DOUBLING CONDITION AT THE ORIGIN FOR NON-NEGATIVE POSITIVE DEFINITE FUNCTIONS

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ABSTRACT. We study upper and lower estimates as well as the asymptotic behavior of the sharp constant $C = C_n(U, V)$ in the doubling-type condition at the origin

$$\frac{1}{|V|} \int_V f(x) \, dx \leq C \frac{1}{|U|} \int_U f(x) \, dx,$$

where $U, V \subset \mathbb{R}^n$ are 0-symmetric convex bodies and $f$ is a non-negative positive definite function.

1. Introduction

Very recently, answering the question posed by Konyagin and Shleinitov related to a problem from number theory [13], the first author proved [1] that for any positive definite function $f: \mathbb{Z}^q \to \mathbb{R}^+$ and for any $n \in \mathbb{Z}^+$ one has

$$\sum_{0 \leq k \leq 2n} f(k) \leq C \sum_{0 \leq k \leq n} f(k),$$

where the positive constant $C$ does not depend on $n$, $f$, and $q$. More precisely, it was proved that $C \leq \pi^2$.

In this paper we study similar inequalities for a non-negative positive definite function $f$ defined on $\mathbb{R}^n$, $n \geq 1$, i.e.,

$$\int_{|x| \leq 2R} f(x) \, dx \leq C \int_{|x| \leq R} f(x) \, dx, \quad R > 0,$$

for some $C > 1$. The latter is the well-known doubling condition at the origin. The doubling condition plays an important role in harmonic and functional analysis, see, e.g., [14]. Note that very recently inequality (1.1) in the one-dimensional case was studied in [3].

Definition 1. A positive definite function $f: \mathbb{R}^n \to \mathbb{R}^+$ is called double positive definite function (denoted $f \geq 0$).

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As usual [11, Chap. 1], a continuous function $f \in C(\mathbb{R}^n)$ is positive definite if for every finite sequence $X \subset \mathbb{R}^n$ and every choice of complex numbers \{c_a: a \in X\}, we have

$$\sum_{a,b \in X} c_a \overline{c_b} f(a-b) \geq 0.$$ 

By Bochner’s theorem [11, Chap. 1], $f \in C(\mathbb{R}^n)$ is positive definite if and only if there is a non-negative finite Borel measure $\mu$ such that

$$f(x) = \int_{\mathbb{R}^n} e(\xi x) \, d\mu(\xi), \quad \xi \in \mathbb{R}^n,$$

where $e(t) = \exp(2\pi it)$. For $f \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ it is equivalent to the fact that the Fourier transform of $f$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e(-\xi x) \, dx$$

is non-negative. Note also that since any positive definite $f$ satisfies $f(-x) = \overline{f(x)}$, a double positive definite function is even.

Throughout the paper we assume that $U, V \subset \mathbb{R}^n$ be 0-symmetric closed convex bodies. For any function $f \geq 0$ we study the inequality

$$\frac{1}{|V|} \int_V f(x) \, dx \leq C \frac{1}{|U|} \int_U f(x) \, dx,$$

where $|A|$ is the volume of $A$ or the cardinality of $A$ if $A$ is a finite set. By $C_n(U, V)$ we denote the sharp constant in (1.3), i.e.,

$$C_n(U, V) := \sup_{f \geq 0, f \neq 0} \frac{1}{|V|} \int_V f(x) \, dx \cdot \frac{1}{|U|} \int_U f(x) \, dx.$$

The fact that $C_n(U, V) < \infty$ for any $U$ and $V$ will follow from Theorem 1 below.

First, we list the following simple properties of $C_n(U, V)$.

(1) A trivial lower bound

$$C_n(U, V) \geq 1,$$

since $1 \geq 0$;

(2) The homogeneity property

$$C_n(\lambda U, \lambda V) = C_n(U, V), \quad \lambda > 0,$$

since $f_\lambda(x) = f(\lambda x) \geq 0$ if and only if $f \geq 0$;

(3) The homogeneity estimate

$$C_n(U, \lambda V) \geq \lambda^{-n} C_n(U, V), \quad \lambda \geq 1,$$

since $V \subset \lambda V$;

