Maximum amplitudes of finite-gap solutions for the focusing Nonlinear Schrödinger Equation

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Abstract

In this paper we prove that the maximum amplitude of a finite-gap solution to the focusing Nonlinear Schrödinger equation with given spectral bands does not exceed half of the sum of the length of all the bands. This maximum will be attained for certain choices of the initial phases. A similar result is also true for the defocusing Nonlinear Schrödinger equation.

1 Introduction

Finite-gap (algebro-geometric) solutions to the focusing Nonlinear Schrödinger Equation (fNLS)

\[ i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0, \tag{1-1} \]

are quasi-periodic solutions that represent nonlinear multi-phase waves. They were first constructed by Its and Kotlyarov in [14] and were extensively studied in the following years. Historically, finite-gap solutions were first constructed for the Korteweg-de Vries (KdV) equation and then were extended to other nonlinear integrable systems, see, for example, the book [1] and references therein. In general, a finite-gap solution is defined by a collection of spectral bands and of real constants (initial phases), associated with the corresponding bands.

Our interest to finite-gap solutions of the fNLS stems from the fact that the fNLS (1-1) is amongst the simplest and most commonly accepted mathematical models that is used to study the rogue wave

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phenomena. Here we refer to the common understanding of rogue waves as exceptionally tall waves with the amplitude $|\psi|^2 \geq 8|\psi_0|^2$, where $|\psi_0|$ is the amplitude of the background waves. Several particular types of solutions to the fNLS expressed through elementary functions (Peregrine, Akhmediev and Kuznetsov-Ma breathers, see, for example, [10]) provide, perhaps, the most known examples of the rogue wave solutions. These breathers can be viewed as degenerate limits of the corresponding finite-gap solutions. Therefore, it appears natural to look for rogue waves in the class of finite-gap solutions to fNLS. This problem is the subject of an ongoing research [4] for finite-gap solutions of any genus (which is the number of the spectral bands minus one). The main goal of the present paper is a new simple formula for the maximal amplitude of a finite-gap solution with given spectral bands. Namely, we proved that the maximal amplitude cannot exceed half of the sum of the length of all the spectral bands, and this maximum will be attained for certain choices of the initial phases. In fact, due to ergodic property of quasi-periodic solutions, this maximum will be approached by a finite-gap solution with a given spectral bands and generic initial phases in a sufficiently large space-time region. In the case of genus two, this result was recently obtained by O. Wright in [21]. It turns out that the obtained formula is also valid for finite-gap solutions of the defocusing NLS (dNLS) and that a somewhat similar statement is valid for KdV. It will be convenient to describe the finite-gap solutions through the corresponding Riemann-Hilbert Problems (RHPs). It is well known that the inverse scattering transform (IST) method of solving nonlinear integrable systems can be reduced to certain matrix RHP (see [17], [16], [22]), where the jump matrices are defined in terms of the scattering data. The RHPs with permutation type piece-wise constant jump matrices correspond to finite-gap solutions ([6], [5]). In the context of the semiclassical (small dispersion limit) analysis, such RHPs (known as model RHPs or outer parametrices) were first studied in [7] for the KdV and in [18], [15] for the fNLS. They represent the leading order term of the original RHP. Model RHPs are usually obtained through the nonlinear steepest descent method of Deift and Zhou. The finite-gap solution of a model problem provides the local (in $x,t$) leading order behavior (in the semiclassical limit) of the corresponding slowly modulated solution.

**Description of results.** The data that characterize a finite-gap solution is: (a) a hyperelliptic Riemann surface $\mathcal{R}$ of genus $g$ with $g + 1$ Schwarz symmetrical vertical branchcuts $\gamma_j = [\bar{\alpha}_j, \alpha_j]$, $j = 0, 1, \ldots, g$, where $\alpha_j = a_j + ib_j$, $b_j > 0$ (they will be referred to as branchpoints); (b) a collection of $g$ real constants $\Omega^0 = (\Omega^0_1, \ldots, \Omega^0_g)$, to be interpreted as a (real) vector in the Jacobian variety $\mathbb{J}_r$ of the Riemann surface $\mathcal{R}$ (see Section A for basic notations). The branchcuts are oriented upwards, see Figure 2. This finite-gap solution is given by (see Section 2)

$$\psi_{\Omega^0}(x,t) = \frac{\Theta(2u_\infty + \Omega(x,t))\Theta(0)}{\Theta(2u_\infty)\Theta(\Omega(x,t))} \sum_{j=0}^{g} b_j,$$

(1-2)

where $\Theta$ is the Riemann Theta function (see (A.1)), $u_\infty$ is the Abel map evaluated at $\infty_+$ (on the main sheet of $\mathcal{R}$), vector $\Omega = \Omega(x,t) = Wt + Vx + \Omega^0$. Here $W, V$ are vectors of $\mathbf{B}$-periods of the normalized
meromorphic differentials of the second kind \( dp, dq \) on \( R \) respectively, which have poles only at \( \infty \) and have the corresponding principal parts \( \mp \frac{1}{\zeta} d\zeta, \mp \frac{2}{\zeta} d\zeta, \zeta = \frac{1}{2} \). Some basic facts about Riemann Theta functions can be found in Appendix 6.

**Remark 1.1.** This type of solution, but with \( \sum_{j=0}^{g} (-1)^j b_j \) instead of \( \sum_{j=0}^{g} b_j \), was constructed in [18]. Same type of solution can also be found in [15].

The goal of this paper is to prove the following sharp estimate

\[
\sup_{x,t \in \mathbb{R}} |\psi_{\Omega^0}(x,t)| \leq |\psi_0(0,0)| = \sum_{j=0}^{g} b_j,
\]

(1-3)

that is valid for any \( x, t \in \mathbb{R} \) and any \( \Omega^0 \in \mathbb{R}^g \). Thus, the amplitude of any finite-gap solution to fNLS (1-1) with vertical spectral bands \( \gamma_j, j = 0, 1 \ldots, g \), cannot exceed one half of the total length (sum) of the bands, and this maximum value will be attained with the proper choice of the initial phases.

This statement, with a proper modification, also holds true for the finite-gap solutions of the defocusing NLS and the KdV. As an illustration, consider the point of gradient catastrophe (see [9]) for a slowly modulated plane wave solution to the semiclassical fNLS. At this point (in the \( x, t \) plane), two new branchpoints \( \alpha_1, \alpha_2 \) instantaneously appear exactly at the branchpoint \( \alpha_0 \) of the existing spectral band of the modulated plane wave (together with their complex conjugate \( \bar{\alpha}_1, \bar{\alpha}_2 \) appearing at \( \bar{\alpha}_0 \)). The chain of scaled Peregrine breathers, appearing immediately beyond the point of gradient catastrophe, have their heights three times higher than the amplitude of the solution at the points of gradient catastrophe, see [2]. Indeed, in accordance with (1-3), we have \( \frac{1}{2} \sum_{j=0}^{2} |\alpha_j - \bar{\alpha}_j| = 3 |\alpha_0 - \bar{\alpha}_0| \). The theory in [2] predicts degenerate gradient catastrophes with higher order Peregrine breathers of the heights 5, 7, etc., which, in accordance with (1-3), would correspond to 5, 7, etc. new spectral bands appearing at the location of the existing band \( [\bar{\alpha}_0, \alpha_0] \). For higher Peregrine breathers see, for example, [8].

In view of (1-2), (1-3), and keeping in mind that the dependence of \( \Omega(x,t) \) on \( x, t \) is linear, we will study the function

\[
f(\Omega) = \frac{\Theta(2u_\infty + \Omega)\Theta(0)}{\Theta(2u_\infty)\Theta(\Omega)} : \mathbb{T}^g \to \mathbb{C}
\]

(1-4)

on the torus \( \mathbb{T}^g = \mathbb{R}^g \mod \mathbb{Z}^g \simeq [0, 1]^g \), with the opposite sides of the cube being identified. In the case of \( g \geq 2 \), the fNLS solution \( \psi_{\Omega^0}(x,t_0) = f(Vx + Wt_0 + \Omega^0) \sum_{j=0}^{g} b_j \) with a fixed \( t_0 \) consists of values of \( f(\Omega) \) over the winding \( \Omega = Vx + Wt_0 + \Omega^0, x \in \mathbb{R} \), of the real torus \( \mathbb{T}^g \). This winding, generically, is irrational so that \( \psi_{\Omega^0}(x,t_0) \) is quasi-periodic and so

\[
\sup_{x \in \mathbb{R}} |\psi_{\Omega^0}(x,t_0)| = \sum_{j=0}^{g} b_j
\]

(1-5)

due to ergodicity. In the case of \( g = 1 \) the solution \( \psi_{\Omega^0}(x,t_0) \) is, obviously, a periodic function. Our results for the function \( f \) in (1-4) are summarized in the following Main Theorem, which implies the main statement of the paper, namely, the inequality (1-3).
Theorem 1.2. The function $f : \mathbb{T}^g \to \mathbb{C}$ in (1-4) has the following properties:

1. The maximum of $|f|$ is attained at $\Omega = 0$ where $f = 1$ and hence (1-3) holds;

2. The nonzero critical points of $|f|$ occur at the half-periods $h = (h_1, \ldots, h_g)^t \in \frac{1}{2} \mathbb{Z}^g$ of $\mathbb{T}^g$, where

   $$f(h) = b_0 + \sum_{j=1}^g (-1)^{2h_j} b_j \sum_{j=0}^g b_j; \quad (1-6)$$

3. If $b_m > \sum_{k=0, k \neq m}^g b_k$ for some $b_m, m = 0, \ldots, g$ then

   $$\min_{\Omega \in \mathbb{T}^g} |f(\Omega)| = \frac{b_m - \sum_{j=0, j \neq m}^g b_j}{\sum_{j=0}^g b_j}. \quad (1-7)$$

The graph of $|f(\Omega)|$ in the case of $g = 2$ is shown on Figure 1, upper left corner. In the cases $g > 2$, one can only graph $|f(\Omega)|$ over two dimensional cross sections of the torus $\mathbb{T}^g$. By choosing cross-sections, defined by the vectors $V, W$, we, in fact, graph $|\psi_{\Omega_0}(x, t)|$ (with different $\Omega^0$) over the $x, t$ plane. The graph of $|\psi_0(x, t)|$ with $g = 4$ is shown in the upper right corner, whereas the graphs of $|\psi_{\Omega_0}(x, t)|$ with $g = 3$ and $\Omega^0 = 0$ (left) and some random $\Omega^0$ (right) are shown below.

