On the classification of laminations associated to quadratic polynomials

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Abstract

Given any rational map \( f \), there is a lamination by Riemann surfaces associated to \( f \). Such laminations were constructed in general by Lyubich and Minsky. In this paper, we classify laminations associated to quadratic polynomials with periodic critical point. In particular, we prove that the topology of such laminations determines the combinatorics of the parameter. We also describe the topology of laminations associated to other types of quadratic polynomials.

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1 Introduction

The “natural extension” $N_f$ (or the “inverse limit”) of a rational map $f$ is a very interesting object whose topology and geometry reflects in an intricate way the dynamics of $f$ (see Lyubich and Minsky in [21]). In this paper, we study the relation between the topology of $N_f$ and the dynamics of $f$, focusing on the case of quadratic polynomials $f_c: z \mapsto z^2 + c$.

The natural extension $N_f$ contains the “regular leaf space” $R_f$ whose connected components (“leaves”) are endowed with a natural conformal structure. When the dynamics of $f$ is simple, the corresponding regular leaf space has a lamination structure; that is, there is an atlas of charts, such that the image of every chart is the product of a disk times a Cantor set. Moreover, the leaves of $R_f$ are simply connected Riemann surfaces conformally isomorphic to the complex plane. This is the case for hyperbolic rational maps, where $R_f$ is obtained from $N_f$ by removing finitely many points. In general, associated to any rational map, Lyubich and Minsky’s construction provides us with an orbifold affine lamination which, in turn, admits an extension to a 3-dimensional hyperbolic lamination.

The main goal of this paper is to prove that, for hyperbolic quadratic polynomials, the topology of the lamination determines the combinatorics of the corresponding parameter. More precisely:

**Main Theorem.** If $h : R_c \to R_{c'}$ is an orientation preserving homeomorphism between the regular leaf spaces associated to the superattracting parameters $c$ and $c'$, then $c = c'$.

Let us now give a more detailed description of the contents of the paper.

In Section 2, we summarize the necessary background in basic holomorphic dynamics. We assume the reader is familiar with the subject, and we highlight only the facts that are important for understanding the structure of the associated laminations. Most of this theory is readily available in various surveys in complex dynamics, e.g., [2], [8], [9], [13] and [24].

From the lamination point of view, the classical theory of linearization around periodic points provides us with suitable uniformizations of the corresponding periodic leaves. We will make use of this application of Königs and Böttcher coordinates (near repelling and superattracting points). This approach has proven useful in the parabolic case (see Tomoki Kawahira [14]).

Although we are focused on the superattracting case, most of our results are valid over a broader class of parameters, specifically those for which the critical point is non-recurrent and has no parabolics. According to [21], a rational map has non-recurrent critical points and no parabolics if and only if the corresponding 3-lamination is convex cocompact. Because of this property, we call this class of quadratic polynomials, or the corresponding parameters, convex cocompact. We describe some basic dynamical features of convex cocompact polynomials (see [7] and [22]).

There are several combinatorial models that describe superattracting quadratic polynomials. For a complete discussion of them see the paper of Henk Bruin and Dierk Schleicher [11]. We describe some of these models, for which we have found topological analogues in the regular leaf space. One of the most informative is the model of ray portraits as presented by John Milnor in [24]. For the proof of our main theorem, we will see that the topology of the affine lamination associated with a superattracting quadratic polynomial determines the ray portrait of the corresponding parameter. The interested reader can also find more about the combinatorics of postcritically finite quadratic polynomials in [11], [8], [11], [13], [18], [25], and [28].

In Section 3, we discuss basic definitions and properties of inverse limits. A classical example of an inverse limit is the dyadic solenoid $S^1$, associated to the map $f_0 : z \mapsto z^2$ on the unit circle $S^1$. As a lamination the dyadic solenoid is one-dimensional; moreover, it is naturally endowed with the structure of a compact topological group.

The map $f_0$ admits a natural extension $\hat{f}_0$ acting on the solenoid $S^1$. It turns out that $\hat{f}_0$-periodic leaves in $S^1$ are in one-to-one correspondence with periodic points of $f_0$ in $S^1$. In turn, the periodic points of $f_0$ are in one-to-one correspondence with rational numbers with odd denominators modulo 1. These observations are the first step towards the proof of the Main Theorem.

To be more precise, let us introduce some objects that will play a key role: We call a solenoidal cone any space homeomorphic to the inverse limit $\text{Con}(S^1) = \lim_{\leftarrow} (f_0, \mathbb{C} \setminus \mathbb{D})$. Due to Böttcher’s coordinate at infinity, for every quadratic polynomial $f_c$ with connected Julia set, $\text{Con}_c$, the lift of the basin of infinity to $\lim(f_c, \hat{\mathbb{C}})$ is homeomorphic to $\text{Con}(S^1) \setminus S^1$. We call $\text{Con}_c$ the solenoidal cone at infinity of $f_c$. 

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Now, note that $Con(S^1)$ is foliated by 1-dimensional solenoids coming from the lifts of the equipotentials in the dynamical plane. By means of Böttcher’s coordinate, we can transfer this foliation to $Con_c$. In $Con_c$ we call any of the corresponding solenoids a solenoidal equipotential. Each solenoidal equipotential is canonically identified with $S^1$. Such identification, allow us to keep track of periodic leaves in $R_c$ in a neighborhood of infinity.

A superattracting parameter $c$ admits a combinatorial model given in terms of external rays, we will see that this combinatorial model persists in the realm of the inverse limit of $f_c$. As a matter of fact, this combinatorial model is the same for all parameters inside a given hyperbolic component in the Mandelbrot set. At the end of Section 3, we prove that the regular leaf space associated to any hyperbolic quadratic polynomial $f_c$ is homeomorphic to the regular leaf space of the center of the hyperbolic component containing $c$. Hence, it is enough to describe the topology of the centers of hyperbolic components in the Mandelbrot set.

In Section 4, we describe the topology of the laminated Julia set, that is, the lift of the Julia set to the regular leaf space. When $f_c$ is convex cocompact, the laminated Julia set is compact, see [21]. We show that if the postcritical set is not the whole Julia set and the Julia set is locally connected, then $f_c$ is convex cocompact if and only if the laminated Julia set is leafwise connected.

By another result in [21], if $f_c$ is convex cocompact, then all the leaves of the regular leaf space $R_c$ are conformally isomorphic to the complex plane. Given a leaf $L \subset R_c$, let us consider the set of unbounded Fatou components in $L$. We show that non-periodic leaves of $R_c$ have at most 2 unbounded Fatou components. On the other hand, the number of unbounded Fatou components on periodic leaves depends on the valence of the corresponding repelling periodic point.

Once we have described the basic topology of the laminated Julia set, we are ready to give some restrictions on homeomorphisms between regular leaf spaces of superattracting parameters. Because the regular leaf space is locally compact and the laminated Julia set is compact, we can compute the number of unbounded Fatou components in some leaf $L$ in terms of the topology at infinity of $L$.

Assume $f_c$ is superattracting, and let $U$ be the Fatou component containing the critical point. The first return map of $U$ has a unique fixed point, say $r_c$, on the boundary of $U$. Following Milnor, $r_c$ is called the dynamic root of $f_c$. Accordingly, we call the cycle of $f_c$ containing the dynamic root, the dynamic root cycle of $f_c$. Since $r_c \in J(f_c)$ and $f_c$ is superattracting, the point $r_c$ must be a repelling periodic point of $f_c$. We prove that the topology at infinity of the leaves containing the lift of the dynamic root cycle is unique among all other leaves. Then, any homeomorphism as in the Main Theorem must send leaves containing the lift of the dynamic root cycle of $f_c$ into the corresponding leaves of $R'$. \textbf{In Section 5, we prove the Main Theorem. The strategy of the proof is to replace the homeomorphism $h: R_c \to R'_c$ by another homeomorphism $\tilde{h}$ with special characteristics.}

First we show that $h$ is isotopic to a homeomorphism $\tilde{h}$ that sends a solenoidal equipotential $S_R$ of $R_c$ homeomorphically into a solenoidal equipotential $S_R'$ of $R'_c$. Now, using the canonical identifications with $S^1$ on $S_R$ and $S_R'$, the map $\tilde{h}$ induces a self-homeomorphism of the dyadic solenoid, which according to Kwapisz [17] is isotopic to an affine transformation of the dyadic solenoid (that is, to the composition of an automorphism and a translation). In order to get $\tilde{h}$, we discuss some isotopic properties of self-homeomorphisms of the dyadic solenoid $S^1$ and its cone $Con(S^1)$.

The last part of the argument is to show that the translation part of the affine transformation of the previous paragraph is isotopic to zero. Thus after a series of improvements of the original homeomorphism, we get a homeomorphism $\tilde{h}$ sending $S_R$ in $R_c$ onto $S_R'$, and such that the restriction of $\tilde{h}$ to the solenoidal equipotentials is the identity. It follows that the orbit portraits of the dynamic root cycles of $c$ and $c'$ are the same, which completes the proof of the Main Theorem.

There is an algebraic approach to laminations given by the theory of Iterated Monodromy Groups developed by Volodymyr Nekrashevych (see [26] and [1]). These Iterated Monodromy Groups are precisely the monodromy groups of the regular spaces of postcritically finite rational maps. For special cases, we were able to prove some of the results in this paper using just properties of the corresponding Iterated Monodromy Group (see [5]). However, in general the combinatorics of these groups is more subtle to describe. A good exposition of combinatorial methods for rational maps can be found in Kevin Pilgrim’s survey [27].
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2 Basic theory

We assume the reader is familiar with the basic theory of dynamics of rational maps. Definitions and properties of the objects involved are widely spread in the literature; a very nice exposition can be found in Milnor’s book [24], and other sources are [6] and [19].

We will review three combinatorial models of postcritically finite quadratic polynomials, each one of which will be useful in this work when we want to prove properties of regular spaces associated to postcritically finite polynomials. For the proof of the Main theorem we are going to use the model of ray portraits as presented by Milnor in [25]; for the interested reader there is a complete discussion of combinatorics of postcritically finite quadratic polynomials in [4]. At the end of the section, we will discuss some properties for critically non-recurrent quadratic polynomials.

2.1 Combinatorics of postcritically finite polynomials

A natural way to classify postcritically finite polynomials is by describing the different arrangements that the critical orbit may have as a subset in the plane. The combinatorics of this arrangement are reflected in both dynamical and topological properties of the Julia set. In fact, we could say that similar properties hold for a slightly larger set of parameters; namely, postcritically non-recurrent parameters. Combinatorics also make explicit the relationship between the parameter and the dynamical plane. So certain combinatorial behavior of a given parameter \(c\) determines a region in the parameter plane where \(c\) must lie.

2.2 Hubbard trees

To begin with, let us present a combinatorial model given by Douady and Hubbard in [9]; for any postcritically finite parameter \(c\) they constructed an abstract graph, called the Hubbard tree, describing the dynamical arrangement of the postcritical set. In [9], Douady and Hubbard also proved that different combinatorics induce different Hubbard trees.

The Hubbard tree is properly embedded in the dynamical plane as a subset of the filled Julia set and can be defined by a finite set of vertices in the Julia set. To see this, we note that between any two points \(z\) and \(ζ\) in the Julia set \(J(f)\) there is an unique arc \(γ\), embedded in the Julia set, connecting \(z\) with \(ζ\). The uniqueness of \(γ\) is subject to the condition that, if the trajectory of \(γ\) intersects a Fatou component, then it goes along internal rays. In this way, the Hubbard tree of a postcritically finite quadratic polynomial \(f_c\) is defined as the smallest collection of arcs, embedded in the Julia set \(J(f_c)\) and connecting the entire critical orbit. This tree is finite and forward invariant under the action of \(f_c\).

2.3 Orbit portraits

The description by orbit portraits of postcritically finite maps is based upon the following proposition:

Proposition 1 (Douady and Hubbard). Every repelling and parabolic periodic point of a quadratic polynomial \(f_c\) is the landing point of an external ray with rational angle. Conversely, every external ray with rational angle lands either at a periodic or preperiodic point in \(J(f_c)\).

From now on, we will follow the exposition of John Milnor in [25], see also the related work of Alfredo Poirier [28] and Dierk Schleicher [31]. By Proposition 1 every parabolic or repelling cycle \(P = \{p_1, ..., p_n\}\) is associated to a family of finite sets \(O_P = \{A_1, ..., A_n\}\), where \(A_i = \{θ ∈ \mathbb{Q}/\mathbb{Z} | Rθ\text{ lands in } p_i\}\). The family \(O_P\) is called the orbit portrait of the cycle \(P\). The doubling map \(D\) on \(T\) permutes the sets \(A_i\) and acts on the angles in \(A_i\) in an order preserving way. Moreover, every angle in \(∪A_i\) is periodic under \(D\) and the
period of such angle only depends on the cycle. As a consequence, all \( A_i \) have the same cardinality. Thus, for \( p \in P \), the \textit{valence} \( v_p \) of \( p \) is the cardinality of any of the sets \( A_i \in O_p \).

Given two angles \( \theta_1 \) and \( \theta_2 \) in the unit circle \( \mathbb{T} \), by \( \hat{\theta}_1 \hat{\theta}_2 \) we denote the \textit{directed arc} in \( \mathbb{T} \) from \( \theta_1 \) to \( \theta_2 \). Now, if \( A \) is a finite set in \( \mathbb{T} \), a \textit{complementary arc} of \( A \) is the closure of any connected component of \( \mathbb{T} \setminus A \). If the valence is bigger than one, each \( A_i \) determines a finite collection of complementary arcs in the circle. Among the union of all complementary arcs of all \( A_i \), see Lemmas 2.5 and 2.6 in [25], there is a unique arc of shortest length called the \textit{characteristic arc}. The preimage of the characteristic arc under doubling is also the longest complementary arc which is called the \textit{critical arc}; it is worth noting that the critical arc can also be defined as the unique complementary arc subtending an angle bigger than 1/2. Since the doubling map preserves the order in the complementary arcs, if \( \hat{\theta}_1 \hat{\theta}_2 \) is the characteristic arc of some orbit portrait \( O_p \), then by a straightforward calculation, the critical arc is \( \eta_1 \eta_2 \) where \( \eta_1 = \theta_1/2 + 1/2 \) and \( \eta_2 = \theta_2/2 \).

### 2.4 The dynamic root point

Let \( c \) be a superattracting parameter of period \( n \), then the first return map of the Fatou component containing \( c \) has a unique fixed point on the boundary. Following Milnor [25] and Schleicher [31], we call \( r_c \) the \textit{dynamic root} of the superattracting parameter \( c \). The orbit portrait of \( r_c \) is called the \textit{critical portrait} of \( c \). It is a Theorem by Milnor [25] that, the critical portrait characterizes the parameter \( c \). In other words, no two superattracting parameters have the same critical portrait (see also [28]). Recall that rotations on the unit circle \( \mathbb{T} \) are given by maps \( r_\theta : \mathbb{T} \to \mathbb{T} \) of the form \( r_\theta(\tau) = \tau + \theta \), mod (\( \mathbb{Z} \)). The following lemma will be important for the closing argument in the Main Theorem:

**Lemma 2.** Let \( O_P \) and \( O_{P'} \) be the orbit portraits of the periodic cycles \( P \) and \( P' \). If there is a rotation of the circle \( r_\theta \), such that \( r_\theta(O_P) = O_{P'} \), then \( O_P = O_{P'} \). In particular, if \( O_c \) and \( O_{c'} \) are the critical portraits of two superattracting quadratic polynomials differing by a rotation, then \( c = c' \).

**Proof.** Since the characteristic arc is the minimal complementary arc, the rotation \( r_\theta \) must send the characteristic arc of \( P \) to the characteristic arc of \( P' \). Analogously, \( r_\theta \) must send the critical arc of \( P \) to the critical arc of \( P' \). If \( \hat{\theta}_1 \hat{\theta}_2 \) and \( \hat{\theta}_1' \hat{\theta}_2' \) are the characteristic arcs of \( P \) and \( P' \) respectively, then the hypothesis yields the following equations:

\[
\theta_i + \theta = \theta_i'
\]

for the characteristic arcs, and

\[
\theta_i/2 + 1/2 + \theta = \theta_i'/2 + 1/2
\]

for the critical arcs. Thus, \( \theta = 0 \) and \( \theta_i = \theta_i' \) for \( i = 1, 2 \), then the critical arc of \( f_c \) is equal to the critical arc of \( f_{c'} \). Since the characteristic arcs generate the orbit portrait, \( P = P' \). The second part of the lemma follows from the fact that the critical portrait determines the parameter \( c \).

