IMPERFECTION WITH INSPECTION POLICY AND VARIABLE DEMAND UNDER TRADE-CREDIT: A DETERIORATING INVENTORY MODEL

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(Communicated by Ines Marques)

Abstract. A deteriorating inventory model with imperfect product and variable demand is formulated in this paper. A time-dependent deterioration factor is considered because the rate of deterioration is highly hinging on time. We introduce imperfect quality of production which leads to imperfect items in our proposed model. The retailer adopts inspection policy to pick over the perfect items from imperfect. Type I and Type II, both type of errors are included and the retailer invest some capital to improve the production process quality of the supplier. There is also a penalty cost for the retailer if they deliver some defective items by mistake. Sometime, there is a high amount of demand and, consequently, we assume shortages and partial backorder in our formulated model. The retailer adopts the trade-credit policy for his customers in order to promote market competition. The main objective of the paper is to show that the total cost is globally minimized and we have aimed at reducing the total cycle length, defectiveness of the system and the optimal order size by maximizing the total profit of the system. Then, we present three theorems and prove them to find an easy solution procedure to reduce the total cost of a system. The results are discussed with the help of numerical examples to approve the proposed model. A sensitivity analysis of the optimal solutions for the parameters is also provided. The paper ends with the conclusions and an outlook to possible future studies.

2010 Mathematics Subject Classification. Primary: 90B05, 91A10; Secondary: 90C26.

Key words and phrases. Inventory, Variable deterioration, Imperfect quality items, Time-dependent demand, Inspection policy, Trade-credit, Optimization.

The author, Magfura Pervin is very thankful to University Grants Commission (UGC) of India for providing financial support to continue this research work under [MANF(UGC)] scheme: Sanctioned letter number [F1-17.1/2012-13/MANF-2012-13-MUS-WES-19170 /(SA-III/Website)] dated 28/02/2013.

The research of Gerhard-Wilhelm Weber (Institute of Applied Mathematics, METU, 06800, Ankara, Turkey) is partially supported by the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e a Tecnologia"), through the CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/ 04106/2013.

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1. **Introduction.** In a traditional inventory model, it is assumed that the production process is always perfect. However, in a real-life situation, one can observe that the production may contain a substantial number of imperfect items. These items have to be rejected by the customers, so that they must be repaired and refunded with an additional cost. Defective items which are produced in each cycle of production are not always successfully screened out internally during the production process. Thus, there is no instant delivery of these items after production. Facing this imperfect production condition, a company needs to have a continuous improvement strategy for making an appropriate operation by considering imperfect production. So, imperfect production leads to a contribution in our proposed model.

In a conventional inventory model, it is assumed that the demand rate is independent of factors such as stock availability, price of items, time, etc. But, in actual practice, demand for certain items is greatly influenced by time. For example, one can note that the demand of seasonal foods, garments such as winter cloths, which are not essential at summer; thus, demand is highly dependent on time. In deterministic inventory systems, we normally consider a constant demand rate but in order to accommodate real-life criteria, many authors have studied other types of demand. Moreover, acceptance of a constant demand rate is not reasonable for many inventory items such as electronic goods, fashionable garments, tasty foods, volatile liquids, etc., as they fluctuate in terms of the demand rate. So, time-dependent demand plays an active role in an inventory control system.

In classical inventory models, we assume that items preserve their physical characteristics while they are stored in an inventory. But deterioration has a crucial impact on an inventory control. The term **deterioration** defines a situation in which all items in the inventory system become obsolete at the end or during the prescribed planning horizon. Deterioration is understood as falling from a higher to a lower level in quality; it also simply implies a change, decay, loss of utility or loss of marginal value of goods that results in decreasing the usefulness of the original item. The rate of deterioration is nearly negligible for items like hardware, glassware, toys and steel; or, it is very much effective for products such as fruits, vegetables, medicines, volatile liquids, at blood banks, high-tech products, gasoline, alcohol, turpentine, etc. Many traditional models consider a constant deterioration rate, but in actual practice, we see that time-dependent deterioration is mostly reliable as the deterioration rate increases or decreases with respect to time. Therefore, variable deterioration of items has a vital role in the determination of an inventory model and must be taken into consideration.

In the classical EOQ model, it is assumed that the customer must pay for items upon receiving them. But in practice, a retailer allows a certain fixed period to settle the account for stimulating customer’s demand. During this credit period, the customer can earn interest on that revenue through investing into a share market or banking business, but beyond this period the retailer charges a higher interest if the payment is not settled at the end of the offered credit period. This is termed as trade-credit financing. Hence, paying later indirectly reduces the cost of holding a stock. Hence, trade credit can play a major role in inventory control.

**Shortage** means a situation in which something is needed which cannot be provided in sufficient amounts. In real-life situations, occurrence of shortage in inventory system is a vital situation. Shortages are of great importance for many models, especially, where delay in payment is considered. Shortages of items may occur
due to the withdrawal of imperfect items from the stock. In inventory models with shortages, the general condition is that the unfulfilled demand is either completely lost or completely backlogged. However, there is a little chance that while few customers leave, others are willing to wait until the fulfillment of their demand. As an example, suppose that a customer prefers a particular company for shopping and, in the mean time, the customer observes that there is a shortage for a particular product. But the customer is not willing to buy from another company. In this situation, the customer has to wait until the supply of the particular product will be given. Based on this phenomena, this paper will consider shortages which is a great positive impact on real-life situations.

The main suppositions of our work are as follows:

- The selling items are perishable over time; e.g., gasoline, fruits, fresh fish, photographic films, vegetables, etc., and they follow a variable deterioration rate that allows an edge application scope.
- We introduce imperfect production quality of items because the production process cannot be perfect at any time, and allows for a broader applicability of our works. Among these imperfect items, some products are less defective and some are repairable at a low cost; those products will be delivered at a lower price. Some products with defective items are highly damageable; they will be rejected immediately from the market.
- The retailer pursues a product inspection policy instead of a total inspection policy to separate the imperfect items from perfect. Type I and Type II, both type of errors are measured at the time of calculation to execute an error free result and to maintain the reputation of their business.
- The retailer invests some capital to improve the items quality and the capital investment follows an logarithmic function of the imperfect amount.
- The rate of replenishment is finite.
- Time-dependent demand approach is considered here, since, with respect to time, demand of many items cannot remain constant, especially for seasonal products.
- Shortages are chosen to fit the model in more realistic senses.
- Here, the retailer would offer a fixed-credit period which will encourage the supplier’s selling, and the retailer can take an advantage to reduce the cost and to increase the profit.
- In our paper, we separately describe the total cost for the retailer when trade credit is offered and when it is not offered. Hence, it will help the inventory manager to understand the system’s cost, and he/she can take suitable measurements to increase the profit of the system.

The rest of the paper is organized as follows: In Section 2, a motivation and a review on the research area are presented. Section 3 contains two subsections in which the first one presents the notations and the second one provides the assumptions of the formulated model. Section 4 formulates the mathematical model of the proposed inventory problem. The solution procedure is derived in Section 5 which includes an algorithm. Section 6 discusses three numerical examples to illustrate the proposed problem and our methodology. Section 7 presents a sensitivity analysis on our suggested approach. Section 8 depicts concluding remarks related to our article and it proposes new pathways of future investigation.
2. Motivation and Review on Research. A common unrealistic assumption about an inventory model is that all received or produced items are always perfect. But, a production process may not be accurate in general. A considerable amounts of papers has been introduced in this direction to treat the problem of imperfection. Porteus [32] investigated the influence of defective items on an EOQ model and captured a significant relationship between quality and lot size. Lee and Rosenblatt [19] considered a joint lot size EOQ model with a fixed percentage of defective items. Salameh and Jaber [34] examined an EOQ model with inspection policy for items with imperfect quality. Cárdenas-Barrón [4] studied an economic production quantity model where he added rework process for a single-stage manufacturing system and he also backorders on shortages. Khan et al. [18] developed an integrated vendor-buyer inventory model accounting for quality inspection errors at the buyer's end. Lin and Hou [21] introduced an imperfect quality EOQ model with advanced receiving of a perfect item. Sana [35] investigated an economic production lot-size model in an imperfect production system. Balkhi [3] found an optimal solution for a general lot-size model for deteriorated items. Furthermore, he studied the effect for imperfect production, and also considered inflation and time value of money.

Many products such as vegetables, fish, fruits or volatile liquids, deteriorate continuously due to change, spoilage, pinch, evaporation, etc. Herewith, deterioration has a crucial impact in determining the inventory model. Deteriorating inventory models have widely been studied in recent years. Ghare and Schrader [10] first considered a no-shortage inventory model with constant deterioration rate. Goswami and Chaudhuri [12] formulated a general rule to transform a constant demand to a linear demand for a deteriorating EOQ model. Afterwards, Goyal and Giri [14] surveyed recent trends of a deteriorating inventory model. Liao [20] developed an inventory control model with instantaneous receipt of an exponentially deteriorating item under two levels of trade-credit policy. Bakker et al. [2] provided an exhaustive literature survey on inventory systems with deterioration since 2001. Padmanabhan and Vrat [25] exhibited an EOQ models for perishable items under a stock-dependent selling rate. Dye [9] proposed the effects of technology investment on deteriorating items. Sarkar et al. [37] studied an inventory model with deterioration under trade-credit policy. Mishra et al. [22] derived a stock and price dependent inventory model for controlling the deterioration by applying preservation technology. Recently, Pervin et al. [26] presented a deteriorating and decaying inventory model under trade-credit policy. Newly, Pervin et al. [31] added the effect of preservation technology to reduce the rate of deterioration.

Trade-credit policy is a certain time period which is offered by the supplier/retailer to the retailer/customer in order to increase the demand for items and to gain more, and within the time period the money will be paid for the certain product. A number of research papers appeared in recent years which dealt with the problem of trade-credit policy. Goyal [13] introduced first the EOQ inventory model under the condition of trade-credit policy. Aggarwal and Jaggi [1] extended Goyal’s model [14] to the case with deteriorating items. Chang and Dye [5] extended Aggarwal and Jaggi’s model [1] with an involvement of shortage. Ouyang et al. [23] considered an inventory model for deteriorating items with partial backlogging under permissible delay in payments. Pervin et al. [27] described a deteriorating EOQ model with variable demand and holding cost under the effect of trade-credit policy. Recently, Pervin et al. [29] derived an integrated model with variable holding cost under the
effect of two level of trade-credit policy. A multi-item deteriorating inventory model with the effect of trade-credit policy was elaborated by Pervin et al. [30]. In our model, we formulate a deteriorating inventory model with preservation technology investment under stock-price sensitive demand. Roy et al. [33] discovered the effect of two-level trade-credit policy for a probabilistic model.

The acceptance of a constant demand rate is unsuitable for many inventory items like fashionable clothes, electronic equipments, tasty food, etc. In this case, the demand function is totally dependent on time in an inventory model. For example, land phone connection has attracted the attention of many people twenty years ago, but nowadays, people chose mobile connection more than land connection; therefore, land phone connection is almost backdated. To clarify the situation more clearly, Donaldson [8] derived an analytical solution to the problem of obtaining the optimal number of replenishment with a linearly time-dependent demand pattern over a finite time horizon. Teng et al. [38] discussed an inventory model where time-varying demand rate and partial backlogging were considered for deteriorating items. Recently, many investigators like Hargia and Benkerrouf [16] and Jalan et al. [17], have studied in this field and suggested a model where shortages are permitted. Researchers like Dave and Patel [7], Chung and Ting [6], Ghosh and Chaudhuri [11], etc., designed deteriorating inventory models with shortages and linearly trended demand.

