FUNDAMENTAL GROUPS OF PEANO CONTINUA

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Abstract. Extending a theorem of Shelah we prove that fundamental groups of Peano continua (locally connected and connected metric compact spaces) are finitely presented if they are countable. The proof uses ideas from geometric group theory.
1. Introduction

This paper is motivated by a question posed to the second author by Mladen Bestvina during his talk at the Spring Topology and Dynamics Conference in Gainesville (March 7-9, 2009):

**Question 1.1.** Is the fundamental group of a Peano continuum finitely presented if it is countable?

It turns out that question was also posed by de la Harpe [10] on p.48 and it is relevant in view of the following:

**Shelah Theorem 1.2.** If $X$ is a Peano continuum and $\pi_1(X)$ is countable, then $\pi_1(X)$ is finitely generated.

Pawlikowski [13] presented another proof of 1.2 from which we extract the following (see the paragraph preceding Lemma 2 in [13] or Theorems 2 and 8 in [8]):

**Theorem 1.3 (Pawlikowski [13]).** If $X$ is a Peano continuum and $\pi_1(X)$ is countable, then $X$ is semi-locally simply connected.

Notice that the second author constructed (see [17]), for each countable group $G$, a 2-dimensional path-connected subcontinuum $X_G$ of $\mathbb{R}^4$ whose fundamental group is $G$ (see [12] and [14] for earlier constructions of compact spaces with a given fundamental group).

Our solution to 1.1 is based on an application of methods of geometric group theory: we construct a geometric action of $\pi_1(X)$ on a coarsely 1-connected proper geodesic space $\tilde{X}$ and we use Švarc-Milnor Lemma ([2, page 140]) plus the fact $G$ is finitely presented if and only if it is coarsely 1-connected.

In geometric group theory, a geometry is any proper, geodesic metric space. An action of a finitely-generated group $G$ on a geometry $X$ is geometric if it satisfies the following conditions:

1. Each element of $G$ acts as an isometry of $X$.
2. The action is cocompact, i.e. the quotient space $X/G$ is a compact space.
3. The action is properly discontinuous, with each point having a finite stabilizer.

Let a group $G$ act on a topological space $X$ by homeomorphisms. Consider a subgroup $H \subseteq G$. One then says that a set $Y$ is precisely invariant under $H$ in $G$ if

$$\forall h \in H, \ h(Y) = Y \quad \text{and} \quad \forall g \in G-H, \ gY \cap Y = \emptyset.$$ 

Then let $G_x$ be the stabilizer of $x$ in $G$. One says that $G$ acts discontinuously at $x$ in $X$ if the stabilizer $G_x$ is finite and there exists a neighborhood $U$ of $x$ that is precisely invariant under $G_x$ in $G$. If $G$ acts discontinuously at every point $x$ in $X$, then one says that $G$ acts properly discontinuously on $X$. 
Theorem 1.4 (Švarc-Milnor [2] or [3]). A group $G$ acting properly discontinuously and cocompactly via isometries on a length space $X$ is finitely generated and induces a quasi-isometry equivalence $g \rightarrow g \cdot x_0$ for any $x_0 \in X$.

Added in proof: We were informed by Greg Conner [7] that Katsuya Eda answered [4] about 5 years ago (unpublished). The argument is contained in [5]: by [4] the space is semi-locally simply-connected and homotopically Hausdorff. Corollary 5.7 says such space has finitely presented fundamental group.

Alternatively, it is pointed out in Lemma A.3 in [6] that this shows that such a space has finitely presented fundamental group. Here is the argument from Lemma A.3. Semilocally simply-connected implies that the space is two-set simple – see [4]. This implies that the fundamental group is group is the fundamental group of the nerve of a finite cover which implies that it’s finitely presented, again from [4].

2. Coarse 1-connectivity of uniformly path connected spaces

In order to complete the proof of our main result [3,4] we need to relate coarse 1-connectivity to simple connectedness.

Recall $(X,d)$ is coarsely 1-connected if it is $t$-chain connected for some $t > 0$ (that is equivalent to $(X,d)$ being coarsely 0-connected - see [11]) and for each $r > 0$ there is $R > 0$ such that the induced map $\text{Rips}_r(X) \rightarrow \text{Rips}_R(X)$ induces the trivial homomorphism of the fundamental groups (see Definition 42 on p.19 of [11]). Here $\text{Rips}_r(X)$ is the Rips complex of $X$, i.e. the complex whose simplices are all finite subsets $A$ of $X$ of diameter at most $r$.

Definition 2.1. A path connected metric space $(X,d)$ is uniformly path connected if there is a function $\alpha : (0,\infty) \rightarrow (0,\infty)$ so that every two points $x, y \in X$ can be connected by a path of diameter at most $\alpha(d(x, y))$.

