FORWARDS DYNAMICS OF NON-AUTONOMOUS DYNAMICAL SYSTEMS: DRIVING SEMIGROUPS WITHOUT BACKWARDS UNIQUENESS AND STRUCTURE OF THE ATTRACTOR

Dedicated to professor Tomás Caraballo on the occasion of his 60th birthday

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Abstract. We investigate the forwards asymptotic dynamics of non-autonomous differential equations. Our approach is centred on those models for which the vector field is only defined for non-negative times, that is, the laws of evolution are not given, or simply not known, for times before a given time (say time $t = 0$). We will be interested in the cases for which the ‘driving’ (time shift) semigroup has a global attractor in which backwards solutions are not necessarily unique. Considering vector fields in the global attractor of the driving semigroup allows for a natural way to extend vector fields, defined only for non-negative times, to the whole real line. These objects play a crucial role in the description of the asymptotic dynamics of our non-autonomous differential equation. We will study, in some particular cases, the isolated invariant sets of the associated skew-product semigroup with the aim of characterising the global attractor. We develop an example for which we derive decomposition for the global attractor of skew-product semigroup from the characterisation of the attractor of the associated driving semigroup.

1. Introduction. The qualitative properties of evolution processes (non-autonomous dynamical systems) in general phase spaces (infinite-dimensional Banach spaces or general metric spaces) has received a lot of attention in recent years (see, for instance, [11, 5, 8, 7, 6, 17, 16]). The study of different notions of attractors (pullback, cocycle, uniform) has formed a wide and deep research area, providing qualitative information for the asymptotics of an increasing number of non-autonomous models of phenomena from different areas of science. The different concepts of ‘attractors’ have proved important for different reasons in the quest for understanding the asymptotics of solutions of a given non-autonomous problem.

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Uniform attractors have the property of attraction whereas pullback attractors have the property of invariance. Both notions are fundamental for the understanding of the asymptotic behavior of a given problem. Pullback attractors require that the vector fields are defined for all times \( t \in \mathbb{R} \) (which is not the case in many models), requiring that the past history of the model is also known. Uniform attractors, in general, do not require the vector fields to be defined for all times but their description as the union of a family of pullback attractors (to take advantage of invariance properties) does. This paper combines, in an appropriate way, the two notions to give the right framework in which the asymptotics of a given non-autonomous problem can be described. In particular our approach establishes, we believe, the correct setting in which to understand the forwards dynamics of non-autonomous differential equations.

To illustrate our approach, we consider a simple non-autonomous ordinary differential equation in \( \mathbb{R}^N \). Let \( f : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N \) be a family of vector fields (one for each \( t \in \mathbb{R}^+ \)). Consider the initial value problem

\[
\begin{cases}
\dot{u} = f(t, u), & t > s \\
u(s) = u_0 \in \mathbb{R}^N,
\end{cases}
\]

for each \( s \in \mathbb{R}^+ \) and \( u_0 \in \mathbb{R}^N \). Assume that \( f \) is such that solutions \( u(\cdot, s, u_0) \) of (1.1), defined in \( [s, \infty) \) for each \( s \in \mathbb{R}^+ \) and \( u_0 \in \mathbb{R}^N \), exist and are unique, and that the map \( \mathcal{P}_{\mathbb{R}^+} \times \mathbb{R}^N \ni (t, s, u_0) \mapsto u(t, s, u_0) \in \mathbb{R}^N \) is continuous, where \( \mathcal{P}_{\mathbb{R}^+} = \{(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+ : t \geq s\} \).

For each \( t \), \( f(t, \cdot) \) is the vector field that drives the solution at time \( t \). Hence, the path described by the solution in \( X \) between \( s \) and \( s + \tau \) will depend on the vector fields \( f(t, \cdot) \) for \( t \in [s, s + \tau] \) and so will depend on both the initial time \( s \) and elapsed time \( \tau \). When \( f(t, \cdot) = f(\cdot) \) is independent of \( t \), that is, when (1.1) is autonomous, the path described by the solution will no longer depend on \( s \) but only on \( \tau \).

The change from a time-independent \( f \) to a time-dependent \( f \) can easily be underestimated in the study of the asymptotics of (1.1). Hence, we insist on calling the reader’s attention to the fact that in the autonomous case there is only one vector field driving the solution at all times whereas in the non-autonomous case there are infinitely many vector fields driving the solution, one for each instant of time, and these vector fields evolve with time, in a prescribed way, just as the solution does (this is a crucial point here). Therefore, the evolution of the vector field (the ‘driving semigroup’) must be taken into account as well. Once one realizes this feature, it becomes clear how rich the “dynamics of non-autonomous dynamical systems” can be.

The basic question we would like to address is how to describe the long-time behaviour (when \( t \to +\infty \)) of solutions of (1.1).

When \( f \) is independent of time (and defined for all times), we have

\[ u(t, s, f, u_0) = u(t - s, 0, f, u_0) \]

and the asymptotics can be seen either by sending \( t \to \infty \) (seeing what happens to the state at the final time \( t \) when \( t \) is driven further and further to the future) or sending \( s \to -\infty \) (seeing what happens to the state at time \( t \) when the initial time \( s \) is driven further and further to the past).

Note that if \( f \) is time dependent and \( t \in \mathbb{R} \), these two approaches give rise to two distinct notions of asymptotic behavior, from which two completely different
scenarios may arise. Indeed, we may study the asymptotics with respect to the elapsed time \( t - s \) (when \( t - s \to +\infty \)) or with respect to \( s \) (when \( s \to -\infty \) and \( t \) is arbitrary but fixed). These are called, respectively, the forwards and pullback dynamics, and are in general unrelated. It is natural that they are unrelated as the set of vector fields driving the solution may be completely different in the two cases.

When \( f \) in (1.1) is given for all \( t \in \mathbb{R} \), we define the solution operator \( S(t, su_0) = u(t, s, u_0) \) for each \((t, s) \in \mathcal{P}_\mathbb{R} = \{(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+ : t \geq s \}\) and the family \( \{S(t, s) : t \geq s\} \subset C(\mathbb{R}^N) \) is an evolution process in \( \mathbb{R}^N \), that is, a family \( \{S(t, s) : t \geq s\} \subset C(\mathbb{R}^N) \) such that \( S(t, t)x = x \), for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^N \), \( S(t, s) = S(t, r)S(r, s) \), for all \( t \geq r \geq s \), and such that the map \( \mathcal{P}_\mathbb{R} \times \mathbb{R}^N \ni (t, s, x) \mapsto S(t, s)x \) is continuous. For an evolution process \( \{S(t, s) : t \geq s, t, s \in \mathbb{R}\} \subset C(\mathbb{R}^N) \), a bounded pullback attractor is a family \( \{A(t) : t \in \mathbb{R}\} \subset C(\mathbb{R}^N) \) such that: \( \bigcup_{t \in \mathbb{R}} A(t) \) is bounded, each \( A(t) \) is compact, \( S(t, s)A(s) = A(t) \) for all \( t \geq s \) and \( \text{dist}_H(S(t, s)B, A(t)) \to 0 \) as \( t \to -\infty \) for each bounded subset \( B \) of \( \mathbb{R}^N \) and for each \( t \in \mathbb{R} \). It is also important to note that \( \{A(t) : t \in \mathbb{R}\} \) may not have any forwards attraction properties (see, for instance, Section 1.3 in [5] or [13]). However, when \( f \) is independent of time, \( A(t) \) coincides with the global attractor for the associated semigroup. We emphasise the fact that, in the theory of pullback attractors, the vector fields must be defined for all times in the real line.

However, in many models the backwards history is simply not available and the vector fields are only known after a given fixed initial time, say, for each time in \( \mathbb{R}^+ \). When \( f(t, u) \) has \( t \in \mathbb{R}^+ \), the pullback attractor does not play a direct role in the asymptotics of (1.1). Nonetheless, as we will see later, it appears again in a natural and essential way, in the understanding of the asymptotic behaviour of solutions of (1.1) within an appropriate framework. If we artificially define the vector fields for negative times the pullback attractors will strongly depend on such extension since most of the pullback asymptotics will be driven by these vector fields artificially introduced.

