On the cycles of components of disconnected Julia sets *

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Abstract
For any integers $d \geq 3$ and $n \geq 1$, we construct a hyperbolic rational map of
degree $d$ such that it has $n$ cycles of the connected components of its Julia set except
single points and Jordan curves.

1 Introduction

Let $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of the Riemann sphere with $\deg f \geq 2$. Denote by $\mathcal{J}_f$
and $\mathcal{F}_f$ the Julia set and the Fatou set of $f$ respectively. Refer to [8] for their definitions
and basic properties. It is classical that $\mathcal{J}_f$ is a non-empty compact set.

Assume that $\mathcal{J}_f$ is disconnected. Let $K$ be a Julia component of $f$, i.e., a connected
component of $\mathcal{J}_f$. Then each component of $f^{-1}(K)$ is also a Julia component. Thus $K$ is either periodic if $f^p(K) = K$ for some integer $p \geq 1$, or eventually periodic if $f^k(K)$ is periodic for
some integer $k \geq 0$, or wandering if $f^n(K)$ is disjoint from $f^m(K)$ for any integers $n > m \geq 0$.

If $K$ is periodic with period $p \geq 1$, then either $\deg(f^p|_K) = 1$ and $K$ is a single point,
or $\deg(f^p|_K) > 1$ and there exists a rational map $g$ with $\deg g = \deg(f^p|_K)$ such that
$(f^p, K)$ is quasi-conformally conjugate to $(g, \mathcal{J}_g)$ in a neighborhood of $K$ (see [9]). If $K$
is wandering and $f$ is a polynomial, then $K$ is a single point (refer to [1, 7, 13]).

The situation for general rational maps is more complicated. There are examples of
rational maps whose wandering components of their Julia sets are Jordan curves [9]. In
fact, a wandering Julia component of a geometrically finite rational map is either a single
point or a Jordan curve [11].

A periodic Julia component $K$ is called simple if either $K$ is a single point or each
component of $f^{-n}(K)$ is a Jordan curve for any $n \geq 0$. It is called complex otherwise. Denote

$N(f) = \# \{\text{cycles of complex periodic Julia components of } f\}$.

Refer to [11] for the next theorem.

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Theorem A. Let $f$ be a geometrically finite rational map with disconnected Julia set. Then $N(f) < \infty$ and each wandering Julia component of $f$ is either a single point or a Jordan curve.

A natural problem is whether Theorem A holds for general rational maps. It is easy to see that $N(f) \leq \deg f - 2$ when $f$ is a polynomial. A related problem is whether $N(f)$ is bounded by a constant depending only upon $\deg f$ for rational maps. In this work, we will construct examples to show that this is not true.

Theorem 1.1. Given any integers $d \geq 3$ and $n \geq 1$. There exists a hyperbolic rational map $g$ such that $\deg g = d$ and $N(g) = n$.

The main tool in the proof of Theorem 1.1 is canonical decompositions. The idea firstly appeared in [3]. However, there are no precise statements in [3] to fit our situation. So we develop the idea in this paper (refer to §3).

To enumerate the cycles of complex Julia components, we will introduce a tree map which characterizes the dynamics on the configuration of Julia components, see §5. In our knowledge, such kind of tree maps firstly appeared in [14] by Shishikura.

For the purpose to construct rational maps with given number of cycles of complex Julia components, we develop a procedure on a tree map to create a new tree map, such that the corresponding new rational map has a new cycle of complex Julia components. Refer to §7 for the procedure.

2 Sub-hyperbolic version of Thurston Theorem

We will recall the sub-hyperbolic version of Thurston Theorem in this section. Let $F$ be a branched covering of $\mathbb{C}$ with $\deg F \geq 2$. Denote by $\Omega_F$ the set of branched points of $F$ and by

$$\mathcal{P}_F = \bigcup_{n>0} F^n(\Omega_F)$$

the postcritical set of $F$. The map $F$ is geometrically finite if the accumulation point set $\mathcal{P}_F'$ of $\mathcal{P}_F$ is finite.

A geometrically finite branched covering $F$ is a (sub-hyperbolic) semi-rational map if $F$ is holomorphic in a neighborhood of $\mathcal{P}_F'$ and each cycle in $\mathcal{P}_F'$ is either attracting or super-attracting.

Two semi-rational maps $F$ and $G$ are c-equivalent if there exist a pair of orientation preserving homeomorphisms $(\phi, \psi)$ of $\mathbb{C}$ and an open set $U \supset \mathcal{P}_F'$ such that $G \circ \psi = \phi \circ F$, $\phi$ is holomorphic in $U$, $\psi = \phi$ in $U$ and $\psi$ is isotopic to $\phi$ rel $\mathcal{P}_F \cup U$.

Let $F$ be a semi-rational map. A Jordan curve $\gamma$ in $\mathbb{C}\setminus \mathcal{P}_F$ is trivial if one component of $\mathbb{C}\setminus \gamma$ is disjoint from $\mathcal{P}_F$; or is peripheral if one component of $\mathbb{C}\setminus \gamma$ contains exactly one point of $\mathcal{P}_F$; or is essential otherwise, i.e. if each component of $\mathbb{C}\setminus \gamma$ contains at least two points of $\mathcal{P}_F$.

Convention. For the simplicity of the writing, we say that two essential curves are isotopic if they are isotopic rel the post-critical set.

A multicurve $\Gamma$ is a non-empty and finite collection of disjoint Jordan curves in $\mathbb{C}\setminus \mathcal{P}_F$, each essential and no two isotopic. It is stable if each essential curve in $F^{-1}(\gamma)$ for $\gamma \in \Gamma$ is isotopic to a curve in $\Gamma$; or pre-stable if each curve $\gamma \in \Gamma$ is isotopic to a
curve in $F^{-1}(\beta)$ for some $\beta \in \Gamma$. A multicurve is **completely stable** if it is stable and pre-stable.

The transition matrix $M(\Gamma) = (a_{\beta\gamma})$ of a multicurve $\Gamma$ is defined by the formula

$$a_{\beta\gamma} = \sum_{\delta} \frac{1}{\deg(F: \delta \rightarrow \gamma)}$$

where the sum is taken over all components $\delta$ of $F^{-1}(\gamma)$ which are isotopic to $\beta$. Let $\lambda(\Gamma) = \lambda(M(\Gamma))$ denote the spectral radius of $M(\Gamma)$. A stable multicurve $\Gamma$ is called a **Thurston obstruction** of $F$ if $\lambda(\Gamma) \geq 1$. Refer to [3, Theorem 1.1] or [6] for the next Theorem.

**Theorem B.** Let $F$ be a semi-rational map with $\mathcal{P}_F \neq \emptyset$. Then $F$ is c-equivalent to a rational map $f$ if and only if it has no Thurston obstruction. Moreover, the rational map $f$ is unique up to holomorphic conjugation.

The following lemmas will be used in this paper.

A multicurve $\Gamma$ is called **irreducible** if for each pair $(\gamma, \beta) \in \Gamma \times \Gamma$, there exists a sequence $\{\gamma = \delta_0, \ldots, \delta_n = \beta\}$ of curves in $\Gamma$ such that $F^{-1}(\delta_k)$ has a component isotopic to $\delta_{k-1}$ for $1 \leq k \leq n$. Refer to [10, Theorem B.6] for the next lemma.

**Lemma 2.1.** Let $F$ be a semi-rational map. For any multicurve $\Gamma$ with $\lambda(\Gamma) > 0$, there is an irreducible multicurve $\Gamma_0 \subset \Gamma$ such that $\lambda(\Gamma_0) = \lambda(\Gamma)$.

Refer to [3, Corollary A.2] for the next lemma.

**Lemma 2.2.** For any non-negative square matrix $M$, its leading eigenvalue satisfies

$$\lambda(M) = \inf \{\lambda : \exists v > 0 \text{ such that } Mv < \lambda v\}.$$ 

**Lemma 2.3.** Let $\Gamma_1 \subset \Gamma$ be multicurves. Then $\lambda(\Gamma_1) \leq \lambda(\Gamma)$.

**Proof.** The transition matrices of $\Gamma$ and $\Gamma_1$ satisfy the following inequality:

$$M(\Gamma) = \begin{pmatrix} M(\Gamma_1) & * \\ * & * \end{pmatrix} \geq M = \begin{pmatrix} M(\Gamma_1) & O_1 \\ O_2 & O_3 \end{pmatrix},$$

where $O_i$ are zero matrices. Thus $\lambda(\Gamma) \geq \lambda(M)$ by [3, Corollary A.3].

By Lemma 2.2, for any $\lambda > \lambda(M)$, there exists a vector $v = (v_1, v_2) > 0$ such that $Mv < \lambda v$. Thus

$$Mv = \begin{pmatrix} M(\Gamma_1) & O_1 \\ O_2 & O_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} M(\Gamma_1)v_1 \\ O \end{pmatrix} < \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix},$$

where $O$ is a zero matrix. So $M(\Gamma_1)v_1 < \lambda v_1$. Therefore $\lambda(\Gamma_1) < \lambda$ by Lemma 2.2. Since $\lambda$ is an arbitrary number with $\lambda > \lambda(M)$, we have $\lambda(\Gamma_1) \leq \lambda(M)$. Now the lemma follows.

**Lemma 2.4.** Let $\Gamma = \Gamma_1 \sqcup \Gamma_2$ be a multicurve of a semi-rational map $F$ such that for each curve $\gamma \in \Gamma_2$, $F^{-1}(\gamma)$ has no component isotopic to a curve in $\Gamma_1$. Then

$$\lambda(\Gamma) = \max\{\lambda(\Gamma_1), \lambda(\Gamma_2)\}.$$
Proof. The transition matrix of $\Gamma$ has the following block decomposition

$$M(\Gamma) = \begin{pmatrix} M(\Gamma_1) & O \\ B & M(\Gamma_2) \end{pmatrix},$$

where $O$ is a zero matrix.

For any $\lambda > \max\{\lambda(\Gamma_1), \lambda(\Gamma_2)\}$, by Lemma 2.2, there exist vectors $v_1, v_2 > 0$ such that $M(\Gamma_1)v_1 < \lambda v_1$ and $M(\Gamma_2)v_2 < \lambda v_2$. Thus there exists $\varepsilon > 0$ such that $M(\Gamma_2)v_2 + \varepsilon Bv_1 < \lambda v_2$. Now we have

$$M(\Gamma) \begin{pmatrix} \varepsilon v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \varepsilon M(\Gamma_1)v_1 \\ M(\Gamma_2)v_2 + \varepsilon Bv_1 \end{pmatrix} < \begin{pmatrix} \varepsilon \lambda v_1 \\ \lambda v_2 \end{pmatrix}.$$

Thus $\lambda(\Gamma) < \lambda$ by Lemma 2.2. So $\lambda(\Gamma) \leq \max\{\lambda(\Gamma_1), \lambda(\Gamma_2)\}$ since $\lambda$ is arbitrary.

By Lemma 2.3, we have $\lambda(\Gamma_1) \leq \lambda(\Gamma)$ and $\lambda(\Gamma_2) \leq \lambda(\Gamma)$. Combining these inequalities, we obtain $\lambda(\Gamma) = \max\{\lambda(\Gamma_1), \lambda(\Gamma_2)\}$. \qed

3 Canonical decompositions

By quasi-conformal surgery, for any sub-hyperbolic rational map $g$, there exist another sub-hyperbolic rational map $f$ and a quasi-conformal map $\phi$ of $\mathbb{C}$ such that $f \circ \phi = \phi \circ g$ holds in a neighborhood of $J_g$, and each super-attracting cycle of $f$ is contained in $P'_f$.

The main object of this work is Julia sets. So we may assume that sub-hyperbolic rational maps $f$ in our consideration satisfy the condition that each super-attracting cycle of $f$ is contained in $P'_f$. This technical assumption will simplify the statements and proofs in this work.

