A Consistency Condition for the Double Series Approximation Method.

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Abstract

The double series approximation method of Bonnor is a means for examining the gravitational radiation from an axisymmetric isolated source that undergoes a finite period of oscillation. It involves an expansion of the metric as a double Taylor series. Here we examine the integration procedure that is used to form an algorithmic solution to the field equations and point out the possibility of the expansion method breaking down and predicting a singularity along the axis of symmetry. We derive a condition on the solutions obtained by the double series method that must be satisfied to avoid this singularity. We then consider a source with only a quadrupole moment and verify that to fourth order in each of the expansion parameters, this condition is satisfied. This is a reassuring test of the consistency of the expansion procedure. We do, however, find that the imposition of this condition makes a physical interpretation of any but the lowest order solutions very difficult. The most obvious decomposition of the solution into a series of independent physical effects is shown not to be valid.

1 Introduction.

In 1959 Bonnor introduced the double series method as a means for looking at the gravitational radiation from an isolated source that oscillates for a finite period. This involved expanding the metric as a double power series in two parameters.

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and solving the vacuum Einstein field equations by successive approximations. This first approach dealt with isolated sources emitting spherical gravitational waves and showed that the source lost mass at a rate equal to that of energy radiated. Following the publication of the Bondi metric in 1960 \[ 2 \], a metric well suited to this problem, Bonnor and Rotenberg refined and generalised the double series method to the axisymmetric case \[ 4 \]. They were then able to reproduce the earlier result concerning mass loss and to show that the momentum generated in the source is equal and opposite to that removed by the gravitational waves. They were also able to assert the existence of wave-tails in Bondi coordinates. These represent the backscattering of the gravitational radiation by the spacetime curvature induced by the source. The tails were interpreted as incoming radiation in a further paper on the double series method by Hunter and Rotenberg \[ 5 \]. Elsewhere, \[ 8 \], I considered the interaction of these incoming wave-tails with the Schwarzschild source. I found that the wave-tails have no permanent effects on the source - all changes in the metric die off at least like \( u^{-2} \) (\( u \) being retarded time). I also showed that the incoming radiation does not alter the mass of the source itself.

The double series method itself has influenced the formalism of Blanchet, Damour and Iyer \[ 6 \], in which the spacetime is divided into a near-zone region, a far-zone region and a region of overlap. In the far-zone a post-Minkowskian expansion is undertaken that is a generalisation of the double series method to more physical, non-axisymmetric sources. A review of the BDI formalism has recently been given by Blanchet \[ 9 \].

In this paper I derive a consistency condition for the double series method. If this condition is not satisfied at a given level of approximation, a singularity appears in the solution along the axis of symmetry. While it might be possible to interpret this singularity as an infinite pipe or strut, this would go against our original criterion
imposed here, namely that the system be isolated. The occurrence of this singularity
would therefore represent a breakdown of the double series method applied to isolated
sources. The importance of the condition lies in the fact that at a given order of
approximation its satisfaction depends not on the solution of the field equations to
that order, but on the previously obtained lower order solutions. The condition is
more a test of the double series method itself than of any particular solution obtained
by the method.

We then extend the work on the double series method using a power series package
developed for SHEEP to carry out the otherwise prohibitive algebraic calculations [7].
We verify that the consistency condition is satisfied at the third order and obtain the
Corresponding solution. We are then able to show that to fourth order the consistency
condition is also satisfied. However, when considering specifically the wave-tail -
wave-tail interaction we find that the condition fails. This component of the solution
generates terms in the metric that are singular along the axis of symmetry. We are
able to show that there is an exact cancellation that removes these singular terms in
the full solution. This suggests that the double series method is not decomposable in
the sense that it cannot be easily separated into a series of individual physical effects
but that there is a complicated mixing of these effects occurring. This in turn leads
to difficulties in the physical interpretation of the solutions beyond the lowest orders.

The order of this paper is as follows. In section 2 we describe the source and the
expansion of the metric coefficients that is characteristic of the double series method.
In section 3 we outline the algorithm that is used to solve the linearised field equations
and in section 4 we describe how this algorithm leads to the possible occurrence
of singularities along the axis of symmetry. We then derive the condition for the
avoidance of these singularities. Section 5 reviews previous work on the double series
method and verifies that the solutions obtained satisfy this consistency condition. In
sections 6 and 7 we extend the solution, for a source with only a quadrupole moment, to the third and fourth orders and verify that these solutions are consistent with our assumption of an isolated source to fourth order.

