A NEW CLASS OF POSITIVE SEMI-DEFINITE TENSORS

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(Communicated by Jin-Yan Fan)

ABSTRACT. In this paper, a new class of positive semi-definite tensors, the MO tensor, is introduced. It is inspired by the structure of Moler matrix, a class of test matrices. Then we focus on two special cases in the MO-tensors: Sup-MO tensor and essential MO tensor. They are proved to be positive definite tensors. Especially, the smallest H-eigenvalue of a Sup-MO tensor is positive and tends to zero as the dimension tends to infinity, and an essential MO tensor is also a completely positive tensor.

1. Introduction. In recent decades, tensors, as the natural extension of matrices, have been more and more ubiquitous in a wide variety of applications, such as data analysis and mining, signal processing, computational biology and so on [3, 6]. The positive (semi-)definiteness of tensors is an intrinsic property for tensors, which is closely related with the nonnegative polynomials. In 2005, the concepts of positive (semi-)definiteness and eigenvalues of matrices were extended to tensors [11]. Positive (semi-)definite tensors have widely applications in polynomial optimization, spectral hypergraph theory, magnetic resonance imaging and so on [11, 12, 16]. Though the verification of positive (semi-)definite tensors has been shown to be NP-hard by Hillar and Lim [5], the positive (semi-)definiteness of some tensors with special structures have been verified, such as even order symmetric diagonally dominated tensors, even order symmetric B tensor, even order symmetric $M$-tensors and so on, see [8, 11, 13, 14, 17].

Furthermore, completely positive tensors, which is connected with nonnegative tensor factorization, also have significant applications in polynomial optimization problems, statistics, data analysis and so on. They were first introduced in [15]. An even order completely positive tensor is a positive semi-definite tensor. In [9], two well-known classes of test matrices, Pascal matrices and Lehmer matrices were

2010 Mathematics Subject Classification. Primary: 15A18, 15A69; Secondary: 15B99.

Key words and phrases. Positive (semi-)definite tensor, completely positive tensor, H-eigenvalue, MO-tensor, MO-like tensor, Sup-MO value.

The first author is supported by National Natural Science Foundation of China Nos. 11501100, 11571178 and 11671082. The third author is supported in part by the Hong Kong Research Grant Council Nos. PolyU 15392114, 15300715, 15301716 and 15300717.

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extended to Pascal tensors and Lehmer tensors. They are easily checkable and were proved to be completely positive tensors [9].

There is another class of test matrices, the Moler matrices. A Moler matrix is a positive definite symmetric matrix. One of its eigenvalues is quite small, and it is proved to be completely positive tensors [9].

934 YI XU, JINJIE LIU AND LIQUN QI

2. Preliminaries. In this part, we will review some basic definitions and lemmas which are useful in our following research.

We call \( \mathcal{A}(n, m) \) a real \( m \)-th order \( n \)-dimensional tensor if

\[
\mathcal{A}(n, m) = (a_{i_1 \cdots i_m}), \quad i_j \in [n], \quad j \in [m],
\]

where \([n] := \{1, \cdots, n\}\). All the real \( m \)-th order \( n \)-dimensional tensors form a \( n^m \)-dimensional linear space denoted by \( T_{n,m} \). Furthermore, if all the entries \( a_{i_1 \cdots i_m} \) of \( \mathcal{A}(n, m) \) are invariant under any index permutation, \( \mathcal{A}(n, m) \) is a symmetric tensor. Denote the set of all the real symmetric tensors of order \( m \) and dimension \( n \) by \( S_{n,m} \). Obviously, \( S_{n,m} \) is a linear subspace of \( T_{n,m} \).

Let \( \mathcal{A}(n, m) = (a_{i_1 \cdots i_m}) \in T_{n,m} \) and \( x \in \mathbb{R}^n \). Then \( \mathcal{A}(n, m)x^m \) is an \( m \)-th degree homogeneous polynomial defined by

\[
\mathcal{A}(n, m)x^m = \sum_{i_1, i_2, \cdots, i_m = 1}^n a_{i_1 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.
\]

We call tensor \( \mathcal{A}(n, m) \) as a positive definite (PD) tensor if \( \mathcal{A}(n, m)x^m > 0 \), for any non-zero vector \( x \in \mathbb{R}^n \), and as positive semi-definite (PSD) tensor if \( \mathcal{A}(n, m)x^m \geq 0 \), for any vector \( x \in \mathbb{R}^n \). Let \( x = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{R}^n \), then \( x^m \) is a vector in \( \mathbb{R}^n \) denoted as

\[
x^{[m]} = (x_1^m, x_2^m, \cdots, x_n^m)^\top.
\]

