The Landau–Lifshitz–Bloch Equation in the Thin Film

Yuxun He and Huaqiao Wang

Communicated by A. L. Mazzucato

Abstract. In this paper, we study the initial-boundary value problem of the Landau–Lifshitz–Bloch equation in three-dimensional ferromagnetic films, where the effective field contains the stray field controlled by the Maxwell equation, and the exchange field contains exchange constant. Firstly, we establish the existence of weak solutions of the equation by using the Faedo–Galerkin approximation. We also derive its two-dimensional limit equation in a mathematically rigorous way when the film thickness tends towards zero under appropriate compactness conditions. Moreover, we obtain an equation that can better describe the magnetic dynamic behavior of ferromagnetic films with negligible thickness at high temperatures.

Keywords. Micromagnetics, Landau–Lifshitz–Bloch equation, Ferromagnetic film, Stray field, Weak solution, Asymptotic limit.

1. Introduction

In 1935, Landau and Lifshitz [23] derived the Landau–Lifshitz (LL for short) equation to describe the evolution of a spin field in continuous ferromagnetic fields below the critical (Curie) temperature. The LL equation is the cornerstone of the dynamic magnetization theory of ferromagnetic materials, and its micromagnetic method is the basis of most theoretical studies of thermal magnetization dynamics. The most important feature is that the magnetization is constant. However, the damping term in the LL equation is only applicable to the case of small damping, and it is no longer applicable when encountering large damping. Glibert [13] proposed the Landau–Lifshitz–Gilbert (LLG for short) equation, based on the LL equation, which can be used in the case of large damping. The LLG equation can be transformed into the LL equation, and they are mathematically identical. The motion of the domain wall is the basic mechanism of ferromagnetic mode dynamics. The internal structure of the domain wall in thin films and its influence on the formation of magnetic mode has always been a topic of general interest. According to the magnetization dynamics described by the LL equation or the LLG equation, physicists and mathematicians [6, 14, 17, 21] have done a lot of research on the asymptotic behavior of the magnetization limit of thin films under different parameter mechanisms by the energy method. In particular, they also discussed the asymptotic behavior of weak solutions of the LL equation or the LLG equation, and derived the limit equation in the thin film (see [3, 12, 28, 29]).

In recent years, research on magnetic materials in the field of thermal excitation has become particularly important. So, more and more studies focus on the dynamic behavior of ferromagnetic materials at high temperatures. Although micromagnetics based on the LL equation or the LLG equation can perfectly describe the micro-nano scale magnetic film system at low temperatures, they cannot accurately describe the magnetodynamic behavior at high temperatures (especially close to the Curie temperature of the material). Therefore, from the perspective of magnetization dynamic modeling, micromagnetism theory needs to be further developed. In 1997, Garanin [9, 10] proposed an effective Landau–Lifshitz–Bloch (LLB) equation at high temperatures, especially when it is close to Curie temperature $T_C$ and...
ultrafast time scale:

$$\frac{\partial \mathbf{u}}{\partial t} = \gamma \mathbf{u} \times \mathbf{H}_{\text{eff}} + L_1 \frac{1}{|\mathbf{u}|^2} (\mathbf{u} \cdot \mathbf{H}_{\text{eff}}) \mathbf{u} - L_2 \frac{1}{|\mathbf{u}|^2} \mathbf{u} \times (\mathbf{u} \times \mathbf{H}_{\text{eff}}),$$  

(1.1)

where the spin polarization $\mathbf{u}(\mathbf{x}, t) = \frac{\mathbf{m}}{m_s^2}(t > 0, \mathbf{x} \in D \subset \mathbb{R}^3)$, $\mathbf{m}$ denotes magnetization, $m_s^0$ is the saturation magnetization value when $T = 0$; $\gamma > 0$ stands for the gyromagnetic ratio; $L_1$ and $L_2$ are the longitudinal and transverse damping coefficients, respectively. Effective field $\mathbf{H}_{\text{eff}}$ in this equation is different from the LL equation or the LLG equation. In form, there will be a term related to longitudinal susceptibility, temperature, and spin polarization. The LLB equation successfully connects ferromagnetism with thermodynamic properties. One of its important properties is that magnetization is no longer conserved, but a dynamic variable. Setting $L_1 = L_2$ when $T \geq T_c$, and the effective field $\mathbf{H}_{\text{eff}} = \Delta \mathbf{u} - \frac{1}{\chi_{11}} \left( 1 + \frac{3}{5} \frac{T}{T_c} |\mathbf{u}|^2 \right) \mathbf{u}$ in the LLB equation (1.1), where $\chi_{11}$ is the longitudinal magnetic susceptibility. Le [24] established the global existence of weak solutions of the LLB equation by using the Faedo-Galerkin approximation. Inspired by the work of Le [24], Jia [20] proved the local existence of strong solutions. Guo et al. [15] further obtained the global existence of smooth solutions. After that, Guo et al. [16] also proved the global existence of weak solutions and smooth solutions of the Landau–Likhtman–Bloch–Maxwell equation. Very recently, Peng and Wang [30] obtained the existence and uniqueness of strong solutions to the three-dimensional Landau–Likhtman–Bloch equation in Besov space.

From the perspective of science and technology, the dynamic behavior of magnetization distribution in ferromagnetic thin films is an interesting and important problem. The dynamic problem of the LLB equation in thin films also has important physical significance and applications. Physicists have done a lot of research on the related problems of magnetic films by the LLB equation, such as using the LLB equation to simulate the magnetization dynamics of a single crystal in the film, and then study the related physical processes (see [2]). Moreover, the LLB equation also has a wide range of practical applications in thin films, including the description of ultrafast demagnetization and subsequent recovery process induced by pulsed laser irradiation of magnetic thin films, the study of the spin-Seebeck effect (SSE) found in the thin films, so as to help the development of new spin thermoelectric devices, and the study of heat-assisted magnetic recording (HAMR) technology to improve the storage density of storage media such as ferromagnetic thin films (see [18,26,31]).

However, it is different from the LL equation or the LLG equation which has a large number of results in mathematics (see [1,4,5,8,22,27]). At present, there are a few mathematical pieces of research about the LLB equation in thin films. However, there is no accurate derivation and proof of its limit equation. In this paper, we want to rigorously derive the limit equation from the LLB equation with exchange fields and stray fields in the thin film when the film thickness tends towards zero. Recently, Le [24] considered the LLB equation with only the exchange field. Based on this work, we couple the simplified Maxwell equation (see [19]) with the LLB equation, add stray fields to the effective field, and then prove the existence of weak solutions of the LLB equation in this case. At the same time, we establish the three-dimensional model of the LLB equation in the thin film and continue the parameter setting in [28], where Melcher studied the limit problem of the LLG equation, that is, considering the case of medium time scale and small damping. More precisely, we consider the LLB equation (1.1) in the film $\Omega(h) : \Omega \times (0, h)$ ($\Omega \subset \mathbb{R}^2$) with the exchange field and the stray field where the longitudinal damping coefficient $L_1$ and lateral damping coefficient $L_2$ are equal when $T \geq T_c$. Noticing that the effective field $\mathbf{H}_{\text{eff}} = \Delta \mathbf{u} - \frac{1}{\chi_{11}} \left( 1 + \frac{3}{5} \frac{T}{T_c} |\mathbf{u}|^2 \right) \mathbf{u} - \nabla U$, where $A$ is the exchange constant, $\chi_{11}$ is the longitudinal susceptibility, and the gradient field $-\nabla U$ is the stray field induced by spin polarization. The spin polarization $\mathbf{u} = \frac{\mathbf{m}}{m_s}$ has the following relationship with its corresponding stray field potential (see [11]):

$$\Delta U = \text{div} (\mathbf{u} \chi_{\Omega(h)}) .$$
For $u \in H^1(\Omega(h))$, $U \in \dot{H}^1(\mathbb{R}^3)$, in weak form this equation reads
\[
\int_{\mathbb{R}^3} \nabla U \cdot \nabla \varphi \, dx = \int_{\Omega(h)} u \cdot \nabla \varphi \, dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3). \tag{1.2}
\]
Noted that the following inequality holds (see [4,25]):
\[
\|\nabla U\|_{L^p(\mathbb{R}^3)} \leq C\|u\|_{L^p(\Omega(h))}, \quad 1 < p < \infty. \tag{1.3}
\]
For the convenience of representation, we define the parameter symbols as $L := L_1 = L_2$, $\mu := \frac{3T}{5(T_0 - T_c)}$, $H := H_{\text{eff}}$. From $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$, we have
\[
u \times (u \times H) = (u \cdot H)u - |u|^2H.
\]
Then, Eq. (1.1) becomes
\[
\frac{\partial u}{\partial t} = \gamma u \times H + LH = L(A\Delta u - \nabla U) + \gamma u \times (A\Delta u - \nabla U) - \frac{L}{\chi_1} (1 + \mu |u|^2) u. \tag{1.4}
\]
We consider the initial-boundary value problem of (1.4) with the initial and Neumann boundary conditions:
\[
u|_{t=0} = u_0, \quad \partial_n u|_{\partial D} = 0, \tag{1.5}
\]
where $n$ is the outward unit normal vector of $\partial D$.

We proceed to adopt the dynamic mechanism in [28], assuming that energy is dominated by the exchange energy, and considering the case of medium time scale and small damping, that is, for some $\varepsilon > 0$ and $a > 0$
\[
\frac{A(h)}{h} \to \varepsilon, \quad \gamma(h)\sqrt{h} \to 1, \quad \frac{L(h)}{\gamma(h)h} \to a, \quad \text{as } h \to 0.
\]
The leading order energy effect of the stray field interaction in the thin film is a quadratic shape anisotropy conducive to in-plane magnetization, resulting in the formation of a forcing term pointing to the film plane and competing with the cyclotron force pushing the magnetization vector away from the plane (see [28]). Therefore, in the limit process ($h \to 0$), some stray field energy will be converted into kinetic energy, and the limit spin polarization $u$ is planar.

The LLB equation is developed from the LL equation and the LLG equation, but its form is more complex. The nonlinear term is increased from a product involving only two terms to three terms, which not only requires more and more precise estimates except $L^2$ estimates, and some existing methods for $L^2$ estimates can no longer be used. Secondly, when considering two important physical quantities in the vertical direction, if we still set $w_1^h = \frac{1}{\sqrt{h}} \int_0^h u_3 \, dx_3$ and $w_2^h = \frac{1}{\sqrt{h}} \int_0^h u_3 \frac{\partial U}{\partial x_3} \, dx_3$, then the final limit equation will show singularity. We put the square of the norm of spin polarization into the physical quantities $w_1^h$ and $w_2^h$ to avoid singularity, that is, $w_1^h = \frac{1}{\sqrt{h}} \int_0^h |u|^2 \, dx_3$, $w_2^h = \frac{1}{\sqrt{h}} \int_0^h \left| u^2 \frac{\partial U}{\partial x_3} \right| \, dx_3$. This improvement produces many nonlinear terms involving the product of three terms, which not only requires more and more precise estimates except $L^2$ estimates, but also requires higher integrability for strong convergence. In order to deal with these problems, we extend $L^2$ estimates on the average of product integrals to $L^p-L^q$ estimates. And then we use the $L^p-L^q$ estimate, energy methods, and Gagliardo-Nirenberg interpolation to obtain more precise estimates of magnetization and stray field potential. Because the space of strong convergences obtained by directly using the compact embedding theorem is poor, we combine the weak compactness argument with Aubin-Lions lemma to obtain the strong convergence in the better space. Finally, we derive the limit equation from the LLB equation (1.4) under some appropriate compactness conditions and prove the limit process in a mathematically rigorous way.
Now, we formally derive the limit equation from equation (1.4) when \( h \to 0 \). Assume that \( u^h \) in the thin film \( \Omega(h) \) satisfies
\[
\frac{\partial u^h}{\partial t} = \gamma u^h \times H^h = \gamma u^h \times \bar{H} + LH^h,
\]
where \( H^h = A\Delta u^h - \frac{1}{\chi_1} (1 + \mu|u^h|^2) u^h - \nabla U^h, \bar{H} = A\Delta u^h - \nabla U^h \). Noted that the effective field \( \bar{H} \) is only composed of the exchange field and the stray field. The energy of effective field \( \bar{H} \) is
\[
\bar{E}(u^h) = \frac{A}{2} \int_{\Omega(h)} |\nabla u^h|^2 dx + \frac{1}{2} \int_{R^3} |\nabla U^h|^2 dx.
\]
For renormalized \( \bar{H}^h = \frac{\bar{H}}{h} \), the renormalized LLB equation reads
\[
\frac{\partial u^h}{\partial t} = \gamma hu^h \times \bar{H}^h + LH^h. \tag{1.6}
\]
The renormalized stray field potential is \( v^h = \frac{V^h}{h} \), then the renormalized energy is given by
\[
\bar{E}_h(u^h) = \frac{1}{h^2} \bar{E}(u^h) = \frac{A}{2h} \int_{\Omega(h)} |\nabla u^h|^2 dx + \frac{1}{2} \int_{R^3} |\nabla v^h|^2 dx.
\]
We temporarily assume that when \( h \to 0 \) \( (x_3 \to 0) \), \( f_0^h u^h dx_3 := (f_0^h u^h dx_3, f_0^h u_3^h dx_3) \to (u, 0) \), \( \bar{H}^h := (H^h, \bar{H}^h) \to H_0 = (\bar{H}_0, 0) \), the spin polarization of the limit field and its self-induced magnetic field are \( u, v \), respectively. Since the limit equations of the first component and the second component are always valid in form, we get the result by considering the third component of (1.6) in a weak sense
\[
\int_0^T \int_{\Omega(h)} \frac{\partial u_3}{\partial t} \cdot \phi dx dt = \int_0^T \int_{\Omega(h)} \left[ \gamma h (u_3^h \bar{H}_2^h - u_2^h \bar{H}_1^h) + LH_3^h \right] \cdot \phi dx dt,
\]
where \( \phi \) is a test function. From the above formula we obtain
\[
\lim_{h \to 0} \int_0^T \int_{\Omega(h)} (u^h \wedge \bar{H}^h) \cdot \phi dx dt = 0,
\]
where \( \wedge \) represents the outer product of a two-dimensional vector. The energy corresponding to the limiting magnetization field \( \bar{H}_0 \) of the effective field \( \bar{H}^h \) is
\[
E_0(u) = \frac{\varepsilon}{2} \int_{\Omega} |\nabla' u|^2 dx + \frac{1}{2} \int_{R^3} |\nabla' v|^2 dx,
\]
where \( \Delta v = \text{div}(u\chi_\Omega) \otimes \delta_{\{x_3=0\}} \). Because a portion of the stray field energy is converted into the kinetic energy in the limit process, the total energy of the limit field is
\[
E_{\text{tot}}(u) = E_0(u) + \frac{\beta}{2} \int_{\Omega} |\omega|^2 dx,
\]
where \( -\omega \) is the angular velocity, \( \frac{1}{\beta} = \lim_{h \to 0} \frac{h}{h} [\gamma(h)]^2 \). Then we consider the kinetic energy term of the limit field. We make a polar transformation of the vector \( u \) in the plane. If \( |u| = 0 \), then \( u \) is the zero vector. It’s easy to know that the angular velocity is zero in this case and the kinetic energy is zero. We consider the case of \( |u| \neq 0 \). Let
\[
\frac{u}{|u|} = (\cos \theta, \sin \theta),
\]
and one has
\[
\partial_t \left( \frac{u}{|u|} \right) = (-\sin \theta \partial_t \theta, \cos \theta \partial_t \theta), \quad \frac{u}{|u|} \wedge \partial_t \left( \frac{u}{|u|} \right) = \cos^2 \theta \partial_t \theta + \sin^2 \theta \partial_t \theta = \partial_t \theta.
\]
Thus the angular velocity is \( -\omega = \frac{u}{|u|} \wedge \partial_t \left( \frac{u}{|u|} \right) \) and
\[
\omega_t = -\partial_t \left( \frac{u}{|u|} \wedge \partial_t \left( \frac{u}{|u|} \right) \right) = -\frac{1}{|u|^2} u \wedge \partial_t^2 u.
\]
Therefore we have
\[
\lim_{h \to 0} \int_0^T \int_{\Omega(h)} (u^h \wedge \tilde{H}^h) \cdot \phi dx dt = \int_0^T \int_{\Omega} [u \wedge H_0 + \beta \omega_t] \phi dx dt
\]
\[
= \int_0^T \int_{\Omega} \left[ u \wedge (\varepsilon \Delta u - \nabla' v) - \frac{\beta}{|u|^2} u \wedge \partial_t^2 u \right] \phi dx dt = 0,
\]
where \( \Delta' \) and \( \nabla' \) represent Laplace operators and gradients of two-dimensional vectors, respectively. In order to avoid singularity, multiplying both sides of the above equation by \(|u|^2\), then the limit equation formally reads
\[
|u|^2 u \wedge (\varepsilon \Delta u - \nabla' v) - \beta u \wedge \partial_t^2 u = 0.
\]

