THE BIJECTION BETWEEN EXCEPTIONAL SUBCATEGORIES AND NON-CROSSING PARTITIONS

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Abstract. This note discusses the bijection between the exceptional subcategories of representations of quivers and generalized non-crossing partitions of Weyl groups. We give a new proof of the Ingalls-Thomas-Igusa-Schiffler bijection by using the exchange property of the Weyl groups of the Kac-Moody Lie algebras.

1. Introduction

Representations of quivers have deep relations with the Kac-Moody Lie algebras. Once we are given an acyclic quiver Q, we can define its representation category repQ over a field k. It is an abelian category. Let A(modΛ) denote the set of all exceptional subcategories of modΛ where Λ = kQ the path algebra of Q and modΛ denotes the finite dimensional (left) Λ-modules. Let K0(repQ) be the Grothendieck group, the symmetric Euler form (−, −) of Euler form is well-defined in K0(repQ). The system {K0(repQ), (−, −)} then can define a generalized Cartan matrix. The correspond Kac-Moody Lie algebra is denoted by g(Λ). And the weyl group of g(Λ) is denoted by W(Λ), the Coxeter element is c(Λ). We define what we call exceptional subcategories and generalized non-crossing partitions. We consider the following

Main Theorem: There is an isomorphism

\[ \text{cox} : A(\text{modΛ}) \to Nc(W(Λ), c(Λ)) \]

We will define the isomorphism in Section 2. This Theorem was first proved by Ingalls and Thomas [4] for Dynkin and tame case. Then Igusa and Schiffler [3] proved it in general case. In this note, we will give an elementary and straightforward proof for this theorem.

In Section 2 we introduce some basic definitions and preliminary results to give a definition of the map cox. In Section 3 we recall the braid group action on exceptional sequences due to Crawley-Boevey[1]. In Section 4 we show that there is a natural action of braid group on weyl group which is called Hurwitz transform and this action is transitive on the generalized non-crossing partitions. With the preparation of Section 3 and 4, Section 5 gives a proof of the bijection between the exceptional sequences and generalized non-crossing partitions.

2. Definitions and preliminary results

An acyclic quiver is an oriented graph Q without oriented cycles. We write it as Q = (Q0, Q1), where Q0 is the set of all vertices and Q1 is the set of all arrows. Consider the representation category of repQ over a field k. Let Λ = kQ the path algebra of Q. There is a canonical category equivalence modΛ ∼= repQ. In the paper we will identify representations of Q over k with Λ-modules.

A representation of Q is denoted by (Vi, vα, i ∈ Q0, α ∈ Q1). Here vα is a linear transform from Vh(α) to Vt(α) where h(α) is the head of the arrow α and t(α) is the tail of it.
Let $K_0(\text{mod}\Lambda)$ be the Grothendieck group. Then $K_0(\text{mod}\Lambda)$ is a free abelian group of rank $n = \#Q_0$. Given $M \in \text{mod}\Lambda$ the dimension vector of it is

$$\dim(M) = (\dim V_1, \ldots, \dim V_n)$$

Given two vectors $v, w \in K_0(\text{mod}\Lambda)$, the bilinear form Euler form is defined as follows:

$$\langle v, w \rangle = \sum_{i \in Q_0} v_i w_i - \sum_{\alpha \in Q_1} v_{\alpha} w_{\alpha}$$

For two modules $M, N \in \text{mod}\Lambda$, we define $\langle M, N \rangle = (\dim(M), \dim(N))$. The symmetric Euler form $(-, -)$ is defined by $\langle v, w \rangle = \langle v, w \rangle + \langle w, v \rangle$. The system $(K_0(\text{mod}\Lambda), (-, -))$ then defines a generalized Cartan matrix and then we obtain the corresponding Kac-Moody Lie algebra $g(\Lambda)$. Let $\Phi$ be its root system. The dimension vectors of simple objects $\{\dim(S_i)\}_{i \in Q_0}$ are exactly the simple roots of $\Phi$. Given an element $v \in K_0(\text{mod}\Lambda)$, if we write $v = \sum_{i=1}^n c_i \dim S_i$, the support of $v$ is the subset of the bases of $K_0(\text{mod}\Lambda)$ such that $c_i \neq 0$. We say $v$ is positive if $c_i > 0$ for $\dim S_i$ in its support and $v \neq 0$. Then $\Phi$ has the decomposition $\Phi = \Phi^+ \cup \Phi^-$ where $\Phi^+$ is the set of positive roots and $\Phi^- = -\Phi^+$. The real roots is the root that can be obtained from simple roots by reflections. We denote the complement of real roots imaginary roots. So $\Phi = \Phi^+ \cup \Phi^\imath$. For every real root $v$, the equality $\langle v, v \rangle = 2$ holds. With this notion we have the reflection transforms for all real roots as

$$\sigma_v(w) = w - \langle v, w \rangle v$$

For each indecomposable module $M$ such that $\dim(M)$ is a real root, we define $\sigma_M = \sigma_{\dim M}$. The equation $\Phi_r = \cup_{i \in Q_0} W(\Lambda) \dim(S_i)$ holds.

