Original Research Paper

W transform and its application in fractional linear systems with rational powers

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Abstract
Fractional linear systems have attracted widespread attention from scholars and researchers due to their excellent performance and potential application prospects. In the analysis and design of fractional linear systems, the solution of fractional linear systems is an important part. So far, the powers of \( s \) in the complex-frequency-domain equations obtained by the existing fractional Laplace transform are fractions, which makes it difficult to solve algebraic equations formed by multiple fractional powers. To solve this problem, based on the traditional Laplace transform, a new fractional Laplace transform—\( W \) transform is proposed, which can make the power of the equation an integer in the \( W \)-domain. The main properties of this transformation are given, and an inversion theorem for \( W \) transform is obtained. When this transformation is applied to fractional linear systems with rational powers, the expansion formula will have a large number of terms if the traditional decomposition method is used, which makes the form of time-domain solutions more complex. Therefore, a partial fraction expansion method in the \( W \)-domain to simplify the form of time-domain solutions is proposed. On this basis, the general steps of circuit analysis in the \( W \)-domain are given. Finally, examples are used to verify the correctness and feasibility of the application.

1  |  INTRODUCTION

With the increase in research on fractional calculus, fractional circuits and systems have been introduced in many fields, including the field of electrical and electronic engineering. System models built using fractional calculus can show different mathematical characteristics than the integer-order model. Considering the applications of fractional circuits and systems, the fractional circuits and systems are studied from different perspectives, such as stability analysis [1, 2], time-domain response analysis [3, 4] and frequency-domain response analysis [5, 6].

In the research of fractional circuits and systems, it is effective to apply the Laplace transform to the fractional derivative. Therefore, the Laplace transform is often used to analyse it in the complex frequency domain, which can greatly simplify the calculation. For commensurate fractional linear systems, if \( s^\alpha \) is replaced by a variable, the original fractional linear systems are mapped to integer-order systems, and the equations can be fractionally expanded and solved using traditional methods. However, when the system contains multiple fractional powers, there is no universal method for fraction expansion, so the Laplace transform is suitable for solving commensurate or special types of equations [7].

To analyse fractional linear systems, some fractional Laplace transforms have been proposed for fractional calculus based on the traditional Laplace transform. The definition of Yang–Laplace is proposed for the case that the fractional derivative of order \( 0 < \alpha < 1 \) exists, but itself is not differentiable [8]. Yang–Laplace uses \( L\alpha \{ f(t) \} = e^{-st} \) instead of \( e^{-st} \) in the traditional Laplace transform, the property of displacement is obtained using \( L\alpha \{ (x + y)^\alpha \} = L\alpha \{ x^\alpha \} L\alpha \{ y^\alpha \} \). However, the generalised Mittage-Leffler function does not have such a relationship. Therefore, the Yang–Laplace transform can only be used to deal with local commensurate fractional differential equations. The conformable fractional Laplace transform [9] is proposed for conformable fractional calculus. The conformable fractional Laplace transform of conformable fractional derivative is \( L\alpha \{ T\alpha f(t) \} = s\alpha F\alpha (s) - f(0) \), which is consistent with the differential properties of the traditional Laplace transform, but the conformable fractional Laplace transform can't solve other forms of fractional derivatives. Watugala [10]...
introduced the Sumudu transform, which is defined as
\[ F(s) = \frac{1}{\Gamma(a)} \int_t^\infty f(t)e^{-\sigma t} dt \]
where \( f(t) \) is the fractional integral operator, \( a \) is the integration order and the value is a positive real number, \( f(t) \) represents the integrated function, \( t \) and \( a \) denote the upper and lower bounds of the integral, and \( \Gamma(a) \) is Euler's Gamma function, its definition is
\[ \Gamma(a) = \int_0^{\infty} e^{-t} t^{a-1} dt \]

Using the definition of Laplace transform \( F(s) = \int_0^{\infty} f(t)e^{-st} dt \) and its convolution theorem, the Laplace transform of (1) is obtained as
\[ \mathcal{L}\{a f(t)\} = \mathcal{L}\left(\frac{e^{\alpha t}}{\Gamma(a)} \ast f(t)\right) = \frac{F(s)}{s^\alpha} \]

where \( \mathcal{L} \) is the Laplace transform symbol, and \( \ast \) represents the convolution symbol.

Fractional derivative is not defined uniformly in mathematic. The Caputo definition of the fractional derivative is mainly used to study fractional system.

The Caputo fractional derivative is defined as [26],
\[ _a^C D_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_t^\infty \frac{f^{(\alpha)}(\tau)}{(\tau - t)^{1 - \alpha}} d\tau \]
where \( _a^C D_t^\alpha \) is fractional differential operator, \( \lfloor \cdot \rfloor \) means floor function, \( \lceil \cdot \rceil \) means ceiling function.

The Laplace transform of (3) is
\[ \mathcal{L}\{_{a}^{C} D_{t}^{\alpha} f(t)\} = s^\alpha F(s) - \sum_{j=0}^{[\alpha]} f^{(j)}(0-)s^{\alpha-j-1} \]

Herein, Section 2 briefly introduces the relevant background knowledge. Section 3 proposes the definition of W transform and inverse transform. Section 4 gives the properties of the W transform. Section 5 gives the partial fraction expansion method for the rational image function in the W-domain. Section 6 discusses the W transform method of the linear constant coefficient fractional differential equations and fractional state equations with rational powers, and verifies them with examples. Section 7 derives the model of fractional order element in the W-domain, and gives W-domain analytical method of fractional circuit with rational powers. Then examples are used to verify the accuracy of the results. Finally, the conclusions are drawn in Section 8.

2 | PRELIMINARIES

2.1 | Fractional calculus

The fractional integral is defined as [26],
\[ _a D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (t - \tau)^{\alpha-1} f(\tau) d\tau \]

where \( _a D_t^\alpha \) is the fractional derivative operator, \( \alpha \) is the integration order and the value is a positive real number, \( f(t) \) represents the divided function, \( t \) and \( \alpha \) denote the upper and lower bounds of the integral, and \( \Gamma(\alpha) \) is Euler's Gamma function, its definition is
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where \( _a^C D_t^\alpha \) is fractional differential operator, \( \lfloor \cdot \rfloor \) means floor function, \( \lceil \cdot \rceil \) means ceiling function.

The Laplace transform of (3) is
\[ \mathcal{L}\{_{a}^{C} D_{t}^{\alpha} f(t)\} = s^\alpha F(s) - \sum_{j=0}^{[\alpha]} f^{(j)}(0-)s^{\alpha-j-1} \]
It can be seen that the image functions contain the fractional powers of \( s \), which brings difficulties to analyse fractional systems in the \( s \)-domain.

### 2.2 Mittag-Leffler function

The one-parameter Mittag-Leffler function is the simplest Mittag-Leffler function, which is defined as

\[
E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}
\]

where \( \alpha \in \mathbb{C}, z \in \mathbb{C}, C \) is a complex set.

The Laplace transform of the one-parameter Mittag-Leffler function is [27].

\[
\mathcal{L}\{E_{\alpha}(\lambda t^\alpha)\} = \frac{s^{\alpha-1}}{s^{\alpha} - \lambda}
\]

The two-parameter Mittag-Leffler function is defined as

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}
\]

If \( \beta = 1 \), then the two-parameter Mittag-Leffler function degenerates into one-parameter Mittag-Leffler function.

