FAMILIES OF MINIMAL SURFACES IN \( \mathbb{H}^2 \times \mathbb{R} \) FOLIATED BY ARCS AND THEIR JACOBI FIELDS

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ABSTRACT. This note provides some new perspectives and calculations regarding an interesting known family of minimal surfaces in \( \mathbb{H}^2 \times \mathbb{R} \). The surfaces in this family are the catenoids, parabolic catenoids and tall rectangles. Each is foliated by either circles, horocycles or circular arcs in horizontal copies of \( \mathbb{H}^2 \). All of these surfaces are well-known, but the emphasis here is on their unifying features and the fact that they lie in a single continuous family. We also initiate a study of the Jacobi operator on the parabolic catenoid, and compute the Jacobi fields associated to deformations to either of the two other types of surfaces in this family.

1. Introduction

In these notes we study properties of an interesting family of minimal surfaces in \( \mathbb{H}^2 \times \mathbb{R} \). The surfaces in this family are foliated by circles or circular arcs in parallel slices of \( \mathbb{H}^2 \times \mathbb{R} \); those surfaces foliated by entire circles are catenoids, those foliated by horocycles are parabolic catenoids, and those foliated by arcs equidistant from a geodesic are the so-called tall rectangles. Somewhat surprisingly, using the Poincaré disk model of \( \mathbb{H}^2 \), these surfaces all appear as intersections of regular surfaces in \( \mathbb{R}^3 \) with the unit cylinder \( \{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 < 1\} \).

2. Preliminaries and notation

To set notation, we shall use both the Poincaré disk and upper half-space models of \( \mathbb{H}^2 \); these are denoted by \( \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \) and \( \mathcal{H} = \{z = x + iy \in \mathbb{C} : y > 0\} \). We also denote by \( o = (0, 0) \in \mathbb{D} \). The isometry between these two models is the Möbius transformation

Date: January 15, 2019.

L. Ferrer, F. Martín and M.M. Rodríguez are partially supported by the MINECO/FEDER grant MTM2014-52368-P and MTM2017-89677-P; R. Mazzeo supported by the NSF grant DMS-1608223; F. Martin is also partially supported by the Leverhulme Trust grant IN-2016-019.
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\[ g : \mathbb{D} \rightarrow \mathcal{H}, \]

\[ g(z) = \frac{1 - z}{1 + z}. \]

These have metrics

\[ d\rho^2 = 4 \frac{|dz|^2}{(1 - |z|^2)^2}, \]

\[ \text{and} \quad \frac{1}{y^2} |dz|^2, \]

respectively; the corresponding metric for \( \mathbb{H}^2 \times \mathbb{R} \) is

\[ d\sigma^2 = d\rho^2 + dt^2. \]

We denote by \( \partial H^2 \) the boundary at infinity of \( H^2 \), usually identified with the boundary \( S^1 = \{z \in \mathbb{C} : |z| = 1\} \) of \( \mathbb{D} \), but sometimes also with the boundary \( \mathbb{R} \) of \( H \) together with the extra point at infinity.

Recall from [4] that there are several interesting compactifications of the boundary at infinity of \( H^2 \times \mathbb{R} \), but for the purposes of this paper we consider only the portion \( \partial H^2 \times \mathbb{R} \), which lies in the boundary of any of these useful compactifications.

3. Catenoids of revolution in \( \mathbb{H}^2 \times \mathbb{R} \)

We begin by studying the minimal surfaces of revolution in \( \mathbb{H}^2 \times \mathbb{R} \). These analogues of catenoids in \( \mathbb{R}^3 \) were originally described in the seminal article [6] of Nelli and Rosenberg. We take a slightly different approach to their construction here. Consider a conformal harmonic parametrization of a minimal annulus \( A \):

\[ X = (F, h) : \Delta_R = \{ R < |z| < 1 \} \rightarrow A \subset \mathbb{H}^2 \times \mathbb{R}; \]

thus

\[ F : \Delta_R \rightarrow \mathbb{H}^2, \quad h : \Delta_R \rightarrow \mathbb{R} \]

are both harmonic maps. We impose the condition that \( h \) is locally constant on \( \partial \Delta_R \), i.e., takes two different constant values on the two boundary components of this annulus; by translation we assume that these two values are 0 and \( h_0 > 0 \).

We now pass to the induced mappings from the universal cover \( M = \{ w = w_1 + iw_2 \in \mathbb{C} : 0 < w_2 < 1 \} \) of \( \Delta_R \). The (holomorphic) covering map is

\[ \varphi : M \rightarrow \Delta_R, \quad \varphi(w) = \exp(-i (\log R) w). \]

