ESTIMATES FOR EIGENVALUES OF
THE PANEITZ OPERATOR*

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ABSTRACT. For an $n$-dimensional compact submanifold $M^n$ in the Euclidean space $\mathbb{R}^N$, we study estimates for eigenvalues of the Paneitz operator on $M^n$. Our estimates for eigenvalues are sharp.

1. Introduction

For compact Riemann surfaces $M^2$, Li and Yau [13] introduced the notion of conformal volume, which is a global invariant of the conformal structure. They determined the conformal volume for a large class of Riemann surfaces, which admit minimal immersions into spheres. In particular, they proved that for a compact Riemann surface $M^2$, if there exists a conformal map from $M^2$ into the unit sphere $S^N(1)$, then the first eigenvalue $\lambda_1$ of the Laplacian satisfies

$$\lambda_1 \text{vol}(M^2) \leq 2V_c(N, M^2)$$

and the equality holds only if $M^2$ is a minimal surface in $S^N(1)$, where $V_c(N, M^2)$ is the conformal volume of $M^2$.

For 4-dimensional compact Riemannian manifolds, Paneitz [15] introduced a fourth order operator $P_g$ defined by, letting $\text{div}$ be the divergence for the metric $g$,

$$P_g f = \Delta^2 f - \text{div} \left[ \left( \frac{2}{3} R_g - 2 \text{Ric} \right) \nabla f \right],$$

for smooth functions $f$ on $M^4$, where $\Delta$ and $\nabla$ denote the Laplacian and the gradient operator with respect to the metric $g$ on $M^4$, respectively, and $R$ and $\text{Ric}$ are the scalar curvature and Ricci curvature tensor with respect to the metric $g$ on $M^4$. Furthermore, Branson [1] has generalized the Paneitz operator to an $n$-dimensional Riemannian manifold. For an $n$-dimensional Riemannian manifold $(M^n, g)$, the operator $P_g$ is defined by

$$P_g f = \Delta^2 f - \text{div} \left[ (a_n R_g + b_n \text{Ric}) \nabla f \right] + \frac{n-4}{2} Q f,$$

where

$$Q = c_n |\text{Ric}|^2 + d_n - \frac{1}{2(n-1)} \Delta R$$


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is called $Q$-curvature with respect to the metric $g$,

\[ a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)}, \quad b_n = -\frac{4}{n-2}, \]

\[ c_n = -\frac{2}{(n-2)^2}, \quad d_n = \frac{n(n-2)^2 - 16}{8(n-1)(n-2)^2}. \]

This operator $P_g$ is also called Paneitz operator or Branson-Paneitz operator. It is known that Paneitz operator is conformally invariant of bi-degree $\left(\frac{n-4}{2}, \frac{n+4}{2}\right)$, that is, under conformal transformation of Riemannian metric $g = e^{2w}g_0$, the Paneitz operator $P_g$ changes into

\[ P_g f = e^{-\frac{n+4}{2}w}P_{g_0}(e^{\frac{n+4}{2}w}f). \]

Let $\mathfrak{M}(M^n)$ be the set of Riemannian metrics on $M^n$. For each $g \in \mathfrak{M}(M^n)$, the total $Q$-curvature for $g$ is defined by

\[ Q[g] = \int_{M^n} Qdv. \]

When $n = 4$, from the Gauss-Bonnet theorem for dimension 4, we have

\[ Q[g] = -\frac{1}{4} \int_{M^4} |W|^2dv + 8\pi^2\chi(M^4), \]

where $W$ is the Weyl conformal curvature tensor and $\chi(M^4)$ is the Euler characteristic of $M^4$. Hence, we know that the total $Q$-curvature is a conformal invariant for dimension 4. In [14], Nishikawa has studied the variation of the total $Q$-curvature for a general dimension $n$. He has proved that a Riemannian metric $g$ on an $n(n \neq 4)$-dimensional compact manifold $M^n$ is a critical point of the total $Q$-curvature functional with respect to a volume preserving conformal variation of the metric $g$, if and only if the $Q$-curvature with respect to the metric $g$ is constant.

Since the Paneitz operator $P_g$ is an elliptic operator and $P_g1 = 0$ for $n = 4$, we know that $\lambda_0 = 0$ is an eigenvalue of $P_g$. Gursky [10] shown that if the Yamabe invariant of $M^4$ is positive and the total $Q$-curvature is positive, the first eigenvalue $\lambda_1$ is positive. For $n \geq 6$, Yang and Xu [16] have proved the Paneitz operator $P_g$ is positive if the scalar curvature is positive and $Q$-curvature is nonnegative. Furthermore, see [2, 4, 6, 12].

For $n \geq 3$, we consider the following closed eigenvalue problem on an $n$-dimensional compact manifold $M^n$:

\[ P_g u = \lambda u. \]

