ON SOME MEAN MATRIX INEQUALITIES OF DYNAMICAL INTEREST

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Abstract. Let $A \in SL(n, \mathbb{R})$. We show that for all $n > 2$ there exist dimensional strictly positive constants $C_n$ such that

$$\int_{O_n} \log \rho(AX) dX \geq C_n \log \|A\|,$$

where $\|A\|$ denotes the operator norm of $A$ (which equals the largest singular value of $A$), $\rho$ denotes the spectral radius, and the integral is with respect to the Haar measure on $O_n$. The same result (with essentially the same proof) holds for the unitary group $U_n$ in place of the orthogonal group. The result does not hold in dimension 2. This answers questions asked in [3, 5, 4]. We also discuss what happens when the integral above is taken with respect to measure other than the Haar measure.

We also give a simple proof that

$$\int_{S^n} \log \|Au\| d\sigma_n \geq 0,$$

with equality if and only if $A'A = I_n$ ($d\sigma_n$ is the standard measure on $S^n$.)

Introduction

A. Wilkinson brought the following questions to the author’s attention:

Question 1. Let $A \in SL(n, \mathbb{R})$. Consider the coset $\mathcal{A} = AO(n)$. Then is it true that

$$\int_{\mathcal{A}} \int_{S^n} \log \|Bu\| d\sigma_n \neq 0,$$
whenever $A$ is not orthogonal? (In the above integral, $d\sigma$ denotes the Haar measure on the orthogonal group, normalized to be a probability measure; $d\sigma_n$ is the standard measure on $S^n$.)

A version of this question is addressed in the paper [5]. In this note we give a very simple proof that the answer is in the affirmative (Theorem 3).

**Question 2.** Let $A \in SL(n, \mathbb{R})$, then

$$\int_{O_n} \log \rho(AX)dX \geq C_n \int _{\mathcal{A}} d\sigma \int_{S^n} \log \|Bu\|d\sigma_n \neq 0,$$

for some positive dimensional constant $C_n$.

The above is Question 6.6 in [3]. The question is answered in dimension 2 in the paper [1] – in fact, the authors show that in that case one can take $C_n = 1$, and replace $\geq$ by $=$.

In the papers [3, 5] the question is also posed as to what happens when the measure on the orthogonal group is *not* the Haar measure (actually, the question is asked in the slightly different context.) Obviously the result is unchanged if a measure $\mu$ is in the same measure class as the Haar measure, but we note that in two dimensions the sign of the inequality in Question 2 can be reversed whenever $\mu$ is supported on a proper subset of $O(2) = S^1$ – see Section 6 for details.

The same question with the orthogonal group replaced by the unitary group is answered in the paper [4] (in arbitrary dimension). The dimensional constant is again equal to 1, though the inequality cannot be replaced by an equality. In [4], Dedieu and Shub also conjecture a positive answer to Question 2. The same conjecture is made in [5].

In this note we give a simple proof of

**Theorem 3.** Let $A \in SL(n, \mathbb{R})$. Then

$$\int_{S^n} \log \|Au\|d\sigma \geq 0,$$

with equality if and only if $A^tA = I_n$, (and $d\sigma_n$ denotes the standard measure on the unit sphere).

and, as one of the consequences, we answer Question 1 in the affirmative.

We also show
Theorem 4. For all $n > 2$ there exist strictly positive constants $C_n$ such that

$$\int_{O_n} \log \rho(AX) dX \geq C_n \log \|A\|,$$

where $\|A\|$ denotes the operator norm of $A$ (which equals the largest singular value of $A$), $\rho$ denotes the spectral radius, and the integral is with respect to the Haar measure on $O_n$. The same result (with essentially the same proof) holds for the unitary group $U_n$ in place of the orthogonal group.

This result easily implies a positive answer to Question 2 in dimensions bigger than 2. Theorem 4 is false in dimension 2. Some discussion of why dimension 2 is different is given in Section 6.

