On the Nonlocal Darboux Transformation for Time-Independent Axially Symmetric Schrödinger and Helmholtz Equations

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The nonlocal Darboux transformation for the time-independent axially symmetric Schrödinger and Helmholtz equations has been considered. New examples of two-dimensional potentials and exact solutions of the time-independent axially symmetric Schrödinger and Helmholtz equations have been obtained on the basis of the formulas of the nonlocal Darboux and generalized Moutard transformations.

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Examples of two-dimensional potentials and exact solutions of the time-independent axially symmetric Schrödinger equation were obtained in [1] on the basis of the generalized Moutard transformation. It was shown in [2] that the generalized Moutard transformation is a particular case of the two-dimensional nonlocal Darboux transformation. The aim of this work is to obtain new examples of two-dimensional potentials and exact solutions for the time-independent axially symmetric Schrödinger and Helmholtz equations using the nonlocal Darboux transformation.

We consider the time-independent Schrödinger equation in the form

\[ (\Delta - u(x,y,z)) Y(x,y,z) = 0. \]

This equation with \( u = -E + V(x,y,z) \) describes a nonrelativistic quantum system with the energy \( E \) [3].

In the case \( u = -\omega^2/c (x,y,z)^2 \), the equation describes the propagation of acoustic waves having a frequency \( \omega \) in an inhomogeneous medium at the speed of sound \( c \) and is widely used in wave theory, being called the Helmholtz equation [4]. The case of the fixed frequency \( \omega \) is interesting for modeling in acoustic tomography [5]. The case of the fixed energy \( E \) for the two-dimensional equation is of interest in the theory of integrable nonlinear systems [6].

In the case of axial symmetry, the time-independent Schrödinger equation in cylindrical coordinates has the form

\[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - u(r,z) \right) Y(r,z) = 0. \] (1)

It is a linear differential equation with the variable coefficient \( u \). The Darboux transformations are useful to analyze such equations. The authors of [7] mentioned that the Darboux transformation allows obtaining new solvable equations from the initial solvable equation. A change in the solution of the initial equation in the Darboux transformation gives solutions of a new equation. The initial equation is exactly solvable if all its solutions can be analytically represented in terms of a set of chosen special functions. If the initial equation is solvable, all solutions of the new equation can be obtained using the Darboux transformation. The Darboux transformation can again be applied to new equations. Thus, an equation is solvable if it can be obtained from another solvable equation by means of a finite number of Darboux transformations. In particular, if the initial equation is the time-independent Schrödinger equation in one-dimensional quantum mechanics, it is reasonable to call solvable equations describing the free motion of particles, harmonic oscillator, Coulomb potential, and Morse potentials. In this work, we state that the time-independent Schrödinger and Helmholtz equations in the two-dimensional case are solvable if they can be obtained from an equation with a constant (e.g., zero) potential by means of a finite number of Darboux transformations.

Applications of the classical Darboux transformation for the one-dimensional Schrödinger equation are considered in [7]. The classical Moutard transformation for the two-dimensional time-independent Schrödinger equation in Cartesian coordinates and the relation of the Moutard transformation with the Darboux transformation are discussed in [8]. Various generalizations of the classical Darboux transformations and their applications to two-dimensional systems were studied (see review [9], recent work [10], and references therein). A nonlocal variable was included in the Darboux transformation in [11, 12]. The nonlocal Darboux transformation of the two-dimensional time-independent Schrödinger equation...
in Cartesian coordinates was obtained and its relation with the Moutard transformation was determined. The main idea of those works was inspired by the approach to nonlocal symmetries in the analysis of groups of symmetry of differential equations [13, 14] (for the variant based on the theory of coverings, see [15] and references therein). In this work, the nonlocal Darboux transformation for the time-independent equation in cylindrical coordinates (1) is analyzed within the approach used in [11, 12].

