeomorphism $\Phi: \Sigma_g \to F_0 := f^{-1}(b_0)$. Since $f$ restricted over $B - \Delta$ is a smooth fiber bundle with fiber $\Sigma_g$, we can define a homomorphism $\rho: \pi_1(B - \Delta, b_0) \to \mathcal{M}_g$

called the monodromy representation of $f$ with respect to $\Phi$ (see Matsumoto [29, §2]). Let $\gamma$ be the loop based at $b_0$ consisting of the boundary circle of a small disk neighborhood of $b \in \Delta$ oriented counterclockwise and a simple path connecting a point on the circle to $b_0$ in $B - \Delta$. It is known that $\rho(\gamma)$ is a Dehn twist along some essential simple closed curve $c$ on $\Sigma_g$. The curve $c$ is called the vanishing cycle of the critical point $p$ on $f^{-1}(b)$. If $p$ is of positive type (respectively of negative type), then the Dehn twist is right-handed (respectively left-handed).

A singular fiber is said to be of type I if the vanishing cycle is non-separating and of type $\Pi_h$ for $h = 1, \ldots, [g/2]$ if the vanishing cycle is separating and it bounds a genus-$h$ subsurface of $\Sigma_g$. A singular fiber is said to be of type $\Gamma^+$ (respectively type $\Gamma^-$, type $\Pi^+_h$, type $\Pi^-_h$) if it is of type I and of positive type (respectively of type I and of negative type, of type $\Pi_h$ and of positive type, of type $\Pi_h$ and of negative type). We denote by $n^+_0(f)$, $n^-_0(f)$, $n^+_h(f)$, and $n^-_h(f)$, the numbers of singular fibers of $f$ of type $\Gamma^+$, $\Gamma^-$, $\Pi^+_h$, and $\Pi^-_h$, respectively. A Lefschetz fibration is called irreducible if every singular fiber is of type I. A Lefschetz fibration is called chiral if every singular fiber is of positive type.

Suppose that the cardinality of $\Delta$ is equal to $n$. A system $A = (A_1, \ldots, A_n)$ of arcs on $B$ is called a Hurwitz arc system for $\Delta$ with base point $b_0$ if each $A_i$ is an embedded arc connecting $b_0$ with a point of $\Delta$ in $B$ such that $A_i \cap A_j = \{b_0\}$ for $i \neq j$, and they appear in this order around $b_0$ (see Kamada [22]). When $B$ is a 2-sphere, the system $A$ determines a system of generators of $\pi_1(B - \Delta, b_0)$, say $(a_1, \ldots, a_n)$. We call $(\rho(a_1), \ldots, \rho(a_n))$ a Hurwitz system of $f$. It is easy to see that $\rho$ is determined by $(\rho(a_1), \ldots, \rho(a_n))$.

2.2. Hyperelliptic structures

We next introduce a notion of hyperellipticity for Lefschetz fibrations (cf. Endo [8, Definition 4.2]).

Let $\iota$ be the isotopy class of a hyperelliptic involution $I$, an involution on $\Sigma_g$ with $2g + 2$ fixed points, and $\mathcal{H}_g$ the centralizer of $\iota$ in $\mathcal{M}_g$, which is called the hyperelliptic mapping class group of $\Sigma_g$ with respect to $\iota$. Let $\zeta_1, \ldots, \zeta_{2g+1}, \sigma_1, \ldots, \sigma_{[g/2]}$ be right-handed Dehn twists along simple closed curves $c_1, \ldots, c_{2g+1}, s_1, \ldots, s_{[g/2]}$ on $\Sigma_g$ depicted in Figure 1, respectively. We suppose that $c_1, \ldots, c_{2g+1}, s_1, \ldots, s_{[g/2]}$ are invariant under $I$ and thus $\zeta_1, \ldots, \zeta_{2g+1}, \sigma_1, \ldots, \sigma_{[g/2]}$ belong to $\mathcal{H}_g$. 


**Definition 2.3.** Let \( f: M \to B \) be a Lefschetz fibration of genus \( g \), \( \Delta \) the set of critical values of \( f \), and \( b_0 \in B - \Delta \) a base point. Take an orientation preserving diffeomorphism \( \Phi: \Sigma_g \to f^{-1}(b_0) \) and consider the monodromy representation \( \rho: \pi_1(B - \Delta, b_0) \to \mathcal{M}_g \) of \( f \) with respect to \( \Phi \). The pair \( (f, \Phi) \) is called a hyperelliptic Lefschetz fibration (and \( \Phi \) is called a hyperelliptic structure on \( f \)) if the image of \( \rho \) is included in \( \mathcal{H}_g \). Let \( \Phi, \Phi': \Sigma_g \to f^{-1}(b_0) \) be hyperelliptic structures on \( f \). We say that \( \Phi \) is equivalent to \( \Phi' \) if the isotopy class of \( (\Phi')^{-1} \circ \Phi \) belongs to \( \mathcal{H}_g \).

**Definition 2.4.** Let \( f: M \to B \) and \( f': M' \to B \) be Lefschetz fibrations of genus \( g \) over the same base space \( B \), \( \Delta \) and \( \Delta' \) the sets of critical values of \( f \) and \( f' \), and \( b_0 \in B - \Delta \) and \( b'_0 \in B - \Delta' \) base points for \( f \) and \( f' \), respectively. Take orientation preserving diffeomorphisms \( \Phi: \Sigma_g \to f^{-1}(b_0) \) and \( \Phi': \Sigma_g \to f'^{-1}(b'_0) \) and suppose that the pairs \( (f, \Phi) \) and \( (f', \Phi') \) are hyperelliptic Lefschetz fibrations. We say that \( (f, \Phi) \) is \( \mathcal{H} \)-isomorphic to \( (f', \Phi') \) if there exist orientation preserving diffeomorphisms \( H: M \to M' \) and \( h: B \to B \) which satisfy the following conditions: (i) \( f' \circ H = h \circ f \); (ii) \( h(b_0) = b'_0 \); (iii) \( H|_{f^{-1}(b_0)} \circ \Phi \) is a hyperelliptic structure on \( f' \) equivalent to \( \Phi' \). If we can choose such an \( h \) isotopic to the identity relative to a given base point \( b_0 \), we say that \( (f, \Phi) \) is strictly \( \mathcal{H} \)-isomorphic to \( (f', \Phi') \). If \( (f, \Phi) \) is \( \mathcal{H} \)-isomorphic (respectively strictly \( \mathcal{H} \)-isomorphic) to \( (f', \Phi') \), then \( f \) is isomorphic (respectively strictly isomorphic) to \( f' \).

The next lemmas easily follow from theorems of Matsumoto [29, Theorems 2.4 and 2.6] (see also Kas [25, Theorem 2.4]).

**Lemma 2.5.** Suppose that \( g \) is greater than 1. Let \( (f, \Phi) \) and \( (f', \Phi') \) be hyperelliptic Lefschetz fibrations of genus \( g \) as in Definition 2.4, and \( \rho: \pi_1(B - \Delta, b_0) \to \mathcal{H}_g \) and \( \rho': \pi_1(B - \Delta', b'_0) \to \mathcal{H}_g \) the monodromy representations of \( f \) and \( f' \) with respect to \( \Phi \) and \( \Phi' \), respectively.

1. \( (f, \Phi) \) is \( \mathcal{H} \)-isomorphic to \( (f', \Phi') \) if and only if there exists an orientation preserving diffeomorphism \( h: B \to B \) and an element \( \alpha \) of \( \mathcal{H}_g \) which satisfies the following conditions: (i) \( h(\Delta) = \Delta' \); (ii) \( h(b_0) = b'_0 \); (iii) \( \rho'(h_#(\gamma)) = \alpha^{-1} \rho(\gamma) \alpha \) for every \( \gamma \in \pi_1(B - \Delta, b_0) \).

2. \( (f, \Phi) \) is strictly \( \mathcal{H} \)-isomorphic to \( (f', \Phi') \) if and only if there exists an orientation preserving diffeomorphism \( h: B \to B \) and an element \( \alpha \) of \( \mathcal{H}_g \), which satisfies the conditions (i), (ii), (iii), and (iv) \( h \), is isotopic to the identity relative to a given base point \( b_0 \).

**Lemma 2.6.** Suppose that \( g \) is greater than 1. Let \( \rho: \pi_1(B - \Delta, b_0) \to \mathcal{H}_g \) be a homomorphism and \( A = (A_1, \ldots, A_n) \) a Hurwitz arc system for \( \Delta \) with base point \( b_0 \). We assume that \( \rho(a_1), \ldots, \rho(a_n) \) are Dehn twists along simple closed curves on \( \Sigma_g \) for the system of generators...
(a_1, \ldots, a_n) of \pi_1(B - \Delta, b_0) determined by \mathcal{A}. Then there exists a hyperelliptic Lefschetz fibration \((f, \Phi)\) of genus \(g\) as in Definition 2.3 with monodromy representation \(\rho\).

For any \((\alpha_1, \ldots, \alpha_n) \in (\mathcal{H}_g)^n\) such that each \(\alpha_i\) is a Dehn twist, there exists a hyperelliptic Lefschetz fibration \((f, \Phi)\) of genus \(g\) over \(S^2\) with Hurwitz system \((\alpha_1, \ldots, \alpha_n)\) by Lemma 2.6. We call such \((f, \Phi)\) a hyperelliptic Lefschetz fibration determined by \((\alpha_1, \ldots, \alpha_n)\).

Two isomorphic hyperelliptic Lefschetz fibrations need not be \(\mathcal{H}\)-isomorphic. We give a sufficient condition for isomorphic hyperelliptic Lefschetz fibrations to be \(\mathcal{H}\)-isomorphic.

Let \((f, \Phi)\) and \((f', \Phi')\) be hyperelliptic Lefschetz fibrations of genus \(g\) as in Definition 2.4, and \(\rho\) and \(\rho'\) their monodromy representations as in Lemma 2.5.

**Proposition 2.7.** Suppose that the image of \(\rho\) is \(\mathcal{H}_g\). Then \(f\) is isomorphic to \(f'\) if and only if \((f, \Phi)\) is \(\mathcal{H}\)-isomorphic to \((f', \Phi')\).

**Proof.** The ‘if’ part is obvious. We show the ‘only if’ part. Since \(f\) is isomorphic to \(f'\), there exists an orientation preserving diffeomorphism \(h: B \to B\) and an element \(\alpha\) of \(\mathcal{M}_g\) which satisfies the following conditions (see Matsumoto [29, Theorem 2.4]): (i) \(h(\Delta) = \Delta'\); (ii) \(h(b_0) = b_0'\); (iii) \(\rho'(h_\#(\gamma)) = \alpha^{-1}\rho(\gamma)\alpha\) for every \(\gamma \in \pi_1(B - \Delta, b_0)\). We will show that \(\alpha\) belongs to \(\mathcal{H}_g\), which implies that \((f, \Phi)\) is \(\mathcal{H}\)-isomorphic to \((f', \Phi')\) by Lemma 2.5.

For any \(i = 1, \ldots, 2g + 1\), there exists an element \(\gamma_i\) of \(\pi_1(B - \Delta, b_0)\) such that \(\rho(\gamma_i) = \zeta_i\) because the image of \(\rho\) is \(\mathcal{H}_g\). Then we have

\[
\alpha^{-1}\zeta_i\alpha = \alpha^{-1}\rho(\gamma_i)\alpha = \rho'(h_\#(\gamma_i)) \in \mathcal{H}_g
\]

from (iii). This implies that \(\alpha^{-1}\zeta_i\alpha = \nu^{-1}\alpha^{-1}\zeta_i\alpha\) and thus \((c_i)A\) is isotopic to \(((c_i)A)I\) by [14, Fact 3.6], where \(A\) is an orientation preserving diffeomorphism on \(\Sigma_g\) representing \(\alpha\). Since \(c_i\) is invariant under \(I\), \((c_i)(AIA^{-1}I^{-1})\) is isotopic to \(c_i\). Hence there exists a diffeomorphism \(F\) on \(\Sigma_g\) isotopic to the identity such that \((c_i)(AIA^{-1}I^{-1}F) = c_i\) for every \(i\) by virtue of [14, Lemma 2.9]. Since \(\Sigma_g - (c_1 \cup \cdots \cup c_{2g+1})\) is a disjoint union of two open disks, \(AIA^{-1}I^{-1}F\) is isotopic to either the identity or \(I\) by [14, Proposition 2.8]. If \(AIA^{-1}I^{-1}F\) is isotopic to \(I\), then we obtain \(\nu = 1\), which is a contradiction. Therefore \(AIA^{-1}I^{-1}F\) is isotopic to the identity and we have \(\alpha \in \mathcal{H}_g\). This completes the proof. \(\square\)

3. **Chart descriptions**

In this section, we introduce chart descriptions for hyperelliptic Lefschetz fibrations by employing finite presentations of hyperelliptic mapping class groups and two other groups. General theories of charts for presentations of groups were developed independently by Kamada [23] and Hasegawa [19]. We use the terminology of chart description in Kamada [23].

