The genesis of two-hump, W-shaped and M-shaped soliton propagations of the coupled Schrödinger-Boussinesq equations with conformable derivative

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Abstract. This work oversees with the coupled Schrödinger-Boussinesq equations with conformable derivative, which have lots of applications in laser and plasma. The said equations are reduced to a coupled stationary form using complex travelling wave transformation. Next Painlevé test applied to derived the integrable cases of the reduced equation, after that using RCAM derived the solution of reduced equations integrable and nonintegrable cases. Few theorems have been presented and proved to ensure their boundedness. All presented boundedness cases have been checked and explained by plotting the solutions for particulars values of parameters satisfying them. The obtained solutions of stationary form utilized to derive solutions of the coupled Schrödinger-Boussinesq equations with conformable derivative. The derived solutions have been plotted and explained. From this, it appears that these solutions propagate by maintaining their two-hump, W-shaped, M-shaped solutions shapes.

Key words. Coupled Schrödinger-Boussinesq equations – conformable derivative – Exact solution – Boundedness – Painlevé test – W-shaped, M-shaped solitons – Rapidly convergent approximation method

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1 Introduction

Diverse wave propagations observed in abundant fields of the environment are modelled by non-linear ordinary or partial differential equations (ODEs/PDEs) involving integer or fractional order derivative. Their chaotic features can be explained by the solutions of the equations which describes them. Due to the existence of interaction terms in such equations, it is not always an easy task to derive an exact solution to these equations. Despite that, a vast amount of literature exists for constructing exact travelling wave and soliton solutions [39,63,64,44,47,30,52,54] and approximation solutions [42,16,24,1,60,3,24,4,5,6,7,56,57,55,58, 59] of ODEs and PDEs involving integer derivative. Also there exists few direct methods [9,10,25,34,4] to derive exact solutions and and analytical methods [59,12,45] to deal with approximate solutions of nonlinear fractional differential equations. These direct methods often need to guess the syntax of solutions and solve a system of nonlinear algebraic equations. Thus these schemes fail when the choice of a form of solution is not suitable or unable to solve the system of equations. And the above mentioned approximate methods are often observed to have slow convergent rate and always unable to provide the close form of the series solution. To overcome the above-mentioned limitations of those methods we adopt the rapidly convergent approximation method (RCAM) [18,17,16,23,19,22,15,21,20]. This current work deals with this scheme to obtained some new solutions for the coupled Schrödinger-Boussinesq systems (SBS) with conformable derivative.

To attained the goal, the paper is organised as follows: in sections 2 and 3 we present the basic properties of conformal derivative and methodology of RCAM to solve a system of ODEs respectively. In section 4 we reduced SBS to coupled ODEs by employing a complex wave transformation. The Painlevé test has been employed to the reduced coupled ODEs and identified its integrable cases in section 5. In section 6 the integrable and nonintegrable cases of the reduced ODEs have been solved by RCAM. Few theorems have been presented to study their boundedness and utilised to plot the stationary form of the solutions. These solutions used to derive explicit solutions of SBS. Furthermore, the
To solve (2), we remodel it in an exponential matrix operator confirmation if linear exponential matrix operator can be recast in the form $X = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix}$, and $N = \begin{bmatrix} N_1 (U_1(x), \cdots, U_k(x)) \\ \vdots \\ N_k (U_1(x), \cdots, U_k(x)) \end{bmatrix}$. The inverse operator $\hat{O}^{-1}$ of $O[\hat{X}] (x)$ is conferred by

\[ \hat{O}^{-1} [\hat{X}] (x) = e^{-A x} \int e^{2 A x'} \int e^{-A x''} [\hat{X}] (x') dx'' dx'. \]
Operating $\mathcal{O}^{-1}$ on $\mathcal{O}[X](x)$ and employing integration by parts yields

$$\mathcal{O}^{-1}\left(\frac{\partial^2}{\partial x^2}X - A^2X\right) = X - Cxe^{Ax} - Dxe^{-Ax},$$

(6)

where $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$ and $D = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{bmatrix}$ are matrices of integration constants. Operating $\mathcal{O}^{-1}$ on (3) and utilising (6), provides

$$X = Cxe^{Ax} + Dxe^{-Ax} + \mathcal{O}^{-1}[N](x),$$

(7)

where $C$, and $D$ are three arbitrary constants matrices. To derive the unknown $X$ terms in the R.H.S of (7), we recast them in the syntax

$$X \cong \begin{bmatrix} U_1(x) \\ U_2(x) \\ \vdots \\ U_k(x) \end{bmatrix} = \sum_{m=0}^{\infty} \begin{bmatrix} U_{1,m}(x) \\ U_{2,m}(x) \\ \vdots \\ U_{k,m}(x) \end{bmatrix}$$

(8)

and $N = \sum_{m=0}^{\infty} \Delta_m(x)$, with

$$\Delta_m \cong \begin{bmatrix} \Delta_{1,m}(x) \\ \Delta_{2,m}(x) \\ \vdots \\ \Delta_{k,m}(x) \end{bmatrix} = \frac{1}{m!} \frac{d^m}{dx^m} \begin{bmatrix} N_1(x) \\ N_2(x) \\ \vdots \\ N_k(x) \end{bmatrix},$$

(9)

and $N_j(x), j = 1, 2, \cdots, k$ are given by

$$N_j(x) = \left(\sum_{k=0}^{\infty} U_{1,k}e^k, \sum_{k=0}^{\infty} U_{2,k}e^k, \cdots, \sum_{k=0}^{\infty} U_{k,k}e^k\right).$$