### 2.5 Combinatorics in the parameter plane

Now, we briefly discuss how the location of a certain parameter \( c \) in the Mandelbrot set affects the combinatorics of periodic orbits in the dynamical plane of \( f_c \). Analogous to Proposition 1 we have:

**Proposition 3** (Douady and Hubbard). \textit{In the parameter plane, every parabolic or Misiurewicz parameter is the landing point of at least one external ray of rational angle. Inversely, every external ray with rational angle \( \theta \) lands at some point \( c \) in the boundary of the Mandelbrot set. Moreover, if \( \theta \) has odd denominator, then \( c \) is a parabolic parameter and Misiurewicz otherwise.}

In this way, every rational angle is associated to either a parabolic or a Misiurewicz parameter. Besides \( c = 1/4 \), which corresponds to the cusp of the main cardioid, every parabolic parameter \( c \) is the landing point of exactly two external rays, say \( R_{\theta_1} \) and \( R_{\theta_2} \). These external rays cut the plane into two parts, one of which contains the Main Cardioid and the other is called the \textit{wake} \( W_c \), determined by \( c \). The \textit{limb at} \( c \) is the intersection \( W_c \cap M \).

Every parabolic parameter \( c \) can be regarded as the root of some hyperbolic component \( H_c \); if \( c \neq 1/4 \), then \( H_c \) is contained in \( W_c \). Thus, except for the hyperbolic component inside the Main Cardioid, the
root of every hyperbolic component \( H \) disconnects \( H \) from the Main Cardioid. Even more, and here is the one of the most beautiful features of the combinatorics of quadratic polynomials, if \( R_{\theta_1} \) and \( R_{\theta_2} \) are the rays determining the wakes of the parabolic parameter \( c \), then \( \theta_1 \theta_2 \) corresponds to the characteristic arc of the parabolic cycle of \( c \). Moreover, \( \theta_1 \theta_2 \) is also the characteristic arc of the dynamic root cycle of \( H_c(0) \), the center of the hyperbolic component \( H_c \). As for the whole wake \( W_c \), it can be described as the set of parameters in \( M \) for which there is a cycle \( P \) with orbit portrait \( O_P \), and such that the characteristic arc of \( O_P \) is \( \theta_1 \theta_2 \). To any given hyperbolic component \( H \), we associate to \( H \) the smallest angle of the external rays landing at the root of \( H \). Another useful fact is that the period of \( \theta_1 \) is the same as the period of the parabolic cycle of \( c \) and the critical cycle of \( H_c(0) \).

When \( c \) is Misiurewicz either \( c \) is the landing point of several external rays or exactly one external ray. On the other hand, the set of Misiurewicz parameters where exactly one ray lands is countable.

### 2.6 Internal address

We now follow the discussion of Eike Lau and Dierk Schleicher in [18] to introduce internal addresses. Although the definition of internal addresses is for hyperbolic components, we can extend this definition to centers of the corresponding parameters.

Given two hyperbolic components or, for this discussion, Misiurewicz points, \( H_1 \) and \( H_2 \), we say that \( H_2 \) is visible from \( H_1 \) if the rays landing at the root of \( H_1 \) separate \( H_2 \) from the Main Cardioid. Visibility induces a partial ordering on the hyperbolic components. More precisely, for every pair of hyperbolic components, or Misiurewicz parameters, \( H_1 \) and \( H_2 \) either one is visible from the other, or there is a hyperbolic component, or Misiurewicz parameter, \( H_0 \) from which both \( H_1 \) and \( H_2 \) are visible.

Let \( H \) be a hyperbolic component, then using visibility between hyperbolic components and Misiurewicz parameters, it is possible to define a path, called the combinatorial arc of \( H \), along the Mandelbrot set connecting the Main Cardioid with \( H \). This imposes a tree structure onto the arrangement of hyperbolic components in the Mandelbrot set. The root of this tree is the Main Cardioid, and the set of vertices is the set of hyperbolic components and Misiurewicz parameters.

In general, the combinatorial arc of \( H_c \) crosses infinitely many hyperbolic components. Nevertheless, one can write down, in an increasing sequence of numbers \( \{a_n\} \) the periods of the hyperbolic components that the combinatorial path of \( H_c \) crosses. It turns out, that this sequence \( \{a_n\} \) is always finite and is called the internal address of the component \( H_c \), or the associated parameter \( c \). Following Lau and Schleicher’s notation, we consider the component inside the Main Cardioid as part of the combinatorial arc, so the internal address always starts with 1, and when we write down the sequence of numbers \( a_n \), we connect consecutive numbers by arrows.

For example, the internal address of the “basilica” map \( f_{-1} = z^2 - 1 \) is \( 1 \to 2 \), while the “airplane” map \( f_c(z) = z^2 + c \) with parameter \( c = -1.7548776662466927601 \) has internal address given by \( 1 \to 2 \to 3 \). It is a Theorem by Schleicher that if two hyperbolic parameters \( c \) and \( c' \) have the same internal address then the maps restricted to the filled Julia sets are conjugate. So, to make internal address a efficient combinatorial model it is necessary to include additional information. Namely, the rotation number around periodic cycles:

Let \( c \) be a parabolic parameter different from 1/4, then the multiplier of the corresponding parabolic cycle is a root of unity of the form \( e^{\frac{2\pi i}{p/q}} \), the number \( p/q \) is called the combinatorial rotation number of \( c \).

If, in the internal address, we label each arrow with the combinatorial rotation number of the hyperbolic components on the sequence, then we obtain the labelled internal address, is a Theorem of Lau and Schleicher [18], that the labelled internal address of a superattracting parameter \( c \) characterizes the parameter.

### 2.7 Postcritically non-recurrent quadratic polynomials

Besides superattracting, hyperbolic and Misiurewicz parameters, there is also a class of parameters such that the orbit of the critical point does not contain the critical point in its accumulation set. More precisely, let \( f : X \to X \) be a dynamical system defined in a metric space \( X \). Let us define the omega limit \( \omega(x) \)-limit set of a point \( x \) as the set of accumulation points of \( \{f^n(x)\}_{n \in \mathbb{N}} \) in \( X \). A non-periodic point \( x_0 \in X \) is said to be a recurrent point of \( f \) if \( x_0 \in \omega(x_0) \). The action of \( f \) on a set \( A \) is said to be non-recurrent if no point \( a \) in \( A \) is a recurrent point of \( f \).
A stronger notion of recurrence in a set $A$ is when every point of $A$ is an accumulation point of its orbit. Let $f : X \to X$ be a dynamical system defined in a metric space $X$. A set $A$ is called minimal if it is closed invariant under $f$ and no proper subset of $A$ has this property.

Note that every point in a minimal set must be recurrent. A Theorem of Birkhoff shows that every dynamical system contains a minimal set.

A postcritically non-recurrent rational map $f$ is a map whose action on the postcritical set is non-recurrent. In the case of quadratic polynomials, it has been proved by Carleson, Jones and Yoccoz \cite{7}, that any postcritically non-recurrent quadratic polynomial has locally connected Julia set. Another non-trivial Theorem by Yoccoz states that critically non-recurrent parameters are locally connected in the Mandelbrot set. In \cite{7} these parameters are called subhyperbolic; however we will introduce later another term related to the associated lamination.

A Theorem by Fatou states that if all the critical points of a rational map $f$ belong to $F(f)$ then, for every $x$ in $J(f)$ there is a number $C > 0$ and $\sigma > 1$ such that $|(f^n)'(x)| > C\sigma^n$. In other words, the map $f$ is expanding on the Julia set. This is the case for all quadratic polynomials with hyperbolic parameter.

Now, the following Theorem by Ricardo Mañe \cite{22} describes, in the general case, those points in the Julia set which are expanding.

**Theorem 4** (Mañe’s Theorem). Let $f : \mathbb{C} \to \mathbb{C}$ be a rational map. A point $z \in J(f)$ is either a parabolic periodic point, or belongs to the $\omega$-limit set of a recurrent critical point, or for every $\epsilon > 0$, there exists a neighborhood $U$ of $x$, such that, $\forall n \geq 0$ every connected component of $f^{-n}(U)$ has diameter $\leq \epsilon$.

A related, and useful, result is the following Lemma:

**Lemma 5** (Shrinking Lemma). Let $f : \mathbb{C} \to \mathbb{C}$ be a rational map. If $K \subset J(f)$ is a compact subset disjoint from parabolic periodic points and $\omega$-limit sets of recurrent critical points, then for every $\epsilon > 0$ there exist $\delta > 0$ such that for every $k \in K$ and every $n \geq 0$, all connected components of $f^{-n}(B(k, \delta))$ have diameter less than $\epsilon$.

The wilder the postcritical set of a function $f$ is, the more intricate becomes the description of the dynamics of $f$. In this work, when speaking of critically recurrent quadratic polynomials $f_c$, we will consider only parameters with locally connected Julia set, and such that the postcritical set $P_c$ is a Cantor set and the action of the map $f_c$ restricted to $P_c$ is minimal.

### 3 Inverse limits and laminations

In this work we follow the notation and definitions as found in the papers \cite{21}, \cite{13} and \cite{19}. A topological space $\mathcal{B}$ is said to have a product structure if it is provided with a homeomorphism $\phi : \mathcal{B} \to \mathbb{D}^n \times T$ where $\mathbb{D}^n$ denotes the open unit disk in $\mathbb{R}^n$, and $T$ is some topological space. Sets of the form $B_x = \phi^{-1}(\mathbb{D}^n \times \{x\})$ are called local leaves or simply plaques, while sets of the form $T_z = \phi^{-1}((z) \times T)$ are called local transversals.

A lamination is a Haussdorf topological space $\mathcal{X}$ which is endowed with an atlas of open charts $(\phi, U)$ where $U$ has a product structure and $\phi$ is a homeomorphism as above. We also require that the change of coordinates are laminar maps, thus change of coordinates are maps of the form $\gamma_{\alpha,\beta} : D \times T \to D' \times T'$ given by $\gamma_{\alpha,\beta}(z, t) = (\sigma(z, t), \psi(t))$, where $\sigma$ and $\psi$ are continuous functions on $t$. The sets $U$ will be called flow boxes. A laminar map between laminations is a continuous map such that when restricted to a flow box it sends plaques into plaques.

Different regularity conditions can be imposed on laminations. This is done by requiring the corresponding regularity of the map $\sigma$ along $z$. For instance, smooth laminations are laminations whose transition maps $\gamma_{\alpha,\beta}$ are smooth in the $z$ variable. Similarly, real and complex analytic laminations can be defined. Laminations are generalizations of the concept of foliations; a foliation is a lamination where $\mathcal{X}$ is a manifold itself.

A lamination $\mathcal{X}$ is decomposed into a disjoint union of connected $n$-manifolds $\sqcup L_\alpha$, where the sets $L_\alpha$ are called global leaves, or just leaves. A global leaf can be characterized as the smallest set $L$ with the property that if it intersects a plaque $B_x$ then $B_x \subset L$. The maps $\phi$ restricted to plaques are, in fact, charts for the leaves. Given a point $z$ in $\mathcal{X}$, we will denote by $L(z)$ the leaf in $\mathcal{X}$ which contains $z$. 

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We define the \textit{dimension of a lamination} \(\mathcal{X}\) as the dimension of any plaque in \(\mathcal{X}\). In dimension two, the concept of a conformal lamination is equivalent to the one of a complex analytic lamination, these type of laminations are also called \textit{Riemann surfaces laminations}.

Other categories of laminations, which play an important role, are affine and hyperbolic laminations. For these, we require the change of coordinates on leaves to be affine and hyperbolic isometries, respectively. In this work, we will be interested in the topology of affine laminations of dimension two arising from dynamics of quadratic polynomials.

### 3.1 Inverse limits

Consider \(\{f_k : X_k \to X_{k-1}\}\), a sequence of \(m\)-to-1 branched covering maps between \(n\)-manifolds \(X_k\). Then, define the inverse limit \(\lim(f_n, X_n)\) as

\[
\lim(f_n, X_n) = \{\hat{x} = (x_1, x_2, \ldots) \in \prod X_n | f_{n+1}(x_{n+1}) = x_n\}.
\]

The space \(\lim(f_n, X_n)\) has a \textit{natural topology} which is induced from the product topology in \(\prod X_n\).

We are interested in inverse limits arising from dynamics; these are particular cases where all coverings \(f_n \equiv f\) and the manifolds \(X_n\) are equivalent to a single phase space \(X\). Such inverse limits are called \textit{solenoids}, or \textit{natural extensions} in ergodic theory, we will denote them by \(\lim(f, X)\) to make emphasis in \(X\).

When \(f\) is a rational function and \(X = \mathbb{C}\) then, following Lyubich and Minsky [21], we will denote \(\lim(f, \mathbb{C})\) by \(N_f\).

The map \(f : X \to X\) has a natural extension \(\hat{f} : \lim(f, X) \to \lim(f, X)\) defined as

\[
\hat{f}(x_0, x_{-1}, \ldots) = (f(x_0), x_0, x_{-1}, \ldots).
\]

Also, there is a family of natural projections \(\pi_n : \lim(f, X) \to X\), given by \(\pi_n(\hat{x}) = x_{-n}\). Each of these maps semiconjugates \(\hat{f}\) to \(f\), so \(\pi_n(\hat{f}(\hat{z})) = f(\pi_n(\hat{z}))\). For simplicity, the subindex of the projection over the first coordinate will be omitted, thus \(\pi \equiv \pi_0\). We are interested in studying properties of dynamics of \(\hat{f}\) and how they are related to the dynamics of the original map \(f\).

Let \(A\) be an invariant set of \(f\), so we have \(f(A) \subset A\). The \textit{invariant lift of \(A\)} in \(\lim(f, \mathbb{C})\) is the set \(\hat{A}\) of all backward orbits \(\hat{z}\) such that \(\pi_n(\hat{z}) \in A\) for every \(n\). Clearly invariant lifts of periodic cycles of \(f\) are periodic cycles of \(\hat{f}\).

Note that there is a natural one-to-one identification of the periodic points of \(f\) with the periodic points of \(\hat{f}\) in \(N_f\). Namely, to every periodic point \(p\) of \(f\), the corresponding point \(\hat{p}\) is the point in the invariant lift of the cycle of \(p\) such that \(\pi(\hat{p}) = p\). Vice versa, given a periodic point \(\hat{p}\) of \(\hat{f}\), the point \(\pi(\hat{p})\) is a periodic point of \(f\). When \(f\) is a rational function, we classify invariant lifts of periodic cycles by borrowing the corresponding classification in the dynamical plane. For instance, a periodic point \(\hat{p}\) in \(\lim(f, \mathbb{C})\) is called parabolic if \(\pi(\hat{p})\) is parabolic in \(\mathbb{C}\).

### 3.2 Lamination structure of solenoids

Assume \(f\) does not have critical points, so it is an \(m\)-to-1 covering map, and let \(\hat{x} = (x_0, x_{-1}, x_{-2}, \ldots)\) be a point in \(\lim(f, X)\); each coordinate \(x_n\) belongs to the set of \(m\) preimages of \(x_{n-1}\). Hence, after a suitable labelling of the branches of \(f^{-1}\), the fiber \(\pi^{-1}(x_0)\) can be identified with \(\{0, \ldots, m-1\}^N\); moreover, by taking the product discrete topology on \(\{0, \ldots, m-1\}^N\), this identification is a homeomorphism. Let \(U\) be an open neighborhood of \(x_0\). The set \(\pi^{-1}(U)\) is an open neighborhood around \(\hat{x}\) and is homeomorphic to \(U \times \{0, \ldots, m-1\}^N\). This endows \(\lim(f, X)\) with a lamination structure. For \(f\) with critical points, whenever an open set \(U \subset \mathbb{C}\) does not intersect the postcritical set of \(f\), the set \(\pi^{-1}(U)\) has a product structure.

