Extending Karger’s randomized min-cut Algorithm for a Synchronous Distributed setting

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ABSTRACT
A min-cut that separates vertices s and t in a network is an edge set of minimum weight whose removal will disconnect s and t. This problem is the dual of the well known s−t max-flow problem. Several algorithms for the min-cut problem are based on max-flow computation although the fastest known min-cut algorithms are not flow based. The well known Karger’s randomized algorithm for min-cut is a non-flow based method for solving the (global) min-cut problem of finding the min s−t cut over all pair of vertices s, t in a weighted undirected graph. This paper presents an adaptation of Karger’s algorithm for a synchronous distributed setting where each node is allowed to perform only local computations. The paper essentially addresses the technicalities involved in circumventing the limitations imposed by a distributed setting to the working of Karger’s algorithm. While the correctness proof follows directly from Karger’s algorithm, the complexity analysis differs significantly. The algorithm achieves the same probability of success as the original algorithm with $O(mn^2)$ message complexity and $O(n^2)$ time complexity, where $n$ and $m$ denote the number of vertices and edges in the graph.

Categories and Subject Descriptors
[Distributed Algorithms]: Metrics—complexity measures;  
[Graph Theory]: Miscellaneous

General Terms
Network-flow

Keywords
Max-flow, Min-cut

1. INTRODUCTION
The problem of computing the minimum-cut in a weighted graph has been classically studied in literature as the dual of the well known max-flow problem for networks [5] and classical solutions to the max-flow problem were used to solve the min-cut problem. These algorithms could be classified as those based on augmenting paths [5, 4], improvements to the augmenting path approach based on blocking flows [3, 12] and those based on pre-flow method introduced by Goldberg and Tarjan [6]. The best known algorithms for the max-flow problem are based on the preflow approach [1, 17, 7]. The max-flow problem also has been recently studied in a distributed setting in [2].

Further investigations revealed that there are more efficient direct solutions to the min-cut problem (without solving max-flow and taking the dual). Nagamochi and Ibaraki [13] published the first deterministic global minimum cut algorithm that is not based on flow, but was rather complicated. Stoer and Wagner [16] presented a simple deterministic global minimum cut algorithm which runs in $O(mn + n^2 \log n)$.

Karger [8] presented the first randomized global min-cut algorithm which runs in $O(mn^2 \log^3 n)$. The running time of a single trial of the algorithm is $O(m \log^2 n)$. The algorithm has to be repeated $n^2 \log n$ times to achieve a high success probability of $1 - \frac{1}{n}$. Karger and Stein [11] further improved its running time to $O(n^2 \log^3 n)$ for the same probability.

Recently there has been revived interest in the min-cut problem owing to its applications to network coding and wireless sensor networks [15, 10, 14]. Sensor networks operate in a distributed setting and motivates a solution to the problem in a distributed setting.

In this paper, we show how Karger’s algorithm [8] can be adapted to efficiently solve the min-cut problem in a distributed setting. We assume a very general model of a graph where each node knows only information about its neighbours. It is assumed that the storage capacity of a node is bounded linearly in the size of the number of its neighbours and the computing capacity of a node is bounded polynomially in the number of its neighbours. The assumption is reasonable as each node must have storage and processing capacity sufficient to keep track of communication with its neighbours. The nodes can perform local computations and can communicate only with its neighbours along the edges of the graph. Our objective is to find the value of the global
min-cut and communicate the same to all the nodes. Moreover, each node must know which among the edges incident on it are present in the min-cut computed. While the correctness proof follows directly from Karger’s algorithm, the complexity analysis differs significantly. We show that for a graph of $n$ vertices and $m$ edges, the algorithm computes the global min-cut with probability at least $1 - \frac{1}{n}$ with $O(mn^2)$ message complexity and $O(n^2)$ time complexity when there is a global clock for synchronization. We note that although the assumption of a global clock may be impractical in applications like sensor networks, there are standard techniques for converting synchronous distributed algorithms to asynchronous algorithms, with some loss in computational efficiency[11]. We pursue the simpler synchronous setting here as it allows a less cumbersome presentation of the algorithm and a simple analysis.

