Goto’s deformation theory of geometric structures, a Lie-theoretical description

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Abstract

In [Go], Ryushi Goto has constructed the deformation space for a manifold equipped with a collection of closed differential forms and showed that in some important cases (Calabi-Yau, $G_2$- and $Spin(7)$-structures) this deformation space is smooth. This result unifies the classical Bogomolov-Tian-Todorov and Joyce theorems about unobstructedness of deformations. Using the work of Fiorenza and Manetti, we show that this deformation space could be obtained as the deformation space associated to a certain $L_\infty$-algebra. We also show that for Calabi-Yau, $G_2$- and $Spin(7)$-structures this $L_\infty$-algebra is homotopy abelian. This gives a new proof of Goto’s theorem.

1 Introduction

The celebrated Berger’s theorem ([Be]) classifies groups occurring as holonomy groups of irreducible simply connected Riemannian manifolds. The full list is

$$SO(n), U(n), Sp(n) \times Sp(1)/\mathbb{Z}_2, SU(n), Sp(n), G_2, Spin(7).$$

Manifold with four last holonomy groups are called, correspondingly, Calabi-Yau manifolds, hyperKähler manifolds, $G_2$-manifolds and $Spin(7)$-manifolds. They share some interesting similar properties: for example, they are all Ricci-flat, their cohomology admit Hodge decomposition into irreducible representations of the holonomy group, and they have smooth local deformation spaces. For Calabi-Yau and hyperKähler manifolds this smoothness was proved by Bogomolov ([Be]), Tian ([Ti]), and Todorov ([To]), and for exceptional holonomy groups this was first proved by Joyce ([Jo]). While Joyce’s proof relies on complicated analytical arguments, proofs of Tian and Todorov are based on investigations of properties of dg-Lie algebra $\Omega^{0\bullet} \otimes T$ of tangent-valued forms. The present work grew out of desire to have the similar Lie-algebraic description of deformations of all manifolds with special holonomy.

Other descriptions of moduli spaces of $G_2$ metrics belong to Hitchin ([Hi]) and to Goto ([Go]). Hitchin’s approach is based on finding extrema of a certain functional defined on the space of forms, while Goto’s approach is based on the following observation. Suppose that tensor $A$ defines some type of geometric structure (for example, in the sense of Cartan, i.e., the reduction of the structure group on the tangent bundle from $GL$ to some smaller group stabilizing $A$), then every other geometric structure (without any
integrability conditions) of this type is defined by \( g \cdot A \) for some \( GL \)-valued function on our base manifold. Thus, instead of deforming tensors, one can deform the element \( g \) of the gauge group.

In ([Go]), Goto writes down conditions, under which it is possible to iteratively solve the equation \( d(g_t \cdot \alpha) = 0 \) for some closed form \( \alpha \). Goto’s arguments are inductive and rather long. In the present work we are interpreting equations of ([Go]) as Maurer-Cartan equations in a certain \( L_\infty \)-algebra. The well-developed theory of \( L_\infty \)-algebras allows to shorten Goto’s calculations.

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2 Geometric structures on manifolds

In this section we define geometric structures and formulate the deformation problem.

Let \( V \) be a vector space, \( \Phi \in \oplus_n \Lambda^n V^* \) be a collection of exterior forms on it, and \( E_\Phi := GL(V) \cdot \Phi \) be its orbit under the \( GL(V) \)-action on the space \( \oplus_n \Lambda^n V^* \). Then \( E_\Phi \) is a homogeneous manifold isomorphic to the \( GL(V)/G \), where \( G \) is the stabilizer of \( \Phi \), with the tangent space at \( \Phi \) isomorphic to the \( \mathfrak{gl}(V)/\mathfrak{g} \).

Now let \( M \) be a differentiable manifold. Locally choosing for each point \( x \in M \) an identification between \( T_x M \) and \( V \), we can form the fiber bundle \( E_\Phi \rightarrow M \), consisting of forms that are pointwise in the \( GL(V) \)-orbit of \( \Phi \). Denote by \( \pi \) the natural projection from \( E_\Phi \) to \( M \).