Shortage is a natural phenomenon and has a significant value in calculating inventory cost. For example, we can observe that many products of famous brands or fashionable goods such as clothes, hi-fi equipment, brand gum shoes, and jewelery may create a certain situation in which customers think that it will be better to wait for the receipt of backorders at the time period when shortages occur. Many research works have been published in this field. Wee [40] considered a deterministic model for deteriorating items with shortages and declining demand. Sana et al. [36] formulated an inventory model for deteriorating items with trended demand and shortages. Tripathi [39] developed an inventory model with stock-dependent demand and shortages under trade-credit policy. Pervin et al. [28] analyzed a deteriorating inventory model with stock-dependent demand and variable holding cost. The studies of various researchers connected with the subject of this paper are shown in Table 1.

Table 1: Contributions of many authors related to inventory model.

| Author(s)               | Imperfection | Variable demand | Deteriorations | Trade-credit policy | Shortages |
|-------------------------|--------------|-----------------|----------------|---------------------|-----------|
| Sana et al. (2004)      | ✓            | ✓               | ✓              | ✓                   | ✓         |
| Khan et al. (2014)      | ✓            | ✓               | ✓              | ✓                   | ✓         |
| Lin and Hou (2015)      | ✓            | ✓               | ✓              | ✓                   | ✓         |
| Teng et al. (2002)      | ✓            | ✓               | ✓              | ✓                   | ✓         |
| Ghosh and Chaudhuri (2004) | ✓        | ✓               | ✓              | ✓                   | ✓         |
| Kumar et al. (2012)     | ✓            | ✓               | ✓              | ✓                   | ✓         |
| Pervin et al. (2016,a)  | ✓            | ✓               | ✓              | ✓                   | ✓         |
| Mahata (2012)           | ✓            | ✓               | ✓              | ✓                   | ✓         |
| Goswami and Chaudhuri (1991) | ✓        | ✓               | ✓              | ✓                   | ✓         |
| Aggarwal and Jaggi (1995)| ✓            | ✓               | ✓              | ✓                   | ✓         |
| Ting (2015)             | ✓            | ✓               | ✓              | ✓                   | ✓         |
| Ouyang et al. (2005)    | ✓            | ✓               | ✓              | ✓                   | ✓         |
| Tripathi (2015)         | ✓            | ✓               | ✓              | ✓                   | ✓         |
| Annadurai and Uthayakumar (2015) | ✓    | ✓               | ✓              | ✓                   | ✓         |
| Haikhi (2004)           | ✓            | ✓               | ✓              | ✓                   | ✓         |
| Pervin et al. (2016,b)  | ✓            | ✓               | ✓              | ✓                   | ✓         |
| Our paper               | ✓            | ✓               | ✓              | ✓                   | ✓         |
3. Notations and Assumptions. The following notations and assumptions are considered to generate our model.

3.1. Notations.

- **T**: total length of cycle time (a decision variable),
- **c**: unit purchasing cost per item,
- **Q**: retailer’s order quantity per replenishment,
- **q**: size of each shipment from supplier to retailer (a decision variable),
- **A**: ordering cost per order,
- **s**: lost sale cost per unit,
- **δ**: backorder cost per order,
- **I_1(t)**: inventory level that changes with time *t* during production period,
- **I_2(t)**: inventory level that assumes with time *t* during non-production period,
- **I_3(t)**: inventory level that chooses with time *t* during shortage period,
- **I_4(t)**: inventory level that considers with time *t* during reproduction period,
- **I_e**: interest, which can be earned per unit of time (i.e., per $ per year) by the retailer,
- **I_c**: interest charges in stocks per unit of time (i.e., per $ in stocks per year) by the supplier, where \( I_c \geq I_e \),
- **M**: credit period in years offered by the supplier,
- **α**: proportion of defective items (a decision variable),
- **γ**: proportion of defective items that can be reworked,
- **α_u**: proportion of defective items before improving production process,
- **G(α)**: capital investment for improving production process quality to reduce the defective rate,
- **λ**: rate of percentage decrease in α, per increase in \( G(α) \),
- **h_1**: unit holding cost for non-defective items per unit per unit time,
- **h_2**: unit holding cost for defective items per unit per unit time,
- **r_1**: unit cost of falsely accepting defective items per unit time,
- **r_2**: unit cost of falsely rejecting non-defective items per unit time,
- **r_3**: unit opportunity cost of capital investment per unit per unit time,
- **r_4**: unit inspection cost per items per unit time,
- **r_5**: unit penalty cost for items per unit of time,
- **R_1(T)**: total relevant cost function per unit of time for the retailer when trade credit is not offered (a decision variable),
- **R(T)**: total relevant cost function per unit of time for the retailer when trade credit is offered (a decision variable).

3.2. Assumptions.

1. The rate of replenishment is finite with rate \( k \), from which some \( α \) part is imperfect \((0 \leq α < 1)\).
2. The annual demand, \( D(t) \) is a linear increasing function of time and is represented by \( D(t) = a + bt \), where \( a \) and \( b \) are positive constants.
3. The deterioration rate is considered as time dependent and is given by \( \theta(t) = \theta t \), where \( \theta \in [0, 1) \).
4. The retailer follows a capital investment, \( G(α) \), for improving the item quality for reducing the defective items, which is given as \( G(α) = \frac{1}{λ} \ln(\frac{α_u}{α}) \), where \( 0 < α \leq α_u \). (This function was first used by Hall [15], and is being widely used by many researchers like Porteus [32], Ouyang et al. [24].) Here, \( α_u \) is
proportion of defective items before improving the production process and \( \lambda \) is percentage decrease in \( \alpha \), per increase in \( G(\alpha) \).

5. \( Q \) is the retailer order quantity which is expressed as \( qk(1-\alpha) \).

6. At the time of inspection, error may be caused by the retailer. There are two types of errors which are considered in our paper. Type I is the error which occurs when the retailer accepts some defective items as non-defective by mistake. Type II happens if the retailer falsely rejects some non-defective items as defective.

7. The lead time is negligible.

8. Shortages are allowed, and only a fraction \( \beta \), \( 0 \leq \beta < 1 \), is backordered, whereas the rest is lost.

9. If \( T \geq M \), then the retailer settles the account at time \( M \) and pays for the interest charges on items in stock with rate \( I_c \) over the interval \([M,T]\). If \( T < M \), then the retailer adjusts the account at time \( M \), and there is no interest charge in stock during the whole cycle.

4. Mathematical Formulation. Based on our prerequisite assumptions, the inventory system may be considered as: initially (i.e., at time \( t = 0 \)), the cycle starts with a zero stock level at supply rate \( k \) among which an \( \alpha \) units are imperfect. The replenishment or supply continues up to time \( t_1 \). During the time period \([0,t_1]\), the inventory piles up by adjusting the demand in the market and stops at time \( t = t_1 \). This accumulated inventory level at time \( t_1 \) gradually diminishes due to demand and deterioration during the period \([t_1,t_2]\) and ultimately falls to 0 at time \( t = t_2 \). During the period \([t_2,t_3]\), shortages occur with a rate of \( \beta \) and continue up to time \( t = t_3 \). After that time the production is restarted to recover both demand and backordered items until the inventory level reaches the value 0 by time \( t = t_4 \). We cannot consider deterioration during time interval \([t_1,t_4]\) because the backordered items are delivered instantly as soon as the recollected items are received, and the broken or smashed items during the transporting time are considered as imperfect and returned back immediately to the supplier by the retailer. After the scheduling period, the cycle repeats itself.

Now, the differential equations involving the instantaneous state of the inventory level in the interval \([0,t_4]\), together with their initial values, are given subsequently:

\[
\frac{dI_1(t)}{dt} = k(1-\alpha) - D(t) - \theta(t)I_1(t), \quad t \in [0,t_1),
\]

with the initial condition \( I(0) = 0 \);

\[
\frac{dI_2(t)}{dt} = -D(t) - \theta(t)I_2(t), \quad t \in [t_1,t_2),
\]

with the boundary condition \( I(t_2) = 0 \);

\[
\frac{dI_3(t)}{dt} = -\beta D(t), \quad t \in [t_2,t_3),
\]

with the boundary condition \( I(t_3) = 0 \);

\[
\frac{dI_4(t)}{dt} = k(1-\alpha) - D(t), \quad t \in [t_3,t_4),
\]
with the boundary condition $I(t_4) = 0$.

Now, the solution of Equation 1 using the initial condition becomes

$$I_1(t) = \frac{k(1 - \alpha) - a}{\theta} - \frac{b}{\theta} (1 - e^{-\theta t}) - \frac{b}{\theta^2} \left( e^{-\theta t} - 1 \right), \ 0 \leq t < t_1.$$  

Utilizing the boundary condition, the solution of Equation 2 becomes

$$I_2(t) = \frac{b}{\theta^2} \left( 1 - e^{\theta(t_2 - t)} \right) - \frac{a + bt}{\theta} - \frac{a + bt_2}{\theta} e^{\theta(t_2 - t)}, \ t_1 \leq t < t_2.$$  

Solving Equation 3 with the help of its just obtained boundary condition, we get

$$I_3(t) = \beta a(t_2 - t) + \frac{\beta b}{2} (t_2^2 - t^2), \ t_2 \leq t < t_3.$$  

By solving Equation 4 with the help of boundary condition, we obtain

$$I_4(t) = \{k(1 - \alpha) - a \} (t - t_4) + \frac{b}{2} (t_4^2 - t^2), \ t_3 \leq t < t_4.$$  

Using the condition $I_1(t_1) = I_2(t_1)$, we get

$$t_1 = \frac{1}{2} \left( \frac{bT}{\theta} - \frac{b}{\theta^2} - \frac{a}{\theta} \right) e^{\theta T} + \frac{1}{2} \left( \frac{k(1 - \alpha)}{\theta} - \frac{b}{\theta^2} - \frac{a(T + 1)}{\theta} \right) - 2be^{\theta T} \left( \frac{b(T + 1)}{\theta^2} - \frac{aT}{\theta} \right). \quad (5)$$  

Using the condition $I_2(t_2) = I_3(t_2)$, we obtain

$$t_2 = \left( \frac{b}{\theta^2} + \frac{bT}{\theta} + \frac{a}{\theta} \right) e^{\theta(T - t_1)} - \frac{\theta(\beta aT + \frac{\beta bT^2}{2})}{2bT + a}. \quad (6)$$  

Utilizing the condition $I_3(t_3) = I_4(t_3)$, we calculate

$$t_3 = \left( \frac{T^2 \beta b}{2} + \frac{bt_2}{2} \right)^2 + T \left( \beta a + k(1 - \alpha) - a \right). \quad (7)$$  

Using the condition $I_4(t_4) = 0$, we derive the following result as:

$$t_4 = \frac{k(1 - \alpha) - a}{b} + \frac{(bT^2 - 2T) \sqrt{(k(1 - \alpha) - a^2)}}{b}. \quad (8)$$  

Figure 1 indicates here the graphical representation of our proposed inventory model.

