Abstract

This paper is a sequel to [Gr]. We present a new construction of gradient-like vector fields in the setting of Morse theory on a complex analytic stratification. We prove that the ascending and descending sets for these vector fields possess cell decompositions satisfying the dimension bounds conjectured by M. Goresky and R. MacPherson in [GM]. The vector fields constructed in [Gr] satisfied the same dimension bounds only “up to fuzz.” The new construction has a number of other advantages. In particular, it is closer to “metric gradient” intuition. The idea is to relate a gradient-like vector field in the neighborhood of a point stratum \{p\} to a gradient-like vector field for the distance to \(p\) on the complex link of \{p\}. Similar results by C.-H. Cho and G. Marelli have recently appeared in [CM].

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1 Introduction

1.1 Main Results

This paper is a sequel to [Gr]. It concerns gradient-like vector fields on complex analytic Whitney stratifications. Some technical notions of Whitney stratification theory, such as control data, are defined somewhat differently by different authors. We will be using the versions from [Gr], with one correction (see Remark 2.6), throughout this paper, and will summarize the basic technical definitions (control data, controlled and weakly controlled vector fields, etc.) in Section 2.2.

Let \((X, S)\) be a Whitney stratified \(C^\infty\) manifold. Given a \(C^\infty\) function \(f : X \to \mathbb{R}\), we denote by \(\Sigma_f \subset X\) the set of stratified critical points of \(f\) (see Definition 2.1). The following definitions introduce the main characters of our story (cf. [Gr, Definitions 2.12-13]).

**Definition 1.1** Let \(f : X \to \mathbb{R}\) be a \(C^\infty\) function. An \(S\)-preserving \(\nabla f\)-like vector field \(V\) on an open subset \(U \subset X\) is a weakly controlled vector field, compatible with some system of control data on \((X, S)\), satisfying:

(a) \(V_p = 0\) for all \(p \in \Sigma_f \cap U\);

(b) \(V_x f > 0\) for all \(x \in U \setminus \Sigma_f\).

Given a weakly controlled vector field \(V\) on an open set \(U \subset X\) and a number \(t \in \mathbb{R}\), we write \(\psi_{V,t} : D(\psi_{V,t}) \to U\) for the time-\(t\) flow of \(V\). Here \(D(\psi_{V,t}) \subset U\) is the largest open set on which this flow can be defined (see [Gr, Proposition 2.11]).

**Definition 1.2** Let \(f : X \to \mathbb{R}\) be a \(C^\infty\) function, let \(U \subset X\) be an open subset, let \(V\) be a \(\nabla f\)-like vector field on \(U\), and let \(p \in \Sigma_f \cap U\). We define the descending set \(M_V^- (p)\) to be the set of all \(x \in U\) such that \(x \in D(\psi_{V,t})\) for all \(t \geq 0\) and

\[
\lim_{t \to +\infty} \psi_{V,t}(x) = p.
\]

The ascending set \(M_V^+(p)\) is defined similarly.

In the situation of Definition 1.2, given an \(\epsilon \in \mathbb{R}\), define

\[
L_{V,\epsilon}^\pm(p) = \{x \in M_V^\pm(p) \mid f(x) - f(p) = \epsilon\}.
\]
**Definition 1.3** The set $L^{-}_{\epsilon}(p)$, for $\epsilon < 0$, is called a descending link of $p$, if for every $\epsilon_1 \in (\epsilon, 0)$, there is a bijection $L^{-}_{\epsilon}(p) \to L^{-}_{\epsilon_1}(p)$ given by identifying points lying on the same trajectory of $V$. An ascending link $L^{+}_{\epsilon}(p)$, for $\epsilon > 0$, is defined similarly.

The existence of ascending and descending links is established by the following proposition.

**Proposition 1.4** In the situation of Definition 1.2, assume that $p$ is an isolated (for example, Morse) critical point of $f$. Then we have the following.

(i) There exists an $\epsilon_0 > 0$, such that $L^{-}_{\epsilon}(p)$ is a descending link for every $\epsilon \in (-\epsilon_0, 0)$ and $L^{+}_{\epsilon}(p)$ is an ascending link for every $\epsilon \in (0, \epsilon_0)$.

(ii) The ascending and descending links $L^{\pm}_{\epsilon}(p)$ are compact.

(iii) For every descending link $L^{-}_{\epsilon}(p)$ and every $\epsilon_1 \in (\epsilon, 0)$, the bijection $L^{-}_{\epsilon}(p) \to L^{-}_{\epsilon_1}(p)$ of Definition 1.3 is a homeomorphism; and similarly for ascending links.

**Proof:** This is an exercise in general topology, using the continuity of the time-$t$ flow $\psi_{V,t} : D(\psi_{V,t}) \to U$ (see [Gr, Proposition 2.11]).

The following definition can be compared to [Gr, Definition 6.1].

**Definition 1.5** (i) A weakly stratified subset $A \subset X$ is a closed subset, presented as a finite disjoint union $A = \bigcup_{i=0}^{N} A_i$ so that:

(a) each $A_i$ is a locally closed smooth submanifold of one of the strata of $S$;

(b) the partial union $A_{\leq n} = \bigcup_{i=0}^{n} A_i$ is closed for every $n \in \{0, \ldots, N\}$.

(ii) A weakly stratified subset $A \subset X$ is called a cellular subset if every $A_i$ is diffeomorphic to an open ball.

Given a weakly stratified subset $A = \bigcup A_i \subset X$, we will refer to the pieces $A_i$ as the strata of $A$. When $A$ is a cellular, we will refer to the $A_i$ as the cells of $A$. Note that an ordering of the strata is part of the structure of a weakly stratified subset. The definition of a weakly stratified subset may seem too weak to have any useful consequences. Nevertheless, it suffices to prove a basic stratified general position result (Lemma 1.16) which can be used in applications to self-indexing (see Section 1.3). Our main result is the following.

**Theorem 1.6** Let $X$ be a nonsingular complex analytic variety. Let $S$ be a complex analytic Whitney stratification of $X$. Let $p \in X$ be a point stratum of $S$. Let $f : X \to \mathbb{R}$ be a smooth function. Assume that $p$ is a stratified Morse critical point of $f$. Then there exist an open neighborhood $U \subset X$ of $p$ and an $S$-preserving $\nabla f$-like vector field $V$ on $U$, such that the following four conditions hold.
(i) The ascending and descending sets $M^V_\pm(p)$ are cellular subsets of $U$.

(ii) The flow of $V$ preserves the cells of $M^V_\pm(p)$.

(iii) The ascending and descending links $L^\pm_{V,c}(p)$ are cellular subsets of $U$, with cell decompositions given by intersecting with the cells of $M^V_\pm(p)$.

(iv) For every stratum $B \in \mathcal{S}$, we have:

$$\dim_r M^V_\pm(p) \cap B \leq \dim_c B.$$
Theorem 1.8 Let \((X, S), f : X \to \mathbb{R}, p \in \Sigma_f, \) and \(A \in S\) be as in Definition 1.7. Then there exist an open neighborhood \(U \subset X\) of \(p\) and an \(S\)-preserving \(\nabla f\)-like vector field \(V\) on \(U\), such that conditions (i)-(iii) as in Theorem 1.6 and condition (iv) below hold.

(iv) For every stratum \(B \in S\), we have:

\[
\dim \mathbb{R} M_V^-(p) \cap B \leq \dim_c B + \text{index}_f(p),
\]

\[
\dim \mathbb{R} M_V^+(p) \cap B \leq \dim_c B - \text{index}_f(p).
\]

The present author’s interest in the Goresky-MacPherson conjecture was re-awakened by an earlier version of the preprint [CM], which claims results very similar to Theorems 1.6, 1.8 (see [CM, Theorem 1.1, Corollary 1.2]). A brief correspondence with Cho and Marelli led to some revisions of their preprint. In the meantime, the author discovered the construction of gradient-like vector fields presented in this paper. Not having studied the proofs in [CM], it appears that the construction of this paper is substantially different from the construction of Cho and Marelli. Furthermore, it seems likely that applications of Theorems 1.6, 1.8, or of the results of [CM], will utilize the specific constructions of gradient-like vector fields used in the proofs, rather than the existence statements alone (see Section 1.3 for some speculations about possible applications). For these reasons, the author hopes that this paper will not be entirely superfluous.

1.2 Conjectured Refinements

Theorem 1.8 can be viewed as an analogue of the classical fact that the gradient flow of a Morse function \(f\) on a compact Riemannian manifold \(X\) gives rise to a cell decomposition of \(X\), whose cells are parameterized by the critical points of \(f\) (see [Th1]). This geometric result admits a homotopy level refinement asserting that \(X\) is homotopy equivalent to a CW-complex, whose cells are likewise parameterized by the critical points of \(f\) (see [Mi, Theorem 3.5]). In fact, it is this refinement that makes the Morse-theoretic cell decomposition so useful in smooth manifold topology. In this section, we state as conjectures two homotopy level refinements of Theorem 1.8 (Conjectures 1.10, 1.11). These conjectures can be viewed as analogues of [Mi, Theorem 3.5]. It seems likely that they can be established by adapting the classical proof of [Mi, Theorem 3.5] to the inductive combinatorics of cells in \(M_V^\pm(p)\) described by Proposition 4.16.

We begin with a point of notation. In the situation of Proposition 1.4, the descending links \(L_{V,\epsilon}^-(p)\), for different \(\epsilon < 0\), are naturally in bijection with each other. These bijections are provided by Definition 1.3. Moreover, in the situation of Theorem 1.8, these bijections preserve all the structures we are interested in: they are cell-preserving homeomorphisms which restrict to diffeomorphisms of the individual cells. For this reason, we will sometimes drop the \(\epsilon\) from the notation, writing \(L_V^-(p) = L_{V,\epsilon}^-(p)\) for the descending link. Similarly, we will write \(L_V^+(p) = L_{V,\epsilon}^+(p)\) for the ascending link.
Theorem 1.8 provides a local topological picture of $M_V(p)$ as a cone over the compact cellular subset $L_V(p) \subseteq U$ (and similarly for $M^+_V(p)$). From a homotopy theory point of view, it is tempting to ask whether the cell decomposition of $L_V(p)$ makes it into a CW-complex. This is more than we can assert. However, we conjecture that there exists a CW-complex $K_V(p)$, whose cells are in one-to-one correspondence with the cells of $L_V(p)$, and a homotopy equivalence $h_V(p) : K_V(p) \to L_V(p)$ respecting the stratification $S$.

To make the above precise, we need the language of $\mathcal{J}$-filtered spaces (see [GM, Part III, §2.1]). Let $\mathcal{J}$ be a partially ordered set with a unique maximal element $J_0 \in \mathcal{J}$.

**Definition 1.9** A $\mathcal{J}$-filtration of a topological space $X$ is a collection of closed subsets $\{X_J\}_{J \in \mathcal{J}}$ such that $X_{J_0} = X$ and $J_1 < J_2 \Rightarrow X_{J_1} \subseteq X_{J_2}$. A $\mathcal{J}$-filtered map $f : X \to Y$ between two $\mathcal{J}$-filtered spaces is a continuous map such that $f(X_J) \subseteq Y_J$ for every $J \in \mathcal{J}$. $\mathcal{J}$-filtered homotopies between $\mathcal{J}$-filtered maps and homotopy equivalences between $\mathcal{J}$-filtered spaces are defined in the obvious way. A pair of $\mathcal{J}$-filtered spaces is a pair $(X, Y)$ of topological spaces plus a $\mathcal{J}$-filtration of $X$. $\mathcal{J}$-filtered maps, homotopies, and homotopy equivalences for pairs are defined in the obvious way.

In the situation of Theorem 1.8, let $\mathcal{I}$ be the set of all closed unions of strata of $S$, partially ordered by inclusion. Note that every closed subset of $X$ is naturally an $\mathcal{I}$-filtered space. Let $L^+_V(p)$ be the set of cells of $L^+_V(p)$.

