Conformal Riemannian morphisms between Riemannian manifolds

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Abstract

In this article we introduce conformal Riemannian morphisms. The idea of conformal Riemannian morphism generalizes the notions of isometric immersions, Riemannian submersions and Riemannian maps.

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1. Introduction

A generalization of the notions of isometric immersion and Riemannian submersion between Riemannian manifolds was introduced by Arthur E. Fischer in 1992 and was named as Riemannian map [1]. The theory of isometric immersions originated from Gauss’s studies on surfaces in Euclidean spaces. On the other hand, the study of Riemannian submersions between Riemannian manifolds was initiated by O’Neil [4] and Gray [2]. Later on, Sahin gave a generalization of Riemannian maps and defined Conformal Riemannian maps [6]. The immersions and submersions are generalized and unified by subimmersions. Fischer introduced Riemannian map as a Riemannian analog of subimmersion.

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In this paper first we introduce conformal Riemannian morphisms and give some examples. Next, we give some necessary and sufficient conditions for a smooth function between Riemannian manifolds to be conformal Riemannian morphism, Theorem 3.2

Also, we prove that a conformal Riemannian morphism on a compact Riemannian manifold is subimmersion (Definition 4.1). Thus conformal Riemannian morphism can be seen as yet another Riemannian analog of the notion of subimmersion. Next, we prove that subimmersions are equivalent to generalized conformal maps (Definition 4.2) if domain has sufficiently large dimension, Theorem 4.4. Finally, we give local structure of conformal Riemannian morphisms, Theorem 4.5.

Throughout this paper $M$ and $N$ denote Riemannian manifolds with Riemannian metrics $g_M$ and $g_N$, respectively. From [1], [6] and [5] we recall following definitions.

**Definition 1.1.** A smooth function $f : M \to N$ is called **conformal Riemannian map** if there exist a linear subspace $H_x \subset T_x M$, a number $\wedge_f(x) \in \mathbb{R}$ and $\text{rank}(d_x f) \neq 0$, for each $x \in M$ such that

1. $H_x = \ker(df_x)^\perp$, for each $x \in M$,
2. $g_N(f(x))(df_x X, df_x Y) = \wedge_f(x) g_M(x)(X, Y)$, for every $X, Y \in H_x$, for each $x \in M$.

From definition, it is clear that $\wedge_f(x) > 0$ and the function $\wedge_f : M \to \mathbb{R}^+$ is smooth. Thus a smooth function $f : M \to N$ with $\text{df}_x > 0$, $\forall \ x \in M$ is **conformal Riemannian map** if there exist a linear subspace $H_x \subset T_x M$, for each $x \in M$ and a smooth function $\wedge_f : M \to \mathbb{R}^+$ such that

1. $H_x = \ker(df_x)^\perp$, for each $x \in M$,
2. $g_N(f(x))(df_x X, df_x Y) = \wedge_f(x) g_M(x)(X, Y)$, for every $X, Y \in H_x$, for each $x \in M$.

If $\wedge_f(x) = 1$, for all $x \in M$, then a conformal Riemannian map is called a **Riemannian map**. If $\ker(df_x) = \{0\}$, for each $x \in M$, then a conformal Riemannian map is called a **conformal immersion**. If $\wedge_f(x) = 1$, for each $x \in M$, then a conformal immersion $f$ is said to be **isometric immersion**. If $df_x$ is surjective, for each $x$ in $M$, then a conformal Riemannian map is called a **horizontally conformal submersion**.
If $df_x$ is surjective, for each $x$ in $M$, then a Riemannian map is called a Riemannian submersion. $f$ is called an isometry if it is isometric immersion as well as Riemannian submersion.

2. Geometric functions and Conformal Riemannian morphisms

**Definition 2.1.** Let $V$ and $W$ be inner–product spaces over the field of real numbers and $f : V \to W$ be a linear transformation. Denote the kernel and range of $f$ by $K$ and $R(f)$, respectively. We say that $f$ is a geometric function if there exists a subspace $C$ of $V$ such that $V = K \oplus C$, and $f|_C : C \to R(f)$ is conformal, that is, for some $r > 0$, $\langle f(u), f(v) \rangle = r\langle u, v \rangle$, for every $u, v \in C$. We call $C$ a Conf subspace corresponding to $f$ and denote it by $\text{Conf}(f)$. We call $r$ conformity factor associated to $f$. We denote the collection of all geometric functions from an inner product space $V$ to another inner product space $W$ by $\text{Geom}(V, W)$.

We now give two general results on the existence of geometric functions.

**Proposition 2.1.** Let $V, W$ be inner product spaces over $\mathbb{R}$ and $T : V \to W$ be a linear map of rank $k$. If $\dim V \geq 2k - 1$, then $T$ is geometric.

**Proof.** Let $\dim V = n \geq 2k - 1$ and $T^*$ be adjoint of $T$. Since $T^*T$ is self adjoint, there exists an orthonormal basis $\{u_i\}_{i=1}^n$ of $V$ such that $T^*T u_i = \lambda_i u_i$ for $\lambda_i \in \mathbb{R}$, for all $i = 1, 2, \ldots, k$ and $T^*T u_{k+1} = \cdots = T^*T u_n = 0$. Since $0 \leq \langle Tu_i, Tu_i \rangle = \langle u_i, T^*Tu_i \rangle = \langle u_i, \lambda_i u_i \rangle = \lambda_i \|u_i\|^2$ and $\text{rank}(T) = k$, $\lambda_i > 0$, for all $1 \leq i \leq k$.

