ADDENDUM TO: ON VOLUMES OF ARITHMETIC QUOTIENTS OF $SO(1,n)$.

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Abstract. There are errors in the proof of the uniqueness of arithmetic subgroups of the smallest covolume. In this note we correct the proof, obtain certain results which were stated as a conjecture, and we give several remarks on further developments.

1.1. Let us recall some notations and basic notions. Following [1] we will assume that $n$ is even and $n \geq 4$. The group of orientation preserving isometries of the hyperbolic $n$-space is isomorphic to $SO(1,n)$, the connected component of identity of the special orthogonal group of signature $(1,n)$, which can be identified with $SO_0(1,n)$, the subgroup of $SO(1,n)$ preserving the upper half space. This group is not Zariski closed in $SL_{n+1}$ thus in order to construct arithmetically defined subgroups of $SO(1,n)$ we consider arithmetic subgroups of the orthogonal group $SO(1,n)$ or, more precisely, of groups $G = SO(f)$ where $f$ is an admissible quadratic form defined over a totally real number field $k$ (see [1, Section 2.1]).

We have an exact sequence of $k$-isogenies:

$$1 \to C \to \tilde{G} \xrightarrow{\phi} G \to 1,$$

(1.1)

where $\tilde{G}(k) \simeq \text{Spin}(f)$ is the simply connected cover of $G$ and $C \simeq \mu_2$ is the center of $\tilde{G}$. This induces an exact sequence in Galois cohomology (see [3, Section 2.2.3])

$$\tilde{G}(k) \xrightarrow{\phi} G(k) \xrightarrow{\delta} H^1(k, C) \to H^1(k, \tilde{G}).$$

(1.2)

The main idea of this note is that by using (1.2) certain questions about arithmetic subgroups of $G$ can be reduced to questions about the Galois cohomology group $H^1(k, C)$.

A coherent collection of parahoric subgroups $P = (P_v)_{v \in V_f}$ of $\tilde{G}$ ($V_f = V_f(k)$ denotes the set of finite places of the field $k$) defines a principal arithmetic subgroup $\Lambda = \tilde{G}(k) \cap \prod_{v \in V_f} P_v \subset \tilde{G}(k)$ (see [2]). We fix an infinite place $v$ of $k$ for which $G(k_v) \simeq SO(1,n)$ and denote it by $Id$. The image of $\Lambda$ under the central $k$-isogeny $\phi$ is an arithmetic subgroup of $G$ and every maximal arithmetic subgroup of $G(k_{Id})$ can be obtained as a normalizer of some $\phi(\Lambda)$ [2, Proposition 1.4]. We will also consider the local stabilizers of $P$ in the adjoint group $G(= \overline{G})$, defining $P_v$ to be the stabilizer of $P_v$ in $G(k_v)$ and $\overline{P} = (\overline{P}_v)_{v \in V_f}$. Clearly, $\overline{P}_v \supset \phi(P_v)$. In the notation of [1] the subgroups $\phi(P_v)$ are called parahoric subgroups of $G$, however this terminology is non-standard and we will avoid using it here.

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1.2. Given a totally real number field $k$ with the group of units $U$, let

$$k^{*}_\infty = \{ a \in k^{*} \mid a_v > 0 \text{ for } v \in V_\infty \setminus \text{Id} \}, \ U_\infty = U \cap k^{*}_\infty.$$

**Lemma 1.1.** $\text{Im}(\delta) \simeq k^{*}_\infty/(k^{*})^2$.

**Proof.** From (1.2) we have $\text{Im}(\delta: G(k) \to H^1(k, \mu_2)) = \text{Ker}(H^1(k, \mu_2) \to H^1(k, \tilde{G}))$. The Hasse principle for simply connected $k$-groups implies that $H^1(k, \tilde{G})$ is isomorphic to $\prod_{v \in V_\infty} H^1(k_v, \tilde{G})$ [5, Theorem 6.6], and hence

$$\text{Im}(\delta) = \text{Ker}(H^1(k, \mu_2) \to \prod_{v \in V_\infty} H^1(k_v, \tilde{G})).$$

