Approximating the Spectral Gap of the
Pólya-Gamma Gibbs Sampler

Bryant Davis
Department of Statistics
University of Florida
davibf11@ufl.edu

James P. Hobert
Department of Statistics
University of Florida
jhobert@stat.ufl.edu

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Abstract

The self-adjoint, positive Markov operator defined by the Pólya-Gamma Gibbs sampler (under a proper normal prior) is shown to be trace-class, which implies that all non-zero elements of its spectrum are eigenvalues. Consequently, the spectral gap is $1 - \lambda^*$, where $\lambda^* \in [0, 1)$ is the second largest eigenvalue. A method of constructing an asymptotically valid confidence interval for an upper bound on $\lambda^*$ is developed by adapting the classical Monte Carlo technique of Qin et al. (2019) to the Pólya-Gamma Gibbs sampler. The results are illustrated using the German credit data. It is also shown that, in general, uniform ergodicity does not imply the trace-class property, nor does the trace-class property imply uniform ergodicity.

1 Introduction

Let $Y_1, \ldots, Y_n$ be independent Bernoulli random variables such that $P(Y_i = 1) = F(x_i^T \beta)$ where $x_i$ is a $p \times 1$ vector of known covariates associated with $Y_i$, $\beta$ is an unknown $p \times 1$ vector of regression parameters, and $F(x) = e^x/(1 + e^x)$, which is the standard logistic distribution function. Under this model, the joint mass function of $Y_1, \ldots, Y_n$ is

$$
\prod_{i=1}^n P(Y_i = y_i | \beta) = \prod_{i=1}^n [F(x_i^T \beta)]^{y_i} [1 - F(x_i^T \beta)]^{1-y_i} I_{\{0,1\}}(y_i).
$$

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Let $y$ denote the $n \times 1$ vector of observed $y_i$ values. We consider a Bayesian analysis of this data under a proper prior for the unknown vector $\beta$. In particular, we take the prior on $\beta$ to be $N_p(b, B)$, where $b \in \mathbb{R}^p$ and $B$ is a $p \times p$ positive definite matrix. The resulting posterior density, $\pi(\beta | y)$, is highly intractable, but there is a simple MCMC algorithm that can be used to explore it. This algorithm was developed by Polson et al. (2013), and is often called the Pólya-Gamma Gibbs sampler. In order to describe the algorithm, we must introduce some notation.

Let $X$ be the $n \times p$ matrix whose $i$th row is $x_i^T$, and define $\mathbb{R}_+ = (0, \infty)$. For $w \in \mathbb{R}_+^n$, let $\Omega(w) = \text{diag}\{w_i\}_{i=1}^n$. We then define
\[
\Sigma(w) = (X^T \Omega(w) X + B^{-1})^{-1},
\]
as well as
\[
\mu(w) = \Sigma(w) \left[ X^T \left( y - \frac{1}{2} 1_n \right) + B^{-1} b \right],
\]
where $1_n$ is an $n \times 1$ vector of 1s. When we write $Z \sim \text{PG}(1, d)$, we mean that the random variable $Z$ follows a Pólya-Gamma distribution with probability density function (pdf) given by
\[
f(z; d) = \cosh(d/2)e^{-d^2z/2}g(z),
\]
where $d \geq 0$, and
\[
g(z) = \sum_{k=0}^{\infty} (-1)^k \frac{2k + 1}{\sqrt{2\pi z^3}} \exp \left\{ -\frac{(2k + 1)^2}{8z} \right\} I_{(0, \infty)}(z).
\]
The function $g(z)$ is a pdf (Biane et al., 2001), and, in particular, it is the pdf of a PG(1,0) random variable (for more on the Pólya-Gamma distribution, including highly efficient methods for simulating from it, see Polson et al. (2013) and Windle et al. (2014)). The Pólya-Gamma (PG) Gibbs sampler simulates a Markov chain $\Gamma = \{\beta(m)\}_{m=0}^\infty$ using the following two-step procedure to move from the current state, $\beta(m)$, to the new state, $\beta(m+1)$.

**Algorithm 1:** Iteration $m + 1$ of the PG Gibbs sampler.

1. Draw $W_1, W_2, \ldots, W_n$ independently with
\[
W_i \sim \text{PG}(1, |x_i^T \beta(m)|),
\]
and call the observed vector $w = (w_1, \ldots, w_n)^T$.

