The modification of the Einstein and Landau-Lifshitz pseudotensors

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Abstract

Deser et al. proposed a combination of the Einstein and Landau-Lifshitz pseudotensors such that the second derivatives in vacuum are proportional to the Bel-Robinson tensor. Stimulated by their work, the present paper discusses the gravitational energy-momentum expression which has the same desirable Bel-Robinson tensor property. We find modifications of the Einstein and Landau-Lifshitz pseudotensors that both give the same coefficient of the Bel-Robinson tensor in vacuum in holonomic frames.

1 Introduction

The classical pseudotensor is not a tensorial object, it is frame dependent. A desirable requirement is that to the lowest non-vanishing order in vacuum the energy density for a gravitating system should be proportional to the Bel-Robinson tensor $B_{\alpha\beta\mu\nu}$ [1]. This will assume a covariant and positive energy on all reference frames, i.e. the associated energy-momentum vector is future pointing and non-spacelike. We find one expression associated with a good, but more complicated boundary condition which gives the Bel-Robinson tensor only [2]. However, it is not easy to obtain the Bel-Robinson tensor alone and in general it is accompanied by certain other tensors, $S_{\alpha\beta\mu\nu}$, $Y_{\alpha\beta\mu\nu}$ and $T_{\alpha\beta\mu\nu}$ [3].

According to the standard textbook [4], the Einstein pseudotensor does not give a positive gravitational energy in vacuum to second order. Recently Deser et al. [3] found the analogous expansion for the Landau-Lifshitz pseudotensor by using the similar method, however this too does not have the expected result. Yet these two classical pseudotensors are good, giving physical sensible results, inside matter (mass density) and at spatial infinity (ADM mass). There comes a natural question can they be modified, so that they have the extra nice result of the positive gravitational energy in vacuum? The answer is yes.

Moreover, the present paper gives the coefficient of the positive gravitational energy for some suitable pseudotensors expressions in holonomic frames, namely the
combination of the Einstein and Landau-Lifshitz pseudotensors \[3\], one of the Chen-Nester four quasilocal expressions \[2\] and Papapetrou pseudotensor \[5\].

## 2 Ingredients

### 2.1 Riemann normal coordinate

In general the metric tensor $g_{\alpha\beta}$, using the Taylor series expansion, can be expanded as

$$g_{\alpha\beta}(x) = g_{\alpha\beta}(0) + (\partial_\mu g_{\alpha\beta})(0)x^\mu + \frac{1}{2}(\partial_\mu^2 g_{\alpha\beta})(0)x^\mu x^\nu + \ldots$$  \hspace{1cm} (1)

At the origin in Riemann normal coordinates

$$g_{\alpha\beta}(0) = \eta_{\alpha\beta}, \quad \partial_\mu g_{\alpha\beta}(0) = 0,$$

$$-3\partial_\mu^2 g_{\alpha\beta}(0) = R_{\alpha\mu\beta\nu} + R_{\alpha\nu\beta\mu}, \quad -3\partial_\nu \Gamma^\mu_{\alpha\beta}(0) = R^\mu_{\alpha\beta\nu} + R^\mu_{\beta\alpha\nu}.$$  \hspace{1cm} (2)

Hence

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} - \frac{1}{3} R_{\alpha\xi\beta\kappa} x^\xi x^\kappa + O(x^3).$$  \hspace{1cm} (4)

### 2.2 The tensors B, S, Y, T and their significance

In order to extract the physical meaning of the tensors $B_{\alpha\beta\mu\nu}$, $T_{\alpha\beta\mu\nu}$, $S_{\alpha\beta\mu\nu}$ and $Y_{\alpha\beta\mu\nu}$, one can use the analog of the “electric” $E_{ij}$ and “magnetic” $H_{ij}$ parts in vacuum of the Weyl tensor.

$$E_{ij} = C_{0i0j}, \quad H_{ij} = *C_{0i0j},$$  \hspace{1cm} (5)

where $C_{\alpha\beta\mu\nu}$ is the Weyl conformal tensor and $*C_{\alpha\beta\mu\nu}$ is its dual,

$$*C_{\alpha\beta\mu\nu} = \frac{1}{2} \sqrt{-g} \epsilon^\alpha_\beta\lambda\sigma C^\lambda\sigma_{\mu\nu}.$$  \hspace{1cm} (6)

