Cooperative conditions
for the existence of rainbow matchings

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Abstract
Let $k > 1$, and let $\mathcal{F}$ be a family of $2n + k - 3$ non-empty sets of edges in a bipartite graph. If the union of every $k$ members of $\mathcal{F}$ contains a matching of size $n$, then there exists an $\mathcal{F}$-rainbow matching of size $n$. Replacing $2n + k - 3$ by $2n + k - 2$, the result is true also for $k = 1$, and it can be proved (for all $k$) both topologically and by a relatively simple combinatorial argument. The main effort is in gaining the last 1, which makes the result sharp.

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1 Introduction

Throughout the paper, “family” means “multiset”, meaning that elements may repeat. To differentiate the notation, we use round brackets for families, and (as usual) curly brackets for sets. For a family \( \mathcal{F} \), we write \( \mathcal{F} \setminus \{ F \} \) and \( \mathcal{F} \cup \{ F \} \) in the family sense. That is, \( \mathcal{F} \setminus \{ F \} \) contains one less copy of \( F \) than \( \mathcal{F} \) if \( F \in \mathcal{F} \), and \( \mathcal{F} \cup \{ F \} \) contains one more copy of \( F \) than \( \mathcal{F} \).

Given a family \( S = (S_1, \ldots, S_m) \) of sets, an \( S \)-rainbow set is the image of a partial choice function of \( S \). So, it is a set \( \{ x_{i_j} \mid j \leq k \} \), where \( 1 \leq i_1 < \cdots < i_k \leq m \) and \( x_{i_j} \in S_{i_j} \).

A complex is a closed down hypergraph, meaning that any subset of any edge is an edge. The injectivity - at most one element from every set \( S_i \) - is a “smallness” condition, in the sense that the set of injective choices is a complex. Hence statements of interest are of the form “there exists a large rainbow set satisfying certain conditions (like being a matching)”.

Below, again, \( S = (S_1, \ldots, S_m) \) is a family of sets. For a set \( I \subseteq [m] \), let \( S_I = \bigcup_{i \in I} S_i \).

**Theorem 1.** If \( |S_J| \geq |J| \) for every \( J \subseteq [m] \) then there is a full rainbow set, that is, a rainbow set of size \( m \).

Another well-known rainbow result is Drisko’s theorem, on rainbow matchings. The following slightly more general version of the original theorem was proved in [1]:

**Theorem 2.** [7] \( 2n - 1 \) matchings in a bipartite graph, of size \( n \) each, have a rainbow matching of size \( n \).

There is a conspicuous difference between the two theorems: in the first the condition is “cooperative”, namely it is on subfamilies of \( S \), whereas in the second it is on singletons - each \( S_i \) is assumed to be large by itself. On the other hand, there is a condition on the number of the sets \( S_i \).

### 1.1 A cooperative version of the Kalai-Meshulam theorem

A complex \( C \) is said to be \( d \)-Leray if \( \tilde{H}_k(C[S]) = 0 \) for all \( S \subseteq V \) and all \( k \geq d \) (\( \tilde{H}_k \) is the reduced \( k \)-th homology group). Let \( \lambda(C) \) be the smallest number \( d \) such that \( C \) is \( d \)-Leray.

A basic result in this direction is a theorem of Kalai and Meshulam [11]:

**Theorem 3.** Let \( \mathcal{M} \) and \( C \) be a matroid and a complex, respectively, on the same ground set. If \( \lambda(\text{lk}_C(S)) < \text{rank}_{\mathcal{M}}(V \setminus S) \) for every \( S \in C \) then \( \mathcal{M} \setminus C = \emptyset \).

Here \( \text{lk}_C(S) = \{ T \subseteq V \setminus S \mid S \cup T \in C \} \). The theorem above is a re-formulation of Theorem 1.6 in [11]. The following was proved in [12]:

**Theorem 4.** For any complex \( C \) and set \( S \in C \), \( \lambda(\text{lk}_C(S)) \leq \lambda(C) \).

Theorems 3 and 4, combined, yield the following:
Theorem 5. If $\lambda(C) \leq d$ and $S = (S_1, \ldots, S_{d+k})$ is a family of subsets of $V(C)$ satisfying $S_i \notin C$ whenever $I \subseteq [d+k]$ is of size $k$, then there exists an $S$-rainbow non-$C$ set.

Proof. By duplicating vertices, if necessary (a vertex having a distinct copy for every set $S_i$ it belongs to), we may assume that the sets $S_i$ are disjoint. Let $\mathcal{M}$ be the partition matroid defined by the sets $S_i$. By Theorems 4 and 3 it suffices to show that if $S \in C$ then $\text{rank}_{\mathcal{M}}(V \setminus S) > d$. This follows from the condition $S_i \notin C \ (|I| \geq k)$ and the fact that $\text{rank}_{\mathcal{M}}(A) = |\{i : A \cap S_i \neq \emptyset\}|$. \qed

This is a “cooperative” version of the Kalai-Meshulam theorem, namely many sets join forces to contain a set not belonging to $C$.

1.2 A cooperative version of Theorem 2

For a set $F$ of edges we denote by $\nu(F)$ the maximal size of a matching in $F$. For a family $\mathcal{F} = (F_1, \ldots, F_m)$ of sets of edges, we denote by $\nu_{\mathcal{F}}(\mathcal{F})$ the maximal size of an $\mathcal{F}$-rainbow matching.

Let $t$ be an integer, and let $n \leq t$. Let $C$ be the complex consisting of all $F \subseteq E(K_{t,t})$, satisfying $\nu(F) < n$. In [3] it was shown that $\lambda(C) \leq 2n - 2$. Together with Theorem 5 this yields:

Theorem 6. $2n + k - 2$ sets of edges in a bipartite graph, the union of any $k$ of which contains a matching of size $n$, have a rainbow matching of size $n$.

Notation 7. We write $(m, k, n) \rightarrow_{\mathcal{B}} q$ for the statement “every $m$ nonempty sets of edges in a bipartite graph, the union of every $k$ of which contains a matching of size $n$, have a rainbow matching of size $q$”.

In this notation, the theorem says that $(2n + k - 2, k, n) \rightarrow_{\mathcal{B}} n$. The case $k = 1$ is Theorem 2. The main result of this paper is that for $k > 1$ this can be improved by 1, thereby obtaining a sharp bound.

Theorem 8. $(2n + k - 3, k, n) \rightarrow_{\mathcal{B}} n$ whenever $1 < k \leq n$.

