PERSISTENCE OF HOMOLOGY OVER COMMUTATIVE NOETHERIAN RINGS

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Abstract. We describe new classes of noetherian local rings R whose finitely generated modules M have the property that Tor\(^R_i(M, M) = 0\) for \(i \gg 0\) implies that M has finite projective dimension, or Ext\(^R_i(M, M) = 0\) for \(i \gg 0\) implies that M has finite projective dimension or finite injective dimension.

Introduction

This work concerns homological dimensions of modules over commutative noetherian rings. Over such a ring R, a finite (meaning, finitely generated) R-module M has finite projective dimension if Tor\(^R_i(M, N) = 0\) for \(i \gg 0\) for each finite R-module N.

We say that R is homologically persistent, or Tor-persistent, if every finite R-module M for which Tor\(^R_i(M, M) = 0\) for \(i \gg 0\) satisfies proj dim\(_R M < \infty\).

Every regular ring is Tor-persistent for then proj dim\(_R M is finite by fiat. Results of Avramov and Buchweitz [4] imply that the same holds for locally complete intersection rings. The work reported in our paper was sparked by that of Segal [44, pp. 1266] who verifies it for certain classes of local rings, and says, in paraphrase: “The author does not know any examples of rings that are not Tor-persistent.” Neither do we, so we ask:

Question. Is every ring Tor-persistent?

The purpose of this paper is to present some new non-trivial examples of Tor-persistent rings. It is not hard to prove that this property can be detected locally—see Proposition 1.6—so for the rest of the introduction R will be a local ring.

A central result of our work is that R is Tor-persistent if its completion has a deformation \(R\) (that is to say, it is of the form \(R/(f)\) where \(f = f_1, \ldots, f_n\) is a \(Q\)-regular sequence) that satisfies any one of the following conditions:

- edim \(Q\) − depth \(Q\) ≤ 3.
- edim \(Q\) − depth \(Q = 4\) and \(Q\) is Gorenstein.
- edim \(Q\) − depth \(Q = 4\) and \(Q\) is Cohen-Macaulay, almost complete intersection, with \(\frac{1}{f} \in Q\).
- \(Q\) is one link from a complete intersection.
- \(Q\) is two links from a complete intersection and is Gorenstein.
- \(Q\) is Golod.

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• mult $Q \leq 7$ and $Q$ is Cohen-Macaulay.

This result is contained in Theorem 5.1. In many of the cases considered above, we deduce the Tor-persistence of $R$ by verifying that $Q$ has the stronger property that if $\text{Tor}^Q_i(M, N) = 0$ for $i \gg 0$, then $M$ or $N$ has finite projective dimension over $Q$. This property, which we call Tor-friendliness, has been studied by Jorgensen [28], Huneke, Seega, and Vraicu [26], Seega [43], Nasseh and Yoshino [42], Gosh and Puthenpurakal [21], Lyle and Montaño [38], among others, and our work extends some of their results. Tor-friendliness of $R$ can also be recast as the statement that each finite $R$-module $M$ of infinite projective dimension is a test module for finite projective dimension, in the sense of Celikbas, Dao, and Takahashi [14].

To prove Theorem 5.1, we draw on a panoply of results concerning the homological properties of the classes of rings appearing in the list above, including structure theorems for their Koszul homology algebra proved by Kustin [29] in collaboration with Miller [31, 32, 33], Jacobsson and Miller [27], Avramov and Miller [9], and Palmer Slattery [34]. To the same end, we also establish a number of general statements for recognizing Tor-persistence, and for tracking that property along homomorphisms of rings. Notable among these is the result below which covers the case of complete intersection rings.

Assume the residue field of $R$ is algebraically closed and that $R$ has a deformation $Q$ such that 0 is the only finite $Q$-module with constant Betti numbers. If $Q$ is Tor-persistent, then so is $R$.

This is the content of Theorem 2.2. To our vexation we have been unable to eliminate the hypothesis on $Q$-modules with constant Betti numbers, except in the case when the generators of the ideal $\text{Ker}(Q \to R)$ can be chosen in $n \setminus n^2$, where $n$ is the maximal ideal of $Q$; see Proposition 3.8(1). This latter result, and some other intermediate ones, have also been proved recently by Celikbas and Holm [15].

In Sections 6 and 7 we turn to cohomological analogs of Tor-persistence. This comes in two flavors: We say $R$ is Ext-persistent if for a finite $R$-module $M$ satisfies $\text{Ext}^i_R(M, M) = 0$ for $i \gg 0$ only when the projective dimension or the injective dimension of $M$ is finite. While not every ring is Ext-persistent (consider rings with nontrivial semidualizing modules), we prove that the class of rings $R$ in Theorem 5.1 are, which implies in particular that there are no nontrivial semidualizing modules, or even complexes, over such rings; see Corollary 6.4.

The ring $R$ has the Auslander-Reiten property if a finite $R$-module $M$ satisfies $\text{Ext}^i_R(M, M \oplus R) = 0$ for $i \gg 0$ only when its projective dimension is finite. The nomenclature is motivated by a conjecture of Auslander and Reiten that asserts that every $R$ has property. Once again, the class of rings $R$ in Theorem 5.1 have the Auslander-Reiten property; see Corollary 7.3.

1. Persistence of homology

In this section we assemble sufficient conditions that ensure that a ring is Tor-persistent. We first recast the definitions in terms of complexes. Throughout, $R$ denotes a commutative noetherian ring.

1.1. An $R$-complex is a family of $R$-linear maps $\partial^U_i : U_i \to U_{i-1}$ with $\partial^U_{i-1} \partial^U_i = 0$. We set

$$\inf U = \inf \{ n \in \mathbb{Z} \mid U_n \neq 0 \} \quad \text{and} \quad \sup U = \sup \{ n \in \mathbb{Z} \mid U_n \neq 0 \}$$
and say that \( U \) is \textit{bounded below} (respectively, \textit{above}) if \( U = 0 \) or \( \inf U \) (respectively, \( \sup U \)) is finite. When both conditions hold \( U \) is said to be \textit{bounded}. It is \textit{finite} if the \( R \)-module \( \bigoplus_{i \in \mathbb{Z}} U_i \) is finite; in particular, then \( U \) is bounded.

The homology of \( U \) is the complex \( H(U) \) with \( H_i(U) \) in degree \( i \) and \( \partial H(U) = 0 \).

We write \( \text{Spec} \, R \) (respectively, \( \text{Max} \, R \)) for the set of prime ideals (respectively, maximal ideals) of \( R \), with the Zariski topology.

\section{1.2.} An \( R \)-complex is said to be \textit{perfect} if it is quasi-isomorphic to a bounded complex of finite projective \( R \)-modules. This property can be verified homologically: An \( R \)-complex \( U \) is perfect if and only if \( H(U) \) is degreewise finite and bounded below and \( \text{Tor}^R(R/\mathfrak{m}, U) \) is bounded for each \( \mathfrak{m} \) in \( \text{Max} \, R \).

Indeed, by [5, 5.5.F] the boundedness implies that the \( R_\mathfrak{m} \)-complex \( U_\mathfrak{m} \) is perfect, and then [7, 4.1] yields that \( U \) is perfect. We refer to [5] for the construction of \( \text{Ext} \) and \( \text{Tor} \) for \( R \)-complexes and related notions.

\textbf{Proposition 1.3.} The following conditions on a ring \( R \) are equivalent.

\begin{enumerate}[(i)]
    \item The ring \( R \) is \textit{Tor-persistent}.
    \item Each \( R \)-complex \( U \) with \( H(U) \) finite and \( \text{Tor}^R(U, U) \) bounded is perfect.
    \item Each \( R \)-complex \( U \) with \( H(U) \) of finite length and \( \text{Tor}^R(U, U) \) bounded is perfect.
\end{enumerate}

\textbf{Proof.} The implications (ii) \( \Rightarrow \) (i) and (ii) \( \Rightarrow \) (iii) are tautologies. In the rest of the proof we repeatedly use 1.2, and without comment.

(i) \( \Rightarrow \) (ii). Replacing \( U \) by an appropriate resolution, we may assume that \( U \) is semiprojective with each \( U_i \) a finite free \( R \)-module. Set \( s = \sup H(U) \) and consider the canonical exact sequence of \( R \)-complexes

\begin{equation}
0 \to U_{<s} \to U \to U_{\geq s} \to 0.
\end{equation}

Observe that \( \Sigma^{-s}U_{\geq s} \) is a free resolution of the finite \( R \)-module \( M = H_s(U_{\geq s}) \). Thus the exact sequence above implies isomorphisms

\[ \text{Tor}^R_i(M, M) \cong \text{Tor}^R_{i+2s}(U, U) \quad \text{for} \quad i \geq 2s + 1. \]

Since \( \text{Tor}^R(U, U) \) is bounded it follows that so is \( \text{Tor}^R(M, M) \), and hence, by hypothesis, \( M \) is perfect. Since the complex \( U_{<s} \) is evidently perfect, it then follows from (1.3.1) that \( U \) is perfect as well, as desired.

(iii) \( \Rightarrow \) (ii). Let \( U \) be a complex as in (ii); we may assume it is semiprojective. It suffices to verify that \( \text{Tor}^R(R/\mathfrak{m}, U) \), that is to say, \( H(R/\mathfrak{m} \otimes_R U) \), is bounded for each maximal ideal \( \mathfrak{m} \) of \( R \). Let \( K \) be the Koszul complex on a finite generating set for \( \mathfrak{m} \). A standard computation shows that \( H(K \otimes_R U) \) has finite length. Moreover, \( K \otimes_R U \) is semiprojective, so one gets the first isomorphism below

\[ \text{Tor}^R(K \otimes_R U, K \otimes_R U) \cong H((K \otimes_R U) \otimes_R (K \otimes_R U)) \]

\[ \cong H((K \otimes_R K) \otimes_R (U \otimes_R U)). \]

The second one is due to the associativity of tensor products. Note that \( K \otimes_R K \) is a bounded complex of free \( R \)-modules. It follows that \( H((K \otimes_R K) \otimes_R (U \otimes_R U)) \) is bounded, along with \( H(U \otimes_R U) \). In view of the preceding isomorphisms and our present hypothesis, \( K \otimes_R U \) is perfect. Therefore, \( H(k \otimes_R (K \otimes_R U)) \) is bounded
when \( k = R/m \). From associativity and the Küneth isomorphism we get
\[
H(k \otimes_R (K \otimes_R U)) \cong H((k \otimes_R K) \otimes_k (k \otimes_R U))
\cong H(k \otimes_R K) \otimes_k H(k \otimes_R U)
\]
Since \( H(k \otimes_R K) \neq 0 \), it follows \( H(k \otimes_R U) \) is bounded, so \( U \) is perfect. \( \square \)

As a first application, we obtain a change of rings property for homological persistence. A homomorphism of rings \( R \to S \) is of finite flat dimension if \( S \) has finite flat dimension when viewed as an \( R \)-module by restriction of scalars. In the next result the hypothesis on the maximal ideals holds when \( R \) and \( S \) are local rings and the homomorphism is local.