We can reorder $\{u_i\}_{i=1}^n$ in such a way that $0 < \lambda_1 < \lambda_i$, for all $2 \leq i \leq k$. Now define $\alpha_i = \sqrt{\frac{\lambda_1}{\lambda_i}}$, $E_1 = u_1$, $E_i = \alpha_i u_i + \sqrt{1 - \alpha_i^2}u_{k+1-i}$, for all $2 \leq i \leq k$. Clearly $\{E_i\}_{i=1}^k$ is orthonormal subset of $V$ and $\|TE_i\| = \lambda_i$, for all $1 \leq i \leq k$. So, $T$ is geometric with $\text{conf}(T) = \text{span}\{E_i\}_{i=1}^k$ and conformity factor $\sqrt{\lambda_i}$. \hfill $\square$

**Proposition 2.2.** Let $V, W$ be inner product spaces over $\mathbb{R}$, $T : V \to W$ be a linear map of rank $k$ and $\dim V \geq 2k$. Then $T$ is geometric. Moreover, there exist uncountably many $\text{conf}(T)$ subspaces such that conformity factors associated to different $\text{conf}(T)$ subspaces are different.
Proof. Let $\dim V = n \geq 2k$ and $T^*$ be adjoint of $T$. As in the proof of Proposition 2.1, there exists an orthonormal basis \{u_i\}_1^n of $V$ such that $T^*T u_i = \lambda_i u_i$ for $0 < \lambda_i \in \mathbb{R}$, for all $i = 1, 2, \cdots, k$ and $T^*T u_{k+1} = \cdots = T^*T u_n = 0$. We can reorder \{u_i\}_1^n in such a way that $0 < \lambda_1 < \lambda_i$, for all $2 \leq i \leq k$. Choose $0 < \alpha < \lambda_1$. Now define

$$E_i = \alpha_i u_i + \sqrt{1-\alpha_i^2}u_{k+i},$$
for all $1 \leq i \leq k$. Clearly \{E_i\}_1^k is orthonormal subset of $V$ and $\|TE_i\| = \alpha$, for all $1 \leq i \leq k$. So, $T$ is geometric with $\text{conf}(T) = \text{span}\{E_i\}_1^k$ and conformality factor $\sqrt{\alpha}$. Clearly for different values of $\alpha$ in the interval $(0, \lambda_1)$, $\text{conf}(T)$ is different. This completes the proof. \qed

Next we define conformal Riemannian morphisms and conformal Riemannian submersions.

Definition 2.2. A smooth function $f : M \to N$ is called \textbf{conformal Riemannian morphism} if there exist a linear subspace $H_x \subset T_x M$, for each $x \in M$ and a smooth function $\wedge f : M \to \mathbb{R}^+$ such that

1. $H_x \oplus \ker(df_x) = T_x M$, for each $x \in M$; and
2. $g_N(f(x))(df_x X, df_x Y) = \wedge f(x)g_M(x)(X, Y)$, for every $X, Y \in H_x$, for each $x \in M$.

Thus a smooth function $f : M \to N$ is called \textbf{conformal Riemannian morphism} if there exists a smooth function $\wedge f : M \to \mathbb{R}^+$ such that $df_x$ is a geometric function with conformality factor $\wedge f(x)$, for all $x \in M$. The function $\wedge f$ is called \textbf{conformality factor} associated to $f$. If $\wedge f(x) = 1$, for each $x \in M$, we call $f$ a \textbf{Riemannian morphism}.

Definition 2.3. A smooth function $f : M \to N$ is called a \textbf{conformal Riemannian submersion} if there exist a linear subspace $H_x \subset T_x M$, for each $x \in M$ and a smooth function $\wedge f : M \to \mathbb{R}^+$ such that

1. $df_x$ is surjective, for each $x$ in $M$;
2. $H_x \oplus \ker(df_x) = T_x M$, for each $x \in M$; and
3. \( g_N(f(x))(df_x X, df_x Y) = \wedge f(x)g_M(x)(X, Y) \), for every \( X, Y \in H_x \), for each \( x \in M \).

Clearly, every conformal Riemannian submersion is a conformal Riemannian morphism.

Next, we give some examples of conformal Riemannian morphisms. We let \( \mathbb{R}^n \) denote the Euclidean n-space taken with its standard flat metric.

**Examples 2.1.** 1. If \( F : M \rightarrow N \) is a conformal Riemannian map, then \( F \) is a conformal Riemannian morphism with \( \ker(df_x) \perp H_x = \text{Conf}(df_x), \forall x \in M_1 \).

