Mathematical Emergency Response Model of Rescue Services

A V Maximov¹, A V Matveev¹, G N Zavodskov¹

¹Saint Petersburg University of State Fire Service of EMERCOM of Russia, 149 Moskovsky Prosp., Saint Petersburg, 196105, Russia

E-mail: he1nze@mail.ru

Abstract. The growing number of incidents and emergencies around the world is raising the issue of security. To improve security, states allocate huge amounts of money, both to support rescue services, and for scientific research in this area. The article reveals an approach to the development of a mathematical model of the rescue services functioning, taking into account the relationship between its individual types of forces and means, allowing simultaneous optimization of various types of material and human resources.

1. Introduction

In recent decades, all over the world there has been a tendency towards an increase in the number and scale of consequences of natural and man-made emergencies, which lead to significant destruction and losses [1-3]. Ensuring safety is a complex task that requires constant improvement, in particular, increasing the efficiency of the functioning of the rescue services (RS) [4,5]. The main goal of managing the functioning of the RS is to maintain a high level of readiness of all resources (forces and means) [6].

2. Methods

In the literature, there are several directions of research in the field of improving the efficiency of the functioning of the RS: at the tactical [7-9], operational [10-12] and strategic [13-15] levels of management. The largest part of scientific research is related to the problems of emergency response. The questions of the development strategy of the RS when the external environment changes have been rarely considered in the literature and require more detailed study. This article is aimed at solving strategic issues in the planning of resources (forces and means) of the RS.

RS is a complex system with a huge number of possible states. In general, the dynamics of the RS functioning process can be described by a vector random process \( X(t) \) with dependent components:

\[
X(t) = \{ Y(t), Z(t), W(t), V(t) \},
\]

\[
Y(t) = \{ Y_1(t), Y_2(t), \ldots, Y_R(t) \},
\]

\[
Z(t) = \{ Z_1(t), Z_2(t), \ldots, Z_L(t) \},
\]

\[
W(t) = \{ W_1(t), W_2(t), \ldots, W_J(t) \},
\]

\[
V(t) = \{ V_1(t), V_2(t), \ldots, V_G(t) \}. \tag{1}
\]
where \( Y(t) \) is an integer random process describing the number and stages of incidents in the territory controlled by the RS unit, where \( r \) is the type of incident, \( r = 1, 2, \ldots, R \) (fires of different ranks, emergencies, road accidents, etc.);

\( Z(t) \) is an integer random process describing the activities of the rescuer, where \( l \) is the assignment (specialization) of the rescuer, \( l = 1, 2, \ldots, L \);

\( W(t) \) – an integer random process describing the use of non-consumable equipment (rescue equipment, communication equipment, etc.), where \( j \) is the type of equipment, \( j = 1, 2, \ldots, J \);

\( V(t) \) – an integer random process describing the use of expendable resources, where \( g \) is the type of expendable resources, \( g = 1, 2, \ldots, G \).

The total amount of available resources in the RS is given by the following vectors:

\[
N^{(i,j,g)} = \left( N_1^{(i,j,g)}, N_2^{(i,j,g)}, \ldots, N_{L,J,G}^{(i,j,g)} \right),
\]

where \( N_i^{(j)} \) is number of rescuers type \( i \), where \( N_i^{(j)} \) – number of non-consumable equipment type \( i \), the \( N_i^{(g)} \) – number of expendable resources type \( i \).

The interaction of the components of the process \( X(t) \) is transitive and is determined by the graph shown in Fig. 1.

![Figure 1. Process graph \( X(t) \)](image)

So, for example, in the event of a fire, the intensity of its extinguishing depends both on the rank (complexity) of the fire [16] and on the productivity and number of rescue units arriving. Therefore, the process \( Y(t) \) is controlled by the process \( W(t) \). On the other hand, the intensity of various fires, emergencies and other incidents affects at the amount of resources used. Therefore, the process \( Y(t) \) controls the process \( W(t) \).

