Convergence of the Linear $\delta$ Expansion in the Critical $O(N)$ Field Theory

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The linear $\delta$ expansion is applied to the 3-dimensional $O(N)$ scalar field theory at its critical point in a way that is compatible with the large-$N$ limit. For a range of the arbitrary mass parameter, the linear $\delta$ expansion for $\langle \phi^2 \rangle$ converges, with errors decreasing like a power of the order $n$ in $\delta$. If the principal of minimal sensitivity is used to optimize the convergence rate, the errors seem to decrease exponentially with $n$.

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Very few systematically improvable methods are available for calculating nonperturbative quantities in field theory. Such a method has the advantages that more accurate results can be obtained at the expense of additional effort and that reliable error estimates can be made. One systematically improvable nonperturbative method that is well-developed is the Monte Carlo method applied to the field theory formulated on a discrete lattice. Since this method is not universally applicable, it is important to develop other nonperturbative methods.

A classic nonperturbative problem that was recently solved definitively is the shift due to interactions in the critical temperature $T_c$ for Bose-Einstein condensation (BEC). If the potential between two bosons is short-ranged, the leading order shift is linear in the $s$-wave scattering length $a$: $\Delta T_c/T_c = c n^{1/3} a$, where $n$ is the number density of the bosons and $c$ is a numerical constant. Baym et al. [1] showed that the coefficient $c$ could be determined by a nonperturbative calculation at the critical point of an effective 3-dimensional statistical field theory with $O(2)$ symmetry. Lattice Monte Carlo calculations by Kashurnikov, Prokof’ev, and Svistunov and by Arnold and Moore [3] give the result $c = 1.32 \pm 0.02$. The second order correction to $\Delta T_c/T_c$ proportional to $(an^{1/3})^2$ has also been calculated [2]. The definitive solution to this problem makes it useful as a testing ground for other nonperturbative methods.

One systematically improvable nonperturbative method that has been applied to the shift in $T_c$ is the 1/N expansion. The $O(2)$ field theory relevant to BEC can be generalized to $O(N)$. Baym, Blaizot and Zinn-Justin calculated the coefficient $c$ analytically in the large-$N$ limit and obtained $c = 2.33$ [2]. The first correction in the 1/N expansion reduces $c$ to 1.72 [3]. These results seem to be converging to the lattice Monte Carlo result. The analytic result in the large-$N$ limit can be used to test other nonperturbative methods. If a method fails to give the correct answer in the large-$N$ limit, one should be suspicious of its predictions for $N=2$.

Another nonperturbative method that has been applied to this problem is the linear $\delta$ expansion [4], also known as optimized perturbation theory [5] or variational perturbation theory [6]. In this method, an arbitrary parameter $m$ is introduced into the theory and calculations are carried out using perturbation theory in a formal expansion parameter $\delta = 1$. The rate of convergence can be improved by adjusting $m$ at each order in $\delta$ using the principal of minimal sensitivity (PMS). It was first applied to the calculation of $\Delta T_c$ by de Souza Cruz, Pinto and Ramos [7]. At 2nd, 3rd, and 4th order in $\delta$, they obtained $c = 3.06, 2.45$, and 1.48, respectively [7], which seem to be converging to the lattice Monte Carlo result.

The fact that corrections can be calculated systematically is no guarantee that they actually improve the result. Hopes for the convergence of the LDE are based largely on studies of its application to the anharmonic oscillator. The LDE has been proven to converge for order-dependent choices of $m$ that include the PMS criterion as a special case. The finite temperature partition function converges exponentially, with the errors at $n^{th}$ order decreasing as $\exp(-bT n^{2/3}/g^{1/3})$ where $g$ is the strength of the anharmonic term in the potential and $b$ is a numerical constant [11]. The energy eigenvalues $E_n$ have been proven to converge uniformly in $g$ as $n \to \infty$ [11]. In particular, the leading term in the strong-coupling expansion for the ground state energy $E_0$ converges exponentially, with the errors decreasing like $\exp(-b' n^{1/3})$ [12].

The anharmonic oscillator is equivalent to a Euclidean field theory with a single real-valued field in 1 space dimension. The statistical field theory relevant to BEC is a generalization to a multicomponent field in 3 space dimensions. The most serious obstacles to generalizing the convergence proofs for the anharmonic oscillator to this more complicated problem come from the infrared (IR) and ultraviolet (UV) regions of momentum space. It is reasonable to expect the convergence behavior to be similar if appropriate IR and UV cutoffs are imposed on the field theory. The coefficient $c$ is insensitive to the UV region, so we do not expect any complications from taking the UV cutoff to $\infty$. However, $c$ is very sensitive
to the IR region, so convergence in the presence of an IR cutoff gives no information about the behavior of the LDE in the limit as the IR cutoff goes to 0.

