Introduction.— The quantum matter under strong electromagnetic (EM) fields is an old but still thriving research area in many disciplines of physics. The strongly coupled quark-gluon plasma (QGP), a new state of matter governed by Quantum Chromodynamics (QCD), has been produced and extensively studied in high energy heavy-ion collisions for decades at Relativistic Heavy Ion Collider (RHIC) of Brookhaven National Lab and at Large Hadron Collider (LHC) of the European Organization for Nuclear Research (CERN). In the early stage of heavy-ion collisions, extremely strong EM fields of the order $10^{18} \sim 10^{19}$ Gauss are generated [1–3], which leave an imprint on the subsequent evolution of the QGP (for recent reviews of heavy ion collisions and QGP, e.g., Refs. [4–6]). Strong EM fields lead to many novel quantum phenomena such as the chiral magnetic effect [7, 8] and the chiral magnetic wave [9, 10] in heavy ion collisions (for recent reviews of these effects, see Refs. [11, 12]).

It requires a self-consistent description of EM fields coupled to the medium to study these effects. For example, the precise information about the evolution of EM fields is crucial to extract the CME signals [12–15] which has been searched for a decade. The EM fields from spectators can be well described in previous studies [3, 16–22] but not for the parts from the medium produced in collisions, because it is difficult to describe the medium effects from first principle with unknown transport properties of the strongly coupled medium and complicated interaction between EM fields and medium particles. The fully self-consistent treatment of EM fields and the interacting medium may help unveil the physics and even puzzles behind these phenomena. So far as we know, due to great numerical challenges, the exact space-time evolution of EM fields has not been achieved.

One of such an example is the puzzle related to the directed flow $v_1$ and the charge-odd directed flow $\Delta v_1$ [23]. The directed flow [24, 25] is defined as $v_1 \equiv \langle \cos(\phi - \Phi_{RP}) \rangle$ and reflects the collective sideward deflection of particles [26, 27], with $\phi$ and $\Phi_{RP}$ denoting the azimuthal angle of an outgoing particle and that of the reaction plane respectively. The charge-odd directed flow is defined as $\Delta v_1 \equiv v_1(h^+) - v_1(h^-)$, which is the difference between the directed flows of charged particles and their anti-particles, is expected to be sensitive to the EM field due to the opposite EM forces exerting on particles with opposite charges. Currently, both the hydrodynamical and transport models give a similar pattern of $v_1$ which agrees with the experiments. However, the results of $\Delta v_1$ from hydrodynamical models [28–33] disagree with the measurement of $\Delta v_1$ — the theoretical results show that both the pion’s $\Delta v_1$ and the proton’s $\Delta v_1$ in Au+Au collisions at $\sqrt{s_{NN}} = 200$ GeV have negative slopes, while the STAR data for $\Delta v_1$ show an almost vanishing slope and those for $\Delta v_1$ have a positive slope [34]. The transport models [35–39] also give consistent results of $\Delta v_1$ for hadrons [34] at high energies, but it is challenging to include the EM effects self-consistently in these models.

To reconcile the disagreement, it is essential to perform a fully self-consistent calculation of the QGP evolution coupled to the Maxwell equations. Most previous studies either treat EM fields as background fields without back reactions from medium particles [28–30] or adopt a perturbation method with simplified distributions of EM fields and QGP [40].

In this work, we carry out a fully self-consistent simulation of the dynamical evolution of the QGP in EM fields by solving the relativistic Boltzmann equations coupled to the Maxwell equations on Graphics Processing Units (GPUs). Our algorithm naturally incorporates all the
electromagnetic effects including the Lorentz, Coulomb and Faraday effects, etc. As a first test of our algorithm, we study the directed flow $v_1$ and its charge-odd component $\Delta v_1$ for pions and protons, trying to unveil the physics behind the $\Delta v_1$ puzzles. With the help of the state of art computing power of GPUs, we are also able to calculate the evolution of the EM fields in heavy-ion collisions in a more realistic and precise way, providing a more reliable baseline for many effects related to EM fields such as the CME effect.