2 The Double Series Method.

We consider an isolated axisymmetric source, such as two particles oscillating along an axis. The system is at rest for \( u < u_1 \), vibrates smoothly for the period \( u_1 < u < u_2 \) and is again at rest for \( u > u_2 \), though not necessarily in the same state as before \( u_1 \).

We are interested in the behaviour of the system at large radial distances and so do not specify the source in detail. We do not describe the mechanism that drives the oscillations since this should not have an effect on the asymptotic behaviour of the solution.

It is assumed that there are two characteristic measurements of the system, \( m \), the source mass, and \( a \), a characteristic length (the separation of the two particles in the example mentioned above). The fundamental assumption of the double series method is that there exist solutions of the vacuum field equations that admit Taylor expansions in \( m \) and \( a \) about \( m = 0, a = 0 \). We do not offer a proof that this is the case, but the linear approximation suggests that this is a valid assumption to make.

We assume that the system can be modelled by Bondi’s radiative metric \[2\] which can be written in the form:

\[
\begin{align*}
\text{ds}^2 &= -r^2(Bd\theta^2 + C \sin^2 \theta d\phi^2) + Ddu^2 + 2Fdrdu + 2rGd\theta du
\end{align*}
\]

(2.1)

with \( C = B^{-1} \) and where the metric coefficients are functions of \( u, r, \theta \). Here we are using Bondi coordinates labelled \((x_1, x_2, x_3, x_4) = (r, \theta, \phi, u)\).
We make two further assumptions in order to start the approximation procedure. The first is that if we remove the mass \( m = 0 \) our metric should reduce to flat spacetime in Bondi’s coordinates:

\[
\begin{align*}
\frac{ds^2}{dr} &= -r^2(d\theta^2 + \sin^2 \theta d\phi^2) + du^2 + 2drdu. \\
\end{align*}
\]  
(2.2)

The second assumption is that if we set \( a = 0 \) then the spacetime should reduce to Schwarzschild space in Bondi coordinates:

\[
\begin{align*}
\frac{ds^2}{dr} &= -r^2(d\theta^2 + \sin^2 \theta d\phi^2) + \left(1 - \frac{2m}{r}\right)du^2 + 2drdu. \\
\end{align*}
\]  
(2.3)

Our fundamental assumption leads to the following representation:

\[
\begin{align*}
B &= 1 + \sum_{p=1}^{\infty} \sum_{s=0}^{\infty} \binom{ps}{g} B m^p a^s \\
C &= 1 + \sum_{p=1}^{\infty} \sum_{s=0}^{\infty} \binom{ps}{g} C m^p a^s \\
D &= 1 + \sum_{p=1}^{\infty} \sum_{s=0}^{\infty} \binom{ps}{g} D m^p a^s \\
F &= 1 + \sum_{p=1}^{\infty} \sum_{s=0}^{\infty} \binom{ps}{g} F m^p a^s \\
G &= \sum_{p=1}^{\infty} \sum_{s=0}^{\infty} \binom{ps}{g} G m^p a^s. \\
\end{align*}
\]  
(2.4)

To zeroth order in \( m \) this corresponds to the metric (2.2). The choice \( D = -2r^{-1} \), with all other \( B, C, D, F, G \), \( (s \geq 0) \) vanishing, ensures that the metric reduces to (2.3) to zeroth order in \( a \). Here again the coefficients in the expansion (2.4) are functions of \( u, r, \theta \). We now introduce the notation \( \binom{ps}{g} \) to represent the terms in the metric proportional to \( m^p a^s \). By the \( (ps) \) case we shall mean the solution of the field equations to order \( m^p a^s \). Where there is ambiguity as to which products give rise to \( m^p a^s \) terms (for example in the (24) case we have \( ma^2 \times ma^2, m \times ma^4, ma \times ma^3 \))

\[
\begin{align*}
ma^2 \times ma^2, m \times ma^4, ma \times ma^3. \\
\end{align*}
\]  
(2.5)
terms), we shall refer to these products explicitly, e.g. the \((12) \times (12)\) case, the \((10) \times (14)\) case or the \((11) \times (13)\) case in the example above.

Having made the expansion (2.4) above, the vacuum field equations are of the form:

\[ \Phi^{(ps)}_{(g_{ab})} = \Psi^{(qr)}_{(g_{ab})} \]  

(2.6)

where the left hand side is linear in \((g_{ab})\) (the unknowns) and the right hand side is non-linear in \((g_{ab})\), \(q < p, r < s\) which are known from previous steps in the approximation. The explicit form of these equations is given in the appendix.