The fundamental group $\pi_1(X,x_0)$ of a path connected metric space $X$ is uniformly generated (see [9]) if it has a generating set of loops of diameter at most $R$ for some $R > 0$. Equivalently, every map $f : (S^1,0) \rightarrow (X,x_0)$ can be extended over 1–skeleton of some subdivision $\tau$ of $(B^2,0)$ to a map $F$ so that the diameter $F(\partial \Delta)$ is at most $R$, for every simplex $\Delta$ of $\tau$.

Theorem 2.2. Suppose $X$ is a uniformly path connected space. $X$ is coarsely 1-connected if and only if $\pi_1(X,x_0)$ is uniformly generated.

Proof. Assume $X$ is coarsely 1-connected. Fix positive numbers $r, R$ so that $\pi_1$ applied to $\text{Rips}_r(X) \rightarrow \text{Rips}_R(X)$ is trivial. Furthermore, let $l$ be a positive number so that every two points of $X$ that are at most $R$ apart can be connected by a path of diameter at most $l$. Let $\alpha : (S^1,0) \rightarrow (X,x_0)$ be a loop. Subdivide $(S^1,0)$ to obtain a subdivision $\tau$ (notation: $S^2_l$) so that the diameter of $\alpha(\Delta)$ is at most $R$ for every edge $\Delta$ of $\tau$. The map $\alpha^{(S^2_l)(0)}$ induces a simplicial map $\tilde{\alpha} : (S^2_l,0) \rightarrow (\text{Rips}_r(X),x_0)$, which extends to a map $\tilde{\beta} : (B^2,0) \rightarrow \text{Rips}_R(X)$. We may assume $\sigma$ is a subdivision of $(B^2,0)$ so that $\tilde{\beta}$ is simplicial and $\sigma|_{S^1}$ is a subdivision of $\tau$. Then $\tilde{\beta}$ induces a map $\beta : ((B^2_\sigma)^{(1)}),0) \rightarrow (X,x_0)$ as follows: $\beta$ equals $\tilde{\beta}$ on vertices and $S^1$ and for every edge $E$ of $B^2_\sigma \setminus S^1$ we can connect two boundary points $\beta(\partial E)$ by a path of diameter at most $l$. Hence we obtain an extension $\beta : ((B^2_\sigma)^{(1)}),0) \rightarrow (X,x_0)$ of $\alpha$ so that diameter of $\beta(\Delta)$ is at most $2 \cdot l$ for every simplex $\Delta$ of $B^2_\sigma$. This means that $\pi_1(X,x_0)$ is $2 \cdot l$-generated.
Assume $\pi_1(X, x_0)$ is uniformly generated by loops of diameter at most $D$. Fix $r > 0, l > 0$ so that every two points of distance at most $r$ can be connected by a path of diameter at most $l$. We can assume $D > l$. Pick any simplicial map $\alpha: (S^1, 0) \to (\text{Rips}_D(X), x_0)$. It induces a map $\tilde{\alpha}: ((S^1)^{(0)}, 0) \to (X, x_0)$ as follows: For every edge $E$ of $S^2$ we connect two boundary points $\partial E$ by a path of diameter at most $\alpha$. We may assume $\partial E$ extends over $1$–skeleton of some subdivision $\sigma$ (containing $\tau$) of $(B^2, 0)$ to a map $\tilde{\beta}$ so that diameter $\tilde{\beta}(\partial \Delta)$ is at most $D$, for every simplex $\Delta$ of $\sigma$. Then $\tilde{\beta}$ induces a map $\beta: ((B^2)^{(0)}, 0) \to \text{Rips}_D(X)$ which extends over $B^2$. Note that $\beta|_{(\partial B^2, 0)} \simeq \alpha$: for every edge $E$ of $r$ the set $\beta(E) \cup \alpha(E)$ is contained in a simplex of $\text{Rips}_D(X)$ because of uniform path connectedness and $D > l$.

3. Main result

Given a Peano continuum $X$ we assume it has a geodesic metric $d_X$ (see [1]). Pick a base point $x_0$ of $X$ and consider the space $\tilde{X}$ of homotopy (rel.endpoints) classes of paths in $X$ originating at $x_0$.

In this section we assume $X$ is semi-locally simply connected.

**Definition 3.1.** Given $[\alpha] \in \tilde{X}$ and a path $\beta$ in $X$ originating at $\alpha(1)$, the canonical lift $\tilde{\beta}$ of $\beta$ is a path in $\tilde{X}$ defined by $\tilde{\beta}(t) = [\alpha \ast (\beta|_{[0, t]})]$, the concatenation of $\alpha$ and $\beta$ restricted to interval $[0, t]$.

