Fractional Repetition Codes for Repair in Distributed Storage Systems

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Abstract—We introduce a new class of exact Minimum-Bandwidth Regenerating (MBR) codes for distributed storage systems, characterized by a low-complexity uncoded repair process that can tolerate multiple node failures. These codes consist of the concatenation of two components: an outer MDS code followed by an inner repair code. We refer to the inner code as a Fractional Repetition code since it consists of splitting the data of each node into several packets and storing multiple replicas of each on different nodes in the system.

Our model for repair is table-based, and thus, differs from the random access model adopted in the literature. We present constructions of Fractional Repetition codes based on regular graphs and Steiner systems for a large set of system parameters. The resulting codes are guaranteed to achieve the storage capacity for random access repair. The considered model motivates a new definition of capacity for distributed storage systems, that we call Fractional Repetition capacity. We provide upper bounds on this capacity while a precise expression remains an open problem.

I. INTRODUCTION

Despite being formed of unreliable nodes having a short lifespan, distributed storage systems (DSS) are required to store data for long periods of time with a very high reliability. Typically, nodes in the system will unexpectedly leave for different reasons such as hardware failures in data centers, or peer churning in peer-to-peer (P2P) systems. To overcome this problem, a two-fold solution can be adopted based on redundancy and repair [1]. Classical erasure codes can be used to introduce redundancy in the system to protect the data from being lost when nodes fail. In addition, to maintain a targeted high reliability, the system is repaired whenever a node fails by replacing it with a new one.

A distributed storage system is formed of \( n \) storage nodes and gives the user the flexibility to recover its stored file by contacting any \( k \) out of the \( n \) nodes, for some \( k < n \). We call this property the MDS property of the DSS in reference to Maximum Distance Separable (MDS) codes. When a node fails, the system is repaired by replacing the failed node with a new “blank” node. The new node contacts \( d \) survivor nodes, downloads encodings of their data and stores it, possibly after processing it. The data stored on the new node should maintain the MDS property of the DSS. In analogy with classical codes defined by the pair \((n,k)\), a DSS is specified by the triplet \((n,k,d)\), where the additional parameter \(d\), referred to as the repair degree, accounts for the repair requirement. Fig. 1 depicts a \((4,2,3)\) DSS where node \(v_1\) has failed and has been replaced by node \(v_1'\) that contacts \(d = 3\) survivor nodes in the system to download its data.

Dimakis et al. introduced and studied in [2], [3], [4] the design of erasure codes with efficient repair capabilities, termed regenerating codes. The authors showed the existence of a tradeoff between storage capacity and repair bandwidth in these systems. In this tradeoff, two regimes are of special interest, the minimum-bandwidth regime and the minimum-storage regime. This original work focused on functional repair where the only requirement on the data regenerated at the replacement node is to maintain the MDS property of the system. Subsequent works focused on the design of exact regenerating codes that can repair the system by reproducing an exact replica of the lost data. Rashmi et al. presented in [5] constructions of exact minimum-bandwidth regenerating (MBR) codes for the case of \(d = n - 1\), and for all feasible values of the repair degree \(d\) in [6]. The existence of exact regenerating codes for the minimum-storage regenerating (MSR) case was demonstrated in [7], [8], and deterministic constructions were investigated in [5], [9], [10], [11].

In this work we are interested in the construction of exact minimum-bandwidth regenerating (MBR) codes that are characterized by a low-complexity repair process. In this regime, a replacement node recovers an exact copy of the lost data by contacting \(d\) survivor nodes and downloading and storing one packet of data from each. To guarantee that the constructed codes have low complexity, we require them to satisfy what we call the uncoded repair property: a survivor node reads the exact amount of data he needs to send to a replacement node and forwards it without any processing. Our motivation is that in practical systems the read/write bandwidth of the storage nodes is the bottleneck since it is much smaller than the network bandwidth [12]. Regenerating codes, such as random network codes [3], do not satisfy the uncoded repair property in general since they require a survivor node to read all his stored data in order to send a linear combination of them to the replacement node. We show the surprising result that even with the two apparently restrictive constraints of exact and unencoded repair, it is possible to construct optimal regenerating codes under a table-based repair model.

