INVERSE MEAN VALUE PROPERTIES (A SURVEY)

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Several mean value identities for harmonic and panharmonic functions are reviewed along with the corresponding inverse properties. The latter characterize balls, annuli, and strips analytically via these functions. Bibliography: 41 titles.

1 Introduction and Notation

Inverse mean value properties of harmonic functions ($C^2$-solutions of the Laplace equation; cf. [1] on the origin of the term harmonic) are well known for balls and spheres (cf. the survey article [2, Sections 7 and 8] respectively). In particular, a simple proof of the following general assertion was obtained in [3].

Theorem 1.1. Let $D$ be a domain (a connected open set) of finite (Lebesgue) measure in the Euclidean space $\mathbb{R}^m$, $m \geq 2$. Assume that there exists a point $P_0$ in $D$ such that for every function $h$ harmonic in $D$ and integrable over $D$ the volume mean of $h$ over $D$ equals $h(P_0)$. Then $D$ is an open ball (a disk if $m = 2$) centered at $P_0$.

At the same time, many interesting results in this area were not reviewed so far. They concern other characterizations of balls via harmonic functions as well as characterization of various other domains: strips, annuli etc. Moreover, it was, until recently, unknown whether an assertion similar to Theorem 1.1 is true if solutions of another partial differential equation are used instead of harmonic functions. Of course, the equality for the arithmetic mean over balls must be adjusted to these solutions. Only in 2021, a characterization of $m$-dimensional balls by solutions to the modified Helmholtz equation

$$\nabla^2 u - \mu^2 u = 0, \quad \mu \in \mathbb{R} \setminus \{0\}, \quad (1.1)$$

was obtained (cf. [4, 5]). Here and below, $\nabla = (\partial_1, \ldots, \partial_m)$, $\partial_i = \partial/\partial x_i$, denotes the gradient operator. In what follows, instead of the cumbersome solution to the modified Helmholtz equation the term panharmonic function is used. This convenient abbreviation was introduced in [8].

There are numerous and diverse results about inverse mean value properties that are not covered in [2], and our goal is to review them. The plan is as follows. Mean value equalities used
in the paper are described in Section 2 with proper references. In Section 3, these equalities are applied for characterizing balls via harmonic and panharmonic functions, whereas characterizations of annuli and strips by quadrature formulae involving mean values are considered in Section 4. It should be said that caloric functions (solutions of the heat equation) are beyond our scope because their mean value properties have rather specific character different from that considered in Section 2 (cf., for example, [6, 7]).

Let us introduce some notation used below. For a point \( x = (x_1, \ldots, x_m) \in \mathbb{R}^m, m \geq 2 \), we denote by \( B_r(x) = \{ y \in \mathbb{R}^m : |y - x| < r \} \) the open ball of radius \( r \) centered at \( x \) (just \( B_r \), if centered at the origin). The ball is called admissible with respect to a domain \( D \subset \mathbb{R}^m \) provided \( B_r(x) \subset D \), and \( \partial B_r(x) \) is called admissible sphere in this case. If \( D \) has finite Lebesgue measure and a function \( f \) is integrable over \( D \), then

\[
M^\circ (f, D) = \frac{1}{|D|} \int_D f(x) \, d x
\]

is its volume mean value over \( D \). Here and below, \( |D| \) is the domain volume (the area if \( D \subset \mathbb{R}^2 \)), and the volume of \( B_r \) is \( |B_r| = \omega_m r^m \), where \( \omega_m = 2\pi^{m/2}/[m\Gamma(m/2)] \) is the volume of the unit ball; as usual \( \Gamma \) denotes the Gamma function. For \( u \in C^0(\partial D) \), the mean value

\[
M^\circ (f, \partial D) = \frac{1}{|\partial D|} \int_{\partial D} f(y) \, d S_y
\]

over a sufficiently smooth \( \partial D \) is another useful notion; here \( |\partial D| \) is the surface area of the domain boundary (\( |\partial B_r| = m \omega_m r^{m-1} \)), and \( d S \) is the surface area measure. The exterior unit normal on \( \partial D \) is denoted by \( n \).

2 Mean Value Equalities

2.1. Spheres and balls. Studies of mean value properties of harmonic functions date back to the Gauss theorem of the arithmetic mean over a sphere (cf. [9, Article 20]). Nowadays, its standard formulation is as follows.

**Theorem 2.1.** Let \( D \) be a domain in \( \mathbb{R}^m, m \geq 2 \). Then \( u \in C^2(D) \) is harmonic in \( D \) if and only if for every \( x \in D \)

\[
M^\circ (u, \partial B_r(x)) = u(x)
\]

for each admissible sphere \( \partial B_r(x) \).

In 1906, it was proved [10] that (2.1) implies that \( u \) is harmonic. Integrating (2.1) with respect to \( r \) over \((0, R)\), one obtains the following assertion.