Definition 2.1: A \( G \)-structure, or simply a geometric structure (in the sense of Goto) is the smooth section of the projection \( \pi \). The geometric structure is called integrable if this section is a collection of closed forms.

Remark 2.2: The notation is misleading in two ways. First, the subgroup \( G \) of \( GL(V) \) could be described by different antisymmetric tensor invariants. Second, usually the name ”\( G \)-structure” means the reduction of the structure group of \( TM \) from \( GL(V) \) to \( G \). The \( G \)-structure in the sense of Goto provides the reduction of the structure group, but not every reduction could be obtained by fixing skew-symmetric tensors. To make things worse, Goto’s notion of integrability does not always agree with other possible notions. However, as the following examples show, this ambiguity should not present a problem, at least for the purposes of this article.
Example 2.3: An $Sp(2n)$-structure on an $2n$-dimensional manifold is the 2-form $\omega$ for which $\omega^n \neq 0$. Integrable $Sp(2n)$-structure is called symplectic.

Example 2.4: An $SL(n, \mathbb{C})$-structure on an $2n$-dimensional manifold is the complex $n$-form $\Omega$, for which $\Omega \wedge \overline{\Omega} \neq 0$. An $SL(n, \mathbb{C})$-structure on $M$ defines an almost complex structure $I \Omega$ on $M$, setting $T^{*1,0}M := \{ \alpha \in T^*M \mid \alpha \wedge \Omega = 0 \}$. Integrability of the given $SL(n, \mathbb{C})$ structure implies the integrability of this complex structure.

Example 2.5: An $SU(n, \mathbb{C})$-structure on an $2n$-dimensional manifold is the pair of forms $(\omega, \Omega) \in \Lambda^2 M \oplus \Lambda^n M$, where $\omega$ is an $Sp(2n)$-structure and $\Omega$ is the $SL(n, \mathbb{C})$-structure, $\Omega \wedge \omega = \overline{\Omega} \wedge \omega = 0$, $\Omega \wedge \overline{\Omega} = c\omega^n$ for some constant $c$, and the bilinear form $\omega(I \Omega, \cdot)$ is positive. A manifold with an integrable $SU(n, \mathbb{C})$-structure is called Calabi-Yau manifold.

Example 2.6: Let $(\omega, \Omega)$ be an $SU(3)$-form on a 6-dimensional vector space $V$. Consider the vector space $V \oplus \mathbb{R}$, and let $\theta$ be a coordinate on $\mathbb{R}$. Consider forms $\varphi = \omega \wedge \theta + \text{Im} \Omega$, $\psi = -\text{Re} \Omega \wedge \theta + \frac{1}{2} \omega \wedge \omega$.

The stabilizer of the pair $(\varphi, \psi)$ is the exceptional Lie group $G_2$. For the 7-dimensional manifold $M$, sections of the corresponding orbit $E_{(\varphi, \psi)}$ are called $G_2$ structures. Integrable $G_2$ structures are in 1-1 correspondence with Riemannian metrics of holonomy $G_2$ (FeGr).

Remark 2.7: By the definition of $E_{\Phi}$, every section $\Phi_t$ could be obtained from $\Phi$ by an action of the gauge group $GL(TM)$, $\Phi_t = g_t \cdot \Phi$, with $g_t \cdot \Phi = h_t \cdot \Phi$ if and only if $g_t h_t^{-1}$ lies in the stabilizer of $\Phi$.

Since geometric structures are differential forms, pullbacks are defined; in particular, the diffeomorphism group $\text{Diff}(M)$ acts on the set of all integrable geometric structures.

Definition 2.8: The set $\Gamma(E_{\Phi}) \cap \text{Ker}(d) / \text{Diff}(M)$ is called the moduli space of $G$-structures.

This is a factor of an infinite-dimensional manifold by an action of a Fréchet-Lie group, so at least it has the structure of a topological space. However, the neighbourhood of $\Phi$ admits a finer description. It makes sense to speak about infinitesimal neighbourhoods of $\Phi$ parametrized by local Artinian commutative $\mathbb{R}$-algebras.

