Bergman kernel and complex singularity exponent

Bo-Yong Chen∗ & Hanjin Lee†

1 Introduction

Let Ω be a bounded $C^\infty$ pseudoconvex domain in $\mathbb{C}^n$. An important subject in complex analysis is to understand the boundary behavior of the Bergman kernel $K_\Omega(z)$ of $\Omega$. The special case of strongly pseudoconvex domains is well-understood through the works of Hörmander[1], Diederich[2, 3] especially the deep work of Fefferman[4]. Precise estimates of $K_\Omega$ for some special pseudoconvex finite type domains are also available in terms of certain embedded polydisks[5, 6]. However, it is generally impossible to construct such embedded polydisks and one only knows $K_\Omega \geq C \delta_\Omega^{-2}$ from the Ohsawa-Takegoshi extension theorem[7] (see also [8] for a slightly weaker result). Diederich-Herbort-Ohsawa[9] proved $K_\Omega \geq C \delta_\Omega^{-2-\epsilon}$, $\epsilon > 0$ for any pseudoconvex $\Omega$ of finite type. On the other side, it was Herbort[10] who firstly noticed that the growth of the Bergman kernel may contain log terms for certain pseudoconvex domain of finite type in $\mathbb{C}^3$. Recently, a far-reaching generalization of this phenomenon was made by Kamimoto[11], who in fact obtained a precise asymptotic expansion of the Bergman kernel non-tangentially at zero on certain model domain of finite type of form $\Omega = \{(z, w) \in \mathbb{C}^{n+1} : r(z, w) = \text{Im } w - |F(z)|^2 > 0\}$, where $F = (f_1, \ldots, f_m)$ is a holomorphic map from $\mathbb{C}^n$ to $\mathbb{C}^m$. We always assume $F(0) = 0$ for the sake of simplicity.

To state our results, let us first recall the following

**Definition 1**[12]. Let $M$ be a complex manifold and $\rho$ be a measurable function on $M$. For any compact set $K \subset M$, the complex singularity exponent of $\rho$ on $K$ is defined by

$$c_K(\rho) = \sup\{c \geq 0 : |\rho|^{-c} \text{ is } L^2 \text{ on a neighborhood of } K\}.$$
Roughly speaking, the complex singularity exponent is a holomorphic invariant which measures the singularity of a function more precisely than the well-known Lelong number. Given a point $p \in M$, we write $c_p(\rho)$ instead of $c_{\{p\}}(\rho)$.

**Theorem 1.** Let $V \subset U$ be two bounded Stein neighborhoods of $0 \in \mathbb{C}^{n+1}$. Given any $p_0 = (z_0, w_0) \in V \cap \Omega_F$ and any non-tangential cone $\Lambda \subset \Omega_F$ with vertex at $p_0$, there exists an integer $1 \leq l \leq n$ such that

$$K_{\Omega_F \cap U}(p) \simeq r(p)^{2-c_{z_0}(|F - F(z_0)|)} | \log r(p) |^{l-1}, \quad \forall p \in V \cap \Lambda.$$ 

Here $A \simeq B$ means that the ratio $A/B$ is pinched between two positive constants (possibly depending on $p_0$).

**Remark 1.** If in addition $|F(z)| \to \infty$ as $|z| \to \infty$, then the above result holds for $K_{\Omega_F}$ itself (compared with [13]).

To get similar results for the Bergman metric $B_{\Omega_F \cap U}$ and the holomorphic sectional curvature $R_{\Omega_F \cap U}$ of $B_{\Omega_F \cap U}$, we introduce the following

**Definition 2.** Let $\rho, \phi$ be non-negative measurable functions in $\mathbb{C}^n$ and $p \in \mathbb{C}^n$. The complex singularity exponent with weight $\phi$ of $\rho$ at $p$ is defined by

$$c_p(\rho; \phi) = \sup \{ c \geq 0 : \rho/\rho^c \text{ is } L^2 \text{ on a neighborhood of } p \}.$$

**Theorem 2.** Under the conditions of Theorem 1, there exist integers $1 \leq l_j', l_j'' \leq n$, $j = 1, \ldots, n$ such that

$$B_{\Omega_F \cap U}(p; X) \simeq |X_{n+1}|^2 r(p)^2 + \sum_{j=1}^n \frac{|X_j|^2}{r(p)^{c_{z_0}(|F - F(z_0)|)} |z_j - z_j^0|} | \log r(p) |^{l_j'-1},$$

for all $p \in V \cap \Lambda$ and non-zero vectors $X \in \mathbb{C}^{n+1}$ and

$$2 - R_{\Omega_F \cap U}[p; (0, \ldots, X_j, \ldots, 0)] \simeq r(p)^{2-c_{z_0}(|F - F(z_0)|)} |z_j - z_j^0|^{2-c_{z_0}(|F - F(z_0)|)} | \log r(p) |^{l_j''-l_j'-1},$$

for all $j = 1, \ldots, n$. Here the implicit constants are independent of $X$.

**Remark 2.** Given $c' < c_{z_0}(\rho)$ and $c'' < c_{z_0}(\rho; |z_j|^2)$, we set $c = \frac{c' + c''}{2}$. Then for some $0 < \eta \ll 1$, the Schwarz inequality implies

$$\int_{B(z_0, \eta)} |\rho|^{-2c} |z_j|^2 dV(z) \leq \left( \int_{B(z_0, \eta)} |\rho|^{-2c'} dV(z) \right)^{1/2} \left( \int_{B(z_0, \eta)} |\rho|^{-2c''} |z_j|^4 dV(z) \right)^{1/2},$$

from which it follows that $2c_{z_0}(\rho; |z_j|) \geq c_{z_0}(\rho) + c_{z_0}(\rho; |z_j|^2)$. Taking $\rho = |F - F(z_0)|$, we see from Theorem 2 that the holomorphic sectional curvature is either bounded below by a constant or is asymptotic to $-\infty$ polynomially w.r.t. the Bergman distance in a non-tangent cone at $p_0$. We conjecture that the latter case actually can’t occur.

