A new method for the calculation of massive multiloop diagrams

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November 1997

Abstract

Starting from the parametric representation of a Feynman diagram, we obtain it’s well defined value in dimensional regularisation by changing the integrals over parameters into contour integrals. That way we eventually arrive at a representation consisting of well-defined compact integrals. The result is a simple transformation of the integrand which gives the analytic continuation of a wide class of Feynman integrals. The algorithm will especially be fit for numerical calculation of general massive multi-loop integrals. An important advantage of this method is that it allows us to calculate both infinite and finite parts independently.

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1 Introduction

In recent years there has been an increasing interest for the evaluation of massive multiloop Feynman diagrams. High precision experiments force the theoretical predictions to reach an equal amount of accuracy. Over the years several methods have been proposed to deal with the problem.

The first successful attempts to calculate Feynman-diagrams systematically were integration by parts [1], IR-rearrangement method [2] and gegenbauer technique [3]. They are not fit though for calculating finite parts of general massive diagrams, although they have led to some quite impressive calculations of beta-functions in different theories [5, 4]. An excellent review of these methods is given in [4].

A first general massive approach [6, 7, 8, 9] is analytic by nature and is based on the following basic principle: by putting a certain number of masses in the diagram equal to zero a gamma-function ansatz in obtained, upon which the masses can be added again by means of the Mellin-Barnes representation. This gives rise to hypergeometric series which have proven to be rather successful, e.g. for the asymptotic expansions in the two loop case [10]. These series have certain drawbacks however: they converge only in certain kinematic regions and although the ansatz needed for the application of this method can easily be found in the case of two loops, more loops will give considerable problems. Applications of these methods are therefore mainly confined to asymptotic 2-loops cases.

Other algorithms avoid the use of Euler-gamma functions and therefore turn out to be numerical. An excellent example of such a method, which even avoids parametric representation and Wick-rotation, relies on
the separation of an orthogonal space of momenta and integrating out this space first. Although this method has been successfully used for some specific cases \cite{11, 12, 13, 14}, it has not yet been expanded up to three loops or more. Other numerical approaches \cite{15, 16} rely on the parametric representation of the diagram. These approaches also differ mainly from ours in this respect that these methods are based on a subtraction of divergences under the parametric integral, while ours is fundamentally based on dimensional regularisation. Analytic continuation in $d$ dimensions is central in our approach: thus we obtain not only the finite parts but also the poles of the Feynman-diagram. It is therefore that it has no trouble in dealing with IR and UV divergences at the same time. Our method is without reservation applicable in a vast number of cases and easily implemented on a computer.

The rest of the article is organised as follows. Section two contains the basic formulas of the contour method and some adaptations desirable for smooth numerical calculations. Section three consists of some examples and section four is a summary and conclusion.

2 Contour method

A scalar diagram $D$ with $L$ loops, $I$ internal lines each labeled by number $l$ and a mass $m_l$, a total external momentum $P_j$ per vertex and a space-time dimension $d$ has a parametric representation \cite{17}

$$I_D(P) = \int_0^\infty \prod_l \alpha_l \exp \left[ -\sum_i \alpha_i m_i^2 + Q_D(P, \alpha) \right] \frac{\prod_{l \notin T} \alpha_l}{(4\pi)^{2L} R_D(\alpha)^{d/2}}$$

(1)

where

$$R_D(\alpha) = \sum_T \prod_{l \notin T} \alpha_l$$

(2)
\[ Q_D(P, \alpha) = \frac{1}{R_D(\alpha)} \sum_{T^2} s_{T^2} \prod_{l \in T^2} \alpha_l \]  

with \( \mathcal{T} \) the set of all the trees of \( D \) and \( \mathcal{T}^2 \) the set of all the two-trees of \( D \) and \( s_{T^2} \) the square of the momentum which passes through the cut. \( R_D \) is a homogeneous polynomial in \( \alpha \) of degree \( L \) and \( Q_D \) of degree 1. If the diagram gives rise to tensor integrals with irreducible numerators we can change these to scalar integrals using for example the general expression in [24, 25].

