THE LEFSCHETZ PROPERTY FOR COMPONENTWISE LINEAR IDEALS
AND GOTZMANN IDEALS

ATTILA WIEBE

Abstract. For standard graded Artinian $K$-algebras defined by componentwise linear
ideals and Gotzmann ideals, we give conditions for the weak Lefschetz property in terms
of numerical invariants of the defining ideals.

1. Introduction

In his paper [S1], R. Stanley proved that the $f$-vector of a simplicial convex polytope
satisfies McMullen’s $g$-condition. The decisive argument in his proof is based on the fact
that the cohomology rings of certain projective $\mathbb{C}$-varieties possess the weak Lefschetz
property (see Section 2 for the definition of the weak and the strong Lefschetz property).

Initiated by this work, the following general question arose: Under which conditions
does a standard graded Artinian $K$-algebra $A$ admit the weak (strong) Lefschetz property?
During the last twenty years, this question has been studied by several authors (see e.g.
[B, H, H-M, N, W1, W2]).

In this paper, we consider an Artinian $K$-algebra $A = S/I$, where $I$ is a component-
wise linear ideal (resp. a Gotzmann ideal) in $S = K[x_1, \ldots, x_n]$. In the case that $I$ is com-
ponentwise linear, we give a necessary and sufficient condition for $S/I$ to have the weak
Lefschetz property in terms of the graded Betti numbers of $I$. Under the stronger assump-
tion that $I$ is even a Gotzmann ideal, we give a necessary and sufficient condition for $S/I$
to have the weak Lefschetz property in terms of the Hilbert function of $I$.

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2. Preparations

We fix the following notation: let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over an
infinite field $K$. The maximal ideal $(x_1, \ldots, x_n)$ will be denoted by $m$.

Let $I \subset S$ be an $m$-primary graded ideal and set $A = S/I$. One says that $A$ has the weak
Lefschetz property, if there is a linear form $l \in A_1$ which satisfies the following condition:
The multiplication map $A_i \to A_{i+1}, f \mapsto lf$, has maximal rank (that means, is injective or
surjective) for all $i \in \mathbb{N}$. Such an element $l$ is called a weak Lefschetz element on $A$.
If there exists an element $l \in A_1$ such that the multiplication map $A_i \to A_{i+k}, f \mapsto l^k f$, has
maximal rank for all $i \in \mathbb{N}$ and all $k \geq 1$, one says that $A$ has the strong Lefschetz property
and calls $l$ a strong Lefschetz element on $A$. It is easy to show that the set of all weak
Lefschetz elements on $A$ is a Zariski-open (but maybe empty) subset of the affine space $A_1$. The same holds for the set of all strong Lefschetz elements on $A$.

We sometimes abuse language and say that $I$ has the weak (resp. strong) Lefschetz property in order to express that $S/I$ has the weak (resp. strong) Lefschetz property.

For a monomial $u \in S$ we define $m(u) = \max \{i \mid x_i \text{ divides } u \}$. If $I \subset S$ is a monomial ideal, the minimal monomial generating set of $I$ will be denoted by $G(I)$. For $j \in \mathbb{N}$ we set $G(I)_j = \{u \in G(I) \mid \deg(u) = j\}$. One says that $I$ is stable (resp. strongly stable) if the following condition holds: For every monomial $u \in I$ we have $(x_i/x_{m(u)})u \in I$ for $i = 1, \ldots, m(u)$ (resp. for every monomial $u \in I$ and each variable $x_k$ that divides $u$, we have $(x_i/x_k)u \in I$ for $i = 1, \ldots, k$). In order to show that $I$ is (strongly) stable, it suffices to verify the condition above for every $u \in G(I)$. The ideal $(x^2, xy, y^2, yz) \subset K[x, y, z]$ is an example of a stable ideal which is not strongly stable.

