An axiomatization of verdict equivalence over regular monitors

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Abstract

Monitors are a key tool in the field of runtime verification, where they are used to check for system properties by analysing execution traces generated by processes. Work on runtime monitoring carried out in a series of papers by Aceto et al. has specified monitors using a variation on the regular fragment of Milner’s CCS and studied two trace-based notions of equivalence over monitors, namely verdict and $\omega$-verdict equivalence. This article is devoted to the study of the equational logic of monitors modulo those two notions of equivalence. It presents complete equational axiomatizations of verdict and $\omega$-verdict equivalence for closed and open terms over recursion-free monitors.

Keywords:
monitors, formal verification, CCS, equational logic, processes, process algebra, axiomatization, trace equivalence, verdicts

1. Introduction

The search for equational axiomatizations of a notion of equivalence over some process description language is one of the classic topics in concurrency theory see, for instance, \cite{1, 2, 3, 4, 5, 6}. Equational axiomatizations provide a purely syntactic description of the chosen notion of equivalence over processes and characterise the essence of a process semantics by means of a few revealing axioms, which can be used to compare a variety of semantics in a model-independent way (as done in \cite{2}). Moreover, such axiomatizations pave the way to the use of theorem-proving techniques to establish that two process descriptions express the same behaviour modulo the chosen notion of behavioural equivalence \cite{7}. In this paper, we study the equational logic of the monitors studied by Aceto et al. in, for instance, \cite{8, 9, 10}. Monitors are a key tool in the field of runtime verification \cite{11}, where they are used to check...
for system properties by analyzing execution traces generated by processes. In the aforementioned papers, Aceto et al. specified monitors using a variation on the regular fragment of Milner’s CCS and studied two trace-based notions of equivalence over monitors, namely verdict and \( \omega \)-verdict equivalence. Intuitively, two monitor descriptions are verdict equivalent when they accept and reject the same finite execution traces of the systems they observe. The notion of \( \omega \)-verdict equivalence is the ‘asymptotic version’ of verdict equivalence, in that it is solely concerned with the infinite traces that are accepted and rejected by monitors.

The main results we present in this paper are complete equational characterizations of verdict equivalence over both closed (that is, variable-free) and open, recursion-free regular monitors. It turns out that those axiomatizations are also complete for \( \omega \)-verdict equivalence if the set of actions monitors may observe is infinite, as in that case the two notions of equivalence coincide. On the other hand, if the set of actions is finite, \( \omega \)-verdict equivalence is strictly coarser than verdict equivalence. We also provide a complete axiomatization of \( \omega \)-verdict equivalence for closed monitors in the setting of a finite set of actions. If selected for presentation, the version of the paper for the symposium will also present a complete axiomatization of \( \omega \)-verdict equivalence for open monitors and any further results we might have obtained in the coming months.

2. Preliminaries

We begin by introducing the recursion-free regular monitors and two notions of verdict equivalence we study in this paper. We refer the interested reader to \[9\] for background motivation and more information.

**Syntax of monitors.** Let \( \text{Act} \) be a set of visible actions, ranged over by \( a, b \). Following Milner \[4\], we use \( \tau \notin \text{Act} \) to denote an unobservable action. We use \( \alpha \) to range over \( \text{Act} \cup \{ \tau \} \). We will denote the set of infinite sequences over \( \text{Act} \) as \( \text{Act}^\omega \). As usual \( \text{Act}^* \) stands for the set of finite sequences over \( \text{Act} \). Let \( \text{Var} \) be a countably infinite set of variables, ranged over by \( x, y, z \).

The collection \( \text{Mon}_F \) of regular, recursion-free monitors is the set of terms generated by the following grammar:

\[
m, n ::= v \mid a.m \mid m + n \mid x
\]

\[
v ::= \text{end} \mid \text{yes} \mid \text{no}
\]

where \( a \in \text{Act} \) and \( x \in \text{Var} \). The terms \( \text{end} \), \( \text{yes} \) and \( \text{no} \) are called **verdicts**. Intuitively, \( \text{yes} \) stands for the acceptance verdict, \( \text{no} \) denotes a rejection verdict and \( \text{end} \) is the inconclusive verdict, namely the state a monitor reaches when, based on the sequence of observations it has processed so far, it realises that it will not be able to issue an acceptance or rejection verdict in the future. See, for instance, \[11\] for a detailed technical discussion. Closed monitors are those that do not contain any occurrences of variables. A (closed) **substitution** is a
mapping \(\sigma\) from variables to (closed) monitors. We write \(\sigma(m)\) for the monitor that results when applying the substitution \(\sigma\) to \(m\). Note that \(\sigma(m)\) is closed, if \(\sigma\) is a closed substitution.

**Definition 1.** We use \(m [+v]\) for a verdict \(v\) to indicate that \(v\) is an optional summand of \(m\), that is, that the term can be either \(m\) or \(m + v\). In addition a monitor will be called \(v\)-free for a verdict \(v\), when it does not contain any occurrences of \(v\).

Here we also introduce the generalized summation \(\sum_{i \in I} m_i\) justified by the fact that \(+\) is associative and commutative in all the semantics we use. \(\sum_{i \in \emptyset} m_i\) stands for \(\text{end}\). This is also consistent equationaly since \(\sum_{a \in A} a.m_a + \sum_{a \in \emptyset} a.m_a = \sum_{a \in A} a.m_a\) is always a valid equation.

We now associate a notion of syntactic depth with each monitor. Intuitively, the decision a monitor \(m\) takes when reading a string \(s \in \text{Act}^*\) only depends on the prefixes of \(s\) whose length are at most the syntactic depth of \(m\).

**Definition 2 (Syntactic Depth).** For any closed monitor \(m \in \text{Mon}_F\), we define \(\text{depth}(m)\) as follows:

- \(\text{depth}(a.m) = 1 + \text{depth}(m)\),
- \(\text{depth}(m_1 + m_2) = \max(m_1, m_2)\) and
- \(\text{depth}(v) = 0\) for a verdict \(v\).

This notion will be useful later on when discussing the asymptotic behavior of monitors.

**Semantics of monitors.** For each \(\alpha \in \text{Act} \cup \{\tau\}\), we define the transition relation \(\alpha \rightarrow \subseteq \text{Mon}_F \times \text{Mon}_F\) as the least one that satisfies the following axioms and rules.

\[
\begin{align*}
(\forall a \in \text{Act}) & \quad a.m \xrightarrow{a} m \\
\frac{m \xrightarrow{\alpha} m'}{m + n \xrightarrow{\alpha} m'} & \quad \frac{n \xrightarrow{\alpha} n'}{m + n \xrightarrow{\alpha} n'} \\
\frac{m \xrightarrow{\sigma} n}{m \xrightarrow{\sigma} n} & \quad \frac{m \xrightarrow{\sigma} n}{\sigma(m) \xrightarrow{\sigma} \sigma(n)}
\end{align*}
\]

Table 1: Operational semantics of processes in \(\text{Mon}_F\).

For \(s = a_1a_2\ldots a_n \in \text{Act}^*, n \geq 0\), we use \(m \xrightarrow{s} m'\) to mean that:

1. \(m(\varepsilon)^*m'\) if \(s = \varepsilon\), where \(\varepsilon\) stands for the empty string,
2. \(m \xrightarrow{a} m_1 \xrightarrow{a} m_2 \xrightarrow{a} m'\) for some \(m_1, m_2\) if \(s = a \in \text{Act}\) and
Lemma 1. For all \( s \in \text{Act}^* \), \( m, n \in \text{Mon}_F \), and verdict \( v \): \( m + n \vdash v \) iff \( m \vdash v \) or \( n \vdash v \)

Proof. In both cases of the implication we will use induction on the length of \( s \):

- For the implication \( m \vdash v \) or \( n \vdash v \) then \( m + n \vdash v \) we proceed as follows: if \( s = \varepsilon \) and \( m = v \) then \( v + n \vdash v \) since \( v \vdash v \). For \( s = a.s' \) if \( m \vdash v \Rightarrow m \vdash m' \) where \( m' = a.s' \) and therefore \( m + n \vdash m' \Rightarrow m + n \vdash v \)

- For the implication \( m + n \vdash v \) if \( m \vdash v \) or \( n \vdash v \) we have that:

\[
\text{If } s = \varepsilon \text{ and } m + n = v \text{ one of the } m, n \text{ must be equal to } v. \text{ The case were } m + n \vdash v \text{ and } s = a.s' \text{ means that } m + n \vdash m' \text{ if and only if one of the } m, n \text{ can do an } a-\text{transition and arrive at } m'. \text{ By the inductive argument either } m \vdash m' \text{ which means } m \vdash v \text{ or, } n \vdash m' \text{ which means that } n \vdash v
\]

□

Verdict and \( \omega \)-Verdict Equivalence. Let \( m \) be a (closed) monitor. We define:

\[
L_a(m) = \{ s \in \text{Act}^* \mid m \vdash s \text{ yes} \} \quad \text{and} \quad L_r(m) = \{ s \in \text{Act}^* \mid m \vdash s \text{ no} \}.
\]

Intuitively, \( L_a(m) \) denotes the set of traces that are accepted by \( m \), whereas \( L_r(m) \) stands for the set of traces that \( m \) rejects. Note that we allow for monitors that may both accept and reject some trace. This is necessary to maintain our monitors closed under \(+\). Of course, in practice, one is interested in monitors that are consistent in their verdicts.

Definition 3. Let \( m \) and \( n \) be closed monitors.

- We say that \( m \) and \( n \) are **verdict equivalent**, written \( m \simeq n \), iff \( L_a(m) = L_a(n) \) and \( L_r(m) = L_r(n) \).

- We say that \( m \) and \( n \) are **\( \omega \)-verdict equivalent**, written \( m \simeq_\omega n \), iff \( L_a(m) \cdot \text{Act}^\omega = L_a(n) \cdot \text{Act}^\omega \) and \( L_r(m) \cdot \text{Act}^\omega = L_r(n) \cdot \text{Act}^\omega \).

For open monitors \( m \) and \( n \), we say that \( m \simeq n \) if \( \sigma(m) \simeq \sigma(n) \), for all closed substitutions \( \sigma \). The relation \( \simeq_\omega \) is extended to open monitors in similar fashion.

One can intuitively see that the notion of \( \omega \)-verdict equivalence refers to a form of asymptotic behaviour. Next we provide a lemma that will make the relations between the two notions of equivalence defined above clearer.

Lemma 2. The following statements hold:
• $\simeq$ and $\simeq_{\omega}$ are both congruences.

• $\simeq \subseteq \simeq_{\omega}$ and the inclusion is strict when $\text{Act}$ is finite.

• If $\text{Act}$ is infinite then $\simeq = \simeq_{\omega}$.

Proof. For the first claim, it suffices to prove that $\simeq$ and $\simeq_{\omega}$ are equivalence relations and that they are preserved by $a_-$ and $\cdot$. The proof is standard and is thus omitted.

For the second claim, the inclusion $\simeq \subseteq \simeq_{\omega}$ is easy to check using the definitions of the two relations. The fact that the inclusion is strict when the set of actions is finite follows from the validity of the equivalence $\text{yes} \simeq_{\omega} \text{yes} \sum_{a \in \text{Act}} a \cdot \text{yes}$.

However, that equivalence is not valid modulo verdict equivalence since the first monitor “accepts” the empty string ($\epsilon$) while the second part can only accept after reading an action. Asymptotically however (that is, after they are extended by infinite sequences over $\text{Act}$) they behave in the same manner.

Finally, suppose that $\text{Act}$ is infinite. Assume that $m$ and $n$ are $\omega$-verdict equivalent and that $s$ is in the set of finite traces accepted by $m$. We will argue that $n$ also accepts $s$. To this end, note that, since $\text{Act}$ is infinite, there is some action $a$ that does not occur in $m$ and $n$. Since $m$ accepts $s$, the infinite trace $sa^\omega$ is in $L_a(m) \cdot \text{Act}^\omega$. By the assumption that $m$ and $n$ are $\omega$-verdict equivalent, we have that $sa^\omega$ is in $L_a(n) \cdot \text{Act}^\omega$. As $a$ does not occur in $n$, it follows that $n$ accepts $s$. Therefore, by symmetry, $m$ and $n$ accept the same traces. The same argument shows that $L_r(m) = L_r(n)$, and therefore $m \simeq n$. \hfill \qed

An axiom system $\mathcal{E}$ over $\text{Mon}_F$ is a collection of equations $t \simeq u$ expressed in the syntax of $\text{Mon}_F$. An equation $t \simeq u$ is derivable from an axiom system $\mathcal{E}$ (notation $\mathcal{E} \vdash t = u$) if it can be proven from the axioms in $\mathcal{E}$ using the rules of equational logic (reflexivity, symmetry, transitivity, substitution and closure under our the $\text{Mon}_F$ context). In the rest of this work we shall always assume that equational axiom systems are closed with respect to symmetry i.e. that if $t \simeq u$ is an axiom, so is $u \simeq t$.

