Is zero-point energy physical? A toy model for Casimir-like effect

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Abstract

Zero-point energy is generally known to be unphysical. Casimir effect, however, is often presented as a counterexample, giving rise to a conceptual confusion. To resolve the confusion we study foundational aspects of Casimir effect at a qualitative level, but also at a quantitative level within a simple toy model with only 3 degrees of freedom. In particular, we point out that Casimir vacuum is not a state without photons, and not a ground state for a Hamiltonian that can describe Casimir force. Instead, Casimir vacuum can be related to the photon vacuum by a non-trivial Bogoliubov transformation, and it is a ground state only for an effective Hamiltonian describing Casimir plates at a fixed distance. At the fundamental microscopic level, Casimir force is best viewed as a manifestation of van der Waals forces.

Keywords: zero-point energy; vacuum; Casimir effect; van der Waals force
So, in the discussion session after Casimir’s lecture I switched topic and asked: “Is the Casimir effect due to the quantum fluctuations of the electromagnetic field, or is it due to the van der Waals forces between the molecules in the two media?” Casimir’s answer began, “I have not made up my mind.”

(I.H. Brevik, from the Foreword in [16].)

1 Introduction

In physics we measure energy differences, not absolute energies. Zero-point energy, that is energy of the ground state, is therefore unphysical and can be removed by a simple subtraction. Yet Casimir effect [1], in its simplest form described as an attractive force between electrically neutral plates, is often presented as a demonstration that zero-point energy is physical [2, 3, 4, 5, 6, 7, 8, 9]. On the other hand, Casimir effect can also be explained as a force that originates from van der Waals forces [10, 11, 12, 13, 14, 15, 16, 17, 18], without referring to zero-point energy. There are arguments that the explanation in terms of van der Waals forces is more fundamental [19, 20] (see also [16, 17]), yet some consider the question of the true nature of Casimir effect as a controversy [21, 22, 23] that still needs to be resolved.

A part of the difficulty stems from the fact that explanation of Casimir effect requires quantum electrodynamics (QED), which involves various technical difficulties coming from the fact that QED is a quantum field theory, i.e. a theory with an infinite number of degrees of freedom. To overcome this technical difficulty, in this paper we shall not be so much concerned with technical details of Casimir effect itself. Instead, our main goal will be to understand in detail how a Casimir-like effect can emerge in general, using only general properties of quantum mechanics. For that purpose we shall study a toy model with only 3 degrees of freedom, which under certain approximations can be reduced to 2 or even 1 degree of freedom. The toy model will be chosen such that it has many conceptual similarities with the real Casimir effect, but is technically much simpler than that. This will enable us to understand relatively easily where the effect comes from, and how is it related to the zero-point energy. In addition, to make a contact with actual Casimir physics, we shall discuss various aspects of Casimir effect at a qualitative non-technical level.

The paper is organized as follows. In Sec. 2 we study various conceptual aspects of Casimir effect. In particular, we explain the difference between ground state and vacuum, point out that Casimir force can be attributed to the interacting vacuum energy but not to the ground state energy, discuss the role of dielectric constant and its relation to van der Waals force, and make a motivation for the toy model discussed in the following sections. In Sec. 3 we introduce our toy model and analyze its classical properties. In Sec. 4 we explain how the toy model leads to a quantum Casimir-like force, using both a Casimir-like and a Lifshitz-like approach. In Sec. 5 we explore the content of the interacting vacuum by using the Bogoliubov-transformation method. In Sec. 6 we discuss how the calculations in our toy model are related to the calculations for real Casimir effect. Finally, the conclusions are drawn in Sec. 7.
As a disclaimer, we also want to remark that in this paper we do not study the relevance of ground-state energy to gravitational physics. For a possible relevance of Casimir energy to gravitational phenomena see e.g. [24] and references therein.

2 Basic conceptual questions

One of the main messages of this paper, in agreement with [19, 20], is that, at the fundamental microscopic level, Casimir effect should be viewed as a manifestation of van der Waals forces, and not as a manifestation of zero-point energy. But why then the effect is often attributed to zero-point energy and why such a description works fine too? And what exactly are drawbacks of the zero-point energy description? In this section we give a non-technical answer to those and many other related conceptual questions.

2.1 What is vacuum?

In physics, the word “vacuum” has many different meanings. It can mean a state without any particles whatsoever, or a state without only one kind of particles such as photons, or a state annihilated by some lowering operators, or a local minimum of a classical potential, or the state with the lowest possible energy. Of course, all these notions of “vacuum” are closely related, but the point is that they are not strictly identical.

Which of those notions of “vacuum” is relevant for the description of Casimir effect? Clearly, Casimir vacuum is not a state without any particles whatsoever, because Casimir effect involves plates made of atoms. As we shall see, Casimir vacuum is also not a local minimum of a classical potential, and perhaps more surprisingly, not even a state without photons. We shall see that Casimir vacuum is a state annihilated by some lowering operators which are not photon lowering operators.

2.2 Is Casimir vacuum a ground state?

A ground state is a state with the lowest possible energy, and energy can be defined as an eigenvalue of a Hamiltonian. But which Hamiltonian? There are many different Hamiltonians used in physics, and hence there are many different notions of “energy”. As a consequence, there are many different notions of “ground state”. The notion of a ground state has no meaning at all if one does not specify the Hamiltonian.

Some Hamiltonians are meant to be fundamental, describing “all” physics, or at least a large part of physics. Other Hamiltonians are merely effective Hamiltonians, describing only a small subset of all physical phenomena. Consequently, some ground states are supposed to be fundamental, while other ground states are merely effective ground states.