The sharpness of this result, namely the fact that $(2n + k - 4, k, n) \not\rightarrow_{\mathcal{B}} n$ for any $k$, is given by the following example. In $C_{2n}$, take the odd edges matching repeated $n - 1$ times, the even edges matching repeated $n - 2$ times, and a singleton set, consisting of an even edge, repeated $k - 1$ times. Explicitly:

Example 9. Consider a complete bipartite graph $K_{n,n}$ with sides $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$. Let

$$S_i = \begin{cases} \{a_1b_1, a_2b_2, \ldots, a_nb_n\} & \text{if } i \in [n-1], \\ \{a_1b_2, a_2b_3, \ldots, a_{n-1}b_n, a_nb_1\} & \text{if } i \in [2n-3] \setminus [n-1], \\ \{a_1b_2\} & \text{if } i \in [2n + k - 4] \setminus [2n - 3]. \end{cases}$$

Let $S = (S_i \mid i = 1, \ldots, 2n + k - 4)$. Then for any $I \subseteq [2n + k - 4]$ with $|I| \geq k$, $\nu(S_I) \geq n$, and $\nu_{\mathcal{B}}(S) < n$. 

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Remark 10. After our result was obtained, Holmsen and Lee [10] gave a topological proof of Theorem 8, using a strong version of Theorem 3. Their result is a somewhat stronger version of Theorem 8.

1.3 Cooperative versions of Colorful Caratheodory

Part of the motivation for Theorem 8 comes from the existence of cooperative versions of a famous rainbow result - Bárány’s Colorful Caratheodory theorem [6]. In fact, as we shall see below (first proof of Theorem 25), the affinity is not merely formal. Theorem 6 follows from a cooperative version of Colorful Caratheodory.

Wegner [13] noted that the complex $C$ of sets of vectors in $\mathbb{R}^d$ not containing a given vector $v$ in their convex hull satisfies $\lambda(C) = d$. Similarly, the complex $D$ of sets not containing $v$ in their cone (set of non-negative combinations) satisfies $\lambda(D) = d - 1$. This, together with Theorem 5, yields:

**Theorem 11.** Let $v \in \mathbb{R}^d$.

1. If $S = (S_1, \ldots, S_{d+k})$ is a family of subsets of $\mathbb{R}^d$ such that $v \in \text{conv}(S_K)$ for every $K \subseteq [d+k]$ of size $k$, then there exists an $S$-rainbow set $S$ such that $v \in \text{conv}(S)$.

2. If $S = (S_1, \ldots, S_{d+k-1})$ is a family of subsets of $\mathbb{R}^d$ such that $v \in \text{cone}(S_K)$ for every $K \subseteq [d+k-1]$ of size $k$, then there exists an $S$-rainbow set $S$ such that $v \in \text{cone}(S)$.

The case $k = 2$ of part (1) of the theorem was strengthened by Holmsen-Pach-Tverberg [9] and Arocha et.al. [5]:

**Theorem 12.** If $S_1, \ldots, S_{d+1}$ are non-empty sets in $\mathbb{R}^d$, and $v \in \text{conv}(S_i \cup S_j)$ whenever $1 \leq i < j \leq d + 1$, then there is a rainbow set $S$ with $v \in \text{conv}(S)$.

Holmsen [8] gave a topological proof of this result, using a notion he called “near $d$-Lerayness”, which means that $lk_C(S)$ is $d$-Leray for every non-empty $S \in C$. The same argument can be used to prove the analogous strengthening for all $k > 1$:

**Theorem 13.** Let $k > 1$, and let $S = (S_1, \ldots, S_{d+k-1})$ be a family of non-empty sets in $\mathbb{R}^d$, such that every $k$ of them contain $v$ in the convex hull of their union. Then there is an $S$-rainbow set containing $v$ in its convex hull.

The analogous strengthening of part (2) of Theorem 11 is false, as witnessed by simple counterexamples.

**Example 14.** Let $v_1, \ldots, v_{d+1}$ be the vertices of a $d$-dimensional simplex $\sigma \subseteq \mathbb{R}^d$ whose barycenter is the origin. Let $v$ be the barycenter of face $\{v_1, \ldots, v_d\}$ of $\sigma$. Consider the family $S = (S_1, \ldots, S_{d+k-2})$ of non-empty sets in $\mathbb{R}^d$, where $S_i = \{v_1, \ldots, v_d\}$ for $1 \leq i \leq d - 1$ and $S_j = \{v_{d+1}\}$ for $d \leq j \leq d + k - 2$. Among any $k$ sets in $S$, at least one is $S_i$ for some $1 \leq i \leq d - 1$, hence the convex cone spanned by their union contains $v$. However, there is no $S$-rainbow set $S$ such that $v \in \text{cone}(S)$. 

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2 Rainbow paths

The proof of Theorem 8 is based on a combinatorial proof of the result $(2n+k-2, k, n) \to B n$, and analysis of the extreme case. This proof, in turn, uses a lemma on rainbow paths in networks. To get the extra 1 we analyze the extreme cases of that lemma. The analysis uses ideas from an analogous lemma in [4], which is the case $k = 1$. But apart from a higher level of complexity, there is the difference that for $k > 1$ the analysis leads to an improvement of 1 in the theorem - which was not the case for $k = 1$.

A network is a triple $\mathcal{N} = (D, s, t)$, where $D$ is a digraph, and $s, t$ are two special vertices in it, called source and target. We assume that there are no edges going out of $t$ or into $s$. We write $V(\mathcal{N})$ for $V(D)$. The set $V(\mathcal{N}) \setminus \{s, t\}$ is denoted by $V^o(\mathcal{N})$, and its elements are called “inner vertices”. For an $s - t$ path $P$ let $V^o(P) = V^o(\mathcal{N}) \cap V(P)$. Two $s - t$ paths $P, Q$ are said to be internally disjoint if $V^o(P) \cap V^o(Q) = \emptyset$.

For an $s - t$ path $Q$ let $B(Q)$ be the set of backward edges on $Q$, namely those directed edges $pq$ where $p, q \in V(Q)$ and $q$ precedes $p$ on $Q$. Let $s_Q$ be the vertex following $s$ in $Q$, and $t_Q$ the vertex preceding $t$ in $Q$. Define $U(Q) = \{vs_Q \mid v \in V^o(\mathcal{N}) \setminus V(Q)\} \cup \{tu \mid u \in V^o(\mathcal{N}) \setminus V(Q)\}$. (“$U$” stands for “useless”, since such edges cannot be used as shortcuts - this will be clarified below).

We shall borrow a term - “regimented” - from [4], but its use is a bit different here.

Definition 15. Let $\mathcal{F}$ be a family of sets of edges in $\mathcal{N}$. A regimentation of $\mathcal{F}$ is a pair $\mathcal{R} = (Q = Q(\mathcal{R}), I = I(\mathcal{R}))$, where $Q$ is a set of internally disjoint $s - t$ paths, and $I$ is a function from a subset $\mathcal{E} = \mathcal{E}(\mathcal{R})$ of $\mathcal{F}$ (the “essential” sets) onto $Q$, satisfying the following conditions:

1. $\bigcup_{Q \in Q} V(Q) = V(\mathcal{N})$,
2. $E(I(F)) \subseteq F$ for every $F \in \mathcal{E}$, and
3. $|I^{-1}(Q)| = |E(Q)| - 1$ for every $Q \in Q$.