We say that $\mathcal{E}$ is sound with respect to $\simeq$ when $m \simeq n$ holds whenever $\mathcal{E} \vdash m = n$. We say that $\mathcal{E}$ is complete with respect to $\simeq$ when $\mathcal{E}$ can prove all the valid equations $m \simeq n$. Similar definitions apply for $\omega$-verdict equivalence.

The notion of completeness, when limited to closed terms, is referred to as ground completeness.

3. A ground-complete axiomatization of verdict equivalence

Our proposed axioms system for verdict equivalence is $\mathcal{E}_v$, whose axioms are:

- **A1**: $x + y = y + x$
- **A2**: $x + (y + z) = (x + y) + z$
- **A3**: $x + x = x$
- **A4**: $x + \text{end} = x$
- **Ea**: $a \cdot \text{end} = \text{end}, \forall a \in \text{Act}$
- **Ya**: $\text{yes} = \text{yes} + a \cdot \text{yes}, \forall a \in \text{Act}$
- **Na**: $\text{no} = \text{no} + a \cdot \text{no}, \forall a \in \text{Act}$
- **Da**: $a \cdot (x + y) = a \cdot x + a \cdot y, \forall a \in \text{Act}$
Lemma 3. When Act is finite, the family of axioms $Y_a$ can be replaced with the axiom

$$Y: \text{yes} = \text{yes} + \sum_{a \in \text{Act}} a.\text{yes}.$$ 

Similarly the family of axioms $N_a$ can be replaced with the axiom $N: \text{no} = \text{no} + \sum_{a \in \text{Act}} a.\text{no}.$

Proof. It is obvious that the $Y$ equation can be proved by using the family of equations $Y_a$. For the converse we can use rule A3 and $Y$ to prove any equations Indeed $\text{yes} = \text{yes} + b.\text{yes}$ of the family $Y_a$.

$$\text{yes} = \text{yes} + \sum_{a \in \text{Act}} a.\text{yes} = \text{yes} + \sum_{a \in \text{Act}} a.\text{yes} + b.\text{yes} = \text{yes} + b.\text{yes}.$$

\[\square\]

Theorem 1. $\mathcal{E}_v$ is sound. That is, if $\mathcal{E}_v \vdash m = n$ then $m \simeq n$, for all $m, n \in \text{Mon}_F$.

Proof. It suffices to prove soundness for each of the proposed axioms separately. The details of the proof are standard and therefore omitted.\[\square\]

In what follows we will consider terms up to axioms A1-A4.

We will now prove that the axiom system $\mathcal{E}_v$ is ground complete for verdict equivalence.

Theorem 2. $\mathcal{E}_v$ is ground complete for $\simeq$ over $\text{Mon}_F$. That is, if $m, n$ are closed monitors in $\text{Mon}_F$ and $m \simeq n$ then $\mathcal{E}_v \vdash m = n$.

As a first step towards proving that $\mathcal{E}_v$ is complete over closes terms, we isolate a notion of normal form for monitors and prove that each closed monitor in $\text{Mon}_F$ can be proved equal to a normal form using the equations in $\mathcal{E}_v$.

Definition 4. (Normal Form) A normal form is a closed term $m \in \text{Mon}_F$ of the form :

$$\sum_{a \in A} a.m_a [+\text{yes}] [+\text{no}]$$

for some finite $A \subseteq \text{Act}$ and $\{m_a | a \in A\}$ where each $m_a$ is a term in normal form that is different from end.

Lemma 4. The only normal form that does not contain occurrences of yes and no is end.

Proof. If $m = \text{end}$ then it is in normal form and it satisfies this lemma. If $m$ is some $\sum_{a \in A} a.m_a$ for some finite $A \subseteq \text{Act}$ and $\{m_a | a \in A\}$. For each $a \in A$ the $m_a$ monitor does not contain any ‘yes or no. Therefore by induction hypothesis
we can say that $\forall a \in A, m_a = \text{end}$. Therefore $m = \sum_{a \in A} a.\text{end}$. However a normal form is not allowed to perform transitions that result to $\text{end}$. For $m$ this means that it is allowed no transitions since we have showed that if it was allowed any they would result in $\text{end}$. Therefore $m = \sum_{a \in \emptyset} a.m_a = \text{end}$.

\[ \text{Lemma 5. (Normalisation)} \] Each closed term $m \in \text{Mem}$ is provably equal to some normal form $m'$ with $\text{depth}(m') \leq \text{depth}(m)$.

Proof. We prove the claim by induction on the size of the monitor. We proceed with a case analysis on the form $m$ may have. Our induction basis will be a verdict $v$. If $v = \text{end}$ then the monitor is already in normal form. Otherwise:

- If $m = v$ for some verdict $v = \text{yes}$ or $\text{no}$ then it is of the form $\sum_{a \in \emptyset} a.m_a + v$ (from axiom $A4$)

Our induction hypothesis is that for all monitors $m \in \text{Mem}$ up to some size, we have that $\text{E}_v \vdash m = m'$ with $m'$ in normal form and $\text{depth}(m') \leq \text{depth}(m)$.

- If $m = m_1 + m_2$ then the monitors $m_1, m_2$ will be a combination of non determinstic choice and action prefixing. When either of $m_1, m_2$ have a verdict “$v$” as an optional summand, then $m$ will also contain “$v$” as an optional summand. Since $m_1, m_2$ are monitors of smaller size than $m$ then each one is provable equal to a normal form $m_1'$ and $m_2'$. It remains to show that the summation of $m_1' + m_2'$ is provably equal to another normal form. After bringing $m_1$ and $m_2$ in their normal forms we can always rewrite the monitor $m$ as

$$m = \sum_{a \in A_1 \setminus A_2} a.m_{1_a}' + \sum_{a \in A_2 \setminus A_1} a.m_{2_a}' + \sum_{b \in A_1 \cap A_2} b.(m_{1_b}' + m_{2_b}')$$

(from axiom $D_a$). Monitors $m_{1_a}'$ and $m_{2_a}'$ are also in normal form, which implies that also $a.m_{1_a}$ ($a \in A_1 \setminus A_2$) and $a.m_{2_a}$ ($a \in A_2 \setminus A_1$) are also normal forms. We can say the same for the monitors $m_{1_b}$ and $m_{2_b}$, since they both occur after some action $b \in A_1 \cap A_2$ is performed and therefore their size must be smaller than the that of $m$. Therefore by using the induction hypothesis we have that their sum is equivalent to some normal form $m_b$. This means that also the summation of all of the normal forms $a.m_{1_a}$ ($a \in A_1 \setminus A_2$), $a.m_{2_a}$ ($a \in A_2 \setminus A_1$), $b.m_b$ ($b \in A_1 \cap A_2$) is also a normal form.

- If $m = a.n$ then by induction $n$ is provably equal to some normal form $n'$. If $n' = \text{end}$ then $m = \text{end}$ (using $E_a$) which is a normal form of smaller depth. Otherwise $a.n'$ is also a normal form.
Lemma 6. The following statements hold for any monitor in $\text{Mon}_F$:

1. For each action $a$, if $m$ is a no-free term then $\mathcal{E}_v \vdash \text{yes} + a.m = \text{yes}$.

2. For each action $a$, if $m$ is a closed monitor that contains occurrences of both yes and no then $\mathcal{E}_v \vdash \text{yes} + a.m = \text{yes} + a.n$ for some yes-free closed monitor $n$.

3. For each action $a$, if $m$ is a yes-free term then $\mathcal{E}_v \vdash \text{no} + a.m = \text{no}$.

4. If $m$ contains occurrences of yes then $\mathcal{E}_v \vdash \text{no} + a.m = \text{no} + a.n$ for some no-free closed monitor $n$.

Proof. We only prove statements 1 and 2 as the proofs of 3 and 4 are similar. We will use structural induction on the form of $m$.

1. If $m$ is a verdict other than no then the claim follows using axioms $E_a$, $Y_a$ and $A4$ appropriately. If $m = b.m'$ where $m'$ is no-free then by axiom $Y_b$:

\[
\text{yes} + a.m = \text{yes} + a.\text{yes} + a.m = \text{yes} + a.(\text{yes} + b.m') = \text{yes} + a.\text{yes} = \text{yes}
\]

1.H.

If $m$ is of the form $m_1 + m_2$ where each $m_1, m_2$ are no-free, then it suffices to apply axiom $D_a$ and the previous argument.

2. If $m = \text{yes} + \text{no}$ ($m$ has to contain occurrences of both yes and no) then by A3 we have the requested result. We start by defining as $m'$ the normal form of $m$. We can rewrite $m'$ as $\sum_{a_1 \in A_1} a_1.m_{a_1} + \sum_{a_2 \in A_2} a_2.m_{a_2}$ where $m_{a_1}$ is no-free and $m_{a_2}$ is yes-free. This form can be constructed by utilizing axiom $D_a$. Note that there might be actions $a$ in both $A_1$ and $A_2$ which means that this is not a normal form. By the previous case of this lemma this reduces to:

\[
\sum_{a_2 \in A_2} a_2.m_{a_2} + \text{yes}
\]

and each $m_{a_2}$ is yes-free and not equal to end.

The above lemma suggests the notion of a reduced normal form.

Definition 5. (Reduced normal form) A reduced normal form is a term

\[
m = \sum_{a \in A} a.m_a \ [+\text{yes}] \ [+\text{no}]
\]

in normal form, where if $v \in \{\text{yes, no}\}$ is a summand of $m$ then each $m_a$ is $v$-free and in reduced normal form.

Lemma 7. For each monitor $m \in \text{Mon}_F$ its normal form is provably equal to a reduced normal form.
Proof. The claim follows from Lemma 6 using induction on the depth of the normal form of $m$.

We are now ready to complete the proof of Theorem 2.

Proof. Since each monitor is provably equal to a reduced normal form (Lemma 6), it suffices to prove the claim for verdict equivalent reduced normal forms $m$ and $n$. We proceed by induction on the sum of the sizes of $m$ and $n$, and a case analysis on the possible form $m$ may have.

1. Assume that $m = \text{yes} + \text{no} \simeq n$. Since $L_a(m) = L_r(m) = \text{Act}^*$, it follows that $n$ has both yes and no as summands. Since $n$ is in reduced normal form it must be $n = \text{yes} + \text{no}$, and we are done.

2. Assume that $m = \sum_{a \in A} a.m_a + \text{yes} \simeq n$. Where for all $a \in A$, $m_a$ is yes-free and in reduced normal form and $n = \sum_{b \in B} b.n_b[+\text{yes}[+\text{no}]]$, where each $n_b$ is in reduced normal form and is $''v''$-free, if $v$ is a summand of $n$. Since $\varepsilon \in L_a(m) \setminus L_r(m)$, we have that yes is a summand of $n$ and no isn’t. Thus $n = \sum_{b \in B} b.n_b + \text{yes}$, and each $n_b$ is yes-free. We claim that:

(C1) $A = B$ and
(C2) for all $a \in A$, $m_a \simeq n_a$.

To prove that $A = B$, we assume that $a \in A$. Since $m_a$ is yes-free and different than end there is some $s \in \text{Act}^*$ such that $a.s \in L_r(m)$. As $m \simeq n$, we have that $a.s \in L_r(n)$. Since $\varepsilon \notin B$ we conclude that $a \in B$ and $s \in L_r(n_a)$. By symmetry, claim (C1) follows.

We now show that $m_a = n_a$ for each $a \in A$. Since $m_a$ and $n_a$ are yes-free, $L_a(m_a) = L_a(n_a) = \emptyset$. We pick now some arbitrary $s \in L_r(m_a)$. This means that $a.s \in L_r(m) = L_r(n)$ and therefore $s \in L_r(n_a)$ because $\varepsilon \notin L_r(n)$. The claim follows by symmetry. By induction, $\mathcal{E}_v \vdash m_a = n_a$ for each $a \in A = B$. So

$$m = \sum_{a \in A} a.m_a + \text{yes} = \sum_{b \in B} b.n_b + \text{yes} = n$$

is provable from $\mathcal{E}_v$ and we are done.

3. We are left with the case where $m = \sum_{a \in A} a.m_a + \text{no} \simeq n$ and the case $m = \sum_{a \in A} a.m_a$. The proofs for those cases are similar to the one for case 2 and are thus omitted.

\[\square\]
Axiomatising $\omega$-verdict equivalence. When $Act$ is infinite, by Lemma 2 and Theorem 2 $E_v$ gives a ground complete axiomatization of $\omega$-verdict equivalence as well. However, when $Act$ is finite, $E_v$ is not powerful enough to prove all the equalities between closed terms that are valid with respect to $\omega$-verdict equivalence. The new axioms needed to achieve a ground complete axiomatization in this setting are:

$$Y_\omega : \text{yes} = \sum_{a \in Act} a \cdot \text{yes}$$

$$N_\omega : \text{no} = \sum_{a \in Act} a \cdot \text{no}.$$ 

The resulting axiom system is called $E_\omega$.

