Brownian motion can feel the shape of a drum

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Abstract

We study the scenery reconstruction problem on the $d$-dimensional torus, proving that a criterion on Fourier coefficients obtained by Matzinger and Lember (2006) for discrete cycles applies also in continuous spaces. In particular, with the right drift, Brownian motion can be used to reconstruct any scenery. To this end, we prove an injectivity property of an infinite Vandermonde matrix.

Keywords: Scenery reconstruction problem; infinite Vandermonde matrix; Brownian motion.

1 Introduction

1.1 Background

In its most general formulation, the scenery reconstruction problem asks the following: Let $C$ be a set, let $f$ be a function on $C$, and $(X_t)_{t \geq 0}$ a stochastic process taking values in $C$. What information can we learn about $f$ from the (infinite) trace $f(X_t)_{t \geq 0}$? Can $f$ be completely reconstructed from this trace?

In one of the most common settings, $C$ is taken to be the discrete integer graph $\mathbb{Z}$, the function $f$ maps $C$ to $\{0, 1\}$, and $X_t$ is a discrete-time random walk. For this model, numerous results exist in the literature for a variety of cases, e.g reconstruction when $f$ is random [1] and when $f$ is periodic. In the latter case, $f$ is essentially defined on a cycle of length $\ell$. Matzinger and Lember showed the following:

Theorem 1 ([5, Theorem 3.2]). Let $f$ be a 2-coloring of the cycle of length $\ell$, and let $X_t$ be a random walk with step distribution $\gamma(x)$. If the Fourier coefficients $\{\hat{f}(k)\}_{k=0}^{\ell-1}$ are all distinct, then $f$ can be reconstructed from the trace $f(X_t)$.

Finucane, Tamuz and Yaari [3] considered the problem for finite Abelian groups, and showed that in many such cases, the above condition on the Fourier coefficients is necessary.

The problem can also be posed for a continuous space $C$, such as $\mathbb{R}^d$ or the torus $\mathbb{T}^d$, with $X_t$ a continuous-time stochastic process. Here there has been considerably less work; to the best of our knowledge, at the time of writing this paper there are only two published results: Detecting “bells” [7] and reconstructing iterated Brownian motion [2]. See [4, 6] and references therein for an overview of the reconstruction problem, with a focus on $\mathbb{Z}$ and $\mathbb{Z}^d$.

1.2 Results

In this paper, we extend Theorem 1 from the discrete cycle to the continuous $d$-dimensional torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$. The discrete-time random walks are replaced by continuous-time processes $X_t$ (such as Brownian motion), and the 2-colorings are replaced by the indicators $f$ of open sets. For an example of how the sample paths $f(X_t)$ might look like, see Figure 1, where $f$ is the indicator of a union of three intervals on the circle, and $X_t$ is Brownian motion. The goal is to reconstruct the size and position of the intervals, up to rotations, from the trace $f(X_t)$.

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Figure 1: **Left**: Scenery reconstruction in one dimension. The black curve is a polar depiction of one-dimensional Brownian motion, with $\theta = X_t$ and $r = \sqrt{t}$ (so that points near the center represent times close 0, and points near the edge represent larger times). The function $f$ is represented by the three shaded sectors. **Right**: The trace $f(X_t)$. It is equal to 1 precisely when the curve on the left is inside one of the shaded sectors.

**Definition 2.** Let $\mathcal{F}$ be a family of functions on the torus. The family $\mathcal{F}$ is said to be **reconstructible** by $X_t$ if there is a function $A : \mathbb{R} \rightarrow \mathbb{T}^d$ such that for every $f \in \mathcal{F}$, with probability 1 there exists a (random) shift $\theta \in \mathbb{T}^d$ such that $A(f(X_t))(x) = f(x + \theta)$ for almost all $x$.

For the particular case of $d = 1$, we also deal with reconstruction up to reflections:

**Definition 3.** The family $\mathcal{F}$ is said to be **reconstructible up to reflections** by $X_t$, if for all $f \in \mathcal{F}$, with probability 1 either $A(f(X_t))(x) = f(x + \theta)$ for almost all $x$, or $A(f(X_t))(x) = f(-x + \theta)$ for almost all $x$.

In order to analyze the trace $f(X_t)$, we must of course have some control over the behavior of $X_t$. In this paper, we assume that $X_t$ is an infinitely-divisible process with independent increments (this is the natural analog of a discrete-time random walk with independent steps). That is, there is a time-dependent distribution $D_t$ on $\mathbb{T}^d$ such that

1. $X_{t_2} - X_{t_1} \sim D_{t_2 - t_1}$ for every $t_2 \geq t_1$;
2. For all $0 \leq t_1 \leq \ldots \leq t_n$, the increments $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent;
3. $D_{t+s} = D_t \ast D_s$ for every $s, t \geq 0$, where $\ast$ is the convolution operator.

We will also assume that $D_t$ is either continuous, or that it is a mixture of an atom at 0 and a continuous distribution. In other words, writing $D_t$ as a function of $x$ for simplicity, we have

$$D_t(x) = \beta_t \delta(x) + (1 - \beta_t) \gamma_t(x),$$

where $\delta(x)$ is the Dirac $\delta$-distribution, $\gamma_t$ is a probability density function, and $\beta_t \in [0, 1]$ is a time dependent factor. We will also assume that $\gamma_t$ is not too wild: $\gamma_t \in L^2(\mathbb{T}^d)$ for all $t > 0$.

**Remark 4.** This class of distributions includes Brownian motion, and any Poisson process whose steps have an $L^2$ probability density function. It also contains the sum of Brownian motion and any arbitrary independent
Poisson process, since the diffusion smooths out any irregularities in the jumps. It does not, however, contain general jump processes with atoms, even if the atoms are dense in \( \mathbb{T}^d \) (e.g. a Poisson process on \( \mathbb{T} \) which jumps by a step size \( \alpha \) rationally independent from \( \pi \)).

The functions we reconstruct will be the indicators of open sets, whose boundary has 0 measure in \( \mathbb{R}^d \). Let
\[
\mathcal{F}_d = \{1_{x \in \Omega}(x) \mid \Omega \subseteq \mathbb{T}^d \text{ is open, Lebesgue}_d(\partial \Omega) = 0\}.
\]
Our main result is as follows:

**Theorem 5** (General reconstruction). Let \( X_t \) be a stochastic process on \( \mathbb{T}^d \) as above. If there exists a time \( t_0 \) such that the Fourier coefficients \( \{\hat{\gamma}_t(k)\}_{k \in \mathbb{Z}^d} \) are all distinct and nonzero, then \( \mathcal{F}_d \) is reconstructible by \( X_t \).

In one dimension, we show that symmetric distributions can reconstruct \( \mathcal{F}_1 \) up to reflections:

**Theorem 6** (Symmetric reconstruction). Let \( X_t \) be a stochastic process on \( \mathbb{T} \) as above, and suppose that \( \gamma_t \) is symmetric, i.e \( \gamma_t(y) = \gamma_t(-y) \) for all \( y \). If there exists a time \( t_0 \) such that the positive-indexed Fourier coefficients \( \{\gamma_t(k)\}_{k \geq 0} \) are all distinct and nonzero, then \( \mathcal{F}_1 \) is reconstructible up to reflections by \( X_t \).

One corollary of Theorem 5, is that with the right drift, Brownian motion can be used to reconstruct \( \mathcal{F}_d \).

**Corollary 7** (Brownian motion can feel the shape of a drum). Let \( X_t \) be Brownian motion with drift \( v \in \mathbb{R}^d \), such that \( \{v_1, \ldots, v_d\} \) are rationally independent. Then \( \mathcal{F}_d \) is reconstructible by \( X_t \).