The rest of the paper is organized as following. In Section 2 we introduce the RHP for finite-gap solutions of the fNLS and sketch the derivation of (1-2). In Section 3 we prove that half integer points of $\mathbb{T}^g$ are critical points of $f$ and we evaluate $f(\Omega)$ at these points. Within the set of critical points, the maximum value of $|f(\Omega)|$ is attained at $\Omega = 0$, where $f(0) = 1$. In Section 4 we prove that half integer points are the only possible critical points of $f$, where $f \neq 0$. That will prove items 1 and 2 of the Main Theorem (1.2). The remaining item 3 of Theorem 1.2 is proven in Section 5. In Section 6 we state and prove an analog of Main Theorem for the defocusing NLS and discuss some similar results for the KdV. Some basic facts about Riemann surfaces as well as proofs of some technical results can be found in the Appendices A - B.

2 RHP representation of finite gap solutions for the focusing NLS: a brief review of the derivation of $f$

Let us briefly review the derivation of (1-2) ([18]). We start with the RHP

$$Y_+ = Y_- i \sigma_2 e^{-2\pi i \Omega_j \sigma_3} \quad \text{on} \quad \gamma_j, \quad j = 0, 1, \ldots, g, \quad Y(z; \Omega) = 1 + \frac{Y_1(\Omega)}{z} + \cdots, \quad \text{as} \quad z \to \infty, \quad (2-1)$$

for the matrix $Y(z; \Omega)$ that is analytic and invertible in $\tilde{\mathbb{C}} \setminus \bigcup_{j=0}^g \gamma_j$, where we take $\Omega_0 = 0$. Solution to this RHP exits and is unique for any choice of symmetrical (with respect to $\mathbb{R}$) vertical branchcuts $\gamma_j$ and for any vector $\Omega \in \mathbb{R}^g$, see [22]. In fact, it will be shown that the existence of solution is equivalent
Figure 1: A plot of $|f(\Omega)|$ for $g = 2$ and branchpoints $\alpha_0 = 0.1 + 2i$, $\alpha_1 = 0.5i$, $\alpha_2 = -0.1 + i$ is in the upper left corner. The maximum $|f(\Omega)| = 1$ is attained at $\Omega = 0 \mod \mathbb{Z}^2$, and the minimum $|f(\Omega)| = \frac{1}{7}$, see (1-6), is attained at $\Omega = \left(\frac{1}{2}, \frac{1}{2}\right) \mod \mathbb{Z}^2$. A plot of $|\psi_0(x,t)|$ with $g = 4$ and $\alpha_0 = 0.2 + i$, $\alpha_1 = 0.1 + i$, $\alpha_2 = i$, $\alpha_3 = -0.1 + i$, $\alpha_4 = -0.2 + i$ is in the upper right corner. The maximum of 5 is achieved at $(x,t) = (0,0)$. Condition 3 of Theorem 1.2 is not satisfied and the minimum is 0. Plots of $|\psi_{\Omega_0}(x,t)|$ with $g = 3$ and $\alpha_0 = 0.15 + i$, $\alpha_1 = 0.05 + i$, $\alpha_2 = -0.05 + i$, $\alpha_3 = -0.15 + i$ are given on the second line. The case $\Omega_0 = 0$ is shown on the left, where the maximum amplitude of 4 is reached at $(x,t) = (0,0)$. In the right picture, the initial vector $\Omega_0$ is chosen randomly; in the shown part of the $x,t$ plane the maximum is smaller than 4. Condition 3 of Theorem 1.2 is again not satisfied and the minimum is 0 for both choices of $\Omega_0$. Notice the different behavior of $|\psi_{\Omega_0}(x,t)|$ for even and odd genera when symmetrical with respect to the imaginary axis branchcuts are located “close” to each other. This difference stems from the fact that in the limiting case (when symmetrical branchcuts collide) we have either an $n$-soliton solution (odd $g$) or $n$-solitons on the plane wave background (even $g$).
to the statement that $\Theta(\Omega) \neq 0$ on $\mathbb{T}^g$, and the latter inequality will be proven in Section B. The same is true for the case of real non intersecting branchcuts $\gamma_j$, see [5].

It is known ([18], [15] [17]) that the solution to fNLS (1-1) is expressed through $Y(z; \Omega)$ by

$$
\psi_{\Omega}^{(0)}(x, t) = -2(Y_1)_{1, 2}(\Omega), \quad \text{where} \quad \Omega = Wt + Vx + \Omega^0
$$

and $(Y_1)_{1, 2}$ denotes the $(1, 2)$ entry of the matrix $Y_1$. The jump contours of the RHP (2-1) for $Y$ coincide with the the branchcuts of the hyperelliptic Riemann surface $\mathcal{R}$, introduced in Section 1. We now remind some standard objects from the theory of Riemann surfaces. Let us define the $A$ and $B$ cycles on $\mathcal{R}$ (the homology basis) of $\mathcal{R}$ as: cycles $A_j$ are negatively oriented loops around $\gamma_j$ on the main sheet of $\mathcal{R}$; cycles $B_j$ are shown on Figure 2.

With this choice of the homology basis, we define the vector $\omega$ of normalized holomorphic differentials on $\mathcal{R}$ in the standard way by

$$
\int_{A_j} \omega_k = \delta_{k,j}, \quad k,j = 1, \ldots, g
$$

where $\delta_{k,j}$ is the Kronecker symbol. We then introduce the function

$$
\lambda(z) = \left( \prod_{j=0}^{g} \frac{z - \alpha_j}{z - \bar{\alpha}_j} \right)^{\frac{i}{4}}
$$

with branch cuts along $\gamma_j$. The determination of $\lambda(z)$ is chosen in such a way that $\lim_{z \to \infty} \lambda(z) = 1$.

It was shown in [18] (and it can be verified directly using the properties of Theta functions described in Section A) that

$$
Y(z; \Omega) = \mathcal{L}^{-1}(\infty)\mathcal{L}(z),
$$

where

$$
\mathcal{L}(z) = \frac{1}{2} \begin{pmatrix}
(\lambda(z) + \lambda^{-1}(z))\mathcal{M}_1(z, d) & -i(\lambda(z) - \lambda^{-1}(z))\mathcal{M}_2(z, d) \\
i(\lambda(z) - \lambda^{-1}(z))\mathcal{M}_1(z, -d) & (\lambda(z) + \lambda^{-1}(z))\mathcal{M}_2(z, -d)
\end{pmatrix}
$$

and

$$
\mathcal{M}(z, d) \equiv (\mathcal{M}_1, \mathcal{M}_2) = \left( \frac{\Theta(u(z) - \Omega + d)}{\Theta(u(z) + d)} , \frac{\Theta(-u(z) - \Omega + d)}{\Theta(-u(z) + d)} \right)
$$

Here $\Theta$ denotes the Theta function on the hyperelliptic Riemann surface $\mathcal{R}$ with the period matrix $\tau$, $u(z) = \int_{\bar{\alpha}_0}^z \omega$ is the Abel map with the base-point $\bar{\alpha}_0$ and a constant vector $d \in \mathbb{C}^g$ is to be determined.
In order for \( \mathcal{L}(z) \) to be non-singular on \( \mathcal{R} \) we need to choose the vector \( d \in \mathbb{C}^g \) in such a way that that the \( g \) finite zeroes \( z_1, \ldots, z_g \) of the meromorphic on \( \mathcal{R} \) function \( \lambda^2(z) - 1 \) cancel the \( g \) zeroes of \( \Theta(u(z) - d) \). If this is the case, then the \( g \) finite zeroes \( \widehat{z}_1, \ldots, \widehat{z}_g \) of the meromorphic function \( \lambda^2(z) + 1 \) on \( \mathcal{R} \), cancel the \( g \) zeroes of \( \Theta(u(z) + d) \), where \( \hat{p} = (\widehat{z}, \widehat{R}) = (z, -R) \) denotes the hyperelliptic involution. Then, according to Theorem A.4,

\[
d = u(D_0) + K \quad \text{and} \quad -d = u(\hat{D}_0) + K,
\]

where the divisor \( D_0 = \sum_j z_j \) and \( K \) denotes the vector of Riemann constants. Here and henceforth we assume that all equations for Abel maps are in the Jacobian \( J_\tau \) (see Section A). Since \( u(\hat{D}_0) = -u(D_0) \) and \( 2K = 0 \), the two equations in (2-7) are equivalent. Observe that: i) the zeroes are at \( \infty \) and at \( D_0 \) while the poles are at the branch-points \( \alpha_j \)'s; ii) according to Proposition A.3, the Abel map of the divisor of the latter points is \( K \). Thus, by the Abel’s Theorem and (2-7), we obtain

\[
u(D_0) + \nu(\infty) = -K \Rightarrow \nu_\infty = -d.
\]

