Balanced-Viscosity solutions for multi-rate systems

Alexander Mielke\textsuperscript{1,2}, Riccarda Rossi\textsuperscript{3}, and Giuseppe Savaré\textsuperscript{4}

\textsuperscript{1} Weierstraß-Institut, Mohrenstraße 39, D–10117 Berlin, Germany.
\textsuperscript{2} Institut für Mathematik, Humboldt-Universität zu Berlin, Rudower Chaussee 25, D–12489 Berlin (Adlershof), Germany.
\textsuperscript{3} DICATAM – Sezione di Matematica, Università di Brescia, via Valotti 9, I–25133 Brescia, Italy.
\textsuperscript{4} Dipartimento di Matematica “F. Casorati”, Università di Pavia. Via Ferrata, I–27100 Pavia, Italy.
E-mail: mielke@wias-berlin.de, riccarda.rossi@unibs.it, giuseppe.savare@unipv.it

Abstract. Several mechanical systems are modeled by the static momentum balance for the displacement $u$ coupled with a rate-independent flow rule for some internal variable $z$. We consider a class of abstract systems of ODEs which have the same structure, albeit in a finite-dimensional setting, and regularize both the static equation and the rate-independent flow rule by adding viscous dissipation terms with coefficients $\varepsilon^\alpha$ and $\varepsilon$, where $0 < \varepsilon \ll 1$ and $\alpha > 0$ is a fixed parameter. Therefore for $\alpha \neq 1$, $u$ and $z$ have different relaxation rates.

We address the vanishing-viscosity analysis as $\varepsilon \downarrow 0$ of the viscous system. We prove that, up to a subsequence, (reparameterized) viscous solutions converge to a parameterized curve yielding a $\textit{Balanced Viscosity}$ solution to the original rate-independent system, and providing an accurate description of the system behavior at jumps. We also give a reformulation of the notion of $\textit{Balanced Viscosity}$ solution in terms of a system of subdifferential inclusions, showing that the viscosity in $u$ and the one in $z$ are involved in the jump dynamics in different ways, according to whether $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0,1)$.

1. Introduction

Several mechanical systems are described by ODE or PDE systems of the type:

\begin{align}
D_u \mathcal{E}(t, u(t), z(t)) &= \mathbb{0} \quad \text{ in } \mathbb{U}^*, \quad \text{ for a.a. } t \in (0, T), \quad (1.1a) \\
\partial \mathcal{R}_0(z'(t)) + D_z \mathcal{E}(t, u(t), z(t)) &\supseteq \mathbb{0} \quad \text{ in } \mathbb{Z}^*, \quad \text{ for a.a. } t \in (0, T), \quad (1.1b)
\end{align}

where $\mathbb{U}$, $\mathbb{Z}$ are Banach spaces, and $\mathcal{E} : [0, T] \times \mathbb{U} \times \mathbb{Z} \to \mathbb{R}$ is an energy functional. For example, within the ansatz of generalized standard materials, $u$ is the displacement, at equilibrium, while changes in the elastic behavior due to dissipative effects are described in terms of an internal variable $z$ in some state space $\mathbb{Z}$. In several mechanical phenomena \cite{1}, dissipation due to inertia and viscosity is negligible, and the system is governed by rate-independent evolution, which means that the (convex, nondegenerate) dissipation potential $\mathcal{R}_0 : \mathbb{Z} \to [0, \infty)$ is \textit{positively homogeneous of degree} 1. Thus system (1.1) is invariant for time-rescalings.

It is well known that, if the map $z \mapsto \mathcal{E}(t, u, z)$ is not uniformly convex, one cannot expect the existence of absolutely continuous solutions to (1.1). This fact has motivated the development of various weak solvability concepts for (1.1), starting with the well-established notion of \textit{energetic solution}. The latter dates back to \cite{2}, and was further developed in \cite{3} (see \cite{4}, as well, in the
context of crack growth), cf. also [1], [5] and the references therein. Despite the several good features of the energetic formulation, it is known that, in the case the energy $z \mapsto \mathcal{E}(t, u, z)$ is nonconvex, the global stability condition may lead to jumps of $z$ as a function of time that are not motivated by, or in accord with, the mechanics of the system, cf. e.g. the discussions in [6, Ex. 6.1], [7, Ex. 6.3], and [8, Ex. 1].

Over the last years, an alternative selection criterion of mechanically feasible weak solution concepts for the rate-independent system [1.1] has been developed, moving from the finite-dimensional analysis in [9]. It is based on the interpretation of (1.1) as originating in the vanishing-viscosity limit of the viscous system

\begin{align}
D_z\mathcal{E}(t, u(t), z(t)) &= 0 \quad \text{in } \mathcal{U}^*, \quad \text{for a.a. } t \in (0, T), \quad (1.2a) \\
\partial\mathcal{R}_0(\dot{z}(t)) + \varepsilon \partial V_z(\dot{z}(t)) + D_z\mathcal{E}(t, u(t), z(t)) &\equiv 0 \quad \text{in } \mathcal{Z}^*, \quad \text{for a.a. } t \in (0, T), \quad (1.2b)
\end{align}

where $V_z : \mathcal{Z} \to [0, \infty)$ is a dissipation potential with superlinear (for instance, quadratic) growth at infinity. Observe that the existence of solutions for the generalized gradient system [1.2] follows from [10, 11], cf. also [12]. This vanishing-viscosity approach leads to a notion of solution featuring a local, rather than global, stability condition for the description of rate-independent evolution, thus avoiding “too early” and “too long” jumps. Furthermore, it provides an accurate description of the energetic behavior of the system at jumps, in particular highlighting how viscosity, neglected in the limit as $\varepsilon \downarrow 0$, comes back into the picture and governs the jump dynamics. This has been demonstrated in [8, 13, 14] within the frame of abstract, finite-dimensional and infinite-dimensional, rate-independent systems, and in [15] for a wide class parabolic equations with a rate-independent term. This analysis has also been developed in several applicative contexts, ranging from crack propagation [7, 16], to plasticity [17, 18, 19, 20], and to damage [21], among others.

In this note, we shall perform the vanishing-viscosity analysis of system [1.1] by considering the viscous approximation of (1.1a), in addition to the viscous approximation of (1.1b). More precisely, we will address the asymptotic analysis as $\varepsilon \downarrow 0$ of the system

\begin{align}
\varepsilon \partial V_u(u'(t)) + D_u\mathcal{E}(t, u(t), z(t)) &= 0 \quad \text{in } \mathcal{U}^*, \quad \text{for a.a. } t \in (0, T), \quad (1.3a) \\
\partial\mathcal{R}_0(\dot{z}(t)) + \varepsilon \partial V_z(\dot{z}(t)) + D_z\mathcal{E}(t, u(t), z(t)) &\equiv 0 \quad \text{in } \mathcal{Z}^*, \quad \text{for a.a. } t \in (0, T), \quad (1.3b)
\end{align}

where $\alpha > 0$ and $V_u$ a quadratic dissipation potential for the variable $u$. Observe that (1.3) models systems with (possibly) very different relaxation times. In fact, the parameter $\alpha > 0$ sets which of the two variables $u$ and $z$ relaxes faster to equilibrium and rate-independent evolution, respectively.

Let us mention that the analysis developed in this paper is in the mainstream of a series of recent papers focused on the coupling between rate-independent and viscous systems. First and foremost, in [22] a wide class of rate-independent processes in viscous solids with inertia has been tackled, while the coupling with temperature has further been considered in [23]. In fact, in these systems the evolution for the internal variable $z$ is purely rate-independent and no vanishing viscosity is added to the equation for $z$: viscosity and inertia only intervene in the evolution for the displacement $u$. For these processes, the author has proposed a notion of solution of energetic type consisting of the weakly formulated momentum equation for the displacements (and also of the weak heat equation in [23]), of an energy balance, and of a semi-stability condition. The latter reflects the mixed rate dependent/independent character of the system. In [22] and [24] a vanishing-viscosity analysis (in the momentum equation) has been performed. As discussed in [24] in the context of delamination, this approach leads to local solutions (cf. also [5]), describing crack initiation (i.e., delamination) in a physically feasible way. In [25], the vanishing-viscosity approach has also been developed in the context of a model for crack growth in the two-dimensional antiplane case, with a pre-assigned crack path,
coupling a viscoelastic momentum equation with a viscous flow rule for the crack tip; again, this procedure leads to solutions jumping later than energetic solutions. With a rescaling technique, a vanishing-viscosity analysis both in the flow rule, and in the momentum equation, has been recently performed in [26] for perfect plasticity, recovering energetic solutions thanks to the convexity of the energy. In [27], the same analysis has led to local solutions for a delamination system.

With the vanishing-viscosity analysis in this paper, besides finding good local conditions for the limit evolution, we want to add as an additional feature a thorough description of the energetic behavior of the solutions at jumps. This shall be deduced from an energy balance. Moreover, in comparison to the aforementioned contributions [25, 26, 27], a greater emphasis shall be put here on how the multi-rate character of system (1.3) enters in the description of the jump dynamics. In particular, we will convey that viscosity in $u$ and viscosity $z$ are involved in the path followed by the system at jumps in (possibly) different ways, depending on whether the parameter $\alpha$ is strictly bigger than, or equal to, or strictly smaller than 1.

To focus on this and to avoid overburdening the paper with technicalities, we shall keep to a simple functional analytic setting. Namely, we shall consider the finite-dimensional and smooth case

$$\mathcal{U} = \mathbb{R}^n, \quad \mathcal{Z} = \mathbb{R}^m, \quad \mathcal{E} \in C^1([0, T] \times \mathbb{R}^n \times \mathbb{R}^m).$$

(1.4)

Obviously, this considerably simplifies the analysis, since the difficulties attached to nonsmoothness of the energy and to infinite-dimensionality are completely avoided. Still, even within such a simple setting (where, however, we will allow for state-dependent dissipation potentials $\mathcal{R}_0, \mathcal{V}_z,$ and $\mathcal{V}_u), the key ideas of our vanishing-viscosity approach can be highlighted.

Let us briefly summarize our results by restricting to a simplified version of (1.3), where we assume (1.4) and use the simple choices

$$\mathcal{V}_u(u') = \frac{1}{2}|u'|^2, \quad \mathcal{V}_z(z') = \frac{1}{2}|z'|^2$$

(cf. $\{\mathcal{V}_u\}$ for the assumptions on state-dependent dissipation potentials), system (1.3) reduces to the ODE system

$$\varepsilon^\alpha u'(t) + D_u \mathcal{E}(t, u(t), z(t)) = 0 \quad \text{in } (0, T), \quad (1.5a)$$

$$\partial \mathcal{R}_0(z'(t)) + \varepsilon z'(t) + D_z \mathcal{E}(t, u(t), z(t)) \ni 0 \quad \text{in } (0, T). \quad (1.5b)$$

First of all, following [8, 13, 14], and along the lines of the variational approach to gradient flows by E. De Giorgi [28, 29], we will pass to the limit as $\varepsilon \downarrow 0$ in the energy-dissipation principle associated (and equivalent, by Fenchel-Moreau duality and the chain rule for $\mathcal{E}$) to (1.5), namely

$$\mathcal{E}(t, u(t), z(t)) + \int_s^t \mathcal{R}_0(z'(r)) + \frac{\varepsilon}{2}|z'(r)|^2 + \frac{1}{2\varepsilon^\alpha}|u'(r)|^2 \, dr$$

$$+ \int_s^t \frac{1}{\varepsilon} \mathcal{W}_z^2(-D_z \mathcal{E}(r, u(r), z(r))) + \frac{1}{2\varepsilon^\alpha}|D_u \mathcal{E}(r, u(r), z(r))|^2 \, dr$$

$$= \mathcal{E}(s, u(s), z(s)) + \int_s^t \partial_t \mathcal{E}(r, u(r), z(r)) \, dr$$

(1.6)

for all $0 \leq s \leq t \leq T,$ where $\mathcal{W}_z^2$ is the Legendre transform of $\mathcal{R}_0 + \mathcal{V}_z.$ As we will see in Section 6, equation (1.6) is well-suited to unveiling the role played by viscosity in the description of the energetic behavior of the system at jumps. Indeed, it reflects the competition between the tendency of the system to be governed by viscous dissipation both for the variable $z$ and for
the variable $u$ (with different rates if $\alpha \neq 1$), and its tendency to be locally stable in $z$, and at equilibrium in $u$, cf. also the discussion in Remark 4.4.

Secondly, to develop the analysis as $\varepsilon \downarrow 0$ for a family of curves $(u_\varepsilon, z_\varepsilon)_\varepsilon \subset W^{1,2}(0, T; \mathbb{R}^n \times \mathbb{R}^m)$ fulfilling (1.6) we will adopt a by now well-established technique from [9]. Namely, to capture the viscous transition paths at jump points, we will reparameterize the curves $(u_\varepsilon, z_\varepsilon)$, for instance by their arc-length. Hence we will address the analysis as $\varepsilon \downarrow 0$ of the parameterized curves $(t_\varepsilon, u_\varepsilon, z_\varepsilon)_\varepsilon$ defined on the interval $[0, S]$ with values in the extended phase space $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m$, with $t_\varepsilon$ the rescaling functions and $u_\varepsilon := u_\varepsilon \circ t_\varepsilon$, $z_\varepsilon := z_\varepsilon \circ t_\varepsilon$. Under suitable conditions it can be proved that, up to a subsequence the curves $(t_\varepsilon, u_\varepsilon, z_\varepsilon)_\varepsilon$ converge to a triple $(t, u, z) \in W^{1,1}([0, S]; [0, T] \times \mathbb{R}^n \times \mathbb{R}^m)$. Its evolution is described by an energy-dissipation principle obtained by passing to the limit in the reparameterized version of (1.6), cf. Theorem 4.5. We will refer to $(t, u, z)$ as a parameterized Balanced Viscosity solution to the rate-independent system $(\mathbb{R}^n \times \mathbb{R}^m, \mathcal{E}; \mathcal{R}_0 + \varepsilon \mathcal{V}_2 + \varepsilon^\alpha \mathcal{V}_u)$.