(4) $C_n(U, U) = 1$ and if $V \subset U$, then

$$C_n(U, V) \leq \frac{|U|}{|V|};$$

(5) The multiplicative estimate

$$C_n(U, V) \leq C_n(\lambda^k U, V)(C_n(U, \lambda U))^k, \quad \lambda \geq 1, \, k \in \mathbb{Z}_+. $$
which follows from the chain of inequalities
\[ C_n(U, V) \leq C_n(\lambda U, V)C_n(U, \lambda U) \]
\[ \leq C_n(\lambda^2 U, V)C_n(U, \lambda^2 U)C_n(U, \lambda U) \]
\[ = C_n(\lambda^2 U, V)(C_n(U, \lambda U))^2 \leq \ldots \]
\[ \leq C_n(\lambda^k U, V)(C_n(U, \lambda U))^k, \]

(6) A trivial upper bound for the doubling constant: for fixed \( \lambda > 1 \) and any \( r > \lambda \)
(1.7) \[ C_n(U, rU) \leq (C_n(U, \lambda U))^{\log_{\lambda} r}. \]
which follows from the multiplicative estimate.

Below we will obtain the upper bound for the constant \( C_n(U, rU) \), which depends only on \( n \).

We will use the following notation. Let \( A + B \) be the Minkowski sum of sets \( A \) and \( B \), \( \lambda A \) be the product of \( A \) and the number \( \lambda \), and \( B_R := \{ x \in \mathbb{R}^n : |x| \leq R \} \) be the Euclidean ball.

2. The upper estimates
In what follows, we set
\[ H := \frac{1}{2} U \quad \text{and} \quad K := V + H. \]

**Theorem 1.** Let \( X \subset \mathbb{R}^n \) be a finite set of points such that
(2.8) \[ K \subseteq H + X. \]
Then
\[ C_n(U, V) \leq \frac{|X||U|}{|V|}. \]

From the geometric point of view, condition (2.8) means that the translates \( \{H + a : a \in X\} \) of the set \( H \) covers the set \( K \).

**Example 1 ([3]).** If \( n = 1 \) and \( r \in \mathbb{N} \), then
\[ C_1(r) := C_1([-1, 1], [-r, r]) \leq 2 + \frac{1}{r}. \]
Indeed, take \( H = [-\frac{1}{2}, \frac{1}{2}] \), \( X = \{-r, -r+1, \ldots, r-1, r\} \), and \( K = [-r - \frac{1}{2}, r + \frac{1}{2}] = H + X. \)

Let \( n \in \mathbb{N} \). There holds ([10] (6))
(2.9) \[ N(K, H) \leq \frac{|K - H|}{|H|} \theta(H). \]
Here \( N(K, H) \) denotes the smallest number of translates of \( H \) required to cover \( K \) and
(2.10) \[ \theta(H) = \inf_{X \subset \mathbb{R}^n} \theta(H, X), \]
where \( \theta(H, X) \) is the covering density of \( \mathbb{R}^n \) by translates of \( H \) [9] p.16. In other words, for a discrete set \( X \) such that \( \mathbb{R}^n \subseteq H + X \) one has \( |X \cap A|/|H|/|A| = \theta(H, X)(1 + o(1)) \) for a convex body \( A \) such that \( |A| \to \infty \).
From (2.9), taking into account that $H = -H$, $K - H = V + 2H = V + U$, and $|U| = 2^n|H|$, we obtain that

$$N(K, H) \leq 2^n \frac{|V + U|}{|V|} \theta(H).$$

Moreover, it is clear that the best possible result in Theorem 1 is when $X$ is such that $|X| = N(K, H)$. Therefore, we have

**Corollary 1.** For $n \geq 1$ and any $U$ and $V$, we have

$$C_n(U, V) \leq 2^n \frac{|V + U|}{|V|} \theta(H).$$

In particular, for $r \geq 1$

$$C_n(U, rU) \leq 2^n(1 + r^{-1})^n \theta(H). \tag{2.11}$$

Estimate (2.11) substantially improves (1.7). For $n = 1$ and $r \geq 1$, we have that $\theta([(-1/2, 1/2] \leq 1$ and $C_1(r) \leq 2(1 + r^{-1})$, which is similar to the estimate from Example 1.

Note that Rogers [8] proved that

$$\theta(H) \leq n \ln n + n \ln \ln n + 5n, \quad n \geq 2. \tag{2.12}$$

Estimate (2.12) was slightly improved in [4] as follows

$$\theta(H) \leq n \ln n + n \ln \ln n + n + o(n) \quad \text{as} \quad n \to \infty.$$

Therefore, we obtain

**Corollary 2.** We have

$$C_n(U, V) \leq 2^n(n \ln n + n \ln \ln n + n + o(n)) \frac{|V + U|}{|V|} \quad \text{as} \quad n \to \infty.$$ 

In particular, taking $V = rU$, $r \geq 1$, we arrive at the following example.

