Empirical Formulas for Estimating Self and Mutual Inductances of Toroidal Field Coils and Structures

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Self and mutual inductances of toroidal-field (TF) coils are empirically expressed by linear combinations of three coil-shape parameters: elongation, aspect ratio, and triangularity, based on their calculation results with the Neumann formula. A regression function was also obtained for calculating rough values of self-inductances of toroidal-shape structures such as a vacuum vessel in a Tokamak-type fusion reactor. An analysis for their eddy currents induced during fast discharge of TF coils is presented for showing as an application example of these formulas.

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1. Introduction

Analyses for electromagnetic dynamics of TF coils are required for estimating thermal and mechanical impacts on related reactor structures during their fast discharge in emergency conditions such as loss of superconductivity (quench) or loss of cyclic symmetry [1, 2]. The most important quantities required for these analyses are self and mutual inductances of TF coils used in their circuit equations [3].

The self inductance of TF coil set becomes larger with increasing its size as in the case of fusion demo reactors [4] that have enormous magnetic stored energy to be thermally and safely dissipated during the fast discharge [2]. These inductances are accurately and numerically evaluated by the well-known Neumann formula [5]. The Neumann formula, however, requires us to carry out many complicated line integrals along conductor-current paths distributed in 3D-space.

Estimation of eddy currents induced in toroidal-shape structures such as a vacuum vessel during the TF-coil fast discharge is also important for consideration of their structural integrity and often analyzed prior to structural analyses with a general purpose numerical calculation code such as ANSYS [6] or a dedicated code for transient electromagnetic analyses such as EDDYCAL [7]. These electromagnetic structural analyses are not easy because they need detailed 3D-modeling of structures and long computation time and therefore it would be necessary to verify and understand their results consisted of massive numerical data by preliminary simple analyses.

In the simplified analysis for eddy currents, they can be estimated by setting up circuit equations that include circuit constants for TF coils and related structures, i.e., their resistances, self and mutual inductances. These circuit constants should also be easily evaluated for convenience from dimensions given in their design drawings.

In this paper, we will present empirical formulas for evaluating self and mutual inductances of TF coils and toroidal-shape structures with defining their cross-sectional shape parameters such as elongation, aspect ratio, and triangularity. For showing an example of application of the simplified inductance calculation, we will also demonstrate an analysis of eddy currents induced in TF-coil structures and the vacuum vessel in the fast discharge of TF coils of a fusion demo reactor, JA DEMO [4], for rough estimations of their thermal and mechanical impacts.

2. Inductance of Toroidal Structure

2.1 General formula

A mutual-inductance between toroidal-shape structures would be approximated by calculating coupled toroidal magnetic flux inside their poloidal current center lines (CCL). The mutual inductance \( M_{ij} \) between the \( i \)-th and \( j \)-th structures with a volume \( V_i \) is then estimated from

\[
M_{ij} I_j \approx N_i \int_i B dS = N_i \int_i B(R)dRdZ = \frac{\mu_0 N_i N_j I_j}{2\pi} (H_i + H'_i) \xi_i,
\]

with the inductance factor

\[
\xi_i = \frac{1}{H_i + H'_i} \int Z(R) + |Z'(R)| \frac{dR}{R} \text{ for } V_i \subseteq V_j,
\]

where \( H_i \) and \( H'_i \) are the external and internal magnetic fields.

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where \( I \) is the poloidal current, \( N \) the number of turns, \( B = \mu_0 N_i I / (2\pi R) \) the toroidal magnetic flux density generated by \( j \)-th current, \( R \) and \( Z \) are radial and axial coordinates of a point on the CCL, respectively, \( H \) the height of CCL (maximum value of \( Z \)) and the prime (‘) denotes quantities below its equatorial planes (see Fig. 1).

We have \( M_i = M_i \approx (\mu_0 N_i I_i) / (2\pi (R_i + H') \xi) \) from Eq. (1) and the self-inductance \( L_i = M_i \). The formula for calculating the inductance factor \( \xi \) was analytically derived based on the geometry shown in Fig. 1, which is presented in Appendix A.

