Interacting Holographic Dark Energy at the Ricci scale and Dynamical system

Nairwita Mazumder
Ritabrata Biswas
Subenoy Chakraborty

Department of Mathematics, Jadavpur University, Kolkata-700 032, India.

Abstract

In this work, we consider homogeneous and isotropic FRW model of the universe, filled with interacting dark matter and dark energy. The dark matter is chosen as usual in the form of dust while dark energy is holographic in nature with IR cut off at the Ricci’s length and it is in the form of a perfect fluid with variable equation of state. We have chosen the interaction term of the following two types: (i) a linear combination of the matter density of the two fluids, (ii) a product of the two matter densities. For both the choices the evolution equations are transformed to an autonomous system and the corresponding critical points are analyzed. Finally, for the first choice of the interaction term the evolution of the ratio of the energy densities has been studied from the point of view of the present coincidence problem.

Keywords : Dynamical System, Phase Plane, Holographic Dark energy.

Pacs no : 04.20.-q, 04.40.-b, 95.35.+d, 98.80.Cq

1 Introduction

In the last decade there are wide variety of modern cosmological observations namely precision measurements of anisotropies in the cosmological microwave background radiation [1], baryon acoustic oscillation [3] and type Ia supernova [1][2]. These observations indicate that at present our universe is composed of nearly 25% cold dark matter (DM), 70% nonbaryonic unknown matter known as dark energy (DE)[4][5][7][8][9] and 5% of radiation and baryonic matter which ia well understood by the standard models of particles. The natural and leading choice of the unknown DE is the cosmological constant($\Lambda CDM$ model) which represents a vacuum energy density having constant equation of state $\omega = -1$. However, its observed value is far below than the estimation from quantum field theory (known as cosmological constant problem). Also there is no expectation why the constant vacuum energy and matter energy densities are precisely of the same order today (coincidence problem). Due to these observational [10] and theoretical [11][12] probes for the cosmological constant there are alternative models for DE (varies from time) in the literature. Scalar field models [13][14][15] (known as quintessence) have attracted special attention compared to the other alternatives [12].

The fact that at present DE and DM are dominant sources of the content of the universe, there has been a lot of interest in studying coupling in the dark sector components [16][17][18][19][20][21][22][23][24][25][26][27][28][29][30][31][32]. It is partly motivated by the fact that one can only extract information of these components through gravitational interaction. Also consideration of interaction is natural in the framework of field theory [18][19]. Recently, it has been shown that an appropriate choice of the interaction between DE and DM can alleviate the coincidence problem [20][21][22][23][24].

To have some inside about the unknown and mysterious nature of DE, many people have suggested that DE should be compatible with Holographic principle namely ”the number of relevant degrees of freedom of a system
dominated by gravity must vary along with the area of the surface bounding the system\textsuperscript{[25]}. Such a DE model is known as Holographic DE (HDE) model. Further the energy density of any given region should be bound by that ascribed to a Schwarzschild black hole (BH) that fills the same volume\textsuperscript{[26]}. Mathematically, we write \( \rho_D \leq M_p^2 L^{-2} \), where \( \rho_D \) is the DE density, \( L \) is the size of the region (or infrared cut off) and \( M_p = (8\pi G)^{-\frac{1}{2}} \) is the reduced Planck mass. Usually, the DE density is written as

\[
\rho_D = \frac{3M_p^2 c^2}{L^2}
\]  

\( \text{(1)} \)

Here the dimensionless parameter ' \( c^2 \)' takes care of the uncertainties of the theory and for mathematical convenience the factor 3 has been introduced. Due to lack of clear idea there are many choices for the infrared cut off of which the most relevant one are the Hubble radius, i.e., \( L = H^{-1} \)\textsuperscript{[21, 27]} and the Ricci’s length, i.e., \( L = \left( \dot{H} + 2H^2 \right)^{-\frac{1}{2}} \)\textsuperscript{[28, 29, 30, 31]}. The argument behind the choice of Ricci’s length as the IR cut-off is that it corresponds to the size of the maximal perturbation, leading to the formation of a black hole\textsuperscript{[32]}. Another commonly used IR cut-off length is the radius of future event horizon, but it suffers from a severe circularity problem.