3.1. **Three finite presentations**

We first review finite presentations of three groups related with hyperelliptic Lefschetz fibrations.

Fadell and Van Buskirk [13] proved that the braid group of a 2-sphere is just the braid group of a 2-disk with a single additional relation.

**Theorem 3.1** (Fadell–Van Buskirk [13]). Suppose that \(g\) is positive. The \((2g + 2)\)-string braid group \(B_{2g+2}(S^2)\) of a 2-sphere is generated by elements \(x_1, x_2, \ldots, x_{2g+1}\) and has defining relations:

1. \(x_ix_j = x_jx_i\) (\(1 \leq i < j - 1 \leq 2g\));
(2) \( x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \) \((i = 1, \ldots, 2g)\);
(3) \( x_1 x_2 \cdots x_{2g+1} x_{2g+1} \cdots x_2 x_1 = 1. \)

Magnus [28] obtained a presentation of the mapping class group of a 2-sphere with marked points, which is essentially the same as the following one.

**Theorem 3.2** (Magnus [28], cf. Birman–Hilden [6]). Suppose that \( g \) is positive. The mapping class group \( \mathcal{M}_{0, 2g+2} \) of a 2-sphere with \( 2g + 2 \) marked points is generated by elements \( \xi_1, \xi_2, \ldots, \xi_{2g+1} \) and has defining relations:

1. \( \xi_i \xi_j = \xi_j \xi_i \) \((1 \leq i < j - 1 \leq 2g)\);
2. \( \xi_i \xi_{i+1} \xi_i = \xi_{i+1} \xi_i \xi_{i+1} \) \((i = 1, \ldots, 2g)\);
3. \( \xi_1 \xi_2 \cdots \xi_{2g+1} \xi_{2g+1} \cdots \xi_2 \xi_1 = 1 \);
4. \( (\xi_1 \xi_2 \cdots \xi_{2g+1})^{2g+2} = 1 \).

Birman and Hilden [6] considered the mapping class group of the orbit space of the involution \( I \) on \( \Sigma_g \), which is a 2-sphere with \( 2g + 2 \) marked points, to obtain a presentation of the hyperelliptic mapping class group of \( \Sigma_g \).

**Theorem 3.3** (Birman–Hilden [6]). Suppose that \( g \) is positive. The hyperelliptic mapping class group \( \mathcal{H}_g \) of \( \Sigma_g \) is generated by elements \( \zeta_1, \zeta_2, \ldots, \zeta_{2g+1} \) and has defining relations:

1. \( \zeta_i \zeta_j = \zeta_j \zeta_i \) \((1 \leq i < j - 1 \leq 2g)\);
2. \( \zeta_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \zeta_{i+1} \) \((i = 1, \ldots, 2g)\);
3. \( \zeta_1 \zeta_2 \cdots \zeta_{2g+1} \zeta_{2g+1} \cdots \zeta_2 \zeta_1 = 1 \);
4. \( (\zeta_1 \zeta_2 \cdots \zeta_{2g+1})^{2g+2} = 1 \);
5. \( [\zeta_1, \zeta_1 \zeta_2 \cdots \zeta_{2g+1} \zeta_{2g+1} \cdots \zeta_2 \zeta_1] = 1 \).

Let \( g \) be a positive integer. Since the homomorphism from \( B_{2g+2}(S^2) \) to \( \mathcal{M}_{0, 2g+2} \) which sends \( x_i \) to \( \xi_i \) is surjective, we have a central extension

\[
1 \rightarrow \mathbb{Z}_2 \rightarrow B_{2g+2}(S^2) \rightarrow \mathcal{M}_{0, 2g+2} \rightarrow 1
\]

of \( \mathcal{M}_{0, 2g+2} \) with kernel generated by the Dirac braid \( \Delta_{2g+2} = (x_1 x_2 \cdots x_{2g+1})^{2g+2} \) (see Gillette and Van Buskirk [16]). We have another central extension

\[
1 \rightarrow \mathbb{Z}_2 \rightarrow \mathcal{H}_g \rightarrow \mathcal{M}_{0, 2g+2} \rightarrow 1
\]

of \( \mathcal{M}_{0, 2g+2} \) with kernel generated by the isotopy class \( \iota = \zeta_1 \zeta_2 \cdots \zeta_{2g+1} \zeta_{2g+1} \cdots \zeta_2 \zeta_1 \) of the hyperelliptic involution \( I \), where the homomorphism \( \pi: \mathcal{H}_g \rightarrow \mathcal{M}_{0, 2g+2} \) sends \( \zeta_i \) to \( \xi_i \).

### 3.2. Chart descriptions

We make use of the above presentations to introduce notions of chart which give graphic description of monodromy representations of Lefschetz fibrations. We first recall a general definition of chart given by Kamada [23].

Let \( \mathcal{X} \) be a set and \( \mathcal{R} \) and \( \mathcal{S} \) sets of words in \( \mathcal{X} \cup \mathcal{X}^{-1} \). Let \( \mathcal{C} := (\mathcal{X}, \mathcal{R}, \mathcal{S}) \) be the triple consisting of \( \mathcal{X} \), \( \mathcal{R} \) and \( \mathcal{S} \), and \( \mathcal{G} \) the group with presentation \( \langle \mathcal{X} \mid \mathcal{R} \rangle \). Let \( B \) be a connected closed oriented surface and \( \Gamma \) a finite graph in \( B \) such that each edge of \( \Gamma \) is oriented and labeled an element of \( \mathcal{X} \). Choose a simple path \( \gamma \) which intersects with edges of \( \Gamma \) transversely and does not intersect with vertices of \( \Gamma \). For such a path \( \gamma \), we obtain a word \( w_\Gamma(\gamma) \) in \( \mathcal{X} \cup \mathcal{X}^{-1} \) by reading off the labels of intersecting edges along \( \gamma \) with exponents as in Figure 2(a). We call the word \( w_\Gamma(\gamma) \) the intersection word of \( \gamma \) with respect to \( \Gamma \). Conversely, we can specify the number, orientations, and labels of consecutive edges in \( \Gamma \) by indicating a (dashed) arrow.
intersecting the edges transversely together with the intersection word of the arrow with respect to $\Gamma$ (see Figure 2(b) and (c)).

For a vertex $v$ of $\Gamma$, a small simple closed curve surrounding $v$ in the counterclockwise direction is called a meridian loop of $v$ and denoted by $m_v$. The vertex $v$ is said to be marked if one of the regions around $v$ is specified by an asterisk. If $v$ is marked, the intersection word $w_{\Gamma}(m_v)$ with respect to $\Gamma$ is well defined. If not, it is determined up to cyclic permutation. See Kamada [23] for details.

**Definition 3.4.** A $C$-chart in $B$ is a finite graph $\Gamma$ in $B$ (possibly being empty or having hoops that are closed edges without vertices) whose edges are labeled an element of $\mathcal{X}$, and oriented so that the following conditions are satisfied:

1. the vertices of $\Gamma$ are classified into two families: **white vertices** and **black vertices**;
2. if $v$ is a white vertex (respectively a black vertex), the word $w_{\Gamma}(m_v)$ is a cyclic permutation of an element of $R \cup R^{-1}$ (respectively of $S$).

A white vertex $v$ is said to be of type $r$ (respectively of type $r^{-1}$) if $w_{\Gamma}(m_v)^{-1}$ is a cyclic permutation of $r \in R$ (respectively of $r^{-1} \in R^{-1}$). A black vertex $v$ is said to be of type $s$ if $w_{\Gamma}(m_v)$ is a cyclic permutation of $s \in S$. A $C$-chart $\Gamma$ is said to be marked if each white vertex (respectively black vertex) $v$ is marked and $w_{\Gamma}(m_v)$ is exactly an element of $R \cup R^{-1}$ (respectively of $S$). If a base point $b_0$ of $B$ is specified, we always assume that a chart $\Gamma$ is disjoint from $b_0$. A chart consisting of two black vertices and one edge connecting them is called a free edge. A subchart of a $C$-chart $\Gamma$ is the intersection of $\Gamma$ with a compact two-dimensional submanifold of $B$.

**Remark 3.5.** It would be worth noting that the intersection word of a ‘clockwise’ meridian of a white vertex of type $r$ is equal to $r$, while that of a ‘counterclockwise’ meridian of a black vertex of type $s$ is equal to $s$ in this paper (see also [10]).

Let $\Gamma$ be a $C$-chart in $B$ with base point $b_0$ and $\Delta_\Gamma$ the set of black vertices of $\Gamma$.

**Definition 3.6.** For a loop $\eta$ in $B - \Delta_\Gamma$ based at $b_0$, the element of $G$ determined by the intersection word $w_{\Gamma}(\eta)$ of $\eta$ with respect to $\Gamma$ does not depend on a choice of representative of the homotopy class of $\eta$. Thus we obtain a homomorphism $\rho_\Gamma: \pi_1(B - \Delta_\Gamma, b_0) \to G$, which is called the **homomorphism determined by $\Gamma$**.

Let $\Delta$ be a finite subset of $B$ and $b_0$ a base point of $B - \Delta$.

**Definition 3.7.** A homomorphism from $\pi_1(B - \Delta, b_0)$ to $G$ is called a **$G$-monodromy representation**. A loop $\ell$ in $B - \Delta$ based at $b_0$ is called a **meridional loop** if it is obtained from a meridian loop $m_v$ of a point $v$ of $\Delta$ by connecting with $b_0$ along a path $n$ in $B - \Delta$, namely: $\ell = n^{-1} \cdot m_v \cdot n$. We denote by $\mathcal{M}(B, \Delta, b_0; C)$ the set of $G$-monodromy representations $\rho: \pi_1(B - \Delta, b_0) \to G$ such that $\rho([\ell])$ is conjugate to the element of $G$ determined by an
element of $S$ for every meridional loop $\ell$. For a $C$-chart $\Gamma$ in $B$ with base point $b_0$ and the black vertices $\Delta_\Gamma$, the homomorphism $\rho_\Gamma$ determined by $\Gamma$ belongs to $\mathcal{M}(B, \Delta_\Gamma, b_0; C)$.

**Theorem 3.8** (Kamada [23], Hasegawa [19]). For any $G$-monodromy representation $\rho: \pi_1(B - \Delta, b_0) \to G$ belonging to $\mathcal{M}(B, \Delta, b_0; C)$, there exists a $C$-chart $\Gamma$ such that $\rho_\Gamma = \rho$.

We next introduce several moves for charts. Let $\Gamma$ and $\Gamma'$ be two $C$-charts on $B$ and $b_0$ a base point of $B$.

Let $D$ be a disk embedded in $B - \{b_0\}$. Suppose that the boundary $\partial D$ of $D$ intersects $\Gamma$ and $\Gamma'$ transversely.

**Definition 3.9.** We say that $\Gamma'$ is obtained from $\Gamma$ by a *chart move of type W* if $\Gamma \cap (B - \text{Int } D) = \Gamma' \cap (B - \text{Int } D)$ and that both $\Gamma \cap D$ and $\Gamma' \cap D$ have no black vertices. We call chart moves of type W shown in Figure 3(a), (b) and (c), a *channel change*, a *birth/death of a hoop*, and a *birth/death of a pair of white vertices*, respectively.

Let $s$ and $s'$ be elements of $S$. Suppose that there exists a word $w$ in $X \cup X^{-1}$ such that two words $s'$ and $wsw^{-1}$ determine the same element of $G$.

**Definition 3.10.** If a $C$-chart $\Gamma$ contains a black vertex of type $s$, then we can change a part of $\Gamma$ near the vertex by using a local replacement depicted in Figure 4 to obtain another $C$-chart $\Gamma'$. We say that $\Gamma'$ is obtained from $\Gamma$ by a *chart move of transition*. Note that the box labeled $T$ can be filled only with edges and white vertices.

