INTRODUCTION

The goal of this paper is to introduce the reader to recent work on some basic arithmetic questions about moduli spaces of vector bundles on curves. In particular, we will focus on the correspondence between rational-point problems on (étale) forms of such moduli spaces and classical problems on the Brauer groups of function fields. This might be thought of as a non-abelian refinement of the following result of Artin and Tate (which we state in a special case).

Fix a global field $K$ with scheme of integers $S$. (If $\text{char } K > 0$, we assume that $S$ is proper so that it is unique.) Let $f : X \to S$ be a proper flat morphism of relative dimension 1 with a section $D \subset X$ and smooth generic fiber $X_\eta$. Let $\text{Br}_\infty(X)$ denote the kernel of the restriction map $\text{Br}(X) \to \prod_{\nu | \infty} \text{Br}(X \otimes K_\nu)$, i.e., the Brauer classes which are trivial at the fibers over the Archimedean places. (If $\text{char } K > 0$ or $K$ is totally imaginary, $\text{Br}_\infty(X) = \text{Br}(X)$.)

**Theorem** (Artin-Tate). *There is a natural isomorphism
\[\text{Br}_\infty(X) \cong \text{III}^1(\text{Spec } K, \text{Jac}(X_\eta)).\]
From a modern point of view, this isomorphism arises by sending a $\mu_n$-gerbe $\mathcal{X} \to X$ to the relative moduli space of invertible twisted sheaves of degree 0. (We have provided a brief review of gerbes, moduli of twisted sheaves, and the Brauer group in the form of an appendix.) Our goal will be to study the properties of higher-rank moduli of twisted sheaves. Each moduli space carries an adelic point (is in the analogue of the Tate-Shafarevich group), but most of the moduli spaces are geometrically rational (or at least rationally connected) with trivial Brauer-Manin obstruction, so that the Hasse principle is conjectured to hold. This yields connections between information about the complexity of Brauer classes on arithmetic surfaces and conjectures on the Hasse principle for geometrically rational varieties.

As the majority of work on moduli spaces of vector bundles on curves is done in a geometric context, we give in Section 1 a somewhat unconventional introduction to the study of forms of moduli. This is done primarily to fix notation and introduce some basic constructions. We also use this section to introduce a central idea: forms of the moduli problem naturally give forms of the stack and not merely of the coarse moduli space, and the relation between these forms captures cohomological information which ultimately is crucial for making the arithmetic connections. In this geometric section, we use this philosophy to prove a silly “non-abelian Torelli theorem”.

Starting with Section 2 we turn our attention to arithmetic problems. In particular, after asking some basic questions in Section 2 we link a standard conjecture on the Hasse principle for 0-cycles of degree 1 (Conjecture 1.5(a) of [4]) to recent conjectures on the period-index problem in Section 3.

We also recast some well-known results due to Lang and de Jong on the Brauer group in terms of the Brauer-Manin obstruction is the only one for this particular variety. Just as the element of $\text{Br}((\mathbb{Q}/\mathbb{P}^1)_K)$ provides in [15] in the form of biquaternion algebras over $\mathbb{Q}$, there are ramified classes whose period and index are unequal. A nice collection of such examples is provided in [20] for twisted sheaves of degree $1$ on geometrically rational varieties.

When $\text{char} K = p > 0$, one can prove that any $\alpha \in \text{Br}(X)$ whose period is relatively prime to $p$ satisfies $\text{per}(\alpha) = \text{ind}(\alpha)$ (see [20]). The proof uses the geometry of moduli spaces of twisted sheaves on surfaces over finite fields, and therefore contributes nothing to our understanding of the mixed-characteristic situation.

In Sections 4 and 5 we start to consider how one might generalize the Theorem to the case of ramified Brauer classes. Things here are significantly more complicated – for example, we know that there are ramified classes whose period and index are unequal. A nice collection of such examples is provided in [15] for the form of biquaternion algebras over $\mathbb{Q}(t)$ with Faddeev index 4 such that for all places $\nu$ of $\mathbb{Q}$ the restriction to $\mathbb{Q}_\nu(t)$ has index 2. Can we view such algebras as giving a violation of some sort of Hasse principle? As we discuss in Section 5 there is a canonical rational-point problem associated to such an algebra, but it fails to produce a counterexample to the Hasse principle because it lacks a local point at some place! In other words, while these algebras seem to violate some sort of “Hasse principle,” the canonically associated moduli problems actually have local obstructions. This phenomenon is intriguing and demands further investigation.
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NOTATION AND ASSUMPTIONS

As in the introduction, given an arithmetic surface $X \rightarrow S$ with $S$ the scheme of integers of $K$, we will write $\text{Br}_\infty(X)$ for the subgroup of $\text{Br}(X)$ of classes whose restriction to $X \otimes K_\nu$ is trivial for all Archimedean places of $K$.

We assume familiarity with the theory of algebraic stacks, as explained in [16]. Given a stack $\mathcal{X}$, we will denote the inertia stack by

$$\mathcal{I}(\mathcal{X}) := \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}} \Delta \mathcal{X};$$

this is the fiber product of the diagonal with itself, and represents the sheaf of groups which assigns to any object its automorphisms. The notation related to moduli is explained in the Appendix. While gerbes, twisted sheaves, and the Brauer group are briefly reviewed in the Appendix, the reader completely unfamiliar with them will probably benefit from consulting [10, 18, 20].

Most of the material described here is derived from the more extensive treatments in [19, 20]. We have thus felt free to merely sketch proofs. The material of Sections 4 and 5 is new, but is not in its final form. Again, we have not spelled out all of the details; they will appear once a more satisfactory understanding of the phenomena described there has been reached.

1. FORMS OF MODULI AND AN ISOTrIVIAL TOrelli THEOREM

Let $C/K$ be a curve over a field with a $K$-rational point $p$ and $L$ an invertible sheaf on $C$. The central objects of study in this paper will be forms of the moduli stack $\mathcal{M}_{C/K}(r, L)$ of stable vector bundles on $C$ of rank $r$ and determinant $L$. In this section we give a geometric introduction to the existence and study of these forms. It turns out that the most interesting forms of $\mathcal{M}_{C/K}(r, L)$ are those which arise from $\mu_r$-gerbes $\mathcal{C} \rightarrow C$. The reader unfamiliar with gerbes should think of these as stacky forms of $C$; the curve $C$ itself corresponds to the trivial gerbe $\mathbb{B}\mu_r \times C$. As a warm-up for the rest of the paper, we give an amusing Torelli-type theorem for such stacky forms of $C$.

We begin by discussing forms of the coarse space $M_C(r, L)$. From a geometric point of view, forms of a variety are simply isotrivial families.

Definition 1.1. Given an algebraically closed field $k$, a morphism of $k$-schemes $f : X \rightarrow Y$ is isotrivial if there is a faithfully flat morphism $g : Z \rightarrow Y$, a $k$-scheme $W$, and isomorphism $X \times_Y Z \cong W \times Z$.

Isotrivial families arise naturally from geometry, as the following example shows.

Example 1.2. Suppose $C$ is an infinitesimal deformation of $C$ over $k[\varepsilon]$ and $L_\varepsilon$ and $L'_\varepsilon$ are two infinitesimal deformations of $L$. We claim that the moduli spaces $M_{C/\kappa[\varepsilon]}(r, L_\varepsilon)$ and $M_{C/\kappa[\varepsilon]}(r, L'_\varepsilon)$ are isomorphic. To see this, note that $L_i \otimes L_i^\ast$ is an infinitesimal deformation of $\mathcal{O}_C$, is thus an $r$th power in $\text{Pic}(C)$. Twisting by an $r$th root gives an isomorphism of moduli problems. This obviously generalizes to deformations of $C$ over Artinian local $k$-algebras $A$. As a consequence, if $R$ is a complete local $k$-algebra with maximal ideal $m$ and residue field $k$ and $\mathcal{L}$ is an invertible sheaf of degree $d$ on $C \otimes R$ with reduction $L$ over the residue field, we see that $M_{C \otimes R/R}(r, \mathcal{L}) \cong M_{C/k}(r, L) \otimes R$. Indeed, since $M_{C \otimes R/R}(r, \mathcal{L})$ and $M_{C/k}(r, L)$ are proper, the Grothendieck existence theorem shows that the natural map

$$\text{Isom}_R(M_{C \otimes R/R}(r, \mathcal{L}), M_{C/k}(r, L) \otimes R) \rightarrow \lim \text{Isom}_{R_n}(M_{C \otimes R_n/R_n}(r, \mathcal{L}_R), M_{C/k}(r, L) \otimes R_n)$$

is an isomorphism, where $R_n := R/m^{n+1}$. Since the identity map over the residue field lifts to each infinitesimal level, we conclude that there is an isomorphism over $R$.

Now, given $r$ and $d$ as above, let $M_C(r, d)$ denote the moduli space of all stable vector bundles of rank $r$ and degree $d$. The determinant defines a morphism $\det : M_C(r, d) \rightarrow \text{Pic}^d_{C/k}$. By the argument of
the previous paragraph, all of the fibers of \( \det \) are mutually isomorphic. This is a basic example of an isotrivial family which arises naturally from geometry.

How can we classify isotrivial families? The classification is familiar from descent theory. Let us illustrate how this works using Example \[ \text{Example 1.2} \]To simplify notation, let \( M \) denote the moduli space \( M_C(r, L) \) for some fixed \( L \in \text{Pic}^d_{C/k} \), and write \( X = C \times \text{Pic}^d_{C/k} \). Consider the \( \text{Pic}^d_{C/k} \)-scheme

\[
T = \text{Isom}_{\text{Pic}^d_{C/k}}(M(r, d), M \times \text{Pic}^d_{C/k});
\]

this is a left torsor under the scheme of automorphisms \( \text{Aut}(M) \). We now make one simplifying assumption which will tie the geometry to algebraic constructions we will do later.

**Hypothesis 1.3.** Suppose that the curve \( C \) has trivial automorphism group, and that both \( r \) and \( d \) are odd.

The purpose of Hypothesis \[ \text{Hypothesis 1.3} \]is to ensure that the automorphism group of \( M_C(r, L) \) is given entirely by the natural image of \( \text{Pic}^d_C[r] \) in \( \text{Aut}(M) \) under the map sending an invertible sheaf \( N \in \text{Pic}^d_C[r] \) to the automorphism given by twisting by \( N \) (see \[ \text{Example 1.3} \]).

Given Hypothesis \[ \text{Hypothesis 1.3} \]the torsor \( T \) corresponds to a class

\[
\pi \in H^1(\text{Pic}^d_{C/k}, R^1\text{pr}_2^*\mu_r),
\]

where \( \text{pr}_2 \) is the second projection of the product \( C \times \text{Pic}^d_{C/k} \). The cohomology class thus arising belongs to a convergent in the Leray spectral sequence for \( \mu_r \) with respect to the morphism \( \text{pr}_2 : C \times \text{Pic}^d_{C/k} \to \text{Pic}^d_{C/k} \). In fact, the Leray spectral sequence gives a surjection

\[
H^2(C \times \text{Pic}^d_{C/k}, \mu_r) \to H^1(\text{Pic}^d_{C/k}, R^1\text{pr}_2^*\mu_r).
\]

(This is surjective because the edge map \( H^3(\text{Pic}^d_{C/k}, \mu_r) \to H^3(C \times \text{Pic}^d_{C/k}, \mu_r) \) is split by any \( k \)-point \( p \in C \).) There is a Chern class map

\[
\text{Pic}(C \times \text{Pic}^d_{C/k}) \to H^2(C \times \text{Pic}^d_{C/k}, \mu_r)
\]

arising in the usual way from the Kummer sequence.

Since \( C \) has a \( k \)-point, there is a tautological sheaf \( \mathcal{L} \) on \( C \times \text{Pic}^d_{C/k} \).

**Claim 1.4.** The class \( \pi \) is the image of \( \mathcal{L} \otimes L^{-1} \) under the composition of the above maps, where \( L \) is the invertible sheaf corresponding to the base point parameterizing \( M \).

