Baire property of space of Baire-one functions

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Abstract

A topological space $X$ is Baire if the Baire Category Theorem holds for $X$, i.e., the intersection of any sequence of open dense subsets of $X$ is dense in $X$. One of the interesting problems for the space $B_1(X)$ of all Baire-one real-valued functions is the Banakh-Gabriyelyan problem of characterization of a topological space $X$ for which the function space $B_1(X)$ is Baire.

In this paper, we solve this problem, namely, we have obtained a characterization when a function space $B_1(X)$ is Baire for a topological space $X$. Also we proved that $B_1(X)$ is Baire for any $\gamma$-space $X$ and we have obtained a characterization of a topological space $X$ for which the function space $B_1(X)$ is a Choquet space. This answers questions posed recently by T. Banakh and S. Gabriyelyan.

We also conclude that, it is consistent there are no uncountable separable metrizable space $X$ such that $B_1(X)$ is countable dense homogeneous.

Keywords: Baire-one function, $\gamma$-space, function space, pseudocomplete, Baire property, topological game, Choquet space, countable dense homogeneity

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1. Introduction

A space is meager (or of the first Baire category) if it can be written as a countable union of closed sets with empty interior. A topological space $X$ is Baire if the Baire Category Theorem holds for $X$, i.e., the intersection of any sequence of open dense subsets of $X$ is dense in $X$. Clearly, if $X$ is
a Baire space, then $X$ is not meager. The reverse implication is in general not true. However, it holds for every homogeneous space $X$ (see Theorem 2.3 in [13]). Being a Baire space is an important topological property for a space and it is therefore natural to ask when function spaces are Baire. The Baire property for continuous mappings was first considered in [32]. Then a paper [13] appeared, where various aspects of this topic were considered. In [13], necessary and, in some cases, sufficient conditions on a space $X$ were obtained under which the space $C_p(X)$ of all continuous real-valued functions $C(X)$ on a space $X$ with the topology of pointwise convergence is Baire.

In general, it is not an easy task to characterize when a function space has the Baire property. The problem for $C_p(X)$ was solved independently by Pytkeev [29], Tkachuk [30] and van Douwen [16].

**Theorem 1.1.** (Pytkeev-Tkachuk-van Douwen) The space $C_p(X)$ is Baire if and only if every pairwise disjoint sequence of non-empty finite subsets of $X$ has a strongly discrete subsequence.

A collection $G$ of subsets of $X$ is *discrete* if each point of $X$ has a neighborhood meeting at most one element of $G$, and is *strongly discrete* if for each $G \in G$ there is an open superset $U_G$ of $G$ such that $\{U_G : G \in G\}$ is discrete.

The problem for $C_k(X)$ was solved for locally compact $X$ by Gruenhage and Ma [21]. In paper [2], Banakh and Gabriyelyan introduced a new class of almost $K$-analytic spaces which properly contains Čech-complete spaces and $K$-analytic spaces and showed that for an almost $K$-analytic space $X$ the following assertions are equivalent: (i) $B_1(X)$ is a Baire space, (ii) $B_1(X)$ is a Choquet space, and (iii) every compact subset of $X$ is scattered.

One of the interesting problems for the space of Baire functions is the Banakh-Gabriyelyan problem (Problem 1.1 in [2]): *Let $\alpha$ be a countable ordinal. Characterize topological spaces $X$ and $Y$ for which the function space $B_\alpha(X,Y)$ is Baire.*

Note that in [2], Corollary 4.2, it is proved that $B_\alpha(X,\mathbb{R})$ is Baire for any space $X$ and every countable ordinal $\alpha \geq 2$.

In this paper, we solve the Banakh-Gabriyelyan problem (in case $Y = \mathbb{R}$), namely, we have obtained a characterization when a function space $B_1(X)$ is Baire for a topological space $X$. 

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2. Main definitions and notation

Throughout this paper, all spaces are assumed to be Tychonoff. This is due to the fact that for any topological space $X$ there exists the Tychonoff space $\tau X$ and the onto map $\tau_X : X \to \tau X$ such that for any map $f$ of $X$ to a Tychonoff space $Y$ there exists a map $g : \tau X \to Y$ such that $f = g \circ \tau_X$. Evidently, for any map $f : X \to Y$ the unique map $\tau f : \tau X \to \tau Y$ is defined such that $\tau f \circ \tau_X = \tau_Y \circ f$. The correspondence $X \to \tau X$ for any space $X$ and $f \to \tau f$ for any map $f$ is called the Tychonoff functor [22, 23].

The set of positive integers is denoted by $\mathbb{N}$ and $\omega = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}$ be the real line, we put $I = [0, 1] \subset \mathbb{R}$, and let $\mathbb{Q}$ be the rational numbers. Let $f : X \to \mathbb{R}$ be a real-valued function, then $\|f\| = \sup\{|f(x)| : x \in X\}$, $S(g, \epsilon) = \{f : \|g - f\| < \epsilon\}$, $B(g, \epsilon) = \{f : \|g - f\| \leq \epsilon\}$, where $g$ is a real valued function and $\epsilon > 0$. Let $V = \{f \in \mathbb{R}^X : f(x_i) \in V_i, i = 1, \ldots, n\}$ where $x_i \in X$, $V_i \subset \mathbb{R}$ are bounded intervals for $i = 1, \ldots, n$, then $\text{supp} V = \{x_1, \ldots, x_n\}$, $\text{diam} V = \max\{\text{diam} V_i : 1 \leq i \leq n\}$. The symbol $0$ stands for the constant function to $0$. A basic open neighborhood of $0$ is of the form $[F, (-\epsilon, \epsilon)] = \{f \in \mathbb{R}^X : f(F) \subset (-\epsilon, \epsilon)\}$, where $F \subseteq X$ and $\epsilon > 0$.

A real-valued function $f$ on a space $X$ is a Baire-one function (or a function of the first Baire class) if $f$ is a pointwise limit of a sequence of continuous functions on $X$. Let $B_1(X)$ denote the space of all Baire-one real-valued functions on a space $X$ with the topology of pointwise convergence. It is well known, the space $B_1(X)$ forms a vector space which contains the space of all continuous functions $C(X)$ and which is closed with respect to the uniform convergence. Moreover, $f \cdot g$ and $\max(f, g)$ are Baire-one functions whenever $f, g \in B_1(X)$.

We recall that a subset of $X$ that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set (or functionally open) of $X$ if $X \setminus O$ is a zero-set of $X$. It is easy to check that zero sets are preserved by finite unions and countable intersections. Hence cozero sets are preserved by finite intersections and countable unions. Countable unions of zero sets will be denoted by $\text{Zer}_\sigma$ (or $\text{Zer}_\sigma(X)$), countable intersection of cozero sets by $\text{Coz}_\delta$ (or $\text{Coz}_\delta(X)$). It is easy to check that $\text{Zer}_\sigma$-sets are preserved by countable unions and finite intersections. Note that if $Y \subseteq X$ and $A \in \text{Coz}_\delta(X)$ then $A \cap Y \in \text{Coz}_\delta(Y)$. Note also that any zero set is $\text{Coz}_\delta$ and any cozero-set is $\text{Zer}_\sigma$. It is well known that $f$ is of the first Baire class if and only if $f^{-1}(U) \in \text{Zer}_\sigma$ for every open $U \subseteq \mathbb{R}$ (see Exercise 3.A.1 in [14]).
If \( f \) is a real-valued function on a set \( X \) and \( a \in \mathbb{R} \), we write \([f \geq a]\) for the set \( \{x \in X : f(x) \geq a\} \). Similarly we use \([f \leq a] \), \([f < a] \), \([f > a] \) and \([f = a]\).

Below, in the main results, we need to apply the following two propositions.

**Proposition 2.1.** (Proposition 3 in [11]) If \( A \) is a Coz\( \delta \)-subset of a space \( X \), then there exists a Baire-one function \( f \) with values in \([0, 1]\) such that \( A = [f = 0] \).

If \( A \) and \( B \) is a pair of disjoint Coz\( \delta \)-subsets of \( X \), then there exists a Baire-one function \( f \) on \( X \) with values in \([0, 1]\) such that \( A = [f = 0] \) and \( B = [f = 1] \).

**Proposition 2.2.** (Proposition 8 in [11]) Let \( X \) be a topological space and \( Y \subset X \). Then the following assertions are equivalent:

(i) For any Baire-one function \( f \) on \( Y \) there is a Baire-one function \( g \) on \( X \) extending \( f \) such that \( \inf f(Y) = \inf g(X) \) and \( \sup f(Y) = \sup g(X) \).

(ii) Any Baire-one function on \( Y \) can be extended to a Baire-one function on \( X \).

(iii) For any Coz\( \delta \)-subset \( A \) of \( Y \) there is a Coz\( \delta \)-subset \( \hat{A} \) of \( X \) with \( A = \hat{A} \cap Y \), and, moreover, for any Coz\( \delta \)-subset \( G \) of \( X \) disjoint with \( Y \) there is a Coz\( \delta \)-set \( H \subset X \) satisfying \( Y \subset H \subset X \setminus G \).

Both Baire and meager space have game characterizations due to Oxtoby [27].

The game \( G_I(X) \) is started by the player ONE who selects a nonempty open set \( V_0 \subseteq X \). Then the player TWO responds selecting a nonempty open set \( V_1 \subseteq V_0 \). At the \( n \)-th inning the player ONE selects a nonempty open set \( V_{2n} \subseteq V_{2n-1} \) and the player TWO responds selecting a nonempty open set \( V_{2n+1} \subseteq V_{2n} \). At the end of the game, the player ONE is declared the winner if \( \bigcap_{n \in \omega} V_n \) is empty. In the opposite case the player TWO wins the game \( G_I(X) \).

The game \( G_{II}(X) \) differ from the game \( G_I(X) \) by the order of the players. The game \( G_{II}(X) \) is started by the player TWO who selects a nonempty open set \( V_0 \subseteq X \). Then player ONE responds selecting a nonempty open set \( V_1 \subseteq V_0 \). At the \( n \)-th inning the player TWO selects a nonempty open
set $V_{2n} \subseteq V_{2n-1}$ and the player ONE responds selecting a nonempty open set $V_{2n+1} \subseteq V_{2n}$. At the end of the game, the player ONE is declared the winner if $\bigcap_{n \in \omega} V_n$ is empty. In the opposite case the player TWO wins the game $G_{\text{II}}(X)$.

The following classical characterizations can be found in [27].

**Theorem 2.3.** (Oxtoby) A topological space $X$ is

1. meager if and only if the player ONE has a winning strategy in the game $G_{\text{II}}(X)$;
2. Baire if and only if the player ONE has no winning strategy in the game $G_{\text{I}}(X)$.

A topological space $X$ is defined to be Choquet if the player TWO has a winning strategy in the game $G_{\text{I}}(X)$. Choquet spaces were introduced in 1975 by White [35] who called them weakly $\alpha$-favorable spaces.

We note the following results obtained in [13].

1. If $Y$ is a homogeneous non-meager space then $Y$ is Baire.

Thus, if $B_1(X)$ (or $C_p(X)$) is a non-meager space then it is Baire.

2. If $X$ is a Tychonoff space, $C_p(X)$ is Baire and $Y \subseteq X$, then $C_p(Y)$ is Baire.

Note also that in [31], a game criterion is given for the fact that a set $Z \subseteq Y^X$ is second category, where $Y$ is a complete metric space.

In [13], there is a game for a topological space $X$, denoted by $\Gamma(X)$. The game $\Gamma(X)$ is a game in which two players, called ONE and TWO, are given an arbitrary finite starting set $S_0$ and then proceed to choose alternate terms in a sequence $S_0, S_1, S_2, \ldots$ of pairwise disjoint finite (possibly empty) subsets of $X$. The resulting sequence $(S_0, S_1, S_2, \ldots)$ is called a play of the game $\Gamma(X)$ and is said to result in a win for player ONE if and only if the set $S_1 \cup S_3 \cup S_5 \cup \ldots$ is not a closed discrete subspace of $X$. If player ONE does not win, then player TWO wins.

A strategy for player ONE in the game $\Gamma(X)$ is a function $\sigma$ which assigns to each pairwise disjoint sequence $(S_0, S_1, \ldots, S_{2k})$, with $k \geq 0$, a finite set $S_{2k+1}$ which is disjoint from $S_0 \cup S_1 \cup \ldots \cup S_{2k}$.

A strategy $\sigma$ for player ONE in the $\Gamma(X)$ is said to be a winning strategy if, whenever $(S_0, S_1, \ldots)$ is a play of the game $\Gamma(X)$ in which $S_{2k+1} = \sigma(S_0, S_1, \ldots, S_{2k})$ for each $k \geq 0$, then player ONE wins that play.
In ([13], Theorem 4.6), it is proved that if $C_p(X)$ is Baire then ONE has a winning strategy in the game $\Gamma(X)$. The converse, in general, is not the case.

In [29], Pytkeev proposed a modification of the game $\Gamma(X)$ - the game $\Gamma'(X)$, in which player ONE wins if the set $S_1 \cup S_3 \cup S_5 \cup \ldots$ is not strongly discrete.

The following characterizations can be found in [29].

