The cylindrical $\delta$-potential and the Dirac equation

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Abstract

In this paper, we discuss the Dirac equation in the presence of an attractive cylindrical $\delta$-shell potential $V(\rho) = -a\delta(\rho - \rho_0)$, where $\rho$ is the radial coordinate and $a > 0$. We present a detailed discussion on the boundary conditions the wavefunction has to satisfy when crossing the support of the potential, proceeding then to explore the dependence of the ground state on the parameter $a$, analyzing the occurrence of supercritical effects. We also apply the Foldy–Wouthuysen transformation, discussing the non-relativistic limit of this problem.

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1. Solution of the Dirac equation for a cylindrical $\delta$-shell

The Dirac equation certainly opened a new world to physics, with marvelous and surprising consequences as, for example, the existence of anti-matter, the fact that $g = 2$ for the electron gyromagnetic moment, etc [1–3]. However, perhaps one of the most interesting facts associated with the Dirac equation is the idea of the existence of a populated vacuum, the Dirac sea, i.e. a vacuum state having a non-trivial structure. Later this concept developed into a cornerstone of our present understanding of modern quantum field theory. Remnants of this exciting idea are, for example, the vacuum condensates in quantum chromodynamics, the thermal vacuum distribution, when finite temperature effects are taken into account in heavy-ion collisions, and, in general, the idea of non-zero vacuum expectation values. Supercritical effects, i.e. an instability that appears when the ground state starts to dive into the Dirac sea, inducing then positron emission, is a prediction of this scenario, which has nevertheless not yet been experimentally confirmed [4].

Singular $\delta$-type potentials have been considered in quantum mechanics from the beginning. In general, singular $\delta$-type potentials can be considered as toy models which allow us to obtain a physical insight, being, at the same time, more easy to deal with in comparison with more realistic extended potentials. An example is the well-known Dirac-comb or Kronig–Penney model [5] in non-relativistic quantum mechanics, which provided us with an understanding and intuition about the emergence of band structures in solid-state
physics. Another situation where these types of interactions have been used is in nuclear physics as a model for residual interactions between nucleons inside an incomplete nuclear shell, [6], providing a correct qualitative picture to this difficult problem.

In the frame of the Dirac equation, singular $\delta$-type interactions have also been considered many times in the literature [7, 8]. Certainly this problem is also attractive from the perspective of mathematical physics. A rigorous construction of self-adjoint extensions for the Dirac operator allowing the handling of matching conditions at the support of the $\delta$-potential for the spherically symmetric case was considered in [9]. These conditions are essential for determining the existence of bound states. The theorem by Svendsen [10] is also a rigorous result which tells us that for the spherically symmetric singular potential $V(\rho) = -a\delta(\rho - \rho_0)$, supercritical effects will be absent in the limit where $\rho_0 \to 0$. This fact is a particular case of the general result of this theorem, namely that contact interactions for the Dirac equation can be constructed only if the supporting manifold has a co-dimension of dimension 1, which is the case in our problem. See also [11] in this context. The cylindrical attractive $V(\rho) = -a\delta(\rho - \rho_0)$ potential, the main subject of this paper, as we will see, presents many interesting features, being worthwhile to go into a detailed discussion of its properties. This case has been discussed in the literature but in the context of the $(2+1)$ Dirac equation, where we have three two by two Dirac matrices [12]. The treatment here is different, and relies on an appropriate unitary transform and the use of the chiral representation for the Dirac matrices.

If we think about the interaction of a neutral Dirac particle, which carries a magnetic moment $\mu$, for example a neutron, with an external magnetic field $\vec{B}$, $H_I = -\vec{\mu} \cdot \vec{B}$, we will have a point-like interaction of a $\delta$-type if the external magnetic field corresponds to an Aharonov–Bohm magnetic vortex. The situation we propose to discuss in this paper corresponds physically to an array of Aharonov–Bohm vortices, distributed along a circle of radius $\rho_0$, which may interact with the magnetic moment of a neutral Dirac particle. Several experiments have been performed on the behavior of electrons under the influence of external Aharonov–Bohm magnetic fields [13], and such an array could in principle be constructed.

Obviously, we are compelled to work in cylindrical coordinates. Since a $\delta$-type potential, which has support in a domain of zero measure, divides the space into two regions, we will start by considering first the free-particle case. Then, in the second step, we will establish the connection between the wavefunctions when crossing the support of the potential.