Let $\mathcal{E}(\mathcal{R}) = \mathcal{F} \setminus \mathcal{E}(\mathcal{R})$ (the “inessential” sets) and $B(\mathcal{R}) = \bigcup_{Q \in Q} B(Q)$.

If such a regimentation $\mathcal{R}$ exists, we say then that $\mathcal{F}$ is regimented by $\mathcal{R}$.

Conditions (1) and (3) imply:

Lemma 16. $|\mathcal{E}(\mathcal{R})| = |V^o(\mathcal{N})|$. 

Proof. Since $\mathcal{E}(\mathcal{R}) = \bigcup_{Q \in Q} I^{-1}(Q)$, we have $|\mathcal{E}(\mathcal{R})| = \sum_{Q \in Q} |I^{-1}(Q)|$. Then by the condition (3) of a regimentation, we have

$$|\mathcal{E}(\mathcal{R})| = \sum_{Q \in Q} |I^{-1}(Q)| = \sum_{Q \in Q} (|E(Q)| - 1) = \sum_{Q \in Q} |V^o(Q)|.$$ 

Since $Q$ is a set of internally disjoint $s - t$ paths, the condition (1) of a regimentation implies $\sum_{Q \in Q} |V^o(Q)| = |V^o(\mathcal{N})|$, and hence we obtain $|\mathcal{E}(\mathcal{R})| = |V^o(\mathcal{N})|$. 

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**Notation 17** (Pruning and concatenation of paths). If $P$ is a directed path and $x \in V(P)$ then $Px$ is the part of $P$ up to and including $x$, and $xP$ is the part of $P$ starting at $x$. If two paths $P$ and $Q$ meet at a vertex $x$, then $P_xQ$ denotes the walk obtained by concatenating $Px$ and $xQ$. If the endpoint of a path $P$ coincides with the initial point in a path $Q$, we write $PQ$ for the walk that is the concatenation of $P$ and $Q$.

**Lemma 18.** Suppose $\mathcal{F}$ is regimented by $\mathcal{R} = (Q, I)$, and let $B = B(\mathcal{R}), \mathcal{IE} = \mathcal{IE}(\mathcal{R})$. If there is no $\mathcal{F}$-rainbow $s - t$ path, then $\bigcup \mathcal{IE} \subseteq B$ and $\bigcup I^{-1}(Q) \subseteq E(Q) \cup B \cup U(Q)$ for every $Q \in \mathcal{Q}$.

(For a set $\mathcal{K}$ of sets $\bigcup \mathcal{K}$ is the union of all sets in $\mathcal{K}$.)

**Proof.** Let $vu$ be an edge belonging to $F$ for some $F \in \mathcal{F}$. Assume that $v \in V(Q_1)$, $u \in V(Q_2)$. Let $P = Q_1vuQ_2$ (see Notation 17).

To obtain the conclusion of the lemma, we will show the following.

1. When $Q_1 = Q_2$, $P$ is an $\mathcal{F}$-rainbow $s - t$ path unless $vu \in B(Q_1)$ or $vu \in E(Q_1)$ and $F \in I^{-1}(Q_1)$.

2. When $Q_1 \neq Q_2$, $P$ is an $\mathcal{F}$-rainbow $s - t$ path unless $v = t_{Q_1}$ and $F \in I^{-1}(Q_1)$, or $u = s_{Q_2}$ and $F \in I^{-1}(Q_2)$.

First suppose that $Q_1 = Q_2$. If $v$ precedes $u$ on $Q_1$ and $vu \notin E(Q_1)$, then $P$ is an $\mathcal{F}$-rainbow $s - t$ path, since by part (3) of Definition 15 it has enough represented sets for its length. If $vu \in E(Q_1)$, then $P$ is an $\mathcal{F}$-rainbow $s - t$ path unless $F \in I^{-1}(Q_1)$. This proves (1).

Now assume $Q_1 \neq Q_2$. We may assume that $v \in V^\circ(Q_1)$ and $u \in V^\circ(Q_2)$ since if not the claim is a special case of (1). Then $Q_1vu$ and $uQ_2$ are rainbow, and they have enough represented sets in $I^{-1}(Q_1)$ and $I^{-1}(Q_2)$, respectively. If $F \notin I^{-1}(Q_1) \cup I^{-1}(Q_2)$, then $P$ is rainbow. If $F \in I^{-1}(Q_1)$ and $v \neq t_{Q_1}$, then $Q_1vu$ is rainbow since it has enough represented sets in $I^{-1}(Q_1)$, since it has length at most $|E(Q_1)| - 1$. Similarly if $F \in I^{-1}(Q_2)$ and $u \neq s_{Q_2}$, then $vuQ_2$ is rainbow since it has enough represented sets in $I^{-1}(Q_2)$. In both cases $P$ is rainbow, which proves (2).

Since we assume there is no $\mathcal{F}$-rainbow $s - t$ path, if $F \in \mathcal{IE}$, then $vu \in B$ by (1) and (2). Thus $\bigcup \mathcal{IE} \subseteq B$. If $F \in I^{-1}(Q)$ for some $Q \in \mathcal{Q}$, then $vu \in E(Q) \cup B \cup U(Q)$ by (1) and (2). Thus $\bigcup I^{-1}(Q) \subseteq E(Q) \cup B \cup U(Q)$. \hfill $\square$

**Corollary 19.** Let $\mathcal{F}$ be regimented by $\mathcal{R}$, and assume that there is no $\mathcal{F}$-rainbow $s - t$ path. If $F \in \mathcal{IE}(\mathcal{R})$ then $F$ does not contain an $s - t$ path.

In fact, $F$ does not even contain an edge $sy$.

**Lemma 20.** Let $P, Q$ be $s - t$ paths in a network $(D, s, t)$. If $E(P) \subseteq E(Q) \cup B(Q) \cup \tilde{B} \cup U(Q)$ for some collection $\tilde{B}$ of edges that are vertex-disjoint from $Q$, then $P = Q$. 

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Proof. The only edge leaving $s$ in $E(Q) \cup B(Q) \cup \tilde{B} \cup U(Q)$ is $ss_Q \in E(Q)$, and the only edge to $t$ is $t_Q t \in E(Q)$. So these are necessarily the first and last edges of $P$. Therefore $P$ has no edges from $U(Q)$, since the in-degree of $s_Q$ and the out-degree of $t_Q$ in $P$ are 1.

