Quasi-4-Connected Components

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Abstract

We introduce a new decomposition of graphs into quasi-4-connected components, where we call a graph quasi-4-connected if it is 3-connected and it only has separations of order 3 that remove a single vertex. Moreover, we give a cubic time algorithm computing the decomposition of a given graph.

Our decomposition into quasi-4-connected components refines the well-known decompositions of graphs into biconnected and triconnected components. We relate our decomposition to Robertson and Seymour’s theory of tangles by establishing a correspondence between the quasi-4-connected components of a graph and its tangles of order 4.

1 Introduction

Decompositions of graphs into their connected, biconnected and triconnected components are fundamental in structural graph theory, and they also belong to the basic toolbox of algorithmic graph theory. The existence of such decompositions goes back to work of MacLane [11] from the 1930s (also see Tutte [21]). In the 1970s, Hopcroft and Tarjan [9, 20] showed that the decompositions can be computed in linear time.

In modern terms, the decompositions into biconnected and triconnected components are best described as tree decompositions. To state the decomposition theorems and also our main results, a few technical definitions are unavoidable. Recall that a tree decomposition of a graph $G$ is a pair $(T,\beta)$, where $T$ is a tree and $\beta$ a mapping that associates a set $\beta(t) \subseteq V(G)$, called the bag at $t$, with every node $t$ of the tree $T$. The adhesion of the decomposition is the maximum of the sizes $|\beta(t) \cap \beta(u)|$ for tree edges $tu$, which intuitively is the order of the separations of the decomposition. Now the decomposition into biconnected components can be phrased as follows: every graph $G$ has a tree decomposition $(T,\beta)$ of adhesion at most 1 such that for all tree nodes $t$ the induced subgraph $G[\beta(t)]$ is either 2-connected or a complete graph of order at most 2. The decomposition into triconnected components is more complicated, mainly because the triconnected components of a graph are no longer induced subgraphs, but just topological subgraphs. We say that the torso of a set $X \subseteq V(G)$ of vertices of a graph $G$ is the graph $G[X]$ obtained from the induced subgraph $G[X]$ by adding edges $vw$ for all distinct $v,w \in X$ such that there is a connected component $C$ of $G \setminus X$ with $v,w \in N(C)$, the neighbourhood of $C$ in $G$. For example, the torso of the set $X = \{x_1, \ldots, x_4\}$ in the graph $G$ shown in Figure 1.1(a) is the complete graph on $X$. Now the decomposition into triconnected components can be phrased as follows: every graph $G$ has a tree decomposition $(T,\beta)$ of adhesion at most 2 such that for all tree nodes $t$ the torso $G[\beta(t)]$ is a topological subgraph of $G$ that is either 3-connected or a complete graph of order at most 3.

How about decompositions into 4-connected components, or $k$-connected components for $k \geq 4$? At least in the clean form of the above decomposition theorems, they simply do not exist. Consider, for example, a hexagonal grid (see Figure 1.2). Even
though the grid is not 4-connected, and it does not even have a nontrivial 4-connected subgraph, there is no good way of decomposing it in a tree like fashion by separations of order 3. However, the only separations of the grid of order 3 are those splitting off a single vertex. If we ignore such separations, we may view the whole grid as one highly connected region. Let us call a graph \( G \) quasi-4-connected if it is 3-connected and for all separations \((Y, S, Z)\) of order 3 (that is, \(|S| = 3\) and \(Y, S, Z\) form a partition of \(V(G)\) and there are no edges between \(Y\) and \(Z\)), either \(|Y| \leq 1\) or \(|Z| \leq 1\). Surprisingly, with this mild relaxation of 4-connectivity we get a nice decomposition theorem along the lines of the decompositions into biconnected and triconnected components.

**Theorem 1.1 (Decomposition Theorem).** Every graph \( G \) has a tree decomposition \((T, \beta)\) of adhesion at most 3 such that for all tree nodes \( t \) the torso \( G[\beta(t)] \) is a minor of \( G \) that is either quasi-4-connected or a complete graph of order at most 4.

Furthermore, this decomposition can be computed in cubic time.

There have been earlier attempts to generalise the decomposition of graphs into triconnected components. The most prominent of these are Robertson and Seymour’s tangles \[17\], which play an important role in the structure theory for graphs with excluded minors \[19\]. Intuitively, a tangle of order \( k \) describes a “\( k \)-connected region” in a graph by “pointing to it”, that is, by assigning a direction to each separation of order less than \( k \) in such a way that “most” of the region described by the tangle is on the side the separation is directed towards. It is known that the tangles of orders 1, 2, 3 are in one-to-one correspondence to the connected, biconnected and triconnected of a graph \[17, 6\]. We establish a similar correspondence between the tangles of order 4 and the quasi-4-connected components. This is our second main theorem, which I think is

![Figure 1.1. A graph and its decomposition into triconnected components](image1)

![Figure 1.2. Hexagonal grids of radius 2 and 3](image2)
interesting in its own right, but is also essential for the proof of Theorem 1.1. We defer the precise technical statement of this Correspondence Theorem to the main part of the paper (Theorem 4.1).

This paper grew out of my work on descriptive complexity theory for graph classes with excluded minors [7, 5], and this may also serve as an illustration of potential applications of our Decomposition Theorem. Separations of order 3 play a special, but somewhat annoying role in the main structure theorems for graph classes with excluded minors such as the “Flat Grid Theorem” of [18] and the structure theorem of [19], and the theorems simplify for quasi-4-connected graphs. In [5] I exploited some of the main ideas underlying our Decomposition Theorem to obtain such simplifications in the context of logical definability, and I believe the Decomposition Theorem proved here may turn out to be similarly useful in an algorithmic context.\footnote{Let me clarify the relation of this work to Chapter 10 of the forthcoming monograph [5]. The basic ideas are the same, and actually my original motivation for the present paper was to make these ideas accessible to readers not interested in logic. However, only when I started to work on this paper I noticed the connection to tangles, and it is this connection that provides the right framework and also makes the decomposition much simpler. On the other hand, the main goal of [5] is to obtain a decomposition that is definable in fixed-point logic with counting, and the decomposition we obtain here is not. So, except for some of the basic lemmas in Section 4.4 the results are incomparable.}

1.1 Related work

It was shown in [17, 1] that for every \(k\), every graph admits a canonical decomposition into its tangles of order \(k\). Related to this is the decomposition into so-called \((k-1)\)-blocks due to [3]. These decompositions (for \(k = 4\)) are related to ours. An important difference between these results and ours, or rather an additional feature of our decomposition, is that the pieces of our decomposition are quasi-4-connected graphs in their own right and can be dealt with separately (for example in an algorithmic context), whereas tangles of order 4 or 3-blocks are only defined within the surrounding graph.

On the algorithmic side, it was shown in [5] that the decomposition into its tangles of order \(k\) can be computed in time \(n^{O(k)}\). I believe that our techniques can be used to improve this to cubic time for \(k = 4\).

There is a different line of work on “\(k\)-connected components” that, as far as I can see, is completely unrelated to ours. There, \(k\)-connected components are simply defined as maximal \(k\)-connected subgraphs (see, for example, [12, 15, 14]). This leads to completely different decompositions. For example, a graph of maximum degree 3 will only have trivial 4-connected components in this framework. However, what I see as the crucial difference between our form of decomposition and this line of work is that we get tree decompositions into independent parts with a small interface (technically, small adhesion). This is important for typical dynamic-programming or divide-and-conquer algorithms on the decomposition.

2 Preliminaries

We assume basic knowledge of graph theory and refer the reader to [4] for background. Our notation is standard, let us just review the most important and frequently used notations. All graphs considered in this paper are finite and simple. The vertex set and edge set of a graph \(G\) are denoted by \(V(G)\) and \(E(G)\), respectively. The order of \(G\) is \(|G| := |V(G)|\). For a set \(W \subseteq V(G)\), we denote the induced subgraph of \(G\) with vertex set \(W\) by \(G[W]\) and the induced subgraph with vertex set \(V(G) \setminus W\) by \(G \setminus W\). For a vertex \(v\), we denote the set of neighbours of \(v\) in \(G\) by \(N^G(v)\). In this and similar notations, we omit the index \(G\) if \(G\) is clear from the context. For a set \(W \subseteq V(G)\), we define \(N^G(W) := \left(\bigcup_{v \in W} N^G(v)\right) \setminus W\), and for a subgraph \(H \subseteq G\) we
let $N^G(H) := N^G(V(H))$.

A minor of $G$ is a graph obtained from $G$ by deleting vertices and edge and contracting edges. An model of $H$ in $G$ consists of a family $(M_w)_{w \in V(H)}$ of mutually disjoint connected subsets of $V(G)$ and a family $(e_f)_{f \in E(H)}$ of edges of $G$ such that for every edge $f = w w'$ of $H$ the edge $e_f$ has one endvertex in $M_w$ and one endvertex in $M_{w'}$. Then $H$ is a minor of $G$ if and only if there is a model of $H$ in $G$. We call the sets $M_w$, for $w \in V(H)$, the branch sets of the model $M$. When reasoning about a model, it is often enough to know the branch sets.

A faithful model of $H$ in $G$ is a model $((M_w)_{w \in V(H)}, (e_f)_{f \in E(H)})$ such that $w \in M_w$ for all $w \in V(H)$. We say that $H$ is a faithful minor of $G$ if $V(H) \subseteq V(G)$ and there is a faithful model of $H$ in $G$.

Separations of a graph $G$ are usually defined as pairs of subgraphs (see the appendix). However, in this paper it will be more convenient to view them as partitions of the vertex set. We say that a separation of $G$ is a triple $(Y, S, Z)$ of (possibly empty) mutually disjoint subsets of $V(G)$ such that $Y \cup S \cup Z = V(G)$ and there is no edge $vw \in E(G)$ such that $v \in Y$ and $z \in Z$. The order of the separation $(Y, S, Z)$ is $|S|$, and the separation is proper if both $Y$ and $Z$ are nonempty. The set of all separations of $G$ is denoted by $\text{Sep}(G)$, and the subset of all separations of order less than $k$ (at most $k$, exactly $k$) by $\text{Sep}_{<k}(G)$ (resp. $\text{Sep}_{\leq k}(G)$, $\text{Sep}_{=k}(G)$).

A set $S \subseteq V(G)$ is a separator of $G$ of order $k := |S|$, or a $k$-separator, if there are two vertices $v, w \in V(G) \setminus S$ such that there is a path from $v$ to $w$ in $G$, but no path from $v$ to $w$ in $G \setminus S$. Note that if $G$ is connected then $S$ is a separator if and only if there is a proper separation $(Y, S, Z)$ of $G$.

A graph $G$ is $k$-connected if $|G| > k$ and $G$ has no proper $(k - 1)$-separation.

A subset $X \subseteq V(G)$ of the vertex set of a graph $G$ is $k$-inseparable if $|X| > k$ and there is no separation $(Y, S, Z)$ of $G$ of order at most $k$ such that $X \cap Y \neq \emptyset$ and $X \cap Z \neq \emptyset$.

### 3 Tangles

Let $G$ be a graph. Deviating from Robertson and Seymour’s original definition, we define tangles as families of separations of the vertex set (as we defined them in Section 2) rather than separations viewed pairs of subgraphs or partitions of the edge set. (In the appendix, we show that the two notions are equivalent.) A $G$-tangle of order $k$ is a family $\mathcal{T} \subseteq \text{Sep}_{<k}(G)$ of separations of $G$ of order less than $k$ satisfying the following conditions.

(T.1) For all separations $(Y, S, Z) \in \text{Sep}_{<k}(G)$ either $(Y, S, Z) \in \mathcal{T}$ or $(Z, Y, S) \in \mathcal{T}$.

(T.2) If $(Y_1, S_1, Z_1), (Y_2, S_2, Z_2), (Y_3, S_3, Z_3) \in \mathcal{T}$ then either $Z_1 \cap Z_2 \cap Z_3 \neq \emptyset$ or there is an edge $e \in E(G)$ that has an endvertex in each $Z_i$.

(T.3) $Z \neq \emptyset$ for all $(Y, S, Z) \in \mathcal{T}$.

In the following, we collect a few basic facts about tangles. For more background and examples, I refer the reader to [17] [16].

#### 3.1 Basic Facts

For $(Y, S, Z), (Y', S', Z') \in \text{Sep}(G)$, we let

$$(Y, S, Z) \cap (Y', S', Z') := (Y \cup Y', (S \cap Z') \cup (S \cap S') \cup (Z \cap S'), Z \cap Z')$$

$$(Y, S, Z) \cup (Y', S', Z') := (Y \cap Y', (S \cap S') \cup (S \cap S') \cup (Y \cap S'), Z \cup Z')$$

(see Figure 3.1 for an illustration). Note that both $(Y, S, Z) \cap (Y', S', Z')$ and $(Y, S, Z) \cup (Y', S', Z')$ are separations of $G$. 


Figure 3.1. Intersection and union of separations

Lemma 3.1. Let $G$ be a graph and $\mathcal{T}$ a $G$-tangle of order $k$.

1. If $(X, Y, Z) \in \text{Sep}(G)$ with $|Y \cup S| < k$ then $(Y, S, Z) \in \mathcal{T}$.

2. If $(Y, S, Z) \in \mathcal{T}$ and $(Y', S', Z') \in \text{Sep}_{<k}(G)$ such that $Z \subseteq Z'$ then $(Y', S', Z') \in \mathcal{T}$.

3. If $(Y, S, Z), (Y', S', Z') \in \mathcal{T}$ such that $(Y, S, Z) \cap (Y', S', Z') \in \text{Sep}_{<k}(G)$ then $(Y, S, Z) \cap (Y', S', Z') \in \mathcal{T}$.

Corollary 3.2. Let $G$ be a graph and $\mathcal{T}$ a $G$-tangle of order $k$. Let $(Y, S, Z), (Y', S', Z') \in \mathcal{T}$. Then $|(S \cup Z) \cap (S' \cup Z')| \geq k$.

The following lemma slightly strengthens Lemma 3.1(1).

Lemma 3.3. Let $G$ be a graph and $\mathcal{T}$ a $G$-tangle of order $k$. Then for all $(Y, S, Z) \in \text{Sep}_{<k}(G)$, if $|Y \cup S| \leq \frac{3}{2} \cdot (k-1)$ then $(Y, S, Z) \in \mathcal{T}$.

Proof. Let $(Y, S, Z) \in \text{Sep}_{<k}(G)$ such that $|Y \cup S| \leq \frac{3}{2} \cdot (k-1)$. By Lemma 3.1(1) we may assume that $|Y \cup S| = k$. Let $S_1 \subseteq Y \cup S$ such that $S \subseteq S_1$ and $|S_1| = k-1$, and let $Y_1 := Y \setminus S_1$. Then it suffices to prove $(Y_1, S_1, Z) \in \mathcal{T}$, because this implies $(Y, S, Z) \in \mathcal{T}$ by Lemma 3.1(2).

As $|Y \cup S| \leq \frac{3}{2} \cdot (k-1)$, we can choose subsets $S_2, S_3 \subseteq Y \cup S$ of cardinality $|S_i| = k-1$ such that for all $x, x' \in Y \cup S$ (not necessarily distinct) there is an $i$ such that $x, x' \in S_i$. Note that $Y_1 \subseteq S_2 \cup S_3$. By Lemma 3.1(1), we have $(0, S_i, V(G) \setminus S_i) \in \mathcal{T}$.

Suppose for contradiction that $(Z, S_1, Y_1) \in \mathcal{T}$. We have $Y_1 \cap (V(G) \setminus S_2) \cap (V(G) \setminus S_3) = Y_1 \setminus (S_2 \cup S_3) = \emptyset$. Furthermore, let $e = xx' \in E(G)$. If $e$ has an endvertex in $Y_1$ then $x, x' \in Y_1 \cup S = Y \cup S$ and not $x, x' \in S_1$. Thus either $x, x' \in S_2$ or $x, x' \in S_3$, and either $e$ has no endvertex in $V(G) \setminus S_2$ or no endvertex in $V(G) \setminus S_3$. This contradicts (1.2).

The next lemma shows that highly connected sets within a graph induce tangles. For a set $X \subseteq V(G)$ and $k \geq 1$, we let

\[ \mathcal{T}^k(X) := \{ (Y, S, Z) \in \text{Sep}_{<k}(G) \mid X \subseteq S \cup Z \} . \]

Of course in general, $\mathcal{T}^k(X)$ is not a tangle, and neither are all $G$-tangles of order $k$ of the form $\mathcal{T}^k(X)$. However, we will see in Section 3.3 that they are if $k \leq 3$. 

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Lemma 3.4. Let \( G \) be a graph and \( k \geq 1 \). Let \( X \subseteq V(G) \) be a \((k-1)\)-inseparable set of cardinality \(|X| > \frac{3}{2}(k-1)\). Then \( T^k(X) \) is a \( G \)-tangle of order \( k \).

Proof. To see that \( T := T^k(X) \) satisfies \( (T.1) \), note that the \((k-1)\)-inseparability of \( X \) implies \( X \subseteq S \cup Y \) or \( X \subseteq S \cup Z \) for every \((Y, S, Z) \in \text{Sep}_{k}(G)\).

To see that \( T \) satisfies \( (T.2) \), let \((Y_i, S_i, Z_i) \in T \) for \( i = 1, 2, 3 \). We have \( X \cap Y_i = \emptyset \) and thus \(|X \setminus Z_i| \leq |S_i| \leq k-1\). As \(|X| > \frac{3}{2}(k-1)\), there is a vertex \( x \in X \) such that \( x \) is contained in at most one of the sets \( S_i \) and hence in at least two of the sets \( Z_j \). Say, \( x \in Z_2 \cap Z_3 \). If \( x \in Z_1 \), then \( Z_1 \cap Z_2 \cap Z_3 = \emptyset \). So let us assume that \( x \in S_1 \).

Let \( x_1, \ldots, x_{k-1} \in X \setminus \{x\} \) be distinct. Such \( x_i \) exists because \(|X| \geq \frac{3}{2}(k-1) + 1 \geq k \). As \( X \) is \((k-1)\)-inseparable, for all \( i \) there is a path \( P_i \) from \( x \) to \( x_i \) such that \( V(P_i) \cap V(P_j) = \{x\} \) for \( i \neq j \). Let \( y_i \) be the last vertex of \( P_i \) (in the direction from \( x \) to \( x_i \)) that is in \( S_1 \cup Y_1 \) (possibly, \( y_i = x_i \)). We claim that \( y_i \in S_1 \). This is the case if \( y_i = x_i \in X \subseteq S_1 \cup Z_1 \). If \( y_i \neq x_i \), let \( z_i \) be the successor of \( y_i \) on \( P_i \). Then \( z_i \in Z_1 \), and as \( y_i, z_i \in E(G) \), it follows that \( y_i \in S_1 \).

Thus \( x, y_1, \ldots, y_{k-1} \in S_1 \), and as \(|S_1| \leq k-1 \) and the \( y_i \) are mutually distinct, it follows that \( y_i = x \) for some \( i \). As \( x \neq x_1 \), the vertex \( z_i \) exists. The edge \( xz_i \) has endvertices \( z_i \in Z_1 \) and \( x \in Z_2 \) and \( Z_3 \).

Finally, \( T \) satisfies \( (T.3) \) because for every \((Y, S, Z) \in T \) we have \( X \cap Z = \emptyset \), because \( X \subseteq S \cup Z \) and \(|X| > k-1 \geq |S| \).

It follows from Lemma 3.3 that the lower bound on \(|X|\) in the Lemma 3.4 is tight.

3.2 Minimal Elements

Let \( G \) be a graph. We define a partial order \( \preceq \) on \( \text{Sep}(G) \) by letting
\[
(Y, S, Z) \preceq (Y', S', Z') \quad \iff \quad S \cup Z \subset S' \cup Z' \quad \text{or} \quad (S \cup Z = S' \cup Z' \quad \text{and} \quad S \subset S').
\] (3.3A)

Note that if \(|S| = |S'|\), then \((Y, S, Z) \preceq (Y', S', Z') \iff (Z, S, Y) \succeq (Z', S', Y')\); this is not necessarily the case if \(|S| \neq |S'|\). For a \( G \)-tangle \( T \), we let \( T_{\min} \) be the set of minimal elements of \( T \) with respect to the partial order \( \preceq \).

Lemma 3.5 (Reed [16]). Let \( T \) be a \( G \)-tangle of order \( k \). Then for every set \( S \subseteq V(G) \) of cardinality \(|S| < k \) there is a connected component \( C_T(S) \) of \( G \setminus S \) such that for all \( Y, Z \) such that \((Y, S, Z) \in \text{Sep}_{k}(G)\),
\[
(Y, S, Z) \in T \iff V(C_T(S)) \subseteq Z.
\]

For a \( G \)-tangle \( T \) of order \( k \) and a set \( S \subseteq V(G) \) of cardinality \(|S| < k \), we let
\[
Z_T(S) := V(C_T(S)), \tag{3.3B}
\]
where \( C_T(S) \) is the connected component of \( G \setminus S \) from Lemma 3.5. Furthermore, we let
\[
Y_T(S) := V(G) \setminus (S \cup Z_T(S)). \tag{3.3C}
\]

Note that \((Y_T(S), S, Z_T(S)) \in T\).

Corollary 3.6. Let \( T \) be a tangle, and let \((Y, S, Z) \in T_{\min} \). Then \( S = N(Z) \) and \( Z = Z_T(S) \).

Corollary 3.7. Let \( T \) be a \( G \)-tangle of order \( k \), and let \((Y_1, S_1, Z_1), (Y_2, S_2, Z_2) \in T_{\min} \) be distinct. Then \(|(S_1 \cap Z_2) \cup (S_i \cap S_j) \cup (Z_i \cap S_j)| \geq k \) and \( S_i \cap Z_{3-i} = \emptyset \) for \( i = 1, 2 \).
Proof. \( S := (S_1 \cap Z_2) \cup (S_1 \cap S_2) \cup (Z_1 \cap S_2) \) is the separator of the separation \((Y, S, Z) := (Y_1, S_1, Z_1) \cap (Y_2, S_2, Z_2)\). As \((Y_1, S_1, Z_1)\) and \((Y_2, S_2, Z_2)\) are distinct, either \((Y, S, Z) \subset (Y_1, S_1, Z_1)\) or \((Y, S, Z) \subset (Y_2, S_2, Z_2)\). By the minimality of \((Y_1, S_1, Z_1)\) this implies \((Y, S, Z) \notin \mathcal{T}\), and thus by Lemma \([3.1][2]\), \(|S| \geq k\).

If \(S_1 \cap Z_{3-i} = \emptyset\) then \(S \subseteq S_{3-i}\), and as \(|S_{3-i}| < k\), this contradicts \(|S| \geq k\). \(\square\)

Corollary 3.8. Let \(\mathcal{T}\) be a \(G\)-tangle and \((Y, S, Z) \in \mathcal{T}_{\text{min}}\). Then for every \((Y', S', Z') \in \text{Sep}(G)\) with \(S' \subseteq S\) it holds that \(Z \cup (S \setminus S') \subseteq Z'\).

Proof. By Corollary \([3.6]\), \(Z \cup (S \setminus S')\) is connected and hence contained in some connected component of \(G \setminus S\). Thus \(Z \cup (S \setminus S') \subseteq Z'\) or \(Z \cup (S \setminus S') \subseteq Y'\). In the latter case, we have \(|Z \cup (S \cap (Z' \cap S'))| = |S'| < k\), which contradicts Corollary \([3.2]\) \(\square\)

3.3 Tangles of Order at Most 3

Let \(G\) be a graph and \(k \geq 1\). Following \([3]\), we call an inclusionwise maximal \(k\)-inseparable set \(X \subseteq V(G)\) a \(k\)-block of \(G\). We call a \(k\)-block \(X\) proper if \(|X| \geq k + 2\).

Observe that if \(X\) is a proper \(k\)-block then the torso \(G[X]\) is \((k + 1)\)-connected.

It can be shown that for \(k = 2, 3\) the torsos \(G[X]\) of the \((k - 1)\)-blocks \(X\), which for \(k \leq 2\) coincide with the induced subgraphs \(G[X]\) and for \(k = 3\) are topological subgraphs of \(G\), are precisely the biconnected and triconnected components appearing the decomposition described in the introduction.

