A 4-choosable Graph that is Not (8 : 2)-choosable

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Abstract: In 1980, Erdős, Rubin and Taylor asked whether for all positive integers $a$, $b$, and $m$, every $(a : b)$-choosable graph is also $(am : bm)$-choosable. We provide a negative answer by exhibiting a 4-choosable graph that is not (8 : 2)-choosable.

Key words and phrases: $(am : bm)$-conjecture, $(a : b)$-choosable graph, list coloring.

Coloring the vertices of a graph with sets of colors (that is, each vertex is assigned a fixed-size subset of the colors such that adjacent vertices are assigned disjoint sets) is a fundamental notion, which in particular captures fractional colorings. The fractional chromatic number of a graph $G$ can indeed be defined to be the infimum (which actually is a minimum) of the ratios $a/b$ such that, if every vertex of $G$ is replaced by a clique of order $b$ and every edge of $G$ is replaced by a complete bipartite graph between the relevant cliques, then the chromatic number of the obtained graph is at most $a$.

In their seminal work on list coloring, Erdős, Rubin and Taylor [2] raised several intriguing questions about the list version of set coloring. Before stating them, let us review the relevant definitions.

Set coloring. A function that assigns a set to each vertex of a graph is a set coloring if the sets assigned to adjacent vertices are disjoint. For positive integers $a$ and $b \leq a$, an $(a : b)$-coloring of a graph $G$ is a set coloring with range $\binom{1 \ldots a}{b}$, i.e., a set coloring that to each vertex of $G$ assigns a $b$-element subset of $\{1, \ldots, a\}$. The concept of $(a : b)$-coloring is a generalization of the conventional vertex coloring. In fact, an $(a : 1)$-coloring is exactly an ordinary proper $a$-coloring.

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A list assignment for a graph $G$ is a function $L$ that to each vertex $v$ of $G$ assigns a set $L(v)$ of colors. A set coloring $\varphi$ of $G$ is an $L$-set coloring if $\varphi(v) \subseteq L(v)$ for every $v \in V(G)$. For a positive integer $b$, we say that $\varphi$ is an $(L:b)$-coloring of $G$ if $\varphi$ is an $L$-set coloring and $|\varphi(v)| = b$ for every $v \in V(G)$. If such an $(L:b)$-coloring exists, then $G$ is $(L:b)$-colorable. For an integer $a \geq b$, we say that $G$ is $(a:b)$-choosable if $G$ is $(L:b)$-colorable for every list assignment $L$ such that $|L(v)| = a$ for each $v \in V(G)$. We abbreviate $(L:1)$-coloring, $(L:1)$-colorable, and $(a:1)$-choosable to $L$-coloring, $L$-colorable, and $a$-choosable, respectively.

**Questions and results.** It is straightforward to see that if a graph is $(a:b)$-colorable, it is also $(am:bm)$-colorable for every positive integer $m$: we can simply replace every color in an $(a:b)$-coloring by $m$ new colors. However, this argument fails in the list coloring setting, leading Erdős, Rubin and Taylor [2] to ask whether every $(a:b)$-choosable graph is also $(am:bm)$-choosable whenever $m \geq 1$. A positive answer to this question is sometimes referred to as “the $(am:bm)$-conjecture”. Using the characterization of $2$-choosable graphs [2], Tuza and Voigt [4] provided a positive answer when $a = 2$ and $b = 1$. In the other direction, Gutner and Tarsi [3] demonstrated that if $k$ and $m$ are positive integers and $k$ is odd, then every $(2mk:mk)$-choosable graph is also $2m$-choosable.

Formulated differently, the question is to know whether every $(a:b)$-choosable graph is also $(c:d)$-choosable whenever $c/d = a/b$ and $c \geq a$. This formulation raises the same question when $c/d > a/b$, which was also asked by Erdős, Rubin and Taylor [2]. About ten years ago, Gutner and Tarsi [3] answered this last question negatively, by studying the $k$th choice number of a graph for large values of $k$. More precisely, the $k$th choice number of a graph $G$ is $\text{ch}_k(G)$, the least integer $a$ for which $G$ is $(a:k)$-choosable. Their result reads as follows.