### 2.2 Shape parameters of toroidal structure

We define the following parameters as being used for a Tokamak plasma to express the poloidal cross-sectional shape: elongation \( \kappa = H / a \), aspect ratio \( A = R / R_0 \), and triangularity \( \theta = (R - R_M) / a \), where \( R_0 \) is the radius at which \( Z = H, R = (R_0 + R_1) / 2 \) the major radius, \( a = (R_0 - R_1) / 2 \) the minor radius with \( R_1 \) and \( R_0 \) being the inboard and outboard radii, respectively, (see Fig. 1). Using the defined parameters and the height \( H \), we inversely obtain these radii as \( R_0 = a(1 + \kappa), R_1 = a(1 - \kappa), R_M = a(1 - \delta) \), and \( R = aH \).

The TF coil shape (or its CCL) is usually expressed by six arcs in its design [8], as shown in Fig. 1. For a limiting case of the shape being vertically symmetric with \( \theta_2 = \pi / 2 - \theta_1 \) and \( \theta_3 = \pi / 2 \) in Fig. 1, radii and center coordinates of these arcs are given with parameters \( \kappa, \delta, A, H, \) and the arc angle \( \theta_1 \) as

\[
R_1 = H \left( \frac{\cos \theta_1 + \epsilon (\sin \theta_1 - 1)}{\cos \theta_1 + \sin \theta_1 - 1} \right),
R_2 = H \left( \frac{\cos \theta_1 - 1 + \epsilon \sin \theta_1}{\cos \theta_1 + \sin \theta_1 - 1} \right),
R_3 = R_M - R_1 = H(1 - \delta)/\kappa,
R_4 = R_O - R_1 = (H/\kappa)(A + 1) - R_1, Z_c = 0,
R_5 = R_2 + (R_1 - R_2) \cos \theta_1, Z_c = (R_1 - R_2) \sin \theta_1,
R_6 = R_3, \quad \text{and} \quad Z_3 = Z_c + R_2 - R_3,
\]

where \( \epsilon = (1 + \delta) / \kappa \).

Note that \( \epsilon < 1 \) and \( R_0 > H \) because \( R_0 \) should be greater than \( R_2 \) and then \( \theta_1 \) is inversely estimated by \( 2 \tan^{-1}\left( (R_1(H - \epsilon) / (R_1 - H)) - \pi / 2 \right) \) for a given radius \( R_1 \) of outboard curvature, which gives \( \theta_1 \to 0 \) for \( R_1 \to \infty \) with \( R_2 \to H_1 = R_O - R_M \).

The cross-sectional shape of toroidal structure is thus defined by parameters \( \kappa, \delta, A, \) and \( \theta_1 \) and therefore the inductance factor \( \xi \) can be expressed as a function of them.

We carried out a regression analysis with randomly generating \( 10^5 \) sets of parameters in ranges of \( 1.5 \leq \kappa \leq 2.0, 1.5 \leq A \leq 2.0, 0.22 \leq \delta \leq 0.5, \) and \( 0 < \theta_1(\kappa) / 90 \leq 0.7 \) and obtained an empirical formula for \( \xi \) as

\[
\xi = c_0 + c_2 \kappa + c_3 \delta + c_4 A + c_5 A^2 + c_6 \theta_1 / 90 + \epsilon, \tag{2}
\]

with \( c_0 = 4.933, c_2 = 0.03728, c_3 = 0.06980, c_4 = -3.551, c_5 = 0.7629, \) and \( c_6 = -0.06298. \) The standard deviation of the relative error \( \varepsilon / \xi \) was then estimated to be 0.12%.

### 2.3 Self inductance of TF coil set

We calculated the self and mutual inductances of TF coils with the Neumann formula for the following various values of parameters: \( \kappa = 1.5, 1.6 \) and \( 7, \delta = 0.2, 0.35 \) and \( 0.5, A = 1.5, 1.6 \) and \( 1.7, \) and \( \theta_1(\kappa) = 40, 50 \) and \( 60, \) all of which totally give 81 parameter combinations and CCL shape variations shown in Fig. 2 for \( H = 9.3 \) m.