**Remark 2.1.** With the choice (2-8), the matrix \( \mathcal{L}(z) \) is non-singular on \( \mathbb{C} \setminus \bigcup \gamma_j \). Therefore, according to (2-4), the existence of the solution \( Y \) of the RHP (2-1) is equivalent to the invertibility of \( \mathcal{L}(\infty) \), which is, according to (2-3), (2-5) and (2-6), equivalent to \( \Theta(\Omega) \neq 0 \) on \( \mathbb{T}^g \). For the benefit of the reader, the inequality \( \Theta(\Omega) > 0 \) for any even \( g \in \mathbb{N} \), any \( \Omega \in \mathbb{T}^g \) and any vertical branchcuts \( \gamma_j \), \( j = 0, 1, \ldots, g \) is proven in Appendix B. In the case of any \( g \in \mathbb{N} \) and all \( \bigcup_{j=0}^g \gamma_j \subset \mathbb{R} \), (i.e. for the defocusing NLS) this statement was proven in [5]. In fact, the inequality \( \Theta(\Omega) > 0 \) for any \( g \in \mathbb{N} \) and any either all real or all vertical Schwarz symmetric branchcuts follows from the results of Chapter VI of [12].

We can now write solution \( Y(z, \Omega) \) by substituting (2-5)-(2-8) into (2-4). Then, according to (2-1),

\[
(Y_{1,2}(\Omega)) = \frac{1}{2} \Theta(2\nu_\infty + \Omega) \Theta(\Omega) \sum_{j=0}^g b_j = \frac{1}{2} f(\Omega) \sum_{j=0}^g b_j,
\]

so that (1-2) for the finite-gap solution follows from (2-9) and (2-2). Note that, taking into account Theorem B.2, zeroes of \( f(\Omega) \) coincide with the zeroes of \( \Theta(2\nu_\infty + \Omega) \) on the real torus \( \Omega \in \mathbb{T}^g \).

**Remark 2.2.** Let \( \mathcal{R} \) be an arbitrary hyperelliptic Riemann surface of genus \( g \) with (oriented) bounded and non intersecting branchcuts \( \gamma_j \), \( j = 0, \ldots, g \). Then solution of the RHP (2-1) with any \( \Omega \in \mathbb{R}^g \), if exists, is still given by (2-4) and

\[
(Y_{1,2}(\Omega)) = \frac{i}{4} f(\Omega) \sum_{j=0}^g (\alpha_j - \beta_j),
\]

where \( \beta_j \) is the beginning and \( \alpha_j \) is the end points of \( \gamma_j \). Solution \( Y \) of the RHP (2-1) exists if and only if \( \Theta(\Omega) \neq 0 \).
Lemma 2.3. The rational function $\lambda^4 - 1$ has $g$ finite simple zeroes $z_1 > z_2 > \ldots > z_g$ on $\mathbb{R}$ and one at $\infty_+$. Zeroes of $\lambda^2(z) - 1$ on $\mathcal{R}$ consists of $z_1 > z_2 > \ldots > z_g$ alternating between the main and the second sheets of $\mathcal{R}$ and of $\infty_+$. Zeroes of $\lambda^2(z) + 1$ on $\mathcal{R}$ consists of the hyperelliptic involutions of the zeroes of $\lambda^2(z) - 1$, that is, of $\hat{z}_1 > \hat{z}_2 > \ldots > \hat{z}_g$ and $\infty_-.

Proof. It follows immediately from (2-3) that $|\lambda^4(z)| = 1$ if and only if $z \in \mathbb{R}$. Then the numerator of

$$\lambda^4(z) - 1 = \prod_{j=0}^{g}(a - \alpha_j) - \prod_{j=0}^{g}(a - \bar{\alpha}_j) = 0$$

is a polynomial of degree $g$ since $\sum_{j=0}^{g}(\alpha_j - \bar{\alpha}_j) = 2i\sum_{j=0}^{g}b_j \neq 0$. Thus, $\lambda^4 - 1 = 0$ has $g$ finite real roots and also a root at $\infty_+$. The rest of the lemma follows from considering the argument of $\lambda^2(z)$ along $\mathbb{R}$. \hfill \Box

Remark 2.4. The statement of Lemma 2.3 is still valid if all the branchcuts $\gamma_j$, $j = 0, \ldots, g$ are on $\mathbb{R}$ and $\lambda(z) = \left(\prod_{j=0}^{g}\frac{z - \alpha_j}{z - \bar{\beta}_j}\right)^{\frac{1}{4}}$, where $\alpha_j, \beta_j$ are defined in Remark 2.2.

Remark 2.5. Since $\sum_{j=0}^{g}b_j > 0$, there is exactly $g$ finite zeroes of $\lambda^2(z) - 1$ and, thus, the divisor $\mathcal{D}_0$ of zeroes of $\Theta(u(z) + u(\infty))$ has only finite points. Therefore, $\Theta(2u_\infty) \neq 0$.

3 Evaluation of $|f|$ at half-integer points

Let $h \in \frac{1}{2}\mathbb{Z}^g$. We want to evaluate $|f(h)|$, since, as we will show in Section 4, these are the only possible nonzero critical points.

We start by discussing deformations of the hyperelliptic Riemann surface $\mathcal{R} = \mathcal{R}(\bar{\alpha})$, where $\bar{\alpha} = (\alpha_0, \ldots, \alpha_g)$ are the endpoints of the branchcuts $\gamma_j$. Let us change the orientation of a branchcut $\gamma_j$, $j = 1, \ldots, g$, by continuously deforming (shrinking and rotation) this branchcut so that we interchange the beginning and the end points of $\gamma_j$. This deformation does not affect $A$ cycles (and, thus, the normalized holomorphic differentials $\omega$), but transforms the cycle $B_j$ into $B_j - A_j$, so that the j-th column $\tau_j$ of the matrix $\tau$ becomes $\tau_j - e_j$, where $e_j \in \mathbb{C}^g$ is the j-th vector of the standard basis.

Let us denote by $Y(z; \Omega, \gamma)$ solution of the RHP (2-1) for a given collection of oriented vertical Schwarz symmetric contours $\gamma$ with jump matrices as in (2-1) defined through a vector of real constants $\Omega$. To keep $Y(z; \Omega, \gamma)$ invariant when reversing the orientation of $\gamma_j$, we need to replace simultaneously the corresponding jump matrix by its inverse, that is, to replace $\Omega_j$ by $\Omega_j + \frac{1}{2}$ in $\Omega$. Now, it is straightforward to check that for any $h \in \frac{1}{2}\mathbb{Z}^g$, the solution $Y(z; \gamma, \Omega)$ is invariant under transformations

$$(\gamma, \Omega) \mapsto \left((-1)^{2h}\gamma, \Omega + h\right), \quad (3-1)$$

where $(-1)^{2h}\gamma$ denotes the contours $\gamma_0, (-1)^{2h_1}\gamma_1, (-1)^{2h_2}\gamma_2, \ldots, (-1)^{2h_g}\gamma_g$ with $h = (h_1, h_2, \ldots, h_g)$. Thus, $Y(z; \gamma, \Omega) = Y(z; (-1)^{2h}\gamma, \Omega + h)$, which implies $(Y_1)_{1,2}(\Omega; \gamma) = (Y_1)_{1,2}(\Omega + h; (-1)^{2h}\gamma)$. The
formula (2.9) for \((Y_1)_{1,2}(\Omega + h; (-1)^{2h}\gamma)\) will have the same form as for \((Y_1)_{1,2}(\Omega; \gamma)\), except that \(\sum_{j=0}^{g} b_j\) must be replaced with \(\sum_{j=0}^{g} (-1)^{2h_j} b_j\), where \(h_0 = 0\). Then we obtain

\[
f(\Omega; \tau) \sum_{j=0}^{g} b_j = f \left( \Omega + h; \tau - 2 \sum_{j=1}^{g} h_j e_j \right) \sum_{j=0}^{g} (-1)^{2h_j} b_j,
\]

where we have emphasized the dependence of \(f\) on the matrix \(\tau\). Since \(h\) is a half-integer vector, we have

\[
f(h; \tau) \cdot \sum_{j=0}^{g} b_j = f \left( 0; \tau - 2 \sum_{j=1}^{g} h_j e_j \right) \cdot \sum_{j=0}^{g} (-1)^{2h_j} b_j \quad \text{or} \quad f(h; \tau) = \frac{\sum_{j=0}^{g} (-1)^{2h_j} b_j}{\sum_{j=0}^{g} b_j}
\]

since for any allowed choice of the \(B\)-cycles (and the corresponding period matrix \(\tau\)) \(f(0; \tau) = 1\). Equation (3-4) shows that maximum of \(|f(h; \tau)|\) among all the half integer points \(h \in \frac{1}{2} \mathbb{Z}^g\) is attained at \(h = 0\) and is equal to 1. Thus we have obtained the following lemma.

