Abstract. In this paper, we study three aspects of the $p$-adic Möbius maps. One is the group $\text{PSL}(2, \mathbb{Q}_p)$, another is the geometrical characterization of the $p$-adic Möbius maps and its application, and the other is different norms of the $p$-adic Möbius maps. Firstly, we give a series of equations of the $p$-adic Möbius maps in $\text{PSL}(2, \mathbb{Q}_p)$ between matrix, chordal, hyperbolic and unitary aspects. Furthermore, the properties of $\text{PSL}(2, \mathbb{Q}_p)$ can be applied to study the geometrical characterization, the norms, the decomposition theorem of $p$-adic Möbius maps, and the convergence and divergence of $p$-adic continued fractions. Secondly, we classify the $p$-adic Möbius maps into four types and study the geometrical characterization of the $p$-adic Möbius maps from the aspects of fixed points in $\mathbb{P}^1_{\text{Ber}}$, and the invariant axes which yields the decomposition theorem of $p$-adic Möbius maps. Furthermore, we prove that if a subgroup of $\text{PSL}(2, \mathbb{C}_p)$ containing elliptic elements only, then all elements fix the same point in $\mathbb{H}_{\text{Ber}}$ without using the famous theorem–Cartan fixed point theorem, and this means that this subgroup has potentially good reduction. In the last part, we extend the inequalities obtained by Gehring and Martin [24,25], Beardon and Short [12] to the non-archimedean settings. These inequalities of $p$-adic Möbius maps are between the matrix, chordal, three-point and unitary norms. This part of work can be applied to study the convergence of the sequence of $p$-adic Möbius maps which can be viewed as a special cases of the work in [20] and the discrete criteria of the subgroups of $\text{PSL}(2, \mathbb{C}_p)$.

1. Introduction

1.1. Statement of results. We call an element in the projective special linear group $\text{PSL}(2, \mathbb{C}_p)$ the $p$-adic Möbius map, where $\mathbb{Q}_p$ is the field of $p$-adic rational numbers and $\mathbb{C}_p$ is the completion of the algebraic closure of $\mathbb{Q}_p$. The projective space $\mathbb{P}^1(\mathbb{C}_p)$ is totally disconnected and not locally compact, which implies that we can not adopt the method used in dealing with the complex settings easily. The main tool that we use is the projective Berkovich space $\mathbb{P}^1_{\text{Ber}}$ (see concrete definitions in section 3), since $\text{PSL}(2, \mathbb{C}_p)$ acts on the hyperbolic Berkovich space $\mathbb{H}_{\text{Ber}}$ isometrically and the projective Berkovich space is compact with respect to the weak topology.
We study the subgroup $\text{PSL}(2, \mathcal{O}_p)$, where $\mathcal{O}_p = \{z \mid ||z|| \leq 1\}$ firstly, since the study of the unitary groups of the projective special linear group $\text{PSL}(2, \mathbb{C})$ is a very important topic in the study of Möbius maps. There exist a lot of equations of Möbius maps in the unitary group. It is natural to generalize the equations to the non-archimedean settings. We give a series of equations of $p$-adic Möbius maps in $\text{PSL}(2, \mathcal{O}_p)$ between matrix, chordal, hyperbolic and unitary aspects. Furthermore, the properties of $\text{PSL}(2, \mathcal{O}_p)$ can be applied to study the geometrical characterization, the norms and the decomposition theorem of $p$-adic Möbius maps and the convergence and divergence of $p$-adic continued fractions.

Let $g$ be a $p$-adic Möbius map, $\rho_v(z, w)$ be the chordal metric on the projective line $\mathbb{P}^1(\mathbb{C}_p)$, $\rho(z, w)$ be the hyperbolic metric on the hyperbolic Berkovich space, $L(g) = p^{\rho(g(\zeta_{\text{Gauss}}), \zeta_{\text{Gauss}})}$, and $\| \cdot \|$ be the matrix norm (see concrete definitions in section 2).

**Theorem 1.1.** For any $p$-adic Möbius map $g$, the following are equivalent:

1. $g \in \text{PSL}(2, \mathcal{O}_p)$;
2. $L(g) = 1$;
3. $\rho(\zeta_{\text{Gauss}}, g(\zeta_{\text{Gauss}})) = 0$;
4. $g$ is a chordal isometry;
5. $\| g \| = 1$;
6. for any $h$ in the Matrix ring $\text{M}(2, \mathbb{C}_p)$, $\| gh \| = \| hg \| = \| ghg^{-1} \| = \| h \|$.

In the second part, we study the characterization of $p$-adic Möbius maps. In [26], Kato introduced the idea of the Kleinian group to study the $p$-adic Möbius maps firstly, and in [6, 30], Vermitage and Parker, Qiu, Yang and Yin gave the discrete criteria of subgroups of $\text{PSL}(2, \mathbb{C}_p)$. The method that they used are also derived from the Kleinian group. We not only follow the philosophy of the Kleinian group to study the $p$-adic Möbius maps, but also we lay emphasis on the study of the difference between them. First, we classify the $p$-adic Möbius maps into four types which is a bit different than those in the Kleinian group, and we study the geometrical characterization of the $p$-adic Möbius maps from the aspects of fixed points in $\mathbb{P}_{\text{Ber}}$ and the invariant axes. Furthermore, we can decompose a $p$-adic Möbius map $g$ into two involutions $\alpha, \beta$ (an elliptic element of order 2), namely $g = \alpha \circ \beta$, and the structure of the fixed points of $\alpha, \beta$ can reflect the type and other properties of $g$. This method can be used to prove that if a subgroup of $\text{PSL}(2, \mathbb{C}_p)$ contains elliptic elements only, then all elements fix the same point in $\mathbb{H}_{\text{Ber}}$ without using the famous theorem–Cartan fixed point theorem, and this means that this subgroup has potentially good reduction (see concrete definitions in section 2). In the proof of the results, we should face three difficulties which do not exist in the archimedean settings. One is that $\mathbb{P}^1(\mathbb{C}_p)$ and $\mathbb{P}_{\text{Ber}}$ are not locally compact, another is that there exists a new kind of $p$-adic Möbius maps—the wild elliptic elements whose geometrical structure of the fixed points are complicated, and the other is that the prime
number \( p \) affects the structure of the fixed points of \( p \)-adic Möbius maps. We give a series of tables to compare the properties of \( p \)-adic Möbius maps and those of the Kleinian group.

| type        | the Kleinian group | \( p \)-adic Möbius maps |
|-------------|--------------------|--------------------------|
| loxodromic  |                    | loxodromic               |
| parabolic   |                    | parabolic                |
| elliptic    |                    | tame elliptic            |
|             |                    | wild elliptic            |

Assuming \( g(z) = \frac{az+b}{cz+d} = \alpha \circ \beta \) with \( ad - bc = 1 \), let \( F_g, F_\alpha, F_\beta \) denote the fixed points of \( g, \alpha, \beta \) in \( \mathbb{P}^1 \cup \mathbb{H}^3 \) respectively.

| the Kleinian group | \( F_g \subset \mathbb{P}^1 \cup \mathbb{H}^3 \) | \( F_\alpha \cap F_\beta \) |
|-------------------|--------------------------------|--------------------------|
| loxodromic        | two points in \( \mathbb{P}^1 \) | \( \emptyset \)          |
| parabolic         | one point in \( \mathbb{P}^1 \)   | unique point in \( \mathbb{P}^1 \) |
| elliptic          | a geodesic line in \( \mathbb{P}^1 \cup \mathbb{H}^3 \) | unique point in \( \mathbb{H}^3 \) |

Let Int(\( A \)) denote the interior of the set \( A \). Assuming \( g(z) = \frac{az+b}{cz+d} = \alpha \circ \beta \) with \( ad - bc = 1 \), let \( F_g, F_\alpha, F_\beta \) denote the fixed points of \( g, \alpha, \beta \) in \( \mathbb{P}^1_{\text{Ber}} \) respectively.

| the \( p \)-adic Möbius map | \( F_g \subset \mathbb{P}^1 \cup \mathbb{H}_{\text{Ber}} \) | \( F_\alpha \cap F_\beta \) |
|-----------------------------|--------------------------------|--------------------------|
| loxodromic                  | two points in \( \mathbb{P}^1 \) | \( \emptyset \)          |
| parabolic                   | \( F_g \cap \mathbb{H}_{\text{Ber}} \neq \emptyset \) | Int(\( F_\alpha \cap F_\beta \)) \neq \emptyset |
| tame elliptic               | a geodesic line in \( \mathbb{P}^1_{\text{Ber}} \) | unique point in \( \mathbb{H}_{\text{Ber}} \) |
| wild elliptic               | Int(\( F_g \)) \neq \emptyset | Int(\( F_\alpha \cap F_\beta \)) \neq \emptyset |

**Theorem 1.2.** If the subgroup \( G \subset \text{PSL}(2, \mathbb{C}_p) \) contains elliptic elements only, then all the elements of \( G \) share at least one fixed point in \( \mathbb{H}_{\text{Ber}} \). Furthermore, \( G \) has potentially good reduction, and \( G \) is equicontinuous on \( \mathbb{P}^1(\mathbb{C}_p) \).

A point \( a \in \mathbb{P}^1(\mathbb{C}_p) \) is called the limit point of a subgroup \( G \) of \( \text{PSL}(2, \mathbb{C}_p) \) if there exists a point \( b \in \mathbb{P}^1(\mathbb{C}_p) \) and an infinite sequence \( \{g_n\}_{n \geq 1} \subset G \), where \( g_n \neq g_m \) if \( n \neq m \) with \( \lim g_n(b) = a \). The set consisting of all limit points is called the limit set. \( G \) is said to act discontinuously at \( x \in \mathbb{P}^1(\mathbb{C}_p) \) if there is a neighborhood \( U \) of \( x \) such that \( g(U) \cap U = \emptyset \) for all but finitely many \( g \in G \). The set of points at which \( G \) acts discontinuously is called the discontinuous set.

**Theorem 1.3.** If the limit set of \( G \) is empty, then \( G \) has potentially good reduction.
Two points \( \alpha, \beta \) are called \textit{antipodal} points if there exists an element \( u \in \text{PSL}(2, \mathbb{C}_p) \) such that \( u(0) = \alpha, u(\infty) = \beta \).

**Theorem 1.4.** For any \( p \)-adic Möbius map \( g \), there exists an element \( u \in \text{PSL}(2, \mathbb{C}_p) \) such that \( g = uf \), where either \( f \) is a loxodromic element with antipodal fixed points, or \( f = I \).

In the last part, we extend the inequalities obtained by Gehring and Martin \cite{2425}, Beardon and Short \cite{12} to the non-archimedean settings. These inequalities of \( p \)-adic Möbius maps are between the matrix, chordal, three-point and unitary norms. This part of work can be applied to study the convergence of the sequence of \( p \)-adic Möbius maps which can be viewed as a special cases of the work in \cite{20} and the discrete criteria of the subgroups of \( \text{PSL}(2, \mathbb{C}_p) \).

For any two \( p \)-adic Möbius maps \( g, h \), we define the \textit{uniformly convergent metric} on the \( \text{PSL}(2, \mathbb{P}^1(\mathbb{C}_p)) \) as follows

\[
\rho_0(g,h) = \sup_{z \in \mathbb{P}^1(\mathbb{C}_p)} |g(z), h(z)|.
\]

Let \( M(g) = \| g - g^{-1} \| / \| g \| \).

**Theorem 1.5.** Let \( g \) be a \( p \)-adic Möbius map.

1. If \( p = 3 \), then \( \rho_0(g, I) = M(g) \).
2. If \( p = 2 \), then \( 2^{-1}M(g) \leq \rho_0(g, I) \leq 2M(g) \).

Let \( \varepsilon(g) = \max\{\rho_v(g(z_0), z_0), \rho_v(g(z_1), z_1), \rho_v(g(z_2), z_2)\} \), where \( z_0, z_1, z_2 \) are three distinct roots of the equation \( x^3 = 1 \).

**Theorem 1.6.** For any \( p \)-adic Möbius map \( g \), we have \( 2^{-1}\varepsilon(g) \leq M(g) \leq 6\varepsilon(g) \).

Let \( \varepsilon_1(g) = \{\rho_v(g(0), 0), \rho_v(g(1), 1), \rho_v(g(\infty), \infty)\} \).

**Theorem 1.7.** For any \( p \)-adic Möbius map, \( 2^{-1}\varepsilon_1(g) \leq M(g) \leq \varepsilon_1(g) \).

If \( g \) is a parabolic element, we can improve the inequality. Let \( \varepsilon_2(g) = \max\{\rho_v(g(0), 0), \rho_v(g(\infty), \infty)\} \).

**Corollary 1.8.** If \( g \) is a parabolic element, then \( 2^{-1}\varepsilon_2(g) \leq M(g) \leq \varepsilon_2(g) \).

Let \( \mathcal{U} = \text{PSL}(2, \mathbb{O}_p) \). We define \( d(g, \mathcal{U}) = \inf\{\rho_0(g, u) | u \in \mathcal{U}\} \). \( d(g, \mathcal{U}) \) measures how far an element from the group \( \mathcal{U} \). This result is quite different from that in the archimedean setting.

**Theorem 1.9.** For any \( p \)-adic Möbius map, either \( d(g, \mathcal{U}) = 0 \), if \( g \in \mathcal{U} \), or \( d(g, \mathcal{U}) = 1 \), if \( g \notin \mathcal{U} \).

As an application of this result, we derive a discrete criteria of subgroups of \( \text{PSL}(2, \mathbb{C}_p) \).