The terms $\Delta_{i,m}(x) = \Delta_{1,m}(U_{1,0}(x), U_{1,1}(x), \ldots, U_{1,m}(x), \ldots, U_{k,0}(x), U_{k,1}(x), \ldots, U_{k,m}(x)), i = 1, 2, \ldots, k$ are Adomian polynomials \cite{244567} outturn from the formula (9). Use of (9) in (11) provides

$$X = \begin{bmatrix} c_1 e^{\lambda_1x} + d_1 e^{-\lambda_1x} \\ c_2 e^{\lambda_2x} + d_2 e^{-\lambda_2x} \\ \vdots \\ c_k e^{\lambda_kx} + d_k e^{-\lambda_kx} \end{bmatrix} + \mathcal{O}^{-1} \begin{bmatrix} \Delta_{1,m}(x) \\ \Delta_{2,m}(x) \\ \vdots \\ \Delta_{k,m}(x) \end{bmatrix},$$

(10)

We obey the footsteps of \cite{10}, to get the higher order iteration terms as

$$X_0 \cong \begin{bmatrix} U_{1,0}(x) \\ U_{2,0}(x) \\ \vdots \\ U_{k,0}(x) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1x} + d_1 e^{-\lambda_1x} \\ c_2 e^{\lambda_2x} + d_2 e^{-\lambda_2x} \\ \vdots \\ c_k e^{\lambda_kx} + d_k e^{-\lambda_kx} \end{bmatrix},$$

(11)

$$X_{n+1} \cong \begin{bmatrix} U_{1,n+1}(x) \\ U_{2,n+1}(x) \\ \vdots \\ U_{k,n+1}(x) \end{bmatrix} = \mathcal{O}^{-1} \begin{bmatrix} \Delta_{1,n}(x) \\ \Delta_{2,n}(x) \\ \vdots \\ \Delta_{k,n}(x) \end{bmatrix},$$

(12)

$n \geq 0$. In case $\lambda_i > 0$, treatment of the vanishing boundary condition $U_i(\infty) = 0$ in (11) for the localized solution leads us to $c_i = 0$, $i = 1, 2, \cdots, k$. Therefore the leading and higher order iteration terms of the series solution are
produced by (12) and

\[
X_0 \cong \begin{bmatrix}
U_{1,0}(x) \\
U_{2,0}(x) \\
\vdots \\
U_{k,0}(x)
\end{bmatrix} = \begin{bmatrix}
d_1 e^{-\lambda_1 x} \\
d_2 e^{-\lambda_2 x} \\
\vdots \\
d_k e^{-\lambda_k x}
\end{bmatrix}.
\]

Further to obtain the localized solution in case \( \lambda_i < 0 \), for the boundary condition \( U_i(-\infty) = 0 \), we go ahead with restraining the term involving \( e^{\lambda_i x} \). In this case, the commanding and subsequent terms of the solution are yields by (12) with

\[
X_0 \cong \begin{bmatrix}
U_{1,0}(x) \\
U_{2,0}(x) \\
\vdots \\
U_{k,0}(x)
\end{bmatrix} = \begin{bmatrix}
c_1 e^{\lambda_1 x} \\
c_2 e^{\lambda_2 x} \\
\vdots \\
c_k e^{\lambda_k x}
\end{bmatrix}.
\]

One can easily obtain the iterative terms and the general term of the series (or generating function) by taking advantage of symbolic software. That leads to the exact solution of the discussed system of ODEs.

### 4 Solution of SBS equations

Consider the generalized Schrödinger-Boussinesq system (SBS) in the form

\[
i \left( \frac{\partial E}{\partial t} + \delta_1 \frac{\partial E}{\partial x} + \delta_2 \frac{\partial^2 E}{\partial x^2} = \delta_3 N E, \right.
\]

\[
\left. \frac{\partial^2 N}{\partial t^2} + \mu_1 \frac{\partial^2 N}{\partial x^2} + \mu_2 \frac{\partial^4 N}{\partial x^4} + \mu_3 \frac{\partial^2 N^2}{\partial x^2} = \mu_4 \frac{|\partial^2 E|^2}{2}. \right\}
\]

where \( E(x, t) \) is complex wave field, \( N(x, t) \) is real wave field, \( \delta_i, i = 1, 2, 3 \) and \( \mu_j, j = 1, 2, 3, 4 \) are arbitrary (real) parameters. The SBS studied in stationary propagation of coupled nonlinear magnetosonic waves and upper-hybrid in amagnetized plasmas in the field of laser and plasma it describes the interaction of long waves with short wave packets in nonlinear dispersive media, it plays important role in diatomic lattice system, and Langmuir soliton formation. Several researchers applied Lie point symmetry, Painlevé Analysis and Backlund transformations finite difference schemes, simplest equation method, extended trial equation method, and direct algebraic method to study its properties and solutions.

In this work, we consider the coupled Schrödinger-Boussinesq systems (SBS) with conformable derivative

\[
i \left( E^{(\beta)}_t + \delta_1 E^{(\alpha)}_x + \delta_2 E^{(\alpha)}_{xx} = \delta_3 N E, \right.
\]

\[
\left. \mu_1 N^{(\alpha)}_t + \mu_2 N^{(\alpha)}_{xx} + \mu_3 (N^2)^{(\alpha)}_{xx} = \mu_4 (|E|^2)^{(\alpha)}_{xx}, \right\}
\]

where \( f^{(\alpha)} \) and \( f^{(\beta)} \) represents \( \alpha \) and \( \beta \) order conformal derivative of \( f \) with respect to suffix variables respectively. Here our aim is to study its integrability and derive its new stationary different shaped exact solutions by employing RCAM. To attain the target we impose the travelling wave transformation

\[
E(x, t) = u(\xi) e^{i\eta}, \quad \eta = k_1 \frac{\mu}{\alpha} + k_2 \frac{\beta}{\beta} + c_0,
\]

\[
N(x, t) = v(\xi), \quad \xi = k_3 \frac{\epsilon}{\alpha} + c \frac{\beta}{\beta} + \xi_0,
\]

