DIVISORS AND CURVES ON LOGARITHMIC MAPPING SPACES

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Abstract. We determine the rational class and Picard groups of the moduli space of stable logarithmic maps in genus zero, with target projective space relative a hyperplane. For the class group we exhibit an explicit basis consisting of boundary divisors. For the Picard group we exhibit a spanning set indexed by piecewise-linear functions on the tropicalisation. In both cases a complete set of boundary relations is obtained by pulling back the WDVV relations from the space of stable curves. Our proofs hinge on a controlled technique for manufacturing test curves in logarithmic mapping spaces, opening up the topology of these spaces to further study.

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Introduction

Fix integers $r \geq 2$, $n \geq 3$, $d \geq 1$ and a non-negative ordered partition $\alpha = (\alpha_1, \ldots, \alpha_n) \vdash d$. This discrete data determines a moduli space

$$\overline{M}_{0,\alpha}(\mathbb{P}^r|H)$$

of stable logarithmic maps, where $H \subseteq \mathbb{P}^r$ is a hyperplane [GS13, Che14, AC14]. It is a logarithmically smooth Deligne–Mumford stack, with dense interior parametrising degree $d$ maps $f : \mathbb{P}^1 \to \mathbb{P}^r$ satisfying

$$f^*H = \sum_{i=1}^n \alpha_i x_i$$

where $x_1, \ldots, x_n \in \mathbb{P}^1$ are marked points (note that $\alpha$ determines both $n$ and $d$). We investigate the divisor theory of this space.

0.1. Faithful tropicalisation. As a preliminary result, we identify the boundary complex of $\overline{M}_{0,\alpha}(\mathbb{P}^r|H)$ with a moduli space of stable tropical maps.

Theorem X (Theorem 1.1). There is a natural isomorphism of cone complexes

$$\Sigma \overline{M}_{0,\alpha}(\mathbb{P}^r|H) = \Sigma_{0,\alpha}$$

where $\Sigma \overline{M}_{0,\alpha}(\mathbb{P}^r|H)$ is the tropicalisation of the moduli space of stable logarithmic maps, and $\Sigma_{0,\alpha}$ is the moduli space of stable tropical maps to $\mathbb{R}_{\geq 0}$ (see Section 1.1).

In particular, the number $N(\alpha)$ of irreducible boundary divisors in $\overline{M}_{0,\alpha}(\mathbb{P}^r|H)$ is equal to the number of one-dimensional combinatorial types of stable tropical map to $\mathbb{R}_{\geq 0}$ with tangency profile $\alpha$. Calculating $N(\alpha)$ for specific values of $\alpha$ is a combinatorial exercise; see Section 1.4 for a discussion in the case $\alpha = (d, 0, \ldots, 0)$. 

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0.2. **Class group.** We begin with Weil divisors. The key geometric input is a method for manufacturing test curves in the moduli space with strongly controlled intersection against the boundary.

**Theorem Y** (Theorem 3.6). The rational class group of \( \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \) has an explicit basis consisting of boundary divisors. It has dimension

\[ N(\alpha) - \left( \frac{n - 1}{2} \right) + 1. \]

In particular, the dimension of the rational class group does not depend on \( r \). Moreover, all relations between boundary divisors are pulled back from the WDVV relations on \( \overline{M}_{0,n} \).

In the above formula, \( N(\alpha) \) counts the number of irreducible boundary divisors (see Theorem X), while the remaining terms count the number of independent linear relations between these divisors.

0.3. **Picard group.** The space \( \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \) is logarithmically smooth, i.e. it has toric singularities. It is almost never smooth, not even as an orbifold; consider e.g. the following combinatorial type of stable tropical map, with \( \alpha = (2, 0, 0) \)

![Diagram of a tropical map](image)

The edge lengths satisfy \( e_1 + e_2 = e_3 + e_4 \) and so the local singularity type of \( \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \) is given by the quadric cone

\[ V(z_1z_2 - z_3z_4) \subseteq \mathbb{A}^4_z. \]

It follows that \( \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \) is not even \( \mathbb{Q} \)-factorial, and most boundary components are not even \( \mathbb{Q} \)-Cartier. This contrasts sharply with the space \( \overline{M}_{0,n}(\mathbb{P}^r, d) \) of ordinary stable maps.

Consequently, the class group and the Picard group of \( \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \) differ. We leverage our complete understanding of the former to obtain a combinatorial description of the latter.

**Theorem Z** (Theorem 3.8). The rational Picard group of \( \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \) is generated by divisors corresponding to integral piecewise-linear functions on \( \Sigma_{0,\alpha} \). A complete set of relations between these divisors is obtained by pulling back the WDVV relations from \( \overline{M}_{0,n} \).

0.4. **Strategy.** Theorems Y and Z are analogous to [Pan99, Theorem 2] (see also [Opr05] for a more general calculation). The basic strategy is similar: to probe divisors in the moduli space by constructing appropriate test curves

\[ \mathbb{P}^1 \rightarrow \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \]

with controlled intersection against the boundary. The details, however, are different. We do not start with an arbitrary test curve and then modify it. Instead, we directly manufacture a roster of test curves which is sufficiently rich and sufficiently controlled to establish the required linear independences. This alternative approach leads directly to an explicit basis for the class group. The test curve construction is considerably more intricate than that of [Pan99] due to the combinatorial complexity of boundary divisors in the logarithmic mapping space, and the strong control we maintain over specific boundary intersections.

The proof proceeds as follows. We first use excision to show that the class group is generated by boundary divisors (Proposition 1.5).\(^1\) We then partition these into **aliens**, **airbornes** and **terrestrials**

\(^1\)In contrast, boundary divisors are insufficient to generate the class group of \( \overline{M}_{0,n}(\mathbb{P}^r, d) \), see Remark 1.6.
(Definition 3.1). An alien divisor has generic point parametrising a curve with two irreducible components, one of which contains two markings and has degree 0, the other of which contains all the other markings and has degree $d$. The airborne divisor has generic point parametrising a smooth curve mapped inside $H$. Boundary divisors which are neither alien nor airborne are referred to as terrestrial.

A consequence of our test curve construction (Section 2) is that for every terrestrial divisor $D$, there exists a test curve which intersects $D$ and does not intersect any other terrestrial divisor. It follows that the terrestrial divisors form a linearly independent subset of the class group. Routine surgery on test curves then shows that the same is true for the set of terrestrial and airborne divisors (Proposition 3.3).

Quotienting, we are left with a space spanned (but not based) by the alien divisors. We show that this quotient has dimension $n$, by establishing upper and lower bounds (Proposition 3.5). The upper bound is obtained by studying pullbacks of relations from the space of stable curves. The lower bound is obtained by exhibiting a linearly independent set (17) of alien divisors of size $n$. As before, linear independence is demonstrated by intersecting with suitable test curves.

This shows that the class group has a basis consisting of the airborne divisor, the terrestrial divisors, and the alien divisors listed in (17). As a corollary we conclude that all relations amongst boundary divisors are pulled back from the WDVV relations on $\overline{M}_{0,n}$. This establishes Theorem Y.

The Picard group embeds in the class group as the set of locally principal divisors. Since the class group is generated by boundary divisors, the Picard group is generated by boundary Cartier divisors, and these are indexed by piecewise-linear functions on the tropicalisation. Finally, all relations in the Picard group come from relations in the class group and hence, by Theorem Y, from relations on $\overline{M}_{0,n}$. This establishes Theorem Z.

It is perhaps surprising that there exist enough test curves to establish all linear independences between divisors. Our construction of test curves shows that

$$\dim A_1(\overline{M}_{0,\alpha}(\mathbb{P}^r | H))_{\mathbb{Q}} \geq \dim A_{n-1}(\overline{M}_{0,\alpha}(\mathbb{P}^r | H))_{\mathbb{Q}}.$$  

There is no a priori reason why this should hold, since Poincaré duality can fail for varieties with toric singularities, see [KP08, Example 4.2] and [Tot14, Section 1].

0.5. Outlook. We expect the test curve construction of Section 2 and the linear independence arguments of Section 3 to open up the study of the topology of logarithmic mapping spaces.

Problem 0.1. Investigate piecewise-linear functions on $\Sigma_{0,\alpha}$.

Theorem Z is less explicit than Theorem Y, because on $\Sigma_{0,\alpha}$ it is harder to enumerate piecewise-linear functions than it is to enumerate rays. A more conceptual description of these piecewise-linear functions is desirable, e.g. in terms of “tautological” functions arising from the modular interpretation of $\Sigma_{0,\alpha}$.