Let $F$ be semi-rational map. It is called generic if $P'_F \neq \emptyset$ and each super-attracting cycle of $F$ is contained in $P'_F$.

A set $E \subset \mathbb{C}$ is D-type if there exists a simply-connected domain $D \subset \mathbb{C}$ such that $E \subset D$ and $D$ contains at most one point of $P_F$; or is A-type if it is not disk type and there exists an annulus $A \subset \mathbb{C}$ such that $E \subset A$ and $A$ is disjoint from $P_F$; or is Q-type otherwise.

From the condition $F(P_F) \subset P_F$, we know that $F(E)$ is Q-type if $E$ is Q-type, and $F(E)$ is A-type or Q-type if $E$ is A-type.

A generic semi-rational map $F$ is called degenerate if for any open set $U \supset P'_F$, there exists $N > 0$ such that for $n > N$, each component of $\mathbb{C} \setminus F^{-n}(U)$ is D-type.

Proposition 3.1. A degenerate and generic semi-rational map is always c-equivalent to a rational map.

Proof. Let $F$ be a degenerate and generic semi-rational map. Let $\Gamma$ be a multicurve with $\lambda(\Gamma) > 0$. Then there is an irreducible multicurve $\Gamma_0 \subset \Gamma$ such that $\lambda(\Gamma_0) = \lambda(\Gamma)$ by Lemma 2.1. Pick an open set $U \supset P'_F$ such that it is disjoint from all the curves in $\Gamma_0$. Then there exists $N > 0$ such that for $n > N$, each component of $\mathbb{C} \setminus F^{-n}(U)$ is D-type. It follows that for each $\gamma \in \Gamma_0$, any curve in $F^{-n}(\gamma)$ is contained in a D-type set and hence is non-essential. This contradicts the condition that $\Gamma_0$ is irreducible. Thus $F$ has no Thurston obstruction and hence it is c-equivalent to a rational map by Theorem B. \qed
By a tame set $U \subset \overline{\mathbb{C}}$, we mean that $U$ is open and has only finitely many components whose closures are disjoint pairwise, and each component is bounded by finitely many Jordan curves.

**Theorem 3.2.** Let $F$ be a non-degenerate and generic semi-rational map. There exist a completely stable multicurve $\Gamma$ and a tame set $U_0 \subset \overline{\mathbb{C}}$ with $P_F' \subset U_0$ and $\partial U_0 \cap P_F = \emptyset$, such that the following conditions hold for $n \geq 0$. Denote $U_n = F^{-n}(U_0)$ and $L_n = \overline{\mathbb{C}} \setminus U_n$.

(a) $U_0$ is compactly contained in $U_1$ (denote $U_0 \subseteq U_1$).

(b) $\{F^k(U_0)\}$ converges to $P_F'$ as $k \to \infty$.

(c) Each essential curve on $\partial U_n$ is isotopic to a curve in $\Gamma$, and vice versa.

(d) Each $Q$-type component $D_{n+1}$ of $U_{n+1}$ contains exactly one $Q$-type component $D_n$ of $U_n$, and each component of $D_{n+1} \setminus D_n$ is not $Q$-type.

(e) Each $Q$-type component of $L_n$ contains exactly one $Q$-type component of $L_{n+1}$.

**Proof.** Pick Koenigs or Böttcher disks at every cycles in $P_F'$ such that their boundaries are disjoint from $P_F$. Denote their union by $V_0$. Then $V_0 \subset F^{-1}(V_0)$ and $\{F^k(V_0)\}$ converges to $P_F'$ as $k \to \infty$. Denote $V_n = F^{-n}(V_0)$ for $n \geq 1$. Then $V_{n-1} \subseteq V_n$.

Note that $\partial V_n$ is disjoint pairwise for all $n \geq 0$. There exists a multicurve $\Lambda_n$ such that each curve in $\Lambda_n$ is contained in $\bigcup_{i=0}^{n} \partial V_i$ and each essential curve on $\bigcup_{i=0}^{n} \partial V_i$ is isotopic to a curve in $\Lambda_n$. It follows that each curve in $\Lambda_n$ is isotopic to a curve in $\Lambda_{n+1}$. In particular, $\# \Lambda_n$ is increasing.

Denote by $m$ the number of components of $V_0 \cup P_F$. Then $\# \Lambda_n \leq m - 3$ for all $n \geq 0$ since each curve in $\Lambda_n$ is disjoint from $V_0 \cup P_F$. So there exists $N \geq 0$ such that $\# \Lambda_n$ is a constant for $n \geq N$. Thus there is a multicurve $\Lambda$ such that for any $n \geq 0$, each essential curve on $\partial V_n$ is isotopic to a curve in $\Lambda$.

Define a sub-multicurve $\Gamma \subset \Lambda$ by $\gamma \in \Gamma$ if for any $N > 0$, there exists a curve $\beta \subset \partial V_n$ with $n > N$ such that $\beta$ is isotopic to $\gamma$. It is well-defined since $\Gamma$ is non-empty by the condition that $F$ is non-degenerate. By the definition of $\Gamma$, there exists $n_1 \geq 0$ such that each essential curve on $\partial V_n$ with $n \geq n_1$ is isotopic to a curve in $\Gamma$.

![Figure 1. A Q-type component of $V_n$](image-url)
Let $\gamma \subset \partial V_n$ be an essential curve with $n \geq n_1$. It is isotopic to a curve in $\Gamma$. By the definition of $\Gamma$, it is also isotopic to a curve $\alpha \subset \partial V_N$ for some integer $N > n + 1$. Let $D_{n+1}$ be the component of $V_{n+1}$ that contains $\gamma$. Since $D_{n+1}$ is disjoint from $\alpha$, there exists a unique curve $\beta \subset \partial D_{n+1}$ such that $\beta$ separates $\gamma$ from $\alpha$. Thus $\beta$ is isotopic to $\gamma$ since $\gamma$ is isotopic to $\alpha$. Refer to Figure 1.

For $n \geq n_1$, let $\Gamma_n \subset \Gamma$ be the multicurve defined by $\gamma \in \Gamma_n$ if $\gamma$ is isotopic to a curve on $\partial V_n$. The above discussion shows that $\Gamma_n \subset \Gamma_{n+1}$. Thus there exists $n_2 \geq n_1$ such that $\Gamma_n = \Gamma_{n_2}$ for $n \geq n_2$. By the definition of $\Gamma$, we have $\Gamma_n = \Gamma$ for $n \geq n_2$.

Let $D_n$ be a Q-type component of $V_n$ with $n \geq n_2$. Let $D_{n+1}$ be the component of $V_{n+1}$ that contains $D_n$. Then $D_{n+1}$ is also Q-type. Let $B$ be a component of $D_{n+1} \setminus D_n$. Then $\gamma = \partial B \cap \partial D_n$ is a curve. If $\gamma$ is non-essential, then $B$ is D-type since $D_n$ is Q-type. If $\gamma$ is essential, then it is isotopic to a curve $\alpha \subset \partial D_{n+1}$ by the above discussion. Since $D_n$ is Q-type, $B$ is contained in the closed annulus bounded by $\gamma$ and $\alpha$. Thus $B$ is not Q-type. It implies that $D_{n+1}$ contains exactly one Q-type component of $V_n$. See Figure 1.

Let $s(n)$ be the number of Q-type components of $V_n$ and $r(n)$ be the number of Q-type components of $\overline{V} \setminus V_n$. Then $s(n) + r(n) = \# \Gamma + 1$ and $s(n)$ is increasing for $n \geq n_2$. So there exists $n_3 \geq n_2$ such that $s(n)$ is a constant for $n \geq n_3$. This shows that each Q-type component of $V_{n+1}$ contains exactly one Q-type component of $V_n$, and each Q-type component of $\overline{V} \setminus V_{n+1}$ contains exactly one Q-type component of $\overline{V} \setminus V_{n+1}$.

Set $U_0 = V_n$ for some $n \geq n_3$. It satisfies the conditions of the theorem. □

We will call $\Gamma$ a canonical multicurve of $F$ and $(U_0, L_0)$ a canonical decomposition of $F$, if they satisfy the conditions of Theorem 3.2.

It is easy to check that for $n \geq 1$, $(F^{-n}(U_0), F^{-n}(L_0))$ is also a canonical decomposition of $F$, and $(U_0, L_0)$ is a canonical decomposition of $F^n$. Let $U'_0$ be the union of A-type or Q-type components of $U_0$, then $(U'_0, \overline{V} \setminus U'_0)$ is also a canonical decomposition of $F$. Moreover, each A-type component of $\overline{V} \setminus U'_0$ is a closed annulus.

Canonical decompositions and canonical multicurves are uniquely determined by semirational maps in the sense of homotopy by the next proposition.

**Proposition 3.3.** Let $(U_0, L_0)$ be a canonical decomposition of $F$. Let $V_0 \in \overline{V}$ be a tame set such that $P'_F \subset V_0$ and $\partial V_0$ is disjoint from $P_F$. Suppose that $V_0 \subset F^{-1}(V_0)$ and $\{F^k(V_0)\}$ converges to $P'_F$ as $k \to \infty$. Set $V_n = F^{-n}(V_0)$ and $U_n = F^{-n}(U_0)$ for $n \geq 0$. Then there exists $N \geq 0$ such that $(V_n, \overline{V} \setminus V_n)$ is a canonical decomposition of $F$ for $n \geq N$. Moreover, the following conditions hold for $n \geq N$.

(a) Each essential curve on $\partial V_n$ is isotopic to an essential curve on $\partial U_0$, and vice versa.

(b) There exist integers $0 < i < j$ such that $U_i \subset V_n \subset U_j$. Each Q-type component of $U_i$ contains exactly one Q-type component of $V_n$ and each Q-type component of $V_n$ contains exactly one Q-type component of $U_i$.

**Proof.** Both $\{F^k(V_0)\}$ and $\{F^k(U_0)\}$ converge to $P'_F$ as $k \to \infty$. So there exist integers $p, q > 0$ such that $U_0 \subset V_p \subset U_{p+q}$. Thus for $n \geq p$,

$$U_{n-p} \subset V_n \subset U_{n+q}.$$

(a) Let $D$ be an A-type or a Q-type component of $V_n$ for some $n \geq p$. Then $D$ is contained in a component $U$ of $U_{n+q}$, which is either an A-type or a Q-type. If $U$ is
A-type, then each essential curve on $\partial D$ is isotopic to a curve on $\partial U$. Thus it is also isotopic to a curve on $\partial U_0$ since $U_0$ is a canonical decomposition.

Suppose that $U$ is Q-type. Then there is a unique Q-type component $U'$ of $U_{n-p}$ such that $U' \subset U$. Moreover, each component of $U \setminus U'$ is not Q-type. If $D \subset U \setminus U'$, then each essential curve on $\partial D$ is isotopic to a curve on $\partial U$. Thus it is also isotopic to a curve on $\partial U_0$.

If $D \cap U' \neq \emptyset$, then $U' \subset D$. For each essential curve $\gamma$ on $\partial D$, there is a unique curve $\beta$ on $\partial U'$ such that it separates $U'$ from $\gamma$. On the other hand, $\beta$ is isotopic to a curve $\alpha$ on $\partial U_{n+q}$. Since $U'$ is Q-type, $\beta'$ must separate $U'$ from $\alpha$. This implies that both $\gamma$ and $\alpha$ are contained in the same complementary component of $\beta$. Thus $\gamma$ is isotopic to $\beta$ and hence is isotopic to a curve on $\partial U_0$.

Conversely, for each essential curve $\gamma$ on $\partial U_0$, it is isotopic to a curve $\beta \subset \partial U_{n-p}$ and a curve $\beta' \subset \partial U_{n+q}$. Let $D$ be the component of $V_n$ that contains $\beta$. Since $\beta'$ is disjoint from $D$, there is a unique curve $\alpha$ on $\partial D$ such that $\alpha$ separates $\beta$ from $\beta'$. Thus $\alpha$ is isotopic to $\beta$ and hence is isotopic to $\gamma$.