2. Draw $\beta^{(m+1)} \sim N_p(\mu(w), \Sigma(w))$. 

The Markov chain $\Gamma$ is irreducible, aperiodic and positive Harris recurrent, and the posterior density, $\pi(\beta \mid y)$, is its unique invariant density (see, e.g., Choi and Hobert, 2013). In this paper, we study the self-adjoint, positive Markov operator associated with $\Gamma$, call it $K$. Our main theoretical result is that this operator is trace-class (for background on the spectral theory of linear operators, see, e.g., Helmberg (2014) and Conway (1990), and for Markov operators in particular, see Mira and Geyer (1999) and Qin et al. (2019)). The trace-class property implies that all non-zero elements of the spectrum of $K$ are eigenvalues, so the spectral gap of $K$, which controls the rate at which $\Gamma$ converges to $\pi(\beta \mid y)$, is equal to $1 - \lambda_\ast$, where $\lambda_\ast \in [0, 1)$ is the second largest eigenvalue. Furthermore, the trace-class property also implies that the sum of the eigenvalues is finite, and this fact allows us to use the results of Qin et al. (2019) to develop a method for constructing an asymptotically valid confidence interval for an upper bound on $\lambda_\ast$. Our point estimator of the upper bound on $\lambda_\ast$ is based on iid random vectors that can be simulated using non-stationary, short runs of the Markov chain $\Gamma$. We illustrate the application of our method using a medium-sized data set concerning 1,000 applications for credit in Germany, the so-called German credit data.

Previous work on the theoretical properties of PG Gibbs samplers includes Choi and Hobert (2013), Choi and Roman (2017), and Wang and Roy (2018b). Choi and Hobert (2013) analyzed the PG Gibbs sampler under the same normal prior on $\beta$ and showed that it is uniformly ergodic. It is well-known that the trace-class property implies geometric ergodicity (see Section 2), and that uniform ergodicity implies geometric ergodicity; however, prior to our work, the relationship between uniform ergodicity and the trace-class property (for ergodic Markov chains with self-adjoint and positive Markov operators) was unknown. In Section 5, we show that, in general, uniform ergodicity does not imply the trace-class property, nor does the trace-class property imply uniform ergodicity. Choi and Roman (2017) and Wang and Roy (2018b) analyzed a slightly different PG Gibbs sampler. In particular, instead of the proper normal prior on $\beta$ that we consider here, these authors used a flat improper prior. Wang and Roy (2018b) showed that this version of the PG Gibbs sampler is geometrically ergodic, and Choi and Roman (2017) showed that, if each row of the $X$ matrix is a unit vector, then the corresponding Markov operator is trace-class. We note that when the rows of $X$ are unit vectors (which corresponds to a one-way ANOVA design) and the prior on $\beta$ is flat, MCMC is actually not required because the $p$ (univariate) components of $\beta$ are a posteriori independent. Indeed, the posterior density factors into a constant times a product of $p$ terms, each having the form $[F(\beta_i)]^{a_i} [1 - F(\beta_i)]^{b_i}$, where $a_i$ and $b_i$ are non-negative integers. Hence, one could presumably design a univariate accept-reject algorithm that would yield exact draws from the
The remainder of the paper is organized as follows. Section 2 describes some theoretical properties of the PG Gibbs sampler, and contains a statement of our main theoretical result. Our implementation of Qin et al.’s (2019) algorithm in the context of the PG Gibbs sampler is described in Section 3. In Section 4, we use our results to approximate the spectral gap of the PG Gibbs sampler for the German credit data. The relationship between uniform ergodicity and the trace class property is the topic of Section 5. Two proofs are relegated to an Appendix.

2 Theoretical Properties of the PG Gibbs Sampler

The Markov transition density (Mtd) of the PG Gibbs sampler can be expressed as

\[ k(\beta, \beta') = \int_{\mathbb{R}_+^p} \pi_2(\beta' | w, y) \pi_1(w | \beta, y) \, dw , \]

where, using notation from the Introduction, \( \pi_1(w | \beta, y) = \prod_{i=1}^n f(w_i; |x_i^T \beta|) \), and \( \pi_2 \) is a \( p \)-variate normal density with mean \( \mu(w) \) and variance \( \Sigma(w) \). Of course, \( \pi_1 \) and \( \pi_2 \) are the conditional pdfs corresponding to an augmented posterior density, \( \pi_a(\beta, w | y) \), which satisfies \( \int_{\mathbb{R}_+^p} \pi_a(\beta, w | y) \, dw = \pi(\beta | y) \). (See Choi and Hobert (2013) for the exact form of \( \pi_a(\beta, w | y) \).) We can also define the conjugate Markov chain, \( \tilde{\Gamma} = \{w^{(m)}\}_{m=0}^\infty \), which lives on \( \mathbb{R}_+^n \), and has Mtd given by

\[ \tilde{k}(w, w') = \int_{\mathbb{R}_+^p} \pi_1(w' | \beta, y) \pi_2(\beta | w, y) \, d\beta . \]

The chain \( \tilde{\Gamma} \) is also irreducible, aperiodic and positive Harris recurrent, and its unique invariant density is \( \int_{\mathbb{R}_+^p} \pi_a(\beta, w | y) \, d\beta \). We will make use of \( \tilde{\Gamma} \) in the sequel.

