Quasi-modular instanton partition function
and elliptic solution of KdV equations

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Abstract

Four dimensional N=2 supersymmetric gauge theories are related to some solvable quantum mechanics models. For SU(2) theory with an adjoint matter, or with 4 fundamental matters, if the mass of matter takes special value then the potential of quantum model is the elliptic solution of KdV equations. We show that the prepotential of gauge theory can be obtained from the average densities of conserved charges of classical KdV solution, the UV gauge coupling dependence is assembled into Eisenstein series. The Eisenstein series come from integration of elliptic functions in KdV Hamiltonians. The gauge theory with adjoint mass is taken as the example.

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1 Introduction

It is well-known that the Korteweg-de Vries equation (KdV) is an integrable Hamiltonian system of infinite dimensional \([2]\). The rapidly decreasing solutions are obtained by the inverse scattering method \([1]\), where the scattering data of the Schrödinger operator with the initial value KdV function \(u(x, t = 0)\) as the potential can be used to reconstruct the exact potential. In the case of reflectionless scattering, the rapidly decreasing soliton solutions are obtained.

The analogous problem of solving KdV equations with periodic initial condition leads to the discovery of relations to some other topics, including the elliptic functions, algebraic geometric methods and finite gap spectrum problem \([3, 4]\). The inverse problem for the periodic potential is related to the spectral problem of Schrödinger operator with Dirichlet boundary condition. The periodic potentials are sometimes called periodic KdV soliton, albeit many aspects of the solution do not parallel with the fast decaying solitons.

The KdV system is a classical Hamiltonian system, a question we can ask is how to quantize the KdV system, and especially how to quantize the soliton solution? For the first half of the question, it was noticed that the Poisson bracket of the KdV hierarchy is the large central charge limit of the Virasoro algebra \([5]\), therefore the conformal field theory (CFT) provides a well defined framework for the quantization of KdV system where the inverse of central charge is the quantum parameter. In fact, the quantum Hamiltonians of some soliton equations, including the closely related sine-Gordon, KdV, and mKdV equations, can be constructed from CFT energy momentum tensor and its derivatives, they are in involution with respect to the Virasoro algebra, and quantum soliton equations can be formally defined in this context \([6, 7, 8]\). These discoveries finally cumulate to a series of papers studying the integrable structure of CFT, starting from \([8]\). The second half of the question is also well motivated. The quantum behavior of soliton shows some remarkable properties such as particle-soliton duality, which are absent at the classical level. In 1+1 dimension we have the example of the sine-Gordon/Thirring model duality \([9]\), in higher dimension we have the Montonen-Olive duality for gauge theories \([10]\).

In this paper we show that some periodic solutions of KdV are related to some particular Liouville CFT conformal blocks, and also related to instanton action \([11]\) of certain four dimensional super Yang-Mills theories \([12, 13]\). These connections are natural if viewed from the perspective of KdV-CFT relation and the recently discovered Alday-Gaiotto-Tachikawa (AGT) correspondence \([14]\) on CFT and instanton partition function. The second order linear differential equation plays a key role in the story, it appears in CFT as the null decoupling equation, in classical KdV theory as the Lax operator. Especially we have a well
defined framework, both in CFT and in gauge theory, to compute quantities of the elliptic KdV solution beyond classical level, and it is reasonable to interpret the subleading terms as quantum effects of the solution.

We restrict to very special elliptic solutions, the Lamé potential which is used to explain the connection in detail, and the Treibich-Verdier potential which can be studied in the same way. At the moment it is not clear to us if the connections can be generalized to other KdV solutions, this is remained for future work.

