SOLUTIONS TO YANG-MILLS EQUATIONS

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ABSTRACT. This article gives explicit solutions to the Yang-Mills equations. The solutions have positive energy that can be made arbitrarily small by selection of a parameter showing that quantum Yang-Mills field theories do not have a mass gap.

1. INTRODUCTION

In the year 2000 the Clay Mathematics Institute (CMI) posed the following problem [1]:

Yang-Mills Existence and Mass Gap. Prove that for any compact simple group $G$, a nontrivial Yang-Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those cited in R. Streamer and A. Wightman (1964) or K. Osterwalder and R. Seiler (1973).

Thus, the existence of a non-trivial Yang-Mills theory involves showing that the theory fills axioms of axiomatic quantum field theory, while the existence of a mass gap seems to be another question that may be shown also in some other way. The problem concerns a pure Yang-Mills Lagrangian, i.e., only the gauge field without spinor fields, Higgs fields, or other fields. The mass gap is expected to arise from self-intersections of the Yang-Mills gauge field. The issue how the mass gap could appear is unclear but [2] has proposed one mechanism. The state of research to this problem up to 2004 is summarized in [3]. After that there have been some efforts to prove the existence of a mass gap, e.g. [4], [5], but the problem is still considered open.

This article presents explicit solutions to the Yang-Mills Euler-Lagrange equations. The solutions give arbitrarily small positive values for energy. This shows that the Hamiltonian has arbitrarily small eigenvalues indicating that there is no mass gap. The solutions can be given on $\mathbb{R}^4$ with Minkowski’s or Euclidean metric and they are simple, natural solutions that should be accepted as gauge fields in any non-trivial quantum field theory for the pure Yang-Mills Lagrangian.

2. DEFINITIONS AND NOTATIONS

We will first describe the problem setting as it can be presented in physics in tensor calculus, and at the end look at the more mathematical formulation with differential forms and the Hodge star operator. Unless otherwise stated, or the sum is written explicitly, there is summation over indices that are repeated on one side of an equation. For notations we refer to [6].

\begin{equation}
\mathcal{L} = -\frac{1}{2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4} F_\mu^a F_\mu^a \tag{2.1}
\end{equation}

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where

\[ F^{\mu\nu} = F_a^{\mu\nu} t_a \]  

and \( t_a \) are the generators of the Lie group satisfying

\[ Tr(t_a t_b) = \frac{1}{2} \delta_{ab} \quad [t_b, t_c] = i f_{abc} t_a \]  

The structure constants \( f_{abc} = f_{a}^{\mu\nu} \) are selected antisymmetric in all indices. The
gauge field

\[ A^\mu = A_a^\mu t_a \]  

defines the curvature \( F^{\mu\nu} \) by

\[ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu] \]  

In component form this gives

\[ F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g f_{abc} A_b^\mu A_c^\nu \]  

Curvature is antisymmetric

\[ F^{\mu\nu} = -F^{\nu\mu} \]  

The number \( g \) is called coupling constant, and

\[ \partial^\mu = \frac{\partial}{\partial x^\mu}, \quad \partial_\mu = \frac{\partial}{\partial x_\mu} \]  

are partial derivatives with respect to the contravariant coordinates \( x^\mu \) and
covariant coordinates \( x_\mu = g_{\mu\nu} x^\nu \). \( x_0 = ct \) and \( x_j, 1 \leq j \leq 3, \) are the space coordinates. The metric \( g_{\mu\nu} = g^{\mu\nu} \) is Minkowski’s metric

\[ (g_{\mu\nu})_{\mu,\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]  

Thus \( x_0 = x^0, x_j = -x^j \) for \( 1 \leq j \leq 3 \). For real vectors and tensors lowering and
raising indices is made by

\[ A_a^\mu = g_{\mu\nu} A_a^\nu, \quad F_a^{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F_a^{\alpha\beta}, \quad \partial_\mu = g_{\mu\nu} \partial^\nu \]  

Therefore (2.6) can also be expressed as

\[ g_{\mu\alpha} g_{\nu\beta} F_a^{\alpha\beta} = g_{\mu\alpha} g_{\nu\beta} \left( \partial^\alpha A_a^\beta \partial^\nu A_a^\alpha - g f_{abc} A_b^\alpha A_c^\nu \right) \]

\[ F_a^{\mu\nu} = \partial_\mu A_a^\nu - \partial_\nu A_a^\mu - g f_{abc} A_b^\mu A_c^\nu \]  

where we have written \( f_{abc} \) instead of \( f_{a}^{\mu\nu} \) to follow the summation convention for
the index \( a \). The natural setting of quantum field theories is that the component
functions of the fields take complex values. Then raising and lowering indices
involves taking complex conjugates but we will only do calculations with real fields.
Complex fields are better treated by the algebraic geometric formulation described
briefly at the end of Section 2.

Let us notice that there is a summation over \( b \) and \( c \) in (2.6) and (2.11). We
formulate this simple observation as a lemma since it is needed in the sequence.

**Lemma 2.1.** Let \( c > b \). The last term in (2.6) can be expressed as

\[ f_{abc} A_b^\mu A_c^\nu = \sum_{c > b} f_{abc} \left( A_b^\mu A_c^\nu - A_c^\nu A_b^\mu \right) \]
Proof. Expanding the commutator \([A^\mu, A^\nu]\)

\[
\left( \sum_b A_b^\mu t_b \right) \left( \sum_c A_c^\nu t_c \right) - \left( \sum_c A_c^\nu t_c \right) \left( \sum_b A_b^\mu t_b \right) = \sum_{b,c} (A_b^\mu A_c^\nu t_b t_c - A_c^\nu A_b^\mu t_c t_b) \tag{2.13}
\]

Since \(A_b^\mu\) are scalars \(A_c^\nu A_b^\mu = A_b^\mu A_c^\nu\). Thus we get

\[
= \sum_{b,c} A_b^\mu A_c^\nu \left[ t_b, t_c \right] = \sum_{b,c} A_b^\mu A_c^\nu i f_{abc} t_a \tag{2.14}
\]

As an example, let the group be SU(2). It has three generators

\[
t_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad t_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad t_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Then

\[
A^\mu = \sum_{a=1}^3 A_a^\mu t_a
\]

\[
[A^2, A^3] = i \left( f_{123} A_2^2 A_3^3 + f_{132} A_3^2 A_2^3 \right) t_1 + i \left( f_{231} A_3^2 A_1^3 + f_{213} A_1^2 A_3^3 \right) t_2 + i \left( f_{312} A_1^2 A_2^3 + f_{321} A_2^2 A_1^3 \right) t_3
\]

showing that Lemma 2.1 holds in this example. The proposed solutions make use of the following lemma.

**Lemma 2.2.** Let the gauge field have the form

\[
A^\mu_a = s_a E^\mu \tag{2.15}
\]

Then \(F_a^{\mu\nu}\) has the form

\[
F_a^{\mu\nu} = s_a G^{\mu\nu} \tag{2.16}
\]

and (2.6) and (2.11) reduce to

\[
F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu
\]

\[
F_a^{\mu\nu} = \partial_\mu A_a^\nu - \partial_\nu A_a^\mu
\]

**Proof.** Because of (2.11) it suffices to show (2.16) and the first equation in (2.17).

From Lemma 2.1

\[
f_{abc} A_b^\mu A_c^\nu = \sum_{c>b} f_{abc} (s_b E^\mu s_c E^\nu - s_c E^\mu s_b E^\nu) = 0 \tag{2.18}
\]

since \(s_a\) and \(E^\mu\) are scalars and commutate.

The Euler-Lagrange equations for \(L = L(A^\mu, \partial^\nu A^\mu)\) are

\[
\partial^\nu \left( \frac{\partial L}{\partial (\partial^\nu A_a^\mu)} \right) = \frac{\partial L}{\partial A_a^\mu} \tag{2.19}
\]
Lemma 2.3. Let
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{\alpha} F_{\mu\nu}^{\alpha} \] (2.20)
and \( A_\mu^a \) be real functions. Then
\[ \frac{\partial F_{\mu\nu}^{\alpha}}{\partial A_\mu^a} = -g f_{dac} A_\nu^c \]
\[ \frac{\partial F_{\mu\nu}^{\alpha}}{\partial A_\mu^a} = -g f_{dac} A_\nu^c g_{\mu\nu} \]
\[ \frac{\partial F_{\mu\nu}^{\alpha}}{\partial A_\mu^a} = \frac{1}{2} g f_{abc} A_\nu^c F_{\mu\nu}^{\alpha} \]
\[ \frac{\partial \mathcal{L}}{\partial A_\mu^a} = \frac{1}{2} F_{\mu\nu}^{\alpha} \]
and the Euler-Lagrange equations are
\[ \partial_{\mu} F_{\mu\nu}^{\alpha} - g f_{abc} A_\mu^b F_{\mu\nu}^{\alpha} = 0 \] (2.22)

Proof. Directly computing
\[ \frac{\partial F_{\mu\nu}^{\alpha}}{\partial A_\mu^a} = \frac{\partial}{\partial A_\mu^a} (-g f_{dbc} A_\nu^b A_\mu^c) = -g \delta_{ab} f_{dbc} A_\nu^c = -g f_{dbc} A_\nu^c \] (2.23)
\[ \frac{\partial F_{\mu\nu}^{\alpha}}{\partial A_\mu^a} = \frac{\partial}{\partial A_\mu^a} (-g f_{dbc} A_\mu^b A_\nu^c) = \frac{\partial}{\partial A_\mu^a} (-g f_{dbc} g_{\mu\nu} A_\rho^c A_\rho^\nu) \]
\[ = -g f_{dbc} g_{\nu\alpha} A_\alpha^c g_{\mu\nu} = -g f_{dbc} g_{\nu\alpha} A_\alpha^c g_{\mu\nu} \] (2.24)
\[ \frac{\partial \mathcal{L}}{\partial A_\mu^a} = -\frac{1}{4} \left( \left( \frac{\partial F_{\mu\nu}^{\alpha}}{\partial A_\mu^a} \right) F_{\mu\nu}^{\alpha} + F_{\mu\nu}^{\alpha} \left( \frac{\partial F_{\mu\nu}^{\alpha}}{\partial A_\mu^a} \right) \right) \]
\[ = \frac{1}{4} g f_{dac} \left( A_\nu^c F_{\mu\nu}^{\alpha} + F_{\mu\nu}^{\alpha} A_\nu^c g_{\mu\nu} \right) \]
\[ = \frac{1}{4} g f_{dac} \left( A_\nu^c F_{\mu\nu}^{\alpha} + A_\nu^c g_{\mu\nu} F_{\mu\nu}^{\alpha} \right) \]
\[ = \frac{1}{4} g f_{dac} \left( A_\nu^c F_{\mu\nu}^{\alpha} + A_\nu^c F_{\mu\nu}^{\alpha} \right) \]
\[ = \frac{1}{2} g f_{dac} A_\nu^c F_{\mu\nu}^{\alpha} = \frac{1}{2} g f_{abc} A_\nu^c F_{\mu\nu}^{\alpha} \] (2.25)
\[ \frac{\partial F_{\mu\nu}^{\alpha}}{\partial (\partial_{\mu} A_\nu^a)} = \frac{\partial}{\partial (\partial_{\mu} A_\nu^a)} (\partial_{\nu} A_\mu^a - \partial_{\nu} A_\mu^a) = -\delta_{ad} \]
\[ \frac{\partial F_{\mu\nu}^{\alpha}}{\partial (\partial_{\nu} A_\mu^a)} = \frac{\partial}{\partial (\partial_{\nu} A_\mu^a)} (\partial_{\mu} A_\nu^a - \partial_{\nu} A_\mu^a) \]
\[ \frac{\partial L}{\partial (\partial^\nu A_\mu^a)} = -\frac{1}{4} \left( -\delta_{ad} F_{\mu\nu}^d - g_{\mu\nu} g_{\alpha\beta} \delta_{ad} F_{\alpha\beta}^d \right) \]

Inserting (2.23) and (2.24) to the Euler-Lagrange equations (2.19) gives

\[ \partial^\nu F_{\mu\nu}^a - g f_{abc} A_{\nu}^b F_{\mu}^c = 0 \]

As \( F_{\mu\nu}^a = -F_{\nu\mu}^c \) we can also write

\[ \partial^\nu F_{\nu\mu}^c - g f_{abc} A_{\mu}^a F_{\nu}^a = 0 \]

and changing \( \nu \) and \( \mu \) yields (2.22)

\[ \partial^\mu F_{\mu\nu}^a - g f_{abc} A_{\nu}^b F_{\mu}^c = 0 \]

\[ \square \]

The Lagrangian can be expressed as

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a = -\frac{1}{2} F_{\mu\nu}^a F_{\mu\nu}^a |_{\nu > \mu} \]

\[ = -\frac{1}{2} \left( F_{01}^a F_{01}^a + F_{02}^a F_{02}^a + F_{03}^a F_{03}^a + F_{12}^a F_{12}^a + F_{13}^a F_{13}^a + F_{23}^a F_{23}^a \right) \]

(2.26)

For Minkowski’s metric

\[ \mathcal{L} = -\frac{1}{2} \left( -\left( F_{01}^a \right)^2 - \left( F_{02}^a \right)^2 - \left( F_{03}^a \right)^2 + \left( F_{12}^a \right)^2 + \left( F_{13}^a \right)^2 + \left( F_{23}^a \right)^2 \right) \]

(2.27)

since \( F_{0j}^a = -F_{aj}^a \) and \( F_{kj}^a = F_{kj}^a \) for \( 1 \leq j, k \leq 3 \).