In the present paper we consider a cosmological model of Holographic DE (HDE) in the form of a perfect fluid interacting with DM in the form of dust. The choice of the IR cut-off is chosen as the Ricci’s length, i.e.,

\[
L = \left( \dot{H} + 2H^2 \right)^{-\frac{1}{2}}
\]  

\( \text{(2)} \)

The evolution equations of the model are formulated into an autonomous system and critical points are analysed. Explicit solutions are obtained and are analysed asymptotically.

## 2 Basic equations for interacting HDE at the Ricci scale

In the present work, the homogeneous and isotropic FRW universe is assumed to fill up with interacting two fluid system-one component is in the form of dust (having energy density \( \rho_m \)) known as dark matter (DM) while perfect fluid having barotropic equation of state \( \rho_D = \omega_D \rho_D, \omega_D, \) a variable is the DE component.

Assuming spatially flat model, the Friedmann equations are (choosing \( 8\pi G = 1 = c \))

\[
3H^2 = \rho_m + \rho_D
\]  

\( \text{(3)} \)

and

\[
2\dot{H} = -\rho_m - (1 + \omega_D) \rho_D
\]  

\( \text{(4)} \)

and we have the conservation equations

\[
\dot{\rho}_m + 3H \rho_m = Q
\]  

\( \text{(5)} \)

\[
\dot{\rho}_D + 3H (1 + \omega_D) \rho_D = -Q
\]  

\( \text{(6)} \)

Here the interaction term \( Q > 0 \) indicates transfer of energy from DE component to DM sector while opposite is the situation for \( Q < 0 \). As \( Q < 0 \) would worsen the coincidence problem so we choose \( Q > 0 \) throughout the work. Also validity of the second law of thermodynamics and Le chatelier’s principle \textsuperscript{[16, 34]} support this choice of positive \( Q \). It should be noted that we have not included baryonic matter in the interaction due to the constraints imposed by local gravity measurements \textsuperscript{[13, 34, 35]}. In the next two sections we shall deal with two different choices of interaction term separately, namely, \((i)\) \( Q = 3b^2 H \rho \) (\( \rho = \rho_m + \rho_D \), the total energy density) and \((ii)\) \( Q = \gamma \rho_m \rho_D \) (\( \gamma > 0 \)). The first choice is the special case of the usual one used in the literature as a linear combination of the energy densities. The second choice is physically more viable in the sense that interaction rate vanishes if one of the densities is zero and increases with each of the densities. Also this choice of interaction for HDE models gives the best fit to observations \textsuperscript{[16, 34]}. Also it should be noted that the constant \( \gamma \) has the dimension of \([L^3 MT]\).

Using the field equations \textsuperscript{[39]} and \textsuperscript{[41]} we have the form \textsuperscript{[11]} the expression for the energy density of HDE as

\[
\rho_D = \frac{c^2}{2} \{ \rho_m + (1 - 3\omega_D) \rho_D \}
\]  

\( \text{(7)} \)
Hence the equation of state parameter can be expressed in terms of the density parameter as
\[ \omega_D = -\frac{2}{3c^2} + \frac{1}{3\Omega_D} \] (8)

Also the deceleration parameter takes the simple form
\[ q = -\frac{\ddot{a}}{a^2 H^2} = 1 - \frac{\Omega_D}{c^2} = 1 - \frac{1}{c^2 (1 + u)} \] (9)

which shows a smooth transition from deceleration to acceleration as universe evolves from the early matter dominated era to the late time DE dominated and here \( u = \frac{\rho_m}{\rho_D} = \frac{1}{\Omega_D} - 1 \).

### 3 Explicit calculations for choice of interaction term

**Case (I) : \( Q = 3b^2 H \rho \)**

The energy conservation equations can be written explicitly as
\[ \dot{\rho}_m = \sqrt{3} (\rho_m + \rho_D) [b^2 \rho_D - (1 - b^2) \rho_m] \] (10)
\[ \dot{\rho}_D = -3H [b^2 (\rho_m + \rho_D) + \rho_D (1 + \omega_D)] = -\sqrt{3} (\rho_m + \rho_D) [b^2 (\rho_m + \rho_D) + \left(1 + \frac{1}{3\Omega_D} - \frac{2}{3c^2}\right) \rho_D] \] (11)