**Example 2.** We have

$$C_n(U, rU) \leq 2^n(n \ln n + n \ln \ln n + n + o(n))(1 + r^{-1})^n \quad \text{as} \quad n \to \infty. \tag{2.13}$$

**Proof of Theorem 1.** Consider the function

$$\varphi := \varphi_H = |H|^{-1} \cdot \chi_H \ast \chi_H,$$

where $\chi_H$ is the characteristic function of $H$ and $(f \ast g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y) \, dy$ is the convolution of $f$ and $g$.

Since $\varphi \geq 0$, supp $\varphi \subset U$, and $\varphi \leq \varphi(0) = 1$, we have for any $f \geq 0$

$$I := \int_{\mathbb{R}^n} f(x) \varphi(x) \, dx = \int_U f(x) \varphi(x) \, dx \leq \int_U f(x) \, dx.$$

Let $X \subset \mathbb{R}^n$ be a finite set and

$$S(x) = \frac{1}{|X|} \sum_{a \in X} \varphi(x - a).$$

Then $S \geq 0$ and $\hat{S} = \hat{\varphi}D$, where

$$D(\xi) = \frac{1}{|X|} \sum_{a \in X} e(a\xi)$$

is the Dirichlet kernel with respect to $X$. 


Let us estimate the integral \( I \) from below. Using \( f(x) = f(-x) \), we get
\[
\int_V f(x)S(x) \, dx \leq \int_{\mathbb{R}^n} f(x)S(x) \, dx = \int_{\mathbb{R}^n} f(x)S_0(x) \, dx := I_1,
\]
where \( S_0(x) = 2^{-1}(S(x) + S(-x)) \). Taking into account that
\[
\hat{S}_0(\xi) = \hat{\varphi}(\xi) \frac{D(\xi) + D(-\xi)}{2} = \hat{\varphi}(\xi) \frac{1}{|X|} \sum_{a \in X} \cos(2\pi a \xi) \leq \hat{\varphi}(\xi), \quad \xi \in \mathbb{R}^n,
\]
and using (1.2), we obtain
\[
I_1 = \int_{\mathbb{R}^n} \hat{S}_0(\xi) \, d\mu(\xi) \leq \int_{\mathbb{R}^n} \hat{\varphi}(\xi) \, d\mu(\xi) = \int_{\mathbb{R}^n} f(x)\varphi(x) \, dx = I,
\]
provided that \( f \) and \( \varphi \) are even.

Let \( K = V + H \subseteq H + X \). This means that for any points \( x \in V \) and \( y \in H \) there is \( a \in X \) such that \( x + y \in H + a \). Hence,
\[
\sum_{a \in X} \chi_H(x + y - a) \geq 1.
\]
Using \( H = -H \), we have
\[
\varphi(x) = \frac{1}{|H|} \int_H \chi_H(x + y) \, dy.
\]
Therefore, for any \( x \in V \)
\[
S(x) = \frac{1}{|X|} \sum_{a \in X} \frac{1}{|H|} \int_H \chi_H(x - a + y) \, dy
\geq \frac{1}{|X||H|} \int_H \sum_{a \in X} \chi_H(x - a + y) \, dy
\geq \frac{1}{|X||H|} \int_H dy = \frac{1}{|X|}.
\]
Thus, combining the estimates above, we arrive at the inequality
\[
\frac{1}{|X|} \int_V f(x) \, dx \leq \int_V f(x)S(x) \, dx \leq I \leq \int_U f(x) \, dx,
\]
which is the desired result. \( \Box \)

3. The lower estimates

Our goal is to improve the trivial lower estimate (1.4). The idea is to consider the functions \( \sum_{a,b \in X \cap B_R} \delta(x + a - b) \), where \( X \) is a packing of \( \mathbb{R}^n \) by \( H \) and \( R \gg 1 \) (see also [2, 3]).

First we consider the one-dimensional result, partially given in Example 1.

Theorem 2 ([3]). For \( r \in \mathbb{N} \), we have
\[
2 \, \frac{1}{r} \leq C_1(r) \leq 2 + \frac{1}{r},
\]
and \( \lim_{r \to \infty} C_1(r) = 2 \).
This is one of the main results of the paper [3]. The upper bound is given in Example 1. The lower bound follows from Theorem 3 below for \(U = [-1, 1]\), \(V = [-r, r]\), and \(\Lambda = \mathbb{Z}\). The fact that \(\lim_{r \to \infty} C_1(r) = 2\) follows from estimates of \(C_1(r)\) for integers \(r\) and \((1, 0)\).