Let \((U_{-n})\) be the pull-back of \(U = U_0\) along \(\hat{x}\), where \(U_{-n}\) is the connected component of \(f^{-n}(U)\) containing \(x_{-n}\). Given a number \(N\), the sets

\[
B(U, \hat{x}, N) = \pi_{-N}^{-1}(U_{-N})
\]
form a local basis of open sets for \( \lim(f, X) \). A plaque, then, can be regarded either as a connected component of a flow box, or as the complete pull back of some open disk \( U_0 \) along \( \hat{x} \); that is, a sequence of the form \((U_0, U_{-1}, U_{-2}, \ldots)\).

3.3 Dyadic solenoid

Consider the polynomial \( f_0(z) = z^2 \) defined on \( X = S^1 \). The solenoid \( S^1 = \lim(f_0, S^1) \) is called the dyadic solenoid, see Figure 1. As \( f_0 \) is a covering map of degree two, for every point \( z \in S^1 \), the fiber \( \pi^{-1}(\pi(z)) \) is homeomorphic to \( \{0, 1\}^\mathbb{N} \). Since \( S^1 \) is a compact topological group, the solenoid \( S^1 \) is a compact topological group, in which the group multiplication is given by the multiplication of \( S^1 \) applied componentwise, so the unit \( \hat{u} \) is the point \((1, 1, 1, \ldots)\). The translations of the solenoid, given by left multiplication of elements in \( S^1 \), will be denoted by \( \tau_\xi(\hat{z}) = \hat{\zeta} \cdot \hat{z} \).

The leaf containing the unit \( \hat{u} \) in \( S^1 \) is a one-parameter subgroup of the dyadic solenoid, parameterized by the map \( \rho : \mathbb{R} \to S^1 \), with \( \rho(t) = (e^{2\pi it}, e^{\pi it}, e^{\pi it/2}, \ldots) \). The image of \( \rho \) is dense in the solenoid, that is \( S^1 = \rho(\mathbb{R}) \), and since \( S^1 \) is a topological group every leaf in the dyadic solenoid is dense. Since in fact, there are uncountable many leaves in the solenoid, let us remark that the dyadic solenoid is connected but not path connected. By transferring the natural order in \( \mathbb{R} \) to the leaves in \( S^1 \), the map \( \rho \) also introduces a leafwise order in \( S^1 \), namely, if \( \hat{z} \) and \( \hat{\zeta} \) are two points in the same leaf, then we say that \( z > \zeta \) whenever \( \rho^{-1}(z \cdot \zeta^{-1}) > 0 \).

![Figure 1: The dyadic solenoid \( S^1 \).](image)

3.3.1 The doubling map

Let us consider the polynomial \( f_0 = z^2 \) which is associated to the center of the component inside the Main Cardioid. The orbit of any point inside the unit circle converges to 0 under iteration of \( f_0 \); while the orbit of any point outside the unit circle tends to \( \infty \). Hence, the Fatou set consists of only two domains and, the Julia set is just the unit circle \( S^1 \) in the complex plane \( \mathbb{C} \).

The action of \( f_0 \) in the unit circle is \( f_0(e^{2\pi it}) = e^{2(2\pi it)} \). That is, it doubles the corresponding angle \( \theta \). So, the map \( f_0 : S^1 \to S^1 \) is conjugate to the doubling map \( \mathcal{D} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \), which sends the angle \( \theta \) to \( 2\theta \) (mod 1). For convenience, whenever we refer to a point in the unit circle, we denote it by its angle in \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \). We also assume the standard orientation on the unit circle. The orbit under doubling of every rational angle is finite, and the set of periodic points of \( \mathcal{D} \) is exactly the set of rational angles \( p/q \) where \( q \) is an odd number.

Using the argument above, it is easy to check that the dyadic solenoid \( S^1 \) is isomorphic to the solenoid \( \lim(\mathcal{D}, \mathbb{T}) \). Thus, the periodic points of \( f_0 \) are in one-to-one correspondence with the reduced rational angles \( p/q \) in \( \mathbb{Q}/\mathbb{Z} \) where \( q \) is an odd number.
By parameterizing external rays with their corresponding angle in $\mathbb{T}$, and as a consequence of Böttcher conjugacy of $f_c$ to $z^2$ on the basin of infinity, we have $f_c(R_0) = R_{2g}$. So, the action of $f_c$ on the set of external rays is conjugate to the action of the doubling map in $\mathbb{T}$. This feature will play a very important role in this work.

3.3.2 The adding machine

The projection $\pi: \lim(f_0, \mathbb{S}^1) \to \mathbb{S}^1$ is a fibration map, the monodromy group of this fibration is isomorphic to $\mathbb{Z}$. As we mentioned above, the fiber of every point in $\mathbb{S}^1$ is homeomorphic to the set $\{0,1\}^N$ which, in turn, can be identified with the set of formal series of the form $\sum_{i=0}^\infty a_i 2^i$ with $a_i \in \{0,1\}$. Given such identification, the action of the Monodromy group is generated by the adding machine action on $\{0,1\}^N$, this is the map given by adding one to an element in $\{0,1\}^N$. Let us take $F$ to be the fiber of $\pi$ over 1. The adding machine action on $F$ extends continuously to the whole solenoid. In fact, its generator is given explicitly by the relation $(1, f) \sim (0, \rho(1) \cdot \hat{z})$. We refer to the action of the group $\sigma > \sigma$ as the adding machine action. Finally, let us note that the solenoid $\mathbb{S}^1$ is the quotient of the product space $S = [0,1] \times F$ by the relation $(1, f) \sim (0, \rho(1) \cdot \hat{z})$.

Two points $\hat{z}$ and $\hat{\zeta}$ in $F$ belong to the same leaf if and only if $\hat{z}$ and $\hat{\zeta}$ belong to the same orbit under the adding machine action. As $F$ is homeomorphic to $\{0,1\}^N$, $F$ is uncountable, but each orbit under the adding machine action has a countable number of points in $F$, so the dyadic solenoid has uncountable many leaves.

3.4 Solenoidal cones

The solenoids $\lim(f_0, \mathbb{D}^*)$ and $\lim(f_0, \mathbb{C} \setminus \mathbb{D})$, are both homeomorphic to $\mathbb{S}^1 \times (0, 1)$. Because $\infty$ is a super-attracting fixed point of $f_0$, the inverse limit $\lim(f_0, \mathbb{C} \setminus \mathbb{D})$ is homeomorphic to the cone over the dyadic solenoid $\mathbb{S}^1$ defined as $\mathbb{S}^1 \times [0,1]/\{(s, 1) \sim (s', 1)\}$ for all $s, s' \in \mathbb{S}^1$. The vertex of $\lim(f_0, \mathbb{C} \setminus \mathbb{D})$ corresponds to the point $\infty = (\infty, \infty, \infty, \ldots)$, we call any space homeomorphic to $\lim(f_0, \mathbb{C} \setminus \mathbb{D})$ a closed solenoidal cone. In particular, the solenoidal cone $\lim(f_0, \mathbb{C} \setminus \mathbb{D})$ will be denoted by $\overline{\text{Con}}(\mathbb{S}^1)$. The dyadic solenoid $\mathbb{S}^1$ is contained in $\overline{\text{Con}}(\mathbb{S}^1)$, and we regard $\mathbb{S}^1$ as the boundary of $\overline{\text{Con}}(\mathbb{S}^1)$ (see figure 2). Since there is no local product structure on $\infty$, the solenoid $\overline{\text{Con}}(\mathbb{S}^1)$ is not a lamination. In general, for dynamical systems with critical points, the situation where critical points occur infinitely many times in the coordinates of a given point is one of the possible obstructions for the inverse limit to be a lamination.

It is important to note that $\lim(f_0, \mathbb{C} \setminus \mathbb{D}) \simeq \mathbb{S}^1 \times (0, 1)$ is not path connected, whereas $\overline{\text{Con}}(\mathbb{S}^1)$ is path connected.

As a consequence of Böttcher’s Theorem, we can associate to every quadratic polynomial $f_c = z^2 + c$ a solenoidal cone. When $c$ belongs to the Mandelbrot set, the Böttcher’s coordinate $\phi_c : (A(\infty), \infty) \to (\mathbb{C} \setminus \mathbb{D}, \infty)$ conjugates $f_c$ to $f_0$. The map $\phi_c$ naturally lifts to a homeomorphism $\hat{\phi}_c : \lim(f_c, A(\infty)) \to \overline{\text{Con}}(\mathbb{S}^1) \setminus \mathbb{S}^1$ given by $\hat{\phi}_c(z_0, z_{-1}, \ldots) = (\phi_c(z_0), \phi_c(z_{-1}), \ldots)$ which conjugates $\hat{f}_c$ to $\hat{f}_0$. We will call the solenoid $\lim(f_c, A(\infty)) \subset N_{f_c}$ the solenoidal cone at infinity of $f_c$. When $c$ does not belong to the Mandelbrot set, we can associate a solenoidal cone at infinity to $f_c$, however this cone does not project to the whole basin of infinity but to a neighborhood of $\infty$. We will construct such a cone in next subsection.

The inverse limit $\lim(f_0, \mathbb{D})$ is also a solenoidal cone, both $\overline{\text{Con}}(\mathbb{S}^1)$ and $\lim(f_0, \mathbb{D})$ share the common boundary $\mathbb{S}^1$. Thus, the natural extension $\mathcal{N}_{f_0}$ can be decomposed into $\overline{\text{Con}}(\mathbb{S}^1)$ and $\lim(f_0, \mathbb{D})$ by cutting along $\mathbb{S}^1$. This shows that $\mathcal{N}_{f_0}$ is homeomorphic to the double cone over the solenoid $\mathbb{S}^1$.

Note that the one-dimensional solenoid $\mathbb{S}^1$ is connected and every leaf in it is dense. Thus, the solenoids $\mathbb{S}^1 \times (0, 1)$ and $\overline{\text{Con}}(\mathbb{S}^1)$ are connected, and every leaf in each of them is dense.
3.5 Subsolenoidal cones

Let $r > 1$, and $D_r = \{ z \in \mathbb{C} | |z| < r \}$, then there is a canonical homeomorphism between $\lim(f_0, \mathbb{C} \setminus D_r)$ and $S^1 \times [0, 1)$ given by

$$(z_0, z_{-1}, ...) \mapsto \left(\frac{z_0}{|z_0|}, \frac{z_{-1}}{|z_{-1}|}, ...\right) \times \left(1 - \frac{r}{|z_0|}\right),$$

which extends to a homeomorphism between $\lim(f_0, \mathbb{C} \setminus D_r)$ and $Con(S^1)$, fixing $\hat{\infty}$. This implies that, for $f_c$ and the space $A_r = \{ z \in A_c(\infty) | |\phi_c(z)| \geq r \}$ outside the equipotential $E_r$ in $A_c(\infty)$, the solenoid $\lim(f_c, A_r)$ is also homeomorphic to $Con(S^1)$.

As we mention above, when $c$ does not belong to the Mandelbrot set, we can not associate a solenoidal cone to the lift of the basin of infinity, however for $r$ big enough, we can take $\lim(f_c, A_r)$ as the solenoidal cone at infinity for $f_c$.

The fiber over $\pi^{-1}(E_r)$ of any equipotential $E_r$ is homeomorphic to the dyadic solenoid $S^1$. We call such a space, the solenoidal equipotential over $E_r$, and denote it by $S_r$. Solenoidal equipotentials form a foliation of the solenoidal cone at infinity in $N_{f_c}$. Similarly, external rays induce a foliation in $\pi^{-1}(A(\infty))$ of solenoidal external rays, or just solenoidal rays for short. The fiber over each external ray is a “Cantor set” of external rays.

In particular, in the fiber of a periodic external ray under $f_c$, exactly one of the components is periodic under $\hat{f}_c$. Moreover, every periodic solenoidal ray must project to a periodic external ray in the dynamical plane. Hence, periodic solenoidal external rays are parameterized by reduced rational numbers $\frac{p}{q} \in \mathbb{T}$ with $q$ odd. Thus, to every periodic solenoidal ray we can associate a periodic point in $S^1$. It turns out, that after identifying a solenoidal equipotential $S_r$ with $S^1$, each periodic solenoidal rays intersect any solenoidal equipotential $S_r$ at a periodic point in $S^1$.

Moreover, every periodic solenoidal ray lands at a periodic point $\hat{p}$ of $\hat{f}_c$ in $N_c$. So if two external rays land at $\pi(\hat{p})$ then the corresponding periodic lifts of these rays belong to the same leaf. In this way, external ray identification corresponds to an identification of different leaves in $S^1$.

3.6 Lyubich-Minsky laminations

Finally, we get to the object of study of this paper. Inverse limits are also defined for branching mappings, however, as we saw in $Con(S^1)$, these solenoids may not have a local lamination structure. The case for rational maps on the sphere was addressed by Lyubich and Minsky in [21]. In this setting, there are two obstructions for the natural extension to be a lamination. The first, as we have noted before, is structural; there are some points which fail to have local product structure. The second comes from the fact that the geometric structure on the leaves may not vary continuously; on the transverse direction. Thus, it is necessary to refine the topology in an appropriate way. The first problem is easy to carry over, simply by
removing all points that don’t have local product charts. The second is of a more delicate nature; in fact, for many rational functions the new topology is hard to describe by intrinsic properties of the natural extension. The way out found by Lyubich and Minsky was to embed the lamination into a universal space where a suitable topology comes naturally.

First let us introduce the regular space. To keep things simple, we return to the assumption that \( f_c \) is a quadratic polynomial defined on the Riemann sphere.

**Definition.** A point \( \hat{z} \in \lim(\hat{\mathbb{C}}, f_c) \) is called regular if there exists a neighborhood \( U_0 \) of \( \pi(\hat{z}) \) such that the pull back of \( U_0 \) along \( \hat{z} \) is eventually univalent. The set \( \mathcal{R}_c \subset \lim(\hat{\mathbb{C}}, f_c) \) of regular points is called the regular part of \( f_c \).

The regular part is a disjoint union of Riemann surfaces. To check this, note that, in terms of plaques \( \hat{U} = (U_0, U_{-1}, U_{-2}, \ldots) \), a point \( \hat{z} \) is regular if there is a number \( N \geq 0 \) such that \( U_{-n} \) does not contain critical points for \( n \geq N \). Then, naturally, for a conformal chart for the plaque containing \( \hat{z} \) we can take any of the maps \( \pi_{-n} : \hat{U} \to U_{-n} \) with \( n > N \). Plaques glue together to form a Riemann surface \( L \). Any such set \( L \) will be called a leaf of the regular part.

Once the leaves are endowed with a conformal structure, the map \( \hat{f}_c \) becomes a conformal bi-holomorphism between leaves sending \( L(\hat{z}) \) to \( L(\hat{f}_c(\hat{z})) \).

A result in [21] states that all leaves in \( \mathcal{R}_c \) are simply connected. Moreover, if \( f \) is a rational function, all the leaves in \( \mathcal{R}_f \) are simply connected except in the case of invariant lifts of Herman rings, which are doubly connected. Also, there are no compact leaves in \( \mathcal{R}_c \). Therefore, by the Uniformization Theorem, from a conformal view point, leaves are either disks or planes. The union \( \mathcal{A}_c \) of the leaves in \( \mathcal{R}_c \) conformally isomorphic to the complex plane \( \mathbb{C} \) is called the affine part of \( \mathcal{R}_c \). We call any such leaf an affine leaf.

A leaf conformally isomorphic to the disk is called a hyperbolic leaf. Such leaves can exist, for instance, invariant lifts of Siegel disks belong to hyperbolic leaves. Examples of non-rotational hyperbolic leaves have been constructed by Jeremy Khan and Juan Rivera-Lettelier. We can say that the geometry of the leaves is related to the dynamical properties of the postcritical set. However, so far there is no characterization of the existence of hyperbolic leaves (see section 4.1 in [21]).