The general philosophy of our method will be as follows: we will isolate the different poles in the integrand and then avoid them by changing the integral in a contour integral flung around the pole. This way we will obtain a well-defined analytic continuation of the Feynman integral in dimensional regularisation.

Our first goal will be to isolate the poles in (1). Since the polynomial \( R_D \) of e.g. the setting-sun diagram has the form

\[ \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_1 \alpha_3 \]

this is not always obvious: no simple poles are visible. There is however a well-known substitution, used in the convergence theorem [17], which does the job. It involves a separation of the integration domain into sectors

\[ 0 \leq \alpha_{\sigma(1)} \leq \alpha_{\sigma(2)} \leq \cdots \leq \alpha_{\sigma(I)} \]

with \( \sigma \) a permutation of \((1, 2, \ldots, I)\). Per sector we preform the following change of variables

\[ \alpha_{\sigma(1)} = \beta_1 \beta_{I-1} \cdots \beta_2 \beta_1 \]
\[ \alpha_{\sigma(2)} = \beta_1 \beta_{I-1} \cdots \beta_2 \]
\[ \alpha_{\sigma(I)} = \beta_I \] 

(4)

where

\[ 0 \leq \beta_I \leq \infty \]

\[ 0 \leq \beta_l \leq 1 \quad l = 1, \ldots, I - 1 \]

Now we can interpret each sector as a family of nested subsets of lines of the diagram \( D \): these nested subsets \( S_i \subset S_j \) are determined by the scaling behaviour of \( \beta_i \) and \( \beta_j \), \( i > j \).

We can prove the following form for \( R_D \):\[ R_D(\beta) = \beta_1^{L_1} \beta_2^{L_2} \ldots \beta_I^{L_I} [1 + \mathcal{O}(\beta)] \]

(5)

where \( L_i \) is the number of independent loops in the subset of lines specified by the lines which disappear if the corresponding \( \beta_i \) is put to zero (remember: each line corresponds to a certain \( \alpha \)). Algebraically we have independent poles in the denominator and now we will be able to preform our "contouration".

The parametric representation (1) becomes

\[
I_D(P) = \sum_{\sigma} \int_0^\infty d\beta_I \beta_I^{I-1} \int_0^{1-I-1} \prod_{l} \left( d\beta_l \beta_l^{l-1} \right) \\
\times \exp \left[ -\sum_{i=1}^I \prod_{j=i}^I \beta_j m_{\sigma(i)}^2 + Q_D(P, \beta) \right] \\
\frac{1}{[R_D(\beta_1, \ldots, \beta_{I-1})]^{d/2}}
\]

thus a sum over permutations \( \sigma \), which means a maximum of \( I! \) terms. Thanks to the symmetry in the diagram a certain collecting of terms will usually be possible. From now on we will omit the summation over \( \sigma \) and
concentrate on one sector only. For notational convenience we take \( \sigma \) to be the identical permutation. Due to homogeneity of \( R_D \) and \( Q_D \) \([7]\) we can immediately perform integration over \( \beta_I \) and get

\[
I_I D(P) = \frac{\Gamma(I - Ld/2)}{(4\pi)^{2L}} \int_0^1 \prod_{l=1}^{I-1} \left( d\beta_l \beta_l^{I-1} \right) \\
\times \frac{[\sum_i \prod_{l=1}^{I-1} \beta_l^m m_l^2 + Q_D(P, \beta_1, \ldots, \beta_{I-1})]^{Ld/2-I}}{[R_D(\beta_1, \ldots, \beta_{I-1})]^{d/2}}
\]

We will omit these independent factors in front of the integration in what follows.

