THE LIMIT SET OF DISCRETE SUBGROUPS OF $PSL(3, \mathbb{C})$

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Abstract. If $\Gamma$ is a discrete subgroup of $PSL(3, \mathbb{C})$, it is determined the equicontinuity region $Eq(\Gamma)$ of the natural action of $\Gamma$ on $\mathbb{P}^2_{\mathbb{C}}$. It is also proved that the action restricted to $Eq(\Gamma)$ is discontinuous, and $Eq(\Gamma)$ agrees with the discontinuity set in the sense of Kulkarni whenever the limit set of $\Gamma$ in the sense of Kulkarni, $\Lambda(\Gamma)$, contains at least three lines in general position. Under some additional hypothesis, it turns out to be the largest open set on which $\Gamma$ acts discontinuously. Moreover, if $\Lambda(\Gamma)$ contains at least four complex lines and $\Gamma$ acts on $\mathbb{P}^2_{\mathbb{C}}$ without fixed points nor invariant lines, then each connected component of $Eq(\Gamma)$ is a holomorphy domain and a complete Kobayashi hyperbolic space.

Introduction

The study of the dynamics of discrete groups of automorphisms of $\mathbb{P}^2_{\mathbb{C}}$ is in its childhood, and one of the first basic problems is understanding how the various possible notions of “limit set” relate among themselves. This is the main topic of this article.

If $\Gamma \subset PSL(2, \mathbb{C})$ is a classical Kleinian group, then its limit set $\Lambda$ is defined as the set of accumulation points of the $\Gamma$ orbits in $\mathbb{P}^1_{\mathbb{C}} \cong S^2$. Well known arguments show that $\Lambda$ is independent of the choice of orbit, whenever $\Gamma$ is non-elementary, and its complement $\Omega$ in $\mathbb{P}^2_{\mathbb{C}}$ is the largest discontinuity domain. Moreover, $\Omega$ coincides also with the region of equicontinuity, admits a complete metric and volume which are invariant under the action of $\Gamma$, facts which are essential for defining the Patterson-Sullivan measure for Kleinian groups (see for instance [6, 9]). When we go look at discrete subgroups of $PSL(3, \mathbb{C})$ acting on $\mathbb{P}^2_{\mathbb{C}}$, there are examples where the action on the complement of the set of accumulation points of the orbits is neither discontinuous nor equicontinuous. There are also examples (see [3]) where there is a largest discontinuity set but it does not agree with the equicontinuity set. Hence it seems that there are different notions of “limit set” for discrete subgroups of $PSL(3, \mathbb{C})$ acting on $\mathbb{P}^2_{\mathbb{C}}$.

For discrete actions in general topological spaces there is a notion of limit set due to Kulkarni, see [3], which has the important property of granting

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that the action on its complement is discontinuous and in the case of conformal automorphisms of the \(n\)-sphere \(S^n\) agrees with the usual notion of limit set. In [7, 8] there are some interesting families of projective groups and their limit set in Kulkarni’s sense. Later in the articles [2, 4, 5], the authors provide sufficient conditions, for particular subgroups \(\Gamma\) of \(\text{PSL}(3, \mathbb{C})\), to grant that the equicontinuity set and Kulkarni’s discontinuity region, \(\Omega(\Gamma)\), agree. It was also shown that, in such particular case, \(\Omega(\Gamma)\) is the largest discontinuity set.

One of the goals of this paper is the following:

**Theorem 0.1.** Let \(\Gamma \subset \text{PSL}(3, \mathbb{C})\) be a discrete group, then the equicontinuity set of \(\Gamma\) is a discontinuity region. Moreover, if \(U\) is an open \(\Gamma\) invariant subset with at least three lines in general position lying on its complement, then \(U\) is contained in the equicontinuity set of \(\Gamma\). Furthermore, if Kulkarni’s limit set \(\Lambda(\Gamma)\) contains at least three lines in general position, then Kulkarni’s discontinuity region \(\Omega(\Gamma) = \mathbb{P}^2_\mathbb{C} \setminus \Lambda(\Gamma)\) is equal to the equicontinuity set of the group \(\Gamma\).

In section 2, we split the Theorem 0.1 in three parts: Theorem 2.4, Theorem 2.5, and Theorem 2.6. The proofs of these theorems rely on pseudo-projective maps, which provide a compactification of the non-compact Lie group \(\text{PSL}(3, \mathbb{C})\).

Kulkarni’s limit set \(\Lambda(\Gamma)\), for a subgroup \(\Gamma\) of \(\text{PSL}(3, \mathbb{C})\) acting on \(\mathbb{P}^2_\mathbb{C}\), may seem a complicated object at first sight. However, when \(\Gamma = \langle \alpha \rangle\) is a cyclic group, generated by the element \(\alpha \in \text{PSL}(3, \mathbb{C})\), then the limit set of this cyclic group, denoted \(\Lambda(\alpha)\), can be found (see [5]) and it is one of the following (according to the Jordan canonical form of \(\alpha\)):

- The empty set,
- all of \(\mathbb{P}^2_\mathbb{C}\),
- one single complex line,
- one complex line and one point,
- two distinct complex lines.

In section 3 we define the set \(C(\Gamma) = \bigcup_{\gamma \in \Gamma} \Lambda(\gamma)\) and we prove the following:

**Theorem 0.2.** Let \(\Gamma \subset \text{PSL}(3, \mathbb{C})\) be a discrete group, if the number of complex lines in general position in \(\Lambda(\Gamma)\) and \(C(\Gamma)\) is at least three, then

\[
\Lambda(\Gamma) = C(\Gamma) = \bigcup_{\gamma \in \Gamma} \Lambda(\gamma),
\]

and \(\Lambda(\Gamma)\) is the union of complex lines. Moreover the discontinuity region according to Kulkarni, \(\Omega(\Gamma)\), agrees with the equicontinuity set. Furthermore, when \(\Gamma\) acts on \(\mathbb{P}^2_\mathbb{C}\) without global fixed points, then \(\Omega(\Gamma)\) is the largest open set where \(\Gamma\) acts properly and discontinuously.

If \(\Gamma \subset \text{PU}(2, 1)\) is a discrete subgroup, then \(\Gamma\) acts on the complex hyperbolic plane \(\mathbb{H}^2_\mathbb{C}\) by isometries. In this case, the limit set according to Chen-Greenberg of \(\Gamma\) is obtained as the set of accumulation points, in \(\partial \mathbb{H}^2_\mathbb{C}\),
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of the $\Gamma$-orbit of any point in $\mathbb{H}^2_\mathbb{C}$. Also, it can be obtained as the closure of the set of fixed points of non-elliptic elements in $\Gamma$. When we consider the action of $\Gamma$ in $\mathbb{P}^2_\mathbb{C}$, the Kulkarni’s limit set $\Lambda(\Gamma)$ is given in the following way (see [4]): Let $F(\Gamma)$ be the set of points $p \in \partial \mathbb{H}^2_\mathbb{C}$ such that there exists a non-elliptic element $\gamma \in \Gamma$ that satisfies $\gamma p = p$. In what follows, $(\mathbb{P}^2_\mathbb{C})^*$, denotes the space of complex lines in $\mathbb{P}^2_\mathbb{C}$. If $\mathcal{E}(\Gamma)$ denotes the subset of $(\mathbb{P}^2_\mathbb{C})^*$ consisting of those complex lines tangent to $\partial \mathbb{H}^2_\mathbb{C}$ at points of $F(\Gamma)$, then

$$\Lambda(\Gamma) = \bigcup_{l \in \mathcal{E}(\Gamma)} l = \bigcup_{l \in \mathcal{E}(\Gamma)} l.$$

Moreover, when $l \in \overline{\mathcal{E}(\Gamma)}$, the closure of the $\Gamma$-orbit of the complex line $l$ in $(\mathbb{P}^2_\mathbb{C})^*$ is equal to $\overline{\mathcal{E}(\Gamma)}$. In other words, the action of $\Gamma$ in the set of lines $\overline{\mathcal{E}(\Gamma)}$ is minimal.

Let $\Gamma \subset PU(2, 1)$ be a discrete subgroup such that $\Lambda(\Gamma) = \mathbb{P}^2_\mathbb{C} \setminus \mathbb{H}^2_\mathbb{C}$, then there exist complex lines contained in $\Lambda(\Gamma)$ not belonging to $\overline{\mathcal{E}(\Gamma)}$, so that $\mathcal{E}(\Gamma)$ gives us a canonical choice of lines to describe the limit set $\Lambda(\Gamma)$ as a union of lines. In section 3, we prove the following generalization of the main theorems in [4, 1]:

**Theorem 0.3.** Let $\Gamma \subset PSL(3, \mathbb{C})$ be an infinite discrete subgroup, without fixed points nor invariant lines. Let $\mathcal{E}(\Gamma)$ be the subset of $(\mathbb{P}^2_\mathbb{C})^*$ consisting of all the complex lines $l$ for which there exists an element $\gamma \in \Gamma$ such that $l \subset \Lambda(\gamma)$.

a) $Eq(\Gamma) = \Omega(\Gamma)$, is the maximal open set on which $\Gamma$ acts properly and discontinuously. Moreover, if $\mathcal{E}(\Gamma)$ contains more than three complex lines then every connected component of $\Omega(\Gamma)$ is complete Kobayashi hyperbolic (compare with [1]).

b) The set

$$\Lambda(\Gamma) = \bigcup_{l \in \mathcal{E}(\Gamma)} l = \bigcup_{l \in \overline{\mathcal{E}(\Gamma)}} l = \bigcup_{\gamma \in \Gamma} \Lambda(\gamma)$$

is path-connected.

c) If $\mathcal{E}(\Gamma)$ contains more than three lines, then $\overline{\mathcal{E}(\Gamma)} \subset (\mathbb{P}^2_\mathbb{C})^*$ is a perfect set. Also, it is the minimal closed $\Gamma$-invariant subset of $(\mathbb{P}^2_\mathbb{C})^*$.