In any case, since the affine part contains all the leaves containing repelling periodic points, there are always infinitely many affine leaves in the affine part of any rational map (see Proposition 4.5 in [21]). Moreover, for any such leaf \( L \) containing a periodic point \( \hat{p} \), the lift of König’s coordinate around \( \pi(\hat{p}) \) provides a conformal isomorphism from \( L \) to \( \mathbb{C} \).

The following is a simple criterion to check if a leaf \( L \) is affine (see Corollary 4.2 in [21]). Given a point \( \hat{z} \in \mathcal{R}_c \), if the coordinates \( \{z_{-n}\} \) of \( \hat{z} \) do not converge to the postcritical set \( P(f_c) \) then the leaf \( L(\hat{z}) \) containing \( \hat{z} \) is affine.

When \( c \) is hyperbolic, there are no critical points in the Julia set, and every backward orbit accumulates on the Julia set, by the previous argument; if \( c \) is hyperbolic then all leaves in \( \mathcal{R}_c \) are affine.

In order to be an affine lamination, the topology on the affine part must be such that locally the affine structure on leaves varies continuously along a transversal. In general, the natural topology on \( \mathcal{R}_c \) is not enough, since local degree of plaques may not depend continuously on the transversal direction.

However, for hyperbolic parameters we have:

**Proposition 6** (Lyubich-Minsky). If \( c \) is hyperbolic, the associated regular part \( \mathcal{R}_c \) is a lamination with the topology induced by the natural topology.

In Lyubich-Minsky [21] the concept of convex cocompactness is introduced in terms of the compactness of the quotient of the laminated Julia set under dynamics. This justifies the terminology, however, we use as a definition a proposition in the same paper which characterizes convex cocompactness.

**Definition.** A parameter \( c \) is called convex cocompact if the critical point is not recurrent and does not converge to a parabolic cycle.

In particular, attracting and superattracting parameters are convex cocompact. If \( c \) is convex cocompact, leaves in the regular part \( \mathcal{R}_c \) are all affine, so \( \mathcal{A}_c = \mathcal{R}_c \); also, the affine part in this case has a simple description: \( \mathcal{R}_c = N_c \setminus \{\text{attracting and parabolic cycles} \} \) (see Proposition 4.5 in [21]).
4 Topology of inverse limits

4.1 The laminated Julia set

Let $\mathcal{J}_c = \pi^{-1}(J_c) \cap \mathcal{R}_c$ be the lift of the Julia set on the regular part. We call $\mathcal{J}_c$ the **laminated Julia set** associated to $f_c$. If $c$ is a convex co-compact parameter, by a theorem in [21], $\mathcal{J}_c$ is compact inside $\mathcal{R}_c$ with the natural topology, so in this case $\mathcal{J}_c = \pi^{-1}(J_c)$.

In this section we discuss the topological properties of $\mathcal{J}_c$. Given a repelling periodic point $q$ in the dynamical plane, let us denote by $\hat{q}$ the periodic point in the regular part which satisfies $\pi(\hat{q}) = q$. Let $\mathcal{P}$ be the set of repelling periodic points in $\mathcal{R}_c$.

4.1.1 Leafwise connectivity of $\mathcal{J}_c$

**Lemma 7.** Let $c$ be a parameter with locally connected Julia set, $\mathcal{P}(f_c) \neq \emptyset$, and such that $\pi^{-1}(J(f_c))$ contains an irregular point in the natural extension $\mathcal{N}_c$. Then, there is a leaf $L$ in the regular part $\mathcal{R}_c$, such that $\mathcal{J}_c \cap L$ is disconnected.

*Proof.* Let $\hat{z}$ be an irregular point in the Julia set in $\mathcal{N}_c$ then, since $J(f_c)$ is completely invariant and locally connected, $\pi_{-j}(\hat{z}) = z_{-j}$ belongs to $J(f_c)$ and there is an external ray $R(j)$ landing at $z_{-j}$ for every $j$. Let $r_0$ be a point in $R(0) \cap A(\infty)$, by pulling back $r_0$ along the backward orbit determined by $\hat{z}$, there is a point $\hat{r} \in \mathcal{R}_c$ such that $\pi(\hat{r}) = r_0$ and $\pi_{-j}(\hat{r}) \subset R(j)$, now by moving $r_0$ along the ray $R(0)$, we construct a line $\hat{R}$ in the regular part, such that $\pi_{-j}(\hat{R}) = R(j)$. Let $L$ be the leaf in the regular part containing $\hat{R}$. By construction the endpoints of $\hat{R}$ are the irregular points $\hat{z}$ and $\infty$, this line can not have accumulation points in $\mathcal{R}_c$ when $r_0$ either tends to $z_0$ or to $\infty$. So, $\hat{R}$ is a line escaping to infinity in both directions and separates $L$ in two pieces.

Now, fix $r_0$ in the external ray $R(0)$, let $a \in J(f_c) \setminus \mathcal{P}(f_c)$, and choose two paths, $\sigma_1$ and $\sigma_2$, from $r_0$ to $a$ starting at different directions with respect to the ray $R(0)$, and such that none of them crosses $R(0)$ again. These two paths lift to paths in $L$, joining $\hat{r}$ with points in $\mathcal{J}_c \cap L$, by construction the points lie on different sides of the line $\hat{R}$. Therefore $\mathcal{J}_c \cap L$ is disconnected. 

As interesting examples of polynomials with $\mathcal{J}$ non leaf-wise connected, consider: quadratic polynomials with parabolic cycles (see Tomoki Kawahira’s paper [14]), or the Feigenbaum quadratic polynomial. In the case of quadratic polynomials with parabolic cycles, leaves in the regular part where the Julia set is disconnected are precisely the periodic leaves corresponding to the parabolic cycle. As for repelling cycles the linearizing coordinate, in this case Fatou’s coordinate, gives a uniformization of the periodic leaves on the regular part. A detailed explanation and construction of such uniformization can be found in [14]. It is worth noting that, when $f_c$ is parabolic, then for any leaf $L$ the set $\mathcal{J}_c \cap L$ consists of finitely many components.

In general, it is not clear whether the intersection of the laminated Julia set consists of finitely many pieces. An interesting example to bear in mind is the natural extension associated to the Feigenbaum polynomial. In this case, there are irregular points in the fiber of the Julia set, and then there are leaves for which the Julia set is not path connected.

As a corollary, if under the conditions of Lemma 7 the laminated Julia set $\mathcal{J}_c$ is leaf-wise connected, then there are no irregular points in $\mathcal{J}_c$, which implies that $f_c$ is convex co-compact. The following Lemma shows that the converse is also true:

**Lemma 8.** If $c$ is a convex cocompact parameter, then the set $\mathcal{J}_c$ is leafwise connected.

*Proof.* Since $f_c$ is convex cocompact, then the Julia set $J(f_c)$ is locally connected and every external ray lands in the Julia set. Moreover, since there are no irregular points in $\pi^{-1}(J(f_c))$ in $\mathcal{N}_c$, every solenoidal ray also lands at $\mathcal{J}_c$. Let $L$ be a leaf in the regular part, and let $\hat{z}$ and $\hat{\zeta}$ be two points in $L \cap \mathcal{J}_c$. Since $\hat{z}$ and $\hat{\zeta}$ belong to the same leaf, there exist a path $\sigma$ in $L$ joining $\hat{z}$ with $\hat{\zeta}$. Then by using the external ray flow in $L$ the path $\sigma$ projects onto a path whose trajectory lies in $\mathcal{J}_c$ and connects $\hat{z}$ with $\hat{\zeta}$, so $\hat{z}$ and $\hat{\zeta}$ lie in the same connected component of $L \cap \mathcal{J}_c$.

Together, Lemma 7 and Lemma 8 imply:
Proposition 9. A quadratic parameter $c$ with locally connected Julia set $J(f_c)$ and $P(f_c) \neq J_c$ has leafwise connected Julia set $J_c$ in its regular part $R_c$, if and only if $c$ is convex cocompact.

4.2 Hyperbolic components

As we pointed on before, combinatorially, the Julia sets associated to parameters within a hyperbolic component $H$ are indistinguishable from that associated to the root of $H$. Here, we present a proposition that topologically ties the regular parts of parameters inside hyperbolic components with the regular part of their center.

Although it was not explicitly stated the proof of the next proposition follows immediately from Lemma 11.1 in Lyubich and Minsky. We include the settings and the statement of that lemma for reference. The interested reader can find the proof in [21].

Let $U$ and $V$ be two open sets with $U \subset V$ and let $f : U \to V$ be an analytic branched covering map. Let us remark that when $U$ and $V$ are disks with the property $U \subset V$ the map $f$ is called polynomial-like or quadratic-like if the degree of $f$ is 2. Let $N_f$ denote the set of backward orbits of $f$. In this setting, iterations of the map $f$ on $N_f$ may not be defined, since $f$ is not defined on $V \setminus U$. For $m = 1, 2, \ldots$, let $N_m \subset N_f$ be the set of backward orbits $\tilde{z}$ that can be iterated under $f$ at most $m$ times. So, we have inclusions $N_m \subset N_{m+1}$ for $m = 1, 2, \ldots$.

The map $f^{-m} : N_f \to N_f$ is an immersion, that maps $N_f$ onto $N_m$ for $m = 1, 2, \ldots$. So, by composing with the inclusions $N_m \to N_{m+1}$, we consider $N_f$ as an extension of $N_m$ and denote these extensions by $N_m$. Make $N_f = N_0$ and identify any point $\tilde{z} \in N_m$ with $f^{-1}(\tilde{z})$, so the map $f^{-1}$ induces the following increasing sequence of sets

$$N^0 \hookrightarrow N^1 \hookrightarrow N^2 \hookrightarrow \ldots$$

let $D_f = \cup N_m$, a set $W$ in $D_f$ is said to be open if $W \cap N_m$ is open for every $m$. The set $D_f$ is called the direct limit of the increasing sequence above. The natural extension $\hat{f}$ of $f$ in $N_f$ induces a homeomorphism of $D_f$ into itself. Now, we can state the following Lemma:

Lemma 10 (Lyubich and Minsky). Assume that a branched covering $f : U \to V$ is the restriction of a rational endomorphism $R : \mathbb{C} \to \mathbb{C}$ such that $\mathbb{C} \setminus V$ is contained in the basin of attraction of a finite attracting set $A$. Then $\hat{f} : D_f \to D_f$ is naturally conjugate to $\hat{R} : \hat{N_R} \setminus \hat{A} \to \hat{N_R} \setminus \hat{A}$.

Let $c$ be a hyperbolic parameter contained in the hyperbolic component $H_c$ in the Mandelbrot set. Then we have the following:

Proposition 11. The regular part $R_c$ is homeomorphic to the regular part of the quadratic polynomial associated to the center of $H_c$.

Proof. First, let us discuss parameters inside the Main Cardioid.

Let $f_c(z) = z^2 + c$ a quadratic polynomial with $c$ a parameter inside the Main Cardioid. Thus $f_c$ has an attracting fixed point $a_c$. The Fatou set consists of two open sets corresponding to the basins of infinity and $a_c$, $A(\infty)$ and $A(a_c)$. The Julia set $J(f_c)$ is a quasicircle, so that the conjugating map $\phi : J(f_c) \to \mathbb{S}^1$ can be quasiconformally extended to a neighborhood of $J(f_c)$. Since $f_c$ is expanding on the Julia set, the map $\phi \circ f_c \circ \phi^{-1}$ is an expanding circle map of degree 2.

By a Theorem of Shub the map $\phi \circ f_c \circ \phi^{-1}$ is topologically conjugate to $z^2$, that is, there is a map $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $f_c = h \circ \phi \circ f_c \circ \phi^{-1} \circ h^{-1}$. Also, $h$ admits an equivariant extension to a neighborhood of $\mathbb{S}^1$. Actually, Böttcher’s coordinate in the basin of infinity extends $h$ to the whole basin of infinity. So we obtain a conjugacy of $f_c$ to $f_0$ defined on a simply connected neighborhood $U$ containing the basin of infinity and the Julia set $J(f_c)$. We can choose $U$ small enough such that the map $f_c : U \to V$ is quadratic-like. This implies that the map $f_c : U \to V$ is topologically equivalent to the map $f_0 : \phi(U) \to \phi(V)$. By construction, $\mathbb{C} \setminus V$ is contained in the basin of attraction of $a_c$.

By Lemma 10, the map $\phi : D_c \to \mathbb{S}^1$ is conjugate to $f_c : \hat{N_0} \setminus \hat{a_c} \to \hat{N}_0 \setminus \hat{a_c}$, also the map $f_0 : D_{f_0} \to \mathbb{S}^1$ is conjugate to $f_0 : N_0 \setminus \hat{0} \to N_0 \setminus \hat{0}$. But the conjugacy $\phi$ from $f_c$ to $f_0$ lifts to a conjugacy from $\hat{f}_c : D_{f_c} \to D_{f_c}$ to $\hat{f}_0 : D_{f_0} \to D_{f_0}$. So, $R_c = N_c \setminus \{\hat{\infty}, \hat{a_c}\}$ is homeomorphic to $R_0 = N_0 \setminus \{\hat{\infty}, \hat{0}\}$.
It follows that $\mathcal{N}_c$ is homeomorphic to the double cone over $S^1$. Let us remark that since the homeomorphism above is a conjugacy, it sends the Julia set $\mathcal{J}_c$ onto $\mathcal{J} = \mathcal{J}_0$, so we can restrict such homeomorphism to the lift of the basin of attraction of $a_c$, therefore $\lim(f_c, A(a_c)) \cup J(f_c)$ is homeomorphic to $\text{Com}(S^1)$.

Now, let $f_c(z) = z^2 + c$ be a quadratic polynomial with $c$ in a hyperbolic component $H$, with attracting cycle $P = \{p_1, \ldots, p_n\}$ of period $n$. Let $U_1, U_2, \ldots, U_n$ be the Fatou components of the basin of attraction of $P$.

The regular part $\mathcal{R}_c$ can be decomposed in several parts by cutting along the Julia set $\mathcal{J}_c$, which are:

- The Julia set $\mathcal{J}_c$.
- The lift of the basin at infinity $\pi^{-1}(A(\infty))$ in $\mathcal{R}_c$.

This set is homeomorphic to $S^1 \times (0,1)$ by means of the lift of Böttcher’s coordinate.

- Fatou components of finite branching.

This set is homeomorphic to a countable union of sets $\{\mathcal{V}_n\}$, where each $\mathcal{V}_n$ is homeomorphic to $\mathbb{D} \times \{0,1\}^n$. Each point $z \in \mathcal{V}_n$ has at most finitely many coordinates in the immediate basin of attraction of the critical cycle $\cup U_j$. Also, each component in $\mathcal{V}_n$ projects onto some $U_i$ for $i$ fixed.

- The invariant lift $\hat{\mathcal{U}}$ of the basin of attraction of $P$.

Let $\hat{U}_i$ be the of points $\hat{z} \in \hat{U}$ such that $\pi(\hat{z}) \in U_i$, with the lift of König’s coordinate on $\hat{U}_i$. $\hat{f}_c^n$ has the form of $z \mapsto \lambda z$, where $\lambda$ is the multiplier of the cycle $P$, so $\hat{f}_c^n$ in $\hat{U}_i$ is topologically equivalent to $f_\lambda$ in $\pi^{-1}(A(\alpha_c))$. By the discussion above, $\hat{U}_i$ is homeomorphic to $S^1 \times (0,1)$.

Let $c_0 = H(0)$ be the center of $H$. Now, the regular part $\mathcal{R}_{c_0}$ has the same decomposition as $\mathcal{R}_c$. This decomposition is such that the corresponding components are homeomorphic. These homeomorphisms glue together to a homeomorphism from $\mathcal{R}_c$ to $\mathcal{R}_{c_0}$.