The Hamiltonian density of a scalar field \( \varphi \) is defined as

\[ \mathcal{H} = \frac{\partial \mathcal{L}}{\partial (\partial_0 \varphi)} \left( \partial_0 \varphi \right) - \mathcal{L} = \pi \partial_0 \varphi - \mathcal{L} \]

(2.28)

where

\[ \pi = \frac{\partial \mathcal{L}}{\partial (\partial_0 \varphi)} \]

The energy of the field is a conserved property

\[ P_0 = \int d^3 x \mathcal{H} \]

(2.29)

In the case of a gauge field \( A_{\mu}^a \) we define the Hamiltonian density as

\[ \mathcal{H} = \frac{\partial \mathcal{L}}{\partial (\partial_0 A_{\mu}^a)} \left( \partial^\mu A_{\mu}^a \right) - \mathcal{L} \]

(2.30)
where summation over \( a \) and \( \mu \) is implied. For the Yang-Mills Lagrangian we have calculated in Lemma 2.2

\[
\frac{\partial L}{\partial (\partial_0 A_\mu^a)} (\partial^0 A_\mu^a) = \frac{1}{2} F_{\mu 0}^a
\]  

(2.31)

Thus

\[
\mathcal{H} = \frac{1}{2} F_{\mu 0}^a \partial^0 A_\mu^a - L
\]  

(2.32)

The energy of the field is a conserved property also in this case

\[
P^0 = \int d^3 x \mathcal{H}
\]

As Minkowski’s metric is indefinite, it is sometimes better to move to either positive or negative definite metric. A convinient choice for computations is the following metric

\[
(g_{\mu \nu})_{\mu, \nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

(2.33)

We will call it negative definite Euclidean metric, though in \( \mathbb{R}^4 \) a proper metric should be positive definite. This negative definite metric has the advantage that if we do not raise or lower the indices for the \( x_0 \) coordinate, all formulas remain valid. When we do lower \( x_0 \) indices, as in (2.27), there is a change of sign. Additionally, the \( x_0 \) coordinate must be replaced by \( ix_0 \). This creates an additional \( i \) when derivating with respect to \( x_0 \).

The problem setting of CMI uses the more modern algebraic geometric formulation where the Yang-Mills action is

\[
S = \frac{1}{4g^2} \int Tr F \wedge *F
\]

(2.34)

Actually [1] calls this action the Lagrangian but the Langrangian is the property that is integrated over the space in action. This terminology is corrected in [3]. The Yang-Mills equations (2.22) can be expressed with the Hodge star operator as

\[
0 = d_A F = d_A * F \quad F = dA + A \wedge A
\]

(2.35)

where \( d_A \) is the gauge-covariant extension of the exterior derivative. This is described in a clearer way in [2]. The gauge field \( A \) is a one-form

\[
A(x) = A_\mu^a(x) t^a dx^\mu
\]

(2.36)

with the values on the Lie algebra of a compact simple Lie group \( G \). The curvature is a two-form

\[
F = dA + A \wedge A
\]

\[
F = F_{\mu \nu}^a t^a dx^\mu \wedge dx^\nu
\]

(2.37)

Instead of the Lagrangian (2.1) we define a four-form

\[
A = Tr F \wedge *F = F_{\mu \nu}^a F_{\mu \nu}^a d^4 x
\]

(2.38)

and the action is

\[
S = \frac{1}{4g^2} \int A
\]

(2.39)
There are differences in the normalization $-\frac{1}{2}$ in (2.1) and in the placement of the coupling constant $g$ in (2.11) and (2.22). There is also a more essential difference in $\mathcal{A}$ compared to (2.1). The summation is $F^a_{\mu\nu} F^a_{\mu\nu}$ and not $F^a_{\mu\nu} F^a_{\mu\nu}$ as in (2.1). This causes a difference in (2.27) and it seems that CMI has wanted to pose the problem in Euclidean metric instead of Minkowski’s metric. This is not essential, we get the same result, apart from a multiplicative constant, for both of the metrics (2.9) and (2.33).

3. Lemmas and Theorems

Lemma 3.1. Let the gauge field satisfy $A^a_3 = 0$ for every $a$. The Euler-Lagrange equations can be expressed as $(0 \leq l, k \leq 2)$

$$\partial^3 A^a_l = F^a_{l3}$$

$$\partial^3 \partial^3 A^a_k = \partial^l F^a_{lk} - g f_{abc} A^b_l F^c_{lk}$$

(3.1)

Proof. Let $l \in \{0, 1, 2\}$. Rewriting (2.22) and inserting the gauge $A^a_3 = 0$ yields

$$\partial^3 F^a_{3\nu} + \partial^l F^a_{lk} - g f_{abc} A^b_l F^c_{lk} = 0$$

(3.4)

As $F^a_{33} = 0$ by (2.7) the case $\nu = 3$ yields

$$\partial^l F^a_{33} - g f_{abc} A^b_l F^c_{33} = 0$$

(3.5)

Inserting $A^a_3 = 0$ to (2.11) yields

$$F^a_{3l} = \partial^3 A^a_l$$

(3.6)

and inserting $\partial^3 = -\partial_3$ and $F^a_{3l} = -F^a_{l3}$ gives (3.2). Inserting (3.1) to (3.5) yields (3.3). The other values $k \in \{0, 1, 2\}$ in (3.4) give

$$-\partial^3 F^a_{3k} = \partial^l F^a_{lk} - g f_{abc} A^b_l F^c_{lk}$$

(3.7)

and inserting (3.6) yields

$$-\partial^3 \partial^3 A^a_k = \partial^l F^a_{lk} - g f_{abc} A^b_l F^c_{lk}$$

Changing $\partial_3 = -\partial^3$ gives (3.2). □

Lemma 3.2. Let the gauge field $A^a_\mu$ be of the form

$$A^a_\mu = s_a E^\mu, \quad E^3 = 0$$

Then the Euler-Lagrange equations are

$$\partial^3 A^a_l = F^a_{l3} \quad 0 \leq l, k \leq 2$$

$$\partial^3 \partial^3 A^a_k = \partial^l F^a_{lk}$$

(3.9)

Proof. Since $E^3 = 0$ Lemma 3.1 applies. By Lemma 2.2 $F^a_{\mu\nu}$ is of the form

$$F^a_{\mu\nu} = s_a G_{\mu\nu}$$

(3.10)

As in Lemma 2.2

$$f_{abc} A^b_l F^c_{\mu\nu} = \sum_{c>b} f_{abc} (s_b E^\nu s_c G_{\mu\nu} - s_c E^\mu s_b G_{\mu\nu}) = 0$$

since $s_a$ and $E^\mu$ are scalars and commute. □
Lemma 3.3. Let the gauge field $A_{a,m}^\mu$ be of the form

$$A_{a,m}^\mu = s_a E_m^\mu, \quad E_3^m = 0$$

for some finite set of indices $m \in B$ and let us assume that each $A_{a,m}^\mu$ is a gauge field such that $A_{a,m}^\mu$ and the corresponding $F_{\mu\nu}^{a,m}$ satisfy the Euler-Lagrange equations (2.22). Then

$$A_a^\mu = \sum_m A_{a,m}^\mu$$

(3.11)

defines the curvature

$$F_{\mu\nu}^a = \sum_m F_{\mu\nu}^{a,m}$$

(3.12)

such that $A_a^\mu$ and $F_{\mu\nu}^a$ satisfy the Euler-Lagrange equations (2.22).

Proof. In this case (2.22) reduces to the linear equations (3.9). Thus, the sums (3.11), (3.12) also satisfy (3.9). The equations (2.17) show that $F_{\mu\nu}^a$ is the sum (3.12).

□

Lemma 3.4. Let the gauge field $A_a^\mu$ of the form

$$A_a^\mu = s_a E^\mu, \quad E^3 = 0$$

be a complex gauge field satisfying (2.22). Let the real and imaginary parts be

$$A_{a,R}^\mu = \text{Re} A_a^\mu$$

(3.13)

and

$$A_{a,I}^\mu = \text{Im} A_a^\mu$$

and the corresponding curvatures be

$$F_{\mu\nu}^{a,R} = \text{Re} F_{\mu\nu}^a$$

(3.14)

and

$$F_{\mu\nu}^{a,I} = \text{Im} F_{\mu\nu}^a$$

are real functions satisfying (3.9).

Proof. The equations (2.22) reduce to (3.9) in this case. The equations (3.9) are linear and the coordinates $x_\mu, x^\mu$ and partial derivatives $\partial^\mu, \partial_\mu$ are all real. Thus the real and imaginary parts of $A_a^\mu$ and $F_{\mu\nu}^a$ satisfy (3.9) separately.

□

Lemma 3.5. Let $\alpha_{ij} \in \mathbb{C}$, $0 \leq i \leq 3$, $j = 1, 2, \ldots$, and $d_k \neq 0$, $0 \leq k \leq 2$, satisfy for every $j$

$$\alpha_{3j}^2 = \sum_{l=0}^2 \alpha_{lj}^2$$

(3.15)

$$\sum_{l=0}^2 d_l \alpha_{lj} = 0$$

(3.16)

The condition

$$\alpha_{3j} \alpha_{3k} = \sum_{l=0}^2 \alpha_{lj} \alpha_{lk}$$

(3.17)
for any $k,j$ with $k > j$ holds if either
\[ \sum_{l=0}^{2} d_l^2 = 0 \]  
(3.18)

or there exists a constant $c$ that for every $j$
\[ \frac{\alpha_{1j}}{\alpha_{2j}} = c \]  
(3.19)

The inverse is also true: if (3.17) holds then either (3.18) or (3.19) must hold.

Proof. If every $d_l = 0$ then (3.18) holds, thus we assume that at least one $d_l \neq 0$. By symmetry we may assume $d_0 \neq 0$. Squaring (3.17) and inserting (3.15) yields
\[ \sum_{l=0}^{2} \sum_{m=0}^{2} \alpha_{lj}^2 \alpha_{mk}^2 = \sum_{l=0}^{2} \sum_{m=0}^{2} \alpha_{lj} \alpha_{lk} \alpha_{mj} \alpha_{mk} \]

Separating $\alpha_{0j}$ terms gives
\[ \alpha_{0j}^2 (\alpha_{1k}^2 + \alpha_{2k}^2) + \alpha_{0k}^2 (\alpha_{1k}^2 + \alpha_{2k}^2) - 2(\alpha_{1j} \alpha_{1k} + \alpha_{2j} \alpha_{2k}) \alpha_{0j} \alpha_{0k} + (\alpha_{ij} \alpha_{2k} - \alpha_{1k} \alpha_{2j})^2 = 0 \]  
(3.20)

Inserting (3.16) in the form
\[ \alpha_{0m} = -\frac{d_1}{d_0} \alpha_{1m} - \frac{d_2}{d_0} \alpha_{2m} \]
for $m \in \{j,k\}$ into (3.20) gives after some calculation
\[ \left( \sum_{l=0}^{2} d_l^2 \right) (\alpha_{1j} \alpha_{2k} - \alpha_{1k} \alpha_{2j})^2 = 0 \]  
(3.21)

proving the lemma. □

Lemma 3.6. Let $d_l, \alpha_{lj} \in \mathbb{C}$, $1 \leq j \leq 3$, $0 \leq l \leq 2$, satisfy
\[ \sum_{l=0}^{2} d_l \alpha_{lj} = 0 \]
for every $j$. The vectors
\[ \rho_j = \sum_{l=0}^{2} \alpha_{lj} x_l \]  
(3.22)

are linearly dependent.

Proof. The determinant of this linear transform
\[ \begin{vmatrix} \alpha_{01} & \alpha_{11} & \alpha_{21} \\ \alpha_{02} & \alpha_{12} & \alpha_{22} \\ \alpha_{03} & \alpha_{13} & \alpha_{23} \end{vmatrix} \]
(3.23)
gives zero when the condition
\[ d_2 \alpha_{2j} = -d_0 \alpha_{0j} - d_1 \alpha_{1j} \]  
(3.24)
is inserted. □
Lemma 3.7. Let $\alpha_{ij} \in \mathbb{C}$, $0 \leq i \leq 3$, $j \geq 0$, $d_i \in \mathbb{C}$, $0 \leq i \leq 3$, satisfy
\begin{align*}
d_3 &= 0 \quad (3.25) \\
2 \sum_{l=0}^{2} d_l^2 &= 0 \quad (3.26) \\
2 \sum_{l=0}^{2} d_l \alpha_{lj} &= 0 \quad (3.27)
\end{align*}
for every $j$, and
\begin{equation}
\alpha_{3j}^2 = 2 \sum_{l=0}^{2} \alpha_{lj}^2 \quad (3.28)
\end{equation}
for every $j$. Let $h : \mathbb{C} \to \mathbb{C}$ be holomorphic in some open set $U$ and
\begin{equation}
r_j = \sum_{\mu=0}^{3} \alpha_{\mu j} x_\mu \quad (3.29)
\end{equation}
Then the gauge field
\begin{equation}
A^a_\mu = s_a d_\mu e^{-\sum_j h(r_j)} \quad , \quad d_3 = 0 \quad (3.30)
\end{equation}
defines $F^a_{\mu
u}$ which satisfies the Euler-Lagrange equations (2.22).

