Definable Tietze extension property in o-minimal expansions of ordered groups

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Abstract
The following two assertions are equivalent for an o-minimal expansion of an ordered group \( \mathcal{M} = (M, <, +, 0, \ldots) \). There exists a definable bijection between a bounded interval and an unbounded interval. Any definable continuous function \( f : A \to M \) defined on a definable closed subset of \( M^n \) has a definable continuous extension \( F : M^n \to M \).

Keywords o-Minimal theory · Definable Tietze extension property

Mathematics Subject Classification Primary 03C64

1 Introduction
In this paper, we investigate the Tietze extension property for functions definable in an o-minimal expansion of an ordered Abelian group. For the basics of o-minimality, good references are [2, 6, 7]. In this context, the Tietze extension property is defined as follows:

Definition 1.1 Consider an expansion \( \mathcal{M} = (M, <, \ldots) \) of a dense linear order without endpoints. The structure \( \mathcal{M} \) enjoys the definable Tietze extension property if, for any positive integer \( n \), any definable closed subset \( A \) of \( M^n \) and any continuous definable function \( f : A \to M \), there exists a definable continuous extension \( F : M^n \to M \) of \( f \).

The definable Tietze extension property is a convenient tool for the geometric study of o-minimal structures. We prove the following theorem in this paper.
Theorem 1.2  Consider an o-minimal expansion $\mathcal{M} = (M, <, +, 0, \ldots)$ of an ordered group. The following are equivalent:

1. There exists a definable bijection between a bounded interval and an unbounded interval.
2. The structure $\mathcal{M}$ enjoys the definable Tietze extension property.

We make a comment on the theorem. Miller and Starchenko studied the asymptotic behavior of o-minimal expansions of an ordered group $\mathcal{M} = (M, <, +, \ldots)$ in [4]. They introduced the notion of linear boundedness. An o-minimal structure is called linearly bounded if, for any definable function $f : M \to M$, there exists a definable automorphism $\lambda : M \to M$ with $|f(x)| \leq \lambda(x)$ for all sufficiently large $x \in M$. Their main theorem is that there exists a definable binary operation $\cdot$ such that $(M, <, +, \cdot)$ is an ordered real closed field when the structure is not linearly bounded.

Peterzil and Edmundo studied the subclass of linearly bounded o-minimal expansions of ordered groups [1, 5]. An o-minimal structure $\mathcal{M}$ is semi-bounded if any set definable in $\mathcal{M}$ is already definable in the o-minimal structure generated by the collection of all bounded sets definable in $\mathcal{M}$. Edmundo gave equivalent conditions for an o-minimal expansion of an ordered group to be semi-bounded in [1, Fact 1.6]. The condition (1) in our theorem is the negation of one of them. Theorem 1.2 gives a new equivalent condition. An o-minimal expansion of an ordered group is semi-bounded if and only if it does not have the definable Tietze extension property. In our proof, we use the following facts:

- In an o-minimal expansion $\mathcal{M} = (M, <, +, 0, \ldots)$ of an ordered group which is not semi-bounded, we can define a real closed field whose universe is an unbounded subinterval of $M$ and whose ordering agrees with $<$ [1, Fact 1.6].
- An o-minimal expansion of an ordered field enjoys the definable Tietze extension property [7, Chapter 8, Corollary 3.10].

We introduce the terms and notations used in this paper. The term ‘definable’ means ‘definable in the given structure with parameters’ in this paper. For a linearly ordered structure $\mathcal{M} = (M, <, \ldots)$, an interval is a nonempty definable set of the form $\{x \in M | a \ast x \ast' b\}$, where $a, b \in M \cup \{\pm \infty\}$ and $\ast, \ast' \in \{<, \leq\}$. The interval is denoted by $[a, b]$ when both $\ast$ and $\ast'$ are the symbol $<$. It is denoted by $[a, b]$ when both $\ast$ and $\ast'$ are $\leq$. We define $[a, b[ \text{ and } ]a, b]$ similarly. An interval is called bounded if both $a$ and $b$ belong to $M$. It is called unbounded otherwise. We consider the order topology on $M$ and its product topology on the Cartesian product $M^n$ in the paper. The notation $M_{>r}$ denotes the set $\{x \in M | x > r\}$ for any $r \in M$.

