RIGIDITY OF MEAN CONVEX SUBSETS IN NON-NEGATIVELY CURVED RCD SPACES AND STABILITY OF MEAN CURVATURE BOUNDS

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ABSTRACT. We prove splitting theorems for mean convex open subsets in RCD (Riemannian curvature-dimension) spaces that extend results by Kasue, Croke and Kleiner for Riemannian manifolds with boundary to a non-smooth setting. A corollary is for instance Frankel's theorem. Then, we prove that our notion of mean curvature bounded from below for the boundary of an open subset is stable w.r.t. to uniform convergence of the corresponding boundary distance function. We apply this to prove almost rigidity theorems for uniform domains whose boundary has a lower mean curvature bound.

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1. Introduction

By the Cheeger-Gromoll splitting theorem a Riemannian manifold with non-negative Ricci curvature which contains a geodesic line splits off a factor $\mathbb{R}$. In [Kas83] Kasue proved a version of this result in the presence of boundary components: A Riemannian manifold with mean convex and compact boundary and nonnegative Ricci curvature that contains a geodesic ray with initial point in the boundary splits off $[0, \infty)$. Kasue also proved that a Riemannian manifold with more than one compact mean convex boundary component and non-negative Ricci curvature is isometric to a product $[0, D] \times \mathbb{N}$. In particular, there are exactly two boundary components and the mean curvature vanishes. Croke and Kleiner [CK92] showed that this is the special case of a more general splitting principle for Riemannian manifolds with boundary. Generalisations for Bakry-Emery Ricci curvature bounds have been obtained by Sakurai [Sak19] and Moore-Woolgar [MW21].

In this article one of our main goals is to generalize Kasue’s rigidity theorems to the nonsmooth context of RCD spaces. The latter is the celebrated synthetic notion of Ricci curvature bounded from below for metric measure spaces. The class of RCD spaces includes Riemannian manifolds with convex boundary. However Riemannian manifolds that admit boundary with only mean curvature bounded from below are in general not in this class: In the presence of boundary components the interior of a Riemannian manifold may not be geodesically convex and therefore will not satisfy any RCD($K, N$) condition. Hence, for a generalization of Kasue’s theorem we consider open subsets inside RCD spaces whose boundary admits a lower mean curvature bound in a generalized sense.

In [Ket20] and in [BKMW20] synthetic notions of lower mean curvature bounds for an open subset $\Omega$ inside an RCD space $(X, d, m)$ were introduced. A similar definition of lower mean curvature bounds in the context of Lorentzian length spaces with synthetic lower Ricci curvature bounds was used in [CM20b]. Geometric consequences that were derived in [BKMW20] are estimates on the inscribed radius of $\Omega$ and rigidity theorems for the corresponding equality cases. One of the key steps in the proof of these rigidity theorems is a comparison estimate for the Laplacian of the boundary distance function $d_{\Omega^c} = \inf_{y \in \Omega^c} d(y, \cdot)$ [BKMW20, Corollary 4.11]:

$$\Delta_{\Omega}(-d_{\Omega^c}) \geq -(N - 1) \frac{s'_{\frac{K}{N-1}} \frac{\mu}{m}(d_{\Omega^c})}{s_{\frac{K}{N-1}} \frac{\mu}{m}(d_{\Omega^c})} m|_{\Omega}. \quad (1)$$

Here $\Delta_{\Omega}$ is the distributional Laplacian in $\Omega$, $m|_{\Omega}$ is the reference measure $m$ restricted to $\Omega$, $H$ is the synthetic lower mean curvature bound and

$$s_{\frac{K}{N-1}} \frac{\mu}{m}(r) = \cos \left( \sqrt{\frac{K}{N-1}r} \right) - \frac{H}{N-1} \sin \left( \sqrt{\frac{K}{N-1}r} \right)$$
for $K > 0$ and appropriately modified for $K \leq 0$. In particular, for $K = 0$ and $H = \delta(N - 1)$ (1) becomes
\[
\Delta_\Omega(-d_{\Omega^c}) \geq \delta(1 - \delta d_{\Omega^c})^{-1}
\]
and by one of the results in [BKMW20] one has $d_{\Omega^c} \leq \frac{1}{\delta}$. Moreover in [MS21] this Laplace estimate for $H = 0$ was derived for perimeter minimizing sets of finite perimeter in an RCD space.

In Section 3 we will show that under general assumptions on $\partial \Omega$ the Laplace estimate (1) is equivalent to the notions of mean curvature bounded from below used in [Ket20, BKMW20]. This is well-known for Riemannian manifolds and justifies the following definition. We will say that the boundary of a general open subset $\Omega \neq \emptyset$ inside some $\text{RCD}(K,N)$ space $(X,d,m)$ has \textit{Laplace mean curvature bounded from below} by $H$ if the corresponding distance function to the complement $d_{\Omega^c}$ satisfies (1). The advantage of this notion for lower mean curvature bounds is that it will work for all open subsets $\Omega$ in RCD spaces without any other a priori assumptions on $\partial \Omega$. Moreover it has nice stability properties.

The first result of this paper is the following theorem.

\textbf{Theorem 1.1.} Let $X$ be an $\text{RCD}(0,N)$ space for $N \geq 1$, and let $\Omega_\alpha \subset X$, $\alpha = 1, \ldots , m$ with $m \geq 2$, be open and connected such that $\Omega_\alpha^c \neq \emptyset$ and $\Omega_\alpha^c \cap \Omega_\beta^c = \emptyset$ for $\alpha \neq \beta$. Assume $\partial \Omega_\alpha$ has Laplace mean curvature bounded from below by $0$ for every $\alpha$ and assume that $\partial \Omega_2$ is compact.

Then, $m = 2$ and there exists a metric measure space space $Y$ such that
\[
(\tilde{\Omega},\tilde{d}_\Omega, m|_{\tilde{\Omega}}) \text{ is isomorphic to } [0,D) \otimes Y \text{ where } D := \inf_{x \in \Omega_1^c, y \in \Omega_2^c} d_X(x,y)
\]
and $\tilde{\Omega} = \Omega_1 \cap \Omega_2$. If $N \geq 2$, then $Y$ is $\text{RCD}(0,N-1)$. If $N \in [1,2)$, then $Y \simeq \{pt\}$.

The distance $\tilde{d}_\Omega$ is the completion of the induced intrinsic distance on $\Omega$ and $(\tilde{\Omega},\tilde{d}_\Omega, m|_{\tilde{\Omega}})$ is the corresponding metric measure space.

\textbf{Remark 1.1.1.} We emphasize that $\tilde{d}_\Omega$ cannot be replaced with $d_X|_{\Omega}$. A simple counterexample is the $\text{RCD}(0,2)$ space $X$ that is constructed by gluing two copies of a disk $\overline{B}_1(0) = D$ to the ends of the cylinder $S^1 \times [0,1]$. For two points in $S^1 \times (0,1) =: \Omega$ that are close to $S^1 \times \{0\}$ the shortest path w.r.t. $d_X$ goes through $D$. But $\Omega$ splits w.r.t. the intrinsic distance.

As a corollary we obtain

\textbf{Corollary 1.2.} Let $X$ be a compact $\text{RCD}(0,N)$ space with $N \geq 2$. There are no open, connected subsets $\Omega_1$ and $\Omega_2$ such that $\partial \Omega_1$ and $\partial \Omega_2$ are disjoint and have Laplace mean curvature bounded from below by $\delta > 0$.

The corollary can be seen as a mean curvature version in context of RCD spaces of the non-existence result of positive scalar curvature metrics on a torus by Schoen-Yau-Gromov-Lawson [SY79a, SY79b, GL80].

Another corollary is a Frankel-type theorem for mean convex subsets in positively curved RCD spaces.
Corollary 1.3. Let $X$ be an RCD$(\delta,N)$ space for $\delta > 0$ and $N \geq 2$. Let $\Omega_1$ and $\Omega_2$ be open connected subsets in $X$ such that $\partial \Omega_1$ and $\partial \Omega_2$ are Laplace mean convex. Then $\Omega_1^c \cap \Omega_2^c \neq \emptyset$.

The proof that is presented in Section 4.2 is close to a proof in the Riemannian setting (see [PW03]). A similar result appears in [MS21] for perimeter minimizing sets.

Putting the boundary of $\Omega_2$ at infinity in Theorem 1.1, we also get the following theorem.

**Theorem 1.4.** Let $X$ be an RCD$(0,N)$ space with $N \geq 1$ and let $\Omega \subset X$ be open and connected with mean curvature bounded from below by 0. Assume there exists a geodesic ray $\gamma : (0,\infty) \to \Omega$ with $\lim_{r \to 0} \gamma(r) = x_0 \in \partial \Omega \neq \emptyset$ and $d_X(x_0,\gamma(r)) = d_{\Omega^c}(\gamma(r))$.

Then, there exists a metric measure space $Y$ such that $(\tilde{\Omega},\tilde{d}_\Omega,m_\Omega)$ is isomorphic to $[0,\infty) \otimes Y$. If $N \geq 2$, then $Y$ is RCD$(0,N-1)$. If $N \in [1,2)$, then $Y \simeq \{pt\}$.

**Remark 1.4.1.** The assumption $d_X(x_0,\gamma(r)) = d_{\Omega^c}(\gamma(r))$ for the geodesic ray $\gamma$ cannot be omitted. A counterexample is $X = \mathbb{R}^2$ with $\Omega = \{(x,y) : y = x^2\}$.

The proof of Theorem 1.1 has two parts. In Section 4.1 we show that $\Omega$ equipped with the reference measure $m$ restricted to $\Omega$ splits as measure space. In Section 4.2 we then see that this implies an isometric splitting for the induced intrinsic geometry of $\Omega$. This part applies methods developed in [KKL23] and we omit details since the steps are identical with the ones in [KKL23]. The proof of Theorem 1.4 follows the same roadmap with obvious modifications where we only provide the details of the first part.

These rigidity results raise the question for corresponding almost rigidity theorems: given a Riemannian manifold that satisfies the assumption of the theorems up to an error $\epsilon$ are we close (and in which sense) to the rigidity case? In absence of extrinsic boundary, that is $\Omega = X$, these questions can be answered by RCD rigidity theorems, stability of RCD curvature bounds w.r.t. measured Gromov-Hausdorff convergence and Gromov’s precompactness theorem.

For domains with lower mean curvature bounds inside of a Riemannian manifold with Ricci curvature bounded from below the problem is more delicate [Per16, Won08]. A sequence of closed domains may not subconverge in Gromov Hausdorff sense to a metric space. This behavior is similar to the one of closed Riemannian manifolds with lower scalar curvature bounds (for instance, see [Sor17, Gro19]).

Our solution to this problem is as follows. Since we study spaces with boundary as subsets of RCD spaces, we consider the function $d_{\Omega^c}$ that is 1-Lipschitz. Then we can apply Gromov’s Arzela-Ascoli theorem as a compactness theorem for this framework. For a family of RCD$(K,N)$ spaces $X_i$ together with functions $d_{\Omega^c_i}$ one obtains a subsequence of metric measure
spaces and distance functions that converge in measured Gromov-Hausdorff sense and uniformly, respectively, to a 1-Lipschitz function $d_{\Omega}$ on a limit RCD space $X$. To quantify uniform convergence we introduce the uniform distance between continuous functions (Definition 5.2). Applied to distance functions to the boundary of subsets $\Omega$ and $\Omega'$ in $X$ and in $Y$, respectively, one can define a distance $D(\Omega, \Omega')$. Moreover Laplace mean curvature bounds are preserved under this convergence (Theorem 5.8). The latter is essentially known to experts. For instance, in [BNS22] the authors prove a sharp Laplace mean curvature bound for the distance function of the intrinsic boundary of Ricci limit spaces.

These notions yield a compactness statement for pairs $(X, \Omega)$ (Corollary 5.10), and our almost rigidity theorem in the class of subsets in smooth Riemannian manifolds reads as follows.

**Theorem 1.5.** Let $L, c, C, \Gamma \in \mathbb{R}_+, N \geq 2$ and $m \in \mathbb{N}\setminus\{1\}$. For every $\epsilon > 0$ there exists $\delta > 0$ such that the following holds.

Let $M$ be a Riemannian manifold with $\text{ric}_M \geq -\delta$, $\text{dim}_M \leq N$ and $\text{diam}_M \leq L$ and let $\Omega_\alpha \subset X$, $\alpha = 1, \ldots, m$, be open subsets with smooth boundary $\partial \Omega_\alpha$ such that $\Omega_\alpha$ is $(c, C)$-uniform, $\partial \Omega_\alpha$ has mean curvature bounded from below by $-\delta$ and $\inf_{x \in \partial \Omega_\alpha, y \in \partial \Omega_\beta} d_M(x, y) \geq \Gamma > 0$ for $\alpha \neq \beta$.

Then, $m = 2$ and there exist an $\text{RCD}(0, N)$ space $Z$, an $\text{RCD}(0, N-1)$ space $Y$ and an open subset $\Omega' \subset Z$ such that $(\tilde{\Omega}', \tilde{d}_{\Omega'}, m_Z|_{\tilde{\Omega}'}) \simeq Y \times [0, D]$ for some $D > 0$ and

$$D(X, Z) \leq \epsilon \quad \text{and} \quad D(\Omega_1 \cap \Omega_2, \Omega') \leq \epsilon.$$

Here $D$ is the Sturm’s transportation distance [Stu06a]. We actually will prove the theorem in the class of RCD spaces.

The main result in [BKMWW20] is that a subset $\Omega$ with mean curvature bounded from below by $N-1$ inside an $\text{RCD}(0, N)$ space $X$ which attains the inscribed radius bound 1, is isomorphic to a truncated cone w.r.t. its intrinsic geometry. The following theorem is now the corresponding almost rigidity theorem.

**Theorem 1.6.** Let $L, c, C, \Gamma > 0$ and $N \geq 2$. For every $\epsilon > 0$ there exists $\delta > 0$ such that the following holds.

Let $M$ be a Riemannian manifold with $\text{dim}_M \leq N$, $\text{ric}_M \geq -\delta$ and $\text{diam}_M \leq L$, and let $\Omega$ be open and $(c, C)$-uniform such that $\partial \Omega \neq \emptyset$ is smooth and has mean curvature bounded from below by $N-1-\delta$. Assume there exists $x \in \Omega$ such that $d_{\Omega'}(x) \geq 1 - \delta$.

Then, there exists an $\text{RCD}(0, N)$ space $Z$, an $\text{RCD}(N-2, N-1)$ space $Y$ and an open subset $\Omega' \subset Z$ such that $(\tilde{\Omega}', \tilde{d}_{\Omega'}, m_Z|_{\tilde{\Omega}'})$ is isomorphic to $Y \times^{N-1}_\Gamma [0, 1]$ and

$$D(X, Z) \leq \epsilon \quad \text{and} \quad D(\Omega, \Omega') \leq \epsilon.$$

Here $Y \times^{N-1}_\Gamma [0, 1]$ denotes the truncated $N$-Euclidean cone over $Y$. 
The notion of \((c, C)\)-uniform domain (Definition 5.6) is well-known in the study of elliptic and parabolic PDEs. In our theorem this property guarantees that connectedness of domains is preserved under uniform convergence of their distance functions to the boundary. In fact one can see that connectedness of the limit domain is necessary to be able to apply the previous rigidity theorems and any assumption on the sequence \(\Omega_i\) that preserves connected in the limit will be enough for the theorem to hold.

Another application of stability of Laplace mean curvature bounds w.r.t. uniform convergence is stability of ”constant mean curvature hypersurfaces”, and in particular ”minimal hypersurfaces”, along a sequence of Riemannian manifold with lower Ricci curvature bounds that converge in measured Gromov-Hausdorff sense. We will discuss this in the Appendix A.

The article is organized as follows. In Section 2 we recall the necessary background about CD spaces, first and second order calculus on metric measure spaces, RCD spaces, and the 1D localisation technique.

