An Axiomatic Approach to Semiclassical Perturbative Gauge Field Theories

O.Yu.Shvedov

Sub-Dept. of Quantum Statistics and Field Theory,
Dept. of Physics, Moscow State University,
119992, Moscow, Vorobiev Gory, Russia

Abstract

Different approaches to axiomatic field theory are investigated. The main notions of semiclassical theory are the following: semiclassical states, Poincare transformations, semiclassical action form, semiclassical gauge equivalence and semiclassical field. If the manifestly covariant approach is used, the notion of semiclassical state is related to Schwinger source, while the semiclassical action is presented via the R-function of Lehmann, Symanzik and Zimmermann. Semiclassical perturbation theory is constructed. Its relation with the S-matrix theory is investigated. Semiclassical electrodynamics and non-Abelian gauge theories are studied, making use of the Gupta-Bleuler and BRST approaches.

Keywords: Maslov semiclassical theory, axiomatic quantum field theory, Bogoliubov S-matrix, Lehmann-Symanzik-Zimmermann approach, Schwinger sources, gauge theories, BRST-quantization.
1 Introduction

A usual way to construct quantum field theory is as follows (see, for example, [1]). First of all, one considers the classical field theory instead of the quantum one. In order to make the classical theory to be manifestly covariant, one uses the Lagrangian approach instead of Hamiltonian. Then one rewrites a manifestly covariant Lagrangian theory to the Hamiltonian language and postulates the "rules" of canonical quantization. Quantum field models with interaction are usually not exactly solvable; therefore, they are investigated with the help of formal perturbation theory. Analogously to quantum mechanical case, one constructs a formal perturbation series for the S-matrix expressed via the T-exponent and for the Green functions, a parameter of expansion is coupling constant.

An alternative way to construct quantum field theory is to use the functional integral approach [2]: quantum field Green functions are written via the functional integrals (of exponent of the classical action) analogously to quantum mechanics [3]. This approach is formally manifestly covariant. However, one should take into account that the non-perturbative definition of a functional interal is not rigorous.

Making use of the calculated Green functions, one can reconstruct all objects of the theory with the help of Whightmann reconstruction theorem [4, 5]. The problem is that for this purpose it is necessary to know the exact non-perturbative Green functions.

The main difficulty of quantum field theory is the problem of divergences. They arise in the functional integral approach, as well as in the T-exponent technique. To eliminate the divergences, one should first introduce the regularization. It may be manifestly covariant (Pauli-Willars, dimensional) or non-covariant (ultraviolet cutoff, dimensional). Then one should perform the renormalization. Counterterms are added to the Lagrangian in order to make the Green functions finite. When the non-covariant regularization is used, one should explicitly check Poincare invariance. For theories with symmetries, it is also necessary to check them in the renormalized theory.

Another group of approaches to construct quantum field perturbation theory is based on the axiomatic approaches [4, 5, 6, 7, 8, 9, 10]. One first formulates the axioms for the Green functions and related objects. This allows to construct the renormalized theory without renormalization. Examples of axiomatic theories are Bogoliubov S-matrix theory [7] with switching on the interaction, Schwinger source theory [10], Lehmann-Symanzik-Zimmermann (LSZ) approach [9] and S-matrix approach of Bogoliubov, Medvedev and Polivanov [7, 8] which is equivalent to LSZ approach.

Making use of the Bogoliubov S-matrix theory, one can obtain the renormalized perturbation theory. One uses the most general properties: Poincare covariance, unitarity and causality. However, there is a non-uniqueness in axiomatic perturbation theory: each order of the S-
matrix is found up to a quasilocal operator.

There are also non-perturbative approaches to quantum field theory. One of them is a semiclassical approximation. Examples of semiclassical results are soliton quantization and instantons [11, 12], quantum field theory in the external background [13] and curved spacetime [14]. Semiclassical theory was developed, making use of the Hamiltonian and functional integral approaches.

The purpose of this paper is to clarify the relationship between Hamiltonian and axiomatic field theories in the semiclassical approach. The semiclassical perturbation theory is constructed, making use of the axiomatic conceptions.

Section 2 deals with general properties of the semiclassical states of the quantum field system. The semiclassical analogs of the QFT axioms are formulated. They should be satisfied for Hamiltonian and axiomatic approaches.

Section 3 is devoted to the manifestly covariant approach to the quantum field theory. Semiclassical states are related to Schwinger sources, while semiclassical action and fields are expressed via LSZ R-functions.

The leading order of semiclassical expansion is developed in section 4. Semiclassical perturbation theory is developed in section 5. Sections 6 and 7 are devoted to semiclassical gauge theories.

2 General Structure of Semiclassical Perturbation Field Theory

2.1 Semiclassical States

A usual way to develop semiclassical field theory (QFT in the external background) is as follows. The field $\varphi$ is presented as a sum of ”classical”, c-number part and fluctuation, ”quantum” part, which is small with respect to the classical one. Action is expanded into a series in powers of quantum fluctuation. In the leading order the quadratic part of action only is taken to account.

Semiclassical methods can be applied iff the Lagrangian of the theory $\mathcal{L}$ depends on the small parameter $\hbar$ (”Planck constant”) as follows (the scalar case is considered for the simplicity):

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - \frac{1}{\hbar} V(\sqrt{\hbar} \varphi).$$

(2.1)

with $V(\Phi)$ being a scalar potential. The classical c-number component is of the order $1/\sqrt{\hbar}$. 

2
In terms of quantum states and equation of motion, this semiclassical method can be reformulated as follows. The following time-dependent state vector \( \Psi(t) \) is considered:

\[
\Psi(t) \simeq e^{\frac{i}{\hbar}S(t)} e^{\frac{i}{\sqrt{\hbar}} \int dx \left[ \Pi(x,t) \hat{\varphi}(x) - \Phi(x,t) \hat{\pi}(x) \right]} f(t).
\]  

(2.2)

Here \( S(t) \) is a real c-number function of \( t \), \( \Phi(x,t) \) and \( \Pi(x,t) \) are classical fields and canonically conjugated momenta, \( \hat{\varphi}(x) \) and \( \hat{\pi}(x) \) are quantum field and momentum operators, \( f(t) \) is a regular as \( \hbar \to 0 \) state vector. Substitute vector (2.2) to Schrodinger equation for the theory (2.1). One obtains in the leading order classical equations of motion for \( \Pi(x,t), \Phi(x,t) \), formula for classical action for \( S(t) \) and equation with quadratic Hamiltonian for \( f(t) \). This is in agreement with method of extracting a c-number component of the field.

In terms of the semiclassical Maslov theory [17, 18, 19], state (2.2) corresponds to the Maslov complex germ in a point. There is also a theory of Lagrangian manifolds with complex germs. It can be also generalized to quantum field case. One considers superpositions of states (2.2) [20]:

\[
\Psi(t) \simeq \text{const} \int d\alpha e^{\frac{i}{\hbar}S(\alpha,t)} e^{\frac{i}{\sqrt{\hbar}} \int dx \left[ \Pi(x,\alpha,t) \hat{\varphi}(x) - \Phi(x,\alpha,t) \hat{\pi}(x) \right]} f(\alpha,t).
\]  

(2.3)

where \( \alpha \) is an additional \( k \)-dimensional parameter.

Semiclassical states (2.3) cannot be considered within the framework of extracting the c-number component. On the other hand, such semiclassical states are very important. When systems with integrals of motion are considered, it is necessary to construct the state satisfying additional conditions. For example, when one quantize kink solution [11], quantum state should be an eigenstate of energy and momentum operators. Projection on the eigenspace of the momentum operator leads to the superposition (2.3). When one uses the simpler semiclassical methods and investigates states of the form (2.2) only, one comes to the well-known difficulty of soliton zero modes [11].

Semiclassical state vectors (2.2) and (2.3) possesses the following geometric interpretation [21]. denote the set of classical variables \( (S,\Pi(x,\Phi(x))) \) as \( X \), the operator that maps vector \( f \) to the state (2.2) is denoted as \( K^h_X \) ("the canonical operator"):

\[
K^h_X f \equiv e^{\frac{i}{\hbar}S} e^{\frac{i}{\sqrt{\hbar}} \int dx \left[ \Pi(x) \hat{\varphi}(x) - \Phi(x) \hat{\pi}(x) \right]} f.
\]  

(2.4)

Then the semiclassical state (2.4) can be viewed as a point on the space of a vector bundle ("semiclassical bundle"). The base of the bundle is set \( \{X\} \) of classical states, fibres \( \{f\} \) are state spaces in the external background \( X \). States of the more general form (2.5) are written as

\[
\int d\alpha K^h_X(\alpha) f(\alpha), \quad \alpha = (\alpha_1, ..., \alpha_k);
\]  

(2.5)

3
they can be identified with $k$-dimensional surfaces on the semiclassical bundle.

Presentation of semiclassical form in the form (2.4) is not manifestly covariant. There are space and time coordinates. It happens that the manifestly covariant form of the state (2.4) is the following:

$$
\Psi \simeq e^{\frac{i}{\sqrt{h}}S_T} \exp\{\frac{i}{\sqrt{h}} \int dx J(x) \hat{\varphi}_h(x) \} \mathcal{F} \equiv e^{\frac{i}{\sqrt{h}}S_T} \mathcal{F}.
$$

(2.6)

Here $S$ is a real number, $J(x)$ is a real function (classical Schwinger source), $\hat{\varphi}_h(x)$ is a Heisenberg field operator, $\mathcal{F}$ is a state vector being regular as $h \to 0$. The Schwinger source $J(x)$ should be rapidly damping at space and time infinity.

One can find that state (2.6) approximately coincides as $h \to 0$ with (2.2), and find a correspondence between $(S, \Pi, \Phi, f)$ and $(\mathcal{S}, J, \mathcal{F})$.

In the covariant approach, the base of the semiclassical bundle is a set of $\{X = (\mathcal{S}, J)\}$, the operator $K^\hbar_X$ has the form $e^{\frac{i}{\sqrt{h}}S_T^\hbar}$. Superpositions of states (2.6) are written as (2.5).

### 2.2 Properties of semiclassical states

The main principles of the axiomatic quantum field theory are [4] Poincare invariance, unitarity and causality. The general structures are introduced: the quantum Poincare transformation operator $U^\hbar_g$ corresponding to the classical Poincare transformation $g \equiv (a, \Lambda)$ of the form $x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$. The group property should be satisfied:

$$
U^\hbar_{g_1 g_2} = U^\hbar_{g_1} U^\hbar_{g_2}.
$$

(2.7)

The Heisenberg field operator $\hat{\varphi}_h(x)$ should satisfy the property of Poincare invariance:

$$
U^\hbar_{g^{-1}} \hat{\varphi}_h(x) U^\hbar_g = \hat{\varphi}_h(w_g x), \quad w_g x = \Lambda^{-1}(x - a)
$$

(2.8)

Formula (2.8) is written for the scalar case; for the more complicated cases it should be modified.

Analogous structures should arise for the semiclassical case as well.

The following commutation relations between quantum field operators and canonical operator $K^\hbar_X$ are satisfied:

$$
U^\hbar_g K^\hbar_X f = K^\hbar_{u_g X} U^\hbar_g (u_g X \leftarrow X) f; \quad \sqrt{h} \hat{\varphi}_h(x) K^\hbar_X f = K^\hbar_X \Phi(x|X) f,
$$

(2.9)

Here $u_g X$ is a Poincare transformation of the classical state, $U^\hbar_g (u_g X \leftarrow X)$ is an unitary operator taking initial state in the external background $X$ to final state in the external background $u_g X$. The operator $U^\hbar_g (u_g X \leftarrow X)$ is supposed to be expanded into an asymptotic series in $\sqrt{h}$. Therefore, in the semiclassical theory the Poincare group is an automorphism group of
the semiclassical bundle. Semiclassical field $\Phi(x|X)$ (distribution) is also viewed as a formal asymptotic series in $\sqrt{h}$. In the leading order, it has a $c$-number form $\Phi(x|X)$ (it is a classical field):

$$\Phi(x|X) = \Phi(x|X) + \sqrt{h}\Phi^{(1)}(x|X) + ... \quad (2.10)$$

It follows from the group property (2.7) and commutation relation (2.9) that

$$U_{g_1g_2}(u_{g_1g_2}X \leftarrow X) = U_{g_1}(u_{g_1}X \leftarrow u_{g_2}X) = U_{g_2}(u_{g_2}X \leftarrow X); \quad (2.11)$$

Making use of (2.8), one finds:

$$\Phi(x|u_gX)U_g(u_gX \leftarrow X) = U_g(u_gX \leftarrow X)\Phi(w_gX|X). \quad (2.12)$$

There are also the following additional structures of semiclassical field theory. To calculate the norm of the semiclassical state, one should evaluate the integral of the form:

$$\int d\alpha d\alpha' \left\{ K_hX(\alpha)f(\alpha), K_hX(\alpha')f(\alpha') \right\} \quad (2.13)$$

If the difference between classical configurations $X(\alpha)$ and $X(\alpha')$ is of the order $O(1)$ as $h \to 0$, the inner product $(K_hX(\alpha)f(\alpha), K_hX(\alpha')f(\alpha'))$ is exponentially small. This means that the state vectors corresponding to different classical configurations are orthogonal with exponential accuracy. Therefore, the integrand in (2.13) is not exponentially small for the case of small $\alpha - \alpha'$ only; more precisely, for $\alpha - \alpha' \sim \sqrt{h}$.

Therefore, it is necessary to expand the states $K_hX^h\delta Xf$ and $K_hX^h\sqrt{\delta X}f$ into series in $\sqrt{h}$ as $h \to 0$. It happens that in the leading order in $h$

$$K_hX^h\delta Xf \simeq e^{-i\omega_X[\delta X]}K_hX f, \quad (2.14)$$

expression $\omega_X[\delta X]$ is an action 1-form on classical phase space. If we consider the case $\omega_X[\delta X] = 0$, the shift of classical variable of the order $\sqrt{h}$ leads to the relation

$$K_hX^h\sqrt{\delta X}f \simeq K_hX e^{-i\omega_X^{(1)}[\delta X]}f; \quad \omega_X[\delta X] = 0 \quad (2.15)$$

with an operator-valued 1-form $\omega_X^{(1)}[\delta X]$.

To write down all the orders of the expansion in $\sqrt{h}$, it is convenient to start from the commutation rule between differentiation operator and canonical operator:

$$ih\frac{\partial}{\partial \alpha_a}K_hX(\alpha) = K_hX(\alpha)\omega_X(\alpha)[\frac{\partial X}{\partial \alpha_a}]. \quad (2.16)$$
Making use of formula (2.16), one introduces to the semiclassical mechanics a new structure, operator-valued 1-form $\omega_X[\delta X]$. It maps a tangent vector $\delta X$ for the base of the semiclassical bundle to the operator $\omega_X[\delta X]$. It is expanded into an asymptotic series in $\sqrt{\hbar}$. In the leading order, it is a c-number:

$$\omega_X[\delta X] = \omega_X[\delta X] + \sqrt{\hbar}\omega_X^{(1)}[\delta X] + ... \tag{2.17}$$

Rules (2.14) and (2.15) are obtained as follows [22]. One expands the operator $K^\hbar_{X(\alpha+\sqrt{\hbar}\beta)}$ in $\sqrt{\hbar}$ as

$$K^\hbar_{X(\alpha+\sqrt{\hbar}\beta)} = K^\hbar_{X(\alpha)}V_\hbar(\alpha, \beta); \tag{2.18}$$

then one differentiates the relation (2.18) with respect to $\beta_a$ and obtains an equation for $V_\hbar$.

Its solution gives a relation (2.15).

Consider the substitution $\alpha' = \alpha + \sqrt{\hbar}\beta$ for eq. (2.13). According to eqs.(2.14) and (2.15), the integrand will be rapidly oscillating, except for the case of the Maslov isotropic condition:

$$\omega_X(\alpha)[\frac{\partial X}{\partial \alpha_a}] = 0. \tag{2.19}$$

The superposition (2.5) can be investigated under condition (2.19) only. Otherwise, the exponentially small norm of the state (2.5) will be larger than the accuracy.

Investigate the properties of the new object $\omega$.

First, notice that for multidimensional $\alpha = (\alpha_1, ..., \alpha_k)$ it is possible to write $[i\hbar \frac{\partial}{\partial \alpha_a}, i\hbar \frac{\partial}{\partial \alpha_b}] = 0$; commuting this relation with the canonical operator $K^\hbar_X$, one finds:

$$\begin{bmatrix} \omega_X(\alpha)[\frac{\partial X}{\partial \alpha_a}]; \omega_X(\alpha)[\frac{\partial X}{\partial \alpha_b}] \end{bmatrix} = -i\hbar \begin{bmatrix} \frac{\partial}{\partial \alpha_a} \omega_X(\alpha)[\frac{\partial X}{\partial \alpha_b}] - \frac{\partial}{\partial \alpha_b} \omega_X(\alpha)[\frac{\partial X}{\partial \alpha_a}] \end{bmatrix}, \tag{2.20}$$

or, in terms of differential forms,

$$[[\omega_X[\delta X_1], \omega_X[\delta X_2]] = -i\hbar d\omega_X(\delta X_1, \delta X_2). \tag{2.21}$$

Furthermore, it follows from the properties

$$[i\hbar \frac{\partial}{\partial \alpha_a}, \phi^\hbar(x)] = 0, \quad [i\hbar \frac{\partial}{\partial \alpha_a}, U^\hbar_g] = 0$$

that

$$U^\hbar_g(u_gX \leftarrow X)\omega_X[\frac{\partial X}{\partial \alpha_a}] = \omega_{u_gX}[\frac{\partial (u_gX)}{\partial \alpha_a}]U^\hbar_g(u_gX \leftarrow X) + i\hbar \frac{\partial}{\partial \alpha_a}U^\hbar_g(u_gX \leftarrow X); \tag{2.22}$$

$$i\hbar \frac{\partial}{\partial \alpha_a}\Phi(x|X) = \left[\Phi(x|X); \omega_X[\frac{\partial X}{\partial \alpha_a}] \right]. \tag{2.23}$$

Relation (2.22) is useful for investigating the properties of the evolution operator.
2.3 Equivalence of semiclassical states

A remarkable feature of the covariant approach to the semiclassical field theory is the following. Starting from the classical Schwinger source $J$, one uniquely reconstructs classical field and momenta entering to eq.(2.4). On the other hand, this correspondence is not one-to-one. Two different sources may correspond to the same configuration $(\Pi(x), \Phi(x))$. Therefore, there is an equivalence relation on the classical space (base of the semiclassical bundle). Moreover, it happens that for each pair of equivalent classical states $X_1 \sim X_2$ semiclassical states

$$K^h_{X_1} f_1 \simeq K^h_{X_2} f_2$$

(approximately coincide under a certain condition between $f_1$ and $f_2$):

$$f_2 = V(X_2 \leftarrow X_1) f_1.$$  

Here $V(X_2 \leftarrow X_1)$ is an unitary operator. Thus, an equivalence relation arises on the semiclassical bundle.

