The effect of retardation in the random networks of excitable nodes, embeddable in the Euclidean space

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Abstract

Some features of random networks with excitable nodes that are embeddable in the Euclidean space are not describable in terms of the conventional integrate and fire model alone, and some further details should be involved. In the present paper we consider the effect of the retardation, i.e. the time that is needed for a signal to traverse between two agents. This effect becomes important to discover the differences between, e.g. the neural networks with low and fast axon conduct times. We show that the inclusion of the retardation effects makes some important changes to the statistical properties of the system. It considerably diminishes the amplitude of the possible oscillations in the random network. Additionally, it causes the critical exponents in the critical regime to considerably change.

Keywords: random networks, retardation effects, embedding in the Euclidean space, oscillatory regime

(Some figures may appear in colour only in the online journal)

1. Introduction

Complex networks today have vast applications in science, ranging from neuroscience [1] and intelligent signal processing [2], to the social networks and world wide web [3]. In the neural networks, the theories of adoptive optimizing control can be served as a basis for the learning process which, in the behavioral sense, is driven by changes in the expectations about the future salient events such as rewards and punishments [4]. There are many models to explain the experimental neuronal avalanches [5] which are based on the Hodgkin–Huxley model [6]. A newly-observed mode of activity in the brain is the (self-organized) criticality, which is realized from the power-law and scaling behaviors in various distribution functions of the brain avalanches [5]. The criticality is a key factor in the brain, since it improves the learning [7], optimizes the dynamic range [8–11], makes information processing efficient [12], and leads to optimal transmission and storage of information [13]. The origin of this criticality has not been understood well.

In the current state of research on the complex networks with excitable agents, the communications between the interacting agents are supposed to be instantaneous [2]. Instantaneous here means that the conduction time is independent of the length of the connection, i.e. two signals traversing two unequal connections in length have the same travel period. An example is the swarm dynamics of flying birds which has been assumed to be instantaneous [14]. Apparently the length should be meaningful here, and the network should have the capability of being embedded in the Euclidean space. Many instantaneous artificial neural networks (which is a type of massively parallel computing architecture based on brain-like information encoding) with a vast range of applications have been invented, like signal processing and also pattern recognition [2]. In the systems which are embedded in the Euclidean space, when the speed of the signal (whatever it is) between two agents is very higher than the characteristic speeds in the system (resulting from the speed of the activities of the agents), this approach...
works well, as can be seen in the partial success of the instantaneous models in describing some experiments on brain [5, 15], like the self-organized criticality mode of brain activity [16, 17], the chaos for balanced excitatory and inhibitory activity [18], the neuronal coherence [19], and the synchronization of cortical activities [20]. Also the theoretical explanation of the neuronal avalanches which are seen in the cerebral cortex (in which the spontaneous neural activities occur at the critical state) are based on such instantaneous dynamics [21]. The instantaneous mechanisms which have also been proposed to explain the signal propagation in neocortical neurons based on the repetitions of spontaneous patterns of synaptic inputs should be added to this list [22]. The same analysis has been done for the edge independent networks [23].

There are however some situations that this speed is not that high, and one should take the retardation effects into account. Here the various time scales play vital role. For example, there are many time scales for the neurons in primary auditory cortex of cats, ranging from hundreds of milliseconds to tens of seconds [24]. For a general argument on the time scales see [25]. In [26] the hierarchy of time scales in the brain has been considered and analyzed. These time scales are not however, necessarily fixed, and in some situations they can be tuned. An example of such a tuning of time scale of neuronal activities has been reported in [27], for which millisecond time scales has been achieved. A tangible example of the importance of the time scales and conduction times (and correspondingly the speed of signal) is the neural systems whose constituents (neurons) are lacking the myelin sheath in which the nerve conduction velocity in the avalanche pulse dynamics are not that high [28]. Axon conduction time is definitely a relevant quantity in these networks. This nerve conduction velocity can also be precisely regulated with internal mechanisms, correct exertion of motor skills, sensory integration and cognitive functions [29]. In a neuronal population if the conduction velocity is low or equivalently the length of the axons is comparable with the speed of signal (v) times (×) the characteristic times of neuronal activities (Δtactivity), the retardation effects become important. The retardation in a nervous system is the effect in which the present activity of a neuron is related to the activity of its neighboring neuron at Δt = r/v times ago in which r is the distance between two neurons and v is the speed of the signal.