**Theorem 1** (Gutner & Tarsi, 2009). *Let $G$ be a graph. For every positive real $\varepsilon$, there exists an integer $k_0$ such that $\text{ch}_k(G) \leq k(\chi(G) + \varepsilon)$ for every $k \geq k_0$.***

As a direct corollary, one deduces that for all integers $m \geq 3$ and $\ell > m$, there exists a graph that is $(a:b)$-choosable and not $(\ell:1)$-choosable with $\frac{a}{b} = m$. (To see this, one can for example apply Theorem 1 with $\varepsilon = 1$ to the disjoint union of a clique of order $m - 1$ and a complete balanced bipartite graph with choice number $\ell + 1$.)

Another related result that should be mentioned here was obtained by Alon, Tuza and Voigt [1]. They proved that for every graph $G$,

$$\inf \left\{ \frac{a}{b} \mid G \text{ is } (a:b)\text{-choosable} \right\} = \inf \left\{ \frac{a}{b} \mid G \text{ is } (a:b)\text{-colorable} \right\}. $$

In other words, the fractional choice number of a graph equals its fractional chromatic number.

The purpose of our work is to provide a negative answer to Erdős, Rubin and Taylor’s question when $a = 4$ and $b = 1$.

**Theorem 2.** There exists a graph $G$ that is $4$-choosable, but not $(8:2)$-choosable.

We build such a graph by incrementally combining pieces with certain properties. Each piece is defined, and its relevant properties established, in the forthcoming lemmas.
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**Gadgets and lemmas.** A gadget is a pair $(G, L_0)$, where $G$ is a graph and $L_0$ is an assignment of lists of even size. Given a gadget $(G, L_0)$, a half-list assignment for $G$ is a list assignment $L$ for $G$ such that $|L(v)| = |L_0(v)|/2$ for every $v \in V(G)$. Let us start the construction by a key observation on list colorings of 5-cycles.

**Lemma 3.** Consider the gadget $(C, L_0)$, presented in Figure 1, where $C = v_1 v_2 v_3 v_4 v_5$ is a 5-cycle, $L_0(v_1) = \{1,2,5,6\}$, $L_0(v_2) = \{1,4,5,6\}$, $L_0(v_3) = L_0(v_4) = \{3,4,5,6\}$ and $L_0(v_5) = \{2,4,5,6\}$. Then $C$ is $L$-colorable for every half-list assignment $L$ such that $|L(v_1) \cap L(v_3)| \leq 1$, but $C$ is not $(L_0 : 2)$-colorable.

**Proof.** The first statement is well known, but let us give the easy proof for completeness: since $|L(v_1) \cap L(v_3)| \leq 1$, we have $|L(v_1) \cup L(v_3)| \geq 3$, and thus $L(v_1)$ or $L(v_3)$ contains a color $c_6$ not belonging to $L(v_2)$. By symmetry, we can assume that $c_6 \in L(v_1)$. We color $v_1$ by $c_6$ and then for $i = 5, 4, 3, 2$ in order, we color $v_i$ by a color $c_i \in L(v_i) \setminus \{c_{i+1}\}$. The resulting coloring is proper—we have $c_2 \neq c_6$, since $c_6 \not\in L(v_2)$.

Suppose now that $C$ has an $(L_0 : 2)$-coloring, and for $c \in \{1, \ldots, 6\}$ let $V_c$ be the set of vertices of $C$ on which the color $c$ is used. Since two colors are used on each vertex of $C$, we have $\sum_{c=1}^{6} |V_c| = 10$. On the other hand, $V_c$ is an independent set of a 5-cycle, and thus $|V_c| \leq 2$ for every color $c$. Furthermore, color 1 only appears in the lists of $v_1$ and $v_2$, which are adjacent in $C$. It follows that $|V_1| \leq 1$. The situation is similar for color 2, which appears only in the lists of $v_1$ and $v_5$, and also for color 3, which only appears in the lists of $v_3$ and $v_4$. Consequently, $\sum_{c=1}^{6} |V_c| \leq 3 \cdot 2 + 3 \cdot 1 = 9$, which is a contradiction. 