**Definition 3.11.** We say that $\Gamma'$ is obtained from $\Gamma$ by a *chart move of conjugacy type* if $\Gamma'$ is obtained from $\Gamma$ by a local replacement depicted in Figure 5.

Let $\Delta$ and $\Delta'$ be finite subsets of $B$, and $b_0$ and $b_0'$ base points of $B - \Delta$ and $B - \Delta'$, respectively.
Definition 3.12. Let $\rho : \pi_1(B - \Delta, b_0) \to G$ and $\rho' : \pi_1(B - \Delta', b_0') \to G$ be $G$-monodromy representations. We say that $\rho$ is equivalent to $\rho'$ if there exists an orientation preserving diffeomorphism $h : B \to B$ and an element $\alpha$ of $G$ which satisfies the following conditions:
(i) $h(\Delta) = \Delta'$; (ii) $h(b_0) = b_0'$; (iii) $\rho'(h_b(\gamma)) = \alpha^{-1}\rho(\gamma)\alpha$ for every $\gamma \in \pi_1(B - \Delta, b_0)$. If we can choose such an $h$ isotopic to the identity relative to a given base point $b_0$, we say that $\rho$ is strictly equivalent to $\rho'$.

We state a classification theorem for $G$-monodromy representations in terms of charts and chart moves.

Theorem 3.13 (Kamada [23], Hasegawa [19]). Let $\Gamma$ and $\Gamma'$ be $C$-charts in $B$, and $\rho_\Gamma : \pi_1(B - \Delta_\Gamma, b_0) \to G$ and $\rho_{\Gamma'} : \pi_1(B - \Delta_{\Gamma'}, b_0') \to G$ the homomorphisms determined by $\Gamma$ and $\Gamma'$, respectively.

1. $\rho_\Gamma$ is equivalent to $\rho_{\Gamma'}$ if and only if $\Gamma$ is transformed into $\Gamma'$ by a finite sequence of the following moves: (i) chart moves of type $W$; (ii) chart moves of transition; (iii) chart moves of conjugacy type; and (iv) sending $\Gamma$ to $\Gamma'$ by orientation preserving diffeomorphisms $h : B \to B$ which satisfies $h(b_0) = b_0'$.

2. $\rho_\Gamma$ is strictly equivalent to $\rho_{\Gamma'}$ if and only if $\Gamma$ is transformed into $\Gamma'$ by a finite sequence of the moves (i), (ii), (iii), and (iv) provided that $h$ is isotopic to the identity relative to a given base point $b_0$.

We now define three explicit $C$s corresponding to three groups $M_{0,2g+2}$, $B_{2g+2}(S^2)$, and $H_g$.

For the group $M_{0,2g+2}$, we set

$C_0 := (X_0, R_0, S_0),$
$X_0 := \{\xi_1, \xi_2, \ldots, \xi_{2g+1}\},$
$R_0 := \{r_1(i, j) \mid 1 \leq i < j - 1 \leq 2g\} \cup \{r_2(i) \mid i = 1, \ldots, 2g\} \cup \{r_3, r_4\},$
$S_0 := \{\ell_0(i)^{\pm 1} \mid i = 1, \ldots, 2g + 1\} \cup \{\ell_h^{\pm 1} \mid h = 1, \ldots, [g/2]\}$

for $g \geq 1$, where

$r_1(i, j) := \xi_i \xi_j \xi_i^{-1} \xi_j^{-1} (1 \leq i < j - 1 \leq 2g),
\quad r_2(i) := \xi_i \xi_{i+1} \xi_{i+1}^{-1} \xi_i^{-1} (i = 1, \ldots, 2g),
\quad r_3 := \xi_1 \xi_2 \cdots \xi_{2g+1} \xi_{2g+1} \cdots \xi_1,
\quad r_4 := (\xi_1 \xi_2 \cdots \xi_{2g+1})^{2g+2},$
\quad $\ell_0(i) := \xi_i (i = 1, \ldots, 2g + 1),
\quad \ell_h := (\xi_1 \xi_2 \cdots \xi_{2h})^{4h+2} (h = 1, \ldots, [g/2]).$

The relator $r_4$ corresponds to the Dirac braid and it is called the Dirac braid relator. Vertices of types $\ell_0(i)^{\pm 1}$, $r_1(i, j)$, $r_2(i)$, $r_3$, $r_4$, $\ell_h$ in $C_0$-charts are depicted in Figures 6 and 7, where the label $\xi_i$ is denoted by $i$ for short.
For the group $B_{2g+2}(S^2)$, we set

$$\tilde{C} := (\tilde{X}, \tilde{R}, \tilde{S}),$$

$$\tilde{X} := \{x_1, x_2, \ldots, x_{2g+1}\},$$

$$\tilde{R} := \{\tilde{r}_1(i, j) \mid 1 \leq i < j - 1 \leq 2g\} \cup \{\tilde{r}_2(i) \mid i = 1, \ldots, 2g\} \cup \{\tilde{r}_3\},$$

$$\tilde{S} := \{\tilde{\ell}_0(i)^\pm | i = 1, \ldots, 2g + 1\} \cup \{\tilde{\ell}_h^\pm | h = 1, \ldots, [g/2]\}$$

for $g \geq 1$, where

$$\tilde{r}_1(i, j) := x_i x_j x_i^{-1} x_j^{-1} \quad (1 \leq i < j - 1 \leq 2g),$$

$$\tilde{r}_2(i) := x_i x_{i+1} x_i^{-1} x_{i+1}^{-1} x_i \quad (i = 1, \ldots, 2g),$$

$$\tilde{r}_3 := x_1 x_2 \cdots x_{2g+1} x_{2g+1} \cdots x_2 x_1,$$

$$\tilde{\ell}_0(i) := x_i \quad (i = 1, \ldots, 2g + 1), \quad \tilde{\ell}_h := (x_1 x_2 \cdots x_{2h})^{4h+2} \quad (h = 1, \ldots, [g/2]).$$

Vertices of types $\tilde{\ell}_0(i)^\pm$, $\tilde{r}_1(i, j)$, $\tilde{r}_2(i)$, $\tilde{r}_3$, $\tilde{\ell}_h$ in $\tilde{C}$-charts are similar to those of types $\ell_0(i)^\pm$, $r_1(i, j)$, $r_2(i)$, $r_3$, $\ell_h$ in $C_0$-charts (cf. Figures 6 and 7), respectively.

For the group $H_g$, we set

$$\tilde{C} := (\tilde{X}, \tilde{R}, \tilde{S}),$$

$$\tilde{X} := \{\zeta_1, \zeta_2, \ldots, \zeta_{2g+1}\},$$

$$\tilde{R} := \{\tilde{r}_1(i, j) \mid 1 \leq i < j - 1 \leq 2g\} \cup \{\tilde{r}_2(i) \mid i = 1, \ldots, 2g\} \cup \{\tilde{r}_3, \tilde{r}_4, \tilde{r}_5\},$$

$$\tilde{S} := \{\tilde{\ell}_0(i)^\pm | i = 1, \ldots, 2g + 1\} \cup \{\tilde{\ell}_h^\pm | h = 1, \ldots, [g/2]\}$$

for $g \geq 1$, where

$$\tilde{r}_1(i, j) := \zeta_i \zeta_j \zeta_i^{-1} \zeta_j^{-1} \quad (1 \leq i < j - 1 \leq 2g),$$

$$\tilde{r}_2(i) := \zeta_i \zeta_{i+1} \zeta_i^{-1} \zeta_{i+1}^{-1} \zeta_i \quad (i = 1, \ldots, 2g),$$

$$\tilde{r}_3 := (\zeta_1 \zeta_2 \cdots \zeta_{2g+1} \zeta_{2g+1} \cdots \zeta_2 \zeta_1)^2, \quad \tilde{r}_4 := (\zeta_1 \zeta_2 \cdots \zeta_{2g+1})^{2g+2},$$

$$\tilde{r}_5 := [\zeta_1, \zeta_1 \zeta_2 \cdots \zeta_{2g+1} \zeta_{2g+1} \cdots \zeta_2 \zeta_1],$$

$$\tilde{\ell}_0(i) := \zeta_i \quad (i = 1, \ldots, 2g + 1), \quad \tilde{\ell}_h := (\zeta_1 \zeta_2 \cdots \zeta_{2h})^{4h+2} \quad (h = 1, \ldots, [g/2]).$$
Vertices of types \( \hat{\ell}_0(i) \pm 1, \hat{r}_1(i, j), \hat{r}_2(i), \hat{r}_4, \hat{\ell}_h \) in \( \hat{C} \)-charts are similar to those of types \( \ell_0(i) \pm 1, r_1(i, j), r_2(i), r_4, \ell_h \) in \( C_0 \)-charts (cf. Figures 6 and 7), respectively. Vertices of types \( \hat{r}_3 \) and \( \hat{r}_5 \) in \( \hat{C} \)-charts are depicted in Figure 8, where the label \( \zeta_i \) is denoted by \( i \) for short.

Let \( B \) be a connected closed oriented surface and \( \Gamma \) a chart in \( B \). We denote the number of white vertices of type \( \hat{r}_1(i, j) \) (respectively \( \hat{r}_2(i), \hat{r}_3, \hat{r}_5 \)) minus the number of white vertices of type \( \ell_1(i, j) \) (respectively \( \ell_2(i), \ell_3, \ell_5 \)) included in \( \Gamma \) by \( m_1(i, j)(\Gamma) \) (respectively \( m_2(i)(\Gamma), m_3(\Gamma), m_4(\Gamma), m_5(\Gamma) \)). Similarly, we denote the number of black vertices of type \( \ell_0(i) \pm 1 \) (respectively \( \ell_0^\pm \)) included in \( \Gamma \) by \( \eta_0^\pm(i)(\Gamma) \) (respectively \( \eta_0^\pm(\Gamma) \)), and set \( n_0(i)(\Gamma) := \eta_0^+(i)(\Gamma) - \eta_0^-(i)(\Gamma) \) (respectively \( n_0(\Gamma) := \eta_0^+(\Gamma) - \eta_0^-(\Gamma) \)) and \( n_0(i)(\Gamma) := \sum_{i=0}^{2g} n_0^\pm(i)(\Gamma) \).

### 3.3. Charts and hyperelliptic Lefschetz fibrations

Combining Lemmas 2.5 and 2.6 with Theorems 3.8 and 3.13 for \( \hat{C} \)-charts, we obtain classification theorems for hyperelliptic Lefschetz fibrations in terms of \( \hat{C} \)-charts.

**Proposition 3.14.** Suppose that \( g \) is greater than 1.

1. Let \((f, \Phi)\) be a hyperelliptic Lefschetz fibration of genus \( g \) over \( B \) and \( \rho \) the monodromy representation of \( f \) with respect to \( \Phi \). Then there exists a \( \hat{C} \)-chart \( \Gamma \) in \( B \) such that the homomorphism \( \rho_\Gamma \) determined by \( \Gamma \) is equal to \( \rho \).

2. For every \( \hat{C} \)-chart \( \Gamma \) in \( B \), there exists a hyperelliptic Lefschetz fibration \((f, \Phi)\) of genus \( g \) over \( B \) such that the monodromy representation of \( f \) with respect to \( \Phi \) is equal to the homomorphism \( \rho_\Gamma \) determined by \( \Gamma \).

We call such \( \Gamma \) as in Proposition 3.14 (1) a \( \hat{C} \)-chart corresponding to \((f, \Phi)\), and such \((f, \Phi)\) as in Proposition 3.14 (2) a hyperelliptic Lefschetz fibration described by \( \Gamma \).

**Proposition 3.15.** Suppose that \( g \) is greater than 1. Let \((f, \Phi)\) and \((f', \Phi')\) be hyperelliptic Lefschetz fibrations of genus \( g \), and \( \Gamma \) and \( \Gamma' \) \( \hat{C} \)-charts in \( B \) corresponding to \((f, \Phi)\) and \((f', \Phi')\), respectively.

1. \((f, \Phi)\) is \( \mathcal{H} \)-isomorphic to \((f', \Phi')\) if and only if \( \Gamma \) is transformed into \( \Gamma' \) by a finite sequence of the following moves: (i) chart moves of type \( W \); (ii) chart moves of transition; (iii) chart moves of conjugacy type; and (iv) sending \( \Gamma \) to \( \Gamma' \) by orientation preserving diffeomorphisms \( \text{id} : B \to B \) which satisfies \( h(b_0) = b'_0 \).