Let $\mathcal{A}$ be a local Artinian unital commutative $\mathbb{R}$-algebra with the maximal ideal $m$. Denote by $\exp(\mathfrak{gl}(TM) \otimes m)$ the exponent of the nilpotent Lie algebra $\mathfrak{gl}(TM) \otimes m$. The
group \(\exp(\mathfrak{gl}(TM) \otimes \mathfrak{m})\) acts on the sections of the bundle of \(\mathcal{A}\)-valued forms. If \(\Phi\) is a section of \(\Lambda^\bullet T^*M \otimes \mathcal{A}\), and we denote the orbit of \(\Phi \otimes 1\) under the action of \(\exp(\mathfrak{gl}(TM) \otimes \mathfrak{m})\) by \(\mathcal{E}_\Phi(\mathcal{A})\). One can think of the sections of \(\mathcal{E}_\Phi(\mathcal{A})\) as of the \(\text{Spec}(\mathcal{A})\)-parametrized families of the sections of \(\mathcal{E}_\Phi\), with the central fiber isomorphic to the original form \(\Phi\). De Rham differential \(d\) extends to \(\Lambda^\bullet T^*(M) \otimes \mathcal{A}\) \(\mathcal{A}\)-linearly. We will denote this extension also by \(d\).

Denote also by \(\exp(\Gamma(TM) \otimes \mathfrak{m})\) the exponent of the nilpotent Lie algebra \(\Gamma(TM) \otimes \mathfrak{m}\). The group \(\exp(\Gamma(TM) \otimes \mathfrak{m})\) acts on the sections of \(\mathcal{E}_\Phi(\mathcal{A})\) preserving closed sections.

**Definition 2.9:** The functor from the category of local commutative Artinian \(\mathbb{R}\)-algebras to the category of sets, defined by

\[
\text{Def}_\Phi : (\mathcal{A}, \mathfrak{m}) \mapsto (\mathcal{E}_\Phi \otimes \mathcal{A}) \cap \ker(d) / \exp(\Gamma(TM) \otimes \mathfrak{m})
\]

is called the deformation functor of \(\Phi\).

**Definition 2.10:** Suppose \(F\) is a functor from local commutative pro-Artinian \(\mathbb{R}\)-algebras to sets. It is called unobstructed if for every square-zero extension

\[
0 \longrightarrow I \longrightarrow A' \longrightarrow A \longrightarrow 0
\]

the induced map \(F(A') \longrightarrow F(A)\) is surjective.

**Theorem 2.11:** (\([\text{Go}]\)) A deformation functor associated to the Calabi-Yau, hyperKähler, \(G_2\)- or \(Spin(7)\)-structure is unobstructed.

The unobstructedness of Calabi-Yau structures is a classical Bogomolov-Tian-Todorov theorem (\([\text{Ti}], [\text{To}]\)). The unobstructedness of \(G_2\) and \(Spin(7)\)-structures was proven by Joyce (\([\text{Jo}]\)) with the help of a lot of hard analysis. Our goal is to describe a deformation functor of a geometric structure as a deformation functor associated to some dg-Lie algebra, giving a conceptual proof of Goto’s theorem.

**Remark 2.12:** Since one can exponentiate pro-nilpotent Lie algebras as well, the deformation functor of \(\Phi\) is in fact defined on the category of local commutative pro-Artinian \(\mathbb{R}\)-algebras, such as the formal series algebra \(\mathbb{R}[[t]]\). The question of convergence arises: is it true that the formal \(\text{Spec} \mathbb{R}[[t]]\)-parametrized deformation actually comes from the actual germ of a family of geometric structures? If the manifold \(M\) is compact, then the answer is often yes. The convergence of the solution of deformation equation considered in this article was proven by Goto in (\([\text{Go}]\)).
3 DGLAs

Definition 3.1: Let \((L, d, [\cdot, \cdot])\) be a differential graded Lie algebra with the inner derivation, that is, there exists an element \(\Delta \in L^1\) such that \(dl = [\Delta, l]\) for every \(l \in L\). An element \(x \in L^1\) is called \textit{Maurer-Cartan element} if \([\Delta + x, \Delta + x] = 0\). The set of all Maurer-Cartan elements is denoted by \(MC(L)\).