With the help of some results of Demailly-Kollár[12], we generalize Theorem 1 to the following
Theorem 3. Let \( \rho \geq 0 \) be a log psh function in \( \mathbb{C}^n \) with \( \rho(0) = 0 \) and let
\[
\Omega_{\rho} := \{(z, w) \in \mathbb{C}^{n+1} : \text{Im} w > \rho(z)\}.
\]
Let \( V \subset U \) be two bounded Stein neighborhoods of the origin. Then for every \( \epsilon > 0 \), there is a constant \( C_\epsilon \gg 1 \) such that
\[
C_\epsilon^{-1}(\text{Im} w)^{-2-2c_0(\rho)+\epsilon} \leq K_{\Omega_{\rho} \cap U}(0, w) \leq C_\epsilon(\text{Im} w)^{-2-2c_0(\rho)-\epsilon}
\]
for \( (0, w) \in \Omega_{\rho} \cap V \).

As an application of Theorem 1, we are able to get asymptotic estimates for the (Euclidean) volume of sublevel sets with parameter \( \zeta \)
\[
D(U', F, \zeta, r) := \{z \in U' : |F(z) - F(\zeta)| < r, |\zeta| < r\}, \quad U' \subset \subset \mathbb{C}^n
\]
where \( F : \mathbb{C}^n \to \mathbb{C}^m \) is a holomorphic map such that \( F(0) = 0 \), which might be useful for other purposes.

Theorem 4. Fix a sufficiently small (Stein) neighborhood \( U' \) of \( 0 \in \mathbb{C}^n \). Then for every \( \epsilon > 0 \), there is a constant \( C_\epsilon = C(U', F, \epsilon) > 0 \) such that
\[
C_\epsilon^{-1}|\log r|^{-1} \leq \text{Vol}(D(U', F, \zeta, r)) \leq C_\epsilon r^{c_0(\rho) - \epsilon}, \quad \text{as } r \to 0.
\]
Here \( 1 \leq l \leq n \) is an integer coming from the resolution of singularity of \( \{F = 0\} \) and \( C_\epsilon \) is a constant independent of \( \epsilon \).

2 Preliminaries

2.1 Bergman invariants and minimum integrals

Assume that \( \Omega \) is a bounded domain in \( \mathbb{C}^n \). Let \( L^2(\Omega) \) denote the space of square-integrable functions on \( \Omega \) and \( \|\cdot\|_\Omega \) the corresponding \( L^2 \)-norm. The Bergman space is given by \( H^2(\Omega) := L^2(\Omega) \cap O(\Omega) \) and the Bergman kernel \( K_\Omega(z) = \sum_j |\phi_j(z)|^2 \) where \( \{\phi_j\} \) is a complete orthogonal basis of \( H^2(\Omega) \). The Bergman metric \( B_\Omega(z; X) \) is given by
\[
B_\Omega(z; X) = \sum_{j,k} g_{jk} X_j \bar{X}_k \text{ where } g_{jk} = \frac{\partial^2 \log K_\Omega(z)}{\partial z_j \partial \bar{z}_k}.
\]
The holomorphic sectional curvature of \( B_\Omega \) is defined by
\[
R_\Omega(z; X) = B_\Omega(z; X)^{-4} \sum_{h,j,k,l} R_{hjkli} X_h X_j X_k \bar{X}_l, \quad X \in \mathbb{C}^n - \{0\},
\]
where
\[
R_{hjkli} = -\frac{\partial^2 g_{jk}}{\partial z_k \partial \bar{z}_l} + \sum_{\mu,\nu} g^{\nu \bar{\mu}} \frac{\partial g_{j\bar{\mu}}}{\partial z_k} \frac{\partial g_{\nu h}}{\partial \bar{z}_l}.
\]
for every open set \( U \). We define the minimum integrals

\[
I^0_{\Omega}(p) = \inf \{ \| f \|_{L^2_{\Omega}}^2 : f \in H^2(\Omega), \; f(p) = 1 \};
\]

\[
I^j_{\Omega}(p; X) = \inf \left\{ \| f \|_{L^2_{\Omega}}^2 : f \in H^2(\Omega), \; f(p) = 0, \; \sum_j X_j \partial f / \partial z_j(p) = 1 \right\};
\]

\[
I^j_{\Omega}(p; X) = \inf \left\{ \| f \|_{L^2_{\Omega}}^2 : f \in H^2(\Omega), \; f(p) = \partial f / \partial z_1(p) = \cdots = \partial f / \partial z_n(p) = 0, \; \sum_{j,k} \partial^2 f / \partial z_j \partial z_k(p) X_j X_k = 1 \right\}.
\]

By the definitions, the minimum integrals increase when the domain does, and they enjoy the following transformation laws under a biholomorphic map \( \Phi : \Omega_1 \to \Omega_2 \):

\[
I^0_{\Omega_1}(p) = |J_{\Phi}(p)|^{-2} I^0_{\Omega_2}(\Phi(p)),
\]

\[
I^j_{\Omega_1}(p; X) = |J_{\Phi}(p)|^{-2} I^j_{\Omega_2}(\Phi(p); \Phi_* X), \quad j = 1, 2,
\]

where \( J_{\Phi} \) denotes the complex Jocobian determinant of \( \Phi \). The following relationships between the Bergman invariants and minimum integrals are well-known (see eg. [14]):

\[
K_\Omega(z) = \frac{1}{I^0_{\Omega}(z)}, \quad B_\Omega(z; X) = \frac{I^0_{\Omega}(z)}{I^2_{\Omega}(z; X)}, \quad R_\Omega(z; X) = 2 - \frac{[I^2_{\Omega}(z; X)]^2}{I^0_{\Omega}(z) I^2_{\Omega}(z; X)}.
\]

### 2.2 Multiplier ideal sheaf

One of the most basic concepts on complex analysis is the multiplier ideal sheaf introduced by Nadel [19].