**Theorem 1.10.** If \( G \) is a subgroup of \( \text{PSL}(2, \mathbb{C}_p) \) with \( G \cap \mathcal{U} = \{I\} \), then \( G \) is a discrete subgroup.
Corollary 1.11. If a subgroup $G \subset \text{PSL}(2, \mathbb{C}_p)$ contains unit element or loxodromic element only, then $G$ is a discrete subgroup.

The other application of these inequalities is to get the convergence theorem of $p$-adic Möbius maps.

Theorem 1.12. Let $\{g_n\}$ be a sequence of $p$-adic Möbius maps, $z_j$, $(j = 1, 2, 3)$ be three distinct points and $g_n(z_j) \to w_j$, where $w_j$ are also three distinct points. Then the sequence $\{g_n\}$ converges to a $p$-adic Möbius map $g$ uniformly, where $g(z_j) = w_j$, $(j = 1, 2, 3)$.

1.2. Motivation. Firstly the study of the Kleinian group in the archimedean case has been well developed for a rather long time. It is natural for one to consider a parallel theory in the non-archimedean settings. Here we wish to give a fairly clear picture of the $p$-adic Möbius maps from the point of view of the Kleinian group. We study three aspects of the $p$-adic Möbius maps. One is the group $\text{PSL}(2; \mathcal{O}_p)$, another is the geometrical characterization of the $p$-adic Möbius maps and its application, and the other is different norms of the $p$-adic Möbius maps, since $\text{PSL}(2; \mathcal{O}_p)$ is similar to the unitary group in the Kleinian group, the geometrical characterization of the $\text{PSL}(2; \mathcal{O}_p)$ maps is useful in the study of the structure and dynamics of the subgroups of $\text{PSL}(2; \mathbb{C}_p)$, and the study of norms of Möbius maps is a very important topic, and many mathematicians such as Gehring and Martin [24, 25], Beardon and Short [12] do a lot of works in this topic.

The other reason is that the three aspects can be viewed as tools to study other related topics. The properties of the norms of $p$-adic Möbius maps have a lot of applications in three topics. One is the discrete criteria of subgroups of $\text{PSL}(2; \mathbb{C}_p)$ and another is the pointwise convergence of $p$-adic Möbius maps(see [26, 32, 34]) and the other is the $p$-adic continued fraction. The geometrical characterization shows that the subgroup $G$ containing elliptic element only shares one unique point, which means that a non-elementary group should have an loxodromic element which is very useful to study the dynamics of the discrete subgroup of $\text{PSL}(2; \mathbb{C}_p)$. The rapid development of the Berkovich space and the arithmetical dynamical system (see [3, 4, 11, 13, 14, 15, 16, 19, 21, 31]) also promote studying the $p$-adic Möbius maps. Especially, the study of $p$-adic Möbius maps can be applied in studying the quantum mechanics and quantum cosmology [22, 23]). In this paper we give the affirmative answers to all these three questions in the non-archimedean settings.

1.3. Outline of the paper. Outline of the paper. In section 1, we present our main results of the paper. In section 2, the basic theories of the $p$-adic analysis and the Berkovich space are briefly reviewed. In section 3, we obtain a few preliminary results of $p$-adic Möbius maps. Section 4 contains proofs of the equations of $p$-adic Möbius maps in $\text{PSL}(2; \mathcal{O}_p)$. In section 5, we give the results of the geometrical characterization of the $p$-adic Möbius maps. In section 6, the inequalities of $p$-adic Möbius maps between matrix,
chordal, three points, hyperbolic, and unitary norms are derived. In section 7, we prove the decomposition theorem of the $p$–adic Möbius maps and discuss its application.

2. Some Preliminary Results

2.1. The non-archimedean space $\mathbb{P}^1(\mathbb{C}_p)$. Let $p \geq 2$ be a prime number. Let $\mathbb{Q}_p$ be the field of $p$-adic numbers and $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$. Denote $|\mathbb{C}_p^*|$ the valuation group of $\mathbb{C}_p$. Then every element $r \in |\mathbb{K}^*|$ can be expressed as $r = p^s$ with $s \in \mathbb{Q}$. We have the strong triangle inequality

$$|x - y| \leq \max\{|x|, |y|\}$$

for $x, y \in \mathbb{C}_p$. If $x, y$ and $z$ are points of $\mathbb{C}_p$ with $|x - y| < |x - z|$, then $|x - z| = |y - z|$. For any $a \in \mathbb{C}_p$, and $r > 0$, we define
denote the closed disk of radius $r$ and centered at $a$. Both $D(a, r)^-$ and $D(a, r)$ are closed and open topologically, and every point in disk $D(a, r)^-$ is the center. This denotes that if $x \in D(a, r)^-$, then $D(a, r)^- = D(x, r)$ (resp. $D(a, r) = D(x, r)$). By the strong triangle inequality, if two disks $D_1$ and $D_2$ in $\mathbb{C}_p$ have non-empty intersection, then either $D_1 \subset D_2$, or $D_2 \subset D_1$.

For any $z, w \in \mathbb{P}^1(\mathbb{C}_p)$, we define the chordal distance

$$\rho_v(z, w) = \frac{|z - w|}{\max\{1, |z|\} \max\{1, |w|\}}$$

for $z, w \in \mathbb{C}_p$, and $\rho_v(z, w) = \frac{1}{\max\{1, |w|\}}$ for $w \in \mathbb{C}_p$ and $z = \infty$, and

$$\rho_v(z, w) = 0$$

for $z = w = \infty$.

It follows the definition of the chordal distance that if $|z| \leq 1, |w| \leq 1$, then $\rho_v(z, w) = |z - w|$, and if $|z| > 1, |w| \leq 1$, then by the strong triangle inequality, we have $|z - w| = |z|$, and hence $\rho_v(z, w) = \frac{|z - w|}{|z|} = 1$, and if $|z| > 1, |w| > 1$, then $\rho_v(z, w) = \frac{|z - w|}{|z||w|} = |\frac{1}{z} - \frac{1}{w}|$.

Lemma 2.1 ([2]). Let $d_0 > 1$ be an integer which is not divisible by $p$, and $d = d_0 p^t$ be a natural number. Let $\zeta$ be the primitive $d$-th root of unity. Then $|\zeta - 1| = 1$. 

Lemma 2.2 (\cite{2}). Let $\zeta$ be the primitive $p^d$-th root of unity. Then $|\zeta - 1| = p^{-d/(p-1)}$.

2.2. The Berkovich space. We shall use a few properties of about the structure of the Berkovich affine line and its topology. Here we give a brief introduction to the Berkovich space. More details can be found in \cite{7}.

The underlying point set for the Berkovich affine line $\mathbb{A}^1_{Ber}$ is the collection of all multiplicative seminorms $[\cdot]_x$ on the polynomial ring $\mathbb{C}_p[z]$ which extend the absolute value on $\mathbb{C}_p$. Recall that a multiplicative seminorm $p$ on the ring $\mathbb{C}_p[z]$ is a function $[\cdot]_x : \mathbb{C}_p[z] \to [0, +\infty)$.

- $[0]_x = 0, [1]_x = 1$;
- $[fg]_x = [f]_x [g]_x$ for all $f, g \in \mathbb{C}_p[z]$;
- $[f + g]_x \leq \max\{[f]_x, [g]_x\}$ for all $f, g \in \mathbb{C}_p[z]$.

It is a norm provided that $[f]_x = 0$ if and only if $f = 0$. The topology on $\mathbb{A}^1_{Ber}$ is the weakest one for which the mapping $x \mapsto [f]_x$ is continuous for all $f \in \mathbb{C}_p[z]$.

Recall the Berkovich’s classification Theorem: Every point $x \in \mathbb{A}^1_{Ber}$ can be viewed as a nested sequence of disks $D(a_1, r_1) \supset D(a_2, r_2) \supset \cdots$. Moreover, points in $\mathbb{A}^1_{Ber}$ can be divided into four types:

- A point in $\mathbb{A}^1_{Ber}$ corresponding to a nested sequence $\{D(a_i, r_i)\}$ of disks with nonempty intersection, for which $r = \lim r_i > 0$ belongs to the valuation group $|C^*|$ of $\mathbb{C}$, is said to be of type I.
- A point in $\mathbb{A}^1_{Ber}$ corresponding to a nested sequence $\{D(a_i, r_i)\}$ of disks with nonempty intersection, for which $r = \lim r_i > 0$ does not belong to the valuation group $|C^*|$ of $\mathbb{C}_p$, is said to be of type II.
- A point in $\mathbb{A}^1_{Ber}$ corresponding to a nested sequence $\{D(a_i, r_i)\}$ of disks with empty intersection is said to be of type IV.

We call points of type I, II, III the nonsingular points, and points of type IV the singular points. Every nonsingular point in $\mathbb{A}^1_{Ber}$ has a representation which is the intersection of the corresponding nested sequence of disks. So a nonsingular point in $\mathbb{A}^1_{Ber}$ can be identifying with a point $a$ (type I) or a disk $D(a, r)$ (type II, III).

We define a partial order on $\mathbb{A}^1_{Ber}$ as follows. For $x, y \in \mathbb{A}^1_{Ber}$, define $x \preceq y$ if and only if $[f]_x \leq [f]_y$ for all $f \in \mathbb{C}_p[z]$. If $x, y$ are two points in $\mathbb{A}^1_{Ber}$ identifying with disks $D(a, r)$ and $D(a', r')$ respectively, then $x \preceq y$ if and only if $D(a, r) \subset D(a', r')$.

For a point $x \in \mathbb{A}^1_{Ber}$, we denote the set of elements larger than $x$ by

$$[x, \infty] = \{ w \in \mathbb{A}^1_{Ber} \mid x \preceq w \}.$$  

Observe that $[x, \infty]$ is isomorphic, as an ordered set, to $[0, +\infty] \subset \mathbb{R}$.

Given two points $x, y$ in $\mathbb{A}^1_{Ber}$, we have that

$$[x, \infty] \cap [y, \infty] = [x \vee y, \infty],$$
where \( x \lor y \) is the smallest element larger than \( x \) and \( y \). If \( x \) is different from \( y \), then the element \( x \lor y \) is a point of type II. We also denote 
\[
[x, y] = \{ w \in A^1_{Ber} \mid x \leq w \leq x \lor y \} \cup \{ w \in A^1_{Ber} \mid y \leq w \leq x \lor y \}.
\]
The sets \([x, y], [x, y[\) and \([x, y[\) are defined in the obvious way.

For a set \( E \subset K \), denote \( \text{diam}(E) = \sup_{z, w \in E} |z - w| \) the diameter of \( E \) in the non-Archimedean metric. For \( x \in A^1_{Ber} \), which is corresponding to the nested sequence \( \{D(a_i, r_i)\} \) of disks, the diameter of \( x \) is given by
\[
\text{diam}(x) = \lim_{i \to \infty} \text{diam}(D(a_i, r_i)).
\]
For a nonsingular element \( x \in A^1_{Ber} \) identifying with the disk \( D(a, r) \), the diameter of \( x \) coincides with the diameter (radius \( r \)) of \( D(a, r) \).

In order to endow the Berkovich affine line with a topology, we define an open disk of \( A^1_{Ber} \) by
\[
D(a, r)^- = \{ x \in A^1_{Ber} \mid \text{diam}(a \vee x) < r \},
\]
for \( a \in K \) and \( r > 0 \). Similarly, a closed disk of \( A^1_{Ber} \) is defined by
\[
D(a, r) = \{ x \in A^1_{Ber} \mid \text{diam}(a \vee x) \leq r \}.
\]

Let \( \mathbb{P}^1(C_p) \) be the projective line over \( C_p \), which can be viewed as \( \mathbb{P}^1(C_p) = C_p \cup \{ \infty \} \). We can also introduce the Berkovich projective line \( \mathbb{P}^1_{Ber} \) over \( \mathbb{P}^1(K) \) similarly. In [7], Baker and Rumely pointed out that \( \mathbb{P}^1_{Ber} \) can be defined as \( A^1_{Ber} \cup \{ \infty \} \), where \( \infty \in \mathbb{P}^1(K) \) is regarded as a point of type I. \( \mathbb{P}^1_{Ber} \) can be also identifying with the disjoint union of a closed set \( \mathcal{X} \) which is homeomorphic to \( D(0, 1) \) and an open set \( \mathcal{Y} \) which is homeomorphic to \( D(0, 1)^- \). This provides a useful way to visualize \( \mathbb{P}^1_{Ber} \).

**Lemma 2.3** ([7]). The space \( \mathbb{P}^1_{Ber} \) is uniquely path-connected. More precisely, given any two distinct points \( x, y \in \mathbb{P}^1_{Ber} \), there is a unique arc \([x, y]\) in \( \mathbb{P}^1_{Ber} \) from \( x \) to \( y \), and if \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \), then the arc contains \( \zeta_{\text{Gauss}} \in A^1_{Ber} \), where \( \zeta_{\text{Gauss}} \) is identifying with the disk \( D(0, 1) \).

We say that a metric space \((X, d)\) is an \( \mathbb{R}\)-tree, if for any two points \( x, y \in X \), there is a unique arc from \( x \) to \( y \) and this arc is the geodesic segment.

**Lemma 2.4** ([7]). Let \( x, y \in D(0, 1) \). Then the metric \( d(x, y) = 2 \text{diam}(x \lor y) - \text{diam}(x) - \text{diam}(y) \) makes \( D(0, 1) \) into an \( \mathbb{R}\)-tree.

Therefore, we can define a metric on \( \mathbb{P}^1_{Ber} \) as \( d_p(x, y) = d(x, y) \) if \( x, y \in \mathcal{X} \) which can be identified with \( D(0, 1) \), and \( d_p(x, y) = d(x, y) \) if \( x, y \in \mathcal{Y} \) which can be identified with \( D(0, 1)^- \), and \( d_p(x, y) = d(x, \zeta_{\text{Gauss}}) + d(\zeta_{\text{Gauss}}, y) \) if \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \).

