A Tale of Two Nekrasov’s Integral Equations

Nikolay Kuznetsov

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Abstract
One hundred years ago, Nekrasov published the widely cited paper (Nekrasov in Izvestia Ivanovo-Voznesensk Politekhn Inst 3:52–65, 1921), in which he derived the first of his two integral equations describing steady periodic waves on the free surface of water. We examine how Nekrasov arrived at these equations and his approach to investigating their solutions. In connection with this, Nekrasov’s life after 1917 is briefly outlined, in particular, how he became a prisoner in Stalin’s Gulag. Further results concerning Nekrasov’s equations and related topics are surveyed.

Keywords Nekrasov’s integral equation · Stokes wave · Solitary wave · Bifurcation theory

1 Introduction

The theory of nonlinear water waves has its origin in the work of George Gabriel Stokes (1819–1903), dating back to his 1847 paper [67] (see also [68], pp. 197–219). His research in this field is well documented; see, for example, the detailed survey [15], where further references are given. The next major achievement in the development of this theory was that of Aleksandr Ivanovich Nekrasov (1883–1957), after whom two integral equations are now named. One of these equations describes Stokes waves on deep water, while the other deals with waves on water of finite depth. Nekrasov investigated his equations with mathematical techniques available in the 1920s, but a comprehensive theory of these and other integral equations arising in water-wave theory was only developed much later, after the invention of abstract global bifurcation theory in the 1970s.

On the occasion of the centenary of Nekrasov’s equation for deep water.

Nikolay Kuznetsov
nikolay.g.kuznetsov@gmail.com

1 Laboratory for Mathematical Modelling of Wave Phenomena, Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, V.O., Bol’shoy pr. 61, St. Petersburg 199178, Russian Federation
Nekrasov’s pioneering paper [48] (published in the Bulletin of the now defunct Ivanovo-Voznesensk Polytechnic Institute) is widely cited, but often without any details concerning its content. The reason was explained by John V. Wehausen (1913–2005) in his review of the memoir [52] published by the Soviet Academy of Sciences in 1951 summarising Nekrasov’s results on both integral equations (see MathSciNet, MR0060363). Wehausen writes: “The author’s work […] appeared in publications with very small distribution outside the USSR […] and consequently has not been well known.” In fact these publications also had very small distribution inside the USSR. The times when Nekrasov carried out and published this research were extremely hectic, rather like those described by Charles Dickens in the opening sentence of *A Tale of Two Cities*.

It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair, we had everything before us, we had nothing before us, we were all going direct to Heaven, we were all going direct the other way—in short, the period was so far like the present period, that some of its noisiest authorities insisted on its being received, for good or for evil, in the superlative degree of comparison only.

The start of 1917 was “the season of Light” for Nekrasov, who had recently turned 33. Indeed, on 18 February 1917, just a few days before the beginning of the February Revolution in Russia, he presented his talk “On waves of permanent type on the surface of a heavy fluid” to the Moscow Physical Society. The accompanying paper [49] was published five (!) years later; its value should however not be underestimated as the first step to Nekrasov’s integral equation for waves on deep water was made in it. The equation itself appeared in 1921 [48], but was not published in a widely available source until 1960 (see the extensive survey [78], pp. 728–730).

J. J. Stoker (1905–1992) cited [48] and [52] in his classical treatise on water waves [66], mentioning only that the problem “of two-dimensional periodic progressing waves of finite amplitude in water of infinite depth […] was first solved by Nekrassov”; see p. 522 of his book. Shortly afterwards the fourth edition of the widely cited textbook [44] by L. M. Milne-Thompson (1891–1974) was published, in which a detailed account of Nekrasov’s equation for waves on deep water was given (see sect. 14 · 84), and the papers [48] and [50] were cited, because these “references […] appear to be useful or appropriate”. Unfortunately some essential points of Nekrasov’s approach to solving his equation are omitted in [44] and an incorrect numerical coefficient is reproduced from [52].

The aim of the present paper is to explore how Nekrasov arrived at his equations and to survey the earlier results upon which he built up his approach. Indeed, one has to trace Nekrasov’s train of thought through the papers [49] and [47] preceding [48] which, containing just six references, provides no real clue. Three of those references concern exact solutions for rotational waves (the well-known articles by F. J. Gerstner [20] and W. J. M. Rankine [60], and the now forgotten note [22]); the others are the classical paper by Stokes [67], Nekrasov’s own paper [49] and Rudzki’s article [61]. Only the last is of real importance for Nekrasov’s approach, but it has now sunk into
oblivion. It should be also mentioned that the final memoir [52], an English translation of which appeared in 1967 as MRC Report no. 813 of the University of Wisconsin (unfortunately now a rarity), similarly contains only one reference (see below), and is thus rather unhelpful for our purposes.

Other sources of information about Nekrasov’s train of thought are therefore needed. One such avenue was provided by Leonid Nikolaevich Sretenskii (1902–1973), Nekrasov’s colleague at Moscow University (both were professors at the Faculty of Mechanics and Mathematics). His book [65], published in 1936, contains many comments about Nekrasov’s work on water waves. These details were however omitted in the second edition, published in 1977, because Sretenskii did not complete the manuscript himself and this task was undertaken by the editors after his death.

Along with describing Nekrasov’s work on water waves, we briefly outline his life after 1917, in particular to demonstrate how easily one could find oneself in Stalin’s Gulag, even as prominent scientist. For further details about the Great Terror see the book [13] by R. Conquest, originally published in 1968 under the title *The Great Terror: Stalin’s Purge of the Thirties*.

Genuine understanding of Nekrasov’s equations was achieved in the 1970s on the basis of modern bifurcation techniques in the framework of nonlinear functional analysis; the corresponding references can be found, for example, in [73]. We survey some results from the global theory of periodic water waves obtained with the help of these techniques, as well as those concerning Stokes’s conjectures for the wave of extreme form, which was also investigated by Nekrasov. (This wave has a stagnation point at each crest, where smooth parts of the wave profile form an angle of $\pi/3$ about the vertical through the crest.) In this connection, it is worth mentioning the opinion of T. Brooke Benjamin (1929–1995) [9] that “[…] the tools available in functional analysis can sometimes be extremely expedient in their applications to physical problems, winning ground that is genuinely valuable by the criteria of good science”. In their pioneering paper [28], Keady and Norbury thanked Benjamin for his supposition that the formulation of Nekrasov’s equation in an operator form “would lead to a successful existence proof”; indeed, Keady and Norbury obtained such an existence proof with the help of the global bifurcation theory developed by Dancer [16] for positive operators in a cone.

