LIE RING ISOMORPHISMS BETWEEN NEST ALGEBRAS ON BANACH SPACES

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Abstract. Let \( N \) and \( M \) be nests on Banach spaces \( X \) and \( Y \) over the (real or complex) field \( F \) and let \( \text{Alg}N \) and \( \text{Alg}M \) be the associated nest algebras, respectively. It is shown that a map \( \Phi : \text{Alg}N \to \text{Alg}M \) is a Lie ring isomorphism (i.e., \( \Phi \) is additive, Lie multiplicative and bijective) if and only if \( \Phi \) has the form \( \Phi(A) = TAT^{-1} + h(A)I \) for all \( A \in \text{Alg}N \) or \( \Phi(A) = -TAT^{-1} + h(A)I \) for all \( A \in \text{Alg}N \), where \( h \) is an additive functional vanishing on all commutators and \( T \) is an invertible bounded linear or conjugate linear operator when \( \dim X = \infty \); \( T \) is a bijective \( \tau \)-linear transformation for some field automorphism \( \tau \) of \( F \) when \( \dim X < \infty \).

1. INTRODUCTION AND MAIN RESULTS

Let \( R \) and \( R' \) be two associative rings. Recall that a map \( \phi : R \to R' \) is called a multiplicative map if \( \phi(AB) = \phi(A)\phi(B) \) for any \( A, B \in R \); is called a Lie multiplicative map if \( \phi([A,B]) = [\phi(A),\phi(B)] \) for any \( A, B \in R \), where \( [A,B] = AB - BA \) is the Lie product of \( A \) and \( B \) which is also called a commutator. In addition, a map \( \phi : R \to R' \) is called a Lie multiplicative isomorphism if \( \phi \) is bijective and Lie multiplicative; is called a Lie ring isomorphism if \( \phi \) is bijective, additive and Lie multiplicative. If \( R \) and \( R' \) are algebras over a field \( F \), \( \phi : R \to R' \) is called a Lie algebraic isomorphism if \( \phi \) is bijective, \( F \)-linear and Lie multiplicative. For the study of Lie ring isomorphisms between rings, see \([3, 5, 10]\) and the references therein. In this paper we focus our attention on Lie ring isomorphisms between nest algebras on general Banach spaces.

Let \( X \) be a Banach space over the (real or complex) field \( F \) with topological dual \( X^* \). \( \mathcal{B}(X) \) stands for the algebra of all bounded linear operators on \( X \). A nest \( \mathcal{N} \) on \( X \) is a complete totally ordered subspace lattice, that is, a chain of closed (under norm topology) subspaces of \( X \) which is closed under the formation of arbitrary closed linear span (denote by \( \bigvee \) ) and intersection (denote by \( \bigwedge \) ), and which includes \{0\} and \( X \). The nest algebra associated with a nest \( \mathcal{N} \), denoted by \( \text{Alg}N \), is the weakly closed operator algebra consisting of all operators

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that leave every subspace \( N \in \mathcal{N} \) invariant. For \( N \in \mathcal{N} \), let \( N_- = \bigvee \{ M \in \mathcal{N} \mid M \subset N \} \) and \( N^\perp = (N_-)^\perp \), where \( N^\perp = \{ f \in X^* \mid N \subseteq \ker(f) \} \). If \( \mathcal{N} \) is a nest on \( X \), then \( \mathcal{N}^\perp = \{ N^\perp \mid N \in \mathcal{N} \} \) is a nest on \( X^* \) and \( (\text{Alg}\mathcal{N})^* \subseteq \text{Alg}\mathcal{N}^\perp \). If \( \mathcal{N} = \{(0), X\} \), we say that \( \mathcal{N} \) is a trivial nest, in this case, \( \text{Alg}\mathcal{N} = \mathcal{B}(X) \). Non-trivial nest algebras are very important reflexive operator algebras that are not semi-simple, not semi-prime and not self-adjoint. If \( \dim X < \infty \), a nest algebra on \( X \) is isomorphic to an algebra of upper triangular block matrices. Nest algebras are studied intensively by a lot of literatures. For more details on basic theory of nest algebras, the readers can refer to [6, 8].

In [9], Marcoux and Sourour proved that every Lie algebraic isomorphism between nest algebras on separable complex Hilbert spaces is a sum \( \alpha + \beta \), where \( \alpha \) is an algebraic isomorphism or the negative of an algebraic anti-isomorphism and \( \beta : \text{Alg}\mathcal{N} \to \mathbb{C} I \) is a linear map vanishing on all commutators, that is, satisfying \( \beta([A, B]) = 0 \) for all \( A, B \in \text{Alg}\mathcal{N} \).

Qi and Hou in [11] generalized the result of Marcoux and Sourour by classifying certain Lie multiplicative isomorphisms. Note that, a Lie multiplicative isomorphism needs not be additive. Let \( \mathcal{N} \) and \( \mathcal{M} \) be nests on Banach spaces \( X \) and \( Y \) over the (real or complex) field \( \mathbb{F} \), respectively, with the property that if \( M \in \mathcal{M} \) such that \( M_- = M \), then \( M \) is complemented in \( Y \) (Obviously, this assumption is not needed if \( Y \) is a Hilbert space or if \( \dim Y < \infty \)).

Let \( \text{Alg}\mathcal{N} \) and \( \text{Alg}\mathcal{M} \) be respectively the associated nest algebras, and let \( \Phi : \text{Alg}\mathcal{N} \to \text{Alg}\mathcal{M} \) be a bijective map. Qi and Hou in [11] proved that, if \( \dim X = \infty \) and if there is a nontrivial element in \( \mathcal{N} \) which is complemented in \( X \), then \( \Phi \) is a Lie multiplicative isomorphism if and only if there exists a map \( h : \text{Alg}\mathcal{N} \to \mathbb{F} I \) with \( h([A, B]) = 0 \) for all \( A, B \in \text{Alg}\mathcal{N} \) such that \( \Phi \) has the form \( \Phi(A) = TAT^{-1} + h(A) \) for all \( A \in \text{Alg}\mathcal{N} \) or \( \Phi(A) = -TA^*T^{-1} + h(A) \) for all \( A \in \text{Alg}\mathcal{N} \), where, in the first form, \( T : X \to Y \) is an invertible bounded linear or conjugate-linear operator so that \( N \mapsto T(N) \) is an order isomorphism from \( \mathcal{N} \) onto \( \mathcal{M} \), while in the second form, \( X \) and \( Y \) are reflexive, \( T : X^* \to Y \) is an invertible bounded linear or conjugate-linear operator so that \( N^\perp \mapsto T(N^\perp) \) is an order isomorphism from \( \mathcal{N}^\perp \) onto \( \mathcal{M} \).

If \( \dim X = n < \infty \), identifying nest algebras with upper triangular block matrix algebras, then \( \Phi \) is a Lie multiplicative isomorphism if and only if there exist a field automorphism \( \tau : \mathbb{F} \to \mathbb{F} \) and certain invertible matrix \( T \) such that either \( \Phi(A) = TAT^{-1} + h(A) \) for all \( A \), or \( \Phi(A) = -T(A_\tau)^{tr}T^{-1} + h(A) \) for all \( A \), where \( A_\tau = (\tau(a_{ij})) \) for \( A = (a_{ij}) \) and \( A^{tr} \) is the transpose of \( A \). Particularly, above results give a characterization of Lie ring isomorphisms between nest algebras for finite-dimensional case, and for infinite-dimensional case under the mentioned assumptions on \( \mathcal{N} \) and \( \mathcal{M} \).
Recently, Wang and Lu in [12] generalized Marcoux and Sourour’s result from another direction, and proved that every Lie algebraic isomorphism between nest algebras $\text{Alg}N$ and $\text{Alg}M$ for any nests $N$ and $M$ on Banach spaces $X$ and $Y$ respectively can be decomposed as $\alpha + \beta$, where $\alpha$ is an algebraic isomorphism or the negative of an algebraic anti-isomorphism and $\beta : \text{Alg}N \to \mathbb{F}I$ is a linear map vanishing on each commutator. Because Lie algebraic isomorphisms were characterized in [4] for the case that the nest $N$ has a nontrivial complemented element, Wang and Lu in [12] mainly dealt with the case that all nontrivial elements of $N$ are not complemented.

The purpose of the present paper is to characterize all Lie ring isomorphisms between nest algebras of Banach space operators for any nests. Note that, the Lie ring isomorphisms are very different from algebraic ones. For example, the method used in [4] to characterize Lie algebraic isomorphisms for the case that the nest $N$ has a nontrivial complemented element is not valid for characterizing Lie ring isomorphisms. Algebraic isomorphisms between nest algebras are continuous, however ring isomorphisms are not necessarily continuous for finite-dimensional case.

The following are the main results of this paper.

**Theorem 1.1.** Let $N$ and $M$ be nests on Banach spaces $X$ and $Y$ over the (real or complex) field $\mathbb{F}$, and, $\text{Alg}N$ and $\text{Alg}M$ be the associated nest algebras, respectively. Then a map $\Phi : \text{Alg}N \to \text{Alg}M$ is a Lie ring isomorphism, that is, $\Phi$ is additive, bijective and satisfies $\Phi([A,B]) = [\Phi(A),\Phi(B)]$ for all $A,B \in \text{Alg}N$, if and only if $\Phi$ has the form $\Phi(A) = \Psi(A) + h(A)I$ for all $A \in \text{Alg}N$, where $\Psi$ is a ring isomorphism or the negative of a ring anti-isomorphism between the nest algebras and $h : \text{Alg}N \to \mathbb{F}$ is an additive functional satisfying $h([A,B]) = 0$ for all $A,B \in \text{Alg}N$.

The ring isomorphisms and the ring anti-isomorphisms between nest algebras of Banach space operators were characterized in [7, Theorem 2.2, Theorem 2.7 and Remark 2.6]. Using these results and Theorem 1.1, we can get more concrete characterization of Lie ring isomorphisms. Recall that a map $S : W \to V$ with $W,V$ linear spaces over a field $\mathbb{F}$ is called $\tau$-linear if $S$ is additive and $S(\lambda x) = \tau(\lambda)Sx$ for all $x \in W$ and $\lambda \in \mathbb{F}$, where $\tau$ is a field automorphism of $\mathbb{F}$.

**Theorem 1.2.** Let $N$ and $M$ be nests on Banach spaces $X$ and $Y$ over the (real or complex) field $\mathbb{F}$, and let $\text{Alg}N$ and $\text{Alg}M$ be the associated nest algebras, respectively. Then a map $\Phi : \text{Alg}N \to \text{Alg}M$ is a Lie ring isomorphism if and only if there exist an additive functional $h : \text{Alg}N \to \mathbb{F}$ satisfying $h([A,B]) = 0$ for all $A,B \in \text{Alg}N$ and a field automorphism $\tau : \mathbb{F} \to \mathbb{F}$ such that one of the following holds.
(1) There exists a \( \tau \)-linear transformation \( T : X \to Y \) such that the map \( N \mapsto T(N) \) is an order isomorphism from \( N \) onto \( M \) and

\[
\Phi(A) = TAT^{-1} + h(A)I \quad \text{for all} \quad A \in \text{Alg}N.
\]

(2) \( X \) and \( Y \) are reflexive, there exists a \( \tau \)-linear transformation \( T : X^* \to Y \) such that the map \( N^\perp \mapsto T(N^\perp) \) is an order isomorphism from \( N^\perp \) onto \( M \) and

\[
\Phi(A) = -TA^*T^{-1} + h(A)I \quad \text{for all} \quad A \in \text{Alg}N.
\]

Moreover, if \( \dim X = \infty \), the above \( T \) is in fact an invertible bounded linear or conjugate-linear operator; if \( F = \mathbb{R} \), \( T \) is linear.

For the finite dimensional case, it is clear that every nest algebra on a finite dimensional space is isomorphic to an upper triangular block matrix algebra. Let \( M_n(F) \) denote the algebra of all \( n \times n \) matrices over \( F \). Recall that an upper triangular block matrix algebra \( T = T(n_1, n_2, \ldots, n_k) \) is a subalgebra of \( M_n(F) \) consisting of all \( n \times n \) matrices of the form

\[
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1k} \\
0 & A_{22} & \cdots & A_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{kk}
\end{pmatrix},
\]

where \( n_1, n_2, \ldots, n_k \) are finite sequence of positive integers satisfying \( n_1 + n_2 + \cdots + n_k = n \) and \( A_{ij} \in M_{n_i \times n_j}(F) \), the space of all \( n_i \times n_j \) matrices over \( F \). Thus by Theorem 1.2, we get a characterization of Lie ring isomorphisms between upper triangular block matrix algebras.

**Corollary 1.3.** Let \( F \) be the real or complex field, and \( m, n \) be positive integers greater than 1. Let \( T = T(n_1, n_2, \ldots, n_k) \subseteq M_n(F) \) and \( S = T(m_1, m_2, \ldots, m_r) \subseteq M_m(F) \) be upper triangular block matrix algebras, and \( \Phi : T \to S \) be a map. Then \( \Phi \) is a Lie ring isomorphism if and only if \( m = n \), and there exist an additive functional \( \phi : T \to F \) satisfying \( \phi([A, B]) = 0 \) for all \( A, B \in T \), a field automorphism \( \tau : F \to F \) such that either

1. \( T = S \), there exists an invertible matrix \( T \in T \) such that

\[
\Phi(A) = TAT^{-1} + \phi(A)I \quad \text{for all} \quad A \in T;
\]

or
(2) \((n_1, n_2, \ldots, n_k) = (m_r, m_{r-1}, \ldots, m_1)\), there exists an invertible block matrix \(T = (T_{ij})_{k \times k}\) with \(T_{ij} \in M_{n_i,n_j}(\mathbb{F})\) and \(T_{ij} = 0\) whenever \(i + j > k + 1\), such that

\[
\Phi(A) = -TA^rT^{-1} + \phi(A)I \quad \text{for all } A \in \mathcal{T}.
\]

Where \(A^r = (\tau(a_{ij}))_{n \times n}\) for \(A = (a_{ij})_{n \times n} \in M_n(\mathbb{F})\) and \(A^r\) is the transpose of \(A\). If \(\mathbb{F} = \mathbb{R}\), then \(\Phi\) is a Lie algebraic isomorphism.

Corollary 1.3 is also a consequence of [11, Corollary 2.2].

Since the Lie ring isomorphisms between nest algebras on finite-dimensional Banach spaces were already characterized in [11, Corollary 2.2], to give a classification of all Lie ring isomorphisms between nest algebras of Banach space operators, it suffices to prove Theorem 1.1 for the infinite-dimensional cases without any additional assumption on the nests. It is clear that \(\dim X = \infty \Leftrightarrow \dim Y = \infty\).