The Casimir vacuum is one such effective ground state. It is the lowest energy state for the system with Casimir plates at a fixed distance $y$. The existence of the attractive Casimir force implies that plates separated by smaller distance $y$ have smaller energy,
so energy of plates at a given non-zero distance \( y \) cannot be the lowest possible energy. Hence, Casimir vacuum is not the ground state for the Hamiltonian describing the change of \( y \). Furthermore, a state without any plates whatsoever has even smaller energy, so Casimir vacuum is certainly not the ground state for the fundamental Hamiltonian that describes all possible physical states, including those with no plates at all. Casimir vacuum is the ground state for an effective Hamiltonian, a Hamiltonian that describes only those phenomena for which (i) the existence of Casimir plates is given and (ii) the distance \( y \) between the plates is fixed.

### 2.3 Can Casimir force be explained in terms of ground-state energy?

No. By the Newton second law, the force in the \( y \)-direction creates acceleration \( \ddot{y}(t) \). Hence, to describe Casimir force, \( y \) needs to be treated as a dynamical variable, not as a fixed parameter. Parameter \( y \) cannot simultaneously be fixed and non-fixed. If it is fixed then Casimir energy can be interpreted as an effective ground-state energy as explained in Sec. 2.2, but in that case there is no Casimir force. If it is not fixed then there is Casimir force, but in that case Casimir energy cannot be interpreted as an effective ground-state energy. To make \( y \) dynamical, one must construct a new Hamiltonian by adding the appropriate kinetic term to the effective Hamiltonian for fixed \( y \). The Casimir energy is not the ground state for the new Hamiltonian because, when \( y \) is dynamical, energy can be further lowered by decreasing \( y \). So to describe Casimir force by a Hamiltonian, Casimir energy cannot be a ground-state energy for that Hamiltonian.

### 2.4 What has Casimir effect to do with vacuum energy?

Many physical systems can be approximated by a series of harmonic oscillators. For such systems the quantum Hamiltonian takes the form

\[
H = \sum_k \hbar \omega_k \left( a_k^\dagger a_k + \frac{1}{2} \right),
\]

where \( a_k^\dagger \) and \( a_k \) are raising and lowering operators satisfying \([a_k, a_{k'}^\dagger] = \delta_{kk'}\). Defining the vacuum \(|0\rangle\) as the state annihilated by the lowering operators, \( a_k |0\rangle = 0 \), we see that vacuum is also the ground state for the Hamiltonian \( (1) \). The corresponding vacuum energy is

\[
E_{\text{vac}} = \langle 0 | H | 0 \rangle = \sum_k \frac{\hbar \omega_k}{2}.
\]

Now assume that, for some reason, \( \omega_k \) depend on some variable \( y \). And assume that \( y \) is a dynamical variable, i.e. the canonical position of some Hamiltonian (say \( H + p_y^2/2M \)) that depends on the canonical momentum \( p_y \). Furthermore, assume that all the dependence on \( y \) comes from \( (1) \). If these assumptions are fulfilled, then the classical Hamilton equation of motion is

\[
\frac{dp_y}{dt} = -\frac{\partial H}{\partial y},
\]
so in the quantum case we can calculate the average force as

\[ F = \langle \psi \left| -\frac{\partial H}{\partial y} \right| \psi \rangle. \]  (4)

If \( y \) can be treated as a macroscopic classical variable, then (4) can be approximated by

\[ F = -\frac{\partial}{\partial y} \langle \psi | H | \psi \rangle. \]  (5)

Finally if \( |\psi\rangle = |0\rangle \), then (5) and (2) give

\[ F = -E'_{\text{vac}}(y) = -\sum_k \frac{\hbar \omega'_k(y)}{2}, \]  (6)

where the prime denotes derivative with respect to \( y \).

Casimir effect can be thought of as an application of (6) to the case where \( y \) is the distance between electrically neutral plates.

2.5 Where does the dependence on \( y \) come from?

In the original analysis [1] Casimir considered an idealized situation in which the plates are made of a perfect conductor with infinite conductivity. (Such an idealized situation is often the only situation considered in textbooks [3, 6, 7, 8, 9].) The electric field vanishes inside the perfect conductor, so electric field between the plates must satisfy a boundary condition that enforces the field to vanish at the plates. Hence the Fourier expansion of the field does not contain wave vectors \( k \) for which \( k_y \) does not satisfy \( k_y = n\pi/y \) (for \( n = 1, 2, 3, \ldots \)). Therefore the frequencies \( \omega_k \) with \( k_y \neq n\pi/y \) do not contribute to (1) and (2). (By Maxwell equations [25] the electric field alone does not oscillate, so the existence of frequency really means that we deal with the electromagnetic field.) For reasons which will become clear soon, instead of saying that those frequencies do not contribute, it is much more appropriate to say that those frequencies are zero. Therefore we can write

\[ \omega_k = \begin{cases} c|k| & \text{for } k_y = n\pi/y, \\ 0 & \text{for } k_y \neq n\pi/y, \end{cases} \]  (7)

where \( c \) is the velocity of light. In this way we see that \( \omega_k \) depend on \( y \).

Of course, realistic materials are usually not perfect conductors. In materials with finite conductivity the electric field does not vanish, so boundary conditions do not remove wave vectors with \( k_y \neq n\pi/y \). A realistic material can be described by a finite dielectric constant \( \epsilon \). The frequency depends on \( \epsilon \) as \( \omega_k = c|k|/\sqrt{\epsilon} \) [25]. In the vacuum between the plates we have \( \epsilon = 1 \), while inside the plates we have \( \epsilon \neq 1 \). Hence \( \epsilon \) is really a function of position which, when Fourier transformed, becomes a function of \( k \) parameterized by the distance \( y \). In this way, instead of (7) we have

\[ \omega_k = \frac{c|k|}{\sqrt{\epsilon_k(y)}}. \]  (8)
That is where the dependence on $y$ comes from in the case of realistic materials.

