Entanglement entropy of a scalar field across a spherical boundary in the Einstein universe

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A scalar field in the ground state, when partially hidden from observation by a spherical boundary, acquires entanglement entropy $S$ proportional to the area of the surface. This area law is well established in flat space, where it follows almost directly from dimensional arguments. We study its validity in an Einstein universe, whose curvature provides an additional physical parameter on which the entropy could, in principle, depend. The surprisingly simple result is that the entanglement entropy still scales linearly with the area. This is supported by other observations to the effect that the entanglement entropy arises mostly from degrees of freedom near the boundary, making it insensitive to the large-scale geometry of the background space.

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I. INTRODUCTION

One of the main open questions in black hole physics concerns the microscopic explanation (if any) of the black hole entropy law:

$$S_{bh} = A_{bh}/(4l_P^2),$$

where $A_{bh}$ is the event horizon area and $l_P = \sqrt{G_N}$ is the Planck length, which equals the square root of the Newtonian constant $G_N$ in $c = \hbar = k_B = 1$ units (see Ref. [1] for a comprehensive review and further references). In contrast to usual thermodynamical systems, the black hole entropy $S_{bh}$ scales with the square rather than with the cube of its linear dimension. Among the various proposals to explain it, the possibility that the black hole entropy has its roots in the entanglement of the background fields \cite{2} is perhaps the most conservative one, since it relies on the same fields which give rise to Hawking radiation. (Nevertheless some points must still be clarified; see Ref. [3] for a recent review.) This line of thought was raised after Bombelli et al. \cite{4} and Srednicki \cite{5} independently concluded that the entanglement entropy $S$ of a vacuum scalar field across a spherical boundary in Minkowski space-time scales with its area $A$ rather than with its volume $V$. Srednicki also showed that the result is independent of any infrared cut off, and obtained an explicit result assuming a massless scalar field:

$$S \approx 0.096 A/(4a^2),$$

where $a$ is a length quantity derived from the theory’s ultraviolet cut off.

Some properties of this law can be derived from physical arguments. For one, the Schmidt decomposition guarantees that the entropy is the same, regardless of whether we trace over the degrees of freedom either inside or outside the sphere, provided that the total state is pure. Thus $S$ can only depend on the area $A$ of the boundary shared by the two regions, but not, for example, on the radius of the sphere (which characterizes only the interior) or the infrared cutoff (exterior). Assuming that the scalar field is massless, the only other quantity on which the entropy may depend is $a$. By demanding that $S$ be dimensionless, we constrain the entropy to be a function of the form $S = S(A/a^2)$. The fact that, among all possibilities, Eq. (2) turns out to be a linear function of $A/a^2$ is quite remarkable. Here we investigate whether the same area law \cite{2} continues to hold if we introduce a curved background, which is characterized by an additional parameter $R_0$, the radius of curvature. Then the most general entropy dependence can be cast as

$$S = S \left[ \frac{A}{4a^2} \frac{\pi R_0}{a^2} \right].$$

The paper is organized as follows. In section II we specify the problem and outline how the entanglement entropy is obtained. Sec. III presents tests of the scheme on regions with arbitrary geometry in flat (Minkowski) space-time. In Sec. IV we turn to the entanglement entropy of a sphere in the Einstein universe, and Sec. V concludes.

II. GENERAL FRAMEWORK

Consider, for the sake of simplicity, a ground-state real scalar field $\phi(\vec{x})$, which lives in a Minkowski space-time. We compute the entanglement entropy $S$ of a region $\Omega$, which arises when one traces out the degrees of freedom inside the region and computes the von Neumann entropy of the reduced state.

In order to perform this computation numerically, we only consider a finite number of degrees of freedom, given...
by the values of the field at the sites of a finite discrete grid. Let \( \alpha \) denote the spacing of the points, which can be interpreted as an ultraviolet cutoff: field modes with wavelengths below this limit are not described by the model. The infrared cutoff, on the other hand, is determined by the total number of points along each axis, denoted \( x_\text{tot}, y_\text{tot} \text{ etc.} \)

Geometrically, the simplest discretization scheme is based on a cubic grid. This puts the Hamiltonian in the form

\[
H_{\text{cubic}} = \frac{\alpha}{2} \sum_{k=1}^{\Omega} \sum_{l=1}^{\Omega} \sum_{m=1}^{\Omega} \sum_{j=1}^{N} \pi_{klm}^2 + \left( \varphi_{k(l+1)m} - \varphi_{klm} \right)^2
\]

for some coupling matrix \( K \). Let \( U \) be the matrix that makes \( K_U \equiv UKU^T \) diagonal, and \( \Omega = UTK_U^{-1/2}U \). Treating the list of values \( \varphi \equiv (\varphi_1, \ldots, \varphi_N) \) as a vector allows us to write the ground-state density operator of such a system simply in terms of matrix products,

\[
\rho_0(\varphi, \varphi') \propto \exp\left(-\frac{1}{2} \varphi^T \Omega \varphi - \frac{1}{2} \varphi'^T \Omega \varphi' \right). \tag{6}
\]