In this calculation, we set geometrically imaginative current cross-sectional area (i.e., winding pack, WP) shown in Fig. 2 (b) to take its finite size into account, which would affect especially on estimations of the self-inductance.
value of a single TF coil and the mutual one between adjacent coils. The typical sensitivity (δL/|L|)/(δω/|ω|) of the self inductance L of the TF-coil set was estimated for the fractional size change (δω/|ω|) of WP to be ~5% in the toroidal direction and 6 - 9% in the radial direction.

The self-inductance of the TF-coil set is then expressed by

\[ L = L_0 + N C L \]

where \( L_0 \) is the inductance factor given by the calculation with the Neumann formula, \( N_{TF} \) is the number of TF coils and \( C_L \) that of turns per coil.

We first compared \( C_L \) given by Eq.(1) to \( C_L \) for all parameter combinations, which is shown in Fig. 3(a). The error of \( C_L \) was then within the range of \(-1.5 < (\xi - C_L)/C_L \leq 0.9\). Next, we assumed \( C_L = C_0 + C_A \kappa + C_B \delta + C_A A \) and had optimum coefficients \( C_0 = 0.05808, C_A = -0.05846, C_B = 0.02906, \) and \( C_A = 0.03008. \) In this case the error is reduced to \( 0.9 < (\xi - C_L)/C_L \leq 0.2\% \) (see Fig. 3(b)). Here we ignored the dependence of inductance on the arc angle \( \theta \), because CCL shapes are hardly changed for its value range.

We also let \( C_L = C_0 + C_A \kappa + C_B \delta + C_A A \) without using \( \xi \), finding optimum coefficients \( C_0 = 0.326, C_A = -0.03238, C_B = 0.1091, \) and \( C_A = -0.1093, \) which gave the error \( |C_L - C_L|/C_L = 1.4\% \) (see Fig. 3(c)). This evaluation is the simplest method to find rough self-inductance values without calculating \( \xi \) whereas the second one (Fig. 3(b)) gives very accurate values.

To verify these regression functions (empirical formulas), we calculated the self inductance of the ITER TF-coil set, which is reported to be 17.3 H in Ref.[2]. The ITER TF-coil CCL parameters are evaluated to be \( \kappa = 1.57, \delta \approx 0.341, A \approx 1.68, \theta_1 \approx 70^\circ, \theta_2 \approx 40^\circ, H \approx 6.31 \text{ m}, N_{TF} = 18, N_C = 134, \) and \( L_0 = 14.7 \text{ H}. \) These parameter values gave the self-inductance as \( L_0/|L| = 17.3 \text{ H}, L_0/|L| = 17.2 \text{ H}, L_0/|L| = 17.3 \text{ H}, \) and \( L_0/|L| = 17.3 \text{ H}. \) Thus, we can immediately find the value of self inductance with the quantity \( L_0 \) and three TF-coil shape parameters \( \kappa, \delta, \) and \( A \).

### 2.4 Mutual inductances between TF coils

Mutual inductances \( M_{ij} = L_0 M_k \) with \( k = |i-j| \) would also be estimated by equating

\[ M_k = M_k' + C''_{ik} A_k + C''_{jk} A_j + C''_{A_1 A_2}, \]

where the normalized value \( M_k \) was found by using the Neumann formula with the number of TF-coils \( N_{TF} = 16 \) selected for JA Demo design[4], i.e., \( i, j = 1 - 16, 0 \leq k = |i-j| \leq 8, \) and \( M_{16,k} = M_{01} \) for \( k > 8 \). Table 1 presents optimized coefficients for each \( k \) and calculated values of \( M_k' \) are compared to \( M_k \) in Fig. 3(d) for all cases, where the error was estimated to be \( |M_k - M_k'|/M_0 = 0.7\% \). (0 \leq k \leq 8).

