On the Symmetry of the Diffusion Coefficient in Asymmetric Simple Exclusion.

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ABSTRACT: We prove the symmetry of the diffusion coefficient that appears in the fluctuation-dissipation theorem for asymmetric simple exclusion processes.

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1 Introduction

The fluctuation-dissipation theorem has been an instrumental tool in the analysis of non-gradient interacting particle systems. In the context of asymmetric simple exclusion processes (ASEP) it has been used to establish the hydrodynamic limit for the mean-zero ASEP\(^7\), as well as the diffusive incompressible limit,\(^2\) the first order corrections to the hydrodynamic limit,\(^3\) the equilibrium fluctuations of the density field,\(^1\) and the diffusive hydrodynamic limit when the initial density profile is constant along the direction of the drift,\(^6\) for the general ASEP in \(d \geq 3\).

The fluctuation-dissipation theorem consists of decomposing the (normalized) particle currents \((w_i)_{1 \leq i \leq d}\) (which are not of gradient form) into gradients of the occupation variable and a rapidly fluctuating term. With a suitable interpretation it can be formulated in the following equation:

\[
w_i = \sum_{j=1}^{d} D_{ij}(\eta(0) - \eta(e_j)) + Lu_i. \tag{1}
\]

The matrix \(D = (D_{ij})_{1 \leq i,j \leq d}\) is called the diffusion coefficient and it naturally appears in the PDEs that arise in the hydrodynamic limit. Explicit and variational formulae for \(D\) are available\(^4\) and it is known to be a smooth function of the particle density.\(^5\) In this article we prove that \(D\) is symmetric, thus answering the question raised by Landim, Olla and Yau in ref. 3 and 4.

2 Notation and Results

Let us fix a finite range probability measure \(p(\cdot)\) on \(\mathbb{Z}^d\), with \(p(0) = 0\). We denote by \(L\) the generator of the simple exclusion process associated to \(p(\cdot)\). \(L\) acts on local functions on the state space \(X = \{0,1\}^{\mathbb{Z}^d}\) according to:

\[
Lf(\xi) = \sum_{x,y} p(y - x) \xi(x) (1 - \xi(y)) \left( f(\xi^{x,y}) - f(\xi) \right) \tag{2}
\]
be the space of local functions $g$.

In order to avoid degeneracies we will assume that the random walk in $\mathbb{Z}^d$ is irreducible, i.e., \{ $x : a(x) > 0$ \} generates the group $\mathbb{Z}^d$. An equivalent formulation of this assumption is that the matrix $S = (S_{ij})_{1 \leq i,j \leq d}$ defined by $S_{ij} = \frac{1}{2} \sum_{i} p(z_{i}z_{j})$ is invertible.

The symmetric part of the generator (denoted by $L^*$) is given by (4) with $p(\cdot)$ replaced by $a(\cdot)$. The measures $\mu_\rho$ ($0 \leq \rho \leq 1$), defined as Bernoulli products of parameter $\rho$ over the sites of $\mathbb{Z}^d$ are invariant under the dynamics. We will denote expectations under $\mu_\rho$ by $\langle \cdot \rangle_\rho$ and inner products in $L^2(\mu_\rho)$ by $\langle \cdot, \cdot \rangle_\rho$.

The adjoint of $L$ in $L^2(\mu_\rho)$ is the generator $L^*$ of the simple exclusion process associated to the law $p^*(x) = p(-x)$. Local functions form a core of both $L$ and $L^*$, and thus $L^*$ extends to a self-adjoint operator in $L^2(\mu_\rho)$.

The particle current along the direction $e_i$ is given by:

$$ W_i = \frac{1}{2} \sum_{z} p(z) z_i \xi(0)(1 - \xi(z)) - p(z) z_i \xi(z)(1 - \xi(0)). $$

Equation (3) is to be understood in the Hilbert space of fluctuations, which we define next. Let $\mathcal{G}_\rho$ be the space of local functions $g$ such that:

$$ \langle g \rangle_\rho = 0, \quad \text{and} \quad \left. \frac{d}{d\theta} \langle g \rangle_\rho \right|_{\theta = \rho} = 0. $$

For a $g \in \mathcal{G}_\rho$ we define $\tau_x g = g(\tau_x \xi)$, where $\tau_x \xi(z) = \xi(x + z)$. For any $f \in \mathcal{G}_\rho$ and $i \in \{1, \ldots, d\}$ we define:

$$ \langle g, f \rangle_{\rho,0} := \sum_x \langle g, \tau_x f \rangle_\rho, \quad t_i(g) = \langle g, \sum_x x_i \xi(x) \rangle_\rho. $$

