ON THE FIRST HOMOLOGY OF PEANO CONTINUA

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Abstract. We show that the first homology group of a locally connected compact metric space is either uncountable or is finitely generated. This is related to Shelah’s well-known result [S] which shows that the fundamental group of such a space satisfies a similar criterion. We give an example of such a space whose fundamental group is uncountable but whose first homology is trivial, showing that our result doesn’t follow from Shelah’s. We clarify a claim made by Pawlikowski [P] and offer a proof of the clarification.

1. Introduction

The classical Hahn-Mazurkiewicz Theorem states that a connected, locally connected compact metric space is the continuous image of an arc – a Peano continuum. Shelah showed [S] that if the first homotopy group, the fundamental group, of a Peano continuum is countable then it is finitely generated. This can be compared to our result:

Theorem 1.1. The first homology group of a compact locally connected metric space is either uncountable or isomorphic to a direct sum of finitely many cyclic groups.

If $X$ is a locally connected compact metric space then $X$ has only finitely many connected components, each of which is a Peano continuum and therefore path connected. Then $H_1(X)$ is the direct sum of the first homology of each of the (path) components of $X$. Thus it suffices to prove Theorem 1.1 for a Peano continuum. For the sake of generality, we note that is countable may be replaced by has cardinality less than the continuum in both the result of Shelah and our result.

Example 1.2. We construct a space whose existence testifies that our result cannot follow immediately from Shelah’s. Let $X$ be a simplicial complex whose fundamental group is the alternating group $A_5$. Recall that each element of $A_5$ is a commutator and thus $X^N$, endowed with the product topology, is a Peano continuum whose fundamental group, $A_5^N$, has the

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property that each of its elements is a commutator. Whence $H_1(X^N)$, which is the abelianization of $A_5^N$ by the classical Hurewicz theorem, is trivial.

2. A construction

We begin with the following lemma.

**Lemma 2.1.** Let $X$ and $Y$ be Peano continua and $f : X \to Y$ be a mapping. If there exists $\epsilon > 0$ such that each loop of diameter less than $\epsilon$ is mapped under $f$ to a nulhomologous loop, then $f_*(H_1(X))$ is a finite sum of cyclic groups.

**Proof.** Consider the composition of the Hurewicz map (which is surjective by the Hurewicz theorem) and the map $f_*$ and apply Lemma 7.6 in [CC] which states that:

If $X$ is a connected, locally path connected separable metric space which is locally trivial with respect to $g : \pi_1(X) \to K$ then $g_*(\pi_1(X))$ is countable, and furthermore is finitely generated if $X$ is compact.  

To finish, we need only remark that a finitely generated Abelian group is a finite sum of cyclic groups. □

Consider a Peano continuum $X$ with metric $d$ such that $H_1(X)$ is not finitely generated. By Lemma 2.1 letting $f : X \to X$ be identity, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of continuous mappings of $S^1$ to $X$, with $\text{diam}(f_n) \leq \frac{1}{2^n}$, each of which is not nulhomologous. By compactness of $X$, there exists a point $x \in X$ and a subsequence (without loss of generality, let the subsequence be the original sequence) such that the sequence of images $f_n(S^1)$ converges to $x$ in the Hausdorff metric. By local path connectedness, we may assume that $x \in f_n(S^1)$ for all $n \in \mathbb{N}$, and by passing to a subsequence (without loss of generality, let the subsequence be the original sequence), we may assume again that $\text{diam}(f_n) \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Thus, each $f_k$ can be thought of as a loop $f_k : [0, 1] \to X$ such that $f(0) = f(1) = x$ (this allows us to concatenate such mappings together).

Recall that the product space $\{0, 1\}^\mathbb{N}$ is the Cantor set, where each factor $\{0, 1\}$ is given the discrete topology.

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1The referee has pointed out that there was a minor flaw in the proof of Lemma 7.6, which we correct here. Select open covers $\{U_\alpha\}$, $C_1$ and $C_2$ as in the original proof. Choose $C$ to be an open refinement of $C_2$ which consists of open path connected sets. If $X$ is compact we may take $C$ to be finite, and if $X$ is merely separable we may take $C$ to be countable (since separable metric spaces are Lindelöf). Now apply Theorem 7.3 (2) of [CC] to conclude that the image of $g$ is finitely generated (resp. countable) in case $X$ is compact (resp. separable).
Let \( f_k^1 \) be \( f_k \) and \( f_k^0 \) be the constant loop at \( x \). For \( \alpha \in \{0, 1\}^\mathbb{N} \), let \( f^\alpha = f_0^{\alpha(0)} * f_1^{\alpha(1)} * f_2^{\alpha(2)} * \cdots \). This can be thought of as the (pointwise) limit of a Cauchy sequence in the complete metric space \( C([0, 1], X) \), the metric being the sup metric. Thus, each concatenation gives a continuous function from \( S^1 \) to \( X \). We define an equivalence relation on the Cantor set \( \{0, 1\}^\mathbb{N} \) as follows: \( \alpha \sim \beta \) iff \( f^\alpha \) is homologous to \( f^\beta \).

