A Holstein-Primakoff and Dyson Realizations for the Lie Superalgebra $\mathfrak{gl}(m/n+1)$

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Abstract. The known Holstein-Primakoff and Dyson realizations for the Lie algebras $\mathfrak{gl}(n+1)$, $n = 1, 2, \ldots$ in terms of Bose operators (Okubo, S.: J. Math. Phys. 16, 528 (1975)) are generalized to the class of the Lie superalgebras $\mathfrak{gl}(m/n+1)$ for any $n$ and $m$. Formally the expressions are the same as for $\mathfrak{gl}(m+n+1)$, however both Bose and $m$ Fermi operators are involved.

Introduction

Recently we wrote down an analogue of the Dyson (D) and of the Holstein-Primakoff (H-P) realization for all Lie superalgebras $\mathfrak{sl}(1/n)$ [1]. In the present note we extend the results to the case of the Lie superalgebras $\mathfrak{gl}(m/n+1)$ for any $m$ and $n$.

Initially the H-P and the D realizations were given for $\mathfrak{sl}(2)$ [2, 3]. The generalization for $\mathfrak{gl}(n)$ is due to Okubo [4]. The extension to the case of quantum algebras is available so far only for $\mathfrak{sl}(2)$ [5] and $\mathfrak{sl}(3)$ [6]. To our best knowledge apart from [1] other results on H-P or D realizations for Lie superalgebras were not published in the literature so far.

The motivation in the present work stems from the various applications of the Holstein-Primakoff and of the Dyson realizations in theoretical physics. Beginning with [2] and [3] the H-P and the D realizations were constantly used in condensed matter physics. Some other early applications can be found in the book of Kittel [7] (more recent results are contained in [8]). For applications in nuclear physics see [9,10] and the references therein, but there are, certainly, several other publications. In view of the importance of the Lie superalgebras for physics, one could expect that extensions of the Dyson and of the Holstein-Primakoff

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realizations to $\mathbb{Z}_2$-graded algebras may be of interest too.

We recall the H-P realization of $gl(n + 1)$. The Weyl generators $E_{AB}$, $A, B = 1, \ldots, n + 1$ of $gl(n + 1)$ satisfy the commutation relations:

$$[E_{AB}, E_{CD}] = \delta_{BC}E_{AD} - \delta_{AD}E_{CD}. \tag{1}$$

Let $b_+^i$, $n = 1, \ldots, n$ be $n$ pairs of Bose creation and annihilation operators (CAOs),

$$[b_+^i, b_+^j] = \delta_{ij}, \quad [b_-^i, b_-^j] = [b_+^i, b_-^j] = 0, \quad i, j = 1, \ldots, n. \tag{2}$$

Then for any nonnegative integer $p$, $p \in \mathbb{Z}_+$, the H-P realization $\pi$ of $gl(n + 1)$ is defined on the generators as follows [4]:

$$\pi(E_{ij}) = b_+^i b_+^j, \quad i, j = 1, \ldots, n, \tag{3a}$$

$$\pi(E_{i,n+1}) = b_+^i p - \sum_{k=1}^n b_+^k b_-^k, \quad \pi(E_{n+1,i}) = p - \sum_{k=1}^n b_+^k b_-^k, \quad \pi(E_{n+1,n+1}) = p - \sum_{k=1}^n b_+^k b_-^k. \tag{3b}$$

(3a) only gives the known Jordan-Schwinger (J-S) realization of $gl(n)$ in terms of $n$ pairs of Bose CAOs. Therefore the H-P (and also the D) realizations are "more economical" than the J-S realization: they allow one to express the higher rank algebra $gl(n + 1)$ also through $n$ pairs of Bose CAOs.

Let us fix some notation. Unless otherwise stated $A, B, C, D = 1, 2, \ldots, m + n + 1$ and $i, j, k, l \in \{1, 2, \ldots, m + n = M\} \equiv \mathbb{M}$; $[x,y] = xy - yx$, $\{x,y\} = xy + yx$; $\mathbb{Z}_2 = \{0, 1\}$; $\langle A \rangle = 1$, if $A \leq m$; $\langle A \rangle = 0$, if $A > m$.

