Quantum adiabatic pumping by modulating tunnel phase in quantum dots

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In mesoscopic system, at zero bias voltage condition, finite charge is transferred by quantum adiabatic pumping with adiabatically and periodically changing two or more control parameters. We obtained the expressions of pumped charge for series two quantum dots (QDs) and three QDs ring by choosing a tunnel phase as one of the control parameters. The pumped time-averaged current is given by dividing the pumped charge per cycle by the period. For the series two QDs, the pumped current is effectively equivalent to the current driven by a finite bias voltage determined by the gauge transformation and hence the pumped current is linearly proportional to the variance of the phase. However, for the three QDs ring, such a gauge transformation cannot be applied. By the gauge invariance, the phase is equivalent to the flux penetrating the ring. Especially we found that the pumped charge shows a step-like behavior with respect to the variance of the flux. We evaluated the maximum values of the pumped charge by changing parameters for various setups.

I. INTRODUCTION

Recently quantum adiabatic pumping gathers much attention1–5. Quantum adiabatic pumping induces finite current in a mesoscopic system at zero bias voltage by adiabatically and periodically changing two or more control parameters. The classical pumping based on the Coulomb charging effect such as single electron transistors6,7 or turnstile devices8 does not require phase coherence. In contrast, the quantum pumping is fundamentally different from the classical pumping. Periodic deformation of two or more parts of the potential induces phase-coherent redistribution of the electron charges in an open quantum system. During this redistribution, electrons can be coherently pumped from one lead to the other leads. In spatially periodic systems, Thouless had shown a quantized charge transport induced by the adiabatic and periodic change of the potentials8. Later, a formulation using scattering matrix appeared9–11, which is called Brouwer’s formula. This formula can only be applied to the quantum adiabatic pump in non-interacting systems. The difference between the modulating phase of control parameters (this phase is not a scattering phase, which will appear in the following) in the Brouwer’s formula determines the magnitude and the sign of the pumped charge. An experimental demonstration of the Brouwer’s formula had been pursued12. In this experiment, two gate voltages controlled the periodic deformations of the shape of the quantum dot (QD), and then the pumped current was observed, where the amplitude of the current changed with the phase difference between the two gate voltages, as the theory predicted. However the result is still open for arguments since the pumped current can also be explained by the rectification effect of the displacement currents generated by the time-dependent gate voltage13,14.

In addition to the experimental studies, there followed several detailed theoretical studies of the Brouwer’s formula. For example, the maximum value of the pumped charge per cycle becomes exactly an elementary charge15 by properly choosing two potentials as control parameters in one QD system. The effect of the resonance on the quantum adiabatic pumping had been analyzed in a double-barrier quantum well16 and a QD in a turnstile geometry17. The effect of dephasing had also been studied18–20. In two terminal systems, the scattering matrix is given by 2 × 2 unitary matrix and there are four independent real parameters in this matrix. Avron et al. clarified the roles of each parameter in transport process21. Additionally, the inverse process, namely the adiabatic quantum motors by applying finite bias, had been analyzed with scattering matrix formalisms22.

As we saw above, there had been many researches of the quantum adiabatic pumping with modulating potentials, however there are no researches choosing a scattering phase as one of the control parameters. Avron et al. had considered the role of the phase in the quantum transport in a general framework22, however there seemed no studies based on an explicit model, system. On the other hand, the quantum mechanical phase plays an important role in the field of quantum transport. The Josephson effect23 is one of the examples, where the current occurs by the difference of the phase of macroscopic wave functions. The electron phase is a quantum mechanical value, and electrons get the phase by transporting in the scattering region, e.g. QDs. Choosing the phase as one of the control parameters is physically interesting because the phase can be related to bias-voltage through the Josephson relation. The effect of time-dependent vector potential on the electrons in the metallic system had been explored24,25. Also, a quantum adiabatic pumping using ac Josephson effect had been proposed26. To investigate how periodicity and quantum property of the phase appear in the transport process,
we analyze the quantum adiabatic pumping by choosing a tunnel phase as one of the control parameters.

We treat series of two QDs in former half of the paper. If we choose the energy levels of the two QDs as control parameters, we find that the maximum value of pumped charge per cycle is exactly an elementary charge. This value is the same as the maximum value per cycle choosing two potentials in one QD system\cite{22}. Moreover, a quantum adiabatic pumping with single control parameter under finite AB flux had been examined\cite{23}. If we choose one QD energy level and the flux penetrating the ring as control parameters, the pumped charge becomes sinusoidal form or step-like form as a function of the variance of the phase depending on the values of system parameters (the energy levels of QDs, the tunnel strengths). For the parameters realizing step-like behavior, there is no upper bound of pumped charge similar to the series of two QDs choosing tunnel phase as one of the control parameters.