Since $P_g$ is an elliptic operator, the spectrum of $P_g$ on $M^n$ is discrete. We assume

\[ 0 < \lambda_1 < \lambda_2 \leq \cdots , \lambda_k \leq \cdots \to +\infty \]

for $n \neq 4$ and for $n = 4$,

\[ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots , \lambda_k \leq \cdots \to +\infty. \]
When $n = 4$, Yang and Xu [17] have introduced an $N$-conformal energy $E_c(N, M^4)$ if $M^4$ can be conformally immersed into the unit sphere $S^N(1)$ and have obtained an upper bound for the first eigenvalue $\lambda_1$:

$$\lambda_1 \text{vol}(M^4) \leq E_c(N, M^4),$$

where $\text{vol}(M^n)$ denotes the volume of $M^n$. Furthermore, Chen and Li [8] have also studied the upper bound on the first eigenvalue $\lambda_1$ when $M^4$ is considered as a compact submanifold in a Euclidean space $\mathbb{R}^N$. They have proved

$$\lambda_1 \leq \frac{\int_{M^4}(16|H|^2 + \frac{2}{3}R)dv \int_{M^4}|H|^2dv}{\text{vol}(M^4)^2}$$

and the equality holds if and only if $M^4$ is a minimal submanifold in a sphere $S^{N-1}(r)$ for $N > 5$ and $M^4$ is a round sphere $S^4(r)$ for $N = 5$. In [9], the second eigenvalue $\lambda_2$ of the Paneitz operator $P_g$ is studied. By making use of the conformal transformation introduced by Li and Yau [13], Chen and Li proved, for $n \geq 7$,

$$\lambda_2 \text{vol}(M^n) \leq \frac{1}{2}n(n^2 - 4) \int_{M^n}|H|^4dv + \frac{n - 4}{2} \int_{M^n}Qdv$$

if $M^n$ is a compact submanifold in the Euclidean space $\mathbb{R}^N$. Here $|H|$ denotes the mean curvature of $M^n$ in $\mathbb{R}^N$. As they remarked, their method does not work for $3 \leq n \leq 6$.

The purpose of this paper is to study eigenvalues of the Paneitz operator $P_g$ in $n$-dimensional compact Riemannian manifolds. Our method is very different from one used by Chen and Li [9] and Xu and Yang [17]. From Nash’s theorem, we know that each compact Riemannian manifold can be isometrically immersed into a Euclidean space $\mathbb{R}^N$. Thus, we can assume $M^n$ is an $n$-dimensional compact submanifold in $\mathbb{R}^N$.

**Theorem 1.1.** Let $(M^4, g)$ be a 4-dimensional compact submanifold with the metric $g$ in $\mathbb{R}^N$. Then, eigenvalues of the Paneitz operator $P_g$ satisfy

$$\sum_{j=1}^{4} \lambda_j^\frac{1}{2} \leq 4 \sqrt{\int_{M^4}(16|H|^2 + \frac{2}{3}R)dv \int_{M^4}|H|^2dv} \text{vol}(M^4)$$

and the equality holds if and only if $M^4$ is a round sphere $S^4(r)$ for $N = 5$ and $M^4$ is a compact minimal submanifold with constant scalar curvature in $S^{N-1}(r)$ for $N > 5$.

**Corollary 1.1.** Let $(M^4, g)$ be a 4-dimensional compact submanifold with the metric $g$ in the unit sphere $S^N(1)$. Then, eigenvalues of the Paneitz operator $P_g$ satisfy

$$\sum_{j=1}^{4} \lambda_j^\frac{1}{2} \leq 4 \sqrt{\int_{M^4}(16|H|^2 + 16 + \frac{2}{3}R)dv \int_{M^4}(|H|^2 + 1)dv} \text{vol}(M^4)$$

and the equality holds if and only if $M^4$ is a compact minimal submanifold with constant scalar curvature in $S^N(1)$. 


Theorem 1.2. Let \((M^n, g)\) \((n > 4)\) be an \(n\)-dimensional compact submanifold with the metric \(g\) in \(\mathbb{R}^N\). Then, eigenvalues of the Paneitz operator \(P_g\) satisfy

\[
\sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1)^{\frac{1}{2}} \leq \sqrt{\int_{M^n} \frac{n(n^2 - 4)|H|^2}{2}u_1^2 dv + 2(n + 2) \int_{M^n} g(\nabla u_1, \nabla u_1) dv} \\
\times \sqrt{\int_{M^n} n^2|H|^2u_1^2 dx + 4 \int_{M^n} g(\nabla u_1, \nabla u_1) dv}
\]

and the equality holds if and only if \(M^n\) is isometric to a sphere \(S^n(r)\), where \(u_1\) is the normalized first eigenfunction of \(P_g\).

Remark 1.1. In our theorem 1.2, we do not need to assume the positivity of the Paneitz operator \(P_g\).

If the Paneitz operator \(P_g\) is a positive operator, we have

Theorem 1.3. Let \((M^n, g)\) \((n \neq 4)\) be an \(n\)-dimensional compact submanifold with the metric \(g\) in the unit sphere \(S^N(1)\). Then, eigenvalues of the Paneitz operator \(P_g\) satisfy

\[
\sum_{j=1}^{n} \lambda_j^{\frac{1}{2}} < n \sqrt{\int_{M^n} ((n^2|H|^2 + n^2) + (na_n + b_n)R + \frac{n - 4}{2}Q) dv \int_{M^n} |H|^2 + 1 dv} \frac{1}{\text{vol}(M^n)}.
\]

2. Eigenvalues of the Paneitz Operator on \(M^4\)

Since \(M^n\) is an \(n\)-dimensional submanifold in \(\mathbb{R}^N\). Let \((x_1, \ldots, x_n)\) be a local coordinate system in a neighborhood \(U\) of \(p \in M^n\). Let \(y\) be the position vector of \(p\) in \(\mathbb{R}^N\), which is defined by

\[
y = (y_1(x_1, \ldots, x_n), \ldots, y_N(x_1, \ldots, x_n)).
\]