In Section 3 we review several notions of mean curvature bounds for open subsets \(\Omega\) in RCD spaces and show that they are equivalent under suitable regularity assumptions on \(\partial \Omega\). In particular, we show equivalence to the Laplace estimate and introduce Laplace mean curvature bounds.

In Section 4 we first prove that open subsets with disconnected boundary and mean curvature bounded from below in essentially non-branching CD spaces admit a measurable splitting. Then, we obtain the isometric splitting in the context of RCD spaces (Theorem 1.1, Theorem 1.4).

In Section 5 we first review uniform convergence of functions on a sequence of compact metric spaces, and define the uniform distance. Then we prove stability of mean curvature bounds under uniform convergence and deduce the almost rigidity theorems (Theorem 1.5 and Theorem 1.6) in the context of RCD spaces.

In the Appendix A we prove Theorem A.3 concerning ”constant mean curvature hypersurfaces”.

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2. Preliminaries.

2.1. **Curvature-dimension condition.** Let \((X, d)\) be a complete and separable metric space equipped with a locally finite Borel measure \(m\). We call the triple \((X, d, m)\) a metric measure space. Sometimes it will be convenient to simplify the notion and to denote a metric or metric measure space just
X and the corresponding distance function or reference measure $d_X$ and $m_X$ respectively. We will frequently use this notation in the following.

Given a metric space $(X, d)$ a geodesic is a length minimizing curve $\gamma : [a, b] \to X$. We denote the set of constant speed geodesics $\gamma : [a, b] \to X$ with $G^{[a, b]}(X)$ equipped with the topology of uniform convergence and set $G^{[0,1]}(X) =: G(X)$. For $t \in [a, b]$ the evaluation map $e_t : G^{[a, b]}(X) \to X$ is defined as $\gamma \mapsto \gamma(t)$ and $e_t$ is continuous. A set of geodesics $F \subset G(X)$ is said to be non-branching if $\forall \epsilon \in (0, 1)$ the map $e_{[0, \epsilon]}|_F$ is one to one.

The set of (Borel) probability measures is $\mathcal{P}(X)$, the subset of probability measures with finite second moment is $\mathcal{P}^2(X)$, the set of probability measures in $\mathcal{P}^2(X)$ that are $m$-absolutely continuous is denoted with $\mathcal{P}^2(X, m)$ and the subset of measures in $\mathcal{P}^2(X, m)$ with bounded support is denoted with $\mathcal{P}^2_b(X, m)$.

The space $\mathcal{P}^2(X)$ is equipped with the $L^2$-Wasserstein distance $W_2$ that is finite on $\mathcal{P}^2(X)$. A dynamical optimal coupling is a probability measure $\Pi \in \mathcal{P}(G(X))$ such that $t \in [0, 1] \mapsto (e_t)_\# \Pi$ is a $W_2$-geodesic in $\mathcal{P}^2(X)$. The set of dynamical optimal couplings $\Pi \in \mathcal{P}(G(X))$ between $\mu_0, \mu_1 \in \mathcal{P}^2(X)$ is denoted with $\text{OptGeo}(\mu_0, \mu_1)$.

A metric measure space $(X, d, m)$ is called essentially nonbranching if for any pair $\mu_0, \mu_1 \in \mathcal{P}^2(X, m)$ every optimal dynamical plan $\Pi \in \text{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of nonbranching geodesics.

**Definition 2.1.** For $\kappa \in \mathbb{R}$ we define $\cos_\kappa : [0, \infty) \to \mathbb{R}$ as the solution of

$$v'' + \kappa v = 0, \quad v(0) = 1 \quad \& \quad v'(0) = 0.$$ 

$sin_\kappa$ is defined as solution of the same ODE with initial value $v(0) = 0 \& v'(0) = 1$. That is

$$\cos_\kappa(x) = \begin{cases} 
\cosh(\sqrt{|\kappa|} x) & \text{if } \kappa < 0 \\
1 & \text{if } \kappa = 0 \\
\cos(\sqrt{|\kappa|} x) & \text{if } \kappa > 0
\end{cases}$$

$$\sin_\kappa(x) = \begin{cases} 
\frac{\sinh(\sqrt{|\kappa|} x)}{\sqrt{|\kappa|}} & \text{if } \kappa < 0 \\
x & \text{if } \kappa = 0 \\
\frac{\sin(\sqrt{|\kappa|} x)}{\sqrt{|\kappa|}} & \text{if } \kappa > 0
\end{cases}$$

Let $\pi_\kappa$ be the diameter of a simply connected space form $S^2_\kappa$ of constant curvature $\kappa$, i.e.

$$\pi_\kappa = \begin{cases} \infty & \text{if } \kappa \leq 0 \\
\frac{\pi}{\sqrt{|\kappa|}} & \text{if } \kappa > 0.
\end{cases}$$

In [Kas83] and [Sak19] the authors define

\begin{equation}
(2) 
\quad s_{\kappa, \lambda}(r) = \cos_\kappa(r) - \lambda \sin_\kappa(r)
\end{equation}

for $\kappa, \lambda \in \mathbb{R}$. The pair $(\kappa, \lambda)$ satisfies the ball condition if the equation $s_{\kappa, \lambda}(r) = 0$ has a positive solution. The latter happens if and only if one of the following three cases holds: (1) $\kappa > 0$ and $\lambda \in \mathbb{R}$, (2) $\kappa = 0$ and $\lambda > 0$ or (3) $\kappa \leq 0$ and $\lambda > \sqrt{|\kappa|}$. For $(\kappa, \lambda) = (\frac{K}{N-1}, \frac{H}{N-1})$ let $r_{K,H,N}$ be the smallest positive zero of $s_{\kappa, \lambda} = s_{K/(N-1), H/(N-1)}$ if any exists; moreover $s_{\kappa, \lambda}(r) < 0$
for all \( r > r_{K,H,N} \) if \( \kappa \leq 0 \), while \( s_{K,N} \) oscillates sinusoidally with mean zero and period greater than \( 2r_{K,H,N} \) if \( \kappa > 0 \). In particular, \( r_{K,H,N} < \infty \) if and only if \( (\frac{K}{N}, \frac{H}{N}) \) satisfies the ball-condition.

For \( K \in \mathbb{R} \), \( N \in (0, \infty) \) and \( \theta \geq 0 \) we define the distortion coefficient as

\[
t \in [0, 1] \mapsto \sigma_{K,N}(\theta) = \begin{cases} \frac{\sin_{K/N}(\theta)}{\sin_{K/N}(\theta)} & \text{if } \theta \in [0, \pi_{K/N}), \\ \infty & \text{otherwise.} \end{cases}
\]

Note that \( \sigma_{K,N}(0) = t \). Moreover, for \( K \in \mathbb{R} \), \( N \in [1, \infty) \) and \( \theta \geq 0 \) the modified distortion coefficient is defined as

\[
t \in [0, 1] \mapsto \tau_{K,N}(\theta) = \begin{cases} \theta \cdot \infty & \text{if } K > 0 \text{ and } N = 1, \\ \tau_{K,N}(\theta) = \left[ \sigma_{K,N-1}(\theta) \right]^{1-\frac{1}{N}} & \text{if } K \leq 0 \text{ and } N \geq 2, \\ \text{otherwise} & \text{otherwise} \end{cases}
\]

where our convention is \( 0 \cdot \infty = 0 \).

**Definition 2.2 (Curvature-Dimension Condition, [Stu06b, LV09, BS10]).** A metric measure space \((X, d, m)\) satisfies the curvature-dimension condition \( \text{CD}(K, N) \), \( K \in \mathbb{R} \), \( N \in [1, \infty) \), if for all \( \mu_0, \mu_1 \in \mathcal{P}_b(X, m) \) there exists an \( L^2 \)-Wasserstein geodesic \((\mu_t)_{t \in [0, 1]}\) and an optimal coupling \( \pi \) between \( \mu_0 \) and \( \mu_1 \) such that

\[
S_N(\mu_i | m) \leq - \int \left[ \tau_{K,N}(\theta) \rho_0(x) - \frac{1}{N} + \tau_{K,N}(\theta) \rho_1(y) - \frac{1}{N} \right] d\pi(x, y)
\]

where \( \mu_i = \rho_i d m, i = 0, 1 \), and \( \theta = d(x, y) \).

We say a metric measure space \((X, d, m)\) satisfies the reduced curvature-dimension condition \( \text{CD}^*(K, N) \) for \( K \in \mathbb{R} \) and \( N \in (0, \infty) \) if we replace the coefficients \( \tau_{K,N}(\theta) \) with \( \sigma_{K,N}(\theta) \).

If \((X, d, m)\) is complete and satisfies the condition \( \text{CD}(K, N) \) for \( N < \infty \), then \((\text{supp} \, m, d)\) is a geodesic space and \((\text{supp} \, m, d, m)\) is \( \text{CD}(K, N) \). In the following we can always assume that \( \text{supp} \, m = X \). The condition \( \text{CD}(K, N) \) implies the condition \( \text{CD}^*(K, N) \).

### 2.2. First order calculus for metric measure spaces.

Let \((X, d, m)\) be a metric measure space. We denote with \( \text{Lip}(X) \) the space of Lipschitz functions \( f : X \to \mathbb{R} \), and with \( \text{Lip}_c(\Omega) \) the space of Lipschitz function with support in \( \Omega \) for an open set \( \Omega \subset X \). For \( f \in \text{Lip}(X) \) the local slope is

\[
\text{Lip}(f)(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}, \quad x \in X.
\]

If \( f \in L^2(m) \), a function \( g \in L^2(m) \) is called relaxed gradient if there exists a sequence of Lipschitz functions \( f_n \), which converges in \( L^2(m) \) to \( f \), and there exists \( h \) such that \( \text{Lip}(f_n) \) weakly converges to \( h \) in \( L^2(m) \) and \( h \leq g \) m-a.e. A function \( g \in L^2(m) \) is called the minimal relaxed gradient of \( f \) and denoted by \( |\nabla f| \) if it is a relaxed gradient and minimal w.r.t. the \( L^2 \)-norm among
all relaxed gradients. The object \(|\nabla f|\) is local in the sense that
\[
|\nabla f| = 0 \text{ m-a.e. on } f^{-1}(N) \forall N \subset \mathbb{R} \text{ s.t. } L^1(N)
\]
and \(|\nabla f| = |\nabla g|\) m-a.e. on \(\{f = g\}\). The space of \(L^2\)-Sobolev functions is
\[
W^{1,2}(X) := \left\{ f \in L^2(m) : \int |\nabla f|^2 dm < \infty \right\}.
\]
The set \(W^{1,2}(X)\) equipped with the norm \(\|f\|^2_{W^{1,2}(X)} = \|f\|^2_{L^2} + \|\nabla f\|^2_{L^2}\) is a Banach space. If \(W^{1,2}(X)\) is a Hilbert space, we say the metric measure space \((X, d, m)\) is \text{infinitesimally Hilbertian}.

For \(f, g \in W^{1,2}(X)\) one defines functions \(D^\pm f(\nabla g) : X \to \mathbb{R}\) by
\[
D^+ f(\nabla g) = \inf_{\epsilon > 0} \frac{|\nabla (f + \epsilon g)|^2 - |\nabla f|^2}{2\epsilon},
D^- f(\nabla g) = \sup_{\epsilon < 0} \frac{|\nabla (f + \epsilon g)|^2 - |\nabla f|^2}{2\epsilon}.
\]
If \((X, d, m)\) is infinitesimally Hilbertian, then \(D^+ f(\nabla g) = D^- f(\nabla g)\) m-a.e. Moreover
\[
(4) \quad \langle \nabla f, \nabla g \rangle := D^+ f(\nabla g) = D^+ g(\nabla f) = \frac{1}{4} \nabla f(\nabla + g) - \frac{1}{4} |\nabla (f - g)|^2
\]
and \(\langle \nabla f, \nabla g \rangle \in L^1(m)\).

2.3. Distributional Laplacian and strong maximum principle. We recall the notion of the distributional Laplacian (cf. [Gig15, CM20a]).

Definition 2.3. Let \((X, d, m)\) be a locally compact metric measure space and \(\Omega \subset X\) be open. Let \(\text{Lip}_c(\Omega)\) denote the set of Lipschitz functions compactly supported in an open subset \(\Omega\). A \text{Radon functional} over \(\Omega\) is a linear map \(T : \text{Lip}_c(\Omega) \to \mathbb{R}\) such that for every compact subset \(W\) in \(\Omega\) there exists a constant \(C_W \geq 0\) such that
\[
(5) \quad |T(f)| \leq C_W \max_W |f| \quad \forall f \in \text{Lip}_c(\Omega) \text{ with supp } f \subset W.
\]
One says \(T\) is non-negative if \(T(f) \geq 0\) for all \(f \in \text{Lip}_c(\Omega)\) satisfying \(f \geq 0\).

Remark 2.4. The Riesz-Markov-Kakutani representation theorem says that for a non-negative Radon functional \(T\) there exists a unique Radon measure \(\mu_T\) such that \(T(f) = \int f \, d\mu_T \forall f \in \text{Lip}_c(\Omega)\).

Recall that \(u \in W^{1,2}_{\text{loc}}(\Omega)\) for an open set \(\Omega \subset X\) if for any Lipschitz function \(\phi\) with compact support in \(\Omega\) we have \(\phi \cdot u \in W^{1,2}(X)\). Thanks to the locality properties of \(|\nabla f|\) for \(f \in W^{1,2}(X)\) the object \(|\nabla u|\) is well defined m-a.e. for \(u \in W^{1,2}_{\text{loc}}(\Omega)\). If \(|\nabla u| \in L^2(m)\), one writes \(u \in W^{1,2}(\Omega)\). If \(u \in \text{Lip}(X)\) then \(u \in W^{1,2}_{\text{loc}}(\Omega)\).
Theorem 2.7 is both sub- and super-harmonic. \( T \) is one such measure \( \mu_T \) with \( \mu_T \in \Delta \Omega u \). If there is only one such measure \( \mu_T \) by abuse of notation we will identify \( \mu_T \) with \( T \) and write \( \mu_T = \Delta \Omega u \).

If \((X, d, m)\) is infinitesimally Hilbertian, \( u \in W^{1,2}(X) \) is in the domain of the \( L^2 \)-Laplacian if there exists \( h \in L^2(m) \) such that

\[
\int (\nabla u, \nabla f) \, d m = \int h f \, d m \quad \forall f \in \text{Lip}(X).
\]

In this case we write \( h = \Delta u \) and \( u \in D_{L^2}(\Delta) \). For a linear subspace \( \mathbb{V} \subset L^2(m) \) we write \( u \in D_{\mathbb{V}}(\Delta) \) whenever \( \Delta u \in \mathbb{V} \).

**Remark 2.6 (Locality and linearity).**

(i) If \( u \in D(\Delta, \Omega) \) and \( \Omega' \subset X \) is open in \( X \) with \( \Omega' \subset \Omega \), then \( u \in D(\Delta, \Omega') \) and for \( \mu \in \Delta \Omega u \) it follows that \( \mu|_{\Omega'} \in \Delta \Omega' u \).

(ii) Assume \((X, d, m)\) is infinitesimally Hilbertian. If \( u, v \in D(\Delta, \Omega) \), then \( u + v \in D(\Delta, \Omega) \) and for \( \mu_u \in \Delta \Omega u \) and \( \mu_v \in \Delta \Omega v \) it follows that \( \mu_u + \mu_v \in \Delta \Omega (u + v) \).

Recall that \( u \in W^{1,2}(\Omega) \) is sub-harmonic if

\[
\int_\Omega |\nabla u|^2 \, d m \leq \int_\Omega |\nabla (u + g)|^2 \, d m \quad \forall g \in W^{1,2}(\Omega) \text{ with } g \leq 0.
\]

One says \( u \) is super-harmonic if \( -u \) is sub-harmonic, and \( u \) is harmonic if it is both sub- and super-harmonic.

**Theorem 2.7 (Characterizing super-harmonicity, \cite[Theorem 4.3]{GM13}).** Let \( X \) be an RCD\((K, N)\) space with \( K \in \mathbb{R} \) and \( N \in [1, \infty) \), let \( \Omega \subset X \) be open and \( u \in W^{1,2}_{\text{loc}}(\Omega) \). Then \( u \) is super-harmonic if and only if \( u \in D(\Delta, \Omega) \) and there exists \( \mu \in \Delta \Omega u \) such that \( \mu \leq 0 \).