**Theorem 2.4.** (Pytkeev). For a Tychonoff space $X$ the following assertions are equivalent:

1. $C_p(X)$ is meager;
2. there is a pairwise disjoint sequence of non-empty finite subsets of $X$ no subsequence of which is strongly discrete;
3. ONE has a winning strategy in the game $\Gamma'(X)$;
4. there is a pairwise disjoint sequence $\{\Delta_n : n \in \mathbb{N}\}$ of finite subsets of $X$ such that $\sup_{n \in \mathbb{N}} \left( \min \{|f(x)| : x \in \Delta_n\} \right) < \infty$ for each $f \in C(X)$.

A subset $A$ of a space $X$ is called a **bounded** subset (in $X$) if every continuous real-valued function on $X$ is bounded on $A$.

**Remark 2.5.** (Proposition 5.1 in [8], Corollary 3.3 in [13] for an infinite pseudocompact subspace $A$) If $X$ is a space containing an infinite bounded subset $A$ (e.g. a non-trivial convergent sequence) then $C_p(X)$ is meager.

### 3. Baireness for space of Baire-one functions

A Coz$_\delta$-subset of $X$ containing $x$ is called a **Coz$_\delta$ neighborhood** of $x$.

**Definition 3.1.** A set $A \subseteq X$ is called **strongly Coz$_\delta$-disjoint**, if there is a pairwise disjoint collection $\{F_a : F_a$ is a Coz$_\delta$ neighborhood of $a, a \in A\}$ such that $\{F_a : a \in A\}$ is a completely Coz$_\delta$-additive system, i.e. $\bigcup_{b \in B} F_b \in Coz_\delta$ for each $B \subseteq A$.

A disjoint sequence $\{\Delta_n : n \in \mathbb{N}\}$ of (finite) sets is **strongly Coz$_\delta$-disjoint** if the set $\bigcup \{\Delta_n : n \in \mathbb{N}\}$ is strongly Coz$_\delta$-disjoint.

Let $Game(X)$ be a modification of the game $\Gamma(X)$ in which player ONE wins if the set $S_1 \cup S_3 \cup S_5 \cup \ldots$ is not strongly Coz$_\delta$-disjoint. If player ONE does not win, then player TWO wins.
Lemma 3.2. Let $X$ be a Hausdorff space and $\{F_i : i \in \mathbb{N}\}$ is a disjoint completely $Coz_\delta$-additive system. Then any function on $\bigcup F_i$ which is constant on each $F_i$ can be extended to a Baire-one function on $X$.

Proof. Step 1: The function is bounded. Proceed as in the proof of Proposition 7, implication (iii) implies (i) in [11].

Let $F = \bigcup F_i$ and $f : F \to \mathbb{R}$ such that $f(F_i) = a_i$ for some $a_i \in \mathbb{R}$ and $\{a_i : i \in \mathbb{N}\}$ is bounded in $\mathbb{R}$.

Set $t := \begin{cases} f & \text{on } F \\ \inf f(F) & \text{on } X \setminus F, \end{cases}$ and $s := \begin{cases} f & \text{on } F \\ \sup f(F) & \text{on } X \setminus F. \end{cases}$

According to Theorem 3.2 in [14], there exists a Baire-one function $g$ on $X$ satisfying $t \leq g \leq s$ if and only if the following condition is satisfied: given a couple of real numbers $a < b$, there is a Baire-one function $\varphi$ on $X$ such that

$\varphi = 0$ on $A := [s \leq a]$ and $\varphi = 1$ on $B := [t \geq b]$.

Using the completely $Coz_\delta$-additivity of $\{F_i : i \in \mathbb{N}\}$ it is easy to check this condition.

So assume that $a < b$ are given. Without loss of generality we may suppose that $\inf f(F) \leq a < b \leq \sup f(F)$.

Then $A = [s \leq a] = \bigcup \{F_{a_i} : a_i \leq a\}$ and $B = [t \geq b] = \bigcup \{F_{a_i} : a_i \geq b\}$. Note that $A \cap B = \emptyset$. Since $\{F_i : i \in \mathbb{N}\}$ is a completely $Coz_\delta$-additivity system, then $A, B \in Coz_\delta(X)$. By Proposition 2.1 there exists a Baire-one function $\varphi$ on $X$ with values in $[0, 1]$ such that $A = [\varphi = 0]$ and $B = [\varphi = 1]$. Thus there is a Baire-one function $g$ on $X$ such that $t \leq g \leq s$. Obviously, $g \upharpoonright F = f$.

Step 2: The function is unbounded. Proceed as in the proof of Proposition 8, implication (iii) implies (i) in [11].

Let $f$ be an unbounded function on $F$ constant on each $F_i$ thus there is a homeomorphism $\varphi : \mathbb{R} \to (-1, 1)$ such that $\inf(\varphi \circ f)(F) < 0 < \sup(\varphi \circ f)(F)$.

The key thing that $\varphi \circ f$ is a bounded function on $F$ constant on each $F_i$ and hence we may use Step 1. Let $h$ be a Baire-one function on $X$ such that $h \upharpoonright F = \varphi \circ f$, $\sup h(X) = \sup(\varphi \circ f)(F) \leq 1$ and $-1 \leq \inf(\varphi \circ f)(F) = \inf h(X)$. 

By setting \( G := h^{-1}(\{-1\} \cup \{1\}) \) we obtain a Coz\( _\delta \)-subset of \( X \) which is disjoint with \( F \). Since \( F \) is a Coz\( _\delta \)-subset of \( X \), by Proposition 2.1 there exists a Baire-one function \( \psi \) on \( X \) with values in \([0, 1] \) such that \( G = [\psi = 0] \) and \( F = [\psi = 1] \). One can readily verify that \( g := \varphi^{-1} \circ (h \cdot \psi) \) is a Baire-one function on \( X \) which satisfies our requirements. This conclude the proof. 

Lemma 3.3. Let \( X \) be a Hausdorff space and let \( \{F_i : i \in \mathbb{N}\} \) forms a disjoint completely Coz\( _\delta \)-additive system. Then any family \( \{L_i : L_i \subseteq F_i, i \in \mathbb{N}\} \) of Coz\( _\delta \) subsets of \( X \) is a completely Coz\( _\delta \)-additive system.

Proof. Let \( F = \bigcup\{F_i : i \in \mathbb{N}\} \). Consider the function \( f : F \to \mathbb{R} \) such that \( f(F_i) = i \) for every \( i \in \mathbb{N} \). Let \( V \) be an open set of \( \mathbb{R} \). Since \( f^{-1}(V) = \bigcup\{F_i : i \in V\} \in \text{Zer}_\sigma(F) \) (because \( \bigcup\{F_j : j \notin V\} \in \text{Coz}_\delta(X) \) and, hence, \( \bigcup\{F_j : j \notin V\} \in \text{Coz}_\delta(F) \)), \( f \in B_1(F) \).

By Lemma 3.2 there is \( \tilde{f} \in B_1(X) \) such that \( \tilde{f} \upharpoonright F = f \). Let \( S_i = \tilde{f}^{-1}\left(i - \frac{1}{2}, i + \frac{1}{2}\right) \) for every \( i \in \mathbb{N} \). Then \( S_i \in \text{Zer}_\sigma(X) \). Since \( L_i \in \text{Coz}_\delta(X) \) for every \( i \in \mathbb{N} \), we get \( S_i \setminus L_i \in \text{Zer}_\sigma(X) \) for every \( i \in \mathbb{N} \).

Let \( \{i_k : k \in \mathbb{N}\} \) be a subsequence of \( \{i : i \in \mathbb{N}\} \). Then \( D = \bigcup\{S_{i_k} \setminus L_{i_k} : k \in \mathbb{N}\} \in \text{Zer}_\sigma(X) \). Since \( \bigcup\{F_{i_k} : k \in \mathbb{N}\} \in \text{Coz}_\delta(X) \), \( \bigcup\{L_{i_k} : k \in \mathbb{N}\} = (\bigcup\{F_{i_k} : k \in \mathbb{N}\}) \setminus D \in \text{Coz}_\delta(X) \). Thus, \( \{L_i : i \in \mathbb{N}\} \) forms a completely Coz\( _\delta \)-additive system. 

Definition 3.4. A family of sets is strongly Coz\( _\delta \)-disjoint if each set has a superset such that the family of all the supersets is disjoint and completely Coz\( _\delta \)-additive.

Proposition 3.5. A pairwise disjoint sequence \( \{\Delta_n : n \in \mathbb{N}\} \) of non-empty finite subsets of \( X \) is strongly Coz\( _\delta \)-disjoint in the new sense if and only if it is strongly Coz\( _\delta \)-disjoint according to Definition 1.

Proof. Let \( \{\Delta_n : n \in \mathbb{N}\} \) be a strongly Coz\( _\delta \)-disjoint family of non-empty finite subsets of \( X \). Then there is a family \( \{B_n : n \in \mathbb{N}\} \) such that \( \Delta_n \subseteq B_n \) for each \( n \), and it is disjoint and completely Coz\( _\delta \)-additive. Let \( h : \bigcup_{n \in \mathbb{N}} B_n \to \mathbb{R} \) be a function such that \( h(B_n) = n \) for every \( n \in \mathbb{N} \). Then, by Lemma 3.2.
there is \( \tilde{h} \in B_1(X) \) such that \( \tilde{h} \upharpoonright \bigcup_{n \in \mathbb{N}} B_n = h \). Let \( S_n = \tilde{h}^{-1}((n-\frac{1}{2},n+\frac{1}{2})) \) and \( P_n = \tilde{h}^{-1}(n) \) for every \( n \in \mathbb{N} \). Since \( \tilde{h} \in B_1(X) \), we get that \( P_n \subseteq S_n \subseteq \text{Zer}_{\sigma} \) and \( \Delta_n \subseteq B_n \subseteq P_n \subseteq \text{Coz}_{\delta} \). Since \( \Delta_n = \{x_{1,n},x_{2,n},\ldots,x_{k(n),n}\} \) is finite, there are co-zero neighborhoods \( W(x_{j,n}) \) of \( x_{j,n}, j = 1,\ldots,k(n) \) such that \( W(x_{i,n}) \cap W(x_{j,n}) = \emptyset \) for \( i \neq j \) and \( i,j \in \{1,\ldots,k(n)\} \).

Let \( F_{j,n} = W(x_{j,n}) \cap B_n \) for \( j = 1,\ldots,k(n) \) and every \( n \in \mathbb{N} \).

We claim that \( \gamma = \{F_{j,n} : j = 1,\ldots,k(n), n \in \mathbb{N}\} \) is completely \( \text{Coz}_{\delta} \)-additive.

1. Since \( W(x_{j,n}) \) is co-zero and \( B_n \in \text{Coz}_{\delta} \) then \( F_{j,n} \in \text{Coz}_{\delta} \).
2. Let \( \gamma_1 \subseteq \gamma \). We show that \( \bigcup \gamma_1 \in \text{Coz}_{\delta} \). Note that \( \gamma = \bigcup_{n \in \mathbb{N}} F_n \) where \( F_n = \{F_{j,n} : j = 1,\ldots,k(n)\} \). For each \( n \), let \( \gamma_{1,n} = \gamma_1 \cap F_n \). Let \( M = \{n \in \mathbb{N} : \gamma_{1,n} \neq \emptyset\} \). Then \( \gamma_1 = \bigcup_{n \in M} \gamma_{1,n} \). For each \( n \in M \), \( \bigcup \gamma_{1,n} \in \text{Coz}_{\delta} \) since \( \gamma_{1,n} \) is finite. Since \( S_n \in \text{Zer}_{\sigma} \) for each \( n \), we have \( S_n \setminus (\bigcup \gamma_{1,n}) \in \text{Zer}_{\sigma} \) for each \( n \in M \). As a result, \( \bigcup_{n \in M} S_n \setminus (\bigcup \gamma_{1,n}) \in \text{Zer}_{\sigma} \). Since \( \{B_n\} \) is completely \( \text{Coz}_{\delta} \)-additive, \( \bigcup_{n \in M} B_n \in \text{Coz}_{\delta} \). Note the following equality.

\[
\bigcup \gamma_1 = (\bigcup_{n \in M} B_n) \setminus \left[ \bigcup_{n \in M} \left( S_n \setminus (\bigcup \gamma_{1,n}) \right) \right] \in \text{Coz}_{\delta}.
\]

Thus, \( \gamma \) is completely \( \text{Coz}_{\delta} \)-additive. It follows that \( \bigcup \{\Delta_n : n \in \mathbb{N}\} \) is strongly \( \text{Coz}_{\delta} \)-disjoint.

The other direction is easy to check, so the proof is not given.

\[\square\]

**Theorem 3.6.** For a Tychonoff space \( X \) the following assertions are equivalent:

1. \( B_1(X) \) is meager;
2. there is a pairwise disjoint sequence \( \{\Delta_n : n \in \mathbb{N}\} \) of non-empty finite subsets of \( X \) such that \( \sup_{n \in \mathbb{N}} \left( \min\{|f(x)| : x \in \Delta_n\} \right) < \infty \) for each \( f \in B_1(X) \);
3. there is a pairwise disjoint sequence of non-empty finite subsets of \( X \) no subsequence of which is strongly \( \text{Coz}_{\delta} \)-disjoint;
4. ONE has a winning strategy in the game \( \text{Game}(X) \).

