ON THE LENGTH OF PIERCE EXPANSIONS

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Abstract. For a given positive integer $n$, how long can the process $x \mapsto n \pmod{x}$ last before reaching 0? We improve Erdős and Shallit’s upper bound of $O(n^{1/3} + \varepsilon)$ to $O(n^{1/3 - 2/177} + \varepsilon)$ for any $\varepsilon > 0$.

1. Introduction

The continued fraction expansion of a real number $x \in (0, 1)$, given by

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \ldots}},$$

plays an important role throughout number theory. The terms $a_i$ can be extracted, for example, from the iterated process $t \mapsto \frac{1}{t}$ (mod 1) beginning with $t = x$. It is well-known and not difficult to see that the continued fraction expansion of a real number $x$ is finite if and only if $x$ is a rational number. And if $x$ is rational, the sequence of terms $a_i$ produced are exactly the quotients produced by the classic Euclidean algorithm applied to the numerator and denominator.

In this paper, we are concerned with the Pierce expansion of a real number $x \in (0, 1)$, introduced by Pierce [5] and named by Shallit [6]. Here, the expansion is of the form

$$x = \frac{1}{b_1} - \frac{1}{b_1b_2} + \frac{1}{b_1b_2b_3} - \ldots,$$

where now the terms $b_i$ can be extracted from the iterated process $t \mapsto 1 \pmod{t}$ beginning with $t = x$. It is also not difficult to see that the Pierce expansion of a real number $x$ is finite if and only if $x$ is a rational number. And if $x$ is rational, the sequence of terms $b_i$ produced are exactly the quotients produced by an algorithm that at first glance appears similar to Euclid’s algorithm.

Let us give an example of the algorithm. Say $x = \frac{13}{35}$. We start with 13 and repeatedly obtain successive integers by reducing 35 modulo the current number. For example,

$$35 = 2 \cdot 13 + 9$$
$$35 = 3 \cdot 9 + 8$$
$$35 = 4 \cdot 8 + 3$$
$$35 = 11 \cdot 3 + 2$$
$$35 = 17 \cdot 2 + 1$$
$$35 = 35 \cdot 1 + 0$$
gives rise to
\[
\frac{13}{35} = \frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{2 \cdot 3 \cdot 4 \cdot 11} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 11 \cdot 17} - \frac{1}{2 \cdot 3 \cdot 4 \cdot 11 \cdot 17 \cdot 35}.
\]

Motivated by the known fact that the Euclidean algorithm used to divide a positive integer \(a\) by a positive integer \(n\) terminates after \(O(\log n)\) steps (which is sharp), it is natural to ask how quickly the above algorithm must terminate for a given denominator, no matter the numerator.

To this end, for positive integers \(a, n \in \mathbb{N}\), define \(P(a, n)\) to be the first positive integer \(k\) such that \(a_k = 0\), where \(a_0 := a\) and \(a_{j+1} = n \pmod{a_j} \in \{0, 1, \ldots, a_j-1\}\) for \(j \geq 0\). In the above example we have \(P(a, n) = P(13, 35) = 6\). Since we only concern ourselves with the “length” of the algorithm, we need not keep track of quotients and may compress for instance the above example to
\[
35 \pmod{13} = 9
35 \pmod{9} = 8
35 \pmod{8} = 3
35 \pmod{3} = 2
35 \pmod{2} = 1
35 \pmod{1} = 0.
\]

Noting \(P(a, n) = 2\) if \(a > n\), we set
\[
P(n) := \max_{1 \leq a \leq n} P(a, n).
\]

The problem we consider that of obtaining bounds on \(P(n)\). Shallit [6] proved, using purely “Archimedean” arguments, that \(P(n) \ll n^{1/2}\) (see §2 for our conventions regarding Vinogradov notation), while
\[
\limsup_{n \to \infty} \frac{P(n)}{\log n / \log \log n} > 0.
\]

The upper bound was improved by Erdős and Shallit [2] who leveraged “arithmetic” arguments to combine with the previous “Archimedean” ones. They established \(P(n) \ll n^{1/3} + \varepsilon\) and also improved the lower bound to \(\limsup_{n \to \infty} P(n) / \log n > 0\). These bounds have since remained the state of the art, with the exponent \(1/3\) representing a natural barrier.