In this letter, we present evidence that the LDE is indeed a systematically improvable method. We show that it can be implemented in a way that is compatible with the large-$N$ limit. We present evidence that the coefficient $c$ in the large-$N$ limit converges to the analytic result of Ref. [2] for a range of $m$, with errors that decrease as a power of the order $n$ in the LDE. If the PMS criterion is used to optimize the convergence rate, the errors seem to decrease exponentially in $n$.

The lagrangian density for the $O(N)$ field theory relevant to BEC is

$$L = -\frac{1}{2} \bar{\phi} \cdot \nabla^2 \phi + \frac{1}{2} r \bar{\phi}^2 + \frac{1}{24} u \left( \bar{\phi}^2 \right)^2,$$

where $\bar{\phi} = (\phi_1, \ldots, \phi_N)$ is an $N$-component real field. The statistical average of the operator $\bar{\phi}^2$ is

$$\langle \bar{\phi}^2 \rangle = 2 \int \left[ p^2 + r + \Sigma(p) \right]^{-1},$$

where $\Sigma(p)$ is the self-energy and $\int_p = \int d^3p/(2\pi)^3$. The integral over $p$ is divergent and requires a UV cutoff. The critical point can be reached by tuning the parameter $r$ to the value $r = -\Sigma(0)$. This condition reduces to $r = 0$ if $u = 0$. The difference $\Delta$ between the critical values of $\langle \bar{\phi}^2 \rangle$ at a nonzero value of $u$ and at $u = 0$ is

$$\Delta = N \int_p \left[ (p^2 + \Sigma(p) - \Sigma(0))^{-1} - (p^2)^{-1} \right].$$

The integral over $p$ is convergent and therefore no longer requires a UV cutoff. At the critical point, the only relevant length scale is set by the parameter $u$. Since $\Delta$ has dimensions of length, it must be proportional to $u$ by dimensional analysis. The determination of the coefficient of $u$ requires a nonperturbative calculation. In the large-$N$ limit defined by $N \to \infty$, $u \to 0$ with $Nu$ fixed, the coefficient is known analytically [3]:

$$\Delta = -N u/(96\pi^2) \quad \text{(large } N).$$

The coefficient in the expression for the shift in $T_c$ is $c = -128\pi^3\zeta(3/2)^{-4/3}\Delta/u$, with $\Delta$ evaluated at $N = 2$.

The linear $\delta$ expansion (LDE) is generated by a lagrangian whose coefficients are linear in a formal expansion parameter $\delta$: $L_\delta = (1 - \delta)L_0 + \delta L$, where $L_0$ is the lagrangian for an exactly solvable theory. The lagrangian $L_\delta$ interpolates between $L_0$ when $\delta = 0$ and $L$ when $\delta = 1$. To apply the LDE to the $O(N)$ statistical field theory defined by the lagrangian (4), we choose the solvable field theory to be the free field theory with mass $m$. The lagrangian can be written $L_\delta = L_0 + L_{\text{int}}$, where

$$L_0 = -\frac{1}{2} \bar{\phi} \cdot \nabla^2 \phi + \frac{1}{2} m^2 \bar{\phi}^2,$$

$$L_{\text{int}} = \frac{1}{2} \delta \left( r - m^2 \right) \bar{\phi}^2 + \frac{1}{24} \delta u \left( \bar{\phi}^2 \right)^2.$$

Calculations are carried out by expanding in powers of $\delta$, truncating at $n$th order, and then setting $\delta = 1$.

In order to apply the LDE to the shift in $T_c$, we need a prescription for generalizing the quantity $\Delta$ defined in (4) to the field theory defined by the lagrangian $L_\delta$. The prescription must have a well-defined expansion in powers of $\delta$, and it must reduce to (4) when $\delta = 1$. The simplest prescription is to use the expression (4), where $\Sigma(p)$ is the self-energy for the field theory with lagrangian $L_\delta$. In the previous application of the LDE to the shift in $T_c$, the authors used an alternative prescription with the additional term $m^2(1 - \delta)$ in the denominator of the first term in the integrand of (4). This prescription has the correct limit as $\delta \to 1$. It can be expressed as an integral with a well-defined large-$N$ limit plus an additional term $-(N/4\pi)m\sqrt{1 - \delta}$. Because of this additional term, the limit $\delta \to 1$ does not commute with the large-$N$ limit. Using this prescription, the prediction for $\Delta$ defined by the PMS criterion at $n$th order in the LDE, scales like $N^{2-1/n}u$ as $N \to \infty$, while the correct result is $Nu$.