Investigate the properties of the operator $V(X_2 \leftarrow X_1)$. Notice that the following relation

$$V(u_g X_2 \leftarrow u_g X_1) \simeq V(u_g X_2 \leftarrow X_2) V(X_2 \leftarrow X_1) \ \text{and} \ \Phi(x|X_2) V(X_2 \leftarrow X_1) = V(X_2 \leftarrow X_1) \Phi(x|X_1).$$

Finally, let $(X_1, \bar{f}_1)$ and $(X_2, \bar{f}_2)$ depend on the parameter $\alpha$. Then one writes

$$ih \frac{\partial}{\partial \alpha} K^h_{X_1} \bar{f}_1 \simeq ih \frac{\partial}{\partial \alpha} K^h_{X_2} \bar{f}_2.$$  

Therefore,

$$V(X_2 \leftarrow X_1) \omega_{X_1} \left[ \frac{\partial X_2}{\partial \alpha} \right] \omega_{X_2} [\frac{\partial X_1}{\partial \alpha}] = \omega_{X_2} [\frac{\partial X_2}{\partial \alpha}] V(X_2 \leftarrow X_1) + ih \frac{\partial}{\partial \alpha} V(X_2 \leftarrow X_1)$$

as $X_1(\alpha) \sim X_2(\alpha)$. 

7
2.4 System of axioms of the semiclassical field theory

Thus, all the problems of semiclassical field theory can be solved within the perturbation framework under certain conditions (axioms). They should be satisfied for both Hamiltonian and covariant approaches. They will be denoted as G1, G2, G3, G4, G5.

**A.** A semiclassical bundle is given. Space of the bundle is set of all semiclassical states. The base \( \mathcal{X} = \{X\} \) is set of classical states \( X \), fibres \( F_X \) are Hilbert spaces of quantum states in the external background \( X \).

**G2.** The Poincare group is presented as a group of automorphisms of the semiclassical bundle. Classical transformations \( u_g : \mathcal{X} \to \mathcal{X} \) and unitary operators \( U_g(u_gX \leftarrow X) : F_X \to F_{u_gX} \) expanded in \( \sqrt{\hbar} \) are given. The properties (2.11) are satisfied.

**G3.** An operator-valued 1- \( \omega \) on the semiclassical bundle is given. It maps tangent vector \( \delta X \) to Hermitian operator \( \omega_X[\delta X] \). It is expanded in \( \sqrt{\hbar} \) according to eq. (2.17) \( (\omega_X[\delta X] \) is a c-number) and satisfied properties (2.20) and (2.22).

**G4.** An equivalence relation may be introduced on the semiclassical bundle. For each pair \( X_1 \sim X_2 \) of equivalent points of the base an unitary operator \( V(X_2 \leftarrow X_1) \) is given. It is expanded in \( \sqrt{\hbar} \). Relations (2.25), (2.26) and (2.28) are satisfied.

**G5.** An operator-valued distribution \( \Phi(x|X) \) ("semiclassical field") is given. It is expanded in \( \sqrt{\hbar} \) (eq.(2.10), \( \Phi(x|X) \) is a c-number) and satisfies the relations (2.12), (2.23) and (2.27).

3 Specific features of the covariant approach to the semiclassical field theory

3.1 Objects of the semiclassical theory and LSZ R-functions

Let us investigate the objects of the covariant formulations of semiclassical field theory.

Semiclassical states in the covariant approach are presented in a form (2.6). Therefore, the base of the semiclassical bundle consists of the classical states \( X = \{S, J\} \), while all the spaces \( F_X \) coincide.

It follows from eq.(2.8) that

\[
U_g^h T^h_j U_g^\dagger = T^h_{u_gJ} U_g^h \dagger,
\]

where \( u_gJ(x) = J(w_gx) \), \( w_g \) has the form (2.8). Therefore, transformation \( u_g \) is known explicitly, satisfy property \( u_{g_1g_2} = u_{g_1}u_{g_2} \), the operator \( U_g(u_gX \leftarrow X) = U_g \equiv \mathcal{U}_g = U_g^h \) does not depend on \( X \) and satisfies the group property and field covariance relation:

\[
U_{g_1g_2} = U_{g_1}U_{g_2}, \quad U_{g^{-1}}\hat{\psi}_h(x)U_g = \hat{\psi}_h(w_gx).
\] (3.1)
It happens that in the covariant approach the 1-form $\omega$ is related to another important object, the LSZ R-function [9, 4] of the form

$$\Phi_R(x|J) \equiv -i\hbar(T^h_J) + \frac{\delta T^h_J}{\delta J(x)}.$$  

(3.2)

To check this statement, notice that the differentiation (or variation) operator commutes with the operator $K^{h}_{S,J} \equiv e^{\frac{i}{\hbar}S}T^h_J$ as

$$i\hbar\delta K^{h}_{S,J} = K^{h}_{S,J}[-\delta S - \int dx\Phi_R(x|J)\delta J(x)];$$

therefore,

$$\omega_X[\delta X] = -\delta S - \int dx\Phi_R(x|J)\delta J(x), \quad (3.3)$$

Here $\Phi_R(x|J)$ and $\omega$, are expanded in $\sqrt{\hbar}$; one writes

$$\Phi_R(x|J) = \Phi(x|J) + \sqrt{\hbar}\Phi^{(1)}(x|J) + \ldots \quad (3.4)$$

The c-number function $\Phi_R(x|J)$ is called as a retarded classical field generated by the Schwinger source $J$.

It is shown in [23] that for the model (2.1) $\Phi_R(x|J)$ is a solution of the equation

$$\partial_{\mu}\partial^{\mu}\Phi_R(x|J) + V'(\Phi_R(x|J)) = J(x), \quad \Phi_R|_{x < supp J} = 0. \quad (3.5)$$

which vanishes as $x^0 \to -\infty$.

The following properties of LSZ R-functions are well-known [9, 4]. They are corollaries of (3.2).

1. The Hermitian property

$$\Phi_R^{\dagger}(x|J) = \Phi_R(x|J). \quad (3.6)$$

2. The Poincare invariance property

$$U_g^{-1}\Phi_R(x|u_g J)U_g = \Phi_R(w_g x|J). \quad (3.7)$$

3. The Bogoliubov causality property: R-function $\Phi_R(x|J)$ depends only on the source $J$ at the preceding time moments. Making use of the standard notations $x > y$ iff $x^0 - y^0 \geq |x-y|$, $x < y$ iff $x^0 - y^0 \leq |x-y|$, $x \sim y$ iff $|x^0 - y^0| < |x-y|$, one rewrites the Bogoliubov condition as

$$\frac{\delta \Phi_R(x|J)}{\delta J(y)} = 0, \quad y \sim x. \quad (3.8)$$
4. Commutation relation

\[
[\Phi_R(x|J); \Phi_R(y|J)] = -i\hbar \left( \frac{\delta \Phi_R(x|J)}{\delta J(y)} - \frac{\delta \Phi_R(y|J)}{\delta J(x)} \right).
\]  

(3.9)

5. Boundary condition at \(-\infty\). If \(x < \sim y\) for all \(y \in \text{supp} J\), the LSZ R-function does not depend on the source:

\[
\Phi_R(x|J) = \hat{\varphi}_h(x) \sqrt{\hbar}, \quad x \sim \text{supp} J.
\]  

(3.10)

In particular, the classical retarded field vanishes as \(x^0 \to -\infty\).

The semiclassical field operator \(\Phi(x|J)\) can be also expressed via the R-function. Namely, at \(+\infty\) the following property is satisfied:

\[
\Phi(x|J) = \Phi_R(x|J), \quad x \sim \text{supp} J.
\]  

(3.11)

In particular, the classical field \(\Phi(x|J)\) corresponding to the source \(J\) coincide with the classical retarded field \(\Phi_R(x|J)\) at \(x \sim \text{supp} J\). The case when the property \(x \sim \text{supp} J\) is not satisfied is considered below.

### 3.2 Equivalence of semiclassical states

Investigate the property of equivalence of semiclassical states. It is convenient to consider first the partial case when the semiclassical state is equivalent to the state \(J = 0\). We say that \(J \sim 0\) iff

\[
T_J^0 \mathcal{J} \simeq e^{i\pi T_J} W_{J,\mathcal{J}}
\]  

for some number \(T_J\) and operator \(W_{J,\mathcal{J}}\) presented as a formal asymptotic series.

It is shown in [23] for the model (2.1) that the source \(J\) is equivalent to zero iff the retarded field generated by \(J\) vanishes at \(+\infty\).

Analogously to [23], one derives the following properties.

1. Poincaré invariance.

\[
U_g W_{J,\mathcal{J}} U_g^{-1} = W_{gJ}, \quad T_{gJ} = T_J;
\]  

(3.13)

2. Unitarity

\[
W_J^+ = W_J^{-1};
\]  

(3.14)

3. Bogoliubov causality: as \(J + \Delta J_2 \sim 0\), \(J + \Delta J_1 + \Delta J_2 \sim 0\) and \(\text{supp} \Delta J_2 \sim \text{supp} \Delta J_1\) the operator \((W_{J+\Delta J_2})^+ W_{J+\Delta J_1+\Delta J_2}\) and number \(-T_{J+\Delta J_2} + T_{J+\Delta J_1+\Delta J_2}\) do not depend on \(\Delta J_2\).
4. Variational property:

\[ \delta I - i\hbar W^+ J \delta W^J = \int dx \Phi_R(x|J) \delta J(x), \]  

(3.15)

which is valid as \( J \sim 0 \) and \( J + \delta J \sim 0 \).

5. Boundary condition at \(+\infty\):

\[ \Phi_R(x|J) = W^+ J \hat{\phi}(x) \sqrt{hW J}, \quad x > \sim \text{supp} J. \]  

(3.16)

It follows from (3.16) that the retarded classical field generated by the source \( J \sim 0 \) will vanish at \(+\infty\). For the model (2.1), an inverse statement is also valid: for any field configuration \( \Phi_c(x) \) with the compact support one can uniquely choose a source \( J \sim 0 \) (denoted as \( J = J_{\Phi_c} = J(x|\Phi_c) \); for example (2.1), it is found from the relation (3.5)) generating \( \Phi_c(x) \) as a retarded classical field: \( \Phi_c(x) = \Phi_R(x|J) \); it satisfies the locality condition \( \frac{\delta J(x|\Phi_c)}{\delta \Phi_c(y)} = 0 \) as \( x \neq y \).

It is possible to treat this statement as a basic postulate of semiclassical field theory. Then the theory may be developed without additional postulating classical stationary action principle and canonical commutation relation.

Namely, it follows from eq. (3.15) in the leading order in \( \hbar \) that the functional

\[ I[\Phi_c] = \mathcal{T} \Phi_c - \int dx J_{\Phi_c}(x) \Phi_c(x) \]  

(3.17)

satisfies the "classical equation of motion"

\[ J_{\Phi_c}(x) = -\frac{\delta I[\Phi_c]}{\delta \Phi_c(x)}. \]  

(3.18)

The functional \( I[\Phi] \) should satisfy the locality condition

\[ \frac{\delta^2 I}{\delta \Phi_c(x) \delta \Phi_c(y)} = 0, \quad x \neq y \]  

(3.19)

This means that it is presented as an integral of a local Lagrangian.

Relation (3.18) allows us to reconstruct the classical retarded field, making use of known source \( J \sim 0 \), since the boundary condition at \(-\infty\) is known. It follows from the Bogoliubov causality condition that the retarded field depends only on \( J \) at the preceding time moments. If the source \( J(x) \) is not equivalent to zero, it can be modified at \(+\infty\) and transformed to the source equivalent to zero. Therefore, the relation

\[ \frac{\delta I}{\delta \Phi_c}[\Phi_R(\cdot|J)] = -J(x), \quad \Phi_R\big|_{x<\text{supp} J} = 0 \]  

(3.20)
is valid for all sources $J$. For the case $x \gtrsim \text{supp} J$, the property (3.20) is taken to the classical field equation

$$\frac{\delta I}{\delta \Phi_c(x)}[\Phi(\cdot | J)] = 0. \quad (3.21)$$

This is a classical stationary action principle. It is viewed as a corollary of other general principles of semiclassical field theory.

Thus, we see that classical action $I[\Phi_c]$ in field theory is related with the phase of the state $T^J_H$ as $J \sim 0$ according to eq. (3.17).

Let us rewrite the properties of the operator $W_J$ via the field $\Phi_c$. Denote $W[\Phi_c] \equiv W_{I_{\Phi_c}}$.

1. Poincare invariance.

$$U_g W[\Phi_c] U_{g^{-1}} = W[u_g \Phi_c]. \quad (3.22)$$

2. Unitarity.

$$W^+[\Phi_c] = (W[\Phi_c])^{-1}. \quad (3.23)$$

3. Bogoliubov causality.

$$\frac{\delta}{\delta \Phi_c(y)} \left( W^+[\Phi_c] \frac{\delta W[\Phi_c]}{\delta \Phi_c(x)} \right) = 0, \quad y \sim x; \quad (3.24)$$

4. Yang-Feldman relation.

$$\int dy \frac{\delta^2 I}{\delta \Phi_c(x) \delta \Phi_c(y)} [\Phi_R(y| J) - \Phi_c(y| J)] = i\hbar W^+[\Phi_c] \frac{\delta W[\Phi_c]}{\delta \Phi_c(x)}. \quad (3.25)$$

5. Boundary condition.

$$W^+[\Phi_c] \hat{\phi}_h(x) \sqrt{\hbar} W[\Phi_c] = \Phi_R(x| J_{\Phi_c}), \quad x \sim \text{supp} \Phi_c, \quad (3.26)$$

Here $\hat{\phi}_h(x) = \Phi_R(x|0)$ is the field operator without source.

### 3.3 Set of axioms of covariant semiclassical field theory

The covariant axioms of semiclassical field theory are as follows.

- **C1.** A Hilbert state space $\mathcal{F}$ is given.

- **C2.** An unitary representation of the Poincare group is given. The operators of the representation $U_g : \mathcal{F} \to \mathcal{F}$ are asymptotic series in $\sqrt{\hbar}$.

- **C3.** To each classical source $J(x)$ with compact support one assigns a retarded field (LSZ $R$-function). It is an operator-valued distribution $\Phi_R(x| J)$ expanded in $\sqrt{\hbar}$ according to (3.4). It satisfies the properties (3.6), (3.7), (3.8), (3.9).
C4. To each classical field configuration $\Phi_c(x)$ with compact support one assigns a c-number. It is a classical action $I[\Phi_c]$ satisfying the locality condition (3.19). The property $\Phi_c(x) = \Phi_R(x|J)$ is valid iff

$$J(x) = -\frac{\delta I[\Phi_c]}{\delta \Phi_c(x)}.$$  

(3.27)

C5. To each classical field configuration $\Phi_c(x)$ with compact support one assigns the operator $W[\Phi_c]$ expanded in $\sqrt{\hbar}$. It satisfies the relations (3.22), (3.23), (3.24), (3.25), (3.26).

4 Leading order of semiclassical expansion: scalar field

4.1 Leading order for semiclassical axioms

Consider the simplest scalar field model with action

$$I[\Phi_c] = \int dx \left[ \frac{1}{2} \partial_\mu \Phi_c \partial_\mu \Phi_c - V(\Phi_c) \right],$$

satisfying the locality condition (3.19). For this model, the classical retarded field is a solution of the Cauchy problem (3.5). Denote by $U_g$ the operator $U_g$ in the leading order of perturbation theory, while $\Phi_R^{(1)}(x|J)$ will be the first correction. Denote also $\hat{\varphi}_0(x) \equiv \Phi_R^{(1)}(x|0)$, let $W_0[\Phi_c]$ be a scattering operator in the external background $W[\Phi_c]$ in the leading order in $\sqrt{\hbar}$:

$$U_g \simeq U_g, \quad W[\Phi_c] \simeq W_0[\Phi_c], \quad \Phi_R(x|J) \simeq \Phi_R^{(1)}(x|J) + \sqrt{\hbar} \Phi_R^{(1)}(x|J), \quad \hat{\varphi}_0(x) = \Phi_R^{(1)}(x|0).$$

Write down postulate C2 and properties (3.6), (3.7), (3.8) of postulate C3 in the leading order in $\sqrt{\hbar}$:

$$U_{g_1g_2} = U_{g_1} U_{g_2}, \quad U_{g^{-1}} \Phi_R^{(1)}(x|u_g J) U_g = \Phi_R^{(1)}(w_g x|J);$$

$$\Phi_R^{(1)}(x|J)^+ = \Phi_R^{(1)}(x|J), \quad \frac{\delta \Phi_R^{(1)}(x|J)}{\delta J(y)} = 0, \quad y \gtrsim x.$$  

(4.1)

Commutation relation (3.9) has the following form in the leading order in $\sqrt{\hbar}$:

$$\left[ \Phi_R^{(1)}(x|J), \Phi_R^{(1)}(y|J) \right] = -i \left( \frac{\delta \Phi_R^{(1)}(x|J)}{\delta J(y)} - \frac{\delta \Phi_R^{(1)}(y|J)}{\delta J(x)} \right)$$

(4.2)

The right-hand side of relation (4.1) is presented via the retarded Green function $D^{ret}(x,y|J)$ of equation

$$(\partial_\mu \partial^\mu + V''(\Phi_R(x|J)))\delta \Phi(x) = \delta J(x), \quad \delta \Phi|_{x<\text{supp} \delta J} = 0$$

(4.3)
as follows:

$$\left[ \Phi_R^{(1)}(x|J), \Phi_R^{(1)}(y|J) \right] = -i \left( D_{0}^{\text{ret}}(x, y|J) - D_{0}^{\text{ret}}(y, x|J) \right)$$ \quad (4.4)

Postulate C5 has the following form in the leading order in $\sqrt{\hbar}$:

$$U_g W_0[\Phi_c][\Phi_c] U_g^{-1} = W_0[u_g \Phi_c], \quad W_0^+[\Phi_c] = (W_0[\Phi_c])^{-1};$$

$$\frac{\delta}{\delta \Phi_c(y)} \left( W_0^+[\Phi_c] \frac{\delta W_0[\Phi_c]}{\delta \Phi_c(x)} \right) = 0, \ y > x; \quad (4.5)$$

$$(\partial_\mu \partial^\mu + V''(\Phi_R(x|J)))\Phi_R^{(1)}(x|J_\Phi_c) = 0; \quad (4.6)$$

$$\Phi_R^{(1)}(x|J_\Phi_c) = W_0^+[\Phi_c] \hat{\phi}_0(x) W_0[\Phi_c], \ x \sim \text{supp} \Phi_c. \quad (4.7)$$

If the objects $U_g, \Phi_R^{(1)}(x|J), W_0[\Phi_c]$ satisfying eqs. (4.1), (4.4), (4.5), (4.6), (4.7) are specified we will say that the leading order of semiclassical field theory is constructed.