Eliahou and Kervaire give in [EK] a formula for the graded Betti numbers of a stable ideal in terms of the monomial generators:

**Theorem 2.1.** Let $I \subset S$ be a stable ideal. Then $\beta_{i,i+j}(I) = \sum_{u \in G(I)_j} (m(u) - 1)^i$ for all $i, j \in \mathbb{N}$.

We consider the natural left action of $GL_n(K)$ on $S$. A monomial ideal $I \subset S$ is called Borel-fixed if $gL = I$ for all $g \in B$, where $B \subset GL_n(K)$ is the Borel group consisting of all invertible upper triangular matrices. It is easy to see that strongly stable ideals are Borel-fixed. In characteristic zero both notions coincide (for a proof see e.g. Section 15.9 of [E]):

**Proposition 2.2.** Assume that $\text{char}(K) = 0$. A monomial ideal $I \subset S$ is strongly stable if and only if it is Borel-fixed.

In general, a Borel-fixed ideal (even if it is stable) is not strongly stable:

**Example 2.3.** Assume that $\text{char}(K) = p > 0$. Let $S = K[x_1, x_2, x_3]$ and let

$$M = \left\{x_1^{v_1}x_2^{v_2}x_3^{v_3} \mid v_1 + v_2 + v_3 = 2p, v_3 < p \right\}.$$  

The ideal $I = (M, x_1^p, x_2^p, x_3^p) \subset S$ is stable and Borel-fixed, but it is not strongly stable.

The following theorem was proved by Galligo in characteristic zero and by Bayer and Stillman in arbitrary characteristic. A proof can be found in [E].

**Theorem 2.4.** Let $I \subset S$ be a graded ideal and let $J$ be the generic initial ideal of $I$ with respect to the reverse lexicographic order. Then $J$ is Borel-fixed.

We recall that a graded ideal $I \subset S$ is said to be componentwise linear if $I_{(j)}$ has a linear resolution for all $j \in \mathbb{N}$. Here $I_{(j)}$ denotes the ideal generated by the elements of $I_j$. Note that for a stable ideal $I$, the ideals $I_{(j)}, j \in \mathbb{N}$, are also stable. Hence Theorem 2.1 shows that a stable ideal is componentwise linear.
In positive characteristic, the generic initial ideal of an ideal need not be stable. For example, take $I = (x^p, y^p) \subseteq K[x, y]$, where char$(K) = p$. The generic initial ideal of $I$ (with respect to the reverse lexicographic order) is $I$ itself, but $I$ is not stable. However, for componentwise linear ideals we have (compare Lemma 1.4 of [CHH]):

**Proposition 2.5.** Let $I \subset S$ be a componentwise linear ideal and let $J$ be the generic initial ideal of $I$ with respect to the reverse lexicographic order. Then $J$ is a stable ideal.

We quote another fact about componentwise linear ideals (see Theorem 1.1 of [AHH]):

**Theorem 2.6.** Let $I \subset S$ be a graded ideal and let $J$ be the generic initial ideal of $I$ with respect to the reverse lexicographic order. If $I$ is componentwise linear, then $\beta_{ij}(I) = \beta_{ij}(J)$ for all $i, j \in \mathbb{N}$. The converse holds if char$(K) = 0$.

The following lemma is simple, but crucial for the whole paper.

**Lemma 2.7.** Let $I \subset S$ be an $m$-primary monomial ideal. If $I$ is stable or Borel-fixed, then the following conditions are equivalent:

(a) $S/I$ has the weak (resp. strong) Lefschetz property.

(b) $x_n$ is a weak (resp. strong) Lefschetz element on $S/I$.