**Conjecture 1.10** The statement of Theorem 1.8 can be strengthened to assert the following. There exists a CW-complex $K_V(p)$, whose set of cells we denote by $K_V(p)$, and a bijection $k^- : K_V(p) \to L_V(p)$, such that the following conditions hold.

(i) For every $C \in K_V(p)$, we have $\dim(C) = \dim k^-(C)$.

(ii) The bijection $k^-$ makes $K_V(p)$ into an $\mathcal{I}$-filtered space.

(iii) There exists a homotopy equivalence of $\mathcal{I}$-filtered spaces

$$h_V(p) : K_V(p) \to L_V(p).$$

Similarly, there exist a CW-complex $K^+_V(p)$, a dimension-preserving bijection $k^+ : K^+_V(p) \to L^+_V(p)$, and a homotopy equivalence of $\mathcal{I}$-filtered spaces

$$h^+_V(p) : K^+_V(p) \to L^+_V(p).$$

Our next conjecture ties in the ascending and descending links $L^+_V(p)$ with a central ingredient of stratified Morse theory: the local Morse data (see [GM, Part I, §3.5]). We recall the definition of the local Morse data in the situation of Definition 1.7. Fix a Riemannian metric $\mu$ on $X$ and a pair of numbers $0 \ll \epsilon \ll \delta \ll 1$. Let $B^\mu_\delta(p) \subseteq X$ be the closed $\delta$-ball around $p$. Define

$$D_f(p) = \{x \in B^\mu_\delta(p) \mid f(x) - f(p) \in [-\epsilon, \epsilon]\},$$

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\[ E_f^\pm(p) = \{ x \in B_\mu^\delta(p) \mid f(x) - f(p) = \pm \epsilon \}. \]

The local Morse data for \( f \) at \( p \) is the pair \((D_f(p), E_f^-(p))\) of closed subspaces of \( X \). Similarly, the pair \((D_f(p), E_f^+(p))\) is the local Morse data for \(-f\) at \( p \). We suppress from the notation the inessential dependence of the pairs \((D_f(p), E_f^\pm(p))\) on \( \mu, \delta, \epsilon \) (see [GM, Part I, Theorem 7.4.1]). In the situation of Theorem 1.8, we have natural inclusion maps:

\[ l_V^\pm(p) : L_V^\pm(p) \rightarrow E_f^\pm(p). \]

**Conjecture 1.11** The statement of Conjecture 1.10 can be strengthened to assert the following. The maps \( l_V^\pm(p) : L_V^\pm(p) \rightarrow E_f^\pm(p) \) are homotopy equivalences of \( \mathcal{I} \)-filtered spaces.

Using a natural generalization of [GM, Part I, Theorem 3.12] to describe the \( \mathcal{I} \)-filtered homotopy type of the local Morse data, we obtain the following corollary of Conjecture 1.11.

**Corollary 1.12** In the situation of Conjecture 1.11, the natural extensions of \( l_V^\pm(p) \) to inclusion maps of pairs:

\[ \tilde{l}_V^\pm(p) : (M_V^\pm(p) \cap D_f(p), L_V^\pm(p)) \rightarrow (D_f(p), E_f^\pm(p)), \]

are homotopy equivalences of pairs of \( \mathcal{I} \)-filtered spaces.

Note that, by combining the dimension bounds of Theorem 1.8 with the homotopy equivalences of \( \mathcal{I} \)-filtered spaces:

\[ l_V^\pm(p) \circ h_V^\pm(p) : K_V^\pm(p) \rightarrow E_f^\pm(p), \]

provided by Conjectures 1.10, 1.11, we obtain dimension bounds for the \( \mathcal{I} \)-filtered homotopy type of \( E_f^\pm(p) \). These dimension bounds are not new. They are essentially contained, minus the language of \( \mathcal{I} \)-filtered homotopy type, in [GM, Introduction, §1.5] and [GM, Part II, §1.1*]. The latter section also contains references to the original work of Kato, Karchyauskas, and Hamm proving that a Stein space of complex dimension \( n \) is homotopy equivalent to a CW-complex of (real) dimension \( n \), which is a closely related phenomenon.

### 1.3 Some Motivation

In this section, we briefly indicate four reasons why one might be interested in the Goresky-MacPherson conjecture (GMC), Theorems 1.6, 1.8 and Conjectures 1.10, 1.11 stated above, and the results of [Gr].

The first reason is that vector fields of the kind constructed in this paper provide a geometric ingredient making some of the main results of complex stratified Morse theory more concrete and geometrically apparent. One example of this is the homotopy dimension.
bounds for the local Morse data discussed at the end of Section 1.2. Another example is the vanishing of intersection homology Morse groups outside of a single degree (see [GM, Part II, Theorem 6.4]). In fact, GMC was introduced as a source of geometric intuition behind this result.

The second reason is that the results of this paper may enable some progress towards another conjecture of Goresky and MacPherson, stated in [GM, Part II, §1.1*]. Here we state a variant of this conjecture which is likely to be accessible using the vector fields provided by the proof of Theorem 1.8.

**Conjecture 1.13** Let \( X \cong \mathbb{C}^d \) be a complex vector space, and let \( S \) be a complex analytic Whitney stratification of \( X \). Let \( \mu \) be a Hermitian metric on \( X \), let \( p \in X \) be the origin, and let \( f : X \to \mathbb{R} \) be the function \( f(x) = \text{dist}_\mu(p, x)^2 \). Assume that \( f \) is Morse. Pick a regular value \( a > 0 \) of \( f \). Let \( X^a = \{ x \in X \mid f(x) \leq a \} \) and let \( \Sigma_f^a = \{ p \in \Sigma_f \mid f(p) < a \} \). Then there exists a \( \nabla f \)-like vector field \( V \) on \( X \) with the following property. Define

\[
X^a_{c} = \bigcup_{p \in \Sigma_f^a} M^V_f(p).
\]

Then \( X^a_\c \) is a cellular subset of \( X \), satisfying \( \dim_{\R} X^a_{c} \cap B \leq \dim_{\C} B \) for every \( B \in S \). Moreover, \( X^a \) deformation retracts to \( X^a_{c} \) by a stratum preserving retraction.

The third reason is that Theorem 1.8 can be used to prove the existence of self-indexing Morse functions for complex analytic stratifications. More precisely, we have the following generalization of [Gr, Theorem 1.6] from the complex algebraic to the complex analytic setting.

**Definition 1.14** Let \( X \) be a compact, nonsingular complex analytic variety, and let \( S \) be a complex analytic Whitney stratification of \( X \). A Morse function \( f : X \to \mathbb{R} \) is called self-indexing if \( f(x) = \text{index}_f(x) \) for every \( x \in \Sigma_f \).

**Theorem 1.15** For every pair \( (X, S) \) as in Definition 1.14, there exists a self-indexing Morse function \( f : X \to \mathbb{R} \).

**Proof:** This is a straightforward adaptation of the proof of [Gr, Theorem 1.6], using Theorem 1.8 in place of [Gr, Theorem 6.3] and Lemma 1.16 in place of [Gr, Lemma 6.5].

The following lemma is analogous to [Gr, Lemma 6.5]. It uses the notion of a time-dependent controlled vector field which is discussed in [Gr, §6.2].

**Lemma 1.16** Let \( (X, S) \) be a Whitney stratified \( C^\infty \) manifold with a fixed system of control data \( D \). Let \( A, B \subset X \) be two weakly stratified subsets. Assume \( A \cap B \) is compact and, for every \( S \in S \), we have:

\[
\dim(A \cap S) + \dim(B \cap S) < \dim S.
\]
Then there exists a time-dependent controlled vector field with compact support \( \{ V_t \}_{t \in (0,1)} \) on \( X \) compatible with \( D \), whose time-1 flow satisfies \( \psi_{V,1}(A) \cap B = \emptyset \).

**Proof:** This is similar to the proof of [Gr, Lemma 6.5], using the linear ordering of the strata of \( A \) and \( B \) given by Definition 1.5 instead of the partial ordering by dimension. \( \square \)

Even in the complex algebraic case, using Theorem 1.8 in place of [Gr, Theorem 6.3] simplifies the proof of Theorem 1.15. This is because the statement of [Gr, Theorem 6.3] is more complicated, involving the choice of an open set informally referred to as “the fuzz.” The reader is referred to [Gr, §1.3] and the lecture notes [MP], [GrM] for a discussion of the role of self-indexing Morse functions in the theory of middle perversity perverse sheaves.

The fourth reason to be interested in the results of this paper is that the sets \( M_{V}^{\pm}(p) \) provided by Theorem 1.8 can serve as building blocks of a useful cycle theory for (the hypercohomology with coefficients in) middle perversity perverse sheaves. For example, in the situation of Conjecture 1.13, let \( \mathcal{P}(X, S) \) be the category of middle perversity perverse sheaves on \( (X, S) \) with coefficients in \( \mathbb{C} \). Let \( P \in \mathcal{P}(X, S) \), and let \( u \in H^k(X^{a}, \partial X^{a}; P) \) be a relative hypercohomology class with coefficients in \( P \). For \( i \in \{-d, -d+1, \ldots, 0\} \), define

\[
\Sigma_{f}^{a}[i] = \{ p \in \Sigma_{f}^{a} | \text{index}_{f}(p) \leq i \},
\]

\[
X_{c}^{a}[i] = \bigcup_{p \in \Sigma_{f}^{a}[i]} M_{V}^{\pm}(p).
\]

Conjecture 1.13 can be strengthened to assert that each \( X_{c}^{a}[i] \subset X \) is a cellular subset satisfying

\[
\dim_{\mathbb{R}} X_{c}^{a}[i] \cap B \leq \dim_{\mathbb{C}} B + i,
\]

for every \( B \in S \). In this situation, we can think of \( X_{c}^{a}[-k] \subset X_{c}^{a} \) as a geometric cycle representing (or supporting) the class \( u \). It seems plausible that a geometric cycle theory along these lines can be useful in understanding the structure of the category \( \mathcal{P}(X, S) \).

### 1.4 Contents of this Paper

The rest of this paper is organized as follows. Section 2 is, in large part, a summary and paraphrase (and in one instance a correction) of the definitions and results of [Gr, §§2-3]. Section 3 is devoted to a model example: the Hermitian metric gradient of a linear function on an affine cone \( X \subset \mathbb{C}^{n} \) over a smooth projective variety \( P \subset \mathbb{C}P^{n-1} \). It turns out that there is a simple geometric reason why the ascending and descending sets have half the dimension of \( X \) in this case. The main idea of this paper is to maneuver the general case of Theorem 1.6 to look like this model example. Section 4 describes an inductive proof of Theorems 1.6 and 1.8 based on this idea. We have tried to keep the notation parallel between Section 3 and 4, to emphasize the analogy.
2 Technical Preliminaries

Whitney stratification theory has a reputation as a rather technical subject. The present author feels that part of the reason for this is the fact that there is no canonical, or intrinsic, notion of an automorphism of a Whitney stratified $C^\infty$ manifold $(X,\mathcal{S})$. Going back to the work of Thom [Th2] and Mather [Ma], in order to construct useful automorphisms, one fixes additional structure on $(X,\mathcal{S})$, called a system of control data. The choice of control data is always somewhat arbitrary, but subsequent constructions must respect it. If the setting includes another geometric ingredient, such as a Morse function, it may be difficult to simultaneously keep track of the chosen control data and the Morse function.

In [Gr, §§2-3] we introduced a technical innovation which allowed us to simultaneously keep track of a Morse function $f$ and a system of control data which is specifically chosen to be “adapted” to $f$. To achieve this, we used a notion of control data which is more flexible than the ones in such references as [G-al] and [dPW]. Specifically, we allowed quasi-distance functions modeled on arbitrary quasi-norms rather than Euclidian or Riemannian distances. Given this flexibility, we used the function $f$ itself in constructing one of the quasi-distance functions making up the control data. Unfortunately, in preparing this paper, we discovered a technical error in [Gr, §3], which is most readily corrected by adding an additional compatibility condition in the definition of control data. This is explained in Remark 2.6 below.

