Abstract. The purpose of this paper is to suggest the construction and study properties of semi-infinite induction, which relates to semi-infinite cohomology the same way induction relates to homology and coinduction to cohomology. We prove a version of the Shapiro Lemma, showing that the semi-infinite cohomology of a module is isomorphic to that of the semi-infinitely induced module. A practical outcome of our construction is a simple construction of the Wakimoto modules, highest-weight modules used in double-sided BGG resolutions of irreducible modules.

Semi-infinite cohomology of Lie algebras, introduced as the appropriate mathematical setting for BRST theory by B. L. Feigin [6] (see also [10, 14] regarding basic facts on semi-infinite cohomology), is a cohomology theory that has properties in common with both cohomology and homology. Semi-infinite cohomology has become an important tool in representation theory of Lie algebras and quantum groups and string theory, see, for instance, [2, 4, 3, 5, 7, 8, 12, 13].

The purpose of this paper is to suggest the construction and study properties of semi-infinite induction, which relates to semi-infinite cohomology the same way induction relates to homology and coinduction to cohomology. The proof of our main theorem (Theorem 1.4, the semi-infinite Shapiro Lemma) is based on the independence of the choice of a resolution, which follows from the machinery of semi-infinite homological algebra developed in [14]. A practical outcome of our construction is a simple construction of Wakimoto modules, which were constructed by Feigin and E. Frenkel [7] in rather roundabout terms: using bosonization and also as $\mathcal{H}^{\infty/2^+}(X, \mathcal{L})$, a hypothetical semi-infinite cohomology, with support on the big Schubert cell, of an invertible sheaf over a semi-infinite flag manifold. The idea that Wakimoto modules might be obtained by some kind of semi-infinite induction goes back to the original paper of Feigin and Frenkel [7]. This idea was implemented by S. M. Arkhipov [2], who suggested an indirect construction of Wakimoto modules, which, in fact, may be considered as representing a different approach to semi-infinite induction. Wakimoto modules play an intermediate role between Verma and contragredient Verma modules: they all have a very similar behavior with respect to semi-infinite cohomology, usual homology, and cohomology, respectively. The three types of modules have the same character (see Proposition 2.2) but a different layout of irreducible pieces.

This work was motivated in part by a construction of N. Berkovits and C. Vafa [5] of $N = 1$ and $N = 2$ string theories out of a given $N = 0$ (bosonic) string theory. In that construction, the bosonic string arose as a particular class of vacua for the $N = 1$ string and the $N = 1$ as a particular class of vacua for the $N = 2$ string. Berkovits and Vafa also suggested that there must be a universal string theory comprising all possible string theories, including, for instance, $W_N$ strings.

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as particular choices of vacua. It was J. Figueroa-O’Farrill [8, 9] who observed that Berkovits-Vafa’s construction presumably had to do with some sort of semi-infinite induction and inquired whether such construction had been known. This paper contains an answer to his question. However, the physically oriented reader must be warned that the physics problem is more complex than the mathematical model. The physical spaces of vacua of two string theories in the hierarchy of the conjectural universal string theory should not only match as vector spaces, but the correlators between different vacua should be equal to each other — this would guarantee that the two theories are physically the same. Mathematically, this amounts to the problem of performing semi-infinite induction in the category of vertex operator algebras: assume that the module over a smaller algebra is a vertex operator algebra and construct a matching vertex operator algebra structure on the semi-induced module. We do not know how to do that.

Throughout the paper we will be working over the ground field $\mathbb{C}$ of complex numbers.

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1. Semi-infinite induction

Induction is a certain construction of “base change” in representation theory. Given a Lie algebra and a subalgebra of it, $\mathfrak{h} \subset \mathfrak{g}$, as well as an $\mathfrak{h}$-module $M$, one constructs a $\mathfrak{g}$-module $\text{Ind}_{\mathfrak{h}}^{\mathfrak{g}} M$, such that the corresponding homology does not change: $H_\bullet(\mathfrak{g}, \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}} M) = H_\bullet(\mathfrak{h}, M)$. Coinduction starts from the same data, but constructs a $\mathfrak{g}$-module $\text{Coind}_{\mathfrak{h}}^{\mathfrak{g}} M$ possessing the similar property with respect to cohomology: $H^\bullet(\mathfrak{g}, \text{Coind}_{\mathfrak{h}}^{\mathfrak{g}} M) = H^\bullet(\mathfrak{h}, M)$. Semi-infinite induction will do the same, pertinent to semi-infinite cohomology: $H^{\infty/2+\bullet}(\mathfrak{g}, \text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}} M) = H^{\infty/2+\bullet}(\mathfrak{h}, M)$.

1.1. Coinduction. Let us briefly recall the coinduction construction. The coinduced module is defined as

$$\text{Coind}_{\mathfrak{h}}^{\mathfrak{g}} M = \text{Hom}_{\mathfrak{h}}(U(\mathfrak{g}), M),$$

where $\mathfrak{g}$ acts on the universal enveloping algebra $U(\mathfrak{g})$ by multiplication on the right and $\mathfrak{h}$ on the left. The following theorem is a standard fact of homological algebra.

Theorem 1.1 (Shapiro Lemma). There is a natural isomorphism of cohomology

$$H^\bullet(\mathfrak{g}, \text{Coind}_{\mathfrak{h}}^{\mathfrak{g}} M) = H^\bullet(\mathfrak{h}, M).$$

Proof. Take a projective resolution

$$P^\bullet : \cdots \to P^{-1} \to P^0 \to 0$$
of the trivial \( g \)-module \( C \). It is a bounded above complex of projective \( g \)-modules such that \( H^*(P^*) = H^0(P^*) = C \). Notice that it is also a projective resolution of \( C \) in the category of \( h \)-modules and \( \operatorname{Hom}(P^*, M) \) is an injective resolution of the \( h \)-module \( M \). Therefore, one produces the cohomology of \( h \) with coefficients in \( M \) by applying the functor of \( h \)-invariants to this resolution and then taking the cohomology of the obtained complex:

\[
H^*(h, M) = H^*(\operatorname{Hom}_h(P^*, M)).
\]

On the other hand, \( \operatorname{Hom}(P^*, \operatorname{Coind}_h^g M) \) is an injective resolution of the \( g \)-module \( \operatorname{Coind}_h^g M \). Thus, if we apply the functor of \( g \)-invariants, we will get the cohomology of \( g \) with coefficients in \( \operatorname{Coind}_h^g M \):

\[
H^*(g, \operatorname{Coind}_h^g M) = H^*(\operatorname{Hom}_g(P^*, \operatorname{Coind}_h^g M)).
\]

It remains to notice that the two complexes are naturally isomorphic because of the universality property of coinduction:

\[
(1.1) \quad \operatorname{Hom}_g(P^*, \operatorname{Coind}_h^g M) = \operatorname{Hom}_h(P^*, M).
\]

1.2. **Semi-infinite structure and semi-invariants.** The semi-infinite induction and Shapiro Lemma are very much parallel to the classical coinduction case, except that each step calls for an entirely new ingredient. The construction of a semi-induced module requires an intermediate object between the universal enveloping algebra and its dual. The semi-infinite analogue of Shapiro Lemma needs a functor, like the one of invariants, to get back to the original module from an induced one. The proof of it also needs the machinery of two-sided resolutions and two-sided derived functors. All these ingredients have been developed in the earlier paper [14].