Main Results

The results of the paper can be divided into two parts. The first part is about the classical elliptic KdV solution and the Seiberg-Witten solution of SU(2) $N = 2^*$ theory. From the conserved charges of the classical KdV solution, in formula (10), where the Eisenstein series already appear, we get the spectrum of the linear problem, given in (15). The spectrum is related to the instanton partition function in the Nekrasov-Shatashvili (NS) limit [15], therefore we obtain the prepotential with all $q$ dependence contained in polynomials of Eisenstein series, in formula (17). The second part continues the line of argument to study quantum aspect of elliptic KdV solution. We use the gauge theory results for generic parameter $\epsilon_1, \epsilon_2$ to obtain a generalized linear spectrum given in (27), then reverse the relation we get KdV charges in (29). The leading order of them are classical charges, and they contain $\hbar$ corrections, therefore interpreted as the quantum charges of elliptic KdV solution. This finally leads to the large $a$-expansion of prepotential whose coefficients are quasi-modular forms, presented in (30).

The organization of the paper is as follows. In Section 2, we compute the average densities of conserved charges for the Lamé elliptic potential, from which we obtain the quasi-modular form of prepotential of $N = 2^*$ SYM in the Omega background in NS limit. In Section 3 we briefly explain the connection of conformal field theory and quantization of KdV system. In Section 4, based on the AGT connection we use CFT/gauge theory technique to compute the spectrum of elliptic solution which is naturally interpreted as quantum spectrum of the solution. Section 5 is devoted to some open questions.

2 Classical spectrum of elliptic solution of KdV equations

In this section we present some results from two subjects seemingly unrelated to each other, one from the classical theory of periodic solution of the KdV equations, the other from the effective action of certain N=2 gauge theory. They contain the same information if we correctly identify parameters from both sides. The reason for this connection is the AGT
correspondence of gauge theory and CFT presented in [14], and the relation of CFT and quantum KdV(qKdV) developed in a series of papers including [6, 7, 8]. The results present in this section is a limit case of the qKdV-CFT-gauge theory connection, it is the classical limit of qKdV, which means the large central charge limit of CFT, or the NS limit of gauge theory. We discuss the full quantum theory in next two sections where it appears that similar to the classical solution, the quantum solution also displays quasi-modular structure in the spectrum.

2.1 Densities of conserved charges of elliptic solutions

The linear operator $L = \partial^2 - u(x, t)$ plays a central role in the theory of KdV hierarchy. In the Lax’s operator formalism of KdV hierarchy, a tower of differential operators with increasing order would give the higher order KdV equations. These operators can be obtained from the formal computation of pseudo-differential operator system of Gelfand and Dickey, the equations appear as $\partial^n u = [L^{(2n-1)/2}, L]$, with $n = 1, 2, 3, \cdots$. A basic fact is that $L$ is isospectral for the KdV solution $u$, therefore it is enough to consider the spectrum of $L = \partial^2 - u(x)$ with $u(x) = u(x, 0)$.

There is a nice way to see the Hamiltonian structure, starting from the linear system of $L$,

$$(\partial^2 - u)\Psi(x) = \lambda \Psi(x).$$

Let $\Psi(x) = \exp(\int^x v(y)dy)$, therefore we introduce the Miura transformation,

$$v' + v^2 = u + \lambda.$$

In accordance with literature, we use $\partial$ or $'$ to denote $\partial_x$. Suppose the spectral parameter $\lambda \gg 1$, perform the expansion for $v(x)$,

$$v = \sqrt{\lambda} + \sum_{k=1}^\infty \frac{v_k}{(\sqrt{\lambda})^k},$$

then we obtain all $v_k$ as functionals of $u$ and its derivatives, $v_k = v_k(u, u', u'', \cdots)$, they can be determined recursively. As $v_{2k}$ are total derivatives, the nontrivial KdV dynamics are driven by $v_{2k-1}$. The KdV Hamiltonians are defined as integration of the densities $v_{2k-1}$,

$$H_k = \int dx v_{2k-1}.$$

The first few of them are

$$H_1 = \frac{1}{2} \int dx u, \quad H_2 = -\frac{1}{8} \int dx (u^2 - u''),$$

$$H_3 = \frac{1}{32} \int dx (2u^2 + u' + (u''' - 6uu')'), \quad \cdots,$$
they are in involution with respect to the Poisson structure of KdV. Restore the time dependence, the equations are $\partial_t u = \{H_n, u\}$. The Miura transformation is related to the bi-Hamiltonian structure of KdV hierarchy.

