A Low-Density Closed Universe

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Abstract

Matter with an equation of state $p = -\rho/3$ may arise in certain scalar field theories, and the energy density of this matter decreases as $a^{-2}$ with the scale factor $a$ of the Universe. In this case, the Universe could be closed but still have a nonrelativistic-matter density $\Omega_0 < 1$. Furthermore, the cosmic microwave background could come from a causally-connected region at the other side of the Universe. This model is currently viable and might be tested by a host of forthcoming observations.

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Of the three possibilities, a closed Universe receives far less attention in the current literature than an open or a flat Universe. Observations that find a matter density less than critical suggest an open Universe. Theoretical arguments, such as the Dicke coincidence and inflation, favor a flat Universe. However, there are heuristic arguments for a closed Universe that involve, for example, consistency of quantum field theories on a compact space or the idea that it is easier to create a finite Universe with zero energy, charge, and angular momentum. Even so, given the observations, it requires some chutzpah to suggest that the matter density is greater than critical. For these reasons, models that are closed by virtue of a cosmological constant (Λ) have been recently considered [1]. In this paper, we consider a variation: a low-density closed Universe, which at low redshifts is entirely indistinguishable from a standard open Friedmann-Robertson-Walker (FRW) Universe with the same non-relativistic matter density.

If some form of matter with an equation of state \( p = -\rho/3 \) exists, then its energy density decreases with the scale factor \( a \) of the Universe as \( a^{-2} \) and thus mimics a negative-curvature term in the Friedmann equation. In this case, the Universe could be closed and still have a nonrelativistic-matter density \( \Omega_0 < 1 \).

In fact, the energy density contributed by a scalar field with a uniform gradient-energy density would scale as \( a^{-2} \). However, such a scalar-field configuration would collapse within a Hubble time unless it was somehow stabilized. Davis [2] pointed out that if there was a manifold of degenerate vacua with nontrivial mappings into the three-sphere [which could be accomplished if there was a global symmetry \( G \) broken to a subgroup \( H \) with \( \pi_3(G/H) \neq 1 \)], then a texture—a topological defect with uniform gradient-energy density—would be stabilized provided that it was wound around a closed Universe [2]. Non-intersecting strings would also provide an energy density that scales as \( a^{-2} \).

Moreover, if the energy density contributed by the texture is chosen properly, the observed cosmic microwave background (CMB) comes from a causally-connected patch at the antipode of the closed Universe [2]. Although the homogeneity problem is still not addressed, we find it illustrative and interesting that one can still construct a viable model, which looks remarkably like an open Universe at low redshifts, even though the largest-scale structure differs dramatically.

The Friedmann equation for a closed Universe with nonrelativistic matter and some other form of matter (perhaps a stable texture) with an equation of state \( p = -\rho/3 \) is

\[
H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho_m + \frac{\gamma - 1}{a^2} = H_0^2 \left[ \Omega_0 (1 + z)^3 + (1 - \Omega_0) (1 + z)^2 \right] \equiv H_0^2 [E(z)]^2, \tag{1}
\]

where \( H = \dot{a}/a \) is the Hubble parameter (and the dot denotes derivative with respect to time), \( z = (a_0/a) - 1 \) is the redshift, \( G \) is Newton’s gravitational constant, \( \rho_m \) is the density of nonrelativistic matter, and \( \gamma \) is a parameter that quantifies the contribution of the energy density of the texture. The second line defines the function \( E(z) \). This is exactly the same

\[1\] This could similarly be accomplished with \( \Lambda \neq 0 \), but these models are likely ruled out by lensing statistics [1].
as the Friedmann equation for an open Universe with the same $\Omega_0$, so this closed Universe has the same expansion dynamics. At the current epoch (denoted by the subscript “0”),

$$\Omega_0 = 1 + \frac{1 - \gamma}{a^2 H^2} = 1 - \Omega_t + \frac{1}{a_0^2 H_0^2},$$

(2)

where $\Omega_t = \gamma (a_0 H_0)^{-2}$ is the contribution of the texture to closure density today. So, $\Omega_0 < 1$ if $\gamma > 1$ even though the Universe is closed, and we require that $\Omega_t + \Omega_0 > 1$.