**Corollary 4.** Consider the gadget $(G_1, L_1)$, presented in Figure 1, where $G_1$ consists of a 5-cycle $C = v_1 v_2 v_3 v_4 v_5$ and a path $v_1 x y v_3$, with $L_1(v_1) = L_1(v_3) = \{1, \ldots, 6\}$, $L_1(v_2) = \{1,4,5,6\}$, $L_1(v_4) = \{3,4,5,6\}$, $L_1(v_5) = \{2,4,5,6\}$, $L_1(x) = \{1,2,3,4\}$ and $L_1(y) = \{1,2\}$. Then $G_1$ is $L$-colorable for every half-list assignment $L$ such that $L(v_1) = L(v_3)$, but $G_1$ is not $(L_1 : 2)$-colorable.

**Proof.** Let $L$ be a half-list assignment for $G_1$. First $L$-color $y$ and $x$ by colors $c_y \in L(y)$ and $c_x \in L(x) \setminus \{c_y\}$, respectively. Since $c_x \neq c_y$ and $L(v_1) = L(v_3)$, there exist sets $L'(v_1) \subseteq L(v_1) \setminus \{c_x\}$ and $L'(v_3) \subseteq L(v_3) \setminus \{c_y\}$.
Consider the gadget Lemma 5. Let \( L'(v_i) = L(v_i) \) for \( i \in \{2, 4, 5\} \). Lemma 3 implies that \( C \) is \( L' \)-colorable, which yields an \( L \)-coloring of \( G \).

In an \( (L_1 : 2) \)-coloring, the vertex \( y \) would have to be assigned \( \{1, 2\} \) and \( x \) would have to be assigned \( \{3, 4\} \), and thus the sets of available colors for \( v_1 \) and for \( v_3 \) would have to be \( \{1, 2, 5, 6\} \) and \( \{3, 4, 5, 6\} \), respectively. However, no such \( (L_1 : 2) \) coloring of \( C \) exists according to Lemma 3.

Next we construct auxiliary gadgets, which will be combined with the gadget from Corollary 4 to deal with the case where \( L(v_1) \neq L(v_3) \). Let \( G \) be a graph, let \( S \) be a subset of vertices of \( G \) and \( L \) a list assignment for \( G \). An \( L \)-coloring of \( S \) is a coloring of the subgraph of \( G \) induced by \( S \). Moreover, if \( S' \) is a subset of vertices of \( G \) that contains \( S \) and \( \phi' \) is an \( L \)-coloring of \( S' \), then \( \phi' \) extends \( \phi \) if \( \phi'|S = \phi \). Let \( (G, L_0) \) be a gadget, let \( v_1 \) and \( v_3 \) be distinct vertices of \( G \), and let \( S \) be a set of vertices of \( G \) not containing \( v_1 \) and \( v_3 \). The gadget is \((v_1, v_3, S)-relaxed\) if every half-list assignment \( L \) satisfies at least one of the following conditions.

(i) There exists an \( L \)-coloring \( \psi_0 \) of \( \{v_1, v_3\} \) such that every \( L \)-coloring of \( S \cup \{v_1, v_3\} \) extending \( \psi_0 \) extends to an \( L \)-coloring of \( G \).

(ii) \( L(v_1) = L(v_3) \) and there exists an \( L \)-coloring \( \psi_0 \) of \( S \) such that every \( L \)-coloring of \( S \cup \{v_1, v_3\} \) extending \( \psi_0 \) extends to an \( L \)-coloring of \( G \).