To prove this claim, we give a geometric interpretation of the maps

\[
\text{Pic}(C \times \text{Pic}^d_{C/k}) \to H^2(C \times \text{Pic}^d_{C/k}, \mu_r)
\]

and

\[
H^2(C \times \text{Pic}^d_{C/k}, \mu_r) \to H^1(\text{Pic}^d_{C/k}, \text{Pic}^d_{C/k}[r]).
\]

To interpret \[ \text{Claim 1.4} \]: given an invertible sheaf \( \mathcal{N} \) on \( C \times \text{Pic}^d/C/k \), we get a \( \mu_r \)-gerbe \( [\mathcal{N}]^{1/r} \) over \( X \). This is a stack, i.e., a moduli problem, which in this case is very explicit. Given an \( X \)-scheme \( T \to X \), an object of \( [\mathcal{N}]^{1/r} \) is given by a pair \( (\Lambda, \psi) \) with \( \Lambda \) an invertible sheaf on \( T \) and \( \psi : \Lambda^{\otimes r} \to \mathcal{N} \) an isomorphism.

To interpret \[ \text{Claim 1.5} \]: given a \( \mu_r \)-gerbe \( \mathcal{X} \to X \), we can consider the pushforward stack \( \text{pr}_2^* \mathcal{X} \to \text{Pic}^d_{C/k} \).

**Claim 1.5.** With the above notation, given a morphism \( f : X \to Y \) the stack \( f_* \mathcal{X} \) is a \( f_* \mu_r \)-gerbe over a \( R^1f_*\mu_r \)-pseudotorsor. If \( \mathcal{X} \times_Y U \) is a neutral gerbe over \( X \times_Y U \) for some étale surjection \( U \to Y \), sheafification of \( f_* \mathcal{X} \) is an étale torsor.

**Sketch of proof.** By definition, an object of \( f_* \mathcal{X} \) over a \( Y \)-scheme \( S \to Y \) is an object of \( \mathcal{X} \) over \( S \times_Y X \). It immediately follows that \( f_* \mathcal{X} \) is a gerbe banded by \( f_* \mu_r \). Let the sheafification of \( f_* \mathcal{X} \) be denoted \( \Theta \to Y \). Twisting by torsors gives an action \( R^1f_*\mu_r \times \Theta \to \Theta \). One checks that this makes \( \Theta \) into a pseudotorsor; the hypothesis of the last sentence of the claim transparently makes \( \Theta \) have local sections in the étale topology, and therefore a torsor. \[ \square \]
Sending \( \mathcal{X} \) to the sheafification of \( pr_2^* \mathcal{X} \) gives an interpretation of (2).

**Proof of Claim 1.4.** Let \( \mathcal{X} \to X \) be the \( \mu_r \)-gerbe \( [\mathcal{L} \otimes L^{-1}]^{1/r} \) defined above, and let \( \Theta \) denote the sheafification of \( \mathcal{X} \) (as a \( Pic^{d}_{C/k} \)-stack). Tensor product defines a morphism of \( Pic^{d}_{C/k} \)-stacks

\[
pr_2^* \mathcal{X} \times \mathcal{M}(r, L)_{Pic^{d}_{C/k}} \to \mathcal{M}(r, d),
\]

where \( \mathcal{M} \) is used in place of \( M \) to denote the stack instead of the coarse moduli space. Passing to sheafifications yields a map

\[
\Theta \times M(r, L) \to M(r, d)
\]

which is compatible with the natural \( R^1pr_2^* \mu_r \)-actions. By adjunction, this gives a \( Pic(C)[r] \)-equivariant map

\[
\Theta \to \text{Isom}(M(r, L), M(r, d)),
\]

yielding the desired result. \( \square \)

This analysis of Example 1.2 fits into a general picture. Let \( f: C \to S \) be a proper smooth relative curve of genus \( g \geq 2 \). Suppose \( \mathcal{C} \to C \) is a \( \mu_r \)-gerbe whose associated cohomology class \( [\mathcal{C}] \in H^2(C, \mu_r) \) is trivial on all geometric fibers of \( f \).

**Proposition 1.6.** There is an étale surjection \( U \to S \) and an isomorphism of stacks \( \tau: \mathcal{M}_{\theta U/U}(r, L) \simeq \mathcal{M}_{\theta U/U}(r, L) \).

In other words, the stack of stable \( \mathcal{C} \)-twisted sheaves of rank \( r \) determinant \( L \) is a form of the stack of stable sheaves on \( C \) of rank \( r \) and determinant \( L \). (As the reader will note from the proof, it is essential that the cohomology class be trivial on geometric fibers for this to be true.)

**Sketch of proof.** The proper and smooth base change theorems and the compatibility of the formation of étale cohomology with limits show that there is an étale surjection \( U \to S \) such that \( [\mathcal{C}]_U = 0 \).

Thus, it suffices to show that if \( [\mathcal{C}] = 0 \) (in other words, if \( \mathcal{C} \cong \mathcal{B}_{\mu_r, C} \)) then there is an isomorphism \( \mathcal{M}_{\theta S}(r, L) \simeq \mathcal{M}_{C/S}(r, L) \). Under this assumption, there is an invertible \( \mathcal{C} \)-sheaf \( \chi \) such that \( \chi^\otimes r \) is isomorphic to \( \mathcal{C} \). Tensoring with \( \chi^{-1} \) gives the isomorphism in question. \( \square \)

The isomorphism \( \tau \) of Proposition 1.6 yields a coarse isomorphism \( \Phi: \mathcal{M}_{\theta U/U}(r, L) \simeq \mathcal{M}_{C/S}(r, L) \), so that \( \mathcal{M}_{\theta S}(r, L) \) is a form of \( \mathcal{M}_{C/S}(r, L) \). Just as above, we can give the cohomology class corresponding to this form (by descent theory): it is precisely the image of \( [\mathcal{C}] \) under the edge map \( \epsilon: H^3(C, \mu_r) \to H^1(S, \mathcal{R}^1 f_* \mu_r) \) in the Leray spectral sequence. There is one mildly interesting consequence of this fact. Since \( \epsilon \) has (in general) a kernel, we see that the coarse moduli space \( \mathcal{M}_{\theta S}(r, L) \) does not (in general) characterize the “curve” \( \mathcal{C} \). In other words, it appears that there is no “stacky Torelli theorem”. It is perhaps illuminating to give an example of the failure.

**Lemma 1.7.** If \( f: X \to S \) is a proper morphism with geometrically connected fibers such that \( Pic_{X/S} = \mathbb{Z} \), then the natural pullback map \( H^2(S, \mu_r) \to H^2(X, \mu_r) \) is injective.

**Proof.** The Leray spectral sequence shows that the kernel of the map is the image of \( H^0(S, \mathcal{R}^1 f_* \mu_r) = H^1(X, Pic_{X/S}[r]) \).

**Example 1.8.** If \( \mathcal{C} = C \times \mathcal{F} \) for a \( \mu_r \)-gerbe \( \mathcal{F} \to S \), then (for example, by the above Leray spectral sequence calculation) there is an isomorphism \( b: \mathcal{M}_{\theta S}(r, L) \simeq \mathcal{M}_{C/S}(r, L) \). However, the stacks \( \mathcal{M}_{\theta S}(r, L) \) and \( \mathcal{M}_{C/S}(r, L) \) are not isomorphic unless \( \mathcal{F} \) is isomorphic to \( \mathcal{B}_{\mu_r, S} \). In fact, viewing (via \( b \)) both of these stacks as \( \mu_r \)-gerbes over \( \mathcal{M}_{C/S}(r, L) \), we have an equation

\[
[\mathcal{M}_{\theta S}(r, L)] - [\mathcal{M}_{C/S}(r, L)] = [\mathcal{F}_{\theta S}(r, L)]
\]

as described in (14). By Lemma 1.7, we see that \( [\mathcal{F}] = 0 \) if and only if \( [\mathcal{F}_{\theta S}(r, L)] = 0 \), as desired.

On the other hand, if we keep track of the stacky structure, we have the following silly “Torelli” theorem. Let \( f: C \to S \) be a proper smooth relative curve of genus \( g \geq 2 \) with a section whose geometric generic fiber has no nontrivial automorphisms. Suppose \( L \) is an invertible sheaf on \( C \) of degree \( d \) on each geometric fiber of \( f \) and \( r \) is a positive integer relatively prime to \( d \) such that \( rd \) is odd. Given
stacks $\mathcal{X}$ and $\mathcal{Y}$ with inclusions $\mu_r \hookrightarrow \mathcal{I}(\mathcal{X})$ and $\mu_r \hookrightarrow \mathcal{I}(\mathcal{Y})$, the notation $\mathcal{X} \cong \mathcal{Y}$ will mean that there is an isomorphism $\iota : \mathcal{X} \cong \mathcal{Y}$ such that the composition $\iota^* \mu_r \cong \mu_r \to \mathcal{I}(\mathcal{X}) \to \iota^* \mathcal{I}(\mathcal{Y})$ is the pullback under $\iota$ of the given inclusion $\mu_r \to \mathcal{I}(\mathcal{Y})$. We will call such an isomorphism "$\mu_r$-linear".

**Theorem 1.9** (Isotrivial Torelli). With the above notation, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are $\mu_r$-gerbes on $C$ whose restrictions to geometric fibers of $f$ are trivial, then $\mathcal{G}_1 \cong \mathcal{G}_2$ if and only if $\mathcal{M}_{\mathcal{G}_1/S}(r,L) \cong \mathcal{M}_{\mathcal{G}_2/S}(r,L)$.

**Proof:** The assumption that $f$ has a section leads (via pullback and the relative cohomology class of the section) to a natural splitting

$$H^2(C, \mu_r) = H^2(S, \mu_r) \oplus H^1(S, R^1 f_\ast \mu_r) \oplus H^0(S, R^2 f_\ast \mu_r)$$

such that the first two summands correspond to classes which are trivial on geometric fibers of $f$. As discussed above, the image of $[\mathcal{G}]$ in $H^1(S, R^1 f_\ast \mu_r)$ is the class associated to $M_{\mathcal{G}/S}(r,L)$. If $\mathcal{M}_{\mathcal{G}_1/S}(r,L)$ is isomorphic to $\mathcal{M}_{\mathcal{G}_2/S}(r,L)$ (say, by an isomorphism $\varphi$) then certainly the same is true for the coarse moduli spaces (isomorphic via $\varphi$), from which we conclude that $[\mathcal{G}_1]$ and $[\mathcal{G}_2]$ have the same image in $H^1(S, R^1 f_\ast \mu_r)$. Thus, there exists some $\mu_r$-gerbe $\mathcal{I} \to S$ such that $[\mathcal{G}_1] - [\mathcal{G}_2] = [\mathcal{I}]_C$ in $H^2(C, \mu_r)$. Using Giraud's theory [10, §IV.2.4], it follows that we can write $\mathcal{G}_1 = \mathcal{G}_2 \wedge \mathcal{I}_C$. In this situation, there is a canonical isomorphism $b : M_{\mathcal{G}_1/S}(r,L) \cong M_{\mathcal{G}_2/S}(r,L)$ with the property that $[\mathcal{M}_{\mathcal{G}_1/S}(r,L)] - b^* [\mathcal{M}_{\mathcal{G}_2/S}(r,L)] = [\mathcal{I}]_{M_{\mathcal{G}_1/S}(r,L)}$ in $H^2(M_{\mathcal{G}_1/S}(r,L), \mu_r)$.