Our codes are based on a generalization of the construction of Rashmi et al. in [5] and are formed by the concatenation of two component codes (see Fig. 2): an outer MDS code to ensure the required MDS property of the DSS, and an
inner repetition code characterized by an efficient uncoded repair process that is resilient to multiple node failures. The design of the inner code represents the challenging task in this construction. We refer to it as Fractional Repetition (FR) code since, in our proposed solution, the data stored on each node is split into \( d \) packets, each of which is repeated a certain number of times in the system. We study the design of FR codes that can achieve the DSS capacity for the MBR point in [3] which assumes random access repair where the new node can contact any \( d \) survivor nodes. For single failures, we provide a construction based on regular graphs for all feasible values of system parameters. For the general case of multiple failures, we propose code constructions based on Steiner systems. The table-based repair model motivates a new concept of storage capacity for distributed storage systems that we call Fractional Repetition (FR) capacity, which we investigate.

The rest of the paper is organized as follows. In Section II, we define our system model and motivate our code design requirements. In Section III, we give two examples of Fractional Repetition codes that are used for constructing optimal exact MBR code with uncoded repair. We describe code constructions based on regular graphs for the single failure case in Section IV. Furthermore, we provide constructions based on Steiner systems for the multiple failures case in Section V. In Section VI, we define the Fractional Repetition capacity of a DSS and provide some bounds. We conclude in Section VII with a summary of our results and discuss related open problems.

II. MOTIVATION AND MODEL

A distributed storage system DSS is defined by the triplet \( (n, k, d) \), where \( n \) is the total number of storage nodes in the system, \( k < n \) is the number of nodes contacted by the user to retrieve his stored file, and \( d \geq k \) is the repair degree that specifies the number of nodes contacted by a replacement node during repair.

We consider distributed storage systems operating in the minimum-bandwidth regime on the storage/bandwidth tradeoff curve described in [3]. Our focus on this regime is motivated by the asymmetrical cost of resources in practical systems where bandwidth is more expensive than storage. In this case, the repair bandwidth of the system, i.e., the total amount of data downloaded by a replacement node is minimized. As a result, the new node needs to download only the amount of data he will store, but no more.

For load-balancing requirements, we assume a symmetric model for repair where the replacement node downloads and stores an equal amount of data, referred to as a packet, from each of the \( d \) nodes it contacts. Therefore, in the minimum bandwidth regime, \( d \) also represents the node storage capacity expressed in packets.

Under this model, the storage capacity \( C_{MBR} \) in packets of the DSS, representing the information-theoretic limit on the maximum file size that can be delivered to any user contacting \( k \) out of the \( n \) nodes, was shown in [3] to be

\[
C_{MBR}(n, k, d) = kd - \left( \frac{k^2}{2} \right).
\]  

This expression assumes a functional repair model where the only constraint on the data regenerated (stored) at the new node is maintaining the MDS property of the system. This allows the regenerated data to be different from the lost data as long as it is “functionally” equivalent. However, a more stringent form of repair, known as exact repair, that is capable of reproducing an exact copy of the lost data, may be required for many system considerations such as maintaining a systematic form of the data, reducing protocol overhead and guaranteeing data security [13], [14]. In this case, regenerating codes are also referred to as being exact.

Recently, Rashmi et al. showed the interesting result that, in the minimum-bandwidth regime, there is no loss in the DSS capacity incurred by requiring exact repair, and constructed exact MBR codes that achieve the capacity \( C_{MBR} \) of (1).

Existing code constructions suffer in general from a high complexity repair process. A survivor node asked to help in repair typically has to read all his \( d \) stored packets and compute a linear combination of them in order to obtain a single packet to be forwarded to the replacement node. In addition to the computational complexity overhead, the repair process results in long delays at the survivor nodes since in general their read/write bandwidth is much smaller than the network bandwidth [12]. This motivates us to study exact MBR codes with fast and low-complexity repair where a survivor node reads only one of his stored packet and forwards it to the replacement node with no additional processing. We refer to this property as uncoded repair.