The contents of this section are as follows. In Section 2.1, we recall some standard definitions pertaining to conormal varieties and stratified Morse functions. In Section 2.2, we define control data, controlled and weakly controlled vector fields, and some related notions. For the most part, these definitions are copied form [Gr, §2]. However, there is one significant distinction in the definition of control data, which gives us a chance to correct an error in [Gr, §3] (see Remark 2.6). In Section 2.3, we introduce the notion of $f$-adapted control data. This is a new way to formalize the technical innovation of [Gr, §§2-3]. It will play a key role in the proofs of our main results in Section 4.

2.1 Stratified Morse Functions

Let $(X,\mathcal{S})$ be a Whitney stratified $C^\infty$ manifold, and let $f : X \to \mathbb{R}$ be a smooth function. The following definitions are standard in the subject (cf. [Gr, Definition 1.2])

**Definition 2.1** (i) Let $p \in X$ be a point contained in a stratum $S$. We say that $p$ is critical for $f$ ($p \in \Sigma_f$) if it is critical for the restriction $f|_S$.

(ii) For every stratum $S \in \mathcal{S}$, let $\Lambda_S$ be the conormal bundle $T^*_S X \subset T^*X$, and let $\Lambda = \Lambda_S = \bigcup_S \Lambda_S$. The set $\Lambda$ is called the conormal variety to $\mathcal{S}$. By Whitney’s condition (a), $\Lambda \subset T^*X$ is a closed subset. Note that $p \in \Sigma_f$ if and only if $d_pf \in \Lambda$. 

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(iii) Let $S \in S$ be a stratum. A covector $\xi \in \Lambda_S$ is said to be generic if it does not annihilate any of the limits of tangent spaces to strata $T \in S$ with $S \subseteq T$. The set of all generic $\xi \in \Lambda_S$ is denoted by $\Lambda_S^0$. We also write $\Lambda^0 = \Lambda^0_S = \bigcup_S \Lambda^0_S$.

(iv) Let $p \in \Sigma_f$, and let $S \in S$ be the stratum containing $p$. We say that $p$ is Morse for $f$ if it is Morse for the restriction $f|_S$ and $d_pf \in \Lambda^0$.

(v) We say that $f$ is a Morse function if every $p \in \Sigma_f$ is Morse for $f$.

2.2 Control Data, etc.

We will use the notation $\mathbb{R}_+ = (0, +\infty) \subseteq \mathbb{R}$ and $\mathbb{R}_{\geq} = [0, +\infty) \subseteq \mathbb{R}$.

**Definition 2.2** Let $X$ be a smooth manifold, let $S \subset X$ be a locally closed smooth submanifold, and let $U \subset X$ be an open neighborhood of $S$. A tubular projection $\Pi : U \to S$ is a smooth submersion restricting to the identity on $S$.

**Definition 2.3** Let $M$ be a smooth manifold, and let $E \to M$ be a vector bundle with zero section $Z$. A quasi-norm on $E$ is a smooth function $\rho : E \setminus Z \to \mathbb{R}_+$ such that $\rho(\lambda e) = \lambda \rho(e)$ for every $e \in E \setminus Z$ and $\lambda \in \mathbb{R}_+$.

Assuming $E \setminus Z \neq \emptyset$, a quasi-norm $\rho : E \setminus Z \to \mathbb{R}_+$ has a unique continuous extension $\hat{\rho} : E \to \mathbb{R}_{\geq}$, which is equal to zero on $Z$. The extension $\hat{\rho}$ is not differentiable on $Z$.

**Definition 2.4** Let $X$ be a smooth manifold, let $S \subset X$ be a locally closed smooth submanifold, let $U \subset X$ be an open neighborhood of $S$, and let $\Pi : U \to S$ be a tubular projection. A quasi-distance function $\rho : U \setminus S \to \mathbb{R}_+$ compatible with $\Pi$ is a smooth function satisfying the following condition. There exist a vector bundle $\pi : E \to S$ with zero section $Z$, an open neighborhood $U' \subset E$ of $Z$, and a diffeomorphism $\phi : U' \to U$, such that $\phi|_Z = \pi|_Z$, $(\Pi \circ \phi)|_{U'} = \pi|_{U'}$, and $\rho \circ \phi : U' \setminus Z \to \mathbb{R}_+$ is the restriction of a quasi-norm on $E$.

Let $(X, S)$ be a Whitney stratified $C^\infty$ manifold.

**Definition 2.5** A system of control data on $(X, S)$ is a collection $\{U_S, \Pi_S, \rho_S\}_{S \in S}$, where $U_S \supset S$ is an open neighborhood, $\Pi_S : U_S \to S$ is a tubular projection, and $\rho_S : U_S \setminus S \to \mathbb{R}_+$ is a quasi-distance function compatible with $\Pi_S$, subject to the following conditions for every pair $S, T \in S$ with $S \neq T$.

1. We have $U_S \cap T = \emptyset$ unless $S \subset \partial T$.
2. Whenever $S \subset \partial T$, the map $(\Pi_S, \rho_S) : U_S \cap T \to S \times \mathbb{R}_+$ is a smooth submersion.
3. Whenever $S \subset \partial T$, we have $\Pi_S \circ \Pi_T = \Pi_S$ on $U_S \cap U_T$.
4. Whenever $S \subset \partial T$, we have $\rho_S \circ \Pi_T = \rho_S$ on $(U_S \cap U_T) \setminus S$.

We will often use single letter notation $\mathcal{D} = \{U_S, \Pi_S, \rho_S\}$ for a system of control data.
Remark 2.6 Definition 2.5 is stronger than the corresponding [Gr, Definition 2.5] in two ways. First, we have added conditions (1) and (2). By [Gr, Lemma 2.4], these conditions can always be satisfied by shrinking the neighborhoods \( \{U_S\} \), so adding them explicitly is not an important distinction. Second, we have imposed the requirement (CR) that \( \rho_S \) be compatible with \( \Pi_S \) for every \( S \in \mathcal{S} \). This distinction is more important. In fact, failure to impose CR in [Gr, Definition 2.5] led us to make a technical error in [Gr, §3]. We take this opportunity to correct this error.

In several places in [Gr, §3] we implicitly made the following assumption. Let \( \mathcal{D} = \{U_S, \Pi_S, \rho_S\} \) be a system of control data. Let \( A \in \mathcal{S} \), \( a \in A \), and \( N = \Pi^{-1}_A(a) \). Then \( \mathcal{D} \) restricts to a system of control data on some open neighborhood \( U_a \subset N \) of \( a \). In part, this assumes that, for some \( U_a \), the restriction \( \rho_A|_{U_a \setminus \{a\}} \) is a quasi-distance function. This is trivially true for control data in the sense of Definition 2.5; but it seems quite subtle and possibly false without CR. Fortunately, there is no need to make this assumption.

The most direct fix is to strengthen the definition of control data by adding the requirement CR. This necessitates two other changes. First, the first sentence of the proof of [Gr, Theorem 3.1] must be extended to assume that the map \( p : \mathcal{U} \to S \) is the restriction of the bundle projection \( X \to M \cong S \). No other changes are needed in the proof of [Gr, Theorem 3.1], and we will make reference to this proof in Section 2.3. Second, we must rephrase the statement and the proof of [Gr, Proposition 3.9], which is an intermediate step for passing between the local and global topological stability results [Gr, Theorems 3.8, 3.10]. We sketch the necessary modification briefly.

We say that a quasi-distance function \( \rho : U_S \to S \) is universally compatible, if it is compatible with every tubular projection \( \Pi : U_S \to S \), after shrinking the neighborhood \( U_S \). The statement of [Gr, Proposition 3.9] must be modified to assert that the control data \( \{U_*, \Pi_*, \rho_*\} \) of [Gr, Theorem 3.8] can be chosen so that \( \rho_{SW} \) is universally compatible, and to proceed with this assumption. The proof of [Gr, Proposition 3.9] then must begin with the following remark. The statement of [Gr, Theorem 3.1] can be strengthened to assert that the quasi-distance to \( B^p \), provided as part of the control data on \( U \), is universally compatible. To ensure this, it is enough to require, in Step 1 of the proof, that the Euclidean norm of the fiber-wise liner function \( \tilde{f} : X \to \mathbb{R} \) be the same in every fiber. It follows that \( \{U_*, \Pi_*, \rho_*\} \) can be chosen so that \( \rho_{SW} \) is universally compatible. The only other change needed in the proof of [Gr, Proposition 3.9] is in the penultimate sentence, where we must ensure that the restriction \( \rho_{SW}|_{U_U} \) is compatible with the tubular projection \( \tilde{x} \mapsto \mu(\tilde{x}) \).

The need for this modification of [Gr, Proposition 3.9] arises because its proof is the only place in [Gr] where we modify a tubular projection after the corresponding quasi-distance function has been fixed. Unrelated to the above error, the proof of [Gr, Proposition 3.9] contains a typo. Namely, the function \( \theta_x \) must be defined as distance squared, rather than simply distance.

The author will evaluate the need for a more formal erratum.
Definition 2.7 Let $\mathcal{D} = \{U_S, \Pi_S, \rho_S\}$ be a system of control data on $(X, S)$, let $U \subset X$ be an open subset, let $A$ be a set, and let $f : U \to A$ be a map of sets. We say that $\mathcal{D}$ is $f$-compatible on $U$ if, for every $S \in S$, there is a neighborhood $U'_S \subset U \cap U$ of $S \cap U$ such that $f \circ \Pi_S = f$ on $U'_S$.

Definition 2.8 Let $(X, S)$, $(\hat{X}, \hat{S})$ be two Whitney stratified $C^\infty$ manifolds, with systems of control data $\mathcal{D} = \{U_S, \Pi_S, \rho_S\}$ on $(X,S)$ and $\hat{\mathcal{D}} = \{U_\hat{S}, \Pi_\hat{S}, \rho_\hat{S}\}$ on $(\hat{X}, \hat{S})$. A controlled homeomorphism $\phi : X \to \hat{X}$, compatible with $\mathcal{D}$ and $\hat{\mathcal{D}}$, is a homeomorphism which takes strata diffeomorphically onto strata, establishing a bijection $S \to \hat{S}$, and satisfies the following condition. For every $S \in S$, there is a neighborhood $U'_S \subset U_S$ of $S$, such that $\phi \circ \Pi_S = \Pi_\hat{S} \circ \phi$ on $U'_S$ and $\rho_\hat{S} = \rho_S \circ \phi$ on $U'_S \setminus S$. A controlled homeomorphism between two open subsets $U \subset X$ and $\hat{U} \subset \hat{X}$ is defined similarly (with compatibility conditions imposed in some neighborhood $U'_S \subset U_S \cap U$ of $S \cap U$, for every $S \in S$).

Definition 2.9 Let $\mathcal{D} = \{U_S, \Pi_S, \rho_S\}$ be a system of control data on $(X, S)$ and let $U \subset X$ be an open set. A controlled vector field $V$ on $U$ compatible with $\mathcal{D}$ is a collection $\{V_S\}_{S \in S}$ of smooth vector fields on the intersections $S \cap U$, satisfying the following condition. For every $S \in S$, there exists a neighborhood $U'_S \subset U_S \cap U$ of $S \cap U$ such that:

(a) $(\Pi_S)_*V_x = V_{\Pi_S(x)}$ for every $x \in U'_S$;
(b) $V_x \rho_S = 0$ for every $x \in U'_S \setminus S$.

Integrating controlled vector fields is a basic technique for constructing controlled homeomorphisms, going back to the work of Thom [Th2] and Mather [Ma]. The following lemma (see [Gr, Lemma 2.9], [Sh, Lemma 4.11], [dP, Theorem 1.1]) is a basic tool for constructing controlled vector fields.

