Exact Nonperturbative Unitary Amplitudes for $1 \to N$ Transitions

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ABSTRACT

I present an extension to arbitrary $N$ of a previously proposed field theoretic model, in which unitary amplitudes for $1 \to 8$ processes were obtained. The Born amplitude in this extension has the behavior $A_{1\to N}^{\text{tree}} = g^{N-1} N!$ expected in a bosonic field theory. Unitarity is violated when $|A_{1\to N}| > 1$, or when $N > N_{\text{crit}} \simeq e/g$. Numerical solutions of the coupled Schrödinger equations shows that for weak coupling and a large range of $N > N_{\text{crit}}$, the exact unitary amplitude is reasonably fit by a factorized expression

$$|A_{1\to N}|_{\text{max}} \simeq 0.73 \cdot \frac{1}{N} \cdot \exp \left( -0.025/g^2 \right).$$

The very small size of the coefficient $1/g^2$, indicative of a very weak exponential suppression, is not in accord with standard discussions based on saddle point analysis, which give a coefficient of $\sim 1$. The weak dependence on $N$ could have experimental implications in theories where the exponential suppression is weak (as in this model). Non-perturbative contributions to few-point correlation functions in this theory would arise at order $K \simeq \left( (0.05/g^2) + 2 \ln N \right) / \ln(1/g^2)$ in an expansion in powers of $g^2$. 
Tree-level calculations of amplitudes for scattering processes in which many bosons are produced fail to obey unitarity when the number produced is too large. Specifically, in either massive scalar [1,2] or vector [3] field theories, the $1 \to N$ tree amplitude for fixed angle scattering has the unacceptable behavior

$$A_{1 \to N} \sim g^{N-1} N!$$

for large $N$, where $g^2$ is the quartic coupling. Cross sections then violate unitarity when $N \gtrsim 1/g^2$ for $E,N$ large, $E/N$ finite. As shown explicitly in refs.[2,3], the problem originates in the coherence of the approximately $N!$ graphs which contribute. The cure surely lies in the summation of all loops, since their contribution is equal to that of the tree graphs precisely when the latter become large. The question remains: does the unitary damping suppress the cross section at an exponentially small value? or do coherence effects allow an experimentally interesting cross section for producing many massive bosons in a high energy collision? Independent of the phenomenological application lies the question as to how to calculate high energy, multiparticle processes in field theory.

Zakharov [4] has given a thoughtful argument for exponential suppression, which I will simply paraphrase. Consider the dispersion relation for the fourier transform of some two point function in field theory or quantum mechanics,with enough subtractions to make it convergent. At zero energy, we have

$$\Pi(0) = \frac{1}{\pi} \int_m^\infty \frac{dE}{E} \text{Im} \, \Pi(E) ,$$

where

$$\text{Im} \, \Pi(E) \sim \sum_N |A_{1 \to N}|^2 \theta(E - mN) .$$

Suppose we now assume that $\Pi(0)$ has an asymptotic expansion in coupling

$$\Pi(0) = \sum_k a_k \, g^{2k} ,$$

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where for large $K$

$$\left| \Pi(0) - \sum_{k=1}^{K} a_k g^{2k} \right| < (g^2/b)^K K! .$$

Then, by the usual arguments, the error in truncating the series is minimized at $K = K_{\text{crit}} = b/g^2$, and the error in omitting the remainder is bounded by

$$\Delta \Pi(0) \lesssim e^{-b/g^2} .$$

At this point the discussion may take the following form:

Suppose we either know or think we know the value of $b$. (For example, in the scalar $\phi^4$ field theory, it is often assumed that $b$ is given correctly as $16\pi^2$ by the saddle point analysis of Lipatov [5].) Then the high order behavior (for $k \geq b/g^2$) of $\text{Im} \Pi$ is bounded by $\exp(-b/g^2)$. Assuming that the high multiplicity $n \geq b/g^2$ piece of Im $\Pi$ is an important component of the high order piece, one concludes that high multiplicity contributions to Im $\Pi$ (or the cross section, in field theories) are bounded by [6]

$$\sum_{N \gtrsim b/g^2} |A_{1 \rightarrow N}|^2 \leq e^{-b/g^2} .$$

One may also turn the argument around: suppose that we have some evidence that multiparticle contributions to Im $\Pi$ are as large as $\exp(-c/g^2)$. Then, from the dispersion relation, we may conclude that the perturbation series for $\Pi(0)$ becomes untrustworthy for $k \gtrsim c/g^2$. If $c \ll 1$, the situation becomes interesting, on two counts: (1) although formally exponentially suppressed, there may be experimentally interesting multiparticle cross sections if $g^2$ is not too small (2) as a consequence of the dispersion relation, the two point function may have interesting non-perturbative contributions at relatively low order $k \sim c/g^2$.

