Measuring Fine Tuning In Supersymmetry

Peter Athron\textsuperscript{1} \textsuperscript{a} and D.J. Miller\textsuperscript{1}

Department of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, UK

\textbf{Abstract.} The solution to fine tuning is one of the principal motivations for supersymmetry. However constraints on the parameter space of the Minimal Supersymmetric Standard Model (MSSM) suggest it may also require fine tuning (although to a much lesser extent). To compare this tuning with different extensions of the Standard Model (including other supersymmetric models) it is essential that we have a reliable, quantitative measure of tuning. We review the measures of tuning used in the literature and propose an alternative measure. We apply this measure to several toy models and the MSSM with some intriguing results.

\textbf{PACS.} 12.60.Jv Supersymmetric models – 11.30.Pb Supersymmetry

1 Introduction

The Little Hierarchy Problem arose when no Beyond the Standard Model (BSM) physics was found at LEP, despite expectations from naturalness that it would. In particular Barbieri and Giudice \cite{BG} argued that to avoid fine tuning the supersymmetric particles of the Constrained Minimal Supersymmetric Standard Model (CMSSM) should be within the mass reach of LEP.

The mass of the Z boson is predicted from the soft supersymmetry (susy) breaking parameters by imposing electroweak symmetry breaking conditions. For tan $\beta = 10$,
\[
M_Z^2 \approx |\mu|^2 + 0.076m_2^2 + 1.97m_3^2 + 0.1A^2 + 0.38Am_4,
\]
where the parameters are $m_0$, the universal scalar mass; $m_{1/2}$, the universal gaugino mass; $A$, the universal trilinear coefficient; sign($\mu$), the undetermined sign of $\mu$, a bilinear soft Higgs mass, and tan $\beta$. Although the correct $M_Z = 91.1876$ GeV can be obtained by fixing $|\mu|$, if $1.97m_{1/2} \approx 500$ then this must cancel with some combination of parameters to $O(1/25)$.

To quantify tuning Barbieri and Guidice applied a measure originally proposed in Ref.\cite{II}. For an observable, $O$, and a parameter, $p_i$, $\Delta_{BG}(p_i) = \frac{p_i}{\partial O(p_i) / \partial p_i}$.

A large value of $\Delta_{BG}(p_i)$ implies that a small change in the parameter results in a large change in the observable, so the parameters must be carefully “tuned” to the observed value. Since there is one $\Delta_{BG}(p_i)$ per parameter, they define the largest of these values to be the tuning for that scenario, $\Delta_{BG} = \text{max}(\{\Delta_{BG}(p_i)\})$. Then they make the aesthetic choice that $\Delta_{BG} > 10$ is fine tuned.

Despite wide use of $\Delta_{BG}$, it has several limitations which may obscure the true picture of tuning:

\begin{itemize}
  \item variations in each parameter are considered separately;
  \item only one observable is considered in the tuning measure, but there may be tunings in several observables;
  \item only infinitesimal variations in the parameters are considered;
  \item there is an implicit assumption that the parameters come from uniform probability distributions.
\end{itemize}

it does not take account of global sensitivity; The final problem can be understood by considering the simple mapping $f : x \rightarrow x^n$, where $n \gg 1$. For this function $\Delta_{BG} = \Delta_{BG}(x) = n$. Since $\Delta_{BG}$ is independent of $x$, we follow the example of \cite{III} and term this \textit{global sensitivity}. Since $\Delta_{BG}(x_1) - \Delta_{BG}(x_2) = 0$ for all $x_1, x_2$, there is no relative sensitivity between points in the parameter space.

If we use $\Delta_{BG}$ as our tuning measure then $f(x)$ appears fine tuned throughout the entire parameter space. This contrasts with our fundamental notion of tuning being a measure of how atypical a scenario is. A true measure of tuning should only be greater than one when there is relative sensitivity between different points in the parameter space.