The remain part of the paper is to prove the main result Theorem 1.1 under the assumption that both \(X\) and \(Y\) are infinite-dimensional. Our approach borrow and combine some ideas developed in [11] and [12]. In Section 2 we give preliminary lemmas, some of them are also parts of the proof of the main result. Section 3 deals with the case that both \((0)\) and \(X\) are limit points of the nest \(\mathcal{N}\), that is, \((0) = (0)_+\) and \(X_- = X\). The case that \(X_- \neq X\) and \(X_-\) is complemented or \((0) \neq (0)_+\) and \((0)_+\) is complemented is discussed in Section 4. And finally, the case that \(X_- \neq X\) and \(X_-\) is not complemented or \((0) \neq (0)_+\) and \((0)_+\) is not complemented is considered in Section 5.

2. Preliminary lemmas

In this section, we give some preliminary lemmas, definitions and symbols which are needed in other sections to prove the main result.

Let \(X\) and \(Y\) be Banach spaces over \(\mathbb{F}\), and let \(\mathcal{N}\) and \(\mathcal{M}\) be nests on \(X\) and \(Y\). Let \(\text{Alg}\mathcal{N}\) and \(\text{Alg}\mathcal{M}\) be associated nest algebras, respectively. It is well known that the commutant of a nest algebra is trivial, i.e., if \(T \in B(X)\) and \(TA = AT\) for every operator \(A \in \text{Alg}\mathcal{N}\), then \(T = \lambda I\) for some scalar \(\lambda \in \mathbb{F}\). This fact will be used in this paper without any specific explanation.

In addition, the symbols \(\text{ran}T\), \(\ker T\) and \(\text{rank}T\) stand for the range, the kernel and the rank (i.e., the dimension of \(\text{ran}T\)) of an operator \(T\), respectively. For \(x \in X\) and \(f \in X^*\), \(x \otimes f\) stands for the operator on \(X\) with rank not greater than 1 defined by \((x \otimes f)y = f(y)x\) for every \(y\). Some times we use \(\langle x, f \rangle\) to present the value \(f(x)\) of \(f\) at \(x\).

The following lemma is also well known which gives a characterization of rank one operators in nest algebras.

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Lemma 2.1. Let $\mathcal{N}$ be a nest on a (real or complex) Banach space $X$ and $x \in X$, $f \in X^*$. Then $x \otimes f \in \text{Alg} \mathcal{N}$ if and only if there exists a subspace $N \in \mathcal{N}$ such that $x \in N$ and $f \in N^\perp$.

For any non-trivial element $E \in \mathcal{N}$, define

$$J(N, E) = \{ A \in \text{Alg} \mathcal{N} : AE = 0 \text{ and } A^*E^\perp = 0 \}. \tag{2.1}$$

In [12], Wang and Lu proved that $\mathcal{L}$ is a proper maximal commutative Lie algebra ideal in $\text{Alg} \mathcal{N}$ if and only if $\mathcal{L} = F I + J(N, E)$ for some unique $E \in \mathcal{N}$. The following lemma shows that any maximal commutative Lie ring ideal also rises in this way.

Lemma 2.2. $J$ is a proper maximal commutative Lie ring ideal in $\text{Alg} \mathcal{N}$ if and only if it is a proper maximal commutative Lie algebra ideal.

Proof. Assume that $J$ is a maximal commutative Lie ring ideal. Then for any $A \in \text{Alg} \mathcal{N}$, any $C \in J$ and any $\lambda \in \mathbb{F}$, we have $[A, \lambda C] = \lambda [A, C] \in \mathbb{F} J$, which implies that $\mathbb{F} J$ is a Lie ring ideal. It is obvious that $\mathbb{F} J$ is also commutative. So $\mathbb{F} J \subseteq J$ as $J$ is maximal. Note that $\mathbb{F} J \supseteq J$. Thus we get $\mathbb{F} J = J$. It follows that $J$ is also a Lie algebra ideal. The converse is obvious. \(\square\)

In the rest part of this paper, we assume that $\Phi : \text{Alg} \mathcal{N} \to \text{Alg} \mathcal{M}$ is a Lie ring isomorphism.

If $\mathcal{N}$ contains at least one nontrivial element, by Lemma 2.2, for any nontrivial element $E \in \mathcal{N}$, $\Phi(F I + J(N, E))$ is a maximal commutative Lie ring ideal in $\text{Alg} \mathcal{M}$. Hence there is a unique nontrivial element $F \in \mathcal{M}$ such that $\Phi(F I + J(N, E)) = F I + J(M, F)$. Define a map

$$\hat{\Phi} : \mathcal{N} \setminus \{(0), X\} \to \mathcal{M} \setminus \{(0), Y\} \tag{2.2}$$

by $\Phi(F I + J(N, E)) = F I + J(M, \hat{\Phi}(E))$.

With the symbols introduced above and by an argument similar to [12] Lemmas 4.1, 4.3, 4.4, one can show that the following lemma is still true for the Lie ring isomorphism $\Phi$.

Lemma 2.3. $\hat{\Phi}$ in Eq.(2.2) is bijective and is either order-preserving or order-reversing, that is, $\hat{\Phi}$ is an order isomorphism or a reverse-order isomorphism from $\mathcal{N}$ onto $\mathcal{M}$ if we extend the definition of $\hat{\Phi}$ so that $\hat{\Phi}((0)) = (0)$ or $Y$ and $\hat{\Phi}(X) = Y$ or $(0)$ accordingly.

By Lemma 2.3, for any $A \in J(N, E)$ with nontrivial $E \in \mathcal{N}$, there exists a unique operator $B \in J(M, \hat{\Phi}(E))$ such that $\Phi(A) - B \in F I$. Thus we can define another map

$$\tilde{\Phi} : \bigcup \{J(N, E) : E \in \mathcal{N} \text{ is nontrivial}\} \to \bigcup \{J(M, F) : F \in \mathcal{M} \text{ is nontrivial}\} \tag{2.3}$$

with the property that $\tilde{\Phi}(A) \in J(M, \tilde{\Phi}(E))$ and $\Phi(A) - \tilde{\Phi}(A) \in F I$ for any $A \in J(N, E)$. Similar to [12] Lemma 4.2, we have
Lemma 2.4. $\Phi$ is a bijective map and $\Phi(J(N, E)) = J(M, \Phi(E))$ for every non-trivial $E \in \mathcal{N}$.

Next we discuss the idempotents in nest algebras. Denote by $\mathcal{E}(\mathcal{N})$ the set of all idempotents in $\text{Alg}\mathcal{N}$.

Lemma 2.5. ([11] Lemma 2.2) Let $\mathcal{N}$ be a nest on a (real or complex) Banach space $X$ and $A \in \text{Alg}\mathcal{N}$.

1. $A \in \mathbb{F}I + \mathcal{E}(\mathcal{N})$ if and only if $[A, [A, A, T]] = [A, T]$ for all $T \in \text{Alg}\mathcal{N}$.

2. $A$ is the sum of a scalar and an idempotent operator with range in $\mathcal{N}$ if and only if $[A, [A, T]] = [A, T]$ for all $T \in \text{Alg}\mathcal{N}$.

By Lemma 2.5, if $P$ is an idempotent operator in $\text{Alg}\mathcal{N}$, then $\Phi(P) = Q + \lambda P I$, where $\lambda P \in \mathbb{F}$ and $Q$ is an idempotent operator in $\text{Alg}\mathcal{M}$. Furthermore, if $\text{ran} P \in \mathcal{N}$, then $\text{ran} Q \in \mathcal{M}$. So we can define a map

$$\hat{\Phi} : \mathcal{E}(\mathcal{N}) \to \mathcal{E}(\mathcal{M})$$

by $\hat{\Phi}(P) = \Phi(P) - \lambda P I$. It is easily seen that $\hat{\Phi}$ is a bijective map from $\mathcal{E}(\mathcal{N})$ onto $\mathcal{E}(\mathcal{M})$; see [11].

Now, for any nontrivial element $E \in \mathcal{N}$, define two sets

$$\Omega_1(\mathcal{N}, E) = \{ P \in \mathcal{E}(\mathcal{N}) : PE = 0 \} \quad \text{and} \quad \Omega_2(\mathcal{N}, E) = \{ P \in \mathcal{E}(\mathcal{N}) : P^\ast E^\perp = 0 \}. \quad (2.5)$$

For any nontrivial element $F \in \mathcal{M}$, the sets $\Omega_1(\mathcal{M}, F)$ and $\Omega_2(\mathcal{M}, F)$ can be analogously defined. Note that, if $P \in \Omega_1(\mathcal{N}, E)$, then one can easily check $P^\ast E^\perp \neq 0$; if $E$ is not complemented, then $PE = 0 \Rightarrow (I - P)^\ast E^\perp \neq 0$ and $P^\ast E^\perp = 0 \Rightarrow (I - P)E \neq 0$. These facts are needed in the proof of Lemma 2.6.

Still, by an argument similar to [12] Lemmas 5.2-5.5, one can show that the following lemma is true for Lie ring isomorphisms $\Phi$, with $\hat{\Phi}$ and $\tilde{\Phi}$ defined in Eq.(2.2) and Eq.(2.4) respectively.

Lemma 2.6. Assume that $E \in \mathcal{N}$ is nontrivial and not complemented in $X$.

1. Either $\hat{\Phi}(\Omega_1(\mathcal{N}, E)) \subseteq \Omega_1(\mathcal{N}, \hat{\Phi}(E))$ or $I - \hat{\Phi}(\Omega_1(\mathcal{N}, E)) \subseteq \Omega_2(\mathcal{N}, \hat{\Phi}(E))$, and either $\hat{\Phi}(\Omega_2(\mathcal{N}, E)) \subseteq \Omega_2(\mathcal{N}, \hat{\Phi}(E))$ or $I - \hat{\Phi}(\Omega_2(\mathcal{N}, E)) \subseteq \Omega_1(\mathcal{N}, \hat{\Phi}(E))$.

2. If $\Omega_1(\mathcal{N}, \hat{\Phi}(E))$ and $\Omega_2(\mathcal{N}, \hat{\Phi}(E))$ are not empty, then $\hat{\Phi}(\Omega_1(\mathcal{N}, E)) \subseteq \Omega_1(\mathcal{N}, \hat{\Phi}(E))$ if and only if $\hat{\Phi}(\Omega_2(\mathcal{N}, E)) \subseteq \Omega_2(\mathcal{N}, \hat{\Phi}(E))$.

3. If $F \in \mathcal{N}$ is also nontrivial and not complemented in $X$ with $F \not< E$, then $\Omega_1(\mathcal{N}, E) \neq \emptyset$ and $\hat{\Phi}(\Omega_1(\mathcal{N}, E)) \subseteq \Omega_1(\mathcal{N}, \hat{\Phi}(E))$ together imply that $\hat{\Phi}(\Omega_1(\mathcal{N}, F)) \subseteq \Omega_1(\mathcal{N}, \hat{\Phi}(F))$.

The following lemma gives a characterization of complemented elements $E \in \mathcal{N}$ by the operators in $J(\mathcal{N}, E)$ and $\mathcal{E}(\mathcal{N})$, which is needed to prove that $\hat{\Phi}$ preserves the complementarity.
Lemma 2.7. Assume that $E \in \mathcal{N}$ is a nontrivial element. The following statements are equivalent.

(1) $E \in \mathcal{N}$ is complemented in $X$.

(2) There exists some idempotent $P \in \text{Alg}\mathcal{N}$ such that $[P, A] = A$ for any $A \in \mathcal{J}(\mathcal{N}, E)$.

(3) There exists some idempotent $P \in \text{Alg}\mathcal{N}$ such that $[P, A] = A$ for any rank-one operators $A \in \mathcal{J}(\mathcal{N}, E)$.

Furthermore, we have $E = \text{ran}P$ for $P$ in (2) and (3).

Proof. (1)$\Rightarrow$(2). If $E \in \mathcal{N}$ is complemented in $X$, there exists an idempotent $P \in \text{Alg}\mathcal{N}$ such that $\text{ran}P = E$. For any $A \in \mathcal{J}(\mathcal{N}, E)$, we have $PA = A$ and $AP = 0$. Hence $[P, A] = A$.

(2)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(1). Assume that there exists some idempotent $P \in \text{Alg}\mathcal{N}$ such that $[P, A] = A$ for any rank-1 operator $A \in \mathcal{J}(\mathcal{N}, E)$. According to the space decomposition $X = \text{ran}P + \text{ker}P$, for any $A \in \mathcal{J}(\mathcal{N}, E)$, we have

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$  

Then $[P, A] = A$ implies that $A = \begin{pmatrix} 0 & A_{12} \\ 0 & 0 \end{pmatrix}$. So $PA = A$ and $AP = 0$ hold for all $A \in \mathcal{J}(\mathcal{N}, E)$. Take any $y \in E$ and any $g \in E^\perp$. It is easy to check that $y \otimes g \in \mathcal{J}(\mathcal{N}, E)$. So $Py \otimes g = y \otimes g$ and $y \otimes gP = 0$. It follows that $Py = y$ and $P^*g = 0$ for all $y \in E$ and all $g \in E^\perp$.

Thus we get $E \subseteq \text{ran}P$ and $E^\perp \subseteq \text{ker}P^*$. If $E \neq \text{ran}P$, then there exist $x \in \text{ran}P$ and $g \in E^\perp$ such that $\langle x, g \rangle = 1$. This leads to a contraction that $0 = \langle x, P^*g \rangle = \langle Px, g \rangle = \langle x, g \rangle = 1$. Hence we must have $\text{ran}P = E$ and $E$ is complemented in $X$. \hfill $\square$

Lemma 2.8. Non-trivial element $E \in \mathcal{N}$ is complemented in $X$ with $\text{ran}P = E$ if and only if $\hat{\Phi}(E)$ is complemented in $Y$ with $\text{ran}\hat{\Phi}(P) = \hat{\Phi}(E)$. Here $P \in \text{Alg}\mathcal{N}$ is an idempotent.