It should also be noted that Theorem 4 immediately implies the following

Corollary 5. Let $\mu$ be an orthogonally invariant measure on $SL(n, \mathbb{R})$, such that $\log \|x\| \in L^1(\mu)$, and $n \geq 3$. Then

$$\int_{SL(n, \mathbb{R})} \log \rho(X) d\mu \geq C_n \int_{SL(n, \mathbb{R})} \log \|A\| d\mu.$$

The same statement holds with $\mathbb{C}$ replacing $\mathbb{R}$ and “unitarily invariant” replacing “orthogonally invariant”.

Proof. Integrate both sides of the inequality in Theorem 4. □

The plan of the paper is as follows: In Section 1 we summarize some necessary facts of linear algebra. In Section 2 we give a proof of Theorem 3. In Section 3 we outline our approach to the proof of Theorem 4. The actual proof falls naturally into two parts – one part covers the case where $A' A$ is far from $I_n$ and is the content of Section 4 and the other part the case where it is close to the identity, and is the content of Section 5. This last section requires some simple results on perturbation theory, and these are covered in Section 7.

It should be remarked that no effort is made to estimate the constant $C_n$ which appears in the statement in the Theorem 4 – in order to have a sharp estimate, the methods of the current paper will need to be combined with detailed analysis of the structure of the orthogonal groups, and this will be the subject of another paper.

1. Generalities on matrices

First,

Definition 6. Let $A \in M_{m \times n}$. Then the nonnegative square roots of the eigenvalues of the (positive semi-definite) matrix $A' A$ are called the singular
values of $A$. We will denote individual singular values by $\sigma_1 \geq \cdots \geq \sigma_m$, and we shall denote the ordered $m$-tuple $(\sigma_1, \ldots, \sigma_m)$ by $\Sigma(A)$.

In this note we will only work with square matrices (although the results go through without change for non-square matrices), and our use of singular values will be localized to the following two lemmas:

**Lemma 7.** Let $A \in M_{n \times n}$, and let $P \in O(n)$. Then

$$\Sigma(A) = \Sigma(AP) = \Sigma(PA).$$

**Proof.** Immediate. \qed

**Lemma 8.** Let $A \in M_{n \times n}$, and let $f: \mathbb{R} \to \mathbb{R}$ be such that

$$\int S^n f(\langle Au, Au \rangle) d\sigma_n < \infty.$$

Then

$$\int S^n f \left( \sum_{i=1}^n \sigma_i^2(A) u_i^2 \right) d\sigma_n.$$

**Proof.** It is sufficient to note that

$$\langle Au, Au \rangle = u^t A^t Au = (Pu)^t \text{Diag} \left( \Sigma^2(A) \right) Pu,$$

where $P$ is an orthogonal matrix (hence an isometry of $S^n$). \qed

**Corollary 9.**

$$\int \int A \log ||Bu|| d\sigma_n = \frac{1}{2} \int \int S^n \log \left( \sum_{i=1}^n \sigma_i^2(A) u_i \right) d\sigma_n.$$

**Proof.** Lemma 7 tells us that the inner integral does not depend on $B \in \mathcal{A}$, and Lemma 8 tells us how to evaluate that integral in terms of singular values of $A$. \qed

Corollary 1 tells us that Question 1 will be answered once we succeed in showing Theorem 3 (for diagonal $A$).

**Theorem-Definition 10** (Gershgorin’s Theorem). Let $A \in M_n(\mathbb{C})$. Let

1. $r_i = \sum_{j \neq i} |a_{ij}|$,
2. $s_i = \sum_{j \neq i} |a_{ji}|$,
and let the row- and column- Gershgorin disks (respectively) be:

\[ R_i = \{ z \in \mathbb{C} \mid |z - a_{ii}| \leq r_i \}, \]
\[ S_i = \{ z \in \mathbb{C} \mid |z - a_{ii}| \leq s_i \}. \]

Let now \( \lambda \) be an eigenvalue of \( A \). Then there exists a \( k \) and an \( l \) such that \( \lambda \in R_k \) and \( \lambda \in S_l \).