Obviously, the arguments are similar for the processes \( Y(t) \) and \( Z(t) \), \( Z(t) \) and \( W(t) \), \( W(t) \) and \( V(t) \). Taking further the principle of linear interaction [17] between the components of the process \( X(t) \), let us consider in more detail the walk of elementary units of the processes \( Y(t) \), \( Z(t) \), \( W(t) \), \( V(t) \) over the plurality of their states in the event of occurrence and elimination of incidents.

To develop a mathematical model of the RS functioning process, it is also necessary to enter the following initial data:

\[
\lambda_{op} = (\lambda_1, \lambda_2, \ldots, \lambda_r) \text{ is frequency of accidents in the territory controlled by the RS, where } \lambda_r \text{ – intensity of accident type } r, \ r = 1, 2, \ldots, R; \]

\( a_{ij} \) is average number of type \( i \) rescuers required at the occurrence of an accident \( r \);

\( b_{ij} \) is average number of required non-consumable equipment type \( j \) when an accident \( r \) occurs;

\( c_{ig} \) is average number of required expendable resources of type \( g \) when an accident \( r \) occurs;

\( t_1 \) is average travel time to the scene (of the type \( r \)); \( t_2 \) is average exploration time; \( t_3 \) is average time for rescue operations; \( t_4 \) is average time to liquidate an accident; \( t_5 \) is average time of special work.

All entered characteristics can be determined from historical statistic data or regulatory requirements.
Taking into account the designations introduced above, consider in detail the components of the process \( X(t) \). The process \( Y_r(t) \) is characterized by the following possible states (Fig.1):

- \( Y_0^{(r)} \) is no accidents;
- \( Y_1^{(r)} \) is occurrence of an incident;
- \( Y_2^{(r)} \) is work is underway to eliminate and conduct RO.

### Figure 2. Process graph \( Y_r(t) \), where: \( \lambda_{01}^{(r)} \) is the intensity of flow of service requests generated by one unit out of the maximum possible number of simultaneous incidents, 
\[
\lambda_{01}^{(r)} = \frac{\lambda_r}{N_r^{(r)}}.
\]

\( \lambda_{12}^{(r)} \) is the intensity of flow of acceptance for service is determined as follows:

\[
\lambda_{12}^{(r)} = \varphi \left( \frac{m_{y_1}^{(r)}}{m_{y_0}^{(r)}} \right) = \begin{cases} 
\frac{1}{t_{r1}}, & \text{если } m_{y_0}^{(r)} \geq b_{r1} \cdot m_{y_1}^{(r)} \\
\frac{1}{\frac{b_{r1}}{t_{r1}}} \cdot \frac{m_{y_0}^{(r)}}{m_{y_1}^{(r)}}, & \text{если } m_{y_0}^{(r)} < b_{r1} \cdot m_{y_1}^{(r)}
\end{cases}
\]

(2)

where \( m_{y_1}^{(r)}, m_{y_0}^{(r)} \) is average number of elements in the corresponding state;

\( \lambda_{20}^{(r)}(t) \) is intensity of elimination of one incident,
\[
\lambda_{20}^{(r)} = \frac{1}{\sum_{i=2}^{6} t_{ril}}.
\]

Each type \( l \) rescuer has the following possible conditions (fig.3): 
- \( z_0^{(l)} \) is on duty;
- \( z_1^{(l)} \) is follows to the scene;
- \( z_2^{(l)} \) is reconnaissance of the scene;
- \( z_3^{(l)} \) is performs OR;
- \( z_4^{(l)} \) is participates in the liquidation;
- \( z_5^{(l)} \) is performs special work;
- \( z_6^{(l)} \) is collection and return to the place of permanent deployment.

### Figure 3. Process graph of type \( l \) rescuer.

Average intensity of the flow of calls of the rescuers of the type \( l \):
The labeled graph of states of the non-consumable equipment unit is represented in fig. 4.