**Definition 3.** Let \( \varphi \) be a psh function on a domain \( \Omega \) in \( \mathbb{C}^n \). The multiplier ideal sheaf \( I(\varphi) \subset \mathcal{O}_\Omega \) is defined by

\[
\Gamma(U, I(\varphi)) = \{ f \in \mathcal{O}_\Omega(U) : |f|^2 e^{-\varphi} \in L^1_{\text{loc}}(U) \}
\]

for every open set \( U \subset \Omega \). For a point \( p \in \Omega \) and an integer \( l \geq 0 \), we set

\[
M^l_p(\Omega) = \{ \varphi \in PSH(\Omega) : \varphi \leq 0, \; \Gamma(U, I(\varphi)) \subset \mathfrak{M}_l^{l+1} \text{ for some neighborhood } U \text{ of } p \},
\]

where \( \mathfrak{M}_{l,p} \) denotes the maximal ideal of \( \mathcal{O}_\Omega \) at \( p \). We have the following

**Proposition 1.** Let \( \Omega \) be a pseudoconvex domain in \( \mathbb{C}^n \) and \( \varphi \in M^l_p(\Omega) \). Then for any \( L^2 \) holomorphic function \( f \) on \{ \varphi < -1 \}, there is an \( L^2 \) holomorphic function \( \tilde{f} \) on \( \Omega \) such that

\[
\frac{\partial^\alpha \tilde{f} / \partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}(p)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}(p)} = \frac{\partial^\alpha f / \partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}(p)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}(p)}, \quad \int_{\Omega} |\tilde{f}|^2 dV \leq C \int_{\{ \varphi < -1 \}} |f|^2 dV,
\]

for all multi-indexes \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( |\alpha| \leq l \). Here \( C \) is a constant depending only on \( l \).
Proof. Take a cut-off function $\chi : \mathbb{R} \to [0, 1]$ such that $\chi\lfloor_{(-\infty,-2\log 2)} = 1$ and $\chi\lfloor_{[-\log 2, +\infty)} = 0$. Set $\psi = -\log(-\varphi + 1)$. By Donnelly-Fefferman type $L^2$ estimate (see eg. [16]), we can solve the equation $\bar{\partial}u = f\bar{\partial}\chi(\psi)$ in the weak sense together with the estimate

$$\int_{\Omega} |u|^2 e^{-\varphi} dV \leq \int_{\Omega} |\bar{\partial}\chi(\psi)|^2 \bar{\partial}\bar{\partial}\chi(\psi) |f|^2 e^{-\varphi} dV \leq C'_2 \int_{\{\varphi < -1\}} |f|^2 dV$$

because $\sqrt{-1}\bar{\partial}\bar{\partial}\psi \geq \sqrt{-1}\bar{\partial}\psi \bar{\partial}\psi$. Here $| \cdot |_{\sqrt{-1}\bar{\partial}\bar{\partial}\psi}$ denotes the pointwise norm with respect to the (singular) metric $\bar{\partial}\bar{\partial}\psi$. Here $\delta^\bullet$ denotes the pointwise norm with respect to the (singular) metric $\bar{\partial}\bar{\partial}\psi$ and $C'_2$ depends only on $n$, $l$ and the choice of $\chi$. Set $\tilde{f} = \chi(\psi) f - u$. This $\tilde{f}$ has the desired properties because $u$ is holomorphic in certain neighborhood of $p$.

### 2.3 Pluricomplex Green function

Given a domain $\Omega$ in $\mathbb{C}^n$. The pluricomplex Green function with pole at $p \in \Omega$ is defined by

$$g_\Omega(z, p) = \sup\{u(z) : u \leq 0, u \in PSH(\Omega), u(z) = \log |z - p| + O(1) \text{ near } p\}.$$  

One basic fact is that the pluricomplex Green function is decreasing under holomorphic maps.

**Proposition 2.** Let $h$ be a holomorphic map from a domain $\Omega \subset \mathbb{C}^n$ to the unit disc $\Delta$. Then

$$\{g_\Omega(\cdot, p) < -1\} \subset \left\{1 - \frac{|h(p)|}{8} \leq 1 - |h| \leq 8(1 - |h(p)|)\right\}.$$  

**Proof.** Observe that

$$-g_\Omega(z, p) \leq -g_\Delta(h(z), h(p)) \leq -\log \frac{|h(z) - h(p)|}{|1 - h(p)|}$$

$$= \frac{1}{2} \log \left(1 + \frac{(1 - |h(z)|^2)(1 - |h(p)|^2)}{|h(z) - h(p)|^2}\right)$$

$$\leq \frac{(1 - |h(z)|^2)(1 - |h(p)|^2)}{2|h(z) - h(p)|^2} \leq 2 \frac{(1 - |h(z)|)(1 - |h(p)|)}{|h(z) - h(p)|^2}. $$

When $1 - |h(z)| \geq 2(1 - |h(p)|)$,

$$|h(z) - h(p)| \geq 1 - |h(z)| - (1 - |h(p)|) \geq \frac{1}{2}(1 - |h(z)|). $$

Thus

$$-g_\Omega(z, p) \leq 8 \frac{1 - |h(p)|}{1 - |h(z)|}, $$

hence $\{g_\Omega(\cdot, p) < -1\} \subset \{1 - |h| \leq 8(1 - |h(p)|)\}$. A similar argument implies

$$\{g_\Omega(\cdot, p) < -1\} \subset \left\{1 - |h| \geq \frac{1 - |h(p)|}{8}\right\}.$$ 

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2.4 Resolution of singularity

The following fundamental theorem will play an essential role in our proofs.

**Hironaka’s Theorem**[^17]. Let $M$ be a complex manifold and $\mathcal{I} \subset \mathcal{O}_M$ be a coherent ideal sheaf. Then there is a log canonical resolution of $\mathcal{I}$, i.e., there exists a proper bimeromorphic morphism $\mu$ from a complex manifold $\tilde{M}$ to $M$ such that $\mu^*\mathcal{I} = \mathcal{O}_{\tilde{M}}(-D)$ is an invertible sheaf associated to a divisor $D$ with $D + E$ being simple normal crossings. Here $E$ denotes the exceptional divisor of $\mu$.

Let $F : \mathbb{C}^n \to \mathbb{C}^m$ be a holomorphic map with $F(0) = 0$, $\mu : \tilde{M} \to \mathbb{C}^n$ a log canonical resolution of $\mathcal{I} = F^{-1}(0)$. Let $U$ be a sufficiently small open neighborhood of $0$. By the Jacobian formula for a change of variable, we have

$$\int_U |F(\zeta)|^{-2c} dV_\zeta = \int_{\mu^{-1}(U)} |F \circ \mu(z)|^{-2c} |J_\mu(z)|^2 \mathrm{d}\tilde{V}_z,$$

where $dV_\zeta, \mathrm{d}\tilde{V}_z$ are volume elements of $\mathbb{C}^n, \tilde{M}$ respectively. According to Hironaka’s theorem, this integral is given by a finite number of integrals of form

$$\int_{\mu^{-1}(U) \cap \tilde{U}} |F \circ \mu(z)|^{-2c} |J_\mu(z)|^2 \mathrm{d}\tilde{V}_z$$

