Quantum Boundaries in Minkowski Space

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Abstract

It is claimed in another paper that the collapse of a quantum mechanical wave function is more than invariant, it is trans-representational. It must occur along a fully invariant surface. The obvious surface available for this purpose is that of the backward time cone of the collapse event as proposed by Hellwig and Kraus. This collapse is widely believed to result in paradoxical causal loops that cannot be removed by special relativistic and/or standard quantum mechanical considerations alone. However, the paradox is resolved when we apply the qRule foundation theory that is developed in the other paper. The causal and temporal orders of state reduction are then found to be in agreement with one another, and the resulting boundaries in Minkowski space are shown to have a novel architecture that limits the range of a Hellwig-Kraus reduction in space and time. Although these boundaries have been worked out using the qRules, they should be the same for any foundation theory that treats the collapse of a wave in an invariant way, and requires that a collapse destroys the possibility of any further influence on itself – as do the qRules. Keywords: measurement, state reduction, wave collapse.

Introduction

The collapse of a wave function is an undeniable feature of individual quantum mechanical systems. However, the collapse of a state along a $t = \text{constant}$ surface of an arbitrary coordinate system is unbelievable, inasmuch as nature does not recognize a surface that is so obviously constructed by humans. For this reason, a foundation theory must provide for the collapse of a wave along an invariant surface that is independent of coordinate representations.

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The collapse of a wave is not just invariant, it is trans-representational; that is, it is independent of any choice of basis states. Furthermore, it will collapse along the surface of the backward time cone that is here called a conic surface. This is the Hellwig and Kraus state reduction \cite{1} that has been widely dismissed as being causally problematic \cite{2}. The supposed paradox cannot be resolved by relativity and quantum mechanics alone because a foundation theory is also required. This will govern the collapse and impose constraints that establish an unambiguous temporal and causal order between reductions. When that is done, it is shown below that a ‘later’ collapse cannot causally penetrate the backward time cone of a ‘former’ collapse. And since a collapse occurs against a background of countless widely disbursed prior collapses, the spatial-temporal extent of any collapse in Minkowski space is limited.

One consequence of a Hellwig-Kraus reduction is that the newly collapsed state must also be the initial state of the next phase of evolution. So initial states as well as collapsed states are conically defined. Therefore the most natural quantum mechanical state is a function mapped onto the conic surface of the backward time cone of an event in space-time. It is shown that the dynamic principle projects these functions forward to successor conic states whose vertices lie along a specified world line.

**Non-Local Correlations**

Claims regarding a Hellwig-Kraus causal ambiguity are usually advanced by referring to non-local correlations. Consider a pair of particles $p_1$ and $p_2$ that are created by a decay to become correlated in the spin zero state.

\[
\psi_0(p_1, p_2) = 2^{-1/2} \{ p_1(\uparrow)p_2(\downarrow) - p_1(\downarrow)p_2(\uparrow) \} \tag{1}
\]

The first particle moves to the left in Fig. 1 and the second particle moves to the right. Initially they both have an uncertain spin direction as indicated by

![Figure 1: First non-local reduction](image)
Imagine that the first particle is measured to have spin-up at an event $A$ in Fig. 1, causing a Hellwig-Kraus state reduction. In that case $p_2$ will go spin-down when it intercepts the backward light cone of event $A$ at event $b$. If $p_2$ is measured later at event $B$ it will of course record spin-down. The result is the state $p_1(\uparrow)p_2(\downarrow)$. Capital bold face letters indicate events that ‘cause’ state reduction like measurements; and lower case bold face letters identify events that are not vertices of reduction sites.

Event $B$ also results in a state reduction. The effect of $B$ on $p_1$ might conceivably be the same as the effect of $A$ on $p_2$, where $p_1$ goes spin-up the moment it intercepts the backward light cone of event $B$ at event $a$ as shown in Fig. 2. So when $p_1$ is later measured at $A$ it will record spin-up. The final results are then the same as those shown in Fig. 1, but this possibility leads to an odd circularity that is characteristic of a causal loop: $A$ causes $b$ causes $B$, and $B$ causes $a$ causes $A$.

These two particles are shown to follow straight lines even though they are waves spreading out over the indicated paths. This does not matter because the rules developed here are not concerned with paths. We are concerned only with the location of reduction events like $A$ and $B$ and their associated conic states. The straight lines in these figures may be thought of as having heuristic value only.