In a simple form using the Riemann tensor in vacuum

$$E_{ab} = R_{0a0b}, \quad H_{ab} = \frac{1}{2} R_{0mann} \epsilon^m_b.$$  \hspace{1cm} (7)

Certain commonly occurring quadratic combinations of the Riemann tensor components in terms of the electric $E_{ab}$ and magnetic $H_{ab}$ parts in vacuum are

$$R_{a0b0} R_{0a0b} = E_{ab} E^{ab}, \quad R_{0abc} R_{0abc} = 2 H_{ab} H^{ab}, \quad R_{abcd} R^{abcd} = 4 E_{ab} E^{ab}.$$  \hspace{1cm} (8)
In particular, the Riemann squared tensor can be written in terms of the electric and
magnetic parts as

\[ R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = 4R_{0a0b} R^{0ab} + 4R_{0abc} R^{0abc} + R_{abcd} R^{abcd} \]
\[ = 8(E_{ab} E^{ab} - H_{ab} H^{ab}), \]  

(9)

where Greek letter means 0, 1, 2, 3 and Latin stands for 1, 2, 3. The Bel-Robinson
tensor \( B_{\alpha\beta\mu\nu} \) and tensor \( S_{\alpha\beta\mu\nu} \) are defined as follows

\[ B_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\nu} R_{\beta\lambda\sigma} - R_{\alpha\lambda\nu\sigma} R_{\beta\lambda\mu} - \frac{1}{8} g_{\alpha\beta} g_{\mu\nu} R_{\rho\tau\lambda\sigma} R^{\rho\tau\lambda\sigma}, \]  

(10)

\[ S_{\alpha\beta\mu\nu} := R_{\alpha\mu\lambda\sigma} R_{\nu\lambda\sigma} + R_{\alpha\nu\lambda\sigma} R_{\beta\mu\lambda\sigma} + \frac{1}{4} g_{\alpha\beta} g_{\mu\nu} R_{\rho\tau\lambda\sigma} R^{\rho\tau\lambda\sigma}. \]  

(11)

For our future analysis, define the tensors \( Y_{\alpha\beta\mu\nu} \) and \( T_{\alpha\beta\mu\nu} \)

\[ Y_{\alpha\beta\mu\nu} := R_{\alpha\lambda\beta\sigma} R_{\mu\lambda\rho\sigma} + R_{\alpha\lambda\beta\rho} R_{\mu\lambda\sigma}, \]  

(12)

\[ T_{\alpha\beta\mu\nu} := -\frac{1}{24} g_{\alpha\beta} g_{\mu\nu} R_{\lambda\sigma\rho\tau} R^{\lambda\sigma\rho\tau}. \]  

(13)

The physical observable of the energy-momentum for the tensors \( B_{\alpha\beta\mu\nu} \), \( T_{\alpha\beta\mu\nu} \), \( S_{\alpha\beta\mu\nu} \) and \( Y_{\alpha\beta\mu\nu} \) are the spatial traces

\[ B_{\mu0l}^l = (E_{ab} E^{ab} + H_{ab} H^{ab}, 2\epsilon^{ab} E_{ad} H_{db}), \]  

(14)

\[ T_{\mu0l}^l = (E_{ab} E^{ab} - H_{ab} H^{ab}, 0), \]  

(15)

\[ S_{\mu0l}^l = -10 T_{\mu0l}^l, \]  

(16)

\[ Y_{\mu0l}^l = 2(E_{ab} E^{ab}, \epsilon^{ab} E_{ad} H^{db}). \]  

(17)

Note that \( B_{\mu0l}^l \) has a form similar to the Maxwell energy-momentum density and in
particular \( B_{00l}^l \geq 0 \). There is one relation which needs our attention throughout the
present text

\[ B_{\mu0l}^l = S_{\mu0l}^l + Y_{\mu0l}^l + 9 T_{\mu0l}^l. \]  

(18)

This means that it is not necessary to obtain the Bel-Robinson tensor \( B_{\mu0l}^l \) for the
positive energy requirement, a combination of the tensors \( Y_{\mu0l}^l \) with \( T_{\mu0l}^l \) or equiva-
lently \( Y_{\mu0l}^l \) with \( S_{\mu0l}^l \) can suffice.