Table 1 Optimized coefficients for calculating mutual inductances of TF coils.

| k=|j · k | C''_{ik} | C''_{jk} | C''_{ik} | C''_{jk} |
|---|---|---|---|---|---|
| 0 | 5.605×10^{-2} | -7.381×10^{-2} | 4.258×10^{-2} | -9.303×10^{-2} |
| 1, 15 | 2.355×10^{-2} | -1.792×10^{-2} | 1.079×10^{-2} | -7.924×10^{-2} |
| 2, 14 | 1.452×10^{-2} | 4.864×10^{-4} | 6.555×10^{-4} | -6.324×10^{-4} |
| 3, 13 | 9.432×10^{-2} | 5.613×10^{-4} | 4.417×10^{-4} | -4.645×10^{-4} |
| 4, 12 | 6.397×10^{-1} | 5.041×10^{-4} | 3.185×10^{-4} | -3.388×10^{-4} |
| 5, 11 | 4.603×10^{-3} | 4.307×10^{-4} | 2.438×10^{-4} | -2.559×10^{-4} |
| 6, 10 | 3.568×10^{-3} | 3.734×10^{-4} | 1.992×10^{-4} | -2.050×10^{-4} |
| 7, 9 | 3.032×10^{-3} | 3.383×10^{-4} | 1.751×10^{-4} | -1.777×10^{-4} |
| 8 | 2.866×10^{-3} | 3.267×10^{-4} | 1.675×10^{-4} | -1.691×10^{-4} |
of the winding pack is decreased by direction and then is slightly higher compared to the result shown in Fig. 3 (d) sectional area of the TF coil current (winding pack) when $I_{N_{TFC}} = 16$ and $M'_{N_{TFC}}$ calculated for $N_{TFC} = 18$ and (b) mutual inductances of ITER TF-coils estimated from this regression curve and those calculated with the Neumann formula.

(4) Calculate the normalized self-inductance $\hat{L}$ of the TF-coil set with $N'_{TFC} = N_{TFC}/2$

$$\hat{L} = \frac{N_{TFC}}{2} \left( M'_{0} + 2(M'_{1} + \cdots + M'_{N_{TFC}-1}) + M'_{N_{TFC}} \right).$$

(5) Calculate mutual inductances with $M_{ij} \approx \hat{M}_{i} L / \hat{L},$ $(k = |i - j|),$ where $\hat{L}$ is the self-inductance of the TF-coil set with $N_{TFC} = 18.$

Figure 4 (a) shows the regression curve for $M'_{N_{TFC}}$ and estimated mutual inductances for ITER TF coils with $N_{TFC} = 18,$ where the relative error $\varepsilon = |M'_{0} - \hat{M}_{0}| / \hat{M}_{0}$ for $i = j$ was 1.4% in comparison to the value calculated with the Neumann formula (see Fig. 4 (b)). This error magnitude of $\hat{M}_{0}$ that is the self inductance of the single TF coil is slightly higher compared to the result shown in Fig. 3 (d) with $\varepsilon < 0.7%.$

One of the reasons would be reduction in the cross-sectional area of the TF coil current (winding pack) when $N_{TFC}$ is increased from 16 to 18 (see Fig. 2 (b)). The size of the winding pack is decreased by \sim 10% in the toroidal direction and then $L$ is increased by \sim 0.5% (see Sec. 2.3), which gives the increment in $\hat{M}_{0}$ of 1.4% as mentioned above with taking account the contribution of the self-inductance $L_{0} \hat{M}_{0}$ to $L$ being \sim 36%.

3. Application

Figure 5 shows an example of conceptual design drawing for JA Demo [4], which shows poloidal cross section of the TF coil and the vacuum vessel (VV). We consider poloidal eddy-current inductions in and their influences on reactor structures in the case of emergency fast discharge of the TF-coil current. Eddy currents are assumed to be induced in TF-coil structures, consisted of coil cases and radial plates (see Fig. 5), and the vacuum vessel.

Circuit equations for solving this problem are written as

$$L_{1} I_{1} + M_{12} I_{2} + M_{10} I_{0} + R_{1} I_{1} = 0$$

$$L_{2} I_{2} + M_{21} I_{1} + M_{20} I_{0} + R_{2} I_{2} = 0$$

where $I$ is the current, $L$ the self-inductance, $M$ the mutual inductance, $R$ the resistance, and subscripts 0, 1, and 2 denote values of the TF-coil conductor, the coil structure, and the vacuum vessel (VV), respectively. Initial conditions are $I_{1}(0) = I_{2}(0) = 0$ and the conductor current of the TF coil is assumed to be exponentially decayed with the time constant $\tau_{d},$ i.e. $I_{0} = I_{OP} \exp(-t/\tau_{d}),$ where $I_{OP} (= 83.2 \, kA)$ is the normal operating current.

