Defect Particle Kinematics in One-Dimensional Cellular Automata

Marcus Pivato*

Dept. of Mathematics & Computer Science, Wesleyan University
and Department of Mathematics, Trent University

Abstract

Let $\mathcal{A}^\mathbb{Z}$ be the Cantor space of bi-infinite sequences in a finite alphabet $\mathcal{A}$, and let $\sigma$ be the shift map on $\mathcal{A}^\mathbb{Z}$. A cellular automaton is a continuous, $\sigma$-commuting self-map $\Phi$ of $\mathcal{A}^\mathbb{Z}$, and a $\Phi$-invariant subshift is a closed, $(\Phi, \sigma)$-invariant subset $S \subset \mathcal{A}^\mathbb{Z}$. Suppose $a \in \mathcal{A}^\mathbb{Z}$ is $S$-admissible everywhere except for some small region we call a defect. It has been empirically observed that such defects persist under iteration of $\Phi$, and often propagate like ‘particles’. We characterize the motion of these particles, and show that it falls into several regimes, ranging from simple deterministic motion, to generalized random walks, to complex motion emulating Turing machines or pushdown automata. One consequence is that some questions about defect behaviour are formally undecidable.

Key words: Cellular automata, subshift, defect, kink, domain boundary, particle.
MSC: 68Q80 (primary), 37B15 (secondary)

A recurring theme in cellular automata is the emergence and persistence of homogeneous ‘domains’ (each characterized by a particular spatial pattern), separated by defects (analogous to ‘domain boundaries’ or ‘kinks’ in a crystalline solid). Defects were first empirically observed by Grassberger in the ‘elementary’ cellular automata or ‘ECA’ (radius-one CA on $\{0,1\}^\mathbb{Z}$) with numbers #18, #122, #126, #146, and #182 [Gra84b, Gra84a] and also noted in ECA #184, which was originally studied as a simple model of surface growth [KS88b, §III.B], and later as a model of single-lane traffic [Fuk99, Bla03, BF05]. Based on Grassberger’s observations, Lind [Lin84, §5] conjectured that the defects of ECA #18 perform random walks. This conjecture was reiterated by

* Trent University, 1600 West Bank Drive, Peterborough, Ontario, Canada, K9J 7B8
Email address: pivato@xarave.trent.ca, marcuspivato@trentu.ca (Marcus Pivato).
Boccara et al., who empirically investigated the motion and interactions of defects in ECA #18 and also #54, #62, and #184 (see Figure 1), and longer range totalistic CA [BR91,BNR91]; see also [Ila01, §3.1.2.2 & §3.1.4.4].

Eloranta developed the first rigorous mathematical theory of cellular automaton defects in [Elo93a,Elo93b,Elo94,Elo95], and, together with Numelin, proved Lind’s conjecture in [EN92]. Meanwhile, Crutchfield and Hanson developed an empirical methodology called Computational Mechanics [Han93], which they applied to ECA#18 [CH92,CH93a] and other CA contrived to act like ECA#18 [CH93b], as well as ECA#54 [CH97,CHS01] and ECA#110 [CHS01]. They also obtained a tight theoretical bound on the number of possible reactions between two defects [CHS01] (improving an earlier result of [PST86]). Finally, using genetic algorithms, they and their collaborators ‘bred’ CA which performed computations such as synchronization or density-classification. A careful analysis then revealed that these CA performed their computations through propagating and interacting defects; this ‘particle-based computation’ had emerged spontaneously through natural selection [DMC94,CHM98,CHS01].

Fig. 1. Spacetime diagrams showing defect dynamics in two cellular automata. Each picture show 120 timesteps on a 300 pixel array (time increases downwards).
In two companion papers [Piv07a, Piv07b], we develop algebraic invariants to explain why defects persist under iteration of the cellular automaton, instead of disappearing. These defects often behave like ‘particles’, which propagate through space until they collide and interact with other defects. In this paper, we characterize the motion of these ‘defect particles’, when the background domain is a one-dimensional subshift of finite type which is invariant under the action of a one-dimensional cellular automaton. In §1 we formally define ‘defect particles’ and introduce a framework to investigate their motion. Depending on the $(\Phi, \sigma)$-dynamical properties of the ambient subshift, the defect particle falls into one of several ‘kinematic regimes’, ranging from ballistic motion (§2), to a generalized random walk (§3), to the emulation of Turing machines or pushdown automata (§4). Sections §2, §3 and §4 are logically independent of one another.

**Preliminaries & Notation**

For any $L \leq R \in \mathbb{Z}$, we define $[L...R] := \{L, L+1, \ldots, R\}$, $[L...R) := \{L+1, \ldots, R\}$, etc. We likewise define $(-\infty...R]$, $[L...\infty)$, etc. Let $A$ be a finite alphabet, and let $A^\mathbb{Z}$ be the set of all doubly-infinite sequences in $A$, which we write as $a = [a_z]_{z \in \mathbb{Z}}$, where $a_z \in A$ for all $z \in \mathbb{Z}$. The Cantor metric on $A^\mathbb{Z}$ is defined by $d(a, b) = 2^{-\Delta(a, b)}$, where $\Delta(a, b) := \min \{|z| : a_z \neq b_z\}$. It follows that $(A^\mathbb{Z}, d)$ is a Cantor space (i.e. a compact, totally disconnected, perfect metric space). If $a \in A^\mathbb{Z}$, and $U \subset \mathbb{Z}$, then we define $a_U \in A^U$ by $a_U := [a_z]_{z \in U}$. If $z \in \mathbb{Z}$, then strictly speaking, $a_{z+U} \in A^{z+U}$; however, it is sometimes convenient to ‘abuse notation’ and treat $a_{z+U}$ as an element of $A^U$ in the obvious way.

We define the shift map $\sigma : A^\mathbb{Z} \rightarrow A^\mathbb{Z}$ by $\sigma(a)_z = a_{z+1}$ for all $a \in A^\mathbb{Z}$ and $z \in \mathbb{Z}$. A cellular automaton is a transformation $\Phi : A^\mathbb{Z} \rightarrow A^\mathbb{Z}$ that is continuous and commutes with $\sigma$. Equivalently, $\Phi$ is determined by a local rule $\phi : A^{[-r...r]} \rightarrow A$ (for some $r \in \mathbb{N}$) such that $\Phi(a)_z = \phi(a_{z-r...z+r})$ for all $a \in A^\mathbb{Z}$ and $z \in \mathbb{Z}$ [Hed69]; we say that $\Phi$ has radius $r$.

A subset $S \subset A^\mathbb{Z}$ is a subshift [LM95, Kit98] if $S$ is closed in the Cantor topology, and $\sigma(S) = S$. For any $U \subset \mathbb{Z}$, we define $S_U := \{s_U : s \in S\}$. In particular, for any $q > 0$, let $S_q := S_{[0...q]}$ be the set of admissible $q$-words for $S$. We say $S$ is subshift of finite type (SFT) if there is some $q > 0$ (the radius of $S$) such that $S$ is entirely described by $S_q$, in the sense that $S = \{s \in A^\mathbb{Z} : s_{[z...z+q]} \in S_q, \forall z \in \mathbb{Z}\}$. If $q = 2$, then $S$ is called a Markov subshift, and the elements of $S_2 \subset A^2$ are called admissible transitions; equivalently, $S$ is the set of all bi-infinite directed paths in a digraph whose vertices are the elements of $A$, with an edge $a \leadsto b$ iff $(a, b) \in S_2$.

If $\Phi : A^\mathbb{Z} \rightarrow A^\mathbb{Z}$ is a cellular automaton, then we say $S$ is (weakly) $\Phi$-invariant.
if \( \Phi(S) \subseteq S \) (i.e. \( \Phi \) is an endomorphism of \( S \)). For example, the set \( \text{Fix}[\Phi] := \{ a \in A^Z ; \Phi(a) = a \} \) of \( \Phi \)-fixed points is a \( \Phi \)-invariant SFT. Likewise, if \( p \in \mathbb{N} \) and \( v \in \mathbb{Z} \), then the set \( \text{Fix}[\Phi^p] \) of \( (\Phi,p) \)-periodic points and the set \( \text{Fix}[\Phi^p \circ \sigma^{-pq}] \) of \( (\Phi,p,v) \)-travelling waves are \( \Phi \)-invariant SFTs. Also, for any \( p \in \mathbb{N} \), the set \( \text{Fix}[\sigma^p] \) of \( p \)-periodic sequences is a \( \Phi \)-invariant SFT.

If \( \Phi \) has radius \( r \), then for any \( q > 0 \), \( \Phi \) induces a function \( \Phi : A^{q+2r} \rightarrow A^q \).

If \( S \subset A^Z \) is an SFT determined by a set \( S_q \subset A^q \) of admissible \( q \)-blocks, then \((\Phi(S) \subseteq S) \iff (\Phi(S_{q+2r}) \subseteq S_q)\). The monoid of endomorphisms of an SFT can be quite huge; see [Kit98, Ch.3] or [LM95, §13.2].

If \( S \subset A^Z \) is a subshift, then we define \( S^- := S_{[-\infty,...,-1]} \subseteq A^{(-\infty,...,-1)} \) to be the set of all left-infinite, \( S \)-admissible sequences, and define \( S^+ := S_{[1,...,\infty]} \subseteq A^{[1,...,\infty)} \) to be the set of all right-infinite, \( S \)-admissible sequences.

**Notation & Font conventions:** Upper case calligraphic letters (\( A, B, C, \ldots \)) denote finite symbolic alphabets (of cellular automata, Turing machines, etc.). Upper-case boldface letters (\( A, B, C, \ldots \)) denote subsets of \( A^Z \) (e.g. subshifts); lowercase bold-faced letters (\( a, b, c, \ldots \)) denote elements of \( A^Z \). Zapf letters (\( a, b, c, \ldots \)) are elements of \( A \); Roman letters (\( a, b, c, \ldots \)) are integers. Upper-case hollow font (\( U, V, W, \ldots \)) are subsets of \( Z \), upper-case Greek letters (\( \Phi, \Psi, \ldots \)) denote functions on \( A^Z \) (e.g. CA), and lower-case Greek letters (\( \phi, \psi, \ldots \)) denote other functions (e.g. local rules, probability measures).

We generally indicate related objects by related letters. For example, if \( L, R \subset A \) are two subalphabets, then a subshift of \( L^Z \) would be denoted by \( L \), with typical element \( l := [l_z]_{z \in Z} \in L \) (where \( l_z \in L \)), whereas a subshift of \( R^Z \) would be denoted by \( R \), with typical element \( r := [r_z]_{z \in Z} \in R \) (where \( r_z \in R \)).

## 1 Defect Particles

Let \( S \subset A^Z \) be a subshift of finite type, and let \( \Phi : A^Z \rightarrow A^Z \) be a one-dimensional cellular automaton with \( \Phi(S) \subseteq S \). By passing to a ‘higher block presentation’, we can assume that \( \Phi \) is a nearest-neighbour CA and that \( S \) is a Markov subshift. To be precise, suppose \( \Phi \) has radius \( r \) and that \( S \) is determined by a set \( S_q \) of admissible \( q \)-blocks. Let \( P := \max\{2r, q\} \), let \( B := A^P \), and let \( \tilde{A} \subset B^Z \) be the \( P \)-block presentation of \( A^Z \) (see [LM95, Defn.1.4.1] or [Kit98, Fig.1.4.1]). That is, \( \tilde{A} \) is the the Markov subshift defined by the digraph with vertex set \( A^P \), with an edge \( [a_1, \ldots, a_P] \sim [b_1, \ldots, b_P] \) iff \( b_p = a_{p+1} \) for all \( p \in [1...P] \) (this is sometimes called the de Bruijn digraph of \( A^P \)). Thus, \( \Phi \) is conjugate to an endomorphism \( \tilde{\Phi} : \tilde{A} \rightarrow \tilde{A} \), which can be extended (in an arbitrary way) to a cellular automaton \( \Psi : B^Z \rightarrow B^Z \) such that \( \Psi(\tilde{A}) \subseteq \tilde{A} \).
The width for some half-infinite, S-admissible sequences:

\[ a^0 = [\ldots a^0_{i-3}, a^0_{i-2}, a^0_{i-1}, a^0_i, a^0_{i+1}, a^0_{i+2}, a^0_{i+3}, \ldots ] \]  

(here we underline the admissible sequences for visibility). If \( i < k \), then we say that \((a_{i+1}, \ldots, a_k)\) is a defect word of width \( w := k - i \):

\[ a^0 = [\ldots a^0_{i-3}, a^0_{i-2}, a^0_{i-1}, a^0_i, a^0_{i+1}, \ldots a^0_k, a^0_{k+1}, a^0_{k+2}, a^0_{k+3}, \ldots ] \]

We want to rewrite this defect word as \((a_{z_0-L_0}, \ldots, a_{z_0}, \ldots, a_{z_0+R_0})\), where \( z_0 \) is roughly in the center of the defect. So let \( L_0 := \lfloor w/2 \rfloor - 1 \) and \( R_0 := \lfloor w/2 \rfloor \), so that \( w = L_0 + R_0 + 1 \). If \( z_0 := i + L_0 + 1 \), then \( z_0 - L_0 = i + 1 \) and \( z_0 + R_0 = k \), as desired. Define \( d^0_z := a^0_{z_0+z} \) for all \( z \in [-L_0...R_0] \), and rewrite eqn.(3) as:

\[ a^0 = [\ldots a^0_{z_0-L_0-2}, a^0_{z_0-L_0-1}, d^0_{z_0-L_0} \ldots d^0_0, a^0_{z_0+R_0+1}, a^0_{z_0+R_0+2}, \ldots ] \]

[If \( w = 0 \), then \( L_0 = -1 \), \( R_0 = 0 \), and \( z_0 = i + 1 \), so eqn.(3) is equivalent to the zero-width defect in eqn.(1).] For all \( t \in \mathbb{N} \), let \( a^t := \Phi^t(a) \). We say the defect is \( \Phi \)-persistent if \( a^t \) has a defect for all \( t \in \mathbb{N} \). In this case,

\[ a^t = [\ldots a^t_{z_1-L_1-2}, a^t_{z_1-L_1-1}, d^t_{z_1-L_1} \ldots d^t_0, a^t_{z_1+R_1+1}, a^t_{z_1+R_1+2}, \ldots ] \]

for some \( z_t \in \mathbb{Z} \), \( R_t \in \mathbb{N} \) and \( L_t \in \{R_t, R_t - 1\} \). The next lemma bounds the growth-rate and displacement of the defect during one \( \Phi \)-iteration.

**Lemma 1.1** Let \( t \in \mathbb{N} \). Then:

(a) \( z_t - L_t - 1 \leq z_{t+1} - L_{t+1} \) and \( z_{t+1} + R_{t+1} \leq z_t + R_t + 1 \).
(b) \( z_t - L_t - 2 \leq z_{t+1} \leq z_t + R_t + 1 \).

**Proof:** (a) For simplicity, set \( t := 1 \). The boundaries of the defect word can advance by at most one unit during each timestep, because \( \Phi \) is a nearest neighbour CA and \( \Phi(S) \subseteq S \). In other words, \( z_0 - L_0 - 1 \leq z_1 - L_1 \) and also \( z_1 + R_1 \leq z_0 + R_0 + 1 \). (b) follows because \( z_1 - L_1 - 1 \leq z_1 \leq z_1 + R_1 \), because \( L_1 \geq -1 \) and \( 0 \leq R_1 \).

The width \( w_t \approx 2L_t \) of the defect word may fluctuate with time. We say that the defect is a particle if \( L := \max_{t \in \mathbb{N}} \{L_t\} \) and \( R := \max_{t \in \mathbb{N}} \{R_t\} \) are finite (possibly
$L = -1$ and $R = 0$). Otherwise the defect is called a blight (i.e. its size grows without bound over time). We will restrict our attention to particles. It will be convenient to treat the particle as having constant width. Hence, we rewrite eqn.(4) as

$$\begin{align*}
a^t &= [\ldots a^t_{z_t-L-2} a^t_{z_t-L-1} a^t_{-L} \cdots a^t_0 a^t_{z_t+R+1} a^t_{z_t+R+2} \ldots],
&= [\ldots l^t_2 l^t_1 d^t_{-L} \cdots d^t_0 r^t_1 r^t_2 \ldots].
\end{align*}$$

That is: we pad the left side (resp. right side) of the defect with $L - L_t$ (resp. $R - R_t$) of the ‘admissible’ symbols, if necessary, and then we define $l^t_n := a^t_{z_t-L-n}$ and $r^t_n := a^t_{z_t+R+n}$ for all $n \in \mathbb{N}$ (note that, for convenience, we reverse the sign of index $n$ in $l^t_n$). We say $W := R + L + 1$ is the width of the defect particle. (If all the defects had zero width, then $L = -1$ and $R = 0$ and $W = 0$, so the non-underlined block is empty.) We can now represent the defect particle as a finite automaton.

A finite automaton is a quintuple $(I, D, O; \Upsilon, \Omega)$, where $I$ is a finite input alphabet, $D$ is a finite state domain, $O$ is a finite output alphabet, $\Upsilon : I \times D \to D$ is an update rule, and $\Omega : I \times D \to O$ is an output rule. Finite automata are models of simple computers: starting from initial state description $d_0 \in D$, and fed an input stream $(i_0, i_1, i_2, \ldots) \in I^\mathbb{N}$, the automaton undergoes a series of state transitions $d_0 \sim d_1 \sim d_2 \sim \ldots$ [where $d_{t+1} := \Upsilon(i_t, d_t)$] and produces an output stream $(o_1, o_2, o_3, \ldots) \in O^\mathbb{N}$, where $o_{t+1} := \Omega(i_t, d_t)$. See [HU79, §2.2].