Set $\chi(\rho) = \rho(1 - \rho)$ and define

$$ \langle \langle g \rangle \rangle_\rho = \sup_{\alpha \in \mathbb{R}^d} \left( 2 \sum_{i=1}^d \alpha_i t_i(g) - \chi(\rho) \alpha \cdot S \alpha \right) + \sup_{f \in \mathcal{G}_\rho} \left( 2 \langle g, f \rangle_{\rho,0} - \langle f, -L^* f \rangle_{\rho,0} \right). $$

The Hilbert space of fluctuations $\mathcal{H}(\rho)$ is defined as the closure of $\mathcal{G}_\rho$ under $\langle \langle \cdot \rangle \rangle_\rho^{1/2}$. If we denote by $\mathcal{H}_0$ the space generated by gradients of the occupation variable: $\mathcal{H}_0 = \{ \sum \alpha_i (\xi(e_i) - \xi(0)) : \alpha \in \mathbb{R}^d \}$, then by Theorem 5.9 in ref. 2 we have:

$$ \mathcal{H}(\rho) = \mathcal{H}_0 + L \mathcal{G}_\rho. $$

Notice that unless $\sum z p(z) = 0$ the currents $W_i$ do not belong to the space $\mathcal{G}_\rho$. Therefore we define the normalized currents $w_i \in \mathcal{G}_\rho$ by:

$$ w_i = W_i - \langle W_i \rangle_\rho - \langle \xi(0) - \rho \rangle \frac{d}{d\theta} \langle W_i \rangle_\theta \bigg|_{\theta = \rho}. $$
According to (5) there exist coefficients \((D_{ij})_{1 \leq i,j \leq d}\) (which depend on \(\rho\)) such that:

\[
w_i - \sum_{j=1}^{d} D_{ij} \times (\xi(0) - \xi(e_j)) \in L\mathcal{G}_\rho.
\] (7)

The matrix \(D = (D_{ij})_{1 \leq i,j \leq d}\) is called the diffusion coefficient of the simple exclusion process. Landim, Olla and Yau proved explicit and variational formulae for \(D\) in ref. 4. In the same paper, as well as in ref. 3, the authors question whether there exists a choice of \(p(\cdot)\) such that \(D\) is asymmetric. The result of this article is the following theorem:

**Theorem 1** The diffusion coefficient \(D\) defined in (7) is always a symmetric matrix.

### 3 Some Properties of \(\mathcal{H}(\rho)\)

In this section we review some properties of the Hilbert space of fluctuations that will be useful in the proof of Theorem 1. We begin with the following lemma.

**Lemma 1** If \(g \in \mathcal{G}_\rho\) and \(h \in \mathbb{Z}^d\), then \(\tau_h g = g\) in \(\mathcal{H}(\rho)\).

**Proof:** In view of (4) it suffices to show that:

\[
\begin{align*}
(i) & \quad \langle \tau_h g - g, f \rangle_{\rho,0} = 0, \forall f \in \mathcal{G}_\rho, \\
(ii) & \quad t_i(\tau_h g - g) = 0, i = 1, \ldots, d.
\end{align*}
\]

Using the translation invariance of \(\mu_\rho\) property (i) follows immediately, while

\[
\sum_x \langle \tau_h g - g, x_i \xi(x) \rangle_{\rho} = h_i \sum_x \langle g, \xi(x) \rangle_{\rho}.
\]

The last expression is trivially zero if \(\rho \in \{0,1\}\), while otherwise by differentiating with respect to \(\theta\) both sides of the following identity

\[
\langle g \theta \rangle = \int g(\xi) \prod_{x \in \text{supp}(g)} \left( \frac{\theta}{\rho} \right)^{\xi(x)} \left( \frac{1-\theta}{1-\rho} \right)^{1-\xi(x)} \, d\mu_\rho(\xi),
\]

we get

\[
\sum_x \langle g, \xi(x) \rangle_{\rho} = \rho \langle g \rangle_{\rho} + \chi(\rho) \frac{d}{d\theta} \langle g \theta \rangle \bigg|_{\theta=\rho} = 0,
\]

thus establishing (ii). \(\square\)

The following lemma is a generalisation of (5.1) in ref. 4 to the general ASEP, and can be proved by polarization of (4). The details are left to the reader.