1. Preliminaries and Notation.

1.1. **Projective Geometry.** We recall that the complex projective plane $\mathbb{P}^2_\mathbb{C}$ is defined as the quotient space

$$(\mathbb{C}^3 \setminus \{(0, 0, 0)\})/\mathbb{C}^*,$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ acts on $\mathbb{C}^3 \setminus \{(0, 0, 0)\}$ by the usual scalar multiplication. This is a complex 2-dimensional Riemannian manifold which is compact, connected and is naturally equipped with the Fubini-Study metric. Let $[\ ]$:
\(\mathbb{C}^3 \setminus \{(0,0,0)\} \to \mathbb{P}^2\) be the quotient map. If \(\beta = \{e_1, e_2, e_3\}\) is the standard basis of \(\mathbb{C}^3\), we write \([e_j] = e_j\) and if \(w = (w_1, w_2, w_3) \in \mathbb{C}^3 \setminus \{(0,0,0)\}\) then we write \([w] = [w_1 : w_2 : w_3]\). Also, the set \(l \subset \mathbb{P}^2\) is said to be a complex line if \([l]^{-1} \cup \{(0,0,0)\}\) is a complex linear subspace of dimension 2. Given two distinct points \(p, q \in \mathbb{P}^2\), there is a unique complex line passing through \(p\) and \(q\), such line is denoted by \(~p, q~\).

Any 2-dimensional complex vector subspace of \(\mathbb{C}^3\) can be expressed as the set of points \((z_1, z_2, z_3)\) satisfying an equation of the form \(A z_1 + B z_2 + C z_3 = 0\). Conversely, any element \((A, B, C) \in \mathbb{C}^3 \setminus \{(0,0,0)\}\) defines, up to a non-zero scalar multiple, the 2-dimensional complex vector subspace of \(\mathbb{C}^3\) consisting of the points \((z_1, z_2, z_3)\) such that \(A z_1 + B z_2 + C z_3 = 0\). In this way, the space of complex lines in \(\mathbb{P}^2\), denoted by \((\mathbb{P}^2)\)\(^*\), can be identified with \(\mathbb{P}^2\).

Consider the action of \(\mathbb{Z}_3\) (the cubic roots of the unity) on \(SL(3, \mathbb{C})\) given by the usual scalar multiplication, then \(PSL(3, \mathbb{C}) = SL(3, \mathbb{C})/\mathbb{Z}_3\) is a Lie group whose elements are called projective transformations. Let \([\cdot] : SL(3, \mathbb{C}) \to PSL(3, \mathbb{C})\) be the quotient map, we say that \(\gamma \in GL(3, \mathbb{C})\) is a lift of \(\gamma \in PSL(3, \mathbb{C})\) if there is a cubic root \(\tau\) of \((Det(\gamma))^{-1}\) such that \([\tau \gamma] = \gamma\), also, we will use the notation \((\gamma_{ij})\) to denote elements in \(SL(3, \mathbb{C})\). One can show that \(PSL(3, \mathbb{C})\) is a Lie group that acts transitively, effectively and by biholomorphisms on \(\mathbb{P}^2\) by \([\gamma](w) = [\gamma(w)]\), where \(w \in \mathbb{C}^3 \setminus \{(0,0,0)\}\) and \(\gamma \in SL(3, \mathbb{C})\). Also it is well known that projective transformations take complex lines into complex lines. Moreover, if the complex line \(l \in (\mathbb{P}^2)\) is represented by \([A : B : C]\) and \(\gamma = [\tilde{\gamma}] \in PSL(3, \mathbb{C})\), then \(\gamma(l) \in (\mathbb{P}^2)\) is represented by \(([A, B, C] \tilde{\gamma}^{-1})\).

### 1.2. Kulkarni’s Limit Set for Subgroups of \(PSL(3, \mathbb{C})\).

The Kulkarni’s limit set is defined for actions of groups on very general topological spaces (see [2]), but we restrict our attention to subgroups of \(PSL(3, \mathbb{C})\) acting on \(\mathbb{P}^2\). Let \(\Gamma \subset PSL(3, \mathbb{C})\) be a subgroup:

1. \(L_0(\Gamma)\) is defined as the closure of the points in \(\mathbb{P}^2\) with infinite isotropy group.
2. \(L_1(\Gamma)\) is the closure of the set of cluster points of the \(\Gamma\)-orbit of the point \(z\), where \(z\) runs over \(\mathbb{P}^2\) \(\setminus L_0(\Gamma)\).

Recall that \(q\) is a cluster point for the family of sets \(\{\gamma(K)\}_{\gamma \in \Gamma}\), where \(K \neq \emptyset\) is a subset of \(\mathbb{P}^2\), if there is a sequence \((k_m)_{m \in \mathbb{N}} \subset K\) and a sequence of distinct elements \((\gamma_m)_{m \in \mathbb{N}} \subset \Gamma\) such that \(\gamma_m(k_m) \overset{m \to \infty}{\to} q\).

3. \(L_2(\Gamma)\) is the closure of cluster points of the family of compact sets \(\{\gamma(K)\}_{\gamma \in \Gamma}\), where \(K\) runs over all the compact subsets of \(\mathbb{P}^2\) \(\setminus (L_0(\Gamma) \cup L_1(\Gamma))\).

4. The Limit Set in the sense of Kulkarni for \(\Gamma\) is defined as:

\[
\Lambda(\Gamma) = L_0(\Gamma) \cup L_1(\Gamma) \cup L_2(\Gamma).
\]
(v) The Discontinuity Region in the sense of Kulkarni of $\Gamma$ is defined as:

$$\Omega(\Gamma) = \mathbb{P}^2_\mathbb{C} \setminus \Lambda(\Gamma).$$

We say that $\Gamma$ is a Complex Kleinian Group if $\Omega(\Gamma) \neq \emptyset$, see \[7\]. The following proposition is obtained from \[3\].

**Proposition 1.1.** Let $\Gamma \subset \text{PSL}(3, \mathbb{C})$ be a complex Kleinian group. Then:

(i) $\Gamma$ is discrete and countable.
(ii) $\Lambda(\Gamma), L_0(\Gamma), L_1(\Gamma), L_2(\Gamma)$ are closed $\Gamma$-invariant sets.
(iii) $\Gamma$ acts discontinuously on $\Omega(\Gamma)$.

As the reader can notice, the computation of the limit set $\Lambda(\Gamma)$ can be very complicated. The following lemma (see \[5\]) helps us in the task of the computation of $L_2(\Gamma)$.

**Lemma 1.2.** Let $\Gamma$ be a subgroup of $\text{PSL}(3, \mathbb{C})$. If $C \subset \mathbb{P}^2_\mathbb{C}$ is a closed set such that for every compact subset $K \subset \mathbb{P}^2_\mathbb{C} \setminus C$, the cluster points of the family of compact sets $\{\gamma(K)\}_{\gamma \in \Gamma}$ are contained in $L_0(\Gamma) \cup L_1(\Gamma)$, then $L_2(\Gamma) \subseteq C$.

In what follows, the limit set of the cyclic group $\langle \gamma \rangle$ will be denoted $\Lambda(\gamma)$.

The non-trivial elements of $\text{PSL}(3, \mathbb{C})$ can be classified as elliptic, parabolic or loxodromic (see \[5\]):

The elliptic elements in $\text{PSL}(3, \mathbb{C})$ are those elements $\gamma$ that have a lift to $\text{SL}(3, \mathbb{C})$ whose Jordan canonical form is

$$\begin{pmatrix}
e^{i\theta_1} & 0 & 0 \\
0 & e^{i\theta_2} & 0 \\
0 & 0 & e^{i\theta_3}
\end{pmatrix}.$$  

The limit set $\Lambda(\gamma)$ for $\gamma$ elliptic is $\emptyset$ or all of $\mathbb{P}^2_\mathbb{C}$ according to whether the order of $\gamma$ is finite or infinite. It is the appropriated moment to remark that those subgroups of $\text{PSL}(3, \mathbb{C})$ containing an elliptic element of infinite order cannot be discrete.

The parabolic elements in $\text{PSL}(3, \mathbb{C})$ are those elements $\gamma$ such that the limit set $\Lambda(\gamma)$ is equal to one single complex line. If $\gamma$ is parabolic then it has a lift to $\text{SL}(3, \mathbb{C})$ whose Jordan canonical form is one of the following matrices:

$$\begin{pmatrix}1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1\end{pmatrix}, \begin{pmatrix}e^{2\pi it} & 1 & 0 \\
0 & e^{2\pi it} & 0 \\
0 & 0 & e^{-4\pi it}\end{pmatrix}, e^{2\pi it} \neq 1$$

In the first case $\Lambda(\gamma)$ is the complex line consisting of all the fixed points of $\gamma$, in the second case $\Lambda(\gamma)$ is the unique $\gamma$-invariant complex line. In the last case $\Lambda(\gamma)$ is the complex line determined by the two fixed points of $\gamma$.

There are four kinds of loxodromic elements in $\text{PSL}(3, \mathbb{C})$: 
• The complex homotheties are those elements $\gamma \in PSL(3, \mathbb{C})$ that have a lift to $SL(3, \mathbb{C})$ whose Jordan canonical form is

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}, \quad |\lambda| \neq 1,$$

and its limit set $\Lambda(\gamma)$ is the set of fixed points of $\gamma$, consisting of a complex line and a point.