**Proposition 1.4.** Let \( R \to S \) be a homomorphism of finite flat dimension such that the induced map \( \text{Max} S \to \text{Max} R \) is surjective.

If \( S \) is Tor-persistent, then so is \( R \).

**Proof.** Let \( M \) be a finite \( R \)-module with Tor\(^R\)(\( M, M \)) bounded. Let \( P \) be a semiprojective resolution of \( M \) and set \( U = S \otimes_R P \). Evidently, the \( S \)-complex \( U \) is semiprojective, which justifies the first isomorphism below.

\[
\text{Tor}^S(U, U) \cong H((S \otimes_R P) \otimes_S (S \otimes_R P))
\cong H((S \otimes_R P) \otimes_R P)
\cong H(S \otimes_R (P \otimes_R P))
\]

The other isomorphisms are standard. As \( H(P \otimes_R P) \) is isomorphic to Tor\(^R\)(\( M, M \)), it is bounded. Given this and the hypothesis that the flat dimension of \( S \) over \( R \) is finite, it follows that the homology of \( S \otimes_R (P \otimes_R P) \), that is to say, Tor\(^S\)(\( U, U \)), is bounded. By the same token, the \( S \)-module \( H(U) \) is finite. Since \( S \) is Tor-persistent, it follows that the \( S \)-complex \( S \otimes_R P \) is perfect; see Proposition 1.3.

Pick a maximal ideal \( m \) of \( R \). By hypothesis, there is a maximal \( n \) of \( S \), so that \( n \cap R = m \). Set \( k = R/m \) and \( l = S/n \). We then have isomorphisms

\[
l \otimes_k \text{Tor}^R(k, M) \cong \text{Tor}^R(l, M) \cong H(l \otimes_R P) \cong H(l \otimes_S (S \otimes_R P))
\]

where \( R \) acts on \( l \) through the composed ring homomorphism \( R \to k \to l \). Since \( S \otimes_R P \) is perfect, the graded module on the right, and hence Tor\(^R\)(\( k, M \), is bounded. As \( m \) was arbitrary, 1.2 yields that \( M \) is perfect as an \( R \)-complex. \( \square \)

**Proposition 1.5.** Let \( I \) be an ideal in the Jacobson radical of \( R \), and let \( \hat{R} \) denote the \( I \)-adic completion of \( R \).

The ring \( R \) is Tor-persistent if and only if so is \( \hat{R} \).

**Proof.** The completion map \( R \to \hat{R} \) is flat and the assignments \( n \mapsto n \cap R \) and \( m \mapsto m\hat{R} \) yield inverse bijections \( \text{Max} \hat{R} \leftrightarrow \text{Max} R \). Now Proposition 1.4 shows that if \( \hat{R} \) is Tor-persistent, then so is \( R \).

Conversely, let \( V \) be an \( \hat{R} \)-complex with homology of finite length and Tor\(^{\hat{R}}\)(\( \hat{R}, V \)) bounded. When an \( \hat{R} \)-module \( L \) has finite length, then it has the same length over \( R \) and the canonical \( \hat{R} \)-linear map \( \hat{R} \otimes_R L \to L \) is bijective. Thus, the map \( \hat{R} \otimes_R V \to V \) is a quasi-isomorphism, so we have isomorphisms

\[
\text{Tor}^{\hat{R}}(\hat{R} \otimes_R V, \hat{R} \otimes_R V) \cong \text{Tor}^{\hat{R}}(\hat{R} \otimes_R V, \hat{R} \otimes_R V) \cong \text{Tor}^{\hat{R}}(V, V)
\]
Since $R$ is Tor-persistent and $R \to \hat{R}$ is faithfully flat, it follows that $V$ is perfect as an $R$-complex. For every maximal ideal $n$ of $\hat{R}$ and for $m = n \cap R$ the isomorphisms
\[
\text{Tor}^\hat{R}(\hat{R}/n, V) = \text{Tor}^\hat{R}(\hat{R}/m\hat{R}, V) \cong \text{Tor}^R((R/m) \hat{R}, V) \cong \text{Tor}^R(R/m, V)
\]
show that $\text{Tor}^\hat{R}(\hat{R}/n, V)$ is bounded, so $V$ is perfect, by 1.2.
\[\square\]

The import of the next result is that Tor-persistence is a local property.

**Proposition 1.6.** The ring $R$ is Tor-persistent if and only if $R_m$ is Tor-persistent for every $m \in \text{Max } R$.

**Proof.** Indeed, let $M$ be a finite $R$-module. If projdim$_{R_m} M_m$ is finite for every $m \in \text{Max } R$, then so is projdim$_R M$; see Bass and Murthy [13, 4.5], or [7, 5.1]. It follows that when $R_m$ is Tor-persistent for each such $m$, then $R$ is Tor-persistent.

Assume $R$ is Tor-persistent and fix an $m \in \text{Max } R$. By Proposition 1.3, it suffices to prove that if $U$ is an $R_m$-complex with $H(U)$ of finite length and Tor$_{R_m}^4(U, U)$ is bounded, then $U$ is perfect. Since the residue field at $m$ is finite as an $R$-module, $H(U)$ is finite also when viewed as an $R$-complex, via restriction of scalars along the localization $R \to R_m$. Moreover, the canonical map is an isomorphism:
\[
\text{Tor}^R(U, U) \xrightarrow{\cong} \text{Tor}^R(U, U).
\]
Thus Tor$_R^R(U, U)$ is bounded. Therefore $U$ is perfect as an $R$-complex and hence also as an $R_m$-complex, since localization induces a quasi-isomorphism $U \xrightarrow{\cong} U_m$. \[\square\]

1.7. As usual, if $R$ is local then $\hat{R}$ denotes its completion in the adic topology of the maximal ideal. Recall that the natural map $R \to \hat{R}$ is a faithfully flat ring homomorphism and that Cohen’s Structure Theorem yields an isomorphism $\hat{R} \cong P/I$ for some regular local ring $(P, p, k)$ and ideal $I$ contained in $p^2$; any such isomorphism is called a minimal Cohen presentation of $\hat{R}$.

We say that $R$ is locally complete intersection if $\hat{R}_m$ has a minimal Cohen presentation with ideal of relations generated by a regular sequence, for each $m \in \text{Max } R$.

**Example 1.8.** If $R$ is locally complete intersection, then $R$ is Tor-persistent.

Indeed, by Proposition 1.6 we may assume $R$ is local. Avramov and Buchweitz [4] attached to every finite $R$-module $M$ a cohomological support variety $V^*_R(M)$. It is a closed subset of some projective space over $k$, and [4, 6.1, 4.7, and 4.9] show that Tor$_R^R(M, M)$ is bounded if and only if $V^*_R(M)$ is empty. By [4, 5.6(9) and 5.6(3)], the latter is equivalent to Ext$_R^i(M, k) = 0$ for $i \gg 0$; that is, to projdim$_R M < \infty$.

2. Deformations

In this section $(R, m, k)$ denotes a local ring and $M$ and $N$ are finite $R$-modules.

**2.1.** By a deformation of $R$ to $Q$, we mean a surjective homomorphism $R \twoheadrightarrow Q$ from a local ring $Q$, with kernel generated by a $Q$-regular sequence; it is said to be embedded if edim $Q = \text{edim } R$. If every embedded deformation of $R$ is bijective, then we say that $R$ has no embedded deformation.

By Proposition 1.4, if $R$ is Tor-persistent, then so is $Q$. It has turned out to be surprisingly difficult to answer the question: Does the converse hold? We have only been able to obtain positive answers under additional assumptions. In the next result the hypothesis concerning the Betti numbers of $Q$ is particularly vexing.
Recall that the $n$th Betti number of a finite module $L$ over a local ring $(Q, n, k)$ can be defined by the equality $\beta_n^Q(L) = \text{rank}_k \text{Tor}_n^Q(L, k)$.

**Theorem 2.2.** Let $R \leftarrow Q$ be a deformation such that $k$ is algebraically closed and there is no nonzero finite $Q$-module with constant Betti numbers.

The ring $R$ is Tor-persistent if and only if the ring $Q$ is.

Another case, when the same conclusion holds, is given by Proposition 3.8(1).

The proof of the theorem is presented in 2.8. It utilizes a number of homological constructions, which we proceed to review, starting with a spectral sequence.

2.3. Let $R \leftarrow Q$ be a deformation and $f = \{f_1, \ldots, f_c\}$ a minimal generating set of $I = \text{Ker}(Q \to R)$. By resolving $R$ over $Q$ by means of the Koszul complex on $f$ one gets isomorphisms $\text{Tor}_q^Q(R, N) \cong N(\mathcal{I})$ for every integer $q$, so the standard change-of-rings spectral sequence with $E_{p,q}^2 = \text{Tor}_p^R(\text{Tor}_q^Q(M, R), N)$ takes the form

\[
E_{p,q}^2 = \text{Tor}_p^R(M, N)(\mathcal{I}) \Rightarrow \text{Tor}_{p+q}^Q(M, N).
\]

It follows immediately that if $\text{Tor}_p^R(M, N)$ is bounded, then so is $\text{Tor}_q^Q(M, N)$.

Next we turn to the multiplicative structure of cohomology. In this section it will be needed only for $R$-modules, but for later use we describe its properties in the more general framework of $R$-complexes.

2.4. Let $U$ and $V$ be $R$-complexes.

Composition products $\circ$ turn $\text{Ext}_R(U, U)$ and $\text{Ext}_R(V, V)$ into $R$-algebras graded by cohomological degree, and endow $\text{Ext}_R(U, V)$ with a structure of graded bimodule on which $\text{Ext}_R(U, U)$ acts on the right and $\text{Ext}_R(V, V)$ on the left.