By Hypothesis [1,3] any automorphism of $M_{\mathcal{G}_1/S}(r,L)$ lifts to a $\mu_r$-linear automorphism of $\mathcal{M}_{\mathcal{G}_1/S}(r,L)$. Applying this to $b \circ \varphi^{-1}$, we see that $b$ lifts to an isomorphism $\mathcal{M}_{\mathcal{G}_1/S}(r,L) \cong \mathcal{M}_{\mathcal{G}_2/S}(r,L)$. We thus conclude that $\mathcal{M}_{\mathcal{G}_1/S}(r,L)$ is a trivial gerbe. By Lemma [1,7] we see that $[\mathcal{I}] = 0$, so $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic $\mu_r$-gerbes. \hfill $\square$

**2. SOME ARITHMETIC QUESTIONS ABOUT BRAUER GROUPS AND RATIONAL POINTS ON VARIETIES OVER GLOBAL FIELDS**

Let $C/k$ be a curve over a field. In this section we describe how the forms arising in the preceding section are related to basic questions about the arithmetic of function fields. The linkage is provided by an interpretation of the group $H^2(C, \mu_r)$. The inclusion $\mu_r \to G_m$ yields classes in $H^2(C, G_m)$, which is equal to the Brauer group of $C$. The reader is referred to the appendix for a review of basic facts about the Brauer group of a scheme.

Given a field $K$, there are several questions about the arithmetic properties of $K$ in which the Brauer group plays a central role.

1. **The period-index problem:** given $\alpha \in \text{Br}(K)$, what is the minimal $g$ such that $\text{ind}(\alpha) \mid \text{per}(\alpha)^g$?
2. **The index-reduction problem:** given a field extension $L/K$ and a class $\alpha \in \text{Br}(K)$, how can we characterize the number $\text{ind}(\alpha)/\text{ind}(\alpha|_L)$?
3. **The Brauer-Manin obstruction to the Hasse principle:** is the Brauer-Manin obstruction the only obstruction to the existence of 0-cycles of degree 1?

In this paper we will focus on questions (1) and (3). Question (2) also has close ties to the geometry of moduli spaces; we refer the reader to [14] for details.

As the third question is the most technical, let us briefly review what it means. Suppose $K$ is a global field with adele ring $A$ and $X$ is a proper geometrically connected $K$-scheme. For example, $X$ could be a smooth quadric hypersurface. For smooth quadric hypersurfaces, a classical theorem of Hasse and Minkowski says that $X(K) \neq \emptyset$ if and only if $X(K_v) \neq \emptyset$ for all places $v$ of $K$ (including the infinite ones). This principle is usually referred to as the "Hasse principle". A natural question which arises from this theorem is whether or not this principle holds for an arbitrary variety. This turns out not to the case [24], but there is often an explanation for the failure of this principle arising from a cohomological obstruction discovered by Manin [21]. To describe this obstruction, few minor technical remarks are in order.

Since $A$ is a $K$-algebra, we can consider the adelic points $X(A)$. Restriction gives a map $X(A) \to \prod_v X(K_v)$. In fact, this map is a bijection. (To prove this, one can use a regular proper model of $X$ over the scheme of integers of $K$ to reduce to the case in which $X$ is affine, where this follows from the universal property of the product.) From this point of view, the Hasse principle says that $X(A) \neq \emptyset$ if
and only if \( X(K) \neq \emptyset \). Moreover, the \( K \)-algebra structure on \( A \) gives a map \( X(K) \to X(A) \). Manin’s idea is to produce a pairing whose “kernel” contains the image of \( X(K) \) in \( X(A) \). The pairing arises as follows: restriction gives a map

\[
\text{Br}(X) \times X(A) \to \text{Br}(A) \to \mathbb{Q}/\mathbb{Z},
\]

where the last map comes from the usual local invariants of class field theory. The standard reciprocity law implies that the Brauer group of \( K \) is in the left kernel of this pairing, yielding an invariant map

\[
X(A) \to \text{Hom}(\text{Br}(X)/\text{Br}(K), \mathbb{Q}/\mathbb{Z}).
\]

Write \( X(A)^{\text{Br}(X)} \) for the “kernel” of this map (i.e., the elements sent to the map \( 0 : \text{Br}(X)/\text{Br}(K) \to \mathbb{Q}/\mathbb{Z} \)). The same reciprocity law shows that \( X(K) \) is contained in \( X(A)^{\text{Br}(X)} \). In particular, if \( X(A)^{\text{Br}(X)} = \emptyset \) then \( X(K) = \emptyset \). The obvious question concerning this pairing is the following.

**Question 2.1.** If \( X(A)^{\text{Br}(X)} \neq \emptyset \) then is \( K(K) \neq \emptyset \)?

As suspected from the beginning, the answer turns out to be “no,” but there was no counterexample until Skorobogatov discovered a bielliptic surface with no rational points and vanishing Brauer-Manin obstruction [25]. There is a refinement of this question due to Colliot-Thélène that is still the subject of much current research. (Cf. Conjecture 1.5(a) of [4] and Conjecture 2.4 of [5].)

**Conjecture 2.2.** If \( X(A)^{\text{Br}(X)} \neq \emptyset \) then there is a 0-cycle of degree 1 (over \( K \)) on \( X \).

We will call this property “the Hasse principle for 0-cycles”. A famous theorem of Saito affirms Conjecture 2.2 when \( X \) is a curve, under the assumption that the Tate-Shafarevich group of the curve is finite (the original is [23] with another account of this result in [3]); the general case is still wide open. According to Colliot-Thélène, it is not known if Skorobogatov’s negative answer to Question 2.1 has a 0-cycle of degree 1. One of our primary goals in this paper will be to link certain cases of Conjecture 2.2 to the period-index problem for function fields of arithmetic surfaces.

There is one case of Conjecture 2.2 which will come up below.

**Conjecture 2.3.** If \( X \) is smooth and geometrically rational and \( \text{Pic}(X \otimes \overline{K}) \) is isomorphic to \( \mathbb{Z} \) then the Hasse principle holds (for 0-cycles) for \( X \).

**Proof that Conjecture 2.3 follows from Conjecture 2.2.** By [11], we know \( \text{Br}(X \otimes \overline{K}) = 0 \), since \( X \) is smooth and geometrically rational. The Leray spectral sequence for \( G_m \), then shows that \( \text{Br}(X)/\text{Br}(K) = \text{H}^1(K, \mathbb{Z}) \), and the latter group is trivial (since the first cohomology of a finite group acting trivially on \( \mathbb{Z} \) is trivial, and the Galois cohomology is a colimit of such). Thus, \( \text{Br}(X) = \text{Br}(X)^{\text{Br}(X)} \).

The statement of Conjecture 2.3 is meant to include both the strong form (classical Hasse principle) and weak form (Hasse principle for 0-cycles). We will discuss two different relationships between Conjecture 2.3 and the period-index problem; one will relate to the strong form, while one will relate to the weak form.

### 3. Moduli Spaces of Stable Twisted Sheaves on Curves and Period-Index Theorems

In this section, we start to explain the connections among the various problems described in the preceding section. In particular, we will use classical theorems about rational points on various kinds of varieties over various \( C_1 \)-fields to solve moduli problems encoding period-index problems. Then we will prove the Theorem from the Introduction.

To begin, we give another result which is a transparent translation of a simple spectral sequence argument.

**Theorem 3.1.** If \( C \) is a proper curve over a finite field \( \mathbb{F}_q \) then \( \text{Br}(C) = 0 \).

**Sketch of proof.** An exercise in deformation theory (see Section A.2) reduces the theorem to the case in which \( C \) is smooth. Let \( \mathcal{C} \to C \) be a \( \mu_n \)-gerbe. Consider the stack \( \mathcal{M}_{q/\mathbb{F}_q}(1,0) \) parameterizing invertible \( \mathcal{C} \)-twisted sheaves of degree 0. Just as in the classical case, \( \mathcal{M}_{q/\mathbb{F}_q}(1,0) \) is a \( G_m \)-gerbe over a \( \text{Jac}(C) \)-torsor \( T \). By Lang’s theorem, \( T \) has a rational point \( p \). By Wedderburn’s theorem, \( p \) lifts to an object of the stack, giving an invertible \( \mathcal{C} \)-twisted sheaf. As described in the appendix, this invertible
twisted sheaf trivializes the Brauer class associated to $[\mathcal{E}]$. Since $\mathcal{E}$ was an arbitrary $\mu_n$-gerbe, this shows that the Brauer group of $C$ is trivial.

Next, we will sketch a proof of the following theorem.

**Theorem 3.2** (de Jong). Let $X$ be a surface over an algebraically closed field $k$. For all $\alpha \in \text{Br}(k(X))$, we have $\text{per}(\alpha) = \text{ind}(\alpha)$.

Unlike Theorem 3.1, this is not merely a geometric realization of a standard cohomological argument. There are various proofs of this result – de Jong’s original proof [7], a proof due to de Jong and Starr [9], and the one we present (found with details in [20]). They each ultimately rest on deformation theory and the definition of a suitable moduli problem. The latter two both reduce the result to the existence of a section for a rationally connected fibration over a curve and use the Graber-Harris-Starr theorem. There are various proof of this result – de Jong’s original proof [7], a proof due to de Jong and Starr [9], and the one we present (found with details in [20]). They each ultimately rest on deformation theory and the definition of a suitable moduli problem. The latter two both reduce the result to the existence of a section for a rationally connected fibration over a curve and use the Graber-Harris-Starr theorem.

**Sketch of proof.** We assume that $\text{char } k = 0$ for the sake of simplicity; the reader will find a reduction to this case in [20]. We proceed in steps. We will write $n$ for the period of $\alpha$.

1. Blowing up in the base locus of a very ample pencil of divisors (one of which contains the ramification divisor of $\alpha$) on $X$, we may assume that there is a fibration $X \to \mathbb{P}^1$ with a section such that the ramification of $\alpha$ is entirely contained in a fiber and the generic fiber is smooth of genus $g \geq 2$. Let $C/k(t)$ be the generic fiber; this is a proper smooth curve of genus $g \geq 2$ with a rational point $p$, and $\alpha$ lies in $\text{Br}(C)$.

2. Choose a $\mu_n$-gerbe $\mathcal{E} \to C$ such that $[\mathcal{E}] = \alpha \in \text{Br}(C)$ and $[\mathcal{E} \otimes k(t)] = 0 \in H^2(C, \mu_n)$. (We can do this because $C$ has a rational point.)

3. Consider the modular $\mu_n$-gerbe $\mathcal{M}_{\mathcal{E}/k(t)}(n, \mathcal{O}(p)) \to M_{\mathcal{E}/k(t)}(n, \mathcal{O}(p))$. As discussed in Section [1] there is an isomorphism

$$M_{\mathcal{E}/k(t)}(n, \mathcal{O}(p)) \otimes k(t) \cong M_{C \otimes k(t)/k(t)}(n, \mathcal{O}(p)),$$

and we know that the latter (hence, the former) is unirational (in fact, rational), and thus rationally connected.

4. The Graber-Harris-Starr theorem implies that there is a rational point $q \in M_{\mathcal{E}/k(t)}(n, \mathcal{O}(p))(k(t))$.

5. Tsen’s theorem implies that $q$ lifts to an object $\mathcal{Y}$: a locally free $\mathcal{E}$-twisted sheaf of rank $n$.

6. The algebra $\mathcal{B}_{\text{ind}(\mathcal{Y})}$ is a central division algebra of degree $n$ with Brauer class $\alpha$, thus proving that the index of $\alpha$ divides $n$. Since we already know the converse divisibility relation, we are done.

**Remark 3.3.** The reader will note that we use the Graber-Harris-Starr theorem in two places in the proof: once to find a rational point on the coarse moduli space, and once (in the guise of Tsen’s theorem) to lift that rational point to an object of the stack. While classical algebraic geometry only “sees” the former, the difference between fine and coarse moduli problems necessitates the latter.

If we start with an arithmetic surface instead of a surface over an algebraically closed field, things get more complicated. (For example, it is no longer true that the period and index are always equal if the class is allowed to ramify.) For the rest of this section, we will discuss unramified classes. In Section 5 below, we will discuss certain ramified classes. In both cases, we will tie the period-index problem to Conjecture 2.3.

Let $K$ be a global field, $S$ the scheme integers of $K$, and $C \to S$ a proper relative curve with a section and smooth generic fiber. (When $\text{char}(K) > 0$, the scheme of integers is assumed to be proper over the prime field; this ensures that it is unique.)

