A COMPLETE SCENARIO ON NODAL RADIAL SOLUTIONS TO THE BREZIS NIRENBERG PROBLEM IN LOW DIMENSIONS

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Abstract. In this paper we consider nodal radial solutions of the problem
\[
\begin{cases}
-\Delta u = |u|^{2^*-2}u + \lambda u & \text{in } B, \\
u = 0 & \text{on } \partial B
\end{cases}
\]
where \(2^* = \frac{2N}{N-2}\) with \(3 \leq N \leq 6\) and \(B\) is the unit ball of \(\mathbb{R}^N\). We compute the asymptotics of the solution \(u\) as well as \(\|u\|_\infty\), its first zero and other relevant quantities as \(\lambda\) goes to a critical value \(\bar{\lambda}\). Also the sign of \(\lambda - \bar{\lambda}\) is established in all cases. This completes an analogous analysis for \(N \geq 7\) given in [12].

1. Introduction

In a celebrated paper, Brezis and Nirenberg [7] proved the existence of positive solutions of the following Sobolev critical problem,
\[
\begin{cases}
-\Delta u = |u|^{2^*-2}u + \lambda u & \text{in } B, \\
u = 0 & \text{on } \partial B
\end{cases}
\]
where \(2^* = \frac{2N}{N-2}\) with \(N \geq 3\), \(B\) is the unit ball of \(\mathbb{R}^N\) centered at the origin and \(\lambda\) is a real positive parameter. In [7] it was proved that

**Theorem A.** The problem \((1.1)\) admits a positive solution if and only if \(\lambda \in (0, \mu_1)\) in dimension \(N \geq 4\), or respectively \(\lambda \in (\mu_1/4, \mu_1)\) in dimension \(N = 3\), where \(\mu_1\) is the first eigenvalue of the Laplacian with Dirichlet boundary conditions.

It is virtually impossible to give an exhaustive bibliography of all works inspired by this paper, so we will limit ourselves to mention only those related to the cases that interest us.

In this paper we consider nodal radial solutions to \((1.1)\). For any fixed \(m \in \mathbb{N}^+\) denote by \(u^m_\lambda\) a classical radial solution to \((1.1)\) corresponding to the parameter \(\lambda\), with \(m\) nodal regions (i.e. \(m-1\) internal zeros) and by \(\mu_m\) the \(m^{th}\) radial eigenvalue of the Laplacian in \(B\) with Dirichlet boundary conditions.

Here another new phenomenon appears, involving the dimension \(N = 7\).

**Theorem B.** In dimension \(N \geq 7\), problem \((1.1)\) admits at least one radial solution \(u^m_\lambda\) with \(m\) nodal zones for \(\lambda \in (0, \mu_m)\). In dimension \(N \) between \(3\) and \(6\), there exists \(\lambda^* = \lambda^*(N,m)\) such that problem \((1.1)\) does not admit any radial solution with \(m\) nodal zones for \(\lambda \in (0, \lambda^*)\) and \(\lambda > \mu_m\).

We refer to [2] for a proof of this facts; see also Figure 2 in the same paper.

Theorem B gives a complete answer to the problem of the existence of nodal radial solutions to \((1.1)\) in the ball.

A first analysis of the associated ordinary differential equation ([2], [3]) brings into light another critical value, say it \(\bar{\lambda} = \bar{\lambda}(N,m)\), with the properties:

(i) there is a sequence of radial solutions with \(m\) nodal zones such that \(\|u^m_\lambda\|_\infty \to \infty\) as \(\lambda \to \bar{\lambda}\).

(ii) if \(\|u^m_\lambda\|_\infty \to C\) as \(\lambda \to \bar{\lambda}\), then \([2, 8, 10]\)

- \(\lambda \to \bar{\lambda} = \left(\frac{2m-1}{2m}\pi^2\right)^{\frac{N-2}{2}}\) in dimension \(N = 3\),
- \(\lambda \to \bar{\lambda} = \mu_{m-1}\) in dimension \(N = 4, 5\),
- \(\lambda \to \bar{\lambda} \in (0, \mu_{m-1})\) in dimension \(N = 6\),
- \(\lambda \to 0\) in dimension \(N \geq 7\).

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seems to depend upon the dimension. Indeed it was proved in [13] that
\[ \lambda \to \infty \]
\[ r_{i,\lambda} \to 0, \]
\[ M_{i,\lambda} \to \infty, \]
\[ -\Delta U = U^{2^*} \quad \text{in } \mathbb{R}^N, \]
\[ U > 0 \quad \text{in } \mathbb{R}^N, \]
\[ U(0) = 1. \]

We shall refer at this number \( \bar{\lambda} \) as a *concentration value* and we recall that in the last decades there has been a lot of work about the properties of the solutions \( u_\lambda \) near \( \bar{\lambda} \).

If \( N \geq 7 \) the radial solutions have a multiple blow up at the origin. The asymptotic profile in this case has been studies first in [17] (concerning solution with two nodal zones) and then completed in [12] in the general case.

In order to state this result, let us introduce the following notations,
\[ M_{i,\lambda} = \sup_{r \in (r_{i-1,\lambda}, r_{i,\lambda})} |u_\lambda^m(r)| \]
and
\[ r_{i,\lambda} \in (0, 1], \ i = 1, \ldots, m-1, \]
be the nodal radii of \( u_\lambda^m \), i.e. \( u_\lambda^m(r_{i,\lambda}) = 0 \).

We have,

**Theorem C.** *(see [12]).* Let \( u_\lambda^m \) be a radial solution to \((1.1)\) with exactly \( m \) nodal zones in dimension \( N \geq 7 \). When \( \lambda \to 0 \) we have
\[ M_{i,\lambda} \to \infty, \quad r_{i,\lambda} \to 0, \]
\[ \frac{1}{M_{i,\lambda}} u_\lambda^m \left( \frac{x}{M_{i,\lambda}^{2/7}} \right) \to (-1)^{i-1} U(x) \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \]
for \( i = 1, \ldots, m \), where \( U \) is the unique solution to \((1.2)\). Furthermore the asymptotic rates of \( M_{i,\lambda} \) and \( r_{i,\lambda} \) with respect to \( \lambda \) are computed.

From this theorem we get that the solution \( u_\lambda^m \) exhibit a *tower of bubbles*, i.e. \( m-1 \) nodal lines collapse to the origin and in any nodal region the solution blows-up like the scaled bubble \( U \).

In this paper we are interested in the remaining open cases \( N = 3, 4, 5, 6 \) and our aim is to prove the corresponding estimates as in [12]. In particular we see that the tower of bubbles behaviour in Theorem C is characteristic of the high dimensions, \( N \geq 7 \). When \( N < 7 \) indeed the solution blows up in the origin as \( \lambda \to \bar{\lambda} \) like the scaled bubble \( U \), but stays bounded in the interval \((r_{1,\lambda}, 1)\). Hence it is needed to analyze only the first zero \( r_{1,\lambda} \), that we denote hereafter by \( r_\lambda \), and the first critical critical value in \((r_{1,\lambda}, 1)\), say \( M_{\lambda} \), since all the others converge to the subsequent zeroes and critical values of the limit problem.

In particular we are interested in the following questions:

**Question 1.** What about the asymptotic behavior of the solution \( u_\lambda^m \) as \( \lambda \to \bar{\lambda} \)?

The answer to this question concerns the computation of the rate of \( \|u_\lambda^m\|_\infty \), \( M_{i,\lambda} \) and \( r_{i,\lambda} \). As we said before it is known that \( \|u_\lambda^m\|_\infty \to +\infty \) but the rate (in term of \( \lambda - \bar{\lambda} \)) is not known as well as for the other quantities.

**Question 2.** What about the sign of the difference \( \lambda - \bar{\lambda} \)?

The motivation of the second question comes from the (quite natural) conjecture that \( \bar{\lambda} \) coincides with the threshold value for existence, that is \( \lambda^* \). Seemingly it is not the case, or better also this issue seems to depend upon the dimension. Indeed it was proved in [13] that \( \lambda - \bar{\lambda} < 0 \) for \( N = 5 \) while \( \lambda - \bar{\lambda} > 0 \) when \( N = 4 \) and in this last case \( \lambda = \lambda^* \), see [1]. Further, when \( N \geq 7 \), \( \lambda > \bar{\lambda} = 0 \) trivially. The problem is still open in the remaining dimensions, namely \( N = 3 \) and \( N = 6 \) and we will give an answer. This point will be discussed in detail later.

We want to stress that, while previously all the results on the asymptotic behavior of \( u_\lambda^m \) used the Emden-Fowler transformation, here a blow-up analysis jointly with some integral identities is used.
This different approach allows us to give an alternative proof of some known facts. We also point out that it has been successfully used in studying the properties of solutions to the classical Yamabe problem in a series of papers, see for example ([10] [11] [22]), and the references therein.

Now we consider separately the different dimensions, since each of them has different features.

### The case \( N=3 \)

In this case, the only known facts are that \( \lambda \to \tilde{\lambda} = (\frac{2m-1}{2} \pi)^2 \) (see introduction in [2]) and, if \( m = 2 \), \( r_\lambda \to \tilde{r} = \frac{4}{5} \), \( u^m_\lambda \to 0 \) in \((0, 1)\) (see [13]). In next result we compute the asymptotics of \( u^m_\lambda \) and we show that \( u^m_\lambda \), up to a suitable normalization, converges to the radial eigenfunction \( z \) of \(-\Delta\) in the annulus \((\tilde{r}, 1)\). This gives that the nodal radii \( r_\lambda \) converge to the zeros of \( z \) as well as for the critical values.

Lastly we show that \( \lambda \to \tilde{\lambda} \) from above.

**Theorem 1.1.** Let \( u^m_\lambda \) be any radial nodal solution to (1.1) with \( m > 1 \) nodal zones in dimension \( N = 3 \) and let \( r_\lambda \) be its first zero in \((0, 1)\). The occurrence \( \| u^m_\lambda \|_{L^\infty(B)} = u^m_\lambda(0) \to \infty \) can happen only if \( \lambda \to \tilde{\lambda} = (\frac{2m-1}{2} \pi)^2 \) from above. Moreover, as \( \lambda \to \tilde{\lambda} \) we have

\[
\begin{align*}
\| u^m_\lambda \|_{L^\infty(B)} &= \sqrt[5]{\frac{(2m-1)\pi}{8m-6}} \left( 1 + o(1) \right) \frac{1}{\sqrt{\lambda - \tilde{\lambda}}}, \\
(1.5) \quad r_\lambda &= \frac{1}{2m-1} + \frac{8(m-1)}{\pi^2(2m-1)^3} (1 + o(1))(\lambda - \tilde{\lambda}), \\
(1.6) \quad u^m_\lambda(x) \to 4 \sqrt[3]{\frac{2(4m-3)}{2m-1}} V((2m-1)x, 0) \text{ in } C^1_\text{loc} \left( \overline{B} \setminus \{0\} \right), \\
(1.7) \quad \lambda - \tilde{\lambda} = 4 \sqrt[3]{\frac{2(4m-3)}{2m-1}} V((2m-1)x, 0) \text{ in } C^1_\text{loc} \left( \overline{B} \setminus \{0\} \right).
\end{align*}
\]

where \( V(x, 0) \) is the Green function of the operator \(-\Delta - \frac{x^2}{r^2} I\) with Dirichlet boundary condition on the unit ball and

\[
\begin{align*}
(1.8) \quad u^m_\lambda(x) \to -4 \sqrt[3]{\frac{2(4m-3)}{2m-1}} V((1, 0) \text{ in } C^1_\text{loc} \left( \overline{B} \setminus \{0\} \right). \\
\end{align*}
\]

### The case \( N=4,5 \)

In this case in [2] it was proved that \( \lambda \to \tilde{\lambda} = \mu_{m-1} \) and \( u^m_\lambda \) converges to the \((m-1)\)-th radial eigenfunction of the Laplacian in \( B \), away from the origin (see the proof of Theorem B, part (a) in [2] for this last claim). Here we compute the asymptotics of \( \| u^m_\lambda \|_{L^\infty(B)} \), \( r_\lambda, M_\lambda \) and we show that \( \lambda \to \mu_{m-1} \) from above in dimension \( N = 4 \) and from below in dimension \( N = 5 \). This last result was already proved by Gazzola and Grunau in [13] using the Emden-Fowler transformation and so we give an alternative proof based only on the blow-up analysis. Our result is the following:

**Theorem 1.2.** Let \( u^m_\lambda \) be any radial nodal solution to (1.1) with \( m > 1 \) nodal zones in dimension \( N = 4 \) or \( N = 5 \) and let \( r_\lambda \) be its first zero in \((0, 1)\). The occurrence \( \| u^m_\lambda \|_{L^\infty(B)} = u^m_\lambda(0) \to \infty \) is equivalent to \( r_\lambda \to 0 \) and can happen only if \( \lambda \to \mu_{m-1} \) from above in dimension \( N = 4 \) and from below in dimension \( N = 5 \). Moreover, as \( \lambda \to \mu_{m-1} \)

\[
\begin{align*}
(1.9) \quad \| u^m_\lambda \|_{L^\infty(B)} &= \left\{ \begin{array}{ll}
\frac{16}{5 \pi^{\mu_{m-1}}} \left( \int_0^1 r^3 |\psi_{m-1}|^2 dr \right)^{\frac{1}{4}} (1 + o(1)) (\lambda - \mu_{m-1})^{-\frac{1}{4}}, & \text{if } N = 4, \\
\frac{8}{\mu_{m-1}^{\mu_{m-1}}} \left( \int_0^1 r^4 |\psi_{m-1}|^2 dr \right)^{\frac{1}{4}} (1 + o(1)) (\mu_{m-1} - \lambda)^{-\frac{1}{4}}, & \text{if } N = 5,
\end{array} \right. \\
(1.10) \quad r_\lambda &= \left\{ \begin{array}{ll}
\frac{2}{\mu_{m-1}} (1 + o(1)) \left( \log (\lambda - \mu_{m-1}) \right)^{-\frac{1}{4}}, & \text{if } N = 4, \\
\frac{8 \sqrt{3}}{\pi^{\mu_{m-1}}} \left( \int_0^1 r^4 |\psi_{m-1}|^2 dr \right)^{\frac{1}{4}} (1 + o(1)) (\mu_{m-1} - \lambda)^{\frac{1}{4}}, & \text{if } N = 5.
\end{array} \right.
\end{align*}
\]
Here \( \psi_h \) denotes the \( h \)-th radial eigenfunction of the Laplacian in \( B \), normalized so that \( \psi_h(0) = -1 \). Furthermore, denoting by \( A_\lambda \) the annulus of radii \( r_\lambda \) and 1, we have

\[
\| u_\lambda^m \|_{L^\infty(A_\lambda)} = \begin{cases} 
\frac{\mu_{m-1}}{4} \int_0^1 r^3 |\psi_{m-1}|^2 \, dr (1 + o(1)) (\lambda - \mu_{m-1}) |\log(\lambda - \mu_{m-1})|, & \text{if } N = 4, \\
\left( \int_0^1 r^4 |\psi_{m-1}|^2 \, dr \right)^{\frac{2}{7}} (1 + o(1)) (\mu_{m-1} - \lambda)^{\frac{2}{7}}, & \text{if } N = 5, 
\end{cases}
\]  

(1.11)

\[
\frac{u_\lambda^m(x)}{\| u_\lambda^m \|_{L^\infty(A_\lambda)}} \to \psi_{m-1}(x) \quad \text{in } C^1_{\text{loc}} (B \setminus \{0\}).
\]  

(1.12)

The case \( N=6 \)

This is the most delicate case. A first question concerns the characterization of the concentration value \( \lambda \). We will see that \( \lambda = \bar{\lambda}(m) \) is characterized as the unique value at which there exists a radial solution \( u^{m-1} \) of the problem

\[
\begin{cases} 
-\Delta u^{m-1} = |u^{m-1}| u^{m-1} + \bar{\lambda} u^{m-1} & \text{in } B \\
u^{m-1}(0) = -\frac{\lambda}{2} \\
u^{m-1} \text{ has } m-1 \text{ nodal zones} \\
u^{m-1} = 0 & \text{on } \partial B.
\end{cases}
\]  

(1.13)

We will show in Proposition 2.11 that (1.13) admits a radial solution only for a unique value of \( \bar{\lambda} \), providing the characterization of \( \lambda \). Moreover since this solution is unique, problem (1.13) characterizes \( u^{m-1} \) as well. One can also see [2] Section 5) or [18] Theorem 4, where the definition of \( \lambda \) is related to the Emden-Fowler transformation. Next we will prove that any radial nodal solution \( u_\lambda^m \) to \( (1.1) \) converges to the solution \( u^{m-1} \) of \( (1.13) \) in \( C^1(B \setminus \{0\}) \), generalizing [18] Theorem 3) to the case of \( m > 1 \), see also [2] Section 5. Observe that, in this case \( M_\lambda \to \frac{1}{2} \). Finally we will compute the asymptotics of the relevant quantities \( \| u_\lambda^m \|_\infty \) and \( r_\lambda \) as in the previous cases, and we characterize the sign of \( \lambda \to \bar{\lambda} \) (in particular it is positive if \( m = 2 \)). In order to get our results we proceed differently from the cases \( N = 3, 4, 5 \). In fact, despite the blow-up procedure shows no differences, the integral identities of the previous cases do not allow us to obtain the desired result. So we argue differently: we first construct a solution to (1.1) using the Ljapunov-Schmidt procedure, next we deduce the asymptotics for this solution and finally we prove the uniqueness of the solution \( u_\lambda^m \) in the class of the blowing-up solutions. A crucial role in our result is played by the solution \( v_0 \) of the problem

\[
\begin{cases} 
-\Delta v_0 - (2|u^{m-1}| + \bar{\lambda}) v_0 = u^{m-1} & \text{in } B \\
v_0 = 0 \text{ on } \partial B
\end{cases}
\]  