The Berkovich hyperbolic space \( \mathbb{H}_{Ber} \) is defined by
\[
\mathbb{H}_{Ber} = \mathbb{P}^1_{Ber} \setminus \mathbb{P}^1(C).
\]
Since $\infty \in \mathbb{P}^1_{\text{Ber}}$ is of type I, $\mathbb{H}_{\text{Ber}}$ can be also viewed as $\mathbb{A}^1_{\text{Ber}} \setminus \mathbb{C}$. Thus $\mathbb{H}_{\text{Ber}}$ has a tree structure induced by $\mathbb{A}^1_{\text{Ber}}$. Over $\mathbb{H}_{\text{Ber}}$ we can define the hyperbolic distance,

$$\rho(x, y) = 2 \log \operatorname{diam}(x \lor y) - \log \operatorname{diam}(x) - \log \operatorname{diam}(y), \quad x, y \in \mathbb{H}_{\text{Ber}}.$$ 

**Lemma 2.5** ([7]). $\mathbb{H}_{\text{Ber}}$ is a complete metric space under $\rho(x, y)$.

**Lemma 2.6** ([7]). Suppose that $w, y \in \mathbb{H}_{\text{Ber}}$. Then $\rho(x, y) = \rho(x, w) + \rho(w, y)$ if and only if $w$ belongs to $[x, y]$.

### 2.3. The action of a rational map $\phi$ over $\mathbb{P}^1_{\text{Ber}}$

Let $\phi \in \mathbb{C}_p(T)$ be a nonconstant rational function of degree $d \geq 1$. Since type I points are dense in $\mathbb{P}^1_{\text{Ber}}$, for any $x \in \mathbb{H}_{\text{Ber}}$, there exists a sequence $x_n$ tending to $x$ with respect to the Berkovich topology. We can define $\phi(z) = \lim_{n \to \infty} \phi(x_n)$.

If $d = 1$, $\phi$ has an algebraic inverse and thus induces an automorphism of $\mathbb{P}^1_{\text{Ber}}$. Define $\text{Aut}(\mathbb{P}^1_{\text{Ber}})$ to be the group of automorphisms of $\mathbb{P}^1_{\text{Ber}}$. The following lemmas can be found in [7].

**Lemma 2.7** ([7]). If $\phi(z) \in \mathbb{C}_p(z)$ is nonconstant, then $\phi : \mathbb{P}^1_{\text{Ber}} \to \mathbb{P}^1_{\text{Ber}}$ takes points of each type (I, II, III, IV) to points of the same type. Thus $\phi(z)$ has a given type if and only if $z$ does.

**Lemma 2.8** ([7]). Let $f(z) \in \mathbb{C}_p(z)$ be a nonconstant rational function, and suppose that $x \in \mathbb{A}^1_{\text{Ber}}$ is a point of type II, corresponding to a disc $D(a, r)$ in $\overline{\mathbb{C}}_p$ under Berkovich’s classification. Then $f(x)$ corresponds to the disc $D(b, R)$ if and only if there exist $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in \mathbb{C}_p$ for which $D(b, R) \setminus \cup_{i=1}^m D(b_i, R^-)$ is the image under $f(z)$ of $D(a, r) \setminus \cup_{i=1}^m D(a_i, r^-)$.

### 2.4. Reduction on rational function over $\mathbb{C}_p$

Let $\mathcal{O}_p = \{z \in \mathbb{C}_p||z| < 1\}$, $\mathcal{O}^* = \{z \in \mathbb{C}_p||z| = 1\}$, $\mathcal{M} = \{z||z| < 1\}$ and $k = \mathcal{O}_p / \mathcal{M}$. We also call $k$ the residue field of $\mathbb{C}_p$. If $x \in \mathcal{O}_p$, we denote the reduction of $x$ modulo $\mathcal{M}$ by $\bar{x}$. For any $z \in \mathbb{C}_p$, there exists a homogeneous coordinate $[x, y]$ for $z$, where $x, y \in \mathcal{O}$ with at least one in $\mathcal{O}^*$. Reduction induces a well-defined map $\mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(k)$ by $[x, y] = [\bar{x}, \bar{y}]$. Any rational function $f(z) \in \mathbb{C}_p(z)$ can be written in homogeneous coordinates as $f([x, y]) = [g(x, y), h(x, y)]$ where $g, h \in \mathcal{O}_p[x, y]$ are relatively prime homogeneous polynomials of degree $d = \deg(f)$. We can ensure that at least one coefficient of either $g$ or $h$ has valuation zero (i.e., absolute value 1). The reduction map induces a map $\mathcal{O}_p[x, y] \to k[x, y]$.

**Definition 2.9.** Let $f(z) \in \mathbb{C}_p(z)$ be a map with homogenous presentation $f([x, y]) = [g(x, y), h(x, y)]$, where $g, h \in \mathcal{O}_p[x, y]$ are relatively prime homogeneous polynomials of degree $d = \deg(f)$, and at least one coefficient of $g$ or $h$ has absolute value 1. We say that $f$ has good reduction if $g$ and $h$ have no common zeros in $k \times k$ besides $(x, y) = (0, 0)$.

If there is some linear fractional transformation $h \in \text{PSL}(2, \mathbb{C}_p)$ such that $h^{-1} \circ f \circ h$ has good reduction, we say that $f$ has potentially good reduction.
Lemma 2.10 ([7]). Let \( f \in \text{PSL}(2, \mathbb{C}_p) \) be a rational function of degree one. Then \( f \) has good reduction if and only if \( f \in \text{PSL}(2, \mathcal{O}) \).

Lemma 2.11 ([7]). Let \( f \in \mathbb{C}_p(z) \) be a non-constant rational function. Then \( f \) has good reduction if and only if \( f^{-1}(\zeta_{\text{Gauss}}) = \zeta_{\text{Gauss}} \).

3. The p-adic Möbius maps

We classify non-unit elements in \( \text{PSL}(2, \mathbb{C}_p) = \text{SL}(2, \mathbb{C}_p)/\{\pm I\} \). Since the product of all eigenvalues of \( g \in \text{PSL}(2, \mathbb{C}_p) \) is one, either the absolute value of each eigenvalue of \( g \) is one or there exists at least one eigenvalue whose absolute value is larger than 1. Thus each non-unit element \( g \in \text{PSL}(2, \mathbb{C}_p) \) falls into the following four classes:

(a) \( g \) is said to be **parabolic** if the absolute value of any eigenvalue of \( g \) is 1, and \( g \) can not be conjugated to a diagonal matrix.

(b) \( g \) is said to be **loxodromic** if there exists at least eigenvalue of \( g \) whose absolute value is larger than 1.

(c) \( g \neq I \) is said to be **elliptic** if the absolute value of any eigenvalue of \( g \) is 1, and \( g \) can be conjugated to a diagonal matrix.

In this paper, we classify the elliptic elements more precisely.

(d) \( g \) is said to be **tame elliptic** if the two eigenvalues \( \lambda_1, \lambda_2 \) of \( g \) satisfy \( |\lambda_1 - 1| = |\lambda_2 - 1| = 1 \).

(e) \( g \) is said to be **wild elliptic** if one of the eigenvalues of \( g \) lies in the disc \( D(1, 1)^- \).

For \( g = (a_{ij}) \) in the matrix ring \( M(m, \mathbb{C}_p) \), the norm of \( g \) is defined by \( \| g \| = \max_{1 \leq i \leq m, 1 \leq j \leq m} \{|a_{ij}|\} \). Obviously, \( \| g \| = 0 \) implies that each \( a_{ij} = 0 \).

It is easy to verify that \( \| \alpha g \| = |\alpha| \| g \| \), \( \| g + h \| \leq \max\{\| g \|, \| h \|\} \) and \( \| gh \| \leq \| g \| \| h \| \).

For any element \( g \in \text{PSL}(2, \mathbb{C}_p) \), there exist two lifts \( g_1, g_2 \) in \( \text{SL}(2, \mathbb{C}_p) \) with \( \| g_1 \| = \| g_2 \| \). We define \( \| g \| = \| g_1 \| = \| g_2 \| \). If \( g, h \in \text{PSL}(2, \mathbb{C}_p) \) correspond the lifts \( g_1, g_2 \in \text{SL}(2, \mathbb{C}_p) \) and \( h_1, h_2 \in \text{SL}(2, \mathbb{C}_p) \) respectively, then we define \( \| g - h \| = \inf_{1 \leq i \leq 2, 1 \leq j \leq 2} \| g_i - h_j \| \).

If \( d = 1 \), \( \phi \) has an algebraic inverse and thus induces an automorphism of \( \mathbb{P}^1_{\text{Ber}} \). Define \( \text{Aut}(\mathbb{P}^1_{\text{Ber}}) \) to be the group of automorphisms of \( \mathbb{P}^1_{\text{Ber}} \). The following lemmas can be found in [7].

Lemma 3.1 ([7]). The path distance metric \( \rho(x, y) \) on \( \mathbb{H}_{\text{Ber}} \) is independent of the choice of homogenous coordinates on \( \mathbb{P}^1_{\text{Ber}} \), in the sense that if \( h(z) \in \mathbb{C}_p(z) \) is a p-adic Möbius map, then \( \rho(h(x), h(y)) = \rho(x, y) \) for all \( x, y \in \mathbb{H}_{\text{Ber}} \).

Lemma 3.2 ([7]). Let \( w \) satisfy \( |w| = \lambda|w - a| \), where \( \lambda \in |\mathbb{C}_p^*| \), and \( w, a \in \mathbb{C}_p \).

1. If \( \lambda > 1 \), then \( w \in D(a, \frac{|w|}{\lambda}) \setminus D(a, \frac{|w|}{\lambda})^- \).
2. If \( \lambda = 1 \), then \( w \in \mathbb{P}^1(\mathbb{C}_p) \setminus D(0, |a|)^- \cup D(a, |a|)^- \).
3. If \( 0 \leq \lambda < 1 \), then \( w \in D(0, \lambda|a|) \setminus D(0, \lambda|a|)^- \).
Proposition 3.3. A p-adic Möbius map $g$ is

1. parabolic if it is conjugate to $z \to z + 1$;
2. elliptic if it is conjugate to $z \to kz$ for some $k$ with $|k| = 1$, $k \neq 1$;
3. loxodromic if it is conjugate to $z \to kz$ for some $k$ with $|k| > 1$.

Proposition 3.4. Let $f$ and $g$ be two p-adic Möbius maps neither of which is the identity. Then $f$ and $g$ are conjugate if and only if $\text{trace} (f) = \text{trace} (g)$.

Proposition 3.5. Let $g \neq 1$ be a p-adic Möbius map. Then

1. $g$ is parabolic if and only if $(a + d)^2 - 4 = 0$;
2. $g$ is elliptic if and only if $0 < |(a + d)^2 - 4| \leq 1$;
3. $g$ is loxodromic if and only if $|(a + d)^2 - 4| > 1$.

Proof. Since the trace of the matrix is invariant under the conjugation, without lose of generality, we can assume that $g(z) = \frac{az + b}{cz + d}$, if $g$ is elliptic or loxodromic which yields that $\text{tr} (g) = a + d = \lambda + 1/\lambda$, where $\lambda$ is the eigenvalue of $g$. Thus $(a + d)^2 - 4 = (\lambda - 1/\lambda)^2$. By the non-archimedean property of the metric, we have $|\lambda - 1/\lambda| > 1$, if $g$ is loxodromic, and $0 < |\lambda - 1/\lambda| < 1$, if $g$ is elliptic. If $g$ is parabolic, we can assume that $g(z) = z + 1$ which yields that $a + d = 2$.

Conversely, if $(a + d)^2 - 4 = 0$, then $\lambda - 1/\lambda = 0$ which denotes $\lambda^2 = 1$. Since $g \in \text{PSL}(2, \mathbb{C}_p)$, we have $\lambda = 1$ which yields that $g$ is parabolic. If $|(a + d)^2 - 4| > 1$, then $|\lambda - 1/\lambda| > 1$ which denotes that $|\lambda| > 1$ or $|1/\lambda| > 1$. Thus $g$ is loxodromic. If $0 < |(a + d)^2 - 4| < 1$, then $0 < |\lambda - 1/\lambda| < 1$ which yields $|\lambda| < 1$. Thus $g$ is elliptic.

4. The properties of $\text{PSL}(2, \mathbb{O}_p)$

Lemma 4.1. If $f \in \text{PSL}(2, \mathbb{O}_p)$, we have $\rho_v (f(z), f(w)) = \rho_v (z, w)$.

Lemma 4.2. Let $g$ be any p-adic Möbius map. Then

$$\rho (g(\zeta_{\text{Gauss}}), \zeta_{\text{Gauss}}) = 2 \log_p |g|.$$  

Lemma 4.3. Let $g$ be any p-adic Möbius map. Then the best Lipschitz constant (relative the chordal metric) for $g$ is given by $L(g) = p^{\rho (g(\zeta_{\text{Gauss}}), g(\zeta_{\text{Gauss}}))}$, namely $\rho_v (g(z), g(w)) \leq L(g) \rho_v (z, w)$. Furthermore, there exist at least two points $z, w \in \mathbb{P}^1 (\mathbb{C}_p)$ such that $\rho_v (g(z), g(w)) = L(g) \rho_v (z, w)$.