**Theorem 3.4.** If Conjecture 2.3 is true, then any $\alpha \in \text{Br}_\infty(C)$ satisfies $\text{per}(\alpha) = \text{ind}(\alpha)$.

**Proof.** Write $n = \text{per}(\alpha)$. Since $\mathcal{E} \to S$ has a section, we can choose a $\mu_n$-gerbe $\mathcal{E} \to C$ such that $[\mathcal{E} \otimes \mathcal{O}_S] = 0 \in H^2(C \otimes \mathcal{O}_S, \mu_n)$. By class field theory, the restriction of $\alpha$ to the point $p \in C(K)$ is trivial.

Consider the stack $\chi : \mathcal{M}_{\mathcal{E}/K}(n, \mathcal{O}(p)) \to M_{\mathcal{E}/K}(n, \mathcal{O}(p))$. We claim that to prove the theorem it suffices to show that for every place $\nu$ of $K$, the category $\mathcal{M}_{\mathcal{E}/K}(n, \mathcal{O}(p))_{\mathcal{O}_{\nu}}$ is nonempty. Indeed, the map $\chi$ is a $\mu_n$-gerbe; let $\beta \in \text{Br}(M_{\mathcal{E}/K}(n, \mathcal{O}(p)))$ be the associated Brauer class. Since $M_{\mathcal{E}/K}(n, \mathcal{O}(p))$ is geometrically rational with Picard group $\mathbb{Z}$ [12] and has a point over every completion of $K$, we
know that the pullback map $\text{Br}(K) \to \text{Br}(M_{\mathcal{E}_K/K}(n, \mathcal{O}(p)))$ is an isomorphism. Thus, $\beta$ is the pullback of a class over $K$. The fact that each $\mathcal{M}_{\mathcal{E}_K/K}(n, \mathcal{O}(p))_{K_\nu}$ is non-empty implies that $\beta_{K_\nu} = 0$; class field theory again shows that $\beta = 0$. But then any rational point of $\mathcal{M}_{\mathcal{E}_K/K}(n, \mathcal{O}(p))$ lifts to an object of $\mathcal{M}_{\mathcal{E}_K/K}(n, \mathcal{O}(p))$.

Let us assume we have found a collection of local objects as in the previous paragraph. The assumption that Conjecture 2.3 holds yields a 0-cycle of degree 1 on the coarse space $\mathcal{M}_{\mathcal{E}_K/K}(n, \mathcal{O}(p))$, which lifts to the stack, producing a complex $P^*$ of locally free $\mathcal{E}_K$-twisted sheaves such that $\text{rk} P^* = n$. Indeed, if there is a $\mathcal{C} \otimes L$-twisted sheaf of rank $n$ for some finite algebra $L/K$ then pushing forward along $\mathcal{C} \otimes L \to \mathcal{C}$ gives a $\mathcal{C} \otimes K$-twisted sheaf of rank $[L : K] n$. A 0-cycle of degree 1 yields two algebras $L_1/K$ and $L_2/K$ such that $[L_1 : K] - [L_2 : K] = 1$ and $\mathcal{M}_{\mathcal{C} \otimes L_1/L_2}(n, \mathcal{O}(p))_{L_1} \neq \emptyset$. There result two $\mathcal{C} \otimes K$-twisted sheaves $V_1$ and $V_2$ such that $\text{rk} V_1 - \text{rk} V_2 = n$, whence we can let $P^*$ be the complex with $V_1$ in degree 0 and $V_2$ in degree 1 (and the trivial differential). Since the category of coherent twisted sheaves over the generic fiber of $C$ is semisimple (it is just the category of finite modules over a division ring), it follows that there is a $\mathcal{C}_\nu$-twisted sheaf of rank $n$, whence $\text{per}(\alpha) = \text{ind}(\alpha)$, as desired.

So it remains to produce local objects of $\mathcal{M}_{\mathcal{E}_K/K}(n, \mathcal{O}(p))_{K_\nu}$ for all places $\nu$. If $\nu$ is Archimedean, then $[\mathcal{C} \otimes K_\nu] = 0 \in H^2(C \otimes K_\nu, \mu_n)$ by assumption, so that stable $\mathcal{C} \otimes K_\nu$-twisted sheaves are equivalent (upon twisting down by an invertible $\mathcal{C} \otimes K_\nu$-twisted sheaf) to stable sheaves on $C \otimes K_\nu$. Since moduli spaces of stable vector bundles with fixed invariants have rational points over every infinite field, we find a local object.

Now assume that $\nu$ is finite, and let $R$ be valuation ring of $K_\nu$, with finite residue field $F$. By Theorem 3.1 and the assumption that $[\mathcal{C} \otimes \overline{F}] = 0 \in H^2(C \otimes \overline{F}, \mu_n)$ we know that $[\mathcal{C} \otimes F] = 0 \in H^2(C \otimes F, \mu_n)$. It follows as in the previous paragraph that it suffices to show the existence of a stable sheaf on $C \otimes F$ of determinant $\mathcal{O}(p)$ and rank $n$. Consider the stack $\mathcal{M}_{C \otimes F/F}(n, \mathcal{O}(p)) \to M_{C \otimes F/F}(n, \mathcal{O}(p))$. Just as above, the space $\mathcal{M}_{C \otimes F/F}(n, \mathcal{O}(p))$ is a smooth projective rationally connected variety. By Esnault’s theorem [9], it has a rational point. Since $\mathcal{M}_{C \otimes F/F}(n, \mathcal{O}(p)) \to M_{C \otimes F/F}(n, \mathcal{O}(p))$ is a $\mu_n$-gerbe and $F$ is finite, the moduli point lifts to an object, giving rise to a stable $\mathcal{C} \otimes F$-twisted sheaf $V$ of rank $n$ and determinant $\mathcal{O}(p)$. Since $\mathcal{M}_{\mathcal{E}_S/S}(n, \mathcal{O}(p))$ is smooth over $S$, the sheaf $V$ deforms to a family over $R$, whose generic fiber gives the desired local object. 

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4. PARABOLIC BUNDLES ON $\mathbb{P}^1$

In this section, we review some basic elements of the moduli theory of parabolic bundles of rank 2 on the projective line. We will focus on the case of interest to us and describe it in stack-theoretic language. We refer the reader to [2] for a general comparison between the classical and stacky descriptions of parabolic bundles.

Let $D = p_1 + \cdots + p_r \subset \mathbb{P}^1$ be a reduced divisor, and let $\pi : \mathcal{P} \to \mathbb{P}^1$ be the stack given by extracting square roots of the points of $D$ as in [3]. The stack has $r$ non-trivial residual gerbes $\xi_1, \ldots, \xi_r$, each isomorphic to $B \mu_2$ over its field of moduli (i.e., the residue field of $p_i$). Recall that the category of quasi-coherent sheaves on $B \mu_2$ is naturally equivalent to the category of representations of $\mu_2$. Given a sheaf $\mathcal{F}$ on $B \mu_2$, we will call the representation arising by this equivalence the associated representation of $\mathcal{F}$.

**Definition 4.1.** A locally free sheaf $V$ on $\mathcal{P}$ is regular if for each $i = 1, \ldots, r$, the associated representation of the restriction $V|_{\xi_i}$ is a direct sum of copies of the regular representation.

**Definition 4.2.** Let $\{a_i \leq b_i\}_{i=1}^r$ be elements of $\{0, 1/2\}$. A parabolic bundle $V_\nu$ of rank $N$ with parabolic weights $\{a_i \leq b_i\}$ is a pair $(W, F)$, where $W$ is a locally free sheaf of rank 2 and $F \subset W_D$ is a subbundle. The parabolic degree of $V_\nu$ is

$$\text{pardeg}(V_\nu) = \deg W + \sum_i a_i(rk W - \text{rk} F_{p_i}) + b_i(\text{rk} F_{p_i})$$

We will only consider parabolic bundles with weights in $\{0, 1/2\}$ in this paper. More general weights in $\{0, 1\}$ are often useful. The stack-theoretic interpretation of this more general situation is slightly more complicated; it is explained clearly in [2].
Vistoli has defined a Chow theory for Deligne-Mumford stacks \([26]\) in which pushforward defines an isomorphism \(A(\mathcal{P}) \otimes Q \cong A(\mathbb{P}^1) \otimes Q\) of Chow rings. In particular, any invertible sheaf \(L\) on \(\mathcal{P}\) has a degree, \(\deg L \in \mathbb{Q}\). One can make a more ad hoc definition of the degree of an invertible sheaf \(L\) on \(\mathcal{P}\) in the following way. The sheaf \(L^{\otimes 2}\) is the pullback of a unique invertible sheaf \(\mathcal{M}\) on \(\mathbb{P}^1\), and we can define \(\deg_{\mathcal{P}} L = \frac{1}{2} \deg_{\mathbb{P}^1} \mathcal{M}\). Thus, for example, \(\deg \mathcal{O}(\xi) = [\kappa_i : k]/2\), where \(\kappa_i\) is the field of moduli of \(\xi\) (the residue field of \(p_i\)).

The following is a special case of a much more general result. The reader is referred to (e.g.) [2] for the generalities.

**Proposition 4.3.** There is an equivalence of categories between locally free sheaves \(V\) on \(\mathcal{P}\) and parabolic sheaves \(V^r\) on \(\mathbb{P}^1\) with parabolic divisor \(D\) and parabolic weights contained in \(\{0, 1/2\}\). Moreover, we have \(\deg V = \deg(\mathcal{V})\).

**Proof.** Given \(V\), define \(V^r\) as follows: the underlying sheaf \(W\) of \(V^r\) is \(\pi_* V\). To define the subbundle \(F \subset W_D\), consider the inclusion \(V(\sum \xi_i) \subset V\). Pushing forward by \(\pi\) yields a subsheaf \(W' \subset W\), and we let \(F\) be the image of the induced map \(W_D' \rightarrow W_D\).

We leave it to the reader as an amusing exercise to check that 1) this defines an equivalence of categories, and 2) this equivalence respects degrees, as claimed. \(\square\)

**Definition 4.4.** Given a non-zero locally free sheaf \(V\) on \(\mathcal{P}\), the slope of \(V\) is

\[
\mu(V) = \frac{\deg V}{\text{rk} V}.
\]

The sheaf \(V\) on \(\mathcal{P}\) is stable if for all locally split proper subsheaves \(W \subset V\) we have \(\mu(W) < \mu(V)\).

The stability condition of Definition 4.4 is identical to the classical notion for sheaves on a proper smooth curve. The reader familiar with the classical definition of stability for parabolic bundles can easily check (using Proposition 4.3) that a sheaf \(V\) on \(\mathcal{P}\) is stable if and only if the associated parabolic bundle \(V^r\) is stable in the parabolic sense. One can check (by the standard methods) that stable parabolic bundles form an Artin stack of finite type over \(\mathbb{A}\).

**Notation 4.5.** Given \(L \in \text{Pic}(\mathcal{P})\), let \(\mathcal{M}^r_{\mathcal{P}/k}(n, L)\) denote the stack of regular locally free sheaves on \(\mathcal{P}\) of rank \(r\) and determinant \(L\), and let \(M^r_{\mathcal{P}/k}(n, L)\) denote the coarse space.

**Proposition 4.6.** If \(\mathcal{M}^r_{\mathcal{P}/k}(n, L)\) is non-empty then it is geometrically unirational and geometrically integral. Moreover, the stack \(\mathcal{M}^r_{\mathcal{P}/k}(2, \mathcal{O}(\sum \xi_i))\) is non-empty if \(r > 3\).

**Proof.** Basic deformation theory shows that \(\mathcal{M}^r_{\mathcal{P}/k}(n, L)\) is smooth. Thus, to show that it is integral, it suffices to show that it is connected. We will do this by showing that it is unirational (i.e., finding a surjection onto \(\mathcal{M}^r_{\mathcal{P}/k}(n, L)\) from an open subset of a projective space. This is a standard trick using a space of extensions. (There are in fact two versions, one using finite cokernels and one using invertible cokernels. We show the reader the former, as it is useful in situations where the latter does not apply and the latter seems to be more easily available in the literature.)