The structure of the paper is following. In Sec. II, we explain theoretical models (series two QDs and three QDs ring) and formal solutions. In Sec. III, we analyze two types of series two QDs. One of them is choosing energies of two QDs as control parameters, and the other is choosing the energy level of one QD and a tunnel phase. In Sec. IV, we analyze three QDs ring, choosing the energy level of one QD and the phase penetrating through the ring. Finally, in Sec. V, we conclude the paper. In Appendix A we derive the kernel of the Brouwer’s formula and the pumped charge of three QDs. In Appendix B we explain an analytical integration appeared in series two QDs. In Appendix C we derive the kernel and the pumped charge of three QDs ring.

II. MODEL

A. Hamiltonian

In this section, we explain the models considered in this study, where QD system being coupled to two leads. The left and right leads couple to QD1 and QD2, respectively. We consider only single level in each QD. We disregarded the spin degree of freedom of electrons and ignore inter-QD Coulomb interactions. The Hamiltonian \( \hat{H} = \hat{H}_{\text{QDs}} + \hat{H}_{\text{leads}} + \hat{H}_T \) is consisted of the QD system part \( \hat{H}_{\text{QDs}} \), non-interacting leads part \( \hat{H}_{\text{leads}} \), and \( \hat{H}_T \).

\[
\hat{H}_{\text{QDs}} = \sum_{i=1}^{N_D} \varepsilon_i \hat{d}_i^\dagger \hat{d}_i + \sum_{i,j=1,i\neq j}^{N_D} t_{ij} \hat{d}_i^\dagger \hat{d}_j, \quad (1)
\]

\[
\hat{H}_{\text{leads}} = \sum_{\alpha=L,R} \sum_{k} \varepsilon_{\alpha k} \hat{C}_{\alpha k}^\dagger \hat{C}_{\alpha k}, \quad (2)
\]

\[
\hat{H}_T = \sum_{k} (V_{Lk} \hat{C}_{Lk}^\dagger \hat{d}_1 + V_{Rk} \hat{C}_{Rk}^\dagger \hat{d}_2) + \text{h.c.}. \quad (3)
\]

Here \( i \) and \( j \) are the indices of QDs and \( N_D \) is the number of QDs. \( \hat{d}_i^\dagger \), \( \hat{d}_i \) is a creation (annihilation) operator of a localized electron in \( i \)-th QD. \( \varepsilon_i \) is the energy level of \( i \)-th QD. The tunnel couplings between \( i \)-th and \( j \)-th QDs, \( t_{ij} \), have a Peierls phase \( \phi_{ij} \) as \( t_{ij} \equiv |t_{ij}| e^{i\phi_{ij}} \). \( \hat{C}_{\alpha k}^\dagger \) \( \hat{C}_{\alpha k} \) is a creation (annihilation) operator of an electron of energy \( \varepsilon_{\alpha k} \) with a wave number \( k \) in lead \( \alpha \). \( V_{\alpha k} \) is the tunnel coupling amplitude.

In the following discussion, we consider two models. The upper of Fig. 1 depicts series of two QDs, corresponds to \( N_D = 2 \). The lower of Fig. 1 is a ring structure of three QDs, corresponds to \( N_D = 3 \). We examine both models at zero bias voltage and at zero temperature condition.

B. Method

Let \( X_1 \) and \( X_2 \) as two independent control parameters. We change two parameters \( X_1, X_2 \) on the rectangular trajectory shown in Fig. 2 along the arrow. The expression of the pumped charge per cycle had been introduced by Büttiker et al.\cite{22} and formulated by Brouwer\cite{23}, which is given by an oriented surface integral of a kernel \( \Pi \) over the area depicted in Fig. 2 in the \( X_1 \times X_2 \) phase space,

\[
\frac{Q}{-e} = q = - \int_{X_1^a}^{X_1^b} \int_{X_2^a}^{X_2^b} dX_1 dX_2 \Pi (X_1, X_2), \quad (4)
\]
where \( q \) means the number of pumped electrons from lead \( R \) to lead \( L \) per cycle and \( e \) is an elementary charge. In the following, we call \( q \) pumped charge. \( \Pi(X_1, X_2) \) is the kernel expressed by the scattering matrix \( S \) which is a function of two control parameters,

\[
\Pi(X_1, X_2) = \frac{1}{\pi} \Im \left\{ \frac{\partial S_{LL}}{\partial X_1} \frac{\partial S_{LL}}{\partial X_2} + \frac{\partial S_{LR}}{\partial X_1} \frac{\partial S_{LR}}{\partial X_2} \right\}.
\]  

(5)

The components of the scattering matrix are given by retarded Green’s functions through the Fisher-Lee relation\(^{32-34}\),

\[
S_{LL} = 1 - i \Gamma_L G_{11}'(\varepsilon_F),
\]

(6)

\[
S_{LR} = -i \sqrt{\Gamma_L \Gamma_R} G_{12}'(\varepsilon_F).
\]

(7)

\( \Gamma_\alpha (\alpha = L \text{ or } R) \) is the line width defined by \( \Gamma_\alpha \equiv 2\pi \rho_\alpha |V_\alpha|^2 \) with wide-band limit (ignoring energy dependence of the line width), where \( \rho_\alpha \) is the density of states of the lead \( \alpha \). The retarded Green’s function of the QD system, \( G_{nm}'(\varepsilon) \), is defined by the Fourier transformation of

\[
G_{nm}'(t, t') \equiv -\frac{i}{\hbar} \theta(t - t') \left\{ \langle \hat{a}_n(t), \hat{a}_m(t') \rangle \right\},
\]

