Abstract. We define a quantum loop group $U^+_Q$ associated to an arbitrary quiver $Q= (I, E)$ and maximal set of deformation parameters, with generators indexed by $I \times \mathbb{Z}$ and some explicit quadratic and cubic relations. We prove that $U^+_Q$ is isomorphic to the (generic, small) shuffle algebra associated to the quiver $Q$ and hence, by [Neg21a], to the localized $K$-theoretic Hall algebra of $Q$. For the quiver with one vertex and $g$ loops, this yields a presentation of the spherical Hall algebra of a (generic) smooth projective curve of genus $g$ (invoking the results of [SV12]). We extend the above results to the case of non-generic parameters satisfying a certain natural metric condition. As an application, we obtain a description by generators and relations of the subalgebra generated by absolutely cuspidal eigenforms of the Hall algebra of an arbitrary smooth projective curve (invoking the results of [KSV17]).

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1. Introduction

1.1. Let $Q$ be a finite quiver, with vertex set $I$ and edge set $E$; edge loops and multiple edges are allowed. The Hall algebra of the category of representations of $Q$ over a finite field is well-known to contain a copy of the quantized enveloping algebra $U^+_q(\mathfrak{g}_Q)$, where $\mathfrak{g}_Q$ is the Kac-Moody Lie algebra associated to $Q$ (or the Bozec-Kac-Moody Lie algebra when $Q$ has edges loops). Cohomological Hall algebras associated to $Q$ for a Borel-Moore homology theory (including $K$-theory) were more recently introduced, in relation to Donaldson-Thomas theory on the one hand and Nakajima quiver varieties on the other hand (see [KS11, SV13, YZ18]). More precisely, the $K$-theoretic Hall algebra of $Q$ is the vector space:

$$K_Q := \bigoplus_{n \in \mathbb{N}^I} K^\mathbb{P}(\text{Rep}_n \Pi_Q)$$

where $\Pi_Q$ is the preprojective algebra of $Q$, and $\text{Rep}_n \Pi_Q$ is the stack of complex $n$-dimensional representations of $\Pi_Q$. The vector space $K_Q$ is equipped with a natural
Hall multiplication making it into an associative algebra \(^1\). Here \(T\) is a torus acting in a Hamiltonian way on \(\text{Rep}_n \Pi_Q\) by appropriately rescaling the maps attached to the arrows \(e \in E\). The algebra \(K_Q\) acts on the \(T\)-equivariant \(K\)-theory groups of Nakajima quiver varieties (see [Neg21b] for a review in our language):

\[
\mathcal{N}_w = \bigsqcup_{v \in \mathbb{N}^I} \mathcal{N}_{v,w}
\]

and is in fact the largest algebra thus acting via Hecke correspondences. When \(Q\) is a finite type quiver, \(K_Q\) is isomorphic to the positive half of the quantum loop algebra (in Drinfeld’s sense) of \(\mathfrak{g}_Q\). The situation of an affine quiver, in which case \(K_Q\) is isomorphic to a quantum toroidal algebra, is studied in detail for the Jordan quiver in [SV13], for cyclic quivers in [Neg15] and for arbitrary affine quivers in [VV20]. More generally, for a quiver without edge loops and a specific one-dimensional torus \(T\), there is an algebra homomorphism:

\[
(1.1) \quad U_q^+(L\mathfrak{g}_Q) \longrightarrow K_Q
\]

which recovers Nakajima’s construction of representations of quantum affinizations of Kac-Moody algebras on the equivariant \(K\)-theory of quiver varieties. The map (1.1) is surjective (under some mild conditions on the torus action on \(\text{Rep}_n \Pi_Q\)), but it is not known to be injective in general (see [VV20]). Beyond these cases, however, very little is known. Moreover, even though \(\text{Rep}_n \Pi_Q\) is equivariantly formal for any \(T\) (and hence \(K^T(\text{Rep}_n \Pi_Q)\) is free as a \(K^T(\text{pt})\)-module), the structure of \(K_Q\) as an algebra depends in a rather subtle way on \(T\). Note that there is a natural gauge action of the group \((\mathbb{C}^*)^E\) on \(T\), but as soon as \(Q\) contains edge loops or multiple edges, the quotient of \(T\) by this gauge group is nontrivial.

1.2. In the present paper, we consider the case when the torus \(T = (\mathbb{C}^*)^{|E|} \times \mathbb{C}^*\) is as large as possible (each of the first \(|E|\) copies of \(\mathbb{C}^*\) scale anti-diagonally the two coordinates of \(\Pi_Q\) corresponding to a given edge, while the last copy of \(\mathbb{C}^*\) scales diagonally one half of the coordinates of \(\Pi_Q\)) and we work over the fraction field:

\[
(1.2) \quad \mathbb{F} = \text{Frac}(K^T(\text{pt})) = \mathbb{Q}(q, t_e)_{e \in E}
\]

Our main result provides an explicit description of:

\[
K_{Q,\text{loc}} := K_Q \otimes_{K^T(\text{pt})} \mathbb{F}
\]

by generators and relations which we will now summarize. Let \(\overline{E}\) be the “double” of the edge set \(E\), i.e. there are two edges \(e = ij\) and \(e^* = ji\) in \(\overline{E}\) for every edge \(e = ij \in E\). The set \(\overline{E}\) is equipped with a canonical involution \(e \leftrightarrow e^*\). We extend the notation \(t_e\) to an arbitrary \(e \in \overline{E}\) by the formula:

\[
(1.3) \quad t_{e^*} = \frac{q}{t_{e}}
\]

\(^1\)There is a specific choice of a line bundle involved in the definition of the multiplication; we refer to [Neg21a] for details.
for any $e \in E$. For any $i,j \in I$, consider the rational function $^2$:

\[ \zeta_{ij}(x) = \left( 1 - xq^{-1} \right)^{\delta_j} \prod_{e=ij \in E} \left( 1 - \frac{t_e}{x} \right) \prod_{e=ji \in E} \left( 1 - \frac{t_e}{qx} \right) \]

and set:

\[ \tilde{\zeta}_{ij}(x) = \zeta_{ij}(x) \cdot (1 - x)^{\delta_j} \]

Let $U_Q^+$ be the $\mathbb{F}$-algebra generated by elements $e_{i,d}$ for $i \in I, d \in \mathbb{Z}$ subject to the following set of quadratic and cubic relations, in which we set

\[ e_i(z) = \sum_{d \in \mathbb{Z}} e_{i,d} z^d \]

- For any pair $(i,j) \in I^2$, the **quadratic** relation:

\[ e_i(z) e_j(w) \zeta_{ij} \left( \frac{w}{z} \right) z^{\delta_j} = e_j(w) e_i(z) \zeta_{ij} \left( \frac{z}{w} \right) (-w)^{\delta_j} \]

- For any edge $e = \overrightarrow{ij}$, the **cubic** relation:

\[ \frac{\tilde{\zeta}_{ii} \left( \frac{x}{x_1} \right) \tilde{\zeta}_{ij} \left( \frac{w}{x_1} \right) \tilde{\zeta}_{ji} \left( \frac{y}{x_1} \right)}{(1 - \frac{x}{x_1q})(1 - \frac{w}{x_2l})} \cdot e_i(x_1) e_i(x_2) e_j(y) \]

\[ + \frac{\tilde{\zeta}_{ii} \left( \frac{x}{x_1} \right) \tilde{\zeta}_{ij} \left( \frac{w}{x_1} \right) \tilde{\zeta}_{ji} \left( \frac{y}{x_1} \right) \left( -\frac{x_1l}{y} \right) \left( -\frac{w}{x_2} \right)^{\delta_j}}{(1 - \frac{x}{x_2l})(1 - \frac{x_1l}{y})} \cdot e_i(x_2) e_j(y) e_i(x_1) \]

\[ + \frac{\tilde{\zeta}_{ii} \left( \frac{x}{x_1} \right) \tilde{\zeta}_{ij} \left( \frac{w}{x_1} \right) \tilde{\zeta}_{ji} \left( \frac{y}{x_1} \right) \left( \frac{x_1l}{w} \right) \left( \frac{x_2l}{w} \right)}{(1 - \frac{x}{x_1q})(1 - \frac{x_1l}{y})} \cdot e_j(y) e_i(x_1) e_i(x_2) = 0 \]

**Theorem 1.3.** There is an algebra isomorphism $K_{Q,\text{loc}} \simeq U_Q^+$. 

When $Q$ is a tree, the quotient of $(C^*)^{|E|} \times C^*$ by the action of the gauge group $(C^*)^{|I|}$ is one-dimensional, hence up to renormalization $K_Q$ can be defined as $C^*$-equivariant $K$-theory (in other words, we do not lose any information by assuming that $t_e = q^\frac{1}{2}$, for all $e \in E$). In addition, one can check that in the case of an $A_2$ quiver, the cubic relations (1.6) are equivalent to the standard $q$-Serre relations (see Example 3.9). With this in mind, Theorem 1.3 implies:

**Theorem 1.4.** Suppose that $Q$ is a tree, and that $\mathbb{T}$ scales the symplectic form on $\text{Rep}_n \Pi_Q$ nontrivially. Then the localization:

\[ U_q^+(Lg_Q) \bigotimes_{K^\times(\text{pt})} \mathbb{F} \longrightarrow K_{Q,\text{loc}} \]

of the map (1.1) is an algebra isomorphism.

---

$^2$Note that this is actually the rational function denoted by $\zeta_{ij}$ in [Neg21a].
Cohomological Hall algebras of quivers are known (at least in the case of Borel-Moore homology and $K$-theory) to embed in a suitable big shuffle algebra $V_Q$, whose multiplication encodes the structure of $Q$ (see [SV20, VV20, YZ18]). In the $K$-theoretic case and for maximal $T$, recent work ([Neg21a, Zha19]) identified the image of this embedding as the small shuffle algebra $S_Q \subset V_Q$ determined by the so-called 3-variable wheel conditions. These wheels conditions feature for instance in [E] where they are derived from purely algebraic formal identities involving products of delta functions (see also [DJ]) and more recently in [NT21].

Theorem 1.3 is a direct corollary of the following theorem, which is the main result of the present paper.

**Theorem 1.5.** There is an algebra isomorphism $U_Q^+ \cong S_Q$.

For a general $Q$ and a general choice of $T$ (which satisfies some mild conditions), there is a chain of algebra homomorphisms:

$$U_Q^+ \rightarrow K_{Q,\text{loc}} \rightarrow S_Q$$

The content of Theorem 1.5 is that these maps are all isomorphisms for $T$ maximal.

Our main tool to prove Theorem 1.5 is the combinatorics of words developed in [Neg21a] (which was in turn influenced by [NT21], and further back, by the seminal work of [LR95, Lec04, Ros02]). It would be interesting to extend the above result to the case of a smaller torus $T$; this would necessitate some more complicated wheel conditions, and result in higher degree relations in $U_Q^+$ (see the last section of [Neg21a]). For instance, when two vertices are joined by more than one edge, it is customary to set the corresponding weights of the torus action to be in a geometric progression, as this yields the $q$-Serre relations (which are of degree two more than the number of edges, so more complicated than cubic in the case of multiple edges).

1.6. Let us mention an important application of Theorem 1.5. When $Q$ is the quiver $S_g$ with one vertex and $g$ loops, it is known by combining [SV12] and [Neg21a] that the spherical subalgebra of the Hall algebra $H^\text{sph}_X$ of the category of coherent sheaves on a genus $g$ curve $X$ defined over the finite field $k$ of $q^{-1}$ elements is isomorphic to $K_Q$ (extended by a commutative Cartan subalgebra). To make the previous statement precise, the equivariant parameters $t_1, \ldots, t_g, q/t_1, \ldots, q/t_g$ must be set equal to the inverses of the Weil numbers $\sigma_1, \ldots, \sigma_g, \P_{\Omega^1}, \ldots, \P_{\Omega^g}$ of $X$. Thus, the rational function (1.4) for the quiver $Q = S_g$ corresponds to the renormalized zeta function of the curve $X$:

$$\zeta_X(x) = \frac{1 - qx^{-1}}{1 - x} \prod_{e=1}^g (\sigma_e - x)(1 - \sigma_e^{-1})$$

For any $e = 1, \ldots, g$, set:

$$Q_e(z_1, z_2, z_3) = \prod_{1 \leq i < j \leq 3} \prod_{f \neq e} \left( \frac{\sigma_f - z_j}{z_i} \right) \left( 1 - \frac{\sigma_f}{z_i} \right) \left( 1 - \frac{\sigma_f}{z_j} \right)$$

Then we have the following result (see Section 6 for details).
Theorem 1.7. When the curve $X$ has distinct Weil numbers (we will refer to this as the “generic” case), its spherical Hall algebra $H^{sph}_X$ is generated by elements $\kappa^{\pm 1}$, $\theta_0, l$, $v_{z, l} \in \mathbb{Z}$ for $l \geq 1$, $d \in \mathbb{Z}$, subject to the following set of relations:

\begin{align*}
H^+(z)H^+(w) &= H^+(w)H^+(z) \\
E(z)H^+(w) &= H^+(w)E(z)\left(\frac{cz}{z}\right) (1.7)
\end{align*}

\begin{align*}
E(z)E(w)\zeta_X\left(\frac{w}{z}\right) &= E(w)E(z)\zeta_X\left(\frac{z}{w}\right) (1.8)
\end{align*}

And for all $e = 1, \ldots, g$ and $m \in \mathbb{Z}$ the relation:

\begin{align*}
\left[(xyz)^m(x + z)(xz - y^2)Q_e(x, y, z)E(x)E(y)E(z)\right]_{ct} &= 0 (1.9)
\end{align*}

(see (3.4) for the notation $[\ldots]_{ct}$). In the formulas above, we set:

\begin{align*}
E(z) &= \sum_{d \in \mathbb{Z}} v^{\text{vec}}\left(d^{-d}z\right), & H^+(z) &= \kappa\left(1 + \sum_{l \geq 1} \theta_0, l z^{-l}\right)
\end{align*}

For $g = 0$, one recovers the defining relations for $U^+_q(Lsl_2)$ (see [Kap97]), while for $g = 1$ one gets the relations describing the elliptic Hall algebra of [BS12]. Note that there is a slight discrepancy between (1.6) and (1.10); this comes from the fact that we added in the Cartan loop generators $\{\theta_0, l\}_{l \in \mathbb{N}}$, see Section 6. From the point of view of function field automorphic forms, Theorem 1.7 says that in the case of the function field of a generic curve, Eisenstein series for the group $GL(n)$ which are induced from the trivial character of the torus satisfy, in addition to the celebrated functional equation (which is equivalent to (1.9)), $g$ families of cubic relations and no higher degree relations.

One might wonder what happens for a non-generic curve, or for the entire Hall algebra $H_X$ rather than the spherical subalgebra $H^{sph}_X$. In Section 7, we answer both of these questions using a version of Theorem 1.5 that holds for equivariant parameters $t_e$ which satisfy a certain metric condition (Assumption B introduced in [Neg21a]). As it turns out, this metric condition applies to the inverse Weil numbers of $X$ due to the (generalized) Riemann hypothesis and the functional equation for the zeta function (and more generally for the Rankin-Selberg L-functions attached to a pair of absolutely cuspidal eigenforms); this also allows us to give, using [KSV17], a complete presentation for an arbitrary curve of the subalgebra $H^{abs}_X$ generated by the coefficients of all absolutely cuspidal eigenforms (Corollary 7.8). As we show, the structure of this algebra only depends on the various orders of vanishing of the Rankin-Selberg L-functions attached to pairs of absolutely cuspidal eigenforms.