(1.14)

which exists and it is unique if the solution \( u^{m-1} \) to (1.13) is nondegenerate. Our result is the following:

**Theorem 1.3.** Let \( u_\lambda^m \) be any radial nodal solution to (1.1) with \( m > 1 \) nodal zones in dimension \( N = 6 \) and let \( r_\lambda \) be its first zero in \( (0, 1) \). The occurrence \( \| u_\lambda^m \|_{L^\infty(B)} = u_\lambda^m(0) \to \infty \) is equivalent to \( r_\lambda \to 0 \) and can happen only if \( \lambda \to \bar{\lambda} \) where \( \bar{\lambda} \) is the unique value such that problem (1.13) has a radial solution. Moreover, denoting by \( A_\lambda \) the annulus of radii \( r_\lambda \) and 1, as \( \lambda \to \bar{\lambda} \) we have

\[
\| u_\lambda^m \|_{L^\infty(A_\lambda)} \to \frac{\bar{\lambda}}{2}
\]  

(1.15)

\[
u_\lambda^m(x) \to u^{m-1}(x) \quad \text{in } C^1_{\text{loc}} (\bar{B} \setminus \{0\}),
\]  

(1.16)

where \( u^{m-1} \) stands for the unique radial solution to (1.13). Furthermore, either if \( m = 2 \) or if \( u^{m-1} \) is nondegenerate, as \( \lambda \to \bar{\lambda} \) we have that

\[
\| u_\lambda^m \|_{L^\infty(B)} = \frac{121 \bar{\lambda}^3}{8(1 + 2 v_0(0))} (1 + o(1)) (\lambda - \bar{\lambda})^{-2}
\]  

(1.17)

\[
r_\lambda = 4 \sqrt[3]{\frac{6 (1 + 2 v_0(0))}{\lambda}} (1 + o(1))(\lambda - \bar{\lambda})^{\frac{1}{2}}
\]  

(1.18)
where \( v_0 \) is the unique solution to (4.8) and the following expansion of the solution \( u_\lambda^m \) holds,

\[
(1.19) \quad u_\lambda^m(r) = u^{m-1}(r) + (\lambda - \bar{\lambda})v(r) + PU_{(\lambda - \bar{\lambda})\Delta}(r) + \phi_\lambda(r)
\]

where \( PU_\delta \) is the projection of the standard bubble \( U_\delta(x) = \delta^{-2}U\left(\frac{x}{\delta}\right) \) onto \( H^1_0(B) \) (see (1.11) and (1.11)), \( d \) is a positive number (see (1.11)) and \( \phi_\lambda \in H^1_0(B) \) is such that \( \|\phi_\lambda\|_{H^1_0(B)} = \mathcal{O}\left(\frac{1}{\sqrt{\lambda - \bar{\lambda}}}\right) \).

Finally if \( m = 2 \) then \( \lambda \to \bar{\lambda} \) from above, while when \( m > 2 \) we have that \( \lambda - \bar{\lambda} > 0 \) if \( 1 + 2v_0(0) > 0 \) while \( \lambda - \bar{\lambda} < 0 \) when \( 1 + 2v_0(0) < 0 \).

**Remark 1.4.** When \( m = 2 \) the nondegeneracy of the positive solution to (1.13) was proved in [23]. So the previous theorem gives a complete scenario of the asymptotics of the solution \( u_\lambda^m \) as \( m = 2 \). Observe that \( 1 + 2v_0(0) \neq 0 \) for every \( m \geq 2 \) and this will be proved in Proposition 4.7.

**Remark 1.5.** We point out the careful construction of the ansatz (1.19) which has to be refined up to a second order and the delicate estimate of the reduced energy (4.29) given in Proposition 4.6 whose leading term arises from the interaction between the bubble and the second order term in the ansatz.

The paper is organized as follows: in Section 2 we recall some known facts about positive solutions of (1.1) and prove some general properties of nodal solutions; in Section 3 we prove Theorems 1.1 and 1.2. Finally in Section 4 we prove Theorem 1.3.

### 2. Known facts and preliminary remarks

In this section we recall some known facts about radial solutions to the Brezis-Nirenberg problem and we fix the notations that will be used in the paper. From now on we will delete the index \( m \) and only write

\[ u^m_\lambda \equiv u_\lambda. \]

We start considering the case of positive solutions that has been extensively studied in the 80’s, mainly by Brezis, Nirenberg, Peletier, Atkison, see [7, 2, 3]. Any positive solution is radial and radially decreasing (by the symmetry result in [14]), and is unique (see [23]), therefore it is a least energy solution and satisfies

\[
(2.1) \quad S_\lambda := \inf_{\phi \in H^1_0(B)} \left\{ \frac{\int_B |\nabla \phi|^2 dx - \lambda \int_B |\phi|^2 dx}{\left(\int_B |\phi|^2 dx\right)^{\frac{N}{2}}} \right\}
\]

for every \( \lambda \in (0, \mu_1) \) in dimension \( N \geq 4 \), or respectively \( \lambda \in (\mu_1/4, \mu_1) \) in dimension \( N = 3 \), from Theorem A. By [7] Lemma 1.1 and 1.3 we get

**Lemma 2.1.** For every \( \lambda > 0 \) when \( N \geq 4 \) or for every \( \lambda > \mu_1/4 = \pi^2/4 \) when \( N = 3 \)

\[
(2.2) \quad S_\lambda < S_N
\]

where \( S_N \) is the best constant for the Sobolev embedding \( H^1_0(B) \subset L^2^N(B) \).

Next we describe the blow up rate of the positive solution. In dimension \( N = 3 \), the profile is linked to the Green function of the operator \(-\Delta - \frac{\pi^2}{4}I\) on the unit ball, namely

\[
(2.3) \quad \begin{cases} 
-\Delta V(x, 0) - \frac{\pi^2}{4}V(x, 0) = \delta_0 & \text{in } B \\
V(x, 0) = 0 & \text{on } \partial B
\end{cases}
\]

where \( \delta_0 \) is the Dirac mass centered at the origin. By [3] Theorem 3 it follows that

**Theorem 2.2.** Let \( u_\lambda \) be the positive solution to (1.1) in dimension \( N = 3 \). Then \( \|u_\lambda\|_\infty \to \infty \) as \( \lambda \to \pi^2/4 \) and precisely

\[
(2.4) \quad u_\lambda(0) = \|u_\lambda\|_\infty = \sqrt{\frac{\pi^2}{2}\sqrt{3}} \left(1 + o(1)\right) \frac{1}{\sqrt{\lambda - \pi^2/4}} \quad \text{as } \lambda \to \pi^2/4
\]

and

\[
(2.5) \quad \frac{u_\lambda(x)}{\sqrt{\lambda - \pi^2/4}} \to 4\sqrt{2}\sqrt{3}V(x, 0) \quad \text{in } C^1_{\text{loc}}(\overline{B} \setminus \{0\}) \quad \text{as } \lambda \to \pi^2/4.
\]
Han [16] dealt with the higher dimensional case, and proved that the limit profile is driven by the
Green function of the Laplacian on the unit ball, i.e.
\begin{align*}
-\Delta G(x, 0) &= \delta_0 \quad \text{in } B \\
G(x, 0) &= 0 \quad \text{on } \partial B
\end{align*}
(2.6)
solving also Conjecture 2 in [8]. Han’s result is the following,

**Theorem 2.3.** Let $u_\lambda$ be the positive solution to (1.1) in dimension $N \geq 4$. Then $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to 0$ and precisely
\begin{align*}
\|u_\lambda\|_\infty &= C_N(1 + o(1)) \lambda^{-\frac{N-2}{2(1+\alpha)}} &&\text{if } N \geq 5, \\
\log \|u_\lambda\|_\infty &= 2(1 + o(1)) \lambda^{-1} &&\text{if } N = 4
\end{align*}
(2.7)
where
$$C_N = \frac{1}{N} \left( \frac{N-2}{2} \right)^{\frac{N-2}{2(1+\alpha)}} \frac{1}{\alpha_N}$$
\[
\alpha_N = \int_0^\infty \frac{r^{N-1}}{(1+r^2)^{N-2}} \mathrm{d}r.
\]
Moreover, letting $\sigma_N$ be the measure of the sphere $S^{N-1} \subset \mathbb{R}^N$
\begin{align*}
\|u_\lambda\|_{L^\infty(B(x_0, r_\lambda \varepsilon \lambda))} &= \frac{N-2}{2(1+\alpha)} (N-2) \sigma_N G(x, 0) \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \quad \text{as } \lambda \to 0.
\end{align*}
(2.8)
Now we consider nodal solutions and, exploiting the results about positive solutions just recalled, we give a rough description of the concentrating phenomenon.

As a preliminary, we recall some general qualitative properties of the radial solutions to (1.1). As usual we write $u_\lambda(r)$ for $u_\lambda(x) = u_\lambda(|x|)$ (meaning $r = |x|$), and $0 < r_{1,\lambda} < r_{2,\lambda} < \ldots < r_{m,\lambda} = 1$ for the zeros of $u_\lambda$. Writing (1.1) in radial coordinates gives an ordinary differential equation with mixed boundary data:
\begin{align*}
-\left( r^{N-1-1} u_\lambda' \right)' &= r^{N-1} \left( |u_\lambda|^2 \lambda u_\lambda \right), \quad \text{in } (0, 1), \\
u_\lambda(0) &= 0, \quad u_\lambda(1) = 0.
\end{align*}
(2.9)
Starting from this, it is easy to see that in each nodal interval the function $u_\lambda$ is alternately strictly positive or strictly negative and has only one critical point, which is respectively a maximum or a minimum. Moreover the extremal values are ordered
\begin{align*}
|u_{\lambda}(0)| &= \max_{[0, r_{1,\lambda}]} |u_\lambda| > \max_{(r_{1,\lambda}, r_{2,\lambda})} |u_\lambda| > \cdots > \max_{(r_{m-1,\lambda}, r_{m,\lambda})} |u_\lambda|,
\end{align*}
(2.10)
see [2] Lemma 1. Henceforth we shall always take that $u_\lambda$ is positive in the first nodal zone $[0, r_{1,\lambda})$
\begin{align*}
u_{\lambda}(0) &= 0,
\end{align*}
(2.11)
and use the notations
- $r_\lambda := r_{1,\lambda}$ for the first zero of $u_\lambda$,
- $s_\lambda$ for point where $u_\lambda$ attains its global minimum, which actually is the first minimum of $u_\lambda$,
- $A_\lambda := \{x : r_{\lambda} < |x| < 1\} = B \setminus \bar{B}_{r_{\lambda}}$.
By the previous consideration we have
\begin{align*}
\|u_\lambda\|_{L^\infty(B)} &= u_{\lambda}(0), \quad \|u_\lambda\|_{L^\infty(A_\lambda)} = -u_{\lambda}(s_\lambda).
\end{align*}
(2.12)
Let us remark that a simple scaling argument and the non-existence result recalled in Theorem [A] implies

**Lemma 2.4.** Let $u_\lambda$ be any radial nodal solution to (1.1) and $r_\lambda$ its first zero in $(0, 1)$. Then
\begin{align*}
0 < \lambda r_{\lambda}^2 < \mu_1 &\quad \text{if } N = 4, 5, 6, \\
\frac{\pi^2}{4} < \lambda r_{\lambda}^2 < \pi^2 &\quad \text{if } N = 3.
\end{align*}
(2.13, 2.14)
**Proof.** Let
\begin{align*}
v_\varepsilon(x) := \frac{r_{\lambda}^{N-2}}{\varepsilon} u_{\lambda}(r_{\lambda} x), \quad \varepsilon = \varepsilon(\lambda) = \lambda r_{\lambda}^2.
\end{align*}
(2.15)
A simple computation shows that $v_\varepsilon$ solves
\begin{equation}
\begin{cases}
-\Delta v = v^{\frac{2N}{N-4}} + \varepsilon v & \text{in } B, \\
v > 0 & \text{in } B, \\
v = 0 & \text{on } \partial B.
\end{cases}
\end{equation}
(2.16)

Then Theorem [A] gives the claim recalling that $\mu_1 = \pi^2$ when $N = 3$.

Starting from the knowledge of the positive solution, one can study the behaviour of the first node $r_\lambda$ for $\lambda$ close to the concentration value $\lambda_1$; we see that the first nodal zone collapses in dimension $N \geq 4$, while it does not vanish in dimension $N = 3$.

**Lemma 2.5.** Let $u_\lambda$ be any radial nodal solution to (1.1) for $N = 3, 4, 5, 6$, $r_\lambda$ its first zero in $(0, 1)$, and $\lambda$ such that $\|u_\lambda\|_\infty = u_\lambda(0) \to \infty$ when $\lambda \to \lambda$, then

\begin{equation}
\lim_{\lambda \to \lambda} r_\lambda = \begin{cases}
0 & \text{if } N \geq 4, \\
\sqrt{\frac{\mu_1}{4\lambda}} = \frac{\pi}{2\sqrt{\lambda}} > 0 & \text{if } N = 3.
\end{cases}
\end{equation}
(2.17)

**Proof.** Recall that, by Theorem [B] $\bar{\lambda} > 0$ when $3 \leq N \leq 6$. As in the previous Lemma we use the function $v_\varepsilon(x) := r_{\lambda r_\lambda}^2 u_\lambda(r_\lambda x)$ that satisfies (2.16) for $\varepsilon = \lambda r_\lambda^2$ and we write $\varepsilon = \lim_{\lambda \to \lambda} \varepsilon$. We claim that $\|v_\varepsilon\|_\infty \to \infty$ as $\varepsilon \to 0$ if and only if either $\varepsilon = \mu_1/4$ when $N = 3$ or $\varepsilon = 0$ when $N \geq 4$. Using this claim we can conclude the proof. Indeed in the case when $N = 3$, (2.14) gives that $\lim_{\lambda \to \lambda} r_\lambda > 0$ so that $\|v_\varepsilon\|_\infty = v_\varepsilon(0) = r_{\lambda r_\lambda}^{\frac{1}{2}} u_\lambda(0) \to +\infty$ as $\varepsilon \to 0$. The previous claim then implies that $\varepsilon = \mu_1/4 = \pi^2/4$ and gives (2.17).

When $N \geq 4$ instead we assume by contradiction that (2.17) does not hold. Then again $\lim_{\lambda \to \lambda} r_\lambda > 0$ and $\|v_\varepsilon\|_\infty = r_{\lambda r_\lambda}^2 u_\lambda(0) \to +\infty$ as $\varepsilon \to 0$. Then the claim gives that $\varepsilon = 0$ which implies $r_\lambda \to 0$ since we know that $\bar{\lambda} > 0$. This contradiction concludes the proof of (2.17).

Finally we prove the claim. Of course it is a known result for positive solutions, but we report a proof for completeness. One implication is already stated in Theorem [2.1] for $N = 3$ and in Theorem [2.3] for $4 \leq N \leq 6$. Next we assume that $\varepsilon > \mu_1/4$ when $N = 3$ and $\varepsilon > 0$ when $N \geq 4$. Let us check first that $S_\varepsilon \to S_\varepsilon^*$ as $\varepsilon \to 0$ where as said in (2.11)

\[ S_\varepsilon = \inf_{\phi \in M_0^{rad}(\varepsilon)} \frac{\int_B \nabla \phi^2 \, dx - \varepsilon \int_B \phi^2 \, dx}{\left(\int_B |\nabla v_\varepsilon|^2 \, dx\right)^{\alpha N}} = \frac{\int_B |\nabla v_\varepsilon|^2 \, dx - \varepsilon \int_B |v_\varepsilon|^2 \, dx}{\left(\int_B |\nabla v_\varepsilon|^2 \, dx\right)^{\alpha N}}.
\]

By definition

\[ S_\varepsilon \leq \frac{\int_B |\nabla v_\varepsilon|^2 \, dx - \varepsilon \int_B |v_\varepsilon|^2 \, dx}{\left(\int_B |v_\varepsilon|^2 \, dx\right)^{\alpha N}} = S_\varepsilon + \frac{(\varepsilon - \varepsilon) \int_B v_\varepsilon^2 \, dx}{\left(\int_B v_\varepsilon^2 \, dx\right)^{\alpha N}} \leq S_\varepsilon + |\varepsilon - \varepsilon| |B|^\varepsilon^*,
\]

where $|B|$ stands for the measure of the ball $B$. Similarly one sees that $S_\varepsilon \leq S_\varepsilon + |\varepsilon - \varepsilon| |B|^\varepsilon^*$, so that $|S_\varepsilon - S_\varepsilon| \leq |\varepsilon - \varepsilon| |B|^\varepsilon^*$ and the claim is proved. In particular, by Lemma [2.1] $\lim_{\lambda \to \lambda} S_\varepsilon = S_\varepsilon < S_N$ and this last fact implies the compactness of $u_\varepsilon$ which ends the proof.

The asymptotics of the positive solutions recalled in Theorems [2.2] and [2.3] together with a scaling argument, allows us to obtain in a simple way an estimate of the $L^\infty$-norm of $u_\lambda$ in term of its first zero $r_\lambda$ which will be very useful in the sequel. First we deal with the case of dimension $N = 3$.