2 A New Tuning Measure

We propose a new measure of tuning. We define two volumes in parameter space for every point $P'(p_i')$. $F$ is the volume formed from dimensionless variations in the parameters over some arbitrary range $[a, b]$, about point $P'$, i.e. $a \leq \frac{p_i}{p_i'} \leq b$. $G$ is the volume in which dimensionless variations of the observables fall into the same range $[a, b]$, i.e. $a \leq \frac{G\{O(p_i')\}}{G\{O(p_i)\}} \leq b$.

We define an unnormalised measure of tuning with, $\Delta = \frac{F}{G}$. This can be used to compare different regions.
of parameter space within a given model as the normalisation factor will be common. However like \( \triangle_{BG} \) it includes global sensitivity. To compare tuning in different models we need to include normalisation, so tuning is given by,

\[
\Delta = \frac{F}{G}, \quad \hat{\Delta} = \frac{1}{G \langle F \rangle},
\]

where

\[
\Delta_{O_j} = \frac{1}{\langle F \rangle_{O_j}} \frac{F}{G_{O_j}}.
\]

with \( \hat{\Delta} = \frac{\langle F \rangle}{G} = \frac{\int dp_1...dp_n \frac{F}{G} \langle \{ p_i \}, \{ O_i \} \rangle}{\int dp_1...dp_n} \). (2)

It is also useful to look at tuning in terms of individual observables, while maintaining our multi-parameter approach. Therefore we define \( G_{O_j} \) to be the volume restricted by \( a \leq \frac{O_j}{\langle O_j \rangle} \leq b \) and \( a \leq \frac{p_i}{\langle p_i \rangle} \leq b \). Tuning is then defined by,

The arbitrary range \([a, b]\) has fallen out of the result.

We also determine \( \hat{\Delta} \) by integrating over the whole parameter range, \( m^2_{0\text{min}} < m^2_0 < m^2_{0\text{max}} \), where \( m^2_{0\text{min}} \) and \( m^2_{0\text{max}} \) are hypothetical upper and lower limits respectively and present results where \( m^2_H > 0 \). These bounds give the total allowed range of the parameter in this model and should not be confused with the range of dimensionless variations which appears in the definition of \( F \). If we take the range of variation to be large, \( m^2_{0\text{max}} - m^2_{0\text{min}} \gg C A^2 \), then \( \hat{\Delta} \approx \frac{m^2_0}{m^2_{\text{Planck}}} = \Delta_{BG} \).

Alternately, if we choose a very narrow range of variation about \( C A^2 + \mu^2_H \), where \( \mu_H \approx 100 \text{ GeV} \), then \( \hat{\Delta} \) is very small.

This is intuitively reasonable. If there was some compelling theoretical reason for the bare mass to be constrained close to the cutoff, the case for new physics at low energies would be dramatically weakened. It is precisely because there is no such compelling reason that we worry about the hierarchy problem and look to low energy BSM physics to explain how we can have \( m_H \ll m^2_{\text{Planck}} \).

Now lets treat the SM Hierarchy Problem in a slightly more sophisticated way. We no longer fix the UV cutoff. Instead we treat \( m^2_H \) as a function of two parameters, \( m^2_0 \) and \( A^2 \) and we have a second observable \( m^2_{\text{Planck}} = A^2 \) (“observed” to be large due to the weakness of gravitation).

There are no new cancellations between the parameters, so we expect the same result for \( \Delta \) as before, but this provides a simple illustration of how our measure works with more than one parameter. Varying the parameters about some point \( \{ m^2_0, A^2 \} \) over the dimensionless interval \([a, b]\) forms an area, \( F \). The bounds from dimensionless variations in \( m^2_H \) introduce two new lines in the parameter space which together with dimensionless variations in \( m^2_{\text{Planck}} \) (the same as those from \( A^2 \)) form the area \( G \).

This is shown schematically in Fig. 1 for two different points. In one, the parameters are of the same order as the observables (since \( m^2_{\text{Planck}} \approx O(m_H) \)), so \( G \) is not much smaller than \( F \). For the other point \( m^2_{\text{Planck}} \gg m^2_H \), resulting in \( F \gg G \) and fine tuning. In general the areas are, \( F = (b-a)^2 m^2_0 A^2 \) and \( G = (b-a)^2 A^2 m^2_H \) so again we obtain, \( \Delta = 1 + \frac{C A^2}{m^2_H} = \Delta_{BG} \).