We have now arrived at a system of linear partial differential equations. In the next section we describe the algorithm that is used to obtain solutions to this system.

3 The Algorithm for Solving the Field Equations.

It can be shown that the field equations (2.6), three of which are identically satisfied due to our imposition of axial symmetry, admit an algorithmic solution. The derivation of this algorithm is given in the appendix. Here we give the bare details that are required for the derivation of the consistency condition. In order to ease the notation we consider the general \(mpas\) case and omit the superscripts \((ps)\). The algorithm consists of three steps:

**Step 1.**

Obtain \(F\), the coefficient of \(g_{12}\) of order \(mpas\), as defined by (2.4) by the relationship:

\[ F = -\frac{1}{4} \int rHdr + \eta(\theta, u) \]  

(3.1)

where \(\eta\) is a function of integration and \(H\) is one of the non-linear terms, i.e. one component of the right-hand side of equations (2.6).

**Step 2.**
Solve for $D$ the ‘pseudo-wave’ equation

$$\Box' D \equiv D_{11} - 2D_{14} + 2r^{-1}(D_1 + D_4) + r^{-2}(D_{22} + D_2 \cot \theta)$$

$$= -K + 2(F_{14} + 2r^{-1}F_4) + 2r^{-2} \left[ \int \{ r^2(N - 2F_{14}) + (F_{22} + F_2 \cot \theta) \} dr + \chi(u, \theta) \right]_4 \quad (3.2)$$

where $K$, $N$ are non-linear terms, $\chi$ is a function of integration and $F$ is defined by (3.1). The pseudo-wave operator denoted $\Box'$ is similar to the usual d’Alembertian wave operator of flat space in the Bondi coordinates used here, the only difference coming in the sign of the $2D_4r^{-1}$ term.

**Step 3.**

Obtain the remaining metric coefficients from the expressions

$$G = r^{-1}\csc \theta \int \sin \theta \left[ \int r^2(N - 2F_{14}) dr + r^2D_1 + \chi \right] d\theta + r^{-1} \int F_2 dr + \nu(r, u) \csc \theta \quad (3.3)$$

$$B = \csc^2 \theta \int \sin^2 \theta \left[ - \int \{ rL + 2r^{-1}(F_2 - G) \} dr + F_2 - G - rG_1 \right] d\theta + \tau(r, u) \csc^2 \theta + \mu(\theta, u) \quad (3.4)$$

where $\nu$, $\tau$, $\mu$ are further functions of integration and $L$ is another non-linear term.

The five functions of integration that arise in the double series method, namely $\eta, \chi, \nu, \tau$ and $\mu$, are chosen to satisfy two criteria. Firstly they should ensure that the solution satisfies the full set of field equations (2.6) (the derivation of the above algorithm depends on only 4 of the 7 non-trivial field equations). Secondly they are used to impose certain boundary conditions on the system, namely that the metric be Minkowskian at spatial infinity, and that the metric remain non-singular on the axis of symmetry except at the origin. A sufficient condition for the latter is that $B \csc^2 \theta, C \csc^2 \theta, D, F, G \csc \theta$ are of class $C^2$ near $\sin \theta = 0$. In the next
section we show that unless a particular condition is satisfied this criterion cannot be achieved. This imposes a consistency condition on the double series method.

4 A Consistency Condition.

We now examine the possibility of a singularity arising along the axis of symmetry during the integration of the field equations contained in the algorithm of the previous section. This singularity cannot be reconciled with our assumption of an isolated source and its existence would imply a lack of consistency in this application of the double series method. In later sections we will show that this condition is indeed satisfied to order $m^4a^4$. Here we derive the condition itself. As in the previous section we consider the general $mpa^4$ case and drop the superscripts ($ps$) to ease the notation.

We suppose that we have obtained a solution $F$ of (3.1) and a solution $D$ of the psuedo-wave equation (3.2). Since the wave operator $\Box'$ defined by (3.2) preserves the Legendre polynomials in $\cos \theta$, it is natural to make the following expansion of the theta dependence of the solution:

$$
D = \sum_{i=0}^{\infty} iD(u, r)P_i(\cos \theta), \quad F = \sum_{i=0}^{\infty} iF(u, r)P_i(\cos \theta)
$$

$$
N = \sum_{i=0}^{\infty} iN(u, r)P_i(\cos \theta), \quad \chi = \sum_{i=0}^{\infty} i\chi(u)P_i(\cos \theta).
$$

(4.1)

where $P_i$ is the $i$th Legendre polynomial.

