Measure of decoherence in quantum error correction for solid-state quantum computing

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We considered the interaction of semiconductor quantum register with noisy environment leading to various types of qubit errors. We analysed both phase and amplitude decays during the process of electron-phonon interaction. The performance of quantum error correction codes (QECC) which will be inevitably used in full scale quantum information processors was studied in realistic conditions in semiconductor nanostructures. As a hardware basis for quantum bit we chose the quantum spatial states of single electron in semiconductor coupled double quantum dot system. The modified 5- and 9-qubit quantum error correction (QEC) algorithms by Shor and DiVincenzo without error syndrome extraction were applied to quantum register. 5-qubit error correction procedures were implemented for Si charge double dot qubits in the presence of acoustic phonon environment. χ-matrix, Choi–Jamiołkowski state and measure of decoherence techniques were used to quantify qubit fault-tolerance. Our results showed that the introduction of above quantum error correction techniques at small phonon noise levels provided quadratic improvement of output error rates. The efficiency of 5-qubits quantum error correction algorithm in semiconductor quantum information processors was demonstrated.

I. INTRODUCTION

In recent years much attention is attracted to the influence of decoherence on quantum communication channels [1–3] and quantum information processing [3–6]. Measure of decoherence approach [7, 8] occurred to be helpful tool for quantitative evaluation of quantum state distortion due to noisy environment. χ-matrix and Choi–Jamiołkowski state representations are very efficient for quantum channel description and study [1–3, 9–12]. In this paper we combine these approaches to analyse the influence of error correction [13–19] on quantum registers. The representative example of solid-state qubit – electron in semiconductor double dot interacting with phonons is also considered.

II. DESCRIBING QUANTUM CHANNELS

We consider a qubit system described by a density matrix ρ of size d × d. By definition, the operator ρ is Hermitian, positive semidefinite and has trace one [1–3, 20, 21]. An evolution of a density matrix is described by a quantum operation E (also called a quantum channel or a stochastic map) [1–3, 9–12]. We use these terms interchangeably. As usual a linear map E is called a quantum operation when it preserves trace (in Schrödinger picture) and is completely positive [1]. In our investigation, a qubit system is placed in an environment that acts on these qubits independently. That means that the quantum operation describing the impact of the environment on the system of qubits is separable. Choi has shown [22] that every quantum channel can be represented as a sum of Kraus operators (Kraus representation [23]):

\[ E[\rho] = \sum_{i=1}^{n} \hat{E}_i \rho \hat{E}_i^\dagger, \quad \sum_{i=1}^{n} \hat{E}_i^\dagger \hat{E}_i = \hat{I}. \]

(1)

This statement is also a sufficient condition for a map to be a quantum channel. Equation (1) shows a way to describe a channel.

There are several ways to describe a quantum channel mathematically. In the sake of convenience, we will use a Choi-Jamiołkowski state [23] \( \tau \) in simulation of quantum circuits and a χ-matrix representation in the calculation of measure of decoherence. Below we consider both of these representations.

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A. \(\chi\)-matrix representation

Every linear map \(\mathcal{E}\) which is a quantum operation can be written in the form of:

\[
\mathcal{E}[\rho] = \sum_{\alpha,\beta=0}^{d^2-1} \chi_{\alpha\beta} \hat{E}_\alpha \rho \hat{E}_\beta^\dagger, \tag{2}
\]

where the set of \(\{\hat{E}_\alpha\}_{\alpha=0}^{d^2-1}\) matrices is the basis in \(\mathcal{H}_d^{\otimes 2}\) space.

The coefficients \(\chi_{\alpha\beta}\) form \(\chi\)-matrix of dimension \(d^2 \times d^2\). From the construction of \(\chi\) it is seen that the matrix is Hermitian. Since \((\mathcal{E}[\rho])^\dagger = \mathcal{E}[\rho]\), we have

\[
\left(\sum_{\alpha,\beta=0}^{d^2-1} \chi_{\alpha\beta} \hat{E}_\alpha \rho \hat{E}_\beta^\dagger\right)^\dagger = \sum_{\alpha,\beta=0}^{d^2-1} \chi_{\alpha\beta}^* \hat{E}_\beta^\dagger \rho \hat{E}_\alpha = \sum_{\alpha,\beta=0}^{d^2-1} \chi_{\alpha\beta} \hat{E}_\alpha \rho \hat{E}_\beta^\dagger = \sum_{\alpha,\beta=0}^{d^2-1} \chi_{\beta\alpha} \hat{E}_\beta \rho \hat{E}_\alpha^\dagger. \tag{3}
\]

Thus \(\chi_{\beta\alpha} = \chi_{\alpha\beta}^*\).