Proof. We can assume that $g(z) = \frac{az + b}{cz + d}$. The element $g$ has at least one fixed point $a_g$. If $|a_g| \leq 1$, let $h(z) = z - a_g$ and $\iota (z) = 1/z$, and then $\iota g h^{-1} l^{-1}$ fixes $\infty$. Since $h$ and $l$ fix the point $\zeta_{\text{Gauss}}$, we have

$$\rho (\iota g h^{-1} l^{-1} (\zeta_{\text{Gauss}}), \zeta_{\text{Gauss}}) = \rho (\iota g (\zeta_{\text{Gauss}}), \zeta_{\text{Gauss}}) = \rho (g(\zeta_{\text{Gauss}}), \zeta_{\text{Gauss}})$$

which yields that $L (g) = L (\iota g h^{-1} l^{-1})$. If $|a_g| > 1$, let $h(z) = z - 1/a_g$ and $\iota (z) = 1/z$. Thus $\iota g h^{-1} l^{-1} \iota$ fixes $\infty$. Similarly, since $h$ and $\iota$ fix the point $\zeta_{\text{Gauss}}$, we have

$$\rho (\iota g h^{-1} l^{-1} (\zeta_{\text{Gauss}}), \zeta_{\text{Gauss}}) = \rho (\iota g (\zeta_{\text{Gauss}}), \zeta_{\text{Gauss}}) = \rho (g(\zeta_{\text{Gauss}}), \zeta_{\text{Gauss}})$$

□
which yields that \( L(g) = L(thg^{-1}, t^{-1}) \). Therefore, without loss of generality, we can assume that \( g(z) = \frac{az + b}{d} \) with \( ad = 1 \). If \(|a| > 1\), we consider the inverse \( g^{-1}(z) = \frac{d}{a}z - \frac{b}{a} \), since \( L(g) = L(g^{-1}) \). Thus we can assume that \(|a| \leq 1\).

For any \( z, w \in \mathbb{P}^1(\mathbb{C}_p) \), we have

\[
\rho_v(g(z), g(w)) = \frac{|g(z) - g(w)|}{\max\{1, |g(z)|\} \max\{1, |g(w)|\}}
\]

\[
= \frac{|z - w|}{\max\{|az + b|, |d|\} \max\{|aw + b|, |d|\}}
\]

\[
= \frac{|z - w|}{\max\{1, |z|\} \max\{1, |w|\}} \frac{\max\{1, |z|\} \max\{1, |w|\}}{\max\{|az + b|, |d|\} \max\{|aw + b|, |d|\}}
\]

\[
= \rho_v(z, w) \frac{\max\{1, |z|\} \max\{1, |w|\}}{\max\{|az + b|, |d|\} \max\{|aw + b|, |d|\}}
\]

\[
= \rho_v(z, w) \frac{|a|^2 \max\{1, |z|\} \max\{1, |w|\}}{\max\{|a^2z + ab|, 1\} \max\{|a^2w + ab|, 1\}}.
\]

When \(|a| = 1\), let \( D_1 = D(-\frac{b}{a}, 1) \) and \( D_2 = D(0, 1) \). If \( D_1 = D_2 \), then \(|\frac{b}{a}| \leq 1\) which yields that \(|b| \leq |a| = 1\), namely \( L(g) = 1 \). If \( z, w \in D_1 = D_2 \), then

\[
\frac{|a|^2 \max\{1, |z|\} \max\{1, |w|\}}{\max\{|a^2z + ab|, 1\} \max\{|a^2w + ab|, 1\}} = 1,
\]

namely \( \rho_v(g(z), g(w)) = L(g)\rho_v(z, w) \). If \( z \in D_1 = D_2 \) and \( w / D_1 = D_2 \),

\[
\frac{|a|^2 \max\{1, |z|\} \max\{1, |w|\}}{\max\{|a^2z + ab|, 1\} \max\{|a^2w + ab|, 1\}} = \frac{|w|}{|w + b/a|}.
\]

Since \(|\frac{b}{a}| \leq 1\), we have \(|w| = |w + \frac{b}{a}|\) which yields that \( \frac{|w|}{|w + b/a|} = 1 \). Thus \( \rho_v(g(z), g(w)) = L(g)\rho_v(z, w) \). If \( z, w \in D_1 = D_2 \), we have

\[
\frac{|a|^2 \max\{1, |z|\} \max\{1, |w|\}}{\max\{|a^2z + ab|, 1\} \max\{|a^2w + ab|, 1\}} = \frac{|w|}{|z + b/a||w + b/a|}.
\]

Since \(|\frac{b}{a}| \leq 1\), we have \(|w| = |w + \frac{b}{a}|\) and \(|z| = |z + \frac{b}{a}|\) which yields that

\[
\frac{|w|}{|z + b/a||w + b/a|} = 1.
\]

Thus \( \rho_v(g(z), g(w)) = L(g)\rho_v(z, w) \). If \( D_1 \cap D_2 = \emptyset \), then \(|\frac{b}{a}| > 1\) which yields that \(|b| > |a| = 1\), namely \( L(g) = |b|^2 \). If \( z, w \in D_1 \), then \(|z| > 1\), \(|w| > 1\), \(|z + b/a| \leq 1\), \(|w + b/a| \leq 1\) which yields that

\[
\frac{|a|^2 \max\{1, |z|\} \max\{1, |w|\}}{\max\{|a^2z + ab|, 1\} \max\{|a^2w + ab|, 1\}} = \frac{|b|^2}{|a|^2} = |b|^2.
\]
Thus \( \rho_v(g(z), g(w)) = L(g)\rho_v(z, w) \). If \( z, w \in D_2 \), then \( |z| \leq 1, |w| \leq 1, |z + b/a| < 1, |w + b/a| < 1 \) which yields that

\[
\frac{|a|^2 \max\{|1, |z|\} \max\{|1, |w|\}}{\max\{|a^2 z + ab, 1\} \max\{|a^2 w + ab, 1\}} = \frac{1}{|z + b/a||w + b/a|} = \left| \frac{a}{b} \right|^2 = |b|^{-2} \leq L(g).
\]

Thus \( \rho_v(g(z), g(w)) \leq L(g)\rho_v(z, w) \). If \( z \in D_1 \) and \( w \in D_2 \), then \( |z| > 1, |w| \leq 1, |z + b/a| \leq 1, |w + b/a| > 1 \) which yields that

\[
\frac{|a|^2 \max\{|1, |z|\} \max\{|1, |w|\}}{\max\{|a^2 z + ab, 1\} \max\{|a^2 w + ab, 1\}} = \frac{|z|}{|w + b/a|} = 1 \leq L(g).
\]

If \( z \notin D_1 \cup D_2 \), then \( |z| > 1, |z + b/a| > 1 \) which yields that

\[
\frac{\max\{|1, |z|\}}{\max\{|a^2 z + ab, 1\}} = \frac{|z|}{|z + b/a|} = 1.
\]

Thus if \( z, w \notin D_1 \cup D_2 \), then \( \rho_v(g(z), g(w)) = \rho_v(z, w) \leq L(g)\rho_v(z, w) \). If \( z \notin D_1 \cup D_2, w \in D_1 \), then \( \rho_v(g(z), g(w)) = \rho_v(z, w)|w| = |b|\rho_v(z, w) \leq L(g)\rho_v(z, w) \). If \( z \notin D_1 \cup D_2, w \in D_2 \), then \( \rho_v(g(z), g(w)) = \rho_v(z, w)/|w + b/a| = |a/b|\rho_v(z, w) \leq L(g)\rho_v(z, w) \). Therefore, \( \rho_v(g(z), g(w)) \leq L(g)\rho_v(z, w) \).

When \( |a| < 1 \), let \( D_1 = D\left(-\frac{b}{a}, |a|^{-1}\right) \) and \( D_2 = D(0, 1) \). If \( D_2 \subset D_1 \), then \( |\frac{b}{a}| \leq |a|^{-1} \) which yields that \( L(g) = |d|^2 = |a|^{-2} \). If \( z, w \in D_1 \), then

\[
\frac{|a|^2 \max\{|1, |z|\} \max\{|1, |w|\}}{\max\{|a^2 z + ab, 1\} \max\{|a^2 w + ab, 1\}} = |a|^2 \max\{|1, |z|\} \max\{|1, |w|\} \leq \frac{1}{|a|^2} = L(g),
\]

namely \( \rho_v(g(z), g(w)) \leq L(g)\rho_v(z, w) \). If \( z, w \notin D_1 \), then

\[
\frac{|a|^2 \max\{|1, |z|\} \max\{|1, |w|\}}{\max\{|a^2 z + ab, 1\} \max\{|a^2 w + ab, 1\}} = \frac{|a|^2 |z||w|}{|a^4||z||w|} = |a|^{-2}
\]

which yields that \( \rho_v(g(z), g(w)) = L(g)\rho_v(z, w) \). If \( z \in D_1, w \notin D_1 \), then

\[
\frac{|a|^2 \max\{|1, |z|\} \max\{|1, |w|\}}{\max\{|a^2 z + ab, 1\} \max\{|a^2 w + ab, 1\}} = \frac{|a|^2 |w| \max\{|1, |z|\}}{|a^2 |w|} = \max\{|1, |z|\} \leq \frac{1}{|a|^2} = L(g).
\]

Thus \( \rho_v(g(z), g(w)) \leq L(g)\rho_v(z, w) \). If \( D_1 \cap D_2 = \emptyset \), then \( |\frac{b}{a}| > |a|^{-1} \) which yields that \( |b| > \frac{1}{|a|} \). Thus \( L(g) = |b|^2 \). If \( z \in D_2 \), then

\[
\frac{\max\{|1, |z|\}}{\max\{|1, |z|\}} \leq 1.
\]

If \( z \in D_1 \), then

\[
\frac{\max\{|1, |z|\}}{\max\{|1, |z|\}} = \frac{|b|}{|a|}.
\]

If \( z \notin D_1 \cup D_2 \), then

\[
\frac{\max\{|1, |z|\}}{\max\{|a^2 z + ab|\}} \leq \frac{|z|}{a^2 z + ab}.
\]
Thus if $z, w \in D_1$, then
\[
\frac{|a|^2 \max\{1, |z|\} \max\{1, |w|\}}{\max\{1, |a^2 z + ab|\} \max\{1, |a^2 w + ab|\}} = |b|^2
\]
which yields that $\rho_v(g(z), g(w)) = L(g)\rho_v(z, w)$. If $z, w \in D_2$, then
\[
\frac{|a|^2 \max\{1, |z|\} \max\{1, |w|\}}{\max\{1, |a^2 z + ab|\} \max\{1, |a^2 w + ab|\}} \leq |a|^2 \leq 1 \leq L(g).
\]
Thus $\rho_v(g(z), g(w)) \leq L(g)\rho_v(z, w)$. If $z_1 \in D_1, w \in D_2$, then
\[
\frac{|a|^2 \max\{1, |z|\} \max\{1, |w|\}}{\max\{1, |a^2 z + ab|\} \max\{1, |a^2 w + ab|\}} \leq |a||b| \leq L(g).
\]
Thus $\rho_v(g(z), g(w)) \leq L(g)\rho_v(z, w)$. If $z, w \notin D_1 \cup D_2$, then
\[
\frac{|a|^2 \max\{1, |z|\} \max\{1, |w|\}}{\max\{1, |a^2 z + ab|\} \max\{1, |a^2 w + ab|\}} = \frac{|a|^2 |z||w|}{|a^2 z + ab||a^2 w + ab|}.
\]
If $|z| = |z + \frac{b}{a}|$, then $\frac{|z|}{|z + \frac{b}{a}|} = 1$. If $|z| < |z + \frac{b}{a}|$, then $\frac{|z|}{|z + \frac{b}{a}|} < 1$. If $|z| > |z + \frac{b}{a}|$, then $|z| = |b/a|$. Therefore, we have
\[
\frac{|a|^2 |z||w|}{|a^2 z + ab||a^2 w + ab|} \leq |b|^2.
\]
Thus $\rho_v(g(z), g(w)) \leq L(g)\rho_v(z, w)$.

\hspace{1cm} \square

**Remark 4.4.** In [33], the Lipschitz constant of the rational map with respect to the chordal metric is derived by the resultant (see definition below). Let $\mathcal{O}_p^* = \{\alpha \in \mathbb{C}_p : |\alpha| = 1\}$. Let $g(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$ and $h(z) = b_0 z^m + b_1 z^{m-1} + \ldots + b_{m-1} z + b_m$. The rational function $\phi(z) = \frac{g(z)}{h(z)} \in \mathbb{C}_p$ is called a normalized form if all the coefficients of $g(z)$ and $h(z)$ are in $\mathcal{O}_p$ and at least one coefficient of $g(z)$ and $h(z)$ are in $\mathcal{O}_p^*$. Let $\alpha_i, 1 \leq i \leq n$ be the roots of $g(z)$, and $\beta_j, 1 \leq j \leq m$ be the roots of $h(z)$. Let
\[
\text{Res}(g(z), h(z)) = a_0 b_m \prod_{i=1}^n \prod_{j=1}^m (\alpha_i - \beta_j)
\]
the resultant of two polynomials, and $\text{Res}(\phi) = \text{Res}(g(z), h(z))$ be the resultant of the rational function $\phi$. The absolute value of the resultant $\text{Res}(\phi)$ depends only on the map $\phi$.

**Lemma 4.5 (33).** Let $\phi : \mathbb{P}^1(\mathbb{C}_p) \longrightarrow \mathbb{P}^1(\mathbb{C}_p)$ be a rational map. Then $\rho_v(\phi(z), \phi(w)) \leq |\text{Res}(\phi)|^{-2} \rho_v(z, w)$ for all $v, w \in \mathbb{P}^1(\mathbb{C}_p)$.