**Corollary 2.1.** Let \( u \in C^2(D) \) be harmonic in a domain \( D \subset \mathbb{R}^m, m \geq 2 \). Then

\[
M^\bullet (u, B_R(x)) = u(x)
\]

for every \( x \in D \) and each admissible ball \( B_R(x) \). The converse is also true. Moreover,

\[
M^\bullet (u, B_R(x)) = M^\circ (u, \partial B_R(x))
\]

for every such ball.
Let us turn to analogous equalities for panharmonic functions. They involve the following coefficients:

\[
a^{\circ}(\mu r) = \Gamma \left( \frac{m}{2} \right) \frac{I_{(m-2)/2}(\mu r)}{(\mu r/2)^{(m-2)/2}},
\]

\[
a^{\bullet}(\mu r) = \Gamma \left( \frac{m}{2} + 1 \right) \frac{I_{m/2}(\mu r)}{(\mu r/2)^{m/2}},
\]

where \( I_\nu \) denotes the modified Bessel function of order \( \nu \).

**Theorem 2.2.** Let \( D \) be a domain in \( \mathbb{R}^m, m \geq 2 \). Then \( u \in C^2(D) \) is panharmonic in \( D \) if and only if for every \( x \in D \)

\[
M^\circ(u, \partial B_r(x)) = a^{\circ}(\mu r) u(x)
\]

for each admissible sphere \( \partial B_r(x) \).

Formula (2.5) has particularly simple form for \( m = 3 \) because \( a^{\circ}(\mu r) = \sinh \mu r / (\mu r) \), which was proved in [11] as early as in 1896. Duffin [8] independently rediscovered this proof, but for the two-dimensional case with \( a^{\circ}(\mu r) = I_0(\mu r) \). Finally, the author [12] derived (2.5) for any \( m \geq 2 \) and announced [5] the converse assertion; its proof is given below.

**Sketch of the proof of Theorem 2.2.** First, we outline the derivation of (2.5) for a panharmonic \( u \). It based on the Euler–Poisson–Darboux equation

\[
M^\circ_{rr} + (m - 1) r^{-1} M^\circ_r = \nabla_x^2 M^\circ
\]

satisfied by \( M^\circ(u, \partial B_r(x)) \) for \( r > 0 \) (cf. [14, Chapter IV]).

By the relations (cf. [15])

\[
zI_{\nu+1}(z) + 2\nu I_\nu(z) - zI_{\nu-1}(z) = 0,
\]

\[
[z^{-\nu} I_\nu(z)]' = z^{-\nu} I_{\nu+1}(z),
\]

it is easy to show that a unique solution to the Cauchy problem

\[
a_{rr} + (m - 1) r^{-1} a_r - \mu^2 a = 0,
\]

\[
a(0) = 1, \quad a_r(0) = 0,
\]

is \( a(r) = a^{\circ}(\mu r) \). Combining (1.1) and (2.6), we see that

\[
a^{\circ}(\mu r) u(x) - M^\circ(u, \partial B_r(x))
\]

satisfies the equation in (2.8) with zero initial conditions. Hence it vanishes identically, thus yielding (2.5).

Prior to proving the converse assertion, let us consider some consequences of (2.5). Since \( a^{\circ}(0) = 1 \), the first initial condition in (2.8) yields that (2.5) turns into (2.1) as \( \mu \to 0 \). To determine how the mean value of a panharmonic function \( u \) depends on radii of admissible spheres centered at an arbitrary point \( x \in D \), we note that the behavior of \( M^\circ(u, \partial B_r(x)) \) is the same as that of \( a^{\circ}(\mu r) \). The latter is continuous and increases monotonically by the second relation in (2.7). Hence

\[
|M^\circ(u, \partial B_r(x))| > |u(x)|, \quad r > 0.
\]

(2.9)
Moreover, the Poisson integral for $I_{\nu}$ (cf. [16]) implies

$$a^\circ(\mu r) = \frac{2\Gamma(m/2)}{\sqrt{\pi}\Gamma((m - 1)/2)} \int_0^1 (1 - s^2)^{(m-3)/2} \cosh(\mu rs) \, ds.$$  \hfill (2.10)

Hence $M^\circ(u, \partial B_r(x))$ is a convex function of $r$.

Integrating the equality (2.5) with respect to $r$ over $(0, R)$ and using the formula

$$\int_0^x x^{1+\nu} I_{\nu}(x) \, dx = x^{1+\nu} I_{\nu+1}(x), \quad \text{Re} \, \nu > -1$$  \hfill (2.11)

(cf. [17, Subsection 1.11.1.5]) with $\nu = (m - 2)/2$, one obtains the following assertion.

**Corollary 2.2.** Let $u \in C^2(D)$ be panharmonic in a domain $D \subset \mathbb{R}^m$, $m \geq 2$. Then

$$M^\bullet(u, B_R(x)) = a^\bullet(\mu R) u(x),$$  \hfill (2.12)

$$a^\circ(\mu R) M^\bullet(u, B_R(x)) = a^\bullet(\mu R) M^\circ(u, \partial B_R(x))$$  \hfill (2.13)

for every $x \in D$ and each admissible ball $B_R(x)$. The converse is also true.

