Lens rigidity in scattering by non-trapping obstacles

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Abstract. We prove that if two non-trapping obstacles in \(\mathbb{R}^n\) satisfy some rather weak non-degeneracy conditions and the scattering rays in their exteriors have (almost) the same travelling times or (almost) the same scattering length spectrum, then they coincide.

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1 Introduction

We consider scattering by obstacles \(K\) in \(\mathbb{R}^n\), \(n \geq 2\). Here \(K\) is compact subset of \(\mathbb{R}^n\) with a \(C^3\) boundary \(\partial K\) such that \(\Omega_K = \mathbb{R}^n \setminus K\) is connected. By a scattering ray in \(\Omega_K\) we mean an unbounded in both directions generalized geodesic (in the sense of Melrose and Sjöstrand [MS1], [MS2]). As is well-known (see Sect. 2 for more information), most of these scattering rays are billiard trajectories with finitely many reflection points at \(\partial K\).

Given an obstacle \(K\) in \(\mathbb{R}^n\), consider a large ball \(M\) containing \(K\) in its interior, and let \(S_0 = \partial M\) be its boundary sphere. Then

\[
\Omega_K = \overline{M \setminus K}
\]

is a compact subset of \(M\) with smooth boundary \(\partial \Omega_K = \partial M \cup \partial K\). Denote by \(F^{(K)}_t\) \((t \in \mathbb{R})\) the generalized geodesic flow on the co-sphere bundle \(S^*(\mathbb{R}^n \setminus K)\). Since most trajectories of \(F^{(K)}_t\) are billiard trajectories, we will sometimes call it the billiard flow in the exterior of \(K\). For any \(q \in \partial \Omega_K\) let \(\nu(q)\) the unit normal to \(\partial \Omega_K\) pointing into the interior of \(\Omega_K\), and let

\[
S^*_+(S_0) = \{x = (q,v) : q \in S_0, \, v \in S^{n-1}, \, \langle v, \nu(q) \rangle \geq 0\}.
\]

Given \(x \in S^*_+(S_0)\), define the travelling time \(t_K(x) \geq 0\) as the maximal number (or \(\infty\)) such that \(\text{pr}_1(F^{(K)}_t(x))\) is in the interior of \(\Omega_K\) for all \(0 < t < t_K(x)\), where \(\text{pr}_1(p,w) = p\). For \(x = (q,v) \in S^*_+(S_0)\) with \(\langle \nu(q), v \rangle = 0\) set \(t(x) = 0\).

It is a natural problem to try to recover information about the obstacle \(K\) from the travelling times \(t_K(x), \, x \in S^*_+(S_0)\). Similar problems have been actively considered in Riemannian geometry – see [SU], [SUV] and the references there for some general information. In scattering by obstacles this kind of problems have been studied as well – see [LP], [M] and [PS] for general information, and also the more recent papers [St2], [St3], [NS1], [NS2] and the references there. In particular, it has been established that for some classes of obstacles \(K\) knowing all (or almost all with respect to the Lebesgue measure) travelling times \(t(x) = t_K(x)\) completely determines \(K\). This is so, for example, in the class of obstacles \(K\) that are finite disjoint unions of strictly convex domains with smooth boundaries ([NS1]). It has been an open problem so far whether a similar result holds for general non-trapping obstacles. Here we prove that this is the case under some mild non-degeneracy assumptions about the obstacle.
A natural impediment in trying to recover information about the obstacle \( K \) from its travelling times is the set \( \text{Trap}(\Omega_K) \) of trapped points. A point \( x = (q, v) \in S(\Omega) \) is called \textit{trapped} if either its forward billiard trajectory
\[
\gamma^K_+(x) = \{ \text{pr}_1(\mathcal{F}^K_t(x)) : t \geq 0 \}
\]
or its backward trajectory \( \gamma^{-}_K(q, v) = \gamma^K_+(q, -v) \) is infinitely long. That is, either the billiard trajectory in the exterior of \( K \) issued from \( q \) in the direction of \( v \) is bounded (contained entirely in \( M \)) or the one issued from \( q \) in the direction of \(-v\) is bounded. The obstacle \( K \) is called \textit{non-trapping} if \( \text{Trap}(\Omega_K) = \emptyset \).

It is known (see \cite{LP} or Proposition 2.3 in \cite{SU}) that \( \text{Trap}(\Omega_K) \cap S^*_+(S_0) \) has Lebesgue measure zero in \( S^*_+(S_0) \). However, as an example of M. Livshits shows (see Figure 1), in general the set of trapped points \( x \in S(\Omega_K) \) may contain a non-trivial open set, and then the recovery of the obstacle from travelling times is impossible. Similar examples in higher dimensions were constructed in \cite{NS3}.

