An optical solution of Olbers’ paradox

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ABSTRACT

Shown is that contrary to common intuition, even an arbitrarily weak attenuating mechanism is sufficient to make the background sky quite dark independently of the size of the universe and the Hubble expansion. Further shown is that such an attenuation already exists in the wave nature of light due to entrapment and diffusion from successive diffractions. This is a fundamentally new mechanism to physics, as illustrated by application to the solar neutrino attenuation, galactic dark matter and gamma ray bursts problems. It not only provides a big bang-like cutoff, but also appears to explain the appearance of primeval, metal-deficient galaxies at high redshifts, without deviating from the Olbers’ premise of an infinite universe.

1. Introduction

This historical paradox highlighted by Olbers almost two centuries ago (Olbers 1826) holds that the night sky should have been as bright as the sun’s disk because we should encounter rays from a stellar surface in any direction we look. More particularly, the argument holds that the stellar light must cover an increasing area \( \propto r^2 \) as its distance \( r \) from its source increases, its brightness \( J \) diminishing as \( J_0/r^2 \), where \( J_0 \) is the stellar surface luminosity. This is the well known inverse-square law for light, and takes only the geometrical spread in 3-dimensional space into account. According to the paradox, as we look at regions of the sky where the stars are very far, they should each appear dimmer, but a given solid angle would cover \( \propto r^2 \) stars, assuming a uniformly populated infinite universe. The luminosity of the sky should therefore be \( \propto (J_0/r^2) \times r^2 = J_0 \), meaning that the background sky should be so bright that the stars should be indistinguishable against it. I present two fundamental results, first, that an infinitesimally small attenuation \( \sigma > 0 \), barely enough to change the propagation law to the form \( J_0 e^{-\sigma r}/r^2 \), suffices to solve the paradox, and second, that such an attenuation happens to be inherent in the wave nature of light.
The results are unintuitive because almost any kind of attenuation leads to this form, and absorption and scattering by dust have been considered inadequate in the past. Harrison argued (Raychaudhuri 1979) that the dust would eventually attain thermal equilibrium with the stars and effectively stop absorbing more energy. Since the radiation too would eventually reach equilibrium, scattering of itself cannot solve the paradox either. The standard model offers three plausible solutions that the Hubble flow causes the light to lose energy well in excess of the inverse-square attenuation, that the universe is too young for thermal equilibrium, and that the universe is as such finite. Wesson has shown that the first would actually contribute less (Wesson et al. 1987), so finiteness of the universe, in both age and extent, is currently believed to be the reason for the darkness of the night sky. I shall show that a similar cutoff occurs because of the inherent attenuation, that it necessarily leads to a further mechanism of spectral modification that would make the most distant galaxies appear primeval, which is again known and currently attributed to the big bang. In both cases, I exploit the smallness of the mechanisms needed for the respective effects. The approach is only possible because of the exponential factor in the attenuated propagation law, but more importantly, it demonstrates that the small orders routinely left out in the approximations applicable on earth could be significant physics on the cosmological scale. To further emphasise the error in letting approximations dictate our reasoning, I shall show that an even smaller order of attenuation due to the same mechanism would explain the missing solar neutrino flux, currently attributed to neutrino oscillation, and could mean new insight of a fundamental kind in the physics of weak interactions.

Unlike the dust effects, the attenuation of present concern depends only on the presence and not the thermal state or other properties of matter, and is therefore immune to thermal equilibrium. It results from a diffuse entrapment of radiation due to successive diffraction and gravitational deflections that continually turn a portion of the wavefront. While the occurrence of successive deflections is known, for example, in Laue diffraction theory, its implications to astrophysics have not been examined in previous treatments, such as in the context of extinction by dust (Spitzer Jr 1978, p149-153), possibly because one ordinarily thinks of diffraction as carrying wave energy around obstructions, increasing rather than diminishing the net power flow. This would be the case in a hypothetical universe where the sources are assumed to be behind a plane of diffracting obstructions, but the notion does not really extend to the three-dimensional universe involved in the paradox, where the stars themselves are the obvious obstructions to each other’s light. The enhancing property turns out to be intransitive because the successive deflections can then keep a fraction of the radiation from ever reaching its original destination.