• The screws are those elements $\gamma \in PSL(3, \mathbb{C})$ that have a lift to $SL(3, \mathbb{C})$ whose Jordan canonical form is

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (\lambda\mu)^{-1} \end{pmatrix}, \quad \lambda \neq \mu, |\lambda| = |\mu| \neq 1,$$

and its limit set $\Lambda(\gamma)$ consists of the complex line, $l$, on which $\gamma$ acts as an elliptic transformation of $PSL(2, \mathbb{C})$ and the fixed point of $\gamma$ not lying in $l$.

• The loxoparabolic elements $\gamma \in PSL(3, \mathbb{C})$ have a lift to $SL(3, \mathbb{C})$ whose Jordan canonical form is

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad |\lambda_1| < |\lambda_2| < |\lambda_3|.$$

This kind of transformation has three fixed points, one of them is attracting, other is repelling and the last one is a saddle. The limit set $\Lambda(\gamma)$ is equal to the union of the complex line determined by the attracting and saddle points and the complex line determined by the saddle and repelling points.

**Theorem 1.3.** [2] Let $\Gamma \subset PSL(3, \mathbb{C})$ be a discrete and infinite group then

(i) there exists $\gamma_0 \in \Gamma$ such that $\gamma_0$ has infinite order;

(ii) If $\Gamma$ acts properly and discontinuously on $U \subset \mathbb{P}_\mathbb{C}^2$, then at least one of the complex lines in $\Lambda(\gamma_0)$, is contained in $\mathbb{P}_\mathbb{C}^2 \setminus U$. In particular, there exists a complex line $l_0$ such that $l_0 \subset \Lambda(\gamma_0)$ and $l_0 \subset \Lambda(\Gamma)$. 

2. Pseudo-Projective Maps and Equicontinuity.

We denote by $M_{3\times 3}(\mathbb{C})$ the space of all $3\times 3$ matrices with complex entries equipped with the standard topology. The quotient space

$$(M_{3\times 3}(\mathbb{C}) \setminus \{0\})/\mathbb{C}^*$$

is called the space of pseudo-projective maps of $\mathbb{P}^2_\mathbb{C}$ and it is naturally identified with the projective complex space $\mathbb{P}^8_\mathbb{C}$. Since $GL(3, \mathbb{C})$ is an open, dense, $\mathbb{C}^*$-invariant set of $M_{3\times 3}(\mathbb{C})\setminus\{0\}$, we obtain that the space of pseudo-projective maps of $\mathbb{P}^2_\mathbb{C}$ is a compactification of $PSL(3, \mathbb{C})$. As in the case of projective maps, if $s \in M_{3\times 3}(\mathbb{C})\setminus\{0\}$, then $[s]$ denotes the equivalence class of the matrix $s$ in the space of pseudo-projective maps of $\mathbb{P}^2_\mathbb{C}$. Also, we say that $s \in M_{3\times 3}(\mathbb{C})\setminus\{0\}$ is a lift of the pseudo-projective map $S$, whenever $[s] = S$.

Let $S$ be an element in $(M_{3\times 3}(\mathbb{C})\setminus\{0\})/\mathbb{C}^*$ and $s$ a lift to $M_{3\times 3}(\mathbb{C})\setminus\{0\}$ of $S$. The matrix $s$ induces a non-zero linear transformation $s : \mathbb{C}^3 \rightarrow \mathbb{C}^3$, which is not necessarily invertible. Let $Ker(s) \subseteq \mathbb{C}^3$ be its kernel and let $Ker(S)$ denote its projectivization to $\mathbb{P}^2_\mathbb{C}$, taking into account that $Ker(s) := \emptyset$ whenever $Ker(s) = \{(0,0,0)\}$. Then $S$ can be considered as a map

$$S : \mathbb{P}^2_\mathbb{C} \setminus Ker(S) \rightarrow \mathbb{P}^2_\mathbb{C}$$

$$S([v]) = [s(v)];$$

this is well defined because $v \notin Ker(s)$. Moreover, the commutative diagram below implies that $S$ is a holomorphic map:

$$\begin{array}{ccc}
\mathbb{C}^3 \setminus Ker(s) & \overset{s}{\longrightarrow} & \mathbb{C}^3 \setminus \{(0,0,0)\} \\
\mid & & \mid \\
\mathbb{P}^2_\mathbb{C} \setminus Ker(M) & \overset{s}{\longrightarrow} & \mathbb{P}^2_\mathbb{C}
\end{array}$$

The image of $S$, denoted $Im(S)$, is defined as the subset of $\mathbb{P}^2_\mathbb{C}$ given by

$$Im(S) := [s(\mathbb{C}^3) \setminus \{(0,0,0)\}],$$

and we have that

$$dim_\mathbb{C}(Ker(S)) + dim_\mathbb{C}(Im(S)) = 1.$$

**Definition 2.1.** The equicontinuity set for a family $\mathcal{F}$ of endomorphisms of $\mathbb{P}^2_\mathbb{C}$, denoted $Eq(\mathcal{F})$, is defined to be the set of points $z \in \mathbb{P}^2_\mathbb{C}$ for which there is an open neighborhood $U$ of $z$ such that $\{f|_U : f \in \mathcal{F}\}$ is a normal family (where normal family means that every sequence of distinct elements has a subsequence which converges uniformly on compact sets).

It is not hard to see that $Eq(\mathcal{F})$ is an open set and, in the particular case when the family $\mathcal{F}$ consists of the elements of a group $\Gamma \subset PSL(3, \mathbb{C})$, the equicontinuity set $Eq(\Gamma)$ is $\Gamma$-invariant. The following lemma helps us to relate the equicontinuity set of a discrete group $\Gamma \subset PSL(3, \mathbb{C})$ to the set of pseudo-projective maps obtained as limits of the elements of $\Gamma$. 
Lemma 2.2. Let \((\gamma_n)\) be a sequence of elements in \(\text{PSL}(3,\mathbb{C})\), then there exists a subsequence, still denoted \((\gamma_n)\), and a pseudo-projective map \(S\) such that:

1) The sequence \((\gamma_n)\) converges uniformly to \(S\) on compact subsets of \(\mathbb{P}^2_\mathbb{C} \setminus \text{Ker}(S)\).

2) If \(\text{Im}(S)\) is a complex line, then there exists a pseudo-projective map \(T\) such that \(\gamma_n^{-1} \xrightarrow{n \to \infty} T\) uniformly on compact subsets of \(\mathbb{P}^2_\mathbb{C} \setminus \text{Ker}(T)\). Moreover, \(\text{Im}(S) = \text{Ker}(T)\) and \(\text{Ker}(S) = \text{Im}(T)\).

3) If \(\text{Ker}(S)\) is a complex line, then there exists a pseudo-projective map \(T\) such that \(\gamma_n^{-1} \xrightarrow{n \to \infty} T\) uniformly on compact subsets of \(\mathbb{P}^2_\mathbb{C} \setminus \text{Ker}(T)\) and \(\text{Im}(S) \subset \text{Ker}(T)\). Moreover, if \(l\) is a complex line not passing through \(\text{Im}(S)\) then the sequence of complex lines \(\gamma_n^{-1}(l)\) goes to the complex line \(\text{Ker}(S)\) as \(n \to \infty\).

Proof. 1) For every \(n \in \mathbb{N}\), let \(g_n = (g^{(n)}_{i,j})\) be a lift to \(GL(3,\mathbb{C})\) of \(\gamma_n\). The unitary ball in \(\text{Mat}_{3 \times 3}(\mathbb{C}) = \mathbb{C}^9\) with respect to the norm \(\|g\|_\infty = \max_{1 \leq i,j \leq 3} |g_{ij}|\) is a compact set. It follows that the sequence \(\tilde{g}_n = \frac{g_n}{\|g_n\|_\infty}\) has a subsequence, still denoted \(\tilde{g}_n\), such that \(\tilde{g}_n \to s \in \text{Mat}_{3 \times 3}(\mathbb{C})\), as \(n \to \infty\). We remark that \(s \neq 0\). Now, if \(K\) is a compact subset of \(\mathbb{P}^2_\mathbb{C} \setminus \text{Ker}(S)\), we consider the sets \(C = \{v \in \mathbb{C}^3 \setminus \{(0,0,0)\} : [v] \in K\}\) and \(\tilde{K} = \{v/\|v\| \in \mathbb{C}^3 : v \in C\}.\) We notice that \(\tilde{K}\) is a compact subset of \(\mathbb{C}^3 \setminus \{(0,0,0)\}\) and \([\tilde{K}] = K\). Since \(\tilde{g}_n \xrightarrow{n \to \infty} s\) uniformly on \(\tilde{K}\) we obtain that \(\gamma_n = [\tilde{g}_n] \xrightarrow{n \to \infty} [s] = S\) uniformly on \(K\).

2) For every \(n \in \mathbb{N}\), let \(g_n = (g^{(n)}_{i,j})\) be a lift to \(GL(3,\mathbb{C})\) of \(\gamma_n\). We can assume that \(g_n \to s\) as \(n \to \infty\), where \(s\) is a lift to \(\text{Mat}_{3 \times 3}(\mathbb{C})\) of the pseudo-projective map \(S\). The cofactor matrix of \(g_n\), denoted \(\text{cof}(g_n)\) is a lift to \(GL(3,\mathbb{C})\) of \(\gamma_n^{-1}\). Since the computation of the cofactor matrix is a continuous function of the matrix entries, we have that \(\text{cof}(g_n) \to \text{cof}(s)\) as \(n \to \infty\). It follows that \(\gamma_n^{-1}\) converges to the pseudo-projective map \([\text{cof}(s)] : T\) uniformly on compact subsets of \(\mathbb{P}^2_\mathbb{C} \setminus \text{Ker}(T)\).