In this paper, we improve the upper bound on \(P(n)\), (slightly) pushing past the \(1/3\) barrier.

**Theorem 1.1.** We have
\[
P(n) \ll n^{1/3 - 2/177 + \varepsilon}.
\]

We did not put substantial effort into optimizing the exponent gain achieved in Theorem 1.1 we could not, however, see a way to improve the upper bound to \(P(n) \ll n^{1/3}\) using our techniques.

Secondly, we establish a lower bound that applies to all \(n \in \mathbb{N}\). As we can tell, the best bound known prior was \(P(n) \gg \log \log n\).
Theorem 1.2. We have the lower bound

\[ P(n) \gtrsim \frac{\log n}{\log \log n} \]

for all sufficiently large \( n \).

As one can see, there is an exponential gap between the best known lower and upper bounds on \( P(n) \). We hope this paper will reignite interest in determining the true asymptotics and related questions.

In §2, we specify the notational conventions we use throughout the paper. In §3, we give the proof of our main theorem, Theorem 1.1. In §4, we give the proof of the lower bound, Theorem 1.2.

2. Notation

Any statement involving \( \varepsilon \) should be read to mean that the statement holds for all \( \varepsilon > 0 \). We use the standard Vinogradov notation, in which we write \( A \ll B \) (and equivalently \( B \gg A \)) to denote that \( |A| \leq CB \) for some implied constant \( C > 0 \) that depends only on \( \varepsilon \) (if \( A, B \) depend on it). We write \( A \approx B \) to denote that both \( A \ll B \) and \( B \ll A \) hold. For a parameter \( \beta \), we write \( \ll \beta \) and \( \approx \beta \) to mean that the implied constant may depend on \( \beta \). For positive integers \( a, A \in \mathbb{N} \), we write \( a \sim A \) to denote \( A < a \leq 2A \). Finally, we use the standard \( e(t) := e^{2\pi i t} \).

3. Proof of Theorem 1.1

In this section, we prove our main theorem, that \( P(n) \ll n^{1/2 - \frac{1}{177} + \varepsilon} \). We do this by establishing bounds for the amount of time the algorithm spends in dyadic intervals.

For the rest of this section, fix a (large) positive integer \( n \) and a positive integer \( a_0 \), letting \( a_{j+1} = n \mod a_j \) for \( j \geq 0 \).

Write

\[ T(A) := \# \{ j \geq 0 : a_j \sim A \} \]

The first bound we present on \( T(A) \) was proven in [6] and is due to “Archimedean” reasons (namely that the \( a_j \) drop quickly near \( n \)).

Lemma 3.1. We have \( T(A) \leq \frac{n}{2A} + 2 \).

Proof. For \( j \geq 0 \), let \( b_j = \lfloor \frac{n}{a_j} \rfloor \), so that \( \frac{n}{b_j} < a_j \leq \frac{n}{b_j} \). We claim that \( b_{j+1} > b_j \) for each \( j \geq 0 \). Indeed, if not, \( n = b_ja_j + a_{j+1} \), so \( a_{j+1} > \frac{n}{b_j} \) implies \( n(b_j + 1) - b_ja_j(b_j + 1) > n \), which yields \( a_j < \frac{n}{b_j+1} \), a contradiction. Therefore, since \( a_j \sim A \) implies \( b_j \in \left[ \frac{n}{2A}, \frac{n}{A} \right) \), the desired bound follows.

Note that Lemma 3.1 combined with the trivial \( T(A) \leq A \) already establishes the bound \( P(n) \ll n^{1/2} \) of Shallit [6]. The second bound we present improves this trivial bound, by taking advantage of “arithmetic” properties of the iterative process. It was proven in [2]. We reproduce this proof in our own notation as many of its features make their way into the proof our improvement.
Lemma 3.2. For $1 \leq A \leq n$, we have the bound

$$T(A) \ll A^{\frac{1}{2}} n^\varepsilon.$$ 

Proof. If $T(A) \leq 1$, we are done, so suppose that $T(A) \geq 2$. Let

$$J := \left\{ j \geq 0: a_j \sim A, a_j - a_{j+1} \leq \frac{4A}{T(A)} \right\}.$$ 

Note that

$$\sum_{j \geq 0} 1 = T(A) - 1 \geq \frac{1}{2} T(A), \quad \sum_{j \geq 0} (a_j - a_{j+1}) \leq A.$$ 