The Laplace transform of the two-parameter Mittag-Leffler function is [27].

\[
\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^{\alpha} - \lambda}
\]

In addition, the three-parameter Mittag-Leffler function is defined as

\[
E'_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta) k!}
\]

Where \( (\gamma)_k = \gamma(\gamma + 1)(\gamma + 2)...(\gamma + k - 1) = \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)} \).

The Laplace transform of the three-parameter Mittag-Leffler function is [27].

\[
\mathcal{L}\{t^{\beta-1}E'_{\alpha,\beta}(\lambda t^\alpha)\} = \frac{s^{\gamma-\beta}}{(s^{\alpha} - \lambda)^{\gamma}}
\]

### 2.3 Riemann sheet

When studying multi-valued functions, we hope to find a generalised field, so that the original multi-valued function becomes a single-valued function on this generalised field. This generalised field is Riemann surface [28].

For a multi-valued function \( P(s^\nu) \), let \( w = \sqrt[n]{s} \), any Riemann sheet in the \( s \)-domain is mapped to a region with an angle of \( \frac{2\pi}{n} \) in the \( W \)-domain. For example, the region of the first Riemann sheet in the \( W \)-domain is \( \Omega = \{ re^{\theta j} \mid r > 0, -\frac{\pi}{n} < \theta < \frac{\pi}{n} \} \).

When \( \nu \) is a positive integer, the analytic domain of the multi-valued function \( P(s^\nu) \) can be regarded as Riemann surface with three Riemann sheets, the \( \nu \) Riemann sheets are mapped into a \( W \)-plane. For the equation \( P(s^\nu) = 0 \), the position of the roots in the \( W \)-domain is misalignment, the relationship between the \( W \)-plane and the Riemann surface is shown in Figure 1.

### 3 W TRANSFORM

#### 3.1 Definition of W transform

**Definition 1** (W transform) Let \( f(t) \) be defined at \( t \geq 0 \), \( w \) is a complex variable and \( \nu \) is a real number. If \( \int_{0}^{\infty} f(t) e^{-\nu wt} dt \) converges in a certain region of the \( W \)-domain, then the function

\[
F(w) = \int_{0}^{\infty} f(t) e^{-\nu wt} dt
\]

is called the W transform of \( f(t) \), recorded as \( F(w) = W\{f(t)\} \). \( F(w) \) is the \( W \)-domain image function of the original function \( f(t) \).

Compared with the traditional Laplace transform, the \( W \) transform only replaces \( s \) in the Laplace transform with \( \omega^\nu \), so the \( s \) in the Laplace transform can be replaced by \( \omega^\nu \) to obtain the corresponding \( W \)-domain image function. Choosing a suitable real number \( \nu \) can make the fractional powers of \( s \) become integer powers of \( \omega \). At the same time, the \( W \) transform can only be used to analyse linear circuits, which is the same as the Laplace transform.

**Example 1** Solving the W transform of \( \frac{e^{-t}}{t^{\nu}} \).
Substituting \( \frac{t^{a-1}}{\Gamma(a)} \) into the definition of \( W \) transformation, then

\[
W \left( \frac{t^{a-1}}{\Gamma(a)} \right) = \int_{0}^{\infty} \frac{u^{a-1}}{\Gamma(a)} e^{-u} \frac{1}{w} \, du = \int_{0}^{\infty} t^{a-1} e^{-w^{t} \cdot dt}
\]

Let \( w^{t} = u \), and applying the Euler’s Gamma function’s definition (1), we get

\[
W \left( \frac{t^{a-1}}{\Gamma(a)} \right) = \frac{1}{\Gamma(a)} \int_{0}^{\infty} \left( \frac{u}{w} \right)^{a-1} e^{-u} \, du = \frac{1}{\Gamma(a)}
\]

When \( a = \frac{k}{b} \) and \( k \) is an integer, the image function of \( \frac{t^{a-1}}{\Gamma(a)} \) is \( \frac{1}{\gamma} \), and the power of \( w \) is an integer.

### 3.2 Inversion of \( W \) transform

**Definition 2** If \( f(t) \) is the original function of \( F(w) \), the \( F(w) \) has no singularities in region \( B \), and the inversion formula is

\[
f(t) = \frac{1}{2\pi i} \lim_{a \to -\infty} \int_{a+ir}^{a-ir} F(w) dw e^{-w^{t} \cdot dt}
\]

recorded as \( f(t) = W^{-1}[F(w)] \). The region \( B \) is the shaded region in Figure 2. In the figure, \( \varphi_{1} \) is the curve that the straight line \( \text{Re}(s) = \sigma \) in the \( s \)-domain mapped to the first Riemann sheet in the \( W \)-domain, \( \varphi_{2} \) is a polyline

\[
\begin{align*}
\{ \varphi_{1} & \} = \{ w = a + re^{\gamma t}, r: \infty \to 0 \} \\
\{ \varphi_{2} & \} = \{ w = a + re^{\gamma t}, r: 0 \to \infty \} \quad (a = \sigma^{b}).
\end{align*}
\]

The proof of Definition 2 is given in Appendix.

When \( v \) is a positive integer \( m \), the analytic domain of the function can be regarded as Riemann surface with \( m \) Riemann sheets, and these \( m \) Riemann sheets exist in a \( W \)-plane. The inverse transform can be obtained using the residue method. The solution process is similar to the Laplace transform. When \( v \) is not a positive integer, because the \( W \) transform is invertible, there is a one-to-one correspondence between the original function and the image function, the inverse transform can be obtained according to the \( W \) transform of some known functions. Using the following lemma get a very useful formula for inverse transform. Taking this as an example to obtain the inverse transform of partial function.

**Lemma 1** if \( \text{Re}(\gamma) > 0 \), \( \text{Re}(\beta) > 0 \) and \( z < 1 \), existing

\[
\frac{1}{\Gamma(a)} = \int_{1}^{\infty} e^{-x} x^{a-1} E_{a}(\beta) (z \cdot x \cdot z) \, dx
\]

Let \( z = \alpha w^{-av} \), \( x = w^{t} \cdot t \), we obtain

\[
\frac{1}{\Gamma(a)} = \int_{1}^{\infty} e^{-x} x^{a-1} E_{a}(\beta) (z \cdot a \cdot w^{-av}) \, dx
\]

That is

\[
W^{-1} \left( \frac{z^{a}(\alpha \beta \gamma)_{a}}{(w^{a \beta} + a)} \right) = \frac{1}{(1 + z)^{v}}
\]

By combining different parameters according to the needs of the functions, the following derivations can be derived, see the following various corollaries.

**Corollary 1** If \( \alpha \gamma = \beta, \alpha v = 1 \) and \( \gamma = k \), Equation (6) can be written as

\[
W^{-1} \left( \frac{z}{w^{a} + a} \right) = \frac{1}{(a + z)^{v}}
\]

where \( k \) is an integer.

**Corollary 2** If \( \alpha \beta = \beta, \alpha v = l \) and \( \gamma = 1 \), Equation (6) can be written as

\[
W^{-1} \left( \frac{z^{k}}{w^{a} + a} \right) = \frac{1}{(a + z)^{v}}
\]

where \( k, l \) are integers.