Denote by \( \mathcal{K} \) be the \textit{class of obstacles with the following property}: for each \( (x, \xi) \in T^*(\partial K) = T^*(\partial K) \setminus \{0\} \) if the curvature of \( \partial K \) at \( x \) vanishes of infinite order in direction \( \xi \), then all points \( (y, \eta) \) sufficiently close to \( (x, \xi) \) are diffractive points, i.e. \( \partial K \) is convex at \( y \) in the direction of \( \eta \) (see e.g. \cite{H} or Ch.1 in \cite{PS} for the formal definition of a diffractive point). Given \( \sigma = (x, \xi) \in S^*_+(S_0) \), we will say that the trajectory \( \gamma_K(\sigma) \) is \textit{regular} (or \textit{non-degenerate}) if for every \( t >> 0 \) the differential of the map \( \mathbb{R}^n \ni y \mapsto \text{pr}_2(\mathcal{F}^K_{\tau}(y, \xi)) \in \mathbb{R}^{n-1} \) has maximal rank \( n - 1 \) at \( y = x \), and also the differential of the map \( \mathbb{R}^{n-1} \ni \eta \mapsto \text{pr}_1(\mathcal{F}^K_{\tau}(x, \eta)) \in \mathbb{R}^n \) has maximal rank \( n - 1 \) at \( \eta = \xi \).

Let \( \mathcal{K}_0 \) be the \textit{class of all obstacles} \( \mathcal{K} \) such that \( \partial K \) does not contain non-trivial open flat subsets and \( \gamma_K(x, u) \) is a regular simply reflecting ray for almost all \( (x, u) \in S^*_+(S_0) \) such that \( \gamma(x, u) \cap \partial K \neq \emptyset \). Using the technique developed in Ch. 3 in \cite{PS} one can show that \( \mathcal{K}_0 \) is of second Baire category in \( \mathcal{K} \) with respect to the \( C^\infty \) Whitney topology in \( \mathcal{K} \). This means that generic obstacles \( K \) in \( \mathbb{R}^n \) belong to the class \( \mathcal{K}_0 \).

Our main result concerns obstacles \( K \) satisfying the following conditions:

(A0): \( K \in \mathcal{K}_0 \),

(A1): The set \( \text{Trap}(\Omega_K) \cap S^*_+(S_0) \) is totally disconnected.

(A2): There exists a finite or countable family \( \{ C_i \} = \{ C_i^K \} \) of codimension two \( C^1 \) submanifolds of \( S^*_+(S_0) \) \( \setminus \text{Trap}(\Omega_K) \) which is locally finite in \( S^*_+(S_0) \) \( \setminus \text{Trap}(\Omega_K) \) and such that for every \( \rho \in S^*_+(S_0) \) \( \setminus (\text{Trap}(\Omega_K) \cup \cup_i C_i) \) the billiard trajectory \( \gamma_K(\rho) \) has no conjugate points that both belong to \( \partial K \).

Clearly (A0) is just a very weak non-degeneracy condition. The conditions in (A1) imply that the set of trapped points is relatively small; as Livshits’ example shows without some condition of this kind complete recovery of the obstacle from travelling times is impossible. The condition (A2) is a bit more subtle, however it appears to be rather general. In fact it might be generic\(^3\), however we do not have a formal proof of this. We refer the reader to \cite{dC} for general information about Jacobi fields and conjugate points.

\(^3\)Indeed, locally at least, it is trivial to ‘destroy’ a pair of conjugate points both lying on \( \partial K \), simply by perturbing slightly \( \partial K \) near one of these points.
in Riemannian geometry and in particular to \cite{W} for these concepts in the case of billiard flow.\footnote{Albeit in the simpler two-dimensional case, however the higher dimensional case is similar.}

In this paper we prove the following.

**Theorem 1.1.** Let \( K \) and \( L \) be two obstacles in \( M \) with \( C^k \) boundaries (\( k \geq 3 \)) so that the billiard flow in \( \Omega_K \) and the one in \( \Omega_L \) satisfy the conditions (A0), (A1) and (A2). Assume that \( t_K(x) = t_L(x) \) for almost all \( x \in S^*_+ (S_0) \). Then \( K = L \).

A similar result holds replacing the assumption about travelling times by a condition about sojourn times of scattering rays – see Sect. 2 for details.

The rest of the paper is devoted to the proof of this theorem. The proof in Sect. 3 below is derived from some results and arguments in \cite{St3} and \cite{NS1}. Some of these are described in Sect. 2.

![Figure 1: Livshits’ Example, adapted from Ch. 5 of M](image)

Here \( K \) is an obstacle in \( \mathbb{R}^2 \) bounded by the closed curve whose part \( E \) is half an ellipse with end points \( P \) and \( Q \) and foci \( f_1 \) and \( f_2 \). Any scattering trajectory entering the area inside the ellipse between the foci \( f_1 \) and \( f_2 \), will reflect at \( E \) and then go out between the foci again. So, no scattering ray ‘coming from infinity’ can have a common point with the bold lines from \( P \) to \( f_1 \) and from \( Q \) to \( f_2 \).

2 Some useful results from previous papers

Here we describe some previous results which will be essentially used in the proof of Theorem 1.1. We also state a more general result which covers the case of pairs of obstacles with almost the same scattering length spectrum.