The matrix \(s\) may have the following Jordan canonical forms.

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\lambda_3 & 1 & 0 \\
0 & \lambda_3 & 0 \\
0 & 0 & 0
\end{pmatrix}, \lambda_j \neq 0 \text{ for } j = 1, 2, 3.
\]

In both cases the matrix \(\text{cof}(s)\) is equal, up to conjugation, to

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \mu
\end{pmatrix}, \quad \mu \neq 0,
\]

and the proof of 2) follows from a straightforward computation.
3) By part 1) we can assume that \( \gamma_n^{-1} \) converges to a pseudo-projective map \( T \) uniformly on compact subsets of \( \mathbb{P}^2_\mathbb{C} \setminus \text{Ker}(T) \). We proceed by contradiction and we assume that \( \text{Im}(S) \) is not contained in \( \text{Ker}(T) \). By hypothesis, \( \text{Ker}(S) \) is a complex line, so that \( \text{Im}(S) \) consists of one single point denoted by \( p \), and \( p \notin \text{Ker}(T) \). Now, if \( x \in \mathbb{P}^2_\mathbb{C} \setminus \text{Ker}(S) \) then \( S(x) = p \), it follows that \( \gamma_n(x) \xrightarrow{n \to \infty} S(x) = p \). Since \( p \notin \text{Ker}(T) \), we can assume that the compact set \( \{ \gamma_n(x) : n \in \mathbb{N} \} \cup \{ p \} \) is contained in \( \mathbb{P}^2_\mathbb{C} \setminus \text{Ker}(T) \). Hence \( x = \gamma_n^{-1}(\gamma_n(x)) \xrightarrow{n \to \infty} T(p) \), which contradicts the fact that \( T \) is a function defined on \( \mathbb{P}^2_\mathbb{C} \setminus \text{Ker}(T) \). Therefore, \( \text{Im}(S) \subset \text{Ker}(T) \).

For every \( n \in \mathbb{N} \), let \( g_n = (g_{ij}^{(n)}) \) be a lift to \( \text{GL}(3, \mathbb{C}) \) of \( \gamma_n \). We can assume that \( g_n \to s \in \text{Mat}_{3 \times 3} \) as \( n \to \infty \), and \( [s] = S \). Without loss of generality we can assume that

\[
s = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

In this case \( \text{Ker}(S) = e_1 \oplus e_2 \) is precisely the set of points in \( \mathbb{P}^2_\mathbb{C} \) that satisfy the equation \( 0z_1 + 0z_2 + 1z_3 = 0 \), then the line \( \text{Ker}(S) \) can be identified with \([0 : 0 : 1]\). Now, if \( I \) is a complex line not passing through \( \text{Im}(S) = [e_3] \) then \( I \) can be identified with \([A : B : C], C \neq 0 \). It follows that the complex line \( \gamma_n^{-1}(I) \) is identified with

\[
[(A, B, C)(g_n^{-1})^{-1}] = [(A, B, C)g_n],
\]

which implies that the sequence of complex lines \( \gamma_n^{-1}(I) \) converges to the complex line identified with

\[
[(A, B, C)g] = [0 : 0 : C] = [0 : 0 : 1],
\]

and this line is \( \text{Ker}(S) \).

\[\square\]

**Lemma 2.3.** Let \( (\gamma_n)_{n \in \mathbb{N}} \subset \text{PSL}(3, \mathbb{C}) \) be a sequence that converges uniformly to the pseudo-projective map \( S \) on compact subsets of \( \mathbb{P}^2_\mathbb{C} \setminus \text{Ker}(S) \). Then

\[
\text{Eq}\{\gamma_n : n \in \mathbb{N}\} = \mathbb{P}^2_\mathbb{C} \setminus \text{Ker}(S).
\]

**Proof.** The inclusion \( \mathbb{P}^2_\mathbb{C} \setminus \text{Ker}(S) \subset \text{Eq}\{\gamma_n : n \in \mathbb{N}\} \) is obtained from part 1) of Lemma 2.2.

Now, we split the proof in two cases according to whether \( \text{Ker}(S) \) is a point or a complex line.

If \( \text{Ker}(S) \) is a point, denoted \( k \), then \( \text{Im}(S) \) is a complex line, and given two distinct points \( p \) and \( q \) in \( \text{Im}(S) \), there exist points \( p' \) and \( q' \) arbitrarily close to \( k \) such that \( S(p') = p \) and \( S(q') = q \) implying that \( S \) cannot be extended to a continuous function in a neighborhood of \( k \), which implies \( k \notin \text{Eq}\{\gamma_n : n \in \mathbb{N}\} \). Therefore \( \text{Eq}\{\gamma_n : n \in \mathbb{N}\} \subset \mathbb{P}^2_\mathbb{C} \setminus \text{Ker}(S) \).

Now, let us assume that \( \text{Ker}(S) \) is a line, and let \( p \) be a point in \( \text{Ker}(S) \setminus \text{Im}(S) \), we must prove that \( p \notin \text{Eq}\{\gamma_n : n \in \mathbb{N}\} \). We choose a complex line

\[
(p'' : 0 : 0)
\]
$l_1$ different from $\text{Ker}(S)$ and passing through $p$. Since $(\mathbb{P}_2^c)^*$ is a compact space, we can assume that there exists a complex line $l_2$ such that $\gamma_n(l_1)$ converges to $l_2$ as $n \to \infty$. Given that $\gamma_n \xrightarrow{n \to \infty} S$ and $\text{Im}(S)$ is a point denoted $q$, we have that $q$ lies in $l_2$.

If $q'$ is any point in $l_2 \setminus \{q\}$ then there exists a sequence of points $x_n$ in $l_1$ such that $\gamma_n(x_n) \to q'$ as $n \to \infty$. Since $l_1$ is compact, we can assume that there exists a point $x \in l_1$ such that $x_n \to x$ as $n \to \infty$. We notice that $x \in \text{Ker}(S)$ because otherwise we can assume that $\{x_n : n \in \mathbb{N}\}$ is a compact subset of $\mathbb{P}_2^c \setminus \text{Ker}(S)$, then $\gamma_n(x_n) \to q$ as $n \to \infty$, so that $q = q'$, which is absurd. Hence $x = p$, then $x_n \to p$ and $\gamma_n(x_n) \to q' \neq q$, which implies that $p \notin \text{Eq}\{\gamma_n : n \in \mathbb{N}\}$. It follows that $\text{Ker}(S) \setminus \text{Im}(S) \subset \text{Eq}\{\gamma_n : n \in \mathbb{N}\}$, since the equicontinuity set is an open set, we have that $\text{Ker}(S) \subset \mathbb{P}_2^c \setminus \text{Eq}\{\gamma_n : n \in \mathbb{N}\}$. Therefore, $\mathbb{P}_2^c \setminus \text{Ker}(S) = \text{Eq}\{\gamma_n : n \in \mathbb{N}\}$.

**Theorem 2.4.** Let $\Gamma \subset \text{PSL}(3, \mathbb{C})$ be a discrete group then $\Gamma$ acts properly and discontinuously on the open $\Gamma$-invariant set $\text{Eq}(\Gamma)$.

**Proof.** By contradiction, let $K$ be a compact subset of $\text{Eq}(\Gamma)$ such that there exists a sequence $(\gamma_n)$ of distinct elements of $\Gamma$ that satisfy $\gamma_n(K) \cap K \neq \emptyset$. For every $n \in \mathbb{N}$ there exists $k_n \in K$ such that $\gamma_n(k_n) \in K$. Without loss of generality we can assume that $k_n \xrightarrow{n \to \infty} k \in K$ and $\gamma_n(k_n) \xrightarrow{n \to \infty} \kappa \in K$.

Without loss of generality, we can assume, by lemma 2.2 1), that there exists a pseudo-projective map $S$ such that $\gamma_n \xrightarrow{n \to \infty} S$ uniformly on compact subsets of $\mathbb{P}_2^c \setminus \text{Ker}(S)$. Since $K \subset \text{Eq}(\Gamma)$, the lemma 2.3 implies that $K \subset \mathbb{P}_2^c \setminus \text{Ker}(S)$. Así que $\gamma_n(k_n) \xrightarrow{n \to \infty} S(k)$ then $S(k) = \kappa \in K$ which means that $\kappa \in \text{Im}(S)$.

The Lemma 2.2 implies that there is a pseudo projective map $T$ such that $\text{Im}(S) \subset \text{Ker}(T)$, and the Lemma 2.3 implies that $\text{Ker}(T) \subset \mathbb{P}_2^c \setminus \text{Eq}(\Gamma)$. Hence $\kappa \in \text{Im}(S) \subset \text{Ker}(T) \subset \mathbb{P}_2^c \setminus \text{Eq}(\Gamma)$, which contradicts that $K \subset \text{Eq}(\Gamma)$. Therefore $\Gamma$ acts properly and discontinuously on $\text{Eq}(\Gamma)$.