Let $g(z) = \frac{az + b}{cz + d} \in \text{PSL}(2, \mathbb{C}_p)$. By Lemma 4.5 we have
\[
\text{Res}(g(z)) = \text{Res}((a/t) z + b/t, (c/t) z + d/t) = 1/\|g\|^2 = L(g),
\]
where $t = \max\{|a|, |b|, |c|, |d|\} = \|g\|$.

In [33], it is shown that there are two points $x, y \in \mathbb{P}^1(\mathbb{C}_p)$ such that
\[
\sup_{x \neq y} \frac{\rho_v(\phi(x), \phi(y))}{\rho_v(x, y)} = |\text{Res}(\phi)|^{-2}. \text{ Since } \mathbb{P}^1(\mathbb{C}_p) \text{ is not compact, the supreme can not be omitted in general cases. However, we get the Lipschitz}
constant by other method and show that we can get the supreme when it is the $p$-adic Möbius map.

Proof of Theorem 1.1

**Proof.** Since $\rho(g(\zeta_{Gauss}),\zeta_{Gauss}) = 2\log_p \| g \|$, $\| g \| = 1$ is equivalent to $\rho(g(\zeta_{Gauss}),\zeta_{Gauss}) = 0$ which yields that $L(g) = 1$. The converse is also true. This means that (1), (2), (3) are equivalent.

(2) $\Rightarrow$ (4) If $L(g) = 1$, $L(g^{-1}) = 1$. Hence by Lemma 4.3 we have $\rho_v(z,w) = \rho_v(g^{-1}(g(z)),g^{-1}(g(w))) \leq L(g^{-1})\rho_v(g(z),g(w)) \leq L(g)\rho_v(z,w)$ which yields that $\rho_v(g(z),g(w)) = \rho_v(z,w)$.

(4) $\Rightarrow$ (2) If $g$ is a chordal isometry, then $\rho_v(g(z),g(w)) = \rho_v(z,w)$.

By Lemma 4.3 we have $L(g) = 1$.

(1) $\Rightarrow$ (5) Since $\| g \| = 1$, let $g = \frac{a2+b}{cz+d}$, we have $\max\{|a|,|b|,|c|,|d|\} = 1$ which yields that $g \in \mathrm{PSL}(2,\mathcal{O}_p)$.

(5) $\Rightarrow$ (1) If $g \in \mathrm{PSL}(2,\mathcal{O}_p)$, let $g = \frac{a2+b}{cz+d}$, then $\max\{|a|,|b|,|c|,|d|\} = 1$ and $ad-bc = 1$. If $\max\{|a|,|b|,|c|,|d|\} < 1$, then $|ad-bc| < \max\{|ad|,|bc|\} < 1$. This is a contradiction. Hence $\max\{|a|,|b|,|c|,|d|\} = 1$ which yields that $\| g \| = 1$.

(1) $\Rightarrow$ (6) Since $\| g \| = 1$, $\| g^{-1} \| = 1$. Since $\| gh \| \leq \| g \| \| h \|$, and $\| h \| = \| g^{-1}gh \| \leq \| g^{-1} \| \| gh \|$, we have $\| gh \| = \| h \|$. Similarly, $\| hg \| = \| h \|$. We can rewrite $h$ as $hg^{-1}$, and then we have $\| ghg^{-1} \| = \| hg^{-1} \| = \| h \|$, since $g \in \mathrm{PSL}(2,\mathcal{O}_p)$.

(6) $\Rightarrow$ (1) Let $h \in \mathrm{PSL}(2,\mathcal{O}_p)$. Then $\| h \| = \| gh \| = \| g \|$ which yields that $\| g \| = 1$.

The metric properties of $\mathrm{PSL}(2,\mathcal{O}_p)$ can be used to study the $p$–adic continued fractions. An infinite $p$–adic continued fraction is a formal expression

$$a_1 + \frac{a_2}{b_1 + \frac{a_3}{b_2 + \cdots}}$$

where $a_i, b_i \in \mathbb{C}_p$ and $a_i \neq 0$. We denote this continued fraction by $\mathbb{K}(a_0,b_n)$. Let $t_n = \frac{a_n}{z+b_n}$ and $T_n = t_1 \circ t_2 \circ t_3 \circ \cdots$ for $n = 1,2,3,\ldots$. The continued fraction is said to be convergent classically if the sequence $\{T_n\}$ converges, else it is said to diverge classically. In the following part, we study the simplest case $\mathbb{K}(1,b_i)$ with $|b_i| \leq 1$. Since $T_{n+1}(\infty) = T_n(0)$, by Theorem 1.1 if the sequence $\{T_n(0)\}$ converges, then $0 = \lim_{n \to \infty} \rho_v(T_n(0),T_n(\infty)) = \rho_v(0,\infty) = 1$. This is a contradiction. This implies that the continued fraction $\mathbb{K}(1,b_i)$ with $|b_i| \leq 1$ diverge classically.
Example 4.6.
\[
\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}
\]
does not converge classically.

This example shows that the convergence and divergence of \(p\)-adic continued fractions are different from those in complex settings.

5. Reduction and \(p\)-adic Möbius maps

We call an elliptic element \(f \in \text{PSL}(2, \mathbb{C}_p)\) of order 2 an involution. If \(g \in \text{PSL}(2, \mathbb{C}_p)\) is a loxodromic element or an elliptic element, \(g\) have two fixed points \(a_g\) and \(r_g\) in \(\mathbb{P}^1(\mathbb{C}_p)\). We call the geodesic line \(A_g\) which connects \(a_g\) and \(r_g\) the axis of \(g\). Let \(A\) be a geodesic line in \(\mathbb{P}^1_{\text{Ber}}\). A geodesic line \(B\) is orthogonal to \(A\) if there exists a \(p\)-adic Möbius transformation \(f\) which is an involution such that endpoints of \(B\) are two fixed points of \(f\), and \(f\) interchanges endpoints of \(A\).

We introduce a new conception, a tailed geodesic line, in order to analyze the geometrical characterization of \(2\)-adic Möbius maps. Let \(A\) be a geodesic line and \(x\) be the point satisfying \(\inf_{y \in A} \rho(x, y) = \log_p 2\). Choosing any point \(y \in A\), \(A_x = A \cup [x, y]\). Since the Berkovich space is a \(\mathbb{R}\)-tree, it is obviously that \(A_x\) is independent of the chosen point \(y\), \(A\) is the geodesic line associated with \(A_x\), and \(x\) is called a tail. If \(A\) is the axis of an involution \(f\), then there exists a unique point \(x\) fixed by \(f\) such that \(A_x\) is a tailed geodesic line. Hence we call \(A_x\) the tailed axis of \(f\).

Lemma 5.1. Let \(g\) be a \(p\)-adic Möbius map. Then there exist two involution \(f, h \in \text{PSL}(2, \mathbb{C}_p)\) such that \(g = f \circ h\). Furthermore

1. The axes \(A\) of \(h\) and \(B\) of \(f\) are orthogonal to the axis \(A_g\) of \(g\).
2. The endpoints of \(A\) are different from the endpoints of \(B\).
3. The element \(g\) is parabolic if and only if \(A\) and \(B\) share the unique endpoint.

When \(p \geq 3\),
4. the element \(g\) is elliptic if and only if \(A \cap B \neq \emptyset\);
5. the element \(g\) is loxodromic if and only if \(A \cap B = \emptyset\).

When \(p = 2\),
6. the element \(g\) is elliptic if and only if the two tailed geodesic lines \(A_x \cap B_y \neq \emptyset\);
7. the element \(g\) is loxodromic if and only if the two tailed axes \(A_x \cap B_y = \emptyset\).

Proof. If \(g\) is loxodromic or elliptic, without loss of generality, let \(g = \lambda^2 z\). The axis \(A_g\) is the geodesic line connecting 0 and \(\infty\). Let \(h(z) = -\frac{\lambda^2}{z}\) and \(f(z) = g \circ h^{-1}(z) = -\frac{\lambda^2 \beta^2}{z}\). Thus \(h\) has two fixed points \(\beta k, -\beta k\) and \(f\) has two fixed points \(\lambda \beta k, -\lambda \beta k\), where \(k^2 = -1\). It is easy to see that \(f\) and
Proposition 5.2. Let \( g \in \text{PSL}(2, \mathbb{C}_p) \).

1. If \( g \) is a loxodromic element, then the set of fixed points of \( g \) contains two points in \( \mathbb{P}^1(\mathbb{C}_p) \).

2. If \( g \) is a tame elliptic element, then the set of fixed points of \( g \) is a geodesic line in \( \mathbb{P}^1(\mathbb{C}_p) \), and \( F_g \cap \mathbb{P}^1(\mathbb{C}_p) \) contains two points.

3. Let \( g \) be a wild elliptic element. Then the interior of the set of the fixed points of \( g \) contains a geodesic line in \( \mathbb{P}^1(\mathbb{C}_p) \).

4. If \( g \) is a parabolic element, then the fixed points of \( g \) is an open disc with its boundary with respect to the Berkovich topology.

Proof. If \( g \) is loxodromic or elliptic, we can assume that \( g(z) = \lambda^2 z \) with fixed points 0, \( \infty \). If \( |\lambda| > 1 \) or \( |\lambda| < 1 \), then by Lemma 4.2 we know that \( g \) can...
Lemma 5.3. If a geodesic line $A$ is orthogonal to the other geodesic line $B$, then $B$ is also orthogonal to $A$.

Proof. Without loss of generality, we can assume that $A$ is a geodesic line with endpoints 0 and $\infty$ and $B$ is a geodesic line with endpoints $-1$ and 1. Then $f(A) = B, f(B) = A$, if $f(z) = \frac{z+1}{z-1}$. □

Lemma 5.4. Let a geodesic line $A$ be orthogonal to the other geodesic line $B$.

(1) If $p \geq 3$, then $A$ intersects $B$ at one unique point.
(2) If $p = 2$, then $A \cap B = \emptyset$.

Proof. Without loss of generality, we may assume that $A$ is the geodesic connecting 0 and $\infty$ and $B$ is a geodesic line with endpoints $-\alpha, \alpha$. Let the point $x$ correspond to the disc $D(\alpha, |2\alpha|)$ which lies on the geodesic line $B$ and contains the points $-\alpha, \alpha$.

If $p \geq 3$, then $|2\alpha| = |\alpha|$ which implies that $D(\alpha, |2\alpha|)$ contains 0. Hence $B$ intersects $A$. Conversely, if $B$ intersects $A$, then there exists a point $x$ corresponding to a disc $D(0, r)$ containing $\alpha$ or $-\alpha$ which implies that $|\alpha| \leq r$. Hence $D(0, r)$ contains both $\alpha$ and $-\alpha$. If there exist two points $x_1, x_2 \in A \cap B$ corresponding to two discs $D(0, r_1)$ and $D(0, r_2)$ respectively, then either $D(0, r_1) \subset D(0, r_2)$ or $D(0, r_2) \subset D(0, r_1)$. Without loss of generality, we can assume that $D(0, r_1) \subset D(0, r_2)$. Let $l_1$ be the geodesic segment connecting $\alpha$ and $x_2$, and $l_2$ be the geodesic segment connecting $-\alpha$ and $x_2$. Hence $l_1 \cup l_2 \subset B$, but $l_1 \cap l_2$ contains a segment containing $x_1$ and $x_2$. This is a contradiction. Hence $A$ intersects $B$ at a uniquely point.

If $p = 2$, then for any $x$ lying on the geodesic line connecting $\alpha, -\alpha$, we have that the disc corresponding to $x$ must contain $\alpha$ or $-\alpha$. Without loss of generality, let $x$ correspond to the disc $D(\alpha, r)$. If $x$ lies on $A$, then $D(\alpha, r)$ contains 0 which implies that $|\alpha| \leq r$. Hence $-\alpha \in D(0, r)$. Since $\zeta$ corresponds to the disc $D(\alpha, |2\alpha|)$ containing both $\alpha$ and $-\alpha$, the geodesic
line contains the segment which connecting ζ and x. This is a contradiction. Hence \( A \cap B = \emptyset \).

\[ \square \]

We say that \( g \) keeps a set \( A \) invariant if \( g(A) = g^{-1}(A) = A \).

**Lemma 5.5.** Let \( A_g \) be the axis of \( g \). If \( g \) is a loxodromic element or an elliptic element, then \( g \) keeps the axis \( A_g \) invariant. Furthermore, \( g \) fixes every point of the axis \( A_g \) if and only if \( g \) is an elliptic element.

**Proof.** Without loss of generality, let \( g = \lambda z \). Hence \( A_g \) is the geodesic line connecting \( 0, \infty \). If \(|\lambda| > 1\), then \( g \) maps each disk \( D(0, r) \) to \( D(0, |\lambda|r) \) which is also on the geodesic line. If \(|\lambda| = 1\), then \( g \) maps each disk \( D(0, r) \) to \( D(0, |\lambda|r) \) which is the disk \( D(0, r) \), namely \( g \) fixes the point \( \zeta_{0,r} \).

\[ \square \]

**Lemma 5.6.** If \( p \geq 3 \), and \( g = h \circ f \) is a tame elliptic element, where \( h, f \) are two involutions, then two axes of \( h \) and \( f \) only intersect at a unique point.

**Proof.** Without loss of generality, we can assume that \( g(z) = \lambda^2(z) \). Thus \( f(z) = -\frac{\lambda^2}{z} \) and \( h(z) = -\frac{\lambda^2}{z} \). By Lemma 2.7, we have \(|\lambda b - b| = |\lambda - 1||b| = |b| \). This implies that two axes of \( h \) and \( f \) only intersect at the point \( \zeta_{0,|b|} \) which corresponds to the disc \( D(0, |b|) \).

\[ \square \]

**Lemma 5.7.** If \( p \geq 3 \), and \( g = h \circ f \) is a wild elliptic element, where \( h, f \) are two involutions, then two axes of \( h \) and \( f \) only intersect on a segment, and this segment belongs to the fixed points of \( g \).