Lemma 2.10 Let $\mathcal{D}$ be a system of control data on $(X, S)$. Let $S$ be a stratum, let $U \subset S$ be open in $S$, and let $V$ be a smooth vector field on $U$. Then there exist an open $U \subset X$, with $U \cap S = U'_S$ and a controlled vector field $\hat{V}$ on $U$ compatible with $\mathcal{D}$, such that $\hat{V}|_U = V$. Furthermore, $\hat{V}$ can be chosen to be continuous as a section of $TU$.

Controlled vector fields are too rigid for discussing ascending and descending sets. Indeed, a trajectory of a controlled vector field can not approach a point on a smaller stratum as time tends to infinity. We therefore need the following definition.

Definition 2.11 Let $\mathcal{D} = \{U_S, \Pi_S, \rho_S\}$ be a system of control data on $(X, S)$ and let $U \subset X$ be an open set. A weakly controlled vector field $V$ on $U$ compatible with $\mathcal{D}$ is a collection $\{V_S\}_{S \in S}$ of smooth vector fields on the intersections $S \cap U$, satisfying the following condition. For every $S \in S$, there exist a neighborhood $U'_S \subset U_S \cap U$ of $S \cap U$ and a number $k > 0$, such that:

(a) $(\Pi_S)_*V_x = V_{\Pi_S(x)}$ for every $x \in U'_S$;
(b) $|V_x \rho_S| < k \cdot \rho_S(x)$ for every $x \in U'_S \setminus S$.

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By [Gr, Proposition 2.11], weakly controlled vector fields integrate to stratum preserving homeomorphisms. Unfortunately, these homeomorphisms do not preserve control data. For this reason, weakly controlled vector fields compatible with a given system of control data do not form a Lie algebra, and we can not use the flow of one weakly controlled vector to transform another. As a result, we need both controlled and weakly controlled vector fields.

2.3 \textit{f-Adapted Control Data}

We continue with a Whitney stratified $C^\infty$ manifold $(X, S)$. Let $f : X \to \mathbb{R}$ be a smooth function. Let $A \in S$ be a stratum, and let $K \subset A$ be a compact subset. In this section, we introduce the notion of a system of control data on $(X, S)$ which is $f$-adapted near $K$. This notion will play an important role in the proofs of our main results in Section 4.

To begin, we fix a “smooth maximum” function

$$s_{\text{max}} : \mathbb{R}_\geq \times \mathbb{R}_\geq \to \mathbb{R}_\geq,$$

satisfying the following properties.

1. The restriction $s_{\text{max}}|_{\mathbb{R}_+ \times \mathbb{R}_+}$ is smooth and (non-strictly) convex.
2. $s_{\text{max}}(x, y) = \max(x, y)$ whenever $\max(x, y) \geq 1.1 \cdot \min(x, y)$.
3. $s_{\text{max}}(x, y) = s_{\text{max}}(y, x)$ for all $x, y \in \mathbb{R}_\geq$.
4. $s_{\text{max}}(a \cdot x, a \cdot y) = a \cdot s_{\text{max}}(x, y)$ for all $a \in \mathbb{R}_+$ and $x, y \in \mathbb{R}_\geq$.

One can check that (1)-(4) imply:

$$\max(x, y) \leq s_{\text{max}}(x, y) \leq 1.05 \cdot \max(x, y),$$

for all $x, y \in \mathbb{R}_\geq$.

**Definition 2.12** A system of control data $D = \{U_S, \Pi_S, \rho_S\}$ on $(X, S)$ is said to be $f$-adapted near $K$ if there exists an open neighborhood $U \subset U_A$ of $K$ such that the following conditions hold.

1. Write $\Pi = \Pi_A$, $\rho = \rho_A$, $A_U = A \cap U$. Let $g : U \to \mathbb{R}$ be the function

$$g(x) = f(x) - f(\Pi(x)).$$

Then we have $\Pi(U) = A_U$, $\Sigma_g = A_U$, and $d_x g \in \Lambda^0_A$ for every $x \in A_U$.

2. Let $\tilde{U} = \{x \in U \mid g(x) = 0\}$, and let $\tilde{\Pi} = \Pi|_{\tilde{U}} : \tilde{U} \to A_U$. Then $\tilde{U}$ is a smooth manifold, and $\tilde{\Pi}$ is a tubular projection. Moreover, there exist a quasi-distance function $\tilde{\rho} : \tilde{U} \setminus A_U \to \mathbb{R}_+$ compatible with $\tilde{\Pi}$, with continuous extension $\rho : \tilde{U} \to \mathbb{R}_\geq$, and a tubular projection $\pi : U \to \tilde{U}$, such that $\tilde{\Pi}(\pi(x)) = \Phi(x)$ for every $x \in U$, and

$$\rho(x) = s_{\text{max}}(\tilde{\rho}(\pi(x)), |g(x)|),$$

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for every \( x \in U \setminus A_U \).

(3) Let

\[
\hat{U} = \{ x \in U \mid \hat{\rho}(\pi(x)) > 0.8 \cdot |g(x)| \}.
\]

Then the map \( \Theta = (\Pi, g, \hat{\rho} \circ \pi) : \hat{U} \to A_U \times \mathbb{R} \times \mathbb{R}_+ \) is a stratified submersion.

(4) The system \( \mathcal{D} \) is \( \Theta \)-compatible on \( \hat{U} \).

The usefulness of the notion of \( f \)-adapted control data comes primarily from the following two propositions.

**Proposition 2.13** Assume that \( f|_A = 0 \) and \( d_x f \in \Lambda^0_A \) for every \( x \in K \). Let \( \Pi : U \to A \) be a tubular projection. Then there exists a system of control data \( \mathcal{D} = \{ U_S, \Pi_S, \rho_S \} \) on \( (X, S) \) which is \( f \)-adapted near \( K \) and satisfies \( \Pi_{A|U} = \Pi|_U \) for some open neighborhood \( U \subset U \cap U_A \) of \( K \).

**Proof:** This is similar to Steps 1-4 of the proof of [Gr, Theorem 3.1].

**Proposition 2.14** Assume that \( f|_A = 0 \) and \( d_x f \in \Lambda^0_A \) for every \( x \in K \). Let \( \mathcal{D} \) be a system of control data on \( (X, S) \) which is \( f \)-adapted near \( K \). Let \( V \) be a smooth vector field on \( A \). Then there exist an open neighborhood \( U \subset X \) of \( K \) and a controlled vector field \( \hat{V} \) on \( U \) compatible with \( \mathcal{D} \), such that \( \hat{V}|_{A \cap U} = V|_{A \cap U} \) and \( \hat{V}_x f = 0 \) for every \( x \in U \).

**Proof:** This is similar to Step 5 of the proof of [Gr, Theorem 3.1].

**Corollary 2.15** Let \( p \in \Sigma_f \cap A \) be a Morse critical point of \( f \). Then there exists a system of control data \( \mathcal{D} \) on \( (X, S) \) which is \( f \)-adapted near \( \{p\} \).

**Proof:** Pick a tubular projection \( \Pi : U \to A \). Pick a smooth function \( g : X \to \mathbb{R} \) such that \( g = f - f \circ \Pi \) in some neighborhood of \( p \). Note that \( d_p g = d_p f \in \Lambda^0_A \). Use Proposition 2.13 to obtain a system of control data \( \mathcal{D} = \{ U_S, \Pi_S, \rho_S \} \) on \( (X, S) \) which is \( g \)-adapted near \( \{p\} \) and satisfies \( \Pi_{A|U} = \Pi|_U \) for some open neighborhood \( U \subset U \cap U_A \) of \( p \). Then \( \mathcal{D} \) is also \( f \)-adapted near \( \{p\} \).

**Definition 2.16** Let \( (X, S) \) be a Whitney stratified \( C^\infty \) manifold, let \( A \in S \) be a stratum, and let \( p \in A \). A normal slice \( N \) to \( A \) passing through \( p \) is a locally closed smooth submanifold of \( X \), such that \( p \in N \), \( \dim A + \dim N = \dim X \), and \( N \) is transverse to the strata of \( S \).

A normal slice \( N \), as in Definition 2.16, is naturally a Whitney stratified space, with a stratification induced from \( S \). The following corollary is a paraphrase of [Gr, Corollary 3.2]. It may be called a “stratified Morse lemma” in the sense that it provides the closest we can come to a local normal form statement for a stratified Morse function near a critical point.
Corollary 2.17 In the situation of Corollary 2.15, let $D = \{U, \Pi, \rho\}$ be a system of control data on $(X, S)$ which is $f$-adapted near $\{p\}$. Let $N = \Pi_A^U(1)$. Then $N$ is a normal slice to $A$. Let $S_1$ be the stratification of $N$ induced from $S$, let $D_1$ be the system of control data on $(N, S_1)$ induced from $D$, and let $f_1 = f|_N : N \to \mathbb{R}$. Then $D_1$ is $f_1$-adapted near $\{p\}$. Moreover, there exists an open neighborhood $U \subset U_A$ of $p$ with the following property. Let $U_1 = U \cap N$ and $U_2 = U \cap A$. Let $D_1$ be the system of control data on $U_1 \times U_2$ induced from $D_1$. Then there exists a controlled homeomorphism

$$\phi : U_1 \times U_2 \to U,$$

compatible with $D_1$ and $D$, such that for every $x_1 \in U_1$ and $x_2 \in U_2$, we have:

(i) $\phi(x_1, p) = x_1$;
(ii) $\Pi_A \circ \phi(x_1, x_2) = x_2$;
(iii) $f \circ \phi(x_1, x_2) = f(x_1) + f(x_2) - f(p)$.

Proof: This is similar to the proof of [Gr, Corollary 3.2] and to Step 6 in the proof of [Gr, Theorem 3.1]. More precisely, pick a smooth function $g : X \to \mathbb{R}$ such that $g = f - f \circ \Pi_A$ in some neighborhood of $p$. Let $a = \dim A$, and let $\{V_i\}_{i=1}^a$ be a collection of vector fields on $A$ such that $\{(V_i)_p\}_{i=1}^a \subset T_pA$ is a basis. Proposition 2.14 provides controlled extensions $\{\tilde{V}_i\}_{i=1}^a$ of $\{V_i\}_{i=1}^a$, defined in some neighborhood of $p$ and preserving the function $g$. The homeomorphism $\phi$ is constructed by integrating the vector fields $\{\tilde{V}_i\}_{i=1}^a$. \[\Box\]

We have the following remarkable “softness” result for $f$-adapted control data.

Proposition 2.18 Let $(X, S)$ be a Whitney stratified $C^\infty$ manifold, and let $A \in S$. Suppose we have two points $p_0, p_1 \in A$, two normal slices $N_0 \ni p_0$ and $N_1 \ni p_1$ to $A$, and two functions $f_0 : N_0 \to \mathbb{R}$ and $f_1 : N_1 \to \mathbb{R}$, with $f_0(p_0) = f_1(p_1) = 0$. Assume that the differentials $d_{p_0}f_0$ and $d_{p_1}f_1$ are both in $\Lambda^1_A$ and, moreover, in the same path-component of $\Lambda^1_A$. Let $S_0, S_1$ be the stratifications of $N_0, N_1$ induced from $S$. Assume that we are given a system of control data $D_0$ on $(N_0, S_0)$ which is $f_0$-adapted near $\{p_0\}$, and similarly for $N_1$. Then there exist open neighborhoods $U_0 \subset N_0$ and $U_1 \subset N_1$ of $p_0$ and $p_1$, and a controlled homeomorphism $\phi : U_0 \to U_1$ compatible with $D_0$ and $D_1$, such that $f_0|_{U_0} = f_1 \circ \phi$.

Proof: The proof consists of two steps. The first step is to establish the proposition in the case when $p_0 = p_1$, $N_0 = N_1$, $f_0 = f_1$, and only the systems of control data $D_0, D_1$ may differ. The second step is to deduce the general case of the proposition. Both steps are similar to the proof [Gr, Corollary 3.5]. \[\Box\]

3 A Model Example

Theorem 1.6 is an existence result, and our proof of it will proceed by construction. This construction will be non-canonical, involving a number of arbitrary choices. However, the
idea of the construction comes from a phenomenon occurring “in nature.” Namely the behavior of the metric gradient of a linear function near an isolated conical singularity. In this section, we describe this “natural phenomenon” as motivation for the proof of Theorem 1.6 in Section 4.

