Hardy inequalities for weighted Dirac operator

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Abstract

An inequality of Hardy type is established for quadratic forms involving Dirac operator and a weight $r^{-b}$ for functions in $\mathbb{R}^n$. The exact Hardy constant $c_b = c_b(n)$ is found and generalized minimizers are given. The constant $c_b$ vanishes on a countable set of $b$, which extends the known case $n = 2$, $b = 0$ which corresponds to the trivial Hardy inequality in $\mathbb{R}^2$. Analogous inequalities are proved in the case $c_b = 0$ under constraints and, with error terms, for a bounded domain.

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1 Introduction

The well-known Hardy inequality in $\mathbb{R}^3$

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2} dx$$

expresses, in the context of classical quantum mechanics the celebrated uncertainty principle. Since the seminal paper of Brezis and Vazquez [4], Hardy inequality has received a renewed attention of numerous authors (here we quote only papers [2, 1, 3, 8, 12] that have an immediate connection to our results).
Similar inequalities are known for relativistic versions of the Schrödinger equation, in particular, for the quadratic form of \( \sqrt{-\Delta} \) (Kato inequality, [11]), and for a quadratic form of Dirac operator \( \int W(x)|\sigma \cdot \nabla u|^2 dx \) for Pauli matrices \( \sigma_i, i = 1, 2, 3 \), by Dolbeault, Esteban, Séré [7] and Dolbeault, Esteban, Loss and Vega [6]. The class of weights in the latter work has a specific decay rate at infinity.

In this paper we prove Hardy inequality, with exact constants, for \( W(x) = |x|^{-b}, b \in \mathbb{R} \), generalizing two known cases, \( b = 0 \) corresponding to the usual Hardy inequality, and \( b = -1 \) corresponding to inequality (4) in [6]. The method, based on reduction to the usual weighted Sobolev inequalities in one dimension, allows to obtain similar estimates for general \( W \). A surprising phenomenon observed here is a “quantization of certainty” - there is a discrete set of values \( b \) for which the exact Hardy constant becomes zero. We have opted here to prove the inequalities for functions in \( C_0^\infty(\mathbb{R}^n \setminus \{0\}) \), since this choice allows to consider all real values of \( b \) rather than \( b < n - 2 \), as well as to extend the range of parameters in the important Caffarelli-Kohn-Nirenberg inequalities (see Appendices A and B). We leave it as an exercise to the reader to show that for \( b < n - 2 \) our inequalities (as well as Caffarelli-Kohn-Nirenberg inequalities for applicable parameters) follow from correspondent inequalities on \( C_0^\infty(\mathbb{R}^n \setminus \{0\}) \) by an elementary approximation argument (multiplication of the function near the origin by a family of cut-off functions).

Let \( n \geq 2, \sigma_i, i = 1, \ldots, m \), be Hermitian \( m \times m \)-matrices satisfying
\[
\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}, \quad i, j = 1, \ldots, m. \tag{1.1}
\]
Such matrices are found, in particular, for \( m = 2^{n/2} \) when \( n \) is even, and for \( m = 2^{(n+1)/2} \) when \( n \) is odd. In particular, for \( n = 3 \) one usually fixes the set of \( \sigma_i \) as Pauli matrices
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

As we study here equations for functions \( \mathbb{R}^n \to \mathbb{C}^m \), we will use distinct notations for the scalar products in the domain and in the range of the functions: \( \langle f, g \rangle \) for the scalar product in \( \mathbb{C}^m \) and \( p \cdot q \) for the scalar product in \( \mathbb{R}^n \). The weighted Dirac operator is induced by the following quadratic form on \( C_0^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^m) \)
\[
Q_b(u) \overset{def}{=} \int_{\mathbb{R}^n} r^{-b}|\sigma \cdot \nabla u|^2 dx, b \in \mathbb{R}. \tag{1.2}
\]
Quadratic form $Q_b(u)$ endows $C_0^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^m)$ with a scalar product, but the completion of $C_0^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^m)$ with respect to the corresponding norm is generally not a function space. An inequality of Hardy type, which is the main objective of this paper, yields an imbedding of the completed Hilbert space into a weighted $L^2$-space. The imbedding fails for a countable subset of $b$, in which case an analogous inequality holds on a subspace of $C_0^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^m)$.