1.1. Dirac free particle in cylindrical coordinates

The Dirac equation in curvilinear coordinates, in general, includes a non-trivial spin connection in the covariant derivative [14]. When considering the free-particle case in cylindrical coordinates, it turns out that the relevant Dirac matrices are coordinate dependent because here we have

$$\sigma_\rho = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}; \quad \sigma_\theta = \begin{pmatrix} 0 & -ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}. \quad (1)$$

This is something we would like to avoid when solving the free Dirac equation $(\gamma \cdot p - m)\psi = 0$. To deal with this, let us consider the unitary transformation:

$$\hat{S} = \frac{1}{\sqrt{\rho}} e^{\frac{i}{\rho} \phi x^1 y^2}, \quad (2)$$

introduced in [15]. Applying this transformation to the Dirac equation and defining $\psi' = \hat{S}\psi$, we obtain

$$\left(\gamma^0 \partial_t + \gamma^1 \partial_\rho + \frac{\gamma^2}{\rho} \partial_\theta + \gamma^3 \partial_z + im\right)\psi' = 0. \quad (3)$$
Note that in our case, due to the form of $\hat{S}$, the spin connection term is trivial. At this point, we can see the effect of the unitary transformation $\hat{S}$. It has rotated our equation so that the $\gamma$ matrices involved are the Cartesian ones, eliminating their dependence on the coordinates.

We now introduce the ansatz $\psi' = \gamma^1 \gamma^2 \phi$, and rewrite equation (3) as

$$(\hat{H}_1 + \hat{H}_2)\phi = 0,$$  \hspace{1cm} (4)

where

$$\hat{H}_1 = (\gamma^0 \partial_t + \gamma^3 \partial_z)\gamma^1 \gamma^2$$  \hspace{1cm} (5)

$$\hat{H}_2 = \left(\gamma^1 \partial_\rho + \frac{\gamma^2}{\rho} \partial_\theta + im\right)\gamma^1 \gamma^2.$$  \hspace{1cm} (6)

It is easy to see that these operators commute. This means that if each one of them satisfies an eigenvalue problem, they have a common basis, i.e.

$$\hat{H}_1 \phi = \lambda_1 \phi,$$  \hspace{1cm} (7)

$$\hat{H}_2 \phi = \lambda_2 \phi.$$  \hspace{1cm} (8)

Putting this into equation (4), it is trivial to note that $\lambda_1 = -\lambda_2 \equiv \lambda$. In this way, we can write two equations for the four-component bispinor $\phi$:

$$\left(-\gamma^2 \partial_\rho + \frac{\gamma^1}{\rho} \partial_\theta + im \gamma^1 \gamma^2 + \lambda\right) \phi = 0,$$  \hspace{1cm} (9)

$$(\gamma^0 \gamma^1 \gamma^2 \partial_t + \gamma^3 \gamma^1 \gamma^2 \partial_z - \lambda) \phi = 0.$$  \hspace{1cm} (10)

It is natural then to propose

$$\phi = e^{-ikz} e^{ikz} e^{iE\phi} \left(\begin{array}{c} \epsilon_1 \\ \epsilon_2 \end{array}\right),$$  \hspace{1cm} (11)

where $\epsilon$ and $\eta$ are both two-component spinors. Note that, because of the symmetry of the problem, we would expect to have a plane wave solution in the $z$ coordinate. This ansatz is quite general, precisely, due to the strong symmetry of our problem. Since we impose no restriction on the radial coordinate $\rho$, this ansatz should represent a general and complete solution of the problem. For our purpose, it is convenient to use the Chiral representation for the Dirac matrices [16] given by

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}.$$  \hspace{1cm} (12)

Substituting then our ansatz in the previous equations, we obtain

$$iE \sigma^1 \epsilon \sigma^2 \eta + ikz \sigma^3 \sigma^1 \sigma^2 \eta - \lambda \epsilon = 0,$$

$$iE \sigma^1 \epsilon - ikz \sigma^3 \sigma^1 \sigma^2 \epsilon - \lambda \eta = 0.$$  \hspace{1cm} (13)

