A diagram-free approach to the stochastic estimates in regularity structures

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Abstract.
In this paper, we explore the version of Hairer’s regularity structures based on a greedier index set than trees, as introduced in [32] and algebraically characterized in [30]. More precisely, we construct and stochastically estimate the renormalized model postulated in [32], avoiding the use of Feynman diagrams but still in a fully automated, i. e. inductive way. This is carried out for a class of quasi-linear parabolic PDEs driven by noise in the full singular but renormalizable range.

We assume a spectral gap inequality on the (not necessarily Gaussian) noise ensemble. The resulting control on the variance of the model naturally complements its vanishing expectation arising from the BPHZ-choice of renormalization. We capture the gain in regularity on the level of the Malliavin derivative of the model by describing it as a modelled distribution. Symmetry is an important guiding principle and built-in on the level of the renormalization Ansatz. Our approach is analytic and top-down rather than combinatorial and bottom-up.

Keywords: Singular SPDE, regularity structures, BPHZ renormalization, Malliavin calculus, quasi-linear PDE

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1 Introduction

We continue the program started in [32] of replacing trees by multi-indices as a more parsimonious but equally natural index set, within the framework of Hairer’s regularity structures. Like in [20], we implement this for quasi-linear parabolic equations of the form

\[(\partial_t - \partial_x^2)u = a(u)\partial_x^2u + \xi,\]  \hspace{1cm} (1.1)

driven by a stationary noise \(\xi\) in the regime where the product \(a(u)\partial_x^2u\) is singular but renormalizable. This is the case when a general solution \(u\) to (1.1) with \(a \equiv 0\) is Hölder continuous with an exponent \(\alpha \in (0, 1)\), which means that \(\xi\) is in the (negative) Hölder space \(C^{\alpha - 2}\). However, we believe that our program applies to a much larger class of nonlinear PDEs.

The investigation of (1.1) started in [33] sparked some activity, at first in the mildly singular range \([12, 1]\) of \(\alpha \in (\frac{1}{4}, 1)\), and then up to the white-noise level \([15, 31]\), and finally including the white-noise level \([14, 2, 32]\). In [32], local Schauder estimates were established, based on the notion of modelled distributions, postulating the existence and estimation of a suitable renormalized model. In [30], the Hopf-theoretic nature of the structure group based on multi-indices was uncovered, which is rather Lie-geometric than combinatorial in the sense that it provides a representation of natural actions on the solution manifold. In this paper, we construct the BPHZ-renormalized model and provide its stochastic estimates, as the input to [32]. Our multi-index approach is analytic, in the sense that it is based on taking derivatives w. r. t. the nonlinearity \(a\) and the noise \(\xi\), whereas the tree-based approach is combinatorial, using Feynman diagrams. Our approach is fully automated in the sense that it proceeds by induction over the index set, with all induction steps having the same structure. Loosely speaking, [32] can be seen as an analogue of [20] in the sense that it deals with the analytic solution theory, this paper corresponds to [8] in the sense that it establishes the stochastic estimates on the model, while [20] works out the Hopf-algebraic structure in the spirit of [5, 4]. Let us stress that this paper is essentially self-contained w. r. t. [32] and [30], and can be read and appreciated independently: It serves as an input for [32], while [20] provides a deeper algebraic understanding not required in this paper.

\footnotesize

1 treating (1.1) as an (anisotropic) elliptic equation, we denote by \(x_1\) the space-like variable and by \(x_2\) the time-like variable; this allows us to use \(r\) for the semi-group convolution (2.1) parameter

2 i. e. invariant in law under space-time shifts

3 as a consequence of the single space dimension, there is an additional constraint

4 In particular, the restriction to a single spatial variable is just for convenience, and has the advantage of making white noise renormalizable with \(\alpha = \frac{1}{2}\). However, as discussed in Subsection 2.2 there is a didactic advantage in allowing for general space dimension, replacing \(\partial_x^2\) by the Laplacian, as done in [32].

5 up to the distinction between singular and regular multi-indices

6 however partial and restricted to (1.1)

7 including aspects of [20] in the sense that the notion of the structure group is less generic than in [20]

8 besides a few finiteness properties and identities explicitly spelled out
In terms of renormalization, the multi-index approach is top-down rather than bottom-up, in the sense that for the renormalized equation
\[(\partial_2 - \partial_1^2)u + h = a(u)\partial_1^2 u + \xi\] 
we postulate on the counter term \(h\):

- \(h\) is local, i.e. it depends on the solution \(u\) only through its value \(u(x)\) at the current space-time point \(x\) and
- \(h\) is homogeneous, i.e. not explicitly dependent on \(x\), and thus deterministic, i.e. not explicitly dependent on \(\xi\). Both conditions imply that \(h = h(u(x))\) for some deterministic nonlinearity \(h\).
- The most subtle postulate relates \(a\) to \(h\): If we replace \(a\) by \(a(\cdot + u)\) for some \(u\)-shift \(u \in \mathbb{R}\), \(h\) is replaced by \(h(\cdot + u)\). This means that the renormalization is independent of the choice of the origin in \(u\)-space. It implies that \(h(u) = c[a(\cdot + u)]\) for \(u \in \mathbb{R}\) for some deterministic functional \(c\) on \(a\)-space, which typically diverges as the regularization of \(\xi\) fades away.

As opposed to the tree-based treatment of quasi-linear equations by regularity structures \([15, 14]\), we thus do not have to show a posteriori that \(h\) is local. Some symmetries are built-in, like the independence from the choice of the origin in \(u\)-space and \(a\)-space. Other symmetries, like the invariance in law under \(\xi \mapsto -\xi\), are easily seen to transmit to the model \([12]\). As a consequence, our more greedy approach based on multi-indices rather than trees reduces the number of divergent constants contained in \(c\). The comparison for the quasi-linear equation (1.1) is complicated by the non-local nature of the tree-based treatment in \([15, 14]\), and the fact that the divergent constants are better treated as functions of a placeholder \(a_0\) for the elliptic coefficient \(1 + a\), see Subsection 2.6. The comparison is easier for the semi-linear multiplicative stochastic heat equation \((\partial_2 - \partial_1^2)u = a(u)\xi\), as treated by tree-based regularity structures in \([24]\) for \(\alpha = \frac{1}{2} - \). Here it is clear that for, e.g., \(\alpha = \frac{1}{4} + \) the number of divergent constants decreases from 85 to 30, see also \([30\text{, Section } 7]\).

The two main conceptual merits of our approach are:

- Spectral gap inequality. Our main assumption on the law of \(\xi\), next to invariance under translation and reflection is a spectral gap (SG) inequality. The SG inequality is specified by a Hilbert norm on \(\xi\)-space, which provides the analogue of the Cameron-Martin space from the Gaussian case, and here is \(L^2(\mathbb{R}^2)\)-based. This Hilbert norm is chosen in agreement with \(\xi \in C^{\alpha - 2}\). While this includes non-Gaussian ensembles, the main benefit is that the SG inequality naturally complements the BPHZ-choice of renormalization: On the one hand, a SG inequality, which we apply to the negative-homogeneity part \(\Pi^-\) of the model, estimates the variance of a random variable. On the other hand, the BPHZ-choice of renormalization is just made to annihilate the expectation \(\mathbb{E}\Pi^-\). We refer to Subsections 4.2 and 5.1 for the details.

- Malliavin derivative as modelled distribution. The use of a SG inequality requires the control of the (first-order) Malliavin derivative of \(\Pi^-\), which is the Fréchet derivative of \(\Pi^- = [\xi]\) w. r. t. the noise \(\xi\). It is convenient to think of it in terms of the directional derivative \(\delta\Pi^-\) for

\[\text{(1.2)}\]
\[\text{for some deterministic functional } c \text{ on } a\text{-space, which typically diverges as the regularization of } \xi \text{ fades away.}\]
some arbitrary infinitesimal noise perturbation\[^5\] $\delta \xi$. Since $\Pi^-$ is multi-linear in $\xi$, passing to $\delta \Pi^-$ amounts to replacing one of the instances of $\xi$ by $\delta \xi$. This leads only to a subtle gain in regularity, which is conveniently expressed after integration\[^16\] i.e. on the level of $\delta \Pi$. It is captured by describing $\delta \Pi$ as a modelled distribution\[^17\] $d \Gamma^\omega$ w. r. t. $\Pi$ itself. Hence surprisingly, the notion of modelled distribution with its intrinsic continuity property, which \[^20\] Definition 3.1] introduced for the deterministic Schauder theory given the model, here plays a role in the stochastic estimation of the model itself. Crucially, as opposed to $\Pi^-$ itself, the representation for $\delta \Pi^-$, or rather of its rough path increment $\delta \Pi^- - d \Gamma^- \Pi^-$, in terms of $\Pi$ and $\delta \Pi - d \Gamma^- \Pi$ does not involve the divergent $c$. This ultimately allows for reconstruction of $\delta \Pi^-$. We refer to Subsection 4.3 for details.

Two more technical merits of our approach are:

- **Scaling as a guiding principle.** In order not to break it, we work on the whole space-time, which because of potential infra-red divergences is interesting in its own right. As a collateral damage, in order to avoid critical cases, we have to generalize from white noise to a more general noise $\xi$ with a fractional (negative) Sobolev norm playing the role of the Cameron-Martin space. Like in \[^9\], this has the positive side effect of allowing to explore the limits of the approach. In order not to break scaling, we work with annealed instead of quenched estimates. By this jargon\[^19\] we mean that the inner norm is an $L^p$-norm in probability, while the outer norm is a Hölder norm in space-time. Subsection 4.2 is the place where this transition is made on the level of $\delta \xi$. Estimates in annealed norms have the advantage of coming without a (marginal) loss in the exponent\[^19\].

- **Hölder vs. $L^2$-topologies.** The SG inequality and Malliavin calculus rely on $L^2$-based space-time norms (the analogue of the Cameron-Martin space is an $L^2$-norm of a negative fractional derivative of $\delta \xi$) whereas the Schauder calculus of modelled distributions like our $d \Gamma^\omega$ is based on Hölder norms. We introduce a weight that is singular (but integrable) in a secondary base point $z$ into the $L^2$-based norms to emulate a Hölder norm localized in $z$. Averaging over $z$ recovers the original Cameron-Martin norm. We refer to Subsection 4.3 for details.

We now comment on related work. Hairer’s regularity structures triggered a rapid development in the field of singular SPDEs. They provide a framework for local well-posedness for a large class of semilinear SPDEs, as worked out in \[^20\] [8, 4]. As mentioned, this approach has been extended to quasi-linear SPDEs in \[^17\] [13, 14]. Gubinelli & Imkeller & Perkowski’s paracontrolled calculus \[^18\] provides an alternative approach, based on Littlewood-Paley decomposition, to (stochastic) estimates and renormalization. While it does not provide a general framework by itself, paracontrolled calculus has been efficiently applied to a variety of singular SPDEs \[^3, 19\] and naturally extends to quasi-linear equations, see \[^12\], \[^1\]. Kupiainen appealed to Wilsonian renormalization \[^27, 28\] to treat some semi-linear SPDEs. Duch \[^10\] used Wilsonian renormalization in the continuum form of the Polchinski flow equation, which naturally extends to fractional Laplacians. This approach is similar to this paper, in the sense that it recursively constructs and estimates multi-linear functionals of the noise based on the flow equation, and avoids trees.

In the use of Malliavin calculus, the spectral gap inequality, and annealed estimates, this paper is inspired by recent developments in quantitative stochastic homogenization \[^16, 11, 26\]. Malliavin calculus has been used, with the framework of regularity structures, in \[^6, 13, 34\], however in its original purpose, namely for the existence of probability densities.

\[^{15}\] an element of what would be the Cameron-Martin space in the Gaussian case

\[^{16}\] in Hairer’s jargon

\[^{17}\] we use the notation $d \Gamma^\omega$ to indicate that it has some structural similarities with the Malliavin derivative $d \Gamma^\omega$ of the change-of-base-point transformation $\Gamma^\omega$; the star is a reminder of the fact that $\Gamma^\omega$ is the transpose of Hairer’s, see the discussion at the beginning of Subsection 2.4

\[^{18}\] from statistical physics and ultimately from metallurgy

\[^{19}\] as is well-known from Brownian motion and its law of iterated logarithm

\[^{20}\] at the expense of loosing the local nature of the problem inside the proof
2 Assumptions and statement of result

2.1 Semi-group convolution

Following [33, Section 2], we use the space-time elliptic operator \( \partial_1^4 - \partial_2^2 \) to introduce the family \( \{ (\cdot)_t \}_{t \in (0, \infty)} \) of convolution operators that respect the parabolic scaling and satisfy the semi-group property

\[
(f_t)_t = (f_t)_{t=0} = \text{Dirac at origin}.
\]

convenient for the dyadic nature of reconstruction arguments. It is defined through the convolution with the Schwartz kernel \( \psi_t \) defined through

\[
\partial_t \psi_t + (\partial_1^4 - \partial_2^2) \psi_t = 0 \quad \text{and} \quad \psi_{t=0} = \text{Dirac at origin}.
\]

Note the scaling \( \psi_t(x) = (\frac{4}{\sqrt{t}})^{-3} \psi_{t=1}(\frac{x_1}{\sqrt{t}}, \frac{x_2}{\sqrt{t}}) \), so that the \( x_1 \)-scale is \( \sqrt{t} \), which explains the appearance of \( \sqrt{t} \) in (2.27) of Theorem 2.2. Finally, because of the factorization

\[
\partial_1^4 - \partial_2^2 = -\left( \partial_2 - \partial_1^2 \right) \left( \partial_2 + \partial_1^2 \right),
\]

the convolution \( (\cdot)_t \) will also be helpful for integration by providing the kernel representation in form of

\[
(\partial_2 - \partial_1^2)^{-1} f = -\int_0^\infty dt (\partial_2 + \partial_1^2) f_t.
\]

Finally, we denote by \( |y - x| \) the parabolic Carnot-Carathéodory distance:

\[
|y - x| := |y_1 - x_1| + \sqrt{|y_2 - x_2|},
\]

which appears in (2.27), (2.28) and (2.29) of Theorem 2.2.

2.2 Spectral gap inequality

In this subsection, we motivate and state our assumptions on the law of Schwartz distributions \( \xi \) on space-time \( \mathbb{R}^2 \). The crucial assumption is that of a spectral gap (SG) inequality. The structure underlying a SG inequality is a Hilbert norm on configuration space, which plays the role of the Cameron-Martin norm in the Gaussian case. In the same way white noise has \( L^2(\mathbb{R}^2) \) as Cameron-Martin space, our norm will be an \( L^2(\mathbb{R}^2) \)-based norm. Because our law is shift-invariant, it is natural to choose a translation invariant norm. Since we are aiming at \( \xi \)’s that are almost surely in the negative Hölder space \( C_{-2} \), by Kolmogorov’s criterion, it should be an \( L^2(\mathbb{R}^2) \)-based Sobolev norm of the fractional order \( D_2 + \alpha - 2 \), where \( D \) is the effective dimension (see [20, Lemma 10.2]). In view of the parabolic nature, both Hölder and Sobolev norms need to be anisotropic: if the spatial variable \( x_1 \) sets the unit, the time variable \( x_2 \) is worth two units. In particular, we have \( D = 1 + 2 = 3 \) so that the order of fractional derivative should be \( \alpha - \frac{1}{2} \). It is convenient to express the anisotropic version of the Sobolev norm in terms of the space-time elliptic operator \( \partial_1^4 - \partial_2^2 \) introduced in Subsection 2.1, which is of order four:

\[
\left( \int_{\mathbb{R}^2} dx \left( (\partial_1^4 - \partial_2^2)^{\frac{1}{2}} \delta \xi \right)^2 \right)^{\frac{1}{2}}.
\]

where here and in the sequel, we think of \( \delta \xi \) as an infinitesimal perturbation of \( \xi \).

Having motivated the Hilbert norm (2.6), we return to the notion of a SG inequality. Loosely speaking, a SG inequality is a Poincaré inequality on configuration space endowed with a probability measure

\footnote{For the Hölder norm that just means that its definition is based on \( |y - x| \).}

\footnote{which amounts to an element of the Cameron-Martin space in the Gaussian case}

\footnote{with mean value zero}
and a Hilbert norm. It is formulated in terms of a generic random variable $F$, which is an integrable function(al) on the space of $\xi$'s, i.e. $F = F[\xi]$. The notion of a gradient of $F$ and its (squared) norm relies on the Hilbertian structure (2.6). We consider $F = F[\xi]$ that are Fréchet differentiable w. r. t. (2.6); which means that the differential $dF[\xi]$ in a configuration $\xi$, which is a linear form on the $\delta\xi$'s, is bounded w. r. t. (2.6). Representing the differential $dF[\xi]$ in terms of

$$\delta F := dF[\xi], \delta \xi = \int_{\mathbb{R}^d} dx \delta \xi \frac{\partial F}{\partial \xi}[\xi], \quad \cdots$$

(2.7)

this implies that the $L^2(\mathbb{R}^d)$-dual norm of the Malliavin derivative $\frac{\partial F}{\partial \xi}[\xi](x)$ is finite:

$$\int_{\mathbb{R}^d} dx ((\partial_1^4 - \partial_2^2)^\frac{1}{2}(1-\alpha)) \left| \frac{\partial F}{\partial \xi}[\xi](x) \right|^2 < \infty.$$

(2.8)

It is the expectation of (2.8) that appears on the r. h. s. of the SG inequality (2.9); the l. h. s. is the variance of $F$. By a (parabolic) rescaling of space-time $x$, we may w. l. o. g. assume that the constant in (2.9) is unity.

A functional-analytic subtlety arises from the fact that Fréchet differentiability of $F$ is not enough to give an a priori meaning to (2.8) for almost every realization $\xi$: Even under the assumption that $\xi$ is smooth, (2.6) with $\delta \xi$ replaced by $\xi$ would be almost surely infinite since $\xi$ as a realization from a stationary ensemble has no decay in space-time. The way out is to restrict to cylinder functions $F$, meaning that there exist $N \in \mathbb{N}$, Schwartz functions $\zeta_1, \ldots, \zeta_N$ on $\mathbb{R}^2$, and a smooth function $\tilde{F}$ on $\mathbb{R}^N$ such that $F[\xi] = \tilde{F}(\int_{\mathbb{R}^2} \zeta_1 \xi, \ldots, \int_{\mathbb{R}^2} \zeta_N \xi)$. Such $F$ are obviously Fréchet differentiable with $\frac{\partial F}{\partial \xi}[\xi] = \sum_{n=1}^N \partial_n \tilde{F}(\int_{\mathbb{R}^2} \zeta_1 \xi, \ldots, \int_{\mathbb{R}^2} \zeta_N \xi) \zeta_n$.

**Assumption 2.1.** The law of $\xi$ is invariant under space-time shift and under the spatial reflection $x_1 \rightarrow -x_1$. For some $\alpha \in (\frac{1}{4}, \frac{1}{2}) \setminus \mathbb{Q}$ it satisfies the spectral gap inequality

$$\mathbb{E}(F - \mathbb{E}F)^2 \leq \mathbb{E} \int_{\mathbb{R}^d} dx ((\partial_1^4 - \partial_2^2)^\frac{1}{2}(1-\alpha)) \left| \frac{\partial F}{\partial \xi}[\xi] \right|^2,$$

(2.9)

for all integrable cylinder functions $F$. Finally, we assume that $\xi$ is almost surely smooth in $x$.

We note that any Gaussian ensemble with a Cameron–Martin norm that dominates (2.6) satisfies (2.9), see [3, Theorem 5.5.1, eq. (5.5.2)]. In particular, this applies to any stationary Gaussian ensemble with a covariance function of which the Fourier transform satisfies $\mathcal{F}c(k) \leq (k_1^2 + k_2^2)^\frac{1}{2}(1-\alpha)$. Let us comment on the constraints on $\alpha$: Recall that white noise corresponds to $\alpha = \frac{1}{2}$; however, because of the Schauder theory involved in integration, we need to avoid rational $\alpha$. For convenience, we restrict ourselves to the more singular side by assuming $\alpha < \frac{1}{2}$. This does include white noise, provided it is tamed by an infra-red cut-off, which can be achieved by cutting off the large-scale Fourier modes to satisfy the above-mentioned estimate (while preserving stationarity). Reconstruction imposes $\alpha > \frac{1}{4}$, a constraint already present on the level of the first counter term, see [33, Proposition 4.2], and specific to the single space dimension. We will discuss in Subsection 4.4 that this is the only reason for restricting the $\alpha$-range away from $\alpha = 0$ and thus the limit of renormalizability.

The assumption of almost-sure smoothness of $\xi$ in $x$, which is purely qualitative, is convenient when constructing the model – no estimate deteriorates for evanescent smoothness. We expect to be not only able to construct but also to characterize the model in the limit of vanishing smoothness, in the spirit of [25, Proposition 1.8]. The assumption of reflection invariance in law is crucial for the (simple) form of the renormalized equation; if omitted, we expect an additional counter term of the form $\tilde{h}(u) \partial_1 u$. 

\[24\] Like the standard Poincaré inequality provides a lower bound on the spectral gap of the (Neumann) Laplacian, the SG inequality provides a lower bound on the spectral gap of the generator of the stochastic process defined through the Dirichlet form arising from the Hilbertian structure, for which the given ensemble is in detailed balance by construction.
2.3 Our notion of model

In this subsection, and following [30], we motivate our notion of an abstract model space \( T \), which amounts to motivating its index set of populated multi-indices \( \beta \), and also motivate our notion of the model \( (\Pi_\beta, \Pi_\gamma) \) centered in a space-time point \( x \in \mathbb{R}^2 \). Since this motivation is logically independent of this paper’s result, we will be brief and refer to [30, Subsection 2.1] for more details. Starting point is a general solution \( u \) of (1.2), considered as a function of the nonlinearity \( a \), and eventually also of the counter term \( h \) constrained by (1.3). If (1.2) were an ODE, imposing homogeneous initial data would suffice to parameterize the entire solution manifold, since inhomogeneous initial data can be obtained by a shift in \( u \)-space, applied to \( a, h, \text{and the solution} \). However, (1.2) being a PDE, we need infinitely more parameters. In the linear case of \( a \equiv h \equiv 0 \), the solution manifold obviously is an affine space over all “caloric” polynomials \( p, i.e. \) those satisfying \( (\partial_x - \partial_t^2)p = 0 \). It is convenient to get rid of this caloricity constraint by relaxing (1.2) to hold only modulo space-time polynomials. However, we would like to keep the ODE aspect that \( u \)-shifts contribute to spanning the solution manifold, modulo constants, so we restrict ourselves to polynomials \( p \) modulo constants. This is crucial for the algebraic aspect of the regularity structure [30 Subsection 2.1].

Following [30, Subsection 2.6], we introduce coordinates on this \((a, p)\)-space, arbitrarily choosing an origin for \( u \)-space and \( x \)-space:

\[
\begin{align*}
z_k[a] &:= \frac{1}{k!} \frac{d^k a}{dx^k} (0) \quad \text{for } k \in \mathbb{N}_0 \quad \text{and} \quad z_n[p] := \frac{1}{n!} \frac{d^n p}{dx^n} (0) \quad \text{for } n \in \mathbb{N}_0^2 \setminus \{(0,0)\}; \\
\end{align*}
\]

(2.10)

note that \( n = 0 \) does not appear here because we consider \( p \) only modulo constants. Returning to our general solution \( u = u[a, p] \), which we anchor by imposing

\[
u[a \equiv 0, p] = u[0,0] + p,
\]

(2.11)

and think of being extended by the implicit function theorem at fixed \( p \), which relies on being able to define a solution operator modulo polynomials,[25] we (formally) take partial derivatives w. r. t. the coordinates (2.10), and evaluate them when these coordinates are set to zero, see also [30 Subsection 3.1]. These partial derivatives are obviously labeled by multi-indices \( \beta \), i.e. functions \( \beta: \mathbb{N}_0 \cup (\mathbb{N}_0^2 \setminus \{(0,0)\}) \to \mathbb{N}_0 \) that vanish for all but finitely many arguments. The resulting random space-time fields \( \Pi_\beta \) are the coefficients of \( u = u[a, p] \) when \( u \) is interpreted as a formal power series in the (infinitely many) coordinates (2.10) with values in random space-time fields. While this is formal, we shall see that for our qualitatively smooth \( \xi \), the \( \Pi_\beta = \Pi_\beta[\xi](x) \) can be rigorously constructed by the hierarchy of PDEs (still modulo polynomials) given by (2.21) and (2.22), where the term \( \sum_{\xi \geq 0} \xi \Pi_\xi \partial_\xi^2 \Pi \), which arises from \( a(u) \partial_\xi^2 u \) in (2.10), and where (2.11) is reflected by (2.23).

It turns out that \( \Pi_\beta \) vanishes unless one of the two (mutually exclusive) cases is satisfied:

\[
\beta = e_n \text{ for some } n \neq 0 \quad \text{or} \quad [\beta] := \sum_{k \geq 0} k \beta(k) - \sum_{n \neq 0} \beta(n) \geq 0 \quad \text{“purely polynomial”}
\]

(2.12)

Note that \([\beta]\) is a natural quantity: Since the monomials \( u^k \) and \( x^n \) are dual to the coordinates (2.10), \( \sum_{k \geq 0} k \beta(k) \) captures the homogeneity in \( u \), whereas \( \sum_{n \neq 0} \beta(n) \) captures the one in \( p \), and \( p \) has the same units as \( u \) in view of (2.11). The constraint (2.12) defines our index set, and thus a linear subspace of the formal power series algebra \( \mathbb{R}[[z_k, z_n]] \) in the infinitely many variables \( \{z_k\}_{k \geq 0} \cup \{z_n\}_{n \neq 0} \), which

\[25\text{with values in random space-time fields}
\[26\text{we are deliberately vague here since for the centered model, we have to proceed differently}
\[27\this is just a fancy way of saying that } \beta \text{ measures the frequency of } k \text{'s and } n \text{'s}
\[28\text{we deliberately omit the index } x \text{ for the moment, since we will address centering later}
\[29\text{where multiplication is in } \mathbb{R}[[z_k, z_n]] \text{ and the sum is effectively finite}
\[30\text{at this stage, the star } \ast \text{ is just notation}
however is not a sub-algebra. The infinite-dimensional space \( T^\ast \), which is a direct product, is the algebraic dual of the direct sum \( T \) over the same index set \( \{2,12\} \). The purely polynomial multi-indices, cf. \( \{2,12\} \), define a linear subspace \( \tilde{T} \cong \mathbb{R}[x_1,x_2]/\mathbb{R} \) of \( T \). We interpret \( \{2,12\} \) as the analogue to Hairer’s abstract model space, and \( \tilde{T} \subset T \) as the analogue of the polynomial sector \( \{20\} \) Remark 2.23. Denoting by \( \tilde{T}^\ast \subset T^\ast \) the direct product over the populated not purely polynomial multi-indices, cf. \( \{2,12\} \), we introduce the projection \( P \) of \( \mathbb{R}[[z_k,z_n]] \) on \( \tilde{T}^\ast \),

\[
P \text{ of } \mathbb{R}[[z_k,z_n]] \text{ on } \tilde{T}^\ast,
\]

which will play an important role, starting with \( \{2,22\} \) in our main result Theorem 2.2.

We now explain how the counter term \( h \) in \( \{1,2\} \) gives rise to the term \( \{31\} \sum_{k>0} \frac{1}{k!} \Pi^k (D^{(0)})^k c \) in \( \{2,22\} \). In view of \( \{1,3\} \), \( h \) is determined by a (deterministic) function \( c = c[a] \), which by definition of \( \{2,10\} \) we may think of as an element of \( \mathbb{R}[[z_k]] \subset \tilde{T}^\ast \). It is thus described by its coefficients \( c_\beta \), where \( \beta \) runs over all multi-indices satisfying the last constraint in \( \{2,20\} \). In view of the \( u \)-shift in \( \{1,3\} \), it is natural that the way \( c \) enters \( \{2,22\} \) is via the infinitesimal generator \( D^{(0)} \) of \( u \)-shift acting on \( \mathbb{R}[[z_k,z_n]] \), which is given by

\[
D^{(0)} := \sum_{k \geq 0} (k + 1)z_{k+1}\partial_{x_k}.
\]

According to \( \{30\} \) Subsection 3.2], \( D^{(0)} \) is well-defined as a derivation on the algebra \( \mathbb{R}[[z_k,z_n]] \). Since the \( u \)-shift in \( \{1,3\} \) is not infinitesimal but finite, it is natural that \( c \) enters via the exponential series with \( D^{(0)} \) as argument, see also \( \{30\} \) eq. (5.18)].