(17)

to (16) and equating real and imaginary parts we get

\[
c + k_3 \delta_1 + 2k_1 k_3 \delta_2 = 0,
\]

\[
k_2^2 \delta_2 u'' - (k_2 + k_1 \delta_1 + k_3 \delta_2)u - \delta_3 u v = 0,
\]

\[
k_3^2 \mu_2 v'' + (c^2 + \mu_1 k_3) u'' - 2k_3^2 \mu_4 (uu')' + 2k_3^2 \mu_3 (vv')' = 0.
\]

Integrating the last equation of the system twice and taking integration constant zero, reduces system (16) to
2. For κγ

6.1 Case-I

Solution of SBS to some special integrable and nontegrable cases by RCAM

for the values of the parameters

\[ c = -k_3(\delta_1 + 2k_1 \delta_2), \quad \alpha_1 = \frac{\delta_3}{\delta_2 k_2^2}, \quad \beta_1 = \frac{\mu_4}{\mu_2 k_2^2}, \quad \gamma_1 = -\frac{\mu_3}{\mu_2 k_2^2}, \]

\[ \lambda_1^2 = \frac{k_1(\delta_1 + 2k_1 k_2)}{\delta_2^2 k_2^4}, \quad \lambda_2 = -\frac{\delta_1^2 + 4\delta_1 \delta_2 k_1 + 4k_1^2 \delta_2^2 + \mu_1}{\mu_2 k_2^4}. \]

Here \( u(\xi) \) and \( v(\xi) \) are real fields, \( \xi \) is the (real) independent variable and all the other remaining quantities are free parameters. It is important to note here that the equations are invariant under the transformations (i) \( \xi \rightarrow -\xi \) (ii) \( u \rightarrow -u \), and (iii) \( \xi \rightarrow \xi + C \), where \( C \) is a constant.

5 The Painlevé test of Eq. (19)

The existence of solutions is a necessary condition of integrability, but it is not sufficient. To confirm the integrability other tests such as the Lax pair or the Painlevé test should be used for the proposed model. The Painlevé analysis is a powerful scheme to check the integrability of a system. Here we apply this important tools Mathematica package PainleveTest.m [10] to examine the integrability of equation (19). Application of the package yields the resonances of the considered equation as

\[ -1; 6; -\frac{\sqrt{-\alpha_1(23\alpha_1 - 48\gamma_1)} - 5\alpha_1}{2\alpha_1}; \frac{\sqrt{-\alpha_1(23\alpha_1 - 48\gamma_1)} + 5\alpha_1}{2\alpha_1}. \]

It can be checked that for resonances \(-1; 6\), this model [19] fails the Painlevé test, and the remaining two resonances are symbolic so we can not proceed further. To move forward we assume that these symbolic resonances are equal to positive integer \( k \) (say), which yields

\[ \pm \frac{\sqrt{-\alpha_1(23\alpha_1 - 48\gamma_1)} \pm 5\alpha_1}{2\alpha_1} = k \quad \text{or} \quad \gamma_1 = \frac{1}{12}(12 + k^2 - 5k) \alpha_1. \]

Subsequent, setting different positive integer values for \( k \) and using the same Mathematica package, we get the following two integrable cases:

1. For \( k = 8 \) compatibility condition is \( \gamma_1 = 3\alpha_1 \).
2. For \( k = 2 \) or 3 compatibility condition is \( \gamma_1 = \frac{\alpha_1}{3}, \quad \lambda_1 = \lambda_2. \)

6 Solution of SBS to some special integrable and nontegrable cases by RCAM

In this section we solve above presented integrable and one nonintegrable cases of [19] (or [10]) by RCAM.

6.1 Case-I \( \gamma_1 = 3\alpha_1 \)

In this case, applying RCAM we get following correction terms

\[
\begin{cases}
  u_0(\xi) = u_+ e^{\lambda_1 \xi}, \\
  v_0(\xi) = v_+ e^{\lambda_2 \xi}, \\
  u_1(\xi) = \frac{\alpha_1 u_+ v_+ e^{(\lambda_1 + \lambda_2)\xi}}{\lambda_1^2 - \lambda_2^2} - \frac{\alpha_2 u_+ v_+ e^{2\lambda_1 \xi}}{4\lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2)} - \frac{4\alpha_1^2 \lambda_1^2}{4\lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2)}, \\
  v_1(\xi) = \frac{\beta_1 u_+ v_+ e^{\lambda_1 \xi}}{\lambda_1^2 - \lambda_2^2} + \frac{\alpha_2 u_+ v_+ e^{2\lambda_1 \xi}}{4\lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2)} - \frac{4\alpha_1^2 \lambda_1^2}{4\lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2)}, \\
  u_2(\xi) = \frac{\alpha_1 u_+ + e^{\lambda_1 \xi}(\beta_1 \lambda_2^2 - 4\alpha_1^2 \lambda_2^2)}{4\lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2)}, \\
  v_2(\xi) = \frac{\beta_1 u_+ + e^{\lambda_1 \xi}(\alpha_2 \lambda_1^2 - 4\alpha_1^2 \lambda_1^2)}{4\lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2)}.
\end{cases}
\]
where