Now let us turn to the periodic solutions of KdV equations, the initial profile $u(x, 0)$ is given by the solution of generalised $n$-th stationary equation $\{\sum_{k=1}^n c_k H_k, u\} = 0$, where $c_k$ are coefficients. We are interested in a class of periodic solutions given by the Weierstrass elliptic function, in the associated linear spectral problem they are called finite gap potentials. One periodic solution is the elliptic Lamé potential,

$$u(x) = n(n-1)\wp(x; \omega_1, \omega_2), \quad n \in \mathbb{Z}^+,$$

it solves the $(n-1)$-th stationary KdV equation, and $n - 1$ would be the number of gaps for the real spectrum of $L$, the arithmetic genus of the associated surface. The nome for the elliptic function is $q = \exp(i2\pi \omega_1^2)$.

For periodic solutions the Hamiltonians $H_k$ are not finite quantities, instead let us compute the average density of Hamiltonians, they can be obtained from the average of $v(x)$ along a period,

$$i\nu = \frac{1}{T} \int_y^{y+T} dx v(x).$$

$\nu$ is called the Floquet exponent in the theory of differential equation with periodic coefficients. For large $\lambda$ expansion the associated period is $T = 2\omega_1$, then the non-vanishing integrals give the energy densities $\varepsilon_k$ by $H_k = 2\omega_1\varepsilon_k$.

Using the basic relations of the $\wp(x; \omega_1, \omega_2)$ function, the integration $\int_y^{y+2\omega_1} dx v_{2k-1}(x)$ is straightforward and the expansion of Floquet exponent would be

$$i\nu = \sqrt{\lambda} + \sum_{k=1}^{\infty} \frac{\varepsilon_k(n, g_{2,3}, \zeta_1)}{(\sqrt{\lambda})^{2k-1}},$$

where $\zeta_{1,2} = \zeta(\omega_{1,2})$ come form the Weierstrass zeta function $\zeta(x) = \int dx \wp(x)$ and its periodic shift $\zeta(x + 2\omega_{1,2}) = \zeta(x) + 2\zeta_{1,2}$. The average density $\varepsilon_k$ can be written in terms of Eisenstein series using

$$\frac{\zeta_1}{\omega_1} = \frac{\pi^2}{3} E_2(q), \quad g_2 = \frac{4\pi^4}{3} E_4(q), \quad g_3 = \frac{8\pi^6}{27} E_6(q).$$

Under the SL($2, \mathbb{Z}$) transformation $E_4, E_6$ are modular forms of weight 4 and 6, respectively,
while \(E_2\) is a quasi-modular form. The first few \(\varepsilon_k\) are

\[
\begin{align*}
\varepsilon_1 &= -\frac{\pi^2}{6}n(n-1)E_2, \\
\varepsilon_2 &= -\frac{\pi^4}{72}n^2(n-1)^2E_4, \\
\varepsilon_3 &= -\frac{\pi^6}{2160}n^3(n-1)^3(9E_2E_4 - 4E_6) + \frac{\pi^6}{180}n^2(n-1)^2(E_2E_4 - E_6), \\
\varepsilon_4 &= -\frac{5\pi^8}{72576}n^4(n-1)^4(15E_2^2 - 8E_2E_6) + \frac{5\pi^8}{1512}n^3(n-1)^3(E_4^2 - E_2E_6) \\
&\quad - \frac{\pi^8}{252}n^2(n-1)^2(E_4^2 - E_2E_6), \\
&\quad \cdots
\end{align*}
\]

(10)

In the next subsection we will show that the spectrum of classical elliptic solution contains the same information of instanton effects in \(SU(2) N = 2^*\) super-Yang-Mills theory.