**Lemma 3.1.** For any \(h \in \frac{1}{2} \mathbb{Z}^g\) we have

\[
\frac{\Theta(2u_{\infty} + h)\Theta(0)}{\Theta(2u_{\infty})\Theta(h)} = \frac{\sum_{j=0}^{g} (-1)^{2h_j} b_j}{\sum_{j=0}^{g} b_j},
\]

so that

\[
\max_{h \in \frac{1}{2} \mathbb{Z}^g} |f(h)| = f(0) = 1.
\]

**Remark 3.2.** Note that Schwarz symmetry of \(R\) is not required for validity of (3-5), where, \(b_j\) in the right hand side should be, according to (2-10), replaced by \(\frac{1}{2}(\beta_j - \alpha_j)\), \(\alpha_j, \beta_j\) being the endpoint and the beginning point of the branchcut \(\gamma_j\), \(j = 0, 1, \ldots, g\). In fact, some general formulae of this type can be found in [20] as a consequence of Thomae formulæ.

**Remark 3.3.** It was shown in Remark 2.5 that \(\Theta(2u_{\infty}) = \Theta(2u_{\infty}; \tau) \neq 0\). However, the equality may occur in the case of a shifted period matrix \(\tau\). Indeed, substituting \(\Omega = 0\) into (3-2), we obtain

\[
\frac{1}{f \left( h; \tau - 2 \sum_{j=1}^{g} h_j e_j \right)} = \frac{\sum_{j=0}^{g} (-1)^{2h_j} b_j}{\sum_{j=0}^{g} b_j}
\]

In the special case when \(\sum_{j=0}^{g} (-1)^{2h_j} b_j = 0\) that implies, according to (1-4) and Theorem B.2, that \(\Theta \left( 2u_{\infty}; \tau - 2 \sum_{j=1}^{g} h_j e_j \right) = 0\).
4 Critical points of $|f(\Omega)|$

The critical points of $|f(\Omega)|$ and $|f(\Omega)|^2$ with nonzero critical value coincide. The Schwarz symmetry of the Riemann surface $R$ plays the central role for the results of this paper. We now assume that $R$ admits an antiholomorphic involution (anti-involution for short): we consider the two cases where all branch-points are either real with non-intersecting branchcuts $\gamma_j = [\beta_j, \alpha_j], j = 0, \ldots, g$, or they come only in complex conjugate pairs. The vertical/horizontal branchcuts are oriented upwards and left to right accordingly. It is straightforward to check (see Figure 2) that in the former case

$$\Re \tau = \frac{1}{2}(1 + L),$$

where $L$ is the $g \times g$ matrix with $L_{ij} = 1$ and $(\tau_{k,j}) = \left( \int_{B_j} \omega_k \right)$ is the standard $B$-period matrix. In the latter case (real branchcuts) we have $\Re \tau = 0$.

Lemma 4.1. If $z \in \mathbb{R}^g$ or $z \in \frac{1}{2}\mathbb{Z}^g + i\mathbb{R}$, then $\Theta(z) \in \mathbb{R}$.

Proof. From (A.1) and (4-1) or $\Re \tau = 0$ we obtain that $e^{i\pi(n, \tau n)} \in \mathbb{R}$. Therefore

$$\overline{\Theta(z)} = \sum_{n \in \mathbb{Z}^g} e^{2\pi i(n, -\bar{z}) + \pi i(n, \tau n)} = \Theta(-\bar{z}).$$

(4-2)

The statement follows from the Proposition A.1. \qed

The normalized holomorphic differentials $\omega(z) = (\omega_1(z), \ldots, \omega_g(z))^t$ have the form $\omega_j = \frac{p_j(z)}{R(z)} dz$, $j = 1, \ldots, g$, where the coefficients of the polynomial $p_j(z) = \lambda_{1,j} z^{g-1} + \ldots + \lambda_{g,j}$ form the $j$-th column of the matrix $(\lambda)_{m,k} = A^{-1}$, where $(A)_{jk} := \int_{A_j} \frac{z^{g-k} dz}{R(z)}$. Since matrix $A$ has purely imaginary entries, the coefficients of all $p_j(z)$ are purely imaginary, so that $\bar{\omega(z)} = -\omega(\bar{z})$. Then, setting the base point of the Abel map at the beginning $\beta_0$ of $\gamma_0$, we obtain

$$\hat{u}_\infty = -u_\infty + h_1 \quad \Rightarrow \quad 2\Re u_\infty = h_1,$$

(4-3)

where the vector $h_1 \in \frac{1}{2}\mathbb{Z}^g$ depends on the location of $\gamma_0$. In particular: if $\gamma_0$ is the rightmost vertical contour, then $h_1 = \frac{1}{2}(1, 1, \ldots, 1)^t$; if all the branchcuts are real, then $h_1 = 0$.

The Abel map $u(z)$ is defined on $R$ up to a vector in $\mathbb{Z}^g + \tau \mathbb{Z}^g$, depending on the path of integration. Choosing $u(z) = u_\infty + \int_\infty^z \omega$, we obtain

$$\overline{u(z)} = -u(\bar{z}) + h_1 \mod \mathbb{Z}^g$$

(4-4)

for any $z$ on the main sheet of $R$. Using $u(\bar{z}) = -u(z)$, we extend (4-4) to the whole $R$.

Lemma 4.2. For any $\Omega \in \frac{1}{2}\mathbb{Z}^g$ we have $f(\Omega) \in \mathbb{R}$ and $\nabla f(\Omega) \in i\mathbb{R}^g$. 

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Proof. The first statement follows from (1-4), Lemma 4.1 and (4-3). If $\Omega \in \frac{1}{2}Z^g$ and $\delta \in \mathbb{C}^g$ then, according to Proposition A.1, $\Theta(\Omega + \delta) = \Theta(-\Omega + \delta) = \Theta(\Omega - \delta)$, i.e., $\Theta(\Omega + \delta)$ is even with respect to $\delta$. Thus, $\nabla \Theta(\Omega) = 0$ for any $\Omega \in \frac{1}{2}Z^g$.

If $h \in \frac{1}{2}Z^g$, then, according to (4-2), for any $w, \delta \in \mathbb{R}^g$ we have $\Theta(\delta) = \Theta(\delta + iw) = \Theta(\delta - iw)$. Taking, according to (4-3), $h + iw = 2u_\infty + \Omega$, we obtain $\Re(\Theta(2u_\infty + \Omega + \delta))$ is an even function of $\delta$. Thus, $\nabla \Theta(\Omega + \delta) = 0$ for any $\Omega \in \frac{1}{2}Z^g$. Therefore,

$$
\nabla \Theta(2u_\infty + \Omega) = \frac{\nabla \Theta(2u_\infty + \Omega) \Theta(\Omega) - \nabla \Theta(\Omega) \Theta(2u_\infty + \Omega)}{\Theta^2(\Omega)} = \frac{i \nabla \Im \Theta(2u_\infty + \Omega)}{\Theta(\Omega)}, \tag{4-5}
$$

which, together with (1-4), proves the lemma. \hfill \Box

Corollary 4.3. Every $\Omega \in \frac{1}{2}Z^g$ is a critical point of $|f(\Omega)|$.

Proof.

$$
2\nabla |f(\Omega)| = \frac{\nabla f(\Omega) \bar{f}(\Omega) + \bar{\nabla f(\Omega)} f(\Omega)}{|f(\Omega)|}. \tag{4-6}
$$

If $\Omega \in \frac{1}{2}Z^g$ then, according to Lemma 4.2, the numerator is zero. In the case $f(\Omega) = 0$, the ratio is understood in the sense of the limit. \hfill \Box

Let $\partial_j = \frac{\partial}{\partial \lambda_j}$. The following theorem implies items 1 and 2 of the Main Theorem (1.2).

Theorem 4.4. If $\Omega \in \mathbb{T}^g$ is such that $\nabla |f(\Omega)| = 0$ then $f(\Omega) = 0$ or $\Omega \in \frac{1}{2}Z^g$.

Proof. To calculate $\partial_j |f(\Omega)|^2 = \bar{f}(\Omega) \partial_j f(\Omega) + f(\Omega) \partial_j \bar{f}(\Omega)$, we start with calculating $\partial_j Y(z; \Omega)$ for some $j \in \{1, \ldots, g\}$. Differentiation of RHP (2-1), yields the following non-homogeneous RHP for $\partial_j Y$:

$$
\partial_j Y_+ = \partial_j Y_- U_k + Y_+ \partial_j U_k \quad \text{on} \quad \gamma_k, \ k = 0, \ldots, g, \quad Y(z; \Omega) = \frac{\partial_j Y_1(\Omega)}{z} + \cdots, \quad \text{as} \quad z \to \infty, \tag{4-7}
$$

where $U_k = i\sigma_2 e^{-2\pi i \lambda_k \sigma_3}$. Since $\partial_j U_k = 0$ when $k \neq j$ and $\partial_j U_j^{-1} = 2\pi i \sigma_3$, the non-homogeneous RHP (4-7) has the solution

$$
\partial_j Y(z) = C_j (Y_+ \partial_j U_j Y_-^{-1}) Y = C_j (Y_+ \partial_j U_j U_j^{-1} Y_-^{-1}) Y = 2\pi i C_j (Y_- \sigma_3 Y_-^{-1}) Y = \int_{\gamma_j} \frac{Y_-(\zeta) \sigma_3 Y_-^{-1}(\zeta) d\zeta}{\zeta - z} Y(z), \tag{4-8}
$$

where $C_j$ denotes the Cauchy operator along the oriented branchcut $\gamma_j$. Then

$$
\partial_j Y_1(\Omega) = -\int_{\gamma_j} Y_-(\zeta; \Omega) \sigma_3 Y_-^{-1}(\zeta; \Omega) d\zeta. \tag{4-9}
$$