1.8. The plan of the present paper is the following. From now on, we will fix the quiver $Q$ and write simply $U^+$ instead of $U^+_Q$.

- In Section 2, we consider the $\mathbb{F}$-algebra $\tilde{U}^+$ generated by elements $\{e_{i,d}\}_{i \in I, d \in \mathbb{Z}}$ modulo the quadratic relations (1.5) and check that it is naturally dual to the so-called big shuffle algebra $\mathcal{V}$, see Proposition 2.9:

\begin{align*}
\tilde{U}^+ \otimes \mathcal{V}^\text{op} \xrightarrow{\cdot \cdot} \mathbb{F}.
\end{align*}
In Section 3, we recall the shuffle algebra $\mathcal{S} \subset \mathcal{V}$ of [Neg21a], which is cut out by the 3-variable wheel conditions (3.1). We show that these wheel conditions arise by pairing with certain cubic elements that will be defined in Proposition 3.5:

(1.12) $\left\{ A_d^{(e)} \right\}_{e \in E, d \in \mathbb{Z}} \in \hat{U}^+$

This allows us to prove that (1.11) descends to a non-degenerate pairing:

(1.13) $\mathcal{U}^+ \otimes \mathcal{S}^{\text{op}} \xrightarrow{\langle \cdot \rangle} F$

where $\mathcal{U}^+$ is the quotient of $\hat{U}^+$ by the ideal generated by the elements (1.12). Comparing (1.13) with the pairing:

$\mathcal{S} \otimes \mathcal{S}^{\text{op}} \xrightarrow{\langle \cdot \rangle} F$

that was studied in [Neg21a] allows us to conclude that $\mathcal{U}^+ \cong \mathcal{S}$, thus establishing Theorem 1.5.

- In Section 4, we provide a definition of a natural Drinfeld-type double $\mathcal{U}$ of $\mathcal{U}^+$.

- In Section 5, we consider specializations of the shuffle algebra $\mathcal{S}$ when the equivariant parameters satisfy Assumption $\mathfrak{b}$ of Definition 5.2. We extend the main results (i.e. the presentation by generators and relations) in this context; this involves some new families of (still cubic) relations.

- In Section 6, we recall the basic notions concerning the spherical Hall algebra $\mathcal{H}^{\text{enh}}_X$ of a generic genus $g$ smooth projective curve $X/\mathbb{k}$ and prove Theorem 1.7.

- Finally, in Section 7, we use Section 5 to extend the results of Section 6 to an arbitrary curve $X$, and to the (much) larger subalgebra $\mathcal{H}^{\text{abs}}_X \subset \mathcal{H}_X$ (corresponding by Langlands duality to all geometrically irreducible local systems).

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2. The big shuffle algebra and the quadratic quantum loop group

2.1. Let us introduce the big shuffle algebra $\mathcal{V}$ and quadratic quantum loop group $\hat{U}^+$ associated to a quiver. We will define a perfect pairing between them, as well as a homomorphism $\tilde{\gamma}: \hat{U}^+ \to \mathcal{V}$.
Remark 2.2. The reader might keep in mind the following analogy: \( \overline{U}^+ \) is like a Verma module, \( V \) is like the corresponding dual Verma module, and \( \overline{\Upsilon} \) is like the canonical map between them. As we will see in Section 3, the image of \( \overline{\Upsilon} \) will be the quantum loop group \( U^+ \) that we are interested in, just like the image of the map from a Verma to the dual Verma is the irreducible highest weight representation.

Alternatively, the more geometry-oriented reader may also think about the inclusion of an affine open subset \( j : O \to X \) of an algebraic variety \( X \) together with a local system \( \mathcal{L} \) on \( O \). \( \overline{U}^+ \) is like \( j_! \mathcal{L} \), \( V \) is like \( j_* \mathcal{L} \), and \( U^+ \) is like the intermediate extension \( j_* \mathcal{L} \), which is the image of the natural morphism \( j_! \mathcal{L} \to j_* \mathcal{L} \).

Throughout the present paper, we fix a finite quiver \( Q \) with vertex set \( I \) and edge set \( E \). The notation \( e = \overrightarrow{i j} \) will mean “\( e \) is an arrow going from \( i \) to \( j \)”. Let:

\[
\begin{align*}
\#_{\overrightarrow{i j}} &= |\text{arrows } \overrightarrow{i j}|, \\
\#_{i j} &= \#_{\overrightarrow{i j}} + \#_{\overleftarrow{i j}}
\end{align*}
\]

We identify \( \mathbb{Z}^I \) with \( \bigoplus_{i \in I} \mathbb{Z} \xi_i \), where \( \xi_i \) is the vector with a single 1 at the \( i \)-th spot, and zeroes everywhere else. In terms of quiver representations, \( \xi_i \) is the dimension vector of the simple quiver representation \( S_i \) supported at \( i \). The set of natural numbers \( \mathbb{N} \) will be considered to include 0.

2.3. We begin by recalling the big shuffle algebra, defined over the field (1.2):

Definition 2.4. The big shuffle algebra is the vector space:

\[
V = \bigoplus_{n \in \mathbb{N}^I} \mathbb{F}[z_{i1}^{\pm 1}, \ldots, z_{in_i}^{\pm 1}]^{\text{sym}}_{i \in I}
\]

endowed with the associative product:

\[
R(\ldots, z_{i1}, \ldots, z_{in_i}, \ldots) * R'(\ldots, z_{i1}, \ldots, z_{in_i'}, \ldots) = 
\]

\[
\text{Sym} \left[ \frac{R(\ldots, z_{i1}, \ldots, z_{in_i}, \ldots) R'(\ldots, z_{i1,n_i+1}, \ldots, z_{i,n_i+n_i'}, \ldots)}{\prod_{i \in I} n_i! \prod_{i \in I} n_i'} \prod_{1 \leq a \leq n_i} \prod_{n_j < b \leq n_j + n_j'} \zeta_{i j} \left( \frac{z_{i a}}{z_{j b}} \right) \right]
\]

and the unit being the function 1 in zero variables (\( \zeta_{i j}(x) \) is defined in (1.4)).

In Definition 2.4, “sym” refers to the set of Laurent polynomials which are symmetric with respect to the variables \( z_{i1}, \ldots, z_{in_i} \) for each \( i \in I \) separately, and “Sym” denotes symmetrization with respect to the variables \( z_{i1}, \ldots, z_{i,n_i+n_i'} \) for each \( i \in I \) separately. Note that even though the right-hand side of (2.3) seemingly has simple poles at \( z_{ia} - z_{jb} \) for all \( i \in I \) and all \( a < b \), these poles vanish when taking the Sym, as the orders of such poles in a symmetric rational function must be even.

The big shuffle algebra is graded by \( \mathbb{N}^I \times \mathbb{Z} \), where \( \mathbb{N}^I \) keeps track of the number of variables \( \{ n_i \}_{i \in I} \) of a Laurent polynomial \( R \), while \( \mathbb{Z} \) keeps track of the homogeneous degree of \( R \). Under these circumstances, we will write:

\[
\text{deg } R = (n, d)
\]

and refer to:

\[
\text{hdeg } R = n \quad \text{and} \quad \text{vdeg } R = d
\]
as the horizontal and vertical degree of $R$, respectively. The graded pieces of the algebra $\mathcal{V}$ will be denoted by $\mathcal{V}_n$ (when grading by horizontal degree only) and by $\mathcal{V}_{n,d}$ (when grading by both horizontal and vertical degree).

2.5. Let us now introduce the quadratic quantum loop group associated to $Q$.

**Definition 2.6.** The quadratic quantum loop group $\tilde{U}^+$ is the $F$-algebra generated by symbols:

$$\{e_{i,d}\}_{i \in I, d \in \mathbb{Z}}$$

modulo the quadratic relations:

$$e_i(z)e_j(w)\zeta_{ij}\left(\frac{w}{z}\right) = e_j(w)e_i(z)\zeta_{ij}\left(\frac{z}{w}\right)$$

for all $i, j \in I$, where $e_i(z) = \sum_{d \in \mathbb{Z}} e_{i,d}z^{-d}$.

The meaning of relation (2.5) is that one cancels denominators (which yields the equivalent relation (1.5)) and then equates the coefficients of every $\{z^kw^l\}_{k,l \in \mathbb{Z}}$ in the left and right-hand sides, thus yielding relations between the generators $e_{i,d}$:

$$(2.6) \quad e_{i,a}e_{j,b} + \sum_{\bullet = -\#_{i,j} - \delta^i_j}^{-1} \text{coeff} \cdot e_{i,a+\bullet}e_{j,b-\bullet} =$$

$$= e_{j,b+\#_{i,j} + \delta^i_j}e_{i,a-\#_{i,j} - \delta^i_j} + \sum_{\bullet = 0}^{\#_{i,j} + \delta^i_j - 1} \text{coeff} \cdot e_{j,b+\bullet}e_{i,a-\bullet}$$

where “coeff” denotes various coefficients in $F$, arising from the numerator of the functions $\zeta_{ij}$ that appear in (2.5); observe that the constant $\gamma$ in the right-hand side of (2.6) is different from 1 if $i = j$, which will be important in Proposition 2.16.

Just like $\mathcal{V}$, the algebra $\tilde{U}^+$ is also $\mathbb{N}^I \times \mathbb{Z}$-graded, with:

$$\text{deg } e_{i,d} = (\varsigma_i, d).$$

Therefore, we will use the terms “horizontal degree” and “vertical degree” pertaining to $\tilde{U}^+$ as well, in accordance with (2.4).

**Proposition 2.7.** The assignment $e_{i,d} \mapsto z_{i1}^d$, for $i \in I, d \in \mathbb{Z}$ induces an algebra homomorphism:

$$\tilde{\Upsilon}: \tilde{U}^+ \rightarrow \mathcal{V}.$$
2.8. As one of our main tools, we will next define a pairing between \( \tilde{U}^+ \) and \( \mathcal{V} \). Let 
\[ Dz = \frac{dz}{2\pi i}. \]
Whenever we write \( \int_{z_1}^{z_n} \) we are referring to a contour integral taken over concentric circles around the origin in the complex plane (i.e. an iterated residue at 0, or \( \infty \)). The following result is close to [Neg21a, Proposition 3.3].

**Proposition 2.9.** There is a well-defined pairing:

\[
\tilde{U}^+ \otimes \mathcal{V}^{\text{op}} \xrightarrow{\cdot \cdot} \mathbb{F}
\]

given for all \( R \in \mathcal{V}_n \) and all \( i_1, \ldots, i_n \in I, d_1, \ldots, d_n \in \mathbb{Z} \) by:

\[
\left\langle e_{i_1, d_1} \cdots e_{i_n, d_n}, R \right\rangle = \int_{|z_1| > \cdots > |z_n|} z_1^{d_1} \cdots z_n^{d_n} R(z_1, \ldots, z_n) \prod_{a=1}^n Dz_a
\]

if \( \zeta_{i_1} \cdots + \cdots + \zeta_{i_n} = n \), and 0 otherwise. Implicit in the notation (2.9) is that the symbol \( z_a \) is plugged into one of the variables \( z_{i_a \bullet} \) of \( R \), for all \( a \in \{1, \ldots, n\} \).

**Remark 2.10.** While we have presented (2.8) as an \( \mathbb{F} \)-linear pairing of vector spaces, note that it naturally extends to a bialgebra pairing once one enlarges the algebras involved according to the standard procedure for quantum loop groups (which we recall in Section 4). The notation \( \mathcal{V}^{\text{op}} \) in (2.8) is meant to underscore this fact.

**Proof.** To check that (2.8) is a well-defined pairing, we need to make sure that any linear relations between the \( e_{i,d} \)'s yield linear relations between the right-hand sides of (2.9), for any \( R \in \mathcal{V}^{\text{op}} \). Let us first observe that because the defining relations in the algebra \( \tilde{U}^+ \) are quadratic, it suffices to treat the case \( n = 2 \). Indeed, if one inserts in the l.h.s. of (2.9) an expression \( e_{i_1, d_1} \cdots e_{i_{k-1}, d_{k-1}} P e_{i_{k+1}, d_{k+1}} \cdots e_{i_n, d_n} \) with \( P = P(e_{i_k \bullet}, \cdots, e_{i_{k-1} \bullet}) \) a quadratic relation, then for any \( R(z_1, \ldots, z_n) \in \mathcal{V}_n \) we may invoke Fubini's theorem to compute the r.h.s. of (2.9) by first performing the integration over the variables \( z_1, z_{k+1} \).

Taking \( n = 2 \), we may rewrite (2.9) in terms of the generating series \( e_i(x) \) and \( e_j(y) \) as follows:

\[
\left\langle e_i(x) e_j(y), R \right\rangle = \int_{|z_1| > |z_2|} \frac{\delta \left( \frac{z_1}{x} \right) \delta \left( \frac{z_2}{y} \right)}{\zeta_{ji} \left( \frac{z_1}{z_2} \right)} R(z_1, z_2) Dz_1 Dz_2
\]

where the \( \delta \) function is \( \delta(x) = \sum_{d \in \mathbb{Z}} x^d \). Therefore, we also have:

\[
\left\langle e_i(x) e_j(y) \zeta_{ji} \left( \frac{y}{x} \right) x^{\delta_j}, R \right\rangle = \int_{|z_1| > |z_2|} \frac{\delta \left( \frac{z_1}{x} \right) \delta \left( \frac{z_2}{y} \right)}{\zeta_{ji} \left( \frac{z_1}{z_2} \right)} x^\delta R(z_1, z_2) Dz_1 Dz_2 =
\]

\[
\int_{|z_1| > |z_2|} \delta \left( \frac{z_1}{x} \right) \delta \left( \frac{z_2}{y} \right) \zeta_{ji} \left( \frac{z_2}{z_1} \right) z_1^\delta R(x, y) \frac{Dz_1 Dz_2}{\zeta_{ji} \left( \frac{z_1}{z_2} \right)} = R(x, y)(x - y)^\delta
\]

\[\text{(2.10)}\]

The choice of \( a \in \{1, \ldots, n_{i_a}\}, \forall a \in \{1, \ldots, n\} \) is immaterial due to the symmetry of \( R \), as long as we ensure that \( \bullet_a \neq \bullet_b \) for all \( a \neq b \) such that \( i_a = i_b \).
(the equalities above are due to the well-known property:
\[ \delta \left( \frac{z}{x} \right) P(z) = \delta \left( \frac{z}{x} \right) P(x) \]
for any Laurent polynomial \( P \)). Analogously, we have:
\[ (2.11) \]
\[ \langle e_j(y) e_i(x) \bar{\zeta}_{ij} \left( \frac{x}{y} \right) (-y)^{\delta_i}, R \rangle = R(x,y)(x-y)^{\delta_i} \]
so the two pairings (2.10) and (2.11) are equal, as we needed to show. \( \square \)

2.11. We will now provide a basis of \( \tilde{U}^+ \), following [Neg21a] (which was inspired by ideas of [LR95, Lec04, Ros02] and [NT21]). Consider the set of \textbf{letters}:
\[ i^{(d)} \quad \forall i \in I, d \in \mathbb{Z} \]
As usual, a \textbf{word} is any sequence of letters:
\[ \left[ i_1^{(d_1)} \ldots i_n^{(d_n)} \right] \quad \forall i_1, \ldots, i_n \in I, d_1, \ldots, d_n \in \mathbb{Z}. \]
If \( w = \left[ i_1^{(d_1)} \ldots i_n^{(d_n)} \right] \) is a word we call \( n \) its \textbf{length}, define its \textbf{degree} as:
\[ \deg w = (\varsigma_{i_1} + \cdots + \varsigma_{i_n}, d_1 + \cdots + d_n) \in \mathbb{N}^I \times \mathbb{Z} \]
set
\[ \mathcal{V} = (d_1, \ldots, d_n) \]
and put
\[ e_w = e_{i_1, d_1} \ldots e_{i_n, d_n} \in \tilde{U}^+ \]
We write \( \mathcal{W} \) for the set of all words. By definition, \( \tilde{U}^+ \) is linearly spanned by the collection of elements \( e_w \) for \( w \in \mathcal{W} \), but there are linear relations among them. We will now point out a subset of words, such that the corresponding elements yield a linear basis of \( \tilde{U}^+ \). For this, we introduce a total order on the set of words. We begin by fixing a total order on the set of vertices \( I \) of \( Q \).