**Remark 8.** The condition on the drift \( v \) is natural: If the components of \( v \) are rationally independent, then the geodesic flow defined by \( v \) is dense in \( \mathbb{T}^d \). Reconstructing a set \( \Omega \) from this geodesic is immediate. In this sense, Corollary 7 states that reconstruction is possible also in the presence of noise which pushes us out of the trajectory. See Section 6 for a question on a related model.

The starting point for our results is a relation, introduced in [5], between two types of \( n \)-point correlations related to \( f \) - one known, and one unknown. After inverting the relation, the latter correlation can be used to reconstruct the function \( f \). In the discrete case, the relation is readily inverted using a finite Vandermonde matrix. In the continuous setting, additional difficulties arise due to both the more complicated nature of the distribution \( D_t \), which mixes together different correlations, and the fact that the function space on \( \mathbb{T}^d \) is infinite-dimensional. To address the latter issue, we prove an injectivity result for infinite Vandermonde matrices, which may be of independent interest.

**Lemma 9** (Infinite Vandermonde). Let \( p, q \in [1, \infty] \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( (z_n)_{n \in \ell^p(\mathbb{C})} \) be a sequence of distinct complex numbers such that \( z_n \to 0 \), and \( z_n \neq 0 \) for all \( n \). Let \( V \) be the infinite Vandermonde matrix with \( z_n \) as generators, i.e
\[
V_{ij} = z_i^j.
\]
If \( \mathbf{x} \in \ell^q(\mathbb{C}) \) is a zero of the infinite system of equations
\[
V \mathbf{x} = 0,
\]
then \( \mathbf{x} = 0 \).

**Remark 10.** The matrix equation \( V \mathbf{x} = 0 \) means that for every index \( i \in \mathbb{N} \) we have
\[
0 = \sum_{j=1}^{\infty} V_{ij} x_j = \sum_{j=1}^{\infty} z_i^j x_j.
\]
By Hölder’s inequality, since \( z \in \ell^p(\mathbb{C}) \) and \( \mathbf{x} \in \ell^q(\mathbb{C}) \), the series \( (z_i^j x_j) \) is absolutely convergent, and the left hand side of (1) is well defined.

Basic preliminaries and the proof of Theorem 5 are given in the next section. Section 3 gives the outline of Theorem 2, relying on the same techniques described in Section 2. Brownian motion is discussed in Section 4, and Lemma 9 is proved in Section 5. We conclude the paper with some open questions.
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2 General reconstruction

2.1 Notation and simple properties of $D_t$

We write $d$-dimensional vectors in standard italics, e.g. $k \in \mathbb{Z}^d$. Tuples of $n$ vectors are written in boldface, e.g. $k = (k_1, \ldots, k_n)$ and $k_i \in \mathbb{Z}^d$.

The Fourier series of a function $g : \mathbb{T}^d \to \mathbb{C}$ is a function $\hat{g} : \mathbb{Z}^d \to \mathbb{C}$, given by

$$\hat{g}(k) = \int_{\mathbb{T}^d} g(x) e^{-ik \cdot x} dx.$$ 

Note that we do not divide by the customary $1/(2\pi)^d$. This simplifies the statement of the convolution theorem: In this setting, we have

$$\hat{f \ast g} = \hat{f} \cdot \hat{g},$$

without any leading factor in the right hand side.

This definition also extends to the Dirac $\delta$-distribution, even though it is not a function, so that for all $k \in \mathbb{Z}^d$,

$$\hat{\delta}(k) = \int_{-\pi}^{\pi} \delta(x) e^{-ik \cdot x} dx = 1.$$ 

Recall that $D_t = \beta_t \delta + (1 - \beta_t) \gamma_t$. Using the fact that $D_{s+t} = D_s \ast D_t$, a short calculation shows that if the parameter $\beta_t$ is not identically 0, it must decay exponentially: $\beta_t = e^{-ct}$ for some constant $c$. We implicitly assume that $\beta_t$ is not identically 1, as in this case $X_t$ does not move and is uninteresting. We thus have that

$$\beta_{\alpha t} = \beta_t^\alpha \quad \forall \alpha, t > 0. \quad (2)$$

Since $D_t$ has a probability density function $\gamma_t$, we have that $D_t \to U(\mathbb{T}^d)$ in distribution as $t \to \infty$, i.e. $X_t$ converges to the uniform distribution no matter its starting point.

The distribution $D_t$ has a Fourier representation $\hat{D}_t$, given by

$$\hat{D}_t(k) = \beta_t \hat{\delta}(k) + (1 - \beta_t) \hat{\gamma}_t(k) = \beta_t + (1 - \beta_t) \hat{\gamma}_t(k) \quad k \in \mathbb{Z}^d. \quad (3)$$

By the convolution theorem, $\hat{D}_{t+s} = \hat{D}_t \ast \hat{D}_s = \hat{D}_t \hat{D}_s$. From this it follows that for any $\alpha, t > 0$, we have

$$\hat{D}_{\alpha t} = \left(\hat{D}_t\right)^\alpha.$$ 

Plugging this into (3) gives

$$\beta_{\alpha t} + (1 - \beta_{\alpha t}) \hat{\gamma}_{\alpha t}(k) = (\beta_t + (1 - \beta_t) \hat{\gamma}_t(k))^\alpha,$$

and so by (2),

$$\hat{\gamma}_{\alpha t} = \frac{(\beta_t + (1 - \beta_t) \hat{\gamma}_t)^\alpha - \beta_t^\alpha}{1 - \beta_t^\alpha}. \quad (4)$$
2.2 Proof of Theorem 5

The proofs of Theorems 5 and 6 use the relation between the spatial correlation and temporal correlation introduced in [5]. Let $f$ be the indicator of an open set $\Omega$. For every integer $n \geq 0$, define the $n$-th spatial correlation of $f$, denoted $S_n : \mathbb{T}^d \to \mathbb{R}$, as

$$S_n (y) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f (x) \prod_{i=1}^n f \left( x + \sum_{i=1}^n y_i \right) dx$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f (x) \prod_{k=1}^n \left[ f \left( x + \sum_{i=1}^k y_i \right) \right] dx,$$

and the $n$-th temporal correlation of $f (X_t)$, denoted $T_n : \mathbb{R}_+ \to \mathbb{R}$, as

$$T_n (t) = E_{X_0 \sim U (\mathbb{T}^d)} \left[ f (X_0) f (X_{t_1}) \cdots f (X_{\sum_{i=1}^n t_i}) \right]$$

(note that $S_0$ and $T_0$ are just constants, equal to the measure of $\Omega$ relative to $\mathbb{T}^d$). The proof involves two parts: The first shows that $S_n (y)$ can be calculated from our knowledge of $T_n (t)$. The second uses $S_n (y)$ to reconstruct $f$ with better and better precision as $n \to \infty$.

**Proposition 11.** Let $n \in \mathbb{N}$. Under the conditions of Theorem 5, the function $S_n$ can be calculated from $f (X_t)$ with probability 1.

We intend to prove Proposition 11 using the Vandermonde lemma (Lemma 9). To this end, we first show the following:

**Proposition 12.** For every positive integers $n, m \in \mathbb{N}$, and every set of times $t_1, \ldots, t_n > 0$, the value of the sum

$$\sum_{k \in \mathbb{Z}^d} \left( \prod_{i=1}^n \hat{\gamma}_{t_i} (k_i) \right) \hat{S} (k)$$

can be calculated from $f (X_t)$ with probability 1.