Using (2-4) - (2-6), we calculate

$$
(Y_-(z; \Omega) \sigma_3 Y_-^{-1}(z; \Omega))_{1,2} = \frac{2i(\lambda^2(z) - \lambda^{-2}(z))}{M_1^2(\infty, \Omega)} M_1(z, d) M_2(z, d), \tag{4-10}
$$

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so that, according to (2-9), (4-9),
\[
\partial_j f = \frac{-2\partial_j(Y_1)_{1,2}}{\sum_{j=0}^{g} b_j} = \frac{2}{\sum_{j=0}^{g} b_j} \int_{\gamma_j} \frac{(Y_{-}(z;\Omega)\sigma_j Y^{-1}_{-}(z;\Omega))_{1,2} dz}{b_j M_1(\infty, d)} = \sum_{j=0}^{g} b_j M_1^2(\infty, d) \int_{\gamma_j} (\lambda^2(z) - \lambda^{-2}(z)) M_1(z, d) M_2(z, d) dz.
\]
(4-11)

Now, using (2-6) and (2-8), we obtain
\[
\partial_j f = \frac{4i\Theta^2(0)}{\sum_{j=0}^{g} b_j \Theta^2(\Omega)} \int_{\gamma_j} (\lambda^2(z) - \lambda^{-2}(z)) \frac{\Theta(u(z) + \Omega + u_{\infty})\Theta(u(z) - \Omega - u_{\infty})}{\Theta(u(z) + u_{\infty})\Theta(u(z) - u_{\infty})} dz.
\]
(4-12)

According to [11], the fraction in the integrand is a meromorphic function on \( R \). In fact, one can use (A.3) from Proposition A.1 to show that this fraction is single valued under analytic continuation along the cycles of \( R \). It follows from (2-11) that
\[
\lambda^2(z) - \lambda^{-2}(z) = \frac{\prod_{j=0}^{g} (z - \alpha_j) - \prod_{j=0}^{g} (z - \bar{\alpha}_j)}{R(z)},
\]
(4-13)

where \( R(z) = \sqrt{\prod_{j=0}^{g} (z - \alpha_j)(z - \bar{\alpha}_j)} \), \( R(\infty) = 1 \), and the \( g \) zeroes (in \( C \)) of the polynomial in the numerator of (4-13) coincide, by construction (see Section 2), with \( 2g \) zeroes (on \( R \)) of the denominator in the integrand in (4-12). Thus, the integrand of (4-12) becomes
\[
-\frac{2i \sum_{j=0}^{g} b_j \Theta(\Omega)\Theta(2u_{\infty} + \Omega + \bar{\Theta})}{\Theta(2u_{\infty})\Theta(\Omega)} \frac{P(z)}{R(z)},
\]
(4-14)

where \( P(z) \) is the monic polynomial of degree \( g \) whose roots (counted on the both sheets of \( R \)) coincide with the zero divisor of \( \Theta(u(z) + \Omega + u_{\infty})\Theta(u(z) - \Omega - u_{\infty}) \). Substituting (4-14) into (4-12) and taking into account (1-4) yields
\[
\partial_j \ln f(\Omega) = 8 \int_{\gamma_j} \frac{P(z)}{R(z)} dz, \quad \text{and, so} \quad \partial_j \ln \bar{f}(\Omega) = -8 \int_{\gamma_j} \frac{\bar{P}(z)}{R(z)} dz.
\]
(4-15)

Therefore, we obtain
\[
\partial_j |f(\Omega)| = 4 |f(\Omega)| \int_{\gamma_j} \frac{Q(z)}{R(z)} dz,
\]
(4-16)

where \( Q(z) = P(z) - \bar{P}(z) \) is a polynomial of degree \( g - 1 \). Thus, \( \nabla |f(\Omega)| = 0 \) implies one of the two following options: i) \( f(\Omega) = 0 \); ii) all the \( A \)–periods of the holomorphic differential \( \frac{Q(z)}{R(z)} dz \) in (4-16) are zero. The latter option would imply that \( Q(z) \equiv 0 \), that is, the polynomial \( \frac{P(z)}{R(z)} \) has real coefficients. It is proved in Lemma B.1, Appendix B, that if \( P(z) \) is the real polynomial satisfying (4-15), then \( \Omega \in \frac{1}{2} \mathbb{Z}^g \).

The proof is completed.

**Corollary 4.5.** The maximum value
\[
\max_{\Omega \in T^g} |f(\Omega)| = 1
\]
(4-17)
is attained at \( \Omega = 0 \), where \( f(\Omega) = 1 \).

**Proof.** According to Theorem 4.4, the local maxima of \( |f(\Omega)| \) can only be attained at some half-integer point \( \hbar \in \frac{1}{2} \mathbb{Z}^g \). Then the statement follows from Lemma 3.1.
5 Minimum of $|f|$

In Section 3 we considered transformations of the RHP (2.1) related to the change of orientation of the branchcuts. Consider now the transformation that interchanges the enumeration of the branchcuts $\gamma_0$ and $\gamma_j$ for some $j = 1, \ldots, g$ whilst the rest of the branchcuts are unchanged. Let $\tilde{\gamma}_k$ be the new enumeration of the branchcuts. Then $\tilde{\gamma}_0 = \gamma_j$, $\tilde{\gamma}_j = \gamma_0$ and $\tilde{\gamma}_k = \gamma_k$ when $k \neq 0, j$. The requirement that the jump matrix on $\tilde{\gamma}_0$ must be $i\sigma_2$ is achieved by transforming the RHP (2.1) for $Y(z; \Omega)$ to the RHP for

$$\tilde{Y}(z; \tilde{\Omega}) = e^{-i\pi\sigma_3}Y(z; \Omega)e^{i\pi\sigma_3},$$

(5-1)

with jump contours $\tilde{\gamma}_k$, $k = 0, \ldots, g$, where the jump matrix on the contour $\tilde{\gamma}_k$ is $i\sigma_2 e^{-2\pi i\tilde{\Omega}_k}\sigma_3$ with $\tilde{\Omega}_0 = 0$, $\tilde{\Omega}_j = -\Omega_j$ and $\tilde{\Omega}_k = \Omega_k - \Omega_j$ for all $k \neq 0, j$. If $\tilde{\Omega}$ denotes the vector of $\tilde{\Omega}_k$, $k = 1, \ldots, g$, then

$$(Y_{1,2}(\Omega)) = (\tilde{Y}_{1,2}(\tilde{\Omega})) e^{-2\pi i\sigma_3},$$

(5-2)

so that, according to (2.9), (local) maxima and minima of $|f|$ do not change if we change the enumeration of the branchcuts $\gamma_j$ (but their locations on $\mathbb{T}^g$ do). Therefore, without loss of generality, we can assume that $\gamma_0$ denotes the largest branchcut, that is, $b_0 \geq b_k$, $k = 1, \ldots, g$. For the rest of this section, we fix the enumeration (upward) of branchcuts $\gamma_j$.

The Riemann Theta function $\Theta(\Omega; \tau)$ is analytic in $\Omega$ and in $\tau$, that is, it depends smoothly on the branchpoints $\alpha_j, \bar{\alpha}_j$, $j = 0, 1, \ldots, g$, provided they are distinct. Let us scale with $\xi \in (0, 1]$ all the branchcuts $\gamma_j$ except $\gamma_0$ by: $\alpha_j(\xi) = a_j + i\xi b_j$, $j = 1, \ldots, g$, whereas $\alpha_0$ stays constant.

Because of the normalization $\int_{A_k} \omega_j = \delta_{k,j}$ (Kronecker delta) and $w_j = \frac{p_j(z)}{R(z)}dz$ with polynomials $p_j$ of degree not exceeding $g - 1$, in the limit $\xi \to 0$ we obtain

$$p_j(a_k) = -\frac{\sqrt{(a_k - a_0)^2 + b_0^2 \prod_{m \neq k, m > 0}(a_k - \alpha_m)}}{2\pi i} \delta_{j,k}.$$  

(5-3)

Then straightforward calculations yield (see also [19], Proposition 4.3)

$$\int_{B_k} \omega = \ln \xi \frac{\sqrt{(a_k - a_0)^2 + b_0^2 \prod_{m \neq k, m > 0}(a_k - \alpha_m)}}{2\pi i} e_k + O(1) = \tau_{0,k} + O(1),$$

(5-4)

where $e_k$ are vectors of the standard basis. Then the matrix

$$\tau(\xi) = \text{diag} (\tau_{0,1}, \ldots, \tau_{0,g}) \ln \xi + O(1).$$

(5-5)

Thus, the imaginary part of the leading order term of $\tau(\xi)$ is of order $O(|\ln \xi|)$ and it is diagonal and positive definite. Therefore

$$\lim_{\xi \to 0^+} \Theta(\xi; \tau(\xi)) = 1$$

(5-6)

uniformly in $\xi \in \mathbb{J}_\tau(\xi)$. Thus, $\lim_{\xi \to 0^+} f(\Omega; \tau(\xi)) = 1$. We can now prove the remaining item 3 from the Main Theorem 1.2.
Proof of Theorem 1.2, item 3. Without any loss of generality, we can assume that the longest branchcut is $\gamma_0$, that is, $m = 0$. As it was shown above, $|f(\Omega, \tau(\xi))| > 0$ for all sufficiently small $\xi > 0$. Then, according to Theorem 4.4 and Lemma 3.1, the minimum of $|f|$ is attained at $h_1 = \frac{1}{2}(1, 1, \ldots, 1)^t$ and is given by (1-7) with $m = 0$ for these values of $\xi$. The function $\phi(\xi) = \min_{\Omega \in \mathcal{T}_g} |f(\Omega, \tau(\xi))|$ is a continuous function of $\xi$. Let $\xi_0 > 0$ be the smallest zero of $\phi(\xi)$. Then $\min_{\Omega \in \mathcal{T}_g} |f(\Omega, \tau(\xi_0))| = 0$ must be attained at $h_1$. Thus, equation (1-7) for the minimum of $|f|$ is valid for all $\xi \leq \xi_0$. Therefore, $\xi \in (0, \xi_0)$ implies $b_0 > \sum_{k=1}^{g} b_k$, which completes the proof.