The defect particle in eqn.(5) behaves like a finite automaton with $I := A^{L+2} \times A^{R+1}$, $D := A^{[L,R]}$, and $O := [-L-2 \ldots R+1]$. The automaton’s inputs are $I^t := [l^t_1, \ldots, l^t_z]$ and $r^t := [r_1, \ldots, r_{R+1}]$, its internal state is $d^t := [d^t_L, \ldots, d^t_0, \ldots, d^t_R] \in D$, and its output is a ‘velocity vector’ in $V := [-L-2 \ldots R+1]$. That is, there is a unique update rule $\Upsilon : A^{L+2} \times D \times A^{R+1} \to D$ and velocity function $\vec{V} : A^{L+2} \times D \times A^{R+1} \to V$ such that

$$d^{t+1} = \Upsilon(I^t, d^t, r^t) \quad \text{and} \quad z_{t+1} = z_t + \vec{V}(I^t, d^t, r^t) \quad (6)$$

Let $L$ and $R$ be the unique $(\sigma, \Phi)$-transitive components of $S$ such that $[\ldots l^t_3 l^t_2 l^t_1]$ is $L$-admissible and $[r^t_1 r^t_2 r^t_3 \ldots]$ is $R$-admissible for all $t > 0$ (possibly $L = R$). We say that $a^t$ has an $(L, R)$ defect particle of width $W$.

**Example 1.2:** (a) (ECA#184) Let $A = \{0, 1\}$. Let $\Phi_{184} : A^Z \to A^Z$ be elementary cellular automaton #184. (Recall: the number ‘184’ encodes the local rule $\phi : A^{[1,0,1]} \to A$ via the formula $\sum_{i=0}^1 \sum_{j=0}^1 \sum_{k=0}^1 \phi(i, j, k)(4i + 2j + k) = 184$.) Let $G^* \subseteq \mathbb{Z}$ be the Markov subshift given by digraph $\blacksquare \Rightarrow \square$ (we use the convention that $\blacksquare = 0$ and $\square = 1$, to ease comparison between equations and figures). Thus $G^* := \{(\blacksquare)^\infty, (\square)^\infty\}$, where $(\blacksquare)^\infty := [\ldots \blacksquare \square \square \ldots]$, etc. (the zeroth coordinate is underlined). Then $\Phi_{184} \big|_{G^*} = \sigma$, as shown in Figure 2(\text{*}). There are two $(G^*, G^*)$-defects of width 0, shown in
Fig. 2. (⋆) The periodic background generated by \( \Phi_{184} \) acting on \( G_e \) in Example 1.2(a). (\( \alpha^\pm, \beta, \omega^\pm, \gamma^\pm \)): Defect particles of \( \Phi_{184} \) acting on \( G \). (Note that the image labelled (\( \beta \)) actually depicts the coalescence of an \( \omega^+ \) and an \( \alpha^- \) into a \( \beta \).

Let \( B \) be unstable, and immediately 'decays' into a two other particles). Again, \( \alpha^- \)eral defect particles of \( \epsilon \) has three (automaton #54. Let \( B := B_0 \cup B_1 \), where \( B_0 \) is the four-element \( \sigma \)-orbit of 0010 and \( B_1 \) is the four-element \( \sigma \)-orbit of 1101. Then \( \Phi_{54}(B_0) = B_1 \) and \( \Phi_{54}(B_1) = B_0 \), so \( B \) is (\( \Phi_{54}, \sigma \))-transitive, so all defects have \( L = B = R \).
Table 1
Seven defect particles for ECA#184 acting on \( G \); see Example 1.2(a).

| \( W = 1 \) | \( W = 2 \) |
| --- | --- |
| \( \alpha^- \) | \( G_1 \) | \( G_0 \) | \( G_0 \) |
| \( \alpha^+ \) | \( G_1 \) | \( G_0 \) | \( G_0 \) |
| \( \omega^- \) | \( G_1 \) | \( G_0 \) | \( G_0 \) |
| \( \omega^+ \) | \( G_1 \) | \( G_0 \) | \( G_0 \) |
| \( \gamma^- \) | \( G_1 \) | \( G_0 \) | \( G_0 \) |
| \( \gamma^+ \) | \( G_1 \) | \( G_0 \) | \( G_0 \) |
| \( \beta \) | \( G_1 \) | \( G_0 \) | \( G_0 \) |

Table 1

Fig. 3. (*) The periodic background generated by \( \varepsilon \Phi_{54} \) acting on \( B \) in Example 1.2(b). \((\alpha, \beta, \gamma^\pm)\): Four defect particles of \( \varepsilon \Phi_{54} \) acting on \( B \). See also [Piv07b, Example 3.5(b)], [CH97, Fig.8], or [BNR91, §III(C)].

Also, \( \varepsilon \Phi_{54}^2|_B = \sigma^2 \) [see Figure 3(*)]. We recode \( B \) as a topological Markov subshift in the alphabet \( A^4 \), with admissible 4-words \( \mathcal{B} := \{\ldots, \ldots, \ldots, \ldots; \ldots, \ldots, \ldots, \ldots; \ldots, \ldots, \ldots, \ldots; \ldots, \ldots, \ldots, \ldots\} \). Figure 4 shows the \( \varepsilon \Phi_{54} \)-evolution of several defect particles in \( B \), along with the relevant values of \( z, R, L, \bar{V}, D, \) and \( \Upsilon \). See also Example 2.3(b).

(c) (ECA#110) Let \( A = \{0, 1\} \). Let \( E \subset A^\mathbb{Z} \) be the 14-element \( \sigma \)-orbit of the 14-periodic sequence \((\ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots)\). If \( \Phi_{110} \) is ECA#110, then \( \varepsilon \Phi_{110}|_E = \sigma^4 \) [see Figure 5(*)], so \( E \) is \( (\Phi_{110}, \sigma) \)-transitive, so all defects have \( L = E = R \). Figure 6 shows the \( \Phi_{110} \)-evolution of two defects (called ‘A’ and ‘B’ in the nomenclature of [Coo04]) along with the relevant values of \( z, R, L, \bar{V}, D, \) and \( \Upsilon \).

Remark 1.3: (a) \( \bar{V} \) takes values in \( \mathbb{V} := [-L-2 \ldots R+1] \) by Lemma 1.1(b). However, the average value of \( \bar{V} \) over time must be in \([-1, 1]\), because the left endpoint of the defect has a minimum velocity of \(-1\), while the right endpoint has a maximum velocity of \(+1\) [by Lemma 1.1(a)]. If \( \bar{V} < 1 \) (resp. \( \bar{V} > 1 \)), this means that the right (resp. left) endpoint is moving leftward (resp. rightward) at speed greater than 1, which means the defect particle is shrinking, which is only sustainable for a short period of time. For example, the particle can achieve an instantaneous velocity \( \bar{V} = R+1 \) only by shrinking from a defect of width \( W \) to one of width 0; it must later remain at velocity \( \bar{V} = 0 \) for \((R+1)\) iterations to grow back to width \( W \).
Fig. 4. Defects in ECA#54. We treat the symbol \( a_z \) as ‘defective’ if the word \((a_{z-2}, a_{z-1}, a_z, a_{z+1})\) is not \( \mathcal{B} \)-admissible. The admissible segments of each sequence are boxed; hence the unboxed segments are the defect words. The table on the right describes the values for \( z, R, L, \) and \( \vec{V} \) and the definition of \( \mathcal{D} \) in each case, as well as the relevant values of the update rule \( \Upsilon : \mathcal{A} \times \mathcal{D} \times \mathcal{A} \rightarrow \mathcal{D} \). See Example 1.2(b).

Fig. 5. (\(*\)) A 30 \times 30 image of the periodic spacetime diagram of \( \mathcal{A}_{110} \) acting on \( \mathcal{E} \) from Example 1.2(c); (A,B) 60 \times 60 images of the \( \mathcal{A}_{110} \)-evolution of two defect particles in \( \mathcal{E} \); See [Lin86,Mcl99a,Mcl99b,CHS01], [Ila01, §3.1.4.4], [Wol02, Chap.11], [Piv07b, Example 3.5(d)], and especially [Coo04].

The ‘constant width’ convention of eqn.(5) masks this shrinkage by ‘padding’ the defect word \( d^{t+1} \) with up to \( R+1 \) admissible characters from \( a^{t+1}_{z+1 \ldots z+1+R+2} \); this is why the function \( \Upsilon \) needs \( \vec{r}^t \) as input. Likewise, possibly rapid leftward
Fig. 6. The $A$ and $B$ defect particles of ECA#110. We treat the symbol $a_z$ as ‘defective’ if the word $(a_{z-6}, \ldots, a_z, a_{z+1}, \ldots, a_{z+7})$ is not $E$-admissible. The admissible segments of each sequence are boxed; hence the unboxed segments are the defect words. The table on the right describes the values for $z$, $R$, $L$, and $V$ in each case. The arrow path is the sequence $(z_t)_t$. The left-hand and right-hand polygonal paths are the sequences $(L_t)_t$ and $(R_t)_t$. See Example 1.2(c).

motion requires $Y$ to incorporate $I'$ as input. In most examples, however, the particle moves slowly, and we can reduce the number of boundary inputs.

(b) If $W = 0$, then $\mathcal{D} = \emptyset$ and $Y$ is trivial, while $\vec{V}$ is a function $\vec{V} : A \times A \rightarrow \{-1, 0, 1\}$.

(c) If $W \geq 1$, then by passing to the $W$th higher power representation [LM95, Defn.1.4.4], we can assume that $L = 0$ and $R = 1$, so that $W = 2$. To see this, replace $A$ with $\hat{A} := A^W$, and represent $a = [\ldots a_{-1} a_0 a_1 a_2 \ldots] \in A^Z$ by

$$\hat{a} := \begin{bmatrix} \ldots & a_{-2W} & a_{-W} & a_0 & a_W & a_{2W} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots & a_{-W-1} & a_{-1} & a_W & a_{2W-1} & a_{3W-1} & \ldots \end{bmatrix} = [\ldots \hat{a}_{-1} \hat{a}_0 \hat{a}_1 \ldots] \in \hat{A}^Z$$

(Note: this is not the same as the higher block recoding described earlier). Thus, if $a_t'$ is as in eqn.(5), and $\hat{z}_t := [z_t/W]$, then $\hat{a}_t' = [\ldots \hat{z}_t \hat{l}_1 \hat{d}_0 \hat{d}_1 \hat{n}_1 \hat{n}_2 \ldots]$, where $\hat{l}_n, \hat{n}_n \in S_W$ for all $n \in \mathbb{N}$, while $\hat{d}_0$ and $\hat{d}_1$ are in $\hat{A}$. The original defect word $d_t'$ is split between $\hat{a}_0$ and $\hat{a}_1$. The particle’s behaviour now depends only on its nearest neighbours, and its speed is never greater than 1. In other words, we have $\hat{Y} : \hat{A} \times \mathcal{D} \times \hat{A} \rightarrow \mathcal{D}$ and $\vec{V} : \hat{A} \times \mathcal{D} \times \hat{A} \rightarrow \{-1, 0, 1\}$ in eqn.(6). The
price of this manoeuvre is that each defect word \(d \in D\) can be represented in \(W\) distinct ways as a pair \((\hat{d}_{-1}, \hat{d}_0)\), depending upon the value of \(z_t \mod W\); this may translate into \(W\) spuriously distinct ‘particle types’ [see Definition 2.2 and Remark 3.5(a) below]. Also, it may make some dislocations look like interfaces [see Remark (d) below]. Nevertheless, it will be useful in the proofs of Proposition 4.1, Theorem 3.3 and Corollary 3.6.

(d) Two sequences \(b, c \in A^\mathbb{Z}\) are homoclinic if there is some \(N > 0\) such that \(b_z = c_z\) for all \(z \in \mathbb{Z}\) with \(|z| > N\). The defect in \(a^t\) is called removable if \(a^t\) is homoclinic to some \(s \in S\); see [Piv07b, §1]. Otherwise the defect in \(a^t\) is essential — i.e. it is impossible to remove the defect by changing \(a^t\) in some finite set. If \(L \neq R\) [e.g. Figure 2(β)] then an \((L, R)\)-defect is called an interface, and is necessarily an essential defect; see [Piv07b, §2]. If \(L = R\) [e.g. Figure 2(γ±)], then an \((L, R)\)-defect is called a dislocation, and may or may not be essential, depending upon whether it induces a ‘phase slip’ in the periodic structure of \(L\); see [Piv07b, Example 3.1].

The kinematics of a one-dimensional defect particle falls into several regimes summarized in Table 2, depending on the \((\sigma, \Phi)\)-dynamical complexity of \(L\) and \(R\). In the Ballistic regime (see §2), the defect acts as a finite automaton driven by periodic input, and moves with a constant average velocity through a periodic background. ECA’s #54, #62, #110, and #184 are all in this regime, which has been studied empirically in [Gra84b,Gra84a,BR91,BNR91,Han93,CH97,CHM98]. At the opposite extreme, in the Diffusive regime (see §3), the defect acts like a finite-state Markov process, and performs a generalized random walk. Diffusive defect dynamics has previously been analyzed by Eloranta [EN92,Elo93a,Elo93b,Elo94]. In the Turing regime (see §4), the defect moves through an inert, positive-entropy background, and modifies this background with its passing; the system acts like a Turing machine, where the particle is the ‘head’ and the inert background is the ‘tape’. In the Autonomous Pushdown Automaton regime (see

| Kinematic Regimes | \(\sigma\)-dynamics | \(\Phi\)-dynamics | Right Side \((\sigma, \Phi)\)-Dynamics |
|-------------------|-------------------|-----------------|-----------------------------------|
| \(\sigma\)-dynamics | \(\Phi\)-dynamics | \(\sigma\)-periodic | Right-regular | Nonzero \(\sigma\)-Entropy, Not \(\sigma\)-periodic |
| \(\Phi\)-dynamics | \(\Phi\)-Periodic or \(\Phi\)-Fixed | Diffusive (Theorem 3.3) | Markov Pushdown Automaton | Complicated |
| \(\Phi\)-dynamics | \(\Phi\)-Periodic or \(\Phi\)-Fixed | Autonomous Pushdown Automaton | Turing Machine (§4) | Complicated |
| \(\Phi\)-Regular | Left-regular | Complicated | Complicated |
| \(\Phi\)-Regular | Left-resolving | Complicated | Complicated |

Table 2
Kinematic regimes for one-dimensional defect particles.
any \( \sigma \)-entropy domain on one side (which we treat as a ‘stack’ memory), and a zero-entropy domain on the other side; the system acts like a pushdown automaton operating autonomously (i.e. without external input). In the Markov Pushdown Automaton regime (see §4), the defect has a \( \Phi \)-fixed, positive \( \sigma \)-entropy domain on one side (which we treat as a ‘stack’), and a \( \Phi \)-resolving subshift on the other; the system acts like a pushdown automaton driven by noise from a Markov process. The \textit{Complicated} regime is none of the above, and is probably too diverse to make any useful generalizations.

\section{The Ballistic Regime}

Let \( \Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z} \) be a cellular automaton, and let \( \mathcal{X} \subseteq \mathcal{A}^\mathbb{Z} \) be a \( \Phi \)-invariant Markov shift. Let \( L, R \subseteq \mathcal{X} \) be \( (\Phi, \sigma) \)-transitive subshifts of \( \mathcal{X} \), let \( W \in \mathbb{N} \), and let \( D^W_{L,R} \) be the set of all sequences in \( \mathcal{A}^\mathbb{Z} \) with a single \((L, R)\) defect particle of width \( W \), such as shown in eqn.(5). By hypothesis, \( \Phi(D^W_{L,R}) \subseteq D^W_{L,R} \).

**Theorem 2.1** Suppose \( L \) and \( R \) are \( \sigma \)-periodic and \( (\sigma, \Phi) \)-transitive. Then the dynamical system \((D^W_{L,R}, \Phi)\) is isomorphic to a dynamical system \((\mathcal{X} \times \mathbb{Z}, \Xi)\), where \( \mathcal{X} \) is a finite set, and where \( \Xi : \mathcal{X} \times \mathbb{Z} \rightarrow \mathcal{X} \times \mathbb{Z} \) is defined by \( \Xi(\mathcal{X}, z) := (\xi(\mathcal{X}), z + \vec{V}(\mathcal{X})) \) for some functions \( \xi : \mathcal{X} \rightarrow \mathcal{X} \) and \( \vec{V} : \mathcal{X} \rightarrow \mathcal{V} := [-L−2\ldots R+1] \).

**Proof idea:** The defect particle’s internal state is a finite automaton driven by a periodic input (because \( L \) and \( R \) are periodic); thus, by incorporating the phase of this periodic input into the state description of the defect, we can treat it as an autonomous finite automaton (i.e. a finite-state dynamical system) \((\mathcal{X}, \xi)\). The defect’s position is then obtained by integrating the velocity signal generated by \((\mathcal{X}, \xi)\).