For perfect conductors the dielectric constant is infinite, so $\epsilon_k(y) \to \infty$ for some values of $k$. In this way (8) contains (7) as a special case. If we simply ignored wave vectors with $k_y \neq n\pi/y$ in (7), then we could not see the relation with the realistic case (8).

### 2.6 Where does $\epsilon$ come from?

We have seen that Casimir force is related to the dielectric constant $\epsilon$, which for perfect conductors is infinite. But dielectric constant is a \textit{phenomenological macroscopic} quantity, so any description of Casimir effect based on $\epsilon$ lacks the fundamental microscopic origin of the effect. A \textit{fundamental microscopic} explanation of Casimir effect must involve a microscopic explanation of $\epsilon$. Even for perfect conductors, where Casimir effect can be explained by vanishing electric field in the conductor, one needs to understand the \textit{microscopic mechanism} by which the field vanishes.

To understand the microscopic origin of $\epsilon$, it is crucial to have in mind that electrically neutral materials are made of particles which are \textit{electrically charged}. When electric field is applied to the material, the charges within the material rearrange their positions. As a consequence, the local charge density $\rho(x)$ is no longer zero everywhere, despite that fact that the total charge $\int d^3x \rho(x)$ vanishes. This means that the electric field induces polarization $P(x)$ – the electric dipole moment per volume. The polarization itself creates additional electric field, so the equations that govern the full electric field $E(x)$ and polarization $P(x)$ become somewhat complicated. It turns out [25] that it is simpler to describe the system in terms of the so-called electric displacement field $D(x)$ defined as

$$D = E + P. \quad (9)$$

(For the sake of notational simplicity, we use units in which permittivity of the vacuum is $\epsilon_0 = 1$.) Since $P$ is induced by the electric field, it is often a good approximation that $P$ is proportional to $E$ [25]. Consequently, $D$ in (9) is also proportional to $E$. The dielectric constant $\epsilon$ is defined as that constant of proportionality, through the relation $D = \epsilon E$. Hence (9) can also be written as

$$P = (\epsilon - 1)E. \quad (10)$$

### 2.7 Where do van der Waals forces come from?

In a dielectric medium, the energy density associated with electric field is [25]

$$\mathcal{H} = \frac{D \cdot E}{2}. \quad (11)$$

Using (9), this can be written as

$$\mathcal{H} = \frac{E^2}{2} + \frac{P \cdot E}{2}. \quad (12)$$
The second term shows that part of energy comes from interaction of electric field with polarized charges. In the absence of an external electric field, the average field $\langle \psi | E | \psi \rangle \equiv \langle E \rangle$ vanishes. Eq. (10) then implies that $\langle P \rangle$ also vanishes, so we have

$$\langle E \rangle = \langle P \rangle = 0. \quad (13)$$

However, unless $|\psi\rangle$ is an electric-field eigenstate, there are quantum fluctuations $\langle E^2 \rangle \neq 0$. Hence (10) implies that average interaction energy in (12) does not vanish

$$\langle H_{\text{int}} \rangle = \langle \frac{P \cdot E}{2} \rangle = \frac{\langle P^2 \rangle}{2(\epsilon - 1)} = \frac{\epsilon - 1}{2} \langle E^2 \rangle. \quad (14)$$

This shows that interaction energy, which really originates from the correlation $\langle P \cdot E \rangle$ between $P$ and $E$, can also be related to polarization fluctuations $\langle P^2 \rangle$, or alternatively, to electric field fluctuations $\langle E^2 \rangle$. Note, however, that the description in terms of fluctuations of $P$ or $E$ involves the phenomenological macroscopic quantity $\epsilon$. The description in terms of correlations between $P$ and $E$ is therefore more fundamental because it does not refer to $\epsilon$.

Forces which originate from interaction between correlated electric fields and polarized charges obeying (13) are known as van der Waals forces [27, 28]. In this sense, the interaction energy (14) is nothing but energy of van der Waals forces.

Casimir effect can be described as a force that originates from van der Waals forces. It is usually described by Lifshitz theory [11, 12, 13, 14, 15, 16, 17] which is technically more involved than calculation based on vacuum energy, but eventually leads to the same results. In a simple toy model which we shall study, a Lifshitz-like calculation of the force will turn out to be no more complicated than the Casimir-like calculation based on vacuum energy.

2.8 So what do we need to model Casimir effect from a microscopic perspective?

Now we see that a microscopic description of Casimir effect requires at least 3 dynamical ingredients. First, we need the electromagnetic field $E(x, t)$ and $B(x, t)$. Second, we need the polarization field $P(x, t)$ originating from microscopic charge density $\rho(x, t)$. The electromagnetic and polarization fields are microscopic. Third, we need one macroscopic ingredient, namely the distance $y$ between the plates treated as a dynamical variable.

However, as we said in the Introduction, in this paper we do not want to deal with all the technical details related to the true Casimir effect. Instead, we want to understand how in general a force can emerge from something which looks like vacuum energy. In other words, we want to understand how all the main ideas discussed in the present section, Sec. 2, are realized in a much simpler model. For that purpose we shall introduce a single degree of freedom $x_1(t)$ which will mimic the electromagnetic degrees $E(x, t)$ and $B(x, t)$. Similarly, another single degree of freedom $x_2(t)$ will mimic the polarization field $P(x, t)$. We shall also have the third degree of freedom $y(t)$ which will mimic the distance between the plates (even though our model will
not describe plates as such). The model will be chosen such that, under appropriate approximations, \( x_1(t) \) and \( x_2(t) \) oscillate with a frequency that depends on \( y \).