Let \(g\) denote the induced metric of \(M^n\) from \(\mathbb{R}^N\) and \(<,>\) is the standard inner product in \(\mathbb{R}^N\). Thus, we have
Lemma 2.1. For any function \( u \in C^\infty(M^n) \), we have
\[
g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \left< \sum_{\alpha=1}^{N} \frac{\partial y_\alpha}{\partial x_i}, \sum_{\beta=1}^{N} \frac{\partial y_\beta}{\partial x_j} \right> = \sum_{\alpha=1}^{N} \frac{\partial y_\alpha}{\partial x_i} \frac{\partial y_\alpha}{\partial x_j},
\]
\[
\sum_{\alpha=1}^{N} (g(\nabla y_\alpha, \nabla u))^2 = |\nabla u|^2,
\]
\[
\sum_{\alpha=1}^{N} g(\nabla y_\alpha, \nabla y_\alpha) = \sum_{\alpha=1}^{N} |\nabla y_\alpha|^2 = n,
\]
\[
\sum_{\alpha=1}^{N} (\Delta y_\alpha)^2 = n^2 |H|^2,
\]
\[
\sum_{\alpha=1}^{N} \Delta y_\alpha \nabla y_\alpha = 0,
\]
where \( \nabla \) denotes the gradient operator on \( M^n \) and \( |H| \) is the mean curvature of \( M^n \).

Proof of Theorem 1. Let \( u_i \) be an eigenfunction corresponding to eigenvalue \( \lambda_i \) such that \( \{u_i\}_{i=0}^\infty \) becomes an orthonormal basis of \( L^2(M^n) \), that is,
\[
\begin{cases}
Pg u_i = \lambda_i u_i, \\
\int_{M^n} u_i u_j dv = \delta_{ij}, \quad i, j = 0, 1, \ldots
\end{cases}
\]
We define an \( N \times N \)-matrix \( A \) as follows:
\[
A := (a_{\alpha\beta})
\]
where \( a_{\alpha\beta} = \int_{M^n} y_\alpha u_0 u_\beta dv \), for \( \alpha, \beta = 1, 2, \ldots, N \), and \( y = (y_\alpha) \) is the position vector of the immersion in \( \mathbb{R}^N \). Using the orthogonalization of Gram and Schmidt, we know that there exist an upper triangle matrix \( T = (T_{\alpha\beta}) \) and an orthogonal matrix \( U = (q_{\alpha\beta}) \) such that \( T = U A \), i.e.,
\[
T_{\alpha\beta} = \sum_{\gamma=1}^{N} q_{\alpha\gamma} a_{\gamma\beta} = \int_{M^n} \sum_{\gamma=1}^{N} q_{\alpha\gamma} y_\gamma u_0 u_\beta dv = 0, \quad 1 \leq \beta < \alpha \leq N.
\]
Defining \( z_\alpha = \sum_{\gamma=1}^{N} q_{\alpha\gamma} y_\gamma \), we get
\[
\int_{M^n} z_\alpha u_0 u_\beta dv = \int_{M^n} \sum_{\gamma=1}^{N} q_{\alpha\gamma} y_\gamma u_0 u_\beta dv = 0, \quad 1 \leq \beta < \alpha \leq N.
\]
Putting
\[
\psi_\alpha := (z_\alpha - b_\alpha) u_0, \quad b_\alpha := \int_{M^n} z_\alpha u_0^2 dv, \quad 1 \leq \alpha \leq N,
\]
we infer
\[
\int_{M^n} \psi_\alpha u_\beta dv = 0, \quad 0 \leq \beta < \alpha \leq N.
\]
Thus, from the Rayleigh-Ritz inequality, we have
\[ \lambda_{\alpha} \int_{M^4} \psi_{\alpha}^2 dv \leq \int_{M^4} \psi_{\alpha} P_g \psi_{\alpha} dv, \quad 1 \leq \alpha \leq N. \]

Since \( u_0 \) is constant and
\[ (2.2) \quad P_g \psi_{\alpha} = \Delta^2 (z_\alpha u_0) - \text{div}\left[ \left( \frac{2}{3} Rg - 2 \text{Ric} \right) \nabla (z_\alpha u_0) \right], \]
according to the Stokes formula, we derive
\[
\int_{M^4} \psi_{\alpha} P_g \psi_{\alpha} dv = \int_{M^4} \left[ (\Delta z_\alpha)^2 u_0^2 + g((\frac{2}{3} Rg - 2 \text{Ric}) \nabla z_\alpha, \nabla z_\alpha) u_0^2 \right] dv.
\]

From the lemma 2.1, we have
\[
\sum_{\alpha=1}^{N} \int_{M^4} \psi_{\alpha} P_g \psi_{\alpha} dv = \sum_{\alpha=1}^{N} \int_{M^4} \left[ (\Delta z_\alpha)^2 u_0^2 + g((\frac{2}{3} Rg - 2 \text{Ric}) \nabla z_\alpha, \nabla z_\alpha) u_0^2 \right] dv = \int_{M^4} (16|H|^2 + \frac{2}{3} R) u_0^2 dv.
\]

Hence,
\[ (2.3) \quad \sum_{\alpha=1}^{N} \lambda_{\alpha} \int_{M^4} \psi_{\alpha}^2 dv \leq \int_{M^4} (16|H|^2 + \frac{2}{3} R) u_0^2 dv. \]

On the other hand,
\[
\int_{M^4} \psi_{\alpha} (u_0 \Delta z_\alpha) dv = \int_{M^4} (z_\alpha u_0 - u_0 b_\alpha)(u_0 \Delta z_\alpha) dv = -\int_{M^4} |\nabla (z_\alpha u_0)|^2 dv.
\]