If the metric of a closed Universe is written as

$$ds^2 = dt^2 - a^2(t) \left[ d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

(3)

then the polar-coordinate distance between a source at a redshift $z_1$ and another source along the same line of sight at a redshift $z_2$ (for $\Omega_0 < 1$) is

$$\chi_2 - \chi_1 = \sqrt{\Omega_0 + \Omega_t - 1} \int_{z_1}^{z_2} \frac{dz}{E(z)}.$$

(4)

If $\Omega_t$ is chosen such that the polar-coordinate distance of the CMB surface of last scatter is $\chi_{LS} \simeq \pi$, then the CMB we observe comes from a causally-connected patch at the antipode of the Universe. Since this Universe expands forever, we could also choose $\chi_{LS} \simeq 2\pi$, in which case the CMB photons have traveled precisely once around the Universe. This introduces the intriguing possibility that when we observe the CMB we are looking at the local (rather than some distant) region of the Universe as it was at a redshift $z \simeq 1100$. In fact, for $\chi_{LS} \simeq n\pi$ with $n = 1, 2, 3, \ldots$, CMB photons have traveled $n/2$ times around the Universe, and the CMB comes from a causally-connected patch on the other side of the Universe (for $n$ odd) or from the local neighborhood (for $n$ even). From Eq. (4) the condition on $\Omega_t$ for $\chi_{LS} = n\pi$ is

$$\Omega_t = \left[ \frac{n\pi \sqrt{1 - \Omega_0}}{\text{arcsinh}(2\sqrt{1 - \Omega_0/\Omega_0})} \right]^2 + 1 - \Omega_0.$$

(5)

For $n = 1$ ($n = 2$), $\Omega_t$ increases from 1.6 to 2.5 (4 to 10) for $\Omega_0$ between 0.1 and 1.

Is this a realistic possibility? For $n \geq 2$, it requires a radius of curvature for the Universe that is probably too small to be consistent with observations. The $n = 1$ case is still consistent with our current knowledge of the Universe. However, forthcoming observations may be used to distinguish it from a standard open Universe, as we now explain.

Since the expansion dynamics is the same as for an open FRW Universe, quantities that depend only on the expansion, such as the deceleration parameter, the age of the Universe, or the distribution of quasar absorption-line redshifts, do not probe $\Omega_t$. Furthermore, the growth of density perturbations is the same as in a standard open Universe, so dynamical measurements of $\Omega_0$ (e.g., from peculiar-velocity flows) will also be insensitive to $\Omega_t$. Effects due to geometry arise only at $O(z^3)$ since $\sin \chi$ and $\sinh \chi$ differ only at $O(\chi^3)$; therefore, this Universe will differ from an open Universe only at $z \gtrsim 1$.

\footnote{For an excellent review of many classical cosmological tests, see Ref. [4].}
Ergo, we now turn to cosmological tests that probe the geometry of the Universe. Underlying these is the angular-diameter distance between a source at a redshift \( z_1 \) and a redshift \( z_1 < z_2 \),

\[
d_A(z_1, z_2) = \frac{\sin(\chi_2 - \chi_1)}{(1 + z_2)H_0\sqrt{\Omega_0 + \Omega_t - 1}}
\]

The angular size of an object of proper length \( l \) at a redshift \( z \) is \( \theta \simeq l/d_A(0, z) \). Consider first the case where \( \Omega_t \) is fixed by \( n = 2 \). Then the antipode \( \chi = \pi \) of the Universe must be at some redshift \( z_a < 1100 \). One finds that \( z_a \lesssim 5 \) for \( \Omega_0 \gtrsim 0.3 \), and therefore, the angular sizes of the highest-redshift quasars must be very large. Additional arguments against an antipode at \( z \lesssim 5 \) for a closed \( \Lambda \) Universe have been given Refs. [5]. These arguments probably apply to the model considered here, although we have not done a complete analysis. Therefore, a closed Universe with \( n \geq 2 \) is highly unlikely and we pursue it no further.

