DUALITY FOR $\kappa$-ADDITIVE COMPLETE ATOMIC MODAL ALGEBRAS

YOSHIHITO TANAKA

Abstract. In this paper, we give a duality theorem between the category of $\kappa$-additive complete atomic modal algebras and the category of $\kappa$-downward directed multi-relational Kripke frames, for any cardinal number $\kappa$. Multi-relational Kripke frames are not Kripke frames for multi-modal logic, but frames for monomodal logics in which the modal operator $\Box$ does not distribute over (possibly infinite) disjunction, in general. We first define homomorphisms of multi-relational Kripke frames, and then show the equivalence between the category of $\kappa$-downward directed multi-relational Kripke frames and the category $\kappa$-complete neighborhood frames, from which the duality theorem follows. We also present another direct proof of this duality based on the technique given by Minari.

1. Introduction

It is proved by Thomason [9] that the category of all completely additive complete atomic modal algebras is dually equivalent to the category of all Kripke frames, where a modal algebra is said to be completely additive, if the modal operator $\Box$ distributes over the joins of every subsets of the algebra. However, there are some modal logics which cannot be characterized by a class of completely additive modal algebras. For example, if we see the existential and universal quantifiers as infinite joins and meets, respectively, the Barcan formula $\forall x \Box \phi \supset \Box \forall x \phi$ corresponds to the complete additivity, but there exist predicate modal logics in which it is not derivable. Moreover, there exists a propositional normal modal logic which is incomplete with respect to any class of completely additive complete modal algebras [9].

Subsequently, Došen [3] gives broad kinds of duality theorems for categories of modal algebras and neighborhood frames, including duality between the category of complete modal algebras and the category of $\omega$-complete neighborhood frames (which are called full filter frames in [3]) and that between the category of completely additive complete atomic modal algebras and the category of complete neighborhood frames (which are called full hyperfilter frames in [3]), as well as equivalence between the category of complete neighborhood frames and the category of Kripke frames. However, it should be remarked that the category of neighborhood frames is not a generalization of the category of Kripke frames, in the following sense: For any Kripke frame $F = \langle W, R \rangle$, we can define the "underlying" neighborhood frame $U(F) = \langle W, V_F \rangle$, where,

$$V_F(x) = \{ \{ y \mid (x, y) \in R \} \},$$

for any $x \in W$. However, as we will see in Theorem 6.1 $U$ does not define the forgetful functor.

In this paper, we give another duality theorem for the category of complete atomic modal algebras between the category of multi-relational Kripke frames.
Multi-relational Kripke frames are not Kripke frames for multi-modal logic, but frames for monomodal logics in which the modal operator $\Diamond$ does not distribute over (possibly infinite) disjunction, in general. For example, in deontic logic (see, e.g., [5–2]),

\begin{equation}
(\Box p \land q) \supset \Box(p \land q)
\end{equation}

should not be derived, as the formula $(\Box \phi \land \Box \neg \phi) \supset \Box \psi$, which means that "if there is any conflict of obligation, then everything is obligatory" ([5], p. 114) can be deduced from it, and in the least infinitary modal logic, it is proved that the countable extension of (1.1) is not derivable [5, 7]. Consequently, these logics are Kripke incomplete, but it is proved that deontic logic $P$ is complete with respect to the class of serial multi-relational Kripke frames [5], and the least infinitary modal logic is complete with respect to the class of $\omega$-downward directed multi-relational Kripke frames [7].

In this paper, we first define homomorphisms of multi-relational Kripke frames so that the category of multi-relational Kripke frames is going to be a generalization of the category of Kripke frames. Then we show that the category of $\kappa$-downward directed multi-relational Kripke frames are equivalent to the category of $\kappa$-complete neighborhood frames for every cardinal number $\kappa$, which is a generalization of Došen's equivalence theorem between the category of Kripke frames and the category of complete neighborhood frames. From this equivalence, duality between the category of $\kappa$-additive complete atomic modal algebras and the category of $\kappa$-downward directed multi-relational Kripke frames follows. In addition, we give another proof for this duality for any regular cardinal $\kappa$. The basic technique of this proof is given by Minari [7]. He proved completeness theorem for the least infinitary modal logic with respect to $\omega$-downward directed multi-relational Kripke frames by constructing a multi-relational Kripke frame such that each binary relation is given in the same way as the canonical frame of a finite fragment of the Lindenbaum algebra. We show that Minari's technique works also for homomorphisms and can be extended for any regular cardinal $\kappa$.

2. Preliminaries

In this section, we fix notation and recall definitions and basic results. For the details, see, e.g., [1, 4].

Let $W$ be a non-empty set and $R$ a binary relation on $W$. For any $w_1$ and $w_2$ in $W$, we write $w_1 <_R w_2$ if $(w_1, w_2) \in R$. For any $X \subseteq W$, $\uparrow_R X$ and $\downarrow_R X$ denote the subsets of $W$ defined by

$\uparrow_R X = \{w \in W \mid \exists x \in X(x <_R w)\}$, $\downarrow_R X = \{w \in W \mid \exists x \in X(w <_R x)\}$,

respectively. If $X$ is a singleton $\{w\}$, we write $\uparrow_R w$ and $\downarrow_R w$ for $\uparrow_R X$ and $\downarrow_R X$, respectively. If $R$ is a partial order $\leq$, we write $\uparrow$ and $\downarrow$ for $\uparrow_{\leq}$ and $\downarrow_{\leq}$, respectively.

Let $f : A \to B$ be a mapping from a set $A$ to a set $B$. For any set $X \subseteq A$ and $Y \subseteq B$, $f[X]$ and $f^{-1}[Y]$ denote the sets

$f[X] = \{f(x) \mid x \in X\}$, $f^{-1}[Y] = \{x \in X \mid f(x) \in Y\}$,

respectively.

**Definition 2.1.** A Boolean algebra $A$ is said to be complete if for any $X \subseteq A$, $\bigvee X$ and $\bigwedge X$ exist in $A$. Let $A$ and $B$ be complete Boolean algebras. A mapping $f : A \to B$ is called a homomorphism of complete Boolean algebras if $f$ is a homomorphism of Boolean algebras which satisfies

$f(\bigvee X) = \bigvee f[X]$, $f(\bigwedge X) = \bigwedge f[X]$,

for any $X \subseteq A$. 


Definition 2.2. For any homomorphism \( f : A \to B \) of complete Boolean algebras, \( f^* \) and \( f_* \) denote mappings from \( B \) to \( A \) which are defined by
\[
f^*(b) = \bigvee f^{-1}[\{b\}], \quad f_*(b) = \bigwedge f^{-1}[\{b\}],
\]
for any \( b \in B \), respectively.

Proposition 2.3. Let \( f : A \to B \) be a homomorphism of complete Boolean algebras. For any \( a \in A \) and \( b \in B \),
\[
(2.1) \quad f(a) \leq b \iff a \leq f^*(b), \quad b \leq f(a) \iff f_*(b) \leq a.
\]
That is, \( f^* \) and \( f_* \) are right and left adjoints of \( f \), respectively.

It follows from (2.1) that \( f^* \) and \( f_* \) are order preserving mappings and
\[
(2.2) \quad f \circ f^*, \quad f_* \circ f \leq \text{Id}_B, \quad \text{Id}_A \leq f^* \circ f, \quad f \circ f_*.
\]

Definition 2.4. Let \( A \) be a Boolean algebra. A non-zero element \( a \in A \) is called an \textit{atom} if \( 0 < x \leq a \) implies \( x = a \). The set of all atoms of \( A \) is denoted by \( A(A) \).

A Boolean algebra \( A \) is said to be \textit{atomic} if every non-zero element \( x \in A \) satisfies
\[
x = \bigvee_{a \in A(A), \ a \leq x} a.
\]

We write \textbf{CABA} for the category whose objects are all complete and atomic Boolean algebras and arrows are all homomorphisms of complete Boolean algebras.

Proposition 2.5. Let \( A \) be a Boolean algebra and \( 0 \neq a \in A \). Then the following conditions are equivalent:

(1) \( a \) is an atom.

(2) For any \( X \subseteq A \), if \( \bigvee X \subseteq A \) and \( a \leq \bigvee X \) then \( a \leq x \) for some \( x \in X \).