2. \((f, \Phi)\) is strictly \( \mathcal{H} \)-isomorphic to \((f', \Phi')\) if and only if \( \Gamma \) is transformed into \( \Gamma' \) by a finite sequence of the moves (i), (ii), (iii), and (iv) provided that \( h \) is isotopic to the identity relative to a given base point \( b_0 \).

We end this subsection with a definition of fiber sums of Lefschetz fibrations. Let \( f : M \to B \) and \( f' : M' \to B' \) be Lefschetz fibrations of genus \( g \), and \( \Delta \) and \( \Delta' \) the sets of critical values of
In this section, we define a Z by a tube, we have a connected sum of f and f', orientation preserving diffeomorphisms Φ: Σg → f−1(b0) and Φ': Σg → f'−1(b'0), and small disks D0 ⊂ B − Δ and D'0 ⊂ B − Δ' near b0 and b'0, respectively.

**Definition 3.16.** Let Ψ: Σg → Σg be an orientation preserving diffeomorphism and r: ∂D0 → ∂D'0 an orientation reversing diffeomorphism. The new manifold M#f M' obtained by gluing M − f−1(∂D0) and M' − f'−1(∂D'0) by (Φ ◦ Ψ ◦ Φ−1) × r admits a Lefschetz fibration f#Ψ f': M#f M' → B#B' of genus g. We call f#Ψ f' the fiber sum of f and f' with respect to Ψ. Although the diffeomorphism type of M#f M' and the isomorphism class of f#Ψ f' depend on a choice of the diffeomorphism Ψ in general, we often abbreviate f#Ψ f' as f#f'.

Let ρ: π1(B − Δ, b0) → Mg and ρ': π1(B' − Δ', b'0) → Mg be the monodromy representations of f and f' with respect to Φ and Φ', respectively. Consider the fiber sum f#Ψ f' of f and f' with respect to Ψ as in Definition 3.16 and the monodromy representation ̂ρ: π1(B#B' − (Δ ∪ Δ'), b0) → Mg with respect to Φ.

**Definition 3.17.** Suppose that (f, Φ) and (f', Φ') are hyperelliptic Lefschetz fibrations and the isotopy class of Ψ belongs to Ηg. We call the pair (f#Ψ f', Φ) the H-fiber sum of (f, Φ) and (f', Φ') with respect to Ψ. The H-fiber sum (f#Ψ f', Φ) is a hyperelliptic Lefschetz fibration because it is easily seen that the image of ̂ρ is included in Ηg.

Let Γ and Γ' be Č-charts corresponding to (f, Φ) and (f', Φ'), and D0 and D'0 small disks near b0 and b'0 disjoint from Γ and Γ', respectively. Connecting B − Int D0 with B' − Int D'0 by a tube, we have a connected sum B#B' of B and B'. Let w be a word in X X−1 which represents the mapping class of Ψ in Mg. Let Γ#wΓ' be the union of Γ, Γ', and hoops on the tube representing w. Then the H-fiber sum (f#Ψ f', Φ) is described by this new chart Γ#wΓ' in B#B' with base point b0. If the word w is trivial, then the chart Γ#wΓ' is denoted also by Γ ⊕ Γ', which is called a product of Γ and Γ'.

4. Counting Dirac braid relators

In this section, we define a Z2-valued invariant for C0-charts in a given surface and prove its invariance under several chart moves.

We first give a precise definition of the invariant. Let B be a connected closed oriented surface and g an integer greater than 1.

**Definition 4.1.** For a C0-chart Γ in B, we denote the number modulo 2 of white vertices of types r4±1 included in Γ by w(Γ). (Note that we named r4 the Dirac braid relator in § 3.2.)

The value of w is obviously invariant under chart moves of conjugacy type. If a C0-chart Γ with base point b0 is sent to another C0-chart Γ' with base point b'0 by an orientation preserving diffeomorphism h : B → B which satisfies h(b0) = b'0, we have w(Γ) = w(Γ').

**Proposition 4.2.** The value of w is invariant under chart moves of type W.

We need a lemma.

**Lemma 4.3.** For any k = 1, . . . , 2g + 1 and a C0-chart depicted in Figure 9, there exists a filling of the box labeled Tk which includes neither black vertices nor white vertices of types r4±1. (Note that the sequence of edges labeled 1, . . . , 2g + 1 in Figure 9 appears 2g + 2 times on each side of the box.)
Figure 9. Subchart without white vertices of types $r_4^{\pm 1}$.

Figure 10. Passing through an edge.

Proof. Since the element of $B_{2g+2}(S^2)$ represented by $(x_1x_2 \cdots x_{2g+1})^{2g+2}$ is included in the center of $B_{2g+2}(S^2)$, it commutes with each of $x_1, \ldots, x_{2g+1}$ in $B_{2g+2}(S^2)$. Thus the word $[x_k, (x_1x_2 \cdots x_{2g+1})^{2g+2}]$ represents the identity element of $B_{2g+2}(S^2)$. Therefore, the word $[x_k, (x_1x_2 \cdots x_{2g+1})^{2g+2}]$ commutes with each of $x_1, \ldots, x_{2g+1}$ in $B_{2g+2}(S^2)$. Thus the word $[x_k, (x_1x_2 \cdots x_{2g+1})^{2g+2}]$ represents the identity element of $B_{2g+2}(S^2)$, and there exists a finite sequence of words in $\tilde{X} \cup \tilde{X}^{-1}$ starting from the word $[x_k, (x_1x_2 \cdots x_{2g+1})^{2g+2}]$ to the empty word such that each word is related to the previous one by one of the following transformations:

(i) insertion or deletion of a trivial relator $x^\varepsilon x^{-\varepsilon}$ for $x \in \tilde{X}$ and $\varepsilon \in \{+1, -1\}$; (ii) insertion of $\tilde{r}^\varepsilon$ for $\tilde{r} \in \tilde{R}$ and $\varepsilon \in \{+1, -1\}$. We first consider a $\tilde{C}$-chart depicted in Figure 9. For each transformation (i) (respectively (ii)), we create an edge labeled $x$ (respectively a vertex of type $\tilde{r}^\varepsilon$). Repeating such creations, we can fill the box labeled $T_k$ with edges and white vertices of types $r_1^{\pm 1}(i,j)$, $r_2^{\pm 1}(i)$, and $r_3^{\pm 1}$. (A similar argument for charts without white vertices of type $r_3^{\pm 1}$ can be found in [21, Lemma 15; 22, §18.2].) Changing all labels $x_i$ of edges into $\xi_i$, we obtain a $C_0$-chart which consists of edges and white vertices of types $r_1^{\pm 1}(i,j)$, $r_2^{\pm 1}(i)$, and $r_3^{\pm 1}$.

Proof of Proposition 4.2. It suffices to show that $w(\Gamma) = 0$ for every $C_0$-chart $\Gamma$ in $S^2$ without black vertices. Let $\Gamma$ be such a $C_0$-chart in $S^2$. We consider a chart move of type $W$ depicted in Figure 10. Suppose that the box labeled $T_k$ is filled with edges and white vertices of types $r_1^{\pm 1}(i,j)$, $r_2^{\pm 1}(i)$, and $r_3^{\pm 1}$. The number of white vertices of type $r_4$ and that of white vertices of type $r_4^{-1}$ included in $\Gamma_1$ are the same as those for $\Gamma$, respectively.

We then apply deaths of pairs of white vertices of types $r_4$ and $r_4^{-1}$ as in Figure 3(c) and sequences of chart moves of type $W$ as in Figure 10 to $\Gamma_1$ repeatedly to obtain a $C_0$-chart $\Gamma_2$. 

□
depicted in Figure 11(b), where the box labeled $\Theta_2$ is filled with edges and white vertices of types $r_1(i, j)\pm^1$, $r_2(i)\pm^1$, and $r_3\pm^1$, and $\varepsilon$ is equal to either $+1$ or $-1$. Let $n$ be the number of white vertices of type $r_3^4$ included in $\Gamma_2$. We replace all the white vertices of type $r_3^4$ included in $\Gamma_2$ with black vertices of types $\ell_0(i)^{-\varepsilon}$ to obtain a $C_0$-chart $\Gamma_3$ depicted in Figure 11(c). The intersection word $w$ of the dashed arrow with respect to $\Gamma_3$ in Figure 11(c) is equal to $(\xi_1\xi_2\cdots\xi_{2g+1})^{\pm(2g+2)n}$. Changing all labels $\xi_i$ of edges of $\Gamma_3$ into $x_i$, we obtain a $\tilde{C}$-chart $\tilde{\Gamma}_3$ because $\Gamma_3$ does not contain white vertices of types $r_3\pm^1$. The intersection word $w$ of the dashed arrow with respect to $\tilde{\Gamma}_3$ in Figure 11(c) is equal to $(x_1x_2\cdots x_{2g+1})^{\pm(2g+2)n}$, which represents the identity element of $B_{2g+2}(S^2)$. Since the element of $B_{2g+2}(S^2)$ represented by $(x_1x_2\cdots x_{2g+1})^{2g+2}$ is of order 2 (see [14, §9.1.4]), $n$ must be even. Therefore we have

$$w(\Gamma) = w(\Gamma_1) = w(\Gamma_2) = w(\Gamma_3) = 0$$

and this completes the proof.

**Proposition 4.4.** Assume that $g$ is odd. Then the value of $w$ is invariant under chart moves of transition.

We divide chart moves of transition for $C_0$-charts into two classes.

**Definition 4.5.** A chart move of transition as in Figure 4 is called an $L_0$-move (respectively $L_+\text{-move}$) if $\mathcal{C} = C_0$, $w \in X_0 \cup X_0^{-1}$, and $s, s' \in \{\ell_0(i)^{\pm} | i = 1, \ldots, 2g + 1\}$ (respectively $s, s' \in \{\ell_h^\pm | h = 1, \ldots, [g/2]\}$). If $s = \xi_i^\varepsilon$, $s' = \xi_j^\varepsilon$ (respectively $s = s' = (\xi_1\xi_2\cdots\xi_{2h})^{\varepsilon(4h+2)}$), and $\varepsilon \in \{+1, -1\}$, then the label $T$ on the box in Figure 4 is also denoted by $T_1(k, \ell; \varepsilon)$ (respectively $T_2(h; \varepsilon)$). Every chart move of transition for $C_0$-charts is either an $L_0$-move or an $L_+\text{-move}$.

We first show a lemma for $L_0$-moves.

**Lemma 4.6.** For any $k, \ell = 1, \ldots, 2g + 1$ and $\varepsilon \in \{+1, -1\}$, there exists a filling of the box labeled $T = T_1(k, \ell; \varepsilon)$ in Figure 4 without black vertices such that the number of white vertices of type $r_3^{\pm 1}$ is even, that is, the filling consists of edges, white vertices of types $r_1(i, j)^\pm, r_2(i)^\pm, r_3^{\pm 1}$, and an even number of white vertices of types $r_4^{\pm 1}$.

**Proof.** We assume that $\varepsilon = +1$ for simplicity. Let $\varphi$ and $\delta_i$ be elements of $\mathcal{M}_{0, 2g+2} \times \mathcal{M}_{0, 2g+2}$ represented by $w$ and $\xi_i$, respectively. Since the intersection word of the boundary of the box labeled $T = T_1(k, \ell; +1)$ with respect to the $\mathcal{C}_0$-chart in Figure 4 is $wsw^{-1} = w\xi_i w^{-1} \xi^{-1}_\ell$, we have a relation $\varphi \delta_i \varphi^{-1} \delta^{-1}_\ell = 1$ in $\mathcal{M}_{0, 2g+2}$.

**Case 1:** Suppose that $k = \ell$. We consider a hyperelliptic involution $I$ on $\Sigma_g$ and think of $S^2$ as the quotient of $\Sigma_g$ by the action of $I$. The image of the simple closed curve $c_i$ in Figure 1 under the double branched covering $\Sigma_g \to S^2$ is a simple arc $a_i$ on $S^2$ depicted in Figure 12, and a right-handed half twist $D_i$ about $a_i$ represents the mapping class $\delta_i$. We also consider an additional arc $a_{2g+2}$ on $S^2$ as in Figure 12, a right-handed half twist $D_{2g+2}$ about $a_{2g+2}$.
and the mapping class $\delta_{2g+2}$ of $D_{2g+2}$. We think of the indices $i$ of $a_i$, $D_i$, and $\delta_i$ as integers modulo $2g+2$. For example, $a_{2g+3} = a_1$.