**Theorem 2.5.** If $\Gamma \subset \text{PSL}(3, \mathbb{C})$ is a discrete group, and $U$ is a $\Gamma$-invariant open subset of $\mathbb{P}_2^c$ such that $\mathbb{P}_2^c \setminus U$ contains at least three complex lines in general position, then

$$U \subset \text{Eq}(\Gamma).$$

**Proof.** It suffices to prove that $\text{Ker}(S) \subset \mathbb{P}_2^c \setminus U$, whenever $S$ is a pseudo-projective map which is the limit of a sequence $(\gamma_n)_{n \in \mathbb{N}} \subset \Gamma$. If $\text{Ker}(S)$ is a point, denoted $p$, then $\text{Im}(S)$ is a complex line. Moreover, there exists a pseudo-projective map $T$ such that $\gamma_n^{-1}$ goes to $T$, as $n \to \infty$, uniformly on compact subset of $\mathbb{P}_2^c \setminus \text{Ker}(T)$, such that $\text{Ker}(S) = \text{Im}(T)$ and $\text{Im}(S) = \text{Ker}(T)$. There is a complex line $l_1 \neq \text{Ker}(T)$ such that $l_1 \subset \mathbb{P}_2^c \setminus U$. It follows that $\gamma_n^{-1}(x)$ goes to $p$ as $n \to \infty$ for every point $x \in l_1 \setminus \text{Ker}(S)$, which implies that $p \in \mathbb{P}_2^c \setminus U$ (because $U$ is a $\Gamma$-invariant open set).
If \( \text{Ker}(S) \) is a complex line, then there exists a complex line \( l_1 \) not passing through \( \text{Im}(S) \) and contained in \( \mathbb{P}^2_C \setminus U \), because \( \mathbb{P}^2_C \setminus U \) contains at least three lines in general position. By the Lemma 2.2 part iii), \( \gamma_n^{-1}(l_1) \) goes to \( \text{Ker}(S) \), as \( n \to \infty \), which implies that \( \text{Ker}(S) \subset \mathbb{P}^2_C \setminus U \). □

**Theorem 2.6.** Let \( \Gamma \subset SL(3, \mathbb{C}) \) be a discrete subgroup such that \( \Lambda(\Gamma) \) contains at least three lines in general position, then

\[ \Omega(\Gamma) = Eq(\Gamma). \]

**Proof.** By Theorem 2.4 we have that \( L_0(\Gamma) \cap Eq(\Gamma) = \emptyset = L_1(\Gamma) \cap Eq(\Gamma) \). Now we prove that \( L_2(\Gamma) \cap Eq(\Gamma) = \emptyset \). Let us denote by \( C \) the complement of \( Eq(\Gamma) \) in \( \mathbb{P}^2_C \), and let \( K \) be a compact subset of \( \mathbb{P}^2_C \setminus C = Eq(\Gamma) \). By Lemma 1.2, it suffices to prove that the accumulation points of the orbit of \( K \) are contained in \( L_0(\Gamma) \cup L_1(\Gamma) \). Let \( x \) be one of such accumulation points, then there exists a sequence, \( k_n \), of points in \( Eq(\Gamma) \) such that \( k_n \to k \in Eq(\Gamma) \) as \( n \to \infty \), and there is a sequence \( \gamma_n \) of different elements in \( \Gamma \), such that \( \gamma_n(k_n) \to x \), as \( n \to \infty \). By the Lemma 2.2 we can assume that there exists a pseudo-projective map \( S \) such that \( \gamma_n \to S \), as \( n \to \infty \), uniformly on compact subsets of \( \mathbb{P}^2_C \setminus \text{Ker}(S) \). We have two cases according to whether \( \text{Ker}(S) \) is a point or a complex line.

If \( \text{Ker}(S) \) is a complex line, then the Lemma 2.2 iii) implies that \( \{k_n : n \in \mathbb{N}\} \cup \{k\} \subset Eq(\Gamma) \subset \mathbb{P}^2_C \setminus \text{Ker}(S) \). Thus, \( \gamma_n(k_n) \to x \) implies that \( \gamma_n(k) \to S(k) = x \), and given that \( k \in Eq(\Gamma) \subset \mathbb{P}^2_C \setminus L_0(\Gamma) \), we deduce that \( x \in L_1(\Gamma) \).

If \( \text{Ker}(S) \) is a point then \( \{k_n \cup \{k\}\} \cap \text{Ker}(S) = \emptyset \), because \( \{k_n \cup \{k\} \subset Eq(\Gamma) \), and the proof follows as in the former case.

Therefore \( L_2(\Gamma) \cap Eq(\Gamma) = \emptyset \), then \( Eq(\Gamma) \subset \Omega(\Gamma) \). The proof of the reverse inclusion follows easily from the Theorem 2.5. □

### 3. \( \Lambda(\Gamma) \) is union of complex lines

In this section \( \Gamma \subset SL(3, \mathbb{C}) \) is a Complex Kleinian Group, and \( C(\Gamma) \) denotes the set

\[ C(\Gamma) = \bigcup_{\gamma \in \Gamma} \Lambda(\gamma) \]

**Lemma 3.1.** If \( \Lambda(\Gamma) \) has at least three complex lines in general position, then \( \Lambda(\gamma) \subset \Lambda(\Gamma) \) for every \( \gamma \in \Gamma \). In particular, \( C(\Gamma) \subset \Lambda(\Gamma) \).

**Proof.** Let \( \gamma \) be an element in \( \Gamma \).

There are three cases depending on whether \( \gamma \) is elliptic, parabolic or loxodromic (5)

1. If \( \gamma \) elliptic then \( \Lambda(\gamma) = \emptyset \) and the result follows immediately.
2. If \( \gamma \) es parabolic, then we have three subcases:
(i) We can assume, without loss of generality, that \( \gamma \) has a lift given by the matrix:

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

then \( \Lambda(\gamma) = L_0(\gamma) = \vec{e}_1, \vec{e}_2 \subset L_0(\Gamma) \subset \Lambda(\Gamma) \).

(ii) We can assume, without loss of generality, that \( \gamma \) has a lift \( \tilde{\gamma} \) which has the form:

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]

It follows that, for any \( n \in \mathbb{N} \),

\[
\tilde{\gamma}^n = \begin{pmatrix}
1 & n & \frac{n(n-1)}{2} \\
0 & 1 & \frac{n}{n-1} \\
0 & 0 & 1
\end{pmatrix}.
\]

If \( \Lambda(\gamma) = \vec{e}_1, \vec{e}_2 \) is not contained in \( \Lambda(\Gamma) \), then there exists \( z \in \mathbb{C} \) such that \( [z : 1 : 0] \in \Omega(\Gamma) \). Let \( w \) be any complex number, then the sequence

\[ a_n(w) := [z : 1 : \frac{2(w-n)}{n(n-1)}], n \in \mathbb{N} \text{ tends to } [z : 1 : 0] \text{ as } n \to \infty. \]

Thus, for \( N > 0 \) large enough (\( N \) depends on \( w \)), we have that \( (a_n(w))_{n \geq N} \subset \Omega(\Gamma) \) for all \( n > N \). We note that

\[ \gamma^n(a_n) = [z + w : \frac{2w - n + 1}{n-1} : \frac{2(w-n)}{n(n-1)}] \longrightarrow [-z - w : 1 : 0]. \]

Hence, \([z - w : 1 : 0] \in \Lambda(\Gamma)\), and taking \( w = -2z \) we obtain a contradiction. Therefore, \( \Lambda(\gamma) \subset \Lambda(\Gamma) \).

(iii) We can assume, without loss of generality, that \( \gamma \) has a lift \( \tilde{\gamma} \) of the form:

\[
\tilde{\gamma} = \begin{pmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda^{-2}
\end{pmatrix}, \quad |\lambda| = 1.
\]

We proceed by contradiction. Let’s assume that \( \Lambda(\gamma) = \vec{e}_1, \vec{e}_3 \) is not contained in \( \Lambda(\Gamma) \), then there exists \( z \in \mathbb{C} \) such that \([z : 0 : 1] \in \Omega(\Gamma) \). We note that

\[ a_n := [z : 1/n^2 : 1] \longrightarrow [z : 0 : 1], \text{ so that for } N > 0 \text{ large enough, we have that } (a_n)_{n \geq N} \subset \Omega(\Gamma) \text{ for all } n > N. \]

The sequence,

\[ \gamma^n(a_n) = [z\lambda^n + \frac{n\lambda^{n-1}}{n^2} : \frac{\lambda^n}{n^2} : \lambda^{-2n}] = [z\lambda^{3n} + \frac{\lambda^{3n-1}}{n} : \frac{\lambda^{3n}}{n^2} : 1], \quad n > N, \]

has cluster point \([z : 0 : 1] \) which implies that \([z : 0 : 1] \in \Lambda(\Gamma)\), a contradiction.

(3) If \( \gamma \) is loxodromic, we have four subcases:
(i) If $\gamma$ has a lift $\tilde{\gamma}$ whose normal Jordan form is:
\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}, \quad |\lambda_1| < |\lambda_2| < |\lambda_3|,
\]
then we can assume that $\tilde{\gamma}$ is such matrix. It follows that $\Lambda(\gamma) = \leftarrow e_1, e_2 \cup \leftarrow e_2, e_3$, and the lambda-lemma implies that $\leftarrow e_1, e_2 \subset \Lambda(\Gamma)$ and $\leftarrow e_2, e_3 \subset \Lambda(\Gamma)$.

Let’s assume, without loss of generality, that $e_1, e_2 \subset \Lambda(\Gamma)$, then there exists $l \subset \Lambda(\Gamma)$ such that $l$ does not pass through the point $[1 : 0 : 0]$ (because $\Lambda(\Gamma)$ contains at least three complex lines in general position). We note that the sequence of complex lines $\gamma^n(l)$ goes to the complex line $\leftarrow e_2, e_3$ as $n \to \infty$, therefore $e_2, e_3 \subset \Lambda(\Gamma)$.

(ii) If $\gamma$ has a lift $\tilde{\gamma}$ whose Jordan normal form is
\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & \lambda_2
\end{pmatrix}, \quad |\lambda_1| \neq 1,
\]
then we can assume that $\tilde{\gamma}$ is such matrix. In this case $\Lambda(\gamma) = L_0(\gamma) \subset L_0(\Gamma) \subset \Lambda(\Gamma)$.

