THE POSITIVE ENERGY THEOREM FOR ASYMPTOTICALLY ANTI-DE SITTER SPACETIMES

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Abstract. We establish the inequality for Henneaux-Teitelboim’s total energy-momentum for asymptotically anti-de Sitter initial data sets which are asymptotic to arbitrary $t$-slice in anti-de Sitter spacetime. In particular, when $t = 0$, it generalizes Chruściel-Maerten-Tod’s inequality in the center of AdS mass coordinates. We also show that the determinant of energy-momentum endomorphism $Q$ is the geometric invariant of asymptotically anti-de Sitter spacetimes.

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1. Introduction

The positive energy theorem plays a fundamental role in general relativity. When the cosmological constant is zero and spacetimes are asymptotically flat, the positive energy theorem for the ADM total energy-momentum \cite{2} was first proved by Schoen and Yau \cite{20, 21, 22}, then by Witten \cite{24, 19}. We refer to \cite{13, 4, 12, 26} for the case of higher dimensional spacetimes.

When the cosmological constant is negative and spacetimes are asymptotically anti-de Sitter, initial data sets are asymptotically hyperbolic and the second fundamental forms are asymptotic to zero. There are a large number of papers to devote to define the total energy-momentum and prove its positivity in a physical manner, see, e.g. \cite{1, 15, 3} and references therein. (It seems the total energy was first defined in \cite{1}, and which also contained the proof of its positivity via SUGRA, exactly as the proof for zero cosmological constant \cite{11}.) However, the mathematical rigorous and complete proofs were given only in \cite{23, 7} for asymptotically anti-de Sitter initial data sets with zero second fundamental form, and in \cite{18, 9} for the initial data sets with nontrivial second fundamental form where the energy-momentum matrix was proved to be positive semi-definite. And some energy-momentum inequalities were proved with respect to certain specific coordinate systems in \cite{9}.

There is also another version of the positive energy theorem for asymptotically hyperbolic manifolds \cite{27, 8, 25} representing initial data sets near null infinity in asymptotically flat spacetimes. In this case both the metrics and the second fundamental forms are asymptotic to the hyperbolic metric.
In particular, the theorem in \cite{27,25} gives a different energy-momentum inequality for asymptotically anti-de Sitter initial data sets with the nontrivial second fundamental form if its trace is nonpositive.

The anti-de Sitter spacetime can be viewed as the hyperboloid
\[
\eta_{\alpha\beta} y^\alpha y^\beta = \frac{3}{\Lambda}, \quad \Lambda = -3\kappa^2 (\kappa > 0)
\]  
(1.1)
in \(\mathbb{R}^{3,2}\) equipped with the metric
\[
\eta_{\alpha\beta} dy^\alpha dy^\beta = -\left(dy^0\right)^2 + \sum_{i=1}^{3} (dy^i)^2 - (dy^4)^2.
\]

There are ten Killing vectors generating rotations for \(\mathbb{R}^{3,2}\)
\[
U_{\alpha\beta} = y_\alpha \frac{\partial}{\partial y^\beta} - y_\beta \frac{\partial}{\partial y^\alpha}.
\]  
(1.2)
Under coordinate transformations
\[
y^0 = \frac{\cos(\kappa t)}{\kappa} \cosh(\kappa r), \quad y^i = \frac{1}{\kappa} \sinh(\kappa r) n^i, \quad y^4 = \frac{\sin(\kappa t)}{\kappa} \cosh(\kappa r),
\]  
(1.3)
where \(n^1 = \sin \theta \cos \psi\), \(n^2 = \sin \theta \sin \psi\), \(n^3 = \cos \theta\), the induced anti-de Sitter metric is
\[
\tilde{g}_{\text{AdS}} = -\cosh^2(\kappa r) dt^2 + dr^2 + \frac{\sinh^2(\kappa r)}{\kappa^2} \left(d\theta^2 + \sin^2 \theta d\psi^2\right).
\]  
(1.4)
Let the coframe of (1.4) be
\[
\tilde{e}^0 = \cosh(\kappa r) dt, \quad \tilde{e}^1 = dr, \quad \tilde{e}^2 = \frac{\sinh(\kappa r)}{\kappa} d\theta, \quad \tilde{e}^3 = \frac{\sinh(\kappa r) \sin \theta}{\kappa} d\psi,
\]
and denote \(\{\tilde{e}_\alpha\}\) as its dual frame.

The metric and the second fundamental form of \(t\)-slice are the same in (1.4) no matter that \(t = 0\) or not. However, \(U_{\alpha\beta}\) are different restricting on different \(t\)-slices and depend on \(t\) (cf. Appendix A). In \cite{15}, Henneaux and Teitelboim defined the total energy-momentum for asymptotically anti-de Sitter spacetimes
\[
J_{ab}^{HT} = \lim_{r \to \infty} \int_{S_r} \tilde{\mathcal{G}}^{ijkl} [U^{-1}_{ab} \tilde{\nabla}_j g_{kl} - \tilde{\nabla}_j U_{ab} a_{kl}] dS_i + \lim_{r \to \infty} \int_{S_r} 2U^{(k)}_{ab} \pi^i_k dS_i, 
\]  
(1.5)
where initial data sets \((M, g, h)\) are asymptotic to \(t\)-slice of (1.4), \(a_{kl} = g_{kl} - \tilde{g}_{kl}\), and \(\tilde{g}, \tilde{\nabla}\) are the metric and the Levi-Civita connection of \(t\)-slice of (1.4) respectively,
\[
\tilde{\mathcal{G}}^{ijkl} = \frac{1}{2} \sqrt{\tilde{g}} \left( \tilde{g}^{ik} \tilde{g}^{jl} + \tilde{g}^{il} \tilde{g}^{jk} - 2 \tilde{g}^{ij} \tilde{g}^{kl} \right), \quad \pi^i_k = h^i_k - \delta_{ki} tr_{\tilde{g}}(h).
\]
These quantities form an energy-momentum endomorphism \(Q\). When \(t = 0\), (1.5) reduce to the definitions provided in \cite{23,7,18,9}.

Recall that, using essentially the explicit forms of \(U_{\alpha\beta}\) for \(t = 0\), Chruściel, Maerten and Tod \cite{9} provided definitions of the total energy \(m_\nu\) \((\nu =}
0, 1, 2, 3), the rest-frame angular momentum \( \vec{j}(i) \) and the center of mass \( c(i) \) \((i = 1, 2, 3)\), with respect to the anti-de Sitter spacetime equipped with the metric

\[
\tilde{g}_{AdS} = -\left(1 + \frac{|x|^2}{1-|x|^2}\right)^2 dt^2 + \frac{4}{(1-|x|^2)^2} \sum_{i=1}^{3} (dx^i)^2.
\]

Denote by \( \nabla \) and \( \tilde{\nabla} \) the Levi-Civita connections of the initial data sets with respect to the metric \( g \) and the background hyperbolic metric \( \tilde{g} \) respectively.

The total energy vector \( m(\nu) \) \((\nu = 0, 1, 2, 3)\) is defined as

\[
m(\nu) = \frac{1}{8\pi} \lim_{|x| \to 1} \int_{S|x|} \sqrt{\det g(V(\nu)g^{ij}Y_j g_{kl} + \nabla[i]V(\nu)g^{ij}(g_{jk} - \tilde{g}_{jk}))} dS_i,
\]

where \( V(0) = \frac{1+|x|^2}{1-|x|^2} \) and \( V(j) = \frac{(-2)x^i}{1-|x|^2} \). Let \( Y \) be a tangential vector to the \( t = 0 \) slice. Denote

\[
H(Y) = \frac{1}{8\pi} \lim_{|x| \to 1} \int_{S|x|} \sqrt{\det g(h_{ij} - h^k\delta^i_j)} Y^j dS_i,
\]

where \( h_{ij} \) is the second fundamental form of the slice in the spacetime. The rest-frame angular momentum vector \( \vec{j}(i) \) \((i = 1, 2, 3)\) is

\[
\vec{j}(i) = \epsilon_{ijk}H(\Omega_{(i)}(t))
\]

where \( \Omega_{(i)}(t) = x_j\partial_t - x_t\partial_j \). And the center of mass \( c(i) \) is

\[
c(i) = H(C(i))
\]

where \( C(t) = \left(\frac{1+|x|^2}{2}\delta^i_j - x^i x^j\right)\partial_j \). Denote

\[
\tilde{m} = (m(1), m(2), m(3)), \quad \tilde{c} = (c(1), c(2), c(3)), \quad \tilde{\vec{j}} = (\vec{j}(1), \vec{j}(2), \vec{j}(3)).
\]

They pointed out that [9], if the total energy 4-vector is timelike, i.e.,

\[
m(0) > (m(1)^2 + m(2)^2 + m(3)^2)^{\frac{1}{2}},
\]

one can make \( SO(3,1) \) coordinate transformations such that

\[
\left(m(0)^2 - m(1)^2 - m(2)^2 - m(3)^2\right)^{\frac{1}{2}} \rightarrow m(0),
\]

\[
m(1), m(2), m(3), c(2), j(1), j(2) \rightarrow 0,
\]

and they proved the energy-momentum inequality

\[
m(0) \geq \sqrt{|\tilde{c}|^2 + |\tilde{\vec{j}}|^2 + 2|\tilde{c} \times \tilde{\vec{j}}|} \quad (1.6)
\]

in this new coordinate system. We refer to the coordinates satisfying

\[
m(1) = m(2) = m(3) = c(2) = j(1) = j(2) = 0 \quad (1.7)
\]

as the “center of AdS mass” coordinates (cf. Appendix B).