Proof. See, eg, [2] or [7]. □

2. Proof of the Theorem

To prove Theorem by Lemma it will be sufficient to prove

**Theorem 11.**

\[ \mathfrak{A}(a_1, \ldots, a_n) = \int_{S^n} \log \left( \sum_{i=1}^{n} a_i x_i^2 \right) d\sigma_n \geq 0, \]

whenever \( a_1, \ldots, a_n > 0 \) and \( \prod_{i=1}^{n} a_i = 1 \). Furthermore, equality holds only when \( a_1 = \cdots = a_n = 1 \).

Proof. By the (weighted version of) the arithmetic-geometric mean inequality (see [6]),

\[ \sum_{i=1}^{n} a_i x_i^2 \geq \prod_{i=1}^{n} a_i x_i^2, \]

with equality if and only if all of the \( a_i \) are equal. (the above uses the fact that \( \sum_{i=1}^{n} x_i^2 = 1 \).) Taking logs, we see that

\[ \log \left( \sum_{a_i x_i^2} \right) \geq \sum_{i=1}^{n} x_i^2 \log(a_i), \]

with equality if and only if all the \( a_i \) are equal. And integrating, we see that

\[ \mathfrak{A}(a_1, \ldots, a_n) \geq \int_{S^n} \sum_{i=1}^{n} x_i^2 \log(a_i) = \]

\[ \sum_{i=1}^{n} \log(a_i) \int_{S^{n}} x_i^2 d\sigma_n = n \log(a_i) \int_{S^n} x_i^2 d\sigma_n = 0, \]

with equality if and only if \( a_i = 1 \) for all \( i \). □
3. A lower bound on average spectral radius

In this section we consider the following average:
\[ \mathcal{A}(A) = \int_{O_n} \log \rho(AX) dX, \]
where \( \rho(M) \) denotes the spectral radius of the matrix \( M \). The main theorem of the section will be the following estimate:

**Theorem 12.** Let \( A \in SL(n, \mathbb{R}) \), let \( n \geq 3 \), and let \( \sigma_1(A) \) be the largest singular value of \( A \). Then there exists a constant \( C_n > 0 \), such that
\[ \mathcal{A}(A) \geq C_n \log \sigma_1(A). \]

**Remark 13.** Theorem 12 is false for \( n = 2 \) and trivially true for \( n = 1 \) (\( C_1 = 1 \)).

To prove Theorem 12 we first make a couple of observations. First, 

**Lemma 14.** Let \( A \in M_n \), let \( \Sigma(A) \) be the singular values of \( A \), and let \( \rho(X) \) be the spectral radius of the matrix \( X \). Then
\[ \int_{O_n} f(\rho(AY)) dY = \int_{O_n} f(\rho(\text{Diag}(\Sigma A))) dY. \]

**Proof.** Form the singular value decomposition of \( A \), that is, write \( A = P_1 \text{Diag}(\Sigma(A)) P_2 \), where \( P_1, P_2 \in O_n \). Then,
\[ \rho(AY) = \rho(\text{Diag}(\sigma(A)) P_2 Y P_1). \]

The statement of the Lemma follows immediately. □

Lemma 14 tells us that in the statement of Theorem 12 it is enough to consider diagonal matrices \( A \) with positive entries. In the rest of this section we will labor under these assumptions. The strategy will be to first demonstrate a bound in any closed set of diagonal matrices not containing the identity matrix \( I_n \) – this will be done in Section 4 (particularly Corollary 17) – and then show a bound in some small neighborhood of the identity – this will be the content of Section 5. Our measure of the distance from the identity will be the quantity
\[ L(A) = \max_{i=1}^n |\log(\sigma_i(A))|. \]

the top singular value \( \sigma_1(A) \). From now on, \( A = \text{Diag}(d_1, \ldots, d_n) \), with \( d_1 \geq d_2 \geq \ldots d_n > 0 \). We will assume that \( \prod_{i=1}^n d_i = 1 \) when necessary – much of the time we will only use the fact that \( d_1 > 1 \).
4. Away from identity

