Algebraicity of metaplectic $L$-functions

Salvatore Mercuri

Abstract

Notable results on the special values of $L$-functions of Siegel modular forms were obtained by J. Sturm in the case when the degree $n$ is even and the weight $k$ is an integer. In this paper we extend this method to half-integer weights $k$ and arbitrary degree $n$, determining the algebraic field in which they lie. This method hinges on the Rankin-Selberg method; our extension of this is aided by the theory of half-integral modular forms developed by G. Shimura. In the second half, an analogue of P. B. Garrett’s conjecture is proved in this setting, a result that is of independent interest but that bears direct applications to our first results. It determines exactly how the decomposition of modular forms into cusp forms and Eisenstein series preserves algebraicity and, ultimately, the full range of special values.

1. Introduction

In the philosophy of the oft-touted Langlands program $L$-functions associated to motives on one side are paired off with $L$-functions attached to automorphic forms on the other. When it comes to Siegel modular forms of integral weight $k$ and arbitrary degree $n$, through Deligne’s conjecture results on the motivic side are known to such an extent that the special values of their automorphic $L$-functions are at this point expected. However, in the case that $k$ is a half-integral weight, the corresponding motives are not even known to exist, so that results on special values of automorphic $L$-functions are in this case less expected.

Of primary interest in this paper are $L$-functions that we associate to Siegel modular forms of half-integral weight – sobriquet metaplectic modular forms – and their special values, with the aim here being a precise determination of the number field in which these values lie. Prior work by Shimura in [Sh00] established that these values belonged to the algebraic closure $\overline{\mathbb{Q}}$, with the Siegel modular forms being defined over an arbitrary totally real number field. In [Boug18], Bouganis works further to specify the exact algebraic number field in which they lie, which kind of precision this paper also pursues, but in so doing some additional conditions on the characters of the modular forms were required (Theorem 6.2 (i)-(ii) of [Boug18]) which we avoid. For concreteness, we limit ourselves to Siegel modular forms defined over $\mathbb{Q}$, though it is believed that these results would easily generalise at least to the case of totally real number fields of class number one.

This is a paper of two halves. In the first, Sections 2 – 5, the method employed by Sturm in [St81] is extended to the present case which is facilitated (the extension) by the integral expression (4.1) of Shimura’s paper [Sh96]. This uses the Rankin-Selberg method to express the $L$-function of an eigenform $f$ as an integral $\langle f, \theta \xi \rangle$ of $f$ against a theta series multiplied by...
a non-holomorphic Eisenstein series. Holomorphic projection is applied to this latter form to produce a holomorphic cusp form $K$ and a subsequent expression of the $L$-function in terms of $\langle f, K \rangle$; this projection is given in Sect. 3 and the altered integral expression in Sect. 4. The special values determined by this method are precisely those values of the Eisenstein series at which holomorphic projection to a cusp form is applicable. Finally, algebraicity of quotients of the form

$$\langle f, g \rangle \mu(f)^{-1},$$

where $g$ is a holomorphic form and $\mu(f)$ is a non-zero constant dependent only on the eigenclass of $f$, is proved. This yields algebraicity of these special values.

Though the algebraicity of the values in the first half is very strong, the actual set of values produced is not optimal; it is smaller than the full range given in Theorem 28.8 of [Sh00]. The set of values has been constrained by the requirement that holomorphic projection produce a cusp form the removal of which is the focus of the second half, Sections 6–7. Most of this is taken up by a proof of a particular case of Paul B. Garrett’s conjecture that if $f$ has algebraic coefficients then its Klingen Eisenstein series $E(f)$ does too, [Garr84]. This is done by a non-trivial extension of Harris’ method found in [Harr81], whose setting is integral weight and full level Siegel modular forms, and which was further extended to more general automorphic forms that are associated to Shimura varieties in [Harr84]. Metaplectic modular forms do not associate to Shimura variety, hence the non-triviality and, in Sect. 6, a precise field extension of $\mathbb{Q}$ is given through which the conjecture holds. The relevant corollary of this work is the means through which to specify an extension $\mathcal{L}/\mathbb{Q}$ whereby the well-known decomposition $\mathcal{M}_k = \mathcal{S}_k \oplus \mathcal{E}_k$ preserves algebraicity of the Fourier coefficients of the forms involved; notationally:

$$\mathcal{M}_k = \mathcal{S}_k(\mathcal{L}) \oplus \mathcal{E}_k(\mathcal{L}). \quad (1.1)$$

Such a decomposition was already shown by Shimura in [Sh00] when $\mathcal{L} = \mathbb{Q}$. As has been alluded to supra, such a decomposition allows the determination of the full set of special values by stipulating that the projection of $\theta \mathcal{E}$ need only be a holomorphic modular form. In splitting up $K = K_S + K_E$ per the decomposition 1.1 we are left with $\langle f, K_S \rangle$ by orthogonality. The methods of the first half now follow giving the full set of special values, but with the slightly weaker algebraicity caused by the addition of $\mathcal{L}$.

2. Modular forms of half-integral weight

To begin with, we run through some general groundwork and notation. Let $\mathbb{A}_\mathbb{Q}$ and $\mathbb{I}_\mathbb{Q}$ denote the adele ring and idele group, respectively, of $\mathbb{Q}$. The set of Archimedean places is denoted by $\infty$ and the non-Archimedean places by $f$. For any fractional ideal $r$ of $\mathbb{Q}$ and any element $t \in \mathbb{I}_F$ we denote by $tr$ the fractional ideal of $\mathbb{Q}$ such that $(tr)_p = tv_p$ for any $p \in f$. We recall the adelic norm

$$|t|_A = \prod_v |tv_v|^v$$
where the valuations $| \cdot |_v$ are normalised. Let $\mathbb{T}$ denote the unit circle; define three characters on $\mathbb{C}$, $\mathbb{Q}_p$, and $\mathbb{A}_\mathbb{Q}$ respectively, with images in $\mathbb{T}$, by
\[
\begin{align*}
eventual_char \colon & z \mapsto e^{2\pi iz}, \\
even_char \colon & x \mapsto e(-\{x\}), \\
ext_char \colon & x \mapsto e(x_\infty) \prod_{p \in \mathfrak{p}} e_p(x_p),
\end{align*}
\]
where $\{x\}$ denotes the fractional part of $x \in \mathbb{Q}_p$. If $x \in \mathbb{A}_\mathbb{Q}$ then we also put $e_\mathfrak{f}(x) = e_\mathfrak{f}(x_\mathfrak{f})$ and $e_\infty(x) = e(x_\infty)$. For any matrix $x \in M_n(\mathbb{C})$ write $x > 0$ ($x \geq 0$) to mean that $x$ is positive definite (respectively positive semi-definite); $|x| = \det(x)$ and $\|x\| = |\det(x)|$; and $\tilde{x} = (x^T)^{-1}$.

If $\alpha \in GL_{2n}(F)$ then put
\[
\alpha = \left( \begin{array}{cc} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{array} \right)
\]
where $a_\alpha, b_\alpha, c_\alpha, d_\alpha \in M_n(F)$. We define an algebraic group $G$, subgroups $P, \Omega \leq G$, and the generalised upper half-plane $\mathbb{H}_n$ by
\[
\begin{align*}
G : &= Sp_n(\mathbb{Q}) = \{ \alpha \in GL_{2n}(\mathbb{Q}) \mid \alpha^T \iota \alpha = \iota \} \\
P : &= \{ \alpha \in G \mid c_\alpha = 0 \} \\
\Omega : &= \{ \alpha \in G_\mathfrak{f} \mid \det(c_\alpha) \in \mathbb{Q}_\mathfrak{p} \} \\
\mathbb{H}_n : &= \{ z = x + iy \in M_n(\mathbb{C}) \mid z^T = z, y > 0 \}
\end{align*}
\]

where $G_\mathfrak{f} := Sp_n(\mathbb{A}_\mathbb{Q})$ denotes the adelization of $G = Sp_n(\mathbb{Q})$.

A half-integral weight is an element $k \in \mathbb{Q}$ such that $k - \frac{1}{2} \in \mathbb{Z}$. The factor of automorphy of a half-integral weight will involve taking a square root, and to choose such a root satisfactorily we make use of the metaplectic group. This is understood as the double cover of $Sp_n$ and is denoted $Mp_n$. Set $M_p := Mp_n(\mathbb{Q}_p)$ for all $p$ and $M_\mathfrak{f}$ to be the adelization.

We have natural projections $p_\mathfrak{f} : \mathfrak{f} \to G_\mathfrak{f}$ and $p_\cdot : \mathfrak{f} \to G_\mathfrak{f}$ which are both denoted by $pr$ when the context is clear. There is a natural lift $r : G \to \mathfrak{f}$ through which we can and do view $G$ as a subgroup of $M_\mathfrak{f}$. There exist further lifts $r_\mathfrak{f} : \mathfrak{f} \to M_\mathfrak{f}$ and $r_\Omega : \Omega \to M_\mathfrak{f}$ which are equal to $r$ on $P$ and $G \cap \Omega$ respectively, and such that
\[
r_\Omega(\alpha \beta \gamma) = r_P(\alpha) r_\Omega(\beta) r_P(\gamma)
\]
for $\alpha, \beta \in P_\mathfrak{f}$ and $\beta \in \Omega$, [Sh95b, p. 24].

Recall that there is a natural action of $Sp_n(\mathbb{R})$ on $\mathbb{H}_n$ given by
\[
\gamma \cdot w : = (a_\gamma w + b_\gamma) (c_\gamma w + d_\gamma)^{-1}
\]
for $\gamma \in Sp_n(\mathbb{R}), w \in \mathbb{H}_n$, and further define
\[
\Delta(w) : = |\text{Im}(w)|, \\
j(\gamma, w) : = |c_\gamma w + d_\gamma|.
\]
If, now, $z \in \mathbb{H}_n$, and $\alpha \in G_\mathfrak{f}$ then $\alpha_\infty \in Sp_n(\mathbb{R})$, so we naturally extend the above
\[
\alpha \cdot z : = \alpha_\infty \cdot z, \\
\Delta(z) : = \Delta(z_\infty), \\
j(\alpha, z) : = j(\alpha_\infty, z_\infty).
\]
For any two fractional ideals $\mathfrak{p}, \mathfrak{q}$ of $F$ such that $\mathfrak{p}\mathfrak{q} \subseteq \mathbb{Z}$, congruence subgroups are defined by the following subsets of $G_p, G_K$, and $G$ respectively, by

$$D_p[\mathfrak{p}, \mathfrak{q}] = \{ x \in G_p \mid a_x, d_x \in M_n(\mathbb{Z}), b_x \in M_n(x_p), c_x \in M_n(y_p) \},$$

$$D[\mathfrak{p}, \mathfrak{q}] = Sp_n(\mathbb{R}) \prod_p D_p[\mathfrak{p}, \mathfrak{q}],$$

$$\Gamma[\mathfrak{p}, \mathfrak{q}] = G \cap D[\mathfrak{p}, \mathfrak{q}].$$

Typically these will take the form $\Gamma[b^{-1}, bc]$ for certain fractional ideals $b$ and integral ideals $c$.

Take a half-integral weight $k$ and put $[k] := k - \frac{1}{2} \in \mathbb{Z}$; if $\ell \in \mathbb{Z}$ then $[\ell] := \ell$. Our factor of automorphy for modular forms of half-integral weight will come in two parts, one of which is the familiar factor of weight $[k]$ and the other acts as a factor of automorphy of weight $\frac{1}{2}$. The major caveat in this setting is that the factor of weight $\frac{1}{2}$ is only definable for a particular subset $\mathcal{M} \subseteq M_k$ given by

$$C^\theta_p = \{ \xi \in D_p[1, 1] \mid (\alpha \xi b^T_\xi)_{ii} \in 2\mathbb{Z}_p, (c_\xi d^T_\xi)_{ii} \in 2\mathbb{Z}_p, 1 \leq i \leq n \},$$

$$C^\theta = Sp_n(\mathbb{R}) \prod_p C^\theta_p,$$

$$\mathcal{M} = \{ \sigma \in M_k \mid \alpha = pr(\sigma) \in P_k C^\theta \}.$$

So in considering modular forms of half-integral weight, we must ensure that $D[b^{-1}, bc] \subseteq \mathcal{M}$.

For any $\sigma \in M_k$ we set $x_\sigma = x_\alpha$ where $x \in \{ a, b, c, d \}$ and $\alpha = pr(\sigma) \in G_K$; define $\sigma \cdot z = \alpha \cdot z$ for $z \in \mathbb{H}_n$. If $\sigma \in \mathcal{M}$ then we may define a holomorphic function $h_\sigma = h(\sigma, \cdot) : \mathbb{H}_n \rightarrow \mathbb{C}$ satisfying the following properties, the proofs for which we refer the reader to [Sh85, pp. 294–295]:

$$h(\sigma, z)^2 = \zeta j(pr(\sigma), z)$$

for a constant $\zeta \in \mathbb{T}$; $h(\sigma, z) \in \mathbb{T}$ if $pr(\sigma)_{\infty} = I_{2n}$; (2.1)

$$h(tr_P(\gamma), z) = t^{-1} \|(d_\gamma)_{\infty}\|_\mathbb{A}^{\frac{1}{2}}$$

if $t \in \mathbb{T}$ and $\gamma \in P_K$; (2.2)

$$h(\rho \sigma \tau, z) = h(\rho, z) h(\sigma, \tau z) h(\tau, z)$$

if $pr(\rho) \in P_K, pr(\tau) \in C^\theta$. (2.3)

The factor of automorphy is then given as

$$j^k_\sigma(z) = j^k(\sigma, z) = h_\sigma(z) j(\alpha, z)^{|k|}$$

where $\sigma \in \mathcal{M}, \alpha = pr(\sigma) \in G_K$, and $z \in \mathbb{H}_n$. Given a function $f : \mathbb{H}_n \rightarrow \mathbb{C}$ and a $\xi \in \mathcal{M}$ we define the slash operator as

$$(f|\kappa \xi)(z) = j^k_\xi(z)^{-1} f(\xi \cdot z)$$

for $z \in \mathbb{H}_n$. If $\Gamma \subseteq G$ is a congruence subgroup such that $\Gamma \subseteq \mathcal{M}$, then let $C^\infty_k(\Gamma)$ denote the set of analytic functions $\mathbb{H}_n \rightarrow \mathbb{C}$ that satisfy $f|\kappa \xi = f$ for any $\xi \in \Gamma$. Let $\mathcal{M}_k(\Gamma) \subseteq C^\infty_k(\Gamma)$ be the subspace of holomorphic functions, $S_k(\Gamma)$ be the subspace of cusp forms, and write

$$\mathcal{M}_k = \bigcup_{\Gamma} \mathcal{M}_k(\Gamma)$$

$$S_k = \bigcup_{\Gamma} S_k(\Gamma)$$

where the union is taken over all congruence subgroups of $G$.

Take a Hecke character $\psi : \mathbb{Q}/\mathbb{Q}^\times \rightarrow \mathbb{T}$ of $\mathbb{Q}$ such that $\psi_p(a) = 1$ if $a \in \mathbb{Z}_p^\times$ and $a - 1 \in \mathfrak{c}_p$, and such that $\psi_\infty(x) = sgn(x_\infty)^{|k|}$. Assume that $b^{-1} \subseteq 2\mathbb{Z}$ and $bc \subseteq 2\mathbb{Z}$, then $\Gamma = \Gamma[b^{-1}, bc] \subseteq \mathcal{M}$. Let $C^\infty_k(\Gamma, \psi)$ denote the space of all $g \in C^\infty_k$ such that

$$g|\kappa \gamma = \psi_\epsilon(|a_\gamma|)g$$

for all $\gamma \in \Gamma$, where $\psi_\epsilon = \prod_{p | \epsilon} \psi_p$. Put $\mathcal{M}_k(\Gamma, \psi) = C_k(\Gamma, \psi) \cap \mathcal{M}_k$ and $S_k(\Gamma, \psi) = C_k(\Gamma, \psi) \cap S_k$. 


Given \( g \in \mathcal{M}_k(\Gamma, \psi) \) define its adelisation \( g_\mathbb{A} : M_\mathbb{A} \to \mathbb{C} \) by
\[
g_\mathbb{A}(\alpha w) = \psi_\mathbb{C}(|d_w|)(g|w)(i)
\]
where \( \alpha \in G, w \in \text{pr}^{-1}(D[b^{-1}, bc]), \) and \( i = \iota_n \in \mathbb{H}_n. \) We have that
\[
g_\mathbb{A}(\alpha xw) = \psi_\mathbb{C}(|d_w|)j^k(w, i)^{-1}g_\mathbb{A}(x)
\]
if \( w \cdot i = i, \text{pr}(w) \in D[b^{-1}, bc], \) and \( \alpha \in G; \) the above goes conversely (\cite{Sh94} p. 537).