Another limitation is imposed by the fact that channel \(\mathcal{E}\) preserves trace because it is a quantum operation. Since

\[
\text{Tr}[\mathcal{E}[\rho]] = \text{Tr}\left[\sum_{\alpha,\beta=0}^{d^2-1} \chi_{\alpha\beta} \hat{E}_\alpha \rho \hat{E}_\beta^\dagger\right] = \sum_{\alpha,\beta=0}^{d^2-1} \text{Tr}\left[\chi_{\alpha\beta} \hat{E}_\alpha^\dagger \hat{E}_\beta \rho \right] = \text{Tr}\left[\rho \sum_{\alpha,\beta=0}^{d^2-1} \chi_{\alpha\beta} \hat{E}_\beta^\dagger \hat{E}_\alpha \right] = \text{Tr}[\rho],
\]

we obtain

\[
\left(\sum_{\alpha,\beta=0}^{d^2-1} \chi_{\alpha\beta} \hat{E}_\beta^\dagger \hat{E}_\alpha \right) = \hat{1}. \tag{4}
\]

Since \(\chi\)-matrix of \(4 \times 4\) dimension is Hermitian, it can be defined by 16 real numbers. The limitations due to trace conservation shown in Eq. (3) allows us to describe \(\chi\) with 12 real parameters. \(\chi\)-matrix formalism provides a simple method of verification of complete positivity. A map \(\mathcal{E}\) is completely positive if and only if \(\chi\)-matrix is positive semidefinite [25].

B. Choi-Jamiolkowski state \(\tau\) representation

Operator \(\hat{\chi}\) is given by

\[
\hat{\chi} = \sum_{\alpha,\beta=0}^{d^2-1} \chi_{\alpha\beta} \hat{E}_\alpha \rangle \langle \hat{E}_\beta|, \tag{5}
\]

where \(\hat{\chi}) = [A_{11}, ..., A_{1d}, A_{21}, ..., A_{2d}, ..., A_{dd}]^T\) is a supervector in Liouville space (L-space). Supervectors \(\{|\hat{E}_\alpha\rangle\}_{\alpha=0}^{d^2-1}\) form the basis in \(\mathcal{H}_d^{\otimes 2}\).

Choi-Jamiolkowski state is defined as

\[
\tau = (\mathcal{E} \otimes \hat{1})|\Omega_{d^2}\rangle \langle \Omega_{d^2}|, \tag{6}
\]

where the state \(|\Omega_{d^2}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle |i\rangle\) is a maximally entangled state. It is worth noting that \(|\sqrt{d}|\hat{1}\rangle \rangle = |\Omega_{d^2}\rangle\rangle\) and \(|\hat{A}\rangle \rangle = (\hat{A} \otimes \hat{1}) |\hat{1}\rangle \rangle\). These properties are followed from the above definitions. The quantum operation is uniquely determined by the Choi-Jamiolkowski state. This fact is known as Choi-Jamiolkowski isomorphism. Let us show that \(\tau = \hat{\chi}/d\):

\[
\hat{\chi} = \sum_{\alpha,\beta=0}^{d^2-1} \chi_{\alpha\beta} \left(\hat{E}_\alpha \otimes \hat{1} \right) \langle \hat{1} | \left(\hat{E}_\beta \otimes \hat{1} \right)^\dagger = \left(\mathcal{E} \otimes \hat{1} \right) |\hat{1}\rangle \langle \hat{1}| = d \left(\mathcal{E} \otimes \hat{1} \right) |\Omega_{d^2}\rangle \langle \Omega_{d^2}| = \tau d. \tag{7}
\]
III. MEASURE OF DECOHERENCE

To estimate the quality of quantum correction we use the concept of measure of decoherence $D$ \cite{10,11,9}. By the definition, measure of decoherence is the maximum over all the states of the operator norm of matrix $\rho_{\text{out}} - \rho_{\text{in}}$:

$$D = \sup_{\rho_{\text{in}}} ||\rho_{\text{out}} - \rho_{\text{in}}||,$$  \hspace{1cm} (8)

where the operator norm of Hermitian matrix $A$ is given by $||A|| = \max_{a \in \text{spec}(A)} |a|$ \cite{11}. By $\text{spec}(A)$ denote the spectrum of operator $A$.