The notion of the centered model relies on a homogeneity, i.e. on assigning a number \( |\beta| \in \mathbb{R} \) to every populated multi-index \( \beta \). It determines the behavior \( \{2,28\} \) one imposes on \( \Pi_\beta \) in terms of the base point \( x \), which in our smooth setting implies the qualitative order of vanishing \( \{2,31\} \). Here, \( |n| \) denotes the parabolic order

\[
|n| := n_1 + 2n_2.
\]

Also the degree of a polynomial is defined w.\,r.\,t. \( \{2,15\} \), which is used to specify the meaning of “modulo polynomial” for the PDE, see \( \{2,21\} \). This is done in such a way that by a parabolic Liouville argument, there is a unique \( \Pi_{\beta,x} \) for given \( \Pi_\beta \). This uniqueness is needed to lift the shift and reflection invariance (as a parity) in law from \( \xi \) to \( \Pi_\beta \), which is essential in the BPHZ-choice of renormalization: In Subsection 5.1 we shall see that the qualitative infra-red part of \( \{2,27\} \), in form of the vanishing ensemble (and space-time average) average

\[
\lim_{t\to\infty} \mathbb{E} \Pi_{\beta,x}(x) = 0 \quad \text{for } |\beta| < 2;
\]

fixes the deterministic \( c_\beta \). This is in line with the population constraint \( \{7\} |\beta| < 2 \) in \( \{2,20\} \) on \( c_\beta \). It is the shift invariance in law of \( \Pi_{\beta,x} \) that allows an \( x \)-independent choice of \( c_\beta \); it is the reflection parity in law of \( \Pi_{\beta,x} \) that ensures that \( c_\beta \) vanishes for odd \( \sum_n n_1 \beta(n) \), which in view of \( |\beta|_p < |\beta| < 2 \), cf. \( \{2,17\} \), yields the second population constraint in \( \{2,20\} \). This is how the BPHZ-choice of renormalization is implemented.

\( \{31\} \) and the product on \( \mathbb{R}[[z_k,z_n]] \) is unrelated to Hairer’s products \( \{20\} \) Section 4.

\( \{32\} \) see however the comment at the beginning of Subsection 2.4.

\( \{33\} \) note that \( \{2,13\} \) differs from \( \{32\} \) and \( \{30\} \) where \( P \) denotes the projection onto the polynomial sector

\( \{34\} \) where the sum is again effectively finite

\( \{35\} \) In the sequel we overload the notation for the symbol \( |\cdot| \), but the meaning will be clear from the context.

\( \{36\} \) since we are working with the centered ensemble, we need to include the space-time average

\( \{37\} \) note that the threshold 2 reflects the order of \( \partial_x - \partial_t^2 \).
The homogeneity of a populated multi-index $\beta$ is defined through

$$|\beta| := \alpha(1 + |\beta|) + |\beta|_p \quad \text{where} \quad |\beta|_p := \sum_{\mathbf{n} \neq 0} |\mathbf{n}| \beta(\mathbf{n}).$$  \hspace{1cm} (2.17)

This expression is motivated by scaling: For any spatial scaling factor $s \in (0, \infty)$, we consider the parabolic rescaling $Sx = (sx_1, sx_2)$ of space-time. The SG assumption (2.9) is easily seen to be invariant under the re-scaling $\xi \mapsto s^{2-\alpha} \xi(S \cdot)$, as is well-known in the white noise case $\alpha = \frac{1}{2}$. Comparing the family of space-time functions $\{\Pi_{s,\beta}[\xi]\}_{s}$ for the given realization $\xi$ with the space-time function $\{\Pi_{s,\beta}[s^{2-\alpha} \xi(S \cdot)]\}_{s}$ for the rescaled realization, it is straightforward to check that they are related by $\Pi_{(sx),\beta}[\xi](S_{x}y) = s^{\beta} \Pi_{s,\beta}[s^{2-\alpha} \xi(S \cdot)](x)$. In agreement with regularity structures [20, Definition 2.1], the set $A$ of homogeneities is locally finite\(^{39}\) and bounded from below. Moreover, (2.15) and (2.17) are consistent in the sense of

$$|e_{n}| = |\mathbf{n}| \quad \text{for} \quad \mathbf{n} \neq \mathbf{0}.$$  \hspace{1cm} (2.18)

Thanks to our assumption of $\alpha \notin \mathbb{Q}$, there is a reverse of (2.18):

$$|\beta| \in \mathbb{N} \quad \implies \quad \beta \text{ is purely polynomial.}$$  \hspace{1cm} (2.19)

Both reproduce [20 Assumption 5.3]; (2.19) plays a role in the Liouville argument and Schauder estimates.

### 2.4 Statement of main result

Our main result Theorem 2.2 provides the construction of a model that, loosely speaking, satisfies Hairer’s axioms and estimates as stated in [20, Definition 2.17], as we shall discuss in Subsection 2.5. Clearly, all the specifics of the type of nonlinearity and its renormalization are contained in (2.22).

**Theorem 2.2.** Under Assumption [2.1], the following holds for every populated $\beta$:

There exists a deterministic $c \in \tilde{T}^\ast$ satisfying

$$c_{\beta} = 0 \quad \text{unless} \quad |\beta| < 2 \quad \text{and} \quad \beta(\mathbf{n}) = 0 \quad \text{for all} \quad \mathbf{n} \neq \mathbf{0},$$  \hspace{1cm} (2.20)

and for every $x \in \mathbb{R}^2$ there exist two random smooth space-time functions $\Pi_{x}^\ast$ and $\Pi_{x}$, the first with values in $\tilde{T}^\ast$, the second in $T^\ast$. These are related by\(^{39}\)

$$\left( (\partial_2 - \partial_1^2) \Pi_{x} - \Pi_{x} \right)_{\beta} \quad \text{is a (random) polynomial of degree} \quad |\beta| - 2,$$  \hspace{1cm} (2.21)

$$\Pi_{x} = P( \sum_{k \geq 0} z_k \Pi_{x}^k \partial_1^k \Pi_{x} - \sum_{k \geq 0} \frac{1}{k!} \Pi_{x}^k (D^{(0)})^k c + \xi_1),$$  \hspace{1cm} (2.22)

$$\Pi_{s,\beta} = (\cdot - x)^n \quad \text{in case of} \quad \beta = e_n \quad \text{for some} \quad \mathbf{n} \neq \mathbf{0}.$$  \hspace{1cm} (2.23)

Moreover, for every $x, y \in \mathbb{R}^2$ there exists a random $\Gamma_{xy}^\ast \in \text{Aut}(T^\ast)$ such that

$$\left( (\Pi_{x}^\ast - \Gamma_{xy}^\ast \Pi_{y}) \right)_{\beta} \quad \text{is a (random) polynomial of degree} \quad |\beta| - 2,$$  \hspace{1cm} (2.24)

$$\Pi_{x} - \Gamma_{xy}^\ast \Pi_{y} \quad \text{is a (random) constant},$$  \hspace{1cm} (2.25)

$$\Gamma_{xy}^{\ast z} = \Gamma_{xz}^\ast \Gamma_{zy}^\ast \quad \text{for all} \quad z \in \mathbb{R}^2.$$  \hspace{1cm} (2.26)

\(^{39}\)by which Hairer and we mean that the intersection with a bounded set is finite

\(^{39}\)stands for the unit element in the algebra $\mathbb{R}[\|z, z_n\|]$
In addition, we have for all $p < \infty$ 
\begin{align}
\mathbb{E}^\frac{1}{p} \left| \Pi_{\alpha, \beta}(y) \right|^p & \lesssim (\sqrt{t})^{n-2} (\sqrt{t} + |y - x|)^{|\beta| - \alpha} \quad \text{for all } t \in (0, \infty), \quad (2.27) \\
\mathbb{E}^\frac{1}{p} \left| \Pi_{\alpha, \beta}(y) \right|^p & \lesssim |y - x|^{\beta}, \quad (2.28) \\
\mathbb{E}^\frac{1}{p} \left| (\Gamma_{xy} - \text{id}) \right|^p & \lesssim |y - x|^{\beta - |\gamma|} \quad \text{for all populated } \gamma, \quad (2.29)
\end{align}
where here and in the sequel, \( \lesssim \) means up to a constant only depending on \( \alpha, \beta \) and \( p \).

Note that the expression in (2.29) vanishes for \( |\gamma| \geq |\beta| \) by strict triangularity of elements of \( G \). Let us point out that the role of \( P \), see (2.13), is rather to project \( z_k \Pi^k_x \partial^2 \Pi_x \) from \( \mathbb{R}^n [z_k, z_n] \) into \( \tilde{T}^\ast \); the further restriction to \( \tilde{T}^\ast \) is automatic due to the presence of \( z_k \), see (2.13). In fact, by (6.3), \( P \) is inactive not just on the third but also on the second contribution:
\[ \Pi_x = P \sum_{k \geq 0} z_k \Pi^k_x \partial^2 \Pi_x - \sum_{k \geq 0} \frac{1}{k!} \Pi^k_x (D^{(0)})^k c + \xi \tag{2.30} \]
Note that (2.23) is consistent with (2.21) and (2.22), and also consistent with (2.28), where we appealed to (2.18) in both instances. Thanks to the qualitative smoothness, the quantitative order of vanishing (2.28) implies the qualitative
\[ \partial^n \Pi_{\alpha, \beta}(x) = 0 \quad \text{for all } |n| < |\beta|, \tag{2.31} \]
which we often will refer to as anchoring. Since \( |\cdot| \geq \alpha > 0 \), we have in particular \( \Pi_c(x) = 0 \).

### 2.5 Relation to Hairer’s model

As we shall discuss now, Theorem 2.2 provides the construction of a model that satisfies Hairer’s axioms and estimates as stated in [20, Definition 2.17]. There are a couple of semantic differences: Hairer’s abstract model space rather corresponds to what is\( \tilde{T} \oplus \mathbb{R} \oplus \mathcal{T} = \tilde{T} \oplus (\mathbb{R} \oplus \tilde{T}) \oplus \tilde{T} \) \( (2.32) \) in our notation, where \( \tilde{T} \) is the direct sum over all populated, not purely polynomial multi-indices\(^{41}\), cf. (2.12). Hairer’s abstract integration operator \( \mathcal{I} \), cf. [20, Definition 5.7], in our notation is given by the identification of the first \( \tilde{T} \)-component with the second \( \tilde{T} \)-component in (2.32), see also [30, Subsection 3.9]. Endowing the \( \beta \)-component of first \( \tilde{T} \)-contribution with the homogeneity \( |\beta| - 2 \) meets [20, Definition 5.7] for our second-order integration operator \( (\partial_2 - \partial_1^2)^{-1} \). The \( \mathbb{R} \)-component, which is endowed with homogeneity 0, stands for the constant functions omitted in our approach to \( \tilde{T} \cong \mathbb{R} [x_1, x_2] / \mathbb{R} \), so that the middle component \( \mathbb{R} \oplus \tilde{T} \) on the r. h. s. of (2.32) is isomorphic to \( \mathbb{R} [x_1, x_2] \). Hence our model comes in the two components \( (\Pi^\ast_x, \Pi_x) \) corresponding to (a linear form on) \( \tilde{T} \oplus \mathcal{T} \), and (2.21) is the analogue of [20, eq. (5.12)]. Note that (2.23) corresponds to [20, Assumption 5.3].

A merit of our approach is that the admissibility relation [20, eq. (5.12)] is expressed in terms of the PDE (modulo polynomials) instead of its truncated – and thus nonlocal and no longer scale-invariant – solution kernel [20, Assumption 5.1]. A further analytic difference lies in the annealed nature of the estimates (2.27) & (2.28) and (2.29), as opposed to the quenched or pathwise [20, eq. (2.15)]. It is however easy to upgrade the annealed estimates to quenched ones, see for instance [33, proof of Lemma 4.1], at the expense of an infinitesimal loss in the exponent. A minor analytic difference lies in the fact that we take shifted\(^{42}\) versions of our semi-group kernel (2.1) to express the distributional bounds on

\(^{41}\)note that there is no dependence on the law, since we normalized the constant in (2.9); the dependence on \( \alpha \) could be subsumed into the one on \( \beta \)

\(^{42}\)of which \( \tilde{T}^\ast \) is the algebraic dual

\(^{42}\)by \( y - x \)
\( \Pi^* \), whereas Hairer allows for a general kernel in [20, first item in eq. (2.15)], which by [33, Lemma A.3] is equivalent. On the other hand, our bounds \([2.28]\) on \( \Pi_x \) are expressed in a (seemingly stronger but equivalent\(^{44}\)) pointwise form.

A second semantic difference lies in the fact that Hairer thinks of the model \( \Pi_x \) as a linear form on \( T \), whereas we think of it as a \( T^* \)-valued function. In line with this, Hairer thinks of the re-centering \( \Gamma_{xy} \) as an automorphism of \( T \), while we work with the dual automorphism \( \Gamma_{xy}^* \) of \( T^* \). We adopt this dual perspective because it reveals additional structure: These elements \( \Gamma^* \) are multiplicative\(^{44}\) on \( T^* \subset \mathbb{R}\{[z_k, z_n]\}: \\
\Gamma^* \pi \pi' = (\Gamma^* \pi)(\Gamma^* \pi') \quad \text{provided} \quad \pi \pi' \in T^* \quad \text{and} \quad \Gamma^* 1 = 1, \quad (2.33) \\
see [30, Proposition 5.1 (ii)]. Multiplicativity arises from the exponential representation \\
\begin{equation} \\
\Gamma^* = \sum_{k \geq 0} \frac{1}{k!} \sum_{n_1, \ldots, n_k} \pi^{(n_1)} \cdot \ldots \cdot \pi^{(n_k)} D^{(n_1)} \cdot \ldots \cdot D^{(n_k)}, \\
\end{equation} \\
(2.34)
where the \( n \)'s range over \( \mathbb{N}^2 \), in particular including \( n = 0 \) (we will always state explicitly if we only mean \( n \neq 0 \)) and thus \( (2.14) \), and we have set for convenience \\
\begin{equation} \\
D^{(n)} := \partial_{\pi_n} \quad \text{for} \quad n \neq 0. \\
\end{equation} \\
(2.35)

The \( \{ \pi^{(n)} \}_n \subset T^* \) satisfy the population condition \\
\begin{equation} \\
\pi^{(n)}_{\beta} = 0 \quad \text{unless} \quad |n| < |\beta|, \\
\end{equation} \\
(2.36)
which in particular ensures that the sum over \( k \) is effectively finite, see [30, Subsection 5.1 and eq. (5.14)] and [29, Lemma 3.8]. Note that \( (2.35) \) defines a derivation on \( \mathbb{R}\{[z_k, z_n]\} \] like \( (2.14) \). The \( \pi^{(n)} \)'s can be recovered from \( \Gamma^* \) via \\
\begin{equation} \\
\Gamma^* z_k = \sum_{k' \geq 0} \binom{k+k'}{k} (\pi^{(0)})^{k'} z_{k+k'} \quad \text{for} \quad k \geq 0, \\
\end{equation} \\
(2.37)
\begin{equation} \\
\Gamma^* z_n = z_n + \pi^{(n)} \quad \text{for} \quad n \neq 0. \\
\end{equation} \\
(2.38)

Formula \( (2.34) \) plays an important role when constructing \( \Gamma_{xy}^* \) to satisfy \( (2.24) \) & \( (2.25) \). For later use, we note that by definitions \( (2.14) \) & \( (2.35) \) and by [30 eq. (3.32)] \\
\begin{equation} \\
D^{(n)} T^* \subset \tilde{T}^* \quad \text{and} \quad \Gamma^* \tilde{T}^* \subset \tilde{T}^*. \\
\end{equation} \\
(2.39)

This structure group \( G \subset \text{Aut}(T) \) constructed in [30, Subsection 5.1] meets the postulates of regularity structures: Its elements are strictly triangular w. r. t. the homogeneity \( (2.17) \), see [30 eq. (5.9)] and [20 eq. (2.1)]; its elements respect the polynomial sector \( \tilde{T} \), see [30 eq. (5.10)]\(^{45}\) and [21, Assumption 3.20]. In order to fully connect our structure group \( G \) to Hairer’s, we would need to extend \( \Gamma \in G \) from \( T \) to \( (2.32) \). This extension hides the polynomial correction terms in the change-of-base-point relations \( (2.24) \) & \( (2.25) \), in line with [20, Definition 2.17]. Indeed, in our proof, we will actually establish the following more specific forms of \( (2.24) \) and \( (2.25) \): \\
\begin{equation} \\
\Pi^*_x = \Gamma_{xy}^* \Pi_y + P \sum_{k \geq 0} z_k (\Gamma_{xy}^* (\text{id} - P) \Pi_y + \pi^{(0)}_{xy})^k D^2 (\Gamma_{xy}^* (\text{id} - P) \Pi_y + \pi^{(0)}_{xy}), \\
\end{equation} \\
(2.40)
\begin{equation} \\
\Pi^*_x = \Gamma_{xy}^* \Pi_y + \pi^{(0)}_{xy}, \\
\end{equation} \\
(2.41)

\(^{43}\)according to Campanato \(^{44}\)due to the presence of the polynomial sector \( \tilde{T} \), cf. \( (2.13) \). \( T^* \) is not closed under multiplication, hence \( \Gamma^* \) cannot be called an algebra endomorphism. \\
\(^{45}\)reproduced in \( (3.5) \).
where \( \pi_{xy}^{(0)} \) is related to \( \Gamma_{xy}^* \) via (2.34). In view of the definition (2.13) of \( P \) and the definition (2.23) of \( \Pi_y \), the second r. h. s. term in (2.40) is (componentwise) a polynomial. In view of the triangularity (6.9) of \( \Pi^* \) w. r. t. \( \cdot \) and the structure (6.7) of \( \sum_{k \geq 0} z_k \pi^k \pi^* \) w. r. t. \( \cdot \), combined with (2.18), the degree of the \( \beta \)-component of the polynomial is \( \leq |\beta| - 2 \). Hence (2.40) is indeed a stronger version of (2.24); it is obvious that (2.41) is a stronger version than (2.25). We learn from evaluating (2.41) that

\[
\Pi_{x}(y) = \pi_{xy}^{(0)},
\]

which will play a major role in the proof.

### 2.6 Relation to model in [32]

While we believe that our approach is also valuable for semi-linear SPDEs, like \((\partial_2 - \partial_1^2)u = a(u)\xi\), the discussion of this subsection is specific to the quasi-linear nature of (1.1). We note that \( z_{k=0} \) plays a special role: Indeed, without changing \( c \) and \( \Pi \), (2.21) and (2.22) can be replaced by

\[
(\partial_2 - (1 + z_0)\partial_1^2)\Pi_{x} - \tilde{\Pi}_{x} \quad \text{is a polynomial of degree } |\beta| - 2,
\]

\[
\tilde{\Pi}_{x} = P\left( \sum_{k \geq 1} z_k \pi_{k}^{\pi} \partial_1^{2k} \Pi_{x} - \sum_{k \geq 1} \frac{1}{k!} \pi_{k}^{\pi}(D^{(0)})^{k} c + \xi \right).
\]

The remainder of this section is a heuristic discussion: The above observation indicates that \( c \) and \( \Pi \) are actual (not just a formal) power series in \( z_0 \), convergent for \( |z_0| < 1 \), since then \( a_0 = 1 + z_0 > 0 \), and thus \( \partial_2 - a_0 \partial_1^2 \) is parabolic. For populated multi-indices \( \beta' \) with \( \beta'(k = 0) = 0 \) and for \( a_0 \in (0, 2) \), this allows us to define

\[
c_{\beta'}(a_0) := \sum_{l \geq 0} c_{l\epsilon_0 + \beta'}(a_0 - 1)^l \quad \text{and} \quad \Pi_{x\beta'}(a_0) := \sum_{l \geq 0} \Pi_{x\epsilon_0 + \beta'}(a_0 - 1)^l.
\]

Once we replace \( z_0 \) and \( \beta \) by \( a_0 \) and \( \beta' \), respectively, we have to re-interpret \( D^{(0)} \), see (2.14), as

\[
D^{(0)} := z_1 \partial_{\epsilon_0} + \sum_{k \geq 1} (k + 1) z_{k+1} \partial_{\epsilon_k},
\]

in line with [32, eq. (36)]. This impacts the interpretation of \( \Gamma^* \), see (2.34), and thus \( \Gamma_{xy} \). Now, the coefficients \( (\Gamma_{xy}^*)_{\beta'}^{\rho} \) are differential operators in \( a_0 \) with coefficients that are analytic functions in \( a_0 \), still determined via the \( \pi_{xy}^{(n)} \).

We note that Theorem 2.2 also provides an estimate of these quantities: Indeed, in the statement and the proof of Theorem 2.2, we may replace \( \partial_2 - \partial_1^2 \) by \( \partial_2 - a_0 \partial_1^2 \) with \( a_0 > 0 \), and obtain for all original \( \beta \) the estimates (2.28) and (2.29), locally uniformly in \( a_0 \). Since also these objects are analytic in \( a_0 \), this also yields an estimate of their \( a_0 \)-derivatives. When restricting to \( \beta' \), by uniqueness, these new objects can be related to the original ones via (2.45).

On the above class of \( \beta' \), \( \cdot \) is actually coercive, meaning that the number of \( \beta' \) with \( |\beta'| < 2 \) is finite. In the case of \( \alpha = \frac{1}{2} \) - relevant for white noise, these 10 multi-indices, ordered by increasing homogeneity, are given in Figure 1.

| homogeneity | multi-indices |
|-------------|--------------|
| \( \alpha \) | 0            |
| 2\( \alpha \) | \( e_1 \)    |
| 3\( \alpha \) | \( e_2, 2e_1 \) |
| \( \alpha + 1 \) | \( e_1 + e_{(1,0)} \) |
| 4\( \alpha \) | \( e_3, e_1 + e_2, 3e_1 \) |
| 2\( \alpha + 1 \) | \( e_2 + e_{(1,0)}, 2e_1 + e_{(1,0)} \) |

Figure 1: Singular multi-indices for \( \alpha \in (\frac{2}{3}, \frac{1}{2}) \).
For the 10 multi-indices in Figure 1, the combination of (2.43) and (2.44) takes the form

$$(\partial_2 - a_0 \partial^2_1)\Pi_{\xi,0} = \xi - c_0,$$

$$(\partial_2 - a_0 \partial^2_1)\Pi_{\xi_1 e_1} = \Pi_{\xi_0} \partial^2_1 \Pi_{\xi_0} - c_{e_1},$$

$$(\partial_2 - a_0 \partial^2_1)\Pi_{\xi_2 e_2} = \Pi_{\xi_0} \partial^2_1 \Pi_{\xi_0} - (c_{e_1} + 2\Pi_{\xi_0} c_{e_1}),$$

$$(\partial_2 - a_0 \partial^2_1)\Pi_{\xi_3 e_3} = \Pi_{\xi_0} \partial^2_1 \Pi_{\xi_0} - (c_{e_1} + 3\Pi_{\xi_0} c_{e_1} + 3\Pi^2_{\xi_0} c_{e_1}),$$

$$(\partial_2 - a_0 \partial^2_1)\Pi_{\xi_0} d_{e_1} = \Pi_{\xi_0} \partial^2_1 \Pi_{\xi_0} - (c_{e_1} + 4\Pi_{\xi_0} c_{e_1} + 3\Pi^2_{\xi_0} d_{e_1}),$$

$$(\partial_2 - a_0 \partial^2_1)\Pi_{\xi_0} d_{e_1} = \Pi_{\xi_0} \partial^2_1 \Pi_{\xi_0} - (c_{e_1} + 5\Pi_{\xi_0} c_{e_1} + 3\Pi^2_{\xi_0} d_{e_1}).$$

Together with the BPHZ-choice of renormalization contained in the large-$\sqrt{t}$ behavior imposed on $\Pi_{\beta'}$ through (2.28), this inductively determines the functions $c_\beta'(a_0)$. Equipped with these, (1.3) takes the form

$$h(u) = \sum_{d < 2} c_\beta'(a(u)) \prod_{k \geq 1} \left( \frac{d^k a}{k! \, du^k}(u) \right)^{\beta'(k)},$$

which reproduces [32, eq. (15)] in the present paper's notation.

It is in this form we may connect to [32] Assumptions 1 and 2. Loosely speaking, the assumptions in [32] are contained in the output of Theorem 2.2 of this paper. More precisely, [32, eq. (5)] is covered by (2.21) in the form of (2.43), and [32, eq. (6)] is covered by (2.23). The estimates [32, eq. (7) and (8)] are covered by (2.27) and (2.28), with the difference that in [32] (like in [20]), they are formulated in terms of a general (though fixed) convolution kernel, and that they are pathwise, with a constant absorbed into a single scaling factor $N_0$ and, as mentioned above, locally uniform in $a_0$, cf. [32, eq. (30)]. Form [32, eq. (9)] is reproduced by (2.41). Estimate [32, eq. (10)] is covered by (2.29); however, in view of (2.46), the entries of $\Gamma_{xy}$ are differential operators in $a_0$. Finally [32, eq. (11)] follows from evaluating (2.44) at $x$ while appealing to (2.31). The crucial population condition (2.20) on $c$ is contained in the text just above [32, eq. (11)], and re-formulated as $D(n)c = 0$ for $n \neq 0$. The polynomial corrections in (2.21) and (2.24), do not appear in [32], since there, the model is (implicitly) truncated beyond $|\beta| < 2$. Because of this truncation, only $n = 0, (1, 0)$ matter in [32]; however, since [32] considers $d$ space dimensions, there are $d$ versions of $n = (1, 0)$. There are some differences in notation: When it comes to $\Gamma$, [32] omits the * but exchanges the order in $xy$, while $c$ is called $q$.

## 3 Structure of proof

In Section 3 all multi-indices $\beta, \beta', \gamma$ are implicitly assumed to be populated, cf. (2.12).

### 3.1 Intertwining of estimates and constructions in induction proof

Working on the whole space-time $\mathbb{R}^2$ instead of a torus, as we do, has many advantages. The most obvious is that we do not introduce an artificial scale, namely the size of the torus, that would break scaling. Another advantage is that we do not need to work on a tensor space of periodic functions and
polynomial \(^{46}\) and that the inversion of \((\partial^2 - \partial^2 x^2)\) does not require a Fredholm alternative and thus a polynomial Lagrangian parameter. However, an inconvenience is that we can’t separate the construction from the estimates: Because of possible infra-red divergences, we need the large-\(\sqrt{t}\) part of the estimate \(^{2.27}\) on \(\Pi^-\) to uniquely solve \(^{2.21}\) for \(\Pi^-\) within the growth and anchoring expressed by \(^{2.28}\). In fact, it is not clear whether one can construct a non-centered model \(\Pi\) in the sense of \([\Sigma]\). Subsection 4.2] on the whole space.

While construction and estimates are logically intertwined, as explained in Subsection 3.4, we choose to separate them in presentation: Section 4 contains the estimates and their proofs, while Section 5 contains the construction. Moreover, the order we present the estimates in Section 4 is not strictly by logical order, but according to the objects estimated. We explain this inherent structure of the estimates in Subsections 3.2 and 3.3. Section 6 establishes the various triangular structures important for the inductive construction.

3.2 The five loops of an induction step: original quantities, expectation, Malliavin derivative, modelled distribution, and back

The structure of an induction step requires the distinction of two cases \(^{47}\)

- The regular case of \(|\beta| \geq 2\).
- The singular case of \(|\beta| < 2\).