(3) For any \( x \) and \( y \) in \( A \), if \( a \leq x \lor y \) then \( a \leq x \) or \( a \leq y \).

(4) For any \( x \in A \), \( a \leq x \) or \( a \leq \neg x \).

Proposition 2.6. Let \( A \) and \( B \) be complete atomic Boolean algebras and \( f : A \to B \) a homomorphism of complete Boolean algebras. If \( b \in A(B) \), then \( f_*(b) \in A(A) \).

Definition 2.7. A \textit{Kripke frame} is a pair \( \langle W, R \rangle \), where \( W \) is a non-empty set and \( R \) is a binary relation on \( W \). Let \( F_1 = \langle W_1, R_1 \rangle \) and \( F_2 = \langle W_2, R_2 \rangle \) be Kripke frames. A \textit{homomorphism} \( f : F_1 \to F_2 \) of Kripke frames is a mapping from \( W_1 \) to \( W_2 \) which satisfies the following:

(1) for any \( v \) and \( w \) in \( W_1 \), if \( v <_{R_1} w \) then \( f(v) <_{R_2} f(w) \);

(2) for any \( w \in W_1 \) and \( u \in W_2 \), if \( f(w) <_{R_2} u \) then there exists \( v \in W_1 \) such that \( w <_{R_1} v \) and \( f(v) = u \).

We write \textbf{KFr} for the category of all Kripke frames.

3. The category of complete atomic modal algebras

Definition 3.1. An algebra \( \langle A; \lor, \land, \neg, \Diamond, 0, 1 \rangle \) is called a \textit{modal algebra} if its reduct \( \langle A; \lor, \land, \neg, 0, 1 \rangle \) is a Boolean algebra and \( \Diamond \) is a unary operator which satisfies \( \Diamond 0 = 0 \) and
\[
\Diamond x \lor \Diamond y = \Diamond (x \lor y)
\]
for any \( x \) and \( y \) in \( A \). A modal algebra \( A \) is said to be \textit{complete} or \textit{atomic} if its Boolean reduct is complete or atomic, respectively. Let \( A \) and \( B \) be modal algebras. A mapping \( f : A \to B \) is called a \textit{homomorphism of modal algebras} if \( f \) is a homomorphism of Boolean algebras which satisfies
\[
f(\Diamond x) = \Diamond f(x)
\]
for any \( x \in A \). A homomorphism \( f \) of modal algebras is called a \textit{homomorphism of complete modal algebras} if it is a homomorphism of complete Boolean algebras.
Definition 3.2. A complete modal algebra $A$ is said to be \textit{completely additive} if
\begin{equation}
\bigvee_{x \in X} \diamond x = \diamond \bigvee X
\end{equation}
holds for any $X \subseteq A$. Let $\kappa$ be a cardinal number. A complete modal algebra $A$ is
said to be $\kappa$-\textit{additive} if the equation (3.1) holds for any $X \subseteq A$ such that $|X| < \kappa$.

Definition 3.3. The objects of the category $\text{CAMA}_\infty$ are all completely additive
complete atomic modal algebras and the arrows of it are all homomorphisms of complete modal algebras between them. Let $\kappa$ be a cardinal number. The objects of the category $\text{CAMA}_\kappa$ are all $\kappa$-additive complete atomic modal algebras and the arrows of it are all homomorphisms of complete modal algebras between them.

Theorem 3.4. (Thomason [9]). $\text{CAMA}_\infty$ and $\text{KFr}$ are dually equivalent.

Proof. First, we define a functor $F : \text{CAMA}_\infty \rightarrow \text{KFr}$. For any object $A$ of $\text{CAMA}_\infty$, define $F(A)$ by
\[
F(A) = \langle A(A), R \rangle,
\]
where,
\[
a <_R b \iff a \leq \diamond b
\]
for any $a$ and $b$ in $A(A)$, and for any arrow $f : A \rightarrow B$ of $\text{CAMA}_\infty$, define $F(f) : F(B) \rightarrow F(A)$ by
\[
F(f)(b) = f_* (b)
\]
for any $b \in A(B)$. Next, we define a functor $G : \text{KFr} \rightarrow \text{CAMA}_\infty$. For any object $K = \langle W, R \rangle$ of $\text{KFr}$, define $G(K)$ by
\[
G(K) = \langle \mathcal{P}(W); \cup, \cap, W \setminus - , \diamond_K, \emptyset, W \rangle,
\]
where
\[
\diamond_K X = \downarrow_R X
\]
for any $X \subseteq W$, and for any arrow $g$ from $K_1 = \langle W_1, R_1 \rangle$ to $K_2 = \langle W_2, R_2 \rangle$ of $\text{KFr}$, define $G(g) : G(K_2) \rightarrow G(K_1)$ by
\[
G(g)(X) = g^{-1}[X]
\]
for any $X \in \mathcal{P}(W_2)$. Then $F : \text{CAMA}_\infty \rightarrow \text{KFr}$ and $G : \text{KFr} \rightarrow \text{CAMA}_\infty$ are
well-defined contravariant functors and
\[
\text{Id}_{\text{CAMA}_\infty} \cong G \circ F, \quad \text{Id}_{\text{KFr}} \cong F \circ G.
\]

\[\square\]

4. The category of neighborhood frames

A \textit{neighborhood frame} is a pair $\langle C, \mathcal{V} \rangle$, where $C$ is a non-empty set and $\mathcal{V}$ is a
mapping from $C$ to $\mathcal{P}(\mathcal{P}(C))$. A neighborhood frame $\langle C, \mathcal{V} \rangle$ is said to include the \textit{whole set} if for any $c \in C$, $C \in \mathcal{V}(c)$, and is said to be \textit{upward closed} if for any $c \in C$, $X \in \mathcal{V}(c)$, and $Y \subseteq C$, if $X \subseteq Y$ then $Y \in \mathcal{V}(c)$. A neighborhood frame $\langle C, \mathcal{V} \rangle$ is said to be \textit{complete} if it includes the whole set, is upward closed, and for any $c \in C$ and non-empty subset $S$ of $\mathcal{V}(c)$,
\begin{equation}
S \subseteq \mathcal{V}(c) \Rightarrow \bigcap S \in \mathcal{V}(c).
\end{equation}
Let $\kappa$ be a cardinal number. A neighborhood frame $\langle C, \mathcal{V} \rangle$ is said to be $\kappa$-\textit{complete} if it includes the whole set, is upward closed, and (4.1) holds for any non-empty subset $S$ of $\mathcal{V}(c)$ such that $|S| < \kappa$. 
Let $Z_1 = \langle C_1, \mathcal{V}_1 \rangle$ and $Z_2 = \langle C_2, \mathcal{V}_2 \rangle$ be neighborhood frames. A mapping $f : C_1 \to C_2$ is called a homomorphism of neighborhood frames from $Z_1$ to $Z_2$ if for any $c \in C_1$ and $X \subseteq C_2$,

$$f^{-1}[X] \in \mathcal{V}_1(c) \iff X \in \mathcal{V}_2(f(c))$$

holds.

We write $\text{NFr}$ for the category of all neighborhood frames. We also write $\text{NFr}_\infty$ and $\text{NFr}_\kappa$ for its full subcategories of all complete neighborhood frames and all $\kappa$-complete neighborhood frames, respectively. The duality theorem between $\text{NFr}_\omega$ and $\text{CAMA}$ and that between $\text{NFr}_\infty$ and $\text{CAMA}_\infty$, which are given in Došen [3], can be generalized to any cardinal number $\kappa$, immediately:

**Theorem 4.1.** (Došen [3]) For any cardinal number $\kappa$, $\text{CAMA}_\kappa$ and $\text{NFr}_\kappa$ are dually equivalent.