Given two elements $[\alpha]$ and $[\beta]$ of $\tilde{X}$ we define the distance $d([\alpha], [\beta])$ as the infimum of lengths $l(\gamma)$ of all paths $\gamma$ from $\alpha(1)$ to $\beta(1)$ such that $\gamma$ is homotopic rel.endpoints to $\alpha^1 \ast \beta$.

**Proposition 3.2.** $(\tilde{X}, d)$ is a proper geodesic space such that the endpoint projection $p: \tilde{X} \to X$ is $1$-Lipschitz and canonical lifts of geodesics in $X$ are geodesics in $\tilde{X}$.

**Proof.** Let $\delta > 0$ be a number such that any loop in $X$ of diameter less than $4 \cdot \delta$ is null-homotopic in $X$. Notice that any two paths at distance less than $\delta$ are homotopic rel.endpoints if they join the same two points.

Given two elements $[\alpha], [\beta]$ of $\tilde{X}$ the path $\alpha^1 \ast \beta$ can be approximated by a piecewise-geodesic path $\gamma$. As $l(\gamma)$ is finite, so is $d([\alpha], [\beta])$. If $d([\alpha], [\beta]) = 0$, then $\alpha(1) = \beta(1)$. As $d([\alpha], [\beta]) = 0$ there is a loop $\gamma$ at $x_1$ of length less than $\delta$ satisfying $\gamma \simeq \alpha^1 \ast \beta$. That means $\alpha \sim \beta$ as $\gamma$ is null-homotopic in $X$. Thus $[\alpha] = [\beta]$ if $d([\alpha], [\beta]) = 0$. It is easy to see $d$ is symmetric and satisfies the Triangle Inequality.

Notice $d([\alpha], [\beta]) \geq d_X(\alpha(1), \beta(1))$, so $p$ is $1$-Lipschitz. Also, it is clear that canonical lifts of geodesics in $X$ are geodesics in $\tilde{X}$.

Suppose $\gamma_n$ is a sequence of paths in $X$ joining $\alpha(1)$ and $\beta(1)$ such that $l(\gamma_n)$ converges to $M = d([\alpha], [\beta])$ and $\gamma_n \sim \alpha^1 \ast \beta$ for all $n \geq 1$. We may assume each $\gamma_n$ is parametrized so that the length of $\gamma_n|_{[0, t]}$ is $t \cdot l(\gamma_n)$. Subdivide the interval $[0, 1]$ into points $y_0 = 0, y_1, \ldots, y_k = 1$ such that $0 < y_{i+1} - y_i < \frac{1}{2^n}$ for all $0 \leq i < k$. We may assume $\gamma_n(y_i)$ converges to $z_i \in X$ for each $0 \leq i \leq k$. The piecewise-geodesic path $\omega$ from $\alpha(1)$ to $\beta(1)$ obtained by connecting points $z_0, z_1, \ldots, z_k$ is homotopic to $\gamma_n$ for $n$ large enough. Also, $l(\omega)$ equals the limit of $l(\gamma_n)$, so $l(\omega) = d([\alpha], [\beta])$. Notice the canonical lift of $\omega$ is a geodesic from $[\alpha]$ to $[\beta]$ in $\tilde{X}$. 
To show $(\tilde{X}, d)$ is a proper metric space assume $\{[\alpha_n]\}_{n \geq 1}$ is a bounded sequence in $\tilde{X}$. We may assume $\alpha_n(1)$ converges to $x_1$ and then alter each $\alpha_n$ by concatenating it with the geodesic from $\alpha_n(1)$ to $x_1$. It suffices to show that the resulting sequence of elements $[\beta_n]$ of $\tilde{X}$ has a convergent subsequence. First of all, we may assume the sequence of lengths $l(\beta_n)$ converges to $M > 0$ (if $M = 0$, then $\beta_n$ converge to $[c]$, each $\beta_n$ is piecewise-geodesic and the length of $\beta_n|_{[0, d]}$ is $t \cdot l(\beta_n)$. Subdivide the interval $[0, 1]$ into points $y_0 = 0, y_1, \ldots, y_k = 1$ such that $0 < y_{i+1} - y_i < \frac{M}{2}$ for all $0 \leq i < k$. We may assume $\beta_n(y_i)$ converges to $z_i \in X$ for each $0 \leq i \leq k$. The piecewise-geodesic path $\omega$ from $x_0$ to $x_1$ obtained by connecting points $z_0, z_1, \ldots, z_k$ is homotopic to $\beta_n$ for $n$ large enough. That means $[\beta_n]$ is constant starting from a sufficiently large $n$.