We define spaces of symmetric matrices as follows
\[
S = \{ \xi \in M_n(\mathbb{Q}) | \xi^T = \xi \} \\
S_+(\mathfrak{r}) = \{ \xi \in S | \xi \geq 0 \} \\
S_+(\mathfrak{r}) = \prod_p S(\mathfrak{r}_p)
\]
for any fractional ideal \( \mathfrak{r} \) of \( \mathbb{Q}. \)

With \( \Gamma = G \cap D[b^{-1}, bc] \subseteq \mathfrak{M}, \ f \in \mathcal{M}_k(\Gamma, \psi), q \in GL_n(\mathbb{A}_\mathbb{Q}), \) and \( s \in S_+, \) then the adelic Fourier expansion is given as
\[
f_\mathbb{A} \left( \begin{pmatrix} q & \tilde{s} \tilde{q} \\ 0 & q \end{pmatrix} \right) = |q_\infty| |q_\infty^\frac{1}{2} \sum_{\tau \in S_+} \mu_f(\tau, q)e_\infty(tr(iq^T\tau q)e_\mathbb{A}(tr(\tau s))
\]
for some \( \mu_f(\tau, q) = \mu(\tau, q; f) \in \mathbb{C} \) satisfying the following properties:

(i) \( \mu_f(\tau, q) \neq 0 \) only if \( e_\mathbb{A}(tr(q^T\tau qs)) = 1 \) for all \( s \in S_+(b^{-1}); \)

(ii) \( \mu_f(\tau, q) = \mu_f(\tau, q); \)

(iii) \( \mu_f(b^T\tau b, q) = |b|^k |b|^\frac{1}{2} \mu_f(\tau, bq) \) for any \( b \in GL_n(\mathbb{Q}); \)

(iv) \( \psi_\mathbb{C}(a)) \mu_f(\tau, qa) = \mu_f(\tau, q) \) for any \( \text{diag}[a, \tilde{a}] \in D[b^{-1}, bc]; \)

(v) if \( \beta \in G \cap \text{diag}[\tilde{r}, \tilde{r}]D[b^{-1}, bc] \) and \( r \in GL_n(\mathbb{A}_F), \) then
\[
j^k(\beta, \beta^{-1})z f(\beta^{-1}z) = \psi_\mathbb{C}(|d_\beta r|) \sum_{\tau \in S_+} \mu_f(\tau, r)e_\infty(tr(\tau z)).
\]

The proof of this expansion and the subsequent properties can be found in Proposition 1.1 of \cite{Sh95b}. The coefficients \( \mu_f(\tau, 1) \) correspond to the usual Fourier coefficients of \( f. \)

For any two \( f, g \in \mathcal{M}_k(\Gamma, \psi) \) define the Petersson inner product
\[
\langle f, g \rangle = \text{Vol}(\Gamma \backslash \mathbb{H}_n)^{-1} \int_{\Gamma \backslash \mathbb{H}_n} f(z)\overline{g(z)}\Delta(z)^k d^\times z
\]
in which
\[
d_\infty = \Delta(z)^{-n-1} \bigwedge_{p \leq q} (dx_{pq} \wedge dy_{pq})
\]
for \( z = (x_{pq} + iy_{pq})_{p,q=1}^n \in \mathbb{H}_n. \)

The elements of \( \text{Aut}(\mathbb{C}) \) act on the space of modular forms in the usual way. That is, if \( f \in \mathcal{M}_k(\Gamma, \psi) \) has Fourier coefficients \( \mu_f(\tau, 1) \) for \( \tau \in S_+, \) then \( f^\sigma \in \mathcal{M}_k(\Gamma, \psi^\sigma) \) is the modular form whose Fourier coefficients are \( \mu_f(\tau, 1)^\sigma \) for all \( \tau \in S_+. \)
3. Holomorphic projection

Assume that \((b^{-1}, bc) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}\) and put \(\Gamma = G \cap D[b^{-1}, bc]\). Suppose that \(F \in C^\infty_k(\Gamma, \psi)\); we say that \(F\) is of bounded growth if for all \(\varepsilon > 0\) we have

\[
\int_X \int_Y |F(z)| \Delta(z)^{k-1-n} e^{-\varepsilon \text{tr}(\mathfrak{m}(z))} dy dx < \infty
\]

where

\[
dy = \bigwedge_{p \leq q} dy_{pq}, \quad dx = \bigwedge_{p \leq q} dx_{pq}, \quad d^x y = \Delta(z)^{-\frac{n+1}{2}} dy,
\]

\(Y = \{y \in M_n(\mathbb{R}) \mid y = y^T, y > 0\}\),

\(X = \{x \in M_n(\mathbb{R}) \mid x = x^T, |x_{ij}| \leq \frac{1}{2} \forall i, j\}\).

Let \(A\) be the set of symmetric and half-integral elements of \(M_n(\mathbb{R})\), and let \(B, C \subseteq A\) be the subset of positive semi-definite and positive definite elements respectively. If \(F \in C^\infty_k(\Gamma)\) then it has an absolutely convergent Fourier expansion of the form

\[
F(z) = \sum_{\tau \in bA} a(\tau, y) e((\text{tr}(sz)),
\]

whereas if \(f\) is holomorphic then it has a Fourier expansion of the form

\[
f(z) = \sum_{\tau \in \mathfrak{b}^*} \mu_f(\tau, 1) e((\text{tr}(sz))
\]

with \(* = C\) or \(B\) according to whether \(f\) is a cusp form or not. The following theorem extends to half-integral \(k\) the notion of holomorphic projection given in Theorem 1 of [St81] when \(k\) is integral.

**Theorem 3.1.** Let \(F \in C^\infty_k(\Gamma, \psi)\) have Fourier coefficients \(a(\tau, y)\) for \(\tau \in \mathfrak{b}A\). Assume that \(k > 2n\) and that \(F\) is of bounded growth. For any \(\tau > 0\) set

\[
c(k, n) = \Gamma_n \left( k - \frac{n+1}{2} \right) \pi^{-n(k-\frac{n+1}{2})},
\]

\[
a(\tau) = c(k, n)^{-1} |4\tau|^{k-\frac{n+1}{2}} \int_Y a(\tau, y) e^{-2\pi \text{tr}(\tau y)} |y|^{k-1-n} dy.
\]

Then define the holomorphic projection map

\[
P : C^\infty_k(\Gamma, \psi) \to \mathcal{S}_k(\Gamma, \psi);
\]

\[
F \mapsto \sum_{\tau \in C} a(\tau) e((\text{tr}(\tau z)).
\]

Furthermore, the projection map satisfies \(\langle F, g \rangle = \langle P(F), g \rangle\) for any \(g \in \mathcal{S}_k(\Gamma', \psi)\) and \(\Gamma' \leq \Gamma\) of finite index.

To prove this we introduce the half-integral weight Poincaré series. Let \(\Gamma\) and \(k > 2n\) be as above, fix \(\tau \in \mathfrak{b}C\), and then define this series by

\[
G_{\tau}(z) := \sum_{M \in \mathfrak{P} \cap \Gamma \setminus \Gamma} \psi^{-1}(|a_M|) x_{M}(z)^{k-1} e((\text{tr}(\tau M \cdot z)).
\]

**Proposition 3.2.** (i) The sum defining \(G_{\tau}\) converges absolutely and uniformly on compact subgroups of \(\mathbb{H}_n\) and we have \(G_{\tau} \in \mathcal{S}_k(\Gamma, \psi)\).
Algebraicity of metaplectic L-functions

(ii) If $F \in C^\infty_k(\Gamma, \psi)$ is as in Theorem 3.1 then

$$N(b) \frac{n(n+1)}{2} \text{Vol}(\Gamma \backslash \mathbb{H}_n) \langle F, G_\tau \rangle = \int_{\mathbb{H}} a(\tau, y) e^{-2\pi \text{tr}(\tau y)} |y|^{k-1-n} dy,$$

with the integral being absolutely convergent.

(iii) If $f \in S_k(\Gamma, \psi)$ then

$$N(b) \frac{n(n+1)}{2} \text{Vol}(\Gamma \backslash \mathbb{H}_n) \langle f, G_\tau \rangle = \mu_f(\tau, 1) |4\tau|^{-\frac{n+1}{2}} c(k, n).$$

Proof. (i) To show $G_\tau \in S_k(\Gamma, \psi)$ we show $G_\tau |_{k, \gamma} = \psi_c(|a_{\gamma}|) G_\tau(z)$ for all $\gamma \in \Gamma$. We have $h(M, \gamma z) = h(M, \gamma z) h(\gamma, z)$ by property 2.3 of the function $h$ and, combined with the usual cocycle relation on $j(M, \gamma z)$, we obtain

$$j^k_M(\gamma z) = j^k_M(z) \frac{1}{j^k_M(\gamma z)}.$$

Write $\gamma = \beta\alpha$ with $\beta \in P \cap \Gamma$ and $\alpha \in P \cap \Gamma \backslash \Gamma$. Then for any $M \in P \cap \Gamma \backslash \Gamma$ we have $M\gamma = M\alpha$ in $P \cap \Gamma \backslash \Gamma$, and further $M \mapsto M\alpha$ is both a bijection and well-defined on $P \cap \Gamma \backslash \Gamma$. With this, and noting $a_{M\gamma} \equiv a_{M\alpha} \pmod{c}$, we get

$$G_\tau(\gamma \cdot z) = j^k_\gamma(z) \psi_c(|a_{\gamma}|) \sum_{M \in P \cap \Gamma \backslash \Gamma} \psi^{-1}_c(|a_{M\gamma}|) j^k_{M\gamma}(z) e(\text{tr}(\tau(M\gamma) z))$$

$$= j^k_\gamma(z) \psi_c(|a_{\gamma}|) \sum_{M\alpha \in P \cap \Gamma \backslash \Gamma} \psi^{-1}_c(|a_{M\alpha}|) j^k_{M\alpha}(z) e(\text{tr}(\tau(M\alpha) z))$$

$$= j^k_\gamma(z) \psi_c(|a_{\gamma}|) G_\tau(z).$$

For convergence, the integral case was proved in [CG58] and, by property 2.1 of the function $h$, we have

$$|j(\gamma, z)| < |j^k_M(z)| < |j(\gamma, z)|^{k''}$$

where $k'' = k' + 1$ if $|j(\gamma, z)| > 1$ and $k'' = k' + 1$ if $|j(\gamma, z)| < 1$. So absolute convergence, uniformly on compact subgroups of $\mathbb{H}_n$, follows from the integral case.

(ii) and (iii) can be proven in precisely the same manner as the integral case in [St81, p. 332] using this adapted Poincaré series in place of the one appearing there. Note that for (iii) we require boundedness of $|y|^\frac{n}{2} |f(z)|$, something which is clarified for the half-integral case later on in the proof of Corollary 3.3 (i).

With Proposition 3.2 above the proof of Theorem 3.1 follows by setting

$$K(z, w) := N(b) \frac{n(n+1)}{2} \text{Vol}(\Gamma \backslash \mathbb{H}_n) c(k, n)^{-1} \sum_{\tau \in bC} |4\tau|^{-\frac{n+1}{2}} G_\tau(z) e(-\text{tr}(\tau w))$$

and proceeding as in [St81, pp. 332–333].

In the rest of the section we extend some bounds found in [St81] pp. 335–336] to our setting: these bounds shall govern when holomorphic projection is applicable in certain cases. Let $\kappa \in \frac{1}{2} \mathbb{Z}$ and $\Gamma_0$ a congruence subgroup that is contained in $\mathfrak{M}$ if $\kappa \notin \mathbb{Z}$. Define, for a variable $z \in \mathbb{H}_n$ and $b \in \mathbb{R}$ such that $b > \frac{n-k}{2}$, the following majorant of the non-holomorphic Eisenstein series

$$H_\kappa(z, b; \Gamma_0) = H_\kappa(z, b) := |y|^b \sum_{\alpha \in P \cap \Gamma_0 \backslash \Gamma_0} \|c_\alpha z + d_\alpha\|^{-\kappa - 2b}.$$

Let $\Omega$ be a fundamental domain for $\text{Sp}_n(\mathbb{Z}) \backslash \mathbb{H}_n$ chosen so that $z = x + iy \in \Omega$ implies that $y > \varepsilon I_n$ for some $\varepsilon > 0$ independent of $z$.  

7
Proposition 3.3. Let \( c_0, a \in \mathbb{R} \) be given with \( c_0 > 0 \) and \( a \geq 0 \). Let \( \varphi : \mathbb{H}_n \to \mathbb{C} \) be such that
\[
|\varphi^2(g \cdot z)| \leq c_0 |y|^a
\]
for all \( z \in \Omega \) and \( g \in Sp_n(\mathbb{Z}) \). Then, taking only positive square roots, we have
\[
|\varphi(z)| \leq c_1 \prod_{j=1}^{n} (\lambda_j^\frac{a}{2} + \lambda_j^{-\frac{a}{2}})
\]
for some constant \( c_1 > 0 \), dependent only on \( \varphi \), and for the eigenvalues \( \lambda_j \) of \( y \).

Proof. Let \( z \in \mathbb{H}_n \) and choose \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_n(\mathbb{Z}) \) such that \( g \cdot z \in \Omega \). Then
\[
|\varphi^2(z)| = |\varphi^2(g^{-1}(g \cdot z))| \leq c_0 |\text{Im}(g \cdot z)|^a = c_0 |y|^a \|cz + d\|^{-2a}.
\]
Let \( r \) be the rank of \( c \); as in [St81, p. 334] there exist \( U_1, U_2 \in GL_n(\mathbb{Z}) \) such that
\[
c = U_1 \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} U_1^T, \quad d = U_1 \begin{pmatrix} d_1 & 0 \\ 0 & I_{n-r} \end{pmatrix} U_2^{-1}
\]
where \( c_1, d_1 \in M_r(\mathbb{Z}) \) are such that \( |c_1| \neq 0 \) and \( c_1 d_1^T \) is symmetric. For \( i = 1, 2 \) put \( U_i = (Q_i \quad Q_i') \) where \( Q_i \in M_{n \times r}(\mathbb{Z}), Q_i' \in M_{n \times (n-r)}(\mathbb{Z}) \). Then we have \( \|cz + d\| \geq \|Q_1 y Q_2^T\| \) so that, from (3.1) above,
\[
|\varphi^2(z)| \leq c_0 |y|^a |y_0|^{-2a}
\]
where \( y_0 = Q_1 y Q_2^T \). Sturm shows, in [St81, p. 334], that there exist \( 1 \leq j_1 < j_2 < \cdots < j_r \leq n \) such that
\[
|y_0| \geq \alpha \prod_{i=1}^{r} \lambda_{j_i}.
\]
Now we have \( \lambda_j > 0 \) for all \( j \) and so \( \prod_{i=1}^{r} \lambda_{j_i}^{-a} \leq \prod_{j=1}^{n} (1 + \lambda_j^{-a}) - \) the left-hand side is just one term in the expansion on the right-hand side, all terms of which are \( \geq 0 \). So for \( c_1 = \sqrt{c_0} \alpha^{-a} \) we do get
\[
|\varphi(z)| \leq c_1 \prod_{j=1}^{n} \lambda_j^\frac{a}{2} (1 + \lambda_j^{-a}) = c_1 \prod_{j=1}^{n} (\lambda_j^\frac{a}{2} + \lambda_j^{-\frac{a}{2}}).
\]

Corollary 3.4. Let \( f \in S_k(\Gamma, \psi); g \in \mathcal{M}_\ell(\Gamma, \psi); \ell, \kappa \in \frac{1}{2}\mathbb{Z} \); and \( b > \frac{n+1-\kappa}{2} \). Then there exists a constant \( 0 < c_1 \in \mathbb{R} \) such that
\[
(i) \quad |f(z)| \leq c_1 |y|^{-\frac{b}{2}};
\]
\[
(ii) \quad |g(z)| \leq c_1 \prod_{j=1}^{n} (1 - \lambda_j^{-\ell});
\]
\[
(iii) \quad |H_\kappa(z, b)| \leq c_1 \prod_{j=1}^{n} (\lambda_j^b + \lambda_j^{-b-\kappa})
\]
for all \( z \in \mathbb{H}_n \).
Proposition 3.3 with 

Proof. (i) Consider \( f^2 \) – a cusp form of integral weight \( 2k \) and level \( \Gamma \). Apply the above Proposition 3.3 to the function \( \varphi(z) = |y|^\frac{k}{2} f(z) \) with \( a = 0 \).

(ii) We take \( \varphi(z) = |y|^\frac{k}{2} |g(z)| \) then \( \varphi^2 \) is of integral weight \( 2\ell \) and satisfies the conditions of Proposition 3.3 with \( a = 2\ell = \ell \).

(iii) \( H^2 \) is a constant multiple of \( H_{2\kappa}(z, 2b) \) which is of integral weight \( 2\kappa \). Sturm shows, in [St81, p. 335], that \(|y|^{\kappa} |H_{2\kappa}(g \cdot z, 2b)| \leq c_0 |y|^{2b+\kappa} \). Hence \( \varphi(z) := |y|^\frac{\kappa}{2} H(z) \) satisfies the conditions of Proposition 3.3 with \( a = 2b + \kappa \).

Let \( k \) be a half-integral weight and \( \ell \in \frac{1}{2}\mathbb{Z} \); take \( a \in \mathbb{R} \) such that

\[
a > \frac{\ell - k + n + 1}{2}.
\]

Corollary 3.5. Let \( g \in \mathcal{M}_\ell(\Gamma, \psi) \) and put \( F^*(z) := g(z) H_{k-\ell}(z, a) \). Then \( F^* \) is of bounded growth provided we have

\[
\frac{\ell - k + n + 1}{2} < a < \begin{cases} 
-n + \frac{\ell}{2} & \text{if } g \in \mathcal{S}_\ell(\Gamma, \psi), \\
-n & \text{otherwise}.
\end{cases}
\]

The proof of the above corollary is precisely as it appears in [St81] pp. 335–336, since we have the same setup with Corollary 3.4.

4. Integral expressions for the standard metaplectic \( L \)-function

The main object of study – the standard, twisted \( L \)-function \( L_\psi(s, f, \eta) \) of an eigenform \( f \) – is introduced here and an integral expression, from [Sh96], is taken and modified for our purposes. Throughout let \( \delta := n \pmod{2} \in \{0, 1\} \).

Though the integral expression we obtain can be stated for any half-integral weight \( k \), for ease of notation we take \( k \geq n + 1 \) – we shall be making this assumption later on anyway. For a prime \( p \), the association of an \( n \)-tuple \( (\lambda_{p,1}, \ldots, \lambda_{p,n}) \in \mathbb{C}^n \) to a non-zero Hecke eigenform \( f \in \mathcal{S}_k(\Gamma, \psi) \) is well-known; this process is briefly outlined later in Sect. 4.1. Then for a Hecke character \( \eta \) of \( \mathbb{Q} \) we define our \( L \)-function by

\[
L_p(t) = \begin{cases} 
\prod_{i=1}^n (1 - p^n \lambda_{p,i} t) & \text{if } p \mid \mathfrak{c}, \\
\prod_{i=1}^n (1 - p^n \lambda_{p,i} t) (1 - p^n \lambda_{p,i}^{-1} t) & \text{if } p \nmid \mathfrak{c};
\end{cases}
\]

\[
L_\psi(s, f, \eta) = \prod_p L_p(\psi^s(\eta(p)p^{-s}))^{-1}
\]

where

\[
\psi^s(x) = \left( \frac{\psi}{\psi_\mathfrak{c}} \right)(x).
\]

Fix \( \tau \in S_+ \) such that \( \mu_f(\tau, 1) \neq 0 \) and let \( \rho_\tau \) be the quadratic character associated to the extension \( \mathbb{Q}(i^{[n/2]}\sqrt{2\tau}) \); choose \( \mu \in \{0, 1\} \) such that \( (\psi \eta)_\infty(x) = \text{sgn}(x_\infty)^{[k] + \mu} \). The integral expression (4.1) in [Sh96] p. 342] is stated there in immense generality and a lot of this simplifies in this setting; in the notation of [Sh96] we can just take \( p = I_n \) and \( D_F = 1 \). The key ingredients
of the integral are three modular forms: the eigenform $f$, a theta series $\theta$, and a normalised Eisenstein series $\mathcal{E}(z, s)$. The definition of the theta series $\theta$, taken from [Sh96 (2.1)], is

$$\theta(z) = \theta^\mu(\eta) (z; \tau) := \sum_{x \in \mathbb{M}_n(Z)} (\eta_\infty \eta^*)^{-1}(|x|) |x|^\mu e_\infty(\text{tr}(x^T \tau x))$$

where $\eta^*(p) = \eta^*(p\mathbb{Z})$ is the ideal Hecke character associated to $\eta$. This has weight $\frac{1}{2} + \mu$, level determined by Proposition 2.1 of [Sh96], character $\rho_p \eta^{-1}$, and coefficients in $\mathbb{Q}(\eta)$.