Now we consider the general case \(n \geq 1\). Our aim is to improve the trivial lower bound \((1, 1)\) respect to \(n\).

Let \(\delta_L(H) = \sup_{\Lambda \subset \mathbb{R}^n} \delta(H, \Lambda)\), where \(\delta(H, \Lambda)\) is the packing density of \(\mathbb{R}^n\) by lattice translates of \(H\) [9] Infr.]. In other words, \(\Lambda = MZ^n \subset \mathbb{R}^n\) is a lattice of rank \(n\) \((M \in \mathbb{R}^{n \times n}\) is a generator matrix of \(\Lambda\), \(\det M \neq 0\)) such that \(a - b \notin \operatorname{int}(2H)\) for any \(a, b \in \Lambda\), \(a \neq b\), and \(|\Lambda \cap H|/|H| = \delta(H, \Lambda)(1 + o(1))\) for a convex body \(A\) such that \(|A| \to \infty\). Note that in this case \(H + \Lambda\) is a lattice packing of \(H\) [6, Sect. 30.1]. Recall that \(H = \frac{1}{2} U\).

**Theorem 3.** Let \(H + \Lambda\) be a lattice packing of \(H\). Then

\[
C_n(U, V) \geq \frac{|\Lambda \cap \operatorname{int} V|}{|V|}.
\]

In particular,

\[
C_n(U, V) \geq 2^n \delta_L(H)(1 + o(1)) \quad \text{as} \quad |V| \to \infty.
\]

**Proof of Theorem 3.** Let \(\Lambda\) be an lattice with the packing density \(\delta(H, \Lambda)\). Denote \(\Lambda_N = \Lambda \cap B_N\) for \(N > 0\). Let \(B_r\) be the smallest ball that contained \(V\). Assume that \(R \geq r\) is sufficiently large number and \(\epsilon\) is sufficiently small.

Define \(\varphi_\epsilon = \varphi_{B_r}\).

We consider the function

\[
f(x) = \sum_{a, b \in \Lambda_R} \varphi_\epsilon(x + a - b).
\]

It is easy to see that

\[
f(x) = \sum_{c \in \Lambda_R} N_c \varphi_\epsilon(x + c),
\]

where

\[
N_c = \sum_{a - b = c} 1 = \sum_{a \in \Lambda_R \cap (\Lambda_R + c)} 1 = |\Lambda_R \cap (\Lambda_R + c)|.
\]

Since \(\Lambda\) is a lattice, we have \(\Lambda = \Lambda + c\) for any \(c \in \Lambda\). Hence, \(N_0 = |\Lambda_R|\) and \(N_c \geq |\Lambda_{R-r}|\) for \(|c| \leq r\), provided \(\Lambda_{R-r} \subset \Lambda_R \cap (\Lambda_R + c)\).

On the one hand, since \(2H = U\) and \(c \notin \operatorname{int} U\) if \(c \in \Lambda \setminus \{0\}\), we have

\[
\int_{(1-\epsilon)U} f(x) \, dx = N_0 = |\Lambda_R|.
\]

On the other hand, since \(V \subset B_r\), we obtain

\[
\int_{(1+\epsilon)V} f(x) \, dx \geq \sum_{c \in \Lambda_R \cap V} N_c \geq |\Lambda_{R-r}| |\Lambda \cap V|.
\]

Therefore,

\[
C_n((1 - \epsilon)U, (1 + \epsilon)V) \geq \frac{(1 - \epsilon)^n |\Lambda_{R-r}|}{(1 + \epsilon)^n} \frac{|\Lambda \cap V| |U|}{|U|}.
\]
Replacing $V$ by $\frac{1}{1 + \varepsilon} V$ and using (1.3) and (1.6) as above, we arrive at

$$C_n(U, V) \geq \frac{|A_{R-r}|}{|A_r|} \frac{|A \cap \frac{1}{1 + \varepsilon} V|}{|V|}.$$ Letting $R \to \infty$ and $\varepsilon \to 0$ concludes the proof of (3.14).

