TG-SUPPLEMENTED MODULES
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Abstract. In this work, we define tg-supplemented modules and investigate some properties of these modules. We prove that the finite t-sum of tg-supplemented modules is tg-supplemented. We also prove that the homomorphic image of a distributive tg-supplemented module is tg-supplemented. We give some examples separating tg-supplemented modules from supplemented and generalized $\oplus$-supplemented modules.

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1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let $R$ be a ring and $M$ be an $R$-module. We will denote a submodule $K$ of $M$ by $K \leq M$. Let $M$ be an $R$-module and $K \leq M$. If $T = M$ for every submodule $T$ of $M$ such that $K + T = M$, then $K$ is called a small submodule of $M$ and denoted by $K \ll M$. Let $M$ be an $R$-module and $K \leq M$. If there exists a submodule $T$ of $M$ such that $K + T = M$ and $K \cap T = 0$, then $K$ is called a direct summand of $M$ and it is denoted by $M = K \oplus T$. For any module $M$, the intersection of maximal submodules of $M$ is called the radical of $M$ and denoted by $\text{Rad}M$. If $M$ have no maximal submodules, then we define $\text{Rad}M = M$. A module $M$ is called distributive [8] if for every submodules $K, L, T$ of $M$, $K \cap (L + T) = K \cap L + K \cap T$ or equivalently $(K + L) \cap (K + T) = K + L \cap T$. Let $U$ and $V$ be submodules of a module $M$. If $U + V = M$ and $V$ is minimal with respect to this property, or equivalently, $U + V = M$ and $U \cap V \ll V$, then $V$ is called a supplement [10] of $U$ in $M$. $M$ is called a supplemented module if every submodule of $M$ has a supplement in $M$. $M$ is called (([5],[6]) $\oplus$-supplemented module if every submodule of $M$ has a supplement that is a direct summand of $M$. Let $M$ be an $R$-module and $U, V$ be submodules of $M$, $V$ is called a generalized supplement ([1],[9],[11]) of $U$ in $M$ if $M = U + V$ and $U \cap V \leq \text{Rad}V$. $M$ is called generalized supplemented or briefly a GS-module if every submodule of $M$ has a generalized supplement in $M$. Clearly
we see that every supplemented module is a generalized supplemented module. $M$ is called a generalized $\oplus$-supplemented ([12],[4],[7],[8]) module if every submodule of $M$ has a generalized supplement that is a direct summand of $M$. In this paper we generalize these modules.

**Lemma 1.** Let $V$ be a supplement of $U$ in $M$ and $L$, $K \leq V$. Then $K$ is a supplement of $L$ in $V$ if and only if $K$ is a supplement of $U + L$ in $M$. ([3], Exercise 20.39)

*Proof.* ($\Rightarrow$) Let $U + L + T = M$, for some $T \leq K$. Since $V$ is a supplement of $U$ in $M$ and $L + T \leq V$, $L + T = V$ and by $K$ being a supplement of $L$ in $V$, $T = K$. Hence $K$ is a supplement of $U + L$ in $M$.

($\Leftarrow$) Let $L + T = V$, for some $T \leq K$. Then $U + L + T = M$, and by $K$ being a supplement of $U + L$ in $M$, $T = K$. Hence $K$ is a supplement of $L$ in $V$. $\square$

**Lemma 2.** Let $M$ be a $\pi$-projective module and $K, L$ be two submodules of $M$. If $K$ and $L$ are mutual supplements in $M$, then $K \cap L = 0$ and $M = K \oplus L$.

*Proof.* See ([10], 41.14(2)). $\square$

2. TG-SUPPLEMENTED MODULES

**Definition 1.** Let $M$ be an $R$-module and $K, L$ be two submodules of $M$. If $K$ and $L$ are mutual supplements in $M$, then $M$ is called a $t$-sum of $K$ and $L$. This equivalent to $M = K + L$, $K \cap L \ll K$ and $K \cap L \ll L$. This case $K$ and $L$ are called $t$-summands of $M$.