![Figure 2: The gadget \((G_2, L_2)\) of Lemma 5](image)

**Lemma 5.** Consider the gadget \((G_2, L_2)\) presented in Figure 2, where \( G_2 \) consists of a 5-cycle \( C_2 = v_1 u_2 v_3 u_4 u_5 \), a vertex \( y_1 \) adjacent to all vertices of \( C_2 \), a triangle \( y_2 y_3 y_4 \), and an edge \( y_1 y_2 \), with \( L_2(v) = \{1, \ldots, 6\} \) for every \( v \in V(C_2) \), \( L_2(y_1) = \{1, \ldots, 8\} \), \( L_2(y_2) = L_2(y_4) = \{1, 2, 3, 4, 7, 8\} \), and \( L_2(y_3) = \{1, 2, 3, 4\} \). The gadget is \((v_1, v_3, \{y_4\})\)-relaxed, and \( \phi(y_4) \cap \{7, 8\} \neq \emptyset \) for every \((L_2 : 2)\)-coloring \( \phi \) of \( G_2 \).

**Proof.** Let \( L \) be a half-list assignment for \( G_2 \). If not all vertices of \( C_2 \) have the same list, then choose a color \( c \in L(y_1) \setminus L(y_2) \), and observe there exists an \( L \)-coloring of \( G_2[V(C_2) \cup \{y_1\}] \) such that \( y_1 \) has
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color $c$. Let $\psi_0$ be the restriction of this coloring to $\{v_1, v_3\}$. Clearly, every $L$-coloring of $\{v_1, v_3, y_4\}$ extending $\psi_0$ extends to an $L$-coloring of $G_2$, and thus (i) holds.

If all the vertices of $C_2$ have the same list (and hence in particular $L(v_1) = L(v_3)$), then let $c$ be a color in $L(y_1) \setminus L(v_1)$. Observe that there exists an $L$-coloring of $G_2 \setminus \{y_1, y_2, y_3, y_4\}$ such that $y_1$ has color $c$. Let $\psi_0$ be the restriction of this coloring to $y_4$. Again, every $L$-coloring of $\{v_1, v_3, y_4\}$ extending $\psi_0$ extends to an $L$-coloring of $G_2$, and thus (ii) holds.

It remains to show that if $\phi$ is an $(L_2 : 2)$-coloring of $G_2$ then $\phi(y_4) \cap \{7, 8\} \neq \emptyset$. Suppose, on the contrary, that $\phi(y_4) \cap \{7, 8\} = \emptyset$. It follows that $\phi(y_4) \cup \phi(y_3) = \{1, 2, 3, 4\}$, and hence $\phi(y_2) = \{7, 8\}$. As a result, $\phi(y_1) \subseteq \{1, \ldots, 6\}$ and, by symmetry, we can assume that $\phi(y_1) = \{5, 6\}$. This implies that $\phi(v) \subseteq \{1, 2, 3, 4\}$ for each $v \in V(C_2)$. In particular, $\phi(V(C_2))$ is a $(4 : 2)$-coloring of $C_2$, which is a contradiction since the 5-cycle $C_2$ has fractional chromatic number $5/2$.

Figure 3: The graph $G_3$ from Lemma 6 is depicted on the left; the gadget $(G_3, L_3)$ is obtained from $(G_2, L_2)$ by adding to $G_1$ a disjoint copy of the graph depicted on the right (with the corresponding lists) for each $i \in \{1, 2\}$, and joining $y_4$ to each of $z_{1, 1}, z_{1, 2}, z_{2, 1}$ and $z_{2, 2}$.