Another reason why this result was unobvious is that Fraunhöfer’s approximation is invariably assumed because of the immense distances involved. A solution to Olbers’ paradox
then appears to be ruled out because the loss of the direct rays from a star due to *en route* diffraction by an angle say $\theta$, would be made up by the rays from another star behind the diffractor at an angle $-\theta$. While the argument is somewhat weaker than enhancement, it reveals the error in our past intuition, because the probability $p(r_s)$ of finding a compensating star at the same or less distance $r_s$ behind the diffractor depends on $r_s^2/r^2$, which is certainly less than unity. The approximation assumes that both the sources and the observer are infinitely far from the diffracting object, i.e. $r_s \to \infty$ and $r \to \infty$, in which case the ratio does not matter. These conditions are, however, applicable only in the vicinity of a given diffractor, and cannot be legitimately applied when the obstructing objects are distributed over the same scale of distances as the sources. In the presence of an attenuation $\sigma$, the distance $r_s$ matters because the compensating source becomes more likely to be farther by the triangle theorem, and therefore likely to be dimmer.

In both the real universe and Olbers’ scenario, therefore, the light from a distance source located on a geometrically unobstructed straight line from us does get diminished by diffraction due to obstructions lying off the straight line. This loss would be compensated, as in the Fraunhöfer case, if enough diffracted light from elsewhere could rejoin the straight line path. Traditional wisdom suggests that the repeated deflections be treated as a random walk leading to a slow diffusion of the photons, which would not yield a net reduction of the average luminosity. However, there are problems with this view, because while it is reasonable to assume that the diffracting objects are randomly distributed, their gross motions are not random and are quite slow in relation to the interstellar distances.

What we have is an essentially static pattern lacking the temporal randomisation of direction needed to qualify it as random walk. More particularly, all efforts to simulate thermalisation with fixed dynamical models have consistently led to persistent oscillatory states, since the very first attempt in 1953 by Fermi, Pasta and Ulam (Fermi et al. 1965; Fillipov et al. 1998), showing that mere complexity of dynamical structure is not enough for assuming diffusion. Moreover, any static pattern necessarily contains circulations, which might not only explain the FPU problem, but in our case, trap some of the light virtually forever. Harrison’s argument cannot be applied to such states because the circulations would be centrally dependent on the individual sources, and each of which presumably has a finite lifetime even in Olbers’ scenario. We do expect most of the “trapped” energy to eventually diffuse out, but we have no basis to assume that all of it will. Rather, we can expect a portion of the energy to get absorbed or turn into matter, given that the attenuated propagation law is already characteristic of the Klein-Gordon equation $(\nabla^2 + \sigma^2 \partial^2/\partial t^2) \psi = 0$. The latter is simply the quantum version of the relativistic argument that radiation when retarded to effective speeds less than $c$ should exhibit rest mass, and relates to our treatment of the solar neutrino problem. We can thus be certain of a net attenuation $\sigma > 0$, and this suffices, as
shown next, to solve the paradox.

2. Solution of the paradox

As stated, our key argument is that any attenuation whatsoever, so long as it operates on the large scale, solves the paradox independently of the standard model. We seek an attenuation $\sigma$ (dB m$^{-1}$), such that the propagation law for light changes from $r^{-2}$ to $r^{-2} e^{-\sigma r}$, which reduces the background brightness of the sky to

$$J_{\sigma} = \int_{\infty}^{0} J_0 e^{-\sigma r} dr = J_0/\sigma. \quad (1)$$

While the integral is well known, the solution is not immediately obvious, largely because the value superficially resembles $J_0$, making it look as if we need a very large $\sigma$, of the order of at least 130 dB $\equiv 10^{13}$ (Roach and Gordon 1973, p.24-25) to get a dark background sky, and hitherto seemed impossible without the big bang theory.