Under certain parameter mechanisms and assumptions, we consider the initial-boundary problem of the LLB equations (1.4) and (1.5) in the thin film. Subsequently, the limit equation is obtained, and the limit process is proved in a mathematically rigorous way when the film thickness tends towards zero. The rest of this paper is arranged as follows. In Sect. 2, we first recall some notations, prove the existence of weak solutions of Eq. (1.4) considering the action of stray fields (see Theorem 2.1), and then obtain the necessary compactness from the energy estimates. We also study the interaction of the static magnetic field (stray field) without considering the setting of time variable and get some lemmas. In Sect. 3, we state the main results (see Theorem 3.1). Finally, we prove the main theorem in Sect. 4.

2. Preliminaries

Let’s first introduce some notations which will be used in the later. Let \( D \) be a whole space or a bounded domain, and we define the function spaces as follows:
\[
W^{k,p}(D) \ (1 \leq p \leq \infty) = \{ f \in L^p(D) : \nabla^\alpha f \in L^p(D), |\alpha| \leq k \},
\]
\[
H^k(D) = W^{k,2}(D),
\]
\[
\dot{H}^1(D) = \{ f \in L^6(D) : \nabla f \in L^2(D) \},
\]
\[
X^\beta = \{ \phi \in L^2 : \| A_1^\beta \phi \|_{L^2} < \infty \}, \quad A_1^\beta \phi = \sum_{i=1}^\infty (1+\lambda_i)^\beta \langle \phi, e_i \rangle e_i \text{ with } \| \cdot \|_{X^\beta} = \| A_1^\beta \|_{L^2}.
\]

Here, \( L^p(D) \ (1 \leq p \leq +\infty) \) stands for p-th power Lebesgue integrable function spaces, \( A_1 = I - \Delta \) and \( \beta \) is any positive real number. \( H^{-1}(\mathbb{R}^n) \) is the dual space of \( H^1(\mathbb{R}^n) \), and \( H^{-1}(D) \) is the dual space of \( H_0^1(D) \) where \( D \) is a bounded domain. \( X^{-\beta} \) is the dual space of \( X^\beta \). \( (\cdot, \cdot) \) denotes the dual product, \( \times \) represents the standard vector product on \( \mathbb{R}^3 \). \( \wedge \) represents the outer product on \( \mathbb{R}^2 \), i.e., for \( p, q \in \mathbb{R}^2 \),
\[
p \wedge q = p_1 q_2 - p_2 q_1.
\]

We use the standard symbols for the gradient \( \nabla \) and the Laplacian \( \Delta = \nabla \cdot \nabla \) acting on functions on \( \mathbb{R}^3 \). The corresponding planar operators are
\[
\nabla' = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right), \quad \Delta' = \nabla' \cdot \nabla'.
\]

Symbol \( \chi_E \) represents the characteristic function of measurable set \( E \subset \mathbb{R}^n \), and the integral average is defined as
\[
\int_E f(y) dy = \frac{1}{|E|} \int_E f(y) dy.
\]
Finally, we mark the vector in $\mathbb{R}^3$ with bold letters to distinguish it from the vector on the plane, such as $X = (X, X_3) \in \mathbb{R}^3$, where $X \in \mathbb{R}^2$.

Now, we prove the existence of weak solutions of Eq. (1.4) with $u_0$ as the initial data and the homogeneous Neumann boundary condition in a bounded open domain $D \subset \mathbb{R}^3$ with $C^2$ boundary. First, we recall the definition of weak solutions.

**Definition 2.1.** Given $T > 0$, we say that $u : [0, T] \rightarrow H^1 \cap L^4$ is a weak solution of (1.4) and (1.5) if the following condition holds:

$$\langle u(t), \phi \rangle = \langle u_0, \phi \rangle - L A \int_0^t < \nabla u(s), \nabla \phi > ds - L \int_0^t < \nabla U(s), \phi > ds$$

$$- \gamma A \int_0^t < u(s) \times \nabla u(s), \nabla \phi > ds - \gamma \int_0^t < u(s) \times \nabla U(s), \phi > ds$$

(2.1)

for every $\phi \in C_0^\infty(D)$ and $t \in [0, T]$.

The global existence of weak solutions is given below.

**Theorem 2.1.** Let $D \subset \mathbb{R}^3$ be an open bounded domain with $C^2$ boundary, for any given $T > 0$ and the initial data $u(0) = u_0 \in H^1$, there exists a weak solution of equations (1.4)–(1.5) such that $u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$.

**Proof.** We apply the Faedo-Galerkin approximation to prove Theorem 2.1. Compared to [24], we consider the equation with the action of the stray field. The method is the same, and here we only give a brief proof.

According to [24], we find that the approximate solution $u_n \in S_n$ and $\Delta u_n \in S_n$. The induced stray field potential is $U_n$ and $\nabla U_n \in S_n$ by (1.3). Then we get

$$\begin{cases}
\frac{\partial u_n}{\partial t} - L A \Delta u_n + L \nabla U_n - \gamma A \Pi_n(u_n \times \Delta u_n) + \gamma \Pi_n(u_n \times \nabla U_n) + \frac{L}{\chi_1} \Pi_n \left((1 + \mu |u_n|^2) u_n\right) = 0, \\
u_n(x, 0) = u_{0n}, \quad u_{0n} \in S_n,
\end{cases}$$

(2.2)

where $u_{0n}$ is the approximation of $u_0$.

In the following, we only consider the terms related to the stray field, and see [24] for details of the proof of other terms. For any $u_1, u_2 \in S_n$, using the fact that the induced stray fields $\nabla U_1, \nabla U_2 \in S_n$, and (1.3), we get

$$\|\nabla U_1 - \nabla U_2\|_{L^2} = \|\nabla (U_1 - U_2)\|_{L^2} \leq C\|u_1 - u_2\|_{L^2},$$

and

$$\|\Pi_n(u_1 \times \nabla U_1) - \Pi_n(u_2 \times \nabla U_2)\|_{L^2} = \|\Pi_n(u_1 \times \nabla U_1 - u_2 \times \nabla U_2)\|_{L^2} \leq \|u_1 \times \nabla U_1 - u_2 \times \nabla U_2\|_{L^2}$$

$$\leq \|u_1 \times (\nabla U_1 - \nabla U_2)\|_{L^2} + \|(u_1 - u_2) \times \nabla U_2\|_{L^2}$$

$$\leq \|u_1\|_{L^\infty} \|\nabla U_1 - \nabla U_2\|_{L^2} + \|u_1 - u_2\|_{L^2} \|\nabla U_2\|_{L^\infty}$$

$$\leq C\|u_1\|_{L^\infty} \|u_1 - u_2\|_{L^2} + \|u_1 - u_2\|_{L^2} \|\nabla U_2\|_{L^\infty}.$$

Employing the existence theorem of solutions of the ordinary differential equations, we can obtain the existence of approximate solutions. From (2.2), we can easily establish the following estimates:

$$\|u_n(t)\|_{L^2}^2 + 2L \int_0^T A\|\nabla u_n(t)\|_{L^2}^2 + \|\nabla U_n(t)\|_{L^2}^2 dt + 2L \int_0^T \left(\|u_n(t)\|_{L^4}^2 + \mu \|u_n(t)\|_{L^4}^{4} \right) dt \leq \|u_n(0)\|_{L^2}^2,$$

(2.3)
and
\[ \|\nabla u_n(t)\|_{L^2}^2 + 2LA \int_0^T \|\Delta u_n(t)\|_{L^2}^2 dt \leq \|\nabla u_n(0)\|_{L^2}^2, \]  
(2.4)
for any \( n \in \mathbb{N} \) and \( t \in [0, T] \).

Finally, we consider the convergence of the terms \( \langle \nabla U_n, \phi \rangle \) and \( \langle \Pi_n(u_n \times \nabla U_n), \phi \rangle \) containing the stray field. From [24], there exist a subsequence of \( \{u_n\} \) (still denoted by \( \{u_n\} \)) and \( u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2) \) such that
\[ u_n \rightharpoonup u \text{ in } L^2(0, T; H^1), \]
which together with (1.2) yields that
\[ \lim_{n \to \infty} \int_0^T \langle \nabla U_n, \phi \rangle dt = \lim_{n \to \infty} \int_0^T \langle u_n, \phi \rangle dt = \int_0^T \langle u, \phi \rangle dt = \int_0^T \langle \nabla U, \phi \rangle dt. \]
(2.5)

According to Hölder’s inequality, (2.3), and (2.4), we have
\[ \int_0^T \|u_n(t) \times \nabla U_n(t)\|_{L^2}^2 dt \leq C \sup_{t \in [0, T]} \|u_n\|_{H^1}^2 \int_0^T \|\nabla U_n\|_{L^2}^2 dt \leq C. \]

Then there exist a subsequence of \( \{u_n \times \nabla U_n\} \) (still denoted by \( \{u_n \times \nabla U_n\} \)) and \( Z \in L^2(0, T; L^\frac{4}{3}) \) such that
\[ u_n \times \nabla U_n \rightharpoonup Z \text{ in } L^2(0, T; L^\frac{4}{3}). \]

Further, we obtain
\[ \int_0^T \|\Pi_n(u_n \times \nabla U_n)\|_{X^{-\beta}}^2 dt \leq \int_0^T \|u_n \times \nabla U_n\|_{X^{-\beta}}^2 dt \leq C \int_0^T \|u_n \times \nabla U_n\|_{L^\frac{4}{3}}^2 dt \leq C. \]

There exist a subsequence of \( \{\Pi_n(u_n \times \nabla U_n)\} \) (still denoted by \( \{\Pi_n(u_n \times \nabla U_n)\} \)) and \( Z \in L^2(0, T; X^{-\beta}) \) such that
\[ \Pi_n(u_n \times \nabla U_n) \rightharpoonup Z \text{ in } L^2(0, T; X^{-\beta}). \]

It follows from [24, Lemma 4.2] that \( Z = \tilde{Z} \) in \( L^2(0, T; X^{-\beta}) \). Then for any \( \phi \in L^4(0, T; L^4) \cap L^2(0, T; X^\beta) \), we have
\[ \lim_{n \to \infty} \int_0^T \langle \Pi_n(u_n \times \nabla U_n), \phi \rangle dt = \lim_{n \to \infty} \int_0^T \langle u_n \times \nabla U_n, \phi \rangle dt. \]

Applying Hölder’s inequality, we find that
\[ \left| \int_0^T \langle u_n \times \nabla U_n, \phi \rangle dt - \int_0^T \langle u \times \nabla U, \phi \rangle dt \right| \]
\[ \leq \left| \int_0^T (u_n - u) \times \nabla U_n, \phi dt \right| + \left| \int_0^T (u \times (\nabla U_n - \nabla U), \phi dt \right| \]
\[ \leq \|u_n - u\|_{L^4(0, T; L^4)} \|\nabla U_n\|_{L^2(0, T; L^2)} \|\nabla \phi\|_{L^4(0, T; L^4)} + \int_0^T \langle \nabla U_n - \nabla U, \phi \times u \rangle dt \]
\[ \leq C \|u_n - u\|_{L^4(0, T; L^4)} + \int_0^T \langle \nabla U_n - \nabla U, \phi \times u \rangle dt \].

