Stability Analysis and Classification of Runge-Kutta Methods for Index 1 Stochastic Differential-Algebraic Equations with Scalar Noise

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Abstract

The problem of solving stochastic differential-algebraic equations (SDAEs) of index one with a scalar driving Brownian motion is considered. Recently, the authors proposed a class of stiffly accurate stochastic Runge-Kutta (SRK) methods that do not involve any pseudo-inverses or projectors for the numerical solution of the problem. Based on this class of approximation methods, a classification for the coefficients of stiffly accurate SRK methods attaining strong order 0.5 as well as strong order 1.0 are calculated. Further, the mean-square stability for the considered class of SRK methods is analysed. As the main result, families of A-stable efficient order 0.5 and 1.0 stiffly accurate SRK methods with a minimal number of stages for SDEs as well as for SDAEs are presented.

Key words: Stochastic differential-algebraic equation, Stochastic Runge-Kutta method, Classification, Mean-square stability, A-stability

1 Introduction

In many applications like, e. g., the simulation of the dynamics of multibody systems, optimal control problems or electric circuit simulation (see [1,10,14] for more details), differential-algebraic equations serve as a model for the dynamical system under consideration. However, often random disturbances,
that can be described by some noise source, have to be taken into account. This
leads to models based on stochastic differential-algebraic equations (SDAEs)
and numerical solutions need to be calculated whenever explicit solutions are
not available. In [9], the authors propose a class of stiffly accurate stochastic
Runge-Kutta (SRK) methods that can be applied for the numerical solution
of nonlinear index 1 SDAEs with scalar noise. The introduced class of SRK
methods contains schemes attaining orders of convergence 0.5 and 1.0 in the
mean-square sense. Compared to well known numerical schemes for SDAEs
(see [9] for details), their main advantages are that they do not need the cal-
culation of any pseudo-inverses or projectors and can be applied directly to
the SDAE system.

In the following, we first give a classification of the space of solutions for
order 0.5 and order 1.0 conditions derived in [9] in case of stiffly accurate
methods that are diagonally implicit in the drift part. Based on this classi-
fication, we determine some coefficients for the SRK method such that the
number of stages is minimal in order to reduce computational costs. Apply-
ing the calculated classification yields the main result: We present families
of stiffly accurate SRK methods for which A-stability is proven explicitly and
that have a minimal number of stages and implicit equations to be solved each
step.

The paper is organized as follows: In Section 2, we present the general class
of SRK methods under consideration, that can be applied to index 1 SDAE
systems with scalar noise. Especially, the strong order conditions for the SRK
methods calculated in [9] are given, representing the basis for the classifica-
tion of order 0.5 SRK methods in Section 3 and of order 1.0 SRK methods in
Section 4. The classification is then used in Section 5 in order to determine
some coefficients for schemes with a minimal number of stages and to analyse
their mean-square stability properties. Finally, some families of A-stable SRK
methods are presented and their A-stability is proved explicitly.

2 Stochastic Runge-Kutta Methods for SDAEs

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space equipped with a filtration \((\mathcal{F}_t)_{t \geq 0}\)
fulfilling the usual conditions. Further, let \((W_t)_{t \geq 0}\) be a real valued Wiener
process adapted to \((\mathcal{F}_t)_{t \geq 0}\) and let \(I = [t_0, T]\) for some \(0 \leq t_0 < T\). Then,
we denote by \((X_t)_{t \in I}\) the \(d\)-dimensional solution of the index 1 Itô stochastic
differential-algebraic equation system

\[
M \, dX_t = f(t, X_t) \, dt + g(t, X_t) \, dW_t \tag{2.1}
\]
with consistent initial value $X_{t_0} \in L^2(\Omega)$. Here, $f, g : \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}^d$ are assumed to be globally Lipschitz continuous functions and $M \in \mathbb{R}^{d \times d}$ is a matrix. If $M$ is non-singular, multiplying by $M^{-1}$ transforms (2.1) to a classical system of stochastic differential equations (SDEs). However, if $M$ is singular, we have a system of SDAEs that can be written as a system of SDEs with some algebraic constraints, see e. g. [9]. In this case, we assume that the noise sources do not appear in the algebraic constraints and that the constraints are globally uniquely solvable for algebraic variables. This guarantees that (2.1) is an index 1 SDAE system [9,15]. In the following, we always assume that the unique solution of (2.1) exists, see [15] for details. Because $f$ and $g$ need not to be linear, we are concerned with a general nonlinear system of index 1 SDAEs driven by a scalar Wiener process.

In order to solve (2.1) numerically, we consider the class of stiffly accurate SRK methods for the strong approximation of $(X_t)_{t \in \mathcal{I}}$ introduced in [9]. The advantage of stiffly accurate SRK methods is that they can be directly applied to the index 1 SDAE system (2.1). We consider a discretization $\mathcal{I}_h = \{t_0, t_1, \ldots, t_N\}$ of $\mathcal{I}$ and we denote by $y_n$ the approximation of $(X_t)_{t \in \mathcal{I}}$ at time $t_n$ using step sizes $h_n = t_{n+1} - t_n > 0$. Further, let $I_{(1),n} = W_{t_{n+1}} - W_{t_n}$ denote an increment of the Wiener process and let $I_{(1,1),n} = \frac{1}{2}(I_{(1),n}^2 - h_n)$ denote the corresponding double integral. Then, the approximations calculated by a stiffly accurate $s$-stages SRK method are defined by

$$y_{n+1} = H_s$$

for $i = 1, \ldots, s$ and $n = 0, 1, \ldots, N - 1$, provided that the coefficient matrix $A = (A_{ij})$ is nonsingular or provided that the first stage of the method is explicit with $M \cdot H_1 = M \cdot y_n$ and $(A_{ij})_{i,j=2}^s$ is nonsingular, see also [8,9]. In general, a SRK method for SDEs, see e. g. [11], is called stiffly accurate if its last stage coincides with the approximation rule, i. e., if $y_{n+1} = H_s$.

The SRK method (2.2) with $s$ stages is defined by its coefficients $A = (A_{ij})$, $B^{(k)} = (B_{ij}^{(k)})$ for $k = 1, 2, 3$ and $c = (c_j)$ for $i, j = 1, \ldots, s$ that are usually given by an extended Butcher tableau:

$$c \begin{array}{c|cccc}
 & A & B^{(1)} & B^{(2)} & B^{(3)}
\hline
1 & A_{1j} & B_{1j}^{(1)} & B_{1j}^{(2)} & B_{1j}^{(3)}
\end{array}$$

In order to analyse the order conditions for an $s$-stages stiffly accurate SRK method (2.2), let $\alpha = (\alpha_j) = (A_{sj})^T$, let $\beta^{(k)} = (\beta_{ij}^{(k)}) = (B_{ij}^{(k)})^T$ for $k = 1, 2, 3$ and define $e = (1, \ldots, 1)^T \in \mathbb{R}^s$. Because the stiffly accurate SRK method (2.2) is a special case of the general class of SRK methods introduced in [11], the
colored rooted tree theory in [11,12] can be applied with Proposition 5.2 in [11] to calculate order conditions for the coefficients of the SRK method (2.2). The strong order 1.0 conditions for (2.2) are calculated in [9] and we print them here since we want to give a full classification based on these order conditions in Sections 3 and 4.

**Theorem 2.1** Let \( f, g \in C^{1,3}(\mathcal{I} \times \mathbb{R}^d, \mathbb{R}^d) \). If the coefficients of the stochastic Runge-Kutta method (2.2) fulfill the equations

1. \( \alpha^T e = 1 \)
2. \( \beta^{(1)} e^T e = 1 \)
3. \( \beta^{(2)} e = 0 \)
4. \( \beta^{(3)} e = 0 \)
5. \( \beta^{(1)} B^{(1)} e = \frac{\lambda}{2} \)
6. \( \beta^{(3)} B^{(3)} e = -\frac{\lambda}{2} \)
7. \( \beta^{(2)} B^{(3)} e + \beta^{(3)} B^{(2)} e = 1 - \lambda \)
8. \( \alpha^T B^{(3)} e = 0 \)
9. \( \beta^{(1)} B^{(3)} e + \beta^{(3)} B^{(1)} e = 0 \)
10. \( \beta^{(2)} B^{(2)} e = 0 \)
11. \( \beta^{(1)} B^{(2)} e + \beta^{(2)} B^{(1)} e = 0 \)
12. \( \beta^{(3)} A e = 0 \)
13. \( 2\beta^{(1)} (B^{(1)} e)(B^{(2)} e) + 2\beta^{(1)} (B^{(1)} e)(B^{(3)} e) + \beta^{(2)} (B^{(1)} e)^2 \)
\[ + \beta^{(3)} (B^{(3)} e)^2 + \beta^{(2)} (B^{(2)} e)(B^{(3)} e) + \beta^{(3)} (B^{(1)} e)^2 \]
\[ + \frac{1}{2} \beta^{(3)} (B^{(3)} e)^2 = 0 \]
14. \( \beta^{(1)} (B^{(1)} (B^{(2)} e)) + \beta^{(1)} (B^{(2)} (B^{(1)} e)) + \beta^{(1)} (B^{(3)} (B^{(1)})) \)
\[ + \beta^{(1)} (B^{(3)} (B^{(2)} e)) + \beta^{(2)} (B^{(1)} (B^{(1)})) + \beta^{(2)} (B^{(3)} (B^{(2)} e)) \]
\[ + \frac{1}{2} \beta^{(2)} (B^{(2)} (B^{(3)} e)) + \frac{1}{2} \beta^{(2)} (B^{(3)} (B^{(2)} e)) + \beta^{(3)} (B^{(1)} (B^{(1)})) \]
\[ + \frac{1}{2} \beta^{(3)} (B^{(3)} (B^{(3)} e)) + \beta^{(3)} (B^{(3)} (B^{(3)} e)) = 0 \]

for some \( \lambda \in \mathbb{R} \) and if \( c = Ae \), then the stochastic Runge-Kutta method (2.2) attains order 1.0 for the strong approximation of the solution of the Itô SDAE (2.1) with scalar noise.