We now turn to the convergence properties of the PG Gibbs sampler. Let \( L^2(\pi) \) denote the Hilbert space of complex valued functions that are square integrable with respect to the target posterior density, \( \pi(\beta | y) \), i.e.,

\[ L^2(\pi) := \left\{ f : \mathbb{R}^p \to \mathbb{C} \ \bigg| \ \int_{\mathbb{R}_+^p} |f(\beta)|^2 \pi(\beta | y) \, d\beta < \infty \right\} . \]

The inner product of \( f, g \in L^2(\pi) \) is given by

\[ \langle f, g \rangle_\pi = \int_{\mathbb{R}_+^p} f(\beta) \overline{g(\beta)} \pi(\beta | y) \, d\beta . \]

The Mtd \( k(\beta, \beta') \) defines a linear (Markov) operator \( K : L^2(\pi) \to L^2(\pi) \) such that if \( f \in L^2(\pi) \), then

\[ Kf(\beta) = \int_{\mathbb{R}_+^p} k(\beta, \beta') f(\beta') \, d\beta' . \]
The Markov chain $\Gamma$ is the $\beta$-marginal of the two-block Gibbs chain (based on $\pi$) that alternates between $\beta$ and $w$. It follows that $K$ is positive (and self-adjoint) (Liu et al., 1994). Our main theoretical result, which is proven in Appendix A, shows that $K$ enjoys additional regularity.

**Proposition 1.** The Markov operator $K$ is trace-class.

The fact that $K$ is positive (and self-adjoint) implies that its spectrum is contained in $[0, 1]$. The trace-class property implies that $K$ is a compact operator, so it possesses a pure eigenvalue spectrum with (at most) a countable number of eigenvalues. Let $\{\lambda_i\}_{i=0}^\kappa$ denote the strictly positive eigenvalues of $K$ in decreasing order, where $0 \leq \kappa \leq \infty$. We know that $\lambda_0 = 1$ and, since there cannot be two eigenvalues equal to 1 (because of irreducibility), we have $\lambda_1 \in [0, 1)$. (In what follows, we use the symbols $\lambda_1$ and $\lambda_*$ interchangeably.) Hence, the spectral gap is $1 - \lambda_* > 0$, so $\Gamma$ is geometrically ergodic and $\lambda_*$ can be viewed as its rate of convergence (Roberts and Rosenthal, 1997). Indeed, let $\Pi$ denote the probability measure defined by $\pi(\beta \mid y)$, and let $\nu$ be any probability measure that is absolutely continuous with respect to $\Pi$ and satisfies $\int_{\mathbb{R}} (d\nu/d\Pi)^2 d\Pi < \infty$. Then there exists a constant $M_\nu < \infty$ such that

$$d_{\text{TV}}(\nu K^m, \Pi) \leq M_\nu \lambda_*^m,$$

where $\nu K^m$ denotes the probability measure of $\beta^{(m)}$ when $\beta^{(0)} \sim \nu$, and $d_{\text{TV}}(\cdot, \cdot)$ is the total variation distance. (As usual, the symbol $K$ is doing double duty, representing both the Markov operator and the Markov transition kernel.)

The trace-class property also implies that the eigenvalues of $K$ are summable, and this means that we can employ the method of Qin et al. (2019) (hereafter QH&K) to estimate $\lambda_*$. The details are presented in the next section.

### 3 Approximating $\lambda_*$

We begin with a high-level description of QH&K’s method. As stated above, $\sum_{i=0}^\kappa \lambda_i < \infty$ is a result of the trace-class property of $K$, and it follows that, if $l$ is any strictly positive integer, then $s_l := \sum_{i=0}^\kappa \lambda_i^l < \infty$ as well. QH&K develop a classical Monte Carlo estimator of $s_l$. Moreover, they show that $u_l := (s_l - 1)^{1/l} \downarrow \lambda_*$ as $l \to \infty$. Thus, for fixed $l$, the estimator of $s_l$ can be converted directly into an estimator of an upper bound on $\lambda_*$. QH&K provide guidance on the choice of $l$, as well as sufficient conditions under which their Monte Carlo estimator of $s_l$ has finite variance, which allows one to calculate a standard error for $u_l$ via the delta method.
In order to use QH&K’s method, an auxiliary density, \( h : \mathbb{R}^p \rightarrow (0, \infty) \), must be specified. This density plays a role similar to that of the importance density in an importance sampling algorithm. The classical Monte Carlo (unbiased) estimator of \( s_l \) is given by

\[
\hat{s}_l = \frac{1}{N} \sum_{i=1}^{N} \frac{\pi(\beta^*_i | w^*_i)}{h(\beta^*_i)},
\]

(1)

where \( N \) is the Monte Carlo sample size, and the random vectors \( \{(\beta^*_i, w^*_i)\}_{i=1}^{N} \) are iid and generated according to Algorithm 2.

