A wavelet-based scheme for optimum measurement /monitoring of dynamic responses of structures

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Abstract. This paper presents an optimum measurement for the dynamic responses of structures from the operation of integration to respective derivation of acceleration’s data using free-scaled wavelet functions. For this purpose, the numerical approach of integration and derivation has been developed for displacement measurement or determination of the third-order derivative (known as quantity of the jerk) from acquired accelerations. A simple improved algorithm is developed in order to optimally measure the dynamic quantities particularly, using Chebyshev and Haar wavelet functions. It is deduced that, stable measurement of dynamic quantities is independently achieved from the structural materials through a satisfactory optimum algorithm; that is capable of online monitoring, while emphasizing on maximum accuracy of the measurement with less computational time.

1. Introduction
In general, the design of mechanical systems is widely influenced by significant innovations in design technology. One of the most affected structural systems has emerged in civil engineering structures, which are increasingly being designed for optimized behaviours and optimum materials to achieve aesthetic and economic benefits. In fact, most of the time, various natural and environmental situations such as earthquakes, wind and tsunamis govern on these structures. As a result, the aforementioned structures are often subjected to abnormal ultimate loadings and may even collapse under such unexpected loadings that were not anticipated at the design stage. Therefore, it is indispensible to evaluate and monitor the structural reliability, integrity and safety [1].

Fundamentally, the main parameter being evaluated for the integrity analysis of a structure is displacement, however, several items of dynamic responses can be monitored. This is because the most important indicators of structural behaviour are short-term and long-term displacements, thus the integrity of structures can be examined for diverse options using displacement quantities. Accordingly, a sufficiently accurate displacement measurement is required to ensure the reliable monitoring of a structure’s situation [2,3]. In addition, for many small-scaled mechanical systems, the precise measurement of other dynamic responses, e.g., velocity or the derivative of acceleration in respect to time (jerk) is essential for an engineering reliability analysis. Practically, the accelerometer is a widely utilized tool for measuring structural accelerations and displacements [4]. For instance, while the displacement is acquired by processing the acceleration data from the accelerometer, and the dynamic displacement and velocity seem precise enough, the process of filtering of pure accelerations can be implemented one step before the displacement measurement [5].

Recently, wavelet solution techniques have attracted much attention in different branches of science and technology. Basically, multi-resolution analysis and preserving time information during time-
scaled-frequency analysis constitute the value of wavelet transforms in contrast to the well-known Fourier analysis [6-10]. Accordingly, wavelet functions have been utilized to numerically solve either ordinary or partial differential equations (ODEs, PDEs). Therefore, it is introduced as a powerful tool in structural dynamics. Moreover, free scales of wavelet functions have been implemented on the direct time history analysis of 2-dimensional 2D frames to calculate rotational and transitional displacements [11]. Furthermore, the direct approach (not inverse scheme) to structural dynamics of framed structures involving 3-dimensional (3D) transitional degrees of freedom has benefitted from using wavelet functions [12]. It was concluded that, as an indirect time integration method the dynamic analysis was optimally accomplished with the maximum accuracy of responses. The emphasis was on de-noising the input data and numerical integration scheme, simultaneously [12]. So far, the implementation of wavelet functions for the solution of invers problems through the aforementioned procedure is not received in the literature.

This paper is organized as follows. The next section provides a brief review of the background of wavelet, which includes 3D and complex Chebyshev wavelet functions and 2D and simple Haar wavelet basis. In Section 3, implementation of the proposed approach is considered for two main operations. For this aim, the proposed scheme is firstly employed for operation of integration to compute velocities and displacements from the acceleration data. Secondly, it is explored for operation of the derivation to calculate the third-ordered derivative of displacement in respect to time (namely, jerk). Finally, in Section 4, the effectiveness of the proposed scheme is experimentally evaluated.

2. Fundamentals of Chebyshev and Haar wavelet functions

As a prefix to this section, the basic notations and definitions of wavelet theory, continuous wavelet transforms (CWT) and characteristics of Haar and Chebyshev wavelet may be found in Refs. [6-16]. In this study, 2M is presented for the number of segmentations (wavelet collocation points) in each global time interval regarding to the proposed scales of wavelet (refers to the segmentation method, SM). Each function f(x) can be decomposed with wavelet transforms as [6-16]:

\[
 f(x) \approx \sum_{n=1}^{2M} \sum_{m=0}^{M-1} C_{n,m} \psi_{n,m}(x) = C^T \psi(x)
\] (1)

where, vectors C and \( \psi(x) \), indicate expansion coefficients and wavelet coefficients, respectively. \( m \) is the degree of polynomials and \( n \) denotes the considered scale of wavelet. Subsequently, 2M squared matrix of Wavelet coefficients (\( \psi(x) \)) are calculated for 2M local nodes \( x_i \) defined as follows [11,12]:

\[
x_i = (2M^{-1})(l - 0.5), \quad l = 1, 2, 3, ..., 2M
\] (2)

It should be noted that, for 2D wavelets, the second expansion in equation (1) is ignored \( (m) \). Eventually, to calculate integration of \( f(x) \), the product matrix of integration (P) is operated on right side of equation (1) [6-11].

