TIME INTEGRATORS FOR DISPERSIVE EQUATIONS 
IN THE LONG WAVE REGIME

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Abstract. We introduce a novel class of time integrators for dispersive equations which allow us to reproduce the dynamics of the solution from the classical ε = 1 up to long wave limit regime ε ≪ 1 on the natural time scale of the PDE t = O(1/ε). Most notably our new schemes converge with rates at order τ ε over long times t = 1/ε.

1. Introduction

As a model problem we consider
\[ \partial_t u(t, x) + \partial_x m_L(\sqrt{\varepsilon} \partial_x)u(t, x) + \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x)u^2(t, x) = 0 \quad (t, x) \in \mathbb{R} \times T \]
with smooth symbols \( m_L, m_Q \) satisfying for \( \xi \in \mathbb{R} \)
\[ m_L(i\xi) \in \mathbb{R}, \quad m_L(-i\xi) = m_L(i\xi), \quad m_Q(i\xi) \in \mathbb{R}, \quad m_Q(-i\xi) = m_Q(i\xi), \]
\[ |m_L^{(4)}(i\xi)| \leq \frac{c_L}{1 + |\xi|^\beta_L}, \quad |m_Q(i\xi)| \leq \frac{1}{1 + |\xi|^\beta_Q}, \quad |m_Q'(i\xi)| \leq \frac{1}{1 + |\xi|} \]
for some \( \beta_L, \beta_Q \geq 0 \). The class of equations (1) includes a large variety of models such as the Benjamin–Bona–Mahony (BBM) equation
\[ m_L(i\xi) = m_Q(i\xi) = \frac{1}{1 + \xi^2}, \]
the Korteweg–de Vries (KdV) equation
\[ m_L(i\xi) = 1 - \xi^2, \quad m_Q(i\xi) = 1 \]
and the Whitham equation
\[ m_L(i\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}}, \quad m_Q(i\xi) = 1. \]

The model (1) can be rigorously derived in the long wave regime from many physical models including water waves, plasma, etc., see, e.g., [1, 4, 5, 11, 12]. In particular, rigorous error estimates between the solution of (1) and the solution of the original model are established on the natural time scale \( t = O(1/\varepsilon) \).

In this paper we introduce a novel class of numerical integrators for (1) based on the long wave behaviour of the dispersion relation
\[ i\xi \left( 1 - \frac{m_L^{(2)}(0)}{2} \xi^2 \right) + \text{higher order terms} \quad \text{with} \quad \xi = \sqrt{\varepsilon} k, \quad k \in \mathbb{Z}. \]

At first order the long wave limit preserving (LWP) scheme takes the form
\[ u^{n+1} = e^{-\tau \partial_x m_L(\sqrt{\varepsilon} \partial_x)} \left[ u^n - \frac{1}{3 \alpha} m_Q(\sqrt{\varepsilon} \partial_x) \left( e^{\tau \varepsilon \partial_x^2} \left( e^{-\tau \varepsilon \partial_x^2} \partial_x^{-1} u^n \right)^2 - \left( \partial_x^{-1} u^n \right)^2 + 2 \varepsilon \tau u^n \partial_x u^n \right] \right] \]
where we have set $\alpha = \frac{m_2(0)}{2}$. Details on its construction will be given in Section 2.1. The scheme (6) (and its second order counterpart, see (23)) will allow us to reproduce the dynamics of the solution $u(t,x)$ of (1) up to long wave regimes $\varepsilon \ll 1$ on the natural long time scale $t = O(\frac{1}{\varepsilon})$.

More precisely, at first ($\sigma = 1$) and second-order ($\sigma = 2$) we will establish the global error estimates

$$\|u(t_n) - u^n\|_{L^2} \leq \tau^\sigma t_n \varepsilon^2 c_0 \varepsilon^{c_1 n \varepsilon}$$

on long time scales $t_n \leq \frac{1}{\varepsilon}$, $\sigma = 1,2$

where $c_0, c_1$ depend on certain Sobolev norms of $u$ (depending on $\beta_L$ and $\beta_Q$). We refer to Theorem 2.5 and Theorem 3.4 for the precise error estimates. Note that the time scale $t = O(\frac{1}{\varepsilon})$ is also the natural time scale on the continuous level, i.e., for the PDE (1) itself.

Compared to classical schemes, e.g., splitting or exponential integrator methods, our long wave limit preserving integrators in particular

- allow for approximations on large natural time scales $t = O(\frac{1}{\varepsilon})$;
- converge with rates at order $\tau^\sigma \varepsilon^2 t$.

Surprisingly, we can even achieve convergence of order $\tau \varepsilon$, i.e., a gain in $\varepsilon$, over long times $t = \frac{1}{\varepsilon}$.

For the analysis of long-time energy conservation for Hamiltonian partial differential equation with the aid of Modulated Fourier Expansion and Birkhoff normal forms we refer to [14, 7, 13, 8, 9, 10] and the references therein. Here we in contrast prove long time error estimates on the solution itself. In case of the nonlinear Klein-Gordon equation with weak nonlinearity $\varepsilon^2 u^3$ long time error estimates of splitting methods were recently established in [2].