**Proof.** By (4), for every $m \in \mathbb{N}$ and $t > 0$ we have

$$\hat{\gamma}_{mt} = \frac{(\beta_t + (1 - \beta_t) \hat{\gamma}_t)^m - \beta_t^m}{1 - \beta_t^m}.$$ 

Rearranging, we can write $\hat{\gamma}_t^m$ as a sum of smaller powers of $\hat{\gamma}_t$:

$$\hat{\gamma}_t^m = \sum_{j=1}^{m-1} c_j \hat{\gamma}_t^j + \hat{\gamma}_{mt},$$

where $c_j$ are some coefficients (in particular, when $D_t$ has no atom, i.e when $\beta_t = 0$ for all $t > 0$, this sum is rather simple: $\hat{\gamma}_t^m = \hat{\gamma}_{mt}$). Reiterating this process, we find that the $m$-th power of $\hat{\gamma}_t$ can be written as some linear combination

$$\hat{\gamma}_t^m = \sum_{j=1}^{m-1} c_j' \hat{\gamma}_{jt}.$$ 

Thus, the product $\left( \prod_{i=1}^n \hat{\gamma}_{t_i} (k_i) \right)^m$ is itself a sum of multilinear monomials in $\{ \hat{\gamma}_{jt_i} \}_{j=1}^{m-1}$, and so to prove the proposition it suffices to prove it for $m = 1$.

The temporal correlation $T_n (t)$ can be computed by the process $f (X_t)$: Since $X_t$ approaches the uniform distribution on $\mathbb{T}^d$ as $t \to 0$, for any fixed times $t_1, \ldots, t_n$, we can choose sampling times $\tau_1, \tau_2, \ldots$ so that $\{ f (X_{\tau_j}) f (X_{\tau_j + t_1}) \cdots f (X_{\tau_j + \sum_{i=1}^n t_i}) \}_{j=1}^{\infty}$ have pairwise correlations that are arbitrarily small, and are
arbitrarily close in distribution to \( f \left( X_0 \right) f \left( X_{t_1} \right) \cdots f \left( X_{\sum_{i=1}^{n-t_0}} \right) \) with \( X_0 \sim U \left( T^d \right) \). The temporal correlation \( T_n \left( t \right) \) is then given, with probability 1, by the sample average at times \( \tau_j \).

A relation between the spatial and temporal correlation can be obtained as follows. First, since \( X_0 \) is uniform on \( T^d \) in the definition of \( T_n \left( t \right) \),

\[
T_n \left( t \right) = \frac{1}{(2\pi)^d} \int_{T^d} f \left( x \right) \mathbb{E} \left[ \prod_{k=1}^{n} f \left( X_{\sum_{i=1}^{k} \tau_i} \right) | X_0 = x \right] \; dx.
\]

By conditioning on the event that between times \( t_{i-1} \) and \( t_i \) the process took a step of size \( y_i \), this is equal to

\[
T_n \left( t \right) = \int_{T^d} \int_{T^{d-n}} \left( \prod_{i=1}^{n} D_{t_i} \left( y_i \right) \right) \left( f \left( x \right) \prod_{k=1}^{n} f \left( x + \sum_{i=1}^{k} y_i \right) \right) \; dy \; dx
\]

\[
= \int_{T^{d-n}} \left( \prod_{i=1}^{n} D_{t_i} \left( y_i \right) \right) S_n \left( y \right) \; dy. \tag{5}
\]

Since \( D_{t} \left( y \right) = \beta_i \delta \left( y \right) + \left( 1-\beta_i \right) \gamma_t \left( y \right) \), the product inside the integral breaks into a sum, where, for each time step \( t_i \), we have to choose whether the process stayed in place (corresponding to the \( \delta \left( y_i \right) \)), or moved according to the density \( \gamma_t \). Whenever we choose to stay in place, we shrink the number of spatial variables in our correlation, since \( S_n \left( y_1, \ldots, y_k, 0, y_{k+2}, \ldots, y_n \right) = S_{n-1} \left( y_1, \ldots, y_k, y_{k+2}, \ldots, y_n \right) \). We can thus go over all choices \( A \subseteq [n] \) of indices of times when the process moved according to \( \gamma_t \), giving

\[
T_n \left( t \right) = \sum_{A \subseteq \left( [n] \right) \setminus \{y \}} \prod_{i \in A} \beta_i \prod_{i \notin A} \left( 1-\beta_i \right) \int_{T^{|A|d}} \left( \prod_{i \in A} \gamma_{t_i} \left( y_i \right) \right) S_{|A|} \left( y \right. \text{ restricted to } A \left. \right) \left( \prod_{i \in A} dy_i \right).
\]

The integral in this expression can be seen as an inner product between \( \prod_{i \in A} \gamma_{t_i} \left( y_i \right) \) and \( S_{|A|} \left( y \right) \) over the torus \( T^{|A|d} \). Since both \( \gamma_t \) and \( S_{|A|} \) are in \( L^2 \left( T^d \right) \), by Parseval’s theorem, we can therefore replace it by a sum over all Fourier coefficients \( \mathbf{k} \in \mathbb{Z}^{|A|d} \):

\[
T_n \left( t \right) = \frac{1}{(2\pi)^d} \sum_{A \subseteq \left( [n] \right) \setminus \{y \}} \prod_{i \in A} \beta_i \prod_{i \notin A} \left( 1-\beta_i \right) \sum_{\mathbf{k} \in \mathbb{Z}^{|A|d}} \left( \prod_{i \in A} \hat{\gamma}_{t_i} \left( k_i \right) \right) \hat{S}_{|A|} \left( \mathbf{k} \text{ restricted to } A \right). \tag{6}
\]

With the above display, Proposition 12 follows quickly by induction on \( n \). For the case \( n = 1 \), we have

\[
(2\pi)^d T_1 \left( t \right) = \beta_t S_0 + \sum_{\mathbf{k} \in \mathbb{Z}^d} \hat{\gamma}_t \left( k \right) \hat{S}_1 \left( k \right).
\]

Since \( S_0 = T_0 \), the value of \( \sum_{\mathbf{k} \in \mathbb{Z}^d} \hat{\gamma}_t \left( k \right) \hat{S}_1 \left( k \right) \) is known. The induction step for general \( n \) is now immediate, since by (6) it is evident that \( T_n \left( t \right) \) is a multilinear polynomial in \( \hat{\gamma}_t \), and all terms with degree strictly smaller than \( n \) are known by the induction hypothesis.

\[
\square
\]

Proof of Proposition 11. We will now show that for every \( n \in \mathbb{N} \), the values

\[
\left\{ \sum_{\mathbf{k} \in \mathbb{Z}^{n\cdot d}} \left( \prod_{i=1}^{n} \hat{\gamma}_t \left( k_i \right) \right)^m \hat{S}_n \left( \mathbf{k} \right) | m \in \mathbb{N}, t_1, \ldots, t_n > 0 \right\}
\]

uniquely determine \( \hat{S}_n \left( \mathbf{k} \right) \). Proposition 11 then follows, since (as a quick, omitted, calculation shows) \( S_n \left( y \right) \) is continuous on \( T^d \), and is thus completely determined by its Fourier coefficients.
Suppose that there exists a bounded function $Q_n(y)$ on $\mathbb{T}^d$ such that for all $m \in \mathbb{N}$ and all times $t_1, \ldots, t_n > 0$,
\[
\sum_{k \in \mathbb{Z}^d} \left( \prod_{i=1}^{n} \hat{\gamma}_{t_i}(k_i) \right)^m \hat{S}_n(k) = \sum_{k \in \mathbb{Z}^d} \left( \prod_{i=1}^{n} \hat{\gamma}_{t_i}(k_i) \right)^m \hat{Q}_n(k).
\]
Then, denoting $x_k = \hat{S}_n(k) - \hat{Q}_n(k)$, we have that
\[
\sum_{k \in \mathbb{Z}^d} \left( \prod_{i=1}^{n} \hat{\gamma}_{t_i}(k_i) \right)^m x_k = 0.
\]
Note that $x_k \in \ell^2(\mathbb{C})$, since both $S_n$ and $Q_n$ are in $L^2(\mathbb{T}^d)$. We wish to use Lemma 9 to show that necessarily $x_k = 0$.