**Algorithm 2: Drawing (\( \beta^*, w^* \))**

1. Draw \( \beta^* \sim h(\cdot) \).

2. Draw \( W_1, W_2, \ldots, W_n \) independently with

\[
W_i \sim PG(1, |x_i^T\beta^*|),
\]

and call the observed vector \( w = (w_1, w_2, \ldots, w_n)^T \).

3. If \( l = 1 \), set \( w^* = w \). If \( l \geq 2 \), draw \( w^* \sim \tilde{k}(l-1)(\cdot | w) \) by running \( l - 1 \) iterations of the conjugate chain \( \tilde{\Gamma} \) initiated at \( w \).

The estimator (1) is strongly consistent no matter what \( h \) is used. However, additional conditions are required to guarantee finite variance, which ensures the existence of asymptotically valid confidence intervals for \( s_l \) and \( u_l \). In particular, QH&K show that the following condition is sufficient for the estimator (1) to have finite variance:

\[
\int_{\mathbb{R}_+^n} \int_{\mathbb{R}^p} \frac{\pi(w | \beta, y) \pi^3(\beta | w, y)}{h^2(\beta)} \, d\beta \, dw < \infty.
\]

The next result, which is proven in Appendix B, shows that this condition is satisfied if \( h \) is taken to be a multivariate Student’s \( t \) density.

**Proposition 2.** Let \( h_\nu(\beta; d, C) \) denote a \( p \)-dimensional Student’s \( t \) density with location parameter \( d \), positive definite scale matrix \( C \), and degrees of freedom \( \nu \). Then

\[
\int_{\mathbb{R}_+^n} \int_{\mathbb{R}^p} \frac{\pi(w | \beta, y) \pi^3(\beta | w, y)}{h^2_\nu(\beta; d, C)} \, d\beta \, dw < \infty.
\]

**Remark 3.** QH&K actually present two different Monte Carlo estimators for \( s_l \) - one is more effective for data sets in which \( n \) is larger than \( p \), while the other tends to work better when \( p \) is larger than \( n \). In this paper, we consider only the former.
In the next section, we use the results above to approximate the spectral gap of the PG Gibbs sampler for a medium-sized real data set.

4 An Application: The German Credit Data

In this section, we apply our method to the so-called German credit data, which are available here: http://archive.ics.uci.edu/ml/index.php. In this data set, there are \( n = 1000 \) binary observations, each one representing the success or failure of a particular loan application. (700 of the 1000 applications were deemed creditworthy.) Associated with each of the 1,000 observations are twenty covariates including loan purpose, demographic information, bank account balances, marital status, and employment status. Seven of the covariates are quantitative and thirteen are categorical. After converting the categorical covariates to indicators, there is a total of \( p = 49 \) regression coefficients. This data set is frequently used to illustrate newly developed statistical and machine learning techniques for binary data (see, e.g., Polson et al., 2013; Jacob et al., 2019).

Our Bayesian model contains the hyper-parameters \( b \) and \( B \), which must be specified. Following Jacob et al. (2019), we place a relatively uninformative prior on \( \beta \) with \( b = 0 \) and \( B = 10I_{49} \). In order to use Algorithm 2, the parameters of the Student’s \( t \) density must be specified, and we used a preliminary run of the PG Gibbs sampler to choose \( d \) and \( C \). In particular, we ran the sampler for 25,000 iterations (with \( \beta^{(0)} \) set equal to the MLE of \( \beta \) based on the frequentist version of our logistic regression model), discarded the first 5,000 draws as burn-in, and then used the remaining 20,000 draws to get estimates (the usual ergodic averages) of the posterior mean and covariance matrix, call these \( \hat{\beta} \) and \( \hat{\Sigma} \). We then set \( d = \hat{\beta} \), \( C = \hat{\Sigma} \) and \( \nu = 5 \). Based on the guidance given in QH&K, and some initial experimentation, it became clear that \( l = 5 \) was a reasonable choice. We utilized a Monte Carlo sample size of \( N = 10^7 \). The simulations yielded \( \hat{u}_5 = 0.787 \) with a standard error of 0.074, resulting in an asymptotically valid 95% confidence interval of (0.639, 0.935) for \( u_5 \). Hence, we can be fairly confident that the unknown spectral gap \( 1 - \lambda_* \) is at least 0.065.

The next section concerns the relationship between uniform ergodicity and the trace-class property.

5 Uniform Ergodicity Versus the Trace-class Property

Consider an ergodic Markov chain whose Markov operator is self-adjoint and positive. As we explained in Section 2 if this Markov chain is trace-class, then it is also geomeetrically ergodic. A
natural question that arose during our study of the Pólya-Gamma Gibbs sampler is this: Is there a similar relationship between uniform ergodicity and the trace-class property? In this section, we show that the answer is “no.” In particular, we show that there exist chains that are uniformly ergodic, but not trace-class, as well as chains that are trace-class, but not uniformly ergodic.