These retardation effects are expected to be important in the neural systems with low-speed neurons. For the neuronal cells with the Myelin sheath around their axons (a fatty insulating later that surrounds the nerve cells of jawed vertebrates, or gnathostomes) the speed of the signal is higher than that ones without myelin sheath. This causes a lot of differences in these systems, which separates jawed vertebrates from the invertebrates. In vertebrates, the rapid transmission of signals along nerve fibers becomes possible by the myelination of axons and the resulting salutatory conduction between nodes of Ranvier [29]. Among the vertebrates also, the speed of neuronal signals is more or less high for most intelligent species. Myelination not only maximizes conduction velocity, but also provides a tool to systematically regulate conduction times in the nervous system, which to date, has not been understood well [29]. Node assembly, internode distance and the diameter of axon, which are controlled by myelination glia, determine the speed of signals along axons. All of these show the importance of the retardation effects in real neural systems.

In this paper we consider these retardation effects for a random network with refractory period, i.e. the agents are prevented to send a signal immediately after spiking. Our numerical results show that the retardation effects, not only change the critical behaviors, but also decrease the oscillatory behaviors of the system. By analyzing the branching ratio and other statistical tests we show that the point (in terms of largest eigenvalue of the adjacency matrix λ) at which the critical behaviors starts, the interval of critical behaviors and the point at which the bifurcation occurs is just the same as the instantaneous random system. The tuning of the signal propagation in such random networks is therefore promising for controlling some undesirable oscillatory responses.

The paper has been organized as follows: in the next section we explain the effects of the retardations and also the method to enter it in the calculations. The section 3 has been devoted to the numerical results and the explanation of the behaviors of the model. We close the paper by a conclusion.

2. Retardation effects

In this section we consider a random undirected graph with N excitable nodes. Each two nodes are connected with the probability q, which results to the average node degree ⟨k⟩ = qN. The connections are weighted with quenched random numbers wij (between nodes i and j which are connected), whose distribution is uniform in the interval [0, 2σ], in which σ is an external parameter. The state of a node (i) at time t is described by Ai(t) which is called the activity, assuming two values: active Ai = 1 or quiescent Ai = 0. The aggregate activity at time t is defined as x(t) = ∑i=1N Ai(t), which is commonly used for analyzing the statistical properties and also the stability of the system. According to the integrate and fire model (IFM) the ith node at time t becomes active depending on the aggregate input signal:

\[ p(A_i(t) = 1) = f \left( \sum_j w_{ij} A_j(t-1) \right) \]  

in which f is a dynamical (monotonically increasing) map which yields the probability that a node becomes active based on the input signal to that node, and is commonly chosen to be:

\[ f(y) = \begin{cases} y, & 0 \leq y \leq 1 \\ 1, & y > 1 \end{cases} \]  

This dynamics is known to be dictated by the largest eigenvalue of the adjacency matrix w, namely λ [11, 30]. For the random graph that has been considered in this work, this eigenvalue is equal to λ = σ ⟨k⟩ = σqN [30–32]. For λ < 1 the system has an attractor x = 0, i.e. some (stochastic) time
after the external local drive, all nodes of the system become inactivate. The completely inverse behavior is seen for \( \lambda > 1 \) in which the perturbation grows with time, reaching \( x = N \) at some stochastic time. The intersection between these two intervals, i.e. \( \lambda = 0 \) is known to be critical for which some power-law behaviors occur for e.g. the avalanches (≡an overall process between starting and ending an activity). Let us define \( S \) as the avalanche size (≡the total number of activities in an avalanche), \( M \) as the avalanche mass (≡the total number of distinct nodes which have been activated (at least once) in an avalanche), \( D \) as the avalanche duration (≡the total time interval of an avalanche), and \( x \) the active nodes at a given time. Then the fingerprint of the criticality can be found in the power-law behavior of the distribution functions, i.e. \( N(\zeta) \sim \zeta^{-\gamma} \) in which \( \zeta = S, M, D, x, \ldots \). Also the critical point is detectable in terms of the branching ratio which is defined as the conditional expectation value \( b(X) \equiv E\left[\frac{x_{i+1}}{X} | x_i = X\right]\), i.e. the expectation value of \( x_{i+1}/X \) conditioned to have \( x_i = X \). For the critical system \( \lim_{\zeta \to 0} b(X) = 0 \), and also \( \lim_{\zeta \to 0} \frac{db(X)}{d\zeta} < 0 \) [32], which is the general property of the nonlinear dynamical systems [33]. These two conditions state that when \( X \to 0, b(X) \) approaches to zero from the negative values. It also determines the possible fixed points of the model in hand by the condition \( b(X^*) = 1 \).