**Definition 2.2.** We say a space \( Y \) is **Polish** if it is completely metrizable and separable (for example, the Cantor set \( \{0, 1\}^\mathbb{N} \)). If \( Y \) is a Polish space, we say that \( A \subseteq Y \) is **analytic** if there exists a Polish space \( Z \) and a closed set \( D \subseteq Y \times Z \) such that \( A \) is the projection of \( D \) in \( Y \). Analytic spaces are preserved under continuous images and preimages, products, and under countable unions and intersections. Closed sets are analytic. If \( \sim \) is an analytic subset of \( X \times X \) we say that \( \sim \) is an analytic relation on \( X \). A subspace of a topological space is **perfect** if it is closed, nonempty and has no isolated points. A perfect subspace of a Polish space has the cardinality of the continuum.

**Lemma 2.3.** The space \( \sim \) is an analytic equivalence relation on the Cantor set which has the property that if \( \alpha \) and \( \beta \) differ at exactly one point then \( \alpha \sim \beta \).

**Proof.** Let \( H \) be the space of all continuous maps of \([0, 1]\) to \( X \) such that \( \{0, 1\} \mapsto x \), with the metric on \( H \) being the sup metric. For each \( n \in \mathbb{N} \) let \( C_n : H^{2n} \to H \) be the map defined by mapping \((l_1, l_2, \ldots, l_{2n})\) to

\[
l_1 * l_2 * (l_1)^{-1} * (l_2)^{-1} * l_3 * l_4 * (l_3)^{-1} * (l_4)^{-1} * \cdots * l_{2n-1} * l_{2n} * (l_{2n-1})^{-1} * (l_{2n})^{-1}
\]

Each such map \( C_n \) is clearly continuous, and so the image \( C_n(H^{2n}) \) is an analytic subset of \( H \). Let \( \mathcal{H} \) be the space of homotopies between loops, also under the sup metric. Let \( D \subseteq H^3 \times \mathcal{H} \) be defined by \( D = \{ (l_1, l_2, l_3, h) : h \text{ homotopes } l_1 \text{ to } l_2 * l_3 \} \). The set \( D \) is obviously closed in \( H^3 \times \mathcal{H} \). Then the set \( D_n = (H \times H \times C_n(H^{2n}) \times \mathcal{H}) \cap D \) is analytic as an intersection of two analytic sets. Letting \( \sim_n \) be the projection of \( D_n \) to \( H \times H \), we see that \( \sim_n \) is analytic. Then \( \bigcup_{n=0}^{\infty} \sim_n \) is also analytic. Letting \( F : \{0, 1\}^\mathbb{N} \to H \) be given by \( F(\alpha) = f^\alpha \), it is clear that \( F \) is continuous. Finally, \( \sim = F^{-1}(\bigcup_{n=0}^{\infty} \sim_n) \), and so \( \sim \) is analytic.

If \( \alpha \) and \( \beta \) differ exactly at \( n \in \mathbb{N} \), then we know that \( f_0^{\alpha(0)} * \cdots * f_{n-1}^{\alpha(n-1)} \) is homotopic to \( f_0^{\beta(0)} * \cdots * f_{n-1}^{\beta(n-1)} \) and \( f_n^{\alpha(n+1)} * f_{n+1}^{\alpha(n+2)} * \cdots \) is homotopic to \( f_n^{\beta(n+1)} * f_{n+1}^{\beta(n+2)} * \cdots \) (the loops are in fact the same). Supposing that \( \alpha \sim \beta \), we apply cancellations on the right and left so that \( f_n^{\alpha(n)} \) is homologous to \( f_n^{\beta(n)} \), so that \( f_n \) is nulhomologous, a contradiction. \( \square \)
3. Proving Theorem 1.1

Following the literature, there are two possible paths on which we might proceed to finish the proof of our result. Shelah [S] and later Pawlikowski [P] both argue that any equivalence relation on the Cantor set satisfying the conclusion of the previous lemma must contain a perfect set, which necessarily has the cardinality of the continuum. However, this final portion of Shelah’s seminal article is extremely terse and apparently uses a sophisticated technique related to forcing not generally available to the naive topologist. For the sake of clarity and completeness we conclude by offering a complete and simplified discussion of Pawlikowski’s method, which has the advantage of using terminology covered in a basic topology course.