As a consequence the evolution of the density parameter \( \Omega_D \) and the ratio of the energy densities \( u \) has the form
\[ \dot{\Omega}_D = -H \left[-\frac{2\Omega_D (1 - \Omega_D)}{c^2} + (1 - \Omega_D) + 3b^2 \right] \] (12)
\[ \dot{u} = H \left[-\frac{2u}{c^2} + u (1 + u) + 3b^2 (1 + u)^2 \right] \] (13)

From equation (12) we see that the DE density gradually decreases with the evolution of the universe until it is in the phantom era \((1 + \omega_D < 0)\). Thus if we assume the DE density to be sufficiently large at the early epochs of the evolution then from equation (11) \( \dot{\rho}_m = 0 \) increases till some intermediate stage and then gradually decreases with \( \dot{\rho}_m = 0 \) along the line \( \frac{\rho_m}{\rho_D} = \frac{b^2}{1 - b^2} \) in the \((\rho_m, \rho_D)\) plane. Thus we have \( u \sim O(1) \) at some intermediate stage of evolution in the neighbourhood of the above line and may be a possible resolution of coincidence problem [33, 34]. Further, the present model of the universe shows a DE dominance at the early epochs and subsequently universe evolves with DM as the dominant component and then again it has DE dominated phase at late time as predicted by observation. Hence the above model is suitable for the present universe.

Now to formulate an autonomous system we rewrite equation (12) using field equation (3) as
\[ \rho_D' = - \left[3H^2 \left(1 + 3b^2\right) + \rho_D \left(3 - \frac{2}{c^2}\right) \right] \] (14)
and the second field equation, i.e., equation (4) can be written as
\[ (H^2)' = -2 \left[2H^2 - \frac{\rho_D}{3c^2}\right] \] (15)

where \( ' = \frac{d}{dx} \), \( x = \ln a \) Thus equations (14) and (15) form a linear homogeneous autonomous system in the phase plane \((\rho_D, H^2)\), having critical point at the origin. For the Jacobi matrix \( A \), \( Tr(A) = -7 + \frac{2}{c^2} \) and \( det(A) = \frac{2}{c^2} \left(6c^2 - 5 - 3b^2\right) \).

The nature of the critical points is characterized [36] in the Table1 and the geometrical features are presented in fig 1(a) and 1(b).
Fig. 1(a) - 1(b) represent the variation of $\rho_D - H$. Though in Fig 1(b) the whole region is not a physically valid region but for better understanding about the system and the nature of the critical point we have drawn graph for the whole region.
Table 1: Nature of the Critical Points for Case-I

| Condition | Nature of the eigen values | Type of critical point |
|-----------|---------------------------|------------------------|
| (i) $6c^2 < 5 + 3b^2$ | (i) real roots of opposite sign | (i) Saddle |
| (ii) $c^2 > \min\left(\frac{2}{7}, \frac{b^2}{2} + \frac{5}{8}\right)$ | (ii) both negative real roots | (ii) Stable nodes. |

From equation (13) if $u_f$ be a fixed point, i.e., $\dot{u}|_{u=u_f} = 0$ then the parameter $b^2$ has the expression

$$b^2 = \frac{u_f \left[\frac{2}{3} - 1 - u_f\right]}{3} \left(1 + u_f\right)^2$$  \hspace{1cm} (16)

Using this value of $b^2$ the other fixed point of equation (13) can be expressed in terms of $u_f$ as

$$u_p = \frac{2}{c^2} - (1 + u_f)$$ \hspace{1cm} (17)

Further from equation (13), differentiating once we obtain

$$\frac{du'}{du} = -\frac{2}{c^2} + 1 + 2u + 6b^2(1 + u)$$  \hspace{1cm} (18)

Now if we assume the fixed points $u_p$ and $u_f$ to be at the far past and at the far future respectively, then $u_f < u_0 \approx 0.45$ and $c^2 < \Omega_{D_0} \approx 0.75$ \cite{33}, where the suffix '0' stands for the present value and the second inequality is obtained from equation (9) with the fact that we are at present in an accelerating phase. Then from (18)

$$\left|\frac{du'}{du}\right|_{u=u_p} = -\frac{\left(2 + c^2\right)}{c^2} + \frac{4}{c^2 (1 + u_f)} > 0$$

and

$$\left|\frac{du'}{du}\right|_{u=u_f} = 1 + \frac{2\left(u_f - 1\right)}{c^2 (u_f + 1)} < 0$$

So the fixed point at far past is an unstable one while the fixed point at far future is a stable one. Moreover, integrating equation (13) we obtain

$$u = \left(\frac{u_p + u_f}{2}\right) + \left(\frac{u_f - u_p}{2}\right) \left[\frac{1 + k(\tilde{a})^\mu}{k(\tilde{a})^\mu - 1}\right]$$  \hspace{1cm} (19)

where $k = \frac{u_0 - u_p}{u_0 - u_f}$, $\tilde{a} = \frac{a}{a_0}$ and $\mu = (1 + 3b^2)(u_p - u_f)$.