**Form of qRule Equations**

The *qRules* are three rules given in another paper that govern the collapse of a wave function. These rules generate *qRule equations* that are of the form
\[ U(t) = u(t) + u'(t) + \ldots \]

where \( t \) is conic time as opposed to conventional Lorentz time. The Lorentz time at an event \( x \) is defined over a horizontal plane that passes through \( x \), whereas conic time is defined over the conic surface with its vertex at \( x \).

It is shown in Appendix I how a conventional Lorentz wave function \( \psi(x, y, z, t) \) with its origin at \( x \) can be mapped into a conic wave function \( \xi(r, t) \) with its vertex at \( x \), where \( r \) represents a variable that is defined (in Appendix I) on the conic surface in 3 + 1 space. We say that the conic wave function \( \xi(r, t) \) mirrors the Lorentz wave function \( \psi(x, y, z, t) \) in a flat Minkowski space. The qRule component \( u(t) \) is then given by the square modulus of \( \xi(r, t) \) with its \( r \) variable integrated out, making a qRule component a function of \( t \) only. It is also shown in Appendix I that a conic dynamic principle that mirrors the standard dynamic principle will project the conic function \( \xi(r, t) \) into successor vertices along the indicated world line.

A qRule equation is always given by a capital \( U \) as a function of time, but the components \( u(t) \) or \( u'(t) \) may be expressed differently. For instance, the component \( u(t) \) consisting of an atom \( a \) and a molecule \( m \) can be written \( u(t) = am(t) \).

Each component in a qRule equation is ‘complete’ in that it implicitly contains all the particles in the universe (Ref. 3). The plus sign in a qRule equation always indicates a discontinuous quantum jump that is also irreversible.

**Decay Application**

The particle pair \( p_1 \) and \( p_2 \) in Eq. 1 is assumed to be created at an event \( 0 \) that results from a decay that begins at a prior conic time \( t_{00} \) along some preferred world line. If the pair is created by the decay of a composite particle \( p_c \), the corresponding qRule equation is given by

\[ U(t \geq t_{00}) = p_c(t) + p_1 p_2(t) \] (2)

where the \( p \)’s in this equation are the qRule values of the designated particles. They represent the square modulus of the conic function of each particle with its independent variables integrated out. In particular, the component \( p_1 p_2(t) \) is the qRule value of the zero spin wave function \( \psi_0 \) in Eq. 1. The fact that the first particle goes to the left (in Fig. 1) and the second particle goes to the right is a distinction that is lost to this component – because it has been integrated out along with the spin distinction. In addition, the component
$p_c(t)$ in Eq. 2 is the qRule value of the composite particle that produces the decay particles. Both of these are trans-representational components that are derived from wave functions on conic surfaces in Minkowski space, where $t$ is the conic time referring to these surfaces. Their evolution is governed by the dynamic principle operating on the underlying wave function. Each component is multiplied by an environmental term representing the rest of the universe, thereby satisfying the requirement that it is ‘complete’. The choice of $t$ in these equations does not affect the outcome as will be demonstrated.

The second component in Eq. 2 is underlined, meaning that it is a ready component. Ready components are not empirically real, so the component $p_1 p_2(t)$ in Eq. 2 refers to particles that do not yet exist at time $t$ in that equation. The non-underlined component $p_c(t)$ is called a realized component. It does exist at time $t$ in that equation.

The second component $p_1 p_2(t)$ is zero at time $t_0$ and increases in time as probability current flows to it from the first component $p_c(t)$ in Eq. 2. The gap between them represented by the + sign is the decay interaction that is discontinuous and irreversible. The qRules tell us that the probability current gives the probability per unit time that the second component will experience a stochastic hit. Only ready components can be stochastically chosen according to the qRules. If a hit occurs at a time $t_0$ (the time of event 0), then the rules tell us that there will be a collapse of Eq. 2 to

$$U(t \geq t_0 > t_0) = p_1 p_2(t)$$

where the first component in Eq. 2 goes to zero. The wave function associated with Eq. 3 is also mapped onto backward time-cone surfaces – in this case on vertices of the successors of event 0, where the time variable is the conic time referring to these vertices.