It is important to emphasize that even though we have a rotated spinor and we are working in the Chiral basis, our energy levels remain unchanged since unitary transformations do not affect the energy spectrum. We define

$$\epsilon = \frac{\epsilon_1}{\epsilon_2}; \quad \eta = \frac{\eta_1}{\eta_2},$$  \hspace{1cm} (14)

where now, $\epsilon_{1,2}$ and $\eta_{1,2}$ are scalar complex functions (i.e. each of them represents one of the components of the bispinor $\phi$). Replacing this on the previous equation, we can obtain the following relations:

$$\lambda^2 = E^2 - k_z^2,$$

$$\epsilon_1 = -\frac{E + k_z}{\lambda} \eta_1,$$

$$\epsilon_2 = \frac{E - k_z}{\lambda} \eta_2.$$  \hspace{1cm} (15)
These relations are quite simple. Their derivation would have been more involved if the normal Dirac representation for the Dirac matrices would have been used. From equations (9) and (10), using the previous relations, we obtain

\[ \begin{align*}
\eta_1' &= -\frac{k_0}{\rho} \eta_2 + \frac{(m + \lambda)(E + k_z)}{\lambda} \eta_1, \\
\eta_2' &= \frac{k_0}{\rho} \eta_1 - \frac{(m - \lambda)(E - k_z)}{\lambda} \eta_2,
\end{align*} \]  

where the prime denotes derivative with respect to \( \rho \).

Without loss of generality, we can take \( k_z = 0 \) decoupling this set of equations. We define \( r \equiv \xi \rho \), where \( \xi^2 \equiv m^2 - \lambda^2 \). It is important to note that \( \xi^2 \) is positive definite, since we are looking for bound states, i.e. \( -m < E < m \). In terms of this new variable, we can write

\[ r^2 \eta_2' - (r^2 + k_0(k_0 + 1)) \eta_2 = 0. \]  

The previous equation is the well-known Bessel equation of second kind. In this way, \( \eta_1 \) and \( \eta_2 \) are given by

\[ \begin{align*}
\eta_2 &= \rho^{\frac{1}{2}}(Alk_{k_{1/2}}(\xi \rho) + BK_{k_{1/2}}(\xi \rho)) \\
\eta_1 &= i\frac{\lambda}{E} \left( \frac{m - \lambda}{m + \lambda} \right)^{\frac{1}{2}} \rho^{\frac{1}{2}}(-Alk_{-k_{1/2}}(\xi \rho) + BK_{k_{1/2}}(\xi \rho)) \\
&= i\rho^{\frac{1}{2}}(-Alk_{-k_{1/2}}(\xi \rho) + BK_{k_{1/2}}(\xi \rho)).
\end{align*} \]  

Once we have found the solution for the Dirac-free particle in cylindrical coordinates, we can return to our original problem, the attractive \( \delta \)-shell potential. The boundary conditions when crossing the support of the \( \delta \) potential, to be discussed in the following section, will provide the energy spectrum.

1.2. Boundary conditions for the Dirac equation in the presence of a cylindrical \( \delta \)-shell

By applying the transformation \( \tilde{S} \) to the Dirac equation, \( (\gamma \cdot p - m + \gamma^0 a \delta(\rho - \rho_0))\psi = 0 \), we obtain

\[ \left( \gamma^0 \partial_t + \gamma^i \partial_i + \frac{\gamma^2}{\rho} \partial_\rho + \gamma^3 \partial_\theta + im - iay^0 \delta(\rho - \rho_0) \right) \psi = 0. \]  

Following exactly the same procedure employed previously, we now obtain

\[ \begin{align*}
\gamma^0 \partial_t + \gamma^i \partial_i + \frac{\gamma^2}{\rho} \partial_\rho + \gamma^3 \partial_\theta + im - iay^0 \delta(\rho - \rho_0) \eta_1 + \lambda \sigma_1 &= 0 \\
\gamma^0 \partial_t + \frac{\gamma^2}{\rho} \partial_\rho + \gamma^3 \partial_\theta + im - iay^0 \delta(\rho - \rho_0) \eta_2 + \lambda \sigma_2 &= 0.
\end{align*} \]  

From here, we obtain

\[ \begin{align*}
\frac{1}{\eta_2^2/\eta_1^2 - 1} \frac{d}{d\rho} \left( \frac{\eta_2}{\eta_1} \right) + i\rho a \delta(\rho - \rho_0) & = \frac{1}{\eta_2^2 - \eta_1^2} \left( \frac{2k_0}{\rho} \eta_1 \eta_2 + im(\sigma_1 \eta_1 + \sigma_2 \eta_2) - \lambda(\sigma_1 \eta_1 + \sigma_2 \eta_2) \right).
\end{align*} \]  

Integrating between the limits \( [\rho_0 - \epsilon, \rho_0 + \epsilon] \), we have

\[ \begin{align*}
- \arctanh \left( \frac{\eta_2}{\eta_1} \right) \bigg|_{\rho_0-\epsilon}^{\rho_0+\epsilon} + ia &= 0.
\end{align*} \]  