With $R \leftarrow Q$ and $f$ as in 2.3, let $R[\chi]$ be a polynomial ring in indeterminates $\chi_1, \ldots, \chi_c$ of cohomological degree 2. By [10, 2.7, p. 700] there are homomorphisms

\[
\text{Ext}_R(U, V) \xleftarrow{\zeta_U} R[\chi] \xrightarrow{\zeta_V} \text{Ext}_R(U, U)
\]

of graded $R$-algebras, with images in the corresponding centers, satisfying

$\zeta_U(\rho) \circ \xi = \xi \circ \zeta_V(\rho)$ for all $\rho \in R[\chi]$ and $\xi \in \text{Ext}_R(U, V)$.

Thus, both maps give the same graded $R[\chi]$-module structure on $\text{Ext}_R(U, V)$.

The $R[\chi]$-module $\text{Ext}_R(U, V)$ is finitely generated if and only if $\text{Ext}_Q(U, V)$ is bounded; see Avramov and Sun, [10, 5.1].

The last result is used to define cohomological support varieties; see Remark 2.6.

It is expedient to first recall a measure of the asymptotic growth of minimal free resolutions, introduced in [1]; see also [3, §4.2].

2.5. The complexity of $M$ over $R$ is the number defined by the equality

\[\text{cx}_R M = \inf\{d \in \mathbb{N} \cup \{0\} \mid \beta_n^R(M) \leq bn^{d-1} \text{ for some } b \in \mathbb{R} \text{ and all } n \geq 0\}\].

In particular, $\text{cx}_R M = 0$ means that $M$ has finite projective dimension, while $\text{cx}_R M = 1$ means that the sequence $(\beta_n^R(M))_{n \geq 0}$ is bounded.

2.6. Let $R \leftarrow (Q, q, k)$ be a deformation and assume that $k$ is algebraically closed.

Set $I = \text{Ker}(R \leftarrow Q)$, let $f = \{f_1, \ldots, f_c\}$ be a regular sequence generating $I$, and write $\mathcal{I}$ for the image in $I/qI$ of $f \in I$. Set $k[\chi] = k \otimes_R R[\chi]$, see 2.4, and identify $k[\chi]$ and the ring of $k$-valued algebraic functions on $\mathcal{I} = I/qI$ by mapping
1 \otimes \chi_i to the $i$th coordinate function of the $k$-basis $\{f_1, \ldots, f_\alpha\}$ of $T$. As in [1, §3], let $V(Q, f, M)$ denote the zero-set in $T$ of the annihilator of $\text{Ext}_\mathbb{R}(M, k)$ in $k[\chi]$.

When $\text{proj dim}_Q M$ is finite, [1, 3.12 and 3.11] yield the following equalities:

\[ (2.6.1) \quad cx_R M = \dim V(Q, f, M) \]
\[ (2.6.2) \quad V(Q, f, M) = \{f \in I \setminus qI \mid \text{proj dim}_{Q/(f)} M = \infty\} \cup \{0\}. \]

Any $f$ in $I \setminus qI$ can be extended to a minimal generating set for $I$. Hence such an $f$ is not a zero divisor on $Q$, and the canonical map $Q/(f) \to R$ is a deformation.

Contained in the preceding formulas is a criterion for finite projective dimension.

**Proposition 2.7.** Let $R \leftarrow (Q, q, k)$ be a deformation such that $I = \text{Ker}(Q \to R)$ is nonzero and $k$ is algebraically closed. When $M$ is a finitely generated $R$-module, $\text{proj dim}_R M$ is finite if and only if $\text{proj dim}_{Q/(f)} M$ is finite for each $f \in I \setminus qI$.

**Proof.** The validity of either hypothesis implies $\text{proj dim}_Q M < \infty$: This follows from $\text{proj dim}_R M < \infty$ because $\text{proj dim}_Q R$ is finite and from $\text{proj dim}_{Q/(f)} M < \infty$ as $\text{proj dim}_Q Q/(f)$ is finite. Thus it suffices to establish the equivalence when $\text{proj dim}_Q M$ is finite.

By (2.6.1), $\text{proj dim}_R M < \infty$ is equivalent to $\dim V(Q, f, M) = 0$. Since it is defined by homogeneous equations, $V(Q, f, M)$ is a cone in $T$ with vertex at 0. As $k$ is infinite, $\dim V(Q, f, M) = 0$ is equivalent to $V(Q, f, M) = \{0\}$. The latter holds if and only if $\text{proj dim}_{Q/(f)} M$ is finite for each $f \in I \setminus q$, due to (2.6.2). \qed

**2.8. Proof of Theorem 2.2.** When $R$ is Tor-persistent, so is $Q$, by Proposition 1.4.

Suppose that $Q$ is Tor-persistent and $M$ is a finite $R$-module with $\text{Tor}^R(M, M)$ finite. We want to prove that $\text{proj dim}_R M$ is finite. By Proposition 2.7, it suffices to show that $\text{proj dim}_{Q/(f)} M$ is finite for every $f \in I \setminus qI$. Fix one such $f$ and set $Q' = Q/(f)$. Recall that $f$ is $Q$-regular and the map $R \leftarrow Q'$ is a deformation.

By 2.3, one has $\text{Tor}^i_Q (M, M) = 0$ for $i > 0$. When $N$ is a sufficiently high syzygy module of $M$ over $Q'$, standard degree-shifting isomorphisms yield

\[ (2.8.1) \quad \text{Tor}^Q_i (N, N) = 0 \quad \text{for} \quad i \geq 1. \]

Then $\text{Tor}^Q(N, N)$ is also bounded, again by 2.3, so $\text{proj dim}_Q N$ is finite because $Q$ is Tor-persistent. The choice of $N$ entails $\text{depth}_{Q'} N = \text{depth} Q'$. By the Auslander-Buchsbaum Equality, $N$ has a minimal $Q$-free resolution of the form

\[ (2.8.2) \quad G = 0 \to Q^b \to Q^b \to 0 \]

To finish the proof, we show that $\text{proj dim}_Q N$ is finite. Set $F = G \otimes_Q G$ and $L = N \otimes_{Q'} N$. In view of (2.8.1), the spectral sequence (2.3.1) for $Q \to Q'$ yields

\[ (2.8.3) \quad H_i(F) \cong \text{Tor}^Q_i (N, N) \cong \begin{cases} \text{Tor}^Q_i (N, N) \cong L & \text{for } i = 0, 1; \\ 0 & \text{otherwise.} \end{cases} \]

Let $F'$ be the subcomplex of $F$ with $F'_i = F_i$ for $i \geq 2$, $F'_i = \text{Ker} \partial_1$, and $F'_i = 0$ for $i \leq 0$. From (2.8.3) we get quasi-isomorphisms $F' \simeq \Sigma L$ and $F/F' \simeq L$, so the exact sequence $0 \to F' \to F \to F/F' \to 0$ of complexes yields an exact sequence

\[ \cdots \to \text{Tor}_{i+2}^Q(F, k) \to \text{Tor}_{i+2}^Q(L, k) \to \text{Tor}_{i}^Q(L, k) \to \text{Tor}_{i+1}^Q(F, k) \to \cdots \]
Set $T_i = \Tor^Q_i(L, k)$. Since $\Tor^Q_i(F, k) \cong F; \otimes_Q k$ holds for each $i$, the resolution (2.8.2) and the exact sequence above yield exact sequences

$$0 \to T_{i+2} \to T_i \to 0 \quad \text{for} \quad i \geq 2$$

$$0 \to T_3 \to T_1 \to k^{b_2} \to T_2 \to T_0 \to k^{2b_2} \to T_1 \to 0 \to k^{b_2} \to T_0 \to 0$$

of vector spaces. As a result, we get equalities

$$\beta^Q_i(L) = \beta^Q_{i+2}(L) \quad \text{for} \quad i \geq 2$$

$$\beta^Q_2(L) = \beta^Q_0(L)$$

We conclude that $\beta^Q_i(L) = \beta^Q_{i+1}(L)$ holds for $i \geq 2$.

In view of the hypothesis on $Q$, this implies $\projdim Q L \leq 1$. If $G'$ is a minimal $Q'$-free resolution of $N$, then $G' \otimes_{Q'} G'$ is one of $L$, by (2.8.1). This gives the first equality in the next string, and [3, 4.2.5(4)] yields the inequality:

$$2 \cx Q' N = \cx Q' L \leq \cx Q L + 1 = 1$$

As a consequence, we get $\cx Q' N = 0$, so $\projdim Q' N$ is finite, as desired. \qed

3. TOR-FRIENDLY RINGS

In this section $(R, m, k)$ denotes a local ring, and $M$ and $N$ are finite $R$-modules.

Outside of the class of locally complete intersection rings, Tor-persistence has been difficult to verify. The following stronger property is easier to work with.

3.1. We say that $R$ is Tor-friendly if for every pair $(M, N)$ of finite $R$-modules $\Tor^R(M, N)$ is bounded only if $\projdim R M$ or $\projdim R N$ is finite.

We record some formal properties of Tor-friendliness. Proofs for the following alternative characterization and descent result are omitted, as they parallel those of the analogous results for Tor-persistence; see Proposition 1.3 and Proposition 1.4.

**Proposition 3.2.** The following conditions on a local ring $R$ are equivalent.

(i) The ring $R$ is Tor-friendly.

(ii) If $U$ and $V$ are $R$-complexes, such that the $R$-modules $H(U)$ and $H(V)$ are finite and $\Tor^R(U, V)$ is bounded, then $U$ or $V$ is perfect.

(iii) If $U$ and $V$ are $R$-complexes, such that the $R$-modules $H(U)$ and $H(V)$ are of finite length and $\Tor^R(U, V)$ is bounded, then $U$ or $V$ is perfect. \qed

**Proposition 3.3.** Let $\varphi: R \to S$ be a local homomorphism of local rings.

If $\varphi$ is of finite flat dimension and $S$ is Tor-friendly, then so is $R$. \qed

**Remark 3.4.** When $R$ is singular, Tor-friendliness need not ascend from $R$ to $S$, even when $\varphi$ is surjective with kernel generated by an $R$-regular element: see Proposition 3.8. This should be compared to Theorems 2.2 and 5.1.

**Proposition 3.5.** Let $I \subseteq m$ be an ideal of $R$, and let $\hat{R}$ denote the $I$-adic completion of $R$. The ring $R$ is Tor-friendly if and only if so is $\hat{R}$. \qed

For the proof of Proposition 3.7, we need a general result of independent interest.