The main result of this paper, Theorem 5.3, provides a more transparent reformulation of the energy-dissipation principle defining a parameterized Balanced Viscosity solution $(t, u, z)$. It is in terms of a system of subdifferential inclusions fulfilled by the curve $(t, u, z)$, namely

$$\theta_u(s)u'(s) + (1 - \theta_u(s))D_u \mathcal{E}(t(s), u(s), z(s)) \geq 0 \quad \text{for a.a. } s \in (0, S),$$

$$\theta_z(s)z'(s) + (1 - \theta_z(s))D_z \mathcal{E}(t(s), u(s), z(s)) \geq 0 \quad \text{for a.a. } s \in (0, S),$$

where the Borel functions $\theta_u, \theta_z : [0, S] \rightarrow [0, 1]$ fulfill

$$t'(s)\theta_u(s) = t'(s)\theta_z(s) = 0 \quad \text{for a.a. } s \in (0, S),$$

plus an extra condition on the triple $(t', \theta_u, \theta_z)$ that depends explicitly on the three cases $\alpha \in (0, 1)$, $\alpha = 1$, and $\alpha > 1$, see (5.13). Condition (1.8) reveals that the viscous terms $u'(s)$ and $z'(s)$ may contribute to (1.7) only at jumps of the system, corresponding to $t'(s) = 0$ as the function $t$ records the (slow) external time scale. In this respect, (1.7)–(1.8) is akin to the (parameterized) subdifferential inclusion

$$D_u \mathcal{E}(t(s), u(s), z(s)) \geq 0 \quad \text{for a.a. } s \in (0, S),$$

$$\partial \mathcal{R}_0(z'(s)) + \theta(s)z'(s) + D_z \mathcal{E}(t(s), u(s), z(s)) \geq 0 \quad \text{for a.a. } s \in (0, S),$$

with the Borel function $\theta : [0, S] \rightarrow [0, \infty)$ fulfilling

$$t'(s)\theta(s) = 0 \quad \text{for a.a. } s \in (0, S).$$

Indeed, (1.9) is the subdifferential reformulation for parameterized Balanced Viscosity solutions obtained by taking the limit $\varepsilon \downarrow 0$ in (1.2), where viscosity is added only to the flow rule.

However, note that (1.7) has a much more complex structure than (1.9). In addition to the switching condition (1.8), the functions $t'$, $\theta_u$, and $\theta_z$ fulfill the additional condition (5.13) that differs in the three cases $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0, 1)$. In particular, for $\theta_u > 0$ or $\theta_z > 0$ the viscosity in $u$ or $z$ pops back into the description of the system behavior at jumps, in a way depending on whether $u$ relaxes faster to equilibrium than $z$, or $u$ and $z$ have the same relaxation rate, or $z$ relaxes faster to local stability than $u$. This explains the reason for the name (parameterized) Balanced Viscosity solutions, see also Definition 4.6. While this limiting system is rate-independent, it still shows a subtle balance between the different dissipative mechanisms, namely the rate-independent one given via $\mathcal{R}_0$ and the viscous ones given via $\mathcal{V}_u$ and $\mathcal{V}_z$. 
Plan of the paper. In Section 2 we set up all the basic assumptions on the dissipation potentials \( R_0, V_u, \) and \( V_z \). Section 3 is devoted to the generalized gradient system driven by \( \mathcal{E} \) and the “viscous” potential \( R_\alpha := R_0 + \varepsilon V_2 + \varepsilon^\alpha V_u \). In particular, we establish a series of estimates on the viscous solutions \( (u_\varepsilon, z_\varepsilon) \) which will be at the core of the vanishing-viscosity analysis, developed in Section 4 with Theorem 4.5. In Section 5 we will prove Theorem 5.3 and explore the mechanical interpretation of parameterized Balanced Viscosity solutions. Finally, in Section 6 we will illustrate this solution notion, focusing on how it varies in the cases \( \alpha > 1, \alpha = 1, \alpha \in (0, 1) \), in two different examples.

Notation. In what follows, we will denote by \( \langle \cdot, \cdot \rangle \) and by \( | \cdot | \) the scalar product and the norm in any Euclidean space \( \mathbb{R}^d \), with \( d = n, m, n + m, \ldots \). Moreover, we will use the same symbol \( C \) to denote a positive constant depending on data, and possibly varying from line to line.

2. Setup
As mentioned in the introduction, we are going to address a more general version of system (1.5), where the 1-positively homogeneous dissipation potential \( R_0 \), as well as the quadratic potentials \( V_u \) and \( V_z \) for \( u' \) and \( z' \), are also depending on the state variable

\[
q := (u, z) \in \Omega := \mathbb{R}^n \times \mathbb{R}^m.
\]

Hence, the rate-independent system is

\[
\partial_t q \cdot (R_0(q(t), z'(t)) + D_q \mathcal{E}(t, q(t))) \geq 0 \quad \text{in } (0, T),
\]

namely

\[
\begin{align*}
D_u \mathcal{E}(t, u(t), z(t)) &= 0, \quad \text{for a.a. } t \in (0, T), \quad (2.2a) \\
\partial_t R_0(q(t), z'(t)) + D_z \mathcal{E}(t, u(t), z(t)) &\geq 0, \quad \text{for a.a. } t \in (0, T). \quad (2.2b)
\end{align*}
\]

We approximate it with the following generalized gradient system

\[
\partial_t q \cdot (R_\alpha(q(t), q'(t)) + D_q \mathcal{E}(t, q(t))) \geq 0 \quad \text{in } (0, T),
\]

where the overall dissipation potential \( R_\alpha \) is of the form

\[
R_\alpha(q, q') = R_\alpha(q, (u', z')) := R_0(q, z') + \varepsilon V_z(q; z') + \varepsilon^\alpha V_u(q; u') \quad \text{with } \alpha > 0. \quad (2.4)
\]

In what follows, let us specify our assumptions on the dissipation potentials \( R_0, V_z, \) and \( V_u \).

Dissipation: We require that

\[
R_0 \in C^0(\Omega \times \mathbb{R}^m), \quad \forall q \in \Omega : \quad R_0(q, \cdot) \text{ is convex, positively 1-homogeneous, and } \exists \, C_{0,R}, C_{1,R} > 0 \forall (q, z') \in \Omega \times \mathbb{R}^m : \quad C_{0,R}|z'| \leq R_0(q, z') \leq C_{1,R}|z'|, \tag{R0}
\]

\[
V_z : \Omega \times \mathbb{R}^m \rightarrow [0, \infty) \text{ is of the form } V_z(q; z') = \frac{1}{2}\langle V_z(q)z', z' \rangle \text{ with } \tag{Vz}
\]

\[
V_z \in C^0(\Omega; \mathbb{R}^{m \times m}) \text{ and } \exists \, C_{0,V}, C_{1,V} > 0 \forall q \in \Omega : \quad C_{0,V}|z'|^2 \leq V_z(q; z') \leq C_{1,V}|z'|^2,
\]

\[
V_u : \Omega \times \mathbb{R}^n \rightarrow [0, \infty) \text{ is of the form } V_u(q; u') = \frac{1}{2}\langle V_u(q)u', u' \rangle \text{ with } \tag{Vu}
\]

\[
V_u \in C^0(\Omega; \mathbb{R}^{n \times n}) \text{ and } \exists \, \tilde{C}_{0,V}, \tilde{C}_{1,V} > 0 \forall q \in \Omega : \quad \tilde{C}_{0,V}|u'|^2 \leq V_u(q; u') \leq \tilde{C}_{1,V}|u'|^2.
\]
For later use, let us recall that, due to the 1-homogeneity of $R_0(q, \cdot)$, for every $q \in \Omega$ the convex analysis subdifferential $\partial R_0(q, \cdot) : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is characterized by

$$
\zeta \in \partial R_0(q, z') \quad \text{if and only if} \quad \begin{cases} 
\langle \zeta, w \rangle \leq R_0(q, w) & \text{for all } w \in \mathbb{R}^m, \\
\langle \zeta, z' \rangle \geq R_0(q, z') .
\end{cases}
$$

(2.5)

Furthermore, observe that $(\nabla_z)$ and $(\nabla u)$ ensure that for every $q \in \Omega$ the matrices $V_z(q) \in \mathbb{R}^{n \times n}$ and $V_u(q) \in \mathbb{R}^{m \times m}$ are positive definite, uniformly with respect to $q$. Furthermore, for later use we remark that the conjugate

$$
V_u^*(q; \eta) = \sup_{v \in \mathbb{R}^n} \langle \eta, v \rangle - V_u(q; v) = \frac{1}{2} \langle V_u(q)^{-1} \eta, \eta \rangle
$$

fulfills

$$
\overline{C}_0 |\eta|^2 \leq V_u^*(q; \eta) \leq \overline{C}_1 |\eta|^2
$$

(2.6)

for some $\overline{C}_0, \overline{C}_1 > 0$. We have the analogous coercivity and growth properties for $V_z^*$.

Our assumptions concerning the energy functional $E$, expounded below, are typical of the variational approach to gradient flows and generalized gradient systems. Since we are in a finite-dimensional setting, to impose coercivity it is sufficient to ask for boundedness of energy sublevels. The power-control condition will allow us to bound $\partial_t E$ in the derivation of the basic energy estimate on system (2.3), cf. Lemma 3.1 later on. The smoothness of $E$ guarantees the validity of two further, key properties, i.e. the continuity of $D_q E$, and the chain rule (cf. (2.10) below), which will play a crucial role for our analysis.

Later on, in Section 3 we will impose that $E$ is uniformly convex with respect to $u$. As we will see, this condition will be at the core of the proof of an estimate for $||u'||_{L^1(0, T; \mathbb{R}^n)}$, uniform with respect to the parameter $\varepsilon$. Observe that, unlike for $z'$ such estimate does not follow from the basic energy estimate on system (2.3), since the overall dissipation potential $R_0$ is degenerate in $u'$ as $\varepsilon \downarrow 0$. It will require additional careful calculations.

**Energy:** We assume that $E \in C^1([0, T] \times \Omega)$ and that it is bounded from below by a positive constant (indeed by adding a constant we can always reduce to this case). Furthermore, we require that

$$
\exists C_{0,E}, \tilde{C}_{0,E} > 0 \forall (t, q) \in [0, T] \times \Omega : \quad E(t, q) \geq C_{0,E} |q|^2 - \tilde{C}_{0,E} \quad \text{(coercivity),}
$$

$$
\exists C_{1,E} > 0 \forall (t, q) \in [0, T] \times \Omega : \quad |\partial_t E(t, q)| \leq C_{1,E} E(t, q) \quad \text{(power control).}
$$

(E)

In view of (2.4), (2.5), and (2.6), the generalized gradient system (2.3) reads

$$
\varepsilon^\alpha V_u(q(t))u'(t) + D_u E(t, u(t), z(t)) = 0 \quad \text{in } (0, T),
$$

(2.7a)

$$
\varepsilon V_z(q(t))z'(t) + \partial R_0(z'(t)) + D_z E(t, u(t), z(t)) = 0 \quad \text{in } (0, T).
$$

(2.7b)

**Existence of solutions to the generalized gradient system (2.3).** It follows from the results in [10] [12] that, under the present assumptions, for every $\varepsilon > 0$ there exists a solution $q_\varepsilon \in W^{1,2}(0, T; \Omega)$ to the Cauchy problem for (2.3). Observe that $q_\varepsilon$ also fulfills the energy-dissipation principle

$$
E(t, q_\varepsilon(t)) + \int_s^t R_\varepsilon(q_\varepsilon(r), q_\varepsilon'(r)) + R^c_\varepsilon(q_\varepsilon(r), -D_q E(r, q_\varepsilon(r))) \, dr = E(s, q_\varepsilon(s)) + \int_s^t \partial_t E(r, q_\varepsilon(r)) \, dr.
$$

(2.8)
In (2.8), the dual dissipation potential $\mathcal{R}_\varepsilon^*: \Omega \times \mathbb{R}^{n+m} \to \mathbb{R}$ is the Fenchel-Moreau conjugate of $\mathcal{R}_\varepsilon$, i.e.

$$\mathcal{R}_\varepsilon^*(q, \xi) := \sup_{v \in \Omega} \langle (\xi, v) - \mathcal{R}_\varepsilon(q, v) \rangle.$$  

(2.9)

In fact, by the Fenchel equivalence the differential inclusion (2.3) reformulates as

$$\mathcal{R}_\varepsilon(q_\varepsilon(t), q_\varepsilon'(t)) + \mathcal{R}_\varepsilon^*(q_\varepsilon(t), -Dq \mathcal{E}(t, q_\varepsilon(t))) = \langle -Dq \mathcal{E}(t, q_\varepsilon(t)), q_\varepsilon'(t) \rangle \quad \text{for a.a. } t \in (0, T).$$

Combining this with the chain rule

$$\frac{d}{dt} \mathcal{E}(t, q(t)) = \partial_t \mathcal{E}(t, q(t)) + \langle Dq \mathcal{E}(t, q(t)), q'(t) \rangle \quad \text{for a.a. } t \in (0, T)$$

(2.10)

along any curve $q \in W^{1,1}(0, T; \Omega)$ and integrating in time, we conclude (2.8).

The energy balance (2.8) will play a crucial role in our analysis: indeed, after deriving in Sec. 3 a series of a priori estimates, uniform with respect to the parameter $\varepsilon > 0$, we shall pass to the limit in the parameterized version of (2.8) as $\varepsilon \downarrow 0$. We will thus obtain a (parameterized) energy-dissipation principle which encodes information on the behavior of the limit system for $\varepsilon = 0$, in particular at the jumps of the limit curve $q$ of the solutions $q_\varepsilon$ to (2.3).

### 3. A priori estimates

In this section, we consider a family $(q_\varepsilon)_\varepsilon \subset W^{1,2}(0, T; \Omega)$ of solutions to the Cauchy problem for (2.3), with a converging sequence of initial data $(q_\varepsilon^0)_\varepsilon$, i.e.

$$q_\varepsilon^0 \to q^0$$

(3.1)

for some $q^0 \in \Omega$. Our first result, Lemma 3.1 provides a series of basic estimates on the functions $(q_\varepsilon)$, as well as a bound for $\|z_\varepsilon^t\|_{L^1(0, T; \mathbb{R}^m)}$, uniform with respect to $\varepsilon$. It holds under conditions $[R_0], [V_u], [V_a], [E]$, as well as (3.1).