**Definition 2.** Let $M$ be an $R$-module and $\{A_i\}_{i \in I}$ be a family of submodules of $M$. $M$ is called a $t$-sum of $\{A_i\}_{i \in I}$, if $A_k$ and $\sum_{j \neq k} A_j$ are mutual supplements in $M$ for every $k \in I$.

**Lemma 3.** Let $M$ be an $R$-module, $V$ be a $t$-summand of $M$ and $K \leq V$. Then $K \ll M$ if and only if $K \ll V$.

*Proof.* Clear from ([12], Lemma 1.1). $\square$

**Lemma 4.** Let $M$ be a $t$-sum of $U$ and $V$. If $K$ is a supplement of $S$ in $U$ and $L$ is a supplement of $T$ in $V$, then $K + L$ is a supplement of $S + T$ in $M$.

*Proof.* Since $U$ is a supplement of $V$ in $M$ and $K$ is a supplement of $S$ in $U$, by Lemma 1.1, $K$ is a supplement of $V + S$ in $M$. Hence $(V + S) \cap K \ll K$. Similarly, we prove that $(U + T) \cap L \ll L$. Then $(S + T) \cap (K + L) \leq (S + T + K) \cap L + (S + T + L) \cap K = (U + T) \cap L + (V + S) \cap K \ll K + L$, and by $M = U + V = S + K + T + L = S + T + K + L$, $K + L$ is a supplement of $S + T$ in $M$. $\square$
Lemma 5. Let \( M \) be a \( t \)-sum of \( U \) and \( V \), and \( L, T \leq V \). Then \( V \) is a \( t \)-sum of \( L \) and \( T \) if and only if \( M \) is a \( t \)-sum of \( U + L \) and \( T \), and \( M \) is a \( t \)-sum of \( U + T \) and \( L \).

Proof. (\( \Rightarrow \)) Let \( V \) be a \( t \)-sum of \( L \) and \( T \). Since \( T \) is a supplement of \( L \) in \( V \) and \( V \) is a supplement of \( U \) in \( M \), then by Lemma 1, \( T \) is a supplement of \( U + L \) in \( M \). Then \( (U + L) \cap T \ll T \). Similarly, we can prove that \( (U + T) \cap L \ll L \). Then by \( U \cap V \ll U \), \( (U + L) \cap T \ll U \cap (L + T) + L \cap (U + T) = U \cap V + (U + T) \cap L \ll U + L \). Since \( U \cap V \ll U \), \( (U + L) \cap T \ll U + L \) and \( M = U + V = U + L + T \), then by Definition 1 \( M \) is a \( t \)-sum of \( U + L \) and \( T \). Similarly, we prove that \( M \) is a \( t \)-sum of \( U + T \) and \( L \).

(\( \Leftarrow \)) Clear from Lemma 1.

Corollary 1. Let \( M \) be a \( t \)-sum of \( U_1, U_2, \ldots, U_n \). If \( K_i \) is a supplement of \( T_i \) in \( U_i \) (\( i = 1, 2, \ldots, n \)), then \( K_1 + K_2 + \cdots + K_n \) is a supplement of \( T_1 + T_2 + \cdots + T_n \) in \( M \).

Proof. Clear from Lemma 5.

Corollary 2. Let \( M \) be a \( t \)-sum of \( U_1, U_2, \ldots, U_n \). If \( U_i \) is a \( t \)-sum of \( K_i \) and \( T_i \) (\( i = 1, 2, \ldots, n \)), then \( M \) is a \( t \)-sum of \( K_1 + K_2 + \cdots + K_n \) and \( T_1 + T_2 + \cdots + T_n \).

Proof. Clear from Corollary 1.

Corollary 3. Let \( M \) be a \( t \)-sum of \( U_1, U_2, \ldots, U_n \). If \( K_i \) is a supplement in \( U_i \) (\( i = 1, 2, \ldots, n \)), then \( K_1 + K_2 + \cdots + K_n \) is a supplement in \( M \).

Proof. Clear from Corollary 1.