2. Define \( f : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) by \( f(x_1, x_2, x_3, x_4) = (e^{x_3}(x_1 - x_2), 0, 0, e^{x_3}(x_4 - x_2)) \), for all \( (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \). Then \( f \) is a smooth function and for any \( a \in \mathbb{R}^4 \), we have,

\[
\begin{pmatrix}
  e^{a_3} & -e^{a_3} & e^{a_3}(a_1 - a_2) & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & -e^{a_3} & e^{a_3}(a_4 - a_2) & e^{a_3}
\end{pmatrix}.
\]

Note that \( df_a(e_1) = e^{a_3}e_1, df_a(e_4) = e^{a_3}e_4, df_a(e_1 + e_4 + e_2) = 0, \) \( df_a(a_2 - a_1, 0, 1, a_2 - a_4) = 0 \). Let \( K = \text{span}\{(1, 1, 0, 1), (a_2 - a_1, 0, 1, a_2 - a_4)\} \) and \( T = \text{span}\{e_1, e_4\} \). Then \( (df_a)|_K = 0 \) and \( f \) is a conformal Riemannian morphism with \( \text{Conf}(df_a) = T \) and \( \wedge f(a) = e^{2a_3} \), for all \( a \in \mathbb{R}^4 \). Note that \( \text{Conf}(df_a) \) and \( \ker(df_a) \) are not orthogonal in this example. So \( f \) is not a conformal Riemannian map.

3. From Proposition 3.1 it is clear that every smooth function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is a conformal Riemannian morphism if and only if it is a constant function or has nowhere zero gradient.

4. Every smooth curve \( c : \mathbb{R} \rightarrow (M, g_M) \) is a conformal Riemannian morphism if and only if it is a constant curve or \( c'(t) \neq 0 \), for all \( t \in \mathbb{R} \). Note that if \( c \) is a curve which is a conformal Riemannian morphism, then \( \wedge_c(t) = g_M(t)(c'(t), c'(t)) \).
5. A linear isomorphism \( A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a conformal Riemannian morphism if and only if \( A \) is a scalar multiple of an orthogonal transformation.

6. A conformal Riemannian morphism need not be a harmonic map, and a harmonic map need not be conformal Riemannian morphism. For example, the curve \( c : \mathbb{R}^+ \rightarrow \mathbb{R}^2 \) given by \( c(t) = (t^3, t^6) \) is conformal Riemannian morphism but not harmonic. Also, \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) defined by \( f(x, y) = (2x, 3y) \) is harmonic map but not conformal Riemannian morphism.

7. From [3], we have following characterization of harmonic morphisms: a smooth map \( \phi : M \rightarrow N \) between Riemannian manifolds is a harmonic morphism if and only if it is both harmonic and a (horizontally) conformal submersion away from points where \( d\phi = 0 \). So we conclude that a harmonic morphism is conformal Riemannian morphism away from points where \( d\phi = 0 \) but converse need not be true because a conformal Riemannian morphism may not be a harmonic map.

8. Composition of conformal Riemannian morphisms need not be a conformal Riemannian morphism. For example, \( f : \mathbb{R} \rightarrow \mathbb{R}^2 \) given by \( f(x) = (x^2 + x, x^2) \), and \( g : \mathbb{R}^2 \rightarrow \mathbb{R} \) given by \( g(x, y) = x \) are conformal Riemannian morphisms but \( g \circ f(x) = x^2 + x \) is not conformal Riemannian morphism, from Example 4 because \( (g \circ f)'(-1/2) = 0 \) and \( g \circ f \) is not constant.

3. Necessary and sufficient conditions for a map to be conformal Riemannian morphism

In this section we give some necessary and sufficient conditions for a map between Riemannian manifolds to be conformal Riemannian morphism.

**Proposition 3.1.** Let \( f : M \rightarrow \mathbb{R} \) be a smooth function on a connected Riemannian manifold \( M \). Then \( f \) is a conformal Riemannian morphism if and only if \( f \) is constant or it has nowhere zero gradient.

**Proof.** If \( f \) is constant, then it is easy to see that \( f \) is a conformal Riemannian morphism, take \( H_x = 0 \), \( \forall x \in M \). Let \( \text{grad}(f) \) denote the gradient of \( f \) which is a vector field defined by \( df_x(X) = g_M(x)(\text{grad}(f)(x), X) \), for each \( X \in T_xM \),
and \( K_x = \{\text{span}(\text{grad}(f))\}^\perp \), for all \( x \in M \). We have, \( df_x(\lambda \text{grad}(f)(x)) = g_M(x)(\text{grad}(f)(x), \lambda \text{grad}(f)(x)) = \lambda \|\text{grad}(f)(x)\|^2 \) and \( df_x|K_x \equiv 0 \). Now
\[
\|\text{grad}(f)(x)\|^2 g_M(x)(\lambda_1 \text{grad}(f)(x), \lambda_2 \text{grad}(f)(x))
\]
\[
= \lambda_1 \lambda_2 \|\text{grad}(f)(x)\|^4
\]
\[
= \lambda_1 \|\text{grad}(f)(x)\|^2 \lambda_2 \|\text{grad}(f)(x)\|^2
\]
\[
= df_x(\lambda_1 \text{grad}(f)(x)).df_x(\lambda_2 \text{grad}(f)(x))
\]

From above discussion, we conclude that \( f \) is a conformal Riemannian morphism with \( \text{Conf}(df_x) = \text{span}(\text{grad}(f)(x)), \wedge f(x) = \|\text{grad}(f)(x)\|^2 \iff f \text{ is constant or } \text{grad}(f)(x) \neq 0, \forall x \in M. \]

**Corollary 3.1.** Let \( f : M \to \mathbb{R} \) be a smooth function. If \( \|\text{grad}(f)(x)\| = 1 \), for each \( x \in M \), then \( f \) is a Riemannian map.