Imposing that the dissipation potential $V_u$ does not depend on the state variable (cf. $[V_u, 1]$ below), assuming uniform convexity of $\mathcal{E}$ with respect to the variable $u$, and requiring an additional condition the initial data $(q_\varepsilon^0)_\varepsilon$ (see (3.5)), in Proposition 3.2 we will derive the following crucial estimate, uniform with respect to $\varepsilon$:

$$\|q_\varepsilon\|_{L^1(0, T; \mathbb{R}^{n+m})} \leq C.$$  

(3.2)

We begin with the following result, which does not require the above mentioned enhanced conditions.

**Lemma 3.1.** Let $\alpha > 0$. Assume $[R_0], [V_u], [V_a], [E]$, and (3.1). Then, there exists a constant $C > 0$ such that for every $\varepsilon > 0$

$$\sup_{t \in [0, T]} \mathcal{E}(t, q_\varepsilon(t)) \leq C,$$  

(3.3a)

$$\sup_{t \in [0, T]} |q_\varepsilon(t)| \leq C,$$  

(3.3b)

$$\int_0^T |z_\varepsilon^t(r)| dr \leq C.$$  

(3.3c)

**Proof.** We exploit the energy identity (2.8). Observe that $\mathcal{R}_\varepsilon^*(q, \xi) \geq 0$ for all $(q, \xi) \in \Omega \times \mathbb{R}^{n+m}$. Therefore, we deduce from (2.8) that

$$\mathcal{E}(t, q_\varepsilon(t)) \leq \mathcal{E}(0, q_\varepsilon(0)) + \int_0^t \partial_t \mathcal{E}(r, q_\varepsilon(r)) dr \leq C_{1, E} \int_0^t \mathcal{E}(r, q_\varepsilon(r)) dr,$$
where we have used the power control from (E), and the fact that $E(0,q_ε(0)) \leq C$, since the $(q_ε(0))_ε$ is bounded. The Gronwall lemma then yields (3.3a), and (3.3b) ensues from the coercivity of $E$. Using again the power control, we ultimately infer from (2.8) that

$$
\int_0^T \mathcal{R}_ε(q_ε(r), q_ε'(r)) + \mathcal{R}_ε'(q_ε(r), -D_qE(r, q_ε(r))) \, dr \leq C.
$$

(3.4)

In particular, $\int_0^T \mathcal{R}_0(q_ε(r), z'_ε(r)) \, dr \leq C$, whence (3.3c) by (R_0).

The derivation of the $L^1(0, T; \mathbb{R}^n)$ estimate for $(u'_ε)_ε$ similar to (3.3c) clearly does not follow from (2.8), which only yields $\int_0^T \varepsilon |u'_ε(r)|^2 \, dr \leq C$ via (3.4) and (V_u). It is indeed more involved, and, as already mentioned, it strongly relies on the uniform convexity of $E$ with respect to $u$. Furthermore, we are able to obtain it only under the simplifying condition that the dissipation potential $\mathcal{V}_u$ in fact does not depend on the state variable $q$, and under an additional well-preparedness condition on the data $(q_0^ε)_ε$, ensuring that the forces $D_uE(0,q_0^ε)$ tend to zero, as $\varepsilon \downarrow 0$, with rate $\varepsilon^\alpha$.

**Proposition 3.2.** Let $\alpha > 0$. Assume (R_0), (V_2), (V_u), and (E). In addition, suppose that

$$
D_q\mathcal{V}_u(q) = 0 \quad \text{for all } q \in \mathcal{Q},
$$

(V_u,1)

$$
E \in C^2([0, T] \times \mathcal{Q}) \quad \text{and}
$$

$$
\exists \mu > 0 \ \forall (t,q) \in [0, T] \times \mathcal{Q} : \ |E_q^2(t, q)| \geq \mu 1_{\mathbb{R}^{n \times n}} \quad \text{(uniform convexity w.r.t. } u),
$$

(E_1)

and that the initial data $(q_0^ε)_ε$ complying with (3.1) also fulfill

$$
|D_uE(0,q_0^ε)| \leq C\varepsilon^\alpha.
$$

(3.5)

Then, there exists a constant $C > 0$ such that for every $\varepsilon > 0$

$$
\|u'_ε(t)\|_{L^1(0,T;\mathbb{R}^n)} \leq C.
$$

(3.6)

**Proof.** By (V_u,1) there exists a matrix $\mathcal{V}_u \in \mathbb{R}^{n \times n}$ such that $\mathcal{V}_u(q) \equiv \mathcal{V}_u$ for all $q \in \mathcal{Q}$ with

$$
\mathcal{V}_u(q; u') = \mathcal{V}_u(u') := \frac{1}{2} (\mathcal{V}_u u', u')
$$

(3.7)

Therefore (2.7a) reduces to

$$
\varepsilon^\alpha \mathcal{V}_u u'_ε(t) + D_uE(t, u_ε(t), z_ε(t)) = 0 \quad \text{for a.a. } t \in (0, T).
$$

(3.8)

We differentiate (3.8) in time, and test the resulting equation by $u'_ε$. Thus we obtain, for almost all $t \in (0, T)$,

$$
0 = \varepsilon^\alpha (\mathcal{V}_u u''_ε(t), u'_ε(t)) + \langle D^2_{u,u}E(t, u_ε(t), z_ε(t))[u'_ε(t)], u'_ε(t) \rangle + \langle D^2_{u,z}E(t, u_ε(t), z_ε(t))[u'_ε(t)], z'_ε(t) \rangle = S_1 + S_2 + S_3,
$$

(3.9)

where $D^2_{u,z}$ denotes the second-order mixed derivative. Observe that

$$
S_1 = \frac{\varepsilon^\alpha}{2} \frac{d}{dt} \mathcal{V}_u(u'_ε), \quad S_2 \geq \mu |u'_ε|^2 \geq \tilde{\mu} \mathcal{V}_u(u'_ε),
$$

$$
S_3 \geq -C|u'_ε||z'_ε| \geq -C \sqrt{\mathcal{V}_u(u'_ε)}|z'_ε|.
$$
Indeed, to estimate $S_2$ we have used the uniform convexity of $\mathcal{E}(t, \cdot, \cdot, z)$, and the growth of $\mathcal{V}_a$ from $V_a$. The estimate for $S_3$ follows from $\sup_{t \in (0, T)} |D_{u,z}^2 \mathcal{E}(u_{\varepsilon}, t, \varepsilon_z(t))| \leq C$, due to (3.3b) and the fact that $D_{u,z}^2 \mathcal{E}$ is continuous on $[0, T] \times \mathcal{Z}$, cf. $V_a$. We thus infer from (3.9) that

$$\frac{1}{2} \frac{d}{dt} \mathcal{V}_u(u'_{\varepsilon}(t)) + \frac{\tilde{\mu}}{\varepsilon^\alpha} \mathcal{V}_u(u'_{\varepsilon}(t)) \leq C \frac{\varepsilon}{\varepsilon^\alpha} |\mathcal{V}_u(u'_{\varepsilon}(t))|^2$$

for a.e. $t \in (0, T)$, which rephrases as

$$\nu_{\varepsilon}(t)u'_{\varepsilon}(t) + \frac{\tilde{\mu}}{\varepsilon^\alpha} \nu_{\varepsilon}(t) \leq C_\alpha \nu_{\varepsilon}(t)$$

where we have used the place-holder $\nu_{\varepsilon} := \sqrt{\mathcal{V}_u(u'_{\varepsilon}(t))}$. We now argue as in [5] and observe that, without loss of generality, we may suppose that $\nu_{\varepsilon}(t) > 0$ (otherwise, we replace it by $\bar{\nu}_{\varepsilon} = \nu_{\varepsilon} + \delta$, which satisfies the same estimate, and then let $\delta \downarrow 0$). Hence, we deduce

$$\nu_{\varepsilon}(t)u'_{\varepsilon}(t) + \frac{\tilde{\mu}}{\varepsilon^\alpha} \nu_{\varepsilon}(t) \leq C \frac{\varepsilon}{\varepsilon^\alpha} |\mathcal{V}_u(u'_{\varepsilon}(t))|^2$$

where we have exploited the a priori estimate (3.3c) for $\mathcal{V}_u(u'_{\varepsilon}(t))$. We thus infer from (3.9) that $\mathcal{V}_u(u'_{\varepsilon}(t)) \leq t$. The estimate for $\mathcal{V}_a$ follows from (3.3c) and the fact that $|\mathcal{V}_u(u'_{\varepsilon}(t))| \leq C \varepsilon^{-\alpha} |\mathcal{E}(u_{\varepsilon}(t)-u_{\varepsilon}(0))| \leq C$. Hence, we find $\parallel a_{\varepsilon}^\alpha \parallel_{L^1(0, T)} \leq C \parallel \mathcal{V}_a(0) \parallel \leq C_1$. To estimate $a_{\varepsilon}^\alpha$ we use Young’s convolution inequality giving

$$\parallel a_{\varepsilon}^\alpha \parallel_{L^1(0, T)} \leq C \int_0^T \int_0^t \exp \left(-\frac{\tilde{\mu}}{\varepsilon^\alpha} (t-r) \right) |\mathcal{V}_u(u'_{\varepsilon}(t))|^2 \, dr \, dt \leq C \left( \int_0^T \exp \left(-\frac{\tilde{\mu}}{\varepsilon^\alpha} (t) \right) \right) \left( \int_0^T |\mathcal{V}_u(u'_{\varepsilon}(t))|^2 \, dt \right) \leq C_2$$

where we have exploited the a priori estimate (3.3c) for $\mathcal{V}_u(u'_{\varepsilon}(t))$. Thus, (3.10) implies (3.6), and we are done.

4. Limit passage with vanishing viscosity

In this section, we assume that we are given a sequence $(q_{\varepsilon})_{\varepsilon} \subset W^{1,2}(0, T; \Omega)$ of solutions to (2.3), satisfying the initial conditions $q_{\varepsilon}(0) = q_0^0$, such that estimate (3.2) holds. As we have shown in Proposition 3.2, the well-preparedness (3.5) of the initial data $(q_{\varepsilon})_{\varepsilon}$, the condition that the dissipation potential $\mathcal{V}_a$ does not depend on the state $q$, and the uniform convexity (3.4) of $\mathcal{E}$ with respect to $u$ guarantee the validity of (3.2). However, these conditions are not needed for the vanishing-viscosity analysis. Therefore, hereafter we will no longer impose (3.5), we will allow for a state-dependent dissipation potential $\mathcal{V}_u = \mathcal{V}_u(q; u')$, and we will stay with the basic conditions (3.3) on $\mathcal{E}$.

The energy-dissipation principle. Following the variational approach of [8, 13, 14], we will pass to the limit in (a parameterized version of) the energy identity (2.8).

Preliminarily, let us explicitly calculate the convex-conjugate of the dissipation potential $\mathcal{R}_\varepsilon$ (2.4).

**Lemma 4.1.** Assume $(\mathcal{R}_\varepsilon), (\mathcal{V}_a)$, and $(\mathcal{V}_b)$. Then, the Fenchel-Moreau conjugate (2.9) of $\mathcal{R}_\varepsilon$ is given by

$$\mathcal{R}_\varepsilon^*(q, \xi) = \frac{1}{\varepsilon} \mathcal{V}_\varepsilon^*(q; \xi) + \frac{1}{\varepsilon^\alpha} \mathcal{V}_b(q; \eta) \quad \text{for all } q \in \Omega \text{ and } \xi = (\eta, \zeta) \in \mathbb{R}^{n+m}, \quad (4.1)$$
where $V_u^*(q; \cdot)$ is the conjugate of $V_u(q; \cdot)$, and
\[
W_z^*(q; \cdot) = \min_{\omega \in R(q)} V_z(q; \cdot - \omega) \quad \text{with} \quad K(q) := \partial R_0(q, 0),
\]
(4.2)

$V_z^*(q; \cdot)$ is the conjugate of $V_z(q; \cdot)$, while $W_z^*$ is the conjugate of $R_0 + V_z$.

Proof. Since $R_\varepsilon(q; \cdot)$ is given by the sum of a contribution in the sole variable $z'$ and another in the sole variable $u'$, we have
\[
R_\varepsilon^*(q, \xi) = (\varepsilon^\alpha V_u)^*(q, \eta) + W_{z,\varepsilon}^*(q; \cdot) \quad \text{for all} \quad (\eta, \xi) \in \mathbb{R}^{n+m}
\]
where we have used the place-holder $W_{z,\varepsilon}^*(q; \cdot) := (R_\varepsilon(q, \cdot) + \varepsilon V_z(q; \cdot))^*(\cdot)$. Now, taking into account that $V_u$ is quadratic, there holds
\[
(\varepsilon^\alpha V_u)^*(q, \eta) = \varepsilon^\alpha V_u^*(q, \frac{1}{\varepsilon^\alpha} \eta) = \frac{1}{\varepsilon} V_u^*(q; \eta),
\]
whereas the inf-sup convolution formula (see e.g. [30]) yields $W_{z,\varepsilon}^*(q; \cdot) = \frac{1}{\varepsilon} W_z^*(q; \cdot)$ with $W_z^*(q; \cdot)$ from (4.2). \qed

In view of (4.1), the energy identity (2.8) rewrites as
\[
\begin{align*}
E(t, q_\varepsilon(t)) + \int_s^t R_\varepsilon(q_\varepsilon(r), z_\varepsilon(r)) + \varepsilon V_z(q_\varepsilon(r); z_\varepsilon(r)) + \varepsilon^\alpha V_u(q_\varepsilon(r); u'_\varepsilon(r)) \, dr \\
+ \int_s^t \frac{1}{\varepsilon} W_z^*(q_\varepsilon(r); -D_z E(r, q_\varepsilon(r))) + \frac{1}{\varepsilon} V_u^*(q_\varepsilon(r); -D_u E(r, q_\varepsilon(r))) \, dr \\
= E(s, q_\varepsilon(s)) + \int_s^t \partial_t E(r, q_\varepsilon(r)) \, dr.
\end{align*}
\]
(4.3)

In fact, the two integral terms on the left-hand side of (4.3) reflect the competition between the tendency of the system to be governed by viscous dissipation both for the variable $z$ and for the variable $u$, and its tendency to fulfill the local stability condition
\[
W_z^*(q(t); -D_z E(t, q(t))) = 0 \quad \text{i.e.} \quad -D_z E(t, q(t)) \in K(q(t)) \quad \text{for a.a.} \quad t \in (0, T)
\]
for $z$, and the equilibrium condition
\[
V_u^*(q(t); -D_u E(t, q(t))) = 0 \quad \text{i.e.} \quad -D_u E(t, q(t)) = 0 \quad \text{for a.a.} \quad t \in (0, T)
\]
for $u$, cf. also the discussion in Remark 4.4.