We now proceed to step 3 of the algorithm and obtain an expression for $G$ from (3.3):

$$
G = \sum_{i=0}^{\infty} r^{-1}\cosec \theta \int \sin \theta P_i(\cos \theta) \left[ \int r^2 (iN - 2iF_{14})dr + r^2 iD_1 + i\chi \right] d\theta
$$

$$
+ \sum_{i=0}^{\infty} r^{-1} P_i(\cos \theta) \int iF dr + \nu(u, r) \cosec \theta,
$$

(4.2)
where \( \dot{\theta} \) here denotes differentiation with respect to \( \theta \). Let us now take the first term \( (i=0) \) of the series in the above expansion. This is

\[
- r^{-1} \cot \theta \left[ \int r^2 (0N - 2F_{14}) dr + r^2 D_1 + \chi \right]
\]

which is singular at \( \sin \theta = 0 \), i.e. along the axis of symmetry. Since

\[
\int P_i(\cos \theta) \sin \theta d\theta
\]

contains a factor \( \sin \theta \) for \( i \geq 1 \), the expression (4.3) is the only singular term in the series contained in (4.2). Moreover, this singularity cannot be removed by a specific choice of the function of integration \( \nu \) in (4.2). The choice

\[
\nu = \pm r^{-1} \left[ \int r^2 (0N - 2F_{14}) dr + r^2 D_1 + \chi \right]
\]

removes the singularity along either the semi-axis \( \theta = 0 \), or the semi-axis \( \theta = \pi \) but not both. Thus we find that the double series method predicts a singularity along the axis of symmetry unless the term (4.3) vanishes, i.e. unless

\[
0D_1 = -r^{-2} \int r^2 (0N - 2F_{14}) dr - r^{-2} \chi.
\]

We now take this expression for \( 0D_1 \) and substitute it into the pseudo-wave equation (3.2) which leads us to an expression for \( 0D_4 \):

\[
0D_4 = r \left[ \frac{1}{2} 0N - \frac{1}{2} 0K \right] + 2F_4.
\]

Now using the fact that partial derivatives commute we can obtain two expressions for \( 0D_{14} \) from (4.6) and (4.7) which must be equivalent. This leads us to the following condition that must be satisfied if a singularity along the axis is to be avoided:

\[
X := r^2 \left[ \frac{1}{2} 0N_{11} - \frac{1}{2} 0K_{11} + \frac{1}{2} 0H_{44} - \frac{1}{2} 0H_{14} \right] + r \left[ 20N_1 - 20K_1 + 0N_4 - \frac{4}{2} 0H_4 \right] + \left[ 0N - 0K \right] \equiv 0.
\]
This is a necessary condition for the avoidance of singularities at the \((ps)\) level. It is also a sufficient condition for the avoidance of singularities in \(G\) in the sense that if \(4.8\) is satisfied then a particular choice of the functions of integration \(\chi, \eta\) can always be made to ensure that there is no singularity along the axis in \(G\). There is the additional possibility of the integration procedure leading to a singularity along the axis in \(B\). The avoidance of this singularity leads to a condition on the \(P_1(\cos \theta)\) coefficients of the non-linear terms and on \(0D_1\). However, this condition turns out not to be as simple as \(4.8\), nor is it relevant to the rest of the work in this paper as we shall see in the next section.

It can now be seen that the condition \(4.8\) depends only on the non-linear terms \(0N, 0H, 0K\), i.e. only on the metric coefficients \(g_{ab}, q < p, r < s\), that have been found by previous approximations. The occurrence of a singularity along the axis would, for the type of sources we have in mind here, appear unphysical. Any interpretation of this singularity as a strut or pipe extending along an infinite semi-axis would contradict our assumption the the source be isolated. Therefore in solving the \((qr), q < p, r < s\) approximations we require a relationship between these solutions to be satisfied at the \((ps)\) order. This relationship does not seem to be manifestly satisfied. We therefore conclude that the condition \(4.8\) is a test of the double series method itself. If \(4.8\) is not satisfied at any particular level of approximation the expansion procedure does not seem to be consistent with our premise of having an isolated source. We now examine the work of Bonnor and Rotenberg \[4\] on the double series method and show that the solutions obtained are consistent in the above sense.

5 The First and Second Order Solutions.

We now briefly review the first and second order \((m)\) solutions obtained by the double series method, paying particular attention to the condition \(4.8\) derived in
the previous section. The first and second order solutions were found and interpreted by Bonnor and Rotenberg [4].