Lemma 6. Consider the gadget $(G_3, L_3)$, obtained from the gadget $(G_2, L_2)$ of Lemma 5 as follows (see Figure 3 for an illustration of $G_3$). The graph $G_3$ consists of $G_2$ and for $i \in \{1, 2\}$, the vertices $z_{i, 1}, \ldots, z_{i, 7}$; the edges $y_4z_{i, 1}$ and $y_4z_{i, 2}$; the edge $z_{j, 3}z_{i, k}$ for every $j$ and every $k$ such that $1 \leq j < k \leq 4$ and $(j, k) \neq (1, 2)$; the edges of the triangle $z_{1, 5}z_{1, 6}z_{1, 7}$ and the edge $z_{1, 4}z_{1, 5}$. Let $L_3(v) = L_2(v)$ for $v \in V(G_2)$, and for $i \in \{1, 2\}$ let $L_3(z_{i, 1}) = \{1, 2, 3, 6 + i\}$, $L_3(z_{i, 2}) = \{4, 5, 6, 6 + i\}$, $L_3(z_{i, 3}) = \{1, \ldots, 6\}$, $L_3(z_{i, 4}) = \{1, \ldots, 8\}$, $L_3(z_{i, 5}) = L_3(z_{i, 7}) = \{1, 2, 3, 4, 7, 8\}$, and $L_3(z_{i, 6}) = \{1, 2, 3, 4\}$. The gadget $(G_3, L_3)$ is $(v_1, v_3, \{z_{1, 7}, z_{2, 7}\})$-relaxed, and $\phi(z_{1, 7}) = \{7, 8\}$ or $\phi(z_{2, 7}) = \{7, 8\}$ for every $(L_3 : 2)$-coloring $\phi$ of $G_3$.

Proof. Let $L$ be a half-list assignment for $G_3$. The gadget $(G_2, L_3[V(G_2))]$ is $(v_1, v_3, \{y_4\})$-relaxed by Lemma 5. Suppose first that (i) holds for the restriction of $L$ to $G_2$ (with $S = \{y_4\}$), and let $\psi_0$ be the corresponding $L$-coloring of $\{v_1, v_3\}$. For $i \in \{1, 2\}$, if $L(z_{i, 1}) \cap L(z_{i, 2}) \neq \emptyset$, then let $c_i$ be a color in $L(z_{i, 1}) \cap L(z_{i, 2})$. Otherwise, $|L(z_{i, 1}) \cup L(z_{i, 2})| = 4 > |L(z_{i, 3})|$, and thus we can choose a color $c_i \in (L(z_{i, 1}) \cup L(z_{i, 2})) \setminus L(z_{i, 3})$. Let $c$ be a color in $L(y_4) \setminus \{c_1, c_2\}$. By (i) for $G_2$, we know that $\psi_0$ extends to an $L$-coloring $\psi$ of $G_2$ such that $\psi(y_4) = c$. If $L(z_{i, 1}) \cap L(z_{i, 2}) \neq \emptyset$, then color both $z_{i, 1}$ and $z_{i, 2}$ by $c_i$, otherwise color one of them by $c_i$ and the other one by an arbitrary color from its list that is different
from \( c \). There are at least two colors in \( L(z_{i,4}) \) distinct from the colors of \( z_{i,1} \) and \( z_{i,2} \), choose such a color \( c'_i \) so that \( L(z_{i,5}) \setminus \{ c'_i \} \neq L(z_{i,6}) \). Color \( z_{i,4} \) by \( c'_i \) and extend the coloring to \( z_{i,3} \), which is possible by the choice of \( c_i \). Observe that any \( L \)-coloring of \( z_{i,7} \) extends to an \( L \)-coloring of the triangle \( z_{i,5}z_{i,6}z_{i,7} \) where the color of \( z_{i,5} \) is not \( c'_i \). We conclude that \((G_3,L_3)\) with the half-list assignment \( L \) satisfies (i).

Suppose next that (ii) holds for the restriction of \( L \) to \( G_2 \) (with \( S = \{ \{y_4\} \} \), and let \( \psi'_0 \) be the corresponding \( L \)-coloring of \( y_4 \). For \( i \in \{ 1, 2 \} \), greedily extend \( \psi'_0 \) to an \( L \)-coloring of \( z_{i,1}, \ldots, z_{i,7} \) in order, and let \( \psi_0 \) be the restriction of the resulting coloring to \( \{ z_{1,7}, z_{2,7} \} \). Observe that \((G_3,L_3)\) with the half-list assignment \( L \) satisfies (ii).

