Rings whose total graphs have small vertex-arboricity and arboricity

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Abstract

Let \( R \) be a commutative ring with non-zero identity, and \( Z(R) \) be its set of all zero-divisors. The total graph of \( R \), denoted by \( T(\Gamma(R)) \), is an undirected graph with all elements of \( R \) as vertices, and two distinct vertices \( x \) and \( y \) are adjacent if and only if \( x + y \in Z(R) \). In this article, we characterize, up to isomorphism, all of finite commutative rings whose total graphs have vertex-arboricity (arboricity) two or three. Also, we show that, for a positive integer \( v \), the number of finite rings whose total graphs have vertex-arboricity (arboricity) \( v \) is finite.

Mathematics Subject Classification (2020). Primary: 05C99, Secondary: 13A99

Keywords. total graph, arboricity, vertex-arboricity

1. Introduction

In [1], D.F. Anderson and A. Badawi introduced the total graph of ring \( R \), denoted by \( T(\Gamma(R)) \), as the graph with all elements of \( R \) as vertices, and for distinct \( x, y \in R \), the vertices \( x \) and \( y \) are adjacent if and only if \( x + y \in Z(R) \), where \( Z(R) \) is the set of zero-divisors of \( R \). They studied some graph theoretical parameters of \( T(\Gamma(R)) \) such as diameter and girth. In addition, they showed that the total graph of a commutative ring is connected if and only if \( Z(R) \) is not an ideal of \( R \). In [7], H.R. Maimani et al. gave the necessary and sufficient conditions for the total graphs of finite commutative rings to be planar or toroidal and in [5] T. Chelvam and T. Asir characterized all commutative rings such that their total graphs have genus two.

Suppose that \( G \) is a graph, and let \( V(G) \) and \( E(G) \) be the vertex set and edge set of \( G \), respectively. The vertex-arboricity of a graph \( G \), denoted by \( va(G) \), is the minimum positive integer \( k \) such that \( V(G) \) can be partitioned into \( k \) sets \( V_1, V_2, \ldots, V_k \) such that \( G[V_i] \) is a forest for each \( i \in \{1, 2, \ldots, k\} \), where \( G[V_i] \) is the induced subgraph of \( G \) whose vertex set is \( V_i \) and its edge set consists of all of the edges in \( E(G) \) that have both endpoints in \( V_i \). This partition is called acyclic partition. The vertex-arboricity can be viewed as a vertex coloring \( f \) with \( k \) colors, where each color class \( V_i \) induces a forest; namely, \( G[f^{-1}(i)] \) is an acyclic graph for each \( i \in \{1, 2, \ldots, k\} \). Vertex-arboricity, also known as point arboricity, was first introduced by G. Chartrand, H.V. Kronk, and C.E.

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Received: 10.07.2019; Accepted: 03.05.2020
Wall [4] in 1968. Note that a graph with no cycles is a forest, and it has vertex-arboricity one.

Likewise, the arboricity of a graph $G$, denoted by $\nu(G)$, is the least number of line-disjoint spanning forests into which $G$ can be partitioned, that is, there is some collection of $\nu(G)$ subgraphs of $G$, where each subgraph is a forest and each edge in $G$ is in exactly one such subgraph. Arboricity of a graph was first introduced by C. St. J. A. Nash-Williams [4] in 1964.

The main purpose of this paper is to characterize all finite commutative rings whose total graph has vertex-arboricity (arboricity) two or three. In addition, we show that, for a positive integer $v$, there are only finitely many finite rings whose total graph has vertex-arboricity (arboricity) $v$.

Now, we recall some definitions of graph theory which are necessary in this article. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We use $n$ and $e$ to denote the number of vertices and the number of edges of $G$, respectively. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. A bipartite graph $G$ is a graph whose vertex set $V(G)$ can be partitioned into two subsets $V_{1}$ and $V_{2}$ such that the edge set of such a graph consists of precisely those edges which join vertices in $V_{1}$ to vertices of $V_{2}$. In particular, if $E(G)$ consists of all possible such edges, then $G$ is called the complete bipartite graph and denoted by the symbol $K_{r,s}$, where $|V_{1}| = r$ and $|V_{2}| = s$.