Indeed, Witten’s argument indicates that \( Q \) is positive semidefinite. But it does not give that the total energy 4-vector is timelike for general nontrivial initial data sets (cf. Remark 4.2). Also the form of (1.6) is not \( SO(3,1) \)
invariant, and it changes when it is transformed back to the non-center of AdS mass coordinates. These motivate us to establish the inequality for Henneaux and Teitelboim’s total energy-momentum in general non-center of AdS mass coordinates. In this paper, we prove (Theorem 4.1)

\[ E_0 \geq \sqrt{L^2 - 2V^2 + 2\left(\max\{A^4 - L^2V^2, 0\}\right)^2} . \]

(See (2.3), (3.5) for the definitions of these notations.) If three vectors \( c, c', J \) or \( \tilde{m}, \tilde{c}, \tilde{j} \) are linearly dependent, i.e, \( V = 0 \), then

\[ E_0 \geq \sqrt{L^2 + 2A^2} . \]

This generalizes the energy-momentum inequality (1.6).

We remark that, unlike the case of non-positive cosmological constant where it always holds and serves as the feature of spacetimes, the positive energy theorem for the positive cosmological constant holds only on certain very restricted spacelike hypersurfaces [17, 16].

The paper is organized as follows: In Section 2, we discuss the relation of the total energy-momenta given in [15] and [9]. In Section 3, we define the energy-momentum endomorphism \( Q \) and compute it explicitly under a fixed Clifford multiplication. In Section 4, we establish the new inequality for Henneaux and Teitelboim’s total energy-momentum. In Section 5, we show that \( Q \) is the geometric invariant of asymptotically anti-de Sitter spacetimes. In Appendix A, we provide the restriction of the ten Killing vectors \( U_{\alpha\beta} \) on the anti-de Sitter spacetime. In Appendix B, we explicitly construct the center of AdS mass coordinate transformations on the \( t = 0 \) slice. In Appendix C, we provide roots of the determinant of \( Q \).

Throughout the paper, repeating indices means taking summation, with Greek indices running from 0 to 3, the lower-case Latin indices running from 1 to 3 and upper-case Latin indices running from 1 to 2.

2. Total energy-momentum

Let \((N, \tilde{g})\) be a spacetime with negative cosmological constant \( \Lambda \), and \( \tilde{g} \) satisfies the Einstein field equations

\[ \tilde{Ric} - \frac{\tilde{R}}{2} \tilde{g} + \Lambda \tilde{g} = T . \]

Suppose that the stress-energy tensor \( T \) satisfies the dominant energy condition

\[ T_{00} \geq \sqrt{\sum_i T_{0i}^2}, \quad T_{00} \geq |T_{\alpha\beta}| . \] (2.2)

Let \((M, g, h)\) be an initial data set where \( M \) is a 3-dimensional spacelike hypersurface with the induced Riemannian metric \( g \) and the second fundamental form \( h \). Let \( \{\tilde{e}_i\} \) be the frame of (1.4). Recall \( \kappa = \sqrt{-\frac{\Lambda}{3}} \). \((M, g, h)\)
is said to be *asymptotically anti-de Sitter* of order $\tau > \frac{3}{2}$ if:

1. There is a compact set $K \subset M$ such that $M \setminus K$ is the disjoint union of a finite number of subsets (ends) $M_i$ and each $M_i$ is diffeomorphic to $\mathbb{R}^3 \setminus B_r$ with $B_r$ the closed ball of radius $r$;
2. Under this diffeomorphism, the metric $g_{ij} = g(\tilde{e}_i, \tilde{e}_j)$ on each end is of the form $g_{ij} = \delta_{ij} + a_{ij}$ where $a_{ij}$ satisfies

$$a_{ij} = O(e^{-\tau kr}), \quad \nabla_k a_{ij} = O(e^{-\tau kr}), \quad \nabla_l \nabla_k a_{ij} = O(e^{-\tau kr});$$

and the second fundamental form $h_{ij} = h(\tilde{e}_i, \tilde{e}_j)$ satisfies

$$h_{ij} = O(e^{-\tau kr}), \quad \nabla_k h_{ij} = O(e^{-\tau kr});$$

3. There exists a distance function $\rho_z$ such that $T_{00} e^{\kappa \rho} = T_{0i} e^{\kappa \rho} \in L^1(M)$. Here $\tilde{\nabla}$, $\{\tilde{e}_i\}$ are the Levi-Civita connection and frame of the hyperbolic metric

$$\tilde{g} = dr^2 + \frac{\sinh^2(\kappa r)}{\kappa^2} (d\theta^2 + \sin^2 \theta d\psi^2)$$

respectively. Denote

$$E_i = \tilde{\nabla}^j g_{ij} - \tilde{\nabla}_i tr_{\tilde{g}}(g) - \kappa (a_{1i} - g_{1i} tr_{\tilde{g}}(a)), \quad P_{ki} = h_{ki} - g_{ki} tr_{\tilde{g}}(h).$$

Let $U_{\alpha\beta}$ be the restrictions of the Killing vectors \((1.2)\) on the $t$-slice. For the convenience of the statement of our main theorem, we introduce the following notions.

$$E_0 = \frac{\kappa}{16\pi} \lim_{r \to \infty} \int_{S_r} \mathcal{E}_i U_{i0}^{(0)} \tilde{\omega},$$

$$c_i = \frac{\kappa}{16\pi} \lim_{r \to \infty} \int_{S_r} \mathcal{E}_i U_{i4}^{(0)} \tilde{\omega} + \frac{1}{8\pi} \sum_{j=2}^{3} \lim_{r \to \infty} \int_{S_r} \mathcal{P}_{j1} U_{j4}^{(j)} \tilde{\omega},$$

$$c'_i = \frac{\kappa}{16\pi} \lim_{r \to \infty} \int_{S_r} \mathcal{E}_i U_{i0}^{(0)} \tilde{\omega} + \frac{1}{8\pi} \sum_{j=2}^{3} \lim_{r \to \infty} \int_{S_r} \mathcal{P}_{j1} U_{j0}^{(j)} \tilde{\omega},$$

$$J_i = \frac{\kappa}{8\pi} \sum_{j=2}^{3} \lim_{r \to \infty} \int_{S_r} \mathcal{P}_{j1} V^{(j)} \tilde{\omega}, \quad J_{jl} = \varepsilon_{ijl} J_i$$

where $\tilde{\omega} = \tilde{e}^2 \wedge \tilde{e}^3$, $U_{\alpha\beta} = U_{\alpha\beta}^{(\gamma)} \tilde{e}_\gamma$, $\varepsilon_{ijkl} V_i = U_{jl}$. In the frame of \((1.4)\),

$$J_{ab}^{HT} = \lim_{r \to \infty} \int_{S_r} \tilde{G}_{ijkl} U_{ab}^{(0)} \tilde{\nabla}_j g_{kl} - \tilde{\nabla}_j U_{ab}^{(0)} a_{kl} \tilde{\omega} + \lim_{r \to \infty} \int_{S_r} 2U_{ab}^{(k)} \mathcal{P}_{k1} \tilde{\omega}. $$
Since

\[
\hat{G}^{i,j,k,l}[U_{ab}^{(0)} \hat{\nabla}_j g_{kl} - \hat{\nabla}_j U_{ab}^{(0)} a_{kl}] = \frac{1}{2} \left( \delta_{1,k} \delta_{j,l} + \delta_{1,l} \delta_{j,k} - 2 \delta_{1,j} \delta_{k,l} \right) \left( U_{ab}^{(0)} \hat{\nabla}_j g_{kl} - \hat{\nabla}_j U_{ab}^{(0)} a_{kl} \right)
\]

we obtain

\[
J^{HT}_{40} = \frac{16\pi}{\kappa} E_0, \quad J^{HT}_{i4} = \frac{16\pi}{\kappa} e_i, \quad J^{HT}_{i0} = \frac{16\pi}{\kappa} c_i, \quad J^{HT}_{j,l} = \frac{16\pi}{\kappa} \varepsilon_{ijl} J_i \quad (2.4)
\]

where \( J^{HT}_{ab} \) is Henneaux-Teitelboim’s total energy-momentum \( (1.5) \).