In Fig. 1, we plot the angular size as a function of redshift fixing \( \Omega_t \) so that the CMB comes from the antipode [i.e., Eq. 5 with \( n = 1 \)]. We also plot the results for a FRW Universe (dashed curves). The Figure shows that the angular sizes in a flat matter-dominated Universe can be roughly similar to those in a low-density closed Universe. Therefore, an analysis of the angular sizes of some compact radio sources, which shows consistency with a flat Universe [6], may also be consistent with a low-density closed Universe. Proper-motion distances of superluminal jets in radio sources at large redshift may provide essentially the same probe as do flux–redshift relations. The common caveat is that evolutionary effects must be understood if these are to provide reliable cosmological tests. It has been proposed that these effects may conceivably be understood well enough to discriminate between open
and flat Λ models [4]. Fig. 4 illustrates, however, that the difference between the angular sizes for the FRW Universe and the closed model for the same value of Ω₀ is quite a bit more dramatic than the difference between open FRW and flat Λ models (c.f., Fig. 13.5 in Ref. [4]). Therefore, if the angle-redshift relation can distinguish open and flat Λ models, then the distinction between these and the closed model will be even clearer.

![Graph showing differential number of galaxies per unit redshift per steradian](image)

**FIG. 2.** The differential number of galaxies per unit redshift per steradian in units of \(n₀H₀⁻³\) for the closed Universe (solid curves) and the open FRW Universe (dashed curves). The upper curves are for \(Ω₀ = 0.3\) and the lower curves are for \(Ω₀ = 1\).

Another classical cosmological test is the number-redshift relation. In the low-density closed Universe, the differential number of galaxies per steradian per unit redshift is,

\[
\frac{dN_{\text{gal}}}{dzdΩ} = \frac{n₀\sin^2[\chi(z)]}{H₀³(Ω₀ + Ω_t - 1)E(z)},
\]

(7)

where \(n₀\) is the local number density of galaxies, and the number per comoving volume is assumed to remain constant. In Fig. 2, we plot the number-redshift relation for the low-density closed Universe with \(Ω_t\) chosen so that the CMB comes from the antipode and for standard open and flat FRW models. The Figure shows that an application of this test, which finds values of \(Ω₀\) near unity in a FRW Universe [5], can also be consistent with a low-density closed Universe. However, galactic evolutionary effects are realistically quite significant, so this remains a controversial test.

A test for Λ discussed by Alcock and Paczyński [9] may also be an especially effective probe of \(Ω_t\). The redshift thickness \(δz\) and angular size \(δθ\) of a roughly spherical structure that grows with the expansion of the Universe will have a ratio

\[
\frac{1}{z} \frac{δz}{δθ} = \frac{E(z)\sin[\chi(z)]}{z\sqrt{Ω₀ + Ω_t - 1}}.
\]

(8)
FIG. 3. Plot of $(\delta z/\delta \theta)/z$ for the closed Universe (solid curves) and the open FRW Universe (dashed curves). Shown are curves for both $\Omega_0 = 0.1$ and $\Omega_0 = 1$.

As shown in Fig. 3, this function is significantly lower in a low-density closed Universe than it is in an open Universe (and in a $\Lambda$ Universe; c.f., Fig. 13.9 in Ref. [4]). Furthermore, it depends only very weakly on the value of $\Omega_0$ and therefore provides an $\Omega_0$-independent determination of the geometry. A precise measurement may be feasible with forthcoming quasar surveys [10].