The operator $\sigma \cdot \nabla$ is well known as a “square root” of the Laplacian, i.e. $(\sigma \cdot \nabla)^2 = \Delta$, and it admits the following representation in polar coordinates (see e.g. [9]):

$$ (\sigma \cdot \nabla)u = (\hat{x} \cdot \sigma)(\hat{x} \cdot \nabla u + \frac{1}{|x|} Lu), \quad (1.3) $$

where $\hat{x} \overset{\text{def}}{=} \frac{x}{|x|}$, and the operator

$$ L \overset{\text{def}}{=} \sum_{j<k} \sigma_j \sigma_k (x_j \partial_{x_k} - x_k \partial_{x_j}) \quad (1.4) $$

involves differentiation only in the directions tangential to a sphere centered at the origin, that is, $[L, \partial_r] = 0$. An elementary calculation based on evaluation of the integral of the squared magnitude in the right and the left hand side of (1.3) shows that

$$ L^2 - (n-2)L = -\Delta_S, \quad (1.5) $$

where $\Delta_S$ is the Laplace-Beltrami operator on the sphere. It is known (see e.g. [9]) that the spectrum $S_L$ of $L$ is discrete and consists of integer values:

$$ S_L = \{Z \setminus \{1, \ldots, n-2\}\}. \quad (1.6) $$

We will denote the eigenspace of $L$ corresponding to the eigenfunction $k \in S_L$ as $E_k$.

Our main results are as follows.

**Theorem 1.1.** Let $b \in \mathbb{R}$. Then for all $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^m)$,

$$ \int_{\mathbb{R}^n} r^{-b}|(\sigma \cdot \nabla)u|^2 dx - c_b \int_{\mathbb{R}^n} r^{-b-2}|u|^2 dx \geq 0, \quad (1.7) $$

where

$$ c_b \overset{\text{def}}{=} \min_{k \in \mathbb{Z} \setminus \{1, \ldots, n-2\}} \left( k - \frac{n - 2 - b}{2} \right)^2 \quad (1.8) $$
is the exact constant for the inequality. Furthermore, the quadratic form in the left hand side of (1.7) has a finite-dimensional space of generalized ground states in the sense of Definition 4.7, that in polar coordinates have the form \( v(r, \omega) = r^{b-\frac{n-2}{2}} \varphi(\omega) \), \( \varphi \in E^{(b)} \), where \( E^{(b)} \) is the span of (at most two) eigenspaces \( E_k \) corresponding to those \( k \) that yield the minimum in (1.8).

Note that the maximal value of the constant \( c_b \) is \( \left( \frac{n-2}{2} \right)^2 \), attained when \( b = 0 \) and the inequality (1.7) becomes the usual Hardy inequality. The constant \( c_b \) equals zero if and only if \( \frac{n-2-b}{2} \in \mathbb{Z} \setminus \{0, \ldots, n-2\} \). This vanishing is not entirely unexpected, as this occurs also in the known case of Hardy inequality in two dimensions (\( b = 0 \) and \( n = 2 \)). In classical quantum mechanics, Hardy inequality (our case \( b = 0 \)) expresses the uncertainty principle (which remarkably fails for \( n = 2 \)).