**Proof.** First, we define a functor $J : \text{CAMA}_\kappa \to \text{NFr}_\kappa$. For any object $A$ of $\text{CAMA}_\kappa$, define $J(A)$ by

$$J(A) = \langle \mathcal{A}(A), \mathcal{V} \rangle,$$

where

$$\mathcal{V}(a) = \{ \mathcal{A}(A) \cap \downarrow x \mid a \not\in \Diamond - x \}$$

for any $a$, and for any arrow $f : A \to B$ of $\text{CAMA}_\kappa$, define $J(f) : J(B) \to J(A)$ by

$$J(f)(b) = f_+(b)$$

for any $b \in \mathcal{A}(B)$. Next, we define a functor $K : \text{NFr}_\kappa \to \text{CAMA}_\kappa$. For any object $Z = \langle C, \mathcal{V} \rangle$ of $\text{NFr}_\kappa$, define $K(Z)$ by

$$K(Z) = \langle \mathcal{P}(C); \cup, \cap, C \setminus -, \Diamond Z, \emptyset, C \rangle,$$

where

$$\Diamond Z X = \{ c \in C \mid C \setminus X \not\in \mathcal{V}(c) \}$$

for any $X \subseteq C$, and for any arrow $g$ from $Z_1 = \langle C_1, \mathcal{V}_1 \rangle$ to $Z_2 = \langle C_2, \mathcal{V}_2 \rangle$ of $\text{NFr}_\kappa$, define $K(g) : K(Z_2) \to K(Z_1)$ by

$$K(g)(X) = g^{-1}[X]$$

for any $X \in \mathcal{P}(C_2)$. Then $J : \text{CAMA}_\kappa \to \text{NFr}_\kappa$ and $K : \text{NFr}_\kappa \to \text{CAMA}_\kappa$ are well-defined contravariant functors and

$$\delta : \text{Id}_{\text{CAMA}_\kappa} \cong K \circ J, \quad \gamma : \text{Id}_{\text{NFr}_\kappa} \cong J \circ K,$$

where the natural isomorphisms $\delta$ and $\gamma$ are defined by

$$\delta_A : x \mapsto \{ a \in \mathcal{A}(A) \mid a \leq x \}, \quad \gamma_Z : y \mapsto \{ y \},$$

for any object $A$ in $\text{CAMA}_\kappa$ and any $Z$ in $\text{NFr}_\kappa$. \hfill \Box

Došen also proved the following equivalence of categories:

**Theorem 4.2.** (Došen [3]) $\text{NFr}_\kappa \cong \text{KFr}$.

For any Kripke frame $F = \langle W, R \rangle$, we can define a neighborhood frame $U(F)$ by $U(F) = \langle W, \uparrow R x \mid x \in W \rangle$. However, as is shown in Theorem 6.1 there exists a Kripke frame $F$ such that $U(F)$ is not a complete neighborhood frame and there exists a homomorphism $f : F_1 \to F_2$ of Kripke frames which is not a homomorphism of neighborhood frames from $U(F_1)$ to $U(F_2)$. In this sense, the neighborhood frames are not a generalization of the Kripke frames, although the two categories are equivalent.
5. The category of multi-relational Kripke frames

**Definition 5.1.** A pair \((W, S)\) is called a multi-relational Kripke frame if \(W\) is a non-empty set and \(S\) is a non-empty set of binary relations on \(W\). A multi-relational Kripke frame \((W, S)\) is said to be completely downward directed if for any \(S' \subseteq S\), there exists \(R \in S\) such that

\[ R \subseteq \bigcap S'. \]

Clearly, \((W, S)\) is completely downward directed if and only if \(\bigcap S \in S\). Let \(\kappa\) be a cardinal number. A multi-relational Kripke frame \((W, S)\) is said to be \(\kappa\)-downward directed if for any \(S' \subseteq S\) such that \(|S'| < \kappa\), there exists \(R \in S\) which satisfies (5.1).

**Proof.** Clear from the definition of the homomorphism of multi-relational Kripke frames. \(\square\)

**Proposition 5.3.** For any Kripke frame \(F = (W, R)\), define \(M(F) = (W, \{R\})\), and for any homomorphism \(f : F_1 \rightarrow F_2\) of Kripke frames, define \(M(f)\) by \(f\). Then, \(M\) is a well-defined functor and the image of \(KFr\) by \(M\) is a full and faithful subcategory of \(MRKF_\kappa\).

**Proof.** Clear from the definition of the homomorphism of multi-relational Kripke frames. \(\square\)

**Definition 5.2.** We write \(MRKF\) for the category of all multi-relational Kripke frames. We also write \(MRKF_\infty\) and \(MRKFa\) for its full subcategories of all completely downward directed multi-relational Kripke frames and all \(\kappa\)-downward directed multi-relational Kripke frames, respectively.

The following theorem states that the multi-relational Kripke frames can be seen as a generalization of the Kripke frames:

**Proposition 6.1.** For any multi-relational Kripke frame \(M = (W, S)\), we can define the “underlying” neighborhood frame \(U(M)\) by \(U(M) = (W, \mathcal{V}_M)\), where

\[ \mathcal{V}_M(x) = \{ \uparrow_R x \mid R \in S\}. \]

However, \(U\) does not define the forgetful functor from \(MRKF\) to \(NFr\) nor that from \(MRKF_\kappa\) to \(NFr_\kappa\). In fact, we have the following:
Theorem 6.1. 

(1) There exists an object $M$ of $\text{MRKF}_\kappa$ such that $U(M)$ is not an object of $\text{NFr}_\kappa$. Moreover, there exists such an object $M$ in $\text{KFr}$ such that $U(M)$ is not an object of $\text{NFr}_\infty$.

(2) There exists an arrow $f : M_1 \to M_2$ of $\text{MRKF}_\kappa$ such that $U(M_1)$ and $U(M_2)$ are objects of $\text{NFr}$ but $f$ is not an arrow of $\text{NFr}$. Moreover, there exists such an arrow $f$ in $\text{KFr}$, either.

(3) There exists an arrow $f : U(M_1) \to U(M_2)$ of $\text{NFr}$ such that $M_1$ and $M_2$ are objects of $\text{MRKF}$ but $f : M_1 \to M_2$ is not an arrow of $\text{MRKF}$. 

Proof. (1): Let $M = \langle \{0\}, \{\emptyset\} \rangle$. Then $M$ is an object of $\text{MRKF}_\kappa$, but not that of $\text{NFr}_\kappa$, since $\mathcal{V}_M(0)$ is not upward closed. If we identify a singleton $\{R\}$ of a relation with $R$, $M$ is a Kripke frame, either.

(2): Let $M_1 = \langle \{0\}, \{\{(0,0)\}\} \rangle$ and $M_2 = \langle \{0,1\}, \{\{(0,0)\}\} \rangle$. Let $f : 0 \to 0$. It is easy to see that $f \in \text{hom}_{\text{MRKF}_\kappa}(M_1, M_2)$. If we identify a singleton $\{R\}$ of a relation with $R$, $f$ is a homomorphism of Kripke frames, either. However, $f$ is not an arrow of $\text{NFr}$ from $U(M_1)$ to $U(M_2)$, since $f^{-1} \{\{0,1\}\} = \{0\} \in \mathcal{V}_{M_1}(0)$, but $\{0,1\} \not\in \mathcal{V}_{M_2}(0)$.

(3): Let $M_1 = \langle \{0,1,2\}, \{R_1, R_2\} \rangle$ and $M_2 = \langle \{0,1\}, \{Q\} \rangle$, where $R_1 = \{0,1\}$, $R_2 = \{(0,0), (0,1), (0,2)\}$, $Q = \{0,0\}, (0,1)\}$. Then

$$\mathcal{V}_{M_1}(0) = \{\{1\}, \{0,1,2\}\}, ~ \mathcal{V}_{M_1}(1) = \mathcal{V}_{M_1}(2) = \{\emptyset\}$$

and

$$\mathcal{V}_{M_2}(0) = \{\{0,1\}\}, ~ \mathcal{V}_{M_2}(1) = \{\emptyset\}.$$ 

Define $f : \{0,1,2\} \to \{0,1\} : f(0) = 0$ and $f(1) = f(2) = 1$. It is easy to see that $f \in \text{hom}_{\text{NFr}}(U(M_1), U(M_2))$. However, $f \not\in \text{hom}_{\text{MRKF}_\kappa}(M_1, M_2)$, since $0 < Q$ 0 but $0 \not\subset R_i$, 0.