III. EXAMPLES

Before introducing our general constructions, we present two examples of exact regenerating codes that can achieve the capacity \( C_{MBR} \) while satisfying the uncoded repair property. The first example is based on the construction of exact MBR codes for \( d = n - 1 \) of Rashmi et al. in [5].
The previous code is a special example of the exact MBR codes devised in [5] where uncoded repair was not a requirement. This construction, however, is limited to systems with a small repair degree $d$ that have an uncoded repair process that can tolerate multiple failures.

To introduce our constructions which will be detailed in the following sections, we give a second example for a DSS with $(n, k, d) = (7, 3, 3)$ having an exact and uncoded repair process that can tolerate up to two failures.

**Example 2:** Consider a $(7, 3, 3)$ DSS where the system storage capacity is $C_{MBR} = 6$ by (1). The code that we propose for this system is also constructed by concatenating two constituent codes: an outer $(7, 6)$ MDS code that outputs coded packets indexed from 1 to 7, followed by the repetition code depicted in Fig. 3(a). It can be seen that each of the 7 packets forming the output of the outer code is replicated on 3 different nodes in the system. Therefore, the system will always have a surviving copy of each packet in the case of two failures. Thus, the code guarantees exact uncoded repair for up to two node failures.

Next, we check that this code indeed achieves the capacity $C_{MBR}$. The structure of the inner repetition code is deduced from the projective plane of order 2, also called the Fano plane, depicted in Fig 3(b) [15, Chap 2]. The Fano plane consists of 7 points indexed from 1 to 7 and the following 7 lines $1234$, $3689$, $25789$, $479$, and $1567$, which implies that any two nodes will have exactly one packet in common. Therefore, any two nodes will have exactly one packet in common. This implies that any user that contacts 3 different nodes can get at least $3 \times 3 - \binom{3}{2} = 6$ distinct packets. For instance, a user contacting nodes $v_2, v_4$ and $v_5$ will get exactly 6 different packets, namely, all the packets except the one with index 6. Whereas another user contacting $v_1, v_3$ and $v_5$ will get all the 7 packets. In a worst-case analysis, the capacity of the system is limited by the user that gets the least number of packets, which is 6 here. Hence, the outer MDS code allows any user to recover the stored file of 6 packets which is exactly the capacity $C_{MBR}(7, 3, 3)$ of (1).

The previous two examples highlight the central role of the inner repetition code that allows us to obtain the desired uncoded and exact repair properties of the code in addition to achieving the capacity $C_{MBR}$ by carefully placing the different copies of the coded packets on different nodes in the system. We call the inner code a Fractional Repetition (FR) code of repetition degree $\rho$ since the content of each node is split into $d$ packets and $\rho$ replicas of each are stored on different nodes in the system. For instance, the inner code in the first example was an FR code with $\rho = 2$, whereas in
of concatenating an outer MDS code with an inner Fractional Repetition code with repetition degree $\rho$ that can tolerate up to $\rho - 1$ nodes failing together. Since MDS codes exist for all feasible parameters provided that the packets are taken from an alphabet of large enough size, the challenging part of the suggested construction is designing the Fractional Repetition code. Assuming all the packets in the system are to be equally protected, we are motivated to provide the following general definition of FR codes:

**Definition 3 (Fractional Repetition Codes):** A Fractional Repetition (FR) code $C$, with repetition degree $\rho$, for an $(n,k,d)$ DSS, is a collection $C$ of $n$ subsets $V_1,V_2,\ldots,V_n$ of a set $\Omega = \{1,\ldots,\theta\}$ and of cardinality $d$ each, satisfying the condition that each element of $\Omega$ belongs to exactly $\rho$ sets in the collection.