We use the relation of the Schrödinger equation with the Fokker–Planck equation [16]. The substitution of

$$Y(r,z) = P(r,z)e^{\phi(r,z)}$$  \(\text{(2)}\)

into Eq. (1) gives the Fokker–Planck equation

$$\frac{\partial}{\partial r} \left( P_r + 2h_P + \frac{1}{r} P \right) + \frac{\partial}{\partial z} \left( P_z + 2h_P \right) = 0 \quad \text{(3)}$$

if

$$u = -h_r + h_r^2 + \frac{1}{r} h_r + \frac{1}{r^2} - h_z + h_z^2. \quad \text{(4)}$$

Equation (3) has the form of a conservation law and provides the following system of equations:

$$P_r + 2h_P + \frac{1}{r} P - Q_z = 0, \quad \text{(5)}$$

$$P_z + 2h_P + Q_r = 0. \quad \text{(6)}$$

The variable $Q$ is a nonlocal variable with respect to Eq. (3) and related Eq. (1). The Darboux transformation of Eqs. (5) and (6) including $Q$ leads to a nonlocal transformation of Eq. (1).

We consider the linear operator corresponding to the system of equations (5) and (6):

$$\hat{L}(h(r,z))f = \left\{ \begin{array}{c}
2h_r + \frac{1}{r} + \frac{\partial}{\partial r} - \frac{\partial}{\partial z} \\
2h_z + \frac{\partial}{\partial z}
\end{array} \right\} \left\{ \begin{array}{c} f_1 \\
 f_2 \end{array} \right\}.$$  \(\text{(8)}\)

The Darboux transformation is sought in the form

$$\hat{L}_D f = \left( g_{11} h_r + a_{11} \frac{\partial}{\partial r} - b_{11} \frac{\partial}{\partial z} + g_{12} h_r + a_{12} \frac{\partial}{\partial r} - b_{12} \frac{\partial}{\partial z} \right) f_1 \left( g_{21} h_z + a_{21} \frac{\partial}{\partial r} - b_{21} \frac{\partial}{\partial z} + g_{22} h_z + a_{22} \frac{\partial}{\partial r} - b_{22} \frac{\partial}{\partial z} \right) f_2. \quad \text{(9)}$$

If the linear operators $\hat{L}$ and $\hat{L}_D$ satisfy the intertwining relation

$$\left( \hat{L} (h(r,z) + s(r,z)) \right) \hat{L}_D f - \hat{L}_D \left( \hat{L} (h(r,z)) \right) f = 0 \quad \text{(7)}$$

for any function $f \in \mathcal{F} \supset \text{Ker} (\hat{L}(h))$, where $\text{Ker} (\hat{L}(h)) = \{ f : \hat{L}(h)f = 0 \}$, then the function $f(r,z) = \hat{L}_D f(r,z)$ for any function $f \in \text{Ker} (\hat{L}(h))$ is a solution of the equation $\hat{L}(\hat{h})f = 0$ with the new potential $\hat{h} = h + s$.

Equations for $s, g_{ij}, a_{ij}$, and $b_{ij}$ are obtained from the intertwining relation (7). The following expression appears:

$$V(r,z) = s(r,z) + 2h(r,z) + \ln r. \quad \text{(8)}$$

The case $V(r,z) = 0$ leads to the generalized Moutard transformation as an important particular case of the nonlocal Darboux transformation [2]. Applications of the generalized Moutard transformation were considered in [1].

In the general case, where $V$ is nonzero, the intertwining relation (7) gives the Darboux transformation operator in the form

$$\hat{L}_D = e^{-s(r,z)} \left( R_1 + \frac{\partial}{\partial z} R_2 \right) \left( s_r - R_2 s_z + R_1 + \frac{\partial}{\partial z} \right), \quad \text{(9)}$$

where

$$R_1 = \frac{1}{2} (V_z - 2s_z + (V_r + V_T)/G),$$

$$R_2 = \frac{1}{2} (s_r + (s_T - s_r)/G),$$

$$G = s_V + s_z V_z, \quad H = V_{zz} + s_r, \quad T = V_{zz} + s_r,$$

and $s$ satisfies the system of two nonlinear partial differential equations [2]. These equations cannot be solved in the general case and are not presented here.