As $\mu \to 0$, the volume mean formula (2.12) for a panharmonic function turns into (2.2) for a harmonic one. Moreover, the dependence on $r$ for the volume mean is the same as for the spherical one. Finally, the equality (2.13) (a consequence of (2.12) and (2.5) with $r = R$) implies

$$M^\bullet(u, B_R(x)) < M^\circ(u, \partial B_R(x)).$$

Indeed,

$$\frac{M^\bullet(u, B_R(x))}{M^\circ(u, \partial B_R(x))} = \frac{a^\bullet(\mu R)}{a^\circ(\mu R)} < 1,$$

which immediately follows from the definition of $a^\bullet$ and $a^\circ$ and the first relation in (2.7).

**Sketch of the proof of Theorem 2.2 (continued).** To show that $u$ is panharmonic in $D$ when (2.5) is valid, we note that a consequence of (2.5), namely, (2.12) (with $R$ changed to $r$) is equivalent to the equality

$$\left(\frac{2\pi r}{\mu}\right)^{m/2} I_{m/2}(\mu r) u(x) = \int_{|y|<r} u(x + y) \, dy.$$  \hfill (2.14)

Applying the Laplacian to the integral on the right-hand side, we find

$$\int_{|y|<r} \nabla^2_x u(x + y) \, dy = \int_{|y|=r} \nabla_x u(x + y) \cdot \frac{y}{r} \, dS_y.$$

Here, the equality is a consequence of the first Green formula. Changing the variables, we can write it as

$$r^{m-1} \frac{\partial}{\partial r} \int_{|\theta|=1} u(x + r\theta) \, dS_{\theta}^{m-1} = |S^{m-1}| r^{m-1} \frac{\partial}{\partial r} M^\circ(u, \partial B_r(x)).$$
By (2.5) and the second relation in (2.7), we have

\[
\frac{\partial}{\partial r} M^\circ(u, \partial B_r(x)) = \frac{\mu I_{m/2}(\mu r)}{(\mu r/2)^{(m-2)/2}} u(x).
\]

Combining the above considerations and (2.14), we conclude that

\[
\int_{|y|<r} [\nabla^2 u - \mu^2 u](x+y) \, dy = 0
\]

for every \(x \in D\) and all \(r\) such that \(B_r(x)\) is admissible. Hence there exists \(y(r,x) \in B_r(x)\) such that \([\nabla^2 u - \mu^2 u](y(r,x)) = 0\). Since \(y(r,x) \to x\) as \(r \to 0\), it follows by continuity that \(u\) is panharmonic in \(D\). \(\square\)

2.2. Annuli. We assume that \(m \geq 3\) and \(r_2 > r_1 \geq 0\). Following [18], we introduce

\[
r_* = \left(\frac{2}{m} \frac{r_2^m - r_1^m}{r_2^m - r_1^m} \right)^{1/(m-2)}.
\]

(2.15)

The convexity of \(t \mapsto t^{m-2}\) implies \(r_* \in (r_1, r_2)\). Therefore, the domain

\[
A(r_1, r_2) = \{x \in \mathbb{R}^m : r_1 < |x| < r_2\},
\]

is called an \(r_*\)-annulus (below, we write \(A\) for brevity). For \(m = 2\), such an annulus has

\[
r_* = \exp \left( \frac{r_2^2 \log r_2 - r_1^2 \log r_1}{r_2 - r_1} - \frac{1}{2} \right)
\]

(2.16)

(cf. [19]). Now, we are in a position to formulate the following assertion.

**Theorem 2.3.** Let \(A \subset \mathbb{R}^m, m \geq 2\), be an \(r_*\)-annulus. If \(u \in C^2(A)\) is harmonic and integrable over \(A\), then

\[
M^\bullet(u, A) = M^\circ(u, \partial B_{r_*}).
\]

(2.17)

**Proof for \(m \geq 3\) (cf. [18]).** For arbitrary \(r'_1\) and \(r'_2\) such that \(r_1 < r'_1 < r'_2 < r_2\) we have

\[
\left[ \int_{\partial B_{r'_2}} - \int_{\partial B_{r'_1}} \right] \frac{\partial u}{\partial n} \, dS = 0
\]

because \(u\) is harmonic. This implies

\[
r^{m-1} \frac{d}{dr} M^\circ(u, \partial B_r) = \text{constant}, \quad r \in (r_1, r_2),
\]

and so \(M^\circ(u, \partial B_r) = c_1 r^{2-m} + c_2\), where \(c_1\) and \(c_2\) are constants. Hence

\[
M^\bullet(u, A) = \frac{m}{r_2^m - r_1^m} \int_{r_1}^{r_2} M^\circ(u, \partial B_r) \, r^{m-1} \, dr = c_1 \frac{m}{2} \frac{r_2^m - r_1^m}{r_2^m - r_1^m} + c_2,
\]

from which (2.17) follows in view of (2.15). \(\square\)

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2.3. Strips. The following mean value property of harmonic functions on an infinite open
strip \( S(a, b) = (a, b) \times \mathbb{R}^m \subset \mathbb{R}^{m+1} \) was obtained in [20] (cf. also [21]). Here, \((a, b)\) is a bounded
open interval on, say, the \( t\)-axis.