3 The interior, ADM and gravitational energy

Note three physical quantities of interest if one considers a massive object in general
relativity. They are the interior mass density, the ADM mass [6] at the spatial
infinity and the gravitational field energy-momentum in vacuum. In order to study the gravitational energy, Einstein proposed the classical pseudotensor $t_\alpha^\mu$ which follows from the superpotential $U_\alpha^{[\mu\nu]}$. Unfortunately the superpotential is not uniquely defined, for example

$$t_\alpha^\mu = \partial_\nu U_\alpha^{[\mu\nu]},$$

but one can introduce a new pseudotensor such as

$$\tilde{t}_\alpha^\mu = t_\alpha^\mu + \partial_\nu \tilde{U}_\alpha^{[\mu\nu]},$$

which is likewise conserved may seem that there is no special way to study the gravitational energy. However, the interior and ADM mass provided some guidelines or restriction, so that one can have some sort of physical energy-momentum components. Recalling the Freud superpotential, one may consider the generalization

$$U_\alpha^{[\mu\nu]} = \sqrt{-g} \left\{ k_1 (\delta_\alpha^\mu \Gamma_\lambda^\gamma \nu - \delta_\alpha^\nu \Gamma_\lambda^\gamma \mu) + k_2 (\delta_\alpha^\mu \Gamma_\nu^\gamma \lambda - \delta_\alpha^\nu \Gamma_\mu^\gamma \lambda) + k_3 (\Gamma_\nu^\mu \alpha - \Gamma_\mu^\nu \alpha) \right\},$$

where $k_1$, $k_2$ and $k_3$ are the extra added constants. Inside matter at the origin in Riemann normal coordinates

$$t_\alpha^\mu = \partial_\nu U_\alpha^{[\mu\nu]}$$

$$= \frac{1}{3} \sqrt{-g} \left\{ (k_1 + 2k_2 + 3k_3)R_\alpha^\mu - (k_1 + 2k_2)\delta_\alpha^\mu R \right\}$$

$$= 2\sqrt{-g}G_\alpha^\mu,$$

provided that

$$k_1 + 2k_2 + 3k_3 = 6,$$

$$k_1 + 2k_2 = 3.$$  \hspace{1cm} (23)  \hspace{1cm} (24)

For the ADM mass at the spatial infinity, let’s use the Schwarzschild metric in Cartesian coordinate

$$ds^2 = - \left( 1 - \frac{2GM}{r} \right) dt^2 + \left( 1 + \frac{2GM}{r} \right) (dx^2 + dy^2 + dz^2).$$

For the energy-momentum four vector

$$2\kappa P_\alpha = -\frac{1}{2} U_\alpha^{[\mu\nu]} \epsilon_{\mu\nu},$$

$$= 2\sqrt{-g}G_\alpha^\mu, \hspace{1cm} (26)$$
the associated ADM mass energy term is
\[ M = -\frac{1}{4\kappa} U_0^{[\mu
u]} \epsilon_{\mu
u} = \frac{1}{2} (k_1 + k_3) M, \] (27)
which generated one more constraint
\[ k_1 + k_3 = 2. \] (28)
Consider (23), (24) and (28), they provide an unique solution
\[ k_1 = k_2 = k_3 = 1. \] (29)
Therefore only the Einstein pseudotensor or other similar ones, such as the Landau-
Lifshitz pseudotensor, have this property. But when one examines the gravitational
energy for Landau-Lifshitz or Einstein, they both do not have the good results in vacuum, namely only the Bel-Robinson tensor [3 4]. It seems hopeless to have the desired result unless one introduces the flat metric tensor \( \eta_{\alpha\beta} \) along with \( g_{\alpha\beta} \). The details will be discussed in the next section.

4 The calculation results

4.1 The Einstein and Landau-Lifshitz pseudotensors

The superpotentials for the Einstein \( E U_\alpha^{[\mu\nu]} \) and Landau-Lifshitz \( L U^{[\alpha[\mu\nu]} \), and Bergman-Thomson \( B U^{[\alpha[\mu\nu]} \) classical pseudotensors are
\begin{align*}
E U_\alpha^{[\mu\nu]} &= -\sqrt{-g} g^\beta\gamma \Gamma^\r_{\gamma\sigma\alpha} \delta^\lambda_{\mu\nu}, \\
L U^{[\alpha[\mu\nu]} &= \sqrt{-g} B U^{\alpha[\mu\nu]} = g g^{\alpha\beta} g^{\pi\sigma} \Gamma^r_{\pi\sigma\r} \delta^\lambda_{\mu\nu}. \tag{31}
\end{align*}