### 4.2 Free case

Consider the case of free field theory: $V(\Phi_c) = \frac{m^2 \Phi_c^2}{2}$. According to eq. (3.5), the semiclassical retarded field is

$$\Phi_R(x|J) = \int D_0^{\text{ret}}(x, y)J(y)dy,$$

where $D_0^{\text{ret}}(x, y) = D_0^{\text{ret}}(x - y)$ is a retarded Green function for the Klein-Gordon equation. The operator $\Phi_R^{(1)}(x|J)$ satisfies equation (4.6)

$$(\partial_\mu \partial^\mu + m^2)\Phi_R^{(1)}(x|J) = 0. \quad (4.8)$$

It follows from the Bogoliubov causality condition that $\Phi_R^{(1)}(x|J)$ does not depend on $J(y)$ as $y \sim x$. Since the solution of Klein-Gordon equation (4.8) is uniquely expressed via the initial conditions at $-\infty$, the causality property implies that $\Phi_R^{(1)}(x|J)$ does not depend on $J$:

$$\Phi_R^{(1)}(x|J) = \hat{\phi}_0(x).$$

The field $\hat{\phi}_0(x)$ can be identified with quantum free field. It satisfies the Klein-Gordon equation and commutation relation (4.4):

$$(\partial_\mu \partial^\mu + m^2)\hat{\phi}_0(x) = 0, \quad \hat{\phi}_0^+(x) = \hat{\phi}_0(x), \quad [\hat{\phi}_0(x), \hat{\phi}_0(y)] = -i(D_0^{\text{ret}}(x, y) - D_0^{\text{ret}}(y, x)). \quad (4.9)$$
It is remarkable that the commutation relations of free field theory are obtained axiomatically, without using any postulates of canonical quantization.

It is well-known that it is possible to construct non-equivalent representations of the canonical commutation relation \[4, 24\]. However, if we suppose in addition that there exists a vacuum state vector invariant under Poincare transformations, the representation of commutation relations \((4.9)\) will be fixed. \(\hat{\varphi}_0\) should be a free scalar field of the mass \(m\). It is presented via creation and annihilation operators \(a^\pm \) of scalar particles with momentum \(p\) as \([1, 25]\)

\[
\hat{\varphi}_0(x) = \frac{1}{(2\pi)^{d/2}} \int \frac{dp}{\sqrt{2\omega_p}} [a^+_p e^{-ipx + i\omega pt} + a^-_p e^{ipx - i\omega pt}].
\]

(4.10)

Here \(d\) is a number of space dimensions, \(\omega_p = \sqrt{p^2 + m^2}\).

The properties of the Poincare transformation operator \(U_g\) are taken to the form

\[
U_{g_1 g_2} = U_{g_1} U_{g_2}, \quad U_{g^{-1}} \hat{\varphi}_0(x) U_g = \hat{\varphi}_0(w_g x).
\]

(4.11)

Therefore, the operator \(U_g\) coincides with Poincare transformation in the free field theory.

Finally, property \((4.7)\) for the scattering operator is taken to the form:

\[
\hat{\varphi}_0(x) W_0[\Phi_c] = W_0[\Phi_c] \hat{\varphi}_0(x).
\]

This means that the operator \(W_0\) is a \(c\)-number \(e^{i\gamma[\Phi_c]}\). The real functional \(\gamma[\Phi_c]\) should be Poincare invariant and satisfy the locality property \(\delta^2 \gamma / \delta \Phi_c(x) \delta \Phi_c(y) = 0\). This arbitrariness is related with the possibility of adding a one-loop correction to the free theory Lagrangian.

Since the renormalization is not necessary, one can set \(\gamma = 0\). Then

\[
W_0[\Phi_c] = 1.
\]

For the free case, the perturbation series for the objects of semiclassical theory contains the finite number of terms:

\[
\Phi_R(x|J) = \int D_0^{ct}(x, y) J(y) dy + \sqrt{\hbar} \hat{\varphi}_0(x);
\]

\[
U_g = U_g, \quad W[\Phi_c] = 1
\]

4.3 Retarded semiclassical field of the interaction theory

Let us investigate the general case, the theory with the potential \(V(\Phi_c)\). As \(J = 0\) and \(\Phi_c = 0\), the properties \((4.1), (4.4), (4.5), (4.6)\) and \((4.7)\) coincided with the free case for the square mass \(m^2 = V''(0)\). Therefore,

\[
\Phi_R^{(1)}(x|0) = \hat{\varphi}_0(x), \quad W[0] = 1,
\]

15
while the leading order of Poincare transformations $U_g$ coincide with free case.

Let $\Phi_c \neq 0$. Denote

$$v(x) \equiv V''(\Phi_c(x)) - m^2, \quad \hat{\varphi}_v(x) \equiv \Phi_R^{(1)}(x|J_{\Phi_c}).$$

Then eq.(4.6) and boundary condition for the field $\hat{\varphi}_v(x)$ will be written as

$$(\partial_\mu \partial^\mu + m^2 + v(x))\hat{\varphi}_v(x) = 0, \quad \hat{\varphi}_v(x)|_{x < \text{suppv}} = \hat{\varphi}_0(x). \quad (4.12)$$

Relation (4.12) specifies the field $\Phi_R^{(1)}(x|J)$ as $J \sim 0$ uniquely. Making use of the Bogoliubov causality property, one extends this definition to the arbitrary case. Properties (4.1) of the field $\Phi_R^{(1)}(x|J)$ are corrolaries of the construction.

Commutation relation (4.4) to be checked can be rewritten as

$$[\hat{\varphi}_v(x), \hat{\varphi}_v(y)] = \frac{1}{i}D_v(x, y), \quad D_v(x, y) = D_v^{\text{ret}}(x, y) - D_v^{\text{adv}}(x, y), \quad (4.13)$$

where $D_v^{\text{ret}}(x, y)$ and $D_v^{\text{adv}}(x, y) \equiv D_v^{\text{ret}}(y, x)$ are retarded and advanced Green functions for eq.(4.12). They satisfy the relation

$$(\partial_\mu \partial^\mu + m^2 + v(x))D_v^{\text{ret}, \text{adv}}(x, y) = \delta(x - y),$$

moreover, the function $D_v^{\text{ret}}(x, y)$ vanishes as $x < y$, while $D_v^{\text{adv}}(x, y)$ vanishes as $x > y$.

To check commutation relation (4.13), consider the difference

$$C_v(x, y) = [\hat{\varphi}_v(x), \hat{\varphi}_v(y)] - \frac{1}{i}D_v(x, y).$$

It satisfies eq. (4.12) with respect to $x$ and $y$:

$$[\partial_\mu \partial^\mu + m^2 + v(x)]C_v(x, y) = 0, \quad [\partial_\mu \partial^\mu + m^2 + v(y)]C_v(x, y) = 0.$$ 

It vanishes as $x, y < \text{suppv}$, since in this domain $\hat{\varphi}_v$ coincides with the free field $\hat{\varphi}_0$. Therefore, $C_v(x, y) = 0$, and property (4.13) is checked.

### 4.4 Asymptotic condition, Relations for scattering operator

Since $v(x)$ is a function with compact support, for the domains $x > \text{suppv}$ and $x < \text{suppv}$ eq.(4.12) coincides with free field equation. Therefore, fields $\hat{\varphi}_v(x)$ as $x > \text{suppv}$ and as $x < \text{suppv}$ coincide
with asymptotic out- and in-free fields \( \hat{\varphi}_{\text{out}}^v(x) \) and \( \hat{\varphi}_{\text{in}}(x) \equiv \hat{\varphi}_0(x) \). Asymptotic fields satisfy Klein-Gordon equations.

To represent field \( \hat{\varphi}_v(x) \) via the asymptotic in-field, notice that the difference satisfy the following equation:

\[
(\partial_\mu \partial^\mu + m^2 + v(x))\left( \hat{\varphi}_v(x) - \hat{\varphi}_{\text{in}}(x) \right) = -v(x)\hat{\varphi}_{\text{in}}(x)
\]

It vanishes as \( x \gg supp v \); therefore,

\[
(\hat{\varphi}_v(x) - \hat{\varphi}_{\text{in}}(x)) = -\int dy D_0^{\text{ret}}(x, y)v(y)\hat{\varphi}_{\text{in}}(y).
\] (4.14)

This formula can be presented in another form. Consider one more equation for the difference:

\[
(\partial_\mu \partial^\mu + m^2)(\hat{\varphi}_v(x) - \hat{\varphi}_{\text{in}}(x)) = -v(x)\hat{\varphi}_v(x).
\]

It implies that

\[
(\hat{\varphi}_v(x) - \hat{\varphi}_{\text{in}}(x)) = -\int dy D_0^{\text{ret}}(x, y)v(y)\hat{\varphi}_v(y).
\] (4.15)

It is convenient to write formulas (4.14) and (4.15) in a symbolic form:

\[
\hat{\varphi}_v = (1 - D_v^{\text{ret}})\hat{\varphi}_{\text{in}}, \quad \hat{\varphi}_{\text{in}} = (1 + D_0^{\text{ret}})\hat{\varphi}_v.
\] (4.16)

Here \( D_v^{\text{ret}} \) is the operator with kernel \( D_v^{\text{ret}}(x, y) \), \( v \) is the operator of multiplication by \( v(x) \). It follows from (4.16) that the operators \( (1 - D_v^{\text{ret}}) \) and \( (1 + D_0^{\text{ret}}) \) are inverse, i.e.

\[
D_v^{\text{ret}} = (1 + D_0^{\text{ret}})^{-1}D_0^{\text{ret}}.
\] (4.17)

Analogs of the formulas (4.16) for asymptotic out-fields are

\[
\hat{\varphi}_v = (1 - D_v^{\text{adv}})\hat{\varphi}_{\text{out}}, \quad \hat{\varphi}_{\text{out}} = (1 + D_0^{\text{adv}})\hat{\varphi}_v.
\] (4.18)

Property (4.7) can be written for the unitary operator \( W_0[\Phi_c] \equiv W_0\{v\} \):

\[
\hat{\varphi}^v_{\text{out}} = W_0^+\{v\}\hat{\varphi}_{\text{in}}W_0\{v\}.
\] (4.19)

We see that the operator \( W_0\{v\} \) is an S-matrix for quantum field theory in the external background \( v(x) \). Let us take eq.(4.19) to the more general form.

Free fields \( \hat{\varphi}^v_{\text{out}}(x) \) and \( \hat{\varphi}_{\text{in}}(x) \) are expanded to positive- and negative-frequency parts:

\[
\hat{\varphi}_{\text{out}}(x) = \hat{\varphi}_{\text{out}}^+(x) + \hat{\varphi}_{\text{out}}^-(x), \quad \hat{\varphi}_{\text{in}}(x) = \hat{\varphi}_{\text{in}}^+(x) + \hat{\varphi}_{\text{in}}^-(x),
\]

\[
\hat{\varphi}_{\text{out}}^\pm(x) = \frac{1}{(2\pi)^2} \int \frac{d^4p}{\sqrt{2\omega_p}} a_{\pm}^\dagger e^{\mp i(\omega_p t - p_x)},
\]

\[
\hat{\varphi}_{\text{in}}^\pm(x) = \frac{1}{(2\pi)^2} \int \frac{d^4p}{\sqrt{2\omega_p}} b_{\pm}^\dagger e^{\pm i(\omega_p t - p_x)}.
\] (4.20)
It follows from eqs. (4.16) and (4.18) that the field \( \hat{\phi}_{\text{out}}(x) \) is a linear combination of \( \hat{\phi}_{\text{in}}(x) \). Therefore, the dependence of creation and annihilation operators for out-particles \( b_+^p \) of \( a_+^p \) is also linear:

\[
\begin{align*}
 b_+^p &= a_+^p + \int dk (A_{pk} a_k^+ + B_{pk} a_k^-), \\
 b_-^p &= a_-^p + \int dk (A_{pk}^* a_k^- + B_{pk}^* a_k^+),
\end{align*}
\]

or in symbolic form

\[
\begin{align*}
 b_+ &= a_+ + A a_+ + B a_-, \\
 b_- &= a_- + A^* a_- + B^* a_+,
\end{align*}
\]

where \( A, B, A^*, B^* \) are operators with kernels \( A_{pk}, B_{pk}, A_{pk}^*, B_{pk}^* \) correspondingly.

Relation (4.19) can be viewed as a transformation of operators \( a_+^p \) and \( b_+^p \) that conserves the canonical commutation relations. According to the Berezin theorem [25], a canonical transformation is unitary iff the function \( B_{pk} \) is square integrable.

Find an explicit form of the function \( B_{pk} \) and show that the Berezin condition is satisfied. It follows from (4.16) and (4.18) that

\[
\begin{align*}
\hat{\phi}_{\text{out}} - \hat{\phi}_{\text{in}} &= [(1 + D_0^{adv}) (1 - D_v^{ret}) - 1] \hat{\phi}_{\text{in}} = -(1 + D_0^{adv}) D_v \hat{\phi}_{\text{in}}.
\end{align*}
\]

In particular, for the case \( x > \text{suppv} \) write:

\[
\hat{\phi}_{\text{out}}(x) = \hat{\phi}_{\text{in}}(x) - \int D_v(x, y) v(y) \hat{\phi}_{\text{in}}(y) dy.
\]

The operator \( b_+^p \) can be expressed via \( \hat{\phi}_{\text{out}} \) and present as an integral over surface \( x^0 = \text{onst} \) as \( x > \text{suppv} \):

\[
\begin{align*}
 b_+^p &= \frac{1}{(2\pi)^{d/2}} \sqrt{\frac{\omega p}{\omega k}} \int_{x^0 = \text{onst}} dx e^{-i(\omega p x^0 - px)} \left\{ \hat{\phi}_{\text{out}}(x) - \frac{i}{\omega p} \frac{\partial}{\partial x^0} \hat{\phi}_{\text{out}}(x) \right\} = \\
 a_+^p - \frac{1}{(2\pi)^{d/2}} \sqrt{\frac{\omega p}{\omega k}} \int_{x^0 = \text{onst}} dx \int dy e^{-i(\omega p x^0 - px)} \left\{ 1 - \frac{i}{\omega p} \right\} D_v(x, y) v(y) \hat{\phi}_{\text{in}}(y).
\end{align*}
\]

Therefore, we find \( B_{pk} \):

\[
B_{pk} = -\frac{1}{(2\pi)^{d/2}} \frac{1}{\sqrt{\omega k}} \int_{x^0 = \text{onst}} dx \int dy e^{-i(\omega p x^0 - px) - i(\omega k y^0 - ky)} \\
\times \left\{ 1 - \frac{i}{\omega p} \right\} D_v(x, y) v(y)
\]

This function rapidly damps at infinity as a Fourier transform of a smooth function with compact support. Therefore, the Berezin condition is satisfied, so that an unitary operator \( W_0 \{ v \} \) is defined from (4.19) up to a c-number multiplier.

To check the properties (4.5) and construct the semiclassical theory, one should develop the calculus of normal symbols.
4.5 Calculus of normal symbols. Propagator

A calculus of normal symbols is often used in quantum field theory [6, 8]. The physical quantities are presented as a series in normal products of the fields. They multiplies according to the Wick theorem.

If there is an external background, the normal products can be defined in different ways. The simplest way is the following: one expresses the field \( \hat{\phi}_v \) via the free in-field at \(-\infty\), expands it into parts with creation and with annihilation operators \( a^\pm \) and puts the creation operators to the left and annihilation operators to the right.

However, in order to make the semiclassical perturbation theory to be analogous to usual Feynman theory, it is more convenient to express field \( \hat{\phi}_v \) via negative-frequency component of in-field \( \hat{\phi}^-_{\text{in}} \) and positive-frequency component of out-field \( \hat{\phi}^+_{\text{out}} \); then one puts the operators \( b^+ \) to the left and \( a^- \) to the right.

Under this definition, the average value of normal product of operators with respect to in and out vacuums will be nonzero. However, the matrix element between vacuums

\[
\langle A \rangle \equiv \frac{\langle 0, \text{out} | A | 0, \text{in} \rangle}{\langle 0, \text{out} | 0, \text{in} \rangle} = \frac{\langle 0 | W_0 \{ v \} | A | 0 \rangle}{\langle 0 | W_0 \{ v \} | 0 \rangle},
\]

will vanish:

\[
\langle : \hat{\phi}_v(x_1) \cdots \hat{\phi}_v(x_k) : \rangle = 0.
\]

4.5.1 Expansion of the field

Express the operator \( \hat{\phi}_v \) via \( \hat{\phi}^-_{\text{in}} \) and \( \hat{\phi}^+_{\text{out}} \). If there are no external fields, one has \( \hat{\phi}_v = \hat{\phi}^-_{\text{in}} + \hat{\phi}^+_{\text{out}} \). Therefore, one should look the expansion in the form

\[
\hat{\phi}_v = \hat{\phi}^-_{\text{in}} + \hat{\phi}^+_{\text{out}} + \Delta \hat{\phi}.
\]

The field \( \Delta \hat{\phi} \) satisfies the following equation:

\[
(\partial^\mu \partial_\mu + m^2 + v(x)) \Delta \hat{\phi}(x) = -v(x)(\hat{\phi}^-_{\text{in}}(x) + \hat{\phi}^+_{\text{out}}(x))
\]

and boundary condition at \(-\infty \) and \(+\infty \):

\[
\Delta \hat{\phi}(x) = \hat{\phi}^+_{\text{out}}(x) - \hat{\phi}^+_{\text{out}}(x), \quad x_{< \text{suppv}},
\]

\[
\Delta \hat{\phi}(x) = \hat{\phi}^-_{\text{in}}(x) - \hat{\phi}^-_{\text{in}}(x), \quad x_{> \text{suppv}}.
\]

Therefore, field \( \Delta \hat{\phi}(x) \) contains only negative-frequency part at \(+\infty \) and positive-frequency at \(-\infty \). These boundary conditions are called Feynman. By \( D^c(x, y) \) we denote the causal Green function of equation (4.12) (or propagator). It satisfies the relation

\[
(\partial^\mu_\mu + m^2 + v(x)) D^c(x, y) = \delta(x - y)
\]
and Feynman boundary conditions. Therefore,
\[ \Delta \hat{\varphi}(x) = - \int dy D_v^c(x, y) v(y) (\hat{\varphi}_{in}^-(y) + \hat{\varphi}_{out}^+(y)), \]

or in symbolic form
\[ \hat{\varphi}_v = (1 - D_v^c v)(\hat{\varphi}_{in}^- + \hat{\varphi}_{out}^+). \] (4.24)

Analogously to formula (4.14), one can write eq.(4.24) as
\[ \hat{\varphi}_v = (1 - D_0^c v)^{-1}(\hat{\varphi}_{in}^- + \hat{\varphi}_{out}^+). \] (4.25)

Therefore,
\[ D_v^c = D_0^c (1 + v D_0^c) - 1. \] (4.26)

Let us show that there exists a Green function that satisfies the Feynman boundary conditions.