The relation $\varphi \delta_k \varphi^{-1} = \delta_k$ implies that $(a_k)F$ is isotopic to $a_k$ by [24, Lemma 4.1], where $F$ is an orientation preserving diffeomorphism on $S^2$ representing $\varphi$. Since $(a_k)D_k = a_k$ and $D_k$ reverses the orientation of $a_k$, we can assume that either $F$ or $FD_k$ fixes $a_k$ pointwise. Cutting $S^2$ along $a_k$, we obtain an orientation preserving diffeomorphism on a $2$-disk which fixes the boundary pointwise. The mapping class group $\mathcal{M}_{g,2g}$ of the $2$-disk with $2g$ marked points is generated by the half twists about arcs corresponding to $a_1, \ldots, a_{2g+2}$ except $a_{k-1}$ and $a_{k+1}$ (see [14, §9.1.3]). Hence $\varphi$ can be factorized as a product of $\delta_1^\pm, \ldots, \delta_{2g+2}^\pm$ except $\delta_{k-1}^\pm$ and $\delta_{k+1}^\pm$, and it can be represented by a word $w'$ in $\xi_1^\pm, \ldots, \xi_{2g+2}^\pm$ except $\xi_{k-1}^\pm$ and $\xi_{k+1}^\pm$, where we put $\xi_{2g+2} := \xi_1 \xi_2 \cdots \xi_{2g} \xi_{2g+1} \xi_{2g}^{-1} \cdots \xi_2^{-1} \xi_1^{-1}$. It is easily checked that the relation $\delta_{2g+2} = \delta_1 \delta_2 \cdots \delta_{2g} \delta_{2g+1} \delta_{2g}^{-1} \cdots \delta_2^{-1} \delta_1^{-1}$ holds in $\mathcal{M}_{0,2g+2}$.

We divide the box labeled $T_1(k,k;+1)$ into three boxes labeled $T_1(k)$, $\Theta_1(k)$, and $\Theta_1(k)^*$ as in Figure 13.

The box labeled $\Theta_1(k)$ can be filled with edges and white vertices because both $w$ and $w'$ represent the same mapping class $\varphi$. The box labeled $\Theta_1(k)^*$ can be filled with the mirror image of the subchart filling the box labeled $\Theta_1(k)$ with edges orientation reversed. Since $w'$ is a word in $\xi_1^\pm, \ldots, \xi_{2g+2}^\pm = \xi_1 \xi_2 \cdots \xi_{2g} \xi_{2g+1} \xi_{2g}^{-1} \cdots \xi_2^{-1} \xi_1^{-1}$ except $\xi_{k-1}^\pm$ and $\xi_{k+1}^\pm$, the box labeled $T_1(k)$ can be filled with copies of three kinds of subcharts depicted in Figure 14. Note that the subcharts depicted in Figure 14 correspond to $\xi_k$, $\xi_i (i \neq k-1,k,k+1,2g+2)$, and $\xi_{2g+2}$, respectively. We also need subcharts corresponding to $\xi_k^{-1}$, $\xi_i^{-1} (i \neq k-1,k,k+1,2g+2)$, and $\xi_{2g+2}^{-1}$.

The box labeled $T_1(k)$ is filled with edges and white vertices of types $r_1(i,j)^\pm$ and $r_2(i)^\pm$. The number of white vertices of type $r_4^\pm$ included in the box labeled $\Theta_1(k)^*$ is equal to that
Figure 15. Inside of the box labeled $T_1(k,\ell;+1)$.

Figure 16. Simple closed curve $b_h$ on $S^2$.

for the box labeled $\Theta_1(k)$. Therefore the box labeled $T_1(k,k;+1)$ is filled with edges, white vertices of types $r_1(i,j)^\pm1$, $r_2(i)^\pm1$, $r_3^\pm1$, and an even number of white vertices of types $r_4^\pm1$.

Case 2: Suppose that $k \neq \ell$. We assume that $k < \ell$ for simplicity. We put $v := (\xi_{k+1} \xi_{k})(\xi_{k+2} \xi_{k+1}) \cdots (\xi_{\ell} \xi_{\ell-1})$. Since $v^{-1} \xi_k v$ and $\xi_\ell$ represent the same element of $\mathcal{M}_{0,2g+2}$, and $w \xi_k v^{-1} \xi_\ell^{-1}$ represents the identity element of $\mathcal{M}_{0,2g+2}$, we fill the box labeled $T_1(k,\ell;+1)$ with edges, white vertices of type $r_2(i)^\pm1$, and a small box labeled $T_1(k,k;+1)'$ as in Figure 15.

By the proof of Case 1, the small box labeled $T_1(k,k;+1)'$ is filled with edges, white vertices of types $r_1(i,j)^\pm1$, $r_2(i)^\pm1$, $r_3^\pm1$, and an even number of white vertices of types $r_4^\pm1$, so is the box labeled $T_1(k,\ell;+1)$.

We next show a lemma for $L_4$-moves.

**Lemma 4.7.** Assume that $g$ is odd. For any $h = 1, \ldots, [g/2]$ and $\varepsilon \in \{+1, -1\}$, there exists a filling of the box labeled $T = T_2(h;\varepsilon)$ in Figure 4 without black vertices such that the number of white vertices of type $r_4^\varepsilon$ is even, that is, the filling consists of edges, white vertices of types $r_1(i,j)^\pm1$, $r_2(i)^\pm1$, $r_3^\pm1$, and an even number of white vertices of types $r_4^\pm1$.

**Proof.** We assume that $\varepsilon = +1$ for simplicity. Let $\varphi$ and $\tau_h$ be elements of $\mathcal{M}_{0,2g+2}$ represented by $w$ and $(\xi_1 \xi_2 \cdots \xi_{2h})^{4h+2}$, respectively. Since the intersection word of the boundary of the box labeled $T = T_2(h;+1)$ with respect to the $C_0$-chart in Figure 4 is $wsw^{-1}s'^{-1} = w(\xi_1 \xi_2 \cdots \xi_{2h})^{4h+2}w^{-1}(\xi_1 \xi_2 \cdots \xi_{2h})^{-4h-2}$, we have a relation $\varphi \tau_h \varphi^{-1} \tau_h^{-1} = 1$ in $\mathcal{M}_{0,2g+2}$.

We consider a hyperelliptic involution $I$ on $\Sigma_g$ and think of $S^2$ as the quotient of $\Sigma_g$ by the action of $I$. The image of the simple closed curve $s_h$ in Figure 1 under the double branched covering $\Sigma_g \to S^2$ is a simple closed curve $b_h$ on $S^2$ depicted in Figure 16, and the square of a right-handed Dehn twist $T_h$ about $b_h$ represents the mapping class $\tau_h$. Note that $a_1, \ldots, a_{2g+1}$ in Figure 16 are the same arcs as those depicted in Figure 12.
The relation $\varphi_\tau \varphi^{-1} = \tau$ implies that $(b_\tau)F$ is isotopic to $b_\tau$ (see [14, Fact 3.6] and [31, Proposition 3.6]), where $F$ is an orientation preserving diffeomorphism on $S^2$ representing $\varphi$. Since $g$ is odd, two components of $S^2 - b_\tau$ include different numbers of branch points. Thus we can assume that $F$ fixes $b_\tau$ pointwise. Cutting $S^2$ along $b_\tau$, we obtain orientation preserving diffeomorphisms on two 2-disks which fix the boundary pointwise. The mapping class group $\mathcal{M}_{10, 2h}^1 (\text{respectively } \mathcal{M}_{10, 2g-2h}^1)$ of the 2-disk with $2h+1$ (respectively $2g-2h+1$) marked points is generated by the half twists about arcs corresponding to $a_1, \ldots, a_{2h}$ (respectively $a_{2h+1}, \ldots, a_{2g+1}$) (see [14, §9.1.3]). Hence $\varphi$ can be factorized as a product of $\delta_{1}^{\pm 1}, \ldots, \delta_{2h}^{\pm 1}, \delta_{2h+2}^{\pm 1}, \ldots, \delta_{2g+1}^{\pm 1}$, and it can be represented by a word $w'$ in $\xi_{1}^{\pm 1}, \ldots, \xi_{2h}^{\pm 1}, \xi_{2h+2}^{\pm 1}, \ldots, \xi_{2g+1}^{\pm 1}$.

We divide the box labeled $T_2(h+1)$ into three boxes labeled $T_2(h)$, $\Theta_2(h)$, and $\Theta_2(h)^*$ as in Figure 17.

The box labeled $\Theta_2(h)$ can be filled with edges and white vertices because both $w$ and $w'$ represent the same mapping class $\varphi$. The box labeled $\Theta_2(h)^*$ can be filled with the mirror image of the subchart filling the box labeled $\Theta_2(h)$ with edges orientation reversed. Since $w'$ is a word in $\xi_{1}^{\pm 1}, \ldots, \xi_{2h}^{\pm 1}, \xi_{2h+2}^{\pm 1}, \ldots, \xi_{2g+1}^{\pm 1}$, the box labeled $T_2(h)$ can be filled with copies of two kinds of subcharts depicted in Figure 18. Note that the subcharts depicted in Figure 18(a) and (b) correspond to $\xi_i (i = 1, \ldots, 2h)$ and $\xi_j (j = 2h+2, \ldots, 2g+1)$, respectively. The boxes labeled $\Omega_k (k = 2, \ldots, 2h)$ and $\Omega_l$ in Figure 18(a) are filled with the subcharts depicted in Figure 18(c) and Figure 19, respectively. We also need subcharts corresponding to $\xi_i^{-1} (i = 1, \ldots, 2h)$ and $\xi_j^{-1} (j = 2h+2, \ldots, 2g+1)$.

The box labeled $T_2(h)$ is filled with edges and white vertices of types $r_1(i,j)^{\pm 1}$ and $r_2(i)^{\pm 1}$. The number of white vertices of type $r_1^{\pm 1}$ included in the box labeled $\Theta_2(h)^*$ is equal to that for the box labeled $\Theta_2(h)$. Therefore the box labeled $T_2(h+1)$ is filled with edges, white vertices of types $r_1(i,j)^{\pm 1}$, $r_2(i)^{\pm 1}$, and an even number of white vertices of types $r_3^{\pm 1}$.

We now prove the invariance of $w$ under chart moves of transition.
Proof of Proposition 4.4. Let $\Gamma$ be a $C_0$-chart $\Gamma$ in $B$ and $\Gamma'$ a $C_0$-chart in $B$ obtained from $\Gamma$ by a chart move of transition. Suppose that the chart move of transition is an $L_0$-move. Two $C_0$-charts $\Gamma$ and $\Gamma'$ are related by a chart move as in Figure 4 and the box is labeled $T = T_1(k, \ell)$ for some $k, \ell = 1, \ldots, 2g + 1$. Let $\Gamma_1$ be a $C_0$-chart in $B$ obtained from $\Gamma'$ by replacing the subchart inside the box labeled $T_1(k, \ell)$ with a subchart satisfying the condition given in Lemma 4.5. By virtue of Proposition 4.2 and Lemma 4.5, we have $w(\Gamma) = w(\Gamma_1) = w(\Gamma')$. It follows from a similar argument together with Proposition 4.2 and Lemma 4.7 that $w(\Gamma) = w(\Gamma')$ if $g$ is odd and $\Gamma'$ is obtained from $\Gamma$ by an $L_1$-move. Thus we have proved the proposition.

\[\Box\]

5. Stable classification

In this section, we define a $\mathbb{Z}_2$-valued invariant of hyperelliptic Lefschetz fibrations of odd genus and show a stable classification theorem for hyperelliptic Lefschetz fibrations.

We first give a definition of the invariant. Let $B$ be a connected closed oriented surface and $g$ an integer greater than 1.

**Definition 5.1.** Let $(f, \Phi)$ be a hyperelliptic Lefschetz fibration of genus $g$ over $B$ and $\rho: \pi_1(B - \Delta, b_0) \to \mathcal{H}_g$ the monodromy representation with respect to $\Phi$. Let $\Gamma$ be a $C_0$-chart in $B$ such that the homomorphism determined by $\Gamma$ is equal to $\rho_0 := \pi \circ \rho: \pi_1(B - \Delta, b_0) \to \mathcal{M}_{0, 2g+2}$. Suppose that $g$ is odd. We set $w(f, \Phi) := w(\Gamma)$, where $w(\Gamma)$ is an element of $\mathbb{Z}_2$ defined in Definition 4.1.