According to Eq. (9),

$$\hat{P} = e^{-s} \left( R_1 P + \frac{\partial}{\partial z} P + R_2 Q \right),$$

and, taking into account that $\hat{\bar{Y}}(r,z) = \hat{P}(r,z)e^{\hat{h}(r,z)}$, we obtain the nonlocal Darboux transformation

$$\hat{\bar{Y}} = \frac{\partial}{\partial z} \bar{Y} + (R_1 - h_r) \bar{Y} + e^h R_2 Q. \quad \text{(10)}$$

In view of Eq. (2), Eqs. (5) and (6) for $Q$ can be presented in the form

$$e^{-h} \left( Y_r + h_r Y + \frac{1}{r} Y \right) - Q_z = 0, \quad \text{(11)}$$

$$e^{-h} (Y_z + h_z Y) + Q_r = 0. \quad \text{(12)}$$

The system of equations for $s$ with a particular form of $h$ can be noticeably simplified. The function $h$ is related to the initial potential $u$ by Eq. (4). It is easy to verify that $h_r = -\ln (g(r,z))$, where $f$ is an arbitrary solution of Eq. (1) with the initial potential $u$, satisfies

JETP LETTERS Vol. 113 No. 6 2021
Eq. (4). If \( u = 0 \) and \( f = \frac{1}{\sqrt{r^2 + z^2}} \), \( h_0 = -\ln(r) + \frac{1}{2} \ln \left( r^2 + z^2 \right) \). The system of equations for \( s \) at \( h = h_0 \) has the particular solution

\[
s_0 = -1 / 2 \ln \left( r^2 + z^2 \right) + \ln \left( \left( r^2 + z^2 \right)^{C_1} + C \right) - \ln \left( \left( 1 - 2C_1 \right) \left( r^2 + z^2 \right)^{C_1} + \left( 1 + 2C_1 \right) C \right),
\]

(13)

where \( C \) and \( C_1 \) are arbitrary constants. Formula (4) at \( \tilde{h} = h_0 + s_0 \) gives the new potential

\[
\tilde{u}(r, z) = -8 \frac{CG^2 \left( r^2 + z^2 \right)^{C_{11} - 1}}{\left( \left( r^2 + z^2 \right)^{C_1} + C \right)^2}.
\]

(14)

This potential satisfies the condition \( \tilde{u} < 0 \) and does not have singularities at \( C > 0, C_1 \geq 1 \). Thus, we obtain a two-parametric family of solvable Helmholtz potentials.

According to Eq. (10), solutions of Eq. (1) with potential (14) are given by the formula

\[
\tilde{Y} = \frac{\partial}{\partial z} Y - M \left( z Y - \sqrt{r^2 + z^2} Q \right),
\]

(15)

where

\[
M = \frac{\left( 1 + 2C_1 \right) \left( r^2 + z^2 \right)^{C_1} + C \left( 1 - 2C_1 \right)}{2 \left( r^2 + z^2 \right) \left( r^2 + z^2 \right)^{C_1} + C}.
\]

The function \( Q \) is determined from the system of equations

\[
\begin{align*}
Y_r + \frac{rY}{r^2 + z^2} - \frac{r}{\sqrt{r^2 + z^2}} - Q_z &= 0, \quad (16) \\
Y_z + \frac{zY}{r^2 + z^2} + \frac{r}{\sqrt{r^2 + z^2}} + Q_r &= 0, \quad (17)
\end{align*}
\]

where \( Y \) is any solution of Eq. (1) with the initial potential \( u = 0 \).

For example, we consider the following simple solutions of Eq. (1): \( 1, z, r^2 - 2z^2, 3r^2 - 2z^3, \frac{1}{\sqrt{r^2 + z^2}} \).

The substitution of these solutions into Eqs. (15)–(17) yields the following solutions of Eq. (1) with potential (14):

\[
\tilde{Y}_1 = \frac{(1 + 2C_1) \left( r^2 + z^2 \right)^{C_1} + \left( 1 - 2C_1 \right) C}{\sqrt{r^2 + z^2} \left( r^2 + z^2 \right)^{C_1} + C},
\]