We define the Eisenstein series in a little more generality. Let $\Gamma = \Gamma[x^{-1}, \tau \eta]$ be a congruence subgroup and let $\varphi$ be a Hecke character such that $\varphi_\infty(x) = \text{sgn}(x_\infty)^{[k]}$ and $\varphi_p(a) = 1$ if $a \in 1 + \eta_p \mathbb{Z}_p^\times$. Then the non-holomorphic Eisenstein series is defined as

$$E(z, s; \kappa, \varphi, \Gamma') := \sum_{\alpha \in \mathbb{P} \cap \Gamma \setminus \Gamma'} \varphi_\eta(|a_\gamma|)(\Delta^{\frac{s}{2}}||\kappa \alpha)(z)$$

for a congruence subgroup $\Gamma'$ (contained in $\mathbb{M}$ if $\kappa \notin \mathbb{Z}$) and variables $z \in \mathbb{H}_n, s \in \mathbb{C}$. This sum is convergent for \Re(s) > n and can be continued analytically to all of $s \in \mathbb{C}$ by a functional equation in $s \mapsto \frac{n+1-k}{2} - s$. This series has weight $\kappa$, level $\Gamma'$, and character $\varphi^{-1}$, and is normalised by a product of Dirichlet $L$-functions as follows. Let $a$ be any integral ideal and define

$$L_a(s, \varphi) = \prod_{p \mid a} (1 - \varphi^*(p)p^{-s})^{-1};$$

$$\Lambda_a^{n, \kappa}(s, \varphi) = \begin{cases} L_a(2s, \varphi) \prod_{i=1}^{[\frac{\kappa}{2}]} L_a(4s - 2i, \varphi^2) & \text{if } \kappa \in \mathbb{Z}; \\ \prod_{i=1}^{[\frac{\kappa}{2}]} L_a(4s - 2i + 1, \varphi^2) & \text{if } \kappa \notin \mathbb{Z}. \end{cases}$$

(4.1)

The normalised Eisenstein series is given by

$$\mathcal{E}(z, s; \kappa, \varphi, \Gamma') := \Lambda_a^{n, \kappa}(s, \varphi)E(z, s; \kappa, \varphi, \Gamma').$$

Then the integral expression of [Sh96 (4.1)] is:

$$L_\psi(s, f, \eta) = \left[ \Gamma_n \left( \frac{s-n+1+k+\mu}{2} \right) 2\mu(f, \tau, 1) \right]^{-1} N(b) \frac{n(n+1)}{2} \left| 4\pi \tau \cdot \frac{s-n+1+k+\mu}{2} \right|$$

$$\times \left( \frac{\Lambda_\eta}{\Lambda_\psi} \right) (2\psi - \eta) \prod_{p \in \mathfrak{b}} g_p((\psi_\eta(p)p^{-s}) \langle f, \theta \mathcal{E}(\cdot, \frac{2s-n}{4}) \rangle)$$

(4.2)

where $\Lambda_\eta(s) = \Lambda_a^{n, k-n/2-\mu}(s, \chi); \eta = \mathfrak{c} \cap \mathfrak{c}'; \mathfrak{b}$ is a finite set of primes and $g_p \in \mathbb{Q}[t]$ is such that $g_p(0) = 1$;

$$\mathcal{E}(z, s) := \mathcal{E}(z, s; k - \frac{\mu}{2} - \mu, \chi, \Gamma, \mathfrak{b}^{-1}, \mathfrak{b} \mathfrak{u})$$

where $\chi = \psi_\eta \rho_{\mathfrak{c}}$; and $V := \text{Vol}(\Gamma \mathfrak{b}^{-1}, \mathfrak{b} \mathfrak{u})\setminus \mathbb{H}_n$.

Notice, by the definitions of 4.1 above, that $\left( \frac{\Lambda_\eta}{\Lambda_\psi} \right) (2\psi - \eta) \mathfrak{b}$ is just a finite product of Euler factors twisted by $\chi$ and denote it by $\epsilon_\eta(s, \chi) = \epsilon_{k, \eta}(s, \chi)$. We have $\epsilon_\eta(s, \chi)^\sigma = \epsilon_{k, \eta}(s, \chi^\sigma)$ for any $\sigma \in \text{Aut}(\mathbb{C})$.

We need some knowledge of algebraicity of our Eisenstein series; this is given by Theorem 3.2 in [Boug18] and is restated next. Define the set

$$\Omega_0 := \left\{ s \in \frac{1}{2} \mathbb{Z} \middle| s - \frac{n+1}{4} + \frac{n+1}{4} - \frac{k-\ell}{2} \in \mathbb{Z}, \frac{n+1-k+\ell}{2} \leq s \leq \frac{k-\ell}{2} \right\}$$

...
and, for any Hecke character \( \varphi \) of conductor \( f \), its Gauss sum to be

\[
G(\varphi) := \sum_{a=1}^{N(\ell)} \varphi^{-1}_f(a)e^{2\pi i a N(\ell)}.
\]

There are exceptional cases where the Eisenstein series has different behaviour. The relevant ones are:

\[m = n + 1 \text{ and } \varphi^2 = 1; \quad \text{(X)}\]
\[n = 1, m = \frac{3}{2}, \text{ and } \varphi = 1; \quad \text{(R1)}\]
\[n > 1, m = n + \frac{3}{2}, \text{ and } \varphi^2 = 1. \quad \text{(R2)}\]

None of these cases affect the first result in algebraicity since the set of special values does not include them. They will affect the second result, of Sect. 7, in which the full range of special values is considered.

**Theorem 4.1 (Bouganis, [Boug18], Th. 3.2).** Fix a half-integral weight \( k \), let \( \ell \in \frac{1}{2}\mathbb{Z} \) satisfy \( k - \ell > \frac{5}{2} \), and let \( \varphi \) be a Hecke character. Exclude case (X). If \( \frac{2m-n}{4} \in \Omega_0 \) we have

\[
\mathcal{E}_\varphi(z, \frac{2m-n}{4}; k - \ell, \varphi, \Gamma) = |y|^{-\frac{\gamma}{2}} \sum_{\tau \in B} P(\tau, \varphi, y)e(\text{tr}(Tz))
\]

where \( r = \frac{k-\ell}{2} - \frac{2m-n}{4} + 1 \) in cases (R1) and (R2), otherwise \( r = \frac{k-\ell}{2} - \frac{2m-n-1}{4} - \frac{n+1}{4} \), and \( P(\tau, \varphi, y) \) is a polynomial with complex coefficients in \( \pi y, \frac{1}{2} \), for \( 1 \leq i \leq j \leq n \). Put

\[
\beta(m) = \beta \left( \begin{array}{l}
\frac{m}{2} (k - \ell + m - n - 2) + m & \text{if } n \text{ is even and } m > n; \\
\frac{m}{2} (k - \ell + m - n) + \frac{\delta}{4} & \text{otherwise};
\end{array} \right.
\]

and define a period \( \omega(m, \varphi) = \omega_\varphi(\varphi) = \omega(\varphi) \) by

\[
\omega(\varphi) := \begin{cases}
\tau |_{\frac{2n-2m+n}{4} + mn - \frac{3n^2-1}{4}} G(\varphi) G(\varphi^{n-1}) & \text{if } k - \ell \in \mathbb{Z} \text{ and } m > n; \\
\tau |_{\frac{2n-2m+n}{4} - \frac{3n+2-2m}{2}} G(\varphi^n) & \text{if } k - \ell \in \mathbb{Z} \text{ and } m \leq n; \\
\tau |_{\frac{2n-2m+n}{4} - \frac{3n+2-2m}{2}} G(\varphi^m) G(\varphi)^n & \text{if } k - \ell \notin \mathbb{Z}, \ n \in 2\mathbb{Z}, \ m > n; \\
\tau |_{\frac{2n-2m+n}{4} - \frac{3n+2-2m}{2}} G(\varphi^n) G(\varphi^m) & \text{if } k - \ell \notin \mathbb{Z}, \ n \notin 2\mathbb{Z}, \ m > n; \\
\end{cases}
\]

where \( \mu_8 \) is a fixed eighth root of unity, \( \zeta \) is the character induced by \( h_\gamma(z)^2 = \zeta(\gamma) j(\gamma, z) \), and \( \delta_4 = 1 \) if \( n \equiv 1 \pmod{4} \) but \( \delta_4 = 0 \) otherwise. Then we have

\[
\left[ \frac{P(\tau, \varphi, y)}{\pi^3 \omega(\varphi)} \right]^\sigma = \frac{P(\tau, \varphi^\sigma, y)}{\pi^3 \omega(\varphi^\sigma)}
\]

for any \( \sigma \in \text{Gal}(\mathbb{Q}_{ab}/\mathbb{Q}) \).

Now fix \( g \in \mathcal{M}_\ell(\Gamma, \psi') \) where \( \ell \) is a either an integral or half-integral weight. Let \( k \) be a half-integral weight and set

\[
\mathcal{O}'(g) = \left\{ \begin{array}{ll}
\{ m \in \mathbb{R} | \frac{n-2m+2k-2\ell}{4} \in \mathbb{Z}, \frac{3n}{2} + 1 < m < k - \ell - \frac{3n}{2} \} & \text{if } g \notin \mathcal{S}_\ell; \\
\{ m \in \mathbb{R} | \frac{n-2m+2k-2\ell}{4} \in \mathbb{Z}, \frac{3n}{2} + 1 < m < k - \frac{3n}{2}, m < k - \ell + \frac{n}{2} \} & \text{if } g \in \mathcal{S}_\ell.
\end{array} \right.
\]

If \( f \in \mathcal{M}_k(\Gamma, \psi) \) then put \( \chi = \psi(\psi')^{-1} \) – assuming as usual that \( \psi_\infty(x) = \text{sgn}(x_\infty)^{|k|} \) and \( \psi_\infty(x) = \text{sgn}(x_\infty)^{|k|} \) – and recall \( \delta = n \pmod{2} \in \{0, 1\} \).
PROPOSITION 4.2. Let $k > 2n$, $\ell$, and $q$ be as above. Set $m_0 := \frac{2k+2\ell+2m-n}{4} - \frac{n+1}{2}$. For every $m \in \Omega'(g)$ there exists $K(m, g) \in S_k(\Gamma, \psi)$ whose Fourier coefficients lie in $Q_{ab}(g)$ such that

$$
\frac{(4\pi)^{nm_0}}{\pi^{\beta(m)} \omega_k(m, \psi \psi')} \Gamma_n(m_0)^{-1} \langle f, g E_{\psi \psi'}(\cdot, \frac{2m-n}{4}) \rangle = \pi^{nk-\frac{3n^2+2n+5}{4}} (f, K(m, g))
$$

for all $f \in S_k(\Gamma, \psi)$, and we also have $K(m, g)^\sigma = K(m, g^\sigma)$ for all $\sigma \in \text{Aut}(\mathbb{C})$.

Before proving this, we require the following lemma, whose statement and proof is adapted from [St81] p. 343).

**LEMMA 4.3.** Let $\tau \in \mathfrak{b}C$ and $P(y) \in \mathbb{Q}[y_{ij} \mid i \leq j]$. If $\nu \in \frac{1}{2} \mathbb{Z} \setminus \mathbb{Z}$ such that $\nu > n$, then we have

$$
|\tau|^2 \Gamma_n \left( \nu - \frac{n+1}{2} \right)^{-1} \int_Y P(y) e^{-\text{tr}(\tau y)} |y|^\nu \frac{n+1}{2} d^x y \in \mathbb{Q}.
$$

**Proof.** We can assume that $P(y) = \prod_{i \leq j} y_{ij}^{a_{ij}}$ where $0 \leq a_{ij} \in \mathbb{Z}$. If $U = (U_{ij}) \in Y$ then by definition

$$
\int_Y |y|^{\nu - \frac{n+1}{2}} e^{-\text{tr}(U y)} d^x y = \Gamma_n(n + \frac{n+1}{2}) |U|^{\frac{n+1}{2} - \nu}
$$

and apply $\prod_{i \leq j} \left( \frac{\partial}{\partial U_{ij}} \right)^{a_{ij}} |U_{ij} = T_{ij}$ to both sides. This gives

$$
\int_Y P(y) e^{-\text{tr}(U y)} |y|^{\nu - \frac{n+1}{2}} d^x y \in \Gamma_n(n + \frac{n+1}{2}) |T|^{\frac{n+1}{2} - \nu} \mathbb{Q}
$$

which, since $\frac{1}{2} - \nu \in \mathbb{Z}$, gives the lemma. \(\square\)

**Proof of Proposition 4.2.** The function $F(z) = g(z) E(z, \frac{2m-n}{4}) \in C^\infty(\Gamma, \psi)$ has bounded growth if and only if $F(z) := g(z) E(z, \frac{2m-n}{4}; k - \ell, \bar{\psi} \psi', \Gamma)$ does. If $m \in \Omega'(g)$ and $r = \frac{n-2m+2\ell-2k}{4}$ then $|E(z, \frac{2m-n}{4}; k - \ell, \bar{\psi} \psi', \Gamma)| \leq H_{k-l}(z, -r)$. Checking further that $-r$ satisfies the inequalities needed for Corollary 3.5 is easy. So we indeed get bounded growth of $F$.

Therefore apply Theorem 3.1 set $	ilde{K}(m, g) = \frac{(4\pi)^{nm_0}}{\pi^{\beta(m)} \omega_k(m, \psi \psi')} \Gamma_n(m_0)^{-1} P(F) \in S_k(\Gamma, \psi)$ and we get

$$
\frac{(4\pi)^{nm_0}}{\pi^{\beta(m)} \omega_k(m, \psi \psi')} \Gamma_n(m_0)^{-1} \langle f, g E_{\psi \psi'}(\cdot, \frac{2m-n}{4}) \rangle = \langle f, \tilde{K}(m, g) \rangle.
$$

The Fourier coefficients $a(\tau, y)$ for each $\tau \in \mathfrak{b}A$ of $F$ are given, by Lemma 4.1 as

$$
a(\tau, y) = \sum_{\tau_1 + \tau_2 = \tau} \mu_g(\tau_1, 1) P(\tau_2, \bar{\psi} \psi', y) |y|^{-r} e^{-2\pi \text{tr}(\tau y)}
$$

and so by Theorem 3.1 we obtain the Fourier expansion

$$
\tilde{K}(m, g) = \sum_{\tau \in \mathfrak{b}C} \left( \sum_{\tau_1 + \tau_2 = \tau} a(\tau_1, \tau_2) \right) e(\text{tr}(\tau z))
$$

$$
a(\tau_1, \tau_2) = c(k, n)^{-1} \mu_g(\tau_1, 1) 4 \tau^{k - \frac{n+1}{2}} \frac{(4\pi)^{nm_0}}{\pi^{\beta(m)} \omega_k(m, \psi \psi')} \Gamma_n(m_0)^{-1}
$$

$$
\times \int_Y P(\tau_2, \bar{\psi} \psi', y) e^{-4\pi \text{tr}(\tau y)} |y|^{k-r-\frac{n+1}{2}} d^x y.
$$

The polynomial $P(\tau_2, \bar{\psi} \psi', y)$ has the form

$$
P(\tau_2, \bar{\psi} \psi', y) = \sum_{1 \leq i \leq j \leq n} \alpha_{ij} (\tau_2, \bar{\psi} \psi') \prod_{i, j}(\pi y_{ij})^{a_{ij}}
$$
for $c_{ij}(\tau_2) = c_{ij}(\tau_2, \psi\psi') \in \mathbb{C}$ satisfying $c_{ij}(\tau_1, \psi\psi')^\sigma = c_{ij}(\tau_2, \psi\sigma(\psi')^\sigma)$, and $\alpha_{ij}$ summing over some finite set of non-negative integers. We have that $c(k, n)^{-1} \in \pi^{n k - 3 n^2 + 2 n + \delta} \mathbb{Q}$ and that $|4\tau|^{k - \frac{n + 1}{2}} \in \mathbb{Q}$.

Now if $m \in \Omega'(\theta)$ then we can make some simplifications to our integral expression. By putting $\ell = \frac{n}{2} + \mu$ into the notation of Proposition 4.2 we get $m_0 = \frac{m - n - 1 + k + \mu}{2} \in \frac{n}{2} + \mathbb{Z}$. Note that $|\tau|^\frac{n}{2} \in 2^\nu |2\tau|^\frac{n}{4} \mathbb{Q}$. Then from the integral expression of 4.2 we obtain

$$L_\psi(m, f, \eta) \in 2^{\frac{\delta}{2}} \mu_f(\tau_1, 1)^{-1} |2\tau|^\frac{\delta}{2} \epsilon_0(m, \chi) \prod_{p \in \mathfrak{b}} g_p \left((\psi^\sigma(\eta)(p))^m\right)$$

$$\times (4\pi)^{m_0} \Gamma(n(m_0)^{-1}(f, \theta E(\cdot, \cdot, 2m_0))) \mathbb{Q}.$$ 

Since $k - \ell \in \mathbb{Z}$ if and only if $n$ is odd we can relabel $\omega_b(\rho) := \omega_b(\rho)$ whenever $\ell = \frac{n}{2} + \mu$. With $\chi = \eta\psi p_r$, multiplying both sides by $\pi^{-\beta(m)} \omega_b(m, \chi)^{-1}$, and then applying Proposition 4.2 gives

$$\frac{2\pi |2\tau|^\frac{n}{2} \mu_f(\tau_1, 1) L_\psi(m, f, \eta)}{\pi^{\beta(m) + n k - 3 n^2 + 2 n + \delta} \omega_b(m, \chi)} \in \epsilon_0(m, \chi) \prod_{p \in \mathfrak{b}} g_p \left((\psi^\sigma(\eta)(p))^m\right) (f, K(m, \theta)) \mathbb{Q}. \quad (4.3)$$

### 5. Algebraicity of the inner product

The algebraicity of the right-hand side of the integral expression 4.3 and subsequently of the special values, is immediate except for the inner product $(f, K(m, \theta))$ – hence this section. For a system of eigenvalues $\Lambda : \mathcal{R}_0 \to \mathbb{C}$, where $\mathcal{R}_0$ is the space of Hecke operators (defined in the next section or in [Sh95b, pp. 39–41]), let $S_k(\Gamma, \psi, \Lambda)$ denote the eigenforms in $S_k(\Gamma, \psi)$ whose eigenvalues are given by $\Lambda$. Let $\rho \in \text{Aut}(\mathbb{C})$ denote the complex conjugation automorphism.