For a vertex $x \in V(G)$, $\deg(x)$ is the degree of vertex $x$, $\delta(G) = \min\{\deg(x) : x \in V(G)\}$, $\Delta(G) = \max\{\deg(x) : x \in V(G)\}$. For a nonnegative integer $d$, a graph is called $d$-regular if every vertex has degree $d$. Let $S \subset V(G)$ be any subset of vertices of $G$. Then the induced subgraph $G[S]$ is the graph whose vertex set is $S$ and whose edge set consists of all of the edges in $E(G)$ that have both endpoints in $S$. A spanning subgraph for $G$ is a subgraph of $G$ which contains every vertex of $G$. A graph without any cycle is called acyclic graph. A forest is an acyclic graph. Let $G_{1}$ and $G_{2}$ be subgraphs of $G$, we say that $G_{1}$ and $G_{2}$ are disjoint if they have no vertex and no edge in common. The union of two disjoint graphs $G_{1}$ and $G_{2}$, which is denoted by $G_{1} \cup G_{2}$ is a graph with $V(G_{1} \cup G_{2}) = V(G_{1}) \cup V(G_{2})$ and $E(G_{1} \cup G_{2}) = E(G_{1}) \cup E(G_{2})$. For any graph $G$, the disjoint union of $k$ copies of $G$ is denoted by $kG$. Graphs $G$ and $H$ are said to be isomorphic to one another, written $G \cong H$, if there exists a one-to-one correspondence $f : V(G) \rightarrow V(H)$ such that for each pair $x, y$ of vertices of $G$, $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$. Also, for a rational number $p$, $[p]$ is the first integer number greater than or equal to $p$, and $[p]$ is the first integer number less than or equal to $p$.

2. Basic properties

First of all, let us recall some of the basic facts about total graphs and vertex arboricity, which we shall use in the rest of the paper.

**Lemma 2.1** ([7, Lemma 1.1]). Let $x$ be a vertex of $T(\Gamma(R))$. Then the following statements are true.

(i) If $2 \in Z(R)$, then $\deg(x) = |Z(R)| - 1$.

(ii) If $2 \notin Z(R)$, then $\deg(x) = |Z(R)| - 1$ for every $x \in Z(R)$ and $\deg(x) = |Z(R)|$ for every vertex $x \notin Z(R)$.

**Remark 2.2.** It is clear that $va(G) = 1$ if and only if $G$ is acyclic. For a few classes of graphs, the vertex-arboricity is easily determined. For example, $va(C_{n}) = 2$, where $C_{n}$ is a cycle graph with $n$ vertices. If $n$ is even, $va(K_{n}) = \frac{n}{2}$; while if $n$ is odd, $va(K_{n}) = \frac{n+1}{2}$. So, in general, $va(K_{n}) = \lceil \frac{n}{2} \rceil$. Also, $va(K_{r,s}) = 1$ if $r = 1$ or $s = 1$, and $va(K_{r,s}) = 2$ otherwise.
Lemma 2.3 ([3, Lemma 1]). Let $G$ be the disjoint union of graphs $G_1, G_2, \ldots, G_k$. Then, for all $i$ with $1 \leq i \leq k$,

$$\text{va}(G) = \max \text{ va}(G_i).$$

Now, we are ready to show that for a positive integer $v$, there are only finitely many finite rings whose total graph has vertex-arboricity $v$.

Theorem 2.4. For any positive integer $v$, the number of finite rings whose total graphs have vertex-arboricity $v$ is finite.

Proof. Let $R$ be a finite ring. We want to obtain a complete subgraph (with vertex set $T$) of $T(\Gamma(R))$. To achieve this, we consider the following two cases:

(a) $R$ is local. In this case $Z(R)$ is the maximal ideal of $R$ and $|R| \leq |Z(R)|^2$ [8]. In this situation, we put $T = Z(R)$.

(b) $R$ is not local. Then there is a natural number $n \geq 2$ and there are local rings $R_1, R_2, \ldots, R_n$ such that $R = R_1 \times R_2 \times \cdots \times R_n$. We may assume that $|R_1| \leq |R_2| \leq \cdots \leq |R_n|$. Now put $R_i' = 0 \times R_2 \times \cdots \times R_n$. Since $|R| = |R_1||R_1'|$, we have $|R| \leq |R_1'|^2$. In this situation, we put $T = R_1'$.

Now, it is easy to see that, for every elements $x$ and $y$ of $T$, $x$ is adjacent to $y$ in $T(\Gamma(R))$. Thus there is an induced subgraph $K_{|T|}$ in $T(\Gamma(R))$. Hence Remark 2.2 implies that $\text{va}(K_{|T|}) \leq v$, and so $\lceil \frac{|T|}{v} \rceil \leq v$. Thus $|R| \leq 4v^2$, and so the proof is complete. \hfill \Box

Let $\text{Reg}(\Gamma(R))$ be the induced subgraph of $T(\Gamma(R))$ with vertices $\text{Reg}(R) = R - Z(R)$, and $Z(\Gamma(R))$ be the induced subgraph of $T(\Gamma(R))$ with vertices $Z(R)$. Next, we record some facts concerning total graphs. If $Z(R)$ is an ideal of $R$, then $Z(\Gamma(R))$ is a complete subgraph of $T(\Gamma(R))$ and is disjoint from $\text{Reg}(\Gamma(R))$. Thus, the following theorem of D.F. Anderson and A. Badawi gives a complete description of $T(\Gamma(R))$.