**Remark 2.2** Let \( f, g \in C^{1,2}(\mathcal{I} \times \mathbb{R}^d, \mathbb{R}^d) \). Then, conditions 1–4 together with the condition \( \beta^{(1)} B^{(1)} e + \frac{1}{2} \beta^{(2)} B^{(2)} e + \beta^{(3)} B^{(3)} e = 0 \) are sufficient for an order 0.5 strong SRK method (2.2) that can be applied to the Itô SDAE (2.1), see also [9].

Using the order conditions, we will analyse the set of solutions in the following sections 3–4. Because diagonally implicit SRK schemes are much more efficient with respect to their computational effort compared to fully implicit SRK schemes, we claim that \( A_{ij} = B_{ij}^{(3)} = 0 \) for \( j > i \) in the following. Further, we
need that $B_{ij}^{(1)} = B_{ij}^{(2)} = 0$ for $j \geq i$ in order to guarantee the existence of a solution for the implicit equations in (2.2) due to the unbounded random variables $I_{(1),n}$ and $I_{(1,1),n}$, i.e., the SRK method has to be explicit in the terms that involve random variables. Taking these restrictions into account, we give a full classification for the coefficients of the SRK method (2.2). Here, we have to point out that in case of a singular matrix $M$ we choose the coefficients within the classification such that either $A$ is regular or such that $A_{1j} = B_{ij}^{(k)} = 0$ and $A_{ii} \neq 0$ for $i \geq 2$. Thus, the classification contains all coefficients such that the SRK method (2.2) can be applied to SDEs and may be explicit as well as the case that it is implicit and can be applied to SDAEs. Finally, the presented classification is the basis for the calculation of coefficients for efficient SRK methods in the sense that they primary possess a minimal number of stages, secondary have a minimal number of implicit stages and finally for Section 5 need a minimum of explicit function evaluations. Under these restrictions, in section 5 we try to find efficient SRK schemes (2.2) that are $A$-stable in the mean-square sense.

3 Classification of order 0.5 stiffly accurate SRK methods

Firstly, we give a full classification of strong order 0.5 stiffly accurate SRK methods (2.2) with a minimal number of stages that can be diagonally implicit. It easily follows that at least two stages are needed for the order 0.5 conditions mentioned in Remark 2.2 to be fulfilled. Therefore, 2-stages SRK methods with coefficient table

\[
\begin{array}{ccc|ccc}
A_{11} & B_{21}^{(1)} & B_{21}^{(2)} & B_{11}^{(3)} & B_{21}^{(3)} & B_{22}^{(3)} \\
A_{21} & A_{22} & & & & \\
\end{array}
\tag{3.1}
\]

are considered in this section. Because the considered SRK schemes have to be explicit in terms involving random variables, the coefficients $B_{11}^{(1)}$, $B_{21}^{(1)}$, $B_{11}^{(2)}$ and $B_{22}^{(2)}$ are set equal to zero. Applying Remark 2.2 to the case $s = 2$ results in the simplified system of order 0.5 conditions

1. $A_{21} + A_{22} = 1$,
2. $B_{21}^{(1)} = 1$,
3. $B_{21}^{(2)} = 0$,
4. $B_{21}^{(3)} + B_{22}^{(3)} = 0$,
5. $B_{21}^{(3)} B_{11}^{(3)} = 0$.

In the following, we denote by capital letters coefficients that can be freely chosen whereas small letters stand for some prescribed values. Solving these
equations, we obviously get by simple calculations the following two classes of order 0.5 stiffly accurate SRK schemes (3.1):

3.1 Strong order 0.5 SRK class I

Choosing the coefficient $B_{21}^{(3)} = 0$ implies that $B_{22}^{(3)} = 0$ and defines class I with

\[
\begin{pmatrix}
A_{11} & a_{22} \\
A_{21} & 1 & 0
\end{pmatrix}
\begin{pmatrix}
B_{11}^{(3)} \\
0 & 0
\end{pmatrix}
\]  

(3.2)

where $a_{22} = 1 - A_{21}$ and $A_{11}, A_{21}, B_{11}^{(3)} \in \mathbb{R}$.

Remark that in case of $A_{11} = B_{11}^{(3)} = 0$, the SRK scheme (2.2) with coefficients (3.2) coincides with the well known stochastic $\theta$-method in [6].

3.2 Strong order 0.5 SRK class II

If we choose $B_{11}^{(3)} = 0$, then we get the class II coefficients with

\[
\begin{pmatrix}
A_{11} & a_{22} \\
A_{21} & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
B_{21}^{(3)} & b_{22}^{(3)}
\end{pmatrix}
\]  

(3.3)

where $a_{22} = 1 - A_{21}$, $b_{22}^{(3)} = -B_{21}^{(3)}$ and $A_{11}, A_{21}, B_{21}^{(3)} \in \mathbb{R}$.

4 Classification of order 1.0 stiffly accurate SRK methods

Next, we search for stiffly accurate diagonally implicit SRK methods of strong order 1.0 with a minimal number of stages. Again, these methods should be explicit in the terms involving random variables. From the order 1.0 conditions given in Theorem 2.1 it follows that a minimum number of $s = 3$ stages are required. This can be seen easily, because for some smaller $s$ there exist no coefficients that fulfill the order conditions 2, 3, 5 and 7 in Theorem 2.1. Thus, at least $s = 3$ stages are needed to assure strong order 1.0 for the SRK method. These 3-stages stiffly accurate diagonally implicit SRK schemes are
determined by the following coefficient table:

\[
\begin{array}{ccc|ccc|ccc}
A_{11} & A_{21} & A_{31} & B_{21}^{(1)} & B_{31}^{(1)} & B_{33}^{(1)} & B_{11}^{(3)} & B_{21}^{(3)} & B_{22}^{(3)} \\
A_{22} & A_{32} & A_{33} & B_{21}^{(2)} & B_{31}^{(2)} & B_{33}^{(2)} & B_{21}^{(3)} & B_{32}^{(3)} & B_{33}^{(3)} \\
 & & & & & & & & (4.1)
\end{array}
\]

Then, the first four order conditions of Theorem 2.1 reduce to

1. \( A_{31} + A_{32} + A_{33} = 1 \),
2. \( B_{31}^{(1)} + B_{32}^{(1)} = 1 \),
3. \( B_{31}^{(2)} + B_{32}^{(2)} = 0 \),
4. \( B_{31}^{(3)} + B_{32}^{(3)} + B_{33}^{(3)} = 0 \).

Taking into account these simplified conditions, the remaining conditions 5–12 can be written as

5. \( B_{32}^{(1)} B_{21}^{(1)} = \frac{\lambda}{2} \),
6. \( B_{31}^{(3)} B_{11}^{(1)} + B_{32}^{(3)} (B_{21}^{(3)} + B_{22}^{(3)}) = -\frac{\lambda}{2} \),
7. \( B_{31}^{(2)} B_{11}^{(1)} + B_{32}^{(2)} (B_{21}^{(3)} + B_{22}^{(3)}) + B_{32}^{(3)} B_{21}^{(2)} = 1 - \lambda \),
8. \( A_{31} B_{11}^{(3)} + A_{32} (B_{21}^{(3)} + B_{22}^{(3)}) = 0 \),
9. \( B_{31}^{(1)} B_{11}^{(1)} + B_{32}^{(1)} (B_{21}^{(3)} + B_{22}^{(3)}) + B_{32}^{(3)} B_{21}^{(1)} + B_{33}^{(3)} = 0 \),
10. \( B_{32}^{(2)} B_{21}^{(2)} = 0 \),
11. \( B_{32}^{(1)} B_{21}^{(2)} + B_{32}^{(2)} B_{21}^{(1)} = 0 \),
12. \( B_{31}^{(3)} A_{11} + B_{32}^{(3)} (A_{21} + A_{22}) + B_{33}^{(3)} = 0 \).

For conditions 13 and 14 we refer to Theorem 2.1. Then, the following result can be derived in the case of \( s = 3 \) from the simplified order conditions.

**Lemma 4.1** For a stiffly accurate order 1.0 SRK method (2.2) with three stages and coefficient scheme (4.1) the following assertions hold:

(i) For the parameter \( \lambda \) which occurs in the order conditions 5–7 follows that \( \lambda \in \{0, 1\} \).
(ii) It holds \( \lambda = 1 \) if and only if \( B^{(2)} = 0 \).

**Proof.** The results follow straight forward from the solution of the order conditions: Assume that \( \lambda \neq 0 \). Then condition 5 yields, that \( B_{32}^{(1)} \neq 0 \) and \( B_{21}^{(1)} \neq 0 \). From condition 10 we get, that \( B_{32}^{(2)} = 0 \) or \( B_{21}^{(2)} = 0 \) and therefore at least one of the terms on the left hand side of condition 11 is equal to 0.
Then the other term on the left hand side of condition 11 also has to be 0 and thus \( B^{(2)}_{32} = 0 \) and \( B^{(2)}_{21} = 0 \). Condition 3 yields, that \( B^{(2)}_{31} = 0 \). Therefore we have \( B^{(2)}_1 = 0 \). Now, the left hand side of condition 7 vanishes, thus we get \( \lambda = 1 \). This proves (i) and (ii).

For the analysis of the set of coefficients that fulfill the strong order 1.0 conditions, we derive the following possible classes of schemes, where we have \( \lambda = 1 \) for the first five classes and \( \lambda = 0 \) for the remaining six classes. Most of the calculations are done using the software Maple. All presented classes are significantly different although not totally disjoint due to our choice of a clear and compact way for their representation. Special attention has to be paid to the signs of some of the coefficients. Whenever positive as well as negative signs are allowed, one has to choose either the upper or the lower sign of the symbols \( \pm \) and \( \mp \), respectively, for all affected coefficients. In the following, we denote all coefficients that can be chosen freely by capital letters, whereas lower case is used to denote more complex expressions.