Now \( \hat{h} := h \circ \varphi : M \rightarrow \mathbb{R} \) is harmonic, and we normalize by assuming that \( \hat{h}|_{w_2=0} \equiv 0, \hat{h}|_{w_2=1} \equiv h_0 \). Since it is bounded in the lateral directions, \( \hat{h}(w) = h_0 \Im w \); this is the imaginary part of the holomorphic function \( z = h_0 w \) which, following notation in [6], we write as \( z = -\theta + it \). This is defined on the strip \( M_{h_0} = \{ w = w_1 + iw_2 \in \mathbb{C} : 0 < w_2 < h_0 \} \).
We now bring in the fact that $A$ is a surface of revolution, so we can write
\[ \hat{F}(\theta, t) := (F \circ \varphi \circ z^{-1})(\theta, t) = r(t) e^{is(\theta)} \]
for some smooth functions $s$ and $r$. The Hopf differential equals $\hat{\Phi} = \frac{1}{4} dz^2$. Since the pair $\left( \hat{F}, \hat{h} \right) = (\hat{F}, t)$ is conformal on $M_{h_0}$, we also have that
\[ \| \hat{F}_t \|_{\mathbb{H}^2}^2 + 1 = \| \hat{F}_\theta \|_{\mathbb{H}^2}^2, \]
which becomes
\[ \frac{4}{(1 - r(t)^2)^2} \left( s'(\theta)^2 r(t)^2 - r'(t)^2 \right) = 1. \]
Rearranging this we obtain
\[ s'(\theta)^2 = \frac{(1 - r(t)^2)^2}{4r(t)^2} + \left( \frac{r'(t)}{r(t)} \right)^2, \]
hence each of the two sides must equal the same constant $\kappa^2$. Write $s'(\theta) = \kappa > 0$ and
\[ \kappa^2 = \frac{(1 - r(t)^2)^2}{4r(t)^2} + \left( \frac{r'(t)}{r(t)} \right)^2. \]
The harmonicity of $\hat{F}$ yields the equation
\[ \hat{F}_{\bar{z}z} + 2(\log \rho \circ \hat{F})_u \hat{F}_z \hat{F}_{\bar{z}} = 0, \]
where $u$ is a (holomorphic) coordinate of $\mathbb{H}^2$ and the metric $d\rho^2$ on $\mathbb{H}^2$ equals $\rho(u)^2|du|^2$. We compute that
\[ 2(\log \rho \circ \hat{F})_u = 2\overline{\hat{F}}/(1 - |\hat{F}|^2), \]
and hence the left hand side of (2) becomes
\[ \frac{1}{4} e^{is(\theta)} \left( r''(t) - r(t) \left( s'(\theta)^2 - is''(\theta) \right) \right) + \frac{r(t)e^{is(\theta)} (r(t)^2 s'(\theta)^2 - r'(t)^2)}{2 (r(t)^2 - 1)} \]
Substituting from (2) and using that $s''(\theta) = 0$, we arrive at the expression
\[ \hat{F}_{\bar{z}z} + 2(\log \rho \circ \hat{F})_u \hat{F}_z \hat{F}_{\bar{z}} = \frac{e^{is(\theta)} (4r(t)r''(t) - 4r'(t)^2 + r(t)^4 - 1)}{16r(t)}. \]
We conclude finally that (3) is equivalent to
\[ 4r(t)r''(t) - 4r'(t)^2 + r(t)^4 - 1 = 0. \]
This is the equation obtained in a different way by Nelli and Rosenberg in [6].

\[ \left\{ (x, y, t) \in \mathbb{R}^3 : x^2 + y^2 < 1 \right\} \]
gives the catenoids in \( \mathbb{H}^2 \times \mathbb{R} \).

From the fact that (2) is a first integral of (5), it is easy to read off that solutions \( r_\kappa(t) \) are defined on the entire real line and are periodic in \( t \), oscillating between two values:

\[ \sqrt{\kappa^2 + 1} - \kappa \leq r_\kappa(t) \leq \sqrt{\kappa^2 + 1} + \kappa. \]

Noting that \( \sqrt{\kappa^2 + 1} - \kappa < 1 < \sqrt{\kappa^2 + 1} + \kappa \), we see that \( (\hat{F}, \hat{h}) \) extends to a map from \( \mathbb{C} \) to \( \mathbb{R}^3 \), with image a complete surface of revolution.
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which we write as $U_\kappa$; this has the conformal type of $C^\ast$. This family of surfaces is similar in many ways to the classical family of Delaunay unduloids.

**Proposition 3.1.** The surfaces $U_\kappa$ converge to the cylinder $\{ |z| = 1 \} \times \mathbb{R}$, as $\kappa \to 0$, and to a foliation of $\mathbb{R}^3$ by parallel planes as $\kappa \to \infty$. The connected components of $U_\kappa \cap \{ |z| < 1 \}$, which are all identified with one another by appropriate vertical translations, are copies of the standard catenoid of revolution, which we write as $C_\kappa$, in $\mathbb{H}^2 \times \mathbb{R}$, see Figure 1. The surface $C_\kappa$ is conformally equivalent to a proper annulus $\Delta_R$ where $R = R_\kappa \in (e^{-\pi}, 1)$. The height $h = h(\kappa)$ of $C_\kappa$ decreases monotonically from $\pi$ to $0$ as $\kappa$ increases from $0$ to $\infty$.

4. PARABOLIC CATENOIDS

We next consider a family of surfaces obtained via a particular limit of horizontal dilations of the catenoids $C_\kappa$.

For any point $p \in \mathbb{H}^2$ and $h \in (0, \pi)$, let $C_{h,p}$ denote the catenoid in $\mathbb{H}^2 \times \mathbb{R}$ which is rotationally symmetric around the axis $\{ p \} \times \mathbb{R}$, symmetric with respect to reflections across $t = 0$ and has height $h \in (0, \pi)$. Observe that $C_{h(\kappa),p}$ is obtained applying an horizontal dilation to $C_\kappa$.

Now take a sequence of these catenoids, $C_j := C_{h_j,p_j}$ such that $p_j \to q \in \partial \mathbb{H}^2$ and $h_j$ remains bounded away from both $0$ and $\pi$. Then $C_j$ converges locally in $C^\infty$ on any compact set of $\mathbb{H}^2 \times \mathbb{R}$ to two horizontal disks $\mathbb{H}^2 \times \{ t_j \}$, $j = 1, 2$, where $|t_2 - t_1| = \lim h_j$, see Figure 2. (If $h_j \to 0$, then $C_j$ converges to one horizontal disk with multiplicity two.)