Let us consider a qubit ($d = 2$). In this article it is shown that if the influence of the environment has the Kraus representation in Pauli basis with $\chi$-matrix in the form of

$$\chi = \begin{pmatrix} \chi_0 & 0 & 0 & 0 \\ 0 & \chi_1 & 0 & 0 \\ 0 & 0 & \chi_2 & 0 \\ 0 & 0 & 0 & \chi_3 \end{pmatrix},$$  \hspace{1cm} (9)

then the expression for measure of decoherence is simplified

$$D = \max\{\chi_1 + \chi_2, \chi_1 + \chi_3, \chi_2 + \chi_3\} = \chi_1 + \chi_2 + \chi_3 - \min\{\chi_1, \chi_2, \chi_3\}.$$  \hspace{1cm} (10)

Let $\chi_{mk} = \chi_{mk}^{\text{re}} + i\chi_{mk}^{\text{im}}$, then using Eq. 2 we get the relations $\chi_{mk}^{\text{re}} = \chi_{km}^{\text{re}}, \chi_{mk}^{\text{im}} = -\chi_{km}^{\text{im}}$. Using Eq. 3 we obtain a system of 4 equations:

$$\begin{align*}
\chi_{00} &= 1 - \chi_{11} - \chi_{22} - \chi_{33} \\
\chi_{12} &= \chi_{21}^{\text{re}} \\
\chi_{13}^{\text{im}} &= \chi_{21}^{\text{re}} \\
\chi_{23}^{\text{im}} &= \chi_{11}^{\text{re}}.
\end{align*}$$  \hspace{1cm} (11)

Consider an arbitrary $\chi$-matrix of $4 \times 4$ dimension:

$$\chi = \begin{pmatrix}
1 - \chi_{11} - \chi_{22} - \chi_{33} & \chi_{01}^{\text{re}} + i\chi_{01}^{\text{im}} & \chi_{02}^{\text{re}} + i\chi_{02}^{\text{im}} & \chi_{03}^{\text{re}} + i\chi_{03}^{\text{im}} \\
\chi_{11}^{\text{re}} & 1 - \chi_{12} - \chi_{33} & \chi_{12}^{\text{re}} - i\chi_{12}^{\text{im}} & \chi_{13}^{\text{re}} - i\chi_{13}^{\text{im}} \\
\chi_{21}^{\text{re}} & \chi_{12}^{\text{re}} - i\chi_{12}^{\text{im}} & 1 - \chi_{22} - \chi_{33} & \chi_{23}^{\text{re}} - i\chi_{23}^{\text{im}} \\
\chi_{31}^{\text{re}} & \chi_{13}^{\text{re}} - i\chi_{13}^{\text{im}} & \chi_{23}^{\text{re}} - i\chi_{23}^{\text{im}} & 1 - \chi_{33}
\end{pmatrix}.$$  \hspace{1cm} (12)

Let us rewrite it in other variables

$$\chi = \begin{pmatrix} 1 - \chi_1 - \chi_2 - \chi_3 & \chi_4 + i\chi_5 & \chi_6 + i\chi_7 & \chi_8 + i\chi_9 \\
\chi_4 - i\chi_5 & \chi_1 + \chi_8 & \chi_10 + i\chi_8 & \chi_11 - i\chi_6 \\
\chi_6 - i\chi_7 & \chi_10 - i\chi_8 & \chi_2 & \chi_12 + i\chi_4 \\
\chi_8 - i\chi_9 & \chi_11 + i\chi_6 & \chi_12 - i\chi_4 & \chi_3
\end{pmatrix}.$$  \hspace{1cm} (13)