As $E(Q) \cup B(Q)$ and $\tilde{B}$ are disconnected, $E(P) \cap \tilde{B} = \emptyset$. It remains to show that $E(P) \cap B(Q) = \emptyset$, which follows from the fact that $P$ does not repeat vertices. $$\square$$

Combining Lemmas 18 and 20 yields:

**Corollary 21.** Let $F$ be regimented by $R$, and having no rainbow $s - t$ path. If $F \in \mathcal{E}(R)$ then $I(F)$ is the only $s - t$ path contained in $F$.

By Corollaries 19 and 21, we can obtain the following corollary.

**Corollary 22.** Let $F$ be regimented by $R$, and having no rainbow $s - t$ path. Then $F \in \mathcal{E}(R)$ if and only if $F$ contains an $s - t$ path, and equivalently, $F \in \mathcal{IE}(R)$ if and only if $F$ does not contain an $s - t$ path.

The following argument will be used twice, and hence it receives separate mention:

**Lemma 23.** Let $G, H$ be two families of sets of edges, none of which possesses a rainbow $s - t$ path. Suppose that $G$ is regimented by $R = (Q, I)$ and $H$ is regimented by $S = (P, J)$. Suppose that $G \setminus H$ consists of a single set of edges $G$, and $H \setminus G$ consists of single set of edges $H$. Then either $G \in \mathcal{IE}(R)$ and $H \in \mathcal{IE}(S)$, or $I(G) = J(H)$.

**Proof.** Let $K = G \cap H$. So $G = K \cup \{G\}$, $H = K \cup \{H\}$.

By Corollary 22, it is obvious that

$$K \cap \mathcal{E}(R) = K \cap \mathcal{E}(S). \tag{1}$$

By Corollary 21, $I(K) = J(K)$ for every $K \in K \cap \mathcal{E}(R)$. Hence

$$\bigcup_{K \in \mathcal{E}(R) \setminus \{G\}} V(I(K)) = \bigcup_{K \in \mathcal{E}(S) \setminus \{H\}} V(J(K)) \tag{2}$$

Let us first show that $G \in \mathcal{IE}(R)$ if and only if $H \in \mathcal{IE}(S)$. Suppose that $G \in \mathcal{IE}(R)$. Then $\mathcal{E}(R) \subseteq K$. By (1) and Lemma 16, it follows that $\mathcal{E}(S) = \mathcal{E}(R)$, so $H \in \mathcal{IE}(S)$. The converse implication is the same.

Assume next that $G \in \mathcal{E}(R)$ and $H \in \mathcal{E}(S)$. Let $Q_0 = I(G)$. Consider first the case that $V^\circ(Q_0)$ consists of a single vertex $v$. We have $\bigcup_{K \in \mathcal{E}(R) \setminus \{G\}} V(I(K)) = V^\circ \setminus \{v\}$, and hence by (2) we have also $\bigcup_{K \in \mathcal{E}(S) \setminus \{H\}} V(J(K)) = V^\circ \setminus \{v\}$. Since the interiors of paths in $P$ partition $V^\circ$, it follows that $J(H)$ is the path $svt$, namely $Q_0$.

It remains to consider the case $|V^\circ(Q_0)| > 1$. Then, not counting multiplicities, $P = Q$, because every path of $Q$ appears as $J(K)$ for some $K \in K$. The only path in $P$ not covered enough times by paths $J(K)$, $K \in \mathcal{E}(S) \setminus \{H\}$, is $Q_0$. So, necessarily $J(H) = Q_0$. $$\square$$

The next theorem is the main step towards the proof of Theorem 8.
Theorem 24. Let $\mathcal{N} = (D, s, t)$ be a network with $n$ inner vertices. Let $\mathcal{F}$ be a family of $n + k − 1$ sets of edges in $\mathcal{N}$, satisfying the condition that $\bigcup \mathcal{K}$ contains an $s − t$ path, for every $\mathcal{K} \subseteq \mathcal{F}$ of size $k$. Then either there exists an $\mathcal{F}$-rainbow $s − t$ path, or $\mathcal{F}$ is regimented.

The case $k = 1$ of the theorem is Theorem 3.3 in [4].

It is worth noting that the weaker result, with $\mathcal{F}$ being of size $n + k$, is not hard. First, the statement:

Theorem 25. Let $\mathcal{N} = (D, s, t)$ be a network with $n$ inner vertices. Let $\mathcal{F}$ be a family of $n + k$ sets of edges in $\mathcal{N}$, satisfying the condition that $\bigcup \mathcal{K}$ contains an $s − t$ path for every $\mathcal{K} \subseteq \mathcal{F}$ of size $k$. Then there exists an $\mathcal{F}$-rainbow $s − t$ path.

Here are two proofs:

Proof 1. Observe that a set $H$ of edges in $\mathcal{N}$ contains an $s − t$ path if and only if the cone of $\{ \chi_b - \chi_a \mid ab \in H \}$ contains the vector $\chi_t - \chi_s$ (here $\chi_v$ is the function that is 1 on $v$ and 0 on all other vertices). Also note that all these vectors reside in an $n + 1$-dimensional space (they are of length $n + 2$, but all are perpendicular to the all-1 vector). Apply now Theorem 11, part (2).

Proof 2. Take a maximal $\mathcal{F}$-rainbow tree $T$ rooted at $s$. Assume, for contradiction, that it does not reach $t$. Then it represents at most $n$ members of $\mathcal{F}$. Hence there are $k$ sets $F \in \mathcal{F}$ not represented in $T$. By assumption, their union contains an $s − t$ path. The first edge leaving $T$ can then be added to $T$ to yield a larger rainbow tree, which contradicts the maximality of $T$.

Definition 26 (contracting an edge $sx$). Let $sx$ be an edge of $\mathcal{N}$. We can contract $sx$ to a newly defined vertex $s'$, that will serve as the source of a new network $\mathcal{N}'$. Here is what this does to sets of edges, and to paths.

1. Let $F$ be a set of edges in a network $\mathcal{N} = (D, s, t)$, and let $sx$ be an edge, where $x$ is an inner vertex. The contracted set of edges $F|_{sx\to s'}$ is obtained from $F$ by replacing every edge $sy$ or $xy$ belonging to $F$ by the edge $s'y$, and removing all edges $yx$.

2. An $s − t$ path $P$ is transformed by the contraction of $sx$ to an $s' − t$ path $P'$, defined as follows. If $x \notin V(P)$ then $P' = P$ with $s'$ replacing $s$. If $x \in V(P)$ then $P' = s'yP$ where $y$ is the vertex following $x$ in $P$ (so, the vertices in $Px$ disappear.) We also write $P' = P|_{sx\to s'}$. Note that in this definition $E(P')$ is not necessarily equal to $E(P)|_{sx\to s'}$.