From the basic energy estimates, we can infer that \( u_n \rightharpoonup u \) in \( L^4(0, T; L^4) \) and \( u \in L^4(0, T; L^4) \). Thus, we obtain
\[ \lim_{n \to \infty} \int_0^T \langle \Pi_n(u_n \times \nabla U_n), \phi \rangle dt = \int_0^T \langle u \times \nabla U, \phi \rangle dt. \]
Combining (2.5) and the above arguments, we can get the desired result. □

Next, we establish the energy estimates of equations (1.4)–(1.5). Let \( u^h \) be a family of weak solutions of the LLB equation (1.4)–(1.5) on \( \Omega(h) \times (0, T) \), and \( U^h \) be the corresponding stray field potential, then there are

\[
\frac{\partial u^h}{\partial t} = \gamma A u^h \times \Delta u^h - \gamma u^h \times \nabla U^h + L A \Delta u^h - \frac{L}{\chi_{11}} u^h - \frac{L \mu}{\chi_{11}} |u^h|^2 u^h - LU^h, \tag{2.6}
\]

and

\[ u^h \in L^\infty(0, T; H^1(\Omega(h))) \cap L^2(0, T; H^2(\Omega(h))). \]

It follows from (1.3) that

\[ U^h \in L^\infty(0, T; \dot{H}^1(\mathbb{R}^3)). \]

By using the above estimates and (2.6), we can conclude that \( u^h \in L^2(0, T; L^2(\Omega(h))) \).

In order to facilitate the proof of the later limit equation, we define the renormalized stray field potential by \( v^h = \frac{U^h}{T} \). Hereafter, without loss of generality, we can assume that \( |\Omega| = 1 \).

Some lemmas about energy estimates are given below.

**Lemma 2.1.** If \( \sup_{h} \frac{1}{T^2} \int_{\Omega(h)} |u^h(0)|^2 \, dx < \infty \), when \( h \rightarrow 0 \), \( \gamma(h)\sqrt{h} \rightarrow 1 \), \( \frac{L(h)}{\gamma(h)h^2} \rightarrow a \), then

\[
\sup_{h,t} \int_{\Omega(h)} |u^h|^2 \, dx < \infty, \quad \sup_{h} \int_{0}^{T} \int_{\Omega(h)} |u^h|^4 \, dx \, dt < \infty, \quad \sup_{h,t} \frac{1}{h} \int_{R^3} |\nabla v^h|^2 \, dx < \infty.
\]

**Proof.** Taking the limit on both sides of (2.3) and using the weak lower semicontinuity of norm, we obtain

\[
\begin{align*}
\int_{\Omega(h)} |u^h|^2 \, dx + 2LA \int_{0}^{T} \int_{\Omega(h)} |\nabla u^h|^2 \, dx \, dt + 2\frac{L}{\chi_{11}} \int_{0}^{T} \int_{\Omega(h)} |u^h|^2 \, dx \, dt + 2\frac{L \mu}{\chi_{11}} \int_{0}^{T} \int_{\Omega(h)} |u^h|^4 \, dx \, dt \\
+ 2Lh \int_{0}^{T} \int_{R^3} |\nabla v^h|^2 \, dx \, dt \leq \int_{\Omega(h)} |u^h(0)|^2 \, dx.
\end{align*}
\tag{2.7}
\]

Then we have

\[ \sup_{h,t} \int_{\Omega(h)} |u^h|^2 \, dx \leq \sup_{h} \int_{\Omega(h)} |u^h(0)|^2 \, dx \leq \sup_{h} \frac{1}{h^2} \int_{\Omega(h)} |u^h(0)|^2 \, dx < \infty, \]

and

\[ 2 \frac{L}{\sqrt{h}} \frac{\mu}{\chi_{11}} \int_{0}^{T} \int_{\Omega(h)} |u^h|^4 \, dx \, dt \leq \frac{1}{\sqrt{h}} \int_{\Omega(h)} |u^h(0)|^2 \, dx \leq \frac{1}{h^2} \int_{\Omega(h)} |u^h(0)|^2 \, dx < \infty. \]

i.e.,

\[ \sup_{h} \int_{0}^{T} \int_{\Omega(h)} |u^h|^4 \, dx \, dt \leq \sup_{h} \frac{1}{h^2} \int_{\Omega(h)} |u^h(0)|^2 \, dx < \infty. \]

We obtain from (1.3) that

\[ \sup_{h,t} \frac{1}{h} \int_{R^3} |\nabla v^h|^2 \, dx \leq \sup_{h,t} \frac{1}{h^2} \int_{\Omega(h)} |u^h|^2 \, dx, \]

which combined with (2.7) yields that

\[ \sup_{h,t} \frac{1}{h} \int_{R^3} |\nabla v^h|^2 \, dx \leq \sup_{h} \frac{1}{h^2} \int_{\Omega(h)} |u^h(0)|^2 \, dx < \infty. \]
Lemma 2.2. If \( \sup_{h} \frac{1}{h^{\gamma}} \int_{\Omega(h)} |\nabla u^{h}(0)|^{2} \, dx < \infty \), when \( h \to 0 \), \( \frac{A(h)}{h} \to \varepsilon \), \( \gamma(h) \sqrt{h} \to 1 \), \( \frac{L(h)}{\gamma(h)} \to a \), then

\[
\sup_{h,t} \frac{1}{h^{\gamma}} \int_{\Omega(h)} |\nabla u^{h}|^{2} \, dx < \infty, \quad \sup_{h} \int_{0}^{T} \int_{\Omega(h)} |\Delta u^{h}|^{2} \, dx dt < \infty, \quad \sup_{h,t} \int_{\Omega(h)} |u^{h}|^{6} \, dx < \infty, \quad \sup_{h} \int_{0}^{T} \left( \int_{\Omega(h)} |\nabla u^{h}|^{6} \, dx \right)^{\frac{1}{3}} \, dt < \infty.
\]

Proof. Taking the limit on both sides of (2.4), we get from the weak lower semicontinuity of norm that

\[
\int_{\Omega(h)} |\nabla u^{h}|^{2} \, dx + 2LA \int_{0}^{T} \int_{\Omega(h)} |\Delta u^{h}|^{2} \, dx dt \leq \int_{\Omega(h)} |\nabla u^{h}(0)|^{2} \, dx.
\]

Then we have

\[
\sup_{h,t} \frac{1}{h^{\gamma}} \int_{\Omega(h)} |\nabla u^{h}|^{2} \, dx \leq \sup_{h} \frac{1}{h^{\gamma}} \int_{\Omega(h)} |\nabla u^{h}(0)|^{2} \, dx \leq \infty, \quad (2.8)
\]

and

\[
\frac{2L}{\sqrt{h}} \frac{A}{h} \int_{0}^{T} \int_{\Omega(h)} |\Delta u^{h}|^{2} \, dx dt \leq \frac{1}{h^{\gamma}} \int_{\Omega(h)} |\nabla u^{h}(0)|^{2} \, dx,
\]

i.e.,

\[
\sup_{h} \int_{0}^{T} \int_{\Omega(h)} |\Delta u^{h}|^{2} \, dx dt \leq \sup_{h} \frac{1}{h^{\gamma}} \int_{\Omega(h)} |\nabla u^{h}(0)|^{2} \, dx \to \infty. \quad (2.9)
\]

From the Gagliardo-Nirenberg (G-N) inequality, we obtain

\[
\|u^{h}\|_{L^{6}} \leq \|\nabla u^{h}\|_{L^{2}},
\]

which together with (2.8) implies that

\[
\sup_{h} \int_{\Omega(h)} |u^{h}|^{6} \, dx \leq \sup_{h} \frac{1}{h} \left( \int_{\Omega(h)} |\nabla u^{h}|^{2} \, dx \right)^{3} = \sup_{h} h^{3} \left( \int_{\Omega(h)} |\nabla u^{h}|^{2} \, dx \right)^{3} < \infty. \quad (2.10)
\]

Utilizing the G-N inequality again, we obtain

\[
\|\nabla u^{h}\|_{L^{6}} \leq \|\Delta u^{h}\|_{L^{2}}.
\]

It holds that

\[
\int_{0}^{T} \left( \int_{\Omega(h)} |\nabla u^{h}|^{6} \, dx \right)^{\frac{1}{3}} \, dt \leq h^{\frac{2}{3}} \int_{0}^{T} \int_{\Omega(h)} |\Delta u^{h}|^{2} \, dx dt.
\]

Combining the above inequality and (2.9), we have

\[
\sup_{h} \int_{0}^{T} \left( \int_{\Omega(h)} |\nabla u^{h}|^{6} \, dx \right)^{\frac{1}{3}} \, dt \leq \sup_{h} \int_{0}^{T} \int_{\Omega(h)} |\Delta u^{h}|^{2} \, dx dt < \infty.
\]

By employing Hölder’s inequality, we also have

\[
\int_{0}^{T} \left( \int_{\Omega(h)} |\nabla u^{h}|^{3} \, dx \right)^{2} \, dt \leq \int_{0}^{T} \left( \int_{\Omega(h)} |\nabla u^{h}|^{6} \, dx \right)^{\frac{1}{3}} \, dt,
\]

then

\[
\sup_{h} \int_{0}^{T} \left( \int_{\Omega(h)} |\nabla u^{h}|^{3} \, dx \right)^{\frac{2}{3}} \, dt \leq \sup_{h} \int_{0}^{T} \left( \int_{\Omega(h)} |\nabla u^{h}|^{6} \, dx \right)^{\frac{1}{3}} \, dt < \infty
\]

holds.
Lemma 2.3. If \( \sup_{h} \frac{1}{h^2} \int_{\Omega(h)} |u^h(0)|^2 \, dx < \infty \) and \( \sup_{h} \frac{1}{h} \int_{\Omega(h)} |\nabla u^h(0)|^2 \, dx < \infty \), when \( h \to 0 \), \( \frac{A(h)}{h} \to \varepsilon \), \( \gamma(h) \sqrt{h} \to 1 \), \( \frac{L(h)}{\gamma(h) h} \to a \), then it holds that
\[
\sup_{h} \int_{0}^{T} \left( \int_{\Omega(h)} |u^h|^\frac{3}{2} \, dx \right)^{\frac{4}{3}} \, dt < \infty.
\]

Proof. Since \( \sup_{h} \int_{0}^{T} \left( \int_{\Omega(h)} |u^h|^\frac{3}{2} \, dx \right)^{\frac{4}{3}} \, dt = \sup_{h} \left\| \frac{1}{h^\frac{3}{2}} u^h \right\|^2_{L^2(0,T;L^\frac{4}{3})} \), we divide both sides of equation (2.6) by \( h^\frac{3}{2} \) and then obtain
\[
\frac{1}{h^{\frac{3}{2}}} u^h = \frac{1}{h^{\frac{3}{2}}} \left( \gamma A u^h \times \Delta u^h - \gamma u^h \times \nabla U^h + L A \Delta u^h - \frac{L}{\chi_{11}} u^h - \frac{L_{11}}{\chi_{11}} |u^h|^2 u^h - L \nabla U^h \right).
\]

Using the G-N inequality and Hölder’s inequality, it is easy to verify that each term on the right-hand side of the above equation belongs to \( L^2(0,T;L^\frac{4}{3}) \) for any \( h > 0 \).

To study the interaction of stray field on a thin magnet in simple environments, only the electrostatic magnetic field needs to be considered. At the end of this section, we give some lemmas of the stray field interaction without considering time variables.

Considering the electrostatic magnetic field
\[
u^h : \Omega(h) \to \mathbb{R}^3, \quad u^h \in H^1(\Omega(h)),
\]
by Hölder’s inequality and the G-N inequality, one has \( u^h \in L^p(\Omega(h)) \) for \( 1 \leq p \leq 6 \).

From the potential equation \( \Delta U^h = \text{div}(u^h \chi_{\Omega(h)}) \), its unique solution \( U^h \in H^1(\mathbb{R}^3) \) satisfies
\[
\int_{\mathbb{R}^3} \nabla U^h \cdot \nabla \phi \, dx = \int_{\Omega(h)} u^h \cdot \nabla \phi \, dx, \quad \phi \in \dot{H}^1(\mathbb{R}^3).
\]

(2.11)

According to (1.3), we have
\[
\left\| \nabla U^h \right\|_{L^6(\mathbb{R}^3)} \leq \left\| u^h \right\|_{L^5(\Omega(h))} < \infty,
\]

which yields that the weak form of potential equation can also be written as
\[
\int_{\mathbb{R}^3} \nabla U^h \cdot \nabla \phi \, dx = \int_{\Omega(h)} u^h \cdot \nabla \phi \, dx, \quad \phi \in W^{1,\frac{6}{5}}(\mathbb{R}^3).
\]

(2.12)

Noticing that \( \nabla \phi \in L^\frac{6}{5}(\mathbb{R}^3) \), and we can naturally obtain \( \phi \in L^2(\mathbb{R}^3) \) by the G-N inequality.