Theorem 2.5 ([1, Theorem 2.2]). Let $R$ be a commutative ring such that $Z(R)$ is an ideal of $R$, and let $|Z(R)| = n$ and $\frac{R}{Z(R)} = m$. Then the following statements hold.

(i) If $2 \in Z(R)$, then $\text{Reg}(\Gamma(R))$ is the union of $m - 1$ disjoint $K_n$'s.

(ii) If $2 \notin Z(R)$, then $\text{Reg}(\Gamma(R))$ is the union of $\frac{m - 1}{2}$ disjoint $K_{n,n}$'s.

Theorem 2.6. Let $R$ be a finite commutative ring with identity and $I$ be a nontrivial ideal contained in $Z(R)$. Set $|I| = n$ and $\frac{I}{Z(R)} = m$. Then the following statements hold.

(i) If $2 \in I$, then $\text{va}(T(\Gamma(R))) \geq \lceil \frac{n}{2} \rceil$.

(ii) If $2 \notin I$, then $\text{va}(T(\Gamma(R))) \geq \max\{\lceil \frac{n}{2} \rceil, 2\}$.

Proof. Let $G$ be the spanning subgraph of $T(\Gamma(R))$ such that, for every two vertices $x, y \in R$, $x$ is adjacent to $y$ in $G$ if $x + y \in I$. Now, since $I$ is an ideal of $R$ contained in $Z(R)$, by making obvious modification to the proof of Theorem 2.5, one can show that

$$G = \left\{ \begin{array}{ll}
    mK_n & \text{if } 2 \in I \\
    K_n \cup (\frac{m-1}{2})K_{n,n} & \text{if } 2 \notin I.
\end{array} \right.$$ 

Now, by Remark 2.2 in conjunction with Lemma 2.3, we have the following equalities

$$\text{va}(G) = \left\{ \begin{array}{ll}
    \lceil \frac{n}{2} \rceil & \text{if } 2 \in I \\
    \max\{\lceil \frac{n}{2} \rceil, 2\} & \text{if } 2 \notin I.
\end{array} \right.$$ 

Now, since $G$ is a subgraph of $T(\Gamma(R))$, we have that $\text{va}(G) \leq \text{va}(T(\Gamma(R)))$, and so the proof is complete. \hfill \Box

The following corollary is immediate from Theorem 2.5.

Corollary 2.7. Let $R$ be a finite commutative ring with identity, $Z(R)$ be nontrivial ideal of $R$ and set $|Z(R)| = n$ and $\frac{R}{Z(R)} = m$. Then the following statements hold.

(i) If $2 \in Z(R)$, then $\text{va}(T(\Gamma(R))) = \lceil \frac{n}{2} \rceil$.

(ii) If $2 \notin Z(R)$, then $\text{va}(T(\Gamma(R))) = \max\{\lceil \frac{n}{2} \rceil, 2\}$. 

Let \( \text{gr} \) denote the girth of a graph. Theorem 3.1 [2, Theorem 4.7] states that for any commutative ring \( R \), the girth of its zero-divisor graph \( \text{gr}(\Gamma(R)) \) is at most 3. Moreover, this girth is \( 2k \) if and only if \( R \) is a \( k \)-fold extension of a field. Theorem 3.2 [10, Theorem 2] further characterizes rings with girth 4 based on their structure. The vertex-arboricity of the total graph \( (\Gamma(R)) \) is a topic of interest, as explored in Remark 3.3 and subsequent results. The authors classify finite commutative rings whose total graphs have vertex-arboricity two or three. The proof of these results involves detailed analysis of the properties of commutative rings and their associated graphs.
Let $3.3$ implies that $6$. If $3.2$ then $3.1$. If $2.2$ and $6$. Theorem $2$, these rings have vertex-arboricity $8$.

Remark two non-isomorphic rings of order $9$.

Theorem $3.6$. Let $R$ be a finite commutative ring such that $va(T(\Gamma(R))) = 2$. Then the following statements hold.

(i) If $R$ is local, then $R$ is isomorphic to one of the following rings:

- $\mathbb{Z}_9$, $\mathbb{Z}_8$, $\mathbb{Z}_6$, $\mathbb{Z}_4$, $\mathbb{Z}_2$, $\mathbb{Z}$
- $\mathbb{Z}_9[x]$, $\mathbb{Z}_8[x]$, $\mathbb{Z}_6[x]$, $\mathbb{Z}_4[x]$, $\mathbb{Z}_2[x]$, $\mathbb{Z}$
- $\mathbb{Z}_9[x, y]$, $\mathbb{Z}_8[x, y]$, $\mathbb{Z}_6[x, y]$, $\mathbb{Z}_4[x, y]$, $\mathbb{Z}_2[x, y]$, $\mathbb{Z}$
- $\mathbb{Z}_9[x, y, z]$, $\mathbb{Z}_8[x, y, z]$, $\mathbb{Z}_6[x, y, z]$, $\mathbb{Z}_4[x, y, z]$, $\mathbb{Z}_2[x, y, z]$, $\mathbb{Z}$
- $\mathbb{Z}_9[x, y, z, w]$, $\mathbb{Z}_8[x, y, z, w]$, $\mathbb{Z}_6[x, y, z, w]$, $\mathbb{Z}_4[x, y, z, w]$, $\mathbb{Z}_2[x, y, z, w]$, $\mathbb{Z}$