### 2.2 Instanton partition function in the NS limit

In our previous paper we notice some finite gap potentials appear in the differential equations of null decoupling condition of Liouville CFT, through the AGT correspondence they are then related to instanton partition function of some \(N=2\) supersymmetric gauge theories. As we proceed, we believe more precise relation between the KdV elliptic solution and quantum gauge theory can be developed.

According the the proposal in [15], \(N=2\) gauge theories in the limit \(\epsilon_1 = \epsilon, \epsilon_2 = 0\) are related to quantum integrable models. In our story, the gauge theory is the mass deformed \(SU(2) N=4\) SYM, i.e. \(N = 2^*\) theory, it is related to the quantum mechanics model described by Schrödinger equation with the Lamé potential. The mass of adjoint matter takes special value

\[
\frac{m}{\epsilon_1} = n \in \mathbb{Z}.
\]

(11)

Recall that the spectral data \(\lambda, \nu\) are related to gauge theory quantities, the moduli \(\hat{u}\) and the v.e.v. of scalar field \(a\), by

\[
\lambda = -\frac{8\pi^2\hat{u}}{\epsilon_1^2}, \quad \nu = \frac{2\pi a}{\epsilon_1},
\]

(12)

and \(\hat{u} = \hat{u}(a, m, q, \epsilon_1)\) is related to the prepotential of gauge theory in the NS limit \(\mathcal{F}(a, m, q, \epsilon_1)\) by

\[
\hat{u} = \frac{1}{2}q \frac{\partial}{\partial q} \mathcal{F}(\epsilon_1) + \frac{m(m-\epsilon_1)}{24}(1 - 2E_2).
\]

(13)

\footnote{Compare to [16] here we have recovered the \(\pi\) factors.}
Here the same $q$ is the complex UV coupling of gauge theory, the instanton expansion parameter.

Therefore to relate spectrum of the KdV solution to the instanton partition function, we need to reverse the relation (8), and get the large $\nu$-expansion for the spectral parameter,

$$\lambda = -\nu^2 + \sum_{k=0}^{\infty} \frac{\lambda_k}{\nu^{2k}},$$

where

$$\lambda_0 = \frac{\pi^2}{3} n(n - 1) E_2,$$

$$\lambda_1 = \frac{\pi^4}{36} n^2 (n - 1)^2 (E_2^2 - E_4),$$

$$\lambda_2 = \frac{\pi^6}{540} n^3 (n - 1)^3 (5 E_2^3 - 3 E_2 E_4 - 2 E_6) - \frac{\pi^6}{90} n^2 (n - 1)^2 (E_2 E_4 - E_6),$$

$$\lambda_3 = \frac{\pi^8}{9072} n^4 (n - 1)^4 (35 E_2^4 - 7 E_2 E_4 - 10 E_4^2 - 18 E_2 E_6) - \frac{\pi^8}{756} n^3 (n - 1)^3 (7 E_2^2 E_4 - 5 E_4^2 - 2 E_2 E_6) + \frac{\pi^8}{126} n^2 (n - 1)^2 (E_2 E_6 - E_4^2)\ldots$$

This is the asymptotic spectrum of the linear Schrödinger operator $L$ [16], but now with the $q$-dependence in terms of quasi-modular functions.

Using the differential relation about the (quasi)modular functions,

$$q \partial_q \ln \eta(q) = \frac{1}{24} E_2, \quad q \partial_q E_2 = \frac{1}{12} (E_2^2 - E_4),$$

$$q \partial_q E_4 = \frac{1}{3} (E_2 E_4 - E_6), \quad q \partial_q E_6 = \frac{1}{2} (E_2 E_6 - E_4^2),$$

where $\eta(q)$ is the Dedekind eta function, we can integrate [13] to get the large $a$-expansion of prepotential with quasi-modular coefficients,

$$F_{\text{mod}}(\epsilon_1) = \left( a^2 - \frac{m(m - \epsilon_1)}{12} \right) \ln q + 2m(m - \epsilon_1) \ln \eta(q) - \frac{m^2(m - \epsilon_1)^2}{48a^2} E_2$$

$$- \frac{1}{5760a^4} [m^3(m - \epsilon_1)^3(5 E_2^2 + E_4) - \epsilon_1^2 m^2(m - \epsilon_1)^2 E_4] + \cdots$$