**Remark 1.** At this point we wish to make an important remark which is concerned with the case when \( f \) is time independent and solutions are defined for \( t \in \mathbb{R}^+ \). In many situations, there is absolutely no reason to extend \( f \) by the same time-independent vector field \( f \), for negative values of time. Yet, we speak about global solutions for such autonomous problems. The explanation of this will come as a consequence of the framework that we will establish for the non-autonomous case.

To treat the asymptotics of a non-autonomous differential equation as (1.1) we consider \( \Sigma^+ = \overline{\{f(s + \cdot, \cdot) : s \in \mathbb{R}^+\}}^\rho \), the closure with respect to a suitable metric \( \rho \) (see Section 2) of the set of all translates of \( f \), a semigroup \( \{\Theta(t) : t \in \mathbb{R}^+\} \) on \( \Sigma^+ \), and, for each \( \sigma \in \Sigma^+ \), the semiflow \( (t, u_0) \in \mathbb{R}^+ \times X \mapsto K(t, \sigma)u_0 \in X \) where, for each \( u_0 \in X \), \( t \in \mathbb{R}^+ \mapsto K(t, \sigma)u_0 \in X \) is the solution of the initial value problem

\[
\begin{align*}
\dot{u} &= \sigma(t, u), t > 0, \\
u(0) &= u_0 \in \mathbb{R}^N.
\end{align*}
\]  

(1.2)

If \( V : \mathbb{R} \to \Sigma^+ \) is a global solution for the semigroup \( \{\Theta(t) : t \geq 0\} \) then, the family \( \{S_V(t, s) : t \geq s\} \subset C(\mathbb{R}^N) \) defined by

\[
S_V(t, s) = K(t - s, V(s)), \quad \text{for } t \geq s,
\]

defines an evolution process.
Together, $\mathcal{K}$ and $\Theta$ determine the non-autonomous problem (1.1) and the pair $(\mathcal{K}, \Theta)_{(\mathbb{R}^N, \Sigma^+)}$ will be called a non-autonomous dynamical system on $(\mathbb{R}^N, \Sigma^+)$, and with this object one can also define an associated autonomous dynamical system or semigroup (see [8, 19, 20]) \( \{ \Pi(t): t \geq 0 \} \) on $X = \mathbb{R}^N \times \Sigma^+$ (with the metric $d_X((x, \sigma), (\bar{x}, \bar{\sigma})) = ||x - \bar{x}|| + \rho(\sigma, \bar{\sigma})$) by setting
\[
\Pi(t)(x, \sigma) = (\mathcal{K}(t, \sigma)x, \Theta(t)\sigma), \quad \text{for } t \geq 0 \text{ and } (x, \sigma) \in X;
\]
the properties of $\Theta$ and $\mathcal{K}$ ensure that $\{ \Pi(t): t \geq 0 \}$ satisfies all properties of a semigroup.

If the semigroup $\Pi(\cdot)$ has a global attractor $A$ then the asymptotic set of states for the solutions of (1.1) is $\mathcal{A} = \Pi_X A$, the projection of $A$ onto $X$, and it is called the uniform attractor for (1.1) in the terminology of Chepyzhov and Vishik [8]. Of course, if $\Pi(\cdot)$ has a global attractor, $\Theta(\cdot)$ also has a global attractor which we will denote by $S$. Thus, if we only look at the uniform attractor we may lose information about the perspective of what is indeed in the asymptotics of (1.1), that is, if $\sigma \in S$ and global solution $V: \mathbb{R} \rightarrow S$ of $\{ \Theta(t): t \geq 0 \}$ such that $V(0) = \sigma$ corresponds a natural (given by the evolution of (1.1)) extension of $\sigma$ for negative times; that is, $\sigma(s + t, x) = V(s)(t, x)$, for all $t \geq 0$. Of course, if $s \geq 0$, $V(s)(t, x) = (\Theta(s)V(0))(t, x) = \sigma(t + s, x)$, for all $t \geq 0$.

In summary, given a non-autonomous differential equation such as (1.1), we need to deal with four different but closely related evolution operators:

(a) the semigroup $\Theta$ on $\Sigma^+$ associated to the dynamics of the time-dependent nonlinearities appearing in the equation;

(b) the non-autonomous dynamical system $(\mathcal{K}, \Theta)_{(\mathbb{R}^N, \Sigma^+)}$.

(c) the skew-product semigroup $\{ \Pi(t): t \geq 0 \}$ defined on the product space $X$;

and lastly, for each global solution $V: \mathbb{R} \rightarrow \Sigma^+$ of $\Theta$, the evolution process $\{ S_V(t, s): t \geq s \}$ given by

(d) $S_V(t, s)x = \mathcal{K}(t - s, V(s))x$, for all $t \geq s$ and $x \in \mathbb{R}^N$.

To each of the objects described above we will present the notions of attractors that are suitable to describe the asymptotic dynamics of (1.1), and they are the following

(a) the global attractor $\mathcal{S}$ for $\Theta$ in $\Sigma^+$;

(b) the uniform attractor $\mathcal{A}$ for $(\mathcal{K}, \Theta)_{(X, \Sigma^+)}$;

(c) the global attractor $A$ for $\{ \Pi(t): t \geq 0 \}$;

(d) for each global solution $V: \mathbb{R} \rightarrow \mathcal{S}$ of $\{ \Theta(t): t \geq 0 \}$ the pullback attractor $\{ A_V(t) \}_{t \in \mathbb{R}}$ associated to $\{ S_V(t, s): t \geq 0 \}$.

In Section 3 we will give the precise definitions of all these objects, as well as conditions ensuring the existence of attractors in each of the cases above. Moreover, we will present the description of the relationship between pullback and uniform attractors, leading to a detailed description of the uniform attractor and providing an understanding of its dynamical structures (associated to limiting evolution processes). The main differences with respect to a more classical approach as, for instance, in Chepyzhov and Vishik [8] or Kloeden and Rasmussen [11] is that, because the non-autonomous system is just defined for positive times, the (compact and invariant) hull is now taken as a positive hull. This is a just a positively invariant metric space for which we suppose there exists a compact global attractor $\mathcal{S}$ associated to the semigroup driving system $\Theta(t)$ (see Section 2). Moreover, we do not assume any backwards uniqueness on this attractor. This important fact leads
to a definition of a pullback attractor for each global solution \( \mathcal{V} : \mathbb{R} \to \mathcal{S} \) which, again as a difference with respect to the classical approach, cannot be interpreted as a cocycle attractor \( \{ \mathcal{A}(\sigma) \}_{\sigma \in \Sigma^+} \) (see [11]), since it is not uniquely defined for negative times. As a consequence, we conclude that the proper indexation for the skew-product attractor and the uniform attractor is by means of global solutions on \( \mathcal{S} \) (see Section 3). Indeed, we will see that

\[
\mathcal{A} = \bigcup_{\mathcal{V}} \mathcal{A}_\mathcal{V}(t) \times \{ \mathcal{V}(t) \}
\]

where the union is taken over all global solutions of \( \{ \Theta(t) : t \geq 0 \} \) in \( \mathcal{S} \).

In summary, the skew-product semigroup appears as the proper dynamical system for the study of the attractors. We are then very interested in the characterization of attractors for the skew-product semigroup. This is a crucial fact in order to prove the robustness of attractors under perturbation ([5]), in particular the continuity of the attractors with respect to parameter perturbation. In Section 4 we present an example in which we can describe the attractor as the union of the unstable sets of isolated invariant sets for the skew-product semigroup (called a gradient-like attractor, see for instance [14, 5]).

2. Positive hull of a non-autonomous vector field.

2.1. Semigroup driving system and its global attractor. We introduce a general method that provides a way to study a non-autonomous differential equation and explain some important features of autonomous equations that are often overlooked. The origin of this method is to consider the family of nonlinearities as a base flow driven by the time-shift operator applied to the nonlinearity \( f(t, \cdot) \) occurring in the original equation (see [19, 20]).