over suitable coordinate charts $\tilde{U} \subset \tilde{M}$ on which one has

$$|F \circ \mu(z)| \asymp \prod_{j=1}^n |z_j|^{a_j}, \quad |J_\mu(z)| \asymp \prod_{j=1}^n |z_j|^{b_j},$$

for certain non-negative integers $a_j, b_j$. Since $|F \circ \mu|^{-2c} |J_\mu|^2$ is $L^1$ on $\mu^{-1}(U) \cap \tilde{U}$ if and only if $c < (b_j + 1)/a_j$ for all $1 \leq j \leq n$, it follows that $c_0(|F|) = \min\{(b_j + 1)/a_j\}$, where the minimum is taken over all $a_j, b_j$ associated to those $\tilde{U}$. In particular, $c_0(|F|)$ is a rational number. Analogously, one can compute $c_0(|F|; |\zeta_k|^\tau)$. We assume $k = 1$ and write

$$\mu = (\mu_1, \ldots, \mu_n), \quad |\mu_1(z)| \asymp \prod_{j=1}^n |z_j|^{c_j},$$

for certain non-negative integers $c_j$ in above $\tilde{U}$. Then

$$\int_U \frac{|\zeta_1|^{2\tau}}{|F(\zeta)|^{2c}} dV_\zeta = \int_{\mu^{-1}(U)} \frac{|\mu_1(z)|^{2\tau} |J_\mu(z)|^2}{|F \circ \mu(z)|^{2c}} \mathrm{d}\tilde{V}_z,$$

therefore, $c_0(|F|; |\zeta_1|^\tau) = \min\{(b_j + c_j\tau + 1)/a_j\}$.
3 Proof of the theorems

Without loss of generality, we assume \( U = U' \times U'' \) with Stein open sets \( U' \subset \mathbb{C}^n \) and \( U'' \subset \mathbb{C} \). Given \( w \in \mathbb{C} \), we set

\[
{\Omega}_w^1 = \{ z \in U' : |F(z)|^2 < 9 \text{Im } w \}, \quad {\Omega}_w^2 = \{ z \in U' : |F(z)|^2 < \frac{1}{9} \text{Im } w \},
\]

\[
S_w = \left\{ \zeta \in \mathbb{C} : \frac{1}{9} \text{Im } w < \text{Im } \zeta < 9 \text{Im } w \right\}.
\]

**Lemma 1.** There exists a universal constant \( C > 0 \) such that for any \( p = (z, w) \in (\Omega_w^2 \times S_w) \cap U \),

\[
I_{\Omega_w^2 \times S_w}^0 (p) \leq I_{\Omega_{F \cap U}}^0 (p) \leq C I_{\Omega_w^2 \times S_w}^0 (p), \quad (1)
\]

\[
I_{\Omega_w^2 \times S_w}^j (p; X) \leq I_{\Omega_{F \cap U}}^j (p; X) \leq C I_{\Omega_w^2 \times S_w}^j (p; X), \quad j = 1, 2. \quad (2)
\]

**Proof.** The first inequalities in (1) and (2) follow directly from the definitions of the minimum integrals. Choosing \( h(z, \zeta) = e^{i\zeta/\text{Im } w} \) for sufficiently small \( \epsilon \) and applying Proposition 2, we obtain

\[
\{ g_{\Omega_{F \cap U}}(\cdot, p) < -1 \} \subset \Omega_w^1 \times S_w.
\]

Since \( (2n+1)g_{\Omega_{F \cap U}}(\cdot, p) \in M_p^0 (\Omega_{F \cap U}) \), the second inequalities in (1) and (2) follow from Proposition 1 and the definitions of the minimum integrals. Note that \( \Phi(\zeta) = e^{i\zeta/\text{Im } w} \) maps the strip \( S_w \) biholomorphically to the ring \( \{ e^{-9} < \text{Im } \zeta < e^{-1/9} \} \) with \( |\Phi(w)| = e^{-1} \), thus

\[
I_{S_w}^0 (w) \asymp (\text{Im } w)^2, \quad I_{S_w}^1 (w; X_{n+1}) \asymp \frac{(\text{Im } w)^2}{|X_{n+1}|^2}, \quad I_{S_w}^2 (w; X_{n+1}) \asymp \frac{(\text{Im } w)^2}{|X_{n+1}|^4},
\]

consequently,

\[
I_{\Omega_w^2 \times S_w}^0 (p) \asymp (\text{Im } w)^2 I_{\Omega_w^2}^0 (z), \quad I_{\Omega_w^2 \times S_w}^j (p; (X', 0)) \asymp (\text{Im } w)^2 I_{\Omega_w}^j (z; X'), \quad (3)
\]

\[
I_{\Omega_w^2 \times S_w}^1 (p; (0, X_{n+1})) \asymp \frac{(\text{Im } w)^2}{|X_{n+1}|^2} I_{\Omega_w}^0 (z), \quad (4)
\]

for \( j, k = 1, 2 \).

**Remark 3.** Actually, the above conclusions still hold if one replaces \( |F|^2 \) in \( \Omega_F \) by any non-negative psh function.

**Lemma 2.** For \( \tau = 0, 1, 2 \), there exist integers \( 1 \leq m_\tau \leq n \) such that

\[
\int_{\Omega_w} |\zeta_1|^{2\tau} dV_\zeta \asymp (\text{Im } w)^{\epsilon_0 (|F| |\zeta|^{\tau})} |\log (\text{Im } w)|^{m_\tau - 1}, \quad j = 1, 2.
\]

Moreover, \( m_\tau \) can be computed through a log canonical resolution.
Proof. Let $\mu : \tilde{M} \to \mathbb{C}^n$ be a log canonical resolution of $F^{-1}(0)$. Fix a small neighborhood $U$ of $0 \in \mathbb{C}^n$. Similar to Subsection 2.4, the integral $\int_{\{\zeta \in U : |F(\zeta)| < t\}} |\zeta|^2 dV_\zeta$ is then determined by integrals of form

$$\int_{\mu^{-1}(U) \cap \{z \in \tilde{U} : |z|^n < t\}} \prod_{j=1}^n |z_j|^{2b_j + 2c_j \tau} d\tilde{V}_z$$

over finite suitable coordinate charts $\tilde{U} \subset \tilde{M}$. Using polar coordinates, it suffices to estimate the integrals

$$\tilde{I}(t) = \int_{\{y \in \mathbb{R}^n : \tau y \leq t\}} y_1^{2b_1 + 2c_1 \tau} \cdots y_n^{2b_n + 2c_n \tau} dy_1 \land \cdots \land dy_n,$$

where $I = [0, 1]$. Set $h(y) = y_1^{a_1} \cdots y_n^{a_n}$. We consider the following two useful integrals:

$$H(s) = \int_{\mathbb{R}^n} h(y)^{-s} y_1^{2b_1 + 2c_1 \tau} \cdots y_n^{2b_n + 2c_n \tau} dy_1 \land \cdots \land dy_n,$$

$$G(t) = \int_{\{y \in \mathbb{R}^n : h(y) = t\}} y_1^{2b_1 + 2c_1 \tau} \cdots y_n^{2b_n + 2c_n \tau} dy_1 \land \cdots \land dy_n/dh.$$