In this definition, each set $V_i$ contains the indices of the coded packets at the output of the outer MDS code that are stored on node $v_i, i = 1, \ldots, n$. The value of $\theta$, which will be determined later, corresponds to the length of the codewords of the outer MDS code. For instance, following this definition, the FR code of Example 2 can be written as $C = \{V_1,\ldots,V_7\}$ with $V_1 = \{1,2,3\}, V_2 = \{3,4,5\}, V_3 = \{1,5,6\}, V_4 = \{1,4,7\}, V_5 = \{2,5,7\}, V_6 = \{3,6,7\}, V_7 = \{2,4,6\}$, where $\theta = 7$ and $\Omega = \{1,\ldots,7\}$.

We focus first on the design of Fractional Repetition codes of repetition degree $\rho = 2$ with an uncoded repair that is tolerant to a single failure. We provide a code construction based on regular graphs that can achieve the capacity $C_{MBR}$ of (1) for all feasible values of $n$ and $d$.

To that end, we define the rate $R_C(k)$ of an FR code $C$ as the maximum file size, i.e., the maximum number of distinct packets, that the code is guaranteed to deliver to any user contacting $k$ nodes.

**Definition 4 (FR Code Rate):** The rate $R_C(k)$ of an FR code $C = \{V_1, V_2, \ldots, V_n\}$ for a DSS with parameters $(n,k,d)$ is defined as

$$R_C(k) := \min_{I \subseteq [n] \atop |I| = k} \bigcup_{i \in I} V_i,$$

with $[n] = \{1,\ldots,n\}$.

As it can be seen from the previous examples and the above definition, the DSS parameter $k$ specifying the number of nodes contacted by a user, is not intrinsically related to the construction of the FR code. An FR code designed for a DSS with parameters $(n,k_1,d)$ can be seamlessly used for another DSS with parameters $(n,k_2,d)$, with $k_1 \neq k_2$. An FR code $C$ is said to be universally good if its rate is guaranteed to be no less than the capacity $C_{MBR}$ of the DSS.
Next, we describe a construction of FR codes with repetition degree \( \rho = 2 \) and \( d < n - 1 \). For \( \rho = 2 \), Prop. 5 gives a necessary condition for the existence of FR codes, that is, \( nd \) should be even. We will show that this is also a sufficient condition and provide a general code construction based on regular graphs.

A \( d \)-regular graph \( R_{n,d} \) on \( n \) vertices is a simple graph where all vertices have the same degree \( d \), i.e., the same number of neighboring nodes. The graph \( R_{n,d} \) has \( \frac{nd}{2} \) vertices, and exists whenever \( nd \) is even [16].

**Construction 1:** An FR code with repetition degree \( \rho = 2 \) can be constructed for an \( (n, k, d) \) DSS, with \( nd \) even, in the following way:

1. Generate a \( d \)-regular graph \( R_{n,d} \) on \( n \) vertices \( u_1, \ldots, u_n \).
2. Index the edges of \( R_{n,d} \) from 1 to \( \frac{nd}{2} \).
3. Store on node \( v_i \) in the DSS the packets indexed by the edges that are adjacent to vertex \( u_i \) in the graph \( R_{n,d} \).

The regular graph in Step 1 can be randomly generated using efficient randomized algorithms that are well-studied in the literature, see for example [17]. The fact that the FR codes obtained by this construction have repetition degree \( \rho = 2 \) is a direct consequence of the graph being simple with each edge being adjacent to exactly two vertices. This also implies that any two nodes cannot have in common more than one packet. Therefore, among any \( k \) nodes observed by a user, there are at most \( \binom{k}{2} \) repeated packets which corresponds to the case when any two nodes share a distinct packet. Therefore, we have the following lemma.

**Lemma 6:** The FR codes with repetition degree \( \rho = 2 \) obtained by Construction 1 are universally good codes.

Fig. 5 shows a 3-regular graph \( R_{6,3} \) and the corresponding universally good FR code obtained by Construction 1 for the DSS with \( n = 6 \) and \( d = 3 \).