The main challenge in the theoretical and numerical analysis of (1) on long time scales $t = O(\frac{1}{\varepsilon})$ lies in the loss of derivative in the nonlinearity. This loss of derivative is clearly seen in case of the KdV equation (1) for which we face a Burger’s type nonlinearity $\varepsilon \partial_x u^2$. However, even in case of the BBM equation (3) where we expect some regularisation through the structure of the leading operators (note that $\beta_L = \beta_Q = 2$), the smoothing only holds with loss in $\varepsilon$

$$\|\varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) u^2\|_r \leq \min(\sqrt{\varepsilon} \|u^2\|_r, 2\|u \partial_x u\|_r).$$

(7)

For BBM this may allow first order error estimates at order $\tau \sqrt{\varepsilon}$ for classical splitting or exponential integrator methods up to time $t = O(\frac{1}{\sqrt{\varepsilon}})$, but not on the natural time scale of the PDE that is $t = O(\frac{1}{\varepsilon})$. Our new long wave limit adapted discretisation (6) in contrast allows for long time error estimates at order $\tau \varepsilon$ on the natural time scale $t = O(\frac{1}{\varepsilon})$.

In case of the BBM equation with a regularising nonlinearity ($\beta_L = \beta_Q = 2$) we can thanks to the estimate (7) play with the gain in $\varepsilon$ and loss of derivatives. This will allow error bounds also for non smooth solutions, however, only on time scales $t = 1$. More precisely, one could prove first-order convergence in $H^r$ for solutions in $H^r$ ($r > 1/2$), i.e., without any loss of derivative, for short times $t = 1$ at the cost of no longer gaining in $\varepsilon$. Such low regularity estimates on short time scales without gain in $\varepsilon$ also hold true for classical schemes, see for instance [3] for the analysis in case of splitting discretisations.

Our idea for LWP schemes can be extended to higher order. We will give details on the second order integrator on long time scales in Section 3. Note that for the classical KdV equation (that is $\varepsilon = 1$ and without transport term $\partial_x u$), and nonlinear Schrödinger equations resonance based schemes were recently introduced in [17, 19] and (short time) error estimates for time $t = 1$ were proven. We also refer to [15, 16, 6, 20] for splitting, finite difference and Lawson-type methods for the classical KdV equation on time scales $t = 1$.

**Outline of the paper.** In Section 2 and Section 3 we introduce the first- and second-order LWP scheme and carry out their convergence analysis over long times $t = O(\frac{1}{\varepsilon})$. Numerical experiments
in Section 4 underline our theoretical findings.

**Notation and assumptions.** In the following we will assume that \( m_L(2)(0) = 2 \) which implies (as \( \alpha = m_L(2)(0)/2 \)) that \( \alpha = 1 \) in (5). Our analysis also holds true for any \( m_L(2)(0) \in \mathbb{R} \). For practical implementation issues we will impose periodic boundary conditions that is \( x \in \mathbb{T} = [-\pi, \pi] \). Our result can be extended to the full space \( x \in \mathbb{R} \). We denote by \((\cdot,\cdot)\) the standard bilinear estimate (see Section 2.2).

2. A first-order long wave limit preserving scheme

In a first section we will formally derive the LWP scheme (6) (see Section 2.1). Then we will carry out its convergence analysis and establish long time error estimate (see Section 2.2).

2.1. Derivation of the scheme. Recall Duhamel’s formula of (1)

\[
u(t) = e^{-t\partial_x m_L(\sqrt{\varepsilon} \partial_x)} u(0) - \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) e^{-t\partial_x m_L(\sqrt{\varepsilon} \partial_x)} \int_0^t e^{s\partial_x m_L(\sqrt{\varepsilon} \partial_x)} u^2(s) ds.\]

Iterating the above formula, i.e., using that

\[
u(s) = e^{-s\partial_x m_L(\sqrt{\varepsilon} \partial_x)} u(t_n) + O(s\varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) u^2)\]

we see that formally

\[
u(t) \approx e^{-t\partial_x m_L(\sqrt{\varepsilon} \partial_x)} \left[ u(0) - \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) \int_0^t e^{s\partial_x m_L(\sqrt{\varepsilon} \partial_x)} \left( e^{-s\partial_x m_L(\sqrt{\varepsilon} \partial_x)} u(0) \right)^2 ds \right].\]

The key point lies in embedding the long wave limit behaviour (cf. (5))

\[
D_L = \partial_x m_L(\sqrt{\varepsilon} \partial_x) + (\partial_x + \varepsilon \partial_x^2) = O\left(\varepsilon^2 \partial_x m_L(4)(\sqrt{\varepsilon} \partial_x)\right)
\]

into our numerical discretisation. This motivates (for sufficiently smooth solutions) the following approximation

\[
u(t) \approx e^{-t\partial_x m_L(\sqrt{\varepsilon} \partial_x)} \left[ u(0) - \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) \int_0^t e^{s(\partial_x + \varepsilon \partial_x^2)} \left( e^{-s(\partial_x + \varepsilon \partial_x^2)} u(0) \right)^2 ds \right].\]

We may solve the oscillatory integral by the observation that

\[
\varepsilon \partial_x \int_0^t e^{s(\partial_x + \varepsilon \partial_x^2)} \left( e^{-s(\partial_x + \varepsilon \partial_x^2)} v \right)^2 ds = \frac{1}{3} e^{t \varepsilon \partial_x^3} \left( e^{-t \varepsilon \partial_x^3} (\partial_x^{-1} v)^2 \right) - \frac{1}{3} (\partial_x^{-1} v)^2 + 2 \varepsilon \partial_x v, \]

see (16). Based on the long wave limit behaviour we thus find the following approximation

\[
u(t) \approx e^{-t\partial_x m_L(\sqrt{\varepsilon} \partial_x)} \left[ u(0) - m_Q(\sqrt{\varepsilon} \partial_x) \left( \frac{1}{3} e^{t \varepsilon \partial_x^3} \left( e^{-t \varepsilon \partial_x^3} (\partial_x^{-1} u(0))^2 \right) - \frac{1}{3} (\partial_x^{-1} u(0))^2 + 2 \varepsilon \partial_x (\partial_x^{-1} u(0)) \right) \right],\]

which builds the basis of our LWP scheme (6).