Equations 2 and 3 reflect the fact that the collapse of a wave function in quantum mechanics is a trans-representational affair. The dynamic principle directs the evolution of a qRule equation, but the qRules govern the stochastic process that interrupts its continuous flow. The foundation theory of Ghirardi et. al. incorporates a collapse directly into the dynamic principle [5], but that theory is not trans-representational.
Hellwig-Kraus Application

When the spin-measuring devices $M_1$ and $M_2$ of particle $p_1$ and $p_2$ are later introduced, Eq. 3 becomes the qRule equation

$$U(t \geq t_0) = p_1 p_2 \otimes M_1 M_2(t) + [p_1(|\uparrow\rangle M_1)p_2(|\downarrow\rangle) \otimes M_2(t)$$

$$+ [p_1(|\downarrow\rangle M_1)p_2(|\uparrow\rangle) \otimes M_2(t)$$

$$+ [p_1(|\downarrow\rangle M_1)p_2(\uparrow) \otimes M_2(t)$$

$$+ [p_1(|\uparrow\rangle M_1)p_2(\downarrow) \otimes M_1(t)]$$

where both measuring devices are on standby in the first component of Eq. 4, and the four ‘ready’ components (on the right) are zero at $t_0$. In the ready component of the first row the first particle engages the spin-measuring device $M_1$ (in square brackets), and in the second row the second particle engages $M_2$. The third and fourth rows are similar except that they provide for the reverse spin measurements.

Probability current begins to flow from the first component to the ready components in the first and third rows when the first particle interacts with the measuring device $M_1$ sometime after $t_0$. Current will begin to flow to the ready components in the second and fourth rows when the second particle interacts with $M_2$. So all four ready components are exposed to the possibility of a stochastic hit. Figure 3a is a graphic description of Eq. 4. The gray area in that figure represents the Minkowski region of interaction where the ready components in Eq. 4 become non-zero (i.e., where the particle interacts with the detector).

Neither one of the world lines in Fig. 3a defines the time $t$ in Eq. 4. That time specifies the chosen orientation (i.e., the chosen Lorentz observer) in the
Minkowski space. Since the probability of any one of these ready states being stochastically chosen is independent of the choice of \( t \), Eq. 4 is invariant under \( t \). If another \( t \) is chosen, then the world lines in Fig. 3 will be rotated to the left or to the right (as with a Lorentz transformation), but the orientation of the conic surfaces will be unchanged. The graphic relationships in Fig. 3 will be unchanged, except event simultaneity.

Suppose the first row in Eq. 4 is stochastically chosen at a time \( t_A \). The system is then reduced to

\[
U(t = t_A \geq t_0) = [p_1(\uparrow)M_1]p_2(\downarrow) \otimes M_2(t) \tag{5}
\]

where \( t_A \) is the time of the stochastic hit at event \( A \) that appears along the world line of \( p_1 \) (see Fig. 3b representing Eq. 5). It is again understood that the wave functions representing \( p_1, p_2, M_1, \) and \( M_2 \) in this figure are spread out over the conic surface of the vertex of event \( A \) in Fig. 3b. Equation 5 not only represents event \( A \), it also represents the cut-off event \( b \) that is simultaneous with \( A \) in conic time. We make no assumption as to the world lines of the measuring devices.

Further evolution carries the resulting conic state forward following event \( A \) until \( p_2 \) interacts with \( M_2 \). A new ready state then emerges giving

\[
U(t \geq t_A > t_0) = [p_1(\uparrow)M_1]p_2(\downarrow) \otimes M_2(t) \tag{6}
\]

\[
+ [p_1(\uparrow)M_1][p_2(\downarrow)M_2](t)
\]

which is graphically represented in Fig. 3c in which the shaded area is the region of the new interaction that gives rise to the ready component in the second row of Eq. 6 when \( p_2 \) encounters \( M_2 \).

The evolution of the second particle after event \( b \) in Fig. 3c repeats part of the evolution that has already occurred in Fig. 3a. The latter part of this evolution involves the interaction \([p_2(\downarrow)M_2] \) between the second particle and its measuring device. This part is not empirically significant because its repetition consists in its appearance in both ‘ready’ components of both Eqs. 4 and 6. An interaction repetition like this would occur even if the time in Eqs. 4 and 6 referred to Lorentz time, so it is not a characteristic of a Hellwig-Kraus reduction. It is a feature of any correlated state reduction. However, the free particle part of the repetition certainly does involve the realized component in Eqs. 4 and 6, so it does (in principle) have empirical significance. It is also characteristic of a Hellwig-Kraus collapse. But this free particle part is not really measurable because it exists independent of a measurement interaction.
Let a second state reduction occur at event $B$ at time $t_B$, giving the final result

$$U(t = t_B) = [p_1(\uparrow)M_1][p_2(\downarrow)M_2](t)$$

Equation 7 appears in Fig. 3d.

