Euclidean formulation of general relativity

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A variational principle is applied to 4D Euclidean space provided with a tensor refractive index, defining what can be seen as 4-dimensional optics (4DO). The geometry of such space is analysed, making no physical assumptions of any kind. However, by assigning geometric entities to physical quantities the paper allows physical predictions to be made. A mechanism is proposed for translation between 4DO and GR, which involves the null subspace of 5D space with signature (−++ + +).

A tensor equation relating the refractive index to sources is established geometrically and the sources tensor is shown to have close relationship to the stress tensor of GR. This equation is solved for the special case of zero sources but the solution that is found is only applicable to Newton mechanics and is inadequate for such predictions as light bending and perihelion advance. It is then argued that testing gravity in the physical world involves the use of a test charge which is itself a source. Solving the new equation, with consideration of the test particle’s inertial mass, produces an exponential refractive index where the Newtonian potential appears in exponent and provides accurate predictions. Resorting to hyperspherical coordinates it becomes possible to show that the Universe’s expansion has a purely geometric explanation without appeal to dark matter.

1 Introduction

According to general consensus any physics theory is based on a set of principles upon which predictions are made using established mathematical derivations; the validity of such theory depends on agreement between predictions and observed physical reality. In that sense this paper does not formulate a physical theory because it does not presume any physical principles; for instance it does not assume speed of light constancy or equivalence between frame acceleration and gravity. This is a paper about geometry. All along the paper, in several occasions, a parallel is made with the physical world by assigning a physical meaning to geometric entities and this allows predictions to be made. However the validity of derivations and overall consistency of the exposition is independent of prediction correctness.

The only postulates in this paper are of a geometrical nature and can be condensed in the definition of the space we are going to work with: 4-dimensional space with Euclidean signature (+++ ). For the sole purpose of making transitions to spacetime we will also consider the null subspace of the 5-dimensional space with signature (−++ ++). This choice of space does not imply any assumption about its physical meaning up to the point where geometric
Table 1: Normalising factors for non-dimensional units used in the text; \( h \rightarrow \) Planck constant divided by \( 2\pi \), \( G \rightarrow \) gravitational constant, \( c \rightarrow \) speed of light and \( e \rightarrow \) proton charge.

| Length        | Time          | Mass          | Charge |
|---------------|---------------|---------------|--------|
| \( \sqrt{\frac{Gh}{c^3}} \) | \( \sqrt{\frac{Gh}{c^5}} \) | \( \frac{hc}{G} \) | \( e \) |

entities like coordinates and geodesics start being assigned to physical quantities like distances and trajectories. Some of those assignments will be made very early in the exposition and will be kept consistently until the end in order to allow the reader some assessment of the proposed geometric model as a tool for the prediction of physical phenomena. Mapping between geometry and physics is facilitated if one chooses to work always with non-dimensional quantities; this is easily done with a suitable choice for standards of the fundamental units. In this work all problems of dimensional homogeneity are avoided through the use of normalising factors for all units, listed in Table 1 defined with recourse to the fundamental constants: Planck constant, gravitational constant, speed of light and proton charge. This normalisation defines a system of non-dimensional units with important consequences, namely: 1) all the fundamental constants, \( h, G, c, e \), become unity; 2) a particle’s Compton frequency, defined by \( \nu = mc^2/h \), becomes equal to the particle’s mass; 3) the frequent term \( GM/(c^2r) \) is simplified to \( M/r \).

The particular space we chose to work with can have amazing structure, providing countless parallels to the physical world; this paper is just a limited introductory look at such structure and parallels. The exposition makes full use of an extraordinary and little known mathematical tool called geometric algebra (GA), a.k.a. Clifford algebra, which received an important thrust with the introduction of geometric calculus by David Hestenes [1]. A good introduction to GA can be found in Gull et al. [2] and the following paragraphs use basically the notation and conventions therein. A complete course on physical applications of GA can be downloaded from the internet [3] with a more comprehensive version published recently in book form [4] while an accessible presentation of mechanics in GA formalism is provided by Hestenes [5].