Proof. We have expressed $A^a_\mu$ in contravariant coordinates $x_\nu$ instead of covariant coordinates $x^\nu$ since the derivations in (3.9) are all $\frac{\partial}{\partial x_\nu}$. From (3.30) follows that $A^3 = 0$ and the gauge field is of the form
\begin{equation}
A^a_\mu = s_a E^\mu 
\end{equation}
By Lemma the Euler-Lagrange equations (2.22) reduce to (3.9). Inserting (3.30) to (2.17) yields
\begin{equation}
F^a_{\mu
u} = s_a e^{-\sum_j h(r_j)} \sum_j (d_\nu \alpha_{\mu j} - d_\mu \alpha_{\nu j}) h'(r_j) \quad (3.31)
\end{equation}
Then
\begin{equation}
\partial^l F^a_{lk} = s_a e^{-\sum_j h(r_j)} d_k \left( \sum_j \left( \sum_{l=0}^{2} \alpha_{lj}^2 \right) h''(r_j) + \sum_{l=0}^{2} \left( \sum_j \alpha_{lj} h'(r_j) \right) \right) + s_a e^{-\sum_j h(r_j)} \sum_j \alpha_{kj} \sum_{l=0}^{2} d_l \alpha_{lj} h''(r_j) + s_a e^{-\sum_j h(r_j)} \sum_j \alpha_{kj} h'(r_j) \sum_{m} h'(r_m) \sum_{l=0}^{2} d_l \alpha_{lm} 
\end{equation}
Simplifying the expression by (3.27) and (3.28)
\begin{equation}
\partial^l F^a_{lk} = s_a e^{-\sum_j h(r_j)} d_k \left( \sum_j \alpha_{3j}^2 h''(r_j) + \sum_{l=0}^{2} \left( \sum_j \alpha_{lj} h'(r_j) \right) \right) 
\end{equation}
Using Lemma 3.5 we can express
\[
\sum_{l=0}^{2} \left( \sum_{j} \alpha_{lj} h'(r_j) \right)^2 = \sum_{l=0}^{2} \sum_{j} \alpha_{lj}^2 (h'(r_j))^2 + 2 \sum_{l=0}^{2} \sum_{j} \sum_{k>j} \alpha_{lj} \alpha_{lk} h'(r_j) h'(r_k)
\]
Thus
\[
\partial^l F_{lk}^a = s_a e \sum_j h'(r_j) d_k \left( \sum_j \alpha_{lj}^2 h''(r_j) + \left( \sum_j \alpha_{lj} h'(r_j) \right)^2 \right)
\]
The first condition in (3.9) is obvious from (2.11) since \( A_3^a = 0 \), and the last condition in (3.9) holds since by (3.27)
\[
\partial^3 \partial^3 A_i^a = \partial^3 \partial^3 A_i^a
\]
Let us select three linearly independent vectors \( r_j = \sum_{\mu=0}^{3} \alpha_{\mu j} x_\mu \) and set the numbers \( d_\mu \) as
\[
d_0 = \sqrt{2}(1 - i) \quad d_1 = d_2 = 1 + i \quad d_3 = 0 \quad (3.32)
\]
\[
r_1 = x_1 - x_2 + \sqrt{2} x_3 \\
r_2 = x_1 - x_2 - \sqrt{2} x_3 \\
r_3 = i \frac{1}{\sqrt{2}} x_0 - x_1 + \frac{1}{\sqrt{2}} x_3 \quad (3.33)
\]
Then
\[
x_1 = \frac{1}{4} r_1 - \frac{1}{4} r_2 - r_3 + i \frac{1}{2} x_0 \\
x_2 = -\frac{1}{4} r_1 - \frac{3}{4} r_2 - r_3 + i \frac{1}{2} x_0 \quad (3.34)
\]
\[
x_3 = \frac{1}{2 \sqrt{2}} r_1 - \frac{1}{2 \sqrt{2}} r_2
\]
These numbers fill the conditions (3.25)-(3.28). We cannot get more than three linearly independent vectors. From 3.6 it follows that there are only two linearly independent linear combinations of \( \{ x_0, x_1, x_2 \} \), and the third vector is obtained from the gauged coordinate \( x_3 \): the condition (3.28) allows two values for \( \alpha_{3j} \). Let
us express \( r_j \) and \( h(r_j) \) as sums of real and imaginary parts.

\[
\begin{align*}
    r_j &= \rho_j + i\sigma_j, \quad \rho_j, \sigma_j \in \mathbb{R} \\
    \rho_1 &= x_1 - x_2 + \sqrt{2} x_3 \\
    \rho_2 &= x_1 - x_2 - \sqrt{2} x_3 \\
    \rho_3 &= -x_1 + \frac{1}{\sqrt{2}} x_3 \\
    \sigma_1 &= \sigma_2 = 0 \\
    \sigma_3 &= \frac{1}{\sqrt{2}} x_0 \\
    h(r_j) &= u(\rho_j, \sigma_j) + iv(\rho_j, \sigma_j)
\end{align*}
\]

(3.35)

Thus, if \( x_0 = 0 \) then \( \sigma_j = 0 \) for every \( j \). As \( h \) is holomorphic, \( u \) and \( v \) are harmonic functions on \( \mathbb{R}^2 \). Thus, \( u \) and \( v \) cannot be bounded on the whole \( \mathbb{R}^2 \), but they can be bounded on a strip \( |x_0| \leq M \) for a finite \( M \). Assuming that \( h(r_j) \) goes sufficiently fast to zero if \( |\rho_j| \) grows, then for any fixed value of \( x_0 \) the integral of the Euclidean norm of the gauge potential (3.30) over the space coordinates \( x_1, x_2, x_3 \) is finite. Also the path integral from finite time \( t' \) to another finite time \( t'' \) is finite. We have much freedom in selecting \( u(\rho, 0) \). We can choose a real analytic function \( f : \mathbb{R} \to \mathbb{R} \) that vanishes when \( |\rho| \) grows, set \( u(\rho, 0) = f(\rho) \) and extend \( u \) to a holomorphic function \( h \). We should expect the solution to behave in the way (3.35) describes. It is a localized gauge field, gauge boson, which moves in the \( x_1, x_2 \) direction with the speed of light as a function of \( x_0 \). We select a concrete case that gives easy calculations. Let

\[
f(\rho_j) = -\beta^2 \rho_j^2
\]

(3.36)

and extend it to

\[
h(r_j) = -\beta^2 r_j^2
\]

(3.37)

The real and imaginary parts of \( d_\mu = c_\mu + ie_\mu \) are

\[
\begin{align*}
    c_0 &= \sqrt{2} \\
    c_1 &= c_2 = 1 \\
    c_3 &= 0 \\
    e_0 &= -\sqrt{2} \\
    e_1 &= e_2 = 1 \\
    e_3 &= 0
\end{align*}
\]

(3.38)

We evaluate the gauge potential at \( x_0 = 0 \) and take the real part.

**Lemma 3.8.** Let the gauge field be

\[
A^a_\mu = s_a d_\mu e^{-\beta^2 \sum_{j=1}^3 \rho_j^2}
\]

(3.39)

where \( r_j \) and \( d_\mu \) are as in (3.32)-(3.33) and \( \beta, s_a \in \mathbb{R} \). Then

\[
A^a_\mu R(0, x_1, x_2, x_3) = Re A^a_\mu(0, x_1, x_2, x_3) = s_a c_\mu e^{-\beta^2 \sum_{j=1}^3 \rho_j^2}
\]

(3.40)

\[
F^{a,R}_{\mu\nu}(0, x_1, x_2, x_3) = Re F^{a}_{\mu\nu}(0, x_1, x_2, x_3)
\]

\[
= -s_a 2\beta^2 e^{-\beta^2 \sum_{j=1}^3 \rho_j^2} \sum_j \Re(d_\nu \alpha_{\mu j} - d_\mu \alpha_{\nu j}) \rho_j
\]

(3.41)

**Proof.** Inserting \( x_0 = 0 \) to (3.39) and (3.31) yields the result. \( \square \)
We need the Gaussian integrals
\[
\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2} = \sqrt{2\pi a}^{-\frac{1}{2}}
\]
\[
\int_{-\infty}^{\infty} xe^{-\frac{1}{2}ax^2} = 0
\]
\[
\int_{-\infty}^{\infty} x^2e^{-\frac{1}{2}ax^2} = \sqrt{2\pi a}^{-\frac{3}{2}}
\]
(3.42)

**Lemma 3.9.** Let the gauge field satisfy
\[
A_{\mu}^a(R)(0, x_1, x_2, x_3) = sa_x e^{-\beta^2 \sum_{j=1}^{3} \rho_j^2}
\]
where \(\rho_j\) and \(c_\mu\) are as in (3.35),(3.38) and \(\beta, s_a \in \mathbb{R}\). Then
\[
\int d^3x(A_k^a(0, x_1, x_2, x_3))^2 = s^2 a^2 c_k^2 \left(\frac{\pi}{2}\right)^\frac{3}{4} \frac{1}{\beta^3}
\]
(3.44)

**Proof.** We change the variables to \(y_1, y_2, y_3\)
\[
y_1 = \sqrt{3}x_1 - \frac{2}{\sqrt{3}}x_2 - \frac{1}{\sqrt{6}}x_3
\]
\[
y_2 = \sqrt{\frac{2}{3}}x_2 - \frac{1}{\sqrt{3}}x_3
\]
\[
y_3 = 2x_3
\]
(3.45)

Then
\[
\sum_{j=1}^{3} \rho_j^2 = y_1^2 + y_2^2 + y_3^2
\]
As \(y_2\) and \(y_3\) are not functions of \(x_1\) we can change the order of integration
\[
\int d^3xe^{-\beta^2 \sum_{j=1}^{3} \rho_j^2} = \int d^3xe^{-\beta^2 \sum_{j=1}^{3} y_j^2}
\]
\[
= \int d^2xe^{-\beta^2(y_3^2+y_3^2)} \int dx_1 e^{-\frac{1}{2}(\sqrt{3}\beta)^2 y_1^2}
\]
\[
= \int d^2xe^{-\beta^2(y_3^2+y_3^2)} \frac{1}{\sqrt{3}} \int dy_1 e^{-\frac{1}{2}(\sqrt{3}\beta)^2 y_1^2}
\]
\[
= \int d^2xe^{-\beta^2(y_3^2+y_3^2)} \frac{1}{\sqrt{3}} \sqrt{2\pi(\sqrt{3}\beta)^{-1}}
\]
As \(y_3\) is not a function of \(x_2\) we can change the order of integration
\[
= \frac{1}{\sqrt{3}} \sqrt{2\pi(\sqrt{2}\beta)^{-1}} \int dx_3 e^{-\beta^2 y_3^2} \int dx_2 e^{-\frac{1}{2}(\sqrt{3}\beta)^2 y_2^2}
\]
\[
= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \sqrt{2\pi(\sqrt{2}\beta)^{-1}} \int dx_3 e^{-\beta^2 y_3^2} \int dy_2 e^{-\frac{1}{2}(\sqrt{3}\beta)^2 y_2^2}
\]
\[
= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \frac{1}{2}(2\pi)(\sqrt{2}\beta)^{-2} \int dx_3 e^{-\beta^2 y_3^2}
\]
\[
= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \frac{1}{2}(2\pi)(\sqrt{2}\beta)^{-2} \int dy_3 e^{-\beta^2 y_3^2}
\]
\[
\begin{align*}
&= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2} \frac{1}{2} (2\pi)^{\frac{3}{2}} (\sqrt{2} \beta)^{-3}} \\
\int d^3 x e^{-2\beta^2 \sum \rho_j^2} &= \left(\frac{\pi}{2}\right)^{\frac{3}{2}} \frac{1}{\beta^3} \\
\end{align*}
\]

Thus the integrand vanishes.