(ii) If $R$ is not local, then $R$ is isomorphic to one of the following rings:

- $\mathbb{F}_2 \times \mathbb{F}_2$, $\mathbb{F}_2 \times \mathbb{F}_4$, $\mathbb{F}_2 \times \mathbb{F}_4$, $\mathbb{F}_2 \times \mathbb{F}_4$.

Proof. (i) Assume that $R$ is a local ring, and let $|Z(R)| = n$ and $|\mathbb{Z}[\bar{x}]| = m$. Then by Theorem $2.5$, $T(\Gamma(R))$ has an induced subgraph isomorphic to $K_n$ and so by Remark $2.2$, $|Z(R)| \leq 4$. Now, we consider the following two cases:

(a) If $2 \in Z(R)$, then by Theorem $3.2$, $|R| = 2^k$ and $k \leq 4$. Since $va(T(\Gamma(R))) = 2$, Theorem $3.1$ implies that $|R| = 16, 8$. According to Corbas and Williams [6] there are two non-isomorphic rings of order $16$ with maximal ideals of order $4$, namely $\mathbb{Z}[x]/(x^2)$ and $\mathbb{Z}[x]/(x^2 + x + 1)$ (see also Redmond [11]), so for these rings have $T(\Gamma(R)) \cong K_4$. Therefore, by Remark $2.2$, these rings have vertex-arboricity $2$. In [6] it is also shown that there are $5$ local rings of order $8$ (except $\mathbb{F}_8$) as follows:

- $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_2[x, y]/(x, y)\mathbb{Z}_2$, $\mathbb{Z}_4[x]/(x^2)$.

In all of these rings we have $|Z(R)| = 4$ and hence $T(\Gamma(R)) \cong K_4$. Then, by Remark $2.2$, these rings have vertex-arboricity $2$.

(b) If $2 \notin Z(R)$, then $|Z(R)| = 3$. According to [6], there are two rings of order $9$ namely, $\mathbb{Z}_9$ and $\mathbb{Z}_4[x]/(x^2)$. For these rings, we have $T(\Gamma(R)) \cong K_3 \cup K_3, 3$. Hence, by Corollary $2.7$, these rings have vertex-arboricity $2$.

(ii) Suppose that $R$ is not local. Since $R$ is finite, there are finite local rings $R_1, \ldots, R_t$ (with $t \geq 2$) such that $R = R_1 \times R_2 \times \cdots \times R_t$. Now, according to Remarks $2.2$ and $3.3$, $x, y \in \mathbb{F}_4$. Hence, each of the sets $V_1$ and $V_2$ has exactly two vertices such that their first components are the same and have exactly two vertices such that the second components are the same. So, each vertex in $V_1$ and $V_2$ has degree $2$, which is a contradiction, since the subgraphs of $T(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))$ induced by the sets $V_1$ and $V_2$ are union of cycles. Thus we have $va(T(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))) > 2$.

Now, according to the Figure 1, we have $va(T(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))) = 3$. 

\[ \begin{array}{c}
\text{(a)} & \text{(b)} & \text{(c)} \\
(0, 0) & \text{(0, a)} & \text{(0, a)} \\
(a^2, 0, a) & (1, a) & (1, a^2) \\
(0, a^2) & \text{(1, a)} & \text{(1, a^2)} \\
\end{array} \]

Figure 1
we have the following candidates:
\[ Z_2 \times Z_2, Z_6, Z_2 \times Z_4, Z_2 \times \frac{Z_2[x]}{[x^2]}, Z_2 \times F_4, Z_3 \times Z_3, Z_3 \times Z_4, Z_3 \times \frac{Z_2[x]}{[x^2]}, Z_3 \times F_4, \\
Z_2 \times Z_2 \times Z_2, Z_4 \times Z_4, Z_4 \times \frac{Z_2[x]}{[x^2]}, Z_4 \times \frac{Z_2[x]}{[x^2]}, Z_4 \times F_4, Z_2 \times \frac{Z_2[x]}{[x^2]}, F_4, F_4 \times F_4. \]

Now we examine each of the above rings.