Finally, we remark that there are numerous papers concerned with numerical solution of Nekrasov’s equation which are out of the scope of this article.

2 A. I. Nekrasov and His Integral Equations

The online biography of Nekrasov by J. J. O’Connor and E. F. Robertson at

[http://mathshistory.st-andrews.ac.uk/Biographies/Nekrasov.html](http://mathshistory.st-andrews.ac.uk/Biographies/Nekrasov.html)

gives rather scanty information about him and does not even mention his imprisonment in 1938–1943. The reason is that this article is based on the note [62] published in 1960, when Soviet censorship was still active—despite the so-called Khrushchev Thaw that lasted several years after the denunciation of Stalin’s crimes in 1956. We now fill in the gaps in this source by outlining Nekrasov’s activities after 1917.
2.1 Twists and Turns of Nekrasov’s Life After 1917

In the book [77] published in 2001 (covering all aspects of Nekrasov’s biography), the authors assign 25 pages to sketching out major events in his life; the most interesting of them concern the Soviet period.

The Tsarist regime was overturned in March 1917. Self-rule was granted to universities, who began re-electing staff, as many progressive professors had been fired by the old government. Nekrasov was re-elected as a docent by the Council of Moscow University and promoted to a professorship in 1918. In the autumn of that year he took leave from Moscow University for four years and moved to Ivanovo-Voznesensk to join the newly formed Polytechnic Institute. His reason was scarce food rations in overcrowded Moscow during the Civil War; the situation was much better in Ivanovo-Voznesensk due to supply from regions down the Volga river. The staff of this first Soviet polytechnic (established by Lenin by decree on 6 August 1918) included many professors evacuated from Riga in 1915 (a consequence of World War I) and several professors from Moscow University, of which Luzin, Nekrasov and Khinchin were the most notable. Along with his professorship in Theoretical Mechanics, Nekrasov was Dean of the Faculty of Civil Engineering for four years and headed the whole institute of six faculties for thirteen months (Figs. 1, 2 and 3).

Nekrasov returned to Moscow University in 1922, simultaneously holding a professorship at Moscow Technical High School. He combined his teaching and administrative duties at Narkompros (the Soviet Ministry of Education) until 1929, but from that year the Central Aero-hydrodynamic Institute (TsAGI, founded in 1918) became his second affiliation. There Nekrasov was drawn into mathematical problems related to aircraft design; he was a deputy of Academician S. A. Chaplygin (1869–
1942), who headed the research at the institute in the 1930s. Nekrasov travelled abroad several times; in particular he was the head of the Soviet delegation at the 14th Air Show in Paris in 1934. The following year he spent six months in the USA with A. N. Tupolev (1888–1972)—a prominent aircraft designer—and other researchers from TsAGI; their aim was to survey aircraft production and commercial operation of airlines in the USA. This was a period of intensive collaboration between the Soviets and Americans: Soviet industry was interested in American technology, while selling to the Soviet Union helped American corporations to recover from the Great Depression.

Tupolev’s design office separated from TsAGI in 1936, but ceased to exist one year later when Tupolev and many of his colleagues were arrested during the Great Terror organised by the People’s Commissariat for Internal Affairs (the interior ministry of the Soviet Union, known as the NKVD). Nekrasov’s turn came in 1938, when he was charged with “participation in anti-Soviet, spy organisation in TsAGI”. He spent the next five years behind bars in Tupolev’s “sharashka” (the slang term for “Special Design Bureau at the NKVD”). This “bureau” was one of hundreds within the special department created in 1938 by Beria (who headed the NKVD at that time) and disbanded in 1953, shortly after Stalin’s death. (For general information about sharashkas see, for example, the Wikipedia article https://en.wikipedia.org/wiki/Sharashka. Tupolev’s sharashka is described in detail by L. L. Kerber [29], who was an aviation specialist with a long professional and personal relationship with Tupolev; see also [26] for a review of [29], where some omissions made by Kerber are mentioned.)

Nekrasov was released in 1943 (but exonerated only in 1955, two years before his death). He again returned to Moscow University, simultaneously holding a position at the Institute for Mechanics of the Soviet Academy of Sciences (Corresponding Member since 1932, promoted to Full Academician in 1946). However he continued to head the theoretical department at Tupolev’s design office until 1949 (which was located in the same building as the former sharashka, but now without armed escorts—
a characteristic feature of the time). Afterwards Nekrasov returned to his studies of water waves and compiled his memoir [53]. His last research note was published in 1953 and he retired from teaching that same year; the reason was asthma contracted during imprisonment.

We conclude with a few lines about Nekrasov’s distinctions. In 1922 he was the first recipient of the Joukowski prize for the paper [48], and received a prize from the Narkompros for his article *Diffusion of a vortex* ten years later. For the memoir [52] he was awarded the Stalin prize in 1952, and the following year he was decorated with the Order of Lenin on the occasion of his 70th birthday.

### 2.2 How Nekrasov Arrived at His Equation for Waves on Deep Water

**Prehistory** In 1906, just before his graduation from Moscow University, Nekrasov had completed a study of the motion of Jupiter’s satellites, for which he was awarded a gold medal. However, he chose water-wave theory for his further studies, presumably due to the influence of N. E. Joukowski (1847–1921), whose own research was focussed on this theory at that time. In his 1907 paper [27] Joukowski obtained an important result on the wave resistance of a ship, which was in fact similar to that published by J. H. Michell (1863–1940) in 1898, but Joukowski found it independently and using a different method.

In his first paper dealing with water waves (see [46], also the first paper in the *Collected Papers* [53]), Nekrasov applied the method of power expansions. He was familiar with this method from his work on Jupiter’s satellites and indeed extended his knowledge of it while translating the second volume of the Goursat *Cours d’Analyse Mathématique* [21] into Russian. In fact Nekrasov follows Stokes’s paper [67], where expansions in powers of the wave amplitude—implicitly supposed to be a small parameter—were used. Nekrasov explicitly introduces such a parameter into the Cauchy-Poisson problem, as it is now called (the problem of determining the motion generated by an impulsive pressure applied to the surface of a fluid initially at rest). Assuming the pressure impulse is given by a convergent series in powers of the parameter (with zero leading coefficient), Nekrasov derives the initial-boundary value problems for four harmonic functions in the leading-order terms of the expansion of the velocity potential and finds particular solutions for two of them.

