Singularity-free orthogonally-transitive cylindrical spacetimes

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In this talk a previous theorem on geodesic completeness of diagonal cylindrical spacetimes will be generalized to cope with the nondiagonal case. A sufficient condition for such spacetimes to be causally geodesically complete will be given.

1 Introduction

This talk will deal to a certain extent with the natural continuation of some work in progress that was introduced last year at the Spanish Relativity Meeting in Salamanca and that has recently been published.

The reason for investigating geodesic completeness of Lorentzian manifolds is twofold. From the mathematical point of view, there is no analogue for the Hopf-Rinow theorem that characterizes geodesic completeness of Riemannian manifolds in terms of metric completeness. Since the Lorentzian metric does not determine a metric structure, just a causal structure, this possibility is hindered.

From the physical point of view, the existence of inhomogeneous cosmological models that are causally geodesically complete and, therefore, singularity-free, and whose role in Cosmology is yet to be determined, induces us to try to characterize when such behaviour arises.

Of course, there are well-known results, the singularity theorems due to Hawking, Penrose, Tipler . . . , that provide sufficient conditions for a Lorentzian manifold to be incomplete. Furthermore, they discern whether there is a Big Bang, Big Crunch or geodesic imprisonment singularity.

On the contrary, sufficient conditions for a spacetime to be causally geodesically complete are not so easily come across in the literature.

Since such nonsingular cosmological models have arisen within the framework of inhomogeneous cosmologies, I shall devote my talk to orthogonally-transitive $G_2$ cylindrical spacetimes.

2 Geodesic equations for an Abelian orthogonally transitive $G_2$ metric.

I shall introduce the metric in isothermal coordinates $t, r$. The other coordinates are cyclic and therefore do not appear explicitly in the expression,

$$ds^2 = e^{2g(t,r)} \left\{-dt^2 + dr^2\right\} + \rho^2(t,r)e^{2f(t,r)}d\phi^2 + e^{-2f(t,r)} \{dz + A(r,t) d\phi\}^2.$$

The coordinates have the usual range and we shall assume from the beginning that every metric function is $C^2$.

For the cylindrical symmetry, I shall take the usual definition. The axis is located where the norm, $\Delta$,

$$\Delta = g(\xi, \xi) = \rho^2(t,r)e^{2f(t,r)} + e^{-2f(t,r)} A^2(r,t),$$

of the axial Killing field vanishes and we shall require the following behaviour for it in the neighbourhood of the axis,
\[
\lim_{r \to 0} \frac{g(\text{grad} \Delta, \text{grad} \Delta)}{4 \Delta} = 1, \tag{3}
\]
assuming that it is located on the locus \( r = 0 \), where it can be set by performing a suitable coordinate transformation.

Now all we have to do is to check the behaviour of the geodesic equations. They form a system of four quasilinear second-order ordinary differential equations plus an additional first-order quadratic differential equation that determines the geodesic parametrization up to an affinity,

\[
\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0,
\]

\[
g(\dot{x}, \dot{x}) = \begin{cases} 
-1 & \text{if } \delta = -1, 0, 1 \\
0 & \text{for spacelike, timelike geodesics} \\
1, & \text{for lightlike geodesics} 
\end{cases} \tag{4}
\]

The existence of isometries leads to integration of two equations, thereby reducing the order of the system. We have two constants of geodesic motion,

\[
L = e^{2f(t, r)} \rho^2(t, r) \dot{\phi} + e^{-2f(t, r)} A(t, r) \left\{ \dot{z} + A(t, r) \dot{\phi} \right\}, \tag{5}
\]

\[
P = e^{-2f(t, r)} \left\{ \dot{z} + A(t, r) \dot{\phi} \right\}, \tag{6}
\]
respectively the angular momentum around the axis and the linear momentum along the axis.