We proceed to define $gl(m/n + 1)$ in a representation independent form. Let $U$ be the (free complex) associative unital (= with unity) algebra of the indeterminants $\{E_{AB}|A, B = 1, \ldots, M + 1\}$ subject to the relations

$$E_{AB}E_{CD} - (-1)^{\langle A \rangle + \langle B \rangle + \langle C \rangle + \langle D \rangle}E_{CD}E_{AB} = \delta_{BC}E_{AD} - (-1)^{\langle A \rangle + \langle B \rangle + \langle C \rangle + \langle D \rangle}E_{CB}. \tag{4}$$

Introduce a $\mathbb{Z}_2$-grading on $U$, induced from

$$\text{deg}(E_{AB}) = \langle A \rangle + \langle B \rangle. \tag{5}$$

Then $U$ is an (infinite-dimensional) associative superalgebra, which is also a Lie superalgebra (LS) with respect to the supercommutator $[\ , \ ]$ defined between every two homogeneous elements $x, y \in U$ as

$$[x, y] = xy - (-1)^{\text{deg}(x)\text{deg}(y)}yx. \tag{7}$$

Its finite-dimensional subspace

$$\text{lin.env.} \{E_{AB}, [E_{AB}, E_{CD}]|A, B, C, D = 1, \ldots, m + n + 1\} \subset U \tag{8}$$
gives the Lie superalgebra \( gl(m/n + 1); U = U[gl(m/n + 1)] \) is its universal enveloping algebra. The relations (4) are the supercommutation relations on \( gl(m/n + 1) \):

\[
[E_{AB}, E_{CD}] = \delta_{BC} E_{AD} - (-1)^{[(A)+(B)]+[(C)+(D)]} E_{CB}.
\] (9)

One can certainly define \( gl(m/n + 1) \) in its matrix representation. In that case \( E_{AB} \) is a \((m+n+1)\times(m+n+1)\) matrix with 1 on the intersection of the \( A^{th} \) row and \( B^{th} \) column and zero elsewhere.

The Dyson and the Holstein-Primakoff realizations are different embeddings of \( gl(m/n + 1) \) into the algebra \( W(m/n) \) of all polynomials of \( m \) pairs of Fermi CAOs and \( n \) pairs of Bose CAOs. The precise definition of \( W(m/n) \) is the following. Let \( A^\pm_i, \ i \in M \) be \( \mathbb{Z}_2 \)-graded indeterminants:

\[
\text{deg}(A^\pm_i) = (i).
\] (10)

Then \( W(m/n) \) is the associative unital superalgebra of all \( A^\pm_i \), subject to the relations

\[
[A^+_i, A^+_j] = \delta_{ij}, \quad [A^+_i, A^-_j] = [A^-_i, A^-_j] = 0.
\] (11)

With respect to the supercommutator (7) \( W(m/n) \) is also a Lie superalgebra.

From (11) one concludes that \( A^+_1, \ldots A^+_m \) are Fermi CAOs, which are odd variables; \( A^\pm_{m+1}, \ldots A^\pm_{m+n} \) are Bose CAOs, which are even. The Bose operators commute with the Fermi operators.

**Proposition 1 (Dyson realization).** The linear map \( \varphi : gl(m/n + 1) \rightarrow W(m/n) \), defined on the generators as

\[
\varphi(E_{ij}) = A^+_i A^-_j, \quad i, j = 1, \ldots, M,
\] (12a)

\[
\varphi(E_{i,M+1}) = A^+_i, \quad \varphi(E_{M+1,i}) = (p - \sum_{k=1}^{M} A^+_k A^-_k) A^-_i, \quad \varphi(E_{M+1,M+1}) = p - \sum_{k=1}^{M} A^+_k A^-_k.
\] (12b)

is an isomorphism of \( gl(m/n + 1) \) into \( W(m/n) \) for any number \( p \).

**Proof.** The images \( \varphi(E_{AB}) \) are linearly independent in \( W(m/n) \). It is straightforward to verify that they preserve the supercommutation relations (9),

\[
[\varphi(E_{AB}), \varphi(E_{CD})] = \delta_{BC} \varphi(E_{AD}) - (-1)^{[(A)+(B)]+[(C)+(D)]} \varphi(E_{CB}).
\] (13)

In the intermediate computations the following relation is useful

\[
[N, A^\pm_i] = \pm A^\pm_i, \quad \text{where} \quad N = \sum_{k=1}^{M} A^+_k A^-_k.
\] (14)