By Lemma \([3.4]\), if \(X\) is a \((k - 1)\)-block for \(k = 1, 2\) or a proper \(k\)-block for \(k = 3\), then \(T^k(X)\) is a \(G\)-tangle of order \(k\). The following theorem shows that all \(G\)-tangles of order at most \(3\) are of this form.

Theorem 3.9 ([17, 8]). Let \(G\) be a graph, and let \(\mathcal{T}\) be a \(G\)-tangle of order \(k \leq 3\). Then the set

\[
X_{\mathcal{T}} := \bigcap_{(Y, S, Z) \in \mathcal{T}} (S \cup Z)
\]

is a \(k\)-block (proper if \(k = 3\)) and \(\mathcal{T} = T^k(X_{\mathcal{T}})\).

The theorem utterly fails for \(k = 4\): a hexagonal grid \(H\) (see Figure \([1, 2]\)) has a unique \(H\)-tangle \(\mathcal{T}\) of order \(4\), but the set \(X_{\mathcal{T}}\) (defined as in \((3.D)\)) is empty, and in fact \(H\) has no \(3\)-inseparable set.

As a motivation for our definition of “quasi-4-connected regions” in Section \([1.2]\), let us give an alternative characterisation of the proper 2-blocks. We have already remarked that they are precisely the vertex sets of the triconnected components. In view of our later terminology, we call them triconnected regions.

Proposition 3.10. Let \(G\) be a graph and \(R \subseteq V(G)\). The the following are equivalent.

1. \(R\) is a triconnected region of \(G\).
2. \(R\) is an inclusionwise maximal subset of \(G\) such that \(G[R]\) is 3-connected and a topological subgraph of \(G\).
3. \(G[R]\) is 3-connected and a topological subgraph of \(G\), and for every connected component \(C\) of \(G \setminus R\) we have \(|N(C)| \leq 2\).

Proof. To prove that (1) implies (3), let \(R\) be a triconnected region of \(G\), that is, an inclusionwise maximal 2-inseparable set \(R \subseteq V(G)\) of cardinality \(|R| \geq 4\). We have already noted that the torso of a proper 2-block is 3-connected.

Suppose for contradiction that \(C\) is a connected component of \(G \setminus R\) such that \(|N(C)| \geq 3\). Let \(C^+\) be the subgraph of \(G\) with vertex set \(V(C) \cup N(C)\) and all edges...
that have at least one endvertex in $C$ (that is, all edges of $C$ and all edges from $C$ to $N(C)$). By the maximality of $R$, for every $y \in V(C)$ there is a separation $(Y, S, Z)$ of order at most 2 such that $y \in Y$ and $R \subseteq S \cup Z$. Let $v_1, v_2, v_3 \in N(C)$ be distinct and $w_1 \in N(v_1) \cap V(C)$. Let $(Y_1, S_1, Z_1) \in \text{Sep}_{\leq 2}(G)$ such that $w_1 \in Y_1$ and $R \subseteq S_1 \cup Z_1$, and subject to these conditions, $(Y_1, S_1, Z_1)$ is $\succeq$-minimal. Then $v_1 \in S_1$, and as $C$ is connected, there is another vertex $x_1 \in S_1$ that separates $w_1$ from $v_2, v_3$ in the graph $C^+$. It follows from the minimality $(Y_1, S_1, Z_1)$ that there is no $x_2$ separating $x_1$ from $v_2, v_3$. Hence there are two internally disjoint paths $P_2, P_3 \subseteq C^+$ from $x_1$ to $v_2, v_3$, respectively. Moreover, there is a path $P_1 \subseteq C^+$ from $x_1$ to $v_1$ (via $w_1$), because $C^+$ is connected. $P_1$ is internally disjoint from $P_2$ and $P_3$, because otherwise $x_1$ would not separate $v_1$ from $v_2, v_3$. But this implies that there is no $(Y_2, S_2, Z_2) \in \text{Sep}_{\leq 2}(G)$ such that $x_1 \in Y_2$ and $R \subseteq S_2 \cup Z_2$. This is a contradiction.

Hence for every connected component $C$ of $G \setminus R$ we have $|N(C)| \leq 2$. This directly implies that $G[R]$ is a topological subgraph of $G$.

To prove that (3) implies (2), suppose that $R$ satisfies (3) and that there is an $R' \supset R$ such that $G[R']$ is 3-connected. Let $C$ be a connected component of $G \setminus R$ that contains a vertex of $R' \setminus R$. Then $|N(C)| \leq 2$ and thus $(V(G) \setminus (V(C) \cup N(C)), N(C), V(C)) \in \text{Sep}_{\leq 2}(G)$. This implies that $(R' \setminus (V(C) \cup N(C)), N(C), V(C) \cap R') \in \text{Sep}_{\leq 2}(G[R'])$, which contradicts $G[R']$ being 3-connected.

Finally, to prove that (2) implies (1), suppose that $R$ satisfies (2). Then $|R| \geq 4$, because $G[R]$ is 3-connected. For every separation $(Y, S, Z) \in \text{Sep}_{\leq 2}(G)$ with $Y \cap R \neq \emptyset$ and $Z \cap R \neq \emptyset$ the triple $(Y \cap R, S, Z \cap R)$ is a proper separation of $G[R]$ of the same order, which implies that $R$ is 2-inseparable. Suppose for contradiction that $R$ is not maximal 2-inseparable, and let $R' \supset R$ be the 2-block that contains $R$. Then by the implications $(1) \implies (3) \implies (2)$, $R'$ satisfies (2) as well, and this is a contradiction. 

### 3.4 Lift and Project

We can “lift” a tangle from a minor of a graph to the original graph. Let $G$ be a graph, $H$ a minor of $G$, and $\mathcal{M}$ a model of $H$ in $G$, say, with branch sets $(M_w)_{w \in V(H)}$. For a separation $(Y, S, Z) \in \text{Sep}(G)$, the $\mathcal{M}$-projection of $(Y, S, Z)$ to $H$ is the triple $(Y', S', Z')$ of subsets of $V(H)$ defined by

$$\begin{align*}
Y' &:= \{ w \in V(H) \mid V(M_w) \subseteq Y \}, \\
S' &:= \{ w \in V(H) \mid V(M_w) \cap S \neq \emptyset \}, \\
Z' &:= \{ w \in V(H) \mid V(M_w) \subseteq Z \}.
\end{align*}$$

(3E)

It is easy to see that $(Y', S', Z')$ is a separation of $H$ of order $|S'| \leq |S|$.

**Lemma 3.11** ([17]). Let $G$ be a graph, $H$ a minor of $G$, and $\mathcal{M}$ a model of $H$ in $G$. Let $T'$ be an $H$-tangle of order $k$. Then the set $\mathcal{T}$ of all separations $(Y, S, Z) \in \text{Sep}_{\leq k}(G)$ such that $\pi_{\mathcal{M}}(Y, S, Z) \in T'$ is a $G$-tangle of order $k$.

We call $\mathcal{T}$ be the lifting of $T'$ to $G$ with respect to the model $\mathcal{M}$. Clearly, the lifting may depend on the model. This is even the case if we only consider faithful minors and models. It is easy to see that the lifting relation is transitive, that is, if we have graphs $G, G', G''$ such that $G'$ a minor of $G$ and $G''$ a minor of $G'$, tangles $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ of $G, G', G''$ such that $\mathcal{T}'$ the lifting of $\mathcal{T}''$ to $G'$ with respect to some model of $G''$ in $G'$ and $\mathcal{T}$ the lifting of $\mathcal{T}'$ to $G$ with respect to some model of $G'$ in $G$, then there is a model of $G''$ in $G$ such that $\mathcal{T}$ is the lifting of $\mathcal{T}''$ to $G$ with respect to this model.

So we can lift tangles from minors of a graph to the graph. What about the converse: do tangles of a graph induce tangles of their minors? Obviously not in general, but in
the following lemma we identify a useful special case where a tangle of graph induces a
angle of some minor that is a torso of a triconnected region. We need some additional
terminology. Let \( T, T' \) be \( G \)-tangles. If \( T' \subseteq T \), we say that \( T \) is an \textit{extension} of \( T' \)
and \( T' \) is a \textit{truncation} of \( T \). Observe that every \( G \)-tangle \( T \) of order \( k \) has a unique
truncation to every order \( k' \leq k \).

\textbf{Lemma 3.12.} Let \( T \) be a \( G \)-tangle of order 4 such that the truncation of \( T \) to ordered
3 is \( T^3(R) \) for some ordered region \( R \) of \( G \). Let \( T[R] \) be the set of all separations
\((Y', S', Z') \in \text{Sep}_{<4}(G[R])\) such that there is a separation \((Y, S, Z) \in T \) with \( Y' = Y \cap X, S' = S \cap X, Z' = Z \cap X \).

Then \( T[R] \) is a \( G[R] \)-tangle of order 4.

\textbf{Proof.} We note first that for all \((Y, S, Z) \in T \) we have \( Z \cap R \neq \emptyset \). To see this, by
Lemma 3.9, we may assume without loss of generality that \( Z \) is connected in \( G \) and that
\( S = N(Z) \). Hence if \( Z \cap R = \emptyset \), there is a connected component \( C \) of \( G \setminus R \) such that \( Z \subseteq V(C) \). By Proposition 3.10, we have \(|N(C)| \leq 2 \). Thus \((V(C), N(C), V(G) \setminus (V(C) \cup N(C))) \in \text{Sep}_{<3}(C) \). As the truncation of \( T \) to order 3 is \( T^3(R) \) and \( R \cap V(C) = \emptyset \), we have \((V(C), N(C), V(G) \setminus (V(C) \cup N(C))) \in T \). But as \((Y, S, Z) \in T \) and \( Z \subseteq V(C) \) and thus \( Z \cup S \subseteq V(C) \cap N(C) \), this contradicts, this contradicts Corollary 3.2.

Let us now prove that \( T' := T[R] \) satisfies the tangle axioms. Let \( G' := G[R] \).
To prove that \( T' \) satisfies \((T.1) \) let \((Y', S', Z') \in \text{Sep}_{<4}(G') \). For every connected component \( C \) of \( G \setminus R \) the set \( N(C) \) is a clique in \( G' \) and thus either \( N(C) \subseteq Y' \cup S' \) or \( N(C) \subseteq S' \cup Z' \). We let \( Y \) be the union of \( Y' \) with the vertex sets of all connected components \( C \) of \( G \setminus R \) such that \( N(C) \subseteq Y' \cup S' \), and we let \( Z \) be the union of \( Z' \) with the
vertex sets of all remaining connected components of \( G \setminus R \). Then \((Y, S, Z) \in T \) and \((Y', S', Z') \in T' \) or \((Z', S', Y') \in T' \).

To prove that \( T' \) satisfies \((T.2) \) let \((Y'_i, S'_i, Z'_i) \in T' \) for \( i = 1, 2, 3 \). Then there are
\((Y_i, S_i, Z_i) \in T \) such that \( Y_i \cap R = Y'_i, S_i \cap R = S'_i, \) and \( Z_i \cap R = Z'_i \). By Lemma 3.5, we may assume that the sets \( Z_i \) are connected in \( G \). By our observation above, they have a nonempty intersection with \( R \). By \((T.3) \) either there is a vertex \( v \in Z_1 \cap Z_2 \cap Z_3 \) or an edge \( e \) that has at least one endvertex in every \( Z_i \). Assume the last, the argument in the former case is similar (and simpler). Let \( z_i \) be the endvertex of \( e \) in \( Z_i \). If all the \( z_i \) are in \( R \), the edge \( e \) is also an edge of \( G' \) which has an endvertex in every \( Z'_i \). Otherwise, there is a connected component \( C \) of \( G \setminus X \) such that all the \( z_i \) are in \( V(C) \cap N(C) \). If \( z_i \in V(C) \), then \( Z_i \cap N(C) \neq \emptyset \), because \( Z_i \) is connected in \( G \) and has a nonempty intersection with \( R \). Thus \( Z'_1, Z'_2, Z'_3 \) all contain at least one of the at most two vertices in \( N(C) \). Thus either they share a vertex, or the edge of \( G' \) that connects the two vertices in \( N(C) \) has an endvertex in all three \( Z'_i \).

Finally, \((T.3) \) follows from the fact that for every vertex separation \((Y, S, Z) \in T \) we have \( Z \cap R \neq \emptyset \).

\end{proof}

\section{Tangles of Order 4}

Let us now look at tangles of order 4. Lemma 3.12 in combination with Theorem 3.9
allows us to focus on 3-connected graphs. The main result of this section is a correspondence
between tangles of order 4 and what we will call \textit{quasi-4-connected regions}
of a graph. This correspondence holds for all but a small number of \textit{exceptional}
regions, which we shall completely characterise. We first state the theorem; the necessary
definitions follow as we go along.

\textbf{Theorem 4.1 (Correspondence Theorem).} Let \( G \) be a 3-connected graph. Then
with every quasi-4-connected region \( R \) of \( G \) we can associate a \( G \)-tangle \( T_R \) of order 4
and with every \( G \)-tangle \( T \) of order 4 a quasi-4-connected region \( R_T \) such that

\[ T = T_{R_T}. \]

We shall call the torsos \( G[R_T] \) for the \( G \)-tangles of order 4 the quasi-4-connected components of \( G \).

In general, the mapping \( R \mapsto T_R \) is not injective; the mapping \( T \mapsto R_T \) is (otherwise the theorem could not hold). The mapping \( R \mapsto T_R \) is canonical (or isomorphism invariant). This means that for any two graphs \( G, G' \) and regions \( R, R' \), if \( f \) is an isomorphism from \( G \) to \( G' \) that maps \( R \) to \( R' \) then \( f \) also maps \( T_R \) to \( T_{R'} \). This will be obvious from the construction. The mapping \( T \mapsto R_T \) is not canonical. However, the mapping from \( T \) to the quasi-4-connected component \( G[R_T] \), viewed as an abstract graph, is (see Corollary 4.4).

### 4.1 Quasi-4-Connected Graphs

Recall from the introduction that a graph \( G \) is quasi-4-connected if \( G \) is 3-connected and for all separations \((Y, S, Z)\) of \( G \) of order 3, either \(|Y| \leq 1\) or \(|Z| \leq 1\). In this section, we will analyse tangles of order 4 of quasi-4-connected graphs.

**Lemma 4.2.** Let \( G \) be a quasi-4-connected graph of order \( |G| \geq 8 \). Let

\[ T := \{ (Y, S, Z) \in \text{Sep}_{<4}(G) \mid |Y| < |Z| \}. \]

Then \( T \) is a \( G \)-tangle of order 4.

**Proof.** To see that \( T \) satisfies (T.1), let \((Y, S, Z) \in \text{Sep}_{<4}(G)\). Without loss of generality, we assume that \(|Y| \leq |Z|\). Then \(|Y| \leq 1\) and thus \(|Z| = V(G) \setminus (Y \cup S) \geq 4\). Hence \((Y, S, Z) \in T\). Similarly, \( T \) satisfies (T.3) because for all \((Y, S, Z) \in T\) it holds that \(|Y \cup S| \leq 4 < |V(G)|\).

It remains to prove that \( T \) satisfies (T.2). For \( i = 1, 2, 3 \), let \((Y_i, S_i, Z_i) \in T\). Suppose for contradiction that \( Z_1 \cap Z_2 \cap Z_3 = \emptyset \) and that there is no edge that has an endvertex in each \( Z_i \).

**Claim 1.** For distinct \( i, j, k \in [3] \) and \( x \in V(G) \), if \( x \in Z_i \cap Z_j \) then \( x \in Y_k \).

**Proof.** We have \( x \notin Z_k \), because \( Z_i \cap Z_j \cap Z_k = \emptyset \). Suppose that \( x \in S_k \), and let \( z \in N(x) \cap Z_k \). Such a \( z \) exists, because \( Z_k \neq \emptyset \) and \( N(Z_k) \subseteq S_k \), and as \(|S_k| \leq 3\) and \( G \) is 3-connected, this implies \( N(Z_k) = S_k \). But the edge \( xz \) has an endvertex in every \( Z_i \), which contradicts our assumption that no such edge exists.

We have \(|Y_i| \leq 1\) and thus \(|V(G) \setminus (Y_1 \cup Y_2 \cup Y_3)| \geq 5\). A simple double counting argument shows that there is a vertex \( x \in V(G) \setminus (Y_1 \cup Y_2 \cup Y_3) \) such that \( x \) is only contained in one of the three sets \( S_1, S_2, S_3 \) (count pairs \((x, S_i)\) where \( x \in V(G) \setminus (Y_1 \cup Y_2 \cup Y_3) \) and \( i = 1, 2, 3 \)). But an \( x \in V(G) \setminus (Y_1 \cup Y_2 \cup Y_3) \) contained in at most one of the sets \( S_i \) is contained in two of the sets \( S_i \), and this contradicts Claim 1.

**Example 4.3.** Figure 4.1 shows a quasi-4-connected graph \( TH_{+3} \) with seven vertices and no tangle of order 4. The name \( TH_{+3} \) is motivated by the fact that this graph can be seen as a tetrahedron with 3 “corners” attached to it.

To see that \( TH_{+3} \) has no tangle of order 4, suppose for contradiction that \( T \) is a \( TH_{+3} \)-tangle of order 4. For \( i = 1, 2, 3 \), let \( S_i = N(w_i) \) and \( Z_i = V(TH_{+3}) \setminus \{\{w_i\} \cup S_i\} \). Then by Lemma 3.3, we have \( \{\{w_i\}, S_i, Z_i\} \in T \). However, \( Z_1 \cap Z_2 \cap Z_3 = \emptyset \), and for every edge \( e = xy \) of \( TH_{+3} \) there is an \( i \) such that \( x, y \in \{w_i\} \cup S_i \). Hence \( T \) violates tangle axiom (T.2).
We now give a precise characterisation of the quasi-4-connected graphs with a tangle of order 4. A quasi-4-connected graph is exceptional if it is either isomorphic to a subgraph of the graph $TH_{+3}$ shown in Figure 4.1 or isomorphic to a subgraph of the graph $TR_{+3}$ shown in Figure 4.2.

**Lemma 4.4.** Let $G$ be an exceptional quasi-4-connected graph. Then $G$ has no tangle of order 4.

**Proof.** As all supergraphs of a graph that has a tangle of order 4 also have a tangle of order 4, we may assume without loss of generality that $G$ is an inclusionwise maximal exceptional quasi-4-connected graph, that is, $TH_{+3}$ or $TR_{+3}$.

We have already seen in Example 4.3 that $TH_{+3}$ has no tangle of order 4.

Suppose for contradiction that there is a $TR_{+3}$-tangle of order 4. By Lemma 3.3, the separations $(Y_i, S, Z_i) := \left(\{w_1\}, \{v_1, v_2, v_3\}, \{w_2, w_2, w_3\} \setminus \{w_i\}\right)$ of order 3 are in $T$. However, we have $Z_1 \cap Z_2 \cap Z_3 = \emptyset$, and for every edge $xy \in E(TR_{+3})$ there is an $i$ such that $x, y \in Y_i \cup S$. This contradicts (1.2). \(\square\)

**Theorem 4.5.** Let $G$ be a quasi-4-connected graph. Then $G$ has a tangle of order 4 if and only if it is not exceptional.

Furthermore, if $G$ has a tangle of order 4, it has exactly one such tangle, which consists of all separations $(Y, S, Z) \in \text{Sep}_{<4}(G)$ such that $|Y| < |Z|$.

**Proof.** We have already seen that if $G$ is exceptional then it has no tangle of order 4. To prove the converse, we assume that $G$ is not exceptional. Let

$$T := \{(Y, S, Z) \in \text{Sep}_{<4}(G) \mid |Y| < |Z|\}.$$

We shall prove that $T$ is a $G$-tangle of order 4.
By Lemma 4.2, we may assume that $|G| \leq 7$. Note that $|G| \geq 5$, because the only quasi-4-connected graph of order at most 4, the tetrahedron, is a subgraph of $TH_{+3}$ (and also of $TR_{+3}$) and hence exceptional. If $G$ is 4-connected, then it follows from Lemma 3.4 that $T$ is a $G$-tangle of order 4. Hence we may assume that $|G|$ is not 4-connected. But then $|G| \geq 6$, because all graphs of order 5 that are not 4-connected are subgraphs of $TH_{+3}$ (and also of $TR_{+3}$) and hence exceptional. From from now on we assume that $6 \leq |G| \leq 7$ and that $G$ is not 4-connected.

$T$ trivially satisfies (T.3). To see that it satisfies (T.1), let $(x, S, Z) \in \text{Sep}_{<4}(G)$. Without loss of generality, we may assume that $|Y| \leq |Z|$. As $G$ is quasi-4-connected, we have $|Y| \leq 1$. Thus $|Z| = |G| \setminus |Y \cup S| \geq 6 - 4 = 2 > |Y|$.

It remains to prove that $T$ satisfies (T.2) for $i = 1, 2, 3$, let $(Y_i, S_i, Z_i) \in T$. Suppose for contradiction that $Z_1 \cap Z_2 \cap Z_3 = \emptyset$ and that there is no edge that has an endvertex in each $Z_i$.

The next claim is the same as Claim 1 in the proof of Lemma 4.2.

**Claim 1.** For distinct $i, j, k \in [3]$ and $x \in V(G)$, if $x \in Z_i \cap Z_j$ then $x \in Y_k$.

**Case 1:** $|G| = 6$.

Then by Claim 2, at least two of the sets $Y_i$ must be nonempty. Say, $Y_1 = \{y_1\}$ and $Y_2 = \{y_2\}$. Then $S_i = N(y_i)$ for $i = 1, 2$.

Suppose for contradiction $y_1 \in S_2$. Then $y_2 \in S_1$. A similar double counting argument as above shows that at least one of the remaining four vertices in $V(G) \setminus (Y_1 \cup Y_2)$ is contained in at most one of the sets $S_i$: there are two pairs $(x, S_1)$ with $x \in V(G) \setminus (Y_1 \cup Y_2)$ and two pairs $(x, S_2)$ with $x \in V(G) \setminus (Y_1 \cup Y_2)$ and at most three pairs $(x, S_i)$ overall at most seven pairs $(x, S_i)$. But if every $x$ was contained in two of the sets $S_i$, there would be eight such pairs. As above, an $x \in V(G) \setminus (Y_1 \cup Y_2 \cup Y_3)$ contained in at most one of the sets $S_i$ is contained in two of the sets $Z_i$, and this contradicts Claim 1.

So $y_1 \not\in S_2$ and $y_2 \not\in S_1$. As $V(G) = 6$, we must have $|S_1 \cap S_2| \geq 2$.

**Case 1a:** $|S_1 \cap S_2| = 3$.

Then $S_1 = S_2$. Let $y_3$ be the unique vertex in $V(G) \setminus (S_1 \cup \{y_1, y_2\})$. Then $N(y_3) = S_1$, because $y_1, y_2 \not\in N(y_3)$ (otherwise $S_1 = S_2$ would not separate $y_1, y_2$ from $y_3$). Thus $G$ is isomorphic to a subgraph of $TR_{+3}$: the three vertices in $S_1$ can be mapped to $v_1, v_2, v_3$, and the vertices $y_1, y_2, y_3$ can be mapped to $w_1, w_2, w_3$.

**Case 1b:** $|S_1 \cap S_2| = 2$.

Say, $S_1 \cap S_2 = \{x_1, x_2\}$. For $i = 1, 2$, let $x_{2+i}$ be the unique vertex in $S_1 \setminus \{x_1, x_2\}$. Figure 4.3(a) shows the situation. As there is no edge from $y_1$ to $y_2$, this shows that $G$ is isomorphic to a subgraph of $TH_{+3}$: the four vertices $x_1, \ldots, x_4$ can be mapped to $v_1, \ldots, v_4$ and $y_1, y_2$ to $w_1, w_2$, respectively.

**Case 2:** $|G| = 7$.

Then all three sets $Y_i$ must be nonempty. Say, $Y_i = \{y_i\}$, and note that $S_i = N(y_i)$.