**Lemma 3.6.** Let $P \to Q$ and $P \to Q'$ be homomorphisms of rings.

If $\Tor^P_i(Q, Q') = 0$ for $i \geq 1$ and $G$ and $G'$ are bounded below complexes of flat modules over $Q$ and $Q'$, respectively, then there are $(Q \otimes_P Q')$-linear isomorphisms

$$H_n(G \otimes_P G') \cong \Tor^P_n(G, G') \quad \text{for each} \quad n \geq 1.$$
Proof. Let $F' \to G'$ be a free resolution of $G'$ over $P$. It suffices to prove that the induced morphism $G \otimes_P F' \to G \otimes_P G'$ is a quasi-isomorphism; equivalently, that the mapping cone $C$ of $F' \to G'$ satisfies

\begin{equation}
H_n(G \otimes_P C) = 0 \quad \text{for each} \quad n \in \mathbb{Z}.
\end{equation}

We proceed by dévissage. Since $F'$ and $G'$ are bounded below, there exists an integer $s$ such that $C_s = 0$ for $j \leq s$, so there are isomorphisms

\[ H_n(G \otimes_P C) \cong H_n(G_{\leq n-s+1} \otimes_P C). \]

Thus, replacing $G$ by $G_{\leq n-s+1}$ we may assume that $G$ is a bounded complex of flat $Q$-modules. Set $b = \min\{j \mid G_j \neq 0\}$. The sequence of complexes of $Q$-modules

\[ 0 \to G_b \to G \to G_{\geq b+1} \to 0 \]

is split-exact as a sequence of graded modules, so the induced sequence of complexes

\[ 0 \to G_b \otimes_P C \to G \otimes_P C \to G_{\geq b+1} \otimes_P C \to 0 \]

is exact. Induction on the number of nonzero terms in $G$ shows that it suffices to prove (3.6.2) when $G$ is a flat $Q$-module. We assume that for the rest of the proof.

Let $F \to G$ be a free resolution of $G$ over $P$. We have standard spectral sequences

\[ E^2_{i,j} = H_j(H_i(F \otimes_P C)) \implies H_{i+j}(F \otimes_P C) \iff H_i(H_j(F \otimes_P C)) = E^2_{i,j}. \]

We have $H_j(F_1 \otimes_P C) \cong F_i \otimes_P H_j(C) = 0$ for all $(j,i)$, so $E^2_{i,j} = 0$ yields $H_i(F \otimes_P C) = 0$.

As $H_i(F \otimes_P C) \cong \text{Tor}_i^P(G, C_j)$ holds for all $(i,j)$, we get a spectral sequence

\[ E^2_{i,j} = H_j(\text{Tor}_i^P(G, C_j)) \implies 0 \quad \text{with} \quad E^2_{0,j} = H_j(G \otimes_P C). \]

The $Q$-module $F'_i$ is free, so for every integer $j$ and each $n \geq 1$ we have isomorphisms

\[ \text{Tor}_n^P(G, C_j) \cong \text{Tor}_n^P(G, F'_i \oplus G'_j) \cong \text{Tor}_n^P(G, G'_j). \]

Therefore it suffices to prove (3.6.1) when $G'$ is a flat $Q'$-module.

By the Govorov-Lazard Theorem, $G$ and $G'$ are filtered colimits of families $G = \{G_i\}_{i \in I}$ and $G' = \{G'_i\}_{i' \in I'}$ of finite free modules over $Q$ and $Q'$, respectively, so

\[ \text{Tor}_n^P(G, G') \cong \text{Tor}_n^P(\colim_{i \in I} G', \colim_{i' \in I'} G) \cong \colim_{(i,i') \in I \times I'} \text{Tor}_n^P(G_i, G_{i'}). \]

Formula (3.6.1) evidently holds for $G \cong Q'$ and $G' \cong Q'^{r'}$. \hfill \Box

Here is a first application of Proposition 3.2.

**Proposition 3.7.** Let $Q$ and $Q'$ be singular residue rings of a regular local ring $P$.

If $\text{Tor}_i^P(Q, Q') = 0$ for $i \geq 1$, then the ring $R = Q \otimes_P Q'$ is not Tor-friendly.

**Proof.** Since $Q$ and $Q'$ are singular, they have finite modules of infinite projective dimension. Let $L$ and $L'$ be such modules, and $G$ and $G'$ be their minimal free resolutions over $Q$ and $Q'$, respectively. Since $G$ and $G'$ are minimal, so are the complexes of finite free $R$-modules $G \otimes_P Q'$ and $Q \otimes_P G'$, and hence of infinite projective dimension over $R$. From Lemma 3.6, applied with $G' = Q'$, and the regularity of $P$ it follows that

\[ H_i(G \otimes_P Q') \cong \text{Tor}_i^P(G, Q') \cong \text{Tor}_i^P(L, Q') = 0 \quad \text{for} \quad i \gg 0. \]
By symmetry, $H(Q \otimes_P G')$ is bounded as well. Consider the following chain of isomorphisms, where the first two are standard.
\[
\text{Tor}_i^R(G \otimes_P Q', Q \otimes_P G') \cong H_i((G \otimes_P Q') \otimes_{(Q \otimes_P Q')} (Q \otimes_P G')) \\
\cong H_i(G \otimes_P G') \\
\cong \text{Tor}_i^R(L, L') \\
\cong 0 \quad \text{for } i \gg 0.
\]

The third isomorphism holds by Lemma 3.6, and the last one because $P$ is regular. It remains to apply Proposition 3.2 to conclude that $R$ is not Tor-friendly. \[\square\]

In part (2) of the next result, the “if” direction is a theorem of Huneke and Wiegand [25, 1.9]; a different proof was given by Miller [40, 1.1]. The converse assertion is due to Šega [43, 4.2]; the proof given below is different.

**Proposition 3.8.** Assume $\tilde{R}$ is isomorphic to $Q/(f)$, where $(Q, q, k)$ is a local ring and $f$ is a $Q$-regular element.

1. When $f$ does not lie in $q^2$, the ring $R$ is Tor-friendly (respectively, Tor-persistent) if and only if $Q$ is.

2. When $f$ lies in $q^2$, the ring $R$ is Tor-friendly if and only if $Q$ is regular.

**Proof.** In view of Proposition 3.5, it suffices to prove the statement for $\tilde{R}$, so we may assume that $R$ is complete and is equal to $Q/(f)$.

1. By Proposition 3.3, we have to show that if $Q$ is Tor-friendly and $M$ and $N$ are finite $R$-modules with Tor$^R(M, N)$ bounded, then proj dim$_R M$ or proj dim$_R N$ is finite. From 2.3 we know that Tor$^Q(M, N)$ is bounded, so proj dim$_Q M$ or proj dim$_Q N$ is finite. As $f \notin q^2$ holds, a classical result due to Nagata shows that proj dim$_R M$ or proj dim$_R N$ is finite; see [3, 2.2.3].

The same argument also settles the case of Tor-persistence.

2. By Proposition 3.5 we may assume $Q$ is complete. Choose a minimal Cohen presentation $Q \cong P/J$ with a regular local ring $(P, p, k)$ and $J \subseteq p^2$. Choose $g$ in $P$ that maps to $f$; as $f$ is in $q^2$ we have $g \in p^2$, so the ring $Q' = P/(g)$ is singular. In addition, we have $R \cong Q \otimes_P Q'$, and Tor$^R_i(Q, Q') = 0$ for $i \geq 1$, as $g$ is $Q$-regular.

If $Q$ is singular, then Proposition 3.7 shows that $R$ is not Tor-friendly.

If $Q$ is regular, then $J = 0$, so $R = Q'$. This is a hypersurface ring, and Huneke and Wiegand [25, 1.9] have shown that hypersurface rings are Tor-friendly. \[\square\]

### 4. Recognizing Friendliness

In this section $(R, m, k)$ denotes a local ring, and $M$ and $N$ are finite $R$-modules. We collect various sufficient conditions for $R$ to be Tor-friendly. All of them are needed in Section 5, for the proof of the main theorem of the paper.

We start by describing those results that bring in the largest haul.

**4.1.** The formal power series with non-negative integer coefficients

\[P_M^R(z) = \sum_{n \geq 0} \beta_n^R(M) z^n\]

is known as the Poincaré series of $M$ over $R$.

A common denominator for Poincaré series over $R$ is a polynomial $d(z) \in \mathbb{Z}[z]$, such that $d(z)P_M^R(z) \in \mathbb{Z}[z]$ holds for every finite $R$-module $N$. 


Following [43] we say that a factorization \( d(z) = p(z)q(z)r(z) \) in \( \mathbb{Z}[z] \) is good if the following conditions hold:
1. \( p(z) = 1 \) or \( p(z) \) is irreducible;
2. \( q(z) \) has non-negative coefficients;
3. \( r(z) = 1 \) or \( r(z) \) is irreducible and none of its complex roots of minimal absolute value is a positive real number.

If the Poincaré series over \( R \) admit a common denominator \( d(z) \) that has a good factorization, then \( R \) is Tor-friendly: This is proved by S\o\e ga [43, 1.4 and 1.5].

4.2. A graded ring \( A = \bigoplus_{i \in \mathbb{Z}} A_i \) is said to be graded-commutative if the identities \( aa' = (-1)^{i'}a' a \) and \( a^2 = 0 \) when \( i \) is odd hold for all \( a \in A_i \) and \( a' \in A_{i'} \).

When \( W = \bigoplus_{j \in \mathbb{Z}} W_j \) is a graded \( A \)-module, the trivial extension \( A \times W \) is the graded ring with underlying graded additive group \( A \oplus W \) and product
\[
(a, w)(a', w') = (aa', aw' + (-1)^{i'}a'w) \quad \text{for} \quad a' \in A_{i'} \quad \text{and} \quad w \in W_j.
\]
Note that \( A \times W \) is also graded-commutative. We identify \( A \) and \( W \) with their images in \( A \times W \); note that \( A \) is a subring and \( W \) is an ideal with \( W^2 = 0 \).

The following result is proved in [8, 5.3] specifically for use in the present paper:

The ring \( R \) is Tor-friendly if some Cohen presentation \( \hat{R} \cong P/I \) satisfies
(a) a minimal free resolution of \( \hat{R} \) over \( P \) has a structure of DG algebra; and
(b) the \( k \)-algebra \( B = \text{Tor}^P(\hat{R}, k) \) is isomorphic to a trivial extension \( A \times W \) of a graded \( k \)-algebra \( A \) by a graded \( A \)-module \( W \neq 0 \) with \( A_{\geq 1} \cdot W = 0 \).

It is easy to see that these conditions do not depend on the choice of presentation.