2.2. Error analysis. In this section we carry out the error analysis of the filtered scheme (6) over long times \( t = O\left(\frac{1}{\varepsilon}\right) \). We start with the local error analysis. For this purpose we will denote by \( \varphi^t \) the exact flow of (1) and by \( \Phi^\tau \) the numerical flow defined by the scheme (6), such that

\[
u(t_n + \tau) = \varphi^\tau(\nu(t_n)) \quad \text{and} \quad \nu^{n+1} = \Phi^\tau(\nu^n).\]
2.2.1. Local error analysis. We will exploit the following estimate which regularises for $\beta_Q > 1/2$.

Lemma 2.1. Let $f \in H^{r+1-\beta_Q}(\mathbb{T})$. It holds that

$$\|\varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) f\|_r \leq \varepsilon^{1-\beta_Q} \|f\|_{r+1-\beta_Q}.$$  

Proof. The assertion follows thanks to the estimate

$$\|\varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) f\|^2 = \sum_{k \in \mathbb{Z}} (1 + |k|)^{2r} \frac{\varepsilon ik}{1 + (\sqrt{\varepsilon} k)^{\beta_Q}} |\hat{f}_k|^2 \leq \varepsilon^{2(1-\beta_Q)} \|f\|^2_{r+1-\beta_Q}.$$  

Lemma 2.2. Fix $r > 1/2$. Then, the local error $\varphi^r(u(t_n)) - \Phi^r(u(t_n))$ satisfies for

$$\beta := \min(2, \beta_L + \beta_Q)$$  

the estimate

$$\|\varphi^r(u(t_n)) - \Phi^r(u(t_n))\|_r \leq \tau^2 \varepsilon^2 c \left( \sup_{t_n \leq t \leq t_{n+1}} \|u(t)\|_{r+2} \right) + c_L \tau^2 \varepsilon^3 \frac{\beta}{2} c \left( \sup_{t_n \leq t \leq t_{n+1}} \|u(t)\|_{r+6-\beta} \right).$$  

Proof. Iterating Duhamel’s formula of (1) yields that

$$u(t_n + \tau) = e^{-\tau \partial_x m_L(\sqrt{\varepsilon} \partial_x)} u(t_n) - \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) e^{-\tau \partial_x m_L(\sqrt{\varepsilon} \partial_x)} \int_0^\tau e^{s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} u^2(t_n + s) ds$$

$$= e^{-\tau \partial_x m_L(\sqrt{\varepsilon} \partial_x)} u(t_n) - \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) e^{-\tau \partial_x m_L(\sqrt{\varepsilon} \partial_x)} \int_0^\tau e^{s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} \left( e^{-s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} u(t_n) \right)^2 ds$$

$$+ \mathcal{R}_1(\varepsilon, \tau, u)$$

with the remainder

$$\mathcal{R}_1(\varepsilon, \tau, u) = \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) \int_0^\tau e^{s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} \left[ \left( e^{-s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} u(t_n) \right)^2 - u^2(t_n + s) \right] ds.$$  

(9)

(10)

Thanks to the observation that

$$u(t_n + s) = e^{-s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} u(t_n) - \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) e^{-s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} \int_0^s e^{s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} u^2(t_n + s_1) ds_1$$

the remainder $\mathcal{R}_1(\varepsilon, \tau, u)$ is of the following form

$$\mathcal{R}_1(\varepsilon, \tau, u) = \mathcal{O} \left( \tau^2 \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) (u(t) \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) (u^2(t))) \right).$$

Thanks to assumption (2) (which guarantees the boundedness of the symbol $m_Q$) we can thus conclude that

$$\|\mathcal{R}_1(\varepsilon, \tau, u)\|_r \leq \tau^2 \varepsilon^2 c \left( \sup_{t_n \leq t \leq t_{n+1}} \|u(t)\|_{r+2} \right).$$  

(11)

Taylor series expansion of the symbol $m_L(\delta)$ around $\delta = 0$ yields that

$$m_L(\delta) = m_L(0) + \xi m_L'(0) + \frac{\delta^2}{2} m_L''(0) + \frac{\delta^3}{3!} m_L^{(3)}(0) + \int_0^\delta \frac{\delta - \tilde{\delta}}{3!} m_L^{(4)}(\tilde{\delta}) d\tilde{\delta}$$

$$= 1 + \delta^2 m_L''(0) + \int_0^\delta \frac{(\delta - \tilde{\delta})^3}{3!} m_L^{(4)}(\tilde{\delta}) d\tilde{\delta}.$$
where in the last step we have used the assumptions (2) (which implies that \( m^{(2\ell+1)}(0) = 0 \) and the assumption that (without loss of generality) \( m^{(2)}(0) = 2 \). Together with the assumption that 
\[
| m^{(4)}_L(i\xi) | \leq \frac{c_L}{1+|\xi|^\beta_L}
\] (see again (2)) we thus find that
\[
D_L = \partial_x m_L(\sqrt{\varepsilon}\partial_x) - (\partial_x + \varepsilon\partial_x^3) = O\left( c_L \frac{\varepsilon^2 \partial_x^5}{1 + (\sqrt{\varepsilon}|\partial_x|)^{\beta_L}} \right).
\] (12)

This allows the following expansion of the oscillations
\[
e^{\pm \varepsilon\partial_x} m_L(\sqrt{\varepsilon}\partial_x) = e^{\pm \varepsilon(\partial_x + \varepsilon\partial_x^3)} + O\left( c_L \frac{\varepsilon^2 \partial_x^5}{1 + (\sqrt{\varepsilon}|\partial_x|)^{\beta_L}} \right).
\] (13)

