Spin squeezing as a measure of entanglement in a two qubit system

A. Messikh\textsuperscript{1}, Z. Ficek\textsuperscript{1,2} and M.R.B. Wahiddin\textsuperscript{1}

\textsuperscript{1} Centre for Computational and Theoretical Sciences, Kulliyyah of Science, International Islamic University Malaysia, 53100 Kuala Lumpur, Malaysia

\textsuperscript{2} Department of Physics, The University of Queensland, Brisbane, QLD 4072, Australia

(October 7, 2018)

We show that two definitions of spin squeezing extensively used in the literature [M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993) and D.J. Wineland et al., Phys. Rev. A 50, 67 (1994)] give different predictions of entanglement in the two-atom Dicke system. We analyze differences between the definitions and show that the Kitagawa and Ueda’s spin squeezing parameter is a better measure of entanglement than the commonly used spectroscopic spin squeezing parameter. We illustrate this relation by examining different examples of a driven two-atom Dicke system in which spin squeezing and entanglement arise dynamically. We give an explanation of the source of the difference in the prediction of entanglement using the negativity criterion for entanglement. For the examples discussed, we find that the Kitagawa and Ueda’s spin squeezing parameter is the sufficient and necessary condition for entanglement.

We illustrate this relation by examining different examples of the commonly used spectroscopic spin squeezing parameter.

Spin squeezing results from quantum correlations between atomic spins and have received a great deal of attention in recent years [1–9]. The interest in spin squeezing arises not only from the fact that it exhibits reduced fluctuations of the collection of atomic spins below the fundamental spin noise limit, but also from the possibility of interesting novel applications in interferometry, high precision spectroscopy and atomic clocks. Recently, Sørensen et al. [10] have proposed spin squeezing as a measure of entanglement in multi-atom systems, which opens further applications in the area of quantum information and quantum computation [11]. The advantage of spin squeezing over the well known entanglement measures, such as concurrence [12] and negativity [13,14] is that spin squeezing can be used as a measure of entanglement in multi-atom systems, whereas the former measures can be applied only to two particle (two qubit) systems. Hald et al. [15] recently reported preparation of an entangled multiatom state via quantum state transfer from squeezed light to a collection of atomic spins. Kuzmich et al. [16] have proposed a scheme to produce spin squeezed states via a quantum nondemolition measurement technique and spin noise reduction using this method has been experimentally observed [17].

There are, however, two different definitions of the spin squeezing parameter frequently used in the literature; the Kitagawa and Ueda’s spin squeezing parameter defined as [1]

\[
\xi_{n_i}^S = \frac{2}{S} \langle (\Delta S_{n_i})^2 \rangle_{\bot}, \quad i = 1, 2, \quad (1)
\]

and the spectroscopic spin squeezing parameter introduced in the context of Ramsey spectroscopy as [2]

\[
\xi_{n_i}^R = \frac{2S \langle (\Delta S_{n_i})^2 \rangle_{\bot}}{\langle S_{n_i} \rangle^2}, \quad (2)
\]

where \( S \) is the total spin of the system, \( n_1, n_2 \) and \( n_3 \) are three mutually orthogonal unit vectors oriented such that the mean value of one of the spin components, assumed here \( \langle S_{n_1} \rangle \), is different from zero, while the other components \( S_{n_1} \) and \( S_{n_2} \) have zero mean values. The variance \( \langle (\Delta S_{n_1})^2 \rangle_{\bot} \) is calculated in the plane orthogonal to the mean spin direction. A multiatom system in a coherent state has variances normal to the mean spin direction equal to the standard quantum limit of \( S/2 \). In this case, \( \xi_{n_i}^S = 1 \). A system with the variance reduced below the standard quantum limit is characterized by \( \xi_{n_i}^S < 1 \), that is spin squeezed in a direction normal to the mean spin direction. With the parameter (2), spin squeezing is manifested by \( \xi_{n_i}^R < 1 \) which indicates a reduction in the frequency noise. Since the mean value \( \langle |S_{n_1}| \rangle \leq S \), it follows that the parameters (1) and (2) do not describe the same spin squeezing, that a spectroscopic spin squeezing \( \xi_{n_i}^R < 1 \) implies \( \xi_{n_i}^S < 1 \), but not vice versa. It should be noted here that in general the spin squeezing parameters (1) and (2) are sufficient but not necessary conditions for entanglement, that one can create entanglement without spin squeezing [18–20].