We refer the reader to Ch. 1 in \cite{PS} for the definition of generalized geodesics in the present (Euclidean) case, and to \cite{MS1}, \cite{MS2} or Sect. 24.3 \cite{H} for the definition and their main properties in more general situations.
Let $K$ be an obstacle in $\mathbb{R}^n$ ($n \geq 2$) as in Sect. 1 and let

$$\widetilde{\Omega}_K = \overline{\mathbb{R}^n \setminus K}.$$ 

Given $\omega, \theta \in S^{n-1}$, a generalized geodesic $\gamma$ in $\widetilde{\Omega}_K$ will be called an $(\omega, \theta)$-ray if $\gamma$ is unbounded in both directions, $\omega \in S^{n-1}$ is its incoming direction and $\theta \in S^{n-1}$ is its outgoing direction.

Next, we define the so called scattering length spectrum associated with an obstacle.

Let again $M$ be a large ball in $\mathbb{R}^n$ containing the obstacle $K$ in its interior and let $S_0 = \partial M$. Given $\xi \in S^{n-1}$ denote by $Z_\xi$ the hyperplane in $\mathbb{R}^n$ orthogonal to $\xi$ and tangent to $S_0$ such that $M$ is contained in the half-space $R_\xi$ determined by $Z_\xi$ and having $\xi$ as an inner normal. For an $(\omega, \theta)$-ray $\gamma$ in $\Omega_K$, the sojourn time $T_\gamma$ of $\gamma$ is defined by $T_\gamma = T_\gamma' - 2a$, where $T_\gamma'$ is the length of that part of $\gamma$ which is contained in $R_\omega \cap R_{-\theta}$ and $a$ is the radius of the ball $M$. It is known (cf. [G]) that this definition does not depend on the choice of the ball $M$.

The scattering length spectrum of $K$ is defined to be the family of sets of real numbers $SL_K = \{SL_K(\omega, \theta)\}_{(\omega, \theta)}$ where $(\omega, \theta)$ runs over $S^{n-1} \times S^{n-1}$ and $SL_K(\omega, \theta)$ is the set of sojourn times $T_\gamma$ of all $(\omega, \theta)$-rays $\gamma$ in $\Omega_K$. It is known (cf. [PS]) that for $n \geq 3$, $n$ odd, and $C^\infty$ boundary $\partial K$, we have $SL_K(\omega, \theta) = \text{sing supp } s_K(t, \theta, \omega)$ for almost all $(\omega, \theta)$.

Here $s_K$ is the scattering kernel related to the scattering operator for the wave equation in $\mathbb{R} \times \Omega_K$ with Dirichlet boundary condition on $\mathbb{R} \times \partial \Omega_K$ (cf. [LP], [M]). Following [St3], we will say that two obstacles $K$ and $L$ have almost the same $SLS$ if there exists a subset $\mathcal{R}$ of full Lebesgue measure in $S^{n-1} \times S^{n-1}$ such that $SL_K(\omega, \theta) = SL_L(\omega, \theta)$ for all $(\omega, \theta) \in \mathcal{R}$.

The flow $\mathcal{F}_t^{(K)}$ can be made continuous using certain natural identifications of points at the boundary. Consider the quotient space $T_0^*(\widetilde{\Omega}_K) = T^*(\widetilde{\Omega}_K)/\sim$ with respect to the equivalence relation: $(x, \xi) \sim (y, \eta)$ iff $x = y$ and either $\xi = \eta$ or $\xi$ and $\eta$ are symmetric with respect to the tangent plane to $\partial K$ at $x$. Let $S^*_0(\widetilde{\Omega}_K)$ be the image of the unit cosphere bundle $S^*(\widetilde{\Omega}_K)$. We will identify $T^*(\widetilde{\Omega}_K)$ and $S^*(\Omega_K)$ with their images in $T_0^*(\widetilde{\Omega}_K)$. It is known that for $K \in \mathcal{K}$ the flow $\mathcal{F}_t^{(K)}$ is well-defined and continuous (cf. [MS2]). Some further regularity properties are established in [ST1] (see also Ch. 11 in [PS]).

**Definition 2.1.** Let $K, L$ be two obstacles in $\mathbb{R}^n$. We will say that $\Omega_K$ and $\Omega_L$ have conjugate flows if there exists a homeomorphism

$$\Phi : T^*(\Omega_K) \setminus \text{Trap}(\Omega_K) \longrightarrow T^*(\Omega_L) \setminus \text{Trap}(\Omega_L)$$

which defines a symplectic map on an open dense subset of $T^*(\Omega_K) \setminus \text{Trap}(\Omega_K)$, it maps $S^*(\Omega_K) \setminus \text{Trap}(\Omega_K)$ onto $S^*(\Omega_L) \setminus \text{Trap}(\Omega_L)$, and satisfies $\mathcal{F}_t^{(L)} \circ \Phi = \Phi \circ \mathcal{F}_t^{(K)}$ for all $t \in \mathbb{R}$ and $\Phi = \text{id}$ on $T^*(\mathbb{R}^n \setminus M) \setminus \text{Trap}(\Omega_K) = T^*(\mathbb{R}^n \setminus M) \setminus \text{Trap}(\Omega_L)$.

The following theorem was proved in [St3] in the case of the scattering length spectrum, and and then in [NS2] similar arguments were used to derive the case involving travelling times.

**Theorem 2.2.** If the obstacles $K, L \in \mathcal{K}_0$ have almost the same scattering length spectrum or almost the same travelling times, then $\Omega_K$ and $\Omega_L$ have conjugate flows.
We can now state the main result in this paper.