Now we discuss the relationship between the quantities \( (2.3) \) and the total energy-momentum defined in \( [9] \). The original definition is given for \( \kappa = 1 \). But we consider the general \( \kappa \) in the followings. The transformations connecting the hyperbolic metric \( b = \frac{4}{\kappa^2(1-|x|^2)^2} dx^2 \) used in \( [9] \) and the metric \( \hat{g} \) used in our setting are

\[
x^1 = \tanh \frac{\kappa r}{2} \sin \theta \cos \psi, \quad x^2 = \tanh \frac{\kappa r}{2} \sin \theta \sin \psi, \quad x^3 = \tanh \frac{\kappa r}{2} \cos \theta.
\]

Straightforward computation yields

\[
\partial_{x^1} = \frac{2}{\kappa} \cosh^2 \frac{\kappa r}{2} \sin \theta \cos \psi \partial_r + \frac{\cos \theta \cos \psi}{\tanh \frac{\kappa r}{2}} \partial_\theta - \frac{\sin \psi}{\tanh \frac{\kappa r}{2} \sin \theta} \partial_\psi,
\]
\[
\partial_{x^2} = \frac{2}{\kappa} \cosh^2 \frac{\kappa r}{2} \sin \theta \sin \psi \partial_r + \frac{\cos \theta \sin \psi}{\tanh \frac{\kappa r}{2}} \partial_\theta + \frac{\cos \psi}{\tanh \frac{\kappa r}{2} \sin \theta} \partial_\psi,
\]
\[
\partial_{x^3} = \frac{2}{\kappa} \cosh^2 \frac{\kappa r}{2} \cos \theta \partial_r - \frac{\sin \theta}{\tanh \frac{\kappa r}{2}} \partial_\theta.
\]
Thus, in the polar coordinates, the vectors used in [9] are

\[ V_0 = \cosh \kappa r, \]
\[ V_1 = -\sinh \kappa r \sin \theta \cos \psi, \]
\[ V_2 = -\sinh \kappa r \sin \theta \sin \psi, \]
\[ V_3 = -\sinh \kappa r \cos \theta, \]

\[ C_1 = \coth \kappa r \left( \cos \theta \cos \psi \partial \theta - \frac{\sin \psi}{\sin \theta} \partial \psi \right) + \frac{1}{\kappa} \sin \theta \cos \psi \partial_r, \]
\[ C_2 = \coth \kappa r \left( \cos \theta \sin \psi \partial \theta + \frac{\cos \psi}{\sin \theta} \partial \psi \right) + \frac{1}{\kappa} \sin \theta \sin \psi \partial_r, \]
\[ C_3 = -\coth \kappa r \sin \theta \partial \theta + \frac{1}{\kappa} \cos \theta \partial_r, \]

\[ \Omega_{(1)(2)} = \partial \psi, \]
\[ \Omega_{(2)(3)} = -\sin \psi \partial \theta - \frac{\cos \theta \cos \psi}{\sin \theta} \partial \psi, \]
\[ \Omega_{(3)(1)} = \cos \psi \partial \theta - \frac{\cos \theta \sin \psi}{\sin \theta} \partial \psi. \]

**Proposition 2.1.** The following relations hold between Henneaux-Teitelboim’s total energy-momentum and Chruściel-Maerten-Tod’s total energy-momentum

\[ E_0 = m_0, \]
\[ c_i = -m_{(i)} \cos \kappa t + c_{(i)} \sin \kappa t, \]
\[ c'_i = m_{(i)} \sin \kappa t + c_{(i)} \cos \kappa t, \]
\[ J_l = j_{(l)}. \]  \hspace{1cm} (2.5)

**Proof:** By the explicit expressions of \( U_{\alpha\beta} \) in Appendix A, we find that \( E_0, J_i \) do not depend on \( t \), and

\[ \frac{dc_i}{dt} = \kappa c'_i, \quad \frac{dc'_i}{dt} = -\kappa c_i. \]

Note the total energy-momentum in [9] is defined on \( t = 0 \) slice. And straightforward computation shows that, at \( t = 0 \),

\[ E_0 = m_0, \quad c_i = -m_{(i)}, \quad c'_i = c_{(i)}, \quad J_l = j_{(l)}. \]

This yields (2.5). Q.E.D.

In [6], Carter obtained a family of solutions for the Einstein field equations.

\[ ds^2 = \frac{\Delta_\mu(d\lambda - \lambda^2 d\psi)^2 - \Delta_\lambda(d\lambda + \mu^2 d\psi)^2}{\lambda^2 + \mu^2} + (\lambda^2 + \mu^2) \left( \frac{d\lambda^2}{\Delta_\lambda} + \frac{d\mu^2}{\Delta_\mu} \right), \]

where

\[ \Delta_\lambda = \frac{1}{3} \Lambda \lambda^4 + h\lambda^2 - 2m\lambda + p + e^2, \quad \Delta_\mu = \frac{1}{3} \Lambda \mu^4 - h\mu^2 + 2q\mu + p \]
and \(-\infty < \chi, \lambda, \mu < \infty\), \(0 \leq \psi < 2\pi\). It provides the Kerr-anti-de Sitter spacetimes if \(\Delta_\lambda, \Delta_\mu\) are given as follows
\[
\Delta_\lambda = (\kappa^2 \lambda^2 + 1)(\lambda^2 + a^2) - 2m\lambda, \quad \Delta_\mu = (\kappa^2 \mu^2 - 1)(\mu^2 - a^2).
\]
The Kerr-anti-de Sitter solution allows \(|\mu| > |\kappa|^{-1}\) and the metric has signature \((-1, 1, 1, 1)\) if \(\Delta_\mu > 0\). If \(m = 0\), it has constant curvature \(-\kappa^2\) and reduces to the anti-de Sitter spacetime.

In the region \(-|\kappa|^{-1} < \mu < |\kappa|^{-1}\), \(\lambda > 0\), we can take the coordinate transformation
\[
\lambda = \hat{r}, \quad \mu = a \cos \hat{\theta}, \quad \chi = t - a \hat{\phi}, \quad \psi = \frac{1}{a} \hat{\phi}
\]
with \(|\mu a| < 1\) and it yields Boyer-Lindquist coordinates for the Kerr-anti-de Sitter spacetime
\[
\tilde{g}_{KAdS} = -\left[1 - \frac{2m\hat{r}}{U} + \kappa^2 (\hat{r}^2 + a^2 \sin^2 \hat{\theta})\right] d\hat{t}^2 + \frac{U}{\Delta_{\hat{r}}} d\hat{r}^2 + \frac{U}{\Delta_{\hat{\theta}}} d\hat{\theta}^2
\]
\[
+ \frac{V}{U} \sin^2 \hat{\theta} d\hat{\phi}^2 - 2a \sin^2 \hat{\theta} \left[\frac{2m\hat{r}}{U} - \kappa^2 (\hat{r}^2 + a^2)\right] d\hat{t} d\hat{\phi},
\]
where
\[
\Delta_{\hat{r}} = (\hat{r}^2 + a^2)(1 + \kappa^{-2} \hat{r}^2) - 2m\hat{r}, \quad \Delta_{\hat{\theta}} = 1 - \kappa^2 a^2 \cos^2 \hat{\theta},
\]
\[
U = \hat{r}^2 + a^2 \cos^2 \hat{\theta}, \quad V = 2m\hat{r}a^2 \sin^2 \hat{\theta} + U(\hat{r}^2 + a^2)(1 - \kappa^2 a^2).
\]

By [15], we can know the total energy-momentum of \(t\)-slices
\[
E_0 = \frac{m}{(1 - \kappa^2 a^2)^2}, \quad c_i = c'_i = 0, \quad J_1 = J_2 = 0, \quad J_3 = \frac{mka}{(1 - \kappa^2 a^2)^2}.
\]

3. Energy-momentum endomorphism

In this section we define energy-momentum endomorphisms for asymptotically anti-de Sitter spacetimes. Recall that the spinor bundle of the anti-de Sitter spacetime is trivial and is \(\mathbb{C}^4\) over the anti-de Sitter spacetime. The anti-de Sitter spacetime is characterized by imaginary Killing spinors satisfying the following equations
\[
\bar{\nabla}_X^{AdS} \Phi_0 + \frac{\kappa \sqrt{-1}}{2} X \cdot \Phi_0 = 0.
\]
Denote \(\mathbb{K}\) the space of imaginary Killing spinors over the anti-de Sitter spacetime. It is a complex linear space with complex dimension 4. There exists a one-to-one complex linear map
\[
\mathcal{K} : \mathbb{C}^4 \rightarrow \mathbb{K}.
\]
For any given complex vector \(\tilde{\lambda}\), \(\mathcal{K}(\tilde{\lambda}) = \Phi_{0}^\lambda\) is the unique corresponding Killing spinor.