Since the numbering of the sites by the index \( j \) is arbitrary, we can choose an ordering such that the partial trace is taken over the degrees of freedom \( \tilde{\varphi} = (\varphi_{n+1}, \ldots, \varphi_N) \), the reduced density matrix over the remaining \( n \) oscillators, \( \tilde{\varphi} = (\varphi_1, \ldots, \varphi_n) \),

\[
\rho_\text{red}(\tilde{\varphi}; \tilde{\varphi}') \equiv \int \prod_{j=n+1}^{N} d\varphi_j \rho_0(\tilde{\varphi}, \tilde{\varphi}'; \varphi). \tag{7}
\]

Given the form of \( \rho_0(\varphi, \varphi') \) in (6), this is an \( n \)-dimensional Gaussian integral. We decompose

\[
\Omega_{N \times N} = \begin{pmatrix} C_{n \times n} & B_{n \times (N-n)}^T \\ B_{(N-n) \times n} & A_{(N-n) \times (N-n)} \end{pmatrix}, \tag{8}
\]

and define the \( n \times n \) matrices

\[
\beta = \frac{1}{2} B^T A^{-1} B \quad \gamma = C - \beta. \tag{9}
\]

In terms of them, the integration in (7) gives

\[
\rho_\text{red}(\tilde{\varphi}; \tilde{\varphi}') \propto \exp\left(-\frac{1}{2} [\tilde{\varphi}^T \gamma \tilde{\varphi} + \varphi'^T \gamma \varphi'] + \tilde{\varphi}^T \beta \varphi' \right). \tag{10}
\]

It can be further simplified with the following substitutions: let \( V \) be the matrix that makes \( \gamma_D \equiv V \gamma V^T \) diagonal and define

\[
\beta' \equiv \gamma_D^{-1/2} V \beta V^T \gamma_D^{-1/2}. \tag{11}
\]

Then let \( \beta_j' \) be the eigenvalues of \( \beta' \), let \( W \) be the matrix that makes \( \beta_D' \equiv \beta \beta^T \) diagonal and define the new vector of variables

\[
\tilde{\varphi} \equiv W_{\text{cubic}}^{1/2} V \varphi. \tag{12}
\]

In terms of them, \( \rho_\text{red} \) takes the form

\[
\rho_\text{red}(\tilde{\varphi}, \tilde{\varphi}') \propto \prod_{j=1}^{n} \exp\left[-\frac{1}{2} (\tilde{\varphi}_j^2 + \tilde{\varphi}_j'^2) + \tilde{\varphi}_j \beta_j \tilde{\varphi}_j' \right]. \tag{13}
\]

Each term in this product is simply a thermal density matrix for a simple harmonic oscillator \( \varphi_j \), whose frequency \( \omega_j \) and temperature \( T_j \) are related to \( \beta_j' \) by

\[
\xi_j \equiv \exp\left(\frac{\omega_j}{T_j}\right) = \frac{\beta_j'}{1 + \sqrt{1 - (\beta_j')^2}}. \tag{14}
\]

Thus it has entropy

\[
S_j = -\ln(1 - \xi_j) - \frac{\xi_j}{1 - \xi_j} \ln \xi_j. \tag{15}
\]

The total entropy of the reduced state, that is, the entanglement entropy, is simply the sum of these terms.

### III. TESTS IN FLAT SPACE-TIME

The cubic discretization scheme allows us to choose for each site independently whether the value of the field at that point is to be traced out or not, thus giving us the freedom to study the entanglement entropy of regions with arbitrary shape. This reveals an area law, not only for cubes and parallelepipeds, but also hollow shells (in which case both the inner and outer surfaces contribute) and in fact arbitrary configurations. One can, for example, choose a set of unit cubes such that they approximate a sphere, and still (16) holds, with the same coefficient. This extends to more convoluted shapes, even when one portion of the surface comes to within one lattice site of another. The fact that there are no long-range effects – and consequently no influence of the global geometry of \( \Omega \) on \( S \) –, but only the linear scaling with the area, is a useful stepping stone towards our main question: how the geometry of the background space-time might affect \( S \).

Before we proceed to the Einstein universe, it is also useful to introduce and test a second discretization scheme. In a spherically symmetric setting, it is
convenient to decompose $\varphi (\vec{x})$ in partial waves, $\varphi_{lm}(r)$. Their angular dependence is given by the real spherical harmonics $Z_{lm}(\theta, \phi)$, so that they are functions only of the radial coordinate. Given the symmetry of the system, the partial waves are not coupled to each other, and their contributions to entanglement entropy can be computed separately: for each $lm$, the radial coordinate is discretized, and the problem again reduces to an array of coupled harmonic oscillators. By the same procedure outlined above, we find the contributions $S_{lm}$, which are summed to obtain the total entanglement entropy $S$. In practice, the convergence of the sum over $l$ allows us to extrapolate the exact value of $S$ after computing only a finite number of terms.