2 Introduction to geometric algebra

We will use Greek characters for the indices that span 1 to 4 and Latin characters for those that exclude the 4 value; in rare cases we will have to use indices spanning 0 to 3 and these will be denoted with Greek characters with an over bar. The geometric algebra of the hyperbolic 5-dimensional space we want to consider \( \mathbb{R}_{4,1} \) is generated by the frame of orthonormal vectors \( \{i, \sigma_\mu\}, \mu = 1 \ldots 4 \), verifying the relations

\[
i^2 = -1, \quad i\sigma_\mu + \sigma_\mu i = 0, \quad \sigma_\mu \sigma_\nu + \sigma_\nu \sigma_\mu = 2\delta_{\mu\nu}.
\]  

We will simplify the notation for basis vector products using multiple indices, i.e. \( \sigma_\mu \sigma_\nu \equiv \sigma_{\mu\nu} \). The algebra is 32-dimensional and is spanned by the basis

\[
\begin{align*}
1, \quad \{i, \sigma_\mu\}, \quad \{i\sigma_\mu, \sigma_\nu\}, \quad \{i\sigma_\mu, \sigma_\mu \sigma_\lambda\}, \quad \{iI, \sigma_\mu I\}, \\
1 \text{ scalar} & \quad 5 \text{ vectors} & \quad 10 \text{ bivectors} & \quad 10 \text{ trivectors} & \quad 5 \text{ tetravectors} & \quad 1 \text{ pentavector}
\end{align*}
\]  

where \( I \equiv i\sigma_1 \sigma_2 \sigma_3 \sigma_4 \) is also called the pseudoscalar unit. Several elements of this basis square to unity:

\[
(\sigma_\mu)^2 = 1, \quad (i\sigma_\mu)^2 = 1, \quad (i\sigma_{\mu\nu})^2 = 1, \quad (iI)^2 = 1;
\]
and the remaining square to $-1$:

$$i^2 = -1, \quad (\sigma_{\mu\nu})^2 = -1, \quad (\sigma_{\mu\nu\lambda})^2 = -1, \quad (\sigma_{\mu} I)^2, \quad I^2 = -1. \quad (4)$$

Note that the symbol $i$ is used here to represent a vector with norm $-1$ and must not be confused with the scalar imaginary, which we don’t usually need.

The geometric product of any two vectors $a = a^0i + a^\mu\sigma_{\mu}$ and $b = b^0i + b^\nu\sigma_{\nu}$ can be decomposed into a symmetric part, a scalar called the inner product, and an anti-symmetric part, a bivector called the exterior product.

$$ab = a \cdot b + a \land b, \quad ba = a \cdot b - a \land b. \quad (5)$$

Reversing the definition one can write internal and exterior products as

$$a \cdot b = \frac{1}{2}(ab + ba), \quad a \land b = \frac{1}{2}(ab - ba). \quad (6)$$

When a vector is operated with a multivector the inner product reduces the grade of each element by one unit and the outer product increases the grade by one. There are two exceptions; when operated with a scalar the inner product does not produce grade $-1$ but grade $1$ instead, and the outer product with a pseudoscalar is disallowed.

### 3 Displacement and velocity

Any displacement in the 5-dimensional hyperbolic space can be defined by the displacement vector

$$ds = idx^0 + \sigma_{\mu}dx^\mu; \quad (7)$$

and the null space condition implies that $ds$ has zero length

$$ds^2 = ds \cdot ds = 0; \quad (8)$$

which is easily seen equivalent to either of the relations

$$(dx^0)^2 = \sum (dx^\mu)^2; \quad (dx^4)^2 = (dx^0)^2 - \sum (dx^\mu)^2. \quad (9)$$

These equations define the metrics of two alternative spaces, one Euclidean the other one Minkowskian, both equivalent to the null 5-dimensional subspace.

A path on null space does not have any affine parameter but we can use Eqs. (9) to express 4 coordinates in terms of the fifth one. We will frequently use the letter $t$ to refer to coordinate $x^0$ and the letter $\tau$ for coordinate $x^4$; total derivatives with respect to $t$ will be denoted by an over dot while total derivatives with respect to $\tau$ will be denoted by a “check”, as in $\check{F}$. Dividing both members of Eq. (7) by $dt$ we get

$$\dot{s} = i + \sigma_{\mu}\dot{x}^\mu = i + v. \quad (10)$$

This is the definition for the velocity vector $v$; it is important to stress again that the velocity vector defined here is a geometrical entity which bears for the moment no relation to physical velocity, be it relativistic or not. The velocity has unit norm because $\dot{s}^2 = 0$; evaluation of $v \cdot v$ yields the relation

$$v \cdot v = \sum (\dot{x}^\mu)^2 = 1. \quad (11)$$
The velocity vector can be obtained by a suitable rotation of any of the $\sigma_\mu$ frame vectors, in particular it can always be expressed as a rotation of the $\sigma_4$ vector.

