Solving RG equations with the Lambert $W$ function

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Abstract

It has been known for some time that 2-loop renormalization group (RG) equations of a dimensionless parameter can be solved in a closed form in terms of the Lambert $W$ function. We apply the method to a generic theory with a Gaussian fixed point to construct RG invariant physical parameters such as a coupling constant and a physical squared mass. As a further application, we speculate a possible exact effective potential for the $O(N)$ linear sigma model in four dimensions.
I. INTRODUCTION

The purpose of this paper is to solve generic 2-loop renormalization group (RG) equations exactly using the Lambert $W$ function. The Lambert $W$ function has been introduced previously to solve analytically the 2- and 3-loop RG equations for QCD. (See also for a review of various applications of the Lambert $W$ function to QCD.) We apply the same function to solve generic 2-loop RG equations for theories with a Gaussian fixed point such as the $O(N)$ non-linear sigma model in four dimensions. (This has been partially done in.)

In the following we wish to justify our purpose by reminding the reader of the generality of 2-loop RG equations. We consider two examples in four dimensions: QCD and the $\phi^4$ theory. Let us consider QCD first.

Let $\Lambda_0$ be the ultraviolet cutoff, and $g_0^2$ be the bare gauge coupling, normalized appropriately. To construct the continuum limit ($\Lambda_0 \to \infty$) we must give a particular $\Lambda_0$ dependence to $g_0^2$:

$$g_0^2 = \frac{1}{\ln \frac{\Lambda_0}{\mu} + c \ln \ln \frac{\Lambda_0}{\mu} - \ln \frac{\Lambda(g^2)}{\mu}}$$

where $c = \frac{6 \times 153}{33}$ for QCD with no quarks. We can introduce a gauge coupling $g^2$, renormalized at a renormalization scale $\mu$, through the $\Lambda_0$ independent constant in the denominator. If we choose

$$\frac{\Lambda(g^2)}{\mu} = e^{-\frac{1}{g^2} \left( \frac{1}{g^2} + c \right)^c}$$

then $g^2$ satisfies the 2-loop RG equation

$$- \mu \frac{\partial}{\partial \mu} g^2 = (g^2)^2 + c(g^2)^3$$

exactly.

We next consider the $\phi^4$ theory defined by the bare action

$$S = \int d^4x \left( \frac{1}{2} \partial_{\mu} \phi \partial_{\mu} \phi + \frac{m_0^2}{2} \phi^2 + \frac{\lambda_0}{4!} \phi^4 \right)$$

with an ultraviolet cutoff $\Lambda_0$. This is a theory with the Gaussian fixed point $m_0^2 = \lambda_0 = 0$. We cannot take $\Lambda_0$ all the way to infinity, but for $\Lambda_0$ large compared with the physical mass, we obtain an almost continuum limit. For a given $\lambda_0$, let the critical squared mass be

$$m_{\text{cr}}^2(\lambda_0) = A_4(\lambda_0) \Lambda_0^2$$
Then, to get an almost continuum limit, we tune the bare squared mass as

\[ m_0^2 = A_4(\lambda_0)\Lambda_0^2 + z_m(\lambda_0) \left( \frac{(4\pi)^2}{3\lambda_0} - c \right)^a m^2 \]  

(6)

where

\[ c = \frac{17}{27}, \quad a = -\frac{1}{3} \]  

(7)

Here, the renormalized coupling \( \lambda \) is defined so that

\[ \frac{\Lambda_0}{\mu} = e^{t_0(\lambda_0) - (4\pi)^2 \frac{3\lambda}{3\lambda_0} \left( \frac{(4\pi)^2}{3\lambda_0} - c \right)^c} \]  

(8)

The \( \lambda_0 \) dependence of \( z_m \) and \( t_0 \) is determined so that the theory with given \( m^2 \) and \( \lambda \) have no \( \lambda_0 \) dependence except for non-universal contributions suppressed by inverse powers of \( \Lambda_0 \). The parameters \( \lambda, m^2 \) renormalized at the scale \( \mu \) satisfy the 2-loop RG equations (1-loop for \( m^2 \)) exactly:

\[
\begin{align*}
-\mu \frac{\partial}{\partial \mu} \left[ 3\lambda \left( \frac{(4\pi)^2}{3\lambda_0} - c \right) \right] &= - \left( \frac{3\lambda}{(4\pi)^2} \right)^2 + c \left( \frac{3\lambda}{(4\pi)^2} \right)^3 \\
-\mu \frac{\partial}{\partial \mu} m^2 &= a \frac{3\lambda}{(4\pi)^2} m^2
\end{align*}
\]  