To keep going, we divide the space in two regions.
Region I ($\rho < \rho_0$):

$$\eta^I_1 = -i \chi \rho^{1/2} A_{k_3 - 1/2}$$

$$\eta^I_2 = \rho^{1/2} \theta^{1/2}$$ (26)

Region II ($\rho > \rho_0$):

$$\eta^{II}_1 = -i \chi \rho^{1/2} B_{k_3 - 1/2}$$

$$\eta^{II}_2 = \rho^{1/2} \theta^{1/2}$$ (27)

The functions in regions I and II are different because the Dirac 4-spinor has to be normalized. In this way, we obtain the relation

$$i \eta^{II}_2 (\rho_0 + \epsilon) \eta^I_1 (\rho_0 + \epsilon) - \eta^{I}_2 (\rho_0 - \epsilon) \eta^I_1 (\rho_0 - \epsilon) = - \tan a.$$ (30)

We are interested in the ground-state energy ($k_0 = 0$). Using the explicit expressions of the relevant Bessel functions \cite{18}, and taking the limit $\epsilon \to 0$, we obtain the transcendental equation that determines the behavior of the ground state

$$\frac{1}{\sqrt{1 - \epsilon}} + \tanh (s_0) \frac{1}{1 + \frac{1}{\sqrt{1 - \epsilon}} \tanh (s_0)} = - \tan a,$$ (31)

where we have defined the following dimensionless variables:

$$\epsilon = \frac{E}{m}$$

$$\psi = m \rho_0$$

$$s_0 = \psi \sqrt{1 - \epsilon}.$$ (32)

For the derivation of this equation, it was important to fix the value of the constant $\lambda$. Its value can either be $\pm |E|$. In order to get rid of this ambiguity, we note that when $a = 0$ (no potential), the energy of the electron should correspond to the free-particle energy. To accomplish this, $\lambda$ has to be equal to $-|E|$. Finally, we can remove any ambiguity by demanding a continuous energy spectrum when $E = 0$. This demands $\lambda = -E$.

The evolution of the ground-state energy as a function of the different parameters involved is shown in figure 1. Numerically, it seems that there will be no critical effects, i.e. $\frac{d\epsilon}{da} \to 0$ when $\epsilon \to -1$. In fact, it is easy to show this property analytically. If we consider equation (31) and define $1 + \epsilon = \tilde{\epsilon}$, then we need to consider the limit when $\tilde{\epsilon} \to 0$. It is easy to find for the leading behavior of the derivative

$$\frac{d\tilde{\epsilon}}{da} = \frac{\tilde{\epsilon}}{\cos^2 a (\tanh(\sqrt{2m\rho_0}) + \tan a)} \to 0,$$ when $\tilde{\epsilon} \to 0$, i.e. supercritical effects in this case are absent, in contrast to what occurs in the case of the spherical delta potential discussed in \cite{7}.

2. Foldy–Wouthuysen transformation for the $\delta$-shell

Now we would like to discuss the Foldy–Wouthuysen \cite{17} transformation in the case of our $\delta$-shell problem. In this section, we will obtain some transcendental equations for the energy of the ground state up to order $1/m$ and $1/m^2$ separately. Later, we will compare this energy with the energy provided by the Schrödinger equation. In addition, we will show how the spin–orbit coupling term appears from the Foldy–Wouthuysen transformation up to order $\frac{1}{m^2}$.
2.1. Foldy–Wouthuysen transformation for the Hamiltonian

We have the Hamiltonian

\[ \hat{H} = \vec{\alpha} \cdot \vec{p} + \beta m - a \delta(\rho - \rho_0). \]  

(33)

We apply the first Foldy–Wouthuysen transformation \( F = e^{iS} \), with \( S = -i\beta \frac{\vec{\alpha} \cdot \vec{p}}{2m} \) and then the second one with \( S' = -i\beta \frac{\vec{\alpha} \cdot \vec{p}}{2m}(\xi_0 - \frac{a}{2m} \delta(\rho - \rho_0)) \) where, similar to the previous one, all odd operators in the Hamiltonian have been included inside \( S' \). We have also defined \( \xi_0 = \frac{\vec{p}^2}{2m} \). In this manner, we arrive to

\[ H'' = \beta \left( m + \frac{\vec{p}^2}{2m} \right) - a \delta(\rho - \rho_0) - \frac{a}{8m^2} (2(\vec{\alpha} \cdot \vec{p}\delta(\rho - \rho_0))\alpha_0 p_0 - \vec{\alpha} \cdot \vec{p}(\vec{\alpha} \cdot \vec{p}\delta(\rho - \rho_0))). \]  

(34)

We can see that \( H'' \) has no odd operators, and so it should be the non-relativistic limit of the Hamiltonian up to order \( 1/m^2 \). It is not difficult to see that, in spite of the presence of two \( \alpha \) matrices, the last two terms in equation (34) correspond to even operators.