**Proof.** Note that for a linear form $l \in S_1$ we have

$$c_l(j, k) := \dim_K(I : S l^k)_j \geq \alpha(j, k) := \max\{ \dim_K I_j, \dim_K S_j - \dim_K(S/I)_{j+k} \}$$

for all $j \in \mathbb{N}$ and all $k \geq 1$. It is clear that $l$ is a weak (resp. strong) Lefschetz element on $S/I$ if and only if $c_l(j, 1) = \alpha(j, 1)$ for all $j \in \mathbb{N}$ (resp. $c_l(j, k) = \alpha(j, k)$ for all $j \in \mathbb{N}$ and all $k \geq 1$). If $I$ is stable, then $(I : S x_n^k) \subseteq (I : S l^k)$ for every $l \in S_1$ and all $k \geq 1$. This implies $c_{x_n}(j, k) \leq c_l(j, k)$ for all $j \in \mathbb{N}$ and all $k \geq 1$, and hence we are done.

Now we assume that $I$ is Borel-fixed. If $S/I$ has the weak (resp. strong) Lefschetz property, then the open set $U \subseteq S_1$ that consists of all weak (resp. strong) Lefschetz elements on $S/I$ is nonempty, and thus it has a nonempty intersection with the open set $Bx_n = \{ \sum^n_{i=1} a_i x_i \mid a_n \neq 0 \}$. Choose $g \in B$ with $g^{-1}(x_n) \in U$. Since $g^{-1}(x_n)$ is a weak (resp. strong) Lefschetz element on $S/I$, we conclude that $x_n$ is a weak (resp. strong) Lefschetz element on $S/gI = S/I$.

The following two statements are essentially consequences of two results in Conca’s paper [C].

**Proposition 2.8.** Let $I \subset S$ be an $m$-primary graded ideal. If $J$ is the generic initial ideal of $I$ with respect to the reverse lexicographic order, then $S/I$ has the weak (resp. strong) Lefschetz property if and only if $S/J$ has the weak (resp. strong) Lefschetz property.

**Proof.** Note that for a linear form $l \in S_1$ we have

$$d_l(j, k) := \dim_K(S/(I, l^k))_{j+k} \geq \gamma(j, k) := \max\{ 0, \dim_K(S/I)_{j+k} - \dim_K(S/I)_{j} \}$$
for all \( j \in \mathbb{N} \) and all \( k \geq 1 \). It is clear that \( l \) is a weak (resp. strong) Lefschetz element on \( S/I \) if and only if \( d_l(j, 1) = \gamma(j, 1) \) for all \( j \in \mathbb{N} \) (resp. \( d_l(j, k) = \gamma(j, k) \) for all \( j \in \mathbb{N} \) and all \( k \geq 1 \)).

Conca proves in [C] Lemma 1.2] that the Hilbert function of \( S/(J, x_n) \) is equal to the Hilbert function of \( S/(I, l) \) for a general linear form \( l \in S_1 \). Together with Theorem 2.4 and Lemma 2.7, this yields the assertion about the weak Lefschetz property.

By slightly generalizing the arguments of Conca’s proof (one has to use the fact that \( \text{in}_{\text{revlex}}(gI + (x_n^k)) = \text{in}_{\text{revlex}}(gI) + (x_n^k) \) for all \( k \geq 1 \)), one obtains that the Hilbert function of \( S/(J, x_n^k) \) is equal to the Hilbert function of \( S/(I, l^k) \) for a general linear form \( l \in S_1 \) and all \( k \geq 1 \). This yields the assertion about the strong Lefschetz property. \( \square \)

In general, an ideal inherits the Lefschetz property from its initial ideal (with respect to any term order):

**Proposition 2.9.** Let \( I \subset S \) be an \( m \)-primary graded ideal and let \( J \) be the initial ideal of \( I \) with respect to a term order \( \tau \). If \( S/J \) has the weak (resp. strong) Lefschetz property, then the same holds for \( S/I \).

