Quark propagator in a covariant gauge

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Using mean-field improved gauge field configurations, we compare the results obtained for the quark propagator from Wilson fermions and Overlap fermions on a $12^3 \times 24$ lattice at a spacing of $a = 0.125(2)$ fm.

1. INTRODUCTION

In QCD, asymptotic freedom states that at short distances the effective coupling constant vanishes. In that regime the interaction between quarks and gluons is largely reduced. The dynamics of these particles can be studied through their propagators in momentum space. The quark propagator is really a description of how the quark propagates in the QCD vacuum and is one of the most fundamental building blocks of QCD. By studying the momentum dependent quark mass function in the infrared region, the scalar part of the propagator, we can gain some insights into the mechanism of chiral symmetry breaking. Chiral symmetry is dynamically broken in the QCD vacuum. This gives rise to mass generation in the infrared.

In the deep infrared region, artifacts associated with the finite size of the lattice spacing become small. This is the most interesting region as non-perturbative physics lies here. However, the ultraviolet behaviour of the propagator at large momenta will in general strongly deviate from the correct continuum behaviour. This behaviour will be action dependent.

In this brief report we compare the quark propagators of Wilson and overlap fermions. Additional details may be found in Ref. [1].

1.1. The Quark Propagator

In the continuum, it is possible to study dynamical chiral symmetry breaking (DCSB) using the renormalized quark Dyson–Schwinger equation (quark–DSE) in Euclidean space. In this equation, information about the renormalized dressed gluon propagator and dressed quark–gluon vertex is embodied. The quark propagator has the general form

$$ S(p) = \frac{1}{i\gamma \cdot p A(p^2; \zeta^2) + B(p^2; \zeta^2)} $$

$$ = \frac{1}{Z(p^2; \zeta^2)} i\gamma \cdot p + M(p^2; \zeta^2), $$

where $M = B/A$ and $Z = 1/A$. Here, the parameter $\zeta$ represents the renormalization point. The functions $A(p^2; \zeta^2)$ and $B(p^2; \zeta^2)$ carry all the effects of vector and scalar quark dressing induced by the quark interactions with the gluon field.

Through this simple form we can extract the quark mass function $M(p)$ and renormalization function, $Z(p)$.

2. QUARK PROPAGATOR ON THE LATTICE

On the lattice we expect the bare quark propagators, in momentum space, to have a similar form as in the continuum \[2-4\]. Hence, the dimensionless inverse lattice bare quark propagator takes the general form

$$ S^{-1}(p) = i \left( \sum_{\mu} C_{\mu}(p) \gamma_{\mu} \right) + B(p) $$

$$ = \frac{i \left( \sum_{\mu} C_{\mu}(p) \gamma_{\mu} \right) + B(p)}{C^2(p) + B^2(p)}, $$

with $C^2(p) = \sum_{\mu}(C_{\mu}(p))^2$. The discrete momentum values for a $t$-antiperiodic lattice of size $N^3_t \times N_t$, with $n_t = 1, \ldots, N_t$ and $n_1 = 1, \ldots, N_1$, $n_2 = 1, \ldots, N_2$, $n_3 = 1, \ldots, N_3$.
are given by $p_i = \frac{2\pi}{N_c a} \left( n_i - \frac{N_c}{2} \right)$ and $p_t = \frac{2\pi}{N_c a} \left( n_t - \frac{1}{2} - \frac{N_c}{2} \right)$. The quark propagator is

$$S(p) \equiv -i \left( \sum \mu C_\mu(p) \gamma_\mu \right) + B(p).$$

(3)

Taking the trace we obtain

$$C_\mu(p) = \frac{i}{4N_c} \text{Tr}[\gamma_\mu S(p)],$$

and

$$B(p) = \frac{1}{4N_c} \text{Tr}[S(p)],$$

(4)

which we use to construct the $C_\mu(p)$ and $B(p)$

$$C_\mu(p) = \frac{C_\mu(p)}{D(p)}, \quad \text{and} \quad B(p) = \frac{B(p)}{D(p)},$$

(5)

where $D(p) = C^2(p) + B^2(p)$. At tree-level the lattice quark propagator takes the same form as in Eq. (6) and we know that the free propagator $(S^{(0)}(p))^{-1} = (Z^{(0)}(p))^{-1} \left[ i\not{k} + M^{(0)}(p) \right]$, where $k_\mu \rightarrow p_\mu$ as $p_\mu \rightarrow 0$. It is then possible to extract the momentum directly from the lattice by calculating

$$q_\mu \equiv C^{(0)}(p) = \frac{C^{(0)}_\mu(p)}{(C^{(0)}(p))^2 + (B^{(0)}(p))^2}.$$

(6)

Results are displayed in Fig. 1.