Definition 3.2: Suppose now that \(L^0\) is a nilpotent Lie algebra. Denote its exponent by \(\exp(L^0)\). The adjoint action \(ad\) of \(L^0\) preserves the quadratic cone \(\{l \in L^1 \mid [l, l] = 0\}\), and so is the action \(e^{ad}\) of \(\exp(L^0)\). The \textit{gauge action} is the action of \(\exp(L^0)\) on \(MC(L)\) given by \(e^a * x := e^{ad(a)}(x + \Delta) - \Delta\).

Remark 3.3: If the differential in \(L\) is not inner, we can nevertheless force it to be so, by constructing the new algebra \(L'\). As a graded vector space, \(L'\) is equal to \(L \oplus \langle \Delta \rangle\), where \(\Delta\) is an element of degree 1. Differential and commutators of \(\Delta\) with other elements of \(L\) are given by the relations \(d\Delta = 0, [\Delta, l] = dl\). This construction allows one to define \(MC(L)\) and the gauge action for algebras with non-inner derivation.

Definition 3.4: The \textit{deformation functor} associated to \(L\) is the functor \(\text{Def}_L\) from the category of local commutative Artinian rings to the category of sets given by

\[
\text{Def}_L(A, m) := \frac{MC(L \otimes m)}{\exp(L^0 \otimes m)}.
\]

Here we list some well-known facts about deformation functors of dg-Lie algebras. For the proofs, see e.g. ([Man1]).

Theorem 3.5: Suppose \(f : L \to M\) is a quasi-isomorphism of dg-Lie algebras. Then the induced morphism of functors \(\text{Def}(f) : \text{Def}_L \to \text{Def}_M\) is an isomorphism.

Definition 3.6: A dg-Lie algebra is called \textit{homotopy abelian} if it is quasi-isomorphic to the Lie algebra with the zero bracket.

Theorem 3.7: If \(M\) is homotopy abelian, then the functor \(\text{Def}_M\) is unobstructed. If \(f : L \to M\) is a dg-Lie morphism, \(\text{Def}_M\) is unobstructed and \(H^\bullet(f)\) is injective, then \(\text{Def}_L\) is unobstructed as well.

Next we describe the relative deformation functor construction, which was invented by Manetti ([Man2]) and later explained by Manetti and Fiorenza ([FiMan]).
Let $f : L \to M$ is the morphism of two dg-Lie algebras. Denote by $MC(f)$ the set
$$\{ x \in L^1, e^m \in \exp(M^0) \mid x \in MC(L), e^m \ast f(x) = 0 \}.$$ 

The group $\exp(L^0 \times M^{-1})$ acts on the set $MC(f)$ by the rule
$$(e^a, e^b) \ast (x, e^m) = (e^a \ast x, e^b e^m e^{f(a)}).$$

This action is also called the gauge action.

**Definition 3.8:** The relative deformation functor associated to the morphism $f : L \to M$ is the functor from the category of local Artinian algebras to the category of sets given by
$$\text{Def}_f(A, m) := \frac{MC(f \otimes m)}{\text{gauge action}}.$$ 

This functor was defined by Manetti in ([Man2]) in order to study deformations of a complex submanifold Lie-theoretically. Later in ([FiMan]) Manetti and Fiorenza obtained this functor as the deformation functor associated to a certain $L_\infty$-structure. We'll state their result, but for the sake of brevity we are not including the treatment of $L_\infty$-algebras into the present article. The reader can refer, for example, to the articles mentioned above.

**Definition 3.9:** Let $f : V \to W$ be the map of complexes. Then the cone of $f$ is the complex $\text{Cone}(f)$ with components $\text{Cone}^i(f) := V^i \oplus W^{i-1}$, and the differential $d_f$, defined as $d_f(v, w) := (-dv, -f(v) + dw)$.