In this paper, I will propose an extension to arbitrary $N$ of a field theoretic model [7] previously applied to a calculation of a unitary $1 \rightarrow 8$ transition amplitude. In this extension, the (time-dependent) amplitudes $A_{1 \rightarrow N}$ may be directly
(numerically) calculated. The result of the present calculation is that in the weak
coupling limit, the amplitude $A_{1 \rightarrow N}$ is very weakly exponentially suppressed:

$$A_{1 \rightarrow N} \sim e^{-0.025/g^2}, \quad (6)$$

with the implications mentioned in the preceding paragraph and in the abstract.

A field theoretic model, loosely based on a \( \phi^3 \) field theory, has been presented
[7] as a laboratory for the study of multiparticle amplitudes. The model is defined
through its Hamiltonian

$$H = \sum_{k=1}^{\infty} a_k^\dagger a_k + \frac{1}{2} g \mathcal{P} \sum_{j,k=1}^{\infty} a_j^\dagger a_k a_j a_k \mathcal{P} + h.c. \quad (7)$$

The modes labeled by \( i, j, k \) will be called “momenta”, and the action of the hermitean
projection operator \( \mathcal{P} \) on a state vector \( |\psi\rangle \) is as follows:

$$\mathcal{P} |\psi\rangle = 0$$

if there is more than one quantum in the state with any given momentum and

$$\mathcal{P} |\psi\rangle = |\psi\rangle$$

otherwise. \( \mathcal{P} \) has been introduced in order to approximate the infinite volume
effect of field theory (in box normalization): namely, one generally omits consider-
eration of amplitudes for transitions to states with more than one particle in a
given (discrete) momentum state. Thus, we exclude “laser” effects. In this sense,
there is an exclusion principle without imposing anticommutation relations and
antisymmetrization. Other than that, \( H \) resembles a \( \phi^3 \) field theory in a cavity,
with no temporal or spatial derivatives in the lagrangian. It is also a kind of matrix
model.
It is an important consequence of (7) that the momentum operator

\[ P = \sum_{k=1}^{\infty} k \ a_k^\dagger a_k \]  

(8)
is a constant of the motion. Thus, the Hilbert space factorizes into subspaces with definite \( P \). Because of the positivity of all of the momenta, \textit{these will be finite dimensional subspaces}. Within a subspace of given \( P \), there will be sectors characterized by different numbers of particles \( n \). A subspace with definite \( P \) may contain a \textbf{maximal} state, namely an \( N \)-particle state with momenta \( k = 1, 2, \ldots N \), such that \( P = N(N + 1)/2 \). I will restrict my study to these subspaces, which can be labeled by \( N \). For \( N = 8 \), the number of states with \( n = 1, 2, \ldots 8 \) particles is \( (1, 17, 91, 206, 221, 110, 21, 1) \) respectively. A numerical analysis of this case [7] proceeded as follows: with the system initially in the one-particle state, the 668 coupled (complex) time-dependent Schrödinger equations were numerically integrated in order to obtain the maximum 8-particle amplitude \( |A_8(t)|_{\text{max}} \). The result indicated an approximate behavior for the \( 1 \rightarrow 8 \) transition

\[ |A_8|_{\text{max}} \simeq e^{-0.20/g^2} \]

for \( g^2 > 0.07 \). This is the value of \( g^2 \) for which the Born amplitude violates unitarity: \( |A_8^{\text{Born}}(t)|_{\text{max}} \geq 1 \).

In general, the total number of states for a given \( P \) can be obtained as the exponent of \( x^P \) in the expansion of the generating function

\[ \prod_{j=1}^{\infty} (1 + x^j) = \sum_{P=0}^{\infty} N_P \ x^P . \]  

(9)

This reveals that even an extension to \( N = 9 \) means quadrupling the subspace from 668 to 2048 states. Thus, exact numerical analysis for larger \( N \) rapidly becomes impractical.
How then to extend the results beyond \( N = 8 \)? Clearly, an approximation to the dynamics of the model at large \( N \) is required; however, it is imperative that the extension to arbitrary \( N \) (\( a \)) retain the unitarity property and (\( b \)) display the \( g^{N-1} N! \) behavior in the tree, or Born approximation.