The following is \cite[Theorem 9.13]{BB11} (see also \cite{GR19}):

**Theorem 2.8 (Strong Maximum Principle).** Let \( X \) be an RCD\((K, N)\) space with \( K \in \mathbb{R} \) and \( N \in [1, \infty) \), let \( U \subset X \) be a connected open set with compact closure and let \( u \in W^{1,2}_{\text{loc}}(\Omega) \cap C(\Omega) \) be sub-harmonic. If there exists \( x_0 \in \Omega \) such that \( u(x_0) = \max_{\bar{\Omega}} u \) then \( u \) is constant.

### 2.4. Riemannian curvature-dimension condition.

**Definition 2.9.** A metric measure space \((X, d, m)\) satisfies the (reduced) Riemannian curvature-dimension condition \( \text{RCD}(K, N) \) \((\text{RCD}^*(K, N))\) for \( K \in \mathbb{R} \) and \( N \in [1, \infty) \) if it satisfies the (reduced) curvature-dimension condition \( \text{CD}(K, N) \) \((\text{CD}^*(K, N))\) and is infinitesimally Hilbertian.
For a brief overview on the history of this definition we refer the reader to the preliminary section of [KK20]. For $N \in [1, \infty)$ an RCD$^*(K, N)$ space $X$ with $m_X$ finite satisfies the condition $\text{RCD}(K, N)$ [CM21] and the converse direction holds without any assumption.

Let $(X, d, m)$ be a metric measure space that is infinitesimally Hilbertian but does not necessarily satisfy a curvature-dimension condition. For $f \in D_{W^{1,2}(X)}(\Delta)$ and $\phi \in D_{L}\infty(\Delta) \cap L\infty(m)$ the carré du champ operator is defined as

$$\Gamma_2(f; \phi) := \int \frac{1}{2} |\nabla f|^2 \Delta \phi \, dm - \int \langle \nabla f, \nabla \Delta f \rangle \phi \, dm.$$ 

A metric measure space $(X, d, m)$ satisfies the Bakry-Émery condition $\text{BE}(K, N)$ for $K \in \mathbb{R}$, $N \in (0, \infty]$ if it satisfies the weak Bochner inequality

$$\Gamma_2(f; \phi) \geq \frac{1}{N} \int (\Delta f)^2 \phi \, dm + K \int |\nabla f|^2 \phi \, dm.$$ 

for any $f \in D_{W^{1,2}(X)}(\Delta)$ and $\phi \in D_{L}\infty(\Delta) \cap L\infty(m)$, $\phi \geq 0$.

A metric measure space satisfies the Sobolev-to-Lipschitz property if every $f \in W^{1,2}(X)$ with $|\nabla f| \in L\infty(m)$ admits a Lipschitz representative $\tilde{f} \in \text{Lip}(X)$ such that the local Lipschitz constant is bounded from above by $\|
abla f\|_{L\infty}$. For RCD spaces the Sobolev-to-Lipschitz property was proved in [AGS14, Theorem 6.2].

**Theorem 2.10** ([EKS15, AGS15, AMS19]). Let $(X, d, m)$ be a metric measure space. The reduced Riemannian curvature-dimension condition $\text{RCD}^*(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty]$ holds if and only if $(X, d, m)$ is infinitesimally Hilbertian, satisfies the Sobolev-to-Lipschitz property and the exponential growth condition $\int e^{-C d(x_0, \cdot)^2} \, dm$ for some $x_0 \in X$, and satisfies the Bakry-Émery condition $\text{BE}(K, N)$.

An important class of functions on an RCD space $(X, d, m)$ is the family $\mathbb{D}_\infty$ of test functions that is defined by

$$\mathbb{D}_\infty = \{ f \in D_{W^{1,2}(X)}(\Delta) \cap L\infty(m) : |\nabla f| \in L\infty(m) \}.$$ 

For $f \in \mathbb{D}_\infty$ one can define a Hessian $\text{Hess}(f)$ via the formula

$$2\text{ Hess}(f)(\nabla g, \nabla h) =$$

$$\langle \nabla g, \nabla \langle \nabla h, \nabla f \rangle \rangle + \langle \nabla h, \nabla \langle \nabla f, \nabla g \rangle \rangle - \langle \nabla f, \nabla \langle \nabla g, \nabla h \rangle \rangle \text{ for } g, h \in \mathbb{D}_\infty.$$ 

One can extend the operator Hess to the bigger class $H^{2,2}(X)$ that contains $\mathbb{D}_\infty$ and $D_{L^2}(\Delta)$. For $f \in H^{2,2}(X)$ the Hessian is a tensorial object and admits a Hilbert-Schmidt norm $||\text{Hess} f||_{HS} \in L^2(m)$.

**Theorem 2.11** ([Sav14, Gig18, Stu18]). If the metric measure space $(X, d, m)$ satisfies the Riemannian curvature-dimension condition $\text{RCD}(K, \infty)$, and $f \in \mathbb{D}_\infty$, then $|\nabla f|^2 \in W^{1,2}(X) \cap D(\Delta)$ and an improved Bochner formula
holds in the sense of measures involving the Hilbert-Schmidt norm of the Hessian of $f$:

$$\Gamma_2(f) := \frac{1}{2} \Delta |\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \ m \geq [K|\nabla f|^2 + |\text{Hess } f|_{HS}^2] \ m$$

where $\Delta |\nabla f|^2$ is given by unique measure, and $\Gamma_2$ is called measure valued $\Gamma_2$-operator. In particular, the singular part of the left hand side in previous inequality is non-negative.

2.5. 1D-localization. In this section we will recall basic facts about the localization technique introduced by Cavalletti and Mondino for 1-Lipschitz functions as a nonsmooth analogue of Klartag’s needle decomposition: needle refers to any geodesic along which the Lipschitz function attains its maximum slope, also called transport rays here and by Klartag and others [EG99, FM02, Kla17]. The presentation follows Sections 3 and 4 in [CM17]. We assume familiarity with basic concepts in optimal transport (for instance [Vil09]).

Let $(X, d, m)$ be a proper metric measure space with $\text{supp } m = X$ as we always assume.

Let $u : X \to \mathbb{R}$ be a 1-Lipschitz function. Then the transport ordering

$$\Gamma_u := \{(x, y) \in X \times X : u(y) - u(x) = d(x, y)\}$$

is a d-cyclically monotone set, and one defines $\Gamma_u^{-1} = \{(x, y) \in X \times X : (y, x) \in \Gamma_u\}$.

Note that we switch orientation in comparison to [CM17] where Cavalletti and Mondino define $\Gamma_u$ as $\Gamma_u^{-1}$.

The union $\Gamma_u \cup \Gamma_u^{-1}$ defines a relation $R_u$ on $X \times X$, and $R_u$ induces the transport set with endpoints and branching points

$$\mathcal{T}_{u,e} := P_1(R_u \backslash \{(x, y) : x = y \in X\}) \subset X$$

where $P_1(x, y) = x$. For $x \in \mathcal{T}_{u,e}$ one defines $\Gamma_u(x) := \{y \in X : (x, y) \in \Gamma_u\}$, and similarly $\Gamma_u^{-1}(x)$ and $R_u(x)$. Since $u$ is 1-Lipschitz, $\Gamma_u, \Gamma_u^{-1}$ and $R_u$ are closed, as are $\Gamma_u(x), \Gamma_u^{-1}(x)$ and $R_u(x)$.

The sets of forward and backward branching points, $A_+ \ & \ A_-$, are defined respectively as

$$A_{+/\ -} := \{x \in \mathcal{T}_{u,e} : \exists z, w \in \Gamma_u(x)/\Gamma_u^{-1}(x) \ \& \ (z, w) \notin R_u\}.$$

Then one considers the (nonbranched) transport set as $\mathcal{T}_u := \mathcal{T}_{u,e} \backslash (A_+ \cup A_-)$ and the (nonbranched) transport relation as the restriction of $R_u$ to $\mathcal{T}_u \times \mathcal{T}_u$.

The sets $\mathcal{T}_{u,e}, A_+$ and $A_-$ are $\sigma$-compact ([CM17, Remark 3.3] and [Cav14, Lemma 4.3] respectively), and $\mathcal{T}_u$ is a Borel set. In [Cav14, Theorem 4.6] Cavalletti shows that the restriction of $R_u$ to $\mathcal{T}_u \times \mathcal{T}_u$ is an equivalence relation. Hence, from $R_u$ one obtains a partition of $\mathcal{T}_u$ into a disjoint family of equivalence classes $\{X_\alpha\}_{\alpha \in \mathcal{Q}}$. A section is a map $s : \mathcal{T}_u \to \mathcal{T}_u$ such that if $(x, s(x)) \in R_u$ and $(y, x) \in R_u$ then $s(x) = s(y)$. By [Cav14, Proposition 5.2] there exists a measurable section $s$, and the quotient space $\mathcal{Q}$ can be identified with the image of $\mathcal{T}_u$ under this map $s$. Hence, we can and will
consider $Q$ as a subset of $X$, namely the image of $s$, equipped with the induced measurable structure.

The quotient map $\Omega : \mathcal{T}_u \to Q$ given by the measurable section $s$ is measurable, and we set $q := \Omega \# [m|_{\mathcal{T}_u}]$. Hence $q$ is a Borel measure on $X$. By inner regularity we replace $Q$ with a Borel set $Q' \subset Q$ such that $q(Q \setminus Q') = 0$ and in the following we denote $Q'$ by $Q$ (compare with [CM17, Proposition 3.5] and the following remarks).

Every $X_\alpha$, $\alpha \in Q$, is isometric to an interval $I_\alpha \subset \mathbb{R}$ (c.f. [CM17, Lemma 3.1] and the comment after Proposition 3.7 in [CM17]) via a distance preserving map $\gamma_\alpha : I_\alpha \to X_\alpha$ where $\gamma_\alpha$ is parametrized such that $d(\gamma_\alpha(t), s(\gamma_\alpha(t))) = \text{sgn}(\gamma_\alpha(t))t$, $t \in I_\alpha$, and where $\text{sgn} x$ is the sign of $u(x) - u(s(x))$. The map $\gamma_\alpha : I_\alpha \to X$ extends to a geodesic also denoted $\gamma_\alpha$ and defined on the closure $\overline{\mathcal{T}}_\alpha$ of $I_\alpha$. We set $\overline{\mathcal{T}}_\alpha = [a(X_\alpha), b(X_\alpha)]$.

In [CM20a, Theorem 3.3], Cavalletti and Mondino prove:

**Theorem 2.12** (Disintegration into needles/transport rays). Let $(X, d, m)$ be a geodesic metric measure space with $\text{supp} m = X$ and $m$ $\sigma$-finite. Let $u : X \to \mathbb{R}$ be a $1$-Lipschitz function, let $\{X_\alpha\}_{\alpha \in Q}$ be the induced partition of $\mathcal{T}_u$ via $R_u$, and let $\Omega : \mathcal{T}_u \to Q$ be the induced quotient map as above. Then, there exists a unique strongly consistent disintegration $\{m_\alpha\}_{\alpha \in Q}$ of $m|_{\mathcal{T}_u}$ with respect to $\Omega$.

The following is [CM20a, Lemma 3.4].

**Lemma 2.13** (Negligibility of branching points). Let $(X, d, m)$ be an essentially nonbranching $\text{MCP}(K, N)$ space, $K \in \mathbb{R}$, $N \in (1, \infty)$, with $\text{supp} m = X$. Then, for any $1$-Lipschitz function $u : X \to \mathbb{R}$, it follows $m(\mathcal{T}_{u,e} \setminus \mathcal{T}_u) = 0$.

The initial and final points are defined by

$$a_u := \{x \in \mathcal{T}_{u,e} : \Gamma_u^{-1}(x) = \{x\}\}, \quad b_u := \{x \in \mathcal{T}_{u,e} : \Gamma_u(x) = \{x\}\}.$$ 

In [CM21, Theorem 7.10] it was proved that under the assumption of the previous lemma there exists $\hat{Q} \subset Q$ with $q(\hat{Q} \setminus \hat{Q}) = 0$ such that for $\alpha \in \hat{Q}$ one has $\overline{X_\alpha \setminus \mathcal{T}_u} \subset a_u \cup b_u$. In particular, for $\alpha \in \hat{Q}$ we have

$$R_u(x) = \overline{X_\alpha \setminus \mathcal{T}_u} \supset \overline{X_\alpha \subset (R_u(x))^\circ} \quad \forall x \in \Omega^{-1}(\alpha) \subset \mathcal{T}_u,$$

where $(R_u(x))^\circ$ denotes the relative interior of the closed set $R_u(x)$.

The following is [CM20a, Theorem 3.5].

**Theorem 2.14** (Factor measures inherit curvature-dimension bounds). Let $K \in \mathbb{R}$, $N \in (1, \infty)$ and let $(X, d, m)$ be essentially nonbranching and $\text{MCP}(K, N)$ with $\text{supp} m = X$. For any $1$-Lipschitz function $u : X \to \mathbb{R}$, let $\{m_\alpha\}_{\alpha \in Q}$ denote the disintegration of $m|_{\mathcal{T}_u}$ from Theorem 2.12 which is strongly consistent with the quotient map $\Omega : \mathcal{T}_u \to Q$.

Then there exists $\hat{Q}$ such that $q(\hat{Q} \setminus \hat{Q}) = 0$ and $\forall \alpha \in \hat{Q}$, $m_\alpha$ is a Radon measure with $\text{d}m_\alpha = h \text{d}H^1|_{X_\alpha}$ and $(X_\alpha, d, m_\alpha)$ satisfies $\text{MCP}(K, N)$. If $(X, d, m)$ satisfies the condition $\text{CD}(K, N)$, then $(X, d, m_\alpha)$ satisfies the condition $\text{CD}(K, N)$ as well.
Remark 2.15. The theorem yields that $h_\alpha$ is locally Lipschitz continuous on $(a(X_\alpha), b(X_\alpha))$ [CM17, Section 4]. In particular, $h_\alpha$ is differentiable for $L^1$-a.e. $r \in (a(X_\alpha), b(X_\alpha))$ and

$$\frac{d}{dr} h_\alpha(r) = \limsup_{h \to 0} \frac{h_\alpha(r+h) - h_\alpha(r)}{h} \quad \text{and} \quad \frac{d^-}{dr} h_\alpha(r) = \limsup_{h \downarrow 0} \frac{h_\alpha(r+h) - h_\alpha(r)}{h}$$

both exist in $\mathbb{R}$ for all $r \in (a(X_\alpha), b(X_\alpha))$. The Bishop-Gromov volume monotonicity implies that $h_\alpha$ can be extended to a continuous function on $[a(X_\alpha), b(X_\alpha)]$ [CM20a, Remark 2.14]. We consider $\frac{d}{dr} h_\alpha : X_\alpha \to \mathbb{R}$ defined a.e. via $\frac{d}{dr} (h_\alpha \circ \gamma_\alpha)(r) =: \frac{d^-}{dr} h_\alpha(\gamma_\alpha(r))$.

Remark 2.16 (Generic geodesics). We set $Q^\dagger := \hat{Q} \cap \check{Q}$, where $\check{Q}$ and $\hat{Q}$ index the transport rays identified between Lemma 2.13 and Theorem 2.14. Then, $q(Q \setminus Q^\dagger) = 0$ and for every $\alpha \in Q^\dagger$ the space $(X, d, h_\alpha H^1)$ is $MCP(K, N)$ (or $CD(K, N)$) and (6) holds. We also set $\Omega^{-1}(Q^\dagger) := T^\dagger_u \subset T_u$ and $\bigcup_{x \in T^\dagger_u} R_u(x) =: T^\dagger_{a, e} \subset T_{a, e}$.