**Theorem 1.2.** Let \( n > 2 \) and \( 2^* = \frac{2n}{n-2} \). Assume that \( \frac{n-2-b}{2} \notin \mathbb{Z} \setminus \{0, \ldots, n-2\} \). Then there is a \( C > 0 \) dependent on \( n \) and \( b \), such that for every \( v \in C_0^\infty (\mathbb{R}^n \setminus \{0\}) \),

\[
\int_{\mathbb{R}^n} |x|^{-b} |(\sigma \cdot \nabla) u|^2 dx \geq C \left( \int_{\mathbb{R}^n} |x|^{-\beta} |u|^{2^*} dx \right)^{2/2^*}, \quad (1.9)
\]

where \( \beta = \frac{bn}{n-2} \).

Note that the case \( b = 0 \) gives the usual Sobolev inequality.

## 2 Proof of Hardy inequality

In this section we derive a representation of the quadratic form (1.2) in polar coordinates and prove Theorem 1.1.

**Lemma 2.1.** For every \( u \in C_0^\infty (\mathbb{R}^n \setminus \{0\}; \mathbb{C}^m) \),

\[
Q_b(u) = \int_{\mathbb{R}^n} r^{-b} \left( |\partial_r u|^2 + r^{-2} \langle (L^2 + (b - n + 2)L) u, u \rangle \right). \quad (2.1)
\]

**Proof.** By (1.3) and taking into account that \( (\hat{x} \cdot \sigma)^2 = I_m \) we have

\[
Q_b(u) = \int_{\mathbb{R}^n} r^{-b} \left( |\partial_r u|^2 + r^{-1} \partial_r \langle Lu, u \rangle + r^{-2} \langle L^2 u, u \rangle \right). \quad (2.2)
\]
The second term in the right hand side can be evaluated by partial integration:

\[ \int_{\mathbb{R}^n} r^{-b-1} \partial_r \langle Lu, u \rangle \, dx = \int_0^\infty \int_{S^{n-1}} r^{-b+n-2} \partial_r \langle Lu, u \rangle \, dr \, d\omega = (b - n + 2) \int_0^\infty \int_{S^{n-1}} r^{-b+n-3} \langle Lu, u \rangle \, dr \, d\omega = (b - n + 2) \int_{\mathbb{R}^n} r^{-b-2} \langle Lu, u \rangle \, dx. \]

Substituting this into (2.2) we obtain (2.1).

**Proof of Theorem 1.1.** Let us expand the function \( u \) in the eigenfunctions of operator \( L \), normalized in \( L^2(S^{n-1}) \):

\[ u = \sum_{k \in S_L} c_k(r) \psi_k(\omega). \]

In order to account for the multiplicity of the eigenvalues we understand the term \( c_k(r) \psi_k(\omega) \) as an implicit finite sum over a basis in the \( k \)-th eigenspace of \( L \). We have from (2.1)

\[ Q_b(u) = \sum_{k \in S_L} \int_0^\infty r^{-b+n-1} (|\partial_r c_k(r)|^2 + r^{-2}(k^2 + (b + 2 - n)k|c_k|^2) \, dr. \tag{2.3} \]

Let us apply one-dimensional Caffarelli-Kohn-Nirenberg inequality (4.10) to the first term in the right hand side. Note that the best constant in (4.10) does not increase when one replaces \( C_0^\infty(\mathbb{R} \setminus \{0\}) \) with \( C_0^\infty((0, \infty)) \).

\[ \int_0^\infty r^{-b+n-1} (|\partial_r c_k(r)|^2 + r^{-2}(k^2 + (b + 2 - n)k|c_k|^2) \, dr \geq \left( \frac{-b + n - 2}{2} \right)^2 \int_0^\infty r^{-b+n-3} |c_k(r)|^2 \, dr. \tag{2.4} \]

Substitution of (2.4) into (2.3) and collection of similar terms gives immediately

\[ Q_b(u) \geq \sum_{k \in S_L} \left( k + \frac{b + 2 - n}{2} \right)^2 \int_0^\infty r^{-b-2}|c_k|^2 r^{n-1} \, dr. \tag{2.5} \]

Inequality (1.7) follows immediately.