Therefore, for any positive \( \delta > 0 \), we obtain from (2.3)
\[
\lambda_{\alpha}^\frac{1}{2} \int_{M^4} |\nabla (z_\alpha u_0)|^2 dv = -\lambda_{\alpha}^\frac{1}{2} \int_{M^4} \psi_{\alpha} (u_0 \Delta z_\alpha) dv \leq \frac{1}{2} (\delta \lambda_{\alpha} \int_{M^4} \psi_{\alpha}^2 dv + \frac{1}{\delta} \int_{M^4} (u_0 \Delta z_\alpha)^2 dv).
\]
\[
\sum_{\alpha=1}^{N} \lambda_{\alpha}^{\frac{1}{2}} \int_{M^4} |\nabla (z_{\alpha} u_{0})|^2 dv \\
\leq \frac{1}{2} (\delta \sum_{\alpha=1}^{N} \lambda_{\alpha} \int_{M^4} \psi_{\alpha}^2 dv + \frac{1}{\delta} \sum_{\alpha=1}^{N} \int_{M^4} (u_{0} \Delta z_{\alpha})^2 dv) \\
\leq \frac{1}{2} (\delta \int_{M^4} (16|H|^2 + \frac{2}{3}R) u_{0}^2 dv + \frac{1}{\delta} \int_{M^4} 16|H|^2 u_{0}^2 dv).
\]

It is not hard to prove that, for any point and for any \(\alpha\),
\[
|\nabla z_{\alpha}|^2 = g(\nabla z_{\alpha}, \nabla z_{\alpha}) \leq 1.
\]

Hence,
\[
\sum_{\alpha=1}^{N} \lambda_{\alpha}^{\frac{1}{2}} |\nabla z_{\alpha}|^2 \\
\geq \sum_{i=1}^{4} \lambda_{i}^{\frac{1}{2}} |\nabla z_{i}|^2 + \frac{3}{5} \sum_{A=5}^{N} |\nabla z_{A}|^2 \\
= \sum_{i=1}^{4} \lambda_{i}^{\frac{1}{2}} |\nabla z_{i}|^2 + \frac{3}{5} (4 - \sum_{j=1}^{4} |\nabla z_{j}|^2) \\
\geq \sum_{i=1}^{4} \lambda_{i}^{\frac{1}{2}} |\nabla z_{i}|^2 + \sum_{j=1}^{4} \lambda_{j}^{\frac{1}{2}} (1 - |\nabla z_{j}|^2) \\
\geq \sum_{j=1}^{4} \lambda_{j}^{\frac{1}{2}}.
\]

We obtain, by (2.4) and (2.5),
\[
\sum_{j=1}^{4} \lambda_{j}^{\frac{1}{2}} \leq \frac{1}{2} (\delta \int_{M^4} (16|H|^2 + \frac{2}{3}R) u_{0}^2 dv + \frac{1}{\delta} \int_{M^4} 16|H|^2 u_{0}^2 dv).
\]

Taking
\[
\frac{1}{\delta} = \sqrt{\frac{\int_{M^4} (16|H|^2 + \frac{2}{3}R) u_{0}^2 dv}{\int_{M^4} 16|H|^2 u_{0}^2 dv}},
\]
we have, because of \(u_{0} = \sqrt{\frac{1}{\text{vol}(M^4)}}\),
\[
\sum_{j=1}^{4} \lambda_{j}^{\frac{1}{2}} \leq 4 \frac{\sqrt{\int_{M^4} (16|H|^2 + \frac{2}{3}R) dv \int_{M^4} |H|^2 dv}}{\text{vol}(M^4)}.
\]

If the equality holds, we have
\[
\lambda_{1} = \lambda_{2} = \cdots = \lambda_{N},
\]
\[ (2.7) \quad \Delta (z_\alpha - b_\alpha) = -\sqrt{\lambda_5} \delta(z_\alpha - b_\alpha). \]

According to Takahashi’s theorem, we know that \( M^4 \) is a round sphere \( S^4(r) \) for \( N = 5 \) and \( M^4 \) is a minimal submanifold in a sphere \( S^{N-1}(r) \) for \( N > 5 \) with \( \sum_{\alpha=1}^{N}(z_\alpha - b_\alpha)^2 = r^2 \). Thus, we have

\[ \lambda_1 = \lambda_2 = \cdots = \lambda_N = \frac{16}{r^4\delta^2}. \]

From the definition of the Paneitz operator \( P_g \), we have

\[ (2.8) \quad P_g(z_\alpha - b_\alpha) = \Delta^2(z_\alpha - b_\alpha) - \text{div}\left[ \left( \frac{2}{3} R_g - 2 \text{Ric} \right) \nabla(z_\alpha - b_\alpha) \right], \]

that is, from (2.7) and (2.8), we have

\[ \lambda_5(1 - \delta^2)(z_\alpha - b_\alpha) = -\text{div}\left[ \left( \frac{2}{3} R_g - 2 \text{Ric} \right) \nabla(z_\alpha - b_\alpha) \right]. \]

According to \( \sum_{\alpha=1}^{N}(z_\alpha - b_\alpha)^2 = r^2 \), we obtain

\[ \lambda_5(1 - \delta^2)r^2 = \sum_{\alpha=1}^{N} g\left( \left( \frac{2}{3} R_g - 2 \text{Ric} \right) \nabla(z_\alpha - b_\alpha), \nabla(z_\alpha - b_\alpha) \right). \]

Hence,

\[ \lambda_5(1 - \delta^2)r^2 = \frac{2}{3} R. \]

Thus, the scalar curvature \( R \) is constant. Hence, \( M^4 \) is a compact minimal submanifold with constant scalar curvature in a sphere \( S^{N-1}(r) \). This finishes the proof of theorem 1.1.