The parameterized energy-dissipation principle. We now consider the parameterized curves $(t_\varepsilon, q_\varepsilon) : [0, S_\varepsilon] \to [0, T] \times Q$, where for every $\varepsilon > 0$ the rescaling function $t_\varepsilon : [0, S_\varepsilon] \to [0, T]$ is strictly increasing, and $q_\varepsilon(s) = q_\varepsilon(t_\varepsilon(s))$. We shall suppose that sup$_{\varepsilon>0} S_\varepsilon < \infty$, and that
\[
\exists C > 0 \quad \forall \varepsilon > 0 \quad \forall s \in [0, S_\varepsilon] : \quad t_\varepsilon'(s) + |q_\varepsilon'(s)| \leq C.
\]
(4.4)

Remark 4.2. For instance, as in [9, 8] we might choose
\[
t_\varepsilon := \sigma_\varepsilon^{-1} \quad \text{with} \quad \sigma_\varepsilon(t) := \int_0^t (1 + |q'(r)|) \, dr,
\]
(4.5)

and set $S_\varepsilon := \sigma_\varepsilon(T)$. In fact, estimate (3.2) ensures that sup$_{\varepsilon>0} S_\varepsilon < \infty$. With the choice (4.5) for $t_\varepsilon$, the functions $(t_\varepsilon, q_\varepsilon)$ fulfill the normalization condition
\[
t_\varepsilon'(s) + |q_\varepsilon'(s)| = 1 \quad \text{for almost all} \quad s \in (0, S_\varepsilon).
\]
For the parameterized curves \((t_\varepsilon, q_\varepsilon)\), the energy-dissipation principle (4.3) reads
\[
E(t_\varepsilon(s_2), q_\varepsilon(s_2)) + \int_{s_1}^{s_2} \mathcal{M}_\varepsilon(q_\varepsilon(r), t'_\varepsilon(r), q'_\varepsilon(r), -D_qE(t_\varepsilon(r), q_\varepsilon(r))) \, dr
\]
\[
= E(t_\varepsilon(s_1), q_\varepsilon(s_1)) + \int_{s_1}^{s_2} \partial_t E(t_\varepsilon(r), q_\varepsilon(r))d(r) \quad \text{for all } 0 \leq s_1 \leq s_2 \leq S,
\]
\[(4.6)\]
where we have used the dissipation functional
\[
\mathcal{M}_\varepsilon(q, \tau, q', \xi) = \mathcal{R}_0(q, \zeta') + \frac{\varepsilon}{\tau} V_\varepsilon(q; \zeta') + \frac{\varepsilon\alpha}{\tau} V_u(q; u') + \frac{\tau}{\varepsilon} W^\alpha_u(q; \zeta) + \frac{\tau}{\varepsilon\alpha} V_\varepsilon^\alpha(q; \eta).
\]
\[(4.7)\]
The passage from (4.3) to (4.6) follows from the change of variables \(t \to t_\varepsilon(r)\), whence \(dt \to t'_\varepsilon(r)dr\), while \(q'_\varepsilon(t) \to \frac{t'_\varepsilon(r)}{t_\varepsilon(r)} q'_\varepsilon(r)\). In order to pass to the limit in (4.6) as \(\varepsilon \downarrow 0\), it is crucial to investigate the \(\Gamma\)-convergence properties of the family of functionals \((\mathcal{M}_\varepsilon)_\varepsilon\). The following result reveals that the \(\Gamma\)-limit of \((\mathcal{M}_\varepsilon)\), depends on whether the parameter \(\alpha\) is above, equal, or below the threshold value 1. Let us point out that, for \(\alpha \in (0, 1)\), setting \(\delta = \varepsilon^\alpha\) we rewrite \(\mathcal{M}_\varepsilon\) as
\[
\mathcal{M}_\varepsilon(q, \tau, (u', z'), (\eta, \zeta)) = \mathcal{R}_0(q, z') + \frac{\delta^{1/\alpha}}{\tau} V_\varepsilon(q; z') + \frac{\delta}{\tau} V_u(q; u') + \frac{\tau}{\delta} W^\alpha_u(q; \zeta) + \frac{\tau}{\delta} V_\varepsilon^\alpha(q; \eta).
\]
\[(4.8)\]
with \(1/\alpha > 1\). It is thus natural to expect that the upcoming results will be specular in the cases \(\alpha \in (0, 1)\) and \(\alpha > 1\).

**Proposition 4.3.** Assume \((\mathcal{R}_0), \{V_\varepsilon\}, \{V_\varepsilon^\alpha\}, \text{ and } \{E\}\). Then, the functionals \((\mathcal{M}_\varepsilon)_\varepsilon\) \(\Gamma\)-converge as \(\varepsilon \downarrow 0\) to \(\mathcal{M}_0 : \Omega \times [0, \infty) \times \Omega \times \mathbb{R}^{m+n} \to [0, \infty]\) defined by
\[
\mathcal{M}_0(q, \tau, (u', z'), (\eta, \zeta)) := \mathcal{R}_0(q, z') + \mathcal{M}^\text{red}_0(q, \tau, (u', z'), (\eta, \zeta)),
\]
\[(4.9)\]
where for \(\tau > 0\) we have
\[
\mathcal{M}^\text{red}_0(q, \tau, (u', z'), (\eta, \zeta)) = \left\{ \begin{array}{ll}
0 & \text{if } W^u(q; \zeta) = V_\varepsilon^\alpha(q; \eta) = 0, \\
\infty & \text{if } W^u(q; \zeta) + V_\varepsilon^\alpha(q; \eta) > 0,
\end{array} \right.
\]
\[(4.10)\]
while for \(\tau = 0\) we have the following cases:

- **For \(\alpha > 1\)**
\[
\mathcal{M}^\text{red}_0(q, 0, (u', z'), (\eta, \zeta)) = \left\{ \begin{array}{ll}
2\sqrt{V_\varepsilon(q; u')} \sqrt{V_\varepsilon^\alpha(q; \eta)} & \text{if } V_\varepsilon(q; z') = 0, \\
2\sqrt{V_\varepsilon(q; z')} \sqrt{W^u_z(q; \zeta)} & \text{if } V_\varepsilon(q; \eta) = 0, \\
\infty & \text{if } V_\varepsilon(q; z') V_\varepsilon^\alpha(q; \eta) > 0,
\end{array} \right.
\]
\[(4.11)\]

- **For \(\alpha = 1\)**
\[
\mathcal{M}^\text{red}_0(q, 0, (u', z'), (\eta, \zeta)) = 2\sqrt{V_\varepsilon(q; z')} + V_\varepsilon(q; u') \sqrt{W^u_z(q; \zeta)} + V_\varepsilon^\alpha(q; \eta),
\]
\[(4.12)\]

- **For \(\alpha \in (0, 1)\)**
\[
\mathcal{M}^\text{red}_0(q, 0, (u', z'), (\eta, \zeta)) = \left\{ \begin{array}{ll}
2\sqrt{V_\varepsilon(q; u')} \sqrt{V_\varepsilon^\alpha(q; \eta)} & \text{if } W^u_z(q; \zeta) = 0, \\
2\sqrt{V_\varepsilon(q; z')} \sqrt{W^u_z(q; \zeta)} & \text{if } V_\varepsilon(q; u') = 0, \\
\infty & \text{if } V_\varepsilon(q; u') W^u_z(q; \zeta) > 0.
\end{array} \right.
\]
\[(4.13)\]
Moreover, if \((\tau_{\varepsilon}, q'_{\varepsilon}) \to (\tau, q')\) in \(L^1(0, S; (0, T) \times \Omega)\) and if \((q_{\varepsilon}, \xi_{\varepsilon}) \to (q, \xi)\) in \(L^1(0, S; \Omega \times \mathbb{R}^{n+m})\), then for every \(0 \leq s_1 \leq s_2 \leq S\)

\[
\liminf_{\varepsilon \to 0} \int_0^S \mathcal{M}_\varepsilon(q_\varepsilon(s), \tau_\varepsilon(s), q'_\varepsilon(s), \xi_\varepsilon(s)) \, ds \geq \int_0^S \mathcal{M}_0(q(s), \tau(s), q'(s), \xi(s)) \, ds.
\]

(4.14)

**Remark 4.4.** Let us briefly comment on the expression (4.9) of the \(\Gamma\)-limit \(M_0\). To do so, we rephrase the constraints arising in the switching conditions for the reduced functional \(M_0^{\text{red}}\), cf. (4.10), (4.11), and (4.13). Indeed, it follows from (V\(\text{red}\)) \((q, \tau, u, \varepsilon)\) that

\[
\mathcal{V}_2(q; z') = 0 \iff z' = 0,
\]

\[
\mathcal{V}_u(q; \eta) = 0 \iff \eta = 0,
\]

\[
\mathcal{W}^*_2(q; \zeta) = 0 \iff \zeta \in K(q) = \partial \mathcal{R}_0(q, 0).
\]

Moreover, if \(q_{\varepsilon}(t, z)\) fulfills the equilibrium condition \(q_{\varepsilon} = D_u \mathcal{E}(t, u_{\varepsilon})\), then for every \((q_{\varepsilon}, \tau_{\varepsilon}, u_{\varepsilon}, \varepsilon)\)

\[
\mathcal{V}_{\text{red}}(q_{\varepsilon}, \tau_{\varepsilon}, u_{\varepsilon}, \varepsilon) = 0.
\]

Therefore, from (4.10) we read that for \(\tau > 0\) the functional \(M_0^{\text{red}}(q, \tau, \cdot, \cdot, \cdot)\) is finite (and indeed equal to 0) only for \(\eta\) and \(\zeta\) fulfilling

\[
\eta = 0, \quad \zeta \in K(q).
\]

For \(\tau = 0\), in the case \(\alpha > 1\), \(M_0^{\text{red}}(q, 0, \cdot, \cdot, \cdot)\) is finite if and only if either \(z' = 0\) or \(\eta = 0\). As we will see when discussing the physical interpretation of our vanishing-viscosity result, this means that, at a jump (i.e. when \(\tau = 0\)), either \(z' = 0\), i.e. \(z\) is frozen, or \(u\) fulfills the equilibrium condition \(\eta = D_u \mathcal{E}(t, u) = 0\).

Also in view of (4.8), the switching conditions for \(\alpha \in (0, 1)\) are specular to the ones for \(\alpha > 1\) in a generalized sense. In fact, \(M_0^{\text{red}}(q, 0, \cdot, \cdot, \cdot)\) is finite if and only if either \(u\) is frozen, or \(\zeta = D_u \mathcal{E}(t, z) \in K(q)\), meaning that \(z\) fulfills the local stability condition.

**Proof of Proposition 4.3.** Observe that

\[
\mathcal{M}_{\varepsilon}(q, \tau, (u', z'), (\eta, \zeta)) = \mathcal{R}_0(q, z') + \mathcal{M}_\varepsilon^{\text{red}}(q, \tau, (u', z'), (\eta, \zeta))
\]

with \(\mathcal{M}_\varepsilon^{\text{red}}(q, \tau, (u', z'), (\eta, \zeta)) := \frac{\varepsilon}{\alpha} \mathcal{V}_2(q; z') + \frac{\varepsilon}{\sqrt{\alpha}} \mathcal{V}_u(q; u') + \frac{\varepsilon}{\alpha} \mathcal{W}_2(q; \zeta) + \frac{\varepsilon}{\sqrt{\alpha}} \mathcal{V}_u(q; \eta).\) Since \(\mathcal{R}_0\) is continuous with respect to both variables \(q\) and \(z\) and does not depend on \(\varepsilon\), it is clearly sufficient to prove that the functionals \(\mathcal{M}_\varepsilon^{\text{red}}\) \(\Gamma\)-converge to \(M_0^{\text{red}}\), namely

\[
\Gamma\text{-lim inf estimate:}
\]

\[
(q_{\varepsilon}, \tau_{\varepsilon}, u'_{\varepsilon}, z'_{\varepsilon}, \eta_{\varepsilon}, \zeta_{\varepsilon}) \to (q, \tau, u', z', \eta, \zeta) \quad \text{for} \quad \varepsilon \to 0
\]

\[
\implies \mathcal{M}_0^{\text{red}}(q, \tau, (u', z'), (\eta, \zeta)) \leq \liminf_{\varepsilon \to 0} \mathcal{M}_\varepsilon^{\text{red}}(q_{\varepsilon}, \tau_{\varepsilon}, (u'_{\varepsilon}, z'_{\varepsilon}), (\eta_{\varepsilon}, \zeta_{\varepsilon})),
\]

(4.15)

\[
\Gamma\text{-lim sup estimate:}
\]

\[
\forall (q, \tau, u', z', \eta, \zeta) \exists (q_{\varepsilon}, \tau_{\varepsilon}, u'_{\varepsilon}, z'_{\varepsilon}, \eta_{\varepsilon}, \zeta_{\varepsilon}) : \quad (q_{\varepsilon}, \tau_{\varepsilon}, u'_{\varepsilon}, z'_{\varepsilon}, \eta_{\varepsilon}, \zeta_{\varepsilon}) \to (q, \tau, u', z', \eta, \zeta)
\]

\[
\mathcal{M}_0^{\text{red}}(q, \tau, (u', z'), (\eta, \zeta)) \geq \limsup_{\varepsilon \to 0} \mathcal{M}_\varepsilon^{\text{red}}(q_{\varepsilon}, \tau_{\varepsilon}, (u'_{\varepsilon}, z'_{\varepsilon}), (\eta_{\varepsilon}, \zeta_{\varepsilon})).
\]