By essentially the same argument as in Case 1, we have $y_i \not\in S_j$ for all $i, j$. Furthermore, $S_i \neq S_j$ for all $i \neq j$, because otherwise $(\{y_i, y_j\}, S_i, V(G) \setminus (S_i \cup \{y_i, y_j\}))$ is separation of $G$ of order 3 where both sides have cardinality at least 2, which contradicts $G$ being quasi-4-connected. Hence $|S_i \cap S_j| \leq 2$. By the usual argument based on Claim 1, each of the four vertices in $V(G) \setminus \{y_1, y_2, y_3\}$ is contained in at least two of the sets $S_i$. It follows that there is one vertex $x_1 \in S_1 \cap S_2 \cap S_3$ and for all distinct $i, j, k$ a vertex $x_{ij} \in S_i \cap S_j \setminus S_k$. So far, the graph $G$ looks like the graph in Figure 4.3(b). This shows that it is a subgraph of $TH_{+3}$. □
4.2 Quasi-4-Connected Regions

For the rest of Section 4, we make the following assumption.

**Assumption 4.6.** $G$ is a 3-connected graph.

A quasi-4-connected region of $G$ is a subset $R \subseteq V(G)$ satisfying the following conditions.

(Q.1) $G[R]$ is a faithful minor of $G$.

(Q.2) $G[R]$ is quasi-4-connected.

(Q.3) For every connected component $C$ of $G \setminus R$ it holds that $N(C) = 3$.

While conditions (Q.1) and (Q.2) are, to some extent, natural, condition (Q.3) may seem less so. It is a (weak) maximality condition: if $R^{'} \supset R$ such that $G[R']$ is quasi-4-connected, then $R^{'} \setminus R$ contains at most one vertex of every connected component of $G \setminus R$ (unless $|R| = 4$). Conditions (Q.1)–(Q.3) are motivated by the characterisation of 3-connected components given in Proposition 3.10(3). The reason for choosing these conditions instead of adding some maximality condition is simply that it works best in combination with tangles and for the Decomposition Theorem; it is condition (Q.3) which guarantees that our decomposition will have adhesion 3.

In the remainder of Section 4.2, we shall prove that we can associate a tangle of order 4 with every quasi-4-connected region, up to a finite number of small exceptional cases. These exceptional cases will be derived from the exceptional quasi-4-connected graphs, but will also have to take the surrounding graph into account. The following example illustrates why.

**Example 4.7.** Consider the graph $G = TH_{+4}$ in Figure 4.3. The dashed edges may or not be present; it makes no difference for the example. Let $R := \{v_1, v_2, v_3, v_4\}$. Then $R$ is a quasi-4-connected region of $G$. The torso $G[R]$ is a tetrahedron, which is an exceptional quasi-4-connected graph. Yet the graph $G$ has a tangle $T$ of order 4. This tangle $T$ consists of all separations $(Y, S, Z) \in \text{Sep}_{<4}(G)$ such that $R \subseteq S \cup Z$, and hence it is fully justified to say that this tangle is “associated with $R$”.

To make the example more interesting, we may replace the vertices $w_i$ by larger 3-connected graphs. Then the resulting graph may have other tangles of order 4. But $R$ remains a quasi-4-connected region and the set of all separations $(Y, S, Z) \in \text{Sep}_{<4}(G)$ such that $R \subseteq S \cup Z$ remains a tangle of order 4.

![Figure 4.3. Proof of Theorem 4.5](image-url)
Figure 4.4. The graph $TH_{+4}$

Let us call subgraph $H$ of $TH_{+4}$ full if it is obtained by deleting some of the dashed edges in Figure 4.4, that is, $V(H) = V(TH_{+4}) = \{v_1, \ldots, v_4, w_1, \ldots, w_4\}$ and

$$\{v_iw_j \mid i,j \in [4], v_iw_j \in E(TH_{+4})\} \subseteq E(H) \subseteq E(TH_{+4}).$$

Observe that every full subgraph of $TH_{+4}$ is non-exceptional quasi-4-connected.

Let $R$ be a quasi-4-connected region of $G$. A non-exceptional extension of $R$ is a graph $\hat{H}$ satisfying the following conditions.

\[(X.1)\] $\hat{H}$ is a faithful minor of $G$.

\[(X.2)\] $\hat{H}$ is non-exceptional quasi-4-connected.

\[(X.3)\] $R \subseteq V(\hat{H})$, and for each connected component $C$ of $G \setminus R$ it holds that $V(\hat{H}) \cap V(C) \leq 1$.

\[(X.4)\] Subject to \[(X.1)\]–\[(X.3)\], $V(\hat{H})$ is inclusionwise minimal.

Note that, by \[(X.1)\] and \[(X.3)\], we have $R \subseteq V(\hat{H}) \subseteq V(G)$. We call the vertices in $V(\hat{H}) \setminus R$ the extension vertices of $\hat{H}$. Further note that if $G[R]$ is non-exceptional, then we have $V(\hat{H}) = R$ for every non-exceptional extension $\hat{H}$ of $R$. This implies $\hat{H} \subseteq G[R]$, but not necessarily $\hat{H} = G[R]$.

**Lemma 4.8.** Let $R$ be a quasi-4-connected region of $G$ such that $G[R]$ is exceptional. Let $\hat{H}$ be a non-exceptional extension of $R$. Then $\hat{H}$ is isomorphic to a full subgraph of $TH_{+4}$.

**Proof.** Let $H := G[R]$ and $\hat{H} := V(\hat{H})$. By \[(Q.3)\] and \[(X.3)\] and since $\hat{H}$ is 3-connected, we have $N(\hat{H}) \subseteq R$ and $|N(\hat{H})| = 3$ for every extension vertex $z \in \hat{H} \setminus R$. We observe next that there are no two extension vertices $z_1, z_2 \in \hat{H} \setminus R$ such that $N(\hat{H}) \cap N(\hat{H}) = N(\hat{H}) \cap N(\hat{H})$. Indeed, if $N(\hat{H}) \cap N(\hat{H}) = N(\hat{H}) \cap N(\hat{H}) = N$ then $|N \setminus (\hat{H} \cup \{z_1, z_2\})| = 1$, because $\hat{H}$ is quasi-4-connected, and thus $\hat{H}$ is isomorphic to a subgraph of $TR_{+3}$, which contradicts $\hat{H}$ being non-exceptional.

**Claim 1.** $H$ is isomorphic to a subgraph of $TH_{+3}$.

**Proof.** Suppose for contradiction that $H$ is not isomorphic to a subgraph of $TH_{+3}$. As $H$ is exceptional, this means that $H$ is isomorphic to a subgraph of $TH_{+3}$ that contains the vertices $w_1, w_2, w_3$. Without loss of generality we assume that $H \subseteq TR_{+3}$.
with \( w_1, \ldots, w_3 \in R \). Then \( v_1, \ldots, v_3 \in R \), because otherwise \( H \) is not 3-connected. For every connected component \( C \) of \( G \setminus R \) we have \( N(C) \leq 3 \) and \( N(C) \) is a clique in \( H \). Thus \( N(C) \) contains at most one of the vertices \( w_i \). Let \( z \in \hat{R} \setminus R \), and let \( C \) be the connected component of \( G' \setminus R \) such that \( z \in V(C) \). Without loss of generality we may assume that \( w_1, w_2 \notin N(C) \). Then there is a separation \((Y, \{v_1, v_2, v_3\}, Z)\) of \( \hat{H} \) with \( w_1, w_2 \in Y \) and \( w_3, z \in Z \), and this contradicts \( \hat{H} \) being quasi-4-connected.

It follows from Claim 1 that \( H \) is a tetrahedron, possibly with some vertices of degree 3 attached. The only way to turn such a graph into a non exceptional quasi-4-connected graph by attaching further vertices of degree 3 with mutually non-adjacent neighbourhoods is to turn it into \( TH + 4 \), possibly with some of the dashed edges missing.

Let us call a quasi-4-connected region \( R \) non-exceptional if it has a non-exceptional extension \( \hat{H} \). Let \( R \) be non-exceptional and \( \hat{H} \) a non-exceptional extension of \( R \). Let \( M \) be a faithful model of \( \hat{H} \) in \( G \), and let \( \hat{T} \) be the unique \( \hat{H} \)-tuple of order 4. Then the lifting \( \hat{T}(H, M) \) of \( \hat{T} \) with respect to \( M \) is a \( G \)-tuple of order 4.

**Lemma 4.9.** Let \( R \) be a non-exceptional quasi-4-connected region of \( G \) such that \( G[R] \) is exceptional. Then for all non-exceptional extensions \( \hat{H}, \hat{H}' \) of \( R \) and all faithful models \( M \) of \( \hat{H} \) and \( M' \) of \( \hat{H}' \) we have \( \hat{T}(H, M) = \hat{T}(H', M') \).

**Proof.** Let \( H := G[R] \). As \( H \) is exceptional, by Lemma 4.8 both \( \hat{H} \) and \( \hat{H}' \) are isomorphic to full subgraphs of \( TH + 4 \).

Without loss of generality we may assume that \( \hat{H} \subseteq TH + 4 \). Then \( v_1, \ldots, v_4 \in R \), because otherwise \( H \) is not quasi-4-connected or we have no way of adding the remaining vertices without violating (X.3). Let \( f \) be an isomorphism from \( \hat{H} \) to a full subgraph of \( TH + 4 \), and let \( v'_i := f^{-1}(v_i) \) and \( w'_i := f^{-1}(w_i) \). Then \( v'_1, \ldots, v'_4 \in R \), and by symmetry we may assume without loss of generality that \( v'_i = v_i \) for all \( i \in [4] \). As the \( w'_1, w'_2, w'_3, w'_4 \) are uniquely determined by their neighbours among the \( v_i \), if \( w_j \in R \), then \( w_j = w'_j \), and if \( w_j \) is in a component \( C \) of \( G \setminus R \) then \( w'_j \) is in a component \( C' \) with \( N(C) = N(C') = N(TH + 4)(w_j) \).

Let \( T := \hat{T}(H, M) \) and \( T' := \hat{T}(H', M') \). Suppose for contradiction that \( T \neq T' \). Then there is a separation \((Y, S, Z)\) in \( \text{Sep}_{\leq 4}(G) \) such that \((Y, S, Z) \in T \) and \((Z, S, Y) \in T' \). Let \((Y_M, S_M, Z_M) := \pi_M(Y, S, Z) \) and \((Y'_M, S'_M, Z'_M) := \pi_M'(Y, S, Z) \). Then

\[
(Y_M, S_M, Z_M) \in \hat{T} \quad \text{and} \quad (Y'_M, S'_M, Z'_M) \in \hat{T}. \tag{4.A}
\]

We have \( Z_M, Z'_M \subseteq Z \) and \( Y_M, Y'_M \subseteq Y \) (see the definition of the projection in (3.E)). Thus

\[
Z_M \cap Y'_M = \emptyset \quad \text{and} \quad Y_M \cap Z'_M = \emptyset. \tag{4.B}
\]

because \( Y \cap Z = \emptyset \).

**Claim 1.** There is no \( j \) such that \( w_j \in Y \) and \( w'_j \in Z \) or vice versa.

**Proof.** Suppose for contradiction that \( w_1 \in Y \) and \( w'_1 \in Z \). Let \( C, C' \) be the connected components of \( G \setminus R \) such that \( w_1 \in V(C) \) and \( w'_1 \in V(C') \) (possibly, \( C = C' \)). Then \( N(C) = N(C') = \{v_1, v_2, v_3\} \). As \( G \) is 3-connected, there are internally disjoint paths \( P_1, P_2, P_3 \) from \( w_1 \) to \( v_1, v_2, v_3 \), respectively, and internally disjoint paths \( P'_1, P'_2, P'_3 \) from \( w'_1 \) to \( v_1, v_2, v_3 \), respectively. The vertex sets of all these paths are contained in \( V(C) \cup V(C') \cup \{v_1, v_2, v_3\} \), and as \( S \) separates \( w_1 \in Y \) from \( w'_1 \in Z \), we have \( S \subseteq V(C) \cup V(C') \cup \{v_1, v_2, v_3\} \). This implies \( S_M, S'_M \subseteq \{v_1, v_2, v_3\} \).

If \( v_4 \in Z \) then \( v_4, w_2, w_3, w_4' \in Z'_M \). Thus \( |Z'_M| \geq 4 > |Y_M| \), and this implies \((Y'_M, S'_M, Z'_M) \in \hat{T} \), contradicting (4.A). Similarly, if \( v_4 \in Y \) then \( v_4, w_2, w_3, w_4 \in Y_M \), and this implies \((Z_M, S_M, Y_M) \in \hat{T} \), contradicting (4.A) again.
As \( \hat{H} \) is quasi-4-connected, \((Y_M,S_M,Z_M) \in \hat{T}\) implies that either \( Y_M = \emptyset \) or \( Y_M = \{w_j\} \) for some \( j \) and thus \( |Z_M| \geq 4 \). Similarly, \((Z_M,S_M,Y_M) \in \hat{T}\) implies \( Z_M = \emptyset \) or \( Z_M = \{w'_j\} \) for some \( j \) and \( |Y_M| \geq 4 \).

**Claim 2.** \( Y_M \neq \emptyset \) and \( Z_M \neq \emptyset \).

**Proof.** Suppose for contradiction that \( Y_M = \emptyset \). Then \( |Y_M| \geq 5 \). As \( |TH_{+4}| = 8 \), it follows that either \( Z_M \subseteq Y_M \neq \emptyset \), which contradicts (4.B), or there is a \( j \) such that \( w_j \in Z_M \subseteq Z \) and \( w'_j \in Y_M \subseteq Y \), which contradicts Claim 1.

Thus \( Y_M = \{w_j\} \) and \( Z_M = \{w'_j\} \), where \( j \neq j' \) by (4.B) and Claim 1. Without loss of generality we assume that \( Y_M = \{w_1\} \) and \( Z_M = \{w'_1\} \). Then \( S_M = N^{TH+4}(w_1) = \{v_1,v_2,v_3\} \) and \( S'_M = N^{TH+4}(w_2) = \{v_1,v_2,v_4\} \). Hence \( w_3 \in Z_M \subseteq Z \) and \( w'_3 \in Y_M \subseteq Y \), and this contradicts Claim 1. \( \square \)

If \( G[R] \) is non-exceptional, then there is no need for non-exceptional extensions, and we can directly work with liftings of the unique \( G[R] \)-tangle of order 4. Surprisingly, it is much harder to prove the uniqueness of the lifting in this case. If \( H := G[R] \) is non-exceptional and \( \mathcal{M} \) is a faithful image of \( H \) in \( G \), then \( T(H,\mathcal{M}) \) is the lifting of the unique \( H \)-tangle of order 4 to \( G \) with respect to \( \mathcal{M} \).

**Lemma 4.10.** Let \( R \) be a non-exceptional quasi-4-connected region of \( G \) such that \( H := G[R] \) is non-exceptional. Then for all faithful models \( \mathcal{M}, \mathcal{N} \) of \( H \) in \( G \) we have \( T(H,\mathcal{M}) = T(H,\mathcal{N}) \).

**Proof.** Let \( \hat{T} \) be the unique \( H \)-tangle of order 4. Let \((\mathcal{M}_w)_{w \in R}\) and \((\mathcal{N}_w)_{w \in R}\) be the branch sets of faith models \( \mathcal{M} \) and \( \mathcal{N} \) of \( G \). Suppose for contradiction that \( T(H,\mathcal{M}) \neq T(H,\mathcal{N}) \). Then there is a separation \((Y,S,Z) \in \text{Sep}_{<4}(G)\) such that

\[ (Y,S,Z) \in T(H,\mathcal{M}) \quad \text{and} \quad (Z,S,Y) \in T(H,\mathcal{N}). \]

Let \((Y_M,S_M,Z_M) := \pi_M(Y,S,Z) \text{ and } (Y_N,S_N,Z_N) := \pi_N(Y,S,Z).\) Then

\[ (Y_M,S_M,Z_M) \in \hat{T} \quad \text{and} \quad (Z_N,S_N,Y_N) \in \hat{T}. \] (4.C)

It follows from the definition of the projections in (3.E) and the assumption that the models \( \mathcal{M} \) and \( \mathcal{N} \) be faithful that \( Y_M \subseteq Y \cap R \subseteq Y_M \cup S_M \) and \( Z_M \subseteq Z \cap R \subseteq Z_M \cup S_M \) and \( Y_N \subseteq Y \cap R \subseteq Y_N \cup S_N \) and \( Z_N \subseteq Z \cap R \subseteq Z_N \cup S_N \). Hence

\[ Y_M \cap Z_N = \emptyset \quad \text{and} \quad Z_M \cap Y_N = \emptyset. \] (4.D)

(see Figure 4.3a).

By (1.C) and Lemma 3.3 we have \( |Z_M \cup S_M| \geq 5 \) and \( |Y_N \cup S_N| \geq 5 \) and thus \( |Z_M|, |Y_N| \geq 2 \). As \( H \) is quasi-4-connected, it follows that

\[ |Y_M| \leq 1 \quad \text{and} \quad |Z_N| \leq 1. \] (4.E)

**Claim 1.**

\[ |(S_M \cap Y_N) \cup (Z_M \cap S_N)| \geq 4. \] (4.F)

**Proof.** Let \( X := (S_M \cap Y_N) \cup (Z_M \cap S_N) \), and suppose for contradiction that \( |X| \leq 3 \). Then \( \emptyset, X, R \setminus X \) \( \in \hat{T} \). Hence by (1.C) and (T.2) either \( Z_M \cap Y_N \cap R \setminus X \neq \emptyset \) or there is an edge that has an endvertex in \( Z_M, Y_N, \) and \( R \setminus X \). However, it follows from (4.D) that neither is the case.
Figure 4.5. Proof of Lemma 4.10
Without loss of generality we assume
\[ |(S_M \cap Y_N)| \geq |(Z_M \cap S_N)| \quad (4.4) \]

Let us call an edge \( yz \in E(H) \) with \( y \in Y \) and \( v \in Z \) a \( yz \)-edge and a connected component \( C \) of \( G \setminus R \) with \( N(C) \cap Y \neq \emptyset \) and \( N(C) \cap Z \neq \emptyset \) a \( yz \)-component. If \( yz \) is a \( yz \)-edge, we have \( yz \notin E(G) \). Thus there must be a \( yz \)-component \( C \) such that \( y, z \in N(C) \). If this is the case, we say that the \( yz \)-component \( C \) covers the edge \( yz \). Note that every \( yz \)-component \( C \) has a nonempty intersection with \( S \), because if \( y \in N(C) \cap Y \) and \( z \in N(C) \cap Z \) then there is a path from \( y \) to \( z \) with all internal vertices in \( C \), and this path must have a nonempty intersection with \( S \). This means that there are at most three \( yz \)-components. It follows from \([Q.3]\) that each \( yz \)-component covers at most two \( yz \)-edges, and if it covers two edges, they have one endvertex in common.

**Case 1:** \( |S_M \cap Y_N| = 3 \).
Then \( S_M \subseteq Y_N \) and thus \( S_M \cap S_N = \emptyset \). Suppose that \( S_M = y_1, y_2, y_3 \). As \( |Y_M \cap S_N| \leq 1 \) by \([4.F]\), we have \( |Z_M \cap S_N| \geq 2 \).

**Case 1a:** \( |Z_M \cap S_N| = 3 \).
Then \( S_N \subseteq Z_M \). Suppose that \( S_N = \{z_1, z_2, z_3\} \) (see Figure 4.5(b)). Whenever \( y_1, z_j \in E(H) \), it is a \( yz \)-edge, and hence there is a \( yz \)-component \( C_{ij} \) that covers it.

**Claim 2.** There is a perfect matching between \( \{y_1, y_2, y_3\} \) and \( \{z_1, z_2, z_3\} \) in \( H \).

**Proof.** We first note that every \( y_i \) has at least one \( z_j \) as a neighbour, because if, say, \( y_1 \) has no neighbour among the \( z_j \)'s, then \( \{y_2, y_3\} \) is a separator of \( H \). Similarly, every \( z_j \) has a neighbour among the \( y_i \)'s.

Now let \( Y \subseteq \{y_1, y_2, y_3\} \), and let \( Z := N^H(Y) \cap \{z_1, z_2, z_3\} \) be the set of neighbours of \( Y \). We shall prove that \( |Y| \leq |Z| \). Then the claim follows from Halls’s Marriage Theorem.

If \( |Y| = 1 \) we have \( |Z| \geq 1 \), because every \( y_i \) has a neighbour among the \( z_j \).
If \( |Y| = 3 \) we have \( |Z| = 3 \), because if there is a \( z \in \{z_1, z_2, z_3\} \setminus Z \) this \( z \) has no neighbour among the \( y_i \)'s. Suppose that \( |Y| = 2 \), and let \( y \) be the unique element of \( \{y_1, y_2, y_3\} \setminus Y \). If \( Z \neq \{z_1, z_2, z_3\} \), then \( Z \cup \{y\} \) is a separator of \( H \), and this implies \(|Z| \geq 2 \).

Without loss of generality we assume that \( y_1z_1, y_2z_2, y_3z_3 \in E(H) \). It follows from \([Q.3]\) that the \( yz \)-components \( C_{11}, C_{22}, C_{33} \) covering these \( yz \)-edges are distinct. Thus these are the only \( yz \)-components. Let \( s_i \in S \cap N(C_{ii}) \).

**Claim 3.** There is no \( y_i \) such that \( z_1, z_2, z_3 \in N^H(y_i) \) and not \( z_j \) such that \( y_1, y_2, y_3 \in N^H(z_j) \).

**Proof.** Suppose for contradiction \( z_1, z_2, z_3 \in N^H(y_1) \).
If \( C_{11} = C_{12} \) then \( C_{11} \) covers the edges \( y_1z_1 \) and \( y_1z_2 \), but not \( y_1z_3 \). Hence \( C_{33} = C_{13} \). It follows \( z_2 \notin N(C_{11}) \cup N(C_{33}) \).

Now we have to analyse the models \( M, N \). As \( y_2 \in S_M \), we must have \( s_2 \in M_{y_2} \). As \( y_3 \notin N(C_{11}) \), we have \( s_3 \in M_{y_3} \). But then the edge \( y_1z_3 \) cannot be realised in \( M \), because \( C_{33} = C_{13} \) is the only component that covers the edge, and \( s_3 \) separates \( y_1 \) from \( z_3 \) in \( C_{33} \).

The case \( C_{11} = C_{13} \) is symmetric.
So suppose that \( C_{11} \neq C_{12}, C_{13} \). Then we have \( C_{22} = C_{12} \) and \( C_{33} = C_{13} \), because we need to cover the edges \( y_1z_2 \) and \( y_1z_3 \), and we cannot have \( C_{22} = C_{13} \) or \( C_{33} = C_{12} \) by \([Q.3]\).
Without loss of generality we may assume that \( y_2 \not\in N(C_{11}) \) (the other case \( y_2 \not\in N(C_{11}) \) is symmetric). As we also have \( y_2 \not\in N(C_{31}) \), we must have \( s_2 \in M_{y_2} \). This implies that the edge \( y_1z_2 \) cannot be realised in \( \mathcal{M} \), because \( s_2 \) separates \( y_1 \) from \( z_2 \) in \( C_{12} \).

By symmetry, we may assume that \( C_{11} = C_{12} \). Suppose that \( C_{22} = C_{23} \). Then if \( C_{33} = C_{13} \), we have \( z_1, z_2, z_3 \in N^H(y_1) \), which contradicts Claim 3. Similarly, if \( C_{33} = C_{23} \), we have \( z_1, z_2, z_3 \in N^H(y_2) \). If \( C_{33} = C_{31} \), we have \( y_1, y_2, y_3 \in N^H(z_1) \), and if \( C_{33} = C_{32} \), we have \( y_1, y_2, y_3 \in N^H(z_2) \). All this contradicts Claim 3.

Suppose next that \( C_{22} = C_{32} \). Then \( y_1, y_2, y_3 \in N^H(z_2) \), which again contradicts Claim 3.

So we must have \( C_{22} = C_{23} \). By symmetry, this implies \( C_{33} = C_{31} \) (just as \( C_{11} = C_{12} \) implies \( C_{22} = C_{23} \)). Then

\[
N^H(C_{11}) = \{y_1, z_1, z_2\}, \quad N^H(C_{22}) = \{y_2, z_2, z_3\}, \quad N^H(C_{11}) = \{y_3, z_3, z_1\}.
\]

Looking at the model \( \mathcal{N} \), we have either \( s_1 \in N_{z_2} \) or \( s_3 \in N_{z_1} \). If \( s_1 \in N_{z_2} \), then the edge \( y_1z_2 \) cannot be realised in the model \( \mathcal{N} \). Thus \( s_3 \in N_{z_1} \). But then the edge \( y_3z_3 \) cannot be realised in the model \( \mathcal{N} \). Either way we have a contradiction.