In studying the relation between entanglement and spin squeezing, we discovered that the two definitions of spin squeezing give somewhat different predictions of entanglement in the two-atom Dicke system. It is the purpose of this Brief Report to point out that the Kitagawa and Ueda’s spin squeezing parameter (1) is a better measure of entanglement than the the spectroscopic spin squeezing parameter (2). Specifically, we will show that there is a large class of processes for which the Kitagawa and Ueda’s parameter is the sufficient and necessary condition for entanglement. It was quite surprising to find this connection, since the spectroscopic spin squeezing parameter (2) is commonly used in the literature to compute spin squeezing and entanglement in multi-atom systems. The spin squeezing is currently the widely accepted measure of multi-atom entanglement, so we believe that a detailed analysis of the relation between entanglement...
and these two definitions of spin squeezing is of general interest.

We illustrate our considerations of the relation between entanglement and the spin squeezing parameters in a simple model of the two-atom (two qubit) Dicke system which consists of two identical atoms confined to a volume with dimensions much smaller than the wavelength of the atomic transitions [21–23]. In this limit, the atomic dipole moments evolve on the time scale much shorter than any $\hat{S}^z$-breaking relaxation mechanism, so that the total spin $\hat{S}^z$ of the system is conserved during the evolution. Each atom is assumed to have only two energy levels, ground level $|g_i\rangle$ and excited level $|e_i\rangle$ ($i = 1, 2$), separated by an energy $\hbar\omega_0 \equiv E_e - E_g$, where $\omega_0$ is the atomic transition frequency. The atomic levels $|g_i\rangle$ and $|e_i\rangle$ are eigenstates of the energy operator $S_i^z$ with eigenvalues $-1/2$ and $1/2$, respectively. Although the two-atom Dicke system is admittedly an elementary model, it offers an insight into the fundamentals of spin squeezing generation process and entanglement creation in multatom systems. Because of its simplicity, we will obtain exact solutions for the density matrix elements of the system, which will allow us to study in details the relation between the atomic coherence properties and the amount of spin squeezing.

In the absence of external driving fields, the two-atom Dicke system is equivalent to a cascade multilevel system composed of three energy levels [21,22]

\begin{align}
|g\rangle &= |g_1\rangle|g_2\rangle, \\
|s\rangle &= (|e_1\rangle|g_2\rangle + |g_1\rangle|e_2\rangle)/\sqrt{2}, \\
|e\rangle &= |e_1\rangle|e_2\rangle, \quad (3)
\end{align}

with energies $E_g = -\hbar\omega_0, E_s = 0$ and $E_e = \hbar\omega_0$, respectively. The energy levels (3) are known as the Dicke collective states [21]. The ground $|g\rangle$ and the upper $|e\rangle$ levels are product states of the individual atoms, whereas the intermediate state $|s\rangle$ is a maximally entangled state of the two-atom system. The state is a linear superposition of the product states which cannot be separated into product states of the individual atoms.

In our analysis, we assume that the atoms are driven by two fields of fixed phases but different statistics. These are a coherent laser field of the (real) Rabi frequency $\Omega$, and a broadband squeezed vacuum field which is known to produce strong two-photon coherences in atomic systems. To keep the mathematical complications to a minimum, we assume that the angular frequency $\omega_L$ of the laser field and the carrier frequency $\omega_s$ of the squeezed field are equal to the atomic transition frequency, i.e. the detunings $\Delta_L = \omega_L - \omega_0$ and $\Delta_s = \omega_s - \omega_0$ are zero. We will examine the relation between entanglement and the spin squeezing parameters in three different models of the interaction in which entanglement and spin squeezing arise dynamically. In the first, the atoms interact only with the squeezed field ($\Omega = 0$). In this case, one achieves spin squeezing and entanglement induced by the two-photon coherences with no contribution from the one-photon coherences. In the second model, the atoms interact only with the coherent field. Here, both the one and two-photon coherences contribute to the spin squeezing and entanglement. In the third model, the atoms interact simultaneously with both fields. In this case the squeezed field enhances the two-photon coherences.