**Proof.** From the proof of the Proposition 3.1, we have \( \text{Conf}(df_x) = \text{span}(\text{grad}(f)(x)) \) and \( \text{Ker}(df_x) = \{\text{span}(\text{grad}(f))\}^\perp \), \( \wedge f(x) = \|\text{grad}(f)(x)\|^2 = 1 \). So \( f \) is a Riemannian map. \( \square \)

**Definition 3.1.** Let \( f : M \to N \) be a smooth function. Let \( H_x \subset T_x M \) be a linear subspace, for each \( x \in M \) such that \( H_x \oplus \text{ker}(df_x) = T_x M \).

- Define, \( \forall x \in M \), a linear function \( (df_{H_x})^\circ : T_f(x)N \to T_x M \) by
\[
(df_{H_x})^\circ(X) = \begin{cases} 
((df_x)|H_x)^*)(X), & \text{if } X \in \text{range}(df_x) \\
0, & \text{if } X \in \text{range}(df_x)^\perp.
\end{cases}
\]

Here \( ((df_x)|H_x)^* \) is adjoint of \( ((df_x)|H_x) : H_x \to \text{range}(df_x). \)

- Define, \( \forall x \in M \), \( P_{H_x} : T_x M \to T_x M \) by \( P_{H_x}(x) = (df_{H_x})^\circ \circ (df_x). \)

- Define, \( \forall x \in M \), \( Q_{H_x} : T_f(x)N \to T_f(x)N \) by \( Q_{H_x}(x) = (df_x) \circ (df_{H_x})^\circ. \)

**Theorem 3.1.** Let \( f : M \to N \) be a smooth function. Then \( f \) is conformal Riemannian morphism if and only if there exist a smooth function \( \wedge f : M \to \mathbb{R}^+ \) and a subspace \( H_x \subset T_x M \) with \( T_x M = H_x \oplus \text{ker} df_x \) such that \( P_{H_x} \circ P_{H_x} = \wedge f(x) P_{H_x} \), for all \( x \in M. \)
Proof. $f$ is conformal Riemannian morphism if and only if there exists a smooth function $\wedge_f : M \to \mathbb{R}^+$ and a subspace $H_x \subset T_x M$ with $T_x M = H_x \oplus \ker d f_x$ such that $\wedge_f (x) g_M (x) (X_1, Y_1) = g_N (f(x)) (d f_x X_1, d f_x Y_1)$, for all $X_1, Y_1 \in H_x$. Since, for all $X_1 \in H_x, Y_1 \in T_x M$, $g_N (f(x)) (d f_x X_1, d f_x Y_1) = g_M (x) (X_1, (df_x)_x H_x)^* \circ d f_x Y_1 = g_M (x, (df_x)_x H_x)^* \circ d f_x Y_1$, $f$ is conformal Riemannian morphism if and only if there exists a smooth function $\wedge_f : M \to \mathbb{R}^+$ and a subspace $H_x \subset T_x M$ with $T_x M = H_x \oplus \ker d f_x$ such that $\wedge_f (x) g_M (x) (X_1, (df_x)_x H_x)^* \circ d f_x Y_1 = g_N (f(x)) (d f_x X_1, d f_x \circ (df_x)_x H_x)^* \circ d f_x Y_1$, (1) for all $X_1 \in H_x, Y_1 \in T_x M$. Equation (1) is equivalent to $\wedge_f (x) g_M (X_1, (df_x)_x H_x)^* \circ d f_x Y_1 = g_M (x) (X_1, (df_x)_x H_x)^* \circ d f_x Y_1$, (2) for all $X_1 \in H_x, Y_1 \in T_x M$. Since $P_{H_x} = (df_x)_x H_x)^* \circ (df_x)_x H_x$, Equation (2) is equivalent to $\wedge_f (x) g_M (x, X_1, P_{H_x} Y_1) = g_M (x, P_{H_x} \circ P_{H_x} Y_1)$, (3) for all $X_1 \in H_x, Y_1 \in T_x M$. Since $\text{range} (P_{H_x}) = H_x$, Equation (3) is equivalent to $P_{H_x} \circ P_{H_x} = \langle x \rangle P_{H_x}$. Hence we conclude the result.

\[\square\]

**Theorem 3.2.** Let $f : M \to N$ be a smooth function. Then $f$ is conformal Riemannian morphism if and only if there exist a smooth function $\wedge_f : M \to \mathbb{R}^+$ and a subspace $H_x \subset T_x M$ with $T_x M = H_x \oplus \ker d f_x$ such that $Q_{H_x} \circ Q_{H_x} = \wedge_f (x) Q_{H_x}$ for all $x \in M$.