3. Notions of synthetic lower mean curvature bounds

Let $(X, d, m)$ be an RCD space with $\text{supp } m = X$ and let $\Omega \subset X$ be an open subset such that $m(\partial \Omega) = 0$. We set $S := \partial \Omega = \overline{\Omega} \setminus \Omega$ and $\Omega^e := X \setminus \Omega$. Since $m(S) = 0$, it holds $\partial \Omega^e = S$. The distance function $d_{\Omega^e} : X \to \mathbb{R}$ is given by

$$d_{\Omega^e}(x) := \inf_{y \in \Omega^e} d(x, y).$$

The signed distance function $d_S$ for $S$ is given by

$$d_S := d_{\overline{\Omega}} - d_{\Omega} : X \to \mathbb{R}.$$

It follows that $d_S(x) = 0$ if and only if $x \in S$, and $d_S \leq 0$ if $x \in \Omega$ and $d_S \geq 0$ if $x \in \Omega^e$. It is clear that $d_S|_\Omega = - d_{\Omega^e}$ and $d_S|_{\Omega^e} = d_{\Omega}$. Setting $v = d_S$ we can also write

$$d_S(x) = \text{sign}(v(x)) d(\{v = 0\}, x), \forall x \in X.$$

Since $(X, d)$ is a proper geodesic space, $d_S$ is 1-Lipschitz [CM20a, Remark 8.4, Remark 8.5].

Let $T_{d_S, e}$ be the transport set of $d_S$ with end- and branching points. We have $T_{d_S, e} \supset X \setminus S$. In particular, we have $m(X \setminus T_{d_S}) = 0$ by Lemma 2.13 and $m(S) = 0$. Therefore, by Theorem 2.14 the 1-Lipschitz function $d_S$ induces a partition $\{X_\alpha\}_{\alpha \in Q}$ of $X$ up to a set of measure zero for a measurable quotient space $Q$, and a disintegration $\{m_\alpha\}_{\alpha \in Q}$ that is strongly consistent with the partition. The subset $X_\alpha$, $\alpha \in Q$, is the image of a distance preserving map $\gamma_\alpha : I_\alpha \to X$ for an interval $I_\alpha \subset \mathbb{R}$ with $T_{T_\alpha} = [a(X_\alpha), b(X_\alpha)] \ni 0$.

We consider $Q^\dagger \subset Q$ as in Remark 2.16. One has the representation

$$m(B) = \int_Q m_\alpha(B) dq(\alpha) = \int_{Q^\dagger} \int_{\gamma^{-1}_\alpha(B)} h_\alpha(r) dr dq(\alpha)$$
for all Borel subsets $B \subset X$. For a transport ray $X_\alpha$ one has $d_S(\gamma_\alpha(b(X_\alpha))) \geq 0$ and $d_S(\gamma_\alpha(a(X_\alpha))) \leq 0$ (for instance compare with [CM20a, Remark 4.12]).

Let us recall another result of Cavalletti-Mondino:

**Theorem 3.1** (Laplacian of signed distance functions [CM20a, Corollary 4.16]). Let $(X, d, m)$ be a CD$(K, N)$ space, and $\Omega$ and $S = \partial \Omega$ as above. Then $d_S|_{X \setminus S} \in D(\Delta, X \setminus S)$, and one element of $\Delta_{X \setminus S}(d_S|_{X \setminus S})$ that we also denote with $\Delta_{X \setminus S}(d_S|_{X \setminus S})$ is the Radon functional on $X \setminus S$ given by the representation formula

$$
\Delta_{X \setminus S}(d_S|_{X \setminus S}) = (\log h_\alpha)' m|_{X \setminus S} + \int_Q (h_\alpha \delta_0(x_a) \cap \{d_S < 0\} - h_\alpha \delta_0(x_a) \cap \{d_S > 0\}) dq(\alpha).
$$

The Radon functional $\Delta_{X \setminus S}(d_S|_{X \setminus S})$ can be represented as the difference of two measures $[\Delta_{X \setminus S}(d_S|_{X \setminus S})]^+$ and $[\Delta_{X \setminus S}(d_S|_{X \setminus S})]^-$ such that

$$
[\Delta_{X \setminus S}(d_S|_{X \setminus S})]^+_{abs} - [\Delta_{X \setminus S}(d_S|_{X \setminus S})]^-_{abs} = (\log h_\alpha)' m \text{-a.e.}
$$

In particular, $-(\log h_\alpha)'$ coincides with a measurable function $m$-a.e.

**Remark 3.2** (Measurability and zero-level selection). It is easy to see that $A := \Omega^{-1}(\Omega(S \cap \mathcal{T}_{d_S})) \subset \mathcal{T}_{d_S}$ is a measurable subset. The reach $A \subset \mathcal{T}_{d_S}$ is defined such that $\forall \alpha \in \Omega(A)$ we have $X_\alpha \cap S = \{\gamma(t_\alpha)\} \neq \emptyset$ for a unique $t_\alpha \in I_\alpha$. Then, the map $\hat{s} : \gamma(t) \in A \mapsto \gamma(t_\alpha) \in S \cap \mathcal{T}_{d_S}$ is a measurable section (i.e. selection) on $A \subset \mathcal{T}_{d_S}$, and one can identify the measurable set $\Omega(A) \subset Q$ with $A \cap S$ and can parameterize $\gamma_\alpha$ such that $t_\alpha = 0$.

This measurable section $\hat{s}$ on $A$ is fixed for the rest of the paper. The reach $A$ is the union of all disjoint needles that intersect with $\partial \Omega$ – eventually in $a(X_\alpha)$ (or in $b(X_\alpha)$) provided $a(X_\alpha)$ (respectively $b(X_\alpha)$) belongs already to $I_\alpha$. We shall also define the inner reach $B_{in}$ as the union of all needles disjoint from $\Omega^c$ and the outer reach $B_{out}$ as the union of all needles disjoint from $\Omega$. The superscript $\dagger$ will be used to indicate intersection with $\mathcal{T}_{d_S}^\dagger$. Thus

$$A \cap \mathcal{T}_{d_S}^\dagger =: A^\dagger \text{ and } \bigcup_{x \in A^\dagger} R_{d_S}(x) =: A^\dagger_e.$$

The sets $A^\dagger$ and $A^\dagger_e$ are measurable, and also

$$B_{in}^\dagger := \Omega^c \cap \mathcal{T}_{d_S}^\dagger \cap A^\dagger \subset \mathcal{T}_{d_S}^\dagger \text{ and } B_{out}^\dagger := \Omega^c \cap \mathcal{T}_{d_S}^\dagger \setminus A^\dagger \subset \mathcal{T}_{d_S}^\dagger$$

as well as $\bigcup_{x \in B_{out}^\dagger} R_{d_S}(x) =: B_{out,e}^\dagger$ and $\bigcup_{x \in B_{in}^\dagger} R_{d_S}(x) =: B_{in,e}^\dagger$ are measurable. The map $\alpha \in \Omega(A^\dagger) \mapsto h_\alpha(0) \in \mathbb{R}$ is measurable (see [CM21, Proposition 10.4]).

**Remark 3.3** (Surface measure via ray maps). Let us briefly explain the previous definition from the viewpoint of the ray map [CM17, Definition 3.6] or its precursor from the smooth setting [FM02]. For the definition we fix a measurable extension $s_0 : \mathcal{T}_{d_S} \rightarrow \mathcal{T}_{d_S}$ such that $s_0|_{A^\dagger} = \hat{s}$ as in Remark
3.2. As was explained in Subsection 2.5 such a section allows us to identify the quotient space $Q$ with a Borel subset in $X$ up to a set of $q$-measure 0. Following [CM17, Definition 3.6] we define the ray map

$$g : \mathcal{V} \subset \Omega(A \cup B_{in}) \times (-\infty, 0] \to \Omega$$

into $\Omega$ and its domain $\mathcal{V}$ via its graph

$$\text{graph}(g) = \{(\alpha, t, x) \in \Omega(A) \times \mathbb{R} \times \Omega : x \in X_\alpha, -d(x, \alpha) = t\} \cup \{(\alpha, t, x) \in \Omega(B_{in}) \times \mathbb{R} \times \Omega : x \in X_\alpha, -d(x, \gamma_\alpha(b(X_\alpha))) = t\}.$$ 

This is exactly the ray map as in [CM17] up to a reparametrisation for $\alpha \in \Omega(B_{in})$. Note that $g(\alpha, 0) = \gamma_\alpha(0) = \alpha$ and $g(\alpha, t) = \gamma_\alpha(t)$ if $\alpha \in \Omega(A)$ but $\gamma_\alpha(t + d(b(X_\alpha), \alpha)) = g(\alpha, t)$ for $\alpha \in \Omega(B_{in})$. Then the disintegration for a non-negative $\phi \in C_b(\Omega)$ takes the form

$$\int_{\Omega} \phi \, dm = \int_{Q} \int_{\mathcal{V}_\alpha} \phi \circ g(\alpha, t) h_\alpha \circ g(\alpha, t) \, d\mathcal{L}^1(t) \, dq(\alpha)$$

where $\mathcal{V}_\alpha = P_2(\mathcal{V} \cap \{\alpha\} \times \mathbb{R}) \subset \mathbb{R}$ and $P_2(\alpha, t) = t$. With Fubini’s theorem the right hand side is

$$\int_{\mathcal{V}} \phi \circ g(\alpha, t) h_\alpha \circ g(\alpha, t) \, d(q \otimes \mathcal{L}^1)(\alpha, t) = \int \int_{\mathcal{V}_t} \phi \circ g(\alpha, t) h_\alpha \circ g(\alpha, t) \, dq(\alpha) \, d\mathcal{L}^1(t)$$

where $\mathcal{V}_t = P_1(\mathcal{V} \cap Q \times \{t\}) \subset Q$ and $P_1(\alpha, t) = \alpha$. In particular, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ the set $\mathcal{V}_t \subset Q$ and the map $\alpha \mapsto h_\alpha \circ g(\alpha, t)$ are measurable. Hence, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we define $d\mathcal{P}_t(\alpha) = h_\alpha \circ g(\alpha, t) \, dq|_{\mathcal{V}_t}(\alpha)$ on $Q$. Then the disintegration takes the form

$$m|_\Omega = m|_{\Omega \cap \mathcal{T}_{dg}} = \int \left(g(\cdot, t) \# \mathcal{P}_t\right) dt.$$

Note that $\mathcal{V}_0 = \mathcal{V} \cap Q \times \{0\} = \Omega(A) \cup \Omega(B_{in})$ is measurable, one has $\mathcal{V}_t \subset \mathcal{V}_0$, $t < 0$, and that $\alpha \in \mathcal{V}_0 \mapsto \lim_{t \to 0} h_\alpha \circ g(\alpha, t) = h_\alpha \circ g(\alpha, 0)$ is measurable. Hence, we set $d\mathcal{P}_0(\alpha) = h_\alpha \circ g(\alpha, t) \, dq|_{\mathcal{V}_0}(\alpha)$.

**Definition 3.4** (Backward mean curvature bounded below). Let $(X, d, m)$ be essentially nonbranching and $MCP(K, N)$ for $K \in \mathbb{R}$ and $N \in (1, \infty)$.

Then $S = \partial \Omega$ has *backward mean curvature bounded from below* by $H \in \mathbb{R}$ if the measure $\mathcal{P}_0$ is a Radon measure, $h_\alpha \circ g(\alpha, 0) > 0$ for $q$-a.e. $\alpha \in Q$ and

$$\left.\frac{d}{dt}\right|_{t=0} \int_Y d\mathcal{P}_t := \limsup_{h \uparrow 0} \frac{1}{h} \left(\int_Y d\mathcal{P}_h - \int_Y d\mathcal{P}_0\right) \geq H \int_Y d\mathcal{P}_0$$

for any bounded measurable subset $Y \subset Q$. Moreover, $S$ has *backward-lower mean curvature bounded from below* by $H$ if the same inequality holds when $\limsup$ is replaced by $\liminf$.

**Remark 3.5.** Since it is not assumed that $(\mathcal{P}_t)_{t \geq 0}$ is a Radon measure, $\int_Y d\mathcal{P}_t$ can be infinite.
**Proposition 3.6** (Rescaling). Let \((X, d, m)\) be \(\text{MCP}(K, N)\) and let \(\Omega \subset X\) with backward mean curvature bounded below by \(H \in \mathbb{R}\) in \(X\). Define \((\tilde{X}, \tilde{d}, \tilde{m})\) with \(\tilde{X} = X\), \(\tilde{m} = m\) and \(\tilde{d} = \epsilon d\). Then \(\tilde{X}\) satisfies \(\text{MCP}(\frac{1}{\epsilon}K, N)\) and \(\Omega\) has mean curvature bounded from below by \(\frac{1}{\epsilon}H\) in \(\tilde{X}\).

**Proof.** The first claim is known. For the second claim observe that \((x, y) \in \Gamma_d S\), satisfies
\[
\tilde{d}_S(y) - \tilde{d}_S(x) = \epsilon (d_S(y) - d_S(x)) = \epsilon d(x, y) = \tilde{d}(x, y).
\]
Hence given transport geodesic \(\gamma_\alpha\) w.r.t. \(d_S\) we have \(r \in [\epsilon a(X_\alpha), \epsilon b(X_\alpha)] \mapsto \gamma(\frac{1}{\epsilon}r)\) is transport geodesic w.r.t. \(\tilde{d}_S\). This implies that \(\partial \Omega = S\) has backward mean curvature bounded below by \(\frac{1}{\epsilon}H\).

**Lemma 3.7.** Let \((X, d, m)\) be an essentially non-branching \(\text{MCP}(K, N)\) space with \(K \in \mathbb{R}\), \(N \in (1, \infty)\), and let \(\Omega \subset X\) such that \(S = \partial \Omega\) has backward mean curvature bounded from below \(H\). Then
\[
\frac{d^-}{dr} \bigg|_{r=0} h_\alpha \circ g(\alpha, r) \geq H h_\alpha(g(\alpha, 0)) \tag{8}
\]
for \(q\)-a.e. \(\alpha \in V_0 = \Omega(A^\uparrow \cup B_{in}^\uparrow)\).

If \(p_0\) is a Radon measure, \(h_\alpha \circ g(\alpha, 0) > 0\) for \(q\)-a.e. \(\alpha \in Q\) and
\[
\frac{d^-}{dr} \bigg|_{r=0} h_\alpha \circ g(\alpha, r) \geq H h_\alpha(g(\alpha, 0)) \tag{9}
\]
for \(q\)-a.e. \(\alpha \in V_0\)

then \(S\) has backward-lower (hence backward) mean curvature bounded from below by \(H\).

If \((X, d, m)\) is a \(\text{CD}(K, N)\) space, then (8) and (9) become
\[
\frac{d^-}{dr} \bigg|_{r=0} h_\alpha \circ g(\alpha, r) \geq H h_\alpha(g(\alpha, 0))
\]
and hence, backward and backward-lower mean curvature bounded from below are equivalent. \(\frac{d^-}{dr}\) is the left derivative.