3. Implementation of the proposed scheme

3.1. Measurement of displacements from accelerations

A linear dynamic equilibrium governing on corresponding mass (\( M^* \)), damping (\( C^* \)) and stiffness (\( K^* \)) of a structure can be expressed as [1,12]:

\[
 M^* \dddot{u}(t) + C^* \ddot{u}(t) + K^* u(t) = F(t)
\] (3)

where, \( \dddot{u}(t) \), \( \ddot{u}(t) \) and \( u(t) \) represent the vectors of acceleration, velocity and displacement of each degree of freedom (DOF) existing in the structure. Furthermore, \( F(t) \) indicates the vector of nodal lateral loads applied to the degrees of freedom. In this study, converse to Mahdavi and Razak [11,12], where the direct dynamic analysis was considered, the \( \dddot{u}(t) \) is assumed to be the known accelerometer’s data (e.g., acquired with piezoelectric sensors). Thus, the unit of input data follows acceleration (\( \text{m/sec}^2 \)). To numerically approximate velocities and displacements we first start to decompose the input data \( (\dddot{u}(t)) \) on the 2M local points of the wavelet basis, with emphasis on the scale of wavelet corresponding to the...
required accuracy of frequency decomposition of the input data. According to equation (1), the known
vector of the accelerometer’s data is numerically approximated using wavelet functions corresponding
to the local fixed 2M points as follows (by any basis function):
\[ \ddot{v}(t) = C^T \psi(t) \]  (4)

Note that, \( \psi(t) \) indicates the corresponding wavelet coefficients matrix. For instance, for Haar
wavelet the \( h(t) \) will be replaced in equations. Accordingly, the coefficient vector corresponding to the
decomposed accelerations is obtained as:
\[ C_{1 \times 2M} = \ddot{u}(t)_{1 \times 2M}(\phi_{2M \times 2M})^{-1} \]  (5)

The vector of velocity is therefore approximated on the global time, multiplying by the operation of
integration (P), as follows (\( d_t = t_{i+1} - t_i \)):
\[ \ddot{u}(t) = d_t C^T P \psi(t) + v_n \]  (6)

Eventually, displacements are numerically expanded as follows:
\[ u(t) = d_t^2 C^T p^2 \psi(t) + u_n \]  (7)

To transfer numerical calculations from 2M local fixed points of the wavelet window to the 2M global
points of the considered time interval, \( d_t \) is operated as a coefficient of the aforesaid transformation. In
other words, quantities of the dynamic system are modified corresponding to local times as:
\[ \ddot{u}(t) = d_t v \]  (8)
\[ \ddot{u}(t) = d_t F(t_n + d_t \tau, u, v) \]  (9)

Moreover, in equations (6) and (7), \( v_n \) and \( u_n \) represent the initial conditions of each global time,
approximated by the wavelet as a constant value. For this aim, the unity is being approximated by the
Chebyshev wavelet as:
\[ 1 \equiv \ddot{I} T \psi(t) \equiv \left( \frac{\sqrt{\pi}}{2} \right) \left[ 1,0,0,\ldots,1,0,0,\ldots \right] \psi(t) \]  (10)

Or alternatively is obtained with Haar wavelet as:
\[ 1 \equiv \ddot{I} h(t) \equiv \left[ 1,0,0,\ldots,0,0,0,\ldots \right] h(t) \]  (11)

Therefore, the initial quantities are numerically developed as:
\[ v_n = S_1^T \psi(t) \]  (12)
\[ u_n = S_2^T \psi(t) \]  (13)

where, \( S_1^T \) and \( S_2^T \) are \( 2M \times 1 \) dimension vectors that may be obtained by Chebyshev wavelet as
follows:
\[ S_1^T \equiv v_n(0) \left( \frac{\sqrt{\pi}}{2} \right) \left[ 1,0,0,\ldots,1,0,0,\ldots \right]^T \]  (14)
\[ S_2^T \equiv u_n(0) \left( \frac{\sqrt{\pi}}{2} \right) \left[ 1,0,0,\ldots,1,0,0,\ldots \right]^T \]  (15)

Substituting equations (14) and (15) into equations (6) and (7) the vectors of velocity and
displacement are defined as:
\[ \ddot{u}(t) = d_t C^T P \psi(t) + S_1^T \psi(t) \]  (16)
\[ u(t) = d_t^2 C^T p^2 \psi(t) + d_t S_1^T P \psi(t) + S_2^T \psi(t) \]  (17)

The robustness of the proposed method is demonstrated through the two steps of computation; firstly,
the quantities of displacements and velocities have been independently approximated from the mass,
damping, stiffness and applied loadings of the considered system. Secondly, the accelerometers’ data
containing wide-frequency contents may be decomposed on the specific range of frequencies.