### 4.5.2 Existence of propagator

Consider the problem of finding the function \( f \) satisfying the Feynman boundary condition and equation
\[ (\partial_\mu \partial^\mu + m^2 + v(x)) f(x) = g(x), \] (4.27)

with fixed right-hand side \( g(x) \) with compact support. Consider the difference \( \tilde{f} = f - D^{ret}_v g \) satisfying equation
\[ (\partial_\mu \partial^\mu + m^2 + v(x)) \tilde{f}(x) = 0. \]

Therefore, the function \( \tilde{f} \) can be expressed as
\[ \tilde{f}(x) = \frac{1}{(2\pi)^{d/2}} \int \frac{dp}{\sqrt{2\omega_p}} \left[ \beta_p e^{i(\omega_p x_0^0 - px)} + \beta^*_p e^{i(\omega_p x_0^0 - px^*)} \right], \quad x_0^0 > supp \nu, \]
\[ \tilde{f}(x) = \frac{1}{(2\pi)^{d/2}} \int \frac{dp}{\sqrt{2\omega_p}} \left[ \alpha_p e^{i(\omega_p x_0^0 - px)} + \alpha^*_p e^{i(\omega_p x_0^0 - px^*)} \right], \quad x_0^0 < supp \nu, \]

with coefficient functions \( \alpha_p, \beta_p, \alpha^*_p, \beta^*_p \). Analogously to derivation of formula (4.22), we find the following relations:
\[ \beta = \alpha + A^* \alpha + B^* \alpha, \quad \beta^* = \alpha + A \alpha^* + B \alpha^*. \] (4.28)

The function \( D_v^{ret} g \) is given. Write it expansion into frequency components as \( x_0^0 > supp \nu \) and \( x_0^0 > supp g \):
\[ \tilde{f}(x) = \frac{1}{(2\pi)^{d/2}} \int \frac{dp}{\sqrt{2\omega_p}} \left[ \gamma_p e^{i(\omega_p x_0^0 - px)} + \gamma^*_p e^{i(\omega_p x_0^0 - px^*)} \right]. \]
The boundary conditions for $f$ will be satisfied iff

$$\beta_p + \gamma_p = 0, \quad \pi_p = 0.$$ 

It follows from (4.28) that $\alpha = -(1 + A^*)^{-1} \gamma$, therefore, the solution of the problem (4.27) and Green function for this problem are uniquely specified. The operator $(1 + A^*)$ is invertible because of general properties of linear canonical transformations [25].

4.5.3 Normal form of the product of operators

Analogously to the free case, the product of operators $\hat{\varphi}_v(x)\hat{\varphi}_v(y)$ can be presented in the following normal form:

$$\hat{\varphi}_v(x)\hat{\varphi}_v(y) =: \hat{\varphi}_v(x)\hat{\varphi}_v(y) + <\hat{\varphi}_v(x)\hat{\varphi}_v(y),$$

hear the ”correlator” of the fields $<\hat{\varphi}_v(x)\hat{\varphi}_v(y)$ is a c-number function

$$<\hat{\varphi}_v(x)\hat{\varphi}_v(y) = [(1 + D^c_0 v)^{-1}\hat{\varphi}_v^m(x), (1 + D^c_0 v)^{-1}\hat{\varphi}_v^+(y)].$$

4.5.4 Normal form of T-product

It is possible to obtain an analogous result for the T-product:

$$T\hat{\varphi}_v(x)\hat{\varphi}_v(y) \equiv \hat{\varphi}_v(x)\hat{\varphi}_v(y) - \theta(y_0 - x_0)[\hat{\varphi}_v(x), \hat{\varphi}_v(y)] =$$

$$\hat{\varphi}_v(x)\hat{\varphi}_v(y) + \frac{1}{i}D^c_v(x, y).$$

4.5.5 Hermite conjugation

It is important to note that the normal product $:\hat{\varphi}_v(x)\hat{\varphi}_v(y) :$ is not a Hermitian operator. Namely, find a conjugated operator $(: \hat{\varphi}_v(x)\hat{\varphi}_v(y) :)^*$. Write the relation (4.29) and its conjugation:

$$\hat{\varphi}_v(x)\hat{\varphi}_v(y) =: \hat{\varphi}_v(x)\hat{\varphi}_v(y) + \frac{1}{i}(D^c_v - D^{adv}_v)(x, y),$$

$$\hat{\varphi}_v(x)\hat{\varphi}_v(y) = (: \hat{\varphi}_v(x)\hat{\varphi}_v(y) :)^* - \frac{1}{i}(D^{cs}_v - D^{adv}_v)(x, y).$$

Therefore,

$$\frac{1}{i}D_v(x, y) =: \hat{\varphi}_v(x)\hat{\varphi}_v(y) - (: \hat{\varphi}_v(x)\hat{\varphi}_v(y) :)^* + \frac{1}{i}(D^c_v + D^{cs}_v - 2D^{adv}_v)(x, y).$$

Thus,

$$(: \hat{\varphi}_v(x)\hat{\varphi}_v(y) :)^* = \hat{\varphi}_v(x)\hat{\varphi}_v(y) + \frac{1}{i}\Delta_v(x, y),$$

$$\Delta_v = D^c_v + D^{cs}_v - D^{ret}_v - D^{adv}_v.$$ (4.31)
4.5.6 Variation of matrix elements

Obtain an expression for the derivative of the matrix element (4.23). Let us variate the relation

\[ <0|W_0\{v\}|0> = <0|W_0\{v\}|0> <A> \]

with respect to \( v(x) \). We obtain:

\[ <0|\frac{\delta W_0}{\delta v(x)}|0> + <0|W_0\frac{\delta A}{\delta v(x)}|0> = <0|\frac{\delta W_0}{\delta v(x)}|0> <A> + <0|W_0|0> \frac{\delta A}{\delta v(x)} <A>. \] (4.32)

It is convenient to introduce the following notations:

\[ \hat{\rho}(x|v) = \frac{1}{i} W^+_0 \{v\} \frac{\delta W_0\{v\}}{\delta v(x)}. \] (4.33)

Within the Bogoliubov S-matrix framework, this is a current operator. Under notations (4.33), relation (4.32) is taken to the form:

\[ <\frac{\delta A}{\delta v(x)}> \frac{\delta}{\delta v(x)} <A> - i(<\hat{\rho}(x|v)A> - <\hat{\rho}(x|v)><A>). \] (4.34)

4.6 Investigation of Poincare covariance, unitarity, causality

Investigate whether the operator \( W_0\{v\} \) specified from the relation

\[ \hat{\varphi}_{in}(x)W_0\{v\} = W_0\{v\}\hat{\varphi}_{out} \] (4.35)

satisfy the properties of Poincare invariance, unitarity, causality (4.5). Denote

\[ <0|W_0\{v\}|0> = e^{iA_1\{v\}}. \] (4.36)

The quantity \( A_1\{v\} \) is a one-loop correction to effective action of the theory. Making use of \( A_1\{v\} \), one finds the operator \( W_0\{v\} \) from relation (4.35) uniquely.

Properties of Poincare invariance, unitarity and causality requires additional conditions on the one-loop effective action \( A_1\{v\} \). Find them.

The property of Poincare invariance of the operator \( W_0 \) leads to Poincare invariance of the effective action:

\[ A_1\{u_gv\} = A_1\{v\}. \] (4.37)
Investigate the unitarity property. It follows from (4.35) that
\[ [\hat{\phi}_{\text{out}}(x), W_0^+ W_0] = 0, \quad [\hat{\phi}_{\text{in}}(x), W_0 W_0^+] = 0. \]
Therefore, the operators \(W_0^+ W_0\) and \(W_0 W_0^+\) are proportional to unit operator. To check the unitarity, it is sufficient to check that
\[ < 0 | W_0^+ \{ v \} W_0 \{ v \} | 0 >= 0. \quad (4.38) \]
It is satisfied as \( v = 0 \). Therefore, it is sufficient to check that
\[ \frac{\delta}{\delta v(y)} < 0 | W_0^+ \{ v \} W_0 \{ v \} | 0 >= 0, \]
or in notations (4.33)
\[ < 0 | \hat{\rho}(y|v) | 0 >= ( < 0 | \hat{\rho}(y|v) | 0 > )^*. \quad (4.39) \]
The causality condition can be presented as
\[ \frac{\delta \hat{\rho}(y|v)}{\delta v(x)} = 0, \quad x > \sim y. \quad (4.40) \]

To present properties (4.39) and (4.40) as conditions on the one-loop effective action \( A_1 \{ v \} \), find an explicit form of the operator \( \hat{\rho}(y|v) \). Consider the variation of relation (4.35) with respect to \( v \). Take into account that the field \( \hat{\phi}_{\text{in}}(x) = \hat{\phi}_0(x) \) does not depend on \( v \), while \( \frac{\delta W_0}{\delta v(y)} = i W_0 \hat{\rho}(y|v) \). Then
\[ \hat{\phi}_0(x) i W_0 \{ v \} \hat{\rho}(y|v) = i W_0 \{ v \} \hat{\rho}(y|v) \hat{\phi}_{\text{out}}(x) + W_0 \{ v \} \frac{\delta \hat{\phi}_{\text{out}}}{\delta v(y)}, \]
or
\[ \frac{\delta \hat{\phi}_{\text{out}}(x)}{\delta v(y)} = i [\hat{\phi}_{\text{out}}(x), \hat{\rho}(y|v)]. \quad (4.41) \]
Let us obtain from (4.41) formula for commutator between \( \hat{\rho}(y|v) \) and \( \hat{\phi}_v(x) \). Use relations (4.16) and (4.18):
\[ \hat{\phi}_v = (1 + D_0^{\text{ret}})^{-1} \hat{\phi}_{\text{in}} (1 + D_0^{\text{adv}})^{-1} \hat{\phi}_{\text{out}}. \]

Variating, one finds:
\[ \delta \hat{\phi}_v = -(1 + D_0^{\text{ret}})^{-1} D_0^{\text{ret}} \delta v (1 + D_0^{\text{ret}})^{-1} \hat{\phi}_{\text{in}} = -D_0^{\text{ret}} \delta v \hat{\phi}_v, \]
\[ \delta \hat{\phi}_v = -D_v^{\text{adv}} \delta v \hat{\phi}_v + (1 + D_0^{\text{adv}})^{-1} \delta \hat{\phi}_{\text{out}}. \]

23
Let us use the property (4.41):

\[
\frac{\delta \hat{\phi}_v(x)}{\delta v(y)} = -D^{ret}_v(x, y) \hat{\phi}_v(y), \\
\frac{\delta \hat{\phi}_v(y)}{\delta v(y)} = -D^{adv}_v(x, y) \hat{\phi}_v(x) + i[\hat{\phi}_v(x), \hat{\rho}(y|v)].
\] (4.42)

Therefore, the commutator \(\hat{\rho}(y|v)\) and field \(\hat{\phi}_v(x)\) linearly depends on the field \(\hat{\phi}_v\):

\[
[\hat{\phi}_v(x), \hat{\rho}(y|v)] = iD_v(x, y) \hat{\phi}_v(y).
\] (4.43)

Property (4.43) specifies the operator \(\hat{\rho}(y|v)\) up to a multiplier. Under notations (4.23),

\[
< \hat{\rho}(y|v) > = \frac{1}{i} < 0 | \frac{\delta W_0}{\delta v(y)} | 0 > = \left( \frac{\delta A_1}{\delta v(y)} \right).
\] (4.44)

Making use of the normal symbol calculus, we find the operator \(\hat{\rho}(y|v)\) satisfying properties (4.43) and (4.44):

\[
\hat{\rho}(y|v) = \frac{1}{2} : \hat{\phi}_v(y) : + \frac{\delta A_1}{\delta v(y)}.
\] (4.45)

To check the unitarity (4.39), consider the difference

\[
\hat{\rho}^+(y|v) - \hat{\rho}(y|v) = -\frac{1}{2i} \Delta_v(y, y) + \frac{\delta}{\delta v(y)} (A_1^* \{ v \} - A_1 \{ v \}).
\]

The properties \(D^{ret}_v(y, y) = 0\) and \(D^{adv}_v(y, y) = 0\) imply that the function \(\Delta_v(y, y)\) (4.31) coincide with \(2ReD_v^c(y, y)\). Therefore, unitarity is satisfied if

\[
\frac{\delta}{\delta v(y)} Im A_1 \{ v \} = \frac{1}{2} ReD_v^c(y, y).
\] (4.46)

To investigate the causality property (4.40), one should show that the commutator

\[
[\hat{\phi}_v(z), \frac{\delta \hat{\rho}(y|v)}{\delta v(x)}] = 0
\] (4.47)

for \(x \succ y\) vanishes. Then one obtains conditions on the one-loop effective action such that

\[
< \frac{\delta \hat{\rho}(y|v)}{\delta v(x)} > = 0.
\] (4.48)
Then properties (4.47) and (4.48) imply the Bogoliubov causality property.

1. It follows from the explicit calculation that

\[
\left[ \hat{\varphi}_v(z), \delta \delta_{v(x)} \right] = \left[ \hat{\varphi}_v(z), \hat{\rho}(y|v) \right] - \left[ \hat{\varphi}_v(z), \hat{\rho}(y|v) \right] = \left[ \hat{\varphi}_v(z), \hat{\rho}(y|v) \right] = \left[ \hat{\varphi}_v(z), \hat{\rho}(y|v) \right] = \delta \delta_{v(x)}(x)
\]

So implies eq. (4.47).

2. Property (4.48) is taken according to (4.34) to the form:

\[
\frac{\delta D_v(z, y)}{\delta_v(x)} = -D_v^{ret}(z, x)D_v^{ret}(x, y), \quad x \gtrsim y
\]

implies eq. (4.47).

Thus, all the axioms of the semiclassical field theory are checked in the leading order of perturbation theory. It is shown that the one-loop effective action allows to reconstruct the operator \(W_0\{v\}\), relations (4.37), (4.46) and (4.49) for the effective action are obtained. These relations shows that the effective action cannot be chosen to be arbitrary. A one-loop contribution \(A_1\) is defined up to a local functional.

Note that formally

\[
A_1\{v\} = \frac{i}{2} \ln\det(1 + D_0^c v); \tag{4.50}
\]

then

\[
\frac{\delta A_1\{v\}}{\delta v(y)} = \frac{i}{2} D_0^c(y, y), \tag{4.51}
\]

and all realtions are satisfied. Formula (4.50) is in agreement with resummation of Feynman graphs method. Namely, the S-matrix in the external scalar field \(v(x)\) can be presented as

\[
W_0\{v\} = Te^{-\frac{i}{2} \int dxv(x)\hat{\varphi}_0^2(x)},
\]

where

\[
\hat{\varphi}_0 = \frac{1}{\sqrt{D_0^c(\Delta)}} \left( \frac{1}{\Delta} \frac{\partial}{\partial v(x)} \right)
\]

and

\[
D_0 = 2\pi^{-\frac{n}{2}}(\Delta^2)^{\frac{n}{2}},
\]

where \(D_0\) is the reduced propagator for the scalar field \(v\) with bare mass \(m\).
and vacuum average of this operator is expressed via determinant (4.50).

However, definition (4.50) is formal due to divergences. Relations (4.37), (4.46) and (4.49) may be viewed as a definition of the renormalized determinant of operator.

Formula (4.51) may be viewed as an additional condition for propagator at coinciding arguments:

\[(D^c_v(y,y))_R \equiv \frac{2}{i} \frac{\delta A_1\{v\}}{\delta v(y)}. \tag{4.52}\]

Relation (4.49) is used for definition of square of distribution \(D^c_v(x,y)\) at \(x = y\):

\[(D^c_v(x,y))^2_R = 2i \frac{\delta^2 A_1\{v\}}{\delta v(x) \delta v(y)}. \]

### 4.7 Generalization to multicomponent fields

Consider the multicomponent fields \(\Phi^c = (\Phi^1_c, ..., \Phi^k_c)\).

Let \(I[\Phi^c]\) be classical action, \(I_0[\Phi^c]\) be action of the free theory, \(K^{ij}\) be operator with the kernel \(-\frac{\delta I\}{\delta \Phi^i(x)\delta \Phi^j(y)}\), \(K'^{ij}\) be operator with the kernel \(-\frac{\delta I_0}{\delta \Phi^i(x)\delta \Phi^j(y)}\), \(K' = K - K_0\), \(\phi(x|\Phi^c) \equiv \Phi^{(1)}_R(x|J_{\Phi^c})\).

Eq.(4.6) and boundary condition for the field \(\phi(x|\Phi^c)\) will be written as

\[(K_0 + K')\phi(x|\Phi^c) = 0, \quad \phi(x|\Phi^c) = \phi_0(x), \quad x \gtrless \text{supp}\Phi^c, \tag{4.53}\]

where \(\phi_0(x)\) is a free field.

Denote by \(D^{ret|ij}(x,y|\Phi^c)\) the retarded Green function for eq.(4.53). It satisfies the relation

\[K^{ij}D^{ret|jl}(x,y|\Phi^c) = \delta^{il}\delta(x - y) \tag{4.54}\]

and the retarded boundary condition \(D^{ret}(x,y|\Phi^c) = 0\) as \(x \gtrless y\). Then the function

\[D^{adv|ij}(x,y|\Phi^c) = D^{ret|ij}(y,x|\Phi^c)\]

will satisfy eq.(4.54) with advanced boundary condition, so that \(D^{adv}(x,y|\Phi^c)\) will call as an advanced Green function.

The commutation relations of the fields are written as

\[\{\phi^i(x|\Phi^c), \phi^j(y|\Phi^c)\} = \frac{1}{i}D^{ij}(x,y|\Phi^c), \quad D^{ij} = D^{ret|ij} - D^{adv|ij}. \tag{4.55}\]
Analogously to (4.16), (4.18), the solution of eq. (4.53) can be expressed via the asymptotic in- and out-fields as follows:

\begin{align*}
\hat{\phi} &= (1 - D^{\text{ret}} K') \hat{\phi}_{\text{in}} = (1 + D^{\text{ret}}_{0} K')^{-1} \hat{\phi}_{\text{in}}, \\
\hat{\phi} &= (1 - D^{\text{adv}} K') \hat{\phi}_{\text{out}} = (1 + D^{\text{adv}}_{0} K')^{-1} \hat{\phi}_{\text{out}},
\end{align*}

(4.56)

where \(D^{\text{ret}}_{0}\) and \(D^{\text{adv}}_{0}\) are advanced and retarded Green functions of the free field.

The operator \(W_{0}[\Phi_{c}]\) is specified from the relation:

\[
\hat{\phi}_{\text{in}}(x) W_{0}[\Phi_{c}] = W_{0}[\Phi_{c}] \hat{\phi}_{\text{out}}(x)
\]

(4.57)

up to a multiplier. It is fixed by a vacuum average:

\[
< 0 | W_{0}[\Phi_{c}] | 0 > = e^{iA_{1}[\Phi_{c}]}.
\]

(4.58)

Conditions of Poincare invariance, unitarity and causality gives us restrictions on the one-loop effective action \(A_{1}[\Phi_{c}]\).