Suppose we wish to sample from the univariate target density

$$\pi(x) = \frac{2}{3}(x + 1)I_{(0,1)}(x).$$

Consider an independence Metropolis algorithm based on a Uniform(0,1) candidate. The corresponding Markov operator is clearly self-adjoint, and Lemma 3.1 of Rudolf and Ullrich (2013) implies that it is also positive. Now letting $q(y)$ denote the Uniform(0,1) candidate density, we can observe that for all $y \in (0,1)$, we have

$$\frac{q(y)}{\pi(y)} = \frac{3}{2(x + 1)} \geq \frac{3}{4}.$$ 

Hence, Theorem 2.1 of Mengersen and Tweedie (1996) implies that the chain is uniformly ergodic. However, the operator cannot be compact (see Chan and Geyer, 1994, p. 1755), and thus cannot be trace-class. Hence, we have a uniformly ergodic chain that is not trace-class.

For the other direction, we turn to a collection of birth-death Markov chains that was analyzed in Tan et al. (2013). Let $\{a_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$ be two sequences of strictly positive real numbers satisfying

$$\sum_{i=1}^\infty a_i + \sum_{i=1}^\infty b_i = 1.$$

Additionally, define $b_0 = 0$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$. We use the two sequences to define a bivariate random vector $(X, Y)$ with support $\mathbb{N} \times \mathbb{N}$ and probability mass function given by

$$\pi(x, y) = \begin{cases} a_x & x = y, y = 1, 2, 3, \ldots \\ b_y & x = y + 1, y = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}.$$ 

The marginal mass functions are $\pi_X(x) = a_x + b_{x-1}$ and $\pi_Y(y) = a_y + b_y$, and the conditional mass functions are given by

$$\pi_{X \mid Y}(x \mid y) = \frac{a_y}{a_y + b_y}I(x = y) + \frac{b_y}{a_y + b_y}I(x = y + 1)$$

for $y \in \mathbb{N}$, and

$$\pi_{Y \mid X}(y \mid x) = \frac{a_x}{a_x + b_{x-1}}I(y = x) + \frac{b_{x-1}}{a_x + b_{x-1}}I(y = x - 1).$$
for \( x \in \mathbb{N} \). Let \( \Gamma = \{X_n\}_{n=0}^{\infty} \) be the Markov chain on \( \mathbb{N} \) with Markov transition probabilities given by

\[
P(X_{n+1} = x' | X_n = x) = k(x, x') = \sum_{y \in \mathbb{N}} \pi_X | Y(x') | y \pi_Y | X(y | x)
\]

Because this Markov chain is the marginal of a two component Gibbs sampler, the corresponding operator is necessarily self-adjoint and positive (Liu et al., 1994). As we now explain, \( \Gamma \) turns out to be a birth-death chain. For \( x \in \mathbb{N} \), define

\[
p_x = \frac{a_x b_x}{(a_x + b_{x-1})(a_x + b_x)},
\]

and, for \( x \in \{2, 3, 4, \ldots \} \), define

\[
q_x = \frac{a_{x-1} b_{x-1}}{(a_x + b_{x-1})(a_{x-1} + b_{x-1})}.
\]

Finally, for \( x \in \{2, 3, 4, \ldots \} \), define \( r_x = 1 - p_x - q_x \). Using this notation, we can express \( k \) as follows:

\[
k(x, x') = \begin{cases} 
1 - p_1 & x' = x = 1 \\
p_x & x' = x + 1, \ x = 1, 2, 3, \ldots \\
q_x & x' = x - 1, \ x = 2, 3, 4, \ldots \\
r_x & x' = x, \ x = 2, 3, 4, \ldots \\
0 & \text{otherwise}.
\end{cases}
\]

Hence, at each step, the chain either goes up one unit, down one unit, or stays at the current state. Now, according to Theorem 2 in Qin et al. (2019), \( \Gamma \) is trace-class if and only if

\[
\sum_{x=1}^{\infty} k(x, x) = 1 - p_1 + \sum_{x=2}^{\infty} r_x < \infty.
\]

The following representation of \( r_x \) will be used below:

\[
r_x = \frac{a_x}{(a_x + b_{x-1})(a_x + b_x)} + \frac{b_x^2}{(a_x + b_{x-1})(a_{x-1} + b_{x-1})} + \frac{1}{(1 + \frac{b_{x-1}}{a_x})(1 + \frac{b_x}{a_x})} + \frac{1}{(1 + \frac{a_x}{b_{x-1}})(1 + \frac{a_{x-1}}{b_x})}.
\]