**Proof.** (1) \( \Rightarrow \) (3). Let \( B_1(X) = \bigcup_{n \in \mathbb{N}} F_n \), where \( F_n \) is nowhere dense in \( B_1(X) \) and \( F_n \subseteq F_{n+1} \) for every \( n \in \mathbb{N} \). Suppose for a contradiction that every sequence \( \{\Delta_n : n \in \mathbb{N}\} \) of pairwise disjoint finite subsets of \( X \) contains a strongly \( \text{Coz}_{\delta} \)-disjoint subsequence.
Claim 1. There are a sequence \( \{ \Delta_i : i \in \mathbb{N} \} \) of pairwise disjoint finite subsets of \( X \), a sequence \( \{ \gamma_i : i \in \mathbb{N} \} \) of finite families of basis open sets in \( \mathbb{R}^X \), and a sequence \( \{ m_i : i \in \mathbb{N} \} \subseteq \mathbb{N} \) such that the following conditions hold for every \( i \in \mathbb{N} \):

(a') \( 1 \leq m_1 \) and \( m_i + 2 \leq m_{i+1} \);

(b') \( \mathbb{R}^X \cap F_i = \emptyset \) and \( \text{supp}(U) \subseteq \bigcup_{j=1}^i \Delta_j \) for every \( U \in \gamma_i \);

(c') if \( f \in U \in \gamma_i \), then \( |f(x)| \leq m_i \) for every \( x \in \text{supp}(U) \);

(d') if \( \varphi : A_i := \bigcup_{j=1}^i \Delta_j \to \mathbb{R} \) is such that \( \| \varphi \|_{A_i} \leq m_i \), then there is a continuous function \( f \in \bigcup_{i=1}^n \gamma_i \) such that \( \| \varphi - f \|_{A_i} < \frac{1}{i} \).

Proof of Claim 1. For \( i = 1 \), choose a basic open set \( U \subseteq \mathbb{R}^X \) such that \( \overline{U}^\mathbb{R}^X \cap F_1 = \emptyset \) (this is possible because \( F_1 \) is nowhere dense). Let \( \Delta_1 := \text{supp}(U) \), \( \gamma_1 := \{ U \} \), and \( m_1 := \sup \{ \| f \|_{\Delta_1} : f \in U \} + 2 \).

Assume that for \( 1 \leq i \leq n \), we have constructed \( \Delta_i, \gamma_i \) and \( m_i \) which satisfy \((a') - (d')\).

Consider the compact set \( K = \{ \varphi : \text{dom}(\varphi) = A_n \) and \( \| \varphi \|_{A_n} \leq m_n \} \).

For each \( \varphi \in K \), choose a basic open subset \( O_\varphi \) of \( \mathbb{R}^X \) such that for each \( \varphi_1 \in O_\varphi \), we have

\[ \| \varphi_1 \upharpoonright A_n - \varphi \|_{A_n} < \frac{1}{2^{i+1}}. \]

Since \( F_{n+1} \) is nowhere dense, each \( O_\varphi \) can be chosen such that \( \overline{O_\varphi} \cap F_{n+1} = \emptyset \) (closure taken in \( \mathbb{R}^X \)). Since \( K \) is compact, there exists \( \{ \psi_1, ..., \psi_k \} \subseteq K \) such that \( K \subseteq O_{\psi_1} \cup ... \cup O_{\psi_k} \). Let \( U_t = O_{\psi_t} \) for each \( t \leq k \).

Let \( \gamma_{n+1} := \{ U_1, ..., U_k \} \), \( \Delta_{n+1} := \bigcup_{t=1}^k \text{supp}(U_t) \setminus \bigcup_{t=1}^n \Delta_i \) and \( m_{n+1} := m_n + \sup \{ \| f \|_{\text{supp}(U)} : f \in U \in \gamma_i \) and \( 1 \leq i \leq n + 1 \} + 1 \).

It is clear that \( \{ \Delta_i \}_{i=1}^{n+1} \) is pairwise disjoint and the conditions \((a') - (c')\) are satisfied. To check \((d')\), fix \( \varphi : A_n \to \mathbb{R} \) such that \( \| \varphi \|_{A_n} \leq m_n \). Choose \( 1 \leq t \leq k \) such that \( \| \psi_t - \varphi \|_{A_n} \leq \frac{1}{2^t} \).

Since \( \text{supp}(U_t) \) is finite, \( U_t \) contains continuous functions, take an arbitrary continuous function \( f \in U_t \). Then, by \((1)\), we have

\[ \| \varphi - f \|_{A_n} \leq \| \varphi - \psi_t \|_{A_n} + \| \psi_t - f \|_{A_n} \leq \frac{2}{2^t+1} = \frac{1}{2^t} < \frac{1}{n}. \]

The claim is proved. \[ \square \]

Now we redefine the sequences in Claim 1 to make simpler and clearer their usage in what follows.

By assumption the sequence \( \{ \Delta_n : n \in \mathbb{N} \} \) constructed in Claim 1 contains a strongly Coz\(\_d\)-disjoint subsequence \( \{ \Delta_{n_k} : k \in \mathbb{N} \} \), where \( 1 < n_1 < \)
Let $n_2 < \ldots$. Put
\[ R_1 := F_1, \Omega_1 := \bigcup_{i=1}^{n_2-1} \Delta_i, \mu_1 := \bigcup_{i=1}^{n_2-1} \gamma_i, l_1 := m_{n_2-1}, \]
and for every $k \in \mathbb{N}$, we define
\[ R_{2k} := F_{n_k}, \Omega_{2k} := \Delta_{n_k}, \mu_{2k} := \gamma_{n_k}, l_{2k} := m_{n_k}, \]
\[ R_{2k+1} := F_{n_k+1}, \Omega_{2k+1} := \bigcup_{i=n_k+1}^{n_{k+1}-1} \Delta_i, \mu_{2k+1} := \bigcup_{i=n_k+1}^{n_{k+1}-1} \gamma_i, \]
and
\[ l_{2k+1} := m_{n_{k+1}-1}. \]
It is clear that $\{R_i : i \in \mathbb{N}\}$ is an increasing sequence of nowhere dense sets in $B_1(X)$ such that $B_1(X) = \bigcup_{n \in \mathbb{N}} R_n$. \hfill \Box

Claim 2. The sequence $\{R_i : i \in \mathbb{N}\}$, the pairwise disjoint sequence $\{\Omega_i, i \in \mathbb{N}\}$, and the sequences $\{\mu_i : i \in \mathbb{N}\}$ and $\{l_i : i \in \mathbb{N}\}$ satisfy the following conditions $(i \in \mathbb{N})$:

(a) $1 \leq l_i$ and $l_i + 2 \leq l_{i+1};$

(b) $\bigcap_{i=1}^{\mathbb{R}^X} R_i = \emptyset$ and $\text{supp}(U) \subseteq \bigcup_{j=1}^{i} \Omega_j$ for every $U \in \mu_i;$

(c) if $f \in U \in \mu_i$, then $|f(x)| \leq l_i$ for every $x \in \text{supp}(U);$

(d) if $\varphi : A_i := \bigcup_{j=1}^{i} \Omega_j \to \mathbb{R}$ is such that $\|\varphi\|_{A_i} \leq l_i$, then there is a continuous function $f \in \bigcup_{i=1}^{\mu_{i+1}}$ such that $\|\varphi - f\|_{A_i} < \frac{1}{i}.$

Moreover, the sequence $\{\Omega_{2i} : i \in \mathbb{N}\}$ is strongly $\text{Coz}_\delta$-disjoint.

Proof of Claim 2. By construction, $\{\Omega_i : i \in \mathbb{N}\}$ is a sequence of pairwise disjoint finite subsets of $X$, all families $\mu_i$ are finite, and the sequence $\{\Omega_{2i} : i \in \mathbb{N}\}$ is strongly $\text{Coz}_\delta$-disjoint by the choice of $\{\Delta_{n_k} : k \in \mathbb{N}\}$. The conditions (a) and (c) are satisfied by (a') and (c'), respectively. Since $F_i \subseteq F_j$ for every $i \leq j$, we have $\text{supp}(U) \subseteq \bigcup_{j=1}^{i} \Omega_j$ for every $U \in \mu_i$, and hence the condition (b) holds true. The condition (d) is satisfied by the definition of $\mu_i$ and the condition (d') for the sets $\gamma_i$. The claim is proved. \hfill \Box

Since, by Claim 2, the sequence $\{\Omega_{2i} : i \in \mathbb{N}\}$ is strongly $\text{Coz}_\delta$-disjoint, there is a completely $\text{Coz}_\delta$-additive system $\{W_x : x \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}\}$ of $\text{Coz}_\delta$-sets of $X$ such that $x \in W_x$ for all $x \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}$, and moreover, the sets $W(i) := \bigcup\{W_x : x \in \Omega_{2i}\}$ $(i \in \mathbb{N})$ are $\text{Coz}_\delta$-sets in $X$.

Claim 3. The completely $\text{Coz}_\delta$-additive system $\{W_x : x \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}\}$ can be chosen such that the following conditions are satisfied:

(i) all sets $W_x$ are zero-sets in $X$;

(ii) if $i \in \mathbb{N}$ and $x \in \Omega_{2i}$, then $W_x \cap \bigcup\{\Omega_j : j < 2i\} = \emptyset$. 

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Consequently, for any countable set \( Z \subseteq \bigcup_{k \in \mathbb{N}} \Omega_{2k} \), the set \( \bigcup_{z \in Z} W_z \) is Zer_\( \sigma \) and Coz_\( \delta \) in \( X \) (in particular, all sets \( W(i) \) are Zer_\( \sigma \) and Coz_\( \delta \) in \( X \)).

Proof of Claim 3. Since \( X \) is a Tychonoff space and the sequence \( \{ \Omega_i : i \in \mathbb{N} \} \) is disjoint, for every \( i \in \mathbb{N} \) there is a continuous function \( f_i : X \to [0, 1] \) such that \( f_i(\bigcup \{ \Omega_j : j < 2i \}) = \{0\} \) and \( f_i(\Omega_{2i}) = \{1\} \). Then \( f_i^{-1}([0, \frac{1}{2})) \) is a Zer_\( \sigma \)-set in \( X \) and \( \bigcup \{ \Omega_j : j < 2i \} \subseteq f_i^{-1}([0, \frac{1}{2})) \). Therefore the set \( W'(i) := W(i) \setminus f_i^{-1}([0, \frac{1}{2})) \) is a Coz_\( \delta \)-set in \( X \) such that \( \Omega_{2i} \subseteq W'(i) \) and \( \bigcup \{ \Omega_j : j < 2i \} \cap W'(i) = \emptyset \). By Lemma 3.3, the sequence \( \{ W'(i) : i \in \mathbb{N} \} \) forms a completely Coz_\( \delta \)-additive system. Replacing \( W(i) \) by \( W'(i) \) if needed, we can assume that the completely Coz_\( \delta \)-additive system \( \{ W(i) : i \in \mathbb{N} \} \) satisfies the following condition

(2) \( W(i) \cap \bigcup \{ \Omega_j : j < 2i \} = \emptyset \) for every \( i \in \mathbb{N} \).

Now we fix an arbitrary \( x \in \bigcup_{k \in \mathbb{N}} \Omega_{2k} \). Then \( W_x \) is a Coz_\( \delta \)-set in \( X \). Therefore \( X \setminus W_x = \bigcup_{i \in \mathbb{N}} Z_i \in \text{Zer}_\( \sigma \)(X) \), where all \( Z_i \) are zero-sets in \( X \).

Since \( X \) is Tychonoff and \( x \notin Z_i \), by Theorem 1.5.13 of [10], for every \( i \in \mathbb{N} \) there is a continuous function \( q_i : X \to [0, 1] \) such that \( q_i(x) = 0 \) and \( q_i(Z_i) = \{1\} \). Then the set \( B_x := \bigcap_{i \in \mathbb{N}} q_i^{-1}(0) \) is a zero-set in \( X \) such that \( x \in B_x \subseteq W_x \). By Lemma 3.3, the family \( \{ B_x : x \in \bigcup_{k \in \mathbb{N}} \Omega_{2k} \} \) forms a disjoint completely Coz_\( \delta \)-additive system of zero-sets in \( X \).

Finally, the claim follows if to replace \( W_x \) by \( B_x \) if needed. \( \square \)

Let \( S = \bigcup_{i \in \mathbb{N}} W(i) \). Then, by Claim 3, \( F := X \setminus S \in \text{Coz}_\( \delta \)(X) \).

Claim 4. There is a sequence \( \{ f_i : i \in \mathbb{N} \} \subset B_1(X) \), a strictly increasing sequence \( \{ r_i : i \in \mathbb{N} \} \subseteq \mathbb{N} \) with \( r_1 = 1 \), a sequence \( \{ b_i : i \in \mathbb{N} \} \), and a sequence \( \{ U_i : i \in \mathbb{N} \} \) of basic open sets in \( \mathbb{R}^X \) such that

(e) \( U_i \in \mu_{2r_i} \) for every \( i \in \mathbb{N} \);
(f) \( \| f_i \|_X \leq l_{2r_i} \) for every \( i \in \mathbb{N} \);
(g) \( \| f_{i+1} - f_i \|_{C_i} \leq \frac{1}{2r_i} \) for every \( i \in \mathbb{N} \) where \( C_i := F \cup \bigcup_{j=1}^{r_i} W(j) \);
(h) \( f_j \in U_i \) for every \( j \geq i \geq 1 \);
(k) \( b_{i+1} - b_i \to \infty \).