**Proof:** Any \( \sigma \)-periodic sequence is automatically \( \Phi \)-periodic. Thus, by hypothesis, every element of \( L \) is \( \sigma^{P_L} \)-fixed and \( \Phi^{Q_L} \)-fixed for some \( P_L, Q_L \in \mathbb{N} \). But \( L \) is \((\Phi, \sigma)\)-transitive, so this means that \( L \) consists of a single finite \((\sigma, \Phi)\)-orbit containing exactly \( P_LQ_L \) elements. Recall that \( L^- \subseteq \mathcal{A}^{(-\infty, -1]} \) is the set of all left-infinite \( L \)-admissible sequences. Since \( L \) is a \( \sigma \)-periodic Markov shift, any element \([\ldots, \ell_1, \ell_2, \ell_3] \] in \( L^- \) is completely determined by the entry \( \ell_1 \). Hence there is a subset \( \mathcal{L} \subseteq \mathcal{A} \) with \#(\mathcal{L}) = \( P_LQ_L \), and a bijection \( \lambda : \mathcal{L} \rightarrow \mathcal{L}^- \) where, for any \( \ell \in \mathcal{L} \), \( \lambda(\ell) \) is the unique sequence \([\ldots, \ell_3, \ell_2, \ell_1] \) in \( \mathcal{L}^- \) with \( \ell_1 = \ell \). Furthermore, there are bijections \( \varphi_L : \mathcal{L} \rightarrow \mathcal{L} \) and \( \varsigma_L : \mathcal{L} \rightarrow \mathcal{L} \) such that \( \varphi_L \circ \lambda = \lambda \circ \varphi_L \) and \( \sigma \circ \lambda = \lambda \circ \varsigma_L \).

Likewise, \( R \) is \( \sigma^{P_R} \)-fixed and \( \Phi^{Q_R} \)-fixed (for some \( P_R, Q_R \in \mathbb{N} \)), so \( R \) has exactly \( P_RQ_R \) elements. Recall that \( R^+ \subseteq \mathcal{A}^{[1, \infty)} \) is the set of right-infinite \( R \)-admissible sequences. There is a subset \( \mathcal{R} \subseteq \mathcal{A} \) with \#(\mathcal{R}) = \( P_RQ_R \), and
a bijection $\rho : \mathcal{R} \to \mathbb{R}^+$ so that, for any $r \in \mathcal{R}$, $\rho(r)$ is the unique sequence $[r_1, r_2, r_3, \ldots]$ in $\mathbb{R}^+$ with $r_1 = r$. There are bijections $\varphi_R : \mathcal{R} \to \mathcal{R}$ and $\varsigma_R : \mathcal{R} \to \mathcal{R}$ such that $\Phi \circ \rho = \rho \circ \varphi_R$ and $\sigma \circ \rho = \rho \circ \varsigma_R$.

Thus, the sequence $a'$ in eqn.(5) is entirely determined by the data $(l'_1, d'_1, r'_1) \in \mathcal{L} \times \mathcal{D} \times \mathcal{R} \times \mathbb{Z}$, because $[\ldots l'_2, l'_3, l'_4] = \lambda(l'_1)$, and $\rho(r'_1) = [r'_1, r'_2, r'_3, \ldots]$. Define $\Psi : \mathcal{L} \times \mathcal{D} \times \mathcal{R} \times \mathbb{Z} \to \mathcal{D}_{l, r}^L$ by $\Psi(l, d, r, z) := [l \ d \ r]$, where $l := \lambda(l) \in \mathbb{L}^r$, $r := \rho(r) \in \mathbb{R}^+$, and we place $d$ so that its center coordinate is at $z$. Then $\Psi$ is a bijection.

Let $\mathcal{X} := \mathcal{L} \times \mathcal{D} \times \mathcal{R}$. If $\Upsilon$ and $\tilde{V}$ are as in eqn.(6), then we can restrict them to functions $\Upsilon_1 : \mathcal{X} \to \mathcal{D}$ and $\tilde{v} := \tilde{V}_1 : \mathcal{X} \to \mathbb{V}$. Define $\xi : \mathcal{X} \to \mathcal{X}$ by $\xi(l, d, r) := (l', d', r')$, where

$$l' := \varsigma_L^* \circ \varphi_L(l), \quad d' := \Upsilon(l, d, r), \quad \text{and} \quad r' := \varsigma_R^* \circ \varphi_R(r),$$

where $v := \tilde{V}(l, d, r)$.

Now define $\Xi : \mathcal{X} \times \mathbb{Z} \to \mathcal{X} \times \mathbb{Z}$ as in the theorem statement. Then $\Phi \circ \Psi = \Psi \circ \Xi$. \hfill \Box

**Definition 2.2:** The (finite) dynamical system $(\mathcal{X}, \xi)$ decomposes into a finite disjoint union of finite $\xi$-orbits, called **particle types**. If $\mathcal{P} \subset \mathcal{X}$ is a particle type, then $P := \#(\mathcal{P})$ is the **period** of type $\mathcal{P}$, and $\tilde{V}(\mathcal{P}) := \frac{1}{P} \sum_{P \in \mathcal{P}} \tilde{V}(P)$ is the **average velocity** of type $\mathcal{P}$. \hfill \diamond

**Example 2.3:** (a) (ECA#184) We continue Example 1.2(a). There are three $(\sigma, \Phi_{184})$-transitive components in $G$, so there are three possible choices for $R$; for each one, we list the corresponding values of $\mathcal{R}, P_R, Q_R, \rho, \varsigma_R$, and $\varphi_R$ in Table 3(A) (the values for $\mathcal{L}, P_L, Q_L, \lambda, \varsigma_L$, and $\varphi_L$ would be exactly the same). In Example 1.2(a), we introduced seven defect particles for $(G, \Phi)$: four of width 1, and three of width 2. In all seven cases, we have $\Upsilon(l, d, r) = d$. Thus, $\xi : \mathcal{L} \times \mathcal{D} \times \mathcal{R} \to \mathcal{L} \times \mathcal{D} \times \mathcal{R}$ is given by $\xi(l, d, r) = (\varsigma_L^* \circ \varphi_L(l), d, \varsigma_R^* \circ \varphi_R(r))$, where $v := \tilde{V}(l, d, r) \in \{-1, 0, 1\}$. The value of $\tilde{V}$ is constant for each particle type, and was shown in the bottom row of Table 1. In all cases, we end up with $\xi = Id$, so all particle types have period 1. Hence, the average velocity of each type is just the value of $\tilde{V}$ on the (unique) member of that type.

(b) (ECA#54) We continue Example 1.2(b). In this case, $L = R = B$, and $L = \mathcal{R} = B := \{\ldots, \ldots, \ldots, \ldots; \ldots, \ldots, \ldots\}$. 

Table 3
Ballistic defects in ECA#184; See Example 2.3(a)

|     | $P_R$ | $Q_R$ | $\mathcal{R}$ | $\rho : \mathcal{R} \to \mathbb{R}$ | $\varsigma_R$ | $\varphi_R$ |
|-----|-------|-------|---------------|------------------------------------|---------------|------------|
| $G_0$ | 1     | 1     | $\{0\}$     | $\rho(0) = (000...)$               | $Id$          | $Id$       |
| $G_1$ | 1     | 1     | $\{1\}$     | $\rho(1) = (111...)$               | $Id$          | $Id$       |
| $G_2$ | 2     | 2     | $\{0, 1\}$  | $\rho(0) = (0101...)$              | $\varsigma_R(0) = 1$ | $\varphi_R(0) = 1$ |
|      |       |       |              | $\rho(1) = (1010...)$              | $\varsigma_R(1) = 0$ | $\varphi_R(1) = 0$ |
The maps \( \varphi_L = \varphi_R \) and \( \varsigma_L = \varsigma_R \) are defined as:

\[
\begin{align*}
\begin{array}{cccccccccccc}
\ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \ldots \\
\ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \ldots \\
\ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \ldots \\
\ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \ldots \\
\ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \ldots \\
\end{array}
\end{align*}
\]

Consider the \( \gamma^\pm \) defects in Figure 4. In this case, \( D = A = \{0, 1\} \), so \( \mathcal{X} = B \times A \times B \). The \( \gamma^\pm \) particle types correspond to 2-periodic orbit classes \( \Gamma^+ \) and \( \Gamma^- \) of the dynamical system \( \xi : \mathcal{X} \to \mathcal{X} \), where

\[
\Gamma^+ := \{ (\ldots, 0, 1, 0), (0, 1, 0, 1, 0) \} \\
\text{and} \quad \Gamma^- := \{ (\ldots, 0, 1, 0), (0, 1, 0, 1, 0) \}.
\]

Note that \( d = d_0 \) is always \( \Box \). Also, \( \tilde{V} \equiv +1 \) on \( \Gamma^+ \), so that \( \xi(f, \Box, r) = (\varsigma \circ \varphi(f), \Box, \varsigma \circ \varphi(r)) \) for both \( (f, \Box, r) \in \Gamma^+ \). Likewise, \( \tilde{V} \equiv -1 \) on \( \Gamma^- \), so that \( \xi(f, \Box, r) = (\varsigma^{-1} \circ \varphi(f), \Box, \varsigma^{-1} \circ \varphi(r)) \) for both \( (f, \Box, r) \in \Gamma^- \).

**Remark:** Theorem 2.1 can easily be generalized to defect particles in \( \mathbb{Z}^D \), where \( X \subset \mathcal{A}^D \) is a transitive, \( \sigma \)-periodic subshift of finite type. However, if \( D \geq 2 \), then such particles cannot be essential defects [see Remark 1.3(d)], because if \( X \) is \( \sigma \)-periodic and \( D \geq 2 \), then any finite defect in \( X \) is removable. Defect particles may still be \( \Phi \)-persistent, however. The most familiar examples of removable, yet persistent, ballistic defect particles are the ‘gliders’ and ‘oscillators’ of Conway’s *Game of Life* [Epp02,Got03,BCG04] and its variants [Bay87,Bay88,Bay90,Bay91,Bay92,Bay94,Eva96,Eva01,Eva03a,Eva03b,Eva05].

\( \Box \)

### 3 The Diffusive Regime

Under certain conditions, a defect particle performs a generalized random walk. To demonstrate this, we first review some elementary probability theory.

**Bernoulli Measures and (hidden) Markov Measures:** Let \( A \) be a discrete set (finite or countable), and let \( \mathcal{M}(\mathcal{A}^\mathbb{N}) \) be the set of Borel probability measures on \( \mathcal{A}^\mathbb{N} \). If \( \mu \in \mathcal{M}(\mathcal{A}^\mathbb{N}) \), then \( \mu \) is \( \sigma \)-**invariant** if \( \sigma(\mu) = \mu \), where \( \sigma(\mu) \in \mathcal{M}(\mathcal{A}^\mathbb{N}) \) is defined by \( \sigma(\mu)[\mathcal{B}] := \mu(\sigma^{-1}(\mathcal{B})) \) for any Borel subset \( \mathcal{B} \subset \mathcal{A}^\mathbb{N} \). The measure-preserving dynamical system \( (\mathcal{A}^\mathbb{N}, \mu, \sigma) \) is then called a **stationary stochastic process**. For any \( m, n \in \mathbb{N} \) and any \( c \in \mathcal{A}^{[0..m]} \), let \( [c]_n := \{ a \in \mathcal{A}^\mathbb{N} : a_{[n..n+m]} = c \} \) be the **cylinder set** defined by \( c \) at position \( n \). Clearly, \( \mu \) is \( \sigma \)-invariant iff \( \mu([c]_n) = \mu([c]_0) \) for all \( m, n \in \mathbb{N} \) and \( c \in \mathcal{A}^{[0..m]} \). Thus, we write “\( \mu[c] \)” to mean \( \mu([c]_0) \). We call \( \mu \) a **Bernoulli measure** if there is a measure \( \mu^0 \in \mathcal{M}(\mathcal{A}) \) (the ‘one-point marginal’ of \( \mu \)) such that, for any \( n \in \mathbb{N} \)
for any \( c \in \mathcal{A}^{[0..m]} \), \( \mu[c] = \mu^0(c_0)\mu^0(c_1) \cdots \mu^0(c_n) \). For example, if \( \#(A) = A \), then the uniform measure is the Bernoulli measure \( \eta \) with \( \eta^0(a) = \frac{1}{A} \) for all \( a \in A \); hence \( \eta(c) = \frac{1}{A^{[0..m]}} \) for all \( c \in \mathcal{A}^{[0..m]} \).

A measure \( \mu \in \mathcal{M}(\mathcal{A}^\mathbb{N}) \) is a Markov measure if there is a measure \( \mu^0 \in \mathcal{M}(\mathcal{A}) \) and a transition probability function \( \tau : \mathcal{A} \rightarrow \mathcal{M}(\mathcal{A}) \) such that, for any \( c \in \mathcal{A}^{[0..m]} \), \( \mu([c]) = \mu^0(c_0)\tau(c_0, c_1)\tau(c_1, c_2) \cdots \tau(c_{n-1}, c_n) \); see [Kit98, §6.2], [LM95, §2.3] or [Var01, §4.4]. In this case, \( \mu \) is \( \sigma \)-invariant iff \( \mu^0 \) is stationary, meaning that \( \mu^0(b) = \sum_{a \in \mathcal{A}} \mu^0(a) \cdot \tau(a, b) \) for all \( b \in \mathcal{A} \). For example, any Bernoulli measure is a \( \sigma \)-invariant Markov measure, with \( \tau(a, b) = \mu^0(b) \) for all \( a, b \in \mathcal{A} \). If \( \mathcal{B} \) is another set, and \( \psi : \mathcal{A} \rightarrow \mathcal{B} \) is any function, we define \( \psi^\mathcal{N} : \mathcal{A}^\mathbb{N} \rightarrow \mathcal{B}^\mathbb{N} \) by \( \psi^\mathcal{N}(a_1, a_2, \ldots) := (\psi(a_1), \psi(a_2), \ldots) \). We say \( \nu \in \mathcal{M}(\mathcal{B}^\mathbb{N}) \) is a hidden Markov measure if \( \nu = \psi^\mathcal{N}(\mu) \), for some Markov measure \( \mu \in \mathcal{M}(\mathcal{A}^\mathbb{N}) \) and function \( \psi : \mathcal{A} \rightarrow \mathcal{B} \). Bernoulli/Markov measures on \( \mathcal{A}^\mathbb{Z} \) are defined analogously.

Random Walks: Let \( \mathcal{V} \subseteq \mathbb{Z} \), and let \( \nu \in \mathcal{M}(\mathcal{V}^\mathbb{N}) \) be a hidden Markov measure. Define \( \Sigma : \mathcal{V}^\mathbb{N} \rightarrow \mathcal{V}^\mathbb{N} \) by \( \Sigma(v_1, v_2, v_3, \ldots) := (0, v_1, v_1 + v_2, v_1 + v_2 + v_3, \ldots) \). The probability measure \( \omega := \Sigma(\nu) \in \mathcal{M}(\mathcal{Z}^\mathbb{N}) \) is called a (generalized) random walk, with increment process \( \nu \). For example, the one-dimensional Simple Random Walk (SRW) is obtained by setting \( \mathcal{V} := \{-1, 1\} \) and letting \( \nu \) be the Bernoulli measure with \( \nu[\pm 1] = \frac{1}{2} \); see [Var01, Example 4.1].

Resolving subshifts: Let \( \mathcal{B} \subseteq \mathcal{A} \), and let \( \mathcal{S} \subseteq \mathcal{B}^\mathbb{Z} \subseteq \mathcal{A}^\mathbb{Z} \) be a Markov subshift. For any \( b \in \mathcal{B} \), let \( \mathcal{P}_\mathcal{S}(b) := \{ a \in \mathcal{B} \; ; \; (a, b) \in \mathcal{S}_2 \} \) be the predecessor set of \( b \), and let \( \mathcal{F}_\mathcal{S}(b) := \{ c \in \mathcal{B} \; ; \; (b, c) \in \mathcal{S}_2 \} \) be the follower set of \( b \). We say that \( \mathcal{S} \) is left-regular if there is some constant \( P_\mathcal{S} \in \mathbb{N} \) such that \( \#(\mathcal{P}_\mathcal{S}(b)) = P_\mathcal{S} \) for all \( b \in \mathcal{B} \). Likewise \( \mathcal{S} \) is right-regular if there is some constant \( F_\mathcal{S} \in \mathbb{N} \) such that \( \#(\mathcal{F}_\mathcal{S}(b)) = F_\mathcal{S} \) for all \( b \in \mathcal{B} \).

The Parry measure \( \eta \in \mathcal{M}(\mathcal{S}) \) is the measure of maximal \( \sigma \)-entropy on \( \mathcal{S} \), and is a Markov measure on \( \mathcal{S} \) which assigns roughly equal probability to all \( \mathcal{S} \)-admissible paths of any given length; see [Par64, Thm.10], [LM95, §13.3], or [Kit98, Thm.6.2.20]. If \( \mathcal{S} \) is left- or right-regular, then \( \eta^0 \) is the uniform measure on \( \mathcal{B} \). If \( \mathcal{S} \) is right-regular, then \( \tau(\mathcal{b}, \bullet) \) is the uniform measure on \( \mathcal{F}(\mathcal{b}) \) for every \( \mathcal{b} \in \mathcal{B} \); that is, \( \tau(\mathcal{b}, c) = 1/F_\mathcal{S} \) for all \( c \in \mathcal{F}(\mathcal{b}) \). Likewise, if \( \mathcal{S} \) is left-regular, and we define the ‘backwards’ transition probability \( \hat{\tau} : \mathcal{A} \rightarrow \mathcal{M}(\mathcal{A}) \) by \( \hat{\tau}(a, b) = \eta[ab]/\eta^0(b) \), then \( \hat{\tau}(\bullet, \mathcal{b}) \) is the uniform measure on \( \mathcal{P}(\mathcal{b}) \) for every \( \mathcal{b} \in \mathcal{B} \); that is \( \hat{\tau}(a, \mathcal{b}) = 1/P_\mathcal{S} \) for all \( a \in \mathcal{P}(\mathcal{b}) \).