We now consider specific versions of the sequences \( \{a_i\}_{i=1}^{\infty} \) and \( \{b_i\}_{i=1}^{\infty} \), which are similar to the ones employed in Example 2.5 of Ha (2016). For \( x \in \mathbb{N} \), take \( a_x' = (2x - 1)^{-(4x-2)} \) and \( b_x' = (2x)^{-4x} \). Let \( c = \sum_{i=1}^{\infty} a_i' + \sum_{i=1}^{\infty} b_i' \), and define \( a_x^* = a_x' / c \) and \( b_x^* = b_x' / c \). Then, as required, we have \( \sum_{i=1}^{\infty} a_i^* + \sum_{i=1}^{\infty} b_i^* = 1 \). For \( x \in \{2, 3, 4, \ldots \} \), we have

\[
\frac{b_x^*}{a_x^*} = \frac{(2x - 1)^{4x-2}}{(2x)^{4x}} = \left[ \frac{2x - 1}{2x} \right]^{4x-2} \left( \frac{1}{(2x)^2} \right) \leq \frac{1}{4x^2}.
\]
and
\[
\frac{a_{x+1}^*}{b_x^*} = \frac{(2x)^{4x}}{(2(x+1) - 1)^{4(x+1)-2}} = \left[ \frac{2x}{2x+1} \right]^{4x} \frac{1}{(2x+1)^2} \leq \frac{1}{4x^2}.
\]

Hence, for \( x \in \{2, 3, 4, \ldots \} \), we see that
\[
\begin{align*}
\sum \frac{1}{4(x-1)^2} + \frac{1}{4(x-1)^2} &= \frac{1}{2(x-1)^2}.
\end{align*}
\]

It follows that \( \sum_{x=2}^{\infty} r_x \leq \frac{1}{2} \sum_{x=2}^{\infty} \frac{1}{(x-1)^2} < \infty \), so this particular version of \( \Gamma \) is, in fact, trace-class. However, Proposition 2.6 of Ha (2016) shows that no version of \( \Gamma \) is uniformly ergodic. Hence, we have a trace-class chain that is not uniformly ergodic.

Appendix

A Proof of Proposition 1

Proof. By Theorem 2 of QH&K, it suffices to establish that
\[
\int_{\mathbb{R}^p} k(\beta, \beta) \, d\beta < \infty.
\]

Fubini’s Theorem implies that
\[
\int_{\mathbb{R}^p} k(\beta, \beta) \, d\beta = \int_{\mathbb{R}^p}^{\int_{\mathbb{R}^p}} \pi(\beta | w, y) \pi(w | \beta, y) \, d\beta \, dw.
\]
Throughout the proof, let $c_i, i = 1, 2, \ldots$, denote finite, positive constants that do not depend on $\beta$ or $w$. First,

\[
\pi(\beta \mid w, y) \pi(w \mid \beta, y) = c_1 |\Sigma(w)|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \left( \beta^T (X^T \Omega(w) X + B^{-1}) \beta - 2 \beta^T [X^T (y - \frac{1}{2} 1_n) + B^{-1} b] \right) \right\} \times \exp \left\{ -\frac{1}{2} \mu(w)^T \Sigma(w)^{-1} \mu(w) \right\} \prod_{i=1}^{n} \cosh \left( \frac{|x_i^T \beta|}{2} \right) \exp \left\{ -\frac{(x_i^T \beta)^2}{2} - w_i \right\} g(w_i) \right\} \right\} \times \prod_{i=1}^{n} \cosh \left( \frac{|x_i^T \beta|}{2} \right) \exp \left\{ -\frac{(x_i^T \beta)^2}{2} - w_i \right\} g(w_i) \right\}.
\]

Now,

\[
\beta^T X^T 1_n = \sum_{i=1}^{n} x_i^T \beta
\]

and

\[
\prod_{i=1}^{n} \cosh \left( \frac{|x_i^T \beta|}{2} \right) = 2^{-n} \prod_{i=1}^{n} \left[ \exp \left\{ -\frac{x_i^T \beta}{2} \right\} + \exp \left\{ \frac{x_i^T \beta}{2} \right\} \right].
\]

Hence,

\[
\exp \left\{ -\frac{1}{2} \beta^T X^T 1_n \right\} \prod_{i=1}^{n} \cosh \left( \frac{|x_i^T \beta|}{2} \right) = 2^{-n} \prod_{i=1}^{n} \left[ 1 + \exp \left\{ -x_i^T \beta \right\} \right].
\]

Also,

\[
\prod_{i=1}^{n} \exp \left\{ -\frac{(x_i^T \beta)^2}{2} - w_i \right\} = \exp \left\{ -\frac{1}{2} \beta^T X^T \Omega(w) X \beta \right\}.
\]