To identify the positive (semi-)definiteness of tensors, the spectral theory of tensors plays an important role for the desired identification. In [11], H-eigenvalues and Z-eigenvalues of tensors were introduced. It was shown there that an even order symmetric tensor is positive (semi-)definite if and only if all of its H-eigenvalues or Z-eigenvalues are positive (nonnegative). Various easily checkable positive (semi-)definite tensors have been discovered consequently [2, 7, 14, 17].

Definition 2.1. [11] Let \( \mathcal{A}(n, m) \in T_{n,m}, \lambda \in \mathbb{R} \). If \( \lambda \) and a nonzero vector \( x \in \mathbb{R}^n \) are the solutions of the following polynomial equation:

\[
\mathcal{A}(n, m)x^{m-1} = \lambda x^{[m-1]},
\]

then we call \( \lambda \) an H-eigenvalue of \( \mathcal{A}(n, m) \), and \( x \) an H-eigenvector of \( \mathcal{A}(n, m) \) associated with the H-eigenvalue \( \lambda \).
Theorem 2.2. \cite{11} Let $A(n, m) \in S_{n,m}$, and $m$ be even. Then $A(n, m)$ is positive definite (positive semi-definite) if and only if all of the H-eigenvalues of $A(n, m)$ are positive (non-negative). Furthermore, we have

1. \[ \lambda_{\min}(A(n, m)) = \min \frac{A(n, m)x^m}{\|x\|_m^m}, \]

2. \[ \lambda_{\min}(A(n, m)) = \min\{A(n, m)x^m : \|x\| = 1\}, \]

where $x \in \mathbb{R}^n$, and $\|x\|_m = (\sum_{i=1}^n |x_i|^m)^{1/m}$.

Definition 2.3. \cite{9} Let $A(n, m) \in S_{n,m}$. We call $A(n, m)$ a completely positive tensor if there exist an integer $r$ and some $u^{(k)} \in \mathbb{R}^n$, $k \in [r]$ such that

\[ A(n, m) = \sum_{k=1}^r (u^{(k)})^m. \]

Theorem 2.4. \cite{9} Let $A(n, m) \in S_{n,m}$. If $A(n, m)$ is a completely positive tensor, then all the H-eigenvalues of $A$ are nonnegative.

Definition 2.5. \cite{10} Let $A(n, 2) \in \mathbb{R}^{n \times n}$. We call $A(n, 2)$ the $n$-dimensional Moler matrix if

\[ A(n, 2)_{i,j} = \begin{cases} i, & i = j \\ \min\{i,j\} - 2, & i \neq j \end{cases}. \]

In the following proposition, we give a proof to show that the Moler matrix is a positive definite matrix, and its smallest eigenvalue tends to zero in decreasing as its dimension tends to infinity.

Proposition 1. Let $A(n, 2) \in \mathbb{R}^{n \times n}$ be an $n$-dimensional Moler matrix. (1) $A(n, 2)$ is positive definite; (2) Let $\lambda_{\min}(A(n, 2))$ be the smallest eigenvalue of $A(n, 2)$. Then $\lambda_{\min}(A(n, 2)) \searrow 0$.

Proof. (1) We note that $A(n, 2) = LL^\top$, where

\[ L_{i,j} = \begin{cases} 1, & i = j \\ -1, & i > j \\ 0, & i < j \end{cases}. \]

Therefore, $A(n, 2)$ is positive definite.

(2) $0 < \lambda_{\min}(A(n+1, 2)) \leq \lambda_{\min}(A(n, 2))$ are easily obtained. Assume that

\[ x = (1, \frac{1}{2}, \frac{1}{2^2}, \cdots, \frac{1}{2^{n-1}})^\top. \]

We have $L^\top x = \left( \begin{array}{c} 1 \\ \vdots \\ \frac{1}{2^{n-1}} \end{array} \right)$, so

\[ \frac{x^\top A(n, 2)x}{x^\top x} = \frac{3n}{4^n - 1} \geq \lambda_{\min}(A(n, 2)), \]

which means $\lambda_{\min}(A(n, 2)) \searrow 0$, when $n \to +\infty$. \qed
3. Main results. In order to introduce a new class of tensors with positive semi-definiteness, we first introduce concepts called the MO value, the MO set and the Sup-MO value.