Employing these expansion for the oscillations in the remaining integral term in (9) yields together with Lemma 2.1 that
\[
u(t_n + \tau) = e^{-\tau \partial_x} m_L(\sqrt{\varepsilon}\partial_x) u(t_n) - \varepsilon \partial_x m_Q(\sqrt{\varepsilon}\partial_x) e^{-\tau \partial_x} m_L(\sqrt{\varepsilon}\partial_x) \int_0^\tau e^{\varepsilon(\partial_x + \varepsilon\partial_x^3)} (e^{-s(\partial_x + \varepsilon\partial_x^3)} u(t_n))^2 \, ds
+ \mathcal{R}_2(\varepsilon, \tau, u)
\] (14)

where the remainder \( \mathcal{R}_2(\varepsilon, \tau, u) \) is thanks to (2) of type
\[
O\left( c_L \tau^2 \varepsilon^\beta \frac{\partial_x^5}{1 + (\sqrt{\varepsilon}|\partial_x|)^{\beta_L}} \varepsilon \partial_x m_Q(\sqrt{\varepsilon}\partial_x) u^2 \right)
= O\left( c_L \tau^2 \varepsilon^\beta \frac{\partial_x^5}{1 + (\sqrt{\varepsilon}|\partial_x|)^{\beta_L}} \frac{\varepsilon^2 \partial_x^5}{1 + (\sqrt{\varepsilon}|\partial_x|)^{\beta_Q}} u^2 \right)
\] such that
\[
\| \mathcal{R}_2(\varepsilon, \tau, u) \|_r \leq c_L \tau^2 \varepsilon^\beta \frac{\partial_x^5}{1 + (\sqrt{\varepsilon}|\partial_x|)^{\beta_L}} \frac{\varepsilon^2 \partial_x^5}{1 + (\sqrt{\varepsilon}|\partial_x|)^{\beta_Q}} \left( \sup_{t_n \leq t \leq t_{n+1}} \| u(t) \|_{r+6-\beta} \right), \quad \beta := \min(2, \beta_L + \beta_Q).\n\] (15)

Next we calculate with the aid of the Fourier transform \( \nu(x) = \sum_{k \in \mathbb{Z}} \hat{\nu}_k e^{ikx} \) and the definition
\[
\partial_x^{-1} \nu(x) = \sum_{k \neq 0} \hat{\nu}_k e^{ikx}
\] that
\[
\mathcal{I}(\tau, \varepsilon, v) = \varepsilon \partial_x \int_0^\tau e^{s(\partial_x + \varepsilon\partial_x^3)} \left( e^{-s(\partial_x + \varepsilon\partial_x^3)} v \right)^2 \, ds
= \varepsilon \sum_{\ell+m=k} \sum_{\ell, m \neq 0} e^{ikx} \hat{\nu}_\ell \hat{\nu}_m(i\kappa) \int_0^\tau e^{-3i\varepsilon\kappa \ell m} ds + 2\varepsilon \tau \hat{\nu}_0 \partial_x v
= \sum_{\ell+m=k} \sum_{\ell, m \neq 0} e^{ikx} \hat{\nu}_\ell \hat{\nu}_m(i\kappa) \left( e^{-3i\varepsilon\kappa \ell m} - 1 \right) \frac{1}{3(\ell)(im)} + 2\varepsilon \tau \hat{\nu}_0 \partial_x v
= \frac{1}{3} \varepsilon \partial_x \left[ e^{-3i\varepsilon\kappa \ell m} (\partial_x^{-1} v)^2 \right] - \frac{1}{3} (\partial_x^{-1} v)^2 + 2\varepsilon \tau \hat{\nu}_0 \partial_x v,
\] (16)

see also [17] in case of no advection term \( \partial_x \). Plugging the above relation into (14) we obtain
\[
\varphi^\tau(u(t_n)) = \Phi^\tau(u(t_n)) + \sum_{i=1,2} \mathcal{R}_i(\varepsilon, \tau, u),
\] where \( \mathcal{R}_1 \) satisfies (11) and \( \mathcal{R}_2 \) satisfies (15). This concludes the proof. \( \square \)
2.2.2. Stability analysis. In order to carry out the stability analysis we need the following Lemma.

**Lemma 2.3.** For \(|m_Q(iξ)| ≤ 1\) and \(|m_Q'(iξ)| ≤ \frac{1}{1+|ξ|}\) it holds that

\[
\|m_Q(\sqrt{ξ}∂_x), w]|∂_xΛ^r v\|_{L^2} ≤ \|w\|_{r+1}\|v\|_{H^r}.
\]

**Proof.** Recall that \(\Lambda = (1 + |ξ|^2)^\frac{1}{2}\). The \(k\)-th Fourier coefficient of \([m_Q, w]|∂_xΛ^r v\) is given by

\[
|m_Q(\sqrt{ξ}il) - m_Q(\sqrt{ξ}ik)| = \sum_l \frac{|\hat{w}(k-l)|}{|l|} \left[ m_Q(\sqrt{ξ}il) - m_Q(\sqrt{ξ}ik) \right] il(1 + l^2)^\frac{1}{2} |\hat{v}(l)|. \tag{17}
\]

Hence we can conclude that

\[
|m_Q(\sqrt{ξ}il) - m_Q(\sqrt{ξ}ik)||l| ≤ |k-l|.
\]

Proof. Fix \(r ≥ 1\). The numerical flow defined by the scheme (6) is \(ε\)-stable in \(H^r\) in the sense that for two functions \(w ∈ H^{r+1}\) and \(v ∈ H^r\) we have that

\[
\|\Phi^r(w) - \Phi^r(v)\|_r ≤ e^{εB}\|w - v\|_r, \quad B = B(\|w\|_{r+1}, \|v\|_r),
\]

where the constant \(B\) depends on the \(H^{r+1}\) norm of \(w\) and \(H^r\) norm of \(v\).