Suppose we have a Lie algebra with a *semi-infinite structure*. This may be understood as a graded Lie algebra \( g = \bigoplus_{n \in \mathbb{Z}} g_n \) with finite-dimensional graded components, along with the following structure. Decompose the algebra \( g \) into the direct sum of two subalgebras:

\[
g = g_+ \oplus g_-,
\]

\[
g_+ = \bigoplus_{n > 0} g_n, \quad g_- = \bigoplus_{n \leq 0} g_n.
\]

Assume that the natural mapping \( g \to \mathfrak{gl}_{\text{res}} \) via the adjoint representation is lifted to a mapping \( g \to \tilde{g}_{\text{res}} \), where \( \tilde{g}_{\text{res}} = g_{\text{res}}(g) \) is a “restricted” general linear algebra of the vector space \( g \), for example, the one consisting of operators whose \( g_+ \) block of a finite rank; \( \tilde{g}_{\text{res}} \) is a nontrivial central extension of \( \mathfrak{gl}_{\text{res}} \), see e.g., [1]. If this lifting is not possible, we should replace \( g \) by the corresponding central extension, which will be lifted canonically. Let \( \beta : g \to C \) be the linear functional defined by this lifting and a splitting of the extension \( 0 \to C \to \tilde{g}_{\text{res}} \to g_{\text{res}} \to 0 \) as an extension of vector spaces. In fact, this splitting can be chosen in such a way that \( \beta \) vanishes on all \( g_n \), see for example, [14, Proposition 2.4]. We will assume this for the sake of simplicity. Notice that \( \beta \) defines a one-cocycle on \( g_- \) and the zero one-cocycle on \( g_+ \). Denote the corresponding one-dimensional modules by one symbol \( L_\beta \). As vector spaces, the modules \( L_\beta \) are canonically isomorphic to \( C \).

We assume the functional \( \beta \) to be part of a semi-infinite structure on a Lie algebra.
Throughout this paper, a \(\mathbb{Z}\)-graded vector space means a vector space \(M\) with a collection of subspaces \(M_n, n \in \mathbb{Z}\), such that \(\bigoplus_{n \in \mathbb{Z}} M_n \subset M \subset \prod_{n \in \mathbb{Z}} M_n\). For two graded vector spaces \(M\) and \(N\), we will define the space \(\text{Hom}(M, N)\) as follows:

\[
\text{Hom}(M, N) = \bigoplus_{n > 0} \text{Hom}(M_n, N) \oplus \prod_{n \leq 0} \text{Hom}(M_n, N).
\]

It has a natural \(\mathbb{Z}\)-bigrading. This definition of Hom is motivated by the following argument. If we consider the inverse-limit topology on \(M\), that is, the topology coming from \(\lim_{n \to \infty} M/F^n M\), where \(F^n M = M \cap (\prod_{k \geq n} M_k)\), and the discrete topology on \(N\), the space of continuous linear maps \(M \to N\) will be identified with \(\text{Hom}(M, N)\) in (1.3).

All Hom’s and duals will be understood in this sense throughout the paper. Note that with this definition of Hom, for \(\mathbb{Z}\)-graded vector spaces \(A, S\), and \(M\), such that \(A = \bigoplus_{n \leq 0} A_n\) and \(M = \bigoplus_{n \leq N} M_n\) and have finite-dimensional graded components \(A_n\) and \(M_n\) for all \(n\), one has canonical isomorphisms

\[
\text{Hom}(S \otimes M, A) \xrightarrow{\sim} \text{Hom}(S, \text{Hom}(M, A)),
\]

\[
\text{Hom}(A, S \otimes M) \xrightarrow{\sim} \text{Hom}(M^* \otimes A, S) \xleftarrow{\sim} \text{Hom}(\text{Hom}(M, A), S).
\]

We will consider \(\mathbb{Z}\)-graded \(\mathfrak{g}\)-modules, as well as objects of the so-called category \(\mathcal{O}_0\), by which we will here mean the category of \(\mathbb{Z}\)-graded \(\mathfrak{g}\)-modules \(M\) semisimple over \(\mathfrak{g}_0\), such that they are direct sums of finite-dimensional graded components \(M_n\) and the grading is bounded above: \(n \leq N\):

\[
M = \bigoplus_{n = -\infty}^N M_n.
\]

Given a semi-infinite structure on a Lie algebra \(\mathfrak{g}\), we can define the functor of semi-invariants of a \(\mathfrak{g}\)-module \(M\):

\[
(M)^+_{\mathfrak{g}^+} = \text{Im} \left( (M \otimes \mathcal{L}_{\beta})^0 \to (M \otimes \mathcal{L}_{\beta})_{\mathfrak{g}^-} \right),
\]

the image of the natural projection of the \(\mathfrak{g}^+\)-invariants \(\{x \in M \otimes \mathcal{L}_{\beta} \mid gx = 0 \text{ for all } g \in \mathfrak{g}^+\}\) onto the \(\mathfrak{g}^-\)-coinvariants \((M \otimes \mathcal{L}_{\beta})_{\mathfrak{g}^-} = M \otimes \mathcal{L}_{\beta}/\mathfrak{g}^- \cdot (M \otimes \mathcal{L}_{\beta})\).

This functor coincides with the semi-infinite cohomology group \(H^{\infty/2+0}(\mathfrak{g}, M)\), see Section 1.3 for a class of modules \(M\) called semijective in [14, Corollary 3.4]. We will recall this notion along with a more general concept of a semijective resolution below in Section 1.6.

1.3. Semi-infinite cohomology. Suppose \(\mathfrak{g}\) is a Lie algebra with a semi-infinite structure. The space \(\Lambda^{\infty/2+}\mathfrak{g}\) of semi-infinite forms on \(\mathfrak{g}\) is an irreducible representation of the Clifford algebra based on the vector space \(\mathfrak{g} \oplus \mathfrak{g}^*\), where \(\mathfrak{g}^* = \text{Hom}(\mathfrak{g}, \mathbb{C})\) in the sense of (1.2). This representation is spanned by a vector \(\omega_0\), called the vacuum, satisfying the conditions

\[
g^* \omega_0 = 0 \quad \text{for } g^* \in \mathfrak{g}^+_+,
\]

\[
g \omega_0 = 0 \quad \text{for } g \in \mathfrak{g}^-.
\]

The space of semi-infinite forms is graded: \(\Lambda^{\infty/2+}\mathfrak{g} = \bigoplus_{p} \Lambda^{\infty/2+p}\mathfrak{g}\), the degree being counted from \(\deg \omega_0 = 0\), \(\deg g = 1\) for \(g \in \mathfrak{g}\) and \(\deg g^* = -1\) for \(g^* \in \mathfrak{g}^*\).