**Theorem 3.** $E_\omega$ is ground complete for $\simeq_\omega$ over closed terms when $Act$ is finite. That is if $m, n$ are closed monitors in $\text{Mon}_F$ and $m \simeq_\omega n$ then $E_\omega \vdash m = n$.

**Proof.** By Lemma 6 we may assume that $m$ and $n$ are in reduced normal form. We will prove the claim by induction on the sizes of $m$ and $n$ for two $\omega$-verdict equivalent monitors $m, n$ in reduced normal form.

We will proceed by a case analysis of the form $m$ may have and limit ourselves to presenting the proof for a few selected cases that did not arise in the proof of Theorem 2.

- Assume that $m = \text{yes} + \text{no} \simeq_\omega \sum_{a \in A} a \cdot n_a = n$. First if all note that $A = Act$.

  Indeed if $a \in Act \setminus A$ then $a^\omega \in (L_a(m) \cdot Act^\omega) \setminus (L_a(n) \cdot Act^\omega)$ which contradicts our assumption that $m \simeq_\omega n$. Moreover, it is not hard to see that for each $a \in A$, $L_a(n_a) \cdot Act^\omega = L_r(n_a) \cdot Act^\omega = Act^\omega$. This means that, for each $a \in Act$, $n_a \simeq_\omega \text{yes} + \text{no}$. By induction, for each $a \in Act$, we have that $E_\omega \vdash n_a = \text{yes} + \text{no}$. Thus, $E_\omega \vdash n = \sum_{a \in Act} a \cdot (\text{yes} + \text{no})$.

  From axiom $D_a$, $E_\omega \vdash n = \sum_{a \in Act} a \cdot \text{yes} + \sum_{a \in Act} a \cdot \text{no}$ which from our two new axioms $Y_\omega, N_\omega$ yields $E_\omega \vdash n = \text{yes} + \text{no} = m$, and we are done.

- $m = \text{yes} + \text{no} \simeq_\omega \sum_{a \in A} a \cdot n_a + \text{yes} = n$, with each $n_a$ being yes-free and different from end. Again, reasoning as in the previous case, we have that $A = Act$. Moreover for each $a \in Act$, $L_r(n_a) \cdot Act^\omega = Act^\omega$. Following the same argument as above only for the no verdict we arrive at the conclusion that $E_\omega \vdash n = \text{yes} + \sum_{a \in Act} a \cdot \text{no} = \text{yes} + \text{no} = m$.

  - The case $m = \text{yes} + \text{no} \simeq_\omega \sum_{a \in A} a \cdot n_a + \text{no} = n$ is exactly symmetrical.

- $m = \text{yes} + \sum_{a \in A} a \cdot m_a \simeq_\omega \sum_{b \in B} b \cdot n_b$ in reduced normal forms is as follows.

  First of all by following the above argument we can arrive to the point where $E_\omega \vdash n \simeq_\omega \text{yes} + \sum_{b \in B'} b \cdot n'_b$.
For the proof of this final case we will use the following, already proven, steps:

(S1) \( B = \text{Act} \)

(S2) For all \( b \in \text{Act}, L_a(n_b) = \text{Act}^\omega \)

(S3) For all \( a \in A, L_r(m_a) = L_r(n_a) \)

So, for each \( a \in A, yes + m_a = n_a \). Since both of these monitors have smaller depth that the original we have that by induction:

\[
\mathcal{E}_\omega \vdash yes + m_a = n_a, \forall a \in A. \tag{1}
\]

For each \( b \in \text{Act} \setminus A \), we have that \( yes \simeq_\omega n_a \) (because \( L_r^\omega(n_b) = \emptyset \)). Again since the depth here is also smaller we have that by induction:

\[
\mathcal{E}_\omega \vdash yes = n_a, \forall b \in \text{Act} \setminus B. \tag{2}
\]

So:

\[
n = \sum_{b \in \text{Act}} b.n_b = \sum_{a \in A} a.n_b + \sum_{b \in \text{Act} \setminus A} b.n_b
\]

By equations (1) and (2):

\[
= \sum_{a \in A} a.(yes + m_a) + \sum_{b \in \text{Act} \setminus A} b.yes
\]

\[
= \sum_{a \in A} a.yes + \sum_{a \in A} a.m_a + \sum_{b \in \text{Act} \setminus A} b.yes
\]

\[
= \sum_{a \in \text{Act}} a.yes + \sum_{a \in A} a.m_a
\]

which by axiom \( Y_\omega \) is equal to \( m \) and we are done.

The above analysis can be applied symmetrically for the cases:

\( m = no + \sum_{a \in A} a.m_a \simeq_\omega \sum_{b \in B} b.n_b = n \)

\( m = \sum_{a \in A} a.m_a \simeq_\omega \sum_{b \in B} b.n_b = n \)

\( \blacksquare \)

**Note 1.** As expected the proof for the \( \omega \)-verdict equivalence involved the extra axioms added and since we assumed that the monitors and not verdict equivalent we did not use any of the old ones.
4. Open Terms

Thus far, we have only been interested in equational axiom systems for $\simeq$ and $\simeq_\omega$ over closed terms. However in our grammar we allow for variables and it is natural to wonder whether the ground-complete axiomatizations we have presented in Theorems 2 and 3 are also complete for verdict equivalence and $\omega$-verdict equivalence over open terms. Unfortunately, this turns out to be false. Indeed, the equation

$$(O1) \quad yes + no = yes + no + x$$

is valid with respect to $\simeq$, but cannot be proved using the equations in $E_v$. This is because all the equations in that axiom system have the same variables on the their left- and right-hand sides. However its soundness is trivial since both sides of the equation trivially accept and reject $Act^*$ and therefore will continue to do so under all possible substitutions. Our goal in the remainder of this section is to study the equational theory of $\simeq$ and $\simeq_\omega$. Section 4.1 will present our results when $Act$ is infinite. We consider the setting of finite actions in Section 4.2.

The initial axiom system for open equations is $E_v$ only expanded with $O_1$ and will be called $E'_v$.

First of all, we modify the notion of normal form, to take variables into account.

**Definition 6.** A term $m \in Mon_F$ is in normal form if it has the form:

$$m = \sum_{a \in A} a.m_a + \sum_{i \in I} x_i [+yes] [+no]$$

where $\{x_i \mid i \in I\}$ is a finite set of variables, $A$ is a finite subset of $Act$ and each $m_a$ is an (open) term in normal form that is different from end.

**Lemma 8.** Each open term $m \in Mon_F$ is provably equal to some normal form $m'$ with $\text{depth}(m') \leq \text{depth}(m)$.

The proof follows the lines of the one for Lemma 5 for closed terms and is thus omitted.

Note that each $m_a$ might contain by itself a set of variables where each one could appear in the set $\{x_i \mid i \in I\}$ or not. Depending on the form of the monitor we might be able to reduce these extra appearances of nested variables and produce what we will refer to as reduced normal form of open terms.

**Definition 7.** A Reduced Normal Form of Open Terms is a term

$$m = \sum_{a \in A} a.m_a + \sum_{i \in I} x_i [+yes] [+no]$$

where if $v \in \{yes, no\}$ is a summand of $m$ then each $m_a$ is $v$-free and in reduced normal form. In addition if both yes and no are summands of $m$ then $m$ is equal to $yes + no$. 

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Lemma 9. For each open monitor \( m \in \text{Mon}_F \), its normal form is provably equal to a reduced normal form.

Proof. We start from the normal form of monitor \( m \in \text{Mon}_F \). The reduced normal form of it will depend on the verdicts \( v \in \{\text{yes}, \text{no}\} \) it has a summands. We isolate the cases:

1. \( m = \sum_{a \in A} a.m_a + \sum_{i \in I} x_i \).

2. \( m = \text{yes} + \sum_{a \in A} a.m_a + \sum_{i \in I} x_i \) in which case \( m \) reduces to \( m' = \text{yes} + \sum_{a \in A'} a.m'_a + \sum_{i \in I} x_i \) where each \( m'_a \) is yes free and is in reduced normal form. In addition if \( m_a \) has \( \text{no} \) as a summand then \( m_a \) is equal to a \( \text{no} \).

3. \( m = \text{no} + \sum_{a \in A} a.m_a + \sum_{i \in I} x_i \) in which case \( m \) reduces to \( m' = \text{no} + \sum_{a \in A'} a.m'_a + \sum_{i \in I} x_i \) where each \( m'_a \) is no free and in reduced normal form.
   In addition if \( m_a \) has \( \text{yes} \) as a summand then \( m_a \) is equal to a \( \text{yes} \).

4. \( m = \text{yes} + \text{no} + \sum_{a \in A} a.m_a + \sum_{i \in I} x_i \) where \( m \) reduces to \( \text{yes} + \text{no} \).

Case 1 accepts no further reducing. By applying an inductive argument each \( m_a \) will be in reduced normal form and we are done. Cases 2 and 3 are symmetric and slightly differ from the proof of Lemma 6. The extra claim in the case of the open terms is that we can remove occurrences of variables in the \( m_a \)'s when they contain a \( \text{no} \) summand (we present here the proof for case 2).

Assume a monitor of the form \( m = \text{yes} + a.(x + \text{no} + m'_a) \). Then we apply our axioms as follows:

\[
\begin{align*}
\text{yes} + a.(x + \text{no} + m'_a) &= \text{yes} + a.\text{yes} + a.(x + \text{no} + m'_a) \\
&= \text{yes} + a.(\text{yes} + \text{no}) = \text{yes} + a.\text{yes} + a.\text{no} = \text{yes} + a.\text{no}
\end{align*}
\]

The final case (4) is a simple application of our new axiom \( O_1 \) to remove all the variables in \( \sum_{i \in I} x_i \) and then repeating the inductive argument for the \( m_a \)'s.

\( \square \)

Note here that the defined normal form depends on the syntax we are analyzing in each case, while the reduced normal form depends on the axioms we have available for application (which in turn depend on our notion of equivalence and the characteristics of the model). Later on, where we will have extra axioms, our notion of reduced normal form will be further refined.
4.1. Infinite set of Actions

When the set of actions \( \text{Act} \) is infinite, it is easy to define a one-to-one mapping from open to closed terms that will help us prove completeness.

**Theorem 4.** (Completeness of open terms) \( \mathcal{E}_v' \) is complete for open monitors in \( \text{Mon}_F \) when \( \text{Act} \) is infinite. That is, for all \( m, n \in \text{Mon}_F \) if \( m \simeq n \), open monitors then \( \mathcal{E}_v' \vdash m = n \).

**Proof.** By lemma 9 we may assume that \( m \) and \( n \) are in reduced normal form.

Let
\[
m = \sum_{a \in A} a.m_a + \sum_{i \in I} x_i [\text{+yes}] [\text{+no}]
\]
and
\[
n = \sum_{b \in B} b.n_b + \sum_{j \in J} y_j [\text{+yes}] [\text{+no}]
\]
be two normal forms for open terms. We will show that \( \mathcal{E}_v' \vdash m = n \) by induction on the sum of the sizes of \( m \) and \( n \). To this end, we will establish a strong structural correspondence between \( m \) and \( n \).

Consider a substitution \( \sigma \) defined as follows:
\[\sigma(x) = a_x.(\text{yes + no})\]
where
- \( \forall x, y, a \) if \( a_x = a_y \) then \( x = y \), and
- \( \{a_x \mid x \in \text{Var}\} \) is disjoint from the set of actions occurring in \( m, n \).

Note that such a substitution \( \sigma \) exists because \( \text{Act} \) is infinite. By induction on the sizes of \( m \) and \( n \), we will prove that if \( \sigma(m) = \sigma(n) \) then:

(C1) \( v \) is a summand of \( m \) iff \( v \) is a summand on \( n \), for \( v \in \{\text{yes, no}\} \)

(C2) \( \{x_i \mid i \in I\} = \{y_j \mid j \in J\} \)

(C3) \( A = B \) and

(C4) for each \( a \in A \), \( \sigma(m_a) = \sigma(n_a) \).

Then assuming two verdict equivalent monitors \( m \) and \( n \) we know by definition that under any substitution \( \sigma \), it will be that \( \sigma(m) = \sigma(n) \) and subsequently using (C1)-(C4), we can prove \( \mathcal{E}_v' \vdash m = n \). This we also do by induction on the sum of the sizes of \( m \) and \( n \).

Having proved that the claims C1–4 follow form \( \sigma(m) \simeq \sigma(n) \), we can apply our inductive argument as follows: From C4 we know that \( \forall a \in A \), \( m_a = n_a \) and from induction hypothesis \( \mathcal{E}_v' \vdash m_a = n_a \). By C1-3 we also have that \( \mathcal{E}_v' \vdash \sum_{i \in I} x_i = \sum_{j \in J} y_j \) and that \( \sum_{a \in A} a.m_a = \sum_{b \in B} b.n_b \), which means that by summation we also have that \( \mathcal{E}_v' \vdash m = n \).

We present now the proof of (C1)-(C4).

