AN EXTREMUM PROPERTY CHARACTERIZING THE
n-DIMENSIONAL REGULAR CROSS-POLYTOPE

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Abstract. In the spirit of the Genetics of the Regular Figures, by L. Fejes Tóth [FT, Part 2], we prove the following theorem: If 2n points are selected in the n-dimensional Euclidean ball B^n so that the smallest distance between any two of them is as large as possible, then the points are the vertices of an inscribed regular cross-polytope. This generalizes a result of R.A. Rankin [R] for 2n points on the surface of the ball. We also generalize, in the same manner, a theorem of Davenport and Hajós [DH] on a set of n + 2 points. As a corollary, we obtain a solution to the problem of packing k unit n-dimensional balls (n + 2 ≤ k ≤ 2n) into a spherical container of minimum radius.

1. Introduction

The regular cross-polytope is the dual to the n-dimensional Euclidean cube. More directly, the regular cross-polytope can be described as the convex hull of the union of n mutually perpendicular line segments of equal length, intersecting at the midpoint of each of them. Obviously, the regular cross-polytope generated by perpendicular segments of length d is inscribed in a sphere of radius r = d/2, and the edge-length of the cross polytope is \( \sqrt{2} r \). Each of the \( 2^n \) facets of the cross-polytope is a regular \((n-1)\)-dimensional simplex. The only distances between vertices of the regular cross-polytope inscribed in a ball of radius r are \( \sqrt{2} r \) and \( 2r \).

The main goal of this note is to prove the following metric characterization of the regular cross-polytope in terms of an extremum property of its vertex set:

**Theorem 1.** If \( V \) is a 2n-point subset \((n \geq 2)\) of the unit ball in n-dimensional Euclidean space such that the shortest distance between points in \( V \) is as large as possible, then \( V \) is the set of vertices of a regular cross-polytope inscribed in the ball.

This characterization of the n-dimensional regular cross-polytope by the extremum property of the distances between its vertices fits well into L. Fejes Tóth’s theory of genetics of regular figures [FT, Part 2]. The theory is supported by a collection of examples illustrating how “[...] regular arrangements are generated from unarranged chaotic sets by ordering effect of an economy principle, in the widest sense of the word” [FT, Preface, p. x].

Theorem 1 is a generalization of a result obtained by R.A. Rankin [R] in 1955. In Rankin’s version it is assumed that the points lie on the surface of the ball. Rankin states the result in a comment at the end of the proof of one of his theorems [R, p. 142]. Before Rankin, K. Schütte and B.L. van der Waerden [SW] solved the 3-dimensional case, also with the assumption that the points lie on the ball’s surface.

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In a similar way we generalize a theorem stated by Davenport and Hajós [DH] (proved in [AS] and in [R]), concerning a set of \(n+2\) points. Again, in our generalization the restriction that the points lie on the surface of the ball is removed:

**Theorem 2.** If \(n+2\) points lie in the \(n\)-dimensional Euclidean unit ball, then at least one of the distances between the points is smaller than or equal to \(\sqrt{2}\).

As a direct application of the above two theorems we obtain the following result on packing balls in a spherical container:

**Theorem 3.** Let \(S\) be a spherical container of minimum radius that can hold \(n+2\) nonoverlapping \(n\)-dimensional balls \((n \geq 2)\) of radius 1 each. Then the radius of \(S\) is \(1 + \sqrt{2}\), and there is enough room in \(S\) to hold as many as \(2n\) such balls. Moreover, the packing configuration of \(2n\) balls in \(S\) is unique up to isometry, the balls’ centers forming the set of vertices of an \(n\)-dimensional cross-polytope. Hence \(S\) cannot accommodate \(2n+1\) unit balls.

