The Bilevel Linear Programming with Fuzzy Relation Constraints

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Abstract. Aiming at the particularity of bilevel linear programming model with fuzzy relationship constraints, a new algorithm is designed based on bilevel linear programming. Firstly, a bilevel linear programming problem model with fuzzy relation constraints is established, and the related definitions are given; Then, the solution process of the model is stated. Finally, an example is given to verify the correctness of the algorithm.

Keywords. Fuzzy relation equation; Bilevel linear programming; Optimum solution

1. Introduction
Currently, fuzzy relational equations have achieved many successful applications in the field of control system fault diagnosis and pattern recognition. Its theory has become an important research content of fuzzy mathematics. Applying the theory of fuzzy relational equations to the optimization domain with uncertainty reasoning is an important research direction of fuzzy optimization. In the field of fuzzy optimization, linear programming, nonlinear programming, geometric programming, etc. with fuzzy relational equation constraints are included in the optimization goal. Linear programming is an important basic planning. It has been widely used in various fields of society. Linear programming problems with fuzzy relational equation constraints in fuzzy optimization have also been proposed, and some good results have been achieved. Based on fuzzy two-layer linear programming, this paper proposes a new optimization model--a two-layer linear programming model with fuzzy relation constraints.

2. Models and definitions
The two-layer linear programming model with fuzzy relational constraints is as follows:

\[(P_1) \quad \max f_1(x, y) = a^T x + b^T y \quad \text{subject to} \quad A x \oplus b \leq r \]

\[y \text{ solution is } (P_2) \quad \max f_2(x, y) = c^T x + d^T y \quad \text{subject to} \quad A x \oplus b \leq r \]

Of which, \(a, c, x \in \mathbb{R}^n\), \(b, d, y \in \mathbb{R}^n\), \(r \in \mathbb{R}^n\), column vectors representing the corresponding dimensions \(A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times n}\), \(A = (a_g), B = (b_g)\) and \(0 \leq a_g \leq 1, 0 \leq b_g \leq 1, 0 \leq r \leq 1\).
Constraint condition $A \oplus b \leq y \leq r$ is a fuzzy relation inequality, and its fuzzy relation is $\vee - \wedge ( \max - \min )$. Synthesis, that is

$$A \oplus b \leq y \leq r$$

can be expressed as

$$[\vee (x_i \wedge a_y)] \vee [\nu (y_i \wedge b_y)] \leq r_i.$$  

And because

$$[\vee (x_i \wedge a_y)] \vee [\nu (y_i \wedge b_y)] \leq r_i$$

is equivalent to

$$\begin{cases}
\vee (x_i \wedge a_y) \leq r_i \\
\nu (y_i \wedge b_y) \leq r_i 
\end{cases}$$

So the constraint can be reduced to

$$\begin{cases}
A \leq x \leq r \\
B \leq y \leq r 
\end{cases}$$

In this way, the bilevel linear programming model with fuzzy relation constraints can be transformed into the following forms

$$\begin{align}
\max f_1(x, y) &= a^T x + b^T y \\
\text{s.t. } &A \leq x \leq r \\
\max f_2(x, y) &= c^T x + d^T y \\
\text{s.t. } &b \leq y \leq r
\end{align}$$  \hspace{1cm} (2)

**Definition 1**

For bilevel linear programming problems with fuzzy relation constraints, Set

$$S = \{(x, y) \mid A \leq x \leq r, 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

is called its admissible set. $S$ is not necessarily a convex set, but it is bounded. Because the set $S$ is the union of several convex sets. Each $(x, y) \in S$ is called an admissible solution.

**Definition 2**

If $(x, y) \in S$, $f_1(x', y') \geq f_1(x, y)$, the optimal solution of bilevel linear programming problem with fuzzy relation constraints is $(x', y') \in S$.

**Lemma 1**

A bilevel linear programming problem with fuzzy relation constraints, if there is an optimal solution, then there must be an optimal solution is a pole in $S$.

**Lemma 2**

Pole of $S$ must be the upper or lower bound of the interval of the solution that in the fuzzy relation inequality of $A \leq x \leq r, 0 \leq x \leq 1, 0 \leq y \leq 1$. 
3. Model solution

For fuzzy relation inequality $x \land a \leq b, 0 \leq a \leq 1, 0 \leq b \leq 1$,

a new operator $\ominus$ is defined in reference [8] as follows:

$$x = b \ominus a = \begin{cases} [0,1] & \text{if } a \leq b \\ [0,b] & \text{if } a > b \end{cases}$$

Using the operator defined above, we can obtain the solution

$$x = b \ominus a$$

for the fuzzy relation inequality $x \land a \leq b$. It's an interval.

For $A \circ x \leq r$, $A = (a_{ij})_{m \times n}, 0 \leq a_{ij} \leq 1, 0 \leq r_i \leq 1$,

Its solution set is $\mathcal{X} = \{x | A \circ x \leq r\}$. The following properties are readily available:

Proposition 1
If $x'_i \in \mathcal{X}, i \in I (I$ is nonempty index set), $\bigcup_{i \in I} x'_i \in \mathcal{X}$.