Only \( R_D \) gives rise to possible poles if the diagram is globally UV-divergent (i.e. \( Ld/2 - I > 0 \)). If the diagram is UV-convergent and some masses are zero then poles can come from the numerator in \([3]\) (which is then in the denominator!): they correspond to IR-divergences. If \( Ld/2 - I > 0 \) and masses are zero, these IR-factors may compensate some UV-poles: this is a well known fact in dimensional regularisation. Now we can write \( I_I D \) in the following form

\[
I_I D(P) = \int_0^1 d\beta_1 \cdots \int_0^1 d\beta_{l+1} \int_0^1 d\beta_{l+1} \cdots \int_0^1 d\beta_{I-1} \\
= \int_0^1 d\beta_1 \cdots \int_0^1 d\beta_{l+1} \int_0^1 d\beta_{l+1} \cdots \int_0^1 d\beta_{I-1} \\
\times \frac{1}{\beta_1^{p_1} \beta_{l+1}^{p_{l+1}} \beta_{l+1}^{p_{l+1}} \cdots \beta_{I-1}^{p_{I-1}}} \left( \int_0^1 d\beta_{I} \frac{f(\beta)}{\beta_{I}^{p_{I}}} \right) \\
\times \frac{1}{\beta_1^{p_1} \beta_{l+1}^{p_{l+1}} \beta_{l+1}^{p_{l+1}} \cdots \beta_{I-1}^{p_{I-1}}} I(t)
\]

where the function \( f(\beta_1, \ldots, \beta_I) \) is written as as \( f(\beta) \) for convenience. We define a unique \( p_i \) and \( f \) by demanding that \( f(\beta_i = 0) \neq 0 \) and that it is analytic at \( \beta_i = 0 \). In dimensional regularisation, i.e. \( d = 4 - 2\varepsilon \),

\[
p_i = n_i - q_i \varepsilon,
\]
Figure 1: The contour for obtaining analytic continuation of $\beta$-integrals

$n_i$ integer and $q_i$ rational. From now on we will omit the cases $n_i \leq 0$ because in these cases the integration can be preformed without regularisation.

Let’s concentrate on the integration of one specific $\beta_l$. In the function $f(\beta)$ all other $\beta_i$, $i \neq l$ will be considered to be parameters for the time being. We will use the notation $f(\beta_l)$. The function $f(\beta_l)$ is analytic at the origin by definition. In dimensional regularization (i.e. $q_l \neq 0$) the integrand as a whole is analytic in a region around the real axis for $\text{Re}\beta_l \geq 0$ minus the cut line along this positive axis. This is true for every $l = 1, \ldots, I$.

We will obtain a meaningful regularized value for the integral $I(\beta_l)$ if we change the integral into a contour-integral, which coincides with the original integral in non-divergent cases. In concreto we define the $U_l$ (un-renormalized) operator (e.g. [18]) to be

$$U_l(I(\beta_l)) = \frac{1}{e^{2\pi i p_l} - 1} \oint_C d\beta_l \frac{f(\beta_l)}{\beta_l^{p_l}}$$

where the contour $C$, as is showed in figure 1, indeed avoids the pole $\beta_l = 0$ by a tiny contour $C_0$ which is a circle with radius $\Delta$, $\lim \Delta \to 0^+$ and the integrand is analytic in the region of the contour as shown before. It is clear that this definition only holds for non-integer $p_l$: our method is intimately
connected to dimensional regularization.

In order to get rid of contours again, we preform integration over $C_0$ explicitly. We are able to do so if we write $f(\beta_l)$ in a Taylor expansion (this is allowed due to analyticity of $f$ at the origin). Since $\lim \Delta \to 0^+$ we keep the $n_l$ first terms of the Taylor expansion. Other terms vanish in this limit, they are namely $\mathcal{O}(\Delta^{p_l \epsilon})$ and higher. Finally we obtain

$$U_l(I_l) = \int_\Delta^1 d\beta_l \beta_l^{p_l - 1} \left[ f(\beta_l) - \sum_{j=0}^{n_l-1} \frac{\beta_l^{j-p_l}}{j!} \left( \frac{\partial^j f(z_l)}{\partial^{j} z_l} \right)_{z_l=0} \right]$$