While our measure does not deviate from \( \Delta_{BG} \) in this simple example, models with additional parameters allow the observable to be obtained from cancellation of more than two terms, complicating the fine tuning picture. For more examples including one with three parameters and four observables please see Ref. 5.

### 3.1 Fine Tuning In the CMSSM

Since the CMSSM contains many parameters and many observables we chose to apply a numerical version of our measure to study the variation of tuning in the CMSSM.

We take random dimensionless fluctuations about a CMSSM point at the GUT scale, \( p' = \{ p_k \} \), to give
new points \{P_i\}. These are passed to a modified version of Softsusy 2.0.5 \[4\]. Each random point \(P_i\) is run down from the GUT scale until electroweak symmetry is broken. An iterative procedure is used to predict \(M_2^2\) and then all the sparticle and Higgs masses are determined.

For every observable \(O_i\), a count, \(N_{O_i}\), is kept of how often the point lies in the volume \(G_{O_i}\) as well as an overall count, \(N_O\), kept of how many points are in \(G\). The tunings are then measured with,

\[
\Delta O_i \approx \frac{N}{N_{O_i}} , \quad \Delta \approx \frac{N}{N_O} . \tag{5}
\]

The set of observables, \\(\{O_i\}\) used in our definition of \(G\) here is the set of \(M_2^2\) and all (masses)\(^2\) predicted in Softsusy.

The parameters we vary simultaneously are the set \(\{m_0, m_{1/2}, \mu_{GUT}, m_3^2, A, y_t, y_b, y_\tau\}\), where \(m_3\) is the soft bilinear Higgs mixing parameter and \(y_t, y_b, y_\tau\) are the Yukawa couplings of the top, bottom and tau respectively.

When using Softsusy to predict the masses for the random points, sometimes the full mass spectrum cannot be predicted as we may have a tachyon, the Higgs potential unbounded from below, or non-perturbativity. Such points don’t belong in \(G\) as they will give dramatically different physics. However it is unclear which volumes, \(G_{O_i}\), the point lies in. Such points never register as hits in any of the \(G_{O_i}\) and this may artificially inflate the individual tunings, including \(\Delta M_2^2\). Keeping the range small reduces such errors, so we chose \(a = 0.9\) and \(b = 1.1\) for our dimensionless variations.

We examine tuning for points in the grid,

\[
A = -100 \text{GeV}, \quad \tan \beta = 10, \quad \text{sign}(\mu) = +, \\
250 \text{GeV} \leq m_+ \leq 500 \text{GeV}, \quad 100 \text{GeV} \leq m_0 \leq 200 \text{GeV}.
\]

\footnote{Note that all CMSSM points have \(|\mu|\) set by \(M_2^2\), so our tuning measure is not sensitive to the \(\mu\)-problem. However for our random variations we do treat \(\mu_{GUT}\) as a parameter because we are predicting \(M_2^2\), not fixing it to it’s observed value.}

Shown in Fig. 2 is the variation in \(\Delta M_2^2\) with respect to \(m_{1/2}\). To reduce statistical errors the \(\Delta M_2^2\) for each \(m_{1/2}\) is averaged over the five different \(m_0\) values. This substantially reduces the errors giving a much more stable picture of tuning increasing linearly with \(m_{1/2}\).

\[\Delta\], which includes all of the masses predicted by Softsusy as well as \(M_2^2\), is shown in Fig. 3. Although the errors are much larger here, a similar pattern to that for \(M_2^2\) can be seen. Since these are unnormalised tunings, the numerical values of the two measures cannot be compared and one should not assume that \(\Delta > \Delta M_2^2\) implies that the tuning is worse than when only \(M_2^2\) was considered. In fact the lack of evidence for distinct patterns of variation in tuning from the Figs. 2 and 3 is consistent with the conjecture that the large cancellation between parameters in \(M_2^2\) is the dominant source of the tuning for these points.