For an element $\omega \in W(\Lambda)$, we define its absolute length $|\omega|_a$ equal to the minimal number $l$ that $\omega$ can be written as product of reflections of real roots. With the absolute length we define a partial order on $W(\Lambda)$ by the following:

$$\omega_1 \leq \omega_2 \iff |\omega_1|_a + |\omega_1^{-1}\omega_2|_a = |\omega_2|_a$$

If $\{i_1, \ldots, i_n\} = Q_0$, then $\sigma_{S_{i_1}} \cdots \sigma_{S_{i_n}}$ is the Coxeter element in $W(\Lambda)$. Choose one Coxeter element $c$, we define the set of generalized non-crossing partitions $N_c(W, c)$ as

$$N_c(W, c) = \{\sigma | W | \sigma \leq c\}$$

For $i \in Q_0$, we have simple modules $S_i$, the indecomposable projective modules $P_i$ with top $P_i = S_i$, the indecomposable injective modules $I_i$ with soc $I_i = S_i$. Then $\{S_i\}_{i \in Q_0}$ is the complete collection of the simple modules, $\{P_i\}_{i \in Q_0}$ is the complete collection of the indecomposable projective modules, $\{I_i\}_{i \in Q_0}$ is the complete collection of the indecomposable injective modules.

A module $M$ is called exceptional if $\text{End}_A(M) = k$ and $\text{Ext}_A(M, M) = 0$. An antichain is a set of modules

$$\{A_1, A_2, \ldots, A_r\}$$

such that $\text{Hom}_A(A_i, A_j) = 0$ for all $i \neq j$ and $\text{Hom}_A(A_i, A_i) = k$. Recall that given an antichain, the Ext-quiver of it is defined as follows: The vertices of the quiver are the elements in the antichain, and there is an arrow from $i$ to $j$ if $\text{Ext}_A(A_i, A_j) \neq 0$. An antichain is called exceptional if its Ext-quiver is acyclic[2]. Then given an exceptional antichain, we can define an exceptional subcategory $A$ as its extension closure. Then we can show that $A$ is closed under extension, kernel of monomorphism and cokernel of epimorphism, which we call it a thick subcategory. By $A \leq B$ we mean $A \subseteq B$.

A sequence $E = (E_1, E_2, \ldots, E_r)$ in $\text{mod}\Lambda$ is called an exceptional sequence if each $E_j$ is an exceptional $\Lambda$ module and we have $\text{Hom}_A(E_j, E_i) = \text{Ext}_A(E_j, E_i) = 0$ for $j > i$. If $r = n$, we call the exceptional sequence a complete exceptional sequence.
Given an exceptional sequence $E$, we can define a full subcategory $\mathcal{A}$ (denote by $\mathcal{C}(E)$) of $\text{mod}\Lambda$ as the thick closure of the sequence. On the other hand, for every exceptional subcategory $\mathcal{A}$, all its simple object $\{S_1, S_2, ..., S_r\}$ is an exceptional antichain, we can relabel it such that $(S_1, ..., S_r)$ is an exceptional sequence.

With the above notions and properties we can define the bijection between the exceptional subcategories and generalized non-crossing partitions.

For every exceptional subcategory $\mathcal{A}$, choose a complete exceptional sequence $E = (E_1, E_2, ..., E_r)$, define a correspondence
\[
\text{cox} : \text{A}(\text{mod}\lambda) \to \Lambda(\text{W}(\lambda), c(\lambda))
\]
\[
\text{cox}(\mathcal{A}) = \sigma_{E_1} \sigma_{E_2} \cdots \sigma_{E_r}
\]

In section 3, we will prove that this is a well defined map and in the last section we will prove that this map is actually a bijection.

3. Braid group action on exceptional sequences

Recall that a braid group $B_n$ is a group generated by $\{\rho_1, \rho_2, ..., \rho_{n-1}\}$ with respect to the following relations:

1) $\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}$
2) $\rho_i \rho_j = \rho_j \rho_i$ for $|j - i| \geq 2$

We introduce some well known lemmas which are taken from [1].

As above, let $Q$ be an acyclic quiver with $n$ vertices. Let $\Lambda = k\mathcal{Q}$. First we define the perpendicular subcategory.

**Definition 3.1.** Given a subcategory $\mathcal{U}$ of $\text{mod}\lambda$. The right (resp. left) perpendicular subcategory of $\mathcal{U}$ which is denoted by $\mathcal{U}^\perp$ (resp. $\mathcal{U}^\perp$) the set
\[
\mathcal{U}^\perp = \{M \in \text{mod}\lambda | \text{Hom}_\Lambda(N, M) = \text{Ext}_\Lambda(N, M) = 0 \forall N \in \mathcal{U}\}
\]

\[
\text{resp. } \mathcal{U}^\perp = \{M \in \text{mod}\lambda | \text{Hom}_\Lambda(M, N) = \text{Ext}_\Lambda(M, N) = 0 \forall N \in \mathcal{U}\}
\]

Now we say a pair $(\mathcal{U}, \mathcal{V})$ is perpendicular pair if $\mathcal{U} = \mathcal{V}^\perp$ and $\mathcal{V} = \mathcal{U}^\perp$.