(iii) If $\gamma$ has a lift $\tilde{\gamma}$ whose Jordan normal form is
\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}, \quad |\lambda_1| = |\lambda_2| < |\lambda_3|,
\]
then we can assume that $\tilde{\gamma}$ is such matrix, so that $\Lambda(\gamma) = \leftarrow e_1, e_2 \cup \{[0 : 0 : 1]\}$.

Let $l$ be a complex line contained in $\Lambda(\Gamma)$ not passing through $[0 : 0 : 1]$ then $\gamma^{-n}(l)$ is a sequence of complex lines that tends to the complex line $\leftarrow e_1, e_2$ as $n$ tends to $\infty$, which implies that $\leftarrow e_1, e_2 \subset \Lambda(\Gamma)$. Moreover, $\gamma^n(z) \to [0 : 0 : 1]$ whenever $z \in P^2 \setminus \leftarrow e_1, e_2$. Therefore $[0 : 0 : 1] \in \Lambda(\Gamma)$.

(iv) If $\gamma$ has a lift $\tilde{\gamma}$ whose Jordan normal form is
\[
\begin{pmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda^{-2}
\end{pmatrix}, \quad |\lambda| < 1,
\]
then we can assume that $\tilde{\gamma}$ is such matrix, in this case, $\Lambda(\gamma) = e_1, e_2 \cup \leftarrow e_1, e_3$.

We have that $\tilde{\gamma}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & 0 \\
0 & \lambda^n & 0 \\
0 & 0 & \lambda^{-2n}\end{pmatrix}$, $n \in \mathbb{N}$. 

\[
\tilde{\gamma}^n = 
\]
Let’s assume that $e_1, e_2$ is not contained in $\Lambda(\Gamma)$, then there exists $z \in \mathbb{C}$ such that $[z : 1 : 0] \in \Omega(\Gamma)$. If $w$ is a non-zero complex number, then the sequence $[wz : w : n\lambda^{3n-1}]$ tends to $[z : 1 : 0]$ as $n \to \infty$. Moreover

$$\gamma^n([z : 1 : \frac{n\lambda^{3n-1}}{w}]) \to [w : 0 : 1],$$

which implies that the complex line $e_1, e_3$ is contained in $\Lambda(\Gamma)$.

Now, assuming $e_1, e_2 \subset \Lambda(\Gamma)$, let $l$ be a complex line different from $e_1, e_2$, which is contained in $\Lambda(\Gamma)$ and does not pass through $[0 : 0 : 1]$, then $l$ has an equation of the form

$$Az_1 + Bz_2 + Cz_3 = 0, \quad |A|^2 + |B|^2 \neq 0, C \neq 0.$$

It is not hard to check that $\gamma^n(l), n \in \mathbb{N}$ has an equation of the same kind, with coefficients $(A'(n), B'(n), C'(n))$ given by the equation

$$\begin{pmatrix} A'(n) \\ B'(n) \\ C'(n) \end{pmatrix} = \left( \begin{pmatrix} \lambda^n & n\lambda^{n-1} & 0 \\ 0 & \lambda^n & 0 \\ 0 & 0 & \lambda^{-2n} \end{pmatrix} \right)^{-1} \begin{pmatrix} A \\ B \\ C \end{pmatrix}.$$

Thus, $(A'(n), B'(n), C'(n)) = (\lambda^{-n}A, \lambda^{-n}B - n\lambda^{-n-1}A, \lambda^{2n}C)$, in other words,

$$(A'(n), B'(n), C'(n)) = (A, B - n\lambda^{-1}A, \lambda^{3n}C),$$

it follows that $\gamma^n(l) \to e_1, e_3$, which implies that $e_1, e_3 \subset \Lambda(\Gamma)$.

If $e_1, e_3 \subset \Lambda(\Gamma)$, let us take a line $l$ contained in $\Lambda(\Gamma)$ not passing through $[0 : 0 : 1]$, then $l$ has an equation of the form

$$Az_1 + Bz_2 + Cz_3 = 0, \quad C \neq 0.$$ 

It is not hard to check that $\gamma^{-n}(l), n \in \mathbb{N}$ has an equation of the same kind, whose coefficients $(A'(-n), B'(-n), C'(-n))$ are given by the equation

$$\begin{pmatrix} A'(-n) \\ B'(-n) \\ C'(-n) \end{pmatrix} = \left( \begin{pmatrix} \lambda^n & 0 & 0 \\ n\lambda^{n-1} & \lambda^n & 0 \\ 0 & 0 & \lambda^{-2n} \end{pmatrix} \right) \begin{pmatrix} A \\ B \\ C \end{pmatrix}.$$

Thus $(A'(-n), B'(-n), C'(-n)) = (\lambda^nA, n\lambda^{n-1}A + \lambda^nB, \lambda^{-2n}C)$, equivalently,

$$(A'(-n), B'(-n), C'(-n)) = (\lambda^{3n}A, n\lambda^{3n-1}A + \lambda^{3n}B, C),$$

since $|\lambda| < 1$ we have that $\gamma^{-n}(l) \to e_1, e_2$, which implies that $e_1, e_2 \subset \Lambda(\Gamma)$. \hfill \Box

**Proof of Theorem 0.2.** First, we notice that Theorem 2.5 implies that $\mathbb{P}^2_{\mathbb{C}} \setminus C(\Gamma) \subset Eq(\Gamma)$, because $\mathbb{P}^2_{\mathbb{C}} \setminus C(\Gamma)$ is a $\Gamma$-invariant open set and $C(\Gamma)$ contains at least three complex lines in general position. It follows from Theorem 2.6 that $\mathbb{P}^2_{\mathbb{C}} \setminus C(\Gamma) \subset Eq(\Gamma) = \Omega(\Gamma)$. Therefore $\Gamma$ acts properly and discontinuously on $\mathbb{P}^2_{\mathbb{C}} \setminus C(\Gamma)$ and $\Lambda(\Gamma) \subset C(\Gamma)$. 


Given that $\Lambda(\Gamma)$ contains at least three complex lines in general position, the Lemma 3.1 implies that $C(\Gamma) \subset \Lambda(\Gamma)$.

In order to prove that $C(\Gamma)$ is a union of complex lines, it suffices to check that for every $p \in \bigcup_{\gamma \in \Gamma} \Lambda(\gamma)$, there exists a complex line contained in $\Lambda(\Gamma)$ passing through $p$. Hence, let us assume that $p \in \Lambda(\gamma)$ for some $\gamma \in \Gamma$. We need only to consider the case when $\Lambda(\gamma)$ consists of one complex line $l$ and one point $q \notin l$. If $p \in l$ then there is nothing to prove. If $p = q$ then there exists a complex line $l_1 \subset \Lambda(\Gamma)$, $l_1 \neq l$, so when $n \to \infty$, $\gamma^n(l_1)$ or $\gamma^{-n}(l_1)$ goes to a complex line passing through $p$ and contained $\Lambda(\Gamma)$.

Since $C(\Gamma)$ contains three complex lines in general position and $\Gamma$ has no fixed point, there exist three complex lines $l_1, l_2, l_3$ and three elements $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ such that $l_1 \subset \Lambda(\gamma_1)$, $l_2 \subset \Lambda(\gamma_2)$, $l_3 \subset \Lambda(\gamma_3)$. Now, let $U$ be a $\Gamma$-invariant open subset of $\mathbb{P}_C^2$ on which $\Gamma$ acts properly and discontinuously.

If $\gamma_1, \gamma_2, \gamma_3$ are pairwise different, then, by Theorem 1.3, we can assume $l_1, l_2, l_3 \subset \mathbb{P}_C^2 \setminus U$. It follows from Theorems 2.5 and 2.6 that $U \subset Eq(\Gamma) = \Omega(\Gamma)$.

If $\gamma_1, \gamma_2, \gamma_3$ are not pairwise different, then without loss of generality we can assume that $\gamma_1 \neq \gamma_2 = \gamma_3$. By Theorem 1.3, we can assume that $l_1 \subset P^2 \setminus U$ and at least one of the lines $l_2$ or $l_3$ is contained in $\mathbb{P}_C^2 \setminus U$, therefore $\mathbb{P}_C^2 \setminus U$ contains at least two complex lines in general position. If there were not more than two complex lines in general position contained in $\mathbb{P}_C^2 \setminus U$, then the intersection point of $l_1$ and $l_2$ or the intersection point of $l_1$ and $l_3$ would be a $\Gamma$-fixed point, and it cannot happen. Therefore $\mathbb{P}_C^2 \setminus U$ contains at least three lines in general position, and it follows from Theorems 2.5 and 2.6 that $U \subset Eq(\Gamma) = \Omega(\Gamma)$.

In what follows, we only consider groups $\Gamma \subset PSL(3, \mathbb{C})$ whose action on $\mathbb{P}^2_C$ have no fixed point nor invariant lines.

**Lemma 3.2.** If $\Gamma \subset PSL(3, \mathbb{C})$ is a infinite discrete group, having no fixed points nor invariant lines then $C(\Gamma) \cap \Lambda(\Gamma)$ contains at least three complex lines in general position.