4.1 Strong order 1.0 SRK class I with \( \lambda = 1 \)

The first class of coefficients is given for \( A_{11}, A_{22}, A_{33}, B^{(3)}_{22} \in \mathbb{R} \) and \( B^{(3)}_{32} \in \mathbb{R} \setminus \{0\} \) by the tableau

\[
\begin{array}{c|ccc|c}
A_{11} & A_{22} & 1 & 0 & 0 \\
\hline
a_{21} & A_{22} & 1/2 & 1/2 & 0 & 0 \\
a_{31} & 0 & A_{33} & 0 & 0 & b_{31}^{(3)} B_{32}^{(3)} B_{33}^{(3)} \\
\end{array}
\]

with fixed coefficients

\[
a_{21} = \frac{A_{11} - 4A_{22}(B_{32}^{(3)})^2 + 4(B_{32}^{(3)})^2}{4(B_{32}^{(3)})^2}, \quad a_{31} = 1 - A_{33}, \quad b_{21}^{(3)} = -\frac{1 + 2B_{32}^{(3)}B_{33}^{(3)}}{2B_{32}^{(3)}},
\]

\[
b_{31}^{(3)} = -\frac{1}{4B_{32}^{(3)}}, \quad b_{32}^{(3)} = \frac{4(B_{32}^{(3)})^2 - 1}{4B_{32}^{(3)}}, \quad b_{33}^{(3)} = \frac{4(B_{32}^{(3)})^2 - 1}{4B_{32}^{(3)}}.
\]
4.2 Strong order 1.0 SRK class II with $\lambda = 1$

The second class is given for $A_{11}, A_{22}, A_{33}, B_{22} \in \mathbb{R}$ and $B_{32} \in \mathbb{R} \setminus \{0\}$ by the tableau

\[
\begin{array}{c|cc|cc|c}
 & A_{11} & A_{22} & b_{21}^{(1)} & 0 & b_{21}^{(3)} & B_{22}^{(3)} \\
 a_{21} & 0 & A_{33} & b_{31}^{(1)} & b_{32}^{(1)} & 0 & b_{31}^{(3)} & B_{32}^{(3)} & 0 \\
 a_{31} & 0 & A_{33} & 0 & 0 & b_{31}^{(3)} & B_{32}^{(3)} & 0
\end{array}
\]

with fixed coefficients

\[
a_{21} = A_{11} - A_{22}, \quad a_{31} = 1 - A_{33}, \quad b_{21}^{(1)} = \pm \frac{1}{2B_{32}^{(3)}}, \quad b_{31}^{(1)} = 1 \mp B_{32}^{(3)},
\]

\[
b_{32}^{(1)} = \pm B_{32}^{(3)}, \quad b_{21}^{(3)} = -\frac{1 + 2B_{32}^{(3)}B_{32}^{(3)}}{2B_{32}^{(3)}}, \quad b_{31}^{(3)} = -B_{32}^{(3)}.
\]

4.3 Strong order 1.0 SRK class III with $\lambda = 1$

The third class of coefficients is determined for $A_{21}, A_{22}, A_{32} \in \mathbb{R}$ and $B_{11}^{(3)} \in \mathbb{R} \setminus \{0\}$ by the tableau

\[
\begin{array}{c|cc|cc|c}
 & 1 & A_{21} & A_{22} & b_{21}^{(1)} & 0 & b_{21}^{(3)} & B_{11}^{(3)} \\
 a_{31} & A_{32} & A_{33} & b_{31}^{(1)} & b_{32}^{(1)} & 0 & b_{31}^{(3)} & 0 & b_{33}^{(3)} \\
 a_{31} & A_{32} & A_{33} & 0 & 0 & b_{31}^{(3)} & 0 & b_{33}^{(3)} & 0
\end{array}
\]

with fixed coefficients

\[
a_{31} = -\frac{A_{32}(B_{11}^{(3)})^2 - 1}{2(B_{11}^{(3)})^2}, \quad a_{33} = -\frac{A_{32}(B_{11}^{(3)})^2 - 2(B_{11}^{(3)})^2 + A_{32}}{2(B_{11}^{(3)})^2}, \quad b_{21}^{(1)} = \frac{1}{2} \frac{(B_{11}^{(3)})^2 + 1}{1 + 2(B_{11}^{(3)})^2},
\]

\[
b_{31}^{(1)} = -\frac{(B_{11}^{(3)})^2}{(B_{11}^{(3)})^2 + 1}, \quad b_{32}^{(1)} = \frac{1 + 2(B_{11}^{(3)})^2}{(B_{11}^{(3)})^2 + 1}, \quad b_{31}^{(3)} = \frac{1}{2B_{11}^{(3)}},
\]

\[
b_{31}^{(3)} = -\frac{1}{2B_{11}^{(3)}}, \quad b_{32}^{(3)} = \frac{1}{2B_{11}^{(3)}}, \quad b_{33}^{(3)} = \frac{1}{2B_{11}^{(3)}}.
\]
4.4 Strong order 1.0 SRK class IV with $\lambda = 1$

For the fourth class, for $A_{11}, A_{22}, A_{32} \in \mathbb{R}$ and $B^{(3)}_{33} \in \mathbb{R} \setminus \{0\}$ the coefficients are given by the tableau

$$
\begin{array}{c|ccc}
A_{11} & a_{21} & A_{22} & b_{21}^{(1)} \\
\hline
a_{31} & A_{32} & a_{33} & b_{31}^{(1)} & b_{32}^{(1)} \\
\hline
& b_{31}^{(3)} & 0 & 0 & b_{21}^{(3)} \\
\end{array}
$$

with fixed coefficients

$$
a_{21} = \frac{2(B^{(3)}_{33})^2 - 2A_{22}(B^{(3)}_{33})^2 - 2 - A_{22} - A_{32}}{1 + 2(B^{(3)}_{33})^2},
\quad a_{31} = \frac{A_{32}}{B^{(3)}_{33}},
\quad a_{33} = \frac{(B^{(3)}_{33})^2 - (B^{(3)}_{33})^2 A_{32} - A_{32}}{(B^{(3)}_{33})^2},
$$

$$
b_{21}^{(1)} = \pm \frac{1 + (B^{(3)}_{33})^2}{B^{(3)}_{33} \sqrt{1 + 2(B^{(3)}_{33})^2}}
\quad b_{31}^{(1)} = -\frac{1}{2} + \frac{(B^{(3)}_{33})^2 \sqrt{1 + 2(B^{(3)}_{33})^2}}{1 + (B^{(3)}_{33})^2},
\quad b_{31}^{(1)} = -\frac{B^{(3)}_{33}}{B^{(3)}_{33}},
\quad b_{21}^{(3)} = \frac{1}{B^{(3)}_{33}},
\quad b_{31}^{(3)} = \frac{-1}{2} \frac{(B^{(3)}_{33})^2}{1 + (B^{(3)}_{33})^2}.
$$

4.5 Strong order 1.0 SRK class V with $\lambda = 1$

The fifth class of coefficients is defined for $A_{11}, A_{22}, A_{32} \in \mathbb{R}$ by the tableau

$$
\begin{array}{c|ccc}
A_{11} & a_{21} & A_{22} & b_{21}^{(1)} \\
\hline
a_{31} & A_{32} & a_{33} & b_{31}^{(1)} & B_{32}^{(1)} \\
\hline
& b_{31}^{(3)} & 0 & 0 & b_{21}^{(3)} \\
\end{array}
$$

with fixed coefficients

$$
a_{21} = \frac{-B^{(3)}_{33} + A_{32}B^{(3)}_{33} - A_{22}B^{(3)}_{33} + A_{11}B^{(3)}_{32}}{B^{(3)}_{32}},
\quad a_{31} = \frac{-B^{(3)}_{31} + B^{(3)}_{33} - 2B^{(3)}_{32}B^{(3)}_{33} - 2B^{(3)}_{32}B^{(3)}_{33} - B^{(3)}_{32}B^{(3)}_{33} - B^{(3)}_{32}B^{(3)}_{33} - (B^{(3)}_{32})^2 A_{32}}{B^{(3)}_{32} - 2B^{(3)}_{32}B^{(3)}_{33} - (B^{(3)}_{32})^2},
\quad a_{33} = \frac{-B^{(3)}_{32} + B^{(3)}_{33} - 2B^{(3)}_{32}B^{(3)}_{33} - 2B^{(3)}_{32}B^{(3)}_{33} - B^{(3)}_{32}B^{(3)}_{33} - B^{(3)}_{32}B^{(3)}_{33} + (B^{(3)}_{32})^2}{B^{(3)}_{32} - 2B^{(3)}_{32}B^{(3)}_{33} - (B^{(3)}_{32})^2},
\quad b_{21}^{(1)} = \frac{1}{2B^{(3)}_{32}},
\quad b_{31}^{(1)} = 1 - B_{32}^{(1)},
\quad b_{31}^{(3)} = \frac{1}{2} \frac{(B^{(3)}_{32})^2 - 2B^{(3)}_{32}B^{(3)}_{33} - (B^{(3)}_{32})^2}{B^{(3)}_{32}B^{(3)}_{33} - (B^{(3)}_{32})^2},
\quad b_{21}^{(3)} = \frac{1}{2} \frac{(B^{(3)}_{32})^2 - 2B^{(3)}_{32}B^{(3)}_{33} - (B^{(3)}_{32})^2}{B^{(3)}_{32}B^{(3)}_{33} - (B^{(3)}_{32})^2},
\quad b_{31}^{(3)} = -B^{(3)}_{32} - B^{(3)}_{33}.
\$$
and all solutions $B_{32}^{(1)}, B_{32}^{(3)}, B_{33}^{(3)} \in \mathbb{R} \setminus \{0\}$ of the equation

\[
4(B_{32}^{(1)})^2 B_{32}^{(3)} (B_{33}^{(3)})^4 + 4(B_{32}^{(1)})^2 (B_{32}^{(3)})^2 (B_{33}^{(3)})^3 + 4(B_{32}^{(1)})^3 B_{32}^{(3)} (B_{33}^{(3)})^2 \\
+ 4(B_{32}^{(1)})^2 (B_{32}^{(3)})^2 (B_{33}^{(3)})^3 + 4B_{32}^{(1)} (B_{32}^{(3)})^3 (B_{33}^{(3)})^2 - 4(B_{32}^{(1)})^3 (B_{33}^{(3)})^3 \\
+ (B_{32}^{(3)})^3 (B_{33}^{(3)})^2 - 2(B_{32}^{(1)})^3 B_{33}^{(3)} - (B_{32}^{(1)})^2 B_{33}^{(3)} - (B_{32}^{(1)})^2 B_{32}^{(3)} (B_{33}^{(3)})^2 - 2(B_{32}^{(1)})^2 B_{32}^{(3)} B_{33}^{(3)} + (B_{32}^{(1)})^4 B_{33}^{(3)} \\
+ (B_{32}^{(3)})^4 B_{33}^{(3)} + 4(B_{32}^{(1)})^4 (B_{33}^{(3)})^3 = 0
\]

where $B_{32}^{(3)} \neq -B_{32}^{(1)} B_{33}^{(3)}$ is needed.