**Proof.** Without loss of generality, we can assume that \( g(z) = \lambda^2 z \), \( f(z) = -\frac{\lambda}{z} \) and \( h(z) = -\frac{\lambda}{z} \). Let \( A \) be the axis of \( f(z) \) and \( B \) be the axis of \( h(z) \). Hence the endpoints of \( A \) are \( \{-1, 1\} \) and the endpoints of \( B \) are \( \{-\sqrt{\lambda}, \sqrt{\lambda}\} \).

Since \( p \geq 3 \), then \( 1 = |\lambda - 1| = |\sqrt{\lambda} - \sqrt{\lambda}| \) which implies that \( \zeta_{\text{Gauss}} \) lies on both the axes \( A \) and \( B \). Since \( |\lambda - 1| < 1 \) and \( |\lambda + 1| < 1 \), we have \( \min\{|\sqrt{\lambda} - 1|, |\sqrt{\lambda} + 1|\} < 1 \) which implies that there exists a point \( x \in A \cap B \cap \mathbb{H}_{\text{Ber}} \) which corresponds to the disc \( D(1, |\sqrt{\lambda} - 1|) \) or the disc \( D(1, |\sqrt{\lambda} + 1|) \). Hence \( A \cap B \) contains either the segment connecting \( \zeta_{1,|\sqrt{\lambda} - 1|} \) and \( \zeta_{0,1} \) or the segment connecting \( \zeta_{1,|\sqrt{\lambda} + 1|} \) and \( \zeta_{0,1} \). This segment belongs to the fixed points of fixed points of \( f \).

\[ \square \]

**Lemma 5.8.** If \( p \geq 3 \), and \( A \) and \( B \) are two geodesic lines with four distinct endpoints, then there exists a unique geodesic line which is orthogonal to \( A \) and \( B \) simultaneously.

**Proof.** Without loss of generality, we can assume that \( A \) is the geodesic with endpoints \( 0 \) and \( \infty \), and \( B \) is the other geodesic line with endpoints \( a \) and \( b \). If \( A \) does not intersect \( B \), then we have \(|a - b| < \max\{|a|, |b|\} \). By the ultrametric property, we have \(|a| = |b| > |a - b| \). Let \( C \) be a geodesic
line with endpoints $\zeta, -\zeta$. By Theorem 5.11, we have that the geodesic line $C$ is orthogonal to the geodesic line $A$. Let $g = \frac{a+b}{c+d}$. Then the geodesic line $B$ is mapped to the geodesic line $A$ by $g$. If the geodesic $g(C)$ is also orthogonal to $A$, then $g(-\zeta) + g(\zeta) = 0$ and $g(C) \cap A \neq \emptyset$. This implies that $\frac{\zeta-a}{c-b} + \frac{-\zeta-b}{d-a} = 0$, namely $\zeta = \sqrt{ab}$. Then the geodesic line $C$ connecting $-\sqrt{ab}, \sqrt{ab}$ is orthogonal to both $A$ and $B$ simultaneously.

$\square$

**Lemma 5.9.** If $p = 2$, and $A_x$ and $B_y$ are two tailed geodesic lines with four distinct endpoints, then there exists a unique geodesic line which is orthogonal to $A$ and $B$ simultaneously.

**Proof.** Without loss of generality, we can assume that the endpoints of $A$ are $-1, 1$, and the endpoints of $B$ are $t, s$. We claim that we can find a $p$-adic Möbius map $f = \frac{az+b}{cz+d}$ such that $f(-1) = -1$, $f(-1) = -1$, $f(t) + f(s) = 0$.

Since $f(-1) = -1$, $f(1) = -1$, we have $a = d, b = c$, and $\frac{at+b}{ct+d} + \frac{as+b}{cs+d} = 0$ which yields that $2abst + (a^2 + b^2)(s + t) + 2ba = 0$. We can lift the solution to the projective space, namely $2ABst + (A^2 + B^2)(s + t) + AB = 0$, and $A^2 - B^2 = C^2$. Since any two curves in the projective space intersect, we have solutions in the projective space. If the solution is $(A : B : 0)$, namely $C = 0$, then $A = B$ or $A = -B$. This implies that $st + s + t + 1 = 0$ or $st - (s + t) + 1 = 0$, namely $s = -1$ or $t = -1$ or $s + 1 = 1$ or $t = 1$. This contradicts that two tailed geodesic line have no common endpoints. Hence $C \neq 0$, namely there exists $p$-adic Möbius map $f$ such that $f(-1) = -1$, $f(-1) = -1$, $f(t) + f(s) = 0$.

Hence we can assume that the tailed geodesic line $A$ has the endpoints $-1, 1$ and the tailed geodesic line has the endpoints $-\lambda, \lambda$. Then the two tailed geodesic line are orthogonal to the line connecting $0, \infty$ simultaneously.

$\square$

**Lemma 5.10.** If $p = 2$, and $A_x$ is a tailed geodesic line with the tail $x \in B$, and $A \cap B = \emptyset$, then there exists a tailed geodesic line $B_y$ such that $B_y$ is orthogonal to $A$, $y \in A$ and $B \subset B_y$.

**Proof.** Let $y$ be the point on the geodesic line $A$ satisfying $\rho(x, y) = \log 2$, and $l$ be the segment connecting $x$ and $y$ such that $B_y = l \cup B$ is the tailed geodesic line satisfying the condition.

$\square$

**Lemma 5.11.** If $p = 2$, and $g = h \circ f$ is a tame elliptic element, where $h, f$ are two involutions, then two tailed geodesic lines of $h$ and $f$ only intersect at a unique point.

**Proof.** By Lemma 5.9, we can assume that the fixed points of $f$ are $-1, 1$ and the fixed points of $h$ are $-\lambda, \lambda$. Since $g$ is a tame elliptic element, we have $|\lambda - 1| = 1$. Hence the tailed point of the tailed geodesic line of $h$ is
We denote the set of the fixed points of \( \zeta \) by \( \text{Fix}(\zeta) \), which is the intersection of the sets of fixed points of \( \zeta \) for \( n \) and \( f, g \). Since \( D(1, 1) \) and \( D(\lambda, 1) \) share the unique point \( \zeta \), the two fixed points intersect the unique point \( \zeta \).

\( \square \)

We give the following lemma without proof, which follows from Lemma 5.7 and Lemma 5.9 directly.

**Lemma 5.12.** If \( p = 2 \), and \( g = h \circ f \) is a wild elliptic element, where \( h, f \) are two involutions, then two tailed axes \( A, B \) of \( h, f \) only intersect on a segment, and this segment belongs to the fixed points of \( g \).

**Proof of Theorem 1.2**

Proof. If \( \bigcap_{g \in G} F_g = \emptyset \), then there exist finitely many elements \( g_1, \ldots, g_n \) such that \( \bigcap_{i=1}^n F_{g_i} = \emptyset \), since the Berkovich space is compact with respect to the weak topology. Hence if we can show that \( \bigcap_{i=1}^n F_{g_i} \neq \emptyset \) for any positive integer \( n \), then we prove the theorem.

Let \( f, g \) be two elliptic elements, and denote the axes of \( f, g \) by \( A_f, A_g \) respectively. When \( p \geq 3 \), by Lemma 5.8, we have that there exists an involution \( a \) whose axis \( A \) is orthogonal to \( A_f, A_g \) simultaneously. By Lemma 5.1, there exist two involutions \( b, c \) such that \( f = a \circ b \) and \( g = a \circ c \). We denote the set of the fixed points of \( f, g \) by \( \text{Fix}(f), \text{Fix}(g) \) respectively, and the axes of \( a, b, c \) by \( A, B, C \) respectively. By Lemma 5.7, we know that \( F_f \supset A \neq \emptyset \), and \( F_g \supset A \neq \emptyset \), and \( F_f \neq \emptyset \), since \( h = f^{-1} \circ g \) is elliptic. Choosing \( x \in A \cap B, y \in A \cap C, z \in B \cap C \), there exists \( w \in [x, z] \cap [y, z] \cap [x, y] \), since \( \mathbb{P}^1_{\text{Ber}} \) is an \( \mathbb{R} - \text{tree} \). This means that \( F_f \cap F_g \neq \emptyset \).

By induction, \( \bigcap_{i=1}^n \bigcap_{j \neq k} F_{g_{i,j}} \neq \emptyset \) for \( k = 1, \ldots, n \), and then we want to show \( \bigcap_{i=1}^n F_{g_i} \neq \emptyset \). Since \( \bigcap_{i=1}^n F_{g_i} \neq \emptyset \), \( \bigcap_{i=1}^n F_{g_i} \neq \emptyset \), and \( \bigcap_{i=1}^n F_{g_n} \neq \emptyset \), choosing \( x \in \bigcap_{i=1}^n F_{g_i} \), \( y \in \bigcap_{i=1}^n F_{g_n} \), \( z \in \bigcap_{i=1}^n F_{g_n} \), there exists \( w \in [x, z] \cap [y, z] \cap [x, y] \) such that \( w \in \bigcap_{i=1}^n F_{g_i} \). This implies that each element in \( G \) share at least one fixed point.

When \( p = 2 \), by Lemma 5.8, we have that there exists an involution \( a \) whose axis \( A \) is orthogonal to \( A_f, A_g \) simultaneously. By Lemma 5.1, there exist two involutions \( b, c \) such that \( f = a \circ b \) and \( g = a \circ c \). We denote the set of the fixed points of \( f, g \) by \( \text{Fix}(f), \text{Fix}(g) \). Thanks to Lemma 5.7 and Lemma 5.10, there exist two tailed geodesic line \( A_x, A_y \) which are orthogonal to two axes \( A_f, A_g \). We denote the tailed axes of \( b, c \) by \( B_x, C_y \). By Lemma 5.12, we know that \( F_f \supset A_x \neq \emptyset \), and \( F_g \supset A_y \neq \emptyset \), and \( B_x \neq \emptyset \), and \( B_x \neq \emptyset \), since \( h = f^{-1} \circ g \) is elliptic. Choosing \( u \in A_x \cap B_x, v \in A_y \cap C_y, w \in B_y \cap C_y \), there exists \( w \in [u, v] \cap [u, w] \cap [v, w] \), since \( \mathbb{P}^1_{\text{Ber}} \) is an \( \mathbb{R} - \text{tree} \). This means that \( F_f \cap F_g \neq \emptyset \).

Following the proof of the case \( p = 2 \), it is obviously that when \( p = 2 \), each element in \( G \) share at least one fixed point.

By conjugation, we can assume that each element in \( G \) shares the unique fixed point \( \zeta_{\text{Gauss}} \). By Lemma 2.11, we know that each element in \( G \) has...
good reduction which yields that $G$ has a potentially good reduction. By Lemma 2.10 we know that if $g$ has good reduction, then $g \in \text{PSL}(2, \mathcal{O})$. Since each element $f \in G$ can be written as $\phi f' \phi^{-1}$, where $f'$ has good reduction and $\phi \in \text{PSL}(2, \mathbb{C}_p)$, $\rho_v(f(x), f(y)) = \rho_v(\phi f' \phi^{-1}(x), \phi f' \phi^{-1}(y)) \leq L_1 \rho_v(f', \phi^{-1}(y)) \leq L_1 \rho_v(\phi^{-1}(x), \phi^{-1}(y)) \leq L_1 L_2 \rho_v(x, y)$, where $L_1, L_2$ depending only on $\phi$. Hence $G$ is equicontinuous on $\mathbb{P}^1(\mathbb{C}_p)$.

□

**Theorem 5.13.** If $G$ is a discrete subgroup of $\text{PSL}(2, \mathbb{C}_p)$ and the limit set of $G$ is empty, then $G$ has potentially good reduction.

*Proof.* We have that $G$ contains no loxodromic element $g$, since the fixed points of $g$ are in the limit set of $G$ which yields that $G$ contains parabolic elements and elliptic elements only. Since $G$ is a discrete subgroup of $\text{PSL}(2, \mathbb{C}_p)$, $G$ contains no parabolic elements. By Theorem 1.12 we know that $G$ has potentially good reduction.

□

**Example 5.14.** Let $f_n(z) = z + p^{-n}$ $(n \geq 1)$, and the group $G$ is generated by each $f_n$. Then $G$ contains parabolic elements only and does not have potentially good reduction.

*Proof.* For any disc $D(a, r)$ which is fixed by $f_n$, we have $r \geq p^n$. Since $n$ is arbitrary, the only point fixed by $G$ is the $\infty$. Since each generator can commutate with each other, we know that each element in $G$ can only fixed the unique point $\infty$ in $\mathbb{P}^1_{\text{Ber}}$.

□

**Example 5.15.** Let $G \subset \text{PSL}(2, \mathbb{C}_p)$, and $\zeta_i$ be the $p^i$-th primitive root of unity. Suppose that $G$ is generated by

$$g_i = \begin{pmatrix} \zeta_i & 0 \\ 0 & \zeta_i^{-1} \end{pmatrix},$$

for all the positive integer $i \geq 1$. Then $G$ is discrete, and the limit set $\Lambda(G)$ of $G$ is $\{0, \infty\}$ is a compact set.

*Proof.* In [30], we have proved that $G$ is a discrete subgroup of $\text{PSL}(2, \mathbb{C}_p)$. Furthermore, the points $\{0, \infty\}$ are the fixed points of all the elements $g_i$, $i \geq 1$, namely $0, \infty$ are the limit sets of $G$.