One can choose a basis \(\{e_i \mid i \in \mathbb{Z}\}\) in \(\mathfrak{g}\) compatible with the \(\mathbb{Z}\)-grading and think of the vacuum as \(\omega_0 = e_0 \wedge e_{-1} \wedge e_{-2} \wedge \ldots\). Then an arbitrary semi-infinite form will be a sum of forms \(\omega = g_1 \wedge g_2 \wedge \ldots\), where \(g_i \in \mathfrak{g}\), and \(\omega\) and \(\omega_0\) have equal
semi-infinite tails, that is to say, terms in the wedge products coincide starting from some point on. The choice of a basis determines an inner product on \( g \) with respect to which the basis is orthonormal.

Define the standard semi-infinite complex for our Lie algebra \( g \) with coefficients in a \( \mathbb{Z} \)-graded \( g \)-module \( M \). This is the complex

\[
C^{\infty/2+n} = \text{Hom}(\Lambda^{\infty/2+n} g, M), \quad n \in \mathbb{Z},
\]

where the “Hom” is understood as in (1.2) with respect to the interior grading on \( \Lambda^{\infty/2+n} g \) given by \( \deg \omega_0 = 0, \deg g = i \) for \( g \in g_i \), and \( \deg g^* = -i \) for \( g^* \in g^*_i := (g_i)^* \). The differential is defined as

\[
(\text{d}\phi)(g_1 \wedge g_2 \wedge \ldots) = \sum_i (-1)^i (g_i + \beta(g_i)) \phi(g_1 \wedge g_2 \wedge \ldots \wedge \widehat{g_i} \wedge \ldots)
\]

\[
+ \sum_{i<j} (-1)^{i+j} \phi([g_i, g_j] \wedge g_1 \wedge \ldots \wedge \widehat{g_i} \wedge \ldots \wedge \widehat{g_j} \wedge \ldots),
\]

where \( g_i \)’s are assumed to be homogeneous, and the normal ordering sign \( :: \) means that whenever \([g_i, g_j] \in g_-\)

\[
:\{g_i, g_j\} \wedge g_1 \wedge \ldots \wedge \widehat{g_i} \wedge \ldots \wedge \widehat{g_j} \wedge \ldots := P_{ij}([g_i, g_j]) \wedge g_1 \wedge \ldots \wedge \widehat{g_i} \wedge \ldots \wedge \widehat{g_j} \wedge \ldots,
\]

\( P_{ij} \) being the orthogonal projection onto the orthogonal complement to the linear span of \( g_i \) and \( g_j \) in the sense of the bilinear inner product determined by the choice of a basis in \( g \). Obviously, either sum in \( d\phi \) will be finite for each \( \phi \). Note that \( M \) needs not lie in \( O_0 \) or even have a bounded above \( \mathbb{Z} \)-grading in this definition. This relaxes the hypothesis usually made in the literature, see [10].

1.4. Universal semijective module. We are going to define a bimodule which plays the same role with respect to semi-infinite cohomology as the universal enveloping algebra plays with respect to homology. Suppose \( g \) is a Lie algebra with a semi-infinite structure. The following \( U(g) \)-bimodule will be called universal semijective:

\[
\text{US} = (\text{Hom}(U(g), U(g)))^{g_-}_{g_+},
\]

where the “anti” \( g \)-semi-invariants are taken with respect to the action

\[
(\text{g}\phi)(u) = g(\phi(u)) + \phi(ug) \quad \text{for } \phi \in \text{Hom}(U(g), U(g)).
\]

Notice that the “anti” semi-invariants with respect to \( g \) are understood as in (1.4), the functional \( \beta \) being replaced by \(-\beta\).

The \( U(g) \)-bimodule structure on \( \text{US} \) is defined through the natural left \( g \)-action

\[
(\text{g}\phi)(u) = f(-gu) \quad \text{for } \phi \in \text{Hom}(U(g), U(g))
\]

and the natural right \( g \)-action

\[
(f\phi)(u) = f(ug) \quad \text{for } \phi \in \text{Hom}(U(g), U(g)).
\]

The universal semijective module \( \text{US} \) was introduced in [14] under the name “standard semijective” and denoted SS. Arkhipov [3] and Soergel [13], who also suggested different constructions for it, used it under the names of semiregular module and semi-regular Bimodul, respectively. The following two important statements regarding the structure of the left and the right actions on \( \text{US} \) were given inaccurate proofs in [14], as was pointed out by Soergel; see also a different proof
of the composite isomorphism $U(g_+)^* \otimes g_+ U(g) = \text{Hom}_{g_-}(U(g), U(g_-) \otimes L_{-\beta})$ of Propositions 1.2 and 1.3 under stronger assumptions on $g$ in Soergel [13].

**Proposition 1.2.** As a right $g$-module, the universal semijective module $US$ is isomorphic to $U(g_+)^* \otimes g_+ U(g)$, where $g_+$ acts as in (1.3) and $g$ acts as in (1.7).

**Proof.** Let us shorten the notation $U(g)$ to $U$ and consider the standard semi-infinite complex $C_{op}^{\infty/2+\bullet}$ for the Lie algebra $g$ with the opposite semi-infinite structure (the opposite $\mathbb{Z}$-grading and $-\beta$ as the structure one-cochain) with coefficients in the $g$-module $M = \text{Hom}(U, U)$, see (1.3). The module $M$ is clearly semijective with respect to this action, and therefore (Corollary 3.4 of [14]), its semi-infinite cohomology is computed as the semi-invariants $M_{g_+}^{\infty/2+\bullet} = US$ in degree 0 and 0 in all other degrees. We will compute the same cohomology group as $U(g_+)^* \otimes g_+ U$, using a spectral sequence. This will prove the proposition.