The group $H^1(k, \mu_2)$ is canonically isomorphic to $k^{*}/(k^{*})^2$ [5, Lemma 2.6]. It is well known that for all $v \in V_\infty$ such that the group $G(k_v)$ is anisotropic, the map $\phi$ in (1.2) is surjective and hence for all such $v$, $\text{Im}(\delta_v) = \text{Ker}(H^1(k_v, \mu_2) \to H^1(k_v, \tilde{G}))$ is trivial. For the remaining one infinite place $v(=\text{Id}) \in V_\infty$, $\phi(G(k_v))$ is a subgroup of index 2 in $G(k_v)$ which consists of the orthogonal transformations with the trivial spinor norm. Collecting this information together we obtain the required isomorphism. □

1.3. The proof of the uniqueness part in [1, Theorem 4.1] contains errors but the result is correct. We will now give another argument for it. In order to do so we first establish a more general fact and then apply it to the cases considered in [1].

Let $P = (P_v)_{v \in V_f}$ and $P' = (P'_v)_{v \in V_f}$ be two coherent collections of parahoric subgroups of $G$ such that for all $v \in V_f$, $P'_v$ is conjugate to $P_v$ under an element of $G(k_v)$. For all but finitely many $v$, $P_v = P'_v$ hence there is an element $g \in G(\mathbb{A}_f)$ ($\mathbb{A}_f$ denotes the ring of finite ad` eles of $k$) such that $P'$ is the conjugate of $P$ under $g$. We have $P' = \prod_{v \in V_f} P'_v$, is the stabilizer of $P$ in $G(\mathbb{A}_f)$. The number of distinct $G(k)$-conjugacy classes of coherent collections $P'$ as above is the cardinality $c(P)$ of $G(\mathbb{A}_f)/G(k)$, which is called the class group of $G$ relative to $P$. The class number $c(P)$ is known to be finite (see e.g. [2, Proposition 3.9]). The following result can be used for obtaining further information about its value.

**Proposition 1.2.** Let $G = \text{SO}(f), \tilde{G} = \text{Spin}(f)$ for an admissible quadratic form $f$ defined over $k$ and let $P = (P_v)_{v \in V_f}$ be a coherent collection of parahoric subgroups of $\tilde{G}$. The class number $c(P)$ divides the order $h_{\infty, 2}$ of a restricted 2-class group of $k$ given by

$$h_{\infty, 2} = 2^{[k: \mathbb{Q}]-1} h_2 [U : U_\infty],$$

where $h_2$ is the order of the 2-class group of $k$.

**Proof.** Recall two isomorphisms (see [5, Proposition 8.8], a minor modification is needed in order to adjust the statement to our setting but the argument remains the same):

$$G(k)/G(\mathbb{A}_f)/\overline{P} \simeq G(\mathbb{A}_f)/\overline{PG}(k);$$

$$G(\mathbb{A}_f)/\overline{PG}(k) \simeq \delta_{h, 2}(G(\mathbb{A}_f))/\delta_{h, 2}(\overline{PG}(k)),$$
where \( \delta_{h_j} \) is the restriction of the product map \( \prod_v G(k_v) \to \prod_v H^1(k_v, C) \) to \( G(\mathcal{A}_f) \).