Let $W \cong \mathbb{C}^n$ be a complex vector space. Let $PW \cong \mathbb{C}^{n-1}$ be the associated projective space. Let $PX \subset PW$ be a smooth algebraic subvariety, and let $X \subset W$ be the affine cone over $PX$. Let $p \in W$ be the origin. Then $X$ has a natural stratification with two strata $X = X^0 \cup \{p\}$, and $W$ has a natural stratification with three strata $W = W^0 \cup X^0 \cup \{p\}$. Fix a Hermitian metric $\mu$ on $W$ and define $r : W \to \mathbb{R}$ by $r(w) = \text{dist}_\mu(p,w)$.

Pick a linear function $\varphi : W \to \mathbb{C}$ such that the real part $f = \text{Re}(\varphi) : W \to \mathbb{R}$ is a generic covector at $p$ (i.e., $f \in \Lambda^0_{\{p\}}$ in the notation of Definition 2.1). Consider a vector field $V$ on $X$ defined by $V(p) = 0$ and

$$V|_{X^0} = r \cdot \nabla_\mu f|_{X^0}.$$  

Then $p$ is a Morse critical point of $f$, and $V$ is the restriction to $X$ of a $\nabla f$-like vector field on $W$. Define the ascending and descending sets $M^+_V(p) \subset X$ by analogy with Definition 1.2. The main result of this section (Theorem 3.1) is a description of $M^+_V(p)$.

Define $W^\pm = \varphi^{-1}(\pm 1) \subset W$. Consider the intersections $Y^\pm = W^\pm \cap X$ and the restrictions

$$g^\pm = r|_{Y^\pm} : Y^\pm \to \mathbb{R}.$$  

Note that each $Y^\pm$ is a smooth affine subvariety of $W$ and each $g^\pm : Y^\pm \to \mathbb{R}$ is a smooth function. Consider the gradient vector fields $G^\pm = \nabla_\mu g^\pm$ on $Y^\pm$. The flow of $G^\pm$ is integrable for all time. Define $Y^\pm_c \subset Y^\pm$ to be the union of all bounded trajectories of $G^\pm$. More precisely, we let $Y^\pm_c = \bigcup \gamma(\mathbb{R})$, where $\gamma$ runs over all parameterized trajectories $\gamma : \mathbb{R} \to Y^\pm$ of $G^\pm$ such that $g^\pm \circ \gamma : \mathbb{R} \to \mathbb{R}$ is bounded. It is not hard to check that $Y^\pm_c \subset Y^\pm$ is compact. Finally, let $\text{Cone}(Y^\pm_c) \subset X$ be the real cone over $Y^\pm_c$. More precisely, for $x \in X^0$, let $R(p,x) \subset X$ be the (closed) straight line ray originating from $p$ and passing through $x$. Also, let $R(\cdot) = R(p,x) \setminus \{p\}$. Then we have:

$$\text{Cone}(Y^\pm_c) = \bigcup_{y \in Y^\pm_c} R(p,y) = \{p\} \cup \bigcup_{y \in Y^\pm_c} R(\cdot).$$

**Theorem 3.1** We have $M^+_V(p) = \text{Cone}(Y^\pm_c)$.

Let $\Omega$ be the cone of all Hermitian metrics on $W$, and let $\Omega^0 = \Omega^0(X,\varphi) \subset \Omega$ be the set of all $\mu \in \Omega$ such that both functions $g^\pm : Y^\pm \to \mathbb{R}$ are Morse.

**Proposition 3.2** The set $\Omega^0$ is open and dense in $\Omega$.

**Proof:** Let us focus on the function $g^- : Y^- \to \mathbb{R}$. Write $\Sigma_{g^-} \subset Y^-$ for its critical locus.

The conical property of $X$ and the generic property of $\varphi$ imply that, for every $\mu_0 \in \Omega$, there

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exist an open neighborhood $U_{\mu_0} \subset \Omega$ of $\mu_0$ and a compact $K \subset Y^-$, such that $\Sigma_{g^-} \subset K$ for every $\mu \in U_{\mu_0}$. It follows that $\Omega^c \subset \Omega$ is open.

Let $\Delta = \text{Ker}(\phi) \subset W$. For a given $\mu \in \Omega$, let $L = \Delta^\perp \subset W$ be the orthogonal complement to $\Delta$ relative to $\mu$. Let $\{q^-\} = L \cap W^-$. Then $g^- : Y^- \to \mathbb{R}$ is Morse if and only if $q^-$ is not a focal point for the submanifold $Y^- \subset W^-$. The density of $\Omega^c \subset \Omega$ follows from the nowhere density of the set $F^- \subset W^-$ of focal points of $Y^-$. More precisely, let $\Omega(\Delta)$ be the cone of all Hermitian metrics on $\Delta$, and let $\delta : \Omega \to \Omega(\Delta)$ be the restriction map. Then the focal set $F^- \subset W^-$ depends on $\mu \in \Omega$ only through the image $\delta(\mu)$, and we can move the point $q^-$ away from $F^-$ by perturbing $\mu$ within the fiber $\delta^{-1}(\delta(\mu))$.

Let $d = \dim_c X$. We will use the term “cellular subset of $X$” to mean “cellular subset of $W$, contained in $X$.”

**Corollary 3.3** For $\mu \in \Omega^c$, the sets $Y_c^\pm$ and $M_V^\pm(p)$ are cellular subsets of $X$, and we have:

$$\dim Y_c^\pm \leq d - 1 \quad \text{and} \quad \dim M_V^\pm(p) \leq d.$$  

**Proof:** We will only consider $Y_c^-$ and $M_V^-(p)$. The statement for $Y^+$ and $M_V^+(p)$ is analogous. Fix a $\mu \in \Omega^c$. The critical locus $\Sigma_{g^-} \subset Y^-$ is compact (see the first paragraph of the proof of Proposition 3.2). Therefore, $\Sigma_{g^-}$ is a finite set. Order $\Sigma_{g^-}$ by critical value. More precisely, fix and ordering $\Sigma_{g^-} = \{y_0, \ldots, y_N\}$, such that:

$$i \leq j \quad \Rightarrow \quad g^-(y_i) \leq g^-(y_j), \quad \text{for all } i, j \in \{0, \ldots, N\}. \quad (1)$$

Note that

$$Y_c^- = \bigcup_{i=0}^N M^-_{G^-}(y_i), \quad (2)$$

where each descending set $M^-_{G^-}(y_i)$ is an open cell of dimension $\text{index}_{g^-}(y_i)$. For every $n \in \{0, \ldots, N\}$, the partial union

$$(Y_c^-)_{\leq n} = \bigcup_{i=0}^n M^-_{G^-}(y_i)$$

is closed because of (1). This proves that equation (2) presents $Y_c^-$ as a cellular subset of $X$. The dimension bound for $Y_c^-$ follows from the inequality:

$$\text{index}_{g^-}(y_i) \leq \dim_c Y^- = d - 1, \quad (3)$$

for every $i \in \{0, \ldots, N\}$ (see [Mi, Part I, §7]).

For every $i \in \{0, \ldots, N\}$, define

$$M^-_{G^-}(y_i)^g = \bigcup_{y \in M^-_{G^-}(y_i)} R^g(p, y).$$

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By Theorem 3.1, we have:

\[ M_V^-(p) = \{p\} \cup \bigcup_{i=0}^{N} M^-_{G_i}(y_i). \]  

Each \( M^-_{G_i}(y_i) \) is a cell of dimension index \( g(y_i) + 1 \). Equation (4) presents \( M_V^-(p) \) as a cellular subset of \( X \). The dimension bound for \( M_V^-(p) \) follows from (3). \( \square \)

We will prove the claim of Theorem 3.1 about the descending set \( M_V^-(p) \) only. The claim about \( M_V^+(p) \) is analogous. Our proof is based on the following three lemmas. Let \( f^\perp = \text{Im}(\varphi) : W \to \mathbb{R} \) be the imaginary part of \( \varphi \).

**Lemma 3.4** We have \( V_x f^\perp = 0 \) for every \( x \in X^\circ \).

**Proof:** This is a standard consequence of the Hermitian property of \( \mu \). \( \square \)

**Lemma 3.5** There exists a \( k > 0 \), such that

\[ r(x) < k \cdot |f(x)|, \]

for every \( x \in M_V^-(p) \).

**Proof:** Let \( \nu : X^\circ \to \mathbb{R} \) be the norm \( \nu = \|\nabla_\mu f|_{X^\circ}\|_\mu \). Since the function \( f \) is linear and the variety \( X \) is conical, the function \( \nu \) is constant on each open ray \( R^\circ(p,x) \) for \( x \in X^\circ \). Therefore, \( \nu \) attains a minimum \( \nu_0 \in \mathbb{R}_{\geq 0} \). Since \( f \in \Lambda^0_{(p)} \), we have \( \nu_0 > 0 \). It is not hard to check that the statement of the lemma holds for every \( k > 1/\nu_0 \). \( \square \)

Let \( Z = X \cap (f^\perp)^{-1}(0) \). Let \( I^- = (-\infty,0) \) and let \( Z^- = Z \cap f^{-1}(I^-) \). Consider the product \( Y^- = Y^- \times I^- \). The decomposition of \( Z^- \) into open rays \( R^\circ(p,y) \), for \( y \in Y^- \), defines an isomorphism \( \chi : Y^- \to Z^- \). More precisely, for \( y \in Y^- \) and \( a \in I^- \), we have

\[ \{\chi(y,a)\} = R^\circ(p,y) \cap f^{-1}(a). \]

Let \( \tilde{G}^- \) be the vector field on \( \tilde{Y}^- \) which is given in components by \( \tilde{G}^-(y,a) = (G_y,0) \). Also, consider the radial vector field \( E = -r \cdot \nabla_\mu r|_{Z^-} \) on \( Z^- \). Let \( \pi : Z^- \to Y^- \) be the projection \( \pi : \chi(y,a) \mapsto y \).

**Lemma 3.6** There exist smooth functions \( \alpha, \beta : Y^- \to \mathbb{R}_+ \), such that:

\[ V|_{Z^-} = (\alpha \circ \pi) \cdot E + (\beta \circ \pi) \cdot \chi_*(\tilde{G}^-). \]
Proof: By \( \mathbb{R}_+ \)-homogeneity, it is enough to find \( \alpha, \beta : Y^- \to \mathbb{R}_+ \), such that

\[
V_y = \alpha(y) \cdot E_y + \beta(y) \cdot G^-_y,
\]

for every \( y \in Y^- \).

Fix a point \( y \in Y^- \). Let \( V^1_y = (\nabla \mu f)_y \in W \). Let \( T_y Y^- \subset T_y Z^- \subset W \) be the tangent spaces to \( Y^- \) and \( Z^- \) at \( y \), considered as a linear subspaces of \( W \). Note that

\[
(\nabla \mu r)_y = -\frac{1}{r(y)} \cdot E_y \in T_y Z^-.
\]

Let \( \zeta : W \to T_y Z^- \) and \( \eta : T_y Z^- \to T_y Y^- \) be the orthogonal projections with respect to \( \mu \). By Lemma 3.4, we have \( V^1_y = \zeta(V^1_y) \). Also, we have \( G^-_y = \eta((\nabla \mu r)_y) \). Furthermore, \( T_y Y^- \subset T_y Z^- \) is the orthogonal complement to \( V_y \). Therefore, writing \( \langle \ , \ \rangle \) for the real inner product given by \( \mu \), we have

\[
G^-_y = -\frac{1}{r(y)} \cdot \eta(E_y) = -\frac{1}{r(y)} \cdot \left( E_y - \frac{\langle E_y, V_y \rangle}{\langle V_y, V_y \rangle} \cdot V_y \right).
\]

Note further that \( \langle E_y, V_y \rangle = \langle E_y, V^1_y \rangle = E_y f = 1 \). Thus, equation (5) holds with \( \alpha(y) = \langle V_y, V_y \rangle \) and \( \beta(y) = r(y) \cdot \langle V_y, V_y \rangle \). It is easy to see that, so defined, the functions \( \alpha, \beta : Y^- \to \mathbb{R}_+ \) are smooth. \( \square \)

Proof of Theorem 3.1: The claim of the theorem for the descending set \( M^-_V(p) \) is proved by putting together Lemmas 3.4, 3.5, 3.6.