It is convenient to introduce the following projection

\[ Q \quad \text{is the projection onto the direct product indexed by } \beta\text{'s with } |\beta| < 2. \]  

\[ Q \Gamma^+ = Q \Gamma^+ Q \quad \text{and} \quad \Gamma^* P = P \Gamma^* P. \]  

(3.1)

We will repeatedly use that by triangularity w. r. t. \(|\cdot|\) and consistency with the polynomial sector \(\tilde{T}\) of the elements \(\Gamma \in \tilde{G}\),

\[ Q \Gamma^+ = Q \Gamma^+ Q \]  

and

\[ \Gamma^* P = P \Gamma^* P. \]  

(3.2)

We will also repeatedly use that by \(^{2.18}\) and \(^{2.23}\),

\[ Q \partial^2 z \Pi^- \in \tilde{T}^*. \]  

(3.3)

The proof of the estimates in Section 4 is structured into five subsections, each having the structure of a loop, and which order by increasing complexity,

- Original quantities: In Subsection 4.1 we estimate \(\Gamma_{xy}, \Pi_{xy}^-, \) and \(\Pi_x\), assuming control of \(Q \Pi_x^-\).
- Expectation: In Subsection 4.2 we show that the BPHZ-choice of renormalization gives control of \(E Q \Pi_x^-\). By the SG inequality, this gives control of \(\Pi_x^-\), assuming control of the (directional) Malliavin derivative \(Q \partial \Pi_x^-\).
- Malliavin derivatives: In Subsection 4.3 we estimate the Malliavin derivatives \(Q \partial \Gamma_{xy} P\) and \(Q \partial \Pi_x\), assuming control of \(Q \partial \Pi_x^-\).
- Modelled distribution: In Subsection 4.4 we introduce \(d \Gamma_{x}^+ \in \text{Hom}(T^+, \tilde{T}^+)\), of which we show that it is continuous in \(^{18}\) in that it is controlling \(Q (d \Gamma_{xy} - d \Gamma_{xy}^+ \Gamma_{xy}) P Q\), that it describes \(Q \partial \Pi_x^-\) in terms of \(Q \Pi_x^-\) in the sense of controlling \(Q \partial \Pi_x^-\) and that it describes \(Q \partial \Pi_x\) in terms of \(Q \Pi_x^-\) in the sense of controlling the rough path increments \(Q (\delta \Pi_x - \delta \Pi_x(z) - d \Gamma_{x}^- Q \Pi_x^-)\). This subsection is the core of the proof.
- Back to the estimate of \(Q \partial \Pi_x^-\) itself. In Subsection 4.5 we provide control of \(Q \partial \Gamma_{x}^+ P\) and then of \(Q \partial \Pi_x^-\), assuming control of \(Q (\delta \Pi_x^- - d \Gamma_{x}^- Q \Pi_x^-)\).

\(^{46}\) more precisely, the space of polynomials with coefficients given by periodic functions, restricted to the diagonal, see \([29]\) Subsection 3.1

\(^{47}\) even if we were just interested in singular \(\beta\)’s, the structure of \(d \Gamma^+\) requires the estimate of regular \(\beta\)’s

\(^{48}\) what we call the secondary base point
3.3 The four types of tasks in a loop: algebraic argument, reconstruction, integration, three-point argument

Subsections 4.1, 4.3, 4.4 and, to some extent, Subsection 4.5 involve the same type of tasks, arranged in a similar type of loop (see Figure 2). An important role is played by the $\pi_{xy}^{(n)}$ that determine the $\Gamma_{xy}^{*}$ via the exponential formula (2.34), and their counterparts $d\pi_{xc}^{(n)}$ for $d\Gamma_{xc}^{*}$, see (4.31). The four tasks are:

- **Algebraic argument.** All four subsections start with an “algebraic argument” (called like this because it is based on an exponential-type formula) to estimate $\Gamma_{xy}^{*} P$, $Q \delta \Gamma_{xy}^{*} P$, $Q (d\Gamma_{xy}^{*} - d\Gamma_{xy}^{*} \Gamma_{xy}^{*} P) Q$, and $Q d\Gamma_{xy}^{*} P$, in terms of $\pi_{xy}^{(n)}$, $Q \delta \pi_{xy}^{(n)}$, $Q (d\pi_{xy}^{(n)} - d\pi_{xy}^{(n)} - d\Gamma_{xc}^{*} \pi_{xy}^{(n)})$, and $Q d\pi_{xy}^{(n)}$, respectively.

- **Reconstruction.** Subsections 4.3 and 4.4 feature a reconstruction argument in order to control $(\text{id} - Q) \Pi_{x}$ and $Q (\delta \Pi_{x} - d\Gamma_{xc}^{*} \Omega_{x})$. By a reconstruction argument, we understand that for a family $\{F_{x}\}_{x \in \mathbb{R}^{2}}$ of Schwartz distributions on $\mathbb{R}^{2}$ that satisfy a continuity condition in the base point $x$, we estimate $F_{x}(x)$ in terms of the diagonal $50 F_{x}(x)$.

- **Integration.** Subsections 4.1, 4.3, 4.4 involve an integration argument to pass from $\Pi_{x}$, $Q \delta \Pi_{x}$, and $Q (\delta \Pi_{x} - d\Gamma_{xc}^{*} \Omega_{x})$, to $\Pi_{x}$, $Q \delta \Pi_{x}$, and $Q (\delta \Pi_{x} - \delta \Pi_{x}(z) - d\Gamma_{xc}^{*} \Omega_{x})$, respectively. By an integration argument, we mean that we pass an annealed Hölder norm anchored in a base point $x$ through an integral representation. It amounts to a Schauder estimate.

- **Three-point argument.** All four subsections appeal to a “three-point argument” (called like this because it involves varying an additional third point in order to control polynomial coefficients) to pass from the estimate of $\Pi_{x}$, $Q \delta \Pi_{x}$, $Q (\delta \Pi_{x} - \delta \Pi_{x}(z) - d\Gamma_{xc}^{*} \Omega_{x})$, and $Q d\Gamma_{xy}^{*} P$, to the estimate of $\pi_{xy}^{(n)}$, $Q \delta \pi_{xy}^{(n)}$, $Q (d\pi_{xy}^{(n)} - d\pi_{xy}^{(n)} - d\Gamma_{xc}^{*} \pi_{xy}^{(n)})$, and $Q d\pi_{xy}^{(n)}$, respectively.

| Original quantities | Expectation | Malliavin derivatives | Path increments | Back to $\delta \Pi_{x}$ |
|---------------------|-------------|-----------------------|-----------------|----------------------|
| $\Gamma_{xy}^{*}$, $\Pi_{x}$, $\pi_{xy}^{(n)}$ | $E \Pi_{x}$ | $\delta \Gamma_{xy}^{*}$, $\delta \Pi_{x}$, $\delta \pi_{xy}^{(n)}$ | $d\Gamma_{xy}^{*} - d\Gamma_{xy}^{*} \Gamma_{xy}^{*} P$, $\delta \Pi_{x} - \delta \Pi_{x}(z) - d\Gamma_{xc}^{*} \Omega_{x}$, $\Pi_{x}$ | $d\Gamma_{xy}^{*}$, $d\pi_{xy}^{(n)}$, and averaging |
| Subsection 4.1 | Subsection 4.2 | Subsection 4.3 | Subsection 4.4 | Subsection 4.5 |
| Algebraic arg. I (i) Proposition 4.1 | Algebraic arg. I (i) Proposition 4.1 | Algebraic arg. II Proposition 4.8 | Algebraic arg. IV Proposition 4.15 | |
| Reconstructed I Proposition 4.2 | Reconstructed I Proposition 4.2 | Reconstructed III Proposition 4.12 | |
| Integration I Proposition 4.3 | Integration I Proposition 4.3 | Integration III Proposition 4.13 | |
| Three-point arg. I Proposition 4.4 | Three-point arg. I Proposition 4.4 | Three-point arg. II Proposition 4.14 | Three-point arg. IV Proposition 4.16 | |
| Averaging Proposition 4.17 | Averaging Proposition 4.17 | Averaging Proposition 4.17 | Averaging Proposition 4.17 | |

Figure 2: Columns correspond to the five loops of an induction step, cf. Subsection 3.2. Rows correspond to the four types of tasks in a loop, cf. Subsection 3.3. The numbers in the lower right corner indicate the logical order within an induction step for singular multi-indices, cf. Subsection 3.4.

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49 and for Subsection 4.1 ends

50 In the sense of $\lim_{n \to \infty} F_{x}(x)$ with the understanding that this limit exists
3.4 The logical order of loops and tasks in one induction step

In the course of Sections 4 and 5 we will add a fairly large number of auxiliary statements. Some have to be logically included into the induction statement, because we refer to them as part of the induction hypothesis. For the convenience of the reader, we list those here:

- Exponential-type formulas: All the statements involving $\pi^{(n)}$, $\delta\pi^{(n)}$, and $d\pi^{(n)}$, namely (4.1), (4.16), (4.35), and (4.88).
- Malliavin derivative: Some of the statements involving $\delta\Pi^-$ and $\delta\Pi$, see (4.13) and (4.25).
- Modelled distribution: Continuity and boundedness of $d\Gamma^*$, namely (4.38) and (4.89), rough-path increment of $\delta\Pi$, that is, (4.73).
- Symmetry: Shift and reflection covariances (5.1) and (5.2), and the anchoring (5.21).

The remaining additional statements are just used inside one induction step and are therefore not listed above: the estimates (4.9) and (4.11) on the expectation, the estimate (4.18) on $\delta\Gamma^*$, the rough-path estimates on $\delta\Pi^-$ (4.43), the shift and reflection covariances (5.4) and (5.5) on the level of $\Pi^-$. The logical order of one induction step depends on whether $\beta$ is regular or singular. For regular $\beta$, we just run through the first Subsection 4.1 (but still most of the construction):

- Construction and estimate of $\Gamma^*P$. Thanks to the projection $P$, $\Gamma^*P$ depends just on the part of the $\pi^{(n)}$'s that is constructed and estimated by induction hypothesis; Proposition 4.1 provides the estimate of $\Gamma^*P$ by the algebraic argument.
- Construction and estimate of $\Pi^-$. Because of $(id - Q)c = 0$, $\Pi^-$ is constructed by induction hypothesis via (2.22), Proposition 4.2 provides control via reconstruction.
- Construction and estimate of $\Pi$. $\Pi$ is constructed and estimated by integration, see Proposition 4.3.
- Construction and estimate of $\pi^{(n)}$. $\pi^{(n)}$ is constructed in Subsection 5.3 and estimated by the three-point argument, see Proposition 4.4.
- Properties and estimate of $\Gamma^*$. The properties are established in Proposition 4.4, and which is estimated in Proposition 4.5 by a second application of the algebraic argument.

The logical order is much more complex for singular $\beta$:

1 By induction hypothesis, the following quantities are defined and controlled by the algebraic argument:
   - $\Gamma_{xy}^-P$, see Proposition 4.1
   - $Q\delta\Gamma_{xy}^+P$, see Proposition 4.8
   - $Qd\Gamma_{xy}^-P$, see Proposition 4.15
   - $Q(d\Gamma_{xy}^+ - d\Gamma_{xy}^+\Gamma_{xy}^-)PQ$, see Proposition 4.11

2 By induction hypothesis, the difference $Q(\Pi^- - c)$ is defined and covariant w. r. t. shift and reflection. As a consequence,
   - the expectation of its space-time average is defined, see Proposition 4.6
   - which allows us construct $c$ and thus $Q\Pi^-$, see Proposition 5.1 but not yet to control $Q\Pi^-$
   - and to control $Q\Pi^-$, see Proposition 4.7

3 In order to get control of $Q\Pi^-$, we control
4 Equipped with control of $\delta \Pi_x$, we construct
- $\Pi_x$ and thereby $\delta \Pi_x$ in Subsection 5.2,
- $\pi_x^{(n)}$ and thereby $\Gamma_x$ in Subsection 5.3,
- $d\pi_x^{(n)}$ and thereby $d\Gamma_x$ in Subsection 5.4.

5 We finish Subsection 4.1 for the control of the original objects:
- Proposition 4.3 estimates $\Pi_x$,
- Proposition 4.4 estimates $\pi_x^{(n)}$,
- Proposition 4.5 estimates (the full) $\Gamma_x$.

6 We finish Subsection 4.3 for the control of Malliavin derivatives:
- Proposition 4.9 estimates $Q\delta \Pi_x$,
- Proposition 4.10 estimates $Q\delta \pi_x^{(n)}$.

7 We finish Subsection 4.4 for the control of rough path increments:
- Proposition 4.13 estimates $Q(\delta \Pi_x - \delta \Pi_x(z) - d\Gamma_x Q\Pi_z)$, and
- Proposition 4.14 estimates $Q(d\pi_x^{(n)} - d\pi_x^{(n)} - d\Gamma_x^\pi z^{(n)})$.

The logical order of the estimates, from Proposition 4.1 to Proposition 4.17, is indicated in Figure 2 by the small number in the lower right corner of each field.

3.5 The purely polynomial case $\beta = e_n$

The purely polynomial multi-indices are easily dealt with before starting the induction: If $\beta = e_n$ for $n \neq 0$, we must set $c_\beta = 0$ and $\Pi_x^{\beta} = 0$ by (2.20) and (2.22), respectively, and define $\Pi_x^{\beta}$ by (2.23). We note that these definitions are consistent with covariance under shift (5.1) and reflection (5.2). Then (2.27) is trivial, and (2.21) and (2.28) are satisfied by (2.18). In view of the exponential formula (2.34), $\Gamma_x$ is determined by $\{\pi_{xy}^{(n)}\}_m$, so that the task is to define $\pi_{xy}^{(m)}$ for $\beta = e_n$ with $n \neq 0$:

$$\pi_{xy}^{(m)} = \begin{cases} \left(\begin{array}{c} n \\ m \end{array}\right)(y-x)^{n-m} & \text{for } m \text{ with } m < n \\ 0 & \text{else} \end{cases}$$

(3.4)

with the understanding that $\left(\begin{array}{c} n \\ m \end{array}\right)$ vanishes if the componentwise $m \leq n$ is violated, and where $m < n$ means that at least one of the components is strictly ordered. Note that this form of (3.4) with a space-time vector (here $y-x$) that is common to all $n, m$ is necessary to ensure the compatibility of $G$ with the polynomial sector $\tilde{T}$ in the sense of [21, Assumption 3.20]. Obviously, (3.4) satisfies the estimate (4.1), which feeds into the estimate (2.29) of $\Gamma_x$. By (2.38), (3.4) implies for $\beta = e_n$

$$\Gamma_x^{\beta} = \begin{cases} \left(\begin{array}{c} n \\ m \end{array}\right)(y-x)^{n-m} & \text{if } \gamma = e_m \text{ for some } m \neq 0 \\ 0 & \text{else} \end{cases}$$

(3.5)

The property (3.5) implies that (3.4) is consistent with (5.7), which is needed for transitivity (2.26).
3.6 The ordering relation \( \prec \) for the induction

At first sight, one would hope for an induction in the homogeneity \( |\beta| \); the set \( A \) of homogeneities, being locally finite and bounded from below, lends itself to an induction argument. However, this is not possible because as opposed to \( \Gamma_{xy} \), or the structurally closer \( \delta \Gamma_{xy} \), \( d\Gamma_{xy} \) is not triangular w. r. t. \( \cdot \).

As we shall see in Subsection 4.4, this lies in the nature of \( d\Gamma_{xy} \): \( \delta \Pi_{\beta} \) is modelled (almost) to order \( \frac{3}{2} + \alpha \), independently of \( \beta \). Here is a simple example for the failure of triangularity: On the one hand, we have\(^{51}\) \( (d\Gamma_{xy})_{e^1+e^1(1,0)} = \partial_1 \delta \Pi_{0}(z) \), which does not vanish for generic \( z \), while on the other hand, \( |e_1| = 2\alpha < \alpha + 1 = |e_1 + e_1(1,0)| \). Even block triangularity with respect to the threshold homogeneity \( 2 \) fails: \( (d\Gamma_{xy})_{2e_1+e_1(1,0)} = 2\partial_1 \delta \Pi_{0}(z) \neq 0 \), while \( |2e_1 + e_1(1,0)| = 2\alpha + 1 < 2 + \alpha = |2e_1 + 2e_1(1,0)| \).

Fortunately, it turns out that \( d\Gamma_{xy} \) does have a triangular structure w. r. t. an ordering that involves \(|\beta|\), cf. (2.12), as a “first digit” and \(|\beta|_p\), cf. (2.17), as a second digit, in the spirit of \([30]\), Subsection 3.5. In the reconstruction argument, based on the structure of the term \( \sum_{k \geq 0} z_k \Pi_k \partial^2 \Pi_k \) (or rather its Malliavin derivative), we need a finer ordering, which involves the component \( \beta(0) \) (to which the other two digits are oblivious) as a third digit. We shall argue in Subsection 6 that at least the triangular effect of this ordering can be captured by the following ordinal

\[
|\beta|_\prec := |\beta| + \frac{1}{2} |\beta|_p + \frac{1}{4} |\beta(0)|, \tag{3.6}
\]

where the weights in this combination are fixed for convenience but could be replaced by any three strictly ordered positive numbers.

For notational convenience, we define

\[
\beta' \prec \beta \iff |\beta'|_\prec < |\beta|_\prec, \tag{3.7}
\]

\[
\beta' \preceq \beta \iff (\beta' \prec \beta \text{ or } \beta' = \beta). \tag{3.8}
\]

A further benefit of (3.6) is that \( \cdot \) is coercive, meaning that the set \( \{ \beta \mid |\beta|_\prec \leq M \} \) is finite for every finite \( M \) (which would not be true when \( \cdot \) is replaced by \( \cdot \) because both \( \cdot \) and \( \cdot \) are oblivious to the \( \beta(0) \) component). This is crucial in the induction since at every step, the stochastic integrability deteriorates due to the unavoidable use of Hölder’s inequality in probability when estimating products of random variables.

While in general, we only\(^{52}\) have

\[
|\beta| \geq -1 \quad \text{and} \quad |\beta|_\prec \geq -\frac{1}{2}. \tag{3.9}
\]

it follows from (2.12) that \(|\beta|_\prec \geq 0\) for all \( \beta \) not purely polynomial with equality iff \( \beta = 0 \). In view of (3.7) and the additivity of \( \cdot \), this implies compatibility of \( \cdot \prec \) and summation:

\[
\beta_1 \preceq \beta_1 + \beta_2 \quad \text{provided } \beta_2 \text{ is not purely polynomial,} \tag{3.10}
\]

which we shall repeatedly use. In fact, the ordering is de facto irrelevant on the purely polynomial \( \beta \)'s, which have been treated with. Among the non-purely polynomial \( \beta \)'s, the base case is given by \( \beta = 0 \).

While \( \cdot \) is additive (but negative on some purely polynomial indices), the homogeneity \( \cdot \) is not (but it is strictly positive, in fact, \( \geq \alpha \)). We will often use that

\[
|\cdot| - \alpha \overset{2.17}{=} \alpha[\cdot] + |\cdot|_p \quad \text{is additive and non-negative} \tag{3.11}
\]

\(^{51}\)as a consequence of the definition (4.31) of \( d\Gamma_{xy} \), and using \( d\Gamma_{xy}(1,0) = \partial_1 \delta \Pi_{0}(z) \), as a consequence of the definition (5.26) of \( d\Gamma_{xy}(1,0) \) and the triangular structure (6.11) of \( d\Gamma_{xy} \) w. r. t. \( \prec \)

\(^{52}\)consider \( \beta = e_1(1,0) \)
on all populated multi-indices.

In order to make the value of the multi-index explicit when referring to a statement like (2.28), we write $(2.28)_\beta$ with the understanding that we refer to the corresponding statement for the multi-index $\beta$. When a statement involves two multi-indices like (2.29), we write $(2.29)_{\beta \gamma}$ when we want to specify also the second multi-index. All statements of the induction hypothesis will be implicitly assumed to hold for every integrability exponent $\rho < \infty$, for every space-time points $x, y, z \in \mathbb{R}^d$, and every convolution parameter $t \in (0, \infty)$, if applicable. For example, when we state $(2.28)_{\beta \gamma}$, we mean the estimate for every $\rho < \infty$ and $x, y \in \mathbb{R}^2$ and for all multi-indices $\beta' < \beta$.

3.7 The base case $\beta = 0$

In fact, the argument for the base case w. r. t. the ordering $\prec$, which reduces to $\beta = 0$, is contained in the argument for the induction step, as we shall explain now, referring to the logical order of the induction in the singular case outlined in Subsection 3.4.

Item 1 is empty: due to the strict triangularity of $\Gamma_{xy}^\gamma - \text{id}$, $\delta \Gamma_{xy}$, $\frac{d\Gamma_{xy}}{dx}$, and $\frac{d\Gamma_{xy}}{dx} \frac{\Gamma_{xy}^\gamma}{\Gamma_{xy}}$, see (6.9), (6.10), (6.11), and (6.13), the $\beta = 0$ row is trivial. Item 2 is unaffected, but much easier with $c_0 = \mathbb{E} \xi$. Item 3 is the only place where the argument for $\beta = 0$ differs from the $\beta \neq 0$ (however only in Proposition 4.12), but is treated in (4.29). Item 4 is unaffected, with $d\gamma_{xy}^{(1,0)} = \partial_1 \delta \Pi_{00}(z)$. Items 5 and 6 are unaffected. Item 7 is unaffected, yielding estimates on $\delta \Pi_{00}(y) - \delta \Pi_{00}(z) - \partial_1 \delta \Pi_{00}(z)(y - z)_1$ and $\partial_1 \delta \Pi_{00}(y) - \partial_1 \delta \Pi_{00}(z)$.

4 Estimates

In this section, we establish the stochastic estimates $(2.27)$ & $(2.28)$ and $(2.29)$ in Theorem 2.2. We fix a non-purely polynomial multi-index $\beta$.

4.1 Estimate of the original quantities $\Gamma_{xy}^\gamma$, $\Pi_{x}^\gamma$, $\Pi_{x}$, and $\pi_{xy}^{(n)}$

The first task is estimate $(2.29)$ on $\Gamma_{xy}^\gamma$, based on the exponential formula $(2.34)$ and the population constraint $(2.36)$. More precisely, we shall estimate $\Gamma_{xy}^\gamma$ in terms of $\{\pi_{xy}^{(n)}\}_n$, and thus include the statement

$$\mathbb{E}^\frac{1}{2} |\pi_{xy}^{(n)}|_\rho \lesssim |y - x|^{\beta - |n|}$$

for $|n| < |\beta|$\hspace{1cm} (4.1)

into our induction. Recall that if not otherwise stated, $n$ includes $n = 0$. We split this first task into a first half, where we treat $\Gamma_{xy}^\gamma P$, see Proposition 4.1, and a second half, where we tackle the full $\Gamma_{xy}^\gamma$, see Proposition 4.2. The reason for this splitting is that according to (6.1), provided $\gamma$ is not purely polynomial, the matrix entry $(\Gamma_{xy}^\gamma)^{\gamma}_{\beta}$ depends on $\pi_{xy}^{(n)}$ only through $\pi_{xy}^{(n)}$, with $\beta' < \beta$. Following the elementary proof of [29, Lemma 4.4] we obtain from Hölder’s inequality in probability:

Proposition 4.1 (Algebraic argument I, first half). Assume that $(4.1)_{\beta \gamma}$ holds. Then $(2.29)_{\gamma \beta}$ holds for all $\gamma$ not purely polynomial.

The second task is the estimate $(2.27)$ of $\Pi_{x}^\gamma$, based on the output of Proposition 4.1. Note that by (2.31), definition (2.30), when evaluated at the base point $x$, collapses to

$$\Pi_{x}^\gamma(x) = Pz_0 \partial_1^2 \Pi_{x}(x) - c + \xi(x)1$$

and thus by $(2.20)$ and $(2.31)$ $(\text{id} - Q)\Pi_{x}(x) = 0$\hspace{1cm} (4.2)

Proposition 4.2 (Reconstruction I). Assume $|\beta| > 2$, that $(2.27)_{\beta \gamma}$ and $(4.1)_{\beta \gamma}$ hold, and that $(2.29)_{\beta \gamma}$ holds for all $\gamma$ not purely polynomial. Then $(2.27)_{\gamma \beta}$ holds.
Proof.** Because of \( \Pi_{\beta'}(x) = 0 \), by general reconstruction, see e. g. \([31\) Proposition 1\)] or \([29\) Lemma 4.8\)], the estimate of \( \Pi_{\beta'} \) follows once we establish its continuity in the base point \( x \) to the order \( |\beta| - 2 > 0 \):

\[
E^x \left( |(\alpha^x - \alpha^y)|_{\beta}(x) \right) \lesssim (\sqrt{t})^{\alpha - 2} (\sqrt{t} + |y - x|)^{\beta - \alpha}. \tag{4.3}
\]

Estimate (4.3) in turn relies on rearranging (2.40) to

\[
P_{\alpha} - \Pi_{\alpha} = (\Gamma - \text{id}) /P_{\alpha} + P \sum_{k \geq 0} c_k (\Gamma_n (\text{id} - P) /\Pi + \pi^{(0)}_n) k \partial_2^2 (\Gamma_n (\text{id} - P) /\Pi + \pi^{(0)}_n),
\]

where \( \Gamma_n (\text{id} - P) /\Pi + \pi^{(0)}_n \) (2.23) (2.38) \( \sum_{n \neq 0} z_n + \pi^{(n)}_x \Pi (\cdot - y)^n \). \tag{4.4}

We note that by the strict triangularity (6.1) of \( \Gamma_n - \text{id} \) w. r. t. \( \cdot \), the first r. h. s. term of (4.4) involves \( \Pi_{\beta'} \), only for \( \beta' < \beta \). We observe that by Hölder’s inequality in probability, the \( E^x (\cdot)^p \)-norm of each constituent \( (\Gamma_n - \text{id}) /P_{\beta'}(x) \) to the matrix-vector product is estimated by \( |y - x|^{\beta - \beta'} ((\sqrt{t})^{\alpha - 2} (\sqrt{t} + |y - x|)^{\beta - \alpha}) \), which because of \( |\beta| - |\beta'| > 0 \) (by the triangularity (6.1) of \( \Gamma_n \)) w. r. t. \( (\cdot) \) is dominated by the r. h. s. of (4.3). By the structure (6.6) of the expression \( \sum_{k \geq 0} c_k \pi^k \pi^t \), the second r. h. s. term of (4.4) involves \( 1_{n \neq 0} z_n + \pi^{(n)}_x \Pi \) for \( \beta' < \beta \). Note that both the population condition (2.36) and the estimate (4.1) extend from \( \pi^{(n)}_x \) to \( 1_{n \neq 0} z_n + \pi^{(n)}_x \Pi \), provided one relaxes \( |\beta| < |n| \) to \( |\beta| \leq |n| \). Hence as a function of the active variable, the second r. h. s. of (4.3) is a linear combination of monomials \( (\cdot - y)^n \) with \( |n| \leq |\beta| - 2 \) with a coefficient estimated by \( |y - x|^{\beta - |n| - 2} \), where we used (6.7).

We now apply the convolution \((\cdot)_t\) to the polynomial in (4.4) and evaluate in \( x \). This calls for moment bounds on the kernel \( \psi_t \), which easily follow from the fact that \( \psi_t \) is the parabolic rescaling of the kernel \( \psi_{t = 1} \), which has Fourier transform \( \exp(-k_1^2 - k_2^2) \) and thus is a Schwartz kernel. Following \([33, \text{eq. (2.4)}]\) and for later use we formulate them in the more general form of \( (\sqrt{t})^{|m|} \int dy^p|\psi_t(y - y^p)|(|\lambda + |y - x|)^p \lesssim (\sqrt{t} + \lambda + |y - x|)^p. \tag{4.5} \)

Using (4.5) for \( m = 0, \lambda = 0, \) and \( p = |n| \) in form of \( |(|\cdot|^{n})_t(x)| \lesssim (\sqrt{t} + |y - x|)^{|n|} \), we see that each summand \([33] \) of the second r. h. s. term of (4.4) is estimated by \( |y - x|^{\beta - |n| - 2} (\sqrt{t} + |y - x|)^{|n|} \), which obviously is dominated by the r. h. s. of (4.3).