Definition 3.1. Let \( m \) be an even number. (1) We call \( \alpha(m) \) as the MO value, if \( \mathcal{A}(n,m) := \mathcal{M}(n,m) - \alpha(m)\mathcal{N}(n,m) \) is positive semi-definite for any \( n \), where

\[
\mathcal{M}(n,m)_{i_1,i_2,\cdots,i_m} = \begin{cases} i_1, & i_1 = i_2 = \cdots = i_m \\ \min\{i_1, i_2, \cdots, i_m\}, & \text{else} \end{cases},
\]

\[
\mathcal{N}(n,m)_{i_1,i_2,\cdots,i_m} = \begin{cases} 0, & i_1 = i_2 = \cdots = i_m \\ 1, & \text{else} \end{cases}.
\]

We call the set of all MO values as the MO set \( \Omega(m) \); (2) We call \( \alpha^*(m) = \sup\{\Omega(m)\} \) as the Sup-MO value. We also define Sub-MO value \( \alpha_*(m) \) of \( \Omega(m) \) as \( \alpha_*(m) = \inf\{\Omega(m)\} \).

It is worth noting that \( \alpha(m) \) is a parameter only related to \( m \). Hence, when we consider to explore its properties, it is necessary to show these properties still hold when \( n \to \infty \).

In this paper, we mainly focus on the properties of \( \alpha^*(m) \). Based on the aforementioned concepts, MO tensors and Sup-MO tensors are given as following.

Definition 3.2. Let \( m \) be an even number. (1) We call \( \mathcal{A}(n,m) \in S_{n,m} \) a MO tensor, if

\[
\mathcal{A}(n,m) = \mathcal{M}(n,m) - \alpha(m)\mathcal{N}(n,m),
\]

where \( \mathcal{M}(n,m) \) and \( \mathcal{N}(n,m) \) are defined in Eq.(1) and Eq.(2), respectively, and \( \alpha(m) \in \Omega(m) \). (2) We call \( \mathcal{A}(n,m) \in S_{n,m} \) a Sup-MO tensor, if

\[
\mathcal{A}(n,m) = \mathcal{M}(n,m) - \alpha^*(m)\mathcal{N}(n,m).
\]

The theorem in the following shows that the Sup-MO tensor \( \mathcal{A}(n,m) \) can be reduced to the Moler matrix when \( m = 2 \).

Theorem 3.3. Let \( \Omega(m) \) be the MO set, \( \mathcal{A}(n,m) \in S_{n,m} \) be a Sup-MO tensor. When \( m = 2 \), we have \( \alpha^*(2) = 2 = \max\{\Omega(2)\} \).

Proof. From the property of the Moler matrix, we get \( 2 \in \Omega(2) \). Then we need to prove that 2 is the Sup-MO value in this case. If \( \alpha^*(2) > 2 \), then there exists an \( \alpha \in (2, \alpha^*(2)) \cap \Omega(2) \) such that

\[
\mathcal{M}(n,2) - \alpha\mathcal{N}(n,2) = \mathcal{M}(n,2) - 2\mathcal{N}(n,2) - (\alpha - 2)\mathcal{N}(n,2),
\]

where \( \mathcal{M}(n,m) \) and \( \mathcal{N}(n,m) \) are defined in Eq.(1) and Eq.(2), respectively.

Let \( x = \left( \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2^{m-1}} \right)^\top \in \mathbb{R}^n \),

\[
x^\top(\mathcal{M}(n,2) - \alpha\mathcal{N}(n,2))x = x^\top(\mathcal{M}(n,2) - 2\mathcal{N}(n,2))x - (\alpha - 2)x^\top\mathcal{N}(n,2)x.
\]

Since

\[
x^\top(\mathcal{M}(n,2) - 2\mathcal{N}(n,2))x = \frac{3n}{4^n - 1}, \quad \text{and} \quad x^\top\mathcal{N}(n,2)x = \frac{8}{3} - \frac{4}{2^n - 1} + \frac{4}{3(4^n - 1)},
\]

when \( n \to +\infty \), we have

\[
x^\top(\mathcal{M}(n,2) - \alpha\mathcal{N}(n,2))x < 0,
\]

which is against to \( \alpha \in \Omega(2) \). Hence \( \alpha^*(2) = 2 \). \( \square \)
In the following work, a special class of MO tensors, the essential MO tensors are discussed. The following theorem shows the relationship between the essential MO tensors and the completely positive tensors. Then some properties of Sup-MO values in MO tensors are given.