Finally, let \( \varphi \) be an \((L_3 : 2)\)-coloring of \( G_3 \). Lemma 5 implies that \( \varphi(\{y_4\}) \cap \{7, 8\} \neq \emptyset \). By symmetry, we can assume that \( 7 \in \varphi(\{y_4\}) \). It follows that \( \varphi(\{z_{1,1}\}) \subset \{1, 2, 3\} \) and \( \varphi(\{z_{1,2}\}) \subset \{4, 5, 6\} \), and thus \( \varphi(\{z_{1,1}\}) \cup \varphi(\{z_{1,2}\}) \cup \varphi(\{z_{1,3}\}) \subset \{1, \ldots, 6\} \). Consequently, \( \varphi(\{z_{1,4}\}) = \{7, 8\} \), and \( \varphi(\{z_{1,5}\}) \) is a subset of \( \{1, 2, 3, 4\} \). This yields that \( \varphi(\{z_{1,5}\}) \cup \varphi(\{z_{1,6}\}) = \{1, 2, 3, 4\} \), and therefore \( \varphi(\{z_{1,7}\}) = \{7, 8\} \).

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Figure 4: The graph \( G_4 \) from Lemma 7: the filled vertices are those added to \( G_3 \) to form \( G_4 \); the red ones have list \( \{1, 2, 3, 4, 7, 8\} \) while the black ones have list \( \{1, 2, 3, 4\} \).

**Lemma 7.** Consider the gadget \((G_4,L_4)\) obtained from the gadget \((G_3,L_3)\) of Lemma 6 as follows (see Figure 4 for an illustration of \( G_4 \)). The graph \( G_4 \) consists of \( G_3 \): the three triangles \( w_1w_2w_3 \) and \( w_1w_2w_3 \) for \( i \in \{1, 2\} \), and the edges \( z_{1,1}w_1, z_{2,1}w_1, w_3w_{1,1} \) and \( w_3w_{2,1} \). Let \( L_4(v) = L_3(v) \) for \( v \in V(G_3) \), and for \( i \in \{1, 2\} \), let \( L_4(w_1) = L_4(w_3) = L_4(w_{1,1}) = L_4(w_{1,3}) = \{1, 2, 3, 4, 7, 8\} \) and \( L_4(w_2) = L_4(w_{2,1}) = \{1, 2, 3, 4\} \). The gadget is \((v_1,v_3,\{v_{1,3},v_{2,3}\})\)-relaxed, and \( \varphi(w_{1,3}) = \varphi(w_{2,3}) = \{7,8\} \) for every \((L_4 : 2)\)-coloring \( \varphi \) of \( G_4 \).

**Proof.** Let \( L \) be a half-list assignment for \( G_4 \). The gadget \((G_3,L_4|V(G_3))\) is \((v_1,v_3,\{z_{1,7},z_{2,7}\})\)-relaxed by Lemma 6. Suppose first that (i) holds for the restriction of \( L \) to \( G_3 \) (with \( S = \{z_{1,7},z_{2,7}\} \), and
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Let $\psi_0$ be the corresponding $L$-coloring of $\{v_1, v_2\}$. Choose a color $c_1 \in L(z_{1,7})$ so that $L(w_1) \setminus \{c_1\} \neq L(w_2)$. If $c_1 \in L(w_1)$, then choose $c_2 \in L(z_{2,7}) \setminus (L(w_1) \setminus \{c_1\})$, otherwise choose $c_2 \in L(z_{2,7})$ so that $L(w_1) \setminus \{c_2\} \neq L(w_2)$. Choose a color $c_3 \in L(w_3)$ so that $L(w_{i,1}) \setminus \{c_3\} \neq L(w_{i,2})$ for $i \in \{1, 2\}$. By (i) for $G_3$, there exists an $L$-coloring of $G_3$ extending $\psi_0$ and assigning $c_i$ to $z_{i,7}$ for each $i \in \{1, 2\}$. Color $w_3$ by $c_3$ and observe that the $L$-coloring can be extended to $w_1$ and $w_2$ thanks to the choice of $c_1$ and $c_2$. Moreover, the choice of $c_3$ ensures that for each $i \in \{1, 2\}$, we can color $w_{i,3}$ with any color in $L(w_{i,3})$ and further extend the coloring to $w_{i,1}$ and $w_{i,2}$. We conclude that $(G_4, L_4)$ with the half-list assignment $L$ satisfies (i).