This is not exactly the prepotential of gauge theory, because the integration of the last three relations in [16] about modular forms would produce some $q$-independent constant terms. The constant part is just the $q = 0$ value of terms involving $E_{2k}$ in $F_{\text{mod}}(\epsilon_1)$, that is

$$F_{\text{con}}(\epsilon_1) = - \frac{m^2(m - \epsilon_1)^2}{48a^2} E_2 - \frac{1}{5760a^4} [m^3(m - \epsilon_1)^3(5 E_2^2 + E_4) - \epsilon_1^2 m^2(m - \epsilon_1)^2 E_4] + \cdots |_{q=0}$$
Expand the Eisenstein series $E_{2k}(q)$ we can compare $F_{\text{mod}} - F_{\text{con}}$ with the prepotential from the instanton partition function, they are of course the same. Take limit $\epsilon_1 = 0$, we get the solution of Seiberg-Witten [13].

3 Quantum KdV hierarchy and CFT

The conformal field theory provides a reliable framework to discuss the quantization of KdV system, because one of the Poisson brackets, given by the Magri-Virasoro bracket, is just the large central charge limit of Virasoro commutation relation [5]. The quantum actions are defined through the normal ordered product of CFT energy-momentum operator and its derivatives [6, 7, 8].

$$I_1 = \int dz T(z), \quad I_2 = \int dz : T^2(z) :, \quad I_3 = \int dz : T^3(z) : + \frac{c + 2}{12} : (T'(z))^2 :, \quad \cdots \quad (19)$$

they commutate with respect to the Virasoro algebra, and the quantum KdV equations can be formally defined. In the classical limit, $c \to \infty$,

$$T(z) \to \frac{c}{6} u(z), \quad [*,*] \to \frac{6i\pi}{c} \{*,*\}, \quad (20)$$

we recover classical KdV theory.

Many ingredients of the classical theory have a corresponding quantum version. In classical theory the Muira transformation [2] maps two Poisson structures satisfied by $u(x)$ and $v(x)$ respectively to each other, in the quantum version it is the Feigin-Fuchs free field realization of CFT. The Schrödinger equation of the classical theory is replaced by the null operator condition of CFT, which corresponds to the surface operator of gauge theory. For the periodic solutions we have the Floquet exponent, i.e. the monodromy, for the wave function $\Psi(x, t)$, in the quantum version it is the monodromy of degenerate operator inserted in the conformal block on punctured surface, which corresponds to the Wilson loop of gauge theory [17, 18].

This construction clarifies the connection of classical and quantum symmetry algebra, but it has not yet addressed some other points of KdV theory. The soliton solution of classical KdV equations is an important aspect, study the quantum version of KdV solitons may provide further insight about them. Generally, a CFT state in the Verma module, measured by its conformal weight, would be a solution to the quantum KdV system. In order to survive in the large $c$ limit, the conformal weight should be of order $O(c)$. In the full quantum theory, the classical profile may be dressed up with light excitations created by
$L_{-n}$ with conformal weight of order $O(1)$. At the moment it is not clear in general what kind of CFT states, in the classical limit, would give various classical solutions of KdV equations.

In the discussion of the relation between N=2 supersymmetric gauge theory and Liouville CFT we notice the null decoupling equations for sphere and torus conformal block in the classical limit take the form of the linear spectral equation (11) with periodic elliptic potential. If the conformal weight of the vertex operators take special value, the potentials would be solution of the classical KdV equations [25]. This leads us to suspect the full quantum conformal block contains quantum effects of the periodic solutions.