The third task is the estimate (2.28) of \( \Pi_{\beta} \), based on the output of Proposition 4.2. It relies on the construction of \( \Pi_{\beta} \) in terms of \( \Pi_{\beta'} \Pi \) with help of the semi-group kernel, see (4.4):

\[
\Pi_{\beta} = - \int_0^\infty dt \left( 1 - T_{\beta'}^{(1)}(t) \frac{\partial_2 + \partial_2^2}{\Pi_{\beta'}} \right) \tag{4.6}
\]

where \( T_{\beta'}^{(1)} \) denotes the operation of taking the Taylor polynomial of degree \( |\beta| \) in the base point \( x \). As specific to integration in regularity structures, the estimate (2.28) can be seen as a Schauder estimate anchored in the base point \( x \), here on an annealed level. As is typical for Schauder theory, one has to avoid integer values, which is ensured by (2.19). The proof distinguishes the far-field range \( \sqrt{t} \geq |y - x| \), where the representation formula (4.78) for Taylor’s remainder \( 1 - T_{\beta'}^{(1)} \) is used, and the near-field range \( \sqrt{t} \leq |y - x| \), where (4.6) is broken up into its two constituents \( T_{\beta'}^{(1)} \) and 1. We refer to \([29, \text{Lemma 4.9}] \) or to Proposition 4.13 for an application of the same tools in a more complicated setting.

Incidentally, (4.6) reproduces Hairer’s form of integration \([20, \text{eq. (8.19)}]\). However, since we are working on the whole space instead of the torus so that there is no a priori decay in \( t \) of the integrand, the polynomial \( T_{\beta'}^{(1)} \int_0^\infty d_2 + d_2^2 \Pi_{\beta'} \) may not be well-defined by itself. Once we establish in Proposition 4.3 that the \( t \)-integral in (4.6) converges (absolutely), it easily follows that \( \Pi_{\beta} \) satisfies both the PDE modulo polynomials (2.21) and the anchoring (2.31).
Proposition 4.3 (Integration I). \((2.27)_{\beta} \text{ implies } (2.28)_{\beta}\).

We now turn to the fourth task of estimating \(\pi_{x}^{(n)}\). This relies on the identity

\[
\sum_{n} \pi_{x}^{(n)}(z - y)^{n} = \Pi_{x}(z) - \Pi_{y}(z) - (\Gamma_{xy} - \text{id})P\Pi_{y}(z)
\]

(4.7)

involving the three points \(x, y, \) and \(z\). Identity \((4.7)\) follows from \((2.41)\), using \((2.38)\) and \((2.23)\) for \(n \neq 0\), and \((2.42)\) for \(n = 0\). Since the invertibility of the Vandermonde matrix of degree \(< |\beta|\) yields the equivalence of annealed norms

\[
\max_{n, |n| < |\beta|} |y - x|^{|n|}E_{y}^{1/2} |\pi_{x}^{(n)}|^{p} \sim \sup_{z}|z - x| \leq |y - x| E_{y}^{1/2} \sum_{n} \pi_{x}^{(n)}(z - y)^{n} |^{p},
\]

formula \((4.7)_{\beta}\) allows to estimate \(\pi_{x}^{(n)}\) by the outputs of Propositions 4.1 and 4.3.

Proposition 4.4 (Three-point argument I). Assume that \((2.28)_{<\gamma} \text{ holds and that } (2.29)_{\beta} \text{ holds for all } \gamma \not\text{ not purely polynomial. Then } (4.1)_{\beta} \text{ holds.}

Equipped with the output of Proposition 4.4 we now may complete our first task. By the same argument as for Proposition 4.1 we have

Proposition 4.5 (Algebraic argument I, second half). \((4.1)_{<\beta} \text{ implies } (2.29)_{\beta}.

4.2 Estimate of the expectation: BPHZ-choice of renormalization, SG inequality, and dualization of Malliavin derivative estimate

For this and the next three subsections, we restrict to singular and not purely polynomial \(\beta\) and start addressing the challenging part of the proof, namely the estimate \((2.27)_{\beta}\) of \(\Pi_{y}\) in this singular case. We will use what is called the \(L^{p}\)-version, for \(p \geq 2\), of the SG inequality

\[
E_{y}^{1/2} |F|^{p} \lesssim \|E F\| + E_{y}^{1/2} \int_{\mathbb{R}^{2}} \left((\partial_{y}^{4} \partial_{z}^{2})^{1/2} \left(\frac{1}{2} - \alpha\right) \frac{\partial F}{\partial z} \right)^{2} dz.
\]

(4.8)

This is a simple consequence of \((2.9)\), using the chain rule for the Malliavin derivative and Hölder’s estimate in probability, and is oblivious to the nature of the underlying Hilbert norm \((2.8)\), see for instance [26, Step 2 in the proof of Lemma 3.1]; the result is classical in the Gaussian case [3, Theorem 5.5.11].

For \(t, x, \) and \(y\) arbitrary yet fixed, we apply \((4.8)\) to \(F = F[\xi]\) given by \(F = \Pi_{y}^{\beta}(y)\). We are cavalier about checking the uniform approximability by cylinder functions, because \(F\) is a multi-linear form of \(\xi\), see for instance [25, Lemma F.1] in the case of quadratic functionals.

We now argue that the first r. h. s. term \(\Xi_{t}^{\beta}(y)\) in \((4.8)\) is estimated as a consequence of the BPHZ-choice of renormalization from Subsection 5.1. In order to pass from the limit \(\lim_{t \to \infty}\) in Proposition 5.1 to a finite value \(\sqrt{T}\) of the (spatial) convolution scale, we need the following proposition.

Proposition 4.6. For \(|\beta| < 2\), suppose that \((2.27)_{<\beta}\) and \((5.4)_{\beta}\) holds, and that \((2.29)_{\beta} \text{ holds for all } \gamma \not\text{ not purely polynomial. Then we have}

\[
\int_{T} \frac{d}{dt} \Xi_{t}^{\beta}(y) \lesssim (\sqrt{T})^{\alpha - 2} (\sqrt{T} + |y - x|)|\beta|^{-\alpha}.
\]

(4.9)

Proof. The proof is based on the identity

\[
\frac{d}{dt} \Xi_{t}^{\beta}(y) = - \int_{\mathbb{R}^{2}} dz (\partial_{y}^{4} - \partial_{z}^{2}) \psi_{-s}(y - z) E(\Gamma_{xy}^{\beta} \Pi_{\psi y}^{\beta})(z)
\]

\[
= - \int_{\mathbb{R}^{2}} dz (\partial_{y}^{4} - \partial_{z}^{2}) \psi_{-s}(y - z) E\left((\Gamma_{x}^{\beta} - \text{id}) \Pi_{\psi y}^{\beta}\right)(z),
\]

(4.10)
for all \( s \in (0, t) \) which can be seen as follows: The first part of the identity is a consequence of \([2,24]^{54}\) the semi-group property \([2,1]\) and the defining property \([2,2]\) of the kernel \( \psi \). The second part is a consequence of shift covariance \([5,4]\) of \( \Pi_x \), see \([5,4]\), and the shift-invariance of \( \xi \) in law, which combine to the independence of \( \mathfrak{E} \Pi_x^\beta (z) \) from \( z \). The merit is that as a consequence of the strict triangularity \([6,1]\) of \( \Gamma \) only features \( \{ \Pi_x^\gamma \}_{\beta' \prec \beta} \). Since by definition, \( \Pi_x^\beta \in \tilde{T}^\ast \), \([4,10]\) only features \( \{ \Pi_x^\gamma \}_{\beta' \prec \beta} \) for \( \beta' \) not purely polynomial.

We use \([4,10]\) with \( s = \frac{1}{4} \). By the Cauchy-Schwarz inequality in probability, the contribution from \( (\Pi_x^\gamma - \Pi_x^\alpha)^\beta \Pi_x^\beta (z) \) is estimated in expectation by \( |z - x|^{\beta - |\beta'|} (\sqrt{T})^{\beta' - 2} \lesssim (|y - z| + |y - x|)^{\beta - |\beta'|} \). After integration in \( y \), by the moment bounds \([4,5]\) on \( \psi \), this contribution to \([4,10]\) is controlled by \( r^{-1} (\sqrt{T} + |y - x|)^{\beta - |\beta'|} (\sqrt{T})^{\beta' - 2} \). Since \( |\beta'| < 2 \), integration in \( t \geq T \) yields control by \( (\sqrt{T} + |y - x|)^{\beta - |\beta'|} (\sqrt{T})^{\beta' - 2} \). By \( \beta' \geq \alpha, (4,9) \) follows.

Equipped with Proposition \([4,6]\) and the choice \([2,16]\) of \( c_\beta \), the qualitative Proposition \([5,1]\) instantly upgrades to the desired estimate of \( \mathfrak{E} \Pi_x^\beta (y) \):

**Proposition 4.7.** For \( |\beta| < 2 \), suppose that \([2,27]\) \( \prec \beta \), that \([4,9]\) \( \beta \), and that \([2,29]\) \( \beta \) holds for all \( \gamma \) not purely polynomial. Then we have

\[
|\mathfrak{E} \Pi_x^\beta (y)| \lesssim (\sqrt{T})^{\alpha - 2} (\sqrt{T} + |y - x|)^{|\beta| - |\alpha|}.
\]

The remaining task of this and the next three subsections is thus to estimate the Malliavin derivative of \( \Pi_x^\beta (y) \), in the norm given by \([4,8]\), by the r. h. s. of \([4,11]\):

\[
\mathbb{E}^\frac{1}{2} \left| \int_{\mathbb{R}^2} \left( (\partial_1^4 - \partial_2^2)^{\frac{1}{2} (\alpha - 1)} \frac{\partial}{\partial \xi} \Pi_x^\beta (y) \right)^2 \right| \lesssim (\sqrt{T})^{\alpha - 2} (\sqrt{T} + |y - x|)^{|\beta| - |\alpha|}.
\]

It is convenient to undo the Riesz representation \([2,7]\) and to return to the derivative \( \delta \Pi_x^\beta (y) \) of \( \Pi_x^\beta (y) \) in direction of the space-time field \( \delta \xi \). It is a straightforward consequence of \( L^q(\mathbb{R}) \)-duality, with \( q \) denoting the conjugate exponent of \( p \), that \([4,11]\) is equivalent

\[
|\mathfrak{E} \delta \Pi_x^\beta (y)| \lesssim (\sqrt{T})^{\alpha - 2} (\sqrt{T} + |y - x|)^{|\beta| - |\alpha|} \mathbb{E}^\frac{1}{2} \left| \int_{\mathbb{R}^2} \left( (\partial_1^4 - \partial_2^2)^{\frac{1}{2} (\alpha - 1)} \delta \xi \right)^2 \right|^{\frac{q}{2}},
\]

provided that this is established for an arbitrary \( \delta \xi \) that is allowed to be random in order to pull the supremum over \( \delta \xi \) out of the \( L^q(\mathbb{R}) \)-norm. For the base case and when introducing a weight, both in Subsection \([4,4]\) it will be convenient to replace the \( L^2(\mathbb{R}^2) \)-based fractional Sobolev norm of \( \delta \xi \) by its equivalent \( L^{q'}(\mathbb{R}^2) \)-based Besov norm. This equivalence is obvious when the Besov side is formulated in terms of our semi-group, since it then follows by Plancherel from the elementary scaling identity

\[
(k_1^4 + k_2^2)^{\frac{1}{2} (\alpha - \frac{1}{2})} \sim \int_0^\infty ds s^{\frac{1}{2} (\alpha - \frac{1}{2})} \exp(-2s(k_1^4 + k_2^2))
\]

in terms of the wave vector \( k = (k_1, k_2) \) (and relies on \( \alpha < \frac{1}{2} \)):

\[
|\mathfrak{E} \delta \Pi_x^\beta (y)| \lesssim (\sqrt{T})^{\alpha - 2} (\sqrt{T} + |y - x|)^{|\beta| - |\alpha|} \mathbb{E}^\frac{1}{2} \left| \int_0^\infty ds s^{\frac{1}{2} (\alpha - \frac{1}{2})} \int_{\mathbb{R}^2} (\delta \xi)^2 \right|^{\frac{q}{2}}.
\]

In fact, we will establish a stronger version of this estimate: It is strengthened on the l. h. s. by replacing the expectation by a \( L^q(\mathbb{R}) \)-norm, and it is strengthened on the r. h. s. by exchanging the spatial and probabilistic norm (which by Minkowski’s inequality is a strengthening due to \( q \leq 2 \))

\[
|\mathfrak{E} \frac{1}{2} | \delta \Pi_x^\beta (y)|^{q'} \lesssim (\sqrt{T})^{\alpha - 2} (\sqrt{T} + |y - x|)^{|\beta| - |\alpha|} \mathbb{E}^\frac{1}{2} \left| \int_0^\infty ds s^{\frac{1}{2} (\alpha - \frac{1}{2})} \int_{\mathbb{R}^2} (\delta \xi)^2 \right|^{\frac{q}{2}}.
\]

\([54]^{54}\) there is no polynomial defect because \( \beta \) is singular
\([55]^{55}\) which transmits to its space-time convolution
where we introduced the following abbreviation for a norm of $\delta \xi$

$$\tilde{w} := \left( \int_0^\infty \frac{ds}{s} \left( \sqrt{s} \right)^{2(\frac{1}{2} - \alpha)} \int \mathbb{E}^{\frac{1}{2}} |\delta \xi_s|^q \right)^{\frac{1}{q}}. \tag{4.14}$$

Estimate (4.13) is an annealed estimate, where an annealed norm of $\delta \Pi_{xy}$ is controlled by the annealed norm $\tilde{w}$ of $\delta \xi$. We shall establish (4.13) for all $q' < q \leq 2$. Hence $\lesssim$ now also acquires a dependence on $q' < q \leq 2$ (next to $\alpha$ and $\beta$) when it comes up in the estimate of a Malliavin derivative. As for the integrability exponent $2 \leq p < \infty$, all estimates will be implicitly assumed to hold for all $q' < q \leq 2$. The strengthening from $q' = 1$ to $q' > 1$ is important when using (4.13) in the induction: One often needs to estimate products where (at most) one of the factors comes from a (directional) Malliavin derivative of the model (as indicated by the appearance of the symbol $\delta$ or $d$) whereas the other factors are one of the model components (i.e. $\Pi_x, \Pi_y, \Gamma_y$). Since the other factors ask for a stochastic $L^p$-norm with $p < \infty$, by H"older’s inequality, we need a stochastic $L^{q'}$-norm with $q' > 1$ on the Malliavin factor, see for instance (4.21) in the proof of Proposition 4.8 below. This reflects a deterioration in the stochastic integrability, which is unavoidable since the homogeneity of $\Pi_{xy}$ in $\xi$ is $|\beta| + 1$.

### 4.3 Estimate of Malliavin derivatives: $\delta \Gamma_{xy}^+, \delta \Pi_x^+, \delta \Pi_y$, and $\delta \pi_{xy}^{(n)}$

For this and the next two subsections, we fix a random $\delta \xi$. This subsection is an interlude: As will become clear only in Subsection 4.5 next to $Q \delta \Pi_x$, we also need to estimate the directional Malliavin derivatives $Q \delta \Pi_x$ and $Q \delta \Gamma_{xy}^+, P$. In fact, in this subsection, we shall proceed like in Subsection 4.1 and assume the estimate (4.13) on $Q \delta \Pi_x$ in order to (inductively) derive the estimates on the remaining objects $Q \delta \Gamma_{xy}^+, P, Q \delta \Pi_x$, and $Q \delta \pi_{xy}^{(n)}$.

Taking the (directional) Malliavin derivative of the exponential formula (2.34) applied to $\Gamma^+ = \Gamma_{xy}^+$, we obtain by Leibniz’ rule

$$\delta \Gamma_{xy}^+ = \sum_n \delta \pi_{xy}^{(n)} \Gamma_{xy}^+ D^{(n)}, \tag{4.15}$$

which motivates to include the estimates

$$\mathbb{E}^{\frac{1}{2}} |\delta \pi_{xy}^{(n)}|^{q'} \lesssim |y - x|^{|\beta| - |n|} \tilde{w} \quad \text{for all } |n| < |\beta|. \tag{4.16}$$

Note that in view of (3.4) we have

$$\delta \pi_{xy}^{(n)} \in \tilde{T}^+. \tag{4.17}$$

In analogy to Proposition 4.1 we have

**Proposition 4.8** (Algebraic argument II). Assume that (4.16) holds, and that (2.29) holds for all $\gamma$ not purely polynomial. Then we have for all $\gamma$ not purely polynomial

$$\mathbb{E}^{\frac{1}{2}} |(\delta \Gamma_{xy}^+)^{\gamma}|^{q'} \lesssim |y - x|^{|\beta| - |\gamma|} \tilde{w}. \tag{4.18}$$

**Proof.** We distinguish the contributions $n = 0$ and $n \neq 0$ to (4.15). By definition (2.14) of $D^{(0)}$, all contributions to (4.15) from $n = 0$ are of the form

$$\delta \pi_{xy}^{(0)} (\Gamma_{xy})^{q_{\gamma} + \epsilon_k + 1} \tag{4.19}$$

with the implicit understanding that this term vanishes if $\gamma(k) = 0$.
for some $k \geq 0$ and (populated) multi-indices $\beta_1, \beta_2$ with $\beta_1 + \beta_2 = \beta$ and $k \geq 0$. Note that by (4.17), $\beta_1$ is not purely polynomial; likewise, since $\gamma - e_k + e_{k+1}$ is obviously neither purely polynomial nor vanishing, this transmits to $\beta_2$ by (2.39) and (6.12). Hence we may apply (3.10) to the desired effect of

$$\beta_1 < \beta \text{ and } \beta_2 \ll \beta.$$  \hspace{1cm} (4.20)

By Hölder’s inequality in probability, we estimate the $E^\frac{1}{p} |\cdot|^{-q'}$-norm of the product (4.19) by the product of the $E^\frac{1}{q'} |\cdot|^q$-norm of the first factor and the $E^\frac{1}{p} |\cdot|^{-q'}$-norm of the second factor; recall that

$$\frac{1}{q} = \frac{1}{q'} + \frac{1}{p} \quad \text{and thus requires } q' < q \text{ because of } p < \infty.$$  \hspace{1cm} (4.21)

By (4.16) $\beta$ and (2.29) $\gamma \leq \beta$ for $\gamma$ not purely polynomial, we thus obtain an estimate by $|y - x| |\beta_1| \bar{w} |y - x||\beta_2|\bar{w} \gamma - e_k + e_{k+1}|$. Since $|\gamma - e_k + e_{k+1}| = |\gamma| + \alpha$ by definition (2.17), we learn from (3.11) that as desired

$$|\beta_1| + |\beta_2| - |\gamma - e_k + e_{k+1}| = |\beta| - |\gamma|.$$  \hspace{1cm} (4.22)

We now address the contributions to (4.15) $\beta$ from some $n \neq 0$, which in view of definition (2.35) of $D(n)$ are of the form

$$\delta \pi_{xy}^{(n)} (\Gamma_{xy}^{\gamma - e_k}).$$  \hspace{1cm} (4.23)

Note that once more, $\gamma - e_n$ is neither purely polynomial nor, by assumption, vanishing. Like above, this transmits to $\beta_2$, and yields (4.20). Now (4.23) is estimated by $|y - x| |\beta_1| \bar{w} |y - x||\beta_2|\gamma - e_n|$. Since $|\gamma - e_n| = |\gamma| + \alpha - |n|$ by definition (2.17), we get once more from (3.11)

$$|\beta_1| - |n| + |\beta_2| - |\gamma - e_n| = |\beta| - |\gamma|.$$  \hspace{1cm} (4.24)

We now pass from $\delta \Pi_x^\beta$ to $\delta \Pi_x^\beta$. To this purpose, we take the Malliavin derivative of (4.6). By an almost identical integration argument to Proposition 4.3, we obtain

**Proposition 4.9** (Integration II). Assume that (4.13) $\beta$ holds. Then we have

$$E^\frac{1}{q} |\delta \Pi_x^\beta (y)|^{q'} \lesssim |y - x| |\beta| \bar{w}.$$  \hspace{1cm} (4.25)

We finally return from $\delta \Pi_x^\beta$ to $\delta \pi_{xy}^{(n)}$, by taking the Malliavin derivative of the three-point identity (4.7), which by Leibniz’ rule assumes the form

$$\sum_n \delta \pi_{xy}^{(n)} (z - y)^n = \delta \Pi_x (z) - \Gamma_{xy}^\alpha \delta \Pi_y (z) \delta \Gamma_{xy}^\alpha \Pi_y (z).$$  \hspace{1cm} (4.26)

We obtain quite analogously to Proposition 4.4 using Hölder’s inequality in probability like in the proof of Proposition 4.8

**Proposition 4.10** (Three-point argument II). Assume that (2.29) $\beta$ and (4.18) $\beta$ hold for $\gamma$ not purely polynomial, and that (2.28) $e_\beta$ and (4.25) $e_\beta$ hold. Then (4.16) $\beta$ holds.
4.4 Estimate of modelled distribution: $\mathrm{d}\Gamma_{xy}^{*} - \mathrm{d}\Gamma_{xy}^{x}, \delta\Pi_x - \mathrm{d}\Pi^{x}, \delta\Pi_x - \delta\Pi_x(z) - \mathrm{d}\Gamma_{xy}^{x}\Pi, \text{ and } \mathrm{d}\pi_{xy}^{(n)} - \mathrm{d}\Pi^{x} - \mathrm{d}\Gamma_{xy}^{x}\Pi_{n}$

This subsection is at the heart of our proof. We return to the estimate \((4.13)_{\beta}\) on $\delta\Pi_x$. Because of a lack of regularity in the singular case, \((4.13)\) cannot be inferred from the estimate \((4.25)\) on $\delta\Pi_x$ via the Malliavin derivative of the formula \((2.22)\). In addition, such a formula would involve the divergent constants $\xi$, at least for $\beta' < \beta$. Instead, we have to capitalize on the gain in regularity that comes with the passage from $\Pi_x$ to $\delta\Pi_x$, which arises from replacing one of the instances of $\xi$ in this multi-linear expression by a $\delta\xi$. However, this gain is subtle for two reasons:

- In terms of derivative count, the passage from $\xi$ to $\delta\xi$ amounts to a gain in regularity by $\frac{D}{2} = \frac{3}{2}$, namely from $\alpha - 2$ to $\alpha - \frac{1}{2}$. However, due to the presence of the other instances of $\xi$ in the multi-linear $\delta\Pi_x$, this does not translate into a plain gain of $\frac{3}{2}$ derivatives when passing from $\Pi_x$ to $\delta\Pi_x$. Still, the degree of modeledness of $\delta\Pi_x$ has a boost from $\alpha$-Hölder continuity to \((\alpha + \frac{1}{2})\)-modeledness. This modeledness w. r. t. $\Pi_x$ is described by a modelled distribution \(\Gamma^{x}\). Indeed, we shall control the rough-path increment $\delta\Pi_x(y) - \delta\Pi_x(z) - \mathrm{d}\Gamma_{xy}^{x}\Pi_x(y)$ and the continuity expression $\mathrm{d}\Gamma_{xy}^{*} - \mathrm{d}\Gamma_{xy}^{x}\Pi_{y}$ to order $\alpha + \frac{1}{2} - \frac{D}{2}$, the former in the sense of Gubinelli’s controlled rough paths \([17, \text{Definition 1}]\), see \((4.73)\), the latter in the sense of \([20, \text{Definition 3.1}]\), see \((4.38)\).\(^{57}\)

- In terms of scaling, there is actually by construction – no difference between the \((\alpha - 2)\)-Hölder norm relevant for $\xi$ and the $L^{2}$-based Sobolev norm \((2.6)\) of (fractional) order $\alpha - 2 + \frac{D}{2} = \alpha - \frac{1}{2}$ relevant for $\delta\xi$, or its (scaling-wise identical) annealed Besov version \((4.14)\). Hence we will resort to a trick that appears like a cheat: In order to control the rough-path increments $\delta\Pi_x(y) - \delta\Pi_x(z) - \mathrm{d}\Gamma_{xy}^{x}\Pi_x(y)$ in terms of the (parabolic) distance $|y - z|$ of the active variable $y$ to the secondary base point $z$ to the desired power of $\alpha + \frac{D}{2}$, we will replace the norm \((4.14)\) of $\delta\xi$ by a norm that involves a weight that diverges in $z$. In order to recover the full (nominal) gain of order of derivatives of $\frac{D}{2}$, one would be tempted to replace $\delta\xi$ in \((4.14)\) by its weighted version $\frac{D}{2} \delta\xi$, which after squaring would result in the weighted integral $\int_{\mathbb{R}^2} |\cdot - z|^{-D}$. Recalling the definition \((2.5)\) of the Carnot-Carathéodory distance, we however see that this integral is borderline divergent. This would, in Subsection \(4.5\) make it impossible to return from the weighted to the unweighted norm $\bar{w}$ by averaging in the base point $z$. Hence we have to marginally tame the weight by replacing $\frac{D}{2} = \frac{3}{2}$ by some exponent

$$\kappa < \frac{3}{2}$$

(and thus in particular $\kappa + \alpha < 2$ by $\alpha \leq \frac{1}{2}$) \(^{(4.27)}\)

and define

$$w(z) := \left( \int_{0}^{\infty} \frac{ds}{s^{1/2 - \alpha}} \int_{\mathbb{R}^2} dy |y - z|^{-2\kappa} \mathbb{E}^{\frac{3}{2}} |\delta\xi_{z}(y)|^{q} \right)^{\frac{1}{q}}.\(^{(4.28)}\)$$

On the one hand, $w(z)$ is strong enough to control an only slightly negative Hölder norm (however quenched and localized in $z$) of $\delta\xi$

$$\mathbb{E}^{1/2} |\delta\xi_{z}(y)|^{q} \lesssim (\sqrt{q})^{\alpha - 2 + \kappa} w(z).\(^{(4.29)}\)$$

as we shall show in Proposition \(4.12\) in the context of the base case. On the other hand, because of \((4.27)\) have that (even square) averages of $w(z)$ reduce to $\bar{w}$:

$$\int_{z:|z - x| \leq \lambda} dz w(z) \leq \left( \int_{z:|z - x| \leq \lambda} dz w^{2}(z) \right)^{\frac{1}{2}} \lesssim \lambda^{-\kappa} \bar{w}.\(^{(4.30)}\)$$

\(^{57}\)which arises from $\mathrm{d}\pi_{xy}^{(1,0)}$\(^{58}\)We remark that this separation between a controlled rough path condition \((4.73)\) and the continuity condition \((4.38)\) is once more due to the fact that our abstract model space $T$ needs to be complemented by a copy of $\mathbb{R}$ capturing constant functions in order to reproduce Hairer’s abstract model space, see the discussion at the beginning of Subsection \(2.4\). If this is done, \([20, \text{Definition 3.1}]\) corresponds to the combination of \((4.73)\) and \((4.38)\).
It is conceivable that one could carry out the tasks of this subsection on the level of Besov spaces, possibly appealing to [23]. However working on the (positive) Hölder level has the advantage that it is well-behaved under taking products, which is amply used in reconstruction.

This stronger norm will indeed result in an (annealed) controlled rough-path estimate of \( \delta \Pi_x \) of order \( \kappa + \alpha \), see Proposition 4.13. This provides sufficient regularity in reconstruction when passing from the rough-path increments of \( \delta \Pi_x \) to those of \( \partial^0 \Pi_x \), see Proposition 4.12. When it comes to the above-mentioned presence of the divergent \( c \) in the formula relating \( \delta \Pi_x \) to \( \partial^0 \Pi_x \), we are saved by the fact that the \( c \) drops out when relating the rough-path increment \( \delta \Pi_x - \partial^0 \Pi_x \) of \( \delta \Pi_x \) to the (second derivative of the) rough-path increment \( \partial^2 \Pi_x - \partial^2 \Pi_x(\pi - \partial^0 \Pi_x) \) of \( \partial^0 \Pi_x \), see the crucial formula (4.41).