V. Fractional Repetition Codes with \( \rho > 2 \)

Practical systems require the repetition degree to be at least 3 [18], and the previous construction based on regular graphs cannot be generalized to this case. We present here two new constructions of FR codes with \( \rho > 2 \) based on a combinatorial structure known as Steiner system that can be thought of as a generalization of the projective plane of Example 2.
A. Steiner Systems

We start by giving a definition of a Steiner system.

**Definition 7 (Steiner System):** A Steiner system $S(t, \alpha, v)$ is a collection of subsets, called blocks, $B_1, \ldots, B_b$, of size $\alpha$ of a set $\mathcal{V}$ containing $v$ elements, called points, with the property that any subset of $t$ points is contained in exactly one block.

It can be shown that in a Steiner system every point belongs to exactly the same number of blocks denoted $r$ [15, p. 60]. One way to guarantee the achievability of the capacity in (1) is to require that nodes do not share more than one packet. For this reason, we will be mostly interested in Steiner systems with $t = 2$. Simple counting arguments give the following two well-known properties of a Steiner system $S(2, \alpha, v)$ [15, Chap. 2].

**Proposition 8:** The parameters $b$ and $r$ of a Steiner system $S(2, \alpha, v)$ are given by:

$$b\alpha = vr, \quad (4)$$
$$v - 1 = r(\alpha - 1). \quad (5)$$

Equation (4) is similar to (3) for FR codes. The Fano plane of Fig. 3(a) is an example of a Steiner system where $\mathcal{V}$ is the set of 7 points and the blocks are the lines (including the circle). The Fano plane is indeed $S(2, 3, 7)$ since there is a single line that goes through any two points. Prop. 8 gives $r = 3$, i.e., each point belongs to exactly 3 lines, and $b = 7$, i.e., the Fano plane contains 7 lines, which can be easily checked on the figure.

For a Steiner system $S(t, \alpha, v)$ to exist, it is necessary that the parameters $b$ and $r$ given in Prop. 8 be integers. Wilson proved in [19] that this condition is also sufficient when $v$ is large enough.

**Theorem 9:** Given a positive integer $\alpha$, Steiner systems $S(2, \alpha, v)$ exist for all sufficiently large integers $v$ for which the congruences $vr \equiv 0 \mod \alpha$ and $v - 1 \equiv 0 \mod \alpha - 1$, are valid.

B. Code Constructions

We present now two constructions of universally good FR codes derived from Steiner systems. Example 2 suggests the following direct construction.

**Construction 2:** Given a Steiner system $S(2, \alpha, v)$ with blocks $B_1, \ldots, B_b \subseteq \mathcal{V} = \{v\}$, an FR code $C$ can be obtained by taking $C = \{B_1, \ldots, B_b\}$. This gives an FR code with $\rho = \frac{v - 1}{\alpha - 1}$ and $\theta = v$ for a DSS with parameters $n = \frac{v(\alpha - 1)}{\alpha(\alpha - 1)}$ and $d = \alpha$ as given by Prop. 8.

By definition, any two blocks in $S(t, \alpha, v)$ cannot intersect in more than $t - 1$ elements. This implies that in the FR codes obtained by Construction 2, two nodes can have at most one packet in common. Therefore, we get the following lemma.

**Lemma 10:** The FR codes obtained by Construction 2 are universally good.

Construction 2 has the disadvantage that the two important code design parameters, $n$ and $\rho$ do not figure explicitly in the Steiner system parameters. Next, we present a second construction where $n$ and $\rho$ directly determine the Steiner system and where the repair degree $d$ is a fraction of the survivor nodes.

**Construction 3:** Given a Steiner system $S(2, \alpha, v)$ with blocks $B_1, \ldots, B_b \subseteq \mathcal{V} = \{v\}$, an FR code $C = \{V_1, \ldots, V_n\}$ can be obtained by taking

$$V_i = \{\{i\} \in B_j\},$$

for $i = 1, \ldots, n$. This gives an FR code with $\rho = \alpha$ and $\theta = \frac{n(n - 1)}{\rho(\rho - 1)}$ for a DSS with parameters $n = v$ and $d = \frac{n - 1}{\rho - 1}$ as given by Prop 8.