A direct computation shows

$$H(s) = C_s \prod_{j=1}^n [(b_j + c_j \tau + 1)/a_j - s]^{-1} = C_s \prod_{j \in J} [(b_j + c_j \tau + 1)/a_j - s]^{-l_j}$$

provided $s < (b_j + c_j \tau + 1)/a_j$, $\forall 1 \leq j \leq n$. Here $C_s \in (0, \infty)$ and $l_j$ is the multiplicity of $H(s)$ at the pole $s = (b_j + c_j \tau + 1)/a_j$. As

$$dy_1 \land \cdots \land dy_n = (a_1 h)^{-1} y_1 dh \land dy_2 \land \cdots \land dy_n,$$

it follows that

$$G(t) = \frac{1}{a_1} t^{\frac{1}{a_1} - 1} \int_{\mathbb{R}^n} y_2^{2b_2 + 2c_2 \tau} \cdots y_n^{2b_n + 2c_n \tau} dy_2 \land \cdots \land dy_n.$$

Therefore, $G(t)$ must be of form

$$\sum_{j=1}^{j_0} \sum_{k=1}^{k_j \leq n} c_{j,k} t^{l_j - 1} (-\log t)^{k-1}$$

for sufficiently small $t$. Noting that $H(s) = \int_0^\infty t^{-s} G(t) dt$, by comparing the multiplicities of poles of both sides, we obtain

$$G(t) \sim t^{\beta - 1} (-\log t)^{\alpha - 1}, \quad t \to 0_+,$$

where

$$\beta = \min_{1 \leq j \leq n} \{(b_j + c_j \tau + 1)/a_j\}, \quad \alpha = \max_{(b_j + c_j \tau + 1)/a_j = \beta, j \in J} l_j.$$
It follows that
\[ \tilde{I}(t) = \int_0^t G(t') dt' \sim t^\beta (-\log t)^{\alpha-1}, \quad t \to 0_+. \]

Since by Subsection 2.4, \( c_0(|F|; |\zeta|^\tau) = \min\{\beta\} \), the lemma is verified with \( l = \max\{\alpha\} \), where the minimum is taken over all \( \widetilde{U} \) and the maximum is taken over those \( \widetilde{U} \) such that the associated \( \beta \) is equal to \( c_0(|F|; |\zeta|^\tau) \).

**Remark 4.** By the definitions, the minimum integrals satisfy
\[
I_{\Omega_w}^0(z) \leq \int_{\Omega_w^1} dV_\zeta, \quad I_{\Omega_w}^1(z; (X_1, 0, \ldots, 0)) \leq |X_1|^{-2} \int_{\Omega_w^1} |\zeta - z_1|^2 dV_\zeta, \quad (6)
\]
\[
I_{\Omega_w}^2(z; (X_1, 0, \ldots, 0)) \leq |X_1|^{-2} \int_{\Omega_w^1} |\zeta - z_1|^4 dV_\zeta. \quad (7)
\]

**Lemma 3.** Let \( m, \tau = 0, 1, 2 \) be as in Lemma 2. Then
\[
I_{\Omega_w}^0(0) \geq C \int_{\Omega_w^1} dV_\zeta, \quad I_{\Omega_w}^1(0; X_1) \geq C|X_1|^{-2} \int_{\Omega_w^1} |\zeta|^2 dV_\zeta, \quad (8)
\]
\[
I_{\Omega_w}^2(0; X_1) \geq C|X_1|^{-4} \int_{\Omega_w^1} |\zeta|^4 dV_\zeta. \quad (9)
\]

**Proof.** Let \( \mu \) be as above. For any \( f \in H^2(\Omega_w^2) \) with \( f(0) = 1 \), we have
\[
\int_{\Omega_w^2} |f(\zeta)|^2 dV_\zeta = \int_{\mu^{-1}(\Omega_w^2)} |f \circ \mu(z)|^2 |J_\mu(z)|^2 d\tilde{V}_z,
\]
which is then given by integrals of form
\[
\int_{\mu^{-1}(U) \cap \{\zeta \in \tilde{U} : \prod_{j=1}^n |z_j|^{\tau_j} < t\}} |f \circ \mu(z)|^2 \prod_{j=1}^n |z_j|^{2\tau_j} d\tilde{V}_z
\]
over finite coordinate charts \( \tilde{U} \subset \tilde{M} \), where \( t = \text{Im} w \). Without loss of generality, we assume that \( \tilde{U} \) contains the unit polydisc \( \Delta^n \). As \( f \circ \mu \) is holomorphic on the Reinhardt domain \( \{z \in \Delta^n : \prod_{j=1}^n |z_j|^\gamma_j < t\} \), it has an expansion
\[
f \circ \mu(z) = 1 + \sum_{\gamma \neq 0} c_\gamma z^\gamma, \quad \gamma = (\gamma_1, \ldots, \gamma_n),
\]
because \( f \circ \mu(0) = 1 \). Therefore,
\[
\int_{\{z \in \Delta^n : \prod_{j=1}^n |z_j|^\gamma_j < t\}} |f \circ \mu(z)|^2 \prod_{j=1}^n |z_j|^{2\tau_j} d\tilde{V}_z
\]
\[= \int_{\{z \in \Delta^n : \prod_{j=1}^n |z_j|^\gamma_j < t\}} \prod_{j=1}^n |z_j|^{2\tau_j} d\tilde{V}_z + \sum_{\gamma \neq 0} |c_\gamma|^2 \int_{\{z \in \Delta^n : \prod_{j=1}^n |z_j|^\gamma_j < t\}} \prod_{j=1}^n |z_j|^{2\tau_j + 2\gamma_j} d\tilde{V}_z,
\]
consequently, \( \|f\|^2_{\Omega_w^2} \) dominates all integrals of form
\[
\int_{\{z \in \Delta^n : \prod_{j=1}^n |z_j|^\gamma_j < t\}} \prod_{j=1}^n |z_j|^{2\tau_j} d\tilde{V}_z,
\]