**Remark.** The problem of packing \(k\) unit \(n\)-dimensional balls in a spherical container of minimum radius \(r = r(k,n)\) for \(k \leq n + 1\) easily reduces to the problem of distributing \(k\) points on the surface of the ball so that the shortest distance between the points is as large as possible. This problem has been solved by Rankin [R]: the points are vertices of a regular \((k-1)\)-dimensional simplex inscribed in the ball and concentric with it. Thus,

\[
r(k,n) = 1 + \sqrt{2 - \frac{2}{k}} \quad \text{for} \quad k \leq n + 1,
\]

while according to Theorem 3

\[
r(k,n) = 1 + \sqrt{2} \quad \text{for} \quad n + 2 \leq k \leq 2n, \quad \text{and} \quad r(2n + 1, n) > 1 + \sqrt{2}.
\]

2. Notation and Preliminary Statements

The cardinality of a set \(A\) is denoted by \(\text{card} A\). \(\mathbb{R}^n\) denotes Euclidean (Cartesian) \(n\)-dimensional space with the usual inner product and the metric thereby generated. The origin \((0,0,\ldots,0)\) is denoted by \(o\). \(B^n\) denotes the unit ball in \(\mathbb{R}^n\), and \(S^{n-1}\) denotes the unit sphere, i.e., the boundary of \(B^n\). For \(x \in S^{n-1}\), the hyperplane containing the origin and normal to \(x\) is denoted by \(H_0(x)\), and \(H(x)\) denotes the half-space of \(H_0(x)\) opposite to \(x\). In other words,

\[
H_0(x) = \{ y \in \mathbb{R}^n : xy = 0 \},
\]

and

\[
H(x) = \{ y \in \mathbb{R}^n : xy \leq 0 \}.
\]

Also, with each \(x \in S^{n-1}\) we associate the set

\[
C(x) = \left\{ y \in B^n : \text{dist}(x,y) \geq \sqrt{2} \right\},
\]

called the crescent determined by \(x\). (For \(n = 2\), the shape of \(C(x)\) resembles a crescent.)

Further, for \(A \subset \mathbb{R}^n\), \(\text{Conv}A\) denotes the convex hull of \(A\) and \(\text{Lin}A\) denotes the linear hull of \(A\), that is, the smallest linear subset of \(\mathbb{R}^n\) containing \(A\). The interior of \(A\) is denoted by \(\text{Int}A\) and \(\text{Int}_L A\) denotes the interior of \(A\) relative to \(\text{Lin}A\).

Next, we state a few propositions, simple enough to have their proofs omitted.
Proposition 1. $C(x) \subset H(x)$ for every $x \in S^{n-1}$.

Proposition 2. $C(x) \cap H_0(x)$ is a great $(n-2)$-sphere of $S^{n-1}$. Specifically, $C(x) \cap H_0(x) = H_0(x) \cap S^{n-1}$.

Proposition 3. If $x \in S^{n-1}$, then for every point $y$ in the closed half-ball $B^n \setminus \text{Int}H(x)$, we have $\text{dist}(x, y) \leq \sqrt{2}$.

Proposition 4. Suppose $A \subset S^{n-1}$. If $o \in \text{Int}(\text{Conv}A)$, then $\bigcap_{x \in A} H(x) = \{o\}$.

The above proposition is immediately generalized to:

Proposition 5. Suppose $A \subset S^{n-1}$ and $k = \dim \text{Lin}A$. If $o \in \text{Int}_{L}(\text{Conv}A)$, then $\bigcap_{x \in A} H(x)$ is the $(n-k)$-dimensional linear subspace of $\mathbb{R}^n$ normal to $\text{Lin}A$. In particular, $\bigcap_{x \in A} H(x) = \bigcap_{x \in A} H_0(x)$.

The above propositions imply directly the following intersection properties of the crescents determined by a subset of $S^{n-1}$:

Proposition 6. Suppose $A \subset S^{n-1}$. If $o \in \text{IntConv}A$, then $\bigcap_{x \in A} C(x) = \emptyset$.

Proposition 7. Suppose $A \subset S^{n-1}$ and $k = \dim \text{Lin}A$. If $o \in \text{Int}_{L}(\text{Conv}A)$, then $\bigcap_{x \in A} C(x)$ is the $(n-k-1)$-dimensional great sphere of $S^{n-1}$ lying in the linear subspace normal to $\text{Lin}A$.

Every bounded subset $A$ of $\mathbb{R}^n$ containing at least two points is contained in a (unique) ball of minimum radius. The radius of the smallest ball containing $A$ is called the circumradius of $A$ and is denoted by $r(A)$. If $A$ is compact, then $A$ contains a finite subset of the same circumradius as $A$. Obviously, among such finite subsets of $A$ there is one (not necessarily unique) of minimum cardinality.