**Case 1b:** \(|Z_M \cap S_N| = 2\).

Then \( Z_N = \emptyset \), because otherwise \( H \) is not 3-connected. Let \( Z_M \cap S_N = \{z_1, z_2\} \) (see Figure 4.5(c)). As \( |Y_M| \leq 1 \), we have \( |R| \leq 6 \).

**Claim 4.** \( N^H(z_i) \supseteq \{y_1, y_2, y_3\} \) for \( i = 1, 2 \).

**Proof.** Suppose for contradiction that \( y_3 \not\in N^H(z_i) \). Then \( N^H(z_i) = \{y_1, y_2, z_j\} \), and the mapping \( \pi \) defined by \( \pi(y_i) := v_i \) for \( i = 1, 2 \), \( \pi(z_j) := v_3 \), \( \pi(z_i) := w_2 \), and if there is a vertex \( x \in Y_M \), \( \pi(x) := w_1 \), is an embedding of \( H \) into \( TH_{3,3} \). Thus \( H \) is exceptional, which is a contradiction.

The six edges \( y_iz_j \) are yz-edges. Thus each edge \( y_iz_j \) needs to be covered by a yz-component \( C_{ij} \). As there are six yz-edges and at most three yz-components and each yz-component covers at most two yz-edges, each yz-component must cover exactly two of the yz-edges \( y_iz_j \).

**Claim 5.** There is an \( i \) such that \( C_{11} = C_{12} \).

**Proof.** Suppose not. Then \( C_{11} \neq C_{12} \) and thus either \( C_{11} = C_{21} \) or \( C_{11} = C_{31} \). By symmetry, we may assume \( C_{11} = C_{21} \). Then the four edges \( y_1z_2, y_2z_3, y_3z_1, y_3z_2 \) must be covered by the remaining two yz-components. We either have \( C_{12} = C_{22} \) or \( C_{12} = C_{32} \). If \( C_{12} = C_{32} \), the two edges \( y_2z_3 \) and \( y_3z_1 \) must be covered by the same yz-component, which is impossible. Hence \( C_{12} = C_{22} \) and thus \( C_{31} = C_{32} \). This proves the claim for \( i = 3 \).

By symmetry, we may assume that \( C_{31} = C_{32} \). Then the four edges \( y_1z_1, y_1z_2, y_2z_1, y_2z_2 \) must be covered by the remaining two yz-components. Suppose first that \( C_{11} = C_{12} \). Then \( C_{22} = C_{21} \), and we have \( N(C_{11}) = \{y_1, z_1, z_2\} \) and \( N(C_{22}) = \{y_2, z_1, z_2\} \). Let \( s_i \) be the unique element of \( S \cap V(C_{ij}) \).

We analyse the model \( \mathcal{N} \). If \( s_i \in N_{z_2} \), then we cannot realise the edge \( z_2y_1 \) in the model \( \mathcal{N} \), because \( s_i \) separates \( y_1 \) and \( z_2 \) in \( C_{11} \) and \( y_1 \not\in N(C_{22}) \).
Similarly, if \( s_2 \in N_{s_1} \), then we cannot realise the edge \( z_2y_2 \) in the model \( N \), because \( s_2 \) separates \( y_2 \) and \( z_2 \) in \( C_{22} \) and \( y_2 \not\in N(C_{11}) \).

If \( C_{11} = C_{21} \) and \( C_{22} = C_{12} \), then we can argue similarly with the model \( M \).

**Case 2:** \( |S_M \cap Y_N| = 2 \).

Then it follows from Claim 1 and \( (4.6) \) that \( |Z_M \cap S_N| = 2 \). Let \( S_M \cap Y_N = \{y_1, y_2\} \) and \( Z_M \cap S_N = \{z_1, z_2\} \).

**Case 2a:** \( S_M \cap S_N \neq \emptyset \).

Then \( |S_M \cap S_N| = 1 \). Let \( x \) be the unique vertex in \( S_M \cap S_N \) (see Figure 4.5(d)). Then \( |R| \leq 7 \). If there is a vertex in \( Y_M \), we call it \( y_3 \), and if there is a vertex in \( Z_N \) we call it \( z_3 \).

**Claim 6.** \( y_iz_j \in E(H) \) for \( i, j = 1, 2 \).

**Proof.** Suppose for contradiction that \( y_1z_1 \not\in E(H) \). Then \( \{x, y_2, z_2\} \) is a separator of \( H \) that separates \( y_1 \) from \( z_1 \). If both \( y_3 \) and \( z_3 \) exist, it even separates \( \{y_1, y_3\} \) from \( \{z_1, z_3\} \), which contradicts \( H \) being quasi-4-connected.

Hence without loss of generality we may assume that \( y_3 \) does not exist, that is, \( Y_M = \emptyset \). Then the following mapping \( \pi \) is an embedding of \( H \) into the graph \( TH_{y_3} \):

\[
\begin{array}{c|c|c|c|c|c|c|c}
 & u & x & y_1 & y_2 & z_1 & z_2 & z_3 \\
\pi(v) & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
\end{array}
\]

Hence \( H \) is exceptional, which is a contradiction.

For all \( i, j \) the edge \( y_iz_j \in E(H) \) is a yz-edge. Hence there is a yz-component \( C_{ij} \) covering it. As one yz-component covers at most two yz-edges, we need at least two yz-components to cover the four edges \( y_iz_j \). Furthermore, the yz-components \( C_{11}, C_{22} \) and the yz-components \( C_{12}, C_{21} \) are distinct. Let \( s_1 \in S \cap V(C_{11}) \) and \( s_2 \in S \cap C_{22} \).

**Claim 7.** \( x \not\in S \).

**Proof.** Suppose for contradiction that \( x \in S \). Then \( |S \setminus R| \leq 2 \) and hence there are at most two yz-components. We can argue exactly as in Case 1b. Let me repeat the argument for the reader’s convenience.

Suppose first that \( C_{11} = C_{12} \). Then \( C_{22} = C_{21} \), and we have \( N(C_{11}) = \{y_1, z_1, z_2\} \) and \( N(C_{22}) = \{y_2, z_1, z_2\} \). We analyse the model \( N \). If \( s_1 \in N_{s_1} \), then we cannot realise the edge \( z_2y_1 \) in the model \( N \), because \( s_1 \) separates \( y_1 \) and \( z_2 \) in \( C_{11} \) and \( y_1 \not\in N(C_{22}) \). Similarly, if \( s_2 \in N_{s_1} \), then we cannot realise the edge \( z_2y_2 \) in the model \( N \), because \( s_2 \) separates \( y_2 \) and \( z_2 \) in \( C_{22} \) and \( y_2 \not\in N(C_{11}) \).

It remains to consider the case \( C_{11} = C_{12} \). But this case is symmetric, and we argue with the roles of \( M \) and \( N \) swapped.

Thus either \( x \in Y \) or \( x \in Z \). By symmetry, we may assume that \( x \in Y \).

**Claim 8.** \( Z_N = \emptyset \), that is, \( z_3 \) does not exist.

**Proof.** Suppose for contradiction that \( z_3 \) exists. Then \( xz_3 \) is a yz-edge, and we need a yz-component \( C \neq C_{11}, C_{12}, C_{21}, C_{22} \) to cover it. We have \( y_1, y_2 \not\in N(C) \), and hence all edges \( y_iz_j \) must be covered by the components \( C_{11}, C_{12}, C_{21}, C_{22} \). Now we can argue as in the proof of Claim 7 to derive a contradiction.

Now we are in the same situation as in Case 1b with \( x \) playing the role of \( y_3 \), and we can argue exactly as we did there.
Example 4.12. Let $G$ be a hexagonal grid (see Figure 1.2) or a cube (see Figure 4.6).

4.3 Degenerate 3-Separations

Figure 4.6. The cube graph

Case 2b: $S_M \cap S_N = \emptyset$.

Then $Y_M \cap Y_N = \emptyset$, because otherwise $Y_M \cap S_N = \emptyset$, and $\{y_1, y_2\}$ separates $Y_M \cap Y_N$ from $\{z_1, z_2\}$, which contradicts $H$ being 3-connected. Similarly, $Z_M \cap Z_N = \emptyset$ (see Figure 4.5(e)).

If $Y_M = \emptyset$ then $Z_N = \emptyset$, because otherwise $\{z_1, z_2\}$ is a separator of $H$.

Then all proper separations $(Y, S, Z)$ of $G$ are degenerate. It follows from Lemma 4.9.

If $V_M \neq \emptyset$ and, by symmetry, $Z_N \neq \emptyset$. Let $y_3$ be the unique vertex in $Y_M$ and $z_3$ the unique vertex in $Z_N$. Then $y_3 \in Y_M \cap S_N$ and $N^H(y_3) = \{y_1, y_2, z_3\}$. Similarly, $z_3 \in S_M \cap Z_N$ and $N^H(z_3) = \{z_1, z_2, y_3\}$.

The $yz$-edge $y_3z_3$ must be covered by some $yz$-component $C$.

If $y_i, z_j \notin E(H)$ for some $i, j \leq 2$, then $H$ can be embedded into $TH_{+3}$ and is exceptional. Thus $y_i, z_j \in E(H)$ for all $i, j \leq 2$. The $yz$-component $C$ covers none of these four $yz$-edges, and we need to cover them with the two remaining $yz$-components. We have seen (several times) that this is impossible.

Theorem 4.11. Let $R$ be a non-exceptional quasi-4-connected region of $G$. If $G[R]$ is non-exceptional, we let $T_R := \mathcal{T}(G[R], \mathcal{M})$ for some faithful model $\mathcal{M}$ of $G[R]$ in $G$, and if $G[R]$ is non-exceptional, we let $T_R := \mathcal{T}(\hat{H}, \mathcal{M})$ for some non-exceptional extension $\hat{H}$ of $R$ and some faithful model $\mathcal{M}$ of $\hat{H}$ in $G$.

Then the mapping $R \mapsto T_R$ is well-defined (that is, $T_R$ only depends on $R$ and not on the extension $\hat{H}$ or the model $\mathcal{M}$), and $T_R$ is a $G$-tangle of order 4.

Proof. If $G[R]$ is non-exceptional, this follows from Lemma 4.10. If $G[R]$ is exceptional, it follows from Lemma 4.9.

This proves the first half of the Correspondence Theorem 4.1. The remainder of Section 4 is devoted to the second half.
For all $z \in Z$ we let $V(M_z) := \{z\}$. For all edges in $f \in E(G) \cap E(H)$ we let $e_f := f$.

It only remains to define the $V(M_{s_i})$ and $e_{s_is_j}$.

Suppose first that $S$ is not independent. Say, $s_1s_2 \in E(G)$. Let $C$ be a connected component of $G[Y]$. As $G$ is 3-connected, we have $N(C) = S$. We let $V(M_{s_i}) := V(C) \cup \{s_3\}$, and for $i = 1, 2$ we let $e_{s_is_j}$ be an edge from $V(C)$ to $s_i$.

Suppose next that $S$ is independent and $G[Y]$ is not connected. Let $C_1$ and $C_2$ be two connected components of $G[Y]$. We let $V(M_{s_i}) := \{s_2\}$. We contract $C_1$ onto $s_1$ to create an edge from $s_1$ to $s_2$. That is, we let $V(M_{s_i}) := V(C_1) \cup \{s_1\}$, and we let $e_{s_is_2}$ be an arbitrary edge from $V(C_1)$ to $s_2 \in N(C_1)$. Then we contract $C_2$ onto $s_3$ to create edges from $s_3$ to $s_1$ and $s_2$. Formally, we let $V(M_{s_i}) := V(C_3) \cup \{s_1\}$, and for $i = 1, 2$ we let $e_{s_is_3}$ be an arbitrary edge from $V(C_2)$ to $s_i \in N(C_2)$.

Finally, suppose that $S$ is an independent set and $G[Y]$. Let $v \in Y$. As $G$ is 3-connected, there are internally disjoint paths $P_i$, for $i \in [3]$, from $v$ to $s_i$. At least one of these paths, say, $P_1$, can be chosen to have length at least 2. To see this, suppose that $P_1, P_2, P_3$ have length 1. Let $w \in N(v) \cap Y$; such a $w$ exists because $G[Y]$ is connected and $|Y| \geq 2$. Then there is a path $Q$ from $w$ to $S$ in $G \setminus \{v\}$. Let $s_i$ be the endvertex of $Q$ in $S$. As for all $j \neq i$ the path $P_j$ only consist of a single edge, $Q$ and $P_j$ have an empty intersection. We can replace $P_i$ by the path from $v$ to $w$ and then along $Q$ to $s_i$; this path has length at least 2.

In the following we assume without loss of generality that $P_1$ has length at least 2. Then $V(P_1) \setminus \{v,s_1\} \neq \emptyset$. Let $Q'$ be a path from $V(P_1) \setminus \{v,s_1\}$ to $(V(P_2) \cup V(P_3)) \setminus \{v\}$ in $G \setminus \{v,s_1\}$. Such a path exists because $G$ is 3-connected. Without loss of generality we may assume that the endvertex of $Q'$ is on $P_2$ and that $Q'$ has no internal vertex on $V(P_1) \cup V(P_2) \cup V(P_3)$. For $i = 1, 2$, let $w_i$ be the endvertex of $Q'$ on $P_i$, and let $P_i'$ be the segment of $P_i$ from $s_i$ to $w_i$. We let $V(M_{s_i}) := V(P_i') \cup V(Q') \setminus \{w_i\}$ and $V(M_{s_3}) := V(P_3') \cup \{s_3\}$, and we let $e_{s_is_2}$ be the edge of $Q'$ incident with $w_i$. We let $V(M_{s_1}) := V(P_1) \cup (V(P_1') \cup V(P_2') \cup V(P_3') \cup (V(P_2) \setminus V(P_3') \cup V(P_3) \setminus V(P_3')))$. For $i = 1, 2$, let $x_i$ be the neighbour of $w_i$ in $V(P_i) \setminus V(P_i')$, and let $e_{s_is_2}$ be the edge from $w_i$ to $x_i$.

\begin{remark}
Assumption 4.15. The converse of Lemma 4.13 holds as well: if $(Y,S,Z) \in \text{Sep}_{\leq 3}(G)$ is a proper separation such that $G[Z \cup S]$ is a minor of $G$, then $(Y,S,Z)$ is non-degenerate.

To see this, suppose that $(Y,S,Z) \in \text{Sep}_{\leq 3}(G)$ is a proper degenerate separation and $H := G[S \cup Z]$ is a minor of $G$. Note that $|H| = |G| - 1$ and $|E(H)| = |E(G)|$, because $S$ is an independent set in $G$. Let $((M_w)_{w \in V(H)}, (e_f)_{f \in E(H)})$ be a model of $H$ in $G$. Then $\{e_f \mid f \in E(H)\} = E(G)$, and this implies that $|M_w| = 1$ for all $w \in V(H)$, because if $|M_w| > 2$ then at least one edge would appear in the connected graph $G[M_w]$. Thus $\sum_{w \in V(H)} |M_w| = |G| - 1$, and hence there is a vertex $v \in V(G) \setminus \bigcup_{w \in V(H)} V(M_w)$. As $G$ is connected, $v$ is incident with at least one edge $e$, and this edge $e$ is not among the $e_f$ for $f \in E(H)$. This is a contradiction.
\end{remark}

\section{Crossing Separations}

Let us call two separations $(Y_1,S_1,Z_1), (Y_2,S_2,Z_2) \in \text{Sep}(G)$ orthogonal if

$$(Y_1 \cup S_1) \cap (Y_2 \cup S_2) \subseteq S_1 \cap S_2.$$ 

(see Figure 4.7(a)). It is not hard to show that the minimal separations of a tangle of order 3 in a graph are mutually orthogonal. This is the key to the proof of Theorem 3.9. The minimal separations of a tangle of order 4 are not necessarily orthogonal, but in this section, we shall prove that they can only "cross" in a very restricted way.

We continue to assume that $G$ is a 3-connected graph and, in addition make the following assumption, which will stay in place until the end of Section 4.

Assumption 4.15. $T$ is a $G$-tangle of order 4.
For all sets \( S \subseteq V(G) \) of cardinality \( |S| \leq 3 \) we let \( Z(S) := Z_{\mathcal{T}}(S) \) and \( \mathcal{Y}(S) := \mathcal{Y}_{\mathcal{T}}(S) \) (see (3.B) and (3.C)).

**Lemma 4.16 (Crossing Lemma).** Let \((Y_1, S_1, Z_1), (Y_2, S_2, Z_2) \in \mathcal{T}_{\text{min}} \) be distinct. Then either \((Y_1, S_1, Z_1)\) and \((Y_2, S_2, Z_2)\) are orthogonal or \(Y_1 \cap Y_2 = \emptyset\) and \(S_1 \cap S_2 = \emptyset\) and there is an edge \( s_1s_2 \in E(G) \) such that for \( i = 1, 2 \) we have \( S_i \cap Y_{3-i} = \{s_i\} \) (see Figure 4.7(b)).

In the latter case, we call the edge \( s_1s_2 \) the crossedge of \((Y_1, S_1, Z_1)\) and \((Y_2, S_2, Z_2)\).

**Proof.** We observe first that \( Y_1 \) and \( Y_2 \) are nonempty. If \( Y_i = \emptyset \) then \( S_i = \emptyset \) by the minimality of \((Y_i, S_i, Z_i)\) and thus \((Y_i, S_i, Y_i) = (\emptyset, \emptyset, V(G)) \succeq (Y_{3-i}, S_{3-i}, Z_{3-i})\). Again by the minimality of \((Y_i, S_i, Z_i)\) this implies that the two separations are equal. This contradicts our assumption that they be distinct.

By Corollary 3.7 we have

\[
|(S_1 \cap Z_2) \cup (S_1 \cap S_2) \cap (Z_1 \cap S_2)| \geq 4
\]  

and

\[
|S_i \cap Z_{3-i}| \geq 1 \quad \text{for } i = 1, 2.
\]  

(4.I)

An easy calculation based on these inequalities and \( |S_i| \leq 3 \) shows that

\[
|(S_1 \cap Y_2) \cup (S_1 \cap S_2) \cup (Y_1 \cap S_2)| \leq 2.
\]  

As \( G \) is 3-connected and \((S_1 \cap Y_2) \cup (S_1 \cap S_2) \cup (Y_1 \cap S_2)\) separates \( Y_1 \cap Y_2 \) from the nonempty set \( S_1 \cap Z_2 \), it follows that

\[
Y_1 \cap Y_2 = \emptyset.
\]

Thus if \( S_1 \cap Y_2 = S_2 \cap Y_2 = \emptyset \), then \((Y_1, S_1, Z_1)\) and \((Y_2, S_2, Z_2)\) are orthogonal.

In the following, we assume without loss of generality that

\[
|S_1 \cap Y_2| \geq 1.
\]  

(4.I)

Suppose for contradiction that \( Y_1 \cap S_2 = \emptyset \). Then \( Y_1 \subseteq Z_2 \). As there is no edge from \( Z_2 \) to \( Y_2 \), we have \( N(Y_1) \subseteq S_1 \setminus Y_2 \). Thus \((Y_1, S_1 \setminus Y_2, Z_1 \cup (S_1 \cap Y_2))\) is a separation of order less than 3, which contradicts \( G \) being 3-connected. Thus

\[
|S_2 \cap Y_1| \geq 1.
\]  

(4.K)
The only solution to the inequalities $|S_i| \leq 3$ and (4.H), (4.J), and (4.K) is

$$|S_i \cap Z_{3-i}| = 2 \quad \text{and} \quad |S_i \cap Y_{3-i}| = 1$$

for $i = 1, 2$.

Let $s_i$ be the unique vertex in $S_i \cap Y_{3-i}$. We have $s_1 s_2 \in E(G)$, because otherwise $S_1 \cap Z_2$ (and also $Z_1 \cap S_2$) separates $s_1$ from $s_2$, which contradicts $G$ being 3-connected. \qed

We say that two separations $(Y_1, S_1, Z_1), (Y_2, S_2, Z_2) \in T_{\min}$ cross if they are not orthogonal. We call $(Y, S, Z) \in T_{\min}$ crossed if there is some $(Y', S', Z') \in T_{\min}$ that crosses $(Y, S, Z)$. We denote the set of all non-degenerate separations $(Y, S, Z) \in T_{\min}$ by $T_{nd}$.

By Corollary 3.6, the minimal separations are determined by their separators: for every 3-separator $S$ of $G$, if $(Y, S, Z) \in T_{\min}$, then $Z = Z(S)$ and $Y = Y(S)$. Therefore, we call $S$ $T$-minimal if $(Y(S), S, Z(S)) \in T_{\min}$, and we denote the set of all $T$-minimal 3-separations by $ST_{\min}$. We say that $S_1, S_2$ are orthogonal (crossing) if the corresponding separations $(Y(S_1), S_1, Z(S_1))$ are orthogonal (crossing), respectively. The crosseage of two crossing separators $S_1, S_2 \in ST_{\min}$ is the crosseage of the corresponding separations. We say that $S \in ST_{\min}$ is crossed if $(Y(S), S, Z(S))$ is crossed. We call $S \in ST_{\min}$ degenerate if $(Y(S), S, Z(S))$ is degenerate and denote the set of all non-degenerate $S \in ST_{\min}$ by $ST_{\min}$.

**Lemma 4.17.** Let $S \in ST_{\min}$ be crossed. Then $Y(S)$ is connected in $G$.

**Proof.** Let $Y := Y(S)$ and $Z := Z(S)$, and let $(Y', S', Z') \in T$ be a vertex separation that crosses $(Y, S, Z)$. Let $ss'$ with $s \in S$ and $s' \in S'$ be the crosseage. Then $S' \cap Y = \{s'\}$ and $S \cap Y' = \{s\}$. Let $s' \in S \cap Z'$.

Suppose for contradiction that $G[Y]$ is not connected. Then there is a connected component $C$ of $G[Y]$ such that $s' \not\in V(C)$ and hence $S' \cap V(C) = \emptyset$. As $G$ is 3-connected, we have $N(C) = S$. Thus there is a path from $s' \in Z'$ to $s \in Y'$ with all internal vertices in $V(C)$. This path has an empty intersection with $S'$, which contradicts $S'$ separating $Z'$ from $Y'$.

**Lemma 4.18 (Crossedge Independence Lemma).** Let $S, S_1, S_2 \in ST_{\min}$ be distinct such that both $S, S_1$ and $S, S_2$ cross, and let $e_i = s_1 s_i, e_2 = s_2 s_i$ with $s_i \in S$ and $s_1, s_2 \in S_i$ be the respective crosseages.

1. $S$ is an independent set.
2. If $S$ is degenerate, then $s_1, s_2, s_1'$ are mutually distinct and $s_1' = s_2'$.
3. If $S$ is non-degenerate, then $s_1, s_2, s_1', s_2'$ are mutually distinct.

**Proof.** Let $(Y, S, Z), (Y_i, S_i, Z_i) \in T_{\min}$ be the separations corresponding to our separators. By the Crossing Lemma 4.16 we have

$$Y \cap Y_1 = Y \cap Y_2 = Y_1 \cap Y_2 = \emptyset$$

(4.L)

$$S \cap Y_1 = \{s_1\} \quad \text{and} \quad S \cap Y_2 = \{s_2\}$$

(4.M)

$$S \cap Y = \{s_1'\} \quad \text{and} \quad S_2 \cap Y = \{s_2'\}$$

(4.N)

$$S \cap S_1 = S \cap S_2 = \emptyset.$$ 

(4.O)

Observe that $s_1 \neq s_2$, because $s_1 \in Y_1$ and $s_2 \in Y_2$ and $Y_1 \cap Y_2 = \emptyset$ by (4.L). Furthermore, $s_1 \neq s_2'$, because $s_1 \in S$ and $s_2' \in Y$ and, similarly, $s_2 \neq s_1'$.