In general, an arbitrary density matrix can be written as $\rho = \begin{pmatrix} 1 + P_x & P_x - iP_y \\ P_x + iP_y & 1 - P_z \end{pmatrix} / 2$. Then measure of decoherence depends on the components of $\chi$-matrix as follows:

$$D(\chi) = \max_{P_x, P_y, P_z} \left| 2\chi_4 P_x + 2\chi_6 P_y + 2\chi_8 P_z \pm \right.$$  \hspace{0.5cm}
$$\left. \pm \left[ (\chi_1^2 + \chi_2^2 + \chi_3^2) + 2\chi_3 \right] \pm \left[ (\chi_4^2 + \chi_6^2 + \chi_8^2) + 2\chi_8 \right] \pm \left[ (\chi_1^2 + \chi_2^2 + \chi_3^2) + 2\chi_3 \right] \right|^{1/2}.$$  \hspace{1cm} (14)
An n-qubit QECC and the scheme for calculating $\tau$-matrix in case of arbitrary error correction. n-qubit QECC is shown on this Figure. $|\Omega\rangle$ is a maximum entangled two-qubit pure state. The first qubit of the state $|\Omega\rangle$ remains unchanged during the process of QEC, the second qubit of $|\Omega\rangle$ is the first qubit of n-qubit QECC.

FIG. 2: Scheme for calculating measure of decoherence in case of an error in a qubit. The first qubit remains unchanged, the second is affected by a quantum channel.

In the case $\chi_4 = \chi_6 = \chi_8 = 0$, the square of the measure of decoherence is a quadratic form, which can be reduced to canonical form by introducing the new variables $Q_x, Q_y, Q_z$:

$$D^2 = \max_{Q_x, Q_y, Q_z} \left[ a_1 Q_x^2 + a_2 Q_y^2 + a_3 Q_z^2 \right].$$

Let us consider the function $f = a_1 Q_x^2 + a_2 Q_y^2 + a_3 Q_z^2$, where the coefficients $\{a_1, a_2, a_3\}$ are the eigenvalues of the quadratic form $f$. With this change of variables:

$$\begin{cases} Q_x^2 + Q_y^2 + Q_z^2 \leq 1 \\ Q_x^2 \geq 0, Q_y^2 \geq 0, Q_z^2 \geq 0 \\ f \geq 0. \end{cases}$$

Since for all $Q_x, Q_y, Q_z$ the function $f$ should be nonnegative, we get $a_1 \geq 0, a_2 \geq 0, a_3 \geq 0$. Hence the function $f$ is nondecreasing, the maximum of this function is achieved on the boundary of Eq. (16). The point $(Q_x, Q_y, Q_z)$ of space at which there is the maximum of measure of decoherence is limited to the set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Therefore, we have

$$D = \sqrt{\max\{\text{spec}(f)\}}.$$
4. We calculate measure of decoherence $D_0(q)$ for qubit in an environment. Here $D_0(q)$ is a probability for $p$ to decohere.

5. We replace $q$ with $p$ in the operator-sum based on the equality $D_0(q) \equiv p$.

6. We obtain the limitation on $p$ based on the positivity of $\chi(p)$-matrix.

7. Next we consider the error correction code. We take $N$ different values of $p$, using the existing limitations on the probability $p$. We calculate Choi-Jamiołkowski state $\tau(p)$ after applying QECC. We get $\tilde{\chi}(p)$ state using $\tau(p)$.

8. From $\tilde{\chi}(p)$ we get $\chi(p)$-matrix.

9. We calculate measure of decoherence $D(p)$ of qubit after interaction with an environment and quantum correction. For $n$ different $p$ we take $n$ values of $D$, $n$ is the number of qubits in QECC.

10. Since the quantum correction code gives a polynomial in $p$ improvement we have

$$D(p) = \sum_{i=1}^{n} \alpha_i p^i.$$  \hspace{1cm} (18)

We solve $n$ equations and calculate coefficients $\{\alpha_i\}_{i=1}^{n}$ of the polynomial of the $n$-th degree. Hence we establish polynomial expression for $D$.

Next we consider the results of the algorithm for calculating $D(p)$.