At this point we are going to make a small detour for the first parallel with physics. In the previous equation we replace $x^0$ by the greek letter $\tau$ and rewrite with $\dot{\tau}^2$ in the first member

$$\dot{\tau}^2 = 1 - \sum (\dot{x}^j)^2.$$  \hfill (12)

The relation above is well known in special relativity, see for instance Martin [6]; see also Almeida [7], Montanus [8] for parallels between special relativity and its Euclidean space counterpart.\footnote{Montanus first proposed the Euclidean alternative to relativity in 1991, nine years before the author started independent work along the same lines.} We note that the operation performed between Eqs. (11) and (12) is a perfectly legitimate algebraic operation since all the elements involved are pure numbers. Obviously we could also divide both members of Eq. (7) by $d\tau$, which is then associated with relativistic proper time;

$$\dot{s} = i\dot{x}^0 + \sigma_j \dot{x}^j + \sigma_4.$$  \hfill (13)

Squaring the second member and noting that it must be null we obtain $(\dot{x}^0)^2 - \sum (\dot{x}^j)^2 = 1$. This means that we can relate the vector $i\dot{x}^0 + \sigma_j \dot{x}^j$ to relativistic 4-velocity, although the norm of this vector is symmetric to what is usual in SR. The relativistic 4-velocity is more conveniently assigned to the 5D bivector $i\sigma_4 \dot{x}^0 + \sigma_j \dot{x}^j$, which has the necessary properties. The method we have used to make the transition between 4D Euclidean space and hyperbolic spacetime involved the transformation of a 5D vector into scalar plus bivector through product with $\sigma_4$; this method will later be extended to curved spaces.

Equation (10) applies to flat space but can be generalised for curved space; we do this in two steps. First of all we can include a scale factor ($v = n\sigma_\mu \dot{x}^\mu$), which can change from point to point

$$\dot{s} = i + n\sigma_\mu \dot{x}^\mu.$$  \hfill (14)

In this way we are introducing the 4-dimensional analogue of a refractive index, that can be seen as a generalisation of the 3-dimensional definition of refractive index for an optical medium: the quotient between the speed of light in vacuum and the speed of light in that medium. The scale factor $n$ used here relates the norm of vector $\sigma_\mu \dot{x}^\mu$ to unity and so it deserves the designation of 4-dimensional refractive index; we will drop the “4-dimensional” qualification because the confusion with the 3-dimensional case can always be resolved easily. The material presented in this paper is, in many respects, a logical generalisation of optics to 4-dimensional space; so, even if the paper is only about geometry, it becomes natural to designate this study as 4-dimensional optics (4DO).

Full generalisation of Eq. (10) implies the consideration of a tensor refractive index, similar to the non-isotropic refractive index of optical media

$$\dot{s} = i + n^\mu_{\nu} \dot{x}^\nu \sigma_\mu;$$  \hfill (15)

the velocity is then generally defined by $v = n^\mu_{\nu} \dot{x}^\nu \sigma_\mu$. The same expression can be used with any orthonormal frame, including for instance spherical coordinates, but for the moment we will restrict our attention to those cases where the frame does not rotate in a displacement; this poses no restriction on the problems to be addressed but is obviously inconvenient when symmetries are involved. Equation (15) can be written with the velocity in the form $v = g_\nu \dot{x}^\nu$ if we define the refractive index vectors

$$g_\nu = n^\mu_{\nu} \sigma_\mu.$$  \hfill (16)
The set of four \( g_\mu \) vectors will be designated the \textit{refractive index frame}. Obviously the velocity is still a unitary vector and we can express this fact evaluating the internal product with itself and noting that the second member in Eq. (15) has zero norm.

\[
v \cdot v = n^\alpha _\mu x^\alpha \delta_\mu_\beta = 1.
\]  

Using Eq. (16) we can rewrite the equation above as \( g_\mu \cdot g_\nu \dot{x}^\mu \dot{x}^\nu = 1 \) and denoting by \( g_{\mu\nu} \) the scalar \( g_\mu \cdot g_\nu \), the equation becomes

\[
g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 1.
\]  

The generalised form of the displacement vector arises from multiplying Eq. (15) by \( dt \), using the definition (16) \( ds = idt + g_\mu dx^\mu \).