**Definition 3.4.** Let $\mathcal{A}(n, m) \in S_{n,m}$. We call $\mathcal{A}(n, m)$ the $n$th order $m$-dimensional essential MO tensor if

$$\mathcal{A}(n, m)_{i_1, \ldots, i_m} = \left\{ \begin{array}{ll} i_1, & i_1 = i_2 = \cdots = i_m = 1 \\
\min\{i_1, i_2, \ldots, i_m\} - 1, & \text{otherwise} \end{array} \right.$$  

**Theorem 3.5.** Let $\mathcal{A}(n, m)$ be an $n$th order $m$-dimensional essential MO tensor. It is positive definite for all $n$, even $m$. Furthermore, it is a completely positive tensor for all $n$ and $m$, such as $\mathcal{A}(n, m) = \sum_{i=1}^{n} \mathbf{e}_i^m + \sum_{i=2}^{n} \mathbf{r}_i^m$, where $(\mathbf{e}_i)_j = \left\{ \begin{array}{ll} 1, & j = i \\
0, & \text{otherwise} \end{array} \right.$

Proof. Let $\mathcal{B}(n, m)_{i_1, \ldots, i_m} = \left\{ \begin{array}{ll} 1, & i_1 = i_2 = \cdots = i_m = 1 \\
1, & i_1, i_2, \ldots, i_m \geq 2 \\
0, & \text{otherwise} \end{array} \right.$

Since $\mathcal{B}(n, m) = (1, 0, \ldots, 0)^m + (0, 1, \ldots, 1)^m$, $\mathcal{B}(n, m)$ is a complete positive tensor. Let $\mathcal{C}(n, m) = \mathcal{A}(n, m) - \mathcal{B}(n, m)$. Then

$$\mathcal{C}(n, m) = \left\{ \begin{array}{ll} i_1 - 1, & i_1 = i_2 = \cdots = i_m \geq 2 \\
0, & \min\{i_1, i_2, \ldots, i_m\} = 1 \\
\min\{i_1, i_2, \ldots, i_m\} - 2, & \text{otherwise} \end{array} \right.$$  

Let $\mathcal{A}(n - 1, m)_{i_1, i_2, \ldots, i_m} = \mathcal{C}(n, m)_{i_1+1, i_2+1, \ldots, i_m+1}, j \in [n-1], j \in [m]$. Then $\mathcal{A}(n - 1, m)$ is an $n$th order $m$-1 dimensional essential MO tensor. Furthermore, if $\mathcal{A}(n - 1, m)$ is the completely positive tensor, $\mathcal{A}(n, m)$ is the completely positive tensor.

By the same way, we could get $\mathcal{A}(i, m), i \in [n]$, are all essential MO tensors. When $n = 1$, $\mathcal{A}(1, m)$ is equal to the positive number 1. By induction, we get that $\mathcal{A}(n, m)$ is a completely positive tensor and also a positive definite tensor. \hfill \Box

**Corollary 1.** (1) Let $\Omega(m)$ be the MO set, and $\mathcal{M}(n, m), \mathcal{N}(n, m)$ be defined in Eq.(1) and Eq.(2), respectively. For all $n$ and even $m$, 1 is always a MO value, i.e. $1 \in \Omega(m)$.

(2) $\mathcal{M}(n, m) - \alpha \mathcal{N}(n, m)$ is completely positive for all $n$ and $m$, while $\alpha \in [0, 1]$.

Proof. (1) Since $\mathcal{M}(n, m) - \mathcal{N}(n, m)$ is an essential MO tensor for all $n$ and even $m$, we get that 1 is a MO value.

(2) Since $\mathcal{N}(n, m) = \mathbf{e}^m - \sum_{i=1}^{n} \mathbf{e}_i^m$, $\mathcal{M}(n, m) = \sum_{i=1}^{n} \mathbf{e}_i^m + \sum_{i=2}^{n} \mathbf{r}_i^m + \mathbf{e}^m - \sum_{i=1}^{n} \mathbf{e}_i^m = \sum_{i=2}^{n} \mathbf{r}_i^m + \mathbf{e}^m$, where $\mathbf{e} = (1, 1, \ldots, 1)^\top$. So if $\alpha \in [0, 1]$, $\mathcal{M}(n, m) - \alpha \mathcal{N}(n, m) = \sum_{i=2}^{n} \mathbf{r}_i^m + (1 - \alpha) \mathbf{e}^m + \alpha \sum_{i=1}^{n} \mathbf{e}_i^m$ is completely positive. \hfill \Box
Corollary 2. \( \alpha_*(m) \) exists, and \(-\frac{1}{2} \leq \alpha_*(m) \leq 0 \).