The resemblance is misleading because the unattenuated $J$ differs in dimensions from $J_0/\sigma$, as the latter has the dimensions of luminosity $\times$ distance. For legitimate comparison, we must express $J$ in exactly same dimensions, hence in the statement of the paradox, the “Olbers luminosity” $J$, which one may informally think of as the brightness of the sun’s disk, corresponds not to $\sigma = 1$ but to $\sigma = 0$, i.e.

$$J \equiv \lim_{\sigma \to 0} \int_{\infty}^{0} J_0 e^{-\sigma r} dr = \lim_{\sigma \to 0} J_0/\sigma, \quad (2)$$

which is infinity. Conversely, when we calibrate with respect to the sun’s disk, the attenuated background sky should be infinitesimally dim, for any $\sigma > 0$. To appreciate why this should be so, consider how bright the background needs to be in order to match the sun’s disk. In Olbers’ argument, whichever direction we look in, our line of sight must meet a stellar surface and at any finite angular resolution $\theta$, the brightness should correspond to the number of stars included within the solid angle $\theta$. Accordingly, the observed brightness would be $J_{\sigma} \equiv J_0 e^{-\sigma r}$ along that direction, and that set of stars then needs to be $e^{\sigma r}$ times brighter than the sun in order to match $J_0$. Clearly, it does not matter if $\sigma$ is very small; as long as it is nonzero, we have an effective cutoff of the observable universe at

$$r_n = \sigma^{-1} \log(J_0/J_n) \quad (3)$$

for a finite $\sigma$, where $J_n$ represents the background noise in the measuring process. By eq. (3), the standard model cutoff, corresponding of an age of the universe of $\sim 15$ Gy, is equivalent
\[ \sigma \approx 130 \text{ dB}/15 \text{ Gy} = 9 \times 10^{-25} \text{ dB m}^{-1}. \] (4)

This is far less than what one might naively expect from the form of \( J_\sigma \) (eq. 1), and amounts to a mere \( 8 \times 10^{-9} \text{ dB per light-year} \). As remarked in the introduction, it is the smallness of the attenuation needed to explain the apparent big bang cutoff that makes it impossible to rule it out on the basis of terrestrial physics. Furthermore, eq. (4) is as yet an upper bound, because we ignored the diffraction from nonluminous bodies as well as the gravitation of these and the visible objects, which contribute substantially to the deflections. We also ignored the dust extinction in our own neighbourhood (Roach and Gordon 1973, ch.4), to which Harrison’s argument again does not apply, and the impact of the Hubble flow, which together account for a good part of the 130 dB. Note that Harrison’s argument remains valid for dust on the large scale, and the subtlety that overcomes it for the diffractive scattering, described in the next section, is the coherence and dependence of the diffuse circulatory states on their respective sources.

3. Scattering approximation

Eq. (2) establishes our principal point that only an extremely small attenuation is needed to reproduce the big bang cutoff. It remains to be shown that such an attenuation is indeed possible and likely from successive diffraction. As explained in §1, we are concerned with multiple interstellar hops that are each much larger than stellar diameters, so that the Fraunhofer theory can be applied to the individual diffractions. We may more particularly treat the stars as point objects, representing the diffraction by an angular spreading function \( f(\phi, \delta) \), \( f \geq 0 \forall \{\phi, \theta\} \), applicable to a parallel beam of incident light, giving

\[ J(\phi, \delta) = J_0 f(\phi, \delta) \text{ and } \int_{\phi=0}^{2\pi} \int_{\delta=0}^{\pi} f(\phi, \delta) \, d\phi \, d\delta = 1, \] (5)

for the diffracted light, \( \phi \) being the azimuthal angle around the beam axis and \( \delta \), the angular spread from the axis. We shall now examine the special treatment necessary to account for successive diffractions.

The spread function \( f \) is rather like the differential scattering function \( \sigma(\Omega) \) of the Rutherford model (Goldstein 1980, §3-20), but several important differences must be noted. Firstly, we are concerned with continuous wavefunctions, not particles, so that the “scattering” itself is not at all probabilistic; the only probability inherent in the model is the stellar distribution. Secondly, the scattering function \( f \) depends on the \( \lambda/D \) ratio, where \( \lambda \) is of
course the wavelength and $D$, the mean stellar diameter. This is treated in more detail in terms of Fresnel-Kirchhoff theory in Appendix A. More importantly, the result of interest is the net attenuation and not a cross-section of interaction. It is common to use $\sigma$ for both, but the second notion is not of concern here. We shall now formalise the cumulative forward loss and the nonzero “backscatter” occurring at each encounter.