Even though the electromagnetic field is different in character from the polarization field, we shall take a model which is symmetric under the exchange \( x_1 \leftrightarrow x_2 \). It is possible to consider a similar model without such a symmetry, but we enforce this symmetry because it greatly simplifies the calculations. Similarly, to make the calculations as simple as possible, the dependence on \( y \) will be introduced in a somewhat *ad hoc* way. It is possible to introduce a more natural dependence on \( y \) as in [29], but the price is a one-dimensional field theory which has an infinite number of degrees of freedom and a divergence that needs to be regularized. By contrast, our simple model with only 3 degrees of freedom will not contain any divergence.

### 3 The model and its classical properties

#### 3.1 Basic properties of the model

We consider a system with 3 degrees of freedom \( x_1(t) \), \( x_2(t) \) and \( y(t) \) described by the Hamiltonian

\[
H = \left( \frac{p_1^2}{2m} + \frac{kx_1^2}{2} \right) + \left( \frac{p_2^2}{2m} + \frac{kx_2^2}{2} \right) + \frac{p_y^2}{2M} + g(y)x_1x_2.
\]  
(15)

Here \( p_1, p_2 \) and \( p_y \) are canonical momenta, while \( m, M \) and \( k \) are positive constants. (The degrees \( x_1, x_2 \) and \( y \) can be interpreted as positions of 3 particles moving in one dimension, in which case \( m \) and \( M \) can be interpreted as particle masses, but our mathematical results will not depend on that interpretation.) The function \( g(y) \) is an arbitrary non-negative function, restricted only by the requirement

\[
g(y) < k,
\]  
(16)

which provides that energy is positive. Our main interest will be the force on the \( y \)-degree given by

\[
F = -\frac{\partial H}{\partial y} = -g'(y)x_1x_2,
\]  
(17)

where the prime denotes derivative with respect to \( y \).

#### 3.2 Normal modes

For a general \( g(y) \), the system cannot be solved exactly in a closed analytic form. Therefore we make an approximation. We assume that \( y(t) \) changes much slower than \( x_1(t) \) and \( x_2(t) \), which can be justified by taking \( M \gg m \). Therefore the motion of \( x_1(t) \) and \( x_2(t) \) can be found by the adiabatic approximation, in which their equations of motion are solved by treating \( g(y) \) as a constant. This leads to the equations of motion

\[
m\ddot{x}_1(t) + kx_1(t) + gx_2(t) = 0,
\]
\[
m\ddot{x}_2(t) + kx_2(t) + gx_1(t) = 0.
\]  
(18)
Defining the quantities
\[ \omega^2 = \frac{k}{m}, \quad \omega_g^2 = \frac{g}{m}, \] (19)
the equations of motion can be written as
\[ \ddot{x}_1(t) + \omega^2 x_1(t) + \omega_g^2 x_2(t) = 0, \]
\[ \ddot{x}_2(t) + \omega^2 x_2(t) + \omega_g^2 x_1(t) = 0. \] (20)
This is a system of two coupled oscillators. It can be solved by the ansatz
\[ x_1(t) = c_1 e^{-i \Omega t}, \quad x_2(t) = c_2 e^{-i \Omega t}, \] (21)
which leads to
\[ -\Omega^2 c_1 + \omega^2 c_1 + \omega_g^2 c_2 = 0, \]
\[ -\Omega^2 c_2 + \omega^2 c_2 + \omega_g^2 c_1 = 0. \] (22)
The first and second equation in (22) give
\[ c_2 = \frac{\omega_g^2}{\Omega^2 - \omega^2} c_1, \quad c_2 = \frac{\Omega^2 - \omega^2}{\omega_g^2} c_1, \] (23)
respectively. These two equations must be consistent, so the factors in front of \( c_1 \) must be the same. This implies \((\Omega^2 - \omega^2)^2 = \omega_g^4\), i.e. \( \Omega^2 - \omega^2 = \pm \omega_g^2 \). Hence we have two possibilities; either \( \Omega^2 = \Omega_+^2 \) or \( \Omega^2 = \Omega_-^2 \), where
\[ \Omega_{\pm}^2 = \omega^2 \pm \omega_g^2 = \frac{k \pm g}{m}, \] (24)
and (19) has been used. We see that \( \Omega_{\pm}^2 > 0 \) due to (16). Now both equations in (23) give the same result \( c_2 = \pm c_1 \). Hence we have two normal modes of oscillation, namely \( x_+(t) \) for which \( x_2(t) = x_1(t) \), and \( x_-(t) \) for which \( x_2(t) = -x_1(t) \). The most general solution is a superposition of normal modes, so in general we have
\[ x_1(t) = c_+ x_+(t) + c_- x_-(t), \]
\[ x_2(t) = c_+ x_+(t) - c_- x_-(t), \] (25)
where \( c_+ \) and \( c_- \) are arbitrary constants. From (25) we see that \( x_1(t) \) and \( x_2(t) \) are not independent; if we know one of them, then we also know the other. Nevertheless, we still have two independent \( x \)-degrees of freedom, namely the normal modes \( x_+(t) \) and \( x_-(t) \) oscillating with frequencies (24).