We have also checked the probability for gravitational lensing of sources at high redshift. This test provides perhaps the strongest constraint on $\Lambda$ models [11], and makes it unlikely that the CMB comes from the antipode of a Universe that is closed with the addition of a cosmological constant [1]. The probability for lensing of a source at redshift $z_s$ for $\Omega_0 < 1$ and $\Omega_t + \Omega_0 > 1$ relative to the fiducial case of a standard flat Universe is

$$P_{\text{lens}} = \frac{15}{4} \left[ 1 - \frac{1}{(1 + z_s)^{1/2}} \right]^{-3}$$

$$\times \int_0^{z_s} \frac{(1 + z)^2}{E(z)} \left[ \frac{dA(0, z) dA(z, z_s)}{dA(0, z_s)} \right]^2 dz.$$  \hspace{1cm} (9)

The current observational constraint is roughly $P_{\text{lens}} \lesssim 5$. If $\Omega_t$ is chosen so that the CMB comes from the antipode, then $P_{\text{lens}} < 2.5$ for $0 < \Omega_0 < 1$. Hence the model is consistent with current data and is likely to remain so.

Finally, if ours is actually a low-density closed Universe, it will probably have a dramatic signature in the anisotropy spectrum of the CMB, especially if the CMB comes from the antipode of the Universe. Although the detailed shape of the anisotropy spectrum depends on a specific model for structure formation, it quite generically has structure (known as “Doppler peaks”) on angular scales smaller than that subtended by the horizon at the
surface of last scatter. The angle subtended by the horizon at last scatter depends on the cosmological model; in a standard FRW Universe, it is \( \theta_{LS} \approx \Omega_0^{1/2} \). Therefore, measurement of the location of the first Doppler peak provides a determination of the geometry of the Universe [12], and with forthcoming all-sky CMB maps with sub-degree angular resolution, this measurement may be quite precise [13].

The angular scale subtended by the horizon in a low-density closed Universe may be approximated by

\[
\theta_{LS} \approx 2^\circ \frac{\sqrt{\Omega_0 + \Omega_t - 1}}{\Omega_0^{1/2} \sin \chi_{LS}},
\]

when \( \theta_{LS} \) evaluates to small angles; otherwise, \( \theta_{LS} = O(\pi) \). Here,

\[
\chi_{LS} = \sqrt{\frac{\Omega_0 + \Omega_t - 1}{1 - \Omega_0}} \arcsinh \left( \frac{2\sqrt{1 - \Omega_0}}{\Omega_0} \right)
\]

is the polar-coordinate distance traversed by the CMB photons since last scatter. As expected, this is always larger than \( \theta_{LS} \) for a flat or open FRW Universe. Moreover, if \( \chi_{LS} \approx \pi \), the Doppler-peak structure of the CMB is shifted to the largest angular scales, and the suppression of CMB anisotropies due to Silk damping is also shifted to larger angular scales. The precise shift depends on exactly how close the last-scattering surface is to the antipode. It is almost certain that these signatures will be distinguishable in forthcoming CMB maps if they are indeed there.

Although there is no horizon problem in this model, at earlier or later epochs, the CMB is not generally at the antipode. Furthermore, the homogeneity of the Universe is not necessarily explained even if the CMB comes from a causally-connected region. Even so, it is worth noting that one can construct a viable model, which is indistinguishable from an open Universe at redshifts \( z \ll 1 \), with a closed geometry. Furthermore, the model will be tested by forthcoming observations of the Universe at large redshifts, especially through angular sizes, \( \delta z/\delta \theta \), and the CMB.

We have focussed in our numerical work on the case where \( \Omega_t \) is such that the CMB comes precisely from the antipode. However, one could explore other values of \( \Omega_t \), perhaps within the context of flat inflationary models.

Finally, what about the homogeneous matter with an energy density which scales as \( a^{-2} \)? If this is due to a topologically stabilized scalar-field configuration, as discussed above, then the symmetry-breaking scale must be of order the Planck scale if \( \Omega_t \) is of order unity. Furthermore, the global symmetry must be exact. This model would therefore have significant implications for Planck-scale physics if verified [14].

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For example, the anisotropy spectrum might resemble those shown in Fig. 6 of Ref. [1] for the analogous case with a cosmological constant for a flat scale-invariant spectrum of density perturbations. However, the overall tilt of the spectrum depends on the model of primordial perturbations and could therefore be considerably different.
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