**Lemma 2.6.** Let $u_\lambda$ be any radial nodal solution to (1.1) with $m \geq 2$ nodal zones in dimension $N = 3$, $r_\lambda$ its first zero in $(0, 1)$, and $\lambda > 0$ such that $\|u_\lambda\|_\infty = u_\lambda(0) \to \infty$ when $\lambda \to \bar{\lambda}$. Then

\begin{equation}
\|u_\lambda\|_\infty = \frac{\pi \sqrt{3\lambda}(1 + o(1))}{\sqrt{\lambda r_\lambda^2} - \pi^2/4} \quad \text{as } \lambda \to \bar{\lambda}.
\end{equation}
(2.18)
Moreover, denoting by $V(x,0)$ the solution of problem (2.13) and by $\bar{r} = \frac{\pi}{2\sqrt{\lambda}} = \lim_{\lambda \to \lambda} r_\lambda$, we have that as $\lambda \to \bar{\lambda}$

$$
(2.19) \quad \frac{u_\lambda(x)}{\sqrt{\lambda r_\lambda^2 - \pi^2/4}} \to \frac{8\sqrt{3\lambda}}{\sqrt{\pi}} V\left(\frac{x}{\bar{r}},0\right) \quad \text{in } C^1_{\text{loc}}(\overline{B}_r \setminus \{0\}).
$$

Finally

$$
(2.20) \quad u'(r_\lambda) = \frac{16\sqrt{3\lambda} V'(1,0)}{\sqrt{\pi}} (1 + o(1)) \sqrt{\lambda r_\lambda^2 - \pi^2/4} \quad \text{as } \lambda \to \bar{\lambda}.
$$

Proof. To get (2.18) and (2.19) it suffices to apply Theorem 2.2 to the function $v_\epsilon$ defined in (2.15). Estimate (2.20) is an easy consequence of (2.19). □

In higher dimension, instead, we have

Lemma 2.7. Let $u_\lambda$ be any radial nodal solution to (1.1) in dimension $N$ between 4 and 6, $r_\lambda$ its first zero in $(0,1)$ and $\lambda > 0$ such that $\|u_\lambda\|_\infty = u_\lambda(0) \to \infty$ when $\lambda \to \lambda$. Then we have

$$
(2.21) \quad \begin{cases}
\log \|u_\lambda\|_\infty = \frac{2}{N} (1 + o(1)) \lambda^{-\frac{2}{N}} \\
\|u_\lambda\|_\infty = 15\pi^\frac{2}{N} (1 + o(1)) \lambda^{-\frac{2}{N}} \\
\|u_\lambda\|_\infty = 11\pi^2 (1 + o(1)) \lambda^{-\frac{1}{N}}
\end{cases}
$$

when $N = 4$, $N = 5$, and $N = 6$ respectively.

$$
(2.22) \quad u'(r_\lambda) = \begin{cases}
-16(1 + o(1)) (\|u_\lambda\|_\infty r_\lambda^2)^{-1} & \text{if } N = 4, \\
-3 + 5\pi \left(\frac{\pi}{N}\right)^\frac{2}{N} (1 + o(1)) r_\lambda^{-\frac{1}{N}} & \text{if } N = 5, \\
-2\lambda(1 + o(1)) r_\lambda^{-1} & \text{if } N = 6,
\end{cases}
$$

for $\lambda \to \bar{\lambda}$.

Proof. We let $\epsilon = \lambda r_\lambda^2$ and $v_\epsilon = r_\lambda^{-\frac{N-2}{2}} u_\lambda(r_\lambda r)$ be as in (2.15). It solves (2.16) and so it is a positive solution to (1.1) corresponding to $\epsilon$. Moreover by (2.17) we know that $r_\lambda \to 0$ so that $\epsilon \to 0$. We can then apply Theorem 2.3 to $v_\epsilon$ getting that, as $\lambda \to \bar{\lambda}$

$$
\|u_\lambda\|_\infty = r_\lambda^{-\frac{N-2}{2}} \|v_\epsilon\|_\infty = C_N r_\lambda^{-\frac{N-2}{2}} (\lambda r_\lambda^2)^{-\frac{N-2}{2} N} (1 + o(1)) \quad \text{if } N = 5, 6,
$$

where $C_N$ is the constant in (2.4). When $N = 4$ instead (2.7) gives

$$
\lambda r_\lambda^2 \log \|u_\lambda\|_\infty + \lambda r_\lambda^2 \log r_\lambda \to 2
$$

as $\lambda \to \bar{\lambda}$, and (2.21) follows recalling that $r_\lambda \to 0$. Further by (2.8)

$$
\|v_\epsilon\|_\infty v_\epsilon(x) \to [N(N - 2)]^\frac{N-2}{2} (N - 2)\sigma_N G(x,0) \quad \text{in } C^1_{\text{loc}}(\overline{B} \setminus \{0\}),
$$

as $\epsilon \to 0$. In particular, using also (2.7)

$$
(2.23) \quad v'_\epsilon(1) \sim \frac{[N(N - 2)]^\frac{N-2}{2} (N - 2)\sigma_N}{\|v_\epsilon\|_\infty} \sum_{i=1}^N x_i \frac{\partial G(x,0)}{\partial x_i} \bigg|_{|x|=1} = \begin{cases}
\left[\frac{N(N - 2)}{2} \frac{\pi}{N} (N - 2)\sigma_N \frac{N - 2}{2} N^\frac{N-2}{2} \right] \sum_{i=1}^N x_i \frac{\partial G(x,0)}{\partial x_i} \bigg|_{|x|=1} & \text{if } N \geq 5, \\
16\pi^2 \sum_{i=1}^N x_i \frac{\partial G(x,0)}{\partial x_i} \bigg|_{|x|=1} & \text{if } N = 4.
\end{cases}
$$

Next recalling that $u'_\lambda(r_\lambda) = r_\lambda^{-\frac{N}{2}} v'_\epsilon(1)$ with $\epsilon = \lambda r_\lambda^2$ we have

$$
(2.22) \quad u'_\lambda(r_\lambda) = r_\lambda^{-\frac{N}{2}} v'_\epsilon(1) \sim \begin{cases}
\left[\frac{N(N - 2)}{2} \frac{\pi}{N} (N - 2)\sigma_N \frac{N - 2}{2} N^\frac{N-2}{2} \right] \sum_{i=1}^N x_i \frac{\partial G(x,0)}{\partial x_i} \bigg|_{|x|=1} & \text{if } N \geq 5, \\
16\pi^2 \sum_{i=1}^N x_i \frac{\partial G(x,0)}{\partial x_i} \bigg|_{|x|=1} & \text{if } N = 4.
\end{cases}
$$

which gives (2.22) using the explicit value for $\sum_{i=1}^N x_i \frac{\partial G(x,0)}{\partial x_i} \bigg|_{|x|=1} = -\frac{1}{\pi}\frac{\partial}{\partial x_i}$ (see [15]). □
To complete the parts of the proofs of Theorems 1.1 and 1.2 concerning the first nodal zone \((0, r_\lambda)\), it is needed to describe the asymptotics of the first zero \(r_\lambda\) in terms of \(\lambda - \bar{\lambda}\). In this matter a role is played by the behaviour of the solution in the subsequent nodal zones. A technical lemma is needed to go further. We state it for any zero of the solution.

**Lemma 2.8.** Let \(u_\lambda\) be any radial nodal solution to (1.1) and \(r_{i,\lambda}\) one of its nodal radii. For every \(0 < a \leq r \leq b \leq 1\) and we have

\[
(2.24) \quad u_\lambda'(r) = \frac{1}{r^{N-1}} b^{N-1} u_\lambda'(b) + \int_a^b s^{N-1} \left( |u_\lambda|^2 t^2 u_\lambda + \lambda u_\lambda \right) ds,
\]

\[
(2.25) \quad u_\lambda(a) = \frac{b^{N-1}}{N-2} u_\lambda'(b) \left( \frac{1}{r_{i,\lambda}^2} - \frac{1}{a^{N-2}} \right) + \frac{1}{N-2} \int_a^{r_{i,\lambda}} r \left( |u_\lambda|^2 t + \lambda u_\lambda \right) dr
\]

\[+ \frac{1}{r_{i,\lambda}^{N-2}} \int_{r_{i,\lambda}}^b r^{N-1} \left( |u_\lambda|^2 t + \lambda u_\lambda \right) dr - \frac{1}{a^{N-2}} \int_a^b r^{N-1} \left( |u_\lambda|^2 t + \lambda u_\lambda \right) dr.
\]

**Proof.** Integrating the equation in (2.24) over \((r, b)\) gives (2.24). Integrating again over \((a, r_{i,\lambda})\) then we get

\[-u_\lambda(a) = \int_a^{r_{i,\lambda}} \frac{1}{r^{N-1}} b^{N-1} u_\lambda'(b) + \int_r^b s^{N-1} \left( |u_\lambda|^2 t + \lambda u_\lambda \right) ds dr.
\]

By simple computations it follows that

\[
u_\lambda(a) = \frac{b^{N-1}}{N-2} u_\lambda'(b) \left( \frac{1}{r_{i,\lambda}^2} - a^{-2N} \right) - \int_a^{r_{i,\lambda}} r^{N-1} \int_r^b s^{N-1} \left( |u_\lambda|^2 t + \lambda u_\lambda \right) ds dr
\]

and integrating by parts

\[= \frac{b^{N-1}}{N-2} u_\lambda'(b) \left( \frac{1}{r_{i,\lambda}^2} - a^{-2N} \right) + \frac{1}{N-2} \int_a^{r_{i,\lambda}} r \left( |u_\lambda|^2 t + \lambda u_\lambda \right) dr
\]

\[+ \frac{1}{r_{i,\lambda}^{N-2}} \int_{r_{i,\lambda}}^b r^{N-1} \left( |u_\lambda|^2 t + \lambda u_\lambda \right) dr - \frac{1}{a^{N-2}} \int_a^b r^{N-1} \left( |u_\lambda|^2 t + \lambda u_\lambda \right) dr.
\]

which gives (2.25).

An immediate, but interesting, consequence of Lemma 2.8 is that the behaviour of the solutions in the annulus \(A_\lambda\), of radii \(r_\lambda\) and 1, is controlled by the derivative of \(u_\lambda\) in the first node \(r_\lambda\), that is

**Corollary 2.9.** Let \(u_\lambda\) be any radial nodal solution to (1.1), \(r_\lambda\) its first zero in \((0, 1)\) and \(A_\lambda\) the annulus of radii \(r_\lambda\) and 1. Then

\[
(2.26) \quad \|u_\lambda\|_{L^\infty(A_\lambda)} \leq -\frac{1}{N-2} r_\lambda u_\lambda'(r_\lambda).
\]

**Proof.** We denote by \(s_\lambda\) the point at which \(u_\lambda\) attains the \(L^\infty\)-norm in \(A_\lambda\), that is the first minimum of \(u_\lambda\) (see (2.12)). So writing (2.25) with \(a = b = s_\lambda\) and \(r_{i,\lambda} = r_\lambda\) gives

\[
\|u_\lambda\|_{L^\infty(A_\lambda)} = -u_\lambda(s_\lambda) = \frac{1}{N-2} \int_s^{s_\lambda} r^{N-1} \left( |u_\lambda|^2 t^2 u_\lambda + \lambda u_\lambda \right) dr
\]

\[+ \int_{r_\lambda}^{s_\lambda} r \left( |u_\lambda|^2 t^2 u_\lambda + \lambda u_\lambda \right) dr
\]

and noticing that \(|u_\lambda|^2 t^2 u_\lambda + \lambda u_\lambda < 0\) on \((r_\lambda, s_\lambda)\) yields

\[
\leq -\frac{1}{(N-2)r_\lambda^{N-2}} \int_{r_\lambda}^{s_\lambda} r^{N-1} \left( |u_\lambda|^2 t^2 u_\lambda + \lambda u_\lambda \right) dr.
\]

On the other hand, choosing \(r = r_\lambda\) and \(b = s_\lambda\) in (2.24) gives

\[
u_\lambda'(r_\lambda) = \frac{1}{r_\lambda^{N-2}} \int_{r_\lambda}^{s_\lambda} r^{N-1} \left( |u_\lambda|^2 t^2 u_\lambda + \lambda u_\lambda \right) dr,
\]

which concludes the proof.

\[\square\]
Corollary 2.9 together with Lemmas 2.6 and 2.7 furnishes an estimate of the norm of $u_\lambda$ on the set $A_\lambda$, which shows that in dimension $N$ between 3 and 6 the solution does not behave like a tower of bubbles.

**Lemma 2.10.** Let $u_\lambda$ be any radial nodal solution to (1.1) in dimension $N$ between 3 and 6, and $\bar{\lambda} > 0$ such that $\|u_\lambda\|_{L^\infty(B)} = u_\lambda(0) \to \infty$ as $\lambda \to \bar{\lambda}$, $r_\lambda$ its first zero in $(0, 1)$ and $A_\lambda$ the annulus of radii $r_\lambda$ and 1. Then

\begin{align*}
\|u_\lambda\|_{L^\infty(B)} & \to 0 \quad \text{as } \lambda \to \bar{\lambda}, \quad \text{if } N = 3, 4, 5, \\
\|u_\lambda\|_{L^\infty(A_\lambda)} & \leq C \quad \text{if } N = 6.
\end{align*}

**Proof.** Inserting (2.20), (2.22) into (2.26) gives

\begin{align*}
0 \leq \|u_\lambda\|_{L^\infty(A_\lambda)} & \leq - \frac{1}{N - 2} r_\lambda u_\lambda'(r_\lambda) \sim \begin{cases} 
C \sqrt{\lambda r_\lambda^3 - \frac{\pi^2}{4}} & \text{if } N = 3, \\
C(r_\lambda^2 \|u_\lambda\|_\infty)^{-1} & \text{if } N = 4, \\
Cr_\lambda^2 & \text{if } N = 5, \\
C & \text{if } N = 6.
\end{cases}
\end{align*}

as $\lambda \to \bar{\lambda}$. So the claim readily follows by (2.17) if $N = 3$ or 5. Otherwise if $N = 4$, (2.21) yields $r_\lambda^2 \|u_\lambda\|_\infty = r_\lambda^2 e^{-\frac{\pi\lambda}{6\lambda^2}}$, and (2.17) allows to conclude also in this case. \hfill $\square$

We end this section with a uniqueness result for solutions to (1.1). It will be used in a crucial way in Section 4.

**Proposition 2.11.** Let $u_1$ and $u_2$ be radial solutions to (1.1) corresponding to $\lambda_1$ and $\lambda_2$ respectively. If $u_1$ and $u_2$ have both $m$ nodal zones and

\begin{align}
\frac{u_1(0)}{\lambda_1^{\frac{N-2}{2}}} = \frac{u_2(0)}{\lambda_2^{\frac{N-2}{2}},}
\end{align}

then

\begin{align}
\lambda_1 = \lambda_2 \text{ and } u_1 \equiv u_2.
\end{align}

**Proof.** Let us consider the functions $\tilde{u}_1 : (0, \sqrt{\frac{\alpha_1}{\lambda_1}}) \to \mathbb{R}$ as

\begin{align}
\tilde{u}_1(r) = \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{N-2}{2}} u_1 \left(\sqrt{\frac{\alpha_2}{\lambda_1}} r\right)
\end{align}

which verifies

\begin{align*}
-\tilde{u}_1'' - \frac{N-1}{r} \tilde{u}_1' = |\tilde{u}_1| \tilde{u}_1 + \lambda_2 \tilde{u}_1 & \quad \text{in } \left(0, \sqrt{\frac{\alpha_1}{\lambda_1}}\right), \\
\tilde{u}_1(0) = \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{N-2}{2}} u_1(0), \\
\tilde{u}_1'(0) = 0.
\end{align*}

So if (2.30) holds then $\tilde{u}_1 \equiv u_2$ by the uniqueness of the solution to the Cauchy problem. In particular the $m^{th}$ zero of $u_2$, that is 1, coincides with the $m^{th}$ zero of $\tilde{u}_1$, that is $\sqrt{\frac{\alpha_1}{\lambda_1}}$, and the claim follows. \hfill $\square$

**Remark 2.12.** Proposition 2.11 is equivalent to the uniqueness of the solution of the transformed (with the Emden-Fowler transformation) problem in [2] Section 2 with a fixed asymptotic value at infinity.

**Remark 2.13.** In dimension $N = 6$, Proposition 2.11 states that the overdetermined problem (1.13) is fulfilled by one couple $(\bar{\lambda}, u^{m-1})$ at most. This allows us to characterize both the concentration value $\bar{\lambda}$ and the asymptotic profile of the solution outside the origin. It will be a key ingredient in the proof of Theorem 1.9.

3. Proof of Theorems 1.1 and 1.2

In this section we compute the rate of $r_\lambda$, $\|u_\lambda\|_{L^\infty(B)}$ and $\|u_\lambda\|_{L^\infty(A_\lambda)}$ in terms of $\lambda - \bar{\lambda}$, and describe the profile of the solution in the set $A_\lambda$ for $N = 3, 4, 5$. Note that we characterize the value of $\bar{\lambda}$ by the asymptotic behaviour of the solution $u_\lambda$ in the annulus $A_\lambda$. This is a completely different approach to the other proof in the literature (see [2, 13] or [18] for example).

We argue separately according to the dimension.
3.1. The case \( N = 3 \).

Let us give a first description of the profile of the solution in the annulus \( A_\lambda \) of radii \( r_\lambda \) and 1.