Inequality (3.15) follows easily from (3.14) and the definition of $\delta_L(H)$. □

**Example 3.** We consider the balls $U = B_1$ and $V = B_r$, $r > 1$. It is known that

$$\delta_L(B_1) \geq c_n 2^{-n},$$

where $c_n \geq 1$ is the Minkowski constant. It was recently proved in [15] that $c_n > 65963n$ for every sufficiently large $n$ and there exist infinitely many dimensions $n$ for which $c_n \geq 0.5n \ln \ln n$.

**Corollary 3.** Let $n \in \mathbb{N}$. We have

$$C_n(B_1, B_r) \geq c_n(1 + o(1)) \quad \text{as} \quad r \to \infty.$$ Comparing (2.13) and (3.16) for fixed $n$ and $r \to \infty$, one observes the exponential gap between the upper and lower estimates of $C_n(B_1, B_r)$ with respect to $n$. Let us give examples of $U$ for which the upper and lower estimates of $C_n(U, V)$ coincide.

**Example 4.** Let $H$ be a convex body and $\Lambda$ be a lattice. The set $H + \Lambda$ is lattice tiling if it is both a packing and a covering [6, Sect. 32]. In this case $H$ is a tile and $\delta_L(H) = \theta_L(H) = 1$, where $\theta_L(H)$ is the lattice covering density, cf. (2.10).

To define $\theta_L(H)$, we take the infimum in (2.10) over all lattices $\Lambda \subset \mathbb{R}^n$ of rank $n$. Note that $\theta(H) \leq \theta_L(H)$.

For example, the Voronoi polytop

$$V(\Lambda) = \{ x \in \mathbb{R}^n : |x| \leq |x - a|, \forall a \in \Lambda \}$$
of a lattice $\Lambda$ is a tile. In particular, $V(\mathbb{Z}^n)$ is the cube $[-\frac{1}{2}, \frac{1}{2}]^n$.

From Corollary 1 and Theorem 3 we have

**Theorem 4.** Let $n \in \mathbb{N}$ and $U$ be a tile. We have

$$C_n(U, V) = 2^n(1 + o(1)) \quad \text{as} \quad |V| \to \infty.$$ 4. Final remarks

1. The inequality

$$\frac{1}{|V|} \int_U f(x) \, dx \leq C_n(U, V) \frac{1}{|U|} \int_U f(x) \, dx$$

holds for any 1-periodic function $f \geq 0$. In this case we assume that $U, V \subset \mathbb{T}^n$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Since a positive definite $f$ is such that $f(-x) = \overline{f(x)}$, then $|f|^p \geq 0$ for any $p = 2k, k \in \mathbb{N}$. Hence, we obtain the following $L^p$-analogue:

$$\frac{1}{|V|} \int_U |f(x)|^p \, dx \leq C_n(U, V) \frac{1}{|U|} \int_U |f(x)|^p \, dx.$$ For $U \subset V = \mathbb{T}^n$, this inequality is the well-known Wiener estimate for positive definite periodic functions (see [12, 14, 2]):

$$\int_{\mathbb{T}^n} |f(x)|^p \, dx \leq W_{n,p}(U) \frac{1}{|U|} \int_U |f(x)|^p \, dx,$$
which is valid only for $p = 2k$, $k \in \mathbb{N}$. Here, $W_{n,p}(U)$ is a sharp constant in (4.17).

It is clear that

$$W_{n,2k}(U) \leq C_n(U, T^n).$$

It is interesting to compare the known upper bounds of $W_{n,2k}(U)$ and $C_n(U, T^n)$. In [2] it was shown that

$$W_{n,2k}(rB_1) \leq 2^{(0,401...+o(1))n}, \quad r \in (0, 1/2).$$

On the other hand, by Corollary 2, we obtain that

$$C_n(rB_1, T^n) \leq 2^{n(1+o(1))(1 + 2r)^n}.$$ 

The exponential gap in the last two bounds is related to the restriction to the class of functions under consideration.

**2.** If $f \geq 0$, then $f^p \geq 0$ for any $p \in \mathbb{N}$. This gives

$$\frac{1}{|V|} \int_V (f(x))^p \, dx \leq C_n(U, V) \frac{1}{|U|} \int_U (f(x))^p \, dx, \quad p \in \mathbb{N}.$$ 

It would be of interest to investigate this inequality for any positive $p$; see in this direction the paper [3].

**3.** As we showed above, any function $f \geq 0$ satisfies the doubling property at the origin (1.1). However, taking any nontrivial function $f \geq 0$ such that $f|_A = 0$, where $A$ is a ball, we can see that the doubling property may fail outside the origin.

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