2.2. Solving the free particle in the non-relativistic limit

Now, we solve the free particle using the Hamiltonian \( H'' \) without the Dirac delta potential. We start from

\[ i\frac{\partial \psi}{\partial t} = \beta \left( m + \frac{\vec{p}^2}{2m} \right) \psi, \]  

(35)

and suggest the ansatz \( \psi = e^{-i\kappa \rho} \), with \( u = e^{i\kappa \rho} \), to obtain

\[ r^2 u'' + ru' - k_0^2 u - \kappa^2 u = 0, \]  

(36)

where \( \kappa^2 \equiv -2m(E - m) \) and \( r \equiv \kappa \rho \).
It is important to note that, since we are looking for bound states, i.e. \(-m < E < m\), \(\kappa^2\) is positive definite.

Equation (36) is the Bessel equation, and its solutions are well known. We can follow a completely similar process for \(v\) to obtain the solutions:

\[
\begin{align*}
  u &= AI_0(\kappa \rho) + BK_0(\kappa \rho) \\
  v &= CI_0(\sqrt{2m(E + m)} \rho) + DK_0(\sqrt{2m(E + m)} \rho).
\end{align*}
\]

Now that we have the solutions, we can apply the boundary conditions, associated with the \(\delta\) potential.

2.3. Boundary conditions for the non-relativistic limit of the \(\delta\)-shell

We have the Dirac equation, up to order \(1/m\):

\[
\frac{\partial \psi}{\partial t} = \left( \beta m + \beta \frac{\vec{p}^2}{2m} - a \delta(\rho - \rho_0) \right) \psi.
\]

For the upper component, we obtain

\[
\begin{align*}
  - \kappa^2 u &= u'' + \frac{1}{\rho} u' - \frac{k_0}{\rho^2} u + 2ma \delta(\rho - \rho_0) u.
\end{align*}
\]

We are interested in the ground state (\(k_0 = 0\)), so the previous equation becomes

\[
\begin{align*}
  - \kappa^2 u &= u'' + \frac{1}{\rho} u' + 2ma \delta(\rho - \rho_0) u.
\end{align*}
\]

We integrate between \([\rho_0 - \epsilon, \rho_0 + \epsilon]\). We should keep in mind that, since the equation is the second-order differential equation, unlike the usual Dirac equation, our wavefunction is continuous in \(\rho_0\):

\[
\left. u' \right|_{\rho_0 - \epsilon}^{\rho_0 + \epsilon} + 2ma u(\rho_0) = 0.
\]

Once again, we divide the space in two regions:

Region I \((\rho < \rho_0)\):

\[
\begin{align*}
  u_I &= AI_0(\kappa \rho) \\
  v_I &= CI_0(\sqrt{2m(E + m)} \rho).
\end{align*}
\]

Region II \((\rho > \rho_0)\):

\[
\begin{align*}
  u_{II} &= BK_0(\kappa \rho) \\
  v_{II} &= CK_0(\sqrt{2m(E + m)} \rho).
\end{align*}
\]

These functions are different in each region in order to ensure the normalization of the Dirac spinor.

With this, we can write, taking the limit \(\epsilon \to 0\):

\[
\begin{align*}
  u_{II} - u_I^* + 2ma u(\rho_0) &= 0.
\end{align*}
\]

Imposing continuity of \(u\) in \(\rho_0\) we can obtain the relation

\[
\begin{align*}
  A &= BK_0(\kappa \rho_0) \\
  C &= BK_0(\kappa \rho_0).
\end{align*}
\]
Using this result and defining the dimensionless variables:

\[ \varepsilon = \frac{E}{m}, \quad \varphi = \rho \alpha m, \]

we can write

\[ \sqrt{2(1 - \varepsilon)}K_1(\kappa \rho_0)I_0(\kappa \rho_0) + \sqrt{2(1 - \varepsilon)}K_0(\kappa \rho_0)I_1(\kappa \rho_0) - 2aK_0(\kappa \rho_0)I_0(\kappa \rho_0) = 0, \]

which is a transcendental equation for the energy of the ground state, up to order \(1/m\).