C1: From the normalization lemma for open terms and lemma 11 it must be the case that if the verdict \( v \in \{\text{yes, no}\} \) is a summand of \( m \) then \( m \) will be in one of the following forms:
\[ m = yes + no \]

\[ m = yes + \sum_{a \in A} a.m_a + \sum_{i \in I} x_i, \] where each \( m_a \) is “yes”-free and in reduced normal form.

\[ m = no + \sum_{a \in A} a.m_a + \sum_{i \in I} x_i, \] where each \( m_a \) is “no”-free and in reduced normal form.

In the first case we have that \( n = yes + no \) as no other reduced normal form is equal to \( yes + no \).

The two following cases are symmetrical so we only discuss the first. We have that \( n \) must be equal to a form \( n = yes + \sum_{b \in B} b.n_b + \sum_{j \in J} y_j \) (where the \( n_b \)'s are “yes”-free) as both monitors need to accept the empty trace.

**C2:** Suppose now that \( \exists x_0 \in \{x_i \mid i \in I\}, x_0 \notin \{y_j \mid j \in J\}. \) Then \( \sigma(m) \) both accepts and rejects the trace \( a_{x_0} \) while \( \sigma(n) \) does not which is a contradiction. Therefore \( \{x_i \mid i \in I\} = \{y_j \mid j \in J\}. \)

**C3:** Similarly if \( \exists a \in A, a \notin B \) then through the mapping \( \sigma \), which is end-free and does not map any variable to a trace starting with \( s \), monitor \( m \) will eventually accept and reject a trace starting with \( a \) while \( n \) will not be able to do so.

**C4:** Our final claim and most complicated one is that \( \forall a \in A, \sigma(m_a) \simeq \sigma(n_a). \) We have two verdict equivalent open monitors \( m \) and \( n \) different than \( yes + no \) (since this case is trivial) in reduced normal form. If they don’t contain a verdict \( v \in \{yes, no\} \) as a summand then the argument is simplified significantly and therefore we will present here the most complicated case where they both contain one verdict \( v \in \{yes, no\} \) as a summand. Without loss of generality we assume that this verdict is \( yes \). Namely:

\[ m = yes + \sum_{a \in A} a.m_a + \sum_{i \in I} x_i \]

and

\[ n = yes + \sum_{a \in A} a.n_a + \sum_{i \in I} x_i \]

We can easily see that the rejection sets \( L_r(\sigma(m_a)) \) and \( L_r(\sigma(n_a)) \) for each \( a \in A \) are equal since any distinction between them would lead to a separation of the monitors \( \sigma(m), \sigma(n) \). It remains to prove the acceptance sets of \( \sigma(m_a), \sigma(n_a) \) are identical.

To that end we present the following argument. It is important here to point out that since both \( m \) and \( n \) contain a \( yes \) verdict as a summand their acceptance sets are equal and \( = Act^* \). However for our inductive argument to work we need to be able to show that there exists a sequence of applications of the axioms that proves the equality \( \sigma(m_a), \sigma(n_a) \). To that end consider a trace \( s \) accepted by monitor \( m_a \) under the substitution \( \sigma \). Consequently monitor \( \sigma(m) \)
accepts the trace a.s. Due both the yes verdict at the top level but also though the \( \sigma(m_a) \). Since monitor \( m_a \) is yes-free as a result of being in reduced normal form, the acceptance of \( s \) must come from a variable \( x \) to \( a_x.(yes + no) \) through the substitution \( \sigma \). I.e monitor \( \sigma(m_a) \) can preform the transitions:

\[
\sigma(m_a) \xrightarrow{s'} \sigma_0(m'_a) \overset{a}{\Rightarrow} yes + no + m'_a \overset{\sigma}{\Rightarrow} yes
\]

where \( s'.a_x \) is a prefix of \( s \).

We argue that \( \sigma(n_a) \) can also accept \( s \). Since \( s'.a_x \) is accepted by \( \sigma(m_a) \) it must also be rejected by it since \( a_x \) is an action that can only be observed after the substitution of a variable \( x \) in \( m_a \). However we have already argued that the rejection sets of \( \sigma(m_a), \sigma(n_a) \) are equal and therefore \( \sigma(n_a) \) also rejects the trace \( s'.a_x \).

The only way this could happen is either by performing a sequence of transitions \( n_a \overset{s'}{\Rightarrow} n'_a + x \) or by rejecting a prefix of \( s' \). The first case where \( n_a \) is performing those transitions would also guarantee that \( \sigma(n_a) \) accepts \( s \) in which case we are done. What remains is the case where \( \sigma(n_a) \) rejects a prefix of \( s' \). In order for \( \sigma(n_a) \) to reject a prefix of \( s'.a_x \) and by the equality of the rejection sets of the two sub-monitors, \( \sigma(m_a) \) would also reject a prefix of it. This can only happen if \( n_a, m_a \) rejected that prefix before the substitution \( \sigma \) takes place. That is because this substitution maps variables to unique actions per variable and therefore it cannot create traces that "match" the actions that were in the closed parts of the monitors. This means that \( m_a, n_a \) reject some prefix \( s' \) of \( s \).

From the reduced normal form lemma for open terms \( \text{[9]} \) then it holds that \( m'_a \) is variable free and therefore \( m_a \) cannot accept \( s \), which is a contradiction. We have then that the acceptance sets of \( m_a, n_a \) are equal and therefore \( m_a \simeq n_a \), which implies that \( E'_v \vdash m_a = n_a \) for all \( a \in A \).

\[\square\]

4.2. Finite set of Actions

The study of the equational theory of \( \simeq \) when \( Act \) is finite turns out to be more interesting and complicated. In this setting, we can identify equations whose validity depends on the cardinality of \( Act \), which is not the case for any of the axioms we used so far. To see this, consider the equation

\[ V_1 : x = x + a.x \]

which is easily proven sound when \( |Act| = 1 \) but cannot be proved by the equations in \( E'_v \).

As a first step in our study of the equational theory of \( \simeq \) when \( Act \) is finite, we characterize some properties of sounds equations.

**Lemma 10.** For a sound equation \( m \simeq n, m, n \in Mon_F \), where \( m, n \) are in reduced normal form, if \( m \overset{a}{\Rightarrow} x \) and for every prefix \( s_p \) of \( s \) we have that \( m \overset{s_p}{\Rightarrow} x \), then \( n \overset{s_p}{\Rightarrow} x \).
Proof. Consider the substitution $\sigma(x) = yes + no$, $\sigma(y) = end$, $\forall y \neq x$. Since $m \xrightarrow{a} x$ we have that $\sigma(m)$ will both accept and reject $s$. Since $m \simeq n$ is sound we have that $\sigma(n)$ must do the same. If $n$ does not contain a $s.x$ summand then there are two cases:

1. $n$ contains a closed $s.(yes + no)$ summand. (Or a $s'.(yes + no)$ summand for $s'$ a prefix of $s$).

2. $n$ contains a $s'.x$ summand for $s'$ a prefix of $s$ (So that $\sigma(n)$ would accept and reject $s'$ and therefore $s$).

In the first case we have that $n \xrightarrow{a} yes + no$ which means that $n$ would accept and reject the trace $s$ even under the substitution $\sigma(x) = end$, $\forall x$. By verdict equivalence of $m$ and $n$ the same should apply for $m$. However this is not allowed since $m \xrightarrow{a} x$ and is in reduced normal form so it cannot be that $m \xrightarrow{a} yes + no$.

In the second case even though now both monitors accept and reject $s$ we have that in addition $\sigma(n)$ accepts and rejects also $s'$. Again since the two monitors are verdict equivalent we know that $\sigma(m)$ must do the same. Since $m$ is in reduced normal form and $m \xrightarrow{a} x$ we have that $\sigma(m)$ can reject $s'$ only due to a closed summand $s'.(yes + no)$ of $m$. This however is not allowed since it contradicts the fact that $m$ is in reduced normal form and $m \xrightarrow{a} x$.

Both cases then are a contradiction and so $n$ must also contain a $s.x$ summand.

Intuitively the above lemma states that on each sound equation (including axioms) the first occurrence of each variable per distinct branch is common for both sides of the equation. This gives us some good insight on what a sound equation looks like. For instance the equation

$$x + a.x + a.a.(yes + no) + a.b.(yes + no) = x + a.a.(yes + no) + a.b.(yes + no)$$

over the set of actions $Act = \{a, b\}$ is sound, but

$$x + a.x + a.a.(yes + no) + a.b.(yes + no) = a.x + a.a.(yes + no) + a.b.(yes + no)$$

could not be. Also notice here the importance of the closed summand $\sum_{a \in Act} a.(yes + no)$ after the trace $s = a$. This typo of summation is crucial for our equations.

In order to continue analyzing them we present some useful notation.

**Definition 8.** (Notation)

1. We use $\text{pre}(s)$ to denote the set of prefixes of $s$ (including $s$).

2. We define $\Xi^\prec(m) = \sum \{s'.m \mid |s'| \leq |s| \text{ and } s' \notin \text{pre}(s)\}$.
3. With the term $\overline{s}(m)$ we denote the summation: \( \overline{s}(m) + \sum_{a \in \text{Act}} a.m \).

Intuitively $\overline{s}$.no stands for the complementary monitor of the trace $s$ that rejects all traces that do not cause a rejection of the string $s$. Those are exactly the traces that are no longer than $s$, but not its prefixes and the ones extending $s$.

4. With the term $\overline{s}^{(k)}(m)$ we will mean the summation: \( \sum_{1 \leq i < k} s_i \overline{s}(m) + s^k.\overline{s}(m) \).

Lemma 11. For any sound equation $m \simeq n$, where $m, n$ are in reduced normal form, if $m \xrightarrow{\sigma} x$ and $n \not\xrightarrow{\sigma} x$ then there exists a prefix of $s$, $s'$ ($s = s'.s''$) for which all $s_b$'s of the form $s.s_b$, where $s_b \not\in \text{pre}(s')$ either:

- $m \xrightarrow{s_b} \text{yes} + \text{no}$ and $n \xrightarrow{s_b} \text{yes} + \text{no}$ or
- $\exists s'_0 \text{ s.t. } m \xrightarrow{s'_0} x$ and $n \xrightarrow{s'_0} x \ x' \in m, n$ and $s'_0, s_b \in \text{pre}(s.s_b)$.

Proof. We have an equation $m \simeq n$ for which we know:

\[
m \simeq m' + s.x \simeq n,
\]

where $n$ does not contain a $s.x$ summand. We will prove the rest of this lemma step by step. We start at the first (in the shortest depth) non-empty possible $s$ we can find. This means that indeed in the monitors $m, n$ all other earlier occurrences of $x$ happen at both sides. By lemma 10 we know that there is a prefix of $s$ called $s_0 = (s = s_0.s_1)$ s.t. $s_0, x \in m, n$, and we can pick the shortest such trace. I.e. our equation looks like:

\[
m \simeq m'' + s_0.(s_1.x + x) \simeq n'' + s_0.(x) \simeq n.
\]

This means that there are no other occurrences of the variable $x$ "between" $s_0$ and $s_1$ (otherwise $s_0.s_1$ would not be the shortest trace). Since this equation is sound we know that under any substitution the resulting monitors are verdict equivalent. Therefore there should not exist a substitution that separates acceptance or rejection sets of the two monitors. We can trivially see that the substitution $\sigma(x) = \text{yes} + \text{no}$ would not separate them. In addition any substitution $\sigma$ that maps $x$ to any $s_b.(\text{yes} + \text{no}) + m_q$ where $s_b$ is a prefix of $s_1$ would also not separate them. How about an arbitrary substitution though? Regardless of the behavior of monitors $m'', n''$ we can see that a disagreement could occur due to the $s_0.s_1.x$ part of monitor $m$. Substitutions that could create this disagreement must map $s$ to some closed monitor $m_x$ such that $m_x \xrightarrow{s_b} \text{yes} + \text{no}$ where $s_b$ is not a prefix of $s_1$. The candidates for $s_b$ therefore are the set:

\[
A = \{ s \mid \vert s \vert \leq \vert s_1 \vert \land s \not\in \text{pre}(s_1) \} \cup s = s_1.s', s' \in \text{Act}^+ \}
\]
Of course if \( m, n \) reject and accept all of these traces then indeed the equation is sound. Could they not reject all of them and yet the equation still be sound?

Take one of the candidate separating substitutions \( \sigma(x) = s_b.(yes + no) \) where \( s_b \) is not a prefix of \( s_1 \). Then we have that:

- \( \sigma(m) \) accepts and rejects \( s_0.s_1.s_b \)
- both monitors accept and reject \( s_0.s_b \) which is not a prefix of \( s_0.s_1.s_b \).