To calculate the variances and the mean values of the spin components appearing in Eqs. (1) and (2), we apply the master equation of the driven two-atom Dicke system, which in the interaction picture is given by [24]$
\frac{\partial \hat{\rho}}{\partial t} = \frac{i}{\hbar} [H_s, \hat{\rho}] - \frac{1}{2}\Gamma (N + 1) (S^+ S^- \hat{\rho} + \hat{\rho} S^+ S^- - 2S^- \hat{\rho} S^+) \\
- \frac{1}{2}\Gamma N (S^- S^+ \hat{\rho} + \hat{\rho} S^- S^+ - 2S+ \hat{\rho} S^-) \\
+ \frac{1}{2}\Gamma M (S^+ S^+ \hat{\rho} + \hat{\rho} S^+ S^+ - 2S^+ \hat{\rho} S^+) \\
+ \frac{1}{2}\Gamma M^* (S^- S^- \hat{\rho} + \hat{\rho} S^- S^- - 2S^- \hat{\rho} S^-), \quad (4)
$

where $\Gamma$ is the spontaneous emission rate of the atoms, $S^\pm = S_1^\pm + S_2^\pm$ are the collective atomic spin operators, and $H_s = -i\hbar/2 (S^+ - S^-)$ is the interaction Hamiltonian between the atoms and the laser field. The parameters $N$ and $M$ characterize the squeezed vacuum field, such that $N$ is the number of photons in the squeezed modes, $M = |M| \exp (i\phi_s)$ is the magnitude of two-photon correlations between the vacuum modes, and $\phi_s$ is the phase of the squeezed field. For simplicity, we set the squeezing phase $\phi_s = 0$ (or $\pi$) so that the squeezing parameter $M$ is real.

In order to analyze the relation between entanglement and the spin squeezing parameters, we express the parameters (1) and (2) in terms of the density matrix elements of the system. Since the driving fields are on resonance with the atomic transition and $M = M^*$, the stationary off-diagonal density matrix elements (coherences) are real, or equivalently, the Bloch vector has the components $\vec{B} = (\langle S_x \rangle, 0, \langle S_z \rangle)$, where $S_x = \langle S^+ S^- \rangle/2$ and $S_z = S_1^z + S_2^z$. Thus, we can study spin squeezing by a single rotation of the nonzero spin components around the y-axis. Let $\vec{n}_3$ be the direction of the total spin in the new (rotated) reference frame. Then the variances calculated in the directions $\vec{n}_1$ and $\vec{n}_2$ perpendicular to the direction of the total spin can be written as

\begin{align}
\langle (\Delta S_{\vec{n}_1})^2 \rangle_\perp &= \langle S_1^2 \rangle \sin^2 \alpha + \langle S_2^2 \rangle \cos^2 \alpha - \langle S_x S_z \rangle \sin 2\alpha, \\
\langle (\Delta S_{\vec{n}_2})^2 \rangle_\perp &= \langle S_1^2 \rangle,
\end{align}

(5)

where $\tan \alpha = \langle S_z \rangle/\langle S_x \rangle$.

A simple calculation using Eqs. (1), (2) and (5) shows that the Kitagawa and Ueda’s parameter becomes

\begin{align}
\xi_{\vec{n}_1}^S &= 2 (1 - \rho_{ss}) \sin^2 \alpha + (1 + \rho_{ss} + 2\rho_{eg}) \cos^2 \alpha, \\
\xi_{\vec{n}_2}^S &= 1 + \rho_{ss} - 2\rho_{eg}, \quad (6)
\end{align}
whereas for the spectroscopic spin squeezing parameter takes the form
\[ \xi_{\text{ss}}^R = \frac{[2(1 - \rho_{ee}) \sin^2 \alpha + (1 + \rho_{ee} + 2\rho_{eg}) \cos^2 \alpha]}{U^2}, \]
\[ \xi_{\text{eg}}^R = \frac{(1 + \rho_{ss} - 2\rho_{eg})}{U^2}, \]
where \( U = (\rho_{ee} - \rho_{gg}) \cos \alpha + 2^{-1/2}(\rho_{ss} + \rho_{eg} + \rho_{se} + \rho_{gs}) \sin \alpha. \)

From the structure of Eqs. (6) and (7) it is clear that the necessary condition to obtain spin squeezing is to create two-photon coherences \( \rho_{ge} \). For \( \rho_{eg} < 0 \), the right-hand sides of \( \xi_{\text{ss}}^S \) and \( \xi_{\text{eg}}^S \) can be less than 1, whereas the right-hand sides of \( \xi_{\text{ss}}^R \) and \( \xi_{\text{eg}}^R \) are always greater than 1. Thus, spin squeezing can be observed only in the \( \xi_{\text{ss}}^S \) and \( \xi_{\text{eg}}^S \) components. On the other hand, for \( \rho_{eg} > 0 \), the right-hand sides of only \( \xi_{\text{ss}}^S \) and \( \xi_{\text{eg}}^S \) can be less than 1.

