Spherically symmetric Jordan-Brans-Dicke quantum gravity with de Broglie Bohm pilot wave perspective

Hossein Ghaffarnejad

Department of Physics, Semnan University, P.O.Box 35195-363, Iran

Abstract

We obtain two dimensional analogue of the Jordan-Brans-Dicke (JBD) gravity action described in four dimensional spherically symmetric curved space time metric. There will be two scalar fields, namely, the Brans Dicke (BD) \( \phi \) and scale factor of 2-sphere part of the space time \( \psi \). We obtained suitable duality transformations between \((\psi, \phi)\) and \((\rho, S)\) where \( \rho \) and \( S \) is respectively amplitude and phase part of the corresponding de Broglie pilot wave function \( \Psi(\rho, S) = \sqrt{\rho}e^{iS} \). Covariant conservation of mass-energy current density of particles ensemble \( J_\alpha = \rho \partial_\alpha S \), is established by applying a particular dynamical conformal frame described by \((\rho, S)\).

1 Introduction

Over half a century of collective study has not diminished the fascination of searching for a consistent theory of quantum gravity [1].

The perturbative quantum field theory in curved space time, [2,3,4] schemes foundered on intractable ultraviolet divergences and gave way to supergravity, the super-symmetric extension of standard general relativity. In spite of initial optimism, this approach succumbed to the same disease and was eventually replaced by the far more ambitious superstring theories [5]. Superstring theory is now the dominant quantum gravity programme in terms of the number of personnel involved and the number of published papers, per year, per unit researcher.

The nonperturbative canonical quantum gravity [6] or quantum geometry attempts for quantizing the metric variables where rather nave and took on various forms according to how the intrinsic constraints of classical general relativity are handled. In the most popular approach, the constraints are imposed on the state vectors and give rise to the famous Wheeler-De Witt

\[1\] E-mail address: hghafarnejad@yahoo.com.
equation arguably one of the most elegant equations in theoretical physics, and certainly one of the most mathematically ill-defined.

The enormous difficulty of the canonical quantum gravity scheme eventually caused it to go into something of a decline, until new life was imparted with Ashtekar’s discovery of a set of variables in which the constraint equations simplify significantly. This scheme slowly morphed into “loop quantum gravity“: an approach which has, for the first time, allowed real insight into what a nonperturbative quantization of general relativity might look like. A number of genuine results were obtained, but it became slowly apparent that the old problems with the Wheeler De Witt equation were still there in transmuted form and the critical Hamiltonian constraint was still ill-defined.

In summary, the canonical quantum gravity and particularly loop quantum gravity has self-contained treaties and it may to be hopeful (see [7] and references therein).

Quantum general relativity or canonical quantum gravity is based to two fundamental building blocks of modern physics, namely, general covariance (the general relativistic principle of background independence) and the uncertainty principle of Copenhagen quantum mechanics. However the well known Copenhagen quantum mechanics has still many fundamental problems: non-locality, non-causality, the measurement problem, the nature of reality and etc. The Bohr-Einstein debate on these problems is followed by other researchers and whose challenge reduces to the de Broglie-Bohm casual quantum mechanics [8-14]. The de Broglie-Bohm ontology of the quantum physics presents new features of quantum gravity theory (see for instance [15-18]). In the standard form of this theory, the classical gravity should be viewed as a classical field containing quantum trajectories originated from quantum potential.

The author exercised conformaly flat space time version of the JBD gravity [19] previously and obtained whose de Broglie-Bohm particle interpretation in [20]. In the present paper, we use JBD scalar tensor gravity described by spherically symmetric curved space-times and obtain whose corresponding de Borglie-Bohm quantum gravity in which amplitude and phase of the pilot wave can be described in terms of the BD scalar field and scale factor of 2-sphere part of the metric. As a future work, results of the paper can be used to study physical features of spherically symmetric dynamical space times such as black holes evaporation, thermodynamics and etc.
2 Spherically symmetric JBD space time

Let us start with JBD gravity theory [19] given by

\[ I_{JBD} = \frac{1}{16\pi} \int d^4 x \sqrt{g} \left\{ \phi R - \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\}. \]  

(2.1)

Using a dimensionless scalar field as

\[ \sigma = (2\omega + 3)^{\frac{1}{2}} \ln G\phi, \]  

(2.2)

and particular conformal metric transformation

\[ \bar{g}_{\mu\nu} = g_{\mu\nu} \exp \left( \frac{\sigma}{\sqrt{2\omega + 3}} \right) = g_{\mu\nu} G\phi \]  