\[
u_\lambda = \frac{1}{\lambda_2 - \lambda_1} \left\{ \begin{aligned}
\alpha_1 \left[ \beta_1 \lambda_1 \lambda_2 \left( \lambda_1 + \lambda_2 \right) e^{2 \lambda_1 \xi} \right. \\
\times v^2 \left( 2 \lambda_1 + \lambda_2 \right) \left( \lambda_1^2 - \lambda_2^2 \right) v^2 e^{2 \lambda_2 \xi} \\
\left. + 16 \beta_1 \lambda_1 \lambda_2 \left( \lambda_1 + \lambda_2 \right) e^{2 \lambda_1 \xi} \right]
\end{aligned} \right. \\
\left[ \begin{aligned}
\alpha_2 \left[ \lambda_1 \lambda_2 \left( \lambda_1 + \lambda_2 \right) e^{2 \lambda_2 \xi} \right. \\
\times v^2 \left( 2 \lambda_1 + \lambda_2 \right) \left( \lambda_1^2 - \lambda_2^2 \right) v^2 e^{2 \lambda_2 \xi} \\
\left. + 16 \beta_1 \lambda_1 \lambda_2 \left( \lambda_1 + \lambda_2 \right) e^{2 \lambda_2 \xi} \right]
\end{aligned} \right. \\
\left[ \begin{aligned}
\alpha_3 \left[ \lambda_1 \lambda_2 \left( \lambda_1 + \lambda_2 \right) e^{2 \lambda_1 \xi} \right. \\
\times v^2 \left( 2 \lambda_1 + \lambda_2 \right) \left( \lambda_1^2 - \lambda_2^2 \right) v^2 e^{2 \lambda_2 \xi} \\
\left. + 16 \beta_1 \lambda_1 \lambda_2 \left( \lambda_1 + \lambda_2 \right) e^{2 \lambda_2 \xi} \right]
\end{aligned} \right. \\
\left[ \begin{aligned}
\alpha_4 \left[ \lambda_1 \lambda_2 \left( \lambda_1 + \lambda_2 \right) e^{2 \lambda_1 \xi} \right. \\
\times v^2 \left( 2 \lambda_1 + \lambda_2 \right) \left( \lambda_1^2 - \lambda_2^2 \right) v^2 e^{2 \lambda_2 \xi} \\
\left. + 16 \beta_1 \lambda_1 \lambda_2 \left( \lambda_1 + \lambda_2 \right) e^{2 \lambda_2 \xi} \right]
\end{aligned} \right. \\
\left[ \begin{aligned}
\alpha_5 \left[ \lambda_1 \lambda_2 \left( \lambda_1 + \lambda_2 \right) e^{2 \lambda_1 \xi} \right. \\
\times v^2 \left( 2 \lambda_1 + \lambda_2 \right) \left( \lambda_1^2 - \lambda_2^2 \right) v^2 e^{2 \lambda_2 \xi} \\
\left. + 16 \beta_1 \lambda_1 \lambda_2 \left( \lambda_1 + \lambda_2 \right) e^{2 \lambda_2 \xi} \right]
\end{aligned} \right. \\
\left[ \begin{aligned}
\alpha_6 \left[ \lambda_1 \lambda_2 \left( \lambda_1 + \lambda_2 \right) e^{2 \lambda_1 \xi} \right. \\
\times v^2 \left( 2 \lambda_1 + \lambda_2 \right) \left( \lambda_1^2 - \lambda_2^2 \right) v^2 e^{2 \lambda_2 \xi} \\
\left. + 16 \beta_1 \lambda_1 \lambda_2 \left( \lambda_1 + \lambda_2 \right) e^{2 \lambda_2 \xi} \right]
\end{aligned} \right. .
\]

Theorem 2 The solution \(u(x, \xi)\) will be bounded if parameters \(\lambda_1, \lambda_2, \alpha_1, \beta_1\) involved in the equation and integration constants \(u_\lambda, v_\lambda\) satisfy any one of the conditions given in the table 7.

Proof Solution \(u(x, \xi)\) and \(v(x, \xi)\) have a common factor in the denominator, which under transformation \(e^\xi = Z\), reduces to a generalized Dirichlet polynomial \(\text{Pol}(Z) = a_1 Z^{2 \lambda_1 + \lambda_2} + a_2 Z^{2 \lambda_1} + a_3 Z^{\lambda_2} + a_4\).
Table 1: Boundedness conditions of theorem

| Case      | Con. No. | $\lambda_2$ | $\alpha_1$ | $\beta_1$ | $u_-$ | $v_-$ |
|-----------|----------|-------------|------------|----------|------|------|
| $\lambda_1 < \frac{\lambda_2}{2} < 0$ | 1(a) | - | + | - | - | - |
|          | 1(b) | - | - | + | - | + |
|          | 1(c) | - | + | - | - | + |
|          | 1(d) | - | - | + | - | + |
| $\lambda_1 > \frac{\lambda_2}{2} > 0$ | 2(a) | - | + | - | - | - |
|          | 2(b) | - | - | + | - | + |
|          | 2(c) | - | - | + | - | + |
|          | 2(d) | - | - | + | - | + |
| $\lambda_1 < -\frac{\lambda_2}{2} < 0$ | 3(a) | + | + | - | - | - |
|          | 3(b) | + | - | + | - | + |
|          | 3(c) | + | + | + | - | - |
|          | 3(d) | + | - | + | - | + |
| $\lambda_1 > \frac{\lambda_2}{2} > 0$ | 4(a) | + | + | - | - | - |
|          | 4(b) | + | - | + | - | + |
|          | 4(c) | + | + | - | - | + |
|          | 4(d) | + | - | + | - | + |
| $\frac{\lambda_2}{2} < \lambda_1 < 0$ | 5(a) | - | + | + | - | - |
|          | 5(b) | - | - | + | - | + |
|          | 5(c) | - | - | + | - | + |
|          | 5(d) | - | - | + | - | + |
| $0 < \lambda_1 < -\frac{\lambda_2}{2}$ | 6(a) | - | + | + | - | - |
|          | 6(b) | - | - | + | - | + |
|          | 6(c) | - | - | + | - | + |
|          | 6(d) | - | - | + | - | + |
| $-\frac{\lambda_2}{2} < \lambda_1 < 0$ | 7(a) | + | + | + | - | - |
|          | 7(b) | + | - | + | - | + |
|          | 7(c) | + | + | + | - | - |
|          | 7(d) | + | - | + | - | + |
| $0 < \lambda_1 < \frac{\lambda_2}{2}$ | 8(a) | + | + | + | - | - |
|          | 8(b) | + | - | + | - | + |
|          | 8(c) | + | + | + | - | - |
|          | 8(d) | + | - | + | - | + |