**Lemma 2** Let \(g, f \in \mathcal{G}_\rho\) and set \(\nabla_{e_k} \xi(0) = \xi(0) - \xi(e_k)\) for \(k = 1, \ldots, d\). Then:

\[
\begin{align*}
(i) & \quad \langle \langle \nabla_{e_k} \xi(0), \nabla_{e_k} \xi(0) \rangle \rangle_{\rho} = \chi(\rho)(S^{-1})_{kk}, \\
(ii) & \quad \langle \langle \nabla_{e_k} \xi(0), Lg \rangle \rangle_{\rho} = -\langle \langle \nabla_{e_k} \xi(0), L^* g \rangle \rangle_{\rho} = \sum_{i=1}^{d} (S^{-1})_{kk} \langle w_i, g \rangle_{\rho,0}, \\
(iii) & \quad \langle \langle L^* g \rangle \rangle_{\rho,0} = 0, \\
(iv) & \quad \langle \langle L^* f \rangle \rangle_{\rho,0} = -\langle g, f \rangle_{\rho,0}.
\end{align*}
\]
4 The Diffusion Matrix

Recall the definition of the normalised currents \(w_i\) given in \([3]\) and \([6]\). It follows by elementary algebra and Lemma 1 that

\[
W^*_i(\xi) = \sum_z z_i \alpha(\xi(0) - \xi(z))
\]

is the current of the symmetric simple exclusion with generator \(L^s\), and

\[
h_i(\xi) = \sum z_i b(\xi(0) - \rho) (\xi(z) - \rho).
\]

Hence, the normalized currents for the reversed process are given by

\[
w^*_i = W^*_i + h_i.
\]

Let now \(C(\rho)\) (resp. \(C^*(\rho)\)) be the real vector space generated by the currents \(\{w_i; \; i = 1, \ldots, d\}\) (resp. \(\{w^*_i; \; i = 1, \ldots, d\}\)).

We define the linear operator \(T\) (resp. \(T^*\)) on \(C(\rho) + LG_\rho\) (resp. \(C^*(\rho) + L^*G_\rho\)) by:

\[
T \left( \sum_{i=1}^d \alpha_i w_i + Lg \right) = \sum_{i,k=1}^d \alpha_i S_{ik} \nabla e_k \xi(0) + L^s g,
\]

\[
T^* \left( \sum_{i=1}^d \alpha_i w^*_i + L^s g \right) = \sum_{i,k=1}^d \alpha_i S_{ik} \nabla e_k \xi(0) + L^s g.
\]

By Theorem 5.9 in ref. 2 we have:

\[
H(\rho) = C(\rho) + LG_\rho = C^*(\rho) + L^*G_\rho.
\]

Now, just as in Lemma 5.4 in ref. 4, \(T\) and \(T^*\) are norm bounded by 1, hence they can be extended to \(H(\rho)\). Furthermore, it follows easily by computations based on Lemma 2 that \(T^*\) is the adjoint of \(T\) with respect to \(\langle\langle \cdot, \cdot \rangle\rangle_{\rho}\) and \(T^* \nabla e_k \xi(0)\) is orthogonal to \(LG_\rho\). Hence by (17):

\[
\langle\langle w_i, T^* \nabla e_k \xi(0) \rangle\rangle_{\rho} = \sum_{i=1}^d D_{ij} \langle\langle \nabla e_j \xi(0), T^* \nabla e_k \xi(0) \rangle\rangle_{\rho},
\]

and thus by Lemma 2(i):

\[
\chi(\rho) I_d = D \cdot Q
\]

where the matrix \(Q = (Q_{jk})_{1 \leq j,k \leq d}\) is given by:

\[
Q_{jk} = \langle\langle T \nabla e_j \xi(0), \nabla e_k \xi(0) \rangle\rangle_{\rho}.
\]

We are now ready to proceed with the proof of Theorem 2.

**Proof** (of Theorem 2): Let us denote the reflection operator on \(\mathbb{X}\) by

\[
R\xi(z) = \xi(-z).
\]

The action of \(R\) is naturally extended to functions as \(Rf(\xi) = f(R\xi)\). Clearly, \(R^2 = 1\). Furthermore, the following commutation relation can be readily verified:

\[
RL = L^* R.
\]

In particular \(R\) commutes with \(L^s\) and hence, \(R\) preserves inner products in \(H(\rho)\).

Notice that \(W^*_i\) are anti-symmetric under \(R\), while \(h_i\) are \(R\)-symmetric. Thus,

\[
Rw_i(\xi) = -W^*_i(\xi) - h_i(\xi) = -w_i^*(\xi).
\]
It is a direct consequence of (9), (10), and the observation that \( R \nabla_{e_k} \xi(0) = -\nabla_{e_k} \xi(0) \) in \( \mathcal{H}(\rho) \) that

\[
RT = T^* R.
\]

Therefore,

\[
Q_{jk} = \langle\langle RT \nabla_{e_j} \xi(0), R \nabla_{e_k} \xi(0) \rangle\rangle_{\rho} \\
= \langle\langle T^* R \nabla_{e_j} \xi(0), R \nabla_{e_k} \xi(0) \rangle\rangle_{\rho} \\
= \langle\langle \nabla_{e_j} \xi(0), T \nabla_{e_k} \xi(0) \rangle\rangle_{\rho} \\
= Q_{kj}.
\]

So \( Q \) and thus the diffusion coefficient \( D \) are symmetric matrices. \( \square \)

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