Inequivalent Quantizations of the Rational Calogero Model

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We show that the self-adjoint extensions of the rational Calogero model with suitable boundary conditions leads to inequivalent quantizations of the system. The corresponding spectrum is non-equispaced, consisting of infinitely many positive energy states and at most a single negative energy state. These new states appear for arbitrary number of particles and for specific range of coupling constant.

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The rational Calogero model is described by \( N \) identical particles interacting with each other through a long-range inverse-square and harmonic interaction on the line [1]. This is one of the most celebrated examples of exactly solvable many-particle quantum mechanical systems [2]. This model and its variants [2] are relevant to the study of many branches of contemporary physics, including generalized exclusion statistics [3], quantum hall effect [4], Tomonaga-Luttinger liquid [5], quantum chaos [6], quantum electric transport in mesoscopic system [7], spin-chain models [8], Seiberg-Witten theory [9] and black holes [10].

The spectrum of the \( N \)-particle rational Calogero model was first obtained almost three decades ago, which has since been analyzed using a variety of different techniques [11]. In his original work [1], Calogero used the boundary condition that the wavefunction and the current vanish when any two or more particles coincide. With this boundary condition the Hamiltonian is self-adjoint, which ensures the reality of eigenvalues as well as the completeness of the states. The central issue that we address in this Letter is whether the spectrum obtained for rational Calogero model is unique or does the system admit inequivalent quantizations leading to different spectra? One way to address this issue is to look for more general boundary conditions for which the Calogero Hamiltonian is self-adjoint. The possible boundary conditions for an operator are encoded in the choice of its domains, which are classified by the self-adjoint extensions [12] of the operator. We are thus naturally led to the study of the self-adjoint extensions of the Calogero model. It may be noted that self-adjoint extensions are known to play important roles in a variety of physical contexts including Aharonov-Bohm effect [13], two and three dimensional delta function potentials [14], anyons [15], anomalies [16], \( \zeta \)-function renormalization [17], particle statistics in one dimension [18] and black holes [19]. Indeed, the self-adjoint extensions of the rational Calogero model in absence of the confining interaction has recently been studied [20].

In this Letter we shall show that the Calogero model in presence of the confining interaction can indeed be consistently quantized with choices of boundary conditions different than what was considered in Ref. [1]. It will be shown that under certain conditions, the corresponding Hamiltonian admits self-adjoint extensions labelled by \( e^{iz} \) where \( z \in \mathbb{R} \, (\text{mod} \, 2\pi) \). The parameter \( z \) classifies the possible boundary conditions for which the Hamiltonian is self-adjoint. In any given situation, the physical interpretation of \( z \) depends on the details of the particular problem [12]. For example, in a description of black holes in terms of the Calogero model [10], \( z \) is related to the mass and entropy of the black hole [19]. As another example, in the realization of generalized exclusion statistics within the framework of Calogero model [3], \( z \) is related to the statistical parameter.

As a consequence of the self-adjoint extensions, we get a new class of bound states for the Calogero model. Unlike the known spectrum obtained by Calogero [1], we find infinitely many energy states which are not equispaced except for special values of \( z \). Moreover, the spectrum in general includes a single negative energy bound state. This is the first time in the literature that the existence of non-equispaced energy levels with a single negative energy state have been found for the the rational Calogero model. The spectrum depends explicitly on the value of \( z \), leading to inequivalent quantizations of this system.
The Hamiltonian of the rational Calogero model is given by

\[ H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \left[ \frac{a^2 - \frac{1}{4}}{(x_i - x_j)^2} + \frac{\Omega^2}{16}(x_i - x_j)^2 \right] \]  

(1)

where \(a, \Omega\) are constants, \(x_i\) is the coordinate of the \(i\)th particle and units have been chosen such that \(2m\hbar^{-2} = 1\). We are interested in finding normalizable solutions of the eigenvalue problem

\[ H\psi = E\psi. \]  

(2)

Following [1], we consider the above eigenvalue equation in a sector of configuration space corresponding to a definite ordering of particles given by \(x_1 \geq x_2 \geq \cdots \geq x_N\). The translation-invariant eigenfunctions of the Hamiltonian \(H\) can be written as

\[ \psi = \prod_{i<j} (x_i - x_j)^{a + \frac{1}{2}} \phi(r) P_k(x), \]  