(9)

We have thus reminded the reader that renormalization schemes exist so that 2-loop RG equations become exact. Hence, solving 2-loop RG equations amounts to solving general RG equations. This paper is organized as follows. In sect. II we solve the generic 2-loop RG equations \([9]\) exactly in terms of the Lambert \( W \) function. (This has actually been done already in sect. II.B.1 of \([6]\).) Then, in sec. III, we give the main results of this paper by constructing two physical parameters: one corresponding to the dimensionless coupling and the other corresponding to a physical squared mass. In sect. IV we invert the construction and express the renormalized parameters in terms of the physical parameters. In sect. V we generalize the exact effective potential in the large \( N \) limit of the \( O(N) \) linear sigma model \([7]\) to construct a trial effective potential for finite \( N \), fully consistent with the 2-loop RG equations.

In this paper we adopt the convention to fix the renormalization scale at \( \mu = 1 \). Hence, a squared mass parameter acquires the canonical dimension 2 in addition to the anomalous dimension in its RG equation.
II. GENERIC 2-LOOP RG EQUATIONS

We consider the following generic 2-loop RG equation:

$$\frac{d}{dt}x = -x^2 + cx^3$$

\hspace{1cm} (10)

and

$$\frac{d}{dt}m^2 = (2 + ax)m^2$$

\hspace{1cm} (11)

For example, in the $O(N)$ linear sigma model in four dimensions, we find

$$c = \frac{9N + 42}{(N + 8)^2}$$

\hspace{1cm} (12)

for the self-coupling, and

$$a = -\frac{N + 2}{N + 8}$$

\hspace{1cm} (13)

for the squared mass parameter.

In the following we assume $c > 0$ and $0 \leq x \ll \frac{1}{c}$. Let us define a mass scale by

$$\Lambda(x) \equiv \left\{ e^{\frac{1}{cx} - 1} \left( \frac{1}{cx} - 1 \right) \right\}^c \gg 1$$

\hspace{1cm} (14)

This satisfies

$$\frac{d}{dt}\Lambda(x) = \Lambda(x)$$

\hspace{1cm} (15)

$\Lambda(x)$ is of the same order as the UV cutoff. We can invert the definition of $\Lambda(x)$ to express $x$ in terms of $\Lambda(x)$. Since

$$\Lambda(x) = e^{\frac{1}{cx} - 1} \left( \frac{1}{cx} - 1 \right)$$

\hspace{1cm} (16)

we obtain

$$\frac{1}{cx} - 1 = W\left( \Lambda(x) \right)$$

\hspace{1cm} (17)

where $W$ is the upper branch of the Lambert $W$ function defined by

$$W(x)e^{W(x)} = x$$

\hspace{1cm} (18)

for $x \geq -\frac{1}{e}$. (See Appendix 1.)

We now define the running parameter $\bar{x}(t; x)$ by

$$\frac{1}{c\bar{x}(t; x)} - 1 = W\left( \left( e^t\Lambda(x) \right) \right)$$

\hspace{1cm} (19)
or equivalently by
\[ \bar{x}(t; x) = \frac{1}{c \left( 1 + W \left( \left( e^{t \Lambda(x)} \right)^{\frac{1}{a}} \right) \right)} \] (20)

so that it satisfies both
\[ \partial_t \bar{x}(t; x) = -\bar{x}(t; x)^2 + c\bar{x}(t; x)^3 \] (21)

and the initial condition
\[ \bar{x}(0; x) = x \] (22)

(Eq. (20) agrees with (26) of [6] which gives the same result for \( c = 1 \).)