Let \( \Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z} \) be a CA with \( \Phi(\mathcal{S}) \subseteq \mathcal{S} \). Suppose \( \Phi \) has local rule \( \phi : \mathcal{A}^{(-1,0,1)} \rightarrow \mathcal{A} \). Then \( \mathcal{S} \) is a left-resolving subshift for \( \Phi \) if, for any fixed \( (b, c, a) \in \mathcal{S}_3 \), with \( e := \phi(b, c, a) \), the function \( \mathcal{P}(\mathcal{b}) \ni a \mapsto \phi(a, b, c) \in \mathcal{P}(e) \) is injective [LM95, Defn.8.1.7]. If \( \mathcal{S} \) is left-regular, then ‘injective’ implies ‘bijective’. In this case, for any \( (b, c) \in \mathcal{S}_2 \), define \( \phi(\mathcal{S}, b, c) := \{ \phi(a, b, c) \; ; \; a \in \mathcal{P}(\mathcal{b}) \} \); then
[# [φ(S, b, c)] = P_S. Likewise, S is a right-resolving subshift for Φ if, for any fixed (a, b, c) ∈ S₃ with e := φ(a, b, c), the function F_S(e) ⊇ d → φ(b, c, d) ∈ F_S(e) is injective. If S is right-regular, then ‘injective’ implies ‘bijective’. In this case, for any (b, c) ∈ S₂, define φ(b, c, S) := {φ(b, c, d) ; d ∈ F(c)}; then # [φ(b, c, S)] = F_S. If S is either left- or right-resolving for Φ, then Φ(S) = S, which implies that the Parry measure on S is Φ-invariant [Kit98, Thm.6.2.21].

Example 3.1: (a) Let B ⊆ A and let S := B^Z. Then B^Z is left- and right-regular, because P_S(b) = B = F_S(b) for all b ∈ B. If Φ : A^Z → A^Z and Φ(B^Z) ⊆ B^Z, then B^Z is left-resolving for Φ iff Φ is left-permutative on B, i.e. for any (b, c) ∈ B^2, the function B ⊇ a → φ(a, b, c) ∈ B is bijective [Hed69]. Likewise, B^Z is right-resolving iff Φ is right-permutative on B, i.e. for any (a, b) ∈ B^2, the function B ⊇ c → φ(a, b, c) ∈ B is bijective. The Parry measure is the uniform measure on B^Z, and is preserved by any permutative cellular automaton. In the terminology of [Elo93a,Elo93b], B is called a permutive subalphabet for Φ.

(b) For example, let (B, +) be a finite abelian group and let S := B^Z. Then Φ is a linear cellular automaton on B^Z if there are endomorphisms φ₋₁, φ₀, φ₁ ∈ End (B) such that, for all b₋₁, b₀, b₁ ∈ B, we have φ(b₋₁, b₀, b₁) = φ₋₁(b₋₁) + φ₀(b₀) + φ₁(b₁). In this case, Φ is left- (resp. right-) permutative on B iff φ₋₁ (resp. φ₁) is an automorphism of B. (Note that we do not require that A be a group, or that Φ be linear on the rest of A^Z.) Under pointwise addition, B^Z is a compact abelian group, and the Parry measure (the uniform measure) is the Haar measure on B^Z.

(c) In particular, if B = Z/n for some n ∈ N, then Φ is linear if there are constants φ₋₁, φ₀, φ₁ ∈ Z such that φ(b₋₁, b₀, b₁) = (φ₋₁ b₋₁ + φ₀ b₀ + φ₁ b₁) mod n. In this case, Φ is left- (resp. right-) permutative on B iff φ₋₁ (resp. φ₁) is relatively prime to n. The Haar measure η on B^Z is the ‘natural’ invariant measure for such linear CA. For example, if μ ∈ M(B^Z) is any measure satisfying broad conditions (e.g. an N-step Markov measure with full support), then Φ asymptotically randomizes μ, meaning that \( \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \Phi^n(\mu) = \eta \) in the weak* topology on M(B^Z); see [Lin84,MM98,MM99,PY02,PY04,PY06]. Furthermore, if n is prime, then η is the only Φ-invariant, σ-ergodic measure with positive entropy [HMM03, Thm. 12]; for some generalizations and related results, see also [Piv05,Sab07].

(d) If (B, +) is a finite group, then a Markov subgroup is a Markov subshift S ⊂ B^Z which is also a subgroup of (B^Z, +); see [Kit87,KSS88a,KSS89] and [Kit98, §6.3]. It follows that F_S(0) and P_S(0) are subgroups of B (see [Kit87, Prop.3(ii)] or [Kit98, Lem.6.3.4(ii,iii)]). Furthermore, S is left- and right- regular, because for any b ∈ B, F_S(b) is a coset of F_S(0), and P_S(b) is a coset of P_S(0) (see [Kit87, Prop.3(iii)] or [Kit98, Lem.6.3.4(iv)]). The Parry measure of S is then the Haar measure on S as a compact group.

If Φ is a linear CA with local rule φ(b₋₁, b₀, b₁) = φ₋₁ b₋₁ + φ₀ b₀ + φ₁ b₁ for
Let $\Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$ be a cellular automaton. A resolving system for $\Phi$ is a quadruple $(L, R; \lambda, \rho)$, where:

1. $L, R \subset \mathcal{A}^\mathbb{Z}$ are Markov subshifts, and $L \cup R$ is also a Markov subshift.
2. $L$ is left-regular, $\Phi(L) \subseteq L$, and $L$ is left-resolving for $\Phi$.
3. $R$ is right-regular, $\Phi(R) \subseteq R$, and $R$ is right-resolving for $\Phi$.
4. $\lambda \in \mathcal{M}(L)$ is the Parry measure on $L$, and $\rho \in \mathcal{M}(R)$ is the Parry measure on $R$.

Example 3.2: If $L, R \subset \mathcal{A}$, then $A := L^\mathbb{Z} \cup R^\mathbb{Z}$ is a subshift of finite type iff either $L = R$ or they are disjoint. In this case, $(L^\mathbb{Z}, R^\mathbb{Z}; \lambda, \rho)$ is a resolving system for $\Phi$ iff:

[i] $\lambda$ (resp. $\rho$) is the uniform measure on $L^\mathbb{Z}$ (resp. $R^\mathbb{Z}$);
[ii] $\Phi(L^\mathbb{Z}) = L^\mathbb{Z}$ and $\Phi$ is left-permutative on $L$; and
[iii] $\Phi(R^\mathbb{Z}) = R^\mathbb{Z}$ and $\Phi$ is right-permutative on $R$, as in Example 3.1(a).

Let $(L, R; \lambda, \rho)$ be a resolving system, let $A := L \cup R$, and let $D_{L,R}^{W,0}$ be the set of all elements in $\mathcal{A}^\mathbb{Z}$ with a single $(L, R)$-defect of width $W$ at zero. Let $D := A^W$. If $\delta \in \mathcal{M}(D)$, then we regard $\lambda \otimes \delta \otimes \rho$ as a probability measure on $D_{L,R}^{W,0}$ in the obvious way. Define $\zeta : D_{L,R}^{W} \rightarrow \mathbb{Z}^\mathbb{N}$ by $\zeta(a) := (z_0, z_1, z_2, \ldots)$, where, for all $t \in \mathbb{N}$, $z_t = z_t(a) \in \mathbb{Z}$ is as in eqn.(5). In other words, $\zeta(a)$ tracks the trajectory of the defect particle over time. If $\mu \in \mathcal{M}(D_{L,R}^{W,0})$, then $\zeta(\mu)$ is a probability measure on $\mathbb{Z}^\mathbb{N}$. The main result of this section is:

Theorem 3.3 Let $\Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$ be a CA, and let $(L, R; \lambda, \rho)$ be a resolving system for $\Phi$. Let $W \in \mathbb{N}$, let $\delta$ be any probability measure on $D := A^W$, and let $\mu := \lambda \otimes \delta \otimes \rho \in \mathcal{M}(D_{L,R}^{W,0})$. Then $\omega := \zeta(\mu) \in \mathcal{M}(\mathbb{Z}^\mathbb{N})$ is a random walk.

Example 3.4: Let $A := \mathbb{Z}/2 \times \{\circ, \bullet\}$. Define $\phi : A^{-1,0,1} \rightarrow A$ by $\phi\left(\begin{array}{c} a \ \ b \ \ c \\ \delta \end{array}\right) := \left(\begin{array}{c} a \ \ b \ \ c \\ \delta \end{array}\right)$, where
measure on \( \lambda \) permutative subalphabet for \( \Phi \) [see Example 3.1(c)]; if 
\[
\begin{cases}
  a_{-1} + a_0 + a_1 & \text{if } b_{-1} = b_0 = b_1 = \circ; \\
  a_{-1} + a_0 & \text{if } b_{-1} = b_0 = \circ \text{ and } b_1 = \bullet; \\
  a_0 + a_1 & \text{if } b_{-1} = \bullet \text{ and } b_0 = b_1 = \circ; \\
  1 - a_0 & \text{if } b_0 = \bullet.
\end{cases}
\]

\[
\begin{cases}
  \bullet & \text{if } b_{-1} = \bullet \text{ and } a_{-1} = a_0 = 0; \\
  \bullet & \text{if } b_1 = \bullet \text{ and } a_0 = a_1 = 1; \\
  \circ & \text{if } b_0 = \bullet \text{ and } a_0 = a_1 = 0; \\
  \circ & \text{if } b_0 = \bullet \text{ and } a_{-1} = a_0 = 1; \\
  b_0 & \text{otherwise.}
\end{cases}
\]

Let \( \mathcal{L} = \mathcal{R} = \mathbb{Z}/2 \times \{\circ\} \), which we identify with \( \mathbb{Z}/2 \); then \( \Phi \) acts on \( \mathbf{L} := \mathcal{L}^\mathbb{Z} = \mathcal{R}^\mathbb{Z} =: \mathbf{R} \) like the linear cellular automaton \( \Psi : (\mathbb{Z}/2)^\mathbb{Z} \rightarrow (\mathbb{Z}/2)^\mathbb{Z} \) with local rule \( \psi(x_{-1}, x_0, x_1) := x_{-1} + x_0 + x_1 \mod 2 \). Thus, \( \mathcal{L} = \mathcal{R} \) is a left- and right-permutative subalphabet for \( \Phi \) [see Example 3.1(c)]; if \( \lambda = \rho \) is the uniform measure on \( \mathbf{L} = \mathbf{R} \), then \( (\mathbf{L}, \mathbf{R}; \lambda, \rho) \) is a resolving system, as in Example 3.2. The set \( \mathbb{Z}/2 \times \{\bullet\} \) is the set of defect states. An element of \( \mathbf{D}_{L,R}^1 \) has the form \( \ldots \hat{\bullet} \hat{\circ} \hat{\circ} \hat{\bullet} \hat{\circ} \hat{\circ} \hat{\bullet} \hat{\bullet} \hat{\circ} \hat{\circ} \hat{\circ} \hat{\bullet} \ldots \) , where \( \ell, n, d_0 \in \mathbb{Z}/2 \). The defect particle \( \bullet \) moves left if \( \ell = d_0 = 1 \), and moves right if \( d_0 = n = 0 \); otherwise it remains stationary. Figure 7(A) shows a close-up spacetime diagram of the resulting random walk, while Figure 7(B) shows a large-scale spacetime diagram of the same walk.

If \( \mu \in \mathcal{M}(\mathcal{A}^\mathbb{N}) \), then we write, “For \( \forall \mu \in \mathcal{A}^\mathbb{N} \), [statement],” or “[statement], (\( \mu\)-ae),” to mean “\( \mu\{a \in \mathcal{A}^\mathbb{N}; [\text{statement}] \text{ is true}\} = 1 \)”. If \( \mathbb{I} \subset \mathbb{N} \), let \( \text{pr}_\mathbb{I} : \mathcal{A}^\mathbb{N} \rightarrow \mathcal{A}^\mathbb{I} \) be the projection map. Thus, if \( \mu \in \mathcal{M}(\mathcal{A}^\mathbb{N}) \), then \( \text{pr}_\mathbb{I}(\mu) \in \mathcal{M}(\mathcal{A}^\mathbb{I}) \).
Fig. 8. A schematic spacetime diagram illustrating the sigma algebras $\mathcal{G}\{t\} \subseteq \mathcal{G}[0...t] \subseteq \mathcal{G}^*$ in Theorem 3.3.

If $J \subset \mathbb{N}$ and $b \in \mathcal{A}^j$, then let $\mu_{|b}$ be the conditional probability measure on $\mathcal{A}^N$ given $b$; in other words, for any $U \subset \mathcal{A}^N$, $\mu_{|b}(U) := \mu(U \cap [b])/\mu([b])$, where $[b]$ is the cylinder set defined by $b$. More generally, if $U \subset \mathcal{A}^N$ is Borel-measurable, and if $\mathcal{G}$ is a $\sigma$-subalgebra of the Borel $\sigma$-algebra on $\mathcal{A}^N$, then let $\mu_{|\mathcal{G}}(U)$ be the conditional probability function of $U$ given $\mathcal{G}$; i.e. $\mu_{|\mathcal{G}}(U)$ is the $\mathcal{G}$-measurable function such that, for any $S \in \mathcal{G}$, $\int_S \mu_{|\mathcal{G}}(U) d\mu = \mu(S \cap U)$.

This function is uniquely defined ($\mu$-æ); see e.g. [Var01, §4.3].

In particular, if $J \subset \mathbb{N}$, let $\mathcal{G}(J)$ be the sigma-algebra generated by all cylinder sets $[c]_j$, where $c \in \mathcal{A}$ and $j \in J$ (hence $\mathcal{G}(\mathbb{N})$ is the Borel sigma-algebra of $\mathcal{A}^N$). A $\sigma$-invariant $\mu \in \mathcal{M}(\mathcal{A}^N)$ is Bernoulli iff, for any disjoint subsets $I, J \subset \mathbb{N}$, and any $b \in \mathcal{A}^I$, $\mu_{|\mathcal{G}(I)}(b) \equiv \mu(b)$ ($\mu$-æ). We say $\mu$ is Markovian iff for any $m \in \mathbb{N}$ and $b \in \mathcal{A}$, $\mu_{|\mathcal{G}[0...m]}([b]_{m+1}) = \mu_{|\mathcal{G}(m)}([b]_{m+1})$. Thus, $\mu$ is a Markov measure if $\mu$ is Markovian and if, furthermore, for any $a \in \mathcal{A}$ and for $\forall \mu$, $x \in [a]_m$, we have $\mu_{|\mathcal{G}(m)}([b]_{m+1})(x) \equiv \tau(a, b)$.

Proof idea for Theorem 3.3: The left-hand measure $\lambda$ and right-hand measure $\rho$ provide a continual influx of ‘random noise’. The ‘$\lambda$-noise’ propagates rightwards with unit speed because $L$ is left-resolving for $\Phi$, whereas the ‘$\rho$-noise’ propagates leftwards with unit speed because $R$ is right-resolving for $\Phi$. As shown in Figure 8, the defect particle’s trajectory from time 0 to time $t$ is entirely determined by the information contained inside of a backwards ‘lightcone’ emanating from its position at time $t$ back to the initial state at time zero. If the particle steps to the left [respectively, right] at time $t$, then it must step into the path of incoming $\lambda$-noise [respectively, $\rho$-noise] which is outside of this lightcone, and hence, statistically independent of the particle’s previ-
ous trajectory; see Figure 9(B) [respectively, Figure 9(C)]. If the particle stays
put at time \( t \), then it is exposed to both fresh \( \lambda \)-noise and fresh \( \rho \)-noise; see
Figure 9(A). In all three cases, the particle is subjected to fresh perturbations
at time \( t + 1 \) which are statistically independent of its previous behaviour.
Furthermore, \( \lambda \) and \( \rho \) are \( \Phi \)-invariant, so the probability distribution of these
perturbations is constant over time; hence they can be treated as a stationary
Markov process, which drives the particle’s motion.

\[ \text{Proof of Theorem 3.3:} \]

Let \( \hat{A} := A^W \), let \( \hat{\Phi} : \hat{A}^2 \to \hat{A}^2 \) be the \( \hat{W} \)-th-power
representation of \( \Phi \); and let \( \hat{L}, \hat{R} \subset \hat{A}^2 \) be the \( \hat{W} \)-th-power representations
of \( L \) and \( R \). Then \( \hat{L} \) (resp. \( \hat{R} \)) is still left-regular (resp. right-regular) and is
still left-resolving (resp. right-resolving) for \( \hat{\Phi} \). Thus, we can replace \( A \) with
\( \hat{A} \), \( L \) with \( \hat{L} \), and \( R \) with \( \hat{R} \) and proceed. By Remark 1.3(c), we can thus
assume that \( W = 2 \) and that \( \hat{Y} : L_1 \times A^2 \times R_1 \	o \hat{A}^2 \) and \( \hat{V} : L_1 \times A^2 \times
R_1 \	o \{-1, 0, 1\} \). A generic element of \( D_{LR}^2 \) has the form

\[ a = [\ldots l_3 l_2 l_1 d_0 d_1 r_1 r_2 r_3 \ldots], \]

where \( l_n := a_{-n} \in L_1 \) and \( r_n := a_{n+1} \in R_1 \) for all \( n \in \mathbb{N} \), while \( d_i := a_{i+1} \in
A \) for \( i = 0, 1 \), with \( z \in \mathbb{Z} \) being the location of the defect.