Corollary 4. Let \( M \) be a \( t \)-sum of \( U_1, U_2, \ldots, U_n \). If \( K_i \) is a \( t \)-summand of \( U_i \) (\( i = 1, 2, \ldots, n \)), then \( K_1 + K_2 + \cdots + K_n \) is a \( t \)-summand of \( M \).

Proof. Clear from Lemma 5.

Lemma 6. Let \( M \) be a distributive \( R \)-module and \( N \leq M \). Then \( (K + N)/N \) is a \( t \)-summand of \( M/N \) for every \( t \)-summand \( K \) of \( M \).

Proof. Let \( K \) be a \( t \)-summand of \( M \). Then there exists a submodule \( L \) of \( M \) such that \( M = L + K, L \cap K \ll L \) and \( L \cap K \ll K \). Since \( M = L + K \), then \( M/N = (L + N)/N + (K + N)/N \). Since \( M \) is distributive, then we have \((L + N) \cap (K + N) = L \cap K + N \). Since \( L \cap K \ll L \) and \( L \cap K \ll K \), then we have \((L + N)/N \cap ((K + N)/N) = (L \cap K + N)/N \ll (L + N)/N \) and \((L + N)/N \cap ((K + N)/N) = (L \cap K + N)/N \ll (K + N)/N \). Hence \( (K + N)/N \) is a \( t \)-summand of \( M/N \).

Theorem 1. Let \( M \) be a \( t \)-sum of \( \{A_i\}_{i \in I} \). Then \( \text{Rad} M = \sum_{i \in I} \text{Rad} A_i \).
Proof. Let $x \in \text{Rad} M$. Since $x \in M = \sum_{i \in I} A_i$, there exist $i_1, i_2, \ldots, i_n \in I$ and $x_{i_1} \in A_{i_1}, x_{i_2} \in A_{i_2}, \ldots, x_{i_n} \in A_{i_n}$ such that $x = x_{i_1} + x_{i_2} + \cdots + x_{i_n}$. Let $k \in \{1, 2, \ldots, n\}, T \leq A_{ik}$ and $Rx_{ik} + T = A_{ik}$. Let $a \in M$. Since $a \in M = \sum_{i \in I, i \neq ik} A_i + A_{ik}$, we can write $a = b + c$ for some $b \in \sum_{i \in I, i \neq ik} A_i$ and $c \in A_{ik}$. Since $c \in A_{ik} = Rx_{ik} + T$, there exist $r \in R$ and $t \in T$ such that $c = rx_{ik} + t$. Then $a = b + c = b + rx_{ik} + t = b + r \left( x - \sum_{s=1, s \neq ik}^{n} x_{is} \right) + t = rx + b - \sum_{s=1, s \neq ik}^{n} rx_{is} + t \in Rx + \sum_{i \in I, i \neq ik} A_i + T$. Hence $M = Rx + \sum_{i \in I, i \neq ik} A_i + T$ and since $Rx \ll M, M = \sum_{i \in I, i \neq ik} A_i + T$. Since $M = \sum_{i \in I, i \neq ik} A_i + T$ and $M$ is a t-sum of $\{A_i\}_{i \in I, T = A_{ik}}$. Thus $Rx_{ik} \ll A_{ik}$ and $x_{ik} \in \text{Rad} A_{ik}$. Consequently, $x \in \sum_{i \in I} \text{Rad} A_i$ and $\text{Rad} M \leq \sum_{i \in I} \text{Rad} A_i$. \[ \sum_{i \in I} \text{Rad} A_i \leq \text{Rad} M \] is clear. Thus $\text{Rad} M = \sum_{i \in I} \text{Rad} A_i$. \(\square\)

Definition 3. Let $M$ be an $R$-module. $M$ is called a tg-supplemented module if every submodule of $M$ has a generalized supplement that is a t-summand of $M$. Clearly generalized $\oplus$-supplemented modules are tg-supplemented. But the converse is not true in general (See Example 4).