**Proposition 3.1.** Let \( u_\lambda \) be any radial nodal solution to (1.1) with \( m \geq 2 \) nodal zones in dimension \( N = 3 \), and \( \lambda \) such that \( \|u_\lambda\|_\infty = u_\lambda(0) \to \infty \) when \( \lambda \to \lambda_\cdot \) Then,

\[
\lambda = \frac{(2m-1)^2}{4}, \quad r_\lambda \to \bar{r} = \frac{1}{2m-1},
\]

\[
\frac{u_\lambda(x)}{\|u_\lambda\|_{L^\infty(A_\lambda)}} \to -\frac{2\theta_o}{(2m-1)\cos \theta_o |x|} \cos \left(\frac{2m-1}{2} \pi |x|\right) \quad \text{in } C^1_{loc}(\bar{B}_1 \setminus \bar{B}_{\frac{1}{2m-1}})
\]

as \( \lambda \to \bar{\lambda} \) where \( \theta_o \approx 2.7984 \) is the unique root of \( 1 + \theta \tan \theta = 0 \) in the interval \( (\frac{\pi}{2}, \frac{3\pi}{2}) \). Furthermore \( \|u_\lambda\|_{L^\infty(A_\lambda)} \to 0 \), more precisely

\[
\|u_\lambda\|_{L^\infty(A_\lambda)} = 4\sqrt{3}V(1,0) \sqrt{2(2m-1)\cos \theta_o} (1 + o(1)) \sqrt{\lambda r_\lambda^2 - \pi^2}/4,
\]

where \( V(r,0) \) is the function defined in (2.5).

**Proof.** Set \( M_\lambda = \|u_\lambda\|_{L^\infty(A_\lambda)} \) and look at the normalized function

\[
\tilde{u}_\lambda(x) := \frac{u_\lambda(x)}{M_\lambda}
\]

which solves

\[
\begin{cases}
-\Delta u = (M_\lambda)^4 |u|^4 u + \lambda u & \text{in } A_\lambda, \\
u = 0 & \text{on } \partial A_\lambda. 
\end{cases}
\]

By construction \( |\tilde{u}_\lambda| \leq 1 \) on \( (r_\lambda, 1) \). We denote by \( s_\lambda \in (r_\lambda, 1) \) the point at which the \( L^\infty \)-norm of \( u_\lambda \) in \( A_\lambda \) is attained, i.e. the first minimum of \( u_\lambda \) according to (2.12). Integrating the equation in (3.3) (written in radial coordinates) gives that, for every \( \rho \in [r_\lambda, 1] \),

\[
\\(\bar{u}_\lambda\)'(r) = \frac{1}{r^2} \left| \int_{r}^{s_\lambda} s^2 ((M_\lambda)^4 \tilde{u}_\lambda^4 \tilde{u}_\lambda + \lambda \tilde{u}_\lambda) \, ds \right| \leq C
\]

since \( r_\lambda \to \bar{r} = \frac{\pi}{2M_\lambda} \) by (2.17) and \( M_\lambda \to 0 \) by (2.25). Then (3.5) gives

\[
(|\tilde{u}_\lambda|)' \leq \frac{2(|\tilde{u}_\lambda|^5 + (M_\lambda)^4 |\tilde{u}_\lambda|^5 + \lambda |\tilde{u}_\lambda|)}{r} \leq C,
\]

for every \( r > \bar{r} > 0 \) and the Ascoli-Arzelá Theorem yields that \( \tilde{u}_\lambda \to w \) in \( C^1[\delta, 1] \), for every \( \delta > \bar{r} \). The limit function \( w \) is radial and solves

\[
\begin{cases}
-\Delta w = \bar{\lambda} w & \text{in } A^*, \\
\|w\|_\infty = 1 \\
w = 0 & \text{on } \partial A^*,
\end{cases}
\]

where \( A^* \) stands for the annulus of radii \( \bar{r} \) and 1. Moreover (3.4) ensures that the minimum point \( s_\lambda \) converges to some point \( \bar{s} > \bar{r} \), because

\[
1 = \tilde{u}_\lambda(r_\lambda) - \tilde{u}_\lambda(s_\lambda) = -\int_{r_\lambda}^{s_\lambda} (\tilde{u}_\lambda)'(r) \, dr \leq C(s_\lambda - r_\lambda).
\]

Hence \( w \) is nontrivial since \( w(\bar{s}) = -1 \). Further \( w < 0 \) on \( (\bar{r}, \bar{s}) \); indeed it is clear that \( w \leq 0 \) on \( (\bar{r}, \bar{s}) \), and if by contradiction \( w(t) = 0 \) at some point \( t \in (\bar{r}, \bar{s}) \), then also \( w'(t) = 0 \) because of the sign condition, implying \( w \equiv 0 \) (since \( w \) solves the ODE obtained by writing (3.7) in radial coordinates). Similarly, one can check that \( w \) has exactly \( m-1 \) nodal zones, that is \( w \) is the \( (m-1)^{th} \) radial eigenfunction of the Laplacian in the annulus \( A^* \). Indeed, assume that a nodal zone \( (r_{1,\lambda}, r_{2,\lambda}) \subset (r_\lambda, 1) \) disappears as \( \lambda \to \bar{\lambda} \), so that \( \lim_{\lambda \to \bar{\lambda}} r_{1,\lambda} = \lim_{\lambda \to \bar{\lambda}} r_{2,\lambda} = r_0 \). Then, \( r_0 \in [\bar{s}, 1] \) and there exists a point \( \bar{x}_\lambda \in (r_{1,\lambda}, r_{2,\lambda}) \) such that \( \bar{u}_\lambda''(\bar{x}_\lambda) = 0 \). The convergence in \( C^1_{loc}(\bar{B}_1 \setminus \bar{B}_{\frac{1}{2m-1}}) \) implies that \( 0 = \lim_{\lambda \to \bar{\lambda}} \bar{u}_\lambda''(\bar{x}_\lambda) = \lim_{\lambda \to \bar{\lambda}} \bar{u}_\lambda'(r_0) = \lim_{\lambda \to \bar{\lambda}} \bar{u}_\lambda(r_1, \lambda) = w(r_0) \) and this is not possible since \( w \) solves the ODE obtained by writing (3.7) in radial coordinates as before. It is easy to see that any radial solution to the equation in (3.7) has the form

\[
w(x) = \frac{a}{|x|} \cos(\sqrt{\bar{\lambda}} |x|) + \frac{b}{|x|} \sin(\sqrt{\bar{\lambda}} |x|).
\]
for suitable $a$ and $b$. Imposing $w(x) = 0$ when $|x| = \bar{r}$ gives $b = 0$ because $\sqrt{\lambda} = \pi/2$ by (2.17). The additional condition $w(x) = 0$ for $|x| = 1$ yields that $\sqrt{\lambda}$ is a zero of the cosine. Then $\sqrt{\lambda} = \frac{2m-1}{2}$ because $w$ has exactly $m - 1$ nodal annuli, and (2.17) implies that $\bar{r} = \frac{1}{2m-1}$, concluding the proof of (3.1). In particular $\bar{s}$ is the minimum point of $\frac{a}{\bar{r}} \cos \left( \frac{2m-1}{2} \pi \cdot \bar{r} \right)$ on $[\bar{r},1] = [1/(2m-1),1]$ which coincide with the minimum point in the first nodal region of $w$, namely $[1/(2m-1),3/(2m-1)]$ and this implies that $\bar{s} = \frac{2m-1}{2m-2}$, where $\theta_o$ is the unique root of $g(\theta) = \theta \tan \theta + 1$ in $\left[ \frac{\pi}{2}, \frac{3\pi}{2} \right]$. Recalling that $w(\bar{s}) = -1$ a straightforward computation allows us to deduce the exact value of the constant $a$, obtaining (3.3).

Lastly from (3.5) we have

$$\frac{w'(r)}{M_\lambda} = \bar{u}'(r) = \frac{1}{r_\lambda} \int_{r_\lambda}^{s} s^2 \left( (M_\lambda)^4 |\bar{u}_\lambda|^4 \bar{u}_\lambda + \lambda \bar{u}_\lambda \right) ds$$

and, since $M_\lambda \to 0$, $r_\lambda \to \bar{r}$, $s_\lambda \to \bar{s}$ and $\bar{u}_\lambda \to w$ pointwise with $|\bar{u}_\lambda| \leq 1$, we can pass to the limit getting

$$\frac{w'(r)}{M_\lambda} = \bar{u}'(r) \to \frac{1}{\bar{r}^2} \int_{\bar{r}}^{r} s^2 wds = w'(\bar{r}) = -\frac{a \sqrt{\lambda}}{\bar{r}}$$

where last equality follows from (3.3) and the fact that $w'(\bar{s}) = 0$. Comparing it with (2.19) we infer

$$\|u_\lambda\|_{L^\infty(A_\lambda)} = M_\lambda \sim \sqrt{3\lambda V'1(1,0)} \sqrt{\lambda r^2 - \frac{\pi^2}{4}}$$

Inserting (3.2) and (3.1) in this last formula gives (3.3). □

In Propositions 2.6 and 3.1 we have depicted the profile of $u_\lambda$ in the sets $(0, r_\lambda)$ and $(r_\lambda, 1)$, respectively, in terms of the quantity $\lambda r_\lambda^2 - \pi^2/4$ which is infinitesimal as $\lambda \to \bar{\lambda}$. Eventually we complete the proof of Theorem 1.1 by describing the asymptotics of $\lambda r_\lambda^2 - \pi^2/4$ in terms of $\lambda - \bar{\lambda}$.

**Proof of Theorem 1.1** Equation (3.1) states that if $\|u_\lambda\| \to \infty$ then $\lambda \to \bar{\lambda} = (\frac{2m-1}{2})^2$ and $r_\lambda \to \bar{r} = 1\frac{1}{2m-1}$. We let $\bar{u}_\lambda$ as in (3.4) and

$$w(x) = -\frac{2\theta_o}{(2m-1)\pi \cos \theta_o} \frac{1}{|x|} \cos \left( \frac{2m-1}{2} \pi |x| \right)$$

the limit function in (3.2), which solves (3.3). Multiplying the equation in (3.5) by $w$, the one in (3.7) by $\bar{u}_\lambda$, integrating by parts on $(r_\lambda, 1)$ and subtracting gives

$$\int_{r_\lambda}^{1} r^2 \bar{u}_\lambda wdr + (M_\lambda)^4 \int_{r_\lambda}^{1} r^2 |\bar{u}_\lambda|^4 \bar{u}_\lambda wdr - r_\lambda^2 (\bar{u}_\lambda)'(r_\lambda)w(r_\lambda) = 0.$$  

Here when $\lambda \to \bar{\lambda}$ we have

$$\int_{r_\lambda}^{1} r^2 \bar{u}_\lambda wdr \to \int_{\bar{r}}^{1} r^2 w^2 dr > 0,$$

$$\int_{r_\lambda}^{1} r^2 |\bar{u}_\lambda|^4 \bar{u}_\lambda wdr \to \int_{\bar{r}}^{1} r^2 w^6 dr > 0,$$

$$r_\lambda^2 \bar{u}_\lambda'(r_\lambda) \to \bar{r}^2 w'(\bar{r}), \quad w(r_\lambda) = (w'(\bar{r}) + o(1)) (r_\lambda - \bar{r})$$

Since $\lambda r_\lambda^2 - \pi^2 = r_\lambda^2 (\lambda - \bar{\lambda}) + \bar{\lambda}(r_\lambda - \bar{r})(r_\lambda - \bar{r})$, then

$$r_\lambda - \bar{r} = \frac{1}{\lambda (r_\lambda + \bar{r})} \left[ (\lambda r_\lambda^2 - \pi^2/4) - r_\lambda^2 (\lambda - \bar{\lambda}) \right]$$

$$= \frac{1}{2 \lambda r} \left( 1 + o(1) \right) \left[ (\lambda r_\lambda^2 - \pi^2/4) - (\bar{r}^2 + o(1)) (\lambda - \bar{\lambda}) \right],$$

so that

$$r_\lambda^2 (\bar{u}_\lambda)'(r_\lambda)w(r_\lambda) = \left( \frac{\bar{r}^2 (w'(\bar{r}))^2}{2\lambda} + o(1) \right) \left( \lambda r_\lambda^2 - \pi^2/4 \right) - \left( \frac{\bar{r}^2 (w'(\bar{r}))^2}{2\lambda} + o(1) \right) (\lambda - \bar{\lambda}).$$
Furthermore \((M_\lambda)^4\) is negligible compared to \(\lambda r_\lambda^2 - \frac{\pi^2}{4}\) by (3.8). Eventually (3.8) and (3.10) imply
\[
\left(\int_0^1 \lambda r_\lambda^2 dr + \frac{\bar{r}^3 (w'(\bar{r}))^2}{2\lambda} + o(1)\right) (\lambda - \bar{\lambda}) = \left(\bar{r} (w'(\bar{r}))^2 + o(1)\right) \left(\lambda r_\lambda^2 - \frac{\pi^2}{4}\right),
\]
and recalling the explicit form of \(w\) in (3.3) we end up with
\[
\lambda r_\lambda^2 - \frac{\pi^2}{4} \sim \frac{4m - 3}{(2m - 1)^2} (\lambda - \bar{\lambda}) \tag{3.11}
\]
Remembering Lemma 2.4 it follows that \(\lambda \to \bar{\lambda}\) from above. Next inserting (3.11) into (2.18), (2.19), (3.9) and (3.3) completes the proof of (1.5), (1.6) and (1.7), respectively. Finally (1.8) follows by (3.2) and (3.11).

3.2. The case \(N = 4, 5\).
In this section we write \(\psi_h\) meaning the \(h\)th radial eigenfunction of \(-\Delta\) normalized so that
\[
\psi_h(0) = -1.
\]
As in the case \(N = 3\), next proposition reduces the rate of \(\|u_\lambda\|_{L^\infty(A_\lambda)}\) to that of \(r_\lambda\). Note that the computation below also shows that
\[
\bar{\lambda} = \mu_{m-1}\tag{3.12}
\]
giving an alternative proof to the same statement in [2, Theorem B].

**Proposition 3.2.** Let \(u_\lambda\) be any radial nodal solution to (1.1) with \(m \geq 2\) nodal zones in dimensions \(N = 4, 5\), \(r_\lambda\) its first zero in \((0, 1)\), \(A_\lambda\) be the annulus of radii \(r_\lambda, 1\) and \(\bar{\lambda}\) such that \(\|u_\lambda\|_\infty = u_\lambda(0) \to \infty\) when \(\lambda \to \bar{\lambda}\). Then
\[
\frac{u_\lambda(x)}{\|u_\lambda\|_{L^\infty(A_\lambda)}} \to \psi_{m-1}(x) \quad \text{in } C^1_{\text{loc}}(B \setminus \{0\}).
\]
Furthermore \(\|u_\lambda\|_{L^\infty(A_\lambda)} \to 0\), more precisely
\[
\|u_\lambda\|_{L^\infty(A_\lambda)} = \begin{cases} 8(1 + o(1))(r_\lambda^2 \|u_\lambda\|_{\infty})^{-1} & \text{if } N = 4, \\ \left(\frac{1}{8}\right)^{\frac{1}{2}} \left(\frac{2M_{m-1}}{\mu_{m-1}}\right)^{\frac{1}{2}} (1 + o(1)) r_\lambda^2 & \text{if } N = 5. \end{cases}\tag{3.14}
\]

**Proof.** We write
\[
M_\lambda := \|u_\lambda\|_{L^\infty(A_\lambda)} = -u_\lambda(s_\lambda),
\]
where \(s_\lambda\) stands for the first minimum of \(u_\lambda\) according to (2.12), and look at the normalized function
\[
\tilde{u}_\lambda(x) = \begin{cases} u_\lambda(x)/M_\lambda & \text{if } r_\lambda \leq |x| \leq 1, \\ 0 & \text{if } |x| < r_\lambda \end{cases}\tag{3.15}
\]
which solves
\[
\begin{cases} -\Delta \tilde{u}_\lambda = (M_\lambda)^{2^*-2} \tilde{u}_\lambda^{2^*-2} \tilde{u}_\lambda + \lambda \tilde{u}_\lambda & \text{in } A_\lambda, \\ |\tilde{u}_\lambda| \leq 1 & \text{in } A_\lambda, \\ \tilde{u}_\lambda = 0 & \text{on } \partial A_\lambda. \end{cases}\tag{3.16}
\]
The function \(\tilde{u}_\lambda\) is uniformly bounded both in \(L^\infty(B)\) and in \(H^1_{0,\text{rad}}(B)\) since we know by (2.23) that \(M_\lambda \to 0\) as \(\lambda \to \bar{\lambda}\). Remembering that also \(r_\lambda \to 0\) by (2.17), it is easy to see that \(\tilde{u}_\lambda\) converges weakly in \(H^1_{0,\text{rad}}(B)\) to a radial bounded function \(\psi \in H^1_{0,\text{rad}}(B)\) which is a weak solution to
\[
\begin{cases} -\Delta \psi = \bar{\lambda} \psi & \text{in } B, \\ \psi = 0 & \text{on } \partial B. \end{cases}\tag{3.17}
\]
Therefore \(\psi(x) = A \psi_n(x)\) for some integer \(n\) and for some constant \(A\), which can possibly be equal to 0. Notice that
\[
|\tilde{u}_\lambda(r)| \leq \frac{1}{r^{N-1}} \left| \int_0^r t^{N-1} \left( (M_\lambda)^{2^*-2} |\tilde{u}_\lambda|^{2^*-2} \tilde{u}_\lambda + \lambda \tilde{u}_\lambda \right) dt \right| \leq Cr^{N/2^* - 1}\tag{3.18}
\]
so that for every $r > 0$, $|\tilde{u}_\lambda'(r)| \leq C$ and the Ascoli-Arzelà Theorem yields that $\tilde{u}_\lambda$ converges to $\psi$ in $C_{\text{loc}}(B \setminus \{0\})$. In the same manner (3.10) gives

$$|\tilde{u}_\lambda(r)^n| \leq \frac{C}{r} |\tilde{u}_\lambda'(r)| + C$$

and the convergence holds in $C^1_{\text{loc}}(B \setminus \{0\})$.