Proof. It is clear that $\hat{\Phi}^{-1} = \hat{\Phi}^{-1}$. So we need only to show that $\hat{\Phi}(E)$ is complemented in $Y$ whenever $E \in \mathcal{N}$ is complemented in $X$. Indeed, if $E \in \mathcal{N}$ is complemented, by Lemma 2.7, there exists some idempotent $P \in \text{Alg}\mathcal{N}$ such that $[P, A] = A$ for any $A \in \mathcal{J}(\mathcal{N}, E)$. By definitions of $\hat{\Phi}$, $\Phi$ and $\tilde{\Phi}$ (ref. (2.2)-(2.4)), there exists some scalar $\gamma$ such that

$$\tilde{\Phi}(A) + \gamma I = \Phi(A) = \Phi([P, A]) = [\Phi(P), \Phi(A)] = [\hat{\Phi}(P), \tilde{\Phi}(A)].$$

Since $\tilde{\Phi}$ maps $\mathcal{J}(\mathcal{N}, E)$ onto $\mathcal{J}(\mathcal{M}, \hat{\Phi}(E))$ by Lemma 2.4, we see that $B + \gamma I = [\hat{\Phi}(P), B]$ holds for any $B \in \mathcal{J}(\mathcal{M}, \hat{\Phi}(E))$. Assume that $B$ is of rank-1. If $\gamma \neq 0$, then $[\tilde{\Phi}(P), B] = B + \gamma I$ is a sum of nonzero scalar and a rank-1 operator, which is impossible since a commutator can not be the sum of a nonzero scalar and a compact operator. Hence $\gamma = 0$, and $B = [\hat{\Phi}(P), B]$. 

for all rank-1 operators \( B \in \mathcal{J}(M, \hat{\Phi}(E)) \). By Lemma 2.7 again, \( \hat{\Phi}(E) \) is complemented in \( Y \) with \( \text{ran} \tilde{\Phi}(P) = \hat{\Phi}(E) \).

The following lemma is obvious.

**Lemma 2.9.** \( \Phi(FI) = FI \).

Finally, we give a lemma, which is needed to prove our main result. Let \( E \) and \( F \) be subspaces of \( X \) and \( X^* \), respectively. Denote by \( E \otimes F \) the set \( \{ x \otimes f : x \in E, f \in F \} \).

**Lemma 2.10.** Let \( X_i \) be an infinite-dimensional Banach space, \( i = 1, 2 \). Let \( E_i \) and \( F_i \) be closed subspaces with dimensions \( > 2 \) of \( X_i \) and \( X_i^* \) respectively. Let \( A_i \) be a unital subalgebra of \( B(X_i) \) containing \( E_i \otimes F_i \). Suppose that \( \Psi : A_1 \to A_2 \) is an additive bijective map satisfying \( \Psi(FI) = FI \) and \( \Phi(FI + E_1 \otimes F_1) = FI + E_2 \otimes F_2 \). Then there is a map \( \gamma : E_1 \otimes F_1 \to F \) and a field automorphism \( \tau : F \to F \) such that either

1. \( \Psi(x \otimes f) = \gamma(x, f)I + Cx \otimes Df \) for all \( x \in E_1 \) and \( f \in F_1 \), where \( C : E_1 \to E_2 \) and \( D : F_1 \to F_2 \) are two \( \tau \)-linear bijective maps; or
2. \( \Psi(x \otimes f) = \gamma(x, f)I + Df \otimes Cx \) for all \( x \in E_1 \) and \( f \in F_1 \), where \( C : E_1 \to F_2 \) and \( D : F_1 \to E_2 \) are two \( \tau \)-linear bijective maps.

Lemma 2.10 can be proved by a similar approach as that in [2] and we omit its proof here.

Since the “if” part of Theorem 1.1 is obvious, we need only to check the “only if” part. If \( \mathcal{N} \) is a trivial nest, that is, \( \mathcal{N} = \{(0), X\} \), then \( \mathcal{M} = \{(0), Y\} \) by Lemma 2.3. So \( \Phi \) is a Lie ring isomorphism from \( B(X) \) onto \( B(Y) \). Bai, Du and Hou showed in [1] that every Lie multiplicative isomorphism between prime rings with a non-trivial idempotent element is of the form \( \psi + \beta \) with \( \psi \) a ring isomorphism or the negative of a ring anti-isomorphism and \( \beta \) a central valued map vanishing on each commutator. Note that \( B(X) \) is prime and contains non-trivial idempotents if \( \dim X \geq 2 \). Hence, for the case that \( \mathcal{N} \) is trivial, Theorem 1.1 follows from [1].

So, in the rest sections we always assume that \( \mathcal{N} \) is nontrivial. Thus \( \mathcal{M} \) is also nontrivial by Lemma 2.3. We shall complete the proof of Theorem 1.1 by considering several cases according to the situations of \( (0)_+ \) and \( X_- \), which will be dealt with separably in Sections 3-5.

### 3. The case that \( (0)_+ = (0) \) and \( X_- = X \)

In this section, we deal with the case that both \( (0) \) and \( X \) are limit points of \( \mathcal{N} \), i.e., \( (0)_+ = (0) \) and \( X_- = X \).

Keep the definitions of \( \Phi, \hat{\Phi} \) in mind; ref. Eqs.(2.2) and (2.3).
Lemma 3.1. Assume that \((0)_+ = (0)\) and \(X_- = X\). Let \(E \in \mathcal{N}\) be a nontrivial element. Then for any \(x \in E\) and \(f \in E^\perp\), \(\Phi(x \otimes f)\) is a rank one operator.

Proof. Since, by Lemma 2.3, the bijective map \(\Phi : \mathcal{N} \to \mathcal{M}\) is either order-preserving or order-reversing, we have \((0)_+ = (0)\) and \(Y_- = Y\) in \(\mathcal{M}\). Write \(R = \Phi(x \otimes f)\) with \(\Phi\) defined in Eq. (2.3). Then \(R \in \mathcal{J}(\mathcal{M}, \Phi(E))\). If, on the contrary, \(\text{rank} R \geq 2\), then, since \(\bigcup \{M : M \in \mathcal{M}\}\) is a dense linear manifold of \(Y\), there exists a nontrivial element \(M_0 \in \mathcal{M}\) and two vectors \(u, v \in M\) such that \(Ru\) and \(Rv\) are linearly independent. As \(\bigcap \{M : (0) < M \in \mathcal{M}\} = \{0\}\), there exists some nontrivial \(L \in \mathcal{M}\) such that \(Ru, Rv \notin L\). Let \(Y_L = \text{span}\{Ru, L\}\). By Hahn-Banach theorem, there exists some \(g \in Y_L^\perp\) such that \(g(Ru) = 0\) and \(g(Rv) \neq 0\). It is easily checked that \(L < \Phi(E) < M_0\). Take \(z \in L\), \(h \in M_{0}^\perp\), and let \(A = \Phi^{-1}(z \otimes g), B = \Phi^{-1}(u \otimes h)\). By Lemma 2.4, \(A \in \mathcal{J}(\mathcal{N}, \Phi^{-1}(L))\) and \(B \in \mathcal{J}(\mathcal{N}, \Phi^{-1}(M_0))\).

If \(\Phi\) is order-preserving, we have \(\Phi^{-1}(L) < E < \Phi^{-1}(M_0)\). So

\[
\Phi(A \otimes f)B = \Phi([A, z \otimes g, R, u \otimes h]) = (z \otimes g)R(u \otimes h) = 0.
\]

It follows from the injectivity of \(\Phi\) that \(A \otimes f)B = 0\), which implies \(A \otimes f = 0\) or \((x \otimes f)B = 0\). If \(A \otimes f = 0\), then \(0 = \Phi([A, x \otimes f]) = (z \otimes g)R \neq 0\), a contradiction; if \((x \otimes f)B = 0\), then \(0 = \Phi([x \otimes f, A]) = (z \otimes g)R(u \otimes h) \neq 0\), a contradiction.

If \(\Phi\) is order-reversing, we have \(\Phi^{-1}(M_0) < E < \Phi^{-1}(L)\). So

\[
\Phi(B \otimes f)A = \Phi([B, z \otimes g, R, u \otimes h]) = (z \otimes g)R(u \otimes h) = 0,
\]

which yields that either \(B \otimes f = 0\) or \((x \otimes f)A = 0\). By a similar argument to that of the above, one can obtain a contradiction.

Therefore, we must have \(\text{rank} \Phi(x \otimes f) = \text{rank} R = 1\). □

Proof of Theorem 1.1 for the case that \((0) = (0)_+\) and \(X_- = X\).

By Lemma 3.1, Lemma 2.9 and the bijectivity of \(\Phi\), we have proved that, for every nontrivial element \(E \in \mathcal{N}\), \(\Phi(FI + E \otimes E^\perp) = FI + \Phi(E) \otimes \Phi(E)^\perp\). So, by Lemma 2.10, there exists a ring automorphism \(\tau_E : F \to F\) and a map \(\gamma_E : E \otimes E^\perp \to F\) such that either

\[
\Phi(x \otimes f) = \gamma_E(x, f)I + C_E x \otimes D_{E^\perp} f
\]

holds for all \(x \in E\) and \(f \in E^\perp\), where \(C_E : E \to \Phi(E)\) and \(D_{E^\perp} : E^\perp \to \Phi(E)^\perp\) are two \(\tau\)-linear bijective maps; or

\[
\Phi(x \otimes f) = \gamma_E(x, f)I + D_{E^\perp} f \otimes C_E x
\]
Combining the above two equations and noting that $I$ and $f$ are two $\tau$-linear bijective maps.

It is easily checked that, if there is a nontrivial $E_0 \in \mathcal{N}$ such that Eq.(3.1) holds, then Eq.(3.1) holds for any nontrivial $E \in \mathcal{N}$; If there is a nontrivial $E_0 \in \mathcal{N}$ such that Eq.(3.2) holds, then Eq.(3.1) holds for any nontrivial $E \in \mathcal{N}$.

Assume that Eq.(3.1) holds for a nontrivial $\tau \in \mathcal{N}$. Then, for any $N \in \mathcal{N}$, any $x \in E \cap N$ and any $f \in N^\perp \cap E^\perp$, we have

$$
\Phi(x \otimes f) = \gamma_E(x, f)I + C_E x \otimes D_E f = \gamma_N(x, f)I + C_N x \otimes D_N f.
$$

Since rank$I = \infty$, the above equation yields $\gamma_N(x, f) = \gamma_E(x, f)$ and $C_N x \otimes D_N f = C_E x \otimes D_E f$ for any $x \in N \cap E$. This entails that there exists an automorphism $\tau : \mathbb{F} \to \mathbb{F}$ so that $\tau_N = \tau_E = \tau$ for any $N, E \in \mathcal{N}$, and if $N \subset E$, then there exists a scalar $\alpha_{EN}$ such that $C_E|_N = \alpha_{EN} C_N$ and $D_N|_{N^\perp} = \alpha_{EN} D_N$. Now fix $E \in \mathcal{N}$. For any $N \in \mathcal{N}$, we define

$$
\begin{align*}
\tilde{C}_N &= C_N, \\
\tilde{D}_N &= D_N, \\
N^\perp &= \tilde{D}_N, \\
{\alpha_{EN}} &= \frac{1}{\alpha_{EN}} C_N, \\
N^\perp &= \tilde{D}_N, \\
{\alpha_{EN}} &= \frac{1}{\alpha_{EN}} D_N.
\end{align*}
$$

It is obvious that $\{\tilde{C}_N : N \neq (0), X\}$ and $\{\tilde{D}_N : N \neq (0), X\}$ are well defined with $\tilde{C}_E|_N = \tilde{C}_E$ and $\tilde{D}_E|_{N^\perp} = \tilde{D}_N$. Hence there exist bijective $\tau$-linear maps $C : \bigcup\{N \in \mathcal{N} : N \neq (0), X\} \to \bigcup\{M \in \mathcal{M} : M \neq (0), Y\}$ and $D : \bigcup\{N^\perp : N \in \mathcal{N}, N \neq (0), X\} \to \bigcup\{M^\perp : M \in \mathcal{M}, M \neq (0), Y\}$ such that $C|_N = \tilde{C}_N$ and $D|_{N^\perp} = \tilde{D}_N$ for any $N \in \mathcal{N}\setminus \{(0), X\}$.

By now, we have shown that, there exist bijective $\tau$-linear maps $C : \bigcup\{N \in \mathcal{N} : N \neq (0), X\} \to \bigcup\{M \in \mathcal{M} : M \neq (0), Y\}$ and $D : \bigcup\{N^\perp : N \in \mathcal{N}, N \neq (0), X\} \to \bigcup\{M^\perp : M \in \mathcal{M}, M \neq (0), Y\}$, and a map $\gamma : \bigcup\{E \times E^\perp : E \in \mathcal{N}\setminus \{(0), X\}\} \to \mathbb{F}$, such that for any $x \in N$ and $f \in N^\perp$ with $N \in \mathcal{N}\setminus \{(0), X\}$, we have

$$
\Phi(x \otimes f) = \gamma(x, f)I + Cx \otimes Df. \tag{3.3}
$$

Therefore, for any $A \in \text{Alg}\mathcal{N}$, any $x \in N$ and any $f \in N^\perp$, by Eq.(3.3), we have

$$
\Phi([A, x \otimes f]) = \Phi(A, \Phi(x \otimes f)) = \Phi(A)Cx \otimes Df - Cx \otimes \Phi(A)^* Df
$$

and

$$
\Phi([A, x \otimes f]) = \Phi(Ax \otimes f - x \otimes A^* f) = (\gamma(Ax, f) - \gamma(x, A^* f))I + C Ax \otimes D f - Cx \otimes DA^* f.
$$

Combining the above two equations and noting that $I$ is of infinite-rank, one obtains that

$$
Cx \otimes \Phi(A)^* D f - Cx \otimes DA^* f = \Phi(A)Cx \otimes D f - C Ax \otimes D f
$$
holds for any \( x \in N, f \in N^\perp \) and any nontrivial \( N \in \mathcal{N} \). Note that \( D \) is bijective. So there exists a scalar \( h(A) \) such that

\[
\Phi(A)Cx = CAx + h(A)Cx
\]

(3.4)

for all \( x \in \bigcup \{ N \in \mathcal{N} : N \neq (0), X \} \). It is clear that \( h \) is additive as a functional of \( \text{Alg} \mathcal{N} \).

Define \( \Psi(A) = \Phi(A) - h(A)I \) for all \( A \in \text{Alg} \mathcal{N} \). Then, by Eq.(3.4), for any \( A, B \in \text{Alg} \mathcal{N} \) and any \( x \in \bigcup \{ N \in \mathcal{N} : N \neq (0), X \} \), we have

\[
\Psi(AB)Cx = CABx = \Psi(A)CBx = \Psi(A)\Psi(B)Cx.
\]

Since \( \bigcup \{ N \in \mathcal{N} : N \neq (0), X \} \) is dense in \( X \) and \( C \) is bijective, it follows that \( \Psi(AB) = \Psi(A)\Psi(B) \) for all \( A, B \in \text{Alg} \mathcal{N} \), that is, \( \Psi \) is a ring isomorphism and \( \Phi(A) = \Psi(A) + h(A)I \) for all \( A \in \text{Alg} \mathcal{N} \).