The CFT, or equivalently the N=2 gauge theory as [14] indicates, provides effective tools to compute the large $c$ expansion, therefore we can interpret the subleading terms as quantum effects for the periodic solutions. The $\Omega$ background parameters of gauge theory are related to the central charge of CFT by

$$c = 1 + \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2}. \quad (21)$$

Therefore the NS limit $\epsilon_1 = \epsilon, \epsilon_2 \to 0$ in gauge theory is the classical limit of CFT, $c \to \infty$. In the case of torus 1-point block, the external state has a conformal weight

$$\Delta_m = \frac{m(\epsilon_1 + \epsilon_2 - m)}{\epsilon_1 \epsilon_2}. \quad (22)$$

In the classical limit $\Delta_m \sim O(n(n-1)c)$, hence it produces the Lamé potential. Instead of performing large $c$ expansion, we define a quantum parameter for the nonlinear KdV system by

$$\hbar = \frac{\epsilon_2}{\epsilon_1}. \quad (23)$$

$\epsilon_1$ is the quantum parameter for the associated linear equation (11), here it is used to make the coefficient of elliptic potential to be triangular numbers $n(n-1)$ in order to satisfy the stationary KdV equations.

4 Quasi-modular instanton partition function

Contrary to the Section 2, here we compute the quantum densities of conserved charges of elliptic solution from the gauge theory. On the one hand we have no a prior quantization scheme in KdV theory because it is a nonlinear system without adjustable couplings, this is helped by the CFT/gauge theory as their computation is straightforward. For N=2 theories in the background of generic $\epsilon_1, \epsilon_2$ the computation is based on the localization method [11]. In this way we can generalize the classical KdV charges to the case $\hbar \neq 0$. On the other hand, the prepotential from localization computation is a $q$-series, quasi-modular form of the
the KdV charges helps us to rewrite the expansion as large $a$-expansion with quasi-modular coefficients.

In order to perform $\hbar$ expansion, we need to slightly change some identifications in Section 2. The same identification is made for the following parameters,

$$
n = \frac{m}{\epsilon_1}, \quad \nu = \frac{2\pi a}{\epsilon_1}, \quad \lambda = -\frac{8\pi^2 \bar{u}}{\epsilon_1^2},
$$

then the moduli is given by the prepotential through a relation that generalizes \[13\],

$$
\bar{u} = \frac{1}{2} q \frac{\partial}{\partial q} F(\epsilon_1, \epsilon_2) + \frac{m(m - \epsilon_1 - \epsilon_2) + \epsilon_1 \epsilon_2}{24} (1 - 2E_2).
$$

The instanton partition function computed from the localization formula, and the conformal block as usually computed, are expanded as $q$-series, the modular property is obscured in this expansion. As in the classical limit the prepotential should reduce to the formula \[17\], it is reasonable to suspect the quasi-modular polynomials are preserved in someway in the quantum case. Indeed, from the Nekrasov partition function we find that for $\lambda_{2k}^q$ in the series

$$
\lambda = -\nu^2 + \sum_{k=0}^{\infty} \frac{\lambda_{2k}^q(h)}{\nu^{2k}},
$$

we can assemble all $q$ into $E_{2k}$, then the polynomials of Eisenstein series appearing in the classical case are preserved, only the coefficients $[n(n-1)]^k$ are corrected by $\hbar$. If we define $N = n(n - 1 - \hbar)$, then

$$
\begin{align*}
\lambda_0^q &= \frac{\pi^2}{3} NE_2, \\
\lambda_1^q &= \frac{\pi^4}{36} N(N + \hbar)(E_2^2 - E_4), \\
\lambda_2^q &= \frac{\pi^6}{540} N(N + \hbar)(N + \frac{3}{2} \hbar)(5E_2^3 - 3E_2E_4 - 2E_6) - \frac{\pi^6}{90} (1 + \hbar)^2 N(N + \hbar)(E_2E_4 - E_6), \\
\lambda_3^q &= \frac{\pi^8}{9072} N(N + \hbar)(N^2 + \frac{17}{5} N\hbar + 3\hbar^2)(35E_2^4 - 7E_2^2 E_4 - 10E_4^2 - 18E_2E_6) \\
&\quad - \frac{\pi^8}{756} N(N + \hbar)[\frac{1}{3} N(5 + 9\hbar + 5\hbar^2) + \frac{1}{4} \hbar(10 + 19\hbar + 10\hbar^2)](7E_2^2 E_4 - 5E_4^2 - 2E_2E_6) \\
&\quad + \frac{\pi^8}{126} (1 + \hbar)^4 N(N + \hbar)(E_2E_6 - E_4^2),
\end{align*}
$$