(8)

where \( n, m = 1, \ldots, N_D \) and \( \theta(t) \) is the step function. The operators are in the Heisenberg picture (e.g. \( \hat{a}_n(t) = e^{iHt/\hbar} \hat{a}_n e^{-iHt/\hbar} \)) and \( \langle \cdot \rangle \) means quantum mechanical and statistical average at zero temperature. Since we consider zero bias voltage and zero temperature condition, the energy of the incident electrons, \( \varepsilon \), is set to the Fermi energy \( \varepsilon_F \). Hence, the kernel can be obtained with the retarded Green’s function:

\[
\Pi(X_1, X_2) = \frac{1}{\pi} \Im \left\{ R \left\{ \frac{\partial G_{11}'(\varepsilon_F) \partial G_{11}'(\varepsilon_F)}{\partial X_1 \partial X_2} + \frac{\partial G_{12}'(\varepsilon_F) \partial G_{12}'(\varepsilon_F)}{\partial X_1 \partial X_2} \right\} \right\}.
\]

(9)

Using the equation of motion method, the Fourier transform of the retarded Green’s function of series two QDs is given by

\[
G'(\varepsilon) = \left( \begin{array}{cc} \varepsilon - \varepsilon_1 + \frac{i}{2} \Gamma_L & -t_{12} \\ -t_{21} & \varepsilon - \varepsilon_2 + \frac{i}{2} \Gamma_R \end{array} \right)^{-1},
\]

(10)

and the retarded Green’s function of three QDs ring is given by

\[
G'(\varepsilon) = \left( \begin{array}{ccc} \varepsilon - \varepsilon_1 + \frac{i}{2} \Gamma_L & -t_{12} & -t_{13} \\ -t_{21} & \varepsilon - \varepsilon_2 + \frac{i}{2} \Gamma_R & -t_{23} \\ -t_{31} & -t_{32} & \varepsilon - \varepsilon_3 \end{array} \right)^{-1}.
\]

(11)

### III. SERIES TWO QDS

In this section, we consider series two QDs (depicted in Fig. 4). We show the results of two sets of control parameters: changing both energies of QD 1, 2 (sub-sec. A), and changing energy of QD 1 and the inter-QD tunnel phase (sub-sec. B).

#### A. \((\varepsilon_1, \varepsilon_2)\)-pump

We calculate the kernel Eq. (6) choosing two control parameters \( X_1 = \varepsilon_1 \) and \( X_2 = \varepsilon_2 \) with the retarded Green’s function of the series two QDs, Eq. (10). The details of the calculation are given in Appendix A. We introduce dimensionless kernel \( \tilde{\Pi}(x_1, x_2) \) by defining dimensionless parameters \( x_1 \equiv \frac{x_{12}}{\Gamma_L}, x_2 \equiv \frac{x_{23}}{\Gamma_R} \), and \( s_{12} \equiv \frac{x_{12}}{\sqrt{\Gamma_L \Gamma_R}} \) from Eq. (10) in Appendix A.

\[
\tilde{\Pi}(x_1, x_2) \equiv \Gamma_L \Gamma_R \Pi(\varepsilon_1, \varepsilon_2)
\]

\[
= -\frac{|s_{12}|^2}{2\pi} \frac{x_1 + x_2}{(x_1 + \frac{i}{2})(x_2 + \frac{i}{2}) - |s_{12}|^2}.
\]

(12)

Figure 3 is a contour plot of the kernel \( \tilde{\Pi}(x_1, x_2) \) as a function of two dimensionless control parameters. We set \( |s_{12}| = 1 \) in this plot. The kernel is symmetric with respect to the line \( x_1 = x_2 = 0 \) and is antisymmetric with respect to the line \( x_1 + x_2 = 0 \). There is an isolated dip at \( x_1 = x_2 = 1 \). For a general choice of \( |s_{12}| \), the location of the dip obeys \( x_1 = x_2 = |s_{12}| \). We can see that the similar resonant behavior appeared in a double-barrier quantum well\(^{18}\) and a QD in a turnstile geometry\(^{19}\). In fact, the transmission probability, \( T(x_1, x_2) = |S_{LL}(\varepsilon_F)|^2 \) through the series two QDs has a common factor in the denominator as \( \tilde{\Pi}(x_1, x_2) \), namely,

\[
T(x_1, x_2) = \frac{|s_{12}|^2}{(x_1 + \frac{i}{2})(x_2 + \frac{i}{2}) - |s_{12}|^2}.
\]

(13)