In order to determine the asymptotic limit, two important physical quantities in the vertical direction are defined by
\[
w_1^h = \frac{1}{\sqrt{h}} \int_{0}^{h} |u^h|^2 u_3^h \, dx_3, \quad w_2^h = \frac{1}{\sqrt{h}} \int_{0}^{h} |u^h|^2 \frac{\partial U^h}{\partial x_3} \, dx_3.
\]

Renormalized stray field potential is given by \( v^h = \frac{U^h}{h} \). The following lemma implies the \( L^2 \) estimates of \( w_1^h \) and \( w_2^h \).

Lemma 2.4. If \( \sup_{h} \int_{\Omega(h)} |\nabla u^h|^2 \, dx < \infty \) and \( \sup_{h} \int_{\mathbb{R}^3} |\nabla v^h|^2 \, dx < \infty \) hold, then \( w_1^h \in L^2(\Omega) \), \( w_2^h \in L^2(\Omega) \).

Proof. Using Hölder’s inequality and (2.10), for any \( h \), we conclude that
\[
\int_{\Omega} |w_2^h|^2 \, dx = \int_{\Omega} \frac{1}{h^\frac{3}{2}} \left| \int_{0}^{h} |u^h|^2 \frac{\partial U^h}{\partial x_3} \, dx_3 \right|^2 \, dx \leq \left( \int_{\Omega(h)} |u^h|^6 \, dx \right)^\frac{2}{3} \left( \int_{\mathbb{R}^3} \left| \frac{\partial v^h}{\partial x_3} \right|^2 \, dx \right)^\frac{1}{3} < \infty.
\]

According to [28, Lemma 2.1], we have
\[
\int_{\Omega(h)} |w_3^h|^2 \, dx \leq \int_{\mathbb{R}^3} |\nabla U^h|^2 \, dx + h \int_{\Omega(h)} |\nabla u^h|^2 \, dx + h^2 |\Omega|.
\]
By Hölder’s inequality and (2.10), for any $h$, we obtain

$$
\int_{\Omega} \left| w_1^h \right|^\frac{\theta}{2} dx = \int_{\Omega} \left( \int_0^h \left| u_3^h \right|^2 u_3^3 dx \right)^\frac{\theta}{2} dx \leq \frac{1}{h^{\frac{\theta}{2}}} \left( \int_{\Omega(h)} \left| u_3^h \right|^6 dx \right)^\frac{\theta}{6} \left( \int_{\Omega(h)} \left| u_3^h \right|^2 dx \right)^\frac{\theta}{2} \left( \int_{\Omega(h)} \left| u_3^h \right|^2 dx \right)^\frac{\theta}{2}
$$

$$
\lesssim \left( \int_{\Omega(h)} \left| u_3^h \right|^6 dx \right)^\frac{\theta}{6} \left( \int_{\Omega(h)} \left| \nabla v^h \right|^2 dx + \int_{\Omega(h)} \left| \nabla u_3^h \right|^2 dx + |\Omega| \right)^\frac{\theta}{2} < \infty.
$$

\[\Box\]

Making some changes to the methods used in Lemma 2.2 and Corollary 2.1 of [28], we can get the following lemma.

**Lemma 2.5.** If $\sup_h \int_{\Omega(h)} \left| \nabla U^h \right|^2 dx < \infty$ and $\sup_h \int_{\Omega(h)} \left| \nabla v^h \right|^2 dx < \infty$ hold, then $w_1^h$ and $w_2^h$ have the same weak limit in $L^4(\Omega)$.

**Proof.** Extending $\varphi \in C_0^\infty(\Omega)$ to $\mathbb{R}^3$, we have

$$
\phi_h(x, x_3) = h \left| u^h \right|^2 \psi \left( \frac{x_3}{h} \right) \rho(x_3) \varphi(x),
$$

where $\psi(x_3) = \int_0^{x_3} \chi_{(0,1)}(z)dz$, $\rho \in C_0^\infty((-1, 1))$ is a positive cut-off function. When $z \in (0, \frac{1}{2})$, one has $\rho(z) = 1$ and $|\rho'(z)| \leq 1$ for any $z \in \mathbb{R}$.

Through simple calculation, we can get $\phi_h \in W^{1, \frac{6}{5}}(\mathbb{R}^3)$, and for $h \in (0, \frac{1}{2})$,

$$
\nabla \phi_h = 2hu^h \cdot \nabla u^h \psi \left( \frac{x_3}{h} \right) \rho(x_3) \varphi(x) + \left| u^h \right|^2 \varphi(x) \chi_{(0,h)}(x_3) \hat{e}_3 + h \left| u^h \right|^2 \psi \left( \frac{x_3}{h} \right) \nabla(\varphi \rho)(x).
$$

Substituting $\phi = \phi_h$ into (2.12), we obtain

$$
\left| \int_{\Omega(h)} \left| u_3^h \right|^2 \left( u_3^3 - U^h \right) \frac{\partial U^h}{\partial x_3} \varphi(x) dx \right|
$$

$$
= -h \int_{\Omega(h)} 2 \left( u^h - \nabla U^h \right) u^h \cdot \nabla u^h \psi \left( \frac{x_3}{h} \right) \rho(x_3) \varphi(x) + \left( u^h - \nabla U^h \right) \left| u^h \right|^2 \psi \left( \frac{x_3}{h} \right) \nabla(\varphi \rho)(x) dx
$$

$$
\lesssim h \int_{\Omega(h)} \left| u_3^h \right|^2 \left| \nabla u^h \right| dx + h \int_{\Omega(h)} \left| \nabla U^h \right| \cdot \left| u^h \right| \cdot \left| \nabla u^h \right| dx + h \int_{\Omega(h)} \left( \left| u_3^h \right|^2 + \left| \nabla U^h \right| \cdot \left| u^h \right|^2 \right) dx,
$$

From (1.3), we have

$$
\left\| \nabla U^h \right\|_{L^4(\mathbb{R}^3)} \lesssim \left\| u^h \right\|_{L^4(\Omega(h))}
$$

We utilize Hölder’s inequality to find that

$$
\left| \int_{\Omega} (w_1^h - w_2^h) \varphi(x) dx \right| \leq \frac{1}{\sqrt{h}} \int_{\Omega(h)} \left| u_3^h \right|^2 \left| \nabla u^h \right| dx + \frac{1}{\sqrt{h}} \int_{\Omega(h)} \left| \nabla U^h \right| \cdot \left| u^h \right| \cdot \left| \nabla u^h \right| dx
$$

$$
+ \frac{1}{\sqrt{h}} \int_{\Omega(h)} \left| u_3^h \right|^3 dx + \frac{1}{\sqrt{h}} \int_{\Omega(h)} \left| \nabla U^h \right| \cdot \left| u^h \right|^2 dx
$$

$$
\lesssim \sqrt{h} \left( \int_{\Omega(h)} \left| u_3^h \right|^4 dx \right)^\frac{1}{2} \left( \int_{\Omega(h)} \left| \nabla u^h \right|^2 dx \right)^\frac{1}{2} + h^\frac{1}{2} \left( \int_{\Omega(h)} \left| u_3^h \right|^4 dx \right)^\frac{1}{4} \left( \int_{\Omega(h)} \left| u^h \right|^4 dx \right)^\frac{1}{4} \left( \int_{\Omega(h)} \left| \nabla u^h \right|^2 dx \right)^\frac{1}{2} + \sqrt{h} \left( \int_{\Omega(h)} \left| u_3^h \right|^3 dx + h \int_{\Omega(h)} \left| \nabla u^h \right|^2 dx \right)^\frac{1}{2} \left( \int_{\Omega(h)} \left| u^h \right|^4 dx \right)^\frac{1}{2}.
$$
Using Hölder’s inequality again, one has
\[
\int_{\Omega(h)} |u_h|^4 \, dx \lesssim \left( \int_{\Omega(h)} |u_h|^6 \, dx \right)^{\frac{2}{3}},
\]
\[
\int_{\Omega(h)} |u_h|^3 \, dx \lesssim \left( \int_{\Omega(h)} |u_h|^6 \, dx \right)^{\frac{1}{2}},
\]
which together with (2.10) and Lemma 2.4 imply that when \( h \to 0 \),
\[
w_1^h \rightharpoonup w_2^h \text{ in } L^2(\Omega).
\]
Then \( w_1^h \) and \( w_2^h \) have the same weak limit in \( L^2(\Omega) \).

In the following, the inequality of Lemma 2.4 in [28] about \( L^2 - L^2 \) is extended to the case of \( L^p - L^q \).

**Lemma 2.6.** If \( f \in L^p(\Omega(h)) \), \( g \in L^q(\Omega(h)) \) and \( \frac{\partial f}{\partial x_3} \in L^p(\Omega(h)) \), then
\[
\left| \int_{\Omega} \left( \int_0^h (f \cdot g) - \int_0^h f \int_0^h g \right) \, dx \right| \leq h \left( \int_{\Omega(h)} \left| \frac{\partial f}{\partial x_3} \right|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega(h)} |g|^q \, dx \right)^{\frac{1}{q}},
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \), \( 1 < p, q < \infty \).

**Proof.** According to Lemma 2.4 in [28], we have
\[
\int_0^h (f \cdot g) \, dx_3 - \int_0^h f \, dx_3 \cdot \int_0^h g \, dx_3 = \int_0^h \int_0^h \left( \int_{z_2}^{z_1} \frac{\partial f}{\partial x_3} (x, y) \, dy \right) g(x, z_1) \, dz_1 \, dz_2.
\]
Using Hölder’s inequality, we obtain
\[
\left| \int_{\Omega} \left( \int_0^h (f \cdot g) \, dx_3 - \int_0^h f \, dx_3 \cdot \int_0^h g \, dx_3 \right) \, dx \right| = \left| \int_{\Omega} \int_0^h \int_0^h \left( \int_{z_2}^{z_1} \frac{\partial f}{\partial x_3} (x, y) \, dy \right) g(x, z_1) \, dz_1 \, dz_2 \, dx \right|
\]
\[
= \left| \int_{\Omega} \int_0^h \int_0^h \left( \int_0^h \frac{\partial f}{\partial x_3} (x, y) \, dz_2 \right) g(x, z_1) \, dz_1 \, dy \, dx \right|
\]
\[
\leq h \left| \int_{\Omega} \int_0^h \int_0^h \frac{\partial f}{\partial x_3} (x, y) \, dz_2 \right| |g(x, z_1)| \, dy \, dz_1 \, dx
\]
\[
= h \left| \int_{\Omega} \left( \int_0^h \frac{\partial f}{\partial x_3} (x, y) \, dz_2 \right) \left( \int_0^h |g(x, z_1)| \, dz_1 \right) \, dx \right|
\]
\[
\leq h \left( \int_{\Omega(h)} \left| \frac{\partial f}{\partial x_3} \right|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega(h)} |g|^q \, dx \right)^{\frac{1}{q}}.
\]

\( \square \)

Similar to the relationship between the magnetization in the limit field and its induced stray field proved by Lemma 2.3 in [28], we can also obtain the relationship between the spin polarization \( u^h \) and its induced stray field potential.

**Lemma 2.7.** If
\[
v^h \rightharpoonup v \text{ in } \dot{H}^1(\mathbb{R}^3), \int_0^h u^h \, dx_3 \to (u, 0) \text{ in } L^2(\Omega),
\]
Proof. We further obtain 
\[ \phi \text{ h} \]
Hence, we obtain
\[ \text{H} \]

If solving \( \Delta \)
and \( \psi \)
then we use H"older's inequality to obtain
\[ (0 \text{ h}, \psi) \leq \sup_{x \in \Omega} (0 \text{ h}) \]
\[ \| \nabla \phi \|^2 + \| \nabla \psi \|^2 \]
Finally, we prove the regularity of the time derivative of the stray field induced by the magnetization field \( u^h = u^h(t) \).

**Lemma 2.8.** If \( \sup_h \int_0^\infty \left( f_{\Omega(h)} | u_t^h |^2 dx \right)^\frac{3}{2} dt < \infty \), then \( v^h_t \) is uniformly bounded about \( h \) in \( L^2(\mathbb{R}^3 \times (0, \infty)) \).

**Proof.** Let \( \psi \in C^\infty_0(\mathbb{R}^3 \times (0, \infty)) \) and \( |\psi| \geq 1 \). Then the associated Newton potential \( \varphi \in C^\infty(\mathbb{R}^3 \times (0, \infty)) \) solving \( \Delta \varphi(\cdot, t) = \psi(\cdot, t) \) is compactly supported in time (see [7]). Since \( \Delta \varphi_t = \varphi_t \), we have
\[ \varphi_t = \Delta^{-1} \varphi_t, \quad \nabla \varphi_t = \nabla^{-1} \varphi_t, \]
and one has \( \varphi_t \in H^1(\mathbb{R}^3) \) by the Hardy-Littlewood-Sobolev inequality. Similarly, \( \varphi \in H^1(\mathbb{R}^3) \). For \( v^h \in L^\infty((0, \infty); H^1(\mathbb{R}^3)) \), we get by (2.11) that
\[ \int_0^\infty \langle v^h, \varphi_t \rangle dt = \int_0^\infty \langle u^h, \Delta \varphi_t \rangle dt = -\int_0^\infty \langle \nabla v^h, \nabla \varphi_t \rangle dt = -\frac{1}{h} \int_0^\infty \langle \nabla U^h, \nabla \varphi_t \rangle dt \]
then we use H"older’s inequality to obtain
\[ \left| \int_0^\infty \langle v^h, \psi_t \rangle dt \right| \leq \left[ \int_0^\infty \left( \int_{\Omega(h)} | u_t^h |^2 dx \right)^\frac{3}{2} dt \right]^{\frac{1}{3}} \left[ \int_0^\infty \left( \int_{\Omega(h)} | \nabla \varphi_t |^3 dx \right)^\frac{3}{2} dt \right]^{\frac{1}{3}}. \]
And one has
\[ \int_0^h \int_{\Omega} | \nabla \varphi |^3 dx \leq \sup_{x \in [0,1]} \int_{\Omega} | \nabla \varphi |^3 dx \leq \int_{\Omega(1)} | \nabla^2 \varphi |^2 + | \nabla \varphi |^3 dx \]
\[ \leq \| \nabla^2 \varphi \|^2_{L^2(\Omega(1))} + C \| \nabla \varphi \|^3_{L^6(\Omega(1))}. \]
Given an open set \( U \supset \Omega(1) \), then \( \varphi \in H^1(U) \) by the fact that \( \varphi \in H^1(\mathbb{R}^3) \). We further obtain \( \varphi \in H^1(U) \) from H"older’s inequality.