It follows that

$$\# \left\{ j \geq 0: a_j \sim A, a_j - a_{j+1} > \frac{4A}{T(A)} \right\} < \frac{1}{4} T(A),$$

so $\# J \geq \frac{1}{4} T(A)$. Now, note that for all $j$,

$$a_{j+1} \equiv n \pmod{a_j} \implies a_j \mid n + a_j - a_{j+1}.$$ 

We obtain that

$$T(A) \ll \# J \leq \sum_{h \leq \frac{1}{4} T(A)} \sum_{a \sim A} 1.$$ 

By the divisor bound, $\sum_{a \sim A} 1 \leq d(n + h) \ll n^\varepsilon$, so we obtain

$$T(A) \ll \frac{A}{T(A)} n^\varepsilon.$$ 

Rearranging yields the desired result. \qed

Together, Lemmas 3.1, 3.2 applied to the ranges $A \geq n^{2/3}, A \leq n^{2/3}$, respectively, give the bound $P(n) \ll n^{3/5 + \varepsilon}$. To obtain a bound of $n^{3/5 - \delta + \varepsilon}$, it suffices to show that $T(A) \ll n^{3/5 - \delta + \varepsilon}$ for $A \in [n^{3/5 - 2\delta}, n^{3/5 + \delta}]$. This is the content of Proposition 3.3 for sufficiently small $\delta > 0$. To do this, we make use of the arithmetic information obtained by analyzing two consecutive jumps. After using Poisson summation, we are reduced, roughly, to obtaining a power saving over the trivial bound for the sum

$$\sum_{b \sim n^{1/3}} e \left( \frac{n}{b} \right).$$

Such bounds follow from standard exponential sum bounds. In our case, we use the exponent pair $(\frac{13}{84} + \varepsilon, \frac{35}{84} + \varepsilon)$ of Bourgain [1]. Much simpler methods would have also worked, to give a slightly worse saving over the trivial bound (the van der Corput A-process, followed by the B-process, for example).

Proposition 3.3. Suppose that $\delta, \lambda > 0$ are such that

$$\delta < \frac{1}{18}, \quad \lambda \leq \frac{1}{3} - \delta.$$ 

Then, for $n^{3/5 - 2\delta} \leq A \leq n^{3/5 + \delta}$, we have

$$T(A) \ll n^{\frac{1}{3} - \gamma + \varepsilon},$$

where $\gamma = \delta + \varepsilon$.
where
\[ \gamma := \min \left( \lambda - 2\delta, \delta, \frac{4}{63} - \frac{349}{84} \delta - \frac{13}{84} \lambda \right). \]

Before proving Proposition 3.3, let us first quickly spell out how Theorem 1.1 follows.

**Proof of Theorem 1.1 assuming Proposition 3.3.** Take
\[ \delta = \frac{2}{177}, \quad \lambda = 3\delta. \]
It is easy to check that \( \delta, \lambda \) satisfy the hypotheses of Proposition 3.3. We have that
\[ P(n) \leq 1 + \sum_{A \leq n} T(A), \]
where the sum over \( A \) runs over only powers of 2. The contribution of \( A > n^{\frac{2\delta}{3}} \) is, by Lemma 3.1,
\[ \ll n^{\frac{2\delta}{3} - \delta}. \]
By Lemma 3.2 the contribution of \( A < n^{\frac{2\delta}{3} - \delta} \) is
\[ \ll n^{\varepsilon} \sum_{A < n^{\frac{2\delta}{3}}} A^{\frac{1}{2}} \ll n^{\frac{1}{3} - \delta + \varepsilon}, \]
For \( n^{\frac{1}{3} - 2\delta} \leq A \leq n^{\frac{1}{3} + \delta} \), by Proposition 3.3, we have that
\[ T(A) \ll n^{\frac{1}{3} - \gamma + \varepsilon}, \]
where
\[ \gamma = \min \left( \lambda - 2\delta, \delta, \frac{4}{63} - \frac{349}{84} \delta - \frac{13}{84} \lambda \right) = \frac{2}{177}. \]
Then, summing over \( A \) in \([n^{\frac{2\delta}{3} - \delta}, n^{\frac{1}{3} + \delta}]\) at the harmless cost of \( O(\log n) \), Theorem 1.1 follows.