Formulas of calculus of normal symbols can be generalized to multicomponent case as follows.

Let \(D^{ij}_{c}\) be causal Green function (propagator) of eq. (4.54). It contains positive-frequency components at \(-\infty\) and negative-frequency components at \(+\infty\). Then generalizations of eqs. (4.24) and (4.25) have the form:

\[
\hat{\phi} = (1 + D^{c} K') (\hat{\phi}_{\text{in}}^{-} + \hat{\phi}_{\text{out}}^{+}) = (1 + D^{c}_{0} K')^{-1} (\hat{\phi}_{\text{in}}^{-} + \hat{\phi}_{\text{out}}^{+}).
\]

(4.59)

One also has:

\[
D^{c} = D^{c}_{0}(1 + K'D^{c}_{0})^{-1}.
\]

Formulas of taking operators to the normal form are:

\[
< \hat{\phi}_{i}(x|\Phi_{c}) \hat{\phi}_{j}(y|\Phi_{c}) > = \frac{1}{i} D^{ij}_{c}(x, y|\Phi_{c}), \quad D^{c} = D^{c} - D^{c^{\text{adv}}},
\]

\[
< T \hat{\phi}_{i}(x|\Phi_{c}) \hat{\phi}_{j}(y|\Phi_{c}) > = \frac{1}{i} D^{ij}_{c}(x, y|\Phi_{c}),
\]

\[
( : \hat{\phi}_{i}(x|\Phi_{c}) \hat{\phi}_{j}(y|\Phi_{c}) : )^{-} = : \hat{\phi}_{i}(x|\Phi_{c}) \hat{\phi}_{j}(y|\Phi_{c}) : + \frac{i}{4} \Delta_{ij}(x, y|\Phi_{c}),
\]

\[
\Delta = D^{c} + D^{c^{*}} - D^{\text{ret}} - D^{\text{adv}}.
\]

Consider the current operator:

\[
j_{0}^{i}(x|\Phi_{c}) = \frac{1}{i} W_{0}[\Phi_{c}] \frac{\delta W_{0}[\Phi_{c}]}{\delta \Phi_{c}(x)}.
\]

(4.59)
Then the properties of unitarity and causality of the operator $W_0[\Phi_c]$ will have the form analogous to (4.39) and (4.40):

\[ <0|j_0^j(y|v)|0> = ( <0|j_0^j(y|v)|0>)^*, \]
\[ \frac{\delta j_0^j(y|v)}{\delta \Phi^x(x)} = 0, \quad x \sim y. \]

Variate the relation (4.57) and make use of eq.(4.56). Analogously to eq.(4.43), we obtain:

\[ [\hat{\varphi}^i(x|\Phi_c), \int j_0^k(y|\Phi_c) \delta \Phi^k(y) dy] = i \int D^{is}(x,y|\Phi_c) (\delta K' \hat{\varphi})^k(y|\Phi_c) dy. \quad (4.60) \]

Property (4.60) specifies the operator $j_0^k(y|\Phi_c)$ up to an additive constant. It is fixed by a one-loop effective action (4.58):

\[ \int dy j_0^k(y|\Phi_c) \delta \Phi^k(y) = -\frac{1}{2} \int dy : \varphi^k(y|\Phi_c)(\delta K' \hat{\varphi})^k(y|\Phi_c) : + \delta A_1[\Phi_c]. \quad (4.61) \]

The Poincare covariance, unitarity and causality properties imply the following relations on the one-loop effective action:

\[ A_1[u_g\Phi_c] = A_1[\Phi_c], \quad Im \delta A_1[\Phi_c] = \frac{1}{4} Tr(\delta K' \Delta), \]
\[ \int dxdy \delta_1 \Phi(x) \frac{\delta^2 A_1}{\delta \Phi^c(x) \delta \Phi^c(y)} \delta_2 \Phi(y) = \frac{1}{2i} Tr[\delta_1 K' D^c \delta_2 K' D^c], \quad (4.62) \]
\[ supp \delta_1 \Phi < supp \delta_2 \Phi. \]

Formally, the solution of eqs.(4.62) have the form:

\[ A_1[\Phi_c] = \frac{i}{2} \text{Indet}[1 + K'D_0^c]; \quad (4.63) \]

definition of the determinant requires renormalization.

### 5 Semiclassical perturbation theory

Investigate the axioms of semiclassical field theory C1-C5 within the perturbation framework. It is convenient to fix a representation. Otherwise, there will be a nonuniqueness for the operators $U_g, \hat{\varphi}_h(x), \Phi_R(x|J), W[\Phi_c]$. 

28
5.1 Asymptotic in-representation. Axioms of semiclassical S-matrix

In quantum field theory, the asymptotic in-representation is often used. One supposes that at \( t \to -\infty \) particles become free and Hilbert state space coincides with the Fock space of free particles. For such a case, Heisenberg field \( \hat{\varphi}_h(x) \) tends as \( t \to \pm \infty \) in a weak sense to the asymptotic free field:

\[
\hat{\varphi}_h(x) \sim_{x^0 \to -\infty} \hat{\varphi}_{\text{in}}(x) \equiv \hat{\varphi}_0(x), \quad \hat{\varphi}_h(x) \sim_{x^0 \to +\infty} \hat{\varphi}_{\text{out}}(x).
\]

The S-matrix (denote it as \( \Sigma[0] \)) is an unitary transformation of asymptotic fields:

\[
\hat{\varphi}_{\text{out}}(x) = \Sigma^+[0] \hat{\varphi}_{\text{in}}(x) \Sigma[0] = \Sigma^+[0] \hat{\varphi}_0(x) \Sigma[0].
\] (5.1)

The operator

\[
\Sigma[\Phi_c] = \Sigma[0] \mathcal{W}[\Phi_c]
\]
i is an S-matrix in the external background. Write the axioms C1-C5, making use of the introduced notations. As \( x^0 \to -\infty \), formulas of the Poincare transformations take the form:

\[
U_{g_1} U_{g_2} = U_{g_1} U_{g_2}, \quad U_{g}^{-1} \hat{\varphi}_0(x) U_g = \hat{\varphi}_0(w_g x).
\] (5.2)

Since the asymptotic in-field is a free scalar field of the mass \( m \), the operator \( U_g = U_g \) coincides with the Poincare transformation of the free theory.

Properties of Poincare invariance, unitarity and causality are presented as:

\[
U_g \Sigma[\Phi_c] U_{g}^{-1} = \Sigma[\Phi_c], \quad (\Sigma[\Phi_c])^+ = (\Sigma[\Phi_c])^{-1}, \quad \frac{\delta}{\delta \Phi_c(x)} j(y|\Phi_c) = 0, \quad x \succ y,
\] (5.3)

where \( j(y|\Phi_c) \) is a current operator

\[
j(y|\Phi_c) = \frac{1}{i} \Sigma^+[\Phi_c] \frac{\delta \Sigma[\Phi_c]}{\delta \Phi_c(y)},
\] (5.4)

entering to the right-hand side of the Yang-Feldman equation (3.25)

\[
(\partial_{\mu} \partial^\mu + V''(\Phi_c(x)))(\Phi_R(x|J_{\Phi_c}) - \Phi_c(x)) = \hbar j(x|\Phi_c).
\] (5.5)

As \( x^0 \to \pm \infty \), semiclassical field \( \Phi_R(x|J_{\Phi_c}) \) will satisfy the asymptotic boundary conditions:

\[
\Phi_R(x|J_{\Phi_c}) \sim_{x^0 \to -\infty} \sqrt{\hbar} \hat{\varphi}_0(x), \quad \Phi_R(x|J_{\Phi_c}) \sim_{x^0 \to +\infty} \sqrt{\hbar} \hat{\varphi}_{\text{out}}(x) \mathcal{W}[\Phi_c] = \Sigma^+[\Phi_c] \sqrt{\hbar} \hat{\varphi}_0(x) \Sigma[\Phi_c].
\] (5.6)
The solution of eq. (5.5) satisfying the first of conditions (5.6) can be expressed via the field \( \hat{\phi}_v(x) \) and retarded Green function:

\[
\Phi_R(x|J_{\Phi_c}) - \Phi_c(x) = \sqrt{\hbar} \hat{\phi}_v(x) + \hbar \int dy D^r_{\text{ret}}(x, y) j(y|\Phi_c).
\] (5.7)

If we use the second condition, the semiclassical field will be expressed via the advanced Green function:

\[
\Phi_R(x|J_{\Phi_c}) - \Phi_c(x) = \sum^+ [\Phi_c] W_0^+ [\Phi_c] \sqrt{\hbar} \hat{\phi}_v(x) \sum^+ [\Phi_c] j(y|\Phi_c).
\] (5.8)

Relations (5.7) and (5.8) imply the following identity:

\[
[\hat{\phi}_v(x), W_0^+ [\Phi_c] \Sigma[\Phi_c]] = \sqrt{\hbar} \int dy D_v(x, y) j(y|\Phi_c).
\] (5.9)

Formula (5.9) can be viewed as a basis for the perturbation theory for \( \Sigma[\Phi_c] \):

\[
\Sigma[\Phi_c] = \Sigma_0[\Phi_c] + \sqrt{\hbar} \Sigma_1[\Phi_c] + ...
\]

Namely, let the \( k - 1 \)-th order of the perturbation theory is constructed. Then \( \Sigma_k[\Phi_c] \) is defined from the commutation relation (5.9) up to an operator of the form \( c_k[\Phi_c] W_0[\Phi_c] \), with a \( c \)-number multiplier \( c_k \). To fix it, introduce the following relation for the vacuum average of the scattering operator in the external field:

\[
< 0|W[\Phi_c]|0 > = < 0|\Sigma[\Phi_c]|0 > = e^{i(A_1[\Phi_c] + \hbar A_2[\Phi_c] + ...)}.
\] (5.10)

Let us call the quantity

\[
\Delta[\Phi_c] = I[\Phi_c] + \hbar A_1[\Phi_c] + \hbar^2 A_2[\Phi_c] + ...
\]

as a semiclassical action of the theory. It differs from the effective action. All connected Feynman graphs are involved to it. If \( \Delta[\Phi_c] \) is given, the operator \( \Sigma[\Phi_c] \) is uniquely defined within the perturbation framework from eq. (5.9). Properties of Poincare invariance, unitarity and causality impose certain conditions on semiclassical action. \( A_k[\Phi_c] \) is defined up to a real local Poincare invariant functional.

Thus, we are coming to the following set of axioms of the S-matrix approach.

**S1.** Hilbert state space \( \mathcal{F} \) is a Fock space for the free field, Poincare transformations coincide with free case.

**S2.** To each classical field configuration \( \Phi_c(x) \) with compact support one assigns S-matrix \( \Sigma[\Phi_c] \) being a formal asymptotic series in \( \sqrt{\hbar} \). It satisfies the Poincare invariance, unitarity and causality properties (5.3), as well as commutation relation (5.9).
The retarded semiclassical field $\Phi_R(x|J)$ is specified from $\Sigma[\Phi_c]$ uniquely, so that there are no additional conditions on $\Phi_R(x|J)$. Properties C3 should be obtained from S2.

Formula (5.7) specifies the classical retarded field $\Phi_R(x|J)$ as $J \sim 0$. Making use of the Bogoliubov causality condition, we extend the definition to all $J$. The properties of Hermitian conjugation, Poincare invariance and Bogoliubov causality for $\Phi_R(x|J)$, as well as commutation relation (3.9) can be checked.

5.2 Perturbation theory

Let us develop the perturbation theory for S-matrix in the external field (the first and the second order). Introduce the notation

$$\tilde{\Sigma}[\Phi_c] = W_0^+[\Phi_c] \Sigma[\Phi_c] = 1 + \sqrt{\hbar} \tilde{\Sigma}_1[\Phi_c] + \hbar \tilde{\Sigma}_2[\Phi_c] + ...$$

(5.11)

Then the current (5.4), with leading order of the form

$$j_0(y|\Phi_c) = \frac{1}{i} W_0^+[\Phi_c] \frac{\delta W_0[\Phi_c]}{\delta \Phi_c(y)} = V'''(\Phi_c(y)) \rho(y|v),$$

will be expressed via the operator (5.11) as:

$$j(y|\Phi_c) = \tilde{\Sigma}^+[\Phi_c] j_0(y|\Phi_c) \tilde{\Sigma}[\Phi_c] + \frac{1}{i} \tilde{\Sigma}^+[\Phi_c] \frac{\delta \tilde{\Sigma}[\Phi_c]}{\delta \Phi_c(y)}$$

$$= j_0(y|\Phi_c) + \sqrt{\hbar} j_1(y|\Phi_c) + ...$$

(5.12)

Write the formulas for constructing the perturbation theory:

$$[\hat{\varphi}_v(x), \tilde{\Sigma}[\Phi_c]] = \sqrt{\hbar} \int dy D_v(x, y) \tilde{\Sigma}[\Phi_c] j(y|\Phi_c);$$

$$< \tilde{\Sigma}[\Phi_c] >= e^{i(h A_2[\Phi_c] + \hbar^2 A_3[\Phi_c] + ...)}$$

(5.13)

5.2.1 First order

In the first order, relations (5.13) take the form

$$[\hat{\varphi}_v(x), \tilde{\Sigma}_1[\Phi_c]] = \sqrt{\hbar} \int dy D_v(x, y) \tilde{\Sigma}[\Phi_c] V'''(\Phi_c(y)) \rho(y|v).$$

$$< \tilde{\Sigma}_1[\Phi_c] >= 0.$$  

(5.14)

To solve these equations, it is convenient to introduce the following notations. For the T-square, set

$$(T^c \tilde{\varphi}_v^2(y))_R \equiv -2 \rho(y|v) =: \tilde{\varphi}_v^2(y) + \frac{1}{i} (D_v^c(y, y))_R.$$  

(5.15)
The operator $\tilde{\Sigma}_1[\Phi_c]$ is rewritten as
\[
\tilde{\Sigma}_1[\Phi_c] = -\frac{i}{6} \int dyV'''(\Phi_c(y))(T\hat{\varphi}_v^2(y))_R,
\] (5.16)

Eq.(5.14) implies the following relation:
\[
[T\hat{\varphi}_v^3(y)]_R = -3iD_v(x,y)(T\hat{\varphi}_v^2(y))_R, \quad <(T\hat{\varphi}_v^3(y))_R >= 0,
\] (5.17)

The quantity $(T\hat{\varphi}_v^3(y))_R$ is uniquely defined.
\[
(T\hat{\varphi}_v^3(y))_R =: \hat{\varphi}_v^3(y) - \frac{3}{i}(D_v^c(y,y))_R\hat{\varphi}_v(y).
\] (5.18)

The Poincare invariance and unitarity properties are taken to the form:
\[
U_g(T\hat{\varphi}_v^3(w_gy))_R U_g^{-1} = (T\hat{\varphi}_v^3(w_{g'y}))_R; \quad (T\hat{\varphi}_v^3(y))_R^\dagger = (T\hat{\varphi}_v^3(y))_R.
\] (5.19)

To check these properties, it is sufficient to notice that differences $(T\hat{\varphi}_v^3(y))_R^\dagger - (T\hat{\varphi}_v^3(y))_R$ and $U_g(T\hat{\varphi}_v^3(w_gy))_R U_g^{-1} - (T\hat{\varphi}_v^3(y))_R$ commute with $\hat{\varphi}_v(x)$ and have zero matrix elements $<\ldots>$. Investigate the causality condition. The current operator in the first order of perturbation theory has the form:
\[
\frac{1}{6} \int dzV'''(\Phi_c(y))V'''(\Phi_c(z)) \left( \delta_{\Phi_c(y)}(T\hat{\varphi}_v^2(z))_R - \frac{1}{2}[(T\hat{\varphi}_v^2(y))_R, (T\hat{\varphi}_v^2(z))_R] \right)
\]

This operator satisfies the causality condition if $(T\hat{\varphi}_v^3(y)_R$ depends on $v(y)$ at the preseeding time moments and the expression $\frac{\delta}{\delta v(y)}(T\hat{\varphi}_v^3(y)_R - \frac{1}{2}[(T\hat{\varphi}_v^2(y))_R, (T\hat{\varphi}_v^2(z))_R] vanishes for $z_s > y$. Represent these properties as
\[
\frac{\delta}{\delta v(y)}(T\hat{\varphi}_v^3(y))_R = 0, \quad x_s > y,
\]

These properties are checked analogously.

5.2.2 The second order

Write down eqs.(5.13) for $\tilde{\Sigma}_2$:
\[
[T\hat{\varphi}_v(x), \tilde{\Sigma}_2[\Phi_c]] = \int dyD_v(x,y)(j_1(y|\Phi_c) + \tilde{\Sigma}_1[\Phi_c]j_0(y|\Phi_c)); \\
<\tilde{\Sigma}_2[\Phi_c] >= iA_2[\Phi_c],
\] (5.20)
Use the explicit forms of \( j_0(y|\Phi_c) \), \( j_1(y|\Phi_c) \), \( \tilde{\Sigma}_1[\Phi_c] \) and take eq.\((5.20)\) to the form:

\[
\tilde{\Sigma}_2[\Phi_c] = -i \int dy D_v(x, y) \left\{ -\frac{i}{6} V^{(IV)}(\Phi_c(y))(T\phi^3_v(y))_R \\
-\frac{1}{12} \int dz V'''(\Phi_c(y))V'''(\Phi_c(z))(T\phi^3_v(y)\phi^3_v(z))_R \right\}.
\]

(5.21)

We use the notation

\[(T\phi^2_v(y)\phi^3_v(z))_R \equiv (T\phi^2_v(y))_R(T\phi^3_v(z))_R + 2i\frac{\delta}{\delta v(y)}(T\phi^3_v(z))_R.\]

(5.22)

The renormalized T-products \((T\phi^2_v(y)\phi^3_v(z))_R\) coinides with \((T\phi^2_v(y))_R(T\phi^3_v(z))_R\) as \(y \sim z\), and with \((T\phi^3_v(z))_R(T\phi^3_v(y))_R\) as \(z \sim y\), so that notation \((5.22)\) is reasonable.

