Regularity and Chaos in low–lying $2^+$ States of Even–Even Nuclei

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March 30, 2022

Abstract

Using all the available empirical information, we analyse the spacing distributions of low-lying $2^+$ levels in even–even nuclei by comparing them with a theoretical distribution characterized by a single parameter (the chaoticity parameter $f$). We use the method of Bayesian inference. We show that the necessary unfolding procedure generally leads to an overestimate of $f$. We find that $f$ varies strongly with the ratio $R_{4/2}$ of the excitation energies of the first $4^+$ and $2^+$ levels and assumes particularly small values in nuclei that have one of the dynamical symmetries of the Interacting Boson Model.

1 Introduction

The interplay between regular and chaotic motion in nuclei has been a long–standing problem in Nuclear Physics. There is, on the one hand, overwhelming evidence in favour of simple dynamical models especially in the ground–state domain. The evidence derives from the agreement between calculated and measured spectral properties. There is, on the other hand, equally strong evidence for the validity of a random–matrix description, especially from the spectral statistics of slow neutron resonances [1, 2]. This success of random–matrix theory negates a dynamical description in terms of simple and (nearly) integrable models and has raised the question: Where in the spectrum of a nucleus with mass number $A$ does the chaotic region start? The statistical analysis of spectra needed to answer this question requires complete (few or no missing levels) and pure (few or no unknown spin–parities) level schemes. Some 15 years ago, complete and pure level schemes were available for only a limited number of nuclei (see, e.g., Refs. [3, 4]). The work of Ref. [5] then suggested that the nearest–neighbour spacing (NNS) distribution of low–lying nuclear levels lies between the Wigner and the Poisson distributions which are characteristic, respectively, of fully regular and fully chaotic
motion. Through the work of Refs. [6, 7, 8, 9, 10, 11, 12], the evidence presented in Ref. [5] has since become an established fact.

The wealth of spectroscopic data now available in the Nuclear Data tables [13] has motivated us to investigate once again the nuclear ground-state domain. We are able to make more definitive and precise statements about regularity versus chaos in this domain than has been possible so far. As in Ref. [5], we focus attention on $2^+$ states of select even–even nuclei. These nuclei are grouped into classes. The classes are defined in terms of the ratio $R_{4/2}$, i.e., the ratio of the excitation energies of the first $4^+$ and the first $2^+$ level in each nucleus. We argue below that the classes define a grouping of nuclei that have common collective behaviour. The sequences of $2^+$ states are unfolded and analysed with the help of Bayesian inference. The chaoticity parameter $f$ defined below is determined for each class. The present paper summarizes two research papers [14, 15] where further details may be found.

## 2 Data Set and Classification of Nuclei

The data on low–lying $2^+$ levels of even–even nuclei are taken from the compilation by Tilley et al. [16] for mass numbers $16 \leq A \leq 20$, from that of Endt [17] for $20 \leq A \leq 44$, and from the Nuclear Data Sheets [13] for heavier nuclei. We considered nuclei for which the spin–parity assignments of at least five consecutive $2^+$-levels are unambiguous. In cases where the spin-parity assignments were uncertain and where the most probable value appeared in brackets, we accepted this value. We terminated the sequence when we arrived at a level with unassigned $J^\pi$, or when an ambiguous assignment involved a $2^+$ spin-parity among several possibilities, as e.g. $J^\pi = (2^+, 4^+)$. We made an exception when only one such level occurred and was followed by several unambiguously assigned levels containing at least two $2^+$ levels, provided that the ambiguous $2^+$ level is found in a similar position in the spectrum of a neighboring nucleus. However, this situation occurred for less than 5% of the levels considered. In this way, we obtained 1306 levels of spin-parity $2^+$ belonging to 169 nuclei. The composition of this ensemble is as follows: 5 levels from each of 47 nuclei, 6 levels from each of 32 nuclei, 7 levels from each of 22 nuclei, 8 levels from each of 22 nuclei, 9 levels from each of 16 nuclei, 10 levels from each of 14 nuclei, 11 levels from each of 5 nuclei, 12 levels from each of 2 nuclei, and sequences of 13, 14, 15, 17, 20, 21, 24, 30, and 32 levels, each belonging to a single nucleus.