3.2. Measurement of derivative of accelerations with respect to time
Physically, the derivative of acceleration in respect to time introduces the quantity of jerk in the
 corresponding DOF. In following section, the proposed scheme is improved for approximation of the
derivatives using free-scaled wavelet functions. For this purpose, integral functions are numerically
developed from local coordinates to the global system. For a differentiable function of $F(t) \in [t_n, t_{n+1}]$, calculation of integral on the interval of integration is considered based on Newton’s theorem as follows:

$$f(t) = f(t_n) + \int_{t_n}^{t} f'(t) \, dt$$

(18)

Equivalently, from our point of consideration we have:

$$\dddot{u}(t) = \dddot{u}(t_n) + \int_{t_n}^{t} \dddot{u}(t) \, dt$$

(19)

For the first step $\dddot{u}(t) = \dddot{u}(t_{n+1})$ is initialized for a given $t_{n+1}$; Using equation (1) to approximate derivative function on global coordinates yields:

$$\dddot{u}(t) = C^T \psi(t)$$

(20)

Substituting into equation (19) yields:

$$\dddot{u}(t_{n+1}) = \dddot{u}(t_n) + \int_{t_n}^{t_{n+1}} C^T \psi(t) \, dt$$

(21)

Multiplying by operational matrices and adding initial constant of integration, we have:

$$\dddot{u}(t_{n+1}) = \dddot{u}(t_n) + d_{C} C^T \psi(t) + \dddot{u}(t_0)$$

(22)

Using the idea proposed in equations (10), (14) and (15), constant quantities are also approximated by the corresponding wavelet in each step (Haar wavelet or Chebyshev wavelet). After substituting the initial quantities, equation (22) is developed as follows:

$$\dddot{u}(t_{n+1})I^T \psi(t) = \dddot{u}(t_n)I^T \psi(t) + d_{C} C^T P \psi(t) + \dddot{u}(t_0)I^T \psi(t)$$

(23)

Subsequently, eliminating $\psi(x)$ from the both sides of equation (23) and after a set of algebraic calculations $C^T$ is calculated. Accordingly, using equation (20) the 3rd ordered derivative of displacement (known as the jerk quantity) is numerically calculated.

4. Experimental verification

A simple SDOF model evaluated as an actual experiment under a regular hammer test. For the considered system, a comprehensive program code was codified in MATLAB, to measure online the dynamic responses. Furthermore, $g = 9.81 \text{ m/sec}^2$ is presumed as the ground acceleration. As shown in Figure 1, the considered system including the two narrow columns are made of aluminum and a rigid Plexiglas, which are only vibrating in the x direction. To invoke only one DOF, the thickness of the columns was chosen to be much narrower than its width ($\equiv 1/37$). The geometry and details of the experiment setup are shown in the figure. As illustrated in the figure, a non-contact laser device was installed on the base in order to identify the accurate time history of displacements (sampling rate 102.4).

![Non-contact laser sensor to pick displacements](image)

**Figure 1.** The ideal SDOF experiment setup and the schematic view, measuring displacements by laser sensor.

Moreover, two piezoelectric sensors (accelerometers) were embedded on the DOF of the system. To evaluate the proposed method, a hammer strikes the structure from 1sec after its initial condition.
Accordingly, the time history of the velocity corresponding to the SDOF has been computed for the 4th scale of Haar and Chebyshev wavelet (designated by Cal-Vel-Cheby and Haar) for the time interval of \(d_t=0.05\) sec as depicted in Figure 2. The results were compared against those picked by the non-contact laser sensor for \(d_t=0.009766\) sec (designated by Orig-Vel).

![Figure 2](image2.png)

**Figure 2.** Time history of velocity for SDOF system shown in Figure 1. (a) Calculated velocity using the 4th scale of Chebyshev wavelet (Cal-Vel-Cheby). (b) Calculated velocity using the 4th scale of Haar wavelet (Cal-Vel-Haar).

To clarify the discrepancy between the computed results through the proposed method, approximation of the original acceleration (accelerometer data) was plotted corresponding to the 4th scale of Haar and Chebyshev wavelet by illustrating the wavelet coefficients for 1-1.2 sec.

![Figure 3](image3.png)

**Figure 3.** Coefficients of wavelet for the 4th scale of (a) ChebyW-Coefficients=Chebyshev wavelet (b) HaarW-Coefficients=Haar wavelet (App-Acc= approximation of original acceleration).