Proof of Claim 4. For \( i = 1 \), take an arbitrary \( U_1 \in \mu_2 = \mu_{2r_1} \). Since \( X \) is Tychonoff and \( \text{supp}(U_1) \) is finite (because \( U_1 \) is a basic open set), (e) implies that there is a continuous function \( \tilde{f}_1 : X \to \mathbb{R} \) such that \( \tilde{f}_1 \in U_1 \) and \( \| \tilde{f}_1 \|_X < l_{2r_1} \); in particular, \( a_1 := \| \tilde{f}_1 \|_{\text{supp}(U_1)} < l_{2r_1} \).

Since \( \{ W_z : z \in \bigcup_{k \in \mathbb{N}} \Omega_{2k} \} \) is pairwise disjoint (see Claim 3), we can define a function \( f_1 : X \to \mathbb{R} \) by
f_1(x) := \begin{cases} \widetilde{f}_1(x), & \text{if } x \in F; \\ \widetilde{f}_1(z), & \text{if } x \in W_z \text{ for some } z \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}. \end{cases}

Since \{W_z : z \in \bigcup_{k \in \mathbb{N}} \Omega_{2k} \} \cup \{F\} is a disjoint completely \(Coz_\delta\)-additive system, \(f_1\) is constant on each \(W_z\) and \(f_1 \restriction F\) is continuous, \(f_1\) is a Baire-one function on \(X\). Let us explain in more detail. Let \(O\) be an open set in \(\mathbb{R}\). Then

\[
f_1^{-1}(O) = \left( \widetilde{f}_1^{-1}(O) \cap F \right) \cup \left( \bigcup \{W_z : f_1^{-1}(O) \cap W_z \neq \emptyset \} \right).
\]

Since \(F\) and \(\widetilde{f}_1^{-1}(O)\) are \(Zer_\sigma\)-sets, the set \(\widetilde{f}_1^{-1}(O) \cap F \in Zer_\sigma\). Since \(\{W_z : z \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}\} \cup \{F\}\) is a disjoint completely \(Coz_\delta\)-additive system, the set \(\bigcup \{W_z : f_1^{-1}(O) \cap W_z \neq \emptyset \} \in Zer_\sigma\). Thus, we get that \(f_1^{-1}(O) \in Zer_\sigma\) and hence \(f_1 \in B_1(X)\).

The inequality \(\|f_1\|_X < l_{2r_1}\) implies that \(\|f_1\|_X < l_{2r_1}\). To check that \(f_1 \in U_1\), we recall that, by (ii) of Claim 3, \(\Omega_1 \cap W_x = \emptyset\) for every \(x \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}\), and hence \(\Omega_1 \subseteq F\). Therefore \(f_1(x) = \widetilde{f}_1(x)\) for every \(x \in \Omega_1 \cup \Omega_2\). Since \(U_1 \in \mu_2\), the condition (b) implies \(\text{supp}(U_1) \subseteq \Omega_1 \cup \Omega_2\). Thus \(f_1 \in U_1\). Recall that \(a_1 < l_{2r_1}\). Now it is clear that for a sufficiently large \(b_1\) (such that \(\frac{1}{2r_1} < l_{2r_1} - a_1\)), we obtain that the condition \(\|f - f_1\|_{F \cap \Omega_2} \leq \frac{1}{2r_1}\) implies \(f \in \tilde{U}_1\).

Let \(i = 2\). Let us construct \(r_2, U_2, f_2\) and \(b_2\), such that (e)-(k) are satisfied as well. Choose \(r_2 \in \mathbb{N}\) such that \(r_2 > r_1\) and \(\frac{1}{2r_2 - 1} < \frac{1}{2r_1}\).

Define \(\varphi := f_1 \restriction D\) where \(D := \bigcup_{i=1}^{2r_2-1} \Omega_i\). Then \(\varphi\) is a function from \(D\) to \(\mathbb{R}\) such that \(\|\varphi\|_D \leq l_{2r_1} < l_{2r_2-1}\), and hence, by (d), there exists a continuous function \(\tilde{f}_2 \in \bigcup \mu_{2r_2}\) such that \(\|\tilde{f}_2\| \leq l_{2r_2-1}\), \(|\varphi(x) - \tilde{f}_2(x)| < \frac{1}{2r_2-1} < \frac{1}{2r_1}\) for each \(x \in D\). Since \(\tilde{f}_1\) is continuous, we can assume that \(|\tilde{f}_2(x) - f_1(x)| = |\tilde{f}_2(x) - \tilde{f}_1(x)| < \frac{1}{2r_1}\) for every \(x \in F\).

Let \(U_2 \in \mu_{2r_2}\) be such that \(\tilde{f}_2 \in U_2\). Since \(\{W_z : z \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}\}\) is pairwise disjoint (see Claim 3), one can define a function \(f_2 : X \to \mathbb{R}\) by

\[
f_2(x) := \begin{cases} \tilde{f}_2(x), & \text{if } x \in F; \\ \tilde{f}_2(z), & \text{if } x \in W_z \text{ for some } z \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}. \end{cases}
\]

Since \(\{W_z : z \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}\} \cup \{F\}\) is a disjoint completely \(Coz_\delta\)-additive system, \(f_2\) is constant on each \(W_z\) and \(f_2 \restriction F\) is continuous, \(f_2\) is a Baire-one function on \(X\).
Since $\|\tilde{f}_2\|_X \leq l_{2r_2-1}$, it follows that $\|f_2\|_X < l_{2r_2}$. Now we check that $f_2 \in U_2$. By (ii) of Claim 3, we have $(\bigcup_{i=1}^{2r_2-1} \Omega_i) \cap W_x = \emptyset$ for every $x \in \bigcup_{i \geq r_2} \Omega_i$, and hence $\bigcup_{i=1}^{2r_2-1} \Omega_i \subseteq F \cup \bigcup_{i=1}^{r_2-1} \Omega_i$.

Therefore $f_2(x) = \tilde{f}_2(x)$ for every $x \in \bigcup_{i=1}^{2r_2} \Omega_i$. Since $U_2 \in \mu_{2r_2}$, the condition (b) implies $\text{supp}(U_2) \subseteq \bigcup_{i=1}^{2r_2} \Omega_i$. Thus $f_2 \in U_2$.

Now we choose $b_2 > b_1 + 1$ (that implies the condition (k) of the claim) as follows. By the previous paragraph we have

$$\text{supp}(U_2) \subseteq \bigcup_{i=1}^{2r_2} \Omega_i \subseteq F \cup \bigcup \{W_z : z \in \bigcup_{i=1}^{r_2-1} \Omega_{2i}\} \cup \bigcup \{W_x : x \in \Omega_{r_2}\}.$$ 

Taking into account that $U_2$ is a basic open set, $f_2 \in U_2$ and $\|f_2\|_X < l_{2r_2}$, it is clear that for a sufficiently small $\varepsilon > 0$, it follows that if $\|g - f_2\|_C_1 < \varepsilon$, then $g \in U_2$. Take $b_2 > b_1 + 1$ such that $\frac{1}{2^{r_2}} < \varepsilon$. Finally, since $r_1 < r_2 - 1$ it follows that $C_1 \subseteq C_2$ and hence $\|f_2 - f_1\|_{C_1} \leq \frac{1}{2r_2}$.

To check the step of induction, assume that we have found $f_1, ..., f_k \in B_1(X)$, $1 = r_1 < ... < r_k$ in $\mathbb{N}$, $b_1 < ... < b_k$, and basic open sets $U_1, ..., U_k$ in $\mathbb{R}^X$ such that (e)-(k) are satisfied. Let us construct $r_{k+1}$, $U_{k+1}$, $f_{k+1}$ and $b_{k+1}$, such that (e)-(k) are satisfied as well. Choose $r_{k+1} \in \mathbb{N}$ such that $r_{k+1} > r_k$ and $\frac{1}{2^{r_{k+1}-1}} < \frac{1}{2^{r_k}}$.

Define $\varphi := f_k \upharpoonright D$ where $D := \bigcup_{i=1}^{2r_{k+1}-1} \Omega_i$. Then $\varphi$ is a function from $D$ to $\mathbb{R}$ such that $\|\varphi\|_D \leq l_{2r_k} \leq l_{2r_{k+1}-1}$, and hence, by (d), there exists a continuous function $\tilde{f}_{k+1} \in \text{supp}(U_{k+1})$ such that $\|\tilde{f}_{k+1}\| \leq l_{2r_{k+1}-1}$, $|\varphi(x) - \tilde{f}_{k+1}(x)| < \frac{1}{2^{r_{k+1}-1}} < \frac{1}{2^{r_k}}$ for each $x \in D$. Since $\tilde{f}_{k+1}$ is continuous, we can assume that $|\tilde{f}_{k+1}(x) - f_k(x)| = |\tilde{f}_{k+1}(x) - \tilde{f}_{k+1}(x)| < \frac{1}{2^{r_k}}$ for every $x \in F$.

Let $U_{k+1} \in \mu_{2r_{k+1}}$ be such that $\tilde{f}_{k+1} \in U_{k+1}$. Since $\{W_z : z \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}\}$ is pairwise disjoint (see Claim 3), one can define a function $f_{k+1} : X \to \mathbb{R}$ by

$$f_{k+1}(x) := \begin{cases} \tilde{f}_{k+1}(x), & \text{if } x \in F; \\ \tilde{f}_{k+1}(z), & \text{if } x \in W_z \text{ for some } z \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}. \end{cases}$$

Since $\{W_z : z \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}\} \cup \{F\}$ is a disjoint completely Coz-$\delta$-additive system, $f_{k+1}$ is constant on each $W_z$ and $f_{k+1} \upharpoonright F$ is continuous, $f_{k+1}$ is a Baire-one function on $X$. Let us explain in more detail. Let $O$ be an open set in $\mathbb{R}$. Then

$$f_{k+1}^{-1}(O) = \left(\tilde{f}_{k+1}^{-1}(O) \cap F\right) \cup \left(\bigcup \{W_z : f_{k+1}^{-1}(O) \cap W_z \neq \emptyset\}\right).$$
Since $F$ and $\tilde{f}_{k+1}(O)$ are Zer$_{\sigma}$-sets, the set $\tilde{f}_{k+1}(O) \cap F \in \text{Zer}_{\sigma}$. Since 
$\{W_z : z \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}\} \cup \{F\}$ is a disjoint completely Coz$_{\delta}$-additive system, the  
set $\bigcup\{W_z : f_{k+1}(O) \cap W_z \neq \emptyset\} \in \text{Zer}_{\sigma}$. Thus, we get that $f_{k+1}(O) \in \text{Zer}_{\sigma}$  
and hence $f_{k+1} \in B_1(X)$.

Since $\|\tilde{f}_{k+1}\| \leq l_{2r_{k+1}}$, it follows that $\|f_{k+1}\|_X < l_{2r_{k+1}}$. Now we check  
that $f_{k+1} \in U_{k+1}$. By (ii) of Claim 3, we have $(\bigcup_{i=1}^{2r_{k+1}} \Omega_i) \cap W_x = \emptyset$ for  
every $x \in \bigcup_{i \geq r_{k+1}} \Omega_{2i}$, and hence $\bigcup_{i=1}^{2r_{k+1}-1} \Omega_i \subseteq F \cup \bigcup_{i=1}^{r_{k+1}-1} \Omega_{2i}$.

Therefore $f_{k+1}(x) = \tilde{f}_{k+1}(x)$ for every $x \in \bigcup_{i=1}^{2r_{k+1}} \Omega_i$. Since $U_{k+1} \in \mu_{2r_{k+1}}$,  
the condition (b) implies $\text{supp}(U_{k+1}) \subseteq \bigcup_{i=1}^{2r_{k+1}} \Omega_i$. Thus $f_{k+1} \in U_{k+1}$.

Now we choose $b_{k+1} > b_k + k$ (that implies the condition (k) of the claim) as follows. By the previous paragraph we have

$$\text{supp}(U_{k+1}) \subseteq \bigcup_{i=1}^{2r_{k+1}} \Omega_i \subseteq F \cup \bigcup_{i=1}^{r_{k+1}-1} \Omega_{2i} \cup \{W_z : z \in \bigcup_{i=1}^{r_{k+1}-1} \Omega_{2i}\}. $$

Taking into account that $U_{k+1}$ is a basic open set, $f_{k+1} \in U_{k+1}$ and  
$\|f_{k+1}\|_X < l_{2r_{k+1}}$, it is clear that for a sufficiently small $\varepsilon > 0$, it follows that  
if $\|g - f_{k+1}\|_{C_k} < \varepsilon$, then $g \in U_{k+1}$. Take $b_{k+1} > b_k + k$ such that $\frac{1}{2^{2r_{k+1}}} < \varepsilon$.  
Finally, since $r_k < r_{k+1} - 1$ it follows that $C_k = F \cup \bigcup_{j=1}^{r_k} W(j) \subseteq C_{k+1}$  
and hence $\|f_{k+1} - f_k\|_{C_k} \leq \frac{1}{2^k}$.

It follows from Claim 4(g) that the sequence $\{f_i : i \in \mathbb{N}\} \subseteq B_1(X)$ converges  
pointwise to some function $f : X \to \mathbb{R}$.

To show that the function $f$ is Baire-one, let $V = (a, b)$ be an open interval of $\mathbb{R}$.