This form still contains $\Delta$ explicitly. If we use the identity for every $p_l \neq 0$

$$\int_\Delta^1 d\beta_l \beta_l^{p_l - 1} = \frac{1}{p_l} - \frac{\Delta^{p_l}}{p_l}$$

and regroup the terms, we find for the case $n_l \geq 1$

$$U_l(I_l) = \int_\Delta^1 d\beta_l \left( f(\beta_l) - \sum_{j=0}^{n_l-1} \frac{\beta_l^{j-p_l}}{j!} \left( \frac{\partial^j f(z_l)}{\partial^{j} z_l} \right)_{z_l=0} \right) + \sum_{j=0}^{n_l-1} \frac{1}{j!} \left( \frac{1}{j - p_l + 1} \right) \left( \frac{\partial^j f(z_l)}{\partial^{j} z_l} \right)_{z_l=0}$$

Noticing that the integrand under the first integral as a whole is of the order $\beta_l^{n_l \epsilon}$, the integral will be convergent in the limit $\Delta \to 0$. We will use the following notation

$$f^{(r)}_{(l)} = \left( f(\beta_l) - \sum_{j=0}^{n_l-1} \frac{\beta_l^{j-p_l}}{j!} \left( \frac{\partial^j f(z_l)}{\partial^{j} z_l} \right)_{z_l=0} \right)$$

and for the other terms

$$f^{(j)}_{(l)} = \frac{1}{j!} \left( \frac{1}{j - p_l + 1} \right) \left( \frac{\partial^j f(z_l)}{\partial^{j} z_l} \right)_{z_l=0}$$

They are independent of $\beta_l$. 

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We can write this result as a simple transformation of the integral

\[ f_{(i)} = U_l(I_{(i)}) \]
\[ = U_l \left( \int_0^1 \frac{d\beta_l f(\beta_l)}{\beta_l^p} \right) \]
\[ = \int_0^1 d\beta_l f^{(s)}_{(i)} + \sum_j f^{(j)}_{(i)} \]  

(9)

The next difficulty we have to face, is whether or not this operation can be repeated for other \( \beta \)'s? Potential problems arise when \( \sum \beta_i m_i^2 + Q_D(P, \beta) \) comes in the denominator: we have not been able to prove an expression like (5) for it, but this appears to be always the case. We have done numerous tests on diagrams up to three loops under different circumstances (several external momenta, masses or massless,...) and have found no counterexamples. Yet a rigorous mathematical proof that this applies for every diagram is still lacking. Here we will assume that the diagrams we are dealing cause no problems.

In that case if we have carried out the \( U_l \)-operation, we can re-establish the other \( \beta_i \) as true variables. Then we choose another \( l_2 \) to focus on, write

\[ I_{(l_1,l_2)} = \int_0^1 d\beta_{l_2} f^{(l_1)}_{l_2} \]

and apply the operator \( U_{l_2} \) to \( I_{(l_1,l_2)} \). We can maintain our notation by adding a number to the sub- and superscript of \( f \), e.g. after \( U_2 \circ U_1 \) we will get

\[ f_{(l_1,l_2)} = f^{(*)}_{l_1,l_2} + \sum_i f^{(i,*)}_{l_1,l_2} + \sum_j f^{(*)}_{l_1,l_2} + \sum_{i,j} f^{(i,j)}_{l_1,l_2} \]

We repeat this procedure until no \( \beta \)-poles are left, thus regulazing the feynman-integral \( I_D(P) \) completely.

Several remarks are in order.
• We can regulaze every variable $\beta_i$ before preforming integration, i.e. $U_{(i)}$ commutes with $\int d\beta_j$ for $i \neq j$. We only have to take care to keep the different parts of $f_{(i)}^{(*)}$ (all other indices are arbitrary: $j$ or $*$) together, because each term on his own has a pole in $\beta_l$, only the sum converges. Before preforming integration we can also expand in $\epsilon$. This allows us to calculate the different coefficients of the laurent-expansion in $\epsilon$ seperately. Thus the $T$ operation of [18] (isolating the divergent part) is easily implemented. Higher pole parts can only come from the $f_{(i)}^{(*)}$ parts.