We need values of self and mutual inductances to solve the circuit equations. To find these values, we roughly drew a (almost handwritten) CCL for each structure along its contour as shown Fig. 5 (indicated by a closed line and dots), measured its dimensions, and estimated values of parameters $\kappa, \delta, A,$ and $H,$ which are presented in Table 2. Note that CCL dimensions of the TF coil and its structure have nearly the same values, i.e., $H_{1} \approx H_{0}, \xi_{1} \approx \xi_{0}$ and the coil structure and the VV are regarded as single-turn coils ($N_{1} = N_{2} = 1$).

Self and mutual inductances of these structures are written from Eq. (1) by

$$L_{0} \approx (\mu_{0}/\pi)N_{0}^{2} H_{0} \xi_{0},$$

| Table 2 Roughly estimated CCL-dimensions of TF coil, its structure, and vacuum vessel of JA DEMO. |
|---------|---------|---------|
| $R_{c}$ (m) | $R_{s}$ (m) | $R_{L}$ (m) |
| 3.55 | 15.62 | 7.66 |
| $H$ (m) | $a$ (m) | $A$ |
| 9.50 | 6.03 | 1.59 |
| $\kappa$ | $\delta$ | $\theta_{i}$ (°) |
| 1.58 | 0.320 | 48.0 |
| $l_{cc}$ (m) | | 51.1 |
| | | 41.4 |
the coil structure is given by structure and the vacuum vessel. The loop resistance of the equations presented in Appendix A or the regression Plasma and Fusion Research: Regular Articles Volume 15, 1405078 (2020) respectively. cross-sectional area of the coil case and the radial plates, factor $CCL$ length, $\mu (S RP)$ approximated that coils and structures are symmetric with respect to their equatorial planes, i.e., $H_1 = H'_1$. The self inductance of each structure can be evaluated by finding arc parameters $R_0, \theta_c, R_c$, and $Z_c$ (j = 1 - 3) described in Sec. 2.2 and then calculating the inductance factor $\xi$ with the equations presented in Appendix A or the regression function Eq. (2). Table 3 presents the inductance matrix $M_{ij}$ of the system Eq. (3).

We also need poloidal loop resistances of the TF-coil structure and the vacuum vessel. The loop resistance of the coil structure is given by $R_1 = \eta_{SSL} (L_{CCL} (N_{TFC} (S_{CC} + S_{RP}))) \approx 1.1 \mu\Omega$, where $\eta_{SSL}$ is the resistivity of SS316 (0.5 $\mu\Omega$m) at low temperature (4.2 K), $\xi_{CCL}$ (51 m) the CCL length, $S_{CC} \approx 0.94 m^2$ and $S_{RP} \approx 0.55 m^2$ are cross-sectional area of the coil case and the radial plates, respectively.

The loop resistance of the VV was estimated from the following poloidal line integral

$$R_2 = \eta_{SSH} \oint_{VV} (2\pi R(t) \Delta l(t))^{-1} dl \sim (\eta_{SSH}/\Delta VV) \varphi, \quad (4)$$

to be $-6.6 \mu\Omega$, where $\eta_{SSH}$ is the resistivity of SS316 (0.84 $\mu\Omega$m) at high temperature (100°C), $\Delta l$~$\Delta VV$ (~2 ~60 mm) the VV thickness, and a formula to calculate the factor $\varphi$ (0.94) for a uniform $\Delta VV$ is given in Appendix A. The conceptually designed VV is double-walled and has 20 mm-thick 64 poloidal ribs with a CCL length of ~41 m. Assuming their averaged width is roughly 1 m, we estimated their total loop resistance to be $27 \mu\Omega$ that reduces the VV resistance from 6.6 $\mu\Omega$ to 5.3 $\mu\Omega$.