**Proof of Corollary 1.1.** Since the unit sphere \( S^N(1) \) is a hypersurface in \( \mathbb{R}^{N+1} \) with the mean curvature \( 1 \), \( M^4 \) can be seen as a compact submanifold in \( \mathbb{R}^{N+1} \) with the mean curvature \( \sqrt{\|H\|^2 + 1} \). According to the theorem 1.1, we complete the proof of the corollary 1.1.

### 3. Eigenvalues of the Paneitz operator on \( M^n \) (\( n \neq 4 \))

**Proof of theorem 1.2.** Since \( n > 4 \), eigenvalues of the Paneitz operator \( P_g \) satisfy

\[ \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty. \]

Let \( u_i \) be an eigenfunction corresponding to eigenvalue \( \lambda_i \) such that \( \{u_i\}_{i=1}^\infty \) becomes an orthonormal basis of \( L^2(M^n) \), that is,

\[
\begin{align*}
\begin{cases}
P_g u_i = \lambda_i u_i, \\
\int_{M^n} u_i u_j dv = \delta_{ij}, \quad i, j = 1, 2, \ldots.
\end{cases}
\end{align*}
\]
We shall use the same idea to prove the theorem 1.2. But, in this case, we need to use the first eigenfunction $u_1$, which is not constant in general. Thus, we need to compute many formulas. We define an $N \times N$-matrix $A$ as follows:

$$A := (a_{\alpha\beta})$$

where $a_{\alpha\beta} = \int_{M^n} y_\alpha u_1 u_{\beta+1} dv$, for $\alpha, \beta = 1, 2, \ldots, N$, and $y = (y_\alpha)$ is the position vector of the immersion in $\mathbb{R}^N$. Thus, there is an orthogonal matrix $U = (q_{\alpha\beta})$ such that

$$\int_{M^n} z_\alpha u_1 u_{\beta+1} dv = 0, \quad 1 \leq \beta < \alpha \leq N,$$

where $z_\alpha = \sum_{\gamma=1}^{N} q_{\alpha\gamma} y_\gamma$. Putting

$$\varphi_\alpha := (z_\alpha - a_\alpha) u_1, \quad a_\alpha := \int_{M^n} z_\alpha u_1^2 dv, \quad 1 \leq \alpha \leq N,$$

we infer

$$\int_{M^n} \varphi_\alpha u_{\beta} dv = 0, \quad 1 \leq \beta < \alpha \leq N.$$

Thus, from the Rayleigh-Ritz inequality, we have

$$\lambda_{\alpha+1} \int_{M^n} \varphi_\alpha^2 dv \leq \int_{M^n} \varphi_\alpha P_g \varphi_\alpha dv, \quad 1 \leq \alpha \leq N,$$

$$P_g(z_\alpha u_1) = P_g(z_\alpha u_1) - a_\alpha P_g u_1 = P_g(z_\alpha u_1) - \lambda_1 a_\alpha u_1.$$

$$P_g(z_\alpha u_1) = \Delta^2(z_\alpha u_1) - \text{div} [(a_n R g + b_n \text{Ric}) \nabla (z_\alpha u_1)] + \frac{n - 4}{2} Q(z_\alpha u_1)$$

$$= \Delta^2 z_\alpha u_1 + 2 \Delta z_\alpha \Delta u_1 + 2 \Delta g(\nabla z_\alpha, \nabla u_1) + 2 g(\nabla z_\alpha, \nabla (\Delta u_1)) + z_\alpha \Delta^2 u_1 + 2 g(\nabla (\Delta z_\alpha), \nabla u_1)$$

$$- \text{div} [u_1(a_n R g + b_n \text{Ric}) \nabla z_\alpha] - \text{div} [z_\alpha (a_n R g + b_n \text{Ric}) \nabla u_1] + \frac{n - 4}{2} Q(z_\alpha u_1)$$

$$= \Delta^2 z_\alpha u_1 + 2 \Delta z_\alpha \Delta u_1 + 2 \Delta g(\nabla z_\alpha, \nabla u_1) + 2 g(\nabla z_\alpha, \nabla (\Delta u_1)) + 2 g(\nabla (\Delta z_\alpha), \nabla u_1)$$

$$- \text{div} [u_1(a_n R g + b_n \text{Ric}) \nabla z_\alpha] - g(\nabla z_\alpha, (a_n R g + b_n \text{Ric}) \nabla u_1) + z_\alpha P u_1$$

$$= r_\alpha + \lambda_1 z_\alpha u_1$$

with

$$r_\alpha = \Delta^2 z_\alpha u_1 + 2 \Delta z_\alpha \Delta u_1 + 2 \Delta g(\nabla z_\alpha, \nabla u_1) + 2 g(\nabla (\Delta z_\alpha), \nabla u_1)$$

$$- \text{div} [u_1(a_n R g + b_n \text{Ric}) \nabla z_\alpha] - g(\nabla z_\alpha, (a_n R g + b_n \text{Ric}) \nabla u_1).$$

According to the Stokes formula, we derive

$$\int_{M^n} r_\alpha u_1 dv = 0.$$
Letting
\[ w_\alpha = \int_{M^n} r_\alpha \varphi_\alpha dv \]
\[ \int_{M^n} \varphi_\alpha P_\alpha \varphi_\alpha dv = \int_{M^n} \varphi_\alpha (P_\alpha(z_\alpha u_1) - \lambda_1 a_\alpha u_1) dv \]
\[ = \int_{M^n} \varphi_\alpha (r_\alpha + \lambda_1 \varphi_\alpha) dv. \]

