Invariants of 4-manifolds from Khovanov-Rozansky link homology

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Motivation

How does Khovanov homology extend to other ambient manifolds?

Hints:
- Functoriality under link cobordisms in 4d.
- Rozansky & Willis invariants for nullhomologous links in $\#^k(S^1 \times S^2)$.
- Rasmussen: Kh sensitive to smooth surfaces in $B^4$.

Proposal: $\text{Kh}(L) = \text{Kh}(B^4; L)$
- 4-manifolds (with (link in) boundary) $\rightarrow$ chain complexes
- 3-manifolds $\rightarrow$ dg categories
- point $\rightarrow$ some 4-category

Today: a few steps in this direction.
Starting in dimension 3...

Link invariants

The $\mathfrak{gl}_N$ link polynomial $P_N : \{\text{framed, oriented links}\} \to \mathbb{Z}[q^{\pm 1}]$:

\[
P_N(\bigcirc) - P_N(\rule{0.5em}{0.1em}) = (q - q^{-1})P_N(\bigcirc)
\]

\[
P_N(\rule{0.5em}{0.1em}) = q^N P_N(\bigcirc), \quad P_N(L_1 \sqcup L_2) = P_N(L_1)P_N(L_2)
\]

Higher categories

Ribbon category $\text{Rep}(U_q(\mathfrak{gl}_N))$, tangle invariants

Manifold invariants

The $\mathfrak{gl}_N$ skein module for compact, oriented $M^3$, $P \subset \partial M^3$:

\[
\text{Sk}_N(M^3; P) := \frac{\mathbb{Z}[q^{\pm 1}]\langle\text{framed, oriented tangles in } (M^3, P)\rangle}{\langle\text{isotopy, local relations in } B^3 \hookrightarrow M^3\rangle}
\]

Part of a 0123ε-dimensional TFT.
...upgrading to dimension 4

Khovanov–Rozansky 2004, Robert–Wagner+Ehrig–Tubbenhauer–W 2017:

**Link invariants**

The $\mathfrak{gl}_N$ Khovanov–Rozansky link homology

$$\text{KhR}_N: \{\text{links/link cobordisms}\} \to K^b(\text{gr}^\mathbb{Z}\text{Vect}), \quad \chi_q \circ \text{KhR}_N = P_N$$

Morrison–Walker–W 2019:

**Higher categories**

A ribbon 2-category / a disk-like 4-category categorifying $\text{Rep}(U_q(\mathfrak{gl}_N))$.

**Manifold invariants**

A ‘skein module’ $S_N(W^4; L)$ for compact, oriented, smooth $W^4$, $L \subset \partial W^4$.

$$S_N(B^4; L) \cong \text{KhR}_N(L).$$

Part of a 01234ε-dimensional TFT?
Approaches

Some routes to Khovanov–Rozansky homology for (links in) 3-manifolds:

- **Categorify Witten–Reshetikhin–Turaev invariants**
  - Categorification at roots of unity
  - Categorification of tensor product reps

- **Categorify skein modules**
  - Via surgery
  - Via Heegaard splitting, categorified skein algebras

- **Extending Witten’s model for Khovanov homology in $\mathbb{R}^3$**

- **Higher skein modules (this talk)**
  - Functorial tangle invariant $\rightarrow$ 4-category $\rightarrow$ skein module
Khovanov–Rozansky homology

\[ \begin{align*}
\{ \text{links diagrams} \} & \xrightarrow{\cong} \{ \text{movies of diagrams/m. moves} \} \\
\{ \text{links embedded in } B^3 \} & \xrightarrow{\cong} \{ \text{cobordisms in } B^3 \times I / \text{isotopy} \}
\end{align*} \]

\[ \begin{align*}
\text{KhR}_N & \quad \rightarrow \quad K^b(\text{gr}\, \mathbb{Z} \text{Vect}) \\
\chi_q & \quad \downarrow
\end{align*} \]

\[ \begin{align*}
P_N \circ K_0 & \quad \rightarrow \quad \mathbb{Z}[q^{\pm 1}]
\end{align*} \]