(4.16)

Preliminarily, observe that minimizing with respect to \(\tau\) we obtain the lower bound

\[
\mathcal{M}_\varepsilon^{\text{red}}(q, \tau, (u', z'), (\eta, \zeta)) \geq 2\sqrt{\varepsilon} \mathcal{V}_2(q; z') + \varepsilon^{\alpha} \mathcal{V}_u(q; u') \sqrt{\frac{1}{\varepsilon} \mathcal{W}_2(q; \zeta) + \frac{1}{\varepsilon^\alpha} \mathcal{V}_u(q; \eta)}.
\]

(4.17)

In all the three cases \(\alpha > 1\), \(\alpha = 1\), and \(\alpha \in (0, 1)\), the expression (4.10) of \(M_0^{\text{red}}\) for \(\tau > 0\) can be easily checked. Indeed, for the \(\Gamma\text{-lim inf estimate, observe that it is trivial in the case} \mathcal{W}_2(q; \zeta) = \mathcal{V}_u(q; \eta) = 0, \) as \(\mathcal{M}_\varepsilon^{\text{red}}\) takes positive values for all \(\varepsilon > 0\). Suppose
now that $W^*_\epsilon(q; \zeta) + V^*_\epsilon(q; \eta) > 0$, e.g. that $V^*_\epsilon(q; \eta) > 0$. Now, $(q_\epsilon, \eta_\epsilon) \to (q, \eta)$ implies that $V^*_\epsilon(q; \eta_\epsilon) \geq \tilde{c} > 0$ for sufficiently small $\epsilon$, and from (4.17) we deduce that

$$\liminf_{\epsilon \downarrow 0} M^\text{red}_\epsilon(q_\epsilon, \tau_\epsilon, (u_\epsilon', z_\epsilon'), (\eta_\epsilon, \zeta_\epsilon)) = \infty = M^\text{red}_0(q, \tau, (u', z'), (\eta, \zeta)).$$

The $\Gamma$-lim sup estimate follows by taking the recovery sequence $(q_\epsilon, \tau_\epsilon, u_\epsilon', z_\epsilon', \eta_\epsilon, \zeta_\epsilon) = (q, \tau, u', z', \eta, \zeta)$. In fact, $W^*_\epsilon(q; \zeta) + V^*_\epsilon(q; \eta) > 0$, then the lim sup-inequality in (4.16) is trivial. If $W^*_\epsilon(q; \zeta) = V^*_\epsilon(q; \eta) = 0$, (4.16) can be checked straightforwardly.

For $\alpha = 1$, in the case $\tau = 0$, (4.17) clearly yields the $\Gamma$-lim inf estimate, whereas the $\Gamma$-lim sup one can be obtained by with the recovery sequence $(q_\epsilon, \tau_\epsilon, u_\epsilon', z_\epsilon', \eta_\epsilon, \zeta_\epsilon) = (q, \tau_\epsilon^*, u', z', \eta, \zeta)$ with

$$\tau_\epsilon^* = \epsilon \frac{\sqrt{V^*_\epsilon(q; z') + V^*_\epsilon(u; u')}}{\sqrt{W^*_\epsilon(q; \zeta)}}.$$

For $\alpha > 1$ and $\tau = 0$, the $\Gamma$-lim inf estimate follows taking into account that (4.17) yields

$$M^\text{red}_\epsilon(q, \tau, (u', z'), (\eta, \zeta)) \geq \frac{2}{\sqrt{\epsilon^{\alpha-1}}} \sqrt{V^*_\epsilon(q; z')V^*_\epsilon(q; \eta)}.$$  

(4.18)

Hence, if both $V^*_\epsilon(q; z') > 0$ and $V^*_\epsilon(q; \eta) > 0$, then $\liminf_{\epsilon \downarrow 0} M^\text{red}_\epsilon(q_\epsilon, \tau_\epsilon, (u_\epsilon', z_\epsilon'), (\eta_\epsilon, \zeta_\epsilon)) = \infty$. In the case when either $V^*_\epsilon(q; z') = 0$ or $V^*_\epsilon(q; \eta) = 0$, we deduce the $\Gamma$-lim inf estimate from (4.17).

For the $\Gamma$-lim sup estimate, we again take the recovery sequence $(t, q, \tau_\epsilon^*, u', z', \eta, \zeta)$, where now

$$\tau_\epsilon^* = \epsilon \frac{\sqrt{V^*_\epsilon(q; z') + z'^{-\alpha}V^*_\epsilon(u; u')}}{\sqrt{W^*_\epsilon(q; \zeta) + \frac{1}{\epsilon^{\alpha}}V^*_\epsilon(q; \eta)}}.$$

The discussion of the case $\alpha \in (0, 1)$ is completely analogous, also in view of (4.8).

Finally, in order to prove (4.14) we use Ioffe’s theorem [31]. We introduce a functional $\overline{M} : [0, \infty) \times \Omega \times [0, \infty) \times \Omega \times \mathbb{R}^{n+m} \to [0, \infty]$ subsuming the functionals $M_\epsilon$ and $M_0$, viz.

$$\overline{M}(\epsilon; q, \tau, q', \xi) := \begin{cases} M_\epsilon(q, \tau, q', \xi) & \text{if } \epsilon > 0, \\ M_0(q, \tau, q', \xi) & \text{if } \epsilon = 0. \end{cases}$$

Arguing in the very same way as in the proof of [8, Lemma 3.1], it follows that the functional $\overline{M}$ is lower semicontinuous on $[0, \infty) \times \Omega \times [0, \infty) \times \Omega \times \mathbb{R}^{n+m}$, and that $(\tau, q') \to \overline{M}(\epsilon; q, \tau, q', \xi)$ is convex for all $(\epsilon, q, \xi) \in [0, \infty) \times \Omega \times \mathbb{R}^{n+m}$. Hence, Ioffe’s theorem yields

$$\liminf_{\epsilon \downarrow 0} \int_0^S \overline{M}(\epsilon; q_\epsilon(s), \tau_\epsilon(s), q'_\epsilon(s), \xi_\epsilon(s)) ds \geq \int_0^S \overline{M}(0; q(s), \tau(s), q'(s), \xi(s)) ds,$$

whence (4.14).

\[\square\]

Observe that the functional $M_0$ in (4.19) fulfills for all $(q, \tau) \in \Omega \times [0, \infty)$ the estimate

$$M_0(q, \tau, q', \xi) \geq \langle q', \xi \rangle = \langle u', \eta \rangle + \langle z', \zeta \rangle \geq \mathcal{R}_0(q, z') = M_0(q, \tau, q', \xi).$$  

Indeed, for $\tau > 0$, the inequality is trivial if either $V^*_\epsilon(q; \eta) > 0$ or $W^*_\epsilon(q; \zeta) > 0$. When both of them equal 0, then $\eta = 0$ and $(q', \xi) = \langle \zeta', \zeta \rangle \leq \mathcal{R}_0(q, z') = M_0(q, \tau, q', \xi)$. For $\tau = 0$, e.g. in the case $\alpha > 1$ we have, if $z' = 0$,

$$\langle q', \xi \rangle = \langle \eta, u' \rangle \leq \sqrt{V^*_\epsilon(q; u')} \sqrt{V^*_\epsilon(q; \eta)} = M^\text{red}_0(q, \tau, q', \xi) + 0 = M_0(q, \tau, q', \xi).$$
while, if \( \eta = 0 \),
\[
\langle q', \xi \rangle = \langle \zeta, z' \rangle = \langle \zeta - \omega, z' \rangle + \langle \omega, z' \rangle
\leq \sqrt{\langle V'_2(q) z', z' \rangle} \sqrt{\langle V'_2(q) (\zeta - \omega), (\zeta - \omega) \rangle} + R_0(z') = M_0(q, \tau, q', \xi)
\]
where we have chosen \( w \in K(q) \) such that \( W^*_2(q; \zeta) = V^*_2(q; \zeta - \omega) = \frac{1}{2} \langle V'_2(q) (\zeta - \omega), (\zeta - \omega) \rangle \),
and used that \( \langle \omega, z' \rangle \leq R_0(z') \).

For the ensuing discussions, the set where \( (4.19) \) holds as an equality shall play a crucial role. We postpone its precise definition right before the statement of Proposition 4.8, cf. \( (4.30) \) ahead.

**The vanishing-viscosity result.** Theorem 4.5 below states that, up to a subsequence the parameterized solutions \((t_\varepsilon, q_\varepsilon)\) of the (Cauchy problems for the) viscous system \( (2.3) \), converge to a parameterized curve \((t, q)\), complying with the analog of the energy balance \( (4.6) \), with \( M_0 \) in place of \( M_0 \).

We postpone after the proof of Theorem 4.5 a thorough analysis of the notion of solution to the rate-independent system \( (2.2) \) thus obtained. Let us instead mention in advance that the line of the argument for proving the limiting parameterized energy balance \( (4.22) \) is by now quite standard, cf. the proofs of [3] Thm. 3.3, [13] Thm. 5.5. In fact, the upper energy estimate (i.e. the inequality \( \leq \) for \( (4.22) \)) shall follow from lower semicontinuity arguments, based on the application of the Ioffe Theorem [31]. The lower energy estimate \( \geq \) will instead ensue from the chain rule \( (2.10) \). We also point out that, for the compactness argument it is actually not necessary to start from parameterized curves for which estimate \( (4.7) \) holds, uniformly w.r.t. time. In fact, the uniform integrability of the sequence \((t'_\varepsilon, q'_\varepsilon)\) is sufficient, cf. \( (4.20) \) below.

**Theorem 4.5.** Assume \((R_0, (V_2), (V_4), (Q), (E))\). Let \((q_\varepsilon) \subset W^{1,2}(0, T; \Omega)\) be a sequence of solutions to the Cauchy problem for \( (2.3) \). Choose nondecreasing surjective parameterizations \( t_\varepsilon : [0, S_\varepsilon] \to [0, T] \) and set \( q_\varepsilon(s) = (u_\varepsilon(s), z_\varepsilon(s)) := q_\varepsilon(t_\varepsilon(s)) \) for \( s \in [0, S_\varepsilon] \). Suppose that \( S_\varepsilon \to S \) as \( \varepsilon \downarrow 0 \) up to a subsequence, and that there exist \( q_0 \in \Omega \) and \( m \in L^1(0, S) \) such that \( q_\varepsilon(0) \to q_0 \), and
\[
m_\varepsilon := \ell'_\varepsilon + |q'_\varepsilon| \to m \quad \text{in } L^1(0, S) \text{ as } \varepsilon \downarrow 0. \tag{4.20}
\]

Then, there exist a (not-relabeled) subsequence and a parameterized curve \((t, q) \in W^{1,1}([0, S]; [0, T] \times \Omega)\) such that as \( \varepsilon \downarrow 0 \)
\[
(t_\varepsilon, q_\varepsilon) \to (t, q) \text{ in } C^0([0, S]; [0, T] \times \Omega), \tag{4.21}
\]
\[
t'_\varepsilon + |q'_\varepsilon| \leq m \text{ a.e. in } (0, S), \text{ and } (t, q) \text{ fulfills the (parameterized) energy identity}
\]
\[
\mathcal{E}(t(s_2), q(s_2)) + \int_{s_1}^{s_2} M_0(q(r), t'(r), q'(r), -D_q \mathcal{E}(t(r), q(r))) \, dr
= \mathcal{E}(t(s_1), q(s_1)) + \int_{s_1}^{s_2} \partial_q \mathcal{E}(t(r), q(r)) t'(r) \, dr \quad \text{for all } 0 \leq s_1 \leq s_2 \leq S. \tag{4.22}
\]

**Proof.** Up to a reparameterization, we may suppose that the curves \((t_\varepsilon, q_\varepsilon)\) are defined on the fixed time interval \([0, S]\). We split the proof in three steps.

**Step 1: compactness.** Observe that for every \( 0 \leq s_1 \leq s_2 \leq S \)
\[
|q_\varepsilon(s_1) - q_\varepsilon(s_2)| \leq \int_{s_1}^{s_2} |q'_\varepsilon(s)| \, ds \leq \int_{s_1}^{s_2} m_\varepsilon(s) \, ds. \tag{4.23}
\]
Since \((q_\varepsilon(0))_\varepsilon\) is bounded, we deduce from (4.23) that \((q_\varepsilon)_\varepsilon \subset C^0([0,S]; \mathcal{Q})\) is bounded as well. What is more, as the family \((m_\varepsilon)_\varepsilon\) is uniformly integrable (4.20), \((q_\varepsilon)_\varepsilon\) complies with the equicontinuity condition of the Ascoli-Arzelà Theorem and so does \((t_\varepsilon)_\varepsilon\), by the analog of estimate (4.23). Hence, (4.21) follows. Taking into account that \(E \in C^1([0,T] \times \mathcal{Q})\), we immediately conclude from (4.21) that

\[
\mathcal{E}(t_\varepsilon, q_\varepsilon) \to \mathcal{E}(t, q), \quad D_q \mathcal{E}(t_\varepsilon, q_\varepsilon) \to D_q \mathcal{E}(t, q), \quad \partial_t \mathcal{E}(t_\varepsilon, q_\varepsilon) \to \partial_t \mathcal{E}(t, q) \text{ uniformly on } [0,S].
\]

(4.24)

Furthermore, (4.20) also yields that the sequences \((t_\varepsilon')_\varepsilon\) and \((q_\varepsilon')_\varepsilon\) are uniformly integrable. Thus, by the Pettis Theorem, up to a further extraction we find

\[
t_\varepsilon' \rightharpoonup t' \quad \text{in } L^1(0,S), \quad q_\varepsilon' \rightharpoonup q' \quad \text{in } L^1(0,S; \mathcal{Q}),
\]

whence \(t' + |q'| \leq m\) a.e. in \((0,S)\).

