Supersymmetry of Demkov-Ostrovsky effective potentials in the $R_0 = 0$ sector

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We present a supersymmetric analysis of the wave problem with a Demkov-Ostrovsky spherically symmetric class of focusing potentials at zero energy. Following a suggestion of Lévai, we work in the so-called $R_0 = 0$ sector in order to obtain the superpartner (fermionic) potentials within Witten’s supersymmetric procedure. General solutions of the superpotential for the known physical cases are given explicitly.

We consider the wave/quantum problem at fixed null energy, (or zero-binding energy), 
\[
-\frac{\hbar^2}{2m} \nabla^2 \psi(r) + U_\kappa(r) \psi(r) = 0,
\]
where $w > 0$, $R > 0$, $\kappa > 0$ are constant parameters, and we introduced an energy scale $E_0 = \hbar^2/2mR^2$ for the potential part, following Daboul and Nieto. Of the $U_\kappa$ class of potentials only the two cases to follow have been studied in the literature. For $\kappa = 1$, one gets the wave problem for the Maxwell fish-eye (MF) lens, whereas the $\kappa = 1/2$ case has been used for the atomic Aufbau (AA) chart since it fulfills the Madelung rules of atomic energy ordering. Demkov and Ostrovsky have shown that for the cases $\kappa = k_1/k_2$, with $k_1$ and $k_2$ integers, i) the classical trajectories of a zero-energy (i.e., zero velocity at infinity) particle close after $k_2$ revolutions around the force centre, and ii) all the trajectories passing through a given point come to a focus after $k_2/2$ revolutions.

The above wave equation can be written in the scaled variable $\rho = r/R$, (hereafter the energy scale is to be understood), as follows
\[
\left[ -\frac{\partial^2}{\partial \rho^2} - \frac{2}{\rho} \frac{\partial}{\partial \rho} + \frac{l(l+1)}{\rho^2} - \frac{w}{\rho^{2(1-\kappa)}(1+\rho^{2\kappa})^2} \right] \psi(\rho) = 0.
\]

It is quite straightforward to solve the Sturm-Liouville problem, Eq. (2). Moreover, it can be turned into an eigenvalue problem of the DO coupling constant, $w$, and also written as a Laplace equation on the four-dimensional sphere. The known results are the following. For $w$ taking the quantized values, $w_{N,\kappa} = (2\kappa)^2[N + (2\kappa)^{-1} - 1][N + (2\kappa)^{-1}]$, the regular, normalizable solutions read
\[
\psi_{Nlm}(\rho) = R_{Nl}(\rho) Y_{lm}(\theta, \phi),
\]
\[
R_{Nl}(\rho) = \frac{N_N^{l}}{\rho^{-l}(1+\rho^{2\kappa})(2l+1)/2\kappa} C_{N-1-l/\kappa}^{(2l+1)/2\kappa+1/2}(\xi),
\]
where $\xi = \frac{1-r^2}{r^2}$, $N = n + (\kappa^{-1} - 1)l$, $n = n_r + l + 1$, $n_r = 0, 1, 2, \ldots$, are the ‘total’, ‘principal’ and ‘radial’ quantum numbers, $l$ and $m$ are the spherical harmonic numbers, $C_p^q(\xi)$ are the Gegenbauer polynomials, i.e., the solutions of the corresponding ultraspherical equation (see Eqs. (7) and (8) below), and $N_{Nl}$ are the normalization constants. What one gets when the $w$ parameter is made bigger and bigger is an increase of the degeneracy of the normalizable state at zero-energy, but only for the quantized values $w_{N,\kappa}$. The degree of degeneracy of such a group of states is $N^2$, $(N = 1, 2, 3 \ldots)$, similar to the electron energy levels in a Coulomb field.