**Theorem 2.4.** Let \( m \geq 2 \). If a real-valued function \( u \) is harmonic and integrable on \( S(a, b) \), then
\[
\int_{S(a,b)} u(t, x) \, dt \, dx = (b - a) \int_{\mathbb{R}^m} u((a + b)/2, x) \, dx.
\]

**Sketch of the proof.** Hyperplane means by the Fubini theorem and the fact that \(|u|\) is
subharmonic were investigated in [22] (cf. also the reference therein). According to these
results, the integral
\[
\int_{\mathbb{R}^m} u(t, x) \, dx, \quad t \in (a, b),
\]
exists and is a first degree polynomial in \( t \), which implies (2.18).

This result was extended to an \( m+l\)-dimensional, \( m, l \geq 2 \), bi-infinite cylinder \( C_r = B_r \times \mathbb{R}_l \)
where \( B_r \) is the \( m\)-dimensional open ball centered at the origin of \( \mathbb{R}^m \); namely, the following
assertion was proved in [23].

**Theorem 2.5.** Let \( C_r \subset \mathbb{R}^{m+l}_{x,y} \) be a bi-infinite open cylinder. If a real-valued \( u \) is harmonic
and integrable on \( C_r \), then
\[
\int_{C_r} u(x, y) \, dx \, dy = |B_r| \int_{\mathbb{R}^l} u(0, y) \, dy.
\]

It is interesting whether the results of Sections 2.2 and 2.3 have analogues for panharmonic
functions.

3 Characterization of Balls via Mean Value Properties

3.1. Harmonic functions. Along with the Kuran theorem [3], there are other characteriza-
tions of balls based on mean value properties. The best known is the following assertion proved
in [24].

**Theorem 3.1.** Let \( D \subset \mathbb{R}^m \), \( m \geq 3 \), be a bounded open set such that \( D = \text{int} \, \overline{D} \) and \( \partial D \)
has zero volume. For some fixed \( x_0 \in D \) the identity
\[
|x_0 - y|^{2-m} = \frac{1}{|D|} \int_D |x - y|^{2-m} \, dx
\]
is valid for every \( y \in \mathbb{R}^m \setminus D \) if and only if \( D = B_r(x_0) \), where \( r \) is such that \( |D| = |B_r(x_0)| \).

The case \( m = 3 \) was considered in [24], but it is straightforward to extend the result of [24]
to higher dimensions. In [2, p. 337], the note [24] was just mentioned as an application of the
Kuran theorem [3] to answering the question: Must a bounded homogeneous solid in \( \mathbb{R}^3 \), "which
gravitationally attracts each point outside it as if all its mass were concentrated at a single point,
[...] be a ball?” (cf. [24, p.331]). Further discussion of this and related results can be found in [25] (cf. also the references therein).

Furthermore, a two-dimensional version of the following general assertion was formulated in [2, Section 8].

**Theorem 3.2.** Let \( D \subset \mathbb{R}^m, m \geq 2 \), be bounded \( C^2 \)-domain. If

\[
\frac{1}{|D|} \int_D h(x) \, dx = \frac{1}{|\partial D|} \int_{\partial D} h(x) \, dS_x
\]

(3.1)

for every harmonic in \( D \) function \( h \in C^1(\overline{D}) \), then \( D \) is a ball.

The equality (3.1) closely resembles the relation between the mean values over a ball and its boundary (cf. (2.3)). The theorem cited in [2] was published [26] five years earlier than Theorem 3.2 was proved in [27] in 1986. However, the results of [26] has two drawbacks: it is essentially two-dimensional and the superfluous assumption that \( D \) is convex is imposed. The latter condition arises in an auxiliary assertion similar to the following theorem (cf. [28, 29]) on which the proof in [27] is based.

**Theorem 3.3** (cf. [28, 29]). Let \( D \subset \mathbb{R}^m \) be a bounded \( C^2 \)-domain, and let

\[
\nabla^2 u = -1 \quad \text{in } D, \\
u = 0, \quad \partial u / \partial n = -c \quad \text{on } \partial D,
\]

(3.2)

for some \( u \in C^2(D) \cap C^1(\overline{D}) \) and constant \( c \). Then \( D \) is a ball and \( u = (b^2 - r^2)/(2m) \), where \( b \) and \( r \) denote the radius of the ball and the distance from its center respectively.

Now we are in a position to present the proof [27] of Theorem 3.2.