We first define globally the energy-momentum endomorphism \(Q\) as a Hermitian transformation over complex space \(\mathbb{C}^4\). Let \(\{\tilde{e}_\alpha\}\) and \(\bar{\nabla}\) be the frame
and Levi-Civita connection of anti-de Sitter metric (1.4) respectively. For each end of an asymptotically anti-de Sitter initial data set,

\[
\Theta = (\bar{\nabla}^{j} g_{ij} - \bar{\nabla}_{1} tr_{g}(g)) \text{Id} + \kappa \sum \limits_{l} (a_{l1} - g_{l1} tr_{\bar{g}}(a)) \sqrt{-1} \hat{e}_{l}
- 2 \sum \limits_{l} (h_{l1} - g_{l1} tr_{\bar{g}}(h)) \hat{e}_{0} \cdot \hat{e}_{l}
\]

serves as an endomorphism of the spinor bundle.

**Definition 3.1.** The energy-momentum endomorphism \( Q \) of an end for an asymptotically anti-de Sitter initial data set is a complex linear map

\[
Q : \mathbb{C}^{4} \rightarrow \mathbb{C}^{4}
\]

such that for any vector \( \bar{x} \in \mathbb{C}^{4} \),

\[
\langle \bar{x}, Q(\bar{x}) \rangle_{C} = \frac{1}{32 \pi} \int_{S_{\infty}} \langle \Phi^{0}_{\lambda}, \Theta \cdot \Phi^{0}_{\lambda} \rangle \hat{\omega}
\]

where \( \langle , \rangle_{C} \) is the Hermitian inner product on \( \mathbb{C}^{4} \), and \( S_{\infty} \) is the 2-sphere at spatial infinity in \( M \) and \( \hat{\omega} \) is the reduced area form of \( S_{\infty} \).

Since \( \Theta \) is Hermitian, \( Q \) is also Hermitian. Now we compute \( Q \) explicitly under the following Clifford representation. (We fix it for convenience throughout the paper although the whole results do not depend on the specific representation.)

\[
\hat{e}_{0} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \hat{e}_{1} \mapsto \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \hat{e}_{2} \mapsto \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \hat{e}_{3} \mapsto \sqrt{-1} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

(3.1)

Under this representation, the imaginary Killing spinor \( \Phi^{0}_{0} \) is of the form

\[
\Phi^{0}_{0} = \begin{pmatrix} u^{+} e^{x_{1}^{'}} + u^{-} e^{-x_{1}^{'}} \\ v^{+} e^{x_{2}^{'}} + v^{-} e^{-x_{2}^{'}} \\ -\sqrt{-1} u^{+} e^{x_{3}^{'}} + \sqrt{-1} u^{-} e^{-x_{3}^{'}} \\ \sqrt{-1} v^{+} e^{x_{4}^{'}} - \sqrt{-1} v^{-} e^{-x_{4}^{'}} \end{pmatrix},
\]

(3.2)
where

\begin{align*}
    u^+ &= \left( \lambda_1 \cos \frac{\kappa t}{2} + \lambda_3 \sin \frac{\kappa t}{2} \right) e^{\frac{\sqrt{-1}}{2} \psi \sin \frac{\theta}{2}} \\
    &\quad + \left( \lambda_2 \cos \frac{\kappa t}{2} + \lambda_4 \sin \frac{\kappa t}{2} \right) e^{\frac{\sqrt{-1}}{2} \psi \cos \frac{\theta}{2}}, \\
    u^- &= \left( - \lambda_1 \sin \frac{\kappa t}{2} + \lambda_3 \cos \frac{\kappa t}{2} \right) e^{\frac{\sqrt{-1}}{2} \psi \sin \frac{\theta}{2}} \\
    &\quad + \left( - \lambda_2 \sin \frac{\kappa t}{2} + \lambda_4 \cos \frac{\kappa t}{2} \right) e^{\frac{\sqrt{-1}}{2} \psi \cos \frac{\theta}{2}}, \\
    v^+ &= - \left( - \lambda_1 \sin \frac{\kappa t}{2} + \lambda_3 \cos \frac{\kappa t}{2} \right) e^{\frac{\sqrt{-1}}{2} \psi \cos \frac{\theta}{2}} \\
    &\quad + \left( - \lambda_2 \sin \frac{\kappa t}{2} + \lambda_4 \cos \frac{\kappa t}{2} \right) e^{\frac{\sqrt{-1}}{2} \psi \sin \frac{\theta}{2}}, \\
    v^- &= - \left( \lambda_1 \cos \frac{\kappa t}{2} + \lambda_3 \sin \frac{\kappa t}{2} \right) e^{\frac{\sqrt{-1}}{2} \psi \cos \frac{\theta}{2}} \\
    &\quad + \left( \lambda_2 \cos \frac{\kappa t}{2} + \lambda_4 \sin \frac{\kappa t}{2} \right) e^{\frac{\sqrt{-1}}{2} \psi \sin \frac{\theta}{2}},
\end{align*}

and \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) are four arbitrary complex numbers.

**Proposition 3.1.** Under the Clifford multiplication (3.1), the energy-momentum endomorphism has the following form

\begin{align*}
    P &= \left( \begin{array}{c}
        E_0 - c_3 \\
        c_1 + \sqrt{-1}c_2
    \end{array} \right), \\
    P &= \left( \begin{array}{c}
        E_0 + c_3 \\
        -c_1 + \sqrt{-1}c_2
    \end{array} \right), \\
    W &= \left( \begin{array}{c}
        w_1 \\
        w_2 \\
        w_1^+ \\
        -w_1
    \end{array} \right), \\
    \hat{P} &= \left( \begin{array}{c}
        E_0 + c_3 \\
        -c_1 + \sqrt{-1}c_2
    \end{array} \right), \\
    \hat{P} &= \left( \begin{array}{c}
        E_0 - c_3 \\
        c_1 + \sqrt{-1}c_2
    \end{array} \right),
\end{align*}

where \( w_1 = \epsilon_3' - \sqrt{-1} J_3, w_2^\pm = -\epsilon_1' \pm J_2 \pm \sqrt{-1}(\epsilon_2' \pm J_1) \).

**Proof:** By [5, 2], [3, 3], we have

\[
\frac{1}{4} \int_{S_{\infty}} (\Phi_0^\lambda, \Theta \cdot \Phi_0^\lambda) \tilde{\omega} = \frac{1}{2} \int_{S_{\infty}} \mathcal{E}_1 (u^+u^+ + v^+v^+) e^{\kappa r} \tilde{\omega} \\
+ \int_{S_{\infty}} \mathcal{P}_{21} (u^+v^+ + v^+u^+) e^{\kappa r} \tilde{\omega} \\
+ \sqrt{-1} \int_{S_{\infty}} \mathcal{P}_{31} (u^+v^+ - v^+u^+) e^{\kappa r} \tilde{\omega},
\]
and

\[
\overline{u^+u^+} + \overline{v^+v^+} = \frac{1}{2}(\lambda_1 \lambda_1 + \lambda_2 \lambda_2 + \lambda_3 \lambda_3 + \lambda_4 \lambda_4)
\]

\[
+ \frac{1}{2} \cos(\kappa t) \sin \theta \cos \varphi (\lambda_1 \lambda_2 + \lambda_2 \lambda_1 - \lambda_3 \lambda_4 - \lambda_4 \lambda_3)
\]

\[
+ \frac{1}{2} \sin(\kappa t) \sin \theta \cos \varphi (\lambda_1 \lambda_4 + \lambda_2 \lambda_3 + \lambda_3 \lambda_2 + \lambda_4 \lambda_1)
\]

\[
+ \sqrt{-1} \cos(\kappa t) \sin \theta \sin \varphi (-\lambda_1 \lambda_2 + \lambda_2 \lambda_1 + \lambda_3 \lambda_4 - \lambda_4 \lambda_3)
\]

\[
+ \sqrt{-1} \sin(\kappa t) \sin \theta \sin \varphi (-\lambda_1 \lambda_4 + \lambda_2 \lambda_3 - \lambda_3 \lambda_2 + \lambda_4 \lambda_1)
\]

\[
+ \frac{1}{2} \cos(\kappa t) \cos \theta (-\lambda_1 \lambda_1 + \lambda_2 \lambda_2 + \lambda_3 \lambda_3 - \lambda_4 \lambda_4)
\]

\[
+ \frac{1}{2} \sin(\kappa t) \cos \theta (-\lambda_1 \lambda_3 + \lambda_2 \lambda_4 - \lambda_3 \lambda_1 + \lambda_4 \lambda_2),
\]

\[
\overline{u^+v^+} + \overline{v^+u^+} = \frac{1}{2} \sin \theta (-\lambda_1 \lambda_3 + \lambda_2 \lambda_4 - \lambda_3 \lambda_1 + \lambda_4 \lambda_2)
\]

\[
+ \frac{1}{2} \cos \varphi (\lambda_1 \lambda_4 - \lambda_2 \lambda_3 - \lambda_3 \lambda_2 + \lambda_4 \lambda_1)
\]

\[
+ \sqrt{-1} \sin \varphi (-\lambda_1 \lambda_4 - \lambda_2 \lambda_3 + \lambda_3 \lambda_2 + \lambda_4 \lambda_1)
\]