Problem 0.2. Investigate divisors on spaces of prestable logarithmic maps to the Artin fan $[\mathbb{A}^1 / \mathbb{G}_m]$.

A minor modification of our test curve construction (simply dropping Section 2.6) also produces test curves in such spaces. Boundary relations on $\mathfrak{m}_{0,n}$ will likely play a role [BS22, BS23]. New ideas are required to handle curve components with negative degrees. This will also open up the divisor theory of spaces of logarithmic maps to general smooth pairs.

Problem 0.3. Investigate divisors on $\overline{M}_{0,\alpha}(\mathbb{P}^r | H_0 + \ldots + H_k)$ for $H_0 + \ldots + H_k$ a subset of the toric boundary.

Such spaces are also logarithmically smooth. The test curve construction must be modified, for which a systematic understanding of the possible shapes of boundary divisors is required, analogous to Proposition 1.4. The case of full toric boundary, at least, is well-understood [Ran17].

Problem 0.4. Investigate higher codimension cycles on logarithmic mapping spaces.
This requires a more systematic understanding of higher codimension boundary strata, and a general method for manufacturing higher dimension test varieties (see [Tar13, Tar15]). First steps towards the higher codimension topology of logarithmic mapping spaces are taken in [Kan23].

**Problem 0.5.** Investigate cycles with \( \mathbb{Z} \) coefficients.

It is still possible to intersect our test curves with boundary divisors in this context. However, care is required around intersection multiplicities and stabiliser groups.

0.6. **Outline.** In Section 1 we establish a faithful tropicalisation result (Theorem 1.1) which allows us to combinatorially enumerate boundary divisors (Corollary 1.3). We then use excision to show that the class group is generated by these boundary divisors (Proposition 1.5).

In Section 2 we outline a general method for manufacturing test curves in the moduli space. Our test curves are morphisms \( \mathbb{P}^1 \to \overline{M}_{0,n}(\mathbb{P}^r|H) \) which intersect the boundary at finitely many points away from strata of codimension \( \geq 2 \). We establish strong control over which boundary strata a test curve can intersect.

In Section 3 we prove the main results. We furnish an explicit basis for the rational class group, consisting of boundary divisors (Theorem 3.6). The test curve construction is crucial for proving that various collections of boundary divisors are linearly independent. As a corollary, we conclude that all relations amongst boundary divisors are pulled back from the WDVV relations on \( \overline{M}_{0,n} \). Finally in Section 3.4 we leverage our control over the class group to give a combinatorial description of the Picard group (Theorem 3.8).

In Appendix A we record for posterity a complete linearly independent set of relations between boundary divisors in \( \overline{M}_{0,n} \).

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1. **Generators**

For background on stable logarithmic maps, stable tropical maps, and combinatorial types thereof, we refer to [ACGS20, Section 2].

1.1. **Tropical moduli.** A **combinatorial type of stable tropical map** to \( \mathbb{R}_{\geq 0} \) with tangency profile \( \alpha \) consists of the following data:

- **Source graph.** A finite tree \( \Gamma \) consisting of vertices \( V(\Gamma) \), finite edges \( E(\Gamma) \), and legs \( L(\Gamma) \) equipped with a bijection

  \[ L(\Gamma) \cong \{1, \ldots, n\}. \]

  Each vertex \( v \in V(\Gamma) \) is equipped with a degree label \( d_v \in \mathbb{N} \) such that \( \sum_{v \in V(\Gamma)} d_v = d \).

- **Image cones.** Faces of \( \mathbb{R}_{\geq 0} \) associated to every vertex and edge of \( \Gamma \)

  \[ v \leadsto \sigma_v, \quad e \leadsto \sigma_e, \]

  such that if \( v \leq e \) then \( \sigma_v \leq \sigma_e \).
• **Slopes.** For each oriented edge $\vec{e} \in \vec{E}(\Gamma)$ a slope $m_{\vec{e}} \in \mathbb{Z}$ satisfying $m_{\vec{e}} = -m_{\vec{e}}$. At each vertex $v \in V(\Gamma)$ these slopes must satisfy the balancing condition

$$d_v = \sum_{v \leq e} m_{\vec{e}} + \sum_i \alpha_i$$

where each edge $e$ is oriented away from $v$, and the second sum is over legs supported at $v$.

Each vertex is required to be stable, meaning that either $d_v > 0$ or $v$ is at least trivalent. To each combinatorial type we associate a strictly convex rational polyhedral cone constituting the associated tropical moduli. This is embedded in an orthant coordinatised by the edge lengths $\ell_e$ and the vertex positions $f(v)$ and cut out by continuity equations

$$f(v_1) + m_{\vec{e}} \ell_e = f(v_2)$$

indexed by oriented edges $\vec{e}$ from $v_1$ to $v_2$. The **dimension** of a combinatorial type is the dimension of the corresponding cone. Specialisations of combinatorial types induce face inclusions, and the colimit of the resulting diagram is a cone complex which we denote

$$\Sigma_{0, \alpha}$$

and refer to as the **moduli space of stable tropical maps**.

### 1.2. Boundary and tropicalisation.

Since the pair $(\mathbb{P}^r | H)$ is convex, the space $\overline{M}_{0, \alpha}(\mathbb{P}^r | H)$ is logarithmically smooth over the trivial logarithmic point, i.e. is a toroidal embedding [KKMSD73, AK00]. It has dense interior

$$M_{0, \alpha}(\mathbb{P}^r | H) \subseteq \overline{M}_{0, \alpha}(\mathbb{P}^r | H)$$

parametrising maps from a smooth source curve that do not factor through the hyperplane $H$. The **boundary** is the complement

$$\partial \overline{M}_{0, \alpha}(\mathbb{P}^r | H) := \overline{M}_{0, \alpha}(\mathbb{P}^r | H) \setminus M_{0, \alpha}(\mathbb{P}^r | H).$$

It is a union of irreducible hypersurfaces. The tropicalisation

$$\Sigma \overline{M}_{0, \alpha}(\mathbb{P}^r | H)$$

is a cone complex whose cones are in bijective, inclusion-reversing correspondence with boundary strata. Tropicalisation of families of stable logarithmic maps [ACGS20, Section 2.5] produces a natural morphism of cone complexes

$$\varphi : \Sigma \overline{M}_{0, \alpha}(\mathbb{P}^r | H) \to \Sigma_{0, \alpha}.$$ 

The following is a case of faithful tropicalisation of moduli spaces [Cap14, ACP15, Uli15, AM16, CMR16, CHMR16, Gro16, Ran17, BBC+20, CCUW20, MR20, LU21, OO21, MW22, MMUV22, BCK23, Ken23, Nab23].

**Theorem 1.1 (Theorem X)**. \( \varphi \) is an isomorphism of cone complexes.

**Proof.** The tropical interpretation of basicness for stable logarithmic maps [GS13, Section 1.4] ensures that $\varphi$ maps every source cone isomorphically onto a target cone. Hence it suffices to show that every target cone is the image of a unique source cone.

A target cone $\tau \leq \Sigma_{0, \alpha}$ corresponds to a combinatorial type of stable tropical map. Let

$$M_\tau(\mathbb{P}^r | H) \subseteq \overline{M}_{0, \alpha}(\mathbb{P}^r | H)$$

denote the locally-closed locus of logarithmic maps which tropicalise to this combinatorial type. A standard lifting argument for smooth pairs (see e.g. [BNR22, Lemma 3.1]) shows that $M_\tau(\mathbb{P}^r | H)$ is nonempty. This ensures that there exists at least one cone in $\Sigma \overline{M}_{0, \alpha}(\mathbb{P}^r | H)$ whose image is $\tau$. To show that this cone is unique, we must show that $M_\tau(\mathbb{P}^r | H)$ is irreducible.

Consider the tower of forgetful morphisms

\begin{equation}
\overline{M}_{0, \alpha}(\mathbb{P}^r | H) \to \mathfrak{M}_{0, \alpha}(\mathcal{A}/\mathcal{D}) \to \mathfrak{M}_{0, n}^{\log} \to \mathfrak{M}_{0, n}.
\end{equation}
where $\mathcal{M}_{0,\alpha}(\mathcal{A}|\mathcal{D})$ is the space of prestable logarithmic maps to the Artin fan and $\mathcal{M}_{0,\alpha}^{\log}$ is the space of logarithmic curves. For background, see [AW18, Section 3], [AMW14, Section 3.5], and [GS13, Appendix A].