Similarly, if Eq.(3.2) holds, one can check that \( \Phi(A) = -\Psi(A) + h(A)I \) for all \( A \in \text{Alg} \mathcal{N} \), where \( \Psi \) is a ring anti-isomorphism and \( h \) is an additive functional.

This completes the proof of Theorem 1.1 for the case that \((0)_+ = (0)\) and \(X_- = X\). \( \square \)

4. The case \( X_- \neq X \) and \( X_- \) is complemented or \((0)_+ \neq 0\) and \((0)_+\) is complemented

We give only the proof in detail for the case that \( X_- \neq X \) and \( X_- \) is complemented. The case that \((0) \neq (0)_+\) and \((0)_+\) is complemented in \( X \) can be dealt with similarly.

**Proof of Theorem 1.1 for the case that \( X_- \neq X \) and \( X_- \) is complemented in \( X \).**

Assume that \( X_- \neq X \) and \( X_- \) is complemented in \( X \).

Since \( X_- \) is complemented, there exists an idempotent \( P_0 \in \text{Alg} \mathcal{N} \) such that \( \text{ran} P_0 = X_- \).

With \( \tilde{\Phi} \) as in Eq.(2.4) and by a similar argument to that in the proof of \[ \text{Theorem 3.1} \], one can show that there exist an idempotent operator \( Q_0 \) and a scalar \( \lambda_{P_0} \) such that \( \Phi(P_0) = Q_0 + \lambda_{P_0}I \) with \( Q_0 = \tilde{\Phi}(P_0) \), \( \text{ran} Q_0 \in \mathcal{M} \) and the following statements hold:

(a) If there is an idempotent \( P_1 \in \text{Alg} \mathcal{N} \) such that \( P_1 < P_0 \) and \( \tilde{\Phi}(P_1) < Q_0 \) (or \( P_1 > P_0 \) and \( \tilde{\Phi}(P_1) > Q_0 \)), then for any \( P \in \text{Alg} \mathcal{N} \), \( P < P_0 \Rightarrow \tilde{\Phi}(P) < Q_0 \) and \( P > P_0 \Rightarrow \tilde{\Phi}(P) > Q_0 \).

(b) If there is an idempotent \( P_1 \in \text{Alg} \mathcal{N} \) such that \( P_1 < P_0 \) and \( \tilde{\Phi}(P_1) > Q_0 \) (or \( P_1 > P_0 \) and \( \tilde{\Phi}(P_1) < Q_0 \)), then for any \( P \in \text{Alg} \mathcal{N} \), \( P < P_0 \Rightarrow \tilde{\Phi}(P) > Q_0 \) and \( P > P_0 \Rightarrow \tilde{\Phi}(P) < Q_0 \).

**Claim 4.1.** If (a) occurs, then \( \Phi = \Psi + h \), where \( \Psi : \text{Alg} \mathcal{N} \to \text{Alg} \mathcal{M} \) is a ring isomorphism and \( h : \text{Alg} \mathcal{N} \to FI \) is an additive map vanishing all commutators.

(a) implies that \( \text{ran} Q_0 = Y_- \) by Lemmas 2.3 and 2.8. For the convenience, let \( A_{11} = P_0(\text{Alg} \mathcal{N})P_0, A_{12} = P_0(\text{Alg} \mathcal{N})(I - P_0), A_{22} = (I - P_0)(\text{Alg} \mathcal{N})(I - P_0), B_{11} = Q_0(\text{Alg} \mathcal{M})Q_0, \).
$B_{12} = Q_0(\text{Alg}M)(I - Q_0)$ and $B_{22} = (I - Q_0)(\text{Alg}M)(I - Q_0)$. Then $\text{Alg}N = A_{11} + A_{12} + A_{22}$ and $\text{Alg}M = B_{11} + B_{12} + B_{22}$.

We will prove Claim 4.1 by several steps.

**Step 1.** $\Phi(A_{12}) = B_{12}$.

The proof is the same as that of [11, Lemma 2.8].

**Step 2.** $\Phi(A_{ii}) \subseteq B_{ii} + FI$, $i = 1, 2$.

For any $A_{11} \in A_{11}$, denote $\Phi(A_{11}) = S_{11} + S_{12} + S_{22}$, where $S_{ij} \in B_{ij}$. Then

$$0 = \Phi([A_{11}, P_0]) = [\Phi(A_{11}), \Phi(P_0)] = [\Phi(A_{11}), Q_0],$$

which implies that $S_{12} = 0$. Let $P \in A_{22}$ be any idempotent with $P \neq I - P_0$. It is clear that $P < (I - P_0)$. Then $I - P > P_0$. As $\Phi$ meets (a), we have $\Phi(I - P) > Q_0$, that is, $\Phi(P) < I - Q_0$. It follows from $[A_{11}, P] = 0$ that $[\Phi(A_{11}), \Phi(P)] = [S_{22}, \Phi(P)] = 0$ by the arbitrariness of $P$ and the bijectivity of $\tilde{\Phi}$, we see that $S_{22}$ commutes with every idempotent in $B_{22}$. Note that $\text{ran}Q_0 = Y_\pi$. So $B_{22} = B(\ker Q_0)$, which implies $S_{22} \in \mathbb{F}(I - Q_0)$. It follows that $\Phi(A_{11}) = S_{11} + \lambda(I - Q_0) = (S_{11} - \lambda Q_0) + \lambda I$ and hence, $\Phi(A_{11}) \subseteq B_{11} + FI$.

Assume that $A_{22} \in A_{22}$. In the same way as above, one can show that, $\Phi(A_{22}) = T_{11} + T_{22}$ for some $T_{ii} \in B_{ii}$, $i = 1, 2$, with $T_{11}$ commuting with every idempotent in $B_{11}$. For any $B_{11} \in B_{11}$, by the surjectivity of $\Phi$, there exists some $A_0 \in \text{Alg}N$ such that $\Phi(A_0) = B_{11}$. Furthermore, $A_0 = A_{11} + \lambda I$ for some $A_{11} \in A_{11}$ and some scalar $\lambda$ by Step 1. Thus we have

$$[B_{11}, T_{11}] = [B_{11}, T_{22} + T_{11}] = [\Phi(A_0), \Phi(A_{22})] = \Phi([A_0, A_{22}]) = \Phi([A_{11} + \lambda I, A_{22}]) = 0$$

for all $B_{11} \in B_{11}$, which implies $T_{11} \in FI$ as $A_{11}$ is a nest algebra. So $\Phi(A_{22}) \subseteq B_{22} + FI$.

By Steps 1-2, for each $A_{12} \in A_{12}$, there exists $B_{12} \in B_{12}$ such that $\Phi(A_{12}) = B_{12}$; for each $A_{ii} \in A_{ii}$, $i = 1, 2$, there exist $B_{ii} \in B_{ii}$ and $\lambda_{ii} \in \mathbb{F}$ such that $\Phi(A_{ii}) = B_{ii} + \lambda_{ii} I$. We claim that $B_{ii}$ and $\lambda_{ii}$ are uniquely determined. In fact, if $\Phi(A_{ii}) = B_{ii} + \lambda_{ii} I = B'_{ii} + \lambda'_{ii} I$, then $B_{ii} - B'_{ii} \in FI$, which implies that $B_{ii} = B'_{ii}$ and $\lambda_{ii} = \lambda'_{ii}$. Let $\Psi(A_{ij}) = B_{ij}$ and $\Psi(A) = \Psi(A_{11}) + \Psi(A_{12}) + \Psi(A_{22})$. Then we define a map $\Psi : \text{Alg}N \to \text{Alg}M$ and a map $h : \text{Alg}N \to FI$ with $h(A) = \Phi(A) - \Psi(A) \in FI$. Now, imitating the proof of [11, Lemmas 2.10-2.13], one can show that $\Psi : \text{Alg}N \to \text{Alg}M$ is a ring isomorphism and $h : \text{Alg}N \to FI$ is an additive map satisfying $h([A, B]) = 0$ for all $A, B$. Hence Claim 4.1 is true.

**Claim 4.2.** If $\Phi$ satisfies (b), then $\Phi = -\Psi + h$, where $\Psi : \text{Alg}N \to \text{Alg}M$ is a ring anti-isomorphism and $h : \text{Alg}N \to FI$ is an additive map vanishing all commutators.

If (b) holds, then $\text{ran}Q_0 = (0)_+ \in M$ and $(0)_+$ is complemented in $M$ by Lemma 2.3 and 2.8. Consider the map $\Phi' : \text{Alg}N \to (\text{Alg}M)^*$ defined by $\Phi'(A) = -\Phi(A)^*$ for all $A \in \text{Alg}N$. 
By imitating the proof of [14, Lemmas 2.14], one can show that $\Phi': \text{Alg}N \to (\text{Alg}M)^\ast$ is a Lie multiplicative bijective map. Since, for any nontrivial idempotent operator $P \in \text{Alg}N$, $\Phi(P) = \tilde{\Phi}(P) + \lambda_P I$ for some $\lambda_P \in \mathbb{C}$, we have $\Phi'(P) = -\tilde{\Phi}(P)^\ast - \lambda_P I$. Now define a map $\Phi': \mathcal{E}(N) \to \mathcal{E}(M)^\ast$ by $\Phi'(P) = I - \tilde{\Phi}(P)^\ast$ for all idempotents $P$. Since $\Phi$ satisfies (b), for any nonzero idempotent $P_1 \in \text{Alg}N$, if $P_1 < P_0$, we have $\tilde{\Phi}'(P_1) = I - \tilde{\Phi}(P_1)^\ast < I - \tilde{\Phi}(P_0)^\ast = \tilde{\Phi}'(P_0)$; if $P_1 > P_0$, we have $\tilde{\Phi}'(P_1) = I - \tilde{\Phi}(P_1)^\ast > I - \tilde{\Phi}(P_0)^\ast = \tilde{\Phi}'(P_0)$. Hence $\Phi'$ satisfies (a).

Note that $\mathcal{M}^\perp = \{ M^\perp : M \in \mathcal{M} \}$ is a nest on $Y^\ast$. Since $\text{ran}Q_0 \in \mathcal{M}$, we have $\text{ker}Q_0^\ast = (\text{ran}Q_0)^\perp \in \mathcal{M}^\perp$, and so $\text{ran}(I - Q_0^\ast) \in \mathcal{M}^\perp$. With respect to the decomposition $Y^\ast = \text{ran}(I - Q_0^\ast) + \text{ran}Q_0^\ast$, we have $\text{Alg}\mathcal{M}^\perp = \mathcal{D}_{11} + \mathcal{D}_{12} + \mathcal{D}_{22}$, where $\mathcal{D}_{11} = (I - Q_0^\ast)(\text{Alg}\mathcal{M}^\perp)(I - Q_0^\ast)$, $\mathcal{D}_{12} = (I - Q_0^\ast)(\text{Alg}\mathcal{M}^\perp)Q_0^\ast$, $\mathcal{D}_{22} = Q_0^\ast(\text{Alg}\mathcal{M}^\perp)Q_0^\ast$. Since $\text{Alg}\mathcal{M} = Q_0(\text{Alg}\mathcal{M})Q_0 + Q_0(\text{Alg}\mathcal{M})(I - Q_0)$, $\mathcal{B}_{11} + \mathcal{B}_{12} + \mathcal{B}_{22} = \mathcal{B}_{11}, \mathcal{B}_{12} \subseteq \mathcal{D}_{12}$ and $\mathcal{B}_{22} \subseteq \mathcal{D}_{11}$. Hence $(\text{Alg}\mathcal{M})^\ast = \mathcal{B}_{22}^* + \mathcal{B}_{12}^* + \mathcal{B}_{11}^* \subseteq \text{Alg}\mathcal{M}^\perp$.

**Step 1.** $\Phi'(A_{12}) = B_{12}^*.$

The proof is the same as that of [14, Lemma 2.15].

**Step 2.** $\Phi'(A_{ii}) \subseteq B_{ij}^* + \mathbb{F}I$, $i, j = 1, 2$ and $i \neq j$.

For any $A_{11} \in A_{11}$, write $\Phi'(A_{11}) = S_{22}^* + S_{12}^* + S_{11}^*$, where $S_{ij} \in B_{ij}$. Then

$$0 = \Phi'([A_{11}, P_0]) = [\Phi'(A_{11}), \Phi'(P_0)] = [\Phi'(A_{11}), -Q_0^*] = [Q_0^*, \Phi'(A_{11})],$$

which implies that $S_{12}^* = 0$.

Let $P \in A_{22}$ be any idempotent with $P < I - P_0$. As $\Phi'$ satisfies (a), we have $\tilde{\Phi}'(P) < \tilde{\Phi}'(I - P_0) = I - \tilde{\Phi}(P_0)^\ast = Q_0^*$. It follows from $[A_{11}, P] = 0$ that $0 = [\Phi'(A_{11}), \Phi'(P)] = [S_{11}^*, \Phi'(P)]$. Since $P$ is arbitrary, we see that $S_{11}^*$ commutes with every idempotent in $B_{11}^*$, which implies that $S_{11}^*$ commutes with every idempotent in $B_{11}$. Noting that $B_{11} = \mathcal{B}(\text{ker}Q_0)$ in this case, so we must have $S_{11} \subseteq \mathcal{F}I_{\text{ker}Q_0}$. By the arbitrariness of $A_{11}$ we obtain that $\Phi'(A_{11}) \subseteq \mathcal{B}_{22}^* + \mathbb{F}I$.

Similarly, one can show that, for any $A_{22} \in A_{22}$, $\Phi'(A_{22}) = T_{22}^* + T_{11}^*$, where $T_{22}$ commutes with every idempotent in $B_{22}$. Taking any $B_{22} \in B_{22}$, by the surjectivity of $\Phi'$, there exists some $A_0 \in \text{Alg}N$ such that $\Phi'(A_0) = B_{22}^*$. Furthermore, $A_0 = A_{11} + \lambda I$ for some $A_{11} \in A_{11}$ and some scalar $\lambda$. Then

$$[B_{22}^*, T_{22}^*] = [B_{22}^*, T_{22}^* + T_{11}^*] = [\Phi'(A_0), \Phi'(A_{22})] = \Phi'([A_0, A_{22}]) = 0$$

for all $B_{22} \in B_{22}$. Thus $[B_{22}, T_{22}] = 0$ for all $B_{22} \in B_{22}$, which implies $T_{22} \in \mathbb{F}I_{\text{ran}Q_0}$ as $B_{22}$ is a nest algebra. Therefore $\Phi'(A_{22}) \subseteq B_{11}^* + \mathbb{F}I$.