**Theorem 5.1.** Assume that $n > 1$ and that $k > \frac{3n}{2} + 1$. If $f \in S_k(\Gamma, \psi, \Lambda)$ is a Hecke eigenform for ideals $(\mathfrak{b}^{-1}, \mathfrak{b}c) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$, a Hecke character $\psi$, and a system of eigenvalues $\Lambda$, then there exists a non-zero constant $\mu'(\Lambda, k, \psi) - \text{dependent only on } \Lambda, k, \psi$ – such that

$$\left(\frac{\langle f, g \rangle}{\mu'(\Lambda, k, \psi)}\right)^\sigma = \frac{\langle f^\sigma, g^{\sigma^\sigma} \rangle}{\mu'(\Lambda^\sigma, k, \psi^\sigma)}$$

for any $g \in S_k(\Gamma, \psi, \Lambda)$, ideals $(\mathfrak{b}^{-1}, \mathfrak{b}c') \subseteq \mathfrak{b}^{-1} \times \mathfrak{b}c$, and $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$.

**Proof.** Take a Hecke character $\varphi$ such that $\varphi_\infty(p) = (-1)^\delta$ (for example $\varphi^\sigma(p\mathbb{Z}) := \left(-\frac{1}{p}\right)^\delta$ where $\varphi^\sigma$ is the ideal character associated to $\varphi$). Then $(\varphi \psi)_\infty(x) = \text{sgn}(x_\infty)|k| + \delta$. The constant to use is

$$\mu'(\Lambda, k, \psi) := 2\pi^{-a} i^{-\frac{\delta}{2}} \omega_\delta(k - 3n, \varphi\psi)^{-1} L_\psi(k - 3n, f, \varphi)$$

13
where \( a = \beta(k - 3n) + nk - \frac{3n^2 + 2n + 3}{4} \). This constant is non-zero as the Euler product of \( L(s, f, \varphi) \) is absolutely convergent for values \( s \) with \( \Re(s) > \frac{3n}{2} + 1 - \) [Sh96, p. 332] – of which \( s = k - 3n \) is such a value by our choice of \( k \). If \( m \) is an integral ideal let \( \varphi_m^* \) denote the ideal Hecke character such that \( \varphi_m^*(d\mathbb{Z}) = \varphi^*(d\mathbb{Z}) \) for any \( (d, N(m)) = 1 \), and let \( \theta_{\varphi_m} \) denote the theta series that we have been using, from [Sh96, (2.1)], but with \( \varphi_m^* \) in place of \( \eta^* \).

If \( m = 2T|bc \), then Proposition 2.1 [Sh96] gives the level \( \Gamma[2, 2T|3\langle bc \rangle^2] \) of \( \theta_{\chi_m} \). Hence, we have \( \theta_{\chi_m} \in \mathcal{M}_{2T} \langle \Gamma[b^{-1}, bc'], \varphi_m^* \rangle \) and \( f \in \mathcal{S}_k(\Gamma[b^{-1}, bc'], \psi) \) with \( c' := 2T|3\langle bc \rangle^2 \). Since

\[
\Omega'(\varphi_m) = \{ m \in \frac{1}{2}\mathbb{Z} \mid \frac{k-m}{2} \in \mathbb{Z}, \frac{3n}{2} + 1 < m < k - 2n - \delta \},
\]

then we can take \( m = k - 3n \in \Omega'_{n,k} \) and apply Proposition 4.2 with \( g = \theta_{\varphi_m} \) to obtain the integral expression 4.3

Notice that

\[
G(\chi)G(\chi^{-1}) = G(\eta\psi)G(\eta\psi^{-1})J(\rho, \rho)^n(\tau, \rho) \frac{G(\rho)^n}{J(\eta\psi, \rho)J(\eta\psi^{-1}, \rho)^{-1}}
\]

where \( J(\chi_1, \chi_2) = \sum_\chi \chi(a)\chi_2(1-a) \) is the Jacobi sum. In the case of \( n \) odd and \( s > n \) this gives

\[
\omega(\eta\psi) = \omega(\chi)G(\rho)^{-n}J(\rho, \rho)^nJ(\eta\psi, \rho)J(\eta\psi^{-1}, \rho)^{-1}.
\]

The other cases are simpler and give \( \omega(\eta\psi) = \omega(\chi)G(\rho)^{-n}J(\eta\psi, \rho)^n \). Denote these products of Jacobi sums by \( J_n(s, \chi) \) and evidently \( J_n(s, \chi)^\sigma = J_n(s, \chi^\sigma) \). So the integral expression 4.3 becomes

\[
\left( f, K(k - 3n, \theta_{\varphi_m}) \right)_{\mu'(\Lambda, k, \psi)} = \left( f, K(k - 3n, \theta_{\varphi_m}) \right)_{\mu'(\Lambda, k, \psi)} = \left( f, K(k - 3n, \theta_{\varphi_m}) \right)_{\mu'(\Lambda, k, \psi)^{-1}}.
\]

For any congruence subgroup \( \Gamma = \Gamma[\mathfrak{g}, \mathfrak{a}] \) let

\[
\Gamma^0 := \{ \gamma \in \Gamma \mid a\gamma \equiv d\gamma \equiv 1 \pmod{\eta} \}.
\]

Suppose that \( \Gamma_2 \leq \Gamma_1 \leq \mathfrak{m} \) are two congruence subgroups, then decompose \( \Gamma_1 = \bigsqcup_{i=1}^d \Gamma_{g_i} \). The trace map is defined, for a Hecke character \( \varphi \), by

\[
\text{Tr}_{\Gamma_1, \varphi} : \mathcal{M}_k(\Gamma_2) \rightarrow \mathcal{M}_k(\Gamma_1, \varphi)
\]

\[
h \mapsto \sum_{i=1}^d \varphi_{\gamma_i}([|a_{g_i}|]^{k-1}h)_{|k, g_i}
\]

where \( \Gamma_1 = \Gamma[b^{-1}, \mathfrak{b}, \mathfrak{c}] \). By Lemma 5.4 of [Boug18] we have that \( \text{Tr}_{\Gamma_1, \varphi}(h) = \text{Tr}_{\Gamma_1, \varphi^*}(h^\sigma) \) for any \( h \in \mathcal{M}_k(\Gamma_2) \) and any \( \sigma \in \text{Aut}(\mathbb{C}) \).

Therefore \( \text{Tr}_{\Gamma, \psi}^{\varphi}(K(k - 3n, \theta_{\varphi_m})) \in \mathcal{S}_k(\Gamma, \psi) \) and so far we have obtained

\[
[(f, g)_{\mu'(\Lambda, k, \psi)^{-1}} = (f, g^\sigma)_{\mu'(\Lambda^\sigma, k, \psi)^{-1}}
\]

for all \( g \in \{ \text{Tr}_{\Gamma_1, \psi}(K(k - 3n, \theta_{\varphi_m})) \mid \tau \in \mathfrak{b}, \mathfrak{c} = |2T|bc \} \). The rest of the proof follows just as in [St81, p. 350] – by extending the above set of \( g \) into a basis for \( \mathcal{S}_k(\Gamma, \psi, \Lambda) \) and using the orthogonal decomposition of \( \mathcal{S}_k(\Gamma, \psi) \) into such eigenspaces.
Theorem 5.2. The above theorem reads almost exactly the same for \( n = 1 \) but with the different bound \( k > \frac{13}{2} + 1 = \frac{13}{2} \). The method above works as well for \( n = 1 \) but only by using a smaller special value \( k - 4 \) (instead of \( k - 3 \)) in the definition of the constant \( \mu(\Lambda, k, \psi) \). This is because \( k - 3 \notin \Omega'(\theta_{\varphi_{\psi}}) \) here.

Theorem 5.3. If \( n = 1 \) assume that \( k > \frac{13}{2} \), otherwise \( k > \frac{9n}{2} + 1 \). Let \( f \in S_k(\Gamma, \psi, \Lambda) \), \( \eta \) be a Hecke character, and choose \( \mu \in \{0, 1\} \) such that \((\psi\eta)_\infty(x) = \text{sgn}(x_\infty)^{|k|+\mu} \). Define the set

\[
\Omega'_{n,k} := \left\{ m \in \frac{1}{2} \mathbb{Z} \middle| \frac{k-m-\mu}{2} \in \mathbb{Z}, \frac{3n}{2} + 1 < m < k - 2n - \mu \right\}.
\]

Now if \( \tau \in S_+ \) is such that \( \mu_f(\tau, 1) \neq 0 \) and \( m \in \Omega'_{n,k} \) then define

\[
Y_\psi(m, f, \eta) := |\tau|^{\frac{k}{2}} \pi^{-b} \mu'(\Lambda, k, \psi)^{-1} \omega_k(m, \tilde{\chi})^{-1} L_\psi(m, f, \eta)
\]

where \( b = \beta(m) + nk - \frac{3n^2 + 2n + \delta}{4} \). We have \( Y_\psi(m, f, \eta)^\sigma = Y_{\psi^\sigma}(m, f^\sigma, \eta^\sigma) \) for all \( \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}) \) and hence

\[
Y_\psi(m, f, \eta) \in \mathbb{Q}(f, \psi, \eta).
\]

Proof. Noting that \( |\tau|^{\frac{k}{2}} \in \mathbb{R}^{2|k|/2} \) and \( \Omega'(\theta) = \Omega'_{n,k} \) we use the integral expression 4.3 to get

\[
Y_\psi(m, f, \eta) \in \mathbb{Q}(f, \psi, \eta).
\]

which, once we consider Theorem 5.1, we see is \( \sigma \)-equivariant for all of \( \text{Aut}(\mathbb{C}/\mathbb{Q}) \). \( \square \)

6. Algebraicity of metaplectic Eisenstein series

So far we have determined the specific algebraicity for only some of the special values given in [Sh00]. The aim of this lengthy section is to investigate and establish the precise algebraicity of the well-known decomposition \( M_k = S_k \oplus E_k \) which will allow the determination of the rest of these special values. Such an algebraicity is equivalent to proving Garrett’s conjecture that the Klingen Eisenstein series \( E(f) \) of a cusp form \( f \) preserves algebraicity, see [Garr84].

Due to its length, this section is split up into two subsections. The first is preliminary and the main aim is to relate the standard \( L \)-function of \( \Phi f \) with \( f \) where \( \Phi \) is the Siegel Phi operator. This relation will be useful in the second subsection, which is the proof of Garrett’s conjecture and the desired decomposition.

6.1 Hecke eigenforms and the Siegel Phi operator

For a real variable \( \rho \) the Siegel Phi operator is defined as

\[
\Phi : M_k^n \to M_k^{n-1}
\]

\[
f(z) \mapsto \lim_{\rho \to \infty} f \left( \begin{array}{cc} w & 0 \\ 0 & i\rho \end{array} \right)
\]

for \( z \in \mathbb{H}_n, w \in \mathbb{H}_{n-1} \).

In order to establish the desired relation of \( L \)-functions of \( \Phi f \) and \( f \) we need to relate the
Satake parameters. Define
\[ x_p := M_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p), \]
\[ \hat{\times} := GL_n(\mathbb{Q}) \prod_p x_p, \]
\[ c_p := GL_n(\mathbb{Z}_p), \]
\[ \hat{c} := \prod_p c_p, \]
\[ Z_0 := \{ \text{diag}[\tilde{q}, q] \mid q \in \hat{\times} \}, \]
\[ Z := D[2, 2]Z_0 D[2, 2], \]
and certain metaplectic lifts
\[ \mathcal{D}[2, 2] := \text{pr}^{-1}(D[2, 2]), \]
\[ \mathcal{D} := \{ \alpha \in \mathcal{D}[2, 2] \mid \text{pr}(\alpha) \in G_\Gamma \cap D \}, \]
\[ \mathcal{Z} := \text{pr}^{-1}(Z), \]
\[ \mathcal{Z}_0 := \{ \alpha \in \mathcal{Z} \mid \text{pr}(\alpha) \in G_\Gamma \cap DZ_0 D \}, \]
\[ \hat{\mathcal{D}} := \{ (\alpha, 1) \in \hat{\mathcal{Z}}_0 \mid \alpha \in \mathcal{D} \}. \]
where \( D = D[b^{-1}, bc] \) for ideals \((b^{-1}, bc) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}\). For any prime \( p \) we use the subscript \( p \)
to denote the \( p \)-th local component of any of the above adelic groups. All of the above sets are
dependent on \( n \) and, when we wish to distinguish this, we shall use \( n \) as a superscript e.g. \( x_p^n \).
The abstract Hecke ring \( \mathcal{R}(\hat{\mathcal{D}}, \hat{\mathcal{Z}}_0) \) comprises all formal, finite sums
\[ \sum_\sigma c_\sigma \hat{\mathcal{D}} \sigma \hat{\mathcal{D}} \]
with \( c_\sigma \in \mathbb{C} \) and \( \sigma \in \hat{\mathcal{Z}}_0 \).

This abstract Hecke ring has a representation on the space of modular forms in \( \mathcal{M}_k(\Gamma, \psi) \),
where \( \Gamma = G \cap D \), which we now describe. For integral weight forms one decomposes the double
cosets into single cosets and let the representatives act on \( f \) by the usual slash operator. This no
longer quite works here due to the weak automorphic property\([2,3]\) of the factor of automorphy. In
\([Sh95b, p. 32]\) Shimura defines a new factor of automorphy \( J^k \) which extends the original \( j^k \)
to \( \mathcal{Z} \) and has strong automorphic properties. Now consider the element \( \hat{\mathcal{D}}(r_P(\sigma), 1) \hat{\mathcal{D}} \in \mathcal{R}(\hat{\mathcal{D}}, \hat{\mathcal{Z}}_0) \)
where \( \sigma = \text{diag}[q, q] \) for \( q \in \hat{\times} \), then we have the decomposition \( G \cap (D \sigma D) = \Gamma \xi \Gamma = \bigsqcup_\alpha \Gamma \alpha \) for
some \( \alpha, \xi \in G \cap Z \). The representation of this element, denoted \( T_{q, \psi} \), on \( f \in \mathcal{M}_k(\Gamma, \psi) \) is given by
\[ (f|T_{q, \psi})(z) = \sum_\alpha \psi_\xi(|a_\alpha|)^{-1} J^k(\alpha, z)^{-1} f(\alpha \cdot z). \]
Let \( \mathcal{R}_0 \) be the factor ring of \( \mathcal{R}(\hat{\mathcal{D}}, \hat{\mathcal{Z}}_0) \) modulo the ideal
\[ \langle \hat{\mathcal{D}}(\alpha, 1) \hat{\mathcal{D}} - t \hat{\mathcal{D}}(\alpha, t) \hat{\mathcal{D}} \mid (\alpha, t) \in \hat{\mathcal{Z}}_0 \rangle. \]
Since the action of this ideal is trivial on \( \mathcal{M}_k(\Gamma, \psi) \) we have an action of this factor ring on
modular forms defined as before. Then denote the element represented by \( \hat{\mathcal{D}}(r_P(\sigma), 1) \hat{\mathcal{D}} \) in \( \mathcal{R}_0 \)
by \( A_q \) where \( \sigma = \text{diag}[q, q] \) with \( q \in \hat{\times} \), that is \( A_q \) is the image of \( T_{q, \psi} \) in \( \mathcal{R}_0 \). Finally we denote
by \( \mathcal{R}_{0p} \), the subalgebra of \( \mathcal{R}_0 \) that is generated by the \( A_q \) for all \( q \in \times_p \). As in \([Sh95b, pp. 41–42]\) we can define a map
\[ \omega_p := \omega_{0p} \circ \Phi_p : \mathcal{R}_{0p} \to \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \]
as follows. If \( \sigma = \text{diag}[\tilde{q}, q] \) with \( q \in \times_p \), then \( A_q \in \mathcal{R}_{0p} \) and we have a decomposition of the form
\[ D_p \sigma D_p = \bigsqcup_{x \in X} \bigsqcup_{s \in Y_x} \bigsqcup_{d \in \mathcal{R}_x} D_p \alpha_{d, s}, \quad \alpha_{d, s} = \begin{pmatrix} d & sd' \\ 0 & d \end{pmatrix} \] (6.1)
with \( X \subseteq GL_n(\mathbb{Q}_p), R_x \subseteq x \circ_p \) representing \( \circ_p \backslash \circ_p x \circ_p \), and \( Y_x \subseteq S_p \). Then extend to all of \( \mathcal{R}_0p \) by \( \mathbb{C} \)-linearity the following map

\[
\Phi_p(A_q) := \sum_{d,s} J(r_p(\alpha_{d,s}))^{-1} \circ_p d \in \mathcal{R}(\circ_p, GL_n(\mathbb{Q}_p))
\]

where \( J(\alpha) = J^\sharp(\alpha,i) \) for \( \alpha \in \mathfrak{a} \). For the second map \( \omega_0p \), note that any coset \( \circ_p d \) with \( d \in GL_n(\mathbb{Q}_p) \) contains an upper triangular matrix of the form

\[
\begin{pmatrix}
p^{a_{d_1}} & \ast & \cdots & \ast \\
0 & p^{a_{d_2}} & \cdots & \ast \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & p^{a_{d_n}}
\end{pmatrix}
\]

with \( a_{d_i} \in \mathbb{Z} \), and then define

\[
\omega_0p(\circ_p d) = \prod_{i=1}^n (p^{-1}x_i)^{a_{d_i}}
\]

which, via decompositions \( \circ_p x \circ_p = \sum_d \circ_p d \) and \( \mathbb{C} \)-linearity, we extend to obtain the map \( \omega_0p : \mathcal{R}(\circ_p, GL_n(\mathbb{Q}_p)) \rightarrow \mathbb{C}[x_1^\pm, \ldots, x_n^\pm] \).