2.1. Mass and Renormalization Functions

The dimensionless inverse lattice bare quark propagator takes the form

$$S^{-1}(p) \equiv i a q A(p) + B(p)$$

$$= [Z^L(p)]^{-1} \left[ i a q + M^L(p) \right],$$

(7)

where $M^L(p) = B(p)/A(p)$ is the lattice quark mass function and $Z^L(p) = 1/A(p)$ the lattice renormalization function. The functions $A(p)$ and $B(p)$ may be written as:

$$A(p) = \frac{A(p)}{D(p)}, \quad \text{and} \quad B(p) = \frac{B(p)}{D(p)},$$

(8)

where $D(p) = A^2(p) q^2 + B^2(p)$. Hence an equivalent definition for the quark propagator is $S(p) \equiv -i a q A(p) + B(p)$. Extracting the functions $A(p)$ and $B(p)$ is done via:

$$A(p) = \frac{i}{4N_c a q^2} \text{Tr}[\not{q} S(p)],$$

and

$$B(p) = \frac{1}{4N_c} \text{Tr}[S(p)].$$

(9)

Figure 1. The lattice momentum $q$ versus the discrete momentum $p$, both in GeV. Overlap ($\Box$) and Wilson fermion ($\circ$) momenta are indicated.

In Fig. 2 we show the uncorrected mass function $M(q^2)$ for Wilson fermions.

2.2. Tree–Level Correction

The tree-level multiplicative correction is given by

$$A^{(c)}(p) = \frac{A(p)}{A^{(0)}(p)}$$

$$B^{(c)}(p) = \frac{B(p)}{B^{(0)}(p)} m_q,$$

(10)
with \( A^{(0)}(p) \) and \( B^{(0)}(p) \) are obtained from \( S^{(0)}(p) \). Then tree-level corrected mass and renormalization functions are constructed using Eq. (10)

\[
M^{(c)}(q^2) = \left( \frac{B^{(c)}(p)}{A^{(c)}(p)} \right), \quad Z^{(c)}(q^2) = \frac{1}{A^{(c)}(p)}. \tag{11}
\]


### 3. FERMIONS ON THE LATTICE

#### 3.1. Wilson Fermions

The Wilson fermion \( \mathcal{W} \) action on the lattice is defined as,

\[
S_W[U, \bar{\psi}, \psi] = \sum_{xy} \bar{\psi}(x) D_W(x, y) \psi(y),
\]

with \( D_W(x, y) \) the usual Wilson fermion operator. The lattice bare mass is related to the hopping parameter via \( \kappa = 1/(2m_qa + 8r) \) and \( m_q \equiv (1/(2\kappa) - 1/(2\kappa_c)) \).

#### 3.2. Overlap Fermions

The overlap fermion \( \mathcal{O} \) realizes exact chiral symmetry on the lattice

\[
D(\mu) = \frac{1}{2} \left[ 1 + \mu + (1 - \mu) \frac{D_W}{\sqrt{D_W D_W}} \right]
\]

with \( 0 \leq \mu \leq 1 \) describing fermions with a positive mass from 0 to \( \infty \). The hopping parameter is given by \( \kappa = 1/(2m + 8r) \) and \( m \equiv (1/(2\kappa) - 1/(2\kappa_c)) \). To describe a single massless Dirac fermion for \( D(0) \) we must have \( 0 \leq m \leq 2 \) at tree-level. The overlap propagator is given by

\[
\tilde{D}^{-1}(\mu) = (1 - \mu)^{-1} [D^{-1}(\mu) - 1], \tag{12}
\]

and is related to the continuum propagator by \( D^{-1}(m_q) = Z_\psi \tilde{D}^{-1}(\mu) \), and similarly for the quark mass \( m_q \equiv Z^{-1}_m \mu \).

At tree-level \( \mathcal{O} \) it is found that \( Z^{(0)}_\psi = Z^{-1}_m = 2m \) and the free propagator becomes

\[
\left[ D^{(0)}_\psi(m_q) \right]^{-1} = \left[ Z^{(0)}_\psi \tilde{D}(\mu) \right]^{-1},
\]

and when the interactions are turned on we have

\[
\left[ D_\psi(m_q) \right]^{-1} = \left[ Z_\psi \tilde{D}(\mu) \right]^{-1},
\]

where \( Z_\psi = Z^{-1}_m \).

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**Figure 3.** The mass function \( M(q^2) \) for overlap fermions (full data) for the ten different masses.

### 4. NUMERICAL RESULTS

Using fifty improved gauge field configurations, on a \( 12^3 \times 24 \) lattice with \( a = 0.125(1) \) fm, the overlap quark propagator is calculated for ten different masses, \( m_q = Z_\psi \mu = \{126, 147, 168, 210, 252, 315, 420, 524, 629, 734\} \) MeV. Results are illustrated in Figs. 3 and 4.

For Wilson fermions, using a hundred configurations, five masses are considered, namely \( m_q = \{221, 181.5, 138.3, 99.91, 62.04\} \) MeV. Results are displayed in Figs. 5 and 6.