Since all functions $\tilde{t}_i$ are continuous, $\{W_z : z \in \bigcup_{k \in \mathbb{N}} \Omega_{2k}\} \cup \{F\}$ is a  
disjoint completely Coz$_{\delta}$-additive system and $f_i$ is constant on each $W_z$ and $C_k \in \text{Zer}_{\sigma}(X)$, the set  
$f^{-1}(V) \cap C_k = \left( \bigcup_{n>\frac{b-a}{2}} \bigcup_{l>n+k} f_i^{-1}([a + \frac{1}{n}, b - \frac{1}{n}]) \right) \cap C_k =  
\left( \bigcup_{n>\frac{b-a}{2}} \bigcup_{l>n+k} \tilde{t}_i^{-1}([a + \frac{1}{n}, b - \frac{1}{n}]) \right) \cap C_k  
\cup \left( \bigcup_{n>\frac{b-a}{2}} \bigcup_{l>n+k} f_i^{-1}([a + \frac{1}{n}, b - \frac{1}{n}]) \right) \cap \bigcup_{j=1}^{r_k} W(j) \subseteq \text{Zer}_{\sigma}(X).$

It follows that $f^{-1}(V) = \bigcup_{k \in \mathbb{N}} (f^{-1}(V) \cap C_k) \in \text{Zer}_{\sigma}(X)$ and, hence, $f \in B_1(X)$.

By Claim 4(h), $f \in \overline{U}_i$ for each $i \in \mathbb{N}$. But, then, by Claim 2(b), $f \not\in B_1(X)$.
\[ \bigcup_{i=1}^{\infty} F_i = B_1(X) \text{, it is a contradiction.} \]

(2) \Rightarrow (1). Let \( \{ \Delta_n : n \in \mathbb{N} \} \) satisfy (2). For every \( m \in \mathbb{N} \), define \( F_m := \left\{ f \in B_1(X) : \sup_{n \in \mathbb{N}} \left( \min\{ |f(x)| : x \in \Delta_n \} \right) \leq m \right\} \).

By (2) we have \( B_1(X) = \bigcup_{m \in \mathbb{N}} F_m \). Therefore it remains to prove that \( F_m \) is nowhere dense in \( B_1(X) \). First we show that \( F_m \) is closed in \( B_1(X) \). Indeed, let \( f \in B_1(X) \) be such that \( |f(z)| > m \) for some \( n \in \mathbb{N} \) and each \( z \in \Delta_n \).

Set \( \varepsilon := \frac{1}{2} \min\{ |f(x)| : x \in \Delta_n \} - m \). Then the standard neighborhood \( f + [\Delta_n; \varepsilon] := \{ g \in B_1(X) : |f(x) - g(x)| < \varepsilon \text{ for every } x \in \Delta_n \} \) of \( f \) does not intersect \( F_m \). Thus \( F_m \) is closed.

To show that the closed set \( F_m \) is nowhere dense in \( B_1(X) \), suppose for a contradiction that \( F_m \) contains a standard neighborhood \( g + [A; \delta] \) of some \( g \in F_m \), where \( A \subset X \) is finite and \( \delta > 0 \). Since the sequence \( \{ \Delta_n : n \in \mathbb{N} \} \) is disjoint, there is \( n_0 \in \mathbb{N} \) such that \( \Delta_{n_0} \cap A = \emptyset \). As \( X \) is Tychonoff, there is \( h \in C(X) \) such that \( h \rest_A = g \) and \( h \rest_{\Delta_{n_0}} = 2m \). It is clear that \( h \in g + [A; \delta] \subseteq F_m \), however

\[
\sup_{n \in \mathbb{N}} \left( \min\{ |h(x)| : x \in \Delta_n \} \right) \geq \min\{ |h(x)| : x \in \Delta_{n_0} \} = 2m > m
\]

and hence \( h \not\in F_m \), a contradiction. Thus \( F_m \) is nowhere dense in \( B_1(X) \) and hence \( B_1(X) \) is meager.

(2) \Rightarrow (3). Let \( \Delta = \{ \Delta_n : n \in \mathbb{N} \} \) be a sequence satisfying (2). Assume that \( \{ \Delta_{n_i} : i \in \mathbb{N} \} \) is a strongly Coz\(_4\)-disjoint subsequence of \( \Delta \). Then, there is a pairwise disjoint collection \( \{ F_a : a \in \bigcup \Delta_{n_i} \} \) of Coz\(_4\) neighborhoods of \( a \in \bigcup \Delta_{n_i} \) such that \( \bigcup_{b \in B} F_b \in \text{Coz}_4 \) for each \( B \subseteq \bigcup \Delta_{n_i} \).

Let \( F = \bigcup \{ F_a : a \in \bigcup \Delta_{n_i} \} \) and \( S = \{ a_j : j \in \mathbb{N} \} = \bigcup \Delta_{n_i} \). Then, the function \( f : F \to \mathbb{R} \) such that \( f(F_a) = i \) for every \( i \in \mathbb{N} \) is constant on each \( F_a \). By Lemma 3.2 there is a Baire-one function \( g \) on \( X \) extending \( f \). Note that \( \sup_{n \in \mathbb{N}} \left( \min\{ |g(x)| : x \in \Delta_n \} \right) = \infty \), a contradiction.

(3) \Rightarrow (2). Suppose for a contradiction that \( \Delta = \{ \Delta_n : n \in \mathbb{N} \} \) does not contain strongly Coz\(_4\)-disjoint subsequence and there is \( f \in B_1(X) \) such that

\[
\sup_{n \in \mathbb{N}} \left( \min\{ |f(x)| : x \in \Delta_n \} \right) = \infty.
\]

Consider \( \{ \Delta_{n_k} : k \in \mathbb{N} \} \) such that \( m_k < \min\{ |f(x)| : x \in \Delta_{n_k} \} \) and

\[
\max\{ |f(x)| : x \in \Delta_{n_k} \} < m_{k+1} \text{ for every } k \in \mathbb{N} \text{ where } m_k \to \infty \text{ (} k \to \infty \).
\]

Then \( (f(\Delta_{n_k}))_k \) is a discrete family of finite (and hence closed) subset of \( \mathbb{R} \). It follows that this family is completely closed-additive (the union of
any subfamily is closed). Since $f$ is Baire-one, we deduce that the family $(f^{-1}(f(\Delta_{n_k})))_k$ is completely $Coz_\delta$-additive. By Proposition 3.5, this immediately shows that the sequence $(\Delta_{n_k})_k$ is strongly $Coz_\delta$-disjoint, a contradiction.

$(4) \Rightarrow (3)$. Let $\sigma$ be a winning strategy for ONE player in the game $Game(X)$. A sequence $(B_0, \ldots, B_n)$ of disjoint finite subsets of $X$ is called correct if $B_{2k+1} = \sigma(B_0, \ldots, B_{2k})$, where $2k + 1 \leq n$. By induction, we construct a disjoint sequence $\{\Delta_n : n \in \mathbb{N}\}$ of finite subsets of $X$.

Let $\Delta_1 = B_0$ be an arbitrary finite subset of $X$. $\Delta_2 = B_1 = \sigma(B_0)$. Assume that $\Delta_0, \ldots, \Delta_n$ are constructed then

$$\Delta_{n+1} := \bigcup\{\sigma(B_0, \ldots, B_{2m}) : (B_0, \ldots, B_{2m}) \text{ is correct and } \bigcup_{j=0}^{2m} B_j = \bigcup_{i=1}^{n} \Delta_i, m \leq n\}.$$ 

We claim that for any subsequence $\{\Delta_{n_k} : k \in \mathbb{N}\}$, $1 \leq n_1 < n_2 < \ldots$ there is a sequence $\{D_k : k \in \mathbb{N}\}$ satisfying the conditions:

(a) $(D_0, \ldots, D_k)$ is a correct sequence for each $k$;
(b) $D_{2p+1} \subseteq \Delta_{n_{p+1}}$ for any $p \geq 0$ and $\bigcup_{i=0}^{2m} D_i = \bigcup_{i=1}^{n_{m+1}} \Delta_i$ for any $m \geq 1$.

By induction, we construct a sequence $\{D_k : k \in \mathbb{N}\}$. Let $D_0 = \bigcup_{i=1}^{n_1-1} \Delta_i$. Assume that $D_0, \ldots, D_{2m}$ are constructed. Let’s construct the sets $D_{2m+1}$, $D_{2m+2}$. By condition (a), $(D_0, \ldots, D_{2m})$ is correct, and, by condition (b), $\bigcup_{i=0}^{2m} D_i = \bigcup_{i=1}^{n_{m+1}-1} \Delta_i$. Then, by constructing the sequence $\{\Delta_n : n \in \mathbb{N}\}$, $\sigma(D_0, \ldots, D_{2m}) \subseteq \Delta_{n_{m+1}}$. Let $D_{2m+1} = \sigma(D_0, \ldots, D_{2m}), D_{2m+2} = \bigcup_{i=n_{m+1}}^{n_{m+2}-1} \Delta_i \setminus D_{2m+1}$. It is not difficult to verify that the sequence $(D_0, \ldots, D_{2m+2})$ also satisfies conditions (a) and (b).

Since $\sigma$ is a winning strategy for ONE player in the game $Game(X)$, the sequence $\{D_{2p+1} : p \in \mathbb{N}\}$ is not strongly $Coz_\delta$-disjoint, hence, by condition (b), $\{\Delta_{n_k} : k \in \mathbb{N}\}$ also is not strongly $Coz_\delta$-disjoint. Since the subsequence $\{\Delta_{n_k} : k \in \mathbb{N}\}$ was arbitrary, it follows that the sequence $\{\Delta_n : n \in \mathbb{N}\}$ satisfies the condition (3).

$(3) \Rightarrow (4)$. Let $\{\Delta_n : n \in \mathbb{N}\}$ be a sequence satisfying (3). For every finite set $P \subseteq X$, put $n(P) = \min\{m : \Delta_m \cap P = \emptyset\}$. Then the strategy of player ONE: $S_{2n+1} = \sigma(S_0, \ldots, S_{2n}) = \Delta_{n(L)}$ where $L = \bigcup_{i=0}^{2n} S_i$, is winning.
Since a non-meager space $B_1(X)$ is Baire, we have the following result.

**Theorem 3.7.** Let $X$ be a topological space. The following assertions are equivalent:

1. $B_1(X)$ is Baire;
2. every pairwise disjoint sequence of non-empty finite subsets of $X$ has a strongly $Coz_δ$-disjoint subsequence;
3. $\text{ONE}$ has no winning strategy in the game $G_1(B_1(X))$;
4. $\text{ONE}$ has no winning strategy in the game $\text{Game}(X)$.

Note that if $Y \subset X$ and $A \in Coz_δ(X)$ then $A \cap Y \in Coz_δ(Y)$.

**Corollary 3.8.** Let $B_1(X)$ be a Baire space and $Y \subseteq X$. Then $B_1(Y)$ is Baire.

It is well-known that there are Baire spaces $X$ and $Y$ such that $X \times Y$ is not Baire [28]. For the product $\prod_{\alpha \in A} B_1(X_\alpha)$ we have the following result.

**Corollary 3.9.** If $B_1(X_\alpha)$ is Baire for all $\alpha \in A$, then $\prod_{\alpha \in A} B_1(X_\alpha)$ is Baire.

**Proof.** It is well-known that $\prod_{\alpha \in A} C_p(X_\alpha) \cong C_p\left(\bigoplus_{\alpha \in A} X_\alpha\right)$ [1]. We claim that $\prod_{\alpha \in A} B_1(X_\alpha) \cong B_1\left(\bigoplus_{\alpha \in A} X_\alpha\right)$. To this end, we define a mapping $f \mapsto f^*$ where $f \in \prod_{\alpha \in A} B_1(X_\alpha)$ and $f^* \in B_1\left(\bigoplus_{\alpha \in A} X_\alpha\right)$.

Let $f \in \prod_{\alpha \in A} B_1(X_\alpha)$. Then $f = (f_\alpha)$ where $f_\alpha \in B_1(X_\alpha)$ for each $\alpha \in A$.

Define $f^* : \bigoplus_{\alpha \in A} X_\alpha \rightarrow \mathbb{R}$ by letting $f^* \upharpoonright X_\alpha = f_\alpha$ for each $\alpha \in A$.

We claim that $f^*$ is a Baire-one function. For each $\alpha \in A$, there is a sequence $\{f_{n,\alpha} : X_\alpha \rightarrow \mathbb{R}\}$ of continuous functions such that $f_{n,\alpha} \rightarrow f_\alpha$ pointwise. For each $n$, define $f_n : \bigoplus_{\alpha \in A} X_\alpha \rightarrow \mathbb{R}$ such that $f_n \upharpoonright X_\alpha = f_{n,\alpha}$ for each $\alpha \in A$. It follows that $f_n \rightarrow f^*$ pointwise. Thus, $f^* \in B_1\left(\bigoplus_{\alpha \in A} X_\alpha\right)$.

The mapping $f \mapsto f^*$ is one-to-one mapping from $\prod_{\alpha \in A} B_1(X_\alpha)$ onto $B_1\left(\bigoplus_{\alpha \in A} X_\alpha\right)$. Furthermore, both the mapping and its inverse are continuous. Thus, the mapping is a homeomorphism.
Let $B_1(X_\alpha)$ be a Baire space for all $\alpha \in A$. We claim that $B_1(\bigoplus_{\alpha \in A} X_\alpha)$ is Baire.