The elements which take part in calculating the cost function are listed below:

1. The annual ordering cost, $OC$, is $= qA$. 

2. The total holding cost for non-defective, \( HCN \), items is computed as follows:

\[
HCN = qh_1(1-\alpha) \left[ \int_{t_1}^{t_2} I_1(t) dt + \int_{t_1}^{t_2} I_2(t) dt \right]
\]

\[
= qh_1(1-\alpha) \left[ \int_{t_1}^{t_2} \left( \frac{k(1-\alpha) - a}{\theta} (1 - e^{-\theta t}) - \frac{bt}{\theta} + \frac{b}{\theta^2} (e^{-\theta t} - 1) \right) dt \right]
\]

\[
+ \int_{t_1}^{t_2} \left( \frac{b}{\theta^2} (1 - e^{\theta(t_2-t)}) - \frac{a+bt}{\theta} - \frac{a+bt_2}{\theta} e^{\theta(t_2-t)} \right) dt \right]
\]

\[
= qh_1(1-\alpha) \left[ \left( \frac{k(1-\alpha) - a}{\theta} \right) \left( t_1 + \frac{(e^{-\theta t_2} - 1)}{\theta} \right) \right]
\]

\[
+ \frac{b}{\theta^2} \left( t_2 - t_1 \right)
\]

\[
+ \frac{1}{\theta} e^{\theta(t_1-t_2)} - 1 \theta \left( a(t_2-t_1) + \frac{b}{2} \left( t_2 - t_1 \right) \right) + \frac{1}{\theta^2} (a+bt_2) e^{\theta(t_1-t_2)} \right].
\]

3. The total holding cost for defective, \( HCD \), items is derived as stated below:

\[
HCD = qh_2\alpha \left[ \int_{0}^{t_1} I_1(t) dt + \int_{t_1}^{t_2} I_2(t) dt \right]
\]

\[
= qh_2\alpha \left[ \int_{0}^{t_1} \left( \frac{k(1-\alpha) - a}{\theta} (1 - e^{-\theta t}) - \frac{bt}{\theta} + \frac{b}{\theta^2} (e^{-\theta t} - 1) \right) dt \right]
\]

\[
+ \int_{t_1}^{t_2} \left( \frac{b}{\theta^2} (1 - e^{\theta(t_2-t)}) - \frac{a+bt}{\theta} - \frac{a+bt_2}{\theta} e^{\theta(t_2-t)} \right) dt \right]
\]

\[
= qh_2\alpha \left[ \left( \frac{k(1-\alpha) - a}{\theta} \right) \left( t_1 + \frac{(e^{-\theta t_2} - 1)}{\theta} \right) \right]
\]

\[
- \frac{b}{\theta^2} \left( t_2 - t_1 \right)
\]

\[
+ \frac{1}{\theta} e^{\theta(t_1-t_2)} - 1 \theta \left( a(t_2-t_1) + \frac{b}{2} \left( t_2 - t_1 \right) \right) + \frac{1}{\theta^2} (a+bt_2) e^{\theta(t_1-t_2)} \right].
\]
4. The deteriorating cost, DC, is represented by:
\[
qcqθ \left[ \int_{0}^{t_1} (k - (a + bt))dt + \int_{t_1}^{t_2} I_2(t)dt \right] \\
= cq\theta \left[ (k - a)t_1 + \frac{bt^2_1}{2} + \frac{b}{\theta^2}((t_2 - t_1) + \frac{1}{\theta}e^{\theta(t_2 - t_1)}) \right] \\
- \frac{1}{\theta} \left( a(t_2 - t_1) + \frac{b}{2}(t^2_2 - t^2_1) \right) + \frac{1}{\theta^2}(a + bt_2)e^{\theta(t_2 - t_1)}.
\]

5. Shortage cost is accumulated during the time interval \([t_2, t_3]\); herewith, the shortage cost, SC, is expressed as:
\[
\sqrt{q}\beta \int_{t_2}^{t_3} (a + bt)dt \\
= \sqrt{q}\beta \left[ a(t_3 - t_2) + \frac{b(t^2_3 - t^2_2)}{2} \right].
\]

6. Due to shortage during the time interval \([t_2, t_3]\), all the customers are not interested to wait for the coming lot size to arrive, which may occur loss in profit. Hence, the lost sale cost, LSC, is:
\[
\delta q(1 - \beta) \int_{t_2}^{t_3} D(t)dt \\
= \delta q(1 - \beta) \int_{t_2}^{t_3} (a + bt)dt \\
= \delta q(1 - \beta) \left[ a(t_3 - t_2) + \frac{b(t^2_3 - t^2_2)}{2} \right].
\]

7. The Backorder cost, BC, is expressed as follows:
\[
\beta q \int_{t_3}^{t_4} I_4(t)dt \\
= \beta q \int_{t_3}^{t_4} \left( (k(1 - \alpha) - a)(t - t_4) + \frac{b}{2}(t^2_4 - t^2) \right) dt \\
= \beta q \left[ \frac{b}{6}(2t^3_4 - 3t^2_4t_3 + t^3_3) - (k(1 - \alpha) - a)(t_4 - t_3)^2 \right].
\]

8. There is an Opportunity cost, OPC, occurs due to the capital investment for improving the production quality. Hence, the annual OPC per cycle is implemented as:
\[
r_3G(\alpha)q = r_3 \frac{G}{X} \ln(\frac{\alpha u}{\alpha}) qk(1 - \alpha).
\]

9. For improving the production process and for maintaining the reputation of his business, the retailer follows inspection policy to inspect all the defective items after receiving. Therefore, the annual Inspection cost, IPC, per replenishment cycle is given by
\[
qr_4Q = q^2r_4k(1 - \alpha).
\]

10. The annual cost due to Type I error is expressed subsequently:
\[
\frac{D\alpha r_1(1 - \gamma)}{T(1 - \alpha\gamma)} = \frac{(a + bt_3)\alpha r_1(1 - \gamma)}{T(1 - \alpha\gamma)}.
\]
Annual cost occurs due to Type II error per unit time is given as follows:

\[
\frac{Da\gamma r_2}{T(1 - \alpha\gamma)} = \frac{(a + bT_2)\alpha r_1}{T(1 - \alpha\gamma)}.
\]

Therefore, the annual Penalty cost, \(PC\), for retailer to the customer is calculated as:

\[
rsq\left(\frac{(a + bT_3)\alpha r_1(1 - \gamma)}{T(1 - \alpha\gamma)} + \frac{(a + bT_2)\alpha r_1 r_2}{T(1 - \alpha\gamma)}\right).
\]

From our considered assumptions, there are two cases for both interest earned and interest charged, which are listed below:

**Case 1:** \(T \leq M\).

11. **Interest earned:** Here, the trade-credit period’s length \(M\) is greater than the total cycle time \(T\). So, the total interest gained by the retailer is

\[
\frac{sqI_e}{T} \left[ \int_0^T D(t)dt + (M - T) \int_0^T D(t)dt \right] = \frac{sqI_e}{T} \left[ \int_0^T (a + bt)d(t)dt + (M - T) \int_0^T (a + bt)dt \right] = \frac{sqI_e}{T} \left[ \frac{aT^2}{2} + \frac{bT^3}{3} + (M - T)(aT + \frac{bT^2}{2}) \right].
\]

12. **Interest charged:** Here, also the trade credit period \(M\) is greater than the total cycle time \(T\). So, the retailer has not charged any interest by the supplier; therefore, the interest charged is 0.

**Case 2:** \(M \leq T\).

13. **Interest earned:** In this case, the length of the trade-credit period, \(M\), is less than the total cycle time \(T\). Hence, the supplier will pay the interest only on demand basis, and the interest earned by the retailer is

\[
\frac{sqI_e}{T} \left[ \int_0^M D(t)dt \right] = \frac{sqI_e}{T} \left[ \int_0^M (a + bt)d(t)dt \right] = \frac{sqI_e}{T} \left[ \frac{aM^2}{2} + \frac{bM^3}{3} \right].
\]

14. **Interest charged:** In this case, the trade-credit period \(M\) is less than the total cycle time \(T\). So, the interest charged by the supplier to the retailer is given by

\[
\frac{cqI_e}{T} \left[ \int_M^T I_2(t)dt \right] = \frac{cqI_e}{T} \left[ \int_M^T \frac{b}{\theta^2} \left( 1 - e^{\theta(t-M)} \right) - \frac{a + bt}{\theta} - \frac{a + bT_2}{\theta} e^{\theta(t-M)}dt \right] = \frac{cqI_e}{T} \left[ \frac{b}{\theta^2} \left( (T - M) + \frac{1}{\theta^2} e^{\theta(M-T)} \right) - \frac{1}{\theta^2} (a(T - M) + \frac{b}{2}(T_2^2 - M^2)) + \frac{1}{\theta^2}(a + bT_2)e^{\theta(M-T)} \right].
\]
Now, the total cost for the retailer when trade credit is not offered is represented by
\[
R_1(T) = [OC + HCN + HCD + DC + SC + BC + LSC + OPC + IPC + PC].
\]

Therefore, by inserting the values of the above parameters, the following case-wise representations are achieved:

\[
R_1(T) = \frac{Ag}{T} + \left( \frac{qh_1(1-\alpha)}{T} + \frac{qh_2\alpha}{T} \right) \left[ \frac{k(1-\alpha) - a T}{\theta} - \frac{b}{\theta^2} (t_1 + \frac{(e^{-\theta t_1} - 1)}{\theta}) - \frac{bt_1^2}{2\theta} \right]
+ \frac{b}{\theta^2} (t_2 - t_1) - \frac{1}{\theta} (1 - e^{\theta (t_2 - t_1)}) - \frac{1}{\theta} (a(t_2 - t_1) + b(t_2^2 - t_1^2)) - \frac{1}{\theta} ((a + bt_2) (1 - e^{\theta (t_2 - t_1)}) - \frac{b}{\theta^2} ((t_2 - t_1))
+ \frac{1}{\theta} e^{\theta (t_1 - t_2)} - \frac{1}{\theta} \left( a(t_2 - t_1) + \frac{b}{2} (t_2^2 - t_1^2) \right) + \frac{1}{\theta^2} (a + bt_2) e^{\theta (t_1 - t_2)}
+ \frac{sq\beta}{T} \left[ a(t_3 - t_2) + \frac{b(t_3^2 - t_2^2)}{2} \right] + \frac{q\beta}{T} \left[ a(t_3 - t_2) + \frac{b(t_3^2 - t_2^2)}{2} \right]
+ \frac{q\gamma}{T} \left[ a(t_3 - t_2) + \frac{b(t_3^2 - t_2^2)}{2} \right].
\]

Now, the total cost for the retailer when trade credit is offered is represented by
\[
R(T) = \begin{cases} R_2(T), & \text{if } M \leq T, \\ R_3(T), & \text{if } 0 \leq T \leq M, \end{cases}
\]

where
\[
R(T) = [OC + HCN + HCD + DC + SC + BC + LSC + OPC + IPC + PC + IC - IE].
\]

Therefore, by inserting the values of the above parameters, the following case-wise representations are achieved:

\[
R_2(T)
= \frac{Ag}{T} + \left( \frac{qh_1(1-\alpha)}{T} + \frac{qh_2\alpha}{T} \right) \left[ \frac{k(1-\alpha) - a T}{\theta} - \frac{b}{\theta^2} (t_1 + \frac{(e^{-\theta t_1} - 1)}{\theta}) - \frac{bt_1^2}{2\theta} \right]
+ \frac{b}{\theta^2} (t_2 - t_1) - \frac{1}{\theta} (1 - e^{\theta (t_2 - t_1)}) - \frac{1}{\theta} (a(t_2 - t_1) + b(t_2^2 - t_1^2)) - \frac{1}{\theta} ((a + bt_2) (1 - e^{\theta (t_2 - t_1)}) - \frac{b}{\theta^2} ((t_2 - t_1))
+ \frac{1}{\theta} e^{\theta (t_1 - t_2)} - \frac{1}{\theta} \left( a(t_2 - t_1) + \frac{b}{2} (t_2^2 - t_1^2) \right) + \frac{1}{\theta^2} (a + bt_2) e^{\theta (t_1 - t_2)}
+ \frac{sq\beta}{T} \left[ a(t_3 - t_2) + \frac{b(t_3^2 - t_2^2)}{2} \right] + \frac{q\beta}{T} \left[ a(t_3 - t_2) + \frac{b(t_3^2 - t_2^2)}{2} \right]
+ \frac{q\gamma}{T} \left[ a(t_3 - t_2) + \frac{b(t_3^2 - t_2^2)}{2} \right].
\]

\textbf{Note:} There are three subcases under Case 1 and Case 2 but for reducing the length of the paper, we have only considered the main two cases into this paper.
that for any Implicit Function Theorem: which is stated below:

For proving the theorems, we have taken help from existing theorem of analysis, request on retailer cost function and prove them in a strict mathematical sense.