To ensure a proper understanding of the framework, we will continue to treat the problem (1.1) and leave the more general abstract framework for next section.

Let \( \mathcal{C} = C_b(\mathbb{R}^+ \times \mathbb{R}^N, \mathbb{R}^N) \). For each pair of positive integers \( n, k \), define the seminorm \( \| \cdot \|_{n, k} \) in \( \mathcal{C} \) by

\[
\| g \|_{n, k} = \sup_{t \in [0, k]} \sup_{x \in B^n} \| g(t, x) \|, \quad \text{for each } g \in \mathcal{C},
\]

where \( B^n = \{ x \in \mathbb{R}^N : \| x \| \leq n \} \). Now define the metric \( \rho \) in \( \mathcal{C} \) of uniform convergence on compact sets by

\[
\rho(g, h) = \sum_{k, n = 1}^{\infty} 2^{-(k+n)} \frac{\| g - h \|_{n, k}}{1 + \| g - h \|_{n, k}}, \quad \text{for all } g, h \in \mathcal{C}.
\]

(2.1)

In this way, \((\mathcal{C}, \rho)\) is a Fréchet space.

2.1.1. Case A: \( t \in \mathbb{R} \). In the case that \( t \in \mathbb{R} \), denote by \( \Sigma \) the closure with respect to \( \rho \) of the set of all translates of \( f \), i.e.

\[
\| g \|_{n, k} = \sup_{t \in [-k, k]} \sup_{x \in B^n} \| g(t, x) \|, \quad \text{for each } g \in \mathcal{C},
\]

and

\[
\Sigma(f) = \{ f(s + \cdot, \cdot) : s \in \mathbb{R} \}^\rho,
\]

known as the hull of the function \( f \) in the metric space \((\mathcal{C}, \rho)\) (see [8, 19, 20] for more details).
We define the shift operator $\Theta(t) : \Sigma(f) \to \Sigma(f)$ by

$$\Theta(t)f(\cdot, \cdot) = f(t + \cdot, \cdot).$$

It is clear that \{\Theta(t) : t \in \mathbb{R}\} defines a group in $\Sigma(f)$.

In some situations the symbol space $\Sigma(f)$ is compact and invariant; it then coincides with the global attractor for the driving group $\Theta(t)$. This approach was followed by Chepyzhov and Vishik in their seminal work [8, Chapter VII] on the forwards dynamics of non-autonomous problems. Assuming the backwards uniqueness property for the driving system $\Theta(t)$, they consider uniform attractors for a family of semiprocesses indexed in the symbol space $\Sigma(f)$. They prove that the uniform attractor can be characterised as the union of the kernel sections at zero corresponding to all the elements in $\Sigma(f)$, for which the driving is naturally defined in a unique way for all negative times.

2.1.2. Case B: $t \in \mathbb{R}^+$. However, it often occurs (and in a natural way) that backwards uniqueness does not hold in the attractor of the driving semigroup. Bearing this in mind, in this paper we present the theory avoiding the requirement of backwards uniqueness does not hold in the attractor of the driving semigroup $\Theta(t)$, with $t \in \mathbb{R}^+$. In fact, we show that non-uniqueness occurs naturally.

Thus, we now consider (1.1) with nonlinearity $f \in C^{0,1}_{\mathbb{R}^+}(\mathbb{R}^+ \times \mathbb{R}^N, \mathbb{R}^N)$, the set of continuous functions $f : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$ such that for each bounded subset $B \subset \mathbb{R}^N$, there exists $L_f(B) > 0$ such that

$$\|f(t, x) - f(t, \bar{x})\| \leq w_f([t-\bar{t}]) + L_f(B)\|x - \bar{x}\|,$$

(2.2)

where $w_f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function with $w_f(0) = 0$. We equip the space $C^{0,1}$ with the metric $\rho$ of uniform convergence in compact subsets of $\mathbb{R}^+ \times \mathbb{R}^N$.

Suppose, in addition, that $f$ is dissipative, that is, there are $M > 0$ and $\delta > 0$ such that

$$f(t, x) \cdot x \leq -\delta < 0$$

(2.3)

for all $t \in \mathbb{R}^+$ and for all $x \in \mathbb{R}^N$ such that $\|x\| > M$.

Denote by $\Sigma^+$ the closure with respect to $\rho$ (now restricted to positive times), of the set of all forwards translates of $f$,

$$\Sigma^+ = \{f(s + \cdot, \cdot) : s \in \mathbb{R}^+\}^\rho,$$

and define the shift operator $\Theta(t) : \Sigma^+ \to \Sigma^+$ by

$$\Theta(t)f(\cdot, \cdot) = f(t + \cdot, \cdot),$$

known as the positive hull of the function $f$ in the metric space $(\mathcal{C}, \rho)$. The family \{\Theta(t) : t \in \mathbb{R}^+\} defines a semigroup in $\Sigma^+$, which we again denote simply as $\Theta$. Also, in this particular case, it follows from the Arzelà-Ascoli Theorem that $\Sigma^+$ is compact and \{\Theta(t) : t \geq 0\} has a global attractor $\mathcal{S}$ in $\Sigma^+$. However, note that, in the general context of Section 3, $\Sigma^+$ is a metric space which we do not assume to be compact.

2.2. Example: positive hull without backwards uniqueness. The following example serves to highlight two important facts. On the one hand, to describe the positive hull of a given vector field $f$ defined for positive times. In particular, the description of the omega limit set of $f$; i.e., the compact global attractor $\mathcal{S}$ of its associated driving semigroup $\Theta(t)$, $t \in \mathbb{R}^+$.
On the other hand, we will show that there is no backwards uniqueness \( \Theta(\cdot) \) on \( S \), which is in contrast to the classical (pullback) treatment of non-autonomous problems in which the vector field is defined for all \( t \in \mathbb{R} \).

Let \( \{a_n^i\}, 1 \leq i \leq 5 \), be sequences of real numbers with

(i) \( a_0^1 = 1 \) and \( a_n^5 = a_{n+1}^1 \), for all \( n \in \mathbb{N} \),
(ii) \( a_n^i < a_{n+1}^i \), \( 1 \leq i \leq 4 \),
(iii) For \( i = 1, 3 \), if \( \tau_n^i = a_{n+1}^i - a_n^i \), then \( \tau_n^i \to \infty \) as \( n \to \infty \).
(iv) For \( i = 2, 4 \), \( \tau_n^i = a_{n+1}^i - a_n^i = 1 \).

Consider

\[
 f(t, x) = \alpha(t)g(x), \tag{2.4}
\]

where \( g \in C_b(\mathbb{R}^N, \mathbb{R}^N) \), satisfies (2.3) and \( \alpha : \mathbb{R}^+ \to [1, 2] \) is a globally Lipschitz function such that \( \alpha(t) = 1 \) if \( t \in (a_n^3, a_n^2) \), \( \alpha(t) = 2 \) if \( t \in (a_n^3, a_n^1), n \in \mathbb{N} \), and linear elsewhere.