**Lemma 3.10.** Let the gauge field satisfy

\[
A_{\mu R}^a(0, x_1, x_2, x_3) = s_a c_\mu e^{-\beta^2 \sum \rho_j^2} 
\]

where \(\rho_j\) and \(c_\mu\) are as in (3.35), (3.38) and \(\beta, s_a \in \mathbb{R}\), and

\[
L_R = -\frac{1}{4} F^{\mu\nu}_{a R} F_{a R}^{\mu \nu}
\]

Then in Minkowski’s metric (2.9) at \(x_0 = 0\)

\[
L_R = 0 
\]

while in the negative definite metric (2.33) at \(x_0 = 0\)

\[
L_R = -\frac{1}{2} \sum a (2\beta^2 s_a)^2 e^{-2\beta^2 \sum \rho_j^2} (4\rho_1^2 + 4\rho_2^2 + \rho_3^2 - 4\rho_1 \rho_3) 
\]

**Proof.** In Minkowski’s metric \(L\) is given by (2.27). Thus

\[
L_R = -\frac{1}{2} \left( -F_{a R}^{\mu R} F_{a R}^{\mu \nu} - F_{a R}^{\nu} F_{a R}^{\mu} - (F_{a R}^{\mu R})^2 - (F_{a R}^{\nu R})^2 - (F_{a R}^{\mu})^2 - (F_{a R}^{\nu})^2 \right) 
\]

From (3.41) we see that

\[
F_{a R}^{\mu R}(0, x_1, x_2, x_3) = -s_a 2\beta^2 e^{-\beta^2 \sum \rho_j^2} \sum_j \text{Re}(d_\alpha \alpha_{\mu j} - d_\mu \alpha_{\alpha j}) \rho_j 
\]

The parameters selected in (3.32)-(3.33) are

\[
\begin{align*}
\alpha_{01} &= 0 & \alpha_{02} &= 0 & \alpha_{03} &= i \frac{1}{\sqrt{2}} \\
\alpha_{11} &= 1 & \alpha_{12} &= 1 & \alpha_{13} &= -1 \\
\alpha_{21} &= -1 & \alpha_{22} &= -1 & \alpha_{23} &= 0 \\
\alpha_{31} &= \sqrt{2} & \alpha_{32} &= -\sqrt{2} & \alpha_{33} &= \frac{1}{\sqrt{2}} \\
c_0 &= \sqrt{2} & c_1 &= c_2 &= 1 & c_3 &= 0 \\
e_0 &= -\sqrt{2} & e_1 &= e_2 &= 1 & e_3 &= 0 
\end{align*}
\]

Let us compute the needed components

\[
\begin{align*}
\sum_{j=1}^3 \alpha_{0j} \rho_j &= i \frac{1}{\sqrt{2}} \rho_3 \\
\sum_{j=1}^3 \alpha_{1j} \rho_j &= \rho_1 + \rho_2 - \rho_3 \\
\sum_{j=1}^3 \alpha_{2j} \rho_j &= -\rho_1 - \rho_2 \\
\sum_{j=1}^3 \alpha_{3j} \rho_j &= \sqrt{2} \rho_1 - \sqrt{2} \rho_2 + \frac{1}{\sqrt{2}} \rho_3 
\end{align*}
\]
\[
\sum_{j=1}^{3} \text{Re}(d_1 \alpha_{0j} - d_0 \alpha_{1j}) \rho_j = -\frac{1}{\sqrt{2}} \rho_3 - c_0 \sum_{j=1}^{3} \alpha_{1j} \rho_j \\
= -\frac{1}{\sqrt{2}} \rho_3 - \sqrt{2}(\rho_1 + \rho_2 - \rho_3) \quad \text{(3.51)}
\]

\[
\sum_{j=1}^{3} \text{Re}(d_2 \alpha_{0j} - d_0 \alpha_{2j}) \rho_j = -\frac{1}{\sqrt{2}} \rho_3 - c_0 \sum_{j=1}^{3} \alpha_{2j} \rho_j \\
= -\frac{1}{\sqrt{2}} \rho_3 - \sqrt{2}(-\rho_1 - \rho_2) \quad \text{(3.52)}
\]

\[
\sum_{j=1}^{3} \text{Re}(d_3 \alpha_{0j} - d_0 \alpha_{3j}) \rho_j = -c_0 \sum_{j=1}^{3} \alpha_{3j} \rho_j \\
= -2\rho_1 + 2\rho_2 - \rho_3 \quad \text{(3.53)}
\]

\[
\sum_{j=1}^{3} \text{Re}(d_2 \alpha_{1j} - d_1 \alpha_{2j}) \rho_j = c_2 \sum_{j=1}^{3} \alpha_{1j} \rho_j - c_1 \sum_{j=1}^{3} \alpha_{2j} \rho_j \\
= 2\rho_1 + 2\rho_2 - \rho_3
\]

\[
\sum_{j=1}^{3} \text{Re}(d_3 \alpha_{1j} - d_1 \alpha_{3j}) \rho_j = c_3 \sum_{j=1}^{3} \alpha_{1j} \rho_j - c_1 \sum_{j=1}^{3} \alpha_{3j} \rho_j \\
= -\sqrt{2}\rho_1 + \sqrt{2}\rho_2 - \frac{1}{\sqrt{2}} \rho_3
\]

\[
\sum_{j=1}^{3} \text{Re}(d_3 \alpha_{2j} - d_2 \alpha_{3j}) \rho_j = c_3 \sum_{j=1}^{3} \alpha_{2j} \rho_j - c_2 \sum_{j=1}^{3} \alpha_{3j} \rho_j \\
= -\sqrt{2}\rho_1 + \sqrt{2}\rho_2 - \frac{1}{\sqrt{2}} \rho_3
\]

The sum of the squares with the signs as in (3.49) is
\[
-\left(-\frac{1}{\sqrt{2}} \rho_3 - \sqrt{2}(\rho_1 + \rho_2 - \rho_3)\right)^2 - \left(-\frac{1}{\sqrt{2}} \rho_3 - \sqrt{2}(-\rho_1 - \rho_2)\right)^2 \\
-\left(-2\rho_1 + 2\rho_2 - \rho_3\right)^2 + (2\rho_1 + 2\rho_2 - \rho_3)^2 \\
+(-\sqrt{2}\rho_1 + \sqrt{2}\rho_2 - \frac{1}{\sqrt{2}} \rho_3)^2 + (-\sqrt{2}\rho_1 + \sqrt{2}\rho_2 - \frac{1}{\sqrt{2}} \rho_3)^2
= 0
\]

Inserting the sum to (3.49) and calculating (3.48) yields
\[
\mathcal{L}_R = \frac{1}{2} \sum_a (2\beta^2 s_a)^2 e^{-2\beta^2 \sum_i \rho_i^2} = 0
\]

In the negative definite metric (2.33) holds \(g_{\mu\nu} = -1\) for all \(\mu\), so
\[
\mathcal{L}_R = -\frac{1}{2} \left((F_{01}^a)^2 + (F_{02}^a)^2 + (F_{03}^a)^2 + (F_{12}^a)^2 + (F_{13}^a)^2 + (F_{23}^a)^2\right) \quad \text{(3.54)}
\]
Then the sum of the terms is

\[ L_R = -\frac{1}{2} \sum_a (2 \beta^2 s_a)^2 e^{-2 \beta^2 \sum_j \rho_j^2} (4 \rho_1^2 + 4 \rho_2^2 + \rho_3^2 - 4 \rho_2 \rho_3) \]

\[ \square \]

**Lemma 3.11.** Let the gauge field satisfy

\[ A_{\mu}^{a,R}(0, x_1, x_2, x_3) = s_a c_\mu e^{-\beta^2 \sum_i j \rho_i^2} \quad (3.55) \]

where \( \rho_j \) and \( c_\mu \) are as in (3.35), (3.38) and \( \beta, s_a \in \mathbb{R} \). Then

\[ \int d^3 L_R = -\frac{1}{\beta} \frac{\pi^{\frac{3}{2}}}{16} \sum_a s_a^2 B \quad (3.56) \]

where in Minkowski’s metric at \( x_0 = 0 \)

\[ B = 0 \]

In the negative definite metric of (2.33)

\[ B = \frac{13}{3} + \frac{2}{3} + 4 \]

**Proof.** From (3.35) and (3.45) follows that

\[ \rho_1 = \frac{1}{\sqrt{3}} y_1 - \frac{1}{\sqrt{6}} y_2 + \frac{1}{\sqrt{2}} y_3 \]

\[ \rho_2 = \frac{1}{\sqrt{3}} y_1 - \frac{1}{\sqrt{6}} y_2 - \frac{1}{\sqrt{2}} y_3 \]

\[ \rho_3 = -\frac{1}{\sqrt{3}} y_1 - \frac{\sqrt{2}}{3} y_2 \]

For Minkowski’s metric

\[ P(\rho) = 0 = B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2 + B_4 y_1 y_2 + B_5 y_1 y_3 + B_6 y_2 y_3 \]

where \( B_k = 0 \) for all \( k \). For the metric in (2.33)

\[ P(\rho) = 4 \rho_1^2 + 4 \rho_2^2 + \rho_3^2 - 4 \rho_2 \rho_3 \]

\[ = B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2 + B_4 y_1 y_2 + B_5 y_1 y_3 + B_6 y_2 y_3 \]

where

\[ B_1 = \frac{13}{3} \quad B_2 = \frac{2}{3} \quad B_3 = 4 \]

\[ B_4 = -\frac{4}{3} \sqrt{2} \quad B_5 = -\frac{4}{\sqrt{6}} \quad B_6 = -\frac{4}{\sqrt{3}} \]

We do the integration with generic parameters \( B_j \). Then

\[ \int d^3 x e^{-\frac{1}{2}(2\beta)^2 (\rho_1^2 + \rho_2^2 + \rho_3^2)} P(\rho) \]

\[ = \int d^3 x e^{-\frac{1}{2}(2\beta)^2 (y_1^2 + y_2^2 + y_3^2)} (B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2 + B_4 y_1 y_2 + B_5 y_1 y_3 + B_6 y_2 y_3) \]
As $y_2$ and $y_3$ are not functions of $x_1$ we can change the order of integration and change the integration parameter $x_1$ to $y_1$.

\[
= \int d^2x e^{-\frac{1}{4}(2\beta)^2(y_2^2+y_3^2)} \left( B_1 \int dy_1 e^{-\frac{1}{2}(2\beta)^2 y_1^2} 
+ (B_2y_2^2 + B_3y_3^2 + B_6y_2y_3) \int dy_1 e^{-\frac{1}{2}(2\beta)^2 y_1^2} 
+ (B_4y_2 + B_5y_3) \int dy_1 e^{-\frac{1}{2}(2\beta)^2 y_1^2} \right)
\]

\[
= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \sqrt{2\pi} \int dx_3 e^{-\frac{1}{4}(2\beta)^2 y_2^2} \left( B_1 \frac{1}{(2\beta)^3} \int dy_2 e^{-\frac{1}{4}(2\beta)^2 y_2^2} 
+ B_2 \frac{1}{2\beta} \int dy_2 y_2 e^{-\frac{1}{4}(2\beta)^2 y_2^2} 
+ B_3y_2^2 \frac{1}{2\beta} \int dy_2 e^{-\frac{1}{4}(2\beta)^2 y_2^2} 
+ B_6y_2y_3 \frac{1}{2\beta} \int dy_2 y_2 e^{-\frac{1}{4}(2\beta)^2 y_2^2} \right)
\]

As $y_3$ is not a function of $x_2$ we can change the order of integration and change the integration parameter $x_2$ to $y_2$.

\[
= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \sqrt{2\pi} \int dx_3 e^{-\frac{1}{4}(2\beta)^2 y_2^2} \left( B_1 \frac{1}{(2\beta)^3} \sqrt{2\pi} \frac{1}{2\beta} 
+ B_2 \frac{1}{2\beta} \sqrt{2\pi} \frac{1}{(2\beta)^3} + B_3y_2^2 \frac{1}{2\beta} \sqrt{2\pi} \frac{1}{2\beta} \right)
\]

\[
= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \sqrt{2\pi} \int dy_3 e^{-\frac{1}{4}(2\beta)^2 y_3^2} \left( B_1 \frac{1}{(2\beta)^4} 
+ B_2 \frac{1}{(2\beta)^4} + B_3y_3^2 \frac{1}{(2\beta)^3} \right)
\]

\[
= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \sqrt{2\pi} \frac{3}{2} \frac{1}{2\beta} \left( B_1 + B_2 + B_3 \right) \frac{1}{(2\beta)^3}
\]
Lemma 3.12. Let the gauge field satisfy
\[ A^a_{\mu}(0, x_1, x_2, x_3) = s_a c_\mu e^{-\beta^2 \sum_{j=1}^{3} \rho_j^2} \]  
(3.57)
where \( \rho_j \) and \( c_\mu \) are as in (3.35), (3.38) and \( \beta, s_a \in \mathbb{R} \). Then at \( x_0 = 0 \)
\[ \int d^3 \mathcal{H}_R = \frac{1}{\beta} \frac{\pi^2}{16} \sum_a s_a^2 B \]  
(3.58)
where in Minkowski’s metric
\( B = 0 \)
and in the metric (2.33) we get
\( B = \frac{13}{3} + \frac{2}{3} + 4 \)