**Proof.** Conca proves in [C] Theorem 1.1] that \( \dim_K(S/(I, l))_j \leq \dim_K(S/(J, l))_j \) for a general linear form \( l \in S_1 \) and all \( j \in \mathbb{N} \). This yields the assertion concerning the weak Lefschetz property (compare the proof of Proposition 2.8). Using virtually the same arguments as in Conca’s proof, one can show that \( \dim_K(S/(I, l^k))_j \leq \dim_K(S/(J, l^k))_j \) for all \( j \in \mathbb{N} \) and all \( k \geq 1 \), where \( l \) is a general linear form. This proves the assertion concerning the strong Lefschetz property. \( \square \)

We close this section by giving an example which shows that the Lefschetz property may depend on the characteristic.

**Example 2.10.** Let \( S = K[x, y, z] \) and \( I = (x^2, y^2, z^2) \subset S \). The Hilbert function of \( S/I \) is \( 1 + 3t + 3t^2 + t^3 \). Let \( l = ax + by + cz \in S_1 \) (with \( a, b, c \in K \)) be a linear form. The determinant of a matrix that represents the multiplication map \( (S/I)_1 \to (S/I)_2, f \mapsto lf \), is (up to a nonzero scalar) equal to \( 2abc \). Therefore \( S/I \) does not have the weak Lefschetz property in case \( \text{char}(K) = 2 \). If \( \text{char}(K) \neq 2 \), then \( l \) is even a strong Lefschetz element on \( S/I \), provided \( abc \neq 0 \).

### 3. Componentwise Linear Ideals

For an \( m \)-primary componentwise linear ideal \( I \), we give a necessary and sufficient condition for \( I \) to have the weak Lefschetz property in terms of the graded Betti numbers of \( I \):

**Theorem 3.1.** Let \( I \subset S \) be an \( m \)-primary componentwise linear ideal and let \( d \) be the minimum of all \( j \in \mathbb{N} \) with \( \beta_{n-1, n-1+j}(I) > 0 \). The following conditions are equivalent:
\begin{enumerate}
\item \textit{S/I has the weak Lefschetz property.}
\item \textit{β}_{n-1,n-1+j}(I) = \beta_{0,j}(I) for all \( j > d \).
\item \textit{β}_{i,i+j}(I) = \binom{n-1}{i} \beta_{0,j}(I) for all \( j > d \) and all \( i \).
\end{enumerate}

\textbf{Proof.} Because of Proposition 2.3, Theorem 2.4 and Proposition 2.8 we can assume that \( I \) is a stable monomial ideal. It follows from Theorem 2.1 that \( d \) is equal to
\[
\min\{ j \in \mathbb{N} \mid \text{there exists } u \in G(I)_j \text{ with } m(u) = n \}.
\]
It also follows that conditions (b) and (c) are both equivalent to the following condition:
\[
m(u) = n \text{ for all } u \in G(I) \text{ with } \deg(u) > d. \quad (\ast)
\]
For \( j > 0 \), the map \((S/I)_{j-1} \twoheadrightarrow (S/I)_j, f \mapsto x_n f\), will be denoted by \( \mu_j \).

(a) \( \Rightarrow \) (b): Let \( t = \min\{ j > 0 \mid \mu_j \text{ is surjective} \} \). Since \( S/I \) has the weak Lefschetz property, we have \((I :_S x_n)_j = I_j \) for all \( j \leq t - 1 \). This means that \( m(u) < n \) for all \( u \in G(I) \) with \( \deg(u) < t \), and hence we get \( d \geq t \). Since \( \mu_j \) is surjective if and only if the ideal \((x_1, \ldots, x_{n-1})^j \) is contained in \( I \), we conclude that \((x_1, \ldots, x_{n-1})^d \subseteq I \). This implies, of course, \( m(u) = n \) for all \( u \in G(I) \) with \( \deg(u) > d \).

(b) \( \Rightarrow \) (a): Since \( m(u) < n \) for all \( u \in G(I) \) with \( \deg(u) < d \), the map \( \mu_j \) is injective for \( 0 < j < d \). It remains to show that \( \mu_j \) is surjective for all \( j \geq d \). There exists a \( t > 0 \) such that \( x_{n-1}^t \in G(I) \). Since (b) holds, we must have \( t \leq d \) (compare (\ast)). But \( x_{n-1}^t \in I \) implies \((x_1, \ldots, x_{n-1})^d \subseteq I \), because \( I \) is stable. Therefore \( \mu_j \) is surjective for \( j \geq t \).