Use these notation, we can describe the following lemmas.

**Lemma 3.2.** If $E = (E_1, E_2, ..., E_r)$ is an exceptional sequence, then $\mathcal{C}(E)^\perp$ (resp. $\mathcal{C}(E^\perp)$) is equivalent to $kQ(E^\perp) - \text{mod} (kQ(E^\perp) - \text{mod})$ category where $Q(E^\perp)$ (resp. $Q(E^\perp)$) is some acyclic quiver with $(n-r)$ vertices.

**Proof.** We refer to Schofield’s paper [5,Theorem 2.3].

**Lemma 3.3.** For a complete exceptional sequence $E = (E_1, E_2, ..., E_n)$, we have $\mathcal{C}(E) = \text{mod}\lambda$.

**Proof.** We prove this lemma by induction on the number of vertices of $Q$. When $n=1$, it is easy to see. Now we suppose for $k < n$, the lemma holds.

Let $X = E_n$. Now $E' = (E_1, ..., E_{n-1})$ is a complete sequence of $kQ(X^\perp)$. By induction we have $\mathcal{C}(E') = X^\perp$.

Suppose that $X$ is not a projective module. Then by Bongartz completion we have $Y \in X^\perp$ such that $T = X \oplus Y$ is a tilting module. Since by the definition of a tilting module, there is an exact sequence
\[
0 \to \Lambda \to T' \to T'' \to 0
\]
where $T', T'' \in \text{add}(T)$, we can conclude that all projectives are in $\mathcal{C}(E)$. Since every module has a projective resolution, the lemma has been proved.
Lemma 3.5. If \( n \) is relative from the representation category of an acyclic quiver of \( n \) vertices. Since we have the exact sequence \( 0 \to \text{rad} X \to X \to S_i \to 0, S_i \in \mathcal{C}(E) \). For \( j \neq i, S_j \in \mathcal{C}(E') \) by induction. So all the simple modules are in the \( \mathcal{C}(E) \). We finish the proof.

**Proof.** So \((\perp)\) is a complete exceptional sequence having the form \((E, F)\) with \( n = 2 \) vertices, by Lemma 3.4, \( \mathcal{E}, \mathcal{F} \) is a complete exceptional sequence of \( \mathcal{C}(E) \). And for exceptional subcategory \( \mathcal{U} \), we have

\[
\perp(\mathcal{U}) = (\perp \mathcal{U}) = \mathcal{U}
\]

**Proof.** Since \( E = (E_1, \ldots, E_r) \) is an exceptional sequence, from Lemma 3.2, \( \mathcal{C}(E) \perp \) is equivalent to the representation category of an acyclic quiver of \( n - r \) vertices. So we can choose a complete exceptional sequence \( F \) of \( kQ(E) - \text{mod} \). Then \((F, E)\) is a complete exceptional sequence of \( \text{mod} \Lambda \). For \( \perp \mathcal{C}(E) \), things are similar. The first statement is proved.

For the second statement, it is obvious that \((\perp \mathcal{U}) \perp \subseteq \mathcal{U}\). We already knew that there is a complete exceptional sequence having the form \((E, F)\) in \( \text{mod} \Lambda \) where \( E \) is a complete sequence of \( \mathcal{U} \) and \( F \) is the complete sequence of \( \perp \mathcal{C}(E) \). Then by Lemma 3.3

\[
\perp \mathcal{C}(E) = \mathcal{C}(F)
\]

So \((\perp \mathcal{U}) \perp = \mathcal{C}(F) \perp \supseteq \mathcal{C}(E)\). Then the second statement is proved.

**Lemma 3.5.** If \( E = (E_1, E_2, \ldots, E_{i-1}, X, E_{i+1}, \ldots, E_n) \) and \( E' = (E_1, E_2, \ldots, E_{i-1}, Y, E_{i+1}, \ldots, E_n) \) both are exceptional sequences, then \( X \cong Y \).

**Proof.** By passing to \( \perp (E_1, E_2, \ldots, E_{i-1}) \) and \( (E_{i+1}, \ldots, E_n) \perp \), we obtain an exceptional subcategory with only one simple object. So \( X \cong Y \).

The following lemma is due to Schofield which is well known, for proof, see [6].

**Lemma 3.6.** For any exceptional module \( M \), if \( M \) is not simple in \( \text{mod} \Lambda \), then there exists two exceptional modules \( X, Y \) such that \( \text{Hom}_\Lambda(X, Y) = \text{Hom}_\Lambda(Y, X) = 0 \) and \( M \) is relative project in \( \mathcal{C}(X, Y) \) and there exists an exact sequence

\[
0 \to Y^b \to M \to X^a \to 0
\]

**Lemma 3.7.** For any exceptional pair \((X, Y)\), there exists a unique exceptional module \( R_Y \in \mathcal{C}(X, Y) (\text{resp.} L_X \in \mathcal{C}(X, Y)) \) such that \((Y, R_Y \cdot X)(\text{resp.}(L_X \cdot Y, X))\) is an exceptional pair.