A. Bit and Phase Flips

Let bit flip environment changes $|0\rangle$ to $|1\rangle$, and $|1\rangle$ to $|0\rangle$ with probability $p'$. This action can be conveniently written as Eq. (2) with the Pauli matrices $\{\hat{I}, \hat{X}, \hat{Y}, \hat{Z}\}$ as the basis $\{\hat{E}_\alpha\}_{d^2-1}^{d^2=2}$ for $d = 2$ (single qubit). Then the operator-sum is written as:

$$\mathcal{E}[\rho] = (1 - p')\hat{I}\rho\hat{I} + p'\hat{X}\rho\hat{X}. \hspace{1cm} (19)$$

This representation in the form of Kraus decomposition allows us to write $\chi$-matrix of the quantum channel as a diagonal matrix in Eq. (9) with parameters $\chi_0 = 1 - p'$, $\chi_1 = p'$ and $\chi_3 = \chi_4 = 0$. It follows from Eq. (10) that measure of decoherence $D_0 \equiv p = p'$. Then $\chi$-matrix of the quantum channel in case of bit flip is equal to

$$\chi_0 = \begin{pmatrix} 1 - p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \hspace{1cm} (20)$$

According to $\chi$-matrix positivity criterion, bit flip channel is completely positive if and only if $0 \leq p \leq 1$.

Let us consider a bit error correction code based on a decoding of type called majority voting [3]. Scheme for finding measure of decoherence in the case of majority voting is represented in Fig. 3.

$\chi$-matrix is calculated following the above algorithm:

$$\chi = \begin{pmatrix} 1 - 3p^2 + 2p^3 & 0 & 0 & 0 \\ 0 & 3p^2 - 2p^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \hspace{1cm} (21)$$
According to Eq. (10), the measure of decoherence

\[ D(p) = 3p^2 - 2p^3. \] (22)

The procedure of finding the result of an action and the result of a correction in case of phase flip is similar to the case of bit error correction. The only difference is the modified scheme of majority voting, shown in Fig. 4.

**B. Depolarizing Channel**

It is convenient to write an operator sum in case of the depolarizing channel in the form of Kraus decomposition:

\[ \mathcal{E}[\rho] = \left(1 - \frac{3}{2}p\right) I \rho I + \frac{p}{2} \left( X \rho X + Y \rho Y + Z \rho Z \right). \] (23)

Since all the matrices in the decomposition are the Pauli matrices, the matrix \( \chi_0 \) is diagonal and is equal to

\[ \chi_0 = \begin{pmatrix} 1 - \frac{3}{2}p & 0 & 0 & 0 \\ 0 & p/2 & 0 & 0 \\ 0 & 0 & p/2 & 0 \\ 0 & 0 & 0 & p/2 \end{pmatrix}. \] (24)

The parameters in the decomposition in Eq. 23 are selected to satisfy \( D_0 = p \). From positivity of \( \chi_0 \)-matrix \( p \in [0, 2/3] \).

Let us consider the result of the 5-qubit DiVincenzo-Shor QEC \[14, 18\], shown in Fig. 5, after the action of the depolarizing channel. In the case of the depolarizing channel it is possible to obtain a general view of \( \chi \)-matrix, which describes the action of the error correction code:

\[ \chi = \begin{pmatrix} 1 - 3\chi_1 & 0 & 0 & 0 \\ 0 & \chi_1 & 0 & 0 \\ 0 & 0 & \chi_1 & 0 \\ 0 & 0 & 0 & \chi_1 \end{pmatrix}. \] (25)

where

\[ \chi_1 = p^2 \left(15 - 50p + 60p^2 - 24p^3\right)/2. \] (26)
It follows from Eq. (10) that measure of decoherence is equal to

\[ D(p) = p^2 \left( 15 - 50p + 60p^2 - 24p^3 \right) . \]  

(27)

Let \( p \to 0 \). For 9-qubit Shor code [13] shown in Fig. 6, measure of decoherence has the form

\[ D(p) = 36p^2. \]  

(28)

Combining Eqs. (27) and (28), we obtain the result of QEC shown in Fig. 7 in case of the depolarizing channel.
V. QEC IN SI DOUBLE DOT CHARGE QUBITS