Next we claim that $s_\lambda \to 0$. Indeed if, instead $\tilde{s} = \lim s_\lambda > 0$ the uniform convergence implies that $\psi(\tilde{s}) = -1$ so that $A \neq 0$ and $\tilde{s} > 0$ is the global maximum of $|\psi|$ in $(0, 1)$ and this contradicts the fact that any nontrivial radial solution to (3.17) has global maximum (or minimum) in the origin.

Hence for every $r > 0$ we may assume that $s_\lambda < r$, so that

$$\tilde{u}_\lambda(r) = \tilde{u}_\lambda(s_\lambda) + \int_{s_\lambda}^r \tilde{u}_\lambda'(t) dt = -1 + \int_{s_\lambda}^r \tilde{u}_\lambda'(t) dt.$$  

On the other hand estimate (3.18) gives

$$(3.19) \quad -1 \leq \tilde{u}_\lambda(r) \leq -1 + C \int_{s_\lambda}^r t dt \leq -1 + \frac{C}{2} r^2,$$

since with the pointwise convergence gives $\psi(0) = -1$, implying at once that $\psi$ is not trivial and $A = 1$. Further $\psi$ admits $n - 1$ nodal regions. Indeed (3.19) ensures also that the first nodal zone do not collapse to $r = 0$, since, denoting by $\tilde{r}_\lambda$ the first zero of $\tilde{u}_\lambda$ in $(r_\lambda, 1)$ we have $0 = \tilde{u}_\lambda(\tilde{r}_\lambda) \leq -1 + C(\tilde{r}_\lambda^2 - s_\lambda^2)$. Next, the $C_{\text{loc}}$ convergence in $B \setminus \{0\}$ and the Rolle Theorem imply that any nodal zone of $\tilde{u}_\lambda$ cannot disappear as one can see repeating the proof in the case when $N = 3$.

Then $\lambda = \mu_{m-1}$, $n = m - 1$ and the proof of (3.12), (3.13) is completed.

Finally we prove (3.14). By (2.27) we have that

$$M_\lambda = -u_\lambda(s_\lambda) = -\frac{1}{N - 2} r_\lambda u'_\lambda(r_\lambda) + \frac{1}{N - 2} \int_{r_\lambda}^{s_\lambda} r |u_\lambda|^2 - 2 u_\lambda + \lambda u_\lambda \, dr$$

and, using $M_\lambda \to 0$

$$\left| \int_{r_\lambda}^{s_\lambda} r |u_\lambda|^2 - 2 u_\lambda + \lambda u_\lambda \, dr \right| \leq C s_\lambda^2 M_\lambda = o(M_\lambda)$$

since we have already proved that $s_\lambda \to 0$ as $\lambda \to \mu_{m-1}$. Then

$$M_\lambda \sim -\frac{1}{N - 2} r_\lambda u'_\lambda(r_\lambda)$$

from which (3.14) follows recalling the behaviour of $u'_\lambda(r_\lambda)$ in (2.22).

Proof of Theorem 1.2 From Proposition 3.2 follows that $\|u_\lambda\|_{\infty} \to \infty$ can happen if and only if $\lambda \to \mu_{m-1}$ and implies that $r_\lambda \to 0$. To get the other estimates we depict first the asymptotics of $r_\lambda$ in terms of the quantity $\mu_{m-1} - \lambda$. We consider the function $\tilde{u}_\lambda$ defined in (3.15) which solves (3.10) where $A_\lambda$ is the annulus of radii $r_\lambda$ and $1$ and $M_\lambda = \|u_\lambda\|_{L^\infty(A_\lambda)}$. Multiplying equation (3.10) by $\psi_{m-1}$, the one in (3.17) by $\tilde{u}_\lambda$, integrating by parts on $(r_\lambda, 1)$ and subtracting gives

$$(3.20) \quad (\mu_{m-1} - \lambda) \int_{r_\lambda}^{1} r^{N-1} \tilde{u}_\lambda \psi_{m-1} \, dr = (M_\lambda)^{2^* - 2} \int_{r_\lambda}^{1} r^{N-1} |\tilde{u}_\lambda|^{2^* - 2} \tilde{u}_\lambda \psi_{m-1} \, dr - r_\lambda^{N-1} \tilde{u}_\lambda(r_\lambda) \psi_{m-1}(r_\lambda).$$

Next, we observe that by (3.15)

$$\int_{r_\lambda}^{1} r^{N-1} \tilde{u}_\lambda \psi_{m-1} \, dr \to \int_{0}^{1} r^{N-1}(\psi_{m-1})^2 \, dr = A_1 > 0$$

and $\psi_{m-1}(r_\lambda) \to -1$ since $r_\lambda \to 0$ by (2.17). Dividing by $\int_{r_\lambda}^{1} r^{N-1} \tilde{u}_\lambda \psi_{m-1} \, dr (3.20)$ gives

$$(3.21) \quad \mu_{m-1} - \lambda = \left( \frac{A_2}{A_1} + o(1) \right)(M_\lambda)^{2^* - 2} + \left( \frac{1}{A_1} + o(1) \right) r_\lambda^{N-1} \tilde{u}_\lambda(r_\lambda).$$
Moreover it is known by (3.14) and (2.22) that, as \( \lambda \to \mu_{m-1} \)

\[
(M\lambda)^{2^*-2} \sim \begin{cases} 
\frac{5}{3} \left( \frac{r\mu_{m-1}}{R} \right)^2 r^2_{\lambda} & \text{if } N = 5, \\
\frac{16}{\lambda_1 \|u\|_{\infty}} & \text{if } N = 4,
\end{cases}
\]

Therefore, depending on the dimension, the first or the second term in (3.21) dominates. Eventually as \( \lambda \to \mu_{m-1} \)

\[
\mu_{m-1} - \lambda \sim \begin{cases} 
\frac{5\lambda_1}{3} \left( \frac{r\mu_{m-1}}{R} \right)^2 r^2_{\lambda} & \text{if } N = 5, \\
\frac{16}{\lambda_1 \|u\|_{\infty}} & \text{if } N = 4,
\end{cases}
\]

showing that \( \lambda \) goes to \( \mu_{m-1} \) from below in dimension \( N = 5 \), and from above in dimension \( N = 4 \). In dimension \( N = 5 \), inverting this relation gives (4.1), next (1.9) and (1.12) easily follow from (2.21) and from (3.13), (3.14), respectively. In dimension \( N = 4 \), instead, inverting (3.22) provides (1.9). Moreover by (2.21) we have

\[
\log(\lambda - \mu_{m-1}) \sim -\log \|u\|_{\infty} \sim -2r^2_{\lambda} \mu_{m-1},
\]

from which follows

\[
r_{\lambda} \sim \sqrt{\frac{-2}{\mu_{m-1} \log(\lambda - \mu_{m-1})}},
\]

that is (1.9). Eventually inserting (1.9) into (3.13), (3.14) gives (1.12).

4. The case \( N = 6 \)

In this section we consider the most delicate case \( N = 6 \). As mentioned in the Introduction, the strategy of the proof of the cases \( N = 3,4,5 \) here does not work. Indeed, although the blow-up estimates of Section 2 hold, the integral identities (3.8) and (3.20) allow to identify the concentration

\[
H_{\lambda}
\]

and the pointwise convergence of \( u_{\lambda} \) to zero in \((0, r_{\lambda})\), and not the rates of the relevant quantities. We start with a result that proves (1.15) and (1.16) and then we will give the details of the proof of main part of this section. Another proof of this first result can be found in [2] Theorem 2 in terms of the Emden-Fowler transformation. This is an alternative proof without using this transformation.

Lemma 4.1 (Proof of (1.15) and (1.16) of Theorem 1.3). Let \( u_{\lambda} \) be any radial solution to (1.1) with \( m \geq 2 \) nodal zones in dimension \( N = 6 \), and assume that \( \|u\|_{\infty} = u_{\lambda}(0) \to \infty \) when \( \lambda \to \lambda \) > 0. Then

\[
(4.1)
\]

where \( u^{m-1} \) is the unique radial solution to (1.13) and \( \tilde{\lambda} \) is the unique value at which (1.13) admits a radial solution.

Proof. By (2.22) we know that the function \( u_{\lambda} \) is bounded in \((r_{\lambda}, 1)\) and satisfies \( u_{\lambda}(r_{\lambda}) = 0 \). We extend \( u_{\lambda} \) to zero in \((0, r_{\lambda})\) so that, denoting by \( \tilde{u}_{\lambda} \) the extended functions, by (1.1) we have

\[
\int_0^1 |\nabla \tilde{u}_{\lambda}|^2 dx = \int_{r_{\lambda}}^1 r^5 (u_{\lambda})^2 dr = \lambda \int_{r_{\lambda}}^1 r^5 (u_{\lambda})^2 dr + \int_{r_{\lambda}}^1 r^5 (u_{\lambda})^2 dr \leq C.
\]

Then \( \tilde{u}_{\lambda} \) converges to a function \( u_* \in H^1_0(B) \) as \( \lambda \to \tilde{\lambda} \), up to a subsequence. The convergence is weak in \( H^1_0(B) \) and a.e. in \( B \). Next, again by (1.1), we have that

\[
\tilde{u}_{\lambda}(r) = \int_r^1 s^{-5} \int_{s_{\lambda}}^s r^5 (\lambda u + |u|_{\lambda}) dt ds,
\]

(4.2)

\[
(\tilde{u}_{\lambda})'(r) = -\frac{1}{r^3} \int_{s_{\lambda}}^s t^5 (\lambda u + |u|_{\lambda}) dt,
\]

for every \( r > r_{\lambda} \). The boundedness of \( u_{\lambda} \) in \((s_{\lambda}, 1)\) shows that

\[
(4.4)
\]

and the pointwise convergence of \( \tilde{u}_{\lambda} \) jointly with \( r_{\lambda} \to 0 \) as \( \lambda \to \tilde{\lambda} \) allows to pass to the limit in the previous integrals implying that the convergence of \( \tilde{u}_{\lambda} \) to \( u_* \) holds in \( C_{loc}^1 (B \setminus \{0\}) \).

Furthermore the weak convergence in \( H^1_0(B) \) of \( \tilde{u}_{\lambda} \) to \( u_* \) gives

\[
\int_B \nabla u_* \cdot \nabla \psi dx = \tilde{\lambda} \int_B u_* \psi dx + \int_B |u_*| u_* \psi dx
\]
for every $\psi \in C^1_0(B)$. The pointwise convergence then implies that $|u_*| \leq C, ||(u_*)'|| \leq C$ and that $u^*$ is a classical solution to $-\Delta u^* = \lambda u^* + |u^*|u^*$ in $B$. Next we show that $M_\lambda \to M > 0$ as $\lambda \to \lambda$. Using (2.24) we have

\begin{equation}
M_\lambda = -\mu_\lambda(s_\lambda) = -\frac{r}{4} u_\lambda'(r_\lambda) - \frac{1}{4} \int_{r_\lambda}^{s_\lambda} r \left(|u_\lambda|^2 + \lambda |u_\lambda|\right) dr.
\end{equation}

Suppose by contradiction that $M_\lambda \to 0$, then $u_* \equiv 0$ and, passing to the limit in (4.2) or in (4.3). Indeed if a nodal domain disappears then passing to the limit into (4.5) and using (2.22) yields

\begin{equation}
0 = \frac{\lambda}{2},
\end{equation}

that gives a contradiction with $\lambda > 0$. Hence $M_\lambda \to M > 0$, and writing (4.2) with $r = s_\lambda$, passing to the limit we get

\begin{equation}
0 < M = \int_{s_1}^1 s^{-5} \int_{s_1}^s \tilde{\rho} \left(\lambda u_* + |u_*|u_*\right) dtds
\end{equation}

where $s_1 = \lim s_\lambda \geq 0$. This shows that $u_* \neq 0$ as well. Then, by the definition of $\tilde{u}_\lambda$ the function $u_*$ has at most $m - 1$ nodal zones in $(0, 1)$. Next we characterize the value of $M = \lim M_\lambda$. Assume that $r_2$ is the first zero of $u_*$ in the interval $(0, 1]$. Then $u_*$ (or $-u_*$) is a positive radial solution to (1.13) in $(0, r_2)$ and so, by the monotonicity result in [13] it satisfies $u_*'(0) = 0$. The convergence of $u_\lambda$ to $u_*$ in $C^1_{\text{loc}}(B \setminus \{0\})$ then implies that $s_\lambda \to 0$ as $\lambda \to \lambda$. The boundedness of $u_\lambda$ in $(r_\lambda, 1)$ and the fact that $r_\lambda, s_\lambda \to 0$ as $\lambda \to \lambda$ gives that

\begin{equation}
\int_{r_\lambda}^{s_\lambda} r \left(|u_\lambda|^2 + \lambda |u_\lambda|\right) dr \to 0
\end{equation}

so that passing to the limit into (4.5) and using (2.22) yields

\begin{equation}
\lim_{\lambda \to \lambda} \tilde{u}_\lambda(s_\lambda) = -\lim_{\lambda \to \lambda} \frac{r_\lambda}{4} u_\lambda'(r_\lambda) = -\frac{\lambda}{2}.
\end{equation}

Afterwards the pointwise convergence of $\tilde{u}_\lambda$ to $u^*$ and (2.24) implies that

\begin{equation}
|u^*(0) - \tilde{u}_\lambda(s_\lambda)| \leq |u^*(0) - u^*(\bar{r})| + |u^*(\bar{r}) - u_\lambda(\bar{r})| + |u_\lambda(s_\lambda) - u_\lambda(\bar{r})| \\
\leq C|\bar{r}| + |u^*(\bar{r}) - u_\lambda(\bar{r})| + \max_{r \in (s_\lambda, \bar{r})} |u_\lambda'(r)||s_\lambda - \bar{r}| \leq 2C|\bar{r}| + |u^*(\bar{r}) - u_\lambda(\bar{r})| < \varepsilon
\end{equation}

for any $\varepsilon > 0$, if $\lambda \to \lambda$ and $\bar{r}$ are sufficiently small. This proves that $u^*(0) = -\frac{\lambda}{2}$. Eventually we prove that $u^*$ has exactly $m - 1$ nodal zones showing that $u^* = u^{m-1}$ as defined in (1.13). It follows passing to the limit in (4.2) or in (4.3). Indeed if a nodal domain disappears then $u_*$ should satisfy $u_*(\bar{r}) = u^*(\bar{r})$ for some $\bar{r} \in [0, 1]$, which is not possible.

As we said before, the main difficulty in the case $N = 6$ is the computation of the rates of $r_\lambda$ and $\|u_\lambda\|_{\infty}$ with respect to $\lambda - \lambda$. Our strategy goes as follows: first, we build a radial nodal solution to (1.1) using the Ljapunov-Schmidt procedure, next we deduce the asymptotics of this solution and finally we prove the theorem using the uniqueness of the radial solution (as proved in Proposition (2.11)).

First of all, it is necessary to introduce some assumptions which are crucial in our argument and whose validity will be discussed later.

Let $\tilde{u}^{m-1}$ be a radial solution to

\begin{equation}
\begin{cases}
-\Delta \tilde{u}^{m-1} = |\tilde{u}^{m-1}|^{\tilde{u}^{m-1}} + \lambda \tilde{u}^{m-1} & \text{in } B, \\
\tilde{u}^{m-1}(0) = \frac{\lambda}{2} \\
\tilde{u}^{m-1} \text{ has } m - 1 \text{ nodal zones} \\
\tilde{u}^{m-1} \neq 0 & \text{on } \partial B.
\end{cases}
\end{equation}

Observe that $\tilde{u}^{m-1} = -u^{m-1}$ where $u^{m-1}$ is as defined in (1.13). We change this notations because it is easier for us to construct the solution to (1.1) using as limit function $\tilde{u}^{m-1}$ instead of $u^{m-1}$. The solution we will find then will satisfies $u_\lambda(0) < 0$ and we will recover our solution just multiplying by $-1$. 

We also need to assume that \( \bar{u}^{m-1} \) is non-degenerate, i.e.

\[
\begin{align*}
\left\{ \begin{array}{l}
- \Delta v = (2|u^{m-1}| + \lambda)v \text{ in } B \\
v = 0 \text{ on } \partial B.
\end{array} \right. \quad \Rightarrow \quad v \equiv 0
\end{align*}
\]

If \( \bar{v}_0 \) is the solution of

\[
\begin{align*}
\left\{ \begin{array}{l}
- \Delta \bar{v}_0 - (2|\bar{u}^{m-1}| + \bar{\lambda})\bar{v}_0 = \bar{u}^{m-1} \text{ in } B \\
\bar{v}_0 = 0 \text{ on } \partial B,
\end{array} \right. \nonumber
\end{align*}
\]

whose existence is due to the non-degeneracy of \( u^{m-1} \) (or of \( \bar{u}^{m-1} \)), we finally need to suppose that

\[
\begin{align*}
2\bar{v}_0(0) \neq 1 \nonumber
\end{align*}
\]

As before we have that \( \bar{v}_0 = -v_0 \) where \( v_0 \) is as defined in (1.14).