The prescription (4) defines $\Delta(u, m, \delta)$ as a function of three variables. The $n$th-order approximation in the LDE is obtained by truncating the expansion in $\delta$ at $n$th order to obtain a function $\Delta^{(n)}(u, m, \delta)$ and then setting $\delta = 1$. At any finite order $n$, the prediction of the LDE depends on $m$. As $m$ varies over its physical range from 0 to $+\infty$, the range of the prediction for $\Delta$ extends out to $\pm \infty$ depending on the order in $\delta$. Some prescription for $m$ is required to obtain a definite prediction. The PMS prescription is $(d/dm)\Delta^{(n)}(u, m, \delta = 1) = 0$. After setting $\delta = 1$, $\Delta^{(n)}$ is a function of $u$ and $m$ only. By dimensional analysis, the value of $\Delta^{(n)}$ at a solution $m$ to the PMS criterion is proportional to $u$. By allowing the variable $m$ to change with the order $n$, the PMS criterion may improve the convergence rate of the LDE.

Using the prescription (4), the LDE for $\Delta$ in the large-$N$ limit can be calculated to all orders in $\delta$. The leading contribution to $\Sigma(p) - \Sigma(0)$ comes from the series of diagrams whose 4th member is shown in Fig. 1. Since $\Sigma(p) - \Sigma(0)$ is of order $1/N$, the leading contribution at large $N$ is obtained by expanding (4) to first order in $\Sigma(p) - \Sigma(0)$. The expression for the large-$N$ diagram for $\Delta$ with $n + 1$ loops can be reduced to a 1-dimensional integral multiplied by $m^{-(n-1)}u^n$. In addition to the diagrams for $\Sigma(p)$ generated by the interaction

FIG. 1: The 4th in the series of diagrams for $\Sigma(p)$ that survive in the large-$N$ limit.
term $\delta u(\delta^2)^2$ in Eq. (6), we must also take into account insertions of $\delta r$ and $-\delta m^2$. The effect of the $\delta r$ insertions is to replace each $\Sigma(p)$ by $\Sigma(p) - \Sigma(0)$. The effect of the $-\delta m^2$ insertions is to replace $m^2$ by $m^2(1 - \delta)$. Summing all the large-$N$ diagrams, we obtain

$$\Delta = \delta N u/(24\pi^3) \sum_{n=2}^\infty \left( -\delta/(\sqrt{1 - \delta \mu}) \right)^{n-1} \times \int_0^\infty dy \, y^2 |A(y)|^{n-1}/(y^2 + 1)^2,$$  \hspace{1cm} (7)

where $\mu = 48\pi m/(N u)$ and $A(y) = (2/y) \arctan(y/2)$. The prediction for $\Delta$ at $n^{th}$ order in the LDE is obtained by expanding Eq. (7) as a power series in $\delta$, truncating after order $\delta^n$, and then setting $\delta = 1$.

It is easy to show that if the LDE for Eq. (6) converges, it converges to the correct analytic result Eq. (4). After interchanging the order of the sum and the integral, the sum can be evaluated. Upon taking the limit $\delta \to 1$, all dependence on $\mu$ disappears. Evaluating the integral over $y$ gives the result Eq. (8). Thus, if the LDE converges at some value of $\mu$, it should converge to Eq. (9).

The manipulations that showed the convergence to Eq. (9) involved several interchanges of limits. It is difficult to translate the conditions for the validity of each step into a condition for the convergence of the LDE. However, the convergence can be easily studied numerically. In Fig. 2, we show $\Delta$ as a function of $\mu$ for several orders in the LDE; $n = 3$ and $n = 2j + 3$, $j = 0, ..., 5$. The horizontal line is the analytic result Eq. (4). The results are consistent with convergence to Eq. (9) for all $\mu$ greater than a critical value $\mu_c$ which we estimate to be $\mu_c \approx 0.71$. If $\mu < \mu_c$, $\Delta$ seems to diverge to $+\infty$ for $n$ even and to $-\infty$ for $n$ odd. For any fixed $\mu > \mu_c$, the convergence with $n$ is very slow. In Fig. 3, we show a log-log plot of the fractional error $\varepsilon_n$ as a function of $n$. The squares lie close to a straight line, indicating that the errors decrease like a power of $n$. The dotted line that goes through the last two points is $\varepsilon_n = 0.70n^{-0.37}$. The errors decrease roughly like $n^{-1/3}$.

The rate of convergence can be improved by using the PMS criterion to choose a value of $\mu$ that depends on the order in the LDE. At $n^{th}$ order, this criterion is a polynomial equation in $\mu$ of order $n - 2$. For $n$ even, there are no real roots. For $n$ odd, there is always one real root that corresponds to the maxima of the curves in Fig. 2. The resulting fractional errors are shown as a function of $n$ in Fig. 3. The diamonds lie close to a straight line, indicating that the errors decrease like a power of $n$. The dashed line that goes through the last two points is $\varepsilon_n = 0.78n^{-0.46}$. The errors decrease roughly as $n^{-1/2}$.