(b) Let $B$ be a Q-type component of $U_{n+q}$. Then there is a unique Q-type component $B'$ of $U_{n-p}$ such that $B' \subset B$. Let $D$ be the component of $V_n$ that contains $B'$, then $D$ is Q-type and $D \subset B$. Obviously, $D$ contains exactly one Q-type component of $U_{n-p}$. Since each component of $B \setminus \overline{B}$ is not Q-type, $D$ is the unique Q-type component of $V_n$ contained in $B$. This proves (b).

Let $D_{n+1}$ be a Q-type component of $V_{n+1}$. Then $D_{n+1}$ contains exactly one Q-type component of $U_{n-p}$. Thus $D_{n+1}$ contains exactly one Q-type component of $V_n$ since each Q-type component of $V_n$ also contains exactly one Q-type component of $U_{n-p}$. Let $B$ be a component of $D_{n+1} \setminus \overline{D_n}$. Then $B$ is contained in a component of $U_{n+q+1} \setminus \overline{U_{n-p}}$. By Theorem 3.2 (d), each component of $U_{k+1} \setminus \overline{U_k}$ is not Q-type for $k > 1$. It deduces that each component of $U_{k+1} \setminus \overline{U_k}$ is not Q-type for any integer $l \geq 1$. Therefore $B$ is not Q-type. It concludes that $(V_n, \overline{\mathbb{C}} \setminus V_n)$ is a canonical decomposition of $F$. Now the proof is complete.

Let $(\mathcal{U}, \mathcal{L})$ be a canonical decomposition of $F$. From Theorem 3.2, we may define a map $\chi_F$ on the collection of Q-type components of $\mathcal{L}$ by $\chi_F(L_i) = L_j$ if the unique Q-type component of $F^{-1}(\mathcal{L})$ in $L_i$ maps to $L_j$ by $F$. Since this collection is finite, each Q-type component of $\mathcal{L}$ is eventually periodic under $\chi_F$. We will call a Q-type component $L$ of $\mathcal{L}$ is periodic if $L$ is periodic under $\chi_F$.

Obviously, for each periodic Q-type component $L$ of $\mathcal{L}$, $F^{-1}(L)$ has exactly one Q-type component contained in periodic Q-type components of $\mathcal{L}$.

Denote by $m \geq 1$ the total number of Q-type components of $\mathcal{L}$. Then for each non-periodic Q-type component $L$ of $\mathcal{L}$, each component of $F^{-m}(L)$ is not Q-type. Otherwise, assume that $L^m$ is a Q-type component of $F^{-m}(L)$, then $F^k(L^m)$ is also Q-type for $0 \leq k \leq m$. Thus at least two of them, denoted by $L^i$ and $L^j$ with $i < j$, are contained in the same component of $\mathcal{L}$. So $L^i \subset L^j$. This implies that $L$ is periodic and hence is a contradiction. Now we have proved the next lemma.

**Lemma 3.4.** (1) Each Q-type component $L$ of $\mathcal{L}$ is eventually periodic under $\chi_F$.

(2) For each periodic Q-type component $L$ of $\mathcal{L}$, $F^{-1}(L)$ has exactly one Q-type component contained in periodic Q-type components of $\mathcal{L}$.

(3) There exists $N \geq 1$ such that for each non-periodic Q-type component $L$ of $\mathcal{L}$ and any $n \geq N$, each component of $F^{-n}(L)$ is not Q-type.
Let $L$ be a Q-type component of $\mathcal{L}$. We say that a multicurve $\Gamma$ of $F$ is \textbf{essentially} contained in $L$ if for each curve $\gamma \in \Gamma$, $\gamma \subset L$ and $\gamma$ is not isotopic to a curve on $\partial L$.

For a multicurve $\Gamma$ of $F$ and an integer $p \geq 1$, we denote by $\lambda(\Gamma, F^p)$ the leading eigenvalue of the transition matrix of $\Gamma$ under $F^p$.

**Theorem 3.5.** Let $F$ be a non-degenerate and generic semi-rational map. Let $(\mathcal{U}, \mathcal{L})$ be a canonical decomposition of $F$ and $\Gamma_F$ be a canonical multicurve of $F$. Then $F$ is c-equivalent to a rational map if and only if $\lambda(\Gamma_F) < 1$ and for each periodic Q-type component $L$ of $\mathcal{L}$ with period $p \geq 1$ and any multicurve $\Gamma$ contained essentially in $L$, we have $\lambda(\Gamma, F^p) < 1$.

**Proof.** The necessity follows directly from Theorem B. In the following, we prove the sufficiency.

Let $\Gamma_1$ be a multicurve of $F$ with $\lambda(\Gamma_1) > 0$. Then there is an irreducible multicurve $\Gamma_0 \subset \Gamma_1$ such that $\lambda(\Gamma_0) = \lambda(\Gamma_1)$ by Lemma 2.4.

There exists $n_0 \geq 0$ such that $F^{n_0}(\mathcal{U})$ is disjoint from $\Gamma_0$. Thus for each $\gamma \in \Gamma_0$, $F^{-n_0}(\gamma)$ is contained in $\mathcal{L}$. Since $\Gamma_0$ is irreducible, we may choose a multicurve $\Gamma'$ in $\mathcal{L}$ such that each curve in $\Gamma'$ is isotopic to a curve in $\Gamma_0$, and vice versa. Thus $\lambda(\Gamma') = \lambda(\Gamma_0) = \lambda(\Gamma_1)$.

Since $\Gamma_F$ is stable and $\Gamma'$ is irreducible, either each curve in $\Gamma'$ is isotopic to a curve in $\Gamma_F$, or every curve in $\Gamma'$ is not isotopic to a curve in $\Gamma_F$. In the former case, $\Gamma'$ is contained in $\Gamma_F$ in the sense of isotopy and hence $\lambda(\Gamma') < \lambda(\Gamma_F) < 1$ by Lemma 2.3.

Now we suppose that every curve in $\Gamma'$ is not isotopic to a curve in $\Gamma_F$. Then each curve in $\Gamma'$ is contained in Q-type components of $\mathcal{L}$.

If a curve $\gamma \in \Gamma'$ is contained in a non-periodic Q-type component $L$ of $\mathcal{L}$, then as $n$ is large enough, each component of $F^{-n}(\gamma)$ is contained in a D-type or an A-type component of $F^{-n}(\mathcal{L})$ by Lemma 3.2 (3). This contradicts the condition that $\Gamma'$ is irreducible. Thus each curve $\gamma \in \Gamma'$ is contained in a periodic Q-type component of $\mathcal{L}$.

Let $L_0$ be a periodic Q-type component of $\mathcal{L}$ with period $p \geq 1$ such that it contains curves of $\Gamma'$. Denote $L_i = \chi_F(L_0)$ for $1 \leq i \leq p$. Then $L_p = L_0$. Let $\Lambda_i \subset \Gamma'$ be the sub-multicurve contained in $L_i$. Since $\Gamma'$ is irreducible, we know that for $0 \leq i < p$, each curve $\beta \in \Lambda_i$ is isotopic to a curve in $F^{-1}(\gamma)$ for some $\gamma \in \Lambda_{i+1}$ by Lemma 3.3 (2). Conversely, if $\gamma \in \Gamma' \setminus \Lambda_{i+1}$, then $F^{-1}(\gamma)$ has no component isotopic to a curve in $\Lambda_i$. This implies that $\Gamma' = \bigcup_{i=0}^{p-1} \Lambda_i$ and

$$
M(\Gamma')^p = \begin{pmatrix} M_0 & \cdots & O \\
\vdots & \ddots & \vdots \\
O & \cdots & M_{p-1} \end{pmatrix},
$$

where $M_i = M(\Lambda_i, F^p)$. By the condition of the theorem, $\lambda(M_i) < 1$ for $0 \leq i < p$. Thus by Lemma 2.4 we have

$$
\lambda(\Gamma')^p = \max\{\lambda(M_0), \cdots, \lambda(M_{p-1})\} < 1.
$$

Therefore $F$ is c-equivalent to a rational map by Theorem B. \qed
4 Complex components of Julia sets

Let $f$ be a generic sub-hyperbolic rational map. Recall that a periodic Julia component $K$ of $f$ is simple if either $K$ is a single point or a Jordan curve disjoint from $\mathcal{P}_f$. It is a complex Julia component otherwise.

**Lemma 4.1.** Let $K$ be a Julia component which is not a single point.

1. If $K$ is wandering, then $f^n(K)$ is $A$-type as $n$ is large enough.
2. If $K$ is a periodic Jordan curve disjoint from $\mathcal{P}_f$, then $K$ is $A$-type.
3. If $K$ is a complex periodic Julia component, then $K$ is $Q$-type.

**Proof.** Denote $K_n = f^n(K)$ for $n \geq 0$. Let $V$ be the union of all the periodic Fatou domains of $f$. Since $f$ is generic, $V$ is non-empty and each component of $V$ contains points of $\mathcal{P}_f$.

1. If $K$ is wandering, then $f^n(K)$ is a Jordan curve for all $n \geq 0$ by Theorem A. Assume by contradiction that $K_n$ is D-type for all $n \geq 0$. Then there is exactly one complementary component $U_n$ of $K_n$ such that $V \subset U_n$. Denote $\tilde{K}_n = \mathbb{C} \setminus U_n$. Then $K_n \subset \tilde{K}_n$ and $\tilde{K}_n$ contains at most one point of $\mathcal{P}_f$. So $f(\tilde{K}_n) = \tilde{K}_{n+1}$. This shows that the forward orbit of $\tilde{K}$ is always disjoint from $V$. Thus the interior of $\tilde{K}$ is contained in the Fatou set. This contradicts the fact that the forward orbit of $\tilde{K}$ is disjoint from $V$.

2. Suppose that $K$ is a periodic Jordan curve disjoint from $\mathcal{P}_f$. Then $K$ is either A-type or D-type. Assume by contradiction that $K$ is D-type. Then $K_n$ is also D-type for all $n \geq 0$. Using the same argument as above, we could deduce again a contradiction. Thus $K$ is A-type.

3. Suppose that $K$ is a complex periodic Julia component with period $p \geq 1$. If $K$ is D-type, then there is a Jordan domain $D \supset K$ such that $D$ contains at most one point of $\mathcal{P}_f$. Denote by $A$ the unique annulus component of $D \setminus K$. We may require that $A$ is disjoint from $\mathcal{P}_f$. Denote by $\tilde{K} = D \setminus A$. Then $K \subset \tilde{K}$ and $\partial K \subset K$.

Let $D_1$ be the component of $f^{-p}(D)$ that contains $K$. Then $D_1$ is also a Jordan domain. Since $f$ is sub-hyperbolic, it is expanding in a neighborhood of $\mathcal{J}_f$ under a degenerate metric. Thus we may choose $D$ such that $D_1 \subset D$. Let $D_n$ be the component of $f^{-pn}(D)$ that contains $K$ for $n \geq 1$. Then $\tilde{K} = \cap_{n=1}^{\infty} D_n$ and hence $f^p(\tilde{K}) = \tilde{K}$.

If $\text{deg}(f^p|_{D_n}) = 1$, then $\tilde{K}$ is a single point by Schwarz Lemma. This is a contradiction.

If $\text{deg}(f^p|_{D_n}) > 1$, then $f$ has a unique critical value $a \in D$ since $D$ contains at most one point of $\mathcal{P}_f$. Moreover $a \in \tilde{K}$ since $A$ is disjoint from $\mathcal{P}_f$. Thus $f^p(a) = a$ and hence the point $a$ is also the unique critical point of $f$ in $D$. So $a$ is a super-attracting point. This contradicts the assumption that $f$ is generic.