**Proof.** Fix \(r ≥ 1\). For the stability analysis we will rewrite the numerical flow back (6) in its integral form. Thanks to (16) we observe that

\[
\Phi^r(v) = e^{-τ\partial_x m_Q(\sqrt{ξ}∂_x) v - ε\partial_x m_Q(\sqrt{ξ}∂_x) e^{-τ\partial_x m_Q(\sqrt{ξ}∂_x)}} \int_0^τ e^{sε\partial_x^2} \left[ e^{-sε\partial_x^2} q^2 \right] ds.
\]

We need to show that

\[
|(\Lambda^r v∂_x m_Q(wv), \Lambda^r v)| ≤ \|w\|_{r+1}\|v\|_r^2 \tag{18}
\]

where for shortness we write \(m_Q = m_Q(\sqrt{ξ}∂_x)\).

Let us note that

\[
\Lambda^r v∂_x m_Q(wv) = \Lambda^r m_Q(w∂_x v + v∂_x w),
\]

where thanks to the boundedness of \(m_Q\) (see (2)) we have that

\[
|(\Lambda^r m_Q(v∂_x w), \Lambda^r v)| ≤ \|v\|_r^2\|v∂_x w\|_r ≤ \|v\|_r^2\|w\|_{r+1}.
\]

Thus we obtain that

\[
|(\Lambda^r v∂_x m_Q(wv), \Lambda^r v)| ≤ |(\Lambda^r m_Q(w∂_x v), \Lambda^r v)| + \|v\|_r^2\|w\|_{r+1} \tag{19}
\]

and it remains to derive a suitable bound on \(|(\Lambda^r m_Q(w∂_x v), \Lambda^r v)|\). For this purpose let us note that

\[
\Lambda^r m_Q(w∂_x v) = m_Q(w\Lambda^r ∂_x v) + m_Q(\Lambda^r w)∂_x v). \tag{20}
\]
For the second term in (20) we see that
\[
\| (m_Q([\lambda^r, w] \partial_x v), \lambda^r v) \| \leq \| \lambda^r, w \partial_x v \|_{L^2} \| v \| \leq (\| \partial_x v \|_{L^2} \| w \|_r + \| w \|_{L^\infty} \| v \|_r) \| v \|_r. \tag{21}
\]
For first term in (20) we see that as \( m_Q = m_Q^* \)
\[
(m_Q(w \lambda^r \partial_x v), \lambda^r v) = (w \lambda^r \partial_x v, m_Q \lambda^r v) = -(\partial_x w)\lambda^r v, m_Q \lambda^r v) - (w \lambda^r v, m_Q \partial_x \lambda^r v)
\]
Hence,
\[
(m_Q(w \lambda^r \partial_x v), \lambda^r v) = -\frac{1}{2} (\partial_x w)\lambda^r v, m_Q \lambda^r v) - \frac{1}{2} (\lambda^r v, [m_Q, w] \partial_x \lambda^r v)
\]
which implies thanks to Lemma 2.3 and the assumptions (2) on \( m_Q \) that
\[
\| (m_Q(w \lambda^r \partial_x v), \lambda^r v) \| \leq \| \partial_x w \|_{L^\infty} \| v \|_r^2 + \| v \|_r \| [m_Q, w] \partial_x \lambda^r v \|_{L^2} \leq c \| v \|_r^2 \| w \|_{r+1}. \tag{22}
\]
Plugging (21) and (22) into (20) yields that
\[
\| (\lambda^r m_Q(w \partial_x v), \lambda^r v) \| \leq c \| v \|_r^2 \| w \|_{r+1}
\]
which by (19) implies the desired estimate (18).

\[\Box\]

2.2.3. Global error estimate.

**Theorem 2.5.** Fix \( \beta := \min(2, \beta_L + \beta_Q) \) and assume that the solution \( u \) of (1) satisfies \( u \in C([0, T]; H^{6-\beta}) \). Then there exists a \( \tau_0 > 0 \) such that for all \( 0 < \tau \leq \tau_0 \) the global error estimate holds for \( u^n \) defined in (6)
\[
\| u(t_n) - u^n \|_{L^2} \leq t_n \tau \varepsilon^2 c \left( \sup_{0 \leq t \leq t_n} \| u(t) \|_2 \right) e^{c t_n \varepsilon} + c_L t_n \tau \varepsilon^2 e^{1-\frac{4}{\beta}} \left( \sup_{0 \leq t \leq t_n} \| u(t) \|_{6-\beta} \right) e^{c t_n \varepsilon}
\]
where \( c \) depends on the \( H^2 \) norm of the solution \( u \).