\[
+ \frac{1}{2} \cos(\kappa t) \cos \theta \cos \varphi (-\lambda_1 \lambda_4 - \lambda_2 \lambda_3 - \lambda_3 \lambda_2 - \lambda_4 \lambda_1)
\]

\[
+ \frac{1}{2} \sin(\kappa t) \cos \theta \cos \varphi (\lambda_1 \lambda_2 + \lambda_2 \lambda_1 - \lambda_3 \lambda_4 - \lambda_4 \lambda_3)
\]

\[
+ \sqrt{-1} \cos(\kappa t) \sin \theta \sin \varphi (\lambda_1 \lambda_4 - \lambda_2 \lambda_3 + \lambda_3 \lambda_2 - \lambda_4 \lambda_1)
\]

\[
+ \sqrt{-1} \sin(\kappa t) \sin \theta \sin \varphi (\lambda_1 \lambda_2 + \lambda_2 \lambda_1 - \lambda_3 \lambda_4 - \lambda_4 \lambda_3)
\]

\[
+ \frac{1}{2} \cos \theta \cos \varphi (-\lambda_1 \lambda_4 - \lambda_2 \lambda_3 + \lambda_3 \lambda_2 + \lambda_4 \lambda_1)
\]

\[
+ \sqrt{-1} \sin \theta \sin \varphi (\lambda_1 \lambda_4 - \lambda_2 \lambda_3 - \lambda_3 \lambda_2 + \lambda_4 \lambda_1),
\]

\[
\overline{u^+v^+} - \overline{v^+u^+} = \frac{1}{2} \sin \theta (-\lambda_1 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_1 - \lambda_4 \lambda_2)
\]

\[
+ \frac{1}{2} \cos(\kappa t) \cos \theta \cos \varphi (\lambda_1 \lambda_4 - \lambda_2 \lambda_3 + \lambda_3 \lambda_2 - \lambda_4 \lambda_1)
\]

\[
+ \frac{1}{2} \sin(\kappa t) \cos \theta \cos \varphi (-\lambda_1 \lambda_2 + \lambda_2 \lambda_1 + \lambda_3 \lambda_4 - \lambda_4 \lambda_3)
\]

\[
+ \sqrt{-1} \cos(\kappa t) \sin \theta \sin \varphi (-\lambda_1 \lambda_4 - \lambda_2 \lambda_3 - \lambda_3 \lambda_2 - \lambda_4 \lambda_1)
\]

\[
+ \sqrt{-1} \sin(\kappa t) \sin \theta \sin \varphi (\lambda_1 \lambda_2 + \lambda_2 \lambda_1 - \lambda_3 \lambda_4 - \lambda_4 \lambda_3)
\]

\[
+ \frac{1}{2} \cos \theta \cos \varphi (-\lambda_1 \lambda_4 - \lambda_2 \lambda_3 + \lambda_3 \lambda_2 + \lambda_4 \lambda_1)
\]

\[
+ \sqrt{-1} \sin \theta \sin \varphi (\lambda_1 \lambda_4 - \lambda_2 \lambda_3 - \lambda_3 \lambda_2 + \lambda_4 \lambda_1).
\]
Thus we obtain
\[
\langle \vec{\lambda}, Q(\vec{\lambda}) \rangle_C = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) Q(\lambda_1, \lambda_2, \lambda_3, \lambda_4)^t,
\]
where $Q$ is given by (3.3).

Q.E.D.

Denote $c = (c_1, c_2, c_3), c' = (c'_1, c'_2, c'_3), J = (J_1, J_2, J_3) = \vec{j}$ and
\[
L = (|c|^2 + |c'|^2 + |J|^2)^{\frac{1}{2}},
\]
\[
A = (|c \times c'|^2 + |c \times J|^2 + |c' \times J|^2)^{\frac{1}{2}},
\]
\[
V = (\varepsilon_{ijl}c_i c'_j J_l)^{\frac{1}{3}},
\]
where $2L$, $2A^2$ and $V^3$ are the (normalized) length, surface area and volume of the parallelepiped spanned by $c$, $c'$ and $J$. Clearly, $L^2 \geq 3V^2$. Using (2.5), we can prove
\[
L = (|\vec{m}|^2 + |\vec{c}|^2 + |\vec{j}|^2)^{\frac{1}{2}},
\]
\[
A = (|\vec{c} \times \vec{m}|^2 + |\vec{c} \times \vec{j}|^2 + |\vec{m} \times \vec{j}|^2)^{\frac{1}{2}},
\]
\[
V = (\varepsilon_{ijl}m_{(i)} c_{(j)} j_{(l)})^{\frac{1}{3}}.
\]

Note that $\mathbb{R}^{3,2}$ has two timelike Killing vectors $\frac{\partial}{\partial y^0}$, $\frac{\partial}{\partial y^4}$ and three spacelike Killing vectors $\frac{\partial}{\partial y^1}$, $\frac{\partial}{\partial y^2}$, $\frac{\partial}{\partial y^3}$. Physically, $E$ measures the rotation on the plane $(y^0, y^4)$, $c_i$ measures the rotation on the plane $(y^i, y^4)$, $c'_j$ measures the rotation on the plane $(y^0, y^j)$ and $J_i$ measures the rotation on the plane $(y^i, y^j)$ where $\{i, j, k\}$ is the even permutation of $\{1, 2, 3\}$. But these rotations are all observed from a curved space, the hyperboloid (1.1), so they contain both translation and rotation of an asymptotically anti-de Sitter spacetime. This indicates that we can not simply refer them as the center of mass as well as the total angular momentum. The total effect of translation and rotation is given by the parallelepiped spanned by $c$, $c'$ and $J$ which can be measured from its length of the edges, surface area and the volume.

Denote by $trQ$, $Q^{(2)}$, $Q^{(3)}$ and $\det Q$ the trace, sum of the second-order minors, sum of the third-order minors and the determinant of $Q$. It is straightforward to prove the following proposition.

**Proposition 3.2.**
\[
trQ = 4E_0, \quad Q^{(2)} = 6E_0^2 - 2L^2, \quad Q^{(3)} = 4E_0(E_0^2 - L^2) + 8V^3, \quad \det Q = (E_0^2 - L^2)^2 + 8E_0V^3 - 4A^4,
\]
and they are independent on $t$. Moreover, they are independent on specific Clifford representation also.
The positive energy theorem

Now we prove the positive energy theorem for Henneaux-Teitelboim’s total energy-momentum. Let \((M, g, h)\) be an asymptotically anti-de Sitter initial data set in \((N, \tilde{g})\) which satisfies the dominant energy condition \((2.2)\).

Let \(\nabla\) and \(\tilde{\nabla}\) be the Levi-Civita connections of \(g\) and \(\tilde{g}\) respectively. Let \(S\) be the locally spinor bundle of \(N\) and we still denote by \(S\) its restriction to \(M\).

Since the hypersurface \(M\) is three dimensional, the restriction \(S\) is globally defined on \(M\). And we lift \(\nabla\) and \(\tilde{\nabla}\) to \(S\) and denote the corresponding spin connections the same as \(\nabla\) and \(\tilde{\nabla}\).

Fix a point \(p \in M\) and an orthonormal basis \(\{e_{\alpha}\}\) of \(T_pN\) with \(e_0\) normal and \(\{e_i\}\) tangent to \(M\). Extend \(\{e_{\alpha}\}\) to a local orthonormal frame in a neighborhood of \(p\) in \(M\) such that \((\tilde{\nabla}g)_{ij}e_0 = 0\).

Extend this to a local orthonormal frame \(\{e_{\alpha}\}\) for \(N\) with \((\tilde{\nabla}0 e_j)_{p} = 0\). Then \((\tilde{\nabla}_i e_j)_{p} = h_{ij}e_0, (\tilde{\nabla}_i e_0)_{p} = h_{ij}e_j\). Define

\[
\hat{\nabla}_i = \tilde{\nabla}_i + \frac{1}{2}e_i e_0, \quad \hat{D} = \sum_{i=1}^{3} e_i \hat{\nabla}_i.
\]

Recall that the Weitzenböck formula gives (e.g. \([25]\))

\[
\int_M |\hat{\nabla}\phi|^2 - |\hat{D}\phi|^2 + \langle \phi, \hat{R}\phi \rangle = \int_{\partial M} \langle \phi, \sum_{j \neq i} e_i \cdot e_j \cdot \tilde{\nabla}_j\phi \rangle \ast e^i
\]

where \(\hat{R} = \frac{1}{2}(T_{00} - T_{0i}e_0 e_i)\) and \(\langle \cdot, \cdot \rangle\) is the positive definite inner product on the spinor bundle \(S\) under which \(e_0\) is Hermitian and \(e_i\) is skew-Hermitian.