**Proof of Theorem 3.2.** It is clear that there exists \( u \in C^2(D) \cap C^1(\overline{D}) \) satisfying the first and second relations in (3.2). Then the first relation yields

\[
\int_D h \, dx = -\int_D h \nabla^2 u \, dx = -\int_D u \nabla^2 h \, dx + \int_{\partial D} \left[ u \frac{\partial h}{\partial n} - h \frac{\partial u}{\partial n} \right] \, dS_x = -\int_{\partial D} h \frac{\partial u}{\partial n} \, dS_x,
\]

where the last equality follows by harmonicity of \( h \) and the second relation in (3.2). Combining this and (3.1), we find

\[
\int_{\partial D} h \frac{\partial u}{\partial n} + \left| \frac{D}{\partial D} \right| dS_x = 0.
\]

Moreover, there exists a harmonic function \( h \) in \( D \) satisfying the condition

\[
h = \frac{\partial u}{\partial n} + \frac{|D|}{|\partial D|} \quad \text{on } \partial D.
\]

Substituting it into the last integral, we see that the third condition in (3.2) is also valid for \( u \) with \( c = |D|/|\partial D| \). Then \( D \) is a ball in view of Theorem 3.3. \(\square\)
An alternative proof of a slightly modified version of Theorem 3.2 was discovered in [30]. Namely, instead of (3.2), it is required that there exists a constant $c$ such that

$$\int_D h(x) \, dx = c \int_{\partial D} h(x) \, dS_x$$

(3.3)

for every harmonic function $h$ in $D$. The proof is based on considerations of the note [29] involving properties of $|\nabla u|^2 + 2u/m$, where $u$ solves (3.2).

One more application of the integral equality (3.3) was given in [31], where the following approach to characterization of balls was proposed. Let $V : \mathbb{R}^m \to \mathbb{R}^m$ be a given vector field with $C^2$-components. Then the problem

$$\nabla^2 h = 0 \text{ in } D,$$

$$h = -(V \cdot n) \frac{\partial h}{\partial n} \text{ on } \partial D$$

(3.4)

has a solution because $h$ is the so-called domain derivative in the direction of $V$ of a unique solution to the Saint-Venant problem in $D$. The latter includes the first two relations in (3.2). A detailed treatment of the concept of differentiation with respect to the domain is given in [32].

Let $\mathcal{V}$ denote the set of harmonic functions in $D$, each being a solution to the problem (3.4) for some possible vector field $V$. The result established in [31] is as follows.

**Theorem 3.4.** Let $D \subseteq \mathbb{R}^m$, $m \geq 2$, be a bounded $C^2$-domain. Then $D$ is a ball if and only if the equality (3.3) is valid for every function $h \in C^2(D) \cap C^1(\overline{D})$ belonging to $\mathcal{V}$. Moreover, $-m[\partial h/\partial n]_{\partial D}$ is constant and equals to the ball radius.

The question whether $\mathcal{V}$ is a proper subset of the whole set of functions harmonic in $D$, in which case Theorem 3.4 improves the result of [30], was not considered in [31].

### 3.2. Panharmonic functions.

A characterization of balls via panharmonic functions was recently obtained by the author [4, 5]. Before giving the precise formulation of the result, we define a dilated copy of a bounded domain $D$ as follows:

$$D_r = D \bigcup \left[ \bigcup_{x \in \partial D} B_r(x) \right].$$

Thus, the distance from $\partial D_r$ to $D$ is equal to $r$. The following general assertion was proved in [5].

**Theorem 3.5.** Let $D$ be a bounded domain in $\mathbb{R}^m$, $m \geq 2$, and let $r$ be a positive number such that $|B_r| \ll |D|$. We assume that there exists a point $x_0 \in D$ such that for some $\mu > 0$ the mean value equality

$$u(x_0) a^*(\mu r) = M^*(u, D)$$

is valid for every positive function $u$ satisfying Equation (1.1) in $D_r$. If also $|D| = |B_r|$ provided $B_r(x_0) \setminus \overline{D} \neq \emptyset$, then $D = B_r(x_0)$.

Prior to proving this theorem, let us consider some properties of the function

$$U(x) = a^*(\mu |x|), \quad x \in \mathbb{R}^m,$$

(3.5)
The second relation in (2.7) shows that this spherically symmetric function monotonically increases from unity to infinity as \( |x| \) goes from zero to infinity. A consequence of the representation (2.10), which is easy to differentiate, is that \( U \) solves Equation (1.1) in \( \mathbb{R}^m \). Since both relations in (2.4) are similar, the Poisson integral allows us to compare these functions, and we immediately obtain the inequality

\[
[U(x)]_{|x|=r} > a^\bullet(\mu r).
\]  

**Proof of Theorem 3.5.** Without loss of generality we assume that the domain \( D \) is located so that \( x_0 \) coincides with the origin. Let us show that the assumption that \( D \neq B_r(0) \) leads to a contradiction.