If $K$ is A-type, then $K$ is disjoint from $\mathcal{P}_f$ and there are exactly two components of $\mathbb{C} \setminus K$ containing points of $\mathcal{P}_f$. Thus there is an annulus $A \subset \mathbb{C}$ such that $K \subset A$ and $A$ is disjoint from $\mathcal{P}_f$. As above, we may choose $A$ such that $A_1 \subset A$, where $A_1$ is the component of $f^{-p}(A)$ that contains $K$. Thus $K$ is also a Jordan curve by a folklore argument. So $K$ is a simple Julia component. This is a contradiction. In conclusion, $K$ is Q-type.

**Corollary 4.2.** The Julia set of a degenerate and generic sub-hyperbolic rational map is a Cantor set.
Proof. Let $f$ be a degenerate and generic sub-hyperbolic rational map. By definition, every Julia component of $f$ is D-type. So each Julia component of $f$ is a single point by the above lemma.

Q-type Julia components or Fatou domains are closely related to canonical decompositions. Let $f$ be a non-degenerate and generic sub-hyperbolic rational map. Let $(\mathcal{U}, \mathcal{L})$ be a canonical decomposition of $f$ and let $\Gamma_f$ be a canonical multicurve of $f$ consisting of curves on $\partial \mathcal{U}$.

Lemma 4.3. (1) Each Q-type Fatou domain contains exactly one Q-type component of $\mathcal{U}$.

(2) Each component of $\overline{\mathcal{U}\setminus \Gamma_f}$ contains exactly one Q-type Julia component or one Q-type component of $\mathcal{U}$.

Proof. (1) Let $D$ be a Q-type Fatou domain. Let $k \geq 0$ be an integer such that $f^k(D)$ is periodic. Note that $f^k(D)$ has only finitely many complementary components containing points of $\mathcal{P}_f$. Denote their union by $E$. Then there exists a domain $V \subset f^k(D)$ bounded by finitely many pairwise disjoint Jordan curves, such that $V \cap \mathcal{P}_f = f^k(D) \cap \mathcal{P}_f$ and each complementary component of $V$ contains at most one component of $E$. In other words, each component $\Omega$ of $f^k(D) \setminus V$ has at most one complementary component $E_1$ containing points of $\mathcal{P}_f$ except the complementary component $E_0$ that contains $V$. Consequently, $V$ is Q-type.

Now $B := \overline{\mathcal{U}\setminus (E_0 \cup E_1)} \supset \Omega$ is an annulus disjoint from $\mathcal{P}_f$. Thus each component of $f^{-n}(B)$ for $n \geq 1$ is also an annulus disjoint from $\mathcal{P}_f$. As a consequence, for each component $\Omega'$ of $f^{-k}(\Omega)$, $\Omega'$ has at most two complementary components containing points of $\mathcal{P}_f$ and $\partial \Omega'$ has exactly one component intersecting with $f^{-k}(V)$. Thus $f^{-k}(V)$ has a unique component $V' \subset D$ and for any domain $K \subset D \setminus \overline{V'}$, $K$ is either D-type or A-type. Since $D$ is Q-type, $V'$ is also Q-type.

There exists an integer $n > 0$ such that $V' \subset f^{-n}(\mathcal{U})$. Let $U_n$ denote the component of $f^{-n}(\mathcal{U})$ that contains $V'$. Then $U_n$ is Q-type, $U_n \subset D$ and for any domain $K \subset D \setminus \overline{U_n}$, $K$ is either D-type or A-type.

Let $U_0$ be the unique Q-type component of $\mathcal{U}$ with $U_0 \subset U_n$. Then $U_0 \subset D$ and $D \setminus U_0$ contains no other Q-type components of $\mathcal{U}$. Thus $D$ contains a unique Q-type component of $\mathcal{U}$.

(2) Let $E$ be a component of $\overline{\mathcal{U}\setminus \Gamma_f}$. Then $E$ is Q-type since each curve in $\Gamma_f$ is essential. We claim that it contains at most one Q-type Julia component or one component of $\mathcal{U}$.

Obviously, $E$ contains at most one Q-type component of $\mathcal{U}$. Assume by contradiction that $E$ contains two Q-type Julia components $K_1$ and $K_2$, then there exists an A-type or a Q-type Fatou domain $B \subset E$ separating $K_1$ from $K_2$.

Let $B_n$ be the union of components of $f^{-n}(\mathcal{U})$ contained in $B$. Then $B_n \subset B_{n+1}$ and $\bigcup_{n \geq 1} B_n = B$ since $\bigcup_{n \geq 1} f^{-n}(\mathcal{U}) = \mathcal{F}_f$. Thus there is an integer $n > 0$ such that $B_n$ separates $K_1$ from $K_2$. Therefore there exists a curve $\beta \subset \partial f^{-n}(\mathcal{U})$ which separates $K_1$ from $K_2$. The curve $\beta$ is not isotopic to any curve in $\Gamma_f$ since both $K_1$ and $K_2$ are Q-type. This is a contradiction.

If $E$ contains a Q-type component $U$ of $\mathcal{U}$ and a Q-type Julia component $K$, then there exists a curve $\beta \subset \partial U$ such that $\beta$ is isotopic to a curve in $\Gamma_f$ and separates $U$ from $K$. This is also a contradiction. Now the claim is proved.
Assume now that $E$ contains no Q-type components of $\mathcal{U}$, then its closure must contain a Q-type component $L$ of $\mathcal{L}$. By Theorem 3.2 (e), $f^{-1}(\mathcal{L})$ has exactly one Q-type component $L_1$ such that $L_1 \subset L$. Inductively, $f^{-n}(\mathcal{L})$ has exactly one Q-type component $L_n$ such that $L_n \subset L_{n-1}$ for $n \geq 2$. Set $K = \cap_{n=0}^{\infty} L_n$. Then $K$ is a Q-type continuum which is disjoint from $\bigcup_{n \geq 1} f^{-n}(\mathcal{U}) = \mathcal{F}_f$. Therefore $K \subset E$ is a Q-type Julia component.

5 The Shishikura tree map

By a tree map we mean a finite tree $T$ with the vertex set $X \subset T$ and a continuous map $\tau : T \to T$ such that $\tau^{-1}(X) \supset X$ is a finite set and $\tau$ is linear on $T \setminus \tau^{-1}(X)$ under some linear metric on $T$.

Let $F$ be a non-degenerate and generic semi-rational map. Let $\Gamma$ be a canonical multicurve of $F$. We want to construct a tree map associated with $\Gamma$ such that it characterizes the configuration and the dynamics of the components of $\Gamma$.

The Shishikura tree $(T_F, X_0)$ of $F$ is the dual tree of $\Gamma$ defined as the following. There exists a bijection $v$ from the collection of components of $\overline{\mathbb{C}} \setminus \Gamma$ to the vertex set $X_0$. For two distinct components $E_1, E_2$ of $\overline{\mathbb{C}} \setminus \Gamma$, the vertices $v(E_1)$ and $v(E_2)$ are connected by an edge of $(T_F, X_0)$ if $E_1$ and $E_2$ have a common boundary component, which is a curve in $\Gamma$. Thus there is a bijection $e$ from $\Gamma$ to the collection of edges of $T_F$.

By the definition, the bijection $v$ is order-preserving, i.e., for distinct components $E_0, E_1$ and $E_2$ of $\overline{\mathbb{C}} \setminus \Gamma$, $E_0$ separates $E_1$ from $E_2$ if and only if $v(E_0)$ separates $v(E_1)$ from $v(E_2)$ in $T_F$.

Let $\Gamma_1$ be the collection of essential curves in $F^{-1}(\Gamma)$. Each component $E_1$ of $\overline{\mathbb{C}} \setminus \Gamma_1$ is either A-type or Q-type. In the latter case, $E_1$ is isotopic to a component $E$ of $\overline{\mathbb{C}} \setminus \Gamma$, i.e. there exists a homeomorphism $\theta : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ isotopic to the identity rel $\mathcal{P}_F$ such that $\theta(E_1) = E$.

There exists a bijection $v_1$ from the collection of components of $\overline{\mathbb{C}} \setminus \Gamma_1$ into $T_F$ such that $v_1(E_1) = v(E)$ if $E_1$ is Q-type, where $E$ is the component of $\overline{\mathbb{C}} \setminus \Gamma$ isotopic to $E_1$.

(1) $v_1(E_1) = v(E)$ if $E_1$ is Q-type, where $E$ is the component of $\overline{\mathbb{C}} \setminus \Gamma$ isotopic to $E_1$.

(2) $v_1(E_1) \in e(\gamma)$ if $E_1$ is A-type, where $\gamma \in \Gamma$ is isotopic to a curve on $\partial E_1$.

(3) $v_1$ is order-preserving, i.e. for distinct components $E_0, E_1$ and $E_2$ of $\overline{\mathbb{C}} \setminus \Gamma_1$, $E_0$ separates $E_1$ from $E_2$ if and only if $v_1(E_0)$ separates $v_1(E_1)$ from $v_1(E_2)$ in $T_F$.

Denote by $X_1$ the image of $\overline{\mathbb{C}} \setminus \Gamma_1$ under $v_1$. Then $X_1 \supset X_0$ and there exists a bijection $e_1$ from $\Gamma_1$ to the collection of edges of $(T_F, X_1)$ such that $e_1(\beta) \subset e(\gamma)$ if $\beta \in \Gamma_1$ is isotopic to $\gamma \in \Gamma$.

Each component $E_1$ of $\overline{\mathbb{C}} \setminus \Gamma_1$ is cut by $F^{-1}(\Gamma)$ into finitely many domains, there is exactly one of them, denoted by $\bar{E}_1$, is not D-type. Define a map $\tau_F : X_1 \to X_0$ by $\tau_F(v_1(E_1)) = v(F(\bar{E}_1))$.

If two points $v_1(E_1)$ and $v_1(E_2)$ in $X_1$ are connected by an edge in $(T_F, X_1)$, then $E_1$ and $E_2$ have a common boundary curve $\beta \in \Gamma_1$. So do $\bar{E}_1$ and $\bar{E}_2$. Thus $F(\bar{E}_1)$ and $F(\bar{E}_2)$ have a common boundary curve $F(\beta) \in \Gamma$. So $\tau_F(v_1(E_1))$ and $\tau_F(v_1(E_2))$ are connected by an edge in $(T_F, X_0)$. Therefore, we can extend the map $\tau_F$ to a continuous map $\tau_F : T_F \to T_F$ such that $\tau_F : e_1(\beta) \to e(F(\beta))$. Moreover, we may equip a linear metric on $T_F$ such that $\tau_F$ is linear on each edge of $(T_F, X_1)$. The tree map $\tau_F : (T_F, X_1) \to (T_F, X_0)$.
will be called the **Shishikura tree map** of $F$.

The transition matrix of the multicurve $\Gamma$ can be expressed by the Shishikura tree map together with the degrees of $F$ on curves in $\Gamma_1$.

By a **weight** of a tree we mean a positive function defined on the collection of edges of the tree.

Let $T$ be a finite tree with vertex set $X \subset T$. Let $\tau : T \to T$ be a tree map and $w$ be a weight on $(T, \tau^{-1}(X))$. Denote by $\{I_1, \cdots, I_n\}$ the edges of $(T, X)$. The transition matrix $M(\tau, w) = (b_{ij})$ of $\tau$ with respect to the weight $w$ is defined by

$$b_{ij} = \sum_j \frac{1}{w(J)},$$

where the sum is taken over all the edges $J$ of $(T, \tau^{-1}(X))$ such that $J \subset I_i$ and $\tau(J) = I_j$. From Lemma 2.2 we have

**Lemma 5.1.** The leading eigenvalue satisfies $\lambda(M(\tau, w)) < 1$ if and only if $\tau$ is contracting with respect to the weight $w$, i.e., there exists a linear metric $\rho$ on $T$ such that for each edge $I$ of $(T, X)$,

$$\sum_j |\tau(J)| < |I|,$$

where the sum is taken over all the edges $J$ of $(T, \tau^{-1}(X))$ with $J \subset I$ and $|\cdot|$ denotes the length of edges under the metric $\rho$.