Proof. From (2.32) for the real gauge field \( A^a_{\mu} \)
\[ \mathcal{H}_R = \frac{1}{2} F^{a,R}_{\mu \nu} \partial^0 A^a_{\mu,R} - \mathcal{L}_R \]
From Lemma 3.8
\[ A^a_{\mu} = s_a c_\mu e^{-\beta^2 \sum_{j=1}^{3} \rho_j^2} \cos \left( \frac{\beta^2}{\sqrt{2}} x_0 \right) + s_a e_\mu e^{-\beta^2 \sum_{j=1}^{3} \rho_j^2} \sin \left( \frac{\beta^2}{\sqrt{2}} x_0 \right) \]
(3.59)
Thus
\[ \partial^0 A^a_{\mu} = s_a c_\mu e^{-\beta^2 \sum_{j=1}^{3} \rho_j^2} \frac{\beta^2}{\sqrt{2}} \left( -c_\mu \sin \left( \frac{\beta^2}{\sqrt{2}} x_0 \right) + e_\mu \cos \left( \frac{\beta^2}{\sqrt{2}} x_0 \right) \right) \]
as \( \partial^0 \sum_j \rho_j^2 = 0 \). At \( x_0 = 0 \)
\[ \partial^0 A^a_{\mu} = s_a c_\mu e^{-\beta^2 \sum_{j=1}^{3} \rho_j^2} \]
Thus
\[ \partial^0 A^a_{\mu}(0, x_1, x_2, x_3) = -s_a \beta^2 e^{-\beta^2 \sum_{j=1}^{3} \rho_j^2} \]
\[ \partial^0 A^a_{1}(0, x_1, x_2, x_3) = s_a \frac{\beta^2}{\sqrt{2}} e^{-\beta^2 \sum_{j=1}^{3} \rho_j^2} \]
\[ \partial^0 A^a_{2}(0, x_1, x_2, x_3) = s_a \frac{\beta^2}{\sqrt{2}} e^{-\beta^2 \sum_{j=1}^{3} \rho_j^2} \]
\[ \partial^0 A^a_{3}(0, x_1, x_2, x_3) = 0 \]  
(3.60)
From (3.50)-(3.53)
\[ \frac{1}{2} F^{a,R}_{00}(0, x_1, x_2, x_3) = 0 \]
\[ \frac{1}{2} F^{a,R}_{10}(0, x_1, x_2, x_3) = -\frac{1}{2} s_a 2 \beta^2 e^{-\beta^2 \sum_j \rho_j^2} \left( \frac{1}{\sqrt{2}} \rho_3 + \sqrt{2} (\rho_1 + \rho_2 - \rho_3) \right) \]
\[ \frac{1}{2} F^{a,R}_{20}(0, x_1, x_2, x_3) = -\frac{1}{2} s_a 2 \beta^2 e^{-\beta^2 \sum_j \rho_j^2} \left( \frac{1}{\sqrt{2}} \rho_3 - \sqrt{2} (\rho_1 + \rho_2) \right) \]
\[ \frac{1}{2} F^{a,R}_{30}(0, x_1, x_2, x_3) = -\frac{1}{2} s_a 2 \beta^2 e^{-\beta^2 \sum_j \rho_j^2} (2 \rho_1 - 2 \rho_2 - \rho_3) \]
Since

\[ A_{\mu,R}^\nu = g^{\mu\nu} A_\nu^R \]

and in the metric (2.9) \( A_0^0 = A^0_0, \ A_0^a = -A^a_0, \ j > 0, \)

\[
\begin{align*}
\partial^0 A_{\alpha,R}^0(0, x_1, x_2, x_3) &= \partial^0 A_{0,R}^0(0, x_1, x_2, x_3) = -s_a \beta^2 e^{-\beta^2 \sum_{j=1}^{3} \rho_j^2} \\
\partial^0 A_{\alpha,R}^1(0, x_1, x_2, x_3) &= \partial^0 A_{1,R}^1(0, x_1, x_2, x_3) = -s_a \frac{\beta^2}{\sqrt{2}} e^{-\beta^2 \sum_{j=1}^{3} \rho_j^2} \\
\partial^0 A_{\alpha,R}^2(0, x_1, x_2, x_3) &= -\partial^0 A_{2,R}^2(0, x_1, x_2, x_3) = -s_a \frac{\beta^2}{\sqrt{2}} e^{-\beta^2 \sum_{j=1}^{3} \rho_j^2} \\
\partial^0 A_{\alpha,R}^3(0, x_1, x_2, x_3) &= 0
\end{align*}
\]

Thus

\[
\begin{align*}
\frac{1}{2} F^{a,R}_{\mu\nu} \partial^0 A_{\mu,R}^a &= \sum_a \sum_{\mu=0}^3 \frac{3}{2} F^{a,R}_{\mu\nu} \partial^0 A_{\mu,R}^a = \sum_a \left( \frac{1}{2} F^{a,R}_{0\nu} \partial^0 A_{R,R}^1 + \frac{1}{2} F^{a,R}_{R0} \partial^0 A_{R,R}^2 \right) \\
&= \beta^4 \left( \sum_a s_a^2 \right) e^{-2\beta^2 \sum_{j=1}^{3} \rho_j^2} \left( \frac{1}{\sqrt{2}} \rho_3 + \sqrt{2}(\rho_1 + \rho_2 - \rho_3) \right) \\
&\quad + \beta^4 \left( \sum_a s_a^2 \right) e^{-2\beta^2 \sum_{j=1}^{3} \rho_j^2} \left( \frac{1}{\sqrt{2}} \rho_3 - \sqrt{2}(\rho_1 + \rho_2) \right) \\
&= \beta^4 \left( \sum_a s_a^2 \right) e^{-2\beta^2 \sum_{j=1}^{3} \rho_j^2} \left( 2 \frac{1}{\sqrt{2}} - 1 \right) \rho_3
\end{align*}
\]

Inserting \( y_1, y_2, y_3 \) from (3.45) allows us to perform the integration

\[
\int d^3 x \frac{1}{2} F^{a,R}_{\mu\nu} \partial^0 A_{\mu,R}^a = \int d^3 x \beta^4 \left( \sum_a s_a^2 \right) e^{-2\beta^2 \sum_{j=1}^{3} \rho_j^2} \left( 2 \frac{1}{\sqrt{2}} - 1 \right) y_3
\]

\[
= \beta^4 \left( \sum_a s_a^2 \right) \left( 2 \frac{1}{\sqrt{2}} - 1 \right) \frac{1}{2} \int d^2 x y_3 e^{-2\beta^2 (y_2^2 + y_3^2)} \int dx_1 e^{-2\beta^2 y_1^2}
\]

\[
= \frac{1}{\sqrt{3}} \beta^4 \left( \sum_a s_a^2 \right) \left( 2 \frac{1}{\sqrt{2}} - 1 \right) \frac{1}{2} \int d^2 x y_3 e^{-2\beta^2 (y_2^2 + y_3^2)} \int dx_1 e^{-2\beta^2 y_1^2}
\]

\[
= \frac{\sqrt{2\pi}}{2\beta} \frac{1}{\sqrt{3}} \beta^4 \left( \sum_a s_a^2 \right) \left( 2 \frac{1}{\sqrt{2}} - 1 \right) \frac{1}{2} \int d^2 x y_3 e^{-2\beta^2 (y_2^2 + y_3^2)} \int dx_1 e^{-2\beta^2 y_1^2}
\]

\[
= \sqrt{\frac{2\pi}{3}} \frac{1}{2\beta} \beta^4 \left( \sum_a s_a^2 \right) \left( 2 \frac{1}{\sqrt{2}} - 1 \right) \frac{1}{2} \int d^2 y_3 e^{-2\beta^2 y_3^2} \int dx_2 e^{-2\beta^2 y_2^2}
\]

\[
= \sqrt{\frac{3}{2}} \frac{1}{2\beta} \beta^4 \left( \sum_a s_a^2 \right) \left( 2 \frac{1}{\sqrt{2}} - 1 \right) \frac{1}{2} \int d^2 y_2 e^{-2\beta^2 y_3^2} \int dx_3 e^{-2\beta^2 y_2^2}
\]

\[
= \sqrt{\frac{3}{2}} \frac{1}{2\beta} \beta^4 \left( \sum_a s_a^2 \right) \left( 2 \frac{1}{\sqrt{2}} - 1 \right) \frac{1}{2} \int d^2 y_2 e^{-2\beta^2 y_2^2} \int dx_3 e^{-2\beta^2 y_2^2}
\]
\[
\sqrt{3} \left( \frac{\sqrt{2\pi}}{2\beta} \right)^2 \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2}} \left( \sum_a s_a^2 \right) \left( 2 \cdot \frac{1}{\sqrt{2}} - 1 \right) \frac{1}{2} \int d\eta_3 y_3 e^{-2\beta^2 y_3^2} = 0
\]

Thus

\[
\int d^3 H_\mathcal{R} = - \int d^3 \mathcal{L}_\mathcal{R}
\]

and (3.58) follows from Lemma 3.11. If the metric is as in (2.33) then \( A^a_0 = -A^a_0 \) but this term disappears and the integral in (3.61) still yields zero. If in addition to changing the metric there has been a replacement \( \varepsilon_0 \rightarrow i\varepsilon_0 \) as is often done in order to move from Minkowski’s metric to Euclidean metric, derivation with respect to \( \varepsilon_0 \) gives an additional \( i \). This changes the coefficients \( c_j \) to \( e^{i c_j} \) in some places but (3.61) still holds because the integral disappears because it is of first order in \( \rho_j \), and that is also true for the metric (2.33) and a change \( \varepsilon_0 \rightarrow i\varepsilon_0 \). Thus, for the metric (2.33) we get another parameters than for Minkowski’s metric but the form is the same.

\[\square\]

**Theorem 3.13.** Let the gauge field be

\[
A^a_\mu = s_a d_\mu e^{-\beta^2 \sum_j r_j^2}
\]

where \( r_j \) and \( d_\mu \) are as in (3.33),(3.32) and \( \beta, s_a \in \mathbb{R} \). The real part is

\[
A^a_\mu R(0, x_1, x_2, x_3) = s_a c_\mu e^{-\beta^2 \sum_j r_j^2}
\]

where \( c_\mu \) and \( c_\mu \) are as in (3.35),(3.38). The following statements hold

\[
\int d^3 x A^a_k(0, x_1, x_2, x_3))^2 = s_a^2 c_k^2 \frac{\pi}{2} \frac{1}{\beta^2}
\]

\[
\int d^3 x A^a_\mu A^a_{\mu} = \sum_a s_a^2 c_k^2 \frac{\pi}{2} \frac{1}{\beta^2}
\]

where \( A^a_\mu \) denotes the complex conjugate of \( A^a_\mu \). The real part of the gauge field \( A^a_\mu R \) and the real part of the curvature \( F^a_{\mu \nu} R \) satisfy the Lagrange-Euler equations

\[
\mathcal{L}_\mathcal{R} = -\frac{1}{4} F^a_{\mu \nu} R F^a_{\mu \nu}
\]

and the energy is

\[
P^{0, R} = \int d^3 x H_\mathcal{R} = \frac{1}{\beta} \frac{\pi^2}{16} \sum_a s_a^2 B
\]

where in Minkowski’s metric

\[
B = 0
\]

and in the metric (2.33)

\[
B = \frac{13}{3} + \frac{2}{3} + 4
\]
Proof. From Lemma 3.8 follows that (3.63) is the real part of (3.62). The claim (3.64) is shown in Lemma 3.9. The imaginary part in $A_\mu^a$ is a phase $e^{-i\sqrt{2}x_0}$ which cancels in $A_\mu^a \ast A_{\mu u}^a$, thus (3.65) holds. The real part of the gauge field and the curvature satisfy the Euler-Lagrange equations by Lemma 3.4, thus (3.66) holds. In Lemma 12 we showed that at $x_0 = 0$ equation (3.67) holds. As $P^0$ is a conserved property, see (2.29), (3.67) holds for all values of $x_0$. □

**Theorem 3.14.** Let $A = (A_{mu})_\mu$, $A_\mu = A_\mu^a t_a$ be a complex gauge field defined by

$$A_\mu^a = s_a d_\mu e^{-\beta^2 \sum_{j=1}^{3} r_j^2}$$  

(3.68)

The numbers $r_j$ and $d_\mu$ are as in (3.33),(3.32) and $\beta$, $s_a \in \mathbb{R}$. The norm is

$$||A|| = \int d^3 x A_\mu^a \ast A_\mu^a = \sum_a s_a^2 \sqrt{2\pi} \frac{1}{\beta^3}$$  

(3.69)

where $A^*$ denotes the complex conjugate of $A$. The gauge field and the corresponding curvature satisfy Euler-Lagrange equations for

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a$$  

(3.70)

In both metrics (2.9) and (2.33)

$$E_\beta = \frac{P^0}{||A||} = \beta^2 C$$  

(3.71)

where

$$P^0 = \int d^3 x \mathcal{H} = \sum_a s_a^2 \sqrt{2\pi} \frac{1}{\beta}$$  

(3.72)

and $C$ is a nonnegative constant.