**Proof.** The assertion in \( H^r, r \geq 1 \), follows by the local error bound given in Lemma 2.2 together with the stability estimate in Lemma 2.4 (with the stronger norm placed on the exact solution \( u(t_n) \)) via a Lady Windermere’s fan argument (see, e.g., [14]). Then under the given regularity assumptions on the exact solution (which imply that \( u \) is at least in \( H^1 \)) we can prove the corresponding \( L^2 \) error bound by first proving convergence (with reduced order in \( \tau \) but full gain of at least one factor \( \varepsilon \)) in \( H^1 \), i.e.,
\[
\| u(t_n) - u^n \|_{H^1} \leq t_n \tau \varepsilon c \left( \sup_{0 \leq t \leq t_n} \| u(t) \|_4 \right) e^{c t_n \varepsilon}
\]
for some \( \delta = \delta(\beta_L, \beta_Q) > 0 \). Thanks to the estimate
\[
\| u^n \|_{H^1} \leq \| u(t_n) - u^n \|_{H^1} + \| u(t_n) \|_{H^1}
\]
this will give us a priori the boundedness of the numerical solution in \( H^1 \) over long times \( t_n = \frac{1}{\varepsilon} \). For details on the latter approach in case of short time \( (t = 1) \) estimates for splitting methods for cubic Schrödinger we refer to [18].

\[\Box\]
3. A SECOND-ORDER LONG WAVE LIMIT PRESERVING SCHEME

Our second order LWP scheme for (1) takes the form
\[ u^{n+1} = e^{-\tau_x m_L(\sqrt{\varepsilon} \partial_x)} \left[ u^n - \frac{1}{3\alpha} m_Q(\sqrt{\varepsilon} \partial_x) \left( e^{\tau_{1\alpha} \partial_x^3} \left( e^{-\tau_{1\alpha} \partial_x^3} \partial_x^{-1} u^n \right)^2 - \left( \partial_x^{-1} u^n \right)^2 + 2 \varepsilon \partial_x \partial_x^2 u^n \right) \right] \\
+ \tau^2 e^{2 \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x)} \Psi_{m_Q} \left( u^n \partial_x m_Q (\sqrt{\varepsilon} \partial_x) (u^n u^n) \right) \\
- \frac{\tau^2}{2} \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) \Psi_{D_L,m_Q} D_L (u^n u^n) + \tau^2 \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) \Psi_{D_L,m_Q} (u^n D_L u^n) \]
\]
where we recall that \( \alpha = m_L^{(2)}(0)/2 \) and
\[ D_L(\sqrt{\varepsilon} \partial_x) = \partial_x m_L(\sqrt{\varepsilon} \partial_x) - (\partial_x + \alpha \partial_x^3). \]

For stability issues we have introduced the filter functions
\[ \Psi_{m_Q}(\sqrt{\varepsilon} \partial_x) \quad \text{and} \quad \Psi_{D_L,m_Q} = \Psi_{D_L,m_Q}(\sqrt{\varepsilon} \partial_x) \]
satisfying
\[ \| \tau \Psi_{m_Q}(\sqrt{\varepsilon} \partial_x) \partial_x m_Q(\sqrt{\varepsilon} \partial_x) v \|_r \leq \| v \|_r, \quad \| \Psi_{m_Q}(\sqrt{\varepsilon} \partial_x) v - v \|_r \leq \tau \| \partial_x m_Q(\sqrt{\varepsilon} \partial_x) v \|_r \]
\[ \| \tau \partial_x m_Q(\sqrt{\varepsilon} \partial_x) D_L \Psi_{D_L,m_Q}(\sqrt{\varepsilon} \partial_x) v \|_r \leq \| v \|_r, \quad \| \Psi_{D_L,m_Q}(\sqrt{\varepsilon} \partial_x) v - v \|_r \leq \tau \| \partial_x m_Q(\sqrt{\varepsilon} \partial_x) D_L v \|_r. \]

For an introduction to filter functions we refer to [14].

In a first section we will derive the LWP scheme (23) (see Section 3.1). Then we will carry out its long time error estimate (see Section 3.2). We will again assume without loss of generality that \( \alpha = 1 \).

3.1. Derived of the scheme. Iterating Duhamel’s formula (1) yields that
\[ u(t + \tau) = e^{-\tau_x m_L(\sqrt{\varepsilon} \partial_x)} u(t) - \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) e^{-\tau_x m_L(\sqrt{\varepsilon} \partial_x)} \int_0^\tau e^{s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} u^n(t + s) ds \]
\[ = e^{-\tau_x m_L(\sqrt{\varepsilon} \partial_x)} u(t) \\
- \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) e^{-\tau_x m_L(\sqrt{\varepsilon} \partial_x)} \int_0^\tau e^{s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} \left( e^{-\tau_x m_L(\sqrt{\varepsilon} \partial_x)} u(t) \right) ds \\
- \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) e^{-\tau_x m_L(\sqrt{\varepsilon} \partial_x)} \int_0^\tau e^{s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} u^n(t + s_1) ds_1 \right)^2 \]
Employing the approximation
\[ \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) e^{-s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} \int_0^s e^{s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} u^n(t + s_1) ds_1 \]
\[ = s \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) u^n(t) + O(s^2 \varepsilon \partial_x m_L(\sqrt{\varepsilon} \partial_x) \partial_x m_Q(\sqrt{\varepsilon} \partial_x) u^n) \]
we obtain that
\[ u(t + \tau) = e^{-\tau_x m_L(\sqrt{\varepsilon} \partial_x)} u(t) \\
- \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) e^{-\tau_x m_L(\sqrt{\varepsilon} \partial_x)} \int_0^\tau e^{s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} \left( e^{-\tau_x m_L(\sqrt{\varepsilon} \partial_x)} u(t) \right) ds \\
+ \tau^2 \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) e^{-\tau_x m_L(\sqrt{\varepsilon} \partial_x)} \left( u(t) \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) u^2(t) \right) + R_1(\tau, \varepsilon, u) \]
The remainder \( R_1(\tau, \varepsilon, u) \) is thereby of order
\[ O(s^2 \varepsilon \partial_x m_L(\sqrt{\varepsilon} \partial_x) \partial_x m_Q(\sqrt{\varepsilon} \partial_x) u^2) \]
which implies by assumption (2) together with the observation (see (12))

\[ D_L = \partial_x m_L(\sqrt{\varepsilon} \partial_x) - (\partial_x + \varepsilon \partial_x^3) = O \left( c_L \frac{\varepsilon^2 \partial_x^5}{1 + (\sqrt{\varepsilon} |\partial_x|)^{\beta_L}} \right) \]

the following bound

\[
\| R_1(\tau, \varepsilon, u) \|_r \leq r^3 \varepsilon^2 c \left( \sup_{t_n \leq t \leq t_{n+1}} \| u \|_{r+5} \right) + r^3 \varepsilon^2 c^{1-\beta_0} \left( \sup_{t_n \leq t \leq t_{n+1}} \| u \|_{r+7-2\beta_0} \right)
\]

(26)

with \( \beta_0 = \min(1, \beta_0) \).