Equations (3) are rewritten as

$$I_1 + g_1 I_1 + \lambda_1 I_1 = \lambda_0 N_0 I_0 \text{ and } I_1 + I_2 + \lambda_2 I_2 = \lambda_0 N_0 I_0,$$

where

$$\lambda_0 = 1/\tau_d \approx 0.033 \text{ s}^{-1} \text{ for } \tau_d = 30 \text{ s},$$

Table 3 Inductance matrix $M_{ij}$ for reactor structures of JA DEMO shown in Fig. 5 (in H).

| $i \ \backslash \ j$ | 0  | 1  | 2  |
|---------------------|----|----|----|
| 0                   | 4.53$\times$10$^{-1}$ | 1.47$\times$10$^{-2}$ | 9.28$\times$10$^{-3}$ |
| 1                   | 1.47$\times$10$^{-2}$ | 4.80$\times$10$^{-6}$ | 3.02$\times$10$^{-6}$ |
| 2                   | 9.28$\times$10$^{-3}$ | 3.02$\times$10$^{-6}$ | 3.02$\times$10$^{-6}$ |

$N_0 = N_{TFC} NC (= 16 \times 192)$ and we assumed or approximated that coils and structures are symmetric with respect to their equatorial planes, i.e., $H_1 = H'_1$. If $f_{RP}$ for reactor structures of JA DEMO show sinusoidal changes with $20 \text{ mm-thick 64 poloidal ribs with a CCL length of 41 m}$, we estimated or approximated solutions are well agreed with accurate ones, i.e., the eddy current of TF-coil structure is hardly reduced.

Sec. 2.2 and then calculating the inductance factor $\xi$ with the equations presented in Appendix A or the regression function Eq. (2). Table 3 presents the inductance matrix $M_{ij}$ of the system Eq. (3).

We also need poloidal loop resistances of the TF-coil structure and the vacuum vessel. The loop resistance of the coil structure is given by $R_1 = \eta_{SSL} (L_{CCL} (N_{TFC} (S_{CC} + S_{RP}))) \approx 1.1 \mu\Omega$, where $\eta_{SSL}$ is the resistivity of SS316 ($-0.5 \mu\Omega$m) at low temperature (4.2 K), $L_{CCL}$ ($51 m$) the CCL length, $S_{CC} \approx 0.94 m^2$ and $S_{RP} \approx 0.55 m^2$ are cross-sectional area of the coil case and the radial plates, respectively.

The loop resistance of the VV was estimated from the following poloidal line integral

$$R_2 = \eta_{SSL} \oint_{VV} (2\pi R(t) \Delta l(t))^{-1} dl \sim (\eta_{SSL}/\Delta VV) \varphi, \quad (4)$$

to be $-6.6 \mu\Omega$, where $\eta_{SSL}$ is the resistivity of SS316 (0.84 $\mu\Omega$m) at high temperature (100°C), $\Delta l$~$\Delta VV$ (~2 ~60 mm) the VV thickness, and a formula to calculate the factor $\varphi$ (0.94) for a uniform $\Delta VV$ is given in Appendix A. The conceptually designed VV is double-walled and has 20 mm-thick 64 poloidal ribs with a CCL length of ~41 m. Assuming their averaged width is roughly 1 m, we estimated their total loop resistance to be $27 \mu\Omega$ that reduces the VV resistance from 6.6 $\mu\Omega$ to 5.3 $\mu\Omega$.

Equations (3) are rewritten as

$$I_1 + g_1 I_1 + \lambda_1 I_1 = \lambda_0 N_0 I_0 \text{ and } I_1 + I_2 + \lambda_2 I_2 = \lambda_0 N_0 I_0,$$

where

$$\lambda_0 = 1/\tau_d \approx 0.033 \text{ s}^{-1} \text{ for } \tau_d = 30 \text{ s},$$

$\lambda_1 = R_1/L_1 = N_0^2 R_1/L_0 \approx 0.22 \text{ s}^{-1}$ and

$$\lambda_2 = R_2/L_2 = N_0^2 R_2/(g L_0) \approx 1.8 \text{ s}^{-1}$$

with $g = L_2/L_1 \approx 0.62$. From these values for this case, we can say that $\lambda_2 \gg \lambda_1, \lambda_0$. If $I_1 \gg I_2$, the equations become

$$I_1 + \lambda_1 I_1 = \lambda_0 N_0 I_0 \text{ and } I_2 + \lambda_2 I_2 = \lambda_1 I_1,$$

and then we have approximated solutions

$$I_1 = \lambda_0 N_0 I_0 (e^{-\lambda_1 t} - e^{-\lambda_0 t})/(\lambda_0 - \lambda_1) \text{ and } I_2 \approx (\lambda_1/\lambda_2) I_1 (\approx 1) \text{ for } \lambda_2 \gg \lambda_1, \lambda_0.$$