2. THE ALGORITHM

2.1 A Brief Description

Assume that given a weighted graph $G = (V, E, w)$ where $E \subseteq V \times V$ and $w : E \rightarrow \mathbb{R}^{+} \cup \{0\}$ is given (We use the terms network and graph interchangeably). In our algorithm $N_u$ represents the neighbourhood of vertex $u$, $weight_u$ represents the present edge weights of $N_u$, that is, for each $v \in N_u$, $weight_u[v]$ indicates the weight of edge $(u, v)$. $rank_u[v]$ is the rank of edge $(u, v)$, a random number which is uniformly chosen between 1 and $m^k$ (for some fixed $k \geq 5$), on each trial. $maxrank_u$ represents the maximum value of rank among all the edges. Initially $maxrank_u$ is defined as the maximum rank of the edges connected to vertex $u$. The algorithm sets $maxrank = \max_{v \in V(G)} maxrank_u$. The status of a vertex may be ACTIVE or INACTIVE (initially ACTIVE). $status_u = INACTIVE$ if all neighbouring edge weights of vertex $u$ are 0, which means that vertex cannot initiate the contraction process. We call an edge active if at least one of its end points is active.

The algorithm proceeds by simulating edge contractions as in[8], by collecting vertices joined together by contraction into vertex groups. Edges within a group are inactive as they cannot be further contracted. At each step, an active edge of maximum rank is chosen for contraction. Since edge ranks are assigned uniformly at random, each active edge has equal probability for getting contracted. The algorithm continues contractions till only two vertex groups remain and the set of edges across the two groups is chosen as the mincut for that trial. The smallest cut found in $n^2 \log n$ trials will be the mincut with probability $1 - \frac{1}{n^2}$.

The variable $lastmsg_u$ stores the last message received at vertex $u$ (used to reduce message flooding) and the boolean variable $stop_u$ is set to true when only two vertex groups are remaining and no more contraction can be made, and set to false otherwise.

The variable $q_u$ represents the present group id of vertex $u$, initially $q_u = u$. Initially there are $n$ groups, one for each vertex. As contractions progress, the number of groups reduces and we set $weight_u[v] = 0$ if $q_u = q_v$ and $weight_u[v] \neq 0$ otherwise. The following description presents a high level view of the algorithm.

2.2 Details of the Algorithm

Algorithm 1 distributed-mincut-in-a-nutshell()

assign a rank (between 1 and $m^k$) to each non-zero weighted edge. {Algorithm 4}
At each node $u$ of the network execute the following: find $maxrank_u$ of each vertex $u$ locally. {Algorithm 6} find the vertex $x$ with largest vertex id having the maximum value of $maxrank$. {Algorithms 6 14}
if there are only two groups then {Algorithms 6 6 14 16 21}
compute local mincut $mc_u$ by summing the non-zero edge weights of vertex $u$. {Algorithm 10}
compute global mincut by summing up all local mincuts. {Algorithms 11 22}
broadcast the mincut to all nodes and stop. {Algorithms 12 23}
else contract two vertex groups by making the edge weights between them zero and group ids equal to the value of $maxrank$ (The contraction process is initiated by the vertex $x$). {Algorithms 7 8 17}
repeat the algorithm end if

Each node in the network executes Algorithm 2 described below. Here, the function $initialize()$ initializes the group id of each vertex with its vertex id. The function $assign-rank()$ assigns a rank to each non-zero weighted edge with in the network, with a random value between 1 and $m^k$. The time complexity for this function is $O(n)$. The function $find-local-maxrank()$ computes the maximum rank within its neighbourhood, with time complexity $O(n)$. The function $find-global-maxrank()$ computes the maximum of all the $local-maxranks$ within the network, with time complexity $O(n)$ and message complexity $O(mn)$.

The function $check-eligibility-and-contract()$ checks whether there are more than two groups within the network and if so, contracts two groups by making all the edge weights between them zero and their group ids the same. This can be accomplished with time complexity $O(n)$ and message complexity $O(m)$. The function $check-termination-status()$ checks whether there are only two groups within the network and if so, invokes mincut computation and halts, otherwise the algorithm is repeated. This can be accomplished with time complexity $O(n)$ and message complexity $O(m)$. All the above mentioned functions except $initialize()$ has to be repeated $n - 2$ times.