Proof. $f$ is conformal Riemannian morphism if and only if there exists a smooth function $\wedge_f : M \to \mathbb{R}^+$ and a subspace $H_x \subset T_x M$ with $T_x M = H_x \oplus \ker d f_x$ such that $\wedge_f (x) g_M (x) (X_1, Y_1) = g_N (f(x)) (d f_x X_1, d f_x Y_1)$, for all $X_1, Y_1 \in H_x$. Since $\text{range} ((df_x)_x H_x)^* = H_x$, $\wedge_f (x) g_M (x) (X_1, Y_1) = g_N (f(x)) (d f_x X_1, d f_x Y_1)$, for all $X_1, Y_1 \in H_x$, if and only if, for all $X_2 \in \text{range} (d f_x)$, $Y_2 \in T_{f(x)} N$

\[\wedge_f (x) g_M (x) ((df_x)_x H_x)^* X_2, (df_x)_x H_x)^* Y_2 = g_N (f(x)) (d f_x \circ (df_x)_x H_x)^* X_2, (df_x)_x H_x)^* Y_2.\] (4) Equation (4) is equivalent to $\wedge_f (x) g_N (f(x)) (X_2, (df_x)_x H_x)^* Y_2 = g_M (x) ((df_x)_x H_x)^* X_2, ((df_x)_x H_x)^* Y_2, (df_x)_x H_x)^* Y_2.\] (5)
for all \( X_2 \in \text{range}(df_x), Y_2 \in T_{f(x)}N \). Equation \( 5 \) is equivalent to

\[
\land_f (x) g_N(f(x))(X_2, df_x \circ (df_{H_x})^* Y_2) = g_N(f(x))(X_2, (df_{H_x})^* df_x \circ (df_{H_x})^* Y_2),
\]

for all \( X_2 \in \text{range}(df_x), Y_2 \in T_{f(x)}N \). Equation \( 6 \) is equivalent to

\[
\land_f (x) g_N(f(x))(X_2, df_x \circ (df_{H_x})^* Y_2) = g_N(f(x))(X_2, df_x \circ (df_{H_x})^* df_x \circ (df_{H_x})^* Y_2),
\]

for all \( X_2 \in \text{range}(df_x), Y_2 \in T_{f(x)}N \).

We have \( Q_{H_x} = (df_x) \circ (df_{H_x})^* \). So, Equation \( 7 \) holds for all \( X_2 \in \text{range}(df_x), Y_2 \in T_{f(x)}N \), if and only if \( Q_{H_x} \circ Q_{H_x} = \land(x)Q_{H_x} \), for each \( x \in M \). Hence we conclude the result.

\[\square\]

**Proposition 3.2.** Let \( f : M \to N \) be a smooth function. If \( H_x = (\ker(df_x))^\perp \), then \((df_{H_x})^\circ = (df_x)^*\).

**Proof.** Suppose that \( H_x = (\ker(df_x))^\perp \). Then, for all \( X \in T_xM \) and \( Y \in T_{f(x)}N \), we have \( X = X_1 + X_2 \) and \( Y = Y_1 + Y_2 \) for unique \( X_1 \in H_x, X_2 \in (\ker(df_x)) \), \( Y_1 \in \text{range}(df_x) \) and \( Y_2 \in (\text{range}(df_x))^\perp \). We have,

\[
g_N(f(x))((df_x)(X), Y) = g_M(X, (df_x)^* (Y)),
\]

for all \( X \in T_xM, Y \in T_{f(x)}N \), implies that \( \text{range}((df_x)^*) = (\ker(df_x))^\perp \) and \( \text{range}(df_x) = (\ker((df_x)^*))^\perp \), for all \( x \in M \). So, we have

\[
g_N(f(x))((df_x)(X), Y) = g_M(X, (df_x)^* (Y)),
\]

for all \( X \in T_xM, Y \in T_{f(x)}N \), if and only if

\[
g_N(f(x))((df_x)(X_1), Y_1) = g_M(X_1, (df_x)^* (Y_1)),
\]

for all \( X_1 \in H_x, Y_1 \in \text{range}(df_x) \). So, \((df_{H_x})^\circ |_{\text{range}(df_x)} = ((df_x)^*) |_{\text{range}(df_x)} \). Since \((df_{H_x})^\circ |_{\text{range}(df_x))^\perp = 0 \), \((df_x)^* |_{\text{range}(df_x))^\perp = 0 \), this implies that \((df_{H_x})^\circ = (df_x)^*\).

\[\square\]

**Remark 3.1.**

1. From Proposition 3.2 we conclude that \((df_{H_x})^\circ \) is a generalization of \((df_x)^*\).
2. From Theorem 3.1, $f$ is Riemannian morphism if and only if there exists a subspace $H_x \subset T_xM$ with $T_xM = H_x \oplus \ker df_x$ such that $P_{H_x}$ is a projection operator, that is, $P_{H_x} \circ P_{H_x} = P_{H_x}$.

3. From Theorem 3.1, $f$ is Riemannian morphism if and only if there exists a subspace $H_x \subset T_xM$ with $T_xM = H_x \oplus \ker df_x$ such that $Q_{H_x}$ is a projection operator, that is, $Q_{H_x} \circ Q_{H_x} = Q_{H_x}$.

4. Subimmersions and generalized-conformal maps

**Definition 4.1.** A smooth function $f : M \to N$ is a subimmersion at $x \in M$, if there is an open set $U$ containing $x$, a manifold $P$, a submersion $S : U \to P$, and an immersion $J : P \to N$ such that $f|U = J \circ S$. A smooth function $f : M \to N$ is a subimmersion if it is a subimmersion at each $x \in M$.