By Steps 1-2, for each $A_{12} \in A_{12}$, there exists $B_{12} \in B_{12}$ such that $\Phi'(A_{12}) = B_{12}^*$; for each $A_{ii} \in A_{ii}$ ($i = 1, 2$), there exist $B_{jj} \in B_{jj}$ and $\lambda_{jj} \in \mathbb{F}$ such that $\Phi'(A_{ii}) = B_{jj}^* + \lambda_{jj}I$, 


i, j = 1, 2 and i ≠ j. It is easily seen that B_{jj} and λ_{jj} are uniquely determined. Define a map \( \Psi' : \text{Alg}N \rightarrow (\text{Alg}M)^{\ast} \) by \( \Psi'(A_{12}) = B_{12}^{\ast}, \Psi'(A_{11}) = B_{22}^{\ast}, \Psi'(A_{22}) = B_{11}^{\ast} \) and \( \Psi'(A) = \Psi'(A_{11}) + \Psi'(A_{12}) + \Psi'(A_{22}) \) for \( A = A_{11} + A_{12} + A_{22} \). Set \( h'(A) = \Phi'(A) - \Psi'(A) \).

It is clear that \( h' \) maps \( \text{Alg}N \) into \( \mathbb{F}I \). By [11, Lemma 2.17], one can show that \( \Psi' \) is a ring isomorphism. Since, for any \( A \in \text{Alg}N \), there exists a unique element \( S \in \text{Alg}M \) such that \( \Psi'(A) = S^{\ast} \), we can define a map \( \Psi : \text{Alg}N \rightarrow \text{Alg}M \) by \( \Psi(A) = S \). Thus \( \Psi(A)^{\ast} = \Psi'(A) \) for every \( A \) and hence \( \Psi \) is a ring anti-isomorphism. Let \( h : \text{Alg}N \rightarrow \mathbb{F}I \) be the map defined by \( h'(A) = -h(A)^{\ast} \) for all \( A \in \text{Alg}N \). Clearly, \( h([A, B]) = 0 \) for all \( A, B \in \text{Alg}N \). Furthermore, we have

\[
-\Phi(A)^{\ast} = \Phi'(A) = \Psi'(A) + h'(A) = \Psi(A)^{\ast} + h'(A) = (\Psi(A) - h(A))^{\ast},
\]

which yields \( \Phi(A) = -\Psi(A) + h(A) \) for every \( A \in \text{Alg}N \). This completes the proof of Claim 4.2.

By Claim 4.1 and Claim 4.2, Theorem 1.1 holds for the case that \( X_{-} \neq X \) and \( X_{-} \) is complemented in \( X \).

\[\square\]

5. The case that \( X_{-} \neq X \) and \( X_{-} \) is not complemented or \( (0) \neq (0)_{+} \) and \( (0)_{+} \) is not complemented

In this section, we deal with the case that \( X_{-} \neq X \) and \( X_{-} \) is not complemented or \( (0) \neq (0)_{+} \) and \( (0)_{+} \) is not complemented. Here we borrow some ideas developed in [12]. Note that, not like [12], we do not assume that all non-trivial elements in the nests are not complemented. Also, we give only the detail of our proof for the case that \( X_{-} \neq X \) and \( X_{-} \) is not complemented in \( X \). The other case can be checked similarly.

Assume that \( X_{-} \neq X \) and \( X_{-} \) is not complemented in \( X \).

Note that, there are three possible situations that \( (0) \) may have, that is, \( (1^{\circ}) (0)_{+} \neq (0) \) and \( (0)_{+} \) is complemented in \( X \), \( (2^{\circ}) (0)_{+} = (0) \) and \( (3^{\circ}) (0)_{+} \neq (0) \) and \( (0)_{+} \) is not complemented in \( X \). By Section 4, Theorem 1.1 is true if the situation \( (1^{\circ}) \) occurs. So what we need to deal with is either \( (2^{\circ}) \) or \( (3^{\circ}) \).

Recall that \( \Phi, \hat{\Phi} \) and \( \bar{\Phi} \) are maps defined in Eqs.(2.2)-(2.4), and \( \Omega_{1}(N, E) \) and \( \Omega_{2}(N, E) \) are defined in Eq.(2.5). By Lemma 2.6, either \( \bar{\Phi}(\Omega_{1}(N, X_{-})) \subseteq \Omega_{1}(N, \Phi(X_{-})) \) or \( \bar{\Phi}(\Omega_{1}(N, X_{-})) \subseteq I - \Omega_{2}(N, \bar{\Phi}(X_{-})) \).

The following lemma is crucial for our purpose.

**Lemma 5.1.** Assume that \( (0) < X_{-} < X \) and \( X_{-} \) is not complemented. The following statements are true:
(1) If \( \tilde{\Phi}(\Omega_1(\mathcal{N}, X_-)) \subseteq \Omega_1(\mathcal{N}, \hat{\Phi}(X_-)) \), then \( \Phi(FI + X \otimes X_\perp) = FI + Y \otimes Y_\perp \), and there exists a ring automorphism \( \tau : F \to F \), a map \( \gamma : X \otimes X_\perp \to F \), bijective \( \tau \)-linear maps \( C : X \to Y \) and \( D : X_\perp \to Y_\perp \) such that

\[
\Phi(x \otimes f) = \gamma(x, f) I + Cx \otimes Df
\]

holds for all \( x \in X \) and \( f \in X_\perp \).

(2) If \( \tilde{\Phi}(\Omega_1(\mathcal{N}, X_-)) \subseteq I - \Omega_2(\mathcal{N}, \hat{\Phi}(X_-)) \), then \( \Phi(FI + X \otimes X_\perp) = FI + (0)_+ \otimes Y^* \), and there exists a ring automorphism \( \tau : F \to F \), a map \( \gamma : X \otimes X_\perp \to F \), bijective \( \tau \)-linear maps \( C : X \to Y^* \) and \( D : X_\perp \to (0)_+ \) such that

\[
\Phi(x \otimes f) = \gamma(x, f) I + Df \otimes Cx
\]

holds for all \( x \in X \) and \( f \in X_\perp \).

To prove Lemma 5.1, we consider two cases, that is, the case that \( \mathcal{N} \) has at least two nontrivial elements and the case that \( \mathcal{N} \) has only one nontrivial element. These will be done by Lemma 5.3 and Lemma 5.4, respectively.

We first consider the case that \( \mathcal{N} \) has at least two nontrivial elements.

The following lemma is crucial for the proof of Lemma 5.3.

**Lemma 5.2.** Assume that \( \mathcal{N} \) has at least two nontrivial elements and \( X_- \) is not complemented in \( X \). Then the following statements hold.

(i) \( \tilde{\Phi}(\Omega_1(\mathcal{N}, X_-)) \subseteq \Omega_1(\mathcal{N}, \hat{\Phi}(X_-)) \) if and only if \( \hat{\Phi} \) is order-preserving.

(ii) \( \tilde{\Phi}(\Omega_1(\mathcal{N}, X_-)) \subseteq I - \Omega_2(\mathcal{N}, \hat{\Phi}(X_-)) \) if and only if \( \hat{\Phi} \) is order-reversing.

**Proof.** By Lemma 2.6, it suffices to show that \( \tilde{\Phi}(\Omega_1(\mathcal{N}, X_-)) \subseteq \Omega_1(\mathcal{N}, \hat{\Phi}(X_-)) \) implies that \( \hat{\Phi} \) is order-preserving and \( \tilde{\Phi}(\Omega_1(\mathcal{N}, X_-)) \subseteq I - \Omega_2(\mathcal{N}, \hat{\Phi}(X_-)) \) implies that \( \hat{\Phi} \) is order-reversing.

Assume that \( \tilde{\Phi}(\Omega_1(\mathcal{N}, X_-)) \subseteq \Omega_1(\mathcal{N}, \hat{\Phi}(X_-)) \). Since \( \mathcal{N} \) has at least two nontrivial elements, we may take a nontrivial element \( E \in \mathcal{N} \) such that \( E \prec X_- \). If, on the contrary, \( \tilde{\Phi} \) is order-reversing, then we have \( \hat{\Phi}(E) > \hat{\Phi}(X_-) \). Fix an idempotent \( P \in \Omega_1(\mathcal{N}, X_-) \). Then we have \( \tilde{\Phi}(P) \in \Omega_1(\mathcal{M}, \hat{\Phi}(X_-)) \). For any \( D_1 \in \mathcal{J}(\mathcal{M}, \tilde{\Phi}(X_-)) \), \( D_2 \in \mathcal{J}(\mathcal{M}, \hat{\Phi}(E)) \), let \( C_1 = \tilde{\Phi}^{-1}(D_1), C_2 = \tilde{\Phi}^{-1}(D_2) \). By the definition of \( \tilde{\Phi} \), we have \( C_1 \in \mathcal{J}(\mathcal{N}, X_-) \) and \( C_2 \in \mathcal{J}(\mathcal{N}, E) \). Furthermore,

\[
0 = \tilde{\Phi}([C_1, [C_2, P]]) = [D_1, [D_2, \tilde{\Phi}(P)]]
\]

\[
= [D_1, D_2 \tilde{\Phi}(P) - \tilde{\Phi}(P)D_2] = D_1 D_2 \tilde{\Phi}(P) - D_1 \tilde{\Phi}(P)D_2.
\]

For any \( y_1 \in \tilde{\Phi}(X_-), y_2 \in \tilde{\Phi}(E) \) and any \( g_1 \in \tilde{\Phi}(X_-), g_2 \in \tilde{\Phi}(E) \), it is obvious that \( y_1 \otimes g_1 \in \mathcal{J}(\mathcal{M}, \tilde{\Phi}(X_-)) \) and \( y_2 \otimes g_2 \in \mathcal{J}(\mathcal{M}, \tilde{\Phi}(E)) \). Letting \( D_i = y_i \otimes g_i \), the above equation
yields
\[
\langle y_2, g_1 \rangle y_1 \otimes \hat{\Phi}(P)^* g_2 = \langle \hat{\Phi}(P)y_2, g_1 \rangle y_1 \otimes g_2.
\] (5.1)

Choose \(y_2 \in \hat{\Phi}(E)\) and \(g_1 \in \hat{\Phi}(X_-)^\perp\) such that \(\langle y_2, g_1 \rangle \neq 0\). Eq.(5.1) implies that there exists some scalar \(\lambda_{g_2}\) such that \(\hat{\Phi}(P)^* g_2 = \lambda_{g_2} g_2\) for each \(g_2 \in \hat{\Phi}(E)^\perp\). It follows that there is a scalar \(\lambda\) such that \(\hat{\Phi}(P)^* g_2 = \lambda g_2\) for all \(g_2 \in \hat{\Phi}(E)^\perp\). Since \(\hat{\Phi}(P)^*\) is an idempotent, either \(\lambda = 0\) or \(\lambda = 1\).

If \(\lambda = 0\), then \(\hat{\Phi}(P)^* \hat{\Phi}(E)^\perp = \{0\}\) and Eq.(5.1) yields \(\langle \hat{\Phi}(P)y_2, g_1 \rangle = 0\) for all \(y_2 \in \hat{\Phi}(E)\) and \(g_1 \in \hat{\Phi}(X_-)^\perp\). It follows that \(\hat{\Phi}(P)^* \hat{\Phi}(X_-)^\perp \subseteq \hat{\Phi}(E)^\perp\). So \(\hat{\Phi}(P)^* \hat{\Phi}(X_-)^\perp = \hat{\Phi}(P)^* (\hat{\Phi}(P)^* \hat{\Phi}(X_-)^\perp) \subseteq \hat{\Phi}(P)^* \hat{\Phi}(E)^\perp = \{0\}\), which is impossible as \(\hat{\Phi}(P) \in \Omega_1(\mathcal{M}, \hat{\Phi}(X_-))\).

If \(\lambda = 1\), then Eq.(5.1) yields \(\langle \hat{\Phi}(P)y_2, g_1 \rangle = \langle y_2, g_1 \rangle\) for all \(y_2 \in \hat{\Phi}(E)\) and \(g_1 \in \hat{\Phi}(X_-)^\perp\). This implies that \((I - \hat{\Phi}(P))^* \hat{\Phi}(X_-)^\perp \subseteq \hat{\Phi}(E)^\perp\), and so

\[
(I - \hat{\Phi}(P))^* \hat{\Phi}(X_-)^\perp = (I - \hat{\Phi}(P))^*((I - \hat{\Phi}(P))^* \hat{\Phi}(X_-)^\perp) \subseteq (I - \hat{\Phi}(P))^* \hat{\Phi}(E)^\perp = \{0\}.
\]

This, together with the fact \(\hat{\Phi}(P) \hat{\Phi}(X_-) = \{0\}\) entails \(\text{ran} \hat{\Phi}(P) = \hat{\Phi}(X_-)\), a contradiction. Hence \(\hat{\Phi}\) is order-preserving.

Similarly, one can show that \(\hat{\Phi}\) is order-reversing if \(\hat{\Phi}(\Omega_1(\mathcal{N}, X_-)) \subseteq I - \Omega_2(\mathcal{N}, X_-)\).

**Lemma 5.3.** Assume that \(\mathcal{N}\) has at least two nontrivial elements and \((0) < X_- < X\) is not complemented in \(X\). Then for any \(x \in X\) and \(f \in X_-^\perp\), \(\Phi(x \otimes f)\) is the sum of a scalar and a rank one operator. Moreover, the statement (1) and (2) of Lemma 5.1 hold.

**Proof.** We will complete the proof of the lemma by considering three cases.

**Case 1.** \(x \in X_-\) and \(f \in X_-^\perp\).

In this case, let \(R = \Phi(x \otimes f)\). Then \(R \in J(\mathcal{M}, \Phi(X_-))\) and \(\Phi(x \otimes f) - \Phi(x \otimes f) = \Phi(x \otimes f) - R \in FI\). We show that \(R\) is of rank one. Assume on the contrary that \(\text{rank} R \geq 2\). We will induce contradiction by considering two subcases.