We pass now to the functions $u_{Nl} = \rho R_{Nl}$ fulfilling the one-dimensional (half-line) radial equation

$$H^- u \equiv \left[-\frac{\partial^2}{\partial \rho^2} + U_{eff}^- (\rho)\right] u = 0,$$

with the effective potential

$$U_{eff}^- = \frac{l(l+1)}{\rho^2} - \frac{(2\kappa)^2[N + (2\kappa)^{-1} - 1][N + (2\kappa)^{-1}]}{\rho^{2(1-\kappa)}(1 + \rho^{2\kappa})^2},$$

where we have already included supersymmetric superscripts. The functions $u_{Nl}$ are of the type $f(\rho)C_{N-1-1/\kappa}^{2l+1}(\xi(\rho))$, where $f(\rho)$ reads

$$f(\rho) = \frac{\rho^{l+1}}{(1 + \rho^{2\kappa})^{(2l+1)/2\kappa}},$$

and the Gegenbauer polynomials, $C_p^q$, of degree $p = n_r$ and parameter $q$ as given above, are the solutions of a second-order differential (ultraspherical) equation of the type

$$P(\xi) \frac{d^2C}{d\xi^2} + Q(\xi) \frac{dC}{d\xi} + R_p(\xi) C(\xi) = 0,$$

with

$$P(\xi) = 1,$$

$$Q(\xi) = \frac{2l + 2\kappa + 1}{\kappa} \frac{\xi}{\xi^2 - 1},$$

and

$$R_p(\xi) = -\frac{p(2q + p)}{\xi^2 - 1}.$$ 

In Eq. (8.3), we emphasized the indexing of the $R$ functions according to the various sectors $p = n_r$ $(0, 1, 2, \ldots)$, which is a well-known characteristic of orthogonal polynomials.

We have now all the requisites for a Natanzon-type approach that we outline here following a recent discussion of Lévai. The method has been first used by Bhattacharjee and Sudarshan and later by other authors, among whom Natanzon is the best known due to his systematic treatment of hypergeometric cases.
scheme deals with the fact that the solutions $\psi$ of any one-dimensional Schrödinger equation can be written as $\psi(x) = f(x)F(z(x))$, where $f(x)$ is a function to be determined and directly related to the superpotential, while $F(z)$ is a special function which satisfies a second-order differential equation of the form

$$\frac{d^2 F}{dz^2} + Q(z) \frac{dF}{dz} + R(z)F(z) = 0.$$  \hspace{1cm} (9)

In our case, $Q(z)$ and $R(z)$ corresponding to the Gegenbauer polynomials have been written above, while $x = \rho$ and $z(x) = \xi(\rho)$. Then, the following equations can be readily obtained

$$\frac{\xi''}{(\xi')^2} + \frac{2f'}{\xi f} = Q(\xi(\rho)), \hspace{1cm} (10)$$

and

$$\frac{f''}{(\xi')^2 f} - \frac{U_{eff}}{(\xi')^2} = R(\xi(\rho)), \hspace{1cm} (11)$$

where $U_{eff}$ is given by Eq. (5). From Eq. (10) the function $f(\rho)$ can be written as follows

$$f(\rho) \sim (\xi'(\rho))^{-1/2} \exp \left[ \frac{1}{2} \int \xi(\rho) Q(\xi(\rho)) d\xi \right]. \hspace{1cm} (12)$$

As suggested by Lévai, one can define a ground state by the $R_0(\xi) = 0$ sector, within which the Gegenbauer polynomials are $C_0^q = 1$ for any parameter $q$. In this simple case, from Eq. (11) one gets

$$U_{eff} = + W^2(\rho) - \frac{dW}{d\rho}, \hspace{1cm} (13)$$

with $W(\rho) = - \frac{d}{d\rho} \ln f(\rho)$. Eq. (13) is just the initial Riccati equation for the DO cases (and for any radial oscillator) in Witten’s supersymmetric quantum mechanics. Hereafter Eq. (13) will be called DORE. Since we actually know from Eq. (6) the function $f(\rho)$ in the DO cases (one can check that Eq. (12) leads to the same function), a short calculation gives the DO superpotential

$$W_\kappa(\rho) = \frac{l}{\rho} - \frac{2l + 1}{\rho(1 + \rho^{2\kappa})}. \hspace{1cm} (14)$$