Proof of Theorem 24. By induction on $n$. The case $n = 0$ is easy. So let $n \geq 1$ and assume that the theorem is valid when $n − 1$ replaces $n$.

Since $n + k − 1 \geq k$, $\bigcup \mathcal{F}$ contains an $s − t$ path. So there exists at least one set $G \in \mathcal{F}$ containing an edge $sx$. If $x = t$ then the path $st$ is rainbow, so we may assume that $x \neq t$.

Now contract $sx$: for each $F \in \mathcal{F}$ let $F' = F|_{sx\to s'}$. Let $\mathcal{K}' = \{ F' \mid F \in \mathcal{F} \}$ for $\mathcal{K} \subseteq \mathcal{F}$. Let $\mathcal{N}'$ be the network with vertex set $V(\mathcal{N}) \setminus \{s, x\} \cup \{s'\}$, source $s'$, target $t$, and edge set $\bigcup(\mathcal{F}' \setminus \{ G' \})$.
Every $\mathcal{K} \subseteq \mathcal{F}$ of size $k$ contains in its union the edge set of an $s - t$ path in $\mathcal{N}$, which is easily seen to imply the same, with $s'$ replacing $s$, for $\mathcal{K}'$ in $\mathcal{N}'$. By the induction hypothesis, either there exists an $\mathcal{F}' \setminus \{G'\}$-rainbow $s' - t$ path $P'$, or $\mathcal{F}' \setminus \{G'\}$ is regimented.

In the first case, let $y$ be the vertex following $s'$ in $P'$. Then either $syP'$ or $sxyP'$ is a rainbow $s - t$ path in $\mathcal{N}$, and we are done. So, we may assume the second possibility. Let $\mathcal{R}' = (Q', I')$ be a regimentation of $\mathcal{F}' \setminus \{G'\}$, and let $\mathcal{E}' = \mathcal{E}(\mathcal{R}')$, $\mathcal{IE}' = \mathcal{IE}(\mathcal{R}')$.

Let $\mathcal{IE} = (F \in \mathcal{F} \setminus \{G\} \mid F' \in \mathcal{IE}')$ and $\tilde{\mathcal{E}} = (F \in \mathcal{F} \setminus \{G\} \mid F' \in \mathcal{E}')$.

By Lemma 16 $|\mathcal{E}'| = n - 1$, so

$$|\mathcal{IE}| = |\mathcal{IE}'| = k - 1. \quad (3)$$

In all claims below we assume that there is no $\mathcal{F}$-rainbow $s - t$ path.

Let $B' = \bigcup_{Q' \in \mathcal{Q}'} B(Q')$. By Lemma 18, $\bigcup \mathcal{IE}' \subseteq B'$ and $\bigcup I'^{-1}(Q') \subseteq E(Q') \cup B' \cup U(Q')$ for every $Q' \in \mathcal{Q}'$.

**Notation 27** (two ways of un-contracting $sx$). Given an $s' - t$ path $Q'$ in $\mathcal{N}'$, let $Q'^{(1)}$ be the path obtained from $Q'$ by replacing $s'$ with $s$ and $Q'^{(2)}$ the path obtained from $Q'$ by expanding its first edge $s'y$ to the path $sxy$.

Our aim is to glean from $\mathcal{R}'$ a regimentation $\mathcal{R} = (Q, I)$ of $\mathcal{F}$. The set $\mathcal{E}(\mathcal{R})$ will contain $G$ and $\mathcal{Q}$ will contain $s - t$ paths $f(Q')$, $Q' \in \mathcal{Q}'$, where $f$ is an injective function defined as follows. Let $Q' \in \mathcal{Q}'$ and let $F \in \mathcal{F} \setminus \{G\}$ be such that $I'(F') = Q'$. By (3) and the condition of the theorem, the set $F \cup \bigcup \mathcal{IE}$ contains an $s - t$ path $Q$. Let $f(Q') = Q$.

**Claim 28.** $Q' = Q|_{s \to s'}$.

**Proof.** By the choice of $Q$, we have $E(Q|_{s \to s'}) \subseteq F' \cup \bigcup \mathcal{IE}'$. By Lemma 18, we have $F' \cup \bigcup \mathcal{IE}' \subseteq E(Q') \cup B' \cup U(Q') = E(Q') \cup B(Q') \cup \bigcup_{T' \in Q' \setminus \{G\}} B(T') \cup U(Q')$. The claim now follows by Lemma 20. \hfill $\Box$

There are two possibilities:

(a) $x \notin V(Q)$. In this case $Q = Q'^{(1)}$.

(b) $x \in V(Q)$. Suppose, in this case, that $Qx$ contains inner vertices. Let $y$ be the first inner vertex of $Qx$. Then $y \in V^o(T')$ for some $T' \in Q' \setminus \{Q'\}$, and then $syT'$ is a rainbow $s - t$ path in $\mathcal{N}$ since it has enough represented sets in $I'^{-1}(T') \cup \{G\}$. So, we may assume that $V^o(Qx) = \emptyset$, meaning that the first edge on $Q$ is $sx$, meaning in turn that $Q = Q'^{(2)}$.

**Claim 29.** $sx \notin \bigcup \mathcal{IE}$.

**Proof.** Let $F_0 \in \mathcal{IE}$ and suppose that $sx \in F_0$. Recall that $\mathcal{F}'$ is the family of sets of edges obtained, where, for every $F \in \mathcal{F}$, $F'$ is the image of $F$ under the contraction of $sx$. By the same argument as above, $\mathcal{F}' \setminus \{F_0\}$ is regimented in $\mathcal{N}'$, by a regimentation $\mathcal{T} = (Q(\mathcal{T}), J)$. Then $G' \in \mathcal{IE}(\mathcal{T})$ by Lemma 23, and hence $G$ do not contain an edge $yt$. But this would imply that $G \cup \mathcal{IE}(\mathcal{R})$ does not contain such an edge, and hence that it does not contain an $s - t$ path, contrary to the assumption of the theorem. \hfill $\Box$
Since $E(Q) \subseteq F \cup \bigcup \mathcal{I}E$ and $\bigcup \mathcal{I}E' \subseteq B'$ by Lemma 18, a corollary of Claim 29 is:

$$E(Q) \subseteq F.$$ \hspace{1cm} (4)