Defining \( \text{KhR}_N \) requires:

- the data of a chain complex for each link diagram (KhR04, RW17)
- the data of a chain map for every elementary movie (KhR04)
- movie move checks (Blanchet10, ETW17)

\[ \implies \text{KhR}_N \text{ can be considered as diagram-independent (MWW19)}. \]
Khovanov–Rozansky homology

\[
\begin{align*}
\{ \text{links diagrams} \\
\text{movies of diagrams/m. moves} \}
\quad \xrightarrow{\text{KhR}_N} \quad K^b(\text{gr}^\mathbb{Z}\text{Vect})
\end{align*}
\]

E.g. this chain map should be homotopic to the identity:

Defining \(\text{KhR}_N\) requires:

- the data of a chain complex for each link diagram (\(\text{KhR}04, \text{RW}17\))
- the data of a chain map for every elementary movie (\(\text{KhR}04\))
- movie move checks (\(\text{Blanchet}10, \text{ETW}17\))

\[\implies\] \(\text{KhR}_N\) can be considered as diagram-independent (\(\text{MWW}19\)).
Functoriality in $S^3$

For $S_N(B^4; L) \cong \text{KhR}_N(L)$ we need $\text{KhR}_N$ for links in $S^3 = B^3 \cup \{\infty\}$.

- links in $S^3$ generically avoid $\infty$
  $\implies$ same chain complexes

- link cobordisms in $S^3 \times I$ generically avoid $\infty \times I$
  $\implies$ same chain maps

- link cobordism isotopies in $S^3 \times I^2$ might intersect $\infty \times I^2$ transversely
  $\implies$ a new movie move to check, non-local if viewed from $B^3$

Theorem (M.-W.-W. 2019)

$\text{KhR}_N$ is invariant under the sweeparound move, thus functorial in $S^3$. 
Proving the sweeparound move

1. Reduce to the case of almost braid closures
2. Compare front and back versions of

\[
\begin{pmatrix}
0 & -(y-1) & -y \\
(1-y) & 0 & y \\
0 & 0 & 1
\end{pmatrix}
\]

3. Consider filtration by homological degree of extra crossings
4. Front and back versions of R1, R2, R3 agree* in associated graded
Ribbon 2-category via KhR$_N$ for tangles

$$\{\text{tangle diagrams}\} \xrightarrow{\mathcal{R}} \{\text{tangles embedded in } B^3\} \xrightarrow{\text{movies of diagrams/m. moves}} \{\text{cobordisms in } B^3 \times I/\text{isotopy}\}$$

$$\mathcal{R} \quad \xrightarrow{[-]_N} \quad K^b(N\text{Foam})$$

$$\chi q \quad \downarrow$$

$$\text{Rep}(U_q(\mathfrak{gl}_N))$$

Theorem (M.-W.-W. 2019)

∃ linear braided monoidal 2-category (Kapranov–Voevodsky, Baez–Neuchl, Day–Street, Crans) with duals (Barrett–Meusburger–Schaumann) with

- Objects: tangle boundary sequences
- 1-morphisms: Morse data for tangle diagrams
- 2-morphisms from $T_1$ to $T_2$: $H^* \text{Ch}(N\text{Foam})([T_1]_N, [T_2]_N)$. 

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Questions

Is this braided monoidal 2-category (or something similar) 4-dualizable and $SO(4)$-fixed in a suitable 5-category of $E_2$ 2-categories? What is the role of the sweeparound move? Can this all be made homotopy-coherent?

$\implies$ a local $01234\varepsilon$-d oriented TFT via the cobordism hypothesis.

Proposed direct construction for the $4\varepsilon$ part (on the level of homology):

**Theorem (M.-W.-W. 2019)**

$\mathcal{K}hR_N$ controls a disk-like 4-category, determines $S_N(W^4; L)$ via the blob complex (Morrison–Walker 2010).