To do this, we must choose the times $t_i$ so that the products $\prod_{i=1}^{n} \hat{\gamma}_{t_i}(k_i)$ are all non-zero and distinct. Then, setting $z_k = \prod_{i=1}^{n} \hat{\gamma}_{t_i}(k_i)$, we will have that $z_k \in \ell^2(\mathbb{C})$ (since $\hat{\gamma}_{t_i}$ are the Fourier coefficients of a square integrable function $\gamma_{t_i}$), $z_k$ are all distinct, and $z_k \neq 0$ for all $k \in \mathbb{Z}^d$, exactly meeting the requirements of the lemma with $p = q = 1/2$.

To choose the times $t_i$, recall that by assumption, there exists a time $t_0$ such that $\hat{\gamma}_{t_0}(k_i) \neq 0$ for all distinct and nonzero. We will show that there exist numbers $\alpha_1, \ldots, \alpha_n > 0$, so that if $t_i = \alpha_i t_0$, then the products $\prod_{i=1}^{n} \hat{\gamma}_{t_i}(k_i)$ satisfy the above requirements. By (4), for every $\alpha$, we have
\[
\hat{\gamma}_{\alpha t_0} = (\beta_{t_0} + (1 - \beta_{t_0}) \hat{\gamma}_{t_0})^{\alpha_0} - \beta_{t_0}^{\alpha_0}.
\]
For a particular $k \in \mathbb{Z}^d$, let $B_k = \{ \alpha > 0 \mid \hat{\gamma}_{\alpha t_0}(k) = 0 \}$ be the set of “bad” multipliers for $\hat{\gamma}_{t_0}(k)$. The coefficient $\hat{\gamma}_{\alpha t_0}(k)$ can be 0 only if
\[
(\beta_{t_0} + (1 - \beta_{t_0}) \hat{\gamma}_{t_0}(k))^{\alpha_0} - \beta_{t_0}^{\alpha_0} = 0,
\]
and since $\hat{\gamma}_{t_0}(k) \neq 0$, the function
\[
z \mapsto (\beta_{t_0} + (1 - \beta_{t_0}) \hat{\gamma}_{t_0}(k))^{\alpha_0} - \beta_{t_0}^{\alpha_0}
\]
is a non-constant holomorphic function of $z$. The set of zeros $B_k$ is thus isolated, and in particular countable.

Now let $k \neq k' \in \mathbb{Z}^d$ be two different vectors, and let $B_{k,k'} = (\cup_{i=1}^{n} B_{k_i}) \cup (\cup_{i=1}^{n} B_{k'_i})$ be the set of bad multipliers which cause one of the $\hat{\gamma}_{\alpha t_0}$ to be 0. Let
\[
R_{k,k'} = \left\{ \alpha \in \mathbb{R}^n \setminus \mathbb{N} \mid \forall j \alpha_j \notin B_{k,k'} \text{ and } \prod_{i=1}^{n} \hat{\gamma}_{\alpha_i t_0}(k_i) = \prod_{i=1}^{n} \hat{\gamma}_{\alpha_i t_0}(k_i') \right\}.
\]
By (7), this means that for every $\alpha \in R_{k,k'}$,
\[
\prod_{i=1}^{n} \frac{(\beta_{t_0} + (1 - \beta_{t_0}) \hat{\gamma}_{t_0}(k_i))^{\alpha_i} - \beta_{t_0}^{\alpha_i}}{(1 - \beta_{t_0}^{\alpha_i})} - \prod_{i=1}^{n} \frac{(\beta_{t_0} + (1 - \beta_{t_0}) \hat{\gamma}_{t_0}(k_i'))^{\alpha_i} - \beta_{t_0}^{\alpha_i}}{(1 - \beta_{t_0}^{\alpha_i})} = 0.
\]
Let $i^*$ be an index so that $k_{i^*} \neq k'_{i^*}$. Since none of the factors $\hat{\gamma}_{\alpha_i t_0}(k_i)$ or $\hat{\gamma}_{\alpha_i t_0}(k_i')$ are 0 by choice of $R_{k,k'}$, we can rearrange the above, yielding
\[
1 - \left( 1 + \frac{1 - \beta_{t_0} \hat{\gamma}_{t_0}(k_{i^*})}{\beta_{t_0}^{\alpha_{i^*}}} \right)^{\alpha_{i^*}} - \prod_{i \neq i^*} \frac{(\beta_{t_0} + (1 - \beta_{t_0}) \hat{\gamma}_{t_0}(k_i'))^{\alpha_i} - \beta_{t_0}^{\alpha_i}}{(1 - \beta_{t_0}^{\alpha_i})} = 0.
\]
For fixed $\{\alpha_i\}_{i \neq i^*}$, the expression on the left-hand side is a function of the form $z \mapsto \frac{1-a^z}{1-b^z} - c$. Since $\{\hat{\gamma}_{t_0}(k)\}_{k \in \mathbb{Z}^d}$ are all distinct, $a \neq b$. Thus this map is a non-constant holomorphic function on $\mathbb{C} \setminus B_{k,k'}$, and
so for any fixed choice of \( \{ \alpha_i \}_{i \neq i^*} \), has only countably many zeros, i.e only countably many choices for \( \alpha_{i^*} \). The Lebesgue measure of \( R_{k,k'} \) in \( \mathbb{R}^n \) is therefore 0. But the set of all \( \alpha_1, \ldots, \alpha_n > 0 \) such that either there are two equal nonzero products in \( \prod_{i=1}^n \tilde{\gamma}_{\alpha_i \tau_0}(k_i) \) or one of the products is itself 0 is the countable union

\[
\bigcup_{k,k' \in \mathbb{Z}^n} R_{k,k'} \cup \bigcup_{k,k' \in \mathbb{Z}^n} \{ (\alpha_1, \ldots, \alpha_k) \mid \alpha_i \in \{B_{k_i} \cup B_{k_i'}\} \},
\]

and so too has measure 0. In particular, there must exist \( \alpha_1, \ldots, \alpha_n > 0 \) so that the products are distinct and nonzero, as needed.

\( \square \)

**Proof of Theorem 5.** We now show that knowledge of \( S_n(y) \) is enough to reconstruct \( f \) up to a translation of the torus. The main idea is this. Suppose that \( S_n(y) > 0 \) for some given \( y \in \mathbb{R}^n \). Then \( \int_{T^d} \prod_{k=0}^n f(x + \sum_{i=1}^k y_i) \, dx > 0 \), which means that there exists a point \( x_0 \in T^d \) such that \( f(x_0 + \sum_{i=1}^k y_i) = 1 \) for all \( k = 0, \ldots, n \). By considering only \( y_i \)'s which partition \( T^d \) into a grid, we can get an approximation of \( \Omega \) by taking the union of the grid blocks.

Let \( \delta_m = 2\pi/m \), let \( n = m^d \), and consider the set \( \mathcal{Y} = \{ y \in \delta \cdot \mathbb{N}^d \mid S_n(y) > 0 \} \). Note that \( (2\pi)^d S_n(\emptyset) = \int_{T^d} f(x) \, dx = \mu(\Omega) \) (where \( \mu \) is the Lebesgue measure), so if \( \mathcal{Y} = \{ \emptyset \} \) then \( \Omega = \emptyset \), and \( f \) is identically zero. We may therefore assume that \( \mathcal{Y} \neq \emptyset \).

Each vector \( y \in \mathcal{Y} \) defines a set of points \( G_y = \left\{ \sum_{i=1}^k y_i \mid k = 0, \ldots, n \right\} \), where the sum \( \sum_{i=1}^k y_i \) is taken to be in the torus \( T^d \). Since \( \delta_m \) divides the side-length of the torus, \( G_y \) can be viewed as a subset of a \( d \)-dimensional grid in \( T^d \), with the individual \( y_i \) serving as “pointer vectors” to the next point in the grid. The number of points in \( G_y \) depends on \( \mathcal{Y} \): If \( y = \emptyset \), for example, then \( G_y = \{ \emptyset \} \); however, we can also choose \( y \) such that \( G_y = \delta \mathbb{Z}^d \cap T^d \).