This computation again leads to an area law:

$$S_{\text{sphere}} = 0.088 \frac{A}{4a^2}. \quad (17)$$

At first glance, this seems incompatible with result (10) by a factor of $\sim 3/2$. However, consider a comparison of the entropy each method yields for a given $\Omega$, namely a sphere of radius $R$: using the spherical discretization scheme, we find simply $0.88 \cdot 4\pi R^2/4$. When this $\Omega$ is decomposed into unit cubes, however, the surface of the actual region is not smooth, and can be shown to have an area of $6\pi R^2$. Thus, the entropy becomes $0.064 \cdot 6\pi R^2/4$, which agrees reasonably well with the previous result.

Besides the geometry of the traced-out region $\Omega$, we also explore how the entropy depends on the total space in which the field lives, $\Omega$ (the discrete array, in our model). We argue in the introduction that $S$ cannot depend on geometric properties of the exterior, such as the infrared cutoff or the shape of $\Omega$. However, if $\Omega$ were to become smaller than $\Omega$, so that all degrees of freedom of the field are traced out, then clearly the entropy must go to zero. Our simulation allows us to explore this transition, revealing that the entropy remains constant until $\Omega$ comes to within $3a$ of $\Omega$, in spherical symmetry, and as little as a single lattice site in the cubic array. Again, there is no evidence of long-range effects in the generation of entanglement entropy.

**IV. ENTROPY IN THE EINSTEIN UNIVERSE**

Now let us replace the Minkowski space-time by an Einstein universe, which has the spatial geometry of a 3-sphere. The corresponding line element can be cast as

$$ds^2 = dt^2 - R_0^2 \left( d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right), \quad (18)$$

with $0 \leq \chi, \theta \leq \pi$, $0 \leq \phi < 2\pi$. Because the Einstein universe is compact, the curvature radius $R_0$ also defines a natural infrared cutoff scale, $L \equiv \pi R_0$. At fixed time $t$, the surface of constant $\chi$ is the boundary of a spherical region $\Omega$, whose radius is $r = R_0 \chi$. Its surface area is $A = 4\pi R_0^2 \sin^2 \chi$, which is upper bounded by $A_{\max} = 4L^2/\pi$ (at $\chi = \pi/2$). When $\chi \ll 1$, the local geometry of Eq. (18) approaches that of Minkowski space-time.

The discretization scheme for spherical symmetry outlined above (see section III) carries over naturally to this case: the radial direction is divided into $N = L/a$ discrete points with spacing $a$, indexed by $j$. For each partial wave $lm$, this leads to a coupling matrix $K_{lm}$, from which one can calculate the contributions $S_{lm}$, and consequently $S$.

For spheres that are small compared to the curvature radius, we recover the flat space area law (17) to good precision (considering the limitations of numerical simulations):

$$S_{\text{curve}} = 0.092 \frac{A}{4a^2}. \quad (19)$$

In fact, it can be shown analytically that, in the low-curvature limit, the coupling matrix reduces to its flat-space counterpart. We also note that $S$ is the same for $r$ and $\pi R_0 - r$ – as expected, since both generate boundaries with the same area.

The question, then, is how $S$ varies as a function of $A$ if the sphere is large enough that effects of curvature become noticeable, i.e., in the limit $r \to R_0/2$, or

$$\frac{A}{4a^2} \to \frac{A_{\max}}{4a^2} = \frac{\pi R_0^2}{a^2}. \quad (20)$$

Fig. 1 gives a clear answer: the area law (19) holds regardless of the large-scale curvature of the background.

![Figure 1: Entanglement entropy $S$ as a function of surface area $A$ in an Einstein universe ($N = 99$). Generally, curvature effects become non-negligible as $A$ approaches its upper bound, but the area law for the entropy is unaffected. (The lower bound $A/(4a^2) \gg 1$ is imposed by the semi-classical approximation; for small areas, the quantum nature of the boundary itself become relevant.)](image-url)

**V. CONCLUSIONS**

The entanglement entropy $S$ of a massless scalar field across a spherical surface in an Einstein universe does
not depend on the radius of the universe, $R_0$. In the flat-space limit - a Minkowski universe, simulated out to a finite radius $R$ - it is not surprising that $S$ does not depend on the infrared cutoff, because $R$ characterizes only the region outside the sphere, whereas the entropy can only depend on shared properties of the interior and the exterior. In curved space, on the other hand, the curvature radius $R_0$ is a meaningful physical parameter of the entire space, and therefore it is non-trivial that it does not affect $S$. We understand this independence as an extension of a trend observed in flat-space studies of the entropy: entanglement entropy is dominated by short-range interactions, and therefore “blind” to large-scale features, including curved background.

The relevant scale is given by the ultraviolet cutoff of the theory, $\alpha$. In our model, it is introduced ad hoc, by discretizing the field, but how it arises from natural laws is a topic of active research. The cutoff is expected to be of the order of the Planck length, both for dimensional reasons and to ensure that the area law for entanglement entropy is compatible with other expressions for the entropy of black holes. If the curvature of space-time becomes noticeable on that scale, which characterizes the regime of quantum gravity, the area law is expected to break down.

Corrections to the area law also arise if one considers flat space, but non-zero spin or excited states \cite{7,8}, suggesting that the entropy is no longer determined solely by short-range interactions. It would be interesting to determine how the entropy responds to background curvature in these cases.

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