### 3.3 Hamiltonian and force in a diagonal form

Eq. (25) can be used to diagonalize the Hamiltonian (15), i.e. to eliminate the term proportional to \( x_1 x_2 \). By choosing \( c_+ = c_- = 1/\sqrt{2} \) in (25) we have
\[ x_1 = \frac{x_+ + x_-}{\sqrt{2}}, \quad x_2 = \frac{x_+ - x_-}{\sqrt{2}}, \] (26)
the inverse of which is
\[ x_\pm = \frac{x_1 \pm x_2}{\sqrt{2}}. \]  
(27)

We can think of (26) and (27) not as solutions to the equations of motion, but as a definition of the new canonical coordinates \( x_+ \) and \( x_- \). We see that
\[ \frac{k(x_1^2 + x_2^2)}{2} + g(y)x_1x_2 = \frac{k_+(y)x_+^2}{2} + \frac{k_-(y)x_-^2}{2}, \]
(28)
where
\[ k_\pm(y) = k \pm g(y). \]  
(29)

Hence (15) can be written as
\[ H = H_+ + H_- + \frac{p_y^2}{2M}, \]
(30)
where
\[ H_\pm = \frac{p_\pm^2}{2m} + \frac{k_\pm(y)x_\pm^2}{2}. \]  
(31)

If we neglect the last term \( p_y^2/2M \) in (30), i.e. treat \( y \) as a non-dynamical constant, we see immediately that (31) leads to the oscillatory solutions with frequencies (24).

In addition, using (27) one can see that
\[ p_\pm = \frac{p_1 \pm p_2}{\sqrt{2}}. \]  
(32)

Now let us discuss the force (17). Inserting (26) into (17) we get
\[ F = -\frac{g'(y) (x_+^2 - x_-^2)}{2}. \]  
(33)

We are interested in the force when \( x_1 \) and \( x_2 \) are in their ground state. From (31) we see that they are in the classical ground state when \( x_\pm = 0 \). Hence (33) implies that \( F = 0 \) when \( x_1 \) and \( x_2 \) are in their classical ground state.

4 Quantum force

The full interacting Hamiltonian (15) has a quantum ground state with some finite ground-state energy \( E_{\text{vac}} \). This energy is a number that does not depend on \( x_1, x_2 \) and \( y \). As such, it does not have any physical consequences so can be subtracted from the Hamiltonian without affecting physics. The ground state of the full Hamiltonian (15) does not contain any interesting physics.

However, interesting physics may appear when the system is in an effective ground state, in which some but not all degrees of freedom are in their ground state. As in the discussion after (33), we shall be interested in the case when \( y \) is not in the ground state, while \( x_1 \) and \( x_2 \) are.
4.1 Quantization of the free Hamiltonian

As a warm up, let us quantize (15) in the free case \( g(y) = 0 \). In this case we have two uncoupled harmonic oscillators with frequency \( \omega \) given in (19), which is a well-known textbook stuff (see e.g. [26]). One introduces the operators

\[
a_j = \sqrt{\frac{m\omega}{2\hbar}} x_j + \frac{i}{\sqrt{2m\hbar\omega}} p_j,
\]

(34)

for \( j = 1, 2 \), which satisfy \([a_j, a_j^\dagger] = \delta_{jj'}\). The position and momentum operators can be expressed from (34) as

\[
x_j = \sqrt{\frac{\hbar}{2m\omega}} (a_j^\dagger + a_j), \quad p_j = i \sqrt{\frac{m\hbar\omega}{2}} (a_j^\dagger - a_j),
\]

(35)

so (15) with \( g(y) = 0 \) can be written as

\[
H^{(\text{free})} = H_1 + H_2 + \frac{p_y^2}{2M}, \quad \text{(36)}
\]

where

\[
H_j = \hbar \omega \left(a_j^\dagger a_j + \frac{1}{2}\right). \quad \text{(37)}
\]

The last term in (36) can also be quantized, but we shall use an approximation in which \( y \) is treated as a classical variable. This can be justified by assuming that \( y \) is a macroscopic degree of freedom, with a large mass \( M \gg m \). This means that we are really doing a semi-classical theory, in which \( y(t) \) is treated as a classical background. Hence we only quantize the Hamiltonian

\[
H^{(\text{free eff})} = H_1 + H_2, \quad \text{(38)}
\]

which is the free effective Hamiltonian for \( x_1 \) and \( x_2 \) degrees. The corresponding free effective vacuum \( |0\rangle \) satisfies

\[
a_j |0\rangle = 0, \quad \text{(39)}
\]

so the free effective-vacuum energy is

\[
E_{\text{vac}}^{(\text{free eff})} = \langle 0|H^{(\text{free eff})}|0\rangle = \frac{\hbar \omega}{2} + \frac{\hbar \omega}{2}. \quad \text{(40)}
\]

Clearly this free effective-vacuum energy is a constant which does not depend on \( y \). As such, it does not have any physical consequences so can be subtracted from the Hamiltonian without affecting physics.

4.2 Force à la Casimir

In the interacting case, the full Hamiltonian is (30). As in the free case in Sec. 4.1 we treat \( y \) as a classical background. Therefore we quantize only the effective Hamiltonian

\[
H^{(\text{eff})} = H_+ + H_-, \quad \text{(41)}
\]
which is the interacting version of (38). Analogously to (34) and (35) we have

$$a_\pm = \sqrt{\frac{m\Omega_\pm}{2\hbar}} x_\pm + \frac{i}{\sqrt{2m\hbar\Omega_\pm}} p_\pm,$$

$$x_\pm = \sqrt{\frac{\hbar}{2m\Omega_\pm}} (a_\pm^\dagger + a_\pm), \quad p_\pm = i\sqrt{\frac{m\hbar\Omega_\pm}{2}} (a_\pm^\dagger - a_\pm),$$

so (31) gives

$$H_\pm = \hbar\Omega_\pm \left(a_\pm^\dagger a_\pm + \frac{1}{2}\right),$$

which is the interacting version of (37). Analogously to (39), the interacting effective vacuum \(|\tilde{0}\rangle\) satisfies

$$a_\pm |\tilde{0}\rangle = 0,$$

so the interacting effective-vacuum energy is

$$E_{\text{vac}}^{(\text{eff})} = \langle 0 | H^{(\text{eff})} | \tilde{0} \rangle = \frac{\hbar\Omega_+ (y)}{2} + \frac{\hbar\Omega_- (y)}{2}.$$