**Subcase 1.1.** \(\hat{\Phi}(\Omega_1(\mathcal{N}, X_-)) \subseteq \Omega_1(\mathcal{M}, \hat{\Phi}(X_-))\).

By Lemma 5.2, \(\hat{\Phi}\) is order-preserving. It is clear in this case that we have \((0) < Y_- = \hat{\Phi}(X_-) < Y\).

Since \(\text{rank} R \geq 2\), there are two vectors \(u, v \in Y \setminus Y_-\) such that \(Ru\) and \(Rv\) are linearly independent. Take \(h \in Y_-^\perp\) such that \(h(u) = 1\). Then \(u \otimes h \in \Omega_1(\mathcal{M}, \phi(X_-))\). Let \(B = \hat{\Phi}^{-1}(u \otimes h)\). By Lemma 5.2(i), \(B \in \Omega_1(\mathcal{N}, X_-)\).

If \((0)_+ = (0) \in \mathcal{N}\), then \((0)_+ = (0) \in \mathcal{M}\) by Lemma 2.3. Thus \((0) = \cap \{M : (0) \neq M \in \mathcal{M}\}\) and there exists some nontrivial element \(M \in \mathcal{M}\) and \(g \in M^\perp\) such that \(g(Ru) = 0\) and \(g(Rv) = 1\). Obviously \(M < \Phi(X_-)\), and so \(\Phi^{-1}(M) < X_-\). Take a nonzero vector \(z \in M\) and let \(A = \Phi^{-1}(z \otimes g)\). Then \(A \in J(\mathcal{N}, \hat{\Phi}^{-1}(M))\), which implies \(A^* \Phi^{-1}(M)^\perp = \{0\}\). Thus we
have
\[
\Phi(A(x \otimes f)B) = \Phi([A, [x \otimes f, B]]) = [z \otimes g, [R, u \otimes h]] = (z \otimes g)R(u \otimes h) = 0.
\]

So either \(A(x \otimes f) = 0\) or \((x \otimes f)B = 0\). If \(A(x \otimes f) = 0\), then \(0 = \Phi([A, x \otimes f]) = [z \otimes g, R] = (z \otimes g)R \neq 0\), which is impossible; If \((x \otimes f)B = 0\), then \(0 = \Phi([x \otimes f, B]) = [R, u \otimes h] = R(u \otimes h) \neq 0\), which is also impossible.

If \((0) < (0)_+ \in \mathcal{N}\), then \((0) < (0)_+ \in \mathcal{M}\) by Lemma 2.3. Let \(M = \hat{\Phi}((0)_+)\). Then \((0) < M = \hat{\Phi}((0)_+) < \hat{\Phi}(X_-)\) as \(\mathcal{M}\) has at least two nontrivial elements. By Lemma 2.8, \(M\) is not complemented and thus infinite-dimensional. So, there is a vector \(z \in M\) and a functional \(g \in Y^*\) such that \(g(Ru) = 0\) and \(g(z) = g(Rv) = 1\). Let \(A = \hat{\Phi}^{-1}(z \otimes g)\). As \(z \otimes g \in \Omega_2(\mathcal{M}, M)\), we have \(A \in \Omega_2(\mathcal{N}, (0)_+)\) by Lemma 2.6(3). It follows that \(A^* \hat{\Phi}^{-1}(M)\) is of rank one. Now by calculating \(\Phi(A(x \otimes f)B)\) in the same way as the above, one can get a contradiction.

So in the case that \(\hat{\Phi}\) is order-preserving, \(R\) is of rank one.

**Subcase 1.2.** \(\hat{\Phi}(\Omega_1(\mathcal{N}, X_-)) \subseteq I - \Omega_2(\mathcal{M}, \hat{\Phi}(X_-))\).

By Lemma 5.1(ii), in this case \(\hat{\Phi}\) is order-reversing. So, we have \((0) < (0)_+ = \hat{\Phi}(X_-) < Y\).

If \(Y = Y_-\), there is a nontrivial element \(M \in \mathcal{M}\) and vectors \(u, v \in M\) such that \(Ru\) and \(Rv\) are linearly independent. Let \(h \in M^\perp\) and \(B = \hat{\Phi}^{-1}(u \otimes h)\). Then \(B \in \mathcal{J}(\mathcal{N}, \hat{\Phi}^{-1}(M))\) by Lemma 2.4. Choose \(g\) in \(Y^*\) such that \(g(Ru) = 0\) and \(g(Rv) = 1\). Let \(z \in \hat{\Phi}(X_-)\) and \(A = \hat{\Phi}^{-1}(z \otimes g)\). By Lemma 2.6(2), \(I - A \in \Omega_1(\mathcal{N}, X_-)\). It follows from \(\hat{\Phi}(X_-) \leq M\) that \(\hat{\Phi}^{-1}(M) \leq X_-\). So we get
\[
\Phi(B(x \otimes f)(I - A)) = \Phi([B, [x \otimes f, I - A]]) = [u \otimes h, [R, -z \otimes g]] = 0,
\]
which implies that either \(B(x \otimes f) = 0\) or \((x \otimes f)(I - A) = 0\). If \(Bx \otimes f = 0\), then we get
\[
0 = \Phi([B, x \otimes f]) = [u \otimes h, R] = -Ru \otimes h \neq 0,
\]
a contradiction. If \(x \otimes f(I - A) = 0\), then
\[
0 = \Phi([x \otimes f, I - A]) = [R, -z \otimes g] = z \otimes R^*g \neq 0,
\]
again a contradiction.

If \(Y_- < Y\), there are two vectors \(u, v \in Y\) such that \(Ru\) and \(Rv\) are linearly independent. If \(u, v \in Y_-\), we take \(h \in Y_-^\perp\) and let \(B = \Phi^{-1}(u \otimes h)\). Then \(B \in \mathcal{J}(\mathcal{N}, \Phi^{-1}(Y_-))\). Choose \(g\) in \(Y^*\) such that \(g(Ru) = 0\) and \(g(Rv) = 1\). Let \(z \in \hat{\Phi}(X_-)\) and \(A = \hat{\Phi}^{-1}(z \otimes g)\). By Lemma 2.6(2), \(I - A \in \Omega_1(\mathcal{N}, X_-)\). Still, by assumption, we have \(\hat{\Phi}(X_-) < Y_-\), and so \(\hat{\Phi}^{-1}(Y_-) < X_-\). Thus by a similar argument to that in the preceding paragraph, one can get a contradiction. So we can assume that \(u \not\in Y_-\). Take \(h \in Y_-^\perp\) such that \(h(u) = 1\) and let \(B = \Phi^{-1}(u \otimes h)\). Then \(I - B \in \Omega_2(\mathcal{N}, \Phi^{-1}(Y_-))\) by Lemma 5.2(ii). In addition, there exists some \(g \in Y^*\) such that \(g(Ru) = 0\) and \(g(Rv) = 1\). Let \(z = Rv\) and \(A = \hat{\Phi}^{-1}(z \otimes g)\). Then, as \(z \in \hat{\Phi}(X_-) = (0)_+\), we see that \(z \otimes g \in \Omega_2(\mathcal{M}, (0)_+)\) and, by Lemma 2.6 and Lemma 5.2, \(I - A \in \Omega_1(\mathcal{N}, X_-)\).
Since \( \hat{\Phi}(X_-) < Y_- \), we have \( \hat{\Phi}^{-1}(Y_-) < X_- \). Calculating \( \Phi(B(x \otimes f)(I - A)) \), one can get a contradiction.

Hence \( \text{rank}R = 1 \) and \( \Phi(x \otimes f) \) is the sum of a scalar and a rank one operator, that is, the lemma is true for the case that \( x \in X_- \) and \( f \in X_-^\perp \). Moreover, \( \hat{\Phi}(X_- \otimes X_-^\perp) = \hat{\Phi}(X_-) \otimes \hat{\Phi}(X_-^\perp) \).

**Case 2.** \( x \in X \setminus X_- \) and \( f \in X_-^\perp \) with \( \langle x, f \rangle = 1 \).

Let \( P = x \otimes f \). Clearly, \( P \in \Omega_1(\mathcal{N}, X_-) \). By Case 1, we have \( \hat{\Phi}(X_- \otimes X_-^\perp) = \hat{\Phi}(X_-) \otimes \hat{\Phi}(X_-^\perp) \) and hence \( \Phi(X_- \otimes X_-^\perp) \subseteq F \hat{\Phi}(X_-) \otimes \hat{\Phi}(X_-^\perp) \). For the seek of convenience, write \( \hat{P} = \hat{\Phi}(P) \) and \( \hat{X}_- = \hat{\Phi}(X_-) \). Then \( \Phi(P) - \hat{P} \in F \) and it suffices to show that \( \hat{P} \) is the sum of a scalar and a rank-1 idempotent operator.

**Subcase 2.1.** \( \hat{\Phi}(\Omega_1(\mathcal{N}, X_-)) \subseteq \Omega_1(\mathcal{M}, \hat{X}_-) \).

In this subcase \( \hat{\Phi} \) is order-preserving by Lemma 5.2(i), and hence \( \hat{X}_- = Y_- \). Note that \( \Phi(FI) = FI \) by Lemma 2.9. So, applying the fact proved in Case 1, we have \( \Phi(FI + X_- \otimes X_-^\perp) = FI + Y_- \otimes Y_-^\perp \). Thus it follows from Lemma 2.10 that there exists a ring automorphism \( \tau : F \to \mathbb{F} \) and a map \( \gamma : X_- \times X_-^\perp \to \mathbb{F} \) such that either

\[
\Phi(y \otimes g) = \gamma(y, g)I + Cy \otimes Dg \quad \text{for all } y \in X_- \text{ and } g \in X_-^\perp, \quad (5.2)
\]

where \( C : X_- \to Y_- \) and \( D : X_-^\perp \to Y_-^\perp \) are two bijective \( \tau \)-linear maps; or

\[
\Phi(y \otimes g) = \gamma(y, g)I + Dg \otimes Cy \quad \text{for all } y \in X_- \text{ and } g \in X_-^\perp, \quad (5.3)
\]

where \( C : X_- \to Y_-^\perp \) and \( D : X_-^\perp \to Y_- \) are two bijective \( \tau \)-linear maps.

We first show that Eq.(5.3) can not occur. On the contrary, if Eq.(5.3) holds, for any \( y \in X_- \) and \( g \in X_-^\perp \), we have

\[
\Phi([y \otimes g, P]) = \Phi(y \otimes P^*g) = \gamma(y, P^*g)I + DP^*g \otimes Cy
\]

and

\[
\Phi([y \otimes g, P]) = [Dg \otimes Cy, \hat{P}] = Dg \otimes \hat{P}^*Cy,
\]

which imply \( Dg \otimes \hat{P}^*Cy = DP^*g \otimes Cy \) for all \( y \in X_- \) and \( g \in X_-^\perp \). Thus there exists some scalar \( \lambda \) such that \( D|_{X_-^\perp} = \lambda DP^*|_{X_-^\perp} \). Since \( P \) is of rank one, we see that \( D|_{X_-^\perp} \) is also of rank one, but this is impossible as \( X_-^\perp \) is infinite-dimensional.

So Eq.(5.2) holds. Then, for any \( y \in X_- \) and \( g \in X_-^\perp \), we have

\[
\Phi([y \otimes g, P]) = \Phi(y \otimes P^*g) = \gamma(y, P^*g)I + Cy \otimes DP^*g
\]

and

\[
\Phi([y \otimes g, P]) = [Cy \otimes Dg, \hat{P}] = Cy \otimes \hat{P}^*Dg.
\]
It follows that \( Cy \otimes \tilde{P}^* D g = \gamma(y, P^* g) I + Cy \otimes \tilde{P}^* D g \) holds for all \( y \in X_- \) and \( g \in X_\perp \). Since \( I \) is of infinite rank, we have \( \gamma(y, P^* g) = 0 \), and \( Cy \otimes \tilde{P}^* D g = Cy \otimes \tilde{P}^* g \). So \( \tilde{P}^* D g = D P^* g \) for all \( g \in X_\perp \). Since \( P \) is of rank one, it follows that the restriction of \( \tilde{P}^* \) to \( Y_\perp \) is of rank one. Note that \( \tilde{P}^* Y_\perp \subseteq Y_\perp \) and \( \tilde{P}^* Y_\perp = \tilde{P}^* (\tilde{P}^* Y_\perp) \subseteq \tilde{P}^* Y_\perp \). So \( \tilde{P}^* \) is of rank one, which implies that \( \tilde{P} \) is also of rank one, as desired.

**Subcase 2.2.** \( \tilde{\Phi}(\Omega_1(N, X_-)) \subseteq I - \Omega_2(\mathcal{M}, \overline{X_-}). \)

By Lemma 5.2(ii), \( \tilde{\Phi} \) is order-reversing, and \( \overline{X_-} = (0)_+ \). Applying the fact proved in Case 1, we have \( \tilde{\Phi}(I + X_- \otimes X_\perp) = I + (0)_+ \otimes (0)_+ \). Thus, by Lemma 2.10 again, there exists a ring automorphism \( \tau : \mathcal{F} \to \mathcal{F} \) and a map \( \gamma : X_- \times X_\perp \to \mathcal{F} \) such that either

\[
\Phi(x \otimes f) = \gamma(x, f) I + Cx \otimes Df \quad \text{for all } x \in X_- \text{ and } f \in X_\perp,
\]

where \( C : X_- \to (0)_+ \) and \( D : X_\perp \to (0)_+ \) are two \( \tau \)-linear bijective maps; or

\[
\Phi(x \otimes f) = \gamma(x, f) I + Df \otimes Cx \quad \text{for all } x \in X_- \text{ and } f \in X_\perp,
\]

where \( C : X_- \to (0)_+ \) and \( D : X_\perp \to (0)_+ \) are two \( \tau \)-linear bijective maps.

If Eq.(5.4) holds, by calculating \( \Phi([y \otimes g, P]) \), one obtains \( (I - \tilde{P}) Cy \otimes D g = Cy \otimes \tilde{P}^* g \) for all \( y \in X_- \) and \( g \in X_\perp \). Since \( P \) is of rank one, we get \( D \) is also of rank one, which is impossible. So we must have that Eq.(5.5) holds. By calculating \( \Phi([y \otimes g, P]) \), one can get \( (I - \tilde{P}) D g = D P^* g \) for all \( f \in X_\perp \), which implies that the restriction of \( I - \tilde{P} \) to \( (0)_+ \) is of rank one. So \( I - \tilde{P} \) is of rank one and \( \tilde{P} \) is the sum of a scalar and a rank one operator, as desired.