Clearly, the denominator becomes small for \( x_1 = x_2 = \pm |s_{12}| \) and resonance behavior occurs.
We then consider maximum pumped charge per cycle. For that purpose, by examining the kernel Fig. 3, we close the modulation trajectory depicted in Fig. 2 in the first quadrant. By integrating Eq. (12) in the region \(0 \leq x_1, x_2 \leq \infty\), we found that the pumped charge per cycle is given by

\[
q = \frac{1}{1 + \frac{1}{4|s_{12}|^2}}, \quad (14)
\]

whose derivation is in Appendix A. This is a monotonically increasing function of \(|s_{12}|\) and has an upper bound. This takes an elementary charge \((q_{\text{max}} = 1)\) as the maximum when tunnel coupling is infinitely strong \((|s_{12}| \to \infty)\). It had been shown that the maximum value of pumped charge per cycle becomes exactly an elementary charge by choosing two potentials as control parameters in one QD system. This makes one to consider that if one chooses the energy of QD or potential as control parameters, maximum pumped charge universally becomes elementary charge. Does the maximum value of pumped charge per cycle always have an upper bound whatever control parameters one chooses? In the following sections, we analyze the basic property and maximum value of pumped charge per cycle by choosing tunnel phase as one of control parameters.

B. \((\varepsilon_1, \phi_{12})\)-pump

We choose the tunneling phase \(\phi_{12}\) instead of the energy of QD2 as one of the control parameters. Here we need to discuss the method to change the tunnel phase in the actual experimental system. One possible system is two QDs formed in a semiconductor nanowire or a carbon nanotube, where the phase can be modulated by mechanically vibrating the nanowire/nanotube containing the QDs (there had been related theoretical propositions). Other possibilities are the electrical or optical control of small magnet on a cantilever, or by shifting the position of the edge channel in two dimensional electron system under strong magnetic field by the side gate.

Choosing two control parameters \(X_1 = \varepsilon_1\) and \(X_2 = \phi_{12}\), from Eq. (A11) in Appendix A the kernel \(\Pi(\varepsilon_1, \phi_{12})\) is given by

\[
\Pi(\varepsilon_1, \phi_{12}) = \frac{|s_{12}|^2}{2\pi \Gamma L} \left( x_1 \left( x_2 + \frac{i}{2} \right)^2 - |s_{12}|^2 x_2 \right)^2. \quad (15)
\]

Surprisingly, this kernel is independent of the phase \(\phi_{12}\). This property, independence of the kernel of the phase \(\phi_{12}\), is maintained even if one chooses control parameters other than \(\varepsilon_1\). For example, as shown in Eq. (A12) in Appendix A, choosing the linear combination of energies of QD 1, 2 instead of energy of QD 1 as one of the control parameters, the kernel is still independent of the tunnel phase \(\phi_{12}\). By integrating Eq. (15), the pumped charge is

\[
q = \frac{\delta \phi_{12}}{2\pi} \left[ T(x_{1a}, x_2) - T(x_{1b}, x_2) \right], \quad (16)
\]

where we defined \(x_{a/b} = x_{1a/b} - \frac{|s_{12}|^2}{x_{12}^2 + \frac{i}{4}}\) and \(\alpha = \frac{1}{2} \left( 1 + \frac{|s_{12}|^2}{x_{12}^2 + \frac{i}{4}} \right) \). The pumped charge depends only on the difference of the start and the end values of phase \(\delta \phi_{12} = \phi_{12b} - \phi_{12a}\). In contrast to the result \(q_{\text{max}} = 1\), in the series two QDs \((\varepsilon_1, \varepsilon_2)\) pump, this has no upper bound and its value is linearly proportional to the change of the phase \(\delta \phi_{12}\).

Remembering that Josephson relation relates the phase to the bias voltage \(\frac{d\phi_{12}}{dt} = \frac{e}{\hbar} V(t)\), we introduce a voltage, \(V(t)\), satisfying

\[
\delta \phi_{12} = \frac{e}{\hbar} \int_0^\tau dt \, V(t), \quad (17)
\]

where \(\tau\) is a time needed to change the phase by \(\delta \phi_{12}\). Then Eq. (16) can be interpreted as follows,

\[
q = \frac{1}{e} \int_0^\tau dt \left( I_a(t) - I_b(t + \tau + \delta t) \right), \quad (18)
\]

where \(\delta t\) is a time needed to change the energy of QD1 from \(\varepsilon_1 = \varepsilon_{1a}\) to \(\varepsilon_1 = \varepsilon_{1b}\). We introduced time-dependent currents at \(\varepsilon_1 = \varepsilon_{1a}\) and \(\varepsilon_1 = \varepsilon_{1b}\) by “Landauer type” expression:

\[
I_{a/b}(t) \equiv G(x_{a/b}) V(t), \quad (19)
\]

Where \(G(x_{a/b}) = \frac{e^2}{2\pi} T(x_{1a/1b}, x_2)\) corresponds to the conductance at the conditions \(\varepsilon_1 = \varepsilon_{1a}\) and \(\varepsilon_1 = \varepsilon_{1b}\).
Figure 4 depicts the pumped current along the trajectory in the series two QDs. The contribution of upper and lower lines to pumped current cancels because the strength is the same and the sign is opposite since the integrand is independent of the phase. Therefore along the trajectory, only the contribution of left and right lines remains. We may interpret Eq. (19) as a steady current, which only presents when one slowly modulates thermodynamic quantities, like bias-voltage or temperature bias. It is quite important that in our setup, we do not modulate any thermodynamic quantities, but only kinetic quantities. Because of the gauge transformation, the variance of the phase corresponds to the variance of the bias voltage which is one of the thermodynamic quantities.