**Proof.** Since \( \mathcal{C}(X, Y) \) can be viewed as the representation category of an acyclic quiver with 2 vertices, by Lemma 3.4, \( X (Y) \) can be extended from the left (right) to a complete exceptional sequence of \( \mathcal{C}(X, Y) \). Then we finish the proof.

Now we can introduce the braid group action on the complete exceptional sequences.

**Definition 3.8.** Given a complete exceptional sequence \( E = (E_1, E_2, \ldots, E_n) \), we define the braid group actions as follows:

\[
\rho_i(E_1, E_2, \ldots, E_n) = (E_1, \ldots, E_{i-1}, E_{i+1}, R_{E_{i+1}E_i}E_i, E_{i+2}, \ldots, E_n)
\]

\[
\rho_i^{-1}(E_1, E_2, \ldots, E_n) = (E_1, \ldots, E_{i-1}, L_{E_{i+1}E_i}E_i, E_{i+2}, \ldots, E_n)
\]

We can check by calculation directly that this is a \( B_n \) action on the complete exceptional sequences. Then Crawley-Boevey proved that this \( B_n \) action is transitive.
Theorem 3.9. The $B_n$ action on the set of complete exceptional sequences is transitive.

Since the antichain of an exceptional subcategory is a complete exceptional sequence, we have the following proposition, see [5].

Proposition 3.10. For any two complete sequences of an exceptional subcategory $\mathcal{A}$: $E = (E_1, \ldots, E_r)$ and $E' = (E'_1, \ldots, E'_r)$, we have $\sigma_{E_1} \sigma_{E_2} \ldots \sigma_{E_r} = \sigma_{E'_1} \sigma_{E'_2} \ldots \sigma_{E'_r}$.

Proof. By Theorem 3.9, the braid group acts transitively on the set of complete sequences. And we have the formulas

$$\dim R_Y X = \pm \sigma_Y (X)$$
$$\dim L_X Y = \pm \sigma_X (Y)$$

(Crawley-Boevey’s paper) [1]

Fix a complete exceptional sequence $E = (E_1, E_2, \ldots, E_r)$. First we prove that for a generator $\rho_i$ of $B_n$ and denote $E^* = (E'_1, E'_2, \ldots, E'_r)$, the equation $\sigma_{E_1} \sigma_{E_2} \ldots \sigma_{E_r} = \sigma_{E'_1}^* \sigma_{E'_2}^* \ldots \sigma_{E'_r}^*$ holds.

By definition,

$$\sigma_{E_1}^* \sigma_{E_2}^* \ldots \sigma_{E_r}^* = \sigma_{E_1} \ldots \sigma_{E_{i_1+1}} \sigma_{E_{i_1+2}} \ldots \sigma_{E_{i_1+i}} \ldots \sigma_{E_{r-1}} \sigma_{E_r} = \sigma_{E_1} \ldots \sigma_{E_{i_1+1}} \sigma_{E_{i_1+2}} \ldots \sigma_{E_{i_1+i}} \ldots \sigma_{E_{r-1}} \sigma_{E_r} = \sigma_{E_1} \sigma_{E_2} \ldots \sigma_{E_r}$$

So we conclude that $B_n$ action does not change the product.

Then for every two complete sequences $E = (E_1, \ldots, E_r)$ and $E' = (E'_1, \ldots, E'_r)$, we have $\sigma_{E_1} \sigma_{E_2} \ldots \sigma_{E_r} = \sigma_{E'_1} \sigma_{E'_2} \ldots \sigma_{E'_r}$ because braid group action on the set of complete exceptional sequences is transitively.

Thus the lemma is proved. $\square$

4. Braid group action on the set of non-crossing partitions

As we introduced in Section 2, from an acyclic quiver we can get a Kac-Moody Lie algebra. The aim of this section is to prove that Hurwitz transformation is transitive in $NC(W, c)$.

By $\Phi$ and $W$ we denote the roots system and Weyl group of the Kac-Moody Lie algebra $\mathfrak{g}(\Lambda)$ respectively. Let $\{S_1, S_2, \ldots, S_n\}$ be the complete collection of non-isomorphic simple modules of $mod\Lambda$. There are natural decompositions $\Phi = \Phi^+ \cup \Phi^-$ and $\Phi = \Phi_{re} \cup \Phi_{im}$ as we discussed in Section 2. According to Kac’s theorem, $\Phi^+$ equal to the set of the dimension vectors of indecomposable modules in $mod\Lambda$. Moreover, $\Phi^− = −\Phi^+$. We write $\alpha > 0$ for an element $\alpha$ in $K_0(mod\Lambda)$ if $\alpha \neq 0$ and $\alpha = \Sigma_{i=1}^n c_i [S_i]$ where $c_i \in \mathbb{Z}_{\geq 0}$ in $K_0(mod\Lambda)$. We write $\alpha > \beta$ if $\alpha - \beta > 0$.

It is well known that $W(\Phi_{re}) = \Phi_{re}$ and $W(\Phi_{im}) = \Phi_{im}$.