The pseudotensors can be obtained respectively as follows
\[ t_\alpha^{\mu} = \partial_\nu U_\alpha^{[\mu\nu]}, \quad t^{\alpha\mu} = \partial_\nu U^{[\alpha[\mu\nu]}. \] (32)
The result inside matter at the origin for the Einstein and Bergmann-Thomson pseudotensors are
\[ 2\kappa t^{[\alpha[\beta}(0) = 2G^{[\alpha[\beta}(0) = 2\kappa T^{[\alpha[\beta}(0), \] (33)
so their corresponding energy densities are
\[ \mathcal{E} = -t_0^{0} = -\frac{G_0^{0}}{\kappa} = -T_0^{0} = \rho, \] (34)
where $\rho$ is the mass-energy density. This feature is important because we have to fulfill this basic requirement according to the equivalent principle. This means the pseudotensor has to match the energy density inside matter at the origin as the metric tensor becomes flat. If the zeroth order is faulty it is not appropriate to study the second derivatives which give the gravitational energy in vacuum. According to [3], the non-vanishing terms of the Einstein and Landau-Lifshitz pseudotensors in vacuum are

\begin{align}
E_{\alpha\beta} &= -2\Gamma_{\lambda\sigma\alpha} \Gamma^{\beta\lambda\sigma} + \delta_{\alpha}^{\beta} \Gamma_{\lambda\sigma\tau} \Gamma^{\tau\lambda\sigma}, \\
L^{\alpha\beta} &= \Gamma^{\alpha}_{\lambda\sigma} (\Gamma^{\beta\lambda\sigma} - \Gamma^{\lambda\beta\sigma}) - \Gamma^{\alpha}_{\lambda\sigma} \Gamma_{\lambda\sigma\tau} \Gamma^{\tau\lambda\sigma} + g^{\alpha\beta} \Gamma_{\lambda\sigma\tau} \Gamma^{\sigma\lambda\tau}.
\end{align}

(35) (36)

The second derivatives of the Einstein pseudotensor [4] in vacuum is

\[ \partial_{\mu\nu} E_{\alpha\beta} = \frac{1}{9} (4B_{\alpha\beta\mu\nu} - S_{\alpha\beta\mu\nu}). \]

(37)

According to [3], a similar calculation gives the non-vanishing second order contribution for the Landau-Lifshitz $L_{\alpha\beta}$ classical pseudotensor in vacuum as

\[ \partial_{\mu\nu} L_{\alpha\beta} = \frac{1}{9} \left( 7B_{\alpha\beta\mu\nu} + \frac{1}{2} S_{\alpha\beta\mu\nu} \right). \]

(38)

In order to obtain a positive multiple of $B_{\alpha\beta\mu\nu}$, they have made the combination

\[ \partial_{\mu\nu} \left( \frac{1}{2} E_{\alpha\beta} + L_{\alpha\beta} \right) = B_{\alpha\beta\mu\nu}. \]

(39)

This result is good because it guarantees positivity energy for all reference frames.

Looking at (37), (38) and (39), what do they mean physically? One way of getting some physical insight is to integrate over a small coordinate sphere at time $t = 0$, the four momentum is

\[ P_{\mu} = (-E, P_i) = \int t^0 d^3 x, \]

(40)

where the energy $E$ should be non-negative. However $P_{\mu}$ is not necessarily a proper four vector, because it refers to the energy-momentum in a unit sphere of $r = \text{constant}$ at $t = \text{constant}$ and needs not refer to the energy-momentum in a unit sphere of $r' = \text{constant}$ at $t' = \text{constant}$. These two are not related by a simple Lorentz
transformation. Consider the four momentum to second order for the Einstein pseudotensor in vacuum within a small sphere, as it appears to the laboratory frame observer; it is

\[
E P_\mu = \frac{1}{2 \kappa} \int \frac{1}{18} (4 B^0_{\mu i j} - S^0_{\mu i j}) x^i x^j d^3 x
\]

\[
= - \frac{r^5}{540 G} (2 B_{\mu 0 l}^l + 5 T_{\mu 0 l}^l)
\]

\[
= - \frac{r^5}{540 G} (7 E_{a b} E^{a b} - 3 H_{a b} H^{a b}, 4 \epsilon_{c a d} E_{a d} H^{d b}),
\]