(3)

where \(x \equiv (x_1, x_2, \ldots, x_N)\),

\[ r^2 = \frac{1}{N} \sum_{i<j} (x_i - x_j)^2 \]  

(4)

and \(P_k(x)\) is a translation-invariant as well as homogeneous polynomial of degree \(k \geq 0\) which satisfies the equation

\[ \left[ \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \frac{2(a + \frac{1}{2})}{(x_i - x_j) \partial x_i} \right] P_k(x) = 0. \]  

(5)

The existence of complete solutions of (5) has been discussed by Calogero [1]. Substituting Eqn. (3) in Eqn. (2) and using Eqns. (4-5) we get

\[ \tilde{H}\phi = E\phi, \]  

(6)

where

\[ \tilde{H} = \left[ -\frac{d^2}{dr^2} - (1 + 2\nu)\frac{d}{dr} + w^2 r^2 \right] \]  

(7)

with \(w^2 = \frac{1}{8}\Omega^2 N\) and

\[ \nu = k + \frac{1}{2}(N - 3) + \frac{1}{2}N(N - 1)(a + \frac{1}{2}). \]  

(8)

\(\tilde{H}\) is the effective Hamiltonian in the “radial” direction. Following [20], it can be easily shown that \(\phi(r) \in L^2[\mathbb{R}^+, d\mu]\) where the measure is given by \(d\mu = r^{1+2\nu} dr\).

The Hamiltonian \(\tilde{H}\) is a symmetric (Hermitian) operator on the domain \(D(\tilde{H}) \equiv \{\phi(0) = \phi'(0) = 0, \ \phi, \ \phi' \text{ absolutely continuous}\}\). To determine whether \(\tilde{H}\) is self-adjoint [12] in \(D(\tilde{H})\), we have to first look for square integrable solutions of the equations

\[ \tilde{H}^*\phi_{\pm} = \pm i\phi_{\pm}, \]  

(9)

where \(\tilde{H}^*\) is the adjoint of \(\tilde{H}\) (note that \(\tilde{H}^*\) is given by the same differential operator as \(\tilde{H}\) although their domains might be different). Let \(n_+, n_-\) be the total number of square-integrable, independent solutions of (9) with the upper (lower) sign in the right hand side. Now \(\tilde{H}\) falls in one of the following categories [12]:

1) \(\tilde{H}\) is (essentially) self-adjoint iff \((n_+, n_-) = (0, 0)\).
2) \(\tilde{H}\) has self-adjoint extensions iff \(n_+ = n_- \neq 0\).
3) If \(n_+ \neq n_-\), then \(\tilde{H}\) has no self-adjoint extensions.
The solutions of Eqn. (9) are given by

$$\phi_{\pm}(r) = e^{-\frac{wr^2}{2}}U(d_{\pm}, c, wr^2),$$  \hspace{1cm} (10)

where $d_{\pm} = \frac{1+\nu}{2} + \frac{i}{2m}, \ c = 1+\nu$ and $U$ denotes the confluent hypergeometric function of the second kind \[21\]. The asymptotic behaviour of $U$ \[21\] together with the exponential factor in Eqn. (10) ensures that $\phi_{\pm}(r)$ vanish at infinity. The solution in Eqn. (10) have different short distance behaviour for $\nu \neq 0$ and $\nu = 0$. From now onwards, we shall restrict our discussion to the case for $\nu \neq 0$, the analysis for $\nu = 0$ being similar. When $\nu \neq 0$, $U(d_{\pm}, c, wr^2)$ can be written as

$$U(d_{\pm}, c, wr^2) = C\left[\frac{M(d_{\pm}, c, wr^2)}{\Gamma(b_{\pm})\Gamma(c)} - (wr^2)^{1-e} \frac{M(b_{\pm}, 2-c, wr^2)}{\Gamma(d_{\pm})\Gamma(2-c)}\right],$$  \hspace{1cm} (11)

where $b_{\pm} = \frac{1+\nu}{2} + \frac{i}{2m}, \ C = \frac{\nu}{\sin(\pi+\nu\pi)}$ and $M$ denotes the confluent hypergeometric function of the first kind \[21\]. In the limit $r \to 0$, $M(d_{\pm}, c, wr^2) \to 1$. This together with Eqns. (10) and (11) implies that as $r \to 0$,