Figures 2 and 3 clearly demonstrate the source of significant errors of Haar wavelet (2M=4) for \(d_t=0.05\) sec; while it is shown that details of the acceleration are accurately captured by the coefficient of Chebyshev wavelet for the same conditions, therefore, a sufficiently precise approximation is satisfied at the first step of the proposed scheme. Furthermore, to evaluate the practice of various scale of Haar wavelet in detail, computed responses for the same \(d_t=0.05\) sec and 8th, 32nd and 64th scales of simple...
Haar wavelet (designated by HA(2M)) were compared with the 4th scale of complex Chebyshev wavelet. The results depicted in Figure 4 are from the computational time involved and percental total average error (PTAE) point of view.

![Figure 4](image)

**Figure 4.** Computational time and PTAE for various scale of Haar (HA) and the 4th scale of Chebyshev wavelet (Cheby).

It is clearly distinguishable from Figure 4 that for the accurate scales of 2D and simple Haar wavelet (2M64) the cost of analysis is dramatically increased; however, the optimum results can be computed even using a low scale of 3D and complex Chebyshev wavelet. Consequently, it is shown that practice of Haar wavelet is not applicable for the online measurement of dynamic responses, however, for an initial evaluation it may be considered where the fastest computation is satisfied by implementation of this basis function. Eventually, the time history of displacement of the only DOF is optimally calculated through the proposed optimal measurement scheme, particularly, using a long time interval of \( \Delta t = 0.05 \) sec by the 4th scale of Chebyshev wavelet (designated by Cal-Disp-Cheby) and is compared with those measured by the laser sensor (Orig-Disp) for the sampling rate of 102.4 (shown in Figure 5).

![Figure 5](image)

**Figure 5.** The first 6 sec time history displacement of SDOF system shown in Figure 1 (Cal-Disp-Cheby = Calculated displacement using Chebyshev wavelet, Orig-Disp=Original displacement).

The capability of the proposed algorithm of measurement is overtly illustrated in Figure 5, using low scales of Chebyshev wavelet. It should be pointed out again that details of the broad-frequency component acceleration are completely captured by the diverse scale of this comprehensive wavelet function in contrast to the simple Haar wavelet. Finally, the third derivative of displacement in respect to time namely, Jerk \( \dddot{u}(t) \) is considered using the 8th scale of Chebyshev wavelet (designated by computed jerk), as depicted in Figure 6 and compared with a normal incremental calculation \( \frac{d(Acc)}{dt} \) for the long interval of \( \Delta t = 0.5 \) sec. Note that the quantity of jerk is often considered for very small scaled or flexible systems. Moreover, measurement of this value is very helpful to make an engineering judgment on the behaviour of a dynamic system.
Figure 6 shows that the proposed method optimally and exactly detects the time of the strike at the first second in contrast with the normal incremental calculation. This is because the time interval of \(d_t=0.5 \text{ sec} \) \(\left( d_n=d_t \text{ for incremental computations} \right) \) does not allow an incremental approach to detect an abrupt derivative of acceleration in respect to time. On the other hand, the 8th scale of Chebyshev wavelet, firstly, saves an accurate decomposition of the broad-frequency of acceleration even for \(d_t=0.5 \text{ sec} \) for the collaborative collocation points, and, secondly, approximates the sharp incremental acceleration. Overall, it is demonstrated that the quantity of jerk as one of the reliable criteria of dynamic behaviour is optimally measured through the proposed which is capable of optimal and online measurement of dynamic responses.

5. Conclusions

This paper introduces a new numerical integration and derivation scheme to optimally measure the dynamic responses of structures in order to monitor structural integrity and reliability. A straightforward algorithm is implemented to measure particularly, displacements with high precision by tracking the existing acceleration data on the structure without requiring the installation of special and extra devices of displacement measurement with very expensive costs and difficult access (i.e., laser sensors or displacement transducers). It is shown that, the dynamic responses are independently obtained from the behavior of the materials, e.g., mass, damping or stiffness. Furthermore, it is demonstrated that this method is also capable of online monitoring in which the optimum and stable calculation have been conducted from de-noising and numerical integration or derivation in a similar sense of analysis for small sampling rates. However, in this study, the practice of free-scaled and simple 2D Haar wavelet and complex and 3D Chebyshev wavelet have been comparatively considered; the implementation of several basis functions is recommended for further study to achieve an adaptive measurement scheme.

Acknowledgement

The authors wish to acknowledge financial supports by the University of Malaya (UM) and Ministry of Education Malaysia (Grant No.: PG078/2013B and UM.C/625/1/HIR /MOHE/ENG/55).

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