**Boltzmann-Maxwell equations.**— The dynamical evolution of the QGP in EM fields is described by the relativistic Boltzmann equation,

$$[p^\mu \partial_\mu + Q_a p_\mu F^{\mu\nu} \partial_\nu] f_a (t, x, p) = C[f_a],$$

where $f_a$ is the spin and color averaged distribution function of the parton $a$ with $a = q, \bar{q}, g$ for the quark, antiquark, and gluon respectively, and $Q_a$ denotes its electric charge. The strong interaction among partons is encoded in the collision term $C[f_a]$ [41]. In the calculation we consider all 2-to-2 scatterings among $u, d, s$ quarks, their antiquarks and gluons [42], and the thermal masses of partons in the matrix elements are chosen to be $m_{u,d,s} = 0.3$ GeV, $m_{u,s} = 0.5$ GeV, and $m_g = 0.5$ GeV. The EM field tensor $F^{\mu\nu}$ is determined by solving the Maxwell equations,

$$\partial_\mu F_{\mu\nu} + \partial_\nu F_{\mu\mu} + \partial_\beta F_{\mu\alpha} = 0,$$

$$\partial_\mu F^{\mu\nu} = j^{\nu}_{\text{ext}} + j^{\nu}_{\text{med}},$$

where the source of the EM field has two parts: the external current $j^{\nu}_{\text{ext}}$ and medium current $j^{\nu}_{\text{med}}$. The external current $j^{\nu}_{\text{ext}}$ is generated by fast-moving partons, including spectators and quarks in the rapidity range $|y| > 1$. The dynamical evolution of $j^{\nu}_{\text{ext}}$ is assumed to be decoupled from the EM field because the trajectories of these fast-moving particles are hardly influenced by the field. The medium current $j^{\nu}_{\text{med}}$ is from quarks in the mid-rapidity

$$j^{\nu}_{\text{med}} = \sum_{a=q,\bar{q}} Q_a N_a \int \frac{d^3p}{(2\pi)^3} \frac{p^\nu}{E_a} f_a (t, x, p).$$

Figure 1. (a) The directed flow $v_1$ and (b) charge-odd directed flow $\Delta v_1$ as functions of the rapidity for protons/antiprotons (red dashed line) and $\pi^\pm$ (blue solid line) in Au+Au collisions at 200 GeV. The values of the parameters are set to: the saturation time $t_0 = 0.2$ fm/c, the impact parameter $b = 8$ fm, the factor $r_q = 0.1$ in Eq. (4), the strong coupling constant $\alpha_s = 0.3$, the range for transverse momenta $p_T \in [0.2, 1.5]$ GeV, and the distribution functions at $t = 5$ fm/c are used.
$f_0^{(d)} = 1.14$, and $f_{s,u,d,s}^{(d)} = 0$. For gluons, we choose $f_g^{(d)} \simeq \alpha_s^{-1}/r_g$ which is inversely proportional to the coupling constant [55].

We emphasize that only 10% of initial partons in the overlapped region, quantified by $r_g \simeq 0.1$, contribute to $f_a$ and $j^{\text{med}}_v$ in the midrapidity region $[-1, 1]$. The remaining 90% of partons are assumed to follow the rapidity distribution $f^{\pm}(y) = e^{\pm y/2}/[4\sinh(y_{\text{beam}}/2)]$ with $1 < |y| < y_{\text{beam}}$, where $\pm$ corresponds to the beam and target direction, respectively. Here $y_{\text{beam}} = 5.36$ is the beam rapidity for collisions at 200 GeV. The motion of these partons and spectators generates the external current $j^{\text{ext}}_v$, and thus provides a background EM field for the dynamical evolution of $f_a$.

In the hadronization stage, partons combine into hadrons in each phase space grid, whose yields agree with experimental data [56–58] for the rapidity density $dN/dy$ for $\pi^\pm$, $K^\pm$, $p$ and $\bar{p}$ at the mid-rapidity $y = 0$.

**Negative slope of $v_1$.** — The calculated results for the directed flows as functions of rapidity for pions and protons in the range $p_T \in [0.2, 1.5]$ GeV are shown in Fig. 1(a) and have negative slopes. We see that $v_1$ for pions and protons have almost the same magnitude. The evolution of the QGP governed by the strong interaction forms an tilted fireball in the reaction plane as shown by the energy density and pressure of all particles in the full range of $y$ and $p_T$ in Fig. 2(a,b). The pressure gradients lead to an antiflow corresponding to a negative $dv_1/dy$ at midrapidity [59–62].