The function $find-local-mincut()$ computes the sum of edge weights within its neighbourhood, with time complexity $O(n)$. The function $find-global-mincut()$ computes the sum of all $local-mincuts$ within the network, with time complexity $O(n^2)$ and message complexity $O(mn)$. Node $u$ messages to node $u + 2i - 1$ in step $i$, for $i \in \{1, ..., \log n\}$ to ensure that the messages propagate to all nodes in $O(n^2)$ time with only $O(mn)$ messages. The function $broadcast-mincut()$ broadcasts the computed mincut value to all the nodes within the network, which is done with time complexity $O(n)$ and message complexity $O(m)$. The function $synchronize()$ allows the nodes to wait for some time so that the same instruction can be executed by each node, in the next time step. This function waits for $O(n)$ steps.
Algorithm 2 distributed-mincut() //To be executed at each node
initialize()
repeat
    assign-rank()
    find-local-maxrank()
    find-global-maxrank()
    synchronize()
    check-eligibility-and-contract()
    synchronize()
    check-termination-status()
    synchronize()
until stopu = true
find-local-mincut()
find-global-mincut()
synchronize()
broadcast-mincut()

Algorithm 3 initialize()
gu ← u

Algorithm 4 assign-rank()
{Rank of an edge to be assigned by higher numbered endpoint}
for each v ∈ Nu do
    if u > v then
        if weightu[v] ≠ 0 then
            ranku[v] ← a random number between 1 and mk
        else
            ranku[v] ← 0
        end if
        send(SET-RANK, ranku[v]) to v. {See Algorithm 13 for receipt of message}
    end if
end for

Algorithm 5 find-local-maxrank()
maxranku ← maxv∈Nu(ranku[v])

Algorithm 6 find-global-maxrank()
send(FIND-MAX-RANK, maxranku) to each v ∈ Nu. {See Algorithm 14 for receipt of message}

Algorithm 7 check-eligibility-and-contract()
stopu ← true
if maxranku = maxv∈Nu(ranku[v]) and u > v then
    if ∃w ∈ Nu with weightu[w] ≠ 0 and v ≠ w and gu ≠ gw then
        stopu ← false
        contract()
    else
        send(IS-ELIGIBLE-CONTRACT, u, gu, gw) to each x ∈ Nu. {See Algorithm 15 for receipt of message}
    end if
else
    send(LOCAL-MC, mcu, u, min(u + 2i−1, n)) to each v ∈ Nu. {See Algorithm 16 for receipt of message}
end if

Algorithm 8 contract()
if maxranku = maxv∈Nu(ranku[v]) and u > v then
    weightu[v] ← 0
    check-active()
    gu ← maxranku
    send(SET-GROUP-ID, gu, gw, maxranku) to each x ∈ Nu with weightu[x] = 0. {See Algorithm 7 for receipt of message}
end if

Algorithm 9 check-termination-status()
send(STOP, stopu) to each x ∈ Nu. {See Algorithm 20 for receipt of message}

Algorithm 10 find-local-mincut()
if statusu = ACTIVE then
    mcu ← ∑v∈Nu weightu[v]
else
    mcu ← 0
end if

Algorithm 11 find-global-mincut()
for i ← 1 to log n step by 1 do
    for j ← 2i−1 to n − 1 step by 2i do
        if u = j then
            send(LIST-MINCUT, mcu, u, min(u + 2i−1, n)) to each v ∈ Nu. {See Algorithm 21 for receipt of message}
        end if
    end for
end for

Algorithm 12 broadcast-mincut()
if u = n then
    mcu ← mcu/2
    send(MINCUT, mcu, u) to each v ∈ Nu. {See Algorithm 22 for receipt of message}
end if