Next we employ the following lemma.

**Lemma 3.1.** It holds that

\[
\varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) e^{-\tau \partial_x m_L(\sqrt{\varepsilon} \partial_x)} \int_0^\tau e^{s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} \left( e^{-s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} v \right)^2 ds
\]

\[
= \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) e^{-\tau \partial_x m_L(\sqrt{\varepsilon} \partial_x)} \int_0^\tau e^{s \partial_x + \varepsilon \partial_x^3} \left( e^{-s \partial_x + \varepsilon \partial_x^3} v \right)^2 ds
\]

\[ + \frac{\tau^2}{2} \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) \Psi_{D_L} D_L v^2 - \tau^2 \varepsilon \partial_x m_Q(\sqrt{\varepsilon} \partial_x) \Psi_{D_L} (v D_L v) + R_2(\tau, \varepsilon, u) \]

with the remainder

\[
\| R_2(\tau, \varepsilon, u) \|_r \leq c_L \tau^3 \varepsilon^2 c^{1-\beta_1/2} \left( \sup_{t_n \leq t \leq t_{n+1}} \| u \|_{r+7-\beta_1} \right) + c_L \tau^3 \varepsilon^2 c^{3-\beta_2/2} \left( \sup_{t_n \leq t \leq t_{n+1}} \| u \|_{r+11-\beta_2} \right)
\]

(27)

where \( \beta_1 = \min(2, \beta_Q + \beta_L) \) and \( \beta_2 = \min(6, \beta_Q + 2\beta_L) \).

**Proof.** Note that

\[
\int_0^\tau e^{s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} \left( e^{-s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} v \right)^2 ds
\]

\[ = \int_0^\tau \left( e^{s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} - e^{s \partial_x + \varepsilon \partial_x^3} \right) \left( e^{-s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} v \right)^2 ds
\]

\[ + \int_0^\tau e^{s \partial_x + \varepsilon \partial_x^3} \left[ \left( e^{-s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} v \right)^2 \right] ds
\]

\[ + \int_0^\tau e^{s \partial_x + \varepsilon \partial_x^3} \left[ \left( e^{-s \partial_x} v \right)^2 \right] ds
\]

Hence, using that (see (12))

\[ D_L = \partial_x m_L(\sqrt{\varepsilon} \partial_x) - (\partial_x + \varepsilon \partial_x^3) = O \left( c_L \frac{\varepsilon^2 \partial_x^5}{1 + (\sqrt{\varepsilon} |\partial_x|)^{\beta_L}} \right) \]

together with the expansions

\[ e^{-s \partial_x m_L(\sqrt{\varepsilon} \partial_x)} = 1 + O(s \partial_x m_L(\sqrt{\varepsilon} \partial_x)), \quad e^{\pm s \partial_x + \varepsilon \partial_x^3} = 1 + O \left( s(\partial_x + \varepsilon \partial_x^3) \right) \]
we obtain that
\[
\int_0^\tau e^{s\partial_x m_L(\sqrt{\varepsilon}\partial_x)} \left( e^{-s\partial_x m_L(\sqrt{\varepsilon}\partial_x)v} \right)^2 ds
\]
\[
= \int_0^\tau e^{s(\partial_x + \varepsilon \partial_x^2)} \left( e^{-s(\partial_x + \varepsilon \partial_x^2)}v \right)^2 ds + \frac{\tau^2}{2} D_L v^2 - \tau^2 v D_L v
\]
\[
+ O\left( c_L \tau^3 \frac{\varepsilon^4 \partial_x^{10}}{1 + (\sqrt{\varepsilon} |\partial_x|)^{2\beta_2}} p(v) \right) + O\left( \tau^3 c_L \frac{\varepsilon^2 \partial_x^6}{1 + (\sqrt{\varepsilon} |\partial_x|)^{2\beta_2}} p(v) \right)
\]
for some polynomials \( p \) of degree 2 in \( v \). This yields together with the properties on the filter functions (cf. (24))
\[
\Psi_{\infty} (\sqrt{\varepsilon} \partial_x) = 1 + O(\tau m_Q (\sqrt{\varepsilon} \partial_x)), \quad \Psi_{D_L} (\sqrt{\varepsilon} \partial_x) = 1 + O(\tau D_L (\sqrt{\varepsilon} \partial_x))
\]
yields that the remainder \( R_\tau (\varepsilon, u) \) is of the form
\[
R_\tau (\varepsilon, u) = O\left( c_L \tau^3 \frac{\varepsilon \partial_x}{1 + (\sqrt{\varepsilon} |\partial_x|)^{\beta_2}} \frac{\varepsilon^4 \partial_x^{10}}{1 + (\sqrt{\varepsilon} |\partial_x|)^{2\beta_2}} p(v) \right)
\]
\[
+ O\left( \tau^3 c_L \frac{\varepsilon \partial_x}{1 + (\sqrt{\varepsilon} |\partial_x|)^{\beta_2}} \frac{\varepsilon^2 \partial_x^6}{1 + (\sqrt{\varepsilon} |\partial_x|)^{2\beta_2}} p(v) \right).
\]
This concludes the proof.