Next, we need to introduce the well-known bubbles (see [4, 9, 24])

\[
U_\delta(x) := \delta^{-2}U\left(\frac{x}{\delta}\right), \text{ with } \delta > 0, 0 \in \mathbb{R} \text{ and } U(y) := \frac{\alpha_6}{(1 + |y|^2)^2}, \alpha_6 := 24,
\]

which are all the radial positive solutions of the Sobolev critical equation

\[-\Delta U = U^2 \text{ in } \mathbb{R}^6.\]

We denote by \( PU_\delta \) the projection onto \( H^1_0(B) \), i.e.

\[
\begin{align*}
\left\{ \begin{array}{l}
- \Delta PU_\delta = U_\delta^2 \text{ in } B \\
PU_\delta = 0 \text{ on } \partial B.
\end{array} \right. \nonumber
\end{align*}
\]

Finally we consider the problem

\[
\begin{align*}
\left\{ \begin{array}{l}
- \Delta u = |u|^r + (\bar{\lambda} + \varepsilon)u \text{ in } B, \\
u = 0 \text{ on } \partial B,
\end{array} \right. \nonumber
\end{align*}
\]

where \( \varepsilon \) is small enough (not necessarily positive).

Our existence result reads as follows.

**Theorem 4.2.** Assume (4.7) and (4.9). There exists \( \varepsilon_0 > 0 \) such that

(1) if \( 1 - 2\bar{v}_0(0) > 0 \) and \( \varepsilon \in (0, \varepsilon_0) \),

or

(2) if \( 1 - 2\bar{v}_0(0) < 0 \) and \( \varepsilon \in (-\varepsilon_0, 0) \)

then there exists a radial nodal solution \( u_\varepsilon \) to (4.12) with \( m \) nodal regions which blows-up in the origin as \( \varepsilon \to 0 \). More precisely

\[
\begin{align*}
u_\varepsilon(x) = \bar{u}^{m-1}(x) + \varepsilon \bar{v}_0(x) - PU_\varepsilon(x) + \phi_\varepsilon(x)
\end{align*}
\]

with as \( \varepsilon \to 0 \)

\[
\begin{align*}
\delta_\varepsilon \varepsilon^{-1} \to d = \frac{8\sqrt{3}}{11} \frac{|1 - 2\bar{v}_0(0)|}{\bar{\lambda}^{\frac{1}{2}}} > 0
\end{align*}
\]

and

\[
\|\phi_\varepsilon\|_{H^1_0(B)} = O\left(\varepsilon^2|\ln|\varepsilon||^{\frac{1}{2}}\right).\nonumber
\]

**4.1. Proof of Theorem 4.2**

4.1.1. Setting of the problem and the choice of the ansatz. In what follows we denote by

\[
(u, v) := \int_B \nabla u \nabla v \, dx, \quad \|u\| := \left(\int_B |\nabla u|^2 \, dx\right)^{\frac{1}{2}}
\]

the inner product and its correspond norm in \( H^1_0(B) \) while we denote by

\[
|u|_r := \left(\int_B |u|^r \, dx\right)^{\frac{1}{r}}.
\]
the $L^*(B)$ standard norm. When $A \neq B$ is any Lebesgue measurable set we specify the domain of integration by using $\| v \|_{A,r}$. Let $(\Delta)^{-1} : L^2(B) \to H^1_0(B)$, be the operator defined as the unique solution of the equation

$$-\Delta u = v \quad \text{in } B \quad u = 0 \quad \text{on } \partial B.$$ 

By the Holder inequality it follows that

$$\| (\Delta)^{-1} (v) \| \leq C \| v \|^{\frac{1}{2}} \quad \forall v \in L^2(B)$$

for some positive constant $C$, which does not depend on $v$. Hence we can rewrite problem (4.12) as

$$(4.15) \quad u = (\Delta)^{-1} [f(u) + (\lambda + \varepsilon)u] \quad u \in H^1_{0,rad}(B)$$

with $f(u) = |u|u$.

Next we remind the expansion of the projection of the bubble defined in (4.11). As before we denote by $G(x,y)$ the Green’s function of the Laplace operator given by

$$(4.16) \quad G(x,y) = \frac{1}{4\sigma_6} \left( \frac{1}{|x-y|^4} - H(x,y) \right)$$

where $\sigma_6$ denotes the surface area of the unit sphere in $\mathbb{R}^6$ and $H$ is the regular part of the Green’s function, namely for all $y \in B$, $H(x,y)$ satisfies

$$\Delta H(x,y) = 0 \quad \text{in } B \quad H(x,y) = \frac{1}{|x-y|^4} \quad x \in \partial B.$$ 

It is known that the following expansion holds (see [21])

$$(4.17) \quad PU_\delta(x) = U_\delta(x) - \alpha_6 \delta^2 H(x,0) + O(\delta^4) \quad \text{as } \delta \to 0$$

uniformly in $B$.

Moreover we recall (see [6]) that all the solutions to the linear equation

$$-\Delta \psi = 2U_\delta \psi \quad \text{in } \mathbb{R}^6$$

in $H^1_{0,rad}(B)$, i.e. the subspace of radial functions in $H^1_0(B)$ are given by

$$\psi(x) = cZ_\delta(x), \quad \text{with } Z_\delta(x) = \partial_\theta U_\delta(x) = 2\alpha_6 \delta \frac{|x|^2 - \delta^2}{\delta^2 + |x|^2} \text{ and } c \in \mathbb{R}$$

for $U_\delta$ and $\alpha_6$ as in (4.10). Let $PZ_\delta$ be the projection of $Z_\delta$ onto $H^1_0(B)$, i.e.

$$(4.18) \quad -\Delta PZ_\delta = f'(U_\delta)Z_\delta \quad \text{in } B \quad \text{and } PZ_\delta = 0 \quad \text{on } \partial B,$$

elliptic estimates give

$$PZ_\delta(x) = Z_\delta(x) - 2\delta \alpha_6 H(x,0) + O(\delta^3) \quad \text{as } \delta \to 0$$

uniformly in $B$, see [21].

We look for a radial solution of (4.12) of the form

$$(4.19) \quad u_\varepsilon(x) = \bar{u}^{m-1}(x) + \varepsilon \bar{v}_0(x) = PU_\delta(x) + \phi_\varepsilon(x)$$

where $\delta$ are chosen so that

$$\delta = |\varepsilon|d \quad \text{with } d \in \left( \sigma, \frac{1}{\sigma} \right) \text{ where } \sigma > 0 \text{ is small}$$

and $\phi_\varepsilon$ is a radial remainder term, which is small as $\varepsilon \to 0$, which belongs to the space $K^+_\delta$ defined by

$$K_\delta := \{ \phi \in H^1_0,rad(B) : \phi = cPZ_\delta, \quad c \in \mathbb{R} \} \text{ and } K^+_\delta := \{ \phi \in H^1_0,rad(B) : (\phi, PZ_\delta) = 0 \}.$$ 

Let us denote by $\Pi_\delta$ and $\Pi^+_\delta$ the projection of $H^1_0,rad(B)$ on $K_\delta$ and $K^+_\delta$ respectively. Then solving problem (4.15) is equivalent to solve the system

$$(4.21) \quad \Pi^+_\delta \{ u_\varepsilon(x) - (\Delta)^{-1} [f(u_\varepsilon) + (\lambda + \varepsilon)u_\varepsilon] \} = 0,$$

$$(4.22) \quad \Pi_\delta \{ u_\varepsilon(x) - (\Delta)^{-1} [f(u_\varepsilon) + (\lambda + \varepsilon)u_\varepsilon] \} = 0.$$
4.1.2. The remainder term: solving equation (4.21). The equation (4.21) can be written as

\[(4.23)\quad \mathcal{L}_\delta(\phi_\varepsilon) + \mathcal{R}_\delta + \mathcal{N}_\delta(\phi_\varepsilon) = 0\]

where

\[(4.24)\quad \mathcal{L}_\delta(\phi_\varepsilon) = \Pi_\delta^1 \left\{ \phi_\varepsilon(x) - (-\Delta)^{-1} [f'(W_\delta)\phi_\varepsilon + \lambda\phi_\varepsilon] \right\}\]

\[(4.25)\quad \mathcal{R}_\delta = \Pi_\delta^1 \left\{ W_\delta(x) - (-\Delta)^{-1} [f(W_\delta) + \lambda W_\delta] \right\}\]

\[(4.26)\quad \mathcal{N}_\delta(\phi_\varepsilon) = \Pi_\delta^1 \left\{-(-\Delta)^{-1} [f(W_\delta + \phi_\varepsilon) - f(W_\delta) - f'(W_\delta)\phi_\varepsilon] \right\}\]

where \(\mathcal{L}_\delta\) is the linearized operator at the approximate solution, \(\mathcal{R}_\delta\) is the error term and \(\mathcal{N}_\delta\) is a quadratic term in \(\phi_\varepsilon\) and, as before \(f(u) = \|u\|u\).

In what follows we estimate the \(H^1_{0,rad}(B)\) - norm of the error term \(\mathcal{R}_\delta\)

**Lemma 4.3.** For any \(\sigma > 0\) there exist \(c > 0\) and \(\varepsilon_0 > 0\) such that for any \(d > 0\) satisfying (4.20) and for any \(\varepsilon \in (-\varepsilon_0, \varepsilon_0)\)

\[\|\mathcal{R}_\delta\| \leq c\varepsilon^2 \ln \|\varepsilon\|^{3/2} \]  

**Proof.** First we remark that

\[\begin{align*}
-\Delta W_\delta - f(W_\delta) &= (\bar{\lambda} + \varepsilon)W_\delta = -\Delta \bar{u}^{m-1} - \varepsilon \Delta \bar{v}_0 - f(U_\delta) - f(\bar{u}^{m-1} + \varepsilon \bar{v}_0 - PU_\delta) \\
-\bar{\lambda} \bar{u}^{m-1} - \bar{\lambda} \varepsilon \bar{v}_0 + (\bar{\lambda} + \varepsilon)PU_\delta - \varepsilon \bar{u}^{m-1} - \varepsilon^2 \bar{v}_0 \\
&= -f(\bar{u}^{m-1} + \varepsilon \bar{v}_0 - PU_\delta) - U_\delta^2 + f(\bar{u}^{m-1}) + \varepsilon (\Delta \bar{v}_0 - \bar{\lambda} \bar{v}_0 - \bar{u}^{m-1}) + (\bar{\lambda} + \varepsilon)PU_\delta - \varepsilon^2 \bar{v}_0.
\end{align*}\]

By the continuity of \(\Pi_\delta^1\) we get that

\[\|\mathcal{R}_\delta\| \leq c \left| -\Delta W_\delta - f(W_\delta) - (\bar{\lambda} + \varepsilon)W_\delta \right|^{3/2} \leq c \left| -f(\bar{u}^{m-1} + \varepsilon \bar{v}_0 - PU_\delta) - (PU_\delta)^2 + f(\bar{u}^{m-1}) + 2\varepsilon |\bar{u}^{m-1}|\bar{v}_0 \right|^{3/2} + c \left| (PU_\delta)^2 - U_\delta^2 \right|^{3/2} + (\bar{\lambda} + \varepsilon) |PU_\delta|^{3/2} + \varepsilon^2 |\bar{v}_0|^{3/2} := O(\varepsilon^2)\]

First of all, we point out that

\[(III) \leq c |U_\delta|^{3/2} \leq c\delta^2 \ln \delta^{3/2}\]

and by (4.17)

\[(II) \leq c \left( \int_B |PU_\delta - U_\delta|^{3/2} |PU_\delta + U_\delta|^{3/2} \right)^{3/2} \leq c\delta^2 \left( \int_B |U_\delta|^{3/2} dx \right)^{3/2} = O(\delta^4 \ln \delta^{3/2}).\]
First let us estimate (I) in $B(0, \sqrt{\delta})$ and $B \setminus B(0, \sqrt{\delta})$:

\[
(I) \leq c \left( \int_{B(0, \sqrt{\delta})} |f(\bar{u}^{m-1} + \varepsilon \bar{v}_0 - PU) + (PU)_\delta|^{\frac{2}{p}} \right)
+ c \left( \int_{B(0, \sqrt{\delta})} |f(\bar{u}^{m-1} + \varepsilon f'(\bar{u}^{m-1})\bar{v}_0|^{\frac{2}{p}} dx \right)
+ c \left( \int_{B \setminus B(0, \sqrt{\delta})} |f(\bar{u}^{m-1} + \varepsilon \bar{v}_0 - PU) - f(\bar{u}^{m-1}) - f'(\bar{u}^{m-1})(\varepsilon \bar{v}_0 - PU)\right|^{\frac{2}{p}} dx)
+ c \left( \int_{B \setminus B(0, \sqrt{\delta})} |(PU)_\delta + f'(\bar{u}^{m-1})PU|^{\frac{2}{p}} dx \right)
= O \left( \delta^2 \ln \delta \right),
\]

since by mean value Theorem (here $\theta \in [0, 1]$)

\[
\int_{B(0, \sqrt{\delta})} |f(\bar{u}^{m-1} + \varepsilon \bar{v}_0 - PU) + (PU)_\delta|^{\frac{2}{p}} \leq 2 \int_{B(0, \sqrt{\delta})} f'(\theta(\bar{u}^{m-1} + \varepsilon \bar{v}_0) - PU) \left(\bar{u}^{m-1} + \varepsilon \bar{v}_0\right)^{\frac{2}{p}} dx,
\]

and by the inequality

\[
|f(a + b) - f(a) - f'(a)b| \leq 7b^2 \text{ for any } a, b \in \mathbb{R}
\]

\[
\int_{B \setminus B(0, \sqrt{\delta})} |(f(\bar{u}^{m-1} + \varepsilon \bar{v}_0 - PU) - f(\bar{u}^{m-1}) - f'(\bar{u}^{m-1})(\varepsilon \bar{v}_0 - PU)|^{\frac{2}{p}} dx
\]

\[
\leq c \left( \int_{B \setminus B(0, \sqrt{\delta})} \varepsilon \bar{v}_0 - PU \right|^{3} dx
+ \int_{B \setminus B(0, \sqrt{\delta})} |U|_3^3 dx \Rightarrow
\left( \int_{B \setminus B(0, \sqrt{\delta})} |f(\bar{u}^{m-1} + \varepsilon \bar{v}_0 - PU) - f(\bar{u}^{m-1}) - f'(\bar{u}^{m-1})(\varepsilon \bar{v}_0 - PU)|^{\frac{2}{p}} \right) \leq O(\varepsilon^2)
\]

and

\[
\int_{B \setminus B(0, \sqrt{\delta})} |f(\bar{u}^{m-1} + \varepsilon \bar{v}_0 - PU) + f'(\bar{u}^{m-1})PU|^{\frac{2}{p}} dx
\]

which ends the proof. □

Next we state the invertibility of the linear operator $L_\phi : \mathcal{K}_c^\phi \rightarrow \mathcal{K}_c^\phi$ defined in (4.24) and provide a uniform estimate of the norm of $L_\phi^{-1}$ (see for example [25], Lemma 2.4 or [22], Lemma 4.2).

**Lemma 4.4.** For any $\sigma > 0$ there exist $c > 0$ and $\varepsilon_0 > 0$ such that for any $d > 0$ satisfying (4.20) and for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$

$$
||L_\phi(\phi)|| \geq c ||\phi|| \text{ for any } \phi \in \mathcal{K}_c^\phi.
$$

Moreover, $L_\phi$ is invertible and $||L_\phi^{-1}|| \leq \frac{1}{c}$. We are in position now to find a solution of the equation (4.27) whose proof relies on a standard contraction mapping argument (see for example [20], Proposition 1.8 and [19], Proposition 2.1).
Proposition 4.5. For any $\sigma > 0$ there exist $c > 0$ and $\varepsilon_0 > 0$ such that for any $d > 0$ satisfying (4.20) and for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ there exists a unique $\phi_\varepsilon = \phi_\varepsilon(d) \in K_\delta^1$ solution to (4.21) which is continuously differentiable with respect to $d$ and such that

\begin{equation}
\|\phi_\varepsilon\| \leq c\varepsilon^2 |\ln |\varepsilon||^{\frac{1}{2}}.
\end{equation}

4.1.3. The reduced problem: solving equation (4.22). To solve equation (4.22), we shall find the parameter $\delta$ as in (4.19), i.e. $d > 0$, so that (4.22) is satisfied.

It is well known that this problem has a variational structure, in the sense that solutions of (4.22) reduces to find critical points to some given explicit function defined on $\mathbb{R}$. Indeed, let $J_\varepsilon : H^1_{rad}(B) \to \mathbb{R}$ defined by

\begin{equation}
J_\varepsilon(u) := \frac{1}{2} \int_B |\nabla u|^2 \, dx - \frac{\lambda + \varepsilon}{2} \int_B u^2 \, dx - \frac{1}{3} \int_B |u|^3 \, dx
\end{equation}

and let $\tilde{J}_\varepsilon : (0, +\infty) \to \mathbb{R}$ be the reduced energy which is defined by

\begin{equation}
\tilde{J}_\varepsilon(d) = J_\varepsilon(W_\delta + \phi_\varepsilon)
\end{equation}

where $W_\delta$ is as defined in (4.19) and $\phi_\varepsilon = \phi_\varepsilon(d)$ is the function found in Proposition 4.5.