We now repeat the previous procedure up to order \(1/m^2\). We have the Dirac equation

\[
\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = \left( \beta \left( \frac{p^2}{2} - a\delta(\rho - \rho_0) \right) - \frac{a}{8m^2} \left( 2(\alpha \cdot p)(\rho - \rho_0) \alpha_\varphi \rho_0 - \alpha \cdot p(\alpha \cdot p)(\rho - \rho_0) \right) \right) \psi,
\]

where

\[ 2(\alpha \cdot p)(\rho - \rho_0) \alpha_\varphi \rho_0 = -2\delta'(\rho - \rho_0)\alpha_\rho \rho_0 \nabla_\rho = -2\delta'(\rho - \rho_0)\alpha_z \nabla_\rho. \]

If we focus in the upper component, this term becomes

\[ -2\delta'(\rho - \rho_0)\alpha_\sigma \nabla_\sigma = -4\delta'(\rho - \rho_0)\hat{L}_z \hat{\sigma}. \]

We recognize the spin–orbit coupling, which is one of the relativistic corrections provided by the Foldy–Wouthuysen transformation. However, this term has no influence on the ground state, where \(L_z = k_\varphi = 0\). We now proceed to solve equation (51). Using the same ansatz as before for the upper component, we obtain

\[
E U = mu - \frac{u^2}{2m} - \frac{u'}{2m\rho} + \frac{k_\rho^2}{2m\rho^2}u - a\delta(\rho - \rho_0)u - \frac{a}{4m^2} \delta'(\rho - \rho_0)u
\]

\[
- \frac{a}{8m^2} \left( \delta''(\rho - \rho_0) + \frac{\delta'(\rho - \rho_0)}{\rho} \right) u.
\]

We solve for the ground state \((k_\rho = 0)\) and the previous equation becomes

\[ \rho U'' + \frac{u'}{2m} + \rho a\delta(\rho - \rho_0)u + \frac{a\rho}{8m^2} \left( \delta''(\rho - \rho_0) + \frac{\delta'(\rho - \rho_0)}{\rho} \right) u + (E - m)u = 0. \]

Integrating between \([\rho_0 - \varepsilon, \rho_0 + \varepsilon]\) and taking the limit \(\varepsilon \to 0\), we obtain

\[ \frac{1}{2m} \left( u_n'(\rho_0) - u_n'(\rho_0) \right) + a\rho(\rho_0) + \frac{a}{8m^2} u_n'(\rho_0) + \frac{a}{8m^2} u_n'(\rho_0) = 0. \]

After inserting solutions (43) and (45), equation (56) becomes

\[ -\sqrt{2(1 - \varepsilon)}I_0(\kappa \rho_0)K_1(\kappa \rho_0) + \left( a + \frac{a}{4} (1 - \varepsilon) \right) I_0(\kappa \rho_0)K_0(\kappa \rho_0)\]

\[ - \sqrt{2(1 - \varepsilon)}I_0(\kappa \rho_0)K_0(\kappa \rho_0) = 0. \]

This is yet another transcendental equation for the energy of the ground state, up to order \(1/m^2\).

From equations (57) and (50), we can see how the ground-state energy depends on the coupling constant \(a\).

In figure 2, we show the behavior of the ground state, as a function of the coupling \(a\), for the Schrödinger case and for the Foldy–Wouthuysen approximation up to order \(1/m^2\). The solid line stands for both the \(1/m\) order of the non-relativistic limit and the Schrödinger case (after the rest energy has been considered). There is no relativistic correction, other than the
Figure 2. Behavior of the ground state as a function of $a$. The solid line is the solution to Schrödinger’s equation and the dashed line is the second-order Foldy–Wouthuysen approximation, both for $\varphi = 1$.

rest energy, to the Schrödinger equation at first order in the Foldy–Wouthuysen transformation. However, up to order $1/m^2$, the first relativistic corrections begin to appear. As we discussed, the spin–orbit coupling can be explicitly written in the equation. Other corrections must be present too, since the energy of the ground state decreases for the same coupling value, with respect to the non-relativistic solution.

3. Conclusions

We have discussed the solutions of the Dirac equation for an attractive cylindrical delta potential. It turns out that there are no supercritical effects, in contrast to what happens in the spherical symmetric case. We also discussed the non-relativistic limit of these problems according to the Foldy–Wouthuysen approximation. No relativistic corrections are found up to order $1/m$, however, up to order $1/m^2$ relativistic corrections are present, such as the spin–orbit coupling.

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