Choose \( t_n^1 = a_n^1 + \frac{\tau_n^1}{\tau_n^2} \), so that

\[
 \Theta(t_n^1)f(s, x) = f(s + t_n^1, x) = g(x) \quad \text{if} \quad s \in \left[ 0, \frac{\tau_n^1}{2} \right].
\]

Actually, \( f(s + t_n^1, x) = g(x) \) for all \( s \in \left[ -\frac{\tau_n^3}{\tau_n^2}, \frac{\tau_n^3}{\tau_n^2} \right] \). As a consequence

\[
 \lim_{n \to \infty} \Theta(t_n^1)f(s, \cdot) = g(\cdot) \quad \text{in} \quad \Sigma^+.
\]

In the same way, if we now choose \( t_n^3 = a_n^3 + \frac{\tau_n^3}{\tau_n^2} \), then

\[
 \lim_{n \to \infty} \Theta(t_n^3)f(s, \cdot) = 2g(\cdot) \quad \text{in} \quad \Sigma^+.
\]

Now, take \( t_n^2 = a_n^2 + \frac{\tau_n^2}{\tau_n^2} = a_n^2 + \frac{1}{2} \). We get

\[
 \Theta(t_n^2)f(s, x) = f(s + t_n^2, x) = \alpha(s + t_n^2)g(x)
\]

and it holds that if \( t \in (-\frac{1}{2}, \frac{1}{2}) \), \( \alpha(t + t_n^2) \in (1, 2) \). We can now define a global solution \( \nu : \mathbb{R} \to \Sigma^+ \). For \( s \in \mathbb{R} \) and \( r \in \mathbb{R}^+ \), take \( t = s + r \), so that we can define \( \nu(s)(r, x) = \nu(s)(t - s, x) = \gamma(t)g(x) \), where \( \gamma : \mathbb{R} \to [1, 2] \) is such that \( \gamma(t) = 1 \) for \( t \leq -\frac{1}{2} \), \( \gamma(t) = 2 \) for \( t \geq \frac{1}{2} \), and linear elsewhere. Note that \( \nu(0)(r, x) = \gamma(r)g(x) \) with \( \gamma = \gamma_{|\mathbb{R}^+} \),

\[
 \lim_{s \to -\infty} \nu(s) = g(\cdot) \quad \text{in} \quad \Sigma^+
\]

and

\[
 \lim_{s \to +\infty} \nu(s) = 2g(\cdot) \quad \text{in} \quad \Sigma^+,
\]

i.e., \( \nu \) is a global solution in the global attractor \( S \subset \Sigma^+ \) joining the limits points \( g \) and \( 2g \). Similar global solutions \( \mu \) also join \( 2g \) and \( g \). (See Figure 2.2).

Another important remark here is that, in general, \( \{\Theta(t) : t \geq 0\} \) is not injective in \( S \). Indeed, \( \sigma(t, x) = 2g(x), t \geq 0 \) is in \( S \) and there are infinitely many global solutions \( V : \mathbb{R} \to \mathbb{R} \) of \( \Theta(\cdot) \) such that \( V(0) = \sigma \). In fact, we consider \( t_n^1 = \frac{a_n^1 + a_n^2}{2} \), \( V_n^1(\cdot) = \Theta(\cdot + t_n^1)f : [-t_n^1, \infty) \to \Sigma(f) \) converges (passing to a subsequence if need be) to \( V_1(t) = 2g \) for all \( t \in \mathbb{R} \) whereas, if \( t_n^2 = a_n^2 \), \( V_n^2(\cdot) = \Theta(\cdot + t_n^2)f : [-t_n^2, \infty) \to \Sigma(f) \) converges (perhaps passing to a subsequence) to \( V_2(t) = 2g \) for \( t \geq 0 \) and \( V_2(t) = g \) if \( t \leq -1 \). Clearly \( V_1(0) = V_2(0) = 2g \) which already gives non-uniqueness but one can pursue this reasoning and construct, for the same example, infinitely many global solutions of \( \{\Theta(t) : t \geq 0\} \) through \( \sigma = 2g \in S \). Having settled this, we will not assume backwards uniqueness for \( \{\Theta(t) : t \geq 0\} \) in \( \Sigma^+ \).
Figure 2.2. Global attractor for vector field (2.4) in its positive hull $\Sigma^+$.

3. **Pullback, uniform and skew-product attractors.** Motivated by this discussion, we now establish an abstract framework in which to treat non-autonomous problems. In order to obtain a full understanding of the asymptotics of such a problem we need to consider the following families of operators: the driving semigroup and the associated cocycle family, the skew-product semigroup, and the evolution processes associated to global solutions of the driving semigroup.

**Definition 3.1.** Let $(X,d_X)$ and $(\Sigma,d_\Sigma)$ be metric spaces. Consider $f_\Theta(t) : t \geq 0 \in C(\Sigma)$ a semigroup in $\Sigma$, called the driving semigroup, and a cocycle family relative to $f_\Theta(t) : t \geq 0$, i.e. a family $\mathcal{K}(t,\sigma) \in C(X) : t \geq 0, \sigma \in \Sigma$. $(K,\Theta)$ is called a *non-autonomous dynamical system* (NADS) if it satisfies

(i) $\mathcal{K}(0,\sigma)x = x$, for all $x \in X$ and $\sigma \in \Sigma$;
(ii) $\mathcal{K}(t+s,\sigma) = \mathcal{K}(t,\Theta(s)\sigma)\mathcal{K}(s,\sigma)$, for all $t,s \geq 0$ and $\sigma \in \Sigma$;
(iii) the map $\mathbb{R}^+ \times \Sigma \times X \ni (t,\sigma,x) \mapsto \mathcal{K}(t,\sigma)x$ is continuous.

Property (ii) is called the cocycle property. Define, in $X = X \times \Sigma$ with the product metric, the skew-product semigroup $\{\Pi(t) : t \geq 0\} \subset C(X)$, that is,

$$
\Pi(t)(x,\sigma) = (\mathcal{K}(t,\sigma),\Theta(t)\sigma) : t \geq 0.
$$

The semigroup composition property for $\Pi(\cdot)$ follows using the semigroup property of $\Theta(\cdot)$ and the cocycle property of $\mathcal{K}(\cdot,\cdot)$: we have

$$
\Pi(t+s)(x,\sigma) = (\mathcal{K}(t+s,\sigma)x,\Theta(t+s)\sigma) = (\mathcal{K}(t,\Theta(s)\sigma)\circ\mathcal{K}(s,\sigma)x,\Theta(t)\Theta(s)\sigma)
$$

$$
= \Pi(t) \circ \Pi(s)(x,\sigma)
$$
and all the other properties of a semigroup are clearly satisfied.

When $\Sigma$ is a one-point set ($\Sigma = \{\sigma_0\}$) and, consequently, $\Theta(t)\sigma_0 = \sigma_0$ for all $t \geq 0$, defining

$$T(t)x = K(t, \sigma_0)x, \text{ for all } t \geq 0 \text{ and } x \in X,$$

(3.1)

the skew-product semigroup reduces to

$$\Pi(t)(x, \sigma_0) = (T(t)x, \sigma_0), \ t \geq 0.$$ 

and $\{T(t) : t \geq 0\}$ defines a semigroup in $X$. This is the situation that arises when we consider an autonomous differential equation. For a non-autonomous differential equation $\Sigma$ will consist of more than one point.

Let us now derive conditions for the existence of an attractor for the skew-product semigroup $\{\Pi(t) : t \geq 0\}$ in $X$. Recall that a semigroup has a global attractor if and only if it is asymptotically compact, bounded and point dissipative ([12, 10, 2]).

Define $\pi_X : X \to X$ as the usual projection onto the first coordinate; that is, $\pi_X(x, \sigma) = x$ for each $(x, \sigma) \in X$. So,

$$\pi_X(\Pi(t)(x, \sigma)) = K(t, \sigma)x, \text{ for all } (t, x, \sigma) \in \mathbb{R}^+ \times X.$$ 

(3.2)

Also define $\pi_\Sigma : X \to \Sigma$ as the usual projection onto the second coordinate; that is, $\pi_\Sigma(x, \sigma) = \sigma$ for each $(x, \sigma) \in X$. So,

$$\pi_\Sigma(\Pi(t)(x, \sigma)) = \Theta(t)\sigma, \text{ for all } (t, x, \sigma) \in \mathbb{R}^+ \times X.$$ 

(3.3)

The point dissipativity property of $\{\Pi(t) : t \geq 0\}$ is equivalent to saying that there are bounded subsets $B_0$ in $X$ and $S_0$ in $\Sigma$ such that, for each $(x, \sigma) \in X$ there exists $t_>(x, \sigma) > 0$ such that $\Pi(t)(x, \sigma) \in B_0 \times S_0$, for all $t \geq t_>(x, \sigma)$. This implies that, for any $\sigma \in \Sigma$ there exists $t_\sigma > 0$ such that $\Theta(t)\sigma \in S_0$ for all $t \geq t_\sigma$. It also implies that, for each $(x, \sigma) \in X$ there exists $t_>(x, \sigma) > 0$ such that $K(t, \sigma)x \in B_0$, for all $t \geq t_>(x, \sigma)$.