**Corollary 3** If \( \alpha \beta = -l, \alpha v = k \) and \( \gamma = p \), Equation (6) can be written as

\[
W^{-1} \left( \frac{z^{k}}{w^{a} + a} \right) = \frac{1}{(a + z)^{v}}
\]
where \( k, l, p \) are integers.

Example 2 Solving the original function of \( \frac{cw+d}{(w-a)^2+b^2} \).

Decomposing \( \frac{cw+d}{(w-a)^2+b^2} \) into the form of two conjugate complex root terms and using (7), we get

\[
W^{-1}\left\{ \frac{cw+d}{(w-a)^2+b^2} \right\} = W^{-1}\left\{ \frac{c_1 + d_1 j}{w-a + bj} + \frac{c_1 - d_1 j}{w-a - bj} \right\}
= (c_1 + d_1 j)\overline{E_{\frac{\nu}{\pi}}}(a - bj)\overline{t}^\nu + (c_1 - d_1 j)\overline{E_{\frac{\nu}{\pi}}}(a + bj)\overline{t}^\nu
\]

According to the two-parameter Mittag-Leffler function's definition (4), we can get

\[
W^{-1}\left\{ \frac{cw+d}{(w-a)^2+b^2} \right\} = c_1 t^{\nu-1} \sum_{l=0}^{\infty} \frac{t^l}{\Gamma\left(\frac{\nu}{\pi} + l + \frac{1}{2}\right)} ((a - bj)^l + (a + bj)^l)\overline{t}^\nu
\]

Because for any natural number \( l \), \((a - bj)^l + (a + bj)^l\) is a real number and \((a - bj)^l - (a + bj)^l\) is a pure imaginary number, the inverse transform of \( \frac{cw+d}{(w-a)^2+b^2} \) does not contain imaginary numbers.

4 | PROPERTIES OF THE W TRANSFORM

Many properties of Laplace transform are also true in the W transform. Similar to the proving process of Laplace transform properties, these properties can be proved using the definition of W transform.

Property 1 (linearity property) Let \( a, b \) be constants, and \( W\{f_1(t)\} = F_1(w) \), \( W\{f_2(t)\} = F_2(w) \), then

\[
W\{af_1(t) + bf_2(t)\} = aF_1(w) + bF_2(w)
\]

This property can be proved by the definition of W transform.

Property 2 (convolution theorem) If the W transform of function \( f_1(t) \), \( f_2(t) \) exist and \( W\{f_1(t)\} = F_1(w) \), \( W\{f_2(t)\} = F_2(w) \), then

\[
W\{f_1(t) * f_2(t)\} = F_1(w) \cdot F_2(w)
\]

or

\[
W\{f_1(t) \ast f_2(t)\} = \frac{1}{2\pi} f_1(t) \cdot f_2(t)
\]

Property 3 (integral property) Let \( W\{f(t)\} = F(w) \), \( \alpha \) is a positive real number, The W transform of the fractional integral is

\[
W\{\alpha f(t)^{\alpha}f(t)\} = \frac{F(w)}{w^{\alpha+1}}
\]

\( \alpha \nu \) can be made to be a positive integer \( \tilde{n}_\alpha \) by selecting the appropriate \( \nu \), that is \( \alpha \nu = \tilde{n}_\alpha \). So the power of \( w \) becomes an integer.

Property 4 (differential property) Let \( W\{f(t)\} = F(w) \), \( \tilde{n}_\alpha \in \mathbb{N}_+ \), then the W transform of the integer-order derivative is

\[
W\{f^{(k)}(t)\} = w^k F(w) - w^{\nu(k-1)} f(0_-) - \ldots - w^{\nu(0)} f^{(k-1)}(0_-)
\]

If \( \alpha \) is a positive real number, then W transform of fractional derivative is

\[
W\{_{\tilde{n}_\alpha}D^{\alpha} f(t)\} = w^{\nu} F(w) - \sum_{j=0}^{[\alpha]} f^{(j)}(0_-) w^{\nu-j(1+\nu)}
\]

In particular, when \( 0 < \alpha < 1 \), the W transform of Caputo fractional derivative is

\[
W\{_{\tilde{n}_\alpha}D^{\alpha} f(t)\} = w^{\nu} F(w) - f(0_-) w^{\nu-\nu}
\]

It can be known from the expression of the differential property that when the initial condition is not zero, to make the power of \( w \) an integer, \( \nu \) in W transform should take a suitable positive integer. When the initial condition is zero, as long as \( \alpha \nu \) is a suitable positive integer, the power of \( w \) is an integer.

Example 3 Considering commensurate irrational differential equation with zero initial value

\[
D^{\nu^2}y(t) + 2D^{\nu^2}y(t) + y(t) = \delta(t)
\]
Let $\alpha = \frac{1}{\sqrt{2}}$, applying W transform to both sides of equation, we get
\[\omega^2 Y(\omega) + 2\omega Y(\omega) + Y(\omega) = 1\]
\[Y(\omega) = \frac{1}{\omega^2 + 2\omega + 1} = \frac{1}{(\omega + 1)^2}\]

Using (11), we can get $y(t)$ as
\[y(t) = W^{-1}\left\{\frac{1}{(\omega + 1)^2}\right\} = \frac{t^{\sqrt{2}}}{\sqrt{2\pi}} e^{-t^2}\]

**Property 5** (initial value theorem) If $W\{f(t)\} = F(\omega)$, and $\lim_{\omega \to \infty} \omega^\alpha F(\omega)$ exists, then
\[f(0) = \lim_{\omega \to \infty} \omega^\alpha F(\omega)\]

**Property 6** (terminal value theorem) If $W\{f(t)\} = F(\omega)$, and all singularities of $\omega^\alpha F(\omega)$ are outside the angle $|\beta|$, then
\[\lim_{t \to \infty} f(t) = \lim_{\omega \to 0} \omega^\alpha F(\omega)\]

### 5 | PARTIAL FRACTION EXPANSION OF RATIONAL IMAGE FUNCTION IN THE W DOMAIN

The powers discussed in the following sections are rational numbers, that is, rational powers, and therefore they can be expressed as the ratio of two positive integers. For example, the rational powers $\alpha, \beta, \gamma$ can be expressed as
\[\alpha = \frac{n_\alpha}{m}, \beta = \frac{n_\beta}{m}, \gamma = \frac{n_\gamma}{m}\]

where $n_\alpha, n_\beta, n_\gamma$ are positive integers, $m$ is the least common multiple of all rational powers' denominators, $n$ is the greatest common divisor of all rational powers' molecules, or 1 etc., which depends on the specific situation. Obviously, $m$ and $n$ must be co-prime. From a practical point of view, the assumption of rational powers can be made without loss of generality.

The number of poles of an integer-order constant coefficient linear differential equation is equal to the order of the differential equation, which does not depend on other coefficients. For fractional systems, if W transform is used to solve linear system equations, and the powers of $\omega$ in the W-domain are integers, then fractional differential equations can be solved using conventional methods in the W-domain.