Since \( \psi \in L^2(U) \), \( \varphi \in H^1(U) \) is the solution of \( \Delta \varphi = \psi \), then \( \varphi \in H^2(\Omega(1)) \) and by using the \( H^2 \)-regularity of the elliptic equation in [7], we conclude that
\[ \| \varphi \|_{H^2(\Omega(1))} \leq C \| \psi \|_{L^2(U)} + \| \varphi \|_{L^2(U)} \leq C \| \psi \|_{L^2(U)}. \]
We further obtain \( \nabla \varphi \in H^1(\Omega(1)) \). One has \( \nabla \varphi \in H^1(\Omega(1)) \) and
\[ \| \nabla^2 \varphi \|_{L^2(\Omega(1))} + \| \nabla \varphi \|_{L^6(\Omega(1))} \leq C \| \psi \|_{L^2(U)}. \]
Hence, we obtain
\[ \int_{\Omega(h)} | \nabla \varphi |^3 dx \leq C \left( \| \psi \|^2_{L^2(U)} + \| \varphi \|^3_{L^2(U)} \right) \leq C \| \psi \|^3_{L^2(U)}. \]
For any \( h \), we have
\[ \left| \int_0^\infty \langle v^h, \psi_t \rangle dt \right| \leq C \left[ \int_0^\infty \left( \psi^3_{L^2(U)} \right)^\frac{3}{2} dt \right]^{\frac{1}{3}} \leq C \left[ \int_0^\infty \| \psi \|^2_{L^2(U)} dt \right]^{\frac{1}{3}}, \]
and
\[ \int_0^\infty \langle v^h_t, \psi \rangle dt \leq \int_0^\infty \langle v^h_t, \psi \rangle dt = \int_0^\infty \langle v^h, \psi_t \rangle dt \leq C \left[ \int_0^\infty \| \psi \|^2_{L^2(U)} dt \right]^{\frac{1}{3}}. \]
Noted that \( C_0^\infty(\mathbb{R}^3 \times (0, \infty)) \) is dense in \( L^2(\mathbb{R}^3 \times (0, \infty)) \), we conclude that \( v^h_t \) is the bounded linear functional in \( L^2(\mathbb{R}^3 \times (0, \infty)) \) for any \( h \). The proof of Lemma 2.8 is completed. \( \square \)

3. The Main Result

We suppose that \( \Omega(h) = \Omega \times (0, h) \) where \( \Omega \subset \mathbb{R}^2 \) is a bounded open domain with \( C^2 \) boundary. Let

\[
\mathbf{u}^h : \Omega(h) \times [0, T] \to \mathbb{R}^3,
\]

and

\[
\mathbf{u}^h \in L^\infty((0, T); H^1(\Omega(h))) \cap L^2((0, T); H^2(\Omega(h))) \cap H^1((0, T); L^2(\Omega(h)))
\]

be a family of weak solution of the LLB equation with the initial data \( \mathbf{u}_0 \) and the Neumann boundary condition \( \frac{\partial \mathbf{u}^h}{\partial n} = 0 \):

\[
\frac{\partial \mathbf{u}^h}{\partial t} = \gamma \mathbf{u}^h \times \mathbf{H}^h + LH^h, \tag{3.1}
\]

where \( \mathbf{H}^h = A\Delta \mathbf{u}^h - \frac{1}{\chi_{11}} (1 + \mu |\mathbf{u}^h|^2) \mathbf{u}^h - \nabla U^h \).

Suppose that the spin magnetic ratio \( \nu = \nu(h) \), the damping parameter \( L = L(h) \) and the exchange constant \( A = A(h) \) are all functions of \( h \). Longitudinal susceptibility \( \chi_{11} \) and \( \mu = \frac{3T}{\Omega(T-T_c)} \) are regarded as constants. The magnetostatic potential \( U^h \) satisfies the Maxwell equation:

\[
\int_{\mathbb{R}^3} \nabla U^h \cdot \nabla \phi dx = \int_{\Omega(h)} \mathbf{u}^h \cdot \nabla \phi dx, \quad \forall \phi \in W^{1, 2}(\mathbb{R}^3). \tag{3.2}
\]

**Definition 3.1.** If for any given finite \( T > 0 \), any \( \Phi \in L^2((0, T); H^1(\Omega(h))) \),

\[
\int_0^T \int_{\Omega(h)} \mathbf{u}^h \cdot \Phi dx dt = \int_0^T \int_{\Omega(h)} \{ -\gamma A (\mathbf{u}^h \times \nabla \mathbf{u}^h) \cdot \nabla \Phi - \gamma (\mathbf{u}^h \times \nabla U^h) \cdot \Phi - L A \nabla \mathbf{u}^h \cdot \nabla \Phi - \frac{L}{\chi_{11}} |\mathbf{u}^h|^2 \Phi - \frac{\mu}{\chi_{11}} |\mathbf{u}^h|^2 \Phi - L \nabla U^h \cdot \Phi \} dx dt,
\]

then \( \mathbf{u}^h \) is a weak solution of (3.1) subject to homogeneous Neumann boundary conditions.

Assume that the initial data satisfies \( \sup \frac{1}{h^2} \int_{\Omega(h)} |\mathbf{u}^h(0)|^2 dx < \infty \), \( \sup \frac{1}{h} \int_{\Omega(h)} |\nabla \mathbf{u}^h(0)|^2 dx < \infty \), from Lemma 2.1 to Lemma 2.3, there are some estimates about \( \mathbf{u}^h \) and \( v^h = \frac{\partial \mathbf{u}^h}{\partial t} \):

\[
\sup \int_{\Omega(h)} |\mathbf{u}^h|^2 dx < \infty, \tag{3.4}
\]

\[
\sup \int_{h \sqrt{h}} \int_{\Omega(h)} |\nabla \mathbf{u}^h|^2 dx < \infty, \quad \text{or} \quad \sup \int_{h \sqrt{h}} \int_{\Omega(h)} |\nabla \mathbf{u}^h|^2 dx < \infty, \tag{3.5}
\]

\[
\sup \int_0^T \int_{\Omega(h)} |\mathbf{u}^h|^4 dx dt < \infty, \tag{3.6}
\]

\[
\sup \int_0^T \int_{\Omega(h)} |\Delta \mathbf{u}^h|^2 dx dt < \infty, \tag{3.7}
\]

\[
\sup \int_0^T \left( \int_{\Omega(h)} |\mathbf{u}^h|^3 dx \right)^\frac{4}{3} dt < \infty, \tag{3.8}
\]

\[
\sup \frac{1}{h^2} \int_{\mathbb{R}^3} |\nabla v^h|^2 dx < \infty, \quad \sup \frac{1}{h} \int_{\mathbb{R}^3} |\nabla v^h|^2 dx < \infty, \tag{3.9}
\]

\[
\sup \int_{h \sqrt{h}} \int_{\Omega(h)} |\mathbf{u}^h|^6 dx < \infty, \tag{3.10}
\]

\[
\sup \frac{1}{h^2} \int_{\mathbb{R}^3} |\nabla v^h|^3 dx < \infty, \quad \sup \frac{1}{h} \int_{\mathbb{R}^3} |\nabla v^h|^3 dx < \infty, \tag{3.9}
\]

\[
\sup \frac{1}{h^2} \int_{\mathbb{R}^3} |\nabla v^h|^6 dx < \infty, \tag{3.10}
\]
\[
\sup_h \int_0^T \left( \int_{\Omega(h)} |\nabla u^h|^6 \, dx \right)^{\frac{1}{6}} \, dt < \infty,
\]
\[
\sup_h \int_0^T \left( \int_{\Omega(h)} |\nabla u^h|^3 \, dx \right)^{\frac{2}{3}} \, dt < \infty.
\]

Then Lemma 2.4 to Lemma 2.8 hold.

Now, we state our main results as follows.

**Theorem 3.1.** Suppose that there exist \( a, \varepsilon > 0 \) such that when \( h \to 0 \),
\[
\frac{L(h)}{\gamma(h) h} \to a, \quad \gamma(h) \sqrt{h} \to 1, \quad \frac{A(h)}{h} \to \varepsilon.
\]

If the weak solutions \( u^h \) of Equation (3.1) and corresponding renormalized magnetostatic potentials \( v^h = \frac{L^h}{h} \) satisfy (3.4–3.12) by assuming initial and boundary conditions, then there is \( h = h_k \to 0 \) such that
\[
\int_0^h u^h dx \to u \text{ in } L^\infty((0, T); H^1(\Omega)) \cap L^2((0, T); H^2(\Omega)) \cap \dot{H}^1((0, T); L^2(\Omega)),
\]
and
\[
v^h \to v \text{ in } L^\infty((0, T); \dot{H}^1(\mathbb{R}^3)) \cap \dot{H}^1((0, T); L^2(\mathbb{R}^3)),
\]
where \( u = (u, 0) : \Omega \times (0, T) \to \mathbb{R}^2 \), \((u, v)\) is a weak solution of
\[
u \land (\partial_t^2 u - \varepsilon |u|^2 \Delta' u + |u|^2 \nabla' v) = 0,
\]
where \( v = v|_{x_3 = 0} \) and
\[
\Delta v = \text{div}(u \chi_\Omega) \otimes \delta_{x_3 = 0} \text{ in } H^{-1}(\mathbb{R}^3) \text{ a.e. } t.
\]

4. **Proof of Theorem 3.1**

For the convenience of calculation, we assume that \( |\Omega| = 1 \). Consider the simplest parameter satisfying (3.13) and set
\[
\gamma = \frac{1}{\sqrt{h}}, \quad L = \sqrt{h}, \quad A = h,
\]

where \( a = 1, \varepsilon = 1 \). Inserting the above parameters into equation (3.3), we get
\[
\int_0^T \int_{\Omega(h)} u^h \cdot \Phi dx dt = \int_0^T \int_{\Omega(h)} \left\{ -\sqrt{h} (u^h \times \nabla u^h) \cdot \nabla \Phi - \sqrt{h} (u^h \times \nabla v^h) \cdot \Phi - h \sqrt{h} \nabla u^h \cdot \nabla \Phi - \frac{1}{\chi_{11}} |u^h|^2 u^h \cdot \Phi \right\} dx dt,
\]
\[
\Rightarrow \frac{1}{\sqrt{h}} \int_0^T \int_{\Omega(h)} u^h \cdot \Phi dx dt = \int_0^T \int_{\Omega(h)} \left\{ - (u^h \times \nabla u^h) \cdot \nabla \Phi - (u^h \times \nabla v^h) \cdot \Phi - h \nabla u^h \cdot \nabla \Phi - \frac{1}{\chi_{11}} |u^h|^2 u^h \cdot \Phi \right\} dx dt,
\]
for any \( \Phi \in L^2((0, T); H^1(\Omega(h))) \), where \( v^h = \frac{L^h}{h} \).

It’s easy to know that \( \int_0^h u^h dx \) and \( v^h \) are uniformly bounded in space-time by (3.4), (3.5), (3.7)–(3.9) and Lemma 2.8, then (3.14) and (3.15) can be obtained from the weak compactness.

According to the compact embedding theorem, we have \( H^1(\Omega) \hookrightarrow L^q(\Omega) \hookrightarrow L^{q_2}(\Omega), \frac{3}{2} \leq q < \infty \). Let
\[
W = \left\{ v \in L^{p_0}(0, T; H^1(\Omega)), v_t \in L^2(0, T; L^\frac{3}{2}(\Omega)), 1 < p_0 < \infty \right\},
\]
then $W \hookrightarrow L^{p_0}(0, T; L^q(\Omega))$ from Aubin-Lions Lemma, and

$$
\int_0^h u^h dx_3 \rightharpoonup u \text{ in } W.
$$

Hence

$$
\int_0^h u^h dx_3 \to u \text{ in } L^{p_0}(0, T; L^q(\Omega)), \quad 1 < p_0 < \infty, \quad \frac{3}{2} \leq q < \infty. \quad (4.2)
$$

To prove the limit equation, we consider

$$
w_1^h = \frac{1}{\sqrt{h}} \int_0^h |u^h|^2 u_3^h dx_3, \quad w_2^h = \frac{1}{\sqrt{h}} \int_0^h |u^h|^2 \frac{\partial U^h}{\partial x_3} dx_3.
$$

It follows from Sect. 2 that they are uniformly bounded in $L^\infty(0, T; L^{\frac{6}{5}}(\Omega))$, and have the same weak limit. Next, we will obtain their weak limits and the weak limits of their time derivatives in Proposition 4.1 and Proposition 4.2, respectively. Finally, we derive the limit equation (3.16).