Let $\tau_F : (T_F, X_1) \to (T_F, X_0)$ be the Shishikura tree map of $F$. Define the weight for edges $J = e_1(\delta)$ of $(T, X_1)$ by $w_F(J) = \deg(F|_J)$. Then the transition matrix $M(\tau_F, w_F)$ is just the transition matrix of the canonical multicurve $\Gamma$.

Now we consider the dynamics of a tree map $\tau : T \to T$. Let $X_0 \subset T$ be the vertex set. Denote $X = \bigcup_{n \geq 0} \tau^{-n}(X_0)$.

**Lemma 5.2.** Suppose that $T \setminus X \neq \emptyset$. Then for every component $J$ of $T \setminus X$, there exists an integer $N \geq 0$ such that $\tau^n(J)$ is an edge of $(T, X_0)$ for all $n \geq N$.

**Proof.** Let $\mathcal{I}$ denote the collection of edges of $(T, X_0)$ which contain points of $X$. Then there exists an integer $m > 0$ such that each edge in $\mathcal{I}$ contains points of $\tau^{-m}(X_0)$. So there is a constant $0 < \lambda < 1$ such that for each edge $I \in \mathcal{I}$, the length of every interval of $I \setminus \tau^{-m}(X_0)$ is less than $\lambda|I|$.

Let $J$ be a component of $T \setminus X$. Assume by contradiction that $\tau^n(J)$ is not an edge of $(T, X_0)$ for all $n \geq 0$. Then $J$ is contained in an edge $I^0 \in \mathcal{I}$. Let $I^1$ be the component of $I^0 \setminus \tau^{-m}(X_0)$ that contains $J$. Then $|I^1| < \lambda|I^0|$.

Now $\tau^m(I^1)$ is an edge of $(T, X_0)$, which is also contained in $\mathcal{I}$ by the assumption. So $I^1$ contains points of $\tau^{-2m}(X_0)$. Let $I^2$ be the component of $I^1 \setminus \tau^{-2m}(X_0)$ that contains $J$. Then $|I^2| < \lambda|I^1|$.

Inductively, we obtain an infinite sequence of intervals $\{I^k\}$ with $I^k \supset I^{k+1} \supset J$ and $|I^k| < \lambda|I^{k-1}|$ for $k \geq 1$. Thus $|J| < \lambda^k|I^0| \to 0$ as $k \to \infty$. This is a contradiction. \qed

**Corollary 5.3.** Each point $x \in \overline{X} \setminus X$ is a double-sides accumulation point, i.e., let $I$ be the edge of $(T, X_0)$ that contains the point $x$, then both of the two components of $I \setminus \{x\}$ contain a sequence of points in $X$ which converges to the point $x$. 

Let \( x \in T \setminus X_0 \) be a periodic point with period \( p \geq 1 \). Then either \( x \in T \setminus \overline{X} \) or \( x \in \overline{X} \setminus X \). In the former case, let \( I \) be the edge of \( T \) that contains the point \( x \), then \( \tau^p(I) = I \) by the above lemma. So \(|(\tau^p)'(x)| = 1 \) on \( I \) since \( \tau \) is a linear map. In the latter case, \(|(\tau^p)'(x)| > 1 \) and hence \( x \) is a repelling periodic point.

6 Jordan curves as components of Julia sets

Let \( f \) be a non-degenerate and generic sub-hyperbolic rational map. A Julia component of \( f \) is called **buried** if it is disjoint from the closure of any Fatou domain of \( f \). Obviously, a Julia component \( K \) is buried if and only if \( f(K) \) is also buried. Denote

\[
\mathcal{A} = \{ \text{A-type buried Julia components of } f \text{ which are Jordan curves} \},
\]

\[
\mathcal{C}_0 = \{ \text{Q-type components of } J_f \text{ or } F_f \},
\]

\[
\mathcal{C}_n = \{ \text{A-type or Q-type components } K \text{ of } J_f \text{ or } F_f \text{ such that } f^n(K) \text{ is Q-type} \},
\]

\[
\mathcal{C} = \bigcup_{n \geq 0} \mathcal{C}_n.
\]

Then \( \mathcal{A} \) is disjoint from \( \mathcal{C} \) and \( f(K) \in \mathcal{A} \) for \( K \in \mathcal{A} \), \( \mathcal{C}_n \supset \mathcal{C}_{n-1} \) and \( f(K) \in \mathcal{C}_{n-1} \) for \( K \in \mathcal{C}_n \) and \( n \geq 1 \).

Let \( \tau: (T, X_1) \to (T, X_0) \) be the Shishikura tree map of \( f \). Denote \( X = \bigcup_{n \geq 0} \tau^{-n}(X_0) \).

**Theorem 6.1.** There exists an order-preserving injection \( \pi: \mathcal{C} \cup \mathcal{A} \to T \) such that \( \pi(\mathcal{C}) = X \), \( \pi(\mathcal{A}) = \overline{X} \setminus X \) and the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{A} \cup \mathcal{C} & \xrightarrow{f} & \mathcal{A} \cup \mathcal{C} \\
\pi \downarrow & & \pi \downarrow \\
X & \xrightarrow{\tau} & X
\end{array}
\]

**Proof.** By Lemma 4.3 for each \( K \in \mathcal{C}_0 \), there exists a unique component \( E \) of \( \overline{C} \setminus \Gamma \) such that \( K \subset E \) if \( K \) is a Julia component, or both \( E \) and \( K \) contain a common Q-type component \( U \) of \( \mathcal{U} \) if \( K \) is a Fatou domain. Thus we have an injection

\[
\pi: \mathcal{C}_0 \to T, \quad \pi(K) = v(E)
\]

such that \( \pi(\mathcal{C}_0) = X_0 \). It is order-preserving since \( v \) is order-preserving.

For each \( K \in \mathcal{C}_1 \), there exists a unique component \( E_1 \) of \( \overline{C} \setminus \Gamma_1 \) such that \( K \subset E_1 \) when \( K \) is a component of \( J_f \), or both \( E_1 \) and \( K \) contain a common Q-type or A-type component \( U \) of \( \mathcal{U} \). Thus \( \pi \) can be extended to

\[
\pi: \mathcal{C}_1 \to T, \quad \pi(K) = v_1(E_1)
\]

such that \( \pi(\mathcal{C}_1) = X_1 \) and \( \tau(\pi(K)) = \pi(f(K)) \) for \( K \in \mathcal{C}_1 \). The injection \( \pi \) is still order-preserving since \( v_1 \) is order-preserving.

Since \( \tau \) is injective on each edge of \((T, X_1)\), for any \( n \geq 2 \), \( \pi \) can be extended to an order-preserving injection \( \pi: \mathcal{C}_n \to T \) such that \( \pi(\mathcal{C}_n) = \tau^{-n}(X_0) \) and the following diagram commutes.

\[
\begin{array}{ccccccccccc}
\mathcal{C}_n & \xrightarrow{f} & \mathcal{C}_{n-1} & \cdots & \xrightarrow{f} & \mathcal{C}_1 & \xrightarrow{f} & \mathcal{C}_0 \\
\pi \downarrow & & \pi \downarrow & & \pi \downarrow & & \pi \downarrow \\
\tau^{-n}(X_0) & \xrightarrow{\tau} & \tau^{-n+1}(X_0) & \cdots & \tau^{-1}(X_0) & \xrightarrow{\tau} & X_0
\end{array}
\]
In conclusion, we obtain an order-preserving injection $\pi : \mathcal{C} \to T$ such that $\pi(\mathcal{C}) = X$ and the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & \mathcal{C} \\
\pi \downarrow & & \downarrow \pi \\
X & \xrightarrow{\tau} & X
\end{array}
\]

For each $K \in \mathcal{A}$, $K$ is Jordan curve as a buried Julia component. So there exists an infinite sequence of pairs of annular Fatou domains $(U_n, V_n)$ such that $K \subset A_{n+1} \subset A_n$ and $A_n$ is disjoint from all the elements in $\mathcal{C}_n$, where $A_n$ is the unique annular component of $\overline{C}(U_n \cup V_n)$. It concludes that both $\pi(U_n)$ and $\pi(V_n)$ converge to the same point $x \in X \setminus X$. Define $\pi(K) = x$. Then we extend $\pi$ from $\mathcal{C}$ to $\mathcal{C} \cup \mathcal{A}$, which is still an order-preserving injection.

For each point $x \in X \setminus X$. Let $I_0$ be the edge of $(T, X_0)$ that contains the point $x$. By Corollary 5.3, $x$ is a double-sides accumulation point. Thus there exists an infinite sequence of intervals $\{I_n = [a_n, b_n]\}$ with $x \in I_n \Subset I_{n-1}$, such that both $a_n$ and $b_n$ are contained in $X$ and $|I_n| \to 0$ as $n \to \infty$.

Let $A_n = \pi^{-1}(a_n)$ and $B_n = \pi^{-1}(b_n)$ be the corresponding components of $J_f$ or $F_f$. Let $C_n$ be the unique annular component of $\overline{C}(A_n \cup B_n)$. Then $C_{n+2} \Subset C_n$ and $C_{n+2}$ separates the two complementary components of $C_n$ for all $n \geq 0$. Thus $K = \bigcap_{n \geq 0} C_n$ is a continuum which has exactly two complementary components. Thus $\partial K$ has at most two components. On the other hand, $\partial K \subset J_f$ and is disjoint from the grand orbit of all the complex periodic Julia components. Thus each component of $\partial K$ is contained in a wandering Julia component or an eventually simple periodic Julia component. In both cases each component of $\partial K$ must be a Jordan curve.

If $\partial K$ has two components, then there exists an A-type Fatou domain $U$ which separates one of them from another. Thus $\pi(U) \Subset I_n$ for all $n \geq 0$. This is a contradiction since $|I_n| \to 0$. Thus $\partial K$ is a Jordan curve and so is $K$.

Note that $K$ is disjoint from the closure of periodic Fatou domains. Applying the above argument for $\tau^n(x)$, we obtain another Jordan curve $K_n$ as a Julia component and $K_n$ is also disjoint from the closure of periodic Fatou domains. From $\pi \circ f = \tau \circ \pi$ on $X$, we obtain $K_n = f^n(K)$. This shows that $K$ is disjoint from the closure of any Fatou domain. Thus $K$ is a buried Julia component. This shows that $\pi(\mathcal{A} \cup \mathcal{C}) = \overline{X}$. Now the proof is complete.

For our purpose, we want to know whether $\mathcal{A}$ is an infinite collection, or equivalently, whether $\overline{X} \setminus X$ is an infinite set by Theorem 6.1. The next lemma provide a necessary and sufficient combinatorial condition. Refer to [14] for the next definition.

Let $\Gamma$ be a multicurve. Denote by $\Gamma_1$ the collection of curves in $F^{-1}(\Gamma)$ isotopic to a curve in $\Gamma$, and denote by $\Gamma_n$ the collection of curves in $F^{-1}(\Gamma_{n-1})$ isotopic to a curve in $\Gamma$ for $n \geq 2$. For each $\gamma \in \Gamma$, denote

$$\kappa_n(\gamma) = \#\{\beta \in \Gamma_n : \beta \text{ is isotopic to } \gamma\}.$$ 

The multicurve $\Gamma$ is a **Cantor multicurve** if $\kappa_n(\gamma) \to \infty$ as $n \to \infty$ for all $\gamma \in \Gamma$. 
Lemma 6.2. Let $f$ be a non-degenerate and generic sub-hyperbolic rational map. Let $\Gamma$ be a canonical multicurve of $f$ and $\tau : (T, X_1) \to (T, X_0)$ be the Shishikura tree map of $f$. The following conditions are equivalent.