Hence,
\[ (3.2) \quad (\lambda_{\alpha+1} - \lambda_1) \int_{M^n} \varphi^2_\alpha dv \leq \int_{M^n} \varphi_\alpha r_\alpha dv = w_\alpha = \int_{M^n} z_\alpha u_1 r_\alpha dv, \quad 1 \leq \alpha \leq N. \]

By a direct calculation, we obtain
\[
2 \int_{M^n} z_\alpha u_1 g(\nabla (\Delta z_\alpha), \nabla u_1) dv = \int_{M^n} (\Delta z_\alpha)^2 u_1^2 dv \\
+ \int_{M^n} \Delta z_\alpha g(\nabla z_\alpha, \nabla u_1^2) dv - \int_{M^n} (z_\alpha \Delta^2 z_\alpha) u_1^2 dv,
\]
\[
2 \int_{M^n} z_\alpha u_1 \Delta g(\nabla z_\alpha, \nabla u_1) dv = 2 \int_{M^n} u_1 \Delta z_\alpha g(\nabla z_\alpha, \nabla u_1) dv \\
+ 2 \int_{M^n} z_\alpha \Delta u_1 g(\nabla z_\alpha, \nabla u_1) dv + 4 \int_{M^n} g(\nabla z_\alpha, \nabla u_1^2) dv \\
2 \int_{M^n} z_\alpha u_1 g(\nabla z_\alpha, \nabla (\Delta u_1)) dv = -2 \int_{M^n} u_1 z_\alpha \Delta z_\alpha \Delta u_1 dv \\
- 2 \int_{M^n} u_1 \Delta u_1 g(\nabla z_\alpha, \nabla z_\alpha) dv - 2 \int_{M^n} z_\alpha g(\nabla z_\alpha, \nabla u_1) \Delta u_1 dv.
\]

Thus, we derive
\[ (\lambda_{\alpha+1} - \lambda_1) \int_{M^n} \varphi^2_\alpha dv \leq w_\alpha = \int_{M^n} z_\alpha u_1 r_\alpha dv \]
\[ (3.3) \quad = \int_{M^n} (u_1 \Delta z_\alpha + 2 g(\nabla z_\alpha, \nabla u_1))^2 dv + \int_{M^n} u_1^2 g((a_n Rg + b_n \text{Ric}) \nabla z_\alpha, \nabla z_\alpha) dv \\
- 2 \int_{M^n} g(\nabla z_\alpha, \nabla z_\alpha) u_1 \Delta u_1 dv, \quad 1 \leq \alpha \leq N. \]

From the lemma 2.1, we have
\[
\sum_{\alpha=1}^{N} (\lambda_{\alpha+1} - \lambda_1) \int_{M^n} \varphi^2_\alpha dv \\
\leq \int_{M^n} (n^2 |H|^2 + (n a_n + b_n) R) u_1^2 dv + 2(n + 2) \int_{M^n} g(\nabla u_1, \nabla u_1) dv.
\]
On the other hand,
\[ \int_{M^n} \varphi_\alpha \left( u_1 \Delta z_\alpha + 2g(\nabla z_\alpha, \nabla u_1) \right) dv \]
(3.5)
\[ = \int_{M^n} (z_\alpha - a_\alpha) u_1 \left( u_1 \Delta z_\alpha + 2g(\nabla z_\alpha, \nabla u_1) \right) dv \]
\[ = - \int_{M^n} |u_1 \nabla z_\alpha|^2 dv. \]
Therefore, for any positive \( \delta > 0 \), we obtain, from (3.5),
\[ (\lambda_{\alpha+1} - \lambda_1)^{\frac{1}{2}} \int_{M^n} |u_1 \nabla z_\alpha|^2 dv \]
(3.6)
\[ = -(\lambda_{\alpha+1} - \lambda_1)^{\frac{1}{2}} \int_{M^n} \varphi_\alpha \left( u_1 \Delta z_\alpha + 2g(\nabla z_\alpha, \nabla u_1) \right) dv \]
\[ \leq \frac{1}{2} \left\{ \delta (\lambda_{\alpha+1} - \lambda_1) \int_{M^n} \varphi_\alpha^2 dv + \frac{1}{\delta} \int_{M^n} \left( u_1 \Delta z_\alpha + 2g(\nabla z_\alpha, \nabla u_1) \right)^2 dv \right\}. \]
According to (3.4) and (3.6), we infer
\[ \sum_{\alpha=1}^{N} (\lambda_{\alpha+1} - \lambda_1)^{\frac{1}{2}} \int_{M^n} |u_1 \nabla z_\alpha|^2 dv \]
(3.7)
\[ \leq \frac{1}{2} \left\{ \delta \sum_{\alpha=1}^{N} (\lambda_{\alpha+1} - \lambda_1) \int_{M^n} \varphi_\alpha^2 dv + \frac{1}{\delta} \sum_{\alpha=1}^{N} \int_{M^n} \left( u_1 \Delta z_\alpha + 2g(\nabla z_\alpha, \nabla u_1) \right)^2 dv \right\} \]
\[ \leq \frac{1}{2} \left\{ \int_{M^n} (n^2|H|^2 + (na_n + b_n)R) u_1^2 dv + 2(n + 2) \int_{M^n} g(\nabla u_1, \nabla u_1) dv \right\} \]
\[ + \frac{1}{2\delta} \left\{ \int_{M^n} n^2|H|^2 u_1^2 dv + 4 \int_{M^n} g(\nabla u_1, \nabla u_1) dv \right\}. \]
By the same proof as the formula (2.5) in the section 2, we have
\[ \sum_{\alpha=1}^{N} (\lambda_{\alpha+1} - \lambda_1)^{\frac{1}{2}} |\nabla z_\alpha|^2 \geq \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1)^{\frac{1}{2}}. \]
(3.8)
Hence, we obtain
\[ \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1)^{\frac{1}{2}} \]
(3.9)
\[ \leq \frac{1}{2} \delta \left( \int_{M^n} (n^2|H|^2 + (na_n + b_n)R) u_1^2 dv + 2(n + 2) \int_{M^n} g(\nabla u_1, \nabla u_1) dv \right) \]
\[ + \frac{1}{2\delta} \left( \int_{M^n} n^2|H|^2 u_1^2 dv + 4 \int_{M^n} g(\nabla u_1, \nabla u_1) dv \right). \]
Letting \( S \) denote the squared norm of the second fundamental form of \( M^n \), from the Gauss equation, we have
\[ R = n(n - 1)|H|^2 - (S - n|H|^2) \leq n(n - 1)|H|^2. \]
Since
\[ na_n + b_n = \frac{n^2 - 2n - 4}{2(n - 1)} > 0, \]
we have
\[ n^2|H|^2 + (na_n + b_n)R \leq \frac{n(n^2 - 4)|H|^2}{2}. \]
Taking
\[ \frac{1}{\delta} = \sqrt{\frac{\int_{M^n} n^2|H|^2u_1^2dv + 4\int_{M^n} g(\nabla u_1, \nabla u_1)dv}{\int_{M^n} (n^2 - 4)|H|^2u_1^2dv + 2(n + 2)\int_{M^n} g(\nabla u_1, \nabla u_1)dv}} \]
we have
\[ \sum_{j=1}^{n}(\lambda_{j+1} - \lambda_1)^{\frac{1}{2}} \leq \sqrt{\frac{\int_{M^n} n(n^2 - 4)|H|^2u_1^2dv + 2(n + 2)\int_{M^n} g(\nabla u_1, \nabla u_1)dv}{\int_{M^n} n^2|H|^2u_1^2dv + 4\int_{M^n} g(\nabla u_1, \nabla u_1)dv}} \]
(3.10)
If the equality holds, we have
\[ \lambda_2 = \lambda_3 = \cdots = \lambda_N, \]
and \( S \equiv n|H|^2 \). Thus, \( M^n \) is totally umbilical, that is, \( M^n \) is isometric to a sphere. It completes the proof of the theorem 1.2.