(27)

We can inverse the relation \[26\] to get the quantum densities of the periodic solution,

$$
iv = \sqrt{\lambda} + \sum_{k=1}^{\infty} \frac{\varepsilon_k^q(n, g_{2,3}, \omega_1, \hbar)}{(\sqrt{\lambda})^{2k-1}},
$$

(28)
where
\[
\begin{align*}
\varepsilon_1^q &= -\frac{\pi^2}{6} \left\{ n(n-1)E_2 - \hbar n E_2 \right\}, \\
\varepsilon_2^q &= -\frac{\pi^4}{72} \left\{ n^2(n-1)^2 E_4 - \hbar n(n-1)(E_2^2 - E_4) + 2n^2(n-1)E_4 \right\} + \hbar^2 [n(E_2^2 - E_4) + n^2 E_4],
\end{align*}
\]

\(\varepsilon_k\) can be interpreted as quantum average densities of conserved charges. This is an exact quantization, in every \(\varepsilon_k\) the quantum effect grows to a finite power of \(\hbar\). As we do not observe interesting pattern in the expansion, we cease to give more detail on this.

Let us move a step backward to retrieve the prepotential by integrating the relation (25), we get its expansion in terms of quasi-modular polynomials. In order to compare with the solution, it can be obtained from (17) by taking the limit \(\epsilon_1 = 0\),

\[
\mathcal{F}_{\text{inst}}^\text{mod}(a, m + \frac{\epsilon_1 + \epsilon_2}{2}, q, \epsilon_{1,2}) = \sum_{g,h=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2g}(\epsilon_1 \epsilon_2)^{h} \mathcal{F}_{g,h}(a, m, E_2, E_4, E_6). \tag{30}
\]

Again some integration constants are introduced which have to be subtracted, the value of the constant part is \(\mathcal{F}_{\text{inst}}^\text{mod}(q = 0)\). \(\mathcal{F}_{0,0}\) gives the instanton contribution in the Seiberg-Witten solution, it can be obtained from (17) by taking the limit \(\epsilon_1 = 0\),

\[
\mathcal{F}_{0,0} = -\frac{m^4}{48a^2} E_2 - \frac{m^6}{5760a^4} (5E_2^2 + E_4) - \frac{m^8}{2903040a^6} (175E_2^3 + 84E_2E_4 + 11E_6) + \mathcal{O}\left(\frac{m^{10}}{a^8}\right). \tag{31}
\]

We list few other \(\mathcal{F}_{g,h}\) up to order \(a^{-6}\) which may be useful, they are