$$|\phi_{\pm}(r)|^2dr \rightarrow \left[A_1r^{(1+2\nu)} + A_2r + A_3r^{(1-2\nu)}\right]dr,$$  \hspace{1cm} (12)

where $A_1, A_2$ and $A_3$ are constants independent of $r$. From Eqn. (12) it is now clear that in the limit $r \to 0$, the functions $\phi_{\pm}(r)$ are not square-integrable if $|\nu| \geq 1$. In that case, $n_+ = n_- = 0$ and $H$ is essentially self-adjoint in the domain $D(H)$. However, if either $0 < \nu < 1$ or $-1 < \nu < 0$, the functions $\phi_{\pm}(r)$ are indeed square-integrable. Thus if $\nu$ lies in these ranges, we have $n_+ = n_- = 1$ and Hamiltonian $\hat{H}$ is not self-adjoint in $D(H)$ but admits self-adjoint extensions. The domain $D_z(H)$ in which $\hat{H}$ is self-adjoint contains all the elements of $D(H)$ together with elements of the form $\phi_+ + e^{iz}\phi_-$, where $z \in R \ (mod\ 2\pi) \ [12]$. We can similarly show that $n_+ = n_- = 1$ for $\nu = 0$ as well. Thus the self-adjoint extensions of this model exist when $-1 < \nu < 1$. It may be noted that the values of $n_+$ and $n_-$ as well as the allowed range of $\nu$ obtained above is the same as that found in Ref. \[20\], which discussed the Calogero model without the confining term. In both these cases, the existence of the self-adjoint extension is essentially determined by the nature of the singularity at $\nu = 0$. However, the domain $D_z(H)$ obtained above is very different from the corresponding domain found in Ref. \[20\]. This is due to the fact that the presence of the confining potential affects the expressions of $\phi_{\pm}(r)$, which in turn determine the allowed domain of $H$. As discussed below, this difference in the structure of the domains leads to a completely different spectrum in presence of the confining potential.

The range of $\nu$ required for the existence of the self-adjoint extension together with Eqn. (8) implies that for given values of $N$ and $k$, $a + \frac{1}{2}$ must lie on the range

$$-\frac{N-1+2k}{N(N-1)} < a + \frac{1}{2} < -\frac{N-5+2k}{N(N-1)}.$$

(13)

For $N \geq 3$, we have the following classifications of the boundary conditions depending on the value of the parameter $a + \frac{1}{2}$.

(i) $a + \frac{1}{2} \geq \frac{1}{2}$: This corresponds to the boundary condition considered by Calogero for which both the wave-function and the current vanish as $x_i \to x_j$. In this case, $\nu > 1$ for all values of $k \geq 0$. The corresponding Hamiltonian is essentially self-adjoint in the domain $D(H)$, leading to a unique quantum theory.

(ii) $0 < a + \frac{1}{2} < \frac{1}{2}$: The wave-function vanishes in the limit $x_i \to x_j$, though the current may show a divergent behaviour in the same limit. Such a boundary condition on the wave-function is quite similar to what one encounters for strongly repulsive $\delta$-function Bose gas. In this case $\nu$ is positive and $k$ must be equal to zero so that $\nu$ may belong to the range $0 < \nu < 1$. The corresponding constraint on $a + \frac{1}{2}$ is given by $0 < a + \frac{1}{2} < \frac{5-3N}{N(N-1)}$, which can only be satisfied for $N = 3$ and $4$.

(iii) $-\frac{1}{2} < a + \frac{1}{2} < 0$: The lower bound on $a + \frac{1}{2}$ is obtained from the condition that the wavefunction be square-integrable. The parameter $a + \frac{1}{2}$ in this range leads to a singularity in the wavefunction resulting from the coincidence of any two or more particles. Using permutation symmetry, such an eigenfunction can be extended to the whole of configuration space, although not in a smooth fashion. The new quantum states in this case exist for arbitrary $N$ and even for non-zero values of $k$. In fact, imposing the condition that the upper bound on $a + \frac{1}{2}$ should be greater than $-\frac{1}{2}$, we find from Eqn. (13) that $k$ is restricted as $k < \frac{1}{5}(N^2 - 3N + 10)$. It can also be shown that there are only two allowed values of $k$ when both $N$ and $a + \frac{1}{2}$ are kept fixed.