Proposition 8. Suppose $F$ is a finite subset of $B^n$ with $r(F) = 1$. If $r(F') < 1$ for every proper subset $F'$ of $F$, then $\dim \text{Lin}F = \text{card}F - 1$, $F$ lies on $S^{n-1}$ and $o \in \text{Int}_{L}(\text{Conv}F)$.

3. Proofs

Proof of Theorem [3]. Let $P = \{p_1, p_2, \ldots, p_{n+2}\}$ be a subset of $B^n$. Since our goal is to show that one of the distances between points in $P$ is smaller than or equal to $\sqrt{2}$, we may assume that the circumradius of $P$ is 1, expanding $P$ homothetically if needed. By this assumption, it follows that $o \in \text{Conv}P$. By a well-known theorem of Carathéodory, $o$ lies in the convex hull of an $(n+1)$-element subset of $P$, say $o \in \text{Conv}P_1$, where $P_1 = P \setminus \{p_1\}$. This implies that $P_1$ is not contained in $\text{Int}H(p_1)$, i.e., some point of $P_1$, say $p_2$, lies in the half-space complementary to $H(p_1)$. Therefore $p_2$ lies in the closed half-ball $B^n \setminus \text{Int}H(p_1)$, which implies $\text{dist}(p_1, p_2) \leq \sqrt{2}$. 

\[\square\]
Proof of Theorem 1. As we noted before, the shortest distance between the 2n vertices of the regular cross-polytope inscribed in $B^n$ is $\sqrt{2}$. It follows that the distance between any two points in $V$ is greater than or equal to $\sqrt{2}$. Observe that $r(V) = 1$, for otherwise all distances between points of $V$ could be enlarged by expanding $V$ homothetically. Let $V_0 \subset V$ be a set of minimum cardinality among all subsets of $V$ whose circumradius is 1, and let $k = \text{card}V_0$. Obviously, $k \geq 2$, and by the theorem of Carathéodory, $k \leq n + 1$. We now proceed inductively:

1° If $n = 2$, then $V$ is a four-point subset of the unit disk $B^2$, and it could well be left to the reader to show that $V$ is the set of vertices of an inscribed square. Nevertheless, we present here the following argument since it will also serve as an illustration to our inductive step 2°. By Proposition 8, $V_0$ is a subset of $S^1$, $\dim \text{Lin}V_0 = k - 1$ is either 2 or 1 and $o \in \text{Int}L(\text{Conv}V_0)$. Now, every point of $V \setminus V_0$ lies in $\bigcap_{x \in V_0} C(x)$. Hence, by Proposition 7, the set $V \setminus V_0$ is contained in a “great $m$-dimensional sphere” of $S^1$, where $m = 2 - k$. Since the set $V \setminus V_0$ is nonempty, $m$ cannot be negative. Thus $k = 2$, and since $o \in \text{Int}L(\text{Conv}V_0)$, the two points of $V_0$ are antipodes. The remaining two points of $V$ lie in the “great 0-dimensional sphere” of $S^1$ determined by the line normal to $L(V_0)$, which concludes this portion of the proof.

2° Assume that $n \geq 3$ and that the conclusion of Theorem 1 is true for balls of dimension smaller than $n$. By Proposition 8, $V_0$ is a subset of $S^{n-1}$, $\dim \text{Lin}V_0 = k - 1$, and $o \in \text{Int}L(\text{Conv}V_0)$. Now, every point of $V \setminus V_0$ lies in $\bigcap_{x \in V_0} C(x)$. Hence, by Proposition 7, the set $V \setminus V_0$ is contained in a great $m$-dimensional sphere of $S^{n-1}$, where $m = n - k$. By the inductive assumption, the cardinality of $V \setminus V_0$ cannot exceed $2(m + 1)$, the number of vertices of the $(m + 1)$-dimensional cross-polytope. That means $2n - k \leq 2(n - k + 1)$, which implies $k \leq 2$. Therefore $V_0$ consists of a pair of antipodes of $S^{n-1}$, and the remaining $2n - 2$ points of $V$ lie on the great $(n - 2)$-dimensional sphere of $S^{n-1}$, in a hyperplane perpendicular to the line containing $V_0$. By the inductive assumption, $V \setminus V_0$ is the set of vertices of an $(n - 1)$-dimensional cross-polytope, hence $V$ is the set of vertices of an $n$-dimensional cross-polytope inscribed in $S^{n-1}$. 

References

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