**Theorem 2.3.** Let the obstacles $K, L \in K_0$ satisfy the conditions (A0), (A1) and (A2). If $\Omega_K$ and $\Omega_L$ have conjugate flows then $K = L$.

Clearly, Theorem 1.1 is an immediate consequence of Theorem 2.3. We prove the latter in Sect. 3.

From now on we will **assume that the obstacles $K$ and $L$ in $M$ belong to the class $K_0$ and that $\Omega_K$ and $\Omega_L$ have conjugate flows**, that is there exists a homeomorphism $\Phi$ with the properties in Definition 2.1.

Next, we describe some propositions from [St1], [St2] and [St3] that are needed in the proof of Theorem 2.3.

**Proposition 2.4.** ([St1], [St2])

(a) There exists a countable family $\{M_i\} = \{M_i^{(K)}\}$ of codimension 1 submanifolds of $S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$ such that every $\sigma \in S^*_+(S_0) \setminus (\text{Trap}(\Omega_K) \cup \bigcup_i M_i)$ generates a simply reflecting ray in $\Omega_K$. Moreover the family $\{M_i\}$ is locally finite, that is any compact subset of $S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$ has common points with only finitely many of the submanifolds $M_i$.

(b) There exists a countable locally finite family $\{P_i\}$ of codimension 2 smooth submanifolds of $S^*_+(S_0)$ such that for any $\sigma \in S^*_+(S_0) \setminus (\bigcup_i \gamma_i)$ the trajectory $\gamma_K(\sigma)$ contains no gliding segments on the boundary $\partial K$ and $\gamma_K(\sigma)$ contains at most one tangent point to $\partial K$.

It follows from the conjugacy of flows and Proposition 4.3 in [St3] that the submanifolds $M_i$ are the same for $K$ and $L$, i.e. $M_i^{(K)} = M_i^{(L)}$ for all $i$. Notice that different submanifolds $M_i$ and $M_j$ may have common points (these generate rays with more than one tangency to $\partial K$) and in general are not transversal to each other. However, as we see from part (b), if $M_i \neq M_j$ and $\sigma \in M_i \cap M_j$, then locally near $\sigma$, $M_i \neq M_j$, i.e. there exist points in $M_i \setminus M_j$ arbitrarily close to $\sigma$.

From now on we will **assume that $K$ and $L$ satisfy the condition (A1)** as well, apart from (A0).

Since $S^*_+(S_0)$ is a manifold and $\text{Trap}(\Omega_K)$ and $\text{Trap}(\Omega_L)$ are compact, using the condition (A1) for both $K$ and $L$, it follows that any two points in $S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$ can be connected by a $C^1$ curve lying entirely in $S^*_+(S_0) \setminus (\text{Trap}(\Omega_K) \cup \text{Trap}(\Omega_L))$.

Let $\Gamma_K$ be the set of the points $\sigma \in S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$ such that $\gamma_K(\sigma)$ is a simply reflecting ray. It follows from [MS2] (cf. also Sect. 24.3 in [H]) and Proposition 2.4 in [St1] that $\Gamma_K$ is open and dense and has full Lebesgue measure in $S^*_+(S_0)$. Moreover, since $K, L$ have conjugate flows, Proposition 4.3 in [St3] implies $\Gamma_K = \Gamma_L$. Finally, Proposition 6.3 in [St3] and the condition (A1) yield the following.

**Proposition 2.5.** Let $K, L$ satisfy the conditions (A0) and (A1). Then

$$\#(\gamma_K(\sigma) \cap \partial K) = \#(\gamma_L(\sigma) \cap \partial L) \quad (2.1)$$

for all $\sigma \in \Gamma_K = \Gamma_L$.

That is, for $\sigma \in \Gamma_K = \Gamma_L$ the number of reflection points of $\gamma_K(\sigma)$ and $\gamma_L(\sigma)$ is the same.
3 Proof of Theorem 2.3

Let $K$ and $L$ be as in Theorem 2.3. We will show that they coincide.

Using the condition $(A2)$ for $K$ and $L$, it follows that there exists a finite or countable family $\{Q_i\}$ of codimension two $C^1$ submanifolds of $S^*_+(S_0) \setminus (\text{Trap}(\Omega_K) \cup \text{Trap}(\Omega_L))$ which is locally finite in $S^*_+(S_0) \setminus (\text{Trap}(\Omega_K) \cup \text{Trap}(\Omega_L))$ and such that for every $\rho \in S^*_+(S_0) \setminus (\text{Trap}(\Omega_K) \cup \text{Trap}(\Omega_L) \cup \cup_i Q_i)$ the billiard trajectory $\gamma_K(\rho)$ has no conjugate points that both belong to $\partial K$, and also the billiard trajectory $\gamma_L(\rho)$ has no conjugate points that both belong to $\partial L$.

Fix a family $\{Q_i\}$ with this property. Fix also a countable family $\{M_i\}$ of codimension one submanifolds of $S^*_+(S_0)$ with the property in Proposition 2.4(a) and a countable family $\{P_i\}$ of codimension 2 smooth submanifolds of $S^*_+(S_0)$ having the property in Proposition 2.4(b) for both $K$ and $L$.