A. Amplitude Damping

Let us consider the basis \( |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \), \( |\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \). In this basis the evolution of the density matrix is written in the form of

\[
\begin{pmatrix}
(1 - \rho_{-\cdots}(0)e^{-\Gamma t}) & \rho_{+\cdots}(0)e^{-\Gamma t/2} \\
\rho_{-\cdots}(0)e^{-\Gamma t/2} & \rho_{+\cdots}(0)e^{-\Gamma t}
\end{pmatrix},
\]

(29)
denote by \( \Gamma(t) \) a relaxation rate. Let us rewrite Eq. (29) to obtain the representation in the form of an operator-sum:

\[
\mathcal{E}[\rho] = \begin{pmatrix}
1 & 0 \\
e^{-\Gamma t/2} & 1
\end{pmatrix} \rho \begin{pmatrix}
1 & 0 \\
e^{-\Gamma t/2} & 1
\end{pmatrix}^\dagger + \begin{pmatrix}
0 & \sqrt{1 - e^{-\Gamma t}} \\
\sqrt{1 - e^{-\Gamma t}} & 0
\end{pmatrix} \rho \begin{pmatrix}
0 & \sqrt{1 - e^{-\Gamma t}} \\
\sqrt{1 - e^{-\Gamma t}} & 0
\end{pmatrix}^\dagger.
\]

(30)

Let us consider the state \( \tau \). The nonzero eigenvalues of the state are \( (1 \pm e^{-\Gamma t})/2 \). Since \( \Gamma t \geq 0 \), we see that all eigenvalues of the matrix \( \tau \) are nonnegative, that is \( \tau \) is positive semidefinite. Therefore, the map in Eq. (30) is completely positive.

We represent the quantum operation shown in Eq. (30) in the form of \( \chi_0 \)-matrix in Pauli matrix basis:

\[
\chi_0 = \begin{pmatrix}
(1 + e^{-\Gamma t})/2 & 0 & 0 & 0 \\
0 & (e^{-\Gamma t} - 1) & \sqrt{1 - e^{-\Gamma t}} & 0 \\
0 & -\sqrt{1 - e^{-\Gamma t}} & (e^{-\Gamma t} - 1) & 0 \\
(1 - e^{-\Gamma t}) & 0 & 0 & (1 - e^{-\Gamma t}/2)^2
\end{pmatrix}/4.
\]

(31)

According to Eq. (8), measure of decoherence is equal to \( D_0 = 1 - e^{\Gamma t} = p \). It follows from the proposed algorithm that measure of decoherence in case of the 5-qubit QECC

\[
D(p) = 5p^2 (3 - 3p + p^2)/8.
\]

(32)

It can be concluded that for small \( p \) the efficiency of the error correction in the case of amplitude damping is 8 times higher than in the case of the depolarizing channel.

B. Phase Damping

The evolution of the density matrix in a single operation can be written as

\[
\begin{pmatrix}
\rho_{00}(0) & \rho_{01}(0)e^{-B^2} \\
\rho_{10}(0)e^{-B^2} & \rho_{11}(0)
\end{pmatrix},
\]

denote by \( B^2(t) \) a spectral function. Using representation in Eq. (33), we can write the operator-sum:

\[
\mathcal{E}[\rho] = \begin{pmatrix}
e^{-B^2/2} & 0 & 0 & 0 \\
0 & e^{-B^2/2} & 0 & 0
\end{pmatrix} \rho \begin{pmatrix}
e^{-B^2/2} & 0 & 0 & 0 \\
0 & e^{-B^2/2} & 0 & 0
\end{pmatrix}^\dagger + \begin{pmatrix}
0 & \sqrt{1 - e^{-B^2}} & 0 & 0 \\
\sqrt{1 - e^{-B^2}} & 0 & 0 & 0
\end{pmatrix} \rho \begin{pmatrix}
0 & \sqrt{1 - e^{-B^2}} & 0 & 0 \\
\sqrt{1 - e^{-B^2}} & 0 & 0 & 0
\end{pmatrix}^\dagger.
\]

(34)

Consider Choi-Jamiołkowski state \( \tau \). The nonzero eigenvalues of the state are \( (1 \pm e^{-B^2})/2 \). Since \( B^2 \geq 0 \), we have nonnegative eigenvalues of matrix \( \tau \), that is the state \( \tau \) is positive semidefinite. Hence, the map in Eq. (34) is completely positive.