Now let us explain the network with refractory period (which has been done in [32]) and also the retardation effects. It has been shown that the inclusion of the refractory period in the dynamics has some non-trivial effects, like the extension of critical interval, bifurcation, and non-trivial fixed points [32], which was considered to be instantaneous. The effect of the retardation in the signal propagation has not been considered in the literature yet. We show that the retardation has non-trivial effects. The retarded IFM is defined by the following nonlinear dynamical equation:

\[
p(A_i(t) = 1) = \delta_{A_i(t-1),0} f(\text{sum}(A_i(t))) \tag{3}
\]

in which \( \text{sum}(A_i(t)) \) is the integrated effect which has arrived to the \( i \)-th site at time \( t \), taking into account the retardation effects. Also \( \delta_{A_i(t-1),0} \) is unity if \( A_i(t-1) = 0 \) and is zero otherwise, i.e. it is the effect of the refractory period in the node. The sum-function sums the integrated retarded weighted signals, and is defined as:

\[
\text{sum}(A_i(t)) \equiv \sum_{\tau=0}^t \sum_{j=1}^N w_{ij} A_j(t') G_{ij}(t, t') \tag{4}
\]

in which we have defined \( G_{ij}(t, t') \equiv \delta\left(t', t - \frac{|i-j|}{v}\right) \) as the retarded Green function, and \(|i-j|\) is the distance of \( i \)-th and \( j \)-th nodes. Therefore, this dynamics works for networks that are embedded in the Euclidean space, that is supposed to be two-dimensional in this study. It is notable that this is not the only way to define \( G_{ij}(t, t') \). For example, one can take into account the dissipation of the signal as a function of the length or the time. By inserting this into equation (3), one finds that:

\[
p(A_i(t) = 1) = \delta_{A_i(t-1),0} \sum_{j=1}^N w_{ij} A_j\left(t - \frac{|i-j|}{v}\right) \tag{5}
\]

This function carries instantaneously the effects of retardation and the refractory period, and \( w_{ij} \) is a priory known quenched stochastic variable that was introduced above.

2.1. Numerical details

For building the host random network, we simply choose randomly two nodes and connect them. We repeat this \( q \frac{N(N-1)}{2} \) times (\( \frac{N(N-1)}{2} \) being the total possible links in the system). In figure 1(a) we have shown schematically a random graph that has been embedded to two dimensions. The
Figure 2. The time series for the activity (with retardation effects) for various rates of $\lambda$, i.e. $\lambda = 0.9, 0.987, 1.5$ and $3.57$ with blue, green, red and orange, respectively.

Figure 1(b) shows the histogram of the lengths of the connections ($P(L)$) for $N = 50^2$, $100^2$ and $150^2$ Erdos–Renyi network (embedded in the Euclidean space) which have their peaks at $L = \frac{1}{2} \sqrt{N}$ as expected. The error bars have been obtained using the least square method.

In [32] it has been shown that the model with refractory period shows three relevant regimes: subcritical regime and period shows three relevant regimes: subcritical regime and orange, respectively. It is seen that for $\lambda < 1$ the stable fixed point is $x^* = 0$, whereas for $1 < \lambda \leq 2$ $x^* \neq 0$ will be attractor of the dynamics (these fixed points should be obtained by the condition $b(x^*) = 1$, and also $\frac{db}{dx}|_{x=x^*} < 0$). Therefore, for $\lambda \leq 1$ we have some well-defined avalanches (avalanche $\equiv$ the process in the time interval in which the activity starts from and ends on zero). For $1 < \lambda \leq 2$ however, we should define the avalanche in another way, since the process does not end, and $x$ fluctuates around $x^* = 0$. In this case we define a threshold $X = x^*$ and define the avalanche as the process which starts from and ends on this threshold. The time series for $\lambda = 0.9, 0.987, 1.5$ and $3.75$ have been shown as an instance in figure 2 for the dynamical system with retardation effects. It is seen that for $\lambda < 1$ the stable fixed point is zero (with some small fluctuations), and for $1 < \lambda = 1.5 < 2$ (the red graph) non-zero stable fixed point arises, and also for $\lambda = 3.75 > 2$ the system is in the oscillatory phase (the orange graph) in which $x$ oscillates between two limits.

We will compare the results of the retarded system with the results for the system without retardation effects.