Next we discuss the maximum value of the pumped charge of Eq. (16) for a fixed $\delta\phi_{12}$. To obtain the maximum value, we first optimize the range of the integration $x_a$ and $x_b$. By inspecting Eq. (16), we can maximize $q$ by choosing $x_a = 0$ and $x_b = -\infty$:

$$q_{\text{max}} = \delta\phi_{12} \frac{2 |s_{12}|^2}{\pi} \frac{x_2^2 + \frac{1}{4}}{(x_2^2 + x_2 + \frac{1}{4}) + |s_{12}|^2}.$$  

We then maximize this by changing $x_2$. As a result when $|s_{12}| < \frac{1}{2}$, the maximum value is

$$q_{\text{max, ext}} = \frac{|s_{12}|^2}{2\pi \left(\frac{1}{4} + |s_{12}|^2\right)} 2 \delta\phi_{12}$$  

at $x_2 = 0$ and when $|s_{12}| > \frac{1}{2}$, the maximum value is

$$q_{\text{max, ext}} = \frac{1}{2\pi} \delta\phi_{12}$$  

at $x_2 = \pm \sqrt{|s_{12}|^2 - \frac{1}{4}}$.

**IV. THREE QDS RING**

In this section, we consider three QDs ring choosing the energy of QD 3 and the flux penetrating through the ring as control parameters. We derive the kernel Eq. (9) using retarded Green’s function of three QDs ring Eq. (11). Details of the calculation are given in Appendix C. We can see that the kernel Eq. (11) is a periodical function of $\phi_{AB}$, where we introduced AB phase, $\phi_{AB}$, from the magnetic flux penetrating through the ring. Regarding the normalized kernel $\tilde{\Pi}(x_3, \phi_{AB})$ as a function of $(x_1, x_2, x_3, \phi_{AB})$, we found following symmetry,

$$\tilde{\Pi}(x_3, \phi_{AB})|_{x_3, x_1, x_2, |s_{23}|, |s_{31}|} = \tilde{\Pi}(x_3, -\phi_{AB})|_{x_2, x_1, |s_{31}|, |s_{23}|},$$  

which is related to the symmetry of time and space reversal. And regarding the kernel $\Pi(x_3, \phi_{AB})$ as a function of $(x_1, x_2, x_3, \phi_{AB})$, there is another symmetry,

$$\Pi(x_3, \phi_{AB})|_{x_1, x_2} = -\Pi(-x_3, -\phi_{AB})|_{-x_1, -x_2},$$  

which is related to the electron-hole symmetry.

In Fig. 4 we show the contour plot of the kernel $\tilde{\Pi}(x_3, \phi_{AB})$ and corresponding pumped charge $q$ as a function of the interval of the phase $\delta\phi_{12}$. We set normalized tunnel couplings $|s_{12}| = |s_{23}| = |s_{31}| = 1$. Upper plots correspond to normalized energies of QD 1, 2 being the same ($x_1 = x_2 = 0.1$), and lower plots correspond to normalized energies of QD 1, 2 being the opposite ($x_1 = -x_2 = 0.1$). In the center two graphs, solid lines mean the integration region $-3 \leq x_3 \leq 3$ of the kernel and dashed lines mean the integration region $-6 \leq x_3 \leq 6$. Positive and negative peaks appear periodically in the direction of phase in the contour plot of kernel. When the signs of energy of QD 1, 2 are the same, peak heights are smaller than the depth of the negative peaks (upper picture of Fig. 5). When the signs of energy of QD 1, 2 are opposite, the peak heights and the depths are the same (lower of Fig. 5). We can understand this behavior for $x_1 = -x_2 = 0.1$ using the two symmetries, Eqs. (23) and (24), $\tilde{\Pi}(x_3, \phi_{AB})|_{x_1} = \tilde{\Pi}(x_3, -\phi_{AB})|_{x_1}$. In the right two graphs, solid lines are the results for the integration region $0 \leq x_3 \leq 3$ of the kernel, and dashed lines are for the integration region $0 \leq x_3 \leq 6$. When we choose the integral region to pick up only negative peaks, corresponding pumped charge shows step-like form as a function of $\delta\phi_{12}$.