Definition 3.1. Recall that a set $A$ in a topological space $Y$ is nowhere dense if its closure has empty interior, is meager if it can be written as a countable union of nowhere dense subsets of $Y$, and comeager if $Y - A$ is meager. The Baire Category Theorem states that a complete metric space is not meager as a subset of itself. A set $A \subseteq Y$ has the property of Baire if there exists an open set $O \subseteq Y$ such that $A \Delta O = (A \cup O) - (A \cap O)$ is meager (and say that $A$ is comeager in $O$.) Analytic spaces have the property of Baire.

To finish, we shall use the following consequence of the Kuratowski-Ulam Theorem:

**Lemma 3.2.** If $Y$ is a Polish space and $A \subseteq Y \times Y$ is comeager in $U \times V$, with $U$ and $V$ open sets in $Y$, then $\{y \in U : \{z \in V : (y, z) \in A\}$ is comeager in $V\}$ is comeager in $U$. Also, if $A \subseteq Y \times Y$ is meager, then $\{y \in Y : \{z \in Y : (y, z) \in A\}$ is meager in $Y\}$ is comeager in $Y$.

For the uncountability of the number of classes of $\sim$ we offer a lemma similar to one in [P] but whose proof is elementary.

**Lemma 3.3.** If $\sim$ is an equivalence relation on the Cantor set satisfying the conclusion of Lemma 2.3 then $\sim$ is meager and has uncountably many classes.

**Proof.** We show first that $\sim$ is meager. Since $\sim$ is analytic it satisfies the property of Baire. If $\sim$ were non-meager, then it would be comeager in a neighborhood in $\{0, 1\}^N \times \{0, 1\}^N$ of the form $U \times V$, with $U$ and $V$ basic open sets in $\{0, 1\}^N$. By Lemma 3.2

$$A = \{\alpha \in U : \{\beta \in V : \alpha \sim \beta\} \text{ is comeager in } V\}$$
is comeager in $U$. Let $n \in \mathbb{N}$ be greater than the length of a generator of $U$. Define $\Phi : U \to U$ by $\Phi(\alpha)(n) = 1 - \alpha(n)$ and $\Phi(\alpha)(i) = \alpha(i)$ for $i \neq n$. Notice that $\Phi$ is a homeomorphism of $U$ to itself. Thus we can choose $\alpha \in A \cap \Phi(A)$, as $A \cap \Phi(A)$ is comeager. Letting $\gamma = \Phi(\alpha)$, we see that $\gamma$ and $\alpha$ differ only at $n$, so that $\alpha \not\sim \gamma$. From the definition of $A$, we have comeagerly many $\beta \in V$ such that $\alpha \sim \beta$. Since $\gamma \in A$, the same can be said of $\gamma$. Then for some $\beta$, we have $\beta \sim \alpha$ and $\beta \sim \gamma$, implying $\gamma \sim \alpha$, a contradiction.

To see that $H_1(X)$ is uncountable we construct a set $Y \subseteq \{0, 1\}^\mathbb{N}$ of cardinality $\aleph_1$ such that for distinct $\alpha$ and $\beta$ in $Y$ we have $\alpha \not\sim \beta$. For $\alpha \in \{0, 1\}^\mathbb{N}$ we write $\sim^\alpha$ as the equivalence class of $\alpha$ in $Y$. As $\sim$ is meager, we have by Lemma 3.2 that $J = \{\alpha \in \{0, 1\}^\mathbb{N} : \sim^\alpha$ is meager$\}$ is comeager in $\{0, 1\}^\mathbb{N}$. By transfinite induction define a sequence $\{\alpha_i\}_{i<\omega_1}$ by picking $\alpha_j$ such $\alpha_j \in J - \bigcup_{i< j} \sim^\alpha_i$, the Baire Category Theorem guaranteeing that this choice is always possible. Let $Y = \{\alpha_i\}_{i<\omega_1}$.

To see that there is in fact a continuum of pairwise non-homologous loops, we invoke the following theorem of Mycielski [M]:

**Theorem 3.4.** Every meager relation on a perfect Polish space admits a perfect, pairwise non-related set.

4. A CLARIFICATION IN THE NON-COMPACT SETTING

In [P], a claim is stated without proof, which we state after first giving some definitions.