Let us represent the quantum operation of Eq. (34) in the form of \( \chi_0 \)-matrix in Pauli matrix basis:

\[
\chi_0 = \begin{pmatrix}
(1 + e^{-B^2})/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & (1 - e^{-B^2})/2
\end{pmatrix}.
\]

(35)
It follows from Eq. (10) that measure of decoherence is equal to $D_0 = \frac{1 - e^{-\frac{B^2(t)}{2}}}{2} = p$. According to the proposed algorithm we calculate measure of decoherence in case of the 5-qubit QECC

$$D(p) = 10p^2(1 - 2p + p^2).$$

(36)

It can be concluded that for small $p$ the efficiency of the error correction in the case of amplitude damping is 1.5 times higher than in the case of the depolarizing channel.

C. QEC Results

During calculations, transmission or information storage there occurs amplitude and phase damping. To fight the process of decoherence one can use error correction algorithms, which have already been discussed in this paper.

Let us consider an error correction in silicon, where the qubit represents as a state of an electron in the double quantum dot [5, 28–30]. To describe the entire system of five qubit we use our results for measure of decoherence in Eqs. (32) and (36). To determine relaxation rate and spectral function, we use the following formulas

$$\Gamma = \frac{\Xi^2 k^3}{4\pi \rho s^2 \hbar} \exp(-a^2 k^2/2) \left(1 - \frac{\sin(kL)}{kL}\right),$$

(37)

$$B^2(t) = \frac{\Xi^2}{\pi^2 \hbar \rho s^3} \int_0^\infty q^2 dq \int_0^{\pi} \sin \Theta d\Theta \sin^2(qL \cos \Theta) \exp(-a^2 q^2/2) \sin^2 qst,$$

(38)

where deformation potential $\Xi = 3.3$ eV, speed of sound $s = 9.0 \times 10^3$ m/s, crystal density $\rho = 2.33$ g/sm$^3$, the distance between points $L = 50$ nm, radius of points $a = 3$ nm.

Depending upon the quantum gate operation required at current algorithmic step, a qubit experience phase or amplitude damping error during quantum computing on double dot qubit. Hence we define measure of decoherence in this case as the maximum of the two measures. Combining Eqs. (32) and (36), we obtain

$$D_0(t) = \max\{p_1, p_2\},$$

(39)

$$D(t) = \max \left\{ 5p_1^2(3 - 3p_1 + p_1^2) / 8, \ 10p_2^2(1 - 2p_2 + p_2^2) \right\},$$

(40)

where $p_1 = 1 - e^{\Gamma(t)t}$ and $p_2 = \left(1 - e^{-\frac{B(t)^2}{2}}\right)/2$.

Let quantum computer perform $N$ operations. By $p$ denote the probability of an error during one cycle time. Substituting $p_1$ for $Np_1$ and $p_2$ for $Np_2$ in Eqs. (39) and (40), we obtain the result of QEC shown in Fig. 8.
VI. CONCLUSIONS

By using $\chi$-matrix, Choi-Jamiołkowski state, and measure of decoherence techniques we analysed the influence of noise on quantum bits. The analytical expressions of measure of decoherence were obtained for practically important wide subset of quantum channels. We have shown that the introduction of DiVincenzo-Shor quantum error correction algorithm is helpful for double dot charge qubits and reduces effective error rate drastically.

ACKNOWLEDGMENTS

The work was supported via the grant No. 07.524.12.4019 of the Ministry of Education and Science of the Russian Federation.