We will use the general framework of the argument in Sect. 3 in [NS1]. Naturally, various modifications will be necessary.

As in [NS1], a point $y \in \partial K$ will be called regular if $\partial K = \partial L$ in an open neighbourhood of $y$ in $\partial K$. Otherwise $y$ will be called irregular. The following definition is similar to the one in Sect. 7 in [St3].

Definition. A $C^1$ path $\sigma(s)$, $0 \leq s \leq a$ (for some $a > 0$), in $S^*_+(S_0) \setminus (\text{Trap}(\Omega_K) \cup \text{Trap}(\Omega_L))$ will be called admissible if it has the following properties:

(a) $\sigma(0)$ generates a free ray in $\Omega_K$ and in $\Omega_L$, i.e. a ray without any common points with $\partial K$ and $\partial L$.

(b) if $\sigma(s) \in M_i$ for some $i$ and $s \in [0,a]$, then $\sigma$ is transversal to $M_i$ at $\sigma(s)$ and $\sigma(s) \notin M_j$ for any submanifold $M_j \neq M_i$.

(c) $\sigma(s)$ does not belong to any of the submanifolds $P_i$ and to any of the submanifolds $Q_i$ for all $s \in [0,a]$.

It follows from Proposition 6.3 in [St3] (and its proof) that for every

$$\rho \in S^*_+(S_0) \setminus (\text{Trap}(\Omega_K) \cup \text{Trap}(\Omega_L) \cup \cup_i P_i \cup \cup_i Q_i)$$

which belongs to at most one of the submanifolds $M_i$ there exists an admissible path $\sigma(s)$, $0 \leq s \leq a$, with $\sigma(a) = \rho$.

Let $m \geq 1$ be an integer. As in [NS1], denote $Z_m$ be the set of irregular points $x \in \partial K$ with the following property: there exists an admissible path $\sigma(s)$, $0 \leq s \leq a$, in $S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$ such that $\sigma(a)$ generates a free ray in $\mathbb{R}^n$, $x$ belongs to the billiard trajectory $\gamma_K^+(\sigma(a))$ and for any $s \in [0,a]$ the trajectory $\gamma_K^+(\sigma(s))$ has at most $m$ irregular common points with $\partial K$.

We will prove by induction on $m$ that $Z_m = \emptyset$ for all $m \geq 1$.

Step 1. $Z_1 = \emptyset$. The proof of this case is the same as the one in [NS1]. We sketch it here for completeness. Assume that $Z_1 \neq \emptyset$. Consider an arbitrary admissible path $\sigma(s)$,
Let \( x_1, \ldots, x_k \) be the common points of \( \gamma_K(\rho) \) with \( \partial K \) and let \( x_i \in Z_1 \) for some \( i \). Then all \( x_j \) with \( j \neq i \) are regular points, so there exists an open neighbourhood \( U_j \) of \( x_j \) in \( \partial K \) such that \( U_j = U_j \cap \partial L \). Setting \( \rho = (x_0, u_0) \), let \( t_{k+1} > 0 \) be the largest number such that \( \rho_{k+1} = F_{t_{k+1}}(\rho) \in S^*(S_0) \). Let \( F_{t_j}^{(K)}(\rho) = (x_j, u_j), 1 \leq j \leq k+1 \). It then follows from the above that \( F_{t_j}^{(K)}(\rho) = F_{t_j}^{(L)}(\rho) \) for \( 0 \leq t < t_i \), and also \( F_{t_i}^{(K)}(\rho_{k+1}) = F_{t_i}^{(L)}(\rho_{k+1}) \) for all \(- (t_{k+1} - t_j) < \tau \leq 0 \). So, the trajectories \( \gamma_K(\rho) \) and \( \gamma_L(\rho) \) both pass through \( x_{i-1} \) with the same (reflected) direction \( u_{i-1} \) and through \( x_{i+1} \) with the same (reflected) direction \( u_{i+1} \). Thus, \( x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k \) are common points of \( \gamma_L(\rho) \) and \( \partial L \). As observed above, \( \gamma_L(\rho) \) must have exactly \( k \) common points with \( \partial L \), so it has a common point \( y_i \) with \( \partial L \) ‘between’ \( x_{i-1} \) and \( x_{i+1} \).

Next, we consider two cases.

**Case 1.** \( x_i \) is a transversal reflection point of \( \gamma \) at \( \partial K \). Then the above shows that \( \gamma' \) has a transversal reflection point at \( x_i \), so in particular \( x_i \in \partial L \). It is also clear that for any \( y \in \partial K \) sufficiently close to \( x_i \) there exists \( \rho' = S_{\gamma}(S_0) \backslash \text{Trap}(\Omega_K) \) close to \( \rho \) so that \( \gamma_{\rho'}(\rho') \) has a proper reflection point at \( y \). A similar argument shows that every \( y \in \partial L \) with \( x_i \) and \( y_i \) is an irregular point. We may assume that \( \partial K = \partial L \) in an open neighbourhood of \( x = x_i \) in \( \partial K \), which is impossible since \( x \) is an irregular point.