**Definition 4.1.** Let $\kappa$ be an infinite cardinal less than continuum. We say a topological space $Y$ is $\kappa$-separable if $Y$ has a dense subset of cardinality less than or equal to $\kappa$ and that $Y$ is $\kappa$-Polish if $Y$ is completely metrizable and $\kappa$-separable.

**Definition 4.2.** [Mo] If $Y$ is a Polish space, we say that $A \subseteq Y$ is $\kappa$-Suslin if there exists a closed set $D \subseteq Y \times \kappa^\mathbb{N}$ such that $A$ is the projection of $D$ in $Y$ (here $\kappa$ is given the discrete topology).

**Observation 4.3.** Since any $\kappa$-Polish space is the continuous image of $\kappa^\mathbb{N}$ we may say $A \subseteq Y$ is $\kappa$-Suslin iff there exists a $\kappa$-Polish space $Z$ and a closed set $D \subseteq Y \times Z$ such that $A$ is the projection of $D$ in $Y$. Thus $\aleph_0$-Polish spaces and $\aleph_0$-Suslin sets are precisely Polish spaces and analytic sets respectively. Also, a metric space is $\kappa$-separable if and only if every open
cover contains a subcover of cardinality at most $\kappa$. This latter condition is often called $\kappa$-Lindelöf.

The aforementioned claim is the following [P]:

**Claim 4.4 (Pawlikowski).** Let $\kappa$ be an infinite cardinal less than continuum. Suppose that $X$ is a path connected locally path connected metric space which is $\kappa$-Lindelöf. Then $\pi_1(X)$ is of cardinality $\leq \kappa$ or of cardinality continuum.

We know of no proof of this claim using the methods of [P]. We state and prove a theorem with more hypotheses but which is nonetheless of interest. Towards this, let $BP(\kappa)$ be the statement that all $\kappa$-Suslin sets have the property of Baire. The claim $BP(\aleph_0)$ is simply true, as we noted earlier. If one assumes extra set theoretic assumptions, for example Martin’s Axiom, then $BP(\kappa)$ holds provided the successor cardinal $\kappa^+$ is less than continuum. This follows from Theorem 2F.2 in [Mo] combined with the consequence of Martin’s Axiom that in a Polish space the collection of sets with the property of Baire is closed under unions of index less than continuum (see [E]). Martin’s Axiom is known to be consistent with the standard axioms of Zermelo-Fraenkel set theory with the axiom of choice [B].

Our theorem is the following:

**Theorem 4.5.** Assume $BP(\kappa)$. Suppose that $X$ is a connected, locally path connected $\kappa$-Polish space. Then $\pi_1(X)$ is either of cardinality $\leq \kappa$ or of cardinality continuum.

**Proof.** There are two cases. Suppose there exists a point $x$ near which we have arbitrarily small essential (non-nilhomotopic) loops. By local path connectedness we have a sequence of essential loops $f_1, f_2, \ldots$ based at $x$ with $\text{diam}(f_n) \leq 2^{-n}$. Define a map from the Cantor set $\{0,1\}^\mathbb{N}$ to the space $L$ of continuous loops based at $x$ under the sup metric as in section 2 by letting $\alpha \mapsto f^\alpha$. This map is continuous. Define an equivalence relation $\approx$ on the Cantor set by letting $\alpha \approx \beta$ if and only if $f^\alpha$ is homotopic to $f^\beta$.

Let $D$ be the space of continuous mappings $H : [0,1] \times [0,1] \to X$ such that $H(s,0) = H(s,1) = x$, also under the sup metric. Letting $\mathcal{D} = \{(f,g,H) \in L \times L \times D: (\forall t \in [0,1])(f(t) = H(0,t) \text{ and } g(t) = H(1,t))\}$, we have that the spaces $L,D$ and $L \times L \times D$ are all $\kappa$-Polish and $\mathcal{D}$ is closed in $L \times L \times D$. Letting $\text{Gr}$ be the graph of the map $(\alpha, \beta) \mapsto (f^\alpha, f^\beta)$, the set $\text{Gr} \times D$ is closed in $\{0,1\}^\mathbb{N} \times \{0,1\}^\mathbb{N} \times L \times L \times D$. The relation $\approx$ is the projection of the closed set $((\{0,1\}^\mathbb{N} \times \{0,1\}^\mathbb{N} \times \mathcal{D}) \cap (\text{Gr} \times D))$ to $\{0,1\}^\mathbb{N} \times \{0,1\}^\mathbb{N}$, and so is $\kappa$-Suslin. By $BP(\kappa)$ the relation $\approx$ has the
property of Baire. If \( \alpha, \beta \in \{0, 1\}^\mathbb{N} \) differ at exactly one point, then \( \alpha \not\approx \beta \) by the same proof as in Lemma \[2.3\] \( \). By Lemma \[3.3\] the set \( \approx \) is meager, and so \( \pi_1(X) \) is of cardinality continuum by Theorem \[3.4\].