The Dyson realization defines an infinite-dimensional representation of \( gl(m/n + 1) \) (for \( m > 0 \)) in the Fock space \( F(m/n) \) with orthonormed basis

\[
|K\rangle \equiv |k_1, \ldots, k_M\rangle = \frac{(A^+_1)^{k_1} \cdots (A^+_M)^{k_M}}{\sqrt{k_1! \cdots k_M!}} |0\rangle, \quad k_1, \ldots, k_m = 0, 1; \ k_{m+1}, \ldots, k_M \in \mathbb{Z}_+.
\] (15)
Let $|K\rangle_{\pm i}$ (resp. $|K\rangle_{i-j}$) be a vector obtained from $|K\rangle$ after a replacement of $k_i$ with $k_i \pm 1$ (resp. $k_i \rightarrow k_i + 1$, $k_j \rightarrow k_j - 1$). The transformations of the basis (15) under the action of the CAOs read:

$$A^+_i |K\rangle = (-1)^{(i)(k_1+...+k_{i-1})} \sqrt{1+(-1)^{(i)}k_i}|K\rangle_i, \quad A^-_i |K\rangle = (-1)^{(i)(k_1+...+k_{i-1})} \sqrt{k_i}|K\rangle_{-i}. \quad (16)$$

As a consequence one obtains the transformations of the $gl(m/n)$ module $F(m/n)$:

$$\varphi(E_{i,M+1}) |K\rangle = (-1)^{(i)(k_1+...+k_{i-1})} \sqrt{1+(-1)^{(i)}k_i}|K\rangle_i, \quad (17a)$$

$$\varphi(E_{M+1,i}) |K\rangle = (-1)^{(i)(k_1+...+k_{i-1})}(p+1-\sum_{j=1}^{M} k_j) \sqrt{k_i}|K\rangle_{-i}, \quad (17b)$$

$$\varphi(E_{i,i}) |K\rangle = k_i|K\rangle, \quad (17c)$$

$$\varphi(E_{i,i}) |K\rangle = (-1)^{(i)(k_1+...+k_{i-1})+(j)(k_1+...+k_{j-1})} \sqrt{k_j(1+(-1)^{(i)}k_i)}|K\rangle_{-j,i}, \quad i < j, \quad (17d)$$

$$\varphi(E_{i,j}) |K\rangle = (-1)^{(i)(k_1+...+k_{i-1})+(j)(k_1+...+k_{j-1})} \sqrt{k_j(1+(-1)^{(i)}k_i)}|K\rangle_{-j,i}, \quad i > j. \quad (17f)$$

If $p$ is not a positive integer, $p \notin \mathbb{N}$, $F(m/n)$ is a simple $gl(m/n+1)$ module. For any positive integer $p$, $p \in \mathbb{N}$, the representation of $gl(m/n+1)$ in $F(m/n)$ is indecomposable. The subspace

$$F(p; m/n)_{inv} = \text{lin.env.} \{ |K\rangle \mid k_1 + \ldots + k_M > p \} \subset F(m/n) \quad (18)$$

is an infinite-dimensional subspace, invariant with respect to $gl(m/n+1)$. The factor spaces

$$F(p; m/n)_0 \equiv F(m/n)/F(p; m/n)_{inv} = \text{lin.env.} \{ |K\rangle \mid p \leq k_1 + \ldots + k_M \}, \quad p = 1, 2, \ldots, \quad (19)$$

are finite-dimensional irreducible $gl(m/n+1)$-modules.

The advantage of the Dyson realization (12) is its simplicity. Its disadvantage - the Fock representation of $gl(m/n+1)$ is not unitarizable. The latter, the representation to be unitarizable, is usually required for physical reasons. We recall that a representation $\varphi$ of a (super)algebra $L$ in a Hilbert space $V$ is unitarizable with respect to an antilinear antiinvolution $\omega : L \rightarrow L$ and a scalar product $(, )$ in $V$, if

$$(\varphi(a)x, y) = (x, \varphi(\omega(a))y), \quad \forall a \in L, \quad \forall x, y \in V. \quad (20)$$

The Dyson representation in $F(m/n)$ is not unitarizable with respect to the "compact" antiinvolution

$$\omega(E_{AB}) = E_{BA}, \quad A, B = 1, \ldots, M + 1. \quad (21)$$

The factor-modules $F_0(p; m/n)$, $p \in \mathbb{N}$, however do carry unitarizable representations for any $p \in \mathbb{N}$. In order to show this it is convenient to introduce a new basis within each $F_0(p; m/n)$, which we postulate to be orthonormed:

$$|K\rangle = \sqrt{(p-\sum_{j=1}^{M} k_j)!} |K\rangle. \quad (22)$$
In this basis the transformation relations (17) read:

\[ \varphi(E_{i,M+1})|K\rangle = (-1)^{(i)(k_i+\ldots+k_{i-1})} \sqrt{(1 + (-1)^{(i)}k_i)(p - \sum_{j=1}^{M} k_j)} |K\rangle_{i}, \tag{23a} \]