**Proposition 3.3.** $(\tilde{X}, d)$ is a simply connected and the endpoint projection $p: \tilde{X} \to X$ is a covering map.

**Proof.** Let $\delta > 0$ be a number such that any loop in $X$ of diameter less than $4 \cdot \delta$ is null-homotopic in $X$.

**Claim:** For any $[\alpha] \in \tilde{X}$ the restriction of $p$ to the ball $B([\alpha], \delta)$ is an isometry onto $B(\alpha(1), \delta)$.

**Proof of Claim:** Given $\beta, \omega \in B([\alpha], \delta)$ let $\gamma$ be the geodesic path from $\beta(1)$ to $\omega(1)$. As $d([\beta], [\omega]) < 2 \cdot \delta$ there is a path $\lambda$ from $\beta(1)$ to $\omega(1)$ of length less than $2 \cdot \delta$ for which $\lambda \sim \beta^{-1} \ast \omega$. Observe $\lambda \sim \gamma$ as both paths are of diameter less than $2 \cdot \delta$. That means $d([\beta], [\omega]) = d_X(\beta(1), \omega(1))$ as the length of $\gamma$ equals $d_X(\beta(1), \omega(1))$ and $d([\beta], [\omega]) \geq d_X(\beta(1), \omega(1))$.

Given $[\beta] \in \tilde{X}$ with $\beta(1) \in B(x_1, \delta)$ let $\gamma$ be the geodesic path from $\beta(1)$ to $x_1$. Observe $d([\beta], [\beta \ast \gamma]) < \delta$ and $p([\beta \ast \gamma]) = x_1$. That means $p^{-1}(B(x_1, \delta))$ is the union of balls $B([\alpha], \delta)$ with $\alpha$ ranging over all paths in $p^{-1}(x_1)$. By Claim we conclude $p$ is a covering projection.

To show $\tilde{X}$ is simply connected suppose $\alpha$ is a loop in $\tilde{X}$ based at the trivial path. Since $p(\alpha)$ can be homotoped to a piecewise-geodesic loop and canonical lifts of piecewise-geodesic loops are paths in $\tilde{X}$, we may assume $\alpha$ is the canonical lift of a piecewise-geodesic loop $\beta$ based at $x_0$. The canonical lift of $\beta$ is a loop if and only if $\beta$ is null-homotopic. As $p$ is a covering projection, $\alpha$ is null-homotopic as well.

**Proposition 3.4.** The action of $G = \pi_1(X, x_0)$ on $\tilde{X}$ (g · [α] being $[\beta \ast \alpha]$, where $[\beta] = g$) is geometric.

**Proof.** $G$ acts by isometries as $d(g \cdot [\alpha], g \cdot [\beta]) = d([\alpha], [\beta])$ for all $\alpha, \beta \in \tilde{X}$.

Let $\delta > 0$ be a number such that any loop in $X$ of diameter less than $4 \cdot \delta$ is null-homotopic in $X$. Given $[\alpha] \in \tilde{X}$ let $U$ be the $\delta$-ball around $[\alpha]$ in $\tilde{X}$. If $[\beta] \in U \cap (g \cdot U)$ there are paths $\gamma_i$, $i = 1, 2$, such that $\beta \sim \alpha \ast \gamma_1$, $\beta \sim g \cdot \alpha \ast \gamma_2$ and $l(\gamma_i) < \delta$ for $i = 1, 2$. Thus $g = [\alpha \ast \gamma_1 \ast \gamma_2^{-1} \ast \alpha^{-1}]$ equals $1$ in $G$ as $\gamma_1 \sim \gamma_2$.

Since $p: \tilde{X} \to X$ is open and, set-theoretically, equals $\tilde{X} \to \tilde{X}/G$, $\tilde{X}/G$ is homeomorphic to $X$ proving that the action of $G$ on $\tilde{X}$ is cocompact.

**Main Theorem 3.5.** The fundamental group of a Peano continuum $X$ is finitely presented if it is countable.
Proof. By the Švarc-Milnor Lemma and the group $G = \pi_1(X, x_0)$ is finitely generated and is quasi-isometric to $\tilde{X}$. As $\tilde{X}$ is coarsely 1-connected (see [2,2]) and coarse 1-connectivity is an invariant of quasi-isometries (see Corollary 47 in [11]), $G$ is also coarsely 1-connected. As $G$ is a finitely generated group it means $G$ is finitely presented (see the proof of Corollary 51 in [11] on p.22 or Proposition 8.24 in [2]). Alternatively, the fundamental group of the Cayley graph $\Gamma(G)$ of $G$ must be uniformly generated by 2.2 which means $G$ is finitely presented. ■

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