\textbf{Corollary 3.2.} Assume that \( \text{char}(K) = 0 \). Let \( I \subseteq S \) be an \( m \)-primary graded ideal and let \( J \) denote the generic initial ideal of \( I \) with respect to the lexicographic order. The following conditions are equivalent:

\begin{enumerate}
\item \textit{S/I has the weak Lefschetz property.}
\item \textit{β}_{n-1,n-1+j}(J) = \beta_{0,j}(J) for all \( j > d \).
\item \textit{β}_{i,i+j}(J) = \binom{n-1}{i} \beta_{0,j}(J) for all \( j > d \) and all \( i \).
\end{enumerate}

\textbf{Proof.} The assertion follows immediately from Proposition 2.2, Theorem 2.4, Proposition 2.8 and Theorem 3.1.

If \( I \subseteq S \) is an \( m \)-primary ideal, we have the following isomorphisms of graded \( K \)-vectorspaces:
\[
\bigoplus_j K(-j)\beta_{n-1,j}(I) \cong \text{Tor}_n^S(S/I,K) \cong H_n(x,S/I) \cong H^0(x,S/I)(-n) = \text{Soc}(S/I)(-n)
\]
(where \( H_n(x, -) \) (resp. \( H^0(x, -) \)) denotes the Koszul homology (resp. Koszul cohomology) of the sequence \( x = (x_1, \ldots, x_n) \)). Hence we get the well-known

\textbf{Fact 3.3.} Let \( I \subseteq S \) be an \( m \)-primary graded ideal. The Hilbert series of \( \text{Soc}(S/I) \) is equal to \( \sum_{j \in \mathbb{N}} \beta_{n-1,n+j}(I)t^j \).
Note that Fact 3.3 is not only of theoretical interest, but also of practical use. In many cases (e.g. if $I$ is generated by monomials) it is possible to compute the socle of $S/I$, and hence to determine the Betti numbers $\beta_{n-1,j}(I)$.

For an arbitrary $m$-primary graded ideal, condition (b) of Theorem 3.1 is neither necessary nor sufficient for the weak Lefschetz property.

Consider the ideal $I = (w^2, wx, wy, wz, x^2, y^2) + m^3$ in $S = K[u, x, y, z]$. The Hilbert function of $S/I$ is equal to $1 + 4t + 4t^2$. Since the residue classes of the elements $w, xy, xz, yz, z^2$ form a $K$-basis of $\text{Soc}(S/I)$, we have $\beta_{3,5}(I) = 1, \beta_{3,6}(I) = 4$, and $\beta_{3,j}(I) = 0$ for $j \neq 5, 6$. There are 4 elements in $G(I)_3$, namely $xyz, xz^2, yz^2, z^3$. Thus $\beta_{0,3}(I) = 4 = \beta_{3,6}(I)$, which means that condition (b) of Theorem 3.1 is satisfied. But since $wl \in I$ for every $l \in S_1$, we see that $S/I$ does not have the weak Lefschetz property.

On the other hand, consider the ideal $I = (x^3, x^2y, y^3)$ in $S = K[x, y]$. The Hilbert function of $S/I$ is equal to $1 + 2t + 3t^2 + t^3$. Every nonzero linear form $l$ is a weak Lefschetz element on $S/I$. We have $\beta_{1,4}(I) = \beta_{1,5}(I) = 1$ and $\beta_{0,4}(I) = 0$. This means that condition (b) of Theorem 3.1 is not satisfied.

The next example shows that the question whether a componentwise linear ideal has the strong Lefschetz property cannot be answered in terms of the graded Betti numbers.