We call this decomposition of $\mathcal{R}_c$, the \textit{laminated decomposition} of the regular part associated to the hyperbolic parameter $c$. Tomoki Kawahira independently proved Proposition\cite{11} in the more general setting of hyperbolic rational maps and in the quasiconformal level. That is, hyperbolic affine laminations are qc-stable in the Lyubich-Minsky setting (see\cite{15}). Let $H$ be a hyperbolic component, and $c_0 = H(0)$ be the center of $H$. For any hyperbolic parameter $c$ in the boundary of $H$, the path from $c_0$ induces a transformation $h_c$ from $\mathcal{R}_{c_0}$ to $\mathcal{R}_c$. If correspondingly $c_0$ denotes the root of $H$, then $f_{c_1}$ is a parabolic quadratic polynomial. Let $\mathcal{L}$ be the regular part of $\mathcal{R}_{c_0}$ with the leaves containing the dynamic root cycle removed. Analogously, let $\mathcal{L}'$ be set obtained by removing from $\mathcal{R}_{c_1}$ the periodic leaves associated to Fatou’s coordinate. Then another result of Kawahira (see\cite{14}), states that the map $h_{c_1} = \lim h_c$, as $c$ tends to $c_1$ in $H$, is a laminar homeomorphism between $\mathcal{L}$ and $\mathcal{L}'$, and moreover, the map $h_{c_1}$ semi-conjugates $\hat{f}_{c_0}|\mathcal{L}$ to $\hat{f}_{c_1}|\mathcal{L}'$.

Let $\mathcal{F}_c = \pi^{-1}(F(f_c)) \cap \mathcal{R}_c$ be the lift of the Fatou set to the regular part; we call $\mathcal{F}_c$ the \textit{laminated Fatou set}. Given an affine leaf $L$ in $\mathcal{R}_c$, consider the uniformization $\phi : L \to \mathbb{C}$. We call a subset $A$ in $L$ \textit{bounded} if the corresponding set $\phi(A)$ is bounded in $\mathbb{C}$. As a consequence of the laminated decomposition of regular parts in the proof of the previous lemma we have the following:

\textbf{Lemma 12.} Let $f_c$ be a convex cocompact quadratic polynomial, and let $A$ be a connected component of $\mathcal{F}_c$ inside an affine leaf $L$. Then $A$ is bounded if and only if the restriction $\pi|A$ has finite degree.

\textbf{Proof.} If $c$ is not hyperbolic, then in the dynamical plane the Fatou set $F(f_c)$ consist only of the basin of infinity. The lift of $F(f_c)$ consist just of the solenoidal cone at infinity; clearly all Fatou components $A$ of $\mathcal{F}_c$ are unbounded and the degree of $\pi$ restricted to $A$ is infinity.

When $c$ is hyperbolic, then by Lemma\cite{11} we can assume that $c$ is superattracting. In this case, all unbounded Fatou components belong either to the solenoidal cone at infinity or to the solenoidal cones at the critical orbit. The degree of $\pi$ restricted to each of these Fatou components is infinity. The remaining Fatou components, the bounded ones, belong by definition to components where the degree of $\pi$ is finite. \hfill $\square$
4.3 Unbounded Fatou components

We want to count how many unbounded Fatou components there are in \( L \). Recall that the valence \( v_p \) of a repelling periodic point \( p \) is the number of external rays landing at \( p \). When \( p \) is the dynamic root point \( r_c \) and \( c \) belongs to a satellite hyperbolic component, there are \( v_p \) Fatou components touching at \( r_c \). If \( c \) belongs to a primitive hyperbolic component, \( r_c \) only touches one Fatou component and \( v_r = 2 \), see Milnor’s [25]. As noted in Lyubich and Minsky [21], the König’s coordinate \( \phi \) around a repelling periodic point \( p \) lifts to the uniformization \( \Phi \) of \( L(\hat{p}) \). Moreover, if \( m \) is the period of \( p \), then \( \Phi \) conjugates \( f^m \mid L(\hat{p}) \) to the affine map \( z \mapsto \lambda_p z \), where \( \lambda_p \) is the multiplier of \( p \). By means of \( \Phi \), local properties of \( p \) are reflected in global properties in \( L(\hat{p}) \). This is the idea behind the proof of the following proposition.

**Proposition 13.** Let \( f_c \) be a quadratic polynomial and let \( L \) be a periodic affine leaf in the regular part \( \mathcal{R}_{f_c} \) containing a periodic point \( \hat{p} \). Then, the number of unbounded Fatou components of \( L \setminus J_c \) is either \( v_p \) if the corresponding periodic point \( p \) does not belongs to the dynamic root cycle, and otherwise there are two cases: There are \( 2v_p \) if \( c \) belongs to a satellite component and exactly 3 unbounded Fatou components if \( c \) belongs to a primitive hyperbolic component.

**Proof.** Let \( m \) be the period of \( \hat{p} \); then \( p = \pi(\hat{p}) \) also has period \( m \). Since Siegel periodic points lift into hyperbolic leaves in \( \mathcal{R}_c \), by Lemma [17] \( p \) must be a repelling periodic point and so, it belongs to the Julia set \( J(f_c) \). In the dynamical plane, there are \( v_p \) rays landing at \( p \) that cut \( \mathbb{C} \) in \( v_p \) sectors. Let \( S \) be one of these sectors, by construction \( S \) is invariant under an appropriate iterate of \( f_c \), say \( f^k_c \), where \( k \) is a multiple of \( m \).

Every ray landing at \( p \) lifts to a landing ray at \( \hat{p} \). As in the dynamical plane, landing rays cut the leaf \( L(\hat{p}) \) in \( v_p \) sectors. We will check first that when \( p \) is not in the dynamic root cycle, every ray landing at \( \hat{p} \) determines an unbounded component of the Fatou set in \( L(\hat{p}) \), to do so we prove that every sector \( S \) in \( L(\hat{p}) \) intersects the Fatou set in two unbounded components. Since \( f^k_c \) is a similarity in \( L(\hat{p}) \), it suffices to prove that the intersection of the Julia set \( J_{f_c} \) with a sector \( S \) grows to infinity in one direction.

To do that, let us construct a fundamental piece to the action of \( f^k_c \) in \( \mathbb{C} \). Given a sector \( S \) in \( \mathbb{C} \), let \( E_S = J(f_c) \cap S \). Let us take a point \( b \) in \( E_S \), very close to \( p \), and such that there is a pair of rays \( R_b \) and \( R_b' \) landing at \( b \), whose images \( R_{f^k_c(b)} \) and \( R_{f^k_c(b)}' \) belong to the wake \( W_S \) determined by \( R_b \) and \( R_b' \). Fix an equipotential \( E_r \) and join consecutive landing rays by arcs of this equipotential. We obtain a region \( P \) around \( p \), since \( p \) is repelling the set \( P \) is compactly contained in \( f^k_c(P) \) and the annulus \( A = f^k_c(P) \setminus \bar{P} \) is the fundamental piece we are looking for (see Figure 3).

![Figure 3: The annulus A when p is not in the dynamic root cycle.](image)

The Julia set intersects the annulus \( A \) at the wake \( W_S \) defined by the rays \( R_b \) and \( R_b' \), thus for every \( S \), we can enclose the subset of the Julia set in \( A \cap W_S \), in a simply connected open set \( V_S \) contained in \( A \cap W_S \).
Now, \( A \) lifts to an annulus \( \hat{A} \) in \( L \), which by construction, is a fundamental region for the action of \( \hat{f}_c^k \). By iterating \( \hat{f}_c^k \) on \( \hat{V}_S \), the lifts of the sets \( V_S \), we obtain a set in \( L \), similar to a “string of pearls” (see Figure 4), which contains the subset of the Julia set \( \hat{E} \) in \( \mathcal{J}_c \). By construction, the lift of this “string of pearls” into \( L(\hat{p}) \) is unbounded and there is one for each sector \( S \) in the dynamical plane. The conclusion follows.

![Figure 4: The set \( V_S \) and some iterates.](image)

In the case when \( p \) belongs to the dynamic root cycle, there are extra unbounded components of the laminated Fatou set in \( L(\hat{p}) \) coming from the lifts of the Fatou components attached to \( p \). We have to modify the previous argument to sectors that contain a Fatou component attached to \( p \). Let \( S \) be such a sector containing a Fatou component, say \( F_S \), having \( p \) in the boundary. Instead of the point \( b \) as above, we consider a pair of points \( q \) and \( q' \) on \( \partial F_S \) and on opposite sides of \( p \). There are two pairs of landing rays, \( \{R_q, I_q\} \) and \( \{R_{q'}, I_{q'}\} \) landing at \( q \) and \( q' \). The \( R \)'s are external and the \( I \)'s are internal rays. On the basin of infinity we follow the same construction as in the case before, whereas in \( F_S \) connect the internal rays \( I_q \) with \( I_{q'} \) by an internal equipotential. So we obtain again a puzzle piece \( P \) around \( p \), see Figure 5. From now on, the argument above goes through either for \( S \) or sectors that do not contain Fatou components attached to \( p \). To complete the computation, notice that if \( c \) belongs to a satellite component, each pair of consecutive rays landing at \( p \) contains a Fatou component with \( p \) on its boundary, so there are \( 2v_p \) components. In the primitive case, there are only 2 rays landing at \( p \) and only 1 Fatou component contains \( p \) on its boundary.

![Figure 5: The annulus \( A \) when \( p \) belongs to the dynamic root cycle.](image)
The following proposition gives another restriction that combinatorics impose on the leaf structure of the Julia set.

**Proposition 14.** Let \( f_c \) be a convex cocompact quadratic polynomial, and let \( L \) be a non-periodic affine leaf. Then, the number of unbounded Fatou components of \( L \setminus J_c \) is either 1 or 2.

**Proof.** Let \( R \) and \( R' \) be two external rays landing at the Julia set such that the wake \( W \) determined by \( R \) and \( R' \) does not contain postcritical points. If \( R \) and \( R' \) are lifts of the rays \( R \) and \( R' \) landing at the same point in a leaf \( L \) in \( \mathcal{R}_c \), then every arc connecting \( R \) and \( R' \) along \( W \) must lift to an arc joining \( R \) and \( R' \). Such an arc, for instance, can be taken to be part of an equipotential. Thus, if \( W \) is the wake determined by \( R \) and \( R' \), then \( W \cap J_c \) is a bounded set in \( L \). So, \( R \) and \( R' \) belong to the same unbounded Fatou component.

Now, let \( L \) be a non-periodic affine leaf. Since the Julia set \( J(f_c) \) is locally connected, the set \( J_c \) is leafwise locally connected and every point in \( J_c \) is the landing point of some external ray in \( \mathcal{R}_c \). If \( L \) has more than 3 unbounded Fatou components, by Lemma 8, \( J_c \cap L \) is path connected, so there is a point \( \hat{z} \) in \( J_c \cap L \) on the boundary of at least 3 unbounded Fatou components in \( L \). This implies that \( \hat{z} \) is the landing point of at least 3 rays, each ray in a different unbounded Fatou component. Hence, each coordinate \( \pi_{-n}(\hat{z}) \in J(f_c) \) is also the landing point of at least 3 rays. By the argument above, the rays landing at \( \pi_{-n}(\hat{z}) \) cut the postcritical set in three disjoint pieces. This implies that \( \pi_{-n}(\hat{z}) \) must be a vertex of the Hubbard tree of \( J(f_c) \) for each \( n \). Since \( L \) is non-periodic, \( \hat{z} \) is non-periodic, and the set of coordinates \( \pi_{-n}(\hat{z}) \) is an infinite set in the dynamical plane. But, this contradicts the fact that the set of vertices in a Hubbard tree with degree at least 3 is finite. \( \square \)

Let us note that when \( L \) is a non-periodic affine leaf, the cases where \( L \) has 1 or 2 unbounded Fatou components can both happen. To get leaves with two unbounded Fatou components, construct a backward history of biaccessible points \( \hat{z} \) in \( J_c \), that is, points where at least 2 rays land, with the property that the two wakes determined by the rays landing at \( \hat{z} \) contain postcritical points. The case where \( L \) has only 1 unbounded component is the most common inside regular parts of convex cocompact parameters \( c \), because almost all repelling periodic points in \( J(f_c) \) are the landing point of exactly one ray. For instance, the leaf containing the lift of the \( \beta \) fixed point is always a leaf with exactly one unbounded component. Non-periodic leaves with one unbounded Fatou components can also be constructed.

In regular parts of convex cocompact parameters, leaves with more than three unbounded components are in one-to-one correspondence with the vertices of the Hubbard tree with degree greater than 2. All other leaves either have one or two unbounded components. So, there are only finitely many leaves with more than three unbounded components. This is how the number of unbounded Fatou components is related to the combinatorics of the parameter \( c \). The following proposition describes the cycle of leaves with more than 3 unbounded components using internal addresses.

**Proposition 15.** Let \( c \) be a superattracting parameter with internal address \( 1 \rightarrow n_1 \rightarrow n_2 \rightarrow ... \rightarrow n_k \). If \( n_j \mid n_{j-1} \) for \( j < k \) then there is a cycle of \( n_j \) periodic leaves such that each leaf has \( n_j \) unbounded Fatou components. When \( j = k \) there are \( n_k \) leaves with \( 2 \frac{n_k}{n_k-1} \) unbounded Fatou components. If \( n_{j-1} \nmid n_j \), there is a cycle of \( n_j \) periodic leaves with 2 unbounded Fatou components for \( j < k \), and 3 unbounded Fatou components if \( j = k \).

**Proof.** The condition of whether \( n_{j-1} \) divides \( n_j \) or not reflects whether the combinatorial arc of \( c \) crosses a satellite or a primitive hyperbolic component in the parameter plane. Let \( j = k \), the number \( n_k \) corresponds to the period of the critical orbit, which is equal to the period of the dynamic root point; hence by Proposition 13, if \( n_k \mid n_k \) the parameter belongs to a satellite component, in which case the value of the dynamic root point is \( \frac{n_k}{n_k-1} \), and there are \( 2 \frac{n}{n_k-1} \) unbounded Fatou components. If \( n_{k-1} \nmid n_k \), then \( c \) is the center of a primitive hyperbolic component and there are \( n_k \) leaves with 3 unbounded Fatou components.

Now, let \( j < k \). If \( n_{j-1} \mid n_j \), then there is a repelling cycle \( P \) of \( f_c \) with period \( n_j \) and valence \( \frac{n_j}{n_{j-1}} \), and by Proposition 13 the lift \( \hat{P} \) belongs to a cycle of \( n_j \) periodic affine leaves with \( \frac{n}{n_{j-1}} \) unbounded Fatou components in \( \mathcal{R}_f \). If \( n_{j-1} \nmid n_j \), the corresponding ray portrait has valence 2 and period \( n_j \). \( \square \)

Leaves containing the periodic lift of the dynamic root cycle of primitive parameters have exactly 3 unbounded Fatou components. If we change the parameter \( c \) to any center in a adjacent satellite component,
one of the unbounded Fatou components collapses to an infinite number of bounded Fatou components. Although $c$ still has a cycle corresponding to the dynamic root cycle of the previous satellite parameter, the leaves containing this cycle will have only 2 unbounded Fatou components, each of them on the lift of the basin of infinity $\pi^{-1}A_c(\infty)$ (see Figure 6).

![Figure 6: Collapsing of unbounded Fatou components by bifurcation.](image)

With all the previous discussion, now we can prove a special case of the Main Theorem, namely when the homeomorphism $h$ is a conjugacy:

**Proposition 16.** Let $c_1$ and $c_2$ be two superattracting parameters. If $h : \mathcal{R}_{c_1} \to \mathcal{R}_{c_2}$ is a homeomorphism conjugating $f_{c_1}$ with $f_{c_2}$, then $c_1 = c_2$.

**Proof.** Any such conjugation sends periodic points into periodic points. Hence, by Lemma 17, $h$ sends the Julia set $J_{c_1}$ into the Julia set $J_{c_2}$. Since $h$ is a homeomorphism it has to leave invariant the number of unbounded Fatou components, and as $h$ is a conjugacy of dynamics, it also leave invariant the combinatorial rotation numbers among the unbounded Fatou components on periodic leaves. This means that $c_1$ and $c_2$ must have the same labelled internal address. But by Lau and Schleicher this implies $c_1 = c_2$. 