Proof. In the case of a complex field, the Lagrangian has two parts, the real and the imaginary. If the field defines a solution to the Euler-Langange equations, the energy (2.29) is conserved. Thus, also the imaginary part is conserved though we only computed the real part. We get the same dependence of $\beta$ for the imaginary part. For the real part of the Lagrangian we get from Theorem 3.13

$$\frac{P_{0,R}}{||A||} = \beta^2 \frac{1}{16\sqrt{2}} B$$  

(3.73)

where we have inserted $\sum_{k=0}^{2} c_k^2 = 4$. Including the imaginary part changes the constant, but it is nonnegative. □

We can find a gauge field that gives positive energy for Minkowski’s metric as sum.

**Lemma 3.15.** Let the gauge field be

$$A_\mu^a = s_a d_\mu e^{-\beta^2 \sum_{j=1}^{3} r_j^2} + s_a d_\mu e^{-\beta^2 \sum_{j=1}^{3} r_j^2}$$  

(3.74)

where $\beta$, $s_a \in \mathbb{R}$ and

$$r_{jk} = \rho_{j,k} + i \sigma_{j,k} = \sum_{\mu=0}^{3} \alpha_{\mu,j,k} x_\mu$$

$$d_k = c_k + i e_k$$
\[\begin{align*}
\alpha_{011} &= 0 \quad \alpha_{021} = 0 \quad \alpha_{031} = \frac{1}{\sqrt{2}} \\
\alpha_{111} &= 1 \quad \alpha_{121} = 1 \quad \alpha_{131} = -1 \\
\alpha_{211} &= -1 \quad \alpha_{221} = -1 \quad \alpha_{231} = 0 \\
\alpha_{311} &= \sqrt{2} \quad \alpha_{321} = -\sqrt{2} \quad \alpha_{331} = \frac{1}{\sqrt{2}} \\
\alpha_{012} &= 0 \quad \alpha_{022} = 0 \quad \alpha_{032} = \frac{1}{\sqrt{2}} \\
\alpha_{112} &= 1 \quad \alpha_{122} = 1 \quad \alpha_{132} = -1 \\
\alpha_{212} &= 1 \quad \alpha_{222} = 1 \quad \alpha_{232} = 0 \\
\alpha_{312} &= \sqrt{2} \quad \alpha_{322} = -\sqrt{2} \quad \alpha_{332} = \frac{1}{\sqrt{2}}
\end{align*}\]

\[
c_{01} = \sqrt{2} \quad c_{11} = 1 \quad c_{21} = 1 \quad c_{31} = 0 \\
c_{02} = \sqrt{2} \quad c_{12} = 1 \quad c_{22} = -1 \quad c_{32} = 0 \\
c_{03} = \sqrt{2} \quad c_{13} = 1 \quad c_{23} = 0 \quad c_{33} = 0
\]

Then
\[A_{a,R}^{\mu}(0, x_1, x_2, x_3) = s_{a}c_{\mu 1}e^{-\beta^2 \sum_{j=1}^{3} \rho_{j}^2} + s_{a}c_{\mu 2}e^{-\beta^2 \sum_{j=1}^{3} \rho_{j}^2} \quad (3.75)\]

and
\[\mathcal{L}_R = -\frac{1}{4} F_{a,R}^{\mu\nu} F_{a,R}^{\mu\nu} \quad (3.76)\]

In Minkowski’s metric (2.9) at \(x_0 = 0\)
\[\mathcal{L}_R = -\frac{1}{2} \sum_{a} (2\beta^2 s_a)^2 e^{-2\beta^2 \sum_{j=1}^{3} \rho_j} \left( \frac{13}{3} y_1^2 + 8y_2^2 - \frac{170}{21} y_3^2 + \frac{8}{21} \sqrt{14} y_1 y_3 \right) \quad (3.77)\]

where
\[y_1 = \sqrt{6} x_1 - \frac{1}{\sqrt{3}} x_3 \quad y_2 = 2x_2 \quad y_3 = \sqrt{\frac{14}{3}} x_3 \quad (3.77)\]

Proof. By Lemma 3.3 the sum of solutions satisfying (3.9) is also a solution satisfying (3.9). Most of the proof is as in Lemma 3.10. We are interested in the cross term that comes from squaring \(F_{a,R}^{\mu\nu}\). As it has two components from the two fields in the sum, the squares of each field give two squares and a cross term (twice the product of the terms). Both of the squares disappear as in Lemma 3.10 but the cross term gives the term in (3.76) and it does not disappear. We compute only this term in detail. Let us notice that \(r_{31} = r_{32}\) and for simplicity we will write
\[\rho_3 = \rho_{31} = \rho_{32} = -x_1 + \frac{1}{\sqrt{2}} x_3\]
We notice that
\[
\sum_{j=1}^{3} \rho_{j1}^2 + \sum_{j=1}^{3} \rho_{j2}^2 = y_1^2 + y_2^2 + y_3^2
\] (3.78)

Let us compute the needed components
\[
\sum_{j=1}^{3} \alpha_{0j1} \rho_{j1} = \frac{i}{\sqrt{2}} \rho_3
\]
\[
\sum_{j=1}^{3} \alpha_{1j1} \rho_{j1} = \rho_{11} + \rho_{21} - \rho_3 = 2x_1 - 2x_2 - \rho_3
\]
\[
\sum_{j=1}^{3} \alpha_{2j1} \rho_{j1} = -\rho_{11} - \rho_{21} = -2x_1 + 2x_2
\]
\[
\sum_{j=1}^{3} \alpha_{3j1} \rho_{j1} = \sqrt{2} \rho_{11} - \sqrt{2} \rho_{21} + \frac{1}{\sqrt{2}} \rho_3 = 4x_3 + \frac{1}{\sqrt{2}} \rho_3
\]
\[
\sum_{j=1}^{3} \alpha_{0j2} \rho_{j2} = \frac{i}{\sqrt{2}} \rho_3
\]
\[
\sum_{j=1}^{3} \alpha_{1j2} \rho_{j2} = \rho_{12} + \rho_{22} - \rho_3 = 2x_1 + 2x_2 - \rho_3
\]
\[
\sum_{j=1}^{3} \alpha_{2j2} \rho_{j2} = \rho_{12} + \rho_{22} = 2x_1 + 2x_2
\]
\[
\sum_{j=1}^{3} \alpha_{3j2} \rho_{j2} = \sqrt{2} \rho_{12} - \sqrt{2} \rho_{22} + \frac{1}{\sqrt{2}} \rho_3 = 4x_3 + \frac{1}{\sqrt{2}} \rho_3
\]

\[
\sum_{j=1}^{3} \text{Re}(d_{11} \alpha_{0j1} - d_{01} \alpha_{1j1}) \rho_{j1} = -\frac{1}{\sqrt{2}} \rho_3 - \sqrt{2}(\rho_{11} + \rho_{21} - \rho_3)
\]
\[
= \frac{1}{\sqrt{2}} x_3 - 2\sqrt{2} x_1 + 2\sqrt{2} x_2
\] (3.79)

\[
\sum_{j=1}^{3} \text{Re}(d_{12} \alpha_{0j2} - d_{02} \alpha_{1j2}) \rho_{j2} = -\frac{1}{\sqrt{2}} \rho_3 - \sqrt{2}(\rho_{12} + \rho_{22} - \rho_3)
\]
\[
= \frac{1}{\sqrt{2}} x_3 - 2\sqrt{2} x_1 - 2\sqrt{2} x_2
\] (3.80)

\[
\sum_{j=1}^{3} \text{Re}(d_{21} \alpha_{0j1} - d_{01} \alpha_{2j1}) \rho_{j1} = -\frac{1}{\sqrt{2}} \rho_3 - \sqrt{2}(-\rho_{11} - \rho_{21})
\]
\[
= -\frac{1}{\sqrt{2}} x_3 + 2\sqrt{2} x_1 - 2\sqrt{2} x_2
\] (3.81)
\[ \sum_{j=1}^{3} Re(d_{22}\alpha_{0j2} - d_{02}\alpha_{2j2})\rho_{j2} = \frac{1}{\sqrt{2}}\rho_3 - \sqrt{2}(\rho_{11} + \rho_{21}) \]
\[ = \frac{1}{\sqrt{2}}x_3 - 2\sqrt{2}x_1 - 2\sqrt{2}x_2 \quad (3.82) \]

\[ \sum_{j=1}^{3} Re(d_{31}\alpha_{0j1} - d_{01}\alpha_{3j1})\rho_{j1} = -2\rho_{11} + 2\rho_{21} - \rho_3 \]
\[ = -4\sqrt{2}x_3 - \rho_3 \quad (3.83) \]

\[ \sum_{j=1}^{3} Re(d_{32}\alpha_{0j2} - d_{02}\alpha_{3j2})\rho_{j2} = -2\rho_{12} + 2\rho_{22} - \rho_3 \]
\[ = -4\sqrt{2}x_3 - \rho_3 \quad (3.84) \]

\[ \sum_{j=1}^{3} Re(d_{21}\alpha_{2j1} - d_{11}\alpha_{2j1})\rho_{j1} = 2\rho_{11} + 2\rho_{21} - \rho_3 \]
\[ = 4x_1 - 4x_2 - \rho_3 \]

\[ \sum_{j=1}^{3} Re(d_{22}\alpha_{2j2} - d_{12}\alpha_{2j2})\rho_{j2} = -2\rho_{12} - 2\rho_{22} + \rho_3 \]
\[ = -4x_1 - 4x_2 + \rho_3 \]

\[ \sum_{j=1}^{3} Re(d_{31}\alpha_{2j1} - d_{11}\alpha_{2j1})\rho_{j1} = -\sqrt{2}\rho_{11} + \sqrt{2}\rho_{21} - \frac{1}{\sqrt{2}}\rho_3 = -4x_3 - \frac{1}{\sqrt{2}}\rho_3 \]

\[ \sum_{j=1}^{3} Re(d_{32}\alpha_{2j2} - d_{12}\alpha_{3j2})\rho_{j2} = \sqrt{2}\rho_{12} + \sqrt{2}\rho_{22} + \frac{1}{\sqrt{2}}\rho_3 = 4x_3 + \frac{1}{\sqrt{2}}\rho_3 \]

\[ \sum_{j=1}^{3} Re(d_{31}\alpha_{2j1} - d_{21}\alpha_{3j1})\rho_{j1} = -\sqrt{2}\rho_{11} + \sqrt{2}\rho_{21} - \frac{1}{\sqrt{2}}\rho_3 = -4x_3 - \frac{1}{\sqrt{2}}\rho_3 \]

\[ \sum_{j=1}^{3} Re(d_{32}\alpha_{2j2} - d_{22}\alpha_{3j2})\rho_{j2} = -\sqrt{2}\rho_{12} + \sqrt{2}\rho_{22} - \frac{1}{\sqrt{2}}\rho_3 = -4x_3 - \frac{1}{\sqrt{2}}\rho_3 \]

The cross term in Minkowski’s metric is
\[ -\left(\frac{1}{\sqrt{2}}\rho_3 - 2\sqrt{2}x_1 + 2\sqrt{2}x_2\right)\left(\frac{1}{\sqrt{2}}\rho_3 - 2\sqrt{2}x_1 - 2\sqrt{2}x_2\right) \]
\[ -\left(\frac{1}{\sqrt{2}}\rho_3 + 2\sqrt{2}x_1 - 2\sqrt{2}x_2\right)\left(\frac{1}{\sqrt{2}}\rho_3 - 2\sqrt{2}x_1 - 2\sqrt{2}x_2\right) \]
\[ -\left(-4\sqrt{2}x_3 - \rho_3\right)^2 \]
\[ + (4x_1 - 4x_2 - \rho_3)(-4x_1 - 4x_2 + \rho_3) \]
\[ + (-4x_3 - \frac{1}{\sqrt{2}}\rho_3)(4x_3 + \frac{1}{\sqrt{2}}\rho_3) \]
SOLUTIONS TO YANG-MILLS EQUATIONS

\[ +(-4x_3 - \frac{1}{\sqrt{2}}\rho_3)^2 \]

\[ = -16x_1^2 + 16x_2^2 - 32x_3^2 - 2\rho_3^2 + \rho_3(8x_1 - 8\sqrt{2}x_3) \]

\[ = -26x_1^2 + 16x_2^2 - 41x_3^2 + 14\sqrt{2}x_1x_3 \]

\[ = -\frac{13}{3}y_1^2 + 8y_3^2 - \frac{170}{21}y_1y_3 + 8 \sqrt{\frac{14}{21}}y_1y_3 \]

Inserting this result as in Lemma 3.10 yields the claim. □

**Theorem 3.16.** Let \( A = (A_{\mu})_\mu, A_\mu = A^a_\mu t_a \) be a complex gauge field as in Lemma 3.15. The gauge field and the corresponding curvature satisfy Euler-Lagrange equations for

\[ \mathcal{L} = -\frac{1}{4}F^\mu_\nu F^a_\mu_\nu \] (3.85)

In Minkowski’s metric

\[ E_\beta = \frac{P^0}{||A||} = \beta^2 C \] (3.86)

where \( C \) is a positive constant.