**Definition 2.12 ([NT21]).** We define a total order on the set of letters as follows:
\[ i^{(d)} < j^{(e)} \quad \text{if} \quad \begin{cases} d > e \\ or \\ d = e \text{ and } i < j \end{cases} \]
We extend this to the total lexicographic order on words by:
\[ \left[ i_1^{(d_1)} \ldots i_n^{(d_n)} \right] < \left[ j_1^{(e_1)} \ldots j_m^{(e_m)} \right] \]
if \( i_1^{(d_1)} = j_1^{(e_1)}, \ldots, i_k^{(d_k)} = j_k^{(e_k)} \) and either \( i_{k+1}^{(d_{k+1})} < j_{k+1}^{(e_{k+1})} \) or \( k = n < m \).

**Definition 2.13 ([Neg21a]).** A word \( w = \left[ i_1^{(d_1)} \ldots i_n^{(d_n)} \right] \) is called \textbf{non-increasing} if:
\[ (2.12) \]
\[ \begin{cases} d_a < d_b + \sum_{a \leq s < b} #_{i_a i_b} \\ or \\ d_a = d_b + \sum_{a \leq s < b} #_{i_a i_b} \text{ and } i_a \geq i_b \end{cases} \]
for all $1 \leq a < b \leq n$ (see (2.1) for the definition of $\#_{ij}$).

2.14. Let $W_{\leq}$ denote the set of non-increasing words. The motivation for introducing them is twofold, and is embodied by Lemma 2.15 and Proposition 2.16.

**Lemma 2.15 ([Neg21a]).** There are finitely many non-increasing words of given degree, which are bounded above by any given word $v$.

**Proof.** Let us assume we are counting non-increasing words $[i_1^{(d_1)} \ldots i_n^{(d_n)}]$ with $d_1 + \cdots + d_n = d$ for fixed $n$ and $d$. The fact that such words are bounded above implies that $d_1$ is bounded below. But then the inequality (2.12) implies that $d_2, \ldots, d_n$ are also bounded below. The fact that $d_1 + \cdots + d_n$ is fixed implies that there can only be finitely many choices for the exponents $d_1, \ldots, d_n$. Since there are also finitely many choices for $i_1, \ldots, i_n \in I$, this concludes the proof. □

**Proposition 2.16 ([Neg21a]).** The set $\{e_w\}_{w \in W_{\leq}}$ is a linear basis of $\tilde{U}^+$.

**Proof (sketch).** A similar result appears in [Neg21a] for a quotient of $\tilde{U}^+$. As the proof given in loc. cit. only uses relations (2.5), we may apply it to our situation. More precisely, loc. cit. shows how to iterate formula (2.6) in order to obtain that for any word $v$, the element $e_v$ belongs to the linear span of elements $e_w$ with $w \in W_{\leq}$ satisfying:

$$w \geq v,$$

where $\beta(n)$ is a universal constant. This implies that:

$$\tilde{U}^+ = \text{span}\{e_w\}_{w \in W_{\leq}}$$

Let us briefly recall how [Neg21a] showed that the elements $\{e_w\}_{w \in W_{\leq}}$ are linearly independent, as this will be useful later. Consider any ordered monomial:

$$\mu = z_{i_1; \bullet_1}^{−k_1} \cdots z_{i_n; \bullet_n}^{−k_n}$$

where we assume that $\bullet_a \neq \bullet_b$ if $a \neq b$ and $i_a = i_b$. The associated word of $\mu$ is:

$$w_{\mu} = [i_1^{(d_1)} \ldots i_n^{(d_n)}]$$

where:

$$d_a = k_a + \sum_{t > a} \#_{i_t; i_1} - \sum_{s < a} \#_{i_s; i_t}$$

The lexicographically largest of the associated words of various orderings of a given monomial $\mu$ will be called the leading word of $\mu$ (it is uniquely determined). It was shown in [Neg21a, Lemma 4.8] that the leading word is the only one among all associated words which is non-increasing in the sense of (2.12). More generally, the leading word of any non-zero $R \in \mathcal{V}$, denoted by $\text{lead}(R)$, will be the lexicographically largest of the leading words (2.15) for all the monomials which appear in $R$ with non-zero coefficient. Conversely, any $w \in W_{\leq}$ appears as the leading word:

$$w = \text{lead}(\text{Sym}_w \mu)$$

of the monomial $\mu$ as in (2.14), chosen such that formula (2.15) holds.
Analogously to [Neg21a, Formula (4.18)], one can show by direct inspection that:

\[(2.18) \langle e_w, R \rangle \text{ is } \begin{cases} \neq 0 & \text{if } w = \text{lead}(R) \\ = 0 & \text{if } w > \text{lead}(R) \end{cases} \]

The formula above immediately shows the linear independence of the elements $e_w$, as $w$ runs over non-increasing words. Indeed, if one were able to write such an element $e_w$ as a linear combination of elements $e_v$ involving only strictly larger non-increasing words $v$ then this would contradict (2.18) for $R = \text{Sym } \mu$ with $\mu$ as in (2.17).

Corollary 2.17. The pairing $\tilde{U}^+ \otimes \mathcal{V}^{\text{op}} \xrightarrow{\langle \cdot, \cdot \rangle} F$ defined in (2.9) is non-degenerate.

Proof. For non-degeneracy in the second factor, we need to show that any $R \in \mathcal{V}^{\text{op}}$ which pairs trivially with the whole of $\tilde{U}^+$ actually vanishes; this is just the obvious fact that if the power series expansion of the rational function:

\[
\frac{R(z_1, \ldots, z_n)}{\prod_{1 \leq a < b \leq n} \zeta_{i_b i_a} \left( \frac{z_a}{z_b} \right)}
\]

(in the domain corresponding to an arbitrary order of the variables $z_1, \ldots, z_n$ and an arbitrary choice of the indices $i_1, \ldots, i_n \in I$) vanishes, then $R = 0$.

Let us now consider some non-zero element:

\[
\phi = \sum_{w \in \mathcal{W}_\leq} a_w \cdot e_w \in \tilde{U}^+
\]

Let $w$ be the smallest word such that $a_w \neq 0$, and choose a monomial $\mu$ whose leading word is $w$ (see (2.14), (2.15) and (2.17)). Then by (2.18), $\langle \phi, \text{Sym } \mu \rangle \neq 0$. This gives the non-degeneracy of the pairing in the first factor.

2.18. For future use, we spell out a “finite support” variant of Corollary 2.17. Let $T \subset \mathcal{W}_\leq$ be a finite set of non-increasing words. We set:

\[\tilde{U}^{+,T} = \bigoplus_{w \in T} F \cdot e_w \subset \tilde{U}^+\]

and let $\mathcal{V}_T \subset \mathcal{V}$ denote the set of symmetric Laurent polynomials spanned by monomials having the property that their leading word (2.15) lies in $T$. Then we claim that the restriction of the pairing (2.8) to:

\[(2.19) \tilde{U}^{+,T} \otimes \mathcal{V}^{\text{op}}_T \xrightarrow{\langle \cdot, \cdot \rangle} F\]

is non-degenerate. Indeed, the two vector spaces above manifestly have the same finite dimension (due to the uniqueness of the leading word (2.15) associated to any given monomial), so it suffices to show that (2.19) is non-degenerate in the second argument. This follows from the fact that the leading word $w$ of any $0 \neq R \in \mathcal{V}_T$ lies in $T$, and (2.18) implies that $\langle e_w, R \rangle \neq 0$. 

□
3. The small shuffle algebra and the quantum loop group

3.1. We will now define a certain subalgebra of $\mathcal{V}$, determined by the so-called wheel conditions. These first arose in the context of elliptic quantum groups in [FO01], and the version herein is inspired by the particular wheel conditions of [FHH$^*$09] (which corresponds to the case when $Q$ is the Jordan quiver).

**Definition 3.2 ([Neg21a]).** The **small shuffle algebra** is the subspace $S \subset \mathcal{V}$ consisting of Laurent polynomials $R(\ldots, z_{i_1}, \ldots, z_{i_n}, \ldots)$ that satisfy the wheel conditions:

$$R\bigg|_{z_{ia}=qz_{ic}, z_{ib}=t, z_{ic}=0} = 0$$

for any edge $E \ni e=i \rightarrow j$ and all $a \neq c$, and further $a \neq b \neq c$ if $i=j$.

It is well-known (and straightforward to show) that $S$ as defined above is a subalgebra of $\mathcal{V}$. Because the wheel conditions are vacuous for $R$ of horizontal degree $\{\varsigma_i\}_{i \in I}$, we conclude that the homomorphism (2.7) actually maps into the small shuffle algebra, i.e.:

$$\bar{\Upsilon}: \bar{\Upsilon}^+ \rightarrow S$$

The map $\bar{\Upsilon}$ was shown to be surjective in [Neg21a]. We will obtain in Theorem 3.8 below a generators-and-relations description of the algebra $\bar{\Upsilon}^+/\text{Ker}\bar{\Upsilon} \cong S$, by describing a set of generators for the kernel of the map $\bar{\Upsilon}$.

**Example 3.3.** Let us consider the quiver $Q$ of type $A_2$, consisting of two vertices $\{i,j\}$ with a single edge, say $e: i \rightarrow j$. Up to gauge transformation, we may specialize the equivariant parameter to $t=q^{1/2}$. In this case, it is known that the small shuffle algebra $S$ is isomorphic to the positive half of the quantum loop group $U_{q^{1/2}}^+(\mathfrak{sl}_3)$, which is the quotient of $\bar{\Upsilon}^+$ by the set of cubic $q$-Serre relations, i.e.:

$$S \cong \bar{\Upsilon}^+/\left(P_{s,t}(x_1, x_2, y) + P_{s,t}(x_2, x_1, y)\right)$$

where for any $s \neq t \in \{i,j\}$, we have set:

$$P_{s,t}(x_1, x_2, y) = e_s(x_1)e_s(x_2)e_t(y) - \left(q^{1/2} + q^{-1/2}\right)e_s(x_1)e_t(y)e_s(x_2) + e_t(y)e_s(x_1)e_s(x_2).$$

3.4. The wheel conditions (3.1) can be interpreted as certain linear conditions on elements $R \in \mathcal{V}$. As can be expected in light of Corollary 2.17, these linear conditions are given by taking the pairing (2.8) with certain elements of $\bar{\Upsilon}^+$, which we now explicitly describe. We first introduce some more notation. Recall that:

$$\zeta_{ij}(x) = \zeta_{ij}(x) \cdot (1 - x)^{d_i} \in \mathbb{F}[x^{\pm 1}]$$

Given a Laurent polynomial $P(x, y, z)$ and three formal series $e_i(x)$, $e_j(y)$, $e_k(z)$ as in Definition 2.6, we will write:

$$\left[P(x, y, z)e_i(x)e_j(y)e_k(z)\right]_{ct} \in \bar{\Upsilon}^+$$
for the constant term of the expression in square brackets in (3.4). For example, if
\[ P(x) = x^a y^b z^c \]
for various integers \( a, b, c \), then (3.4) equals \( e_{i,a} e_{j,b} e_{k,c} \).

For any edge \( \overrightarrow{E} \ni e = i \rightarrow j \) and any triple of integers \( (a, b, c) \in \mathbb{Z}^3 \), we define:

\[
A_{a,b,c}^{(e)} = \left[ \frac{x_1^a \left( \frac{z_{ij}}{x_{ij}} \right)^b \left( \frac{z_{ji}}{x_{ji}} \right)^c}{(1 - q)(1 - t_e)^{\delta_j}(1 - \frac{t_e}{q})^\delta_i} \cdot X^{(e)}(x_1, x_2, y) \right]_{ct} \in \mathbb{U}^+_{2k_i + \varsigma_j, a+b+c}
\]

where:

\[
X^{(e)}(x_1, x_2, y) = \frac{-\zeta_{ii} \left( -\frac{x_i}{x_{ij}} \right) \zeta_{ji} \left( -\frac{x_j}{x_{ji}} \right) \zeta_{ij} \left( -\frac{x_{ij}}{y} \right)}{(1 - \frac{x_i}{x_{ij}})(1 - \frac{y}{x_{ji}})} \cdot e_i(x_1)e_i(x_2)e_j(y) + \frac{-\zeta_{ii} \left( -\frac{x_i}{x_{ij}} \right) \zeta_{ji} \left( -\frac{x_j}{x_{ji}} \right) \zeta_{ij} \left( -\frac{x_{ij}}{y} \right)}{(1 - \frac{x_i}{x_{ij}})(1 - \frac{x_{ij}}{y})} \cdot e_i(x_2)e_j(y)e_i(x_1) + \frac{-\zeta_{ii} \left( -\frac{x_i}{x_{ij}} \right) \zeta_{ij} \left( -\frac{x_{ij}}{x_{ji}} \right) \zeta_{ji} \left( -\frac{x_{ji}}{y} \right)}{(1 - \frac{x_i}{x_{ij}})(1 - \frac{x_{ji}}{y})} \cdot e_j(y)e_i(x_1)e_i(x_2)
\]

The linear factors in \( x_1, x_2, y \) in the denominators above are all canceled by some factors in the numerator, hence the expression in the square brackets of (3.5) is a Laurent polynomial in \( x_1, x_2, y \) times the product of the series \( e_i(x_1), e_i(x_2), e_j(y) \).

**Proposition 3.5.** The elements \( A_{a,b,c}^{(e)} \) only depend on \( d = a + b + c \) (and will henceforth simply be denoted \( A_d^{(e)} \)). Setting:

\[
A_d^{(e)}(x) = \sum_{d \in \mathbb{Z}} A_d^{(e)} x^{-d}
\]

we have:

\[
\langle A_d^{(e)}(x), R \rangle = R \bigg|_{z_{i1} = x, z_{j1} = t_e, z_{i2} = qx} \forall R \in Y_{2k_i + \varsigma_j}^{op}
\]

for any edge \( \overrightarrow{E} \ni e = i \rightarrow j \).