When the W transform is applied to fractional linear system with rational powers, the definition of the W transform (5) can be rewritten as
\[F(\omega) = \int_0^\infty f(t)e^{-\omega \hat{\omega}t}dt\]

In the case of rational powers, the W transform of Caputo fractional derivative can be written as
\[W\{CD^\alpha f(t)\} = \omega^\alpha F(\omega) - \sum_{j=0}^{[\alpha]} \frac{f^{(j)}(0)}{\Gamma(\alpha-j+1)}\omega^{\alpha-j}\]

Obviously, to avoid the fractional powers of $\omega$, we need to take $n = 1$ in the non-zero initial condition.

Differential equations describing fractional linear systems with rational powers can be written as [27].

\[\sum_{k=1}^p a_k D^\alpha y(t) = \sum_{l=1}^q b_l D^\alpha u(t)\]

where $a_1, ..., a_p, b_1, ..., b_q$ are real coefficients, $\alpha, ..., \alpha_p, \beta_1, ..., \beta_q$ are rational positive real numbers. These rational powers can be expressed as $\alpha = \frac{n_\alpha}{m}, \beta = \frac{n_\beta}{m}, (k = 1, 2, ..., p), \beta_1 = \frac{n_\beta}{m}, (l = 1, 2, ..., q)$.

Considering the initial conditions and applying the inverse W transform to both sides of (8), we can obtain
\[\sum_{k=1}^p a_k D^\alpha Y(\omega) - \sum_{j=0}^{[\alpha]} \frac{f^{(j)}(0)}{\Gamma(\alpha-j+1)}\omega^{\alpha-j}\]

where $[\cdot]$ means floor function, $Y(\omega)$ and $U(\omega)$ are the image functions of input function $u(t)$ and output function $y(t)$.

Then the image function of fractional linear system with rational powers is
\[Y(\omega) = \sum_{l=1}^q b_l D^\alpha U(\omega) - \sum_{r=0}^{[\beta]} u^{(r)}(0)\omega^{\beta-r}\]

From the expression of the rational image function (9) in the W-domain, it can be seen that if the zero initial condition of the equation or the function at the right end of the equation $u(t) \neq a\delta(t)$ is to be considered, then $n$ must be taken one to make the powers in the equation all integers. For network function in the W-domain, Equation (9) is simplified as
\[\frac{Y(\omega)}{U(\omega)} = \sum_{k=1}^p b_k \omega^{n_k}\]

\[\sum_{k=1}^p a_k \omega^{n_k}\]
The expansion for \( m \) will have a large number of terms, simplify the time

\[ a \] is a complex number\).

Remarkably, from the expressions of (9) and (10), it can be seen that the multiple fractional powers will cause the powers of image function in the W-domain, which is different from traditional integer-order linear systems.

For the image function \( Y(w) \), if it is an improper fraction, then \( Y(w) \) can be decomposed into the sum of a rational polynomial and a rational proper fraction using polynomial division, that is

\[
Y(w) = g_0 + g_1w + \ldots + g_pw^p + \frac{N(w)}{D(w)} = G(w) + \frac{N(w)}{D(w)}
\]

where, \( p \) is a natural number, \( g_0, g_1, \ldots, g_p \) is a real number, \( G(w) = g_0 + g_1w + \ldots + g_pw^p \) is a rational polynomial, and \( \frac{N(w)}{D(w)} \) is a rational proper fraction.

There are two fundamental types that can be derived from the expression of rational polynomial \( G(w) \): 1, \( w^p \).

The roots of \( H(w) = 0 \) may be single roots, multiple roots or conjugate complex roots. Therefore, using the traditional integer-order method to expand the rational proper fraction \( \frac{N(w)}{D(w)} \) into partial fractions, we can get

\[
\frac{N(w)}{D(w)} = \sum_{p=1}^{n_1} \frac{c_0}{w^p} + \sum_{p=1}^{n_2} \frac{c_p}{w + b_p} + \sum_{q=1}^{n_3} \frac{c_p}{(w + b_p)^q} + \sum_{q=1}^{n_4} \frac{c_q}{(w + b_q)^q} + \sum_{p=1}^{n_5} \frac{d_p + e_p}{w + b_p + c_p} + \frac{d_p - e_p}{w + b_p - c_p}
\]

(11)

The conjugate root \( \frac{d}{{(w + b)^2} + c^2} \) is a combination of two conjugate complex root terms.

There are two fundamental types that can be obtained from the expression of rational proper fraction \( \frac{N(w)}{D(w)} \): \( \frac{w^p}{(w + a)^q} \) (\( a \) is a complex number).

When the power of \( w \) is high and the expansion theorem of partial fraction is used to expand the image function directly, the expansion formula will have a large number of terms, which makes the time-domain form complex. Therefore, to simplify the time-domain expression, we need to fully consider the characteristic that the powers of \( w \) are discontinuous and combine the partial fraction.

According to \( w = \frac{d}{a} \), a point in the s-domain corresponds to \( m \) points in the \( W \)-domain, these points are misalignment, the distances from these points to the origin in the \( W \)-domain are the same, and the phase angle difference between adjacent points is the same. Through this feature, it can be found that if \( H(w) = 0 \) has \( l \) roots, the distances between these roots and the origin are same, and the angle between the two adjacent roots is \( \frac{\pi}{l} \), then these roots can be combined into a fundamental type of \( \frac{w^b}{w^a + b} \) (\( b \) is a real number). It corresponds to the conjugate complex root terms in (11) (when \( l \) is an odd number, this has a real root term).

If \( \frac{N(w)}{D(w)} \) is directly decomposed when there are \( r \) poles at the origin in the \( W \)-domain, there will appear \( r \) terms of \( \frac{w^0}{w^a + b} \). To simplify expressions in the time-domain, a new decomposition method can be used for \( \frac{N(w)}{D(w)} \), and Equation (11) can be written as

\[
\frac{N(w)}{D(w)} = \frac{1}{w^p} \left[ \sum_{p=1}^{n_1} \frac{c_p}{w + b_p} + \sum_{q=1}^{n_2} \frac{c_q}{(w + b_q)^q} + \sum_{p=1}^{n_3} \frac{d_p + e_p}{w + b_p + c_p} + \frac{d_p - e_p}{w + b_p - c_p} \right]
\]

(12)

Combining the terms in (12), Equation (12) can be rewritten as

\[
\frac{N(w)}{D(w)} = \sum_{l=1}^{n_1} \sum_{p=1}^{n_4} \left( \frac{d_p + e_p}{w^l + b_p + c_p} + \frac{d_p - e_p}{w^l + b_p - c_p} \right)
\]

(13)

So we can get a new fundamental type of \( \frac{w^a}{w^b + c} \) (\( a \) is a complex number).

In summary, the partial fraction expansion of the rational image function in the \( W \)-domain can be obtained from the combination of six fundamental types: 1, \( \frac{1}{w^a + b} \) and \( \frac{w^k}{w^a + b^l} \). Where \( k, l, p \) are positive integers, \( a \) is a complex number, and \( b \) is a real number.

From the linearity property of the \( W \) transform, it can be known that the expression of original function \( y(t) \) can be obtained by solving the inverse \( W \) transform of these six types.

The original functions of the above six common image functions are shown in Table 1.

W transform only replaces the complex variable \( s \) in the Laplace transform with \( \frac{w}{w^a + b} \). Therefore, the inverse \( W \) transform method of integer-order can be used when the powers of \( w \) in the decomposed term is an integer multiple of \( \frac{w}{w^a + b} \). For example, the original function of \( \frac{w^b}{w^a + b} \) is \( \sin bt \).