The effective DO superpartners in the $R_0 = 0$ sector can be written as follows

$$U_{eff}^- = - \frac{dW_\kappa(\rho)}{d\rho} + W^2_\kappa(\rho), \hspace{1cm} (15a)$$

and

$$U_{eff}^+ = + \frac{dW_\kappa(\rho)}{d\rho} + W^2_\kappa(\rho). \hspace{1cm} (15b)$$

Thus,

$$U_{eff}^- = \frac{l(l+1)}{\rho^2} - \frac{(2l+1)(2l+2\kappa+1)}{\rho^{2(1-\kappa)}(1 + \rho^{2\kappa})^2}. \hspace{1cm} (16a)$$
and
\[ U_{_{eff}}^+ = \frac{l(l-1)}{\rho^2} - \frac{(2l+1)(2l-2\kappa-1)}{\rho^2(1-\kappa)(1+\rho^2\kappa)^2} + \frac{2(2l+1)}{\rho^2(1+\rho^2\kappa)^2}. \] (16b)

The factorizing operators read
\[ A = \frac{d}{d\rho} + W_\kappa, \] (17a)

and
\[ A^+ = -\frac{d}{d\rho} + W_\kappa. \] (17b)

We have plotted \( U_{_{eff}}^- \) and \( U_{_{eff}}^+ \) for some values of the parameters in Figs. 1 and 2. From the plot of the DO fermionic potentials one can notice their repulsive nature. Consequently, the fermionic equation should be written in the continuum
\[ H^+u_1 \equiv AA^+u_1 \equiv (-\frac{d^2}{d\rho^2} + U_{_{eff}}^+)u_1 = k^2u_1, \; k \in (0, \infty). \] (18)

It will be investigated elsewhere. Here we remark that in order to get the supersymmetric increment in the effective potential we used only the particular solution of the Riccati equation coming out from Eq. (11). On the other hand, it is well-known that the connection with the Gel’fand-Levitan inverse scattering method requires the general solution of the Riccati equation. We construct it in the usual two steps as follows. Firstly, consider \( W = V^{-1} + W(\rho) \) as another solution. Then by substituting in the DORE one gets
\[ \frac{dV}{d\rho} + 2W(\rho)V = -1. \] (19)

This equation can be written as follows
\[ \frac{d}{d\rho}\left[V \exp\left(2 \int W(\rho) d\rho\right)\right] = -\exp\left(2 \int W(\rho) d\rho\right), \] (20)

with the solution
\[ V = -\exp\left(-2 \int W(\rho) d\rho\right) \cdot \int \exp\left(2 \int W(\rho) d\rho\right) d\rho. \] (21)

Since we know that \( W(\rho) = -\frac{d}{d\rho} \ln f(\rho) \) we get \( \int W(\rho) d\rho = -\ln f(\rho). \) Thus
\[ V = -f^2(\rho) \int f^{-2}(\rho) d\rho. \] (22)

The object of interest is now the integral of the inverse square of \( f, \) which reads explicitly
\[ \int \frac{(\rho^{-\kappa} + \rho^{\kappa})^{2l+1}}{\rho} d\rho. \] (23)
Let $\rho^\kappa = \tan(\frac{\alpha}{2})$. Then the integral turns into the form

$$
\frac{2^{2l+1}}{\kappa} \int \frac{d\alpha}{(\sin \alpha)^{(2l+\kappa+1)/\kappa}},
$$

and for the cases of physical interest, MF and AA, the formulas 2.515.1 and 2.515.2, respectively, in Gradshteyn and Ryzhik should be used to express it as a series.

Thus, in the MF case, ($\kappa = 1$), the integral Eq. (24) can be worked out into the series

$$
S_1 = -\frac{2^{2l+1}}{2l+1} \cos \alpha \left\{ (\csc \alpha)^{2l+1} + \sum_{m=1}^{l} \frac{2^m [l(l-1)(l+1-m)]}{(2l-1)(2l-3)...(2l+1-2m)} (\csc \alpha)^{2l+1-2m} \right\},
$$

while for the AA case, ($\kappa = 1/2$), the integral Eq. (24) reads

$$
S_\frac{1}{2} = -\frac{2^{4l+3}}{2l+1} \cos \alpha (\csc^2 \alpha)^{(2l+1)} \left\{ 1 + \sum_{m=1}^{2l} \frac{[(4l+1)(4l-1)...(4l-2m+3)]}{(2 \csc^2 \alpha)^m [(2l-1)(2l-3)...(2l+1-2m)]} \right\} + \frac{4[(4l+1)!!]}{4^{-l}(2l+1)!} \ln \tan \left(\frac{\alpha}{2}\right),
$$

where $\alpha = 2 \arctan \rho$ in the first formula and $\alpha = 2 \arctan \sqrt{\rho}$ in the latter one.