**Proof.** We need to show that for any integers \( a + b + c = d \), we have:

\[
\langle A_d^{(e)}, R \rangle = \text{coefficient of } x^{-d} \text{ in } R \bigg|_{z_{i1} = x, z_{j1} = t_e, z_{i2} = qx}
\]

for all \( R \in Y_{2k_i + \varsigma_j}^{op} \). This will also imply that \( A_{a,b,c}^{(e)} \) only depends on \( d \), because of the non-degeneracy of the pairing (Corollary 2.17). We have:

\[
A_d^{(e)} = C_1 + C_2 + C_3
\]

where \( C_1, C_2, C_3 \) correspond to the three terms defining \( X_{a,b,c}^{(e)} \). Abbreviating:

\[
D = \frac{(z_{i1})^a \left( \frac{z_{i1}}{x_{i1}} \right)^b \left( \frac{z_{i1}}{x_{i2}} \right)^c}{(1 - q)(1 - t_e)^{\delta_j}(1 - \frac{t_e}{q})^\delta_i} D_{z_{i1}} D_{z_{i2}} D_{z_{j1}}
\]
we have, using (2.9):

$$
\langle C_1, R \rangle = \int_{|z_{11}| \gg |z_{12}| \gg |z_{13}|} \frac{R(z_{11}, z_{12}, z_{13})}{(1 - \frac{z_{11}}{z_{12}})(1 - \frac{z_{12}}{z_{13}})(1 - \frac{z_{11}}{z_{13}})} D
$$

$$
\langle C_2, R \rangle = \int_{|z_{12}| \gg |z_{13}| \gg |z_{11}|} \frac{R(z_{11}, z_{12}, z_{13})}{(1 - \frac{z_{11}}{z_{12}})(1 - \frac{z_{12}}{z_{13}})(1 - \frac{z_{11}}{z_{13}})} D
$$

$$
\langle C_3, R \rangle = \int_{|z_{13}| \gg |z_{12}| \gg |z_{11}|} \frac{R(z_{11}, z_{12}, z_{13})}{(1 - \frac{z_{11}}{z_{12}})(1 - \frac{z_{12}}{z_{13}})} D
$$

(we replace $z_{13}$ by $z_{11}$ in the formulas above if $i = j$; note that in the middle formula for $\langle C_2, R \rangle$, we took the liberty of replacing $z_{13} \leftrightarrow z_{12}$ in the arguments of the symmetric Laurent polynomial $R$). Then (3.7) is an immediate consequence of the following identity of formal series:

$$
\delta \left( \frac{z_{12}}{z_{11} q} \right) \delta \left( \frac{z_{11}}{z_{12}} \right) = \text{ev}_{|z_{11}| \gg |z_{12}| \gg |z_{13}|} \left[ \text{ev}_{|z_{11}| \gg |z_{12}| \gg |z_{13}|} \right]
$$

$$
\begin{align*}
&= \text{ev}_{|z_{12}| \gg |z_{13}| \gg |z_{11}|} \left[ \text{ev}_{|z_{12}| \gg |z_{13}| \gg |z_{11}|} \right]
+ \text{ev}_{|z_{13}| \gg |z_{11}| \gg |z_{12}|} \left[ \text{ev}_{|z_{13}| \gg |z_{11}| \gg |z_{12}|} \right]
\end{align*}
$$

(3.8)

where $\text{ev}_{x \gg y \gg z}$ is the operator of taking the Laurent expansion of a rational function in the asymptotic region $|x| \gg |y| \gg |z|$. The formula above is a straightforward computation involving formal power series, which we leave as an exercise to the interested reader. \qed

3.6. By Proposition 3.5, different choices of integers $a, b, c$ yield elements $A_d^{(e)}$ which are equal modulo the ideal generated by relations (2.5). In order to write $A_d^{(e)}$ "canonically", one can expand these elements in the basis $\{ e_w \}_{w \in W'_D}$ of $\bar{U}^+$ (recall Proposition 2.16):

$$
A_d^{(e)} = \sum_{w = [i_1^{(a_1)} i_2^{(a_2)} i_3^{(a_3)}] \in W'_D} c_d^{(e)}(w) \cdot e_i e_\alpha e_j e_\alpha e_i e_j e_\alpha
$$

(3.9)

The coefficients $c_d^{(e)}(w)$ occurring in the expression above are not nice (even in relatively simple cases, such as a vertex with a single loop, or two distinct vertices with two edges between them), but it is easy to see that the transformation:

$$
d \mapsto d + 3 \quad \text{and} \quad w = [i_1^{(a_1)} i_2^{(a_2)} i_3^{(a_3)}] \mapsto [i_1^{(a_1+1)} i_2^{(a_2+1)} i_3^{(a_3+1)}],
$$
rescales $e_{d,w}^{(e)}$ by a monomial in $q,t_e$. This allows us to conclude that there exists a large enough natural number $N$, which only depends on $Q$, such that:

$$|\alpha - \alpha_s| \leq N, \forall s,t \in \{1,2,3\}$$

for all words $i_1^{(\alpha_1)}i_2^{(\alpha_2)}i_3^{(\alpha_3)}$ which appear with non-zero coefficient in (3.9), for all edges $e \in E$ and all $d \in \mathbb{Z}$. We will use this fact in the proof of Theorem 3.8.

3.7. Our main result is that the two-sided ideal $J \subset \tilde{U}^+$ generated by $A_d^{(e)}$, for all $e \in \mathcal{E}$ and $d \in \mathbb{Z}$, coincides with the (positive half of the) generic quantum loop group as:

$$(3.10) \quad \mathbb{U}^+ = \tilde{U}^+/J.$$ 

Note that taking the quotient by $J$ is equivalent to imposing $X^{(e)}(x_1,x_2,y) = 0$ for all edges $e$, which is precisely the content of (1.6). Our main result, which is simply a restatement of Theorem 1.5, is the following.

**Theorem 3.8.** The assignment $e_{i,d} \mapsto z_{i,d}^0$ for $i \in I, d \in \mathbb{Z}$ induces an algebra isomorphism:

$$\Upsilon : \mathbb{U}^+ \xrightarrow{\sim} \mathcal{S}.$$ 

**Example 3.9.** Let us work out the cubic relation (1.6) when $Q$ is the $A_2$ quiver of Example 3.3, with the goal of recovering the $q$-Serre relation (3.3). Recall that we set $t_e = q^{\frac{1}{2}}$, so relation (1.6) states that:

$$(3.11) \quad \left(x_1 - yq^{\frac{1}{2}}\right)e_i(x_1)e_i(x_2)e_j(y) + \left(x_2q^{\frac{1}{2}} - x_1q^{-\frac{1}{2}}\right)e_i(x_2)e_j(y)e_i(x_1)$$

$$+ \left(yq^{\frac{1}{2}} - 1\right)e_j(y)e_i(x_1)e_i(x_2) = 0$$

The quadratic relations (2.5) hold in both $\mathbb{U}^+$ and $\mathbb{U}^+_{q^{\frac{1}{2}}}(L\mathfrak{sl}_3)$, and they read:

$$(3.12) \quad e_s(x)e_s(y) (xq - y) = e_s(y)e_s(x) (x - yq)$$

$$(3.13) \quad e_s(x)e_s(y) (x - yq^{\frac{1}{2}}) = e_s(y)e_s(x) (xq^{\frac{1}{2}} - y)$$

for $s \neq t \in \{i,j\}$. Let us show how to obtain the $q$-Serre relation (for $s = i, t = j$) from (3.11), (3.12), (3.13). As a consequence of (3.13), we have:

$$\left(x_2q^{\frac{1}{2}} - yq\right)e_i(x_2)e_j(y)e_i(x_1) = \left(x_2q - yq^{\frac{1}{2}}\right)e_j(y)e_i(x_2)e_i(x_1)$$

$$\left(yq - x_1q^{-\frac{1}{2}}\right)e_i(x_2)e_j(y)e_i(x_1) = \left(yq^{\frac{1}{2}} - x_1q^{-1}\right)e_i(x_2)e_i(x_1)e_j(y)$$

Adding the two relations together, and then subtracting (3.11) from the result yields:

$$y \left(1 - q\right)e_i(x_2)e_j(y)e_i(x_1)$$

$$= \left(x_2q - yq^{\frac{1}{2}}\right)e_j(y)e_i(x_2)e_i(x_1) + \left(yq^{\frac{1}{2}} - x_1q^{-1}\right)e_i(x_2)e_i(x_1)e_j(y)$$

$$+ \left(x_1 - yq^{\frac{1}{2}}\right)e_i(x_1)e_i(x_2)e_j(y) + \left(yq^{\frac{1}{2}} - x_2\right)e_j(y)e_i(x_1)e_i(x_2)$$
Symmetrizing the relation above with respect to $x_1 \leftrightarrow x_2$ yields:

$$y \left( q^{-1} - q \right) \left[ e_i(x_2)e_j(y)e_i(x_1) + e_i(x_1)e_j(y)e_i(x_2) \right] =$$

$$= \left[ x_2q - yq^{-1} + yq^{-1} - x_1 \right] e_j(y)e_i(x_2)e_i(x_1)$$

$$+ \left[ yq^{-1} - x_1q^{-1} + x_2 - yq^{-1} \right] e_i(x_2)e_i(x_1)e_j(y)$$

$$+ \left[ x_1 - yq^{-1} + yq^{-1} - x_2q^{-1} \right] e_i(x_1)e_i(x_2)e_j(y)$$

$$+ \left[ yq^{-1} - x_2 + x_1q - yq^{-1} \right] e_j(y)e_i(x_1)e_i(x_2)$$

By applying (3.12), the right-hand side of the expression above is equal to:

$$y \left( q^{-\frac{1}{2}} - q^{\frac{1}{2}} \right) \left[ e_j(y)e_i(x_2)e_i(x_1) + e_i(x_2)e_i(x_1)e_j(y) \right.$$ 

$$\left. + e_i(x_1)e_i(x_2)e_j(y) + e_j(y)e_i(x_1)e_i(x_2) \right]$$

Dividing by $y(q^{-\frac{1}{2}} - q^{\frac{1}{2}})$, we obtain precisely the $q$-Serre relation:

$$(3.14) \quad P_{s,t}(x_1, x_2, y) + P_{s,t}(x_2, x_1, y) = 0$$

of (3.3). We have just showed that the usual $q$-Serre relations hold in the generic quantum loop group. The distinction between the cubic relations (3.14) and the cubic relations (1.6) is explained by the fact that the two sets of relations can be obtained from each other by adding appropriate multiples of the quadratic relations.

Proof of Theorem 3.8. Let us recall the non-degenerate pairing:

$$(3.15) \quad \mathcal{S} \otimes \mathcal{S}^{\text{op}} \xrightarrow{\langle \cdot , \cdot \rangle} \mathbb{F}$$

defined in Proposition 3.3 of [Neg21a]. Comparing our formula (2.9) with formula (3.30) of loc. cit., we see that the two pairings are compatible, in the sense that:

$$(3.16) \quad \langle f, \iota(g) \rangle = \langle \widetilde{\Upsilon}(f), g \rangle \in \mathbb{F}, \quad \forall f \in \widetilde{\mathcal{U}}^+, g \in \mathcal{S}^{\text{op}},$$

where $\iota : \mathcal{S}^{\text{op}} \to \mathcal{V}^{\text{op}}$ is the tautological inclusion. To show that the homomorphism $\widetilde{\Upsilon}$ of (3.2) descends to a homomorphism:

$$\Upsilon : \mathcal{U}^+ \to \mathcal{S},$$

one needs to prove that for any $e \in \overline{E}$ and $d \in \mathbb{Z}$, we have:

$$\widetilde{\Upsilon}(A_d^{(e)}) = 0.$$

By the non-degeneracy of the pairing (3.15), it suffices to show that $\widetilde{\Upsilon}(A_d^{(e)})$ pair trivially with anything in $\mathcal{S}^{\text{op}}$. Using (3.16), this is equivalent to showing that $A_d^{(e)}$ pair trivially with anything in $\mathcal{S}^{\text{op}}$ under the pairing (2.8), which follows from (3.6).

Since the homomorphism $\widetilde{\Upsilon}$ of (3.2) is surjective, so is $\Upsilon$. It thus remains to show that $\Upsilon$ is injective, and because the pairing (3.15) is non-degenerate, the task boils down to showing that an element:

$$(3.17) \quad \psi = \sum_{w \in W_{\leq}} c_w \cdot e_w \in \widetilde{\mathcal{U}}^+$$
which pairs trivially with \( S^\op \) lies in \( J \). Clearly, it suffices to do so for a homogeneous \( \psi \). Let us fix a degree \((n, d) \in \mathbb{N}^I \times \mathbb{Z}\), fix some \( \psi \in \bar{U}^+ \) of degree \((n, d)\) as above and define, for a pair of positive integers \( M, m \):

\[
T_{M, m} = \left\{ w = \left[ i_1^{(d_1)}, \ldots, i_n^{(d_n)} \right] \in W, \; \deg(w) = (n, d), \quad \sum_{s \in A} d_s \geq -M|A| + m|A|^2 - \sum_{A \ni s, t \notin A} \#_{i, s, t}, \quad \forall A \subseteq \{1, \ldots, n\} \right\}.
\]

For notational simplicity, we will write \( M \) if \( M \) non-zero coefficient in (3.17). As we are free to pick \( M \) if \( M \) is picked large enough, the set \( T \) will contain all the words which appear with non-zero coefficient in (3.17). As we are free to pick \( M \) and \( m \) arbitrarily large, it might seem strange that we bother to subtract the quantity \( \sum_{s \in A} d_s \) might seem strange that we bother to subtract the quantity \( \sum_{s \in A} d_s \) from the right-hand side of the inequality in (3.18). This is justified by the following claim, whose straightforward proof we leave to the reader:

**Claim 3.10.** The set of inequalities

\[
\sum_{s \in A} d_s \geq -M|A| + m|A|^2 - \sum_{A \ni s, t \notin A} \#_{i, s, t}, \quad \forall A \subseteq \{1, \ldots, n\}
\]

holds for the leading word \( \left[ i_1^{(d_1)}, \ldots, i_n^{(d_n)} \right] \) of a monomial \( \mu \) if and only if it holds for the associated word (2.15) of any other ordering of the variables of \( \mu \).

With this in mind, we have reduced the proof of Theorem 3.8 to the following lemma:

**Lemma 3.11.** Let \( S^\op_T = S^\op \cap V^\op_T \). For \( M \gg m \gg 1 \), the pairing:

\[
\left( \bar{U}^+, J \cap \bar{U}^+T \right) \otimes S^\op_T \xrightarrow{\langle \cdot, \cdot \rangle_T^\prime} F
\]

induced by (2.19) is non-degenerate.

**Proof of Lemma 3.11.** Let us denote by \( \langle \cdot, \cdot \rangle_T \) the restriction of \( \langle \cdot, \cdot \rangle \) to \( \bar{U}^+, J \cap \bar{U}^+T \otimes V^\op_T \), as in (2.19). By construction, the pairings \( \langle \cdot, \cdot \rangle_T^\prime \) and \( \langle \cdot, \cdot \rangle_T \) are compatible in the sense that:

\[
\left\langle f, t(g) \right\rangle_T = \left\langle \bar{\Psi}(f), g \right\rangle_T^\prime \in F, \quad \forall f \in \bar{U}^+, g \in S^\op_T,
\]

where \( t : S^\op_T \to V^\op_T \) is the tautological inclusion; observe that \( \bar{\Psi}(\bar{U}^+, J) \subseteq S_T \). By Section 2.18, \( \langle \cdot, \cdot \rangle_T \) is a non-degenerate pairing of finite-dimensional vector spaces.