The proof strategy is to pass up the tower (2), constructing the locally-closed stratum $M_\tau(\mathcal{P}\mathcal{r}|H)$ inductively. At each step we deduce irreducibility of the given stratum by examining the fibres over the previous stratum. Starting at the bottom, the combinatorial type of stable tropical map indexing $\tau$ contains the data of the marked dual graph $\Gamma$ of the source curve. This defines a locally-closed stratum

$$\mathcal{M}_\Gamma \subseteq \mathcal{M}_{0,\alpha}.$$ 

Note that $\mathcal{M}_\Gamma$ is irreducible. It fits into the following fibre square (see e.g. [CN22, Section 1.4])

$$\begin{array}{ccc}
\mathcal{M}_\Gamma & \longrightarrow & \mathcal{M}_{0,\alpha} \\
\downarrow & \downarrow & \\
BT_\Gamma & \longrightarrow & \mathcal{M}_{0,\alpha}^{\log}
\end{array}$$

where $\sigma_\Gamma \leq \Sigma \mathcal{M}_{0,\alpha}$ is the smooth cone coordinatised by the edge lengths of $\Gamma$. There is a natural morphism of cones $\tau \to \sigma_\Gamma$ and a locally-closed stratum

$$\mathcal{M}_\tau^{\log} \subseteq \mathcal{M}_{0,\alpha}^{\log}$$

parametrising logarithmic curves whose tropicalisation is the pullback of the universal tropical curve along $\tau \to \sigma_\Gamma$. This fits into the following fibre square (see e.g. [Ols03, Corollary 5.25])

$$\begin{array}{ccc}
\mathcal{M}_\tau^{\log} & \longrightarrow & \mathcal{M}_\Gamma \\
\downarrow & \downarrow & \\
BT_\tau & \longrightarrow & BT_{\sigma_\Gamma}
\end{array}$$

The lattice morphism $N_\tau \to N_{\sigma_\Gamma}$ factors through its image

$$N_\tau \twoheadrightarrow N_\tilde{\tau} \hookrightarrow N_{\sigma_\Gamma},$$

and the same holds for the map of classifying stacks

$$BT_\tau \to BT_{\tilde{\tau}} \to BT_{\sigma_\Gamma}.$$ 

A direct argument (see e.g. [CN22, Lemma 4.5]) shows that the first morphism is a gerbe for

$$\ker(N_\tau \to N_{\tilde{\tau}}) \otimes \mathbb{G}_m$$

while the second morphism is a principal bundle for

$$\text{coker}(N_{\tilde{\tau}} \to N_{\sigma_\Gamma}) \otimes \mathbb{G}_m.$$ 

Both these groups are algebraic tori; in the latter case, because the inclusion $N_{\tilde{\tau}} \hookrightarrow N_{\sigma_\Gamma}$ is saturated. In particular, they are irreducible. It follows that the generic fibre of

$$\mathcal{M}_\tau^{\log} \to \mathcal{M}_\Gamma$$

is irreducible, and hence $\mathcal{M}_\tau^{\log}$ is irreducible. Next, consider the locally-closed stratum

$$\mathcal{M}_\tau(\mathcal{A}|\mathcal{D}) \subseteq \mathcal{M}_{0,\alpha}(\mathcal{A}|\mathcal{D})$$

parametrising prestable logarithmic maps to the universal target whose combinatorial type is that indexing $\tau$. We claim that the forgetful morphism

$$\mathcal{M}_\tau(\mathcal{A}|\mathcal{D}) \to \mathcal{M}_\tau^{\log}$$

is bijective on geometric points. Indeed given a logarithmic curve $\mathcal{C}$, a morphism $\mathcal{C} \to (\mathcal{A}|\mathcal{D})$ is equivalent to a piecewise-linear function on $\Sigma \mathcal{C}$ (see e.g. [ACGS20, Proposition 2.10]). However this piecewise-linear function is already determined by the choice of combinatorial type, and hence there is a unique lift. We conclude that $\mathcal{M}_\tau(\mathcal{A}|\mathcal{D})$ is irreducible.
Finally we describe the fibres of
\[ M_\tau(\mathbb{P}^r|H) \to \mathcal{M}_\tau(A|D). \]
Choose coordinates on \( \mathbb{P}^r \) so that \( H = H_0 \) is the first coordinate hyperplane. A geometric point of \( \mathcal{M}_\tau(A|D) \) consists of a marked logarithmic curve \( C \) together with a piecewise-linear function on \( \Sigma C \) with associated line bundle-section pair \( (L, f_0) \). A lift to \( M_\tau(\mathbb{P}^r|H) \) consists of the data of additional sections \( f_1, \ldots, f_r \in H^0(C, L) \) such that the tuple \([f_0, f_1, \ldots, f_r]\) has no basepoints. The space of such sections is a Zariski open subset of affine space, hence is irreducible. We conclude that \( M_\tau(\mathbb{P}^r|H) \) is irreducible. \( \square \)

**Remark 1.2.** The specific geometry of \( (\mathbb{P}^r|H) \) is invoked only in the final step of the preceding proof, where we utilise our understanding of the fibres of (4). For a general smooth pair such fibres are difficult to describe: they may be empty, or may have multiple components. For examples, see [CvGKT21, Section 1.4].

**Corollary 1.3.** The irreducible boundary divisors in \( \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \) are indexed by combinatorial types of stable tropical maps to \( \mathbb{R} \geq 0 \) with tangency profile \( \alpha \), whose associated tropical moduli cone is one-dimensional.

**Proof.** Since \( \varphi \) is an isomorphism, in particular it identifies the sets of rays. \( \square \)

We let \( N(\alpha) \) denote the number of irreducible boundary divisors in \( \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \). By the preceding result, this equals the number of one-dimensional combinatorial types of stable tropical map. This set can be described quite explicitly.

**Proposition 1.4.** The combinatorial types of stable tropical maps with one-dimensional tropical moduli fall into the following families

**Rocket**

\[
\begin{align*}
C_1 & \quad m_1 \\
C_2 & \quad m_2 \\
\vdots & \quad \vdots \\
C_k & \quad m_k \\
C_0 & \quad d_0
\end{align*}
\]

The distribution of the markings \( x_i \) is omitted, but can be arbitrary as long as the resulting map is stable and balanced. For rockets we assume \( k \geq 1 \) and \( d_j \geq m_j > 0 \). The balancing conditions at the vertices are

\[
\begin{align*}
C_j : \quad \sum_{x_i \in C_j} \alpha_i &= d_j - m_j, \\
C_0 : \quad \sum_{x_i \in C_0} \alpha_i &= d_0 + \sum_{j=1}^k m_j.
\end{align*}
\]

For the airborne, the source curve is smooth and mapped entirely inside the divisor. For binaries, we assume

\[
\sum_{x_i \in C_j} \alpha_i = d_j
\]

for \( j \in \{1, 2\} \), in order to guarantee that the connecting edge has weight zero.

**Proof.** If there are two or more vertices mapped into \( \mathbb{R}_{>0} \) then the positions \( f(v) \) provide free parameters, so the tropical moduli has dimension \( \geq 2 \). It follows that there is at most one vertex mapped into \( \mathbb{R}_{>0} \). Given this, it is straightforward to deduce that the only possibilities are the types illustrated above. \( \square \)

The overwhelming majority of combinatorial types are rockets; in other contexts these have been referred to as combs [Gatt02, Lemma 1.12 and Definition 2.2].
1.3. **Excision.** We establish the main generation result.

**Proposition 1.5.** $\text{Cl}_Q(\overline{M}_{0,\alpha}(\mathbb{P}^r|H))$ is generated by classes of boundary divisors.

**Proof.** Consider the dense interior

$$U = M_{0,\alpha}(\mathbb{P}^r|H) = \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \setminus \partial \overline{M}_{0,\alpha}(\mathbb{P}^r|H)$$

parametrising maps from a smooth source curve that do not factor through $H$. We will show that $U$ embeds as an open subset of affine space, and hence has trivial class group.

Consider a map $\mathbb{P}^1 \to \mathbb{P}^r$ intersecting $H$ at the markings with tangency profile $\alpha$. This has a unique logarithmic lift, obtained by equipping the source $\mathbb{P}^1$ with the divisorial logarithmic structure corresponding to the markings, and the base $\text{Spec} \mathbb{k}$ with the trivial logarithmic structure. Consequently we may identify $U$ with a locus in the space of ordinary stable maps.