Having introduced the spin squeezing parameters in terms of the density matrix elements, we now turn to our central problem to determine which of the spin squeezing parameters is a better measure of entanglement and what is the degree of entanglement. Consider first the two-atom Dicke system driven by a broadband squeezed vacuum field alone (\( \Omega = 0 \)). In this case, the master equation (4) leads to the following nonzero steady-state solutions for the density matrix elements [23]
\[ \rho_{ee} = \frac{N^2 (2N + 1) - (2N - 1) |M|^2}{(2N + 1)(3N^2 + 3N + 1 - 3|M|^2)}, \]
\[ \rho_{ss} = \frac{N (N + 1) - |M|^2}{3N^2 + 3N + 1 - 3|M|^2}, \]
\[ \rho_{eg} = \rho_{ge} = \frac{1}{(2N + 1)(3N^2 + 3N + 1 - 3|M|^2)}. \]

This equation expresses the steady-state of the system in terms of the intensity and the two-photon correlations characteristic of a squeezed field. Since the one-photon coherences are zero, we can easily verify that \( \langle S_z \rangle \neq 0 \) and \( \langle S_x \rangle = \langle S_y \rangle = 0 \). This implies that we can determine spin squeezing in the xy-plane without any rotation (\( \alpha = 0 \)). In this case \( \{n_1, n_2, n_3\} = \{x, y, z\}. \)

With the steady state solution (8), the density matrix of the system in the basis of the product states \( \{|e_1, e_2\}, |e_1, g_2\}, |g_1, e_2\}, |g_1, g_2\} \) takes the form
\[ \hat{\rho} = \begin{pmatrix} \rho_{ee} & 0 & 0 & \rho_{eg} \\ 0 & \frac{1}{2}\rho_{ss} & \frac{1}{2}\rho_{gg} & 0 \\ 0 & \frac{1}{2}\rho_{ge} & \frac{1}{2}\rho_{gs} & 0 \\ \rho_{ge} & 0 & 0 & \rho_{gg} \end{pmatrix}, \]
where \( \rho_{gg} = 1 - \rho_{ee} - \rho_{ss}. \)

Given the density matrix, it is possible to calculate the entanglement between the atoms. To quantify the degree of entanglement, we use the negativity criterion for entanglement [13,14] and find that the eigenvalues of the partial transposition of \( \hat{\rho} \) are
\[ \lambda_1 = \frac{1}{2}\rho_{ss} + |\rho_{eg}|, \]
\[ \lambda_2 = \frac{1}{2}((\rho_{ee} + \rho_{gg}) \pm \sqrt{(\rho_{ee} - \rho_{gg})^2 + \rho_{ss}^2}). \]

From this it readily follows that \( \lambda_1 \) and \( \lambda_2 \) are always positive. Moreover, it is easily verified that with the solution (8), the eigenvalue \( \lambda_2 \) is positive for all values of the parameters involved. Thus, the system exhibits entanglement when \( |\rho_{eg}| > \rho_{ss}/2 \), and then the degree of entanglement is
\[ E = \max(0, -2\lambda_1) = 2|\rho_{eg}| - \rho_{ss}. \]

It is evident by comparison of Eq. (11) with Eqs. (6) and (7) that the condition for entanglement (\( E > 0 \)) is completely equivalent to the condition for spin squeezing predicted by the Kitagawa and Ueda’s parameter, and there is a simple relation
\[ E = 1 - \xi_{\text{ss}}^S. \]

A value of \( \xi_{\text{ss}}^S < 1 \) indicates spin squeezing and at the same moment there is entanglement (\( E > 0 \)) between the atoms. In addition, the degree of entanglement is equal to the degree of spin squeezing. Thus, we conclude that the Kitagawa and Ueda’s parameter is the sufficient and necessary condition for entanglement induced by a squeezed vacuum field.