**Theorem 3.10:** ([FiMan]) Suppose $f : L \to M$ is the morphism of dg-Lie algebras. Define the operations $\langle \ldots \rangle_n : \text{Sym}^n(\text{Cone}(f)) \to \text{Cone}(f)$ by the following formulas:

$$\langle (l, m) \rangle_1 = (-dl, -f(l) + dm);$$
$$\langle l_1 \otimes l_2 \rangle_2 = (-1)^{\deg_l(l_1)}[l_1, l_2];$$
$$\langle m \otimes l \rangle_2 = \frac{(-1)^{\deg_M(m)+1}}{2}[m, f(l)];$$
$$\langle m_1 \otimes m_2 \rangle_2 = 0;$$
$$\langle m_1 \otimes \cdots \otimes m_n \otimes l_1 \otimes \cdots \otimes l_k \rangle_{n+k} = 0, \quad n + k \geq 3, k \neq 1;$$

and

$$\langle m_1 \otimes \cdots \otimes m_n \otimes l \rangle_{n+1} =$$
$$= -(-1)^{\sum_{i=1}^n \deg_M(m_i)} \frac{B_n}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) [m_{\sigma(1)}, [m_{\sigma(2)}, \ldots, [m_{\sigma(n)}, f(l)] \ldots]], \quad n \geq 2;$$
where $B_n$ are Bernoulli numbers. Then these operations define the structure of an $L_\infty$-algebra on $\text{Cone}(f)$, and the functors $\text{Def}_{\text{Cone}(f)}$ and $\text{Def}_f$ are isomorphic. ■

We will need the following analogues of Theorem 3.5 and Theorem 3.7.

**Theorem 3.11:** ([FiMan], [Man2]) Suppose $L, M, N, K$ are dg-Lie algebras, and the following square of dg-Lie algebra morphisms commutes:

\[
\begin{array}{ccl}
L & \xrightarrow{i} & N \\
\downarrow f & & \downarrow g \\
M & \xrightarrow{j} & K \\
\downarrow & & \downarrow \\
\text{Cone}(f) & & \text{Cone}(g)
\end{array}
\]

Then there exists a morphism $c : \text{Cone}(f) \rightarrow \text{Cone}(g)$, making the entire diagram commutative. Moreover,

1) If $i$ and $j$ are quasi-isomorphims, then $c$ is also a quasi-isomorphism, and functors $\text{Def}_f$ and $\text{Def}_g$ are isomorphic.

2) If $\text{Cone}(g)$ is homotopy abelian, then $\text{Def}_g$ is unobstructed. Moreover, if the map $H(c)$, induced by $c$ on cohomology groups, is an injection, then $\text{Def}_f$ is also unobstructed.

**Remark 3.12:** Suppose that $f : L \rightarrow M$ is an injection. Then for $(x, e^m) \in \text{MC}(f)$, $x$ could be reconstructed from $e^m$ by the formula $x = e^{-m} \ast 0$. Hence, $\text{MC}(f)$ for an inclusion could be rewritten as $\{e^m \in \exp(M^0) \mid e^{-m} \ast 0 \in L\}$.

**Lemma 3.13:** Let $f : L \rightarrow M$ be the injection of graded Lie algebras (viewed as dg-Lie algebras with the trivial differential). Then $\text{Cone}(f)$ is homotopy abelian, and, consequently, $\text{Def}_f$ is unobstructed.

**Proof:** It is proved in [FiMar], section 5. We present here another proof, based on the homotopy transfer theorem. Reader interested in the homotopy transfer may refer, for example, to [LoVa].

Let us pick a splitting $M = L \oplus M/L$, such that $f$ will be an embedding of the first summand, and denote the corresponding embedding of $M/L$ into $\text{Cone}(f)$ by $e$. Then the complex $M/L$ with the trivial differential is isomorphic to cohomology of the complex $\text{Cone}(f)$. Applying homotopy transfer formulas to this splitting, one can obtain
that transferred operations on $M/L$ vanish. Indeed, all brackets in $\text{Cone}(f)$, applied to the elements of $M \subset \text{Cone}(f)$ vanish, and the image of $e$ lies in $M$. So, the transferred operations on the cohomology of $\text{Cone}(f)$ are zero, thus it is homotopy abelian. Hence the functor $\text{Def}_{\text{Cone}(f)}$ is unobstructed.