There is another constant, \( \delta \), that takes the value \(-1, 0, 1\) for respectively spacelike, lightlike and timelike geodesics,

\[
\delta = e^{2g(t, r)} \left\{ t^2 - \dot{r}^2 \right\} - \left( L - P A(t, r) \right)^2 \rho^{-2}(t, r) e^{-2f(t, r)} - P^2 e^{-2f(t, r)}. \tag{7}
\]

As a matter of convenience, we shall denote by \( \Lambda \) the ‘effective’ angular momentum,

\[
\Lambda(t, r) = L - P A(t, r). \tag{8}
\]

The other equations can be written in a compact form as,

\[
\{e^{2g(t, r)} \dot{t}\} - e^{-2g(t, r)} F(t, r) F_t(t, r) = 0, \tag{9}
\]

\[
\{e^{2g(t, r)} \dot{r}\} + e^{-2g(t, r)} F(t, r) F_r(t, r) = 0, \tag{10}
\]
\[
F(t, r) = e^{g(t, r)} \sqrt[3]{\delta + P^2 e^{2f(t, r)} + \Lambda^2(t, r) \frac{e^{-2f(t, r)}}{\rho^2(t, r)}}, \tag{11}
\]
that suggests a parametrization and reduction of the differential system to a first order one,
\[
\begin{align*}
\dot{t} &= \pm e^{-2g(t,r)} F(t,r) \cosh \xi(t,r), \\
\dot{r} &= e^{-2g(t,r)} F(t,r) \sinh \xi(t,r), \\
\dot{\xi}(t,r) &= -e^{-2g(t,r)} \left\{ \pm F_t(t,r) \sinh \xi(t,r) + F_r(t,r) \cosh \xi(t,r) \right\},
\end{align*}
\]

where the upper sign is meant for future-pointing geodesics and the lower one, for past-pointing geodesics.

The philosophy of the work is just preventing the coordinates \( t, r \) from tending to infinity, and thus leaving the spacetime, at a finite value of the affine parameter.

The results are quoted in the next section. Proofs can be found in \( ^3 \).

### 3 Results

The results can be summarized in two theorems, one for future-pointing and one for past-pointing geodesics,

**Theorem 1:** An orthogonally transitive cylindrical spacetime endowed with a metric whose local expression in terms of \( C^2 \) metric functions \( f, g, A, \rho \) is given by (1) such that the axis is located at \( r = 0 \) has complete future causal geodesics if the following set of conditions is fulfilled:

1. For large values of \( t \) and increasing \( r \),
   
   \[
   \begin{align*}
   (a) & \quad \begin{cases} g_u \geq 0 \\
   h_u \geq 0 \\
   q_u \geq 0, \end{cases} \\
   (b) & \quad \begin{cases} g_r \geq 0 \text{ or } |g_r| \lesssim g_u \\
   h_r \geq 0 \text{ or } |h_r| \lesssim h_u \\
   q_r \geq 0 \text{ or } |q_r| \lesssim q_u. \end{cases}
   \end{align*}
   \]

2. For \( L \neq 0 \) and large values of \( t \) and decreasing \( r \),
   
   \[
   \begin{align*}
   (a) & \quad \delta g_v + P^2 e^{2f} q_v + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_v \geq 0 \\
   (b) & \quad \begin{cases} \delta g_r + P^2 e^{2f} q_r + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_r \leq 0 \text{ or } \delta g_r + P^2 e^{2f} q_r + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_r \lesssim \delta g_v + P^2 e^{2f} q_v + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_v. \end{cases}
   \end{align*}
   \]

3. For large values of the time coordinate \( t \), constants \( a, b \) exist such that,
   
   \[
   \begin{align*}
   \left\{ \begin{array}{c} 2 g(t,r) \\
   g(t,r) + f(t,r) + \ln \rho - \ln |A| \\
   g(t,r) - f(t,r) \end{array} \right\} \geq - \ln |t + a| + b.
   \end{align*}
   \]

4. The limit \( \lim_{r \to 0} \frac{A}{\rho} \) exists.
**Theorem 2:** An orthogonally transitive cylindrical spacetime endowed with a metric whose local expression in terms of $C^2$ metric functions $f, g, A, \rho$ is given by (1) such that the axis is located at $r = 0$ has complete past causal geodesics if the following set of conditions is fulfilled:

1. For small values of $t$ and increasing $r$,
   
   (a) \[
   \begin{align*}
   g_v & \leq 0 \\
   h_v & \leq 0 \\
   q_v & \leq 0,
   \end{align*}
   \]
   
   (b) Either \[
   \begin{align*}
   g_r & \geq 0 \quad \text{or} \quad |g_r| \leq -g_v \\
   h_r & \geq 0 \quad \text{or} \quad |h_r| \leq -h_v \\
   q_r & \geq 0 \quad \text{or} \quad |q_r| \leq -q_v.
   \end{align*}
   \]