(41)

since

\[
\int x^i x^j d^3 x = \frac{1}{3} \delta^{ij} \int r^2 d^3 x = \frac{4 \pi}{15} r^5 \delta^{ij}.
\]

(42)

Similarly for the Landau-Lifshitz pseudotensor

\[
L P_\mu = \frac{1}{2 \kappa} \int \frac{1}{18} \left(7 B^0_{\mu i j} + \frac{1}{2} S^0_{\mu i j} \right) x^i x^j d^3 x
\]

\[
= - \frac{r^5}{1080 G} (7 B_{\mu 0 l}^l - 5 T_{\mu 0 l}^l)
\]

\[
= - \frac{r^5}{540 G} (E_{a b} E^{a b} + 6 H_{a b} H^{a b}, 7 \epsilon_{c a d} E_{a d} H^{d b}).
\]

(43)

Combining these two pseudotensors, the properly normalized superpotential becomes

\[
E L U^{\alpha [\mu \nu]} = \frac{2}{3} \left\{ \frac{1}{2} \eta^{\alpha \lambda} E U^{[\mu \nu]} + B U^{\alpha [\mu \nu]} \right\}.
\]

(44)

The gravitational energy-momentum within a small sphere is

\[
E L P_\mu = \frac{2}{3} \left( \frac{1}{2} E P_\mu + B P_\mu \right)
\]

\[
= - \frac{r^5}{180 G} B_{\mu 0 l}^l
\]

\[
= - \frac{r^5}{180 G} (E_{a b} E^{a b} + H_{a b} H^{a b}, 2 \epsilon_{c a d} E_{a d} H^{d b}).
\]

(45)

This result gives the coefficient of the energy-momentum density should be. In other words, the coefficient of the Bel-Robinson tensor in vacuum in a holonomic frame is

\[
\partial^2_{\mu \nu} t_{\alpha \beta} = \frac{2}{3} B_{\alpha \beta \mu \nu}.
\]

(46)
4.2 The modification of the Einstein pseudotensor

The Einstein pseudotensor offers a good foundation for the gravitational energy expression. One may wonder that if this pseudotensor can be modified so that it has a nice positive gravitational energy result. Based on the simple and natural boundary conditions, analogy with the Dirichlet or Neumann boundary conditions, one of the Chen-Nester four quasilocal expressions can satisfy this requirement [2].

According to [2], there is a two parameters set for the modified Chen-Nester quasilocal boundary expressions. There are infinite number of solutions to achieve a positive Bel-Robinson tensor. The quasilocal boundary expressions, in compact form, are

\[ 2\kappa B_{c_1,c_2}(N) = 2\kappa B_p(N) + c_1 i_N \Delta \Gamma^\alpha_\beta \wedge \Delta \eta^\alpha_\beta - c_2 \Delta \Gamma^\alpha_\beta \wedge i_N \Delta \eta^\alpha_\beta \]

\[ = -\frac{1}{2} N^\alpha \left\{ E U^{[\mu\nu]}_\alpha + c_1 \sqrt{-g} h^{\lambda\pi} \Gamma^\sigma_{\alpha\pi} \delta^{\mu\nu}_{\lambda\sigma} + c_2 \sqrt{-g} h^{\beta\sigma} \Gamma^\tau_{\lambda\beta} \delta^{\lambda\mu\nu}_{\tau\sigma\alpha} \right\} \]

where \( c_1, c_2 \in \mathbb{R} \) and \( h_{\alpha\beta} := g_{\alpha\beta} - \eta_{\alpha\beta} \). When \( (c_1,c_2) = (0,0), (0,1), (1,0) \) and \( (1,1) \), this recovers the original Chen-Nester four expressions. From (47), the superpotential can be extracted as

\[ U^{[\mu\nu]}_\alpha = E U^{[\mu\nu]}_\alpha + c_1 \sqrt{-g} h^{\lambda\pi} \Gamma^\sigma_{\alpha\pi} \delta^{\mu\nu}_{\lambda\sigma} + c_2 \sqrt{-g} h^{\beta\sigma} \Gamma^\tau_{\lambda\beta} \delta^{\lambda\mu\nu}_{\tau\sigma\alpha}. \] (48)

This superpotential looks like a modification of the Freud superpotential. Note that the \( h \Gamma \) terms do not affect the results inside matter and at spatial infinity, but they do affect second order vacuum value. As mentioned before, the pseudotensor can be obtained as

\[ t^\alpha_\mu = \partial_\nu U^{[\mu\nu]}_\alpha. \] (49)