It is not hard to check that the flow of \( V \) is integrable for all time. Let \( \gamma : \mathbb{R} \to X^o \) be a parameterized trajectory of \( V \). Let \( \Gamma = \gamma(\mathbb{R}) \subset X^o \). Suppose \( \Gamma \subset M^-_V(p) \). By Lemma 3.4, we have \( \Gamma \subset Z^- \). By Lemma 3.6, the image \( \pi(\Gamma) \subset Y^- \) is a trajectory of \( G^- \). By Lemma 3.5, the trajectory \( \pi(\Gamma) \) is bounded. This proves the containment \( M^-_V(p) \subset \text{Cone}(Y^-_c) \). The opposite containment is similar.

The claim of the theorem for the ascending set \( M^+_V(p) \) is analogous. \( \square \)

4 Proofs of the Main Results

In this section, we will prove the following enhanced versions of Theorems 1.6 and 1.8.

Theorem 4.1 Theorem 1.6 is true and, moreover, the vector field \( V \) can be chosen to be compatible with a system of control data which is \( f \)-adapted near \( \{p\} \).

Theorem 4.2 Theorem 1.8 is true and, moreover, the vector field \( V \) can be chosen to be compatible with a system of control data which is \( f \)-adapted near \( \{p\} \).
4.1 Reduction to a Point Stratum

The following proposition is a direct consequence of Corollaries 2.15, 2.17 and Proposition 2.18.

**Proposition 4.3** Let \( n > 0 \) be an integer. Assume that Theorem 4.1 is true for \( \dim C X < n \). Then Theorem 4.2 is true for \( \dim C X - \dim C A < n \).

**Proof:** Let \((X, S), f : X \to \mathbb{R}, p \in \Sigma f,\) and \( A \in S\) be as in Theorem 1.8. Assume that \( \dim C X - \dim C A < n \). Use Corollary 2.15 to obtain a system of control data \( D = \{ U_S, \Pi_S, \rho_S \} \) on \((X, S)\) which is \( f \)-adapted near \( \{p\} \). Let \( N = \Pi_A^{-1}(p) \); it is a normal slice to \( A \). Let \( S_1 \) be the stratification of \( N \) induced from \( S \), let \( D_1 \) be the system of control data on \((N, S_1)\) induced from \( D \), and let \( f_1 = f|_N : N \to \mathbb{R} \). Apply Corollary 2.17 to obtain an open neighborhood \( U \subset U_A \) of \( p \) and a controlled homeomorphism \( \phi : (U \cap N) \times (U \cap A) \to U \).

The normal slice \( N \) need not be complex analytic, so we can not apply Theorem 4.1 directly to the function \( f_1 : N \to \mathbb{R} \). However, we can pick a complex analytic normal slice \( N_0 \) to \( A \) through \( p \), apply Theorem 4.1 to the function \( f_a = f|_{N_0} : N_0 \to \mathbb{R} \), then use Proposition 2.18 to relate \( f_1 \) and \( f_a \). In this way, we obtain an open neighborhood \( U_1 \subset N \) of \( p \) and a \( \nabla f_1 \)-like vector field \( V_1 \) on \( U_1 \), compatible with \( D_1 \) and satisfying the conditions of Theorem 1.6. In particular, we obtain presentations of \( L_{V_1}^\pm(p) \) and \( M_{V_1}^\pm(p) \) as cellular subsets of \( U_1 \).

By shrinking the neighborhoods \( U \) and \( U_1 \), we can assume that \( U_1 = U \cap N \). Let \( U_2 = U \cap A \), and let \( f_2 = f|_{U_2} : U_2 \to \mathbb{R} \). Fix a Riemannian metric \( \mu \) on \( A \), and define a vector field \( V_2 \) on \( U_2 \) by \( V_2 = \nabla_\mu f_2 \). The ascending and descending links \( L_{V_2}^\pm(p) \) are both spheres, with

\[
\dim \mathbb{R} L_{V_2}^-(p) = \text{index}_{f_2}(p) - 1, \quad \text{and} \quad \dim \mathbb{R} L_{V_2}^+(p) = \dim \mathbb{R} A - \text{index}_{f_2}(p) - 1.
\]

(By convention, a sphere of dimension \(-1\) is the empty set.)

Let us focus on the descending link \( L_{V_2}^-(p) \). Assuming \( L_{V_2}^-(p) \neq \emptyset \), fix a point \( q^- \in L_{V_2}^-(p) \). Let \( D_1^- = \{ q^- \} \) and \( D_2^- = L_{V_2}^-(p) \setminus D_1^- \). The following equation presents \( L_{V_2}^-(p) \) as a cellular subset of \( U_2 \):

\[
L_{V_2}^-(p) = D_1^- \cup D_2^-.
\]

For \( i = 1, 2 \), let \( (D_i^-)^\# \subset M_{V_2}^-(p) \) be the union of all trajectories of \( V_2 \) passing through \( D_i^- \). By shrinking the neighborhood \( U_2 \), if necessary, we can assume that each \( (D_i^-)^\# \) is diffeomorphic to an open ball. The following equation then presents the descending set \( M_{V_2}^-(p) \) as a cellular subset of \( U_2 \):

\[
M_{V_2}^-(p) = \{ p \} \cup (D_1^-)^\# \cup (D_2^-)^\#.
\]

In the case \( L_{V_2}^-(p) = \emptyset \), we have \( M_{V_2}^-(p) = \{ p \} \), which is also a cellular subset of \( U_2 \). We proceed by analogy to present \( L_{V_2}^+(p) \) and \( M_{V_2}^+(p) \) as cellular subsets of \( U_2 \).
We define \( V = \phi_* (V^\times) \), where \( V^\times \) is the vector field on \( U_1 \times U_2 \) which is given in components by \( V_1 \) and \( V_2 \). Clearly, the vector field \( V \) is compatible with the system of control data \( D \). We have:

\[
M_V^\pm(p) = \phi( M_{V_1}^\pm(p) \times M_{V_2}^\pm(p) ).
\]

A product of cellular subsets can be given a structure of a cellular subset by ordering the pair-wise products of cells lexicographically. This procedure presents the products

\[
M_{V_1}^\pm(p) \times M_{V_2}^\pm(p) \subset U_1 \times U_2
\]
as cellular subsets. By applying the homeomorphism \( \phi \), we obtain structures of cellular subsets on the ascending and descending sets \( M_V^\pm(p) \subset U \). Verification of conditions (ii)-(iv) of Theorem 1.8 is routine.

Proposition 4.3 shows that Theorem 4.1 implies Theorem 4.2. We are going to prove Theorem 4.1 by induction on \( d = \dim_C X \). The case \( d = 0 \) is trivial. Fix an integer \( n > 0 \), and assume that Theorem 4.1 is true for \( d < n \). We will now proceed to establish Theorem 4.1 for \( d = n \), using Proposition 4.3 in the process.

### 4.2 Complex Links and Control Data

Without loss of generality we can assume that \( X \) is a neighborhood of the origin in a complex vector space \( W \cong \mathbb{C}^d \) and that \( p \in W \) is the origin. Furthermore, by Proposition 2.18, we can assume that \( f = \text{Re}(\varphi)|_X \), where \( \varphi : W \to \mathbb{C} \) is a complex linear function.

Pick a Hermitian metric \( \mu \) on \( W \) and define \( r : W \to \mathbb{R} \) by \( r(w) = \text{dist}_\mu(p, w) \). Let \( \Delta = \text{Ker}(\varphi) \subset W \), let \( \theta : W \to \Delta \) be the orthogonal projection with respect to \( \mu \), and let \( L = \theta^{-1}(p) \subset W \). Define \( \hat{r} = r \circ \theta : W \to \mathbb{R} \). Let

\[
\Psi = (\varphi, \hat{r}) : W \to \mathbb{R}^3 (= \mathbb{C} \times \mathbb{R})).
\]

Define \( \alpha : W \setminus L \to \mathbb{R} \) by \( \alpha(w) = |\varphi(w)|/\hat{r}(w) \). Note that \( \hat{r} \) and \( \Psi \) are smooth on \( W \setminus L \), while \( \alpha \) is smooth on \( W \setminus (L \cup \Delta) \).

**Lemma 4.4** There exist \( \delta, \kappa > 0 \) such that the following conditions hold.

(i) Let \( B_\delta = \{ w \in W \mid r(w) < \delta \} \). Then we have \( B_\delta \subset X \).

(ii) The map \( \varphi \) is a stratified submersion on the punctured ball \( B_\delta \setminus \{ p \} \).

(iii) The map \( \Psi \) is a stratified submersion on the region \( \{ x \in B_\delta \mid \alpha(x) < \kappa \} \).

**Proof:** This is similar to the proof of [Gr, Lemma 3.7].

We fix the numbers \( \delta, \kappa > 0 \) provided by Lemma 4.4 for the rest of Section 4. Let \( f^\perp = \text{Im}(\varphi)|_X : X \to \mathbb{R} \). For \( \epsilon > 0 \), define

\[
U_\epsilon = \{ x \in X \mid |f(x)| < \epsilon, \ |f^\perp(x)| < \epsilon, \ \hat{r}(x) < 10 \cdot \epsilon/\kappa \}.
\]
Fix an $\epsilon_0 > 0$ such that $U_{4\epsilon_0} \subset B_{\delta}$. We define the open neighborhood $U$ of Theorems 1.6, 4.1 by $U = U_{2\epsilon_0}$. Also, with a view to constructing a system of control data on $(X, S)$, we let $U_{\{p\}} = U_{3\epsilon_0}$.

Consider the manifolds
\[ Y^\pm = \{ x \in U \mid \varphi(x) = \pm \epsilon_0, \hat{r}(x) < 6 \cdot \epsilon_0/\kappa \}, \]
and the restrictions
\[ g^\pm = r|_{Y^\pm} : Y^\pm \to \mathbb{R}. \]
We will be referring to the spaces $Y^\pm$ as the complex links of $p$. This is slightly different from standard terminology (see [GM, Introduction, §1.5]) in that $Y^\pm$ are not closed in $X$. By condition (ii) of Lemma 4.4, the manifolds $Y^\pm$ meet the strata of $S$ transversely. Therefore, each $Y^\pm$ is Whitney stratified by the intersections with the strata of $S$. We denote these stratifications by $S^\pm$. Consider the stratified critical loci $\Sigma_{g^\pm} \subset Y^\pm$. By condition (iii) of Lemma 4.4, we have
\[ \Sigma_{g^\pm} \subset \{ y \in Y^\pm \mid \hat{r}(y) \leq \epsilon_0/\kappa \}. \]

By perturbing the metric $\mu$ and reducing the numbers $\delta, \kappa, \epsilon_0 > 0$, if necessary, we may assume that $g^\pm$ are stratified Morse functions (this is similar to Proposition 3.2).

Define $\rho_{\{p\}} : U_{\{p\}} \to \mathbb{R}_+$ by
\[ \rho_{\{p\}}(x) = \max(\max((\kappa/10) \cdot \hat{r}(x), |f^+(x)|), |f(x)|). \]

Let $\Pi_{\{p\}} : U_{\{p\}} \to \{p\}$ be the unique such map.
Lemma 4.6 Combining the system $\mathcal{D}^\circ$ of Lemma 4.5 with the triple $(U_{\{p\}}, \Pi_{\{p\}}, \rho_{\{p\}})$ produces a system of control data $\mathcal{D} = \{U_S, \Pi_S, \rho_S\}_{S \in S}$ on $(X, \mathcal{S})$ which is $f$-adapted near $\{p\}$.