The above considerations are illustrated in Figs. 1 and 2, which show the entanglement measure \( E \) and the parameters \( \xi_{\text{ss}}^S \) and \( \xi_{\text{eg}}^S \) for two different types of the squeezed field. In Fig. 1, we plot the entanglement measure and the spin squeezing parameters for a classical squeezed field with the correlations \( M = N \). The classical squeezed field is characterized by an anisotropic distribution of noise with the noise reduced in some directions, but not below the standard vacuum level. The figure shows that \( \xi_{\text{ss}}^S > 1 \) for all \( N \), but \( \xi_{\text{eg}}^S \) is less than 1 for \( N < 1/2 \), and also an entanglement appears in the same range of \( N \). This shows that \( \xi_{\text{ss}}^S \) correctly predicts entanglement, while with the parameter \( \xi_{\text{eg}}^S \) one could observe entanglement without spin squeezing.

**FIG. 1.** Entanglement measure \( E \) (solid line) and the spin squeezing parameters \( \xi_{\text{ss}}^S \) (dashed line) and \( \xi_{\text{eg}}^S \) (dashed-dotted line) as a function of \( N \) for the classical squeezed field with \( M = N \).
In Fig. 2, we plot $E$ and the spin squeezing parameters for a quantum squeezed field with perfect correlations $M^2 = N(N + 1)$. Since $\rho_{ss} = 0$ and the inequality $|\rho_{eg}| > 0$ always holds, both parameters $\xi_{ss}^E$ and $\xi_{ss}^R$ are less than 1 for the entire range of $N$. Thus, both parameters predict entanglement and spin squeezing for all $N$. However, the degree of entanglement is equal to the degree of spin squeezing given by $\xi_{ss}^E$. The entanglement and spin squeezing increase with increasing $N$ and attain their maximal values $E = 1$ and $\xi_{ss}^E = 0$ for large $N$.

![Entanglement parameter $E$ and spin squeezing parameter $\xi_{ss}^E$ as a function of $N$](image)

Fig. 2. Entanglement measure $E$ (solid line) and the spin squeezing parameters $\xi_{ss}^E$ (dashed line), $\xi_{ss}^R$ (dashed-dotted line) as a function of $N$ for the quantum squeezed field with $M = \sqrt{N(N + 1)}$.

It is easy to show that the entanglement created by the quantum squeezed field is related to the pure two-atom squeezed state. Under the squeezed field excitation, there are entangled states generated which can be found by the diagonalization of the density matrix (9). We find that the two-photon coherences produce entangled states

$$\begin{align*}
|\Psi_+\rangle &= (|\Pi_+ - \rho_{cc}| g) + |\rho_{eg}| e]/N_+, \\
|\Psi_\pm\rangle &= [\rho_{gg} g + (\Pi_- - \rho_{gg}) e] / N_-, 
\end{align*}$$

(13)

where $N_\pm$ are the normalization constants, and

$$\Pi_\pm = \frac{1}{2} (\rho_{gg} + \rho_{ee}) \pm \frac{1}{2} \left[(\rho_{gg} - \rho_{ee})^2 + 4 |\rho_{eg}|^2\right]^{1/2}$$

(14)

are the populations of the entangled states.

It is evident from (13), that the two-photon coherences create entangled states which are linear superpositions of the ground state $|g\rangle$ and the upper state $|e\rangle$. Thus, entanglement created by a squeezed field is associated with the two-photon entangled states $|\Psi_\pm\rangle$. Note that the steady-state with the classical squeezed field is a mixed state with the populations $\rho_{ss} \neq 0$ and $\Pi_\pm \neq 0$, whereas for the quantum squeezed field $\rho_{ss} = 0$, $\Pi_\pm = 0$, and then the stationary state of the system is a pure state [25,26]

$$|\Psi_+\rangle = \frac{1}{\sqrt{2N + 1}} \left[\sqrt{N + 1} g + \sqrt{N} e\right].$$

(15)

The pure state is a non-maximally entangled state, and reduces to a maximally entangled state for $N \gg 1$. According to Fig. 2, the pure state admits of the largest amount of entanglement and for $N \gg 1$, the optimum entanglement $E = 1$ can be achieved. In Fig. 1, the mixed state admits of lower level of entanglement, and also the entanglement occurs in a restricted range of the intensity $N$.