The solution of eq.\((5.21)\) can be formally presented as

\[
\tilde{\Sigma}_2[\Phi_c] = -\frac{i}{24} \int dy V^{(IV)}(\Phi_c(y))(T\phi^4_v(y))_R \\
-\frac{1}{72} \int dy dz V'''(\Phi_c(y))V'''(\Phi_c(z))(T\phi^3_v(y)\phi^3_v(z))_R.
\]

(5.23)

New renormalized T-products entering to eq.\((5.23)\) should satisfy the commutation relations:

\[
\hat{\phi}_v(x), (T\phi^4_v(y))_R = \frac{1}{4} D_v(x, y) (T\phi^3_v(y))_R, \\
\hat{\phi}_v(x), (T\phi^3_v(y)\phi^3_v(z))_R = \frac{3}{4} D_v(x, y) (T\phi^3_v(y)\phi^3_v(z))_R + \frac{1}{4} D_v(x, z) (T\phi^3_v(y)\phi^3_v(z))_R.
\]

(5.24)

The renormalized T-products are specified from eq. \((5.24)\) up to a c-number function < ... >. They are related with two-loop semiclassical action:

\[
A_2[\Phi_c] = -\frac{1}{24} \int dy V^{(IV)}(\Phi_c(y)) < (T\phi^4_v(y))_R > + \\
\frac{1}{72} \int dy dz V'''(\Phi_c(y))V'''(\Phi_c(z)) < (T\phi^3_v(y)\phi^3_v(z))_R >.
\]

(5.24) ,:
Investigate the corollaries of properties of Poincare invariance, unitarity and causality. The Poincare invariance and unitarity property imply analogously to (5.19) that
\[
U_g(T\hat{\varphi}^4(w, y))RU_g^{-1} = (T\hat{\varphi}^4(y))R, \quad (T\hat{\varphi}^4(y))_R^I = (T\hat{\varphi}^4(y))_R;
\]
\[
U_g(T\hat{\varphi}^3(w, y)\hat{\varphi}^3(z))RU_g^{-1} = (T\hat{\varphi}^3(y)\hat{\varphi}^3(z))_R;
\]
\[
(T\hat{\varphi}^3(y)\hat{\varphi}^3(z))_R^I + (T\hat{\varphi}^3(y)\hat{\varphi}^3(z))_R = (T\hat{\varphi}^3(y))_R(T\hat{\varphi}^3(z))_R + (T\hat{\varphi}^3(z))_R(T\hat{\varphi}^3(y))_R.
\]
(5.25)
The causality property leads to the following relations on \((T\hat{\varphi}^4(y))_R^I\):
\[
\frac{\delta}{\delta v(x)}(T\hat{\varphi}^4(y))_R = 0, \quad x \succ y,
\]
\[
\frac{\delta}{\delta v(x)}(T\hat{\varphi}^3(y)\hat{\varphi}^3(z))_R = \frac{i}{2}[(T\hat{\varphi}^3(y)_R^I)(T\hat{\varphi}^3(z)_R), \quad x \succ y.
\]
(5.26)
\[
(T\hat{\varphi}^3(y)\hat{\varphi}^3(z))_R^I:
\]
\[
(T\hat{\varphi}^3(y)\hat{\varphi}^3(z))_R = (T\hat{\varphi}^3(y)_R(T\hat{\varphi}^3(z)_R), \quad y \succ z,
\]
\[
\frac{\delta}{\delta v(x)}(T\hat{\varphi}^3(y)\hat{\varphi}^3(z))_R = 0, \quad x \succ y \succ z,
\]
\[
\frac{\delta}{\delta v(x)}(T\hat{\varphi}^3(y)\hat{\varphi}^3(z))_R = \frac{i}{2}[(T\hat{\varphi}^3(y)_R^I)(T\hat{\varphi}^3(z)_R), \quad x \succ y \succ z.
\]
(5.27)
It follows from eq. (5.24) that commutators between left-hand and right-hand sides of relations (5.25), (5.26) and (5.27) with field \(\hat{\varphi}_r(\xi)\) coincide; it is sufficient then to check the equalities for matrix elements < ... >. If we set
\[
<T\hat{\varphi}^3_y> = 3 \left( \frac{1}{4} (D^c_v(y, y, z))_R^2 \right),
\]
\[
<T\hat{\varphi}^3_y\hat{\varphi}^3_z> = \frac{9}{8} (D^c_v(y, y, z))_R^2 + \frac{9}{8} (D^c_v(y, y, z))_R (D^c_v(z, z))_R D^c_v(y, z),
\]
the properties of Poincare invariance, unitarity and causality will be rewritten as:
\[
(D^c_v(y, z))_R^2 = (D^c_v(y, z))_R^3, \quad y \neq z,
\]
\[
(D^c_v(w, y, w, z))_R^2 = (D^c_v(y, z))_R^3, \quad y \neq z,
\]
\[
2iIm(D^c_v(y, z))_R^2 = -3(D^c_v(y, z))_R^2 D^c_v(x, y) D^c_v(x, z),
\]
\[
-3(D^c_v(y, z))_R^2 \Delta_v(x, y) + 3(D^c_v(y, z))_R^2 \Delta_v(x, y) = \Delta_v(x, y) D^c_v(y, z) \Delta_v(x, y) - (\Delta_v(x, y))^3.
\]
Therefore, semiclassical methods in the axiomatic field theory allows us to construct the renormalized perturbation theory for the S-matrix. The obtained results is in agreement with Lagrangian (Hamiltonian) field theory, since
\[
\Sigma[\Phi_c] = T exp \left(-i \int dx \left[ \frac{\sqrt{h}}{6} V'''(\Phi_c(x))\hat{\varphi}^3_v(x) + \frac{h}{24} V''(\Phi_c(x))\hat{\varphi}^4_v(x) + \ldots \right] \right)
\]
6 On semiclassical scalar electrodynamics

Let us generalize the obtained results to the gauge theories. Start from quantum electrodynamics.

6.1 Scalar electrodynamics and its quantization

Consider the scalar electrodynamics, the model of field theory, which consists of vector (electromagnetic) field $A_\mu(x)$ interacting with the complex scalar field. The Lagrangian of the theory has the form:

$$\mathcal{L} = D_\mu \bar{\theta} D^\mu \theta - \frac{1}{\hbar} V(\hbar \bar{\theta} \theta) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

(6.1)

here $D_\mu = \partial_\mu - i\sqrt{\hbar} A_\mu$ is a covariant derivative, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $V$ is a potential of self-interaction of the scalar field. For the Hamiltonian theory, the momenta conjugated to fields $A^k, \theta, \bar{\theta}$ have the form:

$$E_k = F_{k0}, \quad \pi_\theta = D_0 \theta, \quad \bar{\pi}_{\bar{\theta}} = D_0 \bar{\theta}. $$

The momentum canonically conjugated to the field $A_0$ is zero. Therefore, the considered system is a constrained system [26], the Hamiltonian density and constraints have the form:

$$\mathcal{H} = \frac{1}{2} E_k E_k + \frac{1}{4} F_{ij} F_{ij} + \bar{\pi}_\theta \pi_\theta + D_i \bar{\theta}^* D^i \theta + \frac{1}{\hbar} V(\hbar \bar{\theta} \theta),$$

$$\Lambda = \partial_k E_k + i\sqrt{\hbar}(\bar{\pi}_\theta \theta - \pi_{\bar{\theta}}^{\star} \bar{\theta}).$$

(6.2)

To quantize the theory, one should substitute the fields and momenta by operators satisfying the canonical commutation relation. There are different approaches to take the constraints into account.

For the Dirac approach, state $\Psi_D$ is supposed to satisfy not only Schrodinger equation but also additional constrained condition:

$$\hat{\Lambda}_x \Psi_D = 0.$$  

Since the constrained operators commute each other for this model, the Dirac condition does not contradict Schrodinger equation.

The main difficulty of the Dirac approach is to introduce an inner product. To avoid it, one can introduce an additional gauge condition, for example $\partial_k A^k = 0$ [2, 26, 27].

Another way to quantize constrained systems is to use the refined algebraic quantization approach [28]. There are no additional conditions on state vectors $\Psi_A$; instead, the inner
product of the theory is modified:

$$(\Psi_A, \Psi_A)_A = (\Psi_A, \prod_x \delta(\hat{A}_x) \Psi_A).$$

Two state vectors are set to be equivalent if their difference is of zero norm. In particular, state of the form $\int d\mathbf{x} \beta(\mathbf{x}) \Lambda_x Y \sim 0$ is equivalent to zero.

State vectors $\Psi_D$ and $\Psi_A$ are related to each other by the relation:

$$\Psi_D = \prod_x \delta(\hat{A}_x) \Psi_A.$$

A manifestly covariant approach to quantize gauge fields is a BRST-BFV quantization approach [29, 30, 31]. In this approach, additional degrees of freedom should be introduced. These are Lagrange multipliers and Faddeev-Popov ghosts and antighosts [2] and canonically conjugated momenta. When the Abelian case is investigated, one can use the Lagrange multipliers $A_0(\mathbf{x})$ and momenta $E_0(\mathbf{x})$. This is a Gupta-Bleuler approach. If the functional Schrödinger representation is used, states $\Psi_B$ are functionals of the fields $A^k(x), A^0(x), \theta(x), \theta^*(x)$:

$$\Psi_B = \Psi_B[A^k, A^0, \theta, \theta^*], \quad (6.3)$$

while fields and momenta are operators

$$\hat{A}^\mu(x) = A^\mu(x), \quad \hat{\theta}(x) = \theta(x), \quad \hat{\theta}^*(x) = \theta^*(x),$$

$$\hat{E}_\mu(x) = \frac{1}{i} \delta A^\mu(x), \quad \hat{\pi}_\theta(x) = \frac{1}{i} \delta \theta(x), \quad \hat{\pi}^*_\theta(x) = \frac{1}{i} \delta \theta^*(x).$$

An indefinite inner product may be presented via the functional integral of the form [32]

$$(\Psi_B, \Psi_B)_B = \int D\Lambda D\theta D\theta^* (\Psi_B[A^k, -i\lambda, \theta, \theta^*])^* \Psi_B[A^k, -i\lambda, \theta, \theta^*]. \quad (6.4)$$

All functionals (6.3) form a space of virtual states, while physical states should obey the Gupta-Bleuler condition

$$\left[ \frac{1}{i} \delta A^0(\mathbf{x}) - \frac{i}{\sqrt{-\Delta}} \Lambda_x \right] \Psi_B = 0. \quad (6.5)$$

Two physical states with the difference of the form

$$\int d\mathbf{x} \beta(\mathbf{x}) \left[ \frac{1}{i} \delta A^0(\mathbf{x}) + \frac{i}{\sqrt{-\Delta}} \Lambda_x \right] Y_B \sim 0. \quad (6.6)$$

are set to be equivalent. "Zero" states (6.6) are orthogonal to all physical states. Note that Gupta-Bleuler condition and equivalence relation are invariant under evolution.
It follows from eq.(6.5) that
\[ \Psi_B[A^k, A^0, \theta, \theta^*] = e^{-\int dx A^0(x) \sqrt{-\delta} \hat{\Lambda} x \Psi_A[A^k, \theta, \theta^*]}, \]
the inner product (6.4) is taken to the form \( (\Psi_A, \Pi_x \delta(\hat{\Lambda} x \Psi_A)) \), so that the Gupta-Bleuler approach is equivalent to the algebraic quantization.

When the \( \xi \)-gauge is used, the following terms are added to the Hamiltonian density:
\[ H_B = H + A_0 \Lambda x - \frac{\xi}{2} E_0^2 - E_0 \partial_k A^k. \] (6.7)
It follows from eqs.(6.5) and (6.6) that \( H_B \Psi_B \sim H \Psi_B \), so that theories with Hamiltonians \( H \) and \( H_B \) are equivalent.

Starting from the Hamiltonian (6.7), one obtains the following Lagrangian:
\[ L_B = D_\mu \theta^* D^\mu \theta - \frac{1}{\hbar} V(h \theta^* \theta) - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2 \xi} (\partial_\mu A^\mu)^2. \] (6.8)
An additional gauge-fixed term is added. Equations of motion for the model (6.7) (or (6.8)) are the following:
\[ D_\mu D^\mu \theta + V'(h \theta^* \theta) \theta = 0, \quad E_0 = -\frac{1}{\xi} \partial_\alpha A^\alpha, \]
\[ \partial_\nu F^{\mu \nu} + \partial^\mu E^0 = i \sqrt{\hbar} (\theta^* D^\mu \theta - D^\mu \theta^* \theta). \]
This implies that \( E_0(x) \) is a scalar field satisfying the massless Klein-Gordon equation:
\[ \partial_\mu \partial^\mu E_0 = 0, \] (6.9)
the constraint \( \Lambda \) is related with the field \( E_0 \) by the relation \( \dot{E}_0 = -\Lambda \). Therefore, the quantum field \( \hat{E}_0(x) \) can be expanded to positive- and negative-frequency components
\[ \hat{E}_0(x) = \hat{E}_0^+(x) + \hat{E}_0^-(x). \]
The Gupta-Bleuler condition (6.5) is presented as
\[ \hat{E}_0^+(x) \Psi_B = 0. \] (6.10)

### 6.2 Additional axioms for the semiclassical electrodynamics

Let us investigate specific features of semiclassical scalar electrodynamics. As semiclassical virtual states, we consider the vectors (2.4) and their superpositions (2.5). Let \( \Phi(x) \equiv \)
be set of classical fields, \( \Pi(x) \equiv (\mathcal{E}_\mu(x), \Pi_\theta(x), \Pi^*_\theta(x)) \) be canonically conjugated momenta.

Not all semiclassical states are physical. Namely, the Gupta-Bleuler condition (6.10) leads to the nontrivial relations for classical variables \( X \) and vector \( f \in \mathcal{F}_X \).

Analogously to (2.9) and (2.10), for the operator \( \hat{E}_0(x) \) one can write

\[
\sqrt{\hbar} \hat{E}_0(x) K^h_X f \simeq K^h_X \mathcal{E}_0(x|X) f. \quad \mathcal{E}_0(x|X) = \mathcal{E}_0(x|X) + \sqrt{\hbar} \mathcal{E}^{(1)}_0(x|X) + ..., 
\]

with c-number function \( \mathcal{E}_0(x|X) \). Physical states will satisfy the conditions:

\[
\mathcal{E}_0(x|X) = 0, \quad \mathcal{E}^-_0(x|X) f = 0, \tag{6.11}
\]

states of the form

\[
f = \int dx \beta(x) \mathcal{E}^+(x|X) g \sim 0 \tag{6.12}
\]

are equivalent to zero. Conditions \( \mathcal{E}_0(x|X) = 0 \) and \( \dot{\mathcal{E}}_0(x|X) = 0 \) mean that

\[
\mathcal{E}_0 = 0, \quad \partial_k \mathcal{E}_k + i(\Pi^*_\theta \Theta - \Pi_\theta \Theta^*) = 0. \tag{6.13}
\]

Write analogs of the properties (2.12), (2.23) and (2.27) for the semiclassical field \( \mathcal{E}_0(x|X) \):

\[
\mathcal{E}_0(x|u_g X) U_g(u_g X \leftarrow X) = U_g(u_g X \leftarrow X) \mathcal{E}_0(w_g x|X); \quad \partial \mathcal{E}_0 + \frac{\partial}{\partial \alpha} \mathcal{E}_0 = [\mathcal{E}_0(x|X), \omega_X \frac{\partial}{\partial \alpha}], \tag{6.14}
\]

\[
\mathcal{E}_0(x|X_2) V(X_2 \leftarrow X_1) = V(X_2 \leftarrow X_1) \mathcal{E}_0(x|X_1), \quad X_1 \sim X_2.
\]

The relations (6.14) leads to the following important properties of semiclassical transformations:

(a) semiclassical Poincare transformation \( u_g \) takes physical state to physical; semiclassical transformation \( U_g(u_g X \leftarrow X) \) conserves the condition for physical states and equivalence relation.

(b) the operator \( \frac{\partial}{\partial \alpha} + \omega_X \frac{\partial X}{\partial \alpha} \) takes physical semiclassical states to physical and conserves the equivalence relation (this property means that for \( \alpha \)-dependent physical state \( K^h_X(\alpha)f(\alpha) \) state \( \frac{\partial}{\partial \alpha} K^h_X(\alpha)f(\alpha) \) is also physical);

(c) the operator \( V(X_2 \leftarrow X_1) \) takes physical states to physical and conserves the equivalence property.

In semiclassical mechanics of constrained systems [33], there are also equivalent states corresponding to different field configurations. Consider the operator:

\[
e^{-\frac{i}{\sqrt{\hbar}} \int dx \gamma_1(x) \hat{E}_0(x)}, \quad e^{-\frac{i}{\sqrt{\hbar}} \int dx \gamma_1(x) \hat{A}(x)}, \tag{6.15}
\]

38
with real functions $\gamma_{1,2}(x)$. It follows from (6.6) that each operator (6.15) takes a physical state to equivalent one. The classical configuration is transformed as:

$$A^0 \rightarrow A^0 - \gamma_1, \quad A^k \rightarrow A^k + \partial_k \gamma_2,$$
$$\Theta \rightarrow \Theta e^{i\gamma_2}, \quad \Pi_\theta \rightarrow \Pi_\theta e^{i\gamma_2}.$$

Therefore, there is a gauge (physical) equivalence of classical states; for each pair of gauge equivalent classical states $X_1 \sim_{\text{phys}} X_2$ an unitary operator $V_{\text{ph}}(X_2 \leftarrow X_1) : F_{X_1} \rightarrow F_{X_2}$ is specified. It satisfies the relations analogous to (2.25), (2.26), (2.28).

Therefore, the axioms of semiclassical field theory should be modified for QED as follows.

**G1 (QED)** A virtual semiclassical bundle is given, space of the bundle is interpreted as a set of virtual classical states, fibres $F_X$ (indefinite inner product spaces) are spaces of quantum states in the external background.

In axioms G2-G5, semiclassical bundle should be viewed as a virtual semiclassical bundle, field $\Phi$ is a set $(A^\mu, \Theta, \Theta^*)$, property (2.12) should be modified for the vector field $A^\mu$.

In addition, one should postulate the properties of the field $\Sigma_0$.

**G6 (QED).** An operator-valued distribution $\Sigma_0(x|X)$ expanded in $\sqrt{\hbar}$ is given. The leading order of expansion $\Sigma_0(x|X)$ is a $c$-number. The following condition is satisfied:

$$\partial_\mu \partial^\mu \Sigma_0(x|X) = 0 \quad (6.16)$$

Properties (6.14) are obeyed. Elements $X \in \mathcal{X}$ such that $\Sigma_0(x|X) = 0$ forms a set of all classical physical states $X_{\text{phys}}$. Elements of the space of the virtual semiclassical bundle such that $X \in X_{\text{phys}}$ and $\Sigma_0(x|X) f = 0$ are physical semiclassical states. Two physical states $f_1, f_2 \in F_X$ are equivalent iff for some $g \in F_X$ the difference $f_1 - f_2$ has the form (6.12). The scalar square of physical semiclassical state $(X, f)$ is nonzero; it vanishes iff $f \sim 0$.

Take into account the property of gauge invariance.