**Proof.** The Lagrangian is computed in Lemma 3.15. As in Lemma 3.9 the norm \( ||A|| \) is not zero and depends on \( \beta \) as \( \beta^{-3} \). As in Lemma 3.11 the Lagrangian in (3.15) when integrated over the space coordinates is proportional to \( B = B_1 + B_2 + B_3 = -\frac{13}{3} + 8 - \frac{170}{21} \) which is nonzero and the integral over space coordinates does not vanish. As in Lemma 3.12 the first part of the Hamiltonian density (2.30) does not contribute to the integral:

\[ \int d^3H_R = -\int d^3\mathcal{L}_R \]

The rest is as in Theorem 3.14. □

4. Mass Gap and Quantization of Yang-Mills Fields

The first question is what is mass gap. L. Faddeev explains the issue in [2] but let us proceed in a similar way as in [6] from quantum mechanics and scalar quantum field theory to quantum Yang-Mills theory. We take a simple scalar wave function of one variable

\[ \varphi(x_1) = e^{-\frac{1}{2}ax_1^2} \] (4.1)

Then

\[ \left( \frac{1}{a^2} \frac{\partial^2}{\partial x_1^2} + \frac{1}{a} \right) \varphi(x_1) = x_1^2 \varphi(x_1) \] (4.2)

It follows that

\[ \frac{1}{a} \int_{-\infty}^{\infty} dx_1 \varphi(x_1) = \sqrt{2\pi}a^{-\frac{3}{2}} \] (4.3)

while also

\[ \int_{-\infty}^{\infty} dx_1 x_1^2 \varphi(x_1) = \sqrt{2\pi}a^{-\frac{5}{2}} \] (4.4)

The function \( \varphi(x_1) \) is time-independent as it does not depend on \( x_0 \). We can consider it as a state in the Schrödinger picture

\[ |q> = |q>_S \] (4.5)
We can consider
\[ \hat{A} = A(x_1) = \frac{1}{a^2} \frac{\partial^2}{\partial x_1^2} + \frac{1}{a} \] (4.6)

as an operator acting on the state \(|q\rangle\). In order to take an inner product of \(\hat{A}|q\rangle\) with another state \(|q'\rangle\) corresponding to the field \(\phi'(x_1)\) it is more convenient to define the operator as
\[ \hat{H} = H(x', x_1) = \left( \frac{1}{a^2} \frac{\partial}{\partial x'_1} \frac{\partial}{\partial x_1} + \frac{1}{a} \right) \delta(x_1 - x'_1) \] (4.7)

Then
\[ \langle q' | \hat{H} | q \rangle = \int dx'_1 \int dx_1 \phi'(x'_1) H(x'_1, x_1) \phi(x_1) \] (4.8)

Especially
\[ \langle q | \hat{H} | q \rangle = \int dx'_1 \int dx_1 \phi'(x'_1) \delta(x_1 - x'_1) \phi(x_1) \]
\[ = \int dx_1 x_1^2 e^{-2ax_1^2} = \sqrt{2\pi(\sqrt{2}a)}^{-\frac{3}{2}} \] (4.9)

while
\[ \langle q | q \rangle = \int dx'_1 \int dx_1 \phi'(x'_1) \phi(x_1) \]
\[ = \int dx_1 \phi(x_1)^* \phi(x_1) = \int dx_1 e^{-2ax_1^2} = \sqrt{2\pi(\sqrt{2}a)}^{-\frac{1}{2}} \] (4.10)

Thus
\[ \langle q | \hat{H} | q \rangle = E \langle q | q \rangle \quad E = \frac{1}{\sqrt{2a}} \] (4.11)

Thus, \(E\) is the expectation value of the operator \(\hat{H}\) at the state \(|q\rangle\). Let us assume that the state \(|q\rangle\) is expressed as a linear combination of the eigenstates of the Hamiltonian operator \(\hat{H}\). If \(E\) can be selected arbitrarily small then we can select a sequence of states \(|q_n\rangle\) where \(E_n\) goes to zero. This means that either the sequence of the states \(|q_n\rangle\) converges to the vacuum state, or that there is no minimal positive eigenstate for \(\hat{H}\). The state where \(|q_n\rangle\) converges if \(a \to \infty\) is zero, which is not a vacuum state. We conclude that there is no minimal positive eigenvalue for \(\hat{H}\), i.e., there is no mass gap. We can also write the equation with the Hamiltonian density \(H\)
\[ \int dx'_1 \int dx_1 \frac{\partial}{\partial x'_1} \frac{\partial}{\partial x_1} H(x'_1, x_1) \phi(x_1) = \int dx_1 H \] (4.12)

The set of eigenvalues of the Hamiltonian operator forms the energy-mass spectrum of the field. The zero function \(\phi(x_1) = 0\) always satisfies the eigenvalue equation but it is not an acceptable eigenstate since it has zero form. There is assumed to exist an eigenstate with eigenvalue zero, the vacuum. The vacuum is not unique in all theories, but it must be unique in a theory filling Wightman’s axioms. If there is a minimum positive value \(E\) in the energy-mass spectrum, we say that there is a mass gap. The eigenstates are closely related to a parameter called mass because the physical interpretation of the parameter \(m\) in an equation
\[ (\partial_\mu \partial^\mu + m^2) \phi = 0 \] (4.13)
is mass.
Let us now proceed to find the Hamiltonian operator for the Hamiltonian density \( H_R \) in Lemma 3.12. We notice that in Lemma 3.12
\[
\int d^3 \mathcal{H}_R = - \int d^3 \mathcal{L}_R
\]
From Lemmas 3.10 and 3.11 we see that the Lagrangian can be expressed in variables \( y_1, y_2, y_3 \) as
\[
\mathcal{L}_R = \frac{1}{2} \sum_a (2\beta^2 s_a)^2 e^{-2\beta^2 \sum_j \rho_j^2} \mathcal{P}(\rho)
\]
\[
= \frac{1}{2} \sum_a (2\beta^2 s_a)^2 e^{-2\beta^2 \sum_j \rho_j^2} \delta^3 \mathcal{A}(B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2 + B_4 y_1 y_2 + B_5 y_1 y_3 + B_6 y_2 y_3)
\]
We can ignore the terms \((B_4 y_1 y_2 + B_5 y_1 y_3 + B_6 y_2 y_3)\) since they disappear in the integration in Lemma 3.11 and conclude that the Hamiltonian density in the case of this field takes the form
\[
\mathcal{H}_R = C e^{-\frac{1}{2}(2\beta)^2 \sum_j \rho_j^2} \delta^3 \mathcal{A}(B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2) \quad C = 16\sqrt{2} \sum_a s_a^2 \beta^4
\]
Comparing this expression with (4.4) we can write the Hamiltonian operator as
\[
\hat{H} = H(y', y) = C \prod_{j=1}^3 \left( \frac{1}{(2\beta)^4 \partial_j y_{j'} \partial_j y_{j'}} + \frac{1}{(2\beta)^2} \right) \prod_{j=1}^3 \delta(y_j - y'_{j})
\]
where \((y'_1, y'_2, y'_3), y = (y_1, y_2, y_3)\). Let us mention that the Hamiltonian takes this simple form only for the field (3.62), not for every field. Then
\[
<A|\hat{H}|A> = E <A|A>
\]
takes the form
\[
\int d^3 y' \int d^3 y A(y')\hat{H}(y', y)A(y) = E \int d^3 y' \int d^3 y A(y')\delta(y - y')A(y)
\]
which is the same as
\[
\int d^3 y \mathcal{H}_R = E \int d^3 y A(y)^* A(y)
\]
We refer to formulae (2.19), (4.31) and (17.50) in [6] for the connection between the Hamiltonian operator and (4.7) and (4.15). There are of course many approaches but following the approach in [6] the operators (4.7) and (4.15) can be understood to describe the Hamiltonian operator for a field theory.

We see that as \( \beta \) in Theorem 3.14 can be freely selected, either there is no mass gap or vacuum is not unique, provided that the gauge fields \( A^a_\mu \) in (3.68) and (3.74) are acceptable. We obtained \( B = 0 \) for the energy of the real part in Minkowski’s metric in Lemma 3.12 for the gauge field in (3.68). While we gave another gauge field with positive energy in (3.74), let us notice that the result is negative for the CMI problem also if the constant \( C = 0 \). If \( C = 0 \) it implies that the vacuum is not unique and contradicts Wightman’s Axiom II that demands that with the exception of vacuum all states have positive energy.

Let us now continue to the question if \( A^a_\mu \) in (3.68) and (3.74) can be excluded in a non-trivial quantum field theory for the Yang-Mills Lagrangian (2.1).

Quantization of the Yang-Mills theory can be made by two methods; by the path integral method, or by axiomatic quantum field theory. Canonical quantization is also possible but considered difficult. Let us first look at the path integral method.
Basically quantization of a Yang-Mills field is made by writing the ground-state-to-ground-state amplitude $W(J)$ as a path integral

$$W[J] \sim \int D\mathbf{A} \exp \left\{ -i\hbar^{-1} \int d^4x \left( \mathcal{L}_{YM} + J_\mu^a A^a_\mu \right) \right\}$$  \hspace{1cm} (4.19)

However, there are problems in the path integral and the form (4.19) is not to be followed precisely. The path integral may become infinite for a number of reasons and a proper quantization should avoid these pitfalls. The character of such arguments is either mathematical or physical. For instance, the reason why the field should disappear when the space coordinates grow is physical. Mere integrability of a function does not require that it vanishes in infinity as positive and negative parts can cancel.

The discussion in [6] on page 117 mentions the need for fixing the gauge in a case where there are infinitely many $A_\mu^a$ related by a gauge transform, and mentions divergences even in the case that the coupling constant $g = 0$ in (2.6).

However, there are more problems in (4.1) when considering $A_\mu^a$ in (3.68). As can be seen in (3.34), the field $A_\mu^a$ is a localized wave packet, gauge boson, that moves with the speed of light $(x_0)$ in the $(x_1, x_2)$ plane to the direction $e_1 + e_2$ where $e_j$ is the unit vector of the $j$th coordinate. This is very natural behavior for a localized wave packet. It cannot stay in a limited box, and it cannot be bounded in the time dimension $x_0$ because it stays localized and the energy is conserved as $A_\mu^a$ is a solution to the Euler-Lagrangian equations. Thus, the ground-state-to-ground state amplitude

$$W[J] \sim \lim_{t'' \to -\infty, t' \to \infty} < q'', t'' | q', t' >_J$$  \hspace{1cm} (4.20)

is not a proper quantity for this field. We can calculate transitions between any finite times $t'$ to $t''$, and then the path integral is finite.

$$< q'', t'' | q', t' >_J \sim \int D\mathbf{A} \exp \left\{ -i\hbar^{-1} \int_{t'}^{t''} dt \int d^3x \left( \mathcal{L}_{YM} + J_\mu^a A^a_\mu \right) \right\}$$  \hspace{1cm} (4.21)

As essential problem is that as $h : \mathbb{C} \to \mathbb{C}$ in (3.30) must be holomorphic so that differentiation can be made, its real and imaginary parts cannot be bounded. We give a physicality argument.

It is reasonable to require that the field vanishes when the space coordinates go to $\pm \infty$. However, the time coordinate is different. The future cannot effect the past and therefore possible divergences in the future are not an appropriate boundary condition for a physical problem setting. Likewise, there may well be a finite beginning instance of the time and therefore extension of $x_0$ to $-\infty$ is highly speculative. Thus, the integration over $x_0$ in $W[J]$ is physically motivated only between two finite time instances $t'$ and $t''$. Accepting that this argument for avoiding infinities in the path integral is as reasonable as other tricks that have been used to the same goal in the semi-mathematical path integral method, such as cutoffs, renormalization, gauge fixing, etc., the gauge field $A_\mu^a$ in (3.68) is acceptable in a non-trivial quantum Yang-Mills theory created through the path integral method.

There are other possible mechanisms to render (4.21) finite. In perturbation theory the path integral cannot include solutions to the linear Lagrange equations. Thus, (3.68) could be excluded. As there is no other motivation for exclusion than obtaining a finite integral, one should consider the physicality argument above as
a more acceptable way to get a finite (4.21). In any case, there are various ad hoc methods used in the path integral method that have the aim of removing infinities from (4.21).

The gauge fields of the type (3.30) admit a non-trivial quantum field theory for (2.1). We can divide the path integral into two (and later more) parts, where the first part only has fields of the type (3.30). They are easy to handle, sums and real parts also satisfy the Euler-Lagrange equations, as is shown in Lemmas 3.3 and 3.4. Sums of these type of fields with different $s_a$ yield equations that involve the structure coefficients $f_{abc}$ and may have some special solutions. We can briefly look at a sum of solutions of the type (3.30) with different $s_a$.