This spectral sequence, a semi-infinite version of the Hochschild-Serre spectral sequence for the pair $g_- \subset g$, was considered in [14, Theorem 2.2] and comes from the following filtration on the standard semi-infinite complex $C_{op}^{\infty/2+\bullet} = \text{Hom}(\Lambda_{op}^{\infty/2+\bullet} g, M)$, where $op$ in the subscript reminds us to consider the opposite grading on $g$: $F_p C_{op}^{\infty/2+\bullet}$

$$:= \{ \phi \in C_{op}^{\infty/2+p+q} \mid b_1 \ldots b_{q+1} \phi = 0 \text{ for each } b_1, \ldots, b_{q+1} \in g_- \},$$

where $p \leq 1$, $q \geq 0$ and the multiplication $b \phi$ is a contraction of a semi-infinite form. It is evident that $C_{op}^{\infty/2+\bullet} \supset \cdots \supset F_p \supset \cdots \supset F_1 = 0$, that is, the filtration is decreasing and regular, and moreover,

$$\bigcup_p F_p = C_{op}^{\infty/2+\bullet}.$$

One can check that the differential $d$ in $C_{op}^{\infty/2+\bullet}$ is compatible with the filtration. We have a filtered complex, which gives rise to a spectral sequence $E_{p,q}$ converging to $H_{op}^{\infty/2+\bullet}(g, M) = M_{g_+}^{\infty/2+\bullet}$. The first term of this spectral sequence is computed as $E_{1}^{0,q} = H^q(g_-, \text{Hom}(\Lambda_{op}^{\infty/2+p+q}(g/g_-), M))$. Since the module $M$ is $g_-$-injective, $E_{1}^{0,q} = 0$ for any $q > 0$. For $q = 0$, we have the row $E_{1}^{0,0}$:

$$\cdots \to \text{Hom}_{g_-}(\Lambda_{op}^{\infty/2-1}(g/g_-), M) \to \text{Hom}_{g_-}(\Lambda_{op}^{\infty/2+0}(g/g_-), M) \to 0.$$

This complex can be identified with

$$\cdots \to (g/g_- \otimes L_{-\beta} \otimes M)^{g_-} \to (L_{-\beta} \otimes M)^{g_-} \to 0,$$

differential being the $g_+$-homology differential. Since $M = \text{Hom}(U, U)$, this complex can further be identified with

$$\cdots \to g_+ \otimes \text{Hom}(U(g_+), U) \to \text{Hom}(U(g_+), U) \to 0,$$

with the differential being the standard differential in the homology complex of $g_+$ with coefficients in $\text{Hom}(U(g_+), U)$. This shows that the 0th homology $E_{2}^{0,0}$ of this complex is $\text{Hom}(U(g_+), U)_{g_+} = U(g_+)^* \otimes g_+ U$. Since $U(g_+)^* \otimes U$ is free over $g_+$, the higher homology vanishes, i.e., $E_{p,0}^{0} = 0$ for $p > 0$ and, therefore, the spectral sequence collapses at $E_2 = E_\infty$. Thus $U(g_+)^* \otimes g_+ U = E_{2}^{0,0} = H^{\infty/2+0}(g, M) =$
Proposition 1.3. As a left $\mathfrak{g}$-module, the universal semijective module $US$ is isomorphic to $\text{Hom}_{\mathfrak{g}_-}(U(\mathfrak{g}), U(\mathfrak{g}_-) \otimes \mathcal{L}_{-\beta})$, where $\mathfrak{g}_-$ acts as in (1.2) and $\mathfrak{g}$ acts as in (1.6).

Proof. The proof of this proposition is very similar to that of Proposition 1.2. For the module $M = \text{Hom}(U, U)$, we will compute the semi-infinite cohomology $M_{\mathfrak{g}_+}^{\mathfrak{g}_-} = \text{US}$ differently, using a semi-infinite version of the Hochschild-Serre spectral sequence for $\mathfrak{g}_+ \subset \mathfrak{g}$, see [14, Theorem 2.3]. This spectral sequence comes from the following filtration on the standard semi-infinite complex $C_{\mathfrak{g}_+}^{\infty/2+\bullet}$:

$$F^{p\Pi} C_{\mathfrak{g}_+}^{\infty/2+p+q} := \{ b_1^* \cdots b_p^* \phi \in C_{\mathfrak{g}_+}^{\infty/2+p+q} | b_1^*, \ldots, b_p^* \in \mathfrak{g}_+^*, \phi \in C_{\mathfrak{g}_+}^{\infty/2+q} \},$$

where $p \geq 0$, $q \leq 0$ and $b^* \phi$ is the exterior multiplication. We have

$$C_{\mathfrak{g}_+}^{\infty/2+\bullet} = F^{p\Pi} \supset \cdots \supset F^{p\Pi} \supset F^{p\Pi+1} \supset \cdots \supset 0,$$

which in particular means that the filtration is decreasing and coregular; moreover,

$$\bigcap_p F^{p\Pi} = 0.$$

This ensures convergence of the associated spectral sequence to $E'_1 = H_{\mathfrak{g}_+}^{\infty/2+\bullet}(\mathfrak{g}; M) = M_{\mathfrak{g}_+}^{\mathfrak{g}_-}$. The first term of this spectral sequence is $E^{p\Pi,q}_1 = H_{\mathfrak{g}_+}^{\infty/2+p+q}(\mathfrak{g}_+, \text{Hom}(\Lambda^p(\mathfrak{g}/\mathfrak{g}_+), M)) = H_{-q}(\mathfrak{g}_+, \text{Hom}(\Lambda^p(\mathfrak{g}/\mathfrak{g}_+), M) \otimes \mathcal{L}_{-\beta})$. Since the module $M$ is $\mathfrak{g}_+$-projective, $E^{p\Pi,q}_1 = 0$ for any $q < 0$. The $q = 0$ row $E^{0\Pi,0}_1$ is then

$$0 \rightarrow (M \otimes \mathcal{L}_{-\beta})_{\mathfrak{g}_+} \rightarrow (\text{Hom}(\mathfrak{g}/\mathfrak{g}_+, M) \otimes \mathcal{L}_{-\beta})_{\mathfrak{g}_+} \rightarrow \ldots,$$

the differential being the $\mathfrak{g}_-$-cohomology differential. Recalling that $M = \text{Hom}(U, U)$, we identify this complex with

$$0 \rightarrow \text{Hom}(U, U(\mathfrak{g}_-) \otimes \mathcal{L}_{-\beta}) \rightarrow \text{Hom}(\mathfrak{g}_-, \text{Hom}(U, U(\mathfrak{g}_-) \otimes \mathcal{L}_{-\beta})) \rightarrow \ldots,$$

with the standard differential in the cohomology complex of $\mathfrak{g}_-$ with coefficients in $\text{Hom}(U, U(\mathfrak{g}_-) \otimes \mathcal{L}_{-\beta})$. Thus the 0th cohomology $E^{0\Pi,0}_1$ of this complex is $\text{Hom}_{\mathfrak{g}_-}(U, U(\mathfrak{g}_-) \otimes \mathcal{L}_{-\beta})$. The higher cohomology $E^{p\Pi,0}_2$ for $p > 0$ vanishes, because the $\text{Hom}$ module is cofree over $\mathfrak{g}_-$, and, therefore, the spectral sequence collapses at $E'_2 = E'_\infty$. Thus $\text{Hom}_{\mathfrak{g}_-}(U, U(\mathfrak{g}_-) \otimes \mathcal{L}_{-\beta}) = E^{0\Pi,0}_2 = H_{-2+\Pi}(\mathfrak{g}, M) = M_{\mathfrak{g}_+}^{\mathfrak{g}_-}$. As in the proof of the previous proposition, all the identifications made do not affect the left action (1.6) of $\mathfrak{g}$, and, thus, we have an isomorphism of $\mathfrak{g}$-modules.