(a) $\Gamma$ contains a Cantor multicurve.

(b) $\tau$ has infinitely many repelling periodic cycles.

(c) $\tau$ has wandering points.

Proof. (a) $\Rightarrow$ (b) and (c). Let $\Gamma_0 \subset \Gamma$ be a Cantor multicurve. Denote by $I_k$ ($1 \leq k \leq n$) the edges of $(T, X_0)$ corresponding to $\Gamma_0$. Set $T_0 = \bigcup_{k=1}^n I_k$ and $T_1 = T_0 \cap \tau^{-1}(T_0)$. Then $T_1 \subset T_0$ and $\tau(T_1) \subset T_0$. Moreover, $\tau$ maps each component of $T_1$ onto one edge.

Denote $\tau_0 = \tau|_{T_1}$. The multicurve $\Gamma_0$ is a Cantor multicurve implies that for any integer $M > 0$, there is an integer $N > 0$ such that $\tau_0^{-N}(T_0) \cap I_k$ contains at least $M$ components for each $1 \leq k \leq n$.

Take $M = n + 1$. Then for some $N > 0$, $\tau_0^{-N}(T_0) \cap I_k$ contains at least $n + 1$ components for each $1 \leq k \leq n$. Thus at least two of them map to the same edge under $\tau_0^N$. Define a map $\sigma$ on the index set $\{1, \cdots, n\}$ by $\sigma(i) = j$ if $\tau_0^{-N}(T_0) \cap I_i$ has two components mapping to $I_j$. Here we need to point out that the definition of $\sigma$ is not uniquely determined. As the index set is finite, each index is eventually periodic. In particular, there is a periodic index. By relabelling the edges, we may assume that the index 1 is a periodic index with period $p \geq 1$. This implies that $\tau_0^{-pN}(T_0) \cap I_1$ has at least $2^p$ components mapping to $I_1$ by $\tau_0^{pN}$.

Let $J_0$ and $J_1$ be two distinct components of $\tau_0^{-pN}(T_0) \cap I_1$ which maps onto $I_1$ by $\tau_0^{pN}$. Write $\omega = \tau_0^{pN}|_{J_0 \cup J_1}$ for simplicity. It is classical that the linear map $\omega : J_0 \cup J_1 \to I_1$ contains infinitely many repelling periodic cycles and wandering points. So does $\tau$.

(b) $\Rightarrow$ (a). Suppose that $\tau$ has infinitely many repelling periodic cycles. Then there is an edge $I$ of $T$ such that it contains two repelling periodic points $x_1$ and $x_2$ with periods $p_1, p_2 \geq 1$ respectively. For $i = 1, 2$, let $J_i$ be the component of $\tau^{-p_i}(I)$ that contains the point $x_i$. Denote

$$J_i^n = (\tau^{-p_i}|_{J_i})^{-n}(I)$$

Then $J_i^n \to x_i$ as $n \to \infty$. Thus there is an integer $k > 0$ such that $J_1^k$ is disjoint from $J_2^k$. Thus $\tau^{-kp_1p_2}(I)$ has at least two components contained in $I$, one is contained in $J_1^k$, another is contained in $J_2^k$.

Let $J_0$ be the curve corresponding to the edge $I$. Then $f^{-kp_1p_2}(\gamma)$ has at least two curves isotopic to $\gamma$. Let $\Gamma_0 \subset \Gamma$ be a sub-multicurve defined by $\beta \in \Gamma_0$ if $f^{-n}(\gamma)$ has a component isotopic to $\beta$ for some $n > 0$. It is well-defined since $\Gamma$ is completely stable. It is easy to check that $\Gamma_0$ is a Cantor multicurve.

(c) $\Rightarrow$ (a). Suppose that $x \in T$ is a wandering point of $\tau$. Then there exists an edge $I$ of $T$ such that it contains infinitely many points of the forward orbit of $x$. Denote by

$$0 < n_1 < n_2 < \cdots < n_k < \cdots$$

a sequence of integers such that $\tau^{n_k}(x) \in I$. Let $J_1^k$ be the component of $\tau^{-n_1-n_k}(I)$ that contains $\tau^{n_1}(x)$. Then $J_1^k$ is disjoint from $\tau^{-n_1-n_k}(X_0)$. Thus $|J_1^k| \to 0$ as $k \to \infty$ since $\tau^i(x) \in X \setminus X$ is a double-sides accumulation point by Corollary 5.3.

Let $J_2^k$ be the component of $\tau^{-n_2-n_k}(I)$ that contains $\tau^{n_2}(x)$. Then $|J_2^k| \to 0$ as $k \to \infty$. Thus there is an integer $k > 2$ such that $J_1^k$ is disjoint from $J_2^k$. This shows that $\tau^{-(n_1-n_k)(n_2-n_k)}(I)$ has at least two components contained in $I$, one is contained in $J_1^k$, another is contained in $J_2^k$. The same argument as above shows that $\Gamma$ contains a Cantor multicurve.

\qed
7 Self-grafting

Let $\tau : (T, X_1) \to (T, X_0)$ be a tree map. Suppose that $O$ is a repelling periodic cycle in $T \setminus X_0$ with period $p \geq 1$. Then there is a point $x_0 \in O$ and a component $B$ of $T \setminus \{x_0\}$ such that $B \cap O = \emptyset$.

Define a new tree $T' \supset T$ by the following: there are exactly $p$ components $B_i$ of $T' \setminus T$, $0 \leq i \leq p - 1$, each of them is homeomorphic to $B$ by a linear map $\theta_i : B \to B_i$, and $B_i$ is connected to $T$ at the point $x_i = \tau^i(x_0)$. The vertices $X'_0$ on $T'$ is assigned to be the original vertices on $T$ together with $O \cup \bigcup_{i=0}^{p-1} \theta_i(B \cap X_0)$.

Define a tree map $\tilde{\tau} : T' \to T'$ by $\tilde{\tau} = \tau_0 \circ \kappa$, where

$$\tau_0 = \begin{cases} 
\tau : T \to T, \\
\theta_{i+1} \circ \theta_i^{-1} : B_i \to B_{i+1} & \text{for } 0 \leq i < p - 1, \\
\theta_0 \circ \theta_{p-1}^{-1} : B_{p-1} \to B_0, 
\end{cases}$$

and

$$\kappa = \begin{cases} 
id : T' \setminus (B \cup B_0) \to T' \setminus (B \cup B_0), \\
\theta_0 : B \to B_0, \\
\theta_0^{-1} : B_0 \to B.
\end{cases}$$

Set $X'_1 = \tilde{\tau}^{-1}(X'_0)$. We call the new tree map $\tilde{\tau} : (T', X'_1) \to (T', X'_0)$ a self-grafting of the tree map $\tau$ (see Figure 2).

![Figure 2. A self-grafting](image)

By the definition, $\tilde{\tau}^i = \theta_i : B \to B_i$ for $0 < i < p$ and $\tilde{\tau}^p = \theta_0 : B \to B_0$. Thus $\tilde{\tau}^{p+1} = \tau$ on $B$. So the original tree map $\tau$ can be expressed by $\tilde{\tau}$ as

$$\tau = \begin{cases} 
\tilde{\tau} & \text{on } T \setminus B, \\
\tilde{\tau}^{p+1} & \text{on } B.
\end{cases}$$

In conclusion, any periodic point of $\tau$ with period $q \geq 1$ is also a periodic point of $\tilde{\tau}$ with period $q + kp$, where $k \geq 0$ is the number of times at which the cycle passes through $B$. Conversely, any cycle of $\tilde{\tau}$ must pass through $B$ since $\tilde{\tau}^{-i}(B_i) = B$ for $0 < i < p$ and $\tilde{\tau}^{-p}(B_0) = B$. 
For any weight $w$ on the tree $(T, X_1)$, the **induced weight** $\tilde{w}$ on the tree $(T', X'_1)$ is defined as the following: for each edge $J'$ of $(T', X'_1)$,

$$
\tilde{w}(J') = \begin{cases} 
1, & \text{if } J' \subset B \cup \bigcup_{i=1}^{p-1} B_i; \\
w(\kappa(J')), & \text{if } J' \subset B_0; \\
w(J'), & \text{if } J' \subset T \setminus B.
\end{cases}
$$

(1)

Here we need to point out that if $J' \subset B_0$, then $\kappa(J')$ need not to be an edge of $(T, X_1)$. However, it must be contained in an edge of $(T, X_1)$ since $\tau(\kappa(J')) = \tilde{\tau}(J')$ is disjoint from $X_0$. If $J' \subset T \setminus B$, then $J'$ is also contained in an edge $J$ of $(T, X_1)$ since $\tilde{\tau}(J') = \tau(J')$.

We define $w(J') = w(J)$ in both cases.

**Lemma 7.1.** Let $w$ be a weight on the tree $(T, X_1)$ and let $\tilde{w}$ be the induced weight on $(T', X'_1)$. Then $\lambda(M(\tilde{\tau}, \tilde{w})) < 1$ if $\lambda(M(\tau, w)) < 1$.

**Proof.** At first, we consider the tree map $\tau'' = \tau : (T, X_1'' \rightarrow (T, X_1'')$, where $X_1'' = X_0 \cup O$ and $X_1'' = \tau^{-1}(X_0)$.

Denote by $\{I_1, \ldots, I_n\}$ the edges of $(T, X_0)$. Denote by $\{I_1'', \ldots, I_n''\}$ the edges of $(T, X_1'')$. Let $\Theta = \{1, \ldots, m\}$ be the index set. It is divided into $\Theta = \Theta_1 \sqcup \cdots \sqcup \Theta_n$ such that $I_i'' \subset I_k$ if $i \in \Theta_k$.

The weight $w$ on $(T, X_1)$ induces a weight on $(T, X_1'')$, denoted also by $w$, such that $w(J'') = w(J)$ is $J'' \subset J$. Let $M(\tau, w) = (a_{kl})$ and $M(\tau'', w) = (b_{ij})$ be the transition matrices. Then for each pair $(k, l)$ and any $j \in \Theta_k$,

$$
\sum_{i \in \Theta_k} b_{ij} = a_{kl}
$$

(2)

since for all $j \in \Theta_k$, $\tau^{-1}(I''_j)$ have the same number of components in $I_k$.

Let $\lambda$ be the leading eigenvalue of $M(\tau'', w)$. Then there is a non-zero vector $v = (v_i) \geq 0$ such that $M(\tau'', w)v = \lambda v$. Let $u = (u_k)$ be a vector defined by $u_k = \sum_{i \in \Theta_k} v_i$. Then $u$ is also a non-zero vector with $u \geq 0$. Now the equation (2) implies that $M(\tau, w)u = \lambda u$.

This shows that $\lambda$ is also an eigenvalue of $M(\tau, w)$. Thus $\lambda \leq \lambda(M(\tau, w)) < 1$.

Now let us compare the tree maps $\tilde{\tau}$ with $\tau''$. Each edge of $(T, X_0'')$ is also an edge of $(T', X_0'')$. By Lemma 5.1 there exist a linear metric $\rho$ on $(T, X_0'')$ and a constant $0 < \lambda_1 < 1$ such that for each edge $I$ of $(T, X_0'')$,

$$
\sum_{J} |\tau(J)|/w(J) < \lambda_1 |I|,
$$

(3)

where the sum is taken over all the edges $J$ of $(T, X_0'')$ in $I$ and $| \cdot |$ denotes the length with respect to the metric $\rho$.