**Case 2.** \( x_i \) is a tangent point of \( \gamma \) to \( \partial K \). Then each of the trajectories \( \gamma \) and \( \gamma' \) has exactly \( k-1 \) transversal reflection points. Moreover, \( x_i \) lies on the segment \( [x_{i-1}, x_{i+1}] \). The trajectory \( \gamma' \) also has exactly one tangent point to \( \partial L \) and it must be a point \( y_i \) on the segment \( [x_{i-1}, x_{i+1}] \). Assume for a moment that \( y_i \neq x_i \). Then we can choose \( x'_i \in \partial K \) arbitrarily close to \( x_i \) and \( u'_i \in S^{*-1} \) close to \( u_i \) so that \( u'_i \) is tangent to \( \partial K \) at \( x'_i \) and the straight line determined by \( x'_i \) and \( u'_i \) intersects \( \partial L \) transversally near \( y_i \). Let \( \rho' = S_{\gamma}(S_0) \backslash \text{Trap}(\Omega_K) \) be the point close to \( \rho \) which determines a trajectory \( \gamma_{\rho'}(\rho') \) passing through \( x'_i \) in direction \( u'_i \), i.e. tangent to \( \partial K \) at \( x'_i \). Then \( \gamma_{\rho'}(\rho') \) has \( k-1 \) transversal reflections at \( \partial K \) and one tangent point, while \( \gamma_{\rho'}(\rho') \) has \( k \) transversal reflections at \( \partial L \) and no tangent points at all. This impossible, so we must have \( y_i = x_i \). A similar argument shows that every \( x' \in \partial K \) sufficiently close to \( x_i \) belongs to \( \partial L \), as well. So, \( x_i \) is a regular point, a contradiction.

Thus we must have \( Z_1 = \emptyset \).

**Step 2: Inductive Step.** Assume that \( Z_1 = \ldots = Z_m = \emptyset \) for some integer \( m > 1 \). Suppose \( Z_m \neq \emptyset \). Then there exists an admissible \( C^1 \) path \( \sigma(s), 0 \leq s \leq a \), in \( S^*(S_0) \) such that \( \gamma_{\rho}(\sigma(a)) \) has a common point with \( Z_m \) and for each \( s \in [0, a] \) the trajectory \( \gamma_{\rho}(\sigma(s)) \) has at most \( m \) irregular points. We may assume that \( a > 0 \) is minimal with this property. Then for \( s \in [0, a], \gamma_{\rho}(\sigma(s)) \) contains no points of \( Z_m \), so if it passes through any irregular points, they must be from some \( Z_i \) with \( i < m \). However, \( Z_1 = \ldots = Z_{m-1} = \emptyset \) by assumption, so \( \gamma_{\rho}(\sigma(s)) \) contains no irregular points at all for all \( s \in [0, a] \). This implies \( F_{\rho}^{(K)}(\sigma(s)) = F_{\rho}^{(L)}(\sigma(s)) \) for all \( t \geq 0 \) and all \( s \in [0, a] \), and by continuity of the
flows, we derive \( \mathcal{F}^{(K)}_t(\sigma) = \mathcal{F}^{(L)}_t(\sigma) \) for all \( t \geq 0 \).

Set \( \rho = \sigma(a), \gamma = \gamma^+_K(\rho), \gamma' = \gamma^+_L(\rho) \). It follows from the definition of \( Z_m \) and \( \gamma \cap \Omega_m \neq \emptyset \), that \( \gamma \) contains at most \( m \) irregular points. Since \( \gamma^+_K(\sigma(s)) \) has no irregular points at all for \( s < a \), it follows that \( \gamma \) contains exactly \( m \) irregular points; otherwise all irregular points in \( \gamma \) would be in \( Z_i \) for some \( i < m \), which is impossible since \( Z_i = \emptyset \).

Let \( x_1, \ldots, x_m \) be the consecutive irregular common points of \( \gamma \) with \( \partial K \), and let \( y_1, \ldots, y_p \) be its regular common points with \( \partial K \) (if any, i.e. we may have \( p = 0 \)). Then from the definition of a regular point, for each \( i = 1, \ldots, p \), there exists an open neighbourhood \( V_i \) of \( y_i \) in \( \partial K \) with \( V_i \subset \partial L \), i.e. \( V_i \) is an open neighbourhood of \( y_i \) in \( \partial L \) as well.

Since \( \sigma(s) \) is an admissible path, if \( \gamma \) has a tangent point to \( \partial K \), then it is exactly one of the points \( x_1, \ldots, x_m, y_1, \ldots, y_p \). Let \( \text{pr}_1(\mathcal{F}^{(K)}_{\tau_i}) = y_i = \text{pr}_1(\mathcal{F}^{(L)}_{\tau_i}) \) for some \( 0 < \tau_1 < \ldots < \tau_p \). It follows from the continuity of the flows \( \mathcal{F}^{(K)}_t \) and \( \mathcal{F}^{(L)}_t \) that there exists an open neighbourhood \( W \) of \( \rho \) in \( S_1^+(S_0) \) and \( \delta > 0 \) such that for any \( \rho' \in W \) if \( \mathcal{F}^{(K)}_\tau(\rho') \in \partial K \) (or \( \mathcal{F}^{(L)}_\tau(\rho') \in \partial L \)) for some \( \tau \) with \( |\tau - \tau_i| < \delta \), then \( \text{pr}_1(\mathcal{F}^{(K)}_{\tau}(\rho')) \in V_i \) (resp. \( \text{pr}_1(\mathcal{F}^{(L)}_{\tau}(\rho')) \in V_i \)). In particular, taking \( W \) sufficiently small we have that \( \gamma^+_K(\rho') \) contains at most \( m \) irregular points for all \( \rho' \in W \).