Further, equation (13) can be expressed in terms of $u_p$ and $u_f$ as

$$\dot{u} = H \left(1 + 3b^2\right) (u - u_p) (u - u_f),$$  \hspace{1cm} (20)

which clearly shows that $\dot{u} < 0$ between the two fixed points $u_p$ and $u_f$. The continuous decrease of $u$ between the two fixed points is shown in figure 2(a) where $u \approx 1$ near $\tilde{a} = 0.8$. Hence the coincidence problem has some partial solution for the present model, it can not predict $u_0 \sim O(1)$ \cite{33}. The explicit expression for the density parameter $\Omega_D$ is given by

$$\Omega_D = \frac{1 + k(\tilde{a})^\mu - 1}{(1 + u_f) k(\tilde{a})^\mu - (1 + u_p)}$$  \hspace{1cm} (21)

and hence using (8) we have

$$\omega_D = -\frac{2}{3c^2} + \frac{1}{3} \frac{(1 + u_f) k(\tilde{a})^\mu - (1 + u_p)}{k(\tilde{a})^\mu - 1}$$  \hspace{1cm} (22)
Fig. 2(a)-2(b) represent the variation of $u$ and $\omega_D$ against $\tilde{a}$ respectively corresponding to the value of the parameters $u_f = 0.013, c^2 = 0.44, u_0 = 0.4144$ and $b^2 = 0.290$.

The variation of $\omega_D$ over the scale factor is shown in figure 2(b) which shows that we are very close to $\Lambda CDM$ era in the present epoch. At the two extreme limits the limiting values of $\omega_D$ are as follows:

$$\omega_D \rightarrow \omega_D^0 = -\frac{2}{3c^2} + \frac{(1+u_p)}{3} \quad (a \rightarrow 0)$$

$$\omega_D \rightarrow \omega_\infty = -\frac{2}{3c^2} + \frac{(1+u_p)}{3} \quad (a \rightarrow \infty)$$

Note that as $u$ decreases with the evolution so $u_f < u_p$ and hence $\omega_D$ also decreases as the universe grows up. If we choose $u_0 \simeq 1$ then the present value of $\omega_D$ (i.e.,$\omega_D^0$) does not depend on the asymptotic values $u_p$ and $u_f$, i.e.,

$$\omega_D^0 = -\frac{2}{3}\left(-1 + \frac{1}{c^2}\right)$$

which is compatible with recent observation, i.e., $\omega_D^0$ should be very close to $-1$ (as shown in Fig.2(b)) if $c^2 \simeq 0.4$ and is closed to the estimated lower bound of $c^2$. Integrating field equation (4) using equation (21) and (22) we have

$$H = H_0(\tilde{a})\left\\{\frac{1}{c^2(1+u_p)^2}\right\} \times \left[\frac{(1+u_f)}{k(1+u_p)}\right]^{\frac{2}{k+1}}$$

(23)

Now combining equation (21) and (23) the HDE density has the form

$$\rho_D = \frac{3H_0^2}{(u_p-u_f)(u_0+1)} \left[ (u_p-u_0)(\tilde{a})^{\frac{2}{c^2(1+u_p)^2}} + (u_0-u_f)(\tilde{a})^{\frac{2}{c^2(1+u_p)^2}} \right]$$

(24)
Table 2: Nature of the Critical Points for Case-II

| Condition | Nature of the eigen values | Type of critical point |
|-----------|---------------------------|------------------------|
| (i) $\frac{1}{50} < c^2 < \frac{1}{2}$ | (i) pair of complex roots with real part positive. | (i) unstable spiral. |
| (ii) $c^2 > \frac{1}{2}$ | (ii) real roots of opposite sign. | (ii) Saddel. |
| (iii) $c^2 < \frac{1}{50}$ | (iii) real roots of same sign. | (iii) Unstable node. |
| (iv) $c^2 = \frac{1}{50}$ | (iii) equal non-zero real roots. | (iii) degenerate unstable node. |
| (v) $c^2 = \frac{1}{2}$ | (iii) zero roots. | (iii) degenerate point. |

The above expression for $\rho_D$ contains two terms – the first one is dominant at later epochs when $a$ is large while the second term is the dominant one at early phases.