Note that \(\Omega_\pm (y)\) depend on \(y\) because \(\Omega_\pm\) depend on \(g\) due to (24), and \(g\) depends on \(y\) as we assumed already in (15). Hence the force in the interacting effective vacuum can be calculated as

$$F = -\frac{\partial E_{\text{vac}}^{(\text{eff})}}{\partial y} = -\frac{\hbar\Omega_+ (y)}{2} - \frac{\hbar\Omega_- (y)}{2}.$$ 

From (24) we see that

$$\Omega'_\pm = \pm \frac{g'}{2m\Omega_\pm},$$

so (47) becomes

$$F = \frac{-\hbar g'(y)}{4m\Omega_+ (y)} + \frac{\hbar g'(y)}{4m\Omega_- (y)}.$$

### 4.3 Force à la Lifshitz

One may be worried that the calculation of the force based on (47) looks somewhat ad hoc [20], because it is not clear how the force (47) is related to the canonical way to calculate the force by (17) or (33). It is more legitimate to calculate the quantum force as the expectation value of the force operator, namely

$$F = -\langle \Psi | g'(y) x_1 x_2 | \Psi \rangle = -\frac{\langle \Psi | g'(y) (x_1^2 - x_2^2) | \Psi \rangle}{2},$$

where \(|\Psi\rangle\) is the full quantum state of the system. We are using the approximation in which \(y\) is treated as a classical background while \(x_1\) and \(x_2\) are in the interacting effective vacuum \(|\tilde{0}\rangle\), so (50) can be approximated by

$$F = -\frac{g'(y)}{2} \langle \tilde{0} | (x_1^2 - x_2^2) | \tilde{0} \rangle.$$
From the first equation in (43) we see that
\[ \langle \tilde{0} | x^2_\pm | \tilde{0} \rangle = \frac{\hbar}{2m\Omega_\pm}, \] (52)
so (51) becomes
\[ F = -\frac{\hbar g'(y)}{4m\Omega_+(y)} + \frac{\hbar g'(y)}{4m\Omega_-(y)}. \] (53)
We see that this result coincides with (49).

Some remarks are in order. First, in the calculation of (53) we never referred to the energy of the vacuum. We did, however, referred to the vacuum value of \( x^2_\pm \) in (52). If we used the normal ordered product \( :x^2_\pm : \) instead of \( x^2_\pm \) we would get a zero force, which would be a wrong result. The quantum fluctuations described by (52) are physical.

Second, Eq. (50) makes it clear that the force originates from the interaction between \( x_1, x_2, \) and \( y. \) The fact that interaction between \( x_1 \) and \( x_2 \) is important is not so clear from the calculation based on (47). In this sense, even though both calculations lead to the same result, the calculation based on (50) better reflects the true physical origin of the force.

Third, note that in the free vacuum \( |0\rangle \) we have
\[ \frac{\langle 0 | (x^2_+ - x^2_-) | 0 \rangle}{2} = \langle 0 | x_1x_2 | 0 \rangle = 0. \] (54)
This means that \( x_1 \) and \( x_2 \) are not correlated in the free vacuum, which is why the force vanishes in the free vacuum. More generally, if we considered a state of the form \( |\psi\rangle = |0_1\rangle |\psi_2\rangle, \) so that only \( x_1 \) is in the free vacuum while \( x_2 \) is in an arbitrary quantum state, we would again get
\[ \langle \psi | x_1x_2 | \psi \rangle = 0, \] (55)
so the force would vanish again. To get any force at all, it is important that \( x_1 \) is \textit{not} in the free vacuum state. Similar, of course, is also true for \( x_2. \)

5 The content of the interacting vacuum

5.1 General remarks

Any quantum state of \( x_1 \) and \( x_2, \) with or without interaction, can be expanded in the complete basis \( |n, n'\rangle \) defined by
\[ |n, n'\rangle = \frac{(a^\dagger_1)^n(a^\dagger_2)^{n'}|0\rangle}{\sqrt{n!n'}}. \] (56)
The interacting vacuum \( |\tilde{0}\rangle \) is not an exception, so there are some coefficients \( c_{nn'} \) such that
\[ |\tilde{0}\rangle = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} c_{nn'} |n, n'\rangle. \] (57)
5.2 Bogoliubov transformation

The coefficients $c_{nm'}$ can be found by noting that the operators $a_\pm$ and $a_\pm^\dagger$ are related to $a_j$ and $a_j^\dagger$ by a Bogoliubov transformation. To see that, we insert (27) and (32) into (42) and use (35) to obtain

$$a_\pm = \sum_{j=1,2} \alpha_{j\pm} a_1 + \beta_{j\pm} a_j^\dagger,$$

where

$$\alpha_{1\pm} = \frac{\Omega_\pm + \omega}{2\sqrt{2\Omega_\pm\omega}}, \quad \alpha_{2\pm} = \pm \alpha_{1\pm},$$

$$\beta_{1\pm} = \frac{\Omega_\pm - \omega}{2\sqrt{2\Omega_\pm\omega}}, \quad \beta_{2\pm} = \pm \beta_{1\pm},$$

are the Bogoliubov coefficients. Bogoliubov coefficients, in general, can be complex, but in our case they are real. They satisfy

$$\sum_{j=1,2} (\alpha_{j\pm}^2 - \beta_{j\pm}^2) = 1.$$

The inverse transformation of (58) is

$$a_j = \sum_{s=+, -} \alpha_{js} a_s + \beta_{js} a_s^\dagger.$$

This implies $a_j^\dagger = \sum_s \alpha_{js} a_s^\dagger - \beta_{js} a_s$, so one finds

$$\langle \tilde{0} | a_j^\dagger a_j | \tilde{0} \rangle = \sum_{s=+, -} \beta_{js}^2.$$