**Figure 4.** Process graph of non-consumable equipment unit, where: \( W_0^{(j)} \) is the NS unit is in the RS part and is not used; \( W_1^{(j)} \) is NS unit is out of order and is being repaired; \( W_2^{(j)} \) is the NS unit received a call and follows the incident; \( W_3^{(j)} \) is non-consumable equipment unit is used on the incident; \( W_4^{(j)} \) is the NS unit is returned to the part of the CC (in this case, it is assumed that if the unit of funds is out of order in the states \( W_2^{(j)} \), \( W_3^{(j)} \), \( W_4^{(j)} \), it still returns to the unit and then goes for repair).

Intensities of flows of transition between states of the NS unit of the type \( j \):

\[
\lambda_{01}^{(j)} = \lambda_{\text{pem}}^{(j)} \frac{N_{0}^{(j)} - m_{w_0}^{(j)}}{N_{0}^{(j)}}, \quad \lambda_{\text{t}}^{(j)} = 1/t_{\text{pem}}^{(j)}, \quad \lambda_{02}^{(j)} = \varphi \left( m_{w_0}^{(j)}, m_{w_0}^{(2)}, \ldots, m_{w_0}^{(R)}, m_{w_0}^{(l)} \right) = \begin{cases} \sum_{r=1}^{R} \lambda_{r}^{(j)} \cdot b_{r}, & \text{если } m_{w_0}^{(j)} \geq \sum_{r=1}^{R} b_{r} \cdot m_{y_0}^{(r)} \\ \sum_{r=1}^{R} \lambda_{r}^{(j)} \cdot b_{r}, & \text{если } m_{w_0}^{(j)} < \sum_{r=1}^{R} b_{r} \cdot m_{y_0}^{(r)} \end{cases}
\]

(4)
where \( \bar{t} \) is the total average time of use during the departure of an NS unit of the type \( j \).

The labeled state graph of the expendable resources unit is shown in fig. 5.

**Figure 5.** Process graph of expendable resources unit, where: \( v_0 \) is RS unit is in the RS part and is not used; \( v_1 \) is RS unit required; \( v_2 \) is RS unit participates in maintenance; \( v_3 \) is PC unit is put away, \( e_g \) is intensity of expendable resources stock replenishment.

Average intensity of the call flow of the RS unit of the type \( g \):

\[
\lambda_{(g)}^{(g)} = \frac{1}{\bar{t}_g} = \frac{\sum_{r=1}^{R} \bar{t}_r \cdot m^{(r)}_{y_j}}{\sum_{r=1}^{R} \bar{t}_r \cdot m^{(r)}_{y_j}}, \text{ если } m^{(g)}_{y_j} \geq \sum_{r=1}^{R} c_{rg} \cdot m^{(r)}_{y_j},
\]

\[
\lambda_{(g)}^{(g)} = \frac{1}{\bar{t}_g} = \frac{\sum_{r=1}^{R} \bar{t}_r \cdot m^{(r)}_{y_j}}{\sum_{r=1}^{R} c_{rg} \cdot m^{(r)}_{y_j}}, \text{ если } m^{(g)}_{y_j} < \sum_{r=1}^{R} c_{rg} \cdot m^{(r)}_{y_j},
\]

\[
\lambda_{(g)}^{(g)} = 1/\bar{t}^g_{\text{work}} = \frac{\sum_{r=1}^{R} c_{rg} \cdot m^{(r)}_{y_j}}{\sum_{r=1}^{R} (c_{rg} \cdot m^{(r)}_{y_j}) \sum_{i=1}^{6} t_i},
\]

where \( \bar{t}^g_{\text{work}} \) is the total average time of use during the departure of a PC unit of the type \( g \).