Proposition 4.6. For any $\sigma > 0$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$

\begin{equation}
\tilde{J}_\varepsilon(d) = c_0(\varepsilon) + |\varepsilon|^3 \left\{ \text{sgn}(\varepsilon) |1 - 2\varphi_0(0)| d^2a_1 - d^3a_2 \right\} + o \left( |\varepsilon|^3 \right)
\end{equation}

uniformly with respect to $d$ in compact sets of $(0, +\infty)$, where $c_0(\varepsilon)$ only depends on $\varepsilon$ and the $a_i$'s are positive constants. Moreover, if $d$ is a critical point of $\tilde{J}_\varepsilon$, then $W_\delta + \phi_\varepsilon$ is a solution of (4.12).

Proof. It is quite standard to prove that if $d$ satisfies (4.20) and is a critical point of $\tilde{J}_\varepsilon$, then $W_\delta + \phi_\varepsilon$ is a solution of (4.12) (see for example [19], Proposition 2.2). Moreover, it is not difficult to check that $\tilde{J}_\varepsilon(d) = J_\varepsilon(W_\delta) + o \left( |\varepsilon|^3 \right)$ uniformly with respect to $d$ in compact sets of $(0, +\infty)$ (see for example [19], Proposition 2.2).

Let us estimate the main term of the reduced energy, i.e.

\begin{equation}
J_\varepsilon(\bar{u}^{m-1} + \varepsilon\bar{v}_0 - PU_\delta)
\end{equation}

\begin{align}
&= \frac{1}{2} \int_B |\nabla (\bar{u}^{m-1} + \varepsilon\bar{v}_0 - PU_\delta)|^2 - \frac{\lambda + \varepsilon}{2} \int_B (\bar{u}^{m-1} + \varepsilon\bar{v}_0 - PU_\delta)^2 - \frac{1}{3} \int_B |\bar{u}^{m-1} + \varepsilon\bar{v}_0 - PU_\delta|^3 \\
&= \frac{1}{2} \int_B |\nabla (\bar{u}^{m-1} + \varepsilon\bar{v}_0)|^2 + \frac{1}{2} \int_B |\nabla PU_\delta|^2 - \frac{\lambda + \varepsilon}{2} \int_B (\bar{u}^{m-1} + \varepsilon\bar{v}_0)^2 - \frac{\lambda + \varepsilon}{2} \int_B (PU_\delta)^2 \\
&= \left( \int_B |\nabla \bar{u}^{m-1} - \frac{\lambda}{2} \int_B \bar{v}_0 PU_\delta - \frac{\lambda}{2} \int_B \bar{v}_0 PU_\delta \right) - \varepsilon \left( \int_B |\nabla \bar{v}_0 PU_\delta| + \int_B \bar{v}_0 PU_\delta \right) \\
&= \int_B |\nabla \bar{u}^{m-1} + \varepsilon\bar{v}_0 - PU_\delta|^3 \\
&= \int_B |\nabla \bar{u}^{m-1} + \varepsilon\bar{v}_0|^2 - \frac{\lambda + \varepsilon}{2} \int_B (\bar{u}^{m-1} + \varepsilon\bar{v}_0)^2 - \frac{1}{3} \int_B |\bar{u}^{m-1} + \varepsilon\bar{v}_0|^3 \\
&= I_1 \\
&= \int_B |\nabla PU_\delta|^2 - \frac{1}{3} \int_B PU_\delta^3 - \frac{\lambda}{2} \int_B PU_\delta^2 + \int_B \bar{u}^{m-1} PU_\delta^2 - \frac{\varepsilon}{2} \int_B PU_\delta^2 + \varepsilon \int_B \bar{v}_0 PU_\delta^2 \\
&= I_2 + I_3 \\
&= \int_B (|\bar{u}^{m-1} + \varepsilon\bar{v}_0 - PU_\delta|^3 - |\bar{u}^{m-1} + \varepsilon\bar{v}_0|^3 - PU_\delta^3 + 3(\bar{u}^{m-1} + \varepsilon\bar{v}_0) PU_\delta^2 + 3|\bar{u}^{m-1} + \varepsilon\bar{u}^{m-1}| PU_\delta) \\
&= I_4 \\
&= \int_B |\bar{u}^{m-1} + \varepsilon\bar{v}_0| (\bar{u}^{m-1} + \varepsilon\bar{v}_0) - (|\bar{u}^{m-1} + \varepsilon\bar{v}_0|^2 + 2\varepsilon |\bar{u}^{m-1}| \bar{v}_0) PU_\delta + \varepsilon^2 \int_B \bar{v}_0 PU_\delta \\
&= I_5
\end{align}
It is clear that
\[ I_\tau = O \left( \varepsilon^2 \int_B \frac{\delta^2}{|x|^4} \, dx \right) = O \left( \varepsilon^2 \delta^2 \right) = O \left( \varepsilon^4 \right). \]

To estimate \( I_6 \) by (4.27), it follows that
\[ I_6 = O \left( \varepsilon^2 \int_B P\delta \right) = O \left( \varepsilon^2 \delta^2 \right) = O \left( \varepsilon^4 \right). \]

Now, \( I_1 \) does not depend on \( d \) and it will be included in the constant \( c_0 (\varepsilon) \) in (4.29). By (4.17)
\[
I_2 = \frac{1}{2} \int_B U_{\delta}^3 \, dx - \frac{1}{3} \int_B P\delta^3
\]
\[
= \frac{1}{2} \int_B U_{\delta}^3 (U_\delta(x) - \alpha_0 \delta^2 H(x, 0) + O(\delta^4) ) - \frac{1}{3} \int_B (U_\delta(x) - \alpha_0 \delta^2 H(x, 0) + O(\delta^4))^3
\]
\[
= \frac{1}{6} \int_B U_{\delta}^3 + O \left( \delta^2 \int_B U_{\delta}^3 \right) + O(\delta^4) = \frac{1}{6} \int U^3 + O(\delta^4).
\]

Now, setting \( \varphi_{\delta} := P\delta - U_\delta = O(\delta^2) \), by (4.17) and (4.20)
\[
I_3 = \int_B \left( \bar{u}^{m-1}(x) - \frac{1}{2} \right) (U_\delta + \varphi_{\delta})^2
\]
\[
= \int_B \left( \bar{u}^{m-1}(x) - \bar{u}^{m-1}(0) \right) U_{\delta}^2 + O(\delta^4)
\]
\[
= \int_B \left[ \frac{1}{2} (D^2 \bar{u}^{m-1}(0), x) + O(|x|^3) \right] \alpha_0^2 \frac{\delta^4}{(\delta^2 + |x|^2)^4} \, dx + O(\delta^4)
\]
\[
= \alpha_0^2 \int_B \frac{1}{2} (D^2 \bar{u}^{m-1}(0), x) \frac{\delta^4}{(\delta^2 + |x|^2)^4} \, dx + O(\delta^4)
\]
\[
= \alpha_0^2 \delta^2 \int \frac{1}{2} (D^2 \bar{u}^{m-1}(0), \delta y) \frac{1}{(1 + |y|^2)^4} \, dy + O(\delta^4)
\]
\[
= O(\delta^4 \ln \delta) = O(\varepsilon^4 \ln |\varepsilon|).
\]

and analogously
\[
I_4 = \varepsilon \int_B \left( \bar{v}_0(x) - \frac{1}{2} \right) P\delta^2 \, dx = \varepsilon \left( \alpha_0^2 \int \frac{1}{(1 + |y|^2)^2} \, dy \right) \left( \bar{v}_0(0) - \frac{1}{2} \right) + O(1)
\]
\[
= \varepsilon \delta^2 \left( \alpha_0^2 \int \frac{1}{(1 + |y|^2)^2} \, dy \right) \left( \bar{v}_0(0) - \frac{1}{2} \right) + O(1)
\]

Finally, we have to estimate \( I_5 \).

We point out that
\[
|\bar{u}^{m-1} + \varepsilon \bar{v}_0 - P\delta|^3 - |\bar{u}^{m-1} + \varepsilon \bar{v}_0|^3 - P\delta^3 + 3(\bar{u}^{m-1} + \varepsilon \bar{v}_0) P\delta^2 + 3(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^2 P\delta = 0 \text{ if } \bar{u}^{m-1} + \varepsilon \bar{v}_0 \leq 0
\]
and so
\[
I_5 = -\frac{1}{3} \int_{\{\bar{u}^{m-1} + \varepsilon \bar{v}_0 \leq 0\}} \left( |\bar{u}^{m-1} + \varepsilon \bar{v}_0 - P\delta|^3 - (\bar{u}^{m-1} + \varepsilon \bar{v}_0)^3 - P\delta^3 + 3(\bar{u}^{m-1} + \varepsilon \bar{v}_0) P\delta^2 + 3(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^2 P\delta \right) \, dx
\]
\[
= -\frac{1}{3} \int_{\{\bar{u}^{m-1} + \varepsilon \bar{v}_0 \geq 0\}} \left( -2P\delta^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0) P\delta^2 \right)
\]
\[
- \frac{1}{3} \int_{\{0 < \bar{u}^{m-1} + \varepsilon \bar{v}_0 < P\delta\}} \left( -2(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^2 P\delta \right) d\bar{u}^{m-1} + \varepsilon \bar{v}_0.
\]

First of all we claim that for any \( \sigma > 0 \) there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \) and \( d > 0 \) satisfying (4.20)
\[
(4.30) \quad B \left( 0, R_\sigma \sqrt{\delta} \right) \subset \{ x \in B : 0 < \bar{u}^{m-1}(x) + \varepsilon \bar{v}_0(x) < P\delta(x) \} \cap B \left( 0, \delta^2 \right) \subset B \left( 0, R_\sigma \sqrt{\delta} \right)
\]
where

\[
R_3^1, R_3^2 = R_0 + o(1) \text{ with } R_0 := \left(\frac{\alpha_6}{\alpha_{m-1}(0)}\right)^{\frac{1}{2}}. 
\]

We remind that $\delta = O(\varepsilon)$ and also that $PU_{\delta}(x) = \alpha_6 \frac{x^2}{(\varepsilon + |x|^2)^2} + O(\varepsilon^2)$ uniformly in $B$. If $|x| < R_1^1 \sqrt{\delta}$ is small enough then by mean value theorem $\bar{u}^{m-1}(x) + \varepsilon \bar{v}_0(x) = \bar{u}^{m-1}(0) + O_1(\varepsilon)$ and

\[
\bar{u}^{m-1}(x) + \varepsilon \bar{v}_0(x) < PU_{\delta}(x) \iff \frac{\bar{u}^{m-1}(0)}{\alpha_6} + O_1(\varepsilon) < \frac{\delta^2}{(\delta^2 + |x|^2)^2} 
\]

\[
\iff |x| \leq \sqrt{\delta} \left(\frac{1}{\left(\frac{\bar{u}^{m-1}(0)}{\alpha_6} + O_1(\varepsilon)\right)^{\frac{1}{2}}} - \delta\right)^{\frac{1}{2}} 
\]

and the first inclusion in (4.30) together with (4.31) follow. On the other hand, again by mean value theorem we have $\bar{u}^{m-1}(x) + \varepsilon \bar{v}_0(x) = \bar{u}^{m-1}(0) + O_2(\sqrt{\delta})$ for any $x \in B(0, \delta^{\frac{2}{3}})$ and arguing as above we get the second inclusion in (4.30) and (4.31).

It is useful to point out that by (4.30) we immediately get

\[
B^c \left(0, R_3^1 \sqrt{\delta}\right) \supset \{x \in B : \bar{u}^{m-1}(x) + \varepsilon \bar{v}_0(x) \geq PU_{\delta}(x)\} \cup B^c \left(0, \delta^{\frac{2}{3}}\right) \supset B^c \left(0, R_3^2 \sqrt{\delta}\right)
\]

Now by (4.30) and (4.31) we deduce

\[
I_5 = \frac{1}{3} \int_{\{\bar{u}^{m-1} + \varepsilon \bar{v}_0 \geq PU_{\delta}\}} (-2PU_{\delta}^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)PU_{\delta}^2)
\]

\[
- \frac{1}{3} \int_{\{0 < \bar{u}^{m-1} + \varepsilon \bar{v}_0 < PU_{\delta}\}} (-2(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^2PU_{\delta})
\]

\[
= \frac{1}{3} \int_{\{\bar{u}^{m-1} + \varepsilon \bar{v}_0 \geq PU_{\delta}\} \cup B^c \left(0, \delta^{\frac{2}{3}}\right)} (-2PU_{\delta}^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)PU_{\delta}^2)
\]

\[
+ \frac{1}{3} \int_{B^c \left(0, \delta^{\frac{2}{3}}\right) \setminus \{\bar{u}^{m-1} + \varepsilon \bar{v}_0 \geq PU_{\delta}\} \cap B^c \left(0, \delta^{\frac{2}{3}}\right)} (-2PU_{\delta}^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)PU_{\delta}^2)
\]

\[
- \frac{1}{3} \int_{\{0 < \bar{u}^{m-1} + \varepsilon \bar{v}_0 < PU_{\delta}\} \cap B \left(0, \delta^{\frac{2}{3}}\right)} (-2(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^2PU_{\delta})
\]

\[
- \frac{1}{3} \int_{\{0 < \bar{u}^{m-1} + \varepsilon \bar{v}_0 < PU_{\delta}\} \cap B^c \left(0, \delta^{\frac{2}{3}}\right)} (-2(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^2PU_{\delta})
\]

\[
= \frac{1}{3} \int_{\{\bar{u}^{m-1} + \varepsilon \bar{v}_0 \geq PU_{\delta}\} \cup B^c \left(0, \delta^{\frac{2}{3}}\right)} (-2PU_{\delta}^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)PU_{\delta}^2)
\]

\[
- \frac{1}{3} \int_{\{0 < \bar{u}^{m-1} + \varepsilon \bar{v}_0 < PU_{\delta}\} \cap B \left(0, \delta^{\frac{2}{3}}\right)} (-2(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^2PU_{\delta}) + o(\delta^3),
\]

because

\[
\int_{B^c \left(0, \delta^{\frac{2}{3}}\right) \setminus \{\bar{u}^{m-1} + \varepsilon \bar{v}_0 \geq PU_{\delta}\} \cap B^c \left(0, \delta^{\frac{2}{3}}\right)} (-2PU_{\delta}^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)PU_{\delta}^2) = O\left(\int_{B^c \left(0, \delta^{\frac{2}{3}}\right)} (U_{\delta}^3 + U_{\delta}^2)\right) = O\left(\delta^{\frac{2}{3}}\right),
\]

and, since $0 < \bar{u}^{m-1} + \varepsilon \bar{v}_0 < PU_{\delta}$

\[
\int_{\{0 < \bar{u}^{m-1} + \varepsilon \bar{v}_0 < PU_{\delta}\} \cap B \left(0, \delta^{\frac{2}{3}}\right)} (-2(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)^2PU_{\delta}) = O\left(\int_{B^c \left(0, \delta^{\frac{2}{3}}\right)} U_{\delta}^3\right) = O\left(\delta^{\frac{2}{3}}\right) = o(\delta^3)
\]

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We estimate the last two terms in the expansion of $I_5$. Recalling that $P\delta = U_\delta + O(\delta^2)$ uniformly in $B$ we have, for every $\hat{r} > 0$,
\[
\int_{\{\bar{u}^{m-1} + \varepsilon v_0 \geq P\delta \} \cup B(0, \hat{r})} (-2P\delta^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)P\delta^2) = \\
= \int_{\{x \in B_{\hat{r}} : \bar{u}^{m-1} + \varepsilon v_0 \geq P\delta \} \cup \{\delta^{\hat{r}} \leq |x| \leq \frac{\xi_1}{2}\}} (-2P\delta^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)P\delta^2) + O(\delta^4)
\]
Next, we let $\xi_1$ be the first zero of $\bar{u}^{m-1}$ in $(0, 1]$ so that $\bar{u}^{m-1} > 0$ in $(0, \xi_1)$. The function $\bar{u}^{m-1}$ is radially decreasing in $(0, \xi_1)$ (by (4.3)) and so we have $\bar{u}^{m-1} > \bar{u}^{m-1}(\frac{\xi_1}{2}) = c_1 > 0$ in $B_{\frac{\xi_1}{2}}$. Next we claim that
\[
2PU_\delta + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0) > 0 \text{ in } \{x \in B_{\frac{\xi_1}{2}} : \bar{u}^{m-1} + \varepsilon \bar{v}_0 \geq P\delta \} \cup \{\delta^{\hat{r}} \leq |x| \leq \frac{\xi_1}{2}\}
\]
if $\delta$ is small enough. This is easily true in the set $\{\bar{u}^{m-1} + \varepsilon \bar{v}_0 \geq P\delta \}$ since $6(\bar{u}^{m-1} + \varepsilon \bar{v}_0) - 2P\delta > 4P\delta \geq 0$. In the set $\{\delta^{\hat{r}} \leq |x| \leq \frac{\xi_1}{2}\}$ instead we have that $U_\delta \to 0$, and since $\bar{u}^{m-1} < c_1$ we get that (4.34) holds if $\delta$ and $\varepsilon$ are small enough. Moreover, by (4.32)
\[
\left\{ R_3^2 \sqrt{\bar{\sigma}} < |x| < \frac{\xi_1}{2} \right\} \subset \left\{ x \in B_{\frac{\xi_1}{2}} : \bar{u}^{m-1} + \varepsilon \bar{v}_0 \geq P\delta \right\} \cup \{\delta^{\hat{r}} \leq |x| \leq \frac{\xi_1}{2}\},
\]
so that (4.34) gives
\[
\int_{R_3^2 \sqrt{\bar{\sigma}} < |x| < \frac{\xi_1}{2}} (-2P\delta^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)P\delta^2) = \\
\leq \int_{\{x \in B_{\frac{\xi_1}{2}} : \bar{u}^{m-1} + \varepsilon \bar{v}_0 \geq P\delta \} \cup \{\delta^{\hat{r}} \leq |x| \leq \frac{\xi_1}{2}\}} (-2P\delta^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)P\delta^2)
\]
Now if $R_3$ denotes either $R_3^1$ or $R_3^2$ we get
\[
\int_{R_3 \sqrt{\bar{\sigma}} < |x| < \frac{\xi_1}{2}} (-2P\delta^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)P\delta^2)
\]
and by comparison
\[
\int_{\{x \in B_{\frac{\xi_1}{2}} : \bar{u}^{m-1} + \varepsilon \bar{v}_0 \geq P\delta \} \cup \{\delta^{\hat{r}} \leq |x| \leq \frac{\xi_1}{2}\}} (-2P\delta^3 + 6(\bar{u}^{m-1} + \varepsilon \bar{v}_0)P\delta^2) = \\
\leq \int_{|x| < \frac{\xi_1}{2} \sqrt{\bar{\sigma}}} (\bar{u}^{m-1} + \varepsilon \bar{v}_0)^2 P\delta \\
\leq \int_{|x| < \frac{\xi_1}{2} \sqrt{\bar{\sigma}}} (\bar{u}^{m-1} + \varepsilon \bar{v}_0)^2 P\delta
\]
and, when $\delta$ is small enough
\[
-2\int_{|x|<R_\delta} (\tilde{u}^{m-1} + \varepsilon \tilde{v}_0)^3 \leq -2\int_{0<\tilde{u}^{m-1}+\varepsilon \tilde{v}_0<PU_\delta \cap B(0,\delta^\frac{2}{3})} (\tilde{u}^{m-1} + \varepsilon \tilde{v}_0)^3
\]
\[
\leq -2\int_{|x|<R_\delta} (\tilde{u}^{m-1} + \varepsilon \tilde{v}_0)^3
\]
since $\tilde{u}^{m-1} + \varepsilon \tilde{v}_0 > 0$ for $|x|$ sufficiently small.