Monitor \( \sigma(n) \) should still somehow reject and accept the trace \( s_0.s_1.s_b \). This is not done however as \( n \xrightarrow{s_0.s_1.s_b} yes + no \) since that would fall into the previous case. Also \( s_1 \) is the shorter we could find with that property. That only leaves the case were \( s_0 \) has a prefix \( s'_0 \) such that \( n \xrightarrow{s'_0} x \) and \( s'_0.s_b \) is a prefix of \( s_0.s_1.s_b = s.s_b \). In this case we can be sure that \( m \xrightarrow{s'_0} x \) since \( s.x \) is the fist one-sided variable we could find. This concludes the case analysis for the first occurrence of a one sided variable. For the next ones we have the following. If the next one sided variable is on a different branch (not somewhere after a series of transitions \( s_0.s_1 \) has been performed) then we can apply the same analysis. If it is on the same branch the prefix \( s'_0 \) is also a prefix of our current occurrence and therefore we have the conclusion.

**Verdict equivalence for open terms.** We can see from the above discussion that clearly in the setting of finite actions and open terms there are equations that are not provable by our axioms. Namely none of the above Lemmata refer to equations similar to our existing axioms. This means that in order to axiomatize verdict equivalence over open terms, we must expand our set of axioms even more.

Our proposed axioms for open terms and a finite set of axioms are:

\[
(O_{2s,k}): \{x + s.x + \pi^{(k)}(yes + no) = x + \pi^{(k)}(yes + no)\} \quad s \in \text{Act}^*, \quad k \geq 0
\]

We extend our finite axiom set \( \mathcal{E}'_v \) for open terms to the infinite \( \mathcal{E}'_v \cup O_{2s,k} \) which we will call \( \mathcal{E}_{\text{fin}} \).

**Lemma 12.** \( \mathcal{E}_{\text{fin}} \) is sound. That is, if \( \mathcal{E}_{\text{fin}} \vdash m = n \) then \( m \simeq n \), for all \( m, n \in \text{Mon}_F \).

**Proof.** We have to prove soundness only for the new family of equations \( O_{2s,k} \) as the rest have been proven sound in theorem [1]. For simplicity we will call the right hand side of the equation \( m \) and the left hand side \( n \). First of all we observe that for any substitution \( \sigma \), we have that \( \sigma(x + \pi^{(k)}(yes + no)) \) will be a summand of both \( \sigma(m) \) and \( \sigma(n) \) and therefore all the traces accepted and rejected by it will be common for both sides of the equation. The only part of the equation that is not trivially a summand of both sides is \( s.x \). Assume that there is some substitution \( \sigma_0 \) such that the summand \( s.x \) rejects or accepts some
trace \( s_0 \) which is not accepted or rejected by the rest common summands of the equation. We cannot have that \( \sigma_0(x) \) will contain either a yes or a no summand as this would cause the trivial rejection or acceptance of all traces from both sides of the equation. We have now that if \( \sigma_0 \) maps \( x \) to \( s_0, (no + yes) \) which is a prefix of \( s \) then both sides accept and reject \( s_0 \) but also \( s.s_0 \) and therefore not separating them. On the other hand if \( \sigma_0 \) maps \( x \) to \( s' \) which is not a prefix of \( s \) then the trace \( s.s' \) is also not a prefix of \( s.s \) and therefore the summand \( (yes + no) \) for any \( k \) rejects and accepts the trace \( s.s' \) in both sides of the equation.

We provide here some examples on how to use the above to derive some simpler but more intuitive sound equations.

**Lemma 13.** The following equations are derivable from \((O2_{s,k})\):

1. \( x + s.x + s.s_0(yes + no) = x + s.s_0(yes + no) \), with \( s_0 \) a prefix of \( s \).

2. \( yes + x + s_1.s_2(no) = yes + x + s_1.s_2(no) + s_1.x \), where \( s_2 \) is any prefix of \( s_1 \).

3. \( no + x + s_1.s_2(yes) = no + x + s_1.s_2(yes) + s_1.x \), where \( s_2 \) is any prefix of \( s_1 \).

4. \( x + s. \sum_{a \in Act} a.(no + yes) = x + s.(x + \sum_{a \in Act} a.(no + yes)) \)

**Proof.** We first show how to derive the first equation and then we derive the rest form it. We start by picking the equation \( O2_{s,1} \) i.e.

\[
x + s.x + s.s_0(yes + no) + s.s. \sum_{a \in Act} a.(yes + no) =
\]

\[
x + s.s_0(yes + no) + s.s. \sum_{a \in Act} a.(yes + no) .
\]

In addition we have the tautology \( s.s_0 \). \( \sum_{a \in Act} a \), for a the specific prefix \( s_0 \) of \( s \). On the two valid above equations we apply the congruence rule for + and have:

\[
x + s.x + s.s_0(yes + no) + s.s. \sum_{a \in Act} a.(yes + no) + s.s_0 \sum_{a \in Act} a.(yes + no) \]

\[
x + s.s_0(yes + no) + s.s. \sum_{a \in Act} a.(yes + no) + s.s_0 \sum_{a \in Act} a.(yes + no) .
\]

The first simplification that we perform now is by observing that the summand \( s.s_0 \) \( \sum_{a \in Act} a.(yes + no) \) accepts and rejects a prefix of the whole summand
\[ s.s. \sum_{a \in \text{Act}} a.(yes + no) \] and therefore we can eliminate the later from the summation:

\[ x + s.x + s.\sum_{a \in \text{Act}} a.(yes + no) + s.s_0. \sum_{a \in \text{Act}} a.(yes + no) \]

\[ = x + s.\sum_{a \in \text{Act}} a.(yes + no) + s.s_0. \sum_{a \in \text{Act}} a.(yes + no) . \]

In addition the term \( s.s \) can be rewritten as \( s.s_0.s \) \( \sum_{a \in \text{Act}} a.(yes + no) \) = \( x + s.s_0.s \) \( \sum_{a \in \text{Act}} a.(yes + no) . \)

Now we have again that the summand \( s.s_0. \sum_{a \in \text{Act}} a.(yes + no) \) accepts and rejects a prefix of the whole summand \( s.s_0.\sum_{a \in \text{Act}} a.(yes + no) \) and therefore we can omit the later. This gives us the equation:

\[ x + s.x + s.s_0.\sum_{a \in \text{Act}} a.(yes + no) = x + s.s_0.\sum_{a \in \text{Act}} a.(yes + no) , \]

which can be rewritten using our notation as

\[ x + s.x + s.s_0 (yes + no) = x + s.s_0 (yes + no) , \]

giving us the target equation.

Having presented the proof for the first family of equations in detail we give a short description for the rest. For the equations (2) and (3) it suffices to use the congruence rule for + with the equations \( yes = yes \) and \( no = no \) respectively and then simplify the equations by using the distribution axiom for +. For the latter equation (4) it is enough to instantiate the prefix \( s_0 \) in the the family of equations (1) as the empty string \( e \). This is of course allowed since the empty string is a prefix of any string.

\[ \square \]

Now that we have concluded our discussion over the new axioms we proceed to use them in our reduced normal forms definition.
Definition 9. A Reduced Normal Form of Open Terms over finite actions is a term
\[ m = \sum_{a \in A} a.m_a + \sum_{i \in I} x_i \{[yes] + [no]\} \]
where if \( v \in \{yes, no\} \) is a summand of \( m \) then each \( m_a \) is \( v \)-free and in reduced normal form. If both yes and no are summands of \( m \) then \( m \) is equal to \( yes + no \).

In addition if \( m \) contains a summand \( m' \) with \( m' \equiv s^{(k)}(yes + no) \) for some \( k \) then \( m \not\rightarrow x_i \) for any \( i \in I \).

In order to use the above form of the monitors in \( Mon_F \) we need to prove that any term can be rewritten in a reduced normal form. Before doing so will will prove the following useful lemma:

Lemma 14. For a monitor \( m \in Mon_F \), if \( m \not\rightarrow yes \) or \( m \not\rightarrow no \) then \( E_{fin} \vdash m = m + s.yes \) or \( E_{fin} \vdash m = m + s.no \) respectively.

Proof. We will demonstrate the proof for the acceptance case. We are using induction on the length of the trace \( s \).

- If \( s_0 \) is the empty trace then \( m \) accepts the empty trace and therefore it must contain a \( yes \)-syntactic summand. Similarly for \( no \).

- Assume now that we have proved that we can create a syntactic summand in \( m \) for all of the traces up to length \( \ell \).

- Take now a trace \( s_0 \) of length \( \ell + 1 \). Since \( m \) must accept and reject the trace \( s_0 \) it is necessary that \( m \not\rightarrow yes \). If \( m \not\rightarrow yes \) with \( s_0[-1] \) being the trace of all but the last action of \( s_0 \) then we can use the induction hypothesis and say that \( E_{fin} \vdash m = s_0[-1].yes + m \). Then we use axiom \( Y_a \) for \( a \) being the last action of \( s_0 \) to transform said equation to \( E_{fin} \vdash m = s_0[-1].(yes + a.yes) + m \) and then by the distribution axiom \( D_a \) we have the conclusion. If \( m \not\rightarrow yes \) then it must be the case where \( m \) can perform all actions of \( s \) and arrive at a \( yes \). By applying the distribution axiom \( D_a \) for each action of \( s_0 \) we have the necessary syntactic summand.

\[ \square \]

Lemma 15. For each open monitor \( m \in Mon_F \), its normal form is provably equal to a reduced normal form.

Proof. From lemma 8 we can start from a monitor \( m \) already reduced to the normal form defined for open terms and infinite actions. Therefore we have the following cases:

- \( m = yes + no \).

- \( m = yes + \sum_{a \in A} a.m_a + \sum_{i \in int} x_i \), where each \( m_a \) is \( yes \)-free.
\[
m = no + \sum_{a \in A} a.m_a + \sum_{i \in I} x_i,
\]
where each \(m_a\) is no-free.

\[
m = \sum_{a \in A} a.m_a + \sum_{i \in I} x_i.
\]

We begin our analysis from the second case. The same symmetrical analysis can be applied to the third one and the fourth one is as a less complicated version of the same inductive argument. We have therefore a monitor \(m = yes \sum_{a \in A} a.m_a + \sum_{i \in I} x_i\). Our extra assumption for this case is that \(m\) contains a semantic summand \(m'\) with \(m' \simeq \pi^{(k)}(yes + no)\) for some \(k\) and \(m' \not\rightarrow\ x_i\) for any \(i \in I\).

Since \(m'\) is a semantic summand of \(m\), we have that \(m \simeq m + m'\) and that \(m' \not\rightarrow\ yes + no\) for all of the traces \(s_0\) that \(\pi^{(k)}(yes + no) \not\rightarrow\ yes + no\). We call this set of traces \(S\). Therefore by the previous lemma we have that \(E_{fin} \vdash m = m + \sum_{\forall s_0 \in S} s_0.(yes + no)\). Again using the distribution axiom we have \(E_{fin} \vdash \sum_{\forall s_0 \in S} s_0.(yes + no) = \pi^{(k)}(yes + no)\).

For the same monitor \(m\) we now want to argue that if \(m \not\rightarrow\ x\) then we can eliminate this occurrence of variable \(x\). We have already shown that \(m\) contains a syntactic summand \(\pi^{(k)}(yes + no)\). By applying axiom \(D\) for each action of the trace \(s\) in a way similar to the proof of Lemma 14 we have that \(E_{fin} \vdash m = m' + \pi^{(k)}(yes + no + s.x)\) and \(m' \not\rightarrow\ x\). Then on this equality we apply the relevant axiom from the family \(O2_{s,k}\) and we have the conclusion.

\[\square\]

**Lemma 16.** If monitor \(m\) is in reduced normal form and contains an \(x\) summand and \(m \not\rightarrow\ x\) then there is at least one trace \(s_{bad}\) such that \(\forall k: \pi^{(k)}(yes + no) \not\rightarrow\ yes + no\) but \(m \not\rightarrow\ yes + no\).

**Proof.** If no for some \(k_0\), no such trace \(s_{bad}\) exists then the monitor would contain a summand \(m' \simeq \pi^{(k_0)}(yes + no)\) for some \(k\) and still it would be able to perform the transition \(m \not\rightarrow\ x\) which is a contradiction of it being in reduced normal form.

\[\square\]

**4.2.1. Completeness for open terms:**

In the case where \(|\text{Act}|\) is finite the completeness proof must be modified form the infinite actions case, as we can no longer define the convenient substitution. However, through a more complex procedure we can deduce the corresponding result. We distinguish two cases separately, namely when \(|\text{Act}|\geq 2\) and when
*Act* is a singleton. This is necessary because equations such as \( x = x + a.x \) are only sound when \( Act = \{a\} \). For the proof when \(|Act| \geq 2\) it is necessary to utilize at least two actions \( a, b \in Act \), which is the reason why when only one action is available new cases arise.

*Action set with with at least two actions.*

**Theorem 5.** \( \mathcal{E}_v' \) is complete for open terms for a finite \(|Act| \geq 2\). That is, if \( m \simeq n \) then \( \mathcal{E}_{fin} \vdash m = n \)

For each such theorem, in order to show that the monitors can be proven equal by our axioms there are three lemmata that must be proven for their normal forms. I.e. That they have identical sets of variables, that the actions that each one can perform are equal and that after a common action they become monitors that are also identical. Unfortunately for a finite set of actions, a substitution that can cover all three cases is hard to find. We are therefore led to a strategy that focuses on each part of the proof separately. This leads to the need of a more thorough analysis.