**Lemma 4.1.** Let the gauge field

$$A^\mu_a = (s_{a,1}d_\mu,1 + s_{a,2}d_\mu,2)e_{\sum_j h(r_j)} A^3_a = 0$$

be a solution to (2.22). Then $h$ satisfies an equation of the type

$$\sum_j C_{akj}h''(r_j) + \sum_{j,m} D_{akjm}h'(r_j)h'(r_m) + \sum_j E_{akj}h'(r_j) e_{\sum_j h(r_j)} + F_{ak} e_{\sum_j 2h(r_j)}$$

where $C_{akj}, D_{akjm}, E_{akj}, F_{ak}$ are constants.

**Proof.** Calculating $F_{\mu\nu}^a$ yields

$$F_{\mu\nu}^a = \sum_j a_{\mu j}h'(r_j)(s_{a,1}d_\nu,1 + s_{a,2}d_\nu,2)e_{\sum_j h(r_j)}$$

$$- \sum_j a_{\nu j}h'(r_j)(s_{a,1}d_\mu,1 + s_{a,2}d_\mu,2)e_{\sum_j h(r_j)}$$

$$- g \sum_{c>b} f_{abc}(s_{b,1}s_{c,2} - s_{b,2}s_{c,1})(d_\mu,1d_\nu,2 - (d_\mu,2d_\nu,1)e_{\sum_j 2h(r_j)}$$

The last term does not disappear, thus $F_{\mu\nu}^a$ is of the form

$$F_{\mu k}^a = \sum_j a_{ikj}h'(r_j)e_{\sum_j h(r_j)} + b_{ik} e_{\sum_j 2h(r_j)}$$

As $A^3_a = 0$ the equations (2.22) reduce to (3.9). The terms in (3.9) are of the following form, the constants are complex numbers

$$\partial^3 F_{\mu k}^a = \sum_{j,l} a_{ikj}a_{lj}h''(r_j)e_{\sum_j h(r_j)} + \sum_{j,l,m} a_{ikj}a_{lm}h'(r_j)h'(r_m)e_{\sum_j h(r_j)}$$

$$+ \sum_{j,l} b_{ik}2h(r_j)a_{lj}e_{\sum_j 2h(r_j)}$$

$$\partial^3 \partial^3 A_k^a = \sum_j c_{akj}h'(r_j)e_{\sum_j h(r_j)} + \sum_j d_{akjm}h'(r_j)h'(r_m)e_{\sum_j h(r_j)}$$

$$- g f_{abc} A^a_0 F_{ik}^a = \sum_j e_{ikj}h'(r_j)e_{\sum_j 2h(r_j)} + f_{ik} e_{\sum_j 3h(r_j)}$$

$\square$

**Lemma 4.2.** Let $h(r_j) = \beta^2 r_j^2$ in Lemma 4.1. There are no solutions with $F_{\mu k} \neq 0$ of the type in Lemma 4.1.
Proof. The term $e^{-2\beta^2 \sum r_i^2}$ in (4.23) is not cancelled by anything.

Lemma 4.2 shows that there are no interactions for solutions of the type (3.39) but there could be solutions of as in Lemma 4.1 for some other $h(r_j)$. In [4] a type of function is proposed as a solution for (4.23) but it is not explicitly shown that such a solution exists. Even simple solutions of the type (3.30) are not trivial and may give solutions that do not appear in the free field case, i.e., when the coupling constant $g$ is zero.

In any case, the largest group of solutions is surely (3.30) since there we have a free function $h$, while if the structure constants appear in the equations, we get a nonlinear partial differential equation, at least as difficult or worse as in Lemma 4.1, which typically have fewer solutions. If more solution families are found, correction terms can be calculated from the remaining parts of the path integral. Thus, we can make a non-trivial theory for a pure Yang-Mills Lagrangian and compute first order approximations. Let us mention that Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD) are not non-trivial quantum field theories for the pure Yang-Mills Langangian but there the spinor fields interacting with gauge fields create the interesting results. As a conclusion, there is no good reason to exclude $A_a^\mu$ in (3.68) or (3.74) in the path integral approach.

The other approach is axiomatic quantum field theory where Wightman’s axioms, or something as strong, has especially been mentioned in the CMI problem. We do not need to construct a theory filling axioms similar or stronger that Wightman’s but only to investigate if a theory filling such conditions should include $A_a^\mu$ in (3.68), properly normalized, as a state. This involves showing two things. Firstly, that they can be included, and secondly that a theory that does not include them should be called trivial. Let us briefly go through Wightman’s axioms. Wightman does not consider gauge fields at all and so we have to modify the axioms.

Axiom I. The states of a quantum field theory are normalised vectors in a separable Hilbers space, $H$, two such that they differ by a complex phase giving raise to the same state. If we normalize $A_a^\mu$ it is a normalized vector in a separable Hilbert space. Any states that differ by a complex phase give rise to the same state. Thus, $A_a^\mu$ and $A_{a.R}^\mu$ give the same state as these differ by a complex phase. Apparently we can compute the real Lagrangian $L$ as this is what Axiom I seems to imply.

Fortunately, $A_{a.R}^\mu$ is a solution to the real Euler-Lagrange equations.

Axiom II. The space $H$ carries a continuous unitary representation $(a, \Lambda) \mapsto U(a, \Lambda)$ of the restricted orthochronous Poincare group. In $H$ there exists a vector, unique up to a phase, (called the vacuum state) that is invariant under all $U(a, \Lambda)$ and for all other vectors $\Psi \in H$ the energy is positive. The only issue of concern here is that the energy of $A_a^\mu$, and of $A_{a.R}^\mu$, is positive, which is shown in Theorem 3.14 for the metric of (2.33). For Minkowski’s metric it was shown the energy of the real Hamiltonian is zero. If also the imaginary part is zero this means that the vacuum is not unique. We may want to discard (3.68) in Minkowski’s metric but (3.74) gives positive energy and there is no reason to discard that field.

Axiom IIIa. Deals only with the vacuum state and is of no concern to $A_a^\mu$ being an acceptable state or not.

Axiom IIIb. For any pair of vectors $\Phi$ and $\Psi$, the map $f \mapsto \langle \Phi, \phi(f)\Psi \rangle$ is continuous. Here $\phi(f) = \int d^4x \phi(x)f(x)$ is the smeared field. The function $f$ is tempered, i.e., belongs to $S$, the set of infinitely differentiable functions on $\mathbb{R}^4$ which vanish faster than any power of Euclidean distance. It guarantees that the integral
converges. The inner product is given by an integral over $\mathbb{R}^4$. If one of the vectors $\Phi$ and $\Psi$ is $A_\mu^a$ and another one is not, then fulfillment of the axiom depends on the other vector. If both vectors are of the type $A_\mu^a$ then the map $f \mapsto \langle \Phi, \Psi(f) \rangle$ is continuous.

Axiom IV. Suppose that $f, g \in S$ are such that $\text{supp } f$ is space-like to $\text{supp } g$; then $\phi(f) \phi(g) = \phi(g) \phi(f)$. This holds for the free field. $\phi = A_\mu^a$ is a solution to free field equations as the part with the structure constants cancels.

These axioms do not have requirements that exclude $A_\mu^a$. Thus, $A_\mu^a$ can be included in a theory filling the axioms at if the energy is positive. If the energy is zero, there is a problem in the theory. In that case we may exclude $A_\mu^a$ in order to resolve the problem and fill Wightman’s axioms, but it is a bit artificial way. The second part is to show that they must be included in a non-trivial theory. The solutions $A_\mu^a$ are natural solutions to the Euler-Langange equations and especially if the coupling constant $g = 0$ or the group is $U(1)$ they are among the possible solutions. They give arbitrarily small eigenvalues to the Hamiltonian. While it may be possible to create a theory which does not include these solutions and is still valid for $U(1)$ and $g = 0$ cases, such a theory is trivial since it can be made by the following trivial procedure. Take any theory filling the axioms. If it includes the states $A_\mu^a$, then exclude all states that have these states as minimal solutions for the Lagrangian. The resulting theory does not have these eigenstates for the Hamiltonian. Indeed, we can make a theory with two states only, vacuum and an eigenstate of the Hamiltonian with a non-zero eigenvalue. Then all axioms are easily filled. A trick of this type can always be made and it avoids the essential problem of showing that there is a mass gap and has no physical relevance. Thus, we should call trivial any quantum field theory that does not include the solutions of the type $A_\mu^a$ if (more accurately, as) they can be included.

There are two manuscripts [4], [5] arguing that a mass gap exists. Both start by imposing the temporal, or Weyl, gauge $A^0 = 0$. If we impose this gauge and then look at the boundary conditions, the solutions (3.68) cannot be found. This is because when we localize the gauge field we need three linearly independent vectors $r_j$ in (3.33). As can be seen in the selected space gauge $A^3 = 0$, we only get two vectors for the non-gauged coordinates as is shown in Lemma 3.6. The third vector must be obtained from the gauged coordinate. Had we gauged time, then the equations in (3.34) would show that $x_0$ is limited, as now is $x_3$, while $x_1$ and $x_2$ would be linearly dependent on $x_3$. Then the field would not be integrable over the space coordinates, while it would be limited in time. Instead of fixing the gauge first, we must first look at the boundary conditions. This shows that the temporal gauge is not the correct choice, while a space gauge can work.

5. Final comments of the CMI Millennium Prize problem

The CMI problem setting called for mathematical clarity to the area of gauge fields. Much of this lack of clarity has traditionally been caused by mathematical unclarities in the path integral method. Everything is formulated in simple lemmas which are given proofs. This does not imply that the lemmas are considered new, it is only for clarity. The presentation of Yang-Mills fields follows the approach in [6].

There are some final words about the clarity of the CMI problem statement itself.
The problem statement does not specify whether the gauge group should be local or global, and [5] understands that it is global gauge group. It probably must be local gauge group since the relevant issues arise from local gauge invariance. However, this should have been stated.

The metric in the CMI problem setting is unclear. Minkowski’s metric (2.9) is the correct choice for quantum field theory but the CMI problem setting only mentions $\mathbb{R}^4$ and the expert’s explanation in [2], referred to in (2.38), seems to point to the Euclidean metric and to real curvature. We can present the results in $\mathbb{R}^4$ with the Euclidean metric also. The convenient way to do it is to use the negative definite metric (2.33). In Section 3 we have used (2.11). It is still valid for the metric (2.33), as (2.5) is the definition and in (2.11) we have simply multiplied (2.6) by $g_{\mu\beta}g_{\nu\beta}$. We have also used (2.22). In the derivation of (2.22) we have kept the metric explicitly and not used the values of $g_{\mu\nu}$ from (2.9). Thus, (2.22) is also valid for (2.33). There are no raising or lowering $x_0$ indices in Lemmas 3.1-3.9, thus they stay valid. In Lemma 3.10 we use (2.27) but give the result also for the metric in (2.33). Lemma 3.11 has no changes. There is derivation with respect to $x_0$ in Lemma 3.12 but the conclusions remain since they are caused by the disappearance of the integral (3.61) as is mentioned in to proof. It follows that Theorem 3.14 holds also for the metric (2.33) for some other constant $C$. It is assumed that the fields can be complex as it is the situation in the physical problem and the Hodge star operation is defined for differential forms in complex manifolds. But as it is unclear in (2.38) and in the problem setting the calculations were done for the real part of the curvature covering the possibility that the problem statement implies real fields. It would have been much clearer if the CMI problem statement had stated if Minkowski’s metric is assumed, and if the fields are complex or real.

Referring to axiomatic field theory by mentioning axioms that do not as such apply to gauge fields, use of words such as non-trivial, etc. would make any positive solutions to the CMI problem difficult to argue. This would not be an issue if proposed solutions to the CMI problems would be positively received and carefully reviewed. It would be an issue if the opposite were the case.

The results of this article are easier to verify:

It seems that the CMI problem refers to Euclidean metric and real curvature. As there is no minus sign in (2.34) and (2.38) while there is one in (2.1) it seems that the metric is as in (2.33). In this case there is no mass gap since we can by selection of $\beta$ in (3.73) make the eigenvalue of the Hamiltonian as small as desired.

If the problem means Minkowski’s metric and real fields, then the gauge field in (3.74) shows that there is no mass gap. However, the field (3.68) gives zero energy and indicates that vacuum is not unique and Wightman’s axioms cannot be filled. We may want to exclude the field (3.68) in this case but there is no good reason for excluding it.

If the problem means complex fields in either metric, the conclusions are the same.

The results presented here should not be called a trivial free field theory. The coupling constant $g$ is not set to zero. The solutions that have been found are of such a type that the part with structure constants cancel. As Lemmas 4.1 and 4.2 indicate, nontrivial results can be found starting from the solutions in (3.30) and (3.39). Localization of the field in space is not trivial and in general this word should be avoided if clarity is desired because clarity is best achieved by writing
down all steps. This article may be correctly called elementary and easy, but not trivial.

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