**Example 4** Solving the original function of \( F(w) = \frac{5w^3 + w^2 + 7}{w^3 + 2w^2 + 2w} \) when \( \frac{m}{n} = 10 \).
TABLE 1 Common image functions

| Image Function | Original Function |
|----------------|------------------|
| 1              | δ(t)             |
| w^k            | D^k δ(t)         |
| 1/(w-k)        | \( \frac{\delta}{w-k} \) |
| 1/(w^2-k^2)    | \( \frac{1}{w^2-k^2} \) |

It can be seen from the expression of \( F(w) \) that \( F(w) \) has two poles at the origin, so \( F(w) \) can be written as

\[
F(w) = \frac{5w^3 + w + 7}{w^2(w^4 + 2w^3 + w + 2)} = \frac{1}{w^2} Y(w)
\]

where \( Y(w) = \frac{5w^3 + w + 7}{w^2(w^4 + 2w^3 + w + 2)} \).\( Y(w) \) has four poles, which are \( \frac{1}{2} + \frac{\sqrt{3}}{2}, \frac{1}{2} - \frac{\sqrt{3}}{2}, -1 \) and \(-2\). The distances from \( \frac{1}{2} + \frac{\sqrt{3}}{2}, \frac{1}{2} - \frac{\sqrt{3}}{2} \) and \(-1\) to the origin are the same, and the angle between two adjacent points is \( \frac{2\pi}{3} \). So \( F(w) \) can be decomposed into

\[
F(w) = \frac{1}{w^2} \left( \frac{1}{w^3 + 1} + \frac{5}{w + 2} \right) = \frac{1}{w^2(w^3 + 1)} + \frac{5}{w^2(w + 2)}
\]

Using the results in Table 1, \( f(t) \) can be obtained as

\[
f(t) = t^{0.5} E_{0.3,0.5}(-t^{0.3}) + 5t^{0.8} E_{0.1,0.2}(-2t^{0.1})
\]

6 | W TRANSFORM METHOD FOR LINEAR TIME-INVARIANT FRACTIONAL SYSTEMS

6.1 | W transform method for linear constant coefficient fractional differential equation

Linear time-invariant fractional systems can be described by linear constant coefficient fractional differential equations, the expression of which is:

\[
\sum_{k=1}^{p} a_k D^\alpha_k y(t) = \sum_{i=1}^{q} b_i D^\beta_i u(t)
\]  \( (13) \)

where \( a_1, \ldots, a_p, b_1, \ldots, b_q \) are real coefficients, \( \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \) are rational positive real numbers.

By applying the W transform to both sides of (13), the rational image function is expanded to partial fraction and combining them appropriately by the method in the Section 5, we have

\[
Y(w) = e_0 + \sum_{p=1}^{n_1} \frac{c_{1p}}{w^{\alpha_p}} + \sum_{q=1}^{n_2} \sum_{p=1}^{n_3} \frac{c_{pq}}{(w + a_q)^{\beta_q}} + \sum_{p=1}^{n_4} \frac{w^{\alpha_p}}{w^{\beta_p} + a_p}
\]

Then applying the inverse W transform to \( F(w) \) term by term, we can obtain the time-domain solution.

**Example 5** Considering initial value problem in the case of the inhomogeneous Bagley-Torvik equation [29].

\[
D^2 y(t) + D^\gamma y(t) + y(t) = 1 + t
\]

\[
y(0) = 1, \quad y'(0) = 1
\]

According to the order in the equation, we know that \( m = 2 \). Considering the initial value, we should take \( n = 1 \). So \( \alpha = 2 \). According to the linearity property and differential property, applying W transform to the both sides of the equation, we obtain.

\[
w^2 Y(w) - w^3 Y(0) - w^2 y(0) + w^2 Y(w) - w y(0) + Y(w) = \frac{1}{w} + \frac{1}{w^2}
\]

That is

\[
Y(w) = \frac{\frac{1}{w} + \frac{1}{w^2} + \frac{1}{w^3} + \frac{1}{w^4}}{w^3 + w^2 + w + 1} = \frac{\frac{1}{w^2} + \frac{1}{w^3}}{w^3 + w^2 + w + 1}
\]

Using the results in Table 1, we can obtain \( y(t) \) as

\[
y(t) = 1 + t
\]

The calculated result is consistent with the result in reference [29].

6.2 | W transform method for fractional state equation

Linear time-invariant fractional systems can be described using the following state equations.
Simplifying the above formula, we can obtain

$$\begin{bmatrix} d^a x_1(t) \\ \vdots \\ d^a x_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & \ldots & a_{1q} \\ \vdots & \ddots & \vdots \\ a_{q1} & \ldots & a_{qq} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_q(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_q \end{bmatrix} u(t)$$

where $a_{kj}$, $b_k$, $c_{pk}$ and $d_p$ are real numbers, $0 < a_k < 1$, $k, j = 1, 2, \ldots, q, p = 1, 2, \ldots, l$.

Applying $W$ transform to the both sides of (14), we have.

$$\begin{bmatrix} w^{\alpha_1} x_1(w) - x_1(0) w^{\alpha_1 - \frac{\alpha}{2}} \\ \vdots \\ w^{\alpha_q} x_q(w) - x_q(0) w^{\alpha_q - \frac{\alpha}{2}} \end{bmatrix} = \begin{bmatrix} a_{11} & \ldots & a_{1q} \\ \vdots & \ddots & \vdots \\ a_{q1} & \ldots & a_{qq} \end{bmatrix} \begin{bmatrix} x_1(w) \\ \vdots \\ x_q(w) \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_q \end{bmatrix} U(w)$$

Simplifying the above formula, we can obtain

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_l(t) \end{bmatrix} = \begin{bmatrix} c_{11} & \ldots & c_{1q} \\ \vdots & \ddots & \vdots \\ c_{l1} & \ldots & c_{lq} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_q(t) \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_l \end{bmatrix} U(t)$$

The $W$ transform solution of the fractional state equation is the same as the traditional state equation, but the powers of the state variables in the integer state equation is the first derivative. The powers of state variable in the fractional state equation may be different, and the initial condition also contains the variable $w$.

The time-domain solution of the state equation can be obtained by taking the inverse $W$ transform of (15).

**Example 6** Considering the following state equation [30].

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta(t), \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$y(t) = [1 \ 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

According to the order and initial value conditions in the state equation, we can obtain the state equation in the $W$-domain by applying $W$ transform to both sides of the state equation.

$$\begin{bmatrix} w^3 x_1(w) - \frac{1}{w^3} \\ w^2 x_2(w) - 2 \frac{1}{w^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(w) \\ x_2(w) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Y(w) = [1 \ 0] \begin{bmatrix} x_1(w) \\ x_2(w) \end{bmatrix}$$

Simplifying the above formula and using the results in Table 1, we can obtain $y(t)$ as

$$y(t) = E_\frac{1}{8}^1 \left( \frac{1}{8} t \right)$$

This calculation result is consistent with the result in [30], so the method of solving the state equation using the $W$ transform here is correct.