Given a repelling periodic point $q$ in the dynamical plane, let us denote by ̂$q$ the periodic point in the regular part which satisfies $\pi(\hat{q}) = q$. Let $\mathcal{P}$ be the set of repelling periodic points in $\mathcal{R}_c$.

**Lemma 17.** The set $J_c$ is a closed and perfect set in $\mathcal{R}_c$. Every periodic point in $\mathcal{R}_c$ either belongs to $\mathcal{P}$ or is a Siegel periodic point. Moreover, $J_c = \overline{\mathcal{P}}$.

**Proof.** The projection $\pi$ is a continuous function from $\mathcal{R}_c$ to $\mathbb{C}$. Since $J(f_c)$ is closed and perfect, the set $J_c$ inherits these properties from the dynamical plane.

The fact that lifts of Siegel periodic points in the dynamical plane belong to the regular part is a consequence of the existence of linearizing coordinates around Siegel periodic points. By a Theorem of Fatou, parabolic and attracting cycles are the accumulation set of some critical orbit, hence the pull back $\{U_{-n}\}$ of every neighborhood $U_0$ contains critical points at infinitely many times $n$, so parabolic and attracting cycles lift to irregular points.

Now, let us prove that Cremer cycles also lift to irregular points. This is, actually, a consequence of the Shrinking Lemma. To illustrate its use, we will follow the proof given by Lyubich and Minsky [21], see also Proposition 1.10 in Lyubich’s survey [19].

Suppose on the contrary, that a Cremer cycle lifts to a periodic regular point. By considering an appropriate iterate of $f_c$, we can assume that the cycle is a Cremer fixed point $a_0$. As $a_0$ is regular, there exist an open neighborhood $U_0$ of $a_0$ such that no component $U_{-n}$ of the pull back of $U_0$ along $\hat{a}_0$ contains critical points. Without loss of generality we can assume that $U_0$ is a small disk around $a_0$, then $U_{-n}$ is also a topological disk, since $f : U_{-n} \to U_{-n+1}$ is a conformal isomorphism. Hence, there is a sequence of Riemann maps $\phi_n : \mathbb{D} \to U_{-n}$ with $\phi_n(0) = a_0$. Put $\rho_n = f^n \circ \phi_n : \mathbb{D} \to U_0$, then by Montel’s Theorem, the sequence $\{\rho_n\}$ is a normal family. Let $P = \{\alpha_1, \alpha_2, ..., \alpha_p\}$ be any cycle of $f_c$ of period $p \geq 3$. By normality there is
a disk $D' \subset \subset \mathbb{D}$ around 0 such that $\phi_n(D')$ does not intersect $P$ for every $n$. Therefore, $\{\phi_n|_{D'}\}$ is also a normal family.

Let $B(a_0, \delta)$ denote the ball around $a_0$ of radius $\delta$, since $\rho_n$ is a conformal isomorphism, for $\delta$ small enough $\text{mod}(\rho_n^{-1}(U_0 \setminus B(a_0, \delta))) = \text{mod}(U_0 \setminus B(a_0, \delta))$, so the modulus of $\rho_n^{-1}(U_0 \setminus B(a_0, \delta))$ depends only on $\delta$. Because $\text{mod}(U_0 \setminus B(a_0, \delta)) \to \infty$ when $\delta \to 0$, there exist a $\delta$ such that $B = B(a_0, \delta) \subset \rho_n(D')$ for all $n$.

It follows that $\text{diam}(\phi_n^{-1}(B)) \to 0$ uniformly when $n$ tends to infinity, otherwise by taking a converging subsequence from $\{\phi_n\}$ and by normality there would be a limiting open set $B_\infty$ containing $a_0$ such that $f^n(B_\infty) \subset U_0$ for all $k$, contradicting the fact that every Cremer periodic point belongs to the Julia set $J(f)$. If the diameters of $B_n$ tend to zero, there is an $m$ such that $B_m$ is compactly contained in $B$ and $f^n(B_m) = B$, but this would imply that $f$ is repelling at $a_0$, again a contradiction.

Take $\hat{z} \in \mathcal{J}_c$, by continuity of $f_c$ and the density of the set of repelling periodic points in $J(f_c)$, there is a periodic point $p$ in the neighborhood of $z_n$ such that $|f^n_c(p) - z_n| < \epsilon$ for $j = 0, \ldots, n$. Then, $\hat{f}_c^n(p)$ is a repelling periodic point in $B(D(z_0, \epsilon), \hat{z}, n)$.

As part of the proof of the previous lemma let us remark the following:

**Corollary 18.** Every point in the invariant lift of a Cremer cycle is irregular.

### 4.4 Topology of ends

In this section, we will consider the regular part $\mathcal{R}_c$ endowed with the natural topology, so that $\mathcal{R}_c$ is locally compact. Topology is important for local compactness since when we endow affine laminations with Lyubich-Minsky topology, Lyubich and Lasse Rempe [Lyu-Rem] recently found some examples where the affine lamination is not locally compact.

As $\mathcal{R}_c$ is locally compact, we can consider the one point compactification $\hat{\mathcal{R}}_c$ of $\mathcal{R}_c$; let $\ast$ be the point at infinity. A path $\gamma : [0, 1] \to \mathcal{R}_c$ escapes to infinity in $\mathcal{R}_c$ if it eventually leaves every compact set $K \subset \mathcal{R}_c$. Equivalently, $\gamma$ escapes to infinity if admits an extension $\hat{\gamma} : [0, 1] \to \hat{\mathcal{R}}_c$ with $\hat{\gamma}(1) = \ast$. Two paths, $\gamma_1$ and $\gamma_2$, escaping to infinity are homotopic at infinity if for every compact set $K \subset \mathcal{R}_c$ there is an $r \geq 0$ such that the sub-paths $\gamma_1|_r : [r, 1] \to \mathcal{R}_c$ and $\gamma_2|_r : [r, 1] \to \mathcal{R}_c$ are homotopic in $\mathcal{R}_c \setminus K$.

**Definition.** Given a leaf $L$ in $\mathcal{R}_c$, let $E(L)$ denote the number of paths non-homotopic at infinity.

Now, for a convex cocompact parameter $c$, since the laminated Julia set $\mathcal{J}_c$ is compact, we can describe the number unbounded Fatou components in $\mathcal{R}_c$ from a topological point of view:

**Lemma 19.** If $c$ is a convex compact parameter, then for every leaf $L$ in $\mathcal{R}_c$, the number $E(L)$ is equal to the number of unbounded Fatou components in $L$.

**Proof.** This is a direct consequence of the fact, due to Lyubich and Minsky, that when $c$ is a convex cocompact parameter the laminated Julia set $\mathcal{J}_c$ is compact in $\mathcal{R}_c$, so every path escaping to infinity must leave eventually the Julia set $\mathcal{J}_c$ and escape through an unbounded Fatou component. □

Let $LU_n$ be the set of leaves with exactly $n$ unbounded Fatou components.

**Corollary 20.** The cardinality of $LU_n$ is a topological invariant.

**Proof.** The number of non-homotopic paths escaping to infinity is a topological invariant. So, if $h : \mathcal{R}_c \to \mathcal{R}_c'$ is a homeomorphism between regular parts, then $E(L) = E(h(L))$ for every leaf $L$ in $\mathcal{R}_c$. □

**Corollary 21.** Let $f_c$ be a convex cocompact quadratic polynomial such that $c$ does not belong to the interior of the Main Cardioid, then $E(L(\hat{\beta})) = 1$.

**Proof.** The $\beta$ fixed point is, by definition, a repelling fixed point where exactly 1 external ray lands. So, by Proposition 13 the leaf $L(\hat{\beta})$ has exactly one unbounded Fatou component. □

**Corollary 22.** If $f_c$ is a convex cocompact quadratic polynomial such that $c$ does not belong to the Main Cardioid, then $\infty$ is the only disconnectivity point of $\mathcal{N}_c$.
Proof. By going through external rays, all leaves in the regular part have at least one access to $\infty$. But leaves with one unbounded Fatou component have access only to $\infty$. By Corollary 21 at least the leaf corresponding to the $\beta$ fixed point has only one unbounded Fatou component. So, $\infty$ is the only irregular point that can be accessed from every leaf in $N_c$.

Thus, the solenoidal cone at infinity can be characterized as the only solenoidal cone in $N_c$ which connects all of the leaves of the regular part in the natural extension.

Lemma 23. If $c$ is convex cocompact, then every periodic leaf has exactly one periodic point.

Proof. It is clear that every periodic point in $R_c$ lies in a periodic leaf. Now, let $L$ be a periodic leaf in $R_c$ of period $n$, so $f^n_c(L) = L$. Because, in the case of convex cocompact parameters, the affine part coincides with the regular part, for every leaf $L$ in the regular part there is a uniformization $\psi: L \to \mathbb{C}$ which conjugates the map $f^n_c: L \to L$ to an affine map $f^n_c(z) = az + b$ where $a$ is a complex number. We claim that $a \neq 1$. If on the contrary $a = 1$, as the Julia set $J_c$ is compact, there is a finite covering of small flow boxes, with the property that the derivative of $\phi$ is bounded away from zero on the Julia set $J_c$.

Let $\bar{z} \in \pi^{-1}(A(\infty))$ be any point on the lift of the basin of infinity. Take $W$ around $z_0 = \pi(\bar{z})$ as in the Shrinking Lemma and let $W'$ be the plaque containing $\bar{z}$ in the fiber of $W$. By assumption, $f_{n_m}(W')$ has the same diameter for all $m$, since translations are isometries. On the other hand the diameters of $f^{-n_m}(W)$ are shrinking to 0 by the Shrinking Lemma. Moreover, for every neighborhood $V$ around the dynamical Julia set $J(f_c)$ we have $f^{-n_m}(W) \subset V$ for large enough $n$. This means that the derivative of $\pi$, under uniformization $\phi$, shrinks to 0 which is a contradiction. So, the map $f^n_c$ can not be conjugated to a translation in $L$. Therefore $a \neq 1$, which implies the existence of a periodic point in $L$.

At every periodic point in the dynamic root cycle there are at least 2 landing external rays. Also, by definition, any point in the dynamic root cycle is on the boundary of at least 1 Fatou component. Let $L$ be one of the leaves containing a periodic point in the lift of the dynamic root cycle. Then, by Proposition 13 $E(L) \geq 3$.

If $c$ is the center of a primitive hyperbolic component, then any leaf containing a periodic point in the dynamic root cycle has 3 unbounded Fatou components. However, two of the unbounded Fatou components are associated to the solenoidal cone at infinity, whereas the other is associated to a Fatou component in the basin of attraction of the critical cycle. By Proposition 13 and Lemma 19 we obtain the following characterization of regular parts of primitive superattracting parameters:

Corollary 24. A superattracting parameter $c$ is primitive if and only if there is a leaf $L$ in $R_c$ such that $E(L) = 3$, and the classes of paths non-homotopic at infinity in $L$ belong to more than one solenoidal cones.

In order to make the previous corollary more precise, let us introduce the concept of ends of laminated sets. Let $L$ be a locally compact laminated set, and consider the one point compactification $\hat{L} = L \cup \{\ast\}$. Two paths, $\sigma : [0,1) \to L$ and $\tau : [0,1) \to L$ escaping to infinity, are said to be equivalent at infinity if for every compact set $K \subset L$ there is a number $r > 0$ such that $\sigma([r,1))$ and $\tau([r,1))$ belong to the same connected component of $L \setminus K$. This is an equivalence relationship in the set of paths escaping to infinity.

Definition. An end of a locally compact laminated set $L$, is an equivalence class of the relationship above described. Let $\text{End}(L)$ denote the set of ends of $L$, then $L \cup \text{End}(L)$ is the end compactification of $L$.

By definition, each end contains the homotopic class of each of its elements, so equivalence at infinity is a weaker relationship than the equivalence relationship of being homotopic at infinity.

Lemma 25. If the Julia set $J(f_c)$ is locally connected, for every irregular point $\hat{I}$ in $N_c$ there is an end in $\text{End}(R_c)$ associated to $\hat{I}$.

Proof. Since the coordinates of $\hat{I}$ belong to the postcritical set, the coordinates of $\hat{I}$ belong either to the Julia set or to an attracting or superattracting cycle. In any case, by the local connectivity of $J(f_c)$, there is a point $z_0$ in the Fatou set $F(f_c)$ and a path $\gamma$ from $z_0$ to $i_0 = \pi(\hat{I})$ such that the trajectory of the path $\gamma$ intersects either the Julia set or the postcritical set exactly at $i_0$. However, the pullbacks $\{\gamma_n\}$ of $\gamma$ are well defined in $[0,1)$, and altogether define a path $\hat{\gamma} : [0,1) \to R_c$ that escapes to infinity in the regular part.
Hence, we associate the irregular point \( \hat{I} \) with the end \([\hat{\gamma}]\). Let \( \hat{\gamma}' \) be any other path defined as \( \hat{\gamma} \), we want to check that \( \hat{\gamma}' \) is equivalent at infinity with \( \hat{\gamma} \). By definition, \( \hat{\gamma} \) and \( \hat{\gamma}' \) extend to paths from \([0, 1]\) to \( \mathcal{N}_c \), satisfying \( \hat{\gamma}(1) = \hat{\gamma}'(1) = I \), so the trajectories of \( \hat{\gamma} \) and \( \hat{\gamma}' \) eventually belong to any neighborhood of \( I \) in \( \mathcal{N}_c \). Since for every \( t \in [0, 1] \) the points \( \hat{\gamma}(t) \) and \( \hat{\gamma}'(t) \) belong to the regular part, for every compact set \( K \subset \mathcal{R}_c \), the paths \( \gamma \) and \( \gamma' \) eventually belong to the same connected component in \( \mathcal{R}_c \setminus K \).

**Lemma 26.** Let \( c \) be a parameter in the Mandelbrot set. There is one and only one end \( E_\infty \in \text{End}(\mathcal{R}_c) \) associated to \( \infty \) in \( \mathcal{R}_c \). Furthermore, if \( c \) is superattracting. Then every end of \( \mathcal{R}_c \) is associated to a unique irregular point.

**Proof.** Consider the equipotential \( E_c \) in the dynamical plane. Since \( E_c \) is compact and does not contain postcritical points, the corresponding solenoidal equipotential \( S_c = \pi^{-1}(E_c) \) is compact in \( \mathcal{R}_c \). Now, let \( R_0 \) be any external ray in the dynamical plane, then the end \([R]\) of any lift \( \hat{R} \) of \( R_0 \) in \( \mathcal{R}_c \) is associated to \( \infty \). If \( \hat{\gamma} \) is equivalent at infinity to \( \hat{R} \) then \( \hat{\gamma} \) must eventually lie in the same connected component of \( \mathcal{R}_c \setminus S_c \) as \( \hat{R} \). This implies that \( \pi(\hat{\gamma}) \) converges to \( \infty \) in the dynamical plane, and so \( \hat{\gamma} \) must converge to \( \infty \) in \( \mathcal{N}_c \). If \( c \) is superattracting, then instead of the equipotential \( E_c \), we can consider an internal equipotential inside the corresponding Fatou component on the basin of attraction of the critical cycle. The argument goes through using the corresponding internal solenoidal equipotential.

**Lemma 27.** For any quadratic polynomial \( f_c \), every end of \( \mathcal{R}_c \) is associated to an irregular point in \( \mathcal{N}_c \).