Let \(V\) be a parabolic sheaf of rank \(n\) and determinant \(L\). Since \(\mathcal{M}^r_{\mathcal{P}/k}(n, L)\) is of finite type, there exists a positive integer \(N\) such that for every parabolic sheaf of rank \(n\) and determinant \(L\), a general map \(V \rightarrow W(N)\) is injective with cokernel \(Q\) supported on a reduced divisor \(E\) in \(\mathcal{P} \setminus \{\xi_1, \ldots, \xi_r\}\) such that \(E \in |\mathcal{O}(nN)|\). Let \(U \subset |\mathcal{O}(nN)|\) be the open subset parametrizing divisors supported in \(\mathcal{P} \setminus \{\xi_1, \ldots, \xi_r\}\), and let \(\mathcal{E} \subset \mathcal{P} \times U\) be the universal divisor. The sheaf \(R^1(pr_2)_* \mathcal{RHom}(\mathcal{O}_E, pr_1^* V)\) is locally free on \(U\), and gives rise to a geometric vector bundle \(B \rightarrow U\) and an extension

\[
0 \rightarrow V_{\mathcal{P} \times B} \rightarrow \mathcal{H} \rightarrow \mathcal{E}_{\mathcal{P} \times B} \rightarrow 0
\]

such that every extension \(0 \rightarrow V \rightarrow W(N) \rightarrow Q \rightarrow 0\) as above arises is a fiber over \(B\). Passing to the open subscheme \(B^o\) over which the extension \(\mathcal{H}\) is locally free with stable fibers yields a surjective map \(B^o \rightarrow \mathcal{M}^r_{\mathcal{P}/k}(n, L)\) from an open subset of a projective space, proving that \(\mathcal{M}^r_{\mathcal{P}/k}(n, L)\) is geometrically integral and unirational.
To prove that \( \mathcal{M}_{[\omega]}(n, L) \) is nonempty is significantly more subtle. The proof is similar to a result of Biswas \[1\] for parabolic bundles with parabolic degree 0, but including it here would take us too far afield.

5. FORMS OF PARABOLIC MODULI VIA SPLIT RAMIFICATION

5.1. Some generalities. In this section we illustrate how to produce forms of the stack of parabolic bundles on \( \mathbb{P}^1 \) from Brauer classes over \( k(t) \). For the sake of simplicity, we restrict our attention to classes in \( \text{Br}(k(t))[2] \) and parabolic bundles of rank 2. A generalization to higher period/rank should be relatively straightforward.

Let \( \alpha \in \text{Br}(k(t))[2] \) be a Brauer class. Suppose \( D = p_1 + \cdots + p_r \) is the ramification divisor of \( \alpha \), and let \( \mathcal{P} \to \mathbb{P}^1 \) be the stacky branched cover as in Section \[4\]. By Corollary C.2 \( \alpha \) extends to a class \( \alpha' \in \text{Br}(\mathcal{P})[2] \). Suppose \( \mathcal{E} \to \mathcal{P} \) is a \( \mu_2 \)-gerbe representing \( \alpha' \) such that \( [\mathcal{E} \otimes k] = 0 \in H^2(\mathcal{P} \otimes k, \mu_2) \).

(For a proof that \( \mathcal{E} \) is itself an algebraic stack, the reader is referred to \[19\]. We cannot always ensure that the cohomology class of \( [\mathcal{E} \otimes k] \) is trivial; we make that as a simplifying assumption. More general cases can be analyzed by similar methods.)

**Definition 5.1.1.** A regular \( \mathcal{E} \)-twisted sheaf is a locally free \( \mathcal{E} \)-twisted sheaf \( V \) such that for each \( i = 1, \ldots, r \), the restriction \( V_{\mathcal{E} \times_k \text{Spec} \pi_i} \) has the form \( \mathcal{L} \otimes \mathcal{P}^m \) for some integer \( m > 0 \), where \( \mathcal{L} \) is an invertible \( \mathcal{E} \otimes \pi_i \)-twisted sheaf and \( \rho \) is the sheaf on \( \mathbb{P} \mu_2 \) associated to the regular representation of \( \mu_2 \).

Just as in Definition \[4.4\] and Definition A.1.2 we can define stable regular \( \mathcal{E} \)-twisted sheaves.

**Notation 5.1.2.** Let \( \mathcal{M}_{[\omega]}(n, L) \) denote the stack of stable regular \( \mathcal{E} \)-twisted sheaves of rank \( n \) and determinant \( L \), and \( M_{[\omega]}(n, L) \) its coarse moduli space (sheafification).

**Proposition 5.1.3.** For any section \( \sigma \) of \( \mathcal{E} \otimes k \to \mathcal{P} \otimes k \) there is an isomorphism \( \mathcal{M}_{[\omega]}(n, L) \otimes k \sim \mathcal{M}_{[\omega]}(n, L) \).

**Proof.** We may assume that \( k = \overline{k} \). The section \( \sigma \) corresponds to an invertible \( \mathcal{E} \)-twisted sheaf \( \mathcal{L} \) such that \( \mathcal{L} \otimes \mathcal{E} = \mathcal{O}_\mathcal{E} \). Twisting by \( \mathcal{L} \) defines the isomorphism. (Note that the regularity condition implies that \( n \) must be even for either space to be nonempty.)

In other words, the stack \( \mathcal{M}_{[\omega]}(n, L) \) (resp. the quasi-projective coarse moduli space \( M_{[\omega]}(n, L) \)) is a form of \( \mathcal{M}_{[\omega]}(n, L) \) (resp. \( M_{[\omega]}(n, L) \)).

**Corollary 5.1.4.** The space \( M_{[\omega]}(n, L) \) is geometrically (separably) unirational when it is nonempty.

One can use a generalization of Corollary 5.1.4 to higher genus curves and arbitrary period to give another proof of Theorem 3.2 without having to push the ramification into a fiber, by simply taking any pencil and using the generic points of the ramification divisor (and a point of the base locus) to define the parabolic divisor. (This is not substantively different from the proof we give here.) The main interest for us, however, will be for arithmetic surfaces of mixed characteristic.

5.2. An extended example. Let \( \alpha \in \text{Br}(\mathbb{Q}(t))[2] \) be a class whose ramification divisor \( D \subset \mathbb{P}^1 \) is nonempty with simple normal crossings. Let \( \mathcal{P} \to \mathbb{P}^1 \) be the stacky cover branched over \( D \) to order 2 as in the first paragraph of Section \[4\] above. Let \( \mathcal{E} \to \mathcal{P} \) be a \( \mu_2 \)-gerbe with Brauer class \( \alpha \); if \( |D \otimes \mathbb{Q} : \mathbb{Q}| \) is odd, one can ensure that \( \mathcal{E} \) such that \( [\mathcal{E} \otimes \mathbb{Q}] \in H^2(\mathcal{P} \otimes \mathbb{Q}, \mu_2) \) has the form \( [\Lambda]^{1/2} \) for some invertible sheaf \( \Lambda \in \text{Pic}(\mathcal{P} \otimes \mathbb{Q}) \) of half-integral degree.

**Definition 5.2.1.** Given a field extension \( L/K \) and a Brauer class \( \alpha \in \text{Br}(L) \), the Faddeev index of \( \alpha \) is \( \min_{\beta \in \text{Br}(K)} \text{ind}(\alpha + \beta_L) \).

**Proposition 5.2.2.** The class \( \alpha \) has Faddeev index 2 if and only if the space \( M_{[\omega]}(2, L)^{ss} \) has a \( \mathbb{Q} \)-rational point for some invertible sheaf \( L \). If \( |D \otimes \mathbb{Q} : \mathbb{Q}| \) is odd, we need only quantify over \( L \) of half-integral degree and look for points in the stable locus \( M_{[\omega]}(2, L) \).
The point of Proposition 5.2.2 is that the computation of the Faddeev index is reduced to the existence of a rational point on one of a sequence of geometrically rational smooth (projective if $|D \otimes Q : Q|$ is odd) geometrically connected varieties over $Q$.

**Proof:** First, suppose that $P : \text{Spec } Q \to M_{\mathcal{E}/Q}(2, L)$ is a rational point. Pulling back $\mathcal{M}_{\mathcal{E}/Q}(2, L) \to M_{\mathcal{E}/Q}(2, L)$ along $P$ yields a class $\beta = -[\mathcal{F}] \in \text{Br}(Q)[2]$. Just as in [14], we see that

$$\mathcal{M}_{\mathcal{E}/\mathcal{F}/Q}(2, L) \to M_{\mathcal{E}/\mathcal{F}/Q}(2, L) = M_{\mathcal{E}/Q}(2, L)$$

is split over $P$, whence there is a $\mathcal{E} \wedge \mathcal{F}$-twisted sheaf of rank 2. This shows that $\alpha - \beta$ has index 2 and thus that $\alpha$ has Faddeev index dividing 2. Since $\alpha$ is ramified, it cannot have Faddeev index 1.

Now suppose that $\alpha$ has Faddeev index 2, so that there is some $\mathcal{F} \to \text{Spec } Q$ such that there is locally free $\mathcal{E} \wedge \mathcal{F}$-twisted sheaf of rank 2. Since there is a canonical isomorphism $M_{\mathcal{E}/\mathcal{F}/Q}(2, L) = M_{\mathcal{E}/Q}(2, L)$, upon replacing $\mathcal{E}$ by $\mathcal{E} \wedge \mathcal{F}$ we may assume that $\alpha$ has period 2, and is thus represented by a quaternion algebra $[(a, b)] \in \text{Br}(Q(t))$. To prove the result, it suffices to show that in this case there is a regular $\mathcal{E}$-twisted sheaf of rank 2.

By Proposition [A.2.31], it suffices to prove this for $\mathcal{E} \otimes \mathcal{G}_{P_{1, D}}$. Thus, we are reduced to the following: let $R$ be a complete discrete valuation ring with residual characteristic 0 and $(a, b)$ a quaternion algebra over the fraction field $K(R)$. Suppose $(a, b)$ is ramified, and let $\mathcal{E} \to \mathcal{B}_2$ be a $\mu_2$-gerbe with Brauer class $[(a, b)]$. Then there is a regular $\mathcal{E}$-twisted sheaf of rank 2. To prove this, note that the bilinearity and skew-symmetry of the symbol allows us to assume that $b$ is a uniformizer for $R$ and $a$ has valuation at most 1. We will define a $\mu_2$-equivariant Azumaya algebra $A$ in the restriction of $(a, b)$ to $R' = R[\sqrt{b}]$ such that the induced representation of $\mu_2$ on the fiber $A$ over the closed point of $R'$ is $\rho^\beta_2$; it then follows that any twisted sheaf $V$ on $\mathcal{B}_2$ such that $\text{End}(V) = A$ is regular by a simple geometric computation over the residue field.

Let $x$ and $y$ be the standard generators for $(a, b)$, so that $x^2 = a$ and $y^2 = b$, and write $a = ub^\varepsilon$ with $u \in R^\times$ and $\varepsilon \in \{0, 1\}$. Let $\tilde{x} = x/\sqrt{b}$ and $\tilde{y} = y/\sqrt{b}$; we have $\tilde{x}^2 = u$ and $\tilde{y}^2 = 1$, which means that $\tilde{x}$ and $\tilde{y}$ generate an Azumaya algebra with generic fiber $(a, b) \otimes K(R)(\sqrt{b})$. A basis for $A$ as a free $R'$-module is given by $1, \tilde{x}, \tilde{y}, \tilde{x}\tilde{y}$; this also happens to be an eigenbasis for the action of $\mu_2$. Let $\chi^1$ denote the non-trivial character of $\mu_2$, and $\chi^0$ the trivial character. The eigensheaf decomposition of $A$ corresponding to the character can be written as $\chi^0 \otimes \chi^1 \otimes \chi^\varepsilon \otimes \chi^{1+\varepsilon}$, where the last sum is taken modulo 2. For either value of $\varepsilon$ this is isomorphic to $\rho^\beta_2$, as desired.