A class of nuclei is defined by choosing an interval within which the ratio

$$R_{4/2} = \frac{E(4^+_1)}{E(2^+_1)}$$

of excitation energies of the first $4^+$ and the first $2^+$ excited states, must lie. The width of the intervals was taken to be 0.1 when the total number of spacings falling into the corresponding class was about 100 or more. Otherwise, the width of the interval was increased. The use of the parameter $R_{4/2}$ as an indicator of collective dynamics is justified both empirically and by theoretical arguments. We recall the reasons in turn.
(i) Casten et al. [18] plotted $E(4^+_1)$ versus $E(2^+_1)$ for all nuclei with $38 \leq Z \leq 82$ and with $2.05 \leq R_{4/2} \leq 3.15$. The authors found that the data fall on a straight line. This suggests that nuclei in this wide range of $Z$–values behave like anharmonic vibrators with nearly constant anharmonicity. As the ratio $R_{4/2}$ approaches the rotor limit $R_{4/2} = 3.33$, the slope of the curve showing $E(4^+_1)$ versus $E(2^+_1)$ decreases within a narrow range of $E(2^+_1)$–values, asymptotically merging the rotor line of slope 3.33. In a subsequent paper [19] it was found that a linear relation between $E(4^+_1)$ and $E(2^+_1)$ holds for pre–collective nuclei with $R_{4/2} < 2$. Thus, from an empirical perspective, the dynamical structure of medium–weight and heavy nuclei can be quantified in terms of $R_{4/2}$.

(ii) Theoretical calculations based on the Interacting Boson Model (the IBM–1 model [20]) support the conclusion that $R_{4/2}$ is an appropriate measure for collectivity in nuclei. The model has three dynamical symmetries, obtained by constructing the chains of subgroups of the $U(6)$ group that end with the angular momentum group $SO(3)$. The symmetries are labeled by the first subgroup appearing in the chain which are $U(5)$, $SU(3)$, and $O(6)$ corresponding, respectively, to vibrational, rotational and γ–unstable nuclei. Extensive numerical calculations for the classical as well as the quantum-mechanical IBM Hamiltonian by Alhassid et al. [21] indeed showed a considerable reduction of the standard measures of chaoticty when the parameters of the IBM model approach one of the three cases just mentioned. The IBM calculation of energy levels yields values of $R_{4/2} = 2.00, 3.33$, and $2.50$ for the dynamical symmetries $U(5)$, $SU(3)$, and $O(6)$, respectively. Thus, we may expect increased regularity of nuclei having one of these values of $R_{4/2}$.

One might expect that the chaoticy parameter $f$ defined in Eq. (3) below also assumes small values for nuclei near magic numbers. For mass numbers in this domain, our data set is unfortunately too small to allow us to draw definitive conclusions.

### 3 Statistical Analysis

#### 3.1 Chaoticity Parameter $f$

To analyze the data, we need a guess for the form of the NNS distribution $p(s, f)$. Here, $s$ is the spacing of neighboring levels in units of the mean level spacing. The distribution $p(s, f)$ depends on one or more parameters $f$ which describe the transition from Poissonian to Wigner–Dyson form. Several proposals have been advocated for $p(s, f)$. Here we are guided by the following considerations.