![Figure 2. The upper half of three surfaces in the sequence $C_j$. The limit is the union of two horizontal disks.](image)

Suppose however that we let $h_j \uparrow \pi$. Depending on the rates at which $p_j \to q$ and $h_j \to \pi$, various possibilities can occur. We suppose
that these sequences are balanced in such a way that $C_j$ intersects a fixed compact set $K \subset \mathbb{H}^2 \times \mathbb{R}$ for every $j$; this is easy to arrange by an elementary argument. In this case, standard results imply that some subsequence of the $C_j$ converges locally in $\mathcal{C}^\infty$ to a complete properly embedded minimal surface which we write as $\mathcal{D}$. By definition, $\mathcal{D}$ is a parabolic catenoid. It has asymptotic boundary equal to the union of two horizontal circles separated by distance $\pi$, together with a vertical segment joining these two circles. This class of surfaces was discovered independently by Hauswirth [3] and Daniel [1].

We can finesse the construction of the sequence $C_j$: start with a sequence of catenoids $C_{\kappa_j}$, rotationally symmetric around the axis \{o\} $\times$ $\mathbb{R}$, with $\kappa_j \searrow 0$, and lying in the slab $|t| < \frac{1}{2} h_j$. Let $\gamma$ denote the hyperbolic geodesic through $o$ and converging to $q \in \partial \mathbb{H}^2$, and let $\sigma_j$ denote a hyperbolic dilation toward $q$ along this geodesic which has the property that, extending $\sigma_j$ to an isometry of $\mathbb{H}^2 \times \mathbb{R}$ which acts only on the first factor, $\sigma_j(C_{\kappa_j})$ is tangent to the axis \{o\} $\times$ $\mathbb{R}$ at the point $(o,0)$ (i.e., on the central circle of the catenoid), with the neck of the catenoid lying between this point and $q$.

![Figure 3](image)

**Figure 3.** The upper half of four surfaces in the sequence $\sigma_j(C_{\kappa_j})$. The limit as $\kappa_j \searrow 0$ is the parabolic catenoid.

This sequence certainly intersects a fixed compact set for all $j$, hence converges to a parabolic catenoid which we denote by $\mathcal{D}_q$, see Figure 3. Its height is precisely equal to $\pi$. The symmetries of the catenoid
and the naturality of the construction easily imply that these various Daniel surfaces are all related to one another by rotations of $\mathbb{H}^2 \times \mathbb{R}$ around the axis $\{o\} \times \mathbb{R}$. We can also apply horizontal hyperbolic dilations to these surfaces, which is the same as choosing a slightly different normalization of the sequence of surfaces $C_j$ above; denoting the dilation parameter by $\lambda$, we obtain the family of surfaces $D_{q,\lambda}$; $\lambda = 1$ corresponds to the identity dilation. The surfaces in this entire family are all mutually isometric by rotations and hyperbolic dilations. We take as the standard model the surface in this family with $\lambda = 1$. It is an embedded disk with asymptotic boundary the two horizontal circles $S^1 \times \{0\} \sqcup S^1 \times \{\pi\}$ and the vertical segment $\{q\} \times [0, \pi]$.

**Figure 4.** Two different views of the surface in $\mathbb{R}^3$ whose intersection with the solid cylinder $\{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ is the parabolic catenoid.

**Remark 4.1.** If one applies the limit process describe above to the entire unduloid-type surfaces $U_\kappa$, one obtains a complete limiting surface $Q$ in $\mathbb{R}^3$. This limit (see Figure 4) is determined by

$$Q = \{(x, y, t) \in \mathbb{R}^3 : 1 - x^2 - y^2 = \cos(t)((1 + x)^2 + y^2)\}.$$

The components of the intersection

$$Q \cap \{(x, y, t) \in \mathbb{R}^3 : 1 - x^2 - y^2 > 0\}$$

are an infinite union of (translated) copies of the parabolic catenoid.

The simplest analytic representation of $D_{q,\lambda}$ uses the upper half-space model for $\mathbb{H}^2$. Place $q$ at infinity in $\mathcal{H}$ and use the standard coordinates $(x, y, t)$, $\mathcal{H} \times \mathbb{R}$, where $y > 0$. The parabolic catenoid
is invariant under parabolic translations, which in this representation take the form \((x, y, t) \mapsto (x + b, y, t)\) for any \(b \in \mathbb{R}\), and it also intersects each horizontal slice \(t = \text{const}\). transversely. Therefore this surface is a sweep-out of some curve \((0, f(t), t), 0 < t < \pi\) by these parabolic translations. Searching for a minimal surface with these properties leads in a straightforward way to the family of solutions \(f(t) = \lambda \sin t\) for any \(\lambda \in \mathbb{R}^+\). The surface is the image of corresponding family of embeddings of the strip \(M_\pi = \mathbb{R} \times (0, \pi)\) given by

\[
\Psi_\lambda(x, t) = (\lambda x, \lambda \sin t, t).
\]

For simplicity, we write \(\Psi = \Psi_1\).

5. Tall Rectangles

The final family of surfaces in \(\mathbb{H}^2 \times \mathbb{R}\) we consider here is the family of properly embedded minimal disks, described by Sa-Earp and Toubiana in [7]. These have ideal boundary consisting of two parallel arcs \(\sigma \times \{ \pm h/2 \}\), where \(h > \pi\) is arbitrary and \(\sigma\) is an arc in \(\partial \mathbb{H}^2\) with endpoints \(q_1\) and \(q_2\), together with the vertical segments \(\{q_1\} \times [-h/2, h/2]\) and \(\{q_2\} \times [-h/2, h/2]\), see Figure 5. Each of these surfaces, denoted \(\Sigma_{\sigma, h}\), is area minimizing. (A complete surface is area minimizing if any compact piece is area-minimizing among all the surfaces with the same boundary.) The intersections \(\Sigma_{\sigma, h} \cap (\mathbb{H}^2 \times \{ t \})\) foliate \(\Sigma_{\sigma, h}\) by a family of curves which, if all projected down to \(\mathbb{H}^2\), all have the same endpoints \(q_1\) and \(q_2\) and are equidistant to the geodesic \([q_1, q_2]\).