We found very different behavior of the kernel at a special choice of the parameters. In the following, we restrict
ourselves to the symmetric situation: $|s_{12}| = |s_{23}| = |s_{31}| = Z$. When $x_1 = x_2 = Z$, there are series of isolated dips at $x_3 = Z$ and $\phi_{AB} = 2n\pi$ with an integer $n$. Left of Fig. 6 is the contour plot of the kernel $\Pi(x_3, \phi_{AB})$ at this condition $Z = 1$. Right plot of Fig. 6 is the pumped charge which is a surface integral of kernel $\tilde{\Pi}(x_3, \phi_{AB})$ changing energy of QD 3 $x_3$ and phase $\phi_{AB}$. Solid line means the integration region $0.7 \leq x_3 \leq 1.3$, and dashed line means the integration region $0 \leq x_3 \leq 2$. Because of isolated structures in the kernel, pumped charge per cycle becomes step-like as a function of the interval of phase integral $\delta \phi_{AB}$. This behavior means that the pumped charge is quantized and is robust against the uncertainly of the phase. Using the symmetry Eq. (24), we have series of isolated peaks at $\varphi_{AB} = (2n + 1)\pi$ ($n$: integer) and $x_3 = -Z$, when $x_1 = x_2 = -Z$.

The origin of these peaks/dips can be understood as the resonance behavior of the scattering matrix as discussed in the previous section for series two QDs. The transmission probability of this three QDs ring is

$$T = \frac{1}{|\Delta_3|^2} |s_{12}|^2 - |s_{23}s_{31}| e^{-i\phi_{AB}}|^2,$$  \hspace{1cm} (25)

with an integer $n$. In fact, the kernel $\tilde{\Pi}(x_3, \phi_{AB})$ can be well approximated near the peaks/dips as

$$\tilde{\Pi}(\epsilon, \varphi) \sim -\frac{16Z a}{\pi} \frac{(1 - 4Z^2 \varphi^2)}{(1 + 4Z^2 \varphi^2)^2 + 16Z^2} \left(1 - \frac{1}{2} - \frac{\varphi}{\pi} \right)^2 Z \equiv \epsilon, \varphi \equiv \varphi_{AB} - (2n + \frac{1}{2} - \frac{\varphi}{\pi} \right) \pi.$$

where $\epsilon \equiv x_3 - aZ$ and $\varphi \equiv \phi_{AB} - (2n + \frac{1}{2} - \frac{\varphi}{\pi} \right) \pi$. Therefore, as a function of $\varphi$, the kernel has a very sharp structure at $\varphi = 0$ in particular for large $Z$. This means that the top of the steps found in Fig. 6 is very flat for large $Z$. Since the kernel is not so sharply peaked as a function of $\epsilon$, the step value itself depends on the integration range of $x_3$.

**V. CONCLUSION**

We obtained explicit expressions of pumped charge for non-interacting series two quantum dots (QDs) and three QDs ring using the Brouwer’s formula. For series two QDs, by choosing two energy levels of QDs as control parameters $(\epsilon_1, \epsilon_2)$, we find that the maximum value of pumped charge is exactly an elementary charge $-e$. By choosing energy of QD 1 and tunnel phase $(\epsilon_1, \phi_{12})$ as control parameters, we find that the kernel of the Brouwer’s formula is independent of $\phi_{12}$, and the
maximum value of pumped charge is proportional to the phase difference $\delta \phi_{12}$, so there is no upper bound. In addition, we found that the pumped current is effectively equivalent to the current under a finite bias voltage condition due to the gauge transformation.

For three QDs ring, we choose energy of one QD and the flux penetrating through the ring ($\varepsilon_3, \phi_{AB}$) as control parameters. We find pumped charge per cycle shows sinusoidal or step-like behavior depending on the energies of QD 1, 2 and the tunnel couplings. In contrast to the series two QDs, we found that the gauge transformation cannot be applied in this system. For realizing the current standard, our results have an advantage since we can obtain large charges per cycle by changing the phase indefinitely. Moreover from the view point of the stability of the current standard against the fluctuations of the flux in the experiments, the step-like behavior found in three QDs ring may enable precise control of the pumped charge.

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Appendix A: Kernel and pumped charge of series two QDs

We consider the kernel of series two QDs $\Pi (X_1, X_2)$. By definition, the retarded Green’s function has the property: $G^r (G^r)^{-1} = 1$. By differentiating this identity with $X$, we obtain

$$\frac{\partial G^r}{\partial X} = -G^r f_X G^r,$$

where we define a matrix $f_X \equiv \partial (G^r)^{-1} / \partial X$. The kernel needs (1, 1) and (1, 2) components of Eq. (A1).

When we choose $X = \varepsilon_1$,

$$f_{\varepsilon_1} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

then

$$\left( \frac{\partial G^r}{\partial \varepsilon_1} \right)_{11} = (G^r_{11})^2, \quad \left( \frac{\partial G^r}{\partial \varepsilon_1} \right)_{12} = G^r_{11} G^r_{12}.$$ 

When we choose $X = \varepsilon_2$,

$$f_{\varepsilon_2} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

then

$$\left( \frac{\partial G^r}{\partial \varepsilon_2} \right)_{11} = G^r_{12} G^r_{21}, \quad \left( \frac{\partial G^r}{\partial \varepsilon_2} \right)_{12} = G^r_{12} G^r_{22}.$$ 

Therefore, the kernel $\Pi (\varepsilon_1, \varepsilon_2)$ for the control parameters $\varepsilon_1$ and $\varepsilon_2$ becomes Eq. (A10). Dimensionless form is shown in the main text Eq. (12).