The total graph of the ring \( Z_2 \times Z_2 \) is isomorphic to the cycle of size 4. We consider the acyclic partition \( V_1 = \{(0,0),(1,0)\} \) and \( V_2 = \{(0,1),(1,1)\} \) of \( V(T(\Gamma(Z_2 \times Z_2))) \).

Hence, the subgraphs of \( T(\Gamma(Z_2 \times Z_2)) \) induced by sets \( V_1 \) and \( V_2 \) are acyclic. Thus \( va(T(\Gamma(Z_2 \times Z_2))) = 2 \).

For \( Z_6 \), by considering the acyclic partition \( V_1 = \{0,1,3\} \) and \( V_2 = \{2,4,6\} \) of \( V(T(\Gamma(Z_6))) \), we have \( va(T(\Gamma(Z_6))) = 2 \).

For \( Z_2 \times Z_4 \), we put \( V_1 = \{(0,0),(0,2),(1,1),(1,3)\} \) and \( V_2 = \{(0,1),(0,3),(1,0),(1,2)\} \). Now, it is easy to see that \( va(T(\Gamma(Z_2 \times Z_4))) = 2 \). Since \( T(\Gamma(Z_4)) \cong T(\Gamma(\frac{Z_2[x]}{[x^2]})) \), by Remark 3.4, we have \( T(\Gamma(Z_2 \times Z_4)) \cong T(\Gamma(Z_2 \times Z_2)) \). Thus \( va(T(\Gamma(Z_2 \times Z_2))) = 2 \).

For \( Z_2 \times F_4 \), by using the acyclic partition
\[ V_1 = \{(0,0),(0,1),(1,0),(1,a)\} \text{ and } V_2 = \{(0,a),(0,a^2),(1,1),(1,a^2)\} \]
of \( V(T(\Gamma(Z_2 \times F_4))) \), we have \( va(T(\Gamma(Z_2 \times F_4))) = 2 \).

For \( Z_3 \times Z_3 \), we consider the acyclic partition \( V_1 = \{(0,0),(0,1),(1,0),(1,1),(2,1)\} \) and \( V_2 = \{(0,2),(2,0),(1,2),(2,2)\} \) of \( V(T(\Gamma(Z_3 \times Z_3))) \). Hence \( va(T(\Gamma(Z_3 \times Z_3))) = 2 \).

For \( Z_3 \times Z_4 \), the graph \( T(\Gamma(Z_3 \times Z_4)) \) has a complete graph \( K_6 \) as a subgraph with vertex set \( \{(0,0),(1,0),(2,0),(0,2),(1,2),(2,2)\} \), and so, by Remark 2.2, we have \( va(T(\Gamma(Z_3 \times Z_4))) > 2 \). Also by Remark 3.4, we have \( T(\Gamma(Z_3 \times Z_4)) \cong T(\Gamma(Z_3 \times \frac{Z_2[x]}{[x^2]}) \). Thus \( va(T(\Gamma(Z_3 \times Z_4))) > 2 \).

For \( Z_3 \times F_4 \), according to the Figure 2 we have \( va(T(\Gamma(Z_3 \times F_4))) = 2 \).

For \( Z_2 \times Z_2 \times Z_2 \), by Lemma 3.5, we have \( va(T(\Gamma(Z_2 \times Z_2 \times Z_2))) > 2 \).

For \( Z_4 \times Z_4 \), the graph \( T(\Gamma(Z_4 \times Z_4)) \) has a \( K_8 \) as a subgraph with vertex set \( \{(0,0),(1,0),(2,0),(3,0),(0,2),(1,2),(2,2),(3,2)\}, \) and so, by Remark 2.2, we have \( va(T(\Gamma(Z_4 \times Z_4))) > 3 \).

According to Remark 3.4, \( T(\Gamma(Z_4 \times Z_4)) \cong T(\Gamma(Z_4 \times \frac{Z_2[x]}{[x^2]}) \cong T(\Gamma(\frac{Z_2[x]}{[x^2]} \times \frac{Z_2[x]}{[x^2]})). \) So the vertex-arboricity of graphs \( T(\Gamma(Z_4 \times \frac{Z_2[x]}{[x^2]})) \) and \( T(\Gamma(\frac{Z_2[x]}{[x^2]} \times \frac{Z_2[x]}{[x^2]})) \) is greater than three.

For \( Z_4 \times F_4 \), the graph \( T(\Gamma(Z_4 \times F_4)) \) has a \( K_8 \) as a subgraph with vertex set \( \{(0,0),(0,1),(0,a),(0,a^2),(2,0),(2,1),(2,a),(2,a^2)\}. \)
and so, by Remark 2.2, we have \( \text{va}(T(\Gamma(Z_4 \times F_4))) > 3 \). Also by Remark 3.4, \( T(\Gamma(Z_4 \times F_4)) \cong T(\Gamma(\frac{Z_2[z]}{\langle z^2 \rangle} \times F_4)) \). Therefore \( \text{va}(T(\Gamma(\frac{Z_2[z]}{\langle z^2 \rangle} \times F_4))) > 3 \).