Step 2: upper energy estimate. We now take the limit as \(\varepsilon \downarrow 0\) of the (parameterized) energy-dissipation principle (4.6) for every \(0 \leq s_1 \leq s_2 \leq S\):

\[
\mathcal{E}(t(s_2), q(s_2)) + \int_{s_1}^{s_2} \mathcal{M}_0(q(r), t'(r), q'(r), -D_q \mathcal{E}(t(r), q(r))) \, dr \\
\leq \lim_{\varepsilon \downarrow 0} \mathcal{E}(t_\varepsilon(s_2), q_\varepsilon(s_2)) + \limsup_{\varepsilon \downarrow 0} \int_{s_1}^{s_2} \mathcal{M}_\varepsilon(q_\varepsilon(r), t_\varepsilon'(r), q_\varepsilon'(r), -D_q \mathcal{E}(t_\varepsilon(r), q_\varepsilon(r))) \, dr \\
= \lim_{\varepsilon \downarrow 0} \mathcal{E}(t_\varepsilon(s_1), q_\varepsilon(s_1)) + \lim_{\varepsilon \downarrow 0} \int_{s_1}^{s_2} \partial_t \mathcal{E}(t_\varepsilon(r), q_\varepsilon(r)) t_\varepsilon'(r) \, dr \\
\overset{(\alpha)}{=} \mathcal{E}(t(s_1), q(s_1)) + \int_{s_1}^{s_2} \partial_t \mathcal{E}(t(r), q(r)) t'(r) \, dr,
\]

where (1) follows from the energy convergence in (4.24) and the previously proved (4.14), and (2) from (4.24), again, combined with the first of (4.25). This concludes the upper energy estimate.

Step 3: lower energy estimate. We have for all \(0 \leq s_1 \leq s_2 \leq S\) that

\[
\mathcal{E}(t(s_1), q(s_1)) + \int_{s_1}^{s_2} \partial_t \mathcal{E}(t(r), q(r)) t'(r) \, dr \\
\overset{(\alpha)}{=} \mathcal{E}(t(s_2), q(s_2)) + \int_{s_1}^{s_2} \{-D_q \mathcal{E}(t(r), q(r)), q'(r)\} \, dr \\
\overset{(\alpha)}{\leq} \mathcal{E}(t(s_2), q(s_2)) + \int_{s_1}^{s_2} \mathcal{M}_0(q(r), t'(r), q'(r), -D_q \mathcal{E}(t(r), q(r))) \, dr,
\]

where (1) follows from the chain rule, and (2) is due to inequality (4.19). In this way, we conclude (4.22). Finally, combining (4.26) and (4.27) it is easy to deduce that

\[
\lim_{\varepsilon \downarrow 0} \int_{s_1}^{s_2} \mathcal{M}_\varepsilon(q_\varepsilon(r), t_\varepsilon'(r), q_\varepsilon'(r), -D_q \mathcal{E}(t_\varepsilon(r), q_\varepsilon(r))) \, dr = \int_{s_1}^{s_2} \mathcal{M}_0(q(r), t'(r), q'(r), -D_q \mathcal{E}(t(r), q(r))) \, dr
\]

for all \(0 \leq s_1 \leq s_2 \leq S\), whence \(\int_{s_1}^{s_2} \mathcal{R}_0(q_\varepsilon(r), t_\varepsilon'(r)) \, dr \to \int_{s_1}^{s_2} \mathcal{R}_0(q(r), t'(r)) \, dr\).
Parameterized Balanced Viscosity solutions. Let us now gain further insight into the notion of solution to system (1.1) arising from the vanishing-viscosity limit. First of all, we fix its definition.

Definition 4.6. Let \((R_0, V_z, V_u, \mathcal{E})\) comply with \((R_0), (V_z), (V_u), \) and \((E)\). A curve \((t, q) \in W^{1,1}([0, S]; [0, T] \times \Omega)\) is called a parameterized Balanced Viscosity (pBV, for short) solution to the rate-independent system \((\Omega, \mathcal{E}, R_0 + \varepsilon V_z + \varepsilon^\alpha V_u)\) if \(t : [0, S] \rightarrow [0, T]\) is nondecreasing, and the pair \((t, q)\) complies with the energy-dissipation principle (4.22) for all \(0 \leq s_1 \leq s_2 \leq S\).

Furthermore, \((t, q)\) is called

\[ t'(s) + |q'(s)| > 0 \quad \text{for a.a. } s \in (0, S); \quad (4.28) \]

- non-degenerate, if
- surjective, if \(t : [0, S] \rightarrow [0, T]\) is surjective.

The name parameterized Balanced Viscosity solutions derives from the fact that the dissipation potential \(\mathcal{M}_0\) (cf. (4.9)) featuring in (4.22) is defined in a nontrivial way from the three dissipation potentials \(R_0, V_u, \) and \(V_z\) in such a manner that the subtle balance between the different effects is obtained in the limit \(\varepsilon \downarrow 0\).

Remark 4.7. Observe that, even in the case when the function \(m\) in (4.20) is a.e. strictly positive, Theorem 4.5 does not guarantee the existence of non-degenerate pBV solutions. However, any degenerate pBV solution \((t, q)\) can be reparameterized to a non-degenerate one \((\tilde{t}, \tilde{q}) : [0, \tilde{S}] \rightarrow [0, T] \times \Omega\), even fulfilling the normalization condition

\[ \tilde{t}'(\sigma) + \tilde{q}'(\sigma) = 1 \quad \text{for a.a. } \sigma \in (0, \tilde{S}). \quad (4.29) \]

Indeed, following [8 Rmk. 2], starting from a (possibly degenerate) solution \((t, q)\), we set

\[ \sigma(s) := \int_0^s t'(r) + |q'(r)| \, dr \quad \text{and } \tilde{S} := \sigma(S), \]

and define \((\tilde{t}(\sigma), \tilde{q}(\sigma)) := (t(s), q(s))\) if \(\sigma = \sigma(s)\). Then, the very same calculations as in [8 Rmk. 2] lead to (4.29).

We conclude this section with a characterization of pBV solutions in the same spirit as [8 Prop. 2] and [13 Prop. 5.3], [14 Cor. 4.5]. We show that the energy identity (4.22) defining the concept of pBV solutions is equivalent to the corresponding energy inequality on the interval \([0, S]\), and to the energy inequality in a differential form. Finally, (4.31) below provides a further reformulation of this solution concept which involves the contact set (cf. [13 [14]).

\[ \Sigma(q) := \{(\tau, q', \xi) \in [0, \infty) \times \Omega \times \mathbb{R}^{n+m} : \mathcal{M}_0(q, \tau, q', \xi) = \langle q', \xi \rangle\} \quad (4.30) \]

Observe that for all \(q \in \Omega\) the set \(\Sigma(q)\) is closed, as the functional \(\mathcal{M}_0(q, \tau, q', \xi)\) is lower semicontinuous. In Proposition 5.1 we will provide the explicit representation of \(\Sigma(q)\). This and (4.31) will be at the core of the reformulation of pBV solutions in terms of subdifferential inclusions, which we will discuss in Sec. 5.

Proposition 4.8. Let \((R_0, V_z, V_u, \mathcal{E})\) comply with \((R_0), (V_z), (V_u), (E)\). A curve \((t, q) \in W^{1,1}([0, S]; [0, T] \times \Omega)\), with \(t\) nondecreasing, is a pBV solution to the rate-independent system \((\Omega, \mathcal{E}, R_0 + \varepsilon V_z + \varepsilon^\alpha V_u)\) if and only if one of the following equivalent conditions is satisfied:

(i) (4.22) holds as an inequality on \((0, S)\), i.e.

\[
\mathcal{E}(t(S), q(S)) + \int_0^S \mathcal{M}_0(q(r), t'(r), q'(r), -D_q\mathcal{E}(t(r), q(r))) \, dr \\
\leq \mathcal{E}(t(0), q(0)) + \int_0^S \partial_t\mathcal{E}(t(r), q(r))t'(r) \, dr;
\]
The contact set is given by

\[(iii) \text{ the triple } (t', q', -D_q \mathcal{E}(t, q)) \text{ belongs to the contact set, i.e.} \]
\[(t'(s), q'(s), -D_q \mathcal{E}(t(s), q(s))) \in \Sigma(q(s)) \quad \text{for a.a. } s \in (0, S). \quad (4.31)\]

The proof of Proposition 4.8 is omitted: it follows by exploiting the chain rule (2.10), with arguments akin to those in the proof of Theorem 4.5, see also [8] Prop. 2 and 13] Prop. 5.3, [13 Cor. 4.5].

5. Physical interpretation

The following result provides a thorough description of the (closed) contact set \(\Sigma(q)\), cf. (4.30). As we will see, the representation of \(\Sigma(q)\) is substantially different in the three cases \(\alpha > 1, \alpha = 1, \text{ and } \alpha \in (0, 1)\). That is why, in Proposition 5.1 below we will use the notation \(\Sigma_{\alpha>1}(q), \Sigma_{\alpha=1}(q), \text{ and } \Sigma_{\alpha\in(0,1)}(q)\). We will prove that these sets are given by the union of subsets describing the various evolution regimes for the variables \(u\) and \(z\). The notation for these subsets will be of the form

\[A_rB_s \quad \text{with } A, B \in \{E, R, V, B\} \text{ and } r, s \in \{u, z\}.\]

The letters \(E, R, V, B\) stand for Equilibrated, Rate-independent, Viscous, and Blocked, respectively. For instance, \(E_0R_2\) is the set of \((\tau, q', \xi)\) corresponding to equilibrium for \(u\) and rate-independent evolution for \(z\), cf. (5.2); we postpone more comments after the statement of Proposition 5.1. Observe that all of these sets depend on the state variable \(q\), as does \(\Sigma(q)\). However, for simplicity we will not highlight this in their notation. In their description we shall always refer to the representation \(q' = (u', z')\) for the velocity variable, and \(\xi = (\eta, \zeta)\) for the force variable.

**Proposition 5.1.** Assume \((R_0), (V_r), (V_u), \text{ and } (E)\). Then, we have the following results:

**Case \(\alpha > 1\):** The contact set is given by

\[\Sigma_{\alpha>1}(q) = E_0R_2 \cup V_uB_2 \cup E_uV_z \quad \text{with} \quad (5.1)\]

\[E_uR_2 := \{(\tau, q', \xi) : \tau > 0, \quad q' = (u', z'), \quad \xi = (0, \zeta) \text{ and } \partial_0R_0(q, z') \ni \zeta\}, \quad (5.2)\]

\[V_uB_2 := \{(\tau, q', \xi) : (\tau, q', \xi) = (0, (u', 0), (\eta, \zeta)) \text{ and } \exists \theta_u \in [0, 1] : \theta_uV_u(q)u' = (1-\theta_u)\eta\}, \quad (5.3)\]

\[E_uV_z := \{(\tau, q', \xi) : \tau = 0, \quad q' = (u', z'), \quad \xi = (0, \zeta) \text{ and} \]
\[\exists \theta_z \in [0, 1] : (1-\theta_z)\partial_0R_0(q, z') + \theta_zV_z(q)z' \ni (1-\theta_z)\zeta\}. \quad (5.4)\]

**Case \(\alpha = 1\):** The contact set is given by

\[\Sigma_{\alpha=1}(q) = E_0R_2 \cup V_uV_z \quad \text{with} \quad (5.5)\]

\[V_uV_z := \left\{(\tau, q', \xi) : \tau = 0 \text{ and } \exists \theta \in [0, 1] : \right\}
\[\left\{(\thetaV_u(q)u' = (1-\theta)\eta,\right.\]
\[\left.\text{and } (1-\theta)\partial_0R_0(q, z') + \thetaV_z(q)z' \ni (1-\theta)\zeta\right\}. \quad (5.6)\]

**Case \(\alpha \in (0,1)\):** The contact set is given by

\[\Sigma_{\alpha\in(0,1)}(q) = E_0R_2 \cup B_uV_z \cup V_uR_z \quad \text{with} \quad (5.7)\]

\[B_uV_z := \left\{(\tau, q', \xi) : \tau = 0, \quad q' = (0, z'), \quad \xi = (\eta, \zeta) \text{ and} \right\]
\[\exists \theta_z \in [0, 1] : (1-\theta_z)\partial_0R_0(q, z') + \theta_zV_z(q)z' \ni (1-\theta_z)\zeta\}, \quad (5.8)\]

\[V_uR_z := \left\{(\tau, q', \xi) : (\tau, q', \xi) = (0, (u', z'), (\eta, \zeta)), \right\}
\[\left\{\exists \theta_u \in [0, 1] : \theta_uV_u(q)u' = (1-\theta_u)\eta, \text{ and } \right.\]
\[\left.\partial_0R_0(q, z') \ni \zeta\right\}. \quad (5.9)\]
As (4.31) reveals, the contact set encompasses all the relevant information on the evolution of a parameterized Balanced Viscosity solution. The form of the sets $E_u R_2$, $V_u B_2$, ... which constitute the contact set is strictly related to the mechanical interpretation of pBV solutions that shall be explored at the end of this section. Let us just explain here that

- the set $E_u R_2$ corresponds to equilibrium for the variable $u$ (as $\eta = 0$), and a stick-slip regime for $z$, which evolves rate-independently as expressed by $\partial \mathcal{R}_0(q, z') \ni \zeta$. Observe that the stationary state $u' = z' = 0$ is also encompassed.
- The set $V_u B_2$ corresponds to the case in which the variable $u$ still has to relax to an equilibrium and thus is governed by a fast dynamics at a jump $\tau = 0$, while $z$ is “blocked by viscosity” and thus stays constant ($z' = 0$).
- The set $E_u V_z$ corresponds to the regime in which $z$ evolves according to viscosity at a jump $\tau = 0$, and $u$ follows $z$ in such a way that it is at an equilibrium ($\eta = 0$).
- The set $V_u V_z$ corresponds to the case where the evolution of the system at a jump $\tau = 0$ is governed by viscosity both in $u$ and in $z$.
- The set $B_2 V_z$ encompasses the case in which the variable $z$ at a jump $\tau = 0$ evolves according to viscosity, while $u$ is blocked by viscosity ($u' = 0$).
- The set $V_u R_2$ describes viscous evolution for $u$ and rate-independent evolution for $z$.

**Remark 5.2.** Let us stress once more that, as mentioned in advance, in the vanishing-viscosity limit the evolution regimes for $\alpha > 1$ and $\alpha \in (0,1)$ mirror each other. Indeed, formulae (5.1) and (5.7) are specular, up to observing that the analog of the equilibrium regime $E_u$ is indeed the rate-independent regime $R_2$, see also Figure 5.1.