A precursor of [8, 5.3] was proved by Nasseh and Yoshino [42, 3.1]. We give a short, independent proof of that result as part (2) of the next proposition.

Proposition 4.3. When \( \text{proj dim}_R M \) is infinite the following assertions hold.
1. If \( k \) is a direct summand of \( \Omega^R_k(N) \), then \( \text{Tor}_i^R(M, N) \neq 0 \) for \( i \geq h \).
2. If \( (0 : m) \nsubseteq m^2 \), then \( k \) is a direct summand of \( \Omega^R_i(M) \) for each \( i \geq 2 \), and hence \( R \) is Tor-friendly.

Proof. (1) We have \( \text{Tor}_i^R(M, N) \cong \text{Tor}_{i-h}^R(M, \Omega^R_{k}(N)) \cong \text{Tor}_i^R(M, k) \neq 0 \).

(2) We will show that if an \( R \)-linear map \( \delta: L \to L' \) of finite free modules has \( \text{Ker} \delta \subseteq mL \) and \( \text{Im} \delta \subseteq mL' \), then \( \text{Ker} \delta \) has a direct summand isomorphic to \( L/mL \).

Set \( D = \text{Ker} \delta \) and pick \( x \in (0 : m) \setminus m^2 \). Then we have \( \delta(xL) = x\delta(L) = 0 \), and hence \( D \supseteq xL \cong L/mL \). The composed map \( \gamma: xL \to D/mD \) satisfies
\[
\text{Ker} \gamma = mD \cap xL \subseteq m^2L \cap xL = (m^2 \cap xR)L = 0
\]
Choose in \( D \) a submodule \( D' \) containing \( mD \), so that \( D/mD = (D'/mD) \oplus \gamma(xL) \).
It is not hard to verify that \( D = D' \oplus xL \). \( \square \)

4.4. Recall that the formal power series with non-negative integer coefficients
\[
H_M(z) = \sum_{n \geq 0} \text{rank}_k(m^nM/m^{n+1}M)z^n
\]
is called the Hilbert series of \( M \). By the Hilbert-Serre Theorem, it represents a rational function \( h_M(z)/(1-z)^{\dim M} \) with \( h_M(z) \in \mathbb{Z}[z] \) and \( h_M(1) \geq 1 \).

The result below and its proof are in the spirit of those in [26].
Theorem 4.5. Let \((R, \mathfrak{m}, k)\) be a local ring with \(H_R(z) = 1 + e z + s z^2\).

If \(s = 0\), or if \(e^2 - 4s\) is not the square of an integer, then \(R\) is Tor-friendly.

If \(s = 0\), or if \(e^2 - 4s\) is not zero, then \(R\) is Tor-persistent.

Indeed, this is a consequence of the first assertion of the next result.

Proposition 4.6. Let \((R, \mathfrak{m}, k)\) be a local ring with \(H_R(z) = 1 + e z + s z^2\).

If \(M\) and \(N\) are finite \(R\)-modules of infinite projective dimension and \(\text{Tor}^R(M, N)\) is bounded, then there are positive integers \(u\) and \(v\) such that

\[
\text{Tor}^R(M, N) = u \cdot v \cdot (1 + u z)(1 + v z).
\]

Furthermore, there exist finite \(R\)-modules \(M'\) and \(N'\) of infinite projective dimension, such that \(\text{Tor}^R(M', N')\) is bounded and the following equalities hold:

\[
\text{Tor}^R(M, N) = n_1 \cdot n_2 \cdot (1 + n_1 z)(1 + n_2 z) \quad \text{and} \quad \text{Tor}^R(M', N') = n_3 \cdot n_4 \cdot (1 + n_3 z)(1 + n_4 z).
\]

In case \(N = M\), the preceding conclusions hold with \(N' = M'\) and \(u = v\).

Proof. Following [35, 3.1], we say that \(N\) is exceptional if \(m^2 N = 0\) and \(\Omega_j^R(N)\) has no direct summand isomorphic to \(k\), for \(j \geq 1\).

Fix \(h \geq 1\) so that \(\text{Tor}^R(M, N) = 0\) holds for \(i \geq h\). For \(j \geq h\) we then have \(\text{Tor}^R(M, \Omega_j^R(N)) = 0\) for \(i \geq 1\), and also \(m^2 \Omega_j^R(N) = 0\) because \(\Omega_j^R(N)\) is a syzygy module. It follows from Proposition 4.3(1) that \(m \Omega_j^R(N) \neq 0\), and hence \(\Omega_j^R(N)\) is exceptional. By symmetry, \(\Omega_j^R(M)\) has the corresponding properties for \(i \geq 0\).

Thus, suitable syzygy modules \(M'\) of \(M\) and \(N'\) for \(M\) satisfy the conditions:

\[
\text{Tor}^R(M, N') = 0 \quad \text{for} \quad h \geq 1.
\]

\[
\text{Tor}^R(M', N) = 0 \quad \text{for} \quad i \geq 1.
\]

Note that if \(N = M\), then we may choose \(N' = M'\).

As \(M'\) is exceptional, by Lescot [35, 3.4 and 3.6] we have

\[
P_{M'}^R(z) = \frac{H_{M'}(-z)}{H_R(-z)}
\]

The exact sequence \(0 \to mN' \to N' \to N'/mN' \to 0\) induces an exact sequence

\[
\cdots \to \text{Tor}^R_i(M', N') \to \text{Tor}^R_i(M', k)^{n_0} \to \text{Tor}^R_{i-1}(M', k)^{n_1} \to \text{Tor}^R_{i-1}(M', N') \to \cdots
\]

Since \(\text{Tor}^R_i(M', N') = 0\) for \(i \geq 1\), we obtain equalities

\[
n_0 \beta_i^R(M') = n_1 \beta^R_{i-1}(M') \quad \text{for} \quad i \geq 2
\]

and an exact sequence

\[
0 \to \text{Tor}^R_i(M', k)^{n_0} \to \text{Tor}^R_i(M', k)^{n_1} \to M' \otimes_R N' \to \text{Tor}^R_0(M', k)^{n_0} \to 0
\]

Setting \(r = \text{rank}_k \text{Coker}(\delta)\) and \(l = \text{length}_R(M' \otimes_R N')\), we further get

\[
n_0 \beta_i^R(M') = n_1 \beta_0^R(M') - r
\]

\[
n_0 \beta_0^R(M') = l - r
\]

Multiplying (4.6.6.i) by \(z^i\), for \(i \geq 0\), and adding the resulting equalities yields

\[
n_0 P^R_{M'}(z) = n_1 z P^R_{M'}(z) - rz + (l - r)
\]

In view of formulas (4.6.5) and (4.6.4), the preceding equation gives

\[
((l - r) - rz)(1 - ez + sz^2) = (m_0 - m_1 z)(n_0 - n_1 z)
\]
Proposition 4.3(2) gives \( m^2 \neq 0 \), so we have \( s \neq 0 \). Matching degrees and constant terms in the last equality yields \( r = 0 \) and \( 0 \neq l = m_0n_0 \), so we obtain

\[(4.6.7) \quad m_0n_0(1 - ez + sz^2) = (m_0 - m_1z)(n_0 - n_1z)\]

Substituting \( 1/z \) for \( z \), then multiplying both sides by \( z^2/m_0n_0 \) yields

\[z^2 - ez + s = (z - m_1/m_0)(z - n_1/n_0)\]

Thus, \( u = m_1/m_0 \) and \( v = n_1/n_0 \) are integers, and \( m_1 \) and \( n_1 \) are non-zero.

Now formulas (4.6.7) and (4.6.4) turn into (4.6.1) and (4.6.2), respectively. \( \square \)

5. Homologically persistent rings

In this section \((R, m, k)\) denotes a local ring and \( \text{mult} R \) its multiplicity; that is, the positive integer \( h_R(1) \) defined by the Hilbert series \( H_R(t) \) in 4.4. Other notions appearing in the hypotheses of the next result are defined in 5.3, 5.4, and 5.5.

**Theorem 5.1.** Assume that there exist a local homomorphism \( R \to R' \) of finite flat dimension and a deformation \( R' \leftrightarrow Q \), where \( Q \) satisfies one of the conditions

\[(a) \quad \text{edim } Q - \text{depth } Q \leq 3.\]
\[(b) \quad Q \text{ is Gorenstein and } \text{edim } Q - \text{depth } Q = 4.\]
\[(c) \quad Q \text{ is Cohen-Macaulay, almost complete intersection, } \text{edim } Q - \text{depth } Q = 4, \text{ and } 1/2 \in Q.\]
\[(d) \quad Q \text{ is complete intersection.}\]
\[(e) \quad Q \text{ is one link from a complete intersection.}\]
\[(f) \quad Q \text{ is two links from a complete intersection and is Gorenstein.}\]
\[(g) \quad Q \text{ is Golod.}\]
\[(h) \quad Q \text{ is Cohen-Macaulay and } \text{mult } Q \leq 7.\]

The ring \( R \) is Tor-persistent.

The proof of the theorem takes up the balance of this section. It involves several steps, put together in 5.10. Some special cases are known from earlier work.

**Remark 5.2.** The conclusion of the theorem is known when \( R = R' = Q \) and one of the following conditions holds: \( R \) is complete intersection (by [4]; see Example 1.8); \( \text{edim } R - \text{depth } R \leq 1 \) (for then \( R \) is complete intersection); \( R \) is Golod and \( \text{edim } R - \text{depth } R \geq 2 \) (by Jorgensen, [28, 3.1]); \( \text{edim } R - \text{depth } R = 2 \) (for then \( R \) is either complete intersection or Golod; see [3, 5.3.4]); \( R \) is Golod with \( \text{edim } R - \text{depth } R = 3, 4 \) and \( \bar{R} \) has no embedded deformation (by Şega, [43, 2.3]); \( R \) is Golod of minimal multiplicity (by [43, 1.8], and also [24, 3.7]); \( R \) is Cohen-Macaulay with small multiplicity or \( \text{edim } R - \text{dim } R \leq 3 \) (by [38]).

When \( Q \) is Cohen-Macaulay of minimal multiplicity, the ring \( R \) is Tor-persistent by [41, 4.7, 6.5]; see also [21, 1.3]. This result can also be deduced from Theorem 2.2: We can assume \( \text{edim } Q - \text{depth } Q \geq 2 \) (else \( Q \) is a hypersurface), and then, as \( Q \) has minimal multiplicity, the Betti numbers of finitely generated modules are either eventually zero or grow exponentially; see [3, 4.2.6].