Let \( \mathcal{X} := L_2 \times A^2 \times R_2 \), and define \( \xi : D_{LR}^2 \to \mathcal{X} \) so that, if \( a \) is as above,
then \( \xi(a) := (l_2, l_1, d_0, d_1; r_1, r_2) \). For any \( t \in \mathbb{N} \), let \( \xi_t := \xi \circ \Phi^t \). In other words,
\( \xi_t(a) := (l_t^t, l_1^t, d_0^t, d_1^t; r_1^t, r_2^t) \), where \( \Phi(a) = [\ldots l_t^t l_2^t l_1^t d_0^t d_1^t r_1^t r_2^t r_3^t \ldots] \).
Next, define \( \Xi : D_{LR}^{2,0} \to \mathcal{X}^N \) by \( \Xi(a) := (\xi_0(a), \xi_1(a), \xi_2(a), \ldots) \). Clearly,
\( \Xi \circ \Phi = \sigma \circ \Xi \). Recall that \( \hat{V} \) is a function from \( L_1 \times A^2 \times R_1 \) into \( \{-1, 0, 1\} \);
treat this as a function \( \hat{V} : \mathcal{X} \to \{-1, 0, 1\} \), and apply it coordinatewise to
define \( \hat{Y} : \mathcal{X} \to \{-1, 0, 1\}^N \).

If \( \overline{\mu} := \Xi(\mu) \in \mathcal{M}(\mathcal{X}^N) \), and \( \nu := \hat{Y}(\overline{\mu}) \in \mathcal{M}(\{-1, 0, 1\}^N) \), then \( \omega = \Sigma(\nu) \).
Hence, if \( \overline{\mu} \) is Markov, then \( \nu \) is hidden Markov, so that \( \omega \) is a random walk,
as desired. It remains to show that \( \overline{\mu} \) is a Markov measure.

Fix \( t \in \mathbb{N} \). Let \( \mathcal{S}(t) \) be the sigma-algebra on \( \hat{A}^2 \) generated by \( \xi_t \), and
let \( \mathcal{S}[0..t] \) be the sigma-algebra on \( \hat{A}^2 \) generated by \( (\xi_0, \xi_1, \ldots, \xi_t) \). For any
\( \chi \in \mathcal{X} \), let \( U_{t+1}^\chi := \xi_{t+1} \{\chi\} \) be the set of all initial conditions in \( D_{LR}^{2,0} \) such
that the defect particle at time \( t + 1 \) has internal state \( \chi \). To show that \( \overline{\mu} \) is a Markov measure, we must find some transition probability function
\( \tau : \mathcal{X} \to \mathcal{M}(\mathcal{X}) \) such that:

\[ \text{For all } \chi \in \mathcal{X} \text{ and } t \in \mathbb{N}, \quad \mu_{[\mathcal{S}[0..t]} \left[ U_{t+1}^\chi \right] = \mu_{[\mathcal{S}(t)]} \left[ U_{t+1}^\chi \right] = \tau(\xi_t, \chi). \quad (7) \]

For any \( z \in \mathbb{Z} \), let \( D_z^t := \{a \in D_{LR}^{2,0} : z_t(a) = z\} \), and let \( \mathcal{S}_z^* \) be the sigma-algebra on \( D_z^t \) generated by cylinder sets in coordinates \([z - t - 2 \ldots z + t + 3]\).
Then let \( \mathcal{S}_z^* \) be the sigma-algebra on \( D_{LR}^{2,0} \) generated by \( \cup_{z \in \mathbb{Z}} \mathcal{S}_z^* \). Clearly,
\( \mathcal{S}[0..t] \subseteq \mathcal{S}_z^* \), because the information contained in \( \mathcal{S}_z^* \) is sufficient to
determine the first \( t \) positions \( (z_1, \ldots, z_t) \) of the defect particle, and its first \( t \)
internal states \( (\xi_1, \ldots, \xi_t) \); see Figure 8.
CLAIM 1: There exists a function \( \tau : \mathcal{X} \to \mathcal{M}(\mathcal{X}) \) such that, for any \( \chi \in \mathcal{X} \) and \( t \in \mathbb{N} \), we have \( \mu_{\mathcal{E}^t}[U_{\chi}^{t+1}] = \mu_{\mathcal{E}^{t+1}}[U_{\chi}^{t+1}] = \tau(\xi_t, \chi) \).

Proof: Let \( \xi_t := (\ell_t, \ell'_t; d_0, d_1, r_1, r_2) \), where \( (\ell_0, \ell'_0) \in L_2 \), \( (d_0, d'_1) \in \mathcal{A}^2 \), and \( (r_1, r'_1) \in \mathbb{R}_2 \). Fix \( t \in \mathbb{N} \). Let \( \xi_t = (\ell_t, \ell'_t; d'_0, d'_1; r'_1, r'_t) \) and \( \xi_{t+1} = (\ell'_t, \ell''_t; d''_0, d''_1; r''_0, r''_1) \), where we regard these as twelve measurable functions on \( D_{L,R}^{2,0} \). For \( v \in \{-1, 0, 1\} \), let \( D^v := \{ a \in D_{L,R}^{2,0} ; \bar{V}(\xi_t(a)) = v \} \), and let \( \mu^v := \mu|_{D^v} \). Then \( D_{L,R}^{2,0} = D^{-1} \cup D^0 \cup D^1 \), and \( \mu = \sum_{v=-1}^1 \mu[D^v] \cdot \mu^v \). We will thus consider \( \mu^{(-1)}, \mu^0 \) and \( \mu^1 \) separately.

For \( v \in \{-1, 0, 1\} \) and \( z \in \mathbb{Z} \), let \( D^v_z := \{ a \in D^v ; z_t(a) = z \} \), and let \( B^v_z := \text{pr}_{[z-t-2\ldots z+t+3]}[D^v_z] \subset \mathcal{A}^{[z-t-2\ldots z+t+3]} \). Then let \( B^v := \bigsqcup_{z \in \mathbb{Z}} B^v_z \).

CLAIM 1.0: For any \( b \in B^v_z \), let \([b] := \{ a \in D_{L,R}^{2,0} ; a_{[z-t-2\ldots z+t+3]} = b \} \).
Then $D^\nu = \bigsqcup_{b \in B^\nu} [b]$.

Proof: $D^\nu_z = \bigsqcup_{b \in B^\nu_z} [b]$, for any $z \in Z$. Thus, $D^\nu = \bigsqcup_{z \in Z} D^\nu_z = \bigsqcup_{z \in Z} \bigsqcup_{b \in B^\nu_z} [b]$. Q.E.D.

**Claim 1.0:** Let any $\xi \in \mathcal{X}$, $\mu^{(0)}_{(\xi)}[U^0_{\xi}] = \mu^{(0)}_{(\xi)}[U^0_{\xi}] = \tau_0(\xi)$, where

$$\tau_0\left(\ell^t_1, \ell^t_2; d^t_0, d^t_1; r^t_1, r^t_2\right) := \begin{cases} \frac{1}{P_{L,FR}} & \text{if } \ell_2 \in \mathcal{P}_L(\ell^t_1), \ell_1 = \phi(\ell^t_1, l^t_1, d^t_0), \\ d_0 = \phi(\ell^t_1, l^t_1, d^t_0), d_1 = \phi(d^t_0, d^t_1, r^t_1), \\ n = \phi(d^t_1, r^t_1, r^t_2) & \text{and } r_2 \in \mathcal{F}_R(r_2); \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Figure 9(A) shows how the values of $(\ell^t_1, d^t_0, d^t_1, r^t_1)$ are determined by the data in $\mathcal{G}\{t\}$, because $\ell^t_1 = \phi(\ell^t_1, l^t_1, d^t_0)$, $d^t_0 = \phi(\ell^t_1, l^t_1, d^t_0)$, $d^t_1 = \phi(d^t_0, d^t_1, r^t_1)$, and $r^t_1 = \phi(0, r^t_1, r^t_2)$. However, $\ell^t_2 = \phi(\ell^t_2, l^t_2, l^t_1)$ and $r^t_2 = \phi(r^t_2, r^t_2, r^t_2)$ not determined, even by $\mathcal{G}^*$. Instead, for any fixed $z \in Z$ and $b \in B^0_z$, there is a function $\phi_b : \mathcal{P}_L(z_{-t+3}) \times \mathcal{F}_R(z_{-t+3}) \rightarrow \mathcal{P}_L(z_{t+1}) \times \mathcal{F}_R(z_{t+1})$ such that $\phi_b(\ell^t_1, \ell^t_2) = \Phi_b(\ell^t_1, \ell^t_2)$. Furthermore, $\Phi_b$ is bijective, because $L$ is $\Phi$-left-resolving and $R$ is $\Phi$-right-resolving.

The set $\mathcal{P}_L(z_{-t+2}) \times \mathcal{F}_R(z_{t+3})$ has cardinality $P_{L,FR}$, because $L$ is left-regular and $R$ is right-regular. Let $\mu_b$ be the conditional measure on $D^0$ given $b$. If $\eta_b^0 := \text{pr}_{t_{-t+3}}(\mathcal{P}_L(\ell^t_1))$, then $\eta_b^0$ is the uniform measure assigning mass $1/(P_{L,FR})$ to each element of $\mathcal{P}_L(z_{-t+2}) \times \mathcal{F}_R(z_{t+3})$, because $L$ is the Parry measure on $L$ and $\rho$ is the Parry measure on $R$. Note that any $b \in B^0_z$ completely determines the values of $\ell^t_1$ and $\ell^t_2$ (because these are $\mathcal{G}^*$-measurable functions). Let $\eta_b^t$ be the uniform measure assigning mass $1/(P_{L,FR})$ to each element of $\mathcal{P}_L(z_{t+1}) \times \mathcal{F}_R(z_{t+1})$. Then the $\mu_b$-probability distribution of $(\ell^t_1, \ell^t_2)$ is the measure $\Phi_b(\text{pr}_{t_{-t+3}}(\mathcal{P}_L(\ell^t_1))) = \Phi_b(\eta_b^0) \Rightarrow \eta_b^0$ (here $\Rightarrow$ is because $\Phi_b$ is bijective, while both $\eta_b$ and $\eta_b^t$ are uniform measures on sets with $P_{L,FR}$ elements). Thus, $\mu_b[\mathcal{P}_L(\ell^t_1)] = \tau_0(\xi^0_b)$, where $\tau_0$ is as defined above, and where we can treat $\xi^0_b$ as a function of $b$ (because $\xi^t_b$ is $\mathcal{G}^*$-measurable).

This holds for any $b \in B^0_z$, so Claim 1.0 implies that $\mu^{(0)}_{(\xi)}[U^0_{\xi}]$ is the function $D^0 \ni a \mapsto \tau_0(\xi^0_a) \in [0, 1]$. But this function is $\mathcal{G}\{t\}$-measurable (because $\xi^t_b$ is $\mathcal{G}\{t\}$-measurable by definition), so it is also $\mu^{(0)}_{(\xi)}[U^0_{\xi}]$. Q.E.D.

**Claim 1.1:** For any $\xi \in \mathcal{X}$, $\mu^{(-1)}_{(\xi)}[U^t_{\xi}] = \mu^{(-1)}_{(\xi)}[U^t_{\xi}] = \tau_{-1}(\xi)$, where

$$\tau_{-1}\left(\ell^t_1, \ell^t_2; d^t_0, d^t_1; r^t_1, r^t_2\right) := \begin{cases} \frac{1}{(P_L)^2} & \text{if } \ell_2 \in \mathcal{P}_L(\ell^t_1), \ell_1 = \phi(\ell^t_1, l^t_1, d^t_0), \\ d_0 = \phi(\ell^t_1, l^t_1, d^t_0), d_1 = \phi(d^t_0, d^t_1, r^t_1), \\ n = \phi(d^t_1, r^t_1, r^t_2) & \text{and } r_2 = \phi(d^t_1, r^t_1, r^t_2); \\ 0 & \text{otherwise.} \end{cases}$$

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Proof: Figure 9(B) shows how the values of \((d_0^{t+1}, d_1^{t+1}, r_1^{t+1}, r_2^{t+1})\) are determined by the data in \(\mathcal{G}\{t\}\), because \(d_0^{t+1} = \phi(d'_0, l'_0), d_1^{t+1} = \phi(l'_1, d'_1), r_1^{t+1} = \phi(d'_1, r'_1, d'_2), \) and \(r_2^{t+1} = \phi(d'_2, r'_2, r'_3)\). However, \(l_2^{t+1} = \phi(l'_2, l'_3, l'_4)\) and \(l_1^{t+1} = \phi(l'_1, l'_2, l'_3)\) not determined, even by \(\mathcal{G}^*\). Instead, for any fixed \(z \in \mathbb{Z} \) and \(b \in \mathbb{B}^{(z)}\), let

\[
\mathcal{L}_b^0 := \{(l'_{z-4}, l'_{z-3}) \in \mathcal{A}^2 ; l'_{z-3} \in \mathcal{P}_{\mathcal{L}}(l'_{z-4})\} \cap \mathcal{P}_{\mathcal{L}}(l'_{z-3}) \}
\]

Then \(#(\mathcal{L}_b^0) = (P_L)^2\), because \(\mathcal{L}\) is left-regular. Let \(\mu^0_b := \mathbf{pr}(\mathcal{L}_b^0)\) be the uniform probability measure assigning \(1/(P_L)^2\) to each element of \(\mathcal{L}_b^0\), because \(\lambda\) is the Parry measure on \(L\).

For any \((l_2, l_1) \in \mathcal{L}_2\), let

\[
\mathcal{L}(l_2, l_1) := \{(l'_{2}, l'_{1}) \in \mathcal{A}^2 ; l'_{1} \in \phi(L, b, l_1) \text{ and } l'_{1} \in \mathcal{P}_{\mathcal{L}}(l'_{2})\}.
\]

Then \(#[\mathcal{L}(l_2, l_1)] = (P_L)^2\), because \(\mathcal{L}\) is left-regular and \(\Phi\)-left-resolving. Let \(\lambda^0_{b} := \mathbf{pr}(\mathcal{L}(l_2, l_1))\) be the uniform probability measure assigning \(1/(P_L)^2\) to each element of \(\mathcal{L}(l_2, l_1)\). Note that any \(b \in \mathbb{B}^{(z)}\) completely determines the values of \(l'_0\) and \(l'_1\) (because these are \(\mathcal{G}^*\)-measurable functions). Let \(\mathcal{L}_b^{t+1} := \mathcal{L}(l'_0(b), l'_1(b))\) and \(\lambda^{t+1}_b := \eta(l'_0(b), l'_1(b))\).

There is a function \(\Phi_b : \mathcal{L}_b^0 \rightarrow \mathcal{L}_b^{t+1}\) such that \((l'_0(b), l'_1(b)) = \Phi_b(l_{z-4}(b), l_{z-3}(b))\), and \(\Phi_b\) is bijective because \(\mathcal{L}\) is \(\Phi\)-left-resolving. Thus, the \(\mu^{-1}_b\)-conditional probability distribution of \((l'_0, l'_1)\) is the measure \(\Phi_b[\mathbf{pr}(\mathcal{L}_b^0)] = \Phi_b(\lambda^0_b)\) is \(\lambda^{t+1}_b\). Here \((\ast)\) is because \(\Phi_b\) is bijective, while \(\lambda^0_b\) and \(\lambda^{t+1}_b\) are both uniform measures on sets of \((P_L)^2\) elements.

Thus, \(\mu^{-1}_b[U_{\mathcal{X}}^{t+1}] = \tau_{-1}(\xi_{\mathcal{X}})\), where \(\tau_{-1}\) is as defined above, and where we can again treat \(\xi_{\mathcal{X}}\) as a function of \(b\).

This holds for any \(b \in \mathbb{B}^{(z)}\), so Claim 1.0 implies that \(\mu^{-1}_b[U_{\mathcal{X}}^{t+1}]\) is the function \(\mathbf{D}^{(z)}(\mathcal{X}) \ni a \mapsto \tau_{-1}(\xi_{\mathcal{X}}(a)) \in [0, 1]\). But this function is \(\mathcal{G}\{t\}\)-measurable (because \(\xi_{\mathcal{X}}\) is \(\mathcal{G}\{t\}\)-measurable), so it is also \(\mu^{-1}_b[U_{\mathcal{X}}^{t+1}]\).  
\text{\(\nabla\) Claim 1.2}

**Claim 1.3:** For any \(\chi \in \mathcal{X}, \mu^1_{\mathcal{G}}[U_{\mathcal{X}}^{t+1}] = \mu^1_{\mathcal{G}}[U_{\mathcal{X}}^{t+1}] = \tau_1(\xi_{\mathcal{X}}), \) where \(\tau_1 : \mathcal{X} \rightarrow \mathcal{M}(\mathcal{X})\) is defined similarly to \(\tau_{-1}\).