The linear approximation, (1s), is a superposition of terms each involving one of the multipole moments, \( Q_n \), about the axis of symmetry. We use the non-gravitational forces inherent in the system to ensure that these have a suitable form:

\[
Q_n = ma^n h_n(u)
\]

(5.1)

for \( n > 1 \). The dipole moment can be eliminated by a suitable choice of frame. We choose such a frame and then are able to specify that \( g_{ab} = 0 \). The functions \( h_n \) describe the oscillation of the source. Here, for simplicity we assume that only \( h_2 \) is not identically zero so that the source has only a quadrupole moment. This ensures that \( H, K, N, D \) all have only even order dependence on the Legendre polynomials in \( \cos \theta \) and that \( L \) depends only on the derivatives of even order Legendre polynomials in \( \cos \theta \). As a result the condition mentioned in the previous section for the avoidance of singularities in \( B \) will always be trivially satisfied in this work. The function \( h_2 \) is assumed to be constant for \( u < u_1, u > u_2 \) and to vary smoothly for \( u_1 < u < u_2 \). In the double series method all source terms are inserted at the linear approximation. This is a distinct advantage over the Bondi, van den Burg and Metzner formalism [3], in which the news function that generates the solution contains non-linear terms. To ensure that we do not insert higher order source terms we specify that any arbitrary functions of integration at higher orders are set to zero, except where these are required to avoid a singularity occurring along the axis of symmetry.

Bonnor and Rotenberg then solved the (22) approximation. The non-linear terms that arise at the \( m^2 a^2 \) level are due to

\[
g_{ab} \times g_{ab}
\]

(5.2)
products. The (22) solution thus describes the interaction between the Schwarzschild source and the quadrupole wave. The solution contains integrals of the form:

$$S_n = \int_\infty r^{-n} h_2(u + 2r - 2w)dw.$$  (5.3)

These represent wave-tails, describing the back-scattering of the quadrupole radiation by the source.

The (24) solution was solved to $O(r^{-3})$ by Bonnor and Rotenberg and completed by Hunter and Rotenberg [5]. It is here that the loss of mass of the source due to the gravitational radiation is first seen. The occurrence of such integrals as:

$$Y := \int_{-\infty}^u \frac{\ddot{h}_2^2}{h_2} du$$  (5.4)

which are zero for $u < u_1$ and non-zero positive constants for $u > u_2$ indicate a permanent change in the metric due to the period of oscillation. It can be shown that they represent a source losing mass at a rate equal to that at which energy is radiated by gravitational waves.

The (24) approximation is the first at which the consistency condition (4.8) is not trivially satisfied. At the (22) level, $0H, 0K, 0N$ all vanish. In the (24) approximation, Hunter and Rotenberg found that

$$0H = -\frac{2}{15}r^{-4}\dddot{h}_2^2 - \frac{4}{5}r^{-6}\dddot{h}_2 h_2 - \frac{6}{5}r^{-8}h_2^2$$

$$0K = -\frac{2}{15}r^{-2}\dddot{h}_2^2 + r^{-4} \left[ -\frac{4}{15} \dddot{h}_2 h_2 - \frac{4}{15} \dddot{h}_2^2 \right] - \frac{4}{15} r^{-5}\dddot{h}_2 h_2 +$$

$$r^{-6} \left[ -\frac{4}{5} \ddot{h}_2 h_2 + 2\dddot{h}_2^2 \right] + \frac{24}{5} r^{-7}\dddot{h}_2 h_2 + 3r^{-8}h_2^2$$

$$0N = \frac{2}{15}r^{-3}\dddot{h}_2 h_2 + r^{-5} \left[ \frac{2}{5} \dddot{h}_2 h_2 - \frac{2}{5} \dddot{h}_2^2 \right] +$$

$$r^{-6} \left[ -\frac{4}{5} \ddot{h}_2 h_2 + \frac{8}{5} \dddot{h}_2^2 \right] + \frac{26}{5} r^{-7}\dddot{h}_2 h_2 + 3r^{-8}h_2^2$$  (5.5)

It can easily be checked that these values satisfy the condition. Thus to second order the expansion method of the double series method is seen to allow a consistent
specification of an isolated source. It should be noted that the non-linear terms arise only from the product
\[ g_{ab} \times g_{ab} \times g_{ab} \times g_{ab} \].