In his second paper on water waves (recall that [49] was published with a long delay), Nekrasov radically changes his research topic, turning to the two-dimensional problem of steady gravity waves on deep water. He developed a new approach to this problem, which was already 70 years old at that time. Firstly, he formulated it in terms of a *complex velocity potential* \( w(z) = \phi + i\psi \), considered as a function of the complex variable \( z = x + iy \) in the domain under the wave profile \( y = \eta(x) \). (It should be noted that it was Michell who first used methods from complex analysis in water-wave theory; see his pioneering paper [43] on the Stokes wave of extreme form.) Secondly, he proposed a new transformation of the problem, which is indeed one of the key steps on the way to deriving his integral equation. In our presentation of Nekrasov’s results, his terminology and notation (which vary from paper to paper) are unified and updated.
Let the velocity field be $\nabla \varphi$ ($\nabla$ denotes the gradient) in the infinitely deep water and suppose that

$$\lim_{y \to -\infty} \nabla \varphi = (c, 0).$$  \hfill (1)

Nekrasov assumed that the waves are periodic and symmetric about their troughs and crests and denoted their wavelength by $\lambda$ (see, for example, [73] or [12], ch. 10, for a detailed statement of the problem). Let the origin in the $z$-plane be at a trough (this is convenient in what follows; Nekrasov’s placed the origin at a crest, but this is unimportant). The appropriately chosen complex potential $w(z)$ maps the vertical half-strip

$$D_z = \{-\lambda/2 < x < \lambda/2, -\infty < y < \eta(x)\}$$

with unknown upper boundary (corresponding to the water domain under a single wave) conformally onto the fixed half-strip

$$D_w = \{-c\lambda/2 < \phi < c\lambda/2, -\infty < \psi < 0\}$$
in the $w$-plane. The new transformation proposed in [49] is

$$u(w) = \exp\{2\pi w/(ic\lambda)\};$$ \hfill (2)

it maps $D_w$ onto the unit disc cut along the nonpositive real axis, that is

$$D_u = \{|u| < 1\} \setminus \{\text{Re}u \leq 0; \text{Im}u = 0\},$$ \hfill (3)
in the auxiliary $u$-plane. Nekrasov noticed that the mapping $D_u \to D_z$ is defined by the relation

$$\frac{dz}{du} = \frac{i\lambda}{2\pi} \frac{f(u)}{u},$$ \hfill (4)

where

$$f(u) = 1 + a_1u + a_2u^2 + \cdots$$

with real coefficients $a_k$; moreover, $f$ is analytic in the unit disc. This observation allowed him to derive a formula for the potential energy in terms of $a_k$ together with the representation

$$\eta(\theta) = \frac{\lambda}{2\pi} \left( a_1 \cos \theta + \frac{a_2}{2} \cos 2\theta + \frac{a_3}{3} \cos 3\theta + \cdots \right),$$ \hfill (5)

$$x(\theta) = -\frac{\lambda}{2\pi} \left( \theta + a_1 \sin \theta + \frac{a_2}{2} \sin 2\theta + \frac{a_3}{3} \sin 3\theta + \cdots \right)$$ \hfill (6)
of the free surface profile parametrised by $\theta \in [-\pi, \pi]$, which coincides with the polar angle in the $u$-plane. Moreover, differentiating the Bernoulli equation with respect to $\theta$, Nekrasov reduced it to

$$\frac{d}{d\theta} \left[ f(e^{i\theta}) f(e^{-i\theta}) \right]^{-1} = \frac{g\lambda}{\pi c^2} \frac{d\eta}{d\theta}(\theta),$$

(7)

where $g$ is the acceleration due to gravity. Finally, several examples were given in which the leading coefficients $a_k$ can be approximately determined, but they are of little interest.

In his third paper [47] (the first published in the Bulletin of the Ivanovo-Voznesensk Polytechnic), Nekrasov extended his analysis to what he called “the limiting form” of periodic waves, “the possibility of which was first predicted by Stokes”. No reference is given, but he was referring to either Section B of [67], entitled Considerations relative to the greatest height of oscillatory irrotational waves which can be propagated without change of form, or the 1880 collection of papers [68], in which Stokes published three short appendices and a twelve-page supplement to his 1847 paper [67]. In the second appendix, he coined the term “wave of greatest height” to characterize a certain kind of periodic wave. Since the true height $\eta_{\text{max}} - \eta_{\text{min}}$ may not be maximised by it, this wave is now referred to as the “wave of extreme form” in view of the angle of $2\pi/3$ between the tangents to its profile at a crest.

Nekrasov continues by remarking that “a particular form of these waves was found by Michell”. Again, no reference is given, but it is clear that this remark concerns Michell’s paper [43], in which a procedure for calculating the coefficients in a series describing this wave was developed, but convergence was not proved. Nekrasov then formulates the aim of his paper, namely “to show how the general wave can be obtained”. For this purpose he applies a modification of the transformation used in his previous paper for smooth waves. In place of (4) the mapping $D_u \rightarrow D_\xi$ is now described by the relation

$$\frac{d\xi}{du} = -i\lambda \frac{\hat{f}(u)}{2\pi u \sqrt{1-u}},$$

(8)

where

$$\hat{f}(u) = 1 + \hat{a}_1 u + \hat{a}_2 u^2 + \cdots$$

with real coefficients $\hat{a}_k$, in order to capture the singularity. The resulting representation of the free surface profile is

$$\frac{d\hat{\eta}}{d\theta} = \frac{\lambda}{2\pi \sqrt[6]{2} \sin \theta/2} \left( \sin \frac{\pi - \theta}{6} + \sum_{k=1}^{\infty} a_k \sin \frac{\pi + (6k - 1)\theta}{6} \right),$$

(9)

$$\frac{d\eta}{d\theta} = \frac{\lambda}{2\pi \sqrt[6]{2} \sin \theta/2} \left( \cos \frac{\pi - \theta}{6} + \sum_{k=1}^{\infty} a_k \cos \frac{\pi + (6k - 1)\theta}{6} \right)$$

(10)
for $\theta \in [-\pi, \pi]$. Here integration is not as simple as in the case leading to (5) and (6). A complicated nonlinear equation similar to (7) is obtained to determine the coefficients $\hat{a}_k$; it equates the right-hand side of (9) (with an extra constant factor) and the derivative of

$$(\sin \theta/2)^{2/3} \int \left[ \hat{f}(e^{i\theta}) \hat{f}(e^{-i\theta}) \right].$$

After tedious calculations Nekrasov obtained only two leading coefficients, but this was insufficient to achieve Michell’s accuracy. Almost 40 years later, Nekrasov’s method was rediscovered by Yamada [79], whose calculations gave the same accuracy as Michell’s.