Let \( y^* \in \mathcal{Y} \) be such that \(|G_{y^*}| \geq |G_y| \) for all \( y \in \mathcal{Y} \), and let \( G_m^* = G_{y^*} \), so that \( G_m^* \) is a largest possible subset when the pointer vectors are taken from \( \mathcal{Y} \). Using \( G_m^* \), we can now define a rough, shifted approximation \( \Omega_m \) to the domain \( \Omega \): Letting \( C_d = [-\frac{1}{2}, \frac{1}{2}]^d \) be the unit \( d \)-dimensional cube, we cover each point \( x \in G_m^* \) by the scaled cube \( \delta_m C_d \):

\[
\Omega_m = G_m^* + \delta_m C_d
\]

(here we use the Minkowski sum for the addition of two sets / the addition of a point and a set). See Figure 2 for an example of this procedure in 2 dimensions.

![Figure 2: Left: The domain \( \Omega \) (cyan), together with a maximal grid \( G_m^* \) (black dots). The origin is shifted so that all grid points fall in \( \Omega \). The yellow vectors represent a possible choice for \( y_i \). Right: The resultant approximation \( \Omega_m \).](image-url)
We now claim that \( \Omega_m \to \Omega \) up to translations, in the sense that there is a shift \( \theta \in \mathbb{T}^d \) such that the symmetric difference vanishes: \( \mu((\Omega_m + \theta) \setminus \Omega) \to 0 \) as \( m \to \infty \). To see this, let’s look separately at the contribution of \( (\Omega_m + \theta) \setminus \Omega \) and the contribution of \( \Omega \setminus (\Omega_m + \theta) \). First, as mentioned above, since \( S_n(y) > 0 \), there is a point \( x_m \) such that \( G^*_m + x_m \subseteq \Omega \). Adding \( \delta_m C_d \) to both sides gives \( \Omega_m + x_m \subseteq \Omega + \delta_m C_d \). We then have

\[
\mu((\Omega_m + x_m) \setminus \Omega) \leq \mu((\Omega + \delta_m C_d) \setminus \Omega) \leq \mu(\partial \Omega + \delta_m C_d).
\]

The latter expression goes to 0 as \( \delta_m \to 0 \), since

\[
\lim_{m \to \infty} \mu(\partial \Omega + \delta_m C_d) = \mu \left( \bigcap_m (\partial \Omega + \delta_m C_d) \right) \text{ \bigcap \text{closed} } \mu(\partial \Omega) = 0.
\]

Suppose now that \( x \in \Omega \setminus (\Omega_m + x_m) \), and let \( z \in \delta_m \mathbb{Z}^d \cap \mathbb{T}^d \) be a grid-point closest to \( x - x_m \). The point \( z \) cannot be in \( G^*_m \): If it were, then the cube \( z + \delta_m C_d \) (which contains \( x - x_m \)) would be contained in \( \Omega_m \), contradicting the fact that \( x \notin \Omega_m + x_m \). Since \( |G^*_m| \) is maximal, we necessarily have \( z + x_m \notin \Omega \) (otherwise we could add it to \( G^*_m \)). So every point not in \( \Omega_m + x_m \) can be covered by placing the cube \( \delta_m C_d \) on some point in \( (\Omega + x_m)^c \). Thus

\[
\mu(\Omega \setminus (\Omega_m + x_m)) = \mu((\Omega + x_m)^c \setminus \Omega^c) \leq \mu((\Omega + \delta_m C_d) \setminus \Omega^c),
\]

and again the latter goes to 0 as \( \delta_m \to 0 \).

We thus have a sequence of vectors \( x_m \in \mathbb{T}^d \) such that \( \mu((\Omega_m + x_m) \setminus \Omega) \to 0 \) as \( m \to \infty \). Since \( \mathbb{T}^d \) is compact, \( x_m \) has a subsequence converging to some \( \theta \), and it follows that \( \mu((\Omega_m + \theta) \setminus \Omega) \to 0 \) as well. □

### 3 Symmetric reconstruction

The proof of Theorem 6 is similar to that of Theorem 5. The main difference is that since \( \hat{\gamma}_t(k) = \hat{\gamma}_t(-k) \) for all \( k \), we cannot immediately use Lemma 9 to recover \( S_n(y) \) anymore. This can be overcome by working with a completely symmetric version of \( S_n(y) \), denoted \( \sigma_n(y) \) and defined as

\[
\sigma_n(y) = \sum_{\varepsilon \in \{-1,1\}^n} S_n(\varepsilon y_1, \ldots, \varepsilon y_n).
\]

#### Proposition 13.
Under the conditions of Theorem 6, \( \sigma_n(y) \) can be calculated from \( f(X_t) \) with probability 1.

**Proof.** The proof uses the same techniques as that of Proposition 11; we highlight the differences here. Starting with the temporal-spatial relation (5),

\[
T_n(t) = \int_{\mathbb{T}^n} \left( \prod_{i=1}^n D_{\tau_i}(y_i) \right) S_n(y_1, \ldots, y_n) \, dy.
\]

observe that each integral of the form \( \int_{-\pi}^{\pi} D_{\tau_i}(y_i) S_n(y_1, \ldots, y_n) \, dy_i \) can be split into two parts:

\[
\int_{-\pi}^{\pi} D_{\tau_i}(y_i) S_n(y_1, \ldots, y_n) \, dy_i = \int_{-\pi}^{\pi} D_{\tau_i}(y_i) S_n(y_1, \ldots, y_n) \, dy_i + \int_{0}^{\pi} D_{\tau_i}(y_i) S_n(y_1, \ldots, y_n) \, dy_i,
\]

where we use the convention that \( \int_{-\pi}^{0} \delta(y) g(y) \, dy = \frac{1}{2} \lim_{\varepsilon \to 0^+} \int_{-\pi}^{\pi} \delta(y) g(y) \, dy = \frac{1}{2} g(0) \). Making the change of variables \( y_i \to -y_i \) in the first integral and using the fact that \( D_{\tau_i} \) is symmetric, we thus have
\[ \int_{-\pi}^{\pi} D_{t_i} (y_i) S_n (y_1, \ldots, y_n) \, dy_i = \frac{1}{2} \int_{-\pi}^{\pi} D_{t_i} (y_i) [S_n (y_1, \ldots, -y_i, \ldots, y_n) + S_n (y_1, \ldots, y_i, \ldots, y_n)] \, dy_i. \]

Performing this \( n \) times yields

\[ T_n (t) = \frac{1}{2^n} \int_{\mathbb{T}^n} \left( \prod_{i=1}^{n} D_{t_i} (y_i) \right) \sigma_n (y) \, dy. \] (8)

As in the proof of Proposition 12, this allows us to calculate the sum

\[ \sum_{k \in \mathbb{Z}^n} \left( \prod_{i=1}^{n} \gamma_{t_i} (k_i) \right)^m \hat{\sigma}_n (k) \]

for all \( n, m \in \mathbb{N} \) and every set of times \( t_1, \ldots, t_n > 0 \). Now, both \( \left( \prod_{i=1}^{n} \gamma_{t_i} (y_i) \right) \) and \( \sigma_n (y) \) are symmetric in every coordinate \( y_i \), and so the Fourier coefficients are invariant under flipping of individual entries. We can therefore restrict the sum to non-negative \( k \) vectors: Defining

\[ \alpha_n (k) = 2^{\# \{ |k_i| \neq 0 \}} \hat{\sigma}_n (k), \]

we can calculate the sum

\[ \sum_{k \in \mathbb{Z}^n_+} \left( \prod_{i=1}^{n} \gamma_{t_i} (k_i) \right)^m \alpha_n (k) \]

for all \( n, m \in \mathbb{N} \) and times \( t_1, \ldots, t_n > 0 \). As in the proof of Proposition 11, \( \alpha_n \) (and therefore \( \sigma_n \)) can be recovered from these quantities using Lemma 9, since by assumption there is a time \( t_0 \) such that \( \{ \gamma_{t_0} (k) \}_{k \geq 0} \) are all distinct and non-zero.