**Theorem 15.** Let $A = \text{Diag}(d_1, \ldots, d_n)$, with $d_1 \geq d_2, \ldots, \geq d_n \geq 0$, and let $\|M\|_1 \leq 1$. Let $A_t = A(I + tM)$. Then,

$$d_1(1 + 2t) \geq \rho(A_t) \geq d_1(1 - 2nt).$$

**Proof.** Since $(A_t)_{ij} = d_i(\delta_{ij} + tM_{ij})$, it follows that

$$d_i(1 + t) \geq |(A_t)_{ii}| \geq d_i(1 - t),$$

while

$$|r_i(A_t)| \leq d_it.$$

Let

$$G(A_t) = \bigcup_{i=1}^{n} R_i(A_t).$$

By a simple continuity argument (see, for example, [2] for a similar argument), the perturbation of the eigenvalue $d_1$ of $A$ stays in the connected component of the Gershgorin disk $R_{\infty}(A_t)$. Since by the above inequalities Eq. (6) and Eq. (7),

$$\min_{z \in R_{\infty}(A_t)} \geq d_i(1 - 2t),$$

$$\max_{z \in R_{\infty}(A_t)} \leq d_i(1 - 2t),$$

we see that after time $t$ the connected component of $R_{\infty}(A_t)$ can only stay between the advertised bounds. \hfill \box

**Theorem 16.** Let $A = \text{Diag}(d_1, \ldots, d_n)$. For any $\epsilon > 0$, there is a constant $c_\epsilon > 0$, such that

$$\mathcal{A}(A) \geq c_\epsilon(d_1 - \epsilon).$$

**Proof.** First, let $t_0$ be so small that $\log(1 - 2nt_0) > 1 - \epsilon$. Now let

$$S_0 = \{x \in O_n \mid \|x - I\|_1 < t_0\},$$

and let

$$c_\epsilon = \frac{\text{Vol}(S_0)}{\text{Vol}(O_n)}.$$

We see that

$$\mathcal{A}(A) \geq \frac{\int_{S_0} \log \rho(AX)dX}{\text{Vol } O_n} \geq c_\epsilon(d_1 - \epsilon).$$

\hfill \box
Corollary 17. Let $D$ be the set of all diagonal matrices with elements $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, and let $D_t$ be the set of those matrices $M \in D$ with $d_1(M) \geq t > 1$. Then there is a constant $C_t > 0$ such that

$$\frac{\mathfrak{U}(A)}{\log(d_1(A))} > C_t,$$

for all $A \in D_t$.

Proof. Let $s = \log(t)/2$. Then, by Theorem 16

$$\mathfrak{U}(A) \geq c_s (\log(d_1(A)) - s) \geq \frac{c_s \log d_1(A)}{2},$$

where the last inequality holds for all $A \in D_t$. Now set $C_t = c_s/2$. □

5. Close to the Identity

In this section we shall prove the following:

Theorem 18. Let $n \geq 3$. Let $A = \text{Diag}(\exp(d_1 t), \ldots, \exp(d_n t))$, with $d_1 \geq \cdots \geq d_n$, with $\max(|d_1|, |d_n|) = 1$. and

$$\sum_{i=1}^{n} d_i = 0.$$

Then there exist constants $C_n$ and $\epsilon_n$, such that for $t < \epsilon_n$,

$$\mathfrak{U}(A) \geq C_n t d_1$$

for $0 < t \leq \epsilon_n$.

Remark 19. The argument which follows can be made to give quite explicit lower bounds for both $\epsilon_n$ and $C_n$. However, this does not seem useful, since it is quite clear that the bounds will not be close to the truth. It seems plausible that a combination of the methods of this note with a detailed understanding of the structure of the orthogonal group will give a tight result, but we shall not attempt to do this here. It should, on the other hand, be noted that not using any structural property of the orthogonal group means that the arguments in this note work just as well for cosets of unitary groups.