Now we briefly review some basic facts in \([24, 23, 7, 27, 18, 9]\). Note that \(g = \tilde{g} + a\) with \(a = O(e^{-\tau kr})\), \(\tilde{\nabla}a = O(e^{-\tau kr})\), and \(\nabla \tilde{\nabla}a = O(e^{-\tau kr})\). Orthonormalizing \(\tilde{e}_i\) gives a gauge transformation

\[
A : SO(\tilde{g}) \to SO(g)
\]

\[
\tilde{e}_i \mapsto e_i
\]

(and in addition \(e_0 \mapsto \tilde{e}_0\)) which identifies the corresponding spin group and the spinor bundles. Moreover,

\[
e_i = \tilde{e}_i - \frac{1}{2}a_{ik}\tilde{e}_k + o(e^{-\tau kr}).
\]

We extend the imaginary Killing spinors \(\Phi_0\) \((3.2)\) on the end to the inside smoothly. With respect to the metric \(g\), these imaginary Killing spinors \(\Phi_0\) can be written as \(\Phi_0 = A\Phi_0\).

We try to find the unique solution \(\hat{D}\phi = 0\) such that \(\phi\) is asymptotic to the imaginary Killing spinors \(\Phi_0\) on certain end, and to zero on the other ends. Let \(C_0^\infty(S)\) be the space of smooth sections of the spinor bundle \(S\) with compact support. Let the Hilbert space \(H^1(S)\) be the closure of \(C_0^\infty(S)\) with
Obviously, it along a path from \( x \) set of right hand side gives coordinates, the boundary term of the Weitzenböck formula (4.1) in the and the proof of this lemma is complete. Q.E.D.

The energy condition. Then there exists a unique spinor \( \Phi \) in \( H^1(S) \) such that

\[
\tilde{D}(\Phi_1 + \overline{\Phi}_0) = 0.
\]

**Proof:** The proof is essentially similar to that of Lemma 5.1 in [25]. Since \( B(\cdot, \cdot) \) is coercive on \( H^1(S) \), and \( \tilde{D}\overline{\Phi}_0 \in L^2(S) \), \( \tilde{\nabla}\Phi_0 \in L^2(S) \). By the theorem of Lax-Milgram, there exists a spinor \( \Phi_1 \in H^1(S) \) such that \( \tilde{D}^*\tilde{D}\Phi_1 = \tilde{D}\overline{\Phi}_0 \) weakly. Here \( \tilde{D}^* \) is the formal adjoint operator of \( \tilde{D} \). Let \( \Phi = \Phi_1 + \overline{\Phi}_0 \) and \( \psi = \tilde{D}\phi \). The elliptic regularity tells us that \( \psi \in H^1(S) \), and \( \tilde{D}\phi = 0 \) in the classical sense [5]. The Weitzenböck formula implies that \( \tilde{\nabla}\psi = 0 \). We thus have \( |\tilde{\nabla}\log|\psi|^2| \leq \kappa + |h| \) on the complement of the zero set of \( \psi \) on \( M \). If there exists \( x_0 \in M \) such that \( |\psi(x_0)| \neq 0 \), then integrating it along a path from \( x_0 \in M \) gives

\[
|\psi(x)|^2 \geq |\psi(x_0)|^2 e^{\kappa|h|(|x_0| - |x|)}.
\]

Obviously, \( \psi \) is not in \( L^2(S) \) which gives the contradiction. Hence \( \psi = 0 \), and the proof of this lemma is complete. Q.E.D.

Now let \( \phi \) be the solution of the Dirac-type equation \( \tilde{D}\phi = 0 \) as in Lemma 4.1. Plugging this \( \phi \) into the Weitzenböck formula (4.1), we obtain that the boundary term is nonnegative under the dominant energy condition (2.2). Using the Clifford representation (3.1) and (3.2) for \( \Phi_0 \), in the polar coordinates, the boundary term of the Weitzenböck formula (4.1) in the right hand side gives

\[
RHS_{(4.1)} = \frac{1}{4} \lim_{r \to \infty} \int_{S_r} (\tilde{\nabla}^j g_{ij} - \tilde{\nabla}_1 tr_g(a))(\Phi_0)^2 \omega \\
+ \frac{1}{4} \lim_{r \to \infty} \int_{S_r} \kappa(a_k g_k - g_{kk} tr_g(a))(\Phi_0, \sqrt{-1} \tilde{e}_k \cdot \Phi_0) \omega \\
- \frac{1}{2} \lim_{r \to \infty} \int_{S_r} (h_{kk} - g_{kk} tr_g(h))(\Phi_0, \tilde{e}_0 \cdot \tilde{e}_k \cdot \Phi_0) \omega \\
= 8\pi (\check{\lambda}, Q(\check{\lambda}))_C.
\]

Now we prove our main theorem.
Theorem 4.1. Let \((M,g,h)\) be a 3-dimensional asymptotically anti-de Sitter initial data set in spacetime \((N,\tilde{g})\). Suppose \((N,\tilde{g})\) satisfies the dominant energy condition. Then, for each end

\[ E_0 \geq \sqrt{L^2 - 2V^2 + 2\left(\max\{A^4 - L^2V^2, 0\}\right)} \]

(4.2)

If \(E_0 = 0\) for some end, then \(M\) has only one end, \(Q = 0\), and \((N,\tilde{g})\) is anti-de Sitter along \(M\).

Proof: Let \(\phi\) be the solution of the Dirac-type equation \(\hat{D}\phi = 0\) as in Lemma 4.1. The dominant energy condition (2.2) ensures that \(Q\) is positive semidefinite. Now the trace yields

\[ E_0 \geq 0. \]

The sum of the second-order principal minors yields

\[ E_0^2 \geq L^2/3. \]

Therefore

\[ V^3 \leq V^2L/\sqrt{3} \leq V^2E_0, \]

The sum of the third-order principal minors yields

\[ 0 \leq E_0(E_0^2 - L^2) + 2V^3. \]

So, if \(E_0 > 0\), it implies

\[ E_0^2 \geq L^2 - 2V^2 \geq L^2 - 2L^2/3 = L^2/3. \]

Now we use the nonnegativity of the determinant of \(Q\) to prove (4.2). Since

\[ 2E_0V^3 \leq (E_0^2 + V^2)V^2, \]

we obtain

\[ 0 \leq \det Q \leq (E_0^2 - L^2 + 2V^2)^2 - 4(A^4 - L^2V^2). \]

This implies (4.2).

If \(E_0 = 0\) for some end, then it is straightforward that \(M\) has only one end, and \(Q = 0\). This implies that there exists \(\{\phi_\alpha\}\) which forms a basis of the spinor bundle everywhere over \(M\) such that \(\hat{\nabla}\phi_\alpha = 0\). Standard argument gives

\[ \tilde{R}_{ijkl} = (-\kappa^2)(\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk}), \quad \tilde{R}_{0jkl} = 0 \]

along \(M\). The Einstein field equations (2.1) yield

\[ T_{00} = \tilde{R}_{000} + \frac{1}{2}\tilde{R} - \Lambda = \frac{1}{2} \sum_{i,j} R_{ijij} - \Lambda = 0. \]

Then (2.2) implies \(T_{\alpha\beta} = 0\) and furthermore

\[ \tilde{R}_{0j0l} = \kappa^2\tilde{g}_{jl}. \]

Therefore, the curvature tensors of \((N,\tilde{g})\) are

\[ \tilde{R}_{\alpha\beta\gamma\delta} = (-\kappa^2)(\tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta} - \tilde{g}_{\alpha\delta}\tilde{g}_{\beta\gamma}) \]
and \( N \) is anti-de Sitter along \( M \). Q.E.D.

**Corollary 4.1.** If three vectors \( c, c', J \) or \( \vec{m}, \vec{c}, \vec{j} \) are linearly dependent, i.e., \( V = 0 \), then the energy-momentum inequality \( (4.2) \) becomes

\[
E_0 \geq \sqrt{L^2 + 2A^2}.
\]

This corollary generalizes the energy-momentum inequality \( (1.6) \). It also indicates that \( \vec{m}, \vec{c} \) and \( \vec{j} \) play the same role in physics.

**Remark 4.1.** In the above energy-momentum endomorphism \( Q \), nonnegativity of the second-order minor \( K \) gives \( E_0 \geq |c| \). However, this inequality does depend on the Clifford representation. For instance, if we permute \( \vec{e}_1 \rightarrow \vec{e}_2, \vec{e}_2 \rightarrow \vec{e}_3, \vec{e}_3 \rightarrow \vec{e}_1 \) in Clifford representation \( (3.7) \), the energy-momentum endomorphism \( Q \) will change to the new one with

\[
P = \begin{pmatrix}
E_0 + c_3 - c'_1 - J_2 & -c'_2 + J_1 \\
-c'_2 + J_1 & E_0 + c_3 + c'_1 + J_2
\end{pmatrix},
\]

\[
W = \begin{pmatrix}
c_1 + c'_3 & c_2 + J_3 \\
c_2 - J_3 & -c_1 + c'_3
\end{pmatrix},
\]

\[
\hat{P} = \begin{pmatrix}
E_0 - c_3 + c'_1 - J_2 & c'_2 + J_1 \\
c'_2 + J_1 & E_0 - c_3 - c'_1 + J_2
\end{pmatrix}.
\]

The inequality \( E_0 \geq |c| \) does not hold in the new energy-momentum endomorphism.