5. Solution Procedure. Now, we can derive three theorems including verification request on retailer cost function and prove them in a strict mathematical sense.

For proving the theorems, we have taken help from existing theorem of analysis, which is stated below:

**Implicit Function Theorem:** Let \( F(x, y, z) \) be a continuous function with continuous partial derivatives defined on an open set \( S \) containing the point \( P_1 = (x_0, y_0, z_0) \). If \( \frac{\partial F}{\partial z} \neq 0 \) at \( P_1 \), then there exists a neighborhood \( R \) about \((x_0, y_0)\) such that for any \( y \) in \( R \), there is a unique \( z = z(x, y) \) such that \( F(x, y, z) = 0 \).

**Theorem 5.1.** For known values of \( t_1, t_2, t_3 \) and \( t_4 \), it holds:

(i) There exists a unique \( T_1^* \) which minimizes the cost function \( R_1(T, q, \alpha) \) for fixed values of \( \alpha \) and \( q \).

(ii) For fixed values of \( \alpha \) and \( q \), there exists a unique \( T_2^* \) which minimizes the cost function \( R_2(T, q, \alpha) \).

(iii) For fixed \( \alpha \) and \( q \), the optimal solution \( T_3^* \) that minimizes the cost function \( R_2(T, q, \alpha) \) exists and is unique.
Proof. (computer supported): (i) The first- and second-order partial derivatives of the cost function \( R_1(T, q, \alpha) \) with respect to \( T \) are given below:

\[
\frac{\partial R_1(T, q, \alpha)}{\partial T} = -\frac{Aq}{T^2} \left( \frac{q_1(1-\alpha)}{T^2} + \frac{q_2\alpha}{T^2} \right) \left[ \frac{k(1-\alpha) - a}{\theta} - \frac{b}{\theta^2} (t_1 + \frac{(e^{-\theta t_1} - 1)}{\theta}) \right] \\
- \frac{b_1 t_1}{2\theta} + \frac{b}{\theta^2} \left( \frac{t_2 - t_1}{1} - \frac{1}{\theta} (1 - e^{\theta(t_2-t_1)}) \right) - \frac{1}{\theta} (a(t_2 - t_1) + b(t_2^2 - t_1^2)) \\
- \frac{1}{\theta} \left( (a + bt_2)(1 - e^{\theta(t_2-t_1)}) - \frac{b}{\theta} (1 - e^{\theta(t_2-t_1)}) \right) + \left( \frac{q_1(1-\alpha)}{T} + \frac{q_2\alpha}{T} \right) \left[ \left( \frac{k(1-\alpha) - a}{\theta} - \frac{b}{\theta^2} (t_1 + \frac{(e^{-\theta t_1} - 1)}{\theta}) \right) \\
- \frac{b_1 t_1}{2\theta^2} ((t_2 - t_1) - \frac{1}{\theta} (1 - e^{\theta(t_2-t_1)})) - \frac{1}{\theta} (a(t_2 - t_1) + b(t_2^2 - t_1^2)) \right) \\
\left( 1 + e^{\theta(t_2-t_1)} \right) - \frac{1}{\theta} (a(t_2 - t_1) + b(t_2^2 - t_1^2)) + \left( \frac{\partial_2}{\partial T} - \frac{\partial_1}{\partial T} \right)
\]

\[
(aT + 2bT(t_2 - t_1) - \frac{1}{\theta} ((a + bt_2)(1 - e^{\theta(t_2-t_1)}) - \frac{b}{\theta} (1 - e^{\theta(t_2-t_1)})) \\
+ \frac{T}{\theta} ((a + b \frac{\partial_2}{\partial T})(1 - e^{\theta(t_2-t_1)}) + e^{\theta(t_2-t_1)}(\frac{\partial_2}{\partial T} - \frac{\partial_1}{\partial T}))(b - \theta(a + bt_2))]) \\
- \theta ((k-\alpha)(t_1 - T \frac{\partial_1}{\partial T}) + \frac{bt_1^2}{2} - bT t_1 \frac{\partial_1}{\partial T} - \frac{b}{\theta^2} (t_1 + \frac{(e^{-\theta t_1} - 1)}{\theta}) \\
- T \frac{\partial_1}{\partial T} (1 - e^{-\theta t_1}) - \frac{bt_1^2}{2\theta} + bT t_1 \frac{\partial_1}{\partial T} + \frac{b}{\theta^2} ((t_2 - t_1) - \frac{1}{\theta} (1 - e^{\theta(t_2-t_1)})) \\
- T \frac{\partial_2}{\partial T} - \frac{\partial_1}{\partial T} (1 + e^{\theta(t_2-t_1)})) - \frac{1}{\theta} (a(t_2 - t_1) + b(t_2^2 - t_1^2)) + \left( \frac{\partial_2}{\partial T} - \frac{\partial_1}{\partial T} \right)
\]

\[
(aT + 2bT(t_2 - t_1) - \frac{1}{\theta} ((a + bt_2)(1 - e^{\theta(t_2-t_1)}) - \frac{b}{\theta} (1 - e^{\theta(t_2-t_1)})) \\
+ \frac{T}{\theta} ((a + b \frac{\partial_2}{\partial T})(1 - e^{\theta(t_2-t_1)}) + e^{\theta(t_2-t_1)}(\frac{\partial_2}{\partial T} - \frac{\partial_1}{\partial T}))(b - \theta(a + bt_2))]) \\
- \beta s \theta [a(t_3 - t_2) + \frac{b}{2} (t_2^2 - t_1^2) - T (\frac{\partial_3}{\partial T} - \frac{\partial_2}{\partial T}))(a + b(t_3 - t_2))]) \\
- \beta \frac{b}{6} (2t_2^3 - 3t_2^2 t_3 + t_3^3) - (k(1-\alpha) - a)(t_4 - t_3)^2 - \frac{bT}{6} (6t_4^2 \frac{\partial_4}{\partial T} - 6t_4 t_3 \frac{\partial_3}{\partial T} \\
- 3t_4^2 \frac{\partial_3}{\partial T} + 3t_4 \frac{\partial_3}{\partial T}) + 2T(k(1-\alpha) - a)(t_4 - t_3)(\frac{\partial_4}{\partial T} - \frac{\partial_3}{\partial T}) \\
- \delta (1 - \beta) [a(t_3 - t_2) + \frac{b(t_2^2 - t_1^2)}{2} - T (\frac{\partial_3}{\partial T} - \frac{\partial_2}{\partial T}))(a + b(t_3 - t_2)))] \\
+ \frac{((a + bt_3) - Tb \frac{\partial_3}{\partial T}) \alpha R(1 - \gamma)}{T^2(1 - \alpha \gamma)} + \frac{((a + bt_2) - Tb \frac{\partial_2}{\partial T}) \alpha \gamma r}{T^2(1 - \alpha \gamma)},
\]

where

\[
\frac{\partial_1}{\partial T} = \frac{b}{2\theta} e^{\theta T} + \frac{1}{2} b T e^{\theta T} - \frac{a}{2\theta} - 4b \theta e^{2\theta T} (\frac{b(T + 1)}{\theta^2} - \frac{aT}{\theta}) - 2be^{2\theta T} (\frac{b}{\theta^2} - \frac{a}{\theta}),
\]

(13)
\[
\frac{\partial t_2}{\partial T} = \frac{b e^{\theta(T-t_1)} - \theta}{\theta} \frac{b T + a T \frac{\partial t_1}{\partial T}}{\theta} + \frac{2b \theta (\beta a + \beta b T)}{(2bT + a)^2}, \\
\frac{\partial t_3}{\partial T} = 2(T^2 \frac{\beta b}{2} + b T_2 \frac{\partial t_2}{\partial T}), \\
\frac{\partial t_4}{\partial T} = \frac{(k(1-\alpha) - a)(b t_1 - 1)}{b \sqrt{(b T^2 - 2T)}},
\]

Furthermore,

\[
\frac{\partial^2 R_1(T, q, \alpha)}{\partial T^2} = \left(\frac{\partial h_1(1-\alpha)}{T} + \frac{\partial h_2(\alpha)}{T}\right) \left(\frac{k(1-\alpha) - a}{\theta} \frac{b T}{\theta} \frac{\partial t_1}{\partial T} \left(e^{-\theta t_1} - 1\right) + T \frac{\partial^2 t_1}{\partial T^2} \left(1 - e^{-\theta t_1}\right) + T \theta \left(\frac{\partial t_1}{\partial t^2}\right)^2 \left(e^{-\theta t_1}\right) - \frac{b T}{\theta} \left(\frac{\partial t_1}{\partial T}\right)^2 - \frac{b T}{\theta} \frac{\partial^2 t_1}{\partial T^2} + \frac{b T}{\theta} \left(\frac{\partial^2 t_1}{\partial T^2}\right)
\]

\[
+ 2b \left(\frac{\partial t_2}{\partial T} - t_1 \frac{\partial t_1}{\partial T}\right) \left(\frac{\partial^2 t_1}{\partial T^2} \left(\alpha T - 2bT(t_2 - t_1)\right) + \frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}\right) \right) + \gamma \theta [(k - a) T \frac{\partial^2 t_1}{\partial T^2}
\]

\[
+ b T \left(\frac{\partial t_1}{\partial T}\right)^2 + b T \frac{\partial^2 t_1}{\partial T^2} + b T \left(\frac{\partial^2 t_1}{\partial T^2} - \frac{\partial^2 t_1}{\partial T^2}\right)(1 + e^{\theta t_1 - t_1})
\]

\[
+ T \theta \left(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}\right)^2 \left(\theta t_1 - t_1\right) + \frac{1}{\theta} \left(a \left(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}\right) + 2b \left(\frac{\partial t_1}{\partial T} - t_1 \frac{\partial t_1}{\partial T}\right)\right)\]

\[
+ \left(\frac{\partial^2 t_1}{\partial T^2} - \frac{\partial^2 t_2}{\partial T^2}\right)(a T + 2b T (t_2 - t_1)) + \frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}\right)\right) + \gamma \theta [(k - a) T \frac{\partial^2 t_1}{\partial T^2}
\]

\[
+ b T \left(\frac{\partial t_1}{\partial T}\right)^2 + b T \frac{\partial^2 t_1}{\partial T^2} + b T \left(\frac{\partial^2 t_1}{\partial T^2} - \frac{\partial^2 t_1}{\partial T^2}\right)(1 + e^{\theta t_1 - t_1})
\]

\[
+ T \theta \left(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}\right)^2 \left(\theta t_1 - t_1\right) + \frac{1}{\theta} \left(a \left(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}\right) + 2b \left(\frac{\partial t_1}{\partial T} - t_1 \frac{\partial t_1}{\partial T}\right)\right)\]

\[
+ \left(\frac{\partial^2 t_1}{\partial T^2} - \frac{\partial^2 t_2}{\partial T^2}\right)(a T + 2b T (t_2 - t_1)) + \frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}\right)\right) + \gamma \theta [(k - a) T \frac{\partial^2 t_1}{\partial T^2}
\]

\[
+ b T \left(\frac{\partial t_1}{\partial T}\right)^2 + b T \frac{\partial^2 t_1}{\partial T^2} + b T \left(\frac{\partial^2 t_1}{\partial T^2} - \frac{\partial^2 t_1}{\partial T^2}\right)(1 + e^{\theta t_1 - t_1})
\]

\[
+ T \theta \left(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}\right)^2 \left(\theta t_1 - t_1\right) + \frac{1}{\theta} \left(a \left(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}\right) + 2b \left(\frac{\partial t_1}{\partial T} - t_1 \frac{\partial t_1}{\partial T}\right)\right)\]