To prove (1), note that $s_i \not\in S_1 \cup S_2$ by (4.O). As $s_i \in Y_i$ by (4.M), we have $s_i \in Y_i \setminus (S_1 \cup S_2)$. It follows that $s_1 s_2 \not\in E(G)$, because $Y_1$ and $Y_2$ are disjoint and $N(Y_i) \subseteq S_i$. Let $s$ be the unique element in $S \setminus \{s_1, s_2\}$. By (4.M) and (4.O) we have
Figure 4.8. Hexagonal grid with triangles

$s \in Z_1 \cap Z_2$. As $S_i$ separates $Z_i$ from $Y_i$, we have $s_i s \not\in E(G)$. Hence $S$ is an independent set.

To prove (2), suppose that $S$ is degenerate. Then $|Y| = 1$, and as $s'_1, s'_2 \in Y$ this implies $s'_1 = s'_2$.

To prove (3), suppose that $S$ is non-degenerate. Then $|Y| \geq 2$, because $S$ is an independent set. We need to prove that $s'_1 \neq s'_2$. We have $Y \cap Y_1 = \emptyset$ and $Y \cap S_1 = \{s'_1\}$. Hence $Y \cap Z_1 \neq \emptyset$. Let $C$ be a connected component of $G[Y \cap Z_1]$. As $G$ is 3-connected, we have $|N(C)| \geq 3$, and as

$$N(C) \subseteq N(Y \cap Z_1) \subseteq (S \cap Z_1) \cup (S \cap S_1) \cup (Y \cap S_1) = \{s_2, s, s'_1\},$$

it follows that $N(C) = \{s_2, s, s'_1\}$. Hence there is a path from $s_2$ to $s$ with all internal vertices in $C$. As $S_2$ separates $s \in Z_2$ from $s_2 \in Y_2$, it holds that $V(C) \cap S_2 \neq \emptyset$. As $V(C) \subseteq Y$ and $Y \cap S_2 = \{s'_2\}$, we have $s'_2 \in V(C)$. Thus $s'_2 \neq s'_1 \in N(C)$. \hfill \qedsymbol

Example 4.19. The two cases of the Crossedge Independence Lemma are nicely illustrated by a hexagonal grid (see Figure 4.8), where all 3-separators are degenerate, and the graph in Figure 4.8 where we have crossing non-degenerate 3-separators. In fact, a lot of my intuition draws from these two examples.

We call a crossedge of two 3-separators in $S_1, S_2 \in ST_{\text{min}}$ non-degenerate if both $S_1$ and $S_2$ are non-degenerate. Let us denote the set of crossedges of $T$ by $E^x(T)$ and the subset of all non-degenerate crossedges by $E^x_{\text{nd}}(T)$.

Corollary 4.20. $E^x_{\text{nd}}$ is a matching. That is, distinct $e, e' \in E^x_{\text{nd}}(T)$ have no endvertex in common.

Proof. Let $e = st, e' = s't' \in E^x_{\text{nd}}(T)$. Let $S_1, S_2 \in ST_{\text{nd}}$ such that $e$ is the crossedge of $S_1$ and $S_2$ and $s \in S_1$ and $t \in S_2$, and let $S'_1, S'_2 \in ST_{\text{nd}}$ such that $e'$ is the crossedge of $S'_1$ and $S'_2$ and $s' \in S'_1$ and $t' \in S'_2$. Suppose for contradiction that $t = t'$. Then $t \in Y(S_1) \cap S_2 \cap S'_2$, and thus both $S_2$ and $S'_2$ cross $S_1$, and $t$ is an endvertex of both crossedges. This contradicts the Crossedge Independence Lemma 4.18. \hfill \qedsymbol
4.5 Contracting a Crossedge

In this section, we will study what happens if we contract a single non-degenerate crossedge of $G$. We shall prove that the resulting graph $G'$ is still 3-connected and has a tangle $\mathcal{T}'$ of order 4 that is "induced" by $\mathcal{T}$. Technically, we will prove that $\mathcal{T}$ is the lifting of $\mathcal{T}'$ (see Lemma 3.11). Moreover, the minimal separations of $\mathcal{T}'$ are the same as those of $\mathcal{T}$, except of course for the two separations whose crossedge we contract. Before we go to the details, let us put this in a wider perspective. Since the non-degenerate crossedges form a matching, the contraction of one crossedge leaves the remaining ones intact, and we can contract them all, one at a time. This leaves us with a graph and tangle that has no crossedges, which means that all minimal separations in the tangle are orthogonal. We will then remove the "Y-parts" of all non-degenerate minimal separations in the tangle to obtain the quasi-4-connected region associated with our tangle.

The technically most difficult part is the contraction of one crossedge, that is, the present section. The main insight is that if we a have separator $S$ of order 4 of our graph that contains both endvertices of a crossedge, then each connected component of $G \setminus S$ will only have one endvertex of the crossedge in its neighbourhood. Thus there is a subset $S'$ of $S$ of size 3 that is still a separator of $G$. This will allow us to establish a correspondence between the separations of order 3 of the graphs $G$ and $G'$ (obtained from $G$ by contracting the crossedge). It still leaves us with many questions on how exactly we match the separations, and in fact we will only get a reasonable correspondence for the minimal separations in the tangle, but this will be good enough to define a tangle of $G'$.

We still assume that $G$ is a 3-connected graph and $\mathcal{T}$ a $G$-tangle of order 4, and we continue to use the notation and terminology of the previous subsections. In addition, we make the following assumption.

**Assumption 4.21.** $e = s_1s_2$ is the crossedge of $S_1, S_2 \in \mathcal{ST}_{\text{min}}$ and $G'$ is the graph obtained from $G$ by contracting the edge $e$ onto $s_1$. That is, $V(G') := V(G) \setminus \{s_2\}$ and $E(G') := (E(G) \setminus \{vs_2 \mid v \in V(G)\}) \cup \{vs_1 \mid vs_2 \in E(G) \setminus \{e\}\}$.

Let us fix some notation: for $i = 1, 2$, let $Y_i := \mathcal{Y}(S_i)$ and $Z_i := \mathcal{Z}(S_i)$, and we assume $S_i = \{s_i, s'_i, s''_i\}$ (see Figure 4.9).

**Lemma 4.22.** For all $S \in \mathcal{ST}_{\text{min}} \setminus \{S_1, S_2\}$,

$$s_1, s_2 \notin S \cup \mathcal{Y}(S),$$

(4.P)
and $S$ is a separator of $G'$.

Proof. We have $s_j \in Y_{2-j}$ and $Y(S) \cap Y_{2-j} = \emptyset$ by the Crossing Lemma \ref{lem:crossing-lemma}. Hence $s_j \notin Y(S)$. If $s_j \in S$ then $S$ crosses $S_{2-j}$ and the crosedge has an endvertex with the crosedge of $S_1$ and $S_2$ in common. As $S_1$ and $S_2$ are non-degenerate, this contradicts the Crossover Independence Lemma \ref{lem:crossedge-independence}.

To see that $S$ is a separator of $G'$, just note that $(Y(S), S, Z(S) \setminus \{s_2\})$ is a proper separation of $G'$.

\begin{lemma}
Let $S$ be a separator of $G$ such that $s_1, s_2 \in S$ and $|S \setminus \{s_1, s_2\}| \leq 2$. Then $|S \setminus \{s_1, s_2\}| = 2$, and there is a $a$ separator $S^o \subseteq S$ such that $|S^o \setminus \{s_1, s_2\}| = 1$ and $S^o \setminus \{s_1, s_2\} = S \setminus \{s_1, s_2\}$ and there are at least two connected components of $G \setminus S^o$ that have a nonempty intersection with $V(G) \setminus S$.
\end{lemma}

Proof. Choose a set $S^o \subseteq S$ such that there are at least two connected components of $G \setminus S^o$ that have a nonempty intersection with $V(G) \setminus S$, and subject to this condition $|S^o \cap \{s_1, s_2\}|$ is minimum. We shall prove that

$$|S^o \cap \{s_1, s_2\}| = 1. \quad \text{(4.Q)}$$

As $G$ is 3-connected and $S^o$ is a separator, this will imply $|S^o \setminus \{s_1, s_2\}| = |S \setminus \{s_1, s_2\}| = 2$ and thus $S^o \setminus \{s_1, s_2\} = S \setminus \{s_1, s_2\}$.

Suppose for contradiction that $|S^o \cap \{s_1, s_2\}| = 2$, that is, $s_1, s_2 \in S^o$.

Let us call a connected component $C$ of $G \setminus S^o$ relevant if $V(C) \setminus S \neq \emptyset$.

Claim 1. For every relevant component $C$,

$$S^o \cap \{s_1, s_2\} \subseteq N(C).$$

Proof. Suppose that $C$ is a relevant component with $S^o \cap \{s_1, s_2\} \not\subseteq N(C)$. Let $S^o := N(C)$. Then $S^o \subseteq S^o \subseteq S$, and $S^o$ is a separator of $G$ with at least two relevant components. However, $|S^o \cap \{s_1, s_2\}| < |S^o \cap \{s_1, s_2\}|$. This contradicts the minimality of $|S^o \cap \{s_1, s_2\}|$.

Claim 2. Let $C$ be a relevant component. Then $V(C) \setminus (Y_1 \cup Y_2) \neq \emptyset$.

Proof. If $V(C) \subseteq Y_1$, then $S^o = N(C) \subseteq Y_1 \cup S_j$. We have $s_{3-j} \in S^o$, and as $s_{3-j}$ is the only neighbour of $s_j$ in $S_j \cup Y_1$, it follows that $s_j \notin N(C) = S^o$. This is a contradiction.

Hence $V(C) \setminus Y_1 \neq \emptyset$ and $V(C) \setminus Y_2 \neq \emptyset$. Then $V(C) \setminus (Y_1 \cup Y_2) \neq \emptyset$, because $C$ is connected and the only edge from $Y_1$ to $Y_2$ is the crosedge $s_1s_2$, which is not in $E(C)$ by our assumption that $s_1, s_2 \in S^o \subseteq V(G) \setminus V(C)$.
Then there is a unique connected component $C$, let

**Claim 1.**

Thus $S^c \setminus \{s_1, s_2\} = \{t_1, t_2\} \subseteq Y_1 \cup Y_2$, because $|S^c \setminus \{s_1, s_2\}| = |S \setminus \{s_1, s_2\}| \leq 2$. It follows that

$$S^c \cap Z_1 \cap Z_2 = \emptyset.$$ 

**Claim 3.** $Z_1 \cap Z_2 = \emptyset$.

**Proof.** Suppose for contradiction that $Z_1 \cap Z_2 \neq \emptyset$. Let $C$ be a connected component of $G[Z_1 \cup Z_2]$. Then $N(C) \subseteq N(Z_1 \cup Z_2) = \{s_1', s_1'', s_2', s_2''\}$ and $|N(C)| \geq 3$. Hence for some $j \in \{1, 2\}$ we have $s_j', s_j'' \in N(C)$. Then there is a path from $s_j' \in V(C_1)$ to $s_j'' \in V(C_2)$ with all internal vertices in $V(C)$, and as $V(C) \cap S^c = \emptyset$, this is a contradiction.

As $s_1', s_2' \in V(C_1)$ and $s_1'', s_2'' \in V(C_2)$, it follows that the graph $G[\{Z_1 \cup s_1\} \cap \{Z_2 \cup s_2\}]$ has vertex set $\{s_1', s_1'', s_2', s_2''\}$ and edges set $\{s_1's_2', s_1's_2''\}$.

**Claim 4.** $(Y_1', S_1', Z_1') := (Y_1 \cup \{s_1\}, \{s_1', s_2', s_1\}, Z_1 \setminus \{s_2\}) \in \mathcal{T}$. 

**Proof.** The shape of the graph $G[\{Z_1 \cup S_1\} \cap \{Z_2 \cup S_2\}]$ described before the claim implies that $(Y_1', S_1', Z_1')$ is a separation of $G$. Suppose for contradiction that it is not in $\mathcal{T}$. Then $(Z_1', S_1', Y_1') \in \mathcal{T}$. We also have $(Y_1, S_1, Z_1) \in \mathcal{T}$ and $(\emptyset, \{s_1', s_2'\}, V(G) \setminus \{s_1', s_2'\}) \in \mathcal{T}$. As $Y_1' \cap Z_1 \cap V(G) \setminus \{s_1', s_2'\} = (Y_1 \cup \{s_1\}) \cap Z_1 \cap (V(G) \setminus \{s_1', s_2'\}) \subseteq (Y_1 \cup S_1) \cap Z_1 = \emptyset,$

by (T.2) there must be an edge that has an endvertex in $Y_1' = Y_1 \cup \{s_1\}$ and $Z_1$ and $V(G) \setminus \{s_1', s_2'\}$. The only edge that has an endvertex in $Y_1 \cup \{s_1\}$ and $Z_1$ is $s_1's_2'$. However, this edge has no endvertex in $V(G) \setminus \{s_1', s_2'\}$. This is a contradiction.

As $(Y_1, S_1', Z_1')$ is strictly smaller than $(Y_1, S_1, Z_1)$ with respect to the order $\preceq$, this contradicts the minimality of $(Y_1, S_1, Z_1)$.

**Corollary 4.24.** $G'$ is 3-connected.

**Proof.** Let $S'$ be a separator of $G'$ of order at most 2. If $s_1 \not\in S'$, then $S'$ is a separator of $G$ of order at most 2, which contradicts $G$ being 3-connected.

If $s_1 \in S'$, let $S := S' \cup \{s_2\}$. Then $S$ is a separator of $G$ with $|S \setminus \{s_1, s_2\}| \leq 1$. This contradicts the assertion of Lemma 4.23 that $|S \setminus \{s_1, s_2\}| = 2$.

For a separator $S$ of $G$ such that $s_1, s_2 \in S$ and $|S \setminus \{s_1, s_2\}| \leq 2$, we call a subset $S^c \subseteq S$ such that $|S^c \cap \{s_1, s_2\}| = 1$ and $|S^c \setminus \{s_1, s_2\}| = 2$ and there are at least two connected components of $G \setminus S^c$ that have a nonempty intersection with $V(G) \setminus S$ an essential subseparator of $S$. Note that $S$ has at most two essential subseparators: $S \setminus \{s_1\}$ and $S \setminus \{s_2\}$.

**Lemma 4.25.** Let $S$ be a separator of $G$ such that $s_1, s_2 \in S$ and $|S \setminus \{s_1, s_2\}| \leq 2$. Then there is a unique connected component $C^* \subseteq G \setminus S$ such that $V(C^*) \subseteq Z^*$ for every separation $(Y^*, S^*, Z^*) \in \mathcal{T}$ with $S^* \subseteq S$.

**Furthermore,** for every essential subseparator $S^c$ of $S$ it holds that $V(C^*) = Z^c \setminus \{s_1, s_2\}$.

**Proof.** Let $S^c$ be an essential subseparator of $S$, which exists by Lemma 4.23. Without loss of generality we may assume that $s_1 \in S^c$. Let $Z^c := Z(S^c)$ and $Y^c := Y(S^c)$. Moreover, let $(Y^c, S^c, Z^c) \in \mathcal{T}_{\min}$ such that $(Y^c, S^c, Z^c) \preceq (Y^c, S^c, Z^c)$. Then $Y^c \cup S^c \subseteq Y^c \cup S^c$ and thus $s_1 \in Y^c \cup S^c$. By (LP), there is a $j \in \{1, 2\}$ such that $S^c = S_j$.

**Claim 1.** $Z^c \setminus \{s_1, s_2\} = Z^c \setminus \{s_2\}$ is connected.

**Proof.** Note that the equality $Z^c \setminus \{s_1, s_2\} = Z^c \setminus \{s_2\}$ holds because $s_1 \in S^c \subseteq V(G) \setminus Z^c$. 28
We have $Z_j = Z' \subseteq Z^\circ$. As $Z_j$ is connected, it suffices to prove that for every $z \in Z^\circ \setminus (\{s_2\} \cup Z_j)$ there is a path in $G[Z^\circ \setminus \{s_2\}]$ from $z$ to a vertex in $Z_j$. So let

$$z \in Z^\circ \setminus (\{s_2\} \cup Z_j) \subseteq (Y_j \cup S_j) \setminus \{s_1, s_2\}.$$ 

As $N(Z_j) = S_j$, it suffices to find a path from $z$ to a vertex in $S_j$. Let $P$ be a shortest path from $z$ to a vertex in $s \in S_j$ in the connected graph $G[Z^\circ]$. Then $V(P) \setminus \{s\} \subseteq Y_j$. We need to prove that $s_2 \notin V(P)$.

**Case 1:** $j = 1$.

As $N(Y^\circ) = S^\circ$, the vertex $s_1 \in S^\circ$ has a neighbour in $Y^\circ \subseteq Y_1$, and as $s_2$ is the only neighbour of $s_1$ in $Y_1$, we have $s_2 \notin Y^\circ$. Hence $s_2 \notin V(P)$.

**Case 2:** $j = 2$.

The only neighbour of $s_2 \in S_2$ in $Y_2$ is $s_1$, and as $s_1 \notin V(P)$ and $V(P) \setminus \{s\} \subseteq Y_2$, we have $s_2 \notin V(P)$.

We let $C^\circ := G[Z^\circ \setminus S]$. It follows from Claim 1 that this is indeed a connected component of $G \setminus S$. Now let $(Y^\ast, S^\ast, Z^\ast) \in \mathcal{T}$ with $S^\ast \subseteq S$. We need to prove that

$$V(C^\circ) = Z^\circ \setminus S = Z^\circ \setminus \{s_2\} \subseteq Z^\ast.$$  \hspace{1cm} (4.R)

Without loss of generality we may assume that $Z^\ast = Z(S^\ast)$. It follows from Lemma 3.3 that $Z^\ast \subseteq S$, because then $|Z^\ast \cup S^\ast| \leq |S| \leq 4$. If $Y^\ast \subseteq S$ then $Z^\ast \supseteq V(G) \setminus S \supseteq Z^\circ \setminus S = V(C^\circ)$.

Let us assume that neither $Z^\ast \subseteq S$ nor $Y^\ast \subseteq S$. Then $S^\ast$ is an essential subseparator of $S$. Note that the analogue of Claim 1 holds for $S^\ast$: the set $Z^\ast \setminus S$ is connected. If $S^\circ = S^\ast$ then $Z^\circ \subseteq Z^\ast$ and thus $V(C^\circ) \subseteq Z^\ast$. So suppose that $S^\ast \neq S^\circ$. We shall prove the following claim, which implies $V(C^\circ) \subseteq Z^\ast$.

**Claim 2.** $Z^\circ \setminus S = Z^\ast \setminus S$.

**Proof.** It follows from Lemma 4.23 that $|S^\ast \setminus \{s_1, s_2\}| \geq 2$ and thus

$$S^\ast \setminus \{s_1, s_2\} = S \setminus \{s_1, s_2\} = S^\circ \setminus \{s_1, s_2\}. \hspace{1cm} (4.S)$$

Let $(Y'', S'', Z'') \in \mathcal{T}_{\min}$ such that $(Y'', S'', Z'') \preceq (Y^\ast, S^\ast, Z^\ast)$. Then $S'' = S_1$ or $S'' = S_2$. Moreover, $S^\prime = S''$, because $S^\circ \subseteq Y'' \cup S''$ and $S^\ast \subseteq Y'' \cup S''$ and if $S^\prime \neq S''$ we have $S^\prime \cap S^\ast \subseteq (Y'' \cup S'') \cap (Y'' \cup S'') = \{s_1, s_2\}$, which contradicts (4.S).

Since $s_1 \in S^\circ$ we have $s_2 \in S^\ast$. Note the the setting is completely symmetric with respect to $S^\circ$ and $S^\ast$, as both are essential subseparators of $S$. Hence without loss of generality we may assume that $S^\prime = S'' = S_1$.

We prove next that $s_2 \in Y^\circ$ (by a similar argument as in the proof of Claim 1): the vertex $s_1 \in S^\circ$ has a neighbour in $Y^\circ \subseteq Y_1$, and as the only neighbour of $s_1$ in $S_1$ is $s_2$, we have $s_2 \in Y^\circ$.

Thus $S^\ast = (S^\circ \setminus \{s_1\}) \cup \{s_2\} \subseteq Y^\circ \cup S^\circ$. As $s_1 \in S^\prime \subseteq Z^\ast \cup S^\ast$, we have $s_1 \in Z^\ast$, and now $S^\ast \subseteq Y^\circ \cup S^\circ$ implies $Z^\circ \subseteq Z^\ast$, because $Z^\circ \cup \{s_1\}$ is connected. Hence $Z^\circ \setminus S \subseteq Z^\ast \setminus S$. As both $Z^\circ \setminus S$ and $Z^\ast \setminus S$ are vertex sets of connected components of $G \setminus S$, equality holds.

The uniqueness of $C^\circ$ is immediate, because $(Y^\circ, S^\circ, Z^\circ) \in \mathcal{T}$ and $C^\circ$ is the only connected component of $G \setminus S$ with $V(C^\circ) \subseteq Z^\circ$. \hfill \qed

We define the **expansion** of a set $S' \subseteq V(G')$ to be the set

$$S'_{\Delta} := \begin{cases} S' \cup \{s_2\} & \text{if } s_1 \in S', \\ S' & \text{otherwise.} \end{cases}$$
Note that if $S'$ is a 3-separator of $G'$ then either $S'_s = S'$ is a 3-separator of $G$ or $S'_s = S' \cup \{s_2\}$ is a separator of $G$ satisfying the assumptions of Lemmas 4.23 and 4.25.

Next, we define the contraction of a set $S \subseteq V(G)$ to be the set

$$S^\vee := \begin{cases} (S \cup \{s_1\}) \setminus \{s_2\} & \text{if } \{s_1, s_2\} \cap S \neq \emptyset, \\ S & \text{if } \{s_1, s_2\} \cap S = \emptyset. \end{cases}$$

Note that $(S'_s)^\vee = S'$ for all $S' \subseteq V(G')$, but only $(S^\vee)_s \subseteq S$ for $S \subseteq V(G)$, and the inclusion may be strict.

For every set $S' \subseteq V(G')$ of order $|S'| \leq 3$ we define a set $Z'(S)$ as follows.

- If $S'$ is a separator with $s_1 \in S'$, we let $S := S'_s$ be the expansion of $S'$ and $C_0$ the connected component of $G \setminus S$ obtained from Lemma 4.25. Then we let $Z'(S') := V(C_0)$.
- If $S'$ is a separator with $s_1 \notin S'$, we let $Z'(S') := Z(S')^\vee$ be the contraction of the set $Z(S')$.
- If $S'$ is not a separator of $G'$, we let $Z'(S') := V(G') \setminus S'$.

Observe that $Z'(S')$ is the vertex set of a connected component of $G' \setminus S'$. We let $\mathcal{Y}'(S') = V(G') \setminus (S' \cup Z'(S'))$.

We define $\mathcal{T}'$ to be the set of all separations $(Y', S', T') \in \text{Sep}_{<4}(G')$ such that $Z'(S') \subseteq Z'$.

**Lemma 4.26.** $\mathcal{T}'$ is a $G'$-tangle of order 4.

**Proof.** It follows immediately from the definition that $\mathcal{T}'$ satisfies [T.1] and [T.3].

To see that it satisfies [T.2] let $(Y^i, S^i, Z^i) \in \mathcal{T}'$ for $i = 1, 2, 3$. Suppose for contradiction that $Z^1 \cap Z^2 \cap Z^3 = \emptyset$ and there is no edge that has an endvertex in every $Z^i$. For every $i$, we let $S^i_{0} := S^i_s$. We define a separation $(Y^{i, 1}, S^{i, 1}, Z^{i, 1})$ of $G$ as follows.

(i) If $S^i$ is a separator of $G'$ and $s_1 \in S^i$, then we let $S^{i, 1}$ be an essential subseparator of $S^i_{0}$ and $Z^{i, 1} := Z(S^{i, 1})$ and $Y^{i, 1} := \mathcal{Y}(S^{i, 1})$.

(ii) If $S^i$ is a separator of $G'$ and $s_1 \notin S^i$, then we let $S^{i, 1} := S^i$ and $Z^{i, 1} := Z(S^{i, 1})$ and $Y^{i, 1} := \mathcal{Y}(S^{i, 1})$.

(iii) If $S^i$ is not a separator of $G'$ and $s_1 \in S^i$ then we let

$$S^{i, 1} := S^i = S^i_{0} \setminus \{s_2\},$$

and we let $Z^{i, 1} := Z(S^{i, 1})$ and $Y^{i, 1} := \mathcal{Y}(S^{i, 1})$.