• The result of the different operations $U_{l_1} \circ U_{l_2} \circ \ldots \circ U_{l_n}$ is independent of their order. If there are only poles in $\beta_1$ and $\beta_2$, so $U_{l_1} \circ U_{l_2}$ completely regularizes the integral then:

$$U_{l_1} \circ U_{l_2} = U_{l_2} \circ U_{l_1}$$

This follows from the fact that the $U$-operators are in fact analytic continuations (see our primary definition (8)): if $\text{Re} p_l \leq 1$ the contour-integrals coincide with the ordinary integrals which are convergent, so $U_{l_1} \circ U_{l_2} = U_{l_2} \circ U_{l_1}$ in the area $\text{Re} p_l \leq 1$. By the principle of analytic continuation they must coincide for all complex $d$ (which fixes all the $p_i$’s), for which $U_{l_1}$ and $U_{l_2}$ exist. Of course this implies that for every variable $\beta_i$ this analytic continuation has been carried through.

It is clear that an expression like (9) will be especially useful in numerical calculations and therefore it might be useful to examen it’s numerical behaviour. Only $f_{(i)}^{(*)}$ will cause trouble around zero. Although the function as a whole is finite, it is really a subtraction of one or more infinite
values. Moreover it is of the order $\beta_l^{q\epsilon}$ which after expanding in $\epsilon$ gives $1 + q_l \epsilon \ln(\beta_l) + \cdots$ and is thus another thread to numerical stability. Therefore we will approximate $f^{(s)}_{(l)}$ around zero, by the next term in the Taylor expansion, i.e.

$$f^{(s)}_{(l)} \approx \frac{\partial^n f(z_l)}{\partial z_l^n} \bigg|_{z_l=0} \frac{\beta_l^{q\epsilon}}{n!}$$

Introducing this approximation in the interval $[0, \delta]$, $\delta \ll 1$ being a certain numerical value, we obtain

$$\int_0^1 \frac{d\beta_l}{\beta_l^{p_l}} f^{(s)}_{(l)} \approx \frac{\partial^n f(z_l)}{\partial z_l^n} \bigg|_{z_l=0} \frac{\delta^{1-q_l\epsilon}}{(1-q_l\epsilon)n!} + \int_0^1 \frac{d\beta_l}{\beta_l^{p_l}}$$

(10)

The intrinsic error of this approximation depends on the highest order pole. If it is $1/\epsilon^L$, then terms proportional to $\ln(\delta)^{L-1}$ will appear. If $\delta$ is very small then this correction will be large, resulting in a smaller accuracy. Thus there is a best value for $\delta$ typical for every diagram. Typical values are e.g. $\delta = 10^{-5}$. In order to obtain larger values for $\delta$ without losing accuracy one could include more terms of the taylor expansion.

3 Examples

In this section we will discuss one example at length: the famous ”setting-sun” diagram in scalar $\lambda\phi^4$. Then some other results for diagrams with a larger number of propagators are included. In order to be able to compare with analytic results, we begin by choosing some trivial vacumbubbles: no external momenta and equal masses. Of course this is only to make comparison possible: it is the analytic methods that are bound by these limitations, not the contour method as will be shown in some explicit numerical examples.
We put the momentum $p$ equal to zero and choose the masses equal. This makes the diagram is very symmetrical: topologically every leg is equivalent and this will simplify the explicit calculation.

Starting from the well-known parametric representation of the setting-sun diagram, we get only one $\beta$-sector (all masses are equal). Applying formula (6), we get

$$I_{ss} = \frac{(m^2)^{d-3}}{(4\pi)^d} \Gamma(3-d)3! \int_0^1 d\beta_1 \int_0^1 d\beta_2 \frac{(1 + \beta_2 + \beta_1 \beta_2)^{d-3}}{(1 + \beta_1 + \beta_1 \beta_2)^{d/2}} \frac{1}{\beta_2^{d/2-1}}$$

(11)