Thus the universal semijective module $US$ is free over $\mathfrak{g}_-$ and cofree over $\mathfrak{g}_+$ with respect to either action of $\mathfrak{g}$. Its natural $\mathbb{Z}$-grading is also bounded above, and when $\mathfrak{g}_0 = 0$, US lies in $\mathcal{O}_0$. 

$M_{\mathfrak{g}_+}^{\mathfrak{g}_-}$. Note that all the identifications made do not affect the right action (1.7) of $\mathfrak{g}$, and, thus, we have an isomorphism of $\mathfrak{g}$-modules.
1.5. **Semi-infinite induction.** Suppose now we have a graded subalgebra \( \mathfrak{h} \) in \( \mathfrak{g} \). The subalgebra inherits a natural semi-infinite structure from the one on \( \mathfrak{g} \) with \( \mathfrak{h}_+ = \mathfrak{g}_+ \cap \mathfrak{h} \) and \( \mathfrak{h}_- = \mathfrak{g}_- \cap \mathfrak{h} \), so that \( \mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_- \). We also assume that the restriction of the one-cochain \( \beta \) to \( \mathfrak{h} \) defines a morphism \( \mathfrak{h} \to \mathfrak{g}_{\text{res}}(\mathfrak{h}) \). Let \( M \) be an \( \mathfrak{h} \)-module also provided with a semi-infinite structure, see Section 1.2. Then the *semi-induced module* is defined as

\[
\text{S-ind}_{\mathfrak{h}}^g M := \text{US} \otimes_{\mathfrak{h}^+}^g M,
\]

where \( \otimes_{\mathfrak{h}^+}^g \) means taking the semi-invariants of the action of \( \mathfrak{h} \) by

\[
h(u \otimes m) = -uh \otimes m + u \otimes hm \quad \text{on} \ \text{US} \otimes M.
\]

The space \( \text{S-ind}_{\mathfrak{h}}^g M \) inherits the structure of a \( g \)-module from the left \( g \)-action (1.6) on \( \text{US} \). It is easy to see that if \( M \in \mathcal{O}_0 \) as an \( \mathfrak{h} \)-module and \( \mathfrak{h}_0 = \mathfrak{g}_0 \), then the semi-induced module \( \text{S-ind}_{\mathfrak{h}}^g M \) will also lie in the category \( \mathcal{O}_0 \) with respect to the bigger algebra \( \mathfrak{g} \). Otherwise, its \( \mathbb{Z} \)-grading will be just bounded above.

Consider a few degenerations of semi-infinite induction.

**Example 1.1** (Coinduction). Suppose \( \mathfrak{g} = \mathfrak{g}_+ \) and therefore \( \mathfrak{h} = \mathfrak{h}_+ \). Then \( \text{US} = U(\mathfrak{g})^* \) and \( \text{S-ind}_{\mathfrak{h}}^g M = U(\mathfrak{g})^* \otimes_{\mathfrak{h}^+}^g M = \text{Coind}_{\mathfrak{h}}^g M \).

**Example 1.2** (Induction). Suppose \( \mathfrak{g} = \mathfrak{g}_- \) and \( \mathfrak{h} = \mathfrak{h}_- \). Then \( \text{US} = U(\mathfrak{g}) \) and \( \text{S-ind}_{\mathfrak{h}}^g M = U(\mathfrak{g}) \otimes_{\mathfrak{h}^+}^g M = \text{Ind}_{\mathfrak{h}}^g M \).

**Example 1.3** (The universal semijective module). Let \( \mathfrak{h} = 0 \) and \( M = \mathbb{C} \). Then \( \text{S-ind}_{\mathfrak{h}}^g \mathbb{C} = \text{US} \).

1.6. **Semijective resolutions and modules.** Here we are going to recall the notion of a semijective resolution from [14]. By a *resolution* of a module \( M \), we as usual mean a complex \( S^\bullet \) whose cohomology \( H^*(S^\bullet) \) is identified with \( M[0] \), which denotes the complex \( \cdots \to 0 \to M \to 0 \to \cdots \) with \( M \) placed in degree 0. A *semijective complex* is a complex \( S^\bullet \) of \( \mathfrak{g} \)-modules, such that

1. it is \( K \)-injective as a complex of \( \mathfrak{g}_+ \)-modules, i.e., for every acyclic complex \( A^\bullet \) of \( \mathfrak{g}_+ \)-modules, \( \text{Hom}_{K(\mathfrak{g}_+)}(A^\bullet, S^\bullet) = 0 \), where \( K(\mathfrak{g}_+) \) is the homotopy category of complexes of \( \mathfrak{g}_+ \)-modules;
2. it is \( K \)-projective relative to \( \mathfrak{g}_+ \), i.e., for every acyclic complex \( B^\bullet \) of \( \mathfrak{g} \)-modules which is isomorphic to 0 in the category \( K(\mathfrak{g}_+) \), \( \text{Hom}_{K(\mathfrak{g})}(S^\bullet, B^\bullet) = 0 \).

A *semijective module* is nothing but a semijective complex \( 0 \to M \to 0 \). This is equivalent to being injective as a \( \mathfrak{g}_+ \)-module and projective relative to \( \mathfrak{g}_+ \) as a \( \mathfrak{g} \)-module. The universal semijective module \( \text{US} \) is an example, with respect to either action of \( \mathfrak{g} \), see Section 1.4. A serious difficulty in dealing with unbounded complexes comes from the fact that a complex made up of semijective modules will not necessarily be semijective. This difficulty can be gotten round by the above notion of a semijective complex.

Define a *weakly semijective complex* as a complex of \( \mathfrak{g} \)-modules which is \( K \)-injective as a complex of \( \mathfrak{g}_+ \)-modules and \( K \)-projective as a complex of \( \mathfrak{g}_- \)-modules. It is straightforward to see that every semijective complex is weakly semijective. Note also that the functor of semi-invariants takes acyclic weakly semijective complexes to acyclic ones. The proof of this statement is similar to the proof of vanishing theorem [14, Theorem 2.1] for semi-infinite cohomology of a weakly semijective
module, except that now one applies the functor of semi-infinite cohomology to an acyclic weakly semijective complex. Therefore, semi-infinite cohomology may in fact be defined by applying the functor of semi-invariants to weakly semijective resolutions termwise. We will use these facts in the proof of Semi-infinite Shapiro Lemma in the following section.