**Proof.** Let \([\gamma]\) be an end of \( \mathcal{R}_c \), with \( \gamma \) a representative of this end in \( \mathcal{R}_c \). Let \( A_n \) denote the accumulation set of \( \gamma_n = \pi_n(\gamma) \). Let us check that \( f_c(A_n) = A_{n-1} \). By continuity, \( f_c(A_n) \subset A_{n-1} \). Now let \( y \in A_n \). There is a sequence \( t_m \not\rightarrow 1 \) such that \( \gamma_{n-1}(t_m) \) converges to \( y \). That means that \( \gamma_n(t_m) \) is as close as we want to a point in \( f_c^{-1}(y) \). Since \( f_c \) has finite degree, \( \gamma_n(t_m) \) must actually converge to a point in \( f_c^{-1}(y) \). So, \( f_c(A_n) = A_{n-1} \) as we claimed, and this implies that we can construct a backward orbit \( \hat{y} \) in \( \mathcal{N}_c \), such that \( \pi_1(\hat{y}) = y \). Let us check that \( \hat{y} \) must be irregular. If, on the contrary, \( y \) is regular, then there is a \( N \) such that \( y_n \) is outside the postcritical set of \( f_c \) for \( n > N \). Since the postcritical set is closed, there is a neighborhood \( U \) of \( y_n \) such that \( U \) is outside \( P(f_c) \). Let \( K \subset U \) be a compact neighborhood of \( y_n \). Since \( \pi_1^{-1}(K) \) is proper, the set \( \pi_1^{-1}(K) \) is a compact neighborhood of \( y \), but \( \gamma_n(t_m) \) converges to \( y_n \), so \( \gamma(t_m) \) is contained in \( K \) for \( m \) large. This contradicts the fact that \( \gamma \) is escaping to infinity.

Lemma 26 does not rule out the possibility that several irregular points are associated to the same end. It is not clear whether there is a one-to-one correspondence between the irregular points and the ends of the regular part. This would imply that the natural extension of every quadratic parameter \( c \) corresponds to the end compactification of \( \mathcal{R}_c \). However, we have a positive answer for certain parameters.

**Proposition 28.** Let \( c \) be a parameter such that \( J(f_c) \) is locally connected, and \( c \) is either convex cocompact or the postcritical set of \( f_c \) is a Cantor set. Then the set of ends corresponds to the set of irregular points, and the end compactification of \( \mathcal{R}_c \) is \( \mathcal{N}_c \).

**Proof.** If \( f_c \) is convex cocompact then, by Lyubich and Minsky, the Julia set \( J_c \) is compact. Hence, the only irregular points in \( \mathcal{N}_c \) correspond to attracting cycles. By Proposition 11 and Lemma 26, attracting cycles correspond to vertices of solenoidal cones, and vertices of solenoidal cones correspond to ends of the regular part. Thus, if \( c \) is convex cocompact, the end compactification of \( \mathcal{R}_c \) is homeomorphic to the natural extension \( \mathcal{N}_c \).

Now, assume that the postcritical set is a Cantor set. In this case, there are no bounded Fatou components in the dynamical plane. By Lemma 25 and Lemma 27 it is enough to prove that different irregular points are associated to different ends. Let \( I \) and \( I' \) be two irregular points in \( \mathcal{N}_c \). Without loss of generality we can assume that \( \pi(I) = i_0 \neq i'_0 = \pi(I') \), also by Lemma 25 assume that both \( i_0 \) and \( i'_0 \) belong to the Julia set \( J(f_c) \).

By the local connectivity of \( J(f_c) \), there is a path \( \gamma \) embedded in \( J(f_c) \) connecting \( i_0 \) with \( i'_0 \). Let \( U \) and \( U' \) be neighborhoods around \( i_0 \) and \( i'_0 \) small enough that \( \{t \in [0, 1] | \gamma(t) \notin U \cup U' \} \) contains an interval \((t_1, t_2)\). Since \( P(f_c) \) is a Cantor set, there is a \( t' \in (t_1, t_2) \) and an open neighborhood \( V \) around \( \gamma(t') \) and not intersecting \( P(f_c) \). Since \( V \) is open, there are two external rays \( R \) and \( R' \) landing at both sides of \( \gamma \) in \( V \), say at \( z_1 \) and \( z_2 \), and such that the path \( \tau \) embedded in \( J(f_c) \) from \( z_1 \) to \( z_2 \) lies completely in \( V \).
Let $T$ be the image of $\tau$ in the dynamical plane. By construction, the curve whose trajectory is $\sigma = R \cup T \cup R'$ separates $U$ from $V$. Finally, for any equipotential $E_r$, the set that consists of the union of $E_r$ and the part of $\sigma$ inside $E_r$ is a compact set $K$. By construction $K$ does not intersect the postcritical set, so $\pi^{-1}(K)$ is a compact set in $\mathcal{R}_c$ such that $I$ and $I'$ lies in different connected components in $\mathcal{N}_c$ (see Figure 7).

![Figure 7: To the proof of Proposition 28](image_url)

**Corollary 29.** Let $c$ be any parameter as in Proposition 28, then every homeomorphism $h : \mathcal{R}_c \to \mathcal{R}_{c'}$ between regular parts extends to a homeomorphism of the natural extensions $\tilde{h} : \mathcal{N}_c \to \mathcal{N}_{c'}$. Moreover, $\tilde{h}(\hat{\infty}) = \hat{\infty}$

*Proof.* By Proposition 28, the end compactification of $\mathcal{R}_c$ is homeomorphic to the natural extension $\mathcal{N}_c$, since any homeomorphism $h : \mathcal{R}_c \to \mathcal{R}_{c'}$ extends to the end compactification. By Corollary 22, $\hat{\infty}$ is the only disconnection point among the irregular points, and the second part of the corollary follows.

**Corollary 30.** If $h : \mathcal{R}_c \to \mathcal{R}_{c'}$ is a homeomorphism between the regular parts of two superattracting parameters $c$ and $c'$, then the periods of $c$ and $c'$ are equal.

*Proof.* By Corollary 29, $h : \mathcal{R}_c \to \mathcal{R}_{c'}$ extends to a homeomorphism of the natural extensions sending irregular points into irregular point. If $p$ is the period of $c$, there are $p + 1$ irregular points in $\mathcal{N}_c$.

**Lemma 31.** Let $c_1$ and $c_2$ be two superattracting parameters, and $h : \mathcal{R}_{c_1} \to \mathcal{R}_{c_2}$ be a homeomorphism. Then $h$ sends the leaves containing the dynamic root cycle of $c_1$ into the leaves of the dynamic root cycle of $c_2$.

*Proof.* By Proposition 13, there are at least three unbounded Fatou components associated to the leaves containing the dynamic root cycle. On the other hand, these unbounded Fatou components correspond to at least two different ends in the regular part. So, one of the ends is associated to at least two unbounded Fatou components. This end must be $\hat{\infty}$ and no other periodic point in the regular part can have access to different ends.

Given a superattracting parameter $c$, let $v_c$ be the valence of the dynamic root point.
Corollary 32. Let $c_1$ and $c_2$ be two superattracting parameters with the same period. If $v_{c_1} \neq v_{c_2}$, then the corresponding regular parts $R_{c_1}$ and $R_{c_2}$ are not homeomorphic.

Proof. Having different valence, the corresponding leaves containing the lifts of the dynamic root cycles on each regular part must have different numbers of unbounded Fatou components, which implies that the corresponding leaves have different number of accesses to infinity.

5 Isotopies

In the previous section we found some invariants that a homeomorphism between regular spaces must have. In order, to finish the proof of the Main Theorem, we need to discuss the isotopy classes of homeomorphisms of regular spaces. In this section, we will see that among the homeomorphisms isotopic to a given one, there is always a homeomorphism holding special properties.

A self-embedding of a compact topological space $X$ is a continuous injective map of $X$ into itself. A homeomorphism $h: X \to X$ is called isotopic to the identity if there exist a continuous map $\Phi: [0, 1] \times X \to X$ such that $\Phi(0, x) = x$, $\Phi(1, x) = h(x)$ and the restriction $\Phi_t(x) = \Phi(t, x)$ is a homeomorphism of $X$ for every $t \in [0, 1]$. In general, two maps $h_1: X \to Y$ and $h_2: X \to Y$ are called isotopic if there is a homeomorphism $\phi: Y \to Y$ isotopic to the identity, such that $\phi \circ h_1 = h_2$. In this section, we will prove that every self-embedding of the solenoidal cone $Con(S^1)$ is isotopic to a self-embedding of $Con(S^1)$ sending $S^1$ to a solenoid of the form $S^1 \times \{r\}$ in $Con(S^1)$.

5.1 Isotopies of the dyadic solenoid

The group $H(S^1, S^1)$ of self-homeomorphisms of the solenoid, endowed with the uniform topology, is a topological group. An automorphism of $S^1$ is an element in $H(S^1, S^1)$ that preserves the group structure of $S^1$. The set of automorphisms of the solenoid is denoted by $Aut(S^1)$. For $\tau \in S^1$, the map $\zeta \mapsto \tau \cdot \zeta$ is called a left translation of $S^1$. Abusing notation, we identify the map with the element $\tau \in S^1$. The set $Aff(S^1)$ of affine maps of the solenoid is a transformation in $S^1 \times Aut(S^1)$, where the first factor corresponds to the set of left translations in $S^1$.

Let $\ell^2$ be the standard Hilbert space. The following result, due to James Keesling [10], describes the topological embedding of $Aut(S^1)$ inside $H(S^1, S^1)$.

Proposition 33 (Keesling). The group $H(S^1, S^1)$ is homeomorphic to $\ell^2 \times S^1 \times Aut(S^1)$.

Actually, Keesling’s proof shows that $H(S^1, S^1)/Aff(S^1)$ is homeomorphic to $\ell^2$. Since $\ell^2$ is a vector space, this implies that every self-homeomorphism of the solenoid is isotopic to an affine map. According to Jaroslaw Kwapisz [17], the group $Aut(S^1)$ has a simple set of generators:

Proposition 34 (Kwapisz). The group $Aut(S^1)$ is the infinite dihedral group generated by $f_0$ and the inversion $s \mapsto \bar{s}$.

This proposition was proved in [17] in the more general setting of $P$-adic solenoids, where $P$ is an arbitrary sequence of prime numbers. Together these propositions yield the following corollary:

Corollary 35 (Kwapisz). Every homeomorphism of the dyadic solenoid onto itself is isotopic to an affine map of the form $\tau \circ \bar{f}^n \circ r$, where $r$ is the identity if the map is orientation preserving, or the inversion $s \mapsto \bar{s}$ otherwise.

The following is a known topological property of the dyadic solenoid, see [17].

Lemma 36. The image of very continuous map $\phi: S^1 \to S^1$, of the solenoid into itself, is either a point, a closed interval or onto.

Proof. The solenoid is a connected, compact metric Hausdorff space. So is $\phi(S^1)$ by continuity. Now consider a leaf $L \subset S^1$, then its image $\phi(L)$ is contained in a leaf $L'$. We claim that, if $\phi(L)$ is unbounded in $L'$ then $\phi(L) = S^1$. If the image of $L$ is a complete leaf then this is clear because of the density of leaves. Now
assume that \( \phi(L) \) is a half line in the solenoid. Recall that the leaf containing the unit is a one parameter subgroup of the solenoid. By homogeneity, assume that the unit belongs to \( \phi(L) \) and, after identifying with the real numbers \( \mathbb{R} \), that \( \phi(L) \) covers the positive numbers. Now let \(-M\) be some negative number. Since the numbers \( 2^m \) transversally converge to 0 as \( m \) goes to infinity, the numbers \( 2^m - M \) converge to \(-M\) as \( m \) goes to infinity, so the closure of \( \phi(L) \) contains the whole leaf containing the unit and our claim follows.

Assume that \( \phi(L) \) is bounded, then by connectivity \( \phi(S^1) \) is on the connected component of a bounded set, therefore it should be an interval or a point. \( \square \)

### 5.2 Isotopies of solenoidal cones

Recall that we can regard the solenoid \( S^1 \) as the quotient of \( S = I \times F \) by the map \( \sigma \), where \( \sigma \) is the generator of the adding machine action. Here, \( I = [0, 1] \) and \( F \) denotes the fiber over 1 of the projection \( \pi: S^1 \to S^1 \); \( F \) is homeomorphic to the Cantor set. By homogeneity, assume that the unit belongs to \( S \) and \( F \) is homeomorphic to the Cantor set \( \{0, 1\}^\mathbb{N} \). The cylinder \( S^1 \times I \) also can be expressed as the quotient of \( S \times I \) by the map \( \sigma \times Id \). For \( x \in F \), let \( R_x = (0, x) \times I \subset S \times I \), then \( R = \cup R_x \) over all \( x \in F \) is the vertical section of \( S^1 \times I \) which is homeomorphic to the trivial one dimensional lamination \( F \times I \). The set \( R \) can be regarded as the fiber of an external ray in \( Con(S^1) \) minus the point \( \infty \). The goal of this section is to prove:

![Figure 8: Embedding of \( S^1 \) into \( S^1 \times I \).](image)

**Proposition 37.** Let \( \phi: Con(S^1) \to Con(S^1) \) be an orientation preserving self-embedding with \( \phi(S^1) \cap S^1 = \emptyset \), then \( Con(S^1) \setminus \phi(Con(S^1)) \) is homeomorphic to \( S^1 \times I \).

First, we say that two 1-dimensional laminations embedded into a third lamination of dimension 2 intersect transversally if they intersect leafwise transversally. Transversality is a smooth notion, so we need an isotopy that regularizes the embedding \( \phi \) on \( S^1 \). The existence of such an isotopy is given by a generalization of the corresponding theorem about surfaces. Namely, if \( \gamma: I \to I \times I \) is a curve in the unit square, such that \( \gamma(I) \cap \partial(I \times I) = \emptyset \), then there is a map \( h: I \times I \to I \times I \) isotopic to the identity, rel the endpoints of \( \gamma \), such that \( h \circ \gamma \) is piecewise linear, and \( h \) leaves the extremes of \( \gamma \) fixed. See [11].

Moreover, the corresponding map from the set of embeddings \( \text{Emb}(I, I \times I) \) to the set of self-homeomorphisms of the unit square isotopic to identity can be chosen to depend continuously on parameters. That is, if \( X \) is a topological space, and the family of maps \( \phi_x: I \to I \times I \) in \( \text{Top}(I, I \times I) \) depends continuously on \( x \), then there is a family of maps \( h_x \) in \( \text{Top}(I \times I, I \times I) \) depending continuously on \( x \) such that \( h_x \circ \gamma \) is piecewise linear. So, making \( X = F \), we have the following lemma:

**Lemma 38.** Let \( \phi: S \to S \times I \) be a laminar embedding, such that \( \phi(S) \cap \partial(S \times I) = \emptyset \). Then, there is a homeomorphism \( h: S \times I \to S \times I \), isotopic to the identity, such that \( h \circ \phi \) is piecewise linear.

This immediately implies:
**Lemma 39.** Given an embedding $\phi : S^1 \to \partial(S^1 \times I)$ such that $\phi(S^1) \cap \partial(S^1 \times I) = \emptyset$, there exists a homeomorphism $h : S^1 \times I \to S^1 \times I$, isotopic to the identity, such that $h \circ \phi$ is piecewise linear.

**Proof.** Let $\{B_1, B_2, \ldots, B_k\}$ be a partition by square flow boxes in $S^1$ such that each plaque of $B_i$ intersects at most one vertical segment $R_x$. Apply Lemma 38 to each $B_i$. By further local isotopies, we can assure that $\phi(S^1)$ does not contain vertical segments and that $\phi(S^1)$ intersects $R$ transversally.

**Lemma 40.** For every $x \in F$, the intersection $\phi(S^1) \cap R_x$ consists of a finite number of points.

**Proof.** By transversality, every intersection is leafwise isolated. Since every $R_x$ is compact, there are finitely many intersections in every $R_x$.

We can identify $R_x$ with $I$ for each $x \in F$.

**Lemma 41.** Let $m(x) = \min\{r | r \in R_x \cap \phi(S^1)\}$. Then, the function $m : F \to I$ depends continuously on $x$.

**Proof.** By compactness and Lemma 40, there is a covering of $\phi(S^1) \cap R$, by flow boxes $\{B_1, B_2, \ldots, B_k\}$ in $S^1 \times I$ and such that each plaque of $B_i$ contains a single point of $\phi(S^1) \cap R$ for every $i$.

The intersection $\phi(S^1) \cap R \cap B_i$ is transversal for $B_i$. The points where $m$ attains the minimums are arranged into these transversals. By definition, transversals depend continuously on the fiber.