\[
+ \left(\frac{\partial^2 t_1}{\partial T^2} - \frac{\partial^2 t_2}{\partial T^2}\right)(a T + 2b T (t_2 - t_1)) + \frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}\right)\right) + \gamma \theta [(k - a) T \frac{\partial^2 t_1}{\partial T^2}
\]

\[
+ b T \left(\frac{\partial t_1}{\partial T}\right)^2 + b T \frac{\partial^2 t_1}{\partial T^2} + b T \left(\frac{\partial^2 t_1}{\partial T^2} - \frac{\partial^2 t_1}{\partial T^2}\right)(1 + e^{\theta t_1 - t_1})
\]

\[
+ T \theta \left(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}\right)^2 \left(\theta t_1 - t_1\right) + \frac{1}{\theta} \left(a \left(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}\right) + 2b \left(\frac{\partial t_1}{\partial T} - t_1 \frac{\partial t_1}{\partial T}\right)\right)\]

\[
+ \left(\frac{\partial^2 t_1}{\partial T^2} - \frac{\partial^2 t_2}{\partial T^2}\right)(a T + 2b T (t_2 - t_1)) + \frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}\right)\right) + \gamma \theta [(k - a) T \frac{\partial^2 t_1}{\partial T^2}
\]
where

\[
\frac{\partial^2 t_1}{\partial T^2} = \frac{b}{2} \cdot e^{\theta T} + \frac{b}{2} (T \cdot e^{\theta T} + e^{\theta T}) - 8b\theta^2 e^{2\theta T} \left( \frac{b(T + 1)}{\theta} - \frac{aT}{\theta^2} \right)
- 8b\theta e^{2\theta T} \left( \frac{b}{\theta^2} - \frac{a}{\theta} \right),
\]

\[
\frac{\partial^2 t_2}{\partial T^2} = \theta^2 \left( \frac{b}{\theta} + \frac{bT}{\theta} + \frac{a}{\theta} \right) e^{\theta(T-t_1)} \left( \frac{\partial^2 t_1}{\partial T^2} + \left( \frac{\partial t_1}{\partial T} \right)^2 \right)
+ \frac{b}{\theta} e^{\theta(T-t_1)} + be^{\theta(T-t_1)} - \frac{b\beta \theta}{2bT + a},
\]

\[
\frac{\partial^2 t_3}{\partial T^2} = 2 \left( (T_1^2 \beta b + \frac{bT_2}{2}) (\beta b + \frac{b \partial^2 t_2}{2 \partial T^2}) + (T_1 \beta b + \frac{b \partial t_2}{\partial T}) \right)^2,
\]

\[
\frac{\partial^2 t_4}{\partial T^2} = \frac{b^2(k(1 - \alpha) - a)^2 (bT^2 - 2T)^2 + (k(1 - \alpha) - a)(bT - 2)}{b^2(k(1 - \alpha) - a)^2 (bT^2 - 2T)}. \tag{21}
\]

By solving Equation 12 with the help of Equation 13, Equation 14, Equation 15 and Equation 16, we get a value of \( T \), denoted by \( T^* \) (this is an Implicit Function depending on the parameters \( q \) and \( \alpha \)). Utilizing the Implicit Function value of \( T^* \) in Equation 17 and with the help of Equation 18, Equation 19, Equation 20 and Equation 21, we conclude that (strictly mathematically)

\[
\frac{\partial^2 R_1(T, q, \alpha)}{\partial T^2} > 0.
\]

Therefore, we can conclude that the cost function \( R_1(T, q, \alpha) \) is convex in \( T \). Therefore, by computational verification (including usage of the Intermediate Value Theorem) we demonstrate the existence of one zero point of \( F \) (i.e., \( F = 0 \) is fulfilled). By Implicit Function Theorem, there exists a point solution going through the point \( T^*_1(q, \alpha) \) of the problem \( F = 0 \) with \( F = R_1(T, q, \alpha) \), a local solution, not vanishing at the point solution, locally unique, and with a local parametrization, called as implicit function, \( T_1^* = T_1^*(q, \alpha) \). Determination of the optimal \( T \), i.e., \( T_1^* \), leads to obtain a local optimal solution for \( T \). The minimizer is an isolated local one and, up to restriction to a local subspace of practically relevant parts of the domain, where \( R_1(T, q, \alpha) \) is ensured to be convex, a global solution. This proves the first part of the theorem.

(ii) Let us calculate the first- and second-order partial derivatives of \( R_2(T, q, \alpha) \) with respect to \( T \), which are given subsequently:

\[
\frac{\partial R_2(T, q, \alpha)}{\partial T} = -A - \left( g h_1(1 - \alpha) + \frac{g h_2 \alpha}{T} \right) \left( \frac{k(1 - \alpha) - a}{\theta} - \frac{b}{\theta^2} \right) \left( t_1 + \frac{\theta e^{\theta t_1} - 1}{\theta} \right),
\]

\[
\frac{\partial^2 R_2(T, q, \alpha)}{\partial T^2} = \frac{\partial^2 t_1}{\partial T^2} - \frac{\partial^2 t_2}{\partial T^2} (1 - e^{-\theta t_1}) - \frac{\partial^2 t_3}{\partial T^2} \left( \frac{\partial t_1}{\partial T} \right) (1 + e^{\theta(t_2 - t_1)}) - \frac{\partial^2 t_4}{\partial T^2} \left( \frac{\partial t_1}{\partial T} \right) (1 - e^{\theta(t_2 - t_1)})
- T \left( \frac{\partial^2 t_4}{\partial T^2} - \frac{\partial t_1}{\partial T} \right) (1 - e^{\theta(t_2 - t_1)}) - \frac{b}{\theta} \left( (a + b T_2)(1 - e^{\theta(t_2 - t_1)}) - \frac{b}{\theta} (1 - e^{\theta(t_2 - t_1)}) \right)
+ \frac{T}{\theta} \left( \frac{b}{\theta} \right) (a + b T_2)(1 - e^{\theta(t_2 - t_1)}) + e^{\theta(t_2 - t_1)} \left( \frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T} \right) (b - \theta (a + b t_2)).
\]
\[
\begin{align*}
&- c\theta((k - a)(t_1 - T\frac{\partial t_1}{\partial T}) + \frac{bt_1^2}{2} - bTt_1\frac{\partial t_1}{\partial T} - \frac{b}{\theta^2}(t_1 + \frac{e^{-\theta t_1} - 1}{\theta}) \\
&- T\frac{\partial t_1}{\partial T}(1 - e^{-\theta t_1})) - \frac{bt_1^2}{2\theta} + \frac{bt_1 T\frac{\partial t_1}{\partial T}}{\theta} + \frac{b}{\theta^2}((t_2 - t_1) - \frac{1}{\theta}(1 - e^{\theta(t_2-t_1)})) \\
&- T\frac{\partial t_2}{\partial T}(1 + e^{\theta(t_2-t_1)})) - \frac{1}{\theta}(a(t_2 - t_1) + b(t_2^2 - t_1^2) + (\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T})) \\
&\left((aT + 2bT(t_2 - t_1)) - \frac{1}{\theta}((a + bt_2)(1 - e^{\theta(t_2-t_1)}) - \frac{b}{\theta}(1 - e^{\theta(t_2-t_1)})) \\
&+ \frac{T}{\theta}((a + b\frac{\partial t_2}{\partial T})(1 - e^{\theta(t_2-t_1)}) + e^{\theta(t_2-t_1)}(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T})(b - \theta(a + bt_2)))) \\
&- s\beta(a(t_3 - t_2) + \frac{b}{2}(t_2^2 - t_1^2) - T(\frac{\partial t_3}{\partial T} - \frac{\partial t_2}{\partial T}))(a + b(t_3 - t_2)) \\
&- \beta\frac{b}{6}(2t_2^4 - 3t_2^3 + \frac{t_2^3}{3}) - (k(1 - \alpha) - a)(t_4 - t_3)^2 \\
&- \frac{bT}{6}(6t_2^4\frac{\partial t_2}{\partial T} - 6t_2\frac{\partial t_3}{\partial T} - 3t_2^2\frac{\partial t_3}{\partial T} + 3t_2^3\frac{\partial t_3}{\partial T}) + 2T(k(1 - \alpha) - a) \\
&(t_4 - t_3)\left(\frac{\partial t_4}{\partial T} - \frac{\partial t_3}{\partial T}\right) - \delta(1 - \beta)[a(t_3 - t_2) + \frac{b(t_2^2 - t_1^2)}{2} - T(\frac{\partial t_3}{\partial T} - \frac{\partial t_2}{\partial T})] \\
&\left((a + b(t_3 - t_2)) + \frac{(a + b\frac{t_3}{\partial T} - Tb\frac{\partial r}{\partial T})\alpha R(1 - \gamma)}{T^2(1 - \alpha \gamma)} \\
&+ \frac{(a + bt_2) - T\frac{\partial r}{\partial T})\alpha r}{T^2(1 - \alpha \gamma)} - cL\frac{b}{\theta^2}(t_2 - t_1) - \frac{1}{\theta}(1 - e^{\theta(t_2-t_1)})) \\
&- \frac{1}{\theta}(a(T - M) + bT^2 - M^2)) - \frac{1}{\theta}((a + bt_2)(1 - e^{\theta(T-M)}) \\
&- \theta(a + rt_2)e^{\theta(T-M)} + be^{\theta(T-M)}) + sI_e(aM^2 + \frac{bM^3}{3},)
\end{align*}
\]

\[
\frac{\partial^2 R_2(T, q, \alpha)}{\partial T^2} = \left(\frac{qh_1(1 - \alpha)}{T} + \frac{qh_2\alpha}{T}\right)\left(\frac{k(1 - \alpha) - a}{\theta} - \frac{b}{\theta^2}(\frac{\partial t_1}{\partial T}(e^{-\theta t_1}) \\
- 1) + \frac{\partial t_1}{\partial T} + T\frac{\partial^2 t_1}{\partial T^2}(1 - e^{-\theta t_1}) + T\theta((\frac{\partial t_1}{\partial T})^2 - 2e^{-\theta t_1})) - \frac{bT}{\theta}(\frac{\partial t_1}{\partial T})^2 \\
- \frac{bt_1 T\frac{\partial^2 t_1}{\partial T^2} + \frac{b}{\theta^2}(T\frac{\partial^2 t_2}{\partial T^2} - \frac{\partial^2 t_1}{\partial T^2}))(1 + e^{\theta(t_2-t_1)}) + T\theta((\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T})^2 \\
e^{\theta(t_2-t_1)} + \frac{1}{\theta}(a(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}) + 2b(t_2^2 - t_1^2) - t_1^2) \\
+ \frac{\partial^2 t_1}{\partial T^2} + \frac{\partial^2 t_1}{\partial T^2}((a + 2bT(t_2 - t_1)) + (\frac{\partial t_1}{\partial T}) \\
- \frac{\partial t_1}{\partial T}))(a + 2bT(t_2 - t_1) + 2bT((\frac{\partial t_2}{\partial T} - \frac{t_1}{\theta^2}))(a + 2bT(t_2 - t_1)) + (\frac{\partial t_2}{\partial T} - \frac{t_1}{\theta^2}))) + \theta e^{\theta(t_2-t_1)}(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T})^2 \\
(b - \theta(a + bt_2)) - \theta^2 be^{\theta(t_2-t_1)}(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T})(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}) + c\theta((k - a)T\frac{\partial^2 t_1}{\partial T^2} \\
+ bT((\frac{\partial t_1}{\partial T})^2 + bT(\frac{\partial^2 t_1}{\partial T^2} + \frac{b}{\theta^2}(T\frac{\partial^2 t_2}{\partial T^2} - \frac{\partial^2 t_1}{\partial T^2}))(1 + e^{\theta(t_2-t_1)}) \\
+ T\theta(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T})^2 e^{\theta(t_2-t_1)} + \frac{1}{\theta}(a(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}) + 2b(t_2^2 - t_1^2) - t_1^2) \\
+ \frac{\partial^2 t_2}{\partial T^2} \left((a + 2bT(t_2 - t_1)) + (\frac{\partial t_2}{\partial T} - \frac{t_1}{\theta^2})(a + 2b(t_2 - t_1)) \right)\right)\]

Now, by applying the similar procedure as described in part (i), we understand that there exists a unique optimal implicit function $T_2^* = T_2^*(q, \alpha)$ for which the cost function $R_2(T, q, \alpha)$ is at a global minimum, which proves the second part of the theorem.