At the third and fourth orders it is not this simple - the non-linear terms consist of several such products. If the double series method were to be ‘clean’, each of these products could be considered independently. This does not turn out to be the case.

6 Third Order Solutions.

The (32) solution is given in another paper \[8\]. Here it is sufficient to note that at this order the condition \((4.8)\) is trivially satisfied. Here we are interested in the (34) solution.

The non-linear terms of order \(m^3a^4\) are formed by the following products:
\[ g_{ab} \times g_{ab}, g_{ab} \times g_{ab}, g_{ab} \times g_{ab} \times g_{ab} . \]

Let us consider each case in turn. Firstly we consider those non-linear terms that arise through \((10) \times (24)\) products. Using the power series package developed for SHEEP \[7\] we can calculate the corresponding values of \(0H, 0K\) and \(0N\). We do not give these values here since they are not of particular interest in themselves. When we test the consistency condition we find that it is not satisfied. The value of \(X\) in \((4.8)\) is non-zero. We denote this value \(X_1\):
\[ X_1 = \frac{3}{12} r^{-4} \dot{h}_2 \ddot{h}_2 + r^{-6} \left[ -\frac{14}{3} \dddot{h}_2 \dot{h}_2 + \frac{66}{5} \dddot{h}_2 h_2^2 \right] + \]
\[ r^{-7} \left[ 36 \dddot{h}_2 h_2^2 - 48 \dot{h}_2^2 \right] - \frac{654}{5} r^{-8} \dot{h}_2^2 h_2 - 168 r^{-9} h_2^2 \].

\[ (6.2) \]
Now we consider the \((12) \times (22)\) case. Again we find that the consistency condition fails. We denote the values of \(X\) that we find in this case \(X_2\):

\[
X_2 = \frac{2}{15} r^{-4} \dot{h}_2 \ddot{h}_2 - \frac{16}{5} r^{-5} \dddot{h}_2^2 + r^{-6} \left[ \frac{22}{5} \dot{h}_2 \dddot{h}_2 + \frac{146}{15} \dddot{h}_2 \dot{h}_2 \right] \\
-12r^{-7} \ddot{h}_2 h_2 + \frac{342}{5} r^{-8} \dddot{h}_2 h_2 + \frac{336}{5} r^{-9} \dddot{h}_2^2.
\] (6.3)

Finally we look at the non-linear terms that arise through \((10) \times (12) \times (12)\) products. We find a third, non-zero value for \(X\) which we this time denote \(X_3\):

\[
X_3 = -\frac{1}{15} r^{-4} \dddot{h}_2 \dddot{h}_2 + \frac{140}{8} r^{-5} \dddot{h}_2^2 + r^{-6} \left[ -\frac{8}{5} \dddot{h}_2 h_2 - \frac{344}{45} \dddot{h}_2 \dddot{h}_2 \right] \\
+ r^{-7} \left[ -24 \ddot{h}_2 h_2 + 48 \dddot{h}_2^2 \right] + \frac{652}{5} r^{-8} \dddot{h}_2 h_2 + \frac{504}{5} r^{-9} \dddot{h}_2^2.
\] (6.4)

Thus we find that each individual product in (6.1) fails to satisfy the condition (6.8) and hence leads to singularities in the metric. However, if we consider all the products together, i.e. the complete \((34)\) case, we find that the condition is satisfied. This is seen by the relationship:

\[
X_1 + X_2 + X_3 = 0.
\] (6.5)

The singular terms that appear due to the non-zero values of \(X\) given by (6.2) to (6.4) exactly cancel each other. Thus the double series method to this order is mixed in the sense that the decomposition of the problem into the individual cases given by (6.1) leads to a solution that in each case is inconsistent with our assumption that the source is isolated. Only the full solution is valid. We have obtained the solution to the \((34)\) field equations in order to proceed to the fourth order solution but we do not give it here. The fact that it cannot be decomposed into its constituent cases would lead to great difficulties in making a physical interpretation of the solution. This is certainly the case at the \((44)\) level which we consider next.
7 Fourth Order Solutions.