We now consider the second model in which the system is driven by the coherent field $(\Omega \neq 0)$ in the absence of the squeezed field $(N = M = 0)$. This is an interesting example where one can create spin squeezing and entanglement with the linear Hamiltonian $H_s$. Typical schemes considered for the generation of spin squeezing involve quadratic Hamiltonians [1–10]. After straightforward but lengthy calculations, we find the following steady-state solutions for the density matrix elements

$$\begin{align*}
\rho_{cc} &= \Omega^4/D, \\
\rho_{ss} &= (\Omega^4 + 2\Gamma^2\Omega^2)/D, \\
\rho_{sg} &= \rho_{gs} = \sqrt{2\Gamma}\Omega (\Omega^2 + 2\Gamma^2)/D, \\
\rho_{es} &= \rho_{se} = \sqrt{2\Gamma}\Omega^3/D, \\
\rho_{eg} &= \rho_{ge} = 2\Gamma^2\Omega^2/D, 
\end{align*}$$

(16)

where $D = 3\Omega^4 + 4\Gamma^2\Omega^2 + 4\Gamma^4$.

In order to analyze the relationship between entanglement and spin squeezing parameters, we write the density matrix of the system

$$\hat{\rho} = \begin{pmatrix}
\rho_{cc} & \frac{1}{\sqrt{2}}\rho_{cs} & \frac{1}{\sqrt{2}}\rho_{es} & \frac{1}{2}\rho_{eg} \\
\frac{1}{\sqrt{2}}\rho_{se} & \frac{1}{\sqrt{2}}\rho_{ss} & \frac{1}{2}\rho_{ss} & \frac{1}{\sqrt{2}}\rho_{sg} \\
\frac{1}{\sqrt{2}}\rho_{ge} & \frac{1}{2}\rho_{gs} & \frac{1}{\sqrt{2}}\rho_{gs} & \frac{1}{2}\rho_{ss} \\
\frac{1}{2}\rho_{ge} & \frac{1}{\sqrt{2}}\rho_{es} & \frac{1}{\sqrt{2}}\rho_{se} & \rho_{gg} 
\end{pmatrix},$$

(17)

and again make use of the negativity criterion for entanglement. There are obviously four eigenvalues of the partial transposition of the matrix (17). It is straightforward to show that one of the eigenvalues is

$$p_1 = \frac{1}{2}\rho_{ss} - |\rho_{eg}|,$$

(18)

whereas the remaining eigenvalues are the three roots of the cubic equation

$$p^3 - \left(1 - \frac{1}{2}\rho_{ss} + \rho_{eg}\right)p^2$$

$$+ \left[(1 - \rho_{ss}) (\frac{1}{2}\rho_{ss} + \rho_{eg}) + \rho_{cc}\rho_{gg} - \rho_{eg}^2 - \rho_{es}^2 - \rho_{ss}^2\right]p$$

$$- \frac{1}{4}\rho_{ss} + \rho_{eg} (\rho_{cc} + \rho_{gg} - \rho_{eg}^2 + \rho_{es}^2 + \rho_{eg}^2) = 0.$$ 

(19)

It is easily verified from Eqs. (16) and (19) that the roots $p_i$ are real and positive for all values of $\Omega$.

Thus, we conclude that the system is entangled when $|\rho_{eg}| > \rho_{ss}/2$, and again the entanglement is related to the Kitagawa and Ueda’s spin squeezing parameter. Figure 3 shows $E$ and the squeezing parameters as a function of $\Omega$. An entanglement appears for $\Omega < \sqrt{2}\Gamma$ and, as predicted, corresponds to the spin squeezing predicted by the parameter $\xi_{ss}^E$. 
Finally, we turn to the third model in which the atoms are driven simultaneously by coherent and squeezed vacuum fields. In this case the density matrix has the same form as Eq. (17) but with the density matrix elements now dependent on $\Omega$ and the squeezing parameters $N$ and $M$. Hence, the condition for entanglement $|\rho_{eg}| > \rho_{ss}/2$ holds. However, it can be shown that one of the roots of Eq. (19) can be negative indicating that one can observe entanglement without spin squeezing. We have checked numerically that this can happen for $\Omega \neq 0$ and $M > 0$. For $M < 0$ the roots are positive for all values of $\Omega$ and $N$. Hence, the condition for entanglement, $|\rho_{eg}| > \rho_{ss}/2$, also holds in this model and, according to Eq. (6), coincides with the condition for spin squeezing.