2. For $L \neq 0$ and small values of $t$ and decreasing $r$,
   
   (a) \[
   \delta g_u + P^2 e^{2f} q_u + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_u \leq 0
   \]
   
   (b) Either \[
   \delta g_r + P^2 e^{2f} q_r + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_r \leq 0 \quad \text{or} \quad \delta g_r + P^2 e^{2f} q_r + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_r \leq \delta g_u + P^2 e^{2f} q_u + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_u.
   \]

3. For small values of the time coordinate $t$, constants $a, b$ exist such that,
   
   \[
   \begin{align*}
   2g(t, r) + f(t, r) + \ln \rho - \ln |\Lambda| \\
   g(t, r) - f(t, r)
   \end{align*}
   \] \[
   \geq -\ln |t + a| + b.
   \]

4. The limit \[\lim_{r \to 0} \frac{A}{\rho}\] exists.

The coordinates $u, v$ are respectively the usual outgoing and ingoing radial null coordinates. Attention needs be paid to the last condition of the theorems, that just states the fact that the geometry in the vicinity of the axis cannot be determined by the function in the metric related to non-diagonality, $A$. This is the main novelty in the non-diagonal case.

The results for past-pointing geodesics are obtained from the ones for future-pointing geodesics just exchanging some signs and the coordinates $u, v$.

The conditions of the theorems look rather lengthy, but are not difficult to check.

### 4 Nonsingular cosmological models

In order to check whether these theorems are too restrictive sufficient conditions, we shall apply them to all non-diagonal cylindrical perfect fluid cosmological models that are known to be singularity-free in the literature. The diagonal ones were already checked \cite{2}.

1. **Mars**: It is the first known nonsingular non-diagonal cylindrical cosmological model in the literature.

   \[
   \begin{align*}
   g(t, r) & = \frac{1}{2} \ln \cosh(2a t) + \frac{1}{2} \alpha a^2 r^2, \\
   f(t, r) & = \frac{1}{2} \ln \cosh(2a t), \\
   \rho(t, r) & = r, \\
   A(t, r) & = ar^2,
   \end{align*}
   \] (15)
where $a$ is a constant and $\alpha > 1$. If $\alpha = 1$ the pressure and the density of the fluid vanish.

2. Griffiths-Bičak$^8$: The previous model is comprised in this one for $c = 0$ after a redefinition of constants. The metric functions can be written as,

$$
\begin{align*}
g(t, r) &= \frac{1}{2} \ln \cosh(2 at) + \frac{1}{2} a^2 r^2 + \frac{1}{2} \Omega(t, r), \\
f(t, r) &= \frac{1}{2} \ln \cosh(2 at), \\
\rho(t, r) &= r, \\
A(t, r) &= a r^2,
\end{align*}
$$

(16)

where $\Omega$ is a function that is obtained from a solution, $\sigma$, of the wave equation,

$$
\begin{align*}
\Omega_r &= r (\sigma_r^2 + \sigma_t^2), \\
\Omega_t &= 2r \sigma_t \sigma_r, \\
\sigma(t, r) &= bt + \sqrt{2} c \sqrt{\frac{\sqrt{(a^2 + r^2 - t^2)^2 + 4a^2 r^2 + a^2 + r^2 - t^2}}{(a^2 + r^2 - t^2)^2 + 4a^2 r^2}}.
\end{align*}
$$

(17)

Both of them are stiff perfect fluids and can be checked to fulfil the conditions of the theorems.

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References

1. L. Fernández-Jambrina in Proceedings of the 1998 Spanish Relativity Meeting, (World Scientific, Singapore, 1999).
2. L. Fernández-Jambrina, L. M. González-Romero, Class. Quantum Grav. 16, 953 (1999) [arXiv:gr-qc/9812039].
3. L. Fernández-Jambrina, Journ. Math. Phys. 40, 4028 (1999) [arXiv: gr-qc/9906030].
4. S. W. Hawking, G. F. R. Ellis, The Large Scale Structure of Space-time, (Cambridge University Press, Cambridge, 1973).
5. J. Beem, P. Ehrlich, K. Easley, Global Lorentzian Geometry, (Dekker, New York, 1996).
6. A. Romero, M. Sánchez, Geom. Dedicata 53, 103, (1994).
7. M. Mars, Phys. Rev D51, R3989, (1995).
8. J. B. Griffiths, J. Bičak, Class. Quantum Grav. 12, L81, (1995).