Following [7], we construct these surfaces as follows. Using the Poincaré disk model, consider the vertical plane \(\mathcal{P} = \gamma \times \mathbb{R}\) where \(\gamma\) is the geodesic \(\{ \text{Im} z = 0 \} \subset \mathbb{D}\), parametrized either by \(x \in (-1, 1)\) or by signed geodesic distance \(\rho\) from \(\{0\}\). The relationship between the two parameters is \(x = \tanh(\rho/2), \rho = \log(1 + x)/2\). Next, fix \(0 < d < 1\) and consider the curve \(\sigma_d \subset \mathcal{P}\) given as a bigraph of the two functions \(t = \pm \lambda_d(\rho)\), i.e.,

\[
\sigma_d = \{(\rho, \pm \lambda_d(\rho)) : \rho \geq \cosh^{-1}(1/d)\}.
\]

We then determine the conditions under which the surface swept out by this curve with respect to horizontal hyperbolic dilations along the geodesic from \(-i\) to \(i\) is minimal. A standard computations shows that minimality is equivalent to

\[
\frac{d\lambda_d}{d\rho}(\rho) = \frac{1}{\sqrt{d^2 \cosh^2 \rho - 1}}.
\]
(N.B. the treatment in [7] uses \( e = 1/d \) as a parameter instead.) By the chain rule,

\[
\frac{d\lambda_d}{dx} = \frac{2}{\sqrt{d^2(1 + x^2)^2} - (1 - x^2)^2}.
\]

The lower bound \( \rho > \cosh^{-1}(1/d) \) transforms to \( x > d_1 := \frac{\sqrt{1 - d}}{\sqrt{1 + d}} \). Note that the derivative is infinite at \( x = d_1 \), so this graph together with its reflection cross the \( x \)-axis is at least \( C^1 \). A closer analysis shows that this bigraph is in fact \( C^\infty \).

In summary, the complete properly embedded minimal disk \( \Sigma_d \) is the surface swept out by the curve \( \sigma_d \), where

\[
\lambda_d(x) := \int_{d_1}^{x} \frac{2 \, dv}{\sqrt{d^2(1 + v^2)^2} - (1 - v^2)^2}.
\]
It is straightforward to check that:

\[
\lambda_d(x) = -\frac{2}{1-d} \text{Im} \left( F \left( \arcsin \left( \frac{d_1 x}{d} \right) \right) \right),
\]

where

\[
F(\phi|z) := \int_0^\phi (1 - z \sin^2(\theta))^{-1/2} d\theta, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}
\]

is the classical elliptic integral of the first kind. The surface itself is parametrized as a bigraph

\[
\Sigma_d : (d_1, 1) \times (-1, 1) \to \mathbb{H}^2 \times \mathbb{R},
\]

(10)

\[
\Sigma_d(x, y) = \left( \frac{x - xy^2}{x^2y^2 + 1}, \frac{x^2 + 1}{x^2y^2 + 1}, \pm \lambda_d(x) \right).
\]

Note that the top and bottom halves \(\Sigma^\pm_d = \Sigma_d \cap \{ \pm t \geq 0 \}\) are each graphs over the “lunette” region \(L_d \subset \mathbb{D}\) lying between the circular arc \(\gamma_d\) passing through \(\pm i\) and \(d_1\), and the arc on the circumference joining \(-i\) to \(i\) and passing through \(1\).

The curve \(\sigma_d\) and surface \(\Sigma_d\) are contained in the region where

\[
|t| < \frac{1}{2} h_d = \int_{d_1}^1 \frac{2 \, dv}{\sqrt{d^2(v^2 + 1)^2 - (v^2 - 1)^2}} < \infty.
\]

It is not hard from this expression to check that the height \(h_d\) of \(\Sigma_d\) increases monotonically in \(d\) and with the following asymptotic behavior:

- As \(d \to 0\), \(h_d \searrow \pi\) and \(\Sigma_d\) diverges to infinity. Denoting by \(\tilde{T}_d\) the horizontal dilation along \(\gamma\) which maps the point \(d_1\) to \(0\), then the family of disks \(Y_d := (\tilde{T}_d \times \text{Id}_\mathbb{R})(\Sigma_d)\) converges to the parabolic catenoid passing through the origin.
- As \(d \to 1\), \(h_d \to \infty\) and \(\Sigma_d\) limits to the vertical plane \((-i, i) \times \mathbb{R}\).

Observe that, for a fixed \(d \in (0, 1)\), the parametrization \(\Sigma_d\) is defined on \((d_1, 1/d_1) \times (-1, 1)\) and the image of this extension is an annulus in \(\mathbb{R}^3\), see Figure 6. Applying an 180°-degree rotation around the line (geodesic) passing through the points \((-i, h_d/2)\) and \((i, h_d/2)\), then we get another annulus of height \(2h_d\) which is the fundamental piece of a singly periodic surface in \(\mathbb{R}^3\). The intersection of this periodic surface with the solid cylinder \(\{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 < 1\}\) consists of an infinite number copies of the tall rectangle \(\Sigma_d\), see Figure 7.
6. JACOBI FIELDS ON PARABOLIC CATENOIDS

In the remaining sections of this paper we initiate the study of the Jacobi operator on a parabolic catenoid. This is a necessary step before studying the broader space of minimal disks with a similar asymptotic structure. In this section we consider the Jacobi fields on $\mathcal{D}$ which are generated by deformations into catenoids or tall rectangles. The next section is a brief introduction to some more general aspects of the analysis of the Jacobi operator.