\[
\begin{align*}
\mathcal{F}_{1,0} &= \frac{m^2}{96a^2} E_2 + \frac{m^4}{1536a^4} (E_2^2 + E_4) + \frac{m^6}{414720a^6} (25E_2^3 + 48E_2E_4 + 17E_6) + \mathcal{O}\left(\frac{m^8}{a^8}\right), \\
\mathcal{F}_{2,0} &= -\frac{1}{768a^2} E_2 - \frac{m^2}{30720a^4} (5E_2^2 + 9E_4) - \frac{m^4}{1105920a^6} (25E_2^3 + 84E_2E_4 + 101E_6) + \mathcal{O}\left(\frac{m^6}{a^8}\right), \\
\mathcal{F}_{3,0} &= \frac{1}{368640a^4} (5E_2^2 + 13E_4) + \frac{m^2}{9289728a^6} (35E_2^3 + 168E_2E_4 + 355E_6) + \mathcal{O}\left(\frac{m^4}{a^8}\right), \\
\mathcal{F}_{0,1} &= -\frac{m^2}{48a^2} E_2 - \frac{m^4}{2304a^4} (5E_2^2 + E_4) - \frac{m^6}{41472a^6} (11E_2^3 + 6E_2E_4 + E_6) + \mathcal{O}\left(\frac{m^8}{a^8}\right), \\
\mathcal{F}_{1,1} &= \frac{1}{192a^2} E_2 + \frac{m^2}{23040a^4} (25E_2^2 + 17E_4) + \frac{m^4}{276480a^6} (55E_2^3 + 114E_2E_4 + 41E_6) + \mathcal{O}\left(\frac{m^6}{a^8}\right), \\
\mathcal{F}_{2,1} &= -\frac{1}{184320a^4} (25E_2^2 + 29E_4) - \frac{m^2}{7741440a^6} (385E_2^3 + 1386E_2E_4 + 1019E_6) + \mathcal{O}\left(\frac{m^4}{a^8}\right),
\end{align*}
\]

\(\cdots\)
They also give a quasi-modular expansion of the torus conformal block for large $a$, according to [14].

The quasi-modular prepotential, with UV coupling as the modular parameter, has been studied in [19, 20] where they find a recursive relation due to the quasi-modular property of $E_2$. Result for limit case $\epsilon_2 = 0$ can also be found in [21, 22] from classical limit of CFT block. In the holomorphic anomaly approach [23] the argument of modular functions is the IR coupling, for $N = 2^*$ theory in the massless case IR and UV couplings are the same because N=4 SYM is conformal, therefore can be compared in the limit $m \to 0$.

5 Conclusion and discussion

We present an example of relating elliptic KdV solution to 4-dimensional supersymmetric gauge theory. This can be viewed as a natural result from a chain of very interesting connections, the qKdV-CFT relation [8], the CFT/Gauge correspondence [14] and the Gauge/Bethe correspondence [15]. The Schrödinger equation (11) provides a key hint, the Virasoro symmetry is the real reason that makes these connections work. Using all the relations, we obtain the prepotential of gauge theory with the UV coupling dependence assembled into Eisenstein series, which originate from the Weierstrass elliptic function of the KdV solution. On the other hand, we get the quantum spectrum of elliptic KdV solution.

There is another elliptic KdV solution [24],

$$u(x) = \sum_{j=0}^{3} n_j (n_j - 1) \wp(x + \omega_j). \quad (34)$$

It is related to the sphere 4-point conformal block, and SU(2) $N_f = 4$ SYM gauge theory. We emphasis unlike the Lamé potential, in the potential (34) the modulus is not the instanton/conformal block expansion parameter [25]. We believe the spectrum of the solution is related to the quasi-modular expansion of the prepotential for SU(2) $N_f = 4$ SYM. Some formulae about integration of elliptic function, useful to compute the classical spectrum in this case, has been given in [26].

It is not clear if the two examples appear just by coincidence, or in more general case some elliptic KdV solutions are related to some CFT blocks and gauge theories. There is a large class of CFT blocks in [14], on the other hand up to now the most general elliptic solution of KdV is the Picard potential [27]. Superconformal Liouville theory is related to gauge theory on orbifold discussed in [28], while the supersymmetric KdV is constructed in [29].

\[^3\text{However, even in massive case, the polynomials of Eisenstein series in } F^{(0,0)} \text{ in [23] actually are the same polynomials in } F_{0,0}, \text{ this indicates interesting arithmetic property for the prepotential.}\]
simplest extended conformal algebra $W_3$ is connected to the SU(3) gauge theories \cite{30}, and on the other side is connected to the quantum Boussinesq equation \cite{31}, and similarly they have supersymmetric extension. For the connection of general $W$-algebra and generalized classical nonlinear equation of KdV type, see \cite{32}.

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