Assume that \( p \nmid \mathfrak{c} \). For an independent variable \( u \) define a map \( \Psi(\cdot,u) : \mathcal{R}_0p^n \rightarrow \mathcal{R}_0p^{n-1}[u^\pm] \) as follows. Consider the generators \( A_q \in \mathcal{R}_0p^q \) for \( q \in x_p \), with \( D_p \sigma D_p \) having the decomposition \( 6.1 \) and each \( d \) having the form \( 6.2 \) and put

\[
\Psi(A_q, u) = \sum_{x,d,s} J(r_p(\alpha_{d',s'}))J(r_p(\alpha_{d,s}))^{-1}(up^{-n})^{a_{d_n}} D_p \begin{pmatrix} d' & s'd' \\ 0 & d'' \end{pmatrix}
\]

where \( A' \) denotes the upper left \( n-1 \) block of \( A \in M_n \). Extend this to all of \( \mathcal{R}_0p \) by \( \mathbb{C} \)-linearity. The map \( \omega_p^{n-1} \times 1 \) acts as \( \omega_p^{n-1} \) on \( \mathcal{R}_0p^{n-1} \) and as the identity on \( u \). Then

\[
(\omega_p^{n-1} \times 1)(\Psi(A_q, u)) = \sum_{d,s} J(r_p(\alpha_{d',s'}))J(r_p(\alpha_{d,s}))^{-1}(up^{-n})^{a_{d_n}} \prod_{i=1}^{n-1} (p^{-1}x_i)^{a_{d_i}}.
\]

So by defining \( \phi_{n,u}(x_i) = x_i \) for \( 1 \leq i \leq n-1 \) and \( \phi_{n,u}(x_n) = u \), extending \( \mathbb{C} \)-linearly to all of \( \mathbb{C}[x_1^\pm, \ldots, x_n^\pm] \), we get the commuting square

\[
\begin{array}{ccc}
\mathcal{R}_0p^n & \xrightarrow{\omega_p^n} & \mathbb{C}[x_1^\pm, \ldots, x_n^\pm] \\
\downarrow{\Psi(\cdot,u)} & & \downarrow{\phi_{n,u}} \\
\mathcal{R}_0p^{n-1}[u^\pm] & \xrightarrow{\omega_p^{n-1} \times 1} & \mathbb{C}[x_1^\pm, \ldots, x_{n-1}^\pm, u^\pm]
\end{array}
\]

By [Sh94, Lemma 2.6] we are able to choose \( \alpha_{d,s} \) of the form

\[
\alpha_{d,s} = \begin{pmatrix} g^{-1}h & g^{-1}\sigma h \\ 0 & g^T \end{pmatrix}
\]

where \( g, h \in x_p \), and \( \sigma \in gS_l(b^{-1})g^T \). We have \( J^k(r_p(\alpha), z) = J^\sharp(r_p(\alpha), z)J(\alpha, z)^{[k]} \) by (2.1c) of [Sh95a], and by Lemma 2.4 of that same paper \( J^\sharp(\alpha_{d,s}, z) \) is independent of \( z \). So we see that

\[
J^k(\alpha_{d,s}, z) = J(r_p(\alpha_{d,s})),J(r_p(\alpha_{d',s'}))^{-1}p^{a_{d_n}}[k]J^k(\alpha_{d',s'}, z).
\]
Proposition 6.1. If \( f \in \mathcal{M}_k(\Gamma, \psi) \) and \( p \nmid \psi \) then
\[
\Phi(f|A_q) = (\Phi f)|\Psi(A_q, \psi_p^{-1}(p)p^{n-[k]})
\]

Proof. Firstly, since \( \psi_p(p) = \psi_p^{-1}(p) \), we have
\[
f|A_q = \sum_{x,d,s,\tau} \psi_p^{-1}(|d|)J^k(\alpha_d, s, z)^{-1} \mu_f(\tau, 1) e(d^{-1}\tau \tilde{d} \cdot z + \tau s)
\]
and apply \( \Phi \) to the above expression. If \( \tau = (\tau' \tau) \) with \( \tau' \in S_+^n \) and 0 \( \leq t \in \mathbb{Z} \), then we know that the last diagonal entry of \( d^{-1}\tau \tilde{d} \) is \( p^{-2a_{dn}}t \). Thus by writing \( z = (z' 0) \) and letting \( \lambda \to \infty \), any terms involving \( \tau \) for \( t > 0 \) will tend to 0 and we are left only with terms involving \( \tau \in S_+^n \) with \( t = 0 \) – these are precisely the elements of \( S_+^n \). So for \( \tau = (\tau' 0) \) we have
\[
d^{-1}\tau \tilde{d} \cdot z + \tau s = \left( (d')^{-1}\tau' \tilde{d}' \cdot z' + \tau's', 0 \right).
\]
By the identity 6.4 of \( J \) we get
\[
\Phi(f|A_q) = \sum_{x,d,s} J(r_P(\alpha_{d', s'}))J(r_P(\alpha_{d, s}))^{-1}\psi_x^{-1}(p^{a_{dn}})p^{-a_{dn}[k]}
\]
\[
\times \left[ \psi_x(|d'|)J^k(\alpha_{d', s'}, z')^{-1} \sum_{\tau' \in S_+^n} \mu_f \left( \left( \begin{array}{cc} \tau' & 0 \\ 0 & 0 \end{array} \right), 1 \right) e(\tau'(\tilde{d}' \cdot z' + s'd')(d')^{-1}) \right]
\]
which is exactly \((\Phi f)|\Psi(A_q, \psi_p^{-1}(p)p^{n-[k]})\) as \( \Phi f \) has Fourier coefficients \( \mu_f \left( \left( \begin{array}{cc} \tau' & 0 \\ 0 & 0 \end{array} \right), 1 \right) \) for all \( \tau' \in S_+^n \). \( \square \)

For each prime \( p \) an element of \( \mathbb{C}^n \) – the Satake \( p \)-parameters – is associated to a Hecke eigenform in \( S_k \) and this process, taken from [Sh95b], we outline briefly. In doing so, we are able to see how the elements of \( \mathbb{C}^{n-1} \) and \( \mathbb{C}^n \) for \( \Phi f \) and \( f \), respectively, are related.

Assume that \( p \nmid \psi \). Define the operator
\[
T_p^n := \sum_{m=0}^{\infty} A_p^n(p^m)t^m
\]
where \( A_p^n(p^m) \) is the sum of all \( A_q \) with \(|q| = p^m\). If \( f \) be a Hecke eigenform of degree \( n \) with \( f|A_p^n(p^m) = \Lambda(p^m)f \) then
\[
\Phi(f|T_p^n) = \sum_{m=0}^{\infty} \Lambda(p^m)t^m \Phi f. \tag{6.5}
\]

Extend the definitions of \( \Psi, \omega_p^n \), and \( \omega_p^{n-1} \) to \( T_p \) by letting them act linearly on the coefficients. For any \( 1 \leq \ell \in \mathbb{Z} \), Theorem 4.4 in [Sh95b] p. 42] gives
\[
\omega_p^\ell(T_p^\ell) = \prod_{i=1}^{\ell} \frac{1 - p^{2i-1}t^2}{(1 - p^\ell x_i t)(1 - p^\ell x_i^{-1}t)}
\]
and hence, for an \( \ell \)-degree eigenform \( g \), the existence of the Satake \( p \)-parameters \( (\lambda_{p,1}, \ldots, \lambda_{p,\ell}) \) such that
\[
g|T_p^\ell = \prod_{i=1}^{\ell} \frac{1 - p^{2i-1}t^2}{(1 - p^\ell \lambda_{p,i} t)(1 - p^\ell \lambda_{p,i}^{-1}t)}g.
\]
Assume that \( 0 \neq \Phi f \) has Satake \( p \)-parameters \( (\lambda_{p,1}, \ldots, \lambda_{p,n-1}) \) for \( p \nmid \wp \). By the commuting square \([6.3]\) we have

\[
\omega_p^{-1}(\Psi(T_p^n, u)) = \phi_n,u(\omega_p(T_p^n)) = \prod_{i=1}^{n-1} \left( 1 - p^{2i+1}t^2 \right) \frac{1 - p^{2n-1}t^2}{(1 - p^nu)(1 - p^n(1-t))}
\]

so that

\[
(\Phi f)\Psi(T_p^n, u) = \prod_{i=1}^{n-1} \left( 1 - p^{2i-1}t^2 \right) \frac{1 - p^{2n-1}t^2}{(1 - p^nu)(1 - p^n(1-t))} \Phi f
\]

(6.6)

On the other hand Proposition \([6.1]\) with the identity \([6.5]\) above gives

\[
f(\Phi f)|\Psi(T_p^n, (\psi_p^{-1}(p)p^{n-|k|}) = \Phi(f|T_p^n)f = (f|T_p^n)\Phi f
\]

(6.7)

So equating \([6.6]\) and \([6.7]\) with \( u = \psi_p^{-1}(p)p^{n-|k|} \) we have proved the following.

**Proposition 6.2.** Let \( f \in \mathcal{M}_k^\psi(\Gamma, \psi) \) be a non-zero eigenform such that \( \Phi f \neq 0 \). Then \( \Phi f \) is an eigenform of degree \( n-1 \). If \( \Phi f \) has Satake \( p \)-parameters \( (\lambda_{p,1}, \ldots, \lambda_{p,n-1}) \) then the Satake \( p \)-parameters of \( f \) are \( (\lambda_{p,1}, \ldots, \lambda_{p,n-1}, \psi_p^{-1}(p)p^{n-|k|}) \).

Let \( \eta := \psi^{-2} \). We can use the above proposition \([6.2]\) to obtain a relation between \( L^n(s, f, \eta) \)

\[
L^n_p((\psi^s\eta)(p)p^{-s}) = L^{n-1}_p((\psi^s\eta)(p)p^{-s+1})(1 - \eta(p)p^{2n-|k|-s})(1 - p^{k-s})
\]

and the Euler factors at \( p \nmid \wp \) are just 1 by definition of \( \eta \). Therefore

\[
L^n_p(s, f, \eta) = L^{n-1}_p(s - 1, \Phi f, \eta)L(s + [k] - 2n, \eta)\zeta(s - [k])
\]

where \( \zeta \) is the Riemann zeta function with the Euler factors at \( p \mid \wp \) removed. So then, by induction, for any \( 0 \leq r' \leq n \) such that \( \Phi^{n-r'}f \neq 0 \) we get

\[
L^n_p(s, f, \eta) = L^{n-r'}_p(s - n + r', \Phi^{n-r'}f, \eta) \prod_{i=0}^{n-r'-1} L(s + [k] - 2n + i, \eta)\zeta(s - [k] - i).
\]

(6.8)

**6.2 Klingen Eisenstein series**

Let \( G' \) denote the image of \( G \) under the embedding

\[
G \to G_\wp
\]

\[
x \mapsto (x\wp)_u
\]

where \( x_\wp = x \) and \( x_p = I_{2n} \) for all primes \( p \). Let \( \mathfrak{S} = pr^{-1}(G') \subseteq M_\wp \) and we have \( \mathfrak{S} \subseteq \mathfrak{M} \). By \([Sh95a]\) p. 554] the group \( \mathfrak{S} \) can be identified with the group of couples \((\alpha, p)\) where \( \alpha \in G \) and \( p : \mathbb{H}_n \to \mathbb{C} \) is a holomorphic function such that \( p(z)^2/j_\alpha(z) \in \mathbb{T} \) is a constant, with group law \((\alpha, p)(\alpha', p') = (\alpha\alpha', p(\alpha'z)p'(z)) \). This identification is given by \( \alpha \mapsto (\alpha, h_\alpha) \) and \( \mathfrak{S} \) acts on \( f : \mathbb{H}_n \to \mathbb{C} \) as

\[
(f|k\xi)(z) = p(z)j(\alpha, z)^{|k|}f(\alpha z)
\]

where \( \xi = (\alpha, p) \in \mathfrak{S} \).

We have previously been considering congruence subgroups \( \Gamma \) of \( G \) that are contained in \( \mathfrak{M} \), and to such a congruence subgroup we define the group \( \widehat{\Gamma} = \{(\alpha, h_\alpha) \mid \alpha \in \Gamma \} \subseteq \mathfrak{S} \). Indeed,
the definition of a congruence subgroup of $\mathcal{G}$ is given in such a way – it is a subgroup $\Delta \leq \mathcal{G}$ that is isomorphic under projection to a congruence subgroup $\Gamma \leq G$ such that $\Delta = \hat{\Gamma}$. As such, congruence subgroups of $G$ and $\mathcal{G}$ are one and the same and identify $\Gamma = \hat{\Gamma} \leq \mathcal{G}$.

For an integer $r$ such that $0 \leq r \leq n$ and for any $\alpha \in M_n(\mathbb{A}_\mathbb{Q})$ we write

$$\alpha = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \\ b_1 & b_2 \\ b_3 & b_4 \\ c_1 & c_2 \\ c_3 & c_4 \\ d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \quad (6.9)$$

where, for $x \in \{a,b,c,d\}$, we have $x_1 \in M_r(\mathbb{A}_\mathbb{Q}), x_2 \in M_{r,n-r}(\mathbb{A}_\mathbb{Q}), x_3 \in M_{n-r,r}(\mathbb{A}_\mathbb{Q})$, and $x_4 \in M_{n-r}(\mathbb{A}_\mathbb{Q})$. Also write

$$x_\alpha = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1(\alpha) \\ x_2(\alpha) \\ x_3(\alpha) \\ x_4(\alpha) \end{pmatrix}$$

when we wish to emphasis the matrix $\alpha$ to which these blocks belong. If $r = n$ then we make the natural understanding that $x_\alpha = x_1(\alpha)$ and likewise, for $r = 0$, we have $x_\alpha = x_4(\alpha)$. Now for such an $n, r$ we define the following parabolic subgroup $P^{n,r} \leq \text{Sp}_n(\mathbb{Q})$ by

$$P^{n,r} := \{ \alpha \in \text{Sp}_n(\mathbb{Q}) \mid a_2(\alpha) = c_2(\alpha) = 0, c_3(\alpha) = d_3(\alpha) = 0, c_4(\alpha) = 0 \} \quad \text{if } 0 < r < n$$

with $P^{n,0} = P^n$ and $P^{n,n} = \text{Sp}_n(\mathbb{Q})$. With $\alpha$ of the form in $6.9$ we have some maps:

$$\pi_r : M_{2n}(\mathbb{A}_\mathbb{Q}) \to M_{2r}(\mathbb{A}_\mathbb{Q})$$

$$\alpha \mapsto \begin{pmatrix} a_1(\alpha) & b_1(\alpha) \\ c_1(\alpha) & d_1(\alpha) \end{pmatrix}$$

$$\lambda_r : M_{2n}(\mathbb{A}_\mathbb{Q}) \to \mathcal{A}_\mathbb{Q}$$

$$\alpha \mapsto |d_4(\alpha)|.$$}

These define respective homomorphisms $P^{n,r}_k \to \text{Sp}_r(\mathbb{A}_\mathbb{Q})$ and $P^{n,r}_k \to \mathbb{I}_n$. On the metaplectic side let $\mathfrak{P}^{n,r} = \{ (\alpha, p) \in \mathcal{G} \mid \alpha \in P^{n,r} \}$ and extend $\pi_r, \lambda_r$ to $\mathfrak{P}^{n,r}$ by letting

$$\pi_r((\alpha, p)) = (\pi_r(\alpha), |\lambda_r(\alpha)|^{-\frac{1}{2}}p') \in \mathfrak{P}^{r}$$

$$\lambda_r((\alpha, p)) = \lambda_r(\alpha) \in \mathbb{Q}^*$$

where $p'(z) = p((\frac{z}{w}, \frac{z'}{w}))$ does not depend on the choice of $w, z'$.

Supposing $0 \leq r \leq n$, $\Gamma \leq \mathcal{G}^n$ is a congruence subgroup, $\psi$ is a Hecke character, and $K$ is some number field, then for $X \in \{M, S\}$ we denote

$$\mathcal{X}_X^r(\Gamma \cap \mathfrak{P}^{n,r}, \psi) = \{ f \in \mathcal{X}_X^r \mid f \|_k \pi_r(\gamma) = (\text{sgn} |k|^r |\psi(\gamma)|^{-1})(\lambda_r(\gamma))f \text{ for all } \gamma \in \Gamma \cap \mathfrak{P}^{n,r} \}$$

and by $\mathcal{X}_X^r(\Gamma \cap \mathfrak{P}^{n,r}, \psi, K)$, to be forms of the above set with coefficients in $K$. We are now ready to define a certain class of Eisenstein series, the so-called Klingen Eisenstein series. If $0 \leq r \leq n$ and $f \in S_k^r(\Gamma \cap \mathfrak{P}^{n,r}, \psi)$ then in [Sh95a, pp. 547, 554] the Eisenstein series $E_k^{n,r}(z; s; f, \psi, \Gamma)$ is defined for $z \in \mathbb{H}_n$ and $s \in \mathbb{C}$, and is convergent for $\Re(2s) > n + r + 1$. Of more interest to us are the series

$$E_k^{n,r}(z; f, \psi, \Gamma) := E_k^{n,r}(z, \frac{k}{2}; f, \psi, \Gamma) = \sum_{\gamma \in (\Gamma \cap \mathfrak{P}^{n,r}) \backslash \Gamma} \psi_\chi(|a_\gamma|)f\left(z^{(\gamma)}\right)$$

where $z^{(\gamma)}$ is the upper left $r \times r$ block of $z$, and this is convergent provided $k > n + r + 1$. In [Sh96, p. 356] this was extended to all $k > \frac{n + r + 3}{2}$. Put $E_k^{n,r}(z; f, \Gamma) := E_k^{n,r}(z; f, 1, \Gamma)$. By Lemma 8.11 of [Sh95a] these are holomorphic if $k > 2n$. 20
Assume that $k > 2n$. The span of all such Eisenstein series is a space that will play a large role in this section and is denoted

$$E_{n,r}^n := \text{span}_C \{ E_{k}^{n,r}(z; f, \Gamma) | k \alpha | \alpha \in \mathfrak{S}^n, f \in \mathfrak{S}(\Gamma \cap \mathfrak{P}^{n,r}, 1), \Gamma \}.$$ 

As $E_{n,r}^{n,r}(z; f, \Gamma) = f$, we have $E_{n,r}^{n,n} = \mathfrak{S}^n$. Set $E_{k}^{n,r}(\Gamma, \psi) = E_{k}^{n,r} \cap \mathcal{M}^n_k(\Gamma, \psi)$ for any congruence subgroup $\Gamma$ and Hecke character $\psi$. Their eminence in this section comes from a convenient decomposition, in their terms, of the space of modular forms.