Since $n \geq 3$ we may fix the first three markings to be $0, 1, \infty$. The moduli for the remaining markings is

$$\left(\mathbb{P}^1 \setminus \{0, 1, \infty\}\right)^{n-3} \setminus \Delta$$

where $\Delta$ is the large diagonal. Finally the map $\mathbb{P}^1 \to \mathbb{P}^r$ is given by specifying sections $f_0, \ldots, f_r$ of $\mathcal{O}_{\mathbb{P}^1}(d)$ up to overall scaling. We do not need to quotient by automorphisms of the source curve; these have been rigidified by fixing the first three markings. The section $f_0$ agrees up to scaling with

$$f_0 = \prod_{i=1}^n s_i^{\alpha_i}$$

where $s_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ is a section cutting out the marking $x_i$. We fix one such section $f_0$. The remaining moduli is the choice of sections $f_1, \ldots, f_r$. There is no need to quotient by the overall scaling; this has been rigidified by fixing $f_0$. The moduli is the dense open subset of

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))^{\otimes r}$$

consisting of sections $(f_1, \ldots, f_r)$ such that the linear system spanned by $f_0, f_1, \ldots, f_r$ is basepoint-free. We conclude that $U$ is an open subset of

$$\left(\left(\mathbb{P}^1 \setminus \{0, 1, \infty\}\right)^{n-3} \setminus \Delta\right) \times H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))^{\otimes r}$$

which is itself open in an affine space. It follows that $\text{Cl}_Q(U) = 0$. By the excision sequence for Chow groups [Ful98, 1.8] we obtain

$$\bigoplus_D \mathbb{Q} \cdot D \to \text{Cl}_Q(\overline{M}_{0,\alpha}(\mathbb{P}^r|H)) \to \text{Cl}_Q(U) = 0$$

where the direct sum is over irreducible components of the boundary $\overline{M}_{0,\alpha}(\mathbb{P}^r|H) \setminus U$. This proves that $\text{Cl}_Q(\overline{M}_{0,\alpha}(\mathbb{P}^r|H))$ is generated by classes of boundary divisors, as claimed. □

**Remark 1.6.** The above result contrasts with [Pan99, Lemma 1.1.1] where the interior of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ contributes nontrivially to the class group. Recall from Proposition 1.4 that the boundary of $\overline{M}_{0,\alpha}(\mathbb{P}^r|H)$ contains the airborne divisor, over which the source curve is smooth. Conceptually, the logarithmic moduli space therefore has larger boundary, and hence smaller interior, than the ordinary moduli space, explaining the discrepancy.

As we add additional hyperplanes to the logarithmic structure, the interior grows yet smaller; when we reach the full toric boundary, the interior becomes very affine [Ran17].

1.4. **Enumerating tropical types: maximal contact case.** This section is logically independent of the rest of the paper. It illustrates how to work with combinatorial types in practice.

Set $\alpha = (d, 0, \ldots, 0)$ so that all tangency is concentrated at the marking $x_1$. We enumerate the combinatorial types with one-dimensional tropical moduli in this setting.
Proposition 1.7. For $\alpha = (d, 0, \ldots, 0)$ of length $n$ we have

$$N(\alpha) = 2^{n-1} - n + \sum_{k=1}^{d} \sum_{k_1=0}^{\min(k,n-1)} \sum_{d_1=0}^{d-k} \sum_{d_2=0}^{d-k-d_1} \left( \sum_{a=0}^{k_1} (-1)^{k_1+a} \left( \frac{(a+1)^{n-1}}{a!(k_1-a)!} \right) \left( d_1 + k_1 - 1 \right) p_{k-k_1}(d_2 + k - k_1) \right) \right) \right)$$

where $p_k(m)$ is the number of unordered partitions of $m$ into $k$ positive parts, which equals the coefficient of $t^m$ in the power series expansion of

$$t^k \frac{1}{(1-t)(1-t^2) \cdots (1-t^k)}.$$ 

In the above formula the $k_1 = 0$ instances of the binomial coefficient are defined as follows

$$\binom{e}{-1} = \begin{cases} 1 & \text{if } e = -1 \\ 0 & \text{if } e \geq 0 \end{cases}$$

and the $k_1 = k$ instance of the partition function is

$$p_0(e) = \begin{cases} 1 & \text{if } e = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We work through the taxonomy of Proposition 1.4. There is always a single airborne type. By the shape of $\alpha$, any binary type must have degree $d$ on the component containing $x_1$ and degree $0$ on the other component. By stability, there are precisely $2^{n-1} - n$ of these. It remains to enumerate the rocket types. We must have $x_1 \in C_0$. The remaining markings can be distributed arbitrarily, except for the special case $k = 1, d_0 = 0$ where by stability there must be at least one other marking on $C_0$. We will allow for arbitrary marking distributions, and then subtract off at the end to account for this special case.

We enumerate the choices required to determine a rocket type. For integers $p \leq q$ we write $[p, q] = \{p, \ldots, q\}$. We first choose

$$k \in [1, d]$$

the number of external components of the source curve. Note that $k \leq d$ because every external component has degree $\geq 1$. We next choose

$$k_1 \in [0, \min(k, n-1)]$$

the number of external components which carry at least one marking. We must distribute the markings $x_2, \ldots, x_n$ between these external components and the internal component $C_0$.

Letting $T(p, q)$ denote the number of surjective functions $[1, p] \to [1, q]$, the total number of such distributions is

$$(T(n-1, k_1 + 1) + T(n-1, k_1)) / k_1!$$

where the two terms count distributions which do have, respectively do not have, markings assigned to $C_0$. The inclusion-exclusion principle gives

$$T(p, q) = \sum_{a=1}^{q} (-1)^{q+a} \binom{q}{a} a^p$$

from which we compute

$$(T(n-1, k_1 + 1) + T(n-1, k_1)) / k_1! = \sum_{a=0}^{k_1} (-1)^{k_1+a} \frac{(a+1)^{n-1}}{a!(k_1-a)!}. $$
We now consider the distribution of the degree. Each external component has degree \( \geq 1 \) and so the \( k_1 \) external components carrying markings have total degree \( \geq k_1 \). We choose

\begin{equation}
(7) \quad d_1 \in [0, d - k]
\end{equation}

for the total additional degree on these components. This may be distributed arbitrarily; we must count ordered, non-negative partitions of \( d_1 \) into \( k_1 \) parts. A stars and bars argument shows that the number of such partitions is

\[
\binom{d_1 + k_1 - 1}{k_1 - 1}.
\]

Finally, consider the \( k - k_1 \) external components which do not carry any markings. Again the total degree on these must be \( \geq k - k_1 \) and we choose

\begin{equation}
(8) \quad d_2 \in [0, d - k - d_1]
\end{equation}

for the total additional degree. Again this may be distributed arbitrarily, but since the components carry no markings we must count unordered, non-negative partitions of \( d_2 \) into \( k - k_1 \) parts. This is equivalent to counting unordered, positive partitions of \( d_2 + k - k_1 \) into \( k - k_1 \) parts, of which there are precisely

\[
p_{k-k_1}(d_2 + k - k_1).
\]

Summarising, we have made arbitrary choices at (5), (6), (7), (8) and explicitly enumerated all other choices. This gives the total number of rocket types as

\[
\sum_{k=1}^{d} \min(k,n-1) \sum_{k_1=0}^{d-k} \sum_{d_1=0}^{d-k-k_1} \sum_{d_2=0}^{d_1} \binom{k_1}{a} (-1)^{a+1} \frac{(a+1)^{n-1}}{a!(k_1-a)!} \binom{d_1 + k_1 - 1}{k_1 - 1} p_{k-k_1}(d_2 + k - k_1).
\]

Combining this with the number of airborne and binary types, and subtracting 1 for the overcounting of the unstable rocket type, we conclude the result.

\[\square\]

2. Test curves

We come to the main construction: an assembly line for the manufacture of test curves in the moduli space. The geometry of algebraic surfaces plays a fundamental role.

Fix an irreducible boundary divisor \( D \) in the space of stable logarithmic maps, indexed by a combinatorial type of stable tropical map of the form

\[
\begin{array}{c}
C_0 \rightarrow M_0, \alpha(P^r | H) \\
\end{array}
\]

This is a divisor of rocket type (see Proposition 1.4). The distribution of markings is suppressed in the above picture, but understood to be a fixed partition

\[
A_0 \sqcup A_1 \sqcup \ldots \sqcup A_k = \{1, \ldots, n\}
\]

such that the resulting tropical map is stable and balanced. Note that \( d_j \geq m_j > 0 \). We give a process for constructing test curves

\[
\mathbb{P}^1 \to M_{0, \alpha}(P^r | H)
\]
which pass through \( D \) at \( 0 \in \mathbb{P}^1 \) and intersect the boundary of the moduli space transversely and away from codimension-2 strata. Such a test curve corresponds to a family of maps

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & \mathbb{P}^r \\
\pi & \downarrow & \mathbb{P}^1 \\
\end{array}
\]

with \( \pi^{-1}(0) \) of the shape specified by the above combinatorial type. We have taken \( D \) to be a rocket divisor, but the process below can be modified (in fact, simplified) to deal with airborne and binary divisors.