\[\Delta\], as plotted against \(m_{1/2}\). The tunings are then measured with,

\[
\Delta \approx \frac{N}{N_O}.
\]

\[\Delta\], which includes all of the masses predicted by Softsusy as well as \(M_2^2\), is shown in Fig. 3. Although the errors are much larger here, a similar pattern to that for \(M_2^2\) can be seen. Since these are unnormalised tunings, the numerical values of the two measures cannot be compared and one should not assume that \(\Delta > \Delta M_2^2\) implies that the tuning is worse than when only \(M_2^2\) was considered. In fact the lack of evidence for distinct patterns of variation in tuning from the Figs. 2 and 3 is consistent with the conjecture that the large cancellation between parameters in \(M_2^2\) is the dominant source of the tuning for these points.

\[\Delta\], as plotted against \(m_{1/2}\). The tunings are then measured with,

\[
\Delta \approx \frac{N}{N_O}.
\]

\[\Delta\], which includes all of the masses predicted by Softsusy as well as \(M_2^2\), is shown in Fig. 3. Although the errors are much larger here, a similar pattern to that for \(M_2^2\) can be seen. Since these are unnormalised tunings, the numerical values of the two measures cannot be compared and one should not assume that \(\Delta > \Delta M_2^2\) implies that the tuning is worse than when only \(M_2^2\) was considered. In fact the lack of evidence for distinct patterns of variation in tuning from the Figs. 2 and 3 is consistent with the conjecture that the large cancellation between parameters in \(M_2^2\) is the dominant source of the tuning for these points.

Although we can’t easily determine the normalisation using this approach it is nonetheless interesting...
Theoretical Models Contributed Talk

The spectra of these points are displayed in Fig. 4, and Fig. 5. The unnormalised tunings are displayed in Table 1. Note that these are not intended to be complete, and could conclude that \(\hat{\Delta}_N \approx \Delta \) and \(\hat{\Delta}_{M_Z} \approx \hat{\Delta}_{M_Z} \) then we would have demonstrated that \(\hat{\Delta}_{M_Z} > \Delta \) and could conclude that the Little Hierarchy problem is not as severe as has been suggested. Sometimes (e.g. NP1) the lightest neutralino is very light due to large cancellations between the parameters. Similar effects may be present in other masses, so mass hierarchies may appear in a greater proportion of the parameter space than conventional CMSSM wisdom dictates. This would reduce the true tuning in the CMSSM as scenarios with hierarchies would be less atypical than previously thought. A reduction in tuning from this effect can only be measured by using our normalised new measure, \(\hat{\Delta} \).

4 Conclusions

Current measures of tuning have several limitations. They neglect the many parameter nature of fine tuning; ignore additional tunings in other observables; consider local stability only; assume \(\mathcal{L}_{SUSY} \) is parameterised in the same way as \(\mathcal{L}_{GUT} \) and do not account for global sensitivity.

We have presented a new measure of tuning to address these issues. We showed that in the CMSSM both \(\Delta \) and \(\Delta_{BG} \) increase with \(M_{1/2} \). While a naive interpretation suggests \(\Delta_{BG} > \Delta \) normalisation may dramatically change this. If \(\Delta << \Delta_{M_Z} \) we can explain the Little Hierarchy Problem.

References

1. R. Barbieri and G. F. Giudice, Nucl. Phys. B 306, 63 (1988).
2. J. R. Ellis, K. Enqvist, D. V. Nanopoulos and F. Zwirner, Mod. Phys. Lett. A 1 (1986) 57.
3. G. W. Anderson and D. J. Castano, Phys. Lett. B 347, 300 (1995) [arXiv:hep-ph/9409419].
4. B. C. Allanach, Comput. Phys. Commun. 143, 305 (2002) [arXiv:hep-ph/0104145].
5. P. Athron and D. J. Miller, arXiv:0705.2241 [hep-ph].