**Claim 30.** The choice of $f(Q')$ is independent of the choice of $F$.

**Proof.** We have to show that if $F_1, F_2 \in \mathcal{F} \setminus \{G\}$ satisfy $I'(F'_i) = Q'$, $i = 1, 2$ and $Q_i$ are $s-t$ paths whose edge sets are contained in $F_i \cup \mathcal{I}E$ ($i = 1, 2$) then $Q_1 = Q_2$. We know that $Q_i$ are either $Q^{(1)}$ or $Q^{(2)}$. Assume, for contradiction, that $Q_1 \neq Q_2$, say $Q_1 = Q^{(1)}$ and $Q_2 = Q^{(2)}$. Then $sx \in E(Q_2)$ and hence $sx \in F_2$. The set $\mathcal{F} \setminus \{F'_2\}$ lives in $\mathcal{N}'$, and repeating the previous argument we deduce that it has a regimentation $S = (\mathcal{Q}(\mathcal{S}), J)$. By Lemma 23 $J(G') = I'(F'_2) = Q'$. In particular $G' \supseteq E(Q')$. Since $Q_1 = Q^{(1)}$, the edge $ssQ'$ belongs to $E(Q_1) \subseteq F_1$. Then, using an edge from $G$ and edges from the sets $F \in \mathcal{F}$ such that $F' \in I'^{-1}(Q')$ shows that $ssQ'Q' = Q^{(1)}$ is an $\mathcal{F}$-rainbow $s-t$ path (note that edges in $E(sQ'Q')$ are also edges of $F$). This is the desired contradiction. \hfill \Box

**Claim 31.**

1. If $f(Q') = Q^{(2)}$ then $G \supseteq E(f(Q'))$.

2. At most one $Q' \in \mathcal{Q}'$ satisfies $f(Q') = Q^{(2)}$.

3. If $f(Q') = Q^{(1)}$ for all $Q' \in \mathcal{Q}'$ then $G$ contains the edges of the $s-t$ path $sx$.

**Proof.** To prove (1), let $f(Q') = Q^{(2)}$ for some $Q' \in \mathcal{Q}'$.

Then, by Claim 30, $sx \in F$ for every $F' \in I'^{-1}(Q')$. We use the same trick as in the proof of Claim 30, interchanging the roles of $F$ and $G$. Consider $\mathcal{F} \setminus \{F'\}$. As above, we may assume that $\mathcal{F}' \setminus \{F'\}$ is regimented, by a regimentation $(\mathcal{P}', J')$. By Lemma 23, $J'(G') = I'(F') = Q'$, implying that $G' \supseteq E(Q')$. Then $G$ contains either $E(Q^{(1)})$ or $E(Q^{(2)})$. If $G$ contains $E(Q^{(1)})$, then $ssQ'Q'$ (which is just $Q^{(1)}$) is an $\mathcal{F}$-rainbow $s-t$ path: the edge $ssQ'$ represents $G$; since $|I'^{-1}(Q')| = |E(Q')| - 1$, the other edges have enough represented sets $F \in \mathcal{F}$ such that $F' \in I'^{-1}(Q')$ (remember that $G \notin I'^{-1}(Q')$). We have thus shown that $G$ does not contain $E(Q^{(1)})$, so it contains $E(Q^{(2)})$, namely $G \supseteq E(f(Q'))$.

Next we prove (2). Let $f(Q') = Q^{(2)}$ for some $Q' \in \mathcal{Q}'$. By the above argument and Corollary 21, $J'(G') = Q'$ is the only path contained in $G'$. This directly implies (2).

Finally, we prove (3). Assume that $f(Q') = Q^{(1)}$ for all $Q' \in \mathcal{Q}'$. Let $\mathcal{N}$ be the network obtained from $\mathcal{N}$ by deleting the vertex $x$, and let $\tilde{F}$ be the set of edges of $\mathcal{N}$, obtained from $F$ by deleting all edges incident with $x$. Let $\tilde{Q} = \{Q^{(1)} \mid Q' \in \mathcal{Q}'\}$, and $\tilde{I}(\tilde{F}) = f(I'(F'))$. By (4) and the assumption that $f(Q') = Q^{(1)}$ for all $Q' \in \mathcal{Q}'$ the set $\tilde{F} = (\tilde{F} \mid F \in \mathcal{F})$ is regimented by $(\tilde{Q}, \tilde{I})$. The fact that there is no $\mathcal{F}$-rainbow $s-t$ path implies that there is also no $\tilde{F}$-rainbow $s-t$ path. Therefore, by Lemma 18, we have $\tilde{G} \cup \bigcup_{F \in \tilde{I}E} \tilde{F} \subseteq \bigcup_{Q' \in \mathcal{Q}'} B(Q)$. Thus

$$G \cup \bigcup \mathcal{I}E \subseteq \{sx, xt\} \cup \bigcup_{Q' \in \mathcal{Q}'} B(Q^{(1)}) \cup U(sxt).$$
By the assumption of the theorem, \( G \cup \bigcup \tilde{\mathcal{I}} \mathcal{E} \) contains an \( s-t \) path, say \( Q_G \). By Lemma 20 we have \( Q_G = sxt \), and by Claim 29 we obtain \( G \supseteq E(Q_G) \). This concludes the proof of the claim. \( \square \)

**Remark 32.** By the claim the paths \( f(Q') \), \( Q' \in \mathcal{Q} \) are internally disjoint. In particular, there is at most one path \( f(Q') \) containing \( x \).

We can now complete the induction step in the proof of Theorem 24, by showing that \( \mathcal{F} \) is regimented.

**Case I:** \( f(Q') = Q'(1) \) for all \( Q' \in \mathcal{Q} \).

Let \( \mathcal{Q} = \{ f(Q') \mid Q' \in \mathcal{Q} \} \cup \{ Q_0 \} \) where \( Q_0 = sxt \). Let \( \mathcal{E} = (F \mid F' \in \mathcal{E}(\mathcal{R}')) \cup \{ G \} \). Define \( I : \mathcal{E} \to \mathcal{Q} \) by \( I(F) = f(I'(F')) \) for \( F \neq G \), and \( I(G) = Q_0 \).

**Claim 33.** \( (\mathcal{Q}, I) \) is a regimentation of \( \mathcal{F} \).

By Remark 32 and the fact that \( x \notin \bigcup_{Q' \in \mathcal{Q}} V(f(Q')) \), \( \mathcal{Q} \) is a set of internally disjoint \( s-t \) paths.

By (4) \( E(I(F)) \subseteq F \) for all \( F \in \mathcal{E} \setminus \{ G \} \), and by part (3) of Claim 31 \( E(I(G)) = E(Q_0) \subseteq G \). This implies condition (2) in Definition 15.

In addition,

\[
|I^{-1}(Q)| = |I'^{-1}(f^{-1}(Q))| = |E(f^{-1}(Q))| - 1 = |E(f^{-1}(Q)(1))| - 1 = |E(Q)| - 1
\]

for all \( Q \in \mathcal{Q} \setminus \{ Q_0 \} \), and

\[
|I^{-1}(Q_0)| = 1 = |E(Q_0)| - 1.
\]

This yields condition (3) of Definition 15.