**Proof of Proposition 5.2.** In all the three cases $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0,1)$, for $\tau > 0$ the contact condition $\mathcal{M}_0(q, \tau, q', \xi) = \langle \xi, q' \rangle$ can hold only if the constraints $\eta = 0$ and $\zeta \in K(q)$ are satisfied. Then, $\mathcal{M}_0(q, \tau, q', \xi) = \langle \xi, q' \rangle$ reduces to $\mathcal{R}_0(q, z') = \langle \zeta, z' \rangle$. Since $\zeta \in K(q)$, this is equivalent to $\zeta \in \partial \mathcal{R}_0(q, z')$ by (2.5). This gives the set $E_u R_2$, which contributes to the contact set $\Sigma(q)$ in the three cases $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0,1)$.

For $\alpha = 1$, observe that in the case $\tau = 0$ the contact condition is

$$
\mathcal{R}_0(q, z') + 2\sqrt{V_z(q; z')} + V_u(q; u') \sqrt{W_z(q; \zeta)} + V_u(q; \eta) = \langle \zeta, z' \rangle + \langle \eta, u' \rangle.
$$

(5.10)

Let us first address the case in which $\sigma_1 := \sqrt{V_z(q; z')} + V_u(q; u') = 0$ or $\sigma_2 := \sqrt{W_z(q; \zeta)} + V_u(q; \eta) = 0$. The former case corresponds to the stationary state $u' = z' = 0$, which means $\theta = 1$ in (5.6). The latter case corresponds to $\theta = 0$ in (5.6). If $\sigma_1 \sigma_2 > 0$, then we rewrite $2\sigma_1 \sigma_2$ as $\lambda \sigma_1^2 + \frac{1}{\lambda} \sigma_2^2$, with $\lambda > 0$ given by $\lambda = \frac{\sigma_2^2}{\sigma_1^2}$. With such $\lambda$, (5.10) rewrites as

$$
\mathcal{R}_0(q, z') + \lambda (V_z(q; z') + V_u(q; u')) + \frac{1}{\lambda} (W_z(q; \zeta) + V_u(q; \eta)) = \langle \zeta, z' \rangle + \langle \eta, u' \rangle.
$$

Upon multiplying both sides by $\lambda$, using that $V_z$ and $V_u$ are positively homogeneous of degree 2, and rearranging terms, we get

$$
\mathcal{R}_0(q, \lambda z') + V_z(q; \lambda z') + W_z(q; \zeta) - \langle \zeta, \lambda z' \rangle = \langle \eta, \lambda u' \rangle - V_u(q; \lambda u') - V_u(q; \eta).
$$

By the Fenchel-Moreau equivalence, this gives

$$
\mathcal{V}_u(q)(\lambda u') = \eta,
$$

$$
\partial \mathcal{R}_0(q, \lambda z') + V_z(q)(\lambda z') \ni \zeta
$$
with $\lambda > 0$. Then, (5.6) follows with $\theta \in (0,1)$ such that $\lambda = \frac{\theta}{1-\theta}$. All in all, for $\alpha = 1$ we have proved that, if $(\tau, q', \xi) \in \Sigma_{a>1}(q)$, then either $(\tau, q', \xi) \in E_u R_2$, or $(\tau, q', \xi) \in V_u V_2$. This concludes the proof of (5.5) for $\Sigma_{a=1}(q)$.

In the case $\alpha > 1$ and $\tau = 0$, $M_0(q, \tau, q', \xi)$ is finite if and only if either $z' = 0$, or $\eta = 0$. In the former case, the contact condition reduces to $\sqrt{\langle V_u(q) u', u' \rangle} \sqrt{\langle V_u(q)^{-1} \eta, \eta \rangle} = \langle \eta, u' \rangle$, which is equivalent to the fact that there exists $\theta_u \in [0,1]$ with $\theta_u V_u(q) u' = (1 - \theta_u) \eta$. This yields the set $E_u B_2$. Theorem 5.3

5.1. Observe that conditions (5.13a) and (5.13c) are specular (cf. Remark 5.2), revealing once more that the evolution regimes for $\alpha > 1$ and $\alpha < 1$ reflect each other. Nonetheless, a major difference occurs in that, under suitable conditions, for $\alpha > 1$ the regime $V_u B_2$ only occurs at the beginning, when $u$ relaxes fast to equilibrium, cf. Proposition 5.5.

The main result of this paper is the following theorem, which is in fact a direct consequence of the characterization (4.31) of pBV solutions in terms of the contact set, and of Proposition 5.1. Observe that, we confine ourselves to non-degenerate pBV solutions only. This is not restrictive, in view of Remark 4.7.

**Theorem 5.3** (Reformulation as a system of subdifferential inclusions). Assume $[R_0]$, $[V_u]$, $[V_2]$, $[E]$. A curve $(t, q) \in W^{1,1}([0, S]; [0, T] \times \Omega)$ with nondecreasing $t$ is a non-degenerate parameterized Balanced Viscosity solution to the rate-independent system $(\Omega, E, R_0 + \varepsilon V_2 + \varepsilon^a V_u)$ if and only if $t' + |q'| > 0$ a.e. in $(0, S)$ and there exist two Borel functions $\theta_u$, $\theta_z : [0, S] \to [0, 1]$ such that the pair $(t, q)$ with $q = (u, z)$ satisfies the system of equations for a.a. $s \in (0, S)$:

$$
\theta_u(s) V_u(q(s)) u'(s) + (1 - \theta_u(s)) D_u E(t(s), u(s), z(s)) \geq 0,
$$

$$
(1 - \theta_z(s)) \partial \partial R_0(q(s), z'(s)) + \theta_z(s) V_2(q(s)) z'(s) + (1 - \theta_z(s)) D_z E(t(s), u(s), z(s)) \geq 0,
$$

(5.11)

and the following additional conditions depending on $\alpha$:

$$
\alpha > 1: \quad \theta_u(s) (1 - \theta_z(s)) = 0;
$$

(5.13a)

$$
\alpha = 1: \quad \theta_u(s) = \theta_z(s);
$$

(5.13b)

$$
\alpha \in (0, 1): \quad \theta_z(s) (1 - \theta_u(s)) = 0.
$$

(5.13c)

Figure 5.1 displays the structure of the allowed values for the parameters $(t', \theta_u, \theta_z)$ depending on the value of $\alpha$.

**Remark 5.4.** Observe that conditions (5.13a) and (5.13c) are specular (cf. Remark 5.2), revealing once more that the evolution regimes for $\alpha > 1$ and $\alpha < 1$ reflect each other. Nonetheless, a major difference occurs in that, under suitable conditions, for $\alpha > 1$ the regime $V_u B_2$ only occurs at the beginning, when $u$ relaxes fast to equilibrium, cf. Proposition 5.5.
fast relaxation of \( V \) whereas \( \theta \) is constant and \( \alpha > 1 \), are displayed via the allowed combinations of the triples \((t', \theta_u, \theta_z)\).

Finally, let us get further insight into the mechanical interpretation of system (5.11), with the constraints (5.12) and (5.13a)–(5.13c). Preliminarily, let us point out that, as in the case of Proposition 5.5.

**Case \( \alpha > 1 \): fast relaxation of \( u \).** Here \( u \) relaxes faster to equilibrium than \( z \). With (5.12) and (5.13a) we are imposing at a jump that either \( \zeta' = 0 \) (which follows from \( \theta_z = 1 \), i.e. \( V_uB_z \)) or \( u \) is at equilibrium (corresponding to \( \theta_u = 0 \), i.e. \( E_uV_z \)). In fact, \( z \) cannot change until \( u \) has relaxed to equilibrium. When \( u \) has reached the equilibrium, then \( z \) may have either a sliding jump (i.e. \( \theta_z = 0 \)), or a viscous jump (\( \theta_z \in (0, 1) \)).

Our next result shows that, in fact, under the condition that the energy \( \mathcal{E} \) is uniformly convex with respect to the variable \( u \) (cf. Proposition 3.2), after an initial phase in which \( z \) is constant and \( u \) relaxes to an equilibrium evolving by viscosity (i.e. the solution is in regime \( V_uB_z \)), \( u \) never leaves the equilibrium afterwards. In that case the evolution of the system is completely described by \( z \), which turns out to be a parameterized Balanced Viscosity solution to the rate-independent system driven by the reduced energy functional obtained minimizing out the variable \( u \).

**Proposition 5.5.** Assume \([R_0], [V_u], [V_z], [\mathcal{E}]\). Additionally, suppose that \( \mathcal{E} \) complies with the uniform convexity \([E_1]\), and denote by \( u = M(t, z) \) the unique solution of \( D_u\mathcal{E}(t, u, z) = 0 \), i.e. the minimizer of \( \mathcal{E}(t, \cdot, z) \). Let \((t, q) \in W^{1,1}([0, S]; [0, T] \times \Omega) \) be a parameterized Balanced Viscosity solution to the rate-independent system \((\mathcal{Q}, \mathcal{E}, R_0 + \varepsilon V_z + \varepsilon^a V_u) \) with \( \alpha > 1 \). Set

\[
\mathcal{S} := \{s \in [0, S] : D_u\mathcal{E}(t(s), q(s)) = 0\}. \tag{5.15}
\]

Then, \( \mathcal{S} \) is either empty or it has the form \([s_*, S] \) for some \( s_* \in [0, S] \).

(a) Assume \( s_* > 0 \), then for \( s \in [0, s_*] = [0, S] \setminus \mathcal{S} \) we have \( t(s) = t(0) \) and \( z(s) = z(0) \), whereas \( u \) is a solution to the reparameterized gradient flow for \((\mathbb{R}^n, \mathcal{E}(t(0), \cdot, z(0)), V_u) \) (regime \( V_uB_z \)), namely

\[
0 = \theta_u(s) V_u(u(s), z(0)) u(s) + (1 - \theta_u(s)) D_u\mathcal{E}(t(0), u(s), z(0)) \text{ with } u(0) \neq M(t(0), z(0)). \tag{5.16}
\]
(b) Assume $\mathcal{S} = [s_*, S]$ with $s_* < S$, then for $s \in [s_*, S]$ we have $u(s) = M(t(s), z(s))$ whereas the pair $(t, z)$ is a parameterized Balanced Viscosity solution to the reduced rate-independent system $(\mathbb{R}^n, J, \mathcal{R}_0 + \varepsilon V_z)$ with the reduced energy functional $J : [0, T] \times \mathbb{R}^n \to \mathbb{R}; (t, z) \to \min_{u \in \mathbb{R}^n} \mathcal{E}(t, u, z) = \mathcal{E}(t, M(t, z), z)$, which corresponds to the regimes $\mathcal{E}_1 V_z$ and $\mathcal{E}_2 R_z$.

Proof. To avoid overloaded notation we will often omit the state-dependence of the functions $V_u$ and $V_z$. For ease reference we repeat all the conditions for a BV solution $(t, q)$ (cf. Theorem 5.3), in the case $\alpha > 1$:

(i) $0 = \theta_u V_u u' + (1 - \theta_u) D_u \mathcal{E}(t, u, z)$,  
(ii) $0 \in (1 - \theta_z) \partial \mathcal{R}_0(q, z') + \theta_z V_z z' + (1 - \theta_z) D_z \mathcal{E}(t, u, z)$,  
(iii) $t' \theta_u = 0$,  
(iv) $t' \theta_z = 0$,  
(vi) $\theta_u (1 - \theta_z) = 0$,
which have to hold for a.a. $s \in (0, S)$.

Step 1: By the continuity of $(t, z)$ and $D_u \mathcal{E}$ the set $\mathcal{S}$ is closed, hence its complement is relatively open. Consider an interval $(s_1, s_2)$ not intersecting with $\mathcal{S}$. Using (i) we find $\theta_u > 0$ a.e. in $(s_1, s_2)$. Hence, (iii) implies $t' = 0$ a.e., and we obtain $t(s) = t(s_1)$ for $s \in [s_1, s_2]$. By (v) we find $\theta_z = 1$ a.e. Now, (ii) implies $z' = 0$ a.e., which implies $z(s) = z(s_1)$ for $s \in [s_1, s_2]$. From (vi) we conclude $u' \neq 0$ a.e. Thus, we summarize

$$t(s) = t(s_1), \quad z(s) = z(s_1), \quad 0 = V_u(u(s), z(s_1)) u'(s) + \lambda(s) D_u \mathcal{E}(t(s_1), u(s), z(s_1)),$$

where $\lambda(s) = (1 - \theta_u(s))/\theta_u(s) \in (0, \infty)$ a.e. In particular, $u$ satisfies (5.16). From $u \in W^{1,1}([0, S]; \mathbb{R}^n)$ and (i) we obtain $\lambda \in L^1(s_1, s_2)$. Setting $\tau(s) = \int_{s_1}^s \lambda(\sigma) d\sigma$ and defining the inverse $\hat{s}$ via $s = \hat{s}(\tau)$ we find $\hat{s}'(\tau) > 0$ and $\hat{s} \in W^{1,1}(0, \tau(s_2))$. Moreover, the function $\hat{u} : \tau \mapsto u(\hat{s}(\tau))$ is a solution of the gradient flow

$$0 = V_u(\hat{u}(\tau), z(s_1)) \hat{u}'(\tau) + D_u \mathcal{E}(t(s_1), \hat{u}(\tau), z(s_1)).$$

(5.17)

Furthermore, we see that $s \mapsto \mathcal{E}(t(s_1), u(s), z(s_1))$ is strictly decreasing on $[s_1, s_2]$, since its time derivative is given by $-\langle u'(s), V_u u'(s)/\lambda(s) \rangle$ which is negative a.e.