For all \( c_1 \) and \( c_2 \), inside matter at the origin the result is as desired:

\[ 2\kappa t^\alpha_\beta(0) = 2G^\alpha_\beta(0) = 2\kappa T^\alpha_\beta(0), \] (50)

and the corresponding energy density is

\[ \mathcal{E} = -t^0_0 = -\frac{G^0_0}{\kappa} = -T^0_0 = \rho. \] (51)

The second derivatives in vacuum are

\[ \partial^2_{\mu\nu} t^\alpha_\beta(0) = \frac{1}{9} \left\{ (4 + c_1 - 5c_2) B^\beta_{\mu\nu} - (1 - 2c_1 + c_2) S^\beta_{\mu\nu} + (c_1 - 3c_2)(Y^\beta_{\mu\nu} + 9T^\beta_{\mu\nu}) \right\}, \] (52)
and the gravitational energy-momentum is

\[ P_\mu = \frac{1}{2\kappa} \int \frac{1}{18} \left\{ (4 + c_1 - 5c_2)B^0_{\mu ij} - (1 - 2c_1 + c_2)S^0_{\mu ij} \right\} x^i x^j d^3x \]

\[ = -\frac{r^5}{540G} \{(2 + c_1 - 4c_2)B_{\mu 0l}^l + 5(1 - c_1 - 2c_2)T_{\mu 0l}^l \}. \tag{53} \]

In order to obtain the positive gravitational energy, the coefficients of \(B_{\mu 0l}^l\) should be positive and that of \(T_{\mu 0l}^l\) should vanish. Namely

\[ 2 + c_1 - 4c_2 > 0, \quad 1 - c_1 - 2c_2 = 0, \tag{54} \]

or equivalently

\[ c_1 > 0, \quad (c_1, c_2) = \left(c_1, \frac{1 - c_1}{2}\right). \tag{55} \]

Then rewriting (53)

\[ P_\mu = -\frac{c_1 r^5}{180G} B_{\mu 0l}^l \]

\[ = -\frac{c_1 r^5}{180G} (E_{ab} E^{ab} + H_{ab} H^{ab}, 2\epsilon_c E_{ad} H^d_b). \tag{56} \]

There are an infinite number of solutions since there is a parameter \(c_1\) which one can tune. Different solutions represent different boundary conditions which is explained in [2]. However there is one solution with positive energy and a simple boundary condition which is when \((c_1, c_2) = (1, 0)\) in (47). It reduces to one of the original Chen-Nester expressions that they called \(B_c(N)\). In detail

\[ 2\kappa B_c(N) = i_N \Gamma^\alpha_{\beta} \wedge \Delta \eta_\alpha^\beta + \Delta \Gamma^\alpha_{\beta} \wedge i_N \eta_\alpha^\beta, \tag{57} \]

the corresponding superpotential is

\[ U_\alpha^{[\mu\nu]} = E U_\alpha^{[\mu\nu]} + \sqrt{-g} h^\lambda \Gamma^\sigma_{\alpha\nu} \delta^\mu_\sigma. \tag{58} \]

This modification is not just achieving the Bel-Robinson tensor by using an artificial combination, it comes from a simple Lagrangian derivation and a Hamiltonian analysis beginning from the least action principle. Inside matter at the origin and the second derivatives of the small sphere in vacuum results respectively are

\[ 2\kappa t_\alpha^\beta(0) = 2G_\alpha^\beta(0) = 2\kappa T_\alpha^\beta(0), \tag{59} \]

\[ \delta^2_{\mu\nu} t_\alpha^\beta(0) = \frac{1}{9} (5B_{\alpha \beta \mu \nu} + S_{\alpha \beta \mu \nu} + Y_{\alpha \beta \mu \nu} + 9T_{\alpha \beta \mu \nu}). \tag{60} \]
The result inside matter is good. But the second derivatives of the pseudotensor do not give only the Bel-Robinson tensor. However after integration, the gravitational energy-momentum turns out to be what we expect:

\[
P_\mu = \frac{1}{2\kappa} \int \frac{1}{18} \left( 5B^0_{\mu ij} + S^0_{\mu ij} + Y^0_{\mu ij} + 9T^0_{\mu ij} \right) x^i x^j d^3 x
\]

\[
= -\frac{r^5}{180G} B_{\mu \nu l}^l
\]

\[
= -\frac{r^5}{180G} \left( E_{ab} E^{ab} + H_{ab} H^{ab}, 2\epsilon_{c}^{ab} E_{ad} H^{d}_{b} \right),
\]

which is the same as [45]. This result shows that the vector \( P_\mu \) is future pointing and non-spacelike.