The breakthrough In his celebrated paper [48], Nekrasov combined the approach developed in [49] (see (2), (3) and (7) above) with modifications of two earlier results: one obtained by M.P. Rudzki (1862–1916) (whose main field of research was geophysics; see [25]), the other from the dissertation of N.N. Luzin (1883–1950) (a Soviet Academician noted for his numerous contributions to mathematics and his famous doctoral students at Moscow University, known as the “Luzitania” group). This allowed Nekrasov to derive his integral equation for waves on deep water described by the unknown slope of the free surface profile.

In 1898 Rudzki proposed a transformation of the Bernoulli equation (see his paper [61] and also [78], pp. 727–728, where the main idea is explained). In the transformed equation, the sine of the angle between the velocity vector in a two-dimensional flow and the positive $x$-axis is expressed in terms of the harmonic conjugate to the (harmonic) function describing the angle. This allowed Rudzki to apply an inverse procedure (see [54] for a description) to obtain an exact solution for waves over a corrugated bottom; see [78], pp. 737–739, in particular, figure 52 b.

Nekrasov mentioned this solution in his paper [48], remarking that Rudzki failed to consider the case of a horizontal bottom, while his own aim was to investigate waves on deep water. He modified Rudzki’s transformation by introducing two functions $R(\theta)$ and $\Phi(\theta)$ such that

$$\begin{align*}
- \frac{2\pi}{\lambda} \frac{dx}{d\theta}(\theta) &= R(\theta) \cos \Phi(\theta), \\
- \frac{2\pi}{\lambda} \frac{d\eta}{d\theta}(\theta) &= R(\theta) \sin \Phi(\theta),
\end{align*}$$

where $\eta$ and $x$ are defined by (5) and (6), respectively. Thus $\Phi(\theta)$ is the angle between the tangent to the wave profile $\eta$ and the positive $x$-axis, parametrised by $\theta \in [-\pi, \pi]$ over a single period between troughs. In the new variables the differentiated Bernoulli equation (7) takes the form

$$\frac{dR^{-3}}{d\theta} = \frac{3g\lambda}{2\pi c^2} \sin \Phi = \frac{d\exp\{-3 \log R\}}{d\theta},$$

which, as Nekrasov emphasised in his paper, was a crucial point of his considerations. It is worth mentioning that formula (32.88) in Wehausen’s description of Rudzki’s transformation (see [78], p. 728) is similar to the second equality in (12).
The same formula (12) was also crucial for an alternative approach to Stokes waves on deep water developed by Levi-Civita; see equation (I) on p. 280 of his paper [37]. (T. Levi-Civita (1873–1941) is best known for his work in differential geometry, but his interest in the theory of water waves goes back to 1907; see [45] and references cited therein.) Though Levi-Civita’s work was independent of Nekrasov’s (the preliminary note [36] was published in 1924), Levi-Civita acknowledges Nekrasov’s priority; indeed, the introductory section of the article [37] ends as follows (see p. 268):

“Le savant professeur de Moscou a eu la grande amabilité de me transmettre peu avant le Congrès une rédaction française de ses travaux en langue russe (qui ont commencé à paraître dès 1921). Ceci m’a permis de les signaler moi-même au Congrès.”

The meaning of this paragraph becomes clear from a remark in Nekrasov’s *Collected Papers* [53] made by the editor after the paper [48]. According to this remark, Nekrasov prepared a lengthy manuscript in Russian in 1921 entitled *On steady waves* and consisting of chapters I, II and III. The first [48] was published the same year, while the second [50] appeared a year later with the subtitle *On nonlinear integral equations*. The third, concerning the solvability of Nekrasov’s equation, was never published, but its translation into French (presumably abridged) was submitted to the First International Congress on Applied Mechanics, where it was presented by Levi-Civita, and subsequently appeared in the *Proceedings of the Congress* published by Waltman, Delft, in 1925. Unfortunately, the present author was unable to find the exact reference. Despite the fact that Levi-Civita—who was one of the most prominent experts in mechanics at that time—praised Nekrasov’s results, they were neglected by researchers for several decades.

Three scientists from the USSR participated in the Congress held in Delft in April 1924 (see [18], Appendix 2, p. 91). One of them was Aleksandr A. Friedmann, whose contributions to hydrodynamics and meteorology were outstanding, to say nothing about his groundbreaking results in cosmology. He left interesting testimony about the Congress in a letter to his teacher Academician V. A. Steklov sent on 2 May 1924 (see [74], p. 191): “Everything went well at the congress, the attitude towards the Russians was wonderful; in particular, I was included among the members of the committee for convening the next conference. […] Blumenthal, Kármán and Levi-Civita became interested in my own and my colleagues’ work.”

Let us turn to the final—but no less significant—step in deriving an equation for $\Phi$; it concerns the deduction of a further relation involving $\Phi$ and $R$. It was Luzin—one of Nekrasov’s colleagues at the Ivanovo-Voznesensk Polytechnic—who pointed the way to obtaining such a relation (acknowledged in a footnote in [48]). In his outstanding dissertation [40] published in 1915, Luzin proved the following theorem, which involves the singular integral operator $C$ defined almost everywhere on $[-\pi, \pi]$ by the formula

$$CU(\theta) = \frac{1}{2\pi} \text{PV} \int_{-\pi}^{\pi} U(\tau) \cot \frac{\theta - \tau}{2} d\tau,$$

where PV denotes the Cauchy principal value.
Let $U$ be a function harmonic in the open unit disc $D$ and suppose that

$$U(\theta) = \lim_{r \to 1} U(re^{i\theta})$$

belongs to $L^2(-\pi, \pi)$. It follows that $V(\theta) = C U(\theta)$ also belongs to $L^2(-\pi, \pi)$ and is the trace on the unit circle of the harmonic conjugate $V$ to $U$ in $D$.

This theorem inspired numerous generalisations (see, for example, [80] for a description), the first of which was due to Privalov (a member of “Luzitania”), who demonstrated [58] that the operator $C : C^\alpha[-\pi, \pi] \to C^\alpha[-\pi, \pi]$ is bounded for all $\alpha \in (0, 1)$. Dini’s formula (see, for example, [23], pp. 266–267) is another result from the same area of harmonic analysis. Presumably, Luzin realised that a consequence of his formula would be a relation connecting $\Phi_1$ and $R$ and suggested that Nekrasov consider this idea. Indeed, the following corollary of Dini’s formula was obtained in [48].