The second of eqs. (5) represents the conservation of energy at each diffractive encounter, since it means that

$$\int_{\phi=0}^{2\pi} \int_{\delta=0}^{\pi} J(\phi, \theta) \, d\phi \, d\delta = J_0. \quad (6)$$

We start by applying $f$ to a bundle of rays incident on a first star. Since the rays spread as if from a point source, they acquire a $1/4\pi r^2$ spreading loss, even if we had started with a parallel bundle, which would be equivalent to a planar wavefront. A second encounter after a distance $r_1$ therefore yields

$$J(\phi, \theta) = \frac{J_0}{4\pi r_1^2} \int_{\delta=0}^{\theta} f(\phi, \theta - \delta) f(\phi, \delta) \, d\delta$$

$$\equiv \frac{J_0}{r_1^2} g_1(\phi, \theta) \quad (7)$$

where $g_1$ denotes the first cumulative integral

$$g_1(\phi, \theta) = \frac{1}{4\pi} \int_{\delta_1=0}^{\theta} f(\phi, \theta - \delta_1) f(\phi, \delta_1) \, d\delta_1. \quad (8)$$

We thus obtain the recursive set of integrals

$$g_0 \equiv f \quad \text{and} \quad g_n(\phi, \theta) = \frac{1}{4\pi} \int_{0}^{\theta} f(\phi, \theta - \delta) g_{n-1}(\phi, \delta) \, d\delta \quad \text{for } n = 1, 2, \ldots \quad (9)$$

describing $n$ successive encounters as

$$J(\phi, \theta) = J_0 g_n(\phi, \delta) \prod_{j=1}^{n} \frac{1}{r_j^2} \approx J_0 g_n(\phi, \delta) \bar{r}^{-2n} \quad (10)$$

where $r_j$ are intervening distances, and $\bar{r}$, the mean distance between stars. Integrating eq. (10) over $\phi$, and summing over the contributions from one or more encounters, we obtain the total “gain” at $\theta$ from the initial beam as

$$g(\theta) = \frac{g_0(\theta)}{\bar{r}^2} + \frac{n_1 g_1(\theta)}{\bar{r}^4} + \frac{n_2 g_2(\theta)}{\bar{r}^6} + \ldots$$

where $g_n(\theta) \equiv \int_{0}^{2\pi} g_n(\phi, \theta) \, d\phi. \quad (11)$
Here, $n_j > 1$ denote the mean number of parallel paths corresponding to the range of $\phi$ for $j$ encounters, and $g_j(\theta)$ are the per-path contributions. It should be at least intuitively clear that $n_j$ would be an increasing combinatorial function of $j$, offsetting the increasing geometrical attenuation from the denominator ($\bar{r}^{-2j}$).

Eq. (11) reveals an interesting property of the successive diffractions: that the $g_j(\theta)$ increase with $j$ for precisely the reason that non-paraxial angles are ordinarily ignored in terrestrial optics – at large $j$'s, small incremental angles $\delta$ become significant in the integrand, so that

$$\lim_{j \to \infty} g_j(\theta) \equiv \lim_{\delta \to 0} \left[ \frac{1}{2\pi} \int f(\phi, \delta) d\phi \right]^{\theta/\delta} \approx 1$$

(12)

as the integral converges to the zero-th order beam. Eq. (12) represents the observation that a succession of small diffraction angles adds up to a large total deflection with significant amplitude, since the component deflections are paraxial. Three encounters each of $\pi/3$, for example, suffice to make a back contribution, and the direct backscatter $g_0(\theta \approx \pi)$ suffers no $1/\bar{r}^2$ attenuation as it involves no intermediate hops at all. As a result, nonzero total reflection is generally guaranteed and the increasing $n_j$ also partly compensate for the $1/\bar{r}^{2j}$ loss. $g_0(\pi)$ is generally ignored in terrestrial optics because the $\cos^2(\theta)$ factor is more pronounced over small distances; this too is inappropriate over the cosmological scale of distances. As stated in §1, motions of the surface of the diffracting star, although many orders larger than the wavelength of light, is not significant in the present context, principally because $f$ represents angular distribution of the radiant power and is unaffected by phase fluctuations.