**Remark 5.2.** If we have a $\hat{C}$-chart $\hat{\Gamma}$ corresponding to $(f, \Phi)$, a $C_0$-chart $\Gamma$ above is obtained from $\hat{\Gamma}$ by changing the labels $\zeta_i$ of all edges into $\xi_i$ and replacing all white vertices of types $\hat{r}_3^{\pm 1}$ and $\hat{r}_5^{\pm 1}$ with $C_0$-subcharts as in Figures 20 and 21. Therefore $w(f, \Phi)$ is equal to the number modulo 2 of white vertices of types $\hat{r}_3^{\pm 1}$ included in $\hat{\Gamma}$.

Let $(f, \Phi)$ and $(f', \Phi')$ be hyperelliptic Lefschetz fibrations of genus $g$ over $B$.

**Proposition 5.3.** If $g$ is odd and $(f, \Phi)$ is $\mathcal{H}$-isomorphic to $(f', \Phi')$, then we have $w(f, \Phi) = w(f', \Phi')$. 

\[\Box\]
Proof. The statement follows from Propositions 3.14, 3.15, 4.2 and 4.4. □

Thus \( w(f, \Phi) \) turns out to be an invariant of the \( \mathcal{H} \)-isomorphism class of \((f, \Phi)\).

We next prove a stable classification theorem for hyperelliptic Lefschetz fibrations, which improves the stabilization theorem shown in [11]. Let \((f, \Phi)\) and \((f', \Phi')\) be hyperelliptic Lefschetz fibrations of genus \(g\) over \(B\). Let \(\Gamma_0\) be a \(\mathcal{C}\)-chart in \(S^2\) depicted in Figure 22 and \((f_0, \Phi_0)\) a hyperelliptic Lefschetz fibration described by \(\Gamma_0\).

A Hurwitz system of \((f_0, \Phi_0)\) is given by \((\zeta_1, \zeta_2, \ldots, \zeta_{2g+1}, \zeta_{2g+1}, \ldots, \zeta_2, \zeta_1)^2\). The total space of \(f_0\) is diffeomorphic to \(\mathbb{C}P^2 \# (4g + 5)\mathbb{C}P^2\), which is a natural generalization of the rational elliptic surface \(E(1)\) (see Gompf and Stipsicz [18, §8.4] and Ito [20]).

**Definition 5.4.** We say that \((f, \Phi)\) is stably isomorphic to \((f', \Phi')\) if there exists a non-negative integer \(N\) such that an \(\mathcal{H}\)-fiber sum \((f \# Nf_0, \Phi)\) is \(\mathcal{H}\)-isomorphic to an \(\mathcal{H}\)-fiber sum \((f' \# Nf_0, \Phi')\).

**Remark 5.5.** If \(N\) is positive, then the \(\mathcal{H}\)-isomorphism class of an \(\mathcal{H}\)-fiber sum \((f \# Nf_0, \Phi)\) does not depend on choices of orientation preserving diffeomorphisms of \(\Sigma_g\) used for the construction of \((f \# Nf_0, \Phi)\) (see Proof of Theorem 5.6).
We give a complete classification of the stable isomorphism classes of hyperelliptic Lefschetz fibrations of genus $g$ over $B$.

**Theorem 5.6.** Let $(f, \Phi)$ and $(f', \Phi')$ be hyperelliptic Lefschetz fibrations of genus $g$ over $B$. Then $(f, \Phi)$ is stably isomorphic to $(f', \Phi')$ if and only if the following conditions hold:

1. $n_0^+(f) = n_0^+(f')$;
2. $n_h^+(f) = n_h^+(f')$ for every $h = 1, \ldots, [g/2]$;
3. $w(f, \Phi) = w(f', \Phi')$ if $g$ is odd.

**Proof.** We first prove the ‘if’ part. Assume that $(f, \Phi)$ and $(f', \Phi')$ satisfy the conditions (i), (ii), and (iii). Let $\Gamma$ and $\Gamma'$ be $C$-charts in $B$ corresponding to $(f, \Phi)$ and $(f', \Phi')$, respectively. Since every edge has two adjacent vertices, the sum of the signed numbers of adjacent edges for all vertices of $\Gamma$ is equal to zero:

$$4(2g + 1)m_3(\Gamma) + 2(g + 1)(2g + 1)m_4(\Gamma) - \sum_{i=1}^{2g+1} n_0(i)(\Gamma) - 4 \sum_{h=1}^{[g/2]} h(2h + 1)n_h(\Gamma) = 0.$$ 

A similar equality for $\Gamma'$ also holds. Interpreting the conditions (i) and (ii) as conditions on $\Gamma$ and $\Gamma'$, we have

$$\sum_{i=1}^{2g+1} n_0(i)(\Gamma) = \sum_{i=1}^{2g+1} n_0(i)(\Gamma') \quad \text{and} \quad n_h(\Gamma) = n_h(\Gamma') \quad \text{for} \quad h = 1, \ldots, [g/2].$$

Thus we obtain

$$2m_3(\Gamma) + (g + 1)m_4(\Gamma) = 2m_3(\Gamma') + (g + 1)m_4(\Gamma'). \quad (*)$$

Let $N$ be an integer larger than both of the number of edges of $\Gamma$ and that of $\Gamma'$. Choose a base point $b_0 \in B - (\Gamma \cup \Gamma')$. The $H$-fiber sum $(f \# N f_0, \Phi)$ is described by a chart $(\cdots ((\Gamma \# w_5, \Gamma_0) \# w_4, \Gamma_0) \cdots) \# w_1, \Gamma_0$ for some words $w_1, \ldots, w_N$ in $\mathcal{X} \cup \mathcal{X}^{-1}$. Since hoops surrounding $\Gamma_0$ can be removed by use of the edges of $\Gamma_0$, in Figure 23, the chart is transformed into a product $\Gamma \# NT_0$ by channel changes. Similarly, the $H$-fiber sum $f' \# N f_0$ is described by a product $\Gamma' \# NT_0$.

We choose and fix $2g + 1$ edges of $\Gamma_0$ which are labeled with $1, 2, \ldots, 2g + 1$ and adjacent to black vertices. We apply chart moves only to these edges in the following. Since $\Gamma_0$ can pass through any edge of $\Gamma$ as shown in Figure 24, we can move $\Gamma_0$ to any region of $B - \Gamma$ by channel changes.

For each edge of $\Gamma$, we move a copy of $\Gamma_0$ to a region adjacent to the edge and apply a channel change to the edge and $\Gamma_0$ as in Figure 24(a) and (b). Applying chart moves of transition to each component of the chart as in Figure 25, we remove white vertices of type $\tilde{r}_1(i, j)^{\pm 1}, \tilde{r}_2(i)^{\pm 1}, \tilde{r}_5^{\pm 1}$ to obtain a union of copies of $L_0(i), L_h, L_h', R_3, R_4', R_4, \Gamma_0$ shown in Figures 26 and 27, where we use a simplification of diagrams as in Figure 28. Although $R_3$ is the same chart as $\Gamma_0$, we distinguish $R_3$ from $\Gamma_0$s which do not come from vertices of type $r_3$ in $\Gamma$.

If there is a pair of $R_3$ and $R_4'$, we remove them by a death of a pair of white vertices to obtain $4(2g + 1)$ copies of $\Gamma_0$. Similarly, we remove a pair of $R_4$ and $R_4'$ to obtain $2(g + 1)(2g + 1)$ copies of $\Gamma_0$.
HISAAKI ENDO AND SEIICHI KAMADA

Figure 24. Passing through an edge.

Figure 25. Chart moves of transition.

Figure 26. Charts $L_0(i)$, $L_h$, $L_h'$.

copies of $\Gamma_0$. Since there is at least one $\Gamma_0$, any copy of $L_0(i)$ can be transformed into $L_0(1)$ as in Figure 29.

Thus we have a union $\Gamma'_1$ of $n^-(\Gamma)$ copies of $L_0(1)$, $n^+(\Gamma)$ copies of $L_h$, $n^-(\Gamma)$ copies of $L'_h$, $|m_3(\Gamma)|$ copies of $R_3$ (or $R'_3$), $|m_4(\Gamma)|$ copies of $R_4$ (or $R'_4$), and $k$ copies of $\Gamma_0$ for some $k$.

A similar argument implies that $\Gamma' \oplus N\Gamma_0$ is transformed into a union $\Gamma'_1$ of $n^-(\Gamma')$ copies of $L_0(1)$, $n^+(\Gamma')$ copies of $L_h$, $n^-(\Gamma')$ copies of $L'_h$, $|m_3(\Gamma')|$ copies of $R_3$ (or $R'_3$), $|m_4(\Gamma')|$ copies of $R_4$ (or $R'_4$), and $k'$ copies of $\Gamma_0$ for some $k'$ by chart moves of type $W$ and chart moves of transition. By virtue of the conditions (i) and (ii) together with $n^-(\Gamma \oplus N\Gamma_0) = n^-(\Gamma'_1)$, $n^+(\Gamma' \oplus N\Gamma_0) = n^+(\Gamma'_1)$, and the equality ($\ast$), we conclude that $k = k'$ because of $n^-(\Gamma'_1) \neq 0$.

Hence $\Gamma_1$ and $\Gamma'_1$ have the same numbers of copies of $L_0(1)$, $L_h$, $L'_h$, and $\Gamma_0$, whereas they may have different numbers of copies of $R_3$ (or $R'_3$) and $R_4$ (or $R'_4$).

We can suppose that $m_4(\Gamma) \geq m_4(\Gamma')$ without loss of generality. The number $m_4(\Gamma) - m_4(\Gamma')$ is even because of the equality ($\ast$) for even $g$ and because of the condition (iii) for odd $g$. We put $2\ell := m_4(\Gamma) - m_4(\Gamma')$. Taking $N$ large enough, we can assume that $k > 4\ell(g+1)(2g+1)$. 
Applying births of \(2\ell\) pairs of white vertices of type \(\hat{r}^{\pm 1}_4\) to \(4\ell(g+1)(2g+1)\) copies of \(\Gamma_0\) included in \(\Gamma_1\), we obtain \(2\ell\) pairs of \(R_4\) and \(R'_4\). Since \(2R_4\) is transformed into \((g+1)R_3\) by a sequence of chart moves of type W and chart moves of transition (see [9, Lemma 4.1]), we obtain \((g+1)\ell R_3\) from \(2\ell R_4\). We thus have a new chart \(\Gamma_2\). Removing pairs of \(R_3\) and \(R'_3\) and those of \(R_4\) and \(R'_4\) included in \(\Gamma_2\) if necessary, \(\Gamma_2\) and \(\Gamma'_1\) have the same numbers of copies of \(L_0\) (1), \(L_h\), \(L'_h\), \(R_3\) (or \(R'_3\)), \(R_4\) (or \(R'_4\)), and \(\Gamma_0\) because of the equality (\(*\)). Then \(\Gamma_2\) is transformed into \(\Gamma'_1\) by an ambient isotopy of \(B\) relative to \(b_0\), which means that \(\Gamma \oplus \mathcal{N}\Gamma_0\) is transformed into \(\Gamma' \oplus \mathcal{N}\Gamma_0\) by chart moves of type W, chart moves of transition, and ambient isotopies of \(B\) relative to \(b_0\). Therefore \(f \# Nf_0\) is (strictly) \(\mathcal{H}\)-isomorphic to \(f' \# Nf_0\) by Proposition 3.15.

We next prove the ‘only if’ part. Take a non-negative integer \(N\) so that \((f \# Nf_0, \Phi)\) is \(\mathcal{H}\)-isomorphic to \((f' \# Nf_0, \Phi')\). Since an \(\mathcal{H}\)-isomorphism preserves numbers and types of vanishing cycles and the invariant \(w\) (see Proposition 5.3), we have \(n_h^\pm(f \# Nf_0) = n_h^\pm(f' \# Nf_0)\), \(n_h^\pm(f \# Nf_0) = n_h^\pm(f' \# Nf_0)\) for every \(h = 1, \ldots, [g/2]\), and \(w(f \# Nf_0, \Phi) = w(f' \# Nf_0, \Phi')\). The conditions (i), (ii), (iii) follows from additivity of \(n_h^\pm, n_h^\pm, w\) under \(\mathcal{H}\)-fiber sum.