**Proof.** Given that $\Gamma \subset PSL(3, \mathbb{C})$ is an infinite and discrete group, by Theorem 1.3, there exists an infinite order element $\gamma_0 \in \Gamma$, and one complex line, $l_0$, such that $l_0 \subset \Lambda(\gamma_0) \subset C(\Gamma)$, and $l_0 \subset \Lambda(\Gamma)$. Since $l_0$ is not $\Gamma$-invariant, there exists an element $\gamma_1 \in \Gamma$ such that $l_1 := \gamma_1(l_0) \neq l_0$ (we notice that $l_1 \subset \Lambda(\gamma_1 \gamma_0^{-1}) \subset C(\Gamma)$). Let $q_0$ be the intersection point of $l_0$ and $l_1$. If $\gamma(l_0)$ passes through $q_0$ for every $\gamma \in \Gamma$, then $q_0$ is a $\Gamma$-fixed point, which cannot happen. Thus, there exists $\gamma_2 \in \Gamma$ such that $l_2 := \gamma_2(l_0)$, $l_1$ and $l_0$ are in general position, and we see that $l_2 \subset \Lambda(\gamma_2 \gamma_0^{-1}) \subset C(\Gamma)$.

**Example 3.3.** Let $\alpha$ and $\beta$ in $PSL(3, \mathbb{C})$ be the elements induced, respectively, by the matrices

$$
A = \begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
$$
Let $G$ be the group $\langle A, B \rangle \subset GL(3, \mathbb{C})$, and $\Gamma = \langle \alpha, \beta \rangle \subset PSL(3, \mathbb{C})$, then $\Gamma$ is a discrete group, it has no fixed points nor invariant lines and $\Lambda(\Gamma) = C(\Gamma) = e_1 \leftrightarrow e_2 \cup e_2 \leftrightarrow e_3 \cup e_3 \leftrightarrow e_1$.

Proof. a) The group $G$ is a discrete, so $\Gamma$ is a discrete group. In order to proof this statement, we use the norm in $\{g \in GL(3, \mathbb{C}) : \det g = \pm 1\}$ given by $\|g\| = \max |g_{i,j}|$. We notice that each element in $G$ is given by one of the following matrices:

\[
\begin{pmatrix}
2^{m_1} & 0 & 0 \\
0 & 2^{m_2} & 0 \\
0 & 0 & 2^{m_3}
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 2^{m_1} \\
2^{m_2} & 0 & 0 \\
0 & 2^{m_3} & 0
\end{pmatrix},
\begin{pmatrix}
0 & 2^{m_1} & 0 \\
0 & 0 & 2^{m_2} \\
2^{m_3} & 0 & 0
\end{pmatrix},
\]

where $m_1, m_2, m_3 \in \mathbb{Z}$ and $m_1 + m_2 + m_3 = 0$, because the set $H$ of such matrices, is a group, and $A \in H, B \in H$. Since $G \subset H$, it suffices to prove that $H$ is a discrete group, and for this, we only need to check that, for every $k > 0$, the set $\{h \in H | \|h\| < k\}$ is finite. Let us assume that $\|h\| < k$ and, without loss of generality, $m_1 = \max \{m_1, m_2, m_3\} > 0$, then

\[0 < m_1 < \log_2 k,\]

\[-\log_2 k < m_2 + m_3 < 2\log_2 k.\]

We have two cases according to whether $m_2$ and $m_3$ are both negative or $m_1m_2 < 0$. In any case, there are finitely many values of $m_1, m_2, m_3$ under these conditions.

(b) The group $\Gamma$ has no fixed points nor invariant lines, because every fixed point of $\alpha$ is not fixed by $\beta$, and every $\alpha$-invariant complex line is not $\beta$-invariant.

(c) $\Lambda(\Gamma) = C(\Gamma) = e_1 \leftrightarrow e_2 \cup e_2 \leftrightarrow e_3 \cup e_3 \leftrightarrow e_1$.

The Lemma 3.2 implies that $C(\Gamma) \cap \Lambda(\Gamma)$ contains three complex lines in general position, then Theorem 0.2 implies that $\Lambda(\Gamma) = C(\Gamma)$. It is not hard to check that $\Lambda(\gamma) = \Lambda(\gamma^3)$, and $\gamma^3$ is given by a diagonal matrix, it follows that $\Lambda(\gamma^3) \subset e_1 \leftrightarrow e_2 \cup e_2 \leftrightarrow e_3 \cup e_3 \leftrightarrow e_1$. Therefore $C(\Gamma) \subset e_1 \leftrightarrow e_2 \cup e_2 \leftrightarrow e_3 \cup e_3 \leftrightarrow e_1$, and it is not hard to see that $e_1, e_2 \cup e_2, e_3 \cup e_3, e_1 \subset C(\Gamma)$.

Inspired by the last example we state the following Lemma.

**Lemma 3.4.** If $\Gamma \subset PSL(3, \mathbb{C})$ is an infinite discrete group without fixed points nor invariant lines, and $C(\Gamma)$ contains more than three complex lines (not necessarily in general position) then $C(\Gamma) \cap \Lambda(\Gamma)$ has at least four complex lines in general position.

Proof. The proof of this Lemma is an extension of the proof of 3.2 so there exist three complex lines $l_0, l_1, l_2$ (in general position), contained in the set $C(\Gamma) \cap \Lambda(\Gamma)$. Let $q_0, q_1, q_2$ be the intersection points of the pairs of complex lines $l_0$ and $l_1$, $l_1$ and $l_2$, $l_2$ and $l_0$, respectively. Now we proceed by contradiction, we assume that there are not more than three lines (in general position) contained in $C(\Gamma) \cap \Lambda(\Gamma)$, then for every $\gamma \in \Gamma$, the complex lines $\gamma(l_j)$ for $j = 0, 1, 2$ necessarily pass through one of the points $q_0, q_1, q_2$. We
can suppose, without loss of generality, that there exists one such line, \( l_3 \), distinct from \( l_0 \) and \( l_1 \), passing through \( q_0 \). If there are no more lines of the form \( \gamma(l_j) \), \( j = 0, 1, 2 \), then \( q_0 \) is fixed by the whole group \( \Gamma \) which cannot happen. Therefore, there exists another complex line \( l_4 \) such that \( l_4 \) is equal to \( \gamma(l_j) \) for some \( \gamma \in \Gamma \) and for some \( j = 0, 1, 2 \), and \( l_4 \) passes through a point \( q_1 \) distinct from \( q_0 \) (because otherwise \( q_0 \) is fixed by \( \Gamma \)), then \( l_0, l_2, l_3, l_4 \) are in general position.

\[ \square \]

**Lemma 3.5.** Let \( \gamma \in \text{PSL}(3, \mathbb{C}) \) be any element. Let us assume that \( l_1 \) is a complex line contained in \( \Lambda(\gamma) \), then for every complex line \( l \) different from \( l_1 \) (except, maybe, for a family of complex lines in a pencil of complex lines), some of the sequences of distinct complex lines, \( (\gamma^n(l))_{n \in \mathbb{N}} \) or \( (\gamma^{-n}(l))_{n \in \mathbb{N}} \) goes to \( l_1 \), as \( n \to \infty \).

**Proof.** We split the proof in several cases according to the classification of the elements in \( \text{PSL}(3, \mathbb{C}) \) given in [5].

(a) In the first case, we can assume that \( \gamma \) has a lift of the form

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

then \( \Lambda(\gamma) \) is the complex line \( \overrightarrow{e_1,e_3} \), which is represented, in \( (\mathbb{P}_\mathbb{C}^2)^* \), by \([0 : 1 : 0]\). If \( l \) is a complex line represented in \( (\mathbb{P}_\mathbb{C}^2)^* \) by \([A : B : C]\), then \( \gamma^n(l) \), \( n \in \mathbb{N} \) is represented in \( (\mathbb{P}_\mathbb{C}^2)^* \) by

\[
[A(n) : B(n) : C(n)] := [(A, B, C) \begin{pmatrix} 1 & -n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}] = [A : -nA + B : C].
\]

We obtain that \( [A(n) : B(n) : C(n)] \xrightarrow{n \to \infty} [0 : 1 : 0] \), whenever \( A \neq 0 \), which implies that \( \gamma^n(l) \xrightarrow{n \to \infty} \overrightarrow{e_1,e_3} \), whenever \( l \) is a complex line not passing through the point \([1 : 0 : 0]\) \( \in \mathbb{P}_\mathbb{C}^2 \).

b) We can assume that \( \gamma \) has a lift of the form:

\[
\begin{pmatrix}
\zeta & 1 & 0 \\
0 & \zeta & 0 \\
0 & 0 & \zeta^{-2}
\end{pmatrix}, \quad |\zeta| = 1,
\]

then \( \Lambda(\gamma) \) is the complex line \( \overleftrightarrow{e_1,e_3} \) which is represented, in \( (\mathbb{P}_\mathbb{C}^2)^* \), by \([0 : 1 : 0]\). If \( l \) is a complex line represented in \( (\mathbb{P}_\mathbb{C}^2)^* \) by \([A : B : C]\), then \( \gamma^n(l) \), \( n \in \mathbb{N} \) is represented in \( (\mathbb{P}_\mathbb{C}^2)^* \) by

\[
[A(n) : B(n) : C(n)] := [(A, B, C) \begin{pmatrix} \zeta^{-n} & -n\zeta^{-(n+1)} & 0 \\ 0 & \zeta^{-n} & 0 \\ 0 & 0 & \zeta^{2n} \end{pmatrix}]
\]

Hence,

\[
[A(n) : B(n) : C(n)] = [A\zeta^{-n} : -n\zeta^{-(n+1)}A + \zeta^{-n}B : \zeta^{2n}C].
\]
We obtain that $[A(n) : B(n) : C(n)] \xrightarrow{n \to \infty} [0 : 1 : 0]$, whenever $A \neq 0$, which means that $\gamma^n(l) \xrightarrow{n \to \infty} e_1, e_3$, whenever $l$ is a complex line not passing through the point $[1 : 0 : 0] \in \mathbb{P}^2_C$.