To show that \( \langle \cdot, \cdot \rangle_T^\prime \) is also non-degenerate, it hence suffices to check that:

\[
S^\op_T = (J \cap \bar{U}^+, T) \perp
\]

Note that the inclusion \( S^\op_T \subseteq (J \cap \bar{U}^+, T) \perp \) is obvious, so we will prove the opposite:

\[
S^\op_T \supseteq (J \cap \bar{U}^+, T) \perp
\]

Equivalently, this requires us to show that:

(3.19) \( \forall R \in V^\op_T \setminus S^\op_T, \exists \psi \in J \cap \bar{U}^+, T \) such that \( \langle \psi, R \rangle \neq 0 \)

Let us fix \( R \) as above. As it does not satisfy one of the wheel conditions, we have:

(3.20) \( R \bigg|_{z_a = q x, z_b = t x, z_i = x} \neq 0 \)
for some edge $e = ij$. By (2.9) and (3.6), this entails:

$$
(3.21) \quad \left\langle e_{i_1,d_1} \cdots e_{i_{n-3},d_{n-3}} A_d^{(e)}, R \right\rangle \neq 0
$$

for certain $i_1, \ldots, i_{n-3} \in I$ and $d_1, \ldots, d_{n-3}, d \in \mathbb{Z}$ (to see this claim, one must expand the integral (2.9) in the domain where the variables $z_{i_1}, z_{j_1}, z_{i_{n-3}}$ of (3.20) have smaller absolute value than all other variables of $R$). Assume that the word:

$$
(3.22) \quad \left[ i^{(d_1)} \cdots i^{(d_{n-3})} \right]
$$

is maximal such that (3.21) holds. In particular, this implies that the word (3.22) must be non-increasing (as we have seen in the proof of Proposition 2.16, any $e_w$ can be written as a linear combination of $e_w$ as $w > v$ runs over non-increasing words). To establish the required (3.19), it therefore suffices to show that:

$$
(3.23) \quad \psi := e_{i_1,d_1} \cdots e_{i_{n-3},d_{n-3}} A_d^{(e)} \in \tilde{U}^{+,T}
$$

To this end, let us label the variables of $R$ other than $z_{i_a}, z_{j_b}, z_{i_c}$ by $z_1, \ldots, z_{n-3}$ (of colors $i_1, \ldots, i_{n-3}$, respectively). Using formula (2.9), relation (3.21) states:

$$
(3.24) \quad \int_{\mathbb{R}^{n-3}} \frac{Dx \prod_{a=1}^{n-3} Dz_a}{\prod_{1 \leq a < b \leq n-3} s_{i_a i_b} \left( \frac{z_a}{z_b} \right) \prod_{a=1}^{n-3} s_{i_a} \left( \frac{z_a}{z_a} \right) \zeta_{i_a} \left( \frac{z_a}{z_a} \right)} R(z_1, \ldots, z_{n-3}, qx, t_r x, x) \neq 0
$$

As:

$$
\zeta_{i_j}(u) \in u^{-\sum_i F_i[u]} \times, \quad \forall i, j \in I
$$

the non-vanishing (3.24) and the maximality of the word (3.22) imply that:

$$
\left[ k_1 \cdots k_{n-3} x^{d + \sum_{a=1}^{n-3} (2 \# i_a i_a - \# j_a i_a)} \right] R(z_1, \ldots, z_{n-3}, qx, t_r x, x) \neq 0
$$

where we write for all $a \in \{1, \ldots, n-3\}$:

$$
k_a = d_a + \sum_{s < a} \# i_{s} i_a - \sum_{t > a} \# i_{a} i_t - 2 \# i_a i_t - \# i_a j_a
$$

(compare with (2.16)). Thus, we conclude that the Laurent polynomial $R \in V_T^{op}$ must include with non-zero coefficient a monomial of the form:

$$
z_1^{-k_1} \cdots z_{n-3}^{-k_{n-3}} z_{i_a}^{-k_{n-2}} z_{j_b}^{-k_{n-1}} z_{i_c}^{-k_{n}}
$$

where:

$$
k_{n-2} = d_{n-2} + \sum_{a=1}^{n-3} \# i_{a}^{-1} - \# j_a^{-1} - \# i_a^{-1}
$$

$$
k_{n-1} = d_{n-1} + \sum_{a=1}^{n-3} \# i_a^{-1} + \# j_a^{-1} - \# j_a^{-1}
$$

$$
k_n = d_n + \sum_{a=1}^{n-3} \# i_a^{-1} + \# j_a^{-1} + \# i_a^{-1}$$
for some integers $d_{n-2} + d_{n-1} + d_n = d$. We will henceforth write $i_{n-2} = i, i_{n-1} = j$ and $i_n = i$. As explained in the paragraph following (3.18), although the word:

$$[i_1^{(d_1)} \ldots i_{n-3}^{(d_{n-3})} i_{n-2}^{(d_{n-2})} j^{(d_{n-1})} j^{(d_n)}]$$

might fail to be non-increasing (our assumption only states that its prefix obtained by removing the last three letters is non-increasing), its exponents still satisfy the inequality in (3.18). In particular, we have:

$$\sum_{s \in A} d_s \geq -M|A| + m|A|^2 - \sum_{A \supseteq i \neq A} \#_{i_{i_{i}i}}$$

where $A = B$ or $A = B \cup \{n-2, n-1, n\}$, for arbitrary $B \subseteq \{1, \ldots, n-3\}$. However, as explained in Subsection 3.6, the element $\psi$ of (3.23) is a linear combination of products of the form:

$$(3.26) \quad e_{i_{1},d_{1}} \ldots e_{i_{n-3},d_{n-3}} e_{j_{1},d_{1}} e_{j_{2},d_{2}} e_{j_{3},d_{3}}$$

where $\{j_{1}, j_{2}, j_{3}\} = \{i, i, j\}$ and the numbers $\delta_{1}, \delta_{2}, \delta_{3}$ all satisfy:

$$\delta_{1} + \delta_{2} + \delta_{3} = d \quad \text{and} \quad \delta_{u} - d_{|3|} \leq c_{1}, \forall u \in \{1, 2, 3\}$$

for some global constant $c_{1}$. For every term of the form (3.26) that appears in $\psi$, let us consider the largest index $x \in \{0, \ldots, n-3\}$ such that:

$$d_{x} < d_{x+1} - c_{2}$$

where the global constant $c_{2}$ is chosen much larger than the number $\beta(n)$ that appears in (2.13). As explained in Proposition 2.16, we may write:

$$e_{t_{x+1},d_{x+1}} \ldots e_{t_{n-3},d_{n-3}} e_{t_{1},d_{1}} e_{t_{2},d_{2}} e_{t_{3},d_{3}} \in \sum_{[j_{x+1}^{(d_{x+1})} \ldots j_{n}^{(d_{n})}] \in \mathcal{W}_{x}} F \cdot e_{j_{x+1},d_{x+1}} \ldots e_{j_{n},d_{n}}$$

in such a way that the exponents $l_{x+1}, \ldots, l_{n}$ that appear with non-zero coefficient in the right-hand side satisfy the following properties:

- all the numbers $l_{x+1}, \ldots, l_{n}$ are within a global constant $c_{3}$ away from their average, which will be denoted by $o$,
- the large difference between $d_{x}$ and $d_{x+1}$ ensures that all concatenated words

$$\sum_{s \in A} d_s \geq -M|A| + m|A|^2 - 2n^2|E|$$

(3.28)

$$\sum_{s \in B} d_s + \sum_{s=x+1}^{n} l_s \geq -M|A| + m|A|^2 - 2n^2|E|$$

(3.29)

To prove (3.23), it therefore remains to show that the concatenated words that appear in (3.27) are in $T$. Property (3.25) implies that
for \( A = B \) or \( A = B \sqcup \{x+1, \ldots, n\} \) respectively, where \( B \subseteq \{1, \ldots, x\} \) is arbitrary. Assume for the purpose of contradiction that the defining property of \( T \) is violated for

\[
A = B \sqcup C
\]

with \( B \subseteq \{1, \ldots, x\} \) and \( C \) a proper subset of \( \{x+1, \ldots, n\} \), i.e.

\[
(3.30) \sum_{s \in B} d_s + \sum_{s \in C} l_s < -M|A| + m|A|^2
\]

We claim that (3.28)–(3.29) and (3.30) are incompatible (for \( m \) chosen large enough compared to the constant \( c_3 \) mentioned in the first bullet above). Indeed, the properties listed in the first bullet above allow us to obtain the following inequalities from (3.28), (3.29), (3.30):

\[
(3.31) \sum_{s \in B} d_s \geq -M|B| + m|B|^2 - 2n^2|E| \\
(3.32) \sum_{s \in B} d_s + y(o - c_3) < -M(|B| + y) + m(|B| + y)^2 \\
(3.33) \sum_{s \in B} d_s + (n-x)(o + c_3) \geq -M(|B| + n - x) + m(|B| + n - x)^2 - 2n^2|E|
\]

where \( y = |C| \) lies in \( \{1, \ldots, n - x - 1\} \). Subtracting (3.31) from (3.32) yields

\[
o - c_3 < -M + m(y + 2|B|) + 2n^2|E|
\]

and subtracting (3.32) from (3.33) yields

\[
o + c_3 \geq -M + m(n - x + y + 2|B|) - 2n^2|E|
\]

The two inequalities above are incompatible if \( m \) is chosen large enough compared to \( c_3 \) and \( 2n^2|E| \), thus yielding the desired contradiction. We have thus proved Lemma 3.11, hence also Theorem 3.8.

\[\square\]

4. The Drinfeld double of \( U^+ \)

4.1. We will now recall the standard procedure of upgrading \( U^+ \cong S \) to a Hopf algebra isomorphism, by extending and doubling the two algebras involved (we will follow the conventions of [Neg21a]). Consider the opposite algebra:

\[
U^- = U^{+, \text{op}} \cong S^{\text{op}}
\]

whose generators will be denoted by \( \{f_{i,d}\}_{i \in I, d \in \mathbb{Z}} \); we set \( f_i(z) = \sum_{d \in \mathbb{Z}} f_{i,d} z^{-d} \).

**Definition 4.2.** The extended algebras \( U^\geq \) and \( U^\leq \) are defined as the semi-direct products:

\[
U^\geq = U^+ \bigotimes_{\mathbb{F}[h_{i,d}]} \mathbb{F}[h_{i,d}]_{i \in I, d \geq 0} \\
U^\leq = U^- \bigotimes_{\mathbb{F}[h_{i,d}]} \mathbb{F}[h_{i,d}]_{i \in I, d \geq 0}
\]
where the multiplication is governed by the following relations for all \( i, j \in I \):

\[
e_{i}(z)h_{j}^{+}(w) = h_{j}^{+}(w)e_{i}(z) \frac{\zeta_{ij}(\frac{w}{z})}{\zeta_{ji}(\frac{z}{w})}
\]

\[
f_{i}(z)h_{j}^{-}(w) = h_{j}^{-}(w)f_{i}(z) \frac{\zeta_{ji}(\frac{w}{z})}{\zeta_{ij}(\frac{z}{w})}.
\]

Here \( h_{j}^{\pm}(w) = \sum_{d=0}^{\infty} h_{j,d}^{\pm} w^{d} \). The rational functions in the right-hand sides of (4.1) and (4.2) are expanded in the same asymptotic direction of \( w \) as \( h_{j}^{\pm}(w) \).

The algebras \( U^{\ge} \) and \( U^{\le} \) are actually bialgebras, with respect to the coproduct:

\[
\Delta (h_{i}^{\pm}(z)) = h_{i}^{\pm}(z) \otimes h_{i}^{\pm}(z),
\]

\[
\Delta (e_{i}(z)) = e_{i}(z) \otimes 1 + h_{i}^{+}(z) \otimes e_{i}(z),
\]

\[
\Delta (f_{i}(z)) = f_{i}(z) \otimes h_{i}^{-}(z) + 1 \otimes f_{i}(z)
\]

(strictly speaking, the coproduct above consists of infinite sums, meaning that \( U^{\ge} \) and \( U^{\le} \) are topological bialgebras; we will not dwell upon this fact, as it is quite routine in the theory of quantum affinizations). The final step in the construction of the Drinfeld double is to note that there is a (unique) bialgebra pairing:

\[
U^{\ge} \otimes U^{\le} \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{F}
\]

which satisfies the following properties:

i) for all \( i, j \in I \), we have:

\[
\langle h_{i}^{+}(z), h_{j}^{-}(w) \rangle = \frac{\zeta_{ij}(\frac{z}{w})}{\zeta_{ji}(\frac{w}{z})}
\]

where the right-hand side is expanded as \(|z| \gg |w|\), and:

ii) for all \( i, j \in I \) and \( d, k \in \mathbb{Z} \), we have:

\[
\langle e_{i,d}, f_{j,k} \rangle = \delta_{d}^{0} \delta_{i}^{j} \delta_{k}^{d}
\]

This property implies that the restriction of \( \langle \cdot, \cdot \rangle \) to \( U^{+} \otimes U^{-} \) coincides with (3.15) under the identifications \( U^{+} \cong S \) and \( U^{-} \cong S^{op} \).

Summarizing the discussion above, we may define by the usual Drinfeld double procedure (see [Neg21a, Subsection 4.15] for a review) the following algebra.