**Proof.** We start with the first claim. For \(t < 0\) and a bounded measurable set \(Y \subset Q\) we write
\[
\int_Y d\mu_t - \int_Y d\mu_0 = \int_Y h_\alpha(g(\alpha, 0))^{-1} (1_{V_t}(\alpha)h_\alpha \circ g(\alpha, t) - 1_{V_0}(\alpha)h_\alpha \circ g(\alpha, 0)) \ d\mu_0(\alpha).
\]
There exists \(Q^* \subset Q^\uparrow\) with \(q[Q^\uparrow \setminus Q^*] = 0\) such that the map \(\mathcal{M} : \alpha \in Q^* \mapsto -a(X_\alpha)\) is measurable (compare with the proof of Theorem 7.10 in [CM21] or Remark 3.4 in [KKS20]).
Then, we consider measurable sets $Q_m = M^{-1}(\frac{1}{m}, m)$ for $m \in \mathbb{N}$. It holds $\bigcup_{m \in \mathbb{N}_0} Q_m = Q^* \cap \Omega(A^I \cup B_{m}^I)$. From [CM21, Appendix A2] we see

$$h_{\alpha} \circ g(\alpha, 0) - h_{\alpha} \circ g(\alpha, r) \leq (N - 1) \frac{\cos^{-|K|/(N - 1)}(-a(X_{\alpha}))}{\sin^{-|K|/(N - 1)}(-a(X_{\alpha}))} \leq C(K, N, m)$$

for any bounded measurable set $Y \subset Q$. Fatou’s lemma was used in the first inequality together with the backward lower mean curvature bound. It follows that

$$H \int_{Y \cap Q_m \cap V_0} h_{\alpha} \circ g(\alpha, 0) dq(\alpha) = H \int_{Y \cap Q_m} dp_0(\alpha)$$

$$\leq \int_{Y \cap Q_m} \limsup_{t \uparrow 0} \frac{1}{t} \left( 1_{V_t}(\alpha) h_{\alpha} \circ g(\alpha, t) - 1_{V_0}(\alpha) h_{\alpha} \circ g(\alpha, 0) \right) dq(\alpha)$$

$$\leq \int_{Y \cap Q_m} \limsup_{t \uparrow 0} \frac{1}{t} \left( 1_{V_t} \cap V_0(\alpha) h_{\alpha} \circ g(\alpha, t) - 1_{V_0}(\alpha) h_{\alpha} \circ g(\alpha, 0) \right) dq(\alpha)$$

$$= \int_{Y \cap Q_m \cap V_0} \frac{d}{dt} \Big|_{t=0} h_{\alpha} \circ g(\alpha, t) dq(\alpha)$$

for any bounded measurable set $Y \subset Q$. Fatou’s lemma was used in the first inequality together with the backward lower mean curvature bound. It follows that

$$Hh_{\alpha} \circ g(\alpha, 0) \leq \frac{d}{dt} \Big|_{t=0} h_{\alpha} \circ g(\alpha, t)$$

for $q$-a.e. $\alpha \in V_0$.

The second claim follows similarly with Fatou’s Lemma (lim inf version).

**Theorem 3.8.** Let $X$ be an essentially non-branching CD$(K, N)$ space with $K \in \mathbb{R}$, $N \in (1, \infty)$, and let $\Omega \subset X$ be open. Let $u = d_{\Omega} |_{\Omega} = -d_{\Omega^c} |_{\Omega}$. Assume $p_0$ is a Radon measure and $h_{\alpha} \circ g(\alpha, 0) > 0$ for $q$-a.e. $\alpha \in Q$.

Then $\partial \Omega$ has backward mean curvature bounded from below by $H \in \mathbb{R}$ if and only if

$$\Delta_{\Omega} u \geq - (N - 1) \frac{s'_{N-1} \cdot \mu}{s_{N-1} \cdot \mu} (-u)$$

In particular, if $K \leq 0$ and $H = \pm \sqrt{|K|(N - 1)}$, then (11) becomes

$$\Delta_{\Omega} u \geq \mp (N - 1) \sqrt{\frac{|K|}{N - 1}} m |_{\Omega}.$$
\( \frac{d^+}{dt}u(0) \leq -d \). Let \( v : [0, \bar{b}] \to \mathbb{R} \) be the maximal non-negative solution of \( v'' + \kappa v = 0 \) with \( v(0) = 1 \) and \( v'(0) = -d \). That is, \( v = s_{\kappa,d} \) from (2). Then \( \bar{b} \geq b \) and \( \frac{d^+}{dt} \log u \leq (\log v)' \) on \( [0, b) \).

Let \( \{X_\alpha\}_{\alpha \in \Omega} \) be the decomposition of \( \mathcal{T}_u \) and \( \int m_\alpha \, dq(\alpha) \) be the disintegration of \( m \) given by Theorem 2.12 and Remark 3.2. Recall that \( m_\alpha = h_\alpha H^1 \) for \( \text{q.a.e. } \alpha \in \Omega \). We consider \( Q^\dagger \subset \Omega \) that has full \( \text{q-measure} \) as defined in Remark 2.16. For every \( \alpha \in Q^\dagger \) we have that \( m_\alpha = h_\alpha H^1 \), \( X_{\alpha,e} = \widehat{X}_\alpha \) and \( h_\alpha \) is continuous on \( [a(X_\alpha), 0] \) by Remark 2.15 and satisfies

\[
(13) \quad \frac{1}{N-1} h_\alpha'' \leq 0 \quad \text{on} \quad (a(X_\alpha), 0) \quad \forall \alpha \in Q^\dagger,
\]
in the distributional sense. As usual we write \( h_\alpha = h_\alpha \circ \gamma_\alpha \). We also have the properties of \( h_\alpha \) as discussed in Remark 2.15. By the definition of backward mean curvature bounded from below it holds \( h_\alpha(r) > 0 \) for \( \text{q.a.e. } \alpha \).

The function \( r \in [0, -a(X_\alpha)] \mapsto \tilde{h}_\alpha(r) := h_\alpha(-r) \) is also continuous and (13) still holds on \( (0, -a(X_\alpha)) \). Lemma 3.7 implies

\[
\left. \frac{d^+}{dr} \right|_{r=0} \tilde{h} \circ g(\alpha, r) \leq -H \tilde{h} \circ g(\alpha, 0).
\]

and hence with Lemma 3.9

\[
(\log \tilde{h}_\alpha)'(r) \leq \left( \log \left( s_{\frac{K}{N-1}, \frac{H}{N-1}}(r) \right)^{N-1} \right)'.
\]

By Theorem 3.1 we also have

\[
\Delta_{\Omega} u = (\log h_\alpha)' \, m |_{\Omega} + \int_{Q^\dagger} h_\alpha \delta_{a(X_\alpha) \cap \Omega \cap \Omega} \, dq(\alpha) \geq (\log h_\alpha)' \, m |_{\Omega} = -(\log \tilde{h}_\alpha)' \, m |_{\Omega}
\]

where we also used Lemma 4.1 from the next section. This yields the estimate for \( \Delta_{\Omega} u \).

For the estimate (12) we recall that

\[
s_{\frac{K}{N-1}, \frac{H}{N-1}} = -\left( \frac{K}{N-1} \right) \cdot \sin \frac{K}{N-1} - \left( \frac{H}{N-1} \right) \cdot \cos \frac{K}{N-1}.
\]

Using the value \( -H^2 = K(N-1) \) \( \Leftrightarrow \) \( H = \pm \sqrt{K(N-1)} \) \( \Leftrightarrow \) \( \frac{H}{N-1} = \pm \sqrt{\frac{|K|}{N-1}} \), it follows

\[
s_{\frac{|K|}{N-1}, \pm \sqrt{\frac{|K|}{N-1}}} = \frac{|K|}{N-1} \sin \frac{K}{N-1} \pm \sqrt{\frac{|K|}{N-1}} \cos \frac{K}{N-1}.
\]

This proves the claim.
"\[\leftarrow\]": The assumption and Theorem 3.1 imply that \(q\text{-a.e. } \alpha \in Q\) there exists a sequence \((r_n)_{n \in \mathbb{N}}\) in \((0, -a(X_\alpha))\) such that \(r_n \downarrow 0\) and

\[
\frac{d}{dr} \log h_\alpha \circ g(\alpha, -r_n) \geq -(N - 1) \frac{s'_{K,N-1,N-1}}{s_{K,N-1,N-1}}(r_n).
\]

Since \(h_\alpha\) is a semi-concave function for \(q\text{-a.e. } \alpha \in Q\) on \([a(X_\alpha), b(X_\alpha)]\), its right-derivative is right-continuous on \([0, -a(X_\alpha))\). In particular \(\frac{d}{dr} \log h_\alpha \circ g(\alpha, -r_n) \to \frac{d}{dr} \log h_\alpha(r)\) for \(r_n \downarrow 0\). On the other hand, the right hand side of (14) converges to \(H\) for \(r_n \downarrow 0\). One obtains

\[
\left. \frac{d^-}{dr} \right|_{r=0} h_\alpha \circ g(\alpha, r) \geq H h_\alpha(g(\alpha, 0)) \text{ for } q\text{-a.e. } \alpha \in Q.
\]

Hence, by Lemma 3.7 \(S\) has backward mean curvature bounded from below. 

The previous theorem suggests the following definition

**Definition 3.10** (Laplace mean curvature lower bounds). Let \((X, d, m)\) be an \(RCD(K, N)\) space for \(K \in \mathbb{R}, N \in (1, \infty)\), and let \(\Omega \subset X\) be open. We say that \(\partial \Omega\) has Laplace mean curvature bounded from below by \(H \in \mathbb{R}\) if

\[
\Delta_\Omega(-d_{\Omega^c}) \geq -(N - 1) \frac{s'_{K,N-1,N-1}}{s_{K,N-1,N-1}} \circ d_{\Omega^c} \text{ m } |\Omega.
\]

**Remark 3.11.** The direction "\[\leftarrow\]" in Theorem 3.8 holds any open \(\Omega \subset X\) with \(\Omega^c \neq \emptyset\) such that \(\partial \Omega\) has Laplace mean curvature bounded from below.

4. **Splitting**

**Lemma 4.1.** Let \((X, d, m)\) be essentially nonbranching and \(MCP(K, N)\) for \(K \in \mathbb{R}\) and \(N \in (1, \infty)\). Let \(\Omega \subset X\) be open and set \(u := -d_{\Omega^c}\). Then \((\Omega^c)^\circ \cap T_{u,e} = \emptyset\), \(T_{u,e} \supset \Omega\) and \(b_u \subset \partial \Omega\).

**Proof.** First, we observe that for every \(x \in \Omega\) there exists \(y \in \Omega^c\) such that \(-u(x) = d(x, y)\). Indeed, if \(y_n \in \Omega^c\) is a minimal sequence, we have \(y_n \in B_r(x)\) for \(r = -2u(x)\). Since \(B_r(x)\) is compact, there exists a converging subsequence and a limit point \(y \in \Omega^c\).

If \(x \in (\Omega^c)^\circ\), then \(u(x) = 0\) and \((x, y) \in R_u\) only if

\[
d(y, x) = -u(y).
\]

Hence, if \(x \neq y\), it follows that \(y \in \Omega\) and there exists a geodesic \(\gamma : [0, L] \to X\) between \(x\) and \(y\) such that \(\gamma(t) \in \Omega\) for all \(t \in (0, L)\). Consequently \(x \in \partial \Omega\). This contradicts \(x \in (\Omega^c)^\circ\). Therefore \(x = y\) for all \(y \in X\) such that \((x, y) \in R_u\). Hence \(x \notin T_{u,e}\) and \((\Omega^c)^\circ \cap T_{u,e} = \emptyset\).
Assume $x \in \Omega$. There exists $y \in \Omega^c$ and a geodesic $\gamma : [0, L] \to X$ such that $L(\gamma) > 0$ and
\[ d(x, y) = L(\gamma) = u(y) - u(x) = -u(x). \]
Therefore $x \in T_{u,e}$ and $\Omega \subset T_{u,e}$. This also implies $x \notin b$. Consequently $b \subset \partial \Omega$.

**Corollary 4.2.** One has $|\nabla u| = 1$ m.a.e. on $\Omega$.

**Proof.** Let $x \in \Omega$. As in the proof of the previous lemma there exist $y \in \partial \Omega$ and a geodesic $\gamma : [0, L] \to X$ such that $\gamma(0) = x$, $\gamma(L) = y$ and $d(x, y) = L(\gamma)$. Moreover
\[
1 \geq |\nabla u|(x) = \operatorname{Lip} u(x) = \limsup_{y \to x} \frac{|u(x) - u(z)|}{d(x, z)} \geq \lim_{s \to 0} \frac{|u(x) - u(\gamma(s))|}{d(x, \gamma(s))} = 1
\]
where we used the Sobolev-to-Lipschitz property in the first inequality. The first equality holds m.a.e. and is a fundamental result by Cheeger [Che99].

Let $\gamma : [0, \infty) \to \overline{\Omega}$ be the geodesic ray such that $\gamma(0) \in \partial \Omega$, $\gamma((0, \infty)) \subset \Omega$ and $d_{\Omega^c}(\gamma(t)) = t$. The Busemann function of $\gamma$ is defined as
\[
b_x(x) = \lim_{t \to \infty} d(x, \gamma(t)) - t, \quad x \in X.
\]
By triangle inequality the Busemann function is a well-defined and a 1-Lipschitz map from $X$ to $\mathbb{R}$ that satisfies $b \in D(\Delta, \Omega)$. This is proved in [Gig15] and [CM20a]. The statement of the following Lemma appears in [CM20a].

**Lemma 4.3.** $\mathbf{b}_b = \emptyset$.

**Proof.** We pick $x \in X$ and consider the geodesic $\gamma^t : [0, L(\gamma^t)] \to X$ between $x$ and $\gamma(t)$. Clearly $L(\gamma^t) \to \infty$ for $t \to \infty$. Hence, $L(\gamma^t) > s > 0$ for $s > 0$ given and for $t > 0$ sufficiently large. Since $\gamma^t$ is ageodesic we obtain that
\[
s = d(\gamma^t(s), x) = d(\gamma^t(s), \gamma(t)) - t - d(x, \gamma(t)) + t.
\]
Let $z$ be an accumulation point of $\gamma^t(s)$, $t > 0$. Then taking $t \to \infty$ yields $d(x, z) = s = b(z) - b(x)$. Since $s > 0$, it follows that $x \neq z$ and therefore $x \neq \mathbf{a}_b$.

**Lemma 4.4.** Consider $X$ and $\Omega$ as in the previous lemma and assume $X$ is noncompact and $\Omega^c$ is compact. There exists a geodesic ray $\gamma : [0, \infty) \to X$ with $\gamma(0) \in \partial \Omega$, $\gamma((0, \infty)) \subset \Omega$ and $d_X(\gamma(0), \gamma(t)) = d_{\Omega^c}(\gamma(t))$.

**Proof.** Since $X$ is noncompact and $\overline{\Omega^c}$ is compact, there exists a sequence $x_n \in X$ such that $d(x_n, \overline{\Omega^c}) = L_n \to \infty$. Let $\gamma_n : [0, L_n] \to X$ be the constant speed geodesic that connects $y_n \in \overline{\Omega^c}$ and $x_n$ such that $L(\gamma_n) = L_n$. It follows that $\operatorname{Im}(\gamma_n) \subset \Omega$. By compactness of $\Omega^c$ there is a subsequence $(\gamma_{n_i})_{i \in \mathbb{N}}$ such that $\operatorname{Im}(\gamma_{n_i})$ uniformly converges on $[0, L_{n_0}]$ for any $n_0 \in \mathbb{N}$ to a arclength parametrized geodesic ray $\gamma$ with $\gamma(0) \in \Omega^c$. Moreover
\[ \text{Im}(\gamma) \subset \Omega. \] Otherwise there is \( t_0 > 0 \) and a sequence \( (t_n)_{n \in \mathbb{N}} \) such that \( \Omega^c \ni \gamma(t_n) \rightarrow t(t_n) \in \Omega^c \). Since \( t_n = d(\gamma(t_n), \Omega^c) \), it follows \( t_n \rightarrow 0 \) and hence \( t_0 = 0 \) contradicting our assumption. Finally \( \gamma : [0, \infty) \rightarrow X \) also satisfies \( d_X(\gamma(0), \gamma(t)) = d_{\Omega^c}(\gamma(t)) \).

**Proposition 4.5.** Let \((X, d, m)\) be RCD\((K, N)\) and let \( \Omega \subset X \) be connected with backward mean curvature bounded from below by \(-\sqrt{(N-1)|K|}\). Let \( u = d_S|\Omega\) and \( \gamma : (0, \infty) \rightarrow \Omega \) a geodesic ray, such that \( \lim_{t \downarrow 0} \gamma(t) = x \in \partial \Omega \) and \( t = d_X(x, \gamma(t)) = d_{\Omega^c}(\gamma(t)) \). Let \( b \) be the associated Busemann function as before. Assume \( \Omega \) is connected. Then \( b|\Omega = -u \) and

\[
\Delta_\Omega(b|\Omega) = (N-1)\sqrt{\frac{|K|}{N-1}} m|\Omega \quad \text{&} \quad \Delta_\Omega u = -(N-1)\sqrt{\frac{|K|}{N-1}} m|\Omega.
\]

In particular \( a_b = b_u = 0 \).