(2.3)

the action (2.1) leads to the following form.

\[ \bar{I}_{JBD} = \frac{1}{16\pi G} \int \sqrt{\bar{g}} d^4 x \{ \bar{R} - \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \} \]  

(2.4)

which describes minimally coupled scalar-tensor gravity theory where \( \bar{I}_{EH} = \frac{1}{16\pi G} \int \sqrt{\bar{g}} d^4 x \bar{R} \), is well known Einstein-Hilbert action functional. Usually \( \bar{g}_{\mu\nu} \) and \( g_{\mu\nu} \) is called ‘Einstein‘ and ‘Jordan‘ frames respectively [21]. \( G \) is named the Newton’s gravitational coupling constant. Also \( g (\bar{g}) \) is absolute value of determinant of the metric \( g_{\mu\nu} (\bar{g}_{\mu\nu}) \) with Lorentzian signature (\(-,+,+,+)\).

\( \omega \) is dimensionless BD parameter and whose present limits based on time-delay experiments [22-25] require \( \omega \geq 4 \times 10^4 \). When \( \omega \to \infty \), then the JBD gravity theory leads to the Einstein’s general relativity theory and \( \omega = -1 \) is due to a fundamental symmetry of strings [26]. This is a symmetry of string amplitudes which relates large and small radius of compactification. Also negative values of the BD parameter \( \omega \) come from Kauza-Klein theory, when these alternative theories in (4+h) dimensions reduce to a generalized JBD theory after the dimensional reduction in the zero modes approximation such as \( \omega = -(1 + \frac{1}{h}) \) [27].

General form of spherically symmetric curved space time metric is given by

\[ ds^2 = g_{ab} dx^a dx^b + \psi^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \]  

(2.5)

where 2-sphere conformal factor \( \psi^2 \) is called dilaton field and \( g_{ab} \) is described in terms of two dimensional coordinates \( x^a \) with \( a \equiv 1, 2 \). Thomi
et al [28] obtained previously two dimensional analog of the Einstein-Hilbert action by using (2.5) such as follows.

\[
\bar{I}_{EH} = \frac{1}{16\pi G} \int d^4 x \sqrt{|\bar{g}|} \bar{R} = \frac{1}{2G} \int d^2 x \bar{g}\{1 + \bar{g}^{ab} \partial_a \bar{\psi} \partial_b \bar{\psi} + \frac{1}{2} \bar{R} \bar{\psi}^2\} \tag{2.6}
\]

where ‘over-bar’ denotes to the Einstein frame. \([4]g\) (\(g\)) and \([4]R\) (\(R\)) is absolute value of determinant of the four (two) dimensional metric and Ricci scalar in four (two) dimensional space time respectively. Applying (2.5) and (2.6), one can obtain two dimensional analogue of the action (2.4) described in the Einstein frame as

\[
\bar{I}_{JBD} = \frac{1}{2G} \int d^2 x \sqrt{\bar{g}} \{1 + \bar{g}^{ab} \partial_a \bar{\psi} \partial_b \bar{\psi} + \frac{1}{2} \bar{R} \bar{\psi}^2 - \frac{\bar{\psi}^2}{4} g^{ab} \partial_a \bar{\sigma} \partial_b \bar{\sigma}\}. \tag{2.7}
\]

Using the conformal transformation (2.3), we will have

\[
\sqrt{\bar{g}} = \sqrt{g} \exp\left(\frac{\sigma}{\sqrt{2\omega + 3}}\right), \quad \bar{g}^{ab} = g^{ab} \exp\left(\frac{-\sigma}{\sqrt{2\omega + 3}}\right) \tag{2.8}
\]

and

\[
\bar{\psi} = \psi \exp\left(\frac{\sigma}{2\sqrt{2\omega + 3}}\right), \quad \bar{\sigma} = \sigma, \tag{2.9}
\]

with

\[
\bar{R} = \exp\left(-\frac{\sigma}{\sqrt{2\omega + 3}}\right) \left[R - g^{ab} \partial_a \partial_b \sigma\right]. \tag{2.10}
\]

Applying (2.2), (2.8), (2.9) and (2.10), the action (2.7) is given exactly in the Jordan frame such as follows.