Lemma 4.1. The simple reflection $\sigma_{S_i}$ preserves $\Phi^+ - \{\dim S_i\}$

Proof. If $\alpha \in \Phi^+$ and $\alpha \neq \dim S_i$, then $\sigma_{S_i}(\alpha)$ can not be a negative root. Since $\sigma_{S_i}$ transforms roots to roots, then $\sigma_{S_i}(\alpha) \in \Phi^+$. $\square$

The following lemma is the well-known exchange property.

Lemma 4.2. If $\sigma_{S_{i_1}} \sigma_{S_{i_2}} \ldots \sigma_{S_{i_k}} (\alpha) < 0$ for some $\alpha \in \Phi^+ \cap \Phi_{re}$, then there is $1 \leq t \leq k$ such that $\sigma_{S_{i_1}} \sigma_{S_{i_2}} \ldots \sigma_{S_{i_k}} = \sigma_{S_{i_1+1}} \sigma_{S_{i_2+1}} \ldots \sigma_{S_{i_k}} \sigma_{\alpha}$

Proof. Since $\alpha > 0$ there exists $1 \leq t \leq k$ such that $\sigma_{S_{i_1+1}} \sigma_{S_{i_2+2}} \ldots \sigma_{S_{i_k}} (\alpha) > 0$ and $\sigma_{S_{i_1}} \sigma_{S_{i_2+1}} \ldots \sigma_{S_{i_k}} (\alpha) < 0$. By Lemma 4.1, the equality $\sigma_{S_{i_1}} \sigma_{S_{i_2+1}} \sigma_{S_{i_3+2}} \ldots \sigma_{S_{i_k}} (\alpha) = \dim S_{i_t}$ holds. So

$$\sigma_{S_{i_t}} = \sigma_{S_{i_1+1}} \sigma_{S_{i_2+2}} \ldots \sigma_{S_{i_k}} \sigma_{\alpha} (\sigma_{S_{i_1}} \sigma_{S_{i_2+1}} \ldots \sigma_{S_{i_k}})^{-1}$$

The lemma follows. $\square$
Lemma 4.9. An element \( \omega \in W \) has an absolute length \(|\omega|_a\) if \( \omega \) can be written as products of \(|\omega|_a\) reflections but can not be written by product of less number of reflections.

Let \( T \) be the set of all reflections at real roots in \( W \). The following defines the braid group action on \( T^n \), called Hurwitz transformation. The definition of the braid group \( B_n \) is already given at the beginning of Section 3.

Definition 4.4. Given \((\sigma_{a_1}, \sigma_{a_2}, ..., \sigma_{a_n}) \in T^n\) the Hurwitz transformation on \( T^n \) is defined by for the canonical generators \( \rho_i \) of \( B_n \):

\[
\rho_i(\sigma_{a_1}, \sigma_{a_2}, ..., \sigma_{a_n}) = (\sigma_{a_1}, ..., \sigma_{a_{i-1}}, \sigma_{a_{i+1}}, ..., \sigma_{a_n})
\]

It can be checked directly by calculation that this is a group action of \( B_n \) on \( T^n \).

Remark 4.5. From the definition of the action, what should be noticed is that

\[
\sigma_{a_{i+1}} \sigma_{a_{i+1}}(\alpha_i) = \sigma_{a_{i+1}} \sigma_{a_{i+1}} \sigma_{a_i} \sigma_{a_{i+1}} = \sigma_{a_i} \sigma_{a_{i+1}}
\]

so the action of \( B_n \) on \( T^n \) does not change the product of \((\sigma_{a_1}, \sigma_{a_2}, ..., \sigma_{a_n})\). Thus it induces an action of \( B_n \) on \( NC(W, c) \).

We now label the simple objects of \( \text{mod}\Lambda \) in an appropriate order such that \( S = (S_1, S_2, ..., S_n) \) is a complete exceptional sequence.

Theorem 4.6. If \( \sigma_{a_1} \sigma_{a_2}... \sigma_{a_n} = \sigma_{S_1} \sigma_{S_2}... \sigma_{S_n} \) where all \( \alpha_i \) are positive real roots, then \((\sigma_{a_1}, \sigma_{a_2}, ..., \sigma_{a_n})\) and \((\sigma_{S_1}, \sigma_{S_2}, ..., \sigma_{S_n})\) are in the same orbit of the action of \( B_n \).

To prove the theorem, we need the following definition.

Definition 4.7. A sequence \( E = (E_1, E_2, ..., E_n) \) in \( \text{mod}\Lambda \) is called a projective sequence if we have the following properties:

1) The number of simple objects appearing in \( S(E(r)) \) is \( r \).
2) \( E_r \) is a projective object in \( C(S(E(r))) \).
3) \( \text{top}(E_r) \notin S(E(r)) \).

Now we have the following lemma.

Lemma 4.8. A projective sequence is an exceptional sequence.