**Proof.** The CD\((K, N)\) condition yields

\[
\Delta_\Omega(b|\Omega) \leq (N-1)\sqrt{\frac{|K|}{N-1}} m|\Omega.
\]

Hence with the Laplace estimate for \( u = d_S|\Omega = -d_{\Omega^c}|\Omega \) we obtain

\[
\Delta_\Omega(b - u) = \Delta_\Omega b - \Delta_\Omega u \leq (N-1)\sqrt{\frac{|K|}{N-1}} m|\Omega - (N-1)\sqrt{\frac{|K|}{N-1}} m|\Omega = 0.
\]

Pick \( y \in \partial \Omega \) such that \( d(x, y) = d_{\Omega^c}(x) \). Then

\[
d(x, \gamma(t)) - t + d_{\Omega^c}(x) \geq d(y, \gamma(t)) \geq d_{\Omega^c}(\gamma(t)) - t = 0
\]

and it follows \( b(x) - u(x) \geq 0 \) for \( x \in \Omega \) where we used \( d_{\Omega^c}(\gamma(t)) = \inf_{x \in \Omega^c} d_X(x, \gamma(t)) = t = d_X(\gamma(0), \gamma(t)) \) in the last equality. Moreover, equality holds if \( x = \gamma(s) \) for some \( s > 0 \).

By the maximum principle for RCD spaces [GR19, GM13] it follows that \( b = u \) on \( \Omega \) and

\[
\Delta_\Omega(b|\Omega) = \Delta_\Omega u
\]

which by linearity of the Laplacian yields the identity (16).

**Corollary 4.6.** Consider \((X, d, m), b, \Omega \) and \( u \) as before and the 1D localisation \((X_\gamma)_{\gamma \in Q} \) w.r.t. \( u = -b|\Omega^c \) on \( \Omega^c \) where \( \gamma : [0, \infty) \rightarrow \Omega \forall \gamma \in Q \), and the corresponding disintegration of \( m|\Omega^c \) into measures \((m_\gamma)_{\gamma \in Q} \). Then \( m(\Omega^c \setminus T_u^+) = 0 \) and

\[
m_\gamma = h_\gamma(0)s \frac{K}{N-1} \cdot \sqrt{|K|} (r)^{N-1} \mathcal{H}^1_{|[0, \infty)}(r).
\]

In particular

\[
\frac{m(B_R(\Omega^c) \cap \Omega)}{m(B_R(\Omega^c) \cap \Omega)} = \int_0^R s \frac{K}{N-1} \cdot \sqrt{|K|} (t)^{N-1} dt = \int_0^R s \frac{K}{N-1} \cdot \sqrt{|K|} (t)^{N-1} dt.
\]
Let $\Omega \subset X$ be connected, not empty and given by $\Omega = \bigcap_{\alpha=1}^{m} \Omega_\alpha$ for $\Omega_\alpha^c \cap \Omega_\beta^c = \emptyset$ and $d(\Omega_\alpha, \Omega_\beta) = D_{\alpha,\beta} > 0$ for $\alpha \neq \beta$ and $m \in \mathbb{N}$. We set $S_\alpha = \partial \Omega_\alpha$ and $u_\alpha = -d_{\partial \Omega_\alpha(\Omega)}$, $\alpha = 1, \ldots, m$.

**Lemma 4.7.** Let $(X, d, m)$ be an RCD($K,N$) space, and let $\Omega$, $\Omega_\alpha$, $\alpha = 1, \ldots, m$ as before. Assume $\partial \Omega_\alpha$, $\alpha \neq 2$, has backward mean curvature bounded from below by $\sqrt{|K|}$ and $\partial \Omega_2$ has backward mean curvature bounded from below by $-\sqrt{|K|}$. Moreover, assume that $\partial \Omega_2$ is compact. Then $m = 2$ and $-u_1 = d(\Omega_2, \Omega_1) + u_2$ and

$$
\Delta_{\Omega_1} u_1 = -(N-1) \sqrt{\frac{|K|}{N-1}} m|_{\Omega_1} \quad \& \quad \Delta_{\Omega_2} u_2 = (N-1) \sqrt{\frac{|K|}{N-1}} m |_{\Omega_2}.
$$

**Proof.** Consider $\Omega_1$ and $\Omega_2$. Since $\partial \Omega_2$ is compact, there are points $y_i \in \partial \Omega_i$, $i = 1, 2$, such that $d(\Omega_1, \Omega_2) = d(y_1, y_2) = D_{1,2}$. Moreover, the geodesic $\gamma : [0, D_{1,2}] \to \Omega$ from $y_2$ to $y_1$ satisfies

$$
u(\gamma(t)) + u_2(\gamma(t)) = -D_{1,2} \geq u_1(x) + u_2(x) \quad \forall x \in \Omega.
$$

By Theorem 3.8

$$
\Delta_{\Omega_1} u_1 \leq -(N-1) \sqrt{\frac{|K|}{N-1}} m |_{\Omega_1} \quad \& \quad \Delta_{\Omega_2} u_2 \leq (N-1) \sqrt{\frac{|K|}{N-1}} m |_{\Omega_2}.
$$

Hence $\Delta_{\Omega_1 \cap \Omega_2}(u_1 + u_2) \leq 0$. Since we have (17) by the maximum principle it follows

$$
\Delta_{\Omega_1 \cap \Omega_2}(u_1 + u_2) = 0 \quad \& \quad u_1 = -u_2 - D_{1,2} \text{ on } \Omega_1 \cap \Omega_2.
$$

Now assume that $l \geq 3$. Set $d(\Omega_1^c, \Omega_3^c) = D_{i,j}$. Similarly as before one deduces that

$$
u_3 + u_2 = -D_{3,2} \text{ on } \Omega_2 \cap \Omega_3.
$$

Together with the equation for $u_1$ and $u_2$ it follows

$$
u_1 - u_3 = D_{2,3} - D_{1,2} \text{ on } \Omega_1 \cap \Omega_2 \cap \Omega_3.
$$

Note that $\partial \Omega_1, \partial \Omega_3 \subset \Omega_1 \cap \Omega_2 \cap \Omega_3$. Assume w.l.o.g. that $D_{1,2} \geq D_{2,3}$. It holds

$$x \in \partial \Omega_1 \iff u_1(x) = 0 \iff u_3(x) = D_{1,2} - D_{2,3} \geq 0 \iff x \in \Omega_3^c.
$$

Hence $x \in \Omega_3^c \cap \Omega_1^c$. This is a contradiction. □

**Corollary 4.8.**

$$
m(B_R(\Omega_1^c \cap \Omega)) = \frac{f^R_{\frac{K}{N-1}} s_{\frac{K}{N-1}}(t) \sqrt{\frac{|K|}{N-1}}} {f^R_{\frac{K}{N-1}} s_{\frac{K}{N-1}}(t) \sqrt{\frac{|K|}{N-1}}} dt.
$$
4.2. **Isometric splitting.** Recall that \( f \in W^{1,2}(\Omega^c) \) if \( \phi \cdot f \in W^{1,2}(X) \) for every Lipschitz function with support in \( \Omega \). Moreover, we say \( u \in H^{2,2}_{\text{loc}}(\Omega) \) if \( \psi \cdot u \in H^{2,2}(X) \) for every \( \psi \in \mathbb{D}_\infty \) with support in \( \Omega \). Thanks to locality of \( \text{Hess} f \) for \( f \in H^{2,2}(X) \) the Hessian \( \text{Hess}(u) \) for \( u \in H^{2,2}_{\text{loc}}(\Omega) \) is well-defined.

The following theorem is Corollary 4.16 in [KKL23].

**Theorem 4.9.** Let \( X \) be RCD\((0,N)\) and \( \Omega \subset X \) be open. Let \( u : \Omega \to \mathbb{R} \) such that \( |\nabla u| = 1 \) and \( \Delta_{\Omega} u = 0 \). Then \( u \in H^{2,2}_{\text{loc}}(\Omega) \) and

\[
\text{Hess}(u)(\nabla f, \nabla f) = 0 \quad \text{m-a.e. on } \Omega \text{ and } f \in \mathbb{D}_\infty.
\]

**Remark 4.10.** Given an open subset \( \Omega \subset X \) of an RCD space \( X \) we define \((\bar{\Omega}, \bar{d}_\Omega)\) as the completion of \( \Omega \) equipped with the intrinsic distance induced by \( d_X \). We can identify \( \Omega \) as a subset of \( \bar{\Omega} \), but the topology of \((\bar{\Omega}, \bar{d}_\Omega)\) can differ from the topology of \( \Omega \subset X \). An easy example for this scenario is \( X = \mathbb{S}^1 \) and \( \Omega = \mathbb{S}^1 \setminus \{p\} \) with \( p \in \mathbb{S}^1 \). The completion of \( \Omega \) equipped with the intrinsic distance is an interval. But \( \overline{\Omega} = \mathbb{S}^1 \).

Setting \( m|_\alpha = m_\Omega \) the triple \((\bar{\Omega}, \bar{d}_\Omega, m_\Omega)\) is a metric measure space.

A corollary of Theorem 4.9 is the following splitting result.

**Theorem 4.11.** Let \((X, d, m)\), \( \Omega \subset X \) and \( u \) be as in previous theorem. Assume that \( \Omega = u^{-1}((0, D)) \) for \( D > 0 \). Then, there exists an RCD\((0,N-1)\) space \((Y, d_Y, m_Y)\) such that \((\bar{\Omega}, \bar{d}_\Omega, m_\Omega)\) is isomorphic to \([0, D] \otimes Y\).

**Proof.** The proof of the corollary is exactly the content of section 5 and section 6 in [KKL23] that result in the proof of Theorem 6.10 in [KKL23] that corresponds to our statement.

**Remark 4.12.** For the proof of the main theorem in [KKL23] the authors show that the induced intrinsic metric of \( \Omega = f^{-1}((- \min f, \max f)) \) splits off an interval where \( f = \cos^{-1} ou \) with an eigenfunction \( u \) on a compact RCD\((0,N)\) space \( X \).

As consequence of the previous theorem one obtains the following isomorphic splitting statement that generalizes a corresponding theorem in smooth context by Kasue [Kas83] and Croke-Kleiner [CK92].

**Theorem 4.13.** Let \((X, d, m)\) be an RCD\((0,N)\) space, and let \( \Omega, \Omega_\alpha, \alpha = 1, \ldots, m \) as before. Assume \( \partial \Omega_\alpha \) has backward mean curvature bounded from below by \( 0 \) for every \( \alpha = 1, \ldots, m \). Moreover, assume that \( \partial \Omega_2 \) is compact. Then, there exists an RCD\((0,N-1)\) space \((Y, d_Y, m_Y)\) such that \((\bar{\Omega}, \bar{d}_\Omega, m_\Omega)\) is isomorphic to \([0, D_{1,2}] \otimes Y\).

**Proof of Corollary 1.2.** Since mean curvature bounded from below by \( \delta > 0 \) implies nonnegative mean curvature, we can apply Theorem 1.1. It follows that \( \Delta_{\Omega}(-d_{\Omega^c}) = 0 \). But

\[
\Delta_{\Omega}(-d_{\Omega^c}) \geq (N-1) \frac{\delta}{1 - \frac{\delta}{N-1} d_{\Omega^c}} > 0 \text{ on } B_1(\Omega^c) \cap \Omega
\]

by the assumed mean curvature bound. This is a contradiction. \( \square \)
Proof of Corollary 1.3. Recall that for $K \leq K'$ the condition $\text{RCD}(K', N)$ implies $\text{RCD}(K, N)$. Assume $\Omega_1^0$ and $\Omega_2^0$ are disjoint and set $\Omega = \Omega_1 \cap \Omega_2$. Then by Theorem 1.1 $(\Omega, \bar{d}_{\Omega}, m_{\Omega})$ is isomorphic to $[0, D] \times Y$ for some $\text{RCD}(0, N - 1)$ space $Y$. But the product structure contradicts the assumption that $X$ was $\text{RCD}(\delta, N)$ for $\delta > 0$. \hfill\Box

Similarly one can show the following splitting theorem.

**Theorem 4.14.** Let $(X, d, m)$ be $\text{RCD}(0, N)$ and let $\Omega \subset X$ have backward mean curvature bounded from below by 0. Assume $\Omega^0$ is connected and there exists a geodesic ray $\gamma : (0, \infty) \to \Omega$ with $\lim_{t \to 0} \gamma(t) = x_0 \in \partial\Omega$ and $d_X(\gamma(0), \gamma(t)) = d_{\Omega^0}(\gamma(t))$. Then, there exists an $\text{RCD}(0, N - 1)$ space $(Y, d_Y, m_Y)$ such that $(\Omega, d_{\Omega}, m_{\Omega})$ is isomorphic to $[0, \infty) \otimes Y$.

Again, the proof is verbatim the same as for [KKL23, Theorem 610]. Noncompactness only requires minor modifications since the arguments are all of local nature.

5. Almost Rigidity

5.1. Gromov-Hausdorff convergence and the uniform distance. In this and the following sections we will study the stability and almost rigidity properties of lower mean curvature bounds. For simplicity, we will assume that all the involved RCD spaces are compact. An extension of the following concepts for non-compact RCD spaces and pointed Gromov-Hausdorff convergence is omitted but straightforward.

Compact metric spaces $(X_i, d_i)$ converge in Gromov-Hausdorff sense to a compact metric space $(X, d)$ if there exist a compact metric space $(Z, d_Z)$ and distance preserving maps $\iota_i, \iota : X_i \to Z$ such that $\iota_i(X_i)$ converges in Hausdorff sense to $\iota(X)$ in $Z$. The Gromov-Hausdorff distance $d_{GH}(X_i, X)$ is defined as the infimum of Hausdorff distances between $\iota_i(X_i)$ and $\iota(X)$ w.r.t. to all distance preserving maps $\iota_i, \iota$ and metric spaces $Z$. Equivalently, $(X_i, d_i) \overset{GH}{\to} (X, d)$ if there exists a sequence of $\epsilon_i$-isometries $\psi_i : X_i \to X$ such that $\epsilon_i \to 0$. Existence of an $\epsilon$-isometry $\psi : X \to Y$ between compact metric spaces $X$ and $Y$ yields that the Gromov-Hausdorff distance satisfies $d_{GH}(X, Y) \leq 2\epsilon$.

Given a sequence of $\delta_i$-isometries $\psi_i : X_i \to X$ with $\delta_i \to 0$ a sequence of functions $f_i : X_i \to \mathbb{R}^m$ converges uniformly to a function $f : X \to \mathbb{R}^m$ if for every $\epsilon > 0$ there exists $i_\epsilon \in \mathbb{N}$ such that $\|f_i(z_i) - f(z)\|_{\mathbb{R}^m} \leq \epsilon$ for points $z_i \in X_i$ and $z \in X$ with $d_Z(\psi_i(z_i), z) \leq \delta_i$ and $i \geq i_\epsilon$.

The next proposition is Gromov’s Arzela-Ascoli theorem for functions on a Gromov-Hausdorff converging sequence (for instance see [Sor18]).

**Proposition 5.1.** Let $(X_i, d_i)$ be compact metric spaces that converge in $GH$ sense to a compact metric space $(X, d)$, and let $f_i : X_i \to \mathbb{R}^m$ be functions that are $L$-Lipschitz and uniformly bounded. Then there exists a subsequence of $f_i$ that converges uniformly to an $L$-Lipschitz function $f : X \to \mathbb{R}^m$. 

These considerations motivate the following definitions.

Let $X$ and $Y$ be compact metric spaces such that $d_{GH}(X,Y) < r$. Then it is easy to see that there exist $2r$-isometries $\psi : X \to Y$ and $\phi : Y \to X$.