Since the qRules are trans-representational they give us an objective account of what happens. Equation 4 causally precedes Eq. 5, and that causally precedes Eq. 6, and that causally precedes Eq. 7. Causal priority is established by the formal priority among these equations. A collapse equation comes into existence only when its predecessor disappears. We state the corollary.

**Corollary:** The evolution of a qRule equation cannot affect the evolution of a previously ‘collapsed’ qRule equation – since the latter no longer exists.

This means that the evolution described in Eq. 6 can have no influence on the evolution described in Eq. 4; for when Eq. 6 is actively evolving, Eq. 4 no longer exists. Therefore, event $B$ cannot influence event $A$. This is indicated in Fig. 3d where the backward time cone of event $B$ is not allowed to ‘penetrate’ the backward time cone of event $A$.

It is the qRules that govern the causal order of reduction, not relativity or standard quantum mechanics, and these rules are perfectly clear concerning priority. This is shown graphically in the Minkowski space of Fig. 4. The more darkly shaded area (below $A$ in Fig. 4) indicates the region of Minkowski space that evolves according to Eq. 4; and the more lightly shaded area (below $B$) indicates the region that evolves according to Eq. 6, where the lighter area can have no influence on the darker area because the latter is causally prior. The reductions in Fig. 4 go to infinity on the left and on the right as expected of a Hellwig-Kraus collapse. However, the event $B$ collapse is limited in that it cannot penetrate the event $A$ reduction. This will generally be the case. The universe is full of prior reductions that cannot be penetrated by either events $A$ or $B$; and as a result, both reductions will be limited in the space-time extent.

![Figure 4: Causal and temporal order – non-penetration](image-url)
of their influence. This is not just true of correlated reductions, but will also be true of independent reductions (see Appendix II).

Independent Systems

Following event $B$, the measuring device $M_1$ (that now includes particle $p_1$) evolves independent of the measuring device $M_2$ (that now includes $p_2$). These two systems can be described along two different conic world lines designated $t_1$ and $t_2$, where $t_1$ begins with the vertex at event $A$, and a conic time $t_2$ begins with the vertex at event $B$. It is not necessary to introduce these dual times, but doing so helps to emphasize the spatial separation as well as the independence of the two systems. The underlying conic function is then of the form $\xi(t_1; t_2)$ which comes about by using two different origins (on the horizontal $x$-axis) to describe two widely separated systems. As a result, a single cone spreading over two systems of this kind is replaced by an envelope that drapes over both – like the lightly shaded area over events $A$ and $B$ in Fig. 4. The qRule equation following event $B$ is then written

$$U(t_1 \geq t_A; t_2 \geq t_B) = [p_1(\uparrow)M_1(t_1) + i_1(t_1)]\{p_2(\downarrow)M_1(t_2) + i_2(t_2)\}$$

where $i_1$ is an interaction that is located in some part of the lightly shaded envelope above event $A$ in Fig. 4, and $i_2$ is another interaction that is located in the lightly shaded envelope above event $B$. We simplify the equation by dropping the time dependence in each component

$$U(t_1 \geq t_A; t_2 \geq t_B) = [p_1(\uparrow)M_1 + i_1]\{p_2(\downarrow)M_1 + i_2\}$$ (8)

where subscript-1 states are understood to be on the $t_1$ conic surface, and subscript-2 states are understood to be on the $t_2$ conic surface.

Equation 8 might not seem entirely correct according to the qRules. The recipient state $i$ in each bracket of the equation does not appear to be a complete component; and if that were true, it would not be a ready component. However, the equation can be written in the form

$$U(t_1 \geq t_A; t_2 \geq t_B) = p_1(\uparrow)M_1[p_2(\downarrow)M_2 + i_2] + i_1[p_2(\downarrow)M_2 + i_2]$$

The brackets in this equation have a square modulus that is constant in time, so the current from the first component comes solely from $p_1(\uparrow)M_1$ and goes exclusively to $i_1$. Since both components are clearly complete, the first must be a realized component and the second is a ready component. Both $i$’s in Eq. 8
can be understood in this way. We therefore keep the form of Eq. 8. It can be
generalized to any number of independent systems in a way that satisfies the
qRules.