**Theorem 6.3** (Shimura, [Sh95a], pp. 581–582). Let $k > 2n$ be a half-integral weight. Then we have the decompositions

$$\mathcal{M}^n_k(\Gamma, \psi) = \bigoplus_{r=0}^{n} E_{k}^{n,r}(\Gamma, \psi).$$

**Remark 6.4.** Originally in [Sh95a], the above theorem was proven for the bound $k > 2n$; this bound was further improved in [Sh96], p.346. We retain those of the former as later results will require us to take this bound regardless.

For any integral ideal $\mathfrak{a}$ we let $\mathcal{R}_n^0$ (resp. $\mathcal{R}_n(\mathcal{O}, \mathfrak{S}_0)$) denote the subspace of $\mathcal{R}_0$ (resp. $\mathcal{R}(\mathcal{O}, \mathfrak{S}_0)$) generated by all $A_{\mathfrak{a}q}$ (resp. $T_{q,\psi}$) with $q_\mathfrak{p} \in \mathfrak{X}_p$ for all $p$ and $q_\mathfrak{p} \in \mathfrak{O}_p$ if $p \mid \mathfrak{a}$.

**Theorem 6.5.** Let $r, r' \in \mathbb{Z}$, $0 \leq r' < r \leq n$, and assume $[k] > \frac{n}{2} + r' + 1$. Consider two non-zero Hecke eigenforms $f \in E_{k}^{n,r}(\Gamma, \psi), f' \in E_{k}^{n,r'}(\Gamma, \psi)$ with the same eigenvalues for $\mathcal{R}_n^0$. Then $r = r'$.

**Proof.** As in [Harr81, p. 309] we may assume that $r = n$ and therefore that $f$ is a cusp form. Assume for a contradiction that $r' < r$. Since $f$ and $f'$ share the same eigenvalues for $\mathcal{R}_n^0$ we have $L^\psi_n(s, f, \eta) = L^\psi_n(s, f', \eta)$ where we have set $\eta := \psi^{-2}$. The relation (0.8) obtained at the end of the last subsection then gives

$$L^\psi_n(s, f, \eta) = L^{\psi'}_f(s - n + r', \Phi^{n-r'}f', \eta) \prod_{i=0}^{n-r'-1} L(s + [k] - 2n + i, \eta) \zeta(s - [k] - i).$$

Plug $s = [k] + n - r'$ into this; for $i = n - r' - 1$ we have that $\zeta(s - [k] - i) = \zeta(1)$ is a pole; $\zeta(s - [k] - i) \neq 0$ for all other $i$ and $L(s + [k] - 2n + i, \eta) \neq 0$ for all $i$. By Theorem A in [Sh96, p. 332] $L^\psi_n(s, f, \eta)$ is absolutely convergent for $\Re(s) > \frac{3n}{2} + 1$ and by our choice of $s$ and $k$ we indeed have this. So the left-hand side is finite. In the same manner $L^{\psi'}_f(s', \Phi^{n-r'}f', \eta)$ is absolutely convergent for all $\Re(s') > \frac{3n}{2} + 1$, which inequality $s' = s - n + r'$ satisfies by our choice of $s$ and $k$. Thus $L^{\psi'}_f(s - n + r', \Phi^{n-r'}f', \eta)$ is non-zero. We arrive then at a contradiction, as the right-hand side of this expression contains a pole, yet the left does not. So $r' = r$. 

For any $0 \leq r \leq n$ we let $X_r = \mathfrak{P}^{n,r} / \mathfrak{S}^n \Gamma$ be representatives for the $r$-dimensional cusps. For notational purposes let

$$\Phi_{\xi} f = \Phi(f \| \xi^{-1})$$

for any $\xi \in X_{n-1}$ and $f \in \mathcal{M}^n_k(\Gamma)$. Then we define

$$\Phi_{\psi} : \mathcal{M}^n_k(\Gamma, \psi) \to \bigoplus_{\xi \in X_{n-1}} \mathcal{M}^{n-1}_k(\xi \Gamma \xi^{-1} \cap \mathfrak{P}^{n,n-1}, \psi)$$

$$f \mapsto (\Phi_{\xi} f)_{\xi}$$

21
and by definition \( \ker(\Phi_{\ast}) = S^c_k(\Gamma, \psi) \).

**Lemma 6.6.** If \( f \in M_k(\Gamma, \psi) \) then \( (f||_{k\xi^{-1}})|A_q = (f|A_q)||_{k\xi^{-1}} \) for any \( \xi \in X_{n-1} \) and any \( A_q \in R^c_0 \).

**Proof.** Since \( \xi \) is the identity at finite places we have, for \( \sigma = \text{diag}[\tilde{q}, q] \) with \( \tilde{q} \in \hat{\mathbb{Q}} \),

\[
(\xi D\xi^{-1})\sigma(\xi D\xi^{-1}) = \xi Sp_n(\mathbb{R})\xi^{-1} \times \prod_{p|c} D_p\sigma_pD_p.
\]

From this \( G \cap (\xi D\xi^{-1})\sigma(\xi D\xi^{-1}) = \xi G\xi^{-1} \cap (D\sigma D) = \xi (\Gamma\beta\Gamma)\xi^{-1} \) for some \( \beta \in G \cap \mathbb{Z} \). Supposing that \( \Gamma\alpha \) are the single cosets in \( \Gamma\beta\Gamma \), then we see that \( (\xi G\xi^{-1})(\xi\alpha\xi^{-1}) \) are the single cosets of \( G \cap (\xi D\xi^{-1})\sigma(\xi D\xi^{-1}) \). Note that \( \xi \in D[2, 2] \) and that

\[
(f||_{k\xi^{-1}})|A_q = \sum_{\alpha} j^{k}(\xi^{-1}, \xi\alpha\xi^{-1}z^{-1})^{-1}J^{k}(\xi\alpha\xi^{-1}, z^{-1})^{-1} f((\alpha\xi^{-1}) \cdot z)
\]

\[
(f|A_q)||_{k\xi^{-1}} = j^{k}(\xi^{-1}, z^{-1})^{-1} \sum_{\alpha} J^{k}(\alpha, \xi^{-1}z^{-1})^{-1} f((\alpha\xi^{-1}) \cdot z).
\]

So all that remains is to show that

\[
j^{k}(\xi^{-1}, \xi\alpha\xi^{-1}z^{-1})^{-1}J^{k}(\xi\alpha\xi^{-1}, z^{-1})^{-1} = j^{k}(\xi^{-1}, z^{-1})^{-1}J^{k}(\alpha, \xi^{-1}z^{-1})^{-1},
\]

and this follows by various properties of the factors of automorphy involved (\( Sh95b \) (1.9c), (2.1a)] and the usual cocycle relation). \( \square \)

**Proposition 6.7.** The space \( M_k(\Gamma, \psi) \) has a basis consisting of eigenforms for the space \( R^c_0 \).

**Proof.** This is adapted from \( And74 \), Theorem 1.3.4. By \( Sh95b \) Lemma 4.5 we have that \( T_q, \psi \) (or \( A_q \)) is Hermitian on cusp forms provided \( q_p \in \mathbb{O}_p \) for \( p \mid c \). From this it follows immediately that \( S^c_k(\Gamma, \psi) \) has a basis of eigenforms for \( R^c_0 \).

For the Eisenstein series part \( \mathcal{E}^n_k(\Gamma, \psi) \) we use induction on \( n \). By \( Kob84 \) p. 210 the space \( M^1_k(\Gamma, \psi) \) has a basis of eigenforms for \( R^n_0 \).

We make three claims, which hold for all \( 1 \leq n \in \mathbb{Z} \).

(1) The space \( \mathcal{E}^n_k(\Gamma, \psi) \) is invariant under \( (R^n_0)^c \).

(2) There exists an epimorphism \( (R^n_0)^c \to (R^{n-1})^c \); \( A \mapsto A^* \) such that \( \Phi_{\xi}(f|A) = (\Phi_{\xi})|A^* \) for all \( \xi \in X_{n-1} \), \( f \in M^c_k(\Gamma) \).

(3) The space \( \Phi_{\xi}\mathcal{E}^n_k(\Gamma, \psi) \) is invariant under \( (R^n_0)^c \) for all \( \xi \in X_{n-1} \).

Claim (1) follows from the self-adjointness cited in the first paragraph combined with the fact that cusp forms are readily seen to be preserved. Claim (2) is given by the local maps \( \Psi(\cdot, \psi(p)^{-1}p^{n-k}) \) combined with Lemma [6.6] above. Claim (3) follows from the previous two claims; indeed let \( A = A^*_0 \) for \( A \in (R^n_0)^c \) and \( A_0 \in (R^c_0)^c \), then

\[
(\Phi_{\xi}\mathcal{E}^n_k(\Gamma, \psi))|A = \Phi_{\xi}(\mathcal{E}^n_k(\Gamma, \psi)|A_0) \subseteq \Phi_{\xi}\mathcal{E}^n_k(\Gamma, \psi).
\]

So assume the proposition holds for \( n - 1 \). By the induction hypothesis we obtain, for each \( \xi \in X_{n-1} \), a basis of eigenforms for \( M^{n-1}_k(\xi\Gamma\xi^{-1} \cap p^{n,n-1}, \psi) \). Call this \( B_{\xi} \). Since the subspace \( \Phi_{\xi}\mathcal{E}^n_k(\Gamma, \psi) \subseteq M^{n-1}_k(\xi\Gamma\xi^{-1} \cap p^{n,n-1}, \psi) \) is invariant under \( (R^{n-1})^c \) we can obtain (from \( B_{\xi} \)) a basis \( \mathcal{C}_{\xi} \) for \( \Phi_{\xi}\mathcal{E}^n_k(\Gamma, \psi) \) consisting of eigenforms of \( (R^{n-1})^c \). Let \( \mathcal{C}_X \) denote the resultant product basis of \( \Phi_{\ast}\mathcal{E}^n_k(\Gamma, \psi) \). As \( \ker(\Phi_{\ast}) = S^c_k(\Gamma, \psi) \) we have that \( \Phi_{\ast} \) is injective on \( \mathcal{E}^n_k(\Gamma, \psi) \) so that the inverse image of \( \mathcal{C}_X \), call it \( \mathcal{C}_\ast \), gives a basis for \( \mathcal{E}^n_k(\Gamma, \psi) \).
Finally if \( E \in C_* \) then we claim that it is an eigenform for \( (\mathcal{R}_0^n)^t \). By definition \( \Phi \cdot E \) is 0 at all places except one \( \xi_0 \), at which it is some eigenform, say \( F_{\xi_0} \) with eigenvalues \( \Lambda \). Then if \( A_q \in (\mathcal{R}_0^n)^t \) we have
\[
\Phi_{\xi_0}(E|A_q) = (\Phi_{\xi_0}E)|A_q = F_{\xi_0}|A_q = \Lambda(A_q)F_{\xi_0}
\]
so that \( \Phi(E|A_q) = \Lambda(A_q)\Phi(E) \). By injectivity of \( \Phi_* \) we are done.

**Proposition 6.8.** Let \( V \subseteq \mathcal{M}_k(\Gamma, \psi) \) be an eigenspace for \( \mathcal{R}_0^n \) with eigenvalues given by \( \Lambda \), then it is spanned by \( V \cap M_k^n(\mathbb{Q}(\psi, \Lambda)) \).

**Proof.** Write
\[
V = M_k^n(\Gamma, \psi, \Lambda) := \{ f \in M_k^n(\Gamma, \psi) \mid f|A = \Lambda(A)f \text{ for all } A \in \mathcal{R}_0^n \}
\]
From Theorem 10.7 of [Sh00, p. 64] we have \( M_k^n(\Gamma, \psi) = M_k^n(\Gamma, \psi, \mathbb{Q}(\psi_1)) \otimes_{\mathbb{Q}} \mathbb{C} \) and by Lemma 5.1 in [Boug18] the action of \( \mathcal{R}_0^n \) preserves \( M_k^n(\Gamma, \psi, \mathbb{Q}(\psi_1)) \). So using this rational basis we can see that \( M_k^n(\Gamma, \psi, \Lambda)^\sigma = M_k^n(\Gamma, \psi, \Lambda\sigma) \) for any \( \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\psi_1)) \). It follows that any \( \mathbb{C} \)-basis of \( V \) will be fixed pointwise by any element \( \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\psi_1, \Lambda)) \), and so each basis element has coefficients in \( \mathbb{Q}(\psi, \Lambda) \).

So, if \( k > 2n \), we obtain an equality of two different direct sums for \( M_k^n(\Gamma, \psi) \). From Proposition 6.7 one of these consists of eigenspaces for \( \mathcal{R}_0^n \), and by Theorem 6.3 the other one consists of \( \mathcal{E}_k^{n,r}(\Gamma, \psi) \). By Theorem 6.5 we have that \( \mathcal{E}_k^{n,r}(\Gamma, \psi) \) contains entire eigenspaces for \( \mathcal{R}_0^n \). So by the basic properties of direct sums we see that each \( \mathcal{E}_k^{n,r}(\Gamma, \psi) \) is itself a direct sum of eigenspaces.

For any character \( \psi \) let \( \Lambda_{k,\psi} = \Lambda_k^n \subseteq \text{Hom}(\mathcal{R}_0^n, \mathbb{C}) \) be the finite subset such that
\[
M_k^n(\Gamma, \psi, \Lambda_{k,\psi}) = \bigoplus_{\Lambda \in \Lambda_{k,\psi}} M_k^n(\Gamma, \psi, \Lambda),
\]
let \( \mathbb{Q}(\Lambda_{k,\psi})/\mathbb{Q} \) be the field generated by all the values of \( \Lambda \), for all \( \Lambda \in \Lambda_{k,\psi} \). The above discussion in conjunction with Proposition 6.8 gives the following result for \( 0 \leq r \leq n - 1 \) (the cusp form case \( r = n \) is already known).

**Corollary 6.9.** Let \( 0 \leq r \leq n \) be integers and assume that \( k > 2n \). Then \( \mathcal{E}_k^{n,r}(\Gamma, \psi) \) is spanned by \( \mathcal{E}_k^{n,r}(\Gamma, \psi, \mathbb{Q}(\psi, \Lambda_{k,\psi})) := \mathcal{E}_k^{n,r}(\Gamma, \psi) \cap M_k^n(\mathbb{Q}(\psi, \Lambda_{k,\psi})) \).

We need such an algebraic basis at other cusps as well. Let \( \zeta_m := e^{2\pi i \frac{m}{N(m)}} \) denote the \( N(m) \)th root of unity for an integral ideal \( m \) and recall \( \zeta : \mathbb{M} \to \mathbb{T} \) as the character, see property 2.1 such that \( h(\sigma, z)^2 = \zeta(\sigma)j(\text{pr}(\sigma), z) \). Let \( \zeta_* = \zeta|_X \) where \( X = \bigcup_r X_r \).

**Theorem 6.10.** Let \( K/\mathbb{Q} \) be an algebraic field extension and let \( f \in M_k^n(\mathbb{G}, K) \). Then for all \( 0 \leq r \leq n \) and all \( \xi \in X_r \) we have \( f|\xi^{-1} \in M_k^n(\mathbb{G}(\xi)\xi^{-1}, K(\zeta, \zeta_*)) \).

**Proof.** In the integral weight case \( -\ell \in \mathbb{Z} \) and \( g \in M_k(\mathbb{G}, K) \) – Proposition 1.8 in [FC80, p. 146] gives \( g|\xi^{-1} \in M_k(\mathbb{G}(\xi)\xi^{-1}, K(\zeta_*)) \). The half-integral case can be deduced from this via the use of the theta series \( \theta(z) := \sum_{\gamma \in \mathbb{Z}^n} e(\frac{z}{\gamma}) \). This belongs to \( M_\frac{3}{2}(\mathbb{Q}) \) and, by the second equation of Proposition 1.3 in [Sh93], we have that \( \theta|\xi^{-1} = \theta \) has rational coefficients for each \( \xi \).

Take \( f \) as stated in the theorem, then \( \theta f \) has integral weight \( k + \frac{1}{2} = [k + 1] \) and it has
coefficients in $K$. So $(\theta f)|_{[k+1]}\xi^{-1}$ has coefficients in $K(\zeta)$ for any $\xi \in X_r$. We get

$$(\theta f)|_{[k+1]}\xi^{-1}(z) = j(\xi^{-1}, z)^{-[k+1]}\theta(\xi^{-1}z)f(\xi^{-1}z)$$

$$= j(\xi^{-1}, z)^{-1}h(\xi^{-1}, z)^2(\theta|_{[k]}\xi^{-1})(z)(f|_{k}\xi^{-1})(z)$$

$$= \zeta(\xi^{-1})\theta(z)(f|_{k}\xi^{-1})(z)$$

using property 2.11 of the factor $h(\sigma, z)$ in the last line, and the definition of the slash operator for half-integral $k$ along the way. Considering $\theta$ as an element of $\mathbb{Q}[[q]]$ with $q = e^{2\pi i}$ then it is an invertible power series since it has a non-zero constant coefficient. So considering $\theta^{-1} \in \mathbb{Q}[[q]]$ we have

$$f|_{k}\xi^{-1} = \zeta(\xi)\theta^{-1}(\theta f)|_{[k+1]}\xi^{-1} \in \mathcal{M}_k(\xi\Gamma\xi^{-1}, K(\zeta_*, \zeta_*)).$$

\[ \square \]

Remark 6.11. For certain congruence subgroups one can remove the $\zeta_*$. For example, if $\Gamma$ has cusps only at 0 and $\infty$ then $X = \{i2n, i\}$. In this case $\zeta(i) = (-i)^n$ by Proposition 1.1.R of [Sh93] and we see that $\mathbb{Q}(\zeta_*) \subseteq \mathbb{Q}(\zeta)$ since $4 \mid c$. In general, however, it seems to be a necessary addition – something which can be seen by Proposition 1.4 of [Sh93] and the subsequent paragraph detailing this proposition’s non-triviality in contrast to the integral weight case.

Corollary 6.12. Let $0 \leq r \leq n$ be integers and $k > 2n$. Then $\mathcal{E}^{n,r}_k(\xi\Gamma\xi^{-1}, \psi)$ is spanned by $\mathcal{E}^{n,r}_k(\xi\Gamma\xi^{-1}, \psi, \mathbb{Q}(\Lambda_k, \psi, G(\psi), \zeta_*))$.