□

6. **NORMS OF $p$-ADIC MÖBIUS MAPS AND ITS APPLICATIONS**

**Proposition 6.1.** Suppose that $f, g, h \in \text{PSL}(2, \mathbb{C}_p)$. Then

1. $\rho_0(fh, gh) = \rho_0(f, g)$, and $\rho_0(hf, hg) \leq L(h)\rho_0(f, g)$.
2. If $h \in \text{PSL}(2, \mathcal{O}_p)$, $\rho_0(h^{-1}fh, h^{-1}gh) = \rho_0(f, g)$.

*Proof.* Since $h$ is an automorphism on $\mathbb{P}^1(\mathbb{C}_p)$, we have

$$\rho_0(fh, gh) = \sup_{z \in \mathbb{P}^1(\mathbb{C}_p)} \rho_v(fh(z), gh(z)) = \sup_{w = h(z) \in \mathbb{P}^1(\mathbb{C}_p)} \rho_v(f(w), g(w)) = \rho_0(f, g).$$
Since \( \rho_v(h(z), h(w)) \leq L(h)\rho_v(z, w) \), we have \( \rho_v(hf(z), hg(z)) \leq L(h)\rho_v(f(z), g(z)) \) which yields that \( \rho_0(hf, hg) \leq L(h)\rho_0(f, g) \).

Since
\[
\rho_v(h^{-1}fh, h^{-1}gh) = \rho_v(h^{-1}f, h^{-1}g) \leq L(h)\rho_v(f, g) \leq L(h)L(h^{-1})\rho_v(h^{-1}f, h^{-1}g),
\]
we have \( \rho_v(h^{-1}f, h^{-1}g) \leq L(h)\rho_v(f, g) \leq \rho_v(h^{-1}f, h^{-1}g) \) which yields that \( \rho_0(h^{-1}fh, h^{-1}gh) = \rho_0(f, g) \).

\[\square\]

Let \( m(g) = \| g - g^{-1} \| \) and \( M(g) = \frac{\|a-g^{-1}\|}{\|g\|} \).

**Proposition 6.2.** Let \( p \) be a prime number. Then \( p^{- \frac{1}{p-1}} \geq 2^{-1} \).

**Proof.** Let \( f(x) = x^{- \frac{1}{x-1}} \). Then \( f'(x) = f(x)(\ln x - (1 - \frac{1}{x}))/(x-1)^2 \) which yields that \( f'(x) > 0 \) if \( x \geq 3 \). Since \( f(3) = 3^{- \frac{1}{2}} \geq 2^{-1} = f(2) \), we have \( p^{- \frac{1}{p-1}} \geq 2^{-1} \) if \( p \) is a prime number.

\[\square\]

**Theorem 6.3.** Let \( p \geq 3 \), and \( g \) be a \( p \)-adic Möbius map. Then \( \rho_0(g, I) = M(g) \).

**Proof.** By Lemma 4.3 there exists an element \( h \in \text{PSL}(2, \mathbb{O}_p) \) such that \( hgh^{-1} = \frac{az+b}{d} \) with \( ad = 1 \). By Proposition 6.1 we have \( \rho_0(hgh^{-1}, I) = \rho_0(g, I) \). By Theorem 1.1 we know that \( \| g - g^{-1} \| = \| h(g - g^{-1})h^{-1} \| = \| hgh^{-1} - hg^{-1}h^{-1} \| \). Thus we can rewrite \( hgh^{-1} \) as \( g \). Hence
\[
M(g) = \max\{|a-d|, |b|\} / \max\{|a|, |d|, |b|\}.
\]

Moreover
\[
\rho_v(g(z), z) = \frac{|a^2z + ab - z|}{\max\{1, |z|\} \max\{1, |a^2z + ab|\}}.
\]

If \( g \) is parabolic, then
\[
\rho_v(g(z), z) = \frac{|b|}{\max\{1, |z|\} \max\{1, |z + b|\}} \leq M(g),
\]
and
\[
\rho_0(g, I) \geq \rho_v(g(0), 0) = M(g).
\]

Thus \( \rho_0(g, I) = M(g) \).

If \( g \) is loxodromic, we can assume that \( |a| > 1 \). Since
\[
\rho_v(g(0), 0) = \frac{|ab|}{\max\{1, |ab|\}} = \frac{|ab|}{\max\{1, |a^2 + ab|\}},
\]
we have \( M(g) = 1 \geq \rho_0(g, I) \geq \max\{\rho_v(g(0), 0), \rho_v(g(1), 1)\} = 1 = M(g) \).
If $g$ is elliptic, we have $|a^2 - 1| \leq 1$. If $|ab| > |a^2 - 1|$, then $M(g) = |ab| \geq \rho_0(g, I) \geq \rho_v(g(0), 0) = |ab| = M(g)$. If $|ab| \leq |a^2 - 1|$, there exists a number $\omega \in \mathcal{O}_p$ such that $|(a^2 - 1)\omega + ab| = |a^2 - 1|$ which yields that $M(g) = |a^2 - 1| \geq \rho_0(g, I) \geq \rho_v(g(\omega), \omega) = |a^2 - 1| = M(g)$. 

If $p = 2$, then $2^{-1}M(g) \leq \max\{\frac{|a-d|}{|a|}, |b|\} \leq 2M(g)$. Thus we give the following theorem without proof.

**Theorem 6.4.** Let $p = 2$, and $g$ be a $p$-adic Möbius map. Then $2^{-1}M(g) \leq \rho_0(g, I) \leq 2M(g)$.

**Theorem 6.5.** For any $p$-adic Möbius map $g$, $\rho_0(g, I) \leq \| g - I \|$.

**Proof.** Following Theorem 6.3, we can assume that $g(z) = \frac{az+b}{d}$ with $ad=1$. Thus $\| g - I \| = \max\{|a-1|, |b|, |d-1|\}$. Moreover

$$\rho_v(g(z), z) = \frac{|a^2z + ab - z|}{\max\{1, |z|\} \max\{|1, az+b|\}} \leq 1.$$

If $g$ is loxodromic, we can assume $|a| > 1$. Hence $\max\{|a-1|, |d-1|\} > 1$ which yields that $\rho_0(g, I) \leq \| g - I \|.$

If $g$ is parabolic, then $a = 1$. We have

$$\rho_v(g(z), z) = \frac{|b|}{\max\{1, |z|\} \max\{1, |z+b|\}} \leq |b| = \| g - I \|.$$

If $g$ is elliptic, then $|a| = 1$. We have

$$\rho_v(g(z), z) = \frac{|a^2z + ab - z|}{\max\{1, |z|\} \max\{|1, az+b|\}} \leq \frac{|a^2z + ab - z|}{\max\{1, |z|\}} \leq \max\{|a^2-1|, |ab|\} \leq \| g - I \|.$$

Let $\varepsilon(g) = \max\{\rho_v(g(z_0), z_0), \rho_v(g(z_1), z_1), \rho_v(g(z_2), z_2)\}$, where $z_0, z_1, z_2$ are three distinct roots of the equation $z^3 = 1$.

**Theorem 6.6.** For any $p$-adic Möbius map $g$, we have $2^{-1}\varepsilon(g) \leq M(g) \leq 6\varepsilon(g)$.

**Proof.** Since $\varepsilon(g) = \max\{\rho_v(g(z_0), z_0), \rho_v(g(z_1), z_1), \rho_v(g(z_2), z_2)\} \leq \rho_0(g, I)$, by Theorem 6.3 and 6.4 we have $\varepsilon(g) \leq \rho_0(g, I) \leq 2M(g)$.

Since $z_0, z_1, z_2$ are three distinct roots of the equation $z^3 = 1$, and let $z_0 = 1$, we have $z_1 + z_2 + 1 = 0$, and by Lemma 2.1, Lemma 2.2 and Proposition 6.2 we have $2^{-1} \leq |z_1 - z_2| \leq 1$. This implies that

$$|3b| = |cz_1 + (d-a)z_2 - b + cz_2 + (d-a)z_1 - b + cz_0 + (d-a)z_0 - b| 
\leq \max\{|cz_1 + (d-a)z_2 - b|, |cz_2 + (d-a)z_1 - b|, |cz_0 + (d-a)z_0 - b|\}.$$
We denote $\varepsilon'(g)$ by $\max\{|cz_1 + (d-a)z_2 - b|, |cz_2 + (d-a)z_1 - b|, |cz_0 + (d-a)z_0 - b|\}$. Hence $|b| \leq 3\varepsilon'(g)$ which yields that
\[
\max\{|cz_1 + (d-a)z_2|, |cz_2 + (d-a)z_1|, |cz_0 + (d-a)z_0|\} \leq 3\varepsilon'(g).
\]
Thus
\[
\max\{|cz_1 + (d-a)z_2 + c z_2 + (d-a)z_1|, |cz_1 + (d-a)z_2 - c z_2 - (d-a)z_1|\} \leq 3\varepsilon'(g),
\]
namely $|c + (d-a)| \leq 3\varepsilon'(g)$ and $|c(\frac{d^2}{z_2} - 1)| \leq 3\varepsilon'(g)$ which yields that $|c(\frac{d^2}{z_2} - 1)| \leq 3\varepsilon'(g)$. This implies that $|c| \leq 6\varepsilon'(g)$ and $|(d-a)| \leq 6\varepsilon'(g)$, namely $\max\{|a - d|, |b|, |c|\} \leq 6\varepsilon'(g)$. For any $z$ with $|z| = 1$, we have $\max\{|az + b|, |cz + d|\} \leq \max\{|az|, |b|, |cz|, |d|\} = \|g\|$ which yields that $\max\{|a - d|, |2b|, |2c|\} \leq \max\{|a - d|, |b|, |c|\} \leq 6\varepsilon'(g)$. This implies that $\max\{|a - d|, |2b|, |2c|\}/\|g\| \leq \max\{|a - d|, |b|, |c|\}/\|g\| \leq 6\varepsilon'(g)/\max\{|az + b|, |cz + d|\} \max\{1, |z|\}$ for any $z$ with $|z| = 1$. Since
\[
\varepsilon'(g) = \frac{\max\{|cz_0 + (d-a)z_0 - b|\}}{\max\{|az + b|, |cz + d|\} \max\{1, |z|\}} \leq \frac{\max\{|cz_0^2 + (d-a)z_0 - b|\}}{\max\{|az + b|, |cz + d|\} \max\{1, |z|\}},
\]
we have $M(g) \leq 6\varepsilon(g)$. □

**Corollary 6.7.** For any $p$-adic Möbius map $g$, $\frac{1}{3}\varepsilon(g) \leq \rho_0(g, I) \leq \varepsilon(g)$.

**Proof.** By Theorem 6.3 and Theorem 6.6, we get the inequality directly. □

Let $\varepsilon_1(g) = \{\rho_v(g(0), 0), \rho_v(g(1), 1), \rho_v(g(\infty), \infty)\}$.

**Theorem 6.8.** For any $p$-adic Möbius map, $2^{-1}\varepsilon_1(g) \leq M(g) \leq \varepsilon_1(g)$.

**Proof.** Since $\varepsilon(g) = \max\{\rho_v(g(0), 0), \rho_v(g(1), 1), \rho_v(g(\infty), \infty)\} \leq \rho_0(g, I)$, by Theorem 6.3, we have $\varepsilon_1(g) \leq \rho_0(g, I) \leq 2M(g)$.

Since
\[
\rho_v(g(z), z) = \frac{|cz^2 + (d-a)z - b|}{\max\{|az + b|, |cz + d|\} \max\{1, |z|\}},
\]
we have
\[
\rho_v(g(0), 0) = \frac{|b|}{\max\{|b|, |d|\}}, \rho_v(g(\infty), \infty) = \frac{|c|}{\max\{|a|, |c|\}},
\]
and
\[
\rho_v(g(1), 1) = \frac{|c + (d-a) - b|}{\max\{|a + b|, |c + d|\}}.
\]

Since $\max\{|a - d|, |2b|, |2c|\} \leq \max\{|b|, |c|, |e + (d-a) - b|\}$ and $\max\{|a + b|, |c + d|\} \leq \max\{|a|, |c|, |b|, |d|\} = \|g\|$, we have
\[
M(g) = \frac{\max\{|a - d|, |2b|, |2c|\}}{\|g\|} \leq \frac{\max\{|b|, |c|, |e + (d-a) - b|\}}{\|g\|}.
\]
Theorem 6.11. Let \( f \) maps without using the cross ratios of \( p \) when \( U \) contains at least three points. Then a sequence \( \{ f_n \} \) converges pointwisely, and that \( f \) can be a \( p \)-adic Möbius map.

Proof. Since \( \varepsilon_2(g) < \varepsilon_1(g) \), we have \( 2^{-1}\varepsilon_2(g) \leq 2^{-1}\varepsilon_1(g) \leq M(g) \).

By Proposition 6.5, if \( g \) is a parabolic element, then \((a + d)^2 = 4\) which yields that \( 4bc = -(a - d)^2 \). Thus it implies that \( |a - d| = \sqrt{4|bc|} \leq \sqrt{|bc|} \leq \max\{|b|, |c|\} \) which yields that \( \max\{|a - d|, 2|b|, 2|c|\} \leq \max\{|b|, |c|\} \). Hence

\[
M(g) = \max\{|a - d|, 2|b|, 2|c|\} \leq \max\{|a - d|, |b|, |c|\}
\]

\[
\leq \max\{|b|, |c|\} \cdot \frac{|c|(a - d) - b|}{\max\{|b|, |d|\}, \max\{|a|, |c|\}} = \varepsilon_2(g) = \max\{\rho_v(g(0), 0), \rho_v(g(\infty), \infty)\}.
\]

As applications of these inequalities, we can get the convergence theorem of \( p \)-adic Möbius maps.