Suppose next that (ii) holds for the restriction of $L$ to $G_4$ (with $S = \{z_{1,7}, z_{2,7}\}$), and let $\psi_0'$ be the corresponding $L$-coloring of $\{z_{1,7}, z_{2,7}\}$. Greedily extend $\psi_0'$ to an $L$-coloring of $w_1, w_2, w_3$, and $w_{i,1}, w_{i,2}, w_{i,3}$ for $i \in \{1, 2\}$ in order, and let $\psi_0$ be the restriction of the resulting coloring to $\{w_{1,3}, w_{2,3}\}$. Observe that $(G_4, L_4)$ with the half-list assignment $L$ satisfies (ii).

Finally, let $\varphi$ be an $(L_4 : 2)$-coloring of $G_4$. By Lemma 6 and by symmetry, we can assume that $\varphi(z_{1,7}) = \{7, 8\}$. Consequently, $\varphi(w_1) \subseteq \{1, 2, 3, 4\}$, and thus $\varphi(w_1) \cup \varphi(w_2) = \{1, 2, 3, 4\}$, which yields that $\varphi(w_3) = \{7, 8\}$. We conclude analogously that $\varphi(w_{1,3}) = \{7, 8\} = \varphi(w_{2,3})$.

Figure 5: The graph $G_5$ from Lemma 8.
We can now combine \((G_1, L_1)\) with \((G_4, L_4)\) to obtain a gadget \((G_5, L_5)\) that is \(L\)-colorable from each half-list assignment \(L\), but not \((L_5 : 2)\)-colorable.

**Lemma 8.** Consider the gadget \((G_5, L_5)\) obtained from the gadgets \((G_1, L_1)\) of Corollary 4 and \((G_4, L_4)\) of Lemma 7 as follows (see Figure 5 for an illustration of \(G_5\)). The graph \(G_5\) is obtained from the union of the graphs \(G_1\) and \(G_4\) (intersecting in \(\{v_1, v_3\}\)) by adding the edges \(w_{1,3}v_2, w_{1,3}v_4, w_{2,3}x\) and \(w_{2,3}y\).

Let \(L_5(v) = L_4(v)\) for \(v \in V(G_4)\), \(L_5(v) = L_1(v)\) for \(v \in V(G_1) \setminus \{v_2, v_4, x, y\}\), and \(L_5(v) = L_4(v) \cup \{7, 8\}\) for \(v \in \{v_2, v_4, x, y\}\). Then \(G_5\) is \(L\)-colorable for every half-list assignment \(L\), but not \((L_5 : 2)\)-colorable.

**Proof.** Let \(L\) be a half-list assignment for \(G_5\). The gadget \((G_4, L_5(V(G_4)))\) is \((v_1, v_3, \{w_{1,3}, w_{2,3}\})\)-relaxed by Lemma 7. Suppose first that (i) holds for the restriction of \(L\) to \(G_4\) (with \(S = \{w_{1,3}, w_{2,3}\}\)), and let \(\psi_0\) be the corresponding \(L\)-coloring of \(\{v_1, v_3\}\). Greedily extend \(\psi_0\) to an \(L\)-coloring \(\psi\) of \(G_1\). Choose \(c_1 \in L(w_{1,3}) \setminus \{\psi(v_2), \psi(v_4)\}\) and \(c_2 \in L(w_{2,3}) \setminus \{\psi(x), \psi(y)\}\). By (i) for \(G_4\), there exists an \(L\)-coloring of \(G_4\) that extends \(\psi_0\) and assigns to \(w_{1,3}\) the color \(c_i\) for each \(i \in \{1, 2\}\). This yields, together with \(\psi\), an \(L\)-coloring of \(G_5\).