It is clear that either \( B_r(0) \subset D \) or \( B_r(0) \setminus D \neq \emptyset \) (the equality \( |B_r| = |D| \) is assumed in the latter case), and we treat these two cases separately. Let us consider the second case first, for which purpose we introduce the bounded open sets \( G_i = D \setminus B_r(0) \) and \( G_e = B_r(0) \setminus D \), whose nonzero volumes are equal because \( |D| = |B_r| \). The volume mean equality for \( U \) over \( D \) can be written as follows:

\[
|D| a^\bullet(\mu r) = \int_D U(y) \, dy;
\]  

here, the condition \( U(0) = 1 \) is taken into account. Since formula (2.12) is valid for \( U \) over \( B_r(0) \) (\( R = r \) in (2.12) in this case), we write it in the same way:

\[
|B_r| a^\bullet(\mu r) = \int_{B_r} U(y) \, dy.
\]  

Subtracting (3.8) from (3.7), we get

\[
0 = \int_{G_i} U(y) \, dy - \int_{G_e} U(y) \, dy > 0.
\]

Indeed, the difference is positive since \( U(y) \) (positive and monotonically increasing with \( |y| \)) is greater than \( [U(y)]_{|y|=r} \) in \( G_i \) and less than \( [U(y)]_{|y|=r} \) in \( G_e \), whereas \( |G_i| = |G_e| \). This contradiction proves the result in this case.

In the case \( B_r(0) \subset D \), a contradiction must be deduced when \( B_r(0) \neq D \), in which case \( |G_i| = |D| - |B_r| > 0 \). Now, subtracting (3.8) from (3.7), we find

\[
(|D| - |B_r|) a^\bullet(\mu r) = \int_{G_i} U(y) \, dy > |G_i| [U(y)]_{|y|=r},
\]

where the last inequality is again a consequence of positivity and monotonicity of \( U(y) \). This yields \( a^\bullet(\mu r) > [U(y)]_{|y|=r} \), which contradicts (3.6). The proof is complete.

Let us comment on Theorem 3.5. First, the domain \( D \) is supposed to be bounded because it is easy to construct an unbounded domain of finite volume in which \( U \) is not integrable. Thus, the boundedness of \( D \) allows us to avoid formulating rather complicated restrictions on the domain.
Second, one obtains the Laplace equation from (1.1) in the limit \( \mu \to 0 \), and the assumption about \( r \) becomes superfluous in this case. Therefore, Theorem 3.5 turns into an improved version of the Kuran theorem [3] because only positive harmonic functions are involved (cf. also [18]).

Furthermore, the integral
\[
\int_D u(y) \, dy
\]
can be replaced by the flux
\[
\int_{\partial D} \frac{\partial u}{\partial n_y} \, dS_y
\]
in the formulation of Theorem 3.5 provided that \( \partial D \) is sufficiently smooth. Here, \( n \) is the unit outward normal. Indeed, we have
\[
\int_D u(y) \, dy = \mu^{-2} \int_D \nabla^2 u(y) \, dy = \mu^{-2} \int_{\partial D} \frac{\partial u}{\partial n_y} \, dS_y.
\]
These relations are used in [5] (cf. the proof of Theorem 9, which characterizes solutions of equation (1.1) in terms of the mean flux through spheres). This suggests the following.

**Conjecture 3.1.** Let \( D \subset \mathbb{R}^m, m \geq 2 \), be a bounded domain with sufficiently smooth boundary, and let \( r > 0 \) be such that \( |B_r| = |D| \). If there exists \( x_0 \in D \) such that for some \( \mu > 0 \)
\[
\frac{2 \mu}{m} a^*(\mu r) u(x_0) = \frac{1}{|\partial D|} \int_{\partial D} \frac{\partial u}{\partial n_y} \, dS_y
\]
for every panharmonic function \( u \in C^1(\overline{D}) \), then \( D = B_r(x_0) \).

### 4 Characterization of Annuli and Strips via Mean Values

**4.1. Annuli.** An attempt to prove an inverse of Theorem 2.3 was made [33] in 1981. However, the statement and its proof are both erroneous in [33] (cf. the comments in [18, pp. 142 and 145]). The correct formulation (cf. the recent paper [19]) involving the radius \( r_* \) given by (2.15) for \( m \geq 3 \) and by (2.16) for \( m = 2 \), is as follows.

**Theorem 4.1.** Let \( D \subset \mathbb{R}^m, m \geq 2 \), be an open set such that \( |D| < \infty \) and \( \partial B_{r_*} \subset D \). If any function \( u \) harmonic in \( D \) and integrable over \( D \) satisfies the identity
\[
M^*(u, D) = M^o(u, \partial B_{r_*}),
\]
then either \( D \) is the \( r_* \)-annulus with \( 0 \leq r_1 < r_2 \) or \( D \) is an open ball centered at the origin.

A weaker result was obtained [18] in 1989; namely: the identity (4.1) implies that \( \overline{D} \) is either the closed \( r_* \)-annulus or the closed ball centered at the origin. In Problem 3.35 of [34], it is asked whether a similar assertion is true when only the fundamental solution of the Laplace equation is involved in (4.1). An improved version of that problem involving the following solution
\[
g(x, y) = \varphi_m(|x - y|),
\]
where \( \varphi_m(t) = t^{2-m} \) for \( m \geq 3 \) and \( \varphi_2(t) = -\log t \), was formulated in [19] as follows.
Theorem 4.2. Let $D \subset \mathbb{R}^m$, $m \geq 3$, be an open set such that $|D| < \infty$ (let $D$ be bounded in two dimensions) and $\partial B_{r_*} \subset D$. If $g(x,y)$ satisfies the identity (4.1) for any $y \in \mathbb{R}^m \setminus D$, then either $D$ is the $r_*$-annulus with $0 \leq r_1 < r_2$ or $D = B \setminus T$, where $B$ is a ball centered at the origin and $T \subset \partial B_{r_0}$ for some $r_0 < r_*$ ($T$ may be empty).