(c) We can assume that $\gamma$ has a lift of the form:

$$
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
$$

In this case, $\Lambda(\gamma)$ is the complex line $e_1, e_2$ which is represented, in $(\mathbb{P}^2_C)^*$, by $[0 : 0 : 1]$. If $l$ is a complex line represented in $(\mathbb{P}^2_C)^*$ by $[A : B : C]$, then $\gamma^n(l), n \in \mathbb{N}$ is represented in $(\mathbb{P}^2_C)^*$ by

$$
[A(n) : B(n) : C(n)] := [(A, B, C) \begin{pmatrix} 1 & -n & \frac{n(n+1)}{2} \\
0 & 1 & -n \\
0 & 0 & 1 \end{pmatrix}].
$$

Hence,

$$
[A(n) : B(n) : C(n)] = [A : -nA + B : \frac{n(n+1)}{2}A - nB + C],
$$

and it follows that $[A(n) : B(n) : C(n)] \xrightarrow{n \to \infty} [0 : 0 : 1]$, which means that $\gamma^n(l) \xrightarrow{n \to \infty} e_1, e_2$ for any line $l$.

(d) We can assume $\gamma$ has a lift of the form

$$
\begin{pmatrix}
\zeta & 1 & 0 \\
0 & \zeta & 0 \\
0 & 0 & \zeta^{-2}
\end{pmatrix}, \quad |\zeta| > 1.
$$

In this case $\Lambda(\gamma)$ is equal to the union of the complex lines $e_1, e_2$ and $e_1, e_3$ which are represented in $(\mathbb{P}^2_C)^*$, by $[0 : 0 : 1]$ and $[0 : 1 : 0]$, respectively. If $l$ is a complex line represented in $(\mathbb{P}^2_C)^*$ by $[A : B : C]$, then $\gamma^n(l), n \in \mathbb{N}$ is represented in $(\mathbb{P}^2_C)^*$ by

$$
[A(n) : B(n) : C(n)] := [(A, B, C) \begin{pmatrix} \zeta^{-n} & -n\zeta^{-(n+1)} & 0 \\
0 & \zeta^{-n} & 0 \\
0 & 0 & \zeta^{2n} \end{pmatrix}].
$$

Hence,

$$
[A(n) : B(n) : C(n)] = [A\zeta^{-n} : -n\zeta^{-(n+1)}A + \zeta^{-n}B : \zeta^{2n}C] = [A\zeta^{-3n} : -n\zeta^{-3n-1}A + \zeta^{-3n}B : C].
$$

So that $[A(n) : B(n) : C(n)] \xrightarrow{n \to \infty} [0 : 0 : 1]$, whenever $C \neq 0$, which means that $\gamma^n(l) \xrightarrow{n \to \infty} e_1, e_2$, whenever $l$ is a complex line not passing through the point $[0 : 0 : 1] \in \mathbb{P}^2_C$. 

Now, if \( l \) is a complex line represented in \((\mathbb{P}^2_\mathbb{C})^*\) by \([A : B : C]\), then \( \gamma^{-n}(l), n \in \mathbb{N} \) is represented in \((\mathbb{P}^2_\mathbb{C})^*\) by

\[
[A(-n) : B(-n) : C(-n)] := [(A, B, C) \begin{pmatrix} \zeta^n & n\zeta^{n-1} & 0 \\ 0 & \zeta^n & 0 \\ 0 & 0 & \zeta^{-2n} \end{pmatrix}]
\]

Hence,

\[
[A(-n) : B(-n) : C(-n)] = [\zeta^n A : n\zeta^{n-1} A + \zeta^n B : \zeta^{-2n} C] = [A : n\zeta^{-1} A + B : \zeta^{-3n} C] = [A/n : A\zeta^{-1} + B/n : \zeta^{-3n} C/n].
\]

Thus, \( [A(-n) : B(-n) : C(-n)] \xrightarrow{n \to \infty} [0 : 1 : 0] \), whenever \( A \neq 0 \) or \( B \neq 0 \), which implies that \( \gamma^{-n}(l) \xrightarrow{n \to \infty} e_1, e_3 \), whenever \( l \) is a complex line different from \( e_1, e_2 \).

The remaining cases are proved in a similar way. \(\square\)

We define the set \( \mathcal{E}(\Gamma) \) as the subset of \((\mathbb{P}^2_\mathbb{C})^*\) consisting of all the complex lines \( l \) for which there exists an element \( \gamma \in \Gamma \) such that \( l \subset \Lambda(\gamma) \). Also we define \( E(\Gamma) \) as the subset of \( \mathbb{P}^2_\mathbb{C} \) given by \( E(\Gamma) = \bigcup_{l \in \mathcal{E}(\Gamma)} l \). It is not hard to see that \( E(\Gamma) \subset C(\Gamma) \) and \( E(\Gamma) = \bigcup_{l \in \mathcal{E}(\Gamma)} l \).

**Remark 3.6.** If \( \mathcal{E}(\Gamma) \) contains at least two distinct complex lines, then

\[
C(\Gamma) = E(\Gamma) = \bigcup_{l \in \mathcal{E}(\Gamma)} l = \bigcup_{l \in \mathcal{E}(\Gamma)} l.
\]

**Proof.** We need only to consider the case when \( \Lambda(\gamma) \) has the form \( l \cup \{p\} \), where \( l \) is a complex line and \( p \) is a point. By hypothesis, there exists a complex line \( l_1 \in \mathcal{E}(\Gamma) \) such that \( l_1 \neq l \). If \( l_1 \) passes through \( p \) then \( p \in E(\Gamma) \), so we can assume that \( l_1 \) does not pass through \( p \) and it follows that one of the sequences of lines \( \gamma^n(l_1) \) or \( \gamma^{-n}(l_1) \) goes to a complex line in \( \mathcal{E}(\Gamma) \) passing through \( p \), as \( n \to \infty \). Therefore \( p \in E(\Gamma) \) and \( C(\Gamma) \subset E(\Gamma) \). \(\square\)

**Lemma 3.7.** If \( \Gamma \subset PSL(3, \mathbb{C}) \) is a discrete group and \( \mathcal{E}(\Gamma) \) contains at least four complex lines in general position, then \( \overline{\mathcal{E}(\Gamma)} \subset (\mathbb{P}^2_\mathbb{C})^* \) is a perfect set. Hence, \( C(\Gamma) \) is a non-numerable union of complex lines.

**Proof.** It suffices to prove that that each complex line in \( \overline{\mathcal{E}(\Gamma)} \) is an accumulation line of lines lying in \( \mathcal{E}(\Gamma) \). Furthermore, it is sufficient to prove that each complex line in \( \mathcal{E}(\Gamma) \) is an accumulation line of lines lying in \( \mathcal{E}(\Gamma) \). Let \( l_1 \) be a complex line in \( \mathcal{E}(\Gamma) \), then there exists \( \gamma \in \Gamma \) such that \( l_1 \subset \Lambda(\gamma) \). By the Lemma \( \ref{lem:3.5} \) and given that \( \mathcal{E}(\Gamma) \) contains at least four lines in general position, we have that there exists a line \( l \) in \( \mathcal{E}(\Gamma) \) such that some of the sequences of distinct lines in \( \mathcal{E}(\Gamma) \), \((\gamma^n(l))_{n \in \mathbb{N}}\) or \((\gamma^{-n}(l))_{n \in \mathbb{N}}\) goes to \( l_1 \) as \( n \to \infty \). \(\square\)
Proof of Theorem 0.3. a) By Lemma 3.2, the set $\Lambda(\Gamma) \cap C(\Gamma)$ contains at least three complex lines in general position, then by Theorem 2.6 we have that $Eq(\Gamma) = \Omega(\Gamma)$.

If $U \subset \mathbb{P}^2_C$ is a $\Gamma$-invariant open set on which $\Gamma$ acts properly and discontinuously, then, by Theorem 1.3 there exists a complex line $l$ contained in $\mathbb{P}^2_C \setminus U$. Given that $\Gamma$ does not have invariant lines nor fixed points, then $\mathbb{P}^2_C \setminus U$ contains at least three complex lines in general position. Hence, by Theorem 2.5 $U \subset Eq(\Gamma) = \Omega(\Gamma)$. Therefore $\Omega(\Gamma)$ is the maximal open set on which $\Gamma$ acts properly and discontinuously. The proof that every connected component of $\Omega(\Gamma)$ is complete Kobayashi hyperbolic is obtained imitating the proof of Lemma 2.3 in [1], and of the main Theorem in [1].

(b) The equality $\Lambda(\Gamma) = C(\Gamma)$ follows from the Lemma 3.2 and the Theorem 0.2. Conexity follows from the facts that two distinct complex lines always intersect and a complex line is path-connected.

(c) The Lemma 3.7 implies that $E(\Gamma)$ is a perfect set. Now, if $D \subset (\mathbb{P}^2_C)^*$ is a non-empty, closed $\Gamma$-invariant subset, then there is a complex line $l \in D$ and the set $\{\gamma(l) \mid \gamma \in \Gamma\}$ contains at least three complex lines in general position. It cannot happen that $\{\gamma(l) \mid \gamma \in \Gamma\}$ contains only three complex lines because in such case $\Lambda(\Gamma)$ would consist of only three complex lines, a contradiction. Therefore, there are more than three complex lines in the set $\{\gamma(l) \mid \gamma \in \Gamma\}$ and imitating the proof of the Lemma 3.4 we obtain that $\{\gamma(l) \mid \gamma \in \Gamma\}$ contains four lines in general position. Therefore, $D$ contains four lines in general position. Let $\gamma$ be an element in $\Gamma$, applying the lemma 3.5 we deduce that every complex line contained in $\Lambda(\gamma)$ is contained in $D$, then $E(\Gamma) \subset D$.

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