Suppose next that (ii) holds for the restriction of \(L\) to \(G_4\) (with \(S = \{w_{1,3}, w_{2,3}\}\)), and let \(\psi_0\) be the corresponding \(L\)-coloring of \(\{w_{1,3}, w_{2,3}\}\). Note that \(L(v_1) = L(v_3)\) in this case. Corollary 4 implies that \(G_1\) has an \(L\)-coloring \(\psi\) such that \(\psi(v_2) \in L(v_2) \setminus \{\psi_0(w_{1,3})\}\), \(\psi(v_4) \in L(v_4) \setminus \{\psi_0(w_{1,3})\}\), \(\psi(x) \in L(x) \setminus \{\psi_0(w_{2,3})\}\), and \(\psi(y) \in L(y) \setminus \{\psi_0(w_{2,3})\}\). By (ii), the restriction of \(\psi \cup \psi_0\) to \(\{v_1, v_3, w_{1,3}, w_{2,3}\}\) extends to an \(L\)-coloring of \(G_4\), which together with \(\psi\) gives an \(L\)-coloring of \(G_5\).

It remains to show that \(G_5\) is not \((L_5 : 2)\)-colorable. If \(\varphi\) were an \((L_5 : 2)\)-coloring of \(G_5\), then by Lemma 7 we would have \(\varphi(w_{1,3}) = \varphi(w_{2,3}) = \{7, 8\}\), and thus the restriction of \(\varphi\) to \(G_1\) would be an \((L_1 : 2)\)-coloring, thereby contradicting Corollary 4.

The final graph. We are now in a position to prove Theorem 2 by simply using a standard construction to ensure uniform lengths of lists.

**Proof of Theorem 2.** Let \(G\) be a graph and \(L'\) an assignment of lists of size 8 obtained as follows. Let \(K\) be a clique with vertices \(r_1, \ldots, r_4\), and let \(L'(r_1) = \cdots = L'(r_4) = \{9, \ldots, 16\}\). For every \((L' : 2)\)-coloring \(\psi\) of \(K\), let \(G_\psi\) be a copy of the graph \(G_3\) from the gadget \((G_5, L_5)\) of Lemma 8, and for each vertex \(v \in V(G_\psi)\) such that \(|L_\psi(v)| = 2k\) with \(k \in \{2, 3\}\), we add the edges \(vr_{1\ldots k}v_{r_{k+1}}\ldots v_{r_4}\) and let \(L'(v) = L_\psi(v) \cup \bigcup_{i=1}^{k-1} \psi(r_i)\). If \(G_\psi\) had an \((L' : 2)\)-coloring \(\varphi\), then letting \(\psi\) be the restriction of \(\varphi\) to \(K\), the restriction of \(\varphi\) to \(G_\psi\) would be an \((L_5 : 2)\)-coloring of \(G_5\), thereby contradicting Lemma 8.

Consider now a list assignment \(L\) for \(G\) such that \(|L(v)| = 4\) for every \(v \in V(G)\). Let \(\varphi\) be any \(L\)-coloring of \(K\). For each \((L' : 2)\)-coloring \(\psi\) of \(K\), let \(L_\psi\) be the list assignment for \(G_\psi\) obtained by, for each \(v \in V(G_\psi)\), removing the colors of neighbors in \(K\) according to \(\varphi\), and possibly removing further colors to ensure that \(|L_\psi(v)| = |L_\psi(v)|/2\). By Lemma 8, the graph \(G_\psi\) has an \(L_\psi\)-coloring. The union of these colorings and \(\varphi\) yields an \(L\)-coloring of \(G\).

**Concluding remarks.** It follows from Theorem 2 that for each integer \(a \geq 4\), there exists a graph \(G_a\) that is \(a\)-choosable but not \((2a : 2)\)-choosable—if we have such a graph \(G_a\), taking the disjoint union of \(\binom{2a+1}{2}\) copies of \(G_a\) and adding a vertex adjacent to all other vertices yields \(G_{a+1}\), by an argument analogous to the list uniformization procedure used for the proof of Theorem 2. It is natural to ask...
whether there exists a graph that is 3-choosable but not \((6 : 2)\)-choosable. We believe this to be the case; in particular, Corollary 4 only requires lists of size at most 6. However, it does not seem easy to construct a gadget that satisfies the properties stated in Lemma 5 without using a vertex with a list of size 8.

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