If we identify $U(M)$ with $M$ and a singleton $\{R\}$ of a relation with $R$, Proposition 6.2 and Theorem 6.1 can be summarized as follows:

$$\begin{array}{c|c|c}
\text{NFr} & \not\subset & \text{MRKF} \\
\textbf{\#} & \text{arrows} & \textbf{\#} \\
\hline
\text{NFr}_\kappa & \not\subset & \text{MRKF}_\kappa \\
\textbf{\#} & \text{objects} & \textbf{\#} \\
\hline
\text{NFr}_\infty & \not\subset & \text{KFr} \\
\textbf{\#} & \text{objects} & \textbf{\#}
\end{array}$$

In the rest of this section, we show that $\text{NFr}_\kappa$ and $\text{MRKF}_\kappa$ are equivalent.

First, we show the following lemmas:

**Lemma 6.2.** Let $\kappa$ be any cardinal number. For any $\kappa$-downward directed multi-relational Kripke frame $M = (W, S)$, define a $\kappa$-complete neighborhood frame $N(M)$ by $(W, \mathcal{V}_M)$, where $\mathcal{V}_M \subseteq \mathcal{P}(W)$ is defined by

$$\mathcal{V}_M = \uparrow \{\uparrow R \mid R \in S\}$$

for any $x \in W$, and for any homomorphism $f$ of multi-relational Kripke frames, define $N(f)$ by $f$. Then $N$ is a full functor from $\text{MRKF}_\kappa$ to $\text{NFr}_\kappa$.

Proof. It is clear that $N(M)$ is an object of $\text{NFr}_\kappa$. We show that for any $M_1 = \langle W_1, S_1 \rangle$ and $M_2 = \langle W_2, S_2 \rangle$,

$$\text{hom}_{\text{MRKF}_\kappa}(M_1, M_2) = \text{hom}_{\text{NFr}_\kappa}(N(M_1), N(M_2)).$$
Then for any $y \in Y$. Hence, for any $\uparrow v$ and $f$, $H$.

Proof. $\kappa x$ for any $\kappa v$ and $\kappa f$, $V$.

Hence, $\uparrow x \subseteq \uparrow f^{-1} \uparrow f(x) \subseteq f^{-1} \uparrow f(x)$.

Similarly, we have $\uparrow \uparrow f(x) \subseteq \uparrow f(x)$. Hence, for any $y \in Y$. Suppose $\uparrow f(x) \subseteq \uparrow f(x)$.

Then $\uparrow f^{-1} \uparrow f(x)$ is in $V_M(x)$. Since $\uparrow f(x) \subseteq f^{-1} \uparrow f(x)$, there exists an $R \in S$ such that

\[
\uparrow f^{-1} \uparrow f(x) \subseteq f^{-1} \uparrow f(x).
\]

Then for any $y \in Y$, if $x \in R$ then $f(x) < y$. Next, take any $R \in S$. Since $\uparrow f^{-1} \uparrow f(x) \subseteq f^{-1} \uparrow f(x)$, then $\uparrow f^{-1} \uparrow f(x)$ is upward closed,

\[
\uparrow f^{-1} \uparrow f(x) \subseteq f^{-1} \uparrow f(x) \subseteq V_M(x).
\]

Hence, $f(\uparrow f^{-1} \uparrow f(x)) \subseteq V_M(f(x))$. Then there exists a $Q \in S$ such that $\uparrow f^{-1} \uparrow f(x) \subseteq f^{-1} \uparrow f(x)$. Then for any $u \in W$ such that $f(x) < u$, there exists $y \in W$ such that $x < y$ and $f(y) = u$.

Lemma 6.3. Let $Z = (C, V)$ be an $\kappa$-complete neighborhood frame. We write $V_Z$ for the set

\[V_Z = \{ v : C \to P(C) \mid \forall x \in C(v(x) \in V(x)) \},\]

and for any $v \in V_Z$, we write $R_v$ for a binary relation on $C$ defined by

\[R_v = \{ (x, y) \mid x \in C, y \in v(x) \}.\]

Then $H(Z) = (C, S_Z)$, is a $\kappa$-downward directed multi-relational Kripke frame, where $S_Z = \{ R_v \mid v \in V_Z \}$. If we define $H(f)$ by $f$ for any homomorphism $f$ of neighborhood frames, then $H$ is a functor from $\text{NFR}_\kappa$ to $\text{MRKF}_\kappa$.

Proof. We first show that $H(Z)$ is an object of $\text{MRKF}_\kappa$. Since $Z$ includes the whole set, $V_Z \neq \emptyset$. Therefore, $S_Z \neq \emptyset$. Take any subset $\{ R_v \mid v \in V_Z \}$ of $S_Z$. As $Z$ is $\kappa$-complete, there exists $u \in V_Z$ such that

\[u(x) = \bigcap_{v \in \kappa} v(x) \subseteq V(x)\]

for any $x \in C$. Then $R_u = \bigcap_{v \in \kappa} R_v$. Next, we show that $\text{hom}_{\text{NFR}_\kappa}(Z_1, Z_2) \subseteq \text{hom}_{\text{MRKF}_\kappa}(H(Z_1), H(Z_2))$.

For any $\kappa$-complete neighborhood frames $Z_1 = (C_1, V_1)$ and $Z_2 = (C_2, V_2)$. Suppose $f \in \text{hom}_{\text{NFR}_\kappa}(Z_1, Z_2)$. First, take any $x \in C_1$ and $R_v \in S_{Z_2}$. Then $\uparrow R_v f(x) = v(f(x)) \in V_2(f(x))$. Hence, $f^{-1} \uparrow R_v f(x) \subseteq V_1(x)$. By definition of $V_{Z_1}$, there exists $u \in V_{Z_1}$, such that

\[\uparrow R_u x = u(x) = f^{-1} \uparrow R_v f(x).\]

Hence, for any $y \in C_1$,

\[x < R_v y \iff y \subseteq u(x) \iff f(x) < R_v f(y).\]
Next, take any \( x \in C_1 \) and \( R_u \in S_{Z_1} \). Then \( \uparrow_{R_u} x = u(x) \in V_1(x) \). Since \( Z_1 \) is upward closed, \( f^{-1} \[ f(\uparrow_{R_u} x) \] \in V_1(x) \). Therefore, \( f(\uparrow_{R_u} x) \in V_2(f(x)) \). By definition of \( V_{Z_2} \), there exists \( v \in V_{Z_2} \) such that
\[
\uparrow_{R_u} f(x) = v(f(x)) = f(\uparrow_{R_u} x).
\]
Hence, for any \( z \in C_2 \) such that \( f(x) <_{R_u} z \), there exists \( y \in C_1 \) such that \( x <_{R_u} y \) and \( f(y) = z \).

Now, we prove that \( \text{MRKF}_\kappa \) and \( \text{NFr}_\kappa \) are equivalent, which is a generalization of Theorem 4.2.

**Theorem 6.4.** \( N \) and \( H \) are equivalence between \( \text{MRKF}_\kappa \) and \( \text{NFr}_\kappa \), for every cardinal number \( \kappa \).

**Proof.** For any object \( M = \langle W, S \rangle \) of \( \text{MRKF}_\kappa \), define a map \( \gamma_M : M \to H(N(M)) \) by \( \gamma_M(x) = x \) for any \( x \in W \), and for any object \( Z = \langle C, V \rangle \) of \( \text{NFr}_\kappa \), define a map \( \delta_Z : Z \to H(N(Z)) \) by \( \delta_Z(c) = c \) for any \( c \in C \). It is trivial that \( H(N(f)) \circ \gamma_M = \gamma_M \circ f \) holds for any \( f : M_1 \to M_2 \) and \( H(N(g)) \circ \delta_Z = \delta_Z \circ g \) holds for any \( g : Z_1 \to Z_2 \).