Proof. Let \( E = (E_1, E_2, ..., E_n) \) be a projective sequence. Take a \( 1 \leq r \leq n \). Since \( E_r \) is a projective module in \( C(E^r) \), \( \text{Ext}_\Lambda(E_r, E_j) = 0 \) for \( 1 \leq j \leq r \). By property (3), \( \text{top}(E_r) \notin S(E(r - 1)) \), so \( \text{Hom}_\Lambda(E_r, E_j) = 0 \) for \( 1 \leq j < r \). Then it follows that \( E \) is an exceptional sequence. \( \square \)

Lemma 4.9. If \( \omega \in W \) with \(|\omega|_a = 1\), then for each decomposition of \( \omega = \sigma_{S_{i_1}} \sigma_{S_{i_2}}... \sigma_{S_{i_t}} \), we can delete some \( \sigma_{S_{i_j}} \) such that \( \sigma_{S_{i_1}} \sigma_{S_{i_2}}... \sigma_{S_{i_{j-1}}} \sigma_{S_{i_{j+1}}}... \sigma_{S_{i_t}} = 1 \), i.e. \( \sigma_{S_{i_1}} \sigma_{S_{i_2}}... \hat{\sigma}_{S_{i_j}}... \sigma_{S_{i_t}} = 1 \)

Proof. Since \(|\omega|_a = 1\), we have \( \omega = \sigma_{S_{i_1}} \sigma_{S_{i_2}}... \sigma_{S_{i_t}} = \sigma_{\alpha} \) for some real root \( \alpha \). Since \( \sigma_\alpha(\alpha) < 0 \), there must be some \( \sigma_{S_{i_j}} \) such that \( S_{i_j} = \sigma_{S_{i_{j+1}}} \sigma_{S_{i_{j+2}}}... \sigma_{S_{i_t}}(\alpha) \), the following holds:

\[
\sigma_\alpha = \sigma_{S_{i_1}} \sigma_{S_{i_2}}... \sigma_{S_{i_{j-1}}} \sigma_{S_{i_j}} \sigma_{S_{i_{j+1}}}... \sigma_{S_{i_t}}
\]
Thus the lemma is proved.

**Definition 4.10.** For a reflection $\omega \in W$ at some real root, define $l'(\omega)$ to be the minimal length of $\omega_1$ such that $\omega = \omega_1\sigma_j\omega_1^{-1}$ for some simple reflection $\sigma_j$.

The following lemma is well-known. For example, see [5]

**Lemma 4.11.** Given an exceptional sequence $S = (S_1, ..., S_n)$ consisting of simple objects. The Coxeter element $c = \sigma_{S_1}\sigma_{S_2}...\sigma_{S_n}$ has the absolute length $n$.

*Proof.* Set $c = \sigma_{S_1}\sigma_{S_2}...\sigma_{S_n} = \sigma_{a_1}\sigma_{a_2}...\sigma_{a_m}$ where $m = |c|\alpha$. The lemma is a corollary of the following Lemma 4.12.

**Lemma 4.12.** Given an exceptional sequence $S = (S_1, ..., S_n)$ consisting of simple objects. If $\sigma_{S_1}\sigma_{S_2}...\sigma_{S_{i_t}} = \sigma_{a_1}\sigma_{a_2}...\sigma_{a_s}$ ($i_1 < i_2 < ... < i_t$) where $s = |\sigma_{S_1}\sigma_{S_2}...\sigma_{S_{i_t}}|\alpha$, then the following holds:

a) There exists $(\sigma_{\beta_1}, ..., \sigma_{\beta_t})$ in the orbit of $(\sigma_{a_1}, ..., \sigma_{a_t})$ and $l'(\sigma_{\beta_i})$ is minimal in the $B_{i_1+i_1}$ action orbit of $(\sigma_{\beta_1}, ..., \sigma_{\beta_t})$.

b) there exists an $r$ such that $\sigma_{S_1}...\sigma_{S_{i_r}}...\sigma_{S_{i_t}} = \sigma_{\beta_1}\sigma_{\beta_2}...\sigma_{\beta_{r-1}}$.

*Proof.* a) Let $\sigma_{\beta_1}$ be the element in the orbit of $(\sigma_{a_1}, ..., \sigma_{a_t})$ of the action of $B_n$ which has the minimal $l'(\cdot)$. Let $(\sigma_{\beta_1}, \sigma_{a_1}, ..., \sigma_{a_t})$ be the element. Let $\sigma_{\beta_2}$ be the element in the orbit of $(\sigma_{a_1}, ..., \sigma_{a_t})$ of $B_{i_1-i_1}$ action that has the minimal $l'(\cdot)$. We do this process inductively. Finally we will get a sequence $(\sigma_{\beta_1}, ..., \sigma_{\beta_t})$ which satisfies the required properties.

b) For each $\sigma_{\beta_i}$, by the definition of $l'(\sigma_{\beta_i})$, there exists a $\omega_i$ such that $l(\omega_i) = l'(\sigma_{\beta_i})$ and

$$\sigma_{\beta_i} = \omega_i\sigma_{S_j}\omega_i^{-1}$$

Then by the assumption in the lemma, we have

$$\sigma_{\beta_i} = \sigma_{a_1}\sigma_{a_2}...\sigma_{a_s}$$

Now we apply Lemma 4.9. It is divided into four cases:

Case1: If we delete some simple reflection $\sigma_{S_{i_j}}$, then by Lemma 4.9

$$1 = \sigma_{\beta_{i_1}}...\sigma_{\beta_{i_r}}...\sigma_{S_{i_1}}...\sigma_{S_{i_t}}$$

which contradicts the assumption that $s = |\sigma_{S_1}\sigma_{S_2}...\sigma_{S_{i_t}}|\alpha$.