Proof. From Corollary 1(2), \( \alpha_*(m) \leq 0 \). Let \( x = (1, -1, 0, \cdots, 0) \), when \( \alpha < -\frac{1}{2} \),
\[
(M(n, m) - \alpha N(n, m))x^m = 1 + 2\alpha < 0,
\]
so \( \alpha_*(m) = \inf\{\Omega(m)\} \) exists, and \( \alpha_*(m) \geq -\frac{1}{2} \).

Based on Theorem 3.5 and Corollary 1, the MO set \( \Omega(m) \) is nonempty.

Proposition 2. Let \( \Omega(m) \) be the MO set, and \( M(n, m), N(n, m) \) be defined in Eq. (1) and Eq. (2), respectively. Then,
(1) for any \( \alpha_1(m), \alpha_2(m) \in \Omega(m), [\alpha_1(m), \alpha_2(m)] \subseteq \Omega(m); 
(2) for all even \( m \), \( 1 < \alpha^*(m) \leq 2; 
(3) \alpha^*(m) = \max\{\Omega(m)\}; 
(4) \alpha^*(m) \searrow 1, \text{ when } m \to +\infty.

Proof. (1) It is obvious.

(2) Since \( 1 \in \Omega(m) \) for all even \( m \), \( \Omega(m) \neq \emptyset \). Let us consider the tensor
\[
M(n, m) - 2N(n, m) \text{ and } x = \left(1, \frac{1}{2}, \cdots, \frac{1}{2^{n-1}}\right) \in \mathbb{R}^n. 
\]

Then
\[
(M(n, m) - 2N(n, m))x^m = 2\sum_{i=1}^{n} (e_i^\top x)^m + \sum_{i=2}^{n} (r_i^\top x)^m - (e^\top x)^m,
\]
where \( e = (1, \cdots, 1)^\top \). When \( m \geq 4 \) and \( n \geq 2 \), we have
\[
\sum_{i=1}^{n} (e_i^\top x)^m = \sum_{i=1}^{n} \left(\frac{1}{2^{m-1}}\right)^i \leq \frac{2^{m-1}}{2^{m-1} - 1} \leq \frac{16}{15},
\]
\[
\sum_{i=2}^{n} (r_i^\top x)^m \leq \sum_{i=1}^{n-1} \left(\frac{1}{2^{m-1}}\right)^i \leq \frac{2^{m-1}}{2^{m-1} - 1} \leq \frac{16}{15},
\]
and
\[
(e^\top x)^m = \left(\sum_{i=1}^{n} \frac{1}{2^{i-1}}\right)^m = \left(1 - \frac{1}{2^{m-1}}\right)^m \geq \left(\frac{3}{2}\right)^m.
\]

Then
\[
(M(n, m) - 2N(n, m))x^m \leq \frac{48}{15} - \left(\frac{3}{2}\right)^m < 0.
\]

Therefore, for all even \( m \), 2 is an upper bound of \( \Omega(m) \). Then \( \alpha^*(m) \) exists and \( \alpha^*(m) \leq 2 \).

Now we prove \( \alpha^*(m) > 1 \). Let
\[
U(n, m) = M(n, m) - N(n, m), V(n, m; \beta) = U(n, m) - \beta N(n, m), \beta = \alpha - 1.
\]

First, we need to prove that there exists \( \beta \in (0, 1) \) such that
\[
V(n, m; \beta)x^m = (1 + \beta)\sum_{i=1}^{n} (e_i^\top x)^m + \sum_{i=2}^{n} (r_i^\top x)^m - \beta (e^\top x)^m \geq 0,
\]
for all \( \beta \in \mathbb{R} \) and \( x \in \mathbb{R}^n \) satisfying \( ||x||_m = 1 \).