To prove Theorem 18 we shall first need the following trivial but crucial observation:

Observation 20. Let $A \in SL(n, \mathbb{R})$, and let $X \in O_n(\mathbb{R})$. Then $\rho(AX) \geq 1$.

Proof. $\det AX = 1$. □
Now, recall that

\[ \mathcal{A}(A) = \int_{O_n} \log \rho(AX) dx. \]

By the observation above, it follows that the integrand is everywhere nonnegative, and so to prove a lower bound such as that of Theorem 18 it will be enough to show the inequality (8) with \( \mathcal{A}(A) \) replaced by

\[ \mathcal{A}_S(A) = \int_{S \subseteq O_n} \log \rho(AX) dx. \]

Let us pick \( S \) to be a small neighborhood around a matrix \( X_0 \in O_n \).

Notice that all eigenvalues of \( X \in O_n \) have absolute value 1. We will choose \( X_0 \) to have a simple eigenvalue \( \lambda_0 \), such that

\[ \frac{d}{dt} \log |\lambda_0(\text{Diag}(\exp(td_1), \ldots, \exp(td_n))X_0)| \geq C|d_1|, \]

for some constant \( C \). By analyticity of \( \lambda_0 \) (Theorem 23), inequality (9) suffices to show that

\[ \mathfrak{A}(A) \geq C_n t d_1, \]

for \( t \) sufficiently small, and by the Observation 20 this will suffice to show Theorem 18.

5.1. Proof of inequality (9). First, we need to find a suitable orthogonal matrix \( X_0 \). There will be two cases, depending on the parity of \( n \).

First case is that of odd \( n \). In such a case, we pick \( X_0 \) to be a rotation by 180 degrees around the axis \( e_1 = (1, 0, \ldots, 0) \), and our \( \lambda_0(X_0) = 1 \). Its eigenvector is \( v_{\lambda_0} = e_1 \). By Lemma 25

\[ \frac{\lambda(x)}{dx} = \lambda(x)d_1, \]

and we are done.

Second case is that of \( n \) even. In that case, we pick \( X_0 \) to fix the span of \( e_3, \ldots, e_n \) and to rotate the span of \( e_1 \) and \( e_2 \) by 90 degrees. In this case, \( \lambda_0(X_0) = i \), and the eigenvector of \( \lambda_0 \) is

\[ v_{\lambda_0} = \frac{1}{\sqrt{2}}(1, i, 0, \ldots, 0). \]

By Lemma 25

\[ \frac{\lambda(x)}{dx} = \lambda(x) \frac{1}{\sqrt{2}}(d_1 + d_2). \]
If $d_2 > 0$, we are done, since $d_1 + d_2 > d_1$. Since

$$
\sum_{i=1}^{n} d_i = 0,
$$

it follows that

$$
d_1 = -\sum_{i=2}^{n} d_i = \sum_{i=2}^{n} |d_i|.
$$

Since $|d_n| \geq |d_{n-1}| \geq |d_2|$, it follows that $d_1 \geq (n-1)|d_2|$, and so

$$
d_1 + d_2 \geq \frac{n-2}{n-1} d_1,
$$

so as long as $n > 2$ we are done. Notice that the argument breaks down when $n = 2$, and in fact, in that case the result is false.

6. What about dimension 2?

The function $f(d_1, \ldots, d_n) = \mathfrak{U}(\text{Diag}(\exp(d_1), \ldots, \exp(d_n)))$ restricted to the set

$$
\sum_{i=1}^{n} d_1 \ldots d_n = 0
$$

always has a minimum at the origin. In dimension 2, however, the minimum is an actual critical point. This is not hard to see using Lemma 25 together with the observation that the group $O(2)$ is abelian, and thus every $x \in O(2)$ has $(1, i)$ and $(i, 1)$ for an orthonormal basis of eigenvectors. Thus, at $d_1 = 0$,

$$
d \log(\rho(\text{Diag}(\exp(d_1), \exp(-d_1))X))
\frac{dd_1}{dd_1} = 0
$$

for every $X \in O(2)$. In dimension $n > 2$ the function $\mathfrak{U}$ is not smooth at the origin.