**Definition 4.3.** The **generic quantum loop group** \( U \) is the \( \mathbb{F} \)-algebra generated by elements:

\[
\{ e_{i,k}, f_{i,k}, h_{i,l}^{\pm} | i \in I, k \in \mathbb{Z}, l \geq 0 \}
\]

satisfying the fact that \( h_{i,0}^{\pm} \) are invertible, as well as the quadratic relations:

\[
e_{i}(z)e_{j}(w)\zeta_{ji}(\frac{w}{z}) = e_{j}(w)e_{i}(z)\zeta_{ij}(\frac{z}{w})
\]

\[
f_{i}(w)f_{j}(z)\zeta_{ji}(\frac{w}{z}) = f_{j}(w)f_{i}(z)\zeta_{ij}(\frac{z}{w})
\]

for all \( i, j \in I \), the cubic relations:
\[ \frac{\tilde{\zeta}_{ii} \left( \frac{\nu}{x_1} \right) \tilde{\zeta}_{jj} \left( \frac{\nu}{x_1} \right)}{\left( 1 - \frac{\nu}{x_{1q}} \right)} \cdot e_i(x_1)e_i(x_2)e_j(y) + \frac{\tilde{\zeta}_{ii} \left( \frac{\nu}{x_1} \right) \tilde{\zeta}_{jj} \left( \frac{\nu}{x_1} \right)}{\left( 1 - \frac{\nu}{x_{1q}} \right)} \cdot e_i(x_2)e_j(y)e_i(x_1) + \frac{\tilde{\zeta}_{ii} \left( \frac{\nu}{x_1} \right) \tilde{\zeta}_{jj} \left( \frac{\nu}{x_1} \right)}{\left( 1 - \frac{\nu}{x_{1q}} \right)} \cdot e_i(y)e_i(x_1)e_i(x_2) = 0 \]

\[ \frac{\tilde{\zeta}_{ii} \left( \frac{\nu}{x_1} \right) \tilde{\zeta}_{jj} \left( \frac{\nu}{x_1} \right) \tilde{\zeta}_{ij} \left( \frac{\nu}{x_{1q}} \right) \tilde{\zeta}_{ij} \left( \frac{\nu}{x_{1q}} \right)}{\left( 1 - \frac{\nu}{x_{1q}} \right)} \cdot f_j(y)f_i(x_2)f_i(x_1) + \frac{\tilde{\zeta}_{ii} \left( \frac{\nu}{x_1} \right) \tilde{\zeta}_{jj} \left( \frac{\nu}{x_1} \right) \tilde{\zeta}_{ij} \left( \frac{\nu}{x_{1q}} \right) \tilde{\zeta}_{ij} \left( \frac{\nu}{x_{1q}} \right)}{\left( 1 - \frac{\nu}{x_{1q}} \right)} \cdot f_i(x_1)f_j(y)f_i(x_2) + \frac{\tilde{\zeta}_{ii} \left( \frac{\nu}{x_1} \right) \tilde{\zeta}_{jj} \left( \frac{\nu}{x_1} \right) \tilde{\zeta}_{ij} \left( \frac{\nu}{x_{1q}} \right) \tilde{\zeta}_{ij} \left( \frac{\nu}{x_{1q}} \right)}{\left( 1 - \frac{\nu}{x_{1q}} \right)} \cdot f_i(x_2)f_i(x_1)f_j(y) = 0 \]

for any edge \( \mathcal{E} \ni e = ij \), as well as:

\[ e_i(z)h^+_j(w) = h^+_j(w)e_i(z) \tilde{\zeta}_{ij} \left( \frac{\nu}{x_{1q}} \right) \] (for \( |z^\pm| \ll |w^\pm| \))

\[ f_i(z)h^-_j(w) = h^-_j(w)f_i(z) \tilde{\zeta}_{ij} \left( \frac{\nu}{x_{1q}} \right) \] (for \( |z^\pm| \ll |w^\pm| \))

and:

\[ [e_{i,d}, f_{j,k}] = \delta^d_j \cdot \begin{cases} -h^+_{i,d+k} & \text{if } d + k > 0 \\ h^-_{i,0} - h^+_{i,0} & \text{if } d + k = 0 \\ h^-_{i,-d-k} & \text{if } d + k < 0 \end{cases} \]

for all \( i, j \in I \). Formulas (4.3), (4.4), (4.5) endow \( U \) with the structure of a bialgebra. Finally, there is a triangular decomposition as \( F \)-vector spaces:

\[ U \simeq U^+ \otimes U^0 \otimes U^- \]

where \( U^+, U^0, U^- \) are respectively generated by \( \{e_{i,k}\}_{i,k} \), \( \{h^+_i\}_{i,l} \) and \( \{f_{i,k}\}_{i,k} \).

As is standard in the theory of quantum groups, one can write down antipode maps that make \( U^+ \) and \( U^- \) into Hopf algebras, and also define central extensions by making the series \( h^+(z) \) and \( h^-(w) \) “almost” commute with each other (see [Neg20] for a survey in the case of the Jordan quiver). We will not describe these in detail.
Remark 4.4. Since it arises as a Drinfeld double, $U$ has an interesting universal $R$-matrix (up to the usual issues involving completions and central extensions that are required to rigorously define it). This object is studied in [Neg21b], where it is conjectured that it matches the $R$-matrices defined via Nakajima quiver varieties by Aganagic, Maulik, Okounkov and Smirnov (cf. [MO19, OS16, Oko18, AO21]).

5. Special values of the parameters

5.1. An important question is to determine what happens to $U^+$ and $S$ when the parameters $q$ and $\{t_e\}_{e \in E}$ are no longer generic, i.e. when we work over a field different from (1.2). In the present Section, we will provide an answer under Assumption Б of [Neg21a], given as follows:

Definition 5.2. Let $K$ be a field endowed with elements $q$ and $\{t_e\}_{e \in E}$, such that there exists a field homomorphism:

$$\rho : K \rightarrow \mathbb{C}$$

for which $|\rho(q)| < |\rho(t_e)| < 1$ for all $e \in E$.

We emphasize the fact that $K$ will henceforth refer to a choice of a field together with the elements $q, t_e$ as above. With this in mind, we may define:

$$_K\mathcal{V} \text{ and } _K\tilde{U}^+$$

simply by replacing $F$ with $K$ in the definition of $\mathcal{V}$ and $\tilde{U}^+$ of Definitions 2.4 and 2.6, respectively. All the results of Section 2 will continue to hold in the present context, notably the existence of a pairing:

$$_K\tilde{U}^+ \otimes _K\mathcal{V} \rightarrow K$$

analogous to (2.8). This pairing is nondegenerate in the second variable by arguments similar to those in Section 2. It will follow from Theorem 5.8 that this pairing is nondegenerate in the first variable as well (we will not use this fact in the proof of Theorem 5.8).

5.3. However, as soon as we reach Definition 3.2, we notice that (3.1) cannot be the correct definition of the small shuffle algebra anymore (for one thing, it might be the case that $t_e = t_{e'}$ for two distinct edges $e, e'$ between vertices $i$ and $j$, in which case two wheel conditions (3.1) actually impose the same restriction on $R$). As shown in [Neg21a], the way to fix this in the context of Definition 5.2, is to set:

$$_K\mathcal{V} \supset _K\mathcal{S} = \left\{ R(\ldots, z_{i_1}, \ldots, z_{i_m}, \ldots) \right\} \text{ such that } \forall i \in I$$

(5.1) $R \bigg|_{z_{i_2} = q z_{i_1}}$ is divisible by

$$\prod_{(j,b) \notin \{(i_1,1),(i_2,2)\}} \prod_{E \ni e = i j} (z_{j b} - t_e z_{i_1})$$

It is easy to see that (5.1) is equivalent to the following condition for all $i, j \in I$ and all $\gamma \in K$ (below, we replace $z_{i_1}$ by $z_{i_3}$ if $i = j$):

(5.2) $R \bigg|_{z_{i_2} = q z_{i_1}}$ is divisible by $(z_{j_1} - \gamma z_{i_1})^{|b|_{ij}(\gamma)}$
where $b_{ij}(\gamma)$ denotes the number of edges $e = \gamma_{ij}$ in $E$ for which $t_e = \gamma$. This is why if the parameters $\{t_e\}_{e \in E}$ are all distinct, we recover the wheel conditions (3.1).

As shown in [Neg21a], the definition (5.1) ensures that $\mathcal{K}S$ is generated by $\{z^d_i\}_{i \in I}$, and that there is a non-degenerate pairing:

$$\mathcal{K}S \otimes \mathcal{K}S^{\text{op}} \rightarrow \mathbb{K}$$

analogous to (3.15).

5.4. We will now define a quotient:

$$\mathcal{K}\tilde{U}^+ \rightarrow \mathcal{K}U^+$$

for which the analogue of Theorem 3.8 holds. Let us recall the notation of Subsection 3.4, and define for all $i, j \in I$, $0 \leq k \leq b_{ij}(\gamma)$, all $\gamma \in \mathbb{K}$ and all integers $(a, b, c) \in \mathbb{Z}^3$:

$$\mathcal{K}A^{(i,j,\gamma|k)}_{a,b,c} =$$

$$\left[\frac{x^a_1 (\frac{x}{q})_b (\frac{x}{\gamma})_c}{(1 - q)(1 - \gamma)^{\delta_j}(1 - \frac{x}{q})} \cdot \mathcal{K}X^{(i,j,\gamma|k)}(x_1, x_2, y)\right]_{ct} \in \mathcal{K}\tilde{U}^+_{2k, +\gamma, a+b+c}$$

where:

$$\mathcal{K}X^{(i,j,\gamma|k)}(x_1, x_2, y)$$

is defined as the formal series:

$$(-1)^{k-1} \tilde{c}_{ii} (\frac{x_1}{x_2}) \tilde{c}_{ji} (\frac{x}{x_1}) \tilde{c}_{ij} (\frac{x}{y}) (\frac{x}{\gamma})^k \cdot e_i(x_1)e_i(x_2)e_j(y)$$

$$+ \sum_{k'+k'' = k+1} (-1)^{k' \cdot 1} \tilde{c}_{ii} (\frac{x_1}{x_2}) \tilde{c}_{ij} (\frac{x}{y}) \tilde{c}_{ij} (\frac{x}{x_2}) (\frac{x_1}{y})^{k''-1} (\frac{x_1}{x_1\gamma}) (\frac{y}{x_1})^{\delta_j} \cdot e_i(x_2)e_j(y)e_i(x_1)$$

$$+ \frac{\tilde{c}_{ii} (\frac{x_1}{x_2}) \tilde{c}_{ij} (\frac{x}{y}) \tilde{c}_{ij} (\frac{x}{x_2})}{(1 - \frac{x}{x_1\gamma})} \cdot e_j(y)e_i(x_1)e_i(x_2)$$

If $k \leq b_{ij}(\gamma)$, the linear factors in $x_1, x_2, y$ in the denominators above are all canceled by similar linear factors in the numerator, hence (5.5) is a Laurent polynomial in $x_1, x_2, y$ times the product of the series $e_i(x_1), e_i(x_2), e_j(y)$. Note that when $k = b_{ij}(\gamma) = 1$, the expression (5.4) is slightly different from (3.5), but they both yield equivalent sets of cubic relations modulo the ideal of quadratic relations (2.5). As such, it is straightforward to generalize Proposition 3.5 to the following result.

**Proposition 5.5.** For all $i, j \in I$, $R \in Y_{2k, +\gamma, \epsilon}^{\text{op}}$, $0 \leq k \leq b_{ij}(\gamma)$, $\gamma \in \mathbb{K}$ and $(a, b, c) \in \mathbb{Z}^3$, we have:

$$\langle \mathcal{K}A^{(i,j,\gamma|k)}_{a,b,c}, R \rangle = \int \frac{\partial^{k-1} F}{\partial z_{j1}^{k-1}} |_{z_{j1} = z_{i1}} (z_{i1})^{k-1} (k-1)! \cdot Dz_{i1}$$

where $\tilde{c}_{ij} = [\tilde{c}_{ij}]_0$.
where:
\[ F(z_{i1}, z_{j1}) = z_{i1}^{a+b} \left( \frac{z_{i1}}{\gamma} \right)^c R(z_{i1}, z_{i1}q, z_{j1}) \left( 1 - \frac{z_{i1}}{z_{i1}q} \right)^{\delta_i^j} \left( 1 - \frac{z_{j1}}{z_{i1}q} \right)^{\delta_j^i} \]
(we replace \( z_{j1} \) by \( z_{i1} \) in the formulas above if \( i = j \)).

**Proof.** Consider the following formal series, for all \( k \in \mathbb{N} \):
\[
\delta^{(k)}(x) = \sum_{d \in \mathbb{Z}} a^d \left( -d \right)^k
\]
(note that \( \delta^{(0)} = \delta \)). The following property is well-known:
\[
\left[ \delta \left( \frac{z_{i1}}{z_{i1}q} \right) \delta^{(k)} \left( \frac{z_{j1}}{z_{i1}q} \right) P(z_{i1}, z_{i2}, z_{j1}) \right]_{ct} = \int \frac{\partial^k P(z_{i1}, q z_{i1}, z_{j1})}{\partial z_{j1}^k} \bigg|_{z_{j1}=z_{i1}^\gamma} \frac{(z_{i1})^k}{k!} Dz_{i1}
\]
for all Laurent polynomials \( P(z_{i1}, z_{i2}, z_{j1}) \), where \([ \ldots ]_{ct} \) refers to the constant term in the variables \( z_{i1}, z_{i2}, z_{j1} \). With this in mind, formula (5.6) reduces to the following identity of formal power series (whose \( k = 1 \) case is a closely related version of (3.8)):
\[
\begin{align*}
\delta \left( \frac{z_{i2}}{z_{i1}q} \right) \delta^{(k-1)} \left( \frac{z_{j1}}{z_{i1}q} \right) &= ev_{z_{i1}|z_{i2}|z_{j1}|z_{j1}} \left[ (-1)^{k-1} \left( \frac{z_{i2}}{z_{i1}q} \right)^k \left( \frac{z_{j1}}{z_{i1}q} \right)^{k-1} \right] \\
&+ ev_{z_{i1}|z_{i2}|z_{j1}|z_{i2}} \left[ \sum_{k', k'' \in \mathbb{Z} > 0} (-1)^{k'-1} \left( \frac{z_{i2}}{z_{i1}q} \right)^{k'} \left( \frac{z_{j1}}{z_{i1}q} \right)^{k''} \left( \frac{z_{i2}}{z_{i1}q} \right)^{k-1} \left( \frac{z_{i2}}{z_{i1}q} \right)^{k''} \right] \\
&+ ev_{z_{i1}|z_{i2}|z_{j1}|z_{i2}} \left[ \sum_{k', k'' \in \mathbb{Z} > 0} (-1)^{k''-1} \left( \frac{z_{i1}}{z_{i1}q} \right)^{k''} \left( \frac{z_{i1}}{z_{i1}q} \right)^{k} \right]
\end{align*}
\]
(5.7)

If we relabel \( z_{i1} = x, z_{j1} = y, z_{i2} = z, \) then formula (5.7) has left-hand side:
\[
\sum_{d_1, d_2 \in \mathbb{Z}} x^{-d_1-d_2} y^{d_1} z^{d_2} \left( -d_1 \right) \left( k - 1 \right)
\]
(5.8)

Because of the elementary identity
\[
ev_{v|z|u} \left[ \frac{1}{(1 - u^{-1})^k} \right] = \sum_{d=0}^{\infty} \frac{(-u)^d}{d!} \left( -k \right) \left( d \right) = (-1)^{k-1} \sum_{d=0}^{\infty} \frac{u^d}{d!} \left( d - 1 \right)
\]
(5.9)

the right-hand side of (5.7) is equal to:
\[
\begin{align*}
&\sum_{d_1=1}^{\infty} \sum_{d_2=1-1-d_1}^{\infty} x^{-d_1-d_2} y^{d_1} z^{d_2} \left( -d_1 \right) \left( k - 1 \right) \\
&+ \sum_{k', k'' \in \mathbb{Z} > 0} \sum_{k'+k'' = k+1}^{\infty} \sum_{d_1=-\infty}^{d_2=-\infty} x^{-d_1-d_2} y^{d_1} z^{d_2} \left( d_2 \right) \left( k' - 1 \right) \left( -d_1 - d_2 \right) \\
&+ \sum_{d_1=-\infty}^{0} \sum_{d_2=0}^{\infty} x^{-d_1-d_2} y^{d_1} z^{d_2} \left( -d_1 \right) \left( k - 1 \right)
\end{align*}
\]
(5.10)
To obtain (5.10), we relabeled certain indices and used the Taylor series expansion (5.9) for $u$ in place of $u/v$. Using a simple combinatorial identity involving binomial coefficients, expressions (5.8) and (5.10) are easily seen to be equal to each other, thus establishing (5.7).