If \( V > 0 \), \( E_0 \) is very close to \( V \) and \( |c|, |\vec{j}| \) are sufficiently small, the universal inequality \( L^2 \geq 3V^2 \) will give \( |\vec{m}| > m_0 \).

**Remark 4.2.** When three vectors \( \vec{m}, \vec{c}, \vec{j} \) are linearly independent, that \( Q \) is positive semidefinite, in general, does not result that the total energy four vector \( m(\mu) \) is timelike.

**Remark 4.3.** One can construct certain regular initial data sets which are Kerr-anti-de Sitter at infinity with

\[
E_0 = 1, \quad c = (0,0,0), \quad c' = (0,0,0), \quad J = (0,0,\kappa a)
\]

where \( 1 > \kappa |a| \). Thus \( (4.2) \) is optimal in this sense.

If \( M \) has a future/past trapped surface \( (\Sigma, \bar{g}, \bar{h}) \) equipped with the induced metric \( \bar{g} \) and the second fundamental form \( \bar{h} \)

\[
\text{tr}_{\bar{g}}(\bar{h}) \mp \text{tr}_{\bar{g}}(\bar{h}|\Sigma) \geq 0.
\]

Let \( e_3 \) be outward normal and \( e_A \) be tangent to \( \Sigma \). The boundary term involving \( \Sigma \) in the Weitzenböck formula is

\[
\int_{\Sigma} \langle \phi, e_3 e_A \bar{\nabla}_A \phi \rangle = \int_{\Sigma} \langle \phi, e_3 e_A \bar{\nabla}_A \phi \rangle - \int_{\Sigma} \langle \phi, \sqrt{-1} \kappa e_3 \phi \rangle.
\]

**Remark 4.4.** Theorem \(4.1\) also holds for black holes. This is because that, under the local boundary conditions, the term \( \langle \phi, e_3 \phi \rangle \) is both imaginary and real, hence zero. Then it follows by the standard argument \( [13] \).
5. Geometric invariant

We shall show that the determinant of the total energy-momentum endomorphism $Q$ is the geometric invariant which is independent on the choice of admissible asymptotic coordinates.

We omit the upper-case $HT$ and denote $J_{ab}$ ($0 \leq a, b \leq 4$) as Henneaux-Teitelboim’s total energy-momentum in this section. It yields two $O(3, 2)$ Casimir invariants [15]

$$I_1 = \frac{1}{2} J_{ab} J^{ab} = -\frac{1}{2} J_a b J^b a,$$

$$I_2 = \frac{1}{2} J_a b J^c J^d J^a J^b - \frac{1}{4} (J_{a b} J^{a b})^2.$$

**Theorem 5.1.** Denote $\det Q$ as the determinant of the energy-momentum endomorphism $Q$. We have

$$\det Q = \left( \frac{\kappa}{16\pi} \right)^4 (I_1^2 + 2I_2). \quad (5.1)$$

**Proof:** It is straightforward that

$$\det Q = (E_0^2 - L^2)^2 + 8E_0 V^3 - 4A^4$$

$$= E_0^4 + \sum_i [c_i^4 + (c_i')^4 + J_i^4]$$

$$+ \sum_{i \neq j} [c_i^2 c_j^2 + (c_i')^2 (c_j')^2 + J_i^2 J_j^2 - 2c_i^2 J_j^2 - 2(c_i')^2 J_j^2]$$

$$- 2E_0^2 \sum_i [c_i^2 + (c_i')^2] + 2 \sum_i c_i^2 (c_i')^2$$

$$- 2 \sum_{i \neq j} c_i^2 (c_j')^2 - 2E_0^2 \sum_i J_i^2 + 2 \sum_i [c_i^2 J_i^2 + (c_i')^2 J_i^2]$$

$$+ 8E_0 \varepsilon_{ijk} c_i c_j J_k + 4 \sum_{i \neq j} (c_i c_j c_i' c_j' + c_i c_j J_i J_j + c_i' c_j' J_i J_j).$$

By (2.4), we obtain that, in the right hand side of above equality, the sum of the first and the second terms is equal to $I$, the sum of the third, the forth and the fifth terms is equal to $II$, the sum of the sixth, the seventh and the eighth terms is equal to $III$, and the sum of the ninth and the tenth terms is equal to $IV$. Thus

$$\left( \frac{16\pi}{\kappa} \right)^4 \det Q = I + II + III + IV,$$
where
\begin{align*}
I &= \sum_{a < b} J^4_{ab}, \\
II &= \sum_{a, b, c \text{ distinct}} J^b_a J^a_a J^c_c J^a_a, \\
III &= -\frac{1}{4} \sum_{a, b, c, d \text{ distinct}} J^b_a J^a_a J^d_c J^c_d, \\
IV &= \sum_{a, b, c, d \text{ distinct}} J^b_a J^c_c J^d_d J^a_a.
\end{align*}

On the other hand,
\begin{align*}
J^b_a J^c_c J^d_d J^a_a &= 2I + 2II + IV, \\
\frac{1}{4} J^b_a J^a_a J^c_c J^d_d &= I + II - III.
\end{align*}

Therefore we obtain (5.1). Q.E.D.

Proposition 5.1. The admissible coordinate transformations on ends will preserve Henneaux-Teitelboim’s total energy-momentum.

Proof: The proof is essentially the same as that of Theorem 2.3 [10], where it is used that \( X = U_{\alpha \beta} \) is a Killing vector. So the proof goes through no matter that \( t \) is zero or not. Q.E.D.

Theorem 5.2. The determinant \( \det Q \) of the energy-momentum endomorphism \( Q \) is invariant under admissible coordinate transformation (5.2) on ends. It serves as the geometric invariant of asymptotically anti-de Sitter spacetimes.

Remark 5.1. We may define \( \sqrt{\det Q} \) as the total rest mass of asymptotically anti-de Sitter spacetimes.

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6. Appendix A: Ten Killing vectors for AdS spacetime

The followings are ten Killing vectors $U_{\alpha\beta}$ generating rotations for $\mathbb{R}^{3,2}$ along $t$-slices.

$$U_{10} = \frac{\cos(\kappa t)}{\kappa} \left[ \sin \theta \cos \psi \frac{\partial}{\partial r} + \kappa \coth(\kappa r) \left( \cos \theta \cos \psi \frac{\partial}{\partial \theta} - \frac{\sin \psi}{\sin \theta} \frac{\partial}{\partial \psi} \right) \right]$$

$$- \frac{\sin(\kappa t)}{\kappa} \tanh(\kappa r) \sin \theta \cos \psi \frac{\partial}{\partial t},$$

$$U_{20} = \frac{\cos(\kappa t)}{\kappa} \left[ \sin \theta \sin \psi \frac{\partial}{\partial r} + \kappa \coth(\kappa r) \left( \cos \theta \sin \psi \frac{\partial}{\partial \theta} + \cos \psi \frac{\partial}{\partial \psi} \right) \right]$$

$$- \frac{\sin(\kappa t)}{\kappa} \tanh(\kappa r) \sin \theta \sin \psi \frac{\partial}{\partial t},$$

$$U_{30} = \frac{\cos(\kappa t)}{\kappa} \left[ \cos \theta \frac{\partial}{\partial r} - \kappa \coth(\kappa r) \sin \theta \frac{\partial}{\partial \theta} \right] - \frac{\sin(\kappa t)}{\kappa} \tanh(\kappa r) \cos \theta \frac{\partial}{\partial t},$$

$$U_{40} = \frac{1}{\kappa} \frac{\partial}{\partial t},$$

$$U_{14} = \frac{\sin(\kappa t)}{\kappa} \left[ \sin \theta \cos \psi \frac{\partial}{\partial r} + \kappa \coth(\kappa r) \left( \cos \theta \cos \psi \frac{\partial}{\partial \theta} - \frac{\sin \psi}{\sin \theta} \frac{\partial}{\partial \psi} \right) \right]$$

$$+ \frac{\cos(\kappa t)}{\kappa} \tanh(\kappa r) \sin \theta \cos \psi \frac{\partial}{\partial t},$$

$$U_{24} = \frac{\sin(\kappa t)}{\kappa} \left[ \sin \theta \sin \psi \frac{\partial}{\partial r} + \kappa \coth(\kappa r) \left( \cos \theta \sin \psi \frac{\partial}{\partial \theta} + \cos \psi \frac{\partial}{\partial \psi} \right) \right]$$

$$+ \frac{\cos(\kappa t)}{\kappa} \tanh(\kappa r) \sin \theta \sin \psi \frac{\partial}{\partial t},$$

$$U_{34} = \frac{\sin(\kappa t)}{\kappa} \left[ \cos \theta \frac{\partial}{\partial r} - \kappa \coth(\kappa r) \sin \theta \frac{\partial}{\partial \theta} \right] + \frac{\cos(\kappa t)}{\kappa} \tanh(\kappa r) \cos \theta \frac{\partial}{\partial t},$$

$$U_{12} = \frac{\partial}{\partial \psi},$$

$$U_{23} = - \sin \psi \frac{\partial}{\partial \theta} - \frac{\cos \theta \cos \psi}{\sin \theta} \frac{\partial}{\partial \psi},$$

$$U_{31} = \cos \psi \frac{\partial}{\partial \theta} - \frac{\cos \theta \sin \psi}{\sin \theta} \frac{\partial}{\partial \psi}.$$
7. Appendix B: Center of AdS mass coordinates