In order to determine the spectrum we note that the solution to Eqn. (13) which is bounded at infinity is given by

$$\phi(r) = Be^{-\frac{wr^2}{2}}U(d, c, wr^2),$$

(14)
where \( d = \frac{1+\nu}{2} - \frac{E}{4w} \) and \( B \) is a constant. In the limit \( r \to 0 \),

\[
\phi(r) \to BC \left[ \frac{1}{\Gamma(b)} \frac{1}{\Gamma(c)} - \frac{w^{-\nu}r^{-2\nu}}{\Gamma(d)c(2-c)} \right],
\]

(15)

where \( b = \frac{1-\nu}{2} - \frac{E}{4w} \). On the other hand, as \( r \to 0 \),

\[
\phi_+ + e^{iz} \phi_- \to C \left[ \frac{1}{\Gamma(c)} \left( \frac{1}{\Gamma(b)} + \frac{e^{iz}}{\Gamma(c)} \right) - \frac{w^{-\nu}r^{-2\nu}}{\Gamma(d)c} \left( \frac{1}{\Gamma(b)} + \frac{e^{iz}}{\Gamma(c)} \right) \right].
\]

(16)

If \( \phi(r) \in D_2(\tilde{H}) \), then the coefficients of different powers of \( r \) in Eqs. (15) and (16) must match. Comparing the coefficients of the constant term and \( r^{-2\nu} \) in Eqs. (15) and (16) we get

\[
f(E) \equiv \frac{\Gamma \left( \frac{1+\nu}{2} - \frac{E}{4w} \right)}{\Gamma \left( \frac{1-\nu}{2} - \frac{E}{4w} \right)} = \frac{\xi_2 \cos \left( \frac{\pi}{2} - \eta_1 \right)}{\xi_1 \cos \left( \frac{\pi}{2} - \eta_2 \right)},
\]

(17)

where \( \Gamma \left( \frac{1+\nu}{2} + \frac{i}{4w} \right) \equiv \xi_1 e^{i\eta_1} \) and \( \Gamma \left( \frac{1-\nu}{2} + \frac{i}{4w} \right) \equiv \xi_2 e^{i\eta_2} \). For given values of the parameters \( \nu \) and \( w \), the bound state energy \( E \) is obtained from Eqn. (17) as a function of \( z \). The corresponding eigenfunctions are obtained by substituting \( \phi(r) \) from Eqn. (16) into Eqn. (3). Different choices of \( z \) thus leads to inequivalent quantizations of the many-body Calogero model. Moreover from Eqn. (17) we see that for fixed value of \( z \), the Calogero model with parameters \((w,\nu)\) and \((w,-\nu)\) produces identical energy spectrum although the corresponding wavefunctions are different.

The following features about the spectrum may be noted:

1) We have obtained the spectrum analytically when the r.h.s. of Eqn. (17) is either 0 or \( \infty \). When the r.h.s. of Eqn. (17) is 0, we must have the situation where \( \Gamma \left( \frac{1+\nu}{2} - \frac{E}{4w} \right) \) blows up, i.e. \( E_n = 2w(2n + \nu + 1) \) where \( n \) is a positive integer. This happens for the special choice of \( z = z_1 = \pi + 2\eta_1 \). These eigenvalues and the corresponding eigenfunctions are analogous to those found by Calogero although for a different parameter range. Similarly, when the r.h.s. of Eqn. (17) is \( \infty \), an analysis similar to the one above shows that \( E_n = 2w(2n - \nu + 1) \). This happens for the special value of \( z \) given by \( z = z_2 = \pi + 2\eta_2 \).

2) For choices of \( z \) other than \( z_1 \) or \( z_2 \), the nature of the spectrum can be understood from Figure 1, which is a plot of Eqn. (17) for specific values of \( \nu, z \) and \( w \). In that plot, the horizontal straight line corresponds to the r.h.s. of Eqn. (17). The energy eigenvalues are obtained from the intersection of \( f(E) \) with the horizontal straight line. Note that the spectrum generically consists of infinite number of positive energy solutions and at most one negative energy solution. The existence of the negative energy states can be understood in the following way. For large negative values of \( E \), the asymptotic value of \( f(E) \) is given by \((\frac{E}{4w})^{-\nu} \), which monotonically tends to 0 or \( +\infty \) for \( \nu > 0 \) or \( \nu < 0 \) respectively. When \( \nu > 0 \), the negative energy state will exist provided r.h.s. of Eqn. (17) lies between 0 and \( \frac{\Gamma(\frac{1+\nu}{2})}{\Gamma(\frac{1-\nu}{2})} \). Similarly, when \( \nu < 0 \), the negative energy state will exist when the r.h.s. of Eqn. (17) lies between \( \frac{\Gamma(\frac{1-\nu}{2})}{\Gamma(\frac{1+\nu}{2})} \) and \( +\infty \).
For any given values of $\nu$ and $w$, the position of the horizontal straight line in Fig. 1 can always be adjusted to lie anywhere between $-\infty$ and $+\infty$ by suitable choices of $z$. Thus the spectrum would always contain a negative energy state for some choice of the parameter $z$.