We consider a spectrum $S$ containing levels which have the same spin and parity but may differ in other conserved quantum numbers which are either unknown or ignored. The $K$–quantum number serves as an example. The spectrum $S$ can then be broken down into $m$ subspectra $S_j$ of independent sequences of levels. Let $f_j, j = 1 \ldots m$ with $0 < f_j \leq 1$ and $\sum_{j=1}^{m} f_j = 1$ denote the fractional level number, let $p_j(s), j = 1 \ldots m$ denote the NNS distribution for the subspectrum $S_j$ and $p(s)$ the NNS distribution of $S$. Both $p(s)$ and $p_j(s_j)$ are defined for spectra with
unit mean spacing. We assume that each of the distributions $p_j(s)$ is determined by the Gaussian orthogonal ensemble (GOE). To an excellent approximation, the $p_j$’s are then given by Wigner’s surmise \[23\]

$$p_W(s) = \frac{\pi}{2} s \exp \left(-\frac{\pi}{4} s^2 \right). \tag{2}$$

The construction of $p(s, f)$ for the superposition is due to Rosenzweig and Porter \[24\]. It depends on the $(m - 1)$ unknown parameters $f_j, j = 1, ..., (m - 1)$. This fact poses a difficulty because in practice, we do not know the composition of the spectrum. We are not even sure of how many quantum numbers other than spin and parity are conserved. To overcome the difficulty, we use an approximate scheme first proposed in Ref. \[25\]. Effectively, we replace the $(m - 1)$ parameters $f_j$ by a single one, the mean fractional level number $f = \sum_j f_j^2$. This leads to an approximate NNS distribution for $S$,

$$p(s, f) = \left[ 1 - f + f (0.7 + 0.3f) \frac{\pi s}{2} \right] \times \exp \left( - (1 - f) s - f (0.7 + 0.3f) \frac{\pi s^2}{4} \right). \tag{3}$$

We use $f$ as a fit parameter.

For a large number $m$ of subspectra, $f$ is of the order of $1/m$. In this limit, $p(s, f)$ approaches the Poisson distribution as it should. On the other hand, when $f \to 1$ the spectrum approaches the GOE behaviour as it must. This is why we refer to $f$ as to the chaoticity parameter. If the spectrum $S$ is not pure but rather a superposition of subsequences corresponding to different values of an ignored or unknown quantum number then the mean value $f$ of the fractional density of the superimposed sequences is smaller than unity, and the composite sequence looks rather like a sequence of levels with mixed dynamics.

### 3.2 Unfolding

Prior to the actual statistical analysis, every sequence of levels has to be unfolded \[22\] to obtain a new sequence with unit mean level spacing. In the case of a single long spectrum, unfolding is a standard procedure. It consists in fitting a slowly varying function $\epsilon(E, \alpha)$ to the experimental staircase function $N(E)$ of the integrated level density. The fit is obtained by optimizing a set of parameters $\alpha$. The function $\epsilon$ depends monotonically on the energy $E$. Therefore, we can transform $E$ to $\epsilon$. With respect to the new energy variable $\epsilon$, the level density is uniform and equal to unity.

If the available ensemble of spacings consists of many short sequences of levels (we call this a “composite ensemble”), unfolding is not standard nor is it altogether irrelevant. To test the standard unfolding procedure, we have generated short sequences of levels from three artificial ensembles containing 50, 100, and 200 spacings. Construction of the latter involves an artificially chosen chaoticity parameter $f_0$ and is described in the following paragraph. These are referred to as the “initial” ensembles. Each
short sequence is then artificially folded with a monotonically increasing function of energy. An unfolding procedure is subsequently applied to each sequence. The unfolding procedure does not trivially reproduce the initial ensembles and yields the "final" ensembles. The chaoticity parameter \( f \) is then determined for the final ensembles using a \( \chi^2 \) fit and the Bayesian method described below.

The ensembles of spacings are constructed with the help of a random–number generator. We choose average spacing unity and \( f_0 = 0.6 \) for the chaoticity parameter. This value is close to what has been obtained in the previous analysis \cite{9} of low–lying nuclear levels. We generate a set of spacings that obeys the probability distribution \( \chi^2 \) with \( f = f_0 \). In this way, we generate three “initial” artificial ensembles of 50, 100, and 200 spacings. Our procedure is open to the criticism that our construction does not pay attention to the stiffness of GOE spectra. We are in the process of rectifying this shortcoming.