For \( F_4 \times F_4 \), by Lemma 3.5, we have \( \text{va}(T(\Gamma(F_4 \times F_4))) > 2 \).

**Lemma 3.7.** For the ring \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \), \( \text{va}(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))) = 4 \).

**Proof.** First, by Remark 3.3, we have \( \text{va}(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))) > 2 \).

Now, let \( T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = G \) and \( A = A_0 \cup A_1 \), where \( A_0 = \{(0,0,z) : z \in \mathbb{Z}_3\} \) and \( A_1 = \{(0,1,z) : z \in \mathbb{Z}_3\} \). Also put \( B = B_0 \cup B_1 \), where \( B_0 = \{(1,0,z) : z \in \mathbb{Z}_3\} \) and \( B_1 = \{(1,1,z) : z \in \mathbb{Z}_3\} \). It is clear that the two sets \( A \) and \( B \) are partition for \( V(G) \). Let \( \{V_1, V_2, V_3\} \) be an acyclic partition for \( V(G) \). If \( |V_i| \geq 5 \) for some \( j \in \{1,2,3\} \), then \( |A \cap V_j| \geq 3 \) or \( |B \cap V_j| \geq 3 \), which is impossible, since \( G[A] \) and \( G[B] \) are complete graphs isomorphic to \( K_6 \) and \( G[V_i] \) \( (1 \leq i \leq 3) \) are acyclic induced subgraphs of \( G \). Therefore \( |V_i| = 4 \) for some \( i \in \{1,2,3\} \).

We know that every vertex of \( G[A_0] \) \( (G[A_1]) \) are adjacent to every vertex of \( G[B_0] \) \( (G[B_1]) \) and \( G[V_i] \) \( (1 \leq i \leq 3) \) are acyclic induced subgraphs of \( G \). Hence without the loss of generality we can assume that \( |A_0 \cap V_1| = |B_1 \cap V_1| = 2 \) and \( |A_1 \cap V_2| = |B_0 \cap V_2| = 2 \). Then \( V_3 = \{a_0, a_1, b_0, b_1 : a_0 \in A_2, b_1 \in B_1, 0 \leq s, t \leq 1\} \). It follows that \( G[V_3] \) is a cycle of length 4, which is a contradiction and so \( \text{va}(G) > 3 \).

Now, by using the following partition of \( V(G) \), we have that \( \text{va}(G) = 4 \).

\[ V_1 = \{(0,0,0), (1,0,0), (1,1,2)\}, \quad V_2 = \{(0,1,0), (1,1,1), (1,0,1)\}, \]
\[ V_3 = \{(0,1,2), (0,0,2), (1,0,2)\}, \quad V_4 = \{(0,0,1), (0,1,1), (1,1,0)\}. \]

**Theorem 3.8.** Let \( R \) be a finite commutative ring such that \( \text{va}(T(\Gamma(R))) = 3 \). Then the following statements hold.

(i) If \( R \) is local, then \( R \) is isomorphic to \( \mathbb{Z}_{25} \) or \( \frac{\mathbb{Z}[x]}{(x^5)} \).

(ii) If \( R \) is not local, then \( R \) is isomorphic to one of the following rings:

\[ \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \frac{\mathbb{Z}[x]}{(x^5)}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{F}_4 \times \mathbb{F}_4, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_5, \mathbb{F}_4 \times \mathbb{Z}_5, \mathbb{Z}_5 \times \mathbb{Z}_5. \]

**Proof.** (i) Assume that \( R \) is a local ring. We consider the following two cases:

(a) If \( 2 \in Z(R) \), then, by Theorem 2.5, we have \( T(\Gamma(R)) \cong mK_n \). Hence, by Remark 2.2, \( 5 \leq |Z(R)| \leq 6 \). But, in this situation \( 2 \in Z(R) \), and so, there are no such local rings.

(b) If \( 2 \notin Z(R) \), then, by Theorem 2.5, we have \( T(\Gamma(R)) \cong K_n \cup \{\frac{z_1}{x}K_{n,n}\} \). Hence, by Remark 2.2, \( 5 \leq |Z(R)| \leq 6 \). Therefore \( |Z(R)| = 5 \) and so there exist two local rings, \( \mathbb{Z}_{25} \) and \( \frac{\mathbb{Z}[x]}{(x^5)} \) of order 25. For these rings we have \( T(\Gamma(R)) \cong K_5 \cup 2K_{5,5} \). Hence, by Corollary 2.7, we have \( \text{va}(T(\Gamma(R))) = 3 \).