To understand the generation of the antiflow, we plot in Fig. 2(c,d) the contours of the number density for protons and pions in the ranges $y \in [-1, 1]$ and $p_T \in [0.2, 1.5]$ GeV, which are mainly located in the central region. Due to pressure gradients, the protons at forward rapidity, which are mainly located at $x \simeq 0, z > 0$ region, receive a force pointing to the right-bottom direction and leading to a negative $v_1$. Similarly, protons at backward rapidity have a positive $v_1$. Since pions have a smaller mass than protons, their distribution in the $z$-direction is relatively wider than protons, while the spreads of protons and pions in the $x$-direction are similar, since they are dominated by the same $p_T$ range. We see in Fig. 2 that larger pressure gradients appear in the outer region in the $z$-direction, which leads to a larger magnitude of $v_1(\pi^\pm)$ than that of $v_1(p)$ and $v_1(\bar{p})$ in the region off the central rapidity $y = 0$ as shown by the difference between $v_1(\pi^+) + v_1(\pi^-)$ and $v_1(p) + v_1(\bar{p})$ in Fig. 1(a).

**Different behaviors of $\Delta v_1$ for pions and protons.** — The results for charge-odd directed flows $\Delta v_1$ for pions and protons are presented in Fig. 1(b). Since the dynamics of all charged quarks are governed by the same EM fields, a natural expectation is that $\Delta v_1$ for pions and protons as functions of rapidity should be similar, which has been observed in studies of hydrodynamics incorporating the EM fields [28–30]. However, we find in our study that $\Delta v_1^p$ has a positive slope while $\Delta v_1^\pi$ has a very small negative slope.

How to understand such counter-intuitive results of $\Delta v_1$ for pions and protons? In fact, the different behaviors of $\Delta v_1^\pi$ and $\Delta v_1^p$ come from an interplay of pressure gradients and EM fields.

The positive slope for $\Delta v_1^p$ is mainly attributed to

![Figure 2](image-url)  
**Figure 2.** Contour plots for (a) the energy density and (b) the pressure of all particles in the full rapidity and $p_T$ range in the $x–z$ plane. Contour plots for the number density of (c) protons (without anti-protons) and (d) pions in the ranges $y \in [-1, 1]$ and $p_T \in [0.2, 1.5]$ GeV in the $x–z$ plane. The arrows stand for directions of the pressure gradients formed by all particles (same for all plots), and the distribution functions at $t = 2.5$ fm/c are used. Other parameters are the same as in Fig. 1.

![Figure 3](image-url)  
**Figure 3.** Charge-odd directed flow $\Delta v_1^\pi$ and $\Delta v_1^p$ as functions of rapidity with collisions only (without EM fields). The parameters are chosen to be the same as in Fig. 1.
pressure gradients, similar to the difference between \(v_1^\pi^+ + v_1^\pi^-\) and \(v_1^p + v_1^{\bar{p}}\) as shown in Fig. 1. Antiprotons as newly produced particles are more likely to appear in the region with higher energy densities and therefore larger pressure gradients as observed in Fig. 2. Therefore the negativity for \(v_1^{\bar{p}}\) is enhanced relative to \(v_1^p\). Such an effect exists even when the EM fields are switched off. In Fig. 3, we turn off the EM fields and plot \(\Delta v_1\) caused by collisions only. We observe that \(\Delta v_1^\pi\) almost vanish but \(\Delta v_1^p\) still have positive slopes. On the other hand, the EM fields will influence the evolution of the QGP and therefore modify the pressure distribution as well as the number density distribution of hadrons, which finally results in an amplification of the \(\Delta v_1^p\) slope. Our results for \(\Delta v_1^p\) qualitatively agree with the UrQMD simulation [63, 64] and the data of STAR experiment at RHIC [65]. But our results are quantitatively smaller than the data because the pressure induced by \(J_{\text{ext}}^\nu\) in Eq. (2) is neglected in this work.

The approximately vanishing \(\Delta v_1^\pi\) in Fig. 3(a) indicates that the splitting between \(\pi^+\) and \(\pi^-\) in the transverse plane is a cumulative result of the EM fields. The small negative slope of \(\Delta v_1^\pi\) in Fig. 1(b) is consistent with the results from hydrodynamics incorporating the EM fields [28–30]. Unlike the case of protons and antiprotons, \(\pi^+\) and \(\pi^-\) receive similar contributions from pressure gradients since they have almost identical spatial distributions.

\(\Delta v_1\) for quarks.— A widely discussed issue for \(\Delta v_1^\pi\) is whether the contribution from the electric field is more important than that from the magnetic field [28–30]. To answer this question, we take a closer look at \(\Delta v_1\) for quarks. We show the results of \(\Delta v_1^u\) and \(\Delta v_1^d\) as functions of rapidity in collisions with and without the electric (\(E\)) and magnetic (\(B\)) fields in Fig. 4 at three different times \(t = 0.25, 2.5, 5.0\) fm/c.