6.1. Deformations to catenoids. We first analyze the deformations of $\mathcal{D}$ into catenoids $C_{q,h}$. A thorough analysis of the Jacobi operator and local deformation theory for catenoids $C_\kappa$ appears in our earlier paper [2], but we consider here this limiting case and find an explicit expression for the Jacobi field on the parabolic catenoid arising from the ‘regeneration’ of this surface to the degenerating sequence of

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{The annulus $\Upsilon_d((d_1, 1/d_1) \times (-1, 1))$.}
\end{figure}
Figure 7. Tall rectangles can also be seen as the intersection of a cylindrical surface of $\mathbb{R}^3$ with the cylinder \( \{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 < 1\} \).

catenoids. This is a bit complicated because the rotational symmetry of the catenoids is lost in the limit. For this reason, we do all computations in the rectangular coordinates \((x, y, t)\) on $\mathcal{H} \times \mathbb{R}$.

Recall from Section 3 the conformal parametrization

\[
X_\kappa(\theta, t) = (r_\kappa(t)e^{i\sqrt{\kappa}\theta}, t), \quad \kappa > 0,
\]

where

\[
\kappa = \frac{(1 - r_\kappa(t)^2)^2}{4r_\kappa(t)^2} + \left(\frac{r'_\kappa(t)}{r_\kappa(t)}\right)^2.
\]

We choose the solution for which \( r < 1 \) when \( t \in \left(-\frac{h_\kappa}{2}, \frac{h_\kappa}{2}\right) \). We also know that

\[
\min r_\kappa := r_0 = r_\kappa(0) = \sqrt{\kappa + 1} - \sqrt{\kappa} \leq r_\kappa(t) < 1.
\]
Now write the catenoid as a bigraph over the planar annulus $A(r_0, 1)$, see Figure 8 with $t$ a function of $r$. Then

$$\kappa = \frac{(1 - r^2)^2}{4r^2} + \left( \frac{1}{r} \frac{t'(r)}{t'(r)} \right)^2,$$

or equivalently, $t'(r) = 2(4\kappa r^2 - (1 - r^2)^2)^{-1/2}$, whence

$$t_\kappa(r) = \int_{r_0}^{r} \frac{2 \, du}{\sqrt{4\kappa u^2 - (1 - u^2)^2}}.$$

Before differentiating this with respect to $\kappa$, it is convenient to set $u = (r - r_0)s + r_0$, which yields

$$t_\kappa(r) = 2(r - r_0) \int_{0}^{1} \frac{ds}{\sqrt{4\kappa((r - r_0)s + r_0)^2 - (1 - ((r - r_0)s + r_0)^2)^2}}.$$
We next transform to the upper half-space times $\mathbb{R}$, using the conformal diffeomorphism $g : \mathbb{D} \to \mathcal{H}$ from the preliminaries. Write

$$\mu_0 := -ig(r_0) = \frac{1 + \sqrt{\kappa} - \sqrt{1 + \kappa}}{1 - \sqrt{\kappa} + \sqrt{1 + \kappa}} \in (0, 1).$$

We obtain a representation of $C_\kappa$ as a bigraph over the complement of a disk in the half-plane

$$\Omega_\kappa := \mathcal{H} \setminus D(i(\mu_0 + \mu_0^{-1})/2, (\mu_0^{-1} - \mu_0)/2) \subset \mathcal{H},$$

where $D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$.

Next apply the horizontal dilation $T_{1/\mu_0}$ on $\mathcal{H}$ which carries $z \mapsto 1/\mu_0 z$, and write $\tilde{C}_\kappa := (T_{1/\mu_0} \times \text{Id}_\mathbb{R})(C_\kappa)$. As described earlier, the parabolic catenoid is the limit of the family $\tilde{C}_\kappa$, as $\kappa \to 0$. Note that

$$\lim_{\kappa \to 0} T_{1/\mu_0}(\Omega_\kappa) = \tilde{M} = \{z = x + iy \in \mathcal{H} : 0 < y \leq 1\}.$$

Given a point $z \in \tilde{M}$, set $r = ||g^{-1}(T_{\mu_0}(z))||$; the corresponding point on the dilated catenoid is $(x, y, t_\kappa(r))$. This gives the family of minimal immersions

$$\Phi_\kappa(x, y) = (x, y, t_\kappa(x, y)).$$

To compute $d\Phi_\kappa/d\kappa|_{\kappa=0}$, it is first necessary to compute $dt_\kappa/d\kappa|_{\kappa=0}$. Write the integral formula for $t_\kappa$ above as $t_\kappa(x, y) = \int_0^1 G(x, y, s, \kappa)ds$ and expand $G(x, y, s, \kappa)$ in powers of $\sqrt{\kappa}$:

$$G(x, y, s, \kappa) = a_0(x, y, s) + a_1(x, y, s)\sqrt{\kappa} + a_2(x, y, s)\kappa + o(\kappa^{3/2}).$$