We refer to the codes obtained by this construction as Transpose codes since the role of the blocks and points are reversed. The blocks now correspond to packets and the points to the storage nodes. Therefore, any two nodes have exactly one packet in common. Therefore, we get the following lemma.
Lemma 11: The FR codes obtained by Construction 3 are universally good.

To highlight the difference between these two constructions, we give an example in Figure 6 when they are both applied to the unique Steiner system S(2,3,9) [20, p. 27]. Construction 2 gives an FR code with \( \rho = 4 \) for a DSS with \( n = 12 \) and \( d = 3 \), whereas Construction 3 gives an FR code with \( \rho = 3 \) for a DSS with \( n = 9 \) and \( d = 4 \). Note that these two constructions will give the same FR code (up to relabeling) when applied to projective planes such as the Fano plane of Fig. 3(b).

The previous two constructions assume the existence of the Steiner system with the desired parameters, which is not always true. However, Steiner systems \( S(2, \alpha, v) \) are known to exist for small values of \( \alpha \), namely \( \alpha = 2, \ldots, 5 \), and for any \( v \) whenever the integrality conditions given by Th. 9 are satisfied. This result in conjunction with Construction 3 gives the necessary and sufficient conditions for the existence of Transpose codes with low repetition degree.

Corollary 12: Transpose codes with repetition degree \( \rho = 2, \ldots, 5 \) exist if and only if \( n - 1 \equiv 0 \mod \rho - 1 \) and \( n(n - 1) \equiv 0 \mod \rho(\rho - 1) \).

The previous corollary implies that for the important practical case of systems with repetition degree \( \rho = 3 \), universally good FR codes with repair degree \( d = \frac{\rho-1}{3} \) can be obtained by Construction 3 using Steiner systems \( S(2,3,n) \) which exist for all \( n \equiv 1, 3 \mod 6 \). Steiner systems with \( \alpha = 3 \), known in the literature as Steiner triple systems, are historically the most investigated systems and explicit constructions, such as Bose and Skolem constructions, exist for all feasible values of \( n \) [21].

VI. CAPACITY UNDER EXACT UNCODED REPAIR

The universally good FR codes constructed in the previous sections are guaranteed to have a rate greater or equal to the capacity \( C_{MBR} \) of the system under random access and functional repair. However, there exist cases where FR codes can achieve a storage capacity that exceeds \( C_{MBR} \). For instance, consider the FR code \( C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 4, 7, 2, 5, 8, 3, 6, 9\} \) of repetition degree 2 for the (6, 3, 3) DSS depicted in Fig. 7. It can be checked any user contacting 3 nodes observes at least 7 distinct packets. Therefore, this code has a rate \( R_C(3) = 7 > C_{MBR} = 6 \).

We refer to the maximum file size that a DSS with parameters \( (n, k, d) \) can store under exact and uncoded repair as its Fractional Repetition (FR) capacity \( C_{FR} \) defined as follows:

Definition 13 (Fractional Repetition Capacity): The Fractional Repetition (FR) capacity, denoted by \( C_{FR}(k, \rho) \), of a distributed storage system with parameters \( (n, k, d) \) is defined, for all \( \rho \) satisfying \( nd \equiv 0 \mod \rho \), as

\[
C_{FR}(k, \rho) := \max_C R_C(k),
\]

where \( C \) is any FR code with repetition degree \( \rho \) for an \( (n, k, d) \) DSS.

The condition on \( \rho \) in the definition above is needed by Prop. 8 to guarantee the existence of an FR code \( C \). Note that this notion of capacity assumes that a packet is an atomic unit of information that cannot be divided, which is usually true in real applications such the one in [18] where packets are of size 64 MB.

The code constructions of the previous sections imply lower bounds on the FR capacity. Next, we present two upper bounds on \( C_{FR} \). The first is based on an averaging argument and is presented in Lemma 14.