\[
(dt)^2 = g_{\mu\nu} dx^\mu dx^\nu.
\]  

This can be put in the form of a space metric by dotting with itself and noting that the first member vanishes

\[
(dt)^2 - g_{jk} dx^j dx^k = (d\tau)^2.
\]  

When the internal product is performed between the two equations member to member the first member vanishes and the second member produces the result

\[
(d\tau)^2 = g^{44} \left[ (dt)^2 - g_{jk} dx^j dx^k \right].
\]  

If the various \( g_\mu \) are functions only of \( x^j \) the equation is equivalent to a metric definition in general relativity. We will examine the special case when \( g_\mu = n_\mu \sigma_\mu ; \) replacing in Eq. (23) with

\[
(d\tau)^2 = \frac{1}{(n_4)^2} (dt)^2 - \sum \left( \frac{n_j}{n_4} dx^j \right)^2.
\]  

This equation covers a large number of situations in general relativity, including the very important Schwarzschild’s metric, as was shown in Almeida [9] and will be discussed below. Notice that Eq. (20) has more information than Eq. (23) because the structure of \( g_4 \) is kept in the former, through the coefficients \( g_{4\mu} \), but is mostly lost in the \( g^{44} \) coefficient of the latter.

### 4 The sources of space curvature

Equations (20) and (23) define two alternative 4-dimensional spaces; in the former, 4DO, \( t \) is an affine parameter while in the latter, GR, it is \( \tau \) that takes such role. The geodesics of one space can be mapped one to one with those of the other and we can choose to work on the space that best suits us.
The geodesics of 4DO space can be found by consideration of the Lagrangian
\[ L = \frac{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{2} = \frac{1}{2}. \] (25)

The justification for this choice of Lagrangian can be found in several reference books but see for instance Martin [6]. From the Lagrangian one defines immediately the conjugate momenta
\[ v_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = g_{\mu\nu} \dot{x}^\nu. \] (26)
The conjugate momenta are the components of the conjugate momentum vector \( v = g^\mu v_\mu \) and from Eq. (21)
\[ v = g^\mu v_\mu = g^\mu g_{\mu\nu} \dot{x}^\nu = \dot{x}^\nu. \] (27)
The conjugate momentum and velocity are the same but their components are referred to the reverse and refractive index frames, respectively.

The geodesic equations can now be written in the form of Euler-Lagrange equations
\[ \dot{v}_\mu = \partial_\mu L; \] (28)
these equations define those paths that minimise \( t \) when displacements are made with velocity given by Eq. (15). Considering the parallel already made with general relativity we can safely say that geodesics of 4DO spaces have a one to one correspondence to those of GR in the majority of situations.

We are going to need geometric calculus which was introduced by Hestenes and Sobczyk [3] as said earlier; another good reference is provided by Doran and Lasenby [4]. The existence of such references allows us to introduce the vector derivative without further explanation; the reader should search the cited books for full justification of the definition we give below
\[ \Box = g^\mu \partial_\mu. \] (29)
The vector derivative is a vector and can be operated with any multivector using the established rules; in particular the geometric product of \( \Box \) with a multivector can be decomposed into inner and outer products. When applied to vector \( a \) \((\Box a = \Box \cdot a + \Box \wedge a)\) the inner product is the divergence of vector \( a \) and the outer product is the exterior derivative, related to the curl although usable in spaces of arbitrary dimension and expressed as a bivector. We also define the Laplacian as the scalar operator \( \Box^2 = \Box \cdot \Box \). In this work we do not use the conventions of Riemanian geometry for the affine connection, as was already noted in relation to Eq. (21). For this reason we will also need to distinguish between the curved space derivative defined above and the ordinary flat space derivative
\[ \nabla = a^\mu \partial_\mu = \sum \sigma_\mu \partial_\mu. \] (30)
When using spherical coordinates, for instance, the connection will be involved only in the flat space component of the derivative and we will deal with it by explicitly expressing the frame vector derivatives.