We can also clearly see that every supplemented module is tg-supplemented. But the converse of this statement is not always true (See Example 1, 2, 3). Since hollow and local modules are supplemented, they are tg-supplemented modules. Clearly, every tg-supplemented module is generalized supplemented.

Lemma 7. Let $M$ be an $R$-module. If $\text{Rad} M = M$, then $M$ is tg-supplemented.

Proof. Let $N$ be any submodule of $M$. Since $N + M = M$ and $N \cap M \leq M = \text{Rad} M$, we get that $M$ is a generalized supplement of $N$ in $M$. On the other hand $M$ and $0$ are mutual supplements in $M$. Hence $M$ is tg-supplemented. \(\square\)

Lemma 8. Let $M$ be a tg-supplemented R-module and $N \ll M$. Then $M/N$ is tg-supplemented.

Proof. Let $U/N \leq M/N$. Since $M$ is tg-supplemented, $U$ has a generalized supplement $V$ that is a t-summand in $M$. Then by ([9], the proof of Proposition 2.6), $(V + N)/N$ is a generalized supplement of $U/N$ in $M/N$. Since $V$ is a t-summand of $M$, there exists a submodule $L$ of $M$ such that $L$ and $V$ are mutual supplements in $M$. Since $L$ is a supplement of $V$ in $M$ and $N \ll M$, by ([10], 41.1(4)) $L$ is a supplement of $V + N$ in $M$. Then by ([10], 41.1(7)) $(L + N)/N$ is a supplement of $(V + N)/N$ in $M/N$. Similarly, we can prove that $(V + N)/N$ is a supplement of $(L + N)/N$ in $M/N$. Hence $(L + N)/N$ and $(V + N)/N$ are mutual supplements in $M/N$. Thus $M/N$ is tg-supplemented. \(\square\)

Corollary 5. Any small homomorphic image of a tg-supplemented module is tg-supplemented.
Lemma 9. Let $M$ be a $tg$-supplemented module and $N \leq M$. If $(K + N)/N$ is a $t$-summand of $M/N$ for every $t$-summant $K$ of $M$, then $M/N$ is $tg$-supplemented.

Proof. Let $U/N \leq M/N$. Since $M$ is $tg$-supplemented, $U$ has a generalized supplement $K$ in $M$ such that $K$ is a $t$-summand of $M$. Since $K$ is a generalized supplement of $U$ in $M$ and $N \leq U$, we can see that $(K + N)/N$ is a generalized supplement in $M/N$. Since $K$ is a $t$-summand of $M$, then by hypothesis $(K + N)/N$ is a $t$-summand of $M/N$. Hence every submodule of $M/N$ has a generalized supplement that is a $t$-summand of $M/N$, and $M/N$ is $tg$-supplemented. □

Lemma 10. Let $M$ be a distributive $tg$-supplemented $R$-module. Then every factor module of $M$ is $tg$-supplemented.

Proof. Clear from Lemma 6 and Lemma 9. □

Corollary 6. Let $M$ be a distributive $tg$-supplemented $R$-module. Then every homomorphic image of $M$ is $tg$-supplemented.

Proof. Clear from Lemma 10. □

Lemma 11. Let $M$ be an $R$-module and $RadM \ll M$. The following assertions are equivalent.

(i) $M$ is supplemented.

(ii) $M$ is $tg$-supplemented.

Proof. (i)⇒(ii) Clear from definitions.

(ii)⇒(i) Let $U \leq M$. Since $M$ is $tg$-supplemented, there exists a generalized supplement $V$ of $U$ that is a $t$-summand of $M$. Since $V$ is supplement in $M$, then $V \cap RadM = RadV$. Since $RadM \ll M$, $RadV \ll M$ and, by Lemma 3, $U \cap V \leq RadV \ll V$. Thus $V$ is a supplement of $U$ in $M$ and $M$ is supplemented. □

Corollary 7. Let $M$ be a finitely generated $R$-module. The following assertions are equivalent.

(i) $M$ is supplemented.

(ii) $M$ is $tg$-supplemented.