Define a linear metric $\rho_1$ on $T'$ such that for each edge $I$ of $(T', X_0'')$, $|I|_1 = |I|$ if $I$ is contained in an edge of $(T, X_0'')$, and $|I|_1 = \lambda_1 |\theta_1^{-1}(I)|$ if $I \subset B_i$ for $0 < i < p$, and $|I|_1 = \lambda_1 |\theta_1^{-1}(I)|$ if $I \subset B_0$, where $| \cdot |$ denotes the length under the metric $\rho_1$.

For each edge $I$ of $(T', X_0'')$ in $T \setminus B$, $\tilde{\tau}(I) = \tau(I)$ is contained in an edge of $(T, X_0'')$. From (1) and (3), we have

$$
\sum_{J} |\tau(J)|_1/w(J) < |I|_1,
$$

(4)
Then there exists a non-degenerate and generic sub-hyperbolic rational map $\tilde{\tau}(J) = \theta_1(J)$ is contained in $B_1$. Thus
\[
\frac{|\tilde{\tau}(J)|_1}{w(J)} = \lambda_1|\theta_1^{-1}(\tilde{\tau}(J))| = \lambda_1|J| = \lambda_1|I|_1 < |I|_1.
\]
For each edge $I_i$ of $(T', X'_0)$ in $B_i$ with $1 \leq i < p$, $J_i = I_i$ is also an edge of $(T', X'_1)$ with $w(J_i) = 1$ and $\tilde{\tau}(J_i) = \theta_{i+1} \circ \theta_i^{-1}(J_i)$ is contained in $B_{i+1}$ (set $B_p = B_0$ and $\theta_p = \theta_0$). Thus
\[
\frac{|\tilde{\tau}(J_i)|_1}{w(J_i)} = \lambda_1|\theta_{i+1}^{-1} \circ \tilde{\tau}(J_i)| = \lambda_1|\theta_i^{-1}(J_i)| = \lambda_1|I_i|_1 < |I_i|_1.
\]
So the inequality (4) holds for edges in $B$ or $B_o$ for $1 \leq i < p$.

Let $I_0$ be an edge of $(T', X'_0)$ in $B_0$. Let $I$ be the edge of $B$ with $\theta_0(I) = I_0$. Then $\tau(J) = \tilde{\tau}(\theta_0(J))$ for each edge $J$ of $(T, X''_n)$ in $I$. From (3), we have
\[
|I_0|_1 = \lambda_1^0|I| > \sum_j \frac{|\tau(J)|_1}{w(J)} = \sum_j \frac{|\tilde{\tau}(\theta_0(J))|_1}{w(\theta_0(J))},
\]
where the sum is taken over all the edges $J$ of $(T', X'_1)$ in $I$.

The key step in the proof of Theorem L.1 is the next theorem.

**Theorem 7.2.** Let $f$ be a non-degenerate and generic sub-hyperbolic rational map. Suppose that $f$ has a periodic Jordan curve disjoint from $\mathcal{P}_f$ as a buried Julia component. Then there exists a non-degenerate and generic sub-hyperbolic rational map $g$ with $\deg g = \deg f$ such that the Shishikura tree map $\tau_g$ of $g$ is the self-grafting of $\tau_f$. Consequently,

(a) $N(g) = N(f) + 1$, and

(b) if $\tau_f$ has infinitely many repelling periodic cycles, so does $\tau_g$.

Moreover, $g$ can be chosen to be hyperbolic when $f$ is hyperbolic.

**Proof.** Let $C_0$ be a periodic Jordan curve disjoint from $\mathcal{P}_f$ as a buried Julia component of $f$ with period $p \geq 1$. Denote $C_i = f^i(C_0)$ for $0 < i < p$. At least one component of $\overline{\mathbb{C}} \setminus \bigcup_{i=0}^{p-1} C_i$ is a Jordan domain, by relabelling the index, we assume that this Jordan domain is bounded by $C_0$. Then $C_1, \ldots, C_{p-1}$ are contained in the same complementary component of $C_0$.

Let $\mathcal{U}$ be a canonical decomposition of $f$. Since $\bigcup_{n>0} f^{-n}(\mathcal{U}) = \mathcal{F}_f$, as $n$ is large enough, each $C_i$ is contained in an A-type component $L_i$ of $\overline{\mathbb{C}} \setminus f^{-n}(\mathcal{U})$ for $0 \leq i < p$. We may assume $n = 0$ for the simplicity. Drop all the D-type components of $\mathcal{U}$, the remaining is still a canonical decomposition of $f$. Thus we may assume that each component of $\mathcal{U}$ is not D-type. Denote $\mathcal{L} = \overline{\mathbb{C}} \setminus \mathcal{U}$. Let $L_i$ be the component of $\mathcal{L}$ that contains $C_i$. Then $L_i$ is a closed annulus disjoint from $\mathcal{P}_f$.

Pick a Jordan domain $\Delta_0 \subset L_0$ such that $\overline{\Delta_0}$ is disjoint from $\partial L_0$. Then there is a component $\Delta'$ of $f^{-p}(\Delta_0)$ such that $\Delta' \subset L_0$ and $\Delta_i = f^i(\Delta') \subset L_i$ for $1 \leq i < p$.

There exists a homeomorphism $\phi_0$ of $\overline{\mathbb{C}}$ such that $\phi_0 = \text{id}$ in $\overline{\mathbb{C}} \setminus L_0$ and $\phi_0 = (f^p|_{\Delta'})^{-1} : \Delta_0 \to \Delta'$. Set $F = f \circ \phi_0$. Then $F^p = \text{id}$ in $\Delta_0$ and $F = f$ in $\overline{\mathbb{C}} \setminus L_0$. 

Theorem 7.2. Let $f$ be a non-degenerate and generic sub-hyperbolic rational map. Suppose that $f$ has a periodic Jordan curve disjoint from $\mathcal{P}_f$ as a buried Julia component. Then there exists a non-degenerate and generic sub-hyperbolic rational map $g$ with $\deg g = \deg f$ such that the Shishikura tree map $\tau_g$ of $g$ is the self-grafting of $\tau_f$. Consequently,

(a) $N(g) = N(f) + 1$, and

(b) if $\tau_f$ has infinitely many repelling periodic cycles, so does $\tau_g$.

Moreover, $g$ can be chosen to be hyperbolic when $f$ is hyperbolic.
Denote by $\Omega$ and $\Delta$ the two components of $\mathbb{C} \setminus L_0$ such that $\Delta_i \subset \Omega$ for $1 \leq i < p$. Define a homeomorphism $\phi_1$ of $\mathbb{C}$ such that

$$
\begin{cases}
\phi_1 = \text{id} : \Omega \to \Omega, \\
\phi_1 : \Delta \to \Delta_0 \text{ is conformal}, \\
\phi_1 = (\phi_1|_{\Delta})^{-1} : \Delta_0 \to \Delta.
\end{cases}
$$

Set $G = F \circ \phi_1$. Then $G = f$ in $\Omega$. Since $F^p = \text{id}$ in $\Delta_0$, we obtain $G^p = \phi_1$ in $\Delta$. Thus $G^{p+1} = f$ in $\Delta$ (see the next diagram).

Conversely, $f$ can be expressed by $G$ as the following:

$$
f = \begin{cases}
G & \text{on } \Omega, \\
G^{p+1} & \text{on } \Delta.
\end{cases} \quad (5)
$$

By the definition, $G$ and $f$ have the same critical values. From $\mathcal{P}_f \subset \Omega \cup \Delta$, we obtain

$$
\mathcal{P}_G = \mathcal{P}_f \cup \bigcup_{i=1}^{p} P_i \subset \Omega \cup \Delta \cup \Delta_0,
$$

where $P = \Delta \cap \mathcal{P}_f$ and $P_i = G^i(P) \subset \Delta_i$ for $0 < i \leq p$ (set $\Delta_p = \Delta_0$). Moreover,

$$
G^{-1}(P_i) \cap \mathcal{P}_G = P. \quad (6)
$$

Note that $G$ is holomorphic in $\Omega \cup \Delta \cup \Delta_0$. From (5), each periodic point of $f$ in $\mathcal{P}_f'$ with period $p' \geq 1$ is also a periodic point of $G$ with period $q = p' + kp$, where $k \geq 0$ is the number of times at which the cycle passes through $\Delta$. From (6), any cycle of $G$ in $\mathcal{P}_G'$
must pass through $\Delta$. Therefore, each cycle of $G$ in $\mathcal{P}_G$ is attracting or super-attracting. So $G$ is a semi-rational map.

Recall that $(\mathcal{U}, \mathcal{L})$ is a canonical decomposition of $f$. Since $\mathcal{U} \in f^{-1}(\mathcal{U})$, for each component $D$ of $\mathcal{U}$, there exists a tame domain $D' \in D$ such that

(i) $\partial D'$ is disjoint from $\mathcal{P}_f$,
(ii) each component of $D \setminus \overline{D'}$ is an annulus disjoint from $\mathcal{P}_f$, and
(iii) $\mathcal{U} \in f^{-1}(\mathcal{U}')$, where $\mathcal{U}' = \bigcup_{D \in \mathcal{U}} D'$.

Set $\mathcal{L}' = \overline{\mathcal{U}} \setminus \mathcal{U}'$. Then $(\mathcal{U}', \mathcal{L}')$ is also a canonical decomposition of $f$. For each component $D$ of $\mathcal{U}$ in $\Delta$, choose

$$D' \in D^1 \in \cdots \in D^{p-1} \in D^p \in D.$$ Denote $D_i = G^i(D^i)$ for $1 \leq i \leq p$. Then $D_i \in \Delta_i$. Set

$$\mathcal{U}_G = \mathcal{U}' \cup \bigcup_{D \subset \Delta \atop i=1} D_i.$$ Then $\mathcal{U}_G \in G^{-1}(\mathcal{U}_G)$. Denote $\mathcal{L}_G = \overline{\mathcal{U}} \setminus \mathcal{U}_G$. Then each Q-type component $L$ of $\mathcal{L}'$ is also a Q-type component of $\mathcal{L}_G$. Moreover, if $L \subset \Delta$, then $G^i(L)$ is a Q-type component of $\mathcal{L}_G$ for $1 \leq i \leq p$. There is an extra cycle of Q-type components of $\mathcal{L}_G$ consisting of $\{L_0', \ldots, L_{p-1}'\}$ with $L_i' \supset L_i \setminus \Delta_i$.

It is easy to check that $(\mathcal{U}_G, \mathcal{L}_G)$ is a canonical decomposition of $G$. Denote by

$$\tau : (T, X_1) \to (T, X_0) \text{ and } \tilde{\tau} : (T', X_1') \to (T', X_0')$$

the Shishikura tree maps of $f$ and $G$, respectively. Denote

- $x_i \in T$: the point corresponding to the Julia component $C_i$ for $0 \leq i < p$,
- $y_i \in T'$: the point corresponding to the Q-type component $L_i'$ of $\mathcal{L}_G$,
- $B$: the component of $T \setminus \{x_0\}$ such that vertices in $B$ correspond to the Q-type components of $(\mathcal{U}, \mathcal{L})$ contained in $\Delta$.

The above relations between Q-type components of $(\mathcal{U}', \mathcal{L}')$ and Q-type components of $(\mathcal{U}_G, \mathcal{L}_G)$ induce a linear injection $\iota : (T, X_0) \to (T', X_0')$ such that $\iota(x_i) = y_i$ and a linear bijection from $B$ to some component of $T' \setminus \{y_i\}$ for $0 \leq i < p$. Identify the tree $T$ with its image under the injection $\iota$. Then $T \subset T'$. It is easy to check that $\tilde{\tau}$ is a self-grafting of $\tau$.

The weight for the Shishikura tree map of $f$ exactly equals to the induced weight for the Shishikura tree map of $f$. Let $\Gamma_G$ be a canonical multicurve of $G$. By Lemma 7.1, $\lambda(\Gamma_G) < 1$.