## 7 W-DOMAIN ANALYSIS OF LINEAR CIRCUITS

### 7.1 The form of Kirchhoff’s law in W-domain

The expression of Kirchhoff’s current law in the time-domain is
\[ \sum i_k(t) = 0 \] (16)

By applying W transform to both sides of (16) and using the linearity property of W transform, we can get

\[ W\{ \sum i_k(t) \} = \sum W\{i_k(t) \} = 0 \]

That is

\[ \sum I_k(w) = 0 \]

Similarly, the expression of Kirchhoff's voltage law in the W-domain is

\[ \sum U_k(w) = 0 \]

7.2 | The W-domain models of the elements

7.2.1 | Resistance

The resistance is shown in Figure 3(a). In the associated reference direction, its Volt-ampere relationship is

\[ u_R(t) = R i_R(t) \]

Taking W transform on both sides, the Volt-ampere relationship of the resistance in the W-domain is

\[ U_R(w) = R I_R(w) \]

The W-domain model of the resistance is shown in Figure 3(b).

7.2.2 | Fractional capacitor

The fractional capacitor is shown in Figure 4(a). In the associated reference direction, its Volt-ampere relationship is

\[ i_C(t) = C_a \frac{d^\alpha u_C(t)}{dt^\alpha} \] (17)

where \( \alpha \) is the fractional order, \( \alpha \in (0, 1) \), and \( C_a \) with unit \( F/s^{1-\alpha} \) denotes the pseudo-capacitance of the capacitor represented in the form of \( (\alpha, C_a) \).

When the initial condition \( u_C(0^-) \) of the capacitor is zero, \( n \) is the greatest common divisor of all rational powers' molecules. Applying the W transform to both sides of (17), we have

\[ I_C(w) = w^{\alpha} C_a U_C(w) \]

where \( w^{\alpha} C_a \) is the admittance of the fractional capacitor in the W-domain.

When the initial condition \( u_C(0^-) \) of the capacitor is not zero, then \( n = 1 \), the form of the volt-ampere relationship of the fractional capacitor in the W-domain is

\[ I_C(w) = w^{\alpha} C_a U_C(w) - w^{\alpha-n} C_a u_C(0^-) \] (18)

where \( w^{\alpha-n} C_a u_C(0^-) \) is the current of additional current source, which reflects the initial state of the capacitor voltage.

The W-domain model of fractional capacitor is shown in Figure 4(b).

Another volt-ampere relationship of fractional capacitor can be obtained from (18).

\[ U_C(w) = \frac{1}{w^{\alpha} C_a} I_C(w) + \frac{u_C(0^-)}{w^{\alpha-n}} \]

Where \( \frac{1}{w^{\alpha} C_a} \) is the impedance of the fractional capacitor in the W-domain, \( u_C(0^-) \) is the voltage of additional voltage source.

The W-domain model is shown in Figure 4(c).

7.2.3 | Fractional inductor

The fractional inductor is shown in Figure 5(a). In the associated reference direction, its Volt-ampere relationship is

\[ u_L(t) = L_\beta \frac{d^\beta i_L(t)}{dt^\beta} \] (19)

where \( \beta \) is the fractional order, \( \beta \in (0, 1) \), and \( L_\beta \) with unit \( H/s^{1-\beta} \) denotes the pseudo-inductor of the inductor represented in the form of \( (\beta, C_\beta) \).

\[ I_C(w) = \frac{1}{w^{\alpha} C_a} I_C(w) + \frac{u_C(0^-)}{w^{\alpha-n}} \]

Where \( \frac{1}{w^{\alpha} C_a} \) is the impedance of the fractional capacitor in the W-domain, \( u_C(0^-) \) is the voltage of additional voltage source.

The W-domain model is shown in Figure 4(c).

**FIGURE 3** W-domain model of fractional resistance, (a) Time domain model, (b) W-domain model

**FIGURE 4** The W-domain models of fractional order capacitor, (a) Time domain model, (b) Parallel W-domain model, (c) Series W-domain model
When the initial condition $u_L(0^-)$ of the inductor is zero, $n$ is the greatest common divisor of all rational powers' molecules. Applying the W transform on both sides of (19), we have

$$U_L(w) = w^{\beta\theta}L\beta I_L(w)$$

where $w^{\beta\theta}L\beta$ is the impedance of the fractional inductor in the W-domain.

When the initial condition $i_L(0^-)$ of the inductor is not zero, then $n = 1$, the form of the volt-ampere relationship of the fractional inductor in the W-domain is

$$U_L(w) = w^{\beta\theta}L\beta I_L(w) - w^{\beta\theta-m}L\beta i_L(0^-)$$  \hspace{1cm} (20)

where $w^{\beta\theta-m}L\beta i_L(0^-)$ is the voltage of additional voltage source, which reflects the initial state of the inductor current.

The W-domain model of fractional inductor is shown in Figure 5(b).

Another volt-ampere relationship of fractional inductor can be obtained from (20).

$$I_L(w) = \frac{1}{w^{\beta\theta}L\beta}U_L(w) + \frac{i_L(0^-)}{w^{\beta\theta}}$$

where $\frac{1}{w^{\beta\theta}L\beta}$ is the admittance of the fractional inductor in the W-domain, $\frac{i_L(0^-)}{w^{\beta\theta}}$ is the current of additional current source.

The W-domain model is shown in Figure 5(c).

### 7.3 | W-domain analysis of circuits

Based on the previous discussion, the general steps of the W-domain analysis method are:

1. Solving the voltage of capacitor and the current of inductor at $t = 0^-$.
2. Drawing the W-domain circuit.
3. Obtaining the image function of the circuit response.
4. Expanding the partial fraction of the response's image function.
5. Time-domain response is obtained by inverse W transform.

The following examples will illustrate the specific steps.

#### 7.3.1 | Determining the starting state of the circuit

When $t = 0^-$, the capacitor is open-circuit, so

$$u_C(0^-) = u_C(0^-) = 0$$

#### 7.3.2 | Drawing the W-domain circuit

The W-domain circuit is shown in Figure 7.

#### 7.3.3 | Obtaining the image function of the response

\[
\left(\frac{1}{w^3 + 2 + w^4}\right) I(w) = \frac{1}{w^6}
\]

\[
I(w) = \frac{w^4 + 2}{w^3 + w^6 + 2w^3}
\]

#### 7.3.4 | Partial fraction expansion of image function

\[
I(w) = \frac{w^4 + 2}{w^3(w^4 + w^3 + 2)}
\]

\[
= \frac{1}{w^3} \left(1 + \frac{-0.35 - 0.19j}{w + 1.14 - 0.76j} + \frac{-0.35 + 0.19j}{w + 1.14 + 0.76j} + \frac{-0.15 - 0.06j}{w - 0.64 - 0.80j} + \frac{-0.15 + 0.06j}{w - 0.64 + 0.80j} + \frac{-0.35 - 0.19j}{w(1.14 - 0.76j)} + \frac{-0.35 + 0.19j}{w(1.14 + 0.76j)}\right)
\]
7.3.5 | Solving the time-domain response

Taking the inverse transform of (21), we get

\[
i(t) = (-0.35 - 0.19j)t^{-\frac{3}{2}}E_{\frac{1}{2}} \left( (-1.14 + 0.76j)t^\frac{1}{2} \right) + (-0.35 + 0.19j)t^{-\frac{3}{2}}E_{\frac{1}{2}} \left( (-1.14 - 0.76j)t^\frac{1}{2} \right) + (-0.15 - 0.06j)t^{-\frac{3}{2}}E_{\frac{1}{2}} \left( (0.64 + 0.80j)t^\frac{1}{2} \right) + (-0.15 + 0.06j)t^{-\frac{3}{2}}E_{\frac{1}{2}} \left( (0.64 - 0.80j)t^\frac{1}{2} \right) + \frac{t^{-\frac{1}{2}}}{\Gamma \left( \frac{1}{2} \right)}
\]

The voltage excitation are applied to fractional circuit, then gets current by frequency domain analysis. The simulation results are shown in Figure 8(a).