1.7. **Semi-infinite Shapiro Lemma.** In the following theorem, we will assume that $\mathfrak{g}_0 = \mathfrak{h}_0$ to make sure that $S\text{-}\text{ind}^\mathfrak{g}_0 \mathfrak{h} M \in \mathcal{O}_0$ whenever $M \in \mathcal{O}_0$.

**Theorem 1.4** (Semi-infinite Shapiro Lemma). For any $\mathfrak{h}$-module $M$ in the category $\mathcal{O}_0$, there exists a canonical isomorphism

$$H^{\infty/2+*}(\mathfrak{g}, S\text{-}\text{ind}^\mathfrak{g}_0 \mathfrak{h} M) = H^{\infty/2+*}(\mathfrak{h}, M).$$

**Proof.** Take a weakly semijective resolution $S^\bullet = \cdots \to S^{-1} \to S^0 \to S^1 \to \cdots$ of the trivial $\mathfrak{g}$-module $C$. Notice that $S^\bullet$ is also a weakly semijective resolution of $C$ in the category of $\mathfrak{h}$-modules. Indeed, for an acyclic complex $A^\bullet$ of $\mathfrak{h}_+$-modules, the complex $U(\mathfrak{g}_+) \otimes_{\mathfrak{h}_+} A^\bullet$ of $\mathfrak{g}_+$-modules will also be acyclic, and $\text{Hom}_{K(\mathfrak{h}_+)}(A^\bullet, S^\bullet) = \text{Hom}_K(\mathfrak{g}_+)(U(\mathfrak{g}_+) \otimes_{\mathfrak{h}_+} A^\bullet, S^\bullet) = 0$, because $S^\bullet$ is $K$-injective as a complex of $\mathfrak{g}_+$-modules. This shows that $S^\bullet$ is $K$-injective as a complex of $\mathfrak{h}_+$-modules. Similarly, one proves that $S^\bullet$ is $K$-projective as a complex of $\mathfrak{h}_+$-modules.

Now notice that $S^\bullet \otimes M$ will be a weakly semijective resolution of $M$ over $\mathfrak{h}$. To see that $S^\bullet \otimes M$ is a weakly semijective complex of $\mathfrak{h}$-modules, we have to check two things, as in the previous paragraph. The first is that $\text{Hom}_{K(\mathfrak{h}_+)}(A^\bullet, S^\bullet \otimes M) = \text{Hom}_K(\mathfrak{g}_+)(\text{Hom}(M, A^\bullet), S^\bullet) = 0$, see (1.3), whenever $A^\bullet$ is acyclic. This is true because the complex $\text{Hom}(M, A^\bullet)$ is acyclic and $S^\bullet$ is $K$-injective with respect to $\mathfrak{h}_+$. The second is that $\text{Hom}_{K(\mathfrak{h}_-)}(S^\bullet \otimes M, B^\bullet) = \text{Hom}_{K(\mathfrak{h}_-)}(S^\bullet, \text{Hom}(M, B^\bullet)) = 0$ whenever $B^\bullet$ is an acyclic complex in $K(\mathfrak{h}_-)$. That is true because the complex $\text{Hom}(M, B^\bullet) = 0$ is acyclic as well, and $S^\bullet$ is $K$-projective as a complex of $\mathfrak{h}_-$-modules. Thus the complex $(S^\bullet \otimes M)^{\mathfrak{h}_+}$ computes the semi-infinite cohomology $H^{\infty/2+*}(\mathfrak{h}, M)$.

Similarly, $S^\bullet \otimes S\text{-}\text{ind}^\mathfrak{g}_0 \mathfrak{h} M$ is a weakly semijective resolution of $S\text{-}\text{ind}^\mathfrak{g}_0 \mathfrak{h} M$ over $\mathfrak{g}$. Therefore the cohomology of the complex $(S^\bullet \otimes S\text{-}\text{ind}^\mathfrak{g}_0 \mathfrak{h} M)^{\mathfrak{h}_+}$ is equal to $H^{\infty/2+*}(\mathfrak{g}, S\text{-}\text{ind}^\mathfrak{g}_0 \mathfrak{h} M)$.

To conclude the proof, it suffices to observe the canonical isomorphism

$$(S^\bullet \otimes M)^{\mathfrak{h}_+} = (S^\bullet \otimes S\text{-}\text{ind}^\mathfrak{g}_0 \mathfrak{h} M)^{\mathfrak{h}_+},$$

which comes from the functorial isomorphism

$$(N \otimes M)^{\mathfrak{h}_+} = (N \otimes S\text{-}\text{ind}^\mathfrak{g}_0 \mathfrak{h} M)^{\mathfrak{h}_+}$$

for any $\mathfrak{g}$-module $N$. This isomorphism is a universal property of semi-infinite induction; it suffices to establish the following isomorphism of right $\mathfrak{g}$-modules:

(1.9)

$$N \otimes_{\mathfrak{g}_+} US = N$$

as a $\mathfrak{g}$-module. This statement is the computation of the semi-invariants of the $\mathfrak{g}$-module $N \otimes US$, which follows from Propositions 1.2 and 1.3. Indeed, Proposition 1.2 shows that $US \cong U(\mathfrak{g}_+)^* \otimes U(\mathfrak{g}_-)$ as a right $\mathfrak{g}_-$-module. Therefore, taking the $\mathfrak{g}_-$-coinvariants of the tensor product with $N$ and $\mathcal{L}_\beta$ will reduce $N \otimes US \otimes \mathcal{L}_\beta$ to $N \otimes U(\mathfrak{g}_+) \otimes \mathcal{L}_\beta$, which is canonically identified with $N \otimes U(\mathfrak{g}_+)^*$ as a vector space. Similarly, Proposition 1.3 implies that the $\mathfrak{g}_+$-invariants of $N \otimes US$ are computed as
\(N \otimes U(\mathfrak{g}_-)\). And it is obvious that the image of \(N \otimes U(\mathfrak{g}_-)\) in \(N \otimes U(\mathfrak{g}_+)\) is exactly \(N\). The fact that \(\mathfrak{g}\) is in a simple root space of root space corresponding to a negative simple root, \(\deg z\) where \(\mathfrak{g}\) is in a simple root space of root space corresponding to a negative simple root, \(\deg z\) corresponds to the corresponding contragredient \(\mathfrak{v}\) erma module.

2. Wakimoto modules

Wakimoto modules form an interesting class of highest-weight modules over an affine Kac-Moody algebra, playing an intermediate role between Verma and contragredient Verma modules.