Velocity is a vector with very special significance in 4DO space because it is the unitary vector tangent to a geodesic. We therefore attribute high significance to velocity derivatives, since they express the characteristics of the particular space we are considering. When the Laplacian is applied to the velocity vector we obtain a vector
\[ \Box^2 v = T. \] (31)
After evaluation the Laplacian becomes
\[ \Box v = (\Box^2 n^\mu) \sigma^\mu \hat{x}^\nu. \] (32)
The tensor \( T^\mu \nu \) contains the coefficients of the sources vector and we call it the sources tensor; it is very similar to the stress tensor of GR, although its relation to geometry is different. The sources tensor influences the shape of geodesics but we shall not examine here how such influence arises, except for very special cases.

Before we begin searching solutions for Eq. (31) we will show that this equation can be decomposed into a set of equations similar to Maxwell’s. Consider first the velocity derivative equation for \( \mathbf{v} \), arising, except for very special cases.

Sources tensor influences the shape of geodesics but we shall not examine here how such influence arises, except for very special cases.

The Laplacian is the inner product of \( \Box \) with itself but the frame derivatives must be considered
\[ \begin{align*}
\partial_r \sigma_r &= 0, \\
\partial_\theta \sigma_r &= \sigma_\theta, \\
\partial_\varphi \sigma_r &= \sin \theta \sigma_\varphi,
\partial_\varphi \sigma_\theta &= \cos \theta \sigma_\varphi, \\
\partial_r \sigma_\varphi &= 0, \\
\partial_\theta \sigma_\varphi &= 0, \\
\partial_\varphi \sigma_\varphi &= -\sin \theta \sigma_r - \cos \theta \sigma_\theta.
\end{align*} \] (34)

After evaluation the Laplacian becomes
\[ \Box^2 = \frac{1}{(n_r)^2} \left( \partial_{rr} + \frac{2}{r} \partial_r - \frac{n'_r}{n_r} \partial_r + \frac{1}{r^2} \partial_{\theta\theta} + \frac{\cot \theta}{r^2} \partial_\theta + \frac{\csc^2 \theta}{r^2} \partial_\varphi \varphi \right) + \frac{1}{(n_r)^2} \partial_{r\varphi}. \] (35)

In the absence of sources we want the sources tensor to vanish, implying that the Laplacian of both \( n_r \) and \( n_4 \) must be zero; considering that they are functions of \( r \) we get the following equation for \( n_r \)
\[ n''_r + \frac{2n'_r}{r} - \frac{(n'_r)^2}{n_r} = 0, \] (36)
with general solution \( n_r = b \exp(a/r) \). We can make \( b = 1 \) because we want the refractive index to be unity at infinity. Using this solution in Eq. (35) the Laplacian becomes
\[ \Box^2 = e^{-a/r} \left( d^2 + \frac{2}{r} d + \frac{a}{r^2} d \right). \] (37)
When applied to $n_4$ and equated to zero we obtain solutions which impose $n_4 = n_r$ and so the space must be truly isotropic and not relaxed isotropic as we had allowed. The solution we have found for the refractive index components in isotropic space can correctly model Newton dynamics, which led the author to adhere to it for some time \cite{11}. However if inserted into Eq. (24) this solution produces a GR metric which is verifiably in disagreement with observations; consequently it has purely geometric significance.

The inadequacy of the isotropic solution found above for relativistic predictions deserves some thought, so that we can search for solutions guided by the results that are expected to have physical significance. In the physical world we are never in a situation of zero sources because the shape of space or the existence of a refractive index must always be tested with a test particle. A test particle is an abstraction corresponding to a point mass considered so small as to have no influence on the shape of space. But in reality a test particle is always a source of refractive index and its influence on the shape of space may not be negligible in any circumstances. If this is the case the solutions for vanishing sources vector may have only geometric meaning, with no connection to physical reality.

The question is then how do we include the test particle in Eq. (31) in order to find physically meaningful solutions. Here we will make one ad hoc proposal without further justification because the author has not yet completed the work that will provide such justification in geometric terms. The second member of Eq. (31) will not be zero and we will impose the sources vector

$$J = -\nabla^2 n_4 \sigma_4.$$ \hspace{1cm} (38)

Equation (31) becomes

$$\Box^2 v = -\nabla^2 n_4 \sigma_4;$$ \hspace{1cm} (39)

as a result the equation for $n_r$ remains unchanged but the equation for $n_4$ becomes

$$n_4'' + \frac{2n_4'}{r} - \frac{n_r' n_4'}{n_r} = -n_4'' + \frac{2n_4'}{r}.$$ \hspace{1cm} (40)