We defer the (inductive) construction of \( \partial^0 \Pi_x \) to Subsection 5.4 and mention here just what is necessary to explain the estimates: In terms of its form, \( \partial^0 \Pi_x \) is quite similar to \( \delta \Pi_x \), see (4.15), but truncated beyond \( n = 0, (1,0) \), and with the Malliavin derivative \( \delta \pi^{(1,0)}_x \) replaced by some \( \partial^0 \pi^{(1,0)}_x \) in \( Q \tilde{T}^+ \):

\[
d\Gamma^+_{x} = \sum_{n=0} d\pi^{(n)}_x \Gamma^+(n) \text{ with } d\pi^{(n)}_x := d\pi^{(n)}_x .
\]

In line with \( d\pi^{(0)}_x = \delta \pi^{(0)}_x \in \tilde{T}^+ \), see (4.17), we impose

\[
d\pi^{(1,0)}_x \in Q \tilde{T}^+ \quad \text{so that by (2.39) } d\Gamma^+_{x} \subset Q \tilde{T}^+.
\]

In Subsection 5.4, we argue that \( d\pi^{(1,0)}_x \) is determined by imposing qualitative first-order vanishing on every singular component

\[
\frac{\partial^n}{\partial y^n} Q(\delta \Pi_x(y) - \delta \Pi_x(z) - \partial^n \Gamma_x(\pi(x)) = 0 \quad \text{at } y = z \quad \text{for } n = 0, (1,0).
\]

We note that the population pattern of \( (d\Gamma^+_{x})^{\gamma} \) quickly gains in complexity as the homogeneity of \( \beta \) increases, see Figure 3.

\begin{center}
\begin{tabular}{|c|l|}
\hline
\( \beta \) & \( \gamma \)'s for which \( (d\Gamma^+_{x})^{\gamma} \neq 0 \) \\
\hline
0 & \( e_{(1,0)} \) \\
e_1 & \( e_0, e_{(1,0)}, e_1 + e_{(0,0)} \) \\
2e_1 & \( e_0, 2e_0, e_0 + e_1, e_{(1,0)}, e_1 + e_{(1,0)}, e_0 + e_1 + e_{(1,0)}, 2e_1 + e_{(1,0)} \) \\
3e_1 & \( e_0, 2e_0, 3e_0, e_0 + e_1, 2e_0 + e_1, e_{(1,0)}, e_0 + 2e_1, e_1 + e_{(1,0)}, e_0 + e_1 + e_{(1,0)}, 2e_1 + e_{(1,0)}, e_0 + 2e_1 + e_{(1,0)}, 3e_1 + e_{(1,0)} \)
\hline
\end{tabular}
\end{center}

Figure 3: Population pattern of \( (d\Gamma^+_{x})^{\gamma} \) for \( \beta = 0, e_1, 2e_1, 3e_1 \).

Up to these differences, the type and order of tasks will be as in Subsection 4.1. The first task is the algebraic argument relying on the following analogue of (4.15):

\[
(d\Gamma^+_{xy} - d\Gamma^+_{cx} \Gamma^+_{cy})Q = \sum_{n=0} (d\pi^{(n)}_{xy} - d\pi^{(n)}_{cx} - d\pi^{(n)}_{cx} \pi^{(n)}_{cy} \Gamma^{+}_{xy} D^{(n)}Q.
\]

This formula, which will be established in the proof of the upcoming Proposition 4.11 suggests to introduce the following estimate on the rough-path increments of \( d\pi^{(n)}_x \):

\[
\mathbb{E} \left[ |d\pi^{(n)}_{xy} - d\pi^{(n)}_{cx} - d\pi^{(n)}_{cx} \pi^{(n)}_{cy} \right] |^q \right)
\]

\[
\lesssim \gamma - \gamma \gamma - \gamma - \alpha \gamma - \gamma (|y - z| + |z - x|)^{\beta - \gamma} (w_{x}(y) + w_{x}(z)) \quad \text{for } n = 0, (1,0),
\]

\footnote{like for }
which is the analogue of (4.1). We note that (4.42), which we need to impose below, implies in particular
\[ \kappa > 1 - \alpha, \]  
so that the first exponent in (4.35) is strictly positive. As we shall discuss at the beginning of Subsection 4.5 in (4.35) and in this subsection, we do not just need the norm \( w(y) + w(z) \) with a singular weight at the two active points, \( y \) and \( z \), but also a contribution from the unweighted norm \( \bar{w} \). We combine weighted and unweighted norms through
\[ w_\times(z) := w(z) + |z - x|^{-\kappa} \bar{w}. \]  
While this inclusion of \( \bar{w} \) is dimensionally correct, it has the irritating effect of introducing an artificial singularity at \( z = x \), which however does not create problems.

**Proposition 4.11** (Algebraic argument III). Assume that (4.35) \( \gamma \beta \) holds, and that (2.29) \( \gamma \beta \) holds for all \( \gamma \) not purely polynomial. Then we have for all \( \gamma \) not purely polynomial
\[
\mathbb{E}_\gamma \left| (d\Gamma_{xy} - d\Gamma_{x'y}^\alpha \Gamma_{z'y})Q \right|_{\beta}^{\gamma} \lesssim \left( 1_{\gamma(1,0) = 0} |y - z|^{\kappa + \alpha} (|y - z| + |z - x|)^{\beta - |\gamma| - \alpha} \right) (w_\times(y) + w_\times(z)),
\]
with the implicit understanding that all exponents are non-negative unless the l. h. s. vanishes.

Let us interpret (4.38) as a continuity condition on a modelled distribution of the form \( \bar{w} \) of [20, Definition 3.1]: For this, it is convenient to write the l. h. s. in the component-wise fashion of \( (d\Gamma_{xy})^{\gamma}_{\beta} - \sum_{\beta'} (d\Gamma_{z'y})^{\beta'}_{\gamma}(\Gamma_{z'y})^{\gamma}_{\beta'} \), to fix \( \beta \) and \( x \), and to think of \( \gamma, \beta' \) and \( z, y \) as instances of the active index and variable, respectively. Then (4.38) expresses a modelled continuity condition of order \( \kappa + \alpha \), provided the active index \( \gamma \) is graded by \( |\gamma|_{\beta} \), cf. (2.17). There is a second more subtle interpretation which becomes apparent when setting \( z = x \) (and ignoring that \( w_\times(z = x) = \infty \)): Then (4.38) expresses a modelled continuity condition of order \( \kappa + |\beta| \), provided the active index \( \gamma \) is graded by \( |\gamma|_{\beta} \).

**Proof.** We start with the argument for formula (4.34). Applying one of the commuting derivations \( D \in \{ D^{(m)} \}_m \) to (2.34) yields by Leibniz’ rule the operator identity
\[
D\Gamma_{xy} = \Gamma_{xy}D + \sum_n (D\pi_{xy}^{(n)}) \Gamma_{xy}^{(n)} \]  
and thus \( D\Gamma_{xy}Q = \Gamma_{xy}DQ + \sum_{n=0,1,0} (D\pi_{xy}^{(n)}) \Gamma_{xy}^{(n)} Q \).

Using this for \( \Gamma_{xy} = \Gamma_{xy}^{\alpha} \) and then applying \( \Gamma_{xy}^{\alpha} \) to it, we obtain by multiplicativity (2.33), which we may apply since \( D \) and \( \Gamma_{xy}^{\alpha} \) map \( T^* \) into the sub-algebra \( \bar{T}^* \) by (2.39), and by transitivity (2.26)
\[
\Gamma_{xy} \Gamma_{xy}^{\alpha} Q = \Gamma_{xy}DQ + \sum_{n=0,1,0} (\Gamma_{xy}^{\alpha} D\pi_{xy}^{(n)}) \Gamma_{xy}^{(n)} Q.
\]

We now specify to \( D = D^{(m)} \), (left-)multiply by \( d\pi_{xy}^{(m)} \), and sum over \( m = 0,1,0 \) to obtain by definition (4.31),
\[
d\Gamma_{xy}^{\alpha} Q = \sum_{m=0,1,0} d\pi_{xy}^{(m)} \Gamma_{xy} D^{(m)} Q + \sum_{n=0,1,0} (d\Gamma_{xy}^{\alpha} \pi_{xy}^{(n)}) \Gamma_{xy} D^{(n)} Q
\]
\[
= \sum_{n=0,1,0} (d\pi_{xy}^{(n)} + (d\Gamma_{xy}^{\alpha} \pi_{xy}^{(n)})) \Gamma_{xy} D^{(n)} Q.
\]
Subtracting this from \((4.31)\) (with \(z\) replaced by \(y\) and multiplied by \(Q\) from the right) yields \((4.34)\).

We now turn to the estimate \((4.38)\) proper. Since up to the indicator functions, the first r. h. s. term is dominated by the second one, it is enough to establish \((4.38)\) without the first indicator function \(1_{\gamma(1,0)=0}\).

Like in the proof of Proposition \(4.8\), we distinguish the contributions \((n = 0)\) and \((n = (1,0))\). In particular, the \((n = 0)\)-contribution to \((4.34)\) gives rise to the terms of the form

\[
(d\pi_{xy}^{(0)} - d\pi_{xz}^{(0)} - d\Gamma_{xx}^{y} \pi_{xy}^{(0)})_{\beta_1}(\Gamma_{xy})_{\beta_2}^{\gamma - \epsilon_k + \epsilon_{k+1}},
\]

for some \(k \geq 0\) and \(\beta_1 + \beta_2 = \beta\). Since by \((4.17)\) and \((4.32)\), the first factor vanishes when \(\beta_1 < \beta, \beta_2 \leq \beta\), and \(\gamma - \epsilon_k + \epsilon_{k+1}\) not purely polynomial. By Hölder’s inequality in probability space, the \(|\cdot|_{\gamma}^\ast\)-norm of \((4.39)\) is estimated by

\[
|y - z|^\gamma (|y - z| + |z - x|)|\beta_1 - \alpha| (w_x(y) + w_x(z)) |y - x|^\beta_2 |y - e_k + e_{k+1}|.
\]

Using \(|y - x| \leq |y - z| + |z - x|\) on the last factor, and appealing to \((4.22)\), we see that this terms is contained in the first r. h. s. term of \((4.38)\).

The terms coming from the \((n = (1,0))\)-contribution to \((4.34)\) are of the form

\[
(d\pi_{xy}^{(1)} - d\pi_{xz}^{(1)} - d\Gamma_{xx}^{y} \pi_{xy}^{(1)} - d\Gamma_{xx}^{y} \pi_{xz}^{(1)})_{\beta_1}(\Gamma_{xy})_{\beta_2}^{\gamma - \epsilon(1,0)}
\]

for some \(\beta_1 + \beta_2 = \beta\). They are only present for \(\gamma(1,0) > 1\); the presence of \(Q\) on the l. h. s. of \((4.38)\) amounts to the restriction to \(|\gamma| < 2\), which in view of \((2.17)\) only leaves \(\gamma(1,0) = 1\), giving rise to the characteristic function \(1_{\gamma(1,0)=1}\) in the second r. h. s. contribution to \((4.38)\). Again, as in the proof of Proposition \(4.8\), we effectively have \(\beta_1 < \beta, \beta_2 \leq \beta\), and \(\gamma - \epsilon_k\) not purely polynomial. By Hölder’s inequality in probability space, the \(|\cdot|_{\gamma}^\ast\)-norm of \((4.40)\) is estimated by

\[
|y - z|^\gamma + |\beta_1 - \alpha| (w_x(y) + w_x(z)) |y - x|^\beta_2 |y - e_k|.
\]

As before, this time appealing to \((4.24)\), we see that this term is contained in the second r. h. s. term of \((4.38)\).

The second task is to estimate the rough-path increments of \(\delta \Pi_x^{-}\) based on the estimate of the rough-path increments of \(\delta \Pi_x\) stated in Proposition \(4.13\). It relies on the \(c\)-free formula

\[
Q(\delta \Pi_x^{-} - d\Gamma_{xx}^{y} \Pi_x^{-})(z) = Q \sum_{k \geq 0} z_k \Pi_x(z) \partial_x^\gamma (\delta \Pi_x - d\Gamma_{xx}^{y} \Pi_x)(z) + \delta_x(z) 1,
\]

which is the analogue of \((4.2)\), and the argument for which will be given in the proof of Proposition \(4.12\). In order to actually pass from \((4.73)\) to \((4.43)\), we need to free ourselves from the evaluation at \(z \in \delta \Pi_x\), which will be done by a reconstruction argument. Reconstruction requires that the sum of \(\alpha\) (the bare regularity of the first factor \(\Pi_x\)) and of \(\kappa + \alpha\) \((\kappa + \alpha\) the degree of modeledness of the second factor \(\delta \Pi_x\) is larger than 2 \(\text{due to the presence of the second spatial derivatives}\)). This enforces the lower bound

\[
\kappa > 2 - 2\alpha,
\]

which together with the upper bound \((4.27)\) is the sole reason for our assumption \(\alpha > \frac{1}{2}\). Incidentally, an identity analogous to \((4.41)\) would hold for the non-centered model \(\Pi\), i. e. for the first base point \(x\) omitted, and would presumably allow for reconstruction, but would not allow us to derive the estimates of the right homogeneity, quite similar to \((22)\).

In terms of \(Q\), there is a mismatch between the output of Proposition \(4.11\) and the ideal input for the upcoming Proposition \(4.12\). Handling the mismatch requires estimating \(d\Gamma_{xx}^{y}(\id - Q)\), which follows from the boundedness – as opposed to continuity – of \(d\Gamma_{xx}^{y}\) \((\text{see} (4.89)\) in Subsection \(4.5\), where it plays a more important role). Analogously to the second task of Subsection \(4.1\), we obtain
Proposition 4.12 (Reconstruction III). Assume that \((2.27)_{\gamma<\beta}, (2.28)_{\gamma<\beta}, (2.29)_{\gamma<\beta}, \) and \((4.73)_{\gamma<\beta}\) hold, assume that \((4.38)^{\gamma}_{\beta}\) and \((4.89)^{\gamma}_{\beta}\) hold, both for all \(\gamma\) not purely polynomial. Then we have

\[
\mathbb{E}_y \left| (\delta \Pi^c - d\Gamma_{x} \partial \Pi^c \partial y) \right|^{\alpha}
\lesssim (\sqrt{y} + |y-z|)^{\kappa (\sqrt{y} + |y-z| + |z-x|)} \beta^{-\alpha} (w_4(z) + w_4(y)).
\] (4.43)

We remark that these three exponents are natural: The first exponent \(\alpha - 2\) captures the bare (distributional) regularity of \(\delta \Pi^c - d\Gamma_{x} \partial \Pi^c\) at an arbitrary point, which is not better than the one of \(\delta \Pi^c \) or \(\Pi^c\), and does not depend on \(\beta\). If \(y = z\), the sum \(\kappa + \alpha - 2\) of the two first exponents emerges and describes the regularity of \(\delta \Pi^c - d\Gamma_{x} \partial \Pi^c\) near the secondary base point \(z\), which does not depend on \(\beta\); it makes the gain of \(\kappa\) appear that arises from the weight in \(w(z)\). Finally, the sum \(\kappa + |\beta| - 2\) is dictated by scaling: passing from \(w(z)\) back to \(\tilde{w}\) removes a length to the power \(\kappa\), leads to \(|\beta| - 2\), in line with \((4.13)\) and ultimately \((2.27)\).

**Proof.** We start with the proof of \((4.41)\). On the one hand, we evaluate \((2.30)\) at the base point \(z\), appeal to \((2.31)\), and multiply by \(Q\); on the other hand, we take the Malliavin derivative of \((2.30)\):

\[
Q \partial \Pi^c(z) = Q (\sum_{k=0}^{\infty} \Pi^c_k \partial^k \Pi^c + c + \xi) (z),
\] (4.44)

\[
\delta \Pi^c = P \left( \sum_{k=0}^{\infty} \Pi^c_k \partial^k \Pi^c + \sum_{k=0}^{\infty} (k+1) \Pi^c_{k+1} \partial^k \Pi^c \right) - \sum_{k=0}^{\infty} \Pi^c_k \partial^k \Pi^c (D^{(0)})^{k+1} c + \delta \xi.
\] (4.45)

Since by definition \((2.17)\), the \(\beta (k=0)\)-component has no effect on \(|\beta|\), we have \(Q \partial \Pi^c = P \partial \Pi^c\), so that in view of \((3.3)\), \(P\) is inactive in \((4.44)\). Moreover, by the first item in \((2.20)\), \(Q\) is inactive on \(c\) and of course on \(1\). By \((4.17)\) and \((6.5)\), \(P\) is also inactive in \((4.45)\). Hence we learn (using also \(d\Gamma_{x} = 0\)) that \((4.41)\) follows from

\[
Q \partial \Pi^c \partial_\Pi^c(z) = \sum_{k=0}^{\infty} \Pi^c_k \partial^k \Pi^c + \sum_{k=0}^{\infty} (k+1) \Pi^c_{k+1} \partial^k \Pi^c \right) (z),
\] (4.46)

\[
d\Gamma_{x} c = (\delta \Pi^c \sum_{k=0}^{\infty} \Pi^c_k (D^{(0)})^{k+1} c) (z).
\] (4.47)

We start by arguing that \((4.46)\) follows from

\[
d\Gamma_{x} z_0 \pi^r = \sum_{k=0}^{\infty} \Pi^c_k (z) \partial^k \pi^r + \delta \Pi^c (z) \sum_{k=0}^{\infty} (k+1) \Pi^c_{k+1} (z) \partial^k \pi^r \quad \text{for} \quad \pi^r \in \tilde{\Pi}.\]

Indeed, we use \((2.25)\) in form of \(\partial^2 \Pi^c = \Gamma_{x} \partial^2 \Pi^c\) and the triangularity properties \((6.5)\) and \((6.7)\) w. r. t. \(|\cdot|\) to see \(Q \partial \Pi^c = \Gamma_{x} \partial \Pi^c\). Then we use that \((2.23)\) and \((2.31)\) have \(\partial^2 \Pi^c (z) = Q \partial^2 \Pi^c (z) + 2z_{(2,0)}\). Since \(d\Gamma_{x} z_{(2,0)} = 0\), we have \(Q \partial \Pi^c = \partial^2 \Pi^c\). Hence \((4.46)\) indeed follows from \((4.48)\) for \(\pi^r = \partial^2 \Pi^c (z)\). By \((2.42)\) and the second item in \((4.31)\) in form of \(\delta \Pi^c (z) = d\pi^r (0)\), \((4.47)\) and \((4.48)\) take the form of

\[
d\Gamma_{x} z_0 \pi^r = \sum_{k=0}^{\infty} \Pi^c_k (z) \partial^k \pi^r + \pi^r \sum_{k=0}^{\infty} (k+1) \Pi^c_{k+1} (z) \partial^k \pi^r,
\] (4.49)

\[
d\Gamma_{x} c = \partial \pi^r \sum_{k=0}^{\infty} \Pi^c_k (z) (D^{(0)})^{k+1} c.
\] (4.50)

Because of the second item in the population condition \((2.20)\), which we may rewrite as \(D^{(0)} c = 0\) for \(n \neq 0\), it follows from \((2.34)\) that

\[
\Gamma^r (D^{(0)})^k c = \sum_{k=0}^{\infty} \frac{1}{k!} (\pi^r)^k (D^{(0)})^{k+k'} c.
\] (4.51)
Together with (2.37) for \( k = 0, 1 \), we see that the two identities (4.49) & (4.50) can be written as
\[
d\Gamma_{xz}^a z_0 \pi' = (\Gamma_{xz} z_0) d\Gamma_{xz}^a \pi' + d\pi_{xz}^a (\Gamma_{xz} z_1) \Gamma_{xz}^a \pi' \quad \text{and} \quad d\Gamma_{xz}^a c = d\pi_{xz}^a (\Gamma_{xz}^a D(0) c).
\]

By definition (4.31), the first identity follows from the fact that \( D^{(n)} \) is a derivation, mapping \( T^* \) into \( \tilde{T}^* \), see (2.39), from how it acts on \( z_t \), see (2.14), and the multiplicativity (2.33) of \( \Gamma_{xz}^a \), together with the fact that \( T^* \) is closed under multiplication. The second identity follows by definition (4.31) using once more that \( D^{(1,0)} c = 0 \).

Before starting with the estimates, we note that not only \( w(y) \), cf. (4.30), but also \( w_z(y) \) behaves well under (square) averaging in \( y \), in particular by (\( \sqrt{t} \))^\( n \) |\( a^n \psi_t |:
\[
(\sqrt{t})^n \int dy' |\partial^a \psi_t (y - y')| |w_z^2 (y')| \lesssim w_z^2 (z) \quad \text{provided} \ |y - z| \leq \sqrt{t}.
\]
Indeed, in view of the definitions (4.28) and (4.37), (4.52) follows from the elementary fact that also negative moments are preserved by averaging with (\( \sqrt{t} \))^\( n \) |\( a^n \psi_t |:
\[
(\sqrt{t})^n \int dy' |\partial^a \psi_t (y - y')| |x' - y'|^{-2\kappa} \lesssim (\sqrt{t} + |x' - y|)^{-2\kappa}
\]
\[
\sim (2\sqrt{t} + |x' - y|)^{-2\kappa} |y - z| \lesssim (\sqrt{t} + |x' - z|)^{-2\kappa}.
\]

We now introduce the family \( \{F_{xz}\}_{x,z} \) of random space-time Schwartz distributions
\[
F_{xz} := (\delta \Pi_x - d\Gamma_{xz}^a Q_{xz}^a) - \left( \sum_{k \geq 0} z_k \Pi_x^k(z) \partial^2_{\xi} (\delta \Pi_x - d\Gamma_{xz}^a Q_{xz}^a) + \delta \xi 1 \right),
\]
and note that (4.41) amounts to
\[
F_{xz}(z) = 0.
\]

We shall establish the following continuity condition of \( \{F_{xz}\}_{x,z} \) in the secondary base point \( z \):
\[
\mathbb{E}^{\frac{1}{2}} |(F_{xy} - F_{xz})_{\beta \xi}(y)|^q \lesssim (\sqrt{t})^{\theta - 2} (\sqrt{t} + |y - z|)^{\theta - \alpha} (\sqrt{t} + |y - z| + |z - x|)^{|\beta| + \kappa - \theta} (w_x(z) + w_x(y)),
\]
where
\[
\theta := \min \{\kappa + 2\alpha, \inf (A \cap (2, \infty)) \} > 2.
\]

Before proceeding, we note that the l. h. s. of (4.56) vanishes for \( \beta \in \mathbb{N}_0e_0 \). Indeed, we start by observing that \( k \neq 0 \) cannot contribute to such a component of (4.54). Because of the difference on the l. h. s. of (4.56), only the contributions involving \( d\Gamma_{xz}^a \) contribute. Hence in view of (3.3), the claim follows from (6.14). In particular, by (6.13), the l. h. s. of (4.56) vanishes unless \( |\beta| \geq 2\alpha \), which implies for the last exponent in (4.56) that \( |\beta| + \kappa - \theta \geq 0 \) by definition (4.57) of \( \theta \).

Since in view of (4.57), the sum of the first two exponents is positive, (4.56) provides a continuity condition of positive order. Using the reconstruction argument in [29, Lemma 4.8] in combination with the averaging property of the weight (4.52), (4.55) and (4.56) upgrade to
\[
\mathbb{E}^{\frac{1}{2}} |F_{xz\beta}(z)|^q \lesssim (\sqrt{t})^{\theta - 2} (\sqrt{t} + |z - x|)^{|\beta| + \kappa - \theta} w_x(z).
\]

Using once more (4.56), this yields
\[
\mathbb{E}^{\frac{1}{2}} |F_{xz\beta}(y)|^q \lesssim (\sqrt{t})^{\alpha - 2} (\sqrt{t} + |y - z|)^{\theta - \alpha} (\sqrt{t} + |y - z| + |z - x|)^{|\beta| + \kappa - \theta} (w_x(z) + w_x(y)),
\]
which we just use in the weakened form of (since $\kappa \leq \theta - \alpha$ by (4.27) and (4.57))
\[
\mathbb{E}^\frac{1}{2} |F_{x\pi}(y)|^q \lesssim (\sqrt{t})^{\alpha - 2}(\sqrt{t} + |y - z|)^{\kappa}(\sqrt{t} + |y - z| + |z - x|)^{\beta - \alpha}(w_s(z) + w_s(y)).
\]

Once we establish the boundedness of the second contribution to (4.54) in form of
\[
\mathbb{E}^\frac{1}{2} \left| \sum_{k=0}^n \frac{1}{2} \left( \delta \Pi_k - d \Gamma_{x\pi}(Q) \right)_{\beta}(y) \right|^q \lesssim \langle \sqrt{t} \rangle^{\alpha - 2}(\sqrt{t} + |y - z|)^{\kappa}(\sqrt{t} + |y - z| + |z - x|)^{\beta - \alpha}(w_s(z) + w_s(y)),
\]
we obtain the desired (4.43).

The remainder of the proof is devoted to the estimates (4.56) and (4.58). By the triangle inequality, we split (4.56) into
\[
\mathbb{E}^\frac{1}{2} \left| (d \Gamma_{x\pi}(Q)_y - d \Gamma_{x\pi}(Q)_z)_{\beta}(y) \right|^q \lesssim \langle \sqrt{t} \rangle^{\alpha - 2}(\sqrt{t} + |y - z|)^{\kappa}(\sqrt{t} + |y - z| + |z - x|)^{\beta - \alpha}(w_s(z) + w_s(y)),
\]
and the base case, meaning (4.58) for $\beta = 0$. Due to the presence of $z_k$, the l. h. s. of (4.58) $\beta = 0$ collapses to $\delta \xi_k(y)$; we shall establish the stronger version of (4.29). In order to establish (4.29), we appeal to the semi-group property (2.1) in form of $\delta \xi_k(y) = \int dz \psi_{s-y}(y - z) \delta \xi_k(z)$ for $s \in (0, t)$, so that by the triangle inequality w. r. t. $\mathbb{E}^\frac{1}{2} \cdot |^q$ we have $\mathbb{E}^\frac{1}{2} \delta \xi_k(y)|^q \leq \int dz \psi_{s-y}(y - z) \mathbb{E}^\frac{1}{2} \delta \xi_k(z)|^q$, and thus by Cauchy-Schwarz in z (and in view of the obvious sup-bound $|\psi_{s-y}| \lesssim (\sqrt{t} - s)^{-1}$)
\[
\mathbb{E}^\frac{1}{2} \delta \xi_k(y)|^q \leq \int dz \psi_{s-y}(y - z) |z - y|^{2\kappa} \mathbb{E}^\frac{1}{2} \delta \xi_k(z)|^q \lesssim (\sqrt{t} - s)^{-3 + 2\kappa} \int dz |z - y|^{2\kappa} \mathbb{E}^\frac{1}{2} \delta \xi_k(z)|^q.
\]
Applying $\int dz$, so that $(\sqrt{t} - s)^{-3 + 2\kappa} \sim (\sqrt{t})^{(2\alpha - 2 + \kappa)} (\sqrt{t})^{2(1 - \alpha)}$, we obtain the square of (4.29) by definition (4.28).

Turning to (4.61), we note that the l. h. s. has the product structure $(z_k, \pi_{\pi'})_{\beta} = \sum_{\beta_1 + \beta_2 = \beta} (z_k, \pi)_{\beta_1} (\pi')_{\beta_2}$. In view of the presence of $z_k$, $\beta_1$ is neither purely polynomial nor 0; by (4.17) and (4.32) also $\beta_2$ is not purely polynomial. Hence by (3.10) we have $\beta_1 \leq \beta$ and $\beta_2 < \beta$. Therefore (4.61) $\beta$ follows from the two estimates (4.68) $\beta$ and (4.70) $\beta$ stated below via Hölder’s inequality in probability and (3.11).