Proof: Given that $\mathcal{D}^\circ$ is a system of control data on $(X^\circ, \mathcal{S}^\circ)$, in order to check that $\mathcal{D}$ is a system of control data on $(X, \mathcal{S})$, we only need to verify conditions (1)-(4) of Definition 2.5 for $S = \{p\}$. Conditions (1) and (2) follow from Lemma 4.4. Condition (3) is essentially vacuous. And condition (4) follows form conditions (i)-(ii) of Lemma 4.5 and the fact that $\rho_{\{p\}}$ factors through $\Psi$ on $U_{\{p\}}^{\circ,1}$ and through $\phi$ on $U_{\{p\}}^{\circ} \setminus U_{\{p\}}^{\circ,1}$. Verifying that $\mathcal{D}$ is $f$-adapted near $\{p\}$, based on Lemmas 4.4 and 4.5, is routine. \hfill \Box

4.3 Vector Fields on the Complex Links

The following proposition will provide the key inductive input into our construction of the vector field $V$ of Theorems 1.6, 4.1 in Section 4.4. Define 

$$Y_{\pm, a} = \{y \in Y^\pm | \hat{r}(y) > 3 \cdot \epsilon_0 / \kappa\}.$$

Proposition 4.7 There exists a $\nabla g^-$-like vector field $G^-$ on $Y^-$, compatible with the system of control data $\mathcal{D}^\circ$ of Lemma 4.5, such that the following conditions hold.

(i) For every $y \in \Sigma_{g^-}$, there exists an open neighborhood $U_y \subset Y^-$ of $y$ such that the restriction $G^-|_{U_y}$ satisfies conditions (i)-(iv) of Theorem 1.8 with $(X, \mathcal{S}) = (Y^-, \mathcal{S}^-)$, $f = g^-$, $p = y$, $U = U_y$, and $V = G^-|_{U_y}$.

(ii) For every $y \in Y_{\pm, a}$, we have $G_y \hat{r} = \hat{r}(y)$.

Similarly, there exists a $\nabla g^+$-like vector field $G^+$ on $Y^+$ compatible with $\mathcal{D}^+$ and satisfying the analogues of conditions (i)-(ii) above.

Proof: We present the proof for $Y^-$ only. Pick a critical point $y \in \Sigma_{g^-}$. By the induction hypothesis stated at the end of Section 4.1 and by Proposition 4.3, Theorem 4.2 holds for the function $g^- : Y^- \to \mathbb{R}$ and the critical point $y$. This gives us an open neighborhood $U(y) \subset Y^-$ of $y$ and an $\mathcal{S}^-$-preserving $\nabla g^-$-like vector field $G(y)$ on $U(y)$ satisfying conditions (i)-(iv) of Theorem 1.8. Moreover, the vector field $G(y)$ can be chosen to be compatible with a system of control data $\mathcal{D}(y)$ on $(Y^-, \mathcal{S}^-)$ which is $g^-$-adapted near $\{y\}$. By Corollary 2.17, Proposition 2.18, and condition (iii) of Lemma 4.5, we can assume that $\mathcal{D}(y) = \mathcal{D}^\circ$.

Next, let $Y_{-1} = \{y \in Y^- | \hat{r}(y) > 2 \cdot \epsilon_0 / \kappa\}$. By equation (6), we have $\Sigma_{g^-} \cap Y_{-1} = \emptyset$. By condition (ii) of Lemma 4.5, the system of control data $\mathcal{D}^\circ$ is $g^-$-compatible on $Y_{-1}$. Note that the function $\hat{r}$ factors through $g^-$ on $Y_{-1}$. More precisely, let $I_1 = g^-(Y_{-1})$ and $I_2 = \hat{r}(Y_{-1})$. Then both $I_1, I_2 \subset \mathbb{R}$ are open intervals, and there exists an orientation-preserving diffeomorphism $h : I_1 \to I_2$, such that $\hat{r} = h \circ g^-$ on $Y_{-1}$. It follows that there exists a $\nabla g^-$-like vector field $G_{-1}$ on $Y_{-1}$ compatible with $\mathcal{D}^\circ$, such that $G_{y, -1} \hat{r} = \hat{r}(y)$ for every $y \in Y_{-1}$.
The requisite vector field $G^-$ is obtained by “patching together” the vector fields $\{G(y)\}$ for $y \in \Sigma_{g^-}$ and $G^{-1}$, using Lemma 2.10 and a suitable partition of unity on $Y^-$. The only nuance is that $D^-$ must be compatible with the elements of the partition, to ensure that $G^-$ satisfies condition (a) of Definition 2.11.

4.4 Construction of the Vector Field $V$

We are now prepared to construct the requisite $\nabla f$-like vector field $V$ on the open set $U$ defined in Section 4.2. The vector field $V$ will be compatible with the system of control data $D$ of Lemma 4.6. Let $Z = U \cap (f^\perp)^{-1}(0)$. By condition (ii) of Lemma 4.4, the set $Z$ is a smooth manifold with a Whitney stratification $Z$ induced from $S$. By conditions (i)-(ii) of Lemma 4.5, the system of control data $D$ on $(X, S)$ restricts to a system of control data $\tilde{D}$ on $(Z, Z)$. Let $\tilde{f} = f|_Z : Z \to \mathbb{R}$. Define $U^o = U \setminus \{p\}$, $Z^o = Z \setminus \{p\}$, and write $\rho = \rho_{\{p\}} : U^o(p) \to \mathbb{R}_+$ to unclutter the notation.

Lemma 4.8 Let $\tilde{V}$ be a $\nabla f$-like vector field on $Z$ compatible with $\tilde{D}$. Then there exists a $\nabla f$-like vector field $\hat{V}$ on $U$ compatible with $D$, such that:

(i) $\hat{V}|_Z = \tilde{V}$;

(ii) $\hat{V}_x f^\perp = 0$ for every $x \in U$.

Moreover, for every such $\hat{V}$, we have $M^\pm_{\hat{V}}(p) = M^\pm_{\tilde{V}}(p) \subset Z$.

Proof: Cover the set $U^o$ by two open subsets as follows:

$$U^o_{1} = \{ x \in U^o \mid |f^\perp(x)| < \rho(x)/10 \},$$

$$U^o_{2} = \{ x \in U^o \mid |f^\perp(x)| > \rho(x)/20 \}.$$

Let $I = (-2\epsilon_0, 2\epsilon_0) \subset \mathbb{R}$. Define an open subset $\tilde{Z}^o \subset Z^o \times I$ by

$$\tilde{Z}^o = \{(x, a) \in Z^o \times I \mid |a| < \rho(x)/10 \}.$$

Consider $\tilde{Z}^o$ as a smooth manifold with a Whitney stratification $\tilde{Z}^o$ induced from $Z$ and a system of control data $\tilde{D}^o$ induced from $\tilde{D}$. By Lemmas 2.10, 4.4, 4.5 and equation (7), there exists a controlled homeomorphism

$$h : \tilde{Z}^o \to U^o_{1},$$

compatible with $\tilde{D}^o$ and $D$, such that for every $(x, a) \in \tilde{Z}^o$, we have:

(a) $h \left( x, 0 \right) = x$;

(b) $f^\perp \circ h \left( x, a \right) = a$;

(c) $f \circ h \left( x, a \right) = f(x)$;

(d) $\rho \circ h \left( x, a \right) = \rho(x)$.
Define a vector field \( \hat{V}^1 \) on \( \mathcal{U}^{0.1} \) by \( \hat{V}^1_{(x,a)} = h_s(\hat{V}_x, 0) \). Use Lemmas 2.10, 4.4, 4.5 and equation (7) again, to construct a \( \nabla f \)-like vector field \( \hat{V}^2 \) on \( \mathcal{U}^{0.2} \) compatible with \( \mathcal{D} \), such that for some \( k_2 > 0 \) and every \( x \in \mathcal{U}^{0.2} \), we have \( \hat{V}^2_x f^1 = 0 \) and \( |\hat{V}^2_x \rho| < k_2 \cdot \rho(x) \). The restriction of the requisite vector field \( \hat{V} \) to \( \mathcal{U}^0 \) is constructed by combining the vector fields \( \hat{V}^1 \) and \( \hat{V}^2 \), using a suitable partition of unity on \( \mathcal{U}^0 \). Verification of conditions (i) and (ii) is routine. The claim of the lemma about \( M^\pm(p) \) follows immediately from conditions (i) and (ii). □

We will first construct the restriction \( \hat{V} = V|_Z \), then use Lemma 4.8 to obtain the full vector field \( V \). Define

\[
Z^- = \{ x \in Z^o \mid f(x) < 0, \hat{r}(x) < (6/\kappa) \cdot |f(x)| \},
\]

\[
Z^s = \{ x \in Z^o \mid \hat{r}(x) > (3/\kappa) \cdot |f(x)| \},
\]

\[
Z^{ss} = \{ x \in Z^o \mid \hat{r}(x) > (4/\kappa) \cdot |f(x)| \},
\]

\[
Z^+ = \{ x \in Z^o \mid f(x) > 0, \hat{r}(x) < (6/\kappa) \cdot |f(x)| \}.
\]

The superscript “s” stand for “safe.” In the construction that follows, any trajectory of \( \hat{V} \) which enters \( Z^s \) will be safe from approaching the critical point \( p \). We begin by constructing vector fields \( \hat{V}^\pm \) on \( Z^\pm \). They will serve as the main building blocks in the construction of \( \hat{V} \) (see Lemma 4.12). We will only describe the construction of \( \hat{V}^- \). The construction of \( \hat{V}^+ \) is analogous. Lemmas 4.9, 4.10, 4.12, 4.13 below should be seen as Lemmas/Definitions; they introduce objects which will be referred to directly later.

**Lemma 4.9** There exists a controlled vector field \( E \) on \( Z^- \) compatible with \( \hat{D} \), such that the following conditions hold.

(i) For every \( x \in Z^- \), the derivative \( E_x f = -f(x) \).

(ii) For every \( x \in Z^- \cap Z^s \), the derivative \( E_x \hat{r} = -\hat{r}(x) \).

**Proof:** This is similar to the proof of Proposition 4.7, using Lemmas 2.10, 4.4, 4.5 and a partition of unity argument. □

Let \( I^- = (-2\epsilon_0, 0) \). Consider the product \( \bar{Y}^- = Y^- \times I^- \). Let \( \pi_1 : \bar{Y}^- \to Y^- \) and \( \pi_2 : \bar{Y}^- \to I^- \) be the projection maps. Let \( \bar{S}^- \) be the stratification of \( \bar{Y}^- \) induced from \( S^- \), and let \( \bar{D}^- \) be the system of control data on \( (\bar{Y}^-, \bar{S}^-) \) induced from \( D^- \).

**Lemma 4.10** There exists a unique controlled homeomorphism \( \chi : \bar{Y}^- \to Z^- \) compatible with \( \bar{D}^- \) and \( \hat{D} \), such that the following conditions hold.

(i) For every \( y \in Y^- \) and every \( \epsilon \in I^- \), we have \( f(\chi(y, \epsilon)) = \epsilon \).

(ii) For every \( y \in Y^- \), we have \( \chi(y, -\epsilon_0) = y \).

(iii) For every \( y \in Y^- \), the set \( \chi(\pi_1^{-1}(y)) \) is a trajectory of \( E \).

(iv) For every \( y \in Y^- \cap Z^s \) and every \( \epsilon \in I^- \), we have \( \alpha(\chi(y, \epsilon)) = \alpha(y) \).