It remains to show that the constant is exact. Indeed, let \( k \in S_L \) be a value where the minimum of \( \left( k - \frac{b + n - 2}{2} \right)^2 \) is attained. Then, since we used the exact constant for (4.10), a minimizing sequence for (1.7) is given by \( \{c_j(r) \psi_k(\omega)\}_j \), where \( \psi_k \) is an eigenfunction of \( L \) with the eigenvalue \( k \), and \( c_j(r) \) is a minimizing sequence for (4.10), which with necessity converges to a scalar multiple of \( r^{\frac{b+2-n}{2}} \) uniformly on compact subsets of \((0, \infty)\) (see [12] for details.)
3 Proof of the Sobolev inequality

Proof of Theorem 1.2. Let \( \epsilon \in (0, 1) \) be a constant to be specified at a later step. From (2.1), using (1.5), we have

\[
Q_b(u) = \epsilon \int_{\mathbb{R}^n} r^{-b} |\nabla u|^2 dx + \int_{\mathbb{R}^n} r^{-b} \times (1 - \epsilon)|u_r|^2 + (1 - \epsilon)r^{-2}\langle L^2 u, u \rangle + r^{-2}[(b - n + 2) - \epsilon(n - 2)]\langle Lu, u \rangle dx.
\]

Expanding the expression under the second integral in the eigenfunctions of \( L \) we obtain

\[
Q_b(u) = \epsilon \int_{\mathbb{R}^n} r^{-b} |\nabla u|^2 dx + \sum_{k \in S_L} \int_{\mathbb{R}^n} r^{-b} \times (1 - \epsilon)|c_k'|^2 + (1 - \epsilon)r^{-2}[(b - n + 2) - \epsilon(n - 2)]|c_k|^2 r^{n-1} dr.
\]

We apply the one-dimensional inequality (4.10):

\[
\int_{\mathbb{R}^n} r^{-b}|c_k'|^2 r^{n-1} dr \geq \left( \frac{-b + n - 2}{2} \right) \int_{\mathbb{R}^n} r^{-b-2}|c_k|^2 r^{n-1} dr,
\]

and subsequently,

\[
Q_b(u) = \epsilon \int_{\mathbb{R}^n} r^{-b} |\nabla u|^2 dx + \sum_{k \in S_L} \int_{\mathbb{R}^n} r^{-b-2}|c_k|^2 \times (1 - \epsilon) \left\{ \left( \frac{-b + n - 2}{2} \right)^2 + k^2 + [(b - n + 2) - \epsilon(n - 2)/(1 - \epsilon)]k \right\} r^{n-1} dr.
\]

We can estimate the expression inside the large braces as follows, denoting \( h(t) = \frac{-n+2}{2} \) and \( b_\epsilon \overset{def}{=} b - \epsilon(n - 2)/(1 - \epsilon) \):

\[
h(b)^2 + k^2 + 2h(b)k - \epsilon(n - 2)/(1 - \epsilon)k = h(b)^2 + k^2 + 2h(b)k = (k + h(b))h(b) \geq C_{b_\epsilon} - C\epsilon.
\]

Since \( c_0 > 0 \) by assumption on \( b \), and the last expression is continuous with respect to \( \epsilon \), we choose fix an \( \epsilon > 0 \) sufficiently small so that the expression is positive. This immediately implies that

\[
Q_b(u) \geq \epsilon \int_{\mathbb{R}^n} r^{-b} |\nabla u|^2 dx,
\]

and equation (1.9) follows immediately from (4.14) once we note that the condition on the exponent of \( r \) for the latter inequality is \( b \neq n - 2 \), which is included, as \( \frac{n-2-b}{2} \neq 0 \), into the assumptions of the theorem. \( \square \)
4 Further inequalities

Let $P_k, k \in S_L$, denote the orthogonal projectors in $L^2(\mathbb{R}^n; \mathbb{C}^m)$ induced by orthogonal projection on the eigenspace $E_k$ of $L$ in $L^2(S^{n-1}; \mathbb{C}^m)$. Note that while $\dim E_k < \infty$, $\dim P_k L^2(\mathbb{R}^n; \mathbb{C}^m) = +\infty$.