Let $U + iV$ be holomorphic in $D$, and let $V(\theta) = \lim_{r \to 1} V(re^{i\theta})$ be absolutely continuous on $[-\pi, \pi]$ and such that $V(2\pi - \theta) = V(\theta)$ for all $\theta \in [-\pi, \pi]$. It follows that

$$U(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} V'(\tau) \log \left| \frac{\sin(\theta + \tau)/2}{\sin(\theta - \tau)/2} \right| d\tau + \text{const.} \tag{13}$$

Two formulae similar to (13), but expressing $V(\theta)$ in terms of $U'(\theta)$ were also given in [48] for symmetric and antisymmetric $U(\theta)$.

Nekrasov applied this proposition to the function $i \log f(u)$, which is holomorphic in $D$ with $U(\theta) = -\Phi(\theta)$ and $V(\theta) = \log R(\theta)$ (the latter satisfying the symmetry condition). Clearly (13) becomes

$$\Phi(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\log R \log \left| \frac{\sin(\theta - \tau)/2}{\sin(\theta + \tau)/2} \right| d\tau \tag{14}$$

in this case (the constant vanishes because $\Phi(0) = 0$ for a symmetric wave). Eliminating $R$ from this relation using (12), Nekrasov arrived at his integral equation for waves on deep water which are symmetric about the vertical line through a crest (and trough):

$$\Phi(\theta) = \frac{\mu}{6\pi} \int_{-\pi}^{\pi} \frac{\sin \Phi(\tau)}{1 + \mu \int_0^\pi \sin \Phi(\xi) d\xi} \log \left| \frac{\sin(\theta + \tau)/2}{\sin(\theta - \tau)/2} \right| d\tau, \quad \theta \in [-\pi, \pi]. \tag{15}$$

Here $\mu$ is a non-dimensional parameter which initially arose as the constant of integration when determining $R$ from (12). Subsequently, Nekrasov found an expression for this parameter in terms of characteristics of the wave train, namely

$$\mu = \frac{3}{2\pi} \frac{gc\lambda}{q_0^3}, \tag{16}$$
where \( g, c \) and \( \lambda \) are defined above and \( q_0 \neq 0 \) is the velocity at a crest.

What else can be found in the paper [48]? Firstly, a restriction on \( \mu \) is obtained. Assuming that a solution to (15) satisfies \( |\Phi(\theta)| \leq M \), Nekrasov demonstrated that the corresponding \( \mu \) must satisfy the inequality

\[
\mu < \left[ \pi M + \frac{\sin M}{3M} \right]^{-1}.
\]

(17)

Secondly, it is shown that the equality \( \Phi(\theta) = \Phi(\pi - \theta) \) cannot hold for all \( \theta \in (0, \pi/2) \) if \( \Phi \) is a nontrivial solution of (15). This rigorously proves the observation made by Stokes that his approximate wave profile has sharpened crests and flattened troughs; see the corresponding figure in [68], p. 212, reproduced in [15], p. 31. Finally, several pages towards the end of [48] are devoted to various modifications and simplifications of equation (15) obtained under various assumptions about the wave (gentle slope, etc.). In the final paragraph Nekrasov introduces the linearisation

\[
\Phi(\theta) = \frac{\mu}{6\pi} \int_{-\pi}^{\pi} \Phi(\tau) \log \left| \frac{\sin(\theta + \tau)/2}{\sin(\theta - \tau)/2} \right| d\tau, \quad \theta \in [-\pi, \pi]
\]

(18)

of (15). He claims that its characteristic values are 3, 6, 9, \ldots with corresponding eigenfunctions \( \sin \theta, \sin 2\theta, \sin 3\theta, \ldots \).

### 2.3 Nekrasov’s Approach to the Existence of Nontrivial Solutions

At the end of his paper [48], Nekrasov announced that a method to solve the integral equation (15) would appear in a separate paper; a manuscript with the title “On steady waves, III” was prepared, but to date has not been published (see the remark after formula (12)). According to the concluding paragraph of [48], the method was based on theory developed for a certain class of nonlinear integral equations presented in [50] (the last of Nekrasov’s publications in the Bulletin of the Ivanovo-Voznesensk Polytechnic, 1922). The unpublished manuscript was used while preparing [52] for publication almost 30 years later, in the course of which some improvements were proposed by Y.I. Sekerzh-Zenkovich (1899–1985) (each acknowledged in the text). Sekerzh-Zenkovich subsequently became the editor of Nekrasov’s Collected Papers, I, [53], and translated [44] and [66] into Russian.

Let us outline some key points of Nekrasov’s approach to the existence of nontrivial solutions as they are presented in [52], mainly in Sections 5 and 6. He began with a straightforward calculation based on Euler’s formula to transform the kernel in Eq. (15), namely

\[
\log \left| \frac{\sin(\theta + \tau)/2}{\sin(\theta - \tau)/2} \right| = 2 \sum_{k=1}^{\infty} \frac{\sin k\theta \sin k\tau}{k}.
\]
This sum over $3\pi$ was denoted by $K(\theta, \tau)$ and Eq. (15) written in the form

$$\Phi_1(\theta) = \mu \int_{-\pi}^{\pi} \frac{\sin \Phi(\tau)}{1 + \mu \int_0^\tau \sin \Phi(\zeta) \, d\zeta} K(\theta, \tau) \, d\tau, \quad \theta \in [-\pi, \pi]. \tag{19}$$

Nekrasov then noticed that his assertion about the set of solutions to the linearised equation (18) immediately follows from the series for $K(\theta, \tau)$.

Since the right-hand side in inequality (17) is equal to 3 when $M = 0$, Nekrasov concluded that there are no non-trivial solutions when $\mu \leq 3$. It thus seemed reasonable
to introduce $\mu' = 3 - \mu > 0$ and to seek a solution of the form

$$\Phi(\theta, \mu') = \mu' \Phi_1(\theta) + \mu'^2 \Phi_2(\theta) + \mu'^3 \Phi_3(\theta) + \mu'^4 \Phi_4(\theta) + \cdots.$$  

thus reducing (19) to an infinite system of equations for $\Phi_k$. He also emphasised that the series would represent a solution only if its convergence were proved. In modern terms, his aim was to construct the leading-order part of the solution branch bifurcating from the first characteristic value of the linearised operator.