Algorithm 13 upon receipt of (SET-RANK, num) msg from w
ranku[w] ← num

Algorithm 14 upon receipt of (FIND-MAX-RANK, m) msg from w
{find maximum rank among all vertices}
if m > maxranku then
    maxranku ← m
    send(FIND-MAX-RANK, m) to each v ∈ Nu where v ≠ w
end if
Algorithm 15 upon receipt of (IS-ELIGIBLE-CONTRACT, \(v, g', g''\)) msg from \(w\)
\[
\text{\{checks the eligibility of contraction\}}
\]
if (IS-ELIGIBLE-CONTRACT, \(v, g', g''\)) \(\neq\) lastmsg\(_u\)
then
\[
\exists y \in N_u \text{ with } \text{weight} \_u[y] \neq 0 \text{ and } g_y \neq g' \text{ and } g_y \neq g''
\]
send(ELIGIBLE-CONTRACT, \(v, g'\)) to each \(z \in N_u\)
with \(\text{weight} \_u[z] = 0\) or \(\text{weight} \_u[z] \neq 0\) and \(g_z = g'\).
(See Algorithm 16 for receipt of message)
else
send(IS-ELIGIBLE-CONTRACT, \(v, g', g''\)) to each \(z \in N_u\)
with \(\text{weight} \_u[z] = 0\) and \(z \neq w\)
end if
lastmsg\(_u \leftarrow (\text{IS-ELIGIBLE-CONTRACT}, v, g', g'')\)
end if

Algorithm 16 upon receipt of (ELIGIBLE-CONTRACT, \(v, g'\)) msg from \(w\)
if (ELIGIBLE-CONTRACT, \(v, g'\)) \(\neq\) lastmsg\(_u\)
then
if \(u = v\) then
stop\(_u \leftarrow false\)
contract()
else
send(ELIGIBLE-CONTRACT, \(v, g'\)) to each \(z \in N_u\),
\(z \neq w\) with \(\text{weight} \_u[z] = 0\) or \(\text{weight} \_u[z] \neq 0\) and \(g_z = g'\)
end if
lastmsg\(_u \leftarrow (\text{ELIGIBLE-CONTRACT}, v, g')\)
end if

Algorithm 17 upon receipt of (SET-GROUP-ID, \(g', g'', \text{newrank}\)) msg from \(w\)
\[
\text{\{update group id of all vertices in the groups \(g'\) and \(g''\) by maxrank \(_u\) by sending messages\}}
\]
if \(g_u \neq \text{newrank}\)
weight\(_u[w] \leftarrow 0\)
check-active()
g\(_u \leftarrow \text{newrank}\)
if status\(_u = \text{ACTIVE}\)
for all \(v \in N_u\) with \(\text{weight} \_u[v] \neq 0\)
do
if \(g_v = g'\) or \(g_v = g''\) or \(g_v = \text{newrank}\)
weight\(_u[v] \leftarrow 0\)
check-active()
send(SET-WEIGHT) to \(v\). (See Algorithm 19 for receipt of message)
end if
end for
send(SET-GROUP-ID, \(g', g'', \text{newrank}\)) to each \(x \in N_u\) where \(\text{weight} \_u[x] = 0\)
end if

Algorithm 18 synchronize()
\[
\text{\{waits for all nodes to reach the same step of algorithm\}}
\]
wait for \(n\) pulses

Algorithm 19 upon receipt of (SET-WEIGHT) msg from \(w\)
weight\(_u[w] \leftarrow 0\)
check-active()

Algorithm 20 check-active()
if \(\forall v \in N_u, \text{weight} \_u[v] = 0\)
status\(_u = \text{INACTIVE}\)
end if

Algorithm 21 upon receipt of (STOP, \(t\)) msg from \(w\)
\[
\text{\{broadcast the information on the number of groups in the network\}}
\]
if (STOP, \(t\)) \(\neq\) lastmsg\(_u\)
then
if \(t = false\)
stop\(_u \leftarrow false\)
send(STOP, \(t\)) to each \(x \in N_u\)
lastmsg\(_u \leftarrow (\text{STOP}, t)\)
end if
end if

2.3 Correctness

First, we bound the probability of error created by edges getting the same rank.