Potential (14) can be used as the initial potential for new transformations. For the two-dimensional Schrödinger equation in Cartesian coordinates, it was shown in [17] that the double application of the classical Moutard transformation is efficient for obtaining nonsingular potentials. Similarly, nonsingular potentials in cylindrical coordinates can be effectively obtained using the double generalized Moutard transformation [1] that has the form

\[
\tilde{u} = u - 2 \frac{\partial^2 \ln (F)}{\partial r^2} - 2 \frac{\partial^2 \ln (F)}{\partial z^2} + 2 \left( \frac{\partial Y}{\partial z} \right) \left( 2r \frac{\partial Y}{\partial r} + Y \right) F^{-1} - 2 \left( \frac{\partial Y}{\partial z} \right) \left( 2r \frac{\partial Y}{\partial r} + Y \right) F^{-2}
\]

(18)

\[
+ 2r \left( \frac{\partial Y}{\partial z} \right) \left( 2r \frac{\partial Y}{\partial r} + Y \right) F^{-1} - 2r \left( \frac{\partial Y}{\partial z} \right) \left( 2r \frac{\partial Y}{\partial r} + Y \right) F^{-2},
\]

where \( F \) satisfies the system of equations

\[
\frac{\partial}{\partial z} F = r \left( \frac{\partial Y}{\partial r} \right) \left( Y_1 - 2 \frac{\partial Y}{\partial r} \right),
\]

(19)

\[
\frac{\partial}{\partial r} F = -r \left( \frac{\partial Y}{\partial z} \right) \left( Y_2 - 2 \frac{\partial Y}{\partial z} \right),
\]

(20)

and the functions \( Y_1 \) and \( Y_2 \) are solutions of Eq. (1) with the initial potential \( u \).

To avoid lengthy formulas, we consider the simple case \( C_1 = 1 \). In this case, the initial potential (14) has the form

\[
u = -\frac{8C}{\left( r^2 + z^2 + C \right)^2}.
\]

(21)
For the first example, we consider \( \hat{Y}_1 \) and \( \hat{Y}_2 \) at \( C_1 = 1 \) as solutions of the initial equation:

\[
Y_1 = \frac{3\left(r^2 + z^2\right) - C}{\sqrt{r^2 + z^2}\left(r^2 + z^2 + C\right)}, \quad Y_2 = \frac{r^2 + z^2 - 3C}{r^2 + z^2 + C}.
\]

From Eqs. (19) and (20), we obtain

\[
F = \frac{z}{\sqrt{r^2 + z^2}} + K,
\]

where \( K \) is an arbitrary constant. Then, the substitution of this expression into Eq. (18) gives the new solvable potential

\[
\tilde{u} = -\frac{8C}{\left(r^2 + z^2 + C\right)^3} + \frac{2\left(Kz + \sqrt{r^2 + z^2}\right)}{(z + \sqrt{r^2 + z^2}K)^2\sqrt{r^2 + z^2}}.
\]

For the second example, we consider \( \hat{Y}_2 \) and \( \hat{Y}_3 \) at \( C_1 = 1 \) as solutions of the initial equation:

\[
Y_1 = \frac{r^2 + z^2 - 3C}{r^2 + z^2 + C}, \quad Y_2 = \frac{z\left(r^2 + z^2 + 5C\right)}{r^2 + z^2 + C}.
\]

From Eqs. (19) and (20), we obtain

\[
F = \frac{r^4 + \left(z^2 - 15C\right)r^2}{r^2 + z^2 + C} + K,
\]

where \( K \) is an arbitrary constant. The substitution of this expression into Eq. (18) gives the new solvable potential

\[
\tilde{u} = 4N \left/ \left[ \left(r^4 + \left(z^2 + K - 15C\right)r^2 + K\left(z^2 + C\right)\right)^2 \right. \right],
\]

where

\[
N = \left( r^2 - K \right)\left( r^2 + z^2 \right)^2 - C\left( 30z^2 + 22K - 225C \right)r^2 + KC\left( 14z^2 - 2K + 15C \right).
\]

This potential does not have singularities at \( C > 0 \) and \( K \geq 15C \).

The presented examples show that, combining the nonlocal Darboux and generalized Moutard transformations, one can obtain new solvable potentials for the time-independent axially symmetric Schrödinger and Helmholtz equations. The search for partial solutions of the system of nonlinear equations is nontrivial for the nonlocal Darboux transformation, whereas the generalized Moutard transformation requires only the choice of an exact solution of the initial equation.

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