First, we show that for any multi-relational Kripke frame \( M = \langle W, S \rangle \), \( \gamma_M \) is an isomorphism of multi-relational Kripke frames from \( M \) to \( H(N(M)) \). We check the first condition of the homomorphisms of multi-relational Kripke frames: Take any \( x \in W \) and \( R_v \in S_{N(M)} \), where \( v \in V_{N(M)}(x) \). Then there exists \( R \in S \) such that \( \uparrow_{R} x \subseteq v(x) \). For any \( y \in W \), \( R \) satisfies that
\[
x <_{R} y \implies y \in v(x) \iff x <_{R_u} y.
\]
Then we check the second condition: Take any \( x \in W \) and \( R \in S \). As \( \uparrow_{R} x \in V_M(x) \), there exists \( v \in V_{N(M)} \) such that \( \uparrow_{R} x = v(x) \). Then \( R_v \in S_{N(M)} \) satisfies that for any \( y \in W \),
\[
x <_{R_u} y \iff y \in v(x) \iff x <_{R} y.
\]
As \( \gamma_M \) is the identity mapping on \( W \), \( \gamma_M \) is an isomorphism of multi-relational Kripke frames.

Next, we prove that for any neighborhood frame \( Z = \langle C, V \rangle \), \( \delta_Z \) is an isomorphism of neighborhood frames from \( Z \) to \( N(H(Z)) \). Since \( Z \) is upward closed,
\[
\uparrow \{ \uparrow_{R} x \mid v \in V_Z \} = \{ \uparrow_{R} x \mid v \in V_Z \}
\]
for any \( x \in C \). Take any \( c \in C \) and \( X \subseteq C \). Then,
\[
X \in V(x) \iff \exists v \in V_Z(v(x) = X)
\]
\[
\iff \exists R_v \in S_Z(\uparrow_{R} x = X)
\]
\[
\iff X \in V_{H(Z)}(x) \quad \text{(by (6.1))}
\]
As \( \delta_Z \) is the identity mapping on \( C \), \( \delta_Z \) is an isomorphism of neighborhood frames.

By Theorem 4.2 and Theorem 6.3, we have the following:

**Theorem 6.5.** For any cardinal number \( \kappa \), \( \text{CAMA}_\kappa \) and \( \text{MRKF}_\kappa \) are dually equivalent.

By the same argument as Theorem 6.4, it follows that the category of all multi-relational Kripke frames is equivalent to the category of all upward closed neighborhood frames which includes the whole set. These categories are dually equivalent to the category of algebras which is obtained by weakening the definition of the modal operator in \( \text{CAMA} \) to the following: \( \Diamond \emptyset = 0 \) and \( \Diamond x \leq \Diamond y \) whenever \( x \leq y \).
In the rest of the paper, we give another direct proof of duality between \textit{CAMA}_\kappa and \textit{MRKF}_\kappa for every regular cardinal \kappa. First, we define a contravariant functor \( F : \textit{CAMA}_\kappa \rightarrow \textit{MRKF}_\kappa \) for every regular cardinal \( \kappa \). For any object \( A \) of \textit{CAMA}_\kappa, a multi-relational Kripke frame \( F(A) \) is defined by

\[
F(A) = \langle \mathcal{A}(A), \{ R(X) \mid X \subseteq A, |X| < \kappa \} \rangle,
\]

where, for any \( a \in \mathcal{A}(A) \) and \( b \in \mathcal{A}(B) \),

\[
a <_{R(X)} b \iff a \leq \bigwedge \diamond [\neg b \cap X],
\]

and for any arrow \( f : A \rightarrow B \) of \textit{CAMA}_\kappa, the mapping \( F(f) : \mathcal{A}(B) \rightarrow \mathcal{A}(A) \) is defined by

\[
F(f)(b) = f_* (b)
\]

for any \( b \in \mathcal{A}(B) \). Below, we show that \( F \) is a well-defined contravariant functor.

**Proposition 7.1.** Let \( \kappa \) be a regular cardinal. If \( A \) is a \( \kappa \)-additive complete atomic modal algebra, \( F(A) \) is a \( \kappa \)-downward directed multi-relational Kripke frame.

**Proof.** It is clear that \( F(A) \) is a multi-relational Kripke frame. We show that \( F(A) \) is \( \kappa \)-downward directed. Suppose \( X_i \subseteq A \) and \( |X_i| < \kappa \) for any \( i \in I \). If \( |I| < \kappa \), then

\[
| \bigcup_{i \in I} X_i | < \kappa,
\]

since \( \kappa \) is regular. Hence, \( F(A) \) is \( \kappa \)-downward directed, because

\[
R \left( \bigcup_{i \in I} X_i \right) \subseteq \bigcap_{i \in I} R(X_i).
\]

\[ \square \]

**Definition 7.2.** Let \( A \) be a \( \kappa \)-additive complete atomic modal algebra. For any \( X \subseteq A \) and \( a \in \mathcal{A}(A) \), \( p(X, a) \) denotes an element of \( A \) defined by

\[
p(X, a) = \bigvee \diamond^{-1} [\neg a] \cap X.
\]

**Lemma 7.3.** Let \( A \) be a \( \kappa \)-additive complete atomic modal algebra, \( X \) a subset of \( A \) such that \( |X| < \kappa \), and \( a \in \mathcal{A}(A) \). Then for any \( a' \in \mathcal{A}(A) \),

\[
a <_{R(X)} a' \iff a' \not\leq p(X, a).
\]

**Proof.** For any \( a' \in \mathcal{A}(A) \),

\[
a <_{R(X)} a' \iff a \leq \bigwedge \diamond [\neg a' \cap X]
\]

\[ \iff \forall x \in X (a' \leq x \Rightarrow a \leq \diamond x) \]

\[ \iff \forall x \in X (a \not\leq \diamond x \Rightarrow a' \not\leq x) \]

\[ \iff \forall x \in X (a \leq -\diamond x \Rightarrow a' \not\leq x) \quad (a \in \mathcal{A}(A)) \]

\[ \iff \forall x \in X (\diamond x \leq -a \Rightarrow a' \not\leq x) \]

\[ \iff \forall x \in X (x \in \diamond^{-1} [\neg a] \cap X \Rightarrow a' \not\leq x) \]

\[ \iff a' \not\leq \bigvee \diamond^{-1} [\neg a] \cap X \quad (a' \in \mathcal{A}(A)). \]

\[ \square \]
Lemma 7.4. Let $A$ and $B$ be $\kappa$-additive complete atomic modal algebras, $f : A \to B$ a homomorphism of complete modal algebras, $Y \subseteq B$ such that $|Y| < \kappa$, and $b \in A(B)$. Suppose $X = \{f^*(p(Y,b))\}$. Then for any $a \in A(A)$, 
$$f_*(b) <_{R(X)} a \iff a \not\leq f^*(p(Y,b)).$$

Proof. By Lemma (5.3), all we have to prove is 
$$f^*(p(Y,b)) = p(X, f_*(b)).$$

As 
$$p(X, f_*(b)) = \bigvee \diamond^{-1} [\downarrow (-f_*(b))] \cap \{f^*(p(Y,b))\},$$

it is enough to show 
$$f^*(p(Y,b)) \in \diamond^{-1} [\downarrow (-f_*(b))].$$

Since $B$ is $\kappa$-additive
$$\diamond f^*(p(Y,b)) \leq \diamond p(Y,b) \quad \text{(by (2.2))}$$
$$= \diamond \bigvee \diamond^{-1} [\downarrow (-b)] \cap Y$$
$$= \bigvee \diamond \big( \diamond^{-1} [\downarrow (-b)] \cap Y \big) \quad \text{(\kappa-additivity)}$$
$$\leq \bigvee \downarrow (-b)$$
$$= -b.$$

Hence
$$b \leq -\diamond f^*(p(Y,b)) = f(-\diamond f^*(p(Y,b))).$$

By (2.1),
$$f_*(b) \leq -\diamond f^*(p(Y,b)),$$
so
$$\diamond f^*(p(Y,b)) \leq -f_*(b).$$
Hence,
$$f^*(p(Y,b)) \in \diamond^{-1} [\downarrow (-f_*(b))].$$

Proposition 7.5. Let $\kappa$ be a regular cardinal. For any $\kappa$-additive complete atomic modal algebras $A$ and $B$ and for any homomorphism $f : A \to B$ of complete modal algebras, $F(f) : A(B) \to A(A)$ is a homomorphism of multi-relational Kripke frames from $F(B)$ to $F(A)$.