2.1. **Singular total space.** Before giving the construction, we highlight a new phenomenon which appears in the logarithmic setting: we cannot guarantee that the total space \( \mathcal{C} \) is smooth with all marking and special divisors intersecting transversely.

Indeed, suppose we have such a family \( (9) \) with \( \pi^{-1}(0) \) giving an element of \( D \). In a neighbourhood of \( \pi^{-1}(0) \) there is an identification of effective Cartier divisors

\[ f^* H = \sum_{i=1}^n \alpha_i x_i + w C_0 \]

for some \( w > 0 \). For \( j \in \{1, \ldots, k\} \) we then have

\[ m_j = d_j - \sum_{x_i \in C_j} \alpha_i = f^* H \cdot C_j - \sum_{x_i \in C_j} \alpha_i = \left( \sum_{x_i \in C_j} \alpha_i + w C_0 C_j \right) - \sum_{x_i \in C_j} \alpha_i = w C_0 C_j. \]

If \( \mathcal{C} \) is smooth with all marking and special divisors intersecting transversely, then \( C_0 C_j = 1 \) for all \( j \) and we conclude \( m_1 = \ldots = m_k = w \). This is certainly not the case for all \( D \).

In the following construction, the surface \( \mathcal{C} \) will have cyclic quotient singularities arising from non-reduced blowups. This is consistent with the structure of logarithmic curves; the monomial sheaves encode local node smoothings of the form \( xy = t^w \) whose total spaces are singular toric surfaces.

2.2. **Arranging the markings.** We begin the construction. Consider the trivial family

\[ \pi_1: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \]

and note that sections of \( \pi_1 \) correspond to smooth divisors in the linear systems \(|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(b, 1)| \) with \( b \geq 0 \). Working explicitly with homogeneous coordinates, we can choose smooth divisors

\[ x_i \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(b_i, 1)| \]

for \( i \in \{1, \ldots, n\} \), satisfying the following specifications.

**Specification 2.1 (Central fibre).** For those \( j \in \{1, \ldots, k\} \) with \( A_j \neq \emptyset \), the intersection

\[ \pi^{-1}(0) \cap \bigcap_{i \in A_j} x_i \]

is nonempty, and hence consists of a single point. In contrast, for all \( i \in A_0 \) the point \( \pi^{-1}(0) \cap x_i \) is disjoint from the other \( x_i \).

**Specification 2.2 (Genericity).** Away from \( \pi^{-1}(0) \), there is no point where 3 or more of the \( x_i \) intersect, and in each fibre of \( \pi \) there is at most one pairwise intersection point. At every intersection point, including those on the central fibre, the intersections are pairwise transverse and no \( x_i \) is tangent to the fibre of \( \pi \).

For those \( j \in \{1, \ldots, k\} \) with \( A_j \neq \emptyset \), let \( p_j \) denote the intersection point \( (11) \). If \( A_j = \emptyset \), let \( p_j \) be an arbitrary point of \( \pi^{-1}(0) \) disjoint from the \( x_i \). Denote by \( q_1, \ldots, q_l \) the intersection points of the \( x_i \) away from \( \pi^{-1}(0) \). The surface \( \mathbb{P}^1 \times \mathbb{P}^1 \) will be blown up in the points \( p_1, \ldots, p_k, q_1, \ldots, q_l \) to produce the desired curve family.
2.3. **Blowing up.** Define multiplicities

\[ w = \text{lcm}(m_1, \ldots, m_k), \]
\[ w_j = \frac{w}{m_j} \quad \text{for } j \in \{1, \ldots, k\}. \]

Perform the non-reduced blowup of \(\mathbb{P}^1 \times \mathbb{P}^1\) at the points \(p_1, \ldots, p_k\) with multiplicities \(w_1, \ldots, w_k\) in the first factor and \(1, \ldots, 1\) in the second factor. Then perform the ordinary blowup of the resulting surface at the points \(q_1, \ldots, q_l\). The output is a singular surface \(\mathcal{C}\) together with a projection

\[ \pi: \mathcal{C} \to \mathbb{P}^1 \]

which is a flat family of nodal curves.

2.4. **Special fibres.** The family \(\pi\) has singular fibres precisely over the images of the special points \(p_1, \ldots, p_k, q_1, \ldots, q_l\). The fibre over \(0 \in \mathbb{P}^1\) is the reduced union of the following \(\mathbb{Q}\)-Cartier divisors:

- \(C_0\), the strict transform of \(\pi^{-1}(0) \subset \mathbb{P}^1 \times \mathbb{P}^1\).
- \(C_1, \ldots, C_k\), the reduced exceptional divisors over \(p_1, \ldots, p_k\).

These divisors satisfy \(C_0C_j = 1/w_j\) and \(C_j^2 = 0\). Note that both \(wC_0\) and \(w_jC_j\) are Cartier. Moreover we have

\[ wC_0C_j = w/w_j = m_j, \]
\[ wC_0^2 = wC_0(-C_1 - \ldots - C_k) = -(m_1 + \ldots + m_k). \]

The fibre over the image of \(q_j\) is a nodal curve with two components: \(D_j\) the strict transform of the original fibre, and \(E_j\) the exceptional divisor of the blowup. The latter intersects the strict transforms of the two marking sections \(x_{j1}, x_{j2}\) which intersected in \(\mathbb{P}^1 \times \mathbb{P}^1\) at \(q_j\). We will construct a map with degree \(d\) on \(D_j\) and degree 0 on \(E_j\). Orienting the node from \(D_j\) to \(E_j\), the desired tangency order at the node is given by the balancing condition as

\[ u_j := \alpha_{j1} + \alpha_{j2}. \]

Finally, let \(x_1, \ldots, x_n \subset \mathcal{C}\) denote the strict transforms of the marking sections \(x_1, \ldots, x_n \subset \mathbb{P}^1 \times \mathbb{P}^1\). By Specification 2.2, these are pairwise disjoint and contained in the smooth locus of \(\pi\).

The resulting curve family is illustrated below. Divisors coloured red are those which will be mapped into \(H\). The role of the smooth fibres \(B\) will be explained shortly.

2.5. **Constructing \(f_0\).** It remains to construct the map \(\mathcal{C} \to \mathbb{P}^r\). This is achieved by constructing appropriate sections \(f_0, f_1, \ldots, f_r\) of the following line bundle

\[ L := \mathcal{O}_{\mathcal{C}}(\sum_{i=1}^n \alpha_i x_i + wC_0 + \sum_{j=1}^l u_j E_j) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(N) \]
for \( N \geq 0 \) to be determined. We verify that \( L \) has the desired multi-degree on every special fibre:

\[
\deg L_{|C_0} = \sum_{i \in C_0} \alpha_i + wC_0^2 = \sum_{i \in C_0} \alpha_i - (m_1 + \ldots + m_k) = d_0,
\]

\[
\deg L_{|C_j} = \sum_{i \in C_j} \alpha_i + wC_0C_j = \sum_{i \in C_j} \alpha_i + m_j = d_j,
\]

\[
\deg L_{|D_j} = \sum_{i \in D_j} \alpha_i + u_j = d,
\]

\[
\deg L_{|E_j} = \alpha_{j_1} + \alpha_{j_2} - u_j = 0.
\]

Assume without loss of generality that \( H = H_0 \subset \mathbb{P}^r \) so that the section \( f_0 \) governs the tangency with respect to \( H \). Fix a generic section of \( O_{\mathbb{P}^1}(N) \) and let \( B \subset C \) denote the vanishing locus of the pullback; this is a union of smooth fibres of \( \pi \). Define

\[
f_0 \in H^0(C, L)
\]

as the product of the standard section cutting out the effective Cartier divisor on the first factor of (12), together with the section cutting out \( B \) on the second factor of (12).