Here we consider the (44) solution. This would appear to be of significant physical interest since it contains the first interaction between the wave-tails, arising from the product

\[ \text{(22)} \times \text{(22)} g_{ab} \times g_{ab}. \]  

(7.1)

However, when we calculate the non-linear terms due to this product we find that the consistency condition (4.8) is not satisfied. In particular

\[
X = \frac{4}{5} r^{-3} \ddot{h}^2 S_5 + r^{-4} \left[ \ddot{h}^2 S_4 - \frac{12}{5} \dddot{h}^2 S_5 \right] + r^{-5} \left[ \frac{4}{5} \dddot{h}^2 S_3 - 8 \dddot{h}^2 S_4 + \frac{36}{5} \dddot{h}^2 S_5 \right] + \\
- \frac{1}{5} \dddot{h}^2 \dddot{h}^2 S_2 + \frac{2}{5} \dddot{h}^2 S_3 + \frac{3}{15} \dddot{h}^2 S_4 + \frac{196}{15} \dddot{h}^2 S_5 + \frac{9}{5} \dddot{h}^2 S_6 + \frac{24}{5} \dddot{h}^2 S_7 + \\
r^{-7} \left[ 6 \dddot{h}^2 \dddot{h}^2 S_2 - 12 \dddot{h}^2 S_4 + \frac{48}{5} \dddot{h}^2 S_3 - \frac{168}{5} \dddot{h}^2 S_4 \right] + r^{-8} \left[ 3 \dddot{h}^2 \dddot{h}^2 - \frac{9}{5} \dddot{h}^2 S_2 \right] + \\
\left[ \frac{18}{5} \dddot{h}^2 S_2 - \frac{312}{5} \dddot{h}^2 S_3 \right] + r^{-9} \left[ \frac{168}{5} \dddot{h}^2 S_2 - \frac{336}{5} \dddot{h}^2 S_3 \right] + \frac{54}{5} r^{-10} \dddot{h}^2 \dddot{h}^2 S_1. \]  

(7.2)

Thus the (22) \times (22) case leads to singularities along the axis appearing in the metric at the \( m^4 a^4 \) level.

However, just as at the third order, we should also consider all possible products that arise in the non-linear terms:

\[
\text{(10)} g_{ab} \times g_{ab}, \quad \text{(12)} g_{ab} \times g_{ab}, \quad \text{(22)} g_{ab} \times g_{ab}, \quad \text{(32)} g_{ab} \times g_{ab} \]  

(7.3)

as well as triple and quadruple products. We then find that the condition (4.8) is again satisfied. The double series method allows a consistent definition of an isolated source to fourth order. The singular terms that do appear in the wave-tail - wave-tail interaction (7.2) are exactly cancelled in the full (44) solution. This suggests that it is not possible to separate the physically interesting part of the (44) solution from that part with less apparent physical interest. This would make any physical interpretation of the (44) solution very difficult.
8 Discussion.

We have derived a condition for the double series method that must be satisfied at each order for the solution to that order to be consistent with the idea of an isolated source. Failure to satisfy this condition leads to a singularity along the axis of symmetry at that order. We have then verified that this condition is satisfied to order \( m^4 a^4 \) for a source with only a quadrupole moment. This certainly suggests that the particular relationship (4.8) is a feature of the field equations and that the condition is automatically satisfied when we are dealing with isolated sources. However it seems very difficult to prove this. It would appear likely that any generalisation of the double series method to isolated sources without axisymmetry would require an analogous condition to be satisfied.

When we considered the third and fourth order solutions we found that the most straight-forward decomposition of the solution into a series of independent physical effects proves to be impossible. There is a complicated mixing of the individual interactions that ensures that despite the fact that the separate cases each seem to generate singular terms in the metric, the full solution remains non-singular and consistent with the definition of a source that remains isolated. The cancellation of these singular terms that occurs in the full solution at each order, is certainly surprising. It is also very important for the validity of the expansion procedure inherent in the double series method.

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A Appendix

Here we present the vacuum equations used in the double series method and give details of solving these equations. To order $m^p a^s$ the vacuum Einstein equations are given by the vanishing of the Ricci tensor components which are (dropping the labels $(ps)$):