Another of Nekrasov’s aims was to obtain $\Phi_1, \Phi_2$ and $\Phi_3$ explicitly. To this end he noticed that

$$\sin \Phi = \mu' \Phi_1 + \mu'^2 \Phi_2 + \mu'^3 (\Phi_3 - \Phi_1^3/6) + \mu'^4 (\Phi_4 - \Phi_1^2 \Phi_2/2) + \cdots$$

and

$$\frac{(3 + \mu')}{{(1 + (3 + \mu') \int_0^\pi \sin \Phi(\xi) \, d\xi)}^2} = \mu' \Phi_1 + \mu'^2 \left[ 3 \Phi_2 + \Phi_1 - 9 \Phi_1 \int_0^\tau \Phi(\zeta) \, d\zeta \right] + \cdots.$$ (Nekrasov wrote down two further terms in the second expansion, taking three more lines of mathematics and three lines of additional notation.) Substituting these expansions into (19) and equating the coefficients at every $\mu'^k (k = 1, 2, \ldots)$, he would obtain a recurrent sequence of linear integral equations for $\Phi_k$. The first two equations read

$$\Phi_1(\theta) = 3 \int_{-\pi}^\pi \Phi_1(\tau) K(\theta, \tau) \, d\tau, \quad (20)$$

$$\Phi_2(\theta) = 3 \int_{-\pi}^\pi \Phi_2(\tau) K(\theta, \tau) \, d\tau + \int_{-\pi}^\pi \Phi_1(\tau) \left[ 1 - 9 \int_0^\tau \Phi_1(\zeta) \, d\zeta \right] K(\theta, \tau) \, d\tau. \quad (21)$$

The corresponding formulae for $\Phi_3$ and $\Phi_4$ are given in [52]; they look horrendous, but are still tractable.

It is clear that $\Phi_1(\theta) = C_1 \sin \theta$ is a non-trivial solution of (20), where $C_1 \neq 0$ is to be found from the orthogonality of this function and the free term in (21) (the solvability condition for this equation). One finds that $C_1 = 1/9$, so that after some obvious simplification (21) takes the form

$$\Phi_2(\theta) = 3 \int_{-\pi}^\pi \Phi_2(\tau) K(\theta, \tau) \, d\tau + 108^{-1} \sin 2\theta. \quad (22)$$
Hence \( \Phi_2(\theta) = C_2 \sin \theta + 54^{-1} \sin 2\theta \), and the above procedure yields \( C_2 = -8/243 \).

After determining \( \Phi_3(\theta) \), Nekrasov concluded Section 5 with the formula

\[
\Phi(\theta, \mu') = \left( \frac{1}{9} \mu' - \frac{8}{243} \mu'^2 + \frac{115}{13122} \mu'^3 + \ldots \right) \sin \theta \\
+ \left( \frac{1}{54} \mu'^2 - \frac{8}{729} \mu'^3 + \ldots \right) \sin 2\theta + \left( \frac{17}{4374} \mu'^3 + \ldots \right) \sin 3\theta + \ldots, 
\]

accompanied by the remark that it remains to show that this series is convergent. On the basis of this formula, Nekrasov found that the wave height is

\[
\eta_{\text{max}} - \eta_{\text{min}} = \eta(\pi) - \eta(0) = \lambda \left[ a \left( \frac{1}{9} \mu' - \frac{8}{243} \mu'^2 + \frac{71}{6561} \mu'^3 + \ldots \right) \right].
\]

At the beginning of the next section, in which he deals with the question of convergence, Nekrasov pointed out that his method for solving nonlinear integral equations, which is developed in [50] and reproduced in [52], Sections 13 and 14, is not directly applicable to (19). The reason is the presence of \( \int_0^\pi \sin \Phi_1(\tau) \, d\tau \) in the integrand. To overcome this difficulty “an artificial trick was used in the original manuscript”. It is not clear how the trick worked, but Nekrasov preferred to follow the method suggested by Sekerzh-Zenkovich “as more natural”. Its essence is to replace (19) by a system of two integral equations. Setting

\[
\Psi(\theta) = \left[ 1 + \mu \int_0^\theta \sin \Phi_1(\tau) \, d\tau \right]^{-1},
\]

one finds that

\[
\Psi'(\theta) = -\Psi^2(\theta) \sin \Phi_1(\theta),
\]

while (19) takes the form

\[
\Phi(\theta) = \mu \int_{-\pi}^\pi \Psi(\tau) \sin \Phi_1(\tau) K(\theta, \tau) \, d\tau, \quad \theta \in [-\pi, \pi].
\]

Since \( \Psi(0) = 1 \), the above differential equation yields

\[
\Psi(\theta) = 1 - \mu \int_0^\theta \Psi^2(\tau) \sin \Phi_1(\tau) \, d\tau, \quad \theta \in [-\pi, \pi],
\]

which, together with (24), constitutes the required system for \( \Phi \) and \( \Psi \). As in his paper [48], Nekrasov introduced the parameter \( \mu' \), and sought a solution of this system in the form of the series

\[
\Phi(\theta, \mu') = \sum_{k=1}^{\infty} \mu'^k \Phi_k(\theta), \quad \Psi(\theta, \mu') = 1 + \sum_{k=1}^{\infty} \mu'^k \Psi_k(\theta).
\]
Nekrasov’s subsequent considerations are rather vague and can hardly be considered a rigorous proof. They concern the equations forming the infinite system for \( \{ \Phi_k \} \) and \( \{ \Psi_k \} \), the question of solvability of this system and a method of majorising functions to prove the convergence of the series. Indeed, Sekerzh-Zenkovich, the editor of Nekrasov’s *Collected Papers, I* (see [53], around one fifth of whose content is taken up by the article [52]) added five pages of comments aimed at clarifying these points.

Fortunately, a clear presentation of this approach is available in English; see [75], Section 37, where the generalisation

\[
\Phi(\theta) = \mu \int_{-\pi}^{\pi} \frac{\sin \Phi(\tau) + P(\tau) \cos \Phi(\tau)}{1 + \mu \int_0^{\pi} \left[ \sin \Phi(\zeta) + P(\zeta) \cos \Phi(\zeta) \right] d\zeta} K(\theta, \tau) d\tau, \quad \theta \in [-\pi, \pi],
\]

does equation (19) is investigated. This equation, proposed by Sekerzh-Zenkovich [63], describes waves generated by a small-amplitude periodic pressure applied to the free surface of an infinitely deep flow (the variable \( P \) is related to the \( x \)-derivative of the pressure). It is worth noticing that \( \Phi \equiv 0 \) is not a solution of this equation.