(ii) Suppose that \( R \) is not a local ring. Arguments similar to those used in proof of Theorem 3.6 (ii), in conjunction with Remarks 2.2 and 3.3 show that we have the following candidates:

\[ \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_6 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \frac{\mathbb{Z}[x]}{(x^2)}, \mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \frac{\mathbb{Z}[x]}{(x^2)}, \mathbb{Z}_3 \times \mathbb{F}_4, \]
\[ \mathbb{Z}_2 \times \frac{\mathbb{Z}[x]}{(x^2)}, \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \frac{\mathbb{Z}[x]}{(x^2)}, \mathbb{Z}_4 \times \mathbb{F}_4, \]
\[ \mathbb{Z}_2 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_5, \mathbb{Z}_4 \times \mathbb{Z}_5, \mathbb{Z}_5 \times \frac{\mathbb{Z}[x]}{(x^2)}, \mathbb{Z}_5 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{F}_4. \]

According to the proof of Theorem 3.6 (ii), we examine the following cases:

For \( \mathbb{Z}_3 \times \mathbb{Z}_4 \), we consider the partition

\[ V_1 = \{(0,0), (1,1), (1,2), (1,3)\}, \quad V_2 = \{(0,2), (2,0), (2,1), (2,3)\} \]

and

\[ V_3 = \{(0,1), (0,3), (1,0), (2,2)\}. \]
of $V(T(\Gamma(Z_3 \times Z_4)))$. The subgraphs of $T(\Gamma(Z_3 \times Z_4))$ induced by the sets $V_1$, $V_2$ and $V_3$ are acyclic graphs. Hence, we have $va(T(\Gamma(Z_3 \times Z_4))) = 3$. The Remark 3.4 implies that $T(\Gamma(Z_3 \times Z_4)) \cong T(\Gamma(Z_3 \times \frac{Z_2[|x|]}{(x^2)})$ and so $va(T(\Gamma(Z_3 \times \frac{Z_2[|x|]}{(x^2)}))) = 3$.

For rings $Z_2 \times Z_2 \times Z_2$ and $F_4 \times F_4$, by Lemma 3.5, we have $va(T(\Gamma(Z_2 \times Z_2 \times Z_2))) = va(T(\Gamma(F_4 \times F_4))) = 3$.

For rings $Z_2 \times Z_2 \times Z_3$, by Lemma 3.7, we have $va(T(\Gamma(Z_2 \times Z_2 \times Z_3))) > 3$.

For $Z_3 \times Z_5$, by using the acyclic partition

$$V_1 = \{(0, 4), (1, 0), (1, 3), (2, 3)\},$$

$$V_2 = \{(0, 0), (0, 1), (1, 2), (1, 4), (2, 1)\}$$

and

$$V_3 = \{(0, 2), (0, 3), (1, 1), (2, 0), (2, 2), (2, 4)\}$$
of $V(T(\Gamma(Z_3 \times Z_5)))$, we have $va(T(\Gamma(Z_3 \times Z_5))) = 3$.

For $Z_4 \times Z_5$, the graph $T(\Gamma(Z_4 \times Z_5))$ has a complete graph $K_{10}$ as a subgraph with vertex set $\{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (2, 0), (2, 1), (2, 2), (2, 3), (2, 4)\}$, and so, we have $va(T(\Gamma(Z_4 \times Z_5))) \geq 5$. Also, Remark 3.4, $T(\Gamma(Z_4 \times Z_5)) \cong T(\Gamma(\frac{Z_2[|x|]}{(x^2)} \times Z_5))$ and so

$$va(T(\Gamma(\frac{Z_2[|x|]}{(x^2)} \times Z_5))) \geq 5.$$

For $F_4 \times Z_5$, according to Figure 3, we have $va(T(\Gamma(F_4 \times Z_5))) = 3$.

For $Z_5 \times Z_5$, by Figure 4, we conclude that $va(T(\Gamma(Z_5 \times Z_5))) = 3$.

Thus the proof is complete. \hfill $\square$

4. The arboricity of the total graph

In this section, we characterize all finite commutative rings whose total graph has arboricity two or three. In addition, we show that, for a positive integer $v$, there are only finitely many finite rings whose total graph has arboricity $v$. We begin the section with the following result of C. St. J. A. Nash-Williams.

**Theorem 4.1**\ ([9]). For a graph $G$, $\nu(G) = \max \left\lfloor \frac{e_H}{n_H - 1} \right\rfloor$, where $n_H = |V(H)|$, $e_H = |E(H)|$ and $H$ ranges over all non-trivial induced subgraphs of $G$. 

**Figure 3**

For $Z_5 \times Z_5$, by Figure 4, we conclude that $va(T(\Gamma(Z_5 \times Z_5))) = 3$.
Theorem 4.2. For a graph $G$, $\lceil \frac{\delta(G)+1}{2} \rceil \leq \nu(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$. In particular, if $G$ is $d$-regular, then $\nu(G) = \lceil \frac{d+1}{2} \rceil = \lceil \frac{n-1}{2} \rceil$, where $n = |V(G)|$ and $e = |E(G)|$.