\[
I_{JBD}^{Jordan}[\phi, \psi, g_{ab}] = \frac{1}{2} \int d^2 x \sqrt{g}\{\phi + \frac{1}{2} \phi \psi^2 R + \phi g^{ab} \partial_a \psi \partial_b \psi
+ 2\psi g^{ab} \partial_a \psi \partial_b \phi - \frac{\omega \psi^2}{2\phi} g^{ab} \partial_a \phi \partial_b \phi\} \tag{2.11}
\]

in which \(R\) is Ricci scalar of the metric \(g_{ab}\) defined in Jordan frame, and also we eliminated divergence-less terms. Varying the action (2.11), with respect to the fields \(g^{ab}\), \(\psi\) and \(\phi\) the corresponding field equations are obtained respectively as

\[
T_{ab} = \phi \partial_a \psi \partial_b \psi + 2\psi \partial_a \psi \partial_b \phi - \frac{\omega \psi^2}{2\phi} \partial_a \phi \partial_b \phi - \partial_a [\sqrt{g} \partial_b (\phi \psi^2)]/2\sqrt{g}
\]
\[-\frac{1}{2}g_{ab}\left\{ \phi + g^{cd}[\phi \partial_c \psi \partial_d \psi + 2\psi \partial_c \psi \partial_d \phi - \omega \psi^2 \partial_c \phi \partial_d \phi / 2 \phi - \Box(\phi \psi^2)] \right\}, \tag{2.12}\]

\[\frac{1}{2} \phi \psi R - g^{cd}[\partial_c \phi \partial_d \psi + \omega \psi \partial_c \phi \partial_d \phi / 2 \phi] - \psi \Box \phi - \phi \Box \psi = 0 \tag{2.13}\]

and

\[1 + \frac{1}{2} \psi^2 R + g^{cd} [\partial_c \psi \partial_d \psi - \omega \psi^2 \partial_c \phi \partial_d \phi / 2 \phi^2 + 2 \omega \psi \partial_c \psi \partial_d \phi / \phi] + \omega \psi^2 \Box \phi / \phi - \Box \psi^2 = 0 \tag{2.14}\]

where we defined \( \Box = \partial_a (\sqrt{g} g^{ab} \partial_b) / \sqrt{g} \).

Using the transformations

\[\phi \psi^2 = \alpha \rho, \quad \psi = l_p \rho^\beta e^{\gamma S} \tag{2.15}\]

with

\[\alpha = \frac{3 + 2 \omega}{4(2 + \omega)}, \quad \beta = \frac{1 + \omega}{3 + 2 \omega}, \quad \gamma^2 = -\frac{2(2 + \omega)}{(3 + 2 \omega)^2} \tag{2.16}\]

the action (2.11) become

\[I_{JBD}^{Jordan}[\rho, S, g_{ab}] = \frac{1}{2} \int d^2 x \sqrt{g} \left\{ \rho g^{ab} \partial_a S \partial_b S + \frac{g^{ab} \partial_a \rho \partial_b \rho}{4 \rho} + \alpha \rho R + 2 \alpha \rho^{1-2\beta} \right\} \tag{2.17}\]

where \( l_p = (16\pi G)^{1/2} \) with \( c = \hbar = 1 \), is the Planck length, and \((\rho, S)\) are dimensionless real scalar fields. Varying the action (2.17) with respect to the field \( \rho \), one can obtain Hamilton-Jacobi equation as

\[g^{ab} \partial_a S \partial_b S = -\alpha R + \frac{(2\beta - 1)e^{-2\gamma S}}{\rho^{2\beta} l_p^2} + \frac{\Box \sqrt{\rho}}{\sqrt{\rho}} \tag{2.18}\]

Varying the action (2.17) with respect to the field \( S \), we obtain covariant divergence of the mass-energy current density of the particles ensemble \( J_a = \rho \partial_a S \) such that

\[\frac{1}{\sqrt{g}} \partial_a \{ \rho \sqrt{g} g^{ab} \partial_b S \} = -\gamma \rho^{1-2\beta} e^{-2\gamma S} / l_p^2. \tag{2.19}\]