Remark 5.1. In the case of $g = 2$ it was proven in [21] that the condition $b_m > \sum_{k=0, k\neq m}^{g} b_k$ for some $m \in \{0, 1, \ldots, g\}$ is the necessary and sufficient condition for $\min_{\Omega \in \mathcal{T}_g} |f(\Omega)| > 0$. This result, in all likelihood, is true for any $g \in \mathbb{N}$.

6 Defocusing NLS and some other integrable equations

The RHP (2-1) with non intersecting real branchcuts $\gamma_j$ (with natural orientation) defines finite gap solutions to the defocusing NLS

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0,$$

(6-1)
given by (see, for example, [22])

$$\psi_{1,2}(x, t) = 2i(Y_1)_{1,2}(\Omega), \quad \text{where} \quad \Omega = Wt + Vx + \Omega^0$$

(6-2)

and $(Y_1)_{1,2}$ denotes the $(1, 2)$ entry of the matrix $Y_1$ in (2-1) and $\Omega^0$ is the vector of initial phases. Since Corollary 4.5 and Lemma 4.1 are valid for the real branchcuts, so are Lemmas 4.2, B.1 and Theorem 4.4. Thus, Theorem (1.2) can be extended to the case of dNLS. In particular, the following statement is true for finite-gap solutions of the dNLS.

**Theorem 6.1.** Let $\psi_{1,2}(x, t)$ be a finite gap solution for the defocusing NLS (6-1) defined by the RHP (2-1) with real branchcuts $\gamma_j = [\beta_j, \alpha_j], j = 0, \ldots, g$, where $-\infty < \beta_0 < \alpha_0 < \beta_1 < \alpha_1 < \ldots < \beta_g < \alpha_g < \infty$ and arbitrary initial phases $\Omega^0 \in \mathbb{R}^g$. Then: i)

$$|\psi_{1,2}(x, t)| \leq |\psi_0(0, 0)| = \frac{1}{2} \sum_{j=0}^{g} (\alpha_j - \beta_j);$$

(6-3)

ii) if for some $m = 0, \ldots, g$ we have $\alpha_m - \beta_m \geq \sum_{k=0, k\neq m}^{g} (\alpha_k - \beta_k)$, then

$$|\psi_{1,2}(x, t)| \geq \frac{1}{2} \left[ \alpha_m - \beta_m - \sum_{j=0, j\neq m}^{g} (\alpha_j - \beta_j) \right].$$

(6-4)
It is remarkable that the inequality, similar to (6-3), is also valid for finite-gap solutions to the KdV. Indeed, a finite-gap KdV solution \( u(x,t) \), associated with the Riemann surface \( R \) with branchcuts
\[
\gamma_j = [\beta_j, \alpha_j], \ j = 0, \ldots, g,
\]
where \( -\infty < \beta_0 < \alpha_0 < \beta_1 < \alpha_1 < \ldots < \beta_g < \alpha_g = \infty \), is given by ([13])
\[
\begin{align*}
\quad u(x,t) &= \sum_{j=0}^{g-1} (\alpha_j + \beta_j) + \beta_g - 2 \sum_{j=1}^{g} \lambda_j(x,t),
\end{align*}
\]
where the Dirichlet eigenvalues \( \lambda_j(x,t) \in [\alpha_{j-1}, \beta_j] \). Then, the deviation of \( u(x,t) \) from \( \beta_0 \) is bounded by
\[
|u(x,t) - \beta_0| \leq (\beta_g - \beta_0) - \sum_{j=0}^{g-1} (\alpha_j - \beta_j) = \sum_{j=0}^{g-1} (\beta_{j+1} - \alpha_j).
\]

\[ (6-6) \]

\section{A Some basic facts about Theta functions}

The \textbf{Riemann Theta function} associated to a symmetric matrix \( \tau \) with strictly positive imaginary part is the function of the vector argument \( \vec{z} \in \mathbb{C}^g \) given by
\[
\begin{align*}
\Theta(\vec{z}; \tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \exp \left( i\pi \vec{n} \cdot \tau \cdot \vec{n} + 2i\pi \vec{n} \cdot \vec{z} \right).
\end{align*}
\]
\[ (A.1) \]

Often the dependence on \( \tau \) is omitted from the notation.

\textbf{Proposition A.1.} For any \( \lambda, \mu \in \mathbb{Z}^g \), the Theta function has the following properties:
\[
\begin{align*}
\Theta(z; \tau) &= \Theta(-z; \tau); \quad (A.2) \\
\Theta(z + \mu + \tau \lambda; \tau) &= \exp \left( -2i\pi \lambda \cdot z - i\pi \lambda \cdot \tau \lambda \right) \Theta(z; \tau). \quad (A.3)
\end{align*}
\]

We shall denote by \( \Lambda_\tau = \mathbb{Z}^g + \tau \mathbb{Z}^g \subset \mathbb{C}^g \) the \textit{lattice of periods}. The \textbf{Jacobian} \( \mathbb{J}_\tau \) is the quotient \( \mathbb{J}_\tau = \mathbb{C}^g \mod \Lambda_\tau \). It is a compact torus of real dimension \( 2g \) on account that \( \Im \tau \) is a positive definite matrix. Let \( R \) be a Riemann surface with the vector of normalized holomorphic differentials \( \omega \).

\textbf{Theorem A.2. ([11])} The matrix \( \tau \) of \textbf{B} periods of \( \omega \), defined by \( (\tau)_k, j = \int_{\mathbb{J}_\tau} \omega_k \) is symmetric and its imaginary part is strictly positive definite.

The Abel map \( u(z): R \to \mathbb{J}_\tau \) with the base-point \( p_0 \) is defined by
\[
\begin{align*}
u(z) &= \int_{p_0}^z \omega(\zeta).
\end{align*}
\]
\[ (A.4) \]

We choose \( p_0 = \beta_0 \) to be the beginning point of the branchcut \( \gamma_0 \) (Then, in the case of vertical branchcuts) \( p_0 = \bar{\alpha}_0 \).

The general definition of the \textit{vector of Riemann constants} \( K \) can be found in [11]. For the case of a hyperelliptic Riemann surface the following proposition can be considered as the definition of \( K \).
Proposition A.3 ([11], p. 324). Let $\beta_0$ be a base-point of the Abel map $u(z)$ on the hyperelliptic Riemann surface $R$ of $R(z) = \sqrt{\prod_{j=0}^{g}(z - \beta_j)(z - \alpha_j)}$. Then the vector of Riemann constants is

$$K = \sum_{j=1}^{g} u(\beta_j).$$

(A.5)

Theorem A.4 ([11], p. 308). Let $f \in \mathbb{C}^g$ be arbitrary, and denote by $u(p)$ the Abel map (extended to the whole Riemann surface). The (multi-valued) function $\Theta(u(z) - f)$ on the Riemann surface either vanishes identically or vanishes at $g$ points $p_1, \ldots, p_g$ (counted with multiplicity). In the latter case we have

$$f = \sum_{j=1}^{g} u(p_j) + K.$$  

(A.6)

An immediate consequence of Theorem A.4 is the following statement.

Corollary A.5. The function $\Theta$ vanishes at $e \in \mathbb{J}_\tau$ if and only if there exist $g - 1$ points $p_1, \ldots, p_{g-1}$ on the Riemann surface such that

$$e = \sum_{j=1}^{g-1} u(p_j) + K.$$  

(A.7)

Definition A.6. The Theta divisor is the locus $e \in \mathbb{J}_\tau$ such that $\Theta(e) = 0$. It will be denoted by the symbol $(\Theta)$.

Remark A.7. A divisor $D = p_1 + \ldots + p_k$ $(k \leq g)$ on a hyperelliptic Riemann surface $R$ of the genus $g$ is special if and only if at least one pair of points $p_j, p_m$ is of the form $(z, \pm R)$ (i.e. the points are on the two sheets and with the same $z$ value).

B Technical results

Let $D_\Omega \subset R$ denote the divisor of zeroes of $\Theta(u(z) + \Omega + u_\infty)$. The points of $D_\Omega$ have a similar meaning to the Dirichlet eigenvalues $\lambda_j$ in the equation (6-5) for KdV. In this section we will prove that if $\Omega \in T^g$ is a nonzero critical point of $|f|$ then the divisor $D_\Omega$ is Schwarz symmetrical, and, as a consequence, $\Omega \in \frac{1}{2}\mathbb{Z}^g$. The next result of this section is Theorem B.2.