In $2 + 1$ space the mountaintops in Fig. 4 do not appear in front or in back
of one another. Rather, they are superimposed on one another like mountains
on a flat terrain. A new peak will not penetrate an old peak on that terrain
because the above corollary holds for all completed state reductions – including
independent as well as correlated reductions as will be shown in Appendix II.
These mountaintops all build on top of one another without penetration.

Another Lorentz Observer

It is important to understand how Eq. 4 and Fig. 3 play out relative to a Lorentz
frame in which event $A$ occurs (causally) before event $B$ but appears to follow
event $B$ in Lorentz time. Figure 5a is identical with Fig. 3a where the regions
of measurement interaction for $p_1$ and $p_2$ are indicated by the darkened world
lines. Figure 5b looks at the same interactions relative to a Lorentz frame in
which the second particle is at rest. This is called the ‘primed’ frame.

Equation 4 has the same form relative to the primed frame as it does relative
to the unprimed frame. The difference is only in the way that current flows from
the realized component to the four ready components. In the unprimed frame
(Fig. 5a) current flows simultaneously into all four components for as long as
it takes for a stochastic hit on the first component (in our example). In the
primed frame (Fig. 5b) current begins to flow into the second particle before
it begins to flow into the first particle – in Lorentz time. However, this current will

![Figure 5: Reduction in $t'$ frame](image-url)
run out without the second particle being stochastically chosen, while current continues to flow into the first particle. As a result, the first particle is the first to be stochastically chosen at event $\mathbf{A}$ (in Fig. 5c), thereby cutting off the second particle at event $\mathbf{b}$, which is again simultaneously with event $\mathbf{A}$ in conic time. The second particle is subsequently revived at a time that allows it to again interact with the measuring device $M_2$, thereby allowing it to be stochastically chosen at event $\mathbf{B}$ (Fig. 5d) after event $\mathbf{A}$ in conic time – although it is before event $\mathbf{A}$ in Lorentz time. The second particle does not exactly repeat its prior history. The first time that it interacts with the measuring device it is not stochastically chosen, but the second time (following event $\mathbf{b}$) it is stochastically chosen. The first time the probability of a hit on event $\mathbf{B}$ is 0.25, the second time it is 1.0. This repetition has no empirical significance because both interactions appear inside ready components. In general, stochastic interactions are not empirically real in qRule theory. Stochastic hits are empirically real.

Figures 3 and 5 represent two different Lorentz observers, where the causal and the temporal order of events are the same in both frames. Only in Lorentz time does there appear to be some disagreement between the causal and temporal orders. We attribute this to fact that Lorentz time is given along artificiality defined horizontal planar surfaces.

**Other Foundation Theories**

The structure in Figs. 3 and 5 should be the same for any foundational theory that is viewed in an invariant way. If the theory in question provides for the collapse of a wave like Eq. 1, and if the collapse travels backward over a conic surface in a way that destroys the possibility of any further influence on itself, then it should produce the same Minkowski architecture that appears in Fig. 4. For any such theory, in any Lorentz frame, the measurement interaction of the second particle will run its course without result (in our example) allowing event $\mathbf{A}$ to be chosen first. The resulting collapse will re-start the second particle at an event like $\mathbf{b}$ in Fig. 3b that is conically simultaneous with $\mathbf{A}$, so event $\mathbf{B}$ will occur after event $\mathbf{A}$ in conic time. As in our qRule analysis, it is the reduction theory that governs the casual order, not relativity or standard quantum mechanics. The result will be the succession of mountain peaks like those shown in Fig. 4, where a peak in background occurs causally and temporally after a peak in foreground, and where both are limited in space and time by other reductions.
Conclusion

A paradoxical causal loop seems to appear when the collapse of a quantum mechanical wave function is viewed as a Hellwig-Kraus reduction in Minkowski space. This paradox cannot be removed by special relativity and/or standard quantum mechanics alone, but requires the constraints imposed by a suitable foundation theory governing the collapse. In this paper the qRule foundation theory is used to resolve this causal difficulty.