The test of the unfolding procedure leads to the following conclusions. (i) Using several unfolding functions leads to nearly the same values for \( f \). This confirms the insensitivity of the final ensemble of spacings to the form of the unfolding function. (ii) The unfolding procedure introduces a bias towards the GOE, i.e. the best-fit value of \( f \) is larger than \( f_0 \). This is borne out by both, the Bayesian inference and the \( \chi^2 \)-analysis of the spacing histograms for the final distributions. The trend increases as the lengths of the short sequences is decreased. This is simply understood: The unfolding of sequences of just two levels each would give a delta–function peaked at the value of unity (the mean level spacing) and, thus, show strong preference for the GOE. The trend becomes weaker as the sequences become longer but disappears only in the limit of very long sequences. As a consequence, the analysis of the nuclear data set will reliably yield only relative values of \( f \).

The actual unfolding of the data was done by fitting a theoretical expression to the number \( N(E) \) of levels below excitation energy \( E \). The expression used here is the constant–temperature formula \cite{3},

\[
N(E) = N_0 + \exp\left(\frac{E - E_0}{T}\right).
\]

The three parameters \( N_0, E_0 \) and \( T \) obtained for each nucleus vary considerably with mass number. Nevertheless, all three show a clear tendency to decrease with increasing mass number. For the effective temperature, for example, we find, assuming a power–law dependence, the result \( T = (15 \pm 4)A^{-(0.62 \pm 0.05)} \) MeV. This value is consistent with an analysis of the level density of nuclei in the same range of excitation energy carried out by von Egidy \textit{et al.} \cite{4}. These authors find \( T = (19 \pm 2)A^{-(0.68 \pm 0.02)} \) MeV.

### 3.3 Bayesian Analysis

Given Eq. (3) for the proposed distribution, we apply Bayesian analysis to the data. Let \( s = (s_1, s_2, ..., s_N) \) denote a set of spacings \( s_j \). We take the experimental spacings \( s_j \) to be statistically independent. This
assumption does not apply in general. Indeed, the GOE produces significant correlations between subsequent spacings. However, we recall that we are interested only in the NNS distribution. This distribution is only weakly affected by correlations. We calculate the posterior distribution for $f$ given the events $s$. We first determine the conditional probability distribution $p(s|f)$ of the set of spacings $s = (s_1, s_2, \ldots, s_N)$ for a fixed $f$. We accordingly write

$$p(s|f) = \prod_{i=1}^{N} p(s_i, f) ,$$  \hspace{1cm} (5)$$

with $p(s_i, f)$ given by Eq. 3. Bayes' theorem then provides the posterior distribution

$$P(f|s) = \frac{p(s|f)\mu(f)}{M(s)}$$  \hspace{1cm} (6)$$
of the parameter $f$ given the events $s$. Here, $\mu(f)$ is the prior distribution and

$$M(s) = \int_{0}^{1} p(s|f) \mu(f) \, df$$  \hspace{1cm} (7)$$
is the normalization. We use Jeffreys' rule [26]

$$\mu(f) \propto \left| \int p(s|f) \left[ \partial \ln p(s|f) / \partial f \right]^2 \, ds \right|^{1/2}$$  \hspace{1cm} (8)$$
to find the prior distribution. The latter can be interpreted as the distribution ascribed to $f$ in the absence of any observed $s$. It is approximated by

$$\mu(f) = 1.975 - 10.07 f + 48.96 f^2 - 135.6 f^3 + 205.6 f^4 - 158.6 f^5 + 48.63 f^6 .$$  \hspace{1cm} (9)$$

Even for only moderately large $N$, it is useful to write $p(s|f)$ in the form

$$p(s|f) = e^{-N\phi(f)} ,$$  \hspace{1cm} (10)$$

where

$$\phi(f) = (1-f)\langle s \rangle + \frac{\pi}{4} f(0.7+0.3f)\langle s^2 \rangle - \langle \ln[1-f + \frac{\pi}{2} f(0.7+0.3f) s] \rangle .$$  \hspace{1cm} (11)$$