Gluing the local models (as in Proposition [A.2.31]) produces a regular $\mathcal{E}$-twisted sheaf $V$ of rank 2. Since $\alpha$ is non-trivial, we see that this sheaf must be stable, hence geometrically semistable (in fact, geometrically polystable [14]). If $|D \otimes Q : Q|$ is odd, then $V$ is semistable with coprime rank and degree, hence geometrically stable.

Proposition 5.2.2 has an amusing consequence. The starting point is the following (somewhat useful) lemma. Fix a place $\nu$ of $Q$.

**Lemma 5.2.3.** If there is an object of $\mathcal{M}_{\mathcal{E}/Q}(2, L)^{ss}_{Q_{\nu}}$, then there is an object of $\mathcal{M}_{\mathcal{E}/Q}(2, L)^{ss}_{Q_{\nu}}$.

**Proof:** Let $V$ be a regular semistable $\mathcal{E} \otimes Q_{\nu}$-twisted sheaf of rank 2 and determinant $L$, and let $Y$ be an algebriazation of a versal deformation space of $V$. Since deformations of vector bundles on curves are unobstructed, we know that $Y$ is smooth over $Q_{\nu}$. On the other hand, we also know that the field $Q_{\nu}$ has the property that any smooth variety with a rational point has a Zariski-dense set of rational points. Finally, we know that the locus of geometrically stable $\mathcal{E} \otimes Q_{\nu}$-twisted sheaves is open and dense in $Y$. The result follows.

In [15], the authors produce examples of biquaternion algebras $A$ over $Q(t)$ of Faddeev index 4 such that for all places $\nu$ of $Q$ the algebra $A \otimes Q_{\nu}(t)$ has index 2. They describe this as the failure of a sort of “Hasse principle”. However, a careful examination of their examples shows that in fact there is a local obstruction: for each class of such algebras they write down, there is always a place $\nu$ over which there is no regular $\mathcal{E} \otimes Q_{\nu}$-twisted sheaf! (Moreover, the proofs that their examples work use this local failure in an essential way, although the authors do not phrase things this way.) Why doesn’t this contradict Proposition 5.2.2? Because for the place $\nu$ where things fail, one of the original ramification
sections ends up in the locus where the algebra $A \otimes \mathbb{Q}_\nu$ is unramified, and now the condition that the sheaf be regular around the stacky point over that section is non-trivial!

Let us give an explicit example. In Proposition 4.3 of [15], the authors show that the biquaternion algebra $A = (17, t) \otimes (13, (t - 1)(t - 11))$ has Faddeev index 4 while for all places $\nu$ of $Q$ the algebra $A \otimes \mathbb{Q}_\nu$ has index 2. Consider $A_{17} = A \otimes Q_{17}$; since 13 is a square in $Q_{17}$, we have that $A_{17} = (17, t)$ as $Q_{17}$-algebras. Elementary calculations show that $(17, 1) = 0 \in Br(Q_{17})$ and $(17, 11) \neq 0 \in Br(Q_{17})$. It follows that any $Q_{17}(t)$-algebra Faddeev-equivalent to $A_{17}$ has the property that precisely one of its specializations at 1 and 11 will be non-trivial.

**Lemma 5.2.4.** Given a field $F$ and a nontrivial Brauer class $\gamma \in Br(F)[2]$, let $\mathcal{G} \rightarrow B\mathbb{F}_2 \times \text{Spec} F$ be a $\mathbb{F}_2$-gerbe representing the pullback of $\gamma$ in $Br(B\mathbb{F}_2 \times \text{Spec} F)$. There is no regular $\mathcal{G}$-twisted sheaf of rank 2.

**Proof.** The inertia stack of $\mathcal{G}$ is isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$, where the first factor comes from the gerbe structure and the second factor comes from the inertia of $B\mathbb{F}_2$. Given a $\mathcal{G}$-twisted sheaf $F$, the eigen-decomposition of $F$ with respect to the action of the second factor gives rise to $\mathcal{G}$-twisted subsheaves of $F$. Since $\mathcal{G}$ has period 2, if $rk F = 2$ there can be no proper twisted subsheaves. \qed

**Corollary 5.2.5.** In the example above, all of the sets $M_Q^*(2, L)(Q_{17})$ are empty.

**Proof.** Suppose $Q \in M_Q^*(2, L)(Q_{17})$. As above, after replacing $A_{17}$ by a Faddeev-equivalent algebra, we can that $Q$ to an object of $M_Q^*(2, L)(Q_{17})$. In other words, there would be a regular (stable!) $\mathcal{G} \otimes Q_{17}$-twisted sheaf of rank 2. But this contradicts Lemma 5.2.4 and the remarks immediately preceding it. \qed

Combining Proposition 5.2.2 and Corollary 5.2.5, we see that the failure of the “Hasse principle” to which the authors of [15] refer in relation to the algebra $A$ is in fact a failure of the existence of a local point in the associated moduli problem! This is in great contrast to the unramified case, which we saw above was directly related to the Hasse principle. It is somewhat disappointing that the examples we actually have of classes over arithmetic surfaces whose period and index are distinct cannot be directly related to the Hasse principle (except insofar as both sets under consideration are empty!)

There is one mildly interesting question which arises out of this failure.

**Proposition 5.2.6.** If Conjecture 2.3 is true then any element $\alpha \in Br(Q(t))[2]$ such that

1. the ramification of $\alpha$ is a simple normal crossings divisor $D = D_1 + \cdots + D_r$ in $P^1_\mathbb{Z}$ which is a union of fibers and an odd number of sections of $P^1_\mathbb{Z} \to \text{Spec} \mathbb{Z}$,
2. for every crossing point $p \in D_i \cap D_j$, both ramification extensions are either split, non-split, or ramified at $p$, and
3. all points $d$ of $D \times P^1_\mathbb{Z}$ which are not ramification divisors of $\alpha$ give rise to the same element $(\alpha_\mathbb{R}(t))_d \in Br(\mathbb{R})$

satisfies $\text{per}(\alpha) = \text{ind}(\alpha)$.

The second condition of the proposition is almost equivalent to the statement that the restriction of $\alpha$ to $Q_{\nu}(t)$ has no hot points (in the sense of Saltman) on $P^1_{\mathbb{Z}_{\nu}}$; it is not quite equivalent because Saltman’s hot points are all required to lie on intersections of ramification divisors, while some of the ramification divisors of $\alpha$ may no longer be in the ramification divisor of $\alpha_{Q_{\nu}(t)}$. The proof of Proposition 5.2.6 uses Proposition A.2.3 (2) as a starting point for a deformation problem. It is similar in spirit to the proof of Theorem 3.4 and will be omitted.

6. A List of Questions

We record several questions arising from the preceding discussion. Let $C$ be a curve over a field $k$.

1. Are there biquaternion algebras in $Br_{\infty}(Q(t))$ of Faddeev index 4 with non-hot secondary ramification (in the sense of Proposition 5.2.6)? For example, let $p$ be a prime congruent to 1 modulo $3 \cdot 4 \cdot 7 \cdot 13 \cdot 17$ and congruent to 2 modulo 5. What is the Faddeev index of the algebra $(p, t) \otimes (13, 15(t - 1)(t + 13))$?
(2) What is the Brauer-Manin obstruction for $M_{\mathcal{E}/k}(2, L)$? If algebras as in the first question exist, is the resulting failure of the Hasse principle explained by the Brauer-Manin obstruction?

(3) Let $C \to X$ be a $\mu_n$-gerbe. What is the index of the Brauer class $\mathcal{M}_{\mathcal{E}/k}(n, \mathcal{O}) \to M_{\mathcal{E}/k}(n, \mathcal{O})$? If $C$ has a rational point, this index must divide $n$. More generally, it must divide $n \text{ind}(C)$. Is this sharp?

(4) Does every fiber of $M_{\mathcal{C}/k}(n, L) \to \text{Pic}_{\mathcal{C}/k}$ over a rational point contain a rational point? This is (indirectly) related to the index-reduction problem. It is not too hard to see that if the rational point of $\text{Pic}_{\mathcal{C}/k}^d$ comes from an invertible sheaf then there is always a rational point in the fiber.

(5) Suppose $k$ is a global field. Let $C \to S$ be a regular proper model of $C$. Is there a class $\alpha \in \text{Br}_S(C)$ such that $\text{per}(\alpha) \neq \text{ind}(\alpha)$? When $k$ has positive characteristic and the period of $\alpha$ is invertible in $k$, this is impossible [20]. The existence of such a class would disprove Conjecture 2.3 (even the strong form).

**APPENDIX A. GERBES, TWISTED SHEAVES, AND THEIR MODULI**

In this appendix, we remind the reader of the basic facts about twisted sheaves, their moduli, and their applications to the Brauer group. For more comprehensive references, the reader can consult [18, 20]. The basic setup will be the following: let $\mathcal{X} \to X \to S$ be a $\mu_n$-gerbe on a proper flat morphism of finite presentation. We assume $n$ is invertible on $S$. At various points in this appendix, we will impose conditions on the morphism.

We first recall what it means for $\mathcal{X} \to X$ to be a $\mu_n$-gerbe.

**Definition A.1.** A $\mu_n$-gerbe is an $S$-stack $\mathcal{Y}$ along with an isomorphism $\mu_{n, \mathcal{Y}} \to \mathcal{I}(\mathcal{Y})$. We say that $\mathcal{Y}$ is a $\mu_n$-gerbe on $Y$ if there is a morphism $\mathcal{Y} \to Y$ such that the natural map $\text{Sh}(\mathcal{Y}) \to Y$ is an isomorphism, where $\text{Sh}(\mathcal{Y})$ denotes the sheafification of $\mathcal{Y}$ on the big étale site of $S$.

Because $\mathcal{X}$ is a $\mu_n$-gerbe, any quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ admits a decomposition $\mathcal{F} = \mathcal{F}_0 \oplus \cdots \oplus \mathcal{F}_{n-1}$ into eigensheaves, where the natural (left) action of the stabilizer on $\mathcal{F}_i$ is via the $i$th power map.

**Definition A.2.** An $\mathcal{X}$-twisted sheaf is a sheaf $\mathcal{F}$ of $\mathcal{O}_\mathcal{X}$-modules such that the natural left action (induced by inertia) $\mu_n \times \mathcal{F} \to \mathcal{F}$ is equal to scalar multiplication.

**A.1. Moduli.** It is a standard fact (see, for example, [17]) that the stack of flat families of quasi-coherent $\mathcal{X}$-twisted sheaves of finite presentation is an algebraic stack locally of finite presentation over $S$. For the purposes of this paper, we will focus on only a single case, where we will consider stability of twisted sheaves. From now on, we assume that $S = \text{Spec} k$ with $k$ a field, and $X$ is a proper smooth curve over $k$.

To define stability, we need a notion of degree for invertible $\mathcal{X}$-twisted sheaves. For the sake of simplicity, we give an ad hoc definition. Given an invertible $\mathcal{X}$-twisted sheaf $L$, we note that $L^\otimes n$ is the pullback of a unique invertible sheaf $L'$ on $X$. We can thus define $\deg L$ to be $\frac{1}{n} \deg L'$. With this definition, we can define the degree of a locally free $\mathcal{X}$-twisted sheaf $V$ as $\deg V = \deg \text{det } V$. With this notion of degree, we can define stability.