The asymptotic compactness property of $\{\Pi(t) : t \geq 0\}$; that is, given a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that $t_n \to \infty$ and a bounded sequence $\{(x_n, \sigma_n)\}$ in $X$, $\{\Pi(t_n)(x_n, \sigma_n)\}$ has a convergent subsequence in $X$. This is equivalent to say that, given a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that $t_n \to \infty$ bounded sequences $\{x_n\}$ in $X$ and $\{\sigma_n\}$ in $\Sigma$ then,

- $\{\Theta(t_n)\sigma_n\}$ has a convergent subsequence in $\Sigma$ and
- Given a bounded sequence $\{\sigma_n\}$ in $\Sigma$, $\{K(t_n)(x_n, \sigma_n)\}$ has a convergent subsequence in $X$.

Finally, the property that orbits of bounded subsets of $X$ are bounded is equivalent to say that given a bounded subset $B$ in $X$ and a bounded subset $S$ of $\Sigma$, $\bigcup_{t \geq 0} \Theta(t)S$ is a bounded subset of $\Sigma$ and $\bigcup_{t \geq 0} K(t)(B, S)$ is a bounded subset of $X$, where $\bigcup_{t \geq 0} K(t)(B, S) = \bigcup_{t \geq 0} \bigcup_{x \in B} \bigcup_{\sigma \in S} K(t, \sigma)x$.

This reasoning proves the following result on the characterisation of the skew-product flows that possess a global attractor.

**Theorem 3.2.** Let $(X, d_X)$ and $(\Sigma, d_\Sigma)$ be metric spaces, $X = X \times \Sigma$ with the product metric, $\{\Theta(t) : t \geq 0\} \subset C(\Sigma)$ be a semigroup in $\Sigma$ and $\{K(t, \sigma) : t \geq 0, \sigma \in \Sigma) \subset C(X)$ be a cocycle family relative to $\{\Theta(t) : t \geq 0\}$. If $\{\Pi(t) : t \geq 0\}$ is the associated skew-product semigroup then it has a global attractor $\mathcal{A}$ in $X$ if and only if

- $\{\Theta(t) : t \geq 0\}$ has a global attractor in $\Sigma$. 


\begin{itemize}
  \item \( \{ K(t, \sigma) : t \geq 0, \sigma \in \Sigma \} \) is point dissipative, that is, there is a bounded subset \( B_0 \) of \( X \) such that, for each \( (x, \sigma) \in X \), there exists \( t(x, \sigma) > 0 \) such that \( K((t, \sigma)x) \in B_0 \), for all \( t \geq t(x, \sigma) \).
  \item \( \{ K(t, \sigma) : t \geq 0, \sigma \in \Sigma \} \) is asymptotically compact, that is, given sequence \( \{ t_n \} \)
  in \( \mathbb{R}^+ \) with \( t_n \to \infty \), a bounded sequence \( \{ x_n \} \) in \( X \) and a bounded sequence \( \{ \sigma_n \} \) in \( \Sigma \) then, the sequence \( \{ K(t_n)(x_n, \sigma_n) \} \) has a convergent subsequence in \( X \).
  \item \( \{ K(t, \sigma) : t \geq 0, \sigma \in \Sigma \} \) is bounded, that is, given a bounded subset \( B \) of \( X \) and a bounded subset \( S \) of \( \Sigma \), \( \bigcup_{t \geq 0} K(t)(B, S) \) is a bounded subset of \( X \).
\end{itemize}

If we now return to our example \((1.1)\), it is easy to see that if \( f \) satisfies \((2.2)\) and \((2.3)\) then any function in \( \Sigma(f) \) must also satisfy these two conditions. Now if \( \sigma \in \Sigma^+(f) \), \( u_0 \in \mathbb{R}^N \) with \( \|u_0\| > M \), the solution \( K(\cdot, \sigma)u_0 \) of \((1.2)\) must satisfy
\[
\frac{d}{dt}\|K(\cdot, \sigma)u_0\|^2 = 2K(\cdot, \sigma)u_0 \cdot f(t, K(\cdot, \sigma)u_0) \leq -2\delta
\]
and, consequently,
\[
\|K(\cdot, \sigma)u_0\|^2 \leq \|u_0\|^2 - 2\delta t,
\]
for all \( t \) such that \( \|K(t, \sigma)u_0\| > M \).

From this it follows that the solutions of \((1.2)\) exists for all \( t \geq 0 \). Also, if we denote by \( B_M \) the ball \( \{ u_0 \in \mathbb{R}^N : \|u_0\| \leq M \} \), then for any \( M > 0 \) we have\[
\sup_{u_0 \in B_M} \sup_{\sigma \in \Sigma(f)} \text{dist}_H(K(t, \sigma)u_0, B_M) \xrightarrow{t \to \infty} 0
\]
and, since we have already seen that \( \{ \Theta(t) : t \geq 0 \} \) has a global attractor in \( \Sigma(f) \), all the conditions of Theorem 3.2 are satisfied.

3.1. Evolution processes and it relation to NADSs. Another important family of operators that appears naturally in the study of non-autonomous problems are evolution processes, which we define next.

**Definition 3.3.** A family of \( \{ S(t, s) : t \geq s \} \subset C(X) \) is called an evolution process in \( X \) if it satisfies
\begin{itemize}
  \item[(i)] \( S(t, t)x = x \), for all \( t \in \mathbb{R} \) and \( x \in X \);
  \item[(ii)] \( S(t, s) = S(t, r)S(r, s) \), for all \( t \geq r \geq s \);
  \item[(iii)] the map \( \mathcal{P}_\mathbb{R} \times X \ni (t, s, x) \mapsto S(t, s)x \) is continuous, where \( \mathcal{P}_\mathbb{R} = \{ (t, s) \in \mathbb{R}^2 : t \geq s \} \).
\end{itemize}

In fact, if \( \mathcal{V} : \mathbb{R} \to \Sigma \) is a global solution for the semigroup \( \{ \Theta(t) : t \geq 0 \} \) associated to the NADS \( (K, \Theta) \), then, the family \( \{ S_\mathcal{V}(t, s) : t \geq s \} \subset C(X) \) defined by
\[
S_\mathcal{V}(t, s) = K(t - s, \mathcal{V}(s)), \quad t \geq s,
\]
defines an evolution process.

We say that an evolution process \( \{ S(t, s) : t \geq s \} \) is autonomous if the evolution depends only on the elapsed time \( t - s \) rather than on \( t \) and \( s \), that is, if \( S(t, s) = S(t - s, 0) \) for all \( t \geq s \). If an evolution process is not autonomous it is said to be non-autonomous.

Let us now define the notion of invariance for an evolution process \( \{ S(t, s) : t \geq s \} \). This notion is fundamentally connected to the possibility of constructing global solutions. In the autonomous case a set is invariant if it is fixed by the evolution process and that notion is sufficient for the development of a theory for the asymptotic behaviour of an autonomous evolution process (semigroup).
In the non-autonomous case, it may be that no set is fixed by the evolution process. Note that simple objects which are very helpful in the description of the asymptotic behaviour of autonomous systems, such as equilibria, may not exist. The notion of invariance for non-autonomous evolution processes which seems most suitable is the invariance of a time-dependent family of sets.

**Definition 3.4.** Let $X$ be a metric space and $\{S(t, s) : t \geq s\}$ be an evolution process in $X$. A family of sets $\{B(t) : t \in \mathbb{R}\}$ in $X$ is said to be invariant if $S(t, s)B(s) = B(t)$ for all $t \geq s$.

For an autonomous evolution process, the asymptotic behaviour can be studied by sending $t \to +\infty$, the forwards asymptotics, or fixing $t$ and sending $s \to -\infty$, the pullback asymptotics. In the non-autonomous case, these two asymptotic regimes give rise to completely different scenarios. Next we define pullback attraction for an evolution process.

**Definition 3.5.** Let $X$ be a metric space and $\{S(t, s) : t \geq s\}$ be an evolution process in $X$. A set $B_0$ pullback attracts a set $B$, at time $t$, under the action of the evolution process $\{S(t, s) : t \geq s\}$ if

$$\text{dist}_H(S(t, s)B, B_0) \xrightarrow{s \to \infty} 0.$$ 

With these notions we can now define the notion of pullback attractor for an evolution process.

