On the Hyperbolicity Locus of a Real Curve*

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Abstract. Given a real algebraic curve in the projective 3-space, its hyperbolicity locus is the set of lines with respect to which the curve is hyperbolic. We give an example of a smooth irreducible curve whose hyperbolicity locus is disconnected but the connected components are not distinguished by the linking numbers with the connected components of the curve.

Key words: algebraic curve, algebraic knot.

1. Introduction. Let $C$ be a real algebraic curve in $\mathbb{RP}^n$. We say that it is hyperbolic with respect to a projective subspace $L$ of codimension 2 if $C$ is disjoint from $L$ and each hyperplane passing through $L$ has only real intersections with (the complexification of) $C$. Such $L$ was called a witness to the hyperbolicity of $C$ in [5].

Shamovich and Vinnikov asked (see the last part of Question 3.13 in [5]) whether the set of all witnesses to the hyperbolicity of an irreducible curve $C$ is connected (following [2], we refer to this set as the hyperbolicity locus of $C$ and denote it by $\mathcal{H}(C)$). This is evidently true for $n = 2$. However, Kummer and Shaw [2] showed that the answer is negative for $n = 3$. They gave an example of a sextic genus-1 smooth real curve $C$ in $\mathbb{RP}^3$ consisting of two topological circles $A$ and $B$ for which there exist two lines $L$ and $L'$ such that $C$ is hyperbolic with respect to each of them but the linking numbers of $A$ and $B$ with $L$ and $L'$ are different: $(\text{lk}(A, L), \text{lk}(B, L)) \neq (\text{lk}(A, L'), \text{lk}(B, L'))$.

Note that in [3, Theorem 3 and Lemma 3.12] we gave an infinite series of examples with any number of connected components of $\mathcal{H}(C)$, although we did not mention this explicitly. These are the curves $W_g(a_0, \ldots, a_g)$ in the notation of [3]; in particular, the curve constructed in [2] is our $W_1(2, 2)$. As in [2], in all these examples any two components of $\mathcal{H}(C)$ are distinguished by linking numbers. This follows from [3, Proposition 3.13].

Thus, a new question naturally arises [2, Question 2]: Can it happen that $\mathcal{H}(C)$ is disconnected but the elements of different components have the same linking numbers with all components of $C$? Here we give an affirmative answer to this question for $n = 3$.

We construct a rational curve $C$ in $\mathbb{RP}^3$ of degree 8 and two lines $L, L' \in \mathcal{H}(C)$ which belong to different connected components of $\mathcal{H}(C)$, because the links $C \cup L$ and $C \cup L'$ are not isotopic in $\mathbb{RP}^3$. The curve $C$ has only one connected component, and hence the linking numbers cannot distinguish the components of $\mathcal{H}(C)$.

2. The example.

2.1. An auxiliary line arrangement. Let $(x, y, z)$ be coordinates in an affine chart of $\mathbb{RP}^3$. We denote the $z$-axis by $L$ and the common line at infinity of the planes $z = \text{const}$ by $L'$. Let $R$ be the rotation through $90^\circ$ about $L$. We set $p_0 = (3, -1, -1)$, $q_0 = (3, 1, 1)$, $p_k = R^k(p_0)$, $q_k = R^k(q_0)$, $\ell_k = (p_k q_k)$, and $\ell'_k = (p_k q_{k+1})$; see Fig. 1, where the parts of the lines $\ell_k$ and $\ell'_k$ on which the $z$-coordinate is positive and negative are shown in black and gray, respectively.

Figure 2 shows the same line arrangement in the $Oxz$ projection. Here color represents the sign of the $y$-coordinate. Note that since Fig. 2 is obtained from Fig. 1 by a rotation about the $Ox$ axis, the direction of the $y$-axis in Fig. 2 is opposite to that of the $z$-axis in Fig. 1.

We orient the lines $\ell_k$ and $\ell'_k$ so that $dz > 0$ on them. Note that, in this case, we also have $d\theta > 0$ on these lines, where $(x, y) = (r \cos \theta, r \sin \theta)$. This means that (under a suitable choice of the orientation of $L$ and $L'$) both $L$ and $L'$ are positively linked with each of the lines $\ell_k$ and $\ell'_k$.