Let \( \{f_n\} \) be a sequence of \( p \)-adic Möbius maps, and \( U \) be the set of points at which the sequence \( \{f_n\} \) converges pointwisely, and \( f = \lim_{n \to \infty} f_n \) on \( U \).

Write \( f_n \to (U, f) \) to mean that \( U \) is the set of convergence of \( f_n \) and that \( f_n \to f \) on \( U \) (and only on \( U \)). In [34], we proved the following theorem.

**Theorem 6.10** ([34]). Suppose that there exists a sequence \( p \)-adic Möbius maps \( f_n \) such that \( f_n \to (U, f) \) with \( U \neq \emptyset \). Then one of the following possibilities occurs:

(a) \( U = \mathbb{P}^1(\mathbb{C}_p) \), and \( f \) is a \( p \)-adic Möbius map;
(b) \( U = \mathbb{P}^1(\mathbb{C}_p) \), and \( f \) is constant on the complement of one point on \( U \);
(c) \( U = \{z_1, z_2\} \) and \( f(z_1) \neq f(z_2) \); or
(d) \( f \) is constant on \( U \).

We can reprove this theorem by the use of the three-point norms.

If \( U \) contains only one point, it is the case (d), and if \( U \) contains two points only, it is the case (c) or (d). Hence we only need to consider the case when \( U \) contains at least three points.

We prove the following theorem by using different norms of \( p \)-adic Möbius maps without using the cross ratios of \( p \)-adic Möbius maps.

**Theorem 6.11.** Let \( \{f_n\} \) be a sequence of \( p \)-adic Möbius maps and \( z_j, j = 1, 2, 3 \) be three distinct points with \( f_n(z_j) \to w_j \), where \( w_j \) are also three distinct points. Then a sequence \( \{f_n\} \) converge to a \( p \)-adic Möbius map \( f \) uniformly, where \( f(z_j) = w_j, j = 1, 2, 3 \).
Proof. We can find a p-adic Möbius map \( h \) such that \( h(z_1) = u_1, h(z_2) = u_2, h(z_3) = u_3 \), where \( u_i \) are the three distinct roots of \( z^3 = 1 \). Then \( h f_n h^{-1}(u_i) \to u_i \). By Corollary 6.7 we know that \( \frac{1}{\delta} \varepsilon(hf^{-1}f_nh^{-1}) \leq \rho_0(hf^{-1}f_nh^{-1}, I) \leq \varepsilon(hf^{-1}f_nh^{-1}) \). This yields that \( hf^{-1}f_nh^{-1} \) converges to \( I \) uniformly, namely \( f_n \) converges to \( f \) uniformly.

\[ \square \]

**Proposition 6.12** (34). Let \( f \in PSL(2, \mathbb{C}_p) \). Then \( f \) preserves the chordal cross ratio, namely

\[
\frac{\rho_v(f(x), f(y)) \rho_v(f(z), f(w))}{\rho_v(f(x), f(z)) \rho_v(f(y), f(w))} = \frac{\rho_v(x, y) \rho_v(z, w)}{\rho_v(x, z) \rho_v(y, w)}.
\]

**Theorem 6.13.** Let \( \{f_n\} \) be a sequence of p-adic maps, and suppose that there exist three distinct points \( x_1, x_2, x_3 \) in \( \mathbb{P}_1(\mathbb{C}_p) \) such that \( \lim_{n \to \infty} f_n(x_1) = \lim_{n \to \infty} f_n(x_2) = \alpha, \lim_{n \to \infty} f_n(x_3) = \beta \), where \( \alpha \neq \beta \). Then \( f_n \to \alpha \) on \( \mathbb{P}_1(\mathbb{C}_p) \setminus x_3 \), namely \( U = \mathbb{P}_1(\mathbb{C}_p) \), and \( f \) is constant on the complement of one point on \( U \).

Proof. Without loss of generality, we can assume that \( x_1 = 0, x_2 = 1, x_3 = \infty \) and \( \alpha = 0, \beta = \infty \). Assuming \( x_4 \in \mathbb{P}_1(\mathbb{C}_p) \setminus \{0, 1, \infty\} \), if the sequence \( \{f_n(x_4)\} \) does not converge to 0, there exists a subsequence \( \{f_{n_j}(x_4)\} \) and a fixed positive number \( \delta \) such that \( |f_{n_j}(x_4)| > \delta \).

\[
\frac{\rho_v(f_{n_j}(0), f_{n_j}(1)) \rho_v(f_{n_j}(\infty), f_{n_j}(x_4))}{\rho_v(f_{n_j}(0), f_{n_j}(\infty)) \rho_v(f_{n_j}(1), f_{n_j}(x_4))} = \frac{\rho_v(0, 1) \rho_v(\infty, x_4)}{\rho_v(0, \infty) \rho_v(1, x_4)} \neq 0.
\]

Letting \( n_j \) tend to \( \infty \),

\[
0 = \lim_{n_j \to \infty} \frac{\rho_v(f_{n_j}(0), f_{n_j}(1)) \rho_v(f_{n_j}(\infty), f_{n_j}(x_4))}{\rho_v(f_{n_j}(0), f_{n_j}(\infty)) \rho_v(f_{n_j}(1), f_{n_j}(x_4))} = \frac{\rho_v(0, 1) \rho_v(\infty, x_4)}{\rho_v(0, \infty) \rho_v(1, x_4)} \neq 0.
\]

This is a contradiction. Hence \( \lim_{n \to \infty} f_n(x_4) = 0. \) \( \square \)

Combing Theorem 6.11 and Theorem 6.13 we prove Theorem 6.10

7. The decomposition theorem of p-adic Möbius maps

Two points \( \alpha, \beta \) are called antipodal points if there exists an element \( u \in PSL(2, \mathcal{O}_p) \) such that \( u(0) = \alpha, u(\infty) = \beta \).

**Theorem 7.1.** For any p-adic Möbius map \( g \), there exists an element \( u \in PSL(2, \mathcal{O}_p) \) such that \( g = uf \), where either \( f \) is a loxodromic element with antipodal fixed points, or \( f = I \).

Proof. If \( g \) is a loxodromic element or elliptic element, by Lemma 5.4 there exist two involutions \( a, b \) such that \( g = ab \), the (tailed) axes of \( a \) and \( b \) are orthogonal to the axis of \( g \), and the (tailed) axis of \( a \) containing \( \zeta_{\text{Gauss}} \). Let \( \alpha, \beta \) be the fixed points of \( b \). We claim that there exists an element in \( h \in PSL(2, \mathcal{O}_p) \) such that \( h(\alpha) + h(\beta) = 0 \).
**Claim:** Without loss of generality, we can assume that $|\alpha| \leq 1$, otherwise we can consider $1/\alpha$, since $1/z \in \text{PSL}(2, \mathbb{O}_p)$. Let $u(z) = z - \alpha \in \text{PSL}(2, \mathbb{O}_p)$ which yields that $u(\alpha) = 0$. Hence we can assume that $\alpha = 0$. If $|\beta| \leq 1$, then let $h(z) = z - \beta$, which implies that $h(0) + h(\beta) = 0$. If $|\beta| > 1$, then let $h(z) = \frac{az + b}{cz + d}$ with $h(0) + h(\beta) = 0$. Hence

$$\frac{b}{d} + \frac{a\beta + b}{c\beta + d} = 0$$

which yields that $\beta(\bar{a}d + bc) + 2bd = 0$. Let $X_1 = ad, X_2 = bc, X_3 = bd$. Hence we have three equations:

1. $\beta(X_1 + X_2) + 2X_3 = 0$
2. $X_1 - X_2 = 1$
3. $X_1 + X_2 = \lambda$

Thus we have $X_1 = \frac{\beta - 1}{2}, X_2 = \frac{-\beta - 1}{2},$ and $X_3 = -\frac{\beta}{2}$. Since $\frac{\alpha}{\beta} = \frac{X_2}{X_3}$ and $\frac{\beta}{\alpha} = \frac{X_2}{X_3},$ let $a = \frac{\beta + 1}{2}, b = -\frac{\beta}{2}, c = \frac{\beta - 1}{2},$ and $d = -\frac{\beta}{2}$ which yields that

$$-\beta(\bar{a}d + bc) + 2bd = 0$$

This line contains the point $\zeta_{\text{Gauss}}$. Let $l_1$ be the geodesic line which connects the endpoints of involution $a$. Since all the elements in $\text{PSL}(2, \mathbb{O}_p)$ fix the point $\zeta_{\text{Gauss}},$ the point $\zeta_{\text{Gauss}} \in h(l_1)$. By Lemma 5.1, we know that there exists an involution $c$ which fixes the point $\zeta_{\text{Gauss}}$ such that $chbh^{-1}$ is a loxodromic element whose fixed points are $0, \infty$, and $hah^{-1}c$ is an elliptic element in $\text{PSL}(2, \mathbb{O}_p)$, since $hah^{-1}c$ fixes the point $\zeta_{\text{Gauss}}$. Thus $gh^{-1} = hah^{-1}chbh^{-1}$ which yields that $g = h^{-1}hah^{-1}chbh^{-1}h$, where $h^{-1}hah^{-1}ch \in \text{PSL}(2, \mathbb{O}_p)$ and $h^{-1}(chbh^{-1})h$ is a loxodromic element with antipodal fixed points.

If $g$ is a parabolic element, then we can assume the fixed point of $g$ is $\infty$ after conjugating by an element in $\text{PSL}(2, \mathbb{O}_p)$ by the Claim. Thus $g(z) = ab$, where $a(z) = -z, b(z) = -z + b$. Since $a(z) = -z$ contains the point $\zeta_{\text{Gauss}},$ by similar discussion above, we can find an involution $c$ which fixes the point $\zeta_{\text{Gauss}}$ such that $az \in \text{PSL}(2, \mathbb{O}_p)$, and $cb$ is a loxodromic element with antipodal fixed points.

Let $U = \text{PSL}(2, \mathbb{O}_p)$. We define $d(g, U) = \inf\{\rho_0(g, u)|u \in U\}$.

**Theorem 7.2.** For any $p$-adic Möbius map, either $d(g, U) = 0$, if $g \in U$, or $d(g, U) = 1$, if $g \not\in U$. 

\[\square\]
Proof. If \( g \in \mathcal{U} \), then \( d(g, \mathcal{U}) = 0 \). If \( g \notin \mathcal{U} \), then by Theorem 6.5 there exist \( u \in \text{PSL}(2, \mathbb{C}_p) \) and a loxodromic element \( f \) with antipodal fixed points such that \( g = uf \). Hence there exists an element \( h \in \text{PSL}(2, \mathbb{C}_p) \) such that \( v = hfh^{-1} = \lambda z \). Thus \( d(g, \mathcal{U}) = d(uf, \mathcal{U}) = d(v, \mathcal{U}) \).

For any \( s(z) = \frac{az + b}{cz + d} \in \text{PSL}(2, \mathbb{C}_p) \), since \( ad - bc = 1 \) and \( \max\{|a|, |b|, |c|, |d|\} \leq 1 \), we have that \( \frac{|a|}{c} = 1 \), or \( \frac{|b|}{c} = 1 \), or \( \frac{|c|}{c} < 1 \), otherwise, if \( \frac{|a|}{c} > 1 \) and \( \frac{|b|}{c} > 1 \), then \( |d| < |b| \leq 1 \) and \( |c| < |a| \leq 1 \) which yields that \( |ad - bc| \leq \max\{|a|, |b|c|\} < 1 \). This contradicts \( ad - bc = 1 \). Other cases are similar. If \( \frac{|a|}{c} = 1 \), then \( \rho_v(v(0)), s(0)) = 1 \). If \( \frac{|a|}{c} = 1 \), then \( \rho_v(v(\infty), s(\infty)) = 1 \). If \( \frac{|a|}{c} > 1 \), then \( 1 \geq |b| > |d| \) and \( |a| < |c| \leq 1 \). Since \( ad - bc = 1 \), we have \( |b| = |c| = 1 > \max\{|a|, |d|\} \). This implies that \( \frac{|a|}{c} < 1 \).

\[
\rho_v(v(-\frac{d}{c}), s(-\frac{d}{c})) = \frac{|\lambda c(-\frac{d}{c})^2 + (\lambda d - a)(-\frac{d}{c}) - b|}{\max\{1, |\frac{-d}{c}|\} \max\{|a(\frac{-d}{c}) + b|, |c(\frac{-d}{c}) + d|\}} = \frac{|1/c|}{|1/c|} = 1.
\]

This yields that for any \( p \)-adic Möbius map \( g \notin \mathcal{U} \), and any \( s \in \mathcal{U} \), \( \rho_0(g, s) = 1 \), namely \( d(g, \mathcal{U}) = 1 \).

Let \( G \) be a subgroup of \( \text{PSL}(2, \mathbb{C}_p) \). We say that \( G \) a discrete subgroup if there exists a positive number \( \varepsilon \) such that for any non-unit element \( g \) with\( \| g - I \| > \varepsilon \).

**Theorem 7.3.** If \( G \) is a subgroup of \( \text{PSL}(2, \mathbb{C}_p) \) and \( G \cap \mathcal{U} = I \), then \( G \) is a discrete subgroup.

**Proof.** By Theorem 6.5 and Theorem 7.2 we have \( d(f, \mathcal{U}) \leq \rho_0(f, I) \leq \| f - I \| \). Since \( G \cap \mathcal{U} = I \), for any nonunit element \( f \), \( \| f - I \| \geq 1 \).

**Corollary 7.4.** If a subgroup \( G \subseteq \text{PSL}(2, \mathbb{C}_p) \) contains unit element or loxodromic element only, then \( G \) is a discrete subgroup.

**Proof.** Since each loxodromic element does not belong to \( \mathcal{U} \), by Theorem 7.3 we can get the conclusion directly.

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