**Proof:** Figure 9(C) shows how the values of \((l'_0, l'_1, d'_0, d'_1)\) are determined by the data in \(\mathcal{G}\{t\}\), because \(l'_0 = \phi(l'_1, l'_2, l'_3)\), \(l'_1 = \phi(l'_1, d'_1, d'_2)\), \(d'_0 = \phi(d'_1, d'_2, r'_1)\), and \(d'_1 = \phi(d'_1, r'_1, r'_2)\). However, \(r'_1 = \phi(r'_1, r'_2, r'_3)\) not determined, even by \(\mathcal{G}^*\). Now proceed as in Claim 1.2, but replace \(l'_2\) by \(r'_2\), \(l'_3\) by \(r'_3\), \(L\) with \(R\), \(\lambda\) with \(\rho\), and ‘left-resolving’ with ‘right-resolving’.  
\text{\(\nabla\) Claim 1.3}

Finally, define \(\tau : \mathcal{X} \rightarrow \mathcal{M}(\mathcal{X})\) by \(\tau(y, \chi) := \tau_{\mathcal{V}(y)}(\xi_{\mathcal{X}}), \) where \(\mathcal{V}(y) \in \{-1, 0, 1\}\), and where \(\tau_0\) and \(\tau_{\pm 1}\) are defined as in Claims 1.1 to 1.3. Then \(\mu^1_{\mathcal{G}}[U_{\mathcal{X}}^{t+1}] = \mu^1_{\mathcal{G}}[U_{\mathcal{X}}^{t+1}] = \tau(\xi_{\mathcal{X}}). \)  
\text{\(\diamond\) Claim 1}
Claim 2: For any $x \in X$ and $t \in \mathbb{N}$, $\mu_{\sigma[x]t}[U_{x,t}^{t+1}] = \mu_{\sigma} U_{x,t}^{t+1}$.

Proof: Claim 1 implies that $\mu_{\sigma[t]} U_{x,t}^{t+1}$ is actually a $\mathcal{G}(t)$-measurable function, because it is equal to $\mu_{\sigma}[U_{x,t}^{t+1}]$. But this means that $\mu_{\sigma[t]} U_{x,t}^{t+1}$ is also $\mathcal{G}[0...t]$-measurable, because $\mathcal{G}(t) \subseteq \mathcal{G}[0...t]$. Also, for any $C \in \mathcal{G}[0...t]$, we have $\int_C \mu_{\sigma[t]} U_{x,t}^{t+1} \, d\mu = \mu_{\sigma} | C \cap U_{x,t}^{t+1}$, because $C \in \mathcal{G}$, because $\mathcal{G}[0...t] \subseteq \mathcal{G}$. But $\mu_{\sigma[0...t]} U_{x,t}^{t+1}$ is the unique $\mathcal{G}[0...t]$-measurable function with this property (by definition); hence $\mu_{\sigma[t]} U_{x,t}^{t+1} = \mu_{\sigma[0...t]} U_{x,t}^{t+1}$. \(\Box\)

Remark 3.5: (a) Suppose the conditions of Theorem 3.3 are satisfied. Then the measure $\delta$ can always be chosen so that the Markov measure $\overline{\sigma}$ is shift-invariant (because every finite-state Markov chain has a stationary measure). The $\sigma$-ergodic components of $\overline{\sigma}$ are then the stochastic analogs of the particle types of Definition 2.2.

The drift velocity of $\omega$ is the expected value $\tilde{V}_{d_{\text{def}}}(\omega) := \sum_{v \in V} \nu[v] \cdot v$. If $\overline{\sigma}$ is $\sigma$-ergodic (i.e. $\overline{\sigma}$ corresponds to a single particle type), then for all $\omega \in \mathbb{Z}^n, \lim_{n \to \infty} (z_n/n) = \tilde{V}_{d_{\text{def}}}(\omega)$ (by the Birkhoff Ergodic Theorem). Thus, $\tilde{V}_{d_{\text{def}}}(\omega)$ is the long-term average velocity of particles of type $\overline{\sigma}$. For instance, the Markov chain in Example 3.4 has one ergodic component (i.e. one particle type), with $\tilde{V}_{d_{\text{def}}} = 0$.

(b) A special case of Theorem 3.3 was previously proved in [Elo93b, Thm.2.1.1], for when $L = L^Z$ and $R = R^Z$ are permutative subalphabets for $\Phi$ [see Example 3.1(a)] and $W = 0$. In this case we must have $R \neq L$ for an $(L^Z, R^Z)$-defect to be meaningful. We recommend [Elo93a,Elo93b] for further interesting examples of diffusive defect dynamics, as well as analysis of their drift and variance. These methods were extended to defect ensembles in [Elo94], and to the pseudorandom motion of domain boundaries in two-dimensional boolean CA [Elo95].

(c) Empirically, the large $\alpha$ defect particle of ECA#54 [see Figure 3(\alpha)] also performs a random walk, as can perhaps be seen in Figure 1(A). However, this motion is not due to the mechanism of Theorem 3.3, because $\alpha$ belongs to the ‘ballistic’ regime of \S 2, not the ‘diffusive’ regime. Instead, the meandering is due to interactions with neighbouring $\alpha$ particles, mediated by a complex exchange of the tiny $\gamma^\pm$ particles of Figure 3(\gamma^\pm). See [CH97, Fig.13(b)]. \(\Box\)

Corollary 3.6 Let $\Phi : \mathcal{A}^Z \to \mathcal{A}^Z$ be a CA and fix $p, q \in \mathbb{N}$. Suppose that either

[i] $L \subseteq \text{Fix}[\Phi^p, \sigma^q]$ and $\lambda \in \mathcal{M}(L)$, while $R \subseteq \mathcal{A}^Z$ is a right-resolving, right-regular Markov subshift with Parry measure $\rho \in \mathcal{M}(R)$.

or [ii] $L \subseteq \mathcal{A}^Z$ is a left-resolving, left-regular Markov subshift with Parry
measure \( \lambda \in \mathcal{M}(\mathcal{L}) \), while \( \mathbf{R} \subseteq \text{Fix}[\Phi^p, \sigma^q] \) and \( \rho \in \mathcal{M}(\mathcal{R}) \).

Let \( W \in \mathbb{N} \), let \( \delta \) be any probability measure on \( \mathcal{D} := \mathcal{A}^W \), and let \( \mu := \lambda \otimes \delta \otimes \rho \in \mathcal{M}(\mathcal{D}_{\mathcal{L}, \mathcal{R}}^W) \). Define \( \zeta : \mathcal{D}_{\mathcal{L}, \mathcal{R}}^W \to \mathbb{Z}^\mathbb{N} \) by \( \zeta(a) := (z_0, z_p, z_{2p}, \ldots) \), where, for all \( t \in \mathbb{N} \), \( z_{tp} \in \mathbb{Z} \) is as in eqn. (5). Then \( \omega := \zeta(\mu) \in \mathcal{M}(\mathbb{Z}^\mathbb{N}) \) is a random walk.

**Proof:** (Case [i]) By using the \( q \)th higher power representation of \( \mathcal{A}^\mathbb{Z} \) [see Remark 1.3(c)] and replacing \( \Phi \) with \( \Phi^p \), we can assume that \( q = p = 1 \); i.e. that \( \mathcal{L} \subseteq \text{Fix}[\Phi, \sigma] \). Thus \( \mathcal{L} = \{l^\infty\}_{l \in \mathcal{L}} \), where \( \mathcal{L} \subseteq \mathcal{A} \) is some subalphabet, and, for each \( l \in \mathcal{L} \), the point \( l^\infty := [\ldots lll \ldots] \) is \( \Phi \)-fixed. Thus \( \lambda = \sum_{l \in \mathcal{L}} c_l \mathbbm{1}_l \), where, for each \( l \in \mathcal{L} \), \( \mathbbm{1}_l \) is the point-mass on \( l^\infty \), and \( c_l \in [0, 1] \) is a constant. Thus \( \mu = \sum_{l \in \mathcal{L}} c_l \mu_l \), where \( \mu_l := \mathbbm{1}_l \otimes \delta \otimes \rho \). It suffices to prove that each of the measures \( \mu_l \) induces a random walk. Hence, assume that \( \lambda = \mathbbm{1}_l \) for some \( l \in \mathcal{L} \), and redefine \( \mathcal{L} := \{l\} \) and \( \mathcal{L} := \{l^\infty\} \). Then \( \mathcal{L} \) is a left-regular and left-resolving subshift [see Example 3.1(e)]. Now apply Theorem 3.3. The proof of Case [ii] is analogous. \( \square \)

### 4 The Turing Regime and Pushdown Regimes

Recall that a Turing machine [HU79, §7.2] consists of a ‘head’ which deterministically moves back and forth along a ‘tape’, reading and writing symbols from some alphabet. To be precise, let \( T \) be a finite set. A (classical) Turing Machine with tape alphabet \( T \) is a quadruple \( (\mathcal{D}, \tau, \Upsilon, \tilde{V}) \), where \( \mathcal{D} \) is a finite set (called the head state domain), \( \tau : T \times \mathcal{D} \to T \) is a tape rule, \( \Upsilon : T \times \mathcal{D} \to \mathcal{D} \) is an update rule, and \( \tilde{V} : T \times \mathcal{D} \to \{-1, 0, 1\} \) is a velocity rule. The machine statespace of the Turing machine is \( \mathcal{T}^\mathbb{Z} \times \mathcal{D} \times \mathbb{Z} \). If the machine is in state \( (t, d, z) \in \mathcal{T}^\mathbb{Z} \times \mathcal{D} \times \mathbb{Z} \), this means that the tape currently has symbol string \( t \), the head is at position \( z \) on the tape, and the head has state description \( d \). If \( t := [\ldots t_{z-1} t_z t_{z+1} \ldots] \), then define \( t' := [\ldots t_{z-1} \tau(t_z, d) t_{z+1} \ldots] \). The dynamics of the machine is the map \( \Theta : \mathcal{T}^\mathbb{Z} \times \mathcal{D} \times \mathbb{Z} \to \mathcal{T}^\mathbb{Z} \times \mathcal{D} \times \mathbb{Z} \) defined:

\[
\Theta(t, d, z) := (t', \Upsilon(t_z, d), z + \tilde{V}(t_z, d))
\]

We will generalize this definition in two ways. First, we will imagine that the head lies between two tape symbols, rather than over a tape symbol. The head can read the two symbols to its left and two symbols to its right, and can overwrite the symbol immediately left or right. Second, we require that there are Markov subshifts \( \mathcal{L}, \mathcal{R} \subseteq \mathcal{A}^\mathbb{Z} \) such that the symbol sequence on the left half of the tape lies \( \mathcal{L}^- \), while the right half lies in \( \mathcal{R}^+ \). The machine must write new symbols so as to respect the constraints of these subshifts.
Formally, an \((L, R)\)-Turing machine is a sextuple \((D, \tau_L, \tau_C, \tau_R, Y, \bar{V})\), where \(D\) is a finite set, \(\tau_L : A^2 \times D \rightarrow A\), \(\tau_C : A \times D \times A \rightarrow A\), \(\tau_R : D \times A^2 \rightarrow A\), \(Y : A^2 \times D \times A^2 \rightarrow D\), and \(\bar{V} : A \times D \times A \rightarrow \{-1, 0, 1\}\). The statespace of the Turing machine is \(L^- \times D \times R^+ \times Z\). If the machine is in state \((l, d', r, z)\) \(\in L^- \times D \times R^+ \times Z\), this means that the tape currently has symbol string \((l \mid r)\), where the head (indicated by ‘\(\mid\)’) is at position \(z + \frac{1}{2}\) on the tape, and the head has state description \(d'\). The machine dynamical system \(\Theta : L^- \times D \times R^+ \times Z \rightarrow L^- \times D \times R^+ \times Z\) is defined by \(\Theta(l, d', r, z) := (l', d', r', z')\), where \(d' := Y(l_2, l_1, d, r_1, r_2)\) and \(z' := z + \bar{V}(l_1, d', r_1)\) and

\[
(l' \mid r') := \begin{cases} 
\ldots, l_1, l_2, l_3, l_2' | r_0', r_1', r_2, r_3, r_4, \ldots \text{ if } \bar{V}(l_1, d', r_1) = -1; \\
\ldots, l_1, l_2, l_3, l_2', l_1' | r_1', r_2, r_3, r_4, \ldots \text{ if } \bar{V}(l_1, d', r_1) = 0; \\
\ldots, l_1, l_2, l_3, l_2', l_1' | r_2, r_3, r_4, \ldots \text{ if } \bar{V}(l_1, d', r_1) = +1.
\end{cases}
\]

Here, \(l_1' := \tau_L(l_2, l_1; d)\) is such that \((l_2, l_1') \in L_2\);
\(r_1' := \tau_C(d; r_1, r_2)\) is such that \((r_1', r_2) \in R_2\);
and \(l_1' := \tau_C(l_1; d, r_1)\) is such that \((l_1', r_1) \in L_2\), if \(\bar{V}(l_1, d, r_1) = +1\),
whereas \(r_0' := \tau_C(l_1; d, r_1)\) is such that \((r_0', r_1') \in R_2\) if \(\bar{V}(l_1, d, r_1) = -1\).

(If \(\bar{V}(l_1, d, r_1) = 0\), then the value of \(\tau_C(l_1; d, r_1)\) is discarded, so it is irrelevant.) Finally,

- \(Y(l_2, l_1, d, r_1, r_2)\) depends only on \((l_2, l_1, d)\) if \(\bar{V}(l_1, d, r_1) = -1\).
- \(Y(l_2, l_1, d, r_1, r_2)\) depends only on \((l_1, d, r_1)\) if \(\bar{V}(l_1, d, r_1) = 0\).
- \(Y(l_2, l_1, d, r_1, r_2)\) depends only on \((d, r_1, r_2)\) if \(\bar{V}(l_1, d, r_1) = +1\).

**Proposition 4.1** Let \(L, R \subset \mathcal{A}^Z\) be Markov subshifts. Let \(W \in \mathbb{N}\) and let \(D := \mathcal{A}^W\). Let \(L, R \subset \mathcal{D}^Z\) be the \(W\)th higher power representations of \(L\) and \(R\).

(a) Let \(\Phi : \mathcal{A}^Z \rightarrow \mathcal{A}^Z\) be a CA with \(L, R \subseteq \text{Fix}[\Phi]\). Then the dynamical system \((D_{L,R}^W, \Phi)\) is isomorphic to an \((L, R)\)-Turing machine \((D, \tau, Y, \bar{V})\).

(b) Conversely, given any \((L, R)\)-Turing machine \((D, \tau, Y, \bar{V})\), there is a CA \(\Phi : \mathcal{A}^Z \rightarrow \mathcal{A}^Z\), with \(L, R \subseteq \text{Fix}[\Phi]\), such that \((D_{L,R}^W, \Phi)\) is isomorphic to \((D, \tau, Y, \bar{V})\).

*Proof idea:* The defect acts like the Turing machine head. The application of \(\Phi\) changes the head state, and can also modify the adjacent symbols on the \((L, R)\)-tape. However, just as in a Turing machine, the more distant tape symbols remain unchanged, because \(L\) and \(R\) are \(\Phi\)-fixed.

*Proof:* (a) By passing to the \(W\)th higher power recoding, and replacing \(L\)
with \( \hat{L} \) and \( R \) with \( \hat{R} \) and \( A \) with \( \hat{A} := A^W \), Remark 1.3(c) allows us to assume that \( D = \hat{A}^2 \) and that \( \Upsilon : \hat{A} \times D \times \hat{A} \rightarrow D \) and \( \hat{V} : \hat{A} \times D \times \hat{A} \rightarrow \{-1, 0, 1\} \) in eqn.(6). To simplify notation, we will suppress the ‘hats’. Define \( \Psi : D_{L,R}^2 \rightarrow L^- \times D \times R^+ \times Z \) so that, if \( a^t \) is as in eqn.(5), then \( \Psi(a^t) := (l, d, r; z) \), where

\[
1 := [\ldots, l_1^t, l_2^t, l_3^t] \in L^-, \quad d := [d_0^t, d_1^t] \in D = A^2, \\
r := [r_1^t, r_2^t, r_3^t, \ldots] \in R^+, \quad \text{and} \quad z := z_t \in Z.
\]

Let \( \Upsilon \) and \( \hat{V} \) be as in eqn.(6). If \( a^{t+1} := \Phi(a^t) \), then \( \Psi(a^{t+1}) = (l', d', r'; z') \), where \( d' := [d_0^{t+1}, d_1^{t+1}] = \Upsilon(l_t, d_t, n_t) \), and where \( l' \) and \( r' \) are as in eqn.(8), with

\[
\tau_L(l_t, l_t, d_t) := \Phi(l_t, l_t, d_0^t);
\]

\[
\tau_C(l_t, d_t, n_t) := \begin{cases} \\
\Phi(d_0^t, d_1^t, n_t) & \text{if } \hat{V}(l_t, d_t, n_t) = +1; \\
\Phi(d_0^t, d_1^t, n_t) & \text{if } \hat{V}(l_t, d_t, n_t) = -1; \\
\text{irrelevant} & \text{if } \hat{V}(l_t, d_t, n_t) = 0.
\end{cases}
\]

and \( \tau_R(d_t, n_t, r_t) := \Phi(d_1^t, n_t, r_t) \).