**Definition A.1.1.** The slope of a non-zero locally free $\mathcal{X}$-twisted sheaf $V$ is

$$\mu(V) = \frac{\deg V}{\text{rk } V}.$$

**Definition A.1.2.** A locally free $\mathcal{X}$-twisted sheaf $V$ is stable if for all proper locally split subsheaves $W \subset V$ we have

$$\mu(W) < \mu(V).$$

The sheaf is semistable if for all proper locally split subsheaves $W \subset V$ we have

$$\mu(W) \leq \mu(V).$$

By “locally split” we mean that there is a faithfully flat map $Z \to \mathcal{X}$ such that there is a retraction of the inclusion $W_Z \subset V_Z$. (I.e., $W$ is locally a direct summand of $V$.) As in the classical case, the stack of stable sheaves is a $\mathcal{G}_m$-gerbe over an algebraic space. If in addition we fix a determinant, then the resulting stack is a $\mu_r$-gerbe over an algebraic space, where $r$ is the rank of the sheaves in question.
Notation A.1.3. Given an invertible sheaf $L$ on $X$, the stack of stable (resp. semistable) $\mathcal{X}$-twisted sheaves of rank $n$ and determinant $L$ will be denoted $\mathcal{M}_{\mathcal{X}/k}(n, L)$ (resp. $\mathcal{M}_{\mathcal{X}/k}(n, L)^{ss}$). The coarse moduli space (which in the stable case is also the sheafification) will be denoted $M_{\mathcal{X}/k}(n, L)$ (resp. $M_{\mathcal{X}/k}(n, L)^{ss}$).

Given an integer $d$, the stack of stable $\mathcal{X}$-twisted sheaves of rank $n$ and degree $d$ will be denoted $\mathcal{M}_{\mathcal{X}/k}(n, d)$, and its coarse moduli space will be denoted $M_{\mathcal{X}/k}(n, d)$.

As mentioned above, $\mathcal{M}_{\mathcal{X}/k}(n, d) \to M_{\mathcal{X}/k}(n, d)$ is a $\mathbf{G}_m$-gerbe; similarly, $\mathcal{M}_{\mathcal{X}/k}(n, L) \to M_{\mathcal{X}/k}(n, L)$ is a $\mu_n$-gerbe. In fact, the stack $\mathcal{M}_{\mathcal{X}/k}(n, L)$ is the stack theoretic fiber of the determinant morphism $\mathcal{M}_{\mathcal{X}/k}(n, L) \to \text{Pic}_{\mathcal{X}/k}^d$ over the morphism $\text{Spec} \ k \to \text{Pic}_{\mathcal{X}/k}^d$ corresponding to $L$. The comparison results described in Section 7 combined with classical results on the stack of stable sheaves on a curve, show that $\mathcal{M}_{\mathcal{X}/k}(n, d)$ is of finite type over $k$; if we assume that the class $\mathcal{X}$ is zero in $H^2(X \otimes \mathbb{F}, \mu_n)$ we know that $\mathcal{M}_{\mathcal{X}/k}(n, d)$ is quasi-proper (i.e., satisfies existence part of the valuative criterion of properness) whenever $n$ and $d$ are relatively prime.

A.2. Deformation theory. As $\mathcal{O}$-modules on a ringed topos (the étale topos of $\mathcal{X}$), $\mathcal{X}$-twisted sheaves are susceptible to the usual deformation theory of Illusie. The following theorem summarizes the consequences of this fact. Suppose $S = \text{Spec} \ A$ is affine, $I \to A \to \hat{A}$ is a small extension (so that the kernel $I$ is an $A$-module), $X/\hat{A}$ is flat, $\mathcal{X} \to X$ is a $\mu_n$-gerbe with $n$ invertible in $A$, and $\mathcal{F}$ is an $\hat{A}$-flat family of quasi-coherent $\mathcal{X} \otimes \hat{A}$-twisted sheaves of finite presentation.

Theorem A.2.1. There is an element $\varphi \subset \text{Ext}^2(\mathcal{F}, \mathcal{F}_A \otimes I)$ such that

1. $\varphi = 0$ if and only if there is an $\hat{A}$-flat quasi-coherent $\mathcal{X}$-twisted sheaf $\hat{\mathcal{F}}$ and an isomorphism $\hat{\mathcal{F}} \otimes \hat{A} \cong \mathcal{F}$;
2. if such an extension exists, the set of such extensions is a torsor under $\text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes I)$;
3. given one such extension, the group of automorphisms of $\hat{\mathcal{F}}$ which reduce to the identity on $\mathcal{F}$ is identified with $\text{Hom}(\mathcal{F}, \mathcal{F} \otimes I)$.

Corollary A.2.2. If $X/S$ is a relative curve, then the stack of locally free $\mathcal{X}$-twisted sheaves is smooth over $S$.

Deformation theory can also be used to construct global objects from local data. The key technical tool is the following; see [19] for a more detailed proof and further references.

Proposition A.2.3. Let $\mathcal{P}$ be a separated tame Artin stack of finite type over $k$ which is pure of dimension 1. Let $P$ be the coarse moduli space of $\mathcal{P}$ and let $\mathcal{O} \to \mathcal{P}$ be a $\mu_n$-gerbe. Suppose $\mathcal{P}$ is regular away from the (finitely many) closed residual gerbes $\xi_1, \ldots, \xi_r$. Suppose the map $\text{Pic}(\mathcal{P}) \to \prod \text{Pic}(\xi_i)$ has kernel generated by the image of Pic($P$) under pullback.

1. Given locally free $\mathcal{E}_{\xi_i}$-twisted sheaves $V_i$ of rank $m_i$, $i = 1, \ldots, r$, and locally free $\mathcal{E}_{\xi_i}$-twisted sheaf $W_i$ of rank $m$, there is a locally free $\mathcal{E}$-twisted sheaf $W$ of rank $m$ such that $W_i \cong V_i$ and $W_i \cong V_i$.
2. Suppose that $k$ is finite, $P \cong \mathbb{P}^1$, $\mathcal{P} \to P$ is generically a $\mu_n$-gerbe, and the ramification extension $P \to P$ of $\mathcal{E}$ (see Proposition C.2 below) is geometrically connected. Given an invertible sheaf $L \in \text{Pic}(\mathcal{P})$, if there are $\mathcal{E}_{\xi_i}$-twisted sheaves $V_i$ of rank $m$ such that $\text{det} V_i \cong L_{\xi_i}$ and a $\mathcal{E}_{\xi_i}$-twisted sheaf $V_i$ of rank $m$, then there is a locally free $\mathcal{E}$-twisted sheaf $V$ of rank $m$ such that $\text{det} V \cong L$.

Proof: Let $P$ be the coarse moduli space of $\mathcal{P}$, and let $p_i \in P$ be the image of $\xi_i$. Let $\hat{\xi}_i$ denote the localization of $\text{Spec} \ O_{p_i}$ at the set of maximal (i.e., generic) points. Let $\mathcal{O}(1)$ be an ample invertible sheaf on $P$.

To prove the first item, note that the stack $\mathcal{E}_{\xi_i}$ is tame, as its inertia group is an extension of a reductive group by a $\mu_n$. Thus, the infinitesimal deformations of each $V_i$ are unobstructed. Since $\mathcal{E}$ is proper over $P$, the Grothendieck existence theorem implies that for each $i$ there is a $\mathcal{E} \times \text{Spec} \ O_{p_i}$-twisted sheaf $\tilde{V}_i$ of rank $m$ whose restriction to $\mathcal{E}_{\xi_i}$ is $V_i$. On the other hand, since the scheme of generic points of $P$ is 0-dimensional, we know that for each $i$ there is an isomorphism $(\tilde{V}_i)_{\xi_i} \cong (V_i)_{\xi_i}$. (See [19] for a similar situation, with more details.) The basic descent result of [22] shows that we can glue the $\tilde{V}_i$ to $V_i$ to produce $W$, as desired.
To prove the second item, choose any $W$ as in the first part, and let $L' = \det W$. This is an invertible sheaf which is isomorphic to $L$ in a neighborhood of each $\xi$, and therefore (by hypothesis) there is an invertible sheaf $M$ on $P$ such that $L \otimes (L')^{-1} \cong M$. Twisting $W$ by a suitable (negative!) power of $\mathcal{O}(1)$, we may assume that $M$ is ample of arbitrarily large degree. By the Lang-Weil estimates and hypothesis that $P \cong \mathbb{P}^1$, there is a point $q \in R$ whose image $p$ in $P$ is an element of $|M|$. Making an elementary transformation of $W$ along $p$ (over which the Brauer class associated to $\mathcal{C}$ is trivial) produces a locally free $\mathcal{C}$-twisted sheaf with the desired properties. \hfill $\Box$

**APPENDIX B. BASIC FACTS ON THE BRAUER GROUP**

In this appendix, we review a few basic facts about the Brauer group of a scheme. We freely use the technology of twisted sheaves, as introduced in the previous appendix. Let $Z$ be a quasi-compact separated scheme and $\mathcal{X} \to Z$ a $G_m$-gerbe. We start with a (somewhat idiosyncratic) definition of the Brauer group of $Z$.

**Definition B.1.** The cohomology class $[\mathcal{X}] \in H^2(Z, G_m)$ is said to belong to the Brauer group of $Z$ if there is a non-zero locally free $\mathcal{X}$-twisted sheaf of finite rank.

**Lemma B.2.** The Brauer group of $Z$ is a group.

**Proof.** Given two $G_m$-gerbes $\mathcal{X}_1 \to Z$ and $\mathcal{X}_2 \to Z$ which belong to the Brauer group of $Z$, let $V_i$ be a locally free $\mathcal{X}_i$-twisted sheaf. Then $V_1 \otimes V_2$ is a locally free $\mathcal{X}_1 \wedge \mathcal{X}_2$-twisted sheaf, where $\mathcal{X}_1 \wedge \mathcal{X}_2$ is the $G_m$-gerbe considered in \[10, 14\], which represents the cohomology class $[\mathcal{X}_1] + [\mathcal{X}_2]$. The neutral element in the group is represented by the trivial gerbe $\text{BG}_m \times Z \to Z$.

We thus find a distinguished subgroup $\text{Br}(Z) \subset H^2(Z, G_m)$ containing those classes which belong to the Brauer group of $Z$. What are the properties of this group?

**Proposition B.3.** The group $\text{Br}(Z)$ has the following properties.

1. An element $[\mathcal{X}]$ is trivial in $\text{Br}(Z)$ if and only if there is an invertible $\mathcal{X}$-twisted sheaf.
2. $\text{Br}(Z)$ is a torsion abelian group.
3. (Gabber) If $Z$ admits an ample invertible sheaf, then the inclusion $\text{Br}(Z) \subset H^2(Z, G_m)_{\text{tors}}$ is an isomorphism.

**Sketch of proof.** If $\mathcal{X}$ is an invertible $\mathcal{X}$-twisted sheaf, then we can define a morphism of $G_m$-gerbes $\text{BG}_m \to \mathcal{X}$ by sending an invertible sheaf $L$ to the invertible $\mathcal{X}$-twisted sheaf $L \otimes \mathcal{X}$. Since any $G_m$-gerbe admitting a morphism of $G_m$-gerbes from $\text{BG}_m$ is trivial, we see that $\mathcal{X}$ is trivial \[10\].

Now suppose that $\mathcal{X}$ is an arbitrary element of $\text{Br}(Z)$, and let $V$ be a locally free $\mathcal{X}$-twisted sheaf of rank $r$. Writing $\mathcal{X}_r$ for the gerbe corresponding to $r[\mathcal{X}]$, one can show that the sheaf $\det V$ is an invertible $\mathcal{X}_r$-twisted sheaf, thus showing that $r[\mathcal{X}] = 0$, as desired.

The last part of the proposition is due to Gabber; a different proof has been written down by de Jong \[8\].

Given a $G_m$-gerbe $\mathcal{X} \to Z$ and a locally free $\mathcal{X}$-twisted sheaf $V$, the $\mathcal{O}_\mathcal{X}$-algebra $\text{End}(V)$ is acted upon trivially by the inertia stack of $\mathcal{X}$, and thus is the pullback of a unique sheaf of algebras $\mathfrak{A}$ on $Z$. The algebras $\mathfrak{A}$ which arise in this manner are precisely the Azumaya algebras: étale forms of $\text{Mat}_r(\mathcal{O}_Z)$ (for positive integers $r$). Moreover, starting with an Azumaya algebra, we can produce a $G_m$-gerbe by solving the moduli problem of trivializing the algebra (i.e., making it isomorphic to a matrix algebra). Further details about this correspondence may be found in \[10\] \S V.4.