2.1. Contragredient Verma modules. Before going into the subject of Wakimoto modules, we would like to outline certain well-known properties of contragredient Verma modules, mainly for motivational reasons. Let \(\mathfrak{g}\) be a simple finite-dimensional Lie algebra and \(\hat{\mathfrak{g}} = \mathbb{C}[z, z^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d\) the corresponding affine Kac-Moody algebra, \(K\) being its central element and \(d = z\frac{\partial}{\partial z}\) (see V. G. Kac \[1\] for more detail on affine Kac-Moody algebras). Choose a Cartan subalgebra \(\mathfrak{t}\) in \(\mathfrak{g}\) and a system of simple roots for \(\mathfrak{g}\). Let \(\mathfrak{n}_+ (\mathfrak{n}_- )\) be the subalgebra of \(\mathfrak{g}\) spanned by the positive (respectively, negative) root subspaces; then \(\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{t} \oplus \mathfrak{n}_-\). Furthermore, we can define a \(\mathbb{Z}\)-grading on \(\hat{\mathfrak{g}}\) by putting \(\deg 1 \otimes g = 1\) whenever \(g\) is in a simple root space of \(\mathfrak{g}\), \(\deg z = \text{rank}\mathfrak{g} + 1\), \(\deg 1 \otimes g = -1\) if \(g\) is in the root space corresponding to a negative simple root, \(\deg z^{-1} = -\text{rank}\mathfrak{g} - 1\), and \(\hat{\mathfrak{g}}_0 = \mathfrak{t} \oplus \mathbb{C}K \oplus \mathbb{C}d\). Then

\[
\hat{\mathfrak{g}}_+ = \bigoplus_{n>0} \hat{\mathfrak{g}}_n = \mathfrak{n}_+ \oplus (z\mathbb{C}[z] \otimes \mathfrak{g}),
\]

\[
\hat{\mathfrak{g}}_- = \bigoplus_{n\leq 0} \hat{\mathfrak{g}}_n = \mathfrak{n}_- \oplus (z^{-1}\mathbb{C}[z^{-1}] \otimes \mathfrak{g}) \oplus \mathfrak{t} \oplus \mathbb{C}K \oplus \mathbb{C}d.
\]

The splitting \(\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_-\) along with the functional \(\beta\) such that \(\beta(K) = 2h^\vee\), where \(h^\vee\) is the dual Coxeter number, \(\beta|_\mathfrak{t} = 2\rho, \beta(d) = 1\), defines a semi-infinite structure on \(\hat{\mathfrak{g}}\).

Given a character \(\lambda\) of \(\hat{\mathfrak{g}}_0\), one can define a \textit{contragredient Verma module} \(V^\ast(\lambda)\) using coinduction:

\[
V^\ast(\lambda) = \text{Coind}_{\hat{\mathfrak{g}}_-} \mathbb{C}_\lambda,
\]

where \(\mathbb{C}_\lambda\) is the corresponding one-dimensional representation of \(\hat{\mathfrak{g}}_0\) extended by zero to a one-dimensional representation of \(\hat{\mathfrak{g}}_-\).

**Theorem 2.1.** The module \(V^\ast(\lambda)\) can be uniquely characterized as a \(\hat{\mathfrak{g}}_0\)-diagonalizable \(\hat{\mathfrak{g}}\)-module in the category \(\mathcal{O}_0\), such that

\[
H^i(\hat{\mathfrak{g}}_+, V^\ast(\lambda)) = \begin{cases} 
\mathbb{C}_\lambda & \text{if } i = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** The cohomological condition is satisfied for a contragredient Verma module: it is obviously injective as a \(\hat{\mathfrak{g}}_+\)-module.

Conversely, suppose we have a module \(M\) satisfying the cohomological condition. Let us prove it is isomorphic to the corresponding contragredient Verma module.

Diagonalize the \(\hat{\mathfrak{g}}_0\)-action and define a morphism \(M \to \mathbb{C}_\lambda\) of \(\hat{\mathfrak{g}}_0\)-modules which takes a highest-weight vector \(m_\lambda\) (one invariant under \(\hat{\mathfrak{g}}_+\)), defined up to a scalar factor, of \(M\) to the generator of \(\mathbb{C}_\lambda\), mapping all the other weight subspaces to zero. It will automatically be a morphism of \(\hat{\mathfrak{g}}_-\)-modules.
The constructed morphism induces a unique morphism \( f : M \to V^*(\lambda) \) of \( \hat{g} \)-modules by virtue of the universality of coinduction \([3]\). The morphism \( f \) must be injective, otherwise it has a kernel, and we can consider the short exact sequence

\[ 0 \to \ker f \to M \to \im f \to 0. \]

Now let us look at the long exact sequence of cohomology of \( \hat{g}_+ \). Since the highest-weight vector \( m_\lambda \) of \( M \) obviously maps nontrivially to \( \im f \), it follows that \( H^0(\hat{g}_+, \ker f) = 0 \) — no highest-weight vectors, which implies \( \ker f = 0 \), because \( \ker f \) lies in the category \( O_0 \).

The morphism \( f \) should be surjective, otherwise it has a cokernel:

\[ 0 \to M \to V^*(\lambda) \to \coker f \to 0. \]

Again, the long exact sequence of cohomology gives

\[ 0 \to H^0(\hat{g}_+, M) \to H^0(\hat{g}_+, V^*(\lambda)) \to H^0(\hat{g}_+, \coker f) \to 0. \]

Since \( H^0(\hat{g}_+, M) \to H^0(\hat{g}_+, V^*(\lambda)) \) is an isomorphism, \( \coker f \) may not have a highest-weight vector, which means it should be trivial.

**Remark.** From the proof, we can observe that it is enough to require the vanishing of only \( H^1(\hat{g}_+, M) \) to make sure that \( M \cong V^*(\lambda) \).

Analogously, one can homologically characterize the Verma module \( V(\lambda) = \text{Ind}_{\hat{g}_+ \otimes \hat{g}_0}^g C_\lambda \) as a unique \( \hat{g}_0 \)-diagonalizable \( \hat{g} \)-module in the category \( O_0 \), such that

\[ H_i(\hat{g}_0, V(\lambda)) = \begin{cases} C_\lambda & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases} \]

**2.2. Wakimoto modules.** First, let us introduce a setup in which Wakimoto modules arise, see Feigin-Frenkel \([4]\) for more detail. Consider an alternative splitting \( \hat{g} = a \oplus \bar{a}, \) where

\[
\begin{align*}
  a &= (C[z, z^{-1}] \otimes n_-) \oplus (z C[z] \otimes t), \\
  \bar{a} &= (C[z, z^{-1}] \otimes n_+) \oplus (C[z^{-1}] \otimes t) \oplus CK \oplus Cd.
\end{align*}
\]