We verify that \( f_0 \) has the desired behaviour. On a general fibre of \( \pi \) the vanishing locus of \( f_0 \) is

\[
\sum_{i=1}^n \alpha_i x_i
\]

which gives the desired tangency profile. Focusing on \( \pi^{-1}(0) \) we see that \( f_0 \) vanishes identically on \( C_0 \) and has the desired vanishing order at every marking in \( A_1 \cup \ldots \cup A_k \). The vanishing order at the node \( C_0 \cap C_j \) is given by

\[
wC_0C_j = w/w_j = m_j
\]

as required. Hence \( f_0 \) has the desired behaviour on the central fibre. On the other singular fibres, \( f_0 \) vanishes identically on \( E_j \) (assuming \( u_j \neq 0 \)) and its vanishing order at the node \( D_j \cap E_j \) is

\[
u_j D_j E_j = u_j
\]

as required. Finally \( f_0 \) vanishes identically on the smooth special fibres which constitute \( B \). These fibres give points in the airborne boundary divisor (see Proposition 1.4 and Section 3.2). This verifies the behaviour of \( f_0 \).

2.6. Constructing \( f_1, \ldots, f_r \). The remaining sections of \( L \) have no constraints, and may be chosen arbitrarily as long as the resulting tuple \([f_0, f_1, \ldots, f_r]\) has no basepoints. Recall that

\[
L = O_C(\sum_{i=1}^n \alpha_i x_i + wC_0 + \sum_{j=1}^r u_j E_j) \otimes \pi^* O_{\mathbb{P}^1}(N)
\]

where \( N \geq 0 \) is fixed but arbitrary. The following lemma constructs the desired sections of \( L \), and in so doing determines a lower bound for \( N \).

**Lemma 2.3.** Define the integer

\[
M := d_0 + \sum_{j=1}^r d_j w_j - \sum_{i=1}^n \alpha_i b_i - w.
\]

If \( N \geq \max(M, 0) \) then there exist curves

\[
F_1, \ldots, F_r \in |L|
\]

such that the intersection \( V(f_0) \cap F_1 \cap \ldots \cap F_r \subset C \) is empty.

**Proof.** For \( i \in \{1, \ldots, r\} \) and \( j \in \{1, \ldots, k\} \) choose a curve

\[
G_{ij} \in |O_{\mathbb{P}^1 \times \mathbb{P}^1}(w_j, 1)|
\]

that passes through the point \( p_j \) with tangency order \( w_j \) with respect to the horizontal line \( \mathbb{P}^1 \times \{\pi_2(p_j)\} \). For \( i \in \{1, \ldots, r\} \) and \( j = 0 \) choose an arbitrary curve

\[
G_{i0} \in |O_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)|.
\]

Generic choices of such curves satisfy the following.
**Specification 2.4.** For \( i \in \{1, \ldots, r\} \) and \( j \in \{0, 1, \ldots, k\} \) the curve \( G_{ij} \) does not pass through the points \( p_1, \ldots, \tilde{p}_j, \ldots, p_k, q_1, \ldots, q_l \). At \( p_j \), the equations defining the curves

\[
G_{1j}, \ldots, G_{rj}
\]

have distinct partial derivatives of order \( w_j \) with respect to the horizontal coordinate. The points

\[
G_{10} \cap \pi^{-1}(0), \ldots, G_{r0} \cap \pi^{-1}(0)
\]

are all distinct. Away from \( \pi^{-1}(0) \) the \( G_{ij} \) intersect only pairwise and away from \( \varphi(V(f_0)) \), where \( \varphi: \mathcal{C} \to \mathbb{P}^1 \times \mathbb{P}^1 \) is the blowdown.

For \( i \in \{1, \ldots, r\} \) define the (non-reduced) Cartier divisor

\[
G_i := \sum_{j=0}^{k} d_j G_{ij} \in |O_{\mathbb{P}^1 \times \mathbb{P}^1}(d_0 + \Sigma_{j=1}^{k} d_j w_j, d)|
\]

and consider the strict transform

\[
\tilde{G}_i \subseteq \mathcal{C}.
\]

**Specification 2.4** ensures that the intersection \( \varphi(V(f_0)) \cap G_1 \cap \ldots \cap G_r \) consists precisely of the points \( p_1, \ldots, p_k \). Distinctness of the higher partial derivatives ensures that \( G_1, \ldots, G_r \) are separated by the blowup. Since \( r \geq 2 \) we conclude that

\[
V(f_0) \cap \tilde{G}_1 \ldots \cap \tilde{G}_r = \emptyset.
\]

It remains to compare \( O_{\mathcal{C}}(\tilde{G}_i) \) and \( L \). The tangency specification on \( G_{ij} \) gives

\[
\tilde{G}_{ij} = \varphi^* G_{ij} - w_j C_j
\]

and so we have

\[
\tilde{G}_i = \varphi^* G_i - \Sigma_{j=1}^{k} d_j w_j C_j.
\]

Now recall the arrangement of markings from Section 2.2. Combining (10) and (13) together with \( \Sigma_{i=1}^{n} \alpha_i = d \), we have

\[
O_{\mathbb{P}^1 \times \mathbb{P}^1}(G_i) = O_{\mathbb{P}^1 \times \mathbb{P}^1}(d_0 + \Sigma_{j=1}^{k} d_j w_j, d) = O_{\mathbb{P}^1 \times \mathbb{P}^1}(\Sigma_{i=1}^{n} \alpha_i x_i) \otimes \pi^* O_{\mathbb{P}^1}(M')
\]

where \( M' = d_0 + \Sigma_{j=1}^{k} d_j w_j - \Sigma_{i=1}^{n} \alpha_i b_i \). Combining (14) and (15) we obtain

\[
O_{\mathcal{C}}(\tilde{G}_i) = O_{\mathcal{C}}(\varphi^* \Sigma_{i=1}^{n} \alpha_i x_i - \Sigma_{j=1}^{k} d_j w_j C_j) \otimes \pi^* O_{\mathbb{P}^1}(M').
\]

To control the pullback of the marking divisors, we impose the following specification. This can always be achieved by increasing the vertical degrees \( b_i \).

**Specification 2.5.** Consider the marking divisor \( x_i \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \) given in (10). If \( x_i \) passes through the point \( p_j \in \pi^{-1}(0) \) (i.e. \( i \in A_j \)), then at this point the curve \( x_i \) has tangency order \( w_j \) with respect to the horizontal line.

This gives the following comparison between \( x_i \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \) and its strict transform \( x_i \subseteq \mathcal{C} \)

\[
\varphi^* x_i = \begin{cases} 
  x_i + w_j C_j + \sum_{q_j \in x_i} E_j & \text{if } i \in A_j \neq 0 \\
  x_i + \sum_{q_j \in x_i} E_j & \text{if } i \in A_0
\end{cases}
\]

from which we obtain

\[
\varphi^* \Sigma_{i=1}^{n} \alpha_i x_i = \Sigma_{i=1}^{n} \alpha_i x_i + \Sigma_{j=1}^{k} \left( \sum_{i \in A_j} \alpha_i \right) w_j C_j + \Sigma_{j=1}^{l} u_j E_j.
\]
Substituting this into (16) we obtain
\[ O_C(\tilde{G}_i) = O_C(\Sigma_{i=1}^n \alpha_i x_i + \Sigma_{j=1}^k (-d_j + \Sigma_{i \in A_j} \alpha_i) w_j C_j + \Sigma_{j=1}^l u_j E_j) \otimes \pi^* O_{\mathbb{P}^1}(M') = O_C(\Sigma_{i=1}^n \alpha_i x_i + \Sigma_{j=1}^k (-m_j) w_j C_j + \Sigma_{j=1}^l u_j E_j) \otimes \pi^* O_{\mathbb{P}^1}(M'). \]

For all \( j \in \{1, \ldots, k\} \) we have \( m_j w_j = w \). Moreover since the fibre \( \pi^{-1}(0) \) is the reduced union \( C_0 + C_1 + \ldots + C_k \) we have
\[ O_C(\Sigma_{j=1}^k (-m_j w_j) C_j) = O_C(-w \Sigma_{j=1}^k C_j) = O_C(w C_0) \otimes \pi^* O_{\mathbb{P}^1}(-w). \]

Setting \( M = M' - w \) we obtain
\[ O_C(\tilde{G}_i) = O_C(\Sigma_{i=1}^n \alpha_i x_i + w C_0 + \Sigma_{j=1}^l u_j E_j) \otimes \pi^* O_{\mathbb{P}^1}(M) = L \otimes \pi^* O_{\mathbb{P}^1}(M - N). \]

Since \( N \geq M \) we may choose a general fibre \( T_i \in |\pi^* O_{\mathbb{P}^1}(1)| \) and consider the effective divisor
\[ F_i := \tilde{G}_i + (N - M) T_i. \]

If the \( T_i \) are chosen generically then since \( r \geq 2 \) the \( F_i \) still satisfy \( V(f_0) \cap F_1 \cap \ldots \cap F_r = \emptyset \). Moreover we now have
\[ O_C(F_i) = L \]
as required. \( \square \)

2.7. Logarithmic structures. Finally, equip the surface \( C \) with the divisorial logarithmic structure corresponding to the reduced union of: the marking divisors \( x_i \), the singular fibres of \( \pi \), and the smooth fibres \( B \). Equip the base \( \mathbb{P}^1 \) with the divisorial logarithmic structure corresponding to the reduced image of the singular fibres of \( \pi \) and the smooth fibres \( B \). The logarithmic enhancements of the morphisms \( f \) and \( \pi \) are then unique, producing a diagram of logarithmic schemes
\[ (\mathcal{C}, M_C) \xrightarrow{f} (\mathbb{P}^r|H) \]
\[ \downarrow \pi \]
\[ (\mathbb{P}^1, M_{\mathbb{P}^1}). \]
This family of stable logarithmic maps is automatically basic [GS13, Section 1.5], as it only intersects codimension 1 strata which all have basic monoid \( \mathbb{N} \). We thus obtain the desired test curve
\[ \mathbb{P}^1 \to \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \]
and this completes the construction.