4.6 Strong order 1.0 SRK class VI with $\lambda = 0$

For $\lambda = 0$, class six is given by the coefficients $A_{11}, A_{22}, A_{32} \in \mathbb{R}$ with the tableau

\[
\begin{array}{ccc|cc|ccc}
A_{11} & A_{22} & b_{21}^{(1)} & b_{21}^{(2)} & B_{11}^{(3)} & b_{21}^{(3)} & 0 \\
\hline
a_{21} & A_{22} & b_{21}^{(1)} & b_{21}^{(2)} & B_{11}^{(3)} & b_{21}^{(3)} & 0 \\
a_{31} & A_{32} & a_{33} & 1 & 0 & 0 & 0 & b_{31}^{(3)} & B_{32}^{(3)} & b_{33}^{(3)}
\end{array}
\]

where

\[
a_{21} = -\frac{1}{2} \frac{1}{((B_{11}^{(3)})^2 + 1)B_{32}^{(3)}} \left( -A_{11} B_{32}^{(3)} (B_{11}^{(3)})^2 - A_{11} B_{32}^{(3)} + 2A_{11} B_{11}^{(3)} \pm A_{11} \sqrt{D} \right) \\
+ 2A_{22} B_{32}^{(3)} (B_{11}^{(3)})^2 + 2A_{22} B_{32}^{(3)} - 2B_{11}^{-3} - (B_{11}^{(3)})^2 (B_{32}^{(3)} - B_{32}^{(3)} \pm \sqrt{D}),
\]

\[
a_{31} = \frac{1}{2} \frac{1}{((B_{11}^{(3)})^2 + 1)B_{32}^{(3)}} \left( A_{32} (-(B_{11}^{(3)})^2 (B_{32}^{(3)} - B_{32}^{(3)} + 2B_{11}^{(3)} \pm \sqrt{D})) \right),
\]

\[
a_{33} = -\frac{1}{2} \frac{1}{((B_{11}^{(3)})^2 + 1)B_{32}^{(3)}} \left( -2(B_{11}^{(3)})^2 B_{32}^{(3)} - 2B_{32}^{(3)} + 2B_{11}^{(3)} A_{32} + A_{32} (B_{11}^{(3)})^2 B_{32}^{(3)} \right) \\
+ A_{32} B_{32}^{(3)} \pm A_{32} \sqrt{D}),
\]

\[
b_{21}^{(1)} = \frac{1}{2} \frac{1}{((B_{11}^{(3)})^2 + 1)B_{32}^{(3)}} \left( (B_{11}^{(3)})^2 B_{32}^{(3)} + B_{32}^{(3)} - 2(B_{11}^{(3)})^3 \pm \sqrt{D} \right),
\]

\[
b_{21}^{(2)} = \frac{1}{2} \frac{1}{((B_{11}^{(3)})^2 + 1)B_{32}^{(3)}} \left( (B_{11}^{(3)})^2 (B_{32}^{(3)} - B_{32}^{(3)} + 2B_{11}^{(3)} \pm \sqrt{D}) \right),
\]

\[
b_{31}^{(3)} = \frac{1}{2} \frac{1}{((B_{11}^{(3)})^2 + 1)B_{32}^{(3)}} \left( -2B_{11}^{(3)} B_{32}^{(3)} - B_{32}^{(3)} + 2B_{11}^{(3)} \pm \sqrt{D}) \right),
\]

\[
b_{33}^{(3)} = -\frac{1}{2} \frac{1}{((B_{11}^{(3)})^2 + 1)B_{32}^{(3)}} \left( 2B_{11}^{(3)} + (B_{11}^{(3)})^2 B_{32}^{(3)} + B_{32}^{(3)} \pm \sqrt{D} \right)
\]

with $B_{11}^{(3)} \in \mathbb{R}, B_{32}^{(3)} \in \mathbb{R} \setminus \{0\}$ and

\[
D = (B_{11}^{(3)})^4 (B_{32}^{(3)})^2 + 2(B_{11}^{(3)})^2 (B_{32}^{(3)})^2 + (B_{32}^{(3)})^2 + 4(B_{11}^{(3)})^3 (B_{32}^{(3)})^2 \\
- 2(B_{11}^{(3)})^2 - 4(B_{11}^{(3)})^4 + 4B_{11}^{(3)} B_{32}^{(3)} - 2
\]
such that $D \geq 0$ is fulfilled.

4.7 Strong order 1.0 SRK class VII with $\lambda = 0$

Class seven is defined for $A_{11}, A_{22}, A_{32}, A_{33}, B_{22}^{(3)} \in \mathbb{R}$ and $B_{21}^{(1)} \in \mathbb{R} \setminus \{0\}$ by the tableau

\[
\begin{array}{c|c|c|c}
A_{11} & & & \\
A_{21} & A_{22} & B_{21}^{(1)} & b_{21}^{(2)} \\
A_{31} & A_{32} & A_{33} & 1 & 0 \\
& & b_{31}^{(3)} & B_{22}^{(3)} \\
& & b_{32}^{(3)} & b_{33}^{(3)} \\
\end{array}
\]

where

\[
a_{21} = A_{11} - A_{11}B_{21}^{(1)} - A_{22} + B_{21}^{(1)}, \quad a_{31} = 1 - A_{32} - A_{33},
\]

\[
b_{21}^{(2)} = \pm \sqrt{2B_{21}^{(1)} - 2(B_{21}^{(1)})^2},
\]

\[
b_{31}^{(3)} = \mp \frac{1 - B_{21}^{(1)}}{\sqrt{2B_{21}^{(1)} - 2(B_{21}^{(1)})^2}},
\]

\[
b_{33}^{(3)} = \pm \frac{B_{21}^{(1)}}{\sqrt{2B_{21}^{(1)} - 2(B_{21}^{(1)})^2}}.
\]

4.8 Strong order 1.0 SRK class VIII with $\lambda = 0$

For $A_{11}, A_{21}, A_{22}, A_{32}, B_{22}^{(3)} \in \mathbb{R}$ and $B_{32}^{(2)}, B_{11}^{(3)} \in \mathbb{R} \setminus \{0\}$, the eighth class is given by the tableau

\[
\begin{array}{c|c|c|c|c}
A_{11} & & & & B_{11}^{(3)} \\
A_{21} & A_{22} & 0 & 0 & b_{21}^{(3)} & B_{22}^{(3)} \\
A_{31} & A_{32} & a_{33} & b_{31}^{(1)} & b_{32}^{(1)} & b_{31}^{(2)} & B_{32}^{(3)} \\
& & & 0 & 0 & 0 & 0 \\
\end{array}
\]

with

\[
a_{31} = -\frac{A_{32}(1 + B_{32}^{(2)}B_{11}^{(3)})}{B_{32}^{(2)}B_{11}^{(3)}}, \quad a_{33} = \frac{B_{32}^{(2)}B_{11}^{(3)} + A_{32}}{B_{32}^{(2)}B_{11}^{(3)}}, \quad b_{31}^{(1)} = 1 + B_{32}^{(2)}B_{11}^{(3)},
\]

\[
b_{32}^{(1)} = -B_{32}^{(2)}B_{11}^{(3)}, \quad b_{31}^{(2)} = -B_{32}^{(2)}, \quad b_{31}^{(3)} = \frac{1 + B_{32}^{(2)}(B_{11}^{(3)} - B_{22}^{(3)})}{B_{32}^{(2)}}.
\]
4.9 Strong order 1.0 SRK class IX with \( \lambda = 0 \)

Class nine with \( \lambda = 0 \) is given for \( A_{11}, A_{22}, A_{32}, B_{32}^{(3)} \in \mathbb{R} \) and \( B_{11}^{(3)} \in \mathbb{R} \setminus \{0\} \) by the tableau

\[
\begin{array}{c|ccc|c}
A_{11} & 0 & 0 & B_{11}^{(3)} \\
A_{21} & A_{22} & b_{31}^{(1)} & b_{32}^{(1)} \\
A_{31} & A_{32} & a_{33} & b_{31}^{(2)} & b_{32}^{(2)} \\
\hline
B_{21} & 0 & b_{31}^{(3)} & b_{32}^{(3)} & b_{33}^{(3)}
\end{array}
\]

with the coefficients

\[
a_{21} = \frac{(B_{11}^{(3)})^2 - A_{22}(B_{11}^{(3)})^2 - A_{11} + 1}{(B_{11}^{(3)})^2},
\]
\[
a_{33} = \frac{(B_{11}^{(3)})^2 - (B_{11}^{(3)})^2 A_{32} - A_{32}}{(B_{11}^{(3)})^2},
\]
\[
b_{32}^{(1)} = \frac{(B_{11}^{(3)})^3 - (B_{11}^{(3)})^2 B_{32}^{(3)} - B_{32}^{(3)}}{B_{11}^{(3)}((B_{11}^{(3)})^2 + 1)},
\]
\[
b_{32}^{(2)} = -\frac{B_{11}^{(3)}}{(B_{11}^{(3)})^2 + 1},
\]
\[
b_{32}^{(3)} = \frac{B_{32}^{(3)}}{(B_{11}^{(3)})^2}.
\]

4.10 Strong order 1.0 SRK class X with \( \lambda = 0 \)

Class ten is defined for \( A_{11}, A_{21}, A_{22}, A_{33}, B_{22}^{(3)} \in \mathbb{R} \) and \( B_{32}^{(2)} \in \mathbb{R} \setminus \{0\} \) by the tableau

\[
\begin{array}{c|ccc|c}
A_{11} & 0 & 0 & 0 \\
A_{21} & A_{22} & 0 & 0 \\
A_{31} & 0 & A_{33} & 0 & b_{21}^{(3)} & B_{22}^{(3)} \\
\hline
\end{array}
\]

with coefficients

\[
a_{31} = 1 - A_{33}, \quad b_{31}^{(2)} = -B_{32}^{(2)}, \quad b_{21}^{(3)} = \frac{1 - B_{32}^{(2)} B_{22}^{(3)}}{B_{32}^{(2)}}.
\]
4.11 Strong order 1.0 SRK class XI with $\lambda = 0$