**G7 (QED).** A gauge equivalence relation is specified on $X_{\text{phys}}$. For each pair $X_1 \sim_{\text{phys}} X_2$ an unitary operator $V_{\text{ph}} : F_{X_1} \rightarrow F_{X_2}$ taking physical states to physical, equivalent states to equivalent. The following relations are satisfied:

- for $X_1 \sim X_2$, the properties $X_1_{\text{phys}} X_2$ and $V_{\text{ph}}(X_2 \leftarrow X_1) \sim V(X_2 \leftarrow X_1)$ are satisfied;
- for $X_1 \sim_{\text{phys}} X_2$ and $X_2 \sim_{\text{phys}} X_3$, one has $X_1_{\text{phys}} X_3 \sim V_{\text{ph}}(X_3 \leftarrow X_1)_{\text{phys}} X_2 \sim V_{\text{ph}}(X_2 \leftarrow X_1)$;
- let $X_1 \sim_{\text{phys}} X_2$, then $u_g X_1_{\text{phys}} u_g X_2 \sim V_{\text{ph}}(u_g X_2 \leftarrow u_g X_1) \sim V_{\text{ph}}(u_g X_2 \leftarrow X_1)_{\text{phys}} L_g(u_g X_2 \leftarrow X_2) L_g(u_g X_2 \leftarrow X_1)$;
- let $X_1(\alpha) \sim_{\text{phys}} X_2(\alpha)$, then $i \hbar \frac{\partial}{\partial \alpha} V_{\text{ph}}(X_2 \leftarrow X_1)_{\text{phys}} V_{\text{ph}}(X_2 \leftarrow X_1) \sim \omega X_1 [\partial X_1] - \omega X_2 [\partial X_2] \sim V_{\text{ph}}(X_2 \leftarrow X_1)$,
6.3 Specific features of covariant formulation of semiclassical electrodynamics

Let us discuss the axioms of covariant formulation of semiclassical electrodynamics. Let us construct an analog of the state (2.6):

\[ \Psi \approx e^{i \hat{S}} T e^{\frac{i}{\hbar} \int dx [J_\mu(x) A^\mu_\lambda(x) + \sigma(x) \theta^\lambda(x) + \sigma^*(x) \theta^*_\lambda(x) + \kappa(x) E_{0\lambda}(x)]} \equiv e^{i \hat{S}} T f. \]  

(6.17)

Here \( J \equiv (J_\mu, \sigma^*, \sigma, \kappa) \) is a set of real functions, classical Schwinger sources. In addition to the sources \( A^\mu \) and \( \theta \), the source for the field \( E_0 \) is introduced.

Semiclassical retarded fields

\[ \Phi_R \equiv (A^\mu_R, \Theta_R, \Theta^*_R, E_{0R}) \]

are introduced analogously to (3.2). Analogously to [23], we show that classical retarded field vanishes at \(-\infty\) and satisfy the system of equation

\[
\begin{align*}
\partial_\nu F^{\mu\nu}_R + \partial^\mu E_{0R} &= -i(D^\mu_R \Theta^*_R \Theta_R - \Theta^*_R D^\mu_R \Theta_R) + J^\mu, \\
D_{\mu R} D^\mu_R + V'(\Theta^*_R \Theta_R) \Theta_R &= \sigma, \\
\partial_\mu A^\mu_R + \xi E_{0R} &= -\kappa.
\end{align*}
\]

(6.18)

Here the notations \( F_{\mu\nu R} = \partial_\mu A_{\nu R} - \partial_\nu A_{\mu R}, D^\mu_R = \partial^\mu - i A^\mu_R \) are used. Properties of semiclassical retarded fields (3.6), (3.7), (3.8), (3.9), (3.10) are generalized for electrodynamics; when the Poincare invariance property is written, one should take into account that \( A^\mu_R \) is a vector field.

The operator \( W_J \), functional \( T_f \) and equivalence relation \( J \sim 0 \) are defined according to eq.(3.12) which is viewed in virtual sense. It happens that \( J \sim 0 \) iff the classical retarded field generated by \( J \) is zero at \(+\infty\). System (6.18) gives a one-to-one correspondence between field configurations \( \Phi_c \equiv (A^\mu_c, \Theta_c, \Theta^*_c, E_{0c}) \) with compact support and sources equivalent to zero.

Classical action is defined from relation (3.17). It happens that it is presented as a sum of a gauge-invariant part and gauge-fixing term:

\[
\begin{align*}
I[\Phi_c] &= I_{inv}[A^\mu_c, \Theta_c, \Theta^*_c] + I_{gf}[E_{0c}, A^\mu_c], \\
I_{inv} &= \int dx [-\frac{1}{2} F_{\mu\nu c} F^{\mu\nu}_c + D_{\mu c} \Theta^*_c D^\mu_c \Theta_c - V(\Theta^*_c \Theta_c)], \\
I_{gf} &= \int dx [E^2_{0c}]^{\frac{1}{4}} [D_{\mu c} A^\mu_c + \frac{\xi}{2} E^2_{0c}],
\end{align*}
\]

with \( D_{\mu c} = \partial_\mu - i A_{\mu c}, F^{\mu\nu}_c = \partial^\mu A^\nu_c - \partial^\nu A^\mu_c \). The properties of the operator \( W[\Phi_c] \) are also generalized to electrodynamics.

Thus, axioms C1-C5 are modified as follows: \( F \) is a space with indefinite inner product, Poincare invariance should be written for vector case.
Investigate now what additional conditions should be imposed in order to obtain properties G6 and G7.

Relation (6.16) can be rewritten as

$$\partial_\mu \partial^\mu E_0^R(x|J) = 0, \quad x > \sim \text{supp} J. \quad (6.20)$$

Classical source $J$ corresponds to the classical physical state iff

$$E_0^R(x|J) = 0, \quad x > \sim \text{supp} J. \quad (6.21)$$

The Gupta-Bleuler condition can be rewritten as

$$E_0^R(x|J) f = 0. \quad (6.22)$$

Investigate now gauge equivalence of semiclassical states. Notice that the operators (6.15) can be viewed as limit cases of the operator

$$T e^{i \sqrt{\hbar} \int dx \hat{E}_0(x) \Delta \kappa(x)} \quad (6.23)$$

The operator (6.23) takes physical states to equivalent. Therefore, the semiclassical states of the form

$$T J_1 + \Delta \kappa_1 \sim_{\text{phys}} T J_2 + \Delta \kappa_2, \quad \text{supp} \kappa_{1,2} > \text{supp} J$$

are physically equivalent. Here $J + \Delta \kappa \equiv (J_\mu, \sigma^*, \sigma, \kappa + \Delta \kappa)$. Note that classical retarded field generated by sources $J + \Delta \kappa_1$ and $J + \Delta \kappa_2$ are gauge equivalent, since they coincide in the domain $\text{supp} J < x < \text{supp} \Delta \kappa_{1,2}$, while for the $x > \text{supp} J$ case the function $\Delta \kappa$ enters to the gauge condition only $\partial_\mu A^\mu = -\Delta \kappa$.

We say that physical sources $J_1$ and $J_2$ are gauge equivalent, $J_1 \sim_{\text{phys}} J_2$, iff any of the following properties is obeyed:
- there exist sources $\Delta \kappa_{1,2}$ satisfying the conditions $\text{supp} \Delta \kappa_{1,2} > \text{supp} J_{1,2}$, $J_1 + \Delta \kappa_1 \sim J_2 + \Delta \kappa_2$;
- there exist sources $\Delta \kappa_{1,2}$, $J_+$ such that $\text{supp} J_+ > \text{supp} \Delta \kappa_{1,2} > \text{supp} J_{1,2}$, $J_1 + \Delta \kappa_1 + J_+ \sim 0$, $J_2 + \Delta \kappa_2 + J_+ \sim 0$;
- the retarded fields generated by the sources $J_1$ and $J_2$ are gauge equivalent.

We say that $(S_1, J_1) \sim_{\text{phys}} (S_2, J_2)$ iff

$$S_1 + T J_1 + \Delta \kappa_1 + J_+ = S_2 + T J_2 + \Delta \kappa_2 + J_+.$$

Since $E_0^R(x|J_1) = E_0^R(x|J_2)$ in the domain $x > \text{supp} J_{1,2}$, while $I_{\text{inv}}$ is gauge invariant, this definition does not depend on the particular choice $\Delta \kappa_{1,2}$. 

41
Set
\[ V_{ph}(X_2 \leftarrow X_1) = V(J_2 + \Delta \kappa_2 \leftarrow J_1 + \Delta \kappa_1). \]  
(6.24)

If the property
\[ \mathcal{E}_{\partial R}(x|J + \Delta \kappa) = \mathcal{E}_{\partial R}(x|J), \quad \text{supp} \Delta \kappa > \text{supp} J, \]  
(6.25)
is satisfied for any physical source \( J \), the properties G7 will be satisfied. Namely,
\[ \mathcal{E}_{\partial R}(x|J_2)V_{ph}(X_2 \leftarrow X_1) = V_{ph}(X_2 \leftarrow X_1)\mathcal{E}_{\partial R}(x|J_1), \quad x > \text{supp} J_{1,2}. \]

Therefore, the operator \( V_{ph}(X_2 \leftarrow X_1) \) takes physical states to physical and conserves the equivalence property.

Show that the definition (6.24) does not depend on particular choice of \( \Delta \kappa_{1,2} \). Under small variations, one has:
\[
\begin{align*}
ih \delta V_{ph}(X_2 \leftarrow X_1) &= -V_{ph}(X_2 \leftarrow X_1) \int dx \mathcal{E}_{\partial R}(x|J_1 + \Delta \kappa_1)\delta \Delta \kappa_1(x) \\
&\quad + \int dx \mathcal{E}_{\partial R}(x|J_2 + \Delta \kappa_2)\delta \Delta \kappa_2(x)V_{ph}(X_2 \leftarrow X_1) \\
&= -V_{ph}(X_2 \leftarrow X_1) \int dx \mathcal{E}_{\partial R}(x|J_1)\delta \Delta \kappa_1(x) + \int dx \mathcal{E}_{\partial R}(x|J_2)\delta \Delta \kappa_2(x)V_{ph}(X_2 \leftarrow X_1)
\end{align*}
\]

Therefore, \( \delta V_{ph}(X_2 \leftarrow X_1) \) is zero up to equivalence relation. Other properties G7 are also checked by a direct calculation.

Notice that properties (6.20) and (6.25) can be reduced to the one equation
\[ \partial_\mu \partial^\mu \mathcal{E}_{\partial R}(x|J + \Delta \kappa) = 0, \quad x > \text{supp} J, \text{supp} \Delta \kappa > \text{supp} J. \]  
(6.26)
Namely, relation (6.20) is a partial case of (6.26). Moreover, it follows from eqs. (6.26) and (6.20) that \( \mathcal{E}_{D R}(x|J) \) and \( \mathcal{E}_{\partial R}(x|J + \Delta \kappa) \) satisfy the same equation and coincide at \( \text{supp} J < x < \text{supp} \Delta \kappa \). Thus, we obtain property (6.25).

Thus, in addition to properties C1-C5, the following axioms should be satisfied.

\textbf{C6 (QED).} The retarded semiclassical field \( \mathcal{E}_{\partial R}(x|J) \) satisfies the condition (6.26). Physical states satisfying the condition (6.22) have a nonnegative square of norm, which vanishes for the states \( \bar{f} = \int_{x > \supp J} dx \mathcal{E}_{\partial R}(x|J)\beta(x)|\bar{g} \) only.

Represent property (6.26) as a condition on the operator \( W[\Phi_c] \). In the domain \( x > \text{supp} J \), the field \( \Phi_c(x) = \Phi_R(x|J + \Delta \kappa) \) satisfies the equations
\[
\partial_\nu F_c^{\mu
 \nu} + i(D_c^\nu \Theta_c^\nu \Theta_c - \Theta_c^\nu D_c^\nu \Theta_c) = 0, \quad D_\mu D^\mu \Theta_c + V'(\Theta_c^\nu \Theta_c)\Theta_c = 0. \]  
(6.27)
For functionals \( F[\Phi_c] \), introduce the following notations:
\[
\nabla_x F \equiv -\partial^\mu \frac{\delta F}{\delta \Phi_c^\mu(x)} + i\left[ \Theta_c(x)\frac{\delta F}{\delta \Theta_c(x)} - \Theta_c^\nu(x)\frac{\delta F}{\delta \Theta_c^\nu(x)} \right], \quad \Xi F \equiv \int dy \frac{\delta F}{\delta \Phi_c(y)} \{ \Phi_R(y|J_{\Phi_c}) - \Phi_c(y) \}. \]  
(6.28)
The operator $\nabla_x$ is related to the gauge transformation of the functional $F$:

$$\nabla_x F \equiv \frac{\delta}{\delta \alpha(x)} F[\mathcal{A}^\mu_c + \partial^\mu \alpha, \Theta_c e^{i\alpha}, \Theta^*_c e^{-i\alpha}];$$

if the functional $F$ is gauge invariant then $\nabla_x F = 0$.

Under notations (6.28), the Yang-Feldman relations (3.25) has the following form at the domain $x > \text{supp} J$:

$$\Xi \delta I^{\text{inv}} \equiv \frac{\delta I^{\text{inv}}}{\delta \mathcal{A}^\mu_c(x)} + \partial_\mu \mathcal{E}^{\text{int}}(x|J + \Delta \kappa) = i\hbar W^+ \frac{\delta W}{\delta \mathcal{A}^\mu_c(x)},$$

$$\Xi \delta I^{\text{inv}} \equiv i\hbar W^+ \frac{\delta I^{\text{inv}}}{\delta \Theta_c(x)}, \quad \Xi \delta I^{\text{inv}} \equiv i\hbar W^+ \frac{\delta I^{\text{inv}}}{\delta \Theta^*_c(x)}, \quad (6.29)$$

Making use of relations (6.29), we find:

$$\Xi \nabla_x I^{\text{inv}} - i \left( \Xi \Theta_c(x) \frac{\delta I^{\text{inv}}}{\delta \Theta_c(x)} - \Xi \Theta^*_c(x) \frac{\delta I^{\text{inv}}}{\delta \Theta^*_c(x)} \right) = i\hbar W^+ \nabla_x W.$$

Take into account that the functional $I^{\text{inv}}$ is gauge invariant and $\nabla_x I^{\text{inv}} = 0$; $\frac{\delta I^{\text{inv}}}{\delta \Theta_c(x)} = 0$ and $\frac{\delta I^{\text{inv}}}{\delta \Theta^*_c(x)} = 0$ for the classical field at $x > \text{supp} J$. Therefore, the relation (6.26) means that the scattering matrix in the external field should be gauge invariant:

$$\nabla_x W = 0, \quad (6.30)$$

provided that classical equation of motion (6.27) are satisfied.

### 6.4 On the leading order

Fields and S-matrix $W_0$ are constructed, making use of general formulas of subsection 4.7.

For the free case, analogously to subsection 4.2 one obtains equations and commutation relations (4.4) for multicomponent free field $\hat{\varphi}_0(x) = \hat{\varphi}_{in}(x) \equiv (\hat{A}^\mu_{in}(x), \hat{\theta}_{in}(x), \hat{\theta}^*_{in}(x), \hat{E}^0_{in}(x))$, with independent components. Field $\hat{\theta}_{in}$ is a complex scalar free field of the mass $m$. For electromagnetic field, we obtain:

$$\partial_\mu \hat{E}^{\mu\nu}_{in} + \partial^\mu \hat{E}^{\nu}_{in} = 0, \quad \partial_\mu \hat{A}^\mu_{in} + \xi \hat{E}^0_{in} = 0,$$

$$\left[ \hat{A}^\mu_{in}(x), \hat{A}^\nu_{in}(y) \right] = -\frac{i}{4} (g^{\mu\nu} D_0(x - y) - (1 - \xi) \times \partial^\mu \partial^\nu \int dz (D_0^0(x - z) D_0^{\text{et}}(z - y) - D_0^0(x - z) D_0^{\text{et}}(z - y) - D_0^{\text{adv}}(x - z) D_0^{\text{adv}}(z - y)),$$

$$\left[ \hat{A}^\mu_{in}(x), \hat{E}^0_{in}(y) \right] = -\frac{i}{2} \partial_\mu D_0(x - y), \quad \left[ \hat{E}^0_{in}(x), \hat{E}^0_{in}(y) \right] = 0.$$

43
Here $D_0$ is a commutation function for massless scalar field. Commutation relations takes the simplest form at $\xi = 1$; it is this gauge that is used usually in calculations.

Semiclassical field $\hat{\varphi}(x|\Phi_c) \equiv (\hat{A}^\mu(x), \hat{\theta}(x), \hat{\theta}^*(x), \hat{E}_0(x))$ satisfies the system (4.53). Write its explicit form for the case $V(\Theta^*\Theta) = m^2 \Theta^*\Theta$:

$$\partial_\nu \hat{F}^{\mu\nu} + \partial^\mu \hat{E}_0 + i(\hat{D}_\mu^c \theta^* C_c + \hat{D}_c^\mu \Theta^* - \hat{\theta}^* \hat{D}_c^\mu \Theta_c - \Theta^* \hat{D}_c^\mu \hat{\theta}) = 0,$$

$$\partial_\mu \hat{A}^\mu + \hat{E}_0 = 0, \quad \hat{D}_c^\mu \hat{D}_c^{\mu*} \hat{\theta} - i \hat{A}_c^\mu \hat{D}_c^{\mu*} \hat{\theta} = 0,$$

where

$$\hat{D}_c^\mu \hat{\theta} = \partial_\mu \hat{\theta} - i \hat{A}_c^\mu \hat{\theta} - i \hat{A}_c^{\mu*} \hat{\theta}.$$

Components of current vector are

$$j_\mu^A \equiv \frac{i}{2} W_0^+ \frac{\delta W_0}{\delta A_\mu} =: i \hat{\theta}^* \hat{D}_c^\mu \theta - i (\hat{D}_c^\mu \hat{\theta}^*)^* \hat{\theta} + \hat{A}_c^\mu (\Theta^* \hat{\theta} + \Theta_c \hat{\theta}^*) + \frac{\delta A_1}{\delta A^c_\mu};$$

$$j_0^A \equiv \frac{i}{2} W_0^+ \frac{\delta W_0}{\delta \Theta_c} =: -i \hat{A}_c^\mu (\hat{D}_c^\mu \hat{\theta}^*) + i \hat{D}_c^\mu (\hat{A}_c \hat{\theta}^*) + \frac{\delta A_1}{\delta \Theta^*}.$$

Properties of Poincare invariance, unitarity and causality lead to restrictions (4.62) on one-loop effective action. The gauge invariant property (6.30) leads to the following requirement. $A_1[\Phi_c]$ should be gauge invariant:

$$\nabla_x A_1[\Phi_c] = 0$$

under conditions (6.27).

The property of positive definiteness of inner product in physical space is a corollary of general properties of indefinite inner product spaces.

The further analysis of scalar electrodynamics is analogous to the case of scalar field. For each order of perturbation theory, one should additionally check property (6.30) which is related to the gauge invariance of classical action.

## 7 Semiclassical integrals of motion and BRST-charge

The quantizations considered above are applicable only to Abelian theories. A manifestly covariant approach to quantize nonablelian fields is BRST approach.