In all the remaining cases the result is new, even when \( R = R' = Q \).

5.3. Let \( \hat{Q} \cong P/J \) be a minimal Cohen presentation and set

\[e = \text{edim } P \quad \text{and} \quad c = \beta'_1(\hat{Q}).\]
For the residue field $k'$ of $Q$ there are coefficientwise inequalities of power series

$$
(5.3.1) \quad \frac{(1+z)^e}{(1-z^2)^e} \leq P^Q_k(z) \leq \frac{(1+z)^e}{1+z - z^2 P^Q_k(z)}
$$

Equality holds on the left in (5.3.1) if and only if $Q$ is complete intersection. This is equivalent to $\text{edim } Q - \text{dim } Q = c$ and implies that $Q$ is Cohen-Macaulay.

The ring $Q$ is said to be almost complete intersection if $\text{edim } Q - \text{dim } Q = c - 1$.

5.4. When equality holds on the right in (5.3.1), the ring $Q$ is said to be Golod.

If $\text{edim } Q - \text{depth } Q \leq 1$, then $c = \text{edim } Q - \text{depth } Q$ holds, so the upper and lower bounds in (5.3.1) coincide; thus $Q$ is both complete intersection and Golod.

When $\text{edim } Q - \text{depth } Q \geq 2$ and $Q$ is Golod, it has no embedded deformation.

Indeed, assume $Q \cong P''/(g)$ for a local ring $P''$ with $\text{edim } P'' = \text{edim } Q$ and a $P'$-regular sequence $g$. Let $P \to P'$ be a minimal Cohen presentation. Composing it with the induced map $P' \to \tilde{Q}$ yields a minimal Cohen presentation of $\tilde{Q}$. There is then an isomorphism $\text{Tor}_{P'}^i((\tilde{Q}, k) \cong \text{Tor}_{P'}^i(\tilde{P'}, k') \otimes k'(h)$ as graded $k'$-algebras, where $k'(h)$ is an exterior algebra on a generator of degree 1; see [2, 3.3]. We get

$$\text{rank}_{k'} \text{Tor}_{P'}^i((\tilde{Q}, k')^2 \geq \text{rank}_{k'}(\text{Tor}_{P'}^i(\tilde{P'}, k') \otimes_{k'} (k'(h))) = c - 1 \geq 1$$

This is impossible, as Golod’s Theorem [22] shows that all Massey products in $\text{Tor}_{P'}((\tilde{Q}, k')$ are equal to 0; in particular, $\text{Tor}_{P'}((\tilde{Q}, k')^2 = 0$ holds.

5.5. Ideals $J$ and $J'$ in a regular local ring $P$ are said to be linked if there exists a $P$-regular sequence $g$ in $J \cap J'$ such that $J = (Pg : J')$ and $J' = (Pg : J)$.

We say that $Q$ is $s$ links from a complete intersection if in some minimal Cohen presentation $\tilde{Q} \cong P/J$ there is a sequence $J_1, \ldots, J_s$ of ideals with $J = J_1, J_i$ linked to $J_{i+1}$ for $i = 1, \ldots, s - 1$, and $J_s$ generated by a $P$-regular sequence; see [9, 3.3].

We finish the preliminary discussion with a classical construction.

5.6. If $k \to \bar{k}$ is a field extension, then there is a flat ring homomorphism $R \to \bar{R}$, where $\bar{R}$ is a local ring with maximal ideal $\mathfrak{m}\bar{R}$ and residue field $\bar{k}$, and the induced map $R/\mathfrak{m} \to \bar{R}/\mathfrak{m}\bar{R}$ is the given field extension; see [11, IX, App., Theorem 1, Cor.].

The next three items contain the crucial reductions for the proof of the theorem.

**Lemma 5.7.** Under the hypothesis of Theorem 5.1 the ring $Q$ may be chosen to be complete, with algebraically closed residue field, and with no embedded deformation.

**Proof.** Let $R \to R' \leftarrow Q$ be the maps given in Theorem 5.1. Let $\tilde{Q} \cong P/J$ be a minimal Cohen presentation, which in cases (e) and (f) satisfies the properties in 5.5.

Pick a flat homomorphism of local rings $P \to P'$ such that $P'/\mathfrak{p}P'$ is an algebraic closure $l$ of the residue field of $P$; let $\tilde{P}$ be the completion of $P'$; see 5.6. The ring $\tilde{Q} = \tilde{Q} \otimes_P \tilde{P}$ is local and complete with residue field $l$. As $\tilde{P}$ is regular, we get a Cohen presentation $\tilde{Q} \cong P/J\tilde{P}$, and it is minimal due to the equalities

$$\text{edim } \tilde{P} = \text{edim } P = \text{edim } Q = \text{edim } \tilde{Q}.$$

Standard results yield $\text{depth } \tilde{Q} = \text{depth } Q$ and $\text{mult } \tilde{Q} = \text{mult } Q$, and show that $\tilde{Q}$ is Gorenstein, Cohen-Macaulay, (almost) complete intersection, or linked in $s$ steps to a complete intersection whenever $Q$ has the corresponding property. The equalities $P^\tilde{Q}_Q(z) = P^Q_Q(z)$ and $P^\tilde{Q}_Q(z) = P^Q_k(z)$ show that $\tilde{Q}$ is Golod if $Q$ is; see 5.3.
In cases (a)–(g) the surjective map $\bar{Q} \leftarrow \bar{P}$ can be factored as $\bar{Q} \twoheadrightarrow Q' \leftarrow \bar{P}$, where $\bar{Q} \twoheadrightarrow Q'$ is a deformation, $Q'$ satisfies one of properties (a) through (g), and $Q'$ has no embedded deformation. Indeed, the desired factorizations are provided by [2, 3.1] in cases (a), (b), (e), and (f), and by [34, 1.2 and 1.3] in case (c). In case (d), take $Q' = \bar{P}$. By 5.4, this covers case (g) when edim $Q - \text{depth} Q \leq 1$, while $Q' = \bar{Q}$ works when edim $Q - \text{depth} Q \geq 2$.

For case (h), choose an embedded deformation $\tilde{Q} \twoheadrightarrow Q'$ with edim $\tilde{Q} - \text{depth} Q' < 0$. These are covered by our hypothesis that $Q$ has no embedded deformation. Indeed, the desired factorizations are provided by [34, 1.2] in case (a); Kustin and Miller [31, 4.3] (when $Q$ contains $\frac{1}{\alpha}$) and Kustin [29, Theorem] (when $Q$ contains $\frac{1}{\alpha}$) in case (b); Kustin [30, 3.13] in case (c); the Koszul complex on a minimal generating set of $J$ in case (d); Avramov, Kustin, and Miller [9, 4.1] in case (e); Kustin and Miller [32, 1.6] in case (f).

**Lemma 5.9.** Let $Q$ be a complete local ring with algebraically closed residue field. If $Q$ has no embedded deformation in the sense of 2.1, and satisfies one of conditions (a) through (h) in Theorem 5.1, then $Q$ is Tor-friendly.

**Proof.** Cases are tackled one at a time.

(b) This is contained in [43, 2.3].

(d) Since $Q$ is a complete local ring with no embedded deformations, $Q$ is complete intersection precisely when it is regular. This fact will be used again, later in the proof. It settles (d), for regular rings are evidently Tor-friendly.

(g) This is contained in [28, 3.1]; see also [8, Proposition 5.2].

The argument in cases (a), (c), (e), and (f) have a similar structure, which we describe next. Choosing a minimal Cohen presentation $Q \cong P/J$, form the graded $l$-algebras $B = \text{Tor}^P(Q, l)$ where $l$ is the residue field of $Q$. Explicit multiplication tables for these algebras are available in the literature, but sometimes require additional hypotheses on $l$. These are covered by our hypothesis that $l$ is algebraically closed, so we may invoke decompositions $B = A \ltimes W$.

When $W \neq 0$ the ring $Q$ is Tor-friendly by 4.2, which applies because minimal DG algebra resolutions exist, as recalled in 5.8.

When $W = 0$ we utilize the fact that the Poincaré series of finite $Q$-modules admit a common denominator, and that explicit denominators are known in all cases under consideration. We find good factorizations for those polynomials and apply Segre’s criterion to conclude that $Q$ is Tor-friendly; see 4.1.

(a) If edim $Q - \text{depth} Q = 0$, then $Q$ is regular and so Tor-friendly, so we may assume edim $Q - \text{depth} Q \geq 1$. We may further assume edim $Q - \text{depth} Q \geq 2$, because $Q$ cannot to be complete intersection.

If edim $Q - \text{depth} Q = 2$ holds, then $Q$ has to be Golod, and so is covered by (g).
If edim $Q - \text{depth } Q = 3$, then $A$ belongs to one of three types: See [9, 2.1], from where we take the names of the types and the decomposition $B = A \ltimes W$.

Type CI consists of complete intersections, so it does not occur here. In type TE the relations $\text{rank}_Q A_3 = 0 < 1 \leq \text{rank}_k B_3$ imply $W_1 \neq 0$.

In type $B$ the relations $\text{rank}_Q A_1 = 2 < 3 \leq \text{rank}_k B_1$ imply $W_1 \neq 0$.

In type $G(r)$ the algebra $A$ has Poincaré duality in degree 3. If $W = 0$, then $Q$ is Gorenstein by [6, Theorem], and so it is Tor-friendly by [43, 2.3].

In type $H(p, q)$ the algebra $A$ is a free module over a subalgebra isomorphic to $k \ltimes \Sigma k$. As $Q$ has no embedded deformation, this implies $W \neq 0$; see [1, 3.4].

(c) The multiplication table of $A$ belongs to one of 12 types, which are determined in [34, 1.1] and displayed in [34, Table 1, p. 275]; we use the names assigned there and set $t = \text{rank}_k B_3$. The ring $Q$ does not belong to type $B[t], C[t],$ or $C^*$ because [34, 1.2] shows that rings of these types admit embedded deformations.