Proof. Condition 1 of Definition (5.1) Take any $b_1 \in A(B)$ and any $X \subseteq A$ such that $|X| < \kappa$. Then $|f[X]| < \kappa$. Take any $b_2 \in A(B)$. We show that
$$b_1 <_{R(f[X])} b_2 \Rightarrow f_*(b_1) <_{R(X)} f_*(b_2).$$
Suppose $b_1 <_{R(f[X])} b_2$. By definition of $R(f[X])$,
$$b_1 \leq \bigwedge \diamond \uparrow b_2 \cap f[X].$$

Therefore,
$$f_*(b_1) = \bigwedge_{x \in A, b_1 \leq f(x)} x \leq \bigwedge \{x \in A \mid \bigwedge \diamond \uparrow b_2 \cap f[X] \leq f(x)\}.$$  

On the other hand,
$$\diamond \uparrow f_*(b_2) \cap X \subseteq \{x \in A \mid \bigwedge \diamond \uparrow b_2 \cap f[X] \leq f(x)\},$$
because, for any $z \in \diamond \uparrow f_*(b_2) \cap X$, there exists $u \in X$ such that 
$$f_*(b_2) \leq u, \diamond u = z,$$
and, this implies \( b_2 \leq f(u) \) and \( f(u) \in f[X] \), and therefore,
\[
\bigwedge \diamond [\uparrow b_2 \cap f[X]] \leq \diamond f(u) = f(\diamond u) = f(z).
\]
By (7.1) and (7.2),
\[
f_*(b_1) \leq \bigwedge \diamond [\uparrow f_*(b_2) \cap X].
\]
Hence,
\[
f_*(b_2) \leq_{R(X)} f_*(b_1).
\]
Condition 2 of Definition 5.1 Take any \( b \in A(B) \) and any \( Y \subseteq B \) such that \( |Y| < \kappa \). Define \( X \subseteq I \) by
\[
X = \{ f^*(p(Y, b)) \}.
\]
Suppose \( a \in A(A) \) and \( f_*(b) \leq_{R(X)} a \). Then \( a \notin f^*(p(Y, b)) \) by Lemma 7.3. Hence, \( f(a) \notin p(Y, b) \). Since \( B \) is atomic, there exists \( b' \in A(B) \) such that
\[
b' \leq f(a), \quad b' \leq p(Y, b).
\]
Then \( f_*(b') \leq a, \) and \( b \in_{R(Y)} b' \) by Lemma 7.3. Since \( f_*(b') \) and \( a \) are in \( A(A) \), \( f_*(b') = a \).


8. Functor from \( \text{MRKF}_\kappa \) to \( \text{CAMA}_\kappa \)

We define a contravariant functor \( G : \text{MRKF}_\kappa \rightarrow \text{CAMA}_\kappa \) for every cardinal number \( \kappa \). For any object \( M = \langle W, S \rangle \) of \( \text{MRKF}_\kappa \), a complete atomic modal algebra \( G(M) \) is defined by
\[
G(M) = \langle \mathcal{P}(W); \cup, \cap, W \setminus -, \Diamond_M, \emptyset, W \rangle,
\]
where \( \Diamond_M \) is defined by
\[
\Diamond_M X = \bigcap_{R \in S} \downarrow_R X
\]
for any \( X \subseteq W \), and for any multi-relational Kripke frames \( M_1 = \langle W_1, S_1 \rangle, M_2 = \langle W_2, S_2 \rangle \), and any arrow \( g : M_1 \rightarrow M_2 \) of \( \text{MRKF}_\kappa \), the mapping \( G(g) : \mathcal{P}(W_2) \rightarrow \mathcal{P}(W_1) \) is defined by
\[
G(g)[X] = g^{-1}[X]
\]
for any \( X \subseteq W_2 \). Below, we show that \( G \) is a well-defined contravariant functor.

Proposition 8.1. Let \( \kappa \) be a cardinal number. If \( M = \langle W, S \rangle \) is a \( \kappa \)-downward directed multi-relational Kripke frame, \( G(g)(M) \) is a \( \kappa \)-additive complete atomic modal algebra.

Proof. It is clear that \( \langle \mathcal{P}(W); \cup, \cap, W \setminus -, \emptyset \rangle \) is a \( \kappa \)-additive complete atomic Boolean algebra. Since \( \downarrow_R \emptyset = \emptyset \) for any \( R \in S \),
\[
\Diamond_M \emptyset = \bigcap_{R \in S} \downarrow_R \emptyset = \emptyset.
\]
Let \( \{X_i\}_{i \in I} \) be a subset of \( \mathcal{P}(W) \) such that \( |I| < \kappa \). Since \( \Diamond_M \) is order preserving,
\[
\bigcup_{i \in I} \Diamond_M X_i \subseteq \Diamond_M \bigcup_{i \in I} X_i.
\]
We show the converse. For any \( w \in W \),
\[
w \notin \bigcup_{i \in I} \Diamond_M X_i \iff w \notin \bigcup_{i \in I} \bigcap_{R \in S} \downarrow_R X_i
\]
\[
\iff \forall i \in I \left( w \notin \bigcap_{R \in S} \downarrow_R X_i \right)
\]
\[
\iff \forall i \in I \exists R_i \in S \forall x \in X_i (w \notin R_i x).
\]
Since \( M \) is \( \kappa \)-downward directed, there exists \( Q \in S \) such that \( Q \subseteq \bigcap_{i \in I} R_i \).

Then \( \forall i \in I \forall x \in X_i (w \not< Q x) \).

Thus, \( w \not\in \downarrow Q \bigcup_{i \in I} X_i \).

Hence, \( w \not\in \bigcap_{R \in S_1 \downarrow R} \bigcup_{i \in I} X_i = \mathbf{\bullet} M \bigcup_{i \in I} X_i \).

Proposition 8.2. Let \( \kappa \) be a cardinal number. For any \( \kappa \)-downward directed multi-relational Kripke frames \( M_1 = \langle W_1, S_1 \rangle \), \( M_2 = \langle W_2, S_2 \rangle \), and a homomorphism \( g : M_1 \rightarrow M_2 \) of multi-relational Kripke frames, \( G(g) : \mathcal{P}(W_2) \rightarrow \mathcal{P}(W_1) \) is a homomorphism of complete modal algebras from \( G(M_1) \) to \( G(M_2) \).

Proof. We only show that for any \( U \subseteq W_2 \),

\[ \bigcap_{R \in S_1 \downarrow R} g^{-1}[U] = g^{-1} \left( \bigcap_{Q \in S_2 \downarrow Q} U \right). \]

All we have to prove is

\[ \bigcap_{R \in S_1 \downarrow R} g^{-1}[U] = g^{-1} \left( \bigcap_{Q \in S_2 \downarrow Q} U \right). \]

(\( \subseteq \)): Take any \( x \in W_1 \) and suppose \( x \in \bigcap_{R \in S_1 \downarrow R} g^{-1}[U] \). Then \( \forall R \in S_1 \), \( \exists w \in g^{-1}[U] \) with \( x < g^{-1}[R] w \).

Since \( g \) is a homomorphism of multi-relational Kripke frames, for any \( Q \in S_2 \), there exists \( R_Q \in S_1 \) such that for any \( y \in W_1 \),\n
If \( x < R_Q y \Rightarrow g(x) < Q g(y) \).

Therefore, for any \( Q \in S_2 \) there exists \( R_Q \in S_1 \) such that for any \( x \in W_1 \),\n
\( x < R_Q y \Rightarrow g(x) < g(R_Q y) \).

Since \( g \) is a homomorphism of multi-relational Kripke frames, for any \( Q \in S_2 \), there exists \( R_Q \in S_1 \) such that for any \( y \in W_1 \),\n
\( y < g(R_Q y) \Rightarrow g(y) < g(R_Q y) \).

Hence, \( g(x) \in \bigcap_{Q \in S_2 \downarrow Q} U \).

Since \( Q \) is arbitrary,

\( x \in \bigcap_{Q \in S_2 \downarrow Q} U \).

Hence, \( x \in g^{-1} \left( \bigcap_{Q \in S_2 \downarrow Q} U \right) \).