Analogously, the running parameter \( \overline{m^2}(t; x, m^2) \) defined by
\[ \overline{m^2}(t; x, m^2) \equiv e^{2t}m^2 \left( \frac{\frac{1}{c\bar{x}(t; x)} - 1}{\frac{1}{c\bar{x}} - 1} \right)^a \] (23)

satisfies
\[ \partial_t \overline{m^2}(t; x, m^2) = (2 + a\bar{x}(t; x)) \overline{m^2}(t; x, m^2) \] (24)

and the initial condition
\[ \overline{m^2}(0; x, m^2) = m^2 \] (25)

III. PHYSICAL PARAMETERS

We now apply the results of the previous section to construct physical parameters. We first observe that the combination
\[ \frac{m^2}{\left( \frac{1}{c\bar{x}} - 1 \right)^a \Lambda(x)^2} = m^2 \frac{e^{-2c\left( \frac{1}{c\bar{x}} - 1 \right)}}{\left( \frac{1}{c\bar{x}} - 1 \right)^{a+2c}} \] (26)

is an RG invariant. We then note that any RG invariant of \( x \) and \( m^2 \) can be obtained as a function of the above RG invariant.

Let \( x_{ph}(x, m^2) \) be the RG invariant satisfying
\[ x_{ph}(x, 1) = x \] (27)

To obtain an explicit expression for \( x_{ph}(x, m^2) \), let us write it in the form
\[ x_{ph}(x, m^2) = f \left( (m^2)^{-\frac{1}{a+2c}} \frac{2c}{a+2c} \left( \frac{1}{c\bar{x}} - 1 \right) e^{\frac{2c}{a+2c}\left( \frac{1}{c\bar{x}} - 1 \right)} \right) \] (28)
where we assume $m^2 > 0$. The condition (27) implies

$$f \left( \frac{2c}{a + 2c} \left( \frac{1}{cx} - 1 \right) e^{\frac{2c}{a + 2c} \left( \frac{1}{cx} - 1 \right)} \right) = x$$

(29)

We can rewrite this as

$$\frac{2c}{a + 2c} \left( \frac{1}{cf(se^s)} - 1 \right) = s \equiv \frac{2c}{a + 2c} \left( \frac{1}{cx} - 1 \right)$$

(30)

For small $x \ll 1$, we find $s \gg 1$ for $a + 2c > 0$, and $-s \gg 1$ for $a + 2c < 0$. The above equation is solved by the Lambert $W$ function as

$$\frac{2c}{a + 2c} \left( \frac{1}{cf(se^s)} - 1 \right) = \begin{cases} 
W(se^s) & \text{if } a + 2c > 0 \\
W^{-1}(se^s) & \text{if } a + 2c < 0
\end{cases}$$

(31)

We thus obtain the physical coupling $x_{ph}(x, m^2)$ as

$$\frac{2c}{a + 2c} \left( \frac{1}{cx_{ph}(x, m^2)} - 1 \right) = \begin{cases} 
W \left( \frac{(m^2)}{a + 2c} \left( \frac{1}{cx_{ph}(x, m^2)} - 1 \right) e^{\frac{2c}{a + 2c} \left( \frac{1}{cx_{ph}(x, m^2)} - 1 \right)} \right) & \text{if } a + 2c > 0 \\
W^{-1} \left( \frac{(m^2)}{a + 2c} \left( \frac{1}{cx_{ph}(x, m^2)} - 1 \right) e^{\frac{2c}{a + 2c} \left( \frac{1}{cx_{ph}(x, m^2)} - 1 \right)} \right) & \text{if } a + 2c < 0
\end{cases}$$

(32)

We plot the left-hand side as a function of $m^2$ for $x = 0.1$ assuming $c = \frac{17}{27}$ and $a = -\frac{1}{3}$.

![Plots of $x_{ph}(x, m^2)$ and $\frac{34}{25} (\frac{27}{17 x_{ph}(x, m^2)} - 1)$](image)

**FIG. 1.** Plots of $x_{ph}$ (left) and $\frac{34}{25} \left( \frac{27}{17 x_{ph}(x, m^2)} - 1 \right)$ (right) for $x = 0.1$, $c = \frac{17}{27}$, $a = -\frac{1}{3}$: $x_{ph}$ increases monotonically as a function of $m^2$. It vanishes as $\frac{1}{\ln m}$ as $m^2 \to 0$, and approaches $\frac{27}{17}$ as $m^2 \to \infty$.