Figure 6 shows time evolutions of eddy currents induced in TF-coil structure ($I_1$) and vacuum vessel ($I_2$) (red curves) for $\tau_d = 30 \text{ s}$, where blue curves show approximated solutions for $\lambda_2 \gg \lambda_1, \lambda_0$, and the black curve denotes $I_2$ calculated by assuming $I_1 = 0$, i.e., there is no TF-coil structure.

Fig. 6 Time evolutions of eddy currents induced in TF-coil structure ($I_1$) and vacuum vessel ($I_2$) (red curves) for $\tau_d = 30 \text{ s}$, where blue curves show approximated solutions for $\lambda_2 \gg \lambda_1, \lambda_0$, and the black curve denotes $I_2$ calculated by assuming $I_1 = 0$, i.e., there is no TF-coil structure.
Appendix B. Using this, we evaluated it for the peak current of $I_2$ to be 56 MPa for $\tau_d = 30$ s and 74 MPa for $\tau_d = 20$ s, which are lower than the allowable stress (~143 MPa) of the VV material (SS316L).

The estimated Tresca stress of 56 MPa for $\tau_d = 30$ s, however, is smaller that (~100 MPa) obtained by a finite element analysis (FEA) [9] with nearly the same conditions. One of the reasons is that the FEA excluded the eddy current ($I_1$) induced in the TF-coil structure in its modeling, which increases the peak VV eddy current with a factor 1.32 (~0.0177/0.0134, see Fig. 6), i.e., the stress estimated in the FEA should be reduced to 76 MPa. Then the relative estimation error becomes 26%. This residual error would be mainly arisen from the VV model shape drawn in Fig. 5 being quite different from the design drawing of its outboard structure that has maintenance ports.

4. Summary and Conclusions

We have presented a method for easily estimating self and mutual inductances of TF coils and toroidal-shape structures, defining their cross-sectional shape parameters.

In Chap. 2, we tried to estimate the self-inductance of TF-coil set, calculating the toroidal magnetic flux passing through the cross-sectional area enclosed by its CCL. The shape dependence of inductances is then expressed by the inductance factor $\xi$ of a function of shape parameters: the elongation $\kappa$, the triangularity $\delta$, the aspect ratio $A$, and the arc angle $\theta_1$, and we obtained its empirical formula by a regression analysis. This formula would be useful to estimate the inductance of a toroidal-shape structure for its poloidal eddy current analysis.

The inductance factor $\xi$ was compared to that obtained with the Neumann formula for the TF-coil set that has finite-size current-flow areas and a discontinuous structure in the toroidal direction. We found that $\xi$ has relative errors within 1.5% in value ranges of shape parameters treated in this paper. This relative error was reduced to 0.2% by adding the correction term expressed by a linear combination of $\kappa$, $\delta$, and $A$ (without $\theta_1$). The inductance value with this accuracy would be sufficient for using in the TF-coil design and its safety considerations. We also found that their linear combination directly gives the self-inductance without calculating $\xi$, where the relative estimation error was within 1.4% for its optimized coefficients. These empirical formulas were verified by calculating the self-inductance of the ITER TF-coil set to have its known value (~17.3 H).

The mutual inductance $M_{ij}$ ($1 \leq i, j \leq N_{TFC}/2$) was also expressed by a linear combination of $\kappa$, $\delta$, and $A$ with relative errors less than 0.7%. We obtained its optimum coefficients for each pair of TF coils with $N_{TFC} = 16$ and showed that this result can also be applied to the case of $N_{TFC} = 18$ selected for the ITER coil set. A bit of relative error (~1.4%), however, appeared in the self-inductance $M_{00}$ of a single ITER TF-coil because it becomes thinner with increasing $N_{TFC}$, which increases the self-inductance.