We now present some numerical calculations that illustrate the above remarks. Figure 4 shows the entanglement measure and the squeezing parameters as a function of $\Omega$ for $N = 0.1$ and $M = -\sqrt{N(N+1)}$. It is evident that similar to the models considered above, the entanglement is related to the spin squeezing given by $\xi_{ss}^S$. The entanglement induced at $\Omega = 0$ decreases with increasing $\Omega$ and vanishes at $\Omega \approx 2.1\Gamma$, indicating that both entanglement and spin squeezing can be observed only for weak driving fields.

In summary, we have examined the relationship between entanglement and spin squeezing parameters in the two-atom Dicke system. Characterizing the spin squeezing parameters by the density matrix elements, we have examined simple models of driven two-atom Dicke systems in which spin squeezing and entanglement arise dynamically. We have found that the Kitagawa and Ueda’s spin squeezing parameter is a better measure of entanglement than the spectroscopic spin squeezing parameter. For the models discussed we have established that the Kitagawa and Ueda’s parameter is the sufficient and necessary condition for entanglement and the degree of entanglement is equal to the degree of spin squeezing.

[1] M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993).
[2] D.J. Wineland, J.J. Bollinger, W.M. Itano, and D.J. Heinzen, Phys. Rev. A 50, 67 (1994).
[3] A.S. Sørensen and K. Melmer, Phys. Rev. Lett. 86, 4431 (2001).
[4] D. Ulan-Orgilkh and M. Kitagawa, Phys. Rev. A 64, 052106 (2001).
[5] X. Wang and K. Melmer, Eur. Phys. J. D 18, 385 (2002).
[6] L.K. Thomsen, S. Mancini, and H.M. Wiseman, Phys. Rev. A 65, 061801 (2002).
[7] S.D. Jenkins and T.A.B. Kennedy, Phys. Rev. A 66, 043621 (2002).
[8] X. Wang and B.C. Sanders, quant-ph/0302014.
[9] A.R. Usha Devi, X. Wang, and B.C. Sanders, quant-ph/0304051.
[10] A.S. Sørensen, L.M. Duan, J.I. Cirac, and P. Zoller Nature 409, 63 (2001).
[11] M.A. Nielsen and I. Chuang, Quantum Computation and Quantum Information, (Cambridge University Press, Cambridge, 2000).
[12] W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998); S. Hill and W.K. Wootters, Phys. Rev. Lett. 78, 5022 (1997).
[13] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[14] P. Horodecki, Phys. Lett. A 232, 333 (1997).
[15] J. Hald, J.L. Sørensen, C. Schori, and E.S. Polzik, Phys. Rev. Lett. 83, 1319 (1999).
[16] A. Kuzmich, N.P. Bigelow, and L. Mandel, Europhys. Lett. 43, 481 (1998).
[17] A. Kuzmich, L. Mandel, and N.P. Bigelow, Phys. Rev. Lett. 85, 1594 (2000).
[18] A. Banerjee, quant-ph/0110032.
[19] L. Zhou, H.S. Song, and C. Li, J. Opt. B: Quantum Semiclass. Opt. 4, 425 (2002).
[20] A. Messikh, Z. Ficek, and M.R.B. Wahiddin, J. Opt. B: Quantum Semiclass. Opt. 5, L4 (2003).
[21] R.H. Dicke, Phys. Rev. 93, 99 (1954).
[22] R.H. Lehmberg, Phys. Rev. A 2, 883 (1970).
[23] Z. Ficek and R. Tanaš, Phys. Rep. 372, 369 (2002).
[24] B.J. Dalton, Z. Ficek, and S. Swain, J. Mod. Opt. 46, 379 (1999).
[25] G.M. Palma and P.L. Knight, Phys. Rev. A 39, 1962 (1989).
[26] G.S. Agarwal and R.R. Puri, Phys. Rev. A 41, 3782 (1990); Phys. Rev. A 49, 4968 (1994).