[1] Holevo, A., “Quantum systems, channels, information [in Russian],” Moscow Center for Continuous Mathematical Education, Moscow (2010).
[2] Holevo, A., [Probabilistic and statistical aspects of quantum theory], vol. 1, Edizioni della Normale (2011).
[3] Nielsen, M. and Chuang, I., [Quantum computation and quantum information], Cambridge university press (2010).
[4] Preskill, J., “Lecture notes for physics 229: Quantum information and computation,” California Institute of Technology (1998).
[5] Fedichkin, L. and Fedorov, A., “Study of temperature dependence of electron-phonon relaxation and dephasing in semiconductor double-dot nanostructures,” Nanotechnology, IEEE Transactions on 4(1), 65–70 (2005).
[6] Fedichkin, L. and Privman, V., “Quantitative treatment of decoherence,” Electron Spin Resonance and Related Phenomena in Low-Dimensional Structures, 141–167 (2009).
[7] Fedichkin, L., Fedorov, A., and Privman, V., “Measures of decoherence,” in [Proceedings of SPIE], 5105, 243–254 (2003).
[8] Fedichkin, L., Fedorov, A., and Privman, V., “Additivity of decoherence measures for multiqubit quantum systems,” Physics Letters A 328(2), 87–93 (2004).
[9] Sudarshan, E., Mathews, P., and Rau, J., “Stochastic dynamics of quantum-mechanical systems,” Physical Review 121(3), 920 (1961).
[10] Holevo, A., “On the mathematical theory of quantum communication channels,” Problemy Peredachi Informatsii 8(1), 62–71 (1972).
[11] Choi, M., Positive linear maps on C*-algebras, PhD thesis, University of Toronto. (1973).
[12] Caves, C., “Quantum error correction and reversible operations,” Journal of Superconductivity 12(6), 707–718 (1999).
[13] Tomita, H., Nakahara, M., and Tomita, H., “Unitary quantum error correction without error detection,” arXiv preprint arXiv:1101.0413 (2011).
[14] Li, C., Nakahara, M., Poon, Y., Sze, N., and Tomita, H., “Recovery in quantum error correction for general noise without measurement,” Quantum Information & Computation 12(1-2), 149–158 (2012).
[15] Fano, U., “Description of states in quantum mechanics by density matrix and operator techniques,” Reviews of Modern Physics 29(1), 74–93 (1957).
[16] Kraus, K., Böhm, A., Dollard, J., and Wootters, W., “States, effects, and operations: Fundamental notions of quantum theory,” in Lecture Notes in Physics, 190 (1983).
[17] Jamiołkowski, A., “Linear transformations which preserve trace and positive semidefiniteness of operators,” Reports on Mathematical Physics 3(4), 275–278 (1972).
[18] Nambu, Y. and Nakamura, K., “On the matrix representation of quantum operations,” arXiv preprint quant-ph/0504091 (2005).
[19] Choi, M., Positive linear maps on C*-algebras, Springer (2012).
[20] Caves, C., “Quantum error correction and reversible operations,” Journal of Superconductivity 12(6), 707–718 (1999).
[21] Tomita, H., Unitary quantum error correction without error detection,” arXiv preprint arXiv:1101.0413 (2011).
[22] Fano, U., “Description of states in quantum mechanics by density matrix and operator techniques,” Reviews of Modern Physics 29(1), 74–93 (1957).
[23] Kraus, K., Böhm, A., Dollard, J., and Wootters, W., “States, effects, and operations: Fundamental notions of quantum theory,” in Lecture Notes in Physics, 190 (1983).
[24] Jamiołkowski, A., “Linear transformations which preserve trace and positive semidefiniteness of operators,” Reports on Mathematical Physics 3(4), 275–278 (1972).
[25] Nambu, Y. and Nakamura, K., “On the matrix representation of quantum operations,” arXiv preprint quant-ph/0504091 (2005).
[26] Fedorov, A., Choi, M., Positive linear maps on C*-algebras, Springer (2012).
[27] Caves, C., “Quantum error correction and reversible operations,” Journal of Superconductivity 12(6), 707–718 (1999).
[28] Fedichkin, L., Yanchenko, M., and Valiev, K., “Coherent charge qubits based on GaAs quantum dots with a built-in barrier,” Nanotechnology 11(4), 387 (2000).

[29] Fedichkin, L. and Fedorov, A., “Error rate of a charge qubit coupled to an acoustic phonon reservoir,” Physical Review A 69(3), 032311 (2004).

[30] Filippov, S., Vyurkov, V., and Fedichkin, L., “Effect of image charge on double quantum dot evolution,” Physica E: Low-dimensional Systems and Nanostructures 44(2), 501–505 (2011).