When $n_r$ is given the exponential form found above the solution is $n_4 = \sqrt{n_r}$. This can now be entered into Eq. (24) and the coefficients can be expanded in series and compared to Schwarzschild’s for the determination of parameter $a$. The final solution, for a stationary mass $M$ is

$$n_r = e^{2M/r}, \quad n_4 = e^{M/r}.$$ \hspace{1cm} (41)

Equation (31) can be interpreted in physical terms as containing the essence of gravitation. When solved for spherically symmetric solutions, as we have done, the first member provides the definition of a stationary gravitational mass as the factor $M$ appearing in the exponent and the second member defines inertial mass as $\nabla^2 n_4$. Gravitational mass is defined with recourse to some particle which undergoes its influence and is animated with velocity $v$ and inertial mass cannot be defined without some field $n_4$ acting upon it. Complete investigation of the sources tensor elements and their relation to physical quantities is not yet done. It is believed that the 16 terms of this tensor have strong links with homologous elements of stress tensor in GR but this will have to be verified.

Finally we turn our attention to hyperspherical coordinates. The position vector is quite simply $x = \tau \sigma$, where the coordinate is the distance to the hypersphere centre. Differentiating the position vector we obtain the displacement vector, which is a natural generalisation of 3D spherical coordinates case

$$dx = \sigma d\tau + \tau \sigma_\rho d\rho + \tau \sin \rho \sigma_\theta d\theta + \tau \sin \rho \sin \theta \sigma_\phi d\phi;$$ \hspace{1cm} (42)
$\rho$, $\theta$ and $\varphi$ are angles. The velocity in an isotropic medium should now be written as

$$v = n_4 \sigma_r \dot{\tau} + n_r \tau (\sigma_\rho \dot{\rho} + \sin \rho \sigma_\theta \dot{\theta} + \sin \rho \sin \theta \sigma_\varphi \dot{\varphi}).$$  \hfill (43)

In order to replace the angular coordinate $\rho$ with a distance coordinate $r$ we can make $r = \tau \rho$ and derive with respect to time

$$\dot{r} = \rho \dot{\tau} + \tau \dot{\rho} = \frac{r}{\tau} \dot{\tau} + \tau \dot{\rho}.$$  \hfill (44)

Taking $\tau \dot{\rho}$ from this equation and inserting into Eq. (43), assuming that $\sin \rho$ is sufficiently small to be replaced by $\rho$

$$v = n_4 \left( \sigma_r - \frac{r}{\tau} \sigma_r \right) \dot{\tau} + n_r (\sigma_r \dot{\tau} + r \sigma_\theta \dot{\theta} + r \sin \theta \sigma_\varphi \dot{\varphi}).$$  \hfill (45)

we have also replaced $\sigma_\rho$ by $\sigma_r$ for consistency with the new coordinates.

We have just defined a particularly important set of coordinates, which appears to be especially well adapted to describe the physical Universe, with $\tau$ being interpreted as the Universe’s age or its radius; note that time and distance cannot be distinguished in non-dimensional units. When $r \dot{\tau}/\tau$ is small in Eq. (45), the refractive index vectors become orthogonal and we use $n_4$ and $n_r$ in conjunction with Eq. (24) to obtain a GR metric whose coefficients are equivalent so Schwarzschild’s on the first terms of their series expansions. When $r \dot{\tau}/\tau$ cannot be neglected, however, the equation can explain the Universe’s expansion and flat rotation curves in galaxies without dark matter intervention. A more complete discussion of this subject can be found in Ref. [9].

## 5 Conclusions

Euclidean and Minkowskian 4-spaces can be formally linked through the null subspace of 5-dimensional space with signature ($- + + +$). The extension of such formalism to non-flat spaces allows the transition between spaces with both signatures and the paper discusses some conditions for metric and geodesic translation. For its similarities with optics, the geometry of 4-spaces with Euclidean signature is called 4-dimensional optics (4DO). Using only geometric arguments it is possible to define such concepts as velocity and trajectory in 4DO which become physical concepts when proper and natural assignments are made.

One important point which is addressed for the first time in the author’s work is the link between the shape of space and the sources of curvature. This is done on geometrical grounds but it is also placed in the context of physics. The equation pertaining to the test of gravity by a test particle is proposed and solved for the spherically symmetric case providing a solution equivalent to Schwarzschild’s as first approximation. Some mention is made of hyperspherical coordinates and the reader is referred to previous work linking this geometry to the Universe’s expansion in the absence of dark matter.

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