Now we state a proposition from [1] which relates subimmersions and maps of locally constant rank.

**Proposition 4.1.** A smooth function $f : M \to N$ is a subimmersion if and only if the rank function $x \mapsto \text{rank}(df_x)$ (equivalently, if and only if the nullity function $x \mapsto \dim(\ker(df_x))$) is locally constant, and hence constant on the connected components of $M$.

**Proof.** If $f$ is a subimmersion, then it is locally composition of a submersion and an immersion, and hence has locally constant rank. Conversely, if it has locally constant then, by using rank theorem, we conclude that it is a subimmersion. Local constancy of nullity follows from rank nullity theorem. □

We state following theorem about subimmersion [9], [8].

**Theorem 4.1.** Let $f : M \to N$ be a subimmersion.

1. Then for every $y \in f(M) \subset N$, $f^{-1}(y)$ is a smooth closed submanifold of $M$, with $T_x(f^{-1}(y)) = \ker(df_x)$, for every $x \in f^{-1}(y)$.

2. For every $x \in M$, there exists a neighborhood $U$ of $x$ such that $f(U)$ is a submanifold of $N$ with $T_{f(x)}(f(U)) = \text{range}(df_x)$. If, in addition, $f$ is open or closed onto its image, then $f(M)$ is a submanifold of $N$. 

3. If $df_x : T_x M \to T_{f(x)} N$ is not surjective, then $U$ in (2) above may be chosen so that $f(U)$ is nowhere dense in $N$.

**Theorem 4.2.** Let $f : M \to N$ be a subimmersion. Then

1. If $f$ is injective, then $f$ is an immersion, and hence an injective immersion.

2. If $f$ is surjective and $M$ is connected, then $f$ is a submersion, and hence a surjective submersion.

3. More generally, if $M$ is connected and $f$ is open or closed onto its image, then $f(M)$ is a submanifold of $N$, and the range restricted map $f : M \to f(M)$ is a surjective submersion onto $f(M)$.

4. If $f$ is bijective and $M$ is connected, then $f$ is a diffeomorphism.

We have following lemma from the thesis of first author [10].

**Lemma 4.1.** Let $m$ and $n$ be two natural numbers. Further, let $M(m \times n, \mathbb{R})$ denote the inner-product space of all $m \times n$ matrices with real entries. Let $r$ be any integer. Define

$O_r = \{ A | A \in M(m \times n, \mathbb{R}), \text{ rank}(A) \geq r \}$

$C_r = \{ A | A \in M(m \times n, \mathbb{R}), \text{ rank}(A) \leq r \}$

Then, for each $r$, $O_r$ and $C_r$ are respectively open and closed in $M(m \times n, \mathbb{R})$.

**Proof.** Since $C_r = O_{r+1}$, it suffices to show that $O_r$ is open in $M(m \times n, \mathbb{R})$ for each integer $r$. We dispose off the special cases when $r \leq 0$ or $r > \min\{m, n\}$. If $r \leq 0$, then $O_r = M(m \times n, \mathbb{R})$ which is open in $M(m \times n, \mathbb{R})$. On the other hand if $r > \min\{m, n\}$, then $O_r$ is the empty set which too is open in $M(m \times n, \mathbb{R})$.

Next, suppose $0 < r \leq \min\{m, n\}$. In any $m \times n$ matrix the number of minors of size $r$ is given by $\binom{m}{r} \binom{n}{r}$. Denote this number by $k$. Let $p_i : M(m \times n, \mathbb{R}) \to M(r \times r, \mathbb{R})$ for $i \in \{1, 2, \ldots, k\}$ be the projection maps corresponding to these $k$ minors. Let $\delta : M(r \times r, \mathbb{R}) \to \mathbb{R}$ be the determinant function. Since $p_i$ and $\delta$ are continuous functions, the composition $\delta \circ p_i$ is a continuous function from $M(m \times n, \mathbb{R})$ to $\mathbb{R}$ for each $i \in \{1, 2, \ldots, k\}$. Consequently, for each $i \in \{1, 2, \ldots, k\}$, the set $U_i = (\delta \circ p_i)^{-1}(\mathbb{R} \setminus \{0\})$ is open. Now, every matrix in $M(m \times n, \mathbb{R})$ of rank at least
Let the dimension of each \(X\) and \(Y\) be any integer. Define \(T: V \rightarrow W\) be a linear transformation. Suppose that \(\forall k\), \(r_k\) is a conformality factor of \(T_k\) and 0 < \(p \leq r_k\), for some real number \(p\). Then, terms of the sequence \(\{T_k\}\) eventually have the same rank as \(T\).

**Proof.** Let the dimension of \(V\) be \(n\) and rank\((T)\) be \(r\). For each natural number \(k\), let \(X_k\) and \(Y_k\) denote, respectively, the kernel of \(T_k\) and Conf subspace corresponding to \(T_k\).