**Proposition 14.** Let \( \delta > 0 \) divide \( 2\pi \). Given \( \sigma_n (y) \) for all \( n \in \mathbb{N} \) and \( y \in \mathbb{T}^n \), it is possible to calculate \( S_n (y) + S_n (-y) \) for all \( n \in \mathbb{N} \) and all \( y \in \{(k_1 \delta, \ldots, k_n \delta) \mid k_i \in \mathbb{Z}_+ \} \).

**Proof.** The proof is by induction. For \( n = 1 \), we just have \( \sigma_1 (y) = S_1 (y) + S_1 (-y) \), and for general \( n \) and \( k = 0 \), we just have \( \sigma_n (0) = 2^n S_n (0) \). Now let \( n \in \mathbb{N} \) and \( k \in \mathbb{Z}^n_+ \). Assume that the statement holds true for \( n \) for all vectors \( k' \) with \( \sum_{i=1}^{n} k_i' < \sum_{i=1}^{n} k_i \), and also for all \( m < n \). We have

\[ \sigma_n (k_1 \delta, \ldots, k_n \delta) = \sum_{\varepsilon \in \{-1, 1\}^n} \sigma_n (\varepsilon_1 k_1 \delta, \ldots, \varepsilon_n k_n \delta) \]

\[ = S_n (k_1 \delta, \ldots, k_n \delta) + S_n (-k_1 \delta, \ldots, -k_n \delta) + \sum_{\varepsilon \in \{-1, 1\}^n, \varepsilon \not= \text{all equal}} \sigma_n (\ldots). \]

If not all \( \varepsilon_i \) are equal, then \( \sigma_n (\varepsilon_1 k_1 \delta, \ldots, \varepsilon_n k_n \delta) \) is equal to some \( S_m (k'_1 \delta, \ldots, k'_m \delta) \) for \( m \leq n \): The total sum \( \sum \varepsilon_i k_i \) is strictly smaller than \( \sum k_i \), and so the partial sums can be seen as the forward-only jumps of some \( y' \) with corresponding \( k' \) such that \( \sum k'_i < \sum k_i \) (the strict \( m < n \) case is when the partial sums \( \sum \varepsilon_i k_i \) themselves are not unique). See Figure 3 for a visualization. Similarly, \( \sigma_n (-\varepsilon_1 k_1 \delta, \ldots, -\varepsilon_n k_n \delta) = -S_n (k'_1 \delta, \ldots, -k'_m \delta) \). Since the sum over all \( \varepsilon_i \) can be split into polar pairs, by the induction hypothesis we can calculate the sum \( \sum_{\varepsilon \in \{-1, 1\}^n, \varepsilon \not= \text{all equal}} \sigma_n \), and therefore also \( S_n (k_1 \delta, \ldots, k_n \delta) + S_n (-k_1 \delta, \ldots, -k_n \delta) \).
Figure 3: $S_3(y_1, y_2, y_3) = S_3(y_1', y_2', y_3')$. The choice of origin does not matter, since $S_n$ integrates over all $T$ anyway. The total length $y_1' + y_2' + y_3'$ is smaller than $y_1 + y_2 + y_3$.

Proof sketch of Theorem 6. Similarly to the proof of Theorem 5, once we know $S_n(k_1 \delta, \ldots, k_n \delta) + S_n(-k_1 \delta, \ldots, -k_n \delta)$, for every $\delta_n = 2\pi/n$ we can construct a set $\Omega_n$ according a $y \in \{y \in \delta \cdot \mathbb{N}^n \mid S_n(y) + S_n(-y) > 0\}$ which maximizes $G_y$. The resulting $f_n$ contains a subsequence which converges to either $f(x)$ or $f(-x)$, where the ambiguity is because we do not know which of the two of $S_n(y)$ and $S_n(-y)$ was greater than 0.

4 Example: Brownian motion

4.1 Proof of Corollary 7

Proof. For $d = 1$, the step distribution $\gamma_t$ of Brownian motion on $T$ is that of a wrapped normal distribution with drift, and is given by

$$\gamma_t(y) = \frac{1}{\sqrt{2\pi t}} \sum_{m=-\infty}^{\infty} e^{-(y-vt+2\pi m)^2/2t}.$$

The Fourier coefficients of $\gamma_t(y)$ can readily be calculated, by observing that the wrap-around gives the continuous Fourier transform evaluated at integer points:

$$\hat{\gamma}_t(k) = \int_{-\infty}^{\infty} e^{-iky} \frac{1}{\sqrt{2\pi t}} \sum_{m=-\infty}^{\infty} e^{-(y-vt+2\pi m)^2/2t} dy$$

$$= \int_{-\infty}^{\infty} e^{-iky} \frac{1}{\sqrt{2\pi t}} e^{-(y-vt)^2/2t} dy$$

$$= e^{-ivtk} e^{-2\pi^2 k^2}.$$

For standard Brownian motion, without drift, the coefficients are symmetric, but $\{\hat{\gamma}_t(k)\}_{k \geq 0}$ are all distinct, and so by Theorem 6, reconstruction is possible using Brownian motion up to rotations and reflections.

The drift, however, can make the coefficients distinct. For $d = 1$, any non-zero drift will do, and reconstruction is possible up to rotations. In the general case, the Fourier coefficients of $\gamma_t$ are

$$\hat{\gamma}_t(k) = e^{-it \sum_{j=1}^{d} v_j k_j} e^{-2\pi^2 \sum k_j^2} ; \quad k \in \mathbb{Z}^d.$$
In order for the coefficients to be distinct, it suffices to make the factors $e^{-it\sum_{j=1}^{d}v_jk_j}$ all distinct, i.e there should exist a time $t$ such that for every $k \neq k' \in \mathbb{Z}^d$ and every $m \in \mathbb{Z}$,

$$\sum_{j=1}^{d} v_j(k_j - k'_j) \neq 2\pi m/t.$$ 

Choosing $t$ so that $\{v_1, \ldots, v_d, \frac{2\pi}{t}\}$ are all rationally independent completes the proof. \qed

### 4.2 An explicit inversion

As a side note, we would like to mention that for Brownian motion, inverting the symmetric integral (8) (note the changed range of integration),

$$T_n(t) = \int_{[0,\pi]^n} \left( \prod_{i=1}^{n} \gamma_{t_i}(y_i) \right) \sigma_n(y) \, dy,$$

is possible without resorting to Lemma 9: There exists an explicit relation between $\sigma_n(y)$ and $T_n(t)$. The relation appears in (e.g) [8, 3.2-6, item 21]. We repeat the arguments here for completeness.