Proposition 2
If $x^1 \in x^2 \in x^3$ and $x^1 \in \mathcal{X}, x^3 \in \mathcal{X}, x^2 \in \mathcal{X}$.

Because model (1) can be reduced to form (2), and the particularity of fuzzy relation inequality (2), it can be regarded as two linear programming problems with fuzzy relational constraints. That is

$$\max_{x} f_1(x, y) = a^T x + b^T y$$

s.t. $A \circ x \leq r$

and

$$\max_{y} f_2(x, y) = c^T x + d^T y$$

s.t. $B \circ y \leq r$ (3)

These two linear programs can be solved by using the $y$ and $x$ as a known variable, so that the solution can be solved separately.

The first model in (3)

$$\max_{x} f_1(x, y) = a^T x + b^T y$$

s.t. $A \circ x \leq r$ (4)

In the objective function $b, y \in \mathbb{R}^n$, two vectors $a, b$ are represented, that is

$$a^T = (a_1, a_2, \cdots a_n), b^T = (b_1, b_2, \cdots b_n).$$

So the objective function can be written as

$$\max_{x} f_1(x, y) = \sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{m} b_j y_j.$$
Because the constraints in (4) are only with relevant $x$. Now consider the objective function $a_i$ coefficient of $x$.

Method with reference to reference [10]. When the coefficient of the objective function $a_i \geq 0(i = 1, 2, \cdots n)$, then the optimal solution of model (4) must be in the fuzzy relation inequality $A \cdot x \leq r$, the upper limit of the interval of the solution is obtained; When the coefficient of the objective function $a_i \leq 0(i = 1, 2, \cdots n)$, then the optimal solution of model (4) must be in the fuzzy relation inequality $A \cdot x \leq r$, the lower bound of the interval of solution is obtained; When the coefficients of the objective function do not satisfy the above conditions, the

$$a' = \begin{cases} a_i & \text{if } a_i \geq 0 \\ 0 & \text{if } a_i < 0 \end{cases},$$

$$a'' = \begin{cases} 0 & \text{if } a_i \geq 0 \\ a_i & \text{if } a_i < 0 \end{cases}$$

Then $a_i = a'_i + a''_i$. In this way, model (4) can be transformed into two models:

$$\max_{x} f_1(x, y) = \sum_{i=1}^{n} a'_i x_i + b^T y$$

s.t. $A\circ x \leq r$

and

$$\max_{x} f_1(x, y) = \sum_{i=1}^{n} a''_i x_i + b^T y$$

s.t. $A\circ x \leq r$ (6)

The optimal solution of the first model must be the upper bound of the interval of the inequality $A\circ x \leq r$ solution. The optimal solution of the second model must be the lower bound of the interval of the inequality $A\circ x \leq r$ solution.

The second model in (3) is treated in the same way. You can get the value of $x, y$ in this way.

The following steps of model (1) are given:

Step1: Convert model (1) to (2) form.

Step2: Solve the solution set of fuzzy relation inequality constraints, and obtain an interval range of $x$ and $y$.

Step3: Observe the symbols of $a_i$, which the coefficients of the first model objective function in (2), when they are greater than or less than zero, no need to handle; when there are both greater than zero and less than zero, the model is transformed into (6) according to the method in (5). The second model in (2) is treated in the same way.

Step 4: The optimal solution is obtained at the upper limit of the interval $x$ when $a_i$ are all greater than zero in the first model in (2); The optimal solution is obtained at the lower limit of the interval $x$ when $a_i$ are all less than zero; The optimal solution is determined according to (6) method when $a_i$ both greater than zero and less than zero; The second model in (2) is treated in the same way. The value $x, y$ obtained is the optimal solution of model (1).
4. Examples

\[
\max_x f(x, y) = -2x_1 + 3x_2 - 4x_3 + x_4 + y_1 + y_2 + 2y_3 - y_4
\]
\[
y \text{ solution is } \max_y f(y, x) = 3x_1 + 5x_2 - x_3 + 2x_4 - 2y_1 + 4y_2 - y_3 + y_4
\]
\[
s.t. \begin{pmatrix}
0.8 & 0.5 & 0.7 & 1 \\
0.4 & 0.6 & 0.9 & 0.4 \\
0.8 & 0.7 & 1 \\
0.5 & 0.4 & 1 & 0.6
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} \begin{pmatrix}
0 & 1 & 0.8 & 1 \\
0.7 & 0.6 & 0 & 0.4 \\
1 & 0.7 & 0.9 & 1 \\
0.6 & 0.7 & 0 & 0.7
\end{pmatrix} \begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{pmatrix} \leq \begin{pmatrix}
0.7 \\
0.9 \\
0.6 \\
0.5
\end{pmatrix}
\]
\[
0 \leq x_i, y_i \leq 1 \quad i = 1, 2, 3, 4
\]