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2.2. Construction of the curve $C$. We perturb the union of the eight lines constructed in Section 2.1 so that it transforms as in Fig. 3 (left) near the double point $q_0$ and as in Fig. 3 (right) near the other double points $(q_1, q_2, q_3, p_0, \ldots, p_3)$. Such a perturbation is possible due to [1, Theorem 2.4]: one should add lines one by one. It is easy to see that $L, L' \in \mathcal{H}(C)$.

![Fig. 1. Projection on Oxy](image1)

![Fig. 2. Projection on Oxy](image2)

2.3. Proof that $L$ and $L'$ belong to different components of $\mathcal{H}(C)$. It is enough to show that the oriented links $C \cup L$ and $C \cup L'$ are not isotopic in $\mathbb{R}P^3$. Indeed, their lifts to the 3-sphere (we denote them by $\Lambda$ and $\Lambda'$) are distinguished by the link determinant, i.e., the determinant of the symmetrized Seifert matrix (the value at $-1$ of the Alexander polynomial). Namely, we have $\det(\Lambda) = 64$ and $\det(\Lambda') = 0$.

For the computations we used the program from [4, Appendix] available at http://picard.ups-tlse.fr/~orevkov/sm.mat. Its input must be represented in the form of braids. Therefore, for the reader’s convenience, we give here braids $\beta$ and $\beta'$ whose braid closures are $\Lambda$ and $\Lambda'$, respectively.

To find $\beta'$, we rotate a line in Fig. 1 through $360^\circ$ about the origin of the plane and write down the contributions of all crossings consecutively scanned by this line (including the crossings at infinity). We obtain $\beta' = \beta_{1/2}'\tau_9(\beta_{1/2})$, where $\beta_{1/2}'$ is the contribution of the rotation through $180^\circ$ starting at the horizontal position and $\tau_n : B_n \to B_n$ is the braid group automorphism given by $\sigma_i \mapsto \sigma_{n-i}$. We have to apply $\tau_9$ on the second half-turn, because the orientation of the line reverses (see also [3, Secs. 4.3–4.5]). We have

$$\beta_{1/2}' = \sigma_1\Delta_{45}\sigma_8 \pm(2,0), (\infty,0)$$

$$\times \sigma_2^{-1} \quad (3,1) \quad \text{(no contribution of } (-3,-1))$$

$$\times \sigma_3\sigma_6 \quad \pm(5,3)$$

$$\times \sigma_2\Delta_{45}\sigma_7 \quad \pm(3,3), (\infty,\infty)$$

$$\times \sigma_3\sigma_6 \quad \pm(3,5)$$

$$\times \sigma_1\Delta_{45}\sigma_8 \quad \pm(0,2), (0,\infty)$$

$$\times \sigma_3\sigma_6 \quad \pm(-3,5)$$

$$\times \sigma_2\Delta_{45}\sigma_7 \quad \pm(-3,3), (-\infty,\infty)$$

$$\times \sigma_3\sigma_6 \quad \pm(-5,3),$$
where \( \Delta_{45} = \sigma_4 \sigma_5 \sigma_4 \) is the contribution of each triple crossing at infinity (in the comments on the right we refer to the coordinates of the contributing crossings in Fig. 1).

Similarly, \( \beta = \beta_{1/2} \sigma_9 (\beta_{1/2}) \) with

\[
\beta_{1/2} = (\sigma_1 \sigma_4 \sigma_7) \sigma_2^{-1} (\sigma_3 \sigma_5) (\sigma_2 \sigma_4 \sigma_6) (\sigma_3 \sigma_5) \\
\times (\sigma_1 \sigma_4 \sigma_7) (\sigma_3 \sigma_5) (\sigma_2 \sigma_4 \sigma_6) (\sigma_3 \sigma_5) \\
\times (\sigma_8 \sigma_7 \cdots \sigma_1),
\]

where the first two lines represent \( C \) and the third line represents \( L' \).

**Remark.** If we replace the negative crossing in Fig. 3 by a positive one (which corresponds to replacing \( \sigma_2^{-1} \) by \( \sigma_2 \) in our braids), then \( \mathcal{H}(C) \) will become connected by Proposition 3.13 in [3], because the obtained eighth-degree curve will be maximally writhed in this case.

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