We shall explicitly construct a $SO(3,1)$ coordinate transformation on $t = 0$ slice to change arbitrary coordinates $\{y^\alpha\}$ to the center of AdS mass coordinates if the mass vector is timelike. Denote the $SO(3,1)$ matrix $B = (B_\alpha^\beta)$. When $t = 0$, the admissible coordinate transformations reduce to

$$z^\alpha = B_\alpha^\beta y^\beta, \quad z^4 = y^4.$$ 

Denote the lower-bar terms by the corresponding quantities in new coordinates. Since $z_\alpha = B_\beta^\alpha y_\beta$, $B_\alpha^\beta = \eta^{\alpha\gamma} B_\gamma^\delta \eta_{\delta\beta}$ for the flat metric $\eta$ on $\mathbb{R}^{3,2}$, we have

$$U_4^\alpha = z_4 \frac{\partial}{\partial z^\alpha} - z_\alpha \frac{\partial}{\partial z^4} = B_\alpha^4 U_4^\beta,$$

$$U_0^\alpha = z_0 \frac{\partial}{\partial z^\alpha} - z_\alpha \frac{\partial}{\partial z^0} = B_0^\alpha B_i^\beta U_0^\beta,$$

$$U_{ij}^\alpha = z_i \frac{\partial}{\partial z^j} - z_j \frac{\partial}{\partial z^i} = B_{ij}^\alpha B_i^\beta U_j^\beta.$$ 

Consequently, the mass vector $m^{(\mu)}$, the center of mass $c^{(i)}$, and the angular momentum $J_{(i)(j)}$ defined in [9] have the following transformation laws

$$m_\alpha^{(\beta)} = B_\alpha^\beta m^{(\beta)},$$

$$c^{(i)} = (B_i^j B_0^0 - B_i^0 B_0^j) c^{(j)} + B_i^j B_0^k J_{(j)(k)},$$

$$J_{(i)(j)} = (B_i^k B_0^0 - B_i^0 B_0^k) c^{(k)} + B_i^k B_j^l J_{(k)(l)}.$$ 

By (7.1), we find that the following $SO(3,1)$ matrix $B_1$ changing the vector $(m^{(0)}, m^{(1)}, m^{(2)}, m^{(3)})$ to $(\|m_\mu\|, 0, 0, 0)$ if it is timelike,

$$B_1 = \begin{pmatrix}
\frac{m^{(0)}}{\|m_\mu\|} & -\frac{m^{(1)}}{m^{(0)} m^{(1)}} & -\frac{m^{(2)}}{m^{(0)} m^{(2)}} & -\frac{m^{(3)}}{m^{(0)} m^{(3)}} \\
0 & \sqrt{\frac{m^{(1)}_2 + m^{(2)}_2}{m^{(1)}_1 m^{(2)}_2}} & \sqrt{\frac{m^{(1)}_1 + m^{(2)}_1}{m^{(1)}_2 m^{(2)}_1}} & 0 \\
0 & -\frac{m^{(1)} m^{(3)}}{\sqrt{m^{(1)}_1 m^{(2)}_1}} & -\frac{m^{(2)} m^{(3)}}{\sqrt{m^{(1)}_2 m^{(2)}_2}} & -\frac{m^{(3)} m^{(3)}}{\sqrt{m^{(1)}_3 m^{(2)}_3}}
\end{pmatrix}.$$
where \( |m_i| = \sqrt{\sum_{i=1}^{3} m_{(i)}^2} \), \( \|m_\mu\| = \sqrt{m_{(0)}^2 - |m_i|^2} \). And \( B_1 = C_1 C_2 C_3 \),

\[
C_1 = \begin{pmatrix}
\frac{m_{(0)}}{|m_\mu|} & -\frac{|m_i|}{|m_\mu|} & 0 & 0 \\
-\frac{|m_i|}{|m_\mu|} & \frac{m_{(0)}}{|m_\mu|} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
C_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sqrt{m_{(1)}^2 + m_{(2)}^2} & 0 & \frac{m_{(3)}}{|m_i|} \\
0 & 0 & 1 & 0 \\
0 & -\frac{m_{(3)}}{|m_i|} & 0 & \sqrt{m_{(1)}^2 + m_{(2)}^2}
\end{pmatrix},
\]

\[
C_3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{m_{(1)}}{\sqrt{m_{(1)}^2 + m_{(2)}^2}} & 0 & \frac{m_{(2)}}{|m_\mu|} \\
0 & 0 & \frac{m_{(1)}}{\sqrt{m_{(1)}^2 + m_{(2)}^2}} & \frac{m_{(2)}}{|m_\mu|} \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Under this transformation, \( c_{(i)} \) and \( J_{(i)(j)} \) will also be changed under \( B_1 \). Denote by \( c_{(i)}^{(1)} \) and \( J_{(i)(j)}^{(1)} \) the respective new quantities. The following \( SO(3,1) \) matrix \( B_2 \) changes both \( J_{(1)(3)}^{(1)} \) and \( J_{(2)(3)}^{(1)} \) to zero, and \( J_{(1)(2)}^{(1)} \) to

\[
|J^{(1)}| = \sqrt{(J^{(1)}_{(1)(2)})^2 + (J^{(1)}_{(1)(3)})^2 + (J^{(1)}_{(2)(3)})^2}
\]

which is denoted by \( J^{(2)}_{(1)(2)} \),

\[
B_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & J_{(1)(2)}^{(1)} & 0 & 0 \\
0 & J_{(1)(3)}^{(1)} & J_{(1)(2)}^{(1)} & J_{(1)(3)}^{(1)} \\
0 & J_{(2)(3)}^{(1)} & J_{(2)(3)}^{(1)} & J_{(2)(3)}^{(1)}
\end{pmatrix},
\]

where \( |J_{(13)}^{(1)}| = \sqrt{|J^{(1)}|^2 - (J_{(1)(3)}^{(1)})^2} \). Also, \( c_{(i)}^{(1)} \) will be changed, and we denote the corresponding new quantities by \( c_{(i)}^{(2)} \). Now the following \( SO(3,1) \) matrix \( B_3 \) changes \( c_{(2)}^{(2)} \) to zero and preserves \( J_{(1)(2)}^{(1)} \),

\[
B_3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sqrt{c_{(1)}^{(2)}^2 + c_{(2)}^{(2)}^2} & 0 & 0 \\
0 & 0 & \sqrt{c_{(1)}^{(2)}^2 + c_{(2)}^{(2)}^2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
Again $c^{(2)}_{(i)}$ and $J^{(2)}_{(i)(j)}$ will change to the new quantities which are denoted by $c^{(3)}_{(i)}$ and $J^{(3)}_{(i)(j)}$.

Thus the transformation $B = B_3 B_2 B_1$ changes the coordinates $\{y^\alpha\}$ to the center of AdS mass coordinates such that (1.7) holds.

8. Appendix C: Roots of $\text{det} \, Q$

We compute explicitly four formal roots of the determinant $\text{det} \, Q$ of the energy-momentum endomorphism $Q$ given by (3.4). The equation $\text{det} \, Q = 0$ is a quartic equation with the variable $E_0$. Denote

$$\xi_1 = \sqrt{27V_{12}^2 + 4L^6V^6 - 18A^4L^2V^6 - A^8L^4 + 4A^{12}},$$

$$\xi_2 = \sqrt{2L^6 - 9A^4L^2 + 27V^6 + 3\sqrt{3}\xi_1},$$

$$\eta_1 = \sqrt{\frac{4L^2}{3} + \frac{2}{3}2^{2/3}\xi_2 + \frac{4\sqrt{2} (L^4 - 3A^4)}{3\xi_2}},$$

$$\eta_2 = \sqrt{-\frac{16V^3}{\eta_1} + \frac{8L^2}{3} - \frac{2}{3}2^{2/3}\xi_2 - \frac{4\sqrt{2}(L^4 - 3A^4)}{3\xi_2}}.$$

If all of these are well-defined, then $\text{det} \, Q$ has four roots

$$\frac{1}{2} (\pm \eta_1 \pm \eta_2).$$

In this case that $\text{det} \, Q \geq 0$ gives

$$E_0^2 \geq \frac{1}{4} (\eta_1 + \eta_2)^2,$$

or

$$E_0^2 \leq \frac{1}{4} (\eta_1 - \eta_2)^2.$$
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