Using Lemma 3.1 in the expansion of the exact solution (25) yields together with (16) and the definition of the numerical flow \( \Phi^\tau \) in (23) that
\[
u(t_n + \tau) = \Phi^\tau (u(t_n)) + R_1 (\varepsilon, u) + R_2 (\varepsilon, u)
\]
where the remainders \( R_1 \) and \( R_2 \) satisfy the bounds (26) and (27), respectively.

3.2. Error analysis. Let us again denote by \( \varphi^t \) the exact flow of (1) and by \( \Phi^\tau \) the numerical flow defined by the scheme (23), such that
\[
u(t_n + \tau) = \varphi^t (u(t_n)) \quad \text{and} \quad u^{n+1} = \Phi^\tau (u^n).
\]

3.2.1. Local error analysis.

Lemma 3.2. Fix \( \tau \geq 0 \) and let \( \beta_0 = \min(1, \beta_Q) \), \( \beta_1 = \min(2, \beta_Q + \beta_L) \) and \( \beta_2 = \min(6, \beta_Q + 2\beta_L) \). Then, the local error \( \varepsilon \text{.} \)
\[
varepsilon \left( u(t_n) \right) - \Phi^\tau (u(t_n)) \right) \text{ satisfies}
\]
\[
||v^\tau (u(t_n)) - \Phi^\tau (u(t_n))||_\tau \leq \tau^3 \varepsilon^2 c \left( \sup_{t_n \leq t \leq t_{n+1}} ||u||_{r+5} \right) + \tau^3 \varepsilon^2 \varepsilon^{1-\beta_0} c \left( \sup_{t_n \leq t \leq t_{n+1}} ||u||_{r+7-2\beta_0} \right)
\]
\[
+ c_L \tau^3 \varepsilon^2 \varepsilon^{1-\beta_1/2} c \left( \sup_{t_n \leq t \leq t_{n+1}} ||u||_{r+7-\beta_1} \right) + c_L \tau^3 \varepsilon^2 \varepsilon^{3-\beta_2/2} c \left( \sup_{t_n \leq t \leq t_{n+1}} ||u||_{r+11-\beta_2} \right)
\]
Proof. The assertion follows from the expansion of the exact solution given in (28) together with the error bounds on \( R_1 \) and \( R_2 \) in (26) and (27). \( \square \)

3.2.2. Stability analysis.

Lemma 3.3. Fix \( \tau \geq 1 \). The numerical flow \( \Phi^\tau \) defined by the scheme (23) is \( \varepsilon \)-stable in \( H^r \) in the sense that for two functions \( w \in H^{r+1} \) and \( v \in H^r \) we have that
\[
||\Phi^\tau (w) - \Phi^\tau (v)||_r \leq e^{\tau B} ||w - v||_r, \quad B = B(||w||_{r+1}, ||v||_r),
\]
where the constant \( B \) depends on the \( H^{r+1} \) norm of \( w \) and \( H^r \) norm of \( v \).

Proof. Thanks to Lemma 3.3 it remains to prove the stability estimate only on the last three terms in (23). The latter holds true thanks to the properties (24) of the filter functions \( \Psi_{m_Q} \) and \( \Psi_{D_L} \). \( \square \)
3.2.3. Global error estimate.

Theorem 3.4. Let $\beta_0 = \min(1, \beta_Q)$, $\beta_1 = \min(2, \beta_Q + \beta_L)$ and $\beta_2 = \min(6, \beta_Q + 2\beta_L)$. Then there exists a $\tau_0 > 0$ such that for all $0 < \tau \leq \tau_0$ the global error estimate holds for $u_n$ defined in (23)

$$\|u(t_n) - u^n\|_{L^2} \leq \tau^2 t_n \varepsilon^2 \left[ c \left( \sup_{t_n \leq t \leq t_{n+1}} \|u\|_{r+5} \right) + \varepsilon^{1-\beta_0} c \left( \sup_{t_n \leq t \leq t_{n+1}} \|u\|_{r+7-2\beta_0} \right) 
+ c_L \varepsilon^{1-\beta_1/2} c \left( \sup_{t_n \leq t \leq t_{n+1}} \|u\|_{r+7-\beta_1} \right) + c_L \varepsilon^{3-\beta_2/2} c \left( \sup_{t_n \leq t \leq t_{n+1}} \|u\|_{r+11-\beta_2} \right) \right] e^{c_1 t_n \varepsilon},$$

where $c_1$ depends on the $H^2$ norm of $u$.

Proof. The assertion follows by the local error bound given in Lemma 3.2 together with the stability estimate in Lemma 3.3 via a Lady Windermere’s fan argument, see, e.g., [14]. □

4. Numerical experiments

We underline our theoretical results with numerical experiments. As a model problem we take the BBM equation (3) and solve it with our first- and second-order long wave limit preserving schemes (6) and (23), respectively, for various values of $\varepsilon$ on long time scales, i.e., up to $T = \frac{1}{\varepsilon}$. For the spatial discretisation we employ a standard pseudo spectral method. The numerical findings confirm the convergence order stated in Theorem 2.5 and Theorem 3.4, respectively.

![Figure 1](image)

**Figure 1.** Convergence plot ($L^2$ error versus step size) of the first-order LWP scheme (6) (left) and the second-order LWP scheme (23) (right) on long time scales $t = \frac{1}{\varepsilon}$ for various values of $\varepsilon$. The black solid line corresponds to order one (left) and two (right), respectively.

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