These first few coefficients are given by

$$a_0(x, y, s) = \frac{1 - y}{\sqrt{s(1 - y)(2 - s + sy)}},$$

$$a_1(x, y, s) = -s^2y^3 + 3s^2y^2 - 3s^2y + s^2 - 3sy^2 + 6sy - 3s + y^2 - 2y + 1$$

$$2(sy - s + 2)\sqrt{s(1 - y)(sy - s + 2)}$$

and

$$a_2(x, y, s) = -2s^4(y - 1)^5 + 2s^3(y - 5)(y - 1)^4 + s^2(10y - 13)(y - 1)^3$$

$$8(sy - s + 2)^2\sqrt{s(1 - y)(sy - s + 2)}$$

$$+ 2s(y - 1)(y(x^2 + 8y - 6) - 2) + y(4x^2 + (5 - 3y)y + 7) - 9$$

$$8(sy - s + 2)^2\sqrt{s(1 - y)(sy - s + 2)}.$$
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A straightforward but increasingly tedious computation gives

\begin{align*}
(12) \quad \int_0^1 a_0(x, y, s) \, ds &= \arccos y, \\
(13) \quad \int_0^1 a_1(x, y, s) \, ds &= 0, \\
(14) \quad \int_0^1 a_2(x, y, s) \, ds &= \frac{1}{4} \left( \frac{x^2 y}{\sqrt{1 - y^2}} - \arccos y \right).
\end{align*}

These yield

\[ \lim_{\kappa \to 0} \Phi_\kappa(x, y) = \Phi_0(x, y) = (x, y, \arccos y), \]

and

\[ \left. \frac{d\Phi_\kappa}{d\kappa} \right|_{\kappa=0} (x, y) = \left( 0, 0, \frac{1}{4} \left( \frac{x^2 y}{\sqrt{1 - y^2}} - \arccos y \right) \right). \]

Finally, if we parametrize half of the parabolic catenoid as $\Phi_0(x, y) = (x, y, \arccos y)$, then the Gauss map is given by $\nu(x, y) = (0, -y^2, -\sqrt{1 - y^2})$. Hence, the Jacobi field that we are looking for is

\[ w(x, y) = \nu \cdot \left( \left. \frac{d\Phi_\kappa}{d\kappa} \right|_{\kappa=0} \right) = \frac{1}{4} (\sqrt{1 - y^2} \arccos y - x^2 y). \]

Using the parametrization $\hat{F} : \mathbb{R} \times (-\pi/2, \pi/2) \to \mathbb{H}^2 \times \mathbb{R}$,

\[ \hat{F}(x, t) = (x, \cos t, t), \]

then this Jacobi field equals $w(x, t) = \frac{1}{4} (t \sin t - x^2 \cos t)$.

Remark 6.1. The parabolic catenoid in (6) and the one above differ by the vertical translation $t \mapsto t - \pi/2$. The Jacobi field on (6) is

\[ w(x, t) = \frac{1}{8} \left( (\pi - 2t) \cos(t) - 2x^2 \sin(t) \right). \]

6.2. Deformations to tall rectangles. We next compute the Jacobi field on a parabolic catenoid associated to the variation of this surface into the family of tall rectangles. More specifically, we compute the variation associated to the family $Y_d$ defined at the end of section 5 which converge to a given parabolic catenoid $D$.

Recall the parametrization (10) for $\Sigma_d$. The top and bottom halves of $\Sigma_d$ are graphs over the lunette $L_d \subset D$ between the circular arc $\gamma_d$ passing through $\pm i$ and $d_1$, and the boundary arc $\gamma_1$ passing through $\pm i$, 1. Set $\hat{L}_d = g(L_d)$; this is the region in the half-plane $\mathcal{H}$ between
the segment \([-1,1]\) and the circular arc passing through \(\pm 1\) and \(\mu_1\), where

\[
\mu_1 = i \frac{\sqrt{1+d} - \sqrt{1-d}}{\sqrt{1+d} + \sqrt{1-d}}
\]

Defining \(T_s\) as before, then the limit of the domains \(T_{1/\mu_1}(\hat{L}_d)\) as \(d \to 0\) is the strip \(M\) where \(0 < y < 1\).

Given \(z = (x,y) \in M\), \(z \in T_{1/\mu_1}(\hat{L}_d)\) for all \(d < d'\) if \(d'\) is small enough. The transformation of \(\Sigma_d\) is parametrized by the function

\[
G_d(x,y) = (x,y,\lambda_d(X)),
\]

where \(X = X(x,y,d)\) is the positive solution to the quadratic system:

\[
\begin{align*}
\frac{X(1-Y^2)}{1+X^2Y^2} &= \mu_1 x \\
\frac{Y(1+X^2)}{1+X^2Y^2} &= \mu_1 y
\end{align*}
\]

Now recall the formula \(\lambda_d(X)\) for \(\Lambda_d(X)\), and change variables in it by setting \(r = (X - d_1)s + d_1\). This gives

\[
\lambda_d(X) = \int_0^1 H(x,y,s,d) ds.
\]

where

\[
H(x,y,s,d) = \frac{2(X - d_1)ds}{\sqrt{d^2((X - d_1)s + d_1)^2 + 1)} - ((X - d_1)s + d_1)^2 - 1)^2}
\]