where

$$a_1 = \alpha_1^2 \beta_1 (\lambda_2 - 2 \lambda_1)^2 u_+^2 v_-, \quad a_2 = -2 \alpha_1 \beta_1 \lambda_2^2 (2 \lambda_1 + \lambda_2)^2 u_+^2,$$

$$a_3 = -8 \alpha_1 \lambda_2^2 (2 \lambda_1 - \lambda_2) (2 \lambda_1 + \lambda_2)^3 v_-, \quad a_4 = 16 \lambda_2^4 (2 \lambda_1 - \lambda_2) \lambda_2^2 (2 \lambda_1 + \lambda_2)^3.$$

Solution (23)-(24) is unbounded if there exists at least one real positive root. So the boundedness of the solution is ensured by the conditions that the polynomial never have positive real root. Such conditions can be provided by Descartes’ rule of signs [36]. Which states that the polynomial $Pol(Z)$ does not contain any positive real root if its coefficients never changes their signs. That leads us to the condition that either all $a_i > 0$ ($i = 1, 2, 3, 4$) or all $a_i < 0$ ($i = 1, 2, 3, 4$). Restrictions $a_i > 0$ ($i = 1, 2, 3, 4$) yields the condition 1.(a)-4.(d) of the theorem, whereas remaining conditions provided by $a_i < 0$ ($i = 1, 2, 3, 4$). That completes the prove of the theorem.

Next, we are interested in establishing the conditions presented in the above theorem and study the features of the solution (23)-(24). For that, we have taken particular values for different parameter satisfying the conditions of the theorem in table 2 and utilising them to plot the solution shown in figure 1. From the 2D plots, it is clear that the solution is enriched with several one-hump, two-hump, W-shape, M-shape soliton like features.

Solution-I
So in this case solution of (16) can be obtained from (23)-(24) with (17) and (20) in the form

\[
E_1(x,t)=\left\{8\lambda_1^2 (2\lambda_1-\lambda_2) (2\lambda_1+\lambda_2) \lambda_2^2 v_- e^{-\lambda_1 \xi} 
\right. \\
\times \left. v_- e^{\lambda_2 \xi} \right\} / Q(\xi) \ e^{i(k_1 x+\omega t)} \ e^{\alpha t} \ e^{\beta t},
\]
\[
N_1(x,t) = 4 (2\lambda_1-\lambda_2) (2\lambda_1+\lambda_2) \lambda_2 \lambda_1^2 u^2 \ e^{2\lambda_1 \xi} \left(4\lambda_1^2 (2\lambda_1+\lambda_2)^2 \lambda_1^4 + \lambda_2^2 (\lambda_2-2\lambda_1)^2 \right) \\
\times v_- e^{2\lambda_2 \xi} - \alpha_1 (\lambda_2^3 - 4\lambda_1^2 \lambda_2) \lambda_2^2 v_- e^{\lambda_2 \xi} \right\} / Q(\xi)^2
\]

where

\[
Q(\xi) = \alpha_1 \beta_1 u^2 \ e^{2\lambda_1 \xi} \left\{ \alpha_1 (\lambda_2 - 2\lambda_1)^2 v_- e^{\lambda_2 \xi} - 2\lambda_2^2 (2\lambda_1 + \lambda_2)^2 \right\} + 8\lambda_1^2 \\
\times (2\lambda_1-\lambda_2) (2\lambda_1+\lambda_2) \lambda_1^3 (2\lambda_2^2 - \alpha_1 v_- e^{\lambda_2 \xi}), \quad \xi = k_3 \frac{e^{\alpha t}}{\alpha} + \frac{e^{\beta t}}{\beta} + \xi_0.
\]
This solution exists provided
\[
\delta_1 = -\frac{\delta_2 \mu_3}{3 \mu_2}, \quad \frac{k_1 (\delta_1 + \delta_2 k_1) + k_2}{\delta_2 k_3^2} > 0, \quad \text{and} \quad -\frac{\delta_1^2 + 4 \delta_2 \delta_1 k_1 + 4 \delta_2^2 k_1^2 + \mu_1}{\mu_2 k_3^2} > 0,
\]
which are yield from the integrability condition \(\gamma_1 = 3 \alpha_1\) and the requirement that \(\lambda_1, \lambda_2\) are real. The boundedness of solution \((27)-(28)\) can be easily ensured by theorem 2 with the conditions \((29)\). Utilising one of such conditions the solution \((27)-(28)\) has been plotted in figure 2. From the Fig. 1 on the (x,t)-plane, it is clear that absolute value of solution \(E_1\) represents W-shaped soliton wave, real and imaginary parts of \(E_1\) of the wave solution represent the Akhmediev breather (AB) wave, which can evolve periodically along a certain angle with the t axis and component \(N_1\) propagate with W-shaped soliton wave.