For the case of collision only, different spatial distributions of \(u (d)\) and \(\bar{u} (\bar{d})\) give positive slopes for both \(\Delta v_1^u\) and \(\Delta v_1^d\), which leads to the positive slope of \(\Delta v_1^\pi\) via hadronization.

The contributions from electric and magnetic fields to \(\Delta v_1\) are opposite but in the same magnitude, which agrees with the theoretical result of Ref. [28]. Positively charged particles in forward rapidity are mainly influenced by the EM field from spectators with \(B_y < 0\) and \(E_y < 0\). Therefore the magnetic force points to +z direction while the electric force points to the opposite direction, so two forces partially cancel and lead to the net effect that is reflected in the difference of \(\Delta v_1\) between the cases with and without the EM field as shown in Fig. 4. We emphasize that the directed flow is a result of accumulation over time, so the balance of electric and magnetic contributions gradually changes with time. At an earlier time, e.g., \(t = 0.25\) fm/c, \(\Delta v_1\) is almost vanishing which is the result of the cancelation of the electric and magnetic contributions, while the contribution from the electric field becomes larger at later time, e.g., \(t = 5\) fm/c, and eventually \(\Delta v_1\) slightly favors the electric contribution.

Despite the hadronization model used in our simulation, we can understand the results by a simple sum rule in a naive picture of coalescence hadronization, i.e., a hadron’s \(v_1\) is approximately equal to the sum over \(v_1\) of its constituent quarks [66–68]. To this end, we separate the contributions from EM fields and pressure gradients as \(\Delta v_1^u \approx 2\Delta v_1^{\text{EM}} + \Delta v_1^{\text{pressure}}\), and \(\Delta v_1^d \approx\)
The EM field contribution $\Delta v_1^{\text{EM}}$ is proportional to the quark’s charge, while the pressure contribution $\Delta v_1^{\text{pressure}}$ is the same for all quarks, as shown in Fig. 4. Then following the coalescence sum rule, we have $\Delta v_1^u \approx \Delta v_1^s - \Delta v_1^d \approx 3 \Delta v_1^{\text{EM}}$ and $\Delta v_1^u \approx 2 \Delta v_1^+ + \Delta v_1^0 \approx 3 \Delta v_1^{\text{EM}} + 3 \Delta v_1^{\text{pressure}}$, with their slopes in agreement with the results in Fig. 1. Meanwhile, we also find that the slopes of $\Delta v_1$ are insensitive to the coupling constant $\alpha_s$.

**Summary and discussion.**— With the help of the state of art parallel computation algorithm, we are able to calculate the direct flow $v_1$ and charge-odd direct flow $\Delta v_1$ for pions and protons in heavy-ion collisions by solving the coupled Boltzmann-Maxwell equations for QGP self-consistently. The collision configuration is set to Au+Au collisions at 200 GeV and 20-30% centrality. Our results in the slopes of $v_1$ and $\Delta v_1$ in midrapidity are in qualitative agreement with the STAR data.

We found that the positive slope of $\Delta v_1$ for protons comes mainly from pressure gradients in the fireball, while the small negative slope of $\Delta v_1$ for pions reflects the contribution from EM fields over a period of time. The electric and magnetic fields have opposite contributions to $v_1$ and $\Delta v_1$ but with the same magnitude. At a relatively later time, the electric effects will slightly exceed the magnetic effects, which gives rise to the small negative slope of $\Delta v_1$ for pions. Our results are insensitive to the values of the coupling constant and can be understood by a simple sum rule in a naive coalescence picture of hadronization.

To see clear effects from the EM fields, $\Delta v_1$ for $D_0$ and $D_0$ mesons may be a better candidate, which needs to increase the number of momentum grids for heavy quarks. However, restricted by the GPU resources, our current algorithm does not allow such a simple extension. We will address this difficulty in a future study.

Our calculation can also give a prediction for $v_1(\pi^\pm)$ in low energy collisions. At highest RHIC energy, no significant difference between $v_1(\pi^+)$ and $v_1(\pi^-)$ has been observed due to low statistical significance [65]. In lower energy collisions, the EM fields will have longer lifetime and therefore are expected to induce more sideward deflection for charged particles, i.e. a more negative slope of $\Delta v_1^\pi$. This qualitatively agrees with the experimental observation at 7.7, 11.5, and 19.6 GeV [65].

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