The diagram is globally divergent in $d \geq 3$ whence the factor $\Gamma(3-d)$. At this stage we see a simple pole in $\beta_2$ emerging. This corresponds to the two-legged subdivergence present in the diagram (renormalization of the coupling constant). With the notations of section 2, we have

$$f(\beta_2) = \frac{(1 + \beta_2 + \beta_1 \beta_2)^{d-3}}{(1 + \beta_1 + \beta_1 \beta_2)^{d/2}}$$

and $p_2 = n_2 - q_2 \varepsilon = 1 - \varepsilon$. Applying formula (11) we get

$$U_{(2)} \left( \int_0^1 d\beta_2 \frac{(1 + \beta_2 + \beta_1 \beta_2)^{d-3}}{(1 + \beta_1 + \beta_1 \beta_2)^{d/2}} \frac{1}{\beta_2^{d/2-1}} \right)$$

$$= \int_0^1 d\beta_2 \frac{(1 + \beta_2 + \beta_1 \beta_2)^{d-3}}{(1 + \beta_1 + \beta_1 \beta_2)^{d/2}} \frac{1}{(1 + \beta_1)^{d/2}} \frac{1}{\beta_2^{d/2-1}}$$

$$+ \frac{1}{(2 - d/2)(1 + \beta_1)^{d/2}}$$

These integrals are finite, as expected. If we restrict ourselves to the divergent part (without expanding the $\Gamma(3-d)$ though, thus $f_{(2)}^{(s)}$ and the $f_{(2)}^{(j)}$'s up to $\varepsilon^0$) we get

$$f_{(2)}^{(s)} = \frac{1}{\varepsilon(1 + \beta_1)^2} + \frac{\ln(1 + \beta_1)}{(1 + \beta_1)^2}$$

$$f_{(2)}^{(0)} = \frac{1}{\varepsilon(1 + \beta_1)^2} + \frac{\ln(1 + \beta_1)}{(1 + \beta_1)^2}$$
We can perform this integration analytically

\[
\int_0^1 d\beta_2 \int_0^1 d\beta_1 f^{(s)}_{(2)} = \frac{\ln 2}{2} \\
\int_0^1 d\beta_1 f^{(0)}_{(2)} = \frac{1}{2\varepsilon} + \frac{1 - \ln 2}{2}
\]

If we expand further in \(\varepsilon\) and perform the numerical calculations we obtain the finite part as well:

\[
I_{ss} = \frac{(m^2)^{d-3}}{(4\pi)^d} \Gamma(3 - d) \left( \frac{3}{\varepsilon} + 3 - 8.966523919\varepsilon \right)
\]  
(12)

Comparing this result by numerical evaluation of the analytic results in [7] we see that only the last digit is different.

Another test we did was to repeat some of the cases from [6], i.e. a subtracted setting-sun diagram with arbitrary masses and external momentum.

A selection

| \(p^2\) | \(m_1\) | \(m_2\) | \(m_3\) | Contour    | Series     |
|---------|---------|---------|---------|------------|------------|
| 9       | 3       | 3       | 10      | -7.31299   | -7.31298   |
| 49      | 1       | 1       | 10      | -0.316742  | -0.31675   |
| -25     | 2       | 2       | 10      | -1.942844  | -1.94285   |
| -250    | 4       | 4       | 4       | -13.57190  | -13.5719   |
| 100     | 3       | 3       | 3       | -1.282852-20.89960 | -1.28285-20.8996 |
| 49      | 20      | 20      | 10      | -1014.695748 | -1014.695748 |

The last result cannot be calculated with the method of [6] because \(m_1 + m_2 < m_3\) should apply under the threshold.
Another test for our method was the so-called "basketball-diagram", a vacuum diagram in $\lambda\phi^4$ with 3 loops and 4 legs. We also assumed equal masses here in order to be able to compare to known analytic results. We have calculated the diagram in four cases: with one, two, three and four massive legs.

The numerical procedure as given by formula (10) in section two, was implemented in MATHEMATICA [19]: the diagram, expressed in its original form is automatically transformed into the parametric $\alpha$-form and $\beta$-form, then cast into the expression (10) and numerically evaluated. Because we are dealing with a very limited class of functions, it is possible to program a package that performs one integration analytically. This general procedure was used to evaluate all the following results.