**Example 3.4.** Let $S = K[x, y, z]$. We consider the ideals

$I = (x^2, xy, y^3, yz, xz^2, yz^2, z^3, z^4)$ and $I' = (x^2, xy, y^3, xz^2, y^2z^2, yz^3, z^4)$.

Both ideals are strongly stable. The rings $S/I$ and $S/I'$ have the same Hilbert function: $H_{S/I}(t) = H_{S/I'}(t) = 1 + 3t + 4t^2 + 3t^3$. With the help of Theorem 2.1 we can compute the graded Betti numbers of $I$ and $I'$. In both cases the Betti diagram looks like this:

|   | 2 | 2 | 1 |   |
|---|---|---|---|---|
| 3 | 2 | 3 | 1 |   |
| 4 | 3 | 6 | 3 |   |

Theorem 3.1 yields that both ideals have the weak Lefschetz property. The element $z$ is even a strong Lefschetz element on $S/I$. But $z$ is not a strong Lefschetz element on $S/I'$: the element $x \in (S/I')_1$ is nonzero and lies in the kernel of the map $\mu : (S/I')_1 \to (S/I')_3, f \mapsto z^2f$. Since $\dim_K(S/I') = \dim_K(S/I')$, we conclude that $\mu$ is neither injective nor surjective. According to Lemma 2.7, this means that $S/I'$ does not have the strong Lefschetz property.
A monomial ideal $I \subset S$ is said to be a **lexsegment ideal**, if the following condition is satisfied: For every monomial $u \in I$ we have $v \in I$ for all monomials $v \in S$ with $\deg(v) = \deg(u)$ and $u <_{\text{lex}} v$ (where $<_{\text{lex}}$ is the lexicographic order). This condition implies in particular that $I$ is strongly stable. Note that a lexsegment ideal is completely determined by its Hilbert function.

For any graded ideal $I$, there is a (unique) lexsegment ideal, denoted by $I^{\text{lex}}$, which has the same Hilbert function as $I$ (see e.g. Corollary 2.8. of [He]). One can show that $\dim_K(S_1I^j) \leq \dim_K(S_1I^j)$ for all $j \in \mathbb{N}$. If $\dim_K(S_1I^j) = \dim_K(S_1I^j)$ for all $j \in \mathbb{N}$, $I$ is called a **Gotzmann ideal**. Gotzmann ideals are known to be componentwise linear (see [HH]).

Herzog and Hibi give in [HH] the following characterization of Gotzmann ideals:

**Theorem 4.1.** A graded ideal $I \subset S$ is a Gotzmann ideal if and only if $\beta_{ij}(I)$ is equal to $\beta_{ij}(I^{\text{lex}})$ for all $i, j \in \mathbb{N}$.

Before we can state the main result of this section, we have to introduce some notation: Let $d$ be a positive integer. Any integer $a \in \mathbb{N}$ can be written uniquely in the form

$$a = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \ldots + \binom{k(1)}{1},$$

where $k(d) > k(d-1) > \ldots > k(1) \geq 0$ (see e.g. Lemma 4.2.6 of [BH]). Here we use the convention that $\binom{k}{i}$ is zero whenever $i \geq 0$ and $k < i$. The numbers $k(d), \ldots, k(1)$ are called the $d$-th Macaulay coefficients of $a$. We define

$$a^{[d]} = \binom{k(d)-1}{d-1} + \binom{k(d-1)-1}{d-2} + \ldots + \binom{k(1)-1}{0}.$$ 

We now give a necessary and sufficient condition for an $m$-primary Gotzmann ideal to have the weak Lefschetz property in terms of the Hilbert function.

**Theorem 4.2.** Let $I \subset S$ be an $m$-primary Gotzmann ideal and let $t$ be the minimum of all $j \in \mathbb{N}$ with $H(S/I, j) \leq j$. The following conditions are equivalent:

(a) $S/I$ has the weak Lefschetz property.

(b) $H(S/I, j) = H(S/I, j-1)$ for $0 < j < t$.