(iv) If $S^i$ is not a separator of $G'$ and $s_1 \notin S^i$, then we let $S^{i, 1} := S^i$ and $Z^{i, 1} := V(G) \setminus S^i$ and $Y^{i, 1} := \emptyset$.

Note that in cases (i) and (iii) we have $Z^{i, 1} \setminus \{s_1, s_2\} = Z^i$. In cases (ii) and (iv), $Z^i$ is the contraction of $Z^{i, 1}$. Thus in all four cases we have

$$Z^{i, 1} \setminus \{s_1, s_2\} = Z^i \setminus \{s_1\}. \quad (4.1)$$

Moreover, we have $(Y^{i, 1}, S^{i, 1}, Z^{i, 1}) \in \mathcal{T}$. We let $(Y^{i, 2}, S^{i, 2}, Z^{i, 2}) \in \mathcal{T}_{\min}$ such that $(Y^{i, 2}, S^{i, 2}, Z^{i, 2}) \preceq (Y^{i, 1}, S^{i, 1}, Z^{i, 1})$.

**Claim 1.** If $s_1 \in S^i$ then $S^{i, 2} \subseteq \{S_1, S_2\}$. In particular, $s_1, s_2 \in S^{i, 2} \cup Y^{i, 2}$.
Proof. This follows from □.

Claim 2. If \( s_1 \notin S_i \), then \( s_1 \in Z^i \) and \( s_1, s_2 \in Z^{i-1} \).

Proof. If \( s_1 \in Y^i \) then \( s_1, s_2 \in Y^{i-1} \subseteq Y^{i-2} \). Thus for \( j = 1, 2 \) we have \( s_j \in Y^{i-2} \cap Y_{3-j} \).

By the Crossing Lemma 4.16, this implies \( S^{i-2} = S_{3-j} \) for \( j = 1, 2 \), which is impossible.

By Claim 2, either \( Z^{i-2} \cap Z^{i-2} \cap Z^{i-2} \neq \emptyset \) or there is an edge that has an endvertex in every \( Z^i \).

Case 1: \( Z^{i-2} \cap Z^{i-2} \cap Z^{i-2} \neq \emptyset \).

Let \( v \in Z^{i-2} \cap Z^{i-2} \cap Z^{i-2} \). If \( v \in V(G) \setminus \{s_1, s_2\} \) then \( v \in Z^1 \cap Z^2 \cap Z^3 \) by □.

Thus we may assume that \( v = s_j \).

By Claim 1, we have \( s_1 \notin S^i \) for \( i = 1, 2, 3 \), because otherwise \( s_j \notin Z^{i-2} \). Then by Claim 2 we have \( s_1 \in Z^1 \cap Z^2 \cap Z^3 \).

Case 2: \( Z^{i-2} \cap Z^{i-2} \cap Z^{i-2} = \emptyset \).

Then there is an edge \( e = v_1 v_2 \) that has an endvertex in every \( Z^i \). As \( Z^{i-2} \cap Z^{i-2} \cap Z^{i-2} = \emptyset \), we have \( v_1, v_2 \in S^{i-1} \cup S^{i-2} \cup S^{i-3} \). For \( i = 1, 2 \), let \( J_i \) be the set of \( j \) such that \( v_1 \in E_i \). Then \( J_1 \cup J_2 = [3] \).

For \( i = 1, 2 \), if \( v_1 \notin \{s_1, s_2\} \) then for all \( j \in J_i \) we have \( v_1 \in Z^j \) by □.

Thus if \( \{v_1, v_2\} \cap \{s_1, s_2\} = \emptyset \), then \( e \in E(G') \) and \( e \) has an endvertex in each \( Z^i \).

So let us assume that \( v_2 \notin \{s_1, s_2\} \). Say, \( v_2 = s_2 \). Then for all \( j \in J_2 \) we have \( s_j \notin S^j \) by Claim 1 and thus \( s_1 \in Z^j \) by Claim 2.

Case 2a: \( v_1 = s_1 \).

Then for all \( j \in J_1 \) we have \( s_1 \notin S^j \) by Claim 1 and thus \( s_1 \in Z^j \) by Claim 2.

It follows that \( s_1 \in Z^1 \cap Z^2 \cap Z^3 \).

Case 2a: \( v_1 \neq s_1 \).

Then \( v_1 \notin \{s_1, s_2\} \) and thus for all \( j \in J_1 \) we have \( v_1 \in Z^j \) by □.

Furthermore, we have \( e' := v_1 s_1 \in E(G') \). This edge \( e' \) has an endvertex in every \( Z^j \).

\[ \square \]

Lemma 4.27. \( T \) is the lifting of \( T' \) to \( G \) with respect to the contraction of the edge \( e = s_1 s_2 \).

Proof. Let \((Y, S, Z)\) be a separation of \( G \) of order at most 3. Let \((M_w)_{w \in V(G')}\) be the branch sets of the model \( M \) of \( G' \) in \( G \) that corresponds to the contraction of \( e \). Then \( M_{s_1} := \{s_1, s_2\} \) and \( M_w := \{w\} \) for all \( w \neq s_1 \).

Let \((Y', S', Z') := \pi_M(Y, S, Z)\) (see □). Then

\[ (Y', S', Z') = (Y' \setminus S', S', Z' \setminus S'). \tag{4.U} \]

We need to prove

\[ (Y, S, Z) \in T \iff (Y', S', Z') \in T'. \tag{4.V} \]

We prove the backward direction first. Suppose that \((Y', S', Z') \in T' \). If \( S' \) is not a separator of \( G' \), \( Z'(S') = V(G') \setminus S' \), and either \( Z = V(G) \setminus S \) or \( Z = V(G) \setminus (S \cup \{s_2\}) \).

In both cases, \((Y, S, Z) \in T \). (If \( Z = V(G) \setminus (S \cup \{s_2\}) \), this follows from Lemma 3.3.)

So suppose that \( S' \) is a separator of \( G' \). If \( s_1 \notin S' \), then \( S = S' \) is a separator of \( G \), and \( Z' = Z' \) is the contraction of \( Z \).

As \((Y', S', Z') \in T' \), we have \( Z'(S') \subseteq Z' \). Now \( Z'(S') \) is the contraction of \( Z'(S) = Z(S) \), and thus we have \( Z(S) \subseteq Z \). This implies \((Y, S, Z) \in T \).

Finally, suppose that \( s_1 \in S' \). Then \( Z' \supseteq Z'(S') = V(C^0) \) for the connected component \( C^0 \) of \( G \setminus S \) obtained from Lemma 4.25 and we have \( V(C^0) \subseteq Z \) and thus \((Y, S, Z) \in T \).
Lemma 4.28. Either $G'$ is 4-connected and $\mathcal{T}'_{\text{min}} = \{(\emptyset, \emptyset, V(G'))\}$ or

$$\mathcal{T}'_{\text{min}} = \left\{ \left( Y^\vee \setminus S^\vee, S^\vee, Z^\vee \setminus S^\vee \right) \mid (Y, S, Z) \in \mathcal{T}_{\text{min}} \text{ such that } S^\vee \text{ is a separator of } G' \right\}. \quad (4.W)$$

Furthermore, for all $S \in \mathcal{ST}_{\text{nd}}$ such that $S^\vee$ is a separator of $G'$,

$$S \in \mathcal{ST}_{\text{nd}} \iff S^\vee \in \mathcal{ST}'_{\text{nd}}. \quad (4.X)$$

Recall that, by Lemma 4.22, $S^\vee$ is a separator of $G'$ for all $S \in \mathcal{ST}_{\text{min}} \setminus \{S_1, S_2\}$. Thus the clause “such that $S^\vee$ is a separator of $G'$” only refers to the separators $S_1' = S_1$ and $S_2' = (S_2 \setminus \{s_2\}) \cup \{s_1\}$, which may not be separators of $G'$.

Proof of Lemma 4.28. If $G'$ is 4-connected then

$$\mathcal{T}' = \{ (\emptyset, S', V(G') \setminus S') \mid S' \subseteq V(G) \text{ width } |S'| \leq 3 \}. \quad (4.W)$$

The unique minimal element of this set $(\emptyset, \emptyset, V(G'))$. In the following, let us assume that $G'$ is not 4-connected.

To prove the inclusion “$\subseteq$” of $(4.W)$, let $(Y', S', Z') \in \mathcal{T}'_{\text{min}}$. Then $Y' \neq \emptyset$, because $G'$ is not 4-connected. Hence the expansion $S_1'$ is a separator of $G$.

**Case 1:** $s_1 \in S'$.

Let $S'$ be an essential subseparator of $S_1'$. Then $|S' \cap \{s_1, s_2\}| = 1$. Say, $s_1 \in S'$.

Let $Z := Z(S')$ and $Y := Y(S')$. Then $Z' = Z \setminus \{s_1, s_2\}$ and $Y' = Y \setminus \{s_1, s_2\}$.

Let $(Y, S, Z) \in \mathcal{T}_{\text{min}}$ such that $(Y, S, Z) \preceq (Y^\vee, S^\vee, Z^\vee)$. Then $s_1 \in S \subseteq Y \cup S$ and thus by $(4.P)$, $S = S_1$ or $S = S_2$. Say, $S = S_j$. Note that $Z_j = Z \subseteq Z \setminus \{s_1, s_2\} = Z'$. Thus

$$(Y', S', Z') \preceq (Y_j \setminus \{s_{j-1}\}, S'_j, Z_j) = (Y_j^\vee \setminus S'_j, Z'_j \setminus S'_j).$$

By the minimality of $(Y', S', Z')$, equality holds.

**Case 2:** $s_1 \not\in S'$.

Then $S_1' = S'$ and $(Y_1', S_1', Z_1') \in \mathcal{T}$. Suppose for contradiction that $(Y_1', S_1', Z_1')$ is not minimal in $T$ and let $(Y, S, Z) \in \mathcal{T}_{\text{min}}$ such that $(Y, S, Z) \prec (Y_1', S_1', Z_1')$.

**Case 2a:** $S \cap \{s_1, s_2\} = \emptyset$.

Then $(Y', S, Z') \in \mathcal{T}'$ is strictly smaller than $(Y', S', Z')$, which contradicts the minimality of $(Y', S', Z')$.

**Case 2b:** $S \cap \{s_1, s_2\} \neq \emptyset$.

Then $S \in \{S_1, S_2\}$. Say, $S = S_1$. Then $(Y_2 \setminus \{s_1\}, S_2, Z_2) \in \mathcal{T}'$ is strictly smaller than $(Y', S', Z')$, again contradicting the minimality of the latter.

To prove the converse inclusion of $(4.W)$, let $(Y, S, Z) \in \mathcal{T}_{\text{min}}$. 

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Case 1: $S \in \{S_1, S_2\}$.

Say, $S = S_2$. Then $Y = Y_2$ and $Z = Z_2$. Clearly,

$$(Y^\vee \setminus S^\vee, S^\vee, Z^\vee \setminus S^\vee) = (Y_2 \setminus \{s_1\}, (S_2 \setminus \{s_2\}) \cup \{s_1\}, Z_2) \in T'.$$

Suppose for contradiction that it is not minimal. Let $(Y', S', Z') \in T'_{\min}$ such that $(Y', S', Z') \prec (Y_2 \setminus \{s_1\}, (S_2 \setminus \{s_2\}) \cup \{s_1\}, Z_2)$. By the converse inclusion (proved above), there is a $(Y'', S'', Z'') \in T'_{\min}$ such that

$$(Y''^\vee \setminus (S'')^\vee, (S'')^\vee, (Z'')^\vee \setminus (S'')^\vee) = (Y', S', Z').$$

Then $(Z'')^\vee \setminus (S'')^\vee \subset Z_2$, which implies $Z'' \setminus \{s_1, s_2\} \subset Z_2$. By the minimality of $(Y_2, S_2, Z_2)$, we have $Z'' \not\subset Z_2$. Thus $Z'' \cap \{s_1, s_2\} \neq \emptyset$. Say, $s_j \in Z''$. Furthermore, $(Y'')^\vee \setminus (S'')^\vee \supset (Y_2 \setminus S_2) \setminus \{s_2\}$ and thus $(Y'' \cup S'_3) \cap \{s_1, s_2\} \neq \emptyset$. As $s_j \in Z''$ and $s_1, s_2 \in E(G)$ it follows that $s_{3-j} \in S''$. By $\text{(4.P)}$, it follows that $S'' = S_{3-j}$. But then $s_j \in Y'' = Y_{3-j}$, which is a contradiction.

Case 2: $S \not\in \{S_1, S_2\}$.

Then $s_1, s_2 \not\in Y \cup S$ by $\text{(4.P)}$, and thus

$$(Y^\vee \setminus S^\vee, S^\vee, Z^\vee \setminus S^\vee) = (Y, S, Z^\vee) \in T'.$$

Suppose for contradiction that $(Y, S, Z^\vee)$ is not minimal in $T'$. Let $(Y', S', Z') \in T'_{\min}$ such that $(Y', S', Z') \prec (Y, S, Z^\vee)$. By the converse inclusion (proved above), there is a $(Y'', S'', Z'') \in T'_{\min}$ such that

$$(Y''^\vee \setminus (S'')^\vee, (S'')^\vee, (Z'')^\vee \setminus (S'')^\vee) = (Y', S', Z').$$

Then $(Y'')^\vee \cup (S'')^\vee = Y' \cup S' \supset Y \cup S$, which implies $Y'' \cup S'' \supset Y \cup S$ and thus $(Y'', S'', Z'') \prec (Y, S, Z)$. This contradicts the minimality of $(Y, S, Z)$.

It remains to prove $\text{(4.X)}$. Let $S \in ST'_{\min}$ such that $S^\vee$ is a separator of $G'$. Then $S^\vee \in ST'_{\min}$. Let $Y := \mathcal{Y}(S)$ and $Z := \mathcal{Z}(S)$. Then $(Y, S, Z) \in T'_{\min}$ and $(Y^\vee \setminus S^\vee, S^\vee, Z^\vee \setminus S^\vee) \in ST'_{\min}$.

Suppose first that $S \not\in \{S_1, S_2\}$. Then $S^\vee = S$ and $Y^\vee \setminus S^\vee = Y^\vee = Y$ and $Z^\vee \setminus S^\vee = Z^\vee$, because $s_1, s_2 \in Z$ by $\text{(4.P)}$. It follows that $S$ is degenerate in $G$ if and only if $S^\vee$ is degenerate in $G'$.

If $S \in \{S_1, S_2\}$, say, $S = S_1$, we need to prove that $S^\vee$ is non-degenerate. We have

$$(Y^\vee \setminus S^\vee, S^\vee, Z^\vee \setminus S^\vee) = (Y_j \setminus \{s_{3-j}\}, S_j^\vee, Z_j).$$

As $S^\vee$ is a separator of $G'$, we have $Y_j \setminus \{s_{3-j}\} \neq \emptyset$. If $|Y_j \setminus \{s_{3-j}\}| > 1$, then $(Y_j \setminus \{s_{3-j}\}, S_j^\vee, Z_j)$ is non-degenerate. Suppose that $|Y_j \setminus \{s_{3-j}\}| = 1$. Then $s_{3-j}$ is adjacent to $s'_i$ or $s'_j$, because $s_{3-j}$ has degree at least 3 in $G$. Say, $s_{3-j} s'_i \in E(G)$. Then $s_1 s'_i \in E(G')$, and thus $S^\vee_j$ is not an independent set in $G'$, which again means that $(Y_j \setminus \{s_{3-j}\}, S_j^\vee, Z_j)$ is non-degenerate.

$S^\vee_j$ is not necessarily a separator of $G'$. However, if $S_i$ is crossed by some separator $S \in ST'_{\min} \setminus \{S_1, S_2\}$ then $S^\vee_i$ remains a separator of $G'$. Hence we get the following corollary.

**Corollary 4.29.** $E^\times(T') = E^\times(T) \setminus \{e\}$ and $E^\times_{\text{nd}}(T') = E^\times_{\text{nd}}(T) \setminus \{e\}$.
4.6 The Region of the Tangle

We still assume that $G$ is a 3-connected graph and $T$ is a $G$-tangle of order 4 (but we drop Assumption 4.21). Let $e^1, \ldots, e^m$ be an enumeration of all non-degenerate crossedges of $T$. Recall that, by Corollary 4.20, $\{e^1, \ldots, e^m\}$ is a matching of $G$. Say, $e^i = s^i_1 s^i_2$. Let $G^{(0)} := G$, and for $i \in [m]$, let $G^{(i)}$ be the graph obtained from $G$ by contracting the edges $e^1, \ldots, e^i$ to the vertices $s^i_j$. We inductively define for all $i$ a $G^{(i)}$-tangle $T^{(i)}$ of order 3 as follows.

We let $T^{(0)} := T$. To define $T^{(i+1)}$, we assume that $G^{(i)}$ is 3-connected and $T^{(i)}$ is a $G^{(i)}$-tangle of order 3 and $e^{i+1}$ is a non-degenerate crossedge of $T^{(i)}$. Then Assumptions 4.6, 4.15, 4.21 are satisfied with $G := G^{(i)}, T := T^{(i)}, e := e^{i+1}, G^* := G^{(i+1)}$, and we can apply the results of Section 4.5. We let $T^{(i+1)}$ be the tangle $(T^{(i)})'$ (see Lemma 4.26).

For every $i \in [m]$ and $v \in V(G)$ we let
\[ e^{i,v} := \begin{cases} s^i_j & \text{if } v \in \{s^i_1, s^i_2\} \text{ for some } j \leq i, \\ v & \text{otherwise}. \end{cases} \]

For $W \subseteq V(G)$, we let $W^{i,v} := \{e^{i,v} | w \in W\}$.

**Lemma 4.30.** Let $i \in [m]$.

1. $G^{(i)}$ is 3-connected.
2. $T^{(i)}$ is a $G^{(i)}$-tangle of order 3 with
   \[ T^{(i)}_{\min} = \left\{ (Y^{i,v}, S^{i,v}, Z^{i,v}) \mid S^{i,v} \text{ is a separator of } G^{(i)} \right\}. \]
   For $i = m$, it may also happen that $G^{(m)}$ is 4-connected and $(T^{(m)})_{\min} = \{(\emptyset, \emptyset, V(G^{(m)})\}$.
3. $E_{\text{nd}}(T^{(i)}) = \{e^{i+1}, \ldots, e^m\}$.
4. $T$ is the lifting of $T^{(i)}$ from $G^{(i)}$ to $G$ with respect to the contraction of $e^1, \ldots, e^i$.
5. The graph $G^{(i)}$ and the tangle $T^{(i)}$ do not depend on the order in which the edges $e^1, \ldots, e^i$ are contracted.
   Up to isomorphism, $G^{(i)}$ and $T^{(i)}$ also do not depend on whether $e^i$ is contracted to $s^i_1$ or $s^i_2$.

**Proof.** Assertions (1)–(4) follow by induction from Corollary 4.24, Lemma 4.26 and Lemma 4.28, Corollary 4.29, Lemma 4.27, respectively. Assertion (5) is obvious as far as the graph $G^{(i)}$ is concerned, and for the tangle $T^{(i)}$ it follows from (4). \qed

We let
\[ R^{(0)} := \bigcap_{(Y,S,Z) \in T_{\text{nd}}} Z \cup \bigcup_{S \in T_{\text{nd}}} S, \]
and
\[ R^{(i)} := R^{(0)} \setminus \{s^i_2, \ldots, s^i_1\}. \]

We shall prove that $R^{(m)}$ is a non-exceptional quasi-4-connected region of $G$ and that $T$ is equal to $T_{R^{(m)}}$, the tangle associated with $R^{(m)}$.

For all $i \in [m]$ we let
\[ H^{(i)} := G[R^{(i)}]. \]
The fence of a separator $S \in \mathcal{ST}_{nd}$ is the set $fc(S)$ consisting of all vertices in $S$ that are not endvertices of a non-degenerate crossedge of $S$ and all vertices in $\mathcal{Y}(S)$ that are endvertices of non-degenerate crossedges. For example, if $S = \{s_1, s_2, s_3\}$, and $S$ is crossed by $S_1, S_2 \in \mathcal{ST}_{nd}$ with crossedges $s_1s'_1$ and $s_2s'_2$, respectively, then $fc(S) = \{s'_1, s'_2, s_1, s_2\}$. Note that $fc(S) \subseteq R(0)$ and that $fc(S) \cap S = \mathcal{Y}(S) \cap R(0)$.

Lemma 4.31. For every connected component $C$ of $G \setminus R(0)$ there is a unique $S \in \mathcal{ST}_{nd}$ such that $V(C) \subseteq \mathcal{Y}(S) \setminus fc(S)$ and $N(C) = fc(S)$.

Proof. It follows from the definition of $R(0)$ and the connectedness of $C$ that there is an $S \in \mathcal{ST}_{nd}$ such that $V(C) \subseteq \mathcal{Y}(S)$. This is unique, because $\mathcal{Y}(S) \cap \mathcal{Y}(S') = \emptyset$ for distinct $S, S' \in \mathcal{ST}_{min}$ (by the Crossing Lemma 4.16).

Let $S_1, \ldots, S_k \in \mathcal{ST}_{nd}$ be the non-degenerate separators crossing $S$, and let $s_i, s'_i$ with $s_i \in S$ and $s'_i \in S_i$ be the crossedge of $S$ and $S_i$. Then $k \leq 3$. Let $s_{k+1}, \ldots, s_3$ be the elements of $S \setminus \{s_1, \ldots, s_k\}$. Then $fc(S) = \{s'_1, \ldots, s'_k, s_{k+1}, \ldots, s_3\}$ and $C$ is a connected component of $G[S \setminus \{s'_1, \ldots, s'_k\}]$. Hence $N(C) \subseteq \{s_1, s_2, s_3\} \cup \{s'_1, \ldots, s'_k\}$. However, it follows from the Crossing Lemma 4.16 that $s'_i$ is the only neighbour of $s_i$ in $\mathcal{Y}(S)$, for $i \leq k$. Hence $s_i \notin N(C)$, and we have $N(C) \subseteq fc(S)$. As $G$ is $3$-connected and $fc(S) = 3$, we have $N(C) = fc(S)$.

Lemma 4.32. $H(0)$ is a faithful minor of $G$.

Proof. By Lemma 4.31 for every connected component $C$ of $G \setminus R(0)$ there is an $S \in \mathcal{ST}_{nd}$ such that $V(C) \subseteq \mathcal{Y}(S) \setminus fc(S)$ and $N(C) = fc(S)$. Thus by Lemma 4.15 it suffices to prove that for every $S \in \mathcal{ST}_{nd}$ such that $\mathcal{Y}(S) \setminus fc(S) \neq \emptyset$, either $fc(S)$ is not an independent set or $|\mathcal{Y}(S) \setminus fc(S)| \geq 2$.

So let $S = \{s_1, s_2, s_3\} \in \mathcal{ST}_{nd}$ such that $\mathcal{Y}(S) \setminus fc(S) \neq \emptyset$, and let $k$ be the number of $S' \in \mathcal{ST}_{min}$ crossing $S$. Then $0 \leq k \leq 3$. If $k \geq 1$, let $S_1, \ldots, S_k \in \mathcal{ST}_{min}$ be the separations crossing $S$ and let $s_1, s'_1$ be the crossedges.

Case 1: $k = 0$.
Then $fc(S) = S$ and $\mathcal{Y}(S) \setminus fc(S) = \mathcal{Y}(S)$ and thus either $fc(S)$ is not an independent set or $|\mathcal{Y}(S) \setminus fc(S)| \geq 2$, because $S$ is non-degenerate.

Case 2: $k = 1$.
Then $fc(S) = \{s'_1, s_2, s_3\}$ and thus $\mathcal{Y}(S) \setminus fc(S) = \mathcal{Y}(S) \setminus \{s'_1\}$. Suppose that $|\mathcal{Y}(S) \setminus fc(S)| = 1$. The degree of $s'_1$ is at least $3$, and as all its neighbours are in $S \cup \mathcal{Y}(S)$, it must have at least one neighbour in $\{s_2, s_3\}$. Thus $fc(S)$ is not an independent set.