Fix $x \in F$. By the action of $\sigma$, the segments $\{R_{\sigma^{-n}(x)}\}$ with $n \in \mathbb{Z}$ are precisely the segments in $R$ that belong to the same leaf in $S^1 \times I$. Let $L$ be the corresponding leaf in $S^1$ that contains $\phi^{-1}(x)$. By a suitable parametrization, identify $L$ with $\mathbb{R}$ in an order preserving way.

**Lemma 42.** Assume that $\phi$ is orientation preserving and let $t_x = \phi^{-1}(x, m(x))$, then $t_x < t_{\sigma(x)}$.

**Proof.** Every leaf $S$ in $S^1 \times [0, 1]$ is an infinite horizontal strip. By Lemma 38, composition of $\phi$ with the vertical projection over the solenoid is onto. This implies that of every $x \in F$, there is a first time $t_0$ such that $\phi(t_0) \in R_x$ and a last time $t_1$ such that $\phi(t_1) \in R_{\sigma(x)}$. Suppose, on the contrary, that $t_{\sigma(x)} < t_x$, also we have $t_1 > t_{\sigma(x)}$. By definition $\phi(t_0) > m(x)$, and $\phi(t_1) > m(\sigma(x))$ in $R_x$ and $R_{\sigma(x)}$ respectively. Now, from $m(x)$ there is no way that the trajectory of $\phi$ gets to $\phi(t_1)$ without self-intersecting or crossing $R_{\sigma(x)}$ in a lower point than $m(\sigma(x))$, see Figure 9. Therefore $t_x < t_{\sigma(x)}$. 

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![Figure 9: To the proof of Lemma 42](image-url)
Thus, there is a solenoidal cone at \( S \in N \) equipotentials define a local basis of neighborhoods homeomorphic to solenoidal cones around \( \hat{S} \). The solenoidal cone at infinity admits a foliation by solenoidal equipotentials. These solenoidal

**Proof.**

must send this solenoidal local basis of neighborhoods into a local basis of neighborhoods around \( \hat{S} \). To a homeomorphism sending a solenoidal equipotential \( \phi \) to a homeomorphism sending the boundary onto the boundary.

**Proposition 46.** Assume that \( \phi : \text{Con}(S^1) \to \text{Con}(S^1) \) be an embedding with \( \phi(S^1) \cap S^1 = \emptyset \), then there is an isotopy of \( \phi \) to a map that sends the boundary onto the boundary.

**Proof.** The external ray foliation in \( S^1 \times I \) gives the track of the desired isotopy. \( \square \)

**Lemma 44.** A map \( h : S_R \to S_R \), isotopic to the identity, extends to a map \( \tilde{h} : \text{Con}(S^1) \to \text{Con}(S^1) \) isotopic to the identity. This extension can be done in such a way that \( \tilde{h} \) restricted to the complement \( \text{Con}(S^1) \setminus N \) of some neighborhood \( N \) of \( S_R \) is the identity.

**Proof.** Consider \( N = \cup_{t \in (R-\epsilon, R+\epsilon)} S_t \), clearly \( N \) is a neighborhood of \( S_{E_R} \). Let \( \Phi : I \times S_{E_R} \to S_{E_R} \) be an isotopy of \( h \) to the identity. Define \( b : I \to I \) by \( b(t) = \max \{1, \frac{1}{2} \left( t - R \right) \} \). The map \( \tilde{h} : \text{Con}(S^1) \to \text{Con}(S^1) \) given by \( \tilde{h}(t,s) = (t, \Phi(b(t), s)) \) satisfies the conditions of the lemma. \( \square \)

**Corollary 45.** Let \( \phi : S^1 \to \text{Con}(S^1) \) be an embedding. Assume that \( \phi \) admits an extension to a map \( S^1 \times (-\epsilon, \epsilon) \to I \times S^1 \) for some positive \( \epsilon \). Then, the image of \( \phi \) is isotopic to \( S^1 \times 0 \).

**Proposition 46.** Assume that \( h : N_c \to N_{c'} \) is a homeomorphism such that \( h(\infty) = \infty \). Then \( h \) is isotopic to a homeomorphism sending a solenoidal equipotential \( S_R(c) \) onto \( S_R(c') \).

**Proof.** The solenoidal cone at infinity admits a foliation by solenoidal equipotentials. These solenoidal equipotentials define a local basis of neighborhoods homeomorphic to solenoidal cones around \( \infty \). Locally, \( h \) must send this solenoidal local basis of neighborhoods into a local basis of neighborhoods around \( \infty \) in \( N_{c'} \). Thus, there is a solenoidal cone at \( S_R(c) \) in \( N_c \) which is embedded by \( h \) into some solenoidal cone at \( S_R(c') \) in \( N_{c'} \). Now, the proposition follows from Proposition 46. \( \square \)

**5.3 Proof of the Main theorem**

In this section, we will prove the central theorem of this work. The idea is to recognize the topological imprints that combinatorics of parameters impose over the regular parts. We will prove the following:
Theorem 47. Let \( h : \mathcal{R}_{f_c} \to \mathcal{R}_{f_c} \) be an orientation preserving homeomorphism between the regular parts of two superattracting quadratic polynomials, \( f_c \) and \( f_{c'} \). Then, \( f_c = f_{c'} \).

Proof. By Corollary 29 the homeomorphism \( h \) admits an extension \( \hat{h} : \mathcal{N}_c \to \mathcal{N}_{c'} \) such that \( \hat{h}(\infty) = \infty \). By Proposition 46 \( \hat{h} \) is isotopic to a homeomorphism that sends a solenoidal equipotential \( \hat{S}'_h \) in \( \mathcal{N}_c \) onto a solenoidal equipotential \( S'_h \) in \( \mathcal{N}_{c'} \). Using the lift of Böttcher’s coordinate at the solenoidal cones at infinity in \( \mathcal{N}_c \) and \( \mathcal{N}_{c'} \), the map \( \hat{h} \) restricted to \( \mathcal{S}_R \), becomes a homeomorphism of \( S^1 \) into itself. By, Corollary 30 there exists a map, say \( \psi : S^1 \to S^1 \), isotopic to the identity such that \( \psi \circ \hat{h} \) is an affine transformation of \( S^1 \) of the form \( \tau \circ \hat{f}_n^t \). By Lemma 31 \( \psi \) extends to a map \( \hat{\psi} \) isotopic to the identity, defined on a neighborhood \( N \) of \( \mathcal{S}_R \), so that \( \hat{\psi} \circ \hat{h} \) coincides with \( \hat{h} \) outside \( N \). The map \( \hat{f}_n^t \circ \hat{\psi} \circ \hat{h} \) is conjugate to \( \tau \) in \( \mathcal{S}_R \). So, by means of the previous normalizations we can assume that \( \hat{h} \) restricted to \( \mathcal{S}_R \) is already the translation \( \tau \). On the other hand, by Lemma 31 \( \hat{h} \) must send the leaves containing the dynamic root cycle of \( \hat{f}_c \) into the leaves containing the dynamic root cycle of \( \hat{f}_{c'} \). The lift of the dynamic root cycle of \( f_c \) may not be mapped to the lift of the dynamic root cycle of \( f_{c'} \). However, on the level of solenoidal equipotentials, \( \hat{h} \) maps the periodic leaves in the solenoid \( \hat{S}_R \) (under doubling) associated to the ray portrait of \( r_c \), into the corresponding periodic leaves of \( \hat{S}'_R \). Remind that there is a one-to-one correspondence between periodic leaves and periodic points on \( S^1 \). Thus, possibly after another isotopic deformation of \( \hat{h} \), we can assume that \( \hat{h} \) restricted to \( \mathcal{S}_R \) is already the translation \( \tau \). By Corollary 30 the periods of \( c \) and \( c' \) must be the same.

Therefore, \( \tau \) sends every periodic point in \( \mathcal{S}_R \cap L(\hat{r}_c) \) into every periodic point in \( \mathcal{S}'_R \cap L(\hat{r}_{c'}) \). That happens in every leaf containing the lift of the dynamic root cycle. So, by projecting onto \( S^1 \) by \( \pi \), the action of \( \tau \) becomes a rotation in \( S^1 \) that sends the ray portrait of \( r_c \) onto the ray portrait of \( r_{c'} \). By Lemma 2 the dynamic root cycle must be the same and then \( c = c' \).

We can extend the previous theorem to the following:

Theorem 48. Let \( c \) be a convex cocompact parameter with \( \text{Im}(c) \neq 0 \), if \( h : \mathcal{R}_c \to \mathcal{R}_c' \) is an orientation preserving homeomorphism between regular spaces then \( c = c' \).

Proof. If \( c \) is hyperbolic, by Proposition 11 the regular part \( \mathcal{R}_c \) is homeomorphic to \( \mathcal{R}_{c_0} \), where \( c_0 \) is the center of the hyperbolic component containing \( c \), so we are in the same situation of Theorem 47.

If \( c \) is not hyperbolic, then \( c \in \partial M \) and since \( c \) is not a real number, \( f_c \) has a cycle with valence bigger than 2.

Since the critical point belongs to the Julia set and there are no irregular points in \( \pi^{-1}(J(f)) \), the point \( \infty \) is the only irregular point in \( \mathcal{N}_c \). Moreover, the laminated Julia set \( \mathcal{J}_c \) is compact, so it follows that the end compactification of \( \mathcal{R}_c \) is homeomorphic to \( \mathcal{N}_c \). Because \( h \) is a homeomorphism, the end compactification of \( \mathcal{R}_{c'} \) also coincides with \( \mathcal{N}_{c'} \), and \( \mathcal{N}_{c'} \) has only one irregular point, thus \( c' \) is also a convex cocompact parameter. Moreover, \( h \) extends to a homeomorphism sending \( \infty_c \) to \( \infty_{c'} \).

We are now in the situation of the hypothesis of Proposition 10 and we can find a homeomorphism \( \hat{h} \) in the isotopy class of \( h \), such that \( \hat{h} \) sends a solenoidal equipotential in \( \mathcal{R}_c \) onto another in \( \mathcal{R}_{c'} \). When we restrict to these solenoidal equipotentials, the map acts as a translation \( \tau \).

By the remark above, there is a cycle with valence bigger than 2, so the lift of this cycle belongs to a cycle of leaves with at least 3 unbounded Fatou components. The image of any of these leaves is also a leaf \( L \) with at least 3 unbounded Fatou components; by Proposition 13 \( L \) is periodic under \( f_{c'} \). Using a similar argument as in the proof of Theorem 17 we can check that \( \hat{h} \) must preserve the ray portraits associated to the cycle of \( L \). Moreover, the translation \( \tau \) is equal to the identity. Hence, preserves the orbit portrait of every cycle of \( f_{c} \). The same holds for \( h^{-1} \), then \( c \) and \( c' \) are combinatorially equivalent. But if \( c \) is convex cocompact, by Carleson, Jones and Yoccoz 7, \( c \) is rigid which implies \( c = c' \).

When \( c \) is real and convex cocompact, all periodic leaves of \( \mathcal{R}_c \) have either 1 or 2 unbounded components. However, there are many non-periodic leaves with 2 unbounded components. So, the argument of unbounded components does not guarantee that \( h \) must send periodic leaves into periodic leaves, and there is no reason for the ray portraits to be preserved. In this case, we would need another argument to justify that the induced translation on solenoidal equipotentials is the identity.
References

[1] L. Bartholdi, R. Grigorchuk, and V. Nekrashevych, *Trends in mathematics: Fractals in graz*, ch. From fractal groups to fractal sets, pp. 25–118, Birkhauser and Verlag-Basel, Switzerland, 2002.

[2] A. Beardon, *Iterations of rational function*, Grad. Texts Math, Springer, 1991.

[3] B. Bielefeld, J.H. Hubbard, and Y. Fisher, *The classification of critically preperiodic polynomials as dynamical systems*, J. Amer. Math. Soc. 5 (1992), no. 4, 721–762.

[4] H. Bruin and D. Schleicher, *Symbolic dynamics of quadratic polynomials*, Mittag-Leffler Preprint Series (2002).

[5] C. Cabrera, *Towards classification of laminations associated to quadratic polynomials*, Ph.d thesis, Stony Brook, 2005, http://www.math.sunysb.edu/cgi-bin/thesis.pl?thesis05-3.

[6] L. Carleson and T. Gamelin, *Complex dynamics*, Springer, 1993.

[7] L. Carleson, P.W. Jones, and J.C. Yoccoz, *Julia and John*, Bol. Soc. Brasil. Mat. N.S (1994), no. 24, 1–30.

[8] A. Douady, *Chaotic dynamics and fractals*, ch. Algorithms for computing angles in the Mandelbrot set, pp. 155–168, Acad. Press, 1986.

[9] A. Douady and J. H. Hubbard, *Étude dynamique des polynômes complexes I & II*, Publ. Math. Orsay (1984).

[10] ______, *On the dynamics of polynomial-like mappings*, Ann. Ec. Norm. Sup. (1985).

[11] D. B. A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. 115 (1966), 83–107.

[12] J.H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: Three theorems of J.-C. Yoccoz*, Topological Methods in Modern Mathematics (1993).

[13] V. Kaimanovich and M. Lyubich, *Conformal and harmonic measures on laminations associated with rational maps*, AMS, 2005.

[14] T. Kawahira, *On the regular leaf space of the cauliflower*, Kodai Math. Journal (2002).

[15] ______, *Tessellation and Lyubich-Minsky laminations associated with rabbits*, Preprint, 2005.

[16] J. Keesling, *The group of homeomorphisms of the solenoid*, Trans. Amer. Math. Soc. 172 (1972), 119–131.

[17] J. Kwapisz, *Homotopy and dynamics for homeomorphisms of solenoids and Knaster continua*, Fund. Math. (2000).

[18] E. Lau and D. Schleicher, *Internal addresses of the Mandelbrot set and irreducibility of polynomials*, Preprint, Institute for Mathematical Sciences, Stony Brook (1994).

[19] M. Lyubich, *Dynamics of the rational transforms; the topological picture*, Russian Math. Surveys (1986).

[20] ______, *Laminations and holomorphic dynamics*, Lecture Notes of the mini-course given at the Conference “New Directions in Dynamical Systems” in Kyoto (2002).

[21] M. Lyubich and Y. Minsky, *Laminations in holomorphic dynamics*, J. Diff. Geom. 47 (1997), 17 – 94.

[22] R. Mañé, *On a theorem of Fatou*, Bol. Soc. Brasil Mat. NS (1993), no. 24, 1–12.

[23] R. Mañé, P. Sad, and D. Sullivan., *On the dynamics of rational maps*, Ann. Scien. Ec. Norm. Sup. Paris(4) (1983).
[24] J. Milnor, *Dynamics of one complex variable*, vieweg, 1999.

[25] ________, *Periodic orbits, external rays and the Mandelbrot set: An expository account*, Asterisque 261 (2000).

[26] V. Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, vol. 117, AMS, 2005.

[27] K. Pilgrim, *Combinatorics of complex dynamical systems*, Lecture notes in mathematics, no. 1827, Springer, 2003.

[28] A. Poirier, *The classification of postcritically finite polynomials I: Critical portraits*, Stony Brook IMS Preprint (1993).

[29] ________, *The classification of postcritically finite polynomials II: Hubbard trees*, Stony Brook IMS Preprint (1993).

[30] W. Scheffer, *Maps between topological groups that are homotopic to homomorphisms*, Proc. Amer. Math. Soc 33 (1972), 562–567.

[31] D. Schleicher, *Rational external rays of the Mandelbrot set*, Asterisque 261 (2000), 405–443.

[32] D. Sullivan, *Quasiconformal homeomorphisms and dynamics I, solution of the Fatou-Julia problem on wandering domains.*, Ann. Math. 122 (1985), 401–418.

[33] W. Thurston, *On the combinatorics of iterations of rational maps*, Preprint, Princeton University.

[34] M. Urbansky, *Rational functions with no recurrent critical points*, Ergod. Th. & Dynam. Sys. 14 (1994), 319–414.

[35] ________, *Geometry and ergodic theory of conformal non-recurrent dynamics*, Ergod. Th. & Dynam. Sys. 17 (1997), 1449–1476.