When we choose $X = \phi_{12}$,

$$f_{\phi_{12}} = \begin{pmatrix} 0 & -i t_{12} \\ i t_{21} & 0 \end{pmatrix},$$

then

$$\left( \frac{\partial G^r}{\partial \phi_{12}} \right)_{11} = i G^r_{11} (t_{12} G^r_{51} - t_{21} G^r_{12}) = 0,$$

where we used $t_{12} \equiv |t_{12}| e^{i \phi_{12}}$, $t_{12} = t_{21} = G^r_{12} = -G^r_{21} = |t_{12}| e^{i \phi_{12}} / \Delta_2$, $\Delta_2 = (\varepsilon_F - \varepsilon_1 + \frac{1}{2} \Gamma_L) (\varepsilon_F - \varepsilon_2 + \frac{1}{2} \Gamma_R) - |t_{12}|^2$ in last equation. Similarly,

$$\left( \frac{\partial G^r}{\partial \phi_{12}} \right)_{12} = G^r_{11} G^r_{22} i t_{12} - (G^r_{12})^2 i t_{21},$$

therefore the kernel with the control parameters $\varepsilon_1$ and $\phi_{12}$ becomes Eq. (A11). We obtain dimensionless form in Eq. (15).
Moreover we consider general cases that instead of the energy of QD 1, we use the linear combination of energies of QD 1, 2 \( X_1 (\varepsilon_1, \varepsilon_2) = \alpha \varepsilon_1 + \beta \varepsilon_2 \) as a control parameter (note that above case \( (\varepsilon_1, \phi_{12}) \) corresponds to \( (\alpha, \beta) = (1, 0) \)). Then using Eq. (A1),

\[
\frac{\partial G^r}{\partial X_1} = \frac{\partial \varepsilon_1}{\partial X_1} \frac{\partial G^r}{\partial \varepsilon_1} + \frac{\partial \varepsilon_2}{\partial X_1} \frac{\partial G^r}{\partial \varepsilon_2} \\
= -G^r \left[ \frac{\partial \varepsilon_1}{\partial X_1} \frac{\partial (G^r)^{-1}}{\partial \varepsilon_1} + \frac{\partial \varepsilon_2}{\partial X_1} \frac{\partial (G^r)^{-1}}{\partial \varepsilon_2} \right] G^r \\
\equiv -G^r \Phi G^r, \quad (A7)
\]

where

\[
f = \left( \begin{array}{cc} -\frac{\partial \varepsilon_1}{\partial X_1} & 0 \\ 0 & -\frac{\partial \varepsilon_2}{\partial X_1} \end{array} \right). \quad (A8)
\]

Since \( \left( \frac{\partial G^r}{\partial \phi_{12}} \right)_{11} = 0 \), it is enough to evaluate only

\[
\left( \frac{\partial G^r}{\partial \phi_{12}} \right)_{12} = \frac{\partial \varepsilon_1}{\partial X_1} G^r_{11} + \frac{\partial \varepsilon_1}{\partial X_1} G^r_{12}, \quad (A9)
\]

therefore, kernel \( \Pi (X_1, \phi_{12}) \) becomes Eq. (A12) and is also independent of phase \( \phi_{12} \).

**Appendix B: Integral of series two QDs : \((\varepsilon_1, \varepsilon_2)\)-pump**

To obtain the pumped charge, we need to integrate the kernel Eq. (12). We can integrate analytically when the integral region is \( 0 < x_1, x_2 < \infty \),

\[
q = \int_0^\infty dx_1 \int_0^\infty dx_2 \frac{x_1 + x_2}{2 \pi |x_1 + \frac{i}{2}| (x_2 + \frac{i}{2}) - s^2|}, \quad (B1)
\]

we set \( s = |s_{12}| \). Changing variables \( u = -\frac{i}{2} \ln \frac{x_1}{x_2} \) and \( v = \sqrt{x_1 x_2} \), then we obtain \( x_1 = s v e^u \) and \( x_2 = s v e^{-u} \). Integral regions of \( u \) and \( v \) become \( -\infty < u < \infty \) and \( 0 \leq v < \infty \). Then, integral of Eq. (B1) reduces to,

\[
q = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \frac{v^2 \cosh u}{2 \pi s^3 \left[ (v^2 - 1 - \frac{1}{4s^2}) + (\cosh u) \frac{v^2}{s^2} \right]^2} \\
= \frac{1}{\pi s^3} \int_{-\infty}^{\infty} du \; J_s (u) \cosh u, \quad (B2)
\]

where

\[
J_s (u) \equiv \int_{-\infty}^{\infty} dv \frac{v^2}{\left[ (v^2 - 1 - \frac{1}{4s^2}) + (\cosh u) \frac{v^2}{s^2} \right]^2}, \quad (B3)
\]

and we change the integral region of \( v \) to \( -\infty < v < \infty \) because the integrand is an even function of \( v \). We can perform the integral of Eq. (B3) using the residue theorem. There are two poles \( v_1 \) and \( v_2 \) for \( \text{Im} (v) > 0 \),

\[
v_{1,2} = \frac{1}{2} \left[ \cosh u \frac{s}{v} \pm \sqrt{4 \left( 1 + \frac{1}{4s^2} \right) - \left( \cosh u \frac{s}{v} \right)^2} \right]. \quad (B4)
\]