Though it is not obvious, the physical coupling $x_{ph}$ admits an asymptotic expansion in powers of $x$. This is because $x_{ph}$ can be defined by the differential equation

$$\frac{d}{dt} x_{ph} \equiv \left[ (-x^2 + cx^3) \partial_x + (2 + ax)m^2 \partial_{m^2} \right] x_{ph} = 0$$

(33)
and the initial condition (27). For small \( x \ll 1 \), we can expand \( x_{\text{ph}} \) asymptotically in powers of \( x \) in the form

\[
x_{\text{ph}}(x, m^2) = x \left[ 1 + \sum_{n=1}^{\infty} x^n p_n(\ln m^2) \right]
\]

(34)

where \( p_n \) is a polynomial of degree \( n \) satisfying \( p_n(0) = 0 \). In principle, this can be shown directly from (32), but it is more easily shown from the differential equation (33).

The physical coupling can also be given in the form of a running parameter:

\[
x_{\text{ph}}(x, m^2) = \bar{x}(-t_{\text{ph}}(x, m^2); x)
\]

(35)

where \( t_{\text{ph}}(x, m^2) \) satisfies

\[
\frac{d}{dt} t_{\text{ph}}(x, m^2) = 1
\]

(36)

and

\[
t_{\text{ph}}(x, 1) = 0
\]

(37)

To find \( t_{\text{ph}}(x, m^2) \), we use the defining equality \( W(xe^x) = x \) to obtain

\[
\frac{1}{cx_{\text{ph}}(x, m^2)} - 1 = W \left( \frac{1}{cx_{\text{ph}}(x, m^2)} - 1 \right) e^{\frac{1}{cx_{\text{ph}}(x, m^2)} - 1} = W \left( \frac{\Lambda(x)^{\frac{1}{2}} \left( \frac{1}{cx_{\text{ph}}(x, m^2)} - 1 \right) e^{\frac{1}{cx_{\text{ph}}(x, m^2)} - 1} \cdot \Lambda(x)^{\frac{1}{2}}} {\left( e^{\frac{1}{cx_{\text{ph}}(x, m^2)} - 1} \right)^{\frac{1}{2}}} \right)
\]

(38)

Hence, we obtain

\[
t_{\text{ph}}(x, m^2) = \ln \Lambda(x) - c \ln \left\{ \left( \frac{1}{cx_{\text{ph}}(x, m^2)} - 1 \right) e^{\frac{1}{cx_{\text{ph}}(x, m^2)} - 1} \right\}
\]

(39)

In addition to the physical coupling, we can introduce a physical squared mass by

\[
m_{\text{ph}}^2(x, m^2) \equiv m^2 \left( \frac{1}{cx_{\text{ph}}(x, m^2)} - 1 \right)^a
\]

(40)

This satisfies

\[
\frac{d}{dt} m_{\text{ph}}^2(x, m^2) = 2m_{\text{ph}}^2(x, m^2)
\]

(41)

and the initial condition

\[
m_{\text{ph}}^2(x, 1) = 1
\]

(42)

Using (32), we can rewrite the physical squared mass as

\[
m_{\text{ph}}^2(x, m^2) = \frac{m^2}{\left( \frac{1}{x} - c \right)^a} \left\{ \frac{a + 2c}{2} W \left( \frac{2c - \frac{2c}{a + 2c} \left( \frac{m^2}{\left( \frac{1}{x} - c \right)^a \Lambda(x)} \right)^{-\frac{1}{a + 2c}}} {a + 2c} \right) \right\}^a
\]

(43)
\( W \) should be replaced by \( W_{-1} \) if \( a + 2c < 0 \). The physical squared mass also admits an asymptotic expansion in \( x \) just as \( x_{\text{ph}} \):

\[
m^2_{\text{ph}}(x, m^2) = m^2 \left[ 1 + \sum_{n=1}^{\infty} x^n q_n(\ln m^2) \right] \tag{44}
\]

where \( q_n \) is a polynomial of degree \( n \) satisfying \( q_n(0) = 0 \).

![Graph](image)

**FIG. 2.** Plot of \( m^2_{\text{ph}} \) and \( m^2 \): We plot \( m^2_{\text{ph}} \) for \( x = 0.1, c = \frac{17}{27}, a = -\frac{1}{3} \). \( m^2_{\text{ph}} \) is essentially a monotonically increasing function of \( m^2 \), even though it eventually starts decreasing when \( m^2 \) reaches the UV cutoff scale.