**Proof.** In the case where \( m = yes + no \) then \( n \) must also explicitly contain a *yes* and a *no* summand and therefore from lemma 6 and the new axiom \( O_1 \) both monitors are proven equal to *yes + no*.

Assume now that

\[
m = yes + \sum_{a \in A} a.m_a + \sum_{i \in I} x_i ,
\]

where \( \{x_i \mid i \in I\} \) is the set of variables occurring as summands of \( m \) and each \( m_a \) is *yes*-free and different from *end* (as a reduced normal form). Since \( \sigma(m) \) accepts \( \epsilon \) for each \( \sigma \) and \( m \simeq n \), monitor \( n \) is bound to have a similar form since it must contain the verdict *yes* as a summand, but not a *no* one. Therefore:

\[
n = yes + \sum_{b \in B} b.n_b + \sum_{j \in J} y_j
\]

and we need to show that there is a way to apply our axioms to show that monitor \( n \) is provably equal to \( m \).

We start by proving that \( \{x_i \mid i \in I\} = \{y_j \mid j \in J\} \). By symmetry, it suffices to only show that \( \{x_i \mid i \in I\} \subseteq \{y_j \mid j \in J\} \). To this end, assume \( x \in \{x_i \mid i \in I\} \). Consider the substitution \( \sigma \) mapping \( x \) to *no* every other variable to *end*, i.e:

\[
\sigma(x) = \begin{cases} no, & x \in Var(m) \setminus Var(n) \\ yes, & otherwise \end{cases}
\]

This means that if there is at least one variable in \( m \) that does not appear in \( n \) then \( \sigma(m) \) rejects the empty trace while \( \sigma(n) \) can’t. Therefore the set of variables of \( m \) is a subset of the variables of \( n \). By constructing the symmetric
substitution, the set of variables of \( n \) is proved a subset of the variables of \( m \) which makes them identical.

Next, we o prove that the action sets \( A, B \) are identical. Assume, towards a contradiction, that \( a \in A \setminus B \). Since \( \text{Act} \) contains at least two actions, there is some action \( b \neq a \). Consider the substitution \( \sigma_1 \) defined by \( \sigma_1(x) = b.n.o \) for each \( x \in \text{Var} \). Since \( a \in A \) and \( m_a \) is yes-free and different from \( \text{end} \), it is easy to see that \( a.s \in L_r(\sigma_1(m)) \) for some \( s \in \text{Act}^* \). On the other hand, no trace in \( L_r(\sigma_1(n)) \) begins with \( a \). This contradicts the assumption that \( m \) is verdict equivalent to \( n \). Hence, by symmetry, \( A = B \).

For the final part of the proof we must show that \( m_a \simeq n_a \) for each \( a \in A \). Towards a contradiction we will assume that the two monitors \( m_a, n_a \) are not verdict equivalent. Therefore there exists a substitution \( \sigma_0 \) that separates them, that is without loss of generality, there is a trace \( s_0 \) such that \( s_0 \in L_r(\sigma_0(m_a)), s_0 \notin L_r(\sigma_0(n_a)) \) or \( \exists s_0 \in L_a(\sigma_0(m_a)), s_0 \notin L_a(\sigma_0(n_a)) \). We are aware that if \( m_a \) is not verdict equivalent to \( n_a \) then various substitutions would cause different sorts of disagreements. For our convenience we will later on pick a specific one. As for now we continue without assuming something about the substitution. We will analyze first the case of rejection of the string \( s_0 \). The substitution \( \sigma_0 \) must be a closed one for \( m_a, n_a \) i.e. it must map to a closed monitor all variables in \( (\text{Var}(m_a) \cup \text{Var}(n_a)) \). We will use this substitution to create a new one \( \sigma_{bad} \) that would also separate the original monitors \( m, n \).

The first step towards this is:

\[
\sigma_{bad}(x) = \begin{cases} 
\text{end, } x \in \text{Var}(m) \setminus [\text{Var}(m_a) \cup \text{Var}(n_a)] & \\
\sigma_0(x), \text{otherwise} & 
\end{cases}
\]

Now since \( s_0 \in L_r(\sigma_0(m_a)) \) and \( \sigma_{bad}(m_a) = \sigma_0(m_a) \) we also know that \( a.s_0 \in L_r(\sigma_{bad}(a.m_a)) \). Our aim is to show that \( a.s_0 \notin L_r(\sigma_{bad}(n)) \). Following the definition \( \sigma_{bad}(n_a) = \sigma_0(n_a) \) and therefore \( s_0 \notin L_r(\sigma_{bad}(n_a)) \).

The only case where \( \sigma_{bad}(n) \) could reject \( a.s_0 \) like \( \sigma_{bad}(m) \) does, would be if it was rejected by the mapping of one the variables contained in the set \( \{x_i \mid i \in I\} \). It is useful to make here apparent that in order for \( \sigma_{bad}(n) \) to reject \( a.s_0 \), it must do so completely independently of the summand \( \sigma_{bad}(a.n_a) \), since the latter cannot reject any of the prefixes of \( a.s_0 \) as well.

By the definition of \( \sigma_{bad} \), the variables that did not appear at all in \( n_a \) or \( m_a \) were mapped to \( \text{end} \) and therefore cannot reject any string starting from \( a \). Therefore the only way for \( n \) to reject \( a.s_0 \) is for one of the variables appearing in \( \text{Var}(n_a) \cup \text{Var}(m_a) \) to have been mapped to a closed term \( m_x \) that can reject \( a.s_0.n.o \) or one of its prefixes (this does not contradict the fact that \( \sigma_{bad}(n_a) \) does not reject \( s_0 \)). Therefore there is at least one \( x_0 \in \text{Var}(m_a) \cup \text{Var}(n_a) \) and \( x_0 \in \{x_i \mid i \in I\} \) such that \( a.s_0 \in L_r(\sigma_{bad}(x_0)) \) or a prefix of it.

This leads to the case were \( m, n \) reject a prefix of \( a.s_0 \) because of the mapping of \( x_0 \). However this implies that we have the following situation:

\[
m = \text{yes} + x_0 + a.m_a + \sum_{b \in A \setminus \{a\}} b.m_b + \sum_{i \in I \setminus \{0\}} x_i \simeq
\]

25
\[
\text{yes} + x_0 + a.n_a + \sum_{b \in A \setminus \{a\}} b.m_b + \sum_{i \in I \setminus \{0\}} x_i = n
\]

and that monitor \(m_a\) can perform the transitions:

\[
\sigma_{\text{bad}}(m_a) s' \xrightarrow{a.s'} \sigma_{\text{bad}}(m'_a + x_0) \xrightarrow{\sigma_a(x_0)} \text{no}
\]

where \(s'\) is a prefix of \(s_0\) and in addition monitor \(n_a\) cannot arrive at the variable \(x_0\) after reading the trace \(s'\). This means respectively that

\[
\sigma_{\text{bad}}(m) a.s' \xrightarrow{a.s'} \sigma_{\text{bad}}(m'_a + x_0) \xrightarrow{\sigma_a(x_0)} \text{no}.
\]

We choose here to work with the shortest \(s'\) available in the monitors. Namely of all the substitutions that could separate the monitors \(m_a, n_a\), we pick to work with the one for which the trace \(s'\) that leads to this "one-sided" occurrence of the variable \(x_0\) is the shortest. This guarantees that there are no other one-sided occurrences of this variable between the top level one in \(m\) and the one occurring after the trace \(s'\).

By Lemma 10 we have that there exists at least one trace \(s_b\) such that \(m \xrightarrow{s_b} \text{yes} + \text{no}\) but \(s_b \in L_r(a.s^1(k))\) for all \(k \geq 0\). What would happen if we modified \(\sigma_{\text{bad}}\) to map the variable \(x_0\) to \(s_b\).\text{no}\)? We have that then \(s_b\) and \(a.s'.s_b\) \(\in L_r(\sigma_{\text{bad}}(m))\). In addition \(s_b \in L_r(\sigma_{\text{bad}}(n))\). However the traces that are rejected by the term \(a.s^1(k)\) are exactly traces that their rejection does not cause a rejection of the \(a.s'\) trace. This means that under the modified substitution \(\sigma_b\), monitor \(n\) cannot reject the trace \(a.s'.s_b\). This deems the monitors \(m, n\) not verdict equivalent which is a contradiction. We conclude then that the rejection sets of \(m_a\) is equal to the rejection set of \(n_a\) for each \(a\).

It remains to show that they also have identical acceptance sets. Towards contradiction suppose they don’t and assume an \(s\) that under the substitution \(\sigma\) separates them, i.e. \(s \in L_\alpha(\sigma(m_a)), s \notin L_\alpha(\sigma(n_a))\). In addition assume that \(s\) is of minimum length, meaning both that no prefix of \(s\) has this property but also that its acceptance is the result of a variable \(x\) occurring in \(m_a\) as \(m_a \xrightarrow{s} x\) and \(n_a \xrightarrow{s} x\). We know that this is exactly the case since if the variable \(x\) occurred earlier in \(m_a\) then by mapping it the \(\text{yes}\) we would have a shortest trace being accepted by \(\sigma(m_a)\) but not \(\sigma(n_a)\). We are sure now that monitor \(n_a\) cannot perform the transition \(n_a \xrightarrow{s} \text{yes}\). Which means that not only it does not arrive at the variable \(x\) after reading the trace \(s\), but also does not arrive to the \(\text{yes}\) verdict for any of its prefixes. Note here that any variable mapped to \(\text{no}\) immediately makes any sequence following it both accepted and rejected. Since we have already proved that the rejections sets of \(m_a, n_a\) under any substitution must be identical we can also conclude that as monitor \(n_a\) is reading trace \(s\) it also does not arrive to a \(\text{no}\) verdict, since this would mean that \(m_a\) must also arrive at that verdict earlier on while reading \(s\) and therefore from the reduced normal form lemma it would be equal to \(\text{no}\) which we know is not the case as
it \( m_a \xrightarrow{s} yes \). Given all of the above we can now construct a substitution \( \sigma' \) that would separate the rejection sets of \( n_a, m_a \) which is enough to prove the contradiction as the case where such a substitution exists has already been covered.

The situation we have at hand is as follows:

Monitor \( \sigma(m_a) \) can arrive to the verdict \( yes \) after reading the trace \( s \) while \( \sigma(n_a) \) cannot and also neither \( n_a \) nor \( m_a \) can produce a \( no \) verdict for the trace \( s \). Therefore if we switch the mapping of \( x \) to \( no \) in \( \sigma' \) and the verdicts of all other variables that where mapped to a \( no \) verdict to \( end \) we have produced a substitution that causes \( s \) to be rejected by \( \sigma'(m_a) \) but not from \( \sigma'(n_a) \). By utilizing our previous construction there exists another one that separates the monitors \( n, m \) as well which is a contradiction.

We have concluded then that the \( L_a(m_a) = L_a(n_a) \) and \( L_r(m_a) = L_r(n_a) \) which means that they are verdict equivalent.

This means that we can apply the inductive hypothesis and have that \( \mathcal{E}_{fin} \vdash m_a = n_a \). Using now congruence rules we have that \( \mathcal{E}_{fin} \vdash m = n \). All other possible forms of monitors \( m, n \) are sub-cases that the relative analysis can be applied symmetrically and therefore they are omitted.

Unary Action Set. If: \( Act = \{a\} \) then the proposed axiom set \( \mathcal{E}_{fin} \) is not complete for open terms. To see this, consider the equation \( V_1 : x = x + a.x \) which is easily proven sound but cannot be proved by the equations in \( \mathcal{E}_{v'} \). Notice that all equations in \( \mathcal{E}_v \) and therefore all those that can be derived form them are valid, regardless of the cardinality of \( Act \), which is why \( V_1 \) cannot be derived form them since it is only valid when \( |Act| = 1 \). Since this equation is valid and yet not provable we will add them to \( \mathcal{E}_v \).

**Theorem 6.** \( \mathcal{E}_{v1} = \mathcal{E}_{v} \cup V_1 \) is complete for open monitors when \( |Act| = 1 \). That is, if \( m \sim n \Rightarrow \mathcal{E}_{v1} \vdash m \sim n \)

**Proof.** First of all the new axiom \( V_1 \) can prove the family of equations \( X = a^+.X + X \). We will present the argument only for \( m = yes + a.m_a + \sum_{i \in I} x_i \) and \n respectively \( n = yes + a.n_a + \sum_{j \in J} y_j \). The variable sets are proved equal by the same substitution as above. The only case where the action sets are not equal is when \( A = \{a\} \) and \( B = \emptyset \). That is \( m = yes + a.m_a + X \), \( n = yes + X \). Our claim is that \( m_a \) is equal to \( a^+.no \). Otherwise \( m_a = a^k.no \) for some \( k \). Then by the substitution \( \sigma(X) = a^{2^k}.no \) the two monitors are proven not verdict equivalent. Therefore \( m \) must be of the form \( yes + a^+.X + X \) which by the use of our new axioms is proved equal to \( n = yes + X \).