A linear extrapolation is used to compare the two actions. In Fig. 6 we show the linearly extrapolated quark mass function for both actions plotted versus the discrete momentum \( p \). When plotting \( M \) and \( Z \) versus the lattice momentum, \( q \), the points are pushed away from the origin in the case of overlap fermions as opposed to Wilson fermions where the points are pulled towards the origin. The overlap action (plotted as \( \Box \)) produces in the deep infrared \( M(0) = 297(11) \) MeV. Comparing with the Wilson fermion action (\( \times \)) we see a clear superiority of the overlap action over the Wilson action. Wilson fermions show a significant dip between 0.8 GeV and 2.0 GeV.

The linearly extrapolated renormalization function is shown in Fig. 8. In the deep infrared...
for the overlap action we have $Z(0) = 0.48(2)$ with the renormalization point located at $\zeta = 3.9$ GeV.

5. SUMMARY

Tree-level correction represents a crucial step and a powerful tool for correcting the Wilson action results. The division method produces a smooth quark mass function throughout the momentum spectrum. We are able to gain some insights into the analytic structure of the overlap quark propagator: $B^{(0)}(p) = Z^{(0)} \mu$ and $A^{(0)}(p) = 1$. The overlap produces very good results for $M(p)$ and $Z(p)$. No tree-level correction is required for overlap fermions.

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Figure 6. The tree–level corrected renormalization function for Wilson fermions (half cut data).

Figure 7. Comparison of the linearly extrapolated mass function $M$ for both Wilson fermions ($\times$) and overlap fermions ($\square$), (cylinder cut data).

Figure 8. Comparison of the linearly extrapolated renormalization function for Wilson fermions ($\times$) and overlap fermions ($\square$), (cylinder cut data).
$\Lambda_{\text{GeV}}$ vs $b$ as $(b)(r)Z_{\text{EXT}}$. 

$\mathcal{O}_{5}$
Elsevier instructions for the preparation of a 2-column format camera-ready paper in \LaTeX

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3.2. Useful table packages

Modern \LaTeX comes with several packages for tables that provide additional functionality. Below we mention a few. See the documentation of the individual packages for more details. The

| Table 2: The next-to-leading order (NLO) results without the pion field. | \( \Lambda \) (MeV) | 140 | 150 | 175 | 200 | 225 | \( v_{\text{as}} \) [2] |
|---|---|---|---|---|---|---|---|
| \( r_d \) (fm) | 1.976 | 1.973 | 1.978 | 1.983 | 1.978 | 1.967 |
| \( Q_d \) (fm\(^2\)) | 0.302 | 0.312 | 0.319 | 0.326 | 0.320 | 0.318 |
| \( P_d \) (%) | 6.09 | 8.06 | 9.90 | 11.76 | 13.50 | 15.24 |
| \( M_{\text{LO}} \) (fm) | 3.856 | 3.846 | 3.836 | 3.826 | 3.816 | 3.806 |
| \( M_{\text{GT}} \) (fm) | 4.887 | 4.877 | 4.867 | 4.857 | 4.847 | 4.837 |
| \( \delta_{\text{VP}}^{1B} \) (%) | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 |
| \( \delta_{\text{C2:C}}^{1B} \) (%) | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 |
| \( \delta_{\text{C2:N}}^{1B} \) (%) | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 |

The experimental values are given in ref. [4].
Table 1
The next-to-leading order (NLO) results without the pion field.

| \( \Lambda \) (MeV) | 140 | 150 | 175 | 200 |
|---------------------|-----|-----|-----|-----|
| \( r_d \) (fm)     | 1.973 | 1.972 | 1.974 | 1.978 |
| \( Q_d \) (fm\(^2\)) | 0.259 | 0.268 | 0.287 | 0.302 |
| \( P_D \) (%)      | 2.32 | 2.83 | 4.34 | 6.14 |
| \( \mu_d \)        | 0.867 | 0.864 | 0.855 | 0.845 |
| \( M_{M1} \) (fm)  | 3.995 | 3.989 | 3.973 | 3.955 |
| \( M_{GT} \) (fm)  | 4.887 | 4.881 | 4.864 | 4.846 |
| \( \delta_{VP} \)  (%) | -0.45 | -0.45 | -0.45 | -0.45 |
| \( \delta_{C2:1B} \) (%) | 0.03 | 0.03 | 0.03 | 0.03 |
| \( \delta_{C2:N} \) (%) | -0.19 | -0.19 | -0.18 | -0.15 |

The experimental values are given in ref. [4].

packages can be found in \LaTeX’s tools directory.

array Various extensions to \LaTeX’s array and tabular environments.

longtable Automatically break tables over several pages. Put the table in the longtable environment instead of the table environment.

dcolumn Define your own type of column. Among others, this is one way to obtain alignment on the decimal point.

tabularx Smart column width calculation within a specified table width.

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\includegraphics[angle=90,width=20pc]{file}

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H_{\alpha\beta}(\omega) = E^{(0)}_{\alpha}(\omega)\delta_{\alpha\beta} + \langle \alpha|W_{\pi}|\beta\rangle
\end{equation}

You need not put in equation numbers, since this is taken care of automatically. The equation numbers are always consecutive and are printed in parentheses flush with the right-hand margin of the text and level with the last line of the equation. For multi-line equations, use the \texttt{eqnarray} environment.

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