**Proof.** Because of Theorem 3.1, Theorem 4.1, and the fact that $I$ is componentwise linear, we may assume that $I$ is a lexsegment ideal.

If $n = 1$, both conditions are fulfilled by trivial reasons. Hence we can assume that $n > 1$. For $j > 0$ we denote the map $(S/I)_{j-1} \to (S/I)_j, f \mapsto x_nf$, by $\mu_j$. Since $I$ is a...
lexsegment ideal, we have
\[ t = \min \{ j > 0 \mid x^j_{n-1} \in I \} = \min \{ j > 0 \mid (x_1, \ldots, x_{n-1})^j \subseteq I \} = \]
\[ \min \{ j > 0 \mid \mu_j \text{ is surjective} \}. \]

From Lemma 2.7 we know that condition (a) holds if and only if \( x_n \) is a weak Lefschetz element on \( S/I \). This is the case if and only if \( \mu_j \) is injective for all \( j \in \{1, \ldots, t-1\} \).

For a finite-dimensional subvector space \( V \subseteq S \) that is generated by monomials, we let \( m_n(V) \) be the number of monomials \( u \in V \) with \( m(u) = n \). Since \( I \) is stable, we have \( m_n(S/I_j) = H(I, j) \) for all \( j \in \mathbb{N} \) (see e.g. [He] Lemma 2.9). It is clear that \( \mu_j \) is injective for all \( j \in \{1, \ldots, t-1\} \) if and only if \( m_n(I_j) = m_n(S/I_j) \) for all \( j \in \{1, \ldots, t-1\} \).

In the following lemma we show that \( m_n(I_j) \) is equal to \( H(S, j-1) - H(S/I, j)^{[\bar{j}]} \) for \( j > 0 \). Summing up, we get: (a) \( \iff \) \( H(S, j-1) - H(S/I, j)^{[\bar{j}]} = H(I, j-1) \) for all \( j \in \{1, \ldots, t-1\} \) \( \iff \) (b) \( \square \)

Lemma 4.3. If \( I \subseteq S \) is a lexsegment ideal, then \( m_n(I_j) = H(S, j-1) - H(S/I, j)^{[\bar{j}]} \) for all \( j > 0 \).

Proof. Since the case \( I_j = 0 \) is trivial, we may assume \( I_j \neq 0 \). Choose \( u \in I_j \) such that \( v \notin I \) for all monomials \( v \in S_j \) with \( v <_{\text{ lex}} u \). We write \( u = x_{\alpha(1)} \cdots x_{\alpha(j)} \) with \( 1 \leq \alpha(1) \leq \ldots \leq \alpha(j) \leq n \). The set of all monomials \( v \in S_j \) with \( v <_{\text{ lex}} u \) is equal to the (disjoint!) union
\[ \bigcup_{i=1}^{j} x_{\alpha(1)} \cdots x_{\alpha(i-1)}[x_{\alpha(i)+1}, \ldots, x_n]_{j+1-i}, \]
where \([x_{\alpha(i)+1}, \ldots, x_n]_{j+1-i}\) denotes the set of all monomials in \( K[x_{\alpha(i)+1}, \ldots, x_n] \) that have degree \( j+1-i \) (compare the proof of Lemma 4.2.5 in [BH]). Hence we have
\[ H(S/I, j) = \sum_{i=1}^{j} \binom{n-\alpha(i)+j-i}{j+1-i} = \sum_{i=1}^{j} \binom{k(i)}{i}, \]
where \( k(i) = n-\alpha(j+1-i)+i-1 \) for \( i = 1, \ldots, j \). Since \( k(j) > \ldots > k(1) \geq 0 \), the numbers \( k(j), \ldots, k(1) \) are the \( j \)-th Macaulay coefficients of \( H(S/I, j) \). The set \( Y \) consisting of all monomials \( v \in S_j \) with \( v <_{\text{ lex}} u \) and \( m(v) < n \) is equal to
\[ \bigcup_{i=1}^{j} x_{\alpha(1)} \cdots x_{\alpha(i-1)}[x_{\alpha(i)+1}, \ldots, x_{n-1}]_{j+1-i}, \]