\[ \varphi(E_{M+1,i})|K\rangle = (-1)^{(i)(k_1+\ldots+k_{i-1})} \sqrt{k_i(p + 1 - \sum_{j=1}^{M} k_j)} |K\rangle_{-i}, \tag{23b} \]

\[ \varphi(E_{M+1,M+1})|K\rangle = (p + 1 - \sum_{j=1}^{M} k_j)|K\rangle, \tag{23c} \]

\[ \varphi(E_{ii})|K\rangle = k_i|K\rangle, \tag{23d} \]

\[ \varphi(E_{ij})|K\rangle = (-1)^{(i)(k_i+\ldots+k_{i-1})+(j)(k_j+\ldots+k_{j-1})} \sqrt{k_i(1 + (-1)^{(i)}k_i)} |K\rangle_{-j,i}, \quad i < j, \tag{23e} \]

\[ \varphi(E_{ij})|K\rangle = (-1)^{(i)(k_i+\ldots+k_{i-1})+(j)(k_j+\ldots+k_{j-1})} \sqrt{k_j(1 + (-1)^{(j)}k_j)} |K\rangle_{-j,i}, \quad i > j. \tag{23f} \]

It is straightforward to check that (20) hold with respect to the antiinvolution (21). Hence the representation related to the result we have just obtained.

**Proposition 2 (Holstein-Primakoff realization).** The linear map \( \pi : gl(m/n + 1) \rightarrow W(m/n) \), defined on the generators as

\[ \pi(E_{ij}) = A_i^+ A_j^-, \quad i, j = 1, \ldots, M, \tag{24a} \]

\[ \pi(E_{i,M+1}) = A_i^+ \sqrt{p - \sum_{j=1}^{M} A_j^+ A_j^-}, \quad \pi(E_{M+1,i}) = \sqrt{p - \sum_{k=1}^{M} A_k^+ A_k^-}, \quad \pi(E_{M+1,M+1}) = p - \sum_{k=1}^{M} A_k^+ A_k^- . \tag{24b} \]

is an isomorphism of \( gl(m/n + 1) \) into \( W(m/n) \) for any positive integer \( p \).

**Proof.** Acting with the \( gl(m/n + 1) \) generators on the basis (15) one obtains the same transformation relations (23) with the only difference that everywhere in (23) \( |K\rangle \) have to be replaced with \( |K\rangle \). The proof can be carried out also purely algebraically, using the supercommutation relations (11). To this end the following formula is useful:

\[ f(N)A_i^\pm = A_i^\pm f(N \pm 1), \quad N = \sum_{j=1}^{M} A_j^+ A_j^-. \tag{25} \]

where \( f(z) \) is any (analytical) function in \( z \).

The representation \( \pi \) is defined in the entire Fock space. Observe that with respect to \( \pi(E_{AB}) \), \( A, B = 1, \ldots, M + 1 \), the Fock space resolves into a direct sum of two invariant (and moreover irreducible) subspaces (which was not the case with the Dyson representation):

\[ F(p;m/n)_{\text{lin.env.}} = \{ |K\rangle | p \leq k_1 + \ldots + k_M \}, \quad F(p;m/n)_{\text{inv}} = \{ |K\rangle | k_1 + \ldots + k_M > p \}. \tag{26} \]

This property is due to the factors \( \sqrt{p - \sum_{j=1}^{M} k_j} \) and \( \sqrt{p + 1 - \sum_{j=1}^{M} k_j} \) in (23a) and (23b), respectively.
In the case \( m = 0 \) the Holstein-Primakoff realization (24) reduces to the Holstein-Primakoff realization (3) of \( gl(n+1) \) in terms of only Bose operators. Taking in (24) all \( A_i^\pm \) to be Bose CAOs, one obtains the H-P realization of \( gl(m+n+1) \). The case \( n = 0 \) yields a Fermi realization of the Lie superalgebras \( gl(m/1) \). Its restriction to \( sl(m/1) \) coincides with the results announced in [1].

Let us note in conclusion that explicit expressions for all finite-dimensional irreducible representations of \( gl(m/1) \) and a large class of representations of \( gl(m/n+1) \) are available [12]. They have been generalized also to the quantum case [13]. The formulae are however extremely involved. The Dyson and the Holstein-Primakoff representations lead to a small part of all representations. Their description is however simple and it is given in familiar for physics Fock spaces.

Acknowledgments. The author is thankful to Prof. Randjbar-Daemi for the kind hospitality at the High Energy Section of ICTP. It is a pleasure to thank Prof. C. Reina for making it possible to visit the Section on Mathematical physics in Sissa, where most of the results were obtained. The work was supported by the Grant \( \Phi - 416 \) of the Bulgarian Foundation for Scientific Research.

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