We turn to (4.59), which relies on (2.24) in form of $Q \Pi_z = Q \Pi_z \Pi_y$. In preparation for the use of (4.38) we split the increment:
\[
(d \Gamma_{x\pi}(Q)_y - d \Gamma_{x\pi}(Q)_z)_{\beta}(y) = (d \Gamma_{x\pi}(Q)_y - d \Gamma_{x\pi}(Q)_y)_{\beta}(y) + d \Gamma_{x\pi}(Q)_y(\beta \Pi_z - \beta \Pi_y). (4.62)
\]
By the triangular structure (6.9) of $\Gamma^*$ and the strict triangular structure (6.11) of $d \Gamma^*$ and (6.13) of its increments, we learn that in order to estimate (4.62) $\beta$, we only need (2.27) $\beta$ and (2.29) $\beta$. Because
of the presence of $P$ in the definition (2.22) and by the second item in (3.2), we only need (2.29)_{Y<\rho=p}, (4.38)_{Y<\rho=p}, \text{ and } (4.89)_{Y<\rho=p}. \text{ We thus have term-by-term}

$$
\mathbb{E}^Y \left| (d\Gamma_{x\gamma} Q\Pi - d\Gamma_{x\gamma} \Phi\Pi)_{\beta} (y) \right|^{\gamma}
\lesssim \sum_{y|\in \mathbb{A}} (|y - z|^{\kappa+\alpha}) |(|y - z| + |y - x|)|^{\beta - |y| - \alpha} (w_x(z) + w_y(y)) (\sqrt{\gamma}) |\gamma - 2|
\lesssim \sum_{y|\in \mathbb{A}} (\sqrt{\gamma}) |\gamma - 2| (|y - z|^{\kappa+\alpha}) (\sqrt{\gamma} + |y - z| + |x - y|)|^{\beta - |y| - \alpha}
\lesssim \sum_{y|\in \mathbb{A}} (\sqrt{\gamma}) |\gamma - 2| \sum_{y|\in \mathbb{A}} (\sqrt{\gamma} + |y - z|)|^{\beta - |y| - \alpha}
\times (w_x(z) + w_y(y)).
$$

The term (4.63) is absorbed into the r. h. s. of (4.59) because the first exponent decreases from $|\gamma| - 2$ to $\alpha - 2$, because the sum of the first two exponents decreases from $|\gamma| - 2 + \kappa + \alpha$ to $\theta - 2$ by definition (4.57) of $\theta$, and because the sum $|\beta| - 2 + \kappa$ of the three exponents agrees. The term (4.64) is also absorbed because once more, the first exponent decreases and the sum of all three exponents agrees, and because for our not purely polynomial $\gamma$, the constraint $\gamma(1,0) = 1$ implies $|\gamma| \geq 1 + \alpha$ by definition (2.17) of $|\cdot|$, leading to $|\gamma| - 2 + \kappa + \alpha - 1 \geq \theta - 2$ on the sum of the first two exponents. Finally, the term (4.65) is also absorbed, since for the not purely polynomial $\gamma$, the constraint $|\gamma| \geq 2$ by (2.19) and thus $|\gamma| \geq \theta$ by definition (4.57) of $\theta$, leading once more to $|\gamma| - 2 \geq \theta - 2$.

We now turn to (4.60). As usual, we will estimate this difference of a product by the sum of two products, where each summand contains a difference as one of its factors. For the same reason as for (4.61) we have the structure $(z_{k1} \pi_{\beta k})_{\beta} = \sum_{\beta_1 + \beta_2 = \beta} (z_{k1} \pi_{\beta k})_{\beta_1} \pi_{\beta_2}$ with $\beta_1 \ll \beta$ and $\beta_2 \ll \beta$. Hence we see that (4.60)_{\beta} follows from the four estimates (4.66)_{\beta}, (4.67)_{\beta}, (4.68)_{\beta}, and (4.70)_{\beta}, stated below, by the triangle inequality and Hölder’s inequality in probability and (3.11).

As the first ingredient to (4.60), we estimate the components $\ll \beta$ of $d\Gamma_{x\gamma} Q\partial_{\gamma}^{2} \Pi - d\Gamma_{x\gamma} \Phi\partial_{\gamma}^{2} \Pi$. In view of (3.3), the arguments is similar to (4.59). More precisely, considering (4.63)

$$
\mathbb{E}^Y \left| (d\Gamma_{x\gamma} Q\partial_{\gamma}^{2} \Pi - d\Gamma_{x\gamma} \Phi\partial_{\gamma}^{2} \Pi)_{\beta} (y) \right|^{\gamma}
\lesssim (\sqrt{\gamma})^{\beta - |\gamma| - \alpha} (\sqrt{\gamma} + |y - z| + |z - x|)|^{\beta - |y| - \alpha} (w_x(z) + w_y(y)),
$$

(4.66)_{\beta} follows from (2.28)_{\beta}, (2.29)_{\beta}, (4.38)_{\beta}, \text{ and } (4.89)_{\beta}. \text{ As for (4.56), we note that the l. h. s. of (4.66)_{\beta} vanishes unless } |\beta| \geq 2\alpha \text{ because of (6.14), so that } |\beta| + \kappa - \theta > 0.

As the second ingredient to (4.60), we need to estimate all $\ll \beta$-components of the increment of the factor $z_{k1} \pi_{\beta k}$. Considering for all $k \geq 0$

$$
\mathbb{E}^Y \left| (z_{k1} \pi_{\beta k}(y) - z_{k1} \pi_{\beta k}(z))^{p} \right|^{\gamma}
\lesssim |y - z|^{\alpha} (|y - z| + |z - x|)|^{|\beta| - 2\alpha},
$$

(4.67) where $\beta$ here is generic and in fact will be applied to a preceding multi-index.
$|\beta| - 2\alpha < 0$ implies $\beta \in \mathbb{N}_0\theta_0$; due to the presence of $z_k$, the term $(z_k \Pi_{x}^k)_{\beta}$ vanishes unless $k = 0$, in which case the l. h. s. obviously vanishes. In particular, in establishing (4.67), we may restrict to $k \geq 1$ and write with the help of (2.41) and (2.42)

$$\Pi_{x}^k(y) - \Pi_{x}^k(z) = \sum_{k' + k'' = k - 1} \Pi_{x}^{k'}(y) \Pi_{x}^{k''}(z) \Gamma_{x}^{k'} \Gamma_{x}^{k''}(y),$$

so that we obtain componentwise

$$(z_k \Pi_{x}^k(y) - z_k \Pi_{x}^k(z))_{\beta} = \sum_{k' + k'' = k - 1} \sum_{e_k + \beta_1 + \cdots + \beta_k = \beta} \Pi_{x}^{k'}(y) \cdots \Pi_{x}^{k''}(z) \cdots \Pi_{x}^{\beta_{k'}}(z) \sum_{\gamma} (\Gamma_{x}^{\gamma})_{\beta_{k'}} \Gamma_{x}^{\gamma}(y).$$

By (6.6) we have $\beta_1, \ldots, \beta_k < \beta$, and then by (6.9) also $\gamma < \beta$, so that (2.28) and (2.29) are indeed sufficient to conclude by Hölder’s inequality in probability

$$\mathbb{E}_y \left| (z_k \Pi_{x}^k(y) - z_k \Pi_{x}^k(z))_{\beta} \right|^p \lesssim \sum_{k' + k'' = k - 1} \sum_{e_k + \beta_1 + \cdots + \beta_k = \beta} |y - x|^{|\beta_1| + \cdots + |\beta_k|} |z - x|^{|\Gamma_{x}^{\gamma}}_{\beta_{k'}} |\gamma| |y - z|^{|\gamma|}.$$

Using $|y - x| \leq |y - z| + |z - x|$ and (3.11), this collapses to (4.67).

As the third ingredient to (4.60), and the first ingredient to (4.61), we estimate the $\approx \beta$-components of $\partial_{y}^2 (\delta \Pi_{x} - d\Gamma_{x}^{\gamma} \partial_{y}^2 \Pi_{x})_{\beta}(y)$. Introducing

$$\mathbb{E}_y \left| (\partial_{y}^2 (\delta \Pi_{x} - d\Gamma_{x}^{\gamma} \partial_{y}^2 \Pi_{x})_{\beta}(y)) \right|^q \lesssim (\sqrt{\gamma})^{\alpha - 2} (\gamma + |y - z|)^\kappa (\gamma + |y - z| + |z - x|)^{|\beta| - \alpha} (w_{x}(z) + w_{x}(y))$$

(4.68)

we claim that (4.73) $\beta$ implies (4.68) $\beta$. Indeed by (4.66) in the weakened form of

$$\mathbb{E}_y \left| (d\Gamma_{x}^{\gamma} \partial_{y}^2 \Pi_{x} - d\Gamma_{x}^{\gamma} \partial_{y}^2 \Pi_{x})_{\beta}(y)) \right|^q \lesssim (\sqrt{\gamma})^{\alpha - 2} (\gamma + |y - z|)^\kappa (\gamma + |y - z| + |z - x|)^{|\beta| - \alpha} (w_{x}(z) + w_{x}(y)),$$

it is enough to establish (4.68) for $z = y$,

$$\mathbb{E}_y \left| \partial_{y}^2 (\delta \Pi_{x} - d\Gamma_{x}^{\gamma} \partial_{y}^2 \Pi_{x})_{\beta}(y) \right|^q \lesssim (\sqrt{\gamma})^{\alpha - 2 + \kappa} (\gamma + |y - x|)^{|\beta| - \alpha} w_{x}(y).$$

(4.69)

Writing

$$\partial_{y}^2 (\delta \Pi_{x} - d\Gamma_{x}^{\gamma} \partial_{y}^2 \Pi_{x})_{\beta}(y) = \int dy' \partial_{y}^2 \psi_{y}(y - y') (\delta \Pi_{x} - \delta \Pi_{x}(y) - d\Gamma_{x}^{\gamma} \partial_{y}^2 \Pi_{x})_{\beta}(y'),$$

we obtain (4.69) $\beta$ from (4.72) $\beta$ via Hölder’s inequality in $y'$ and the moment bounds (4.5) and (4.52) (both with $n = (2, 0)$ and the latter with $z = y$).

The last ingredient to (4.60), and the second ingredient for (4.61), is the estimate of the $\approx \beta$-components $z_k \Pi_{x}^k(y)$. Assuming just (2.28) $\beta$ and with a subset of the arguments for (4.67), we obtain (4.70) $\beta$, where

$$\mathbb{E}_y \left| (z_k \Pi_{x}^k(z))_{\beta} \right|^p \lesssim |z - x|^{|\beta| - \alpha}. $$

(4.70)

\footnote{once more $\beta$ here denotes a generic multi-index}
The third task is to pass from the estimate (4.43) of the rough-path increment of $\delta \Pi_\tau^-$ to the estimate of the rough-path increment of $\Pi_\tau$. We obtain from (2.21) and its Malliavin derivative that
\[
(\partial_2 - \partial_1^2) Q(\delta \Pi_\tau - \delta \Pi_\tau(z) - d\Gamma_{xz}^+ \Pi_\tau) = Q(\delta \Pi_\tau - d\Gamma_{xz}^+ \Pi_\tau^-),
\] (4.71)
where there is no polynomial thanks to the (double) presence of $Q$. By (4.33), $Q(\delta \Pi_\tau - \delta \Pi_\tau(z) - d\Gamma_{xz}^+ \Pi_\tau)_\beta$ vanishes to first order in $\tau$, where we could smuggle in the second $Q$ by (2.31). Thanks to the presence of $Q$, and the control of (2.28), $(d\Gamma_{xz}^+ \Pi_\tau)_\beta$ grows sub-quadratically at infinity, by the control (4.25) $\beta$, the same applies to $\delta \Pi_\tau$. Hence by the Liouville argument from the proof of Proposition 5.3, $(\delta \Pi_\tau - \delta \Pi_\tau(z) - d\Gamma_{xz}^+ \Pi_\tau)_\beta$ is thus uniquely determined by $(\delta \Pi_\tau - d\Gamma_{xz}^+ \Pi_\tau^-)_\beta$. Therefore, provided the $t$-integral below converges as we shall presently establish, we have the representation
\[
(\delta \Pi_\tau - \delta \Pi_\tau(z) - d\Gamma_{xz}^+ \Pi_\tau)_\beta = - \int_0^\infty dt (1 - T_2^\beta)(\partial_2 + \partial_1^2)(\delta \Pi_\tau - d\Gamma_{xz}^+ \Pi_\tau^-)_\beta_t. \tag{4.72}
\]
Compared to the analogous integration task of Proposition 4.3, there are now three length scales involved, namely $\sqrt{t}$, $|y - z|$, and $|z - x|$. In the near-field range $\sqrt{t} \leq |y - z|$, we split $1 - T_2^\beta$; in the far-field range $\sqrt{t} \geq \max\{|y - z|, |z - x|\}$, we split $\delta \Pi_\tau - d\Gamma_{xz}^+ \Pi_\tau^-$. Only on the intermediate range $|y - z| \leq \sqrt{t} \leq \max\{|y - z|, |z - x|\}$ we use the cancellations in both the Taylor remainder and the rough-path increment.

**Proposition 4.13** (Integration III). Assume that (2.27) $\gamma$, (4.13) and (4.43) $\beta$, and that (4.89) $\beta$ hold, and that (4.43) $\gamma$ holds for all $\gamma$ not purely polynomial. Then we have
\[
E^x_T |(\delta \Pi_\tau - \delta \Pi_\tau(z) - d\Gamma_{xz}^+ \Pi_\tau)_\beta(y)|^{q'} \lesssim |y - z|^{k + \alpha}(|y - z| + |z - x|)^{|\beta| - \alpha}(w_\alpha(z) + w_\alpha(y)). \tag{4.73}
\]

**Proof.** In preparation for the near-field range $\sqrt{t} \leq |y - z|$, we pre-process (4.43) $\beta$. By the semigroup property (2.1) followed by Jensen’s inequality we have for any random space-time function $f$ that $E^x_T |\partial^n f(y)|^{q'} \leq \int d\psi(y') |\partial^n \psi(y')E^y_T |f(y')|^{q'}$. We use this for $f = (\delta \Pi_\tau - d\Gamma_{xz}^+ \Pi_\tau^-)_\beta$, insert (4.43) $\beta$ (with $t$ replaced by $\frac{t}{2}$), and appeal to Hölder’s inequality on the r. h. s. of (4.43) $\beta$ (with $y'$ replacing $y$), in order to use both the positive moment bounds (4.5) and the negative moment bounds (4.52) (for $z = y$ and with $t$ replaced by $\frac{t}{2}$). This leads to
\[
E^x_T |\partial^n (\delta \Pi_\tau - d\Gamma_{xz}^+ \Pi_\tau^-)_\beta(y)|^{q'} \lesssim (\sqrt{t})^{\alpha - 2 - \alpha}(|y - z| + |z - x|)^{|\beta| - \alpha}(w_\alpha(z) + w_\alpha(y)). \tag{4.74}
\]
We restrict (4.74), once to $y = z$, and once to the near-field range:
\[
E^x_T |\partial^n (\delta \Pi_\tau - d\Gamma_{xz}^+ \Pi_\tau^-)_\beta(z)|^{q'} \lesssim (\sqrt{t})^{\alpha - 2 - \alpha + \kappa}(|y - z| + |z - x|)^{|\beta| - \alpha}w_\alpha(z),
\]
\[
E^x_T |\partial^n (\delta \Pi_\tau - d\Gamma_{xz}^+ \Pi_\tau^-)_\beta(y)|^{q'} \lesssim (\sqrt{t})^{\alpha - 2 - \alpha}(|y - z| + |z - x|)^{|\beta| - \alpha}w_\alpha(z) + w_\alpha(y) \quad \text{providing} \sqrt{t} \leq |y - z|.
\]
We use this in two ways:
\[
E^x_T |(\partial^2 + \partial_1)(\delta \Pi_\tau - d\Gamma_{xz}^+ \Pi_\tau)_\beta(y)|^{q'} \lesssim t^{-1} \sum_{n=0,1,0} |y - z|^n (\sqrt{t})^{\alpha - 2 - \alpha + \kappa}(|y - z| + |z - x|)^{|\beta| - \alpha}w_\alpha(z),
\]
\[
E^x_T |(\partial^2 + \partial_1)(\delta \Pi_\tau - d\Gamma_{xz}^+ \Pi_\tau^-)_\beta(y)|^{q'} \lesssim t^{-1} (\sqrt{t})^{\alpha} |y - z|^{\kappa}(|y - z| + |z - x|)^{|\beta| - \alpha}w_\alpha(z) + w_\alpha(y) \quad \text{providing} \sqrt{t} \leq |y - z|.
\]
Applying $\int_0^{|y-z|^2} dt$, using $\alpha - 1 + \kappa > 0$ by (4.36) on the first integral, and $\alpha > 0$ on the second, we obtain

\[
\mathbb{E}^{\hat{T}} \left| \int_0^{|y-z|^2} dt T_z^2 (\partial_t^2 + \partial_2) (\delta \Pi_z^+ - d \Gamma_{x}^+ Q \Pi_z^-) \beta_t(y) \right|^{q'} \\
\lesssim |y-z|^{\alpha + \kappa} ((y-z) + |z-x|)^{|\beta|} w_x(z),
\]

(4.75)

\[
\mathbb{E}^{\hat{T}} \left| \int_0^{|y-z|^2} dt (\partial_t^2 + \partial_2) (\delta \Pi_z^+ - d \Gamma_{x}^+ Q \Pi_z^-) \beta_t(y) \right|^{q'} \\
\lesssim |y-z|^{\alpha + \kappa} ((y-z) + |z-x|)^{|\beta|} w_x(z) + w_x(y),
\]

(4.76)

which takes care of the near-field contribution.

We now turn to the far-field contribution $\sqrt{t} \geq \max \{ |y-z|, |z-x| \}$, which we split into the one coming from $d \Gamma_{x}^+ Q \Pi_z^-$ and the one from $\delta \Pi_z^+$. For the first one, we note that by $\Pi_z^- \in \hat{T}^+$, see (2.22), and the strict triangularity (6.11) of $d \Gamma_{x}^+$, only (4.89) and (2.27) are needed for $T_{x}^+$

\[
\mathbb{E}^{\hat{T}} \left| (d \Gamma_{x}^+ Q \Pi_z^-) \beta_t(y) \right|^{q'} \\
\lesssim \sum_{|\gamma| \in \mathbb{C} \cap (-\infty,0) \cap [\alpha, \kappa + |\beta|]} |z-x|^{\kappa + |\beta| - |\gamma|} w_x(z) (\sqrt{t})^{\alpha - 2} (\sqrt{t} + |y-z|)^{|\gamma| - \alpha},
\]

which, by a similar but simpler argument as for (4.74), we pre-process to

\[
\mathbb{E}^{\hat{T}} \left| \partial^n (d \Gamma_{x}^+ Q \Pi_z^-) \beta_t(y) \right|^{q'} \\
\lesssim \sum_{|\gamma| \in \mathbb{C} \cap (-\infty,0) \cap [\alpha, \kappa + |\beta|]} (\sqrt{t})^{|\gamma| - |n|} |z-x|^{\kappa + |\beta| - |\gamma|} w_x(z) \quad \text{provided} \; |y-z| \leq \sqrt{t}.
\]

(4.77)

Representing Taylor’s remainder in a way suitable for our parabolic scaling, namely

\[
(1 - T_{x}^2)^{f}(y) = \int_0^1 ds (1-s) \frac{d^2 h}{ds^2}(s) \quad \text{with} \quad h(s) = f(sy_1 + (1-s)z_1, s^2 y_2 + (1-s^2)z_2),
\]

(4.78)

so that the l. h. s. involves the four partial derivatives $\partial^n f$ with $|n| \geq 2$ and $n_1 + n_2 \leq 2$. Applying this to $f = (\partial_t^2 + \partial_2) (d \Gamma_{x}^+ Q \Pi_z^-) \beta_t$, we learn from (4.77) that

\[
\mathbb{E}^{\hat{T}} \left| (1 - T_{x}^2)^{\partial^n (d \Gamma_{x}^+ Q \Pi_z^-) \beta_t(y) \right|^{q'} \\
\lesssim \sum_{|\gamma| \in \mathbb{C} \cap (-\infty,0) \cap [\alpha, \kappa + |\beta|]} |z-x|^{|n|} (\sqrt{t})^{|\gamma| - |n|} |z-x|^{\kappa + |\beta| - |\gamma|} w_x(z) \quad \text{provided} \; |y-z| \leq \sqrt{t}.
\]

(4.79)

Applying $\int_{\max\{|y-z|^2,|z-x|^2\}}^\infty dt$ we obtain because of $|\gamma| - |n| < 2 - 2 = 0$

\[
\mathbb{E}^{\hat{T}} \left| \int_{\max\{|y-z|^2,|z-x|^2\}}^\infty dt (1 - T_{x}^2)^{\partial^n (d \Gamma_{x}^+ Q \Pi_z^-) \beta_t(y) \right|^{q'} \\
\lesssim \sum_{|\gamma| \in \mathbb{C} \cap (-\infty,0) \cap [\alpha, \kappa + |\beta|]} |z-x|^{|n|} (\sqrt{t})^{|\gamma| - |n|} |z-x|^{\kappa + |\beta| - |\gamma|} w_x(z) \\
\lesssim |y-z|^{\kappa + \alpha} (|y-z| + |z-x|)^{|\beta| - |\gamma|} w_x(z) \quad \text{by} \; |n| \geq 2 \geq \kappa + \alpha.
\]

(4.79)

For the second part of the integrand, we pre-process (4.13) to

\[
\mathbb{E}^{\hat{T}} \left| \partial^n \delta \Pi_x^+ \beta_t(y) \right|^{q'} \lesssim (\sqrt{t})^{\alpha - 2 - |n|} (\sqrt{t} + |y-x|)^{|\beta| - |\gamma|} w_x,
\]
which in turn implies by Taylor and \(|y - x| + |z - x| \lesssim |y - z| + |z - x|

\[
\mathbb{E}^\frac{1}{t} \left| (1 - T^2_x) (\partial^2_1 + \partial_2) \delta \Pi^{\alpha \beta} (y) \right|^q \lesssim t^{-1} \sum_{n_1 + n_2 \leq 2} |y - z|^n (\sqrt{t})^{\alpha - |n|} (\sqrt{t} + |y - z| + |z - x|)^{\beta - \alpha} \hat{w}.
\]

Applying \( \int_{\max \{|y - z|^4, |z - x|^4\}}^\infty dt \) we obtain because of \(|\beta| - |n| < 2 - 2 = 0 \) and as in (4.79)

\[
\mathbb{E}^\frac{1}{t} \left| \int_{\max \{|y - z|^4, |z - x|^4\}}^\infty dt (1 - T^2_x) (\partial^2_1 + \partial_2) \delta \Pi^{\alpha \beta} (y) \right|^q \lesssim \sum_{n_1 + n_2 \leq 2} |y - z|^n (|y - z| + |z - x|)^{\alpha - |n|} (|y - z| + |z - x|)^{\beta - \alpha} \hat{w}.
\]

In view of (4.75), (4.76), (4.79), and (4.80), it remains to consider the case \(|y - z| \leq |z - x|\) and to estimate the intermediate range

\[
\mathbb{E}^\frac{1}{t} \left| \int_{|y - z|^4}^{|z - x|^4} dt (1 - T^2_x) (\partial^2_1 + \partial_2) (\delta \Pi_x - d\Gamma^*_{x^*} \Pi_x^0) \beta_t (y) \right|^q \lesssim |y - z|^{\alpha + \kappa} |z - x|^{\beta - \alpha} w_x (z). \quad (4.81)
\]

To this purpose, we pre-process (4.74) to: We apply the semi-group property (2.1) in form of \( \mathbb{E}^\frac{1}{t} |f_\gamma (y)|^{q'} \leq \int dt |\psi_\gamma (y - t')| \mathbb{E}^\frac{1}{t'} |f_\gamma (y')|^{q'} \) if \( f = \partial^n (\delta \Pi_x - d\Gamma^*_{x^*} \Pi_x^0) \beta_t \), insert Hölder’s inequality in \( y' \) in order to access (4.5) and (4.52) (both with \( n = 0 \) and \( t \) replaced by \( \frac{t}{2} \)), thereby obtaining

\[
\mathbb{E}^\frac{1}{t} |\partial^n (\delta \Pi_x - d\Gamma^*_{x^*} \Pi_x^0) \beta_t (y)|^{q'} \lesssim (\sqrt{t})^{\alpha - 2 - |n| + \kappa} (\sqrt{t} + |z - x|)^{\beta - \alpha} w_x (z) \quad \text{provided} \ |y - z| \leq \sqrt{t}. \quad (4.82)
\]

By Taylor, this implies

\[
\mathbb{E}^\frac{1}{t} \left| (1 - T^2_x) (\partial^2_1 + \partial_2) (\delta \Pi_x - d\Gamma^*_{x^*} \Pi_x^0) \beta_t (y) \right|^q \lesssim t^{-1} \sum_{n_1 + n_2 \leq 2} |y - z|^n (\sqrt{t})^{\alpha - |n| + \kappa} |z - x|^{\beta - \alpha} w_x (z) \quad \text{provided} \ |y - z| \leq \sqrt{t} \leq |z - x|.
\]

Integration gives (4.81). \( \square \)

The fourth task is to pass from the output (4.75) \( \beta \) of Proposition 4.13 to (4.35) \( \beta \). The proof is based on formula

\[
(d\pi_x^{(0)} - d\pi_{x^*}^{(0)} - d\Gamma_{x^*}^* Q \pi_x^{(0)}) + (d\pi_{x^*}^{(1)} - d\pi_{x^*}^{(1)} - d\Gamma_{x^*}^{1} Q \pi_{x^*}^{(1)}) (\cdot - y) = (\delta \Pi_x - \delta \Pi_x (z) - d\Gamma_{x^*}^* \Pi_x^0) - (\delta \Pi_x - \delta \Pi_x (y) - d\Gamma_{x^*}^* \Pi_x^0) - (d\Gamma_{x^*}^* Q - d\Gamma_{x^*}^{1} Q \pi_{x^*}^{(1)}) \Pi_y. \quad (4.83)
\]

which we shall establish in the proof of the upcoming Proposition 4.14. Its merit is that it connects an affine polynomial with the coefficients given by the rough-path increments of \( \{d\pi_x^{(n)}\}_{n=0,(1,0)} \) to the rough-path increments of \( \delta \Pi_x \) (in the secondary base points \( y \) and \( z \)) and the continuity expression for \( d\Gamma_{x^*}^* \). In analogy to the fourth task in Subsection 4.1 we have

**Proposition 4.14** (Three-point argument III). Assume that \( (2.28)_{\beta}, (4.11)_{\beta}, (4.73)_{\beta} \) hold, and that \( (2.29)_{\beta}, (4.38)_{\beta}, (4.89)_{\beta} \) hold for all \( \gamma \) not purely polynomial. Then (4.35) \( \beta \) holds.
Proof. The formula \((4.83)\) follows from substituting \(\Pi_z\) according to \((2.41)\) by \(\Gamma_z^* \Pi_y + \pi_z^{(0)}\), and then appealing to
\[
\begin{align*}
(\delta \Pi_y(y), \delta \Pi_z(z)) &= (d \pi_y^{(0)}, d \pi_z^{(0)}), \\
Q^*_{g y} (\text{id} - \Pi_y) \Gamma_z^* &= (z(1.0) + Q \pi_y^{(1)})(-y), \\
(d \Gamma_y^* Q - d \Gamma_y^* Q \Gamma_y^*) (\text{id} - \Pi_y) \Gamma_z^* &= (d \pi_y^{(1)} - d \pi_z^{(1)} - d \Gamma_y^* Q \pi_y^{(1)})(-y). 
\end{align*}
\]
Evaluating at \(y\) makes formula \((4.83)\) collapse to
\[
d \pi_y^{(0)} - d \pi_z^{(0)} - d \Gamma_y^* \pi_y^{(1)} = \delta \Pi_y(y) - \delta \Pi_z(z) - d \Gamma_y^* \Pi_y(y). \tag{4.85}
\]
Since \(d \Gamma_y^*\) vanishes on \(z\) unless \(n = (1, 0)\), we may rewrite \((4.85)\) as
\[
d \pi_y^{(0)} - d \pi_z^{(0)} - d \Gamma_y^* \pi_y^{(1)} = \delta \Pi_y(y) - \delta \Pi_z(z) - d \Gamma_y^* \Pi_y(y) - d \Gamma_y^* P (\text{id} - Q) \Pi_y(y).
\]
Taking the \(E^\frac{1}{2} |\cdot|^\gamma\) of the \(\beta\)-component of this identity, appealing to the strictly triangular structure \((6.11)\) of \(d \Gamma_y^*\) w. r. t. \(\prec\), we obtain by our assumptions \((2.28)_{<\beta}, (4.73)\), and \((4.89)^{\gamma \neq p.p.}\),
\[
E^\frac{1}{2} |d \pi_y^{(0)} - d \pi_z^{(0)} - d \Gamma_y^* \pi_y^{(1)}|^\gamma \lesssim |y - z|^\kappa + |y - z| + |z - x| |\beta| - |\gamma| (w_x(y) + w_z(z)) + \sum_{|\gamma| \in \mathbb{A}^0 / (2, \infty)} |z - x|^\kappa - |\gamma| \cdot |\gamma| (w_x(y) - w_z(z)).
\]
By the second item in \((4.27)\), the second r. h. s. term can be absorbed into the first. This establishes \((4.35)\) for \(n = 0\).