**Theorem 3.14:** Let $(V, d)$ be a complex, $\text{End}(V)$ its automorphism dg-Lie algebra (that is, $\text{End}'(V) := \prod_{k \in \mathbb{Z}} \text{Hom}(V^k, V^{k+i})$ and the differential is given by the graded commutator with $d$), $v \in V$ a closed (possibly non-homogeneous) vector, $dv = 0$. Let $\text{Ann}(v)$ be the subalgebra of endomorphisms annulating $v$, and denote the embedding $\text{Ann}(v) \subset \text{End}(V)$ by $\varepsilon$. Then $\text{Def}_\varepsilon$ is unobstructed.

**Proof:** Let us pick a splitting of $V$ into an acyclic complex and a complex with the zero differential, such that the projection of $v$ onto the acyclic summand is zero. This splitting induces splittings of $\text{End}(V)$ and $\text{Ann}(v)$. Then the statement of the theorem follows from Theorem 3.11 and Lemma 3.13.

**4 Deformation of geometric structures**

Let $M$ be a smooth manifold, $\Omega^\bullet(M)$ its de Rham dg-algebra, and $\Phi \in \Omega^\bullet(M)$ an integrable geometric structure on $M$. Our goal is to describe functor from Definition 2.9 as the deformation functor associated to some $L_\infty$-algebra. We will describe it as the relative deformation functor.

Denote by $\text{Der} \Omega^\bullet(M)$ the dg-Lie algebra of derivations of $\Omega^\bullet(M)$. That is, $\text{Der} \Omega^\bullet(M)$ is the subcomplex of $\text{End} \Omega^\bullet(M)$ consisting of those linear morphisms $D \in \prod_{k \in \mathbb{Z}} \text{Hom}(\Omega^k, \Omega^{k+i})$ which satisfy the graded Leibniz rule: $D(\alpha \wedge \beta) = D(\alpha) \wedge \beta + (-1)^{k|\alpha|} \alpha \wedge D(\beta)$.

**Theorem 4.1:** ([FrNi]) Every degree $k$ derivation on $\Omega^\bullet(M)$ is of the form $i_A + \{d, i_B\}$, where $A \in \Omega^{k+1}(M, TM)$, $B \in \Omega^k(M, TM)$, and $i_A$ is the inner derivation: $i_{A \otimes \beta}(\beta) := \alpha \wedge i_v(\beta)$. In particular, every derivation of degree 0 is the sum of a Lie derivative with respect to some vector field $v$ and an infinitesimal gauge automorphism $A \in \mathfrak{gl}(TM)$.

Let $\text{Stab} \Phi$ be a subalgebra of $\text{Der} \Omega^\bullet(M)$ consisting of derivations vanishing on $\Phi$. Denote the inclusion $\text{Stab} \Phi \longrightarrow \text{Der} \Omega^\bullet(M)$ by $j$.

**Theorem 4.2:** The functor $\text{Def}_j$ is isomorphic to the functor $\text{Def}_\Phi$.

**Proof:** Remind that $MC_j(m)$ is the set of those $m \in \text{Der}(\Omega^0) \otimes \mathfrak{m}$ for which $e^{ad(-m)}d - d$ lies in $\text{Stab}(\Phi)$, or, equivalently,

$$e^{-m}de^m \Phi - d\Phi = 0.$$
Since $\Phi$ is integrable, and $e^{-m}$ is invertible, this equation is equivalent to the equation $de^m \Phi = 0$. By Theorem 4.1, $m$ uniquely decomposes into a sum $A + \text{Lie}_v$, where $A \in \mathfrak{gl}(TM) \otimes m$ and $v \in \Gamma(TM) \otimes m$.