Since \(T_k \rightarrow T\), there exists a natural number \(K_1\) such that for any natural number \(k > K_1\), we have \(\|T_k - T\| < \frac{\sqrt{p}}{2}\). We claim that for each natural number \(k > K_1\), the intersection \(\ker(T) \cap Y_k = \{0\}\). To prove this, it suffices to show that if \(\alpha \in \ker(T)\) with \(\|\alpha\| = 1\), then \(\alpha \not\in Y_k\), \(\forall k > K_1\). For \(\alpha \in \ker(T)\) with \(\|\alpha\| = 1\), we have \(\forall k > K_1\), \(\|T_k(\alpha)\| = \|T_k(\alpha) - T(\alpha)\| \leq \|T_k - T\| \|\alpha\| < \frac{\sqrt{p}}{2}\). Also, \(\alpha \in Y_k\) implies that \(\|T_k(\alpha)\| \geq \sqrt{p}\). So, we conclude that \(\alpha \not\in Y_k\), \(\forall k > K_1\).

Fix a natural number \(k > K_1\) and suppose that \(\{u_i\}_{i=1}^{n-r}\) is a basis of \(\ker(T)\). Since each \(T_k\) is a geometric function, we have \(V = X_k \oplus Y_k\). Thus, we can find \(v_i \in X_k\) and \(w_i \in Y_k\) such that \(u_i = v_i + w_i\) for all \(i \in \{1, 2, \ldots, n - r\}\). We claim that \(\{v_i\}_{i=1}^{n-r}\) is linearly independent. Suppose that for some scalars \(\beta_i\), we have \(\sum_{i=1}^{n-r} \beta_i v_i = 0\). Since each \(v_i = u_i - w_i\), we get \(\sum_{i=1}^{n-r} \beta_i u_i = \sum_{i=1}^{n-r} \beta_i w_i\) which implies that \(\sum_{i=1}^{n-r} \beta_i u_i = 0\) which by linear independence implies that \(\beta_i = 0\) for all \(i \in \{1, 2, \ldots, n - r\}\).
Conclude that \( \{v_i\}_{i=1}^{n-r} \) is linearly independent and hence \( \text{rank}(T_k) \leq \text{rank}(T) \). Since, from Theorem 4.3 \( \{S : \text{rank}(T) \leq \text{rank}(S)\} \) is open, there exists \( \epsilon > 0 \) such that \( B_\epsilon(T) \subset \{S : \text{rank}(T) \leq \text{rank}(S)\} \). Since \( \{T_n\} \) is eventually in \( B_\epsilon(T) \), for some natural number \( K_2, k > K_2 \Rightarrow \text{rank}(T_k) \geq \text{rank}(T) \). Hence there exists a natural number \( K = \max\{K_1, K_2\} \) such that for all \( k > K \), we have \( \text{rank}(T_k) = \text{rank}(T) \).

**Proposition 4.2.** Let \( f : M \to N \) be a conformal Riemannian morphism. If the conformality factor \( \lambda_f \) is bounded below by a positive number, then \( f \) is a subimmersion. In particular, if \( M \) is compact, then \( f \) is a subimmersion.

**Proof.** Let the dimensions of \( M \) and \( N \) be \( m \) and \( n \) respectively and \( \lambda_f : M \to \mathbb{R} \) be a smooth function such that such that \( c < \lambda_f(x), \forall x \in M, \) for some number \( c > 0 \).

Define \( \rho : M \to \mathbb{Z} \) by \( x \mapsto \text{rank}(df_x) \). We shall show that \( \rho \) is a locally constant function on \( M \).

On the contrary suppose that \( \rho \) is not locally constant at a \( p \in M \). Choose co–ordinate charts \((U, \phi)\) and \((V, \psi)\) around \( p \) and \( f(p) \) respectively. Let \( g \) denote the function \( \psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V) \). Since \( f \) is a conformal Riemannian morphism, for each \( y \in \phi(U) \) we may view \( dg_y : \mathbb{R}^m \to \mathbb{R}^n \) as a geometric function between real inner–product spaces. And since \( \phi \) and \( \psi \) are diffeomorphisms, the function \( y \mapsto \text{rank}(dg_y) \) is not locally constant at \( q = \phi(p) \). Hence, there exists a sequence \( \{q_k\} \) in \( \phi(U) \) such that \( q_k \to q \) but has infinitely many terms \( q_l \) with \( \text{rank}(dg_{q_l}) \neq \text{rank}(dg_q) \), i.e. the sequence \( \text{rank}(dg_{q_k}) \) is not eventually constant. This contradicts Lemma 4.2 and makes untenable our assumption that \( \rho \) is not locally constant at \( p \). If \( M \) is compact then the conformality factor \( \lambda_f \) being a continuous real valued function on a compact space is bounded below by a positive number.

**Definition 4.2.** We define a smooth function \( f : M \to N \) to be **generalized conformal map** if there exists a positive real number \( \lambda_f \) such that \( df_x \) is a geometric function with conformality factor \( \lambda_f \), for all \( x \in M \). Clearly, a generalized conformal map \( f : M \to N \) is a conformal Riemannian morphism with a constant conformality factor...
\[ \wedge f(x) = \lambda_f, \forall x \in M. \] Converse is not true because conformality factor of \( f, \wedge f(x) \) may not be a constant function.

**Theorem 4.4.** Let \( f : M \to N \) be a smooth function, \( M \) be compact and \( \dim M \geq 2 \text{rank}(df_x), \forall x \in M \). Then \( f \) is subimmersion if and only if it is a generalized conformal map.