3. Result and discussion

Having considered in detail all the components of the process \( X(t) \), it is possible for each of the labeled graphs to compose the following equations for the dynamics of means with the corresponding initial conditions. Systems of differential equations (9), (12), (15) for the mean numbers of states \( m_{y_i,j,g}^{(i,j,g)} \) are written for those homogeneous (wandering in the same states) elements of a specific type, of which there are quite a lot in the system. In cases for unique (unit) elements, it is necessary to write differential equations similar to (9), (12), (15), but relatively to the probabilities \( P_{y_i,j,g}^{(i,j,g)} \).
For the process $Y(t)$:
\[
\begin{align*}
\left( \frac{dm_{y_0}^{(r)}}{dt} \right) &= -\lambda_{01}^{(r)} \cdot m_{y_0}^{(r)} + \lambda_{20}^{(r)} \cdot m_{y_2}^{(r)} \\
\left( \frac{dm_{y_1}^{(r)}}{dt} \right) &= \lambda_{01}^{(r)} \cdot m_{y_0}^{(r)} - \lambda_{12}^{(r)} \cdot m_{y_1}^{(r)} \\
\left( \frac{dm_{y_2}^{(r)}}{dt} \right) &= \lambda_{12}^{(r)} \cdot m_{y_1}^{(r)} - \lambda_{20}^{(r)} \cdot m_{y_2}^{(r)}
\end{align*}
\]
\[m_{y_0}^{(r)} + m_{y_1}^{(r)} + m_{y_2}^{(r)} = N_r, \quad (6)\]
\[m_{y_0}^{(r)}(0) = N_r, \quad m_{y_1}^{(r)}(0) = m_{y_2}^{(r)}(0) = 0, \quad r=1,2,\ldots, R. \quad (7)\]

For the process $V_g(t)$:
\[
\begin{align*}
\left( \frac{dm_{v_0}^{(g)}}{dt} \right) &= \varepsilon_g - \lambda_{01}^{(g)} \cdot m_{v_0}^{(g)} \\
\left( \frac{dm_{v_1}^{(g)}}{dt} \right) &= \lambda_{01}^{(g)} \cdot m_{v_0}^{(g)} - \lambda_{21}^{(g)} \cdot m_{v_1}^{(g)} \\
\left( \frac{dm_{v_2}^{(g)}}{dt} \right) &= \lambda_{21}^{(g)} \cdot m_{v_1}^{(g)} - \lambda_{22}^{(g)} \cdot m_{v_2}^{(g)} \\
\left( \frac{dm_{v_3}^{(g)}}{dt} \right) &= \lambda_{22}^{(g)} \cdot m_{v_2}^{(g)} - \lambda_{23}^{(g)} \cdot m_{v_3}^{(g)} \\
\left( \frac{dm_{v_4}^{(g)}}{dt} \right) &= \lambda_{23}^{(g)} \cdot m_{v_3}^{(g)} - \lambda_{24}^{(g)} \cdot m_{v_4}^{(g)} \\
\left( \frac{dm_{v_5}^{(g)}}{dt} \right) &= \lambda_{24}^{(g)} \cdot m_{v_4}^{(g)} - \lambda_{30}^{(g)} \cdot m_{v_5}^{(g)}
\end{align*}
\]
\[\sum_{i=0}^{3} m_{v_i}^{(g)}(T) = N_{v_i}^{(g)}(0) + \int_0^T \varepsilon_g(t) dt \quad (9)\]
\[m_{v_i}^{(g)}(0) = N_{v_i}^{(g)}, \quad i=0,1,2,3. \quad (10)\]

\[m_{v_0}^{(g)}(0) = m_{v_1}^{(g)}(0) = m_{v_2}^{(g)}(0) = 0, \quad g=1,2,\ldots, G. \quad (11)\]

Relations (7) - (8), (10) - (11), (13) - (14), (16) - (17) are normalizing and determine the initial conditions for the integration of systems of differential equations, as well as the total number of elements of each type of resources at the disposal of the RS.