For every finite place \( v \), \( H^1(k_v, C) \) is trivial (see [5, Theorem 6.4]) which implies \( \delta_v : G(k_v) \to H^1(k_v, C) \) is surjective. Thus the image of \( \delta_{h_j}(G(\mathcal{A}_f)) \) can be identified with the restricted direct product \( \prod_v H^1(k_v, C) \) with respect to the subgroups \( \delta_v(P_v) \). Also \( \delta_{h_j}(G(k)) \) naturally identifies with the image of \( \delta(G(k)) \) in \( H^1(k, C) \) under the embedding \( \psi : H^1(k, C) \to \prod_v H^1(k_v, C) \). Hence we have an isomorphism

\[
\delta_{h_j}(G(\mathcal{A}_f))/\delta_{h_j}(P_G(k)) \cong \prod_v H^1(k_v, C)/\left( \prod_v \delta_v(P_v) \cdot \psi(\text{Im} \delta(G(k))) \right).
\]

The group \( H^1(k_v, \mu_2) \) is canonically isomorphic to \( k_v^*/(k_v^*)^2 \), by Lemma [11] \( \text{Im} \delta(G(k)) \cong k_v^*/(k_v^*)^2 \), so we obtain

\[
\frac{\prod_v H^1(k_v, C)}{\prod_v \delta_v(P_v) \cdot \psi(\text{Im} \delta(G(k)))} \cong \frac{\prod_v k_v^*/(k_v^*)^2}{\delta_P \cdot k_v^*/(k_v^*)^2} \cong \frac{J_f \cdot k_v^*/k_v^*}{U/U_\infty},
\]

where \( J_f \) is the ring of finite ideles of \( k \) and \( \delta_P \) denotes \( \prod_v \delta_v(P_v) \).

Now, \( \#(J_f/k_f^*) = h_2 \), the group \( k^*/k^*_\infty \) splits as a product of local factors and \( \#(k^*/k^*_\infty) = 2^{[k:Q]-1} \) (see [4, Chapter 6]). This implies the proposition. \( \square \)

In order to give a precise formula for the class number \( c(P) \) one has to analyze the image of \( \prod_v \delta_v(P_v) \) in \( \prod_v H^1(k_v, C) \). Still in many practical cases this appears to be unnecessary. Thus in order to prove the uniqueness of the minimal hyperbolic orbifolds we need to consider \( k = \mathbb{Q}[\sqrt{5}] \) (in the compact case) and \( k = \mathbb{Q} \) (for the non-compact orbifolds). In both cases \( h_2 = h = 1 \). For \( k = \mathbb{Q}[\sqrt{5}] \), \( U/U_\infty = \{ U_\infty, \frac{1+\sqrt{5}}{2} U_\infty \} \) and \( [U : U_\infty] = 2 \), which implies \( h_{\infty, 2} = 1 \). For \( k = \mathbb{Q} \), clearly, \( h_{\infty, 2} = 1 \) as well. So in all the cases \( c(P) = 1 \) which implies that the corresponding arithmetic subgroups are defined uniquely up to a conjugation by \( g \in SO(1, n) \). It is clear that we can always chose \( g \in SO_0(1, n) \) and therefore the smallest orbifolds constructed in [4] are unique up to an (orientation preserving) isometry.

1.4. We now turn to Conjecture 4.1 and its analogue for the non-cocompact orbifolds in [11, Section 4.4]. Recall that in [11] the numbers \( N(r), N'(r) \) were defined for every \( r \geq 2 \) and estimated from above. These numbers are related to the index of the principal arithmetic subgroups in their normalizers. We now prove

**Proposition 1.3.** For every \( r \geq 2 \), \( N(r) = N'(r) = 1 \).

**Proof.** Let \( \Lambda \) be a principal arithmetic subgroup of \( \bar{G} \) which corresponds to a compact or non-compact hyperbolic \( n \)-orbifold of the minimal volume, \( \Lambda' = \phi(\Lambda) \) and \( \Gamma = N_G(\Lambda') \).