**Theorem 4.1.** Assume that $j = \frac{n-2-b}{2} \in \mathbb{Z} \setminus \{1, \ldots, n-2\}$. Then for all $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^m)$ satisfying $P_j u = 0$,

$$Q_b(u) = \int_{\mathbb{R}^n} r^{-b}|(\sigma \cdot \nabla)u|^2 dx \geq \int_{\mathbb{R}^n} r^{-b-2}|u|^2 dx, \quad (4.1)$$

and the constant 1 in the right hand side cannot be improved.

**Proof.** Consider relation (2.5) under the orthogonality condition $P_j u = 0$, that is, assuming that the term corresponding to $k = j$ is excluded from the sum representing $Q_b(u)$ in (2.5). Then $k = j - 1$ or $k = j + 1$ lies in $S_L$ and the smallest coefficient $(k - j)^2$ remaining in the expansion of (2.5) corresponds to $k = j \pm 1$ and equals 1. An argument repetitive of that in the proof of Theorem 1.1 shows that this coefficient is exact. \qed

An analog of the theorem above using a finite-dimensional projector gives a somewhat weaker inequality, connected the fact that $\int |\nabla u|^2$ cannot dominate any weighted $L^2$-norm in $\mathbb{R}^2$, and that in restriction to subspace of codimension 1 it still cannot dominate the $L^2$-norm with the Hardy weight $r^{-2}$.

**Theorem 4.2.** Assume that $j = \frac{n-2-b}{2} \in \mathbb{Z} \setminus \{1, \ldots, n-2\}$. Then there is a $C > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^m)$ satisfying

$$\int_1^2 \int_{S^{n-1}} P_j u(r, \cdot)rdrd\omega = 0, \quad (4.2)$$

the following inequality holds true:

$$Q_b(u) = \int_{\mathbb{R}^n} r^{-b}|(\sigma \cdot \nabla)u|^2 dx \geq C \int_{\mathbb{R}^n} r^{-b-2}(1 + |\log r|)^{-2}|u|^2 dx. \quad (4.3)$$

**Proof.** Repeating the proof of Theorem 4.1, it suffices to estimate from below, under the orthogonality condition

$$\int_1^2 c_j(r)dr = 0, \quad (4.4)$$
the term in the expansion (2.3) corresponding to \( k = j \), that is, the expression

\[
I \overset{\text{def}}{=} \int_0^\infty r^{-2j} \left( |\partial_r c_j(r)|^2 - j^2 |c_j|^2 \right) r dr.
\]

Substituting \( w(r) = c_j(r)r^j \) we get

\[
I = \int_0^\infty |\partial_r w(r)|^2 r dr.
\]

Then by (4.19)

\[
I \geq C \int_0^\infty r^{-b} (1 + |\log r|)^{-2} |w(r)|^2 r dr = C \int_0^\infty r^{-b-2} (1 + |\log r|)^{-2} |c_j|^2 r^{n-1} dr.
\]

This yields (4.3).

**Theorem 4.3.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain containing the origin and let \( R = \sup_{x \in \Omega} |x| \). Let \( \eta_1(r) \overset{\text{def}}{=} \log(R/r) \) and define recursively \( \eta_j(r) \overset{\text{def}}{=} \eta_1 \circ \eta_{j-1} \). Then for all \( u \in C_0^\infty(\Omega \setminus \{0\}; C^m) \),

\[
Q_b(u) = \int_\Omega r^{-b} |(\sigma \cdot \nabla)u|^2 dx \geq c_b \int_\Omega r^{-b-2} |u|^2 dx + c_b \sum_{k=1}^\infty \int_\Omega r^{-b} \eta_1(r) \ldots \eta_k(r) |u|^2 dx
\]

