Klaus Kirsten · Paul Loya · Jinsung Park

Functional determinants for general self-adjoint extensions of Laplace-type operators resulting from the generalized cone

Abstract. In this article we consider the zeta regularized determinant of Laplace-type operators on the generalized cone. For arbitrary self-adjoint extensions of a matrix of singular ordinary differential operators modelled on the generalized cone, a closed expression for the determinant is given. The result involves a determinant of an endomorphism of a finite-dimensional vector space, the endomorphism encoding the self-adjoint extension chosen. For particular examples, like the Friedrich’s extension, the answer is easily extracted from the general result. In combination with (Bordag et al. in Commun. Math. Phys. 182(2):371–393, 1996), a closed expression for the determinant of an arbitrary self-adjoint extension of the full Laplace-type operator on the generalized cone can be obtained.

1. Introduction

Motivated by endeavors to give answers to some fundamental questions in quantum field theory there has been significant interest in the problem of calculating the determinants of second order Laplace-type elliptic differential operators; see for example [6, 58, 96, 97, 100]. In case the operator $\Delta_1$ in question has regular coefficients and is acting on sections of a vector bundle over a smooth compact manifold, it will have a discrete eigenvalue spectrum $\lambda_1 \leq \lambda_2 \leq \cdots \rightarrow \infty$. If all eigenvalues are different from zero the determinant, formally defined by $\det \Delta_1 = \prod \lambda_i$, is generally divergent. In order to make sense out of it different procedures like Pauli-Villars regularization [93] or dimensional regularization [104] have been invented. Mathematically the probably most pleasing regularization is the zeta function prescription introduced by Ray and Singer [98] (see also [48, 72]) in the context of analytic torsion; see i.e. [7–9, 89, 90].
In this method, one uses the zeta function $\zeta(s, \Delta)$ associated with the spectrum $\lambda_i$ of $\Delta$. In detail, for the real part of $s$ large enough one has

$$\zeta(s, \Delta) = \sum_{i=1}^{\infty} \lambda_i^{-s}.$$ 

In the briefly described smooth setting, one can show that $\zeta(s, \Delta)$ is analytic about $s = 0$ [67,101,108], which allows to define a zeta regularized determinant via

$$\det_\zeta(\Delta) = e^{-\zeta'(0, \Delta)}.$$ 

This definition has been used extensively in quantum field theory, see i.e. [18,22,51–54,72,74], as well as in the context of the Reidemeister–Franz torsion [98,99]. In particular, in one dimension rather general and elegant results may be obtained, which has attracted the interest of mathematicians especially in the last decade or so [20,21,50,59,60,83–85]. In higher dimensions known results are restricted to highly symmetric configurations [13–15,22,43–45,49] or conformally related ones [10,11,15,46,47].

Whereas most analysis has been done in the smooth setting, relevant situations do not fall into this category. For example, in order to compute quantum corrections to classical solutions in Euclidean Yang–Mills theory [25,103] singular potentials need to be considered. They also serve for the description of physical systems like the Calogero Model [3,4,26–28,55,92] and conformal invariant quantum mechanical models [2,19,29,30,38,61,95]. More recently they became popular among physicists working on space-times with horizons. There, for a variety of black holes, singular potentials are used to describe the dynamics of quantum particles in the asymptotic near-horizon region [5,35,63,68,88].

A similar situation occurs when manifolds are allowed to have conical singularities [31,34]. Under these circumstances, in general, $\zeta'(0, \Delta)$ will not be defined, although for special instances this definition still makes sense; nearly all of the literature has concentrated on these special instances. In order to describe these instances in more detail, let us consider a bounded generalized cone. As we will see below, the Laplacian on a bounded generalized cone has the form

$$\Delta = -\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} A_\Gamma,$$

where $A_\Gamma$ is defined on the base of the cone. If $A_\Gamma$ has eigenvalues in the interval $[\frac{3}{4}, \infty)$ only, one can show that $\Delta$ is essentially self-adjoint and no choices for self-adjoint extensions exist. Spectral functions, in particular the determinant, have been analyzed in detail in [13]. In case $A_\Gamma$ has one or more eigenvalues in the interval $[-\frac{1}{4}, \frac{3}{4})$ different self-adjoint extensions exist; see for example [34,87]. Most literature is concerned with the so-called Friedrich’s extension [16,23,24,34,36,37,41,42,83,102] and homogeneous or scale-invariant extensions [34,82,86]. Exceptions are [55–57] where general self-adjoint extensions associated with one eigenvalue in $[-\frac{1}{4}, \frac{3}{4})$ have been considered. For recent and ongoing work involving resolvents of general self-adjoint extensions of cone operators, see e.g. Gil et al. [65,66] and Coriasco et al. [40]. Only recently, properties of spectral functions for