Translate the original title into

\[
\max_x f(x, y) = -2x_1 + 3x_2 - 4x_3 + x_4 + y_1 + y_2 + 2y_3 - y_4
\]
\[
s.t. \begin{pmatrix}
0.8 & 0.5 & 0.7 & 1 \\
0.4 & 0.6 & 0.9 & 0.4 \\
0.8 & 0.7 & 1 \\
0.5 & 0.4 & 1 & 0.6
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} \begin{pmatrix}
0 & 1 & 0.8 & 1 \\
0.7 & 0.6 & 0 & 0.4 \\
1 & 0.7 & 0.9 & 1 \\
0.6 & 0.7 & 0 & 0.7
\end{pmatrix} \begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{pmatrix} \leq \begin{pmatrix}
0.7 \\
0.9 \\
0.6 \\
0.5
\end{pmatrix}
\]
\[
0 \leq x_i \leq 1 \quad i = 1, 2, 3, 4
\]

and

\[
\max_y f(y, x) = 3x_1 + 5x_2 - x_3 + 2x_4 - 2y_1 + 4y_2 - y_3 + y_4
\]
\[
s.t. \begin{pmatrix}
0 & 1 & 0.8 & 1 \\
0.7 & 0.6 & 0 & 0.4 \\
1 & 0.7 & 0.9 & 1 \\
0.6 & 0.7 & 0 & 0.7
\end{pmatrix} \begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{pmatrix} \leq \begin{pmatrix}
0.7 \\
0.9 \\
0.6 \\
0.5
\end{pmatrix}
\]
\[
0 \leq y_i \leq 1 \quad i = 1, 2, 3, 4
\]

Solving Fuzzy Relational Inequalities in the First

Model
\[
\begin{pmatrix}
0.8 & 0.5 & 0.7 & 1 \\
0.4 & 0.6 & 0.9 & 0.4 \\
0.8 & 0.7 & 1 \\
0.5 & 0.4 & 1 & 0.6
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} \leq \begin{pmatrix}
0.7 \\
0.9 \\
0.6 \\
0.5
\end{pmatrix}
\]

obtain
\[
\begin{pmatrix}
0.8 & 0.5 & 0.7 & 1 \Rightarrow [0.7] \\
0.4 & 0.6 & 0.9 & 0.4 \Rightarrow [0.4, 0.9] \\
0.8 & 0.7 & 1 & 0.6 \Rightarrow [0.6, 1] \\
0.5 & 0.4 & 1 & 0.6 \Rightarrow [0.5, 0.6]
\end{pmatrix}
\]

So
\[
x_1 = ([0, 0.7] \land [0, 1] \land [0, 1] \land [0, 0.5]), x_2 = ([0, 0.7] \land [0, 1] \land [0, 0.6] \land [0, 1]),
\]
\[
x_3 = ([0, 1] \land [0, 1] \land [0, 0.6] \land [0, 0.5]), x_4 = ([0, 0.7] \land [0, 1] \land [0, 0.6] \land [0, 0.5])
\]
then
\[
x_1 = [0, 0.5], x_2 = [0, 0.6], x_3 = [0, 0.5], x_4 = [0, 0.5]
\]
The coefficients of $x$ in the objective function of the first model are $-2, 3, -4, 1$. So the optimal solution of the first model are $x_1 = 0, x_2 = 0.6, x_3 = 0, x_4 = 0.5$.

In the same way, to solve the fuzzy relation inequality in the second model,

$y_1 = ([0, 1] \land [0, 1] \land [0, 0.6] \land [0, 0.5])$,

$y_2 = ([0, 0.7] \land [0, 1] \land [0, 0.6] \land [0, 0.5])$,

$y_3 = ([0, 0.7] \land [0, 1] \land [0, 0.6] \land [0, 1])$,

$y_4 = ([0, 0.7] \land [0, 1] \land [0, 0.6] \land [0, 0.5])$,

then $y_1 = [0, 0.5], y_2 = [0, 0.5], y_3 = [0, 0.6], y_4 = [0, 0.5]$.

The coefficients of $y$ in the objective function of the second model are $-2, 4 - 1, 1$. So the optimal solution of the first model are $y_1 = 0, y_2 = 0.5, y_3 = 0, y_4 = 0.5$.

So, the model achieves the optimal solution, when

$x_1 = 0, x_2 = 0.6, x_3 = 0, x_4 = 0.5$

and

$y_1 = 0, y_2 = 0.5, y_3 = 0, y_4 = 0.5$.

The value of the upper objective function is $f_1(x, y) = 2.3$, The value of the lower level objective function values $f_2(x, y) = 6.5$.

5. Summary

In this paper, the two-layer linear programming problem model with fuzzy relation constraints is given on the basis of two-layer linear programming, and the correlation definition is given. Based on the particularity of the two-layer linear programming model with fuzzy relation constraints, an algorithm is given. An example proves that the algorithm is correct.

6. References

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