These were the results we obtained up to order $\varepsilon^0$. We compare these values to analytic results in [20]. All results are given up to a factor $\frac{(m^2)^{4-3d/2}}{(4\pi)^{3d/2}}\Gamma(4-3d/2)$. Numerical evaluations of the analytic results are only shown in relevant cases.

| Type  | Contour | Analytic |
|-------|---------|----------|
|       | $\varepsilon^{-2}$ | $\varepsilon^{-1}$ | $\varepsilon^0$ | $\varepsilon^1$ | $\varepsilon^{-2}$ | $\varepsilon^{-1}$ | $\varepsilon^0$ | $\varepsilon^1$ |
| 1 mass| -0.5    | -1.5     | -5.967401126 | -1.5 | -5.967401100 | -1.5 | -5.967401100 |
| 2 masses| 2     | -2 | -19.86960443 | 22.33542372 | -19.86960440 | 22.335425302 |
| 3 masses| 6 | -4.5 | -55.10881324 | 44.04808956 | -55.10881320 | 44.04808897 |
| 4 masses| 12 | -8 | -107.2176265 | 26.87588164 | -107.2176264 | 26.87588031 |

These results were obtained with $\delta = 10^{-5}$ and relative errors are of order $10^{-8}$.

Other diagrams we evaluated are the 3-loop, 5 propagator diagram with
2 masses

\[ I_5 = \int dk \frac{1}{(k_1 - k_3)^2(k_1 - k_2)^2(k_2 - k_3)^2(k_3^2 + m^2)(k_2^2 + m^2)} \]

and the mercedes diagram with 3 masses in a very asymmetric configuration

\[ I_6 = \int dk \frac{1}{k_1^2(k_2^2 + m^2)k_3^2((k_1 - k_3)^2 + m^2)(k_1 - k_2)^2((k_2 - k_3)^2 + m^2)} \]

where \( dk \) stands for integration over all internal momenta \( k_i \). As a result we found:

\[ I_5 = \frac{(m^2)^{5-3d/2}\Gamma(5-3d/2)}{(4\pi)^{3d/2}} \left[ \frac{1}{\varepsilon^2} + \frac{3}{\varepsilon} + 2.065197729 + 19.42850404 \varepsilon \right] \]

and

\[ I_6 = \frac{(m^2)^{6-3d/2}\Gamma(6-3d/2)}{(4\pi)^{3d/2}} \left[ 7.21234140 - 9.081028480 \varepsilon \right] \]

In the case of \( I_5 \) we encounter relative errors of order \( 10^{-8} \) in comparison with analytic results. According to [20], \( I_6 \) has not yet been evaluated analytically. Our numerical results differ only by order \( 10^{-9} \) from theirs. We can conclude that the numerical behaviour of the method is fine.

4 Summary and conclusion

The main merit of our contour-method as introduced here, lies evidently in its general character. Indeed it is applicable for a wide class of Feynman-diagrams scalar and tensorial (although the borders of this class are not yet quite clear). The computer-algorithm based on expression (10) is easy to implement and virtually independent of the actual diagram fed to the system.

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In the section examples we confined ourselves mainly to diagrams for which analytic expressions are known, in order to investigate numerical behaviour. As shown in the case of the setting-sun diagram our contour-method is without further trouble applicable in the cases of external momenta and arbitrary masses and that is where her strength lies.

As remarked our contour-method is a numerical method in the first place. The necessary calculations involve especially a lot of numerical integrations, which are known not to be an easy problem. Even at this point our method has some advantages: the integration intervals are compact at all time \( f \) and -thanks to our numerical adjustment \( g \)- all integrands are finite. Such calculations can best be done by some adaptive Monte-Carlo method. The method will also give rise to a large number of integrals, but this drawback can partially be met using the recursion algorithms in \( h, i, j \), forcing us only to calculate a small number of so-called master-integrals. This is a common practice: such techniques have to be used in any method involving higher-order Feynman-diagrams in order to reduce the number of calculations.

To sum up, we have a method which is easy the implement and applicable in a vast number of cases, even those where traditional methods fail, such as finite parts of complicated diagrams. Furthermore we are able to calculate the different coefficients of the laurent-expansion in \( \varepsilon \) independently. Therefore it is a useful tool for high-precision calculation of general massive Feynman-diagrams.

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