Case2: If we delete some simple reflection in $\omega_i$. We write $\omega_i = \sigma_{S_{i_1}}...\sigma_{S_{i_p}}$ where $p = l(\omega)$. Suppose $\sigma_{S_{kq}}$ is deleted. By Lemma 4.9

$$1 = \sigma_{\beta_{i_1}}...\sigma_{\beta_{i_r}}...\sigma_{S_{i_1}}...\sigma_{S_{i_2}}...\omega_i^{-1}\sigma_{F_{i_1}}...\sigma_{F_{i_1}}...\sigma_{S_{i_1}}...\sigma_{S_{i_t}}$$

where $\sigma_{\beta_{i_1}}...\sigma_{\beta_{i_r}}...\sigma_{S_{i_1}}...\sigma_{S_{i_2}}...\omega_i^{-1}\sigma_{F_{i_1}}...\sigma_{F_{i_1}}...\sigma_{S_{i_1}}...\sigma_{S_{i_t}}$

But by $B_s$ action $(\sigma_{\beta_1}, ..., \sigma_{\beta_1}, \sigma_{\beta_2}, ..., \sigma_{\beta_2})$ and $(\sigma_{\beta_1}, ..., \sigma_{\beta_2})$ are in the same orbit. And by the expression of $l'(\sigma_{\beta_1}) < l'(\sigma_{\beta_2})$. Which is a contradiction.

Case3: If we delete some simple reflection in $\omega_i^{-1}$. This is similar to Case2.

Case4: If we delete some $\sigma_{S_{i_r}}$ it is exactly what the lemma says.
Apply Lemma 4.12 b) inductively we can get \( s = t \). So apply this to Lemma 4.11 then it holds.

Now we come to prove Theorem 4.6.

Proof. By Lemma 4.12 a), there exists \((\sigma_{\beta_1}, \ldots, \sigma_{\beta_n})\) in the orbit of \((\sigma_{\alpha_1}, \ldots, \sigma_{\alpha_n})\) and \(l'(\sigma_{\beta_i})\) is minimal in the \(B_{i-i+1}\) action orbit of \((\sigma_{\beta_1}, \ldots, \sigma_{\beta_n})\). Since \(\sigma_{\beta_i}, 1 \leq i \leq n\) are real roots, there exists a unique indecomposable module \(F_i\) satisfying \(\dim F_i = \beta_i\).

Now we have

\[
\sigma_{F_1} \sigma_{F_2} \cdots \sigma_{F_n} = \sigma_{S_1} \cdots \sigma_{S_n}
\]

By Lemma 4.8, what we need to prove firstly is that \(F = (F_1, F_2, \ldots, F_n)\) is a projective sequence.

Denote \(S(F(r))\) be the set of composition factors of \(F_1, \ldots, F_r\).

By applying Lemma 4.12 inductively, the following properties of \(F = (F_1, F_2, \ldots, F_n)\) follow:

1) The number of simple objects appearing in \(S(F(r))\) is \(r\).
2) \(\text{top}(F_r) \notin S(F(r))\).

By our assumption, \(S = (S_1, \ldots, S_n)\) is a complete exceptional sequence. If \(S(F(r)) = \{S_{i_1}^{(r)}, \ldots, S_{i_r}^{(r)}\}\) where \(i_1^{(r)} < i_2^{(r)} < \ldots < i_r^{(r)}\), \((S_{i_1}^{(r)}, \ldots, S_{i_r}^{(r)})\) is a complete exceptional sequence of \(\mathcal{C}(S(F(r)))\). By Lemma 4.12, \(\dim(F_r) = \sigma_{S_{i_1}^{(r)}} \sigma_{S_{i_2}^{(r)}} \cdots \sigma_{S_{i_k}^{(r)}} (\dim S_{i_k}^{(r)})\) for some \(k, 1 \leq k \leq r\). Thus \(F_r\) is a projective object in \(\mathcal{C}(S(F(r)))\). So \(F\) is a projective sequence. By Lemma 4.8, \(F\) is an exceptional sequence. By Theorem 3.9, \(F\) and \(S\) are in the same orbit. If \(g \cdot S = F\) for some \(g \in B_n\), by \(\dim R_XY = \pm \sigma_Y(X), \dim L_XY = \pm \sigma_X(Y)\), we get \(g \cdot (\sigma_{S_1}, \sigma_{S_2}, \ldots, \sigma_{S_n}) = (\sigma_{F_1}, \sigma_{F_2}, \ldots, \sigma_{F_n})\). The theorem is proved.

\[\square\]

5. INGALLS-THOMAS-IGUSA-SCHIFFLER BIJECTION

In the previous section, we label the simple modules in an appropriate order such that \(S\) is a complete exceptional sequence. In this section, we write \(c(\Lambda) = \sigma_{S_1} \cdots \sigma_{S_n}\). By then we have two posets: \(A(\text{mod}\Lambda)\) and \(Nc(W(\Lambda), c(\Lambda))\).
By Theorem 4.6 and Theorem 3.9 we have

**Lemma 5.1.** If $S = (S_1, S_2, ..., S_n)$ is the complete exceptional sequence consisting of simple modules, and for $n$ exceptional modules $\{E_1, E_2, ..., E_n\}$ we have

$$\sigma_{S_1} \sigma_{S_2} \cdots \sigma_{S_n} = \sigma_{E_1} \sigma_{E_2} \cdots \sigma_{E_n}$$

then $E = (E_1, E_2, ..., E_n)$ is a complete exceptional sequence.