3) Contrary to the spectrum of the rational Calogero model, the energy spectrum obtained from Eqn. (17) is not equispaced for finite values of $E$ and for generic values of $z$. For example, it is seen from Eqn. (17) that the ratio

$$\frac{f(E + 4w)}{f(E)} = \frac{E - 2 + \frac{1-\nu}{2}}{4w + \frac{1}{2}z^2}$$

in general is not unity except when $E \to \infty$. This may seem surprising with the presence of $SU(1,1)$ as the spectrum generating algebra in this system [22], which demands that the eigenvalues be evenly spaced. In order to address this issue, we consider the action of the dilatation generator $D = \frac{1}{2} (r \frac{d}{dr} - \frac{1}{r})$ on an element $\phi(r) = \phi_+ (r) + e^{iz} \phi_- (r)$. In the limit $r \to 0$, we have

$$D \phi = \frac{C}{2} \left[ \frac{1}{\Gamma(c)} \left( \frac{1}{\Gamma(b_+)} + \frac{e^{iz}}{\Gamma(b_-)} \right) - \frac{r^{-2\nu}(1-4\nu)}{\Gamma(2-c)} \left( \frac{1}{\Gamma(d_+)} + \frac{e^{iz}}{\Gamma(d_-)} \right) \right].$$

We therefore see that $D \phi(r) \in D(\tilde{H})$ only for $z = z_1$ or $z = z_2$. Thus the generator of dilatations does not in general leave the domain of the Hamiltonian invariant [14, 16, 20, 23]. Consequently, $SU(1,1)$ cannot be implemented as the spectrum generating algebra except for $z = z_1, z_2$.

4) For $N \geq 3$, the range of $\alpha + \frac{1}{2}$ for which the new quantum states have been found is different from what was used in Ref. [1]. The $N = 2$ Calogero model however admits new quantum states even in the range of $\alpha + \frac{1}{2}$ considered in Ref. [1]. When $N = 2$, $k$ must be equal to zero and Eqn. (8) gives $\nu = \alpha$. In this case, the system therefore admits self-adjoint extensions and new quantum states when $-1 < \alpha < 1$. It may be noted that the eigenvalue problem for $N = 2$ was solved in Ref. [1] with the condition that $\alpha > 0$. Thus when $0 < \alpha < 1$, our analysis predicts a larger family of solutions labelled by the parameter $z$. This set of solutions reduces to that found in Ref. [1] for $z = z_1$.

5) It may be interesting to compare the spectrum obtained above with that found in Ref. [20], where the self-adjoint extension of the Calogero model without the confining term was discussed. In the latter case, the spectrum consists of at most one negative energy bound state and infinite number of scattering states with momentum dependent phase shifts. In the presence of the confining potential, as discussed above, we get at most one negative energy bound state and an infinite number of positive energy bound states which are in general not equispaced. It may also be noted that the spectrum found in Ref. [20] cannot be obtained as the $w \to 0$ limit of that obtained in this paper. This is due to the fact that Eqn. (17), which determines the spectrum in the present case, becomes singular in the $w \to 0$ limit.

In conclusion, we have presented a new quantization scheme for the rational Calogero model. The non-equispaced nature of the energy levels and the existence of a negative energy bound state are some of the salient features that emerge from our analysis. It is expected that the generalized exclusion statistics parameter of this model would be a function of both $z$ and $\nu$, since the energy spectrum depends on these parameters. We can ascertain this for the special case of $z = z_1$ and $z_2$, when Eq. (17) can be solved exactly. The generalized exclusion statistics is believed to play an important role in one dimensional non-fermi liquids as well as in the edge excitations in the fractional quantum Hall effect. Thus, it would be interesting to investigate the generalized exclusion statistics and the thermodynamic properties of Calogero models for arbitrary $z$ with the spectrum as described here.