**IV.** \( x, m^2 \) in Terms of \( x_{\text{ph}}, m^2_{\text{ph}} \)

In the above we have introduced two physical parameters \( x_{\text{ph}}, m^2_{\text{ph}} \) as functions of \( x, m^2 \). We can invert their relations to express \( x, m^2 \) in terms of \( x_{\text{ph}}, m^2_{\text{ph}} \). We first rewrite \( (32) \) and \( (40) \) as

\[
\left( \frac{1}{c x} - 1 \right) e^{\frac{2c}{x - a + 2c}}(m^2)^{-\frac{1}{a + 2c}} = \left( \frac{1}{c x_{\text{ph}}} - 1 \right) e^{\frac{2c}{x_{\text{ph}}}}\left( \frac{1}{c x_{\text{ph}}} - 1 \right)^{-a} \tag{45}
\]

\[
m^2 \left( \frac{1}{c x} - 1 \right)^{-a} = m^2_{\text{ph}} \left( \frac{1}{c x_{\text{ph}}} - 1 \right)^{-a} \tag{46}
\]

where we assume \( m^2 > 0 \), and \( W \) should be replaced by \( W_{-1} \) if \( a + 2c < 0 \). Substituting the second equation into the first to eliminate \( m^2 \), we obtain

\[
\left( \frac{1}{c x} - 1 \right) e^{\frac{1}{c x_{\text{ph}}} - 1} = \left( m^2_{\text{ph}} \right)^{\frac{1}{x_{\text{ph}}}} \left( \frac{1}{c x_{\text{ph}}} - 1 \right) e^{\frac{1}{c x_{\text{ph}}} - 1} \tag{47}
\]
This gives
\[ \frac{1}{c^X} - 1 = W \left( \left( \frac{m_{ph}^2}{c^X} \right)^{\frac{1}{2}} \left( \frac{1}{c} \right)^{\frac{1}{2}} e^{\frac{1}{2} c^X - 1} \right) \] (48)
which is valid irrespective of the sign of \( a + 2c \). Hence, we obtain
\[ x = \frac{1}{c} \frac{1}{1 + W \left( \left( \frac{m_{ph}^2}{c^X} \right)^{\frac{1}{2}} \left( \frac{1}{c} \right)^{\frac{1}{2}} e^{\frac{1}{2} c^X - 1} \right)} \] (49)

Using this result, we then obtain
\[ m^2 = m_{ph}^2 \left( \left( \frac{1}{c^X} - 1 \right) \right)^{-a} \]
\[ = m_{ph}^2 \left\{ \frac{1}{c^X} - 1 \right\}^{-a} \]
\[ = m_{ph}^2 \left\{ \frac{1}{c^X} - 1 \right\}^{-a} \]
(50)

V. A TRIAL EFFECTIVE POTENTIAL CONSISTENT WITH RG

In [7], the effective potential for the large \( N \) limit of the \( O(N) \) linear sigma model in four dimensions has been obtained as
\[ \frac{d}{d \ln^2 v} V_{\text{eff}}(v) = m_{ph}^2 \left( v, m^2 + (4\pi)^2 x \frac{v^2}{2} \right) \] (51)
where \( v \) is the VEV of the scalar field with no anomalous dimension:
\[ \frac{d}{dt} v = v \] (52)

In the large \( N \) limit, we obtain
\[ c = 0, \quad a = -1 \] (53)
so that
\[ \left\{ \begin{array}{l}
\Lambda(x) = e^{\frac{1}{x}} \\
m_{ph}^2(x, m^2) = \Lambda(x)^2 \exp \left[ W_{-1} \left( -2 \frac{m^2}{x \Lambda(x)x} \right) \right]
\end{array} \right. \] (54)