Proof. As $\mathcal{E}^{n,r}_k(\xi\Gamma\xi^{-1}, \psi) = \mathcal{E}^{n,r}_k(\Gamma, \psi)|_{k}\xi^{-1}$ then this follows from Corollary 6.9 and Theorem 6.10 above.

\[ \square \]

The previously defined map $\Phi_*$ provides a useful isomorphism from which we can determine the rationality of the of $\Phi_*f$, given that of $f$.

Theorem 6.13 ([Sh5a], p. 582; [Sh6], p. 347). Let $k > 2n$ and fix $r < n$. Then

$$\Phi_*^{n-r} : \mathcal{E}^{n,r}_k(\Gamma, \psi) \rightarrow \prod_{\xi \in X_r} S^r_k(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,r}, \psi)$$

is a $\mathbb{C}$-linear isomorphism.

Corollary 6.14. If $f \in \mathcal{E}^{n,r}_k(\Gamma, \psi)$ with $k > 2n$, then $f \in \mathcal{M}_k(\Gamma, \mathbb{Q}(\Lambda_k, \psi, G(\psi), \zeta_*))$ if and only if

$$\Phi_*^{n-r}f \in \prod_{\xi \in X_r} S^r_k(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,r}, \psi, \mathbb{Q}(\Lambda_k, \psi, G(\psi), \zeta_*)).$$

Proof. Theorem 6.10 and the fact that $\Phi_*^{n-r}(\mathcal{M}^{n,r}_k(\Gamma, \psi, L)) \subseteq \mathcal{M}_k^r(\Gamma, \psi, L)$ for any subfield $L \subseteq \mathbb{C}$ gives necessity.

For sufficiency, let $\{g_1^n, \ldots, g_m^n\}$ be a $\mathbb{Q}(\Lambda_k, \psi, G(\psi), \zeta_*)$-rational basis for $\mathcal{E}^{n,r}_k(\Gamma, \psi)$. By Corollary 6.12 there also exists a $\mathbb{Q}(\Lambda_k, \psi, G(\psi), \zeta_*)$-rational basis for each $S^r_k(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,r}, \psi)$, so let $\{g_1^n, \ldots, g_m^n\}$ be the product basis for $\prod_{\xi \in X_r} S^r_k(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,r}, \psi)$ obtained out of this. Thus $g_i^n$ is 0 for all of $X_r$ except for one $\xi$ whereby it is some element of $S^r_k(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,r}, \psi, \mathbb{Q}(\Lambda_k, \psi, G(\psi), \zeta_*)$. Assume further that it is ordered so that $\Phi_*^{n-r}(g_i^n) = g_i^n$ for all $i$. Writing $f = \sum_{i=1}^m \alpha_i g_i^n$ then we claim that $\alpha_i \in \mathbb{Q}(\Lambda_k, \psi, G(\psi), \zeta_*)$. By assumption

$$\Phi_*^{n-r}f = \sum_{i=1}^m \alpha_i g_i^n \in \prod_{\xi \in X_r} S^r_k(\xi\Gamma\xi^{-1} \cap \mathfrak{P}^{n,r}, \psi, \mathbb{Q}(\Lambda_k, \psi, G(\psi), \zeta_*)).$$
If $m = 1$ and assuming $f \neq 0$ then by Lemma 8.2 (2) and Lemma 8.11 (4) in [Sh95a] there exists $\xi \in X_r$ whereby $\Phi^{\alpha} (f||_k \xi^{-1}) \neq 0$ and thus $\Phi^{\alpha} (g_1^n ||_k \xi^{-1}) \neq 0$. Then as $\alpha_1 = (g_1^n)^{-1} \Phi^{\alpha} f$ we immediately see that $\alpha_1 \in \mathbb{Q} (L_k, G(\psi), \zeta_s)$. The rest follows by induction on $m$. 

All of the above results allow us to now prove a particular case of Garrett’s conjecture in Theorem 6.17 below.

**Lemma 6.15** (Shimura, [Sh95a], 5.78). If $f \in \mathcal{S}^n_k (\Gamma, \psi)$, $\xi \in X_r$, and $k > n + r + 1$ then we have

$$\Phi^{\alpha} (f; \xi, G(\psi)) = \Phi^{\alpha} (f; \xi, G(\psi), \zeta_s).$$

**Remark 6.16.** The above lemma is given in [Sh95a] with trivial character and the proof then follows directly from Lemma 8.5 of that paper. This lemma (8.5) clearly applies for non-trivial character Klingen Eisenstein series, hence the above formulation.

For any $f \in \mathcal{E}^{n,r}_k (\Gamma, \psi)$ and any $0 \leq r \leq n$ define

$$\mathcal{F}^{n,r}_k (z; f, \psi, \Gamma) := \sum_{\xi \in X_r} E^{n,r}_k (z; \Phi^{\alpha}_\xi f, \psi, \xi \Gamma \xi^{-1}) \xi.$$

**Theorem 6.17.** Let $0 \leq r \leq n$ and $f \in \mathcal{E}^{n,r}_k (\Gamma, \psi, \mathbb{Q} (\Lambda_k, \psi, G(\psi), \zeta_s))$ with $k > n + r + 1$. Then $\mathcal{F}^{n,r}_k (z; f, \psi, \Gamma) \in \mathcal{E}^{n,r}_k (\Gamma, \mathbb{Q} (\Lambda_k, \psi, G(\psi), \zeta_s)).$

**Proof.** For any $\nu \in X_r$ we have $\Phi^{\alpha} \mathcal{F}^{n,r}_k (z; f, \psi, \Gamma) = \Phi^{\alpha} f$ by Lemma 6.15. By Theorem 6.10 $\Phi^\alpha f$ has coefficients in $\mathbb{Q} (\Lambda_k, \psi, G(\psi), \zeta_s)$ for each $\nu \in X_r$. If $0 \leq r \leq n - 1$ then by Corollary 6.14 $\mathcal{F}^{n,r}_k (z; f, \psi, \Gamma)$ also has coefficients in $\mathbb{Q} (\Lambda_k, \psi, G(\psi), \zeta_s)$. If $r = n$ then this is given immediately by Theorem 6.10.

Let $\mathcal{E}^{n}_k := \prod_{r=0}^{n-1} \mathcal{E}^{n,r}_k$. The decomposition $\mathcal{M}^{n}_k (\Gamma, \psi) = \mathcal{S}^{n}_k (\Gamma, \psi) \oplus \mathcal{E}^{n}_k (\Gamma, \psi)$ of Theorem 6.3 is proven inductively and each step involves the use of the Eisenstein series $\mathcal{F}^{n,r}_k$. Observing this proof along with Theorem 6.17 gives the following corollary.

**Corollary 6.18.** Assume that $k > 2n$. Then

$$\mathcal{M}^{n}_k (\Gamma, \psi, \mathbb{Q} (\Lambda_k, \psi, G(\psi), \zeta_s)) = \mathcal{S}^{n}_k (\Gamma, \psi, \mathbb{Q} (\Lambda_k, \psi, G(\psi), \zeta_s)) \oplus \mathcal{E}^{n}_k (\Gamma, \psi, \mathbb{Q} (\Lambda_k, \psi, G(\psi), \zeta_s)).$$

**Proof.** That follows easily from Theorem 6.17 above will be clear after outlining the proof, taken from [Sh95a] pp. 581 – 582 of the decomposition of Theorem 6.3. Let $f \in \mathcal{M}_k (\Gamma, \psi)$ such that $\lambda_r (\Gamma \cap \mathcal{P}^{n,r}) |k| = 1$ for all $r$. Put $f_0 = \mathcal{F}^{n,0}_k (z; f, \psi, \Gamma)$. Then $\Phi^\alpha (f - f_0) = 0$ by Lemma 6.15 so that $\Phi^{\alpha} (f - f_0)$ is a cusp form for each $\nu$. Then put $f_1 := \mathcal{F}^{n,1}_k (z; f - f_0, \psi, \Gamma)$ and repeat the above procedure to get $f_2 = \mathcal{F}^{n,2}_k (z; f - f_0 - f_1, \psi, \Gamma)$ and so on. At the final step we obtain $0 = \Phi^\alpha (f - f_0 - f_1 - \cdots - f_n) = f - f_0 - \cdots - f_n$ and this gives Theorem 6.3. So we see that if $f$ has coefficients in $\mathbb{Q} (\Lambda_k, G(\psi), \zeta_s)$ then, by Theorem 6.3 so do each of $f_0, f_1, \ldots, f_n$.

**7. Special values**

The results of the previous section allow more special values to be determined via the method used earlier. This is done by relaxing growth conditions on holomorphic projection. We make use of the notation $z^{-k-|2s|} := z^{-k}|z|^{-2s}$. 25
**Definition 7.1.** If $F \in C_k^\infty(\Gamma, \psi)$ then we say that $F$ is of moderate growth if, for all $z \in \mathbb{H}_n$ and sufficiently large $\Re(s) \gg 0$, we have that the integral

$$
\int_{\mathbb{H}_n} f(w)|\bar{w} - z|^{-k-|2s|}\Delta(w)^{k+s}d^{\times}w
$$

is absolutely convergent and admits an analytic continuation over $s$ to the point $s = 0$.

Now forms of moderate growth are sent by the projection of Theorem 3.1 to $\mathcal{M}_k$ instead of $\mathcal{S}_k$.

**Theorem 7.2.** Let $F \in C_k^\infty(\Gamma, \psi)$ have Fourier coefficients $a(\tau, y)$ for $\tau \in \mathfrak{ba}$, where $k > 2n$. Assume that $F$ is of moderate growth. Then with $c(k, n), a(\tau)$, and the projection map defined as in Theorem 3.1, we have $P(F) \in \mathcal{M}_k(\Gamma, \psi)$. Furthermore $\langle F, g \rangle = \langle P(F), g \rangle$ for any $g \in \mathcal{S}_k(\Gamma, \psi)$ and $\Gamma' \leq \Gamma$ such that $[\Gamma : \Gamma'] < \infty$.

The proof of this is also given by the study of a certain Poincaré series which is defined for variables $z, w \in \mathbb{H}_n$ and $s \in \mathbb{C}$ as

$$
P(z, w, s) := (\Delta(z)\Delta(w))^s \sum_{\gamma \in \Gamma} \psi^{-1}(|a_\gamma|) j^k_\gamma(z)^{-1}|j(\gamma, z)|^{-2s}|\gamma z + w|^{-k-|2s|}.
$$

This converges absolutely and uniformly on products $V(d) \times V(d)$ for $\Re(2s) > 2m - k + 1$, $d > 0$, and $V(d) := \{z \in \mathbb{H}_n \mid y > dI_n, \text{tr}(a^T x) \leq d^{-1}\}$, see [Pan91, p. 72]. This series has been altered from the definition of the integral weight version found in [Pan91] only by the change in the factor of automorphy $j^k_\gamma(z)^{-1}$. Once it is shown that this series exhibits the analogous three properties to (4.9), (4.10), and (4.12) of [Pan91] p. 72, then the proof of Theorem 7.2 follows precisely as is found there.

**Proposition 7.3.** For any $\gamma \in \Gamma$ let $\gamma' := (I_n \ 0 \ -I_n \ 0 \ -I_n)$, Then

$$
j^k_\gamma(z)|\gamma z + w|^k = j^k_{\gamma'}(w)|\gamma' w + z|^k.
$$

**Proof.** It is easy to see that $|j(\gamma, z)||\gamma z + w|^\kappa| = |j(\gamma', w)||\gamma' w + z|^\kappa|$ for any $\kappa \in \frac{1}{2}\mathbb{Z}$. We claim that

$$
\frac{h_{\gamma'}(w)}{|h_{\gamma'}(w)|} = \frac{h_\gamma(z)}{|h_\gamma(z)|} \in \mathbb{T}.
$$

These are constants independent of $w$ and $z$ respectively. Lemma 2.2 of [Sh93] tells us that

$$
\lim_{z \to 0} \frac{h_\gamma(z)}{|h_\gamma(z)|} = g(d^{-1}_\gamma c_\gamma)
$$

$$
\lim_{w \to 0} \frac{h_{\gamma'}(w)}{|h_{\gamma'}(w)|} = g(a^{-1}_\gamma c_\gamma)
$$

where $g(s) = \gamma(s)|\gamma(s)|^{-1}$ and $\gamma(s) = \prod_p \int_{\mathbb{Z}_p} e_p\left(\frac{z x + t}{2}\right) d_p x$ for any symmetric matrix $s$. For a symmetric matrix $s \in S_p$ there exists $u \in GL_n(\mathbb{Z}_p)$ such that $usu^T$ is diagonal – see, for example, Lemma A1.5 of [Sh00]. Therefore, in the calculation of $g(y^{-1}_\gamma c_\gamma)$ for $y \in \{a, d\}$, we may assume that $c_\gamma = \text{diag}[c_1, \ldots, c_n]$ and $y_\gamma = \text{diag}[y_1, \ldots, y_n]$, and we obtain

$$
\gamma(y^{-1}_\gamma c_\gamma) = \prod_p \prod_{\vert y_p \vert} n \frac{\text{ord}_p(y_p)}{2} c_p^{\text{ord}_p(y_p)}\left(\frac{c_p}{p}\right)^{\text{ord}_p(y_p)} = |y_\gamma|^{-\frac{1}{2}}\pi_{y_\gamma}\left(\frac{|c_\gamma|}{|y_\gamma|}\right)
$$
where \( \pi_{\gamma'} = \prod_{p \mid \eta_{\gamma}} \prod_{\nu=1}^n \epsilon_{p^{ou(p_{\nu})}} \) and from which \( g(y_{\gamma}^{-1}c_{\gamma}) = \pi_{\gamma'} \left( \frac{|c_{\gamma}|}{|y_{\gamma}|} \right) \). If two numbers \( m \) and \( n \) satisfy \( mn \equiv 1 \pmod{4} \) then the primes occurring in their decomposition that are \( 3 \pmod{4} \) must occur with the same multiplicity. Since \( |a_{\gamma}d_{\gamma}| \equiv 1 \pmod{4} \) we obtain \( \pi_{d_{\gamma}} = \pi_{a_{\gamma}} \). Now suppose that \( -1 = \left( \frac{|a_{\gamma}d_{\gamma}|}{|c_{\gamma}|} \right) \), then

\[
-1 = \left( \frac{|a_{\gamma}d_{\gamma}|}{|c_{\gamma}|} \right) = \left( \frac{1}{|c_{\gamma}|} \right)
\]

which is a contradiction. So \( \left( \frac{|c_{\gamma}|}{|a_{\gamma}|} \right) = \left( \frac{|c_{\gamma}|}{|a_{\gamma}|} \right) \) and we see that \( g(d_{\gamma}^{-1}c_{\gamma}) = g(a_{\gamma}^{-1}c_{\gamma}) \). This gives the claim, and so

\[
\frac{h_{\gamma}(z)}{|h_{\gamma}(z)|} \left| j(\gamma, z)\gamma z + w \right|^{\frac{1}{2}} = \frac{h_{\gamma}(w)}{|h_{\gamma}(w)|} \left| j(\gamma', w)\gamma' w + z \right|^{\frac{1}{2}}
\]

which gives the proposition.

The above proposition proves the first two of the following three properties

\[
P(z, w, s) = P(w, z, s) \tag{7.1}
\]

\[
P(\gamma_1 z, \gamma_2 w, s) = \psi_{\epsilon}(a_{\gamma_1}a_{\gamma_2})j_{\gamma_1}(z)j_{\gamma_2}(w)P(z, w, s) \tag{7.2}
\]

\[
(F(w), P(-\bar{z}, w, s)) = \mu F(z) \tag{7.3}
\]

for any \( F \in \mathcal{C}_k^{\infty}(\Gamma, \psi) \) such that the integral of \( \mathbf{7.3} \) converges, and for some constant \( \mu \) given in \([\text{Pan91}]\ p. 73\). By definition the left-hand side of \( \mathbf{7.3} \) is

\[
(\mathbf{7.3}) = \int_{\Gamma \backslash \mathbb{H}} \sum_{\gamma \in \Gamma} \psi_{\epsilon}(a_{\gamma})F(z)\left| j(z)^{-1}j(\gamma, z)^{-2s} \gamma \bar{z} - w \right|^{-k-2s} \Delta(z)^{k+s} d^\times z.
\]

Now use that \( \psi_{\epsilon}(a_{\gamma})F(z) = j_{\gamma}^{-k}(z)^{-1}F(\gamma \cdot z) \) and \( \Delta(z) = j(\gamma, z)j(\gamma, \bar{z})\Delta(\gamma z) \) to get

\[
(\mathbf{7.3}) = \left( \frac{-1}{n} \right)^s \int_{\Gamma \backslash \mathbb{H}} \sum_{\gamma \in \Gamma} F(\gamma w)\left| \gamma w - z \right|^{-k-2s} \Delta(\gamma z)^{k+s} d^\times z.
\]

which is exactly of the form found in (4.14) of \([\text{Pan91}]\ p. 73\). So the rest of that proof using Cayley transforms applies, and we get property \( \mathbf{7.3} \). Note that the above integral is convergent and has analytic continuation to \( s = 0 \) precisely when \( F \) is of moderate growth.

To finish the proof of the projection in this case, set \( K(z, w, s) := \mu^{-1}P(-\bar{z}, w, s) \) and then define \( P(F)(z) := \langle F(w), K(z, w, s) \rangle_{s=0} \). The reader is referred to \([\text{Pan91}]\ pp. 74–75\) for the details here.