**Proof of Proposition 3.3.** Suppose that \( T(A) \geq T_0 = n^{\frac{1}{3} - \delta} \), for we are done otherwise. Let \( m \) be so that \( a_{m+T(A)} \leq A < a_{m+T(A)} - 1 < \cdots < a_m \leq 2A \). Then, as in the proof of Lemma 3.2 for a positive proportion of \( m + 2 \leq j < m + T(A) \), we have that
\[ a_{j-2} - a_j \leq H := \frac{10A}{T_0}. \]
We record the bound \( n^{\frac{1}{3} - \delta} \ll H \ll n^{\frac{1}{3} + 2\delta} \). Write
\[ \mathcal{J} = \{ m + 2 \leq j < m + T(A) : a_{j-2} - a_j \leq H \}. \]
Consider some \( j \in \mathcal{J} \), and write \( a = a_{j-2}, a-h = a_{j-1}, a-h-h' = a_j \). Then, as in the proof of Lemma 3.2 we have
\[ a|n+h, a-h|n+h'. \]
In particular, there exist \( b \approx n/A, k \) such that \( ab = n+h, (a-h)(b+k) = n+h' \).
Also, note that
\[ |(b+k)h-ak| = |ab - (a-h)(b+k)| \ll H, \]
so rearranging, we have
\[ h = \frac{ak}{b+k} + O\left(\frac{AH}{n}\right) = \frac{abk}{b(b+k)} + O\left(\frac{AH}{n}\right) = \frac{nk}{b(b+k)} + O\left(\frac{AH}{n}\right) \]
since \( Hk/B^2 \ll H/B = AH/n \). Write \( H_0(b,k) := \frac{nk}{b(b+k)} \). Recall that \( \lambda \leq \frac{1}{3} - \delta \), so for \( b \approx \frac{n}{A} \)
\[ H_0(b,k) n^{-\lambda} \geq \frac{H_0(b,k)}{T_0} \gg \frac{A^2}{nT_0} \leq \frac{AH}{n}. \]

It follows for some sufficiently large \( C > 0 \) that
\[ |h - H_0(b,k)| \ll AH/n. \]

By Poisson summation,
\[ \sum_{|k| \ll Hn/A^2} \sum_{b \ll H} \sum_{b \equiv n/A} 1_{|h - H_0(b,k)| \ll L} \ll \sum_{|k| \ll Hn/A^2} \sum_{b \ll H} \sum_{b \equiv n/A} w \left( \frac{h - H_0(b,k)}{L} \right). \]

The contribution of the zero frequency, \( r = 0 \), is
\[ \ll \frac{Hn}{A^2} \cdot \frac{n}{A} \cdot \frac{Hn^{-\lambda}}{n^\lambda} \ll n^{\frac{1}{3} + 2\delta - \lambda}, \]
which is acceptable. It remains to bound the contribution when \(|r| > 0\), so we restrict to that case from now on.

A quick computation shows that for \( x_0 \approx n/A \), we have that for some constant
\[ \frac{d^3}{dx^3} \left| \frac{r(n + H_0(x,k))}{x} \right| \bigg|_{x=x_0} \asymp j^3 r x_0^{-j} \]
uniformly in \(|k| \ll Hn/A^2\).
\( \hat{w}, (\hat{w})' \) are Schwartz, and that \( |r| > 0 \) (which implies that \( |r| A \gg n/A \), so (8.56) of [3] holds), we have for some \( c > 0 \) that
\[
\left| \sum_{b \leq n/A} e\left( \frac{r(n + H_0(b,k))}{b} \right) \frac{n/A}{b} \hat{w}\left( \frac{L_r}{b} \right) \right| \\
\ll \left( 1 + \frac{L_r}{n/A} \right)^{-2022} \sup_{1 \leq n/A < b \leq t} \left| \sum_{c n/A < b \leq t} e\left( \frac{r(n + H_0(b,k))}{b} \right) \right| \\
\ll \left( 1 + \frac{L_r}{n/A} \right)^{-2022} \left( \frac{A^2 |r|}{n} \right)^{\frac{13}{84} + \varepsilon} \left( \frac{n}{A} \right)^{\frac{55}{84} + \varepsilon}.
\]