Here the notation $\langle x \rangle = (1/N) \sum_{i=1}^{N} x_i$ has been used. By calculating the mean values $\langle \cdots \rangle$ in Eq. (11) for various spectra, one finds that the function $\phi(f)$ has a deep minimum, say at $f = f_1$. One can therefore represent the numerical results in analytical form by parametrizing $\phi$ as

$$\phi(f) = A + B(f - f_1)^2 + C(f - f_1)^3 .$$  \hspace{1cm} (12)$$

We then obtain

$$P(f|s) = c \mu(f) \exp(-N[2B(f - f_1)^2 + C(f - f_1)^3]) ,$$  \hspace{1cm} (13)$$
where $c = e^{-N A / M(s)}$ is a normalization constant.
The last step of the Bayesian analysis consists in determining the best-fit value of the chaoticity parameter $f$ and its error for each NNS distribution. When $P(f|s)$ is not Gaussian, the best-fit value of $f$ cannot be taken as the most probable value. Rather we take the best-fit value to be the mean value $\bar{f}$ and measure the error by the standard deviation $\sigma$ of the posterior distribution (6), i.e.

$$\bar{f} = \int_0^1 f P(f|s) \, df \quad \text{and} \quad \sigma^2 = \int_0^1 (f - \bar{f})^2 P(f|s) \, df \, .$$

(14)

This is not optimal but provides a useful approximation.

4 Results and Discussion

The results obtained for $\bar{f}$ and $\sigma$ are given in Figure 1 of Ref. [14]. Figure 2 of that reference shows a comparison of the spacing distributions conditioned by $\bar{f}$ and the histograms for each class of nuclei. In view of the small number of spacings within each class, the agreement seems satisfactory.

We recall that the analysis of many short sequences of levels tends to overestimate $\bar{f}$. Therefore, we focus attention not on the absolute values of $\bar{f}$ but on the way $\bar{f}$ changes with $R_{4/2}$. The graph of $\bar{f}$ against $R_{4/2}$ in the Ref. [14] has deep minima at $R_{4/2} = 2.0, 2.5, \text{and} 3.3$. These values of $R_{4/2}$ are associated with the dynamical symmetries of the Interacting Boson Model mentioned above. Another minimum of statistical significance occurs for $2.25 \leq R_{4/2} \leq 2.35$. This minimum may indicate that nuclei which lie between the limiting cases of the $U(5)$ and $O(6)$ dynamical symmetries, are relatively regular. One may associate this region with the critical point of the $U(5)$-$O(6)$ shape transition in nuclei. Iachello [27] has recently shown that this transition is approximately governed by the “critical” $E(5)$ dynamical symmetry. Nuclei with $E(5)$ dynamical symmetry have $R_{4/2} = 2.2$. Experimental examples of this critical symmetry have been found by Casten and Zamfir [28].

In summary, we have determined the chaoticity parameter $f$ for $2^+$ levels of even–even nuclei at low excitation energy with the help of a systematic analysis of the NNS distributions. While in a single nucleus the number of states with reliable spin–parity assignments is not sufficient for a meaningful statistical analysis, a combination of sequences of levels taken from similar nuclei provides a sufficiently large ensemble. As the measure of similarity we have taken the ratio $R_{4/2}$ of the excitation energies of the lowest $4^+$ and $2^+$ levels in each nucleus. The mean chaoticity parameter $\bar{f}$ is found to be indeed dependent on $R_{4/2}$. It has deep minima at $R_{4/2} = 2.0, 2.5, \text{and} 3.3$. These minima correspond, respectively, to the $U(5), SO(6), \text{and} SU(3)$ dynamical symmetries of the IBM. A further minimum may relate to the critical $E(5)$ symmetry.
Acknowledgments

The authors thank Professor J. Hufner for useful discussions. A. Y. A.-M. and M. H. S. acknowledge the financial support granted by Internationales Büro, Forschungszentrum Jülich which permitted their stay at the Max–Planck–Institut für Kernphysik, Heidelberg.

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