If \( e^\top x = 0 \), then \( V(n, m; \beta)x^m \geq 0 \) for all \( \beta \in [0, 1] \). If \( e^\top x \neq 0 \), then there exists \( y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n \) and \( z = (z_1, z_2, \cdots, z_n) \in \mathbb{R}^n \) such that
\[
y_1 = e^\top x, y_i = r_i^\top x, i = 2, \cdots, n,
\]
then
\[
\sum_{i=2}^{n} (r_i^\top x)^m \leq \frac{16}{15}, 
\]
and
\[
(e^\top x)^m \geq \left(\frac{3}{2}\right)^m > 0.
\]

Therefore,
\[
V(n, m; \beta)x^m = (1 + \beta)\sum_{i=1}^{n} (e_i^\top x)^m + \sum_{i=2}^{n} (r_i^\top x)^m - \beta (e^\top x)^m \geq 0,
\]
for all \( n \in \mathbb{R} \) and \( x \in \mathbb{R}^n \) satisfying \( ||x||_m = 1 \).
and
\[ z_i = \frac{y_i}{y_1}, \quad i = 1, \ldots, n. \]

Then \( z_1 = 1 \) and
\[ V(n, m; \beta)x^m = y_1^m \left[ (1 + \beta) \left( \sum_{i=1}^{n-1} (z_i - z_{i+1})^m + z_n^m \right) + \sum_{i=2}^{n} z_i^m - \beta \right]. \]

Let
\[ g_{n,m}(z, \beta) = (1 + \beta) \left( \sum_{i=1}^{n-1} (z_i - z_{i+1})^m + z_n^m \right) + \sum_{i=2}^{n} z_i^m, \quad (3) \]

and
\[ f_{n,m}(\beta) = \min_{z \in \mathbb{R}^n, z_1 = 1} g_{n,m}(z, \beta). \quad (4) \]

It is easy to get
\[ 0 \leq f_{n+1,m}(\beta) \leq f_{n,m}(\beta), \text{ for all } \beta \in [0, 1]. \]

Hence,
\[ f_m(\beta) = \lim_{n \to +\infty} f_{n,m}(\beta) \quad (5) \]

exists for all \( \beta \in [0, 1] \).

In fact, if there exists \( \beta \in [0, 1] \) such that \( f_m(\beta) = 0 \), then by the definition of \( f_{n,m}(\beta) \), there exists \( z^* \in \mathbb{R}^n \) with \( z^*_1 = 1 \) such that \((1 - z^2_j)^m < \varepsilon \) and \( z^*_2 < \varepsilon \), for any \( \varepsilon > 0 \), which is impossible. Then \( f_m(\beta) > 0 \). Furthermore, \( f_m(0) > 0 \).

Additionally, when \( m \geq 4 \), assume that \( z^* = \left(1, \frac{1}{2}, 0, \ldots, 0\right) \), we get
\[ g_{n,m}(z^*, 1) = \frac{5}{2m}. \] Thus \( f_m(1) \leq f_{n,m}(1) \leq \frac{5}{2m} < 1. \)

Moreover, since \( m \) is even,
\[ g_{n,m}(z^*, \beta) \leq g_{n,m}(z, \beta_2), \text{ for all } z \in \mathbb{R}^n, \beta_1, \beta_2 \text{ with } 0 < \beta_1 \leq \beta_2. \]

Then \( f_{n,m}(\beta_1) \leq f_{n,m}(\beta_2) \) and \( f_m(\beta_1) \leq f_m(\beta_2) \), which means that \( f_m(\beta) \) is a nondecreasing function in \( \beta \) on \([0, 1]\).

Then we prove that \( f_m(\beta) \) is a continuous function in \( \beta \) on \((0, 1)\). Denote \( f_m(\beta^+), f_m(\beta^-) \) as the right-hand and left-hand limit on \( \beta \in (0, 1) \), \( f_m(0^+), f_m(1^-) \) as the right-hand limit on 0, and left-hand limit on 1 of the function \( f_m(\beta) \), respectively. Since \( f_m(\beta) \) is a nondecreasing function, for any \( \beta \in (0, 1) \), \( f_m(\beta^+), f_m(\beta^-), f_m(0^+), f_m(1^-) \) exist and \( f_m(\beta^+) \geq f_m(\beta^-) \). Assuming that \( f_m(\beta^+) > f_m(\beta^-) \), and \( \delta = \frac{f_m(\beta^+) - f_m(\beta^-)}{2} > 0 \), for \( 0 < \beta_1 < \beta < \beta_2 \), there exists \( N^* \), when \( n > N^* \), we have
\[ f_{n,m}(\beta_1) = f_m(\beta_1) + \frac{f_m(\beta^+ - f_m(\beta^-))}{2} \leq f_m(\beta^-) + \frac{f_m(\beta^+ - f_m(\beta^-))}{2} = \frac{f_m(\beta^+) + f_m(\beta^-)}{2}, \]

and
\[ f_{n,m}(\beta_2) \geq f_m(\beta_2) \geq f_m(\beta^+). \]