6.2 Case-II \(\alpha_1 = 2 \gamma_1, \quad \lambda_1 = \lambda_2 = \lambda \) (say)

In this case, RCAM gives the following correction terms

\[
\begin{align*}
u_0(\xi) &= u_- e^{-\lambda \xi}, \\
u_1(\xi) &= v_- e^{-\lambda \xi}, \\
u_2(\xi) &= \frac{2 \gamma_1 u_- - e^{-2 \lambda \xi}}{3 \lambda} , \\
u_3(\xi) &= \frac{e^{-2 \lambda \xi} (\beta_1 u_+^2 + \gamma_1 v_+^2)}{3 \lambda}, \\
u_4(\xi) &= \frac{\gamma_1 u_- - e^{-3 \lambda \xi} (\beta_1 u_+^2 + 3 \gamma_1 v_+^2)}{12 \lambda^3}, \\
u_5(\xi) &= \frac{e^{-3 \lambda \xi} (2 \lambda^3 (\beta_1 u_+^2 + 3 \gamma_1 v_+^2))}{54 \lambda^6}.
\end{align*}
\]

where \(u_-\) and \(v_-\) are integration constants. Summing the above series terms one can obtain close form solution of \((19)\) in the form

\[
\begin{align*}
u(\xi) &= \frac{36 \lambda^4 u_- e^{\lambda \xi} (36 \lambda^4 e^{2 \lambda \xi} - \beta_1 \gamma_1 u_+^2 - \gamma_1 v_+^2)}{(36 \lambda^4 e^{2 \lambda \xi} - \beta_1 \gamma_1 u_+^2 + \gamma_1 v_+^2 - 12 \gamma_1 \lambda^2 e^{2 \lambda \xi})^2}, \\
u(\xi) &= \frac{36 \lambda^4 e^{\lambda \xi} (12 \beta_1 \lambda^2 u_+^2 + \gamma_1 u_+^2 - 12 \gamma_1 \lambda^2 v_+^2 e^{2 \lambda \xi})}{(36 \lambda^4 e^{2 \lambda \xi} - \beta_1 \gamma_1 u_+^2 + \gamma_1 v_+^2 - 12 \gamma_1 \lambda^2 e^{2 \lambda \xi})^2}.
\end{align*}
\]

In the following, a theorem has been proposed and proved to ensure the boundedness of this derived solution.

**Theorem 3** The solution \((28)\) will be bounded if real parameters \(\lambda, \beta_1, \gamma_1\) present in the equation and integration constants \(v_-, u_-\) involved in solution satisfies any one of the following conditions

\(Ia.\) \(\beta_1 > 0 \& \gamma_1 < 0\)
\[ E_2(x,t) = \frac{36\lambda^4e^{-\alpha t}\left(36\lambda^4e^{2\alpha t} + \beta_1\gamma_1u^2 - \gamma_1^2v^2\right)}{\left(36\lambda^4e^{2\alpha t} - \beta_1\gamma_1u^2 - \gamma_1^2v^2 - 12\gamma_1\lambda^2v_\alpha e^{\alpha t}\right)^2} e^{ik_1x + k_2t + \alpha t}, \]

\[ N_2(x,t) = 36\lambda^4e^{\alpha t}\left[12\beta_1\lambda^2u^2e^{\alpha t} - \beta_1\gamma_1u^2v^2 - \gamma_1^2v^2 - 12\gamma_1\lambda^2v_\alpha e^{\alpha t}\right] / \left[36\lambda^4e^{2\alpha t} - \beta_1\gamma_1u^2 - \gamma_1^2v^2 - 12\gamma_1\lambda^2v_\alpha e^{\alpha t}\right]^2, \]
Table 3: Particular values of parameters satisfying conditions presented in Theorem \(3\) & \(4\) used in Fig. \(3\)

| \(\lambda\) | \(c_1\) | \(\delta_1\) | \(\gamma_1\) | \(\omega\) | \(\varphi\) | Fig. |
|---|---|---|---|---|---|---|
| .3 | 30 | -20 | 2 | -15 | Ia |
| .3 | -30 | 20 | -2 | 15 | Ib |
| .3 | 30 | 20 | 2.44 | -3 | IIA |
| .8 | 30 | -20 | 18.7 | -3 | IIb |
| .8 | 1 | -3 | 3 | -2 | IIIa |
| 1 | -0.1 | 3 | -3 | 2 | IIIb |
| 1 | -1 | 3 | 2 | | IIIc |
| 1 | 3 | 2 | | | IId |

where \(\xi = k \sqrt{\alpha} + c \frac{a}{\sqrt{\beta} + \xi_0}\). This solution exist provided

\[
\lambda_3 = -\frac{2\delta_2\mu_3}{\mu_2}, \quad k_2 = -\frac{\delta_2\delta_1^2 + \delta_1\kappa_1\left(4\delta_2^2 + \mu_2\right) + \delta_2\left(k_1^2 - 4\delta_2^2 + \mu_2 + \mu_1\right)}{\mu_2 \kappa_3^2}, \quad \text{and}
\]

\[
-\frac{\delta_2^2 + 4\delta_2\delta_1\kappa_1 + 4\delta_2^2\kappa_1^2 + \mu_1}{\mu_2 \kappa_3^2} > 0,
\]

which are derived using the integrability condition \(\alpha_1 = 2\gamma_1, \quad \lambda_1 = \lambda_2 = \lambda\) and demand that \(\lambda\) is real.