The proof in [19] of Theorem 4.2 is based on the main result of [35] dealing with the mean value property of $U_E(y) = \int_E g(x,y) \, dx$, where $E \subset \mathbb{R}^m$ is Lebesgue measurable and $|E| \in (0, \infty)$.

Theorem 4.3. Let $B \subset \mathbb{R}^m$, $m \geq 2$, be the open ball centered at the origin and such that $|B| = |E| \in (0, \infty)$. If for any compact $F,G \subset \mathbb{R}^m \setminus E$ the identity

$$|E|^{-1} \int_E [U_F(y) - U_G(y)] \, dy = U_F(0) - U_G(0)$$

(4.2)

is valid provided $U_F - U_G$ is bounded, then $|B \setminus E| = 0$.

In particular, the proof of Theorem 4.3 implies that the requirement $U_E(y) = |E| g(0,y)$ almost everywhere outside $E$ may be used instead of the identity (4.2) provided that $m \geq 3$. In the case $m = 2$, the analogous requirement is slightly more complicated.

The derivation [19] of Theorem 4.2 is highly technical and rather long. Subsequently, to prove Theorem 4.1, it remains to show that the set $T$ permissible by Theorem 4.2 is empty. This is achieved by choosing some particular harmonic function and demonstrating that it satisfies an inequality which contradicts (4.1).

An alternative approach to characterizations of two-dimensional annuli was developed in [36]. In some sense, it is similar to the result of [27] for balls (cf. Theorems 3.2 and 3.3). It was assumed that $D \subset \mathbb{R}^2$ is a bounded, finitely connected domain, whose boundary $\partial D$ consists of two or more pairwise disjoint closed analytic curves. Let $\Gamma_0$ denote the curve separating $D$ from infinity, and let $\Gamma_1 = \partial D \setminus \Gamma_0$. The simply connected domain within $\Gamma_0$ is denoted by $D_0$ and $D_1 = D_0 \setminus \overline{D}$. Let $c_0 = |D_0|/|\Gamma_0|$ and $c_1 = -|D_1|/|\Gamma_1|$. Now, we are in a position to formulate the result obtained in [36].

Theorem 4.4. Let $D \subset \mathbb{R}^2$ be a bounded domain, and let $\partial D$ consist of finitely many (two or more) pairwise disjoint closed analytic curves. Then the following assertions are equivalent.

(i) For some $u \in C^2(D) \cap C^1(\overline{D})$ the relations

$$\nabla^2 u = -1 \quad \text{in } D,$$

$$u = -c_0^2, \quad \frac{\partial u}{\partial n} = -c_0 \quad \text{on } \Gamma_0,$$

$$u = -c_1^2, \quad \frac{\partial u}{\partial n} = -c_1 \quad \text{on } \Gamma_1,$$

are fulfilled with $c_0$ and $c_1$ defined above.

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(ii) The quadrature identity
\[
\int_D h \, d\mathbf{x} = c_0 \int_{r_0} h \, dS + c_1 \int_{r_1} h \, dS - c_0^2 \int_{r_0} \frac{\partial h}{\partial n} \, dS - c_1^2 \int_{r_1} \frac{\partial h}{\partial n} \, dS
\]
is valid for every harmonic function \( h \in C^1(D) \) in \( D \).

(iii) \( D \) is an annulus centered at the origin with the smaller radius \(-2c_1\) and the larger radius \(2c_0\).

As in the case of the quadrature identity (3.3) yielding that \( D \) is a ball, the proof is based on the consideration of [29] involving properties of \(|\nabla u|^2 + 2u/m\), where \( u \) satisfies the first relation in (3.2).

4.2. Strips. Some early partial results concerning characterization of strips by harmonic functions were surveyed in 1992 (cf. [21, 37]). The following general converse of Theorem 2.4 was obtained in [38] next year.

**Theorem 4.5.** Let \( D \subset \mathbb{R}^{m+1} \), \( m \geq 2 \), be an open subset of \( S(-a,a) \) for some \( a \in (0, \infty) \), and let \( \{0\} \times \mathbb{R}^m \) be a proper subset of \( D \). If the identity
\[
\int_D h(t,x) \, dt \, dx = 2 \int_{\mathbb{R}^m} h(0,x) \, dx
\]
is valid for every positive function \( h \) that is harmonic in \( D \) and integrable over \( D \), then \( D = S(-1,1) \).

The requirement that \( D \subset S(-a,a) \) is essential because (4.3) is fulfilled vacuously when \( D = S(\mathbb{R},+\infty) \) with \( c \in [-\infty,0) \).

Theorem 4.5 improves the result obtained in [39] under the following additional assumptions:
\[
D \subset S(-3,3), \quad \partial S(-1,1) \setminus D \neq \emptyset,
\]
(4.3) is fulfilled for all harmonic in \( D \) functions integrable over \( D \).