The last class eleven is given for $A_{21}, A_{22}, A_{33} \in \mathbb{R}$ and $B_{33}^{(3)} \in \mathbb{R} \setminus \{0\}$ by the tableau

| $a_{11}$ | $A_{21}$ | $A_{22}$ | $0$ | $b_{21}^{(2)}$ | $b_{21}^{(3)}$ | $0$ | $b_{11}^{(3)}$ |
|----------|----------|----------|-----|----------------|----------------|-----|----------------|
| $a_{31}$ | $a_{32}$ | $A_{33}$ | $1$ | $0$            | $0$            | $0$ | $b_{31}^{(3)}$ |
| $a_{32}$ | $0$      |          |     |                |                |     | $b_{32}^{(3)}$ |
| $A_{33}$ | $B_{33}^{(3)}$ |           |     |                |                |     | $B_{33}^{(3)}$ |

and the coefficients

$$a_{11} = \frac{2A_{21}(B_{33}^{(3)})^4 + A_{21} + 2A_{22}(B_{33}^{(3)})^4 + A_{22} - 2(B_{33}^{(3)})^4 - 2(B_{33}^{(3)})^2}{1 - 2(B_{33}^{(3)})^2},$$

$$a_{31} = 1 - a_{32} - A_{33}, \quad a_{32} = \frac{2A_{33}(B_{33}^{(3)})^4 - 2(B_{33}^{(3)})^4 - 1 + A_{33}}{2(B_{33}^{(3)})^2((B_{33}^{(3)})^2 + 1)},$$

$$b_{21}^{(2)} = \frac{\sqrt{-2(B_{33}^{(3)})^4b_{32}^{(3)} - 2b_{32}^{(3)}B_{33}^{(3)} - 2(B_{33}^{(3)})^4}}{b_{32}^{(3)}},$$

$$b_{11}^{(3)} = -B_{33}^{(3)}, \quad b_{21}^{(3)} = \frac{b_{32}^{(3)}B_{33}^{(3)} + (B_{33}^{(3)})^2}{b_{32}^{(3)}},$$

$$b_{31}^{(3)} = -b_{32}^{(3)} - B_{33}^{(3)}, \quad b_{32}^{(3)} = \frac{2(B_{33}^{(3)})^4 + 1}{2B_{33}^{(3)}((B_{33}^{(3)})^2 + 1)}.$$

5 Efficient drift-implicit SRK schemes and stability analysis

The aim of this section is to determine efficient drift-implicit SRK schemes that are included in the previously presented classification with respect to a minimal number of implicit stages and explicit function evaluations needed each step as well as good stability properties. First, we briefly summarize the concept of mean–square stability for SDEs. Therefore, we consider the scalar linear test equation with multiplicative noise

$$dX_t = \lambda X_t \, dt + \mu X_t \, dW_t,$$

for $t \geq t_0$ with initial value $X_{t_0} = x_0 \in \mathbb{R} \setminus \{0\}$ and with some constants $\lambda, \mu \in \mathbb{C}$. In order to analyse the mean–square stability (MS–stability), we have to consider the second moment of the solution process of SDE (5.1) and of the corresponding numerical approximation process, respectively. The
solution of SDE (5.1) is said to be (asymptotically) MS–stable if

\[
\lim_{t \to \infty} E(|X_t|^2) = 0 \iff 2 \Re(\lambda) + |\mu|^2 < 0 \tag{5.2}
\]

holds for the coefficients \( \lambda, \mu \in \mathbb{C} \), see e. g. [2,3,4,6,7,13] for further details. We call \( \mathcal{D}_{\text{SDE}} = \{ (\lambda, \mu) \in \mathbb{C}^2 : 2 \Re(\lambda) + |\mu|^2 < 0 \} \subset \mathbb{C}^2 \) the domain of MS–stability of SDE (5.1). Here, we point out that for \( \mu = 0 \) the stability condition (5.2) reduces to the well known deterministic stability condition \( \Re(\lambda) < 0 \).

In order to analyse the stability of the SRK method (2.2), we apply the method to the test problem (5.1). We are looking for conditions such that the SRK method yields numerically stable solutions whenever (5.2) is fulfilled. A numerical method is said to be numerically MS–stable if the approximations \( y_n \) satisfy \( \lim_{n \to \infty} E(|y_n|^2) = 0 \). Applying the numerical method to (5.1), we obtain the recursion

\[
y_{n+1} = R_n(\hat{h},k) y_n, \tag{5.3}
\]

with a stability function \( R_n(\hat{h},k) \) using the parametrization \( \hat{h} = \lambda h \) and \( k = \mu \sqrt{h} \) for \( h > 0 \) [4,6]. Then, calculating the mean–square norm of (5.3), we obviously yield MS–stability, if

\[
\hat{R}(\hat{h},k) := E(|R_n(\hat{h},k)|^2) < 1. \tag{5.4}
\]

Now, we call \( \mathcal{D}_{\text{SRK}} = \{ (\hat{h},k) \in \mathbb{C}^2 : \hat{R}(\hat{h},k) < 1 \} \subset \mathbb{C}^2 \) the domain of MS–stability of the SRK method. The numerical method is said to be A–stable if \( \mathcal{D}_{\text{SDE}} \subseteq \mathcal{D}_{\text{SRK}} \). Because the domain of stability for \( \lambda, \mu \in \mathbb{R} \) is not easy to visualize, we have to restrict the figures to presenting the region of stability for \( \lambda, \mu \in \mathbb{R} \) in the \( \hat{h}–k^2 \)–plane. Then, for fixed values of \( \lambda \) and \( \mu \), the set \( \{ (\lambda h, \mu^2 h) \subset \mathbb{R}^2 : h > 0 \} \) is a straight ray starting at the origin and going through the point \( (\lambda, \mu^2) \). Varying the step size \( h \) corresponds to moving along this ray. For \( \lambda, \mu \in \mathbb{R} \), the region of MS–stability for SDE (5.1) reduces to the area of the \( \hat{h}–k^2 \)–plane with the \( \hat{h} \)–axis as the lower bound and \( k^2 < -2\hat{h} \) giving the upper bound for \( \hat{h} < 0 \).

Next, we calculate the stability function \( R_n(\hat{h},k) \) for the s-stages SRK method (2.2). Let \( H = (H_1, \ldots, H_s)^T \). Then (2.2) applied to (5.1) with equidistant step size \( h = h_n \) becomes

\[
H = e y_n + \lambda h A H + \mu \left( I_{(1),n} B^{(1)} + \frac{I_{(1),n}}{\sqrt{h}} B^{(2)} + \sqrt{h} B^{(3)} \right) H.
\]

Together with \( I_{(1),n} = \sqrt{h_n} \xi_n \) where \( \xi_n \sim N(0,1) \) and the parametrization \( \hat{h} = \lambda h \) and \( k = \mu \sqrt{h} \) this can be reformulated to

\[
H = \left( I_s - \hat{h} A - k \left( \xi_n B^{(1)} + \frac{1}{2} (\xi_n^2 - 1) B^{(2)} + B^{(3)} \right) \right)^{-1} e y_n.
\]
Fig. 1. Mean-square stability region for class I/II with \( a_2 = 0 \) and with \( a_1 = 1, \ a_1 = \frac{1}{16}, \ a_1 = \frac{1}{32} \) and \( a_1 = \frac{1}{64} \), respectively.

Since the methods are stiffly accurate, that is \( y_{n+1} = H_s \), the stability function is given as

\[
R_n(\hat{h}, k) = \varepsilon_s^T \left( I_s - \hat{h}A - k \left( \xi_n B^{(1)} + \frac{1}{2} (\xi_n^2 - 1) B^{(2)} + B^{(3)} \right) \right)^{-1} e
\]

where \( \varepsilon_s^T = (0, \ldots, 0, 1) \in \mathbb{R}^s \).

5.1 A-stable strong order 0.5 SRK schemes

In the following, the computational costs are measured as the number of function evaluations that are necessary in each step and we try to minimize them. Therefore, the following coefficients for drift-implicit order 0.5 SRK schemes are considered for both classes I and II:

\[
\begin{array}{c|c|c|c}
   & a_1 & 1 - a_2 & 1 \\
\hline
   a_2 & 0 & 0 & 0 \\
\end{array}
\]

where we choose \( B_{11}^{(3)} = B_{21}^{(3)} = 0 \) and \( a_1, a_2 \in \mathbb{R} \). First, we consider the case of diagonally drift-implicit SRK methods where we choose \( a_2 = 0 \).
Lemma 5.1 The order 0.5 SRK scheme with coefficients (5.6) and $a_2 = 0$ is $A$-stable for equation (5.1), i.e. $\mathcal{D}_{SDE} \subseteq \mathcal{D}_{SRK}$, if and only if $a_1 \geq 0$.

Proof. Calculating $\hat{R}(\hat{h},k)$ from the stability function (5.5) using the coefficients (5.6) yields

$$\hat{R}(\hat{h},k) = \frac{|1 - a_1 \hat{h}|^2 + |k|^2}{|1 - \hat{h}|^2|1 - a_1 \hat{h}|^2}. \quad (5.7)$$

Now, we obtain that $\mathcal{D}_{SDE} \subseteq \mathcal{D}_{SRK}$ if $\hat{R}(\hat{h},k) < 1$ for all $\hat{h}, k \in \mathbb{C}^2$ with $2\Re(\hat{h}) + |k|^2 < 0$. Assuming that $|k|^2 < -2\Re(\hat{h})$, we have to prove that $\hat{R}(\hat{h},k) - 1 < 0$ holds. Using this assumption, we get

$$\hat{R}(\hat{h},k) - 1 < \frac{\phi(\hat{h}, a_1)}{|1 - \hat{h}|^2|1 - a_1 \hat{h}|^2} \quad (5.8)$$

with

$$\phi(\hat{h}, a_1) := (-4a_1 - 1)\Re(\hat{h})^2 + (2a_1^2 + 2a_1)\Re(\hat{h})^3 - a_1^2\Re(\hat{h})^4 - \Im(\hat{h})^2$$

$$+ (2a_1^2 + 2a_1)\Re(\hat{h})\Im(\hat{h})^2 - 2a_1^2\Re(\hat{h})^2\Im(\hat{h})^2 - a_1^2\Im(\hat{h})^4. \quad (5.9)$$

Now, for $\Re(\hat{h}) < 0$ and $a_1 \geq 0$ the expression (5.9) is obviously not positive, i.e., the order 0.5 scheme (5.6) is $A$-stable.