Consider the functor morphism $T : \text{MC}_j \rightarrow \text{Def}_\Phi$ given by the formula

$$T(m) = T(A + \text{Lie}_v) = e^A \Phi.$$ 

Since $A$ could be any element of $\mathfrak{gl}(TM) \otimes m$, this functor morphism is surjective. If $A$ and $B$ are two elements of $\mathfrak{gl}(TM) \otimes m$, $e^A \Phi = e^B \Phi$ in $\text{Def}_\Phi$ if and only if there exist an element $v \in \Gamma(TM) \otimes m$ such that $e^{-B} e^A \Phi = e^{\text{Lie}_v \Phi}$. This is the equivalence under gauge action introduced in Definition 3.8.

One advantage of working with $\text{Def}_j$ rather than with $\text{Def}_\Phi$ is the access to Theorem 3.5 and Theorem 3.7.

Consider the following commutative diagram, where horizontal arrows are natural embeddings:

$$\begin{array}{ccc}
\text{Stab}(\Phi) & \longrightarrow & \text{Ann}(\Phi) \\
\downarrow j & & \downarrow i \\
\text{Der } \Omega^\bullet(M) & \longrightarrow & \text{End}(\Omega^\bullet(M))
\end{array}$$

By naturality of Fiorenza-Manetti construction, this commutative square provides an $L_\infty$-map from $\text{Cone}(j)$ to $\text{Cone}(i)$. By Theorem 3.7, $\text{Cone}(i)$ is homotopy abelian, so the functor $\text{Def}_i$ is unobstructed. If we will be able to prove that the induced map $H^\bullet(\text{Cone}(j)) \longrightarrow H^\bullet(\text{Cone}(i))$ is injective, we will prove the unobstructedness of the functor $\text{Def}_\Phi$.

**Theorem 4.3:** If $\Phi$ is a Calabi-Yau structure, hyperKähler structure, $G_2$-structure or $\text{Spin}(7)$-structure, then the map $H^\bullet(\text{Cone}(j)) \longrightarrow H^\bullet(\text{Cone}(i))$ is injective.

**Proof:** Both cones $\text{Cone}(j)$ and $\text{Cone}(i)$ are quasi-isomorphic to corresponding quotients $\text{Der } \Omega^\bullet(M)/\text{Stab}(\Phi)$ and $\text{End}(\Omega^\bullet(M))/\text{Ann}(\Phi)$. Both factors are isomorphic to orbits of $\Phi$ under the action of, correspondingly, $\text{Der } \Omega^\bullet(M)$ and $\text{Ann}(\Phi)$. Since the action of $\text{End}(\Omega^\bullet(M))$ can map any non-zero form onto any other non-zero form, the quotient $\text{End}(\Omega^\bullet(M))/\text{Ann}(\Phi)$ is isomorphic to the sum of several shifted de Rham complexes, with the number of complexes being the number of non-zero components in $\Phi$ and the shifts being degrees of the corresponding components. Since $d\Phi = 0$, from Theorem 4.1 we obtain that the orbit of $\Phi$ under the action of $\text{Der } \Omega^\bullet(M)$ is isomorphic to the following complex:

$$0 \longrightarrow i_{TM} \Phi \longrightarrow \mathfrak{gl}(TM) \cdot \Phi \longrightarrow \Omega^1 \wedge \mathfrak{gl}(TM) \cdot \Phi \longrightarrow \Omega^2 \wedge \mathfrak{gl}(TM) \cdot \Phi \longrightarrow \ldots$$

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This is precisely the complex # of Goto ([Go]). Cohomological injectivity of the obvious map from \( \text{Der} \Omega^\bullet(M)/\text{Stab}(\Phi) \) to \( \text{End}(\Omega^\bullet(M))/\text{Ann}(\Phi) \) in the cases of Calabi-Yau, hyperKähler, \( G_2 \)- and \( \text{Spin}(7) \)-structures are proven in chapters 4, 5, 6 and 7 of [Go]. In former two cases this injectivity is just a consequence of \( \partial \overline{\partial} \)-lemma, and in the latter, it is a consequence of some \( \partial \overline{\partial} \)-lemma-like statements about forms on manifolds with special holonomy.

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