Remark 5.7. Theorem 5.6 is also valid for any hyperelliptic Lefschetz fibration \((f_0, \Phi_0)\) which satisfies the following conditions: (i) \(f_0\) is chiral and irreducible; (ii) the base space of
$f_0$ is $S^2$; (iii) a Hurwitz system of $f_0$ with respect to $\Phi_0$ includes at least one $\zeta_i$ for every $i = 1, \ldots, 2g + 1$ (cf. Auroux [1] and [10, Definition 4.1]).

Remark 5.8. Auroux and Smith [3, §2.1] remarked that the existence of a certain $\mathbb{Z}_2$-valued invariant and a stabilization theorem for hyperelliptic Lefschetz fibrations follow from a result of Kharlamov and Kulikov [26] about braid monodromy factorizations. Although they did not give any precise definition of the invariant, we expect that their invariant would be the same as the invariant $w$ and their theorem would be similar to (a part of) Theorem 5.6.

6. Examples and remarks

In this section, we exhibit examples of stabilizations of hyperelliptic Lefschetz fibrations and examples of pairs of hyperelliptic Lefschetz fibrations which are not stably equivalent, and make a few remarks. We do not specify hyperelliptic structures of Lefschetz fibrations in the following because they are obvious from given Hurwitz systems.

We first show an example of non-isomorphic pair of irreducible chiral hyperelliptic Lefschetz fibrations which become isomorphic after one stabilization.

Example 6.1. Let $g$ be an integer greater than 1. We consider the following Hurwitz systems:

$$C_{11} := (\zeta_1, \zeta_2, \ldots, \zeta_{2g})^{4g+2}, \quad I^{2g} := (\zeta_1, \zeta_2, \ldots, \zeta_{2g+1}, \zeta_{2g+1}, \ldots, \zeta_2, \zeta_1)^{2g}.$$  

Let $f_{C_{11}} : M_{C_{11}} \to S^2$ and $f_{I^{2g}} : M_{I^{2g}} \to S^2$ be the hyperelliptic Lefschetz fibrations of genus $g$ determined by $C_{11}$ and $I^{2g}$, respectively. $f_{I^{2g}}$ is nothing but the (untwisted) $H$-fiber sum of $g$ copies of $f_0$ in Section 5. Both $f_{C_{11}}$ and $f_{I^{2g}}$ are irreducible, chiral and have $4g(2g + 1)$ singular fibers. They are not $H$-isomorphic, because the images of their monodromy representations are different (see [8, Example 4.14]). Moreover, the manifolds $M_{C_{11}}$ and $M_{I^{2g}}$ are homeomorphic but not diffeomorphic by Freedman’s theorem and a theorem of Usher [39] (see [9, Remark 4.9]). In particular $f_{C_{11}}$ and $f_{I^{2g}}$ are not isomorphic. On the other hand, the (untwisted) $H$-fiber sums $f_{C_{11}} \# f_0$ and $f_{I^{2g}} \# f_0$ are $H$-isomorphic because a Hurwitz system of $f_{C_{11}} \# f_0$ is transformed into that of $f_{I^{2g}} \# f_0$ by elementary transformations and simultaneous conjugations (see [9, Lemma 4.1]).

We next show an example of non-isomorphic pair of chiral hyperelliptic Lefschetz fibrations with singular fiber of type $\Pi_{g/2}$ which become isomorphic after one stabilization.

Example 6.2. Let $g$ be an even integer greater than 1. We consider the following Hurwitz systems:

$$P J := ((\zeta_1 \zeta_2 \ldots \zeta_{2g})^{2g+2}, (\zeta_1', \ldots, \zeta_{2g}'), \ldots, (\zeta_{2g}', \ldots, \zeta_2), (\zeta_{g+1}, \ldots, \zeta_3, \zeta_2), \ldots, (\zeta_2, \ldots, \zeta_{g+2}, \zeta_{g+1}), (\zeta_1, \zeta_2, \ldots, \zeta_{2g})^{2g+1},$$

where $\zeta_i' := (\zeta_g^{-1} \cdots \zeta_{i-1}^{-1})^{g+1} \zeta_i \zeta_1 \cdots \zeta_{g},$ and

$$R I^{g-1} := ((\zeta_1 \zeta_2)^{2g+2}, \zeta_1^{-1} \zeta_2^{-1} \cdots \zeta_1^{-1} \zeta_{g+1} \zeta_g \cdots \zeta_{2} \zeta_1, \zeta_2^{-1} \zeta_3 \cdots \zeta_{g+1} \zeta_g \cdots \zeta_{2} \zeta_1, \zeta_3^{-1} \zeta_4 \cdots \zeta_{g+1} \zeta_g \cdots \zeta_{2} \zeta_1, \ldots, \zeta_{g+1}^{-1} \zeta_{g+2} \zeta_g \cdots \zeta_{2} \zeta_1, \zeta_{g+1}^{-1} \zeta_{g+2} \zeta_g \cdots \zeta_{2} \zeta_1, \zeta_{g+2}^{-1} \zeta_{g+3} \zeta_g \cdots \zeta_{2} \zeta_1, \ldots, \zeta_1, \zeta_2, \ldots, \zeta_{g+1}, \zeta_2) \zeta_{g+1} \ldots \zeta_{2})^{g-1}.$$

Let $f_{P J} : M_{P J} \to S^2$ and $f_{R I^{g-1}} : M_{R I^{g-1}} \to S^2$ be the hyperelliptic Lefschetz fibrations of genus $g$ determined by $P J$ and $R I^{g-1}$, respectively. Both $P J$ and $R I^{g-1}$ are chiral and have $6g^2 + 2g + 1$ singular fibers. One singular fiber is of type $\Pi_{g/2}$ and the others are of
type I. The manifolds $M_{P,I}$ and $M_{R_{g-1}}$ are homeomorphic but not diffeomorphic (see [9, Theorem 4.8] for $g \geq 4$ and Sato [34, Answer to Question 5.1] for $g = 2$). In particular $f_{P,I}$ and $f_{R_{g-1}}$ are not isomorphic. On the other hand, the (untwisted) \( \mathcal{H} \)-fiber sums $f_{P,I} \# f_0$ and $f_{R_{g-1}} \# f_0$ are \( \mathcal{H} \)-isomorphic because a Hurwitz system of $f_{P,I} \# f_0$ is transformed into that of $f_{R_{g-1}} \# f_0$ by elementary transformations and simultaneous conjugations (see [9, Theorem 4.10]).

We then show an example of pair of irreducible chiral hyperelliptic Lefschetz fibrations with the same number of singular fibers which are not stably isomorphic.

**Example 6.3.** Let $g$ be an odd integer greater than 1. We consider the following Hurwitz systems:

\[
C_1 := (\zeta_1, \zeta_2, \ldots, \zeta_{2g+1})^{2g+2}, \quad I^{g+1} := (\zeta_1, \zeta_2, \ldots, \zeta_{2g+1}, \zeta_2, \zeta_1)^{g+1}.
\]

Let $f_{C_1} : M_{C_1} \to S^2$ and $f_{I^{g+1}} : M_{I^{g+1}} \to S^2$ be the hyperelliptic Lefschetz fibrations of genus $g$ determined by $C_1$ and $I^{g+1}$, respectively. Both $f_{C_1}$ and $f_{I^{g+1}}$ are irreducible, chiral and have $2(g + 1)(2g + 1)$ singular fibers. \( \mathcal{C} \)-charts corresponding to $f_{C_1}$ and $f_{I^{g+1}}$ are $R_1$ and $((g + 1)/2)R_3 = ((g + 1)/2)G_0$, respectively (see Figures 22 and 27). The values of the invariant \( \tau \) for $f_{C_1}$ and $f_{I^{g+1}}$ are computed as follows (see Definitions 4.1 and 5.1):

\[
w(f_{C_1}) = w(R_4) = 1, \quad w(f_{I^{g+1}}) = w\left(\left(\frac{g + 1}{2}\right)R_3\right) = \frac{g + 1}{2}w(R_3) = 0.
\]

Therefore $f_{C_1}$ and $f_{I^{g+1}}$ are not stably isomorphic by Theorem 5.6. Since the images of monodromy representations of the (untwisted) \( \mathcal{H} \)-fiber sums $f_{C_1} \# N f_0$ and $f_{I^{g+1}} \# N f_0$ coincide with \( \mathcal{H}_g \) for any positive integer $N$, they are not even isomorphic by virtue of Proposition 2.7.

Both manifolds $M_{C_1}$ and $M_{I^{g+1}}$ are simply-connected and have the Euler characteristic $2(2g^2 + g + 3)$ and signature $-2(g + 1)^2$. If $g \equiv 3 \pmod{4}$, then $M_{I^{g+1}}$ is spin while $M_{C_1}$ is not. Hence they are not homeomorphic. If $g \equiv 1 \pmod{4}$, then both $M_{C_1}$ and $M_{I^{g+1}}$ are not spin, and they are homeomorphic by Freedman’s theorem. Since $M_{C_1}$ has a $(-1)$-section and $M_{I^{g+1}}$ is a non-trivial fiber sum, they are not diffeomorphic by a theorem of Usher [39]. The fact that $f_{C_1}$ and $f_{I^{g+1}}$ are not isomorphic also follows from this observation, or from a result of Smith [37] and Stipsicz [38].

We lastly show an example of pair of chiral hyperelliptic Lefschetz fibrations with the same numbers of singular fibers of each type which have a singular fiber of type $\Pi_{(g-1)/2}$ and are not stably isomorphic.

**Example 6.4.** Let $g$ be an odd integer greater than 1. We consider the following Hurwitz systems:

\[
Q := ((\zeta_1, \zeta_2, \ldots, \zeta_{2g+1})^{2g+2}, (\zeta_{g-1}, \ldots, \zeta_2, \zeta_1)^2, \\
\zeta_1^{-1}\zeta_2^{-1}\cdots\zeta_{g-1}\zeta_2\zeta_{g-1}\cdots\zeta_2\zeta_1, \\
\zeta_2^{-1}\zeta_3^{-1}\cdots\zeta_{g}\zeta_{g+1}\zeta_1\cdots\zeta_3\zeta_2, \\
\cdots, \\
\zeta_{g+1}\zeta_{g+2}\zeta_{g+3}\cdots\zeta_{g+2}\zeta_{g+1}\zeta_2\cdots\zeta_3\zeta_2),
\]

\[
R := (\zeta_1, \zeta_2, \ldots, \zeta_{2g+1}, (\zeta_{g-1}, \ldots, \zeta_2, \zeta_1)^2, \\
\zeta_1^{-1}\zeta_2^{-1}\cdots\zeta_{g-1}\zeta_2\zeta_{g-1}\cdots\zeta_2\zeta_1, \\
\zeta_2^{-1}\zeta_3^{-1}\cdots\zeta_{g}\zeta_{g+1}\zeta_1\cdots\zeta_3\zeta_2, \\
\cdots, \\
\zeta_{g+1}\zeta_{g+2}\zeta_{g+3}\cdots\zeta_{g+2}\zeta_{g+1}\zeta_2\cdots\zeta_3\zeta_2).
\]
Let \( f_Q : M_Q \rightarrow S^2 \) and \( f_R : M_R \rightarrow S^2 \) be the hyperelliptic Lefschetz fibrations of genus \( g \) determined by \( Q \) and \( R \), respectively. Both \( f_Q \) and \( f_R \) are chiral and have \( 2g^2 + 8g + 3 \) singular fibers. One singular fiber is of type \( \Pi_{(g-1)/2} \) and the others are of type \( \Gamma \). From the constructions of \( Q \) and \( R \) shown in [9, Lemma 3.4, Theorem 3.5] together with correspondence between moves for Hurwitz systems and those for \( \check{C} \)-charts (see [11, Lemma 20]), we can see that a \( \check{C} \)-chart corresponding to \( f_Q \) has one white vertex of type \( \check{r}_4 \) while that for \( f_R \) has no white vertices of type \( \check{r}_4 \). Thus we conclude that \( w(f_Q) = 1 \) and \( w(f_R) = 0 \), and \( f_Q \) and \( f_R \) are not stably isomorphic by Theorem 5.6. Since the images of monodromy representations of the (untwisted) \( \mathcal{H} \)-fiber sums \( f_Q \# N f_0 \) and \( f_R \# N f_0 \) coincide with \( \mathcal{H}_g \) for any positive integer \( N \), they are not even isomorphic by virtue of Proposition 2.7.