\[9\]
hence the integral \( \int_{\Omega^2_w} dV_\zeta \). Next, let \( g \) be a given \( L^2 \) holomorphic function on \( \Omega^2_w \) satisfying \( g(0) = 0 \) and \( X_1 \partial g/\partial \zeta_1(0) = 1 \). Then

\[
g \circ \mu(z) = \sum_{\gamma \neq 0} \delta_{\gamma} z^\gamma, \quad \text{for } z \in \Delta^n, \quad \prod_{j=1}^n |z_j|^{a_j} < t,
\]

because \( g \circ \mu(0) = 0 \). Since \( \mu \) is locally biholomorphic on \( \tilde{M} - D - E \) where \( \mu^* \circ F^{-1}(0) = O_M(-D) \) and \( E \) is the exceptional divisor of \( \mu \), it follows that for any \( k \geq 2 \),

\[
|\mu_k(z)| \asymp |z_1|^{\gamma_1} \cdots |z_n|^{\gamma_n} \quad \text{with } (\gamma_1, \ldots, \gamma_n) \neq (c_1, \ldots, c_n),
\]

consequently,

\[
\frac{\partial |c_1 + \cdots + c_n| \mu_k}{\partial z_1^{c_1} \cdots \partial z_n^{c_n}} \bigg|_{z=0} = 0, \quad k \geq 2,
\]

which implies

\[
\delta_{c_1 - c_n} = \frac{\partial |c_1 + \cdots + c_n| g \circ \mu}{\partial z_1^{c_1} \cdots \partial z_n^{c_n}} \bigg|_{z=0} = \frac{\partial g}{\partial \zeta_1}(0) = \frac{1}{X_1}.
\]

Thus \( \|g\|_{\Omega^2_w}^2 \) must dominate the \( |X_1|^{-2} \) multiple of the sum of integrals

\[
\int_{\{z \in \Delta^n : \prod_{j=1}^n |z_j|^{a_j} < \epsilon\}} \prod_{j=1}^n |z_j|^{2b_j + 2c_j} dV_z
\]

over those coordinate charts \( \tilde{U} \subset \tilde{M} \), which is equivalent to \( \int_{\Omega^2_w} |\zeta_1|^2 dV_\zeta \). Inequality (9) can be verified similarly.

**Proof of Theorems 1, 2.** First we assume \( p_0 = 0 \). The theorems follow directly from Subsection 2.1 and equations (1)–(9). The general case follows from the transformation laws of the minimum integrals by noting that the transformation \( \Phi \) maps \( \Omega_F \) biholomorphically to the domain

\[
\Omega'_F = \{(z', w') \in \mathbb{C}^{n+1} : \text{Im } w' > |F(z_0 + z') - F(z_0)|^2\},
\]

where

\[
z' = z - z_0, \quad w' = w - w_0 - 2i \sum_{k=1}^m \frac{f_k(z_0)(f_k(z) - f_k(z_0))}{f_k(z_0)}.
\]

**4 Newton polyhedron and holomorphic sectional curvature**

There is an effective way to compute the complex singularity exponent in terms of the Newton polyhedron by using toric resolution of singularity. Given a power series \( f(z) = \sum_{\alpha} c_\alpha z^\alpha \) over \( \mathbb{C}^n \) with \( f(0) = 0 \). The Newton polyhedron \( \Gamma(f) \) of \( f \) is the convex hull of the set \( \bigcup(\gamma + \mathbb{R}_+^n) \), where \( \mathbb{R}_+^n \) is the positive octant and the union is taken over multi-indexes \( \gamma \) such that \( c_\gamma \neq 0 \). We associate every compact face \( \Delta \) of \( \Gamma(f) \) with a polynomial \( f_\Delta(z) = \sum_{\gamma \in \Delta} c_\gamma z^\gamma \). We say that \( f \) is non-degenerate on \( \Delta \) if...
\[ df_\Delta = 0 \] has no solution in \((\mathbb{C}^*)^n\) where \(\mathbb{C}^* = \mathbb{C} - \{0\}\). We say that \(f\) is non-degenerate if \(df_\Delta\) is non-degenerate for any \(\Delta\). Assume that the line \(\{(1+\tau)t, t, \ldots, t\}\) intersects the boundary of \(\Gamma(f)\) at point \(Q_\tau = ((1+\tau)d_\tau, d_\tau, \ldots, d_\tau)\). Let \(\hat{m}_\tau\) denote the number of compact faces of \(\Gamma(f)\) containing the point \(Q_\tau\). Then the following fact is well known (see eg. [18]).

**Proposition 3.** Assume \(f\) is a non-degenerate entire function in \(\mathbb{C}^n\) with \(d_0 > 1\). Then

\[
\begin{align*}
\kappa_0(|f|) &= 1/d_0, \quad \kappa_0(|f|, |z|) = 1/d_\tau, \quad \kappa = 1, 2; \\
\hat{m}_\tau &= \min\{\hat{m}_\tau, n\}, \quad \kappa = 0, 1, 2. \tag{11}
\end{align*}
\]

As Subsection 2.1 shows, the holomorphic sectional curvature is always bounded above by 2. It is natural to ask whether this upper bound is optimal. It is also interesting to ask whether the holomorphic sectional curvature is bounded below. For the special case \(\Omega_f = \{(z, w) \in \mathbb{C}^{n+1} : \text{Im} \ w > |f(z)|^2\}\) with \(f\) being a non-degenerate entire function, we have the following self-contained characterization.

**Proposition 4.** Let \(f\) be as in Proposition 3, \(U\) be a bounded Stein neighborhood of \(0 \in \mathbb{C}^{n+1}\).

(i) If \(Q_\tau, \kappa = 0, 1, 2\) are not contained in the same supporting hyperplane of \(\Gamma(f)\), then \(R_{\Omega_f \cap U}(p; (X_1, 0, \ldots, 0)) \to 2\) as \(p \to 0\) non-tangentially.

(ii) Otherwise, \(R_{\Omega_f \cap U}(p; (X_1, 0, \ldots, 0))\) is bounded below by a constant in a non-tangent cone \(\Lambda\) at \(0\).