If we treat $\sigma_n(y)$ as a periodic function over all $\mathbb{R}^n$, we can replace the folded normal distribution $\gamma_{t_i}$ with a normal distribution over all $\mathbb{R}^n$: 

$$T_n(t) = \int_{y \in \mathbb{R}^n} \left( \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi t_i}} e^{-\frac{y^2}{2t_i}} \right) \sigma_n(y) \, dy.$$ 

Let $p \in \mathbb{R}^n_+$. Multiplying both sides by $e^{-pt_1 - \cdots - p_{n-1}t_{n-1}}$ and integrating all $t_i$s from 0 to $\infty$, we get

$$\mathcal{L}\{T_n\}(p) = \int_{y \in \mathbb{R}^n_+} \int_{t \in \mathbb{R}^n_+} \left( \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi t_i}} e^{-\frac{y^2}{2t_i}} e^{-p_{i}t_{i}} \right) \sigma_n(y) \, dy dt,$$

where $\mathcal{L}\{f\}$ is the Laplace transform of $f$. The individual integrals over $t_i$ in the right hand side can be readily calculated to be:

$$\int_{0}^{\infty} e^{-pt - y^2/2t} \, \frac{1}{\sqrt{2\pi t}} dt = e^{-\sqrt{2py}}.$$

This gives

$$\mathcal{L}\{T_n\}(p) = \int_{\mathbb{R}^n_+} \left( \prod_{j=1}^{n} \frac{1}{\sqrt{2p_j}} e^{-\frac{y^2}{2p_j}} \right) \sigma_n(y) \, dy.$$

Up to a change of variables $s_i = \sqrt{2p_i}$, the right hand side is the Laplace transform of $\sigma_n(y)$. Thus

$$\sigma_n = \mathcal{L}^{-1}\left\{ \left( \prod_{i=1}^{n} s_i \right) \mathcal{L}\{T_n\}\left(\frac{1}{2}s_1^2, \ldots, \frac{1}{2}s_n^2\right) \right\}.$$

### 5 Proof of Lemma 9

**Proof.** As noted in Remark 10, the equality $Vx = 0$ means that for every index $i \in \mathbb{N}$ we have

$$0 = \sum_{j=1}^{\infty} V_{i,j}x_j = \sum_{j=1}^{\infty} z_{i,j}^2 x_j.$$ 

(9)
Since \( \mathbf{z} \in \ell^p(\mathbb{C}) \) and \( \mathbf{x} \in \ell^q(\mathbb{C}) \), by Hölder’s inequality the series \( \left( z^i_j x_j \right) \) is absolutely convergent for all \( i \), and we can change the order of summation; without loss of generality we can assume that \( z_n \) are ordered so that \( |z_1| \geq |z_2| \geq \ldots \).

Assume by induction that \( x_1, \ldots, x_k \) have already been shown to be equal to 0, and let \( \ell > 0 \) be such that \( |z_{k+1}|, \ldots, |z_{k+\ell}| \) are all of equal magnitudes, but \( |z_{k+\ell}| > |z_{k+\ell+1}| \). Since \( z_n \to 0 \), \( \ell \) is necessarily finite. For any fixed \( i \), the sum \((9)\) can then be split into two parts:

\[
\sum_{s=1}^{\ell} z^i_k x_{k+s} + \sum_{j>k+\ell} z^i_j x_j = 0.
\]

Dividing by \( z^i_{k+\ell} \) and denoting \( \omega_s = \frac{z^i_{k+s}}{z^i_{k+\ell}} \), we have

\[
\sum_{s=1}^{\ell} \omega^i_s x_{k+s} + \sum_{j>k+\ell} \left( \frac{z^i_j}{z^i_{k+\ell}} \right)^i x_j = 0.
\]

Consider the \( \ell \) equations of the above form for \( i = r \cdot m \), where \( r = 1, \ldots, \ell \) and \( m \) is a large number. In matrix form, this system of \( \ell \) equations can be written as

\[
V_m \tilde{\mathbf{x}} + \mathbf{u} = 0, \quad (10)
\]

where:

1. \( V_m \) is a finite \( \ell \times \ell \) Vandermonde matrix with generators \( \omega^m_s = \left( \frac{z^i_{k+s}}{z^i_{k+\ell}} \right)^m \), \( s = 1, \ldots, \ell \).
2. \( \tilde{\mathbf{x}} \) is a vector of size \( \ell \) with \( \tilde{x}_s = x_{k+s} \) for \( s = 1, \ldots, \ell \).
3. \( \mathbf{u} \) is a vector of length \( \ell \) with entries \( u_r = \sum_{j>k+\ell} \left( \frac{z^i_j}{z^i_{k+\ell}} \right)^m x_j \) for \( r = 1, \ldots, \ell \). By our ordering and choice of \( \ell \), \( |z_{k+\ell}| > |z_j| \) for \( j > k + \ell \), and we can factor out an exponentially decreasing term from each summand:

\[
|u_r| \leq \frac{|z_{k+\ell+1}|}{z^i_{k+\ell}} \sum_{j>k+\ell} \left| \frac{z^i_j}{z^i_{k+\ell}} \right|^m |x_j|.
\]

(Hölder’s inequality) \[
\leq \frac{|z_{k+\ell+1}|}{z^i_{k+\ell}} \left( \sum_{j>k+\ell} \left| \frac{z^i_j}{z^i_{k+\ell}} \right|^{pm} \right)^{1/p} \|\mathbf{x}\|_q.
\]

The sum \( \sum_{j>k+\ell} \left| \frac{z^i_j}{z^i_{k+\ell}} \right|^{pm} \) is finite since \( \mathbf{z} \in \ell^p(\mathbb{C}) \), and is in fact uniformly bounded as a function of \( m \), since every summand has magnitude less than or equal to 1. Since \( |z_{k+\ell+1}| < |z_{k+\ell}| \), the term \( \left| \frac{z^i_{k+\ell+1}}{z^i_{k+\ell}} \right|^m \) goes to 0 as \( m \to \infty \), and so the entries of \( \mathbf{u} \) also decrease to 0 as \( m \to \infty \).

The generators of \( V_m \) are all distinct, and so \( V_m \) is invertible. Equation \((10)\) thus gives

\[
\tilde{\mathbf{x}} = -V_m^{-1} \mathbf{u}.
\]

As mentioned in item (3), the entries of \( \mathbf{u} \) decay to 0 in \( m \). To show that \( \tilde{\mathbf{x}} = 0 \) (and thus all of \( x_{k+s} = 0 \) for \( s = 1, \ldots, \ell \)), it therefore suffices to uniformly bound the infinity-norm \( \|V_m^{-1}\|_{\infty} \) for infinitely many \( m \). Recall that the inverse of the \( \ell \times \ell \) Vandermonde matrix with generators \( \omega^m_s \) has entries

\[
(V_m^{-1})_{ij} = (-1)^{j-1} \prod_{k < s < \ell} \frac{C_{ij}}{\omega_{ij}^{m} \prod_{k \neq i} (\omega_{ik}^{m} - \omega_{kj}^{m})}, \quad (11)
\]
that for every set of distinct points \(y\) will do. Otherwise, let 

**Proof.** The proof is by induction on \(\ell\). This does not give any bound on \(|\omega^n - \omega^m|\) uniformly bounded away from 0 for all \(i \neq j\).

**Lemma 15** (Recurrent rotations). Let \(\ell > 0\) be an integer and let \(\varepsilon > 0\). There exists a constant \(C_{\varepsilon, \ell}\) such that for every set of distinct points \(\omega_1, \ldots, \omega_\ell\) on the unit circle, there is an integer \(1 \leq m \leq C_{\varepsilon, \ell}\) such that

\[
\frac{1}{\pi} |\arg (\omega_i^m)| \leq \varepsilon \quad i = 1, \ldots, \ell,
\]

where \(\arg : \mathbb{C} \to (-\pi, \pi)\) returns the angle with the origin.