5.6. Formula (5.6) shows that (5.2) holds for $R$ if and only if:

$$(5.11) \quad \left\langle \mathcal{K} A_{i,j}^{(k)} , R \right\rangle = 0 \quad \forall k \leq b_{ij}(\gamma) \text{ and } a,b,c \in \mathbb{Z}$$

Therefore, generalizing (3.10), we define:

$$(5.12) \quad \mathcal{K} U^+ = \mathcal{K} \tilde{U}^+ / \mathcal{K} J$$

where $\mathcal{K} J$ denotes the two-sided ideal of the quadratic quantum loop group $\mathcal{K} \tilde{U}^+$ generated by the elements (5.4), for all $i,j \in I, \gamma \in \mathbb{K}, k \in \{1, \ldots, b_{ij}(\gamma)\}$ and $a,b,c \in \mathbb{Z}$.

Remark 5.7. One could (and should) define $\mathcal{K} J$ in (5.12) to be the two-sided ideal generated by finitely many of the elements (5.4) in every fixed degree. Indeed, we simply need to use enough $a,b,c$ in (5.11) to ensure that $R$ satisfies property (5.2). The latter property is equivalent\(^\text{4}\) to the vanishing of the r.h.s. of equation (5.6) (for all applicable $i,j,k,\gamma$) for a single triple $(a,b,c)$ of any fixed $d = a + b + c \in \mathbb{Z}$. Thus, in (5.11), it is enough to consider (for all applicable $i,j,k,\gamma$) a single triple $(a,b,c)$ of any fixed $d = a + b + c \in \mathbb{Z}$. Compare with Proposition 3.5.

The analogue of Theorem 3.8 in the setting at hand is the following.

Theorem 5.8. The assignment $e_{i,d} \mapsto z_i^d$ for all $i \in I, d \in \mathbb{Z}$ induces an algebra isomorphism:

$$\mathcal{K} U^+ \xrightarrow{\sim} \mathcal{K} S.$$ 

We will now sketch the main points of the proof of Theorem 5.8, leaving the details to the interested reader. Just like the proof of Theorem 3.8 started with the non-degenerate pairing (3.15), in the case at hand we start from the pairing (5.3), whose non-degeneracy was the main reason for introducing the specific conditions (5.2) in [Neg21a]. Then the proof of Theorem 3.8 carries through as stated, once one replaces the elements (3.5) of the quadratic quantum loop group by the elements (5.4) (in both situations, one needs only consider finitely many such elements in any given degree; see Remark 5.7).

\(^{4}\)We are not saying anything deep here, simply the fact that for all constants $\alpha, \beta, \gamma$, we have $f(z\alpha, z\beta, z\gamma) = 0$ if and only if

$$\int \left( z_1^a z_2^b z_3^c f(z_1, z_2, z_3) \right) \bigg|_{z_1 = z\alpha, z_2 = z\beta, z_3 = z\gamma} \ dz = 0$$

for a single triple $(a,b,c)$ of any fixed $d = a + b + c \in \mathbb{Z}$.
6. The spherical Hall algebra of a generic curve

6.1. Let us fix \( g \in \mathbb{N} \) and specialize the quiver to \( Q = S_g \), the quiver with one vertex and \( g \) loops. In this special case, we will show how to connect the extended quantum loop group \( U^\mathbb{Z} \) of Definition 4.2 with the Hall algebra of a smooth projective curve of genus \( g \) over a finite field. In the present Section, we will treat the case of a generic curve (in the sense of having distinct Weil numbers), and then in the next Section we will show how to adapt to the case of an arbitrary curve.

In more detail, let \( X \) be a smooth, geometrically connected projective curve of genus \( g \) defined over the finite field \( k = \mathbb{F}_{q^{-1}} \) (here the unusual choice of notation for the cardinality of the finite field is made for the purpose of compatibility with (1.4)). Let \( \text{Coh}_r,d(k) \) denote the groupoid of coherent sheaves on \( X \) of rank \( r \) and degree \( d \). We refer to [Sch12b, Lectures 1, 4] for the definition of the Hall algebra of the category of coherent sheaves on \( X \).

As a vector space:

\[
H_X = \bigoplus_{(r,d) \in (\mathbb{Z}^2)^+} \text{Fun}_0(\text{Coh}_{r,d}(k), \mathbb{Q}) \otimes \mathbb{C}[\kappa^\pm 1]
\]

is the space of finitely supported functions on \( \text{Coh}(k) = \bigcup_{r,d} \text{Coh}_{r,d}(k) \), where:

\[(\mathbb{Z}^2)^+ := \{(r,d) \in \mathbb{Z}^2 \mid r \geq 0, d \in \mathbb{Z} \text{ such that } d \geq 0 \text{ if } r = 0\}.
\]

The element \( \kappa \) is usually denoted by \( \kappa_{1,0} \) in the literature, and satisfies the following commutation relations:

\[ \kappa f \kappa^{-1} = q^{(g-1)} f, \quad f \in \text{Fun}_0(\text{Coh}_{r,d}(k), \mathbb{Q}); \]

one often adds a central element (denoted by \( \kappa_{0,1} \)) to \( H_X \), although we will not do so, as it only plays a significant part when considering the double Hall algebra.

**Definition 6.2.** The spherical Hall algebra \( H^\text{sph}_X \) is the subalgebra of \( H_X \) generated by \( \kappa^\pm 1 \) together with the following elements:

\[
1_{\text{vec}1,d} = \chi_{\text{Pic}_d(k)}, \quad 1_{0,l} = \chi_{\text{Coh}_{0,l}(k)}, \quad (d \in \mathbb{Z}, l \geq 1)
\]

where \( \chi_Y \) stands for the characteristic function of a subgroupoid \( Y \subset \text{Coh}(k) \).

Let us consider the subalgebras:

\[
H^\text{sph}_X \supset H^\text{sph,+}_X = \mathbb{C}(1_{\text{vec}1,d} \mid d \in \mathbb{Z})
\]

\[
H^\text{sph}_X \supset H^\text{sph,0}_X = \mathbb{C}(\kappa^{\pm 1}, 1_{0,l} \mid l \geq 1)
\]

In Proposition 6.4, we will recall the way to realize \( H^\text{sph}_X \) as a semidirect product of \( H^\text{sph,+}_X \) with \( H^\text{sph,0}_X \). In order to set this up, it is convenient to introduce new generators \( \{T_{0,l}\}, \{\theta_{0,l}\}, l \geq 1 \) of \( H^\text{sph,0}_X \) through the relations:

\[
1 + \sum_{l=1}^{\infty} 1_{0,l}s^l = \exp\left(\sum_{l=1}^{\infty} T_{0,l}s^l / [l]\right)
\]

\[
1 + \sum_{l=1}^{\infty} \theta_{0,l}s^l = \exp\left(\sum_{l=1}^{\infty} (q^{-1/2} - q^{1/2})T_{0,l}s^l\right)
\]
where \( l = \left( q^{1/2} - q^{-1/2} \right) / \left( q^{1/2} - q^{-1/2} \right) \). It is known that \( H_{\text{sph}}^{sph,0} \) is a free commutative polynomial algebra in \( \kappa^{\pm 1} \) and any one of the families of generators \( \{1_{0,l}\}_{l \geq 1} \) or \( \{\theta_{0,l}\}_{l \geq 1} \).

6.3. Let \( \sigma_1, \sigma_1, \ldots, \sigma_g, \sigma_g \) denote the Weil numbers of \( X \), paired up such that \( \sigma_e \sigma_e = q^{-1} \) for all \( e = 1, \ldots, g \). Therefore, we may write:

\[
(6.2) \quad t_e = \frac{1}{\sigma_e} \quad \text{and} \quad t_{e^*} = \frac{1}{\sigma_e}
\]

for all \( e = 1, \ldots, g \), and this would be compatible with (1.3). With this notation, the particular case of the rational function (1.4) when \( Q \) is the \( g \)-loop quiver (one vertex with \( g \) loops) is a renormalized form of the zeta function of \( X \):

\[
(6.3) \quad \zeta_X(x) = \frac{1 - x q^{-1}}{1 - x} \prod_{e=1}^{g} (\sigma_e - x)(1 - \sigma_e x^{-1})
\]

(explicitly, (6.3) is equal to \([SV12, (1.22)] \times (-1)^{g-1}\)). Finally, set:

\[
E(z) = \sum_{d \in \mathbb{Z}} 1^e_{1,d} z^{-d}, \quad H^+(z) = \kappa \left( 1 + \sum_{l=1}^\infty \theta_{0,l} z^{-l} \right)
\]

Proposition 6.4. The multiplication map \( H_{\text{sph}}^{sph,+} \otimes H_{\text{sph}}^{sph,0} \rightarrow H_{\text{sph}}^{sph} \) is a vector space isomorphism. In addition, as an algebra, \( H_{\text{sph}}^{sph} \) is generated by \( H_{\text{sph}}^{sph,+} \) and \( H_{\text{sph}}^{sph,0} \) modulo the following relations:

\[
E(z)H^+(w) = H^+(w)E(z) \zeta_X \left( \frac{z}{w} \right) \quad \text{for} \ |w| \gg |z|
\]

Proof. See [SV12, Corollary 1.4 and Section 1.5]. Note that \( 1^e_{1,d} \kappa = q^{-1} \kappa 1^{e^*}_{1,d} \). \( \square \)

Concerning the structure of \( H_{\text{sph}}^{sph,+} \), we have the following result. Let \( \mathcal{V}_X \) be the \( \mathbb{C} \)-algebra defined by the relations (2.3) when \( Q \) is the \( g \)-loop quiver (and the equivariant parameters \( t_e \) are specialized as in (6.2)), and let \( \mathcal{V}_{sph}^X \) be the subalgebra of \( \mathcal{V}_X \) generated by its horizontal degree 1 pieces \( \{\mathcal{V}_X^{[1,d]}\}_{d \in \mathbb{Z}} \).

Theorem 6.5 ([SV12, Theorem 1]). The assignment \( 1^e_{1,d} \kappa \rightarrow z^d \in \mathcal{V}_X^{[1,d]} \) for \( d \in \mathbb{Z} \) extends to an algebra isomorphism \( H_{\text{sph}}^{sph,+} \cong \mathcal{V}_{sph}^X \).

6.6. As long as the Weil numbers of \( X \) are distinct (which implies that \( \{t_e\}_{e \in \mathcal{E}} \) are distinct complex numbers), the small shuffle algebra behaves just like in the case of generic parameters. In the language of Section 5, this is because conditions (5.1) are still none other than the usual 3-variable wheel conditions (3.1). Thus, the following is proved just like [Neg21a, Theorem 1.1].

Theorem 6.7. We have \( \mathcal{V}_{sph}^X = S_X \), namely the subalgebra of \( \mathcal{V}_X \) determined by the 3-variable wheel conditions (3.1) (but with \( t_e \) specialized as in (6.2)).
Proof of Theorem 1.7. By combining Theorem 6.5, Theorem 6.7 and Theorem 1.5, we only need to check that the collections of relations (1.6) and (1.10) are equivalent (assuming relations (1.7), (1.8), (1.9) hold). By (1.8), taking commutators with the symbols $\theta_{0,1}$ means that formula (1.10) is equivalent to:

$$\left[p(x, y, z)(x + z)(xz - y^2)Q_e(x, y, z)E(x)E(y)E(z)\right]_{ct} = 0$$

for any symmetric Laurent polynomial $p$. Thus, we need to prove that for any given $R(z_1, z_2, z_3) \in V_X$, the wheel conditions (3.1) are equivalent to the relations:

$$\left< \left[ P(x, y, z)(x + z) \left( \frac{1}{y} - \frac{x}{xz} \right) Q_e(x, y, z)E(x)E(y)E(z) \right]_{ct}, R \right> = 0$$

for any symmetric Laurent polynomial $P(x, y, z) = p(x, y, z)xyz$. By definition of the pairing (2.9), the condition above reads precisely:

$$(6.4) \quad \int_{|z_1| \gg |z_2| \gg |z_3|} \frac{Q_e(z_1, z_2, z_3)}{\xi_X \left( \frac{z_1}{z_1} \right) \xi_X \left( \frac{z_2}{z_2} \right) \xi_X \left( \frac{z_3}{z_3} \right)} (z_1 + z_3) \left( \frac{1}{2} - \frac{z_2}{z_1 z_3} \right) (PR)(z_1, z_2, z_3)Dz_1Dz_2Dz_3 = 0$$

By the definition of $Q_e$, the rational function on the first row above is equal to:

$$\frac{1}{\xi_1^{(e)}(\frac{z_1}{z_1}) \xi_1^{(e)}(\frac{z_2}{z_2}) \xi_1^{(e)}(\frac{z_3}{z_3})}$$

where $\xi_1^{(e)}$ is the rational function (1.4) for the Jordan quiver, associated to the single equivariant parameter $t_e$ (besides $q$). This reduces the problem to the case $g = 1$, in which case the equivalence of (6.4) and the wheel conditions (3.1) follows from the main results of [Sch12a] and [Neg14]. In more detail, it is straightforward to show that for any symmetric Laurent polynomials $P$ and $R$, we have:

$$\int_{|z_1| \gg |z_2| \gg |z_3|} \frac{(z_1 + z_3) \left( \frac{1}{z_1} - \frac{z_2}{z_1 z_3} \right) (PR)(z_1, z_2, z_3)}{\xi_1^{(e)}(\frac{z_1}{z_1}) \xi_1^{(e)}(\frac{z_2}{z_2}) \xi_1^{(e)}(\frac{z_3}{z_3})}Dz_1Dz_2Dz_3 =$$

$$= \int (PR)(x, xt_e, xq) - (PR)(x, \frac{xt_e}{t_e}, \frac{xq}{q})DX \left( \frac{t_e}{t_e} - t_e \right) \left( \frac{t_e}{q} - \frac{t_e}{t_e} \right) q^{-3}$$

(it suffices to do so when $PR(z_1, z_2, z_3) = \text{Sym} z_1^a z_2^b z_3^c$ for various $a, b, c \in \mathbb{Z}$, in which case the formula above is a straightforward computation involving power series). Clearly, the vanishing of the right-hand side of the equation above (for all symmetric Laurent polynomials $P$) is equivalent to (3.1).

7. The whole Hall algebra of an arbitrary curve

7.1. We now briefly explain how one can adapt and mix the results in the previous two Sections to give a presentation, first of the spherical Hall algebra of an arbitrary smooth projective curve of genus $g$, and then of a much larger subalgebra of the Hall algebra of such a curve (under the technical hypothesis that the cotangent bundle $\Omega_X$ admits a square root). Recall that $X$ is a smooth, projective, geometrically
Let us first consider the spherical Hall algebra of $X$. Observe that because: $|\sigma_e| = q^{-\frac{1}{2}} \Rightarrow |t_e| = q^\frac{1}{2}$ for all $e$, we are in the situation of Assumption B (i.e. Definition 5.2). We can therefore apply the results of Theorem 5.8 and obtain a presentation of $H_{sph}^X$ by generators and relations, the form of which only depends on the multiplicities of the various Weil numbers, i.e. on the potential numerical coincidences between the elements of the multiset $\{t_e, t_e^*\}_{e=1,...,g}$. This leads to the following result. We use the same definitions as in Theorem 1.7 for the generating series $E(z), H(z)$, and we let $X(\gamma_k|x_1,x_2,x_3)$ be defined as in (5.5), but with $\tilde{\zeta}_{ij}(u)$ replaced by $(1-u)\zeta_X(u)$ and $e_i(u), e_j(u)$ both replaced by $E(u)$.