We define conic time $t$ over the surface of a backward time cone of some event $x$. A quantum mechanical wave function $\xi(t)$ is also mapped onto that surface. This function is used instead of the usual quantum mechanical wave function $\psi$ that is defined on a horizontal surface going through $x$.

A Hellwig-Kraus reduction describes two correlated space-like reduction events $A$ and $B$, and it is found that the qRules establish a definite causal priority between them. There is no causal loop or temporal ambiguity between these two events when the qRules govern the collapse. The causal and conic temporal orders of events $A$ and $B$ are the same. The resulting landscape in Minkowski space is shown in Fig. 4, where each mountaintop in that figure is a separate reduction. The peak in the background of Fig. 4 will occur causally and temporally after the peak in the foreground. The background reduction cannot ‘penetrate’ or causally influence the interior of the foreground reduction, as shown in the figure. Any reduction will take place against a background of countless prior reductions that cannot be penetrated, thereby limiting its range in space and time. This will be true even if the reduction is independent (i.e., not a correlation) as is shown in Appendix II.

It is my belief that the structure of Fig. 4 will be similar for any foundation theory that is viewed in an invariant way, so long as the theory requires that a collapse destroys the possibility of any further influence on itself – as do the qRules.

The dynamic principle is understood to influence the temporal development of a qRule component through the underlying function $\xi(r,t)$. This function is defined in this paper and evolves in time as described in Appendix I. However, the qRules govern state reduction and therefore interrupt the dynamic process under the right conditions. More will be said in a subsequent paper about the relationship between the dynamic principle, the qRule equations, and the architecture of state reduction in Minkowski space.
Appendix I

A particle is initially located at \( t_0 \) between \( x_1 \) and \( x_2 \) in Fig. 6. Its wave function \( \psi(x, t) \) advances along the +x-axis occupying the shaded area in flat Minkowski space. This function is mapped onto the dark line on the surface of the conic section with its vertex at \( t_0 \). Each event on that surface has assigned coordinates \( r \) (perpendicular to the \( t \)-axis) and \( t_0 \), so we project \( \psi(x, t_0) \) onto the conic surface giving

\[
\xi(r_2 \geq r \geq r_1, t_0) = \xi(r, t_0)
\]

where \( x - x_1 = r - r_1 \) holds for all \( x \) and \( r \). If \( x_1 \) is negative, this function can be extended over the vertex to the rising side of the conic surface without difficulty. More generally we say that any wave function in flat space that is mapped onto a conic section with its vertex at \( t \) is given by \( \xi(r, t) \).

We want to know how to write the dynamic principle for \( \xi(r, t) \). Assume that it is analogous to the Schrödinger equation in the non-relativistic case, with a dynamic operator \( \hat{D} \) in place of the Hamiltonian.

\[
\hat{D}\xi(r, t) = -ih\partial_t\xi(r, t)
\]

This equation is correct if we let

\[
\hat{D} = \frac{\hbar^2}{2m}\partial_r^2
\]
The function that is mapped onto successive conic surfaces is therefore propagated in time by a dynamic principle that mirrors the Schrödinger equation for a non-relativistic system. Equations 9 and 10 applied to \( \xi(r,t) \) will advance it from one conic surface to another.

The same construction can be applied to each of the four components of a Dirac wave function, in which case the dynamic operator \( \hat{D} \) mirrors the Dirac Hamiltonian. The amplitude in the above equations then has the required four components.

In \( 2 + 1 \) space the surface is a cone with a vertex time \( t \). A function \( \psi \) defined on a horizontal plane with its origin at the vertex is projected onto the cone’s surface. Its \( x, y \) coordinates are mapped onto \( r_x, r_y \) coordinates that intercept the conic surface and are perpendicular to the \( t \)-axis. The dynamic principle causes these surfaces to evolve along the specified world line.

In \( 3 + 1 \) space the incoming ‘spherical’ conic surface will converge on a vertex time \( t \). The particle volume that is simultaneous with \( t \) contains the initial condition \( \psi_0(x, y, z, t) \), and each of these values can be mapped onto the incoming spherical surface when (prior to \( t \)) the surface passes through each particle part at its time \( t \) location.

Our claim is that the \( x \)-axis is superfluous, and that the only coordinates we need are those associated with a conic surface. If given the wave function \( \xi(r, t) \) of a particle along the surface with vertex \( t_0 \) in Fig. 7, the dynamic principle will carry it into all succeeding surfaces like that of \( t_1 \) in that figure.