It is clear that $B_1(\bigoplus_{\alpha \in A} X_\alpha)$ is Baire for a finite set $A$. Suppose that $|A| \geq \aleph_0$.

Let $\gamma = \{\Delta_n : n \in \mathbb{N}\} \subseteq \bigoplus_{\alpha \in A} X_\alpha$ be a pairwise disjoint sequence of non-empty finite subsets of $\bigoplus_{\alpha \in A} X_\alpha$. Let $A_1$ be a finite subset of $A$ such that $\Delta_1 \subseteq \bigoplus_{\alpha \in A_1} \{X_\alpha : \alpha \in A_1\} = Y_1$. Consider $\gamma \cap Y_1 = \{\Delta_n \cap Y_1 : n \in \mathbb{N}\}$. There is $\gamma_1 \subseteq \gamma$ such that $|\gamma_1| = \aleph_0$ and $\gamma_1 \cap Y_1$ is strongly $\text{Coz}_\delta$-disjoint in $Y_1$ and hence in $\bigoplus_{\alpha \in A} X_\alpha$. Suppose that $\Delta_1 \in \gamma_1$. Let $\Delta_{n_2} \in \gamma_1 \setminus \Delta_1$. Choose $A_2 \subseteq A$, $|A_2| < \aleph_0$ such that $\Delta_{n_2} \subseteq Y_1 \cup Y_2$, where $Y_2 = \bigoplus_{\alpha \in A_2} \{X_\alpha : \alpha \in A_2\}$. Consider $\gamma_1 \cap Y_2$. There is $\gamma_2 \subseteq \gamma_1$, $|\gamma_2| = \aleph_0$ and $\gamma_2 \cap Y_2$ is strongly $\text{Coz}_\delta$-disjoint in $Y_2$. Choose $\Delta_{n_3} \in \gamma_2 \setminus \{\Delta_1, \Delta_{n_2}\}$ and so on. We get the strongly $\text{Coz}_\delta$-disjoint sequence $\{\Delta_{n_i} : i \in \mathbb{N}\}$ in $\bigoplus_{\alpha \in A} X_\alpha$ which satisfies our requirements. \hfill $\Box$

**Proposition 3.10.** Let $X$ be a Hausdorff space and let $\{F_i : i \in \mathbb{N}\}$ be a pairwise disjoint family of $\text{Coz}_\delta$ subsets of $X$ such that $\bigcup F_i$ is a $\text{Coz}_\delta$-subset of $X$. Then the following assertions are equivalent:

1. $\{F_i : i \in \mathbb{N}\}$ forms a completely $\text{Coz}_\delta$-additive system;
2. there is a pairwise disjoint family $\{D_i : i \in \mathbb{N}\}$ of $\text{Zer}_\sigma$ subsets of $X$ such that $F_i \subseteq D_i$ for each $i \in \mathbb{N}$.

**Proof.** (1) $\Rightarrow$ (2). By Lemma 3.2, there is $f \in B_1(X)$ such that $f(F_i) = i$ for each $i \in \mathbb{N}$. Let $D_i = f^{-1}(i - \frac{1}{2}, i + \frac{1}{2})$ for each $i \in \mathbb{N}$. Then $\{D_i : i \in \mathbb{N}\}$ satisfies our requirements.

(2) $\Rightarrow$ (1). Let $\{i_k : k \in \mathbb{N}\} \subseteq \mathbb{N}$. Then $\bigcup\{F_{i_k} : k \in \mathbb{N}\} = \bigcup F_i \setminus (\bigcup\{D_j : j \neq i_k, j, k \in \mathbb{N}\})$. Since $\bigcup F_i \in \text{Coz}_\delta(X)$ and $\bigcup\{D_j : j \neq i_k, j, k \in \mathbb{N}\} \in \text{Zer}_\sigma(X)$, the set $\bigcup\{F_{i_k} : k \in \mathbb{N}\} \subseteq \text{Coz}_\delta(X)$. \hfill $\Box$

Recall that the pseudocharacter $\psi(X)$ of $X$ is the smallest cardinal $\kappa$ such that every singleton $\{x\} \subseteq X$ is the intersection of $\leq \kappa$ open sets in $X$.

**Lemma 3.11.** Let $X$ be a space of countable pseudocharacter and let $A = \{a_i : i \in \mathbb{N}\}$ be a strongly $\text{Coz}_\delta$-disjoint countable subset of $X$. Then $\{a_i : i \in \mathbb{N}\}$ forms a completely $\text{Coz}_\delta$-additive system.
Proof. Since \( X \) is a space of countable pseudocharacter, \( \{ x \} \in Coz_\delta(X) \) for every \( x \in X \). Then, by Lemma 3.3 \( \{ a_i : i \in \mathbb{N} \} \) forms a completely \( Coz_\delta \)-additive system.

Note that any Lindelöf \( G_\delta \)-set is \( Coz_\delta \) (Proposition 4 (b) in [11]). Then, by Lemma 3.11 and Theorem 3.7, we have the following result.

**Theorem 3.12.** Let \( X \) be a space of countable pseudocharacter. A space \( B_1(X) \) is Baire if, and only if, every pairwise disjoint sequence \( \{ \Delta_i : i \in \mathbb{N} \} \) of non-empty finite subsets of \( X \) has a subsequence \( \{ \Delta_{i_k} : k \in \mathbb{N} \} \) such that \( \bigcup \{ \Delta_{i_k} : k \in \mathbb{N} \} \) is \( G_\delta \).

A set \( A \) is called *locally non-\( G_\delta \) if no nonempty relatively open subset of \( A \) is \( G_\delta \).

**Proposition 3.13.** Let \( B_1(X) \) be a Baire space for a metrizable space \( X \). Then for any countable locally non-\( G_\delta \) subset \( A \subseteq X \) there is a disjoint family \( \{ B_i : i \in \mathbb{N} \} \) of countable subsets of \( A \) such that \( A = \bigcup B_i \), \( \overline{B_i} = \overline{A} \) and \( B_i \) is a \( G_\delta \)-set in \( X \) for all \( i \in \mathbb{N} \).

Proof. Since \( \overline{A} \) is a separable metrizable space, by Theorem 4.3.5 from [10], \( \overline{A} \) is metrizable by a totally bounded metric \( \rho \). For each \( \epsilon = \frac{1}{2^n} \) there exists a finite set \( F_n \) of \( \overline{A} \) which is \( \epsilon \)-dense in \( (\overline{A}, \rho) \). It follows that for each \( n \in \mathbb{N} \) there exists a finite set \( A_n \subseteq A \) which is \( \frac{1}{2^n} \)-dense in \( (\overline{A}, \rho) \). Since \( A \) is locally non-\( G_\delta \), then we can assume that \( A_i \cap A_j = \emptyset \) for \( i \neq j, i, j \in \mathbb{N} \).

By Corollary 3.7 there exists a subsequence \( \{ A_{n_k} : k \in \mathbb{N} \} \subseteq \{ A_n : n \in \mathbb{N} \} \) such that \( \{ A_{n_k} : k \in \mathbb{N} \} \) is strongly \( Coz_\delta \)-disjoint. Let \( B_1 = \bigcup \{ A_{n_k} : k \in \mathbb{N} \} \) and, by Lemma 3.11, \( B_1 \) is a \( G_\delta \)-set in \( X \). Consider \( A \setminus B_1 \). Since \( A \) is not-\( G_\delta \) then \( A \setminus B_1 = \{ A_j : j \neq n_k, k \in \mathbb{N} \} \) is not \( G_\delta \), too. Since \( A \) is locally not-\( G_\delta \) then \( A \setminus B_1 = \overline{A} \). Analogously, choose \( B_2 \subseteq (A \setminus B_1) \) such that \( B_2 = \{ A_{j_s} : s \in \mathbb{N} \} \) is a \( G_\delta \)-set in \( X \) and \( \overline{B_2} = \overline{A} \). By induction, choose \( B_j \subseteq (A \setminus \bigcup \{ B_i : i = 1, \ldots, j-1 \}) \) such that \( B_j \) is a \( G_\delta \)-set in \( X \) and \( \overline{B_j} = \overline{A} \). Note that the induction is an infinite because \( A \) is not \( G_\delta \). We get the family \( \{ B_i : i \in \mathbb{N} \} \) which satisfies our requirements. This conclude the proof.

**Theorem 3.14.** If \( X \) is metrizable and \( B_1(X) \) is a Baire space, then each separable subset of \( X \) without isolated points is meager (of first category) in itself.
Proof. Let $Y$ be a separable subset of $X$ without isolated points. Assume that $Y$ is of second category in itself. Then there is $Z \subseteq Y$ relatively open which is a Baire space. In particular, $Z$ is locally uncountable (as it has no isolated points). Let $A \subseteq Z$ be a countable dense subset.

We claim that $A$ is locally non-$G_\delta$. Assume there is a nonempty relatively open $B \subseteq A$ which is $G_\delta$. Let $U \subseteq X$ be open such that $U \cap A = B$. Then $B$ is dense in $U \cap Z$. Moreover, $U \cap (Z \setminus B)$ is also dense in $U \cap Z$. Since $U \cap (Z \setminus B)$ is $G_\delta$ in $U \cap Z$ (as $B$ is countable), we deduce that the Baire space $U \cap Z$ has two disjoint dense $G_\delta$ sets, a contradiction. Thus, $A$ is locally non-$G_\delta$.

By Proposition 3.13, there is a disjoint family $\{B_i : i \in \mathbb{N}\}$ of countable subsets of $A$ such that $A = \bigcup B_i$, $\overline{B_i} = \overline{A}$ and $B_i$ is a $G_\delta$-set in $X$ for all $i \in \mathbb{N}$. Then each $B_i$ is a $G_\delta$-set in $Z$. Thus, we deduce that the Baire space $Z$ has disjoint dense $G_\delta$ sets $B_i$, a contradiction. 

\[\square\]

Corollary 3.15. If $X$ is a separable metrizable space without isolated points such that $B_1(X)$ is Baire, then $X$ is meager.

In particular, a space $B_1(X)$ is meager for any uncountable Polish space $X$ without isolated points.

Note that if $X = \mathbb{N}$ is a set of natural numbers ($X$ is Polish) then $B_1(X) = \mathbb{R}^\omega$ and, hence, $B_1(X)$ is Baire.

Let us recall that a cover $\mathcal{U}$ of a set $X$ is called

- an $\omega$-cover if each finite set $F \subseteq X$ is contained in some $U \in \mathcal{U}$;
- a $\gamma$-cover if for any $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite.

A topological space $X$ is called a $\gamma$-space if each countable open $\omega$-cover $\mathcal{U}$ of $X$ contains a $\gamma$-subcover of $X$. $\gamma$-Spaces were introduced by Gerlits and Nagy in [19] and are important in the theory of function spaces as they are exactly those $X$ for which the space $C_p(X)$ has the Fréchet-Urysohn property [20].

Theorem 3.16. Let $X$ be a $\gamma$-space. Then $B_1(X)$ is Baire.

Proof. We can assume that $X$ is a Tychonoff space otherwise we apply the Tychonoff functor to the space $X$. 

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By Theorem 3.7, it suffices to prove that every pairwise disjoint sequence of non-empty finite subsets of \( X \) has a strongly \( Coz_\delta \)-disjoint subsequence. Let \( \{ \Delta_n : n \in \mathbb{N} \} \) be a pairwise disjoint sequence of non-empty finite subsets of \( X \). Since \( X \) is a Tychonoff space, there exists a pairwise disjoint family \( \{ F_a : a \in \bigcup \Delta_n \} \) of zero-sets of \( X \) such that \( a \in F_a \) for each \( a \in \bigcup \Delta_n \).

Indeed, let \( a, b \in \bigcup \Delta_n \) and \( a \neq b \). Then, there is a continuous function \( f_{a,b} : X \to \mathbb{R} \) such that \( f_{a,b}(a) \neq f_{a,b}(b) \). Note that \( \{ f_{a,b} : a, b \in \bigcup \Delta_n \} \) is a countable family. Let \( F_a = \bigcap \{ f_{a,b}^{-1}(f_{a,b}(a)) : b \in \bigcup \Delta_n \setminus \{ a \} \} \). Note that \( F_a \) is a zero-set of \( X \) and \( F_a \cap F_b = \emptyset \) for \( a \neq b \).

Let \( Q_n = \bigcup \{ F_a : a \in \Delta_n \} \). Clearly, \( Q_n \) is a zero-set of \( X \), \( \Delta_n \subseteq Q_n \) for each \( n \in \mathbb{N} \) and \( Q_n \cap Q_{n'} = \emptyset \) for \( n \neq n' \).

For every \( n \in \mathbb{N} \) the zero-set \( Q_n \) can be represented as \( Q_n = \bigcap \{ W_{n,i} : i \in \mathbb{N} \} = \bigcap \{ S_{n,i} : i \in \mathbb{N} \} \) where \( W_{n,i+1} \subseteq S_{n,i+1} \subseteq W_{n,i} \), \( W_{n,i} \) is a cozero-set of \( X \) and \( S_{n,i+1} \) is a zero-set of \( X \) for each \( i \in \mathbb{N} \).