(4.8)

**Proof.** The proof is repetitive of that of Theorem 1.1 except for instead of substituting into (2.3) the estimate (2.4), one uses a refinement of (2.4) for functions supported on a ball \( B_R(0) \). Modifying the reduction to the standard Hardy inequality by means of Theorem 4.4, one replaces (4.13) with the following well-known Hardy inequality with remainder terms ([8],[3]):

\[
\int_0^1 |w'|^2 dr \geq \frac{1}{4} \int_0^R r^{-2} |w|^2 dr + \frac{1}{4} \sum_{k=1}^\infty \int_0^R r^{-2} \eta_1(r) \ldots \eta_k(r) |w|^2 dr
\]

(4.9)

Several other error expressions refining (4.13) found in literature can be used to provide refinements of (2.4), and subsequently, of (1.7) as well. We leave it as an exercise for the reader.
Appendix A: Caffarelli-Kohn-Nirenberg inequality

**Theorem 4.4.** Let $a \in \mathbb{R}$, $n \in \mathbb{N}$. Then for every $u \in C_0^\infty(\mathbb{R}^n) \setminus \{0\}$,

$$
\int_{\mathbb{R}^n} |x|^a |\nabla u|^2 \, dx \geq \left( \frac{a + n - 2}{2} \right)^2 \int_{\mathbb{R}^n} |x|^{a-2} |u|^2 \, dx, \quad (4.10)
$$

and the constant in the right hand side is exact.

When $a > 2 - n$ the inequality is due to [5], where it is established for all $u \in C_0^\infty(\mathbb{R}^n)$. However once one considers only the functions vanishing near the origin the restriction $a > 2 - n$ can be be removed and a significantly shorter proof can be given.

**Proof.** The following well-known identity holds for any positive $\psi \in C^2(\mathbb{R}^n \setminus \{u\})$ and any $u \in C_0^\infty(\mathbb{R}^n) \setminus \{0\}$:

$$
\int_{\mathbb{R}^n} \psi^2 |\nabla u|^2 \, dx = \int_{\mathbb{R}^n} (|\nabla (u\psi)|^2 + (\Delta \psi)\psi |u|^2) \, dx. \quad (4.11)
$$

Applying it to $\psi = |x|^{a/2}$, we have

$$
\int_{\mathbb{R}^n} |x|^a |\nabla u|^2 \, dx = \int_{\mathbb{R}^n} \left( |\nabla (u\psi)|^2 + \frac{a}{2} (a/2 + n - 2) |x|^{a-2} |u|^2 \right) \, dx. \quad (4.12)
$$

Applying to the first term the standard Hardy inequality with the exact constant

$$
\int_{\mathbb{R}^n} |\nabla w|^2 \, dx \geq \left( \frac{n - 2}{2} \right)^2 \int_{\mathbb{R}^n} |x|^{-2} |w|^2 \, dx \quad (4.13)
$$

and collecting similar terms in (4.12), we arrive at (4.10).

Appendix B: Weighted Sobolev inequality

The inequality below for $\alpha > 2 - n$ is due to [5] and [10][2.1.6, Cor.2]. For a smaller class of functions $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ the inequality also extends to $\alpha < 2 - n$.

**Proposition 4.5.** Let $n > 2$ and $2^* = 2n/(n-2)$. There is a $C > 0$ such that for every $v \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$,

$$
\int_{\mathbb{R}^n} |x|^\alpha |\nabla \beta|^2 \, dx \geq C \left( \int_{\mathbb{R}^n} |x|^\beta |v|^2^* \, dx \right)^{2/2^*}, \quad (4.14)
$$

where $\alpha \in \mathbb{R} \setminus \{2 - n\}$ and $\beta = \frac{\alpha n}{n-2}$.
Proof. By Sobolev inequality, there exists $C > 0$ such that for all $v \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$,

$$\left(\int_{1 <|x| \leq 2} |v|^{2^*} dx\right)^{2/2^*} \leq C \int_{1 <|x| \leq 2} |\nabla v|^2 dx + C \int_{1 <|x| \leq 2} |v|^2 dx. \quad (4.15)$$