The presence of a net backscatter $g(\pi)$ means that light from sources in the observer's own neighbourhood will tend to lighten the sky. This is an expected effect of scattering due to dust, for example: Roach and Gordon estimate that for an observer near the centre of our galaxy, the night sky should be relatively opaque (Roach and Gordon 1973). This does not contradict either the paradox or Harrison's argument, but the diffractive backscatter potentially does because it would be spread over cosmological distances. Our result remains intact because the backscattered light would itself be again subject to repeated diffractions and suffer the same attenuation. The totality of "local" sources we need to consider would cumulatively increase as $\bar{r}^3$, but gets overcome by the exponential attenuation $e^{-\sigma r}$ within a limited $r$. 
4. Applications

As mentioned in §1, our mechanism has several interesting applications, beginning with the manifestation of rest mass of the photons because of the Klein-Gordon form of the attenuated propagation law. This suggests that our trapped circulatory photon states could be responsible for a portion of the dark matter especially if the universe were very old. A related observation is that the density of circulatory states would be proportional to the density of particulate matter, because the deflections would become available at smaller distances in dense neighbourhoods. We would expect the attenuation to be greater within galaxies than in the sparser regions between them, and the resulting “optical dark matter” distribution to be consistent with the overall galactic dark matter indicated by their rotation profiles, as the circulatory states would presumably add up with radial distance from the galactic centre. At present, we do not know how to estimate this optical component of dark matter.

The attenuation might also explain the intensity of gamma ray bursts from galactic regions. For a given aperture between the stars and other objects surrounding the line of sight, the basic principles of diffraction dictate that gamma rays would suffer far less diffractive loss than visible light because of their shorter wavelengths. The observed gamma ray burst intensities should therefore be much more than we might expect from the visible brightness. While the reasoning appears to be in the right direction, it remains to be substantiated.

Another high energy scenario consistent with our theory is the known attenuation of the solar neutrino flux, which, as mentioned in §1, is currently attributed to quantum oscillation. Given that the mass of the sun is $M_\odot \approx 2 \times 10^{30}$ kg and the average mass of a nucleon, $1.67 \times 10^{-27}$ kg, we estimate there are $1.2 \times 10^{57}$ nucleons within the sun’s volume $\approx 1.4 \times 10^{27}$ m$^3$, which allows roughly $1.2 \times 10^{-30}$ m$^3$ per nucleon. This means that a neutrino would encounter a nucleon every $1.06 \times 10^{-10}$ m, or up to $6.6 \times 10^{18}$ times from the centre to the surface. This mean free path between nucleons is the de Broglie wavelength for a particle of about 12 keV, so some diffraction appears to be inevitable. The observed attenuation, about two-thirds, must result from $O(10^{18})$ compoundings, assuming, as an approximation, that all neutrinos originate from near the centre; this yields $\sigma \approx \exp[10^{-18} \log(2/3)] \approx \exp(4 \times 10^{-19})$ or $10^{-19}$ dB per encounter, as the diffractive attenuation needed to explain the observed loss. Once again, it is the smallness of the necessary attenuation that makes it plausible as the cause, and we speculate that this could be the wave-theoretic reason for the breaking of symmetry that leads to weak interactions.

Our last application is to the appearance of primeval galaxies at high redshift factors $z$, and comes from observing that the selective absorption of certain frequencies by matter close to the line of sight should trigger diffractive loss $\sigma' \approx \sigma$ at the absorbed frequencies. This
should result in a subtractive spectral modification of the light from more distant matter, and the spectral subtraction by our own galactic neighbourhood, which is well evolved and metal-rich, should make more distant galaxies appear metal-deficient and primeval. Only an extremely small $\sigma'$ is needed, once again, to explain the almost total metal-deficiency of the primeval galaxies. As $\sigma'$ must result from selective absorption by obstructions that must be transparent at other frequencies, i.e. from a different set of obstructions, $\sigma'$ cannot be identical to $\sigma$. But it would be of the same order, leading to a similar cutoff $r'_n \approx r_n$ (eq. 3), because denser regions of dust that invariably surround opaque bodies are likely causes of selective absorption. Importantly, we may argue that the most distant galaxies can appear mature only if there were a uniform mix of young and old galaxies all the way right up to our own galaxy, but this would contradict the standard model. This does not mean unexpected support for the big bang theory, however, because we could merely be in a relatively small region of the Olbers’ universe, of radius at least $r_n$, that formed at or before the big bang indicated by the Hubble redshift.