Supposing that no such point exists, select for each \( x \in X \) an open neighborhood \( U_x \) such that any loop which lies in \( U_x \) is nullhomotopic in \( X \). Since paracompactness follows from metrizability, select an open star refinement \( \mathcal{U}_1 \) of \( \{U_x\}_{x \in X} \) that is locally finite. By local path connectedness, let \( \mathcal{U}_2 \) be the open cover consisting of path components of elements of \( \mathcal{U}_1 \). As \( X \) is \( \kappa \)-separable and metric we may pick a subcover \( \mathcal{U} \subseteq \mathcal{U}_2 \) whose cardinality is less than or equal to \( \kappa \). Notice that the identity mapping \( \pi_1(X) \rightarrow \pi_1(X) \) is 2-set simple relative to \( \mathcal{U} \) (see [CC] Definition 7.1). Letting \( \mathcal{N} \) be the nerve of \( \mathcal{U} \) we have that \( \pi_1(X) \) is a factor group of \( \pi_1(\mathcal{N}) \) by Theorem 7.3 (2) of [CC]. As \( \mathcal{N} \) is a simplicial complex of at most \( \kappa \) many vertices and 1-cells, we are done.

The comparable statement for first homology holds, via a similar argument.

**Theorem 4.6.** Suppose \( BP(\kappa) \). Suppose that \( X \) is a connected, locally path connected \( \kappa \)-Polish space. Then \( H_1(X) \) is either of cardinality \( \leq \kappa \) or of cardinality continuum.

**Proof.** The proof treats the two analogous cases as in the previous theorem. Each element of \( H_1(X) \) has a representative that is a mapping of a circle. Suppose there exists a point \( x \) near which there exist arbitrarily small mappings of \( S^1 \) that aren’t nullhomologous. Then as in the proof of Theorem 1.1 we treat these mappings as paths that are based at \( x \) by local path connectedness and select paths \( f_n \) at \( x \) that aren’t nullhomologous such that \( \text{diam}(f_n) \leq \frac{1}{2^n} \), define \( f_\alpha \) for each \( \alpha \in \{0, 1\}^\mathbb{N} \) and relations \( \sim_n \) in the same way. Each \( \sim_n \) has the property of Baire since \( \sim_n \) is \( \kappa \)-Suslin by a proof as in the previous theorem, and letting \( \sim = \bigcup_{n \in \mathbb{N}} \sim_n \) we have that \( \sim \) has the property of Baire, and \( \alpha \sim \beta \) if and only if \( f_\alpha \) is homologous to \( f_\beta \). The relation \( \sim \) also enjoys the property that if \( \alpha \) and \( \beta \) differ at exactly one point then \( \alpha \approx \beta \). By Theorem \[3.3\] and Lemma \[3.4\] we get continuum many classes.

If no such point exists, then for each \( x \in X \) we have an open neighborhood \( U_x \) such that any loop in \( U_x \) is nullhomologous. This gives an open cover \( \{U_x\}_{x \in X} \) of \( X \), which we refine as in the previous theorem to a cover \( \mathcal{U} \) of cardinality at most \( \kappa \) each of whose elements is path connected and
such that given any mapping of $S^1$ to $X$ whose image lies entirely in the union of two elements of $\mathcal{U}$ is nulhomologous. Let $g : \pi_1(X) \to H_1(X)$ be the Hurewitz map, which is onto. This map $g$ is 2-set simple relative to $\mathcal{U}$ ([CC] Definition 7.1), and so $g(\pi_1(X))$ is a factor group of the fundamental group of the nerve of $\mathcal{U}$, and so is at most of cardinality $\kappa$ by Theorem 7.3 (2) of [CC].

□

Since $BP(\aleph_0)$ is simply true, we may state special cases of the above two theorems:

**Theorem 4.7.** Suppose that $X$ is a connected, locally path connected Polish space. Then $\pi_1(X)$ is either countable or of cardinality continuum. Also, $H_1(X)$ is either countable or of cardinality continuum.

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