3. Relations

3.1. Pulling back divisors. Given a test curve
\[ \mathbb{P}^1 \to \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \]
we wish to pull back elements of the class group of \( \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \). This requires some care, since the divisors may not be Cartier and the test curve may not be a regular embedding (or even a closed embedding). Let
\[ W \subseteq \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \]
denote the complement of the boundary strata of codimension \( \geq 2 \). Since \( \overline{M}_{0,\alpha}(\mathbb{P}^r|H) \) is logarithmically smooth, it follows that \( W \) is smooth (because all normal toric varieties of dimension \( \leq 1 \) are smooth). By the construction in Section 2, the test curve factors through this open subset
\[ \mathbb{P}^1 \to W \subseteq \overline{M}_{0,\alpha}(\mathbb{P}^r|H). \]
Given an element in the class group of $\overline{M}_{0,\alpha}(\mathbb{P}^r|H)$, we first pull it back along the flat morphism $W \rightarrow \overline{M}_{0,\alpha}(\mathbb{P}^r|H)$. Since $W$ is smooth the resulting divisor is Cartier, so we can pull it back along the morphism $\mathbb{P}^1 \rightarrow W$.

Equivalently, $\mathbb{P}^1 \rightarrow W$ is l.c.i. (since both source and target are smooth) and $W \leftarrow \overline{M}_{0,\alpha}(\mathbb{P}^r|H)$ is l.c.i. (since it is in fact smooth). The composite $\mathbb{P}^1 \rightarrow \overline{M}_{0,\alpha}(\mathbb{P}^r|H)$ is therefore l.c.i. and induces a pullback on Chow groups [Ful98, Section 6.6].

We will refer to the above process as “intersecting a divisor on $\overline{M}_{0,\alpha}(\mathbb{P}^r|H)$ with a test curve”. The result agrees with the intuitive expectation.

3.2. **Terrestrial, airborne and alien divisors.** The test curves constructed above intersect the boundary of $\overline{M}_{0,\alpha}(\mathbb{P}^r|H)$ at finitely many points away from the central point 0. These additional intersections occur only at a special class of boundary divisors, which play a key role in the forthcoming analysis.

**Definition 3.1.** The airborne divisor is the boundary divisor in $\overline{M}_{0,\alpha}(\mathbb{P}^r|H)$ whose generic point parametrises a smooth curve mapped inside $H$. The combinatorial type is

```
\[ \begin{array}{c}
  d \\
  f \\
\end{array} \]
```

The alien divisors are the boundary divisors in $\overline{M}_{0,\alpha}(\mathbb{P}^r|H)$ whose generic point parametrises a curve with two irreducible components, one of which contains two markings and has degree 0, the other of which contains all the other markings and has degree $d$. Since $d \geq 1$ there are precisely \( \binom{n}{2} \) of these, and they are denoted $D_{ij}$ where $1 \leq i < j \leq n$ index the two markings. Their combinatorial types are:

```
\[ \begin{array}{c}
  d & 0 & x_i \\
  \downarrow f \\
  x_j \\
\end{array} \quad \begin{array}{c}
  x_j & x_i \\
  \downarrow f \\
  d \\
\end{array} \]
```

\[ \alpha_i + \alpha_j > 0 \quad \alpha_i + \alpha_j = 0 \]

Boundary divisors which are neither airborne nor alien are referred to as terrestrial. Note that this taxonomy of boundary divisors differs from the one given in Proposition 1.4; aliens and terrestrials may be either rockets or binaries.

The test curves of Section 2 are constructed in such a way that they do not intersect any terrestrial divisor away from the central point 0.

3.3. **Linear independencies.** We now leverage our control over test curves to deduce the main theorem.

**Proposition 3.2.** The airborne divisor has coefficient zero in any linear relation in $\text{Cl}_Q(\overline{M}_{0,\alpha}(\mathbb{P}^r|H))$ between boundary divisors.

**Proof.** Take such a linear relation. Construct any test curve in $\overline{M}_{0,\alpha}(\mathbb{P}^r|H)$ using the techniques of Section 2. We then modify this test curve as follows. Replace $L$ by $L \otimes \pi^*O_{\mathbb{P}^1}(1)$ and replace every $f_i$ by $f_i t_i$ for $t_i \in H^0(C, \pi^*O_{\mathbb{P}^1}(1))$ a section cutting out a fibre of $\pi$. If the $t_i$ are chosen generically then the map $[f_0 t_0, \ldots, f_r t_r]$ will be basepoint-free. We obtain a new test curve whose intersection with the boundary is the same as the original test curve, except that it intersects the airborne divisor in one more point than before. Intersecting both test curves with the given linear relation and taking the difference, we conclude that the coefficient of the airborne divisor is zero. \qed
Proposition 3.3. The union of the terrestrial divisors and the airborne divisor forms a linearly independent subset of \( \text{Cl}_Q(\overline{M}_{0,n}(\mathbb{P}^r|H)) \).

Proof. By the previous proposition, it suffices to show that the set of terrestrial divisors forms a linearly independent subset. Suppose we are given a linear relation amongst the terrestrial divisors, and fix an arbitrary terrestrial divisor \( D \). The construction in Section 2 produces a test curve which intersects \( D \) in a single point and does not intersect any other terrestrial divisors. Intersecting this test curve with the given linear relation, we conclude that the coefficient of \( D \) is zero. Since \( D \) was arbitrary, this completes the proof. \( \square \)

Definition 3.4. Let \( S \subseteq \text{Cl}_Q(\overline{M}_{0,n}(\mathbb{P}^r|H)) \) denote the subspace with basis the set of terrestrial and airborne divisors, and denote the quotient \( Q := \text{Cl}_Q(\overline{M}_{0,n}(\mathbb{P}^r|H))/S. \)

Proposition 3.5. The quotient \( Q \) is \( n \)-dimensional. All boundary relations in \( Q \) arise by pullback from \( \overline{M}_{0,n} \).

Proof. By Proposition 1.5, \( Q \) is spanned by the alien divisors. There are precisely \( \binom{n}{2} \) of these. On the other hand, Proposition A.2 below exhibits a set of \( \binom{n-1}{2}-1 \) independent relations between the boundary divisors in \( \overline{M}_{0,n} \). These pull back to relations in \( \text{Cl}_Q(\overline{M}_{0,n}(\mathbb{P}^r|H)) \) which induce relations in the quotient \( Q \). We claim that these relations are independent.

Indeed, the divisor \( E_{ij} \subseteq \overline{M}_{0,n} \) defined in Appendix A pulls back to the class of the alien divisor \( D_{ij} \) in \( Q \). The same argument as in the proof of Proposition A.2 then shows that the relations are independent. It follows that the dimension of \( Q \) is bounded from above:

\[
\dim Q \leq \binom{n}{2} - \left( \binom{n-1}{2} - 1 \right) = n.
\]

We claim that these are all the relations in \( Q \). For this, we will show that \( \dim Q \geq n \), by exhibiting a linearly independent subset of size \( n \). Consider the following alien divisors

\[
D_{12}, D_{13}, \ldots, D_{1n}, D_{23}.
\]

Suppose that there is a linear relation between these in \( Q \), with \( D_{ij} \) having coefficient \( a_{ij} \). This implies that the corresponding linear combination in \( \overline{M}_{0,n}(\mathbb{P}^r|H) \) belongs to the span of the airborne and terrestrial divisors. However, by Proposition 3.2, it in fact belongs to the span of the terrestrial divisors, giving the following relation in \( \text{Cl}_Q(\overline{M}_{0,n}(\mathbb{P}^r|H)) \)

\[
a_{12}D_{12} + a_{13}D_{13} + \ldots + a_{1n}D_{1n} + a_{23}D_{23} = \sum_D a_D D
\]

where the sum on the right-hand side is over the set of terrestrial divisors.