**Proof.** Let \( f \) be a subimmersion. Without any loss of generality we assume that \( M \) is connected. Hence \( \text{rank}(df_x) = k \), for some fixed integer \( k \), \( \forall x \in M \). By Wielandt-Hoffman inequality for Hermitian matrices \( S \) defined on the set of \( n \times n \) real symmetric matrices by \( S(A) = (\lambda_1(A), \ldots, \lambda_n(A)) \), where \( \lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n \) are eigen values of \( A \), is continuous. Since \( x \to df_x^* df_x \) is smooth and \( S \) is continuous, \( x \to S(df_x^* df_x) \) is continuous. We define \( E_f : M \to \mathbb{R} \) by \( E_f(x) = \) smallest positive eigen value of \( (df_x^* df_x) \), \( \forall x \in M \). Since \( df_x^* df_x \) has all its eigen values nonnegative, \( S(df_x^* df_x) = (0, 0, \cdots, 0, \lambda_1(df_x^* df_x), \cdots, \lambda_k(df_x^* df_x)) \). Clearly \( E_f(x) = \lambda_1(df_x^* df_x) \). So \( E_f \) is continuous. So, since \( M \) is compact, \( E_f \) is bounded below by a positive number \( \beta_f \). So \( 0 < \beta_f \leq \lambda_1(df_x^* df_x), \forall x \in M \). As in the proof of Proposition 2.2 if we choose \( 0 < \alpha < \beta_f \leq \lambda_1(df_x^* df_x) \) then \( df_x \) is geometric with conformality factor \( \sqrt{\alpha}, \forall x \in M \). Converse part follows from the Proposition 2.2. \( \square \)

We state following proposition from [**1**]

**Proposition 4.3.** A map \( f : M_1 \to N \) from a manifold \( M_1 \) to a Riemannian manifold \( N \) is an immersion iff the pullback tensor \( f^* g_N \) is a Riemannian metric on \( M \). In this case, \( f : (M_1, f^* g_N) \to (N, g_N) \) is an isometric immersion.

Now we show that locally a conformal Riemannian morphism is composition of a conformal Riemannian submersion and isometric immersion. This gives information about local structure of conformal Riemannian morphisms.

**Proposition 4.4.** Let \( f : M \to N \) be a conformal Riemannian morphism such that conformality factor \( \wedge f \) is bounded below by a positive number. Then \( f \) is composition of a conformal Riemannian submersion and isometric immersion.
Proof. From Proposition 4.2, f is a submersion. For \( x \in M \), let \( U, P, S \) and \( J \) be as in Definition 4.1 for a submersion, so that \( f_U = J \circ S \). Let \( g_U = (g_M)|U \) denote the restriction of the Riemannian metric to the open set \( U \) of \( M \), and let \( g_P = J^*g_N \), be the pullback tensor on \( P \). We show that \((U, g_U)\) and \((P, g_P)\) are Riemannian manifolds, the submersion \( S : (U, g_U) \to (P, g_P) \) is a conformal Riemannian submersion, and the immersion \( J : (P, g_P) \to (N, g_N) \) is an isometric immersion. Since \( U \) is open in \( M \), \((U, g_U)\) is an open Riemannian submanifold of \((M, g_M)\). Since \( J \) is an immersion by Proposition 4.3, \( g_P \) is a Riemannian metric on \( P \), and \( J : (P, g_P) \to (N, g_N) \) is an isometric immersion. Let \( \wedge \) be the conformality factor associated to \( f \). Since \( f \) is a conformal Riemannian morphism, \( g_N(f(x))(df_x X, df_x Y) = \wedge(x)g_M(x)(X, Y) \), for all \( X, Y \in \text{Conf}(df_x) \). Now \( g_N(f(x))(df_x X, df_x Y) = \wedge(x)g_M(x)(X, Y) \), for all \( X, Y \in \text{Conf}(df_x) \), if and only if \( \wedge(x)g_M(x)(X, Y) = g_N(f(x))(dJ_{S(x)} \circ dS_x X, dJ_{S(x)} \circ dS_x Y) \), for all \( X, Y \in \text{Conf}(df_x) \), if and only if \( \wedge(x)g_M(x)(X, Y) = g_P(S(x))(dS_x X, dS_x Y) \), for all \( X, Y \in \text{Conf}(df_x) \). Hence \( S \) is a conformal Riemannian submersion.

Theorem 4.5. Let \( f : M \to N \) be a conformal Riemannian morphism such that conformality factor \( \wedge_f \) is bounded below by a positive number. Then

1. If \( f \) is injective, then \( f \) is an injective conformal immersion.
2. If \( f \) is surjective and \( M \) is connected, then \( f \) is a surjective conformal Riemannian submersion.
3. More generally, if \( f \) is open or closed onto its image, then \( f(M) \) is submanifold of \( N \). Let \( g^N_{f(M)} \) denote the Riemannian metric induced on \( f(M) \) by the metric \( g_N \). Then \( f : (M, g_M) \to (f(M), g^N_{f(M)}) \) is a surjective conformal submersion onto \( f(M) \).
4. If \( f \) is bijective and \( M \) is connected, then \( f \) is a conformal map.

Proof. From Proposition 4.4 and Theorem 4.2 we conclude (1), (2) and (3). From (1) and (2), we conclude (4).
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