Proof. Since $M$ is finitely generated, $RadM \ll M$. Then clearly this assertions is derived from Lemma 11. □

Lemma 12. Let $M$ be a $t$-sum of $M_1$ and $M_2$. If $M_1$ and $M_2$ are $tg$-supplemented, then $M$ is $tg$-supplemented.
Proof. Let $U \subseteq M$. Since $M_1$ is tg-supplemented, $(M_2 + U) \cap M_1$ has a generalized supplement $X$ that is a t-summand in $M_1$. Since $M_2$ is tg-supplemented, $(U + X) \cap M_2$ has a generalized supplement $Y$ that is a t-summand in $M_2$. Then we get $M = M_1 + M_2 = M_2 + U + X = U + X + Y$ and $U \cap (X + Y) \subseteq (U + Y) \cap X + (U + X) \cap Y \leq \text{Rad}(X + Y)$. Hence $X + Y$ is a generalized supplement of $U$ in $M$. Since $M$ is a t-sum of $M_1$ and $M_2$, and $X$ is a t-summand of $M_1$, and $Y$ is a t-summand of $M_2$, then by Corollary 3, $X + Y$ is a t-summand of $M$. Thus $M$ is tg-supplemented. □

Corollary 8. Let $M$ be a t-sum of $M_1, M_2, \ldots, M_n$. If $M_i$ is tg-supplemented ($i = 1, 2, \ldots, n$), then $M$ is tg-supplemented.

Proof. Clear from Lemma 12. □

Example 1. Consider the $\mathbb{Z}$-module $Q$. Since $Q$ has no maximal submodule, we have $\text{Rad}(Q) = Q$. By Lemma 2.13, $Q$ is a tg-supplemented module. But it is well known that $Q$ is not supplemented (See [3], Example 20.12).

Example 2. Let $M$ be a non-torsion $\mathbb{Z}$-module with $\text{Rad}(M) = M$, then by Lemma 2.13, $M$ is tg-supplemented. But $M$ is not supplemented ([12]).

Example 3. Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}/p\mathbb{Z}$, for any prime $p$. In this case $\text{Rad}(M) \neq M$. Since $\mathbb{Q}$ and $p\mathbb{Z}$ are tg-supplemented, then by Lemma 12, $M$ is tg-supplemented. But $M$ is not supplemented.

Example 4. Let $R$ be a commutative local ring which is not a valuation ring. Let $a$ and $b$ be elements of $R$, where neither of them divides the other. By taking a suitable quotient ring, we may assume that $(a) \cap (b) = 0$ and $am = bm = 0$ where $m$ is the maximal ideal of $R$. Let $F$ be a free $R$-module with generators $x_1, x_2$, and $x_3$. $K$ be the submodule generated by $ax_1 - bx_2$ and $M = F/K$. Thus, $M = \frac{Rx_1 \oplus Rx_2 \oplus Rx_3}{R(ax_1 - bx_2)} = (Rx_1 + Rx_2) \oplus Rx_3$. Here $M$ is not $\oplus$-supplemented. But $F = Rx_1 \oplus Rx_2 \oplus Rx_3$ is completely $\oplus$-supplemented ([5]).

Since $F$ is completely $\oplus$-supplemented, $F$ is supplemented. Since a factor module of a supplemented module is supplemented, we have $M$ is supplemented. So $M$ is tg- supplemented. But since $M$ is finitely generated and not $\oplus$-supplemented, $M$ is not generalized $\oplus$-supplemented.

Lemma 13. Let $M$ be a t-sum of $M_1$ and $M_2$. Then $M_2$ is tg-supplemented if and only if for every submodule $N$ of $M$ such that $M_1 \leq N \leq M$, there exists a t-summand $K$ of $M_2$ such that $M = K + N$ and $N \cap K \leq \text{Rad}(M)$. 