Furthermore, since \( \bigcup_{Q' \in \mathcal{Q}} V^\circ(Q') = V^\circ(\mathcal{N}) \setminus \{ x \} \) and \( V^\circ(Q'(1)) = V^\circ(Q') \), we have

\[
\bigcup_{Q \in \mathcal{Q}} V^\circ(Q) = \bigcup_{Q' \in \mathcal{Q}'} V^\circ(Q'(1)) \cup \{ x \} = V^\circ(\mathcal{N}).
\]

This implies condition (1) of Definition 15, thus completing the proof of the claim.

**Case II:** \( f(Q_0) = Q_0''(2) \) for some \( Q_0' \in \mathcal{Q} \).

Let \( \mathcal{Q} = \{ f(Q') \mid Q' \in \mathcal{Q} \} \) and \( \mathcal{E} = (F \mid F' \in \mathcal{E}(\mathcal{R}')) \cup \{ G \} \). Define \( I : \mathcal{E} \to \mathcal{Q} \) by \( I(F) = f(I'(F')) \) for all \( F \in \mathcal{F} \setminus \{ G \} \) and \( I(G) = f(Q_0') \).

**Claim 34.** \( (\mathcal{Q}, I) \) is (here, too) a regimentation of \( \mathcal{F} \).

By Remark 32, \( \mathcal{Q} \) is a set of internally disjoint \( s-t \) paths.

By (4) \( E(I(F)) \subseteq F \) for \( F \in \mathcal{E} \setminus \{ G \} \), and by (1) of Claim 31 \( E(I(G)) = E(f(Q_0')) \subseteq G \), so condition (2) of Definition 15 is fulfilled.

In addition,

\[
|I^{-1}(Q)| = |I'^{-1}(f^{-1}(Q))| = |E(f^{-1}(Q))| - 1 = |E(f^{-1}(Q)(1))| - 1 = |E(Q)| - 1
\]
for all \( Q \neq f(Q'_0) \). On the other hand, for \( Q = f(Q'_0) \),
\[
|I^{-1}(Q)| = |I^{-1}(f^{-1}(Q))| + 1 = |E(f^{-1}(Q))| = |E(f^{-1}(Q)(2))| - 1 = |E(Q)| - 1.
\]

This proves condition (3) in Definition 15.

Furthermore, since \( \bigcup_{Q' \in \mathcal{Q}} V^o(Q') = V^o(N) \setminus \{x\} \), \( V^o(Q'(1)) = V^o(Q') \) and \( V^o(Q'(2)) = V^o(Q') \cup \{x\} \), we have
\[
\bigcup_{Q \in \mathcal{Q}} V^o(Q) = \bigcup_{Q' \in \mathcal{Q} \setminus \{Q'_0\}} V^o(Q'(1)) \cup V^o(Q'_0(2)) = V^o(N).
\]

So, condition (1) of Definition 15 is also valid, completing the proof of the theorem. \( \square \)

3 Proof of Theorem 8

Let us first state the theorem in a slightly stronger form, that allows some of the edge sets to be empty.

**Theorem 35.** Let \( S \) be a family of \( 2n+k-3 \) sets of edges in a bipartite graph \( G \), at most \( k-2 \) of them being empty. If \( \nu(\bigcup \mathcal{K}) \geq n \) for every \( K \subseteq S \) of size \( k \) then \( \nu_R(S) \geq n \).

Before proving the theorem, we need the following definition.

**Definition 36.** For a matching \( N \) in a graph, a path is called \( N \)-alternating if every other edge in it belongs to \( N \) and it is called \( \text{augmenting} \) if its starting edge and ending edge are not in \( N \).

**Proof.** Suppose, for contradiction, that \( \nu_R(S) =: m < n \). Let \( M = \{f_S \mid S \in S_0\} \) be a maximal size \( S \)-rainbow matching, where \( f_S \in S \). Let \( S_0^c = S \setminus S_0 \).

Let \( A, B \) be the two sides of \( G \). For every \( h \in E(G) \) let \( h_A \) be the \( A \)-vertex of \( h \), and \( h_B \) the \( B \) vertex.

We construct a network \( N \), having the property that its paths correspond to \( M \)-alternating paths, and its source-target paths correspond to augmenting \( M \)-alternating paths. Let \( V(N) = M \cup \{s,t\} \), where \( s \) represents \( U_A := A \setminus \bigcup M \), and \( t \) represents \( U_B := B \setminus \bigcup M \).

To every edge \( h = ab \in E(G) \setminus M \) (\( a \in A, b \in B \)) we assign an edge \( F(h) \) of \( N \), as follows.

1. If \( a \in f \in M, b \in g \in M \) then \( F(h) = fg \).
2. If \( a \in U_A \) and \( b \in g \in M \) then \( F(h) = sg \).
3. If \( b \in U_B \) and \( a \in f \in M \) then \( F(h) = ft \).
4. If \( a \in U_A \) and \( b \in U_B \) then \( F(h) = st \).
For a set $S$ of edges in $G$, let $F(S)$ be the set of edges in $N$, defined by $F(S) = \{ F(h) \mid h \in S \setminus M \}$. The function $F$ is not one-to-one, because the inverse image of an edge $sh$ ($h \in M$) can be any edge $ah_B, a \in U_A$.

Clearly, if $M \cup S$ contains an augmenting $M$-alternating path, then $F(S)$ contains an $s-t$ path in $N$, and vice versa. Let $F = \{ F(S) \mid S \in S_0 \}$.

Since, by assumption, $m < n$, $|S_0| = 2n - m + k - 3 \geq m + k - 1$. If $N$ is a matching of size $n$, then $M \cup N$ contains an augmenting $M$-alternating path, and hence $F(N)$ contains an $s-t$ path. Hence, by Theorem 24 and Theorem 25, either

(i) there exists an $F$-rainbow $s-t$ path $P$, or
(ii) $|S_0| = m + k - 1$ and $F$ is regimented.

In case (i), as mentioned above, $P$ yields an augmenting $M$-alternating path, whose application yields a larger rainbow matching. So we may assume (ii). Let $R = (Q, I)$ be the regimentation of $F$. Let $F^{-1}(IE(R)) = (S \in S_0 \mid F(S) \in IE(R))$. Since at most $k-2$ sets $S \in S$ are empty and $|IE(R)| = |S_0| - |E| = k - 1$ by Lemma 16, $\bigcup F^{-1}(IE(R))$ is non-empty.

**Claim 37. It is possible to choose $M$ so that $\bigcup IE(R) \neq \emptyset$.**

This means that $\bigcup F^{-1}(IE(R)) \setminus M \neq \emptyset$.