For the process $W_j(t)$:
\[
\begin{align*}
\left( \frac{dm_{w_0}^{(j)}}{dt} \right) &= -\left( \lambda_{01}^{(j)} + \lambda_{22}^{(j)} \right) \cdot m_{w_0}^{(j)} + \lambda_{02}^{(j)} \cdot m_{w_2}^{(j)} + \lambda_{20}^{(j)} \cdot m_{w_4}^{(j)} \\
\left( \frac{dm_{w_1}^{(j)}}{dt} \right) &= \lambda_{01}^{(j)} \cdot m_{w_0}^{(j)} - \lambda_{12}^{(j)} \cdot m_{w_1}^{(j)} \\
\left( \frac{dm_{w_2}^{(j)}}{dt} \right) &= \lambda_{12}^{(j)} \cdot m_{w_1}^{(j)} - \lambda_{22}^{(j)} \cdot m_{w_2}^{(j)} \\
\left( \frac{dm_{w_3}^{(j)}}{dt} \right) &= \lambda_{20}^{(j)} \cdot m_{w_2}^{(j)} - \lambda_{34}^{(j)} \cdot m_{w_4}^{(j)} \\
\left( \frac{dm_{w_4}^{(j)}}{dt} \right) &= \lambda_{34}^{(j)} \cdot m_{w_4}^{(j)} - \lambda_{40}^{(j)} \cdot m_{w_5}^{(j)}
\end{align*}
\]
\[\sum_{j=0}^{4} m_{w_i}^{(j)} = N_{w_i}^{(j)} \quad (12)\]

\[m_{w_0}^{(j)}(0) = N_{w_0}^{(j)}, \quad m_{w_1}^{(j)}(0) = \ldots = m_{w_4}^{(j)}(0) = 0, \quad j=1,2,\ldots, J. \quad (13)\]

For the process $Z_l(t)$:
\[
\begin{align*}
\left( \frac{dm_{z_0}^{(l)}}{dt} \right) &= \lambda_{01}^{(l)} \cdot m_{z_0}^{(l)} - \lambda_{12}^{(l)} \cdot m_{z_1}^{(l)} \\
\left( \frac{dm_{z_1}^{(l)}}{dt} \right) &= \lambda_{12}^{(l)} \cdot m_{z_1}^{(l)} - \lambda_{22}^{(l)} \cdot m_{z_2}^{(l)} \\
\left( \frac{dm_{z_2}^{(l)}}{dt} \right) &= \lambda_{22}^{(l)} \cdot m_{z_2}^{(l)} - \lambda_{48}^{(l)} \cdot m_{z_4}^{(l)} \\
\left( \frac{dm_{z_3}^{(l)}}{dt} \right) &= \lambda_{48}^{(l)} \cdot m_{z_4}^{(l)} - \lambda_{60}^{(l)} \cdot m_{z_6}^{(l)} \\
\left( \frac{dm_{z_4}^{(l)}}{dt} \right) &= \lambda_{60}^{(l)} \cdot m_{z_6}^{(l)} - \lambda_{10}^{(l)} \cdot m_{z_{10}}^{(l)} \\
\left( \frac{dm_{z_5}^{(l)}}{dt} \right) &= \lambda_{10}^{(l)} \cdot m_{z_{10}}^{(l)} - \lambda_{80}^{(l)} \cdot m_{z_{80}}^{(l)}
\end{align*}
\]
\[\sum_{l=0}^{6} m_{z_i}^{(l)} = N_{z_i}^{(l)} \quad (15)\]

\[m_{z_0}^{(l)}(0) = N_{z_0}^{(l)}, \quad m_{z_1}^{(l)}(0) = \ldots = m_{z_6}^{(l)}(0) = 0, \quad l=1,2,\ldots, L. \quad (16)\]
By integrating this system of differential equations, it is possible to obtain the dynamics of all the numbers of states of the elements of the entire service system, and from them determine the necessary characteristics of the efficiency of its activities. In addition, this model allows one to study the stability of the solutions obtained and the dynamics of the moments (variances and correlation functions) of the considered service processes.

4. Conclusion
The resulting model makes it possible to obtain the main analytical regularities between the indicators of the operational activity of the RS, the amount of resources of the RS, indicators of the operational situation, as well as the indicators of the effectiveness of the functioning of the RS. Knowing the dynamics of calls to RS units, you can predict the effectiveness of its forecasting. It is also possible to solve the inverse problem: substantiation of the required amount of RS resources to ensure a given level of operational efficiency.

5. References
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