Case 3: $k = 2$.
Then $fc(S) = \{s'_1, s'_2, s_3\}$ and thus $\mathcal{Y}(S) \setminus fc(S) = \mathcal{Y}(S) \setminus \{s'_1, s'_2\}$. Suppose that $|\mathcal{Y}(S) \setminus fc(S)| = 1$. The degree of $s'_1$ is at least $3$, and all its neighbours are in $S \cup \mathcal{Y}(S)$. It follows from the Crossing Lemma 4.16 that the only neighbour of $s_2$ in $\mathcal{Y}(S)$ is $s'_2$. Thus $s_2$ is not a neighbour of $s'_1$, and $s'_1$ must have at least one neighbour in $\{s'_2, s_3\}$. Hence $fc(S)$ is not an independent set.

Case 4: $k = 3$.
Similar to the previous cases.

Lemma 4.33. For all $S \in \mathcal{ST}_{nd}$ the set $fc(S)$ is a clique in $H(0)$.

Proof. Let $S = \{s_1, s_2, s_3\} \in \mathcal{ST}_{nd}$. As in the proof of the previous lemma, let $k$ be the number of $S' \in \mathcal{ST}_{nd}$ crossing $S$. Let $S_1, \ldots, S_k \in \mathcal{ST}_{nd}$ be the separations crossing $S$ and let $s_i, s'_i$ be the crossedges. If $\mathcal{Y}(S) \setminus fc(S) \neq \emptyset$, the claim is immediate from the definition of the torso. So suppose that $\mathcal{Y}(S) \setminus fc(S) = \emptyset$. Then $\mathcal{Y}(S) \subseteq fc(S) \setminus S =$
\{s'_1, \ldots, s'_k\}, and as \( \mathcal{Y}(S) \neq \emptyset \), this implies that \( k \geq 1 \). Now we argue similarly to the proof of the previous lemma.

**Case 1:** \( k = 1 \).

Then \( \text{fc}(S) = \{s'_1, s_2, s_3\} \) and \( \mathcal{Y}(S) = \{s'_1\} \) and hence \( N^G(s'_1) = \{s_1, s_2, s_3\} \). In particular, \( s'_1, s_2, s_3 \in E(G) \). As \( |\mathcal{Y}(S)| = 1 \) and \( S \) is non-degenerate, \( S \) is not an independent set. There is no edge from \( \{s_2, s_3\} \subseteq \mathcal{Z}(S_1) \) to \( s_1 \in \mathcal{Y}(S_1) \). Hence \( s_2 s_3 \in E(G) \), which implies \( s_2 s_3 \in E(H^{(0)}) \). Thus \( \text{fc}(S) \) is a clique in \( H^{(0)} \).

**Case 2:** \( k = 2 \).

Then \( \text{fc}(S) = \{s'_1, s'_2, s_3\} \) and thus \( \mathcal{Y}(S) = \{s'_1, s'_2\} \). The degree of \( s'_1 \) is at least 3, and all its neighbours are in \( S \cup \mathcal{Y}(S) \setminus \text{fc}(S) \). It follows from the Crossing Lemma that the only neighbour of \( s_2 \) in \( \mathcal{Y}(S) \) is \( s'_2 \). Thus \( s_2 \) is not a neighbour of \( s'_1 \), and therefore \( N(s'_1) = \{s_1, s'_2, s_3\} \). Similarly, \( N(s'_2) = \{s'_1, s_2, s_3\} \). Hence \( \text{fc}(S) = \{s'_1, s'_2, s_3\} \) is a clique.

**Case 3:** \( k = 3 \).

Then \( \text{fc}(S) = \{s'_1, s'_2, s'_3\} \) and thus \( \mathcal{Y}(S) = \{s'_1, s'_2, s'_3\} \). The degree of \( s'_1 \) is at least 3, and all its neighbours are in \( S \cup \mathcal{Y}(S) \setminus \text{fc}(S) \). As \( s_2, s_3 \) are not neighbours of \( s'_1 \), we have \( N(s'_1) = \{s_1, s'_2, s'_3\} \). Similarly, \( N(s'_2) = \{s'_1, s_2, s'_3\} \). Hence \( \text{fc}(S) \) is a clique.

The next lemma is a little bit surprising. It says that instead of deleting the endpoints \( s'_1 \) of the crossovers \( e^j \), we could have contracted the crossovers to \( s'_1 \) with the same result.

**Lemma 4.34.** For all \( i \geq 0 \), the graph \( H^{(i)} \) is equal to the graph obtained from \( H^{(0)} \) by contracting the edges \( e^1, \ldots, e^i \) to \( s'_1, \ldots, s'_i \), respectively.

**Proof.** The proof is by induction on \( i \). The base step \( i = 0 \) is trivial.

For the inductive step \( i \rightarrow i + 1 \), suppose that \( e^{i+1} \) is the crossover of \( S_1, S_2 \in \mathcal{ST}_{md} \) with \( s_j := s^{i+1}_j \in S_j \). Then \( s_j \in \text{fc}(S_{3-j}) \). Suppose that \( \text{fc}(S_j) = \{s_{3-j}, t_j, u_j\} \). By a similar analysis as in the proofs of the previous two lemmas, we see that

\[
N^{H^{(0)}}(s_j) = \{s_{3-j}\} \cup (\text{fc}(S_{3-j}) \setminus \{s_j\}) = \{s_{3-j}, t_{3-j}, u_{3-j}\}.
\]

By the induction hypothesis, \( H^{(i)} \) is the graph obtained from \( H^{(0)} \) by contracting \( e^1, \ldots, e^i \) to \( s'_1, \ldots, s'_i \), respectively. Hence

\[
N^{H^{(i)}}(s_j) = (N^{H^{(0)}}(s_j))^{\backslash i} = \{s_{3-j}, t_{3-j}, u_{3-j}\}.
\]

Here we use the fact that \( s_{3-j}^{\backslash i} = s_{3-j} \) because the edges \( e^1, \ldots, e^{i+1} \) form a matching.

Hence the neighbours of \( s_1 \) in the graph obtained from \( H^{(i)} \) by contracting the edge \( e^{i+1} = s_1 s_2 \) to \( s_1 \) are \( t_1^{\backslash i}, u_1^{\backslash i}, t_2^{\backslash i}, u_2^{\backslash i} \). I claim that the neighbours of \( s_1 \) in the graph \( H^{(i+1)} \) are \( t_1^{\backslash i}, u_1^{\backslash i}, t_2^{\backslash i}, u_2^{\backslash i} \) as well. Observe that

\[
H^{(i+1)} = H^{(i)}[R^{(i+1)}] = H^{(i)}[R^{(i)} \setminus \{s_2\}].
\]

The vertices \( t_2^{\backslash i}, u_2^{\backslash i} \) are neighbours of \( s_1 \) in \( H^{(i)} \) and hence in \( H^{(i+1)} = H^{(i)}[R^{(i)} \setminus \{s_2\}] \).

To see that \( t_1^{\backslash i}, u_1^{\backslash i} \) are neighbours of \( s_1 \) in \( H^{(i+1)} \), note that the unique connected component of \( H^{(i)} \setminus R^{(i)} \) consists of the vertex \( s_2 \), and \( N^{H^{(i)}}(s_2) = \{s_1, t_1^{\backslash i}, u_1^{\backslash i}\} \). Thus the set \( \{s_1, t_1^{\backslash i}, u_1^{\backslash i}\} \) is a clique in the torso \( H^{(i+1)} = H^{(i)}[R^{(i+1)}] \). This implies that \( s_1 t_1^{\backslash i} \) and \( s_1 u_1^{\backslash i} \) are edges of \( H^{(i+1)} \).

Hence indeed \( H^{(i+1)} \) is the graph obtained from \( H^{(i)} \) by contracting the edge \( e^{i+1} \) to \( s_1 = s_1^{i+1} \). \( \square \)
Corollary 4.35. For $0 \leq i \leq m$, the graph $H^{(i)}$ is a faithful minor of $G$.

Corollary 4.36. For $0 \leq i \leq m$,

$$H^{(i)} = G^{(i)} \langle R^{(i)} \rangle.$$ 

Lemma 4.37.

$$R^{(m)} = V(G^{(m)}) \setminus \bigcup_{(Y,S,Z) \in T^{(m)}_{ad}} Y.$$ 

Proof. This follows from Lemmas 4.30(2) and 4.34 and the definition of $R^{(m)}$. □

Corollary 4.38. For all connected components $C$ of $G \setminus R^{(m)}$ there is an $S \in ST_{ad}$ such that $N(C) = S^{4}_{ad}$.

Lemma 4.39. Let $(Y,S,Z) \in Sep_{4}(H^{(m)})$ be a proper separation. Then $(Y,S,Z)$ or $(Z,S,Y)$ is degenerate. Furthermore, either $(Y,S,V(G^{(m)}) \setminus (Y \cup S))$ or $(Z,S,V(G^{(m)}) \setminus (Z \cup S))$ is a degenerate separation of $G^{(m)}$ contained in $T^{(m)}_{min}$.

Proof. $(Y,S,Z)$ gives rise to a separation $(Y',S,Z')$ of $G^{(m)}$ with $Y = Y' \cap R^{(m)}$ and $Z = Z' \cap R^{(m)}$. Without loss of generality we assume that $(Y',S,Z') \in T^{(m)}$. Let $(Y'',S'',Z'') \in T_{min}$ such that $(Y'',S'',Z'') \subset (Y',S,Z')$. Then $Y \subset Y' \subset Y''$. If $(Y'',S'',Z'')$ is non-degenerate, then by Lemma 4.37, $Y'' \cap R^{(m)} = \emptyset$, and thus $Y \subset Y'' \cap R^{(m)} = \emptyset$, which contradicts $(Y,S,Z)$ being a proper separation. Thus $(Y'',S'',Z'')$ is degenerate and therefore $|Y''| = 1$. But then $Y = Y' = Y''$ and thus $S' = S$ and $Z' = Z'' = V(G^{(m)}) \setminus (Y \cup S)$. □

Lemma 4.40. $R^{(m)}$ is a quasi-4-connected region of $G$.

Proof. We have seen that $H^{(m)} = G[R^{(m)}]$ is a faithful minor of $G$ (Corollary 4.35). Thus $R^{(m)}$ satisfies (Q.1). By Lemma 4.39, $H^{(m)}$ is quasi-4-connected. Thus $R^{(m)}$ satisfies (Q.2). It follows from Corollary 4.38 that $R^{(m)}$ satisfies (Q.3). □

It remains to prove that $R^{(m)}$ is non-exceptional and that $T = T^{(m)}$ (Recall the definition of $T^{(m)}$ from Theorem 4.11).

Lemma 4.41. If $H^{(m)}$ is non-exceptional then $T = T^{(m)}$.

Proof. Suppose that $H^{(m)}$ is non-exceptional. Let $\tilde{T}$ be the unique $H^{(m)}$-tangle of order 4. By Theorem 4.15, $\tilde{T}$ consist of all separations $(Y',S',Z') \in Sep_{4}(H^{(m)})$ such that $|Y'| < |Z'|$. By the transitivity of the lifting relation, Lemma 4.34, and Corollary 4.36, it suffices to prove that $T^{(m)}$ is the lifting of $\tilde{T}$ to $G^{(m)}$ with respect to some faithful model of $H^{(m)}$ in $G^{(m)}$.

So let $\tilde{M}$ be a faithful model of $H^{(m)}$ in $G^{(m)}$. Let $(Y,S,Z) \in T^{(m)}$ and $(Y',S',Z') := \pi_{\tilde{M}}(Y,S,Z)$. Recall from the definition 3.1) of the projection operation that $Y' \subset Y \cap R^{(m)}$ and $Z' \subset Z \cap R^{(m)}$. We need to prove that $(Y',S',Z') \in \tilde{T}$. Let $(Y'',S'',Z'') \in T^{(m)}_{min}$ such that $(Y'',S'',Z'') \subset (Y,S,Z)$. Then $Y' \subset Y \subset Y''$. If $(Y'',S'',Z'')$ is non-degenerate, then by Lemma 4.37, we have $Y'' \cap R^{(m)} = \emptyset$. Thus $Y'' = \emptyset$, which implies $(Y',S',Z') \in \tilde{T}$. If $(Y'',S'',Z'')$ is degenerate, then $|Y''| = 1$ and thus $|Y'| \leq 1$. Again, this implies $(Y',S',Z') \in \tilde{T}$: either $Y' = \emptyset$, or $|Y'| = 1$, but then $|R^{(m)}| \geq 6$, because $(Y',S',Z')$ is a proper separation of $H^{(m)}$ and the only non-exceptional quasi-4-connected graph of order 5 is the complete graph, which has no proper separation. □
Lemma 4.42. Suppose that $H^{(m)}$ is exceptional. Then it is isomorphic to a subgraph of $TH_{+3}$. Furthermore, there is an embedding $f$ of $H^{(m)}$ into $TH_{+3}$ such that $v_1, \ldots , v_4 \in f(R^{(m)})$, and if $w_j \in f(R^{(m)})$, say with $w_j = f(w'_j)$, then $N^{G^{(m)}}(w'_j) = H^{(m)}(w'_j)$ and
\[
\{ \{ w'_j \}, N^{G^{(m)}}(w'_j), V(G^{(m)}) \setminus (\{ w'_j \} \cup N^{G^{(m)}}(w'_j)) \}
\]
is a degenerate separation of $G^{(m)}$ in $T^{(m)}_{\min}$.

Proof. Suppose that $H^{(m)}$ is not isomorphic to a subgraph of $TH_{+3}$. Then, without loss of generality, it is a subgraph of $TR_{+3}$ with $w_1, w_2, w_3 \in V(H^{(m)})$. We apply Lemma 4.43 to the separation $(\{w_1\}, \{v_1, v_2, v_3\}, \{w_2, w_3\})$ of $H^{(m)}$. Then
\[
(\{w_1\}, \{v_1, v_2, v_3\}, V(G^{(m)}) \setminus \{v_1, v_2, v_3, w_1\}) \in T^{(m)}_{\min}.
\]
However, $V(G^{(m)}) \setminus \{v_1, v_2, v_3, w_1\}$ is not connected in $G^{(m)}$, because $w_2$ and $w_3$ belong to different connected components. This contradicts Corollary 3.3.

Thus $H^{(m)}$ is isomorphic to a subgraph of $TH_{+3}$. Without loss of generality, we may assume that $H^{(m)} \subseteq TH_{+3}$. We may further assume that $v_1, \ldots , v_4 \in V(H^{(m)})$, because the only quasi-4-connected subgraphs of $TH_{+3}$ that do not contain $v_1, \ldots , v_4$ contain one $w_i$, and the three adjacent $v_i$s, and these subgraphs are isomorphic to the subgraph induced by $v_1, \ldots , v_4$.

Suppose that $w_j \in V(H^{(m)})$ for some $j$, say, $j = 1$. Let $S := N^{H^{(m)}}(w_1) = \{v_1, v_2, v_3\}$. Then $(\{w_1\}, S, V(H^{(m)}) \setminus (S \cup \{w_1\}))$ is a proper separation of $H^{(m)}$. By Lemma 4.43 either
\[
(\{w_1\}, S, V(G^{(m)}) \setminus (S \cup \{w_1\}))
\]
is a degenerate separation of $G^{(m)}$ in $T^{(m)}_{\min}$, or $|V(H^{(m)}) \setminus (S \cup \{w_1\})| = 1$, which implies that $V(H^{(m)}) \setminus (S \cup \{w_1\}) = \{v_4\}$, and
\[
(\{v_4\}, S, V(G^{(m)}) \setminus (S \cup \{v_4\}))
\]
is a degenerate separation of $G^{(m)}$ in $T^{(m)}_{\min}$. In the former case, we are done, and in the latter case the mapping $f : V(H^{(m)}) \to V(TH_{+3})$ defined by $f(w_1) := v_4$, $f(v_i) = w_i$, $f(v_i) = v_i$ for $i = 1, 2, 3$ is an embedding of $H^{(m)}$ into $TH_{+3}$ with the desired properties. \hfill \square

Lemma 4.43. Suppose that $H^{(m)}$ is exceptional. Then there is a non-exceptional extension $\hat{H}$ of $R^{(m)}$ and a faithful model $\mathcal{M}$ of $\hat{H}$ in $G$ such that $T = T(\hat{H}, \mathcal{M})$.

Proof. We first prove the assertion for $G^{(m)}$ instead of $G$; it will then be easy to lift it to $G$.

Claim 1. There is a non-exceptional extension $\hat{H}$ of $R^{(m)}$ in $G^{(m)}$ and a faithful model $\mathcal{M}$ of $\hat{H}$ in $G^{(m)}$ such that $T^{(m)}$ is the lifting of the unique $\hat{H}$-tangle of order 4 to $G^{(m)}$ with respect to $\mathcal{M}$.

Proof. By Lemma 4.42 we may assume without loss of generality that $H^{(m)} \subseteq TH_{+3}$ with $v_1, \ldots , v_4 \in R^{(m)}$. We actually view $H^{(m)}$ as a subgraph of $TH_{+4}$, and for $j = 1, \ldots , 4$, let $S_j := N^{TH_{+3}}(w_j)$. Then, by Lemma 4.42 if $w_j \in R^{(m)}$ then $(\{w_j\}, S_j, V(G) \setminus (\{w_j\} \cup S_j))$ is a degenerate 3-separation of $G^{(m)}$.

For $i \in [4]$, let $Z_i$ be the connected component of $G^{(m)} \setminus S_i$ that contains the unique element in $\{v_1, \ldots , v_4\} \setminus S_i$, and we let $Y_i := V(G^{(m)}) \setminus (S_i \cup Z_i)$. Then $(Y_i, S_i, Z_i) \in T^{(m)}$. This can be seen as follows. If $w_i \in R^{(m)}$, then $(\{w_i\}, S_i, V(G) \setminus (\{w_i\} \cup S_i))$ is a
Lemma 4.41 and 4.43. That need to prove that \((\hat{Y}, S, Z)\) implies \(Z_i = V(G) \setminus ((\{w_i\} \cup S_i)\) and \(Y_i = \{w_i\}\) and \((Y_i, S_i, Z_i) = \{(w_i), S_i, \emptyset. V(G) \setminus ((\{w_i\} \cup S_i)\) ∈ \(T(m)\). Otherwise, \(Y_i \cap R(m) = \emptyset\), and either \(Y_i = \emptyset\), which trivially implies \((Y_i, S_i, Z_i) ∈ T(m)\), or it follows from Lemma 4.37 that \((Y_i, S_i, Z_i) ∈ T_{ad}\).

Now it follows from \([T.3]\) that if \(w_j \notin R(m)\) then there is a connected component \(C_j \subseteq (G(m) \setminus R(m))\) such that \(N(C_j) = S_j\), because otherwise for the three separations \((Y_i, S_i, Z_i) ∈ T(m)\), where \(i ∈ [4] \setminus \{j\}\), the intersection of the \(Z_i\) is empty, and there is no edge that has an endvertex in each \(Z_i\). Contracting \(C_j\) to a single vertex gives us the desired faithful model \(\hat{M}\) of a full subgraph \(\hat{H}\) of \(TH_{44}\) in \(G(m)\).

Let \(\hat{T}\) be the unique \(\hat{H}\)-tangle of order 4. It remains to prove that \(\hat{T}(m)\) is the lifting of \(\hat{T}\) with respect to \(\hat{M}\). So let \((Y, S, Z) ∈ T(m)\), and let \((\hat{Y}, \hat{S}, \hat{Z}) := \pi_M(Y, S, Z)\). We need to prove that \((\hat{Y}, \hat{S}, \hat{Z}) ∈ \hat{T}\). As \(\hat{Y} ⊆ Y\), we may assume without loss of generality that \(|Y| ≥ 2\). Let \((Y', S', Z') ∈ T_{min}(m)\) such that \((Y', S', Z') ⊆ (Y, S, Z)\). Then \(Y ⊆ Y'\), and thus \((Y', S', Z')\) is non-degenerate. By Lemma 4.37, \((Y' \cap R(m)) = \emptyset\) and thus

\[|\hat{Y}| ≤ |Y ∩ V(\hat{H})| ≤ 1\]

By Lemma 3.3, it follows that \((\hat{Y}, \hat{S}, \hat{Z}) ∈ \hat{T}\).

Now choose \(\hat{H}\) according to the claim, and let \(\hat{T}\) be the unique \(\hat{H}\)-tangle of order 4. Then \(\hat{H}\) is also a non-exceptional extension of \(H\) in \(G\), and by Lemma 4.30 and the transitivity of the lifting relation, \(\hat{T}\) is the lifting of \(\hat{T}\) to \(G\) with respect to some faithful model of \(\hat{H}\) in \(G\).

\[R_{\hat{T}} := R(m)\]

**Proof of the Correspondence Theorem 4.4** The theorem follows from Theorem 4.11 and Lemmas 4.41 and 4.43.

So far, we have only talked about the quasi-4-connected regions of a graph. It is natural to call the graphs \(G[R]\) for the \(G\)-tangles \(T\) of order 4 the **quasi-4-connected components** of \(G\). While the regions \(R\) are not canonical, the following corollary (to Lemma 4.30) says that the quasi-4-connected components \(G[R]\) are canonical if viewed as abstract graphs (that is, up to isomorphism).

**Corollary 4.44.** Let \(G, G'\) be a 3-connected graphs, and let \(T, T'\) be tangles of order 4 of the these graphs. Suppose that there is an isomorphism \(f\) from \(G\) to \(G'\) that maps \(T\) to \(T'\), that is, such that for all \((Y, S, Z) ∈ \text{Sep}_{c4}(G)\) we have \((Y, S, Z) ∈ T \iff (f(Y), f(S), f(Z)) ∈ T'\). Then there is an isomorphism from \(G[R]\) to \(G'[R']\).

**5 Decomposition into Quasi-4-Connected Components**

With the Correspondence Theorem at hand, it is now relatively easy to prove the Decomposition Theorem 1.1.

**Theorem 5.1.** Let \(G\) be a 3-connected graph. Then \(G\) has a tree decomposition \((T, \beta)\) of adhesion at most 3 such that for all \(t ∈ T\), the torso \(G[\beta(t)]\) is either is a complete graph \(K_3\) or \(K_4\) or a quasi-4-connected component of \(G\).

Furthermore, such a decomposition can be computed in time \(O(n^2(n + m))\).

Here, and throughout this section, we denote the numbers of vertices and edges of the input graph \(G\) of our algorithms by \(n\) and \(m\), respectively.

The Decomposition Theorem 1.1 follows by combining the decomposition of Theorem 5.1 with the standard decomposition of a graph into its three connected components.
The proof of Theorem 5.1 requires some preparation. For the rest of this section, we assume that $G$ is a 3-connected graph. Let $(Y, S, Z) \in \text{Sep}_{=3}(G)$ be non-degenerate. A split vertex of $(Y, S, Z)$ is a vertex $z \in Z$ such that for every connected component $C$ of $G \setminus (S \cup \{z\})$ it holds that $|N(C)| = 3$.

**Lemma 5.2.** Let $(Y_0, S_0, Z_0) \in \text{Sep}_{=3}(G)$ be a non-degenerate proper separation such that $Z_0$ is connected and $(Y_0, S_0, Z_0)$ has no split vertex. Then the set $T(Y_0, S_0, Z_0)$ of all separation $(Y, S, Z) \in \text{Sep}_{<4}(G)$ such that either $Z_0 \subseteq Z$ or $|Z \cap S_0| > |Y \cap S_0|$ is a $G$-tangle of order 4.

**Proof.** Let $T := T(Y_0, S_0, Z_0)$. To see that $T$ satisfies (T.1), let $(Y, S, Z) \in \text{Sep}_{<4}(G)$. If $S \subseteq Y_0 \cup S_0$, then the connected set $Z_0$ is either a subset of $Z$ or of $Y$, and thus either $(Y, S, Z) \in T$ or $(Z, S, Y) \in T$. Suppose next that $|S \cap Z_0| = 1$. Let $z$ be the unique vertex in $S \cap Z_0$. Then $z$ is not a split vertex of $(Y_0, S_0, Z_0)$, and hence there is a connected component $C$ of $G \setminus (S_0 \cup \{z\})$ such that $N(C) = S_0 \cup \{z\}$. Then $V(C) \subseteq Z_0$, because $z \in Z_0$, and thus $V(C) \cap S = \emptyset$. It follows that either $V(C) \subseteq Y$ or $V(C) \subseteq Z$. Without loss of generality we may assume that $V(C) \subseteq Z$. As $S_0 \subseteq N(C)$, this implies $S_0 \setminus S \subseteq Z$. As $S_0 \setminus S \neq \emptyset$, it follows that $(Y, S, Z) \in T$. Finally, suppose that $|S \cap Z_0| \geq 2$. If $S \cap S_0 = \emptyset$, then either $|Z \cap S_0| \geq 2$ or $|Y \cap S_0| \geq 2$, and thus either $(Y, S, Z) \in T$ or $(Z, S, Y) \in T$. If $|S \cap S_0| = 1$, then $S \cap Y_0 = \emptyset$, and as $G$ is 3-connected and $Y_0 \neq \emptyset$, the vertices in $S_0 \setminus S$ belong to the same connected component of $G \setminus S$. Hence either both are in $Z$ or both are in $Y$, and again it follows that either $(Y, S, Z) \in T$ or $(Z, S, Y) \in T$.