In the remaining cases [34, Table 3, p. 281] yields $W \neq 0$, unless $Q$ is of type $D(2)$ with $t = 1$, or $E(3)$ with $t = 1$, or $F(4)$ with $t = 1$, or $F^*$ with $t = 2$. From the proof of [34, 4.2] one sees that for the first three types a common denominator $d(z)$ can be chosen from [34, Table 2, p. 280]. A common denominator for the fourth type is given at the bottom of [34, 289]. Here are the explicit values:

$$D(2) \quad (t = 1): \quad d(z) = (1 - 2z - 2z^2 + 5z^3 - 2z^4 - 2z^5 + z^6)(1 + z)^2$$

$$E(3) \quad (t = 1): \quad d(z) = (1 - 2z - 2z^2 + 5z^3 - 2z^4 - 4z^5 + z^6 + z^7)(1 + z)^2$$

$$F(4) \quad (t = 1): \quad d(z) = (1 - 2z - 2z^2 + 5z^3 - 2z^4 - 7z^5 + z^6 + 4z^7 - z^9)(1 + z)^2$$

$$F^* \quad (t = 2): \quad d(z) = (1 - 2z - 2z^2 + 5z^3 - 3z^4 - 9z^5 + z^6 + 2z^7 - z^8)(1 + z)^2 = (1 - 5z + 10z^2 - 11z^3 + 5z^4 - z^5)(1 + z)^5$$

Let $p(z)$ denote the first factor in the final form of an expression for $d(z)$. Verifications with Mathematica show that $p(z)$ is irreducible over $Q$. Thus, we get good factorizations $d(z) = p(z)q(z)r(z)$ with $q(z)$ a power of $1 + z$ and $r(z) = 1$.

When discussing the next two cases we set $m = \text{grade}_P Q$.

(e) As $Q$ is not complete intersection, $m \geq 2$ holds. The DG algebra structure on a minimal free resolution of $Q$ over $P$, constructed in the proof of [9, 4.4], gives $B = A \ltimes W$ with $\text{rank}_k W_i = \binom{m}{i - 1}$ for $i = 1, \ldots m - 1$; in particular, $W \neq 0$.

(f) We have $m \geq 3$ because otherwise $Q$ is a complete intersection. It is proved in [9, 6.3 and 5.18] that there exists a common denominator $d(z)$ for Poincaré series over $Q$, and it satisfies $d(z)P_k^Q(z) = (1 + z)^e$ with $e = \text{edim } Q$. By using the expression for $P_k^Q(z)$, obtained in [23, Theorem 3] (see also [27, 2.4]), we get

$$d(z) = ((1 - z)^m - z)(1 + z)^{e - m}.$$

Set $f(z) = (1 - z)^m - z$ and $q(z) = (1 + z)^{e - m}$.

The substitution $y = z - 1$ turns $f(z)$ into $g(y) = (-1)^m y^m - y - 1$. The polynomial $(-1)^m g(y)$ is factored by Selmer [45, Theorem 1]; Ljunggren [37, Theorem 3] greatly simplified the proof. The result is that if $m \not\equiv 5 \pmod{6}$, then $(-1)^m g(y)$ is irreducible; else, $(-1)^m g(y) = (y^2 + y + 1)h(y)$, and $h(y)$ is irreducible.

If $m \equiv 5 \pmod{6}$, set $p(z) = f(z)$ and $r(z) = 1$; else, set $p(z) = (-1)^m h(z - 1)$ and $r(z) = z^2 - z + 1$. In either case, $d(z) = p(z)q(z)r(z)$ is a good factorization.
(h) Since $k$ is infinite, there is a $Q$-regular sequence $g$ that is linearly independent modulo $q^2$ and such that the ring $S = Q/(g)$ has length $S = \text{mult} Q$; thus, length $S \leq 7$ holds. By Proposition 3.8(1), it suffices to show that $S$ is Tor-friendly. Set $n = qS$, $e = \text{edim} S$ and $s = \text{rank}_k(n^2/n^3)$.

If $e \leq 3$, then $S$ is Tor-friendly by the already proved case (a) of the theorem, so we may assume $e \geq 4$. When $n^3 = 0$ we have $s = \text{length} S - e - 1 \leq 2$; this implies $s = 0$ or $e^2 - 4s$ is not the square of an integer, so $S$ is Tor-friendly by Theorem 4.5. When $n^3 \neq 0$ the only possibility is $H_S(z) = 1 + 4z + z^2 + z^3$. If $S$ is Gorenstein, then it is Tor-friendly, by case (b). Else, $(0 : n) \not\subseteq n^2$ holds, so $S$ is Tor-friendly by Proposition 4.3(2).

5.10. Proof of Theorem 5.1. In view of Proposition 1.4, it suffices to show that $R'$ is Tor-persistent. Thus, for the rest of the proof we assume $R' = R$.

Due to Lemma 5.7, we may assume that $Q$ is complete, has no embedded deformation, and its residue field is algebraically closed. Now Lemma 5.9 shows that $Q$ is Tor-friendly. In view of Theorem 2.2, the desired assertion will follow once we show that there is no finite $Q$-module $L$ with $c_{Q}L = 1$.

In case (d) this is evident: Complete intersections with no embedded deformations are regular, so $c_{Q}L = 0$. When $Q$ is Golod and not complete intersection, Lescot [36, 6.5] proved that $\text{projdim}_{Q} M = \infty$ implies $\beta_{n+1}^{Q}(M) < \beta_{n+1}^{Q}(M)$ for $n \gg 0$, and this settles case (g); see also [3, 5.3.3]. In case (h) the same conclusion was obtained by Gasharov and Peeva [21, 1.1], provided $\text{edim} Q - \text{depth} Q \geq 4$; when $\text{edim} Q - \text{depth} Q \leq 3$ holds case (a) applies; it is treated below.

In the remaining cases we argue by contradiction. Assume that there exists a finite $Q$-module $L$ with $c_{Q}L = 1$. Replacing it with a sufficiently high syzygy module, we may further obtain $\text{depth}_{Q}L = \text{depth} Q$; see [3, 1.2.8]. From [2, 1.6.II] we see that in cases (a), (b), (c), or (f) the virtual projective dimension of $L$ (defined in [1, 3.3]) is equal to 1: by [34, 5.2(3)] the same conclusion holds in case (c) as well. Since $Q$ is complete with infinite residue field, [1, 3.4(c)] yields a deformation $Q \rightarrow P$ with $\text{edim} P = \text{edim} Q$ and $\text{projdim}_{P} L = 1$. Since $Q$ has no embedded deformation we must have $P = Q$. This gives $\text{projdim}_{Q}L = 1$, and hence $c_{Q}L = 0$. We have produced the desired contradiction, so the proof of the theorem is complete.

5.11. The preceding result gives a direction to a search for rings that may not be Tor-persistent. Naturally, the first example to check is the artinian ring in [20, 3.4] (reproduced in [3, 5.1.4]), which has embedding dimension 4 and length 8. However its Hilbert series is $1 + 4t + 3t^2$ and so Theorem 4.5(2) yields that the ring is Tor-persistent. In fact, any local ring $(R, m, k)$ with $m^3 = 0$ is Tor-persistent; this is proved in [39].

6. Cohomological persistence

In this section we explore the cohomology analogue of Tor-persistence. Throughout, $(R, m, k)$ will be a local ring.

6.1. We say that $R$ is cohomologically persistent, or Ext-persistent, if every $R$-complex $M$ for which $H(M)$ is finite and $\text{Ext}_{R}(M, M)$ is bounded is either perfect or quasi-isomorphic to a bounded complex of injective $R$-modules.

In contrast with Tor-persistence, not every ring can be Ext-persistent.
6.2. An $R$-complex $M$ is **semidualizing** if $H(M)$ is finite and the canonical morphism $R \to \text{Ext}_R(M, M)$ is bijective; equivalently, $\text{Ext}_R^i(M, M) = 0$ for $i \neq 0$ and there is an isomorphism $\text{Ext}_R^0(M, M) \cong R$.

The ring $R$ itself, viewed as an $R$-module is semidualizing, as is any dualizing complex for $R$. There exist rings that admit semidualizing complexes besides these obvious ones; see, for example, [17, 7.8]. Such a ring is not Ext-persistent.

The next result contains the analogue of Proposition 1.4 and Theorem 2.2 for Ext-persistence; unlike in the latter, there are no additional hypothesis on $Q$.

**Theorem 6.3.** Assume that there exist a local homomorphism $R \to R'$ of finite flat dimension and a deformation $R' \leftarrow Q$.

If $Q$ is Ext-persistent, then so is $R$.

**Corollary 6.4.** If $R$ satisfies the hypotheses of Theorem 5.1, then it is Ext-persistent. In particular, such an $R$ has no nontrivial semidualizing complexes.

The proofs are given in 6.9 and 6.10, respectively, following some preparation.

The next result complements Proposition 3.2. It follows from the implication (i) $\implies$ (ii) that Tor-friendly rings are Ext-persistent.

**Proposition 6.5.** The following conditions on a local ring $R$ are equivalent.

(i) The ring $R$ is Tor-friendly.

(ii) If $U$ and $V$ are $R$-complexes, such that $H(U)$ and $H(V)$ are finite and $\text{Ext}_R(U, V)$ is bounded, then $U$ is perfect or $V$ is quasi-isomorphic to a bounded complex of injective $R$-modules.

(iii) If $U$ and $V$ are $R$-complexes, such that $H(U)$ and $H(V)$ have finite length and $\text{Ext}_R(U, V)$ is bounded, then $U$ is perfect or quasi-isomorphic to a bounded complex of injective $R$-modules.

**Proof.** The implication (ii) $\implies$ (iii) is a tautology, while (iii) $\implies$ (ii) is verified by an argument similar to the one for the corresponding implication in Proposition 3.2.

Let $E$ be the injective hull of the residue field of $R$. The standard isomorphisms

$$\text{Hom}_R(\text{Tor}_R^n(U, V), E) \cong \text{Ext}_R^n(U, \text{Hom}_R(V, E)) \quad \text{for} \quad n \in \mathbb{Z}$$

show that $\text{Tor}_R^n(U, V)$ is bounded if and only if $\text{Ext}_R(U, \text{Hom}_R(V, E))$ is bounded. When $H(V)$ has finite length, Matlis duality yields the following assertions:

(a) The length of $H(\text{Hom}_R(V, E))$ is finite.
(b) The canonical map $V \to \text{Hom}_R(\text{Hom}_R(V, E), E)$ is an quasi-isomorphism.
(c) $V$ is perfect if, and only if, $\text{Hom}_R(V, E)$ is quasi-isomorphic to a bounded complex of injective $R$-modules.

In view of these properties, condition (iii) above is equivalent to the corresponding condition in Proposition 3.2, so that result gives (iii) $\iff$ (i). \qed

6.6. Let $R = Q/(f)$ where $Q$ is a local ring and $f$ is a $Q$-regular set.

If $\text{Ext}_R(M, N)$ is bounded, then so is $\text{Ext}_Q(M, N)$.