For $a_1 < 0$, we restrict our analysis to the case of $\Im(\hat{h}) = \Im(\hat{k}) = 0$ in the following. Since $\hat{R}(\hat{h},k)$ has a singularity at $\hat{h} = \frac{1}{a_1}$, we restrict our considerations to the case where $\hat{h} < \frac{1}{a_1}$. Then, considering the boundary $|k| = \sqrt{-2\hat{h}}$ of the domain of stability of the test equation (5.1), we get

$$\hat{R}(\hat{h}, \sqrt{-2\hat{h}}) - 1 = \frac{\hat{h}^2 \left(\frac{\hat{h}^2 - 2a_1}{a_1} + \frac{a_1 + 1}{a_1^2}\right)}{|1 - \hat{h}|^2|1 - a_1 \hat{h}|^2}. \quad (5.10)$$

By calculating the roots of (5.10) we get that $\hat{R}(\hat{h}, \sqrt{-2\hat{h}}) - 1 > 0$ for

$$\hat{h} \in I(a_1) := \left[ \frac{1 + a_1 + \sqrt{a_1^2 - 2a_1}}{a_1}, \frac{1}{a_1} \right]. \quad (5.11)$$

Due to the continuity of $\hat{R}(\hat{h}, k)$ on $]-\infty, \frac{1}{a_1}[ \times \mathbb{R}$ there exists some $\varepsilon > 0$ such that $\hat{R}(\hat{h}, k) - 1 > 0$ on some open ball $B_\varepsilon(\hat{h}, \sqrt{-2\hat{h}})$ with radius $\varepsilon$ and center $(\hat{h}, \sqrt{-2\hat{h}})$ with $\hat{h} \in I(a_1)$. Since $B_\varepsilon(\hat{h}, \sqrt{-2\hat{h}}) \cap \mathcal{D}_{SDE} \neq \emptyset$ it follows that the scheme can not be $A$-stable. \hfill \Box

Considering the regions of MS-stability for the SRK schemes with $a_2 = 0$ and different values $a_1 \in \{ \frac{1}{10}, \frac{1}{32}, \frac{1}{64} \}$, we can see in Figure 1 that $\mathcal{D}_{SDE} \subseteq \mathcal{D}_{SRK}$ is always fulfilled.
Fig. 2. Mean-square stability region for class I/II with $a_1 = 0$ and $a_2 = 0$, $a_2 = \frac{15}{32}$, $a_2 = \frac{31}{64}$ and with $a_2 = \frac{1}{2}$, respectively.

**Remark 5.2** If we choose $a_1 = 1$ and $a_2 = 0$ in (5.6), then the resulting order 0.5 scheme is $A$-stable and a singly diagonally drift-implicit stiffly accurate SRK scheme. Especially, the calculation of only one $LU$ decomposition is needed each step if a simplified Newton method is applied to solve the implicit equations.

As another class of schemes, we consider the case of an explicit first stage, i.e. where $a_1 = 0$. However, then we need $a_2 \neq 1$ if the SRK method is applied to an SDAE, see [9].

**Lemma 5.3** The order 0.5 SRK scheme with coefficients (5.6) is $A$-stable for equation (5.1), i.e. $\mathcal{D}_{SDE} \subseteq \mathcal{D}_{SRK}$, if $a_1 \geq 0$ and $a_2 \leq \min\{\frac{1+a_1}{2(1+a_1)}; \frac{1}{2(1-a_1)}\}; 1$.

**Proof.** Calculating $\hat{R}(\hat{h}, k)$ from the stability function (5.5) using the coefficients (5.6) yields

\[
\hat{R}(\hat{h}, k) = \frac{|1 + (a_2 - a_1)\hat{h}|^2 + |k|^2}{|1 - (1 - a_2)\hat{h}|^2|1 - a_1\hat{h}|^2}. \tag{5.12}
\]

Now, we obtain that $\mathcal{D}_{SDE} \subseteq \mathcal{D}_{SRK}$ if $\hat{R}(\hat{h}, k) < 1$ for all $\hat{h}, k \in \mathbb{C}^2$ with $2\Re(\hat{h}) + |k|^2 < 0$. Assuming that $|k|^2 < -2\Re(\hat{h})$, we have to prove that...
\( \hat{R}(\hat{h}, k) - 1 < 0 \) holds. Using this assumption, we get

\[
\hat{R}(\hat{h}, k) - 1 < \frac{\phi(\hat{h}, a_1, a_2)}{|1 - (1 - a_2)\hat{h}|^2|1 - a_1\hat{h}|^2}
\]

with

\[
\phi(\hat{h}, a_1, a_2) := (-4a_1 - 1 + 2a_2 + 2a_1a_2)\Re(\hat{h})^2 + (2a_2 - 2a_1a_2 - 1)\Im(\hat{h})^2 \\
+ 2(a_1^2(1 - a_2) + a_1(1 - a_2)^2)|\hat{h}|^2\Re(\hat{h}) - a_1^2(1 - a_2)^2|\hat{h}|^4.
\]

Thus, for \( \Re(\hat{h}) < 0 \) the expression (5.14) is obviously not positive if \( a_1 \geq 0 \) and if

\[
a_2 \leq \min \left\{ \frac{1 + 4a_1}{2(1 + a_1)}, \frac{1}{2(1 - a_1)}, 1 \right\}.
\]

Then, the order 0.5 scheme (5.6) is \( A \)-stable.

In case of \( a_1 = 0 \), the regions of MS-stability for the SRK method with \( a_2 \in \{0, \frac{15}{32}, \frac{21}{64}, \frac{1}{2}\} \) are presented in Figure 2 where \( D_{SDE} \subseteq D_{SRK} \) is fulfilled. For \( a_1 = 0 \) and \( a_2 = \frac{1}{2} \), the region of MS-stability for the SRK scheme coincides perfectly with the region of MS-stability for the test SDE.

**Remark 5.4** In the case of \( a_1 = 0 \) and \( a_2 \neq 1 \) in (5.6), the order 0.5 stiffly accurate drift-implicit SRK scheme coincides with the well known \( \theta \)-method [6] and needs only one stage-evaluation of the drift function \( f \) and one of the diffusion function \( g \) each step due to the FSAL (first same as last) property [5]. Further, only one implicit equation has to be solved each step.

### 5.2 \( A \)-stable strong order 1.0 SRK schemes

Next, we want to find some \( A \)-stable order 1.0 SRK schemes. As mentioned in Section 4 the smallest number of stages for order 1.0 schemes is \( s = 3 \). Within this case of 3-stages schemes, it turns out that the Classes II and X are the ones with the lowest number of function evaluations, i.e. with minimal computational costs. This is due to the fact that these are the classes including schemes that are explicit in the diffusion.

Thus, choosing the coefficients for Class II such that the computational effort is minimized, i.e. with \( A_{11} = a_1, A_{22} = a_2, A_{33} = a_3, B^{(3)}_{22} = 0 \) and
\( B_{32}^{(3)} = \pm \frac{1}{2b} \), we get the tableau

\[
\begin{array}{ccc|cc|cc}
& a_1 & & b & 0 & 0 \\
\hline
a_1 - a_2 & a_2 & 1 - \frac{1}{2b} & 0 & 0 & \mp b & 0 \\
1 - a_3 & 0 & a_3 & 0 & 0 & \mp \frac{1}{2b} & \pm \frac{1}{2b} & 0 \\
\end{array}
\tag{5.15}
\]

with \( a_1, a_2, a_3 \in \mathbb{R} \) and \( b \in \mathbb{R} \setminus \{0\} \).

Further, choosing the coefficients for Class X such that the computational effort is minimized, i.e. with \( A_{11} = a_1, A_{22} = a_2, A_{33} = a_3, A_{21} = a_4, B_{22}^{(3)} = 0 \) and \( B_{32}^{(2)} = \frac{1}{b} \) results in the tableau

\[
\begin{array}{ccc|cc|cc}
& a_1 & & 0 & 0 \\
\hline
a_4 & a_2 & 0 & 0 & b & 0 \\
1 - a_3 & 0 & a_3 & -\frac{1}{b} & \frac{1}{b} & 0 & 0 & 0 \\
\end{array}
\tag{5.16}
\]

with \( a_1, a_2, a_3, a_4 \in \mathbb{R} \) and \( b \in \mathbb{R} \setminus \{0\} \).

Now, in the case of \( s = 3 \) with \( A_{ij} = B_{ij}^{(3)} = 0 \) for \( j > i \) and \( B_{ij}^{(1)} = B_{ij}^{(2)} = 0 \) for \( j \geq i \), by rearranging the terms with respect to powers of \( \xi_n \) the stability function (5.5) has a representation of type

\[
R_n(\hat{h}, k) = \Gamma + \Sigma_1 \xi_n + \Sigma_2 \xi_n^2 + \Sigma_3 \xi_n^3 + \Sigma_4 \xi_n^4
\]

with some suitable coefficients \( \Gamma, \Sigma_1, \ldots, \Sigma_4 \) independent of \( \xi_n \), see also [9].