Small solutions of equation (19) are also considered in the book [75] (see Sections 13.4 and 13.5); this approach is based on the expansion

\[
\left[ 1 + \mu \int_0^{\theta} \sin \Phi(\tau) d\tau \right]^{-1} = \sum_{k=0}^{\infty} \left[ -\mu \int_0^{\theta} \sin \Phi(\tau) d\tau \right]^k.
\]

Further studies of equation (19) are outlined in Sect. 3 below, where methods of nonlinear functional analysis are applied; these tools have been actively developed since the 1950s.

### 2.4 Nekrasov’s Integral Equation for Waves on Water of Finite Depth

Nekrasov’s studies of water waves were interrupted by an overload of teaching, administrative duties and preparation of Joukowski’s papers for publication after his death in 1921. However in 1927 he presented his second integral equation at the All-Russian Congress of Mathematicians; see the enlarged abstract [51], reproduced (with the author’s permission) in the survey monograph [65] published in 1936.

The presentation in [51] follows that in the paper [48] dealing with waves on deep water, but with appropriate amendments. The stream function \( \psi \) (the imaginary part of the complex velocity potential \( w \)) is now chosen so that it vanishes on the free surface \( y = \eta(x) \) (the location of the origin is however not specified), while the domain

\[
D_z = \{-\lambda/2 < x < \lambda/2, -h < y < \eta(x)\}
\]

corresponding to a single wave is mapped conformally onto the annulus

\[
D_u = \{r_0 < |u| < 1\} \setminus \{\text{Re} u < 0; \text{Im} u = 0\},
\]
where \( \log r_0 = -2\pi h \). On the basis of a formula analogous to (4) Nekrasov concluded that

\[
\frac{dw}{dz} = -\frac{c}{e^{\Omega(u)}},
\]

where

\[
\Omega(u) = a_0 + \sum_{k=1}^{\infty} a_k \left[ \left( \frac{u}{r_0} \right)^k + \left( \frac{r_0}{u} \right)^k \right]
\]

has real coefficients in view of the bottom boundary condition. (It should be mentioned that the meaning of \( c \), referred to as the wave velocity in [51], was clarified in [52], where, at the end of Section 8, it was demonstrated that \( c \) is the mean velocity of the flow at the bottom.) Nekrasov obtained formulae similar to (11) for \( \Phi_1(\theta) \) and \( \Psi_1(\theta) \) (the imaginary and real parts of \( \Omega(e^{i\theta}) \)), namely

\[
\frac{dx}{d\theta}(\theta) = -\frac{\lambda}{2\pi} \frac{e^{\Psi(\theta)}}{\Psi_1(\theta)} \cos \Phi_1(\theta),
\]

\[
\frac{d\eta}{d\theta}(\theta) = -\frac{\lambda}{2\pi} \frac{e^{\Psi(\theta)}}{\Psi_1(\theta)} \sin \Phi_1(\theta),
\]

which imply that \( \Phi_1(\theta) \) is the angle between the tangent to the wave profile \( \eta \) and the positive \( x \)-axis. Moreover, since

\[
\Psi(\theta) = a_0 + \sum_{k=1}^{\infty} a_k \left( r_0^{-k} + r_0^k \right) \cos k\theta,
\]

where \( a_0 \) and \( a_k \left( r_0^{-k} + r_0^k \right) \) are the Fourier cosine coefficients of \( \Psi \), we have that

\[
\Omega(u) = \frac{1}{\pi} \int_{-\pi}^{\pi} \Psi(\theta) \left( \frac{1}{2} + \sum_{k=1}^{\infty} \left[ \frac{u^k \cos k\theta}{1 + r_0^{2k}} + \frac{r_0^{2k} \cos k\theta}{u^k (1 + r_0^{2k})} \right] \right) d\theta.
\]

Nekrasov noticed that

\[
\sum_{k=1}^{\infty} \frac{r_0^{2k} \cos k\theta}{u^k (1 + r_0^{2k})} = \frac{u}{2} \frac{d}{du} \sum_{k=1}^{\infty} (-1)^{k-1} \log \left[ 1 - \frac{2r_0^{2k}}{u} \cos \theta + \frac{r_0^{4k}}{u^2} \right],
\]

and a similar formula holds for the first sum in (27). This sum of logarithms is equal to the logarithm of an infinite product naturally representable in terms of Weierstrass sigma functions whose periods \( \omega \) and \( \omega' \) satisfy \( \omega' / i\omega = 4h / \lambda \). These considerations reduce (27) “after tedious calculations” to

\[
\Phi(\theta) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi'(\tau) \log \frac{\sigma(\omega(\theta + \tau)/\pi) \sigma_3(\omega(\theta - \tau)/\pi)}{\sigma_3(\omega(\theta + \tau)/\pi) \sigma(\omega(\theta - \tau)/\pi)} d\tau, \quad \theta \in [-\pi, \pi]
\]

(28)
(most of the details are also omitted in [52]). Note that (28) is analogous to the deep-water formula (14).

On the other hand, differentiating Bernoulli’s equation with respect to $\theta$ yields

$$\frac{d e^{-3\Psi}}{d\theta} = \frac{3 g \lambda}{2\pi c^2} \sin \Phi,$$

so that

$$\Psi' (\theta) = \frac{-\mu}{3} \frac{\sin \Phi(\theta)}{1 + \mu \int_0^\theta \sin \Phi(\tau) \, d\tau},$$

where $\theta \in [-\pi, \pi]$ and $\mu$ is the same as for deep water; see (16). Here it is appropriate to mention that Levi-Civita’s method was extended by Struik to the finite-depth case; see his article [70] and the preliminary note [69] published in 1925, which gives him priority over Nekrasov, who announced his finite-depth equation in 1928. (D. J. Struik (1894–2000) obtained a Rockefeller Fellowship to work abroad and came to Levi-Civita, who was famous for his results in differential geometry, the topic of Struik’s PhD thesis. However, he was encouraged to try water waves instead; see [45].) Following Levi-Civita’s approach, based in particular on (12), Struik used an analogue of (29) in his considerations; see formula (F) in [70], p. 601.