**Proposition 4.1.** Under the assumptions of Theorem 3.1, there exists $h = h_k \to 0$ such that

$$w_2^h \rightharpoonup u \wedge u_t \text{ in } L^2(0, T; L^{\frac{6}{5}}(\Omega)).$$

**Proof.** The proof of Proposition 4.1 is divided into two steps.

**Step 1:** $w_2^h \rightharpoonup \int_0^h u^h \wedge u_t^h dx_3$ in $L^2(0, T; L^{\frac{6}{5}}(\Omega))$.

Substituting $\Phi = u^h \perp \phi = (-u_2^h, u_1^h, 0) \phi$ into the equation (4.1), where $\phi \in C^\infty_0(\Omega \times (0, T))$, we obtain

$$
\frac{1}{\sqrt{h}} \int_0^T \int_{\Omega(h)} u_t^h \cdot u_\perp^h \phi dxdt = \frac{1}{\sqrt{h}} \int_0^T \int_{\Omega(h)} \left\{ - (u^h \times \nabla u^h) \cdot \nabla (u_\perp^h \phi) - (u^h \times \nabla v^h) \cdot u_\perp^h \phi - h\nabla u^h \cdot u_\perp^h \nabla \phi - h\nabla v^h \cdot u_\perp^h \phi \right\} dxdt. \quad (4.3)
$$

If $X \in \mathbb{R}^3$,

$$u_\perp^h \cdot (u^h \times X) = (u^h \cdot X) u_3^h - |u^h|^2 X_3,$

we have

$$u_\perp^h (u^h \times \nabla v^h) = (u^h \cdot \nabla v^h) u_3^h - |u^h|^2 \frac{\partial v^h}{\partial x_3}.$$

Next, we obtain from (4.3) that

$$
\int_0^T \int_{\Omega(h)} (u^h \wedge u_t^h) \phi dxdt = \int_0^T \int_{\Omega(h)} \left\{ \sqrt{h} (u^h \times \Delta u^h) \cdot u_\perp^h \phi - \sqrt{h} u_3^h (u^h \cdot \nabla v^h) \phi \right\} dxdt + \frac{1}{\sqrt{h}} |u^h|^2 \frac{\partial U^h}{\partial x_3} \phi - h\nabla u^h \cdot u_\perp^h \nabla \phi - h\nabla v^h \cdot u_\perp^h \phi \right\} dxdt. \quad (4.4)
$$
By Hölder’s inequality, the right-hand side of equation (4.4) can be controlled by
\[
\leq \frac{1}{\sqrt{h}} \int_0^T \int_{\Omega(h)} |u^h|^2 \frac{\partial U^h}{\partial x_3} \phi dx dt + \left| \int_0^T \int_{\Omega(h)} \sqrt{h} (u^h \times \Delta u^h) \cdot u^h_1 \phi dx dt \right|
\]
\[
+ \left| \int_0^T \int_{\Omega(h)} \sqrt{h} u^h_3 (u^h \cdot \nabla v^h) \phi dx dt \right| + \left| \int_0^T \int_{\Omega(h)} h \sqrt{h} \nabla u^h \cdot u^h_1 \nabla \phi dx dt \right| + \left| \int_0^T \int_{\Omega(h)} h \sqrt{h} \nabla v^h \cdot u^h_1 \phi dx dt \right|
\]
\[
\lesssim \frac{1}{\sqrt{h}} \int_0^T \int_{\Omega(h)} |u^h|^2 \frac{\partial U^h}{\partial x_3} \phi dx dt + \sqrt{h} \left( \int_0^T \int_{\Omega(h)} |u^h|^4 dx dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\Omega(h)} |\Delta u^h|^2 dx dt \right)^{\frac{1}{2}}
\]
\[
+ \sqrt{h} \left( \int_0^T \int_{\Omega(h)} (u^h \cdot \nabla v^h) u^h_3 dx dt \right) + h \sqrt{h} \left( \int_0^T \int_{\Omega(h)} |\nabla u^h|^2 dx dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\Omega(h)} |u^h|^2 dx dt \right)^{\frac{1}{2}}
\]
\[
+ h \left( \int_0^T \int_{\mathbb{R}^3} |\nabla v^h|^2 dx dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\Omega(h)} |u^h|^2 dx dt \right)^{\frac{1}{2}},
\]
and the third term of (4.5) is bounded by
\[
\left| \int_0^T \int_{\Omega(h)} (u^h \cdot \nabla v^h) u^h_3 dx dt \right| = \frac{1}{h} \left| \int_0^T \int_{\Omega(h)} \nabla u^h_3 \cdot u^h \cdot v^h + u^h_3 \cdot \text{div} u^h \cdot v^h dx dt \right|
\]
\[
\lesssim \frac{1}{h^{\frac{1}{2}}} \left( \int_0^T \int_{\Omega(h)} |\nabla u^h_3|^2 dx dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\Omega(h)} |u^h|^3 dx dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |v^h|^6 dx dt \right)^{\frac{1}{6}}
\]
\[
+ \frac{1}{h^{\frac{1}{2}}} \left( \int_0^T \int_{\Omega(h)} |u^h_3|^3 dx dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\Omega(h)} |\nabla u^h|^2 dx dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |v^h|^6 dx dt \right)^{\frac{1}{6}}.
\]
Using Hölder’s inequality and the G-N inequality, for any fixed $h$, we have
\[
\int_0^T \int_{\Omega(h)} |u^h|^3 dx dt \lesssim \left( \int_0^T \int_{\Omega(h)} |u^h|^4 dx dt \right)^{\frac{2}{3}} \left( \int_0^T \int_{\mathbb{R}^3} |v^h|^6 dx dt \right)^{\frac{1}{6}} \lesssim \left( \int_0^T \int_{\mathbb{R}^3} |\nabla v^h|^2 dx dt \right)^{\frac{1}{2}}.
\]
By using (3.5), (3.6) and (3.9), we can obtain
\[-\sqrt{h} \int_0^T \int_{\Omega(h)} (u^h \cdot \nabla v^h) u^h_3 dx dt = O(h^{\frac{1}{2}}).
\]
Combining (4.5), (3.4)–(3.7) and (3.9), we have
\[
\int_0^T \int_{\Omega(h)} (u^h \wedge u^h_t) \phi dx dt = \frac{1}{\sqrt{h}} \int_0^T \int_{\Omega(h)} |u^h|^2 \frac{\partial U^h}{\partial x_3} \phi dx dt + O(h^{\frac{1}{2}}).
\]
Therefore, when $h \to 0$, one has
\[
\int_0^T \int_{\Omega(h)} (u^h \wedge u^h_t) \phi dx dt \to \frac{1}{\sqrt{h}} \int_0^T \int_{\Omega(h)} |u^h|^2 \frac{\partial U^h}{\partial x_3} \phi dx dt.
\]
The conclusion of Step 1 is proved.

**Step 2:** $\int_0^h u^h \wedge u^h_t dx_3 \to u \wedge u_t$ in $L^2(0, T; L^{\frac{7}{2}}(\Omega))$, i.e., $\int_0^T \int_{\Omega(h)} u^h \wedge u^h_t \cdot \phi dx dt \to \int_0^T \int_{\Omega} u \wedge u_t \cdot \phi dx dt.$
From Lemma 2.6, Hölder’s inequality, (3.5), and (3.8), we obtain as \( h \to 0 \)
\[
\left| \int_0^T \left[ \int_{\Omega(h)} u_i^h \partial_i u_2^h \phi dx - \int_\Omega \left( \int_0^h u_i^h dx_3 \right) \left( \int_0^h \partial_i u_2^h dx_3 \right) \phi dx \right] dt \right|
\leq h \left[ \int_0^T \left( \int_{\Omega(h)} \left| \partial_i u_1^h \right|^3 dx \right) dt \right]^{\frac{1}{3}} \left[ \int_0^T \left( \int_{\Omega(h)} \left| \partial_i u_2^h \right|^2 \left| \phi \right|^\frac{3}{2} dx \right) dt \right]^{\frac{1}{2}} \to 0,
\]
that is,
\[
\int_0^T \int_{\Omega(h)} u_i^h \partial_i u_2^h \phi dx dt \to \int_0^T \int_\Omega \left( \int_0^h u_i^h dx_3 \right) \left( \int_0^h \partial_i u_2^h dx_3 \right) \phi dx dt.
\]
Similarly,
\[
\int_0^T \int_{\Omega(h)} u_2^h \partial_i u_1^h \phi dx dt \to \int_0^T \int_\Omega \left( \int_0^h u_2^h dx_3 \right) \left( \int_0^h \partial_i u_1^h dx_3 \right) \phi dx dt.
\]
By virtue of (3.14) and (4.2), we can infer that \( \int_0^h \partial_i u^h dx_3 \to u_t \) in \( L^2(0,T;L^2(\Omega)) \), \( \int_0^h u^h dx_3 \to u \) in \( L^2(0,T;L^3(\Omega)) \), which combined with (3.8) yields that
\[
\int_0^T \int_{\Omega(h)} u^h \cdot u^h \cdot \phi dx dt \to \int_0^T \int_{\Omega} u \cdot u_t \cdot \phi dx dt.
\]
Thus
\[
\int_0^T \int_{\Omega(h)} u^h \cdot u^h \cdot \phi dx dt \to \int_0^T \int_{\Omega} u \cdot u_t \cdot \phi dx dt.
\]
The conclusion of Step 2 is proved. Combining the conclusions of Step 1 and Step 2, the proof of Proposition 4.1 is completed. \( \square \)

**Proposition 4.2.** Under the assumptions of Theorem 3.1, there exists \( h = h_k \to 0 \) such that
\[
\partial_t u_1^h \to |u|^2 \nabla'(u \wedge \nabla'u) - |u|^2 u \wedge \nabla'v \text{ in } H^{-1}(0,T;L^2(\Omega)).
\]

**Proof.** Noticing that
\[
\partial_t u_1^h = \frac{1}{\sqrt{h}} \int_0^h 2u^h \cdot u^h \cdot u_3^h dx_3 + \frac{1}{\sqrt{h}} \int_0^h |u^h|^2 \partial_t u_3^h dx_3,
\]
we prove the weak limits of the two terms on the right-hand side of the above equality, respectively.

**Step 1:** \( \frac{1}{\sqrt{h}} \int_0^h u^h \cdot u_1^h \cdot u_3^h dx_3 \to 0 \) in \( H^{-1}(0,T;L^2(\Omega)) \), that is, for any \( \phi \in C_0^\infty(\Omega \times (0,T)) \), \( \frac{1}{\sqrt{h}} \int_0^T \int_{\Omega(h)} u^h \cdot u_1^h \cdot \phi dx dt \to 0 \).

According to Lemma 2.6, Hölder’s inequality, (3.8), (3.10), and (3.11), we obtain
\[
\frac{1}{\sqrt{h}} \left| \int_0^T \left[ \int_{\Omega(h)} \left( \int_0^h u^h \cdot u^h \cdot \phi \cdot u_3^h dx_3 - \int_0^h u^h \cdot u_1^h \cdot u_3^h dx_3 \right) \cdot \phi dx dt \right] \right|
\leq \sqrt{h} \sup_t \left( \int_{\Omega(h)} |u^h|^6 |\phi|^6 dx \right)^\frac{1}{6} \left[ \int_0^T \left( \int_{\Omega(h)} |u^h|^3 dx \right)^\frac{2}{3} dt \right]^\frac{1}{2} \left[ \int_0^T \left( \int_{\Omega(h)} \left| \partial u_3^h \right|^6 dx \right)^\frac{1}{3} dt \right]^{\frac{1}{2}} \to 0,
\]
that is,
\[
\frac{1}{\sqrt{h}}\int_0^T \int_{\Omega(h)} u^h \cdot u^h dx_3 \cdot \phi dx dt \rightarrow \frac{1}{\sqrt{h}}\int_0^T \int_{\Omega} u^h \cdot u^h dx_3 \cdot \int_0^h u_3^h dx_3 \cdot \phi dx dt.
\]

By Hölder’s inequality, we have
\[
\int_0^T \left( \int_{\Omega} \left| \frac{1}{\sqrt{h}} \int_0^h u^h \cdot u^h dx_3 \right|^2 dx \right)^{\frac{2}{3}} dt \leq \left( \sup_{t} \frac{1}{h^3} \int_{\Omega(h)} |u^h|^6 dx \right)^{\frac{2}{3}} \int_0^T \left( \int_{\Omega(h)} |u^h|^2 dx \right)^{\frac{4}{3}} dt,
\]
and using the G-N inequality and (3.5), we can further get
\[
\left( \sup_{h,t} \frac{1}{h^3} \int_{\Omega(h)} |u^h|^6 dx \right)^{\frac{1}{3}} \leq \sup_{h,t} \frac{1}{h^2} \int_{\Omega(h)} |\nabla u^h|^2 dx = \sup_{h,t} \left( h^{-\frac{2}{3}} \cdot \frac{1}{h^{\frac{1}{2}}} \int_{\Omega(h)} |\nabla u^h|^2 dx \right) < \infty.
\]

We know that \( \frac{1}{\sqrt{h}} \int_0^h u^h \cdot u^h dx_3 \) is uniformly bounded in \( L^2(0,T; L^\infty(\Omega)) \) in combination with (3.8), and \( \int_0^h u_3^h dx_3 \rightarrow 0 \) in \( L^2(0,T; L^6(\Omega)) \) by (4.2). Then we find that
\[
\frac{1}{\sqrt{h}}\int_0^T \int_{\Omega} u^h \cdot u^h dx_3 \cdot \int_0^h u_3^h dx_3 \cdot \phi dx dt \rightarrow 0.
\]

The conclusion of Step 1 is proved.