Therefore, we calculate the mean-square stability function \( \hat{R}(\hat{h}, k) \) for the diagonally implicit SRK method (2.2) as

\[
\hat{R}(\hat{h}, k) = |\Gamma|^2 + \Gamma \Sigma_2 + \Gamma \Sigma_2 + 3 \Gamma \Sigma_4 + 3 \Gamma \Sigma_1^2 + 3 \Sigma_1 \Sigma_3 + 3 \Sigma_1 \Sigma_4 + 3 |\Sigma_2|^2 + 15 |\Sigma_2 \Sigma_4 + 15 |\Sigma_2 \Sigma_4 + 15 |\Sigma_3|^2 + 105 |\Sigma_4|^2
\tag{5.17}
\]

Especially, for class II with the coefficients (5.15), we get

\[
\Gamma = \frac{1 - \frac{1}{2} k^2 - (a_1 + a_2 + a_3 - 1) \hat{h} + a_2 (a_1 + a_3 - 1) \hat{h}^2}{(1 - a_1 \hat{h}) (1 - a_2 \hat{h}) (1 - a_3 \hat{h})}, \tag{5.18}
\]

\[
\Sigma_1 = \frac{k}{(1 - a_1 \hat{h}) (1 - a_3 \hat{h})}, \quad \Sigma_2 = \frac{k^2}{2 (1 - a_1 \hat{h}) (1 - a_2 \hat{h}) (1 - a_3 \hat{h})}, \tag{5.19}
\]

and \( \Sigma_3 = \Sigma_4 = 0 \). Here, we would like to point out, that the stability function does not depend on the parameter \( b \). Further, for class X with the coefficients
First, we consider the case of diagonally drift-implicit stiffly accurate SRK methods. Therefore, we analyse class II with \( a_1 = a_2 = a \) for some \( a \in \mathbb{R} \) and \( a_3 = 1 \). Here, we have to point out, that we need \( a \neq 0 \) if the SRK method is applied to SDAEs, see [9]. Then, we need three stage-evaluations of the drift function \( f \) and two stage-evaluations of the diffusion function \( g \) for the diagonally implicit SRK method (2.2) each step.

**Lemma 5.5** The family of order 1.0 SRK schemes with coefficients (5.15) in case of \( a_1 = a_2 = a \) and \( a_3 = 1 \) is \( A \)-stable for equation (5.1), i.e. it holds \( \mathcal{D}_{SDE} \subseteq \mathcal{D}_{SRK} \), if and only if \( a \geq \frac{1}{4} \) and \( b \in \mathbb{R} \setminus \{0\} \).
Proof. Inserting (5.18) and (5.19) into (5.17) we have to prove that
\[
\hat{R}(\hat{h}, k) = \frac{|a\hat{h} - 1|^4 + \frac{1}{2}|k|^4 + |k|^2 |a\hat{h} - 1|^2}{|ah - 1|^4 |\hat{h} - 1|^2} < 1,
\] (5.22)
because \( \mathcal{D}_{SDE} \subseteq \mathcal{D}_{SRK} \) if \( \hat{R}(\hat{h}, k) < 1 \) for all \( \hat{h}, k \in \mathbb{C}^2 \) with \( 2\Re(\hat{h}) + |k|^2 < 0 \). Assuming that \( \Re(\hat{h}) < 0 \) and \( |k|^2 < -2\Re(\hat{h}) \), we prove that \( \hat{R}(\hat{h}, k) - 1 < 0 \) holds. Using this assumption, we get
\[
\hat{R}(\hat{h}, k) - 1 < \frac{\phi(\hat{h}, a)}{|a\hat{h} - 1|^4 \cdot |\hat{h} - 1|^2}
\] (5.23)
with
\[
\phi(\hat{h}, a) := |a\hat{h} - 1|^4 + 2\Re(\hat{h})^2 - 2\Re(\hat{h}) \cdot |a\hat{h} - 1|^2 - |a\hat{h} - 1|^4 \cdot |\hat{h} - 1|^2.
\] (5.24)
Since the denominator in (5.23) is positive, it is sufficient to prove that \( \phi(\hat{h}, a) \leq 0 \). Considering (5.24) and collecting for the real part of \( \hat{h} \) results in
\[
\phi(\hat{h}, a) = -a^4\Re(\hat{h})^6 + (4a^3 + 2a^4)\Re(\hat{h})^5
+ (-8a^3 - 3a^4\Im(\hat{h})^2 - 6a^2)\Re(\hat{h})^4
+ (4a^4\Im(\hat{h})^2 + 10a^2 + 4a + 8a^3\Im(\hat{h})^2)\Re(\hat{h})^3
+ (1 - 3a^4\Im(\hat{h})^4 - 8a^3\Im(\hat{h})^2 - 8a^2\Im(\hat{h})^2 - 4a)\Re(\hat{h})^2
+ (4a\Im(\hat{h})^2 + 4a^2\Im(\hat{h})^4 + 2a^4\Im(\hat{h})^4 + 2a^2\Im(\hat{h})^2)\Re(\hat{h})
- \Im(\hat{h})^2 - a^4\Im(\hat{h})^6 - 2a^2\Im(\hat{h})^4.
\] (5.25)

Due to our assumption \( \Re(\hat{h}) < 0 \), it is easy to see that \( \phi(\hat{h}, a) \leq 0 \) if \( a \geq \frac{1}{4} \). Thus, we get \( A \)-stability for \( a \geq \frac{1}{4} \).

As the final step, we prove that this bound for \( a \) is also sharp. Let us choose \( \Im(\hat{h}) = \Im(k) = 0 \). For the proof, we consider the boundary of the set \( \mathcal{D}_{SDE} \) in the real case, which reduces to the half-line \( \partial \mathcal{D}_{SDE} := \{(\hat{h}, \sqrt{-2\hat{h}}) : \hat{h} \in ] - \infty, 0[\}. \) Let \( \psi(\hat{h}, a) := \hat{R}(\hat{h}, \sqrt{-2\hat{h}}) - 1 \). Then, we get from (5.22) that
\[
\lim_{\hat{h} \to 0} \psi(\hat{h}, a) = 0.
\] (5.26)

Now, the idea is to show that \( \psi(\hat{h}, a) \) is strictly decreasing on the open set \( S_\varepsilon := \{(\hat{h}, \sqrt{-2\hat{h}}) : \hat{h} \in ] - \varepsilon, 0[\} \) for some \( \varepsilon = \varepsilon(a) > 0 \), i.e. \( \frac{\partial \psi(\hat{h}, a)}{\partial \hat{h}} < 0 \). Since \( \hat{R}(\hat{h}, k) \) is continuous on \( ] - \infty, 0[ \times \mathbb{R} \), for each point \( P \in S_\varepsilon \) there exists an open ball \( B_\delta(P) \) for some \( \delta > 0 \), such that \( \hat{R}(\hat{h}, k) > 1 \) on \( B_\delta(P) \cap \mathcal{D}_{SDE} \), i.e.
the scheme is not $A$-stable. Thus, we consider
\[
\frac{\partial \psi(\hat{h},a)}{\partial \hat{h}} = \frac{-2\hat{h}}{(\hat{h} - 1)^5(\hat{h} - 1)^3} 
\times (\hat{h}^4 a^5 - 5\hat{h}^3 a^4 + (11\hat{h}^2 - 3\hat{h}^3)a^3 + (7\hat{h}^2 - 11\hat{h})a^2 + (4 + 4\hat{h}^2 - 7\hat{h})a - 1).
\]
(5.27)

We distinguish the cases $a \in [0, \frac{1}{4}]$ and $a < 0$. Let $-1 < \hat{h} < 0$. First, we consider $a \in [0, \frac{1}{4}]$. Then, using the estimates $\hat{h}^4 < -\hat{h}$, $-\hat{h}^3 < -\hat{h}$ and $\hat{h}^2 < -\hat{h}$ we obtain
\[
\frac{\partial \psi(\hat{h},a)}{\partial \hat{h}} < \frac{-2\hat{h}(4a - 1 - \hat{h}(a^5 + 5a^4 + 14a^3 + 18a^2 + 11a)))}{(ha - 1)^5(\hat{h} - 1)^3}.
\]
(5.28)

Since $\frac{-2\hat{h}}{(ha-1)^5(\hat{h}-1)^3} > 0$ it obviously follows from (5.28) that $\frac{\partial \psi(\hat{h},a)}{\partial \hat{h}} < 0$ for $a = 0$. Further, for $a \in [0, \frac{1}{4}]$ we have $\frac{\partial \psi(\hat{h},a)}{\partial \hat{h}} < 0$ if
\[
\max \left\{ \frac{4a-1}{a^5+5a^4+14a^3+18a^2+11a}, -1 \right\} < \hat{h} < 0.
\]
(5.29)

For the case $a < 0$, let $\max \left\{ \frac{1}{a}, -1 \right\} < \hat{h} < 0$. Then, using the estimate $-\hat{h}^3 < \hat{h}^2$ and neglecting some negative terms, we get from (5.27) that
\[
\frac{\partial \psi(\hat{h},a)}{\partial \hat{h}} < \frac{-2\hat{h}(\hat{h}^2(5a^4 + 18a^2) - 1)}{(ha - 1)^5(\hat{h} - 1)^3}.
\]
(5.30)

Since $\frac{-2\hat{h}}{(ha-1)^5(\hat{h}-1)^3} > 0$ it follows that $\frac{\partial \psi(\hat{h},a)}{\partial \hat{h}} < 0$ if
\[
\max \left\{ \sqrt[3]{\frac{a}{a^4+5a^3+14a^2+18a+11a}}, \frac{1}{a}, -1 \right\} < \hat{h} < 0,
\]
(5.31)

which completes the proof. \qed

Considering class X in case of a diagonally drift implicit stiffly accurate SRK method, we get with $a_1 = a_2 = a$ for some $a \in \mathbb{R}$, $a_3 = 1$ and $a_4 = 0$ a family of SRK schemes (2.2) that need three stage-evaluations of the drift $f$ and two stage-evaluations of the diffusion $g$ each step. Again, we need $a \neq 0$ if the SRK method is applied to SDAEs.

**Lemma 5.6** The family of order 1.0 SRK schemes with coefficients (5.16) in the case of $a_1 = a_2$, $a_3 = 1$ and $a_4 = 0$ is $A$-stable for equation (5.1), i.e. $\mathcal{D}_{SDE} \subseteq \mathcal{D}_{SRK}$, if and only if $a_1 \geq \frac{1}{4}$ and $b \in \mathbb{R} \setminus \{0\}$.

**Proof.** The assertion follows from the fact that for $a_1 = a_2$, $a_3 = 1$ and $a_4 = 0$ the stability function (5.17) with (5.20) and (5.21) coincides with the
stability function for the coefficients (5.15) of class II under the assumptions of Lemma 5.5. Therefore, the result follows from the proof of Lemma 5.5. □

Remark 5.7 With the choice \( a_1 = a_2 = a_3 = 1 \) for both classes II and X, we get families of \( A \)-stable stiffly accurate singly diagonally drift-implicit SRK schemes. Therefore, the calculation of only one LU decomposition is needed each step if a simplified Newton method is applied to solve the implicit equations (see also [5]).