The boundedness of solution \((29)-(30)\) can be easily ensured by theorem \(3\) with the conditions \((20)\). Using one among those conditions solution \((20)-(30)\) is plotted in figure \(3\). Fig. \(3\) shows that the absolute value of solution \(E_2\) have M-shaped soliton profile, whereas real and imaginary parts of \(E_2\) of the wave solution constitute the Akhmediev breather (AB) wave, the wave is not the space-periodic breather but the time-periodic breather and component \(N_2\) propagate with keeping M-shaped form soliton wave. In the previous cases, we have derived exact solutions of SBS by eliminating one or more free parameters using compatibility conditions of the Painlevé test. Then it is of natural curiosity to explore whether it is possible to obtain an exact solution of the equations without using compatibility conditions (nonintegrable case) containing all parameters involved in the equation? To answer that here we consider a nonintegrable case \(\lambda_1 = \lambda_2\). To solve the considered system of equations rather eliminating any other additional parameters, we eliminate one integration constant.

6.3 Case-III \(\upsilon_\pm = \sqrt{\alpha_1 - \gamma_1} \sqrt{\alpha_1 - \gamma_1}, \quad \lambda_1 = \lambda_2 = \lambda \text{ (say)}\)

In this case, RCAM gives the following correction terms

\[
\begin{cases}
  u_0(\xi) = u_\pm e^{-\lambda \xi}, \\
  v_0(\xi) = \frac{\sqrt{\alpha_1 - \gamma_1}}{\sqrt{\alpha_1 - \gamma_1}} e^{-\lambda \xi},
\end{cases}
\]

\[
\begin{cases}
  u_1(\xi) = \frac{\alpha_1 \beta_1 u^2}{3\lambda^3 (\alpha_1 - \gamma_1)} e^{-2\lambda \xi}, \\
  v_1(\xi) = \frac{\alpha_1 \beta_1 u_\pm^2}{3\lambda^3 (\alpha_1 - \gamma_1)} e^{-2\lambda \xi},
\end{cases}
\]

\[
\begin{cases}
  u_2(\xi) = \frac{\alpha_1 \beta_1 u^3}{12\lambda^4 (\alpha_1 - \gamma_1)} e^{-3\lambda \xi}, \\
  v_2(\xi) = \frac{\alpha_1 \beta_1 u_\pm^3}{12\lambda^4 (\alpha_1 - \gamma_1)} e^{-3\lambda \xi},
\end{cases}
\]

\[
\begin{cases}
  u_3(\xi) = \frac{\alpha_1 \beta_1^2 u^4}{54\lambda^5 (\alpha_1 - \gamma_1)^2} e^{-4\lambda \xi}, \\
  v_3(\xi) = \frac{\alpha_1 \beta_1^2 u_\pm^4}{54\lambda^5 (\alpha_1 - \gamma_1)^2} e^{-4\lambda \xi},
\end{cases}
\]
So the solution of (16) can be obtained from (31) with (17) and (20) in the form

\[
\begin{align*}
    u_m(\xi) &= \frac{(m+1)}{6^m} u_- e^{-\xi} \left[ \frac{a_1 \sqrt{a_1 - \xi}}{\lambda^2 \sqrt{\alpha_1 + 1}} \right]^m, \\
v_m(\xi) &= \frac{3^2 (m+1) \alpha}{6^m \alpha_2} \left[ \frac{a_1 \sqrt{a_1 - \xi}}{\lambda^2 \sqrt{\alpha_1 + 1}} \right]^m,
\end{align*}
\]

where \( u_- \) and \( v_- \) are integration constants. Summing the above series terms one can derive the close form solution of (19) in the form

\[
\begin{align*}
    u(\xi) &= \frac{36 u_- \lambda^4 (a_1 - \gamma_1)^2 e^{2\lambda \xi}}{(6 \lambda^2 (a_1 - \gamma_1) e^{\lambda \xi} - u_- a_1 \sqrt{a_1 - 1})^2}, \\
v(\xi) &= \frac{36 u_- \lambda^4 \sqrt{a_1 - \gamma_1} \beta_1 e^{\lambda \xi}}{(6 \lambda^2 (a_1 - \gamma_1) e^{\lambda \xi} - u_- a_1 \sqrt{a_1 - 1})^2}.
\end{align*}
\]  (31)

To make the solution (31) physically relevant below we present a theorem to derive its bounded cases.

**Theorem 4** The solution (31) will be bounded if parameters \( \alpha_1, \beta_1, \gamma_1 \) involved in the equation and integration constant \( u_- \) involve in solution satisfies any one of the following conditions

**IIIa** \( \alpha_1 > 0, \beta_1 > 0, \ u_- < 0 \) and \( \alpha_1 > \gamma_1 \)

**IIIb** \( \alpha_1 > 0, \beta_1 < 0, \ u_- < 0 \) and \( \alpha_1 < \gamma_1 \)

**IIIc** \( \alpha_1 < 0, \beta_1 > 0, \ u_- > 0 \) and \( \alpha_1 > \gamma_1 \)

**IIId** \( \alpha_1 < 0, \beta_1 < 0, \ u_- > 0 \) and \( \alpha_1 < \gamma_1 \)

**Proof** The components \( u(\xi) \) and \( v(\xi) \) of the solution (31) have a common denominator given by the quartic polynomial

\[
(6 \lambda^2 (a_1 - \gamma_1) e^{\lambda \xi} - u_- a_1 \sqrt{a_1 - 1})^2.
\]

The repeated roots of the polynomial given by

\[
\xi = \frac{1}{\lambda} \log \left[ \frac{a_1 u_- \sqrt{a_1 - 1}}{6 \lambda^2 \sqrt{a_1 + 1}} \right].
\]

The positive real values of these roots make denominator zero that leads to an unbounded solution. So the requirement of boundedness of solution suggests that the argument of the \( \log \) has to be complex or negative. The complex case is not admissible here because it makes solutions complex, so the real bounded solutions given by the conditions among parameters

\[
\beta_1 (a_1 - \gamma_1) > 0 \quad \text{and} \quad \alpha_1 u_- < 0.
\]

These restrictions can be split to derive the conditions of the theorem.