In the symmetric phase \( m^2 > 0 \), \( m_{ph}^2(x, m^2) \) gives the physical squared mass of the scalar fields \( \phi^I (I = 1, \cdots, N) \). In the broken phase \( m^2 < 0 \), the physical squared mass vanishes at
\[ v^2 = \frac{-m^2}{(4\pi)^2 x} \] (55)
To generalize (51) for a finite $N$, for which $c$ and $a$ are given by (12) and (13), we may try
\[
\frac{d}{d^2 v} V_{\text{eff}}(v) = z(x'_{\text{ph}}) m^2_{\text{ph}} \left( x, m^2 + g(x'_{\text{ph}}) \left( \frac{1}{x} - c \right) \frac{v^2}{2} \right) \geq 0
\]  
which is a monotonically increasing function of $v^2$. Here $z$ and $g$ are positive functions of the RG invariant
\[
x'_{\text{ph}} \equiv x_{\text{ph}}(x, |m^2|)
\]
which is well-defined irrespective of the sign of $m^2$. Note that the term added to $m^2$ satisfies the same RG equation as $m^2$:
\[
\frac{d}{dt} \left[ \left( \frac{1}{x} - c \right) \frac{v^2}{2} \right] = (2 + ax) \left[ \left( \frac{1}{x} - c \right) \frac{v^2}{2} \right]
\]
There is no justification for (56) except that it is fully consistent with RG, and that it gives the correct result in the large $N$ limit where $z$ and $g$ are mere constants:
\[
z = 1, \quad g = (4\pi)^2
\]
Note that in the broken phase $m^2 < 0$, the effective potential (or equivalently (56)) is defined only for
\[
v^2 \geq v^2_{\text{min}} \equiv -\frac{2m^2}{g(x'_{\text{ph}})} \left( \frac{1}{x} - c \right)^{-a} > 0
\]
Since the right-hand side of (56) is monotonically increasing with $v^2$, the effective potential is minimized at $v^2 = v^2_{\text{min}}$.

The main advantage of the assumption (56) is its integrability. To integrate (56) with respect to $v$, we use (13) to write (56) as
\[
\frac{d}{d^2 v} V_{\text{eff}}(v) = z(x'_{\text{ph}}) \left( \frac{m^2}{(\frac{1}{x} - c)^a} + g(x'_{\text{ph}}) \frac{v^2}{2} \right)
\]
\[
\times \left\{ \frac{a + 2c}{2} W \left[ \frac{2c^{-\frac{2c}{a+2c}}}{a + 2c} \left( \frac{1}{\Lambda(x)^2} \left( \frac{m^2}{(\frac{1}{x} - c)^a} + g(x'_{\text{ph}}) \frac{v^2}{2} \right) \right) \right] \right\}^a
\]
where $W$ should be $W_{-1}$ for $a + 2c < 0$. Denoting
\[
\eta \equiv \frac{1}{\Lambda(x)^2} \left( \frac{m^2}{(\frac{1}{x} - c)^a} + g(x'_{\text{ph}}) \frac{v^2}{2} \right) \ll 1
\]
we can rewrite the differential equation for $V_{\text{eff}}$ as
\[
\frac{d}{d\eta} V_{\text{eff}} = z(x'_{\text{ph}}) \frac{1}{g(x'_{\text{ph}}) \Lambda(x)^4} \eta \left\{ \frac{a + 2c}{2} W \left[ \frac{2c^{-\frac{2c}{a+2c}}}{a + 2c} \eta \left( \frac{1}{\eta^{\frac{a+2c}{a}}(\frac{1}{x} - c)^{a+2c}} \right) \right] \right\}^a
\]
We thus obtain
\[
V_{\text{eff}}(v) = \frac{z(x_{\text{ph}}')}{g(x_{\text{ph}}')} \Lambda(x)^4 \int_0^\eta d\eta \left\{ \frac{a + 2c}{2} W \left( \frac{2c^{\frac{a}{a+2c}}}{a+2c} \eta^{\frac{1}{a+2c}} \right) \right\}^a
\]  
(64)

Using the formulas
\[
\int_s^\infty ds s^{\beta-1} W(s)^\alpha = -(-\beta)^{1-\alpha-\beta} \left[ \beta \Gamma (\alpha + \beta, -\beta W(s)) - \Gamma (\alpha + \beta + 1, -\beta W(s)) \right]
\]  
(65)

\[
\int_0^s ds (-s)^{\beta-1} (-W_{-1}(s))\alpha = -\beta^{1-\alpha-\beta} \left[ \beta \Gamma (\alpha + \beta, -\beta W_{-1}(s)) - \Gamma (\alpha + \beta + 1, -\beta W_{-1}(s)) \right]
\]  
(66)

where
\[
\Gamma(a, z) \equiv \int_z^\infty dt t^{a-1} e^{-t}
\]  
(67)

is the incomplete gamma function, we finally obtain
\[
V_{\text{eff}}(v) = \frac{z(x_{\text{ph}}')}{g(x_{\text{ph}}')} 2^{a+8c-1} c^4 \Lambda(x)^4 \times \left\{ 2(a + 2c) \Gamma (-a - 4c, 2(a + 2c)W(s)) + \Gamma (-a - 4c + 1, 2(a + 2c)W(s)) \right\}
\]  
(68)

where
\[
s \equiv \frac{2}{a + 2c} c^{-\frac{a}{a+2c}} \eta^{-\frac{1}{a+2c}}
\]  
(69)

Note that $W$ should be replaced by $W_{-1}$ for $a + 2c < 0$.