Now proceed as in the previous subsection by expanding

\[
H(x,y,s,\sqrt{d}) = h_0(x,y,s) + h_1(x,y,s)\sqrt{d} + h_2(x,y,s)d + o(d^{3/2}),
\]

where

\[
\begin{align*}
h_0(x,y,s) &= \frac{1-y}{\sqrt{s(1-y)(2-s+sy)}}, \\
h_1(x,y,s) &= \frac{(1-y)^{3/2}(s^2y-s^2+3s-1)}{2\sqrt{s(sy-s+2)^{3/2}}},
\end{align*}
\]

and

\[
\begin{align*}
h_2(x,y,s) &= \frac{-2s^4(y-1)^5 + 2s^3(y-5)(y-1)^4 + s^2(10y-17)(y-1)^3}{8(s(y-1)+2)^2\sqrt{s(1-y)(s(y-1)+2)}} + \\
&\frac{2s(y-1)(-(x^2+14)y+8y^2+6) - y(4x^2+y(3y-5)+9) + 7}{8(s(y-1)+2)^2\sqrt{s(1-y)(s(y-1)+2)}}.
\end{align*}
\]
By a straightforward computation,
\[ \int_0^1 h_1(x, y, s)ds = 0, \quad \text{and} \]
\[ \int_0^1 h_2(x, y, s)ds = \frac{1}{4} \left( \arccos(y) - \frac{x^2y}{\sqrt{1 - y^2}} \right). \]
Therefore, just as at the end of the previous subsection, the corresponding Jacobi field is
\[ \hat{w}(x, y) = -\frac{1}{4} \left( \sqrt{1 - y^2} \arccos y - x^2y \right). \]
Note the unexpected fact that this is equal to \(-w(x, y)\), where \(w\) is the Jacobi field associated to the deformation of \(\mathcal{D}\) into catenoids.

7. The Jacobi operator on parabolic catenoids

We finally turn to the task of describing the beginnings of the analytic theory of the Jacobi operator on the parabolic catenoid \(\mathcal{D}_{\infty, 1}\) given by parametrization (6) for \(\lambda = 1\).

**Coordinate vector fields and metric:** Via the parametrization \(\Psi\), the coordinates \((x, t)\) on \(\Sigma\) induce the coordinate vector fields
\[ \Psi_*(\partial_x) = X_1 = (1, 0, 0), \quad \Psi_*(\partial_t) = X_2 = (0, \cos t, 1). \]
The metric coefficients are
\[ g_{11} = X_1 \cdot X_1 = 1/\sin^2 t, \quad g_{12} = X_1 \cdot X_2 = 0, \]
and \[ g_{22} = X_2 \cdot X_2 = (\cos^2 t/\sin^2 t) + 1 = 1/\sin^2 t, \]
(thus displaying the conformality of \(\Psi\)).

**Jacobi operator:** The unit normal to \(\Sigma\) at \(\Psi(x, t)\) is
\[ \nu(x, t) = (0, \sin^2 t, -\cos t), \]
whence \(\text{Ric}_{\mathbb{H}^2 \times \mathbb{R}}(\nu, \nu) = -g_{\mathbb{H}^2}((0, \sin^2 t), (0, \sin^2 t)) = -\sin^2 t\). To compute the Jacobi operator we can avoid computing \(|A|^2\) directly by the following observation.

We first compute the Jacobi field corresponding to the family \(\lambda \mapsto \Sigma_{\infty, 0, \lambda}\). Indeed,
\[ \frac{d}{d\lambda}\bigg|_{\lambda=1} \Psi_\lambda(x, t) = (x, \sin t, 0), \]
so the normal component of this, which is the Jacobi field we seek, equals
\[ \psi = \nu \cdot (x, \sin t, 0) = \sin t. \]
This vanishes simply at both $t = 0$ and $t = \pi$ and is $L^2$ on any portion $|x| \leq C$ since the measure equals $dxdt/\sin^2 t$, but is not $L^2$ on $\Sigma$. Now, $\Delta_g = \sin^2 t(\partial_x^2 + \partial_t^2)$ and the Jacobi operator equals

$$L = - (\Delta_g + \text{Ric}(\nu, \nu) + |A|^2).$$

Writing out the equality $L\psi = 0$ with $\psi = \sin t$, we obtain

$$\sin^2 t(- \sin t) + (- \sin^2 t + |A|^2)\sin t = 0 \Rightarrow |A|^2 = 2 \sin^2 t$$

This shows that $L = - \sin^2 t(\partial_x^2 + \partial_t^2 + 1)$.

This is a nonnegative operator: indeed, $L = \sin^2 tL_0$, where $L_0 = -\partial_x^2 - \partial_t^2 - 1$, and its action on $L^2((\sin t)^{-2}dxdt)$ is equivalent to the action of $L_0$ on $L^2(dxdt)$ with Dirichlet boundary conditions at $t = 0, \pi$. This latter operator is nonnegative since $-\partial_t^2 - 1$ with Dirichlet conditions is nonnegative on $[0, \pi]$. The fact that the spectrum of $L$ lies in $\mathbb{R}^+$ also follows from the existence of the nonnegative solution $\psi$ to $L\psi = 0$.

The function $\tilde{u}(x, t) = x\sin t$ is another non-$L^2$ solution to $L\tilde{u} = 0$.

It is not hard to show that $u$ and $\tilde{u}$ span the full space of tempered solutions to the Jacobi equation which vanish at $t = 0, \pi$.

**Mapping properties of the Jacobi operator:** We next describe some aspects of the mapping properties of $L$ on the infinite strip $S = \mathbb{R} \times [0, \pi]_t$. The remarks here are meant to be preparatory to a deeper study of the deformation theory of tall rectangles, to which we shall return in a work in progress. We consider two main questions:

i) Find classes of functions $\phi_{\pm}(x)$ such that the problem $Lu = 0$ on $S$, $u(x, \pi) = \phi_{\pm}(x)$, $u(x, 0) = \phi_{\pm}(x)$ is solvable;

ii) Find a class of functions $f$ on $S$ for which we can solve $Lu = f$ with $u = 0$ at $t = 0, \pi$, and $u \to 0$ as $x \to \pm \infty$.