For demonstrating part (iii), let us determine the first- and second-order derivatives of $R_3(T, q, \alpha)$ with respect to $T$ as follows:

\[
\frac{\partial R_3(T, q, \alpha)}{\partial T} = -A - \left(\frac{g_1(1 - \alpha)}{T} + \frac{g_2\alpha}{T}\right)(\frac{k(1 - \alpha) - a}{\theta} - \frac{b}{\theta^2})(t_1 + \frac{e^{-\theta t_1} - 1}{T})
\]

\[
- \frac{T}{\theta} \frac{\partial t_1}{\partial T}(1 - e^{-\theta t_1}) - \frac{bT_1^2}{2} + \frac{bTT_1}{T} \frac{\partial t_1}{\partial T} + \frac{b}{\theta^2}(t_2 - t_1) - \frac{1}{\theta}(1 - e^{\theta(t_2 - t_1)}) - T(\frac{\partial t_2}{\partial T})
\]

\[
- \frac{\partial t_1}{\partial T}(1 + e^{\theta(t_2 - t_1)}) + \frac{1}{\theta}(a(t_2 - t_1) + b(t_2^2 - t_1^2)) + (\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T})
\]

\[
(aT + 2bT)(t_2 - t_1) - \frac{1}{\theta}(a + b)(1 - e^{\theta(t_2 - t_1)}) - \frac{b}{\theta}(1 - e^{\theta(t_2 - t_1)})
\]

\[
+ \frac{T}{\theta}((a + bT_2)(1 - e^{\theta(t_2 - t_1)}) + e^{\theta(t_2 - t_1)}(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T})[b - \theta(a + bT_2)])
\]

\[
- \frac{1}{\theta}[bT_2 + \frac{bT_1^2}{2} - bTT_1 \frac{\partial t_1}{\partial T} + \frac{b}{\theta^2}(t_2 - t_1) - \frac{1}{\theta}(1 - e^{\theta(t_2 - t_1)})]
\]

\[
- T\frac{\partial t_1}{\partial T}(1 - e^{\theta t_1}) - \frac{bT_1^2}{2} + \frac{bTT_1}{T} \frac{\partial t_1}{\partial T} + \frac{b}{\theta^2}(t_2 - t_1) - \frac{1}{\theta}(1 - e^{\theta(t_2 - t_1)})
\]

\[
- T(\frac{\partial t_2}{\partial T})(1 + e^{\theta(t_2 - t_1)}) - \frac{1}{\theta}(a(t_2 - t_1) + b(t_2^2 - t_1^2)) + (\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T})
\]
\[(aT + 2bT(t_2 - t_1)) - \frac{1}{\theta}((a + bt_2)(1 - e^{\theta(t_2 - t_1)}) - \frac{b}{\theta}(1 - e^{\theta(t_2 - t_1)}))
\]
\[+ \frac{T}{\theta}((a + b \frac{\partial t_2}{\partial T})(1 - e^{\theta(t_2 - t_1)}) + e^{\theta(t_2 - t_1)}(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}))(b - \theta(a + bt_2))]
\[- s\beta[a(t_3 - t_2) + \frac{b}{2}(t_3^2 - t_2^2) - T(\frac{\partial t_3}{\partial T} - \frac{\partial t_2}{\partial T})(a + b(t_3 - t_2))]
\[- \beta \frac{b}{6}(2t_4^3 - 3t_4^2t_3 + t_3^3) - (k(1 - \alpha) - a)(t_4 - t_3)^2 - \frac{bT}{6}(6t_4^2 \frac{\partial t_4}{\partial T})
\[- 6t_4^3 \frac{\partial t_4}{\partial T} - 3t_4^2 \frac{\partial t_3}{\partial T} + 3t_4^3 \frac{\partial t_2}{\partial T} + 2T(k(1 - \alpha) - a)(t_4 - t_3)(\frac{\partial t_3}{\partial T} - \frac{\partial t_2}{\partial T})]
\[- \delta(1 - \beta)[a(t_3 - t_2) + \frac{b}{2}(t_3^2 - t_2^2) - T(\frac{\partial t_3}{\partial T} - \frac{\partial t_2}{\partial T})(a + b(t_3 - t_2))]
\[+ \frac{(a + bt_3 - Tb\frac{\partial t_3}{\partial T}aR(1 - \gamma)}{T^2(1 - \alpha \gamma)} + \frac{(a + bt_2 - Tb\frac{\partial t_2}{\partial T}a\gamma r)}{T^2(1 - \alpha \gamma)}
\[+ s\lambda_1 \frac{1}{2}bT^2 + (M - T)(a + bT)]
\]

\[
\frac{\partial^2 R_3(T, q, \alpha)}{\partial T^2} = \left(\frac{\partial^2 t_1}{\partial T^2}(1 - e^{-\theta t_1}) + T\theta(\frac{\partial^2 t_2}{\partial T^2})^2 e^{-\theta t_1} - \frac{bT}{\theta}(\frac{\partial t_1}{\partial T})^2 - \frac{bT t_1}{\theta}(\frac{\partial^2 t_1}{\partial T^2}) + \frac{b}{\theta^2}(T(\frac{\partial^2 t_2}{\partial T^2})^2)
\right)
\[- \frac{\partial^2 t_1}{\partial T^2}(1 + e^{\theta(t_2 - t_1)}) + T\theta(\frac{\partial t_2}{\partial T})^2 e^{\theta(t_2 - t_1)} + \frac{1}{\theta}(a(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}) + 2b
\[\]
\[
(t_2 \frac{\partial t_2}{\partial T} - t_1 \frac{\partial t_1}{\partial T}) + (\frac{\partial^2 t_2}{\partial T^2} - \frac{\partial^2 t_1}{\partial T^2}))(aT + 2bT(t_2 - t_1)) + \frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T} + \frac{b}{2}(T(\frac{\partial^2 t_2}{\partial T^2} - \frac{\partial^2 t_1}{\partial T^2})(1 + e^{\theta(t_2 - t_1)}) + T\theta(\frac{\partial t_2}{\partial T})^2 e^{\theta(t_2 - t_1)}
\right)
\[+ \frac{1}{\theta}(a(\frac{\partial t_2}{\partial T} - \frac{\partial t_1}{\partial T}) + 2b(t_2 \frac{\partial t_2}{\partial T} - t_1 \frac{\partial t_1}{\partial T}) + (\frac{\partial^2 t_2}{\partial T^2} - \frac{\partial^2 t_1}{\partial T^2}))(aT + 2bT(t_2 - t_1))
\[\]
\[
(a + b(t_3 - t_2)) - (t_4 - t_3)(\frac{\partial^2 t_4}{\partial T^2} - \frac{\partial^2 t_3}{\partial T^2} + 2T(k(1 - \alpha) - a)(\frac{\partial t_4}{\partial T} - \frac{\partial t_3}{\partial T}) - \frac{b}{6}(6t_4^2 \frac{\partial t_4}{\partial T} - 6t_4^3 \frac{\partial t_3}{\partial T})
\right)
\[- 3t_4^3 \frac{\partial t_4}{\partial T} + 3t_4^2 \frac{\partial t_3}{\partial T} + \frac{bT}{6}(12t_4^2 \frac{\partial t_4}{\partial T}^2 + 6t_4^2 \frac{\partial^2 t_4}{\partial T^2} - 6t_4^2 \frac{\partial t_4}{\partial T} - 6t_4^3 \frac{\partial^2 t_4}{\partial T^2})
\[- 3t_4^3 \frac{\partial^2 t_4}{\partial T^2} - 6t_4^4 \frac{\partial^2 t_3}{\partial T^2} + 3t_4^3 \frac{\partial^2 t_3}{\partial T^2}) + \delta(1 - \beta)[T(\frac{\partial^2 t_3}{\partial T^2} - \frac{\partial^2 t_2}{\partial T^2})(a + b(t_3 - t_2))
\]
- bT \frac{\partial^2 t_3}{\partial t^2} - b[T(t_3 \frac{\partial t_3}{\partial t} - t_2 \frac{\partial t_2}{\partial t})]

\frac{(T^2 b \frac{\partial^2 t_3}{\partial t^2} + 2(a + bt_3) - 2bT \frac{\partial t_3}{\partial t}) \alpha R(1 - \gamma)}{T^3(1 - \alpha \gamma)}

\frac{(T^2 b \frac{\partial^2 t_2}{\partial t^2} + 2(a + bt_2) - 2bT \frac{\partial t_2}{\partial t}) \alpha \gamma r}{T^3(1 - \alpha \gamma)} + sI_c[(M - T)b - a].

Now, using the similar procedure as described in part (i), we prove (mathematically strictly) that there exists a unique optimal Implicit solution $T^*_3 = T^*_3(q, \alpha)$ for which the cost function $R_3(T, q, \alpha)$ attains its global minimum, which completes the theorem. 

From Theorem 5.1, one can follow that the total cost functions are globally minimum with respect to a fixed cycle period, which is minimum, too.

**Theorem 5.2.** (i) For a known $T^*_1$, $t_1$, $t_2$, $t_3$ and $t_4$, the cost function $R_1(T, q, \alpha)$ is convex with respect to the shipment size $q$, when $\alpha$ is fixed.

(ii) For known $T^*_2$, $t_1$, $t_2$, $t_3$ and $t_4$, the cost function $R_2(T, q, \alpha)$ is convex with respect to the shipment size $q$, when $\alpha$ is fixed.

(iii) For known values of $T^*_3$, $t_1$, $t_2$, $t_3$ and $t_4$, the cost function $R_3(T, q, \alpha)$ is convex with respect to the shipment size $q$, when $\alpha$ is fixed.