The above equation shows covariant conservation of the mass-energy current density \( J_a \) only at the particular Einstein’s regime \( \gamma = 0 \), namely \( \omega \rightarrow +\infty \). In other words nonzero value of RHS of the above equation presents particle creation coming from Bohm quantum potential. However, with arbitrary
nonzero value of $\alpha \ (\omega \nrightarrow \infty)$, the covariant conservation of this current density can be still established by applying conformal frame $\tilde{g}_{ab} = \Omega^2 g_{ab}$ where
\[ \Box \ln \Omega = -\rho^{-2\beta} e^{-2\gamma S} / l_p^2. \] (2.20)
In the latter frame the action (2.17) is transformed to
\[ \tilde{I}_{JBD}[^{Jordan}[\rho, S, \tilde{g}_{ab}] = \frac{1}{2} \int d^2x \sqrt{\tilde{g}} \left\{ \rho \tilde{g}^{ab} \partial_a S \partial_b S + \frac{\tilde{g}^{ab} \partial_a \rho \partial_b \rho}{4\rho} + \alpha \rho \tilde{R} \right\} \] (2.21)
in which the corresponding Hamilton-Jacobi equation and the covariant conservation equation are obtained as
\[ \tilde{g}^{ab} \partial_a S \partial_b S = -\alpha \tilde{R} + \frac{\Box \sqrt{\rho}}{\sqrt{\rho}} \] (2.22)
and
\[ \frac{1}{\sqrt{\tilde{g}}} \partial_a \left\{ \rho \sqrt{\tilde{g}} \tilde{g}^{ab} \partial_b S \right\} = 0. \] (2.23)
Defining de Borglie-Bohm pilot wave of the gravitational system as $\Psi = \sqrt{\rho} e^{iS}$, the action (2.21) can be rewritten as
\[ \tilde{I}_{JBD}[^{Jordan}[\Psi, \tilde{g}_{ab}] = \frac{1}{2} \int d^2x \sqrt{\tilde{g}} \left\{ \tilde{g}^{ab} \partial_a \Psi \partial_b \Psi^* + \alpha \Psi \Psi^* \tilde{R} \right\} \] (2.24)
where $\Psi^* = \sqrt{\rho} e^{-iS}$ is complex conjugate of the scalar field $\Psi$. In order to obtain the WKB approximation, in the classical limits where there is a wave packet of width much greater than the wave length, the quantum potential term $\Box \sqrt{\rho} / \sqrt{\rho}$ will be very small compared with the classical kinetic term $\tilde{g}^{ab} \partial_a S \partial_b S$. We therefore can be neglect it in the WKB approximation. In other words in the de Broglie-Bohm particle interpretation the quantum potential causes to trajectories on the classical paths of the particles ensemble. These particles move normal to the wave front $S = constant$. It follows then that the Hamilton-Jacobi equation (2.22) can be regarded as a conservation equation for the probability in an ensemble of such particles, all moving normal to the same wave front with a probability density $\rho$. One should be note that the metric function $\tilde{g}_{ab}$ is still treats as classical metric field and whose quantum trajectories come from quantum potential term $\Box \sqrt{\rho} / \sqrt{\rho}$ such as follows.
The Hamilton Jacobi equation (2.22) can be rewritten as
\[ g^{ab} \partial_a S \partial_b S = -\alpha \tilde{R}, \] (2.25)
where we defined quantum perturbed metric function as

\[ g_{ab} = \tilde{g}_{ab} \left\{ 1 - \frac{\Box \sqrt{\rho}/ \sqrt{\rho}}{\alpha R} \right\}. \tag{2.26} \]

Thus quantum trajectories of the classical metric will be

\[ \Delta_{ab} = g_{ab} - \tilde{g}_{ab} = -\tilde{g}_{ab} \left[ \frac{\Box \sqrt{\rho}/ \sqrt{\rho}}{\alpha R} \right] \tag{2.27} \]

which are negligible in the WKB approximation. Interaction of the particles ensemble with the classical background metric \( \tilde{g}_{ab} \) causes to create these quantum trajectories. We should be point that the covariant conservation condition (2.23) described in terms of the quantum perturbed metric function \( g_{ab} \) will be violated. This violation originates from creation of quantum particles interacting with curved space time (Hawking radiation). Physically it seems that the pointed creation of particles should be related to the conformal anomaly derived from renormalization of stress tensor operator expectation value of the quantum fields propagating on a curved space time (see [29] and references therein). As a future work this problem may to be a suitable exercise leading to a duality picture between de Broglie-Bohm anomaly derived from quantum potential and conformal anomaly obtained from renormalization theory of quantum fields.