Lemma B.1. If $\partial_j \ln f(\Omega)$ is given by (4-15), where $f(\Omega) \neq 0$ and the polynomial $P(z)$ has real coefficients, then $\Omega \in \frac{1}{2}\mathbb{Z}^g$.

Proof. By construction, zeroes of $P(z)$ coincide with the zeroes of the product

$$\Theta(u(z) + \Omega + u_\infty)\Theta(-u(z) + \Omega + u_\infty),$$

(B.1)
which is clearly invariant under the hyperelliptic involution \( \widehat{(z,R)} = (z,-R) \). Thus, zeroes of the product (B.1) are Schwarz and involution invariant. Note that, according to Theorem B.2, both factors in (B.1) are not identically zero. Indeed, evaluating the first at \( \infty_- \) and the second at \( \infty_+ \) yields \( \Theta(\Omega) \neq 0 \).

The divisor \( \mathcal{D}_\Omega \) is of degree \( g \) and, according to Theorem A.4, is given by

\[
\mathbf{u}(\mathcal{D}_\Omega) = -\Omega - \mathbf{u}_\infty + \mathcal{K}.
\] (B.2)

Let us show that \( \mathcal{D}_\Omega \) is non-special. Indeed, if that would be the case, then, in view of Remark A.7, the divisor \( \mathcal{D}_\Omega \) would be of degree \( g-2 \). Then, according to (B.2), we would have \( \Theta(\mathbf{u}(z) + \Omega + \mathbf{u}_\infty) = \Theta(\mathbf{u}(z) - \mathbf{u}(\mathcal{D}_\Omega) + \mathcal{K}) \equiv 0 \) on \( \mathcal{R} \), which is a consequence of Corollary A.5. The obtained contradiction with Theorem B.2 proves that \( \mathcal{D}_\Omega \) is nonspecial.

The divisor of zeroes of the second factor is simply \( \widehat{\mathcal{D}}_\Omega \) obtained by the reflection of all points to the other sheet. Equations (B.2) and (2-8) imply

\[
\mathbf{u}(\mathcal{D}_\Omega - \mathcal{D}_0) = -\Omega \quad \iff \quad \mathbf{u}(\mathcal{D}_\Omega) - \mathbf{u}(\mathcal{D}_0) = -\Omega,
\] (B.3)

so that we have expressed \( \Omega \) as the Abel map of the divisor \(-\mathcal{D}_\Omega + \mathcal{D}_0\) of degree zero. On the other hand, since \( P(z) \) is a real polynomial, we have

\[
\overline{\mathcal{D}_\Omega} + \overline{\mathcal{D}_\Omega} = \mathcal{D}_\Omega + \widehat{\mathcal{D}}_\Omega.
\] (B.4)

Thus, according to Lemma 2.3 and (4-4),

\[
\mathbf{u}(\mathcal{D}_\Omega - \mathcal{D}_0) = -\Omega \quad \Rightarrow \quad \mathbf{u}(\overline{\mathcal{D}_\Omega} - \overline{\mathcal{D}_0}) = -\Omega \quad \Rightarrow \quad -\mathbf{u}(\overline{\mathcal{D}_\Omega} - \overline{\mathcal{D}_0}) = -\Omega \quad \Rightarrow \quad \mathbf{u}(\overline{\mathcal{D}_\Omega} - \overline{\mathcal{D}_0}) = \Omega. \tag{B.5}
\]

Suppose that we knew that \( \mathcal{D}_\Omega = \mathcal{D}_\Omega \): then (B.5) would imply \( \Omega = -\Omega \) as an equation in \( \mathcal{J}_\tau \), and hence \( 2\Omega = 0 \), or, equivalently, \( \Omega \in \mathbb{Z}^g \). Thus, it only remains to prove that \( \mathcal{D}_\Omega = \overline{\mathcal{D}_\Omega} \).

Let \( \mathcal{D}_\Omega = \mathcal{D}_s + \mathcal{D}_n \), where \( \mathcal{D}_s \) denotes the Schwartz symmetric part of \( \mathcal{D}_\Omega \) and \( \mathcal{D}_n \) is the remaining part; obviously, \( \mathcal{D}_n \cap \overline{\mathcal{D}_n} = \emptyset \). Then, by (B.5), we obtain

\[
2\mathbf{u}(\mathcal{D}_s) + \mathbf{u}(\mathcal{D}_n) + \mathbf{u}(\overline{\mathcal{D}_n}) = \mathbf{u}(\mathcal{D}_\Omega) + \mathbf{u}(\overline{\mathcal{D}_\Omega}) = 2\mathbf{u}(\mathcal{D}_0). \tag{B.6}
\]

We aim at showing that \( \mathcal{D}_n = 0 \). Equation

\[
\mathcal{D}_n - \overline{\mathcal{D}_n} = \mathcal{D}_\Omega - \overline{\mathcal{D}_\Omega} = \overline{\mathcal{D}_\Omega - \mathcal{D}_\Omega} = \overline{\mathcal{D}_n - \widehat{\mathcal{D}}_n}
\] (B.7)

follows from (B.4). Since \( \mathcal{D}_\Omega \) is non-special, equation (B.7) implies \( \mathcal{D}_n = \overline{\mathcal{D}_n} \). We have thus established that \( \mathcal{D}_n + \overline{\mathcal{D}_n} = \mathcal{D}_n + \widehat{\mathcal{D}}_n \) and hence

\[
\mathbf{u}(\mathcal{D}_n) + \mathbf{u}(\overline{\mathcal{D}_n}) = 0. \tag{B.8}
\]

Inserting (B.8) into (B.6) we obtain

\[
2\mathbf{u}(\mathcal{D}_s) = 2\mathbf{u}(\mathcal{D}_0). \tag{B.9}
\]
and hence \( u(D_s) = u(D_0) + \text{half period} \).

We also observe that \( \deg D_s = g - 2k \) for some \( k \geq 1 \) because \( D_n \) contains an even number of points. Indeed, if it were odd, then at least one \( p \in D_n \) must be such that \( p + \bar{p} \) is Schwarz symmetric, which can only happen if \( p \) is on the real axis, against the hypothesis \( D_n \cap \overline{D}_n = \emptyset \). Now recall that \( \lambda(z) \) (2.3) is such that (the bracket indicating the divisor of zeroes)

\[
(\lambda^2(z) - 1) = D_0 + \infty_+ - B,
\]

where \( B \) is the divisor consisting of the \( g + 1 \) branchpoints \( \tilde{a}_j, j = 0, 1, \ldots, g \) in the lower half-plane. By Abel’s theorem, \( u(D_0) = -u_\infty + u(B) \). Plugging this into (B.9) yields

\[ u(2D_s) = u(-2\infty_+ + 2B). \]  

(B.11)

Now note that the polynomial \( \prod_{j=0}^g (z - \bar{a}_j) \) has double zeroes at the branchpoints \( B \) and a pole of order \( g + 1 \) at \( \infty_+ \) and \( \infty_- \). Thus, again by Abel’s theorem, \( u(2B) = (g + 1)u(\infty_+ + \infty_-) \) and (B.11) becomes

\[ u(2D_s) = u((g - 1)\infty_+ + (g + 1)\infty_-) = u((g - 2k - 1)\infty_+ + (g - 2k + 1)\infty_-). \]  

(B.12)

Equation (B.12) is an identity between the Abel maps of two divisors of the same total degree (which is \( 2g - 4k \)). Hence, Abel’s theorem guarantees the existence of a meromorphic function with poles only at \( \infty_\pm \) of the indicated degrees and double zeroes at the points of \( D_s \). Such a function is necessarily of the form

\[ F(z) = Q_0(z)R(z) + P_0(z) \]  

for some polynomials \( P_0, Q_0 \); since the zeroes are Schwarz symmetric, \( P_0, Q_0 \) should be real polynomials. However, the maximal degree of poles at infinity is \( g + 1 - 2k < g + 1 \) and since \( R \) has a pole of degree \( g + 1 \) at both infinities, we are forced to conclude \( Q_0 \equiv 0 \). But then \( 2D_s \) would be the zeros of a real polynomial \( P_0(z) \) and hence be invariant under the involution \( \bar{\cdot} \). This is impossible because \( D_\Omega \) (and thus also \( D_s \)) was already established to be non-special. The proof is complete. \( \square \)

**Theorem B.2.** \( \Theta(\Omega) > 0 \) for any \( \Omega \in \mathbb{R}^g \).

The proof can be extracted from [12], Ch. VI but it requires a considerable effort for the un-initiated reader (and for the present authors). For this reason we include here a complete proof that requires slightly less advanced knowledge of properties of Theta functions and divisors on Riemann surfaces.

**Remark B.3.** The reader that wishes to read directly loc. cit. may benefit from the following reading tips: Fay normalizes the matrix of periods as \( 2i\pi \delta_{jk} \) on the \( a \)-cycles and thus the normalized matrix of \( b \)-periods has negative definite real part. Second, his choice of cycles is different; it would correspond to choosing \( a \) and \( b \) cycles entirely contained in the two upper/lower half planes. In his notation, our situation corresponds to a number of real ovals \( n = 1 \) for even genus, and \( n = 2 \) for odd genus. In either cases the real oval(s) is(are) the real axis on both sheets.
We shall give only the proof for even genus, because the case of odd genus requires slightly more discussion, but can be found in full generality in [12].