**Step 2:**

We substitute \( \Phi = |u^h|^2 \hat{e}_3 \phi, \phi \in C_0^\infty(\Omega \times (0,T)) \) into the equation (4.1)
\[
\frac{1}{\sqrt{h}}\int_0^T \int_{\Omega(h)} |u^h|^2 \partial_t u_3^h \phi dx dt \rightarrow \int_0^T \int_{\Omega} \left[ |u|^2 \nabla' (u \wedge \nabla u) - |u|^2 u \wedge \nabla' v \right] \phi dx dt.
\]

Substitute \( \Phi = |u^h|^2 \hat{e}_3 \phi, \phi \in C_0^\infty(\Omega \times (0,T)) \) into the equation (4.1)
\[
\frac{1}{\sqrt{h}}\int_0^T \int_{\Omega(h)} |u^h|^2 \partial_t u_3^h \phi dx dt = \int_0^T \int_{\Omega(h)} \left\{ - (u^h \wedge \nabla u^h) \cdot \nabla \left( |u^h|^2 \phi \right) - (u^h \wedge \nabla v^h) \cdot |u^h|^2 \phi \right. - h \nabla u^h \cdot \nabla \left( |u^h|^2 \phi \right) - \frac{1}{\lambda_{11}} u_3^h \cdot |u^h|^2 \phi - \frac{\mu}{\lambda_{11}} |u^h|^2 u_3^h \cdot |u^h|^2 \phi
\]
\[
- h \frac{\partial u^h}{\partial x_3} \cdot |u^h|^2 \phi \right\} dx dt
\]
\[
= \int_0^T \int_{\Omega(h)} \left[ - (u^h \wedge \nabla u^h) \cdot \nabla \left( |u^h|^2 \phi \right) - (u^h \wedge \nabla v^h) \cdot |u^h|^2 \phi \right] dx dt
\]
\[
+ A + B + C + D,
\]

where
\[
A = -h \int_0^T \int_{\Omega(h)} \nabla u_3^h \cdot \nabla \left( |u^h|^2 \phi \right) dx dt, \quad B = -\frac{1}{\lambda_{11}} \int_0^T \int_{\Omega(h)} u_3^h \cdot |u^h|^2 \phi dx dt,
\]
\[
C = -\frac{\mu}{\lambda_{11}} \int_0^T \int_{\Omega(h)} |u^h|^4 u_3^h \phi dx dt, \quad D = -h \int_0^T \int_{\Omega(h)} \frac{\partial u^h}{\partial x_3} \cdot |u^h|^2 \phi dx dt.
\]
Next, we consider the limit problem of the right-hand side of (4.6) when \( h \to 0 \). For the term \( A \), applying Hölder’s inequality, (3.6), and (3.7), we obtain

\[
\left| \int_0^T \int_{\Omega(h)} \nabla u_3^h \cdot \nabla \left( |u^h|^2 \phi \right) \, dx \, dt \right| = \left| \int_0^T \int_{\Omega(h)} \Delta u_3^h \cdot |u^h|^2 \phi \, dx \, dt \right|
\leq \left( \int_0^T \int_{\Omega(h)} |\Delta u_3^h|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\Omega(h)} |u^h|^4 |\phi|^2 \, dx \, dt \right)^{\frac{1}{2}} < \infty,
\]

for any fixed \( h > 0 \) and then \( A \to 0 \).

For the term \( B \), employing Lemma 2.6, Hölder’s inequality, (3.6), and (3.7), we have

\[
\left| \int_0^T \left[ \int_{\Omega(h)} \left( \int_0^h u_3^h \cdot |u^h|^2 \phi \, dx_3 - \int_0^h u_3^h dx_3 \cdot \int_0^h |u^h|^2 \phi \, dx_3 \right) \, dx \right] \, dt \right|
\leq h \left( \int_0^T \int_{\Omega(h)} \left| \frac{\partial u_3^h}{\partial x_3} \right|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\Omega(h)} |u^h|^4 |\phi|^2 \, dx \, dt \right)^{\frac{1}{2}} \to 0,
\]

that is,

\[
\int_0^T \int_{\Omega(h)} u_3^h \cdot |u^h|^2 \phi \, dx \, dt \to \int_0^T \int_{\Omega} u_3^h \cdot \int_{\Omega} |u^h|^2 \phi \, dx_3 \, dx \, dt.
\]

We utilize Hölder’s inequality, (3.6) and (4.2) to get

\[
\int_0^T \int_{\Omega} \int_{\Omega} u_3^h \cdot \int_{\Omega} |u^h|^2 \phi \, dx_3 \, dx \, dt \to 0,
\]

which yields \( B \to 0 \).

For the term \( C \), using Lemma 2.6, Hölder’s inequality, (3.10), and (3.12), we deduce that

\[
\left| \int_0^T \left[ \int_{\Omega(h)} \left( \int_0^h |u^h|^4 u_3^h \phi \, dx_3 - \int_0^h |u^h|^4 \phi \, dx_3 \cdot \int_0^h u_3^h \, dx_3 \right) \, dx \right] \, dt \right|
\leq h \left[ \int_0^T \left( \int_{\Omega(h)} |u^h|^6 |\phi|^2 \, dx \right)^{\frac{4}{3}} \right]^{\frac{3}{2}} \left[ \int_0^T \left( \int_{\Omega(h)} \left| \frac{\partial u_3^h}{\partial x_3} \right|^3 \, dx \right)^{\frac{2}{3}} \right]^{\frac{3}{2}} \to 0,
\]

that is,

\[
\int_0^T \int_{\Omega(h)} |u^h|^4 u_3^h \phi \, dx \, dt \to \int_0^T \int_{\Omega} |u^h|^4 \phi \, dx_3 \cdot \int_0^h u_3^h \, dx_3 \, dx \, dt.
\]

Combining (4.2) with (3.10), we can easily get that

\[
\int_0^T \int_{\Omega} |u^h|^4 \phi \, dx_3 \cdot \int_0^h u_3^h \, dx_3 \, dx \, dt \to 0,
\]

that is, \( C \to 0 \).

For the term \( D \), we have

\[
|D| = \sqrt{h} \cdot \sqrt{h} \int_0^T \int_{\Omega(h)} \frac{\partial u^h}{\partial x_3} \cdot |u^h|^2 \phi \, dx \, dt.
\]

By Hölder’s inequality, (3.6) and (3.9), for any \( h \), one has

\[
\sqrt{h} \int_0^T \int_{\Omega(h)} \frac{\partial u^h}{\partial x_3} \cdot |u^h|^2 \phi \, dx \, dt \leq \left( \int_0^T \int_{\Omega(h)} \left| \frac{\partial u^h}{\partial x_3} \right|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\Omega(h)} |u^h|^4 |\phi|^2 \, dx \, dt \right)^{\frac{1}{2}} < \infty,
\]
and it follows that $D \to 0$.

Inserting the above estimates into (4.6), we get
\[
\frac{1}{\sqrt{h}} \int_0^T \int_{\Omega(t)} |u^h|^2 \partial_t u^h \phi \, dx \, dt \to \int_0^T \int_{\Omega(t)} \left[ - (u^h \wedge \nabla u^h) \cdot \nabla \left( |u^h|^2 \phi \right) - (u^h \wedge \nabla v^h) \cdot |u^h|^2 \phi \right] \, dx \, dt.
\]

Now, we prove
\[
\int_0^T \int_{\Omega(t)} \left[ - (u^h \wedge \nabla u^h) \cdot \nabla \left( |u^h|^2 \phi \right) - (u^h \wedge \nabla v^h) \cdot |u^h|^2 \phi \right] \, dx \, dt
\]
\[
\to \int_0^T \int_\Omega \left( u \wedge \Delta' u \right) \cdot |u|^2 \phi - (u \wedge \nabla v) \cdot |u|^2 \phi \right] \, dx \, dt.
\]

First, we prove that
\[
- \int_0^T \int_{\Omega(t)} (u^h \wedge \nabla u^h) \cdot \nabla \left( |u^h|^2 \phi \right) \, dx \, dt \to \int_0^T \int_\Omega (u \wedge \Delta' u) \cdot |u|^2 \phi \, dx \, dt.
\]

By virtue of Lemma 2.6, Hölder’s inequality, (3.7), (3.10), and (3.11), we obtain
\[
\left| \int_0^T \int_\Omega \left( \int_0^h u_1^h \Delta u_2^h |u|^2 \phi _{dx_3} - \int_0^h u_1^h dx_3 \cdot \int_0^h u_2^h |u|^2 \phi _{dx_3} \right) \, dx \, dt \right|
\]
\[
\lesssim h \left( \sup_t \int_{\Omega(t)} |u|^6 \, dx \right) \frac{1}{2} \left[ \int_0^T \left( \int_0^h \left| \frac{\partial u_1^h}{\partial x_3} \right|^6 \, dx \right) ^{\frac{1}{3}} \, dt \right] \left( \int_0^T \int_{\Omega(t)} |\Delta u_2^h|^2 \, dx \, dt \right)^{\frac{1}{4}} \to 0,
\]

that is,
\[
\int_0^T \int_{\Omega(t)} u_1^h \Delta u_2^h |u|^2 \phi _{dx_3} \to \int_0^T \int_{\Omega(t)} \left( \int_0^h u_1^h \, dx_3 \right) \cdot \left( \int_0^h u_2^h |u|^2 \phi _{dx_3} \right) \, dx \, dt.
\]

By (4.2), we have
\[
\int_0^h u_1^h \, dx_3 \to u_1 \text{ in } L^6(0, T; L^6(\Omega)).
\]

Using Lemma 2.6, Hölder’s inequality, (3.7), (3.10), and (3.12), we can also get
\[
\left| \int_0^T \int_{\Omega(t)} \left( \int_0^h \Delta u_2^h |u|^2 \, dx_3 - \int_0^h \Delta u_2^h dx_3 \cdot \int_0^h |u|^2 \, dx_3 \right) \, dx \, dt \right|
\]
\[
\lesssim h \left( \int_0^T \int_{\Omega(t)} |\Delta u_2^h|^2 \, dx \, dt \right) \frac{1}{2} \left( \sup_t \int_{\Omega(t)} |u|^6 \, dx \right) \frac{1}{2} \left[ \int_0^T \left( \int_0^h \left| \frac{\partial u_1^h}{\partial x_3} \right|^3 \, dx \right) ^{\frac{1}{2}} \, dt \right] \frac{1}{2} \to 0,
\]

that is,
\[
\int_0^T \int_{\Omega(t)} \Delta u_2^h |u|^2 \, dx_3 \, dx \, dt \to \int_0^T \int_{\Omega(t)} \left( \int_0^h \Delta u_2^h \, dx_3 \right) \cdot \left( \int_0^h |u|^2 \, dx_3 \right) \, dx \, dt.
\]

By (3.14), we have
\[
\int_0^h \Delta u_2^h \, dx_3 \to \Delta u_2 \text{ in } L^2(0, T; L^2(\Omega)).
\]

Let $W = \left\{ v \in L^3(0, T; W^{1,2}(\Omega)), v_t \in L^2(0, T; L^2(\Omega)) \right\}$, by Aubin-Lions Lemma and the compact embedding theorem, we conclude that
\[
W \hookrightarrow L^3(0, T; L^3(\Omega)).
\]
Through simple calculation and using Hölder’s inequality, (3.5), (3.8), (3.10), we can know that $\int_0^h |u_h|^2 dx_3$ is uniformly bounded about $h$ in $W$. Then $\int_0^h |u_h|^2 dx_3$ converges weakly in $W$ by the weak compactness argument.

Utilizing Lemma 2.6, (3.4), and (3.5), we obtain
\[
\left| \int_0^T \int_\Omega \left( \int_0^h u_h dx_3 \right)^2 \phi dxdt - \int_0^T \int_\Omega \left( \int_0^h u_h dx_3 \right)^2 \phi dxdt \right| \\
\leq h \left( \int_0^T \int_\Omega (\int_0^h |u_h|^2 dx_3)^{1/2} \left( \int_0^T \int_\Omega \frac{\partial u_h}{\partial x_3} dx_3 \right)^{1/2} \right) \\
\rightarrow 0.
\]

Hence, $\int_0^h |u_h|^2 dx_3 \rightarrow (\int_0^h u_h dx_3)^2$ in $W$. By Hölder’s inequality, we have
\[
\left| \int_0^T \int_\Omega \left( \int_0^h u_h dx_3 \right)^2 - |u|^2 \right| = \left| \int_0^T \int_\Omega \left( \int_0^h u_h dx_3 - u \right) \left( \int_0^h u_h dx_3 + u \right) \phi dxdt \right| \\
\leq \left( \int_0^T \int_\Omega \left( \int_0^h u_h dx_3 - u \right)^2 dxdt \right)^{1/2} \left( \int_0^T \int_\Omega \left( \int_0^h u_h dx_3 + u \right)^2 |\phi|^2 dxdt \right)^{1/2},
\]
which implies that $(\int_0^h u_h dx_3)^2 \rightarrow |u|^2$ in $W$ by (4.2). Therefore, we have
\[
\int_0^h |u_h|^2 dx_3 \rightarrow |u|^2 \text{ in } W,
\]
and further get
\[
\int_0^h |u_h|^2 dx_3 \rightarrow |u|^2 \text{ in } L^3(0, T; L^3(\Omega)).
\]
(4.12)

Combining (4.11) and (4.12), we obtain that
\[
\int_0^h \Delta u_2 u_h^2 dx_3 \rightarrow \Delta u_2 |u|^2 \text{ in } L^6(0, T; L^6(\Omega)).
\]
(4.13)

Utilizing (4.9) and (4.13), one has
\[
\int_0^T \int_\Omega \int_0^h u_h dx_3 \cdot \int_0^h \Delta u_2 |u_h|^2 \phi dx_3 dxdt \rightarrow \int_0^T \int_\Omega \int_0^h u_1 \Delta u_2 |u|^2 \phi dxdt,
\]
i.e.,
\[
\int_0^T \int_{\Omega(h)} \int_0^h u_1 \Delta u_2 |u_h|^2 \phi dxdt \rightarrow \int_0^T \int_{\Omega(h)} \int_0^h u_1 \Delta u_2 |u|^2 \phi dxdt.
\]