The number of free quanta is counted by the operators $N_j = a_j^\dagger a_j$, so (62) tells us that the average number of free quanta in $| \tilde{0} \rangle$ is non-zero

$$\langle \tilde{0} | N_j | \tilde{0} \rangle = \beta_{j+}^2 + \beta_{j-}^2.$$

5.3 The expansion coefficients

In our case $\beta_{j+}$ and $\beta_{j-}$ are different and both non-zero. This makes the explicit calculation of $c_{nm'}$ in (57) somewhat complicated. Physically, this is related to the fact that both $\Omega_+$ and $\Omega_-$ contribute to (46). As we shall see in Sec. 6 in the real Casimir effect there are reasons to ignore the contribution from one frequency (say $\Omega_-$) and consider only the contribution from the other (say $\Omega_+$). From this perspective it makes sense to ignore $\beta_{j-}$ and $\alpha_{j-}$ and to consider only $\beta_{j+} \equiv \beta$ and $\alpha_{j+} \equiv \alpha$. Hence, instead of (58) we study a simplified Bogoliubov transformation

$$a = \alpha (a_1 + a_2) + \beta (a_1^\dagger + a_2^\dagger).$$
Such a Bogoliubov transformation is known to lead to the so-called two-mode squeezed states, but here we shall study it from scratch without referring to the results in the literature.

The interacting vacuum $|\bar{0}\rangle$ is defined by

$$a|\bar{0}\rangle = 0.$$  \hfill (65)

Instead of starting from the general expansion (57), we make the ansatz

$$|\bar{0}\rangle = \sum_{n=0}^{\infty} c_n |n,n\rangle.$$  \hfill (66)

By inserting (64) and (66) into (65) and using

\begin{align*}
a_1 |n,n\rangle &= \sqrt{n} |n-1,n\rangle, \\
a_2 |n,n\rangle &= \sqrt{n} |n-1\rangle, \\
a_1^\dagger |n,n\rangle &= \sqrt{n+1} |n+1,n\rangle, \\
a_2^\dagger |n,n\rangle &= \sqrt{n+1} |n,n+1\rangle,
\end{align*}

we obtain

\begin{align*}
\sum_{n=1}^{\infty} \alpha c_n \sqrt{n} (|n-1,n\rangle + |n,n-1\rangle) \\
+ \sum_{n=0}^{\infty} \beta c_n \sqrt{n+1} (|n+1,n\rangle + |n,n+1\rangle) &= 0.
\end{align*}

(68)

In the first sum we introduce a new variable $n' = n - 1$ and then remove the prime from $n'$ because it is a dummy variable. This leads to

\begin{align*}
\sum_{n=0}^{\infty} \left[ \alpha c_{n+1} + \beta c_n \right] \sqrt{n+1} (|n+1,n\rangle + |n,n+1\rangle) &= 0.
\end{align*}

(69)

Hence the expression in the square brackets must vanish, which leads to the simple recursion relation $c_{n+1} = - (\beta/\alpha) c_n$ with the solution

$$c_n = \left( -\frac{\beta}{\alpha} \right)^n c_0.$$  \hfill (70)

Therefore (66) becomes

$$|\bar{0}\rangle = c_0 \sum_{n=0}^{\infty} \left( -\frac{\beta}{\alpha} \right)^n |n,n\rangle.$$  \hfill (71)

The constant $c_0$ can be determined from the normalization condition $\langle \bar{0}|\bar{0}\rangle = 1$. This gives

$$|c_0|^2 \sum_{n=0}^{\infty} \left( -\frac{\beta}{\alpha} \right)^{2n} = 1,$$

(72)
so applying the geometric series formula $\sum_{n=0}^{\infty} z^n = (1 - z)^{-1}$ to $z = (-\beta/\alpha)^2 = (\beta/\alpha)^2$, we get

$$c_0 = \sqrt{1 - (\beta/\alpha)^2}.$$  \hfill (73)

Note that $|n, n\rangle$ in (71) is a state in which the number of $x_1$-quanta is always equal to the number of $x_2$-quanta. In other words, $|n, n\rangle$ describes $n$ pairs, where each pair contains one $x_1$-quantum and one $x_2$-quantum. The interacting vacuum $|\tilde{0}\rangle$ is a state with an uncertain number of such pairs. In the real Casimir effect, to which we turn in the next section, $x_1$-quantum corresponds to a photon and $x_2$-quantum corresponds to a quantum of polarization.

## 6 Relation to the real Casimir effect

Let us now discuss how the calculations in our toy model are related to the calculations for the real Casimir effect. Our discussion will be non-technical, qualitative and hopefully intuitive.

As we already said in Sec. 2.8, $x_1$ mimics electromagnetic field and $x_2$ mimics polarization field. For instance, the first bracket in (15) is analogous to the pure electromagnetic Hamiltonian, i.e. we have the analogy

$$\frac{p_1^2/m + kx_2^2}{2} \leftrightarrow \int d^3x \frac{E^2 + B^2}{2}.$$  \hfill (74)

Similarly, the second bracket in (15) is analogous to the pure matter term. For an appropriate model of polarized matter in a dielectric material see e.g. [31]. The last term in (15) is similar to the interaction between charges and electromagnetic field, i.e.

$$gx_1 x_2 \leftrightarrow \int d^3x A_\mu j^\mu,$$  \hfill (75)

where $A_\mu$ is the electromagnetic 4-potential and $j^\mu$ is the charged 4-current.