A detailed study of the coupling structure of the Hamiltonian (7) for \( N = 8 \) yields a strong clue as to how this extension may be accomplished. If one asks, how many states (on the average) with \( n - 1 \) particles does a particular state with \( n \) particles couple to, then one arrives at the following list:

- \( 8 \to 7 : 16 \) states
- \( 7 \to 6 : 15 \pm 1 \) states
- \( 6 \to 5 : 12 \pm 1 \) states
- \( 5 \to 4 : 9 \pm 1 \) states
- \( 4 \to 3 : 6 \) states
- \( 3 \to 2 : 3 \) states
- \( 2 \to 1 : 1 \) state

If we ignore the \( 8 \to 7 \) result, and are liberal with the \( 7 \to 6 \), then it is not too far amiss to conclude that, approximately, each state with \( n \) particles couples to an average of \( n(n - 1)/2 \) states with \( n - 1 \) particles. [8] This appeals to intuition: adding each (unequal) pair of ‘momenta’ in the given state with \( n \) particles will, in large probability, give one of the possible states with \( n - 1 \) particles. Thus, I propose as an extension to (7) a quantum mechanical system in which there are simply \( N \) states labelled by \( n = 1, \ldots, N \), differing in energy by a constant amount (taken to be 1 in our units), and in which each state \( |n\rangle \) is coupled with strength \( g n(n - 1)/2 \) to the state \( |n - 1\rangle \). In the interaction representation the dynamics of this model is embodied in the system of time dependent Schrödinger equations

\[
i \dot{A}_n(t) = g \left( \frac{n(n-1)}{2} A_{n-1}(t) e^{it} + \frac{n(n+1)}{2} A_{n+1}(t) e^{-it} \right)
\]

\[n = 1 \ldots N\]

where \( A_0(t) = A_{N+1}(t) \equiv 0 \). For a given \( N \), these equations may be solved by first
diagonalizing the $N \times N$ Hamiltonian, or by direct numerical integration. But first, I state without details of proof a result which establishes this system as a viable laboratory in which to study the large-$N$ problem:

If $A_n(0) = \delta_{n1}$, then in Born approximation

$$A_n^{\text{Born}}(t) = g^{n-1} \left( \frac{1 - e^{it}}{2} \right)^{n-1} n! . \tag{11}$$

This can be proven by direct substitution into Eq.(10) with no rescattering (i.e., dropping the second term on the right). It was actually arrived at constructively by using Laplace transforms.

As a result of (11),

$$|A_N^{\text{Born}}|_{\text{max}} = g^{N-1} N! . \tag{12}$$

I note that in the exact version of the model, with $N = 8$, [7] it was found that

$$|A_8^{\text{Born}}|_{\text{max}} \simeq 0.27 g^7 8! . \tag{13}$$

Note also that in Born approximation the anharmonic oscillator with coupling $\frac{1}{4}g^2 x^4$ gives [9,10]

$$|A_N^{\text{Born}}|_{\text{max}} \sim g^{N-1} \sqrt{N!} . \tag{14}$$

In both the present case and in the case of the anharmonic oscillator the state $|N\rangle$ with unperturbed energy $N$ is normalized to unity.

Since there is a normalization condition $\sum_n |A_n(t)|^2 = 1$, it is seen from Eq.(12) that unitarity will certainly be violated in Born approximation when

$$N \gtrsim N_{\text{crit}} = \frac{e}{g} . \tag{15}$$

For a given $g^2$, this defines the value of $N$ for which rescattering terms are essential.
It is then a simple matter to go back to Eq. (10), and find (numerically) the maximum value $|A_N|_{\text{max}}$ attained by $A_N(t)$ as a function of $N, g^2$. [11]

**Results:** The results of this work are encompassed in Figs. (1) and (2). In Fig. (1), I display on a log–log plot the $N$ behavior of $|A_N|_{\text{max}}$, for a large range of values of $g^2$. The range of $N$ for each $g^2$ is chosen to lie above $N_{\text{crit}}$. Two striking observations may be made from the graph: (1) The $N$-behavior is universal over the whole array of $g^2$ — the amplitude factorizes; (2) The curves are excellently fit with a simple inverse proportionality $|A_N|_{\text{max}} \sim 1/N$. The wavy oscillations in the curves for the larger values of $g^2$ are real: they can be seen in detail in Fig. (3) as an odd-even effect.

In Fig. (2), I plot $|A_N|_{\text{max}}$ vs. $1/g^2$ on a log plot for a range of values of $N$. Again, one notes the factorizability. For the smallest values of $g^2$, the curves are well fit by the exponential form $|A_N|_{\text{max}} \sim \exp(-0.025/g^2)$. In sum, therefore, the results of the numerical study of the equations (10) is that

$$|A_{1\rightarrow N}|_{\text{max}} \simeq 0.73 \cdot \frac{1}{N} \cdot \exp(-0.025/g^2) \ .$$

I close with some discussion of these results.

**Remarks and Conclusions:**

(1) First, the factorizability. The results of Lipatov [5] and the graphical analysis of Parisi [12] support the notion that the contribution of a high order $K$ of perturbation theory to an $N$-legged Green’s function should be (roughly) independent of $N$, for $N$ not too large. What I find here is that the functional dependence on coupling constant of the exact non-perturbative $N$-point function is independent of $N$, in the region where $N$ is large ($N > e/g$.)