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[1] F. Calogero, Jour. Math. Phys. 10, 2191 (1969); Jour. Math. Phys. 10, 2197 (1969); Jour. Math. Phys. 12, 419 (1971).
[2] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. 71, 314 (1981); ibid 94, 6 (1983).
[3] M. V. N. Murthy and R. Shankar, Phys. Rev. Lett. 73, 3331 (1994); Z. N. C. Ha, Quantum Many-Body Systems in One Dimension, Series on Advances in Statistical Mechanics, Vol. 12, (World Scientific, 1996); A. P. Polychronakos, hep-th/9902157.
[4] H. Azuma and S. Iso, Phys. Lett. B331, 107 (1994).
[5] N. Kawakami and S.-K. Yang, Phys. Rev. Lett. 67, 2493 (1991).
[6] B. D. Simons, P. A. Lee, and B. L. Altschuler, Phys. Rev. Lett. 72, 64(1994); S. Jain, Mod. Phys. Lett. A11, 1201(1996).
[7] C. W. J. Beenakker and B. Rejaee, Phys. Rev. B49, 7499 (1994); M. Caselle, Phys. Rev. Lett. 74, 2776 (1995).
[8] F. D. M. Haldane, Phys. Rev. Lett. 60, 635 (1988); B. S. Shastry, Phys. Rev. Lett. 60, 639 (1988); A. P. Polychronakos, Phys. Rev. Lett. 70, 2329 (1993).
[9] E. D’Hoker and D. H. Phong, hep-th/9912271; A. Gorsky and A. Mironov; hep-th/0011197 A. J. Bordner, E. Corrigan and R. Sasaki, Prog. Theor. Phys. 100, 1107 (1998).

[10] G. W. Gibbons and P. K. Townsend, Phys. Lett. B454, 187 (1999).

[11] A. P. Polychronakos, Phys. Rev. Lett. 69, 703 (1992); L. Brink, T. H. Hansson and M. A. Vasiliev, Phys. Lett. B286, 109 (1992); N. Gurappa and P. K. Panigrahi, Phys. Rev. B 59, R2490 (1999).

[12] M. Reed and B. Simon, Methods of Modern Mathematical Physics, volume 2, (Academic Press, New York, 1972).

[13] P. Gerbert, Phys. Rev. D 40, 1346 (1989).

[14] R. Jackiw in M.A.B. Beg Memorial Volume, A. Alka and P. Hoo dbhoy, eds. (World Scientific, Singapore, 1991).

[15] C. Manuel and R. Tarrach, Phys. Lett. B 268, 222 (1991); M. Bourdeau and R. D. Sorkin, Phys. Rev. D 45, 687 (1992).

[16] J. G. Esteve, Phys. Rev. D 34, 674 (1986); J. G. Esteve, Phys. Rev. D 66, 125013 (2002).

[17] H. Falomir, P. A. G. Pisani and A. Wipf, Jour. Phys. A 35, 5427 (2002).

[18] C. Aneziris, A. P. Balachandran and Diptiman Sen, Int. Jour. Mod. Phys. 6, 4721 (1991).

[19] T. R. Govindarajan, V. Suneeta and S. Vaidya, Nucl. Phys. B583, 291 (2000); D. Birmingham, Kumar S. Gupta and Siddhartha Sen, Phys. Lett. B505, 191 (2001); Kumar S. Gupta and Siddhartha Sen, Phys. Lett. B 526, 121 (2002); Kumar S. Gupta, hep-th/0204137.

[20] B. Basu-Mallick and K. S. Gupta, Phys. Lett. A292, 36(2001); B. Basu-Mallick, P. K. Ghosh and K. S. Gupta, hep-th/0207040 (to appear in Nucl. Phys. B.)

[21] Handbook of Mathematical Functions, M. Abromowitz and I. A. Stegun (Dover Publications, New York, 1974).

[22] V. de Alfaro, S. Fubini and G. Furlan, Nuovo Cim. 34A, 569 (1976).

[23] E. D’Hoker and L. Vinet, Comm. Math. Phys. 97, 391 (1985).