Thus, when \( n > N^* \),
\[ f_{n,m}(\beta_2) - f_{n,m}(\beta_1) \geq \frac{f_m(\beta^+) - f_m(\beta^-)}{2} = \delta > 0. \]
Algorithm 3.6 (Computing $\alpha^*(m)$).

We could use the following algorithm to compute $\alpha^*(m)$.
Theorem 3.7. Let $\Omega(m)$ be an MO set. For all even $m \geq 4$ and $\alpha^*(m) = \max\{\Omega(m)\}$, $A(n,m;\alpha^*(m))$ is a Sup-MO tensor, i.e.,

$$A_{i_1,\ldots,i_m}(n,m;\alpha^*(m)) = M(n,m) - \alpha^*(m)N(n,m)$$

$$= \begin{cases} i_1, & i_1 = i_2 = \cdots = i_m, \\ \min\{i_1,i_2,\ldots,i_m\} - \alpha^*(m), & \text{otherwise.} \end{cases}$$

Assume that $\lambda_{\min}(A(n,m;\alpha^*(m)))$ is the smallest eigenvalue of $A(n,m;\alpha^*(m))$. Then, $\lambda_{\min}(A(n,m;\alpha^*(m)))$ strictly decreases to 0, when $n \to \infty$. Furthermore, $A(n,m;\alpha^*(m))$ is positive definite.

Proof. By Theorem 2.2, it is easy to see that $\lambda_{\min}(A(n,m;\alpha^*(m)))$ decreases in $n$, for all even $m$. In the following, we prove that it strictly decreases to 0.

By the definition of $g_{n,m}(z,\beta), f_{n,m}(\beta)$ and $f_m(\beta)$ are defined as (3-5).

Since $1 > f_{n,m}(\beta^*(m)) \geq \beta^*(m) > 0$, $g_{n,m}(\beta^*(m)) \to \beta^*(m)$. Suppose that $z^* \in \arg \min_{z \in \mathbb{R}^m} g_{n,m}(z,\beta^*(m))$. Because $z_1^* = 1$, $||z^*||_m \geq 1$. Let $w_i^* = z_i^* - z_i^{i+1}, i = 1, \ldots, n - 1, w_n^* = z_n^*$. Then, when $m \geq 4$,

$$\beta^*(m) \leq g_{n,m}(z^*,\beta^*(m)) = f_{n,m}(\beta^*) \leq f_{n,m}(1) < \frac{5}{2m} < 1.$$ 

By the definition of $g_{n,m}(z^*,\beta^*(m))$, $\sum_{i=2}^n (z_i^*)^m \leq \frac{5}{2m} < 1$, which means that $|z_2^*| \leq \frac{5\frac{m}{2}}{2} < 1$. Thus $w_i^* = 1 - z_2^* \geq 1 - \frac{5\frac{m}{2}}{2}$. Hence, when $m \geq 4$, $||w^*||_m \geq 1 - \frac{5\frac{m}{2}}{2}$.

According to Theorem 2.2

$$0 \leq \lambda_{\min}(A(n,m;\alpha^*(m))) \leq A(n,m;\alpha^*(m))(w^*)^m.$$ 

By the definition of $w^*, A(n,m;\alpha^*(m))(w^*)^m \to 0$, when $n \to +\infty$. Since $||w^*||_m \geq 1 - \frac{5\frac{m}{2}}{2}$, we get that $\lambda_{\min}(A(n,m;\alpha^*(m))) \to 0$, when $n \to \infty$.