Few particular values of the free parameters which satisfy the conditions of the theorem presented in table III and using them solution have been plotted in figure 3. The 2D plots ensure that solution always has a one-hump soliton-like profile.

**Solution-III**

So the solution of (16) can be obtained from (31) with (17) and (20) in the form

\[
E_3(x,t) = \frac{36 u_- \lambda^4 (a_1 - \gamma_1)^2 e^{\lambda \xi}}{(6 \lambda^2 (a_1 - \gamma_1) e^{\lambda \xi} - u_- a_1 \sqrt{a_1 - 1})^2} e^{i(k_1 \frac{x}{\alpha} + k_2 \frac{t}{\beta} + c_0)},
\]

\[
N_3(x,t) = \frac{36 u_- \lambda^4 \beta_1 (a_1 - \gamma_1)^2 e^\xi}{(6 \lambda^2 (a_1 - \gamma_1) e^{\lambda \xi} - u_- a_1 \sqrt{a_1 - 1})^2}, \quad \xi = k_3 \frac{x}{\alpha} + c \frac{t}{\beta} + \xi_0,
\]  (32)

provided

\[
k_2 = - \frac{\delta_2 \xi^2 + \delta_1 k_3 (4 \delta_2^2 + \mu_2) + \delta_2 (k_1^2 (4 \delta_2^2 + \mu_2) + \mu_1)}{\mu_2},
\]

\[
- \frac{\delta_2^2 + 4 \delta_2 \delta_1 k_3 + 4 \delta_2^2 k_1^2 + \mu_1}{\mu_2 k_1^2} > 0,
\]  (33)

which are given by condition \( \lambda_1 = \lambda_2 = \lambda \) and assertion that \( \lambda \) is real. One can easily derive the boundedness conditions of solution (32)- (33) by using theorem 3 with the conditions (20). Using one among those conditions graphical representations of the solution (32)- (33) are presented in figure 3. The figure shows that the absolute value of solution \( E_3 \) has always soliton profile, whereas real and imaginary parts of \( E_3 \) of the wave solution constitute a space-periodic breather and component \( N_3 \) propagate with maintaining soliton shaped form.
7 Conclusions

In this work, a few new exact solutions of three integrable/nonintegrable cases of SBS with space-time conformable derivative have been derived. Here we have applied a complex travelling wave transformation and a modified RCAM to solve the SBS. Painlevé test is used to identify the integrable cases of the stationary form of SBS. In general Painlevé analysis cannot handle the case when considered equation contains parameter coefficients. In this case, the arbitrariness of parameters leads to the symbolic resonances and one can not execute the remaining steps of Painlevé test. But here we have proposed an alternative technique to handle this limitation and derived integrable cases. We have classified all the bounded physically relevant cases of the solutions and presented them in three theorems. General theories of algebra have been utilized to prove the theorems. In addition to that, all bounded cases have been checked and used in plots to establish our claims. The presented 2D and 3D plots of solutions of SBS reflect the appearance of a few new two-hump, W-shaped, M-shaped of solution propagation state. We believe these findings are new, not available in the literature for the considered equation. The results can be helpful for analyzing the dynamics of nonlinear localized waves in the generalized coupled SBS and other coupled systems.

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Fig. 1: Plots of the solution $u(\xi), v(\xi)$ for the conditions presented in Theorem 2, using values of the parameters submitted in Table 2.
Fig. 2: Plots of the solution (25)-(26) for values of parameters values $k_1 = 0.3$, $k_2 = -0.1585$, $k_3 = 2$, $\delta_1 = 0.5$, $\delta_2 = 0.5$, $\mu_1 = -0.515$, $\mu_2 = -0.5$, $\mu_3 = 28.5$, $\mu_4 = 0.075$, $\alpha = 0.8$, $\beta = 0.9$, $c_0 = 9$, $\xi_0 = 0.4$, $u_\rightarrow = -0.5$, $v_\rightarrow = -0.6$.

Fig. 3: Plot of solutions (28) and (31) for the conditions presented in Theorem 3 & 4, using values of parameters given in Table 3.
Fig. 4: Plot of the solution (29)-(30) for values of parameters values $k_1 = 0.4$, $k_3 = 1.4$, $\delta_1 = 0.5$, $\delta_2 = 0.6$, $\mu_1 = 0.2$, $\mu_2 = -1$, $\mu_3 = 0.4$, $\mu_4 = 0.5$, $\nu_- = 0.4$, $\nu_+ = 0.99$, $\alpha = .7$, $\beta = .5$, $c_0 = 1$, $\xi_0 = -4$. 
Fig. 5: Plot of solution (32) for values of parameters values $k_1 = 0.4$, $k_3 = 0.4$, $\delta_1 = -1$, $\delta_2 = 0.6$, $\delta_3 = 0.6$, $\mu_1 = 0.2$, $\mu_2 = -2$, $\mu_3 = -0.4$, $\mu_4 = -0.5$, $\alpha = 0.8$, $\beta = 0.9$, $c_0 = 9$, $\xi_0 = 3$, $u_- = -37.4$. 