Then \( \{ X \setminus S_{n,k} : k, n \in \mathbb{N} \} \) is an open \( \omega \)-cover of \( X \). Since \( X \) is a \( \gamma \)-space, there is a \( \gamma \)-subcover \( \mathcal{V} = \{ X \setminus S_{n,i_k} : i \in \mathbb{N} \} \). The \( \gamma \)-subcover \( \mathcal{V} \) leads to the subsequence \( \{ \Delta_{n_i} : i \in \mathbb{N} \} \) and the subsequence \( \{ Q_{n_i} : i \in \mathbb{N} \} \) with \( \Delta_{n_i} \subseteq Q_{n_i} \) for each \( i \).

Since any infinite subset of \( \mathcal{V} \) is a \( \gamma \)-cover of \( X \), shrinking \( \mathcal{V} \), if necessary, we may assume that \( n_{i+1} > n_i \) for all \( i \in \mathbb{N} \). Also, we may assume that \( k_{i+1} > k_i \) for all \( i \) by enlarging the elements of \( \mathcal{V} \).

Consider the \( Coz_\delta \)-set \( G := \bigcap \bigcup \bigcup W_{n,i,k_i} \).

Note that \( A = \bigcup \{ Q_{n_i} : i \in \mathbb{N} \} \subseteq G \). We claim that \( A = G \). Fix any \( x \in X \setminus A \). Since \( \mathcal{V} \) is a \( \gamma \)-cover of \( X \), there is \( j_0 \in \mathbb{N} \) such that \( x \not\in S_{n_i,k_i} \) for all \( l \geq j_0 \). Find \( k \in \mathbb{N} \) such that \( x \not\in W_{n_i,k} \) for all \( l \leq j \), and let \( j \geq j_0 \) be such that \( k_j \geq k \). Thus for all \( l \geq j \) and \( i \leq l \) we have \( x \not\in W_{n_i,k_i} \), because if \( i \geq j_0 \) we have \( x \not\in W_{n_i,k_i} \supseteq W_{n_i,k_j} \) by the choice of \( j_0 \), and if \( i < j_0 \) we have \( x \not\in W_{n_i,k_j} \supseteq W_{n_i,k_i} \) by the choice of \( k \). This shows that \( x \not\in G \) and thus \( A = G \) is a \( Coz_\delta \)-set of \( X \).

Let \( \mathcal{Q} = \{ Q_{n_1}, Q_{n_2}, Q_{n_3}, \ldots \} \). Let \( W \subset \mathcal{Q} \) and \( W_c = \mathcal{Q} \setminus W \). Since each \( Q_n \) is a zero-set, \( \bigcup W_c \) is a \( \text{Zer}_\sigma \). Since \( A = \bigcup \{ Q_{n_i} : i \in \mathbb{N} \} = G \) is a \( Coz_\delta \), \( \bigcup W = A \setminus \bigcup W_c \) is \( Coz_\delta \). Thus, the family \( \mathcal{Q} \) has the property of completely \( Coz_\delta \)-additivity. By Proposition 6, the sequence \( \{ \Delta_{n_i} : i \in \mathbb{N} \} \) is strongly \( Coz_\delta \)-disjoint.

\( \square \)

Recall that \( C_p(X) \) is a Fréchet-Urysohn space if and only if \( X \) is a \( \gamma \)-space.
This and Theorem 3.16 imply the following corollary.

**Corollary 3.17.** Let $C_p(X)$ be a Fréchet-Urysohn space. Then $B_1(X)$ is Baire.

On the other hand, when $X$ is a Lindelöf scattered space or a Lindelöf $P$-space, $C_p(X)$ has the Fréchet-Urysohn property (Theorems II.7.15 and II.7.16 in [1]). Thus, we get the following corollaries.

**Corollary 3.18.** Let $X$ be a scattered Lindelöf space. Then $B_1(X)$ is Baire.

**Corollary 3.19.** Let $X$ be a Lindelöf $P$-space. Then $B_1(X)$ is Baire.

In [2], T. Banakh and S. Gabriyelyan introduced new class of space ($G^N_\delta$-winning) and showed that it has meager function space $B_1(X)$.

We say that two subsets $A, B$ of a topological space $X$ can be separated by $G_\delta$-sets if there exist disjoint $G_\delta$-sets $G_A, G_B \subseteq X$ such that $A \subseteq G_A$ and $B \subseteq G_B$.

Now we consider a topological game $G_\delta(X)$ played by two players $S$ and $N$ (abbreviated from Separating and Nonseparating) on a topological space $X$. The player $S$ starts the game $G_\delta(X)$ selecting a finite set $S_0 \subseteq X$. The player $N$ responds selecting two disjoint finite sets $A_0, B_0 \subseteq X \setminus S_0$. At the $n$-th inning the player $S$ chooses a finite set $S_n \subseteq X$ containing $S_{n-1} \cup A_{n-1} \cup B_{n-1}$ and the player $N$ responds selecting two disjoint finite sets $A_n, B_n \subseteq X \setminus S_n$. At the end of the game the player $N$ is declared the winner if the countable sets $A := \bigcup_{n \in \mathbb{N}} A_n$ and $B := \bigcup_{n \in \mathbb{N}} B_n$ cannot be separated by $G_\delta$-sets in $X$. Otherwise the player $S$ wins the game.

A topological space $X$ is defined to be

- $G^N_\delta$-winning if the player $N$ has a winning strategy in the game $G_\delta(X)$;
- $G^N_\delta$-loosing if the player $N$ has no winning strategy in the game $G_\delta(X)$;
- $G^S_\delta$-winning if the player $S$ has a winning strategy in the game $G_\delta(X)$.

Thus we have the following implications:

$\lambda$-space $\Rightarrow$ $G^S_\delta$-winning $\Rightarrow$ $G^N_\delta$-loosing $\Leftrightarrow$ not $G^N_\delta$-winning.

Since Baireness of $B_1(X)$ implies that $X$ is $G^N_\delta$-loosing (Theorem 7.2 in [2]), we get a positive answer to Problem 7.12 in [2]: Is each metrizable $\gamma$-space $G^N_\delta$-loosing?
Corollary 3.20. Each $\gamma$-space is $G_\delta^N$-loosing.

A set of reals $X$ is concentrated on a set $D$ if and only if for any open set $G$ if $D \subseteq G$, then $X \setminus G$ is countable [5].

In 1914 Luzin constructed, using the continuum hypothesis, an uncountable set of reals having countable intersection with every meager set. A set of reals $X$ is a Luzin set if and only if $X$ is uncountable and concentrated on every countable dense set of reals.

Proposition 3.21. Let $X$ be a Luzin set. Then $B_1(X)$ is meager.

Proof. Assume that $B_1(X)$ is Baire. Let $A$ be a countable dense subset of $X$. By Proposition 3.13, there is $B \subset A$ such that $B$ is a dense subset of $X$ and $B$ is a $G_\delta$ set in $X$. Then $X$ is countable, a contradiction. \hfill $\square$

4. Pseudocompleteness for space of Baire-one functions

The sequence $\{C_n : n \in \mathbb{N}\}$ is called pseudocomplete if, for any family $\{U_n : n \in \mathbb{N}\}$ such that $\overline{U_{n+1}} \subseteq U_n$ and we have $U_n \in C_n$ for each $n \in \mathbb{N}$, we have $\bigcap\{U_n : n \in \mathbb{N}\} \neq \emptyset$. A space $X$ is called pseudocomplete if there is a pseudocomplete sequence $\{B_n : n \in \mathbb{N}\}$ of $\pi$-bases in $X$.

It is a well-known that any pseudocomplete space is Baire and any Čech-complete space is pseudocomplete. Note that if $X$ has a dense pseudocomplete subspace (in particular, if $X$ has a dense Čech-complete subspace) then $X$ is pseudocomplete (p. 47 in [31]).

Lemma 4.1. Let $Y$ be a topological vector space and $L \subseteq Y$ be a dense Čech-complete subspace. Then the linear span of $L$ equal to $Y$.

Proof. Note that $L$ is $G_\delta$ in $Y$. Let $Z$ be the linear span of $L$. If $y \in Y \setminus Z$, then $y + L$ is a dense Čech-complete subspace of $Y$ (hence $G_\delta$) disjoint with $L$. But this is a contradiction, since $L$ is Baire and hence $Y$ is also Baire (by density of $L$). \hfill $\square$

For $A \subseteq X$, denote by $\pi_A$ the projection $\pi_A : B_1(X) \rightarrow B_1(A)$, i.e., $\pi_A(f) = f \restriction A$ for $f \in B_1(X)$.

By Lemma 4.1, if $A \subseteq X$ and $\pi_A(B_1(X))$ contains a dense Čech-complete subspace, then $\pi_A(B_1(X)) = \mathbb{R}^A$. 24
Lemma 4.2. ([29]) Let \( Y \subseteq Z \) and \( Y \) is a dense pseudocomplete subspace of a regular space \( Z \). If \( \pi : Z \to M \) is an open continuous mapping from \( Z \) onto a complete metric space \( M \) then there is \( M_1 \subseteq \pi(Y) \) such that \( M_1 \) is a dense \( G_\delta \)-set in \( M \).

Theorem 4.3. For a space \( X \) the following assertions are equivalent:

1. \( B_1(X) \) is pseudocomplete.
2. Every countable subset of \( X \) is strongly \( Coz_\delta \)-disjoint.

Proof. (1) \( \Rightarrow \) (2). Let \( Y = B_1(X) \), \( Z = \mathbb{R}^X \), \( \pi = \pi_A \) where \( A = \{a_i : i \in \mathbb{N}\} \) is a countable subset of \( X \). Then, by Lemma 4.2 there is \( M_1 \subseteq \pi_A(B_1(X)) \) such that \( M_1 \) is a dense \( \check{C}ech \)-complete subspace of \( \mathbb{R}^A \). By Lemma 4.1, \( \pi_A(B_1(X)) = \mathbb{R}^A \). Let \( h : A \to \mathbb{R} \) such that \( h(a_i) = i \). Then \( h \) can be extended to a Baire-one function \( g \) on \( X \). The family \( \{g^{-1}(i) : i \in \mathbb{N}\} \) is a completely \( Coz_\delta \)-additive system and \( a_i \in g^{-1}(i) \) for each \( i \in \mathbb{N} \).

(2) \( \Rightarrow \) (1). For each \( n \), let \( \Psi_n \) be the collection of all basic neighborhoods of the form \( \langle f, S, \varepsilon \rangle := \{g \in B_1(X) : |g(s) - f(s)| < \varepsilon, s \in S\} \) where \( f \in B_1(X) \), \( S \) is finite, and \( \varepsilon < \frac{1}{n^2} \). Note that each \( \Psi_n \) is a \( \pi \)-base in \( B_1(X) \). Suppose, for each \( n \geq 1 \), \( \langle f_n, S_n, \varepsilon_n \rangle \in \Psi_n \) with \( \langle f_n, S_n, \varepsilon_n \rangle \supseteq cl(\langle f_{n+1}, S_{n+1}, \varepsilon_{n+1}\rangle) \). Let \( T = \bigcup \{S_n : n \in \mathbb{N}\} \). Then \( T \) is countable, and hence a strongly \( Coz_\delta \)-disjoint set. On the set \( T \) the sequence \( (f_n)_n \) converges pointwise to some function \( g \) on \( T \). Since \( T \) is strongly \( Coz_\delta \)-disjoint, there is a pairwise disjoint collection \( \{F_t : F_t \) is a \( Coz_\delta \) neighborhood of \( t \), \( t \in T\} \) such that \( \{F_t : t \in T\} \) is completely \( Coz_\delta \)-additive.

Consider \( h : \bigcup_{t \in T} F_t \to \mathbb{R} \) such that \( h(F_t) = g(t) \) for every \( t \in T \).

By Lemma 3.2, the function \( h \) can be extended to a Baire-one function \( \tilde{h} \) on \( X \). But then, \( \tilde{h} \in \bigcap \{(f_n, S_n, \varepsilon_n) : n \in \mathbb{N}\} \).

A family \( \mathcal{F} \subseteq Y^X \) of functions from a set \( X \) to a set \( Y \) is called \( \omega \)-full in \( Y^X \) if each function \( f : Z \to Y \) defined on a countable subset \( Z \subseteq X \) has an extension \( \tilde{f} \in \mathcal{F} \).

The following theorem uses notions (\( G_\delta \)-dense, countably base-compact, strong Choquet) not defined in this paper although we do not use them in the paper, but we recommend seeing the definitions of these notions in [2].

Theorem 4.4. (Theorem 2.20 in [2]) For any cardinal \( \kappa \) and each dense subgroup \( X \) of \( \mathbb{R}^\kappa \), the following assertions are equivalent:
Combining Theorem 4.3 with Theorem 4.4, we get the following characterization of a topological space $X$ when $B_1(X)$ is a Choquet space.

**Theorem 4.5.** For a space $X$ the following assertions are equivalent:

1. $B_1(X)$ is pseudocomplete;
2. $B_1(X)$ is $\omega$-full in $\mathbb{R}^X$;
3. $B_1(X)$ is $G_\delta$-dense in $\mathbb{R}^X$;
4. $B_1(X)$ is countably base-compact;
5. $B_1(X)$ is strong Choquet;
6. $B_1(X)$ is Choquet;
7. Every countable subset of $X$ is strongly Coz$_\delta$-disjoint.