Lemma 14: For a DSS with parameters \( (n, k, d) \),

\[
C_{FR}(k, \rho) \leq \left\lfloor \frac{nd}{\rho} \left(1 - \frac{(n-\rho)}{\binom{n}{k}}\right)\right\rfloor.
\]

Proof: Let \( C = \{V_1, \ldots, V_n\} \) be an FR code with repetition degree \( \rho \), where \( V_i \subset [\theta], |V_i| = d \) and \( \theta = \frac{nd}{\rho} \) as given by Prop. 5.

Define the set \( U \) as

\[
U := \{U_I = \cup_{i \in I} V_i : I \subset [n], |I| = k\}.
\]

The set \( U_I \) represents the set of packets observed by a user contacting the nodes in the DSS indexed by the elements in \( I \). We want to show that the term on the right in the inequality is the average cardinality of the sets in \( U \) under uniform distribution. We denote this average by \( \bar{U} \). To find \( \bar{U} \), we count the following quantity \( \sum_{U_I \in U} |U_I| \) in two ways.

First, we have by definition

\[
\sum_{U_I \in U} |U_I| = \left(\binom{n}{k}\right) \bar{U}.
\]

But, each element in \( [\theta] \) belongs to exactly \( \binom{k}{\theta} - \binom{n-\rho}{k} \) sets in \( U \). Therefore,

\[
\sum_{U_I \in U} |U_I| = \theta \left(\binom{n}{k} - \binom{n-\rho}{k}\right).
\]

The upper bounds follows then from the fact that there must be in \( U \) at least one set of cardinality less that the average.

For instance, for the DSS \( (7, 3, 3) \), Lem. 14 implies that \( R(3,3) \leq 6 \). Therefore, the FR code of Example 2 is optimal and \( C_{FR}(3,3) = 6 \). However, the above upper bound
has the disadvantage of becoming loose for large values of \( n \) and \( k \) since the FR capacity is by definition a worst case measure.

We also give a second bound on the FR capacity of a DSS which is defined using a recursive function and is tighter than the previous one.

**Lemma 15:** For a DSS \((n, k, d)\), the FR capacity is upper bounded by the function \( g(k) \), i.e., \( C_{FR}(k, \rho) \leq g(k) \), where \( g(k) \) is defined recursively as
\[
\begin{align*}
g(1) &= d, \\
g(k + 1) &= g(k) + d - \left\lfloor \frac{\rho g(k) - kd}{n - k} \right\rfloor.
\end{align*}
\]

The proof for this lemma is omitted due to space restrictions and can be found in [22].

**VII. CONCLUSION AND OPEN PROBLEMS**

We proposed a new class of Exact Minimum-Bandwidth Regenerating (MBR) codes for distributed storage systems characterized by a low complexity uncoded repair process. The main component of our construction is a new code that we call Fractional Repetition (FR) code. An FR code with repetition degree \( \rho \) guarantees uncoded repair for up to \( \rho - 1 \) failures. It consists of splitting the data on each node into multiple packets and storing \( \rho \) replicas of each on distinct nodes in the system. An additional outer MDS code guarantees that a user contacting a sufficient number of storage nodes will be able to retrieve the stored file.

For single node failures, i.e., \( \rho = 2 \), we presented a construction of FR codes based on regular graphs for all feasible system parameters. For the multiple failures case, i.e., \( \rho > 2 \), we presented two code constructions based on Steiner systems. Of particular importance are the constructed Transpose codes where the nodes contacted for repair are just a fraction of the surviving ones. All the obtained codes are guaranteed to achieve the storage capacity under random-access repair. The adopted table-based repair model motivates a new concept of capacity for distributed storage systems, referred to as of Fractional Repetition (FR) capacity, which we studied and derived corresponding bounds.

This work constitutes the first step in the study of Fractional Repetition codes and many important questions remain open. For instance, it is not known whether FR codes with \( \rho > 2 \) exist for system parameters not covered by our constructions. Moreover, a general expression of the FR capacity is still an open problem, as well as codes that can achieve it.

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