As a matter of fact, in dimension 2 many of the computations can be made much more explicit. Indeed, a general element $R_\theta$ of $SO(2)$ has the form:

$$
R_\theta = \begin{pmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{pmatrix},
$$

and so if $A = \text{Diag}(a, 1/a)$, then, denoting $t = (a + 1/a)/2$,

$$
\text{tr} AR_\theta = 2t \cos(\theta),
$$

and so

$$
\rho(AR_\theta) = \max\left(|t \cos(\theta) + \sqrt{t^2 \cos^2(\theta) - 1}|, |t \cos(\theta) - \sqrt{t^2 \cos^2(\theta) - 1}|\right).
$$

A simple computation shows:
Lemma 21. When $|\cos(\theta)| < 1/t$, then $\rho(AR_\theta) = 1$. (in other words, the matrix $AR_\theta$ is elliptic.

We have the following corollary (which answers a question of [5]):

Corollary 22. Let $\mu$ be a measure on $SO(2)$ such that the support of $\mu$ omits some interval. Then, there exists a matrix $A \in SL(2, \mathbb{R})$, such that

$$\int_{SO(2)} \log \rho(AR_\theta)d\mu = 0.$$ 

Proof. Suppose that the support of $\mu$ omits $[a, b]$. Let $c = (a + b)/2$ $(a,b,c$ are all modulo $2\pi$.) Let $s = |(a - b)/2|$, let $\alpha = 1/\cos(s)$, and let $\beta$ be positive, and such that $\beta + 1/\beta < 2\alpha$ (in other words, $\beta < \exp(\text{arccosh}(\alpha))$. Let $A = \text{Diag}(\beta, 1/\beta)R_{-c}$. Then Lemma 21 shows that $A$ is the matrix whose existence is asserted in the corollary. \hfill \Box

7. Some perturbation theory

In this section we will recall some formulas of perturbation theory (an exhaustive treatment can be found in the classic [8]), and derive some estimates needed in the rest of this paper.

Theorem 23. Let $T$ be an $n \times n$ matrix (thought of as a point in $\mathbb{C}^n$, and let $\lambda$ be a simple eigenvalue of $T$. Then there is a neighborhood of $T$ where $\lambda$ is a holomorphic function of $T$. In particular, $\lambda$ is $C^\infty$ in a neighborhood of $T$.

Proof. This follows from generalities on analyticity of (simple) roots of polynomials as functions of their coefficients. See [8] for some further details. \hfill \Box

Theorem 24. Let $\lambda$ be a simple eigenvalue of $T(0)$. Then

$$\frac{d\lambda}{dx}|_{x=0} = \text{tr}\left(\frac{\partial T}{\partial x}|_{x=0} P_\lambda\right),$$

where $P_\lambda$ is the orthogonal projection onto the eigenspace of $\lambda$.

Proof. See [8, p. 79]. \hfill \Box

Lemma 25. Let $T(x) = \text{Diag}(\exp(d_1x), \ldots, \exp(d_nx))T$, and let $\lambda(x)$ be a simple eigenvalue of $T(x)$. Let the unit eigenvector of $\lambda$ be $v_\lambda = (v_1, \ldots, v_n)$. Then,

$$\frac{\lambda(x)}{dx} = \lambda(x) \sum_{i=1}^n d_i|v_i|^2.$$

In particular, the logarithmic derivative of $\lambda(x)$ is real.
Proof. Use Theorem 24 and the observation that
\[ T(x)P_{\lambda(x)} = \lambda(x)P_{\lambda(x)}, \]
to write
\[ \frac{\lambda(x)}{dx} = \text{tr}\left(\text{Diag}(d_1, \ldots, d_n)P_{\lambda(x)}\right). \]
Now, note that \((P_{\lambda(x)})_{ij} = v_i \overline{v_j} \). □

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