**Corollary 3.1.** Let \((M^n, g) (n > 4)\) be an \( n \)-dimensional compact submanifold with the metric \( g \) in the unit sphere \( S^n(1) \). Then, eigenvalues of the Paneitz operator \( P_g \) satisfy
\[ \sum_{j=1}^{n}(\lambda_{j+1} - \lambda_1)^{\frac{1}{2}} \leq \sqrt{\frac{\int_{M^n} n(n^2 - 4)(|H|^2 + 1)u_1^2dv + 2(n + 2)\int_{M^n} g(\nabla u_1, \nabla u_1)dv}{\int_{M^n} n^2(|H|^2 + 1)u_1^2dv + 4\int_{M^n} g(\nabla u_1, \nabla u_1)dv}} \]
(3.11)
and the equality holds if and only if \( M^n \) is isometric to a sphere \( S^n(r) \), where \( u_1 \) is the normalized first eigenfunction of \( P_g \).

**Proof of Theorem 1.3.** Since \( n \neq 4 \), we assume that eigenvalues of the Paneitz operator \( P_g \) satisfy
\[ 0 < \lambda_1 < \lambda_2 \leq \cdots, \lambda_k \leq \cdots \to +\infty. \]
Let $u_i$ be an eigenfunction corresponding to eigenvalue $\lambda_i$ such that $\{u_i\}_{i=1}^{\infty}$ becomes an orthonormal basis of $L^2(M^n)$, that is,
\[
\begin{cases}
P_g u_i = \lambda_i u_i, \\
\int_{M^n} u_i u_j dv = \delta_{ij}, \quad i, j = 1, 2, \ldots.
\end{cases}
\]
We shall use the similar method to prove the theorem 1.3. We define an $(N+1) \times (N+1)$-matrix $A$ as follows:
\[
A := (a_{\alpha \beta})
\]
where $a_{\alpha \beta} = \int_{M^n} y_\alpha u_\beta dv$, for $\alpha, \beta = 1, 2, \ldots, N+1$, and $y = (y_\alpha)$ is the position vector of the immersion in $\mathbb{R}^{N+1}$ with $|y|^2 = \sum_{\alpha=1}^{N+1} y_\alpha^2 = 1$. Thus, there is an orthogonal matrix $U = (q_{\alpha \beta})$ such that
\[
\int_{M^n} z_\alpha u_\beta dv = 0, \quad 1 \leq \beta < \alpha \leq N+1,
\]
where $z_\alpha = \sum_{\gamma=1}^{N+1} q_{\alpha \gamma} y_\gamma$. Since $U$ is an orthogonal matrix, we have
\[
\sum_{\alpha=1}^{N+1} z_\alpha^2 = 1.
\]
Putting $\psi_\alpha := z_\alpha, \quad 1 \leq \alpha \leq N+1,$ we infer
\[
\int_{M^n} \psi_\alpha u_\beta dv = 0, \quad 1 \leq \beta < \alpha \leq N+1.
\]
Thus, from the Rayleigh-Ritz inequality, we have
\[
\lambda_\alpha \int_{M^n} \psi_\alpha^2 dv \leq \int_{M^n} \psi_\alpha P_g \psi_\alpha dv, \quad 1 \leq \alpha \leq N+1.
\]
(3.12)
\[
P_g \psi_\alpha = P_g(z_\alpha).
\]
According to the Stokes formula, we derive
\[
\int_{M^n} \psi_\alpha P_g \psi_\alpha dv = \int_{M^n} \left[ (\Delta z_\alpha)^2 + g((a_n Rg + b_n Ric) \nabla z_\alpha, \nabla z_\alpha) + \frac{n-4}{2} Q(z_\alpha)^2 \right] dv
\]
From the lemma 2.1, we have
\[
\sum_{\alpha=1}^{N+1} \int_{M^n} \psi_\alpha P_g \psi_\alpha dv
\]
(3.13)
\[
= \sum_{\alpha=1}^{N+1} \int_{M^n} \left[ (\Delta z_\alpha)^2 + g((a_n Rg + b_n Ric) \nabla z_\alpha, \nabla z_\alpha) + \frac{n-4}{2} Q(z_\alpha)^2 \right] dv
\]
\[
= \int_{M^n} \left( (n^2|H|^2 + n^2) + (na_n + b_n) R + \frac{n-4}{2} Q \right) dv.
\]
Hence,