**Theorem 7.2.** Let $\gamma_1, \ldots, \gamma_s$ be the distinct elements of $\{t_1, \ldots, t_g, q/t_1, \ldots, q/t_g\}$, and let $k_1, \ldots, k_s$ stand for their respective multiplicities. Then $H_{sph}^X$ is isomorphic to the algebra generated by $\kappa_{\pm1}, \theta_0, l, \vec{v}_x$ for $l \geq 1, d \in \mathbb{Z}$ subject to the relations (1.7), (1.8), (1.9) and the relations:

$$X(\gamma_k|x_1,x_2,x_3) = 0,$$

for all $i \in \{1, \ldots, s\}$ and $k \in \{1, \ldots, k_i\}$.

### 7.3. Let us keep the same notation as above concerning the curve $X$, and consider the entire Hall algebra $H_X$ (i.e. not just the spherical part). For the most part, we will drop the subscript $X$ from the notation of the Hall algebra, as the curve will be fixed. We will recall a few features of $H$ and refer to [KSV17] for precise definitions and more details. We will assume that $X$ has a theta characteristic, i.e. that $\Omega_X$ admits a square root $\Omega_X^{1/2}$ (see [KSV17, Definition 3.9]). Let $H^+, H^0$ be the subalgebras of functions (6.1) on the stacks of vector bundles and torsion sheaves on $X$, respectively. We will write $H_{r,d}, H_{r,d}^+$, etc, for the graded pieces of the aforementioned Hall algebras, corresponding to sheaves of rank $r$ and degree $d$. There is a canonical decomposition of the commutative algebra $H^0$ according to support:

$$H^0 = \bigotimes_{x \in X(k)} H_x, \quad \text{where} \quad H_x = \mathbb{C}[T_{x,1}, T_{x,2}, \ldots]$$

is generated by primitive elements $T_{x,l} \in H_{0,l,\deg(x)}$, for $l \geq 1$. Next, there is a left action $\cdot$ of $H^0$ on $H^+$ by *Hecke operators*, which is defined as the composition:

$$H^0 \otimes H^+ \overset{m}{\longrightarrow} H \overset{\pi}{\longrightarrow} H^+$$

where $m$ is the multiplication map and $\pi$ is the natural projection map. It satisfies:

$$T_{x,l} \cdot f = [T_{x,l}, f] \quad \forall \ x \in X(k), \ l \geq 1.$$
7.4. An element \( f \in H^+ \) is called cuspidal if the standard coproduct on \( H_X \) (cf. [Kap97, Theorem 3.3]) satisfies:

\[
\Delta(f) \in (f \otimes 1) \oplus (H^0 \otimes H^+).
\]

We will denote by \( H^{\text{cusp}} \) the subspace of cuspidal functions. It is finite-dimensional in any fixed rank \( r \) and degree \( d \). It is known that \( H^{\text{cusp}} \) is a minimal generating subspace of \( H^+ \), and that \( H^{\text{cusp}} \) is stable under the Hecke action of \( H^0 \). Cuspidal functions are also closely related to the following standard construction in the theory of automorphic forms. Let \( \chi : H^0 \rightarrow \mathbb{C} \) be an algebra character. A cuspidal eigenform of rank \( r \) and of eigenvalue \( \chi \) is a nontrivial formal infinite sum \( f = \sum_d f_d \in \prod_d H^{\text{cusp},\text{cusp}}_r \) such that:

\[
h \cdot f = \chi(h) f, \quad \forall h \in H^0.
\]

There is an action of \( \mathbb{C}^* \) on the set of pairs \( (\chi, f) \) as above given by:

\[
t \cdot \chi(h) = t^{\deg(h)} \chi(h), \quad t \cdot \sum_d f_d = \sum t^d f_d.
\]

Let \( \Sigma_r \) be the set of all eigenvalues of cuspidal eigenforms of rank \( r \). The strong multiplicity one theorem says that \( \Sigma_r \) is a finite union of \( \mathbb{C}^* \)-orbits, and that for any \( \chi \in \Sigma_r \) there exists a unique (up to scalar) eigenform \( f_\chi \) of eigenvalue \( \chi \). To a pair of characters \( (\chi, \chi') \in \Sigma_r \times \Sigma_s \) one associates a Rankin-Selberg L-function \( L\text{Hom}(\chi, \chi'; z) \) which is known to enjoy the following properties (see [Laf02, App. B] for the formulas and notation below; recall that our base field is \( k = \mathbb{F}_{q-1} \)):

i) \( L\text{Hom}(t^{-\deg} \chi, \chi'; z) = L\text{Hom}(\chi, t^{\deg} \chi'; z) = L\text{Hom}(\chi, \chi'; t z) \).

ii) If \( \mathbb{C}^* \cdot \chi \neq \mathbb{C}^* \cdot \chi' \) then \( L\text{Hom}(\chi, \chi'; z) \) is a polynomial in \( z \) of degree \( 2(g-1)rs \) and constant term \( 1 \), all of whose zeros are of complex norm \( q^{-1/2} \).

iii) If \( \chi = \chi' \) then there exists a positive integer \( d_\chi \) such that:

\[
L\text{Hom}(\chi, \chi'; z) = \frac{P(z)}{(1 - z^{d_\chi})(1 - (z/q)^{d_\chi})}
\]

where \( P(z) \) is a polynomial in \( z \) of degree \( 2(g-1)rs + 2d_\chi \) and constant term \( 1 \), all of whose zeros are of complex norm \( q^{-1/2} \).

In i) above, \( t^{-\deg} \) refers to the function on \( H^0 \) given by \( h \mapsto t^{-\deg(h)} \). The statements in ii) and iii) concerning the norms of the zeros of \( P(z) \) is known as the generalized Riemann hypothesis and is proved by Lafforgue (see [Laf02, Thm. VI.10]).

7.5. A cuspidal eigenform \( f_\chi \) is called absolutely cuspidal if \( d_\chi = 1 \). We will denote by \( H^{\text{abs}} \) the subalgebra of \( H \) generated by \( H_x \) for \( x \in X(k) \) and the collection of Fourier modes of the absolutely cuspidal eigenforms. We let \( \Sigma^{\text{abs}}_r \subset \Sigma_r \) be the collection of eigenvalues of absolutely cuspidal eigenforms. We will now give a full presentation of \( H^{\text{abs}} \) (see Remark 7.9 for some motivation in considering this specific subalgebra). To this end, for \( \chi \in \Sigma_r \) consider the generating series:

\[
E_\chi(z) = \sum_{d \in \mathbb{Z}} f_d z^{-d} \in H^{\text{abs}}[[z^{\pm 1}]]
\]

where \( f \) is the associated cuspidal eigenform (well-defined up to a scalar). The relevance of Rankin-Selberg L-functions in our context stems from the following: as in [Kap97, Theorem 3.3], there exists a series \( \Psi_\chi(z) \in H^0[[z^{-1}]] \) such that:

\[
\Delta(E_\chi(z)) = E_\chi(z) \otimes 1 + \kappa^r \Psi_\chi(z) \otimes E_\chi(z)
\]
and for any $\chi, \chi' \in \Sigma := \bigsqcup_r \Sigma_r$,

$$E_\chi(z)\psi_{\chi'}(w) = \psi_{\chi'}(w)E_\chi(z) \frac{\operatorname{LHom}(\chi', \chi; qz/w)}{\operatorname{LHom}(\chi, \chi; z/w)}.$$  

The properties above and the shape of the L-functions $\operatorname{LHom}(\chi, \chi; z)$ for absolutely cuspidal characters $\chi$ suggest an isomorphism between $H_{\text{abs}}$ and the shuffle algebras considered in the present paper. To make this precise, consider the following renormalization of $\operatorname{LHom}$, for a pair of characters $(\chi, \chi') \in \Sigma_{\text{abs}}^r \times \Sigma_{\text{abs}}^s$:

$$\zeta_{\chi\chi'}(z) = \begin{cases} 
\theta_{\chi,\chi'}z^{(g-1)rs}\operatorname{LHom}(\chi, \chi'; z^{-1}) & \text{if } \mathbb{C}^* \cdot \chi \neq \mathbb{C}^* \cdot \chi' \\
(1 - \frac{z}{q})(1 - \frac{1}{zq}) z^{(g-1)rs} \operatorname{LHom}(\chi, \chi'; z^{-1}) & \text{if } \chi = \chi'
\end{cases}$$

where

$$\theta_{\chi,\chi'} = \pi(\chi^* \boxtimes \chi')(\Omega_X^{1/2})$$

and $\pi(\chi^* \boxtimes \chi')$ is defined as in [KSV17, Remark 3.3]. Moreover, we have for all $t$:

$$\zeta_{t^{-\deg \chi, \deg \chi'}}(z) = \zeta_{\chi, t^{\deg \chi'}}(z) = \zeta_{\chi\chi'}(t^{-1}z)$$

Using the functional equation for Rankin-Selberg L-functions ([KSV17, Proposition 3.7]), it is straightforward to check that:

$$\frac{\zeta_{\chi\chi'}(z)}{\zeta_{\chi'\chi}(z^{-1})} = q^{(g-1)rs} \frac{\operatorname{LHom}(\chi, \chi'; z^{-1})}{\operatorname{LHom}(\chi, \chi'; qz^{-1})}$$

and that the sets of zeros $\{u_1^{\chi,\chi'}, \ldots, u_{2(g-1)rs}^{\chi,\chi'}\}$ of the polynomials $\zeta_{\chi\chi'}(z)$ satisfy the following relations for all $\chi, \chi' \in \Sigma_{\text{abs}}$:

$$\{u_1^{\chi,\chi'}, \ldots, u_{2(g-1)rs}^{\chi,\chi'}\} = \left\{\frac{1}{qu_1^{\chi',\chi}}, \ldots, \frac{1}{qu_{2(g-1)rs}^{\chi',\chi}}\right\}$$

Special care must be taken with the formula above when $\chi = \chi'$, in which case the zeroes of the polynomial $\zeta_{\chi\chi}(z)$, $\frac{1-z}{1-qz}$, enjoy the symmetry property above.

7.6. Let $\Sigma_{\text{abs}} = \{\chi\}$ be a fixed set of representatives of $\Sigma_{\text{abs}}/\mathbb{C}^*$, and fix a corresponding set $\{f_\chi\}$ of cuspidal eigenforms (choosing different representatives will modify the rational functions defined in the previous Subsection according to (7.1)). To this datum, we associate the quiver $Q_{\chi_{\text{abs}}}$, with vertex set:

$$I = \bigsqcup_{r=1}^{\infty} \Sigma_{\text{abs}}^r$$

and edge set $E$ defined as follows:

- if $(\chi, \chi') \in \Sigma_{\text{abs}}^r \times \Sigma_{\text{abs}}^s$ are distinct, then we draw $(g-1)rs$ arrows going from $\chi$ to $\chi'$
- if $\chi \in \Sigma_{\text{abs}}^r$, then there are $(g-1)r^2 + 1$ loops at $\chi$.

\footnote{Even though the set $I$ is countable, the results in the present paper still hold as stated, because all direct summands of (2.2) and all relations (1.6) only involve finitely many elements of $I$.}
From the definition of $\zeta_{\chi,\chi'}(z)$ and properties ii), iii) and (7.2) of Rankin-Selberg L-functions, we see that the function $\zeta_{\chi,\chi'}(z)$ is, up to the (nonzero) constant factor $\theta_{\chi,\chi'}$, exactly of the form (1.4), upon the specialization of the parameters:

$$\left\{ \zeta_{\chi,\chi'} \right\} \cup \left\{ q/\zeta_{\chi,\chi'} \right\} = \left\{ 1/u \zeta_{\chi,\chi'} \right\}, \quad (\chi, \chi') \in (\Sigma_{\text{abs}})^2.$$

Let us denote by $\mathcal{S}$ the associated (small) shuffle algebra, as in (5.1).

**Theorem 7.7.** The assignment $\chi \mapsto f_{\chi,d}$ for $\chi \in \Sigma_{\text{abs}}$ extends to an algebra isomorphism:

$$\mathcal{S} \xrightarrow{\sim} \mathcal{H}_{+,\text{abs}}.$$

**Proof.** This is essentially [KSV17, Theorem 3.10] (the shuffle kernels differ by the factors $\theta_{\chi,\chi'}$, but this does not affect any of the arguments in the present paper). □

Note that [KSV17] does not single out absolutely cuspidal eigenforms; however, the kernels for the shuffle algebras considered there are, for not absolutely cuspidal eigenforms, different from the ones considered in the current paper.

In combination with Theorem 5.8, we deduce the following presentation for $\mathcal{H}_{\text{abs}}$. Let us set, for $x \in X(k)$ and $\chi \in \Sigma_{\text{abs}}$:

$$E_{\chi}(z) = \sum_{d \in \mathbb{Z}} f_{\chi,d} z^{-d}, \quad T_{x}(z) = \sum_{l=1}^{\infty} T_{x,l} z^{-l}.$$

**Corollary 7.8.** The algebra $\mathcal{H}_{\text{abs}}$ is isomorphic to the algebra generated by elements $\kappa$, $\{T_{x,l} \mid x \in X(k), l \geq 1\}$ and $\{f_{\chi,d} \mid \chi \in \Sigma_{\text{abs}}, d \in \mathbb{Z}\}$ modulo the following set of relations:

- $\kappa$ and $T_{x,l}$ all commute ($x \in X(k), l \geq 1$)
- $\kappa E_{\chi}(z) = q^{(g-1)\chi} E_{\chi}(z) \kappa$, $\chi \in \Sigma_{\text{abs}}$
- $[T_{x}(z), E_{\chi}(w)] = \left( \sum_{l=1}^{\infty} \chi(T_{x,l}) \left( \frac{w}{z} \right)^{l} \right) E_{\chi}(w)$
- $E_{\chi}(z) E_{\chi'}(w) \zeta_{\chi,\chi'} \left( \frac{w}{z} \right) = E_{\chi'}(z) E_{\chi}(w) \zeta_{\chi,\chi'} \left( \frac{z}{w} \right)$

and the collection of cubic relations determined by setting (5.5) equal to 0.

In particular, all the relations satisfied by Eisenstein series attached to absolutely cuspidal eigenforms are a consequence, in addition to the usual functional equation, of certain cubic relations (and the structure of these only depend on the multiplicities of the zeros of the corresponding Rankin-Selberg L-functions).

**Remark 7.9.** One might wonder if there is an analogue for $\mathcal{H}_{X}$ of the generic spherical Hall algebra. In the context of a quiver $Q$, such an analogue takes the form of a quantized graded Borcherds algebra, whose Cartan datum encodes the dimensions of the spaces of cuspidal functions in $\mathcal{H}_{Q}$, see [BS19]. By a recent theorem of H. Yu, [Yu18], the dimensions of the spaces of absolutely cuspidal functions in $\mathcal{H}_{X}$ are given by some universal polynomials in the Weil numbers of $X$ (depending
on the rank of the sheaves considered). This suggests that it is $H_{abs}^X$ rather than $H^X$ which admits a natural generic form, and that the classical limit of such a generic form of $H_{abs}^X$ would be a Lie algebra in the category of $GSp_{2g}(C)$-modules. However, we do not know at the moment how to encode the zeros of the various Rankin-Selberg $L$-functions (which account for the “quantum” parameters).

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