Then, with a renamed $C$,

$$\left(\int_{1 <|x| \leq 2} |x|^{\beta} |v|^{2^*} dx\right)^{2/2^*} \leq C \int_{1 <|x| \leq 2} |x|^\alpha |\nabla v|^2 dx + C \int_{1 <|x| \leq 2} |x|^\alpha - 2 |v|^2 dx. \quad (4.16)$$

Replacing $v$ with $2^{j(N + \alpha - 2)/2} v(2^j \cdot)$, $j \in \mathbb{Z}$, we obtain

$$\left(\int_{2^j <|x| \leq 2^{j+1}} |x|^{\beta} |v|^{2^*} dx\right)^{2/2^*} \leq C \int_{2^j <|x| \leq 2^{j+1}} |x|^\alpha |\nabla v|^2 dx + C \int_{2^j <|x| \leq 2^{j+1}} |x|^\alpha - 2 |v|^2 dx. \quad (4.17)$$

Addition over $j \in \mathbb{Z}$ provides, taking into account subadditivity in the left hand side,

$$\left(\int_{\mathbb{R}^n} |x|^{\beta} |v|^{2^*} dx\right)^{2/2^*} \leq C \int_{\mathbb{R}^n} |x|^\alpha |\nabla v|^2 dx + C \int_{\mathbb{R}^n} |x|^\alpha - 2 |v|^2 dx. \quad (4.18)$$

By (4.10) the second term in the right hand side is dominated by the first term, from which, once we note that $\alpha \neq 2$, (4.14) is immediate. \qed

Appendix C: Hardy inequality in $\mathbb{R}^2$

**Theorem 4.6.** Let $\Psi(u) = \int_{1 <|x| < 2} u(x)dx$. Then there exists $C > 0$ such that for ever $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ satisfying $\Psi(u) = 0$, the following inequality holds true:

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx \geq C \int_{\mathbb{R}^2} r^{-2}(1 + |\log r|)^{-2} |u|^2 dx. \quad (4.19)$$

**Proof.** Assume first that $u \in C_0^\infty(B_R(0) \setminus \{0\})$, $R > 0$ and is radially symmetric. Then it is easy to show that (4.19) holds unconditionally by using the change of variable $t = \log(2R/|x|)$, which reduces the inequality to the standard one-dimensional Hardy inequality. This argument can be repeated for radially symmetric functions $u \in u \in C_0^\infty(\mathbb{R}^2 \setminus B_1(0))$ (or by applying Kelvin transformation to the inequality in the ball). From this and the
elementary density argument follows that \((4.19)\) holds for all radially symmetric functions \(u \in C^1_0(\mathbb{R}^2 \setminus \{0\})\) satisfying \(u(R) > 0\) or \(\Psi_0(u) = 0\), where \(\Psi_0(u) = \int_{|x|=R} u dx\). This can be equivalently rewritten (with a different constant \(C\)) as

\[
\Psi_0(u) + \int_{\mathbb{R}^2} |\nabla u|^2 dx \geq C \int_{\mathbb{R}^2} r^{-2}(1 + |\log r|)^{-2}|u|^2 dx \tag{4.20}
\]

for all radially symmetric functions \(u \in C^1_0(\mathbb{R}^2 \setminus \{0\})\). Assume now that

\[
\inf \{\int_{\mathbb{R}^2} |\nabla u|^2 : \int_{\mathbb{R}^2} r^{-2}(1 + |\log r|)^{-2}|u|^2 dx = 1, \Psi(u) = 0\} = 0, \tag{4.21}
\]