Summing up, we have proved that \( \Phi(x \otimes f) = \Phi(P) \) is the sum of a scalar and a rank one operator if \( x \in X, f \in X_\perp \) with \( \langle x, f \rangle = 1 \). Moreover, \( \tilde{\Phi}(\Omega_1(N, X_-)) \subseteq \Omega_1(\mathcal{M}, \overline{X_-}) \) implies that Eq.(5.2) holds, while \( \tilde{\Phi}(\Omega_1(N, X_-)) \subseteq I - \Omega_2(\mathcal{M}, \overline{X_-}) \) implies that Eq.(5.5) holds.

**Case 3.** \( x \in X \setminus X_- \) and \( f \in X_\perp \).

Now assume that \( x \in X \setminus X_- \) and \( f \in X_\perp \). We need still consider two subcases.

**Subcase 3.1.** \( \langle x, f \rangle = \lambda \neq 0 \).

Let \( P = \lambda^{-1} x \otimes f \). Then the rank-one idempotent \( P \in \Omega_1(N, X_-) \) and \( x \otimes f = \lambda P \). Write \( \hat{P} = \tilde{\Phi}(P) \). By what proved in Case 2, \( \hat{P} \) is a rank-1 idempotent.

**Subcase 3.1.1.** \( \hat{\Phi}(\Omega_1(N, X_-)) \subseteq \Omega_1(\mathcal{M}, \overline{X_-}) \).

By Lemma 5.2(i), \( \hat{\Phi} \) is order-preserving, and then \( \overline{X_-} = Y_- \). So we still have that either Eq.(5.2) or Eq.(5.3) holds.

If Eq.(5.2) holds, then for any \( y \in X_- \) and \( g \in X_\perp \), we have

\[
\Phi([y \otimes g, \lambda P]) = [\Phi(y \otimes g), \Phi(\lambda P)] = C y \otimes \Phi(\lambda P)^* D g - \Phi(\lambda P) C y \otimes D g
\]
and

\[ \Phi([y \otimes g, \lambda P]) = [\tau(\lambda)C y \otimes D g, \tilde{P}] = \tau(\lambda)C y \otimes \tilde{P}^* D g. \]

It follows that \( \tau(\lambda)C y \otimes \tilde{P}^* D g = C y \otimes \Phi(\lambda P)^* D g - \Phi(\lambda P) C y \otimes D g, \) that is,

\[ C y \otimes (\tau(\lambda)\tilde{P}^* D g - \Phi(\lambda P)^* D g) = -\Phi(\lambda P) C y \otimes D g \text{ for all } y \in X_\pm \text{ and } g \in X_\pm. \]

Since \( C \) is bijective, there exists some scalar \( \alpha \) such that \( \tau(\lambda)\tilde{P}^* D g - \Phi(\lambda P)^* D g = \alpha D g \) for all \( g \in X_\pm, \) that is, \( (\Phi(\lambda P)^* + \alpha I)|_{Y_\pm} = \tau(\lambda)\tilde{P}^*|_{Y_\pm} \) as \( D : X_\pm \to Y_\pm \) is bijective. Let \( \Phi(\lambda P) = \Phi(\lambda P) + \alpha I. \) Note that \( [\lambda P, A] = [\lambda P, [P, A]] \) holds for all \( A \in \text{Alg}^N. \) So we get

\[ [\Phi(\lambda P), \Phi(A)] = [\Phi(\lambda P), [\tilde{P}, [\tilde{P}, \Phi(A)]]], \quad \forall A \in \text{Alg}^N. \quad (5.6) \]

By the bijectivity of \( \Phi, \) for any \( y \in Y \) and \( g \in Y_\pm, \) there exists some \( A \in \text{Alg}^N \) such that \( \Phi(A) = y \otimes g. \) Thus Eq.(5.6) entails

\[ (\Phi(\lambda P) - \Phi(\lambda P)\tilde{P}) y \otimes g = (\tau(\lambda)\tilde{P} + \Phi(\lambda P) - 2\Phi(\lambda P)\tilde{P}) y \otimes \tilde{P}^* g \]

for all \( y \in Y \) and \( g \in Y_\pm. \) By Case 2, we can write \( \tilde{P} = u \otimes h, \) where \( u \in Y \) and \( h \in Y_\pm \) with \( \langle u, h \rangle = 1. \) Since \( \text{dim}(Y_\pm) > 2, \) there exists some \( g_1 \in Y_\pm \) such that \( g_1 \) is linearly independent of \( h. \) If \( \langle u, g_1 \rangle \neq 0, \) let \( g = g_1. \) If \( \langle u, g_1 \rangle = 0, \) let \( g = h + g_1. \) Then \( g \) and \( h \) are linearly independent and \( \langle u, g \rangle \neq 0. \) So \( g \) and \( \tilde{P}^* g \) are also linearly independent. By Eq.(5.7), we get

\[ (\Phi(\lambda P) - \Phi(\lambda P)\tilde{P}) y = 0 \text{ for all } y \in Y. \]

It follows that \( \Phi(\lambda P) = \Phi(\lambda P)\tilde{P}, \) which implies \( \Phi(\lambda P) \) is of rank one. So \( \Phi(\lambda P) = \Phi(\lambda P) - \alpha I \) is the sum of a scalar and a rank one operator.

We claim that Eq.(5.3) can not occur. If, on the contrary, Eq.(5.3) holds, then for any \( y \in X_\pm \text{ and } g \in X_\pm, \) we have

\[ \Phi([y \otimes g, \lambda P]) = [\Phi(y \otimes g), \Phi(\lambda P)] = D g \otimes \Phi(\lambda P)^* C y - \Phi(\lambda P) D g \otimes C y \]

and

\[ \Phi([y \otimes g, \lambda P]) = \Phi(\lambda y \otimes P^* g) = \tau(\lambda) D(P^* g) \otimes C y. \]

It follows that \( D g \otimes \Phi(\lambda P)^* C y = (\Phi(\lambda P) D g + \tau(\lambda) D(P^* g)) \otimes C y \) for all \( y \in X_\pm \text{ and } g \in X_\pm. \) So there exists some scalar \( \gamma \) such that \( \Phi(\lambda P)^* C y = \gamma C y, \) which implies \( \Phi(\lambda P)^* = \gamma I \) on \( Y_\pm \) as \( C \) is bijective. Now, for any \( A \in \text{Alg}^N, \) by the relation \( [\lambda P, A] = [\lambda P, [P, A]] \), we get

\[ [\Phi(\lambda P), \Phi(A)] = [\Phi(\lambda P), [\tilde{P}, [\tilde{P}, \Phi(A)]]]. \]

Particularly, for any \( y \in Y \) and \( g \in Y_\pm, \) there exists some \( A \in \text{Alg}^N \) such that \( \Phi(A) = y \otimes g. \) Thus we have \([\Phi(\lambda P), y \otimes g] = [\Phi(\lambda P), [\tilde{P}, [\tilde{P}, y \otimes g]]], \) that is,

\[ (\Phi(\lambda P) - \Phi(\lambda P)\tilde{P} - \gamma I + \gamma \tilde{P}) y \otimes g = (\Phi(\lambda P) - 2\Phi(\lambda P)\tilde{P} + 2\gamma \tilde{P} - \gamma I) y \otimes \tilde{P}^* g \]
holds for all \( y \in Y \) and \( g \in Y_\perp \). Still, we can choose \( g \) such that \( g \) and \( \tilde{P}^*g \) are linearly independent. The above equation yields \( (\Phi(\lambda P) - \Phi(\lambda P)\tilde{P} - \gamma I_+ + \gamma \tilde{P})y = (\Phi(\lambda P) - 2\Phi(\lambda P)\tilde{P} + 2\gamma \tilde{P} - \gamma I)y = 0 \) for all \( y \in Y \). This implies \( \Phi(\lambda P) = \gamma I \), which is contradicting to \( \Phi(\mathbb{F} I) = \mathbb{F} I \) and the bijectivity of \( \Phi \). So Eq.(5.3) can not occur.

**Subcase 3.1.2.** \( \hat{\Phi}(\Omega_1(N, X_-)) \subseteq I - \Omega_2(M, X_-) \)

By Lemma 5.2(ii), \( \hat{\Phi} \) is order-reversing and thus \( \hat{X}_- = (0)_+ \). It follows that either Eq.(5.4) or Eq.(5.5) holds. By a similar argument to the Subcase 3.1.1 above, one can check that, Eq.(5.4) can not occur and if Eq.(5.5) holds, then \( \Phi(x \otimes f) = \Phi(\lambda P) \) is the sum of a scalar and a rank one operator.

**Subcase 3.2.** \( \langle x, f \rangle = 0 \).

In this case, take \( x_1 \in X \) such that \( \langle x_1, f \rangle = 1 \) and let \( x_2 = x - 2x_1, x_3 = x - x_1 \). Then \( \langle x_i, f \rangle \neq 0 \) for \( i = 1,2,3 \). By Subcase 3.1, \( \Phi(x_i \otimes f) \) is the sum of a scalar and a rank one operator. So we can assume \( \Phi(x_i \otimes f) = \lambda_i I + u_i \otimes h_i \) for some \( \lambda_i, i = 1,2,3 \). Note that

\[
(\lambda_1 + \lambda_2)I + u_1 \otimes h_1 + u_2 \otimes h_2 = \Phi((x_1 + x_2) \otimes f) = \Phi(x_3 \otimes f) = \lambda_3 I + u_3 \otimes h_3.
\]

Since \( I \) is of infinite rank, we must have \( u_1 \otimes h_1 + u_2 \otimes h_2 = u_3 \otimes h_3 \). This forces that \( \{u_1, u_3\} \) or \( \{h_1, h_3\} \) is a linearly dependent set. So \( u_1 \otimes h_1 + u_3 \otimes h_3 \) is of rank one. Thus \( \Phi(x \otimes f) = \Phi((x_1 + x_3) \otimes f) = (\lambda_1 + \lambda_3)I + u_1 \otimes h_1 + u_3 \otimes h_3 \) is the sum of a scalar and a rank one operator.

Combining Cases 1-3 and the bijectivity of \( \Phi \), we have shown that

(i) \( \Phi(\mathbb{F}I + X \otimes X_\perp) = \mathbb{F}I + Y \otimes Y_\perp \) if \( \hat{\Phi}(\Omega_1(N, X_-)) \subseteq \Omega_1(M, X_-) \);

(ii) \( \Phi(\mathbb{F}I + X \otimes X_\perp) = \mathbb{F}I + (0)_+ \otimes Y^* \) if \( \hat{\Phi}(\Omega_1(N, X_-)) \subseteq I - \Omega_2(M, X_-) \).

So Lemma 2.10 is applicable. Observing the arguments in Case 2 and Case 3, it is easily seen that (i) implies Lemma 5.1(1) and (ii) implies Lemma 5.1(2).

The proof of Lemma 5.3 is finished.

For the case that the nest has only one nontrivial element, we have

**Lemma 5.4.** Assume that \( N = \{(0), X_-, X\} \) and \( X_- \) is not complemented in \( X \). Then for any \( x \in X \) and \( f \in X_\perp \), \( \Phi(x \otimes f) \) is the sum of a scalar and a rank one operator. Moreover, the statements (1) and (2) of Lemma 5.1 hold.

**Proof.** Not that \( (0)_+ = X_- \) in this situation. By Lemma 2.3, \( M = \{(0), Y_-, Y\} \) and \( (0)_+ = Y_- \) is not complemented in \( Y \). Let \( R \in \mathcal{J}(M, Y_-) \) so that \( \Phi(x \otimes f) - R \in \mathbb{F}I \) (Lemma 2.2). We remark here that, since \( N \) only contains one nontrivial element, Lemma 5.2 and Lemma 2.6(3) are not applicable. Similar to the proof of Lemma 5.3, we consider three cases.

**Case 1.** \( x \in X_- \) and \( f \in X_\perp \).
In this case, $x \otimes f \in J(N, X_-)$. We'll prove $\text{rank} R = 1$. Assume on the contrary that $\text{rank} R \geq 2$. Then there are two vectors $u, v \in Y \setminus Y_-$ such that $Ru$ and $Rv$ are linearly independent. Take $h \in Y_+^\perp$ such that $h(u) = 1$. Then $u \otimes h \in \Omega_1(M, Y_-)$. Note that $Y_-$ is infinite-dimensional. There is a vector $z \in Y_-$ and a functional $g \in Y^*$ such that $g(Ru) = 0$ and $g(z) = g(Rv) = 1$. Let $A = \tilde{\Phi}^{-1}(z \otimes g)$ and $B = \tilde{\Phi}^{-1}(u \otimes h)$.

By Lemma 2.6(1), either $\tilde{\Phi}(\Omega_1(N, X_-)) \subseteq \Omega_1(M, Y_-)$ or $I - \tilde{\Phi}(\Omega_1(N, X_-)) \subseteq \Omega_2(M, Y_-)$. If $\tilde{\Phi}(\Omega_1(N, X_-)) \subseteq \Omega_1(M, Y_-)$, then $B \in \Omega_1(N, X_-)$, and by Lemma 2.6(2), $A \in \Omega_2(N, X_-)$. Thus we get $\Phi(A(x \otimes f)B) = \Phi([A, [x \otimes f, B]]) = [z \otimes g, [R, u \otimes h]] = (z \otimes g)R(u \otimes h) = 0$, which implies either $A(x \otimes f) = 0$ or $(x \otimes f)B = 0$. If $A(x \otimes f) = 0$, then $0 = \Phi([A, x \otimes f]) = [z \otimes g, R] = (z \otimes g)R$, which is impossible; if $(x \otimes f)B = 0$, then $0 = \Phi([x \otimes f, B]) = [R, u \otimes h] = R(u \otimes h)$, which is also impossible. If $I - \tilde{\Phi}(\Omega_1(N, X_-)) \subseteq \Omega_2(M, Y_-)$, then $I - B \in \Omega_2(N, X_-)$, and by Lemma 2.6(2), $I - A \in \Omega_1(N, X_-)$. Thus $\Phi((I - B)(x \otimes f)(I - A)) = \Phi([I - B, [x \otimes f, I - A]]) = [-u \otimes h, [R, -z \otimes g]] = 0$. So we get either $(I - B)(x \otimes f) = 0$ or $(x \otimes f)(I - A) = 0$. Still, this is impossible.

Hence rank $R = 1$.