We analyze these questions using the Fourier transform in $x$. Writing the dual variable as $\xi$, then question i) leads to the study of the two families of problems

$$\hat{L}_{\xi}\hat{u}(\xi, t) := \sin^2 t(-\partial_t^2 + \xi^2 - 1)\hat{u} = 0,$$

$$\hat{u}(\xi, \pi) = \hat{\phi}_+(\xi), \quad \hat{u}(\xi, 0) = \hat{\phi}_-(\xi)$$

and

$$\hat{L}_{\xi}\hat{u}(\xi, t) = \hat{f}(\xi, t), \quad \hat{u}(\xi, 0) = \hat{u}(\xi, \pi) = 0.$$

Consider (15) first. For $\xi \neq 0$, there exist two functions $v_{\pm}(\xi, t)$ which satisfy $L_{\xi}v_{\pm}(\xi, t) = 0$ and

$$v_+(\xi, 0) = 0, \quad v_+(\xi, \pi) = 1, \quad v_-(\xi, 0) = v_-(\xi, \pi) = 0,$$
FAMILIES OF MINIMAL SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$ FOLIATED BY ARCS

namely

\[
\begin{align*}
v_+(\xi, t) &= \sin((1 - \xi^2)^{1/2}t) / \sin((1 - \xi^2)^{1/2}\pi), \\
v_-(\xi, t) &= \sin((1 - \xi^2)^{1/2}(\pi - t)) / \sin((1 - \xi^2)^{1/2}\pi)
\end{align*}
\]

when $|\xi| < 1$,

\[
v_+(\pm 1, t) = t / \pi, \quad v_-(\pm 1, t) = 1 - t / \pi,
\]

and

\[
\begin{align*}
v_+(\xi, t) &= \sinh((\xi^2 - 1)^{1/2}t) / \sinh((\xi^2 - 1)^{1/2}\pi), \\
v_-(\xi, t) &= \sinh((\xi^2 - 1)^{1/2}(\pi - t)) / \sinh((\xi^2 - 1)^{1/2}\pi)
\end{align*}
\]

for $|\xi| > 1$. These functions are clearly holomorphic when $\xi \in \mathbb{C} \setminus \{0, \pm 1\}$, but although they appear to be branched at $\xi = \pm 1$, they are single-valued at these points so are holomorphic away from $\xi = 0$, where they have a double pole.

Now suppose, for example, that $\hat{\phi}_\pm(\xi)$ are functions in $L^1$ such that $\phi_\pm(\xi) \xi^{-2} \in L^1_{\text{loc}}$. Then the solution to problem i) is

\[
u(x, t) = \mathcal{F}^{-1} \left( \hat{\phi}_+(\xi)v_+(\xi, t) + \hat{\phi}_-(\xi)v_-(\xi, t) \right),
\]

where $\mathcal{F}$ is the Fourier transform. It is not hard to check that under these hypotheses, $u(x, t)$ is continuous on $\mathbb{R} \times [0, \pi]$, $u(x, t) \to 0$ uniformly in $t \in [0, \pi]$ as $x \to \pm \infty$, and furthermore, that

\[
\int_{\mathbb{R}} u(x, \pi) \, dx = \int_{\mathbb{R}} u(x, 0) \, dx = 0.
\]

It is clearly possible to choose functions $\phi_\pm$ satisfying these constraints but so that $u(x, \pi) \neq u(x, 0)$ for every $x \in \mathbb{R}$. This implies that there are infinitesimal deformations where the difference of the heights of the two boundary curves may vary, though the average of the difference of their heights equals $\pi$.

There are also some interesting constraints on Jacobi fields. Indeed, let

\[S_r = \{ \Psi(x, t) : (x, t) \in [-r, r] \times [0, \pi] \}\]

denote the truncated surface. Consider the basic Jacobi field $u(x, t) = \sin t$, and suppose that $v$ is any other Jacobi field, i.e., $Lv = 0$, which has sufficient decay as $|x| \to \infty$ for the following computations to make sense (such Jacobi fields certainly exist by virtue of the preceding
calculations.) We then compute that
\[
0 = \int_{S_r} ((Lu)w - u(Lw)) \frac{1}{\sin^2 t} \, dx \, dt = \\
\int_{-r}^{r} (w(x, \pi) + w(x, 0)) \, dx + \int_{0}^{\pi} \sin t \left( w_x(r, t) - w_x(-r, t) \right) \, dt.
\]

We conclude that Jacobi fields (at least the well-behaved ones) must satisfy the ‘moment condition’
\[
\int_{-\infty}^{\infty} \left( w(x, \pi) + w(x, 0) \right) \, dx + \int_{0}^{\pi} \sin t \left( w_x^+(t) - w_x^-(t) \right) \, dt = 0,
\]
where \( w_x^\pm(t) := \lim_{x \to \pm \infty} w_x(x, t) \). The precise geometric meaning of this is not evident.

To convert these infinitesimal statements into statements about minimal surfaces near to \( \Sigma_{q,\tau,\lambda} \), it is necessary to solve the inhomogeneous problem ii). The details of this proceed in an unsurprising fashion: passing to the Fourier transform again, there is a Green function \( \hat{G} (\xi, t, t') \) for \( \hat{L_\xi} \) for \( \xi \neq 0 \), and this can be used to solve \( Lu = f \) for a broad collection of functions \( f \). By this linear theory and standard use of the implicit function theorem, the Jacobi fields discussed earlier can be integrated to nearby minimal surfaces.

The point of all of this is the following. There exist deformations of \( \Sigma_{q,\tau,\lambda} \) which deform the top and bottom boundary curves to \( \Gamma_\pm \), but which fix the vertical line connecting them. Although it might be natural to conjecture that any such deformation has top and bottom boundary separated at exactly distance \( \pi \), i.e., \( \Gamma_+(x) - \Gamma_-(x) = \pi \) for all \( x \), we have shown that this is not true.

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