**Case (II) : $Q = \gamma \rho_m \rho_D$**

As before the explicit form of the energy conservation equations are

$$\dot{\rho}_m = \rho_m [\gamma \rho_D - 3H]$$  \hspace{1cm} (25)

and

$$\dot{\rho}_D = -\rho_D [\gamma \rho_m + 3H(1 + \omega_D)]$$  \hspace{1cm} (26)

and hence the ratio of the energy densities has the evolution equation

$$\dot{u} = 3H u [\gamma H + \omega_D] = 3H u \left[ \gamma H + \frac{(1 + u)}{3} - \frac{2}{3c^2} \right]$$  \hspace{1cm} (27)

The field equation (4) can be written as

$$\dot{H} = -\frac{3H^2}{2} \left[ \frac{4}{3} - \frac{2}{3c^2(1 + u)} \right]$$  \hspace{1cm} (28)

Thus equations (27) and (28) form an autonomous system in the $(u, H)$-phase plane. The only critical point which is of physical interest is \((\frac{1}{\frac{\gamma}{3}}, 1, \frac{1}{2})\) in the $(u, H)$ phase plane. It is to be noted that the other critical points correspond to static model of the universe or a degenerate line [36] representing universe filled up with DE only. The critical point is characterized in the Table 2 where for the linearized matrix $A$, $p = tr(A) = \frac{1}{2c^2} (\frac{1}{2} - c^2)$, $q = detA = -\frac{1}{2c^2} (\frac{1}{2} - c^2)$ and $\Delta = p^2 - 4q = \frac{25}{2c^2} (\frac{1}{2} - c^2)$ (\frac{1}{50} - c^2). The geometric nature of the equilibrium point in each of the above five cases are presented in figures 3(a)-(e).

**4 Discussion:**

The paper analyzes the HDE model interacting with DM(in the form of dust). Here the IR cut off is chosen at the Ricci’s length with the justification that it corresponds to the size of the maximal perturbation corresponding to formation of a black hole. The interaction between the two fluids is either a linear combination or in product form of the two energy densities of which the product form is physically more variable one. In both the cases, the evolution equations are transformed to an autonomous system for which the nature of the critical points are presented in tabular form and are graphically analyzed. Finally, the coincidence problem is discussed by studying the evolution of the energy ratio for the first case only.
Phase Portrait for $\gamma = 0.232$ and $\frac{1}{50} < c^2 < \frac{1}{2}$

Phase Portrait for $\gamma = 0.232$ and $c^2 > \frac{1}{2}$

Phase Portrait for $\gamma = 0.232$ and $\frac{1}{50} < c^2 < \frac{1}{2}$

Phase Portrait for $\gamma = 0.232$ and $c^2 = \frac{1}{50}$

Phase Portrait for $\gamma = 0.232$ and $c^2 = \frac{1}{2}$

Fig. 3(a) - Fig. 3(e) represent the variation of $\Omega_H - H_0$ under the given conditions. $H_0$ and $\Omega_H$...
Acknowledgement:
RB and NM want to thank West Bengal State Govt. and CSIR, India respectively for awarding JRF. Authors are thankful to IUCAA, Pune as this work was done there during a visit.