![Figure 7: General conic wave](image)

The ‘real’ location of the particle – in the shaded area of Fig. 6 or in the shaded area of Fig. 7 – is our choice. If we say the particle is initially specified at time \( t_0 \), then it will occupy the shaded area of Fig. 6 when we understand ‘simultaneous’ to be the horizontal surface through \( t_0 \). But it will occupy the shaded area of Fig. 7 when we understand ‘simultaneous’ to be the conic surface
with \( t_0 \) at its vertex.

**Appendix II**

It is possible to draw a Minkowski diagram using one vertex time \( t \), with all the particles in the system mapped onto the surface of its backward time cone. We have seen that it is also possible to use a diagram with two or more vertices as will be done in the present case of two independent particles.

Consider a nucleus \( n_1 \) with a vertex \( t_1 \) as shown in the diagram of Fig. 8a, together with a second nucleus \( n_2 \) with a vertex \( t_2 \). Both nuclei are radioactive, where the first decays at event \( A \) in Fig. 8b. Multiple time descriptions like this are possible only when the vertex events so identified have a space-like relationship to each other. In this case the vertices are chosen to follow the separate particles, since there is no reason why the time scale must be the same as that of the Minkowski observer. The total wave function in Fig. 8a will be spread over both peaks. It will be mapped onto the dashed lines joined at their intersection as shown in the figure.

The qRule equation follows Eq. 8 for these independent nuclei.

\[
U(t_1; t_2) = [n_1 + p_1p'_1][n_2 + p_2p'_2]
\]

where \( n_1 \) and \( p_1p'_1 \) are specified relative to the \( t_1 \) vertex, and \( n_2 \) and \( p_2p'_2 \) are specified relative to the \( t_2 \) vertex. The component \( p_1p'_1 \) represents the decay products of \( n_1 \), and \( p_2p'_2 \) represents the decay products of \( n_2 \). The two ready components are initially zero. Probability current will flow from both \( n_1 \) and \( n_2 \) in this equation to the corresponding ready components, marking both candidates for a possible stochastic choice.

The first ready component is stochastically chosen at Event \( A \), after which

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8}
\caption{Two independent systems – first reduction}
\end{figure}
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the qRule equation is

\[ U(t_1 = t_A; t_2) = p_1 p'_1 (n_2 + p_2 p'_2) \]  

(11)
as shown in Fig. 8b. At event A the first nucleus is reduced to its decay products, although they do not yet appear in Fig. 8b. The second nucleus in Eq. 11 continues unaffected in its unstable state, and is represented by the line that extends beyond the shaded area. That line is the same length in Fig. 8b as it is in Fig. 8a because it is not reduced by the first decay.

Further evolution of the system is described by same equation

\[ U(t_1 \geq t_A; t_2) = p_1 p'_1 (n_2 + p_2 p'_2) \]  

(12)except that the magnitude of the component \(p_2 p'_2\) increases in time as represented by the longer line in Fig. 9a. The decay particles \(p_1\) and \(p_2\) are also shown in Fig. 9a. They are actually waves spreading out in all directions from the vertex, but they are shown as two distinct world lines as are all other particle paths in these diagrams. The qRules do not specify paths, only reduction events and their associated states.

With a stochastic hit on \(p_2 p'_2\) in Eq. 12, the reduction at event B will yield

\[ U(t_1 \geq t_A; t_2 = t_B) = p_1 p'_1 p_2 p'_2 \]  

(13)
This equation is shown in the Minkowski diagram of Fig. 9b, where the decay products \(p_2 p'_2\) do not yet appear.

As before, the lighter shaded reduction under event B in Fig. 9b does not penetrate the darker shaded reduction defined by event A. This is because the B reduction in Eq. 13 occurs causally after the A reduction in Eq. 11, so it can have no influence on the evolution leading to Eq. 11.

![Figure 9: Two independent systems – second reduction](image-url)
It was shown in the text that correlated reductions are causally sequenced so that a later reduction will not penetrate a former reduction. We see here that the same is true when the reductions are independent of one another. A distinct causal order is therefore characteristic of all reductions. As before, the reduction in Fig. 8b may appear to go back infinitely far in time and extend infinitely far in space, but that will not happen. Every reduction will find a floor of other mountaintops that will support it and limit its range in space and time.

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