Putting this all together, we obtain that
\[
\left| \sum_{|k| \leq H n/A^2} \sum_{b \leq n/A} \frac{L}{n/A} \sum_{r \neq 0} e\left( \frac{r(n + H_0(b,k))}{b} \right) \frac{n/A}{b} \hat{w}\left( \frac{L_r}{b} \right) \right| \\
\ll \frac{H n}{A^2} \left( \frac{A n^\lambda}{H} \right)^{\frac{13}{84}} \left( \frac{n}{A} \right)^{\frac{55}{84}} \cdot n^\varepsilon \ll \frac{n}{A T_0} \left( \frac{n}{A} \right)^{\frac{13}{84}} \left( \frac{n}{A} \right)^{\frac{55}{84}} n^\varepsilon \\
\ll n^{34} \cdot n^{\frac{13}{84} \cdot \frac{55}{84} + \frac{13}{84} \cdot \frac{55}{84} - 2s} \\
\ll n^{\frac{13}{84} + \frac{55}{84} + \frac{13}{84} + \frac{55}{84} - 2s}.
\]

The desired result follows. \( \square \)

4. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2, repeated below for the reader’s convenience.

Theorem 1.2 We have the lower bound
\[
P(n) \gg \frac{\log n}{\log \log n}
\]
for all sufficiently large \( n \).

Shallit [6] and Erdős-Shallit [2] established lower bounds for \( P(n) \) of \( c \log n \) for positive integers \( n \) such that \( n + 1 \) is divisible by all sufficiently small positive integers. Such positive integers \( n \) will cause the process \( x \mapsto n \pmod{x} \) to repeatedly decrement by 1 at the end. We establish a lower bound that is valid for all positive integers by choosing a starting number based on \( n \) that causes the process \( x \mapsto n \pmod{x} \) to repeatedly decrement by 1 at the beginning, for “Archimedean” reasons rather than “arithmetic” ones.

We will need the following elementary lemma.

Lemma 4.1. There exists \( c > 0 \) so that the following holds for sufficiently large \( n \in \mathbb{N} \). For any \( k \in \mathbb{N} \) with \( k \leq c \log n \), one has
\[
(-1)^k k! \left( \sum_{j=0}^{k} \frac{(-1)^j}{j!} - \frac{1}{e} \right) n > \frac{n}{k + 2} + k!.
\]
Proof. Note, from the power series for $e^{-1}$, that

$$(-1)^k (k+2)! \left( \sum_{j=0}^{k} \frac{(-1)^j}{j!} - \frac{1}{e} \right) = 1 + \frac{1}{(k+3)(k+1)} - O\left( \frac{1}{k^3} \right) = 1 + \frac{1}{k^2} - O\left( \frac{1}{k^3} \right),$$

which is greater than $1 + \frac{k!(k+2)}{n}$ for sufficiently large $n$, by assumption. \qed

Proof of Theorem 5.2. By adjusting the implied constant, we may assume $n$ is sufficiently large. Let $a = \lfloor (1 - \frac{1}{e})n \rfloor$, $a_0 = a$, and $a_{k+1} = n \pmod{a_k}$ for $k \geq 0$. Let

$$b_0 = (1 - \frac{1}{e})n \text{ and } b_k = (-1)^k k! \left( \sum_{j=0}^{k} \frac{(-1)^j}{j!} - \frac{1}{e} \right) n \text{ for } k \geq 1.$$

We show $P(a, n) \geq c \frac{\log n}{\log \log n}$, where $c > 0$ is as in Lemma 4.3.

We prove inductively that $|a_k - b_k| \leq k!$ and $a_k = n - ka_{k-1}$. For $k = 0$, the first is clearly true. The second is true for $k = 1$ and thus so is the first. Now assume they are both true for some $k \geq 1$. We have by Lemma 4.1 that $a_k \geq b_k - k! > \frac{n}{k+2}$. Since $\lfloor \frac{n}{a_k} \rfloor$ must strictly increase, we have $a_k < \frac{n}{k+2}$. Therefore, $a_{k+1} = n - (k+1)a_k$ and thus $|a_{k+1} - b_{k+1}| = |(n - (k+1)a_k) - (n - (k+1)b_k)| = (k+1) |a_k - b_k| \leq (k+1)!$. We have thus shown $a_k > \frac{n}{k+2} > 0$ as long as $k \leq c \frac{\log n}{\log \log n}$. \qed

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