**Example.** Take \(f(z_1, z_2) = z_1^4 + z_1^2 z_2 + z_1 z_2^2 + z_2^4\). It is easy to verify that \(Q_\tau, \kappa = 0, 1, 2\) are not contained in any hyperplane supporting the Newton polyhedron. Proof. Suppose that \(H\) is a hyperplane which supports \(\Gamma(f)\) at \(Q_1\), given by the equation \(\sum_{k=1}^n x_k/a_k = 1\) where \(a_k > 0\). Since \(\Gamma(f)\) is convex,

\[
\frac{1}{d_0} \leq \sum_{k=1}^n \frac{1}{a_k}, \quad \frac{1}{d_2} \leq \sum_{k=2}^n \frac{1}{a_k} + \frac{3}{a_1},
\]

hence

\[
\frac{1}{d_2} + \frac{1}{d_0} \leq 2\left(\sum_{k=2}^n \frac{1}{a_k} + \frac{2}{a_1}\right) = \frac{2}{a_1}.
\]

It follows that the equality holds if and only if \(Q_\tau, \kappa = 0, 1, 2\) are all contained in \(H\). Combining Theorem 2 with Proposition 3, (i) is verified. On the other hand, if \(Q_\tau, \kappa = 0, 1, 2\) are all contained in a supporting hyperplane supporting \(\Gamma(f)\), then any compact face of \(\Gamma(f)\) containing \(Q_1\) must contain \(Q_0, Q_2\), consequently, \(2m_1 \leq m_2 + m_0\) and (ii) follows from Theorem 2.

## 5 Proofs of Theorems 3,4

*Proof of Theorem 3.* We keep the notions as above. Without loss of generality, we assume \(U = U' \times U''\) with Stein open sets \(U' \subset \mathbb{C}^n\), and \(U'' \subset \mathbb{C}\). Given \(w \in U''\), we
set
\[ \Omega^1_w = \{ z \in U' : \rho(z) < 9 \text{Im } w \} \]
\[ \Omega^2_w = \{ z \in U' : \rho(z) < \frac{1}{9} \text{Im } w \} \]
\[ S_w = \{ \zeta \in \mathbb{C} : \frac{1}{9} \text{Im } w < \text{Im } \zeta < 9 \text{Im } w \} . \]

Without any change of the above argument, we can prove the following
\[ CK_{\Omega^1_w \times S_w}((0, w)) \leq K_{\Omega^2_w \times S_w}((0, w)), \quad w \in U'' \]
for suitable constant \( C > 0 \). It is easy to see
\[ K_{\Omega^1_w \times S_w}((0, w)) \simeq (\text{Im } w)^{-2} K_{\Omega^1_w} (0), \quad j = 1, 2. \quad (13) \]
Hence it suffices to estimate \( K_{D_r}(0) \) in terms of certain power of \( r \) where
\[ D_r := \{ z \in U' : \rho(z) < r \}, \quad r \ll 1. \]
First we have the trivial inequality
\[ K_{D_r}(0) \geq \frac{1}{\text{Vol } (D_r)}. \quad (14) \]
Since \( \varphi := \log \rho \) is psh, we infer from Proposition 4.3 (1) in [12] that for all positive real number \( c < c_0(\rho) \) there is an estimate
\[ \text{Vol } (D_r) \leq C(c) r^{2c} \quad (15) \]
(Here we remark that the authors of [12] use the notion \( c_0(\varphi) \) for \( c_0(e^{\varphi}) \) when \( \varphi \) is psh).

The first inequality in Theorem 3 is then an consequence of (13)–(15). For the second inequality, we shall use the celebrated Demailly’s approximation: there is a constant \( C > 0 \) independent of \( m \) and \( \varphi \) such that
\[ \varphi(z) - \frac{C}{m} \leq \psi_m(z) := \frac{1}{2m} \log \sum_{k} |g_{m,k}(z)|^2 \]
where \( \{g_{m,k}\} \) is an orthonormal basis of \( H_{m\varphi}(U') \), the Hilbert space of holomorphic functions \( f \) on \( U' \) such that
\[ \int_{U'} |f|^2 e^{-2m\varphi} dV < \infty. \]
It follows from the strong Noetherian property that there exists an integer \( k_0(m) \) and a constant \( C_{m,1} > 0 \) such that
\[ \psi_m - C_{m,1} \leq \psi_{m,0} := \frac{1}{2m} \log \sum_{0 \leq k \leq k_0(m)} |g_{m,k}|^2 \leq \psi_m \quad \text{on } U'. \]
Thus
\[ D_r = \{ z \in U' : \varphi(z) < \log r \} \supset \{ z \in U' : \psi_{m,0}(z) < \log r - C_{m,2} \} := D_{r,m} \]
for some constant \( C_{m,2} > 0 \). Note that
\[ K_{D_r}(0) \leq K_{D_{r,m}}(0). \tag{16} \]
By Theorem 1,
\[ K_{D_{r,m}}(0) \leq C_{m,3} r^{-2c_0(e^{\psi_{m,0}})}|\log r|^{1-l_m} \tag{17} \]
where \( 1 \leq l_m \leq n \) is certain integer coming from the resolution of the singularity of \( \psi_{m,0} \). As
\[ c_0(e^{\psi_{m,0}}) = c_0(e^{\psi_m}) \to c_0(e^{\varphi}) = c_0(\rho) \]
by Theorem 4.2 (3) in [12], the second inequality follows from (13), (16) and (17). The proof is complete.

**Proof of Theorem 4.** We fix a sufficiently small Stein neighborhood \( U = U' \times U'' \) of \( 0 \in \mathbb{C}^{n+1} \). Fix arbitrary \((z_0, w_0) \in \Omega_F \cap U\) such that \( \text{Im } w_0 - |F(z_0)|^2 = r/9 \) and \( |z_0| \leq r \). Take a holomorphic transformation \( \Phi \) as (10), we have
\[ K_{\Omega_F \cap U}((z_0, w_0)) = K_{\Omega_F \cap \Phi(U)}(\Phi(z_0, w_0)) \geq C \frac{1}{\text{Vol}(D(U', F, z_0, r))} \]
where the second inequality follows from Lemma 1 with \( C \) a universal constant.

Now \((z_0, w_0)\) lies in a non-tangential cone with vertex at the origin, we have
\[ K_{\Omega_F \cap U}((z_0, w_0)) \asymp r^{-c_0(|F|)}|\log r|^{1-l} \]
by Theorem 1. Thus we get the first inequality in Theorem 4.