VI. CONCLUSIONS

In this paper we have constructed two physical parameters $x_{\text{ph}}$ and $m_{\text{ph}}^2$ by solving generic 2-loop RG equations analytically in terms of the Lambert $W$ function. In addition we have constructed explicitly a trial effective action, which is fully consistent with RG, by generalizing the analytic expression for the large $N$ limit of the $O(N)$ linear sigma model in four dimensions. The trial effective potential is, however, at best a wild guess at the true effective potential. Its only merit may be that it gives an intriguing example of what RG improved perturbation theory can produce.
The closed-form analytic expressions for $x_{ph}$ (given by (32)) and $m^2_{ph}$ (given by (40)) sum the corresponding perturbative series. Further studies may elucidate the precise asymptotic nature of the perturbative expansions, as has been done for QCD.\cite{2,3}

Appendix A: The Lambert $W$ function

The Lambert $W$ function is defined implicitly by

$$W(x)e^{W(x)} = x \quad (A1)$$

or equivalently by

$$W(xe^x) = x \quad (A2)$$

Restricted to real values, the function has two branches: the upper $W_0(x) > -1$ defined for $x \in [-e^{-1}, +\infty)$ and the lower $W_{-1}(x) < -1$ for $x \in [-e^{-1}, 0)$. (See Fig. 3) For simplicity, we denote $W_0$ as $W$ in this paper. We obtain the following asymptotic expansions:

1. For $x \gg 1$,

$$W_0(x) = \ln x - \ln \ln x + O\left(\frac{\ln \ln x}{\ln x}\right) \quad (A3)$$

2. For $-x \ll 1$,

$$W_{-1}(x) = \ln (-x) - \ln (-\ln (-x)) + O\left(\frac{\ln (-\ln (-x))}{\ln (-x)}\right) \quad (A4)$$
Appendix B: Asymptotic free theories

Let us quickly summarize the applications of the Lambert W function to asymptotic free theories.[2, 3] For asymptotic free theories, the generic 2-loop RG equation is

\[
\frac{d}{dt} x = x^2 + c x^3
\]  

(B1)

For example, in QCD with \( n_f \) flavors, we find

\[
c = \frac{6 (153 - 19n_f)}{33 - 2n_f}
\]  

(B2)

and in the \( O(N) \) non-linear sigma model in two dimensions, we find

\[
c = \frac{1}{N - 2}
\]  

(B3)

The scale parameter in this case is defined by

\[
\Lambda(x) \equiv \left( e^{-\frac{1}{cx} - 1} \left( \frac{1}{cx} + 1 \right) \right)^c
\]  

(B4)

Inverting this, we obtain

\[
x = \frac{1}{-c \left\{ 1 + W \left( -(\Lambda(x)^{\frac{1}{c}}) \right) \right\}}
\]  

(B5)

Hence, the running parameter is given by

\[
\bar{x}(t; x) = \frac{1}{-c \left\{ 1 + W \left( -(e^{t\Lambda(x)})^{\frac{1}{c}} \right) \right\}}
\]  

(B6)

which satisfies

\[
\partial_t \bar{x}(t; x) = \bar{x}(t; x)^2 + c\bar{x}(t; x)^3
\]  

(B7)

and \( \bar{x}(0; x) = x \).

The large \( t \) behavior of \( \bar{x}(t; x) \) depends on the sign of \( c \):

- For \( c > 0 \), \( \bar{x}(t; x) \) diverges as \( t \to t_{\text{max}} \), where \( t_{\text{max}} \) is given by

\[
e^{-t_{\text{max}}} = e^c \Lambda(x)
\]  

(B8)

- For \( c < 0 \), \( \bar{x}(t; x) \) approaches \( \frac{1}{-c} \) as \( t \to +\infty \).

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