**Proof.** (7) $\Rightarrow$ (2). Let $Z$ be a countable subset of $X$ and $f : Z \to \mathbb{R}$ be a function on $Z$. Since $Z$ is strongly Coz$_\delta$-disjoint, there exists a pairwise disjoint family $\alpha = \{F_z : z \in Z\}$ of Coz$_\delta$ subsets of $X$ such $\alpha$ forms a completely Coz$_\delta$-additive system. Consider the function $h : \bigcup F_z \to \mathbb{R}$ such that $h(x) = f(z)$ for $x \in F_z$. Since $\alpha$ forms a completely Coz$_\delta$-additive system (by Lemma 3.2), there exists $\tilde{f} \in B_1(X)$ such that $\tilde{f} \upharpoonright \bigcup F_z = h$. Hence, $\tilde{f} \upharpoonright Z = f$.

(2) $\Rightarrow$ (7). Let $Z = \{z_i : i \in \mathbb{N}\}$ be a countable subset of $X$. Consider the function $h : Z \to \mathbb{R}$ such that $h(z_i) = i$ for each $i \in \mathbb{N}$. Since $B_1(X)$ is $\omega$-full in $\mathbb{R}^X$ then there is $\tilde{h} \in B_1(X)$ such that $\tilde{h} \upharpoonright Z = h$. Then $\{h^{-1}([i - \frac{1}{3}, i + \frac{1}{3}]) : i \in \mathbb{N}\}$ forms a disjoint completely Coz$_\delta$-additive system and $z_i \in h^{-1}([i - \frac{1}{3}, i + \frac{1}{3}])$ for each $i \in \mathbb{N}$. Thus, $Z$ is strongly Coz$_\delta$-disjoint.

Recall that a topological space $X$ is called a $\lambda$-space if every countable subset is of type $G_\delta$ in $X$. A subset $X$ of the real line $\mathbb{R}$ is called a $\lambda$-set if each countable subset $A \subset X$ is $G_\delta$ in $\mathbb{R}$.

**Corollary 4.6.** Let $X$ be a space of countable pseudocharacter. A space $B_1(X)$ is pseudocomplete if and only if $X$ is a $\lambda$-space.
Let us recall the definition of some small uncountable cardinal (see [9], p.149):
\[ b = \min\{|X| : X \text{ has a countable pseudocharacter but } X \text{ is not a } \lambda\text{-space}\} \]

Note that the small cardinal \( b \) has a different standard definition (see [9], p.115), the given formula is one of the equivalent descriptions.

**Proposition 4.7.** Let \( X \) be a space \( X \) of countable pseudocharacter. If \(|X| < b\) then \( B_1(X) \) is Choquet (pseudocomplete).

*Proof.* By definition of \( b \), if \(|X| < b\) then \( X \) is a \( \lambda \)-space.

By Theorem 1.3 in [2], for any space \( X \) of countable pseudocharacter, \( B_1(X) \) is Choquet if and only if \( X \) is a \( \lambda \)-space. \( \square \)

Recall that a space \( X \) is called a \( Q \)-space if every subset is an \( F_{\sigma} \)-set in \( X \).

By Theorem 2.1 in [26], Lemma 3.2 and Theorem 4.3 we have the following result.

**Corollary 4.8.** Let \( X \) be a space of countable pseudocharacter. A space \( B_1(X) \) is pseudocomplete and realcompact if and only if \( X \) is a \( Q \)-space.

The following example we consider under the model of set theory in which every \( b \)-scale set is a \( \gamma \)-space (Corollary 1.5 in [6]).

**Example 4.9.** It is consistent with ZFC there is a zero-dimensional metrizable separable space \( X \) such that
1. \( B_1(X) \) is not pseudocomplete;
2. \( B_1(X) \) is Baire;
3. \(|X| = b\);
4. \( X \) is a \( \gamma \)-space;
5. \( X \) is not a \( \lambda \)-space.

*Proof.* Let \( X \) be the space \( H \) from Theorem 10 of [4] or \( X \) from Example 8.4 of [2]. Let us recall its construction.

Given two functions \( f, g : \omega \to \omega \), we write \( f \leq^* g \) if the set \( \{ n \in \mathbb{N} : f(n) \leq g(n) \} \) is finite (empty). A subset \( B \subseteq \omega^\omega \) is unbounded
if for any \( y \in \omega^\omega \) there is \( x \in B \) such that \( x \not\leq^* y \). The cardinal \( b \) can be equivalently defined as the smallest cardinality of an unbounded subset of \( \omega^\omega \).

By \( \omega^\omega \) we denote the family of increasing functions from \( \omega \) to \( \omega \). There exists a transfinite sequence \( \{ f_\alpha \}_{\alpha < b} \) of increasing functions \( f_\alpha : \omega \to \omega \) such that the set \( S = \{ f_\alpha \}_{\alpha < b} \) is unbounded in \( \omega^\omega \) and \( f_\alpha \leq^* f_\beta \) for every \( \alpha < \beta \) in \( b \) (see Example 8.4 of [2]).

Consider the closure \( \overline{\omega} = [0, \omega] \) of \( \omega \) in the ordinal \( \omega + 1 \) endowed with the order topology. Let \( C \subset \overline{\omega} \) be the countable set of functions \( f : \omega \to \overline{\omega} \) such that there exists \( n \in \mathbb{N} \) so that \( f(i) < f(j) \) for \( i < j < n \) and \( f(k) = \omega \) for all \( k \geq n \). Note that the subspace \( \overline{\omega} \uparrow \omega := \omega \uparrow \omega \cup C \) in \( \overline{\omega} \) is closed and hence compact. The subspace \( X := S \cup C \) of \( \omega^\omega \) has the property (1)-(5).

- \( |X| = b \).
- By the proof (ii) in Example 8.4 of [2], the set \( C \) is not a \( G_\delta \)-set in \( X \), hence, \( X \) is not a \( \lambda \)-space. By Corollary 4.6, \( B_1(X) \) is not pseudocomplete.
- Galvin and Miller [18] constructed under \( p = c \) a \( b \)-scale set which is a \( \gamma \)-space. According to [6], it is consistent that for any \( b \)-scale \( S = \{ f_\alpha \}_{\alpha < b} \) the space \( X = S \cup C \) is a \( \gamma \)-space and, hence, by Theorem 3.16, \( B_1(X) \) is Baire.

Let us give several examples of subsets \( X \) of the real line \( \mathbb{R} \) for which \( B_1(X) \) is Baire.

**ZFC Examples**

1. \( B_1(\mathbb{Q}) \), where \( \mathbb{Q} \) is the space of all rational numbers (or any countable subset of \( \mathbb{R} \)), is Baire. This is because \( b \) is uncountable.

2. Let \( X \) be an uncountable \( \lambda \)-set that is a subset of the real line. Then \( B_1(X) \) is Baire.

**Consistent Examples**

3. For any uncountable subset \( X \) of the real line of cardinality \( < b \), \( B_1(X) \) is Baire.

4. Let \( X \) be an uncountable \( \gamma \)-set that is a subset of the real line. Then \( B_1(X) \) is Baire.

**Remark.** There is a ZFC example of an uncountable subset of the real line that is a \( \lambda \)-space (see [2] page 3). It is consistent with ZFC that \( \omega_1 < b \).
When this is the case, any subset of the real line of cardinality \(\omega_1\) would be a \(\lambda\)-set.

According to Gerlits and Nagy [19], under \(MA+\) not \(CH\), uncountable \(\gamma\)-sets exist since any subset of the real line of cardinality less than continuum is a \(\gamma\)-set. On the other hand, in Laver’s model for the Borel conjecture, all \(\gamma\)-sets are countable [12].

**Comment.** All of these examples (1)-(4) are examples of Baire \(B_1(X)\) such that \(C_p(X)\) is not Baire. Note that there exists a non-trivial convergent sequence in each of the \(X\). The example of \(B_1(\mathbb{Q})\) is a clear illustration of the difference between strong discrete sequence and strongly \(Coz_\delta\)-disjoint sequence.

There is a \(ZFC\) example for Baire \(C_p(X)\) that is not pseudocomplete (Ex. 469 in [31]). Example 1 is an example of a Baire \(B_1(X)\) that is not pseudocomplete. This a consistent example since it involves an uncountable \(\gamma\)-set.

**Question 1.** Is there a \(ZFC\) example for a Baire \(B_1(X)\) that is not pseudocomplete?

5. Countable dense homogeneity of space of Baire-one functions

A space \(X\) is *countable dense homogeneous* (CDH) if \(X\) is separable and given countable dense subsets \(D, D' \subseteq X\), there is a homeomorphism \(h : X \to X\) such that \(h[D] = D'\). Canonical examples of CDH spaces include the Cantor set, Hilbert cube, space of irrationals, and all separable complete metric linear spaces and manifolds modeled on them. An easy example of a non-metrizable CDH space is the Sorgenfrey line.

In [3], it is proved that every CDH topological vector space is a Baire space.

If \(X\) is a space and \(A \subseteq X\), then the sequential closure of \(A\), denote by \([A]_{seq}\), is the set of all limits of sequences from \(A\). A set \(D \subseteq X\) is said to be sequentially dense if \(X = [D]_{seq}\).

Note that if \(B_1(X)\) is CDH, then it is strongly sequentially separable, i.e. \(B_1(X)\) is separable and every countable dense subset of \(B_1(X)\) is sequentially dense.

Many topological properties are defined or characterized in terms of the following selection principle. Let \(\mathcal{A}\) and \(\mathcal{B}\) be sets consisting of families of subsets of an infinite set \(X\). Then:
$S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each $n$, $b_n \in A_n$, and \{b_n : n \in \mathbb{N}\} is an element of $\mathcal{B}$.

For a topological space $X$ we denote:

- $\mathcal{B}_\Omega$ – the family of countable Baire $\omega$-covers of $X$;
- $\mathcal{B}_\Gamma$ – the family of countable Baire $\gamma$-covers of $X$.

Then we have the following results.

- If $\mathcal{B}_1(X)$ is strongly sequentially separable, then $X$ has the property $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ [25].
- If $X$ has the property $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ then it is a $\sigma$-space, i.e. each $G_\delta$-subset of $X$ is an $F_\sigma$-subset [34].
- If $\mathcal{B}_1(X)$ is separable, then $X$ is submetrizable [24].

Thus, we get that if $\mathcal{B}_1(X)$ is CDH then $\mathcal{B}_1(X)$ is Baire and a submetrizable space $X$ has the property $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ (hence, is a $\sigma$-space).

Let us recall the definitions of some small uncountable cardinals:

- $q_0 := \min\{|X| : X \subseteq \mathbb{R}, X \text{ is not a } Q\text{-space}\}$;
- $q := \min\{\kappa : \text{if } X \subseteq \mathbb{R} \text{ and } |X| \geq \kappa, \text{ then } X \text{ is not a } Q\text{-space}\}$;
- $b := \min\{|X| : X \text{ is perfect but is not a } \sigma\text{-set}\}$;
- $p$ is the smallest cardinality of a family $\mathcal{F}$ of infinite subsets of $\omega$ such that every finite subfamily $\mathcal{E} \subseteq \mathcal{F}$ has infinite intersection $\bigcap \mathcal{E}$ and for any infinite set $I \subseteq \omega$ there exists a set $F \in \mathcal{F}$ such that $I \setminus F$ is finite.

It is known (see [3]) that $p \leq q_0 \leq \min\{b, q\} \leq q \leq c$.

It is consistent that there are no uncountable $\sigma$-space (Theorem 22 in [15]).

**Proposition 5.1.** It is consistent that there are no uncountable separable metrizable space $X$ such that $\mathcal{B}_1(X)$ is CDH.

On the other hand, if $|X| < q_0$ then $X$ is a $Q$-space. For a perfect normal space $X$, if $|X| < q_0$ then $\mathcal{B}_1(X) = \mathbb{R}^X$. But, $\mathbb{R}^X$ is CDH if, and only if, $|X| < p$ [33]. Since $p \leq q_0$, we have the following result.

**Proposition 5.2.** Let $X$ be a perfect normal space and $|X| < p$. Then $\mathcal{B}_1(X)$ is CDH.

In particular, if $X$ is a countable metrizable space, then $\mathcal{B}_1(X)$ is CDH.
Proposition 5.3. \((p < q_0)\). It is consistent that there are a set of reals \(X\) such that \(B_1(X)\) is Baire, but is not CDH.

Proof. Let \(X\) be a set of real of cardinality \(p\). Since \(p < q_0\), \(X\) is a \(Q\)-set and, hence, \(B_1(X) = \mathbb{R}^X\). But, \(\mathbb{R}^X\) is CDH if and only if \(|X| < p\). Hence, \(B_1(X)\) is not CDH. Since \(p < q_0 \leq b\), \(X\) is a \(\sigma\)-set (hence, \(\lambda\)-space, i.e. every countable subset is \(G_\delta\) in \(X\)) and \(B_1(X)\) is Baire (see [2]).

Proposition 5.4. If an uncountable set of real numbers is concentrated on a countable subset of itself, then \(B_1(X)\) is not CDH.

Proof. If an uncountable set of real numbers is concentrated on a countable subset of itself, then it does not have property \(S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)\) (see Corollary 5 in [34]) and hence \(B_1(X)\) is not CDH.

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