If $R_\delta$ denotes either $R_3$ or $R_5^2$ we get
\[
-2\int_{|x|<R_\delta} (\tilde{u}^{m-1} + \varepsilon \tilde{v}_0)^3 = -2\delta^3 (1 + O(\delta)) \int_{|y|<\sqrt{\delta}} (\tilde{u}^{m-1}(\delta y))^3 + O(\delta^5)
\]
\[
= -2\delta^3 (\tilde{u}^{m-1}(0))^3 + O\left(\sqrt{\delta}\right) \delta^6 \sigma_0 \int_0^{R_\delta} r^5 + O(\delta^5)
\]
\[
= -2\delta^3 (\tilde{u}^{m-1}(0))^3 \sigma_0 R_\delta^6 + O\left(\delta^5\right)
\]
\[
= -2\delta^3 (\tilde{u}^{m-1}(0))^3 \sigma_0 R_\delta^6 + o(\delta^3), \text{ because of (4.31)}
\]
and
\[
6\int_{|x|<R_\delta} (\tilde{u}^{m-1} + \varepsilon \tilde{v}_0)^2 PU_\delta = 6\delta^4 (1 + O(\delta)) \int_{|y|<\sqrt{\delta}} (\tilde{u}^{m-1}(\delta y))^2 \frac{\sigma_6}{(1 + |y|^2)^2} + O(\delta^5)
\]
\[
= 6\sigma_6 \left((\tilde{u}^{m-1}(0))^2 + O\left(\sqrt{\delta}\right)\right) \delta^4 \sigma_0 \int_0^{R_\delta} \frac{r^5}{(1 + r^2)^2} + O(\delta^5)
\]
\[
= 3\sigma_6 \delta^3 (\tilde{u}^{m-1}(0))^2 \sigma_6 R_\delta^2 + O(\delta^5)
\]
\[
= 3\sigma_6 \delta^3 (\tilde{u}^{m-1}(0))^2 \sigma_6 R_\delta^2 + o(\delta^3), \text{ because of (4.31)}
\]
and by comparison
\[
\int_{\tilde{u}^{m-1}+\varepsilon \tilde{v}_0<PU_\delta \cap B(0,\delta^\frac{2}{3})} (-2(\tilde{u}^{m-1} + \varepsilon \tilde{v}_0)^3 + 6(\tilde{u}^{m-1} + \varepsilon \tilde{v}_0)^2 PU_\delta)
\]
\[
= -2\delta^3 (\tilde{u}^{m-1}(0))^3 \sigma_6 R_\delta^6 + 3\sigma_6 \delta^3 (\tilde{u}^{m-1}(0))^2 \sigma_6 R_\delta^2 + o(\delta^3)
\]
Finally, by (4.35) and (4.36)
\[
I_5 = |\varepsilon|^3 d^3 \left(-\frac{11}{9} \sigma_6 \alpha_0^2 (\tilde{u}^{m-1}(0))^\frac{2}{3} + o(1)\right).
\]
Collecting all the previous estimates we get (4.29) with
\[
(4.37) \quad \alpha_1 = \alpha_0^2 \int_{|y|<\delta^\frac{2}{3}} \frac{1}{(1 + |y|^2)^2} dy = 96\sigma_0 \text{ and } \alpha_2 = \frac{11}{9} \sigma_6 \alpha_0^2 (\tilde{u}^{m-1}(0))^\frac{2}{3}
\]
and
\[
\alpha_0 (\varepsilon) = I_1 + \frac{1}{6} \int_{|y|<\delta^\frac{2}{3}} U^3
\]
that ends the proof. ⊤

We are now in position to prove Theorem 4.2.

Proof of Theorem 4.2 completed. The claim follows by Proposition 4.3 taking into account that if $\text{sgn}(\varepsilon) (1 - 2\tilde{v}_0(0)) > 0$ the function $Y$ has always an isolated maximum point
\[
d_0 := \frac{2\alpha_1}{3\alpha_2} \text{sgn}(\varepsilon) (1 - 2\tilde{v}_0(0))
\]
which is stable under uniform perturbations.

⊓⊔
4.2. **Proof of Theorem 1.3** Here we give the proof of Theorem 1.3. We start analyzing the assumptions (4.7) and (4.9) and the asymptotics of the solution $u_\varepsilon$. In the case $m = 2$ the condition (4.7) states the nondegeneracy of a solution to the Brezis-Nirenberg problem (1.1) with fixed sign, which was proved in [23]. In next proposition we check the validity of the condition (4.9).

**Proposition 4.7.** Assume $\bar{u}^{m-1}$ is nondegenerate and let $\bar{v}_0$ solves (4.7). We have that

\begin{align}
(4.38) & \quad \text{for every } m \geq 2, \text{ then } 1 - 2\bar{v}_0(0) \neq 0, \\
(4.39) & \quad \text{if } m = 2, \text{ then } 1 - 2\bar{v}_0(0) > 0.
\end{align}

**Proof.** Let us start to prove (4.38). A straightforward computation shows that

$$w_0(r) = \frac{r}{2} (\bar{u}^{m-1})'(r) + \bar{u}^{m-1}(r)$$

solves

$$w_0(1) = \frac{1}{2} (\bar{u}^{m-1})'(1)$$

Therefore, the function $z_0 = w_0 - \lambda\bar{v}_0$ solves (in radial coordinates)

\begin{align}
(4.41) & \quad \begin{cases}
-\Delta z_0 - (2|\bar{u}^{m-1}| + \lambda)w_0 = \bar{\lambda}\bar{u}^{m-1} & \text{in } B \\
0 & \text{on } \partial B.
\end{cases}
\end{align}

It is immediate to check that $z_0(0) \neq 0$, otherwise by the uniqueness of solutions to the Cauchy problem, we get $z_0 \equiv 0$ which is not possible because $z_0(1) \neq 0$.

Next we prove (4.39). This is the case where $m = 2$ and $u^1 > 0$ in $(0,1)$, and the claim $1 - 2\bar{v}_0(0) > 0$ is equivalent to $z_0(0) > 0$. By contradiction suppose that $z_0(0) < 0$ and, since $z_0(1) < 0$, only two possibilities occur

1. $z_0(r) \leq 0$ for any $r \in [0,1]$
2. there exist $a, b \in (0,1)$ such that $z_0(a) = z_0(b) = 0$, $z_0(r) < 0$ for any $r \in [0,a)$ and $z_0(r) > 0$ for any $r \in (a,b)$

We will prove that both of them lead to a contradiction.

If $z_0(r) \leq 0$ for any $r \in [0,1]$ then by the maximum principle immediately we deduce that $z_0 < 0$ in $[0,1]$. Therefore, it follows that the first eigenvalue of the linear operator $-\Delta - (2u^{m} + \lambda)I$ is strictly positive (see, for example [2], p.48) and a contradiction arises since the Morse index of $u_0$ is 1.

In case 2, set

$$z_1(r) = \begin{cases} z_0(r) & \text{if } r \in [0,a] \\
0 & \text{if } r \in [a,1]
\end{cases}$$

and $z_2(r) = \begin{cases} 0 & \text{if } r \in [0,a] \cup [b,1] \\
z_0(r) & \text{if } r \in [a,b]
\end{cases}$

It is clear then $z_1, z_2 \in H^1_{0,rad}(B)$ and they are linearly independent. If $E = \text{span}\{z_1, z_2\}$ then $\dim E = 2$ and the quadratic form

$$Q(z) := \int_B (|\nabla z|^2 - (2|\bar{u}^{m-1}| + \lambda)z^2) \, dx$$

vanishes over $E$. By the variational characterization of the Morse index, we deduce that the second eigenvalue of $-\Delta - (2|\bar{u}^{m-1}| + \lambda)I$ is less or equal than zero. Since $\bar{u}^{m-1}$ is non-degenerate and it has Morse index 1 we get a contradiction.

The behavior of $||u_\varepsilon||_{\infty}$ is described in the following lemma.

**Lemma 4.8.** With the same assumption as in Theorem 4.2 we have that

$$||u_\varepsilon||_{\infty} = -u_\varepsilon(0) \sim \frac{\bar{\lambda}^3}{8(1 - 2\bar{v}_0(0))^2} \frac{1}{\varepsilon^2}.$$
Proof. We have to compute $||u_\varepsilon||_\infty = ||W_\delta + \phi_\varepsilon||_\infty$ where $W_\delta$ is as defined in (4.19), $\delta$ as in (4.20) and $d$ as in (4.13). First let us write the equation satisfied by $\phi_\varepsilon$,

$$
\begin{align*}
-\Delta (\tilde{u}^{m-1} + \varepsilon \tilde{v}_0 - PU_\delta + \phi_\varepsilon) &= (\tilde{u}^{m-1} + \varepsilon \tilde{v}_0 - PU_\delta + \phi_\varepsilon)(\tilde{u}^{m-1} + \varepsilon \tilde{v}_0 - PU_\delta + \phi_\varepsilon) + \\
\lambda + \varepsilon(\tilde{u}^{m-1} + \varepsilon \tilde{v}_0 - PU_\delta + \phi_\varepsilon) &\Rightarrow \\
-\Delta \phi_\varepsilon - (\lambda + \varepsilon) \phi_\varepsilon &= (\tilde{u}^{m-1} + \varepsilon \tilde{v}_0 - PU_\delta + \phi_\varepsilon)(\tilde{u}^{m-1} + \varepsilon \tilde{v}_0 - PU_\delta + \phi_\varepsilon) + U^2_\delta \\
- |\tilde{u}^{m-1}|\tilde{u}^{m-1} - (\lambda + \varepsilon) PU_\delta + \varepsilon \tilde{u}^{m-1} + \varepsilon^2 \tilde{v}_0.
\end{align*}
$$

Next we check that $\phi_\varepsilon(x) = \tilde{u}^2 \phi_\varepsilon(\tilde{d}x)$, $\tilde{d} \in B(0, \frac{1}{2}) \rightarrow \mathbb{R}$. It verifies the equation

$$
-\Delta \tilde{\phi}_\varepsilon - \delta^2 \Delta \phi_\varepsilon(\tilde{d}x) = (\lambda + \varepsilon)\delta^2 \tilde{\phi}_\varepsilon + O(\delta^2) - \delta^2 PU_\delta(\tilde{d}x) + \tilde{\phi}_\varepsilon
$$

Using (4.27) with $a = -U$ and $b = \tilde{\phi}_\varepsilon + o(1)$ we get

$$
-\Delta \tilde{\phi}_\varepsilon = (\lambda + \varepsilon)\delta^2 \tilde{\phi}_\varepsilon + 2U(\tilde{d}x) \tilde{\phi}_\varepsilon + o(1) + O(\tilde{\phi}_\varepsilon + o(1))^2
$$

and then

$$
-\Delta \tilde{\phi}_\varepsilon = (\lambda + \varepsilon)\delta^2 \tilde{\phi}_\varepsilon + 2U(\tilde{d}x) \tilde{\phi}_\varepsilon + O(\tilde{\phi}_\varepsilon + o(1))^2.
$$

We claim that the RHS of (4.45) goes to 0 in $L^2(B(0, R))$ for any $R > 0$. Indeed, using the the Poincaré inequality,

$$
\int_{B_R} |\phi_\varepsilon|^2 dy = \delta^4 \int_{B_R} |\phi_\varepsilon(\delta y)|^2 dy = \frac{1}{\delta^2} \int_{\delta B_R} \phi_\varepsilon^2 dx \leq (\text{by (4.28)}) \leq \frac{\delta^4 |\ln |\delta||^2}{\delta^2} = o(1)
$$

and as before using that $0 \leq PU_\delta \leq U_\delta$,

$$
\delta^4 \left( \int_{B_R} |PU_\delta(\tilde{d}x)|^2 \right)^{\frac{1}{2}} \leq \delta^2 \left( \int_{B_R} U^2(x) \right)^{\frac{1}{2}} = o(1).
$$

Then by the standard regularity theory we get that $\tilde{\phi}_\varepsilon \rightarrow 0$ uniformly on $B(0, R)$ and then

$$
\tilde{\phi}_\varepsilon(0) \rightarrow 0 \Rightarrow \phi_\varepsilon(0) = o\left(\frac{1}{\delta^2}\right).
$$

Of course this estimate is far to be sharp but it is enough for our aims. Indeed we get from (4.19)

$$
\delta^3 ||u_\varepsilon||_\infty = -\delta^2 u_\varepsilon(0) = -\delta^3 \tilde{u}^{m-1}(0) = \alpha_6 \delta^2 \phi_\varepsilon(0) + O(\delta^3)
$$

(recalling that $PU_\delta(x) = \frac{\alpha_6 \delta^2}{(\delta^2 + |x|^2)^2} + O(\delta^3)$)

$$
\alpha_6 + o(1) - \delta^3 \phi_\varepsilon(0) = \alpha_6 + o(1) \Rightarrow
$$

$$
||u_\varepsilon||_\infty \sim \frac{\alpha_6 d^2 e^2}{8} = \frac{1}{(1 - 2\tilde{v}_0(0))^2} \varepsilon^2
$$

which ends the proof.

We have now all the ingredients to conclude the proof of Theorem 1.3

Proof of Theorem 1.3 If $u_\lambda$ is any radial solution to (1.1) which is positive in the origin and has $m > 1$ nodal zones, the assumption $||u_\lambda||_\infty = u_\lambda(0) = \frac{\lambda - \lambda_0}{\varepsilon_0} \rightarrow \infty$ is equivalent to $r_\lambda \rightarrow 0$, thanks to Lemma 2.7. In Lemma 4.7 we have showed that $\lambda$ is characterized by (1.13), proving (1.15) and (1.16). Next we check that $u_\lambda$ coincides with $-u_\varepsilon$, defined in (4.13), for $\varepsilon = \lambda - \lambda_0$. Observe that Proposition 4.7 states that $2\tilde{v}_0(0) \neq 1$ (which given by (1.14)), hence the only assumption that $u^{m-1}$ is nondegenerate is needed to apply Theorem 4.2. Recall that the nondegeneracy of $u^{m-1}$ for $m = 2$ was proved in [23]. Let us denote by

$$
K = \left\{ \begin{array}{l}
-\frac{u_\varepsilon(0)}{\lambda + \varepsilon} \text{ with } |\varepsilon| \in (0, \varepsilon_0) \text{ and } \varepsilon_0, \varepsilon \text{ as in Theorem 4.2} \\
= (K_0, +\infty)
\end{array} \right.
$$
for some $K_0 > 0$ by (4.12). In the present setting $\frac{u_0(0)}{\lambda} \subset K$ for $\lambda$ close to $\bar{\lambda}$, hence Proposition 2.11 yields that $\lambda = \bar{\lambda} + \varepsilon$ and $u_\lambda \equiv -u_\varepsilon$.

So (4.17) follows by Lemma 1.8 and (4.18), recalling that $v_0 = -\bar{v}_0$. Inserting (4.17) into (2.21) gives

$$r_4^2 = \frac{1152}{\lambda ||u_\lambda||_\infty^2} \sim \frac{9216 (1 + 2v_0(0))^2}{121 \lambda^2},$$

which brings to (4.18). Eventually Theorem 4.2 also states that $\varepsilon = \lambda - \bar{\lambda}$ has the same sign of $1 - 2v_0(0) = 1 + 2v_\lambda(0)$. In particular when $m = 2$ Proposition 5.7 guarantees that $1 - 2v_\lambda(0) > 0$, so that $\lambda \to \bar{\lambda}$ from above. Moreover in this particular case the nondegeneracy of $u^{m-1}$ is known by 23.

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