The interaction between charged matter and electromagnetic field in terms of normal modes and their frequencies is discussed in many solid-state textbooks [32, 33, 34, 35, 36, 37, 38]. Instead of two frequencies $\Omega_\pm$ in (24) one gets two branches of the dispersion relation $\omega_\pm(k)$. The upper branch $\omega_+(k)$ is phonon-like (i.e. varies very slowly with $|k|$) for small $|k|$ and photon-like (i.e. behaves approximately as $\omega_+(k) \approx c|k|$) for large $|k|$. The lower branch $\omega_-(k)$ has the opposite behavior, i.e. it is photon-like for small $|k|$ and phonon-like for large $|k|$. Thus, ignoring a small range of intermediate $|k|$, we can use an approximation according to which, for each $k$, we have only one branch that significantly varies with $k$. Since Casimir force (6) is proportional to

$$\int d^3k \frac{\partial \omega(k)}{\partial y} = \int d^3k \frac{\partial \omega(k)}{\partial k} \frac{\partial k}{\partial y},$$  \hfill (76)

it follows that for each $k$ there is only one branch with dispersion relation $\omega(k) \approx c|k|$ that significantly contributes to Casimir force. Indeed, $\partial \omega(k)/\partial k$ is the group velocity
of waves, which for phonons is much smaller than that for photons. A calculation for perfect conductors (see e.g. \([2]\)) gives the Casimir force

\[
F(y) = -\frac{\pi^2}{240} \frac{\hbar c}{y^4},
\]

(77)

so heuristically one may expect that a contribution from the phonon-like branch would give a similar result with \(c \to c_s\), which would be negligible because the velocity of sound \(c_s\) is much smaller than the velocity of light \(c\). More details about calculations of Casimir force from vacuum energy of realistic materials can be found in \([39, 40, 41, 4]\). Not let us say a few words about the physical nature of normal modes. Just like \(x_{\pm}\) in (27) is a mixture of \(x_1\) and \(x_2\), a normal mode in a real material is a mixture of electromagnetic field and polarization field \([31]\). For a photon-like branch, (27) is roughly analogous to (9), as studied in more detail in \([42]\) and applied to Casimir effect in \([43]\). As analyzed in \([31]\), the number of quanta of such mixed fields are lowered and raised by operators analogous to (42). Such mixed quanta are often referred to as polaritons \([44, 32, 33, 34, 35, 36, 37, 38]\). (Note, however, that the word “polariton” was first introduced in \([31]\) where it meant the quantum of pure polarization field, not the quantum of a mixture.) The lowering and raising operators for mixed fields are related to lowering and raising operators for electromagnetic and polarization fields by a Bogoliubov transformation \([31]\) analogous to that in Sec. 5. The Casimir vacuum can be expressed in terms of photons and polarization quanta by a Bogoliubov transformation \([45, 46]\) leading to a state analogous to (71). This means that Casimir vacuum can be thought of as a state with a zero number of polaritons \([47]\), but one should not forget that this vacuum is really a state with an uncertain number of pairs, with each pair containing one photon and one quantum of polarization. Nevertheless, these photons and polarization quanta cannot be directly observed because they are not energy eigenstates of the interacting Hamiltonian. The energy eigenstates are polaritons. For other roles of polaritons in Casimir physics see also \([48, 49, 50, 51, 52]\).

Finally a few words on Lifshitz theory. Analogously to (17), one can start from the classical Lorentz force on charges derived from electromagnetic interaction (75). In the quantum case, analogously to the first equality in (50), the average Lorentz force is

\[
\mathbf{F} = \int d^3x \langle \rho \mathbf{E} + \mathbf{j} \times \mathbf{B} \rangle.
\]

(78)

By Maxwell equations, \(\rho\) and \(\mathbf{j}\) can be expressed in terms of \(\mathbf{E}\) and \(\mathbf{B}\). In this way the calculation of force reduces to a calculation of the expectation value of an operator quadratic in \(\mathbf{E}\) and \(\mathbf{B}\). This operator turns out to be proportional to a derivative of the energy-momentum tensor of electromagnetic field \([16, 17]\). The explicit calculation is technically involved \([11, 12, 13, 4, 5, 15, 16, 17]\), but is conceptually analogous to the simple calculation \([50] - (53)\). The result \(55\) is analogous to the result \([20]\) that \(\langle A_\mu j^\mu \rangle\) vanishes in any state proportional to the photon vacuum, implying that Casimir force is impossible in photon vacuum \([20]\).
7 Conclusions

There is a general physical principle telling that zero-point energy is unphysical. Casimir effect is perfectly compatible with this principle. In this paper we have seen that this can be understood at several levels.

First, Casimir vacuum is not a ground state for the full Hamiltonian, but only a ground state for an effective Hamiltonian that describes physics at a fixed distance $y$ between Casimir plates. A description of Casimir force requires $y$ to be a dynamical variable, and Casimir vacuum is not a ground state for a Hamiltonian in which $y$ is dynamical.

Second, at a fundamental microscopic level, Casimir force should be viewed as a manifestation of van der Waals forces, which involves correlated fluctuations of polarization and associated electromagnetic field.

Third, Casimir vacuum is not a state without photons. It can be related to the photon vacuum by a non-trivial Bogoliubov transformation, leading to the picture of Casimir vacuum as a state with a zero number of certain quasi-particle excitations (polaritons), but uncertain number of photons and polarization quanta.

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