3. Measures and results

We first start with the activity-dependent branching ratio $b(x)$. In figure 3(a) we have shown this function for both retarded and instantaneous dynamical systems (IDSs). We have defined $M_\ell$ by the relation $b(M_\ell) = 1$. The main panel shows $b(M)$ in terms of $M - M_\ell$, from which we see that the slopes at the points in which the graphs cross $b(M) = 1$ is negative. This confirms that the found points are stable fixed points. In the left inset the $\lambda$ dependence of $M_\ell$ has been compared for $N = 50^2, 100^2$ and $150^2$ networks. It is notable that the mean field results for the instantaneous (retardation-free) dynamical systems reveals that $M_\ell^{\text{spontaneous}} = N\left(1 - \frac{1}{2}\right)$ [32]. The data in this inset is consistent with the mean field result for IDS, but the (absolute value of) slope of the graphs are very higher, demonstrating that the dynamics (towards the fixed points) is faster.

One of the most serious differences of the retarded and IDSs arises from their behaviors in the oscillatory regime $\lambda > 2$. The figures 3(b) and (c) show the distribution of the activity $x$ for instantaneous and retarded dynamical systems (RDSs) respectively. In the figure 3(b) the two branches of the oscillations are evident which rapidly grow with $\lambda$. To quantify this, we have plotted $\hat{x}(\lambda) \equiv \hat{x}_{\text{upper branch}} - \hat{x}_{\text{lower branch}}$, whose numerical value shows the amplitude of the oscillations. $\hat{x}$ is the average value of $x$. We see that it rapidly grows with increasing $\lambda$. The figure 3(c) however shows a different behavior for the same $\lambda$s. The growth of this amplitude is meaningfully smaller than that for figure 3(b). For example, for $\lambda = 35$ the gap between two
branching ratio is less than 100, whereas for IDS, it becomes of order 7000. Here we see that the inclusion of the retardation, diminishes the oscillatory behaviors of the random networks with excitable nodes. For neural networks (when it is modelled by random excitable nodes) this finding sounds promising for controlling undesirable activity oscillations.

The same graphs have been shown for the $\lambda$s in the onset of criticality, i.e. in the vicinity of $\lambda = 1$ (figure 4(a)) and in the critical interval, i.e. $1 < \lambda \leq 2$ (figure 4(b)). We see from figure 4(a) that for $\lambda = 1$ (or in its vicinity) the exponents of the retarded and IDSs are nearly the same for the $x$ variable. In the subcritical case however, (the inset), $P(x)$ behaves logarithmically for both systems with different (non-universal) slopes. The same has been sketched for $1 < \lambda \leq 2$ in figure 4(b), whose left inset shows that the standard deviation $\zeta$ grows monotonically with $\lambda$. This increase is faster for larger $N$s.

The retardation is a relevant factor for the statistics of the avalanche duration $D$. More precisely $\tau_D(\lambda = 1)$ is different for retarded and instantaneous dynamics. In the figure 5(a) we see that $\tau_D(\lambda = 1)^{\text{retarded}} = 2.61 \pm 0.1$, whereas $\tau_D(\lambda = 1)^{\text{simultaneous}} = 1.78 \pm 0.1$. Finite size effects have been analyzed by repeating the calculations for $N = 50^2$, $100^2$ and $N = 150^2$ for this and other observables. A clean finite size scaling has not been seen, therefore we...
Figure 4. (a) The log–log plot of $P(x)$ for both retarded and instantaneous dynamics for $\lambda$ in the onset of the critical region. Inset: the same for the subcritical regime. (b) The plot of $P(x)$ for both retarded and instantaneous dynamics for $1 < \lambda \leq 2$. Lower inset: the same for $N = 50^2$. Upper inset: the standard deviation $\zeta$ in terms of $\lambda$.

Figure 5. The log–log plot of the distribution functions of (a) the avalanche duration $D$, (b) the avalanche mass $M$, and (c) the avalanche size $S$ for $N = 100^2$. Lower insets: the same graph for $N = 50^2$ and $N = 150^2$. Upper insets: The same for IDS. (d) The log–log plot of $S–T$ diagram. Upper inset: The log–log plot of $S–M$ diagram. Lower inset: the log–log plot of $M–T$ diagram.
have reported the exponents for the largest system size, i.e. $N = 150^2$.