One can think of \( \bar{a} \) as a Borel subalgebra of \( \hat{g} \), obtained as the limit under the action by the elements \( 2m \rho, \rho \) being the half-sum of positive roots of \( g, m \to \infty, \) of the affine Weyl group on the Borel subalgebra \( \hat{g}_- \). The subalgebras \( a \) and \( \bar{a} \) are obviously \( \mathbb{Z} \)-graded subalgebras of \( \hat{g}, \) and the decomposition \( a = a_+ \oplus \bar{a}_- \), where

\[
\begin{align*}
  \bar{a}_+ &= \bar{a} \cap \hat{g}_+ = C[z] \otimes n_+,
  \\
  \bar{a}_- &= \bar{a} \cap \hat{g}_- = (C[z^{-1}] \otimes n_+) \oplus (C[z^{-1}] \otimes t) \oplus CK \oplus Cd,
\end{align*}
\]

along with the same functional \( \beta \) induces a semi-infinite structure on the subalgebra \( \bar{a} \). Similarly, the decomposition \( a = a_+ \oplus a_- \), where

\[
\begin{align*}
  a_+ &= a \cap \hat{g}_+ = (z C[z] \otimes n_-) \oplus (z C[z] \otimes t),
  \\
  a_- &= a \cap \hat{g}_- = C[z^{-1}] \otimes n_-,
\end{align*}
\]

with \( \beta = 0 \) defines a semi-infinite structure on the subalgebra \( a \).

Now we are ready to give a constructive definition of a Wakimoto module.
Definition 2.1. Let $\lambda$ be a character of $\hat{g}_0$ and $\mathbb{C}_\lambda$ the corresponding one-dimensional $\hat{g}_0$-module, extended trivially to $\hat{a}$. Then the Wakimoto module $W(\lambda)$ is the semi-induced module

$$W(\lambda) = \text{S-ind}_{\hat{a}}^{\hat{g}} \mathbb{C}_\lambda.$$  

2.3. The structure of a Wakimoto module. Let $\alpha_1, \alpha_2, \ldots$ be all the roots of the Lie algebra $\hat{g}$ and $e_{\alpha_1}, e_{\alpha_2}, \ldots$ a basis of the root subspace $\hat{g}_{\alpha_i}$, $i = 1, \ldots, m$, where $m$ is the multiplicity $\text{mult} \alpha_i$, which is one when the root $\alpha_i$ is real, rank $g$ when $\alpha_i$ is imaginary, and rank $g + 2$ when $\alpha_i = 0$. The Poincaré-Birkhoff-Witt Theorem states that the products

$$e_{\alpha_1}^{n_1} e_{\alpha_2}^{n_2} \cdots e_{\alpha_m}^{n_m} e_{\alpha_{m+1}}^{n_{m+1}} \cdots e_{\alpha_{2m}}^{n_{2m}} \cdots$$

containing only a finite number of terms form a basis of the universal enveloping algebra $U(\hat{g})$. Let

$$(e_{\alpha_1}^{n_1} e_{\alpha_2}^{n_2} \cdots e_{\alpha_m}^{n_m} e_{\alpha_{m+1}}^{n_{m+1}} \cdots e_{\alpha_{2m}}^{n_{2m}} \cdots)^*$$

denote the corresponding element of the dual basis in $U(\hat{g})^*$. Let $\beta_1, \beta_2, \ldots$ be the roots of $\hat{a}_+$ with the corresponding multiplicities $k_1, k_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$ the roots of $\hat{a}_-$. The roots of $\hat{a}_-$ are all real, therefore, the multiplicities are all one. Let $w_\lambda$ be the generator of $\mathbb{C}_\lambda$.

Proposition 2.2. 1. In the notation of the previous paragraph, the vectors

$$(e_{\beta_1}^{n_1} e_{\beta_2}^{n_2} \cdots e_{\beta_{k_1}} e_{\gamma_1}^{n_1} \cdots e_{\gamma_{k_2}} \cdots \cdots)^* \otimes (e_{\gamma_1}^{l_1} \cdots e_{\gamma_{k_2}} e_{\gamma_2}^{l_2} \cdots \cdots) \otimes w_\lambda$$

form a basis of the Wakimoto module $W(\lambda)$.

2. The formal character $\text{ch} W(\lambda) = \sum_{\mu \in \hat{g}_0} \dim W(\lambda)_\mu e^\mu$, where $W(\lambda)_\mu$ is the weight space, of $W(\lambda)$ is equal to

$$\text{ch} W(\lambda) = e^\lambda \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{-\text{mult} \alpha},$$

where $\Delta_+$ are the positive roots of $\hat{g}$. Thus, $\text{ch} W(\lambda) = \text{ch} V(\lambda) = \text{ch} V^*(\lambda)$.

Proof. Since $U(\hat{g}) = U(\hat{a}_+) \otimes U(\hat{a}_-) \otimes U(\hat{a}_+) \otimes U(\hat{a}_-) \otimes U(\hat{a}_-)$ as a vector space, a Wakimoto module $W(\lambda)$ is isomorphic to $U(\hat{a}_+)^* \otimes U(\hat{a}_-) \otimes \mathbb{C}_\lambda$, and the Poincaré-Birkhoff-Witt Theorem for $\hat{a}_+$ and $\hat{a}_-$ proves $\Box$.

The weight of a vector (2.1) is obviously $-((n_1,1 + \cdots + n_{1,k_1})\beta_1 + (n_{2,1} + \cdots + n_{2,k_2})\beta_2 + \cdots) + (l_1 \gamma_1 + l_2 \gamma_2 + \cdots) + \lambda$. Notice that $-\beta_1, -\beta_2, \ldots, \gamma_1, \gamma_2, \ldots$ comprise all the negative roots of $\hat{g}$. Therefore,

$$\text{ch} W(\lambda) = e^\lambda \prod_{\alpha \in \Delta_+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots)^{\text{mult} \alpha},$$

which implies the character formula.

2.4. Semi-infinite cohomology of Wakimoto modules.

Theorem 2.3. The Wakimoto module $W(\lambda)$ is a $\hat{g}_0$-diagonalizable $\hat{g}$-module from the category $O_0$, such that

$$H^{i/2+1}(a, W(\lambda)) = \begin{cases} \mathbb{C}_\lambda & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$
Proof. It follows from Proposition 2.2, that as an $\alpha$-module, $W(\lambda) = US(\hat{\mathfrak{g}}) \otimes_{\mathfrak{a}}^{\hat{\mathfrak{a}}} \mathbb{C}_\lambda \cong US(\mathfrak{a}) \otimes \mathbb{C}_\lambda$. By the semi-infinite Shapiro Lemma applied to $\mathfrak{g} = \mathfrak{a}$ and $\mathfrak{h} = 0$, we have the required computation of the semi-infinite cohomology of $W(\lambda)$. □

This Theorem was used by Feigin and Frenkel as the definition of a Wakimoto module, without proving that that cohomological property defined it uniquely. This uniqueness of a Wakimoto module is presumably true, but it would be more cautious to call it an open problem. In a recent e-mail message to the author, S. M. Arkhipov has suggested an outline of a prospective solution, using the fact that a Wakimoto module is a projective limit of twisted contragredient Verma modules.

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