Since \( \sigma(s) \) is an admissible path, does not contain any points from \( \cup_i Q_i \). In particular, \( \rho \notin \cup_i Q_i \), so \( \gamma \) does not have conjugate points both belonging to \( \partial K \).

Let \( \text{pr}_1(\mathcal{F}^{(K)}_{t_j}(\rho)) = x_j = \text{pr}_1(\mathcal{F}^{(L)}_{t_j}(\rho)), j = 1, \ldots, m \), for some \( 0 < t_1 < \ldots < t_m \), and let \( u_j \) be the reflected direction of the trajectory \( \gamma \) at \( x_j \).

**Case 1.** The points \( x_1, \ldots, x_m \) are all transversal reflection points of \( \gamma \) (some of the points \( y_i \) might be a tangent point of \( \gamma \) to \( \partial K \)). Then for \( s < a \) close to \( a \), \( \gamma^+_K(\sigma(s)) \) has transversal reflection points \( x_1(s), \ldots, x_m(s) \) close to \( x_1, \ldots, x_m \), respectively, that is \( x_i(s) \to x_i \) as \( s \nearrow a \). Since \( \gamma^+_K(\sigma(s)) \) has only regular reflection points, we have \( \partial K = \partial L \) in an open neighbourhood of \( x_i(s) \) in \( \partial K \) for \( s < a \) close to \( a \). Hence there exists an open subset \( U_i \) of \( \partial K \) with \( x_i \in U_i \) and \( \partial K \cap U_i = \partial L \cap U_i \) for all \( i = 1, \ldots, m \).

Set \( \rho_j = \mathcal{F}^{(K)}_{t_j}(\rho) \). Take a small number \( \delta > 0 \) and consider \( \mathcal{O} = \{ u \in S^{n-1} : \|u - u_1\| < \delta \} \).

If \( \delta \) is small enough, for every \( u \in \mathcal{O} \) there exists \( t = t(u) \in \mathbb{R} \) close to \( t_2 - t_1 \) such that \( G(u) = \text{pr}_1(\mathcal{F}^{(K)}_{t(u)}(\rho_1)) \in \partial K \). This defines a smooth local map \( G : \mathcal{O} \to \partial K \) taking values near \( x_2 \). Since \( \gamma \) has no conjugate points both belonging to \( \partial K \), \( x_1 \) and \( x_2 \) are not conjugate points along \( \gamma \). This implies that the linear map \( dG(u_1) \) has full rank \( n - 1 \), and therefore \( G(\mathcal{O}) \) covers a whole open neighbourhood of \( x_2 \) in \( \partial K \). Since \( x_2 \in \overline{U_2} \), it follows from the continuity of \( G \) at \( u_1 \) that \( G(u) = \overline{U_2} \). Given such an \( u \in \mathcal{O} \), consider the billiard trajectory in \( \Omega_K \) generated by \( (x_1, u) \), and let \( \rho' \in S_1^+(S_0) \) be the point that belongs to this billiard trajectory. Then \( \gamma^+_K(\rho') \) has transversal reflection points \( x'_1, x'_2, \ldots, x'_m \) close to the points \( x_1, x_2, \ldots, x_m \), respectively. If \( u \) is sufficiently close to \( u_1 \), then \( \rho' \in W \), so \( \gamma^+_K(\rho') \) contains at most \( m \) irregular points. On the other hand, for such \( u \) we have \( x'_1 = G(u) \in U_2 \) and we have \( \partial K = \partial L \) on \( U_2 \). Thus, \( x'_2 \) cannot be an irregular point. There are no other places from which \( \gamma^+_K(\rho') \) can gain an irregular point, therefore it turns out \( \gamma^+_K(\rho') \) has at most \( m - 1 \) irregular points. Now the assumption
\(Z_1 = Z_2 = \ldots = Z_{m-1} = \emptyset\) gives that \(\gamma^+_K(\rho')\) contains no irregular points at all. However \(x_1\) is one of the reflection points of \(\gamma^+_K(\rho')\) and \(x_1\) is irregular, a contradiction.

This proves that Case 1 is impossible.