I thank Shai Ronen, Gyan Bhanot, A Joseph Hoane and most of all, Bruce Elmegreen, for valuable discussions in the context.

A. Point diffraction

The usual treatment for diffraction due to a small region concerns apertures in an opaque screen, but our stars are in effect obstructions in an otherwise empty space. If $U_s$ represents the diffracted amplitude due to a star, and $U_a$, that due to an aperture of the same size, we have by Babinet’s principle that $U_s = U_0 - U_a$, where $U_0$ represents the unobstructed incident wavefront. In holography, for instance, we can use this theorem to compute the resulting interference between the direct rays ($U_0$) and the diffracted rays ($U_s$) from a point obstruction. In the present context, however, we are not interested in the local interference patterns, but only in the direction of the power flow, so the interference between direct and diffracted wavefronts is irrelevant. We do sum over rays going in the same direction $\theta$ in eq. (11), but $\bar{r} \gg \lambda$, and in any case, there is enormous thermal jitter from our stellar obstructions, which would randomise the phase between the summed paths, hence interference effects can again be ignored, even if our initial rays happened to be quite coherent.

We do need to consider phase within narrow bundles of rays, however, in order to obtain the wavelength dependence of the diffraction. Under the far-field assumption $\bar{r} \gg D$, $D$ being the mean diameter of our stellar obstructions, the incident wavefronts before diffraction
would be not only almost planar, but likely to be temporally coherent across a sizeable fraction of the stellar surface. We shall limit ourselves to pure Fourier components, avoiding questions of incoherence, because issues of decoherence and wavepacket dispersal can be considered in terms of these components. With these assumptions, we may apply Kirchhoff’s condition that the conditions in a stellar diffracting region $A$ will not be much affected by the surrounding matter, which yields the usual Fresnel-Kirchhoff diffraction formula (Born and Wolf 1959, §8.3.2)

$$U(\theta) = -\frac{i}{2\lambda} \int_A [\cos(\hat{r}) - \cos(\hat{s})] \frac{e^{i k (r+s)}}{rs} dS,$$

(A1)

where $s$ and $r$ are distances to the source and to the point of observation, respectively. We cannot, however, afford to replace the cosines with a single $\cos(\theta)$ factor as in the traditional theory, since this would be proper only for paraxial rays. To correctly compute $U$ at large diffraction angles $\delta$, we need to retain the two-cosine form in our Fraunhöfer approximation, obtaining

$$U(\theta) = -\frac{i}{2\lambda rs} \int_A e^{ik(r+s)} dS$$

$$\equiv -\frac{i \omega}{2\pi c} \frac{[\cos(\hat{r}) - \cos(\hat{s})]}{rs} \int_A e^{ik(\xi+\eta)} dS,$$

(A2)

$\xi$ and $\eta$ being the direction cosines as in prior theory. A remaining problem is, of course, that the above formulae are meant for diffraction from an aperture, and not an opaque disk presented by each star, but Babinet’s principle assures us that the resulting $f$ will be identical. Assuming our stars to be generally circular, we find that the integrand would have the form $2 J_1(ka\theta)/ka\theta$, where $k \equiv 2\pi/\lambda$ and $a$ is the radius of the aperture, yielding

$$f(\theta) \sim |U(\theta)|^2 = \frac{\omega^2 [\cos(\hat{r}) - \cos(\hat{s})]^2}{(2\pi c)^2} \left[ \frac{2 J_1(ka\theta)}{ka\theta} \right]^2,$$

(A3)

since the $(rs)^2$ factor is separately accounted for by $\bar{r}$ in §3.

As a result, $f(\theta)$ is non-zero almost everywhere, over the entire range of $\theta$, which includes $\pi$. This “backscatter” is a reflection from space, due to the local perturbation in the impedance of space caused by the off-axis diffracting star, and is definitely nonzero for any nonzero solid angle around $\theta = \pi$. The angular spread is more strongly governed by the phase factor than by $\omega$, so that a given gap between nearer obstructions will be more open to gamma rays than to visible light, as remarked in §1.

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