Thus,\n
\[ \bigcap_{R \in S_1 \downarrow R} g^{-1}[U] \subseteq \bigcap_{Q \in S_2 \downarrow Q} U. \]

Hence, \( \bigcap_{R \in S_1 \downarrow R} g^{-1}[U] = g^{-1} \left( \bigcap_{Q \in S_2 \downarrow Q} U \right) \).
(⊇): Take any \( x \in W_1 \). Then
\[
x \in g^{-1} \left[ \bigcap_{Q \in S_2} \downarrow_Q U \right] \iff g(x) \in \bigcap_{Q \in S_2} \downarrow_Q U
\]
\[
\iff \forall Q \in S_2 \exists u_Q \in U \ (g(x) <_Q u_Q).
\]
Since \( g \) is a homomorphism of multi-relational Kripke frames, for any \( R \in S_1 \), there exists \( Q_R \in S_2 \) such that for any \( u \in W_2 \),
\[
g(x) <_{Q_R} u \Rightarrow \exists y \in W_1 \text{ such that } x <_R y \text{ and } g(y) = u.
\]
Therefore, for any \( R \in S_1 \), there exist \( Q_R \in S_2 \), \( u_{Q_R} \in U \), and \( y \in W_1 \) such that
\[
x <_R y, \ g(y) = u_{Q_R} \in U.
\]
Hence,
\[
x \in \downarrow_R g^{-1} [U].
\]
Since \( R \) is arbitrary,
\[
x \in \bigcap_{R \in S_1} \downarrow_R g^{-1} [U].
\]
\[\Box\]

9. Duality between \( \text{CAMA}_\kappa \) and \( \text{MRKF}_\kappa \)

In this section, we show that for any regular cardinal \( \kappa \),
\[
\text{Id}_{\text{CAMA}_\kappa} \cong G \circ F, \ \text{Id}_{\text{MRKF}_\kappa} \cong F \circ G.
\]

**Proposition 9.1.** Let \( \kappa \) be a regular cardinal. For any object \( A \) of \( \text{CAMA}_\kappa \), define a mapping \( \tau_A : A \to G(F(A)) \) by
\[
\tau_A(x) = \{ a \in \mathcal{A}(A) \mid a \leq x \}
\]
for any \( x \in A \). Then \( \tau \) is a natural transformation from \( \text{Id}_{\text{CAMA}_\kappa} \) to \( G \circ F \).

**Proof.** Let \( f : A \to B \) be an arrow of \( \text{CAMA}_\kappa \). Then for any \( x \in A \) and \( b \in \mathcal{A}(B) \),
\[
b \in G(F(f)) \circ \tau_A(x) \iff b \in (f_\ast)^{-1} \left[ \{ a \in \mathcal{A}(A) \mid a \leq x \} \right]
\]
\[
\iff f_\ast(b) \leq x
\]
\[
\iff b \leq f(x)
\]
\[
\iff b \in \tau_B \circ f(x).
\]
Hence,
\[
G(F(f)) \circ \tau_A = \tau_B \circ f.
\]
\[\Box\]

**Theorem 9.2.** Let \( \kappa \) be a regular cardinal. For any object \( A \) of \( \text{CAMA}_\kappa \), \( \tau_A : A \to G(F(A)) \) is an isomorphism of complete modal algebras.

**Proof.** It is clear that \( \tau_A \) is an isomorphism of complete Boolean algebras. We show that
\[
\tau_A(\Diamond x) = \Diamond_{F(A)} \tau_A(x)
\]
for any \( x \in A \). What we have to show is
\[
\{ a \in \mathcal{A}(A) \mid a \leq \Diamond x \} = \bigcap_{X \subseteq A, |X| < \kappa} \downarrow_{R(X)} \{ a \in \mathcal{A}(A) \mid a \leq x \}.
\]
DUALITY FOR $\kappa$-ADDITIVE COMPLETE ATOMIC MODAL ALGEBRAS

(⊆): Suppose $a \leq \Diamond x$. Take any $X \subseteq A$ such that $|X| < \kappa$. If $x \leq p(X, a)$, then

$$
a \leq \Diamond x
\leq \Diamond p(X, a)
= \Diamond \bigvee \Diamond^{-1} [\downarrow (-a)] \cap X
= \bigvee \Diamond \left[ \Diamond^{-1} [\downarrow (-a)] \cap X \right] \quad \text{(κ-additivity)}
\leq \bigvee \downarrow (-a)
= -a,
$$

which contradicts to $a \in A(A)$. Hence, $x \not\leq p(X, a)$. As $A$ is atomic, there exists $b \in A(A)$ such that $b \leq x$ and $b \not\leq p(X, a)$. Then $a \leq^R X \{ b \in A(A) \mid b \leq x \}$.

As $X$ is taken arbitrarily,

$$
a \in \bigcap_{X \subseteq A, |X| < \kappa} \downarrow^R X \{ b \in A(A) \mid b \leq x \}.
$$

(⊇): Suppose $a \not\leq \Diamond x$. Then for any $b \in A(A)$ such that $b \leq x$,

$$
a \not\leq \Diamond x = \bigwedge \Diamond \left[ [\downarrow h] \cap \{x\} \right]
$$

Hence,

$$
a \not\in \downarrow^R \{ b \in A(A) \mid b \leq x \}.
$$

Thus,

$$
a \not\in \bigcap_{X \subseteq A, |X| < \kappa} \downarrow^R X \{ b \in A(A) \mid b \leq x \}.
$$

Proposition 9.3. Let $\kappa$ be a regular cardinal. For any object $M = \langle W, S \rangle$ of $\text{MRKF}_\kappa$, define $\theta_M : M \to F(G(M))$ by

$$
\theta_M(w) = \{w\}
$$

for any $w \in W$. Then $\theta$ is a natural transformation from $\text{Id}_{\text{MRKF}_\kappa}$ to $F \circ G$.

Proof. For any $M$, $\theta_M$ is well-defined as a mapping, since

$$
A(G(M)) = \{ \{w\} \mid w \in W \}.
$$

Let $M_1 = \langle W_1, S_1 \rangle$ and $M_2 = \langle W_2, S_2 \rangle$ be objects of $\text{MRKF}_\kappa$, and $g : M_1 \to M_2$ an arrow of $\text{MRKF}_\kappa$. Then for any $w \in W_1$,

$$
F(G(g)) \circ \theta_{M_1}(w) = G(g)(\{w\})
= \bigcap \{ X \subseteq W_2 \mid w \in G(g)(X) \}
= \bigcap \{ X \subseteq W_2 \mid w \in g^{-1}[X] \}
= \bigcap \{ X \subseteq W_2 \mid g(w) \in X \}
= \{g(w)\}
= \theta_{M_2}(g(w)).
$$

Hence,

$$
F(G(g)) \circ \theta_{M_1} = \theta_{M_2} \circ g.
$$

Theorem 9.4. Let $\kappa$ be a regular cardinal. For any object $M = \langle W, S \rangle$ of $\text{MRKF}_\kappa$, $\theta_M : M \to F(G(M))$ is an isomorphism of multi-relational Kripke frames.
Proof. It is clear that $\theta_M$ is a set-theoretical bijection. We show that it is a homomorphism of multi-relational Kripke frames. By definition of $G$ and $F$, $$F(G(M)) = \{ \{ w \mid w \in W \} , \{ R(U) \mid U \subseteq \mathcal{P}(W) , |U| < \kappa \} ,$$ where $$\{ w_1 \} \prec \{ w_2 \} \iff \{ w_1 \} \subseteq \cap \Diamond_M [ \uparrow \{ w_2 \} \cap U ] .$$ By definition of $\Diamond_M$ in $G(M)$, $$\{ w_1 \} \prec \{ w_2 \} \iff \{ w_1 \} \subseteq \cap \left\{ \bigcap_{R \in S} \downarrow R X \mid X \in \uparrow \{ w_2 \} \cap U \right\} ,$$ where $$\{ w_1 \} \prec \{ w_2 \} \iff \{ w_1 \} \subseteq \bigcap_{R \in S} \downarrow R X = \{ w_2 \} \cap U .$$ Condition 1 of Definition 5.1: Take any $w \in W$ and any $U \subseteq \mathcal{P}(W)$ such that $|U| < \kappa$. For any $X \in U$, if $w \notin \bigcap_{R \in S} \downarrow R X$, then we can fix one $R_X \in S$ such that $w \notin \downarrow R_X X$. Since $M$ is $\kappa$-downward directed, there exists $Q \subseteq S$ such that $$Q \subseteq \bigcap_{R \in S} \left\{ R_X \mid X \in U , w \notin \bigcap_{R \in S} \downarrow R X \right\} .$$ We claim that for any $w' \in W$, $$w \prec Q w' \Rightarrow \{ w \} \prec \{ w' \} .$$ Suppose $w \prec Q w'$. Take any $X \in U$ and suppose $w \notin \bigcap_{R \in S} \downarrow R X$. Then $w \notin \downarrow R X$. As $w \prec R_X w'$ by definition of $Q$, $w' \notin X$.