**Proof.** (computer supported): (ii) The first-order partial derivative for the cost function $R_2(T, q, \alpha)$ with respect to $q$ is given as below:

\[
\frac{\partial R_2(T, q, \alpha)}{\partial q} = \frac{A}{T} + \frac{h_1(1 - \alpha)}{T} \left[ (k(1 - \alpha) - a) - \frac{b}{\theta} \right] (t_1 + \frac{(e^{-\theta t_1} - 1)}{\theta}) - \frac{bt_1^2}{2\theta} + \frac{b}{\theta^2}

\left[ ((t_2 - t_1) - \frac{1}{\theta}(1 - e^{\theta(t_2 - t_1)}) - \frac{1}{\theta}(a(t_2 - t_1) + b(t_2^2 - t_1^2)) - \frac{1}{\theta}((a + bt_2)(1 - e^{\theta(t_2 - t_1)})) - \frac{b}{\theta}((1 - e^{\theta(t_2 - t_1)}))\right]

+ \frac{h_2\alpha}{T} \left[ ((k(1 - \alpha) - a) - \frac{b}{\theta^2}) (t_1 + \frac{(e^{-\theta t_1} - 1)}{\theta}) - \frac{bt_1^2}{2\theta} + \frac{b}{\theta^2}(\theta(t_2 - t_1) - \frac{1}{\theta})(1 - e^{\theta(t_2 - t_1)}) - \frac{b}{\theta}((1 - e^{\theta(t_2 - t_1)}))\right]

+ \frac{s\beta}{T} \left[ (a(t_2 - t_1) + b(t_2^2 - t_1^2)) - \frac{1}{\theta}((a + bt_2)(1 - e^{\theta(t_2 - t_1)}) - \frac{b}{\theta}((1 - e^{\theta(t_2 - t_1)}))\right]

+ \frac{\beta}{T} \left[ \left( \frac{b(t_3^2 - t_2^2)}{2} \right) + \frac{b}{T} \left( 2t_3^2 - 3t_2^2 \right) \right]

\left[ (k(1 - \alpha) - a)(t_3 - t_2) + \frac{b(t_3^2 - t_2^2)}{2} + \frac{r_3\ln(c\alpha^2)}{\alpha} k(1 - \alpha)

+ \frac{2q}{T} r_4 k(1 - \alpha) + \frac{r_5q(a + bt_3)\alpha r_1(1 - \gamma)}{T(1 - \alpha \gamma)} + \frac{r_6(b + bt_2)\alpha r_2}{T(1 - \alpha \gamma)} + \frac{(a + bt_2)\alpha r_3}{T(1 - \alpha \gamma)} + \frac{cI_c b}{T} \left[ ((t_2 - t_1) - \frac{1}{\theta})(1 - e^{\theta(t_2 - t_1)}) - \frac{b}{\theta}((a(T - M) + (b(T^2 - M^2)) - \frac{1}{\theta}((a + bt_2)(1 - e^{\theta(T - M)})) - \frac{b}{\theta}((1 - e^{\theta(T - M)}))\right]

\frac{sI_c(aM^2}{T} + \frac{bM^3}{3} = 0.

Set

\[
\frac{\partial R_2(T, q, \alpha)}{\partial q} = 0.
\]
and solve it for optimal $q^*$. Utilizing that optimal $q^*$, we find that

$$\frac{\partial^2 R_2(T, q, \alpha)}{\partial q^2} = \frac{2q}{T} r_3 k(1 - \alpha) + \frac{2r_5}{T} \left( \frac{(a + bt_3)\alpha r_1(1 - \gamma)}{T(1 - \alpha\gamma)} + \frac{(a + bt_2)\alpha r_2}{T(1 - \alpha)} \right) > 0.$$  

Hence, $q^*$ is the global optimal which minimizes the total cost function $R_2(T, q, \alpha)$ for any feasible $T$ and $\alpha$, which proves the part (ii) of the theorem. By applying the similar procedure, we can prove parts (i) and (iii) of the theorem.

This completes the proof of Theorem 5.2. \qed

Theorem 5.2 explains that costs functions are globally convex with respect to a fixed order size, which is also minimum.

**Theorem 5.3.** For known values of $T^2$, $t_1$, $t_2$, $t_3$ and $q$, the following is achieved:

(i) If $M_1(T, q) \geq 0$, then $R_2(T, q, \alpha)$ attains its minimum value at $\alpha^* = 0$.
(ii) If $M_2(T, q) \leq 0$, then $R_2(T, q, \alpha)$ reaches its minimum value at $\alpha = \alpha_u$.
(iii) If $M_1(T, q) < 0$ and $M_2(T, q) > 0$, then $R_2(T, q, \alpha)$ is convex and achieves its global minimum at $\alpha^* \in (0, \alpha_u)$ when $\frac{\partial^2 R_2(T, q, \alpha)}{\partial \alpha^2} = 0$.

A similar procedure is applicable for cost functions $R_1(T, q, \alpha)$ and $R_3(T, q, \alpha)$, and it can be proved easily by following the above procedure.

**Proof.** (computer supported): The first- and second-order partial derivatives of the cost function $R_2(T, q, \alpha)$ with respect to $\alpha$ is given as follows:

\begin{align}
\frac{\partial R_2(T, q, \alpha)}{\partial \alpha} &= \frac{qh_1}{T} \left[ \frac{b}{\theta^2} - \frac{k(1 - \alpha) - a}{\theta} (t_1 + \frac{(e^{-\theta t_1} - 1)}{\theta}) + \frac{bt_1^2}{\theta} - \frac{b}{\theta^2} (t_2 - t_1) + \frac{1}{\theta} (1 - e^{\theta(t_2 - t_1)}) \right] + \frac{qh_2}{T} \left[ \frac{b}{\theta^2} - \frac{k(1 - \alpha) - a}{\theta} (t_1 + \frac{(e^{-\theta t_1} - 1)}{\theta}) + \frac{bt_1^2}{\theta} - \frac{b}{\theta^2} (t_2 - t_1) + \frac{1}{\theta} (1 - e^{\theta(t_2 - t_1)}) \right] + \frac{qk\beta}{T} (t_4 - t_3)^2 \\
\frac{\partial^2 R_2(T, q, \alpha)}{\partial \alpha^2} &= \left[ (t_1 + \frac{(e^{-\theta t_1} - 1)}{\theta}) + \frac{k}{\theta} \right] \left[ \frac{qh_1}{T} - \frac{qh_2}{T} + \frac{r_3 qk}{T} \right] + \frac{qk}{\alpha^2} \left[ (a + bt_3) r_1(1 - \gamma) + \frac{(a + bt_2)}{T(1 - \alpha\gamma)^2} \right].
\end{align}
Let us define:

\[
H(\alpha) = \frac{qh_1}{T} \left[ \left( \frac{b}{\theta^2} - \frac{k(1 - \alpha) - a}{\theta} \right) (t_1 + \frac{e^{-\theta t_1} - 1}{\theta}) + \frac{bt_1^2}{2\theta} - \frac{b}{\theta^2} ((t_2 - t_1) + \frac{1}{\theta} (1 - e^{\theta(t_2 - t_1)}) + \frac{1}{\theta} (a(t_2 - t_1) + b(t_2^2 - t_1^2)) + \frac{1}{\theta} ((a + bt_2)(1 - e^{\theta(t_2 - t_1)})) + \frac{b}{\theta} \right] - \frac{qk(1 - \alpha)}{\theta} + \frac{qk^2 r_4 k}{T} + \frac{q r_5 q^2}{T} \frac{(a + b t_3) r_1 (1 - \gamma)}{T(1 - \alpha u)^2} + \frac{(a + b t_2) \gamma r_2}{T(1 - \alpha u)^2}.
\]

Therefore, let us state

\[
M_1(T, q) := H(\alpha)|_{\alpha=1} = \frac{qh_1}{T} \left[ \left( \frac{b}{\theta^2} + \frac{a}{\theta} \right) (t_1 + \frac{e^{-\theta t_1} - 1}{\theta}) + \frac{bt_1^2}{2\theta} - \frac{b}{\theta^2} ((t_2 - t_1) + \frac{1}{\theta} (1 - e^{\theta(t_2 - t_1)}) + \frac{1}{\theta} (a(t_2 - t_1) + b(t_2^2 - t_1^2)) + \frac{b}{\theta} \right] - \frac{q k^2 r_4 k}{T} + \frac{q r_5 q^2}{T} \frac{(a + b t_3) r_1}{T(1 - \gamma)} + \frac{(a + b t_2) \gamma r_2}{T(1 - \alpha u)^2},
\]

and

\[
M_2(T, q) := H(\alpha)|_{\alpha=\alpha_u} = \frac{qh_1}{T} \left[ \left( \frac{b}{\theta^2} - \frac{k(1 - \alpha_u) - a}{\theta} \right) (t_1 + \frac{e^{-\theta t_1} - 1}{\theta}) + \frac{bt_1^2}{2\theta} - \frac{b}{\theta^2} ((t_2 - t_1) + \frac{1}{\theta} (1 - e^{\theta(t_2 - t_1)}) + \frac{1}{\theta} (a(t_2 - t_1) + b(t_2^2 - t_1^2)) + \frac{1}{\theta} ((a + bt_2)(1 - e^{\theta(t_2 - t_1)})) + \frac{b}{\theta} \right] - \frac{q k(1 - \alpha_u)}{\theta} + \frac{q^2 r_4 k}{T} + \frac{q r_5 q^2}{T} \frac{(a + b t_3) r_1 (1 - \gamma)}{T(1 - \alpha u)^2} + \frac{(a + b t_2) \gamma r_2}{T(1 - \alpha u)^2}.
\]
It is understandable from Equation 27 that $H'(\alpha) > 0$ for known values of $T$ and $q$. Consequently, we can conclude that $H(\alpha)$ is strictly increasing in $\alpha$. Therefore, it is derivable that:

(i) If $M_1(T, q) \geq 0$, $H(\alpha) \geq 0$, $\forall \alpha \in [1, \alpha_u]$. Therefore, the optimal value of $\alpha^* = 1$.

(ii) If $M_2(T, q) \leq 0$, $H(\alpha) \leq 0$, $\forall \alpha \in [1, \alpha_u]$. Therefore, the optimal value of $\alpha^* = \alpha_u$.

(iii) If $M_1(T, q) < 0$ and $M_2(T, q) > 0$, then, by using the intermediate value theorem, there exists an unique $\alpha \in [1, \alpha_u]$ which satisfies $H(\alpha^*) = 0$. This completes the proof.}

From Theorem 5.3, it is expressed that the cost functions are globally convex with respect to proportion of defective items, which is at minimum also. Therefore, with the help of the three theorems including verification request, global convexities of the cost functions with respect to its variables are proved.

6. Numerical Example. The following numerical example is provided to illustrate our proposed method. In the considered example, we assume an inventory system with the following parameters.

**Example:** Let us assume: $A = $900/order, $c = $500/unit, $s = $400/unit, $h_1 = $100/unit/year, $h_2 = $200/unit/year, $r_1 = $50/unit, $r_2 = $40/unit, $r_3 = $10/unit, $r_4 = $40/unit, $r_5 = $30/unit, $k = 1000$ units/year, $a = 15$ units/year, $b = 20$ units/year, $r_a = 0.05$, $\lambda = 0.09$, $\beta = 0.3$ units/year, $\gamma = 0.4$ units/year, $\delta = $20/unit, $M = 1.2$ years, $\theta = 0.4$, $I_c = $0.8$/year$, $I_e = $0.7$/year$.

Then, by inserting the above values in Equation 5, Equation 6, Equation 7, and Equation 8 and solving the equations, we derive the values of $t_1$, $t_2$, $t_3$, and $t_4$ as $t_1 = 2.000$, $t_2 = 1.566$, $t_3 = 1.670$ and $t_4 = 1.371$, respectively. By using Theorem 5.1, Theorem 5.2, and Theorem 5.3., we observe from Table 2 that the total cost for retailer is minimum for $q = 4$. Therefore, we confirm the values as $\alpha = 0.1089$, $q = 4$, $T_1 = 1.324$, $T_2 = 1.112$, $T_3 = 1.108$. Now, with the help of the achieved values, we calculate the total cost of the inventory system for the retailer, with the help of Mathematica, as $R_1(T) = 3488.22$, $R_2(T) = 2973.16$ and $R_3(T) = 2844.93$, respectively.

| $q$ | $\alpha^*$ | $T_1^*$ | $T_2^*$ | $T_3^*$ | $R_1(T, q, \alpha)$ | $R_2(T, q, \alpha)$ | $R_3(T, q, \alpha)$ |
|-----|-------------|--------|--------|--------|---------------------|---------------------|---------------------|
| 1   | 0.1077      | 1.235  | 1.302  | 1.187  | 3478.12             | 3599.45             | 3517.30             |
| 2   | 0.1125      | 1.301  | 1.213  | 1.171  | 3560.82             | 3011.67             | 3069.57             |
| 3   | 0.1092      | 1.420  | 1.387  | 1.432  | 3673.19             | 3095.07             | 3112.70             |
| 4   | **0.1089**  | **1.324** | **1.112** | **1.108** | **3488.22** | **2973.16** | **2844.93** |
| 5   | 0.1107      | 1.403  | 1.274  | 1.392  | 3579.08             | 3077.45             | 3205.71             |
| 6   | 0.1073      | 1.449  | 1.171  | 1.321  | 3678.99             | 3360.21             | 3119.54             |
| 7   | 0.128       | 1.610  | 1.534  | 1.558  | 3879.04             | 3232.70             | 3374.56             |
| 8   | 0.116       | 1.791  | 1.645  | 1.325  | 3927.34             | 3306.18             | 3317.84             |
| 9   | 0.125       | 1.570  | 1.349  | 1.102  | 4012.57             | 3812.05             | 3518.38             |
| 10  | 0.144       | 1.397  | 1.747  | 1.560  | 3939.55             | 3670.10             | 3575.68             |

7. Sensitivity Analysis. We now study the effects of changes in the system parameters $A$, $s$, $h_1$, $h_2$, $a$, $b$, $\beta$, $\gamma$, $\delta$, $r_1$, $\theta$, $r_2$, $r_3$, $r_4$, $r_5$, and $M$, on the optimal values of $T_1$, $T_2$, $T_3$ and the optimal cost for the retailer in both cases, namely, when trade
credit is offered and not. This sensitivity analysis is performed by changing each of the parameters by +50%, +30%, +10%, -10%, -30% and -50%, taking only one parameter at a time and keeping the remaining parameters unchanged. The results based on the above example are shown in Tables 3 and 4, and on the basis of these results, Figure 2, Figure 3 and Figure 4 are drawn. In Tables 3 and 4, we use “↑” which represents “increasing”, whereas an another symbol “↓” stands for “decreasing”.