**Proof.** By Theorem 4.6, $(\sigma_{S_1}, ..., \sigma_{S_n})$ and $(\sigma_{E_1}, ..., \sigma_{E_n})$ are in the same orbit under braid group action. So

$$g \cdot (\sigma_{S_1}, ..., \sigma_{S_n}) = (\sigma_{E_1}, ..., \sigma_{E_n})$$

for some $g \in B_n$. By the formulas in Lemma 3.10:

$$\dim R_{XY} = \pm \sigma_X(Y)$$

$$\dim L_{XY} = \pm \sigma_Y(X)$$

If $\rho_i(\sigma_{E_1}, ..., \sigma_{E_n}) = (\sigma_{E_1'} \cdots \sigma_{E_n'})$, $E_i, F_j$ in $T^n$, then for modules $\rho_i(E_1', ..., E_n') = (F_1', ..., F_n')$ in the level of exceptional sequence. So $(E_1, ..., E_n) = g^{-1} \cdot (S_1, ..., S_n)$ is a complete exceptional sequence. \[\square\]

Now we can prove as same as in [5].

**Main Theorem (Ingalls-Thomas-Igusa-Schiffler)** The map

$$\text{cox} : A(\text{mod} \Lambda) \to Nc(W(\Lambda), c(\Lambda))$$

is a poest isomorphism.

**Proof.** If $\text{cox}(A) = \text{cox}(B)$, then we can choose a complete exceptional sequence $A = (A_1, ..., A_r)(B = (B_1, ..., B_t))$ for $A(B)$, then we must have

$$\sigma_{A_1} \sigma_{A_2} \cdots \sigma_{A_r} = \sigma_{B_1} \sigma_{B_2} \cdots \sigma_{B_t}$$

By Lemma 3.4, we can extend $A$ to $A' = (A_1, ..., A_r, A_{r+1}, ..., A_n)$, so we can see that $\sigma_{B_1} \sigma_{B_2} \cdots \sigma_{B_t} \sigma_{A_{r+1}} \cdots \sigma_{A_n}$ is the Coxeter element. By Lemma 5.1, $(B_1, ..., B_t, A_{r+1}, ..., A_n)$ is a complete exceptional sequence. So we have

$$B = (\ominus A)^\perp = A$$

which proves the injection.

For surjection, if $\omega = \sigma_{E_1} \sigma_{E_2} \cdots \sigma_{E_r} \in Nc(W(\Lambda), c(\Lambda))$, then we can extend it to $\sigma_{E_1} \sigma_{E_2} \cdots \sigma_{E_n}$ which is the Coxeter element. By Lemma 5.1, $(E_1, E_2, ..., E_n)$ is a complete sequence. We just need to choose $A = \mathcal{C}(E_1, ..., E_r)$. 
Finally we prove that this is a poset morphism, i.e. \( A \leq B \Leftrightarrow \cox(A) \leq \cox(B) \). If \( A \leq B \), choose a complete exceptional sequence \( A = (A_1, \ldots, A_r) \) of \( A \). By Lemma 3.4, it can be extended to a complete exceptional sequence \( B = (A_1, \ldots, A_r, B_{r+1}, \ldots, B_t) \) of \( B \), thus \( \cox(A) \leq \cox(B) \) by the definition. Conversely, if \( \cox(A) \leq \cox(B) \), i.e. \( \sigma_{A_1}\sigma_{A_2}\ldots\sigma_{A_r} \leq \sigma_{B_1}\sigma_{B_2}\ldots\sigma_{B_t} \) where \( A = (A_1, \ldots, A_r) \) (resp.\( B = (B_1, \ldots, B_t) \)) is a complete exceptional sequence of \( A \) (resp.\( B \)). So there exists \( C_{r+1}, \ldots, C_t \in B \) such that \( \sigma_{A_1}\sigma_{A_2}\ldots\sigma_{A_r}\sigma_{C_{r+1}}\ldots\sigma_{C_t} = \sigma_{B_1}\sigma_{B_2}\ldots\sigma_{B_t} \). Thus \( A \leq B \). We finish the proof. \( \square \)

**Remark 5.2.** In fact, all the consequence above can be generalized to the Artin hereditary algebras. Given an Artin hereditary algebra \( \Lambda \), we can define the Cartan matrix associated to \( K_0(\Lambda \text{-mod}) \). It is a symmetrizable generalized Cartan matrix. So there is a Kac-Moody Lie algebra \( g(\Lambda) \). Like what we have done in this note, there is a poset isomorphism

\[
\cox : \mathbf{A}(\text{mod}\Lambda) \rightarrow \mathcal{Nc}(W(\Lambda), c(\Lambda))
\]

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