### 4.3 The modification of the Landau-Lifshitz pseudotensor

The superpotentials for the Papapetrou classical pseudotensor is

\[
p H^{[\mu \nu] [\alpha \beta]} = -\sqrt{-g} g^{ma} n^{b \mu} \delta^{a \nu} \delta_{mn},
\]

or equivalently

\[
p U^{[\alpha [\mu [\nu] = \partial_\beta \left( p H^{[\mu \nu] [\alpha \beta]} \right)
\]

\[
= B U^{[\alpha [\mu [\nu] - \sqrt{-g} \left( g^{\lambda \sigma} h_{\pi \beta} \Gamma^\alpha_{\lambda \pi} \delta^{\mu \nu} + g^{\alpha \beta} h_{\pi \sigma} \Gamma^\tau_{\lambda \pi} \delta^{\lambda \mu \nu} \right),
\]

At this point, this superpotential looks like a modification of the Bergmann-Thomson or Landau-Lifshitz pseudotensor. Once again, the \( h \Gamma \) terms do not affect the results inside matter and at spatial infinity, they would contribute however to the second order vacuum value. As mentioned before, the pseudotensor can be treated as

\[
t^{\alpha \mu} = \partial_\nu U^{[\alpha [\mu].
\]

Inside matter at the origin and the second derivatives in vacuum [5] are respectively

\[
2\kappa t^{\alpha \beta}(0) = 2 G^{\alpha \beta}(0) = 2\kappa T^{\alpha \beta}(0),
\]

\[
\partial^2_{\mu \nu} P_{\alpha \beta}(0) = \frac{2}{9} (4B_{\alpha \mu \nu} - S_{\alpha \mu \nu} - Y_{\alpha \mu \nu} - 9T_{\alpha \beta \mu \nu}).
\]

The result inside matter is good but the second derivative term does not give the desired result. However after integration, the gravitational energy-momentum becomes
good:

\[ P_\mu = \frac{1}{2\kappa} \int \frac{1}{9} \left( 4B^0_{\mu ij} - S^0_{\mu ij} - Y^0_{\mu ij} - 9T^0_{\mu ij} \right) x^i x^j d^3x \]

\[ = -\frac{r^5}{180G} B_{\mu 0l}^l \]

\[ = -\frac{r^5}{180G} (E_{ab} E^{ab} + H_{ab} H^{ab}, 2\epsilon^{cd}_{\mu} E_{ad} H^{d}_b). \]  

(67)

Once again, it is the same as (45) and, as said previously, the vector \( P_\mu \) is future pointing and non-spacelike.

5 Conclusion

The Einstein and Landau-Lifshitz pseudotensors do not give a positive gravitational energy in vacuum in terms to second order. In order to obtain the expected result of the Bel-Robinson tensor in vacuum, one may consider the modification of these two pseudotensors. One of the possible modifications of the Einstein pseudotensor turns out to be one of the Chen-Nester quasilocal expressions. For the modification of the Landau-Lifshitz pseudotensor, one of the choices is the Papapretou pseudotensor. There are two points that need to be emphasized, they both have the same magnitude of the \( B_{\mu 0l}^l \) term and they both correspond to natural boundary conditions.

References

[1] L.B. Szabados, Living Rev. Relativity, 7 (2004) 4

http://www.livingreviews.org/lrr-2004-4.

[2] L.L. So, A modification of the Chen-Nester quasilocal expressions, to appear in Int. J. Mod. Phys. D, arXiv:gr-qc/0605149.

[3] S. Deser, J.S. Franklin and D. Seminac, Class. Quantum Grav. 16, 2815, (1999).

[4] C.W. Misner, K.S. Thorne and J.A. Wheeler 1973 Gravitation (San Francisco, CA: Freeman).

[5] L.L. So, J.M. Nester and H. Chen, Classical pseudotensors and positive energy in small regions, to appear in Proceedings of the 7th International Conference on Gravitation and Astrophysics, arXiv:gr-qc/0605150.
[6] R. Arnowitt, S. Deser and C.W. Misner, *Phys. Rev.* 122, 997, (1961).