The cycle of Q-type components of $\mathcal{L}_G$ consisting of $\{L_0', \ldots, L_{p-1}'\}$ contains essentially no multicurve of $G$ since each $L_i'$ is disjoint from $\mathcal{P}_G$ and has only three complementary components.

Let $\Gamma$ be a multicurve contained essentially in a periodic Q-type component $L$ of $\mathcal{L}_G$, where $L \neq L_i$ for all $0 \leq i \leq p - 1$. Let $q \geq 1$ be its period. The orbit of $L$ either is disjoint from $\Delta$ or passes $k$ times through $\Delta$. In the former case, $L$ is also a cycle of $\mathcal{L}$ with the same period. Thus $\lambda(\Gamma, G^q) = \lambda(\Gamma, f^q) < 1$.

In the latter case, it contains a cycle of Q-type components of $\mathcal{L}$ with period $q_1 \geq 1$ and $q = q_1 + kp$. Thus when $L$ is a component of $\mathcal{L}$, $\lambda(\Gamma, G^q) = \lambda(\Gamma, f^q) < 1$. When $L$ is contained in $\Delta_i$ for $1 \leq i \leq p$, $\Gamma' = \{G^{-i}(\gamma), \gamma \in \Gamma\}$ is a multicurve essentially contained
in $\Delta$ and $\lambda(\Gamma, G^q) = \lambda(\Gamma', f^q) < 1$. From Theorem 3.5, $G$ has no Thurston obstruction. Thus $G$ is $c$-equivalent to a rational map $g$.

It is obvious that (a) $N(g) = N(f) + 1$ and (b) if $\tau_f$ has infinitely many repelling periodic cycles, so does $\tau_g$. Moreover, $g$ is hyperbolic when $f$ is hyperbolic. \qed

8 Proof of Theorem 1.1

At first, we want to construct a hyperbolic rational map $f$ with $\deg f = 3$ such that $N(f) = 1$ and it has infinitely many periodic Jordan curves as components of $\mathcal{J}_f$. Refer to Figure 4 for the construction.

We begin with the quadratic rational map

$$h(z) = \frac{1}{(z - 1)^2}.$$ 

It has two critical points $z_1 = 1$ and $z_2 = \infty$. Both of them are contained in the cycle

$$z_1 \mapsto z_2 \mapsto 0 \mapsto z_1.$$ 

So $\mathcal{P}_h = \{0, 1, \infty\}$ and $\mathcal{J}_h$ is connected.

Pick a Böttcher disk $z_0 \in V_0 \subset \mathcal{F}_h$. Then there are Böttcher disks $V_1 \ni z_1$ and $V_2 \ni z_2$ such that $V \ni h^{-1}(V)$, where $V = \bigcup_{i=0}^{2} V_i$.

Denote $z_3 = 2$. Then $h(z_3) = z_1$. The set $h^{-1}(V)$ has 4 components, denote them by $D_i$ such that $z_i \in D_i$.

Pick a Jordan curve $\alpha \subset D_0$ such that it separates the point $z_0$ from $\partial D_0$. Then $D_0 \setminus \alpha$ has two components: an annulus $A$ and a Jordan domain $D$.

Define a branched covering $F$ by the following:

(1) $F = h$ on $\overline{\mathbb{C} \setminus D_0}$,
(2) $F : A \to V_1$ is a branched covering with degree 2, and
(3) $F : D \to \overline{\mathbb{C} \setminus V_1}$ is a homeomorphism such that $F(z_0) = z_3$ and $F$ is holomorphic in a neighborhood of $z_0$.
Now \( \{z_0, z_1, z_2, z_3\} \) is a super-attracting cycle of \( F \). We may require that the two critical values of \( F : A \rightarrow V_1 \) are contained in a Böttcher disk \( U_1 \) at \( z_1 \). Then \( F \) is a semi-rational map with \( \mathcal{P}_F' = \{z_0, z_1, z_2, z_3\} \).

There exist Böttcher disks \( U_i \ni z_i \) (i = 0, 2, 3) such that \( \mathcal{U} \subseteq h^{-1}(\mathcal{U}) \), where \( \mathcal{U} = \bigcup_{i=0}^3 U_i \). Since the two critical values of \( F : A \rightarrow V_1 \) are contained in \( U_1 \), we have \( \mathcal{P}_F \subseteq \mathcal{U} \).

Set \( \mathcal{L} = \overline{\mathcal{C}} \setminus \mathcal{U} \). It is easy to check that \((\mathcal{U}, \mathcal{L})\) is a canonical decomposition of \( F \). The set \( F^{-1}(\mathcal{L}) \) has two components, one is Q-type and the other is A-type.

Denote \( \gamma_i = \partial U_i \) (i = 0, 1, 2, 3). Then \( \Gamma_F = \{\gamma_i\}_{i=0}^3 \) is a canonical multicurve. Its transition matrix is:

\[
M = \begin{pmatrix}
0 & 2 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\frac{1}{2} & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

By a direct computation, we have

\[
M^3 v = \begin{pmatrix}
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{4} \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{4} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} v_1 + \frac{1}{2} v_3 \\
\frac{1}{2} v_2 + \frac{1}{4} v_4 \\
\frac{1}{4} v_3 + \frac{1}{2} v_2 \\
\frac{1}{4} v_1
\end{pmatrix}.
\]

Choose the positive vector \( v \) such that \( v_4/2 < v_2 < v_3 < v_1 < 4v_4 \). Then \( Mv < v \). So \( \lambda(\Gamma_F) = \lambda(M) < 1 \).

Let \( \Gamma \) be a multicurve of \( F \) contained essentially in \( \mathcal{L} \). Then \( \Gamma \) contains exactly one curve \( \gamma \). If \( \gamma \) separates \( z_0 \) from \( z_2 \), then \( F^{-1}(\gamma) \) has only one component \( \delta \) in the Q-type component of \( F^{-1}(\mathcal{L}) \) and \( \deg(F|_\delta) = 2 \). So \( \lambda(\Gamma) < 1 \). If \( \gamma \) does not separate \( z_0 \) from \( z_2 \), then each component of \( F^{-1}(\gamma) \) in the Q-type component of \( F^{-1}(\mathcal{L}) \) does not separate \( z_1 \) from \( z_2 \). On the other hand, \( \gamma \) separates \( z_1 \) from \( z_2 \). Thus \( \lambda(\Gamma) = 0 \). Therefore \( F \) is \( c \)-equivalent to a rational map \( f \) by Theorem 8.3.

One may also apply [2, Theorem 2.1] to show \( \lambda(\Gamma) < 1 \).

The Shishikura tree map \( \tau : (T, X_1) \rightarrow (T, X_0) \) of \( f \) is shown in Figure 5, where

\[
X_0 = \{a_0, a_1, a_2, a_3, b\}, \quad X_1 = \{a_0, a_1, a_2, a_3, b, b_{-1}, a_0'\},
\]

and the tree map is uniquely determined by its definition on vertices:

\[
\tau : a_0 \mapsto a_3 \mapsto a_1 \mapsto a_2 \mapsto a_0, a_0' \mapsto a_1, b_{-1} \mapsto b, b \mapsto b.
\]

One may refer to [5] for a formula of the rational map \( f \).

**Proof of Theorem 7.1**

Applying Theorem 7.2 successively, we obtain a sequence of rational maps \( \{f_n\} \) such that \( \deg f_n = 3 \) and \( N(f_n) = n \).

Fix \( n \geq 1 \), for any integer \( d > 3 \), applying the disc-annulus surgery in [12], we could obtain a rational map \( g_n \) such that \( \deg g_n = d \) and \( N(g_n) = n \). The following is a detailed construction of \( g_n \).

Let \((\mathcal{U}, \mathcal{L})\) be a canonical decomposition of \( f_n \). Let \( U \) be a Q-type periodic component of \( \mathcal{U} \) and \( U_1 \) be a non-periodic component of \( f_n^{-1}(\mathcal{U}) \). Take a quasi-disk \( \Omega \subseteq U_1 \setminus \mathcal{P}_f_n \) such
that \( f_n \) is injective on \( \overline{\Omega} \). Pick another quasi-disk \( \Delta \in \Omega \). Then there is a quasi-regular branched covering \( G \) of \( \overline{\mathbb{C}} \) with \( \text{deg} \ G = d \) such that

1. \( G = f_n \) on \( \overline{\mathbb{C}} \setminus \overline{\Omega} \),
2. \( G : \Delta \to \overline{\mathbb{C}} \setminus f_n(\Omega) \) is a holomorphic proper map with degree \( d - 3 \), and
3. \( G : \Omega \setminus \Delta \to f_n(\Omega) \) is a quasi-regular branched covering with degree \( d - 2 \).

Refer to Figure 6 for the construction of \( G \). It is clear that the forward orbit of any point under \( G \) passes through \( \overline{\Omega} \setminus \Delta \) at most once. Thus by Lemma 1 in [14], \( G \) is quasi-conformally conjugated to a rational map \( g_n \).

\[ \begin{array}{c}
U \\
\times
\end{array} \]

\[ f_n(\Omega) \]

\[ \begin{array}{c}
U_1 \\
\Delta \\
\bigcirc
\end{array} \]

\[ \Omega \]

Figure 6. The construction of \( G \).

Obviously, \((\mathcal{U}, \mathcal{L})\) is still a canonical decomposition of \( G \) and the Shishikura tree map of \( G \) is the same as \( \tau_{f_n} \). Thus \( N(g_n) = N(f_n) = n \). \qed

References

[1] B. Branner and J. Hubbard, The iteration of cubic polynomials, I, The global topology of parameter space, Acta Math., 160 (1988), 143-206.

[2] X. Buff, G. Cui and L. Tan, Teichmüller spaces and holomorphic dynamics, Handbook of Teichmüller theory, Vol. IV, ed. Athanase Papadopoulos, Societ Mathmatique Europenne (2014), 717-756.
[3] G. Cui and L. Tan, A characterization of hyperbolic rational maps, Invent. Math., 183 (2011), 451-516.

[4] G. Cui, W. Peng and L. Tan, Renormalization and wandering Jordan curves of rational maps, Comm. Math. Phy., 344 (2016), 67-115.

[5] S. Godillon, A family of rational maps with buried Julia components, Ergodic Theory Dynam. Systems, 35 (2015), 1846-1879.

[6] Y. Jiang and G. Zhang, Combinatorial characterization of sub-hyperbolic rational maps, Adv. Math., 221 (2009), 1990-2018.

[7] O. Kozlovski and S. van Strien, Local connectivity and quasi-conformal rigidity of non-renormalizable polynomials, Proc. London Math. Soc., 99 (2009), 275-296.

[8] J. Milnor, Dynamics in One Complex Variable, 3rd edition, Princeton University Press, 2006.

[9] C. T. McMullen, Automorphisms of rational maps, In Holomorphic functions and moduli I, 31-60. Springer-Verlag, 1988.

[10] C. T. McMullen, Complex Dynamics and Renormalizations, Ann. of Math. Stud., No. 135, Princeton University Press, 1994.

[11] K. Pilgrim and L. Tan, Rational maps with disconnected Julia set, Astérisque 261 (2000), volume spécial en l’honneur d’A. Douady, 349-384.

[12] K. Pilgrim and L. Tan, Disc-annulus surgery on rational maps, a section in: B. Branner and N. Fagella, Quasiconformal surgery in holomorphic dynamics, Cambridge Stud. Adv. Math. 141, Cambridge University Press, Cambridge 2014, 267-282.

[13] W. Qiu and Y. Yin, Proof of the Branner-Hubbard conjecture on Cantor Julia sets, Sci. China Ser. A, 52 (2009), 45-65.

[14] M. Shishikura, On the quasiconformal surgery of rational functions, Ann. Sci. Éc. Norm. Sup., 20 (1987), 1-29.

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