Similarly, we obtain
\[
\int_0^T \int_{\Omega(h)} \int_0^h u_2 \Delta u_1 |u_h|^2 \phi dxdt \rightarrow \int_0^T \int_{\Omega(h)} \int_0^h u_2 \Delta u_1 |u|^2 \phi dxdt.
\]

Noticing that
\[
- \int_0^T \int_{\Omega(h)} (u_h \wedge \nabla u_h) \cdot \nabla |u_h|^2 \phi dxdt = \int_0^T \int_{\Omega(h)} (u_h \wedge \Delta u_h) \cdot |u_h|^2 \phi dxdt
\]
and \(|u|^2 = |u|^2\), we get
\[
- \int_0^T \int_{\Omega(h)} (u^h \wedge \nabla u^h) \cdot \nabla \left( |u|^2 \phi \right) \, dx \, dt \rightarrow \int_0^T \int_{\Omega} (u \wedge \Delta u) \cdot |u|^2 \phi \, dx \, dt.
\]
Similar to (4.7), we will prove
\[
\int_0^T \int_{\Omega(h)} (u^h \wedge \nabla u^h) \cdot |u|^2 \phi \, dx \, dt \rightarrow \int_0^T \int_{\Omega} u \wedge \nabla' v \cdot |u|^2 \phi \, dx \, dt.
\]
(4.14)

According to Lemma 2.6, Hölder’s inequality, (3.9), (3.10) and (3.11), we obtain
\[
\left| \int_0^T \int_{\Omega} \left( \int_0^h u_1^h \frac{\partial u^h}{\partial x_2} |u|^2 \phi \, dx_3 - \int_0^h u_1^h \cdot \int_0^h \frac{\partial u^h}{\partial x_2} |u|^2 \phi \, dx_3 \right) \, dx \, dt \right|
\leq \sqrt{h} \left( \sup_t \int_{\Omega(h)} |u|^6 \, dx \right)^{1/4} \left[ \int_t^T \left( \int_{\Omega(h)} \left| \frac{\partial u^h}{\partial x_3} \right|^6 \, dx \right)^{1/4} \, dt \right]^{1/2} \left( \int_0^T \int_{\Omega(h)} \left| \frac{\partial u^h}{\partial x_2} \right|^2 \, dx \, dt \right)^{1/2} \rightarrow 0,
\]
then
\[
\int_0^T \int_{\Omega(h)} u_1^h \frac{\partial u^h}{\partial x_2} |u|^2 \phi \, dx \, dt \rightarrow \int_0^T \int_{\Omega} \int_0^h u_1^h dx_3 \cdot \int_0^h \frac{\partial u^h}{\partial x_2} |u|^2 \phi \, dx_3 \, dx \, dt.
\]
Similarly, by Lemma 2.6, Hölder’s inequality, (3.9), (3.10) and (3.12), we get
\[
\left| \int_0^T \int_{\Omega} \left( \int_0^h \frac{\partial u^h}{\partial x_2} \left| \frac{\partial u^h}{\partial x_2} \right|^2 \, dx_3 - \int_0^h \frac{\partial u^h}{\partial x_2} \cdot \int_0^h \left| \frac{\partial u^h}{\partial x_3} \right|^2 \, dx_3 \right) \, dx \, dt \right|
\leq \sqrt{h} \left( \int_0^T \int_{\Omega} \left| \frac{\partial u^h}{\partial x_2} \right|^2 \, dx \, dt \right)^{1/2} \sup_t \left( \int_{\Omega(h)} |u|^6 \, dx \right)^{1/2} \left[ \int_0^T \left( \int_{\Omega(h)} \left| \frac{\partial u^h}{\partial x_3} \right|^3 \, dx \right)^{1/2} \, dt \right]^{1/2} \rightarrow 0.
\]

Then
\[
\int_0^T \int_{\Omega(h)} \frac{\partial u^h}{\partial x_2} |u|^2 \, dx_3 \, dx \rightarrow \int_0^T \int_{\Omega} \int_0^h \frac{\partial u^h}{\partial x_2} \cdot \int_0^h |u|^2 \, dx_3 \, dx \, dt.
\]

Now, we prove
\[
\int_0^h \nabla v^h dx_3 \rightarrow \nabla v(x_1, x_2, 0, t) \text{ in } L^2(0, T; L^2(\Omega)).
\]
(4.15)

It follows from Hölder’s inequality and (3.9) that \(f_0^h \nabla v^h dx_3\) is uniformly bounded in \(L^2(0, T; L^2(\Omega))\). Then \(f_0^h \nabla v^h dx_3\) weakly converges in \(L^2(0, T; L^2(\Omega))\) by the weak compactness argument. From Proposition 4.2 in [28], we deduce \(f_0^h v^h dx_3 \rightarrow v(x_1, x_2, 0, t) \text{ in } L^2(0, T; L^2(\Omega))\). By the fact that \(L^2(\Omega \times (0, T)) \subset H^{-1}(\Omega \times (0, T))\), we infer that \(f_0^h \nabla v^h dx_3\) weakly converges in \(H^{-1}(\Omega \times (0, T))\). Then for any \(\psi \in H_0^1(\Omega \times (0, T))\), we have
\[
\int_0^T \int_{\Omega} \int_0^h \nabla v^h dx_3 \cdot \psi \, dx \, dt = - \int_0^T \int_{\Omega} f_0^h v^h dx_3 \cdot \nabla \psi \, dx \, dt
\]
\[
- \int_0^T \int_{\Omega} v(x_1, x_2, 0, t) \cdot \nabla \psi \, dx \, dt = \int_0^T \int_{\Omega} \nabla v(x_1, x_2, 0, t) \cdot \psi \, dx \, dt,
\]
thus (4.15) holds by the uniqueness of the limit. This combined with (4.12) yields that
\[
\int_0^h \frac{\partial v}{\partial x_2} |u|^2 \, dx_3 \rightarrow \frac{\partial v}{\partial x_2} |u|^2 \text{ in } L^2(0, T; L^2(\Omega)).
\]
It follows from (4.9) that
\[
\int_0^T \int_{\Omega(h)} u_h^1 \frac{\partial v}{\partial x_2} |u_h^1|^2 \phi dx dt \to \int_0^T \int_{\Omega} u_1 \frac{\partial v}{\partial x_2} |u_1|^2 \phi dx dt.
\]
Similarly,
\[
\int_0^T \int_{\Omega(h)} u_h^2 \frac{\partial v}{\partial x_1} |u_h^2|^2 \phi dx dt \to \int_0^T \int_{\Omega} u_2 \frac{\partial v}{\partial x_1} |u_2|^2 \phi dx dt.
\]
Therefore, (4.14) is proved by $|u|^2 = |u|^2$, then (4.7) holds. The conclusion of Step 2 is proved. Combining the conclusions of Step 1 and Step 2, the proof of Proposition 4.2 is completed.

Finally, we use Proposition 4.1 and Proposition 4.2 to derive the limit equation (3.16).

**Proof.** From Lemma 2.5 and Proposition 4.1, we have
\[
w_h^1 \rightharpoonup u \land u_t \text{ in } L^2(0,T; L^2(\Omega)).
\]
This combined with Proposition 4.2 yields that
\[
\int_0^T \int_{\Omega} w_1^1 \cdot \phi_t + \partial_t w_1^1 \cdot \phi dx dt \to \int_0^T \int_{\Omega} u \land u_t \cdot \phi + |u|^2 [\nabla'(u \land \nabla'u) - u \land \nabla'v] \cdot \phi dx dt
\]
\[
= \int_0^T \int_{\Omega} -u \land u_{tt} \cdot \phi + |u|^2 [\nabla'(u \land \nabla'u) - u \land \nabla'v] \cdot \phi dx dt
\]
\[
= -\int_0^T \int_{\Omega} u \land (u_{tt} - |u|^2 \Delta' u + |u|^2 \nabla' v) \cdot \phi dx dt.
\]
Since
\[
\int_0^T \int_{\Omega} w_1^1 \cdot \phi_t + \partial_t w_1^1 \cdot \phi dx dt = \int_0^T \int_{\Omega} w_1^1 \phi_t - w_1^1 \cdot \phi_t dx dt = 0,
\]
then
\[
\int_0^T \int_{\Omega} u \land (u_{tt} - |u|^2 \Delta' u + |u|^2 \nabla' v) \cdot \phi dx dt = 0.
\]
We get the desired results from Lemma 2.7. The proof of Theorem 3.1 is completed.

**Acknowledgements.** H. Wang’s research was supported by the National Natural Science Foundation of China (No. 11901066), the Natural Science Foundation of Chongqing (No. cstc2019jcyj-msxmX0167), projects Nos. 2022CDJXY-001 and 2020CDJQY-A040 supported by the Fundamental Research Funds for the Central Universities.

**Data availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
References

[1] Alouges, F., Soyeur, A.: On global weak solutions for Landau-Lifshitz equations: existence and nonuniqueness. Nonlinear Anal. Theory Methods Appl. 18(11), 1071–1084 (1992)
[2] Atkinson, J.L., Evans, R.F.L., Chantrell, R.W.: Micromagnetic modeling of the heat-assisted switching process in high anisotropy FePt granular thin films. J. Appl. Phys. 128(7), 073907 (2020)
[3] Capella, A., Melcher, C., Otto, F.: Wave-type dynamics in ferromagnetic thin films and the motion of Néel walls. Nonlinearity 20(11), 2519–2537 (2007)
[4] Carbou, G., Fabrie, P.: Regular solution for Landau-Lifshitz equations in a bounded domain. Differ. Integral Equ. 14(2), 213–229 (2001)
[5] Chmiel, P.: A survey on the numerics and computations for the Landau-Lifshitz equation of micromagnetism. Arch. Comp. Methods Eng. 15(3), 1–37 (2007)
[6] DeSimone, A., Kohn, R.V., Müller, S., Otto, F.: A reduced theory for thin-film micromagnetics. Commun. Pure Appl. Math. 55(11), 1408–1460 (2002)
[7] Evans, L.C.: Partial Differential Equations. American Mathematical Society, Providence (2010)
[8] Feischl, M., Tran, T.: Existence of regular solutions of the Landau–Lifshitz–Gilbert equation in 3D with natural boundary conditions. SIAM J. Math. Anal. 49(6), 4470–4490 (2017)
[9] Garanin, D.A.: Generalized equation of motion for a ferromagnet. Phys. A Stat. Mech. Its Appl. 172(3), 470–491 (1991)
[10] Garanin, D.A.: Fokker-Planck and Landau–Lifshitz–Bloch equations for classical ferromagnets. Phys. Rev. B 55(5), 3050–3057 (1997)
[11] García-Cervera, C.J.: Numerical micromagnetics: a review. SeMA J. Bol. Soc. Esp. Mat. Apl. 39, 103–136 (2007)
[12] García-Cervera, C.J., Carlos, J., Weinan, E.: Effective dynamics for ferromagnetic thin films. J. Appl. Phys. 90(1), 370–374 (2001)
[13] Gilbert, T.L.: A phenomenological theory of damping in ferromagnetic materials. Magn. IEEE Trans. Magn. 40(6), 3343–3449 (2004)
[14] Gioia, G., James, R.D.: Micromagnetics of very thin films. In: Proceedings of the Royal Society of London. Series A: Mathematical. Physical and Engineering Sciences. 453(1956), 213–223 (1997)
[15] Guo, B.L., Li, Q., Zeng, M.: Global smooth solutions of the Landau–Lifshitz–Gilbert equation. Preprint
[16] Guo, B.L., Han, Y.Q., Huang, D.W.: Weak and smooth global solution for Landau–Lifshitz–Bloch–Maxwell equation. Ann. Appl. Math. 36(1), 1–30 (2020)
[17] Hadda, M., Tiloua, M.: Thin film limits in magnetoelastic interactions. Math. Probl. Eng. 2012, 165962 (2012)
[18] Hinz, D., Nowak, U.: Domain wall motion by the magnonic spin Seebeck effect. Phys. Rev. Lett. 107(2), 027205 (2011)
[19] Jackson, J.D.: Classical Electrodynamics, 2nd edn. John Wiley and Sons Inc., New York (1975)
[20] Jia, Z.: Local strong solution to general Landau–Lifshitz–Bloch equation, arXiv:1802.00414 (2018)
[21] Kohn, R.V., Slastikov, V.V.: Another thin film limit in micromagnetics. Arch. Ration. Mech. Anal. 178(2), 227–245 (2005)
[22] Kruzik, M., Prohl, A.: Recent developments in the modeling, analysis, and numerics of ferromagnetism. SIAM Rev. 48(3), 439–483 (2006)
[23] Landau, L., Lifshitz, E.: On the theory of the dispersion of magnetic permeability in ferromagnetic bodies. Phys. Zeitsch. Sowjet. 8(2), 153–169 (1935)
[24] Le, K.N.: Weak solution of the Landau–Lifshitz–Bloch equation. J. Differ. Equ. 261(12), 6699–6717 (2016)
[25] Liu, X.G.: Elliptic Partial Differential Equation. Higher Education Press, Beijing (2015)
[26] McDaniel, T.W.: Application of Landau–Lifshitz–Bloch dynamics to grain switching in heat-assisted magnetic recording. J. Appl. Phys. 112(1), 013914 (2012)
[27] Moser, R.: Boundary vortices for thin ferromagnetic films. Arch. Ration. Mech. Anal. 174(2), 267–300 (2004)
[28] Peng, Y., Wang, H.Q.: Strong solutions of the Landau–Lifshitz–Bloch equation in Besov space, arXiv:2211.03056, (2022)
[29] Sultan, M., Atxitia, U., Melnikov, A.: Electron- and phonon-mediated ultrafast magnetization dynamics of Gd (0001). Phys. Rev. B 85(18), 184407 (2012)