We now address the \(n = (1, 0)\) contribution to \((4.83)\). Appealing to \((3.2)\) and \((4.85)\), we may rewrite \((4.83)\) as
\[
\begin{align*}
&\quad (d \pi_y^{(1)} - d \pi_z^{(1)} - d \Gamma_y^* \pi_y^{(1)})(-y), \\
&= (\delta \Pi_y - \delta \Pi_z(z) - d \Gamma_y^* \Pi_y(z)) (y) - (\delta \Pi_y - \delta \Pi_z(y) - d \Gamma_y^* \Pi_y(y)) \\
&- (d \pi_y^{(1)} - d \Gamma_y^* \pi_y^{(1)} P Q \Pi_y - d \Gamma_y^* P (\text{id} - Q) \pi_y^{(1)})(y).
\end{align*}
\]
We then take the \(E^\frac{1}{2} |\cdot|^\gamma\) of the \(\beta\)-component of this identity. The first three r. h. s. terms are controlled by \((4.73)\). For the fourth term we use the strict triangularity \((6.13)\), so that \((2.28)_{<\beta}\) is sufficient, next to \((4.38)^{\gamma \neq p.p.}\). For the fifth term we use the strict triangularity of \(d \Gamma_y^*\), and the triangularity of \(\Gamma_y^*\), so that \((2.29)^{\gamma \neq p.p.}\) and once more \((2.28)_{<\beta}\) are sufficient, next to \((4.89)^{\gamma \neq p.p.}\). For the sixth term we just use the strict triangularity of \(d \Gamma_y^*\), so that \((4.9)_{<\beta}\) is sufficient, once more next to \((4.89)^{\gamma \neq p.p.}\). We obtain term by term, using that \(\gamma(1, 0) \neq 0\) implies \(|\gamma| \geq 1\) on the fifth r. h. s. term below,
Restricting the active variable to the (parabolic) ball $|\cdot - y| \leq |y - z|$, and using the second item in (4.27) on the last two terms, this estimate collapses to

$$
|\langle \cdot - y \rangle_1 |\mathbb{E}_1^{\gamma} | d\pi^{(1,0)}_{x^y} - d\pi^{(1,0)}_{x^z} - d\Gamma^\times_{\beta}(z) \nu \rangle^{(1,0)}| \nu' \rangle
\lesssim |y - z|^{k+i+1}(\gamma - e_k^1) - |y - z|^{k+i+1}(w_{y_1}(\cdot) + w_y(\cdot) + w_y(z)).
$$

(4.86)

We now average the active variable over this ball in order to recover (4.35) for $n = (1,0)$. Indeed, for the l. h. s. of (4.86), we appeal to the obvious \( f_{|\cdot - y| \leq |y - z|} \langle \cdot - y \rangle_1 \sim |y - z| \). For the r. h. s. of (4.86), by definition (4.57) of $w_x$, it suffices to establish

$$
\int_{|\cdot - y| \leq \lambda} |\cdot - x|^{-k} \lesssim |y - z|^{-k} \quad \text{for } \lambda = |y - z|,
$$

(4.87)

which is an easy version of (4.53). \( \square \)

4.5 From $\delta \Pi^\times_{x^y} - d\Gamma^\times_{x^z, \beta}$ back to $\delta \Pi^\times_x$ via boundedness of $d\Gamma^\times_{x^z}$ and $d\pi^{(n)}_{x^z}$, and by averaging in $z$

The aim of this last subsection is to pass from the estimate (4.43) of the rough-path increment of $\delta \Pi^\times_x$ to the estimate (4.13) of $\delta \Pi^\times_x$ itself. Clearly, in view of the structure of the rough-path increment, this will require an estimate of $d\Gamma^\times_{x^z} P$, see Proposition 4.15. The proof of Proposition 4.15 will be similar to the first task of Subsections 4.1 and 4.4, and rely on the estimate of $d\pi^{(n)}_{x^z}$ for $n = 0, (1,0)$. By (4.84) the estimate of $d\pi^{(0)}_{x^z}$ is already part of the induction hypothesis. However, the estimate of $d\pi^{(1,0)}_{x^z}$ needs to be included into the induction:

$$
\mathbb{E}_1^{\gamma} \langle d\pi^{(1,0)}_{x^z, \beta} \nu \rangle \lesssim |z - x|^{k+1} \omega(z).
$$

(4.88)

Note that the exponent is strictly positive, see (4.36). The first task of this subsection is based on the formula (4.31) for $d\Gamma^\times_{x^z}$.

Proposition 4.15 (Algebraic argument IV). Assume that (4.25) and (4.88) hold, and that (2.29) holds for all $\gamma$ not purely polynomial. Then we have for $\gamma$ not purely polynomial\(^\footnote{\text{with the understanding that the l. h. s. vanishes when the exponent is non-positive}}\)

$$
\mathbb{E}_1^{\gamma} \langle (d\Gamma^\times_{x^z, \beta} \nu \rangle \lesssim |z - x|^{k+1} \omega(z).
$$

(4.89)

This includes the statement that $(d\Gamma^\times_{x^z} P)_{\beta}$ depends only on $d\pi^{(1,0)}_{x^z}$ and $\delta \Pi^\times_{x^z, \beta'}$ with $\beta' \sim \beta$, and on $(\Gamma^\times_{x^z} P)_{\beta'}$ with $\beta' \ll \beta$.

Proof. The structure is very similar to the proof of Proposition 4.11. Again, we distinguish the contributions from $n = 0$ and $n = (1,0)$ to (4.31)\(^\footnote{\text{with the understanding that the l. h. s. vanishes when the exponent is non-positive}}\). In view of (4.84), all terms in the $(n = 0)$-contribution are of the form

$$
\delta \Pi^\times_{x^z, \beta}(z)(\Gamma^\times_{x^z, \beta'} \nu \rangle^{\gamma - e_k^1})
$$

(4.90)

for some $k \geq 0$ and multi-indices $\beta_1, \beta_2$ constrained by $\beta_1 < \beta$, $\beta_2 \ll \beta$, $\beta_1 + \beta_2 = \beta$, and noting that $\gamma - e_k^1$ is not purely polynomial. Hence under our assumptions, the $\mathbb{E}_1^{\gamma} |\cdot \nu \rangle$-norm of (4.90) is estimated by

$$
|z - x|^{\beta_1} |\omega(z)| |z - x|^{\gamma - e_k^1} \lesssim |z - x|^{\gamma - e_k^1} |\omega(z)| |z - x|^{\beta_1}.
$$

(4.37)

It follows from (4.22) that this is contained in the r. h. s. of (4.89).
All terms in the \((n = (1, 0))\)-contribution are of the form
\[
d\mathbf{p}^{(1,0)}_{z}(0) (\Gamma^{+}_{\varepsilon})^{y-\varepsilon(1,0)}_{0},
\]
for some multi-indices \(\beta_{1}, \beta_{2}\) constrained by \(\beta_{1} < \beta, \beta_{2} \leq \beta, \beta_{1} + \beta_{2} = \beta\), and noting that \(\gamma - e(1,0)\) is not purely polynomial. Hence under our assumptions, the \(\frac{1}{2} \cdot |\cdot|\beta\)-norm of (4.91) is estimated by
\[
|z - x|^{\kappa + |\beta_{1}| - 1} w_{x}(z) |z - x|^{\beta_{2}| - |\gamma - e(1,0)|}.
\]
It follows from (4.24) that also this is contained in the r. h. s. of (4.89).

We will establish (4.88) based on the three-point formula
\[
(y - z)_{1}d\mathbf{x}_{z}^{(-1,0)} = - (\delta \Pi_{x}(y) - \delta \Pi_{x}(z) - d\Gamma^{+}_{\varepsilon}Q\Pi_{x}(y)) + \delta \Pi_{x}(y) - \delta \Pi_{x}(z) - d\Gamma^{+}_{\varepsilon}PQ\Pi_{x}(y),
\]
which follows from (2.23) in form of \((\text{id} - P)\Pi_{x}(y) = (y - z)_{1}z_{(1,0)}\) and (4.25). By an argument similar to the fourth task of Subsections 4.1 and 4.4 (see Propositions 4.4 and 4.14) we obtain

**Proposition 4.16** (Three-point argument IV). Assume that (2.28) in \(\beta\) holds, that (4.89) holds for all \(\gamma\) not purely polynomial, and that (4.73) in \(\beta\) and (4.25) in \(\beta\) hold. Then (4.88) holds.

**Proof.** Taking the \(\frac{1}{2} \cdot |\cdot|\beta\) of (4.92), and appealing to the strict triangularity (6.11) of \(d\Pi^{+}\), we obtain by our assumptions
\[
|(y - z)_{1}| \frac{1}{2} \cdot |\cdot|\beta_{y} \lesssim |y - z|^{\kappa + \alpha}|y - z| + |z - x| |\beta - \alpha| w_{x}(y) + w_{x}(z))
\] 
\[+ |y - x| |\beta| w + |z - x| |\beta| w + \sum_{|\gamma| \in A_{\gamma}([\beta, \kappa + |\beta|])} |z - x|^{\kappa + |\beta| - |\gamma|} |w_{x}(z) - z - y|\]
\[
\overset{(4.37)}{\lesssim} (|y - z| + |z - x|)^{\kappa + |\beta|} (w(y) + |y - x|^{-\kappa} w + w_{x}(z)).
\]
We now average over all \(y\) with \(|y - z| \leq \frac{1}{2}|z - x|\), where the factor of \(\frac{1}{2}\) ensures that \(|y - x| \geq \frac{1}{2}|z - x|\), so that in this range (4.93) simplifies to
\[
|(y - z)_{1}| \frac{1}{2} \cdot |\cdot|\beta_{y} \lesssim |z - x|^{\kappa + |\beta|} (w(y) + |z - x|^{-\kappa} w + w_{x}(z)).
\]
The averaging of (4.94) over \(\{y : |y - z| \leq \frac{1}{2}|z - x|\}\) ensures that on the one hand, we have for the l. h. s. that \(f_{y:|y - z| \leq \frac{1}{2}|z - x|} dy(y - z)_{1} \sim |z - x|\), and that on the other hand, we may use (4.30) for the r. h. s. to the effect of \(f_{y:|y - z| \leq \frac{1}{2}|z - x|} d\gamma(y) \lesssim |z - x|^{-\kappa} w\). Hence, once more appealing to definition (4.37) of \(w_{x}(z)\), we see that (4.94) turns into (4.88)\(\beta\).

Passing from (4.43) to (4.13) also means replacing the weighted norms of \(\delta \xi\) by the original norm \(\bar{\omega}\). This will be done starting from the obvious identity
\[
\delta \Pi_{z}^{-} = (\delta \Pi_{z}^{-} - d\Gamma^{+}_{\varepsilon}Q\Pi_{z}^{-}) + d\Gamma^{+}_{\varepsilon}Q\Pi_{z}^{-}
\]
and averaging in the secondary base point \(z\). In particular, we have the following proposition

**Proposition 4.17.** Assume that (2.27) in \(\beta\), (4.43) in \(\beta\), (4.13) in \(\beta\), and (4.89) in \(\beta\) hold, assume that (2.29) in \(\beta\) and (4.18) in \(\beta\) for \(\gamma\) not purely polynomial hold. Then (4.13)\(\beta\) holds.
Proof. We apply (\ref{eq:4.95}) to \((4.95)_{\beta}\) and that the \(E^T|\cdot|^q\)-norm; on the first term, we use the upgrade \((4.82)_{\beta}\) (with \(n=0\)) of \((4.43)_{\beta}\); on the second term, we use \((4.89)_{\beta}\), and \((2.27)_{\pi}\) (thanks to the strict triangularity \((6.11)\) of \(d\Gamma_{xc}^*\) w. r. t. \(\prec\)), obtaining

\[
E^T \frac{1}{2} |\delta \Pi_{xc}^{-}\beta}(y)|^{q} \lesssim (\sqrt{T})^{|\beta|+2}(w(z) + (\sqrt{T})^{|\beta|-2}w(z)\big)
\]

provided \(|y - z| \leq \sqrt{T}|\).

By definition \((4.37)\), this yields

\[
E^T \frac{1}{2} |\delta \Pi_{xc}^{-}\beta}(y)|^{q} \lesssim (\sqrt{T})^{|\beta|-2}w.
\]

(4.96)

In order to replace in \((4.96)\) the evaluation at \(x\) by a generic point \(y\), we take the Malliavin derivative of \((2.24)_{\beta}\), noting that there is no polynomial defect because of \(|\beta| < 2:\)

\[
\delta \Pi_{xc}^{-}\beta = (\delta \Gamma_{xc}^{-}\beta + \Gamma_{xc}^{*}\delta \Pi_{y}^{-}\beta).
\]

Applying (\ref{eq:4.18}), evaluating at \(y\), and taking the \(E^T|\cdot|^q\)-norm we obtain using \((4.18)^{\gamma p,p,p}_{\beta}, (2.27)_{\pi \beta}, (2.29)^{\gamma p,p}_{\beta}, (4.13)_{\beta}, \) and \((4.96)_{\beta}\) (with \(x\) replaced by \(y\)) that

\[
E^T \frac{1}{2} |\delta \Pi_{xc}^{-}\beta}(y)|^{q} \lesssim \sum_{|\gamma| \leq 2} |x-y|^{|\gamma|-2}(\sqrt{T})^{2|\gamma|-2}\big)
\]

\[
\lesssim (\sqrt{T})^{|\beta|-2}(\sqrt{T} + |y - x|)^{|\beta|-2}w.
\]

5 Constructions

In this section, we carry out one step of the inductive construction of \(c, \Gamma_{x}, \Pi_{x},\) and \(\Gamma^{*}_{xy}\) such that, next to the population properties, the axioms \((2.21), (2.22), (2.24), (2.25)\) and \((2.26)\) in Theorem 2.2 are satisfied. In fact, instead of \((2.24), (2.25)\) we shall establish the stronger \((2.40), (2.41)\). In line with the logical order of an induction step, see Subsection 5.4, we proceed as follows

- In Subsection 5.1, we construct \(\Pi_{x}\) by the choice of \(c\) that amounts to the BPHZ-choice of renormalization.

- In Subsection 5.2, we construct \(\Pi_{x}\) by integration of \(\Pi_{x}\).

- In Subsection 5.3, we construct \(\Gamma_{xy}\) via the re-centering when passing from \(\Pi_{x}\) to \(\Pi_{x}\) by a polynomial with coefficients \(\pi_{xy}^{(n)}\).

- In Subsection 5.4, we construct \(d\Gamma_{xc}^{*}\) via \(d\pi_{xc}^{(n)}\) (with \(n = (1,0)\)).
5.1 Construction of $c$ and thus $\Pi^-_x$ via BPHZ-choice of renormalization

According to (6.6) and (4.4), the r. h. s. of (2.22) depends on $\Pi_x$ only through $\Pi_{x\beta'}$ with $\beta' \prec \beta$. Removing the $(k = 0)$-contribution to the renormalization term, which coincides with $c_\beta$, in $\Pi^-_x$ leads to

$$\tilde{\Pi}^-_x := \sum_{k \geq 0} z_k \Pi^k_x \partial^2 \Pi_x - \sum_{k \geq 1} \frac{1}{k!} \Pi^k_1 (D^{(0)})^k c + \xi;$$

the benefit is that $\tilde{\Pi}^-_x$ also depends on $c$ only through $c_{\beta'}$ with $\beta' \prec \beta$, see (6.4). Hence $\tilde{\Pi}^-_x$ is constructed by induction hypothesis. We now distinguish regular $\beta$ and the singular $\beta$. In the regular case, in view of (2.20) we have to set $c_\beta = 0$ and thus are done by $\Pi^-_x = \tilde{\Pi}^-_x$. The remainder of this subsection is devoted to the construction of $c_\beta$, and thus of $\Pi^-_x = \tilde{\Pi}^-_x - c_\beta$, in the singular case. In line with BPHZ-choice of renormalization, we shall choose $c_\beta$ such that (2.16) is satisfied. Note that according to Proposition 4.6 (where $\Pi^-_x$ can be replaced by $\tilde{\Pi}^-_x$ since the statement (4.9) is oblivious to adding a space-time constant) the limit $\lim_{\beta \to 0} E \Pi^-_{x\beta}(x)$ does at least exist, but could a priori depend on $x$.

As alluded to at the end of Subsection 2.3, we have to include shift covariance and reflection parity of $\Pi_x$ into the induction in order for $c$ to be independent of $x$ and $z_n$, thus meeting the population condition (2.20). By covariance under a space-time shift $z + \cdot$, we mean

$$\Pi^-_{x_1+}(\xi)(z + y) = \Pi^-_x(\xi(\cdot))(y).$$

(5.1)

By parity, we understand the following covariance under the spatial reflection $R_x = (-x_1, x_2)$:

$$\Pi^-_{R_x}(\xi)(R_y) = (-1)^{\sum_n n_1 \beta(n)} \Pi^-_x(\xi(R)(\cdot))(y).$$

(5.2)

Note that in view of (2.23), (5.1) and (5.2) are tautologically satisfied for $\beta$ purely polynomial, see Subsection 3.5. We note that as long as $|\beta| < 2$ the population condition (2.20) $\beta$ is equivalent to

$$c_\beta = 0 \quad \text{unless} \sum_n n_1 \beta(n) \text{ is even.}$$

(5.3)

Since $\{c_\beta\}_{\beta \prec \beta}$ is independent of $x$, deterministic and satisfies (5.3) $\prec \beta$, one easily checks that the properties (5.1) $\prec \beta$ and (5.2) $\prec \beta$ transmit to $\tilde{\Pi}^-_x$ on level $\beta$:

$$\tilde{\Pi}^-_{x_1+}(\xi)(z + y) = \tilde{\Pi}^-_x(\xi(\cdot))(y),$$

(5.4)

$$\tilde{\Pi}^-_{R_x}(\xi)(R_y) = (-1)^{\sum_n n_1 \beta(n)} \tilde{\Pi}^-_x(\xi(R)(\cdot))(y),$$

(5.5)

which also extends to the convoluted versions $\tilde{\Pi}^-_{x\beta'}$. Together with the shift and reflection invariance of the law of $\xi$, see Assumption 2.1, this extends to the expectation $E \Pi^-_{x\beta}(x)$, which as a consequence does not depend on $x$ and vanishes unless $\sum_n n_1 \beta(n)$ is even. This establishes

**Proposition 5.1.** Suppose that (2.20) $\prec \beta$, (5.1) $\prec \beta$, (5.2) $\prec \beta$ and (4.9) $\beta$ hold. Then there exists a deterministic constant $c_\beta$ satisfying (2.20) $\beta$ such that (2.16) $\beta$ holds. Moreover (5.4) $\beta$ and (5.5) $\beta$ hold with $\tilde{\Pi}^-$ replaced by $\Pi^-$. 

5.2 Construction of $\Pi_x$ via integration

The construction of $\Pi_{x\beta}$ given $\Pi^-_x$ is provided by formula (4.6), the convergence of which is part of Proposition 4.3. It follows from the representation (4.6) that (5.4) and (5.5), with $\tilde{\Pi}^-$ replaced by $\Pi^-$, transmit to (5.1) and (5.2).
5.3 Construction of $\Gamma_{xy}$ via re-centering encoded through $\pi_{xy}(n)$

Via the exponential formula (2.34), $\Gamma_{xy}$ is determined by $\{\pi_{xy}(n)\}_n$, so that the task is to construct the latter. In terms of the induction, this means constructing $\{\pi_{xy}(n)\}_{n:|n|<|\beta|}$. Let us recall how $\pi_{xy}(n)$ affects the entries $(\Gamma_{xy})^T_{\beta}$, in terms of the order $<$ relevant for the induction: According to (6.2), the row $\Gamma_{xy}\beta'$ does not depend on $\pi_{xy}(n)$ unless $\beta' \lesssim \beta$. In other words, the rows $\{\Gamma_{xy}\beta'\}_{\beta' \lesssim \beta}$ are not affected by $\pi_{xy}(n)$.

So while the “current” row of $\Gamma_{xy}$ is affected by constructing the current $\pi_{xy}(n)$'s, the previous rows have “stabilized”. The core of the construction is to choose $\pi_{xy}(n)$ such that the re-centering properties (2.40) & (2.41) hold. By the above stabilization, it is enough to verify (2.40) & (2.41). We shall first deal with (2.40) and then with (2.41).

In addition, we need to take care of the transitivity property (2.26). Note that by the triangularity (6.9) of $\Gamma_{xy}$ w. r. t. $<$, also the second factor on the r. h. s. of (2.26) involves $(\Gamma_{xy})^T_{\beta}$, only for $\beta' \lesssim \beta$. Hence also (2.26) is stable under the induction, so that we just have to verify the current (2.26). Note that while (2.26) is consistent with the re-centering (2.40) & (2.41), it is not a consequence of it. In order to independently guarantee (2.26), we shall single out one base point, and construct $\{\pi_{0\beta}(n)\}_{n:|n|<|\beta|}$ to satisfy (2.41) when changing base-point from $0$ to $x$. We then use the general structure of $G^* \subset \text{End}(T^*)$ to extend this definition to a general pair $(y,x)$ of base-points, thereby enforcing (2.26) while preserving (2.41). This intricacy arises because we don’t have access to a model $\Pi$ in the sense of [21, Subsection 4.2], so that we need to emulate it by $\Pi_0$. Likewise, we cannot follow [20, eq. (8.20)], because we have no access to the polynomial $-T_y^{|\beta|} \int_0^\infty dt (\partial_t^2 + \partial_t^2)\pi_{xy}$, see the discussion after (4.6). Incidentally, our treatment is closer to [21, eq. (4.8)] than to [20, eq. (8.20)] in the sense that polynomial coefficients are defined w. r. t. to the base point 0.

More precisely, we proceed in three steps:

- Proposition 5.2 passes from the $\Pi$-statement (2.41) to the $\Pi^-$-statement (2.40). It is based on the definition (2.22) of $\Pi^-$ in terms of $\Pi$ and the multiplicativity (2.33) of $\Gamma^*$. The renormalization term transforms as desired because $c$ is independent of $x$ and $z_n$. A difficulty in appealing to multiplicativity for (2.22) lies in the presence of the projection $P$, cf. (2.13), in that formula. This key step is algebraic.

- Proposition 5.3 passes from the $\Pi^-$-statement (2.40) for $y = 0$, or rather the weaker (2.24), to the $\Pi$-statement (2.41) for $y = 0$. It is based on the uniqueness for the PDE (2.21) under the growth condition contained in (2.28), and the anchoring (2.31) (which also follows from (2.28)). Because of the latter, it requires the following definition of $\{\pi_{0\beta}(n)\}_{n:|n|<|\beta|}$

$$\frac{\partial^m}{\partial y^m} \left( (\Gamma_{xy}^* P\Pi_0)_\beta(y) + \sum_{n:|n|<|\beta|} \pi_{0\beta}(n) y^n \right) = 0 \quad \text{at} \quad y = x \quad \text{provided} \ |m| < |\beta|. \quad \text{(5.6)}$$

Hence $\{\pi_{0\beta}(n)\}_n$ characterizes the Taylor polynomial of $\Gamma_{xy}^* P\Pi_0$ of degree $|\beta|$ in $x$, which is required in re-centering when passing from the base point 0 to the base point $x$. This key step is analytic.

- Proposition 5.4 extends the $\Pi$-statement (2.41) from $y = 0$ to all $y$; as a way of ensuring (2.26) for $y = 0$. It requires the following definition of $\{\pi_{xy}(n)\}_{n:|n|<|\beta|}$:

$$\pi_{xy}(n) := (\pi_{xy}(n) - \Gamma_{xy}^* \pi_{0\beta}(n))_{\beta} \quad \text{(5.7)}$$

and relies on the structure of the exponential map $\exp \{\pi(n)\}_n \mapsto \Gamma^* \in G^*$. This step is again algebraic.

\footnote{by which we mean that the $\pi(n)$'s give rise to $\Gamma^*$ via the exponential formula (2.34)
Before we address Proposition 5.2, we need to argue that the r. h. s. term in (2.40)_\beta is well-defined at this stage of the induction step. Indeed, because of $\Pi^* \in \tilde{T}^*$ and by the triangular structure (6.9) of $\Gamma^*$ w. r. t. $\prec$, the first r. h. s. term $(\Gamma^*_y \Pi_y)_\beta$ involves $(\Gamma^*_y)_\beta$ only for $\gamma$ not purely polynomial, which has stabilized, and $\Pi_y$, for $\beta' \ll \beta$, which has been constructed. Because of the triangular structure (6.6) of $\sum_{k \geq 0} z_k \pi^k \pi'$, the second r. h. s. term of (2.40)_\beta, namely the product $(z_k(\Gamma^*_y \Pi_y + \pi^{(0)}_y)^k \partial^k_{\gamma}(\Gamma^*_y \Pi_y + \pi^{(0)}_y))_\beta$, depends on its factors only in terms of $(\Gamma^*_x \Pi_x + \pi^{(0)}_x)_\beta \ll \beta$. Note that $(\Gamma^*_x \Pi_x + \pi^{(0)}_x)_\beta \ll \beta$ has stabilized for all $\gamma$. Again, by the triangular structure (6.9) of $\Gamma^*_x \Pi_x$, $\pi^{(0)}_x \ll \beta$ is involved only for $\beta' \ll \beta$, which has been constructed.