Proof. First, it is clear that, if $G$ has some isolated vertices, say $X = \{x_1, x_2, \ldots, x_k\}$, then $\nu(G) = \nu(G[V(G) \setminus X])$. So, we can assume that $G$ has no isolated vertices. Let $H$ be a subgraph of $G$ with $|V(H)| = n'$ and $|E(H)| = e'$. Then we have

$$\frac{e'}{n'-1} \leq \Delta(H)n' \leq \frac{\Delta(H)+1}{2},$$

Since $\Delta(H) \leq \min\{\Delta(G), n' - 1\}$, we have $\frac{e'-1}{n'-1} \leq \frac{\Delta(G)+1}{2}$, and hence, by Theorem 4.1, $\nu(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$. On the other hand $\frac{e}{n-1} \geq \frac{\delta(G)n}{2(n-1)} > \frac{\delta(G)}{2}$. Since $\nu(G)$ is an integer, $\nu(G) \geq \lceil \frac{\delta(G)+1}{2} \rceil$, as required.

Clearly, in view of the above theorem, $\nu(K_n) = \lceil \frac{n}{2} \rceil$. So, by arguing as in the proof of Theorem 2.4, we have the following theorem.

Theorem 4.3. For any positive integer $v$, the number of finite rings $R$ whose total graph has arboricity $v$ is finite.

Theorem 3.1 implies that $T(\Gamma(R))$ has arboricity one if and only if either $R$ is an integral domain or $R$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. Now, we will classify, up to isomorphism, all the finite commutative rings whose total graph has arboricity two or three.

Theorem 4.4. Let $R$ be a finite ring such that $\nu(T(\Gamma(R))) = 2$. Then the following statements hold.

(i) If $R$ is local, then $R$ is isomorphic to one of the following rings:

$$\mathbb{Z}_9, \mathbb{Z}[x]/(x^2), \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_2[x, y]/(x^2, y^2), \mathbb{Z}_4[x]/(x^2), \mathbb{F}_4[x]/(x^2), \mathbb{Z}_4[x]/(x^2 + x + 1)$$

(ii) If $R$ is not local, then $R$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_6$.

Proof. (i) Assume that $R$ is a local ring. If $2 \in Z(R)$, then, by Lemma 2.1 and Theorem 4.2, we have $|Z(R)| = 4$. Then by Theorem 3.2, $|R| = 16, 8$. Now, by same argument of
Theorem 3.6. Let $R$ be a finite ring. If
\[ Z_8, \frac{Z_4[x]}{(x^3)}, \frac{Z_4[x]}{(x^2 + 2)}, \frac{Z_4[x]}{(x,y)^2}, \frac{F_4[x]}{(x^2)}, \frac{Z_4[x]}{(x^2 + x + 1)}. \]
If $2 \notin Z(R)$, then $|Z(R)| = 3$. So, $R$ is isomorphic to $Z_9$ or $\frac{Z_2[x]}{(x^2)}$.

(ii) If $R$ is not a local ring, then, by Theorem 4.2, we have $3 \leq |Z(R)| \leq 4$. When $|Z(R)| = 3$, it is clear that $R$ is isomorphic to $Z_2 \times Z_2$. Moreover, if $|Z(R)| = 4$, then $R$ is isomorphic to $Z_6$, and so the proof is complete.

By slight modifications in the proof of Theorem 4.4, one can prove the following theorem.

Theorem 4.5. Let $R$ be a finite ring such that $\nu(T(\Gamma(R))) = 3$. Then the following statements hold.

(i) If $R$ is local, then $R$ is isomorphic to $Z_{25}$ or $\frac{Z_5[x]}{(x^2)}$.

(ii) If $R$ is not local, then $R$ is isomorphic to one of the following rings:
\[ Z_2 \times F_4, Z_3 \times Z_3, Z_2 \times Z_4, Z_2 \times \frac{Z_2[x]}{(x^2)}, Z_2 \times Z_5, Z_3 \times F_4. \]

In general, we can determine the arboricity of the total graph as in the following theorem.

Theorem 4.6. Let $R$ be a finite ring.

(i) If $2 \in Z(R)$, then $\nu(T(\Gamma(R))) = \left\lfloor \frac{|Z(R)|}{2} \right\rfloor$.

(ii) If $2 \notin Z(R)$, then the following statements hold.

(1) If $|Z(R)| = 2k + 1$, then $\nu(T(\Gamma(R))) = k + 1$.

(2) If $|Z(R)| = 2k$, then $k \leq \nu(T(\Gamma(R))) \leq k + 1$.

Proof. It follows from Lemma 2.1 and Theorem 4.2. \qed

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