Although the PMS criterion at $n^{th}$ order has at most one real solution, there are always $n - 2$ complex-valued solutions. Studies of the anharmonic oscillator have revealed that there are families of complex solutions with much better convergence properties than families of real solutions Eq. (10). In our problem, at any odd order $n$, the real solution always gives the value of $\text{Re}\Delta$ that is farthest from the correct result Eq. (4). Thus this family of solutions gives the slowest possible convergence rate. However there is a strong anticorrelation between the errors in $\text{Re}\Delta$ and $\text{Im}\Delta$. This is illustrated in Fig. 4, which is a scatter plot of $|\text{Im}\Delta|$ vs. $\text{Re}\Delta$ for the solutions to the PMS criterion for $n = 35$. The solutions with the most accurate values for $\text{Re}\Delta$ are those with the largest values for $|\text{Im}\Delta|$. Thus we can define a nearly optimal family of solutions by choosing those with the maximal values of $|\text{Im}\Delta|$. While $\text{Im}\Delta$ for these solutions shows no sign of converging to 0, there is dramatic improvement in the convergence of $\text{Re}\Delta$. A log-log plot of the fractional error $\varepsilon_n$ as a function of $n$ in Fig. 3 shows that the errors decrease like a power of $n$. The dotted line that goes through the last two points is $\varepsilon_n = 0.70n^{-0.37}$. The errors decrease roughly like $n^{-1/3}$.
The errors in the 3rd order predictions in the large-$N$ limit is 62%, which is larger than the errors of 52% and 44% in the lattice Monte Carlo result. The percentage improvement in going from 3rd to 4th order is only half as large as in the large-$N$ limit. The slow convergence in going from 3rd to 4th order in the LDE may also be a systematically improvable approximation scheme in the case $N = 2$ relevant to BEC.

In conclusion, we have shown that the LDE for the quantity $\Delta$ in the 3-dimensional critical $O(N)$ field theory in the large $N$ limit converges if we use an appropriate prescription. If the PMS criterion is used to optimize the convergence, the errors seem to decrease exponentially in the order of the LDE. This provides some hope that the LDE may also be a systematically improvable approximation scheme in superrenormalizable field theories, even at a critical point.

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\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{scatter_plot.png}
\caption{Scatter plot of $|\text{Im}\Delta|$ vs. $\text{Re}\Delta$ for the solutions $\mu$ of the PMS criterion at 35th order in the LDE.}
\end{figure}

The downward curvature of the points indicates that the errors decrease faster than any power of $n$. The solid line that goes through the last two points is $\varepsilon_n = 0.32(0.93)^n$. The errors are decreasing faster than this exponential.

We have calculated $\Delta$ for finite $N$ to 4th order in the LDE using our prescription. Setting $N = 2$ and using the PMS criterion, we obtain a real result $\Delta(3) = 0.192$ at 3rd order and a complex result $\Delta(4) = 0.214 \pm 0.084i$ at 4th order. Taking $\Delta(3)$ and $\text{Re}\Delta(4)$ as the predictions, the corresponding values for the coefficient in $\Delta T_c$ are $c = 0.447$ and 0.492. They seem to be slowly approaching the lattice Monte Carlo result $c = 1.32 \pm 0.02$ from below.

The errors in the 3rd and 4th order results are 66% and 62%, which is larger than the errors of 52% and 44% in the corresponding predictions in the large-$N$ limit. The percentage improvement in going from 3rd to 4th order is only half as large as in the large-$N$ limit. The 3rd and 4th order predictions for $N = 2$ using the prescription of Ref. have errors of 85% and 20%. The small error in their 4th order prediction may be fortuitous. The low order predictions using their prescription are skewed by the term in $\Delta$ that does not have a well-behaved large-$N$ limit. The prescription for $\Delta$ obtained by deleting that term is equally valid, and it gives values below the lattice Monte Carlo result with errors of 68% and 65%.

Although the convergence rate of the optimized LDE appears to be exponential in the large-$N$ limit, it is still rather slow. One must calculate to about 18th order in $\delta$ to achieve 10% accuracy. For general $N$, it may be feasible to calculate $\Delta$ to 5th order, but it would be difficult to go to much higher order. The slow convergence in the large-$N$ limit suggests that even if the LDE also converges for $N = 2$, a strict expansion in $\delta$ is not useful for quantitative calculations. It may however be possible to use order-dependent mappings to change the expansion in $\delta$ into a more rapidly converging expansion.

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