**Definition 5.2 (Uniform distance).** For functions $f : X \to \mathbb{R}^m$ and $g : Y \to \mathbb{R}^m$ we define

$$\sup \{ \| f(x) - g(y) \|_{\mathbb{R}^m} : x \in X, y \in Y \text{ s.t. } d_X(\psi(x), y) \leq 2r \} =: S_{\psi}(f,g).$$

The uniform distance between $f$ and $g$ is then defined via

$$\inf_{(\psi, \phi)} \max \{ S_{\psi}(f,g), S_{\phi}(f,g) \} =: d^*(f, g)$$

where the infimum is taken w.r.t. any pair $(\psi, \phi)$ such that $\psi : X \to Y$ and $\phi : Y \to X$ are $2r$-isometries for $r > d_{GH}(X,Y)$.

By definition we have $d^*(f,g) = d^*(g,f)$, and $d_{GH}(X,Y) + d^*(f,g) = 0$ if and only if $X \simeq Y$ and $f = g$ pointwise as functions on $X \simeq Y$. Moreover, for compact metric spaces $X$, $Y$ and $Z$, and continuous functions $f : X \to \mathbb{R}^m$, $g : Y \to \mathbb{R}^m$ and $h : Z \to \mathbb{R}^m$ we have

$$d^*(f,h) \leq d^*(f,g) + d^*(g,h).$$

**Lemma 5.3.** Consider compact metric spaces $(X_i, d_i)$ for $i \in \mathbb{N}$ and $(X,d)$ such that $d_{GH}(X_i,X) < r_i \to 0$. Then $f_i : X_i \to \mathbb{R}^m$ converges uniformly to $f : X \to \mathbb{R}$ if and only if $d^*(f_i,f) \to 0$.

**Proof.** Let $\epsilon > 0$, then we can pick $i_\epsilon > 0$ such that $d^*(f_i,f) \leq \epsilon$ for $i \geq i_\epsilon$. In particular, there exists a sequence of $2r_i$-isometries $\psi_i : X_i \to X$ such that

$$\| f_i(x_i) - f(x) \|_{\mathbb{R}^m} \leq \epsilon \quad \forall x_i \in X_i, x \in X \text{ with } d(\psi_i(x_i), x) \leq 2r_i \quad \forall i \geq i_\epsilon.$$ 

Hence, $f_i$ converges uniformly to $f$. On the other hand, the definition of uniform convergence implies $d^*(f_i,f) \to 0$. \qed

Let $i = 1, 2$. Given families of open sets $\Omega_{i,\alpha} \subset X_i$, $\alpha = 1, \ldots, m$ such that $\Omega_{i,\alpha}^c$ is connected for all $\alpha$ and $d_{X_i}(\Omega_{i,\alpha}^c, \Omega_{i,\beta}^c) = \inf_{x \in \Omega_{i,\alpha}^c, y \in \Omega_{i,\beta}^c} d(x,y) > 0$ for $\alpha \neq \beta$, we consider $\Omega_i = \bigcap_{\ell=1}^m \Omega_{i,\ell}$ and $f_i = (d_{\Omega_{i,1}}^c, \ldots, d_{\Omega_{i,m}}^c) : X_i \to \mathbb{R}^m$. Then we define

$$D(\Omega_1, \Omega_2) := d^*(f_1, f_2).$$

A sequence of compact metric measure spaces $(X_i, d_i, m_i)$ converges in measured Gromov-Hausdorff sense to a compact metric measure spaces $(X,d, m)$ if $(X_i, d_i) \xrightarrow{GH} (X,d)$ and $m_i$ converges to $m$ in duality with $C_b(Z)$ where $(Z, d_Z)$ is a metric space where GH convergence is realized. A distance that metrizes measured GH convergence is given for instance by Sturm’s transportation distance $D$ [Stu06a]. Actually $D$ is a distance on the set of isomorphism classes $[X]$ of metric measure spaces $X$ with finite measure $m_X$.

But after normalisation of $m_X$, that is replacing $m_X$ with $m_X(X)^{-1} m_X = \bar{m}_X$, we can see $D$ is a distance on the family of normalized metric measure
Lemma 5.4. Let \( f_i \in L^2(m) \) converges in \( L^2 \)-weak sense to \( f \in L^2(m) \) if \( f_i \to f \) m in duality with \( C_b(Z) \) and \( \sup_{i \in \mathbb{N}} \| f_i \|_{L^2(m)} < \infty \). If
\[
\lim_{i \to \infty} \| f_i \|_{L^2(m)} = \| f \|_{L^2(m)}
\]
holds, then one says the sequence \( f_i \) converges \( L^2 \)-strongly to \( f \).

Lemma 5.5. Stability and almost rigidity results.

5.2. Stability and almost rigidity results.

Lemma 5.5. Let \( K \in \mathbb{R} \) and \( N \in (1, \infty) \). Let \( (X_i, d_i, m_i)_{i \in \mathbb{N}} \), be a sequence of \( RCD(K, N) \) spaces that converges in measured Gromov-Hausdorff sense to a compact \( RCD(K, N) \) space \( (X, d, m) \), and let \( \Omega_i \subset X_i \) be open sets. Then, \(-d_{\Omega_i} : X_i \to \mathbb{R} \) subconverges in Arzela-Ascoli sense to a 1-Lipschitz function \( u : X \to \mathbb{R} \) such that \( |\nabla u| = 1 \) m-a.e. on \( \Omega = u^{-1}((-\infty, 0)) \) and \( \Omega^c = u^{-1}(\{0\}) \neq \emptyset \). Moreover \( u = -d_{\Omega^c} \) if \( \Omega \neq \emptyset \). Otherwise \( u \equiv 0 \).

Proof. The existence of a 1-Lipschitz function \( u : X \to \mathbb{R} \) that arises as the limit of a subsequence of \( d_{\Omega_i} \) is guaranteed by Gromov’s Arzela-Ascoli theorem.

We embed \( (X_i, d_i) \) and \( (X, d) \) into a metric space \((Z, d_Z)\) where measured Gromov-Hausdorff convergence is realized. Assume \( \Omega = u^{-1}((-\infty, 0)) \neq \emptyset \). Then we pick \( x \in \Omega \) and a sequence of points \( x_i \in X_i \) such that \( x_i \in \Omega_i \) and \( x_i \to x \) in \( Z \). There exists a sequence of geodesics \( \gamma_i : [-L_i, 0] \to \Omega_i \) that are arclength parametrized such that \( \gamma_i(-L_i) = x_i \), \( u(\gamma_i(0)) = 0 \) and \( L_i = u(x_i) \). After extracting another subsequence \( (\gamma_i)_{i \in \mathbb{N}} \) converges uniformly to a geodesic \( \gamma : [-L, 0] \to X \) in \( Z \) such that \( \gamma(-L) = x \), \( L = u(x) > 0 \) and \( \gamma((0, L)) \subset \Omega \). It holds
\[
u(\gamma(-L)) - u(\gamma(0)) = d_X(x, \gamma(0)).
\]
Hence \( \gamma \) is a transport geodesic of \( u \) and \( x \) is contained in the transport set \( T_u \). Hence \( \Omega \subset T_u \) and \( \text{Lip } u = |\nabla u| = 1 \) on \( T_u \).

If we assume there exists \( y \in \Omega^c \) such that \( d(x, y) < L \), then there exist \( y_i \in \Omega_i^c \) such that \( y_i \to y \) and \( d_i(x_i, y_i) \to d(x, y) \). This would contradict the choice of \( \gamma_i \) before. Hence \( -u(x) = L = d_{\Omega^c}(x) \). Hence \( -u = d_{\Omega^c} \). □
**Definition 5.6** (uniform domain). Let $X$ be a geodesic metric space. An open subset $\Omega \subset X$ is called $(c,C)$-uniform if for any two points $x, y \in \Omega$ there exists a rectifiable curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$ that satisfies

1. $d_{\Omega}(\gamma(t)) \geq c \min\{d_X(x, \gamma(t)), d_X(\gamma(t), y)\}$ \quad \forall t \in [0,1],
2. $\text{length}(\gamma) \leq C d_X(x, y)$.

In particular, a $(c,C)$-uniform domain is connected.

**Lemma 5.7.** Consider $X_i$, $X$, $\Omega_i$ and $\Omega$ as in Lemma 5.5. If $\Omega_i$ is $(c,C)$-uniform for all $i \in \mathbb{N}$, then $\Omega$ is $(c,C)$-uniform. If $\Omega_i \neq \emptyset$ for all $i$, then $\Omega \neq \emptyset$.

**Proof.** Pick two points $x, y \in \Omega$ and $x_i, y_i \in \Omega_i$ such that $x_i \to x$ and $y_i \to x$ after embedding $X_i, X$ into a common metric space $Z$.

Since $\Omega_i$ is $(c,C)$-uniform, there exists a sequence of rectifiable curves $\gamma_i : [0,1] \to \Omega_i$ that connects $x_i$ and $y_i$ and such that $\text{length}(\gamma_i) \leq C d(x_i, y_i)$. We apply the Arzela-Ascoli theorem to extract a subsequence that converges uniformly in $Z$ to a curve $\gamma : [0,1] \to X$. Lower semi-continuity of the length implies that $\gamma$ is rectifiable and

$$\text{length}(\gamma) \leq C d(x, y)$$

Moreover, uniform convergence of $d_{\Omega_i}$ and convergence of $\gamma_i$ implies

$$d_{\Omega_i}(\gamma(t)) \geq c \min\{d_X(x, \gamma(t)), d_X(\gamma(t), y)\}.$$  

Hence $\Omega$ is $(c,C)$-uniform.

The second claim is clear. \qed

**Theorem 5.8.** Consider $X_i$, $X$, $\Omega_i$, $\Omega$ as in Lemma 5.5 such that $\Omega \neq \emptyset$. We set $u_i := d_{\Omega_i} |_{\Omega_i}$ and $u := d_{\Omega} |_{\Omega}$. Assume $u_i$ satisfies

$$\Delta_{\Omega_i} u_i \leq (N-1) \frac{s'_{\lambda_1, \lambda_{N-1}}(u_i)}{s_{\lambda_1, \lambda_{N-1}}(u_i)} m |_{\Omega_i},$$

where $H_i \in \mathbb{R}$ with $H_i \to H$. Then $u$ satisfies

$$\Delta_{\Omega} u \leq (N-1) \frac{s'_{\lambda_1, \lambda_{N-1}}(u)}{s_{\lambda_1, \lambda_{N-1}}(u)} m |_{\Omega}.$$  

**Proof.** By measured Gromov-Hausdorff convergence there exists a compact metric space $(Z, d_Z)$, distance preserving maps $\iota_i, \iota : X_i, X \to Z$ and couplings $\pi_i$ between $m$ and $m$ such that $d_Z(x, y) \leq \delta$ for $\pi_i$-almost every $(x, z) \in X_i \times X$ if $i \geq i_\delta$. Let $\phi \in C_b(Z)$ and define $g_i = \phi \cdot d_{\Omega_i}$. Then $g_i$ converges uniformly to $g = \phi \cdot d_{\Omega}$, and we can choose $i_\delta \in \mathbb{N}$ such that
\[ |g_i(x_i) - g(x)| < \epsilon \] whenever \( |x_i - x| \leq \delta \) and \( i \geq i_\delta \). Indeed, we observe

\[
\begin{align*}
|\phi(x) \cdot d_{\Omega_i}(x) - \phi(y) \cdot d_{\Omega_i}(y)| \\
\leq |\phi(x)||d_{\Omega_i}(x) - d_{\Omega_i}(y)| + |\phi(x) - \phi(y)||d_{\Omega_i}(y) \\
\leq \sup_{z \in \Omega} |\phi(z)| \epsilon + \epsilon \cdot \text{diam} x_i
\end{align*}
\]

whenever \( i \geq i_\delta \) is sufficiently large and \( d_{\Omega_i}(x, y) \leq \delta \).

It follows that \( d_i \rightarrow \infty \) for \( i \rightarrow \infty \) and \( \delta \rightarrow 0 \) if \( \epsilon > 0 \). Indeed, we observe

\[
\int g_i d m_i - \int g d m = \int |g_i - g| d \pi_i \leq \epsilon \text{ for } i \geq i_\delta.
\]

It follows that \( d_{\Omega_i} m_i \rightarrow d_{\Omega} m \) in duality with \( C_b(Z) \). Moreover

\[
\left| \int d_{\Omega_i}^2 d m_i - \int d_{\Omega}^2 d m \right| = \int \left| 2(d_{\Omega_i} - d_{\Omega}) + (d_{\Omega_i} - d_{\Omega})^2 \right| d \pi_i \leq 2\epsilon + \epsilon^2
\]

if \( i \geq j_\delta \) for \( j_\delta \in \mathbb{N} \) sufficiently large. Hence \( d_{\Omega_i} \) converges \( L^2 \)-strongly to \( d_{\Omega} \).

Let \( \varphi^k \in C_b(\mathbb{R}) \) be sequence of continuous functions such that \( \varphi^k \uparrow 1_{[\eta, \infty)} \) pointwise for \( \eta > 0 \). One can check that \( h_i^k = \varphi^k \circ d_{\Omega_i} \in C_b(X) \) converges uniformly to \( h^k = \varphi^k \circ d_{\Omega} \), and in particular there exists \( i_\epsilon \in \mathbb{N} \) such that

\[
\int h_i^k d m_i \leq \int h^k d m + \epsilon \leq \int 1_{[\eta, \infty)} \circ d_{\Omega} d m + \epsilon = m(d_{\Omega_i}^{-1}([\eta, \infty))) + \epsilon
\]

for \( i \geq i_\epsilon \). For \( k \rightarrow \infty \) we obtain \( h_i^k \rightarrow 1_{[\eta, \infty)} \circ d_{\Omega} = d_{\Omega_i}^{-1}([\eta, \infty)) \) and

\[
m_i(d_{\Omega_i}^{-1}([\eta, \infty))) \leq m(d_{\Omega_i}^{-1}([\eta, \infty))) + \epsilon.
\]

Finally, we take \( \eta \downarrow 0 \), \( i \rightarrow \infty \) and \( \epsilon \downarrow 0 \) in this order. It follows

\[
\limsup_{i \rightarrow \infty} m_i(\Omega_i) \leq m(\Omega).
\]

Corollary 4.2 implies

\[
\limsup_{i \rightarrow \infty} \int |\nabla d_{\Omega_i}|^2 d m_i = \limsup_{i \rightarrow \infty} m_i(\Omega_i) \leq m(\Omega) = \int |\nabla d_{\Omega}|^2 d m.
\]

Hence \( d_{\Omega_i} \) converges \( H^{1,2} \)-strongly to \( d_{\Omega} \).

Let \( x \in \Omega \) be arbitrary. Then, there exists \( \delta > 0 \) such that \( B_\delta(x) \subset \Omega \) and there exists a sequence \( x_i \in \Omega_i \) such that \( x_i \rightarrow x \), \( B_\delta(x_i) \subset \Omega_i \) and \( B_\delta(x_i) \) converges in Gromov-Hausdorff sense to \( B_\delta(x) \).

We recall the following lemma [AH18, Lemma 2.10].

**Lemma 5.9.** For any \( \phi \in \text{Lip}(X) \) with \( \text{supp} \phi \subset B_\delta(x) \) there exists a sequence \( \phi_i \in \text{Lip}(X_i) \) with \( \text{supp} \phi_i \subset B_\delta(x_i) \) such that \( \text{sup} \text{Lip} \phi_i < \infty \) and \( \phi_i \) converges \( H^{1,2} \)-strongly to \( \phi \).
Hence, given $\phi$ and $\phi_i$ as in the previous lemma $H^{1,2}$-strong convergence of $d_{\Omega_i}$ to $d_{\Omega}$ together with (4) yields
\[
\int \langle \nabla d_{S_i}, \nabla \phi_i \rangle d m_{X_i} \to \int \langle \nabla d_S, \nabla \phi \rangle d m_X.
\]
Set $f_{K,N,H} = \frac{s_{K,N,H}}{\alpha_{K,N,H}}$. Since $H_i \to H$, it follows $f_{K,N,H_i} \to f_{K,N,H}$ locally uniformly. Hence, the composition $f_{K,N,H} \circ d_{\Omega_i}$ converges uniformly to $f_{K,N,H} \circ d_{\Omega'}$, and hence $L^2$-strongly. Therefore
\[
\int \phi_i \frac{s'_{K,N,H}}{s_{K,N,H}} (u_i) d m_{X_i} \to \int \phi \frac{s'_{K,N,H}}{s_{K,N,H}} (u) d m_X
\]
By locality of the distributional Laplacian, this implies the desired estimate. \hfill \Box

Remark. As the referee pointed out to the author that a similar strategy as in the previous proof is applied in [AHT18] where it is proved that for sequences of uniformly continuous functions, $L^2$-convergence and uniform convergence are equivalent.