When $Z = \text{Spec } K$ for some field $K$, an Azumaya algebra is precisely a central simple algebra over $K$, and thus we recover the classical Brauer group of the field. Note that in this case if $\mathcal{X} \to Z$ is a $G_m$-gerbe, any nonzero coherent $\mathcal{X}$-twisted sheaf is locally free. Moreover, we know that $\mathcal{X} \otimes \overline{K}$ is isomorphic to $\text{BG}_m \otimes \overline{K}$, and therefore that there is an invertible $\mathcal{X} \otimes \overline{K}$-twisted sheaf. Pushing forward to $\mathcal{X}$, we see that there is a nontrivial quasi-coherent $\mathcal{X}$-twisted sheaf $\mathcal{F}$. Since $\mathcal{X}$ is Noetherian, $\mathcal{F}$ is a colimit of coherent $\mathcal{X}$-twisted subsheaves. This shows that $\mathcal{F}$ belongs to the Brauer group of $Z$. We have just given a geometric proof of the classical Galois cohomological result $\text{Br}(K) = H^2(\text{Spec } K, G_m)$.

Now assume that $Z$ is integral and Noetherian with generic point $\eta$, and let $\mathcal{X} \to Z$ in arbitrary $G_m$-gerbe. By the preceding, there is a coherent $\mathcal{X}$-twisted sheaf of positive rank.
Definition B.4. The index of $[\mathcal{Z}]$, denoted $\text{ind}([\mathcal{Z}])$, is the minimal nonzero rank of a coherent $\mathcal{Z}$-twisted sheaf. The period of $[\mathcal{Z}]$, denoted $\text{per}([\mathcal{Z}])$, is the order of $[\mathcal{Z}]_\eta$ in $H^2(\eta, G_m)$.

We have written Definition B.4 so that it only pertains to generic properties of $\mathcal{Z}$ and $Z$. One can imagine more general definitions (e.g., using locally free $\mathcal{Z}$-twisted sheaves; when $Z$ is regular of dimension 1 or 2, the basic properties of reflexive sheaves tell us that the natural global definition will actually equal the generic definition. When $Z = \text{Spec } K$, is easy to say what the index of a Brauer class $\alpha$ is: $\alpha$ parameterizes a unique central division algebra over $K$, whose dimension over $K$ is $n^2$ for some positive integer $n$. The index of $\alpha$ is then $n$.

The basic fact governing the period and index is the following.

Proposition B.5. For any $\alpha \in \text{Br}(K)$, there is a positive integer $h$ such that $\text{per}(\alpha) | \text{ind}(\alpha)$ and $\text{ind}(\alpha) | \text{per}(\alpha)^h$.

The proof of Proposition B.5 is an exercise in Galois cohomology; the reader is referred to (for example) [20] Lemma 2.1.1.3. One immediate consequence of Proposition B.5 is the question: how can we understand $h$? For example, does $h$ depend only on $K$ (in the sense that there is a value of $h$ which works for all $\alpha \in \text{Br}(K)$)? If so, what properties of $K$ are being measured by $h$? And so on. Much work has gone into this problem; for a summary of our current expectation the reader is referred to [20].

Appendix C. Ramification of Brauer Classes

In this section, we recall the basic facts about the ramification theory of Brauer classes. We also describe how to split ramification by a stack. To begin with, we consider the case $X = \text{Spec } R$ with $R$ a complete discrete valuation ring with valuation $v$. Fix a uniformizer $t$ of $R$; let $K$ and $\kappa$ denote the fraction field and residue field of $R$, respectively. Write $j : \text{Spec } K \rightarrow \text{Spec } R$ for the natural inclusion. Throughout, we only consider Brauer classes $\alpha \in \text{Br}(K)$ of period relatively prime to $\text{char}(\kappa)$; we write $\text{Br}(K)^\prime$ for the subgroup of classes satisfying this condition.

The usual theory of divisors yields an exact sequence of étale sheaves on $\text{Spec } R$

$$0 \rightarrow G_m \rightarrow j_! G_m \rightarrow \mathbb{Z}(t) \rightarrow 0,$$

which yields a map

$$H^2(\text{Spec } R, j_! G_m) \rightarrow H^2(\text{Spec } R, \mathbb{Z}(t)) = H^2(\kappa, \mathbb{Z}) = H^1(\kappa, \mathbb{Q}/\mathbb{Z}).$$

Since any $a \in \text{Br}(K)^\prime$ has an unramified splitting field, the Leray spectral sequence for $G_m$ on $j$ shows that $H^2(\text{Spec } R, j_! G_m)^\prime = \text{Br}(K)^\prime$ (where the $'$ denotes classes with orders invertible in $R$). Putting this together yields the ramification sequence

$$0 \rightarrow \text{Br}(R)^\prime \rightarrow \text{Br}(K)^\prime \rightarrow H^1(\kappa, \mathbb{Q}/\mathbb{Z}).$$

The last group in the sequence parameterizes cyclic extensions of the residue field $\kappa$. Suppose for the sake of simplicity that $\kappa$ contains a primitive $n$th root of unity. The ramification of a cyclic algebra $(a, b)$ is given by the extension of $\kappa$ generated by the $n$th root of $(-1)^{\text{v}(a)\text{v}(b)}a^{\text{v}(b)}/b^{\text{v}(a)}$. In particular, given any element $\eta \in \kappa^*$, the algebra $(u, t)$ has ramification extension $\kappa(\eta^{1/n})$, where $u$ is any lift of $\eta$ in $R^*$. With our assumption about roots of unity, any cyclic extension of degree $n$ is given by extracting roots of some $\eta$. This has the following two useful consequences. Given a positive integer $n$, let $\mathbb{A}_n \rightarrow \text{Spec } R$ denote the stack of $n$th roots of the closed point of $\text{Spec } R$, as in [3].

Proposition C.1. Assume that $\kappa$ contains a primitive $n$th root of unity $\zeta$. Fix an element $\alpha \in \text{Br}(K)[n]$.

1. There exists $u \in R^*$ and $\alpha' \in \text{Br}(R)$ such that $\alpha = \alpha' + (u, t) \in \text{Br}(K)$.
2. There exists $\beta \in \text{Br}(\mathbb{A}_n)$ whose image in $\text{Br}(K)$ under the restriction map is $\alpha$.

Proof: The first item follows immediately from the paragraph preceding this proposition: we can find $u$ such that $(u, t)$ has the same ramification as $\alpha$, and subtracting this class yields an element of $\text{Br}(R)$. To prove the second item, it follows in the first that it suffices to prove it for the class $(u, t)$. Recall that $\mathbb{A}_n$ is the stacky quotient of $\text{Spec } R[t^{1/n}]$ by the natural action of $\mu_n$; to extend $(u, t)$ to $\mathbb{A}_n$, it suffices to find a $\mu_n$-equivariant Azumaya algebra in $(u, t)K_{(t^{1/n})}$. Recall that $(u, t)$ is generated by $x$ and $y$
such that $x^n = u$, $y^n = t$, and $xy = \zeta yx$. Letting $\tilde{y} = y/t^{1/n}$, the natural action of $\mu_n$ on $t^{1/n}$ yields an equivariant Azumaya algebra in $(u, t)_{K(t^{1/n})}$, as desired. 

The following corollary is an example of how one applies Proposition C.1 in a global setting. More general results are true, using various purity theorems.

**Corollary C.2.** Let $C$ be a proper regular curve over a field $k$ which contains a primitive $n$th root of unity for some $n$ invertible in $k$. Suppose $\alpha \in Br(k(C))[n]$ is ramified at $p_1, \ldots, p_r$, and let $\mathcal{E} \rightarrow C$ is a stack of $n$th roots of $p_1 + \cdots + p_r$. There is an element $\alpha' \in Br(\mathcal{E})[n]$ whose restriction to the generic point is $\alpha$.

**Proof.** Let $A$ be a central simple algebra over $k(C)$ with Brauer class $\alpha$. For any point $q \in C$, we know (at the very least) that there is a positive integer $m$ and an Azumaya algebra over the localization $\mathcal{E}_q$ which is contained in $M_m(A)$. Suppose $U \subset \mathcal{E}$ is an open substack over which there is an Azumaya algebra $B$ with generic Brauer class $\alpha$. Given a closed point $q \in C \setminus U$, we can choose some $m$ such that there is an Azumaya algebra $B'$ over $\mathcal{E}_q$ whose restriction to the generic point $\tilde{q}$ of $\mathcal{E}_q$ is isomorphic to $M_m(B)_\tilde{q}$. Since $U \times_C \mathcal{E}_q = \tilde{q}$, we can glue $B'$ to $M_m(B)$ as in the proof of Proposition [A.2.3](using [22]) to produce an Azumaya algebra over $U \cup \{q\}$. Since $C \setminus U$ is finite, we can find some $m$ such that $M_m(A)$ extends to an Azumaya algebra over $\mathcal{E}$, as desired. 

The following is a more complicated corollary of Proposition C.1 using purity of the Brauer group on a surface. We omit the proof.

**Corollary C.3.** Let $X$ be a connected regular Noetherian scheme pure of dimension 2 with function field $K$. Suppose $U \subset X$ is the complement of a simple normal crossings divisor $D = D_1 + \cdots + D_r$ and $\alpha \in Br(U)[n]$ is a Brauer class such that $n$ is invertible in $\kappa(D_i)$ for $i = 1, \ldots, r$. If $\mathcal{F} \rightarrow X$ is the root construction of order $n$ over each $D_i$, then there is a class $\tilde{\alpha} \in Br(\mathcal{F})$ such that $\tilde{\alpha}_U = \alpha$.

We end this discussion with an intrinsic characterization of the ramification extension as a moduli space.

**Proposition C.4.** Let $X$ be a scheme on which $n$ is invertible such that $\Gamma(X, \mathcal{O})$ contains a primitive $n$th root of unity. Suppose $\pi : \mathcal{F} \rightarrow X$ is a $\mu_n$-gerbe. Given a further $\mu_n$-gerbe $\mathcal{Y} \rightarrow \mathcal{F}$, the relative Picard stack $\mathcal{Pic}_{\mathcal{Y}/X}$ of invertible $\mathcal{Y}$-twisted sheaves is a $G_m$-gerbe over a $\mathbb{Z}/n\mathbb{Z}$-torsor $T$. Moreover, the Brauer class of $\mathcal{Y}$ in $\Pi^2(\mathcal{F}, G_m)$ is the pullback of a class from $X$ if and only if $T$ is trivial.

**Proof.** Standard methods show that $\mathcal{Pic}_{\mathcal{Y}/X}$ is a $G_m$-gerbe over a flat algebraic space of finite presentation $P \rightarrow X$. Since the relative Picard space $Pic_{\mathcal{Y}/X}$ is isomorphic to the constant group scheme $\mathbb{Z}/n\mathbb{Z}$, tensoring with invertible sheaves on $\mathcal{F}$ gives an action

$$\mathbb{Z}/n\mathbb{Z} \times P \rightarrow P.$$

To check that this makes $B$ a torsor, it suffices (by the obvious functoriality of the construction) to treat the case in which $X$ is the spectrum of an algebraically closed field $k$. In this case, $Br(\mathcal{F}) = 0$; choosing an invertible $\mathcal{Y}$-twisted sheaf $L$, we see that all invertible $\mathcal{Y}$-twisted sheaves $M$ of the form $L \otimes \Lambda$, where $\Lambda$ is invertible sheaf on $\mathcal{F}$. This gives the desired result. 

Starting with a complete discrete valuation ring $R$ with fraction of $K$, an element $\alpha \in Br(K)$ gives rise to two cyclic extensions of the residue field $\kappa$ of $R$: the classical ramification extension and the moduli space produced in Proposition C.4. In fact, these two extensions are isomorphic. We will use this fact, but we omit the details.

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