(2) The behavior of $A_{1\rightarrow N}$ in the model being examined is totally different from its behavior in the case of the anharmonic oscillator. In that case, for $Ng^2$ not too large, $|A_N|_{\text{max}} \sim \exp(-N)$. [9,10]
(3) Another surprising result is the generation of a small dimensionless number, namely 0.025, the coefficient of \(1/g^2\) in the exponential suppression. Typical saddle point analyses (even in the \(\phi^3\) theory [13]) give coefficients of \(O(1)\) \((\text{modulo factors of } (4\pi)^{d-2} \text{ for } d \text{ dimensions})\). The implication is that for \(N > N_{\text{crit}}\), the theory becomes strongly coupled for very small values of \(g^2\). The next point is related to this one.

(4) The contribution of the \(1 \rightarrow N\) excitation to a few-point function (such as the \(1 \rightarrow 1\) transition amplitude) can be estimated as

\[
\langle 1, t = T | 1, t = 0 \rangle_N \sim |A_N|_{\text{max}}^2 .
\]

If we wish to see at what order \(K\) the non-perturbative contribution of the right-handed side competes with a perturbative development of the left-hand side, we can use the parameterization (16) and set

\[
g^{2K_{\text{crit}}} \simeq \exp \left(-0.05/g^2\right)/N^2 , \tag{18}
\]

with \(N \geq e/g\). This gives the formula in the abstract. Even for \(g^2 = 0.005\) and \(N = 100\), this gives \(K_{\text{crit}} = 3.6\), a very low value.

(3) Finally, a comment on the limitations of the numerical study. As \(N\) becomes large and/or \(g^2\) becomes small, the roundoff errors become more important. At some point, \(|A_N|_{\text{max}}^2\) is smaller than the deviation from unitarity \(\delta = |\sum_n |A_n(t)|^2 - 1|\) caused by finite numerical accuracy. This occurred for \(g^2 < 0.005\) (and \(N > N_{\text{crit}}\)). The results for such small values of coupling may therefore be untrustworthy, and at present I am examining alternate methods of exploring the full range of \(g^2\) in the non-perturbative region.

To conclude, I have found in a unitary model a large \(1 \rightarrow N\) amplitude which is only very weakly exponentially suppressed. The dependence on \(N\) is also very weak. This may hold out some possibility for observing large multiplicity central events in high energy collisions.
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Figure Captions

Fig. 1 The maximum value attained by $|A_N(t)|$ as a function of $N$, for various values of $g^2$. The Born approximation fails for $N \geq N_{\text{crit}} = e/g$. For small $g^2$, large $N$, $|A_N|_{\text{max}} \sim 1/N$.

Fig. 2 $|A_{1\rightarrow N}|_{\text{max}}$ vs. $1/g^2$ for three values of $N$. The behavior in $1/g^2$ is independent of $N$: $A_{1\rightarrow N} \sim \exp(-0.025/g^2)$.

Fig. 3 Detail of Fig. (1) for $g^2 = 0.10$, showing origin of waviness as an odd-even effect.
REFERENCES

1. J. M. Cornwall, *Phys. Lett.* **243B**, 271 (1990).

2. H. Goldberg, *Phys. Lett.* **246B**, 445 (1990).

3. H. Goldberg, *Phys. Rev.* **D45**, 2945 (1992).

4. V. I. Zakharov, Max Planck preprint MPI-PAE-PTH-11-91, March 1991.

5. L. N. Lipatov, Sov. Phys. JETP **45** (2), 216 (1977).

6. It must be noted that we generally do not know the value of $b$. This is the case in the spontaneously broken electroweak theory.

7. H. Goldberg and M. T. Vaughn, Northeastern preprint NUB-3043/92-TH.

8. The end-point problem with $8 \rightarrow 7$ is obvious: many pairwise additions in the maximal state $|1, 2, 3, 4, 5, 6, 7, 8\rangle$ will give a momentum already present in the spectator particles.

9. M. B. Voloshin, *Phys. Rev.* **D43** (1991) 1726. See also a recent discussion by J. M. Cornwall and G. Tiktopoulos, UCLA preprint UCLA/91/TEP/55.

10. C. Bachas, Ecole Polytechnique preprint A089.1191, December 1991.

11. I find for all cases of interest,

$$\left( \frac{d}{dt} |A_N(t)|^2 \right)_{\max} \sim |A_N(t)|^2_{\max} .$$

Thus, I will phrase the discussion in terms of $A_{1\rightarrow N}$.

12. G. Parisi, *Phys. Lett.* **68B**, 117 (1977).

13. A. Houghton, J. S. Reeve, and D. J. Wallace, *Phys. Rev.* **B17**, 2956 (1978).