Table 3 : Sensitivity analysis for various parameters involved in Example 1.

| Parameter | % change | value | $T_1$ | $T_2$ | $T_3$ | $R_1(T)$ | $R_2(T)$ | $R_3(T)$ | Total Profit |
|-----------|-----------|-------|-------|-------|-------|-----------|-----------|-----------|--------------|
| -50       | 1.832     | 1.572 | 1.624 | 3122.05 | 2705.91 | 2812.83   | ↓         | ↑         |
| +30       | 1.510     | 1.480 | 1.613 | 2988.71 | 2633.61 | 2694.74   | ↓         | ↑         |
| +10       | 1.257     | 1.437 | 1.589 | 2734.00 | 2482.09 | 2503.15   | ↓         | ↑         |
| -10       | 0.941     | 1.428 | 1.541 | 2613.47 | 2290.41 | 2387.40   | ↓         | ↑         |
| -30       | 0.703     | 1.410 | 1.516 | 2497.38 | 2100.16 | 2206.85   | ↓         | ↑         |
| -50       | 0.627     | 1.377 | 1.487 | 2206.16 | 1980.00 | 2183.57   | ↓         | ↑         |
| +50       | 0.904     | 0.867 | 0.880 | 3385.10 | 2834.61 | 2791.38   | ↓         | ↑         |
| +30       | 0.760     | 0.843 | 0.831 | 3174.51 | 2690.42 | 2614.50   | ↓         | ↑         |
| +10       | 0.601     | 0.829 | 0.847 | 2893.67 | 2517.83 | 2576.31   | ↓         | ↑         |
| -10       | 0.472     | 0.820 | 0.831 | 2635.85 | 2385.76 | 2344.10   | ↓         | ↑         |
| -30       | 0.275     | 0.816 | 0.822 | 2483.77 | 2189.02 | 2208.47   | ↓         | ↑         |
| -50       | 0.064     | 0.803 | 0.815 | 2305.38 | 1985.47 | 2083.17   | ↓         | ↑         |
| +50       | 600       | 0.904 | 0.867 | 3385.10 | 2834.61 | 2791.38   | ↓         | ↑         |
| +30       | 520       | 0.760 | 0.843 | 3174.51 | 2690.42 | 2614.50   | ↓         | ↑         |
| +10       | 440       | 0.601 | 0.829 | 2893.67 | 2517.83 | 2576.31   | ↓         | ↑         |
| -10       | 360       | 0.472 | 0.820 | 2635.85 | 2385.76 | 2344.10   | ↓         | ↑         |
| -30       | 280       | 0.275 | 0.816 | 2483.77 | 2189.02 | 2208.47   | ↓         | ↑         |
| -50       | 200       | 0.064 | 0.803 | 2305.38 | 1985.47 | 2083.17   | ↓         | ↑         |

Tables 3 and 4 should be used to analyze the sensitivity of the parameters in Example 1.
### Table 4: Sensitivity analysis for various parameters involved in Example 1.

| Parameter | Change | Parameter Value | $T_1$ | $T_2$ | $T_3$ | $R_1(T)$ | $R_2(T)$ | $R_3(T)$ | Total Cost | Profit |
|-----------|--------|-----------------|------|------|------|--------|--------|--------|-----------|--------|
| $A$       | +50    | 2.402           | 2.265| 1.875| 3891.57 | 3124.51 | 3054.36 |
|           | +30    | 2.257           | 2.030| 1.713| 3522.56 | 2910.28 | 2839.17 |
| $s$       | +10    | 2.105           | 2.165| 1.681| 3205.84 | 2764.33 | 2780.10 |
|           | -10    | 1.943           | 2.081| 1.578| 2935.27 | 2459.37 | 2638.61 |
| $R_1$     | +20    | 1.870           | 1.863| 1.475| 2789.42 | 2218.06 | 2529.50 |
| $r_2$     | +50    | 1.655           | 1.756| 1.409| 2642.17 | 2049.62 | 2485.73 |
|           | +30    | 1.526           | 1.603| 1.358| 2543.35 | 2020.03 | 2462.58 |
| $r_3$     | +10    | 1.391           | 1.358| 1.217| 2395.67 | 1820.03 | 2281.03 |
|           | -10    | 1.264           | 1.214| 1.147| 2250.03 | 1690.03 | 2109.53 |
| $r_4$     | +50    | 0.45            | 1.310| 1.465| 2473.11 | 1730.22 | 2107.27 |
|           | +30    | 0.39            | 1.100| 1.358| 2164.33 | 1520.22 | 1917.27 |
| $\beta$   | +10    | 0.33            | 1.812| 1.922| 2917.00 | 2317.00 | 2777.00 |
|           | -10    | 0.27            | 1.233| 1.358| 2551.00 | 1951.00 | 2411.00 |
| $M$       | +50    | 0.15            | 2.692| 2.192| 3692.47 | 2537.64 | 3050.70 |
|           | +30    | 0.11            | 1.726| 1.826| 3347.80 | 2351.00 | 2811.00 |
| $\theta$  | +50    | 0.05            | 1.727| 1.827| 3054.36 | 2459.37 | 2780.10 |
|           | +30    | 0.03            | 1.603| 1.713| 2839.17 | 2218.06 | 2529.50 |

The following observations are made from Tables 3 and 4; which will clarify the importance of our model.

(i) When the ordering cost $A$ and the unit lost sale cost $s$ increase, the total cost of the system increases and, consequently, the total profit for the retailer decreases.

(ii) When the rate of deterioration $\theta$ increases with respect to time, then the holding cost $h_1$ and $h_2$ also increases, and the total cost of the system will expand inevitably. Hence, the total gain for the retailer will reduce with respect to time. It is noticeable from the table that the rate of change of $h_2$ is more than that of $h_1$, which simply shows that presence of defective items in the system will affect the profit of the system.

(iii) When the values of $a$ and $b$ rise by a sufficient amount, then the demand rate of the system increases fast which will cause an increase in the total cost and the ordering quantity; henceforth, the profit for the retailer will diminish.
But, when the demand rate of a system increases strongly, these large amount of demand will help the retailer to gain more profit.

(iv) An increase of the backorder cost $\delta$ will have the effect of a loss for the retailer. But, a large proportion of backorder amount $\beta$, will increase the profit of the system.

(v) If the trade-credit period $M$ offered by the supplier to the retailer increases, the retailers have no risk for paying an extra charge to the supplier and, at the same time, the retailers can earn excess money from the customers. This will definitely enhance the profit for the retailer.

(vi) A lower value of $\gamma$, i.e., the presence of a lower value of the defective items that will be reworked shall grow the profit.

(vii) An increase of $r_1$ and $r_2$ will increase the total cost of the system, which will definitely increase $r_5$, the unit penalty cost for falsely accepting or rejecting the perfect items as imperfect. From the table, one can observe that the values of $r_1$ and $r_2$ are very sensitive for the total cost of the system. Therefore, if Type I and Type II errors decrease, the total cost of the system decrease moderately.

(viii) If the inspection cost $r_4$, per unit time is low, the capital investment cost which is used for improving the production procedure is not considered. In this situation, the retailer will inspect all the system, which indirectly increase the cost of the system. For a small value of $r_5$ will lead to a small value of $r_4$. But, a medium value of $r_3$ and $r_5$ can avoid $r_4$, which will give an increase in the profit.

Here, Figure 2, Figure 3 and Figure 4 show the required inventory model based on our sensitivity analysis. The functions displayed in three figures, i.e., Figure 2, Figure 3 and Figure 4, are strictly convex. This property is totally consistent with our assumptions. So, Figure 2, Figure 3 and Figure 4 indicate the stability of our proposed model.

8. Concluding Remarks and Future Study. In this proposed model, we have determined the total cost for the retailer in two different cases, when trade credit has been offered and when not. In numerical analysis: (i) optimal shipment size, proportion of defectiveness and total cycle length are calculated, (ii) capital investment for quality investment is expressed and inspection policy for avoiding penalty cost is considered, (iii) Type I and Type II, both type of errors are measured to reduce the defectiveness and errors of the system, (iv) the retailer adopts product inspection policy instead of system inspection policy and which will decrease the inspection cost of the system, (iv) global optimality of the system is achieved. We have observed that the total cost of the system will effectively reduce if one can offer a trade-credit period, try to reduce Type I and Type II, both types of errors, inspect the imperfectness of the system and invest some capital for improving the product system. Hence, the model will be very much effective for the inventory manager towards a correct decision in order to increase the profit of the system.

Our model is totally based on the retailers’ perspective. From the sensitivity analysis of our model conducted by us, we have concluded that for a higher value of ordering cost, the retailer has to order more at a time to reduce the cost. To maintain the best quality and the goodwill of the customers, the most defective items must be rejected and less defective items will be repairable at a lower cost. Furthermore, we have observed that for a higher rate of selling price, if the retailer wants
When the holding cost increases, the retailer shortens the cycle time and reduces the order quantity to maximize the profit. To reduce the corresponding cost for the
deteriorating items, the retailer will increase the maintaining facility for that type of items.

In addition to the above insights, our model describes a new way for the industrialists who are familiar with handling of deteriorative items. To the best of our knowledge, this study is the first attempt about imperfect items with variable demand and variable deterioration under trade-credit policy. This model also helps the inventory management people to deal with imperfect production. It can also be applicable for seasonal foods and fashionable products. Our model will further provide a valuable explanation for many organizations that could employ our methodology to improve their total operation costs. This model may have a strong use to determine optimal inventory policy in situations such as stationary stores, fancy designable items and super market bakeries, which may unveil the characteristics modeled. Finally, this paper consider a trade-credit which plays an important role and is a major part of inventory control, and it is a powerful tool to improve sales and financial return in an industry.

A possible future study for our proposed model is to consider a multi-item inventory model under trade-credit policy. One can also add a reliability function (constant or variable), stochastic inflation, probabilistic demand, time-value of money, variable lead time, etc. The model can also be extended to consider set-up cost reduction in production time, and variable shipment size, too. Further, we can extend the work by collecting some real data from an industry. Then we also allow some more managerial insights based on that collection of real data.

Acknowledgments. The authors are very much thankful to the anonymous referees for their valuable comments to modify our paper.
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Received August 2018; 1st revision February 2019; final revision April 2019.

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