Combining (2), (3), and (4), we have

\[
\pi(\beta \mid w, y) \pi(w \mid \beta, y) \leq c_2 |\Sigma(w)|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \left( \beta^T (2X^T \Omega(w) X + B^{-1}) \beta - 2 \beta^T [X^T y + B^{-1} b] \right) \right\} \times \prod_{i=1}^{n} \left[ 1 + \exp \left\{ -x_i^T \beta \right\} \right] g(w_i).
\]

Now, adopting an argument used in Choi and Român (2017), let $A \subseteq N_n := \{1, 2, \ldots, n\}$, and define $X_A$ to be the $n \times p$ matrix whose $i$th row is equal to

\[
x_{A,i} = \begin{cases} x_i & i \in A \\ 0 & \text{otherwise} \end{cases}.
\]

If we let $I_A$ denote a diagonal matrix whose $i$th diagonal element is 1 if $i \in A$, and 0 otherwise, then we can write $X_A = I_A X$. Clearly, if $A = \emptyset$, then $X_A$ is a null matrix. We can see that

\[
\exp \left\{ -\frac{1}{n} X_A \beta \right\} = \begin{cases} \exp \left\{ -\sum_{i \in A} x_i^T \beta \right\} & \text{A nonempty} \\ 1 & \text{otherwise} \end{cases}.
\]
It then follows that
\[
\prod_{i=1}^{n} \left[ 1 + \exp\left\{ -x_i^T \beta \right\} \right] = 1 + \sum_{A \subseteq \mathbb{N}_n : A \neq \emptyset} \exp\left\{ - \sum_{i \in A} x_i^T \beta \right\}
\]
\[= \sum_{A \subseteq \mathbb{N}_n} \exp\left\{ - \beta^T X_A 1_n \right\}
\]
\[= \sum_{A \subseteq \mathbb{N}_n} \exp\left\{ - \beta^T X^T (I_A 1_n) \right\}.\]
(6)

Combining (5) and (6), and recalling that \(\Sigma(w) = (X^T \Omega(w) X + B^{-1})^{-1}\), we have
\[
\pi(\beta \mid w, y) \pi(w \mid \beta, y) \leq c_2 |\Sigma(w)|^{-\frac{1}{2}} \exp\left\{ - \frac{1}{2} \beta^T (2X^T \Omega(w) X + B^{-1}) \beta \right\}
\]
\[\times \left[ \prod_{i=1}^{n} g(w_i) \right] \sum_{A \subseteq \mathbb{N}_n} \exp\left\{ - \beta^T X^T (I_A 1_n) \right\} \exp\left\{ \beta^T [X^T y + B^{-1} b] \right\}
\]
\[= c_2 \left[ \frac{|X^T \Omega(w) X + B^{-1}|}{|2X^T \Omega(w) X + B^{-1}|} \right]^{\frac{1}{2}} \exp\left\{ - \frac{1}{2} \beta^T (2X^T \Omega(w) X + B^{-1}) \beta \right\}
\]
\[\times \left[ \prod_{i=1}^{n} g(w_i) \right] \sum_{A \subseteq \mathbb{N}_n} \exp\left\{ \beta^T [X^T (y - I_A 1_n) + B^{-1} b] \right\}
\]
\[\leq c_2 \frac{\exp\left\{ - \frac{1}{2} \beta^T (2X^T \Omega(w) X + B^{-1}) \beta \right\}}{|(2X^T \Omega(w) X + B^{-1})^{-1}|^{\frac{1}{2}}}
\]
\[\times \left[ \prod_{i=1}^{n} g(w_i) \right] \sum_{A \subseteq \mathbb{N}_n} \exp\left\{ \beta^T [X^T (y - I_A 1_n) + B^{-1} b] \right\},\]
(7)

where the second inequality is a result of the fact that
\[
|2X^T \Omega(w) X + B^{-1}| \geq |X^T \Omega(w) X + B^{-1}| + |X^T \Omega(w) X| \geq |X^T \Omega(w) X + B^{-1}|,
\]
which follows from the Minkowski determinant inequality (see, e.g., Horn and Johnson, 1985, Theorem 7.8.8). Letting \(l_A = X^T (y - I_A 1_n) + B^{-1} b\), and using the formula for the moment generating function of the multivariate normal distribution, we have
\[
\int_{\mathbb{R}^p} e^{\beta^T l_A} \frac{\exp\left\{ - \frac{1}{2} \beta^T (2X^T \Omega(w) X + B^{-1}) \beta \right\}}{(2\pi)^{\frac{n}{2}} |(2X^T \Omega(w) X + B^{-1})^{-1}|^{\frac{1}{2}}} d\beta = \exp\left\{ l_A^T (2X^T \Omega(w) X + B^{-1})^{-1} l_A / 2 \right\}.
\]
(8)