Proof of Theorem B.2. Using the symmetry $\omega_j(z) = -\omega_j(\overline{z})$, we denote, with Fay, by $\phi$ the induced anti-involution on $J = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. If $\mathcal{A}, \mathcal{B}$ are positive divisors of the same degree then

$$\phi(\mathcal{u}(\mathcal{A} - \mathcal{B})) := \mathcal{u}(\mathcal{A}) - \mathcal{u}(\mathcal{B}) = -\mathcal{u}(\mathcal{A}) + \mathcal{u}(\mathcal{B})$$  \hspace{1cm} (B.14)

and hence (cf. formula above (126) in [12])

$$\phi(z) = -z, \quad z \in J.$$  \hspace{1cm} (B.15)

The situation which is relevant for us is that of Proposition 6.8 and Corollary 6.13 of [12]; the latter states directly $\Theta(\Omega) > 0$. In the interest of being self-contained we are going to prove simply $\Theta(\Omega) \neq 0$.

A deformation argument, similar to the one used in [12] (see also the proof of Main Theorem in Section 5 and (5-6)) can then be used to show that $\Theta(\Omega) > 0$.

Fix $\Omega \in \mathbb{R}^g$ and a point $a = (z_0, R(z_0))$ with $z_0$ in the upper half plane. By Jacobi’s inversion theorem there is a positive divisor $\mathcal{D} = \sum_{j=1}^g p_j$ of degree $g$ such that $\mathcal{u}(\mathcal{D} - a) = \Omega - \mathcal{K}$. Then, using that $\mathcal{K} = -\mathcal{K} + \frac{g-1}{2}\mathbf{1}$ we also obtain that $\mathcal{u}(\mathcal{D} - \mathcal{K}) = -\Omega - \mathcal{K}$. Recall that $2\mathcal{K}$ is the image of the class of the canonical divisor (in hyperelliptic case it is a period, but in general it is not) [11]. Therefore there is a (meromorphic) differential $\eta$ with at most two simple poles at $a, \overline{a}$ and zeroes at $\mathcal{D}, \overline{\mathcal{D}}$.

We want to show that this differential is unique; this is the same as saying that $\mathcal{D} - a$ is non-special. Note that since the zero divisor is $\mathcal{D} + \overline{\mathcal{D}}$, this differential has only zeroes of even multiplicities on $\mathbb{R}$ (the boundary of the bordered Riemann surface, denoted $\partial \mathbb{R}$ in Fay). It could happen that one of the zeroes in $\mathcal{D}$ cancels the pole $a$; we need to show that this does not happen. To this end, since the residues are opposite, we can assume that the residue is normalized to be imaginary (which we can always accomplish by multiplication since the two residues are opposite to each other), then $\eta$ has a definite sign on $\mathbb{R}$, which we can assume without loss of generality to be $\geq 0$. Then

$$0 \leq \int_{\Gamma_0} \eta = 2i\pi \text{ res } \eta(z) = -2i\pi \text{ res } \eta(z).$$  \hspace{1cm} (B.16)

Thus the residue being zero (i.e. $\eta$ being holomorphic) forces $\eta$ to be identically zero (because it would have to vanish identically on the real axis given the fact that it has a definite sign on $\mathbb{R}$). We have concluded that:

the divisor $\mathcal{D} - a$ necessarily is not positive (i.e. the point $-a$ is not canceled by a point in $\mathcal{D}$).

We now show that both $\mathcal{D}, \overline{\mathcal{D}}$ are non-special. Suppose that $\mathcal{D}$ is special; then Riemann–Roch theorem implies immediately that there is a non-constant meromorphic function $F$ with $(F) \geq -\mathcal{D}$; adding a constant, we can assume $(F) \geq -\mathcal{D} + a$ (i.e. the function has a zero at $z = a$). The function $F^*(z) = \overline{F(\overline{z})}$
has similarly \( (F^*) \geq -\mathcal{D} + \pi \). Then \( \omega(z) := F(z)F^*(z)\eta(z) \) must be a holomorphic differential which is

- Schwartz-symmetric;
- has zeroes of even multiplicities on \( \mathbb{R} \).

Therefore its sign on \( \Gamma_0 \) is definite and we can assume is nonnegative; but then Cauchy’s theorem (note that \( \Gamma_0 \) splits the Riemann surface in two disjoint halves) implies \( \int_{\Gamma_0} \omega = 0 \) which in turn implies that \( \omega \) is identically zero. This means that the assumption of having a non-constant \( (F) \geq -\mathcal{D} + a \) (i.e. \( \mathcal{D} \) special) has lead to a contradiction.

Now that we have established that the positive divisor \( \mathcal{D} \) of degree \( g \) is non-special, we know that \( \Theta(u(z) - u(\mathcal{D}) - \mathcal{K}) \) is not identically zero, and similarly also \( \Theta(u(z) - u(\mathcal{D}) - \mathcal{K}) \). We construct \( \eta \) directly in the following way choose \( g - 1 \) branchpoints with indices in \( J = \{ j_1, \ldots, j_{g-1} \} \) and define

\[
H(z) = \sqrt{\prod_{j \in J} (z - \mu_j)} \prod_{j \notin J} (z - \nu_j). \quad \text{Then} \quad h(z) \sim \frac{1}{z} . \quad \text{Here} \quad \mu_j \text{'s denote generically the } 2g+2 \text{ branch-points } \{ \alpha_j, \overline{\alpha}_j \}_{j=0}^g \quad \text{(cf. pag. 13 of [12])}. \quad \text{Let } W_0 = \frac{1}{2} (\vec{m} + \tau \vec{n}) \text{ with } \vec{n}, \vec{m} \in \mathbb{Z}^g \text{ be the Abel map of these points; } \sum_{j \in J} u(p_j) + \mathcal{K} = W_0. \quad \text{It is known ([12] pag. 13-14, or a direct but tedious computation) that it is an odd half period, namely } \vec{n} \cdot \vec{m} \in 2\mathbb{Z} + 1. \quad \text{Consider the function (called “theta function with characteristics } \vec{n}, \vec{m})
\]

\[
\Theta \left[ \frac{\vec{n}}{\vec{m}} \right] (\vec{z}) := \exp \left[ \frac{i\pi}{4} \vec{n}^2 \tau \vec{n} - i\pi \vec{n} \cdot \vec{z} + \frac{i\pi}{2} \vec{n} \cdot \vec{m} \right] \Theta (\vec{z} - W_0) . \tag{B.17}
\]

Then one verifies by the periodicities of \( \Theta \) that this is an odd function \( \Theta \left[ \frac{\vec{n}}{\vec{m}} \right] (-\vec{z}) = e^{i\pi \vec{n} \cdot \vec{m}} \Theta \left[ \frac{\vec{n}}{\vec{m}} \right] (\vec{z}). \) Thus \( \Theta \left[ \frac{\vec{n}}{\vec{m}} \right] (0) = 0 \) vanishes at \( z = 0 \), namely \( \Theta(-W_0) = 0 = \Theta(W_0) \). It is also known that the gradient of \( \Theta \left[ \frac{\vec{n}}{\vec{m}} \right] (\vec{z}) \) at \( \vec{z} = 0 \) is not zero. Then one can check directly that the following differential has simple poles at \( a, \pi \) and zeroes at \( \mathcal{D}, -\mathcal{D}: \)

\[
\eta(z) = e^{i\theta} \frac{\Theta \left( \int_a^z \omega + \Omega \right) \Theta \left( \int_{\pi}^z \omega - \Omega \right)}{\Theta \left( \int_a^z \omega - W_0 \right) \Theta \left( \int_{\pi}^z \omega + W_0 \right)} \sqrt{\prod_{j \in J} (z - \mu_j)} \prod_{j \notin J} (z - \mu_j) \, dz. \tag{B.18}
\]

Here \( \theta \in \mathbb{R} \) is a constant chosen so that the residue at \( z = a \) is imaginary (and thus makes the differential \( \eta \) Schwartz symmetric, \( \eta(z) = \overline{\eta(\overline{z})} \)). The functions in the denominator \( \Theta \left( \int_a^z \omega \pm W_0 \right) \) have both simple zeroes at the \( \mu_j \), \( j \in J \). These double zeroes in the denominator are cancelled by the double zeroes of \( H(z)dz \) in the numerator and hence there are no poles other than \( a, \pi \). Computing directly the residue at \( z = a \) gives

\[
0 \neq \operatorname{res}_{z=a} \eta = e^{i\theta} \frac{\Theta (\Omega) \Theta \left( \int_a^a \omega - \Omega \right)}{\frac{1}{da} \tilde{\omega}(a) \nabla \Theta (-W_0) \Theta \left( \int_a^a \omega + W_0 \right)} \sqrt{\prod_{j \in J} (a - \mu_j)} \prod_{j \notin J} (a - \mu_j). \tag{B.19}
\]

The expression in the denominator cannot vanish (for simplicity, choose \( a \) not a branch-point); indeed the differential

\[
\tilde{\omega}(z) \nabla \Theta (-W_0) \propto H(z)dz. \tag{B.20}
\]
This is proved in ([12], p. 13, or in the appendix of [3]). Thus we conclude that

$$\forall \Omega \in \mathbb{R}^n, \forall a = (z_0, R(z_0)), \ z_0 \in \mathbb{C} \setminus \mathbb{R}, \ \Theta(\Omega) \neq 0 \neq \Theta \left( \int_a^z \bar{\omega} - \Omega \right).$$

(B.21)

The proof is complete. 

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