In the case were \( A = B \) we are bound to prove that \( m_a \sim n_a \). Towards contradiction if \( \exists \sigma_0 \) separating the monitors then \( \exists s = a^{k_0} \in L_r(\sigma_0(m_a)), a^{k_0} \not\in L_r(\sigma_0(n_a)) \) (as well as all of its prefixes). This means that \( a^{k_0+1} \in L_r(\sigma_0(m)) \), \( a^{k_0+1} \not\in L_r(\sigma_0(n_a)) \). 

\[ \square \]
\textit{\(\omega\)-verdict equivalence for open terms.} This section presents our axiomatization results for monitors that include variables, over \(\omega\) verdict equivalence. We have already presented the necessary axioms that capture \(\omega\) verdict equivalence over closed terms, as well as the necessary ones to capture equivalence of terms that include variables. We will show here that these two cases are independent and their combination does not produce any new kinds of equalities. This means that besides the equations that will be produced from the axioms of each case combined, there is no other type of equality between these monitors that holds.

\textbf{Theorem 7.} \(\mathcal{E}_\omega' = \mathcal{E}_\omega \cup \mathcal{E}_{fin}\) is complete up to \(\omega\)-verdict equivalence for open terms for a finite \(|\text{Act}| \geq 2\). That is, if \(m \sim n \Rightarrow \mathcal{E}_\omega' \vdash m \sim n\)

\textbf{Proof.} We begin with \(m = \sum_{a \in A} a.m_a + \sum_{i \in I} x_i \ [+\text{yes}] \ [+\text{no}]\) in normal form. We will analyze the following cases of \(m\) and \(n\):

- \(m = \text{yes} + n = \sum_{a \in A} a.n_a + \sum_{j \in J} y_j = n\). First if all note that \(A = \text{Act}\).

  Indeed if \(t \in \text{Act}^\omega \setminus L_a(n_a) \cdot \text{Act}^\omega\) then for substitution \(\sigma\) that maps all variables in \(\{y_j \mid j \in J\}\) to \(b\text{.no}\), \(b \neq a\), we have that \(a.t \in L_a(\text{yes} + \text{no}) \cdot \text{Act}^\omega\) but \(a.t \notin L_a(\sigma(n)) \cdot \text{Act}^\omega\). Moreover, for each \(a \in A\), \(L_a(n_a) \cdot \text{Act}^\omega = L_{\sigma}(n_a) \cdot \text{Act}^\omega = \text{Act}^\omega\). This means that, for each \(a \in \text{Act}\), \(n_a \simeq_\omega \text{yes} + \text{no}\). By induction for each \(a \in \text{Act}\), \(\mathcal{E}_\omega' \vdash n \simeq_\omega \sum_{a \in \text{Act}} a.(\text{yes} + \text{no}),\) from axiom

  \(D_a : \mathcal{E}_\omega' \vdash n \simeq_\omega \sum_{a \in \text{Act}} a.\text{yes} + \sum_{a \in \text{Act}} a.\text{no} + \sum_{j \in J} y_j\) which from out two axioms

  \(Y_\omega, N_\omega\) leads to \(\mathcal{E}_\omega' \vdash n \simeq_\omega \text{yes} + \text{no} + \sum_{j \in J} y_j\) which from \(O_1\) reduces to \(\mathcal{E}_\omega' \vdash n \simeq_\omega \text{yes} + \text{no}\).

- \(m = \text{yes} + n \simeq_\omega \sum_{a \in A} a.n_a + \sum_{j \in J} y_j + \text{yes} = n\), with each \(n_a\) being \text{yes}- and \text{end-free}. As above \(A = \text{Act}\). Moreover for each \(a \in \text{Act}\), \(L_a(n_a) \cdot \text{Act}^\omega = \text{Act}^\omega\). Following the same argument as above only for the \text{no} verdict we arrive at the conclusion that \(\mathcal{E}_\omega' \vdash n \simeq_\omega \text{yes} + \sum_{a \in \text{Act}} a.\text{no} \simeq_\omega \text{yes} + \text{no} = m\).

The case \(m = \text{yes} + \text{no} = \sum_{a \in A} a_n a + \text{yes} = n\) is exactly symmetrical.

- The final case where \(m = \text{yes} + \sum_{a \in A} a.m_a + \sum_{i \in I} x_i \simeq_\omega \sum_{b \in B} b.n_b + \sum_{j \in J} y_j\) in reduced normal forms is as follows. First of all by following the above argument we can arrive to the point where \(\mathcal{E}_\omega' \vdash n \simeq_\omega \text{yes} + \sum_{b \in B'} b.n'_b + \sum_{j \in J} y_j\)

To prove that the two reduced normal forms \(\text{yes} + \sum_{a \in A} a.m_a + \sum_{i \in I} x_i\) and
\[ \text{yes} + \sum_{b \in B} b.n_b + \sum_{j \in J} y_j \] are provably \( \omega \)-verdict equivalent we follow the argument presented below: If they are verdict equivalent as well, then they can be proven as so by the use of the axioms in \( E'_\omega \). We will explore the case where the two monitors are not verdict equivalent. Therefore we start with the assumption that \( \exists \sigma_0 \) substitution and \( \exists s \in L_a(\sigma_0(m)), s \notin L_a(\sigma_0(n)) \). The rejection set follows the same rules. Now since \( s \notin L_a(\sigma_0(n)) \) none of its prefixes are in \( L_a(\sigma_0(n)) \) either. We will prove that for all such \( s \) there is some \( i_0 : s.T^{i_0}(\text{Act}) \in L_a(\sigma_0(n)) \).

At first one can see that using the axioms \( Y_a \) and \( N_a \) a \( k \) number of times, we can prove the family of equations : \( \text{yes} = \sum_{a \in \text{Act}} a. \sum_{a \in \text{Act}} a. \ldots .a.\text{yes} \).

Towards a contradiction assume that there is some \( s \) such that for all \( i \) none of the terms \( s.T^{i}(\text{Act}) \) is in \( L_a(\sigma_0(n)) \). Then for a \( k \) large enough we have that \( |s| + k \geq \text{depth}(\sigma_0(m)) \). Therefore

\[ s.T^{k}(\text{Act}) \cdot \text{Act}^\omega \notin L_a(\sigma_0(n)) \cdot \text{Act}^\omega \]

Since \( s.\text{Act}^\omega \in L_a(\sigma_0(m)) \cdot \text{Act}^\omega \Rightarrow L_a(\sigma_0(m)) \cdot \text{Act}^\omega \neq L_a(\sigma_0(n)) \cdot \text{Act}^\omega \). Which is a contradiction since \( \forall s \in L_a(\sigma_0(m)), s \notin L_a(\sigma_0(n)) \), \( \exists k_k \) such that of the traces \( s.\text{Act}^{k_k} \) are in \( L_a(\sigma_0(n)) \). We can then conclude that for each such \( s \) of \( m \), the relative \( s.T^{k_k}(\text{Act}) \) exists as a summand in \( \sigma_0(n) \) (respectfully for accepting and rejecting verdicts). Then by using our new axioms \( Y_\omega, N_\omega \) we can prove the two monitors \( \omega \) verdict equivalent.

The above analysis can be applied symmetrically for the cases:

- \( m = na + \sum_{a \in A} a.m_a + \sum_{i \in I} x_i = \sum_{b \in B} b.n_b + \sum_{j \in J} y_j \)
- \( m = \sum_{a \in A} a.m_a + \sum_{i \in I} x_i = \sum_{b \in B} b.n_b + \sum_{j \in J} y_j \)

**Note 2.** As expected the proof for the \( \omega \)-verdict equivalence involved the extra axioms added and since we assumed that the monitors and not verdict equivalent we did not use any of the old ones.

\[ \square \]

4.2.2. Nonexistence of a finite axiom set for open terms and finite actions

Observe that the axioms described as the family \( OZ_{s,k} \) even though we have a finite amount of actions available are not finite. This "breaks" our finite axiomatizability claim by itself. The only case where this is not a problem is the case where \( \text{Act} \) is a singleton alphabet \( \{a\} \). When we have at least two actions available things change. Could it be that not all of these equations are necessary? This section is dedicated to proving that no finite subset of these equations is enough to prove them all. Specifically we will show that for an
arbitrary subset of the equations in $O_{2,s,k}$ there is always an infinite number of these families that we cannot prove.

A short intuition behind the proof of the above is the following:

For all of the axioms we have that some variables occur in one or both of its sides. Namely we have two kinds of sound equations. Some were the occurrences of variables are identical and some that they are not. In the later case we have that these occurrences are always accompanied with some closed term of some maximum depth of the equation in both sides. These one sided occurrences of variables happen at arbitrary depths for the family of equations $O_{2,s,k}$. If we assume that some finite subset of them is complete, then we have that there is a way to apply the axioms such that this distance is increasing. Though a case analysis we see that the only way to do so also forces these closed terms that accompany these variables to also occur and therefore it forbids us from proving any of the equations that do not contain any such terms.

The following definition does not apply over all open monitors in $Mon_F$. We have made this distinction in order to keep the case analysis a bit more clear. The case considered is that of monitors that only contain one variable. We will see later on how this is indeed enough for our result. In the following definitions the expression $m \in n, m, n \in Mon_F$ stands for $m$ being a semantic summand of $n$.

**Definition 10.** For an open monitor $m \in Mon_F$ with $m$.

- The **Maximal difference** ($MD(m)$) as:
  \[
  \max \{ |s_1| \mid \exists s_0 : m \xrightarrow{s_0} x \& m \xrightarrow{s_0,s_1} x \& m \notxrightarrow{s_0,s_1} yes + no \}
  \]

For an open equation $e : m = n$ with $m, n \in Mon_F$ in.

- The **Characteristic Variable Difference** ($CVD(e)$) as:
  \[
  |MD(m) - MD(n)|
  \]

For monitors with only one occurrence of variable $x$ we see that $MD(m) = 0$.

Of course we have that $CVD(e) \leq \max\{MD(n), MD(m)\}$ for all equations. We can now prove the first lemma necessary for our claim.

Another useful observation here that will come in handy later is that for all of the equations of the family $O_{s,k}$ their $CVD$ is equal to the length of $s$. For all of the other equations of $E_{fin}$ we have that their $CVD$ is zero.

**Lemma 17.** For each $n \in \mathcal{N}$ there exists a sound equation $e$ of $CVD(e) = n$.

The proof is by example on one of the members of the family $O_{2,s,k}$. Namely take for instance the equation:

\[
  x + a^u.x + \sum_{a \in \text{Act}} a.(yes + no) = x + \sum_{a \in \text{Act}} a.(yes + no).
\]
We can clearly see that the left hand side can perform \( n \) actions and become \( x \), but not \( yes + no \). At the same time both sides have an \( x \) summand. Therefore the \( MD \) of the left hand side is \( n \) and of the right hand side is 0, which means that the \( CVD \) of this equation is \( n \).

**The Axiom Set.** We have from the previous section proved that \( E_{fin} \) is complete for open terms and finite actions. Therefore when describing any finite basis we can assume that this basis is in fact a subset of the equations in \( E_{fin} \). To see this consider any equation that could be involved in an arbitrary axiom set. Since \( E_{fin} \) is complete this equation is derivable form it. In addition since every finite it there is a finite number of axioms of \( E_{fin} \) involved in this proof. Therefore any finite family of equations is derivable from a finite subset of the equations in \( E_{fin} \). From now on then when considering a finite equational basis we will always mean a subset of the equations in \( E_{fin} \).

The main idea is that every proof sequence is either decreasing the maximum \( CVD \) of the involved axioms or that it is creating equations that contain unwanted closed terms. A basic point of this proof is that as long as a closed term is introduced in some open equation, it can no longer be removed. Therefore if a specific has a greater \( CVD \) than the axioms and does not contain any of the closed terms occurring in those it cannot be proven. We will use induction over the size of the proof that results in an arbitrary equation \( e \). The last step of a proof can be:

- Congruence rule for \(+\).
- Congruence rule for action prefixing \( a . \).
- Variable substitution (for an open substitution \( \sigma \) that complies with the existence of only one variable).
- Axiom application on some sub-term of the existing equation.

We will prove that the \( CVD \) function over equations is a upper bounded function when applied to the steps of a sound proof, i.e. that by starting from a sound equation \( e \) and performing any of the above listed options to arrive at an equation \( e' \) we have that \( CVD(e') \leq k \) for some fixed constant \( k \) that only depends on the axioms.