**Definition 3.6.** Let $X$ be a metric space and $\{S(t, s) : t \geq s\}$ be an evolution process in $X$. A family $\{A(t) : t \in \mathbb{R}\}$ is called a pullback attractor if it is invariant, and for each $t \in \mathbb{R}$ the set $A(t)$ is compact and pullback attracts bounded subsets of $X$ at time $t$. We say that $\{A(t) : t \in \mathbb{R}\}$ is bounded if $\bigcup_{t \in \mathbb{R}} A(t)$ is bounded.

A bounded pullback attractor is unique. In fact, if $\{\tilde{A}(t) : t \in \mathbb{R}\}$ and $\{A(t) : t \in \mathbb{R}\}$ are both bounded pullback attractors for $\{S(t, s) : t \geq s\}$, since $B = \bigcup_{t \in \mathbb{R}} \tilde{A}(t)$ is bounded and $A(t)$ pullback attracts bounded subsets of $X$ at time $t$ and from the invariance property of pullback attractors, we have that

$$0 \leq \text{dist}_H(\tilde{A}(t), A(t)) = \text{dist}_H(S(t, s)\tilde{A}(s), A(t)) \leq \text{dist}_H(S(t, s)B, A(t)) \xrightarrow{s \to \infty} 0.$$ 

That implies $\tilde{A}(t) \subseteq A(t)$. The other inclusion follows exactly in the same way.

There is a vast literature on pullback attractors with results that characterise the evolution processes that posses pullback attractors (see [5] and [11]). In the context of this paper the conditions for existence of bounded pullback attractors will be drawn from the conditions for the existence of a global attractor for the skew-product semigroup.

**Theorem 3.7** ([5]). Let $X$ be a metric space and $\{S(t, s) : t \geq s\} \subset C(X)$ be an evolution process in $X$. If $\{S(t, s) : t \geq s\}$ has a bounded pullback attractor $\{A(t) : t \in \mathbb{R}\}$ then, for all $t \in \mathbb{R},$

$$A(t) = \{\xi(t) : \xi : \mathbb{R} \to X \text{ is a global bounded solution of } \{S(t, s) : t \geq s\}\}.$$ 

**Proof.** From the invariance of $\{A(t) : t \in \mathbb{R}\}$, for each $x \in A(t_0)$, we can construct a global solution $\xi : \mathbb{R} \to X$ such that $\xi(t_0) = x$ and $\xi(t) \in A(t)$ for all $t \in \mathbb{R}$ (see, for instance, Section 16.6 in [5] or Chapter VIII in [8]). Since $\{A(t) : t \in \mathbb{R}\}$ is bounded we have that $\xi : \mathbb{R} \to X$ is bounded.

On the other hand, if $\xi : \mathbb{R} \to X$ is a global bounded solution of $\{S(t, s) : t \geq s\}$ then, for $t \in \mathbb{R}$ fixed, since $A(t)$ pullback attracts $B = \{\xi(t) : t \in \mathbb{R}\}$, for each
n ∈ N*, there exists t_n ≤ t such that dist_H(S(t,s)B,A(t)) ≤ 1/n, for all s ≤ t_n.
In particular dist_H(S(t,t_n)ξ(t_n), A(t)) = dist_H(ξ(t), A(t)) ≤ 1/n for all n ∈ N*. It
follows that ξ(t) ∈ A(t). This completes the proof. □

We construct pullback attractors as pullback ω-limit sets.

**Definition 3.8.** Let X be a metric space and \{S(t,s) : t ≥ s\} ⊂ C(X) be an
evolution process in X. The pullback ω-limit set at time t of B ⊂ X is defined by

\[
ω(B,t) := \bigcap_{σ ≤ t} \bigcup_{s ≤ σ} S(t,s)B.
\]

Clearly ω(B,t) is closed and it is easy to see that,

\[
ω(B,t) = \{y ∈ X : \text{there are sequences } \{s_k\} \text{ in } (-∞,t], \ s_k \xrightarrow{k→∞} -∞,
\text{and } \{x_k\} \text{ in } B, \text{ such that } y = \lim_{k→∞} S(t,s_k)x_k\}. \tag{3.4}
\]

**Theorem 3.9 (\cite{5}).** Let X be a metric space and \{S(t,s) : t ≥ s\} ⊂ C(X) be an
evolution process in X. Assume that there exists a compact subset K of X that
pullback attracts bounded subsets of X at time t for all t ∈ R. Then, \{S(t,s) : t ≥ s\}
has a bounded pullback attractor \{A(t) = ω(K,t) : t ∈ R\} and \bigcup_{t∈R} A(t) is a
compact subset of X.

**Proof.** Let us show that ω(K,t) is non-empty. Take a sequence \{s_k\} in (-∞,t]
with s_k \xrightarrow{k→∞} -∞ and a sequence \{x_k\} in K. Since K is compact and attracts
itself at time t, the sequence \{S(t,s_k)x_k\} must have a convergent subsequence, consequently, ω(K,t) is non-empty.

Clearly ω(K,t) ⊂ K for all t ∈ R and \bigcup_{t∈R} ω(K,t) ⊂ K, since ω(K,t) is closed,
we conclude that ω(K,t) is compact for all t ∈ R and that \bigcup_{t∈R} ω(K,t) is compact.

We now prove the invariance of \{ω(K,t) : t ∈ R\}. If t ≥ s and x ∈ ω(K,t),
there are sequences \{s_k\} in (-∞,t] with s_k \xrightarrow{k→∞} -∞ and \{x_k\} in K such that
S(t,s_k)x_k \xrightarrow{k→∞} x. Since s_k \xrightarrow{k→∞} -∞, we may assume that s_k ≤ s for all k ∈ N.
Now since K attracts itself at time t the sequence \{S(s,s_k)x_k\} must have a
convergent subsequence (which we relabel) with limit y, say. Clearly, y ∈ ω(K,s)
and from the continuity of S(t,s) we must have that S(t,s)y = x. It follows that
S(t,s)ω(K,s) ⊃ ω(K,t). The proof of the other inclusion is simpler and is left to
the reader.

Next, we prove that ω(K,t) pullback attracts bounded subsets of X at time t.
If there is a bounded subset B of X that is not pullback attracted by ω(K,t) at
time t then, there is an δ > 0 and sequences \{s_k\} in (-∞,t] with s_k \xrightarrow{k→∞} -∞ and
\{x_k\} in B such that dist(S(t,s_k)x_k,ω(K,t)) ≥ δ for all k. Since K is compact and pullback attracts B \{S(t,s_k)x_k\} must have a convergent subsequence with limit in
ω(K,t). This contradiction proves that ω(K,t) pullback attracts bounded subsets
of X, which completes the proof. □

The existence of pullback attractors for \(SV(\cdot,\cdot)\) when \(V\) is a global bounded
solution of \(Θ(\cdot)\) now follows directly from Theorem 3.9.

**Corollary 1.** Let \((X,d_X)\) and \((Σ,d_Σ)\) be metric spaces, \(X = X × Σ\) with the
product metric, \(Θ(t) : t ≥ 0\) ⊂ C(Σ) be a semigroup in Σ and \(K(t,σ) ∈ C(X) : t ≥ 0, σ ∈ Σ\)
be a cocycle family relative to \(Θ(t) : t ≥ 0\), so that \((K,Θ)\) is a NADS. If the associated skew-product semigroup \(Π(t) : t ≥ 0\) has a global
Let \( \Pi(\cdot) \) be a cocycle family relative to \( \theta(\cdot) \). Moreover, \( \Pi(\cdot) \) possesses a pullback attractor \( A_\Pi(t) : t \in \mathbb{R} \) which satisfies

\[
A_\Pi(t) = \{ x \in X : (x, \theta(t)) \in A \}
\]

where the first union is taken over all global bounded solutions \( \theta(t) : t \geq 0 \).