(b) is a straightforward generalization of the method of Lindgren and Nordahl [LN90] for simulating a classical Turing machine with a cellular automaton. \(\square\)

Proposition 4.1 applies even when \( L \) and \( R \) are \( \sigma \)-periodic subshifts, but in this case it isn’t very interesting, because an \((L, R)\)-admissible ‘tape’ can’t encode any information, so the resulting Turing machine is rather trivial, and is described in \( \S 2 \). To perform useful computation, we need \( L \) and \( R \) to have nonzero entropy. If \( B \subset A^Z \) is a subshift, then the topological entropy of \( B \) is defined

\[
h(B, \sigma) := \lim_{N \to \infty} \frac{\log_2(\#B_{[0,N]})}{n}.
\]

If \( B \) is a subshift of finite type, then \( h(B, \sigma) > 0 \) iff \( B \) is not \( \sigma \)-periodic. In particular, if \( B \) is a Markov subshift defined by a digraph on the vertex set \( A \), then \( h(B, \sigma) > 0 \) iff this digraph is not just a disjoint union of cycles. Equivalently, there is a choice point vertex \( c \in A \), meaning that \( c \) belongs to at least two distinct cycles. See [LM95, Ch.4], [Kit98, §1.4] or [Kur03, §3.6.2]. A \( B \)-admissible sequence \( b \) can then encode nontrivial information, because for every \( z \in Z \) with \( b_z = c \), there are at least two \( B \)-admissible possibilities for \( b_{z+1} \), and a choice between these encodes at least one bit of information.

A pushdown automaton [HU79, §5.2] is a finite automaton augmented with a ‘stack’ or ‘last in, first out’ (‘LIFO’) memory model. To be precise, a pushdown automaton is a septuple \((I, D, O, T; \Upsilon, \Omega, \Sigma)\), where \( I, D, \) and \( O \) are a
finite input space, state domain, and output space, respectively (as in a finite automaton), and \( T \) is a finite \emph{stack alphabet}. Now \( \Upsilon : I \times T \times D \rightarrow D \) is the \emph{update rule}, \( \Omega : I \times T \times D \rightarrow \mathcal{O} \) is the \emph{output rule}, and \( \Sigma : I \times T \times D \rightarrow T \cup \{ \emptyset, \triangle \} \) is a \emph{stack rule}. The machine statespace of the pushdown automaton is \( D \times T^\mathbb{N} \).

The machine behaviour is defined by the map \( \Theta : I \times D \times T^\mathbb{N} \rightarrow D \times T^\mathbb{N} \times \mathcal{O} \) defined \( \Theta(i, d, t) := (d', t', o) \), where \( d' = \Upsilon(i, d, t_0) \), \( o := \Omega(i, d, t_0) \), and where

\[
\begin{align*}
t' &= \begin{cases}
(t_1, t_2, \ldots) & \text{if } \Sigma(i, d, t_0) = \triangle \text{ (i.e. 'pop' the symbol } t_0 \text{ off the stack)}; \\
(t_0, t_1, \ldots) & \text{if } \Sigma(i, d, t_0) = \emptyset \text{ (i.e. do not touch the stack)}; \\
(t', t_0, t_1, \ldots) & \text{if } \Sigma(i, d, t_0) = t' \text{ (i.e. 'push' the symbol } t' \text{ onto the stack).}
\end{cases}
\end{align*}
\]

An \emph{autonomous finite automaton} is a finite automaton with no input or output; i.e. a dynamical system \( \Upsilon : D \rightarrow D \) where \( D \) is a finite set. Similarly, an \emph{autonomous pushdown automaton} (APDA) is a pushdown automaton with no input or output; i.e. \( I = \emptyset = \mathcal{O} \). Thus, the function \( \Omega \) is trivial, \( \Upsilon : T \times D \rightarrow D \), and \( \Sigma : T \times D \rightarrow T \cup \{ \emptyset, \triangle \} \). The APDA’s future behaviour is entirely determined by the initial stack state. It is easy to see that an APDA is equivalent to an \( (L, R) \)-Turing machine where \( R = A^Z \) and \( L = \{ \emptyset \} \) where 0 is some ‘null’ symbol. Thus, we will treat it as such.

Let \( M \) and \( M' \) be two machine-classes. We write \( M \preceq M' \) if any machine in \( M \) can be simulated by one in \( M' \) (possibly not in real time). We say that \( M \) and \( M' \) are \emph{computationally equivalent} (and write \( M \approx M' \)) if \( M \preceq M' \) and \( M' \preceq M \). Let \( TM \) be the class of (classical) Turing machines, and let \( TM_{LR} \) be the class of \( (L, R) \)-Turing machines. Let \( APDA \) be the class of autonomous pushdown automata, and let \( AFA \) be the class of autonomous finite automata.

**Proposition 4.2** Let \( L, R \subset A^Z \) be Markov subshifts.

(a) If \( h(L, \sigma) > 0 \) and \( h(R, \sigma) > 0 \), then \( TM_{LR} \approx TM \).

(b) If \( h(L, \sigma) > 0 = h(R, \sigma) \), or \( h(L, \sigma) = 0 < h(R, \sigma) \), then \( TM_{LR} \approx APDA \).

(c) If \( h(L, \sigma) = 0 = h(R, \sigma) \), then \( TM_{LR} \approx AFA \).

**Proof:** By a cycle of length \( P \) in \( R \), we mean a word \( c = (c_1, \ldots, c_P) \in R_P \), such that \((c_P, c_1) \in R_2\); hence the infinite sequence \([\ldots c c c \ldots]\) is \( R \)-admissible.

**Claim 1:** Suppose \( h(R, \sigma) > 0 \). Then there is some \( P \in \mathbb{N} \) and some \( c \in A \) such that \( c \) begins two different cycles \( c_0 \) and \( c_1 \) in \( R \), both of length \( P \).

**Proof:** \( h(R, \sigma) > 0 \), so there is some \( c \in A \) which belongs to two different cycles in \( R \); say \( b_0 = (b_{01}^0, b_{02}^0, \ldots, b_{0P}^0) \) and \( b^1 = (b_{11}^1, b_{12}^1, \ldots, b_{1Q_1}^1) \), where \( Q_0, Q_1 \in \mathbb{N} \) and \( b_{1t}^1 = c = b_{0t}^0 \). Let \( P := \text{lcm}(Q_0, Q_1) \). Let \( c_0 \) (resp. \( c_1 \)) be the cycle obtained by chaining together \( P/Q_0 \) copies of \( b_0 \) (resp. \( P/Q_1 \) copies of \( b^1 \)). Then \( c_0 \) and \( c_1 \) are distinct cycles of length \( P \), both starting with \( c \). \( \diamondsuit \) **Claim 1**
Claim 2: Suppose \( h(R, \sigma) > 0 \) and \( h(L, \sigma) > 0 \). Then there is some \( P \in \mathbb{N} \) and some \( r, t \in \mathcal{A} \) such that \( r \) begins two different cycles \( r_0 \) and \( r_1 \) in \( R \), and \( t \) begins two different cycles \( l_0 \) and \( l_1 \) in \( L \), with all four cycles having length \( P \).

Proof: Claim 1 yields two cycles \( c^R_0, c^R_1 \) in \( R \), say of length \( P_R \), beginning with the same symbol, say \( r \). The same argument also yields two cycles \( c^L_0, c^L_1 \) in \( L \), say of length \( P_L \), beginning with the same symbol, say \( t \). Let \( P := \text{lcm}(P_R, P_L) \). Let \( r_0 \) (resp. \( r_1 \)) be obtained by chaining together \( P/P_R \) copies of \( c^R_0 \) (resp. \( c^R_1 \)). Let \( l_0 \) (resp. \( l_1 \)) be obtained by chaining together \( P/P_L \) copies of \( c^L_0 \) (resp. \( c^L_1 \)).

Let \( R^+_1 \subseteq R^+ \) be the set of all right-infinite sequences made by concatenating copies of \( r_0 \) and \( r_1 \), and let \( T := \{0, 1\} \). Define \( \beta_R : T \rightarrow \{r_0, r_1\} \) by \( \beta_R(t) := r_t \) for \( t = 0, 1 \). Define bijection \( \beta^R_D : \mathcal{T}^D_R \rightarrow \mathbb{R}_+^1 \) by \( \beta^R_1(t_1, t_2, t_3, \ldots) := [\beta_R(t_1) \beta_R(t_2) \beta_R(t_3) \ldots] \). Clearly, \( \beta_R \circ \sigma = \sigma^P \circ \beta_R \). In this way, we can encode any binary sequence with an element of \( R^+_1 \). Likewise, let \( L^-_1 \subseteq L^- \) be the set of all left-infinite sequences made from \( l_0 \) and \( l_1 \), define \( \beta_L : \mathcal{T} \rightarrow \{l_0, l_1\} \) by \( \beta_L(t) := l_t \) for \( t = 0, 1 \), and define bijection \( \beta^L_D : \mathcal{T}^D_L \rightarrow L^-_1 \) by \( \beta^L_1(t_1, t_2, t_3, t_4, \ldots) := [\ldots \beta_L(t_3) \beta_L(t_2) \beta_L(t_1)] \). Clearly, \( \beta_L \circ \sigma^{-1} = \sigma^{-P} \circ \beta_L \).

(a) "\( \mathcal{TM}_{LR} \leq \mathcal{TM} \)" Every \((L, R)\)-Turing machine is clearly also an \((\mathcal{A}^2, \mathcal{A}^2)\)-Turing machine, which can clearly be simulated by a classical Turing machine with tape alphabet \( \mathcal{A} \).

"\( \mathcal{TM} \leq \mathcal{TM}_{LR} \)" If \( M = (D, \tau, \Gamma, V) \) is a classical Turing machine with tape alphabet \( \mathcal{T} \), then one tape symbol \( t_0 \) lies directly 'underneath' the head. In contrast, in an \((L, R)\)-Turing machine \( M = (D_s, \tau_L, \tau_C, \tau_R, \Gamma_s, V_s) \), we only have tape symbols to the left and right sides. However, if the head state domain \( D_s \) is large enough, then the head of \( M \) can temporarily 'remember' the value of the symbol \( t_0 \), even though \( t_0 \) is not written anywhere on the tape.

So, let \( D_s := D \times T \times L \times R \). If \( t := (\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots) \) and \( M \) has headstate \( d \in D \), then let \( l := \beta^L_1(t_0, t_1, t_2, \ldots) \), let \( r := \beta^R_1(t_0, t_1, t_2, \ldots) \), and let \( M \) have head state description \( d_s := (d, l_0, l_s, r_s) \in D_s \). The third and fourth entries of \( d_s \) are input buffers; their values (represented by \( l_s \) and \( r_s \)) are currently irrelevant.

Now, suppose \( M \) moves right by one step, overwriting \( t_0 \) with \( t'_0 \) and changing its head state to \( d' \). Then \( M_s \) moves right by \( P \) steps; during which time it reads \([r_1, \ldots, r_P]\) and stores this \( P \)-tuple in the fourth entry of \( d_s \) (labelled \( r_s \) above), while writing the \( P \) symbols of \( \beta_L(t'_0) \) to the tape. Finally \( M_s \) computes \( t_1 := \beta^{-1}_L[r_1, \ldots, r_P] \) and changes its headstate to \( d'_s := (d', t_1, l_s, r_s) \) (where \( l_s \) and \( r_s \) are again irrelevant).

Suppose \( M \) moves left by one step, overwriting \( t_0 \) with \( t'_0 \) and changing its head state to \( d' \). Then \( M_s \) moves left by \( P \) steps, during which time it reads \([l_0, \ldots, l_P]\) (in reverse order) and stores this \( P \)-tuple in the third entry of \( d'_s \) (labelled \( l_s \) above), while writing the \( P \) symbols of \( \beta_R(t'_0) \) to the tape. Finally \( M_s \) computes \( t_{-1} := \beta^{-1}_R[l_P, \ldots, l_1] \) and changes its headstate to \( d'_s := (d', t_{-1}, l'_s, r_s) \) (where \( l'_s \) and \( r_s \) are again irrelevant).
Thus, the update rule $\Upsilon_*$ not only must emulate $\Upsilon$, but also must implicitly compute $\beta_L^{-1}$ and $\beta_R^{-1}$. Also, the tape rules $\tau_L$ and $\tau_C$ not only must emulate $\tau$, but also must implicitly compute $\beta_L$; likewise, the tape rules $\tau_C$ and $\tau_R$ must implicitly compute $\beta_R$.

(b) Suppose $h(L, \sigma) = 0 < h(R, \sigma)$ (the case “$h(L, \sigma) > 0 = h(R, \sigma)$” is analogous).

\textit{“APDA \preceq TM_{LR}”:} Let $c_0, c_1$ be as in Claim 1, and let $R^+_1 \subseteq R^+$ be the set of all right-infinite sequences made by concatenating copies of $c_0$ and $c_1$. Now define a bijection $\beta^N: T^N \rightarrow R^+_1$, and use $\beta^N$ to build an $(L, R)$-Turing machine which can emulate a given APDA with stack alphabet $T$, as in part (a).

\textit{“TM_{LR} \preceq APDA”:} $L$ is $\sigma$-periodic, so by passing to a higher power presentation, we can assume $L$ contains only constant sequences. At this point, any $(L, R)$-Turing machine is clearly an APDA.

(c) \textit{“TM_{LR} \preceq AFA”:} If $h(L, \sigma) = 0 = h(R, \sigma)$, then both $L$ and $R$ are periodic, so by passing to a higher power presentation, we can assume that both $L$ and $R$ contain only constant sequences. Thus, the only computation performed by an $(L, R)$-Turing machine is computation of the update rule $\Upsilon: D^{-} \rightarrow D$; i.e. it is an autonomous finite automaton.

\textit{“AFA \preceq TM_{LR}”:} Conversely, if $\Upsilon: D \rightarrow D$ is an autonomous finite automaton, then let $M = (D, \tau, \Upsilon_*, \vec{V})$ be the $(L, R)$-Turing machine where $\Upsilon: A \times D \times A \rightarrow A$ is defined by $\Upsilon_*(\ell, d, r) := \Upsilon(d)$, and the functions $\vec{V}, \tau_L, \tau_C, \tau_R$ are not important (for simplicity, assume they are constants). Then clearly, the ‘head dynamics’ of $(D, \tau, \Upsilon_*, \vec{V})$ is an emulation of $\Upsilon: D \rightarrow D$. \hfill \square

Remarks: (a) If $h(L, \sigma) > 0$ and $h(R, \sigma) > 0$, then Propositions 4.1(b) and 4.2(a) imply that some questions about the long-term behaviour of an $(L, R)$-defect particle are formally undecidable. For example, the question of whether the defect particle eventually stops moving is equivalent to the Halting Problem. Sutner [Sut03] has identified similar undecidability issues for defect behaviour.

(b) The $(\beta_L, \beta_R)$-encoding mechanism in Proposition 4.2 is quite crude; a much more efficient encoding could be obtained using finite state codes [LM95, Ch.5].

(c) In the standard definition, a Turing machine tape has only a finite segment of nontrivial information; we do not assume this. Likewise, in a standard pushdown automaton, the stack has finite (but unbounded) height, whereas our definition allows an infinitely high stack.

(d) Let $M$ be an APDA. When moving to the right (i.e. into the $R$-segment), $M$ acts like a finite automaton with state domain $D$, reading an $A$-valued input stream and producing no output. When moving to the left (i.e. into the constant $0$-segment), $M$ acts like an autonomous finite automaton with state domain $D \times A$ and update rule $\Upsilon: D \times A \rightarrow D \times A$ defined by $\Upsilon(d, r) :=$
A runaway cycle for $\mathcal{M}$ is an $\Upsilon$-periodic orbit $\{(d_p, r_p)\}_{p=1}^P$ [i.e. $\Upsilon(d_p, r_p) = (d_{p+1}, r_{p+1})$ and $\Upsilon(d_P, r_P) = (d_1, r_1)$] such that $\vec{V}(d_p, r_p) = -1$ for all $p \in [1...P]$. In this case, $\mathcal{M}$ moves leftwards forever, and essentially belongs to the Ballistic regime of §2. Not every APDA has a runaway cycle. However, a variation of the Pumping Lemma shows that, if $\mathcal{M}$ moves leftward for long enough, it must enter a runaway cycle. Also, if $\mathcal{M}$ has a runaway cycle which is reachable from any initial conditions, and if $\rho \in \mathcal{M}(\mathcal{R}^+)$ is a Bernoulli measure with full support, then for $\forall \rho \ r \in \mathcal{R}^+$, an APDA with stack $r$ will eventually enter a runaway state (see §3 for definitions of ‘Bernoulli’ and ‘$\forall \rho$’).

(e) By combining the arguments of Theorem 3.3 and Propositions 4.1(a) and 4.2(b), we can show that a defect with a $\Phi$-fixed domain on one side and a $\Phi$-resolving subshift on the other behaves like a pushdown automaton driven by a Markov process. This is the ‘Markov Pushdown Automaton’ regime in Table 2.