2 Proof

We now begin to prove Theorem 1.2. An o-minimal structure is always definably complete. We use this fact without notice. We first prove two lemmas.

Lemma 2.1  Consider an o-minimal structure. The structure has a definable bijection between a bounded interval and an unbounded interval if and only if it has a definable homeomorphism between a bounded interval and an unbounded interval.
Proof It is immediate from the monotonicity theorem [7, Chapter 3, Theorem 1.2]. □

Lemma 2.2 Consider a definably complete expansion of a densely linearly ordered abelian group \( \mathcal{M} = (M, \cdot, +, 0, \ldots) \). If the structure \( \mathcal{M} \) has a strictly monotone definable homeomorphism between a bounded open interval and an unbounded open interval, any two open intervals are definably homeomorphic and there exists a definable strictly increasing homeomorphism between them.

Proof By the assumption, there exists a strictly monotone definable homeomorphism \( \varphi : I \rightarrow J \), where \( I \) is a bounded open interval and \( J \) is an unbounded open interval.

We may assume that \( \tau : ]0, u[ \rightarrow ]0, 0[ \) defined by \( \tau(t) = u - t \) is a definable homeomorphism.

We next reduce to the case in which \( J = ]0, \infty[ \). We have only three possibilities; that is \( J = ]v, +\infty[ \), \( J = ]-\infty, v[ \) and \( J = M \) for some \( v \in M \). In the first and second cases, we may assume that \( J = ]0, \infty[ \) because \( J = ]v, +\infty[ \) and \( J = ]-\infty, v[ \) are obviously definably homeomorphic to \( ]0, \infty[ \). In the last case, set \( u' = \varphi^{-1}(0) \). Then the restriction of \( \varphi \) to the open interval \( ]0, u'[ \) is a definable homeomorphism between \( ]0, u'[ \) and \( ]-\infty, 0[ \). Hence, we can reduce to the second case. We have constructed a strictly increasing definable homeomorphism \( \varphi : ]0, u[ \rightarrow ]0, \infty[ \). We fix such a homeomorphism.

We next construct a definable strictly increasing homeomorphism between an arbitrary bounded open interval and \( ]0, \infty[ \). We may assume that the bounded interval is of the form \( ]0, v[ \). We have nothing to do when \( v = u \). When \( v < u \), the map defined by \( \varphi(t + u - v) - \varphi(u - v) \) for all \( t \in ]0, v[ \) is a definable homeomorphism between \( ]0, v[ \) and \( ]0, \infty[ \). When \( v > u \), consider the map \( \psi : ]0, v[ \rightarrow ]0, \infty[ \) given by \( \psi(t) = t \) for all \( t \leq v - u \) and \( \psi(t) = \varphi(t + u - v) + v - u \) for the other case. The map \( \psi \) is the desired definable homeomorphism. We have constructed a definable homeomorphism between \( ]0, u[ \) and all open intervals other than \( M \).

The remaining task is to construct a definable homeomorphism between \( ]0, u[ \) and \( M \). There exists a strictly increasing definable homeomorphisms \( \psi_1 : ]0, u/2[ \rightarrow ]-\infty, 0[ \) and \( \psi_2 : ]u/2, u[ \rightarrow ]0, \infty[ \). The definable map \( \psi : ]0, u[ \rightarrow M \) given by \( \psi(t) = \psi_1(t) \) for \( t < u/2 \), \( \psi(t) = 0 \) for \( t = u/2 \) and \( \psi(t) = \psi_2(t) \) for \( t > u/2 \) is a definable homeomorphism. The function \( \psi \) is well defined because \( (M, +) \) is a divisible group by [3, Proposition 2.2]. □

The following proposition is a part of [1, Fact 1.6].