Theorem 5.8. Lemma 5.5, compactness of RCD spaces w.r.t. $\mathcal{D}$, the Arzela-Ascoli theorem and the definition of the uniform distance $\mathcal{D}$ imply the following compactness theorem.

Corollary 5.10. Given $K,H \in \mathbb{R}, N \in [1, \infty)$ and $D > 0$ the family $\mathcal{M}(K,N,D,H)$ of pairs $(X,\Omega)$ for a compact, normalized RCD$(K,N)$ space $X$ with $\diam_X \leq D$ and an open subset $\Omega \subset X$ with $\partial \Omega$ having Laplace mean curvature bounded from below by $H$ is compact w.r.t. $\mathcal{D}$ where $(\mathcal{D} + \mathcal{D})((X,\Omega), (\tilde{X},\tilde{\Omega})) = \mathcal{D}(X,\tilde{X}) + \mathcal{D}(\Omega,\tilde{\Omega})$.

Theorem 5.11. Let $\Gamma, D, c, C > 0$, $N > 1$ and $m \in \mathbb{N}\backslash\{1\}$. For every $\epsilon > 0$ there exists $\delta > 0$ such that the following holds.

Let $X$ be a normalized RCD$(-\delta,N)$ space with $\diam_X \leq D$ and let $\Omega_\alpha \subset X$ be open subsets $\Omega_\alpha \subset X$, $\alpha = 1,\ldots,m$, such that $\Omega_\alpha$ is $(c,C)$-uniform, $\Omega_\alpha$ has Laplace mean curvature bounded from below by $-\delta$ and $d(\Omega_\alpha, \Omega_\beta) \geq \Gamma > 0$ for $\alpha \neq \beta$. Set $\Omega = \bigcap_{\alpha=1}^m \Omega_\alpha$.

Then, $m = 2$ and there exist $D > 0$, an RCD$(0,N)$ space $Z$, an RCD$(0,N-1)$ space $Y$ and an open subset $\Omega' \subset Z$ such that $(\Omega', d_{\Omega'}, m_Z |_{\Omega'}) \simeq Y \otimes [0,D]$ and
\[
\mathcal{D}(X,Z) \leq \epsilon \quad \text{and} \quad \mathcal{D}(\Omega,\Omega') \leq \epsilon.
\]

Proof. We assume, there exists a sequence of RCD$(-\frac{1}{m},N)$ spaces $X_i$ with subsets $\Omega_{\alpha,i}$ that satisfy the assumptions in the theorem but fail the second claim in (19) for $\epsilon > 0$. 
By stability and compactness of the class of RCD spaces w.r.t. measured GH convergence there exists an RCD(0,N) space Z such that a sub-
sequence of $X_i$, that by abuse of notation we also call $X_i$, converges in measured Gromov-Hausdorff sense to Z. Hence, there exists $i_\epsilon \in \mathbb{N}$ such that $\mathcal{D}(X, Z) < \epsilon$ for $i \geq i_\epsilon$. After extracting another subsequence $d_{\Omega_\alpha}^{\epsilon}$, $\Omega_\alpha$ converges uniformly to $d_{\Omega_\alpha}$ for open subsets $\Omega_\alpha \subset Z$, $\alpha = 1, \ldots, k$ where $k \leq m$. By Theorem 5.8 $d_{\Omega_\alpha}^{\epsilon} \big|_{\Omega_\alpha} =: u^{\alpha}$ satisfies

$$\Delta_{\Omega} u^{\alpha} \geq 0$$

i.e. $\Omega_\alpha$ has Laplace mean curvature bounded from below.

By Lemma 5.7 $\Omega_\alpha$ is a $(c, C)$-uniform domain and in particular connected. Hence $\Omega = \bigcap_\alpha \Omega_\alpha$ is connected. Moreover $d(\Omega_\alpha, \Omega_\beta) \geq \Gamma$ for all $\alpha \neq \beta$. As in Lemma 4.7 we derive that $k = 2$ and that $u^{\alpha}$ is harmonic on $\Omega$. Hence, $(\Omega, d_\Omega, m_\Omega)$ is isomorphic to $Y \otimes [0, D]$ for an RCD$(0, N - 1)$ space $Y$.

On the other hand, uniform convergence of $d_{\Omega_\alpha}^{\epsilon} \to d_{\Omega_\alpha}$ for all $\alpha = 1, \ldots, m$ implies $m = 2$ and

$$\mathcal{D}(\Omega_i, \Omega) \leq \epsilon$$

for $i$ sufficiently large by definition of $\mathcal{D}$. This is a contradiction. \hfill \Box

Very similarly one can prove the following result which is an almost rigidity statement that corresponds to the main rigidity theorem in [BKMW20].

**Theorem 5.12.** Let $D, c, C > 0$ and $N > 1$. For every $\epsilon > 0$ there exists $\delta > 0$ such that the following holds.

Let $X$ be a normalized RCD$(-\delta, N)$ space with $\text{diam}_X \leq D$ and let $\Omega$ be open and $(c, C)$-uniform with Laplace mean curvature bounded from below by $N - 1 - \delta$. Assume there exists $x \in \Omega$ such that $d_{\Omega'}(x) \geq 1 - \delta$.

Then, there exists an RCD$(0, N)$ space $Z$, an RCD$(N - 2, N - 1)$ space $Y$ and an open subset $\Omega' \subset Z$ such that $(\Omega', d_{\Omega'}, m_{\Omega'} |_{\Omega'})$ is isomorphic to $Y \times_{N-1} [0, 1]$ and

$$\mathcal{D}(X, Z) \leq \epsilon \quad \text{and} \quad \mathcal{D}(\Omega, \Omega') \leq \epsilon.$$

**Appendix A. Stability of constant mean curvature sets**

The definition of Laplace mean curvature bounds motivates us to say that the boundary $\partial \Omega$ of an open subset $\Omega$ in a compact RCD space $X$ is a generalized CMC hypersurface with curvature $H \in \mathbb{R}$ (a generalized minimal hypersurface if $K = 0$) if $m(\partial \Omega) = 0$ and the signed distance function $d_{\partial \Omega} := d_{\Omega} - d_{\Omega^c}$ satisfies

$$\Delta_{\Omega} (d_{\partial \Omega}) \geq -(N - 1) \frac{\mu}{s_{N-1}} \frac{(- d_{\partial \Omega})}{\mu (- d_{\partial \Omega})} m |_{\Omega} \quad \text{on} \ \Omega$$

(20)
and

\[
\Delta_{X\setminus\overline{\Omega}}(-d_{\partial\Omega}) \geq -(N-1)\frac{s_k^\prime}{N-1} \frac{-\mu}{s_{\frac{N-2}{N-1}}(d_{\partial\Omega})} m|_{X\setminus\overline{\Omega}} \text{ on } X\setminus\overline{\Omega}.
\]

By symmetry in \(\Omega\) and \((\Omega^c)^o\), \(\partial\Omega\) has constant mean curvature \(H\) if and only if \(\partial\Omega^c\) has constant mean curvature \(-H\).

When \(\Omega\) is a subset with smooth boundary in a Riemannian manifold with Ricci curvature bounded from below by \(K\) \((20)\) and \((21)\) are equivalent to \(\partial\Omega\) being a CMC hypersurface, as recently discussed in \([MS21]\) for \(K = 0\). In nonsmooth setting one can find examples that satisfy these estimates for every \(H \in [-1,1]\) \(\text{(Example A.5)}\). Therefore it is suggested by the authors in \([APPS22]\) to say the boundary of \(\Omega\) has a \textit{mean curvature barrier} \(H\). We will adapt this in the following.

For stability of this notion we encounter the following problem: The uniform limit of a signed distance functions \(d_{\partial\Omega}\) on \(\text{RCD}(K,N)\) spaces \(X_i\) may not be a signed distance function of a set \(\Omega\) with \(m(\partial\Omega) = 0\). But assuming a uniform inner/outer ball condition for \(\Omega\) \(\text{(Definition A.2)}\) one can prove the following lemma.

**Lemma A.1.** Let \(K \in \mathbb{R},\ N \in (1,\infty)\) and \(\delta > 0\). Let \((X_i,d_i,m_i)\)\(_{i \in \mathbb{N}}\), be a sequence of \(\text{RCD}(K,N)\) spaces that converges in measured Gromov-Hausdorff sense to a compact metric measure space \((X,d,m)\), and let \(\Omega_i \subset X_i\) be open sets with \(m_i(\partial S) = 0\) that satisfy a \(\delta\)-uniform outer/inner ball condition. Set \(\partial\Omega_i = S_i\). Then, \(d_{S_i} : X_i \to \mathbb{R}\) subconverges in Arzela-Ascoli sense to a \(1\)-Lipschitz function \(u : X \to \mathbb{R}\) that is the signed distance function of \(\partial\Omega\) with \(\Omega = u^{-1}((-\infty,0))\).

**Definition A.2** \(\text{(Outer and inner ball condition)}\). Let \((X,d)\) be a metric space. Let \(\Omega \subset X\) and \(\partial\Omega = S\). We say \(S\) satisfies an outer ball condition in \(x \in S\) if there exists \(r_x > 0\) and \(p_x \in \Omega^c\) such that \(d(x,p_x) = r_x\) and \(B_{r_x}(p_x) \subset \Omega^c\). We say \(S\) satisfies an outer ball condition if it satisfies an exterior ball condition in every \(x \in S\). Moreover \(S\) satisfies a uniform \(\delta\)-outer ball condition if there exists \(\delta > 0\) such that \(r_x \geq \delta\) for all \(x \in S\).

Similar, \(\Omega\) satisfies an inner \(\text{(uniform \(\delta\)-inner)}\) ball condition if the previous definition holds with \(\Omega^c\) replaces with \(\Omega\).

**Proof of Lemma A.1.** Form the Lemma 5.5 we conclude that \(d_{\partial\Omega_i}\) subconverges uniformly to a function \(u\) such that \(u = -d_{\partial\Omega_i}^c\) on \(\Omega_1\) and \(u = d_{\partial\Omega_i}^c\) on \(\Omega_2\) where \(\Omega_1 = u^{-1}((-\infty,0))\) and \(\Omega_2 = u^{-1}((0,\infty))\).

We only have to show \(\partial\Omega_1 = \partial\Omega_2 = u^{-1}(\{0\})\). By symmetry we only have to prove the first equality. For that we set \(\Omega_1 = \Omega\). We know that \(\partial\Omega \subset u^{-1}(0)\). Pick \(x \in u^{-1}(0)\). Then, there exist \(x_i\) with \(d_{\partial\Omega_i}(x_i) = 0\) such that \(x_i \to x\). Since \(\Omega_i\) satisfies a \(\delta\)-uniform outer/inner ball condition there exist geodesics \(\gamma_i : [-\delta,\delta] \to X_i\) with \(\gamma_i(0) = x_i\), \(\gamma_i([-\delta,0]) \subset \Omega\) and \(\gamma_i((0,\delta]) \subset \Omega_i^c\). Moreover \(\gamma_i\) converges uniformly to geodesic \(\gamma : [-\delta,\delta] \to X\).
with $\gamma([-\delta,0)) \subset u^{-1}((-\infty,0))$ and $\gamma((0,\delta]) \subset u^{-1}((0,\infty))$. Hence $x \in \partial \Omega$. □

Theorem A.3. Let $K \in \mathbb{R}$, $D, \eta > 0$ and $N \in [2, \infty)$. For $\epsilon > 0$ there exists $\delta > 0$ such that the following holds.

Let $X_i$ be a sequence of RCD($K,N$) spaces with $\operatorname{diam}_X \leq D$ and let $\Omega_i \subset X_i$ be open subsets that satisfy a $\eta$-uniform inner-outer ball condition and such that $\partial \Omega_i$ have a mean curvature barrier $H \in \mathbb{R}$ in the sense of (20) and (21).

Then, there exists a measured GH converging subsequence of $X_i$ with a limit RCD($K,N$) space $X$ such that a subsequence of $\operatorname{d} \partial \Omega_i$ uniformly converges to $\operatorname{d} \partial \Omega$ for an open subset $\Omega$ in $X$ that has a mean curvature barrier $H$.

Proof of Theorem A.3. The Theorem follows now from stability of mean curvature bounds together with the previous lemma. □

Remark A.4. In general CMC hypersurfaces don’t satisfy an effective $\delta$-uniform outer/inner ball condition with $\delta$ only depending on geometric information of $X$ and the mean curvature $H$. Counter-examples are families of catenoids in $\mathbb{R}^3$ with increasingly big second fundamental form. On the other hand a regularity theory for perimeter minimizing set $s$ and for isoperimetric sets in the context of RCD spaces was developped in recent work by Mondino and Semola [MS21], and Antonelli, Pasqualetto, Pozzetta and Semola [APPS22].

Example A.5. In the following we give two examples: (1) The first example was suggested to the author by Daniele Semola. One can consider the metric (measure) space that is the result of gluing together two copies of $B_1(0) \subset \mathbb{R}^2$ along their boundaries. This doubling space $X$ has Alexandrov curvature bounded from below by 0 and is therefore an RCD(0,2) space by theorems of Perelman-Petrunin [Per, Pet97, Pet11] and Lytchak-Stadler [LS22]. There is an isometric copy of $B_1(0) = \Omega$ inside of $X$ such that $\Omega^c = B_1(0)$ and $\partial \Omega \simeq \partial \Omega^c \simeq \partial B_1(0) \subset S$. Then $S$ has Laplace mean curvature bounded from below by 1, seen both as boundary of $\Omega$ and as boundary of $\Omega^c$. Hence $S$ has a mean curvature barrier $H$ for every $H \in [-1, 1]$ in the sense that (20) and (21) hold for every $H \in [-1, 1]$. In particular, it is a generalized minimal surface because one can choose $H = 0$. The space $X$ can be obtained as a limit of smooth Riemannian manifolds $M_i$, and the distance function $\operatorname{d} \partial \Omega$ as the limit of distance functions on $M_i$ corresponding to smooth domains $\Omega_i \subset M_i$. More precisely, as consequence of the proof of the double space theorem in smooth context $X$ can be constructed as the $C^0$-limit of Riemannian spheres with curvature bounded from below by 0, and $\Omega$ is the $\mathcal{D}$-limit of balls with constant mean curvature $H$ for a given $H \in [-1, 1]$.

(2) Another example the referee suggested is the double space $X$ of two copies $D_1$ and $D_2$ of a convex domain $D$ with smooth boundary in $\mathbb{R}^2$ such that the second fundamental form of $\partial D$ is non-negative and not necessarily
positive. Again \( X \) is an \( \text{RCD}(0,2) \) space. In this situation \( \partial D_1 \simeq \partial D_2 = S \)
is Laplace mean convex as the boundary of \( D_1 \) but also as the boundary of \( D_2 \). Hence, \( S \subset X \) has a mean curvature barrier 0. Again one may obtain \( X \) and \( S \) as the limit of smooth Riemannian manifolds with Ricci curvature bounded from below and as the \( D \)-limit of smooth minimal hypersurfaces, respectively. The same construction also works in higher dimensions.

The hypersurfaces presented in (1) and (2) are not minimal hypersurfaces in the classical sense or locally perimeter minimizing in the sense of [MS21]. But they are "equatorial" inside of the ambient space and may emerge as the solution of a variational problem, for instance a min-max problem, like the equator in a sphere of constant curvature.

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