References
[1] Spergel, D. N. et al. :- [WMAP Collaboration], Astron. J. Suppl 148, 175(2003) [arXiv : astro-ph/0302209]; Astron. J. Suppl 170, 377(2007).
[2] Einstein, D.J. et. al. :- [SDSS Collaboration], Astrophys. J. 148, 175(2003).
[3] Perieval et. al. :- MNRAS 381 1053(2007).
[4] Riess, A. G. et al. :- [Supernova Search Team Collaboration], Astron. J. 116, 1009(1998)[arXiv:9805201(astro-ph)].
[5] Perlmutter, S. et al. :- [Supernova Cosmology Project Collaboration], ApJ 517, 565(1999)[arXiv:9812133(astro-ph)].
[6] Amanullah, R. et. al. :-[Supernova Cosmology Project Collaboration], Astrophys. J 716 712(2010).
[7] Perlmutter, S. et al. :- Nature 391, 51(1998).
[8] Tonry, S.L. et. al. :- Astrophys. J 594 1(2003).
[9] We now refer to references [9] in the text.
[10] Peebles, P. J. E., Musser, A :- Nature 465 565(2010).
[11] Weinberg, S. :- Rev. Mod. Phys. 61, 1(1989).
[12] Copeland, E. J., Sami, M., Tsujikawa, S. :- Int. J. Mod. Phys. D 1 1753(2006) [arxiv:hep-th/0603057].
[13] Ratra, B., Peebles, P. J. E. :- Phys. Rev. D 37, 3406(1988).
[14] Caldwell, R. R., Dave, R., Steinhardt, P. J. :- Phys. Rev. Lett. 80, 1582(1998). [arXiv : astro-ph/9708069].
[15] Copeland, E. J., Liddle, A.R., Wands, D.:- Phys. Rev. D 57, 4686(1998).
[16] Pavon, D. and Wang,B. Gen. Rel. Grav. 41(2009) 1;Ma, Y.-Z., Gong, Y. :- Euro. Phys. J. C 60 (2009) 303 .
[17] Mangano, G. , Miele, G., Pettorino, V. :- Mod. Phys. Lett. A 18 (2003)831; He, J.-H., Wang, B. :- J. Cosmo. Astropart. Phys. 06 (2007)010.
[18] Micheletti, S., Abdalla, E., Wang, B. :- Phys. Rev. D 79, 123506(2009).
[19] Sandro, M.R., Micheletti, J., :- JCAP 05 009(2010).
[20] Amendola, L. :- Phys. Rev. D 62, 043511(2000); Amendola, L., Quercellini :- Phys. Rev. D 68, 023514(2003); Amendola, L., Tsujikawa, S., Sami, M :- Phys. Lett. B 632, 155(2006).
[21] Pavon, D., Zimdahl, W. :-Phys. Lett. B 628, 206(2005); Campo, S., Herrera, R. , Pavon, D :-Phys. Rev. D 78, 021302(R)(2008).
[22] Boehmer, C.G., Caldera-Cabral, G., Lazkoz, R., Maartens, R. :- Phys. Rev. D 78 023505(2008).
[23] Olives, G., Atrio-Barandela, F., Pavon, D. :- Phys. Rev. D 74 043521(2006).
[24] Chen, S. B., Wang, B., Jing, J.L. :- Phys. Rev. D 78 123503(2008).
[25] Hooft, G. T. :- [arXiv : gr-qc/9310026]; Susskind, L. :- J. Math. Phys. 36, 6377(1994).[arXiv : hep-th/9409089].
[26] Cohen, A. G., Kaplan, D. B., Nelson, A. E. :- Phys. Rev. Lett. 82, 4971(1999).[arXiv : hep-th/9803132].
[27] Hsu, S.D.H. :- Phys. Lett. B 594, 13(2004); Guberind, B., Horvat, R., Nikolic, H. :- JCAP 01 012(2007); Xu, L. :- JCAP 09 016(2009).
[28] Gao, C., Wu, F., Chen, X., Shen, Y.G. :- *Phys. Rev. D* 79 043511(2009).
[29] Xu, L., Li, W., Lu, J. :- *MPLA* 24 1355(2009).
[30] Suwa, M., Nihei, T. :- *Phys. Rev. D* 81 023519(2010).
[31] Lepe, S. and Pena, F. :- *Eur. Phys. J.C* 69 575 (2010).
[32] Brustein, R., In String Theories and Fundamental Instructions, Lecture Notes in Physics vol 737 edited by Gasperini, M. and Maharana, J., Springer-Verlag (NY), 619, (2008).
[33] Duran, I., Pavon, D. :- *Phys. Rev. D* 83 023504 (2011).
[34] Lip, S.Z.W. :- *Phys. Rev. D* 83 023528(2011)
[35] Hagiwara, K. et. al. :- *Phys. Rev. D* 66 (2002)010001.
[36] Perko, L. :- *Differential Equations and Dynamical systems* Springer-Verlag; N.Y. (1991).