We obtain,

\[
J_s (u) = \frac{2 \pi i}{(v_1 + v_2) (v_1 - v_2) (v_1 + v_2) - \frac{1}{v_1 + v_2}} \left( \frac{v_1 + v_2}{4v_1 v_2} - \frac{1}{v_1 + v_2} \right) \\
= \frac{\pi s^3}{2 \cosh^3 u \left( 1 + \frac{1}{4s^2} \right)}, \quad (B5)
\]
where the definition of introduce dimensionless parameters similar way as in series two QDs in Appendix A. We substitute this into Eq. (B2) and using the formula
\[ \int du \frac{1}{\cos^2 u} = \tanh u, \]
we reach to
\[ q = \frac{1}{1 + \frac{1}{4\pi^2}}. \] (B6)

Appendix C: Kernel and pumped charge of three QDs ring

We can obtain the kernel of three QDs ring with a similar way as in series two QDs in Appendix A. We introduce dimensionless parameters \( x_1/2 = \frac{e^{F/2} - 1}{F_{L/R}}, \quad x_3 = \frac{e^{F} - e^{F_1}}{\gamma}, \quad s_{12} = \frac{t_{12}}{\sqrt{t_{12} t_{23}}}, \quad s_{23} = \frac{t_{23}}{\sqrt{t_{12} t_{23}}}, \) and \( s_{31} = \frac{s_{31}}{\sqrt{t_{12} t_{23}}} \) (we introduce a parameter \( \gamma \) to normalize energy of QD 3 \( \varepsilon_3 \)). We found that the kernel depends only on \( \phi = \phi_{12} + \phi_{23} + \phi_{31} \) (gauge invariance). We can divide the phase \( \phi \) into two parts, AB phase from the magnetic flux penetrating through the ring \( \phi_{\text{AB}} \) (\( \phi_{\text{AB}} = 2\pi \frac{\Phi_0}{\Phi_0} = \frac{1}{2} \)) and phase independent of the flux \( \phi^*; \phi = \phi_{\text{AB}} + \phi^* \). Assuming the QDs made of \( s \)-orbitals, tunnel coupling provides lower energy for a symmetric orbital without nodes, and the tunnel amplitude should take a negative value \( t_{12} = -|t_{12}| \). This corresponds to \( e^{i\phi^*} = -1, \) i.e. \( \phi^* = \pi \). The phase which we can change freely is \( \phi_{\text{AB}} \) from the flux penetrating through the ring and we regard \( \phi_{\text{AB}} \) as a control parameter, so the final form of the kernel is
\[
\hat{\Pi}(x_3, \phi_{\text{AB}}) = \frac{|s_{12}|}{\pi |\Delta_3|^4} \times \text{Im} \left[ 2\sin\phi_{\text{AB}} |s_{23}s_{31}| \left\{ |s_{12}s_{23}| e^{-i\phi_{\text{AB}}} - |s_{31}| \left( x_2 - i \frac{1}{2} \right) \right\} \left\{ |s_{12}s_{23}| e^{i\phi_{\text{AB}}} - |s_{31}| \left( x_2 + i \frac{1}{2} \right) \right\} \left\{ x_3 - |s_{23}|^2 \right\} \right]
- i e^{i\phi_{\text{AB}}} \left\{ |s_{12}s_{23}| e^{-i\phi_{\text{AB}}} - |s_{31}| \left( x_2 - i \frac{1}{2} \right) \right\} \left\{ |s_{12}s_{23}| e^{i\phi_{\text{AB}}} - |s_{31}| \left( x_2 + i \frac{1}{2} \right) \right\} \left\{ x_3 - |s_{23}|^2 \right\}.
\]
\[ \times \left\{ \left( x_2 - i \frac{1}{2} \right) x_3 - |s_{23}|^2 \right\} \left\{ \left( x_1 + i \frac{1}{2} \right) x_3 - |s_{31}|^2 \right\} \left\{ \left( x_2 + i \frac{1}{2} \right) x_3 - |s_{12}|^2 \right\} \left( |s_{31}s_{23}| e^{-i\phi_{\text{AB}}} - |s_{12}| x_3 \right)^2 \right], \] (C1)
where the definition of \( \Delta_3 \) is
\[ \Delta_3 = \left\{ \left( x_1 + i \frac{1}{2} \right) x_2 - i \frac{1}{2} x_3 + 2 |s_{12}s_{23}s_{31}| \cos\phi_{\text{AB}} - |s_{23}|^2 \left( x_1 + i \frac{1}{2} \right) - |s_{31}|^2 \left( x_2 + i \frac{1}{2} \right) - |s_{12}|^2 x_3 \right\}. \] (C2)
10

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