It can be seen from Figure 8(a) that “Analytical Calculation” and “Circuit Simulation” are consistent, so it proves that the method of solving the circuit by W transform is correct.

When the circuit given in Figure 6 is a traditional circuit, \( \alpha = \beta = 1 \), the circuit current \( I(t) \) is

\[
I(t) = W^{-1} \left[ \frac{w + 2}{2w + 2} \right] = \frac{1}{2} \delta(t) + \frac{1}{2} e^{-t}.
\]

Comparing the solution of traditional circuit with the solution of fractional circuits, it can be seen from Figure 8(b) that two curves have the same trend, but the rate of decline is different. The higher the powers in the circuit, the faster the response will change. Therefore the fractional circuit is more affected by past, compared with the traditional circuit which show a rapid descent.

8 | CONCLUSION

The W transform is proposed and applied to fractional linear systems with rational powers. First, we give the definition of W transform and its inverse transform. The W transform uses \( e^{\pi w^{-t}} \) to replace the kernel function \( e^{\pi s t} \) of the traditional Laplace transform, which can make the fractional calculus with rational powers become integer powers of \( w \) in the W domain. Based on these definitions, we derive the properties of the W transform and this transformation can effectively solve fractional differential equations with rational or irrational powers.

To simplify the time-domain solutions of differential equations, a partial fraction expansion method in the W domain is presented. On this basis, we solve linear constant coefficient fractional differential equations and fractional state equations.

FIGURE 7 W-domain circuit

FIGURE 8 The simulation results, (a) Analytical Calculation and Circuit Simulation of fractional circuit, (b) the solution of fractional circuit and traditional circuit.
Finally, the W transform is applied to the analysis of fractional circuits with rational powers, the forms of the fractional-order elements in the W-domain are derived, and the general steps of the W-domain analysis method are given. The W transform proposed here provides a new direction for the analysis of fractional circuits. This paper mainly gives the idea of W transform and applied it to the solution of fractional linear system with rational powers, so further research should be conducted on study the application of W transform in the analysis of fractional linear circuits containing the passive criteria of fractional circuits with rational order elements.

ACKNOWLEDGEMENTS
The authors express their gratitude to the Natural Science Foundation of Hebei Province (grant no. E2018502121).

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REFERENCES
1. Chen, Y.Q., Moore, K.L.: Analytical stability bound for a class of delayed fractional-order dynamic systems. In: Proceedings of the 40th IEEE Conference on Decision and Control (Cat. No.01CH37228), Orlando, pp. 1421-1426 vol. 2 (2001)
2. Radwan, A.G.: Resonance and quality factor of the fractional circuit. IEEE J. Emerg. Select. Top. Circ. Syst. 3(3), 377–385 (2013)
3. Alsaedi, A., Nieto, J.J., Venktesh, V.: Fractional electrical circuits. Adv. Mech. Eng. 7(12), 1–7 (2015)
4. Radwan, A.G., Elwakil, A.S.: An expression for the voltage response of an current-excited fractance device based on fractional-order trigonometric identities. Int. J. Circ. Theor. Appl. 40(5), 533–538 (2012)
5. Guia, M., Gómez, E., Rosales, J.: Analysis on the time and frequency domain for the RC electric circuit of fractional order. Cent. Eur. J. Phys. 11(10), 1366–1371 (2013)
6. Radwan, A.G., Salama, K.N.: Passive and active elements using fractional circuit. IEEE Trans. Circ. Syst. I Reg. Papers. 58(10), 2388–2397 (Oct. 2011)
7. Podlubny, I.: Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Elsevier (1998)
8. Jumarie, G.: On the fractional solution of the equation f(x+y)=f(x)f(y) and its application to fractional Laplace's transform. Appl. Math. Comput. 219(4), 1625–1643 (2012)
9. Abdeljawad, T.: On conformable fractional calculus. J. Comput. Appl. Math. 279, 57–66 (2015)
10. Watugala, G.K.: Sumudu transform: a new integral transform to solve differential equations and control engineering problems. Int. J. Math. Educ. Sci. Technol. 24(1), 35–43 (1993)
11. Pei, S.C., Ding, J.J.: Eigenfunctions of Fourier and fractional Fourier transforms with complex offsets and parameters. IEEE Trans. Circ. and Syst. I Reg. Papers. 54(7), 1599–1611 (2007)
12. Torre, A.: Linear and radial canonical transforms of fractional order. J. Comput. Appl. Math. 153(1-2), 477–486 (2003)
13. Sharma, K.K.: Fractional Laplace transform. Signal Image Video Process. 4(3), 377–379 (2010)
14. Salahshour, S., Allahviranloo, T., Abbasbandy, S.: Solving fuzzy fractional differential equations by fuzzy Laplace transforms. Commun. Nonlinear Sci. Numer. Simulat. 17(3), 1372–1381 (2012)
15. Atıcı, F., Eloe, P.W.: A transform method in discrete fractional calculus. Int. J. Differ. Equ. 2(2), 165–176 (2007)
16. Doha, H.E., El-Maoûni, S., Bhrawy, A.H.: Composite shifted Legendre-Padé method for solving high-index singular fractional differential equations. Int. J. Numer. Methods Eng. 101(2), 96–120 (2015)
17. Liang, G.S., Liu, C.: Positive-real property of passive fractional circuits in W-domain. Int. J. Circ. Theor. Appl. 2(1), 73–81 (2018)
18. Radwan, A.G., et al.: On the stability of linear systems with fractional-order elements. Chaos Solit. Fract. 40(5), 2317–2328 (2009)
19. Farhadi, M.B., Mahdi, A.: Extending the root-Locus method to fractional-order systems. J. Appl. Math. 2008, 1–13 (2008)
20. Jakubowska-Ciszek, A., Walczak, J.: Analysis of the transient state in a parallel circuit of the class. Appl. Math. Comput. 319, 287–300 (2018)
21. Liang, G.S., Li, T.W.: Fractional order W-domain passive comprehensive method for the impedance of two component circuits. J. North China Electr. Power Univ. (Soc. Sci.)6, 6 (2019)
22. Mourad, S.S., Radwan, A.G., Hany, N.H.: Fundamentals of fractional-order LTI circuits and systems: number of poles, stability, time and frequency responses. Int. J. Circ. Theor. Appl. 44(12), 2114–2133 (2016)
23. Sene, N., Gómez-Aguilar, J.F.: Analytical solutions of electrical circuits considering certain generalized fractional derivatives. Eur. J. Phys. 31(4), 260 (2019)
24. Alsaedi, A., Nieto, J.J., Venktesh, V.: Fractional electrical circuits. Adv. Mech. Eng. 7(12), 1687814015618127 (2015)
25. Gómez-Aguilar, J.F., Atangana, A., Morales-Delgado, V. F.: Electrical circuits RC, LC, and RL described by Atangana-Baleanu fractional derivatives. Int. J. Circ. Theor. Appl. 45(11), 1514–1533 (2017)
26. Diethelm, K.: The analysis of fractional differential equations: an application-oriented exposition using differential operators of Caputo type. J. Math. Anal. Appl. 265(2), 229–248 (2002)
27. Xue, D.Y.: Fractional calculus and fractional order control, pp. 78–81. Science Press, Beijing (2018)
28. Hitchin, N.J.: The Self-Duality equations on a Riemann surface. Proc. Lond. Math. Soc.(3), 1 (1987)
29. Diebarnu, K., Ford, J.: Numerical solution of the Bagley-Torvik equation. BIT Numer. Math. 42(3), 490–507 (2002)
30. Odibat, Z.M.: Analytic study on linear systems of fractional differential equations. Comput. Math. Appl. 59(3), 1171–1183 (2010)
31. Freedholm, T.J.: A Survey of fractional-order circuit models for Biology and Biomedicine. IEEE J. Emerg. Select. Top. Circ. Syst. 3(3), 416–424 (2013)
32. Song, S.N., Sun, T., Zhang, G.W.: Complex function and integral transform, pp. 31–54. Science Press, Beijing (2006)