where the infimum is taken over all radially symmetric functions \(u \in C^1_0(\mathbb{R}^2 \setminus \{0\})\) satisfying the constraints. Then there exists a sequence of radial functions \(u_k \in C^1_0(\mathbb{R}^2 \setminus \{0\})\) such that \(\int_{\mathbb{R}^2} |\nabla u_k|^2 \to 0\), \(\int_{\mathbb{R}^2} r^{-2}(1 + |\log r|)^{-2}|u_k|^2 dx = 1\) and \(\Psi(u_k) = 0\). Then \(u_k\) is bounded in \(H^1_{loc}(\mathbb{R}^2)\) and, on a renumbered subsequence, it converges weakly in \(H^1_{loc}(\mathbb{R}^2)\) to some constant \(\lambda\) satisfying \(\Psi(\lambda) = 0\). Then \(\lambda = 0\), which implies that \(\Psi_0(u_k) \to 0\), and therefore it follows from \((4.20)\) that \(\int_{\mathbb{R}^2} r^{-2}(1 + |\log r|)^{-2}|u_k|^2 dx \to 0\), a contradiction. From \((4.21)\) follows

\[
\Psi(u) + \int_{\mathbb{R}^2} |\nabla u|^2 dx \geq C \int_{\mathbb{R}^2} r^{-2}(1 + |\log r|)^{-2}|u|^2 dx \tag{4.22}
\]

for all radially symmetric functions \(u \in C^1_0(\mathbb{R}^2 \setminus \{0\})\). The inequality for general functions \(u \in C^1_0(\mathbb{R}^2 \setminus \{0\})\) follows from decomposition into spherical harmonics once one observes that \(\Psi(u)\) depends only the radial component of \(u\) and that the the spherical Laplacian for any higher harmonics yields the lower bound of the form \(C \int_{\mathbb{R}^2} r^{-2}|u|^2 dx\).

Appendix D: Definition of generalized ground state

We give a generalization of definition of \([12]\) for real valued functionals on the normed (not necessarily complete) vector space.

**Definition 4.7.** Let \(X\) be a normed vector space and assume that a functional \(Q : X \to \mathbb{R}\) is positively homogogeneous of a positive degree and that

\[
\inf_{\langle \xi, x \rangle = 1} Q(x) = 0 \tag{4.23}
\]

for every \(\xi \in X^*\). One says that \(v \in X^{**}\) is a generalized ground state of \(Q\) if for every \(\xi \in X^*\) there exists a minimizing sequence \(x_k\) for \((4.23)\), whose weak-* limit is a scalar multiple of \(v\).
This definition allows to define a ground state when (4.23) has no minimizer even under an appropriate extension of \( Q \) (the functional \( Q \) is not required to be semicontinuous and \( X \) is not required to be complete.) When \( Q \) is a positive quadratic form of the Schrödinger operator with a potential term, defined on \( C_0^\infty(\Omega) \), the classical ground state of \( Q \) in \( L^2(\Omega) \) is also the ground state in the sense of the definition above. On the other hand, if \( Q \) is the difference between the right and the left hand side in (4.13), it has a generalized ground state \( v(x) = |x|^{\frac{2-n}2} \) which is neither in \( L^2(\mathbb{R}^n) \), nor in \( \mathcal{D}^{1,2}(\mathbb{R}^n) \), nor in \( L^2(\mathbb{R}^n; |x|^{-2}) \). In this case, for \( n > 1 \), \( X^{**} \) is the subspace of \( u \in W^{1,2}_\text{loc}(\mathbb{R}^n) \) such that

\[
\int_{\mathbb{R}^n} |x|^{2-n} |\nabla (|x|^{-\frac{n-2}2} u(x))|^2 dx + \left( \int_{1<|x|<2} u dx \right)^2 < \infty,
\]

denoted in [12] as \( \mathcal{D}^{1,2}_V(\mathbb{R}^n) \) with \( V(x) = - (\frac{n-2}2)^2 \frac{1}{|x|^2} \).

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