**Proposition 5.2.** Suppose (2.41)_\beta and (2.22)_\beta hold. Then (2.40)_\beta holds.

**Proof.** We deal with the three terms on the r. h. s. of (2.30) one by one. In preparation of the first one, we claim that (2.33) implies for any $\pi, \pi' \in T^*$

$$P \sum_{k \geq 0} z_k (\pi^* + \pi^{(0)}_k)^k (\pi^* \pi') = \Gamma^* P \sum_{k \geq 0} z_k \pi^k \pi' + P \sum_{k \geq 0} z_k (\pi^* (id - P) \pi + \pi^{(0)}_k)^k (\pi^* (id - P) \pi'),$$  \hspace{1cm} (5.8)

where $\pi^{(0)}_k$ is related to $\pi^*$ by (2.34). Note that $P$ plays the role of the projection of $\mathbb{R}[[z_k, z_n]]$ onto $\tilde{T}^*$ and $id - P$ the one of $\tilde{T}^*$ onto $\tilde{T}^*$. Here comes the argument for (5.8): The $(k+1)$-fold iteration of (2.33) component-wise reads$^{66}$

$$\Gamma^*_\gamma = \sum_{\beta_0 + \cdots + \beta_{k+1} = \gamma} (\Gamma^*)^\gamma_0 \cdots (\Gamma^*)^\gamma_{k+1} \beta_0 \cdots \beta_{k+1} \text{ provided } \gamma = \gamma_0 + \cdots + \gamma_{k+1}. $$

This allows to characterize the commutator of $\pi^*$ and $P$ on a product of arbitrary $\pi^{(0)}_k \in \tilde{T}^*$, $\pi^{(1)}_k$, $\pi^{(k+1)}_k \in T^*$ (so that below, $P$ acts like a projection from $\mathbb{R}[[z_k, z_n]]$ onto $T^*$):

$$P(\Gamma^* \pi^{(0)}) \cdots (\Gamma^* \pi^{(k+1)})_\beta = (\Gamma^* P \pi^{(0)} \cdots \pi^{(k+1)})_\beta + \sum_{\beta_0 + \cdots + \beta_{k+1} = \gamma} (\Gamma^*_\beta_0 \pi^{(0)}_0 \cdots (\Gamma^*_\beta_{k+1} \pi^{(k+1)}_{k+1}) \beta_0 \cdots \beta_{k+1}, \text{ not populated}) \hspace{1cm} (5.9)$$

We use (5.9) for $\pi^{(0)}_k = z_k$, $\pi^{(1)}_k = \cdots = \pi^{(k)}_k = \pi$, and $\pi^{(k+1)}_k = \pi'$, and combine it with the following consequence$^{67}$ of the definition (2.12)

$$e_k + \gamma_1 + \cdots + \gamma_{k+1} \text{ not populated } \iff \gamma_1, \ldots, \gamma_{k+1} \text{ purely polynomial.}$$

Hence (5.9) assumes the form of

$$P(\Gamma^* z_k)(\Gamma^* \pi^k)(\Gamma^* \pi') = \Gamma^* P z_k \pi^k \pi' + P(\Gamma^* z_k)(\Gamma^* (id - P) \pi)^k (\Gamma^* (id - P) \pi'),$$

which we sum in $k$,

$$P \sum_{k \geq 0} (\Gamma^* z_k)(\Gamma^* \pi^k)(\Gamma^* \pi') = \Gamma^* P \sum_{k \geq 0} z_k \pi^k \pi' + P \sum_{k \geq 0} (\Gamma^* z_k)(\Gamma^* (id - P) \pi)^k (\Gamma^* (id - P) \pi').$$  \hspace{1cm} (5.10)

By (2.37) followed by the binomial formula, we obtain

$$\sum_{k \geq 0} (\Gamma^* z_k)(\Gamma^* \pi^k)(\Gamma^* \pi') = \sum_{k \geq 0} z_k^e (\Gamma^* \pi + \pi^{(0)}_e)^k (\Gamma^* \pi').$$

---

$^{66}$ with the understanding that all multi-indices are populated

$^{67}$ recalling our implicit assumption that $\gamma_1, \ldots, \gamma_{k+1}$ are populated
and the same identity with $\pi, \pi'$ replaced by $(id - P)\pi, (id - P)\pi'$. Inserting these two identities in (5.11) yields (5.8). We apply (5.8) with $\Gamma = \Gamma^*_{xy}$ (noting that the corresponding $\pi^{(0)}_{xy}$ is a constant in space-time), $\pi = \Pi_y$, and $\pi' = \partial^2_t \Pi_y$, which results in

$$
P \sum_{k \geq 0} z_k (\Gamma^*_{xy} \Pi_y + \pi^{(0)}_{xy} \Pi_y) \partial^2_t (\Gamma^*_{xy} \Pi_y + \pi^{(0)}_{xy})
= \Gamma^*_{xy} P \sum_{k \geq 0} z_k (\Gamma^*_{xy} (id - P) \Pi_y + \pi^{(0)}_{xy}) \partial^2_t (\Gamma^*_{xy} (id - P) \Pi_y + \pi^{(0)}_{xy}).$$

(5.11)

We now turn to the second r. h. s. contribution to (2.30), and claim that for any $\pi \in T^*$ and $c \in \tilde{T}^*$ satisfying the population condition (2.20), we have

$$
\sum_{k \geq 0} \frac{1}{k!} (\Gamma^* \pi + \pi^{(0)})^k (D^{(0)})^k c = \Gamma^* \sum_{k \geq 0} \frac{1}{k!} \pi^k (D^{(0)})^k c.
$$

(5.12)

Because of this and (2.33), it remains to show

$$
\sum_{k \geq 0} \frac{1}{k!} (\Gamma^* \pi + \pi^{(0)})^k (D^{(0)})^k c = \sum_{k \geq 0} \frac{1}{k!} (\Gamma^* \pi)^k (\Gamma^* (D^{(0)}))^k c.
$$

(5.13)

Note that the second item in the population condition (2.20) on $c$ can be re-expressed as $D^{(0)}c = 0$ for $n \neq 0$, cf. (2.35). We now appeal to (4.51), which implies (5.13), once more by the binomial formula. We apply (5.12) to $\Gamma^* = \Gamma^*_{xy}$ and $\pi = \Pi_y$, to the effect of

$$
\sum_{k \geq 0} \frac{1}{k!} (\Gamma^*_{xy} \Pi_y + \pi^{(0)}_{xy})^k (D^{(0)})^k c = \Gamma^*_{xy} \sum_{k \geq 0} \frac{1}{k!} \Pi_y^k (D^{(0)})^k c.
$$

(5.14)

We finally turn to the third r. h. s. contribution to (2.30) and note that by the second item in (2.33) we have

$$
\xi 1 = \Gamma^*_{xy} \xi 1.
$$

The sum of (5.11), (5.14) and the last identity yields (2.40) by definition (2.22).

Before addressing Proposition 5.3, we need to argue that $\pi^{(n)}_{0,\beta}$ is well-defined at this stage of the induction step: Thanks to the presence of $P$, cf. (2.13), $(\Gamma^*_{0,\beta} P \Pi_0)_{\beta}$ only involves $(\Gamma^*_{0,\beta})_\gamma$ with $\gamma$ not purely polynomial, which has stabilized. By the triangularity of (6.9) of $\Gamma^*_{0,\beta}$ w. r. t. $|\cdot|_{\infty}$, (5.6) only involves $\Pi_0$’s with $\gamma \leq \beta$, and thus already constructed. Hence $(\Gamma^*_{0,\beta} P \Pi_0)_{\beta}$ is well-defined at this stage of the induction, and thus also $\pi^{(n)}_{0,\beta}$ via (5.6). As always in an integration integration step, both for the estimates in Proposition 4.3 and for uniqueness here, we have to stay away from integers, cf. (2.19).

**Proposition 5.3** (Liouville). Suppose (2.28) $\leq \beta$ and (2.24) $\leq \beta$ hold. Define $\{\pi^{(n)}_{0,\beta}\}_{n \leq |\beta|}$ by (5.6). Then (2.41) $\beta$ holds for $y = 0$.

**Proof.** Applying $\partial_2 - \partial^2_t$ to $(\Gamma^*_{0,\beta} P \Pi_0)_{\beta}$ results in

$$(\partial_2 - \partial^2_t)(\Gamma^*_{0,\beta} P \Pi_0)_{\beta} = (\Gamma^*_{0,\beta} P (\partial_2 - \partial^2_t) \Pi_0)_{\beta}$$

(2.28) $\leq$ (2.24) $\leq \beta$ $\leq \beta$

(2.24) $\leq$ (6.9) $\leq \beta$

(5.15)
where we used the triangularity w. r. t. both \( \prec \) and homogeneity \( | \cdot | \), the latter in order to control the degree of the polynomial. Hence \( (\Gamma_{x_0} P \Pi_0)_\beta \) solves the same PDE \((2.21)\) modulo polynomials as \( \Pi_{x_\beta} \), so that

\[
(\partial_2 - \partial_1^2)\partial^n(\Gamma_{x_0} P \Pi_0 - \Pi_\beta) = 0 \quad \text{provided } |n| > |\beta| - 2. \tag{5.16}
\]

In view of the kernel estimate \((4.5)\), the quantitative \((2.28)_{\leq \beta} \) yields the qualitative \((5.17)_{\leq \beta} \), where

\[
\lim_{t \to \infty} \mathbb{E}^\beta_t |\partial^n \Pi_{x_\beta}(y)|^p = 0 \quad \text{provided } |n| > |\beta|. \tag{5.17}
\]

Hence by the triangularity of \( \Gamma_{x_0} \) w. r. t. both \( \prec \) and \( | \cdot | \), this implies

\[
\lim_{t \to \infty} \mathbb{E}^\beta_t |\partial^n(\Gamma_{x_0} P \Pi_0 - \Pi_\beta)(y)|^p = 0 \quad \text{provided } |n| > |\beta|. \tag{5.18}
\]

We now argue that \((5.16)\) and \((5.18)\) imply

\[
\partial^n(\Gamma_{x_0} P \Pi_0 - \Pi_\beta) = 0 \quad \text{provided } |n| > |\beta|. \tag{5.19}
\]

Indeed, according to the factorization \((2.3)\) and the definition of \((\cdot)_t\), \((5.16)\) implies

\[
\partial_t \partial^n(\Gamma_{x_0} P \Pi_0 - \Pi_\beta) = 0 \quad \text{provided } |n| > |\beta|.
\]

This permits to pass from \((5.18)\) for \( t \uparrow \infty \) to \((5.19)\) for \( t \downarrow 0 \).

We now consider

\[
\hat{\Pi}_{x_\beta}(y) := (\Gamma_{x_0} P \Pi_0)_\beta(y) + \sum_{n:|n|\leq |\beta|} \pi_n^{(n)} y^n;
\]

obviously, \((5.19)\) transfers to \( \hat{\Pi}_{x_\beta} \):

\[
\partial^n(\hat{\Pi}_{x} - \Pi_x) = 0 \quad \text{provided } |n| > |\beta|. \tag{5.20}
\]

By construction \((5.6)\), \( \hat{\Pi}_{x_\beta} \) satisfies the same anchoring \((2.31)\) as \( \Pi_{x_\beta} \), which means that

\[
\partial^n(\hat{\Pi}_{x} - \Pi_x) = 0 \quad \text{provided } |n| < |\beta|.
\]

Since \( \beta \) is not purely polynomial, by \((2.19)\), this automatically extends to \( |n| \leq |\beta| \). Together with \((5.20)\) this implies

\[
\Pi_{x_\beta}(y) = \hat{\Pi}_{x_\beta}(y) = (\Gamma_{x_0} P \Pi_0)_\beta(y) + \sum_{n \neq 0:|n| < |\beta|} \pi_n^{(n)} y^n + \pi_0^{(0)}.
\]

By definition \((2.23)\), and because of \( \Gamma^* z_n \stackrel{2.34}{=} z_n + \pi^{(n)} \) in form of \( (\gamma^* \gamma)^{\alpha} = \pi^{(n)} \beta \), this turns into the desired \((2.41)\) for \( y = 0 \).

In preparation of Proposition \( 5.4 \), we argue that \( \pi_n^{(n)} \) is well-defined at this stage of the induction step. First of all, because of \( G^* \subset \text{Aut}(\Gamma^*) \), the inverse \( \Gamma_{0^*}^{-1} = (\Gamma_{0^*})^{-1} \) exists and thus the expression \((5.7)\) makes sense. Moreover, by the triangular structure \((6.9)\) of \( \Gamma_{x_0} \in G^* \), and the strictly triangular structure of \( \Gamma_{0}^{-1} \in G^* \), the row \( (\Gamma_{x_0} \Gamma_{0^-})_{\beta} \) depends only on the rows \( \Gamma_{x_0} \) and \( \Gamma_{0\gamma} \), for \( \beta' \ll \beta \). These rows have been constructed via the exponential formula \((2.34)\) through \((5.6)\), and in the previous induction steps. Finally, the triangular structure \((6.9)\) of \( \Gamma_{x_0} \Gamma_{0^-} \in G^* \) ensures that \((5.7)\) involves \( \pi_{x_0\beta'}^{(n)} \) and \( \pi_{x_0\beta'}^{(n)} \) only for \( \beta' \ll \beta \), which have been constructed through \((5.6)\) and in the previous induction steps. We also have to make sure that \((5.7)\) is consistent for \( y = 0 \), which follows provided we have

\[
\pi_n^{(n)} = 0, \tag{5.21}
\]

which we need to include into the induction.
**Proposition 5.4.** Suppose \((4.41)_{\beta}\) holds for \(y = 0\), and that \((5.21)_{\beta}\) holds. Then \((4.41)_{\beta}, (2.26)_{\beta},\) and \((5.21)_{\beta}\) hold.

**Proof.** Since the row \((\Gamma_{00}P)_{\beta}\) depends on \(\pi_{\alpha}(n)_{\beta}\) only through \(\beta' \prec \beta\), see the discussion before Proposition 5.3, it follows from \((5.21)_{\beta}\) that \((\Gamma_{00}P)_{\beta} = P_{\beta}\). It thus follows from \((2.31)\) via \((5.6)\) that we must have \((5.21)_{\beta}\).

We give ourselves a general pair \((x, y)\) of points. We have \((4.41)_{\beta, \gamma}\) for \((x, y)\) replaced by \((x, 0)\) and \((y, 0)\) at our disposal. By the triangular structure of \(\Gamma_{00}\), solving the latter for \(\Pi_{y}\) and inserting it into the former, we obtain

\[
\Pi_{y\beta} = (\Gamma_{00}^{-1}\Pi_{y} + \pi_{n}(0) - \Gamma_{00}^{-1}\pi_{n}(0))_{\beta}.
\]

By the property \((6.1)\), the \(\Gamma_{xy} \in G^{+}\) associated to the \(\pi_{xy}(n)\) defined in \((5.7)_{\beta'} \prec \beta\) actually satisfies

\[
\Gamma_{xy} = \Gamma_{x0}^{+} \Gamma_{y0}^{-}.
\]

Inserting this and \((5.7)_{\beta}\) into \((5.22)_{\beta}\) yields \((4.41)_{\beta}\). Note that \((5.23)\) immediately yields \((2.26)\).

**5.4 Construction of \(d\Gamma_{x\zeta}^{+}\) through \(d\pi_{x\zeta}^{(n)}\)**

Recall the Ansatz \((4.31)\) for \(d\Gamma_{x\zeta}^{+}\). We seek \(d\pi_{x\zeta}^{(1,0)} \in Q\tilde{T}^{+}\) such that \((4.33)\) holds, which is obviously equivalent to

\[
Q(\partial_{1}\delta\Pi_{x}(z) - d\Gamma_{x\zeta}^{+}d_{1}\Pi_{x}(z)) = 0.
\]

Evaluating \((4.31)\) on \(z_{(1,0)}\), and using the definitions \((2.14)\) & \((2.35)\) of \(D(n)\) (together with the last item in \((2.33)\)), we obtain \(d\Gamma_{x\zeta}^{+}z_{(1,0)} = d\pi_{x\zeta}^{(1,0)}\). Applying \(Q\) and feeding in the postulate \(d\pi_{x\zeta}^{(1,0)} \in Q\tilde{T}^{+}\), we have

\[
d\pi_{x\zeta}^{(1,0)} = Qd\Gamma_{x\zeta}^{+}z_{(1,0)}.
\]

Hence in view of \((id - P)\partial_{1}\Pi_{x}(z) = z_{(1,0)}\), which follows from \((2.23)\), \((5.24)\) can be re-arranged to

\[
d\pi_{x\zeta}^{(1,0)} = Q(\partial_{1}\delta\Pi_{x}(z) - d\Gamma_{x\zeta}^{+}Pd_{1}\Pi_{x}(z)).
\]

According to the order within an induction step stated in Subsection \((3.1)\), \((\Gamma_{x\zeta}^{+}P)_{\beta}\) has been constructed (Proposition 4.1), and based on this and the induction hypothesis, \((d\Gamma_{x\zeta}^{+}P)_{\beta}\) has been constructed (Proposition 4.15). Also \(Q\Pi_{\beta}\) and thus \(Q\delta\Pi_{\beta}\) have been constructed (Proposition 4.1). In view of the strict triangularity \((6.1)\) of \(d\Gamma_{x\zeta}^{+}\) w. r. t. \(\prec\), \((5.26)\) involves only \((P\Pi_{\beta})_{\prec}\), which has been constructed by induction hypothesis. Hence \(d\pi_{x\zeta}^{(1,0)}\) is well-defined through \((5.26)\).

**6 Triangular structures and dependencies**

The entire induction argument relies on the (strict) triangular structure of \(\Gamma^{+} - id\) and \(d\Gamma^{+}\) w. r. t. \(|\cdot|\) and \(|\cdot|_{<}\). Equally important are the strict triangular dependencies of \(\Gamma^{+}\) and \(d\Gamma^{+}\) on \(\pi^{(n)}\) and \(dx^{(n)}\), respectively. The same applies to the dependencies of the expressions \(z_{k}x_{k}^{n}x_{n}'\) and \(x_{k}'(D(\theta))^{k}c\) on \(x, x', c\).

**Lemma 6.1 (Triangular dependencies).** For \(\gamma\) not purely polynomial,

\[
(\Gamma^{+})_{\beta}^{\gamma} \text{ does not depend on } \pi_{\beta}^{(n)} \text{ unless } \beta' \prec \beta.
\]

(6.1)
Lemma 6.2 (Triangular structures). For any $\Gamma^* \in G^*$,

$$(\Gamma^* - \text{id})^\gamma_{\beta} \neq 0 \implies \gamma < \beta \quad \text{and} \quad |\gamma| < |\beta|, \quad (6.9)$$

$$(\delta \Gamma^*)^\gamma_{\beta} \neq 0 \implies \gamma < \beta \quad \text{and} \quad |\gamma| < |\beta|, \quad (6.10)$$

$$(d\Gamma^*)^\gamma_{\beta} \neq 0 \implies \gamma < \beta \quad \text{and} \quad |\gamma| \leq |\beta| + 1 - \alpha. \quad (6.11)$$

In particular,

$$(\Gamma^*)^\gamma_{0} \neq 0 \implies \gamma = 0, \quad (6.12)$$

$$(d\Gamma^*_{xy} - d\Gamma^*_{x^*y^*})^\gamma_{\beta} \neq 0 \implies \gamma \prec \beta. \quad (6.13)$$

We state (and instantly prove) the two further properties (6.14) and (6.16):

$$(d\Gamma^*_{x^*})^\gamma_{\beta} = 0 \quad \text{for} \quad \beta \in \mathbb{N}_0 e_0 \quad \text{and} \quad \gamma \text{ not purely polynomial.} \quad (6.14)$$

By definition (4.31), for (6.14) it suffices to show that $$(\Gamma^* D^{(n)})^\gamma_{\beta}$$ vanishes for such $\beta, \gamma$. Since by (2.17) and $\alpha \in (0, \frac{1}{2}]$,

$$\beta \in \mathbb{N}_0 e_0 \iff |\beta| = \min A \iff |\beta| < 2\alpha, \quad (6.15)$$

the latter follows from the triangularity of $\Gamma^*$ w. r. t. $| \cdot |$ and the fact that $$(D^{(n)})^\gamma_{\beta}$$ vanishes for such $\beta, \gamma$, see (2.14) and (2.35).

We also recall from [30, Proposition 5.1 (iii)] that

$$(\pi^{(n)} + \Gamma^* \pi^{(n)})_n \mapsto \Gamma^* \bar{\Gamma}^* \quad \text{provided} \quad \{\pi^{(n)}\}_n \mapsto \Gamma^* \quad \text{and} \quad \{\pi^{(n)}\}_n \mapsto \bar{\Gamma}^*, \quad (6.16)$$

which by $\{0\}_n \mapsto \text{id}$ yields $\{-(\Gamma^{-\ast} \pi^{(n)})\}_n \mapsto \bar{\Gamma}^{-\ast}$; hence

$$(\pi^{(n)} - \Gamma^* \bar{\Gamma}^{-\ast} \pi^{(n)})_n \mapsto \Gamma^* \bar{\Gamma}^{-\ast} \quad \text{provided} \quad \{\pi^{(n)}\}_n \mapsto \Gamma^* \quad \text{and} \quad \{\pi^{(n)}\}_n \mapsto \bar{\Gamma}^{-\ast}. \quad (6.16)$$
Proof of Lemma 6.1. Proof of (6.1) & (6.2). Indeed, (6.2) is an immediate consequence of (6.1) by (2.38). By (2.34) in its component-wise version, we see that for (6.1) we have to show
\[ \pi_{\beta_1}^{(n_1)} \cdots \pi_{\beta_k}^{(n_k)} (D^{(n_1)} \cdots D^{(n_k)})_{\beta_{k+1}} > 0 \implies \beta_1, \ldots, \beta_k \prec \beta, \]  
(6.17)
where \( k \geq 1 \) and \( \beta_1, \ldots, \beta_{k+1} \) satisfy
\[ \sum_{k'=1}^{k+1} \beta_{k'} = \beta \quad \text{and} \quad |n_{k'}| < |\beta_{k'}| \quad \text{for} \quad k'=1, \ldots, k, \]  
(6.18)
where the second part comes from (2.36). From the definitions (2.14) and (2.35) we infer
\[ (D^{(0)})_{\beta} > 0 \implies \left( |\gamma| = |\beta| - 1 \quad \text{and} \quad |\gamma|_p = |\beta|_p \quad \text{and} \quad \gamma(0) \leq \beta(0) + 1 \right), \]  
(6.19)
and for \( n \neq 0 \)
\[ (D^{(n)})_{\beta} > 0 \implies \left( |\gamma| = |\beta| - 1 \quad \text{and} \quad |\gamma|_p = |\beta|_p + |n| \quad \text{and} \quad \gamma(0) = \beta(0) \right). \]  
(6.20)
This yields by iteration
\[ (D^{(n_1)} \cdots D^{(n_k)})_{\beta_{k+1}} > 0 \implies |\gamma| = |\beta_{k+1} - k \quad \text{and} \quad |\gamma|_p = |\beta_{k+1}|_p + \sum_{k'=1}^{k} |n_{k'}| \quad \text{and} \quad \gamma(0) \leq \beta_{k+1}(0) + \sum_{k'=1}^{k} \delta_{0}^{n_{k'}}, \]  
(6.21)
which by definition (3.6) of \( | \cdot |_\prec \) implies
\[ (D^{(n_1)} \cdots D^{(n_k)})_{\beta_{k+1}} > 0 \implies |\gamma|_\prec \leq |\beta_{k+1}|_\prec + \sum_{k'=1}^{k} \left( |n_{k'}|_\prec + \frac{1}{4} \delta_{0}^{n_{k'}} - 1 \right). \]  
(6.22)
Hence by the first item in (6.18) and the additivity of \( | \cdot |_\prec \), the l. h. s. of (6.17) yields
\[ |\beta|_\prec = \sum_{k'=1}^{k+1} |\beta_{k'}|_\prec \geq |\gamma|_\prec + \sum_{k'=1}^{k} \left( |\beta_{k'}|_\prec - (\frac{1}{4} |n_{k'}|_\prec + \frac{1}{4} \delta_{0}^{n_{k'}}) \right). \]  
(6.23)
We now argue that each summand in the last term of (6.23) is positive
\[ \frac{1}{4} |n_{k'}| + \frac{1}{4} \delta_{0}^{n_{k'}} - 1 < |\beta_{k'}|_\prec \quad \text{for} \quad k'=1, \ldots, k. \]  
(6.24)
Indeed, for \( n_{k'} = 0 \) we have as desired
\[ \frac{1}{4} |n_{k'}| + \frac{1}{4} \delta_{0}^{n_{k'}} - 1 = \frac{1}{4} - 1 < -\frac{3}{2} \]  
(3.9)
For \( n_{k'} \neq 0 \), we have
\[ \frac{1}{4} |n_{k'}| + \frac{1}{4} \delta_{0}^{n_{k'}} - 1 = \frac{1}{4} |n_{k'}| - 1 < \frac{1}{2} |\beta_{k'}| - 1, \]  
(6.18)
and conclude by
\[ \frac{1}{2} |\beta_{k'}| - 1 > \frac{1}{2} \left( (\frac{1}{2} |\beta_{k'}| + 1) + |\beta_{k'}|_p \right) - 1 \geq |\beta_{k'}| + \frac{1}{2} |\beta_{k'}|_p + \frac{1}{2} \beta_{k'}(0) \]  
(6.6)
\[ |\beta|_\prec \geq |\beta_{k'}|_\prec + |\gamma|_\prec - (\frac{1}{4} |n_{k'}| + \frac{1}{4} \delta_{0}^{n_{k'}} - 1) \quad \text{for any} \quad k'=1, \ldots, k. \]  
(6.25)
In case \( n_{k'} = 0 \), we obtain from (6.25) as desired
\[
|\beta|_\prec \geq |\beta_{k'}|_\prec + |\gamma|_\prec - \frac{1}{2} + 1 = |\beta_{k'}|_\prec.
\]
In case \( n_{k'} \neq 0 \) we obtain from (6.25)
\[
|\beta|_\prec \geq |\beta_{k'}|_\prec + |\gamma|_\prec - \frac{1}{2} n_{k'} + 1 \overset{\text{(6.6)}}{=} |\beta|_\prec + |\gamma|_\prec + \frac{1}{2} (|\gamma|_{p} - |n_{k'}|) + \frac{1}{2} \gamma(0) + 1.
\]
Since \(|\gamma|_{p} - |n_{k'}| \geq 0\) by (6.21), \(|\gamma| \geq 0\) by the assumption that \( \gamma \) is not purely polynomial, and \( \gamma(0) \geq 0\), this yields again \(|\beta|_\prec > |\beta_{k'}|_\prec\), which finishes the argument for (6.1).

Proof of (6.9). Recall that \( c_{\pi} \) equals (6.26). From (6.21) we obtain
\[
|\beta|_\prec = |\beta_{1}|_\prec + \cdots + |\beta_{k+1}|_\prec \geq |\beta_{1}|_\prec + \cdots + |\beta_{k}|_\prec + |\gamma|_\prec + \frac{1}{2} k.
\]
Since \(|\cdot|_\prec \geq -1\), we obtain for any \( \beta' \in \{\beta_{1}, \ldots, \beta_{k}, \gamma\}\)
\[
|\beta|_\prec \geq |\beta'|_\prec + \frac{1}{2} k > |\beta'|_\prec,
\]
which finishes the proof of (6.4).

Proof of (6.6) and (6.7). We have to show that
\[
\left| e_{k} + \beta_{1} + \cdots + \beta_{k+1} = \beta \right| \implies \begin{cases} 
\beta_{1}, \ldots, \beta_{k+1} < \beta, \\
|\beta_{1}| + \cdots + |\beta_{k+1}| = |\beta|.
\end{cases}
\]
For the upper item we distinguish \( k = 0 \) from \( k \neq 0 \). In the first case we have \( |\beta|_\prec = |\beta_{1}|_\prec + \frac{1}{2} > |\beta_{1}|_\prec\).
In the latter one, we have by additivity \(|\beta|_\prec = k + |\beta_{1}|_\prec + \cdots + |\beta_{k+1}|_\prec\), and since \(|\cdot|_\prec \geq -1/2\), we obtain \(|\beta|_\prec \geq k + 2/\pi_{k+1}|_{p} > |\beta_{1}|_\prec\) for all \( i = 1, \ldots, k + 1\). The lower item is an immediate consequence of the definition of \(|\cdot|_\prec\), cf. (2.17).

Proof of (6.8). The \( \beta \)-component of (6.8) equals (6.5), hence \( |\beta| = |e_{k}| + |\beta_{1}| + \cdots + |\beta_{k+1}|\). Since \( |e_{k}| = k, |\cdot| \geq -1\) and \( |\beta_{1}, \ldots, \beta_{k+1}| \geq 0\) by \( \pi_{k+1} \in T^{*}\), see (2.12), we obtain as desired \(|\beta| \geq 0\).

**Proof of Lemma 6.2.** Proof of (6.9). Recall that \( (\Gamma_{\beta}^{*} - \text{id})_{\beta}^{\gamma} \) is a linear combination of terms of the form of the l. h. s. of (6.17), involving multi-indices \( \beta_{1}, \ldots, \beta_{k+1} \) for \( k \geq 1 \) subject to (6.18). Putting together (6.23) and (6.24), we obtain \( \gamma < \beta \). From (6.21) and
\[
\sum_{k'=1}^{k+1} |\beta_{k'}| = |\beta|, \quad \sum_{k'=1}^{k+1} |\beta_{k'}|_{p} = |\beta|_{p},
\]
which is an immediate consequence of (6.18), we see that the l. h. s. of (6.17) implies
\[
|\gamma| = |\beta| - \sum_{k'=1}^{k} |\beta_{k'}| - k \quad \text{and} \quad |\gamma|_{p} < |\beta|_{p} + \sum_{k'=1}^{k} (|\beta_{k'}| - |\beta_{k'}|_{p}).
\]

This yields

$$|γ| < |β| - \sum_{k=1}^{\beta} (\alpha(1 + [β_k]) + |β_k| - |β_k|^p) = |β|,$$

establishing the last item in (6.9).

Proof of (6.10). This is an immediate consequence of (6.9).

Proof of (6.11). We first note that

$$D_{γ}^γ γ' ≠ 0 \implies γ ≺ γ' \text{ and } |γ| ≤ |γ'| + 1 - α \text{ for } D \in \{D(0), D(1,0)\},$$

which is an immediate consequence of (6.19) and (6.20). By (6.9) we therefore obtain

$$(Γ^p D)^γ γ' ≠ 0 \implies γ ≺ β'' \text{ and } |γ| ≤ |β''| + 1 - α \text{ for } D \in \{D(0), D(1,0)\}.$$  

To establish the last item in (6.11), we also note that by β' + β'' = β and |·| ≥ α, we have |β''| = |β'| + |β''| + α ≤ |β|, which yields as desired |γ| ≤ |β| + 1 - α. The first item in (6.11) follows from β'' ≈ β, which we shall establish now. From the definition (4.31) of dΓ^γ, we see that (dΓ^γ)^γ is a linear combination of terms of the form dΓ^γ, with β' + β'' = β and n = 0, (1, 0). By (4.17) and the first item in (4.32) we see that purely polynomial β' do not contribute, hence β'' ≈ β by (3.7).

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