**Lemma 18.** The following congruence rules preserve are only decreasing the \( CVD \) of all involved equations:

- \( CVD(a.m = a.n) = CVD(m = n) \)
- Given equations \( m_1 = n_1 \) and \( m_2 = n_2 \). \( \Rightarrow \) \( CVD(m_1 + m_2 = n_1 + n_2) \leq \max \{ CVD(m_1 = n_1), \ CVD(m_2 = n_2) \} \)
- \( CVD(\sigma(m) = \sigma(n)) = CVD(m = n) \)

**Proof.** They are all pretty standard
• ok - Same occurrences must still be there a bit to the side
• do some basic arithmetic on the size of the term that replaced an occurrence. Must have happened in both sides ech.
• ok - First occurrence of the variable on each branch must happen at the same point. Largest occurrence on each side is either common or occurred in one of the equations that contributed to the new one and was counted for the CVD. (To prove assume that CVD increased and arrive on a contradiction since one variable must have gone away.)

All of the above cases have one thing in common. Since no axioms are involved, all variable occurrences are preserved, either by being more nested in the terms or just being added to the occurrences of some other equation. As we have shown above this type of proof step can only cause the CVD to decrease. However when we apply an axiom more complicated things can happen. As we have axioms that some occurrence of a variable takes place only in one side this could somehow affect the CVD. However we clearly see that for all of the axioms that indeed fall in this case, the ”one-sided” occurrence of the variable is accompanied with a very specific closed term. Namely in order to apply our axioms this specific closed term needs to be a syntactic summand of some sub-term of our equation. This leads us to the proof of the final universal result.

**Theorem 8.** For any finite subset of $E$ of maximum $CVD$, $k$ we have that $E \not\vdash Os, 1$ for any $s$ of length bigger than $k$.

**Proof.** We will use induction in the number of steps of the proof. We have that first of all none of the equations of $E_{fin}$ of $CVD$ larger than $k$ are in $E$. Consequently no proof of size 1 can prove one of them. Assume now that the stated result holds for any proof of number of steps up to $n$, i.e. that no proof of $n$ steps can give us one of the stated equations. We show that no proof of $n + 1$ steps can prove them either. We will call the already proved equation (after $n$ proof steps $e$ and the result one $e'$). First of all we have that if the final step of the proof is any congruence rule or variable substitution, then by Lemma [15] the resulting equation will have $CVD \leq k$. Since all of the target equations have $CVD$ of $k + 1$ we have the result.

What’s left is to examine the case of axiom application as a last step of the proof. First of all we see that any of the axioms in $E_v$ contain the same variable occurrences in both sides. Therefore if the application of one of those is the final step of the proof, then the $CVD$ is preserved. The axioms left are $O1$ and the ones of the family $O_{s,k}$ of $CVD$ up to $k$.

Assume then that we apply one of these axioms to create (or remove) a variable occurrence. First of all we see that the application of $O1$ from left to right or from right to left does not interfere with the $CVD$, as the variable that is either added or removed coincides with the occurrence of a yes + no term and therefore not counted for the $MD$ of the relevant side of $e'$. Again we see that
the \( CVD \) is preserved and it cannot be that we proved an equation of \( CVD \) larger than \( k \). As a final case we have now the application of an axiom of the family \( O_{s,k} \) of bounded \( CVD \). Lets call the axiom we are applying \( O \). Since we had a syntactic sub-term of the equation equal to one of the sides of our axiom \( O \), immediately we know that the closed terms appearing in the axiom also appear in the this equation. However we can clearly see that the closed terms involved in the finite subset of axioms we are working with, do not occur also in an arbitrary member of the \( O2_{s,k} \) that we are trying to prove. Namely for all of the axioms of \( \mathcal{E} \) we are working with, we have that after \( k \) actions the monitor (both sides of the axioms) can arrive at a \( yes + no \) for all but one trace. This of course is not the case for an equation of the family \( O_{s,k} \) with \( CVD \ k + 1 \) or larger.

Therefore we might somehow have increased the \( CVD \) of the current equation but in the cost of also having specific closed terms occurring inside it along with the variables. As we can see form lemma (insert lemma here) these terms cannot be removed from the equation now. This means that there is no way from this current equation to prove one of the axioms that where not included in \( \mathcal{E} \).

The proof of theorem 5 follows from the above.

5. Complexity for Normalization and Equality Testing

An arbitrary monitor of the syntax we are given will look like a rooted finite tree. The root node represents the monitor we are given. Each child node of any node contains a label. That label can be a verdict if the node is a leaf node or an action from \( \text{Act} \) if it is not. All leaves are labeled with verdicts. Under the same node there might be many copies of the same label, both verdicts and actions. For example the monitor described by the term

\[
m = a.(a.yes + b.c.yes) + b.(a.yes + c.end) + a.c.no + a.b.yes
\]

can be though of as the following tree:

```
   m
  / \   /
 a   b /   /
 a   c /   /
 a   d   /
 yes yes
```

Figure 1: A representation of a regular monitor (not in normal form)

The LTS for that monitor would be figure 2.

We will choose the representation of a graph by a slightly modified adjacency list as shown in figure 3. The monitor represented by that data structure is

\[
m = a.(yes + no) + b.yes + no + end.
\]
The first node of each line is called a header node and does not contain any labels. It represents a sub-monitor of the original one. It does contain two empty slots which we will utilize later to deal with the verdicts. Those slots are called headers. From each header node starts a list of nodes we call children nodes.

A children node has one label annotating the action that the parent node can perform. This label can also be a verdict. It also contains a link to another header node that is the representation of the sub-monitor that follows that label. When the label is a verdict that pointer is null. Therefore each sub-tree is represented by its root as a header node on the list and the list starting from a node has labels to indicate with which action the header monitor goes to each sub-monitor. The number of actions (and the three verdicts) is represented as $|Act| = k$.

5.1. Determining a normal form

Minimizing the actions. This partial normal form is a kind of determinization of the input monitor. It is not the complete reduced normal form. The focus
here is to remove duplicate actions under the same node without affecting the verdicts sets of the original monitor.

**PartialNF**(m)

**Result:** The adjacency list of the partial normal form of the input monitor.

Start at root node(id);
Keep a set A of size k;

**while** Exists next child do
  Get next child (u);
  Read the label (a);
  if a \(\not\in\) A then
    add a to A
  else
    if a is not a verdict then
      Find earlier child with same label (v);
      Join(u, v);
    end
    Delete u;
  end
end

Start at the first Child;

**while** Exists next child (u) do
  Get the sub-monitor of u (m₁);
  \(m₁ = \text{PartialNF}(m₁)\)
end

**Algorithm 1:** Partial Normal Form

*Correctness.*: Yes

*Complexity Analysis.*: For each list node we have two possible scenarios. Either its label is added to the set A or a *join* operation is performed that results in its deletion.

The *join*(u,v) operation takes place as follows: We extend the list of children of u with the list of children of v. By having each head node knowing what its first and last children are we can reduce the cost of this operation to \(O(1)\) time since we are just switching were some pointers in the adjacency list point to. Instead of being a separate sub-monitor now one of the heads in the list becomes part of the list of another header. The complexity does not depend on the size of the lists. The cost of deleting duplicates is counted in the number of possible *join* operations. Assuming that each check that a label has been seen before in the set of children takes \(\log(k)\) time by using some efficient heap data structure and the deletion of it happens in constant time the overall running time of this part for each time PartialNF is called is \(O(\log(k))\).

Since we are already utilizing a heap data structure we can request the second while loop to take place with respect to some known order regarding the labels. Assume therefore that all of the verdicts occur first in the list of the
children of the node and the rest of the labels follows in an ordered format. This assumption causes an increment in the overall running time to $O(n \cdot k \cdot \log k)$ since for each header node we are constructing a shorted list of $k + 3$ elements (in the worst case).

The recursive function PartialNF will be called at most one time per node of the adjacency list (i.e. the size of the list denoted $n$) and each time a $k \cdot \log(k)$ steps are performed. Therefore we have an overall running time of $O(n \cdot k \cdot \log(k))$.

The resulting monitor would now look like:

```
  m
   / \  \
 a   b
   \ /  \
 c   c
```

The example monitor in partial normal form

**Determining the minimal Verdicts.** The main question regarding minimality of verdicts that we have to ask ourselves is whether a branch contributes anything to $L_a(m)$ or $L_c(m)$. Since we have removed ambiguity regarding the actions the only "redundant" information that remains in the monitor are branches that do not contribute to our verdicts sets. Those branches are exactly those that stop at an end verdict and those that end with a verdict already encountered earlier in the branch.

We will now utilize the header slots of the root nodes of each subtree. Those headers will be filled with the verdicts that have been encountered earlier on. The input monitor is the output of the previous algorithm and therefore each label occurs once at most as a child of a node. In addition verdicts appear first
Algorithm 2: Verdicts Normal Form
After running this procedure the example would look like:

![Diagram of a regular monitor in normal form]

Figure 5: A regular monitor in normal form

**Correctness:** This doesn’t seem necessary but I could argue about it.

**Complexity Analysis:** Each time we call the Reduced Verdicts function we perform some constant number of steps to check the situation regarding verdicts and headers. This comes as a result of the verdicts appearing first in the list of the children of a node. Each node is examined once and for its examination we need a constant amount of time. Therefore for a monitor of \( n \) nodes we have that:

\[
O(\text{verdictsNF}(n)) = O(1) + O(\text{verdictsNF}(n - 1)) \Rightarrow O(n)
\]

5.2. **Equality of normal Forms**

Since the normalization algorithm also makes sure that the labels on the Reduced list of a header are given under some sorting agreement. We could also run this sorting after the normalization takes place with the same cost of the **PartialNF** algorithm.

Equality testing for two adjacency lists representing trees is linear in the size of the lists.

6. **Conclusion**

In this study we have shown, that the hole family of monitors we considered is finitely axiomatizable in the case we have finitely many actions. In case were \( \text{Act} \) is infinite we have still determined the axioms needed to prove all valid equations over \( \text{Mon}_F \) but they are infinitely many. In addition since all of the axioms are sound equations in \( \text{Mon}_F \) we can deduce that there is no finite equational basis for verdict and \( \omega \) verdict equivalence when \( \text{Act} \) is infinite. The syntax taken into account in this work can me expanded first of all with recursion and then with other auxiliary operators. A resulting grammar would still be in need of analysis in order to determine similar results.

Another interesting question in equational logic would be the decidability properties of equational theories, and the characterization of the computational complexity of decidable theories. It is an interesting question to determine whether the equational theory of verdict equivalence is decidable over \( \text{Mon}_F \) and, if so, to find out what its structural complexity is.
References

[1] J. C. Baeten, T. Basten, M. A. Reniers, Process Algebra: Equational Theories of Communicating Processes, Vol. 50 of Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 2009.

[2] R. J. van Glabbeek, The linear time – branching time spectrum I: The semantics of concrete, sequential processes, in: J. A. Bergstra, A. Ponse, S. A. Smolka (Eds.), Handbook of Process Algebra, Elsevier, 2001, Ch. 1, pp. 3–99.

[3] M. Hennessy, R. Milner, Algebraic laws for nondeterminism and concurrency, Journal of the ACM 32 (1) (1985) 137–161. URL https://doi.org/10.1145/2455.2460

[4] R. Milner, Communication and Concurrency, Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1989.

[5] J. C. M. Baeten, J. A. Bergstra, Process algebra with a zero object in: Baeten and Klop [6], pp. 83–98. doi:10.1007/BFb0039045. URL https://doi.org/10.1007/BFb0039053

[6] J. C. M. Baeten, J. W. Klop (Eds.), CONCUR ‘90, Theories of Concurrency: Unification and Extension, Vol. 458 of Lecture Notes in Computer Science, Springer, 1990. doi:10.1007/BFb0039045. URL https://doi.org/10.1007/BFb0039045

[7] A. Lazrek, P. Lescanne, J. Thiel, Tools for proving inductive equalities, relative completeness, and ω-completeness, Inf. Comput. 84 (1) (1990) 47–70.

[8] L. Aceto, A. Achilleos, A. Francalanza, A. Ingólfsdóttir, A framework for parameterized monitorability in: C. Baier, U. Dal Lago (Eds.), Foundations of Software Science and Computation Structures - 21st International Conference, FOSSACS 2018, Vol. 10803 of Lecture Notes in Computer Science, Springer, 2018, pp. 203–220. URL https://doi.org/10.1007/978-3-319-89366-2_11

[9] L. Aceto, A. Achilleos, A. Francalanza, A. Ingólfsdóttir, K. Lehtinen, Adventures in monitorability: From branching to linear time and back again Proceedings of the ACM on Programming Languages 3 (POPL) (2019) 52:1–52:29. URL https://dl.acm.org/citation.cfm?id=3290365

[10] A. Francalanza, L. Aceto, A. Ingolfsdottir, Monitorability for the Hennessy–Milner Logic with recursion Formal Methods in System Design 51 (1) (2017) 87–116. URL http://dx.doi.org/10.1007/s10703-017-0273-z
[11] E. Bartocci, Y. Falcone (Eds.), *Lectures on Runtime Verification - Introductory and Advanced Topics* Vol. 10457 of Lecture Notes in Computer Science, Springer, 2018. URL https://doi.org/10.1007/978-3-319-75632-8