Rest of the talk: focus on degree zero blob homology $S^0_N(W^4; L)$. 
A skein module for 4-manifolds

In analogy to

$$\text{Sk}_N(M^3; P) := \frac{\mathbb{Z}[q^{\pm 1}] \langle \text{framed, oriented tangles in } (M^3, P) \rangle}{\langle \ker RT_N \text{ in } B^3 \hookrightarrow M^3 \rangle}$$

we would like to define $S^0_N(W^4; L)$ as:

$$\mathbb{Z} \langle \text{framed, oriented surfaces in } (W^4, L) \rangle \langle \ker [-]_N \text{ in } B^4 \hookrightarrow W^4 \rangle$$

Problem: Want $S_N(B^4; L) \cong S^0_N(B^4; L) \cong \text{KhR}_N(L)$, but this is not always spanned by images of cobordisms maps.

\[\Rightarrow\] consider decorated framed, oriented surfaces.
A lasagna filling of $W^4$ with a link $L \subset \partial W^4$ is the data of:

- $B_i^4$: finitely many disjoint 4-balls in $W^4$.
- $L_i$: input links in $\partial B_i^4$.
- $\Sigma$: f., o. surface in $(W^4 \setminus \sqcup i B_i^4; L \sqcup i L_i)$.
- $v_i \in \text{KhR}_N(\partial B_i^4, L_i)$.
Khovanov–Rozansky homology is an algebra for the lasagna operad

\[ \begin{align*}
\Sigma &
\rightarrow \left( \begin{array}{c}
\text{KhR}_N(S^3, L) \\
\uparrow \text{KhR}(D) \\
\bigotimes_i \text{KhR}_N(S_i, L_i)
\end{array} \right)
\end{align*} \]

A lasagna diagram \( D \) with \( L_i, L \) : input/output links
\( \Sigma \) : f., o. surface in \( (B^4 \setminus \bigcup_i B^4_i; L \sqcup_i L_i) \)

Note: A lasagna filling of \( (B^4, L) \) is a lasagna diagram \( D \) plus \( (v_1, \ldots, v_r) \).
\( \Rightarrow \) evaluates to \( \text{KhR}_N(D)(v_1 \otimes \cdots \otimes v_r) \in \text{KhR}(\partial B^4, L) \).
Definition of $S^0_N(W^4; L)$

**Definition**

We define the $H_2(W^4, L) \times \mathbb{Z}_q \times \mathbb{Z}_t$-graded abelian group

$$S^0_N(W^4; L) := \mathbb{Z} \langle \text{lasagna fillings of } (W^4, L) \rangle / \sim$$

where the ‘skein relations’ $\sim$ are generated by

$$v \sim \text{KhR}(D)(v_j \otimes \cdots \otimes v_j).$$
To finish, some examples

Example \((B^4)\)

\[
S_N(B^4; L) \cong S_N^0(B^4; L) \cong \text{KhR}(L)
\]
by construction.

Example \((B^3 \times S^1)\)

\[
S_2(B^3 \times S^1; L)
\]
is related to the Hochschild homology of Khovanov’s arc algebra and to Rozansky’s homology theory for links \(L\) in \(S^2 \times S^1\).

Theorem (Manolescu–Neithalath 2020)

If \(W^4\) is a 2-handle body with a single 0-handle, \(L \subset S^3\) the attaching link of the 2-handles, then

\[
S_N^0(W^4; \emptyset) \cong \text{KhR}_N(L)
\]
where \(\text{KhR}_N(L)\) depends on \(\text{KhR}_N\) of cables of \(L\).

E.g. \(\dim_q(S_N^0(S^2 \times D^2; \emptyset, \alpha)) = \prod_{k=1}^{N-1} \frac{1}{1-q^{2k}}, \) results for \(\mathbb{C}P^2\) and \(\overline{\mathbb{C}P^2}\).