**Proof.** Note that, by Corollary 1, \( \{S_V(t,s) := K(t-s,\theta(s)) : t \geq s \} \) has a pullback attractor \( A_V(t) : t \in \mathbb{R} \) and

\[
\bigcup_{t \in \mathbb{R}} A_V(t) \subset \pi_X(A).
\]

We can now characterise the attractor of the skew-product semigroup in terms of pullback attractors of processes associated to global bounded solutions of the driving semigroup.

**Theorem 3.10.** Let \( (X,d_X) \) and \( (\Sigma,d_\Sigma) \) be metric spaces, \( X = X \times \Sigma \) with the product metric, \( \{\theta(t) : t \geq 0 \} \subset C(\Sigma) \) be a semigroup in \( \Sigma \) and \( \{K(t,\sigma) \in C(X) : t \geq 0, \sigma \in \Sigma \} \) be a cocycle family relative to \( \theta(t) : t \geq 0 \). If the associated skew-product semigroup \( \{\Pi(t) : t \geq 0 \} \) has a global attractor \( A \) in \( \mathbb{R} \), then, the driving system \( \{\theta(t) : t \geq 0 \} \) has a global attractor \( S \). If \( \theta(\cdot) : \mathbb{R} \to S \) is a global bounded solution for \( \{\theta(t) : t \geq 0 \} \) then the evolution process \( \{S_V(t,s) : t \geq s \} \) given by

\[
S_V(t,s)x = K(t-s,\theta(s))x, \quad x \in X,
\]

possesses a pullback attractor \( \{A_V(t) : t \in \mathbb{R} \} \), which satisfies

\[
A_V(t) = \{ x \in X : (x, V(t)) \in A \}
\]

Moreover,

\[
A = \bigcup_{V \in \mathbb{R}} \bigcup_{t \in \mathbb{R}} A_V(t) \times \{ V(t) \}.
\]

where the first union is taken over all global bounded solutions \( V \) for \( \{ \theta(t) : t \geq 0 \} \).

On the other hand, if \( V : \mathbb{R} \to S \) is a global solution for \( \{ \theta(t) : t \geq 0 \} \) and \( x_0 \in A_V(t_0) \) then, there is a global solution \( x : \mathbb{R} \to X \) of \( \{S_V(t,s) = K(t-s,\theta(s)) : t \geq s \} \) such that \( x(t_0) = x_0 \) and \( x(t) \in A_V(t) \) for all \( t \in \mathbb{R} \). Since \( \bigcup_{t \in \mathbb{R}} A_V(t) \) is bounded we have that \( x : \mathbb{R} \to X \) is a global bounded solution of \( \{S_V(t,s) = K(t-s,\theta(s)) : t \geq s \} \) and consequently \( (x, V) : \mathbb{R} \to X \) is a global bounded solution of \( \{\Pi(t) : t \geq 0 \} \), showing that \( (x(t), V(t)) \in A \) for all \( t \in \mathbb{R} \) and, in particular, \( (x_0, V(t_0)) \in A \) and

\[
\bigcup_{V \in \mathbb{R}} \bigcup_{t \in \mathbb{R}} A_V(t) \times \{ V(t) \} \subset A.
\]

where the first union is taken over all global bounded solutions \( V \) for \( \{ \theta(t) : t \geq 0 \} \). This completes the proof. \( \square \)
Let $fK$ noting that, if $2.2$ and Theorem $3.10$ product metric, develop an example for which we can describe part of the structure of the attractor $A$ for the non-autonomous dynamical system $t_0$. From Theorem $3.2$ and Theorem $3.10$ the following result holds.

\textbf{Theorem 3.12.} Let $(X,d_X)$ and $(\Sigma,d_\Sigma)$ be metric spaces, $X = X \times \Sigma$ with the product metric, $\{\Theta(t): t \geq 0\} \subset C(\Sigma)$ be a semigroup in $\Sigma$ and $\{K(t,\sigma) \subset C(X) : t \geq 0, \sigma \in \Sigma\}$ be a cocycle family relative to $\{\Theta(t): t \geq 0\}$. The uniform attractor for the non-autonomous dynamical system $(K, \Theta)_{X, \Sigma}$ is the minimal closed subset $\bar{A}$ of $X$ such that, for each bounded subset $B$ of $X$ and each bounded subset $S$ of $\Sigma$

$$\lim_{t \rightarrow \infty} \sup_{\sigma \in S} \text{dist}_H(\Theta(t,\sigma)B, \bar{A}) = 0.$$

From Theorem $3.2$ and Theorem $3.10$ the following result holds.

\textbf{Theorem 3.12.} Let $(X,d_X)$ and $(\Sigma,d_\Sigma)$ be metric spaces, $X = X \times \Sigma$ with the product metric, $\{\Theta(t): t \geq 0\} \subset C(\Sigma)$ be a semigroup in $\Sigma$ and $\{K(t,\sigma) \subset C(X) : t \geq 0, \sigma \in \Sigma\}$ be a cocycle family relative to $\{\Theta(t): t \geq 0\}$. If

- $\{\Theta(t): t \geq 0\}$ has a global attractor in $\Sigma$.
- $\{K(t,\sigma): t \geq 0, \sigma \in \Sigma\}$ is point dissipative, that is, there is a bounded subset $B_0$ of $X$ such that, for each $(x,\sigma) \in X$, there exists $t(x,\sigma) > 0$ such that $K(t,\sigma)x \in B_0$, for all $t \geq t(x,\sigma)$.
- $\{K(t,\sigma): t \geq 0, \sigma \in \Sigma\}$ is asymptotically compact, that is, given sequence $\{t_n\}$ in $\mathbb{R}^+$ with $t_n \rightarrow \infty$, a bounded sequence $\{x_n\}$ in $X$ and a bounded sequence $\{\sigma_n\}$ in $\Sigma$ then, the sequence $\{K(t_n)(x_n,\sigma_n)\}$ has a convergent subsequence in $X$.
- $\{K(t,\sigma): t \geq 0, \sigma \in \Sigma\}$ is bounded, that is, given a bounded subset $B$ of $X$ and a bounded subset $S$ of $\Sigma$, $\bigcup_{t \geq 0} K(t)(B,S)$ is a bounded subset of $X$.

Then the non-autonomous dynamical system $(K, \Theta)_{X, \Sigma}$ has a compact uniform attractor $\bar{A}$ and, if $\kappa$ is the global attractor for the associated skew-product semigroup $\Pi(t): t \geq 0$ then, $\bar{A} = \pi_X(\kappa)$. Furthermore

$$\bar{A} = \bigcup_{\mathcal{V}} A_{\mathcal{V}}(0),$$

where the union is taken over all global bounded solutions $\mathcal{V}$ for $\{\Theta(t): t \geq 0\}$.

This result is straightforward from Theorem $3.2$ noting that, if $\mathcal{V}: \mathbb{R} \rightarrow \Sigma$ is a global bounded solution for $\{\Theta(t): t \geq 0\}$ then, $\mathcal{V}(\cdot + r) = \mathcal{V}(\cdot)$ is also a global bounded solution for $\{\Theta(t): t \geq 0\}$. Hence $A_{\mathcal{V}}(0) = A_{\mathcal{V}}(r)$.

\section{Structure of the attractor for the skew-product semigroup from the structure of the attractor for the driving system.}

In this subsection we develop an example for which we can describe part of the structure of the attractor for a driving semigroup of translations in the positive hull of a given non-autonomous function.

Inspired in the example in subsection $2.2$, and given two continuous functions $r, g: \mathbb{R}^N \rightarrow \mathbb{R}^N$, we define $f: \mathbb{R}^+ \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ by

$$f(t,x) = h(t)r(x) + (1 - h(t))g(x),$$

where $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined as follows: let $\{a_n\}$ be a sequence defined by $a_1 = 0$ and $a_{n+1} = a_n + 4n$ for each $n \geq 1$ and define $b_n = a_n + n$, $c_n = a_n + 2n$ and $d_n = a_n + 3n$ for all $n \geq 1$. Hence
\[ a_n < b_n < c_n < d_n < a_{n+1} \text{ for all } n \in \mathbb{N}; \]
\[ t_n = b_n - a_n = c_n - b_n = d_n - c_n = a_{n+1} - d_n = n; \]

Now, we define the function \( h \) in such a way that is smooth, \( 0 \leq h \leq 1, h(t) = 1 \) if \( t \in [a_n, b_n] \) for some \( n \in \mathbb{N} \) and \( h(t) = 0 \) if \( t \in [c_n, d_n] \) for some \( n \in \mathbb{N} \).