We now construct test curves. This requires a minor retooling of the assembly line of Section 2. Namely, we drop Specification 2.1 governing the behaviour of the marking divisors on the central fibre. In this way we produce a test curve which does not intersect any terrestrial divisor. This also enables us to drop Specification 2.5, and as such we may choose the \( b_i \) in (10) arbitrarily (previously each \( b_i \) was bounded from below by the corresponding \( w_j \)).

Fix \( i \in \{1, \ldots, n\} \) and arrange markings as in Section 2.2 with

\[
b_i = 1, \quad b_{\neq i} = 0.
\]

This ensures that \( x_i \cdot x_{\neq i} = 1 \) whilst \( x_j \cdot x_k = 0 \) otherwise. The resulting test curve does not intersect any terrestrial divisors, and intersects \( D_{jk} \) if and only if \( i \in \{j, k\} \). Intersecting against (18) we obtain
the following equations

\[
\begin{align*}
i = 1 & : \quad a_{12} + a_{13} + \ldots + a_{1n} = 0 \\
i = 2 & : \quad a_{12} + a_{23} = 0 \\
i = 3 & : \quad a_{13} + a_{23} = 0 \\
i = 4 & : \quad a_{14} = 0 \\
\vdots & \\
i = n & : \quad a_{1n} = 0
\end{align*}
\]

which are easily solved to give all \(a_{ij} = 0\). We conclude that the set (17) is linearly independent, as required. \(\square\)

Combining Propositions 3.3 and 3.5 we conclude the main result.

**Theorem 3.6 (Theorem Y).** For \(r \geq 2, n \geq 3, d \geq 1, \) and \(\alpha \vdash d\), the dimension of \(\text{Cl}_Q(\overline{M}_{0,\alpha}(\mathbb{P}^r|H))\) is

\[
N(\alpha) - \binom{n - 1}{2} + 1.
\]

In particular, it does not depend on \(r\). A basis is given by the union of the airborne and terrestrial divisors and the subset (17) of the alien divisors. Moreover, all relations amongst boundary divisors are pulled back from \(\overline{M}_{0,n}\).

### 3.4. Picard group

The space \(\overline{M}_{0,\alpha}(\mathbb{P}^r|H)\) is logarithmically smooth, hence normal. It follows that the Picard group embeds in the class group

\[
\text{Pic}_Q\overline{M}_{0,\alpha}(\mathbb{P}^r|H) \hookrightarrow \text{Cl}_Q\overline{M}_{0,\alpha}(\mathbb{P}^r|H)
\]

as the locally principal divisor classes, see [GW10, Theorem 11.38(1)] and [Har77, Remark II.6.1.12]. Applying Proposition 1.5 we see that the Picard group is generated by Cartier divisors supported on the boundary. These admit the following description.

**Lemma 3.7.** Boundary Cartier divisors on \(\overline{M}_{0,\alpha}(\mathbb{P}^r|H)\) are indexed by integral piecewise-linear functions on the moduli space \(\Sigma_{0,\alpha}\) of stable tropical maps to \(\mathbb{R}^n \geq 0\).

**Proof.** By Theorem 1.1 we have a natural isomorphism \(\Sigma\overline{M}_{0,\alpha}(\mathbb{P}^r|H) = \Sigma_{0,\alpha}\). It is then a general fact about toroidal embeddings that boundary Cartier divisors correspond to integral piecewise-linear functions on the tropicalisation. \(\square\)

We arrive at a combinatorial description of the Picard group.

**Theorem 3.8 (Theorem Z).** The rational Picard group of \(\overline{M}_{0,\alpha}(\mathbb{P}^r|H)\) is generated by divisors corresponding to integral piecewise-linear functions on \(\Sigma_{0,\alpha}\). A complete set of relations between these divisors is obtained by pulling back the WDVV relations from \(\overline{M}_{0,n}\).

**Proof.** The description of the generators follows from Lemma 3.7. The description of the relations follows from the injectivity of (19) and the corresponding result for the class group (Theorem 3.6). Note that since \(\overline{M}_{0,n}\) is smooth, every boundary relation pulls back to a relation between boundary Cartier divisors on \(\overline{M}_{0,\alpha}(\mathbb{P}^r|H)\). \(\square\)

### Appendix A. Divisors on \(\overline{M}_{0,n}\)

The following facts can all be found in [Kee92]. Fix \(n \geq 3\). The Grothendieck–Knudsen space \(\overline{M}_{0,n}\) is a smooth projective variety of dimension \(n - 3\). Its Picard group has dimension \(2^{n-1} - \binom{n-1}{2} - n\) and is generated by the set of boundary divisors, of which there are \(2^{n-1} - 1 - n\). It follows that there are

\[
\binom{n - 1}{2} - 1
\]
independent relations amongst the boundary divisors. All such relations are obtained by pulling back the WDVV relations from $\overline{M}_{0,4}$. However, there are $2\binom{n}{2}$ of these, so while they generate all relations, they are not independent.

We now exhibit a complete and independent set of boundary relations. This is used in the proof of Proposition 3.5 above. By the preceding discussion, it suffices to exhibit a linearly independent set of relations of size $\binom{n-1}{2} - 1$. We let

$$\mathcal{R} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$$

denote the WDVV relation

$$D(ab|cd) = D(ac|bd).$$

Note that $\mathcal{R}(M) = \mathcal{R}(M^T)$. For $1 \leq i < j \leq n$ let $E_{ij} \subseteq \overline{M}_{0,n}$ denote the boundary divisor whose generic point parametrises a curve with two components, one of which contains precisely the markings $x_i, x_j$.

**Lemma A.1.** The boundary divisor $E_{ij}$ appears in the relation $\mathcal{R}(M)$ if and only if $(ij)$ appears as a row or a column of $M$.

**Proof.** The irreducible components of $D(ab|cd)$ are indexed by splittings of the markings into two parts, with $x_a, x_b$ in the first part and $x_c, x_d$ in the second part. We conclude that $E_{ij}$ is an irreducible component of $D(ab|cd)$ if and only if $\{i, j\} = \{a, b\}$ or $\{i, j\} = \{c, d\}$. The same applies to $D(ac|bd)$, and the result follows.

With this, the search for a set of linearly independent relations reduces to a game of sudoku.

**Proposition A.2.** The $\mathcal{R}(M)$ in the following table form a basis for the boundary relations in $\overline{M}_{0,n}$.

| $\mathcal{R}(M)$ | $\begin{array}{cccc} \frac{1}{3} & 2 & 4 \\ \frac{1}{4} & 3 & 5 \end{array}$ | $\begin{array}{cccc} \frac{1}{n-1} & \frac{n-2}{n} \end{array}$ | $\begin{array}{cccc} \frac{2}{n-1} & \frac{n-2}{n} \end{array}$ | $\begin{array}{cccc} \frac{3}{n-1} & \frac{4}{n-1} \end{array}$ |
|------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $E(M)$           | $E_{12}$                        | $E_{13}$                        | $E_{1,n-2}$                     | $E_{23}$                        |
| $\mathcal{R}(M)$ | $\begin{array}{cccc} \frac{3}{n-1} & \frac{n-2}{n} \end{array}$ | $\begin{array}{cccc} \frac{n-3}{n-1} & \frac{n-2}{n} \end{array}$ | $\begin{array}{cccc} \frac{n}{n-2} & \frac{1}{n-1} \end{array}$ | $\begin{array}{cccc} \frac{n}{n-2} & \frac{n-3}{n-1} \end{array}$ |
| $E(M)$           | $E_{3,n-2}$                     | $E_{n-3,n-2}$                  | $E_{1n}$                        | $E_{n-3,n}$                     |

**Proof.** It is straightforward to check that there are precisely $\binom{n-1}{2} - 1$ relations. It therefore suffices to demonstrate linear independence.

In the above table, every relation $\mathcal{R}(M)$ is paired with a boundary divisor $E(M)$. The following property is easily checked using Lemma A.1: $E(M)$ appears in $\mathcal{R}(M)$, and $E(M)$ does not appear in any $\mathcal{R}(M')$ occurring to the right of $\mathcal{R}(M)$ in the table.

This implies that the set of relations is linearly independent: projecting onto the subspace based by the $E(M)$, the resulting square matrix is upper triangular with 1s on the diagonal. □

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