**Proof.** The proof is by induction on \(\ell\). For \(\ell = 1\), denote \(\omega = e^{\pi i \alpha}\) for \(\alpha \in [-1, 1]\). If \(|\alpha| \leq \varepsilon\), then \(m = 1\) will do. Otherwise, let \(k > 0\) be the smallest integer such that \(\alpha \in \left[-2^k \varepsilon, 2^k \varepsilon\right]\). Then one of the points \(y \in \{\omega, \omega^2, \ldots, \omega^{\lceil 1/\alpha \rceil + 1}\}\) satisfies \(y = e^{\pi i \beta}\) with \(\beta \in \left[-2^{k-1} \varepsilon, 2^{k-1} \varepsilon\right]\). The points \(\omega, \omega^2, \ldots, \omega^{\lceil 1/\alpha \rceil + 1}\) make at least one complete revolution around the unit circle, but since the angle between two consecutive points is at most \(2^k \varepsilon\), one of these points must fall in the interval \([-2^{k-1} \varepsilon, 2^{k-1} \varepsilon]\). Repeating this procedure iteratively, we obtain a sequence of points \(y_1 = e^{\pi i \beta_1}, \ldots, y_q = e^{\pi i \beta_q}\), where:

1. \(y_1 = \omega = e^{\pi i \alpha}\).
2. \(y_{i+1} = y_i^{\ell_i}\) for some \(\ell_i \leq \lceil 1/ |\beta_i| \rceil + 1\).
3. \(\frac{1}{\ell_i} |\arg (y_i)| \leq \varepsilon\)
4. The number of iterations \(q\) is bounded by \(\log_2 1/\varepsilon + 1\).

Apart from \(\beta_q\), the magnitude of each \(\beta_i\) is larger than \(\varepsilon\). Thus each \(\ell_i \leq \lceil 1/\varepsilon \rceil + 1\), and we have

\[
\varepsilon \geq \frac{1}{\pi} |\arg (y_q)| = \frac{1}{\pi} |\arg \omega^{\ell_q^{q-1}}| = \frac{1}{\pi} |\arg \omega^{\ell_1 \cdots \ell_{q-1}}|,
\]

yielding an \(m \leq \prod_{j=1}^{q-1} \ell_j \leq (\lceil 1/\varepsilon \rceil + 1)^{\log_2 1/\varepsilon}\). We therefore have \(C_{\varepsilon, 1} = (\lceil 1/\varepsilon \rceil + 1)^{\log_2 1/\varepsilon}\).

Now assume by induction that the statement holds for all \(n < \ell\). By applying the induction hypothesis on the first \(\ell - 1\) points with \(\varepsilon' = \varepsilon/C_{\varepsilon, \ell - 1}\), we obtain an integer \(m_1 \leq C_{\varepsilon', \ell - 1}\) such that for \(i = 1, \ldots, \ell - 1\),

\[
\frac{1}{\pi} |\arg (\omega_i^{m_1})| \leq \varepsilon/C_{\varepsilon, 1}.
\]

(12)

This does not give any bound on \(\arg (\omega_\ell^{m_1})\). However, we can now apply the lemma for a single point \(\omega_\ell^{m_1}\) and \(\varepsilon\), obtaining an \(m_2 \leq C_{\varepsilon, 1}\) such that

\[
\frac{1}{\pi} |\arg ((\omega_\ell^{m_1})^{m_2})| \leq \varepsilon.
\]

Choosing \(m = m_1 m_2 \leq C_{\varepsilon, 1} C_{\varepsilon', \ell - 1}\) yields the required bound on \(\omega_\ell\); as for \(i < \ell\), using (12), we have

\[
\frac{1}{\pi} |\arg ((\omega_i^{m_1})^{m_2})| \leq \frac{\varepsilon}{C_{\varepsilon, 1}} \cdot m_2 \leq \varepsilon
\]

as well.

\[\square\]
We can now finish the proof of Lemma 9. If all the ratios \( \{\arg (\omega_s)/\pi\}_{s=1}^\ell \) are rational, then there are infinitely many \( m \)'s such that
\[
\omega^m_s = \omega_s
\]
for all \( s = 1, \ldots, \ell \). The denominator in (11) stays the same in this case for all such \( m \). Otherwise, there is an \( s^* \) such that \( \arg (\omega_{s^*}) \) is an irrational multiple of \( \pi \). Let \( \varepsilon_n \to 0 \) be a positive sequence and let \( m_n \) be the number of rotations given by Lemma 15 applied to \( \{\omega_s\}_{s=1}^\ell \) with \( \varepsilon_n \). Then \( m_n \) has a subsequence which diverges to infinity, since for any finite set of values of \( m \), \( \arg (\omega^m_s) \) is bounded below. For all large enough \( n \), we necessarily have
\[
|\omega^{m_n+1}_i - \omega^{m_n+1}_j| \geq \frac{1}{2} |\omega_i - \omega_j|
\]
for all \( i \neq j \), and so the denominator in (11) is bounded.

6 Other directions and open questions

At least for processes with continuous paths on the circle \( \mathbb{T} \) (such as Brownian motion), it seems reasonable that it’s possible to reconstruct functions which are more complicated than indicators.

**Question 16.** What classes of functions are reconstructible from Brownian motion, with or without drift? Given a nice enough function \( f: \mathbb{T} \to \mathbb{R} \), is it perhaps possible to stitch together its level sets \( \{f \geq \alpha\} \), which we know are reconstructible due to Theorems 5 / 6, to gain knowledge about the entire function?

**Question 17.** Find another algorithm which shows directly that Brownian motion can reconstruct \( \mathcal{F}_d \), using its local properties.

Theorem 6 suggests that stochastic processes with symmetries should allow reconstruction, up to some symmetry of \( \mathbb{T}^d \) itself.

**Question 18.** Suppose that some coordinates of \( X_t \) are independent from others, or that \( X_t \) is invariant under some orthogonal transformation in \( O(\mathbb{R}^d) \). What can be said about reconstruction on \( \mathbb{T}^d \)?

It is natural to consider larger spaces of functions, larger spaces, and more general distributions.

**Question 19.** What can be said for sets \( \Omega \) with fat boundary, i.e \( \mu(\partial \Omega) > 0 \)?

**Question 20.** Can Theorem 5 be extended to general compact Riemannian manifolds?

**Question 21.** How do the results extend to processes whose step sizes are allowed to contain atoms at \( x \neq 0 \)? (consider, for example, a Poisson process which, when its clock fires, jumps either by \( \alpha_1 \) or \( \alpha_2 \), with \( \{\alpha_1, \alpha_2, \pi\} \) rationally independent).

Finally, the independence condition of the drift in Corollary 7 gives rise to a slightly different model for reconstruction on the torus, where we try to learn \( f \) from its values on a random (irrational) geodesic.

**Question 22.** Let \( v \) be a uniformly random unit vector in \( \mathbb{R}^d \). Which classes of functions \( f \) can be reconstructed (with probability 1) from \( f(t \cdot v)_{t\in\mathbb{R}} \)? How about \( f(X_t) \), where \( X_t \) is Brownian motion with random drift \( v_t \)?

References

[1] Itai Benjamini and Harry Kesten. Distinguishing sceneries by observing the scenery along a random walk path. *Journal d’Analyse Mathématique*, 69(1):97–135, 1996.

[2] Krzysztof Burdzy. Some path properties of iterated Brownian motion. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)*, volume 33 of *Progr. Probab.*, pages 67–87. Birkhäuser Boston, Boston, MA, 1993.
[3] Hilary Finucane, Omer Tamuz, and Yariv Yaari. Scenery reconstruction on finite abelian groups. *Stochastic Process. Appl.*, 124(8):2754–2770, 2014.

[4] Harry Kesten. Distinguishing and reconstructing sceneries from observations along random walk paths. *Microsurveys in discrete probability (Princeton, NJ, 1997)*, 41:75–83, 1998.

[5] Heinrich Matzinger and Jüri Lember. Reconstruction of periodic sceneries seen along a random walk. *Stochastic Process. Appl.*, 116(11):1584–1599, 2006.

[6] Heinrich Matzinger, Jüri Lember, and J Liivi. Scenery reconstruction: an overview.

[7] Heinrich Matzinger and Serguei Popov. Detecting a local perturbation in a continuous scenery. *Electron. J. Probab.*, 12:no. 22, 637–660, 2007.

[8] Andrei D Polyanin and Alexander V Manzhirov. *Handbook of integral equations*. CRC press, 2008.