Without these assumptions the proof in [38] of Theorem 4.5 is completely different from the reasoning in [39] based on the refined technique applied earlier in [40].

Let us outline their proof for \( m \geq 2 \). It involves an investigation of the Green potential on the half-space \( S(b, +\infty) \), where \( b = -a - 1 \). The Green kernel of \( S(b, +\infty) \) is
\[
G(t,x;\tau,\xi) = |(t,x) - (\tau,\xi)|^{1-m} - |(t,x) - (\tau,\xi)^*|^{1-m}, \quad (t,x),(\tau,\xi) \in S(b, +\infty),
\]
where \(|(t,x) - (\tau,\xi)|^2 = |t - \tau|^2 + |x - \xi|^2 \) and \((\tau,\xi)^* \) denotes the mirror-image of \((\tau,\xi)\) with respect to the hyperplane \( \{b\} \times \mathbb{R}^m \), i.e., \((\tau,\xi)^* = (2b - \tau,\xi_1,\ldots,\xi_m) \). The Green potential is defined by
\[
U(t,x) = \int_D G(t,x;\tau,\xi) \, d\tau d\xi, \quad (t,x) \in S(b, +\infty).
\]
The properties of \( U \) used in the proof are as follows.

**Lemma 4.1** (cf. [41, Subsection 1.1.7]). The following assertions hold.
\[(i) \quad U \in C^2(D) \cap C^1(S(b, +\infty)).\]

(ii) \(|\nabla U|\) is bounded on \(D\).

(iii) \(U_{tt} + \nabla^2 U = (m^2 - 1) \omega_{m+1}\) in \(D\).

The proof in [38] begins by demonstrating that \(D\) is connected. For this purpose the function \(G(t, x; a, 0)(\chi_{D_c} + k\chi_{D \setminus D_c})\) serves with \(k = 1\) and \(k = 2\), where \(\chi_E\) is the characteristic function of a set \(E\) and \(D_c\) is a connected component of \(D\). It is clear that these two functions cannot satisfy (4.3) simultaneously. Extensively using Lemma 4.1, at the next step, we can show that there exists a real number \(g\) such that \(D \subset S(g - 2, g)\) and \(|S(g - 2, g) \setminus D| = 0\). Finally, the following expression obtained in [22] for \(
\int_{\mathbb{R}^m} G(t, x; c, \xi) \, d\xi, \quad c \in (b, +\infty),
\)
is applied to demonstrate that \(g = 1\) and the inclusion is, in fact, an equality. The same proof remains valid for \(m = 1\) provided that \(|(t, x) - (\tau, \xi)|^{1-m}\) is changed to \(-\log |(t, x) - (\tau, \xi)|\).

Now, we turn to characterization of strips, which is similar to Theorem 3.1 for balls, but involves the Green kernel (4.4) instead of the fundamental solution of the Laplace equation.

**Theorem 4.6.** Let \(D\) be an open set such that \(\{(0, \ldots, 0)\} \times \mathbb{R}^l \subset D \subset S(-a, a)\) for some \(a \in (0, \infty)\). If for \(G\) defined by (4.4) the identity

\[
\int_D G(t, x; \tau, \xi) \, d\tau d\xi = 2 \int_{\mathbb{R}^m} G(t, x; 0, \xi) \, d\xi,
\]
is valid for every \((t, x) \in [(a - 1, \infty) \times \mathbb{R}^m] \setminus D\), then \(D = S(-1, 1)\).

The proof in [7] of Theorem 4.6 repeats, to a large extent, the proof of Theorem 4.5 in [38].

It occurs that a bi-infinite cylinder is also characterized by harmonic quadrature; namely, the following converse of Theorem 2.5 was obtained in [23].

**Theorem 4.7.** Let \(D \subset \mathbb{R}^{m+l}\) be an open subset of some (arbitrarily large) cylinder such that \(\{(0, \ldots, 0)\} \times \mathbb{R}^l \subset D\) and \(\text{int} \, D = D\). If for every positive function \(h\) that is harmonic and integrable on \(D\)

\[
\int_D h(x, y) \, dx \, dy = \int_{\mathbb{R}^l} h(0, y) \, dy,
\]

then \(D = B_r \times \mathbb{R}^l\).

The proof in [23] of Theorem 4.7 is based on the relationship between the Green functions of \(B_r \times \mathbb{R}^l\) and \(B_r\) (its tedious derivation occupies four pages) and the following result of independent interest.

**Proposition 4.1.** Let \(D \subset \mathbb{R}^m, \ m \geq 2, \) be an unbounded open set such that

\[
|D \cap B_r| = o(r^m) \quad r \to \infty. \quad (4.5)
\]

If \(h \in C(\overline{D}) \cap L^\infty(D)\) is harmonic in \(D\) and vanishes on \(\partial D\), then \(h\) vanishes on \(\overline{D}\).

To prove Proposition 4.1, the mean value of the subharmonic function \(|h|\) was estimated by using the property (4.5).
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