Next, we try to find within classes II and X some \( A \)-stable stiffly accurate SRK schemes with a minimized number of stage-evaluations for the drift function \( f \) and the diffusion function \( g \) needed each step. Therefore, we analyse some stiffly accurate SRK methods with an explicit first stage, i.e., we choose \( a_1 = 0 \) in the following. These schemes can be applied to SDAEs as well, provided that the sub-matrix \((A_{ij})_{2 \leq i,j \leq s}\) is nonsingular [9].

Lemma 5.8 The family of order 1.0 SRK schemes with coefficients (5.15) is \( A \)-stable for equation (5.1) in case of \( a_1 = 0 \), i.e. it holds \( \mathcal{D}_{SDE} \subseteq \mathcal{D}_{SRK} \), if \( a_2 \geq 0, \ a_3 \geq \frac{3}{2} \) and \( b \in \mathbb{R} \setminus \{0\} \).
Fig. 5. Mean-square stability region for class II and class X applicable to SDAEs with $a_1 = 0$, $a_3 = \frac{3}{2}$, $a_4 = -a_2$, $b = 1$ and with $a_2 = \frac{3}{2}$, $a_2 = \frac{1}{4}$, in the lower figures with $a_2 = \frac{1}{16}$ and $a_2 = \frac{1}{64}$, respectively.

**Proof.** First, calculate the stability function $\hat{R}(\hat{h}, k)$ from (5.17). As a result of this, we have to prove that

$$\hat{R}(\hat{h}, k) = \frac{1}{2} |k|^4 + |a_2 \hat{h} - 1|^2 \left( |k|^2 + |\hat{h}|^2 (1 - a_3)^2 + (\hat{h} + \overline{\hat{h}})(1 - a_3) + 1 \right) < 1,$$

(5.32)

for all $\hat{h}, k \in \mathbb{C}^2$ with $2\Re(\hat{h}) + |k|^2 < 0$. Therefore, we assume that $\Re(\hat{h}) < 0$ and $|k|^2 < -2\Re(\hat{h})$ and we prove that $\hat{R}(\hat{h}, k) - 1 < 0$ is fulfilled under the assumptions of Lemma 5.8. Using these assumptions, we get

$$\hat{R}(\hat{h}, k) - 1 = \frac{1}{2} |k|^4 + |a_2 \hat{h} - 1|^2 \left( |k|^2 + |\hat{h}|^2 - 2a_3|\hat{h}|^2 + 2\Re(\hat{h}) \right)$$

$$< \frac{\phi(\hat{h}, a_2, a_3)}{|a_2 \hat{h} - 1|^2 \cdot |a_3 \hat{h} - 1|^2}$$

(5.33)

where

$$\phi(\hat{h}, a_2, a_3) := 2\Re(\hat{h})^2 + (\Re(\hat{h})^2 + 3(\hat{h})^2)(1 - 2a_3)|a_2 \hat{h} - 1|^2.$$

(5.34)
The denominator in (5.33) is positive, thus it is sufficient to prove \( \phi(\hat{h}, a) \leq 0 \). Collecting for the real part of \( \hat{h} \) in (5.34), we get

\[
\phi(h, a_2, a_3) = a_2^2(1 - 2a_3)\Re(\hat{h})^4 + (2a_2(1 - 2a_3))\Re(\hat{h})^3 \\
+ (3 - 2a_3 + 2(1 - 2a_3)a_2^2\Im(\hat{h})^2)\Re(\hat{h})^2 \\
- 2a_2(1 - 2a_3)\Im(\hat{h})^2\Re(\hat{h}) \\
+ (1 - 2a_3)\Im(\hat{h})^2 + (1 - 2a_3)a_2^2\Im(\hat{h})^4.
\] (5.35)

Due to our assumption \( \Re(\hat{h}) < 0 \), it is easy to see that \( \phi(h, a_2, a_3) \leq 0 \) if
\( 3 - 2a_3 \leq 0, 1 - 2a_3 \leq 0 \) and \( a_2 \geq 0 \). Thus, we need \( a_2 \geq 0 \) and \( a_3 \geq \frac{3}{2} \) for A-stability. \( \square \)

Considering now class X with an explicit first stage, that is in case of \( a_1 = 0 \), then similar results can be obtained if the simplifying assumption \( a_4 = -a_2 \) is fulfilled.

**Lemma 5.9** The family of order 1.0 SRK schemes with coefficients (5.16) is A-stable for equation (5.1) in case of \( a_1 = 0 \) and \( a_4 = -a_2 \), i.e. it holds \( D_{SDE} \subseteq D_{SRK} \), if \( a_2 \geq 0, a_3 \geq \frac{3}{2} \) and \( b \in \mathbb{R} \setminus \{0\} \).

**Proof.** We calculate the stability function \( \hat{R}(\hat{h}, k) \) from (5.17) in the case of \( a_1 = 0 \). Therefore, we have to prove that

\[
\hat{R}(\hat{h}, k) = \frac{\frac{1}{4}|k|^4 + |1 - a_2\hat{h}|^2 (|k|^2 + |\hat{h}|^2(1 - 2a_3) + \hat{h} - 1| - |a_3\hat{h} - 1|)}{|a_2\hat{h} - 1|^2 |a_3\hat{h} - 1|^2} \\
+ \frac{\frac{1}{26}|k|^2(a_2 + a_4)(\Re\hat{h} + k\hat{h}) + \frac{1}{25^2}|k|^2|\hat{h}|^2(a_2 + a_4)^2}{|a_2\hat{h} - 1|^2 |a_3\hat{h} - 1|^2} < 1,
\] (5.36)

for all \( \hat{h}, k \in \mathbb{C}^2 \) with \( 2\Re(\hat{h}) + |k|^2 < 0 \). Thus, we assume that \( \Re(\hat{h}) < 0 \) and \( |k|^2 < -2\Re(\hat{h}) \) and we prove that \( \hat{R}(\hat{h}, k) - 1 < 0 \) is fulfilled under the assumptions of Lemma 5.9. Using the assumption that additionally \( a_4 = -a_2 \), we calculate that

\[
\hat{R}(\hat{h}, k) - 1 = \frac{\frac{1}{4}|k|^4 + |a_2\hat{h} - 1|^2(|k|^2 + |\hat{h}|^2 - 2a_3|\hat{h}|^2 + 2\Re(\hat{h}))}{|a_2\hat{h} - 1|^2 |a_3\hat{h} - 1|^2}.
\] (5.37)

Here, it turns out that (5.37) is the same as (5.33), i.e., the rest of the proof is the same as in the proof of Lemma 5.8. \( \square \)

Figure 4 shows the region of mean-square stability for classes II and X with \( a_1 = a_2 = a_4 = 0, b = 1 \) and for \( a_3 \in \{1, \frac{3}{2}, 2, 4\} \). We point out that the SRK schemes with \( a_1 = a_2 = 0 \) can be applied to SDEs but not to SDAEs. Here,
the SRK schemes SADIRK12II and SADIRK12X for SDEs with coefficients \( a_1 = a_2 = a_4 = 0, b = 1 \) and \( a_3 = \frac{3}{2} \) for class II and X, respectively, cover the region of stability for the SDE best. However, for SDAEs we need \( a_2 \neq 0 \) and \( a_3 \neq 0 \) if \( a_1 = 0 \) holds. Some regions of mean-square stability for the coefficients \( a_1 = 0, a_3 = \frac{3}{2}, b = 1 \) and \( a_2 \in \{ \frac{2}{3}, \frac{1}{4}, \frac{1}{16}, \frac{1}{64} \} \) are given in Figure 5. Here, it can be observed that the region of mean-square stability fits better to the region of the test equation the smaller the values of \( a_2 \) are.

**Remark 5.10** If we choose \( a_1 = a_2 = 0 \) and \( a_3 \geq \frac{3}{2} \) for class II or for class X with additionally \( a_4 = 0 \), then only one stage-evaluation of the drift \( f \) and two stage-evaluations of the diffusion \( g \) are necessary each step for the \( A \)-stable stiffly accurate SRK scheme 2.2 applicable to SDEs. This is due to the FSAL (first same as last) property and due to an explicit first stage, see also e. g. [5]. Further, we get a family of drift-implicit SRK schemes that need only one implicit equation to be solved each step. However, to apply the stiffly accurate SRK schemes to SDAEs, we need \( a_2 \neq 0 \) and thus two stage-evaluations of the drift \( f \) and two stage-evaluations of the diffusion \( g \) due to FSAL. Especially, in case of \( a_2 = a_3 \) only one \( LU \) decomposition has to be calculated each step if a simplified Newton method is applied to solve the implicit equations (see [5]).

6 Conclusions

We have calculated a classification of the set of solutions for the order conditions of stiffly accurate strong order 0.5 and order 1.0 SRK methods for SDAEs with a scalar driving Wiener process introduced in [9]. As the main advantages of the considered SRK method compared to well known schemes, no projectors and no pseudo-inverses have to be calculated and the considered SRK methods are derivative-free what makes them easy to be implemented. Based on this classification, a mean-square stability analysis is carried out for the two classes II and X. These two classes allow to minimize the computational costs in the sense that a minimum number of stage-evaluations are needed as well as a minimum number of implicit equations that have to be solved each step. Further, the two classes represent both cases where \( B^{(2)} = 0 \) and \( B^{(2)} \neq 0 \), i.e., where the random variables \( I_{(1,1),n} \) do not appear and do appear explicitly within the scheme, respectively. For both classes II and X, conditions for the coefficients such that the SRK method is \( A \)-stable in the mean-square sense are proved for diagonally drift-implicit schemes as well as for schemes with an explicit first stage. Especially, a family of \( A \)-stable stiffly accurate drift-implicit order 1.0 SRK schemes for SDEs has been found that needs only one stage-evaluation of the drift function \( f \), two stage-evaluations of the diffusion function \( g \) and one implicit equation to be solved each step. However, for the SDAE case at least two stage-evaluations of the drift \( f \) and two stage-evaluations of the diffusion \( g \) are needed for an \( A \)-stable SRK method. For future research it would be
interesting to analyse not only mean-square stability, but maybe to find some further concepts of stability that are of importance especially for SDAEs.

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