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Proof: It is not hard to check that the requisite homeomorphism $\chi$ is given by:

$$\chi(y, \epsilon) = \psi_{E,t(\epsilon)}(y),$$

where $t(\epsilon) = \ln(\epsilon_0/\epsilon)$ and $\psi_{E,t(\epsilon)}$ is the flow of the vector field $E$. \qed

Recall the vector field $G^-$ on $Y^-$, provided by Proposition 4.7. Let $\hat{G}^-$ be the weakly controlled vector field on $\hat{Y}^-$ which is given in components by $\hat{G}^-(y,\epsilon) = (G^- y, 0)$. We define

$$\hat{V}^- = E + \chi^*(\hat{G}^-).$$

(8)

Lemma 4.11 The vector field $\hat{V}^-$ is $\nabla \hat{f}$-like on $Z^-$. Furthermore, it satisfies the following conditions.

(i) For every $x \in Z^-$, we have $\hat{V}_x^- f = \rho(x)$.

(ii) For every $x \in Z^- \cap Z^s$, we have $\hat{V}_x^- \hat{r} = 0$.

Proof: By condition (i) of Lemma 4.10, we have $\chi^* (\hat{G}^-) x f = 0$ for every $x \in Z^-$. Combining this with condition (i) of Lemma 4.9, we obtain:

$$\hat{V}_x^- f = E_x f = -f(x) = |f(x)|.$$

By equation (7), we have $\rho(x) = |f(x)|$ for every $x \in Z^-$. This verifies condition (i) and the fact that $\hat{V}^-$ is $\nabla \hat{f}$-like.

Next, by condition (ii) of Proposition 4.7 and conditions (i), (iv) of Lemma 4.10, we have $\chi^* (\hat{G}^-) x \hat{r} = \hat{r}(x)$ for every $x \in Z^- \cap Z^s$. Combining this with condition (ii) of Lemma 4.9, we obtain:

$$\hat{V}_x^- \hat{r} = \chi^* (\hat{G}^-) x \hat{r} + E_x \hat{r} = \hat{r}(x) - \hat{r}(x) = 0.$$

This verifies condition (ii). \qed

At this point, we assume that we have carried out the analogous construction to obtain a $\nabla \hat{f}$-like vector field $\hat{V}^+$ on $Z^+$, satisfying $\hat{V}_x^+ f = \rho(x)$ for every $x \in Z^+$ and $\hat{V}_x^+ \hat{r} = 0$ for every $x \in Z^+ \cap Z^s$.

Lemma 4.12 There exists a $\nabla \hat{f}$-like vector field $\hat{V}^\circ$ on $Z^\circ$ compatible with $\hat{D}$, such that the following conditions hold.

(i) For every $x \in Z^\pm \setminus Z^{ss}$, we have $\hat{V}_x^\circ = \hat{V}_x^\pm$.

(ii) For every $x \in Z^s$, we have $\hat{V}_x^\circ f = \rho(x)$.

(iii) For every $x \in Z^s$, we have $\hat{V}_x^\circ \hat{r} = 0$.

Proof: This is another partition of unity argument, similar to the proof of Proposition 4.7. By Lemma 2.10, condition (iii) of Lemma 4.4, condition (ii) of Lemma 4.5, and equation (7), there exists a weakly controlled vector field $V^{ss}$ on $Z^{ss}$ compatible $\hat{D}$, such that $\hat{V}_x^{ss} f = \rho(x)$.
and \( \hat{V}^{ss} \hat{r} = 0 \) for every \( x \in Z^{ss} \). Pick a partition of unity \( \eta^{-} + \eta^{ss} + \eta^{+} = 1 \) on \( Z^{o} \), such that \( \text{supp}(\eta^{\pm}) \subset Z^{\pm} \), \( \text{supp}(\eta^{ss}) \subset Z^{ss} \), and each of the functions \( \eta^{\pm}, \eta^{ss} : Z^{o} \to [0, 1] \) factors through \( \alpha : W \setminus L \to \mathbb{R} \) on the overlaps \( Z^{\pm} \cap Z^{ss} \subset W \setminus L \). We define \( \hat{V}^{o} \) by

\[
\hat{V}^{o}_{x} = \eta^{-}(x) \cdot \hat{V}^{-} + \eta^{ss}(x) \cdot \hat{V}^{ss} + \eta^{+}(x) \cdot \hat{V}^{+},
\]

for every \( x \in Z^{o} \). In the above equation, each of the terms \( \eta^{*}(x) \cdot \hat{V}^{*} \) is understood to be zero if \( \eta^{*}(x) = 0 \) and \( \hat{V}^{*} \) is undefined. Verification of the properties of \( \hat{V}^{o} \) is routine. \( \square \)

**Lemma 4.13** Define a vector field \( \hat{V} \) on \( Z \) by \( \hat{V}|_{Z^{o}} = \hat{V}^{o} \) and \( \hat{V}_{p} = 0 \). Then \( \hat{V} \) is a \( \nabla \hat{f} \)-like vector field on \( Z \) compatible with \( \hat{D} \).

**Proof:** By Lemma 4.12, the restriction \( \hat{V}|_{Z^{o}} \) is a \( \nabla \hat{f} \)-like vector field compatible with \( \hat{D} \). The only property of \( \hat{V} \) left to verify is condition (b) of Definition 2.11 for the stratum \( \{p\} \in \mathcal{Z} \). We will show that there exists a \( k > 0 \), such that

\[
|\hat{V}_{x} p| < k \cdot \rho(x), \quad (9)
\]

for every \( x \in Z^{o} \). Recall the function \( \text{smax} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) introduced in Section 3.3 and used in equation (7). By property (1) of \( \text{smax} \), we can consider the partial derivative

\[
\zeta(x,y) = \frac{\partial \text{smax}(x,y)}{\partial x},
\]

as a function \( \zeta : \mathbb{R}_{+} \times \mathbb{R}_{+} \to \mathbb{R} \). By property (4) of \( \text{smax} \), we have \( \zeta(a \cdot x, a \cdot y) = \zeta(x,y) \) for all \( a, x, y \in \mathbb{R}_{+} \). By property (2) of \( \text{smax} \), we have \( \zeta(x,y) = 1 \) for all \( x, y \in \mathbb{R}_{+} \) with \( x > 1.1 \cdot y \), and \( \zeta(x,y) = 0 \) for all \( x, y \in \mathbb{R}_{+} \) with \( y > 1.1 \cdot x \). It follows that the function \( \zeta \) attains a maximum \( c = \max(\zeta) \in \mathbb{R} \). Conditions (ii) and (iii) of Lemma 4.12 then imply that inequality (9) holds for every \( k > c \). It is a pleasant exercise, not necessary for this proof, to check that, in fact, \( c = 1 \). \( \square \)

We use Lemma 4.8 to obtain an extension \( \hat{V} \) of \( \hat{V} \) to \( \mathcal{U} \); then let \( V = \hat{V} \). This completes our construction of the vector field \( V \).

### 4.5 Identification of the Sets \( M^{+}_{V}(p) \)

In this section, we identify the sets \( M^{\pm}_{V}(p) \) for the vector field \( V \) constructed in Section 4.4, and verify conditions (i)-(iv) of Theorem 1.6. We will limit our discussion to the descending set \( M^{-}_{V}(p) \). The discussion for the ascending set \( M^{+}_{V}(p) \) is analogous and will be omitted.

Recall the vector field \( G^{-} \) on the complex link \( Y^{-} \), provided by Proposition 4.7. For every \( y \in \Sigma_{g^{-}} \), consider the descending set \( M^{-}_{G^{-}}(y) \subset Y^{-} \). Define

\[
Y_{c}^{-} = \bigcup_{y \in \Sigma_{g^{-}}} M^{-}_{G^{-}}(y). \quad (10)
\]
By the continuity of the flow of $G^-$, the set $Y_c^- \subset Y^-$ is compact. Moreover, by equation (6), we have
\[ Y_c^- \subset \{ y \in Y^- \mid \hat{r}(y) \leq \epsilon_0 / \kappa \}. \]
Recall the controlled homeomorphism $\chi : \tilde{Y}^- \to Z^-$ of Lemma 4.10. The following proposition is analogous to Theorem 3.1.

**Proposition 4.14** We have $M_{\tilde{V}}(p) = \{ p \} \cup \chi(Y_c^- \times I^-)$.

**Proof:** This is analogous to the proof of Theorem 3.1. The role of Lemma 3.4 in that proof is played by condition (ii) of Lemma 4.8. The role of Lemma 3.5 is played by condition (iii) of Lemma 4.12, which implies that $M_{\tilde{V}}(p) \cap Z^- = \emptyset$. The role of Lemma 3.6 is played by equation (8).

Proposition 4.14 asserts that $Y_c^-$ is a descending link of $p$. We will now show that $Y_c^-$ is naturally a cellular subset of $Y^-$. As in the proof of Corollary 3.3, order the set $\Sigma_{g^-}$ by critical value. More precisely, fix an ordering $\Sigma_{g^-} = \{ y_0, \ldots, y_N \}$ satisfying implication (1) from that proof. For each $y \in \Sigma_{g^-}$, recall the open neighborhood $U_y \subset Y^-$ of Proposition 4.7, pick a small number $\nu < 0$, and consider the descending link $L^-(y) = L_{G^-,\nu}(y) \subset U_y$. By Proposition 4.7, the set $L^-(y)$ is a cellular subset of $U_y$. Let $L^-(y) = \{ C_0(y), \ldots, C_n(y) \}$ be the ordered set of cells of $L^-(y)$. For $C \in L^-(y)$, define $C^\sharp \subset M_{G^-}(y)$ to be the union of all trajectories $\Gamma$ of $G^-$ with $\Gamma \cap C \neq \emptyset$. It is not hard to check that $C^\sharp$ is diffeomorphic to an open ball with $\dim R C^\sharp = \dim R C + 1$. Furthermore, we have
\[ M_{G^-}(y) = \{ y \} \cup \bigcup_{C \in L^-(y)} C^\sharp. \] (11)

**Proposition 4.15** (i) We have:
\[ Y_c^- = \bigcup_{i=0}^N \left( \{ y_i \} \cup \bigcup_{j=0}^{n(y_i)} C_j(y_i)^2 \right). \] (12)
Moreover, the above equation endows $Y_c^- \subset Y^-$ with a structure of a cellular subset.

(ii) For every stratum $S \in \mathcal{S}^-$, we have $\dim_R Y_c^- \cap S \leq \dim_C S$.

**Proof:** Equation (12) follows from equations (10) and (11). The right-hand side of (12) is an ordered union of cells. Condition (b) of Definition 1.5 follows from the fact each $L^-(y_i) \subset Y^-$ is a cellular subset and the continuity of the flow of $G^-$. This verifies claim (i).

Claim (ii) follows from condition (i) of Proposition 4.7 and the inequality $\text{index}_{g^-}(y_i) \leq 0$ for every $y_i \in \Sigma_{g^-}$ (cf. inequality (3) in the proof of Corollary 3.3).

We are now prepared to describe a cell decomposition of $M_{\tilde{V}}(p)$. For every $y \in \Sigma_{g^-}$ and $C \in L^-(y)$, define $\{ y \}^2 = \chi(\{ y \} \times I^-)$ and $C^\sharp^2 = \chi(C^\sharp \times I^-)$.  

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Proposition 4.16  We have

\[ M_V(p) = \{ p \} \cup \bigcup_{i=0}^{N} \left( \{ y_i \}^{\#} \cup \bigcup_{j=0}^{n(y_i)} C_j(y_i)^{\#} \right). \]

Moreover, the above equation endows \( M_V(p) \subset U \) with a structure of a cellular subset.

Proof: This follows from Proposition 4.14, claim (i) of Proposition 4.15, and the continuity of the flow of \( V \).

Condition (i) of Theorem 1.6 for the vector field \( V \) follows from Proposition 4.16. Condition (ii) follows from equation (8) and condition (iii) of Lemma 4.10. Conditions (iii) and (iv) follow from claims (i) and (ii) of Proposition 4.15, respectively. Since the vector field \( V \) is compatible with the system of control data \( \mathcal{D} \) of Lemma 4.6, which is \( f \)-adapted near \( \{ p \} \), this completes our proof of Theorem 4.1. Theorem 4.2 follows by Proposition 4.3.

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