How to cite this article: Liang G., Jiang M. W transform and its application in fractional linear systems with rational powers. IET Circuits Devices Syst. 2021;15:209–223. https://doi.org/10.1049/eds2.12013

APPENDIX

Proof of Definition 2
Two lemmas are useful in the proof:

Lemma 2 [32] The sum, difference, product, and quotient of two complex functions which are analytic in region D (except for the point where the denominator is zero) are still analytic in D. Composite functions of analytic functions are still analytic functions.

From Lemma 2, we know that all polynomials are analytic everywhere in the complex plane.

For any rational function $f(w) = \frac{P(w)}{Q(w)}$, $P(w)$, $Q(w)$ are polynomials, its derivative is
\[
\frac{dF(w)}{dw} = \frac{d}{dw} \left[ \frac{P(w)}{Q(w)} \right] = \frac{P'(w)Q(w) - Q'(w)P(w)}{(Q(w))^2}
\]

Except for the point of \(Q(w) = 0\), \(\frac{dF(w)}{dw}\) exists, so \(F(w)\) is analytic everywhere.

**Lemma 3** [32] If \(F(w)\) is analytic everywhere in the simply connected domain \(D_s\), then the integral \(\int_C F(w)dw\) along the curve \(C\) in region \(D\) is independent of the path connecting the starting point to the ending point, and is only related to the starting point \(w_1\) and the ending point \(w_2\).

The lemma can be expressed as

\[
\int_C F(w)dw = \int_C F(w)dw = \int_{w_1}^{w_2} F(w)dw
\]

Let \(w = \xi\), then the first Riemann sheet in the \(W\)-domain is located in the region of \(|\theta| < \frac{\pi}{2}\). Any complex number in \(s\)-domain has an unique corresponding point in the first Riemann sheet in the \(W\)-domain. Mapping the straight line \(\text{Re}(s) = \sigma\) in the \(s\)-domain to the first Riemann sheet in the \(W\)-domain, and selecting the curve \(\phi_1\) located in the region of \(|\theta| < \frac{\pi}{2}\), as shown in Figure A1.

Integrating \(\frac{1}{2\pi j} F(w)\nu w^{-1} e^{\nu \omega t} dw\) along curve \(\phi_1\), we obtain

\[
\frac{1}{2\pi j} \int_{\phi_1} F(w)\nu w^{-1} e^{\nu \omega t} dw = \frac{1}{2\pi j} \int_{\phi_1} \nu w^{\phi - 1} e^{\nu \omega t} \left[ \int_{0}^{\infty} f(u) e^{-\nu u} du \right] dw
\]

Let \(\nu = \nu^\phi\), Equation (A1) can be rewritten as

\[
\frac{1}{2\pi j} \int_{\phi_1} F(w)\nu w^{\phi - 1} e^{\nu \omega t} dw = \frac{1}{2\pi j} \int_{\phi_1} \nu w^{\phi - 1} e^{\nu \omega t} \left[ \int_{0}^{\infty} f(u) e^{-\nu u} du \right] dw
\]

Let \(x = \frac{\nu \omega}{\imath}\), Equation (A2) can be written as

\[
\frac{1}{2\pi j} \int_{\phi_1} F(w)\nu w^{\phi - 1} e^{\nu \omega t} dw = \frac{1}{2\pi j} \int_{0}^{\infty} f(u) \left[ \int_{-\infty}^{\sigma + j\phi} e^{\nu \omega (t-u)} dx \right] du
\]

\[
= \frac{1}{2\pi j} \int_{0}^{\infty} f(u) \left[ \int_{-\infty}^{\sigma + j\phi} e^{\nu \omega (t-u)} dx \right] du
\]

\[
= \frac{1}{2\pi j} \int_{0}^{\infty} f(u) e^{\nu (t-u)} \left[ \int_{-\infty}^{\sigma + j\phi} e^{\nu \omega (t-u)} dx \right] du
\]

**FIGURE A1** The correspondence between \(s\)-domain and \(W\)-domain

Since

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega t} dt = \delta(\omega)
\]

And when \(f(t)\) be continuous at \(t = 0\), we have

\[
\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0)
\]

So Equation (A3) is simplified as

\[
\frac{1}{2\pi j} \int_{\phi_1} F(w)\nu w^{\phi - 1} e^{\nu \omega t} dw = \int_{0}^{\infty} f(u) e^{\nu (t-u)} \delta(t-u) du = f(t)
\]

**FIGURE A2** \(\phi_2\) in The \(W\)-domain

\(F(w)\) is no singularity in region \(B\). From Lemma 2, we know that \(F(w)\nu w^{\phi - 1} e^{\nu \omega t}\) is analytic everywhere in region \(B\). If \(F(w)\) is a rational fraction function, as long as \(F(w)\) does not have a pole in region \(B\), \(F(w)\nu w^{\phi - 1} e^{\nu \omega t}\) is analytic everywhere.

By Lemma 3, it can be seen that the function is analytic everywhere in the region \(B\), the integral is independent of the
path, so the integral curve \( \varphi_1 \) can be replaced by the polyline

\[
\varphi_2 : \begin{cases}
w = a + re^{\frac{\pi}{2}}, r : \infty \to 0 \\
w = a + re^{\frac{\pi}{2}}, r : 0 \to \infty
\end{cases}
\]

as shown in Figure A2.

Therefore, the inversion formula (A4) can be rewritten as

\[
f(t) = \frac{1}{2\pi i} \int_{\varphi_2} F(w) v e^{\omega t} w^{-1} dw
\]

\[
= \frac{1}{2\pi i} \lim_{r \to \infty} \int_{a + re^{\frac{\pi}{2}}}^{a + re^{\frac{\pi}{2}}} F(w) v w^{-1} e^{\omega t} dw
\]