**Proposition 7.4.** Let \( k \) be a half-integral weight, \( \ell \in \left\lfloor \frac{1}{2} \right\rfloor \mathbb{Z} \), and \( a > \frac{\ell + n + 1}{2} \). If \( g \in \mathcal{M}_\ell(\Gamma, \psi) \) then \( F^*(z) := g(z)H_{k-\ell}(z, a) \) is of moderate growth provided

\[
\ell - n - nk - 2 < a < nk - k + 2 + n.
\]

**Proof.** Set \( s = 0 \) in the integral characterising moderate growth in Definition 7.1. Fixing \( z \in \mathbb{H}_n \), then let \( w = x + iy \) with \( \lambda_j \) being the eigenvalues of \( y \). Notice that \( |\bar{w} - z| \) is a polynomial in \( x_{ij}, y_{ij} \) of degree \( n > 0 \) which is \( |iy + z| \) as \( x \to 0 \). Hence \( |\bar{w} - z|^{-k} \) decays as \( |x| \to \pm \infty \) and is finite as \( x \to 0 \). Then, by Corollary 3.3, we may write for some constant \( \nu \)

\[
\int_{\mathbb{H}_n} |F^*(w)||\bar{w} - z|^{-k}|y|^{k-n-1}dydx \leq \nu \int_{Y} f(y)|P(y)|^{-k}dy
\]
where $P(y)$ is a polynomial in $y_{ij}$ of degree $n$, and
\[
v(y) := \prod_{j=1}^{n} (1 - \lambda_j^{-\ell})(\lambda_j^2 + \lambda_j^{-a-k+\ell})\lambda_j^{k-1-n}.\]

Let $\bar{\Lambda} := \{\text{diag}[\lambda_1, \ldots, \lambda_n] \mid 0 < \lambda_1 \leq \cdots \leq \lambda_n\}$. As is done in the proof of Corollary 2 of \[\text{[SNS]}\] we may make the (unique up to multiplication by diag($\pm 1, \ldots, \pm 1$)) substitution $y = UAU^T$, where $U \in O_n(\mathbb{R})$ and $\Lambda \in \bar{\Lambda}$. The integral over $O_n(\mathbb{R})$ is evidently finite, so it is enough to show that
\[
\int_{\bar{\Lambda}} \tau(y) |P(\Lambda)|^{-k} |J(\lambda_1, \ldots, \lambda_n)| d\lambda_1 \cdots d\lambda_n < \infty
\]
where $J$ is the determinant of the jacobian matrix which is independent of $U$. To do this we check the limits $\lambda_j \to 0$ and $\lambda_j \to \infty$. Firstly, as $\lambda_j \to 0$, then $|P(\Lambda)|^{-k} \to \|z\|^{-k}$ is just finite so we require the exponent of each $\lambda_j$ to be greater than $-1$ (and $a > \frac{\ell-k+n+1}{2}$ in order for $H$ to be defined). This just gives us the original bounds found in Corollary 3.5 for bounded growth. For the limit $\lambda_j \to \infty$ we have that $|P(\Lambda)|^{-k}$ decays to order $nk$, so as long as the exponent of $\lambda_j$ in $\tau(y)$ is $\leq nk$ we obtain convergence. That is
\[
a + k - 1 - n \leq nk; \quad -a - k + \ell + k - 1 - n \leq nk,
\]
giving
\[
\ell - n - nk - 2 < a < nk - k + 2 + n.
\]

\[\Box\]

If $g \in \mathcal{M}_\ell(\Gamma, \psi')$ for an $\ell \in \frac{1}{2}\mathbb{Z}$ then define
\[
\Omega^+(g) := \{m \in \frac{1}{2}\mathbb{Z} \mid \frac{n-2m+2k-2\ell}{4} \in \mathbb{Z}, n < m \leq k - \ell + \frac{n}{2}\}, \quad \Omega^-(g) := \{m \in \frac{1}{2}\mathbb{Z} \mid \frac{2m-3n+2k-2\ell-2}{4} \in \mathbb{Z}, \frac{3n}{2} + 1 - k + \ell < m \leq n\},
\]
and put $\Omega(g) := \Omega^-(g) \cup \Omega^+(g)$.

**Proposition 7.5.** Exclude case \[\text{[X]}\]. Let $\ell \in \frac{1}{2}\mathbb{Z}$ and $g \in \mathcal{M}_\ell(\Gamma, \psi')$. Assume that $k > 2n$. In case \[\text{[R1]}\] set $m_0 := \frac{k+\ell-3}{2}$ and in case \[\text{[R2]}\] set $m_0 := \frac{2k+2\ell+n-1}{4} - \frac{n+1}{2}$. For all other cases
\[
m_0 := \begin{cases} 
\frac{2k+2\ell+2m-n}{4} - \frac{n+1}{2} & \text{if } m > n \\
\frac{2k+2\ell+3m-2n+2}{4} - \frac{n+1}{2} & \text{if } m \leq n.
\end{cases}
\]
For every $m \in \Omega(g)$ there exists $K_S(m, g) \in \mathcal{S}_k(\Gamma, \psi)$, whose Fourier coefficients belong to $\mathbb{Q}_{ab}(g, \Lambda_{k, \psi}, G(\psi), \zeta_*)$, such that
\[
\frac{(4\pi)^{nm_0}}{\pi^{\beta(m)}\omega(\ell, m, \psi')} \Gamma(n(m_0))^{-1} \langle f, g \mathcal{E}_{\psi'}(\cdot, \frac{2m-n}{4}) \rangle = \pi^{nk-\frac{an^2+4m+5}{4}} \langle f, K_S(m, g) \rangle
\]
for all $f \in \mathcal{S}_k(\Gamma, \psi)$. Moreover $K_S(m, g)^\sigma = K_S(m, g^\sigma)$ for all $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\Lambda_{k, \psi}, G(\psi), \zeta_*)$.

**Proof.** Much of this remains the same as the proof of Proposition 4.2. To apply holomorphic projection we ensure moderate growth of $g(z)H_{k-\ell}(z, -r)$ where $r = \frac{k-\ell}{2} - \frac{2m-n}{4} + 1$ in cases \[\text{[R1]}\] and \[\text{[R2]}\] and $r = \frac{k-\ell}{2} - \frac{2m-n}{4} - \frac{n+1}{4} - \frac{n+1}{4}$ otherwise. By definition of $\Omega(g)$ it is easy to check that $-r$ satisfies the bounds in Proposition \[\text{[7.3]}\] hence we have moderate growth. Moreover $\frac{2m-n}{4} \in \Omega_0$ – allowing the application of Theorem \[\text{[4.1]}\] and $k - r > n$ – allowing the application
of Lemma 4.3. The changes in the definition of $m_0$ is a result of the change to the order $r$ of the non-holomorphic Eisenstein series.

Therefore in applying holomorphic projection, and replicating the proof of Proposition 4.2 we obtain a holomorphic modular form $K(m, g) \in \mathcal{M}_k(\Gamma, \psi)$ with coefficients in $\mathbb{Q}_{ab}(g)$. By Corollary 5.18 this splits up as $K(m, g) = K_S(m, g) + K_E(m, g)$ where $K_S(m, g) \in \mathcal{X}_k(\Gamma, \psi)$, for $\mathcal{X} \in \{S, E\}$, has coefficients in $\mathbb{Q}_{ab}(g, \Lambda_k, G(\psi), \zeta_s)$. Since $\langle f, K_E(m, g) \rangle = 0$ we are done. □

Now set $\ell = \frac{n}{2} + \mu$ and assume $k > 2n$ in all cases. If $m \in \Omega^+(\theta)$ and we are not in cases [R1] or [R2] then we obtain the same integral expression 4.3 for $L_\psi(m, f, \eta)$. On the other hand, if [R1], [R2], or $m \in \Omega^-(\theta)$ then this will be slightly different since here the value $m_0$ required to apply Proposition 7.5 above is no longer occurring naturally from the original expression in 4.2.

If $m \in \Omega^-(\theta)$ then

$$m_0 = \frac{k+n+\mu-m}{2} = \left(\frac{m-n-1+k+\mu}{2}\right) + n - m + \frac{1}{2}$$

from which

$$(4\pi)^{n\left(\frac{m-n-1+k+\mu}{2}\right)} = (4\pi)^{nm_0}(4\pi)^{\frac{1}{2}(2m-2n-1)},$$

$$\Gamma_n \left(\frac{m-n-1+k+\mu}{2}\right) \Gamma_n(m_0)^{-1} \in \mathbb{Q}.$$ 

Therefore

$$L_\psi(m, f, \eta) \in 2^{-\frac{k}{2}}(2m-2n-1)\mu_f(\tau, 1)^{-1}|2\tau|^{\frac{3}{2}}\epsilon_\theta(m, \chi) \prod_{p \in \mathfrak{b}} g_p \left((\psi^\eta)(p)p^{-m}\right) \times \left(4\pi\right)^{nm_0} \Gamma_n(m_0)^{-1} \left(f, \theta \mathcal{E}(\cdot, \frac{2n-\mu}{4})\right) \mathbb{Q}$$

and, as before, multiplying both sides by $\pi^{-\beta(m)}\omega_\delta(m, \chi)^{-1}$ and applying Proposition 7.5 gives

$$\frac{2^{\frac{k}{2}}|2\tau|^{\frac{3}{2}}\mu_f(\tau, 1)L_\psi(m, f, \eta)}{\pi^{\beta(m)+n(k+m)-\frac{3n^2+4n+\delta}{4}}\omega_\delta(m, \chi)} \in \epsilon_\theta(m, \chi) \prod_{p \in \mathfrak{b}} g_p \left((\psi^\eta)(p)p^{-m}\right) \left(f, K_S(m, \theta)\right) \mathbb{Q}. \quad (7.6)$$

If we are in cases [R1] or [R2] then $(4\pi)^{n\left(\frac{m-n-1+k+\mu}{2}\right)} = (4\pi)^{nm_0+\frac{1}{2}}$ and rationality of the $\Gamma$-factors is, again, preserved. Hence

$$\frac{2^{\frac{k}{2}}|2\tau|^{\frac{3}{2}}\mu_f(\tau, 1)L_\psi(m, f, \eta)}{\pi^{\beta(m)+n(k+m)-\frac{3n^2+4n+\delta}{4}}\omega_\delta(m, \chi)} \in \epsilon_\theta(m, \chi) \prod_{p \in \mathfrak{b}} g_p \left((\psi^\eta)(p)p^{-m}\right) \left(f, K_S(m, \theta)\right) \mathbb{Q}. \quad (7.7)$$

We can also make some improvements on the bounds for $k$ in Theorem 5.1. Let

$$c(m) = \beta(m) + nk - \frac{3n^2+4n+\delta}{4}$$

in cases [R1] and [R2]. Otherwise let

$$c(m) = \begin{cases} \beta(m) + nk - \frac{3n^2+4n+\delta}{4} & \text{if } m > n \\ \beta(m) + n(k+m) - \frac{7n^2-4n+\delta}{4} & \text{if } m \leq n. \end{cases}$$

**Theorem 7.6.** Let $1 \leq n \in \mathbb{Z}$ and assume that $k > \frac{5n}{2} + 1$. If $f \in S_k(\Gamma[b^{-1}, bc], \psi, \Lambda)$ is a Hecke eigenform for ideals $(b^{-1}, bc) \subseteq 2\mathbb{Z} \times 2\mathbb{Z}$, a Hecke character $\psi$, and a system of eigenvalues $\Lambda$, then there exists a non-zero constant $\mu(\Lambda, k, \psi)$ dependent only on $\Lambda, k, \psi$ such that

$$\frac{\langle f, g \rangle}{\mu(\Lambda, k, \psi)} = \frac{\langle f^\sigma, g^{\rho^*\rho} \rangle}{\mu(\Lambda^\sigma, k, \psi^\sigma)}$$

for any $g \in S_k(\Gamma[(b'^{-1})^{-1}, b'c'], \psi)$, ideals $((b')^{-1}, b'c') \subseteq b^{-1} \times bc$, and $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$. 29
Proof. Since we have a larger set $\Omega_{n,k}$ of special values, we can change the special value of the $L$-function that defined our original constant $\mu'(\Lambda, k, \psi)$. With $\varphi$ as in the proof of Theorem 5.1 we let

$$\mu(\Lambda, k, \psi) := 2\frac{\delta_2}{\pi} (-c(k-n))^{-\frac{1}{2}} \omega_\delta(k-n, \varphi)^{-1} L_\psi(k-n, f, \psi).$$

In order to guarantee non-vanishing of the $L$-function we need $k-n$ to be strictly greater than $\frac{3n}{2} + 1$ whence our bound on $k$. With $m$ as in the proof of Theorem 5.1 we have

$$\Omega^+(\theta_{\varphi_m}) = \{ m \in \frac{1}{2} \mathbb{Z} \mid \frac{k-m-\delta}{2}, n < m \leq k - \delta \}.$$ 

Then since $k-n \in \Omega(\theta_{\varphi_m})$ the rest of this proof follows exactly as that of Theorem 5.1, but using the integral expressions 7.6 (resp. 7.7) above when $m \leq n$ (resp. cases (R1) or (R2)) instead.

Remark 7.7. Unlike Theorem 5.1, the above is also true for $n = 1$ due to the non-strict upper bound of $\Omega(\theta_{\varphi_m})$. Notice that the occurrence of case (X) is not possible in this proof – and likewise in the next theorem – since $\Omega^+(\theta_{\varphi_m})$ consists of strict half-integers.

Theorem 7.8. Assume that $k > \frac{5n}{2} + 1$ and let $f \in \mathcal{S}_k(\Gamma, \psi, \Lambda)$, $\eta$ be a Hecke character, and choose $\mu \in \{0, 1\}$ such that $(\psi \eta)_\infty(x) = \text{sgn}(x)|k|^{\mu}$. Define the set

$$\Omega_{n,k}^+ := \{ m \in \frac{1}{2} \mathbb{Z} \mid \frac{m-k-\mu}{2} \in \mathbb{Z}, n < m \leq k - \mu \},$$

$$\Omega_{n,k}^- := \{ m \in \frac{1}{2} \mathbb{Z} \mid \frac{m-k+\mu}{2} \in \mathbb{Z}, 2n + 1 - k + \mu \leq m \leq n \},$$

$$\Omega_{n,k} := \Omega_{n,k}^- \cup \Omega_{n,k}^+.$$ 

Now if $\tau \in S_+$ is such that $\mu_f(\tau, 1) \neq 0$ and $m \in \Omega_{n,k}$ then define

$$Z_\psi(m, f, \eta) := |\tau|^\frac{\delta}{2} \pi^{-(m)} \mu(\Lambda, k, \psi)^{-1} \omega_\delta(m, \bar{\chi})^{-1} L_\psi(m, f, \eta).$$

We have $Z_\psi(m, f, \eta)^\sigma = Z_\psi(m, f^{\sigma}, \eta^\sigma)$ for any $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\Lambda_k, \psi, G(\psi), \zeta_*)$) and hence

$$Z_\psi(m, f, \eta) \in \mathbb{Q}(f, \eta, \bar{\Lambda}_k, \psi, G(\psi), \zeta_*).$$

Proof. Note that $\Omega^\pm(\theta) = \Omega_{n,k}^\pm$. If $m \in \Omega_{n,k}^+$ and we are not in cases (R1) or (R2) then combine the integral expression of 7.3 with Proposition 7.5 whereas, if $m \in \Omega_{n,k}^-$ (resp. (R1) or (R2)), then use the integral expression of 7.6 (resp. 7.7) directly. This gives

$$Z_\psi(m, f, \eta) \in \mu_f(\tau, 1)^{-1} G_9(m, \chi) \prod_{p \in b} g_p((p)\eta(p)p^{-m}) \frac{\langle f, K_S(m, \theta) \rangle}{\mu(\Lambda, k, \psi)} \mathbb{Q}$$

which is evidently $\sigma$-equivariant over $\text{Aut}(\mathbb{C}/\mathbb{Q}(\Lambda_k, \psi, G(\psi), \zeta_*))$ by Theorem 7.6.

Acknowledgements

I would like to thank my PhD supervisor Thanasis Bouganis for the direction and guidance of this paper, and for generally keeping my head screwed on through its intricacies. Funding was provided by Engineering and Physical Sciences Research Council (Grant No. 000118421).

References

And74 A. Andrianov, ‘Euler Products Corresponding to Siegel Modular Forms of Genus 2’, Russian Mathematical Surveys, 29 (3) (1974), 45–116.

And79 A. Andrianov, ‘The Multiplicative Arithmetic of Siegel Modular Forms’, Russian Mathematical Surveys, 34 (1) (1979), 75–148.
Algebraicity of metaplectic $L$-functions

Boug18 T. Bouganis, ‘On Special $L$-Values Attached to Metaplectic Modular Forms’, *Mathematische Zeitschrift*, **3-4** (2018), 725–740.

Garr84 P. B. Garrett, ‘Pullbacks of Eisenstein series: Applications’, *Progress in Mathematics*, **46** (1984), 114–137.

CG58 H. Cartan, R. Godement, *et al.*, ‘Fonctions Automorphes’, *Séminaire Cartan*, 10 (1958).

FC80 G. Faltings & C.-L. Chai, *Degeneration of Abelian Varieties*, Springer-Verlag, Berlin Heidelberg (1980).

Harr81 M. Harris, ‘The Rationality of Holomorphic Eisenstein Series’, *Inventiones Mathematicae*, **63** (1981), 305–310.

Harr84 M. Harris, ‘Eisenstein Series on Shimura Varieties’, *Annals of Mathematics*, **119** (1984), 59–94.

Kob84 N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer-Verlag, New York, (1984).

Pan91 A. Panchishkin, ‘Non-Archimedean $L$-Functions of Siegel and Hilbert Modular Forms’, *Lecture Notes in Mathematics*, **1471**, Springer-Verlag, Berlin Heidelberg, (1991).

Sh85 G. Shimura, ‘On Eisenstein Series of Half-integral Weight’, *Duke Mathematical Journal*, **52** (1985), 281–314.

Sh93 G. Shimura, ‘On the Transformation Formulas of Theta Series’, *American Journal of Mathematics*, **115** (5) (1993), 1011–1052.

Sh94 G. Shimura, ‘Euler Products and Fourier Coefficients of Automorphic Forms on Symplectic Groups’, *Inventiones Mathematicae*, **116** (1994), 531–576.

Sh95a G. Shimura, ‘Eisenstein Series and Zeta Functions on Symplectic Groups’, *Inventiones Mathematicae*, **119** (1995), 539–584.

Sh95b G. Shimura, ‘Zeta Functions and Eisenstein Series on Metaplectic Groups’, *Inventiones Mathematicae*, **121** (1995), 21–60.

Sh96 G. Shimura, ‘Convergence of Zeta Functions on Symplectic and Metaplectic Groups’, *Duke Mathematical Journal*, **82** (2) (1996), 327–347.

Sh00 G. Shimura, ‘Arithmeticity of Automorphic Forms’, *Mathematical Surveys and Monographs*, **82**, Amer. Math. Soc. (2000).

St81 J. Sturm, ‘The Critical Values of Zeta Functions Associated to the Symplectic Group’, *Duke Mathematical Journal*, **48** (2) (1981), 327–350.

Salvatore Mercuri salvatore.mercuri@dur.ac.uk
Department of Mathematical Sciences, Durham University,
Stockton Road,
Durham DH1 3LE,
UK.