If \( \Theta(t)f(\cdot, \cdot) = f(t + \cdot, \cdot) \) for each \( t \geq 0 \), let \( \Sigma^+ \) be the positive hull of \( f \) in \( C = C_0(\mathbb{R}^+ \times \mathbb{R}^N, \mathbb{R}^N) \) with the metric \( \rho; \{ \Theta(t) : t \geq 0 \} \) is a semigroup in \( \Sigma^+ \).

First, we choose the sequence \( t_n^* = a_n + \frac{\gamma_n}{2}, \) thus
\[ \Theta(t_n^*)f(s, x) = f(s + t_n^*, x) = r(x), \]
if \( s \in [0, \frac{\gamma_n}{2}] \). Hence \( \Theta(t_n^*)f \rightarrow r \) as \( n \rightarrow \infty \) in the metric \( \rho \), which shows that \( r \in \omega(f) \), the omega-limit set of \( f \). Choosing \( \tau_n^* = c_n + \frac{\gamma_n}{2} \) we can see that \( \Theta(\tau_n^*)f \rightarrow g \) as \( n \rightarrow \infty \), which shows that \( g \in \omega(f) \) in a similar way. Choosing \( \gamma_n^* = b_n + \frac{\gamma_n}{2} \) we see that
\[ \Theta(\gamma_n^*)f(t, x) = f(t + \gamma_n^*, x). \]

Using the Arzelà-Ascoli theorem and a diagonalization process, we can see that there exists a function \( \xi^* : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) and a subsequence \( \{ \gamma_n^* \} \) of \( \{ \gamma_n \} \) such that \( \Theta(\gamma_n^*)f \rightarrow \xi^* \) in the metric \( \rho \). Also, since \( \Theta(\gamma_n^*)f(t, x) = f(t + \gamma_n^*, x) \), we can see that if \( t \in [-\gamma_n^*, b_n - \gamma_n^*] \), then \( \Theta(\gamma_n^*)f(t, x) = r(x) \) for all \( x \in \mathbb{R}^N \). Thus \( \xi(t) \xrightarrow{t \rightarrow \infty} r \) where \( \xi(t) = \xi^*(t, \cdot) \) and, analogously, \( \xi(t) \xrightarrow{t \rightarrow \infty} g \). It is clear that \( \xi \) is a global solution \( \Theta(t) \), therefore \( \xi \) is a connection between \( r \) and \( g \). With the same reasoning, we can construct a connection \( \psi \) between \( g \) and \( r \). In this way we can have some description of what happens in \( \Sigma^+ \). Note that \( \Sigma^+ \) is composed by connecting orbits between \( r \) and \( g \), and between \( g \) and \( r \). In particular, we have the same structure as in Figure 2.2 for \( r \) and \( g \) instead \( g \) and \( 2g \).

Consider now the following non-autonomous ordinary differential equation
\[
\begin{cases}
\dot{x} = f(t, x), & t > 0, \\
x(0) = x_0 \in \mathbb{R}^N.
\end{cases}
\tag{4.1}
\]

We suppose that the autonomous equations
\[
\begin{cases}
\dot{x} = r(x), & t > 0, \\
\dot{x} = g(x), & t > 0,
\end{cases}
\tag{4.2}
\]
possess global attractors \( A_r \) and \( A_g \) associated to semigroups \( S_r, S_g : X \rightarrow X \) respectively.

Let \( X = C^0_b(\mathbb{R}^N, \mathbb{R}^N) \) and \( \mathbb{R}^+ \ni t \mapsto K(t, \gamma)u_0 \in X \) be the unique solution of (4.1). The skew-product semigroup in \( X = X \times \Sigma^+ \) (with the product metric) is defined by
\[ \Pi(t)(x, \gamma) = (K(t, \gamma)x, \Theta_t\gamma), \quad t \geq 0, \quad \gamma \in \Sigma^+ \text{ and } x \in X. \]
If \( \mathcal{V} : \mathbb{R} \rightarrow \Sigma^+ \) is a global solution of \( \{ \Theta_t : t \geq 0 \} \) \( (\Theta_t(V(s)) = V(t + s) \) for all \( t \geq 0, \ s \in \mathbb{R} \) through \( \gamma \) \( \mathcal{V}(0) = \gamma \)), we define the evolution process associated to (4.1) by \( S_{\mathcal{V}}(t, s) = K(t - s, \mathcal{V}(s)), t \geq s. \)

Suppose the skew-product semigroup \( \{ \Pi(t) : t \geq 0 \} \) associated to (4.1) has a global attractor \( \mathcal{A} \) (hence \( \Theta_t \) is a semigroup with a global attractor \( \mathcal{S} \) in \( \Sigma^+ \)). Thus, from Theorem 3.10, if \( \mathcal{V} : \mathbb{R} \rightarrow \mathcal{S} \) is a global solution for \( \Theta_t \), the evolution process \( S_{\mathcal{V}}(t, \tau)x = K(t - \tau, \mathcal{V}(\tau))x, t \geq \tau \), has a pullback attractor \( \{ A_{\mathcal{V}}(t) : t \in \mathbb{R} \} \) and
\[ \mathcal{A} = \bigcup_{\mathcal{V} \in \mathbb{R}} A_{\mathcal{V}}(t) \times \{ \mathcal{V}(t) \}, \]
where the first union is taken over all solutions \( \mathcal{V} : \mathbb{R} \to \mathcal{S} \) of \( \Theta_x \).

We now want to describe the structure of the attractor for the skew-product flow. For this, suppose that both the equations in (4.2) are generalised gradient systems (see [5]), i.e., the attractors \( \mathcal{A}_r \) and \( \mathcal{A}_g \) of (4.2) have a decomposition in a finite set of invariant subsets given by \( \{ M^r_1, \ldots, M^r_m \} \) in \( \mathcal{A}_r \), and \( \{ M^g_1, \ldots, M^g_p \} \) in \( \mathcal{A}_g \), respectively. In particular, the global attractors are just the union of the unstable manifolds of these invariants sets (see [1, 5]), i.e.,

\[
\mathcal{A}_r = \bigcup_{i=1}^{m} W^u(M^r_i) \quad \text{and} \quad \mathcal{A}_g = \bigcup_{j=1}^{p} W^u(M^g_j),
\]

where, for a given bounded set \( D \subset X \) and \( l = r, g \)

\[
W^u(M^l_i) = \{ x \in \mathcal{A}_l : \text{there exist sequences } t_n^l \to -\infty \text{ as } n \to \infty \text{ and } x_n^l \in D \text{ such that } \lim_{n \to \infty} \text{dist}(S(t_n^l)x_n^l, x) = 0 \}.
\]

We now consider, for the skew-product semigroup, the sets

\[
M^l_i = \{ M^l_i \} \times \{ r \}, i = 1, \ldots, m, \quad \text{and} \quad M^g_j = \{ M^g_j \} \times \{ g \}, j = 1, \ldots, p,
\]

which are isolated and invariant for \( \Pi(t) \). Now we see how the structure of \( \Sigma^+ \) induces the structure of the attractor \( \mathcal{A} \) for the skew-product semiflow. Indeed, since, by Theorem 3.10, through any point \( (x, \sigma) \in \mathcal{A} \) passes a unique bounded global solution \( (\xi(t), \mathcal{V}) \) with \( \xi(0) = x, \mathcal{V}(0) = \sigma \), and the unique global solutions in \( \Sigma^+ \) are \( r, g \) or connections between them, we can conclude that

\[
\mathcal{A} = \bigcup_{i=1}^{m} W^u(M^r_i) \cup \bigcup_{j=1}^{p} W^u(M^g_j).
\]

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