Since \( |Y| = \sum_{i=1}^{j} \binom{k(i)-1}{i-1} \), we finally get \( m_n(I_j) = m_n(S_j) - (H(S/I, j) - |Y|) = H(S, j-1) - \sum_{i=1}^{j} \binom{k(i)-1}{i-1} = H(S, j-1) - H(S/I, j)^{[\bar{j}]} \). \( \square \)

One easily checks that the ideals \( I \) and \( I' \) in Example 3.4 are both Gotzmann ideals. The ideal \( I' \) is even a lexsegment ideal. The rings \( S/I \) and \( S/I' \) possess the same graded
Betti numbers (and hence the same Hilbert function), but only one of them has the strong Lefschetz property – namely $S/I$. This shows that the question whether a Gotzmann ideal has the strong Lefschetz property cannot be answered in terms of the Hilbert function.

Nevertheless, for lexsegment ideals we have:

**Theorem 4.4.** Let $I \subset S$ be an $m$-primary lexsegment ideal and let $t$ be the minimum of all $j \in \mathbb{N}$ with $H(S/I, j) \leq j$. If $H(S/I, 1) \leq 2$, then $I$ has the strong Lefschetz property. If $H(S/I, 1) > 2$, the following conditions are equivalent:

(a) $S/I$ has the strong Lefschetz property.

(b) $H(S/I, t) \leq 2$ and $H(S/I, j)[j] = H(S/I, j - 1)$ for $0 < j < t$.

**Proof.** It is easy to see that $S/I$ has the strong Lefschetz property in case $H(S/I, 1) \leq 2$ (compare the proof of Proposition 4.4 in [HMNW]). So we can assume $H(S/I, 1) > 2$, that is, the variables $x_{n-2}, x_{n-1}, x_n$ are not in $I$.

In the proof of Theorem 4.2, we showed that $t$ is equal to

$$\min\{ j > 0 \mid (S/I)_{j-1} \to (S/I)_j, f \mapsto x_n f, \text{ is surjective}\}.$$ 

Since $I$ is a lexsegment ideal and $x_{n-1}^2 \in I$, we also have $x_{n-2} x_{n-1}^2 \in I$. Therefore the map $\mu : (S/I)_1 \to (S/I)_t, f \mapsto x_{n-1}^2 f$, is not injective.

(a) $\Rightarrow$ (b): The strong Lefschetz property implies that the map $\mu$ is surjective (see Lemma 2.7). Therefore $x_{n-1}^2 x_{n-2}^2 \in I$, and hence $H(S/I, t) \leq 2$. From Theorem 4.2, we obtain that $H(S/I, j)[j] = H(S/I, j - 1)$ for $0 < j < t$.

(b) $\Rightarrow$ (a): Since $S/I$ has the weak Lefschetz property (see Theorem 4.2) and since $(S/I)_{j-1} \to (S/I)_j, f \mapsto x_n f$, is not surjective for $0 < j < t$, we conclude that the map $(S/I)_{j-k} \to (S/I)_j, f \mapsto x_n^k f$, is injective for $0 < j < t$ and $k \geq 1$.

Since $H(S/I, t) \leq 2$, we have $x_{n-1}^2 x_{n-2} \in I$ for $j \geq t$. This implies that the map $(S/I)_{j-k} \to (S/I)_j, f \mapsto x_n^k f$, is surjective for $j \geq t$ and $1 \leq k < j$. Combining these arguments, we see that $S/I$ has the strong Lefschetz property.

$\square$

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Fachbereich Mathematik und Informatik, Universität Duisburg-Essen, 45117 Essen, Germany

E-mail address: attila.wiebe@uni-essen.de