**Lemma 2.3.1.** The probability that two edges get the same rank in \(n\) trials is \(O(n^{-2})\).

**Proof.** The rank is a value from the set \(\{1...m^k\}\). The probability that two edges \(m\) and \(m'\) having the same rank, \(Pr[\text{rank}(m) = \text{rank}(m')]\) \(\leq \frac{1}{m^2}\)
Hence, \(Pr[\exists (m, m'): \text{rank}(m) = \text{rank}(m')] \leq \sum_{(m, m') \in E \times E} Pr[\text{rank}(m) = \text{rank}(m')] \leq \frac{m}{m^2} = \frac{1}{m}\)
Thus, using the union bound, probability that there exists two edges \(m\) and \(m'\) having the same rank in \(n\) iterations is \(\leq \frac{m}{m^2} \leq \frac{n}{m} = \frac{1}{m}.\) Now choose \(k \geq 5\). Then,
\(Pr[\text{rank}(m) = \text{rank}(m')] \leq \frac{1}{m^2} = O(n^{-2}).\) \(\square\)

The following Lemma proceeds exactly as in [8].

**Lemma 2.3.2.** A particular min-cut in \(G\) is produced by the contraction algorithm with probability \(\Omega(n^{-2})\).

**Proof.** Let \(c\) be the value of the mincut in \(G\). Each contraction reduces the number of vertices in the graph by one. Consider the contraction executed when the graph has \(r\) vertices. Since the contracted graph has a min-cut of at least \(c\), it must have minimum degree \(c\), and thus at least \(\frac{c}{r}\) edges. However, only \(c\) of these edges are in min-cut. Thus, a randomly chosen edge is in the min-cut with probability at most \(\frac{2}{c}\). The probability that we never contract a min-cut edge through all \(n-2\) contractions is at least \((1-\frac{2}{c}(1-\frac{2}{n-1})(1-\frac{2}{n-2})...(1-\frac{2}{3}) = \frac{n}{2} = \Omega(n^{-2})\) \(\square\)
Thus, the total number of messages is $\sum_{i=1}^{n} \frac{2m}{n}$. Hence the overall message complexity is $O(n^2)$. 

### 2.4 Complexity Analysis

#### 2.4.1 Message complexity

**Theorem 2.4.1.** The Karger’s distributed algorithm uses $O(mn^2)$ messages, in a single trial.

**Proof.** It is not hard to see that the most expensive steps in a trial are those of determination of $maxrank$ from local $maxrank$s($find\text{-}global\text{-}maxrank()$) and that of computing the mincut at the end($find\text{-}global\text{-}mincut()$). In $find\text{-}global\text{-}maxrank()$, each node sends its local $maxrank$ value to its neighbours and this is repeated atmost $n$ times(number of times equal to the diameter of the graph suffices). Hence the total number of messages is bounded by $nO(m + n) = O(mn)$. Thus the message complexity for $n - 2$ iterations per trial is $O(mn^2)$. Finally, in step $i$ of $find\text{-}global\text{-}mincut()$, $\frac{2m}{n}$ nodes send messages to its neighbours. The total number of messages sent at each step is bounded by $O(m)$. Thus, the total number of messages is $\sum_{i=1}^{n} \frac{2m}{n} = O(mn)$. Hence the overall message complexity is $O(mn^2) + O(mn) = O(mn^2)$. 

#### 2.4.2 Time complexity

**Theorem 2.4.2.** The Karger’s distributed algorithm computes mincut in $O(n^2)$ time, in a single trial.

**Proof.** Before contraction, the algorithm assigns a rank (random number) to each edge and finds the max-rank among all the vertices in the graph. This requires atmost $n - 1$ steps(strictly, number of steps equal to the diameter of the graph). For contraction, a message is sent from a vertex within one group to other group and the message is propagated to all the vertices within the group, which also takes atmost $n - 1$ pulses. Since only one contraction can take place at any time and there are $n - 2$ such contractions, the running time is $O(n^2)$. To estimate time for computing the mincut, the function $find\text{-}global\text{-}mincut()$ runs $O(\log n)$ steps and in step $i$, $\frac{2m}{n}$ nodes flood the network. Thus the time per step is $\frac{2m}{n^2}$. Hence the total complexity is $\sum_{i=1}^{n} \frac{2m}{n^2} = O(n^2)$. 

### 3. Conclusion and Future Work

A synchronous distributed version of the Karger’s randomized algorithm under network setting is presented in this paper with a proof of correctness and complexity analysis. The present algorithm appears not to make use of the full power of parallelism available. It is interesting to look at how to efficiently reduce time and message complexity by conducting edge contractions in parallel.

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