Step 2: Since $\mathcal{S}$ is closed the complement is an at most countable disjoint union of intervals of the form $(s_1, S]$, $(s_2, s_3]$, $[0, s_1)$, or $(0, S]$, which are maximal in the sense that they cannot be extended without meeting $\mathcal{S}$. Thus, for the “open” sides $s_j$ this means $s_j \in \mathcal{S}$. In the first two cases this implies $u(s_j) = M(t(s_j), z(s_j))$, i.e. we start a gradient flow with initial condition in the global minimizer. Hence, the solution stays constant for all future times, i.e. $u(s) = u(s_1)$ for $s \in (s_1, S]$ or $(s_2, s_3)$, respectively. But this contradicts the fact that $s \mapsto \mathcal{E}(t(s_j), u(s), z(s_j))$ is strictly decreasing (cf. Step 1). Hence, the first two cases cannot occur, and we conclude $\mathcal{S} = [s_*, S]$ with $s_* = s_4$ or $\mathcal{S} = \emptyset$. In particular, assertion (a) is established.

Step 3: To show (b) assume $s \in \mathcal{S} = [s_*, S]$, then $u(s) = M(t(s), z(s))$ by the definition of $\mathcal{S}$. Observe that $D_z J(t, z) = D_z \mathcal{E}(t, M(t, z), z) + D_z M(t, z)$ integrates $D_z \mathcal{E}(t, M(t, z), z) = D_z \mathcal{E}(t, M(t, z), z) + 0$. Thus, $(t, z)$ solves

(ii$'$) $0 \in (1 - \theta_z) \partial \mathcal{R}_0(z, z') + \theta_z V_z z' + (1 - \theta_z) D_z J(t, z), \quad$ (iv$'$) $t' \theta_z = 0,$  
(vi$'$) $t' + |z'| > 0$
which proves that $(t, z)$ is a BV solution of the reduced system. For the latter relation note that $t'(s) + |z'(s)| = 0$ implies $u'(s) = \frac{d}{ds} M(t(s), z(s)) = 0$ so that (vi$'$) follows from (vi).

Our approach in Step 1 of the above proof uses the qualitative ideas from [32, 33], but our reduction to the simpler case makes the analysis much easier.

**Case $\alpha = 1$: comparable relaxation times.** Here $u$ and $z$ relax at the same rate. At a jump, the system may switch to the viscous regime $V_u V_z$, where both in the evolution of $u$, and in the evolution for $z$, viscous dissipation intervenes, modulated by the same coefficient $\theta = \theta_u = \theta_z$. 
Figure 6.1. Solutions for (6.1) for the cases $\alpha = 2$ (blue), $\alpha = 1$ (green), and $\alpha = 1/2$ (red).

Case $\alpha \in (0,1)$: fast relaxation of $z$. Here $z$ relaxes faster than $u$, and jumps in the $z$-component are faster than jumps in the $u$-component. If $z$ jumps (possibly governed by viscous dissipation), then $u$ is blocked while $z$ moves viscously (regime $B_uV_z$). But then $u$ has still to relax to equilibrium, and it will do it on a faster scale than the rate-independent motion of $z$, if $z$ stays in locally stable states (regime $V_uR_z$). Finally, full rate-independent behavior in the regime $E_uR_z$ will occur, where $t'(s) > 0$. Unlike in the case $\alpha > 1$, all three regimes may occur more than once in the evolution of the system, see Section 6.2 for an example.

6. Examples

To illustrate the difference between the three limit models (namely for $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0,1)$), we discuss two examples. The first one treats a quadratic energy and emphasizes the different initial behavior before the solution converges to a truly rate-independent regime. In the second example we show that solutions that start in a rate-independent regime and coincide for the three different limit models may separate if viscous jumps start, leading to different rate-independent behavior afterwards.

6.1. Initial relaxation for a system with quadratic energy

We consider the energy functional $E(t,u,z) = \frac{1}{2}(u - z)^2 + \frac{1}{2}z^2 - tu$ and trivial viscous energies leading to the ODE system

$$\begin{cases} 0 = \varepsilon^\alpha \dot{u} + u - z - t, \\ 0 \in \text{Sign}(\dot{z}) + \varepsilon \dot{z} + 2z - u \end{cases} \quad \text{with } (u(0),z(0)) = (2,-3/2). \quad (6.1)$$

We show simulations for the three cases $\alpha = 2$ (blue), $\alpha = 1$ (green), and $\alpha = 1/2$ (red) with sufficiently small $\varepsilon$ (typically 0.001...0.03). The components $u$ and $z$ as functions of time are depicted in Figure 6.1.

However, to detect different jump behavior at $t \approx 0$ it is advantageous to look at the parameterized solutions, which are depicted in Figure 6.2, showing $(t,q)$ for the three different cases. The parameterization was calculated using $s(t) = \max\{0.5,|\dot{u}(t)|,\dot{z}(t)|\}$. In the parameterized form we fully see the structure of the jump for $t \approx 0$. For $\alpha = 2$ we obtain first a jump from the initial datum $(u,z) = (2,-1.5)$ to $(u,z) = (-1.5,-1.5)$ on the timescale $\varepsilon^2$, which is the regime $V_uB_2$. Then, $u$ is equilibrated, and a jump to $(-1,-1)$ along the diagonal $u = z$ occurs on the timescale $\varepsilon$, which is the regime $E_uV_2$. Finally, the solution finds the rate-independent regime $E_uR_2$ with $(u(t),z(t)) = q(t) := (2t-1,t-1)$.

For $\alpha = 1/2$ the solution first jumps to $(2,0.5)$ on the time scale $\varepsilon$, which is the regime $B_uV_2$. Next, there is a jump to $(0.5,0.5)$ in the time scale $\varepsilon^{1/2}$, which is regime $V_uR_2$. Then, the
Figure 6.2. Solutions \((t, u, z)\) for (6.1) with dotted \(t\), full \(u\), and dashed \(z\). Left \(\alpha = 2\), middle \(\alpha = 1\), right \(\alpha = 1/2\).

Figure 6.3. Solutions \((z(t), u(t))\) for (6.1). The dotted line is the diagonal \(u = z\), while the yellow area is the locally stable region \(|2z−u| ≤ 1\).

The different behavior and the different regimes are also nicely seen by plotting the trajectories in the \((u, z)\)-plane, see Figure 6.3, where the three different cases for \(\alpha\) are depicted.
6.2. Different jumps starting from the rate-independent regime

Finally we provide an example where the jumps start out of a rate-independent motion, i.e. we first have the regime $E_uR_z$, and then the system becomes unstable and develops a jump. For this purpose we use the nonconvex energy

$$E(t,u,z) = \frac{1}{2} (u-g(z))^2 + F(z) - tu \quad \text{with} \quad g(z) = 4z^3 - 4z$$

and $F'(z) = -1 + (z+1)^2(-40 + 10(z+1)^2 + 38e^{-10(z+0.5)^2})$.

Using the standard viscous potentials as above, the ODE system reads

$$\begin{cases}
0 = \varepsilon \dot{u} + u - g(z) - t, \\
0 \in \text{Sign}(\dot{z}) + \varepsilon \dot{z} + F'(z) + g'(z)(g(z)-u)
\end{cases} \quad \text{with} \quad (u(-0.2),z(-0.2)) = (-2.4,-1.2). \quad (6.2)
$$

Figure 6.4 shows simulation results of $u(t)$ and $z(t)$ for the three cases $\alpha = 2$ (blue), $\alpha = 1$ (green), and $\alpha = 1/2$ (red) with sufficiently small $\varepsilon$. We see that the solutions stay together for $t \in [-0.2,-0.1]$, which is exactly the time they stay in regime $E_uR_z$. Then, in all three cases a jump develops, but this is quite different for different $\alpha$. In Figure 6.5 we provide graphics of the same solutions, but now in the reparameterized form $(t,u,z)$ for the three $\alpha$-values 2, 1, and $1/2$, where again the parameterization $s$ is chosen such that $s(t) = \max\{0.5,|\dot{u}(t)|,|\dot{z}(t)|\}$. However, for this example numerical instabilities prevented us from taking $\varepsilon$ small enough to have a better separation of time scales. Even in the viscous regimes we still see $t' > 0$ but small. Nevertheless, Figure 6.5 clearly shows the different regimes.

Figure 6.6 shows the trajectories in the $(z,u)$-plane.

Acknowledgments

A.M. has been partially supported by DFG via SFB 1114, subproject C5. R.R. and G.S. have been partially supported by a MIUR-PRIN’10-11 grant for the project “Calculus of Variations”. R.R. also acknowledges support from the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] Mielke A 2005 Evolution in rate-independent systems. Ch. 6 in “Handbook of Differential Equations, Evolutionary Equations, vol. 2” ed Dafermos C and Feireisl E (Amsterdam: Elsevier B.V.) pp 461–559

[2] Mielke A and Theil F 1999 A mathematical model for rate-independent phase transformations with hysteresis. In “Proceedings of the Workshop on Models of Continuum Mechanics in Analysis and Engineering” ed Alber H D, Bailean R and Farwig R (Aachen: Shaker-Verlag) pp 117–129
Figure 6.5. Solutions \((t,u,z)\) for (6.2) with dotted \(t\), full \(u\), and dashed \(z\). Left \(\alpha = 2\), middle \(\alpha = 1\), right \(\alpha = 1/2\).

Figure 6.6. Solutions \((z(t),u(t))\) for (6.2). The dashed magenta line is \(u = g(z)\), while the black curves display the boundaries of the locally stable domain \(|F'(z) + g'(z)(g(z) - u)| \leq 1\).

[3] Mielke A and Theil F 2004 On rate–independent hysterisis models Nonl. Diff. Equas. Appl. (NoDEA) 11 151–189 (Accepted July 2001)
[4] Dal Maso G, Francfort G and Toader R 2005 Quasistatic crack growth in nonlinear elasticity Arch. Rational Mech. Anal. 176 165–225
[5] Mielke A 2011 Differential, energetic and metric formulations for rate-independent processes. Ch. 3 in “Nonlinear PDEs and Applications.C.I.M.E. Summer School, Cetraro, Italy 2008” ed Ambrosio L and Savaré G (Heidelberg: Springer) pp 87–170
[6] Mielke A 2003 Energetic formulation of multiplicative elasto–plasticity using dissipation distances Cont. Mech. Thermodynamics 15 351–382
[7] Knees D, Mielke A and Zanini C 2008 On the inviscid limit of a model for crack propagation Math. Models Methods Appl. Sci. 18 1529–1569
[8] Mielke A, Rossi R and Savaré G 2009 Modeling solutions with jumps for rate-independent systems on metric spaces Discrete Contin. Dyn. Syst. 25 585–615
[9] Efendiev M and Mielke A 2006 On the rate–independent limit of systems with dry friction and small viscosity J. Convex Analysis 13 151–167
[10] Colli P and Visintin A 1990 On a class of doubly nonlinear evolution equations Comm. Partial Differential Equations 15 737–756
[11] Colli P 1992 On some doubly nonlinear evolution equations in Banach spaces Japan J. Indust. Appl. Math. 9 181–203
Mielke A, Rossi R and Savaré G 2013 Nonsmooth analysis of doubly nonlinear equations Calc. Var. Partial Differential Equations 46 253–310

Mielke A, Rossi R and Savaré G 2012 BV solutions and viscosity approximations of rate-independent systems ESAIM Control Optim. Calc. Var. 18 36–80

Mielke A, Rossi R and Savaré G 2014 Balanced Viscosity (BV) solutions to infinite-dimensional rate-independent systems J. Eur. Math. Soc. To appear. (WIAS preprint 1845. http://arxiv.org/abs/1309.6291)

Mielke A and Zelik S 2014 On the vanishing viscosity limit in parabolic systems with rate-independent dissipation terms Ann. Sc. Norm. Sup. Pisa Cl. Sci. (5) XIII 67–135

Toader R and Zanini C 2009 An artificial viscosity approach to quasistatic crack growth Boll. Unione Mat. Ital. (9) 2 1–35

Dal Maso G, DeSimone A and Solombrino F 2011 Quasistatic evolution for Cam-Clay plasticity: a weak formulation via viscoplastic regularization and time rescaling Calc. Var. Partial Differential Equations 40 125–181

Dal Maso G, DeSimone A and Solombrino F 2012 Quasistatic evolution for Cam-Clay plasticity: properties of the viscosity solution Calc. Var. Partial Differential Equations 44 495–541

Babadjian J F, Francfort G and Mora M 2012 Quasistatic evolution in non-associative plasticity - the cap model SIAM J. Math. Anal. 44 245–292

Francfort G A and Stefanelli U 2013 Quasi-static evolution for the Armstrong-Frederick hardening-plasticity model Appl. Math. Res. Express. AMRX 2013 297–344

Knees D, Rossi R and Zanini C 2013 A vanishing viscosity approach to a rate-independent damage model Math. Models Methods Appl. Sci. 23 565–616

Roubíček T 2009 Rate-independent processes in viscous solids at small strains Math. Meth. Appl. Sci. 32 825–862

Roubíček T 2010 Thermodynamics of rate-independent processes in viscous solids at small strains SIAM J. Math. Anal. 40 256–297

Roubíček T 2013 Adhesive contact of visco-elastic bodies and defect measures arising by vanishing viscosity SIAM J. Math. Anal. 45 101–126

Racca S 2012 A viscosity-driven crack evolution Adv. Calc. Var. 5 433–483

Dal Maso G and Scala R 2014 Quasistatic evolution in perfect plasticity as limit of dynamic processes J. Dynam. Differential Equations 26 915–954

Scala R 2014 Limit of viscous dynamic processes in delamination as the viscosity and inertia vanish Preprint Available at http://cvgmt.sns.it/paper/2434/

Ambrosio L 1995 Minimizing movements Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) 19 191–246

Ambrosio L, Gigli N and Savaré G 2008 Gradient flows in metric spaces and in the space of probability measures 2nd ed Lectures in Mathematics ETH Zürich (Basel: Birkhäuser Verlag)

Ioffe A D and Tihomirov V M 1979 Theory of extremal problems (Studies in Mathematics and its Applications vol 6) (Amsterdam-New York: North-Holland Publishing Co.)

Ioffe A D 1977 On lower semicontinuity of integral functionals I SIAM J. Control Optimization 15 521–538

Zanini C 2007 Singular perturbations of finite dimensional gradient flows Discr. Cont. Dynam. Syst. 18 657–675

Agostiniani V, Rossi R and Savaré G 2014 Balanced viscosity solutions of singularly perturbed gradient flows in infinite dimension In preparation