**Proof.** Assume, for contradiction, that $\bigcup F^{-1}(IE(R)) \subseteq M$. Since $\bigcup F^{-1}(IE(R))$ is non-empty, there is an element $S_0 \in S_0$ such that $f_{S_0} \in M \cap \bigcup F^{-1}(IE(R))$. Let $S_1$ be a set in $F^{-1}(IE(R))$ containing $f_{S_0}$. By the condition of the theorem, $\bigcup F^{-1}(IE(R)) \cup S_0$ contains a matching of size $n$. This, in turn, means that there exists an edge $f \in \bigcup F^{-1}(IE(R)) \cup S_0 \setminus M$. Since by assumption $\bigcup F^{-1}(IE(R)) \subseteq M$, we have $f \in S_0$.

Now we can consider $S_1 = (S_0 \setminus \{S_0\}) \cup \{S_1\}$ as a represented set of $M$ by changing the roles of $S_0$ and $S_1$. Let $\tilde{F} = (F(S) \mid S \in S_1)$. Then by the same reasoning as above, we may assume that $\tilde{F}$ is regimented by $\tilde{R} = (\tilde{Q}, \tilde{I})$. By Lemma 23, we have $F(S_0) \in IE(\tilde{R})$ and $f \in S_0 \setminus M$, which implies $\bigcup IE(\tilde{R}) \neq \emptyset$. \hfill \Box

So, we assume $\bigcup IE(R) \neq \emptyset$. Let $pq$ be an edge in $F(S)$ for some $F(S) \in IE(R)$. By Lemma 18, $pq$ is a backward edge on some path $Q \in Q$. Let $Q = s y_1 y_2 \ldots y_c t$. For each $1 \leq i < c$ let $e_i$ be the edge connecting the $(y_i)A$ with $(y_{i+1})B$, in $G$ (these are the $F^{-1}$ images of the edges of $Q$).

Let $l$ be such that $p = ye$. As $p$ is an edge in $M$, $p$ is contained in a set $S_p \in S_0$. By the condition of the theorem, the set $S_p \cup \bigcup F^{-1}(IE(R))$ contains a matching $N$ of size $n$. Since $|M| < n$, $N$ contains an edge $ax$, where $a \in U_A$ (recall that $U_A = A \setminus M$). Suppose $x \in U_B$. If $ax \in \bigcup F^{-1}(IE(R))$, then $M \cup \{ax\}$ is a rainbow matching, contradicting the maximality of $M$. Thus we have $ax \in S_p$. Let $q = y_{\ell'}$ for some $\ell' < \ell$. Now consider

$$N = (M \cup \{ax, p_Aq_B \} \cup \{(y_i)_A(y_{i+1})_B \mid 1 \leq i \leq \ell - 1\}) \setminus \{ye, ye_{\ell+1}, \ldots, ye_{\ell}\}.$$ 

Since $p_Aq_B \in S$ and $\{(y_i)_A(y_{i+1})_B \mid 1 \leq i \leq \ell - 1\}$ has enough represented sets in $I^{-1}(Q)$, then $N$ is a rainbow matching. However, it is a contradiction to the maximality of $M$ since $N$ has size $|M| + 1$. 

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Hence, we may assume that $x$ lies on an edge $h$ of $M$, meaning that $sh$ is an edge in $F(S_p) \cup \bigcup \mathcal{I}(R)$. Since all edges in $\bigcup \mathcal{I}(R)$ are backwards, and $sh$ is not a backward edge on any path, $sh$ belongs to $F(S_p)$.

Let $h \in V(Q_h)$ for $Q_h \in Q$, and let $P$ be the $s-t$ path $shQ_h$. Let $\hat{P}$ be a path in $F^{-1}(P)$, whose first vertex is $a$, meaning that its first edge belongs to $S_p$. Let $X \triangle Y$ be the symmetric difference of $X$ and $Y$, that is, $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. Let $N = M \triangle E(\hat{P})$.

Consider two possibilities:

**Possibility I:** $h = y_d$ for $d \leq \ell$.

In this case $N$ is an $S$-rainbow matching of size $m + 1$: we let the first edge, $ah_B$, represents $S_p$, and the other edges in $E(\hat{P}) \setminus M$ has a represented sets in $I^{-1}(Q)$ and keep all other representations as they are. Since the edge in $M$ representing $S_p$ is removed by the symmetric difference, this assignment of representation yields an $S$-rainbow matching.

**Possibility II:** Either $h \notin V(Q)$ or $h = y_d$ for $d > \ell$.

In this case, $N$ is not $S$-rainbow, since there are two edges representing $S_p$, namely $p$ and $ah_B$. But this is rectifiable, using the edge $pq$. Suppose that $q = y_b$, where $b < \ell$. Let $C$ be the cycle whose edges are $p,q_b,q,e_b,y_{b+1},e_{b+1},\ldots,e_{\ell-1},p$. Let $N' = N \triangle E(C)$. Then $N'$ is a matching of size $m + 1$, and it is $S$-rainbow, since $S_p$ is represented in it just once - by the edge $ah_B$. \hfill \Box

4 Somewhere over the rainbow - two possible strengthenings

It is possible that Theorem 8 can be given a strong cooperation generalisation.

**Conjecture 38.** Let $\mathcal{F}$ be a family of $2k - 1$ sets of edges in a bipartite graph. If $\nu(\bigcup \mathcal{K}) \geq \min(|\mathcal{K}|,k)$ for every $\mathcal{K} \subseteq \mathcal{F}$ then $\nu_{R}(\mathcal{F}) \geq k$.

This generalises the following theorem from [2]:

**Theorem 39.** If $\mathcal{F} = (F_1, \ldots, F_{2k-1})$ is a family of matchings in a bipartite graph, and $|F_i| = \min(i,k)$ for all $i$, then there exists an $\mathcal{F}$-rainbow matching of size $k$.

Here is another possible strong version of Theorem 8.

**Conjecture 40.** Let $\mathcal{F} = (F_1, \ldots, F_{2k-1})$ be a system of bipartite sets of edges, sharing the same bipartition, and suppose that $\nu(F_i) \geq k$ for all $i \leq 2k - 1$. Let $V'$ be a copy of $V$ disjoint from $V$, let $F'_i$ be a copy of $F_i$ on $V'$ ($i \leq 2k - 1$) and let $\tilde{F}_i = F_i \cup F'_i$ for $i \leq 2k - 1$. Then the system $(\tilde{F}_i | i \leq 2k - 1)$ has a full rainbow matching.

This implies Theorem 2, since by the pigeonhole principle either $V$ or $V'$ contains a rainbow matching of size $k$. Conjecture 40 would follow from the following conjecture of the first author and Eli Berger [1].

**Conjecture 41.** $n$ matchings of size $n$ in any graph have a rainbow matching of size $n - 1$. 

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