In the MF case, the radial factor $f_2(\rho)$ can be written trigonometrically as $2^{-2l}(\sin \alpha)^{2l} \sin^2(\frac{\alpha}{2})$ implying

$$
V_1 = 2 \cos \alpha \frac{\alpha}{2l+1} \tan \left(\frac{\alpha}{2}\right) \left\{ 1 + \sum_{m=1}^{l} \frac{(2 \sin^2 \alpha)^m [l(l-1)...(l-m)]}{[(2l-1)(2l-3)...(2l+1-2m)]} \right\},
$$

In the AA case, the square radial factor is $2^{-4l}(\sin \alpha)^{4l} \sin^2(\frac{\alpha}{2})$ and one can work out easily a formula for $V_\frac{1}{2}$

$$
V_\frac{1}{2} = 2 \cos \alpha \frac{\alpha}{2l+1} \tan^2 \left(\frac{\alpha}{2}\right) \left\{ 1 + \sum_{m=1}^{2l} \frac{(\sin^2 \alpha)^m [(4l+1)(4l-1)...(4l-2m+3)]}{(2l-1)(2l-3)...(2l+1-2m)} \right\} + \frac{4[(4l+1)!!](\sin^2 \alpha)^{2l}}{(2l+1)!(\csc^2 \alpha)} \ln \tan \left(\frac{\alpha}{2}\right).
$$

We have in this way all the ingredients for the second step, which is to write down the general solution of DORE, containing a constant $\lambda$-parameter in Eq. (22), see Refs. 8,10. The general solution means Eq. (22) modified as follows

$$
V_{\lambda,\kappa} = -f_2^2(\rho) \left( \lambda + \int_0^\infty f_\kappa^{-2}(\rho) d\rho \right).
$$

In this case, the integral equation (24) will have the lower limit $\alpha$ and the upper one $\pi$. Thus, $V_{\lambda,\kappa}$ can be calculated from our formulas as $V_{\lambda,\kappa} = V_\kappa - \lambda f_2^2(\rho)$. 
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Figure Captions

Fig. 1. Effective DO superpartners in the $R_0 = 0$ sector, $U_{\pm \text{eff}}(\rho)$, (a) and (b), respectively, in units of $\mathcal{E}_0$, for $l = 2$, and $\kappa = 1/2, 1, \text{and } 3/2$.

Fig. 2. Effective superpartners of the $R_0 = 0$ sector in $\mathcal{E}_0$ units: (a), $U_{\text{eff}}^-$ for $l= 1, 5, 10$, and (b), $U_{\text{eff}}^+$ for $l=6, 7, 8$, in the case of Maxwell fish-eye, $\kappa = 1$. We have plotted $U_{\text{eff}}^+$ in the region of the critical (inflexion) angular number, $l_{cr}$, that we have found numerically to be $l_{cr}=6.876$ for $\rho_{cr}=1.599$. The critical $l$ is the entry point toward a pocket (trapping) region of $U_{\text{eff}}^+$ for $l > l_{cr}$.

Note added on January 21st, 1996

The mathematics in the last two pages of the paper (pp. 38, 39), though not wrong, might be considered as misleading with respect to the literature, since we obtained the general solution of the initial ‘bosonic’ Riccati equation. In this way, one can introduce the one-parameter family of fermionic potentials with the same bosonic superpartner. This is what we have done. However, for the usual connection with the Gel’fand-Levitan method, one should obtain the general solution of the ‘fermionic’ Riccati equation, and thus, the one-parameter family of bosonic potentials with the same fermionic superpartner. The task can be readily done on the base of the results at pp. 38, 39.

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