### 7.1 On BRST-quantization

In BRST-quantization, additional degrees of freedom are introduced to the theory. To each constraint $\Lambda^a$ one assigns the Lagrange multiplier $\lambda^a$, canonically conjugated momentum $\pi^a$ and fermionic variables: ghost $C^a$ and antighost $\bar{C}^a$ and canonically conjugated momenta $\Pi^a$. 
and $\Pi^a$. The operators $\overline{C}_a$ and $\Pi_a$ are anti-Hermitian, others are Hermitian. The nontrivial commutation relations are:

$$[\lambda^a, \pi_b] = i\delta^a_b, \quad [C^a, \Pi_b]_+ = \delta^a_b, \quad [\overline{C}_a, \Pi^b]_+ = \delta^b_a.$$ 

The main object of the theory is BRST-charge $\hat{Q}$. If the constraints form a Lie algebra with structure constants $f^{abc}$

$$[\hat{\Lambda}_a, \hat{\Lambda}_b] = -if^{abc}\hat{\Lambda}_c,$$

the BRST-charge has the following form:

$$\hat{Q} = C^a\hat{\Lambda}_a + \frac{i}{2}f^{abc}\Pi_a C^b C^c - i\pi_a\Pi^a. \quad (7.1)$$

It satisfies the Hermitian and nilpotent conditions:

$$\hat{Q}^+ = \hat{Q}, \quad \hat{Q}^2 = 0.$$

The ghost number operator is also introduced to the theory:

$$\hat{N} = \Pi^a\overline{C}_a - \Pi_a C^a,$$

It is the difference between number of ghosts and antighosts. It satisfies the commutation relations:

$$[\hat{N}, C^a] = C^a, \quad [\hat{N}, \Pi^a] = \Pi^a,$$

$$[\hat{N}, \overline{C}_a] = -\overline{C}_a, \quad [\hat{N}, \Pi_a] = -\Pi_a, \quad [\hat{N}, \hat{Q}] = \hat{Q}.$$ 

The following conditions are imposed for physical states:

$$\hat{Q}\Psi_B = 0, \quad \hat{N}\Psi_B = 0; \quad (7.2)$$

physical states of the form

$$\hat{Q}\Psi_B \sim 0, \quad \hat{N}\Psi_B = -\Psi_B,$$

being orthogonal to physical subspace are set to be equivalent to zero.

For the scalar electrodynamics, the BRST-quantization method is equivalent to the Gupta-Bleuler approach. Namely, for system with Hamiltonian and constraints (6.2) the BRST-charge has the form:

$$\hat{Q} = \int d\mathbf{x}[\mathcal{L}_0(\mathbf{x})\Lambda_\mathbf{x} - iE_0(\mathbf{x})\Pi(\mathbf{x})]. \quad (7.3)$$

45
The operator \([\hat{Q}, K]_+\) being equivalent to zero is added to the Hamiltonian of the system. The operator \(K\) is chosen to be as follows:

\[
K = \int d\mathbf{x}[A_0(\mathbf{x})\Pi(\mathbf{x}) - i\overline{C}(\mathbf{x})\partial_k A_k(\mathbf{x}) - \frac{i}{2}\xi\overline{C}(\mathbf{x})E_0(\mathbf{x})].
\]  

(7.4)

Then the full Hamiltonian will take the form:

\[
\mathcal{H}_B = \mathcal{H} + A_0\Lambda - \frac{\xi}{2}E_0^2 + A_k\partial_k E_0 + \Pi\Pi + \overline{C}(\Delta)C,
\]  

(7.5)

An additional term with ghost fields is added. For the model under consideration, ghosts do not interact with other fields, they satisfy the equations:

\[
\begin{align*}
\partial_\mu \partial^\mu C(x) &= 0, \\
\partial_\mu \partial^\mu \overline{C}(x) &= 0,
\end{align*}
\]

\[
\begin{align*}
[C(x), C(y)]_+ &= 0, \\
[\overline{C}(x), \overline{C}(y)]_+ &= 0,
\end{align*}
\]

\[
[C(x), \overline{C}(y)]_+ = -iD_0(x - y),
\]

(7.6)

while the BRST-charge is taken to the form:

\[
\hat{Q} = \int d\sigma^\mu[\partial_\mu C E_0 - C\partial_\mu E_0].
\]

(7.7)

One supposes that ghosts are in vacuum; then condition (7.2) is taken to the form (6.10).

Notice that BRST-approach allows us to construct a covariant formulation of Abelian Higgs model with spontaneous symmetry breaking in ‘t Hooft gauge (cf. [34]). The ghosts interact with scalar field, so that they should be taken into account.

The BRST-approach is applicable to non-Abelian theories as well [29].

To investigate semiclassical gauge transformations, we need some properties of integrals of motion.

### 7.2 Noether integrals of motion

Investigate properties of the Noether integrals of motion \(\hat{Q}\) in semiclassical field theory. Commutation relations between \(\hat{Q}\) and \(K^h_X\) may be written analogously to (2.9) as

\[
\hat{Q}K^h_X f = K^h_X \overline{Q}_X f,
\]

(7.8)

semiclassical charge \(\overline{Q}_X\) is expanded in \(\sqrt{h}\):

\[
\overline{Q}_X = Q_X + \sqrt{h}Q^{(1)}_X + ...
\]

46
and coincide in the leading order with classical integral of motion $Q_X$.

Analogously to properties of the field operator (2.12), (2.23) and (2.27), one obtains the relations

$$Q_{u^g} X \leftarrow X = U_{u^g} (u_g X \leftarrow X) Q_X,$$

$$\frac{i h}{\partial \alpha} Q_X = \left[ Q_X, \omega_X \left( \frac{\partial X}{\partial \alpha} \right) \right],$$

$$Q_{X_2} V (X_2 \leftarrow X_1) = V (X_2 \leftarrow X_1) Q_{X_1}.$$  \hspace{1cm} (7.9)

When the covariant formulation is used, the relations takes the form:

$$Q_{u^g} J \leftarrow J = U_{u^g} Q_J,$$  \hspace{1cm} (7.10)

$$i h \frac{\delta Q_J}{\delta J (x)} = \left[ \Phi_R (x|J), Q_J \right].$$  \hspace{1cm} (7.11)

and

$$W_{J_2 + J_+} Q_{J_2} W_{J_2 + J_+}^+ = W_{J_1 + J_+} Q_{J_1} W_{J_1 + J_+}^+,$$

here $\text{supp} J_+ > \text{supp} J_{1,2}$, $J_1 + J_+ \sim 0$, $J_2 + J_+ \sim 0$. It follows from (7.11) that when one variates $J \rightarrow J + \delta J$ the variation $W_{J_1 + J_+} Q_{J_1} W_{J_1 + J_+}^+$ will vanish. This means that the third property (7.9) is a corollary of others.

It happens that integrals of motion corresponding to symmetry transformations

$$\Phi (x) \rightarrow \Phi (x) + \delta \Phi (x|\Phi),$$

have the following form [23] in the leading order:

$$Q_J = \int dx J (x) \delta \Phi (x|\Phi_R (\cdot|J)).$$  \hspace{1cm} (7.12)

### 7.3 Properties of semiclassical BRST-charge

Let us discuss specific features of covariant formulation of gauge theories. First, introduce semiclassical fermionic fields: antighost $\overline{C} (x|J)$ and ghost $C_R (x|J)$.

**C6 (BRST).** To each classical source $J (x)$ with compact support one assigns operators of semiclassical fields of ghosts and antighosts $C_R (x|J)$ (Hermitian) and $\overline{C} (x|J)$ (anti-Hermitian) expanded in $\sqrt{h}$. The field $C_R (x|J)$ depends on $J$ at preceding time moments only; the following properties are satisfied:

$$U_{u^g - 1} C_R (x|u_g J) U_{u^g} = C_R (u_g x|J),$$

$$i h \frac{\delta C_R (x|J)}{\delta J (y)} = \left[ \Phi_R (y|J), C_R (x|J) \right], \hspace{1cm} x \sim y,$$

$$i h \frac{\delta \overline{C} (x|J)}{\delta J (y)} = \left[ \Phi_R (y|J), \overline{C} (x|J) \right].$$  \hspace{1cm} (7.13)
The operator $W[\Phi_c]$ satisfies the following commutation relations:

$$W[\Phi_c]C_R(x|0)W[\Phi_c] = C_R(x|J_{\Phi_c}), \quad x > supp\Phi_c,$$

$$W[\Phi_c]\overline{C}(x|0)W[\Phi_c] = \overline{C}(x|J_{\Phi_c}).$$  \hfill (7.14)

Take into account the relations imposed on the BRST-charge and ghost number.

**C7 (BRST).** To each classical source $J(x)$ with compact support one assigns the operators of semiclassical BRST-charge $Q_J$ and ghost number $N_J$, expanded in $\sqrt{h}$. They satisfy the property:

$$Q_{\mu,\nu}L_\nu = L_\nu Q_{\mu}, \quad i\hbar \delta Q_J = [\Phi_R(y|J), Q_J],$$

$$N_{\mu,\nu} = N_{\nu,\mu}, \quad i\hbar \delta N_J = [\Phi_R(y|J), N_J],$$

$$[N_J, C_R(x|J)] = C_R(x|J), \quad [N_J, \overline{C}(x|J)] = \overline{C}(x|J), \quad [N_J, \Phi_R(x|J)] = 0.$$  \hfill (7.15)

$$[Q_J, \overline{C}(x|J)] = -i\mathcal{E}_0(x|J).$$  \hfill (7.16)

Find an explicit form of the BRST-charge in the leading order of the expansion. Consider the scalar electrodynamics as an example. It follows from (7.7) that

$$Q_J^{(1)} = \int d\sigma^\mu [\partial_\mu C_R^{(1)} \mathcal{E}_{0R}^{(1)} - C_R^{(1)} \partial_\mu \mathcal{E}_{0R}^{(1)}];$$

the integral is taken over any space-like surface at the domain $x > suppJ$. Making use of relations

$$\partial_\mu \partial^\mu C_R^{(1)} = 0,$$

$$\partial_\mu \partial^\mu \mathcal{E}_{0R} = \partial_\mu J^\mu - i(\sigma^* \Theta_R - \Theta_R^* \sigma),$$

take BRST-charge to the form

$$Q_J^{(1)} = \int dx [J^\mu \partial_\mu C_R^{(1)} + i(\sigma^* \Theta_R - \Theta_R^* \sigma)C_R^{(1)}].$$  \hfill (7.17)

We see that formula (7.17) is analogous to relation (7.12) for Noether integral of motion, provided that $\delta \Phi$ is an infinitesimal transformation with anticommuting variables

$$\delta A_\mu = \partial_\mu C_R^{(1)}, \quad \delta \Theta = i\Theta C_R^{(1)}, \quad \delta \Theta^* = -i\Theta^* C_R^{(1)},$$  \hfill (7.18)

which is analogous to gauge transformation. Property (7.17) can be viewed as a basis for the semiclassical perturbation theory for BRST-charge.
Consider the conditions on the physical states.

**C8 (BRST)** Physical states satisfying conditions $Q_f \bar{f} = 0$, $N_f \bar{f} = 0$, have non-negative square of norm. $(\bar{f}, \bar{f}) = 0 \iff \bar{f} = Q_f \bar{f}$.

Analogously to electrodynamics, introduce equivalence relation for sources and semiclassical states. An analog of property (6.25) will have the following form.

**C9 (BRST).** The operator of semiclassical BRST-charge obeys the condition

$$Q_{f+\Delta \kappa} = Q_f, \quad \text{supp} \Delta \kappa > \text{supp} J.$$ (7.19)

### 7.4 Leading order of semiclassical expansion

Investigate the properties of BRST-charge in the leading order of semiclassical expansion. Suppose that zero order of expansion for the BRST-charge, classical action, ghost and antighost fields $c_0(x)$ and $\overline{c_0}(x)$ as $J = 0$ is known. These are free fields.

Denote $\Phi_R^{(1)}(x|J) \equiv (\hat{A}^\mu(x), \hat{E}_0(x), \hat{\theta}(x), \hat{\theta}^*(x)), c(x) \equiv C_R^{(1)}(x|J)$. It follows from commutation relation (7.13) that the operators $c(x)$ and $\overline{c}^{(1)}(x|J)$ commute with $(\hat{A}^\mu(x), \hat{E}_0(x), \hat{\theta}(x), \hat{\theta}^*(x))$.

It follows from the nilpotent property of the BRST-charge that the operators $c(x)$ should anti-commute

$$[c(x), c(y)]_+ = 0.$$

Write down the commutation relation (7.15) in the second order of perturbation theory. For electrodynamics, one obtains:

$$\left[ \hat{A}^\mu(x), Q_J^{(2)} \right] = i \partial^\mu c(x), \quad \left[ \hat{E}_0(x), Q_J^{(2)} \right] = 0,$$

$$\left[ \hat{\theta}(x), Q_J^{(2)} \right] = -\Theta_R(y|J)c(y), \quad \left[ \hat{\theta}^*(x), Q_J^{(2)} \right] = \Theta_R(y|J)c(y).$$ (7.20)

Making use of eq.(6.31), we find that

$$\partial_\mu \hat{A}^\mu + \xi \hat{E}_0 = 0,$$

therefore, the ghost field $c(x)$ satisfies the free equation

$$\partial_\mu \partial^\mu c(x) = 0.$$

At $x < \text{supp} J$, field $c(x)$ should coincide to the free field $c_0(x)$; therefore, property $c(x) = c_0(x)$ is valid for all $x$.  

49
It follows from property (7.16) in the leading order that

\[ [c_0(x), \overline{c}^{(1)}(y|J)] = -iD_0(x - y). \]

Then

\[ \overline{c}^{(1)}(y|J) = \overline{c}_0(y), \]

since the operator \( \overline{c}^{(1)}(y|J) \) should be linear combination of \( \overline{c}_0 \).

It follows from commutation relations (7.16) and (7.20) that

\[ Q^{(2)}_j = \int dx [\dot{c}(x)E_0(x) - c(x)\dot{E}_0(x)]. \]

Properties (7.14) mean that the operator \( W[\Phi_c] \) should commute with ghost and antighosts. Higher orders are constructed analogously.
References

[1] S. Schweber. An Introduction to relativistic Quantum Field Theory. Evanston, 1961.

[2] A.A. Slavnov, L.D. Faddeev. Introduction to Quantum Theory of Gauge Fields. Moscow, Nauka, 1988.

[3] R. Feynmann, A. Hibbs. Quantum Mechanics and Path Integrals. N.Y. 1965.

[4] N.N. Bogoliubov, A.A. Logunov, A.I. Oksak, I.T. Todorov. General Principles of Quantum Field Theory. Moscow, Nauka, 1987.

[5] R.F. Streater, A.S. Wightman. PCT, Spin and Statistics and All That, N.Y., 1964. R. Jost. The General Theory of Quantized Fields. Providence, 1965.

[6] N.N. Bogoliubov, D.V. Shirkov. Introduction to the Theory of Quantized Fields. N.-Y., Interscience Publishers, 1959.

[7] N.N. Bogoliubov, B.V. Medvedev, M.K. Polivanov. Questions of the Theory of Dispersion Relations. Moscow, Fizmatgiz, 1958; B.V. Medvedev, M.K. Polivanov, V.P. Pavlov, A.D. Sukhanov, Teor. Mat. Fiz, 13 (1972) 3.

[8] O.I. Zavialov. Renormalized Feynman Graphs. Moscow, Nauka, 1979.

[9] H. Lehmann, K. Symanzik, W. Zimmermann. Nuovo Cim. 6 (1957) 319; K. Nishijima. Phys. Rev. 119 (1960) 485.

[10] J. Schwinger. Particles, Sources and Fields. Addison-Wesley. 1970.

[11] R. Rajaraman, Solitons and Instantons. An Introduction to solitons and instantons in quantum field theory, North-Holland, Amsterdam, Netherlands, 1982; J. Coldstone, R. Jackiw, Phys. Rev. D11 (1975), 1486; R. Jackiw, Rev. Mod. Phys. 49 (1977), 681.

[12] V.A. Rubakov. Classical Gauge Fields, Moscow, Editorial URSS, 1999.

[13] A.A. Grib, S.G. Mamaev, V.M. Mostepanenko, Vacuum Quantum Effects in Strong Fields, Atomizdat, Moscow, 1988; Friedmann Laboratory Publishing, St. Petersburg 1994.

[14] N.D. Birrell, P.C.W. Davies, Quantum Fields in Curved Space, Cambridge, UK: Univ. Pr., 1982.
[15] V.P. Maslov, O.Yu. Shvedov. The Complex Germ Method for Many-Particle and Quantum Field Theory Problems. Moscow, Editorial URSS, 2000.

[16] V.P. Maslov, O.Yu. Shvedov. Teor. Mat. Fiz. 104 (1995) 310.

[17] V.P. Maslov. Perturbation Theory and Asymptotic Methods. Moscow, Moscow University Press, 1965.

[18] V.P. Maslov. Operational Methods. Moscow, Mir publishers, 1976.

[19] V.P. Maslov. The Complex-WKB Method for Nonlinear Equations. Moscow, Nauka, 1977.

[20] V.P. Maslov, O.Yu. Shvedov. Teor. Mat. Fiz. 104 (1995) 479.

[21] O.Yu. Shvedov. Mat. Zametki. 65 (1999) 437; O.Yu. Shvedov. Mat. Sbornik. 190(10) (1999) 123.

[22] O.Yu. Shvedov. Ann. Phys. 296 (2002) 51.

[23] O.Yu. Shvedov. Teor. Mat. Fiz. 144 (2005) 492.

[24] G. Emch. Algebraic Methods in Statistical Mechanics and Quantum Field Theory. Wiley, 1972.

[25] F.A. Berezin. The Method of Second Quantization, N.Y., 1966.

[26] P.A.M. Dirac. Lectures on Quantum Mechanics, Yeshiva Univ., New York, 1965.

[27] L.D. Faddeev. Teor. Mat. Fiz. 1 (1969) 1.

[28] A. Astekar, J. Lewandowski, D. Marolf, J. Mourao, T. Thiemann. J. Math. Phys. 36 (1995) 6456; D. Giulini, D. Marolf. Class. Q. Grav. 16 (1999) 2489; O.Yu. Shvedov. Ann. Phys. 302 (2002) 2.

[29] C. Becchi, A. Rouet and R. Stora. Phys. Lett. B52 (1974) 344; C. Becchi, A. Rouet and R. Stora. Ann. Phys. 98 (1976) 287; I.V. Tyutin. FIAN preprint 39 (1975).

[30] E.S. Fradkin and G.A. Vilkovisky. Phys. Lett. B55 (1975) 224; I.A. Batalin and G.A. Vilkovisky. Phys. Lett. B69 (1977) 309; T. Kugo and I. Ojima. Suppl. Prog. Theor. Phys., No 66 (1979) 1.

[31] M. Henneaux. Phys. Reports 126 (1985) 1.
[32] H. Arisue, T. Fujiwara, T. Inoue and K. Ogawa, J. Math. Phys. 22 (1981) 2055.

[33] O. Yu. Shvedov. Teor. Mat. Fiz. 136 (2003) 418; hep-th/0111265.

[34] J. C. Collins. Renormalization. Cambridge University Press, 1987.