**Case 2.** One of the points \(x_1, \ldots, x_m\) is a tangent point of \(\gamma\) to \(\partial K\). Then all other \(x_i\) are transversal reflections, and since \(m > 1\), we have at least one such point. We will assume that one of the points \(x_1\) or \(x_2\) is a tangent point of \(\gamma\) to \(\partial K\); the other cases are considered similarly. Assume e.g. that \(\gamma\) is tangent to \(\partial K\) at \(x_2\) and has a transversal reflection at \(x_1\); otherwise we will change the roles of \(x_1\) and \(x_2\) and reverse the motion along the trajectory. Since \(\gamma\) can have only one tangent point to \(\partial K\), all points \(y_i\) (if any) are transversal reflection points. Moreover we have \(\partial K = \partial L\) in an open neighbourhood \(V_i\) of \(y_i\) in \(\partial K\). Take again a small number \(\delta > 0\) and consider \(\mathcal{O} = \{u \in S^{n-1} : \|u - u_1\| < \delta\}\). Given \(u \in \mathcal{O}\), consider the billiard trajectory in \(\Omega_K\) generated by \((x_1, u)\), and let \(\rho(u) \in S^*_+ (S_0)\) be the point that belongs to this billiard trajectory. Assuming that \(\delta\) is small enough, for all \(u \in \mathcal{O}\) the trajectory \(\gamma^+_K(\rho(u))\) has transversal reflection points \(x_j(u)\) close to \(x_j\) for \(j \neq 2\) and \(y_i(u)\) close to \(y_i\) for all \(i\). It may not have a common point with \(\partial K\) near \(x_2\). In fact, we will show that we can choose \(u \in \mathcal{O}\) arbitrarily close to \(u_1\) so that \(\gamma^+_K(\rho(u))\) has no common point with \(\partial K\) near \(x_2\). This is obvious when there are no reflection points of \(\gamma\) between \(x_1\) and \(x_2\), so assume that some of the reflections at the points \(y_i\) occur between \(x_1\) and \(x_2\). Let \(y_i\) be the last reflection point of \(\gamma\) before \(x_2\), i.e. \(t_1 < t_i < t_2 < t_{i+1}\). Recall the open neighbourhood \(V_i\) of \(y_i\) in \(\partial K\) with \(\partial K = \partial L\) on \(V_i\) (i.e. \(V_i \subset \partial L\)).

Denote by \(\Sigma\) the hyperplane in \(\mathbf{R}^n\) passing through \(x_2\) and perpendicular to \(u_2\). Define the map \(G : \mathcal{O} \rightarrow \Sigma\) as follows: given \(u \in \mathcal{O}\), the trajectory \(\gamma^+_K(\rho(u))\) reflects at \(y_i(u)\) on \(V_i\) with a reflected direction \(\eta_i(u)\), and the straight-line ray issued from \(y_i(u)\) in direction \(\eta_i(u)\) intersects \(\Sigma\) at some point which we call \(G(u)\). It is clear that \(G\) is a smooth map (only transversal reflections occur between \(x_1\) and \(x_2\)) and \(G(u_1) = x_2\). Since \(\gamma\) does not have conjugate points both belonging to \(\partial K\), we have \(\text{rank}(dG(u_1)) = n - 1\), so \(G(\mathcal{O})\) contains a whole open neighbourhood \(V\) of \(x_2\) in \(\Sigma\). Assuming that the neighbourhood \(V\) of \(x_2\) is sufficiently small, \(V \setminus K\) contains a non-trivial open subset whose closure contains \(x_2\). Thus, there exist \(u \in \mathcal{O}\) arbitrarily close to \(u_1\) for which \(G(u) \in V \setminus K\), which means that the trajectory \(\gamma^+_K(\rho(u))\) will intersect \(\Sigma\) in \(V \setminus K\) and so it will not have a common point with \(\partial K\) near \(x_2\). In particular, \(\gamma^+_K(\rho(u))\) will have at most \(m - 1\) irregular points, and the assumption now yields that \(\gamma^+_K(\rho(u))\) has no irregular points at all. This is a contradiction, since \(x_1\) is an irregular point and it belongs to \(\gamma^+_K(\rho(u))\).

Hence Case 2 is impossible as well. This proves that we must have \(Z_m = \emptyset\).

By induction \(Z_m = \emptyset\) for all \(m \geq 1\), so there are no irregular points at all.

It is now easy to prove that \(\partial K \subset \partial L\). Let \(A\) be the set of those \(\rho \in S^*_+(S_0) \setminus (\text{Trap}(\Omega_K) \cup \text{Trap}(\Omega_L) \cup \cup_i P_i \cup \cup_i Q_i)\)

such that \(\rho\) belongs to at most one of the submanifolds \(M_i\). Clearly, \(A\) is a dense subset of \(S^*_+(S_0)\). Moreover, the set \(B\) of the points \(x \in \partial K\) that belong to \(\gamma^+_K(\rho)\) for some \(\rho \in A\) is dense in \(\partial K\). As we mentioned earlier, it follows from Proposition 6.3 in [St3] (and its proof) that for every \(\rho \in A\) there exists an admissible path \(\sigma(s), 0 \leq s \leq a\), with
\[ \sigma(a) = \rho. \] Since \( Z_m = \emptyset \) for all \( m \), we derive \( \gamma^+_K(\rho) = \gamma^+_L(\rho) \). The latter is then true for all \( \rho \in A \), and therefore \( B \subset \partial L \). Since \( B \) is dense in \( \partial K \), it follows that \( \partial K \subset \partial L \).

By symmetry we get \( \partial L \subset \partial K \) as well, so \( \partial K = \partial L \). ■

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