**Case 2.** $x \in X \setminus X_-$ and $f \in X_+^\perp$ with $\langle x, f \rangle = 1$.

In this case $P = x \otimes f \in \Omega_1(N, X_-)$. Write $\tilde{P} = \tilde{\Phi}(P)$. By Case 1 and Lemma 2.9, we have $\Phi(\mathbb{F}I + X_- \otimes X_-^\perp) = \mathbb{F}I + Y_- \otimes Y_-^\perp$. So, by Lemma 2.10, there exists a ring automorphism $\tau : \mathbb{F} \to \mathbb{F}$ and a map $\gamma : X_- \times X_-^\perp \to \mathbb{F}$ such that either

$$\Phi(y \otimes g) = \gamma(y, g)I + Cy \otimes Dg$$

for all $y \in X_-$ and $g \in X_-^\perp$, (5.8)

where $C : X_- \to Y_-$ and $D : X_-^\perp \to Y_-^\perp$ are two $\tau$-linear bijective maps; or

$$\Phi(y \otimes g) = \gamma(y, g)I + Dg \otimes Cy$$

for all $y \in X_-$ and $g \in X_-^\perp$, (5.9)

where $C : X_- \to Y_-^\perp$ and $D : X_-^\perp \to Y_-$ are two $\tau$-linear bijective maps.

Assume first that Eq.(5.8) holds. If $\tilde{\Phi}(\Omega_1(N, X_-)) \subseteq \Omega_1(M, Y_-)$, then $\tilde{P} \in \Omega_1(M, Y_-)$. Thus, for any $y \in Y$ and any $g \in Y_-^\perp$, we have

$$\Phi([y \otimes g, P]) = \Phi(y \otimes P^*g) = \gamma(y, P^*g)I + Cy \otimes DP^*g$$

and

$$\Phi([y \otimes g, P]) = [Cy \otimes Dg, \tilde{P}] = Cy \otimes \tilde{P}^*Dg.$$  

Since $I$ is of infinite rank, the above two equations yields $Cy \otimes \tilde{P}^*Dg = Cy \otimes DP^*g$, and so $\tilde{P}^*Dg = DP^*g$ for all $g \in X_-^\perp$. Since $P$ is of rank one, the restriction of $\tilde{P}^*$ to $Y_-^\perp$ is of rank one. Note that $\tilde{P}^*Y^* \subseteq Y_-^\perp$ and $\tilde{P}^*Y^* = \tilde{P}^*(\tilde{P}^*Y^*) \subseteq \tilde{P}^*Y_-^\perp$. So $\tilde{P}^*$ is of rank one, which implies that $\tilde{P}$ is also of rank one. Hence $\Phi(P)$ is the sum of a scalar and a rank one operator.
If \( I - \Phi(\Omega_1(\mathcal{N}, X_-)) \subseteq \Omega_2(\mathcal{M}, Y_-) \), then \( I - \tilde{P} \in \Omega_2(\mathcal{M}, Y_-) \). For any \( y \in Y \) and any \( g \in Y_\perp \), we have

\[
\Phi([y \otimes g, P]) = \Phi(y \otimes P^*g) = \gamma(y, P^*g)I + Cy \otimes DP^*g
\]

and

\[
\Phi([y \otimes g, P]) = [Cy \otimes Dg, \tilde{P}] = -[Cy \otimes Dg, I - \tilde{P}] = (I - \tilde{P})Cy \otimes Dg.
\]

The above two equations yield \((I - \tilde{P})Cy \otimes Dg = Cy \otimes DP^*g \) for all \( y \in X_- \) and \( g \in X_\perp \), and hence \( D \) and \( DP^* \) are linearly dependent. Since \( P \) is of rank one, \( D \) is also of rank one, which is impossible as \( X_\perp \) is infinite-dimensional. Therefore, this case can not occur.

Similarly, if Eq.(5.9) holds, one can show that \( I - \Phi(\Omega_1(\mathcal{N}, X_-)) \subseteq \Omega_2(\mathcal{M}, Y_-) \) and \( \Phi(P) \) is the sum of a scalar and a rank one operator.

Case 3. \( x \in X \setminus X_- \) and \( f \in X_\perp \).

Note that, we still have that either Eq.(5.8) or Eq.(5.9) holds.

Subcase 3.1. \( \langle x, f \rangle = \lambda \neq 0 \).

Then there exists a rank-one idempotent \( P \in \Omega_1(\mathcal{N}, X_-) \) such that \( x \otimes f = \lambda P \). Write \( \tilde{P} = \Phi(P) \). By Case 2, \( \tilde{P} \) is a rank-1 idempotent.

Assume that Eq.(5.8) holds. If \( \tilde{P} = \Phi(\Omega_1(\mathcal{N}, X_-)) \subseteq \Omega_1(\mathcal{M}, Y_-) \), then \( \tilde{P} \in \Omega_1(\mathcal{M}, Y_-) \). So, for any \( y \in X_- \) and \( g \in X_\perp \), we have

\[
\Phi([y \otimes g, \lambda P]) = [\Phi(y \otimes g), \Phi(\lambda P)] = Cy \otimes \Phi(\lambda P)^*Dg - \Phi(\lambda P)Cy \otimes Dg
\]

and

\[
\Phi([y \otimes g, \lambda P]) = \Phi([\lambda y \otimes g, P]) = [\tau(\lambda)Cy \otimes Dg, \tilde{P}] = \tau(\lambda)Cy \otimes \tilde{P}^*Dg.
\]

It follows that

\[
Cy \otimes (\tau(\lambda)\tilde{P}^*Dg - \Phi(\lambda P)^*Dg) = -\Phi(\lambda P)Cy \otimes Dg \text{ for all } y \in X_- \text{ and } g \in X_\perp.
\]

Hence there exists some scalar \( \alpha \) such that \( \tau(\lambda)\tilde{P}^*Dg - \Phi(\lambda P)^*Dg = \alpha Dg \) for all \( g \in X_\perp \), which implies that \((\Phi(\lambda P)^* + \alpha I)|_{Y_\perp} = \tau(\lambda)\tilde{P}^*|_{Y_\perp} \) as \( D \) is bijective. Let \( \Phi(\lambda P) = \Phi(\lambda P) + \alpha I \). For any \( A \in \text{Alg}\mathcal{N} \), by the relation \( [\lambda P, A] = [\lambda P, [P, [P, A]]] \), we get \([\Phi(\lambda P), \Phi(A)] = [\Phi(\lambda P), \tilde{P}, [\tilde{P}, [\tilde{P}, y \otimes g]]]\). Particularly, for any \( y \in Y \) and \( g \in Y_\perp \), By the bijectivity of \( \Phi \), there exists some \( A \in \text{Alg}\mathcal{N} \) such that \( \Phi(A) = y \otimes g \). So we have \([\Phi(\lambda P), y \otimes g] = [\Phi(\lambda P), [\tilde{P}, [\tilde{P}, y \otimes g]]]\), that is,

\[
(\Phi(\lambda P) - \Phi(\lambda P)\tilde{P})y \otimes g = (\tau(\lambda)\tilde{P} + \Phi(\lambda P) - 2\Phi(\lambda P)\tilde{P})y \otimes \tilde{P}^*g \quad (5.10)
\]

holds for all \( y \in Y \) and \( g \in Y_\perp \). By Case 2, we can write \( \tilde{P} = u \otimes h \), where \( u \in Y \) and \( h \in Y_\perp \) with \( \langle u, h \rangle = 1 \). Since \( \dim(Y_\perp) > 2 \), there exists \( g \) with \( \langle u, g \rangle \neq 0 \) such that \( g \) and \( h \)
are linearly independent. So \( g \) and \( \tilde{P}^*g \) are also linearly independent. By Eq.(5.10), we get 
\[
(\Phi(\lambda P) - \Phi(\lambda P)\tilde{P})y = 0 \text{ for all } y \in Y.
\]
So \( \Phi(\lambda P) = \Phi(\lambda P)\tilde{P} \), which implies \( \Phi(\lambda P) \) is of rank one. So \( \Phi(\lambda P) = \Phi(\lambda P) - \alpha I \) is the sum of a scalar and a rank one operator.

We claim that the case \( I - \tilde{\Phi}(\Omega_1(N, X_\perp)) \subseteq \Omega_2(M, Y_\perp) \) does not happen. Otherwise, we have \( I - \tilde{P} \in \Omega_2(M, Y_\perp) \). Then for any \( y \in X_\perp \) and \( g \in X_\perp^\perp \), by calculating \( \Phi([y \otimes g, \lambda P]) \), one can obtain
\[
Cy \otimes \Phi(\lambda P)^*Dg = (\Phi(\lambda P) - \tau(\lambda)(I - \tilde{P}))Cy \otimes Dg \text{ for all } y \in X_\perp \text{ and } g \in X_\perp^\perp.
\]
It follows that there exists some scalar \( \gamma \) such that \( \Phi(\lambda P)^*Dg = \gamma Dg \), which implies \( \Phi(\lambda P)^* = \gamma I \) on \( Y_\perp^\perp \) as \( D \) is bijective. Now, for any \( A \in \text{Alg}N \), by the relation \( [\lambda P, A] = [\lambda P, [P, [P, A]]] \), we get \([\Phi(\lambda P), \Phi(A)] = [\Phi(\lambda P), [\tilde{P}, [\tilde{P}, \Phi(A)]]] \). Particularly, for any \( y \in Y \) and \( g \in Y_\perp^\perp \), there exists some \( A \in \text{Alg}N \) such that \( \Phi(A) = y \otimes g \). So we have \([\Phi(\lambda P), y \otimes g] = [\Phi(\lambda P), [\tilde{P}, [\tilde{P}, y \otimes g]]] \), that is,
\[
(\Phi(\lambda P) - \Phi(\lambda P)\tilde{P} - \gamma I + \gamma \tilde{P})y \otimes g = (\Phi(\lambda P) - 2\Phi(\lambda P)\tilde{P} + 2\gamma \tilde{P} - \gamma I)y \otimes \tilde{P}^*g 
\]
holds for all \( y \in Y \) and \( g \in Y_\perp^\perp \). Still, we can choose \( g \) such that \( g \) and \( \tilde{P}^*g \) are linearly independent. By Eq.(5.11), we get 
\[
(\Phi(\lambda P) - \Phi(\lambda P)\tilde{P} - \gamma I + \gamma \tilde{P})y = (\Phi(\lambda P) - 2\Phi(\lambda P)\tilde{P} + 2\gamma \tilde{P} - \gamma I)y = 0 \text{ for all } y \in Y.
\]
This leads to a contradiction \( \Phi(\lambda P) = \gamma I \).

If Eq.(5.9) holds, by a similar argument to that of the above, one can show that \( I - \tilde{\Phi}(\Omega_1(N, X_\perp)) \subseteq \Omega_2(M, Y_\perp) \) and \( \Phi(\lambda P) \) is also the sum of a scalar and a rank one operator.

Subcase 3.2. \( \langle x, f \rangle = 0 \).

The proof is the same to that of Subcase 3.2 in Lemma 5.3. We omit it here.

Combining Cases 1-3, we see that the statements (i) and (ii) in the proof of Lemma 5.3 still hold, and this completes the proof of Lemma 5.4. \( \square \)

Proof of Lemma 5.1. It is immediate from Lemma 5.3 and Lemma 5.4. \( \square \)

Proof of Theorem 1.1 for the case \( X_\perp \neq X \) and \( X_\perp \) is not complemented.

Now let us show that Theorem 1.1 is true for the case that \( X_\perp \neq X \) and \( X_\perp \) is not complemented. By Lemma 5.1, (1) or (2) holds.

If Lemma 5.1(1) holds, then \( \Phi(FI + X \otimes X_\perp) = FI + Y \otimes Y_\perp \), and there exists a ring automorphism \( \tau : F \rightarrow F \) and a map \( \gamma : X \times X_\perp \rightarrow F \) such that
\[
\Phi(x \otimes f) = \gamma(x, f)I + Cx \otimes Df 
\]
holds for all \( x \in X \) and \( f \in X_\perp \), where \( C : X \rightarrow Y \) and \( D : X_\perp \rightarrow Y_\perp \) are two \( \tau \)-linear bijective maps. Thus for any \( A \in \text{Alg}N \), any \( x \in X \) and any \( f \in X_\perp \), as \( \Phi([A, x \otimes f]) = \)
\[ [\Phi(A), \Phi(x \otimes f)] \text{ and } I \text{ is of infinite rank, one obtains that} \]

\[ Cx \otimes (\Phi(A)^* Df - DA^* f) = (\Phi(A)Cx - CAx) \otimes Df. \]

As \( D \) is injective, the above equation entails that there exists a scalar \( h(A) \) such that

\[ \Phi(A)C = CA + h(A)C. \tag{5.13} \]

It is clear that \( h \) is an additive functional on \( \text{Alg}\mathcal{N} \). Define \( \Psi : \text{Alg}\mathcal{N} \to \text{Alg}\mathcal{M} \) by \( \Psi(A) = \Phi(A) - h(A)I \) for all \( A \in \text{Alg}\mathcal{N} \). Then, \( \Psi \) is an additive bijection. By Eq.(5.13), for any \( A, B \in \text{Alg}\mathcal{N} \), we have

\[ \Psi(AB)C = CAB = \Psi(A)CB = \Psi(A)\Psi(B)C. \]

Since \( \text{ran}C = Y \), we see that \( \Psi(AB) = \Psi(A)\Psi(B) \) for all \( A, B \in \text{Alg}\mathcal{N} \), that is, \( \Psi \) is a ring isomorphism.

Assume that Lemma 5.1(2) holds. Then \( \Phi(FI + X \otimes X^\perp) = FI + (0)_+ \otimes Y^* \) and there exists a ring automorphism \( \tau : F \to F \) and a map \( \gamma : X \otimes X^\perp \to F \) such that

\[ \Phi(x \otimes f) = \gamma(x, f)I + Df \otimes Cx \tag{5.14} \]

holds for all \( x \in X \) and \( f \in X^\perp \), where \( C : X \to Y^* \) and \( D : X^\perp \to (0)_+ \) are two \( \tau \)-linear bijective maps. A similar argument to the above, one can check that there exists an additive functional \( h \) and a ring anti-isomorphism \( \Psi \) such that \( \Phi(A) = -\Psi(A) + h(A)I \) for all \( A \in \text{Alg}\mathcal{N} \).

This completes the proof of Theorem 1.1 for the case that \( (0) < X^- < X \) and \( X^- \) is not complemented. \( \square \)

The proof of Theorem 1.1 for the case that \( (0) < (0)_+ < X \) and \( (0)_+ \) is not complemented is similar and we omit it here.

Now, combining Sections 3-5, the proof of Theorem 1.1 is finished.

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