(f) The obvious multidimensional analogy of Theorem 4.1 involves a multidimensional Turing machine [HU79, §7.5]. However, the problem of encoding a multidimensional bit array using a multidimensional subshift of finite type (analogous to the $(\beta_L, \beta_R)$-encoding mechanism in Proposition 4.2) becomes much more complex.

(g) A completely different mechanism for universal computation has been implemented using the (ballistic) defect dynamics of ECA #110; see [Coo04], [McI99b] or [Wol02, Chap.11].

Conclusion

We have described the propagation of defects under the action of cellular automata, but many questions remain. For example, we assumed that the defects remain bounded in size, and act like ‘particles’, as is the case in well-known examples such as ECAs #54, #62, #110, and #184. In general, however, defects may grow over time like ‘blights’ which invade the whole lattice. What are necessary/sufficient conditions for the defect to remain bounded? (In general, this is probably formally undecidable; see [Sut03, Thm.3.2].)

Our theory is limited to one-dimensional subshifts of finite type. This excludes some important cases (such as ECA #18), where the invariant subshift is sofic. Can our theory be extended to sofic shifts? (Eloranta’s ‘invariant subalphabet’ approach covers some sofic shifts by passing to a higher power presentation; see [EN92,Elo93a,Elo93b]).

Even when $L$ and $R$ are subshifts of finite type, we only understand defect dynamics in the polar opposite cases of ‘extreme order’ (i.e. $L$ and/or $R$ are $\Phi$-periodic) and ‘extreme chaos’ (i.e. $L$ and/or $R$ are $\Phi$-resolving, and endowed with Parry measures). We have been conspicuously silent about the so-called
‘complicated’ regime in Table 2. In this regime, pretty much anything can happen. To see this, let \( \mathcal{L} \) and \( \mathcal{R} \) be two disjoint finite alphabets, and let \( \Phi_\mathcal{L} : \mathcal{L}^\mathbb{Z} \rightarrow \mathcal{L}^\mathbb{Z} \) and \( \Phi_\mathcal{R} : \mathcal{R}^\mathbb{Z} \rightarrow \mathcal{R}^\mathbb{Z} \) be any two cellular automata with local rules of radius 1. Let \( \mathcal{A} := \mathcal{L} \sqcup \mathcal{R} \), and let \( \Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z} \) be any radius-1 cellular automata such that \( \Phi|_{\mathcal{L}^\mathbb{Z}} = \Phi_\mathcal{L} \) and \( \Phi|_{\mathcal{R}^\mathbb{Z}} = \Phi_\mathcal{R} \). Let \( a := [l \ r] \) where \( l \in \mathcal{L}^{(-\infty...0)} \) and \( r \in \mathcal{R}^{[0...\infty)} \); then \( a \) has a zero-width \((\mathcal{L}, \mathcal{R})\)-defect, where \( \mathcal{L} := \mathcal{L}^\mathbb{Z} \) and \( \mathcal{R} := \mathcal{R}^\mathbb{Z} \). This defect must persist over time and can move either left or right with unit speed. If \( \Phi \) has local rule \( \phi : \mathcal{A}^{[-1..1]} \rightarrow \mathcal{A} \), then the defect’s next move is determined by the restriction of \( \phi \) to the set \( \mathcal{A}^{[-1..1]} \setminus (\mathcal{L}^{[-1..1]} \sqcup \mathcal{R}^{[-1..1]}) \). However, the long-term behaviour of the defect also depends on the dynamics of the CA \((\mathcal{L}^\mathbb{Z}, \Phi_\mathcal{L})\) and \((\mathcal{R}^\mathbb{Z}, \Phi_\mathcal{R})\), which determine the ‘input signals’ which drive the defect. Thus, the defect’s behaviour is potentially at least as complicated as the dynamics of any one-dimensional CA, which could be very complicated indeed.

However, perhaps if we control the topological dynamics of \((\mathcal{L}^\mathbb{Z}, \Phi_\mathcal{L})\) and \((\mathcal{R}^\mathbb{Z}, \Phi_\mathcal{R})\), we can extend the classification of Table 2. For example, perhaps we could weaken the assumption of ‘\( \Phi \)-periodic’ to ‘equicontinuous’ in the Ballistic and machine-emulating regimes, or perhaps we could replace ‘right/left-resolving’ with ‘positively expansive’ in the Diffusive regime. Also, if \((\mathcal{L}^\mathbb{Z}, \Phi_\mathcal{L})\) and \((\mathcal{R}^\mathbb{Z}, \Phi_\mathcal{R})\) themselves manifest emergent defect dynamics, then perhaps we can analyze the behaviour of the \((\mathcal{L}, \mathcal{R})\)-defect through its interaction with these other defect particles (just as the Brownian motion of a macromolecule is driven by a continual bombardment of micromolecules).

Finally, can a comparable theory of defect particle kinematics be developed for subshifts of \( \mathbb{Z}^D \) for \( D > 1 \)? Higher-dimensional shifts also admit infinitely extended defects shaped like ‘curves’ or ‘surfaces’ [Elo95,Piv07a,Piv07b]; what sort of motion do they exhibit? A general theory is probably hopeless: even interface curves in a two-dimensional boolean CA exhibit a bewildering variety and complexity of behaviour [GG98, §3-§6]. However, perhaps some special cases are tractable.

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References

[Bay87] Carter Bays. Candidates for the Game of Life in three dimensions. *Complex Systems*, 1(3):373–400, 1987.

[Bay88] Carter Bays. A note on the discovery of a new game of three-dimensional Life. *Complex Systems*, 2(3):255–258, 1988.

[Bay90] Carter Bays. The discovery of a new glider for the game of three-dimensional Life. *Complex Systems*, 4(6):599–602, 1990.

[Bay91] Carter Bays. A new game of three-dimensional Life. *Complex Systems*, 5(1):15–18, 1991.

[Bay92] Carter Bays. A new candidate rule for the game of three-dimensional Life. *Complex Systems*, 6(5):433–441, 1992.

[Bay94] Carter Bays. Further notes on the game of three-dimensional Life. *Complex Systems*, 8(1):67–73, 1994.

[BCG04] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning ways for your mathematical plays. Vol. 4*, chapter 25: ‘What is Life?’ A K Peters Ltd., Wellesley, MA, second edition, 2004.

[BF05] Vladimir Belitsky and Pablo A. Ferrari. Invariant measures and convergence properties for cellular automaton 184 and related processes. *J. Stat. Phys.*, 118(3-4):589–623, 2005.

[Bla03] Michael Blank. Ergodic properties of a simple deterministic traffic flow model. *J. Statist. Phys.*, 111(3-4):903–930, 2003.

[BNR91] N. Boccara, J. Naser, and M. Roger. Particle-like structures and their interactions in spatiotemporal patterns generated by one-dimensional deterministic cellular automata. *Physical Review A*, 44(2):866–875, 1991.

[BR91] N. Boccara and M. Roger. Block transformations of one-dimensional deterministic cellular automaton rules. *J. Phys. A*, 24(8):1849–1865, 1991.

[CH92] James P. Crutchfield and James E. Hanson. The attractor-basin portrait of a cellular automaton. *J. Statist. Phys.*, 66(5-6):1415–1462, 1992.

[CH93a] James P. Crutchfield and James E. Hanson. Attractor vicinity decay for a cellular automaton. *Chaos*, 3(2):215–224, 1993.

[CH93b] James P. Crutchfield and James E. Hanson. Turbulent pattern bases for cellular automata. *Phys. D*, 69(3-4):279–301, 1993.

[CH97] James P. Crutchfield and James E. Hanson. Computational mechanics of cellular automata: an example. *Phys. D*, 103(1-4):169–189, 1997. Lattice dynamics (Paris, 1995).
[CHM98] James P. Crutchfield, Wim Hordijk, and Melanie Mitchell. Mechanisms of emergent computation in cellular automata. In A.E. Eiben T. Bäck M. Schoenauer and H.-P. Schwefel, editors, Parallel Problem Solving in Nature, volume vol. V of Lecture notes in computer science, pages 613–622. Springer Verlag, Berlin, 1998.

[CHS01] James P. Crutchfield, Wim Hordijk, and Cosma Rohilla Shalizi. Upper bound on the products of particle interactions in cellular automata. Phys. D, 154(3-4):240–258, 2001.

[Coo04] Matthew Cook. Universality in elementary cellular automata. Complex Systems, 15(1):1–40, 2004.

[DMC94] R. Das, M. Mitchell, and J. P. Crutchfield. A genetic algorithm discovers particle-based computation in cellular automata. In Y. Davidor, H.-P. Schwefel, and R. Manner, editors, Parallel Problem Solving from Nature III, pages 244–353, 1994.

[Elo93a] Kari Eloranta. Partially permutive cellular automata. Nonlinearity, 6(6):1009–1023, 1993.

[Elo93b] Kari Eloranta. Random walks in cellular automata. Nonlinearity, 6(6):1025–1036, 1993.

[Elo94] Kari Eloranta. The dynamics of defect ensembles in one-dimensional cellular automata. J. Statist. Phys., 76(5-6):1377–1398, 1994.

[Elo95] Kari Eloranta. Cellular automata for contour dynamics. Phys. D, 89(1-2):184–203, 1995.

[EN92] Kari Eloranta and Esa Nummelin. The kink of cellular automaton Rule 18 performs a random walk. J. Statist. Phys., 69(5-6):1131–1136, 1992.

[Epp02] David Eppstein. Searching for spaceships. In More games of no chance (Berkeley, CA, 2000), volume 42 of Math. Sci. Res. Inst. Publ., pages 433–453. Cambridge Univ. Press, Cambridge, 2002.

[Eva96] Kellie Michele Evans. Larger than Life: It's so nonlinear. PhD thesis, University of Wisconsin Madison, 1996. http://www.csun.edu/~kme52026/thesis.html.

[Eva01] Kellie Michele Evans. Larger than Life: digital creatures in a family of two-dimensional cellular automata. In Discrete models: combinatorics, computation, and geometry (Paris, 2001), Discrete Math. Theor. Comput. Sci. Proc., AA, pages 177–191 (electronic). Maison Inform. Math. Discrèt. (MIMD), Paris, 2001.

[Eva03a] Kellie Michele Evans. Larger than Life: threshold-range scaling of Life’s coherent structures. Phys. D, 183(1-2):45–67, 2003.

[Eva03b] Kellie Michele Evans. Replicators and Larger-than-Life examples. In New constructions in cellular automata, St. Fe Inst. Stud. Sci. Complex., pages 119–159. Oxford Univ. Press, New York, 2003.
[Eva05] Kellie Michele Evans. Bug guns and logic in a family of two-dimensional cellular automata. (in press), 2005.

[Fuk99] Henryk Fukś. Exact results for deterministic cellular automata traffic models. Physical Review E, 60:197–202, 1999.

[GG98] Janko Gravner and David Griffeath. Cellular automaton growth on $\mathbb{Z}^2$: theorems, examples, and problems. Adv. in Appl. Math., 21(2):241–304, 1998.

[Got03] Nicholas M. Gotts. Self-organized construction in sparse random arrays of Conway’s Game of Life. In New constructions in cellular automata, St. Fe Inst. Stud. Sci. Complex., pages 1–53. Oxford Univ. Press, New York, 2003.

[Gra84a] P. Grassberger. Chaos and diffusion in deterministic cellular automata. Phys. D, 10(1-2):52–58, 1984. Cellular automata (Los Alamos, N.M., 1983).

[Gra84b] P. Grassberger. New mechanism for deterministic diffusion. Phys. Rev. A, 28(6):3666–3667, 1984.

[Han93] James E. Hanson. Computational Mechanics of Cellular Automata. PhD thesis, Univ. of California, Berkeley, Ann Arbor, MI, 1993. published by University Microfilms.

[Hed69] G. Hedlund. Endomorphisms and automorphisms of the shift dynamical systems. Mathematical Systems Theory, 3:320–375, 1969.

[HMM03] Bernard Host, Alejandro Maass, and Servet Martínez. Uniform Bernoulli measure in dynamics of permutative cellular automata with algebraic local rules. Discrete Contin. Dyn. Syst., 9(6):1423–1446, 2003.

[HU79] John E. Hopcroft and Jeffrey D. Ullman. Introduction to automata theory, languages, and computation. Addison-Wesley Publishing Co., Reading, Mass., 1979. Addison-Wesley Series in Computer Science.

[Ila01] Andrew Ilachinski. Cellular automata. World Scientific Publishing Co. Inc., River Edge, NJ, 2001.

[Kit87] Bruce Kitchens. Expansive dynamics in zero-dimensional groups. Ergodic Theory & Dynamical Systems, 7:249–261, 1987.

[Kit98] Bruce Kitchens. Symbolic dynamics : one-sided, two-sided, and countable state Markov shifts. Springer-Verlag, Berlin, 1998.

[KS88a] Bruce Kitchens and Klaus Schmidt. Periodic points, decidability, and Markov subgroups. In J. C. Alexander, editor, Dynamical Systems, proceedings of the special year, volume 1342 of Springer Lecture Notes in Math., pages 440–454. Springer-Verlag, 1988.
[KS88b] J. Krug and H. Spohn. Universality classes for deterministic surface growth. *Phys. Rev. A* (3), 38(8):4271–4283, 1988.

[KS89] Bruce Kitchens and Klaus Schmidt. Automorphisms of compact groups. *Ergodic Theory & Dynamical Systems*, 9:691–735, 1989.

[Kur03] Petr Kůrka. *Topological and symbolic dynamics*, volume 11 of *Cours Spécialisés*. Société Mathématique de France, Paris, 2003.

[Lin84] D. A. Lind. Applications of ergodic theory and sofic systems to cellular automata. *Phys. D*, 10(1-2):36–44, 1984. Cellular automata (Los Alamos, N.M., 1983).

[Lin86] Douglas Lind. Table 15. In Stephen Wolfram, editor, *Theory and Applications of Cellular Automata*. World Scientific, Singapore, 1986.

[LM95] Douglas Lind and Brian Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge UP, New York, 1995.

[LN90] Kristian Lindgren and Mats G. Nordahl. Universal computation in simple one-dimensional cellular automata. *Complex Systems*, 4(3):299–318, 1990.

[McI99a] Harold V. McIntosh. Rule 110 as it relates to the presence of gliders. http://delta.cs.cinvestav.mx/~mcintosh/comun/RULE110W/rule110.pdf, 1999.

[McI99b] Harold V. McIntosh. Rule 110 is universal: A gamin’s guide to Goldbergean gadgeteering. http://delta.cs.cinvestav.mx/~mcintosh/comun/texlet/texlet.pdf, 1999.

[MM98] Alejandro Maass and Servet Martínez. On Cesàro limit distribution of a class of permutative cellular automata. *J. Statist. Phys.*, 90(1-2):435–452, 1998.

[MM99] Alejandro Maass and Servet Martínez. Time averages for some classes of expansive one-dimensional cellular automata. In *Cellular automata and complex systems (Santiago, 1996)*, volume 3 of *Nonlinear Phenom. Complex Systems*, pages 37–54. Kluwer Acad. Publ., Dordrecht, 1999.

[MMPY06a] Alejandro Maass, Servet Martínez, Marcus Pivato, and Reem Yassawi. Asymptotic randomization of subgroup shifts by linear cellular automata. *Ergodic Theory Dynam. Systems*, 26(4):1203–1224, 2006.

[MMPY06b] Alejandro Maass, Servet Martínez, Marcus Pivato, and Reem Yassawi. Attractiveness of the Haar measure for the action of linear cellular automata in abelian topological Markov chains. In *Dynamics and Stochastics: Festschrift in honour of Michael Keane volume 48 of Lecture Notes Monograph Series of the IMS*, pages 100–108. Institute for Mathematical Statistics, 2006.

[Par64] William Parry. Intrinsic Markov chains. *Transactions of the AMS*, 112:55–66, 1964.
M. Pivato. Invariant measures for bipermutative cellular automata. *Discrete and Continuous Dynamical Systems, Series A*, 12(4):723–736, April 2005.

M. Pivato. Algebraic invariants for crystallographic defects in cellular automata. *Ergodic Theory & Dynamical Systems*, 27(1):199–240, February 2007.

M. Pivato. Spectral domain boundaries cellular automata. *Fundamenta Informaticae*, 77(special issue), 2007.

James K. Park, Kenneth Steiglitz, and William P. Thurston. Soliton-like behavior in automata. *Phys. D*, 19(3):423–432, 1986.

M. Pivato and Reem Yassawi. Limit measures for affine cellular automata. *Ergodic Theory Dynam. Systems*, 22(4):1269–1287, 2002.

M. Pivato and Reem Yassawi. Limit measures for affine cellular automata. II. *Ergodic Theory Dynam. Systems*, 24(6):1961–1980, 2004.

Marcus Pivato and Reem Yassawi. Asymptotic randomization of sofic shifts by linear cellular automata. *Ergodic Theory Dynam. Systems*, 26(4):1177–1201, 2006.

Mathieu Sablik. Measure rigidity for algebraic bipermutative cellular automata. *Ergodic Theory Dynam. Systems*, 27, 2007.

Klaus Sutner. Almost periodic configurations on linear cellular automata. *Fund. Inform.*, 58(3-4):223–240, 2003.

S. R. S. Varadhan. *Probability theory*, volume 7 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2001.

Stephen Wolfram. *A new kind of science*. Wolfram Media, Inc., Champaign, IL, 2002.