Condition 2 of Definition 5.1: Take any $w \in W$ and any $R \in S$. Let $$U = \{ W \setminus \uparrow_R w \} .$$ Clearly, $$w \notin \downarrow_R (W \setminus \uparrow_R w) .$$ Therefore, $$w \notin \bigcap_{Q \in S} \downarrow Q (W \setminus \uparrow_R w) .$$ Hence, for any $v \in W$, $$\{ v \} \prec \{ w \} \iff v \notin W \setminus \uparrow_R w$$ $$\iff w < R v .$$

Theorem 9.5. For any regular cardinal $\kappa$, $\text{CAMA}_\kappa$ and $\text{MRKF}_\kappa$ are dually equivalent. 

Proof. Theorem 9.2 and Theorem 9.4.

Corollary 9.6. Let $M_1 = (W_1 , S_1)$ and $M_2 = (W_2 , S_2)$ be multi-relational Kripke frames. A mapping $f : W_1 \to W_2$ is a homomorphism of multi-relational Kripke frames from $M_1$ to $M_2$ if and only if the mapping $g : \mathcal{P}(M_2) \to \mathcal{P}(M_1)$ which is defined by $$g : S \mapsto f^{-1}[S]$$ for any $S \subseteq W_2$ is a homomorphism of complete modal algebras from $G(M_2)$ to $G(M_1)$. 

\[ \square \]
Proof. We only show the if-part. Suppose that $g$ is a homomorphism of complete modal algebras. Then $F(g) : FG(M_2) \to FG(M_2)$ is a homomorphism of multi-relational Kripke frames. Let

$$h = \theta_{M_2}^{-1} \circ F(g) \circ \theta_{M_1}.$$

By definition of $\theta$ and $\tau$, the composite of $G\theta$ and $\tau_G$ is the identity natural transformation on $G$. Hence, for any $S \subseteq \mathcal{P}(W_2)$,

$$h^{-1}[S] = G(h)(S) = G(\theta_{M_1}) \circ GF(g) \circ G(\theta_{M_2}^{-1}) = \tau_{G(M_1)} \circ GF(g) \circ \tau_{G(M_2)} = g(S) = f^{-1}[S].$$

Thus, $f = h$ is a homomorphism of multi-relational Kripke frames.

10. Application

As an application of the duality theorem, we show that for any regular cardinals $\kappa$ and $\kappa'$ with $\kappa < \kappa'$, the inclusion functor from $\text{CAMA}_{\kappa'}$ to $\text{CAMA}_\kappa$ and that from $\text{MRKF}_{\kappa'}$ to $\text{MRKF}_\kappa$ are not essentially surjective, where a functor $F$ from a category $C$ to a category $D$ is said to be essentially surjective, if for any object $d$ of $D$, there exists an object $c$ of $C$ such that $F(c)$ is isomorphic to $d$.

The following proposition is based on Fact 4.5 of [7].

**Proposition 10.1.** Let $\kappa$ and $\kappa'$ be regular cardinals. If $\kappa < \kappa'$, there exists a complete atomic modal algebra $A$ which is $\kappa$-additive but not $\kappa'$-additive.

**Proof.** Consider a multi-relational Kripke frame $M$ defined by

$$M = \langle \kappa \cup \{\infty\}, \{Q_X \mid X \subseteq \kappa, |X| < \kappa \} \rangle$$

where

$$Q_X = \{(\infty, \alpha) \mid \alpha \notin X\}.$$

Suppose $|I| < \kappa$, and for any $i \in I$, suppose $X_i \subseteq \kappa$ and $|X_i| < \kappa$. Then $|\bigcup_{i \in I} X_i| < \kappa$ and

$$Q_{\bigcup_{i \in I} X_i} = \bigcap_{i \in I} Q_{X_i}.$$

Hence, $M$ is an object of $\text{MRKF}_{\kappa}$. Therefore, by the duality theorem, $G(M)$ is an object of $\text{CAMA}_\kappa$. We show that in $G(M)$,

$$\diamond_M \bigvee_{i \in \kappa} \{i\} \nleq \bigvee_{i \in \kappa} \diamond_M \{i\}.$$
For any $X \subseteq \kappa$ such that $|X| < \kappa$, there exists $i \in \kappa$ such that $i \not\in X$. Hence, 

$$\infty \in \bigcap_{X \subseteq \kappa, |X| < \kappa} \downarrow Q_X \bigcup_{i \in \kappa} \{i\}.$$ 

Thus, 

$$\infty \in \diamond_M \bigcup_{i \in \kappa} \{i\}.$$ 

On the other side, for any $i \in \kappa$, 

$$\infty \not\in \downarrow Q_{(i)} \{i\}.$$ 

Therefore, 

$$\infty \not\in \bigcap_{X \subseteq \kappa, |X| < \kappa} \downarrow Q_X \{i\}.$$ 

Since $i$ is taken arbitrarily 

$$\infty \not\in \bigcup_{i \in I, X \subseteq \kappa, |X| < \kappa} \downarrow Q_X \{i\}.$$ 

Hence, 

$$\infty \not\in \bigvee_{i \in \kappa} \diamond_M \{i\}.$$ 

\[\blacklozenge\]

**Theorem 10.2.** Let $\kappa$ and $\kappa'$ be regular cardinals such that $\kappa < \kappa'$. Then the inclusion functor from $\text{CAMA}_{\kappa'}$ to $\text{CAMA}_{\kappa}$ and that from $\text{MRKF}_{\kappa'}$ to $\text{MRKF}_{\kappa}$ are not essentially surjective.

**Proof.** Let $M$ be the multi-relational Kripke frame defined in Proposition 10.1. Then $G(M)$ is an object of $\text{CAMA}_{\kappa}$, and it is clear that no objects of $\text{CAMA}_{\kappa'}$ are isomorphic to $G(M)$. Hence, by Theorem 10.1, no objects of $\text{MRKF}_{\kappa'}$ are isomorphic to $M$. \[\blacklozenge\]

**References**

[1] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge, third edition, 2001.

[2] Erica Calardo. *Non-normal modal logics, quantification and deontic dilemmas. A study in multi-relational semantics*. PhD thesis, University of Bologna, 2013.

[3] Kosta Došen. Duality between modal algebras and neighbourhood frames. *Studia Logica*, 48:219–234, 1989.

[4] Steven Givant and Paul Halmos. *Introduction to Boolean Algebras*. Springer, 2009.

[5] Lou Goble. Multiplex semantics for deontic logic. *Nordic Journal of Philosophical Logic*, 5(2):113–134, 2000.

[6] Wesley H Holliday and Tadeusz Litak. Complete additivity and modal incompleteness. *The Review of Symbolic Logic*, 12(3):487–535, 2019.

[7] Pierluigi Minari. Some remarks on the proof-theory and the semantics of infinitary logic. In Reinhard Kahle, Thomas Strahm, and Thomas Studer, editors, *Advances in Proof Theory*, pages 291–318. Springer, 2016.

[8] Yoshihito Tanaka. Cut-elimination theorems for some infinitary modal logics. *Mathematical Logic Quarterly*, 47:327–339, 2001.

[9] Steven K. Thomason. Categories of frames for modal logic. *The Journal of Symbolic Logic*, 40(3):439–442, 1975.

**Department of Economics, Kyushu Sangyo University, Fukuoka 813-0015, JAPAN**

*E-mail address: ytanaka@ip.kyusan-u.ac.jp*