Both manifolds \( M_Q \) and \( M_R \) are simply-connected, non-spin and have the Euler characteristic \( 2g^2 + 4g + 7 \) and signature \(- (g + 1)^2 \). Hence they are homeomorphic by Freedman’s theorem. It does not seem to be known whether they are diffeomorphic.

We end this section by making two remarks.

**Remark 6.5.** For a pair of stably isomorphic hyperelliptic Lefschetz fibrations, it is not easy to determine the minimal number of copies of \( f_0 \) which make them \( \mathcal{H} \)-isomorphic. In particular, it does not seem to be known whether there exists a pair of stably isomorphic hyperelliptic Lefschetz fibrations which do not become \( \mathcal{H} \)-isomorphic after taking an \( \mathcal{H} \)-fiber sum with one copy of \( f_0 \). On the other hand, the following observation shows that there are many examples of non-\( \mathcal{H} \)-isomorphic hyperelliptic Lefschetz fibrations with the same base, the same fiber, and the same numbers of singular fibers of each type which become \( \mathcal{H} \)-isomorphic after one stabilization.

Let \( g \) be an integer greater than 1 and \( B_1, \ldots, B_r \) connected closed oriented surfaces. We consider a hyperelliptic Lefschetz fibration \( f_i : M_i \rightarrow B_i \) of genus \( g \) for each \( i \in \{1, \ldots, r\} \), and (possibly different) \( \mathcal{H} \)-fiber sums \( f \) and \( f' \) of \( f_1, \ldots, f_r \). It is shown by the same argument as the proof of [10, Proposition 4.10] that \( \mathcal{H} \)-fiber sums \( f \# f_0 \) and \( f' \# f_0 \) are \( \mathcal{H} \)-isomorphic to each other. For example, various \( \mathcal{H} \)-fiber sums of (generalizations of) Matsumoto’s fibration studied by Ozbagci and Stipsicz [30] and Korkmaz [27] become \( \mathcal{H} \)-isomorphic after one stabilization.

**Remark 6.6.** We consider the following Hurwitz systems in \( \mathcal{H}_3 \):

\[
U := ((\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7, \zeta_8, \zeta_9), \\
\zeta_3, \zeta_2, \zeta_1, \zeta_4, \zeta_5, \zeta_6, \zeta_7, \zeta_8, \zeta_9, \zeta_0, \zeta_1, \ldots, \zeta_7, \zeta_1, \ldots, \zeta_7),
\]

\[
V := ((\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7, \zeta_8, \zeta_9), \\
\zeta_3, \zeta_2, \zeta_1, \zeta_4, \zeta_5, \zeta_6, \zeta_7, \zeta_8, \zeta_9, \zeta_0, \zeta_1, \ldots, \zeta_7, \zeta_1, \ldots, \zeta_7),
\]
where $\zeta'_i := (\zeta_2^{-1} \zeta_1^{-1})^3 \zeta_0 (\zeta_1 \zeta_2)^3$ and $\zeta''_i := (\zeta_7^{-1} \cdots \zeta_1^{-1})^3 \zeta_0 (\zeta_1 \cdots \zeta_7)^3$. Let $f_U : M_U \to S^2$ and $f_V : M_V \to S^2$ be the hyperelliptic Lefschetz fibrations of genus 3 determined by $U$ and $V$, respectively. Both $f_U$ and $f_V$ are chiral and have 45 singular fibers. One singular fiber is of type $I_{II}$ and the others are of type $I$. Both manifolds $M_U$ and $M_V$ are simply-connected, non-spin and have the Euler characteristic 37 and signature $-25$. Hence they are homeomorphic by Freedman’s theorem. $\hat{C}$-charts corresponding to $f_U$ and $f_V$ are $\Gamma_U$ and $\Gamma_V$ depicted in Figures 30 and 31, respectively (see also Figure 32).

Since both $\Gamma_U$ and $\Gamma_V$ have a white vertex of type $\hat{r}_4$, we conclude that $w(f_U) = w(f_V) = 1$, and that $f_U$ and $f_V$ are stably isomorphic by Theorem 5.6. In fact it is not difficult to see that the (untwisted) $\mathcal{H}$-fiber sums $f_U \# f_0$ and $f_V \# f_0$ are $\mathcal{H}$-isomorphic (and hence isomorphic by Proposition 2.7) by applying chart moves to $\Gamma_U \oplus \Gamma_0$ and $\Gamma_V \oplus \Gamma_0$. It is not clear whether $f_U$ and $f_V$ are $\mathcal{H}$-isomorphic, and whether $M_U$ and $M_V$ are diffeomorphic. It would be worth noting that $M_U$ and $M_V$ cannot be distinguished by Usher’s theorem [39] because both $f_U$
and $f_V$ have a $(-1)$-section. Two Hurwitz systems $U$ and $V$ are related by a ‘partial twisting’ operation, namely, $V$ is obtained from $U$ by replacing $(\zeta_1, \ldots, \zeta_7)$ with $(\zeta_1', \ldots, \zeta_7')$. Such pairs of Hurwitz systems often yield pairs of 4-manifolds with subtle difference (see works of Auroux [2] and Yasui [40]).

Acknowledgements. The first author would like to thank Kokoro Tanaka and Isao Hasegawa for helpful discussions on charts and central extensions. The authors would like to thank the referee for helpful suggestions and comments.

References

1. D. Auroux, ‘A stable classification of Lefschetz fibrations’, Geom. Topol. 9 (2005) 203–217.
2. D. Auroux, ‘The canonical pencils on Horikawa surfaces’, Geom. Topol. 10 (2006) 2173–2217.
3. D. Auroux and I. Smith, ‘Lefschetz pencils, branched covers and symplectic invariants’, Symplectic 4-manifolds and algebraic surfaces, Lecture Notes in Mathematics 1938 (Springer, Berlin, 2008) 1–53.
4. R. I. Baykur, ‘Inequivalent Lefschetz fibrations and surgery equivalence of symplectic 4-manifolds’, J. Symplectic Geom. 14 (2016) 671–686.
5. R. I. Baykur and M. Korkmaz, ‘Small Lefschetz fibrations and exotic 4-manifolds’, Preprint, arXiv:1510.00089.
6. J. Birman and H.ilden, ‘On mapping class groups of closed surfaces as covering spaces’, Advances in the Theory of Riemann surfaces, Ann. of Math. Stud. 66 (Princeton Univ. Press, Princeton, NJ, 1971) 81–115.
7. S. K. Donaldson, ‘Lefschetz pencils on symplectic manifolds’, J. Differential Geom. 53 (1999) 205–236.
8. H. Endo, ‘Meyer’s signature cocycle and hyperelliptic fibrations’, (with Appendix written by T. Terasoma), Math. Ann. 316 (2000) 237–257.
9. H. Endo, ‘A generalization of Chakiris’ fibrations’, Groups of diffeomorphisms, Advanced Studies in Pure Mathematics 52 (Mathematical Society of Japan, Tokyo, 2008) 251–282.
10. H. Endo, I. Hasegawa, S. Kamada and K. Tanaka, ‘Charts, signatures, and stabilizations of Lefschetz fibrations’, Geom. Topol. Monogr. 19 (2015) 237–267.
11. H. Endo and S. Kamada, ‘Chart description for hyperelliptic Lefschetz fibrations and their stabilization’, Topology Appl. 196 (2015) 369–393.
12. H. Endo and S. Nagami, ‘Signature of relations in mapping class groups and non-holomorphic Lefschetz fibrations’, Trans. Amer. Math. Soc. 357 (2004) 3179–3199.
13. E. Fadell and J. Van Buskirk, ‘The braid groups of $E^n$ and $S^3$’, Duke Math. J. 29 (1962) 243–258.
14. B. Farb and D. Margalit, A primer on mapping class groups, Princeton Mathematical Series 49 (Princeton University Press, Princeton, NJ, 2011).
15. T. Fuller, ‘Hyperelliptic Lefschetz fibrations and branched covering spaces’, Pacific J. Math. 196 (2000) 369–393.
16. R. Gillette and J. Van Buskirk, ‘The word problem and consequences for the braid groups and mapping class groups of the 2-sphere’, Trans. Amer. Math. Soc. 131 (1968) 277–296.
17. R. E. Gompf, ‘A new construction of symplectic manifolds’, Ann. of Math. 142 (1995) 527–595.
18. R. E. Gompf and A. I. Stipsicz, 4-Manifolds and Kirby calculus, Graduate Studies Mathematics 20 (American Mathematical Society, Providence, RI, 1999).
19. I. Hasegawa, ‘Chart descriptions of monodromy representations on oriented closed surfaces’, Thesis, University of Tokyo, 2006.
20. T. Ito, ‘Splitting of singular fibers in certain holomorphic fibrations’, J. Math. Sci. Univ. Tokyo 9 (2002) 425–480.
21. S. Kamada, ‘Surfaces in $R^4$ of braid index three are ribbon’, J. Knot Theory Ramifications 1 (1992) 137–160.
22. S. Kamada, Braid and knot theory in dimension four, Mathematical Surveys and Monographs 95 (American Mathematical Society, Providence, RI, 2002).
23. S. Kamada, ‘Graphical descriptions of monodromy representations’, Topology Appl. 154 (2007) 1430–1446.
24. S. Kamada and Y. Matsumoto, ‘Certain racks associated with the braid groups’, Knots in Hellas ’98, Series on Knots and Everything 24 (World Scientific, Hackensack, NJ, 2000) 118–130.
25. A. Kas, ‘On the handlebody decomposition associated to a Lefschetz fibration’, Pacific J. Math. 89 (1980) 89–104.
26. V. M. Kharchakov and V. Kulikov, ‘On braid monodromy factorizations’, Izv. Math. 67 (2003) 499–534.
27. M. Korkmaz, ‘Noncomplex smooth 4-manifolds with Lefschetz fibrations’, Int. Math. Res. Not. IMRN 2001 (2001) 115–128.
28. W. Magnus, ‘Über Automorphismen von Fundamentalgruppen berandeter Flächen’, Math. Ann. 109 (1934) 617–646.
29. Y. Matsumoto, ‘Lefschetz fibrations of genus two — a topological approach’, Topology and Teichmüller spaces, Proceedings of the 37th Taniguchi Symposium (eds S. Kojima et al.; World Scientific Publishing, River Edge, NJ, 1996) 123–148.
COUNTING DIRAC BRAID RELATORS

30. B. Ozbagci and A. I. Stipsicz, ‘Noncomplex smooth 4-manifolds with genus-2 Lefschetz fibrations’, Proc. Amer. Math. Soc. 128 (2000) 3125–3128.
31. L. Paris and D. Rolfsen, ‘Geometric subgroups of mapping class groups’, J. reine angew. Math. 521 (2000) 47–83.
32. J. Park and K.-H. Yun, ‘Nonisomorphic Lefschetz fibrations on knot surgery 4-manifolds’, Math. Ann. 345 (2009) 581–597.
33. J. Park and K.-H. Yun, ‘Lefschetz fibrations on knot surgery 4-manifolds via Stallings twist’, Preprint, arXiv:1503.06272.
34. Y. Sato, ‘Canonical classes and the geography of nonminimal Lefschetz fibrations over $S^2$’, Pacific J. Math. 262 (2013) 191–226.
35. B. Siebert and G. Tian, ‘On hyperelliptic $C^\infty$-Lefschetz fibrations of four-manifolds’, Commun. Contemp. Math. 1 (1999) 466–488.
36. B. Siebert and G. Tian, ‘On the holomorphicity of genus two Lefschetz fibrations’, Ann. of Math. (2) 161 (2005) 959–1020.
37. I. Smith, ‘Geometric monodromy and the hyperbolic disc’, Quart. J. Math. 52 (2001) 217–228.
38. A. I. Stipsicz, ‘Indecomposability of certain Lefschetz fibrations’, Proc. Amer. Math. Soc. 129 (2001) 1499–1502.
39. M. Usher, ‘Minimality and symplectic sums’, Int. Math. Res. Not. IMRN 2006, Article ID 49857, 1–17.
40. K. Yasui, ‘Partial twists and exotic Stein fillings’, Preprint, arXiv:1406.0050.

Hisaaki Endo
Department of Mathematics
Tokyo Institute of Technology
2-12-1 Oh-okayama, Meguro-ku
Tokyo 152-8551
Japan
endo@math.titech.ac.jp

Seiichi Kamada
Department of Mathematics
Osaka City University
3-3-138 Sugimoto, Sumiyoshi-ku
Osaka 558-8585
Japan
skamada@sci.osaka-cu.ac.jp