For the second inequality, we use Lemma 3.2 (2) in [3] that for any \( c < c_0(|F|) \), there exists a neighborhood \( U'_c \) of 0 such that
\[ \int_{U'} |F(z) - F(z_0)|^{-c} dV(z) \leq C(c), \quad z_0 \in U'_c \]
(Shrinking \( U' \) if necessary). Since
\[ \int_{U'} |F(z) - F(z_0)|^{-c} dV(z) \geq \frac{1}{r^c} \text{Vol}(D(U', F, z_0, r)) \]
for \( r \ll 1 \), we are done.

### 6 Remarks and questions

**Remark 5.** Generally, the conclusion of Theorem 1 fails for domains \( \Omega_\rho = \{(z, w) \in \mathbb{C}^{n+1} : \text{Im } w > \rho(z)\} \) when \( \rho \) is a non-negative real-analytic psh function in \( \mathbb{C}^n \). Consider a power series \( \rho = \sum c_\gamma x^{2\gamma_1}y^{2\gamma_2} \) in \( \mathbb{R}^2 \) with all \( c_\gamma \geq 0 \). Write \( z = x + iy \). Then \( \rho \)
is a non-negative subharmonic function. Assume that the Newton polyhedron $\Gamma(\rho)$ of $\rho$ (over $\mathbb{R}$) intersects the $x, y$ axes. Then $c_0(\rho) = 1/d_0$ where $d_0$ denotes the distance to $\Gamma(\rho)$. Set $\delta = \inf\{ |\gamma| : c_\gamma > 0 \}$. Note that the domains

$$D_t = \{ z \in \mathbb{C} : \rho(z) < t \}, \quad D'_t = \{ \zeta = (\xi, \eta) \in \mathbb{C} : \rho(t^{\frac{1}{\delta}} \zeta) < t \}$$

are biholomorphically equivalent, therefore $K_{D_t}(0) = t^{-\frac{1}{\delta}} K_{D'_t}(0) \asymp t^{-\frac{1}{\delta}}$, because $D'_t$ is pinched between two planar domains

$$\left\{ \zeta \in \mathbb{C} : \sum_{c_\gamma > 0, |\gamma| = \delta} \xi^{2\gamma_1} \eta^{2\gamma_2} < \epsilon, |\zeta| < \epsilon \right\}, \quad \left\{ \zeta \in \mathbb{C} : \sum_{c_\gamma > 0, |\gamma| = \delta} \xi^{2\gamma_1} \eta^{2\gamma_2} < \frac{1}{\epsilon} \right\}$$

for some $0 < \epsilon \ll 1$ independent of $t$. By the remarks under Theorem 1 and Lemma 1, we have

$$K_{\Omega_p}((0, w)) \asymp (\text{Im} w)^{-2 - \frac{1}{\delta}}.$$  

Nevertheless, $c_0(\rho) \neq 1/\delta$ (i.e., $d_0 \neq \delta$) in general, for instance, one can take $\rho(z) = x^4 + x^3 y^2 + x^2 y^6 + y^{10}$, then $d_0 = 10/3 > 3 = \delta$.

**Remark 6.** Given a point $p_0 \in \mathbb{C}^n$. Let $\mathcal{S}$ denote the space of all bounded $C^2$ pseudoconvex domains in $\mathbb{C}^n$ whose boundary contains $p_0$. For any $\Omega \in \mathcal{S}$, we define the growth exponent of the Bergman kernel of $\Omega$ at $p_0$ by

$$b_{p_0}(\Omega) = \sup \left\{ b \geq 0 : \lim_{p \to p_0, p \in \Lambda} \delta_{\Omega}(p)^b K_{\Omega}(p) = \infty \right\},$$

where $\Lambda$ is some non-tangent cone at $p_0$ and $\delta_{\Omega}$ denotes the boundary distance function. Clearly, $b_{p_0}$ defines a map from $\mathcal{S}$ to $[2, n + 1]$. Note that for those domains considered in Theorem 1, the values of $b_{p_0}$ are always rational numbers. Thus it is natural to ask

**Question 1.** Is the image of $b_{p_0}$ dense in $[2, n + 1]$? Is $b_{p_0}$ surjective?

**Remark 7.** We can’t get global uniform estimates of the Bergman invariants as in the case of strongly pseudoconvex domains or finite type domains in $\mathbb{C}^2$. The difficulty is that we do not know how the log canonical resolution of the ideal sheaf $\{ z \in \mathbb{C}^n : F(z) = F(z_0) \}$ depends on the parameter $z_0$. On the other hand, the parameter dependence of the complex singularity exponent is clear from the work of Demailly-Kollár (see also [19] for weaker results).

**Proposition 5**[12]. Let $M$ be a complex manifold. Let $\mathcal{P}(M)$ be the set of locally $L^1$ psh functions on $M$, equipped with the topology of $L^1$ convergence on compact subsets. Let $p \in M$ and $\varphi \in \mathcal{P}(M)$ be given. If $c < c_\rho(e^{-\varphi})$ and $\psi$ converges to $\varphi$ in $\mathcal{P}(M)$, then $e^{-c\psi}$ converges to $e^{-c\varphi}$ in $L^2$ norm over some neighborhood $V$ of $p$.

Theorem 1 shows $b_{p_0}(\Omega_F \cap U) = 2 + c_{z_0}(|F - F(z_0)|)$ for all $p_0 = (z_0, w_0) \in \partial \Omega_F \cap U$, while Proposition 5 implies that for any $c < c_0(|F|)$ there exists a neighborhood $V$ of $0 \in \mathbb{C}^n$ such that $|F - F(z_0)|^{-c}$ is $L^2$ on $V$ provided $|z_0|$ sufficiently small, consequently, $c_{z_0}(|F - F(z_0)|) \geq c$ and it follows that the map $p \to b_p(\Omega_F \cap U)$ is lower semi-continuous on $\partial \Omega_F \cap U$. 

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Question 2. Is the map $p \to b_p(\Omega)$ lower semi-continuous on $\partial \Omega$ for any bounded $C^2$ pseudoconvex domain in $\mathbb{C}^n$?

Remark 8. We do not know whether there exists a bounded $C^2$ pseudoconvex domain such that the holomorphic sectional curvature of the Bergman metric is unbounded.

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