Example 5. Let $R$ be a commutative local ring which is not a valuation ring. Let $a$ and $b$ be elements of $R$, where neither of them divides the other. By taking a suitable quotient ring, we may assume that $(a) \cap (b) = 0$ and $am = bm = 0$ where $m$ is the maximal ideal of $R$. Let $F$ be a free $R$-module with generators $x_1, x_2$, and $x_3$. $K$ be the submodule generated by $ax_1 - bx_2$ and $M = F/K$. Thus, $M = \frac{Rx_1 \oplus Rx_2 \oplus Rx_3}{R(ax_1 - bx_2)} = (Rx_1 + Rx_2) \oplus Rx_3$. Here $M$ is not $\oplus$-supplemented. But $F = Rx_1 \oplus Rx_2 \oplus Rx_3$ is completely $\oplus$-supplemented ([5]).

Since $F$ is completely $\oplus$-supplemented, $F$ is supplemented. Since a factor module of a supplemented module is supplemented, we have $M$ is supplemented. So $M$ is tg- supplemented. But since $M$ is finitely generated and not $\oplus$-supplemented, $M$ is not generalized $\oplus$-supplemented.
Proof. ($\Rightarrow$) Let $M_1 \leq N \leq M$. Since $M_2$ is tg-supplemented, there exists a generalized supplement $K$ of $N \cap M_2$ in $M_2$ such that $K$ is a t-summand of $M_2$. Then $M = M_1 + M_2 = N + N \cap M_2 + K = K + N$ and $N \cap K = N \cap M_2 \cap K \leq \Rad K \leq \Rad M$.

($\Leftarrow$) Let $L \leq M_2$ and $N = M_1 + L$. By hypothesis, there exists a t-summand $K$ of $M_2$ such that $M = K + N$ and $N \cap K \leq \Rad M$. Since $K, L \leq M_2$, by Modular law, $M_2 = M_2 \cap M = M_2 \cap (K + N) = K + M_2 \cap N = K + M_2 \cap (M_1 + L) = L + K + M_2 \cap M_1$, and then by $M_2 \cap M_1 \ll M_2$, $M_2 = L + K$. Since $K$ is a t-summand of $M_2$, then by Corollary 3, $K$ is a t-summand of $M$. Then $\Rad K = K \cap \Rad M$ and by $N \cap K \leq \Rad M, L \cap K \leq N \cap K = K \cap (N \cap K) \leq K \cap \Rad M = \Rad K$. Hence $K$ is a generalized supplement of $L$ in $M_2$. Thus, $M_2$ is tg-supplemented. □

Theorem 2. Let $M$ be a tg-supplemented module. Assume that $M$ is a t-sum of $M_1$ and $M_2$. If $K \cap M_2$ is a t-summand of $M_2$ for every t-summand $K$ of $M$ such that $M = K + M_2$, then $M_2$ is tg-supplemented.

Proof. Let $M_1 \leq N \leq M$. Since $M$ is tg-supplemented, $N \cap M_2$ has a generalized supplement $K$ in $M$ such that $K$ is a t-summand of $M$. From this we have $M = N \cap M_2 + K$ and $N \cap M_2 \cap K \leq \Rad K \leq \Rad M$. Since $M = N \cap M_2 + K$, then by Modular law $M_2 = N \cap M_2 + M_2 \cap K$. Since $M_1 \leq N$, $M = M_1 + M_2 = M_1 + N \cap M_2 + M_2 \cap K = N + M_2 \cap K$. Since $M = K + M_2$ and $K$ is a t-summand of $M$, then by hypothesis $M_2 \cap K$ is a t-summand of $M_2$. Hence by Lemma 13, $M_2$ is tg-supplemented. □

Lemma 14. Let $M$ be a $\pi$-projective module. Then $M$ is tg-supplemented if and only if $M$ is generalized $\oplus$-supplemented.

Proof. Clear from Lemma 2. □

Theorem 3. Let $M$ be a projective module. The following assertions are equivalent.

(i) $M$ is semiperfect.
(ii) $M$ is generalized $\oplus$-supplemented.
(iii) $M$ is tg-supplemented.

Proof. (i)$\Leftrightarrow$(ii) Clear from ([10], 42.1).
(ii)$\Leftrightarrow$(iii) Clear from Lemma 14. □

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