At $\lambda = \lambda_c$ we actually observed scaling behaviors. At the sub- and super-critical regimes, these scaling behaviors become more limited. These scaling behaviors are valid up to some points which depend on $\lambda$ and $N$ and that whether the model is retarded or not (apparently for $\lambda = \lambda_c$ the range of the scaling behavior is most). As an example, the comparison between the subcritical (blue graph) and the critical (orange graph) in figure 5(a) shows the difference.

The same exponents for $M$ show an agreement with the IDS for $\lambda$ in the onset of critical region, i.e. $\tau_M(\lambda = 1)^{\text{instantaneous}} = 1.47 \pm 0.2$ and $\tau_M(\lambda = 1)^{\text{simulated}} = 1.45 \pm 0.2$ (figure 5(b)). For the avalanche size $S$ we have $\tau_S(\lambda = 1)^{\text{instantaneous}} = 2.02 \pm 0.1$ and $\tau_S(\lambda = 1)^{\text{instantaneous}} = 1.48 \pm 0.2$ (figure 5(c)). Note that the determination of the onset of the critical region for a given $N$ has itself an uncertainty and should be determined by analyzing the branching ratio. For example, as the system size $N$ decreases, this value also decreases, e.g. For $N = 50^2$ $\chi^{\text{onset}} = 0.95 \pm 0.02$. This itself generates a systematic error in the determination of the exponents on the onset of the criticality.

Now let us consider the scaling properties of the statistical variables. This has been done in figure 5(d) for all possible scaling quantities. As is explicit in this graph, the scaling between $S$ and $T$ (the duration of avalanche here) is displaced and the corresponding exponents changes from $1.87 \pm 0.05$ (for IDS) to $1.74 \pm 0.05$ (for RDS) which is out of its error bar, and the change is meaningful. The same occurs for the $\gamma_{MT}$ (lower inset), i.e. it changes from $1.77 \pm 0.05$ (for IDS) to $1.64 \pm 0.05$ (for RDS). Interestingly the $\gamma_{SM}$ does not change considerably and remains on $1.03 \pm 0.05$.

The critical exponents have been gathered in table 1 and are compared to the IDS.

4. Conclusion

In this paper we have addressed the problem of the effect of retardation in random networks with excitatory nodes. The retardation effects have been brought into the calculations using the equation (5), which is mixed by the refractory period. We analyzed the branching ratio which yields the possible intervals of distinct behaviors, like the subcritical, critical and oscillatory behaviors. Our calculations demonstrated that the oscillations are remarkably suppressed by the retardation. This can be a promising effect in the systems that such oscillations are undesirable.

Also the critical exponents of the RDSs are meaningfully different from that for the IDSs. The numerical amounts of these exponents can be found in table 1. Interestingly the exponents of avalanche masses $\gamma_M$ and also $\gamma_S$ do not change considerably with respect to the system without delay. However $\tau_D$ and $\tau_S$ change considerably, i.e. $\tau_{RDS} - \tau_{IDS} \approx 0.83$, $\tau_{RDS} - \tau_{IDS} \approx 0.5$, $\gamma_{IDS} - \gamma_{RDS} \approx 0.13$ and $\gamma_{IDS} - \gamma_{RDS} \approx 0.13$. Although these changes show that the universal properties of the IFM model changes, it should be tested by some other methods and simulations. Also the change of the universal properties of the model on the other random lattices, like the small world network and finite range random link lattices is good ideas for future works.

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Table 1. The critical exponents in the onset of criticality of two models: RDS (for $N = 150^2$) and IDS (for $N = 100^2$).

| Exponent | Definition | RDS | IDS |
|----------|------------|-----|-----|
| $\tau_D$ | $P(D) \sim D^{-\tau_D}$ | 2.61 $\pm$ 0.1 | 1.78 $\pm$ 0.1 |
| $\tau_M$ | $P(M) \sim M^{-\tau_M}$ | 1.49 $\pm$ 0.2 | 1.45 $\pm$ 0.2 |
| $\tau_S$ | $P(S) \sim S^{-\tau_S}$ | 2.02 $\pm$ 0.1 | 1.48 $\pm$ 0.2 |
| $\gamma_{ST}$ | $S \sim T^\gamma_{ST}$ | 1.74 $\pm$ 0.05 | 1.87 $\pm$ 0.05 |
| $\gamma_{SM}$ | $S \sim M^{\gamma_{SM}}$ | 1.03 $\pm$ 0.05 | 1.03 $\pm$ 0.05 |
| $\gamma_{MT}$ | $M \sim T^{\gamma_{MT}}$ | 1.64 $\pm$ 0.05 | 1.77 $\pm$ 0.05 |
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