This follows from the standard change-of-rings spectral sequence

$$E_2^{p,q} = \text{Ext}_R^p(M, \text{Ext}_Q^q(R, N)) \implies \text{Ext}_Q^{p+q}(M, N)$$

Indeed, resolving $R$ over $Q$ by means of the Koszul complex on $f$ one gets an isomorphisms $\text{Ext}_Q^q(R, N) \cong N(\cdot)_q$ for every integer $q$, whence $E_2^{p,q} \cong \text{Ext}_R^p(M, N)(\cdot)_q$. 

The statement about projective dimensions in the following result is equivalent to [4, 4.2], where the proof relies on minimal free resolutions. The argument given below works equally well for injective dimension and for projective dimension.

**Proposition 6.7.** Let \( R \leftarrow Q \) be a deformation and \( U \) an \( R \)-complex with \( H(U) \) finite, such that \( \text{Ext}^2_R(U, U) = 0 \) for some positive integer \( j \).

1. If \( U \) is perfect over \( Q \), then it is perfect over \( R \).
2. If \( U \) is quasi-isomorphic to a bounded complex of injective \( Q \)-modules, then it is quasi-isomorphic to a bounded complex of injective \( R \)-modules.

**Proof.** (2) Let \( \{\xi_1, \ldots, \xi_s\} \) be a set of generators of \( \text{Ext}_R(k, U) \) as graded module over \( R[\chi] \); see 2.4. For \( m = \max\{\deg(\xi_r)\}_{1 \leq r \leq s} \) and each \( i \geq m + 2j \) we then have

\[
\text{Ext}_R^i(k, U) = \sum_{h \geq j} (\chi)^h \text{Ext}_{R}^{i-2h}(k, U)
\]

\[
\subseteq \sum_{h \geq j} (\text{Ext}_R^2(U, U))^j(\chi)^{h-j} \text{Ext}_{R}^{i-2h}(k, U)
\]

\[
\subseteq \sum_{h \geq j} \text{Ext}_R^2(U, U) \circ \text{Ext}_R^{i-2j}(k, U)
\]

Thus, \( \text{Ext}_R^2(U, U) = 0 \) implies \( \text{Ext}_R^{>0}(k, U) = 0 \), so by [5, 5.5(I)] it is quasi-isomorphic to a bounded complex of injective \( R \)-modules.

(1) One uses the graded bimodule \( \text{Ext}_R(U, k) \) and argues as above. \( \square \)

**Proposition 6.8.** Let \( R \rightarrow S \) be a local homomorphism of finite flat dimension and \( F \) a semifree \( R \)-complex with \( H(F) \) finite.

If the \( S \)-complex \( S \otimes_R F \) is perfect, then so is \( F \).

If the \( S \)-complex \( S \otimes_R F \) has finite injective dimension, then so does \( F \).

**Proof.** We may assume that the \( R \)-complex \( F \) is minimal, that is to say, \( \partial(F) \subseteq mF \), where \( m \) is the maximal ideal of \( R \). Then \( F \) is perfect if and only if \( F_i = 0 \) for \( |i| = 0 \) for \( i \gg 0 \). It is evident that the \( S \)-complex \( S \otimes_R F \) is minimal, so it follows that when it is perfect so is \( F \).

In the remainder of the proof, let \( k \) denote the the residue field of \( R \) and \( E \) its resolution by finite free \( R \)-modules. Let \( I \) be a semiinjective resolution of \( F \) and \( G \) a finite resolution of \( S \) by flat \( R \)-modules. One then has the following quasi-isomorphisms of \( R \)-complexes:

\[
G \otimes_R I \leftarrow \sim G \otimes_R F \rightarrow \sim S \otimes_R F.
\]

These induce the quasi-isomorphisms below

\[
\text{Hom}_R(E, I) \otimes_R G \cong \text{Hom}_R(E, G \otimes_R I)
\]

\[
\cong \text{Hom}_R(E, G \otimes_R F)
\]

\[
\cong \text{Hom}_R(E, S \otimes_R F)
\]

\[
\cong \text{Hom}_S(S \otimes_R E, S \otimes_R F)
\]

The isomorphisms are standard. Note that \( \text{H}(S \otimes_R E) \) is bounded, for it is isomorphic to \( \text{Tor}^R(S, k) \) and the flat dimension of \( S \) over \( R \) is finite. The finiteness of the injective dimension of \( S \otimes_R F \) thus implies that the homology of the complex \( \text{Hom}_S(S \otimes_R E, S \otimes_R F) \) is bounded. Therefore the quasi-isomorphisms above yields the same conclusion for the complex \( \text{Hom}_R(E, I) \otimes_R G \). The homology of the
$R$-complex $\text{Hom}_R(E, I)$ is degreewise finite, as it is isomorphic to $\text{Ext}_R(k, F)$, so the version of the Amplitude Inequality from [19, 3.1] (see [18, 5.12] for a different proof) implies then that $\text{Ext}_R(k, F)$ is bounded. This implies that the injective dimension of $F$ is finite, by see [5, 5.5(I)].

\section*{6.9. Proof of Theorem 6.3.} Assume that $\text{Ext}_R(M, M)$ is bounded.

Let $F \xrightarrow{\sim} M$ be a free resolution by finite free $R$-modules. The graded module $\text{H}(\text{Hom}_R(F, M))$ is isomorphic to $\text{Ext}_R(M, M)$, and thus bounded. Let $G \xrightarrow{\sim} R'$ be a finite resolution by flat $R$-modules. In view of the quasi-isomorphisms

$$G \otimes_R \text{Hom}_R(F, M) \cong \text{Hom}_R(F, G \otimes_R M)$$

$$\xleftarrow{\sim} \text{Hom}_R(F, G \otimes_R F)$$

$$\xrightarrow{\sim} \text{Hom}_R(F, R' \otimes_R F)$$

$$\cong \text{Hom}_R(R' \otimes_R F, R' \otimes_R F)$$

all the complexes involved have bounded homology. Since $R' \otimes_R F$ is semifree over $R'$, the homology of the last complex is $\text{Ext}_R(R' \otimes_R F, R' \otimes_R F)$.

The hypothesis is inherited by the $R'$-module $M' = R' \otimes_R F$ and the finiteness of $\text{proj} \dim_R(M')$ (respectively, $\text{inj} \dim_R(M')$) implies that of $\text{proj} \dim_R M$ (respectively, $\text{inj} \dim_R(M')$); see Proposition 6.8. Thus, we replace $R$ and $M$ by $R'$ and $M'$, and assume that $R$ itself has a deformation $Q \to R$, where $Q$ is Ext-persistent.

Now from 6.6 one gets that $\text{Ext}_Q(M, M)$ is bounded, and hence that $\text{proj} \dim_Q M$ or $\text{inj} \dim_Q M$ is finite. Since $\text{Ext}_R(M, M)$ is bounded, it then follows from Proposition 6.7 that $\text{proj} \dim_R M$ or $\text{inj} \dim_R M$ is finite, as desired.

\section*{6.10. Proof of Corollary 6.4} When $R$ satisfies the hypotheses of Theorem 5.1, Lemmas 5.7 and 5.9 yield local ring homomorphisms $R \to R' \xleftarrow{Q}$, such that the first one is of finite flat dimension, the second one is a deformation, and the ring $Q$ is Tor-friendly. Therefore $Q$ is Ext-persistent, by Proposition 6.5, and hence so is $R$, by Theorem 6.3.

\section*{7. Rings with the Auslander-Reiten Property}

As before, $R$ denotes a commutative noetherian ring.

\section*{7.1. We say that $R$ has the Auslander-Reiten property if every finite $R$-module $M$ satisfies the equality

$$(7.1.1) \quad \text{proj} \dim_R M = \sup\{i \in \mathbb{Z} \mid \text{Ext}_R^i(M, R \oplus M) \neq 0\}$$

This is equivalent to the condition that the graded module $\text{Ext}_R(M, R \oplus M)$ is bounded only if proj $\dim_R M < \infty$. For when $p = \text{proj} \dim_R M$ is finite, using a minimal free resolution of $M$ we get $\text{Ext}_R^p(M, R) \neq 0$ and $\text{Ext}_R^n(M, N) = 0$ for all $n > p$ and any $R$-module $M$, so that (7.1.1) holds.

\section*{Theorem 7.2. Each Ext-persistent local ring has the Auslander-Reiten property.}

\textbf{Proof.} Let $R$ be an Ext-persistent local ring and $M$ a finite $R$-module such that $\text{Ext}_R(M, R \oplus M)$ is bounded. Then $\text{Ext}_R(R \oplus M, R \oplus M)$ is also bounded, and hence the Ext-persistence of $R$ implies that $R \oplus M$, equivalently, $M$, has finite projective dimension or $R \oplus M$ has finite injective dimension. It remains to note that, in the latter case, inj $\dim_R R$ is finite, so $R$ is Gorenstein, and so the finiteness of inj $\dim_R M$ implies that of proj $\dim_R M$. \hfill $\Box$
The next result is a direct consequence of the preceding one and Corollary 6.4.

**Corollary 7.3.** If $R$ satisfies the hypotheses of Theorem 5.1, then it has the Auslander-Reiten property. □

To wrap up this discussion we record an analogue of Theorem 6.3; the proof is also similar, even a little easier for one does not have to contend with finiteness of injective dimension, and so is omitted.

**Theorem 7.4.** Assume that there exist a local homomorphism $R \rightarrow R'$ of finite flat dimension and a deformation $R' \equiv Q$.

If $Q$ has the Auslander-Reiten property, then so does $R$. □

In contrast with (7.1.1) the projective dimension of $M$ cannot be determined from where $\text{Tor}_i^R(M, M)$ vanishes, though the latter does provide bounds.

**Example 7.5.** If $M$ is a finite $R$-module of finite projective dimension, then with $s = \sup \{i \mid \text{Tor}_i^R(M, M) \neq 0\}$ there are bounds

$$s \leq \text{proj dim}_R M \leq (\text{depth } R + s)/2.$$ 

The inequality on the left is clear, whereas one on the right follows from [19, 2.4.2.7]. These inequalities can be strict: Let $R$ be a regular local ring of Krull dimension $2d$. For each integer $1 \leq j \leq d$, the $R$-module $M_j = \Omega_{d+j}^R(k)$ satisfies

$$\text{proj dim}_R M_j = d - j$$

$$\text{Tor}_i^R(M_j, M_j) = 0 \quad \text{for } i \geq 1$$

For a proof of these assertions see, for example, [16, 2.2].

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