### 3 Concluding remarks

Applying JBD gravity theory described by spherically symmetric curved space time we seek corresponding de Broglie Bohm pilot wave perspective, where amplitude and phase part of the pilot wave is defined by the Brans Dicke scalar field and 2-sphere scale factor. This approach of quantum gravity is useful to study spherically symmetric space time with black hole topology. Also quantum trajectory of the background metric is obtained in terms of the Bohm quantum potential. There is obtained a suitable conformal frame where the covariant conservation of the matter-energy current density of particles ensemble is established. However it violates in the presence of the quantum potential effects.

**References**
1. B. Fauser, J. Tolksdorf and E. Zeidler, *Quantum Gravity-Mathematical Models And Experimental Bounds*, Birkhäuser verlag, Basel-Boston-Berlin (2007).

2. N. D. Birrell and P. C. W. Davies, *Quantum Fields In Curved Space*, Cambridge University press (1984).

3. L. Parker and D. Toms, *Quantum Field Theory In Curved Space Time-Quantized Fields And Gravity*, Cambridge University press (2009).

4. F. Bastianelli and P. Van Nieuwenhuizen, *Path Integrals And Anomalies In Curved Space*, Cambridge University press (2006).

5. B. R. Iyer, N. Mukunda and C. V. Vishveshwara, *Gravitation, Gauge Theories And The Early Universe*, Kluwer academic publishers (1989).

6. B. S. De Witt, Phys. Rev. D160, 1113, (1967).

7. T. Thiemann, *Modern Canonical Quantum General Relativity*, Cambridge University press (2008).

8. D. Bohm and B. J. Hiley, *The Undivided Universe-An Ontological Interpretation Of Quantum Theory*, Routledge, 11 New Fetter Lane, London EC4P, 4EE (1993).

9. P. R. Holland, *The Quantum Theory of Motion-An Account Of The de Broglie Bohm Causal Interpretation Of Quantum Mechanics*, Cambridge University press (1993).

10. L. de Broglie, *Non Linear Wave Mechanics*, translated to English, by A. J. Kondel, Elsevier publishing company (1960).

11. M. Bell, K. Gottfried and M. Veltman, *John S. Bell On The Foundations Of Quantum Mechanics*, World Scientific publishing Co. Pte. Ltd (2001).

12. G. Greenstein and A. G. Zajonc *The Quantum Challenge-Modern Research On The Foundations Of Quantum Mechanics*, Jones and Bartlett publishers, Inc (2006).
13. R. Vasudevan, K. V. Parthasarathy and R. Ramanathan, *Quantum Mechanics-A Stochastic Approach*, Alpha science international LTD, 7200. The Quorum, Oxford Business Park North, Garsington Road, Oxford OX4 2JZ, U.K. (2008).

14. R. E. Wyatt, *Quantum Dynamics With Trajectories-Introduction To Quantum Hydrodynamics*, Springer Science+Business Media, Inc (2005).

15. J. Soda, H. Ishihara and O. Iguchi, arXiv: gr-qc/9509008 v1, 6 Sep. (1995).

16. M. Kenmoku, H. Kubotani, E. Takasugi and Y. Yamazaki, arXiv: gr-qc/ 9810039 v1, 10 Oct (1998).

17. A. Blaut and J. K. Glikman, Class. Quantum Grav. 13, 39 (1996).

18. El-Nabulsi Ahmad Rami, Rom. Journ. Phys. Vol.53, Nos. 7-8, P. 933-940, Bucharest, (2008).

19. C. Brans and R. Dicke, Phys. Rev. 124, 925 (1961).

20. H. Ghafarnejad, Astrophys. Space Sci. 301, 145-148, (2006).

21. Y. M. Cho, Class. Quantum. Grav. 10, 2963 (1997).

22. C. M. Will, *Theory And Experiment In Gravitational Physics*, Cambridge University press (1993); revised version: gr-qc/9811036.

23. E. Gaztanaga and J. A. Lobo, Astrophys. J., 548, 47 (2001).

24. R. D. Reasenberg et al, Astrophys. J., 234, 925 (1961).

25. C. M. Will, Living Rev. Rel. 9 (2006); [http://WWW.livingreviews.org/lrr-2006-3](http://WWW.livingreviews.org/lrr-2006-3).

26. L. J. Garay and J. G. Bellido, gr-qc/9209015 (1992).

27. J. M. Overdid and P. S. Wesson, gr-qc/9805018 (1998).

28. P. Thomi, B. Isaac and P. Hajicek, Phys. Rev. D30, 1168 (1984).

29. H. Ghafarnejad and H. Salehi, Phys. Rev. D56, 4633 (1997); Phys. Rev. D57, 5311 (1998).