Observe next $|V(G)| \geq 6$, because $|Y_0| \geq 1$ and $|S_0| = 3$ and $|Z_0| \geq 2$ (otherwise the unique vertex in $Z_0$ would be a split vertex).

**Claim 1.** For all $(Y, S, Z) \in T$ we have $|S \cup Z| \geq 4$.

**Proof.** It follows from the definition of $T$ that $Z \neq \emptyset$. If $Y = \emptyset$, then $|S \cup Z| = |V(G)| \geq 6$. Otherwise, $(Y, S, Z)$ is a proper separation and thus $|S| = 3$, which implies $|S \cup Z| \geq 4$.

The claim implies that $T$ satisfies (T.3).

To prove that $T$ satisfies (T.2), let $(Y_i, S_i, Z_i) \in T$ for $i = 1, 2, 3$. Suppose for contradiction $Z_1 \cap Z_2 \cap Z_3 = \emptyset$ and that there is no edge that has an endvertex in each $Z_i$. Once more we recycle Claim 1 of Lemma 4.2.

**Claim 2.** For distinct $i, j, k \in [3]$ and $x \in V(G)$, if $x \in Z_i \cap Z_j$ then $x \in Y_k$.

**Case 1:** There is an $i \in [3]$ such that $S_i \subseteq Y_0 \cup S_0$.

Without loss of generality, we may assume that $i = 1$ and $(Y_1, S_1, Z_1) = (Y_0, S_0, Z_0)$. We may further assume that $S_i \not\subseteq Y_0 \cup S_0$ for $i = 2, 3$. Then $|Z_i \cap S_0| > |Y_i \cap S_0|$.

By Claim 1 we have $Z_2 \cap Z_3 \cap S_0 = Z_2 \cap Z_3 \cap S_1 = \emptyset$. Thus for some $i \in \{2, 3\}$ $|Z_i \cap S_0| < 2$. Without loss of generality we assume $|Z_2 \cap S_0| < 2$. Then $|Z_2 \cap S_0| = \emptyset$ and thus $|S_2 \cap S_0| = 2$. Since $S_2 \not\subseteq Y_0 \cup S_0$, we have $|S_2 \cap Z_0| = 1$. As the vertex in $S_2 \setminus Z_0$ is not a split vertex, there is a connected component $C$ of $G \setminus (S_0 \cup S_2)$ such that $N(C) = S_0 \cup S_2$. Then $V(C) \subseteq Z_0 \cap Z_2 = Z_1 \cap Z_2$. Now let $v \in Z_3 \cap S_0$, and let $w \in V(C)$ be adjacent to $v$. Then the edge $vw$ has an endvertex in each $Z_i$.

**Case 2:** $|Z_i \cap S_0| \neq \emptyset$ for all $i \in [3]$.

Then $|Z_i \cap S_0| > |Y_i \cap S_0|$. If $|Z_i \cap Z_j \cap S_0| = \emptyset$ for all $i \neq j$, then $|Z_i \cap S_0| = 1$ and thus $|Y_i \cap S_0| = 0$ for all $i$. Thus $|S_i \cap S_0| = 2$ and $|S_i \cap Y_0| = 0$, because $S_i \not\subseteq S_0 \cup Y_0$. But this implies $Y_0 \subseteq Z_1 \cap Z_2 \cap Z_3$, which is a contradiction.
Lemma 5.3 ([13]). Let $s \in Z_1 \cap Z_2 \cap S_0$. Then by Claim 1, $s \in Y_3$. Then $|Y_3 \cap S_0| \geq 1$, and this implies $|Z_3 \cap S_0| \geq 2$. Let $s', s'' \in Z_3 \cap S_0$. Then $S_0 = \{s, s', s''\}$. If $|S_3 \cap Z_0| \leq 1$, there is a connected component $C$ of $G \setminus (S_0 \cup S_3)$ such that $N(C) = S_0 \cup S_3$. But then there is a path from $s \in Y_3$ to $s' \in Z_3$ in $G \setminus S_3$, which is impossible. Hence $|S_3 \cap Z_0| \geq 2$.

Thus $|S_3 \cap Y_0| \leq 1$. Since the graph $G[Y_0 \cup S_0]$ is connected, we have $|Y_0 \cap S_3| = 1$, and the unique vertex $y \in Y_0 \cap S_3$ separates $s$ from $\{s', s''\}$. Then $ss', ss'' \notin E(G)$. Furthermore, $sy \in E(G)$ and $y$ is the only neighbour of $s$ in $Y_0 \cup S_0$, because otherwise $\{y, s\}$ would be separator of $G$. By Claim 1, $y \notin Z_1 \cap Z_2$. Say, $y \notin Z_2$, then $y \in Z_2$, because $y$ is adjacent to $s \in Z_2$. As $S_2 \subseteq Y_0 \cup S_0$, it now follows that $s'$ and $s''$ are not both in $S_2$. As $|Z_2 \cap S_0| > |Y_2 \cap S_2|$, one of these vertices, say, $s'$ is in $Z_2$.

By Claim 1, $s' \in Z_2 \cap Z_3$ implies $s' \in Y_1$. Arguing as above with $(Y_1, S_1, Z_1)$ instead of $(Y_3, S_3, Z_3)$, we see that $Z_1 \cap S_0 = \{s, s''\}$ and $|S_1 \cap Z_0| = 2$ and $|S_1 \cap Y_0| = 1$, and the unique vertex $y' \in S_1 \cap Y_0$ separates $s'$ from $s''$ in $G$. Furthermore, $ss', ss'' \notin E(G)$, and $y' \in E(G)$ and $y'$ is the only neighbour of $s'$ in $Y_0 \cup S_0$.

Now we have $s'' \in Z_1 \cap Z_3$, and again by the same argument we see that $s'' \in Y_2$ and $Z_2 \cap S_0 = \{s, s'\}$ and $|S_2 \cap Z_0| = 2$ and $|S_2 \cap Y_0| = 1$ and the unique vertex $y'' \in S_1 \cap Y_0$ separates $s'$ from $s''$ in $G$. Furthermore, $ss', ss'' \notin E(G)$, and $y' \in E(G)$ and $y'' \in E(G)$ and $y''$ is the only neighbour of $s''$ in $Y_0 \cup S_0$.

Let us rename the vertices $s, s', s''$ to $s_{12}, s_{23}, s_{13}$ and the vertices $y', y''$ to $y_{12}, y_{23}, y_{13}$. Then for distinct $i,j,k$ we have $s_{ij} \in S_0 \cap Z_i \cap Z_j \cap Y_k$ and $S_k \cap Y_0 = \{y_{ij}\}$ and $N(s_{ij}) \cap (Y_0 \cup S_0) = \{y_{ij}\}$. Note that this implies that $S_0 = \{s_{12}, s_{13}, s_{23}\}$ is an independent set. It follows that $Y_0 \setminus \{y_{ij}\} \subseteq Z_k$, \hspace{1cm} (5.A)

because all $y \in Y_0 \setminus \{y_{ij}\}$ are reachable in $G \setminus \{y_{ij}\}$ by a path from $s_{ik}, s_{jk} \subseteq Z_k$.

As the separation $(Y_0, S_0, Z_0)$ is non-degenerate and $S_0$ is an independent set, we have $|Y_0| > 1$. Since $N(S_0) = \{y_{12}, y_{23}, y_{13}\}$ and $N(Y_0) \cap S_0 = \{s_{ij}\}$ and $G$ is 3-connected, it is easy to see that this implies that the vertices $y_{ij}$ are mutually distinct. Now let $e = vw$ be an arbitrary edge of $G[Y_0]$. Such an edge exists, and it follows from (5.A) that the edge has an endvertex in each $Z_k$. Again, this is a contradiction. \hfill \Box

Let $W, X \subseteq V(G)$. Then a $(W, X)$-separation is a vertex separation $(Y, S, Z)$ such that $W \subseteq Y \cup S$ and $X \subseteq Z \cup S$. A $(W, X)$-separation $(Y, S, Z)$ is minimum if its order is minimal, that is, there is no $(W, X)$-separation $(Y', S', Z')$ such that $|S'| < |S|$. It is leftmost minimum if it is minimum and, subject to this condition, $Y$ is inclusionwise minimal.

The following lemma follows from Lemma 2.4 of [13].

**Lemma 5.3 ([13]).** There is an algorithm that, given a graph $G$ and sets $W, X \subseteq V(G)$, computes a leftmost minimum $(W, X)$-separation in time $O(k(n + m))$, where $n := |G|$, $m := |E(G)|$, and $k$ is the order of a minimum $(X, Y)$-separation.

A $(W, X)$-separation $(Y, S, Z)$ is proper if $W \cap Y \neq \emptyset$ and $X \cap Z \neq \emptyset$. Note that there is a proper $(W, X)$-separation if and only if there is a $w \in W \setminus X$ and an $x \in X \setminus W$ such that $wx \notin E(G)$. A (leftmost) minimum proper $(W, X)$-separation is a proper $(W, X)$-separation that is (leftmost) minimum among all proper $(W, X)$-separations. While there always is a unique leftmost minimum $(W, X)$-separation, as can be a proved by
a straightforward submodularity argument, there is not necessarily a unique leftmost minimum proper \((W, X)\)-separation. However, the proof of the following lemma shows that there are at most \(k^2\) leftmost minimum proper \((W, X)\)-separations, where \(k\) is the order of a leftmost minimum \((W, X)\)-separation.

**Lemma 5.4.** Let \(k \geq 1\). Then there is a linear time algorithm that, given a graph \(G\) and sets \(W, X \subseteq V(G)\), decides if there is a proper \((W, X)\)-separation of order at most \(k\), and if there is computes the set of all leftmost minimum proper \((W, X)\)-separations.

Note that we treat \(k\) as a constant here. In fact, we will only apply the lemma for \(k = 3\).

**Proof.** Let \(G\) be a graph and \(W, X \subseteq V(G)\). Let us first assume that \(|W|, |X| \leq k\). For a vertex \(v \in V(G)\), we let \(G_v\) be the graph obtained from \(G\) by replacing \(v\) by fresh vertices \(v_1, \ldots, v_{k+1}\) and adding edges from \(v_i\) to \(v_j\) for all \(i \neq j\) and from \(v_i\) to \(w\) for all \(i\) and all \(w \in N_G(v)\).

Now let \(w \in W \setminus X\) and \(x \in X \setminus W\) such that \(w \neq x\), and consider the graph \(G_{w, x} := (G_w)_x\). Let \(W_w := (W \setminus \{w\}) \cup \{w_1, \ldots, w_{k+1}\}\) and \(X_x := (X \setminus \{x\}) \cup \{x_1, \ldots, x_{k+1}\}\). Observe that if \((Y, S, Z)\) is a minimum \((W_w, X_x)\)-separation in \(G_{w, x}\) of order \(|S| \leq k\), then \(\{w_1, \ldots, w_{k+1}\} \subseteq Y\) and \(\{x_1, \ldots, x_{k+1}\} \subseteq Z\). Thus \((Y, S, Z)\) “projects” to a proper \((W, X)\)-separation

\[
P(Y, S, Z) := \left((Y \setminus \{w_1, \ldots, w_{k+1}\}) \cup \{w\}, S, (Z \setminus \{x_1, \ldots, x_{k+1}\}) \cup \{x\}\right)
\]

of \(G\). Moreover, if \((Y, S, Z)\) is leftmost minimum, then \(P(Y, S, Z)\) is leftmost minimum among all \((W, X)\)-separations \((Y', S', Z')\) with \(w \in Y'\) and \(x \in Z'\).

Now we let \(P\) be the set of all \(P(Y, S, Z)\), where \((Y, S, Z)\) is a leftmost minimum \((W_w, X_x)\)-separation in \(G_{w, x}\) for some \(w \in W, x \in X\) with \(w \neq x\). All separations in \(P\) are proper \((W, X)\)-separations, and provided there is a proper \((W, X)\)-separation of order at most \(k\), all leftmost minimum proper \((W, X)\)-separations are in the set \(P\). In fact, the leftmost minimum proper \((W, X)\)-separations are precisely the \((Y, S, Z) \in P\) with minimum \(|S|\) and, subject to this, inclusionwise minimal \(Y\).

By Lemma 5.3 and the assumption \(|W|, |X| \leq k\), the set \(P\) can be computed in linear time, and then we can filter out those separations that are actually leftmost minimum.

It remains to deal with the case that \(|W| > k\) or \(|X| > k\). If both \(|W| > k\) and \(|X| > k\), every \((W, X)\)-separation of order at most \(k\) is proper. Thus the assertion of the lemma follows directly from Lemma 5.3. If \(|W| \leq k\) and \(|X| > k\), we consider \((W_w, X_x)\)-separations in the graph \(G_w\), for all \(w \in W\), and if \(|W| > k\) and \(|X| \leq k\), we consider \((W, X_x)\)-separations in the graph \(G_x\), for all \(x \in X\).

Let us say that a separation \((Y_0, S_0, Z_0) \in \operatorname{Sep}_{=3}(G)\) defines a tangle if \((Y_0, S_0, Z_0)\) is non-degenerate and \(Z_0\) is connected in \(G\) and \((Y_0, S_0, Z_0)\) has no split vertex. Then the **tangle defined by** \((Y_0, S_0, Z_0)\) is \(\mathcal{T}(Y_0, S_0, Z_0)\) (of Lemma 5.2).

**Lemma 5.5.** There is an algorithm that, given a 3-connected graph \(G\) and a separation \((Y_0, S_0, Z_0)\) of \(G\) of order 3 defining the tangle \(\mathcal{T} = \mathcal{T}(Y_0, S_0, Z_0)\), computes the set \(\mathcal{T}_{\text{nd}}\) and the set of all non-degenerate crossedges of \(\mathcal{T}\) in time \(O(n^2 + m^4)\).

**Proof.** We show how to compute the set \(\mathcal{T}_{\text{min}}\); then we can easily filter out the non-degenerate separations in \(\mathcal{T}_{\text{min}}\) to obtain \(\mathcal{T}_{\text{nd}}\).

Let \(x \in Z_0\). Observe that if \((Z, S, Y)\) is a proper \((S_0, \{x\})\)-separation of order at most 3, then \((Y, S, Z) \in \mathcal{T}\). This follows immediately from the definition of \(\mathcal{T}\). It implies the following equivalence for every separation \((Y, S, Z)\) of \(G\) of order 3.

(i) \((Y, S, Z) \in \mathcal{T}_{\text{min}}\) and \((Y, S, Z)\) does not cross \((Y_0, S_0, Z_0)\).
(ii) There is an $x \in Z_0$ such that $(Z, S, Y)$ is a leftmost minimum proper $(S_0, \{x\})$-separation.

We can use this equivalence to compute the set of all $(Y, S, Z) \in T_{\text{min}}$ such that $(Y, S, Z)$ does not cross $(Y_0, S_0, Z_0)$ (repeatedly applying the algorithm of Lemma 5.4 to all $x \in Z_0$). Note that the equivalence also gives us a linear bound on the number of such $(Y, S, Z)$.

It remains to deal with the $(Y, S, Z) \in T_{\text{min}}$ crossing $(Y_0, S_0, Z_0)$. For each $s \in S_0$ that has a unique neighbour $y \in Y_0 \cup S_0$, the edge $sy$ may be a crossedge. This gives us at most three potential crossedges, and we deal with them separately. So let $s \in S$ and $y \in Y_0$ such that $N(s) \cap (Y_0 \cap S_0) = \{y\}$. Then for every separation $(Y, S, Z) \in \text{Sep}_{\equiv3}(G)$ the following are equivalent.

(iii) $y \in S$ and $(Z \cap (S_0 \cup Z_0), S \cap (S_0 \cup Z_0), Y \cap (S_0 \cup Z_0))$ is a leftmost minimum proper $(S \setminus \{s\}, \{s\})$-separation in the graph $G[S_0 \cup Z_0]$.

(iv) $(Y, S, Z) \in T_{\text{min}}$ and $(Y, S, Z)$ crosses $(Y_0, S_0, Z_0)$ with crossedge $ys$.

To see this, note that (iii) implies that $|S \cap Z_0| = 2$, because $(Y_0, S_0, Z_0)$ has no split vertex. The equivalence between (iii) and (iv) allows us to compute the remaining separations in $T_{\text{min}}$.

As we have an overall linear bound on the number of separations in $T_{\text{min}}$ and hence in $T_{\text{nd}}$, we can easily compute the set of non-degenerate crossedges.

Let us call a 3-separator $S$ of $G$ degenerate if there is a connected component $C$ of $G \setminus S$ such that the separation $(G \setminus (S \cup V(C)), S, V(C))$ is degenerate. It is easy to see that this is the case if and only if $S$ is an independent set and $G \setminus S$ has exactly two connected components, one of which has order 1.

**Lemma 5.6.** There is an a algorithm that, given a 3-connected graph $G$, decides if $G$ has a non-degenerate 3-separator and computes one if there is in time $O(n^2(n + m))$.

**Proof.** We first test if there is an $S \subseteq V(G)$ such that $|S| = 3$ and all connected components of $G \setminus S$ have order 1. In this case, $S$ is a non-degenerate 3-separator if $|G| \geq 6$ or if $|G| \geq 5$ and $S$ is not an independent set.

In the following, we assume that for every $S \subseteq V(G)$ such that $|S| = 3$ there is at least one connected components $C$ of $G \setminus S$ such that $|C| \geq 2$. Now suppose that $S$ is a non-degenerate 3-separator of $G$. Let $Y$ be the vertex set of a connected component of $G$ of size $|Y| \geq 2$, and let $Z := V(G) \setminus (S \cup Y)$. Let $y \in Y$ and $z \in Z$.

Then there is a leftmost minimum proper $(\{y\}, \{z\})$-separation $(Y', S', Z')$ with $Y' \cup S' \subseteq Y \cup S$, because $(Y, S, Z)$ is a minimum proper $(\{y\}, \{z\})$-separation. The separator $S'$ is non-degenerate unless $S'$ is an independent set and $S' = N(y)$. However, in this case there is a leftmost minimum proper $(S', \{z\})$-separation $(Y'', S'', Z'')$ such that $S''$ is non-degenerate. To see this, let $y' \in N(y) \cap Y$. Then there is a leftmost minimum proper $(S', \{z\})$-separation $(Y'', S'', Z'')$ with $y, y' \in Y''$ and $Y'' \cup S'' \subseteq Y \cup S$, because $(Y, S, Z)$ is a minimum proper $(S', \{z\})$-separation with $y, y' \in Y$. The set $S''$ is a non-degenerate 3-separator.

Thus we can find a non-degenerate 3-separator as follows. For all pairs $y, z$ of distinct vertices, we compute all leftmost minimum proper $(\{y\}, \{z\})$-separations $(Y', S', Z')$ and check if there is one such that $S'$ is a non-degenerate 3-separator. If $y$ has degree 3 and $S' := N(y)$ is an independent set, we also compute all leftmost minimum proper $(S', \{z\})$-separations $(Y'', S'', Z'')$ and check if $S''$ is a non-degenerate 3-separator.

**Proof of Theorem 5.7.** If $G$ has no non-degenerate 3-separator, then $G$ is quasi-4-connected, and we return the trivial tree decomposition with a one-node tree. In the following, we assume that $G$ has at least one non-degenerate 3-separator.
Remark 5.7. There is one minor issue that we ignored so far in order to not complicate things unnecessarily. It may happen that some quasi-4-connected components $G[R_T]$ of $G$ do not appear as torsors $G[\beta(t)]$ in the decomposition, because if $G[R_T]$ is a subgraph of $\mathcal{TH}_{4+}$ (see Figure 4.4), it will be treated in Case 2 of the construction: the vertex $v_t$ is a split vertex with respect to the separation $\{v_1, v_2, v_3\}$, and the vertices $w_i$ will be split off. But we can easily detect this during the construction and avoid to split off those vertices if we want to.

If we carry out the construction exactly as in the proof of the theorem, then the $G$-tangles of order 4 are associated with all nodes $t$ such that...
• either $|\beta(t)| \geq 5$
• or $|\beta(t)| = 4$ and for each subset $S \subseteq \beta(t)$ of size $|S| = 3$ there is a neighbour $u$ of $t$ such that $\beta(u) \cap \beta(t) = S$.

In the second case, the neighbours of $t$ allow us to find a non-exceptional extension of the quasi-4-connected region $\beta(t)$.

6 Conclusions

Relaxing 4-connectedness, we introduce the notion of quasi-4-connectedness of graphs and prove that every graph has a decomposition into quasi-4-connected components. We show that these quasi-4-connected component correspond to the tangles of order 4, putting our result in the context of recent work on tangles and decompositions [1, 2, 3, 8, 10, 17]. Furthermore, we prove that our decomposition can be computed in cubic time. I think that our decomposition generalises the decomposition of a graph into its 3-connected components in a natural way and as such is a fundamental and interesting result in structural graph theory. Although we did not explore this in the present paper, I also believe that the result may turn out to be a useful algorithmic tool, just like the decomposition into 3-connected components (though maybe not quite as broadly applicable).

The most obvious question is whether our result has a generalisation to “quasi-$k$-connected components”, whatever they may be, for $k \geq 5$. I am skeptical, because we exploit many special properties of separators of order 3 here, most importantly the limited way in which they can cross. However, our decomposition is not a straightforward generalisation of the decomposition into 3-connected components either, and arguably, our main contributions are the conceptual ideas related to quasi-4-connectedness. It may well be that new conceptual ideas lead to perfectly nice decompositions of higher order.

Finally, in particularly when thinking of applications, it would be desirable to have a decomposition algorithm working in quadratic or even in linear time. I see no fundamental obstructions to the existence of such an algorithm.

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Appendix

In this appendix, we discuss how our notion of tangles based on separations of the vertex set relates to Robertson and Seymour’s original definition of tangles [17] based on partitions of the edge set. Let us say that an RS-separation of a graph $G$ is a pair $(A, B)$ of subgraphs of $G$ such that $A \cup B = G$ and $E(A) \cap E(B) = \emptyset$. The order of the RS-separation $(A, B)$ is $\text{ord}(A, B) := |V(A) \cap V(B)|$.

Robertson and Seymour define a $G$-tangle of order $k$ to be a family $T$ of RS-separations of $G$ of order less than $k$ satisfying the following conditions.

$(T'.1)$ For all RS-separations $(A, B)$ of $G$ of order less than $k$, either $(A, B) \in T$ or $(B, A) \in T$.

$(T'.2)$ If $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in T$ then $A_1 \cup A_2 \cup A_3 \neq G$.

$(T'.3)$ $V(A) \neq V(G)$ for all $(A, B) \in T$.

Let us call such tangles RS-tangles in the following.

With every RS-separation $(A, B)$ we associate the separation $\langle A, B \rangle = (Y, S, Z)$ with $Y := V(A) \setminus V(B)$, $S := V(A) \cap V(B)$, and $Z := V(B) \setminus V(A)$.

**Proposition 1.** Let $G$ be a graph.

1. Let $T$ be a $G$-tangle of order $k$. Then
   $$T' := \{(A, B) \mid \text{RS-separation of } G \text{ of order } < k \text{ with } \langle A, B \rangle \in T\}$$
   is an RS-tangle of $G$ of order $k$.

2. Let $T'$ be a an RS-tangle of $G$ of order $k$. Then
   $$T := \{(A, B) \mid (A, B) \in T'\}$$
   is a $G$-tangle of order $k$.

We omit the straightforward proof. the proposition shows that our version of tangle and Robertson and Seymour’s original version are essentially the same, and it also shows how to translate our results to the original framework.