In Chap. 3 we demonstrated an analysis for eddy current inductions during the fast discharge of the TF coil set of JA DEMO, calculating self and mutual inductances of the TF-coil structure and the vacuum vessel (VV), and estimated the peak Joule heat generated in the TF-coil structure and the Tresca stress in the VV inboard wall. Although the result for the VV stress was not well agreed with the FEA due to a poor modeling of the VV structure, this simplified analysis would be useful to verify the analysis model, understand the result, and make a plan for the large scale FEA.

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Appendix A Inductance Factor

The inductance factor $\xi$ in Eq. (1) is written by

$$\xi := \frac{1}{H + H'} \sum_{k=1}^{6} \int_{R_{ik}}^{R_{2k}} \frac{|Z_k(R)|}{R} dR = \sum_{k=1}^{6} \xi_k,$$

with $Z_k(R) = Z_{ck} + R_{ck} \sqrt{1 - \left(\frac{R}{R_{ck}}\right)^2}$, which becomes

$$\xi_k = \frac{1}{H + H'} \left( R_{ck} \int_{\tau_{ik}}^{\tau_{2k}} \sqrt{1 - \tau^2} \right)^{1/2} \int_{\tau_{ik}}^{\tau_{2k}} \frac{Z_{ck}}{R_{ck}} \ln \left( \frac{R_{ck}}{R_{ik}} \right),$$

where $\omega_k = R_{ck}/R_{ik}$, $\tau_{sk} = (R_{sk} - R_{ik})/R_{ck} = \cos \Theta_{sk}$, $(s = 1, 2)$, with

$$\Theta_{11} = \theta_1, \quad \Theta_{21} = \theta_1 + \theta_2, \quad \Theta_{22} = \theta_1,$$

$\Theta_{13} = \theta_1 + \theta_2 + \theta_3 = \pi$, and $\Theta_{23} = \theta_1 + \theta_2$.

The integral in $\xi_k$ is carried out analytically [10] to be

$$\xi_k(H + H') = R_k \left[ \omega_k \sin^{-1} \frac{\tau}{\sqrt{1 - \tau^2}} + \frac{p_k (\Lambda(\omega_k) - \Lambda(\tau))}{\tau} \right]_{\tau_{ik}}^{\tau_{2k}}$$

with

$$\left| \frac{1}{\sqrt{|p_k|}} \sin^{-1} \left( \frac{1 + \omega_k \tau}{\tau + \omega_k} \right) \right|$$

for $p_k < 0$, $\left| \frac{\sqrt{1 - \tau^2}}{\omega_k (\tau + \omega_k)} \right|$ for $p_k = 0$, and

$$\left| \frac{1}{\sqrt{|p_k|}} \ln \left( \frac{1 + \tau \omega_k - \sqrt{p_k (1 - \tau^2)}}{\tau + \omega_k} \right) \right|$$

for $p_k > 0$.

The quantity $\varphi$ in Eq. (4) that gives the VV loop resistance is also expressed by using $\Lambda$ as

$$\varphi = \int \frac{dl}{2\pi R(l)} \approx \frac{1}{2\pi} \int \sqrt{1 + (dZ(R)/dR)^2} \frac{dR}{R}.$$
\[
\sigma_{\theta} \approx \frac{B_{\text{VVI}} J_{\text{VVI}} \Delta_{\text{VVI}}}{\Delta_{\text{VVI}} R_{\text{VVI}}} \
\approx -B_{\text{VVI}} J_{\text{VVI}} R_{\text{VVI}}.
\]

using the cylindrical thin shell model.

We thus obtain the Tresca stress as

\[
\sigma_{\text{Tresca}} = |\sigma_{\theta} - \sigma_Z| = \zeta B_{\text{VVI}} J_{\text{VVI}} R_{\text{VVI}},
\]

where \(\zeta = 1 + (A_{\text{VVO}} - 1) \ln(A_{\text{VVO}} + 1) / (A_{\text{VVO}} - 1) / 2A_{\text{VVO}}\) with the VV aspect ratio \(A_{\text{VVO}} = (R_{\text{VVO}} + R_{\text{VVI}}) / (R_{\text{VVO}} - R_{\text{VVI}})\).

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