From [2, Proposition 2.9], which in turn follows from the work of J. Rohlf, using the fact that the center of our group \( G \) is trivial, we obtain:

\[
[\Gamma : \Lambda'] = \#(H^1(k, \mu_2) \cap \delta(G(k))) = \#\text{Im}(\delta : G(k) \to H^1(k, \mu_2))\).
\]

We can identify the image of \( \delta \) by Lemma [11] and then compute \( \text{Im}(\delta) \) using [2, Section 5.1]. The cases we are interested in are

\[
k = \mathbb{Q} : \text{Im}(\delta) = \left\{ k^2, (-1)k^2 \right\};
k = \mathbb{Q}[\sqrt{5}] : \text{Im}(\delta) = \left\{ k^2, \frac{1 - \sqrt{5}}{2} k^2 \right\}.
\]
In both cases $[\Gamma : \Lambda'] = \#\text{Im}(\delta) = 2$. Now it is easy to see that $\Lambda' = \phi(\Lambda) \subset \text{SO}_0(1, n)$. From the other side there always exists $g \in \text{SO}(1, n) \setminus \text{SO}_0(1, n)$ which normalizes $\phi(\Lambda)$. For example take $g = \text{diag}(-1, -1, 1, \ldots, 1)$. As in all the cases under consideration the quadratic form associated to $\Lambda$ is diagonal [1, Sections 4.3, 4.4], $g$ stabilizes $\Lambda$ and clearly $g \in \text{SO}(1, n) \setminus \text{SO}_0(1, n)$. From these facts it follows that $\Lambda'$ is a maximal arithmetic subgroup in $\text{SO}_0(1, n)$ and thus $N(r)/(or N'(r)) = 1$.

This proposition makes precise the statements of Theorem 4.1 and 4.4 of [1]. It also implies that Table 2 of loc. cit. gives the precise values of the covolumes of the smallest $n$-dimensional hyperbolic orbifolds in even dimensions up to 18.

One other corollary is that cocompact and non-cocompact arithmetic subgroups of $\text{SO}(1, 2r)^o$ of the smallest covolumes can be obtained as the stabilizers of certain lattices described in [1, Section 4.3]. We remark that since the fields of definition of the groups have class number 1, the lattices in both cases are free as $\mathcal{O}_k$-modules.

1.5. Correction: on p. 765, l. 9 one should read “grow super-exponentially” instead of “grow exponentially”. (It follows from [1] that the Euler characteristic is bounded from below by $\text{const} \cdot \left(\prod_{i=1}^{r} \left(\frac{(2i-1)!}{(2\pi)^{2i}}\right)^{[k:Q]}\right)$ which for large enough $r$ is $\geq \text{const} \cdot (2r-1)!$)

We conclude this addendum with a few remarks on related results which appeared after the paper was published.

1.6. In [1, Section 4.5] we observed that for $r > 2$ the minimal covolume among the arithmetic lattices in $\text{SO}(1, 2r)$ is attained on a non-uniform lattice. This interesting phenomenon was first discovered by A. Lubotzky for $\text{SL}_2$ over local fields of positive characteristic. Recently, in [6] A. Salehi Golsefidy proved that lattices of minimal covolume in classical Chevalley groups over local fields of characteristic $p > 7$ are all non-uniform. This result gives further support to a conjecture that generically (i.e. for groups of high enough rank or fields of large enough positive characteristic) the minimal covolume is always attained on a non-uniform lattice.

1.7. In [3] M. Conder and C. Maclachlan constructed a compact orientable hyperbolic 4-manifold which has Euler characteristic 16. The previously known smallest example which was used in order to formulate the main result in [1, Section 5] had $\chi = 26$. The construction of [3] agrees with our Theorem 5.5 and it also allows us to give a more precise formulation of the theorem:

**Theorem 5.5'.** If there exists a compact orientable arithmetic hyperbolic 4-manifold $M$ with $\chi(M) \leq 16$, then $M$ is defined over $\mathbb{Q}[\sqrt{5}]$ and has the form $\Gamma_M \backslash \mathcal{H}^4$ with $\Gamma_M$ being a torsion-free subgroup of index $7200\chi(M)$ of the group $\Gamma_1$ of the smallest arithmetic hyperbolic 4-orbifold.

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