Proposition 2.3 Consider an o-minimal expansion of an ordered group \( \mathcal{M} = (M, <, +, 0, \ldots) \). The followings are equivalent:

1. There exists a definable bijection between a bounded interval and an unbounded interval.
2. In \( \mathcal{M} \), we can define a real closed field whose universe is an unbounded subinterval of \( M \) and whose ordering agrees with \( < \).

We now begin to prove Theorem 1.2.
Proof of Theorem 1.2 We first show that the condition (1) implies the condition (2).

There exist an unbounded subinterval $I$ of $M$, two elements $0^*$ and $1^*$ in $I$, and definable functions $\oplus$, $\otimes : I \times I \to I$ such that $(I, 0^*, 1^*, \oplus, \otimes)$ is a real closed field with the ordering $<$ by Proposition 2.3. The subinterval $I$ is obviously an open interval.

If $I = M$, the assertion (2) directly follows from the original definable Tietze extension theorem [7, Chapter 8, Corollary 3.10].

We next consider the other case. Consider a definable continuous function $f : A \to M$ defined on a definable closed subset $A$ of $M^n$. We construct a definable continuous extension $F : M^n \to M$ of the function $f$. There exists a definable homeomorphism $\sigma : M \to I$ by Lemmas 2.1 and 2.2. The notation $\sigma_n$ denotes the homeomorphism from $M^n$ onto $I^n$ induced by $\sigma$. The definable set $\sigma_n(A)$ is contained in $I^n$. Consider the definable continuous function $f_\sigma : \sigma_n(A) \to I$ defined by $f_\sigma(x) = (\sigma \circ f \circ \sigma_n^{-1})(x)$. Its graph is obviously contained in $I^{n+1}$.

We consider a new structure $\mathcal{I}$ whose universe is $I$. Let $\mathcal{G}_n$ be the set of all subset of $I^n$ definable in $\mathcal{M}$. Set $\mathcal{G} = \bigcup_{n \geq 0} \mathcal{G}_n$. For any $S \in \mathcal{G}$, we introduce new predicate symbol $R_S$ and define $\mathcal{I} \models R_S(x)$ by $x \in S$. The structure $\mathcal{I} = (I, <, \{R_S\}_{S \in \mathcal{G}})$ is obviously an o-minimal structure. Since the operators $\oplus$ and $\otimes$ are definable in $\mathcal{I}$, the structure $\mathcal{I}$ is an o-minimal expansion of an ordered field. The $\mathcal{M}$-definable set $\sigma_n(A)$ and the $\mathcal{M}$-definable function $f_\sigma$ are also definable in the structure $\mathcal{I}$. Note that the function $f_\sigma$ is also continuous under the topology induced by the ordering of the real closed field $(I, 0^*, 1^*, \oplus, \otimes)$ because the two structures $\mathcal{I}$ and $\mathcal{M}$ share the same order $<$. There exists a continuous extension $F_\sigma : I^n \to I$ of $f_\sigma$ definable in $\mathcal{I}$ by the original definable Tietze extension theorem [7, Chapter 8, Corollary 3.10]. The function $F_\sigma$ is also definable in $\mathcal{M}$ by the definition of the structure $\mathcal{I}$. The function $F = \sigma^{-1} \circ F_\sigma \circ \sigma_n$ is the desired definable continuous extension of $f$ definable in $\mathcal{M}$.

We next show that the condition (2) implies the condition (1). We construct a definable bijection between a bounded interval and an unbounded interval. Take a positive element $c$ in $M$. Consider the definable closed set $A = \{ (x, y) \in M^2 \mid x \leq 0 \text{ or } x \geq c \}$ and the definable continuous function $f : A \to M$ given by $f(x, y) = y$ if $x \geq c$ and $f(x, y) = 0$ otherwise. By the condition (2), there exists a definable continuous extension $F : M^2 \to M$ of $f$. The notation $g$ denotes the restriction of $F$ to $[0, c] \times M$.