Then we prove that the decreasing of $\lambda_{\min}(A(n,m;\alpha^*(m)))$ is strict. Consider the following program:

$$\min A(n,m;\alpha^*(m))x^m$$

$$s.t. ||x||_m = 1.$$ 

Then its KKT conditions are

$$\begin{cases} A(n,m;\alpha^*(m))x^{m-1} = \lambda x^{m-1} \\ ||x||_m = 1. \end{cases}$$ (6)
The smallest solution \( \lambda_{n,m} \) and the corresponding vector \( x \in \mathbb{R}^n \) of above program are the smallest H-eigenvalue and H-eigenvector of \( A(n, m; \alpha^*(m)) \). If \( \lambda_{n,m} = \lambda_{n+1,m} \) for some \( n \), then there exist \( x \in \mathbb{R}^n \) and \( \bar{x} \in \mathbb{R}^{n+1} \) with \( \bar{x} = (x^\top, 0)^\top \) satisfying

\[
A(n, m; \alpha^*(m)) x^{m-1} = \lambda_{n,m} x^{[m-1]},
\]

\[
A(n + 1, m; \alpha^*(m)) \bar{x}^{m-1} = \lambda_{n+1,m} \bar{x}^{[m-1]}.
\]

Hence,

\[
\sum_{i_2, \cdots, i_m = 1}^{n+1} A(n + 1, m; \alpha^*(m))_{n+1,i_2,\cdots,i_m} \bar{x}_{i_2} \cdots \bar{x}_{i_m} = \lambda_{n+1,m} \bar{x}^{m-1}_{n+1}.
\]

Since \( \bar{x} = (x^\top, 0)^\top \), the above equation is

\[
\sum_{i_2, \cdots, i_m = 1}^{n} A(n + 1, m; \alpha^*(m))_{n+1,i_2,\cdots,i_m} x_{i_2} \cdots x_{i_m} = 0.
\]

Because

\[
\sum_{i_2, \cdots, i_m = 1}^{n} A(n, m; \alpha^*(m))_{n,i_2,\cdots,i_m} x_{i_2} \cdots x_{i_m} = \lambda_{n,m} x^{m-1}_n,
\]

we have

\[
\sum_{i_2, \cdots, i_m = 1}^{n} A(n, m; \alpha^*(m))_{n,i_2,\cdots,i_m} x_{i_2} \cdots x_{i_m} - \sum_{i_2, \cdots, i_m = 1}^{n} A(n + 1, m; \alpha^*(m))_{n+1,i_2,\cdots,i_m} x_{i_2} \cdots x_{i_m} = \alpha^*(m) x^{m-1}_n = \lambda_{n,m} x^{m-1}_n.
\]

According to the above proof, \( \alpha^*(m) > 1 > \lambda_{n,m} \). Therefore, \( x_n = 0 \). By the same discussion, we get \( x = 0 \), which is against \( \|x\|_m = 1 \).

Thus, \( \lambda_{min}(A(n, m; \alpha^*(m))) \) strictly decreases. Finally, together with Corollary 1, we have \( A(n, m; \alpha^*(m)) \) is positive definite.

\[\Box\]

4. Final remarks. In this paper, we construct the MO tensor with introducing the concepts of the MO value and the MO set. Then we mainly discuss two special cases of the MO tensor: the Sup-MO tensor and the essential MO tensor. We prove that an even order essential MO tensor is a completely positive tensor and positive definite. Then, some related properties of the Sup-MO value of an even order Sup-MO tensor are given. Furthermore, the positive definiteness of an even order Sup-MO tensor is proved, since the minimal H-eigenvalue of the Sup-MO tensor strictly decreases to 0, when \( n \to \infty \). In the future work, some of the applications of the Sup-MO tensor and the properties of Sub-MO value and Sub-MO tensor will be discussed.

There are three further research questions for the MO tensor, the Sup-MO tensor and the Sub-MO tensor.

1. Are the Sup-MO tensors SOS (sum-of-squares) tensors? For the definition of SOS tensors, see [2]. An SOS tensor is a PSD tensor, but not vice versa. This theory can be traced back to David Hilbert [4]. We randomly tested some Sup-MO tensors, and found that they were SOS tensors. We will study this in the future study.
2. We cannot verify whether $\alpha_*(m)$ can be reached or not. Hence, we do not know whether $\Omega(m)$ is compact or not. If the $\Omega(m)$ is compact, how to get the length of the MO set $\Omega(m)$ is also an interesting work. We will continue to explore the properties of $\alpha_*(m)$ and $\Omega(m)$.

3. Since the properties of the Moler matrices make them to be good test matrices for the linear equations and eigensystems, we will try to find that if the Sup-MO tensor can also be a good candidate for testing in some tensor computation software packages or not.

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Received March 2018; 1st revision June 2018; 2nd revision September 2018.

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