\begin{align*}
2R_{11} & \equiv -4r^{-1}F_1 - H \\
2r^{-2}R_{22} & \equiv B_{11} - 2B_{14} + 2r^{-1}(B_1 - B_4 + D_1 - F_1 - G_{12}) + \\
& \quad r^{-2}(-B_{22} - 3B_2 \cot \theta + 2B + 2D + 2F_{22} - 4F - 4G_2 - \\
& \quad 2G \cot \theta) - I \\
-2R_3^2 & \equiv -B_{11} + 2B_{14} + 2r^{-1}(-B_1 + B_4 + D_1 - F_1 - G_1 \cot \theta) + \\
& \quad r^{-2}(-B_{22} - 3B_2 \cot \theta + 2B + 2D + 2F_2 \cot \theta - 4F - \\
& \quad 2G_2 - 4G \cot \theta) - J \\
2R_{44} & \equiv -D_{11} + 2F_{14} + 2r^{-1}(-D_1 - D_4 + 2F_4 + G_{24} + G_4 \cot \theta) - \\
& \quad r^{-2}(D_{22} + D_2 \cot \theta) - K \\
2r^{-1}R_{12} & \equiv -G_{11} + r^{-1}(-B_{12} - 2B_1 \cot \theta + F_{12} - 2G_1) + \\
& \quad 2r^{-2}(-F_2 + G) - L \\
2R_{14} & \equiv -D_{11} + 2F_{14} + r^{-1}(-2D_1 + G_{12} + G_1 \cot \theta) + \\
& \quad r^{-2}(-F_{22} + F_2 \cot \theta + G_2 + G \cot \theta) - N \\
2r^{-1}R_{24} & \equiv -G_{11} + G_{14} + r^{-1}(-B_{24} - 2B_4 \cot \theta - D_{12} + F_{12} + F_{24} - \\
& \quad 2G_1 - G_4) - P
\end{align*}

(A.1)

where $H, I, J, K, L, N, P$ are the non-linear parts of the equations (the right hand sides of (2.6)).
The first equation is directly integrable with respect to $r$ to give:

$$ F = -\frac{1}{4} \int rHdr + \eta(\theta, u) \quad (A.2) $$

where $\eta$ is a function of integration. We can also integrate the $R_{14}$ component with respect to $r$ to give:

$$ r(G_2 + G \cot \theta) = r^2D_1 + \int \{r^2(N - 2F_{14}) + (F_{22} + F_2 \cot \theta)\}dr + \chi(\theta, u) \quad (A.3) $$

where $\chi$ is also a function of integration. We can now use this to eliminate $G$ from the $R_{44}$ component. This leads to the pseudo-wave equation (3.2):

$$ \Box' D = D_{11} - 2D_{14} + 2r^{-1}(D_1 + D_4) + r^{-2}(D_{22} + D_2 \cot \theta) $$

$$ = -K + 2(F_{14} + 2r^{-1}F_4) + 2r^{-2} \left[ \int \{r^2(N - 2F_{14}) + (F_{22} + F_2 \cot \theta)\}dr + \chi \right]_4. \quad (A.4) $$

To find $G$ we integrate (A.3) with respect to $\theta$:

$$ G = r^{-1} \int F_2dr + r^{-1}\csc \theta \int \sin \theta \left[ \int r^2(N - 2F_{14})dr + r^2D_1 + \chi \right]d\theta $$

$$ + \nu(r, u) \csc \theta \quad (A.5) $$

with $\nu$ being another function of integration. Finally we integrate the $R_{12}$ component with respect to $r$ and $\theta$ to obtain an expression for $B$:

$$ B = \csc^2 \theta \int \sin^2 \theta \left[ -\int \{rL + 2r^{-1}(F_2 - G)\}dr + F_2 - G - rG_1 \right]d\theta $$

$$ + \tau(r, u) \csc^2 \theta + \mu(\theta, u) \quad (A.6) $$

where $\tau, \mu$ are two further functions of integration.
References

[1] Bonnor, W.B. Phil. Trans. R. Soc. Lond. A, 251, 233 (1959).

[2] Bondi, H. Nature, Lond. 186, 535 (1960).

[3] Bondi, H., van der Burg, M.G.J. and Metzner, A.W.K. Proc. R. Soc. Lond. A 269, 21 (1962).

[4] Bonnor, W.B. and Rotenberg, M.A. Proc. R. Soc. Lond. A, 289, 247 (1966).

[5] Hunter, A.J. and Rotenberg, M.A. J. Phys. A, 2.2, 34 (1969).

[6] L.Blanchet and T.Damour, Phil. Trans. R. Soc. London A, 320, 379, (1986); Phys. Rev. D, 37, 1410, (1988); Ann. Inst. H. Poincaré (Phys. Théorique), 50, 377, (1989); L. Blanchet, Proc. R. Soc. Lond. A, 409, 383, (1987); T. Damour and B.R. Iyer, Ann. Inst. H. Poincaré (Phys. Théorique), 54, 115, (1991).

[7] Piper, M.S. A Power Series Package for SHEEP. Manual (in preparation).

[8] Piper, M.S. The Wave-Tail - Source Interaction in the Double Series Method. (in preparation).

[9] Blanchet, L. Gravitational radiation from relativistic sources. gr-qc/9607023.