Consider the sets $S_{t, y} = \{ x \in [0, c] \mid g(x, y) = t \}$ for all $t \geq 0$ and $y \geq 0$. The definable sets $S_{t, y}$ is not empty for $y > t$ by the intermediate value theorem [3, Corollary 1.5]. The definable function $\varphi_t : M_{> t} \to [0, c]$ is given by $\varphi_t(y) = \sup S_{t, y}$. For any $t > 0$, there exists a nonnegative $u_t$ such that the restriction $\varphi_t|_{M_{> u_t}}$ of the function $\varphi_t(y)$ to $M_{> u_t} = \{ y \in M \mid y > u_t \}$ is continuous and strictly monotone or constant for $y > u_t$ by the monotonicity theorem.

We consider the following two cases separately.

(a) The restriction $\varphi_t|_{M_{> u_t}}$ is continuous and strictly monotone for some $t > 0$.
(b) The restriction $\varphi_t|_{M_{> u_t}}$ is constant for any $t > 0$.

In the case (a), the restriction $\varphi_t|_{M_{> u_t}}$ gives a bijection between a bounded interval and an unbounded interval. We have finished the proof in this case. We next consider the case (b). By the definition of the function $\varphi_t$, the following assertion holds true:
For any \( t > 0 \), there exist a point \( x_t \in [0, c] \) and a nonnegative \( u_t \) such that 
\[
g(x_t, y) = t \quad \text{for all} \quad y > u_t.
\]
In fact, we have only to take \( y' > u_t \) and set \( x_t = \varphi_t(y') \). Consider the definable map \( \psi : [0, \infty[ \to [0, c] \) given by \( \psi(t) = x_t \), where \( x_t \) is the point defined above. Since \( \varphi_t|_{M_{>u_t}} \) is constant, the point \( x_t \) is independent of the choice of \( y' \). It means that \( \psi \) is well defined. The map \( \psi \) is injective. In fact, if \( \psi(t) = \psi(t') \), we have \( t = g(\psi(t), y') = g(\psi(t'), y') = t' \) for a sufficiently large \( y' \). By the monotonicity theorem, there exists \( c > 0 \) such that the restriction \( \psi|_{]c, \infty[} \) of \( \psi \) to \( ]c, \infty[ \) is continuous and monotone. The restriction \( \psi|_{]c, \infty[} \) is strictly monotone because \( \psi \) is injective. Therefore, it gives a definable bijection between a bounded interval and an unbounded interval.

**Corollary 2.4** An o-minimal expansion of an ordered group is semi-bounded if and only if it does not have the definable Tietze extension property.

**Proof** The corollary follows from Theorem 1.2 and [1, Fact 1.6].

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**References**

1. Edmundo, M.J.: Structure theorems for o-minimal expansions of groups. Ann. Pure Appl. Logic 102, 159–181 (2000)
2. Knight, J., Pillay, A., Steinhorn, C.: Definable sets in ordered structures II. Trans. Am. Math. Soc. 295, 593–605 (1986)
3. Miller, C.: Expansions of dense linear orders with the intermediate value property. J. Symb. Log. 66, 1783–1790 (2001)
4. Miller, C., Starchenko, S.: A growth dichotomy for o-minimal expansions of ordered groups. Trans. Am. Math. Soc. 350, 3505–3521 (1998)
5. Peterzil, Y.: A structure theorem for semi-bounded sets in the reals. J. Symb. Log. 57, 779–794 (1992)
6. Pillay, A., Steinhorn, C.: Definable sets in ordered structures I. Trans. Am. Math. Soc. 295, 565–592 (1986)
7. van den Dries, L.: Tame Topology and O-Minimal Structures. London Mathematical Society Lecture Note Series, vol. 248. Cambridge University Press, Cambridge (1998)

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