Combining (7) and (8) yields
\[
\int_{\mathbb{R}^p} \pi(\beta \mid w, y) \pi(w \mid \beta, y) d\beta \leq c_3 \left[ \prod_{i=1}^{n} g(w_i) \right] \sum_{A \subseteq \mathbb{N}_n} \exp\left\{ l_A^T (2X^T \Omega(w) X + B^{-1})^{-1} l_A / 2 \right\}.
\]
(9)

Now, since \(2X^T \Omega(w) X + B^{-1} \geq B^{-1}\), it follows that \((2X^T \Omega(w) X + B^{-1})^{-1} \leq (B^{-1})^{-1} = B\). Thus,
\[
l_A^T (2X^T \Omega(w) X + B^{-1})^{-1} l_A \leq l_A^T B l_A.
\]
Combining this with (9), we have
\[ \int_{\mathbb{R}^p} \pi(\beta \mid w, y) \pi(w \mid \beta, y) \, d\beta \leq c_4 \prod_{i=1}^{n} g(w_i), \]  
and it follows that
\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^p} \pi(\beta \mid w, y) \pi(w \mid \beta, y) \, d\beta \, dw \leq c_4 \int_{\mathbb{R}^n} \prod_{i=1}^{n} g(w_i) \, dw = c_4 \prod_{i=1}^{n} g(w_i) \, dw = c_4 < \infty. \]

\[ \square \]

**B Proof of Proposition 2**

Proof. Throughout the proof, let \( c_i, i = 1, 2, \ldots, \) denote finite, positive constants that do not depend on \( \beta \) or \( w \). We begin by showing that there exists a constant \( M \in (0, \infty) \) such that
\[ \pi(\beta \mid w, y) \leq M |X^T \Omega(w) X + B^{-1}|^{\frac{1}{2}}. \]  
Letting \( z = y - \frac{1}{2} 1_n \), and recalling that \( \Sigma(w) = (X^T \Omega(w) X + B^{-1})^{-1} \) and \( \Sigma^{-1}(w) \mu(w) = X^T z + B^{-1} b \), we have
\[ -\frac{1}{2} (\beta - \mu(w))^T \Sigma^{-1}(w) (\beta - \mu(w)) \leq \frac{1}{2} \beta^T \Sigma^{-1}(w) \beta + \beta^T \Sigma^{-1}(w) \mu(w) \]
\[ \leq \frac{1}{2} \beta^T B^{-1} \beta + \beta^T B^{-1} (B X^T z + b) \]
\[ = \frac{1}{2} (\beta - (B X^T z + b))^T B^{-1} (\beta - (B X^T z + b)) + c_1. \]
It follows that
\[ \frac{\pi(\beta \mid w, y)}{h_{\nu}(\beta; d, C)} = c_2 |X^T \Omega(w) X + B^{-1}|^{\frac{1}{2}} \exp \left\{ -\frac{1}{2} (\beta - \mu(w))^T \Sigma^{-1}(w) (\beta - \mu(w)) \right\} \]
\[ \times \left( 1 + \frac{1}{\nu} (\beta - d)^T C^{-1} (\beta - d) \right)^{\frac{\nu + p}{2}} \]
\[ \leq c_3 |X^T \Omega(w) X + B^{-1}|^{\frac{1}{2}} \exp \left\{ -\frac{1}{2} (\beta - (B X^T z + b))^T B^{-1} (\beta - (B X^T z + b)) \right\} \]
\[ \times \left( 1 + \frac{1}{\nu} (\beta - d)^T C^{-1} (\beta - d) \right)^{\frac{\nu + p}{2}} \]
\[ \leq c_4 |X^T \Omega(w) X + B^{-1}|^{\frac{1}{2}}, \]
where the last inequality is a consequence of the fact that the product of the exponential term and the polynomial term converges to 0 as \( \beta \) diverges. Hence, (11) holds. Now using (11) and then (10),
\[ 13 \]
we have
\[
\int_{\mathbb{R}_+^n} \int_{\mathbb{R}^p} \frac{\pi(w | \beta, y) \pi^3(\beta | w, y)}{h_P(\beta; d, C)} \, d\beta \, dw \leq \int_{\mathbb{R}_+^n} M^2 |X^T \Omega(w)X + B^{-1}| \int_{\mathbb{R}^p} \pi(w | \beta, y) \pi(\beta | w, y) \, d\beta \, dw
\]
\[
\leq c_5 \int_{\mathbb{R}_+^n} |X^T \Omega(w)X + B^{-1}| \prod_{i=1}^n g(w_i) \, dw .
\]

Once expanded, the determinant will result in a finite sum of products of polynomials of the \(w_i\)s, and since \(\int_{\mathbb{R}_+^a} u^a g(u) \, du < \infty\) for all positive integers \(a\) (see, e.g., Biane et al. [2001]), the proof is complete. □

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