\[ \sum_{\alpha=1}^{N+1} \lambda_{\alpha} \int_{M^n} \psi_{\alpha}^2 dv \leq \int_{M^n} \left( (n^2|H|^2 + n^2) + (n\alpha_n + b_n)R + \frac{n-4}{2}Q \right) dv. \]  

On the other hand,

\[ \int_{M^n} \psi_{\alpha}(\Delta z_{\alpha}) dv = \int_{M^n} z_{\alpha} \Delta z_{\alpha} dv = -\int_{M^n} |\nabla z_{\alpha}|^2 dv. \]

Therefore, for any positive \( \delta > 0 \), we obtain

\[ \lambda_{\alpha}^{\frac{1}{2}} \int_{M^n} |\nabla z_{\alpha}|^2 dv = -\lambda_{\alpha}^{\frac{1}{2}} \int_{M^n} \psi_{\alpha}(\Delta z_{\alpha}) dv \leq \frac{1}{2} (\delta \lambda_{\alpha} \int_{M^n} \psi_{\alpha}^2 dv + \frac{1}{\delta} \int_{M^n} (\Delta z_{\alpha})^2 dv) \]

and

\[ \sum_{\alpha=1}^{N+1} \lambda_{\alpha}^{\frac{1}{2}} \int_{M^n} |\nabla z_{\alpha}|^2 dv \leq \frac{1}{2} \left( \delta \sum_{\alpha=1}^{N+1} \lambda_{\alpha} \int_{M^n} \psi_{\alpha}^2 dv + \frac{1}{\delta} \sum_{\alpha=1}^{N+1} \int_{M^n} (\Delta z_{\alpha})^2 dv \right) \]

\[ \leq \frac{1}{2} \left[ \delta \int_{M^n} \left( (n^2|H|^2 + n^2) + (n\alpha_n + b_n)R + \frac{n-4}{2}Q \right) dv \right. \]

\[ \left. + \frac{1}{\delta} \int_{M^n} (n^2|H|^2 + n^2) dv \right]. \]

By using the same proof as the formula (2.5) in the section 2, we have

\[ \sum_{\alpha=1}^{N+1} \lambda_{\alpha}^{\frac{1}{2}} |\nabla z_{\alpha}|^2 \geq \sum_{j=1}^{n} \lambda_{j}^{\frac{1}{2}}. \]

Thus, we obtain

\[ \sum_{j=1}^{n} \lambda_{j}^{\frac{1}{2}} \text{vol}(M^n) \leq \frac{1}{2} \left[ \delta \int_{M^n} \left( (n^2|H|^2 + n^2) + (n\alpha_n + b_n)R + \frac{n-4}{2}Q \right) dv \right. \]

\[ \left. + \frac{1}{\delta} \int_{M^n} (n^2|H|^2 + n^2) dv \right]. \]

Taking

\[ \frac{1}{\delta} = \sqrt{\int_{M^n} \left( (n^2|H|^2 + n^2) + (n\alpha_n + b_n)R + \frac{n-4}{2}Q \right) dv \int_{M^n} (n^2|H|^2 + n^2) dv} \]
we have
\[ \sum_{j=1}^{n} \frac{1}{\lambda_j^2} \leq \frac{n \sqrt{\int_{M^n} \left( (n^2|H|^2 + n^2) + (na_n + b_n)R + \frac{n-4}{2}Q \right) dv \int_{M^n} (|H|^2 + 1) dv \text{vol}(M^n)}}. \]

If the equality holds, we have
\[ \lambda_2 = \lambda_3 = \cdots = \lambda_{N+1}, \]
\[ |\nabla z_1| \equiv 1 \]
because of \( \lambda_1 < \lambda_2 \) and
\[ \Delta z_1 = -\sqrt{\lambda_1} \delta z_1, \quad \Delta z_\alpha = -\sqrt{\lambda_n} \delta z_\alpha \quad \text{for} \quad \alpha > 1. \]

Since
\[ \sum_{\alpha=1}^{N+1} z_\alpha^2 = 1, \]
we have
\[ n - \sqrt{\lambda_n} \delta + (\sqrt{\lambda_n} \delta - \sqrt{\lambda_1} \delta) z_1^2 = 0. \]
Thus,
\[ \sqrt{\lambda_n} \delta = \sqrt{\lambda_1} \delta \]
or \( z_1^2 \) is constant. It is impossible because \( |\nabla z_1| \equiv 1 \) and \( \lambda_1 < \lambda_2 \). Therefore, the equality does not hold. It completes the proof of the theorem 1.3.

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