Let us return to Nekrasov’s considerations. Nekrasov combined (28) and (29), thus arriving at his integral equation

$$\Phi (\theta) = \frac{\mu}{6\pi} \int_{-\pi}^\pi \frac{\sin \Phi(\tau)}{1 + \mu \int_0^{\pi/2} \sin \Phi(\tau) \, d\tau} \log \left| \frac{\sigma(\omega(\theta + \tau)/\pi) \sigma_3(\omega(\theta - \tau)/\pi)}{\sigma_3(\omega(\theta + \tau)/\pi) \sigma(\omega(\theta - \tau)/\pi)} \right| \, d\tau, \quad \theta \in [-\pi, \pi]$$

for waves on water of finite depth. This equation differs from the deep-water equation (15) in the expression in the logarithm. At the end of the note [51], Nekrasov pointed out that

$$\log \left| \frac{\sigma(\omega(\theta + \tau)/\pi) \sigma_3(\omega(\theta - \tau)/\pi)}{\sigma_3(\omega(\theta + \tau)/\pi) \sigma(\omega(\theta - \tau)/\pi)} \right| = 2 \sum_{k=1}^{\infty} \frac{1 - r_0^{2k} \sin k\theta \sin k\tau}{1 + r_0^{2k}} \frac{\sin k\theta \sin k\tau}{k},$$

which was demonstrated in [52] with the help of some formulae from [71]. (The presentation of the theory of elliptic functions in this 1895 book was quite modern and advanced for that time). Furthermore, the equation

$$\frac{1 - r_0^{2k}}{1 + r_0^{2k}} = \tanh (2\pi k h/\lambda)$$

expresses the sum in terms of geometric characteristics of the wave.
Denoting the above sum over $3\pi$ by $K(\theta, \tau)$, we see that Eq. (30) takes exactly the same form as (19). This similarity was emphasised by Nekrasov, and for this reason he restricted himself to calculating just the first two terms in the expansion analogous to (23), which expresses a solution $\Phi$ of (30) as a series in powers of a small parameter $\mu'$. For $h < \infty$, he used $\mu' = \mu_1 - \mu$, where $\mu_1$ is the first characteristic value of $K(\theta, \tau)$, which coincides with the of the kernel of the linearisation of the operator in Eq. (30). The set of these characteristic values, namely \{$(3k \coth(2\pi kh/\lambda))_{k=1}^{\infty}$\}, was announced in [51].

In the monograph [52], Nekrasov concludes his considerations of Eq. (30) with the brief Section 11, in which the following important property of this equation is formulated. Since the kernel satisfies $K(\theta, 2\pi - \tau) = -K(\theta, \tau)$, a solution has the symmetry $\Phi(2\pi - \theta) = -\Phi(\theta)$; it therefore suffices to consider an equivalent equation on the interval $[0, \pi]$. This property is also mentioned for a solution of Eq. (19) at the beginning of Sect. 4. The advantage of reducing the equation to $[0, \pi]$ demonstrated by subsequent studies (see below) was however not used in [52].

The tale of Nekrasov and his work on water waves ends here, but it is not the end of the tale of Nekrasov’s integral equations.

### 3 Nekrasov’s Equations and Functional Analysis

Nekrasov’s equations are the oldest examples of nonlinear integral equations describing water waves, but there are many others, some of which are mentioned below. The first survey of this topic seems to have been published in 1964; see [24].

#### 3.1 On Local and Global Branches of Solutions

**Existence theorems for equation (19)** Mark Alexandrovich Krasnosel’skii (1920–1997) (a pioneer of nonlinear functional analysis in the Soviet Union) was the first to consider (19) as an operator equation in a Banach space. His brief note [32] on this topic was published in 1956, just five years after [52]. A general approach to local bifurcation theory for nonlinear operator equations can be found in his monograph [33]; see, in particular, the following assertion applicable to (19) and (30).

Let $A_\mu (\mu > 0)$ be a family of operators defined in a neighbourhood of the zero element $0$ of a Banach space $X$; each operator is assumed to be completely continuous and such that $A_\mu 0 = 0$. Suppose also that the Fréchet derivative of $A_\mu$ is $\mu B$, where $B$ is linear, completely continuous and independent of $\mu$. If $\mu_0$ is a characteristic value of $B$ of odd multiplicity, then the equation $\Phi = A_\mu \Phi$ has a continuous branch of nontrivial solutions $(\mu, \Phi_\mu)$ in a neighbourhood of $(\mu_0, 0)$ and $\|\Phi_\mu\|_X \to 0$ as $\mu \to \mu_0$.

In his note [32], Krasnosel’skii demonstrated that the operator $A_\mu$ defined by the right-hand side of either (19) or (30) is completely continuous in a neighbourhood of $0 \in C[-\pi, \pi]$ and its Fréchet derivative is $\mu B$, where

$$
(B \phi)(\theta) = \int_{-\pi}^{\pi} \phi(\tau) K(\theta, \tau), \ d\tau, \ \theta \in [-\pi, \pi]. \quad (31)
$$
Since all characteristic values of $B$ (namely $\{3k\}_{k=1}^{\infty}$ for infinite depth or $\{3k \coth(2\pi kh/\lambda)\}_{k=1}^{\infty}$ for finite depth) have odd multiplicity, each is a bifurcation point of $A_\mu$; that is, in a neighbourhood of a characteristic value $\mu^*$ equation (19) or (30) has a continuous branch of solutions $\Phi_\mu \in C[-\pi, \pi]$ satisfying $\|\Phi_\mu\|_\infty \to 0$ as $\mu \to \mu^*$.

Although the question of local solution branches for Nekrasov’s equation was thus answered, the existence of global branches was only established more than twenty years later. The first global result for periodic waves was given by Krasovskii [34] in 1961. He reduced the Levi-Civita formulation of the water-wave problem (see [37]) to a nonlinear operator equation in a cone of nonnegative functions in a Banach space. This formulation allowed him to apply Krasnosel’skiı’s theorem on positive operators with monotonic minorants (see [33], Chapter 5, Section 2.6), thus demonstrating the existence of a branch of waves with the property that each value in $(0, \pi/6)$ serves as $\max \Phi$ for some wave profile on the branch. Krasovskii’s approach also proved to be applicable to the case of finite depth with a periodic bottom.

The story continues with the existence of a global branch of solutions to Nekrasov’s equation. The abstract bifurcation theory developed by Rabinowitz [59] and Dancer [16] in the 1970s provided tools for this purpose. A self-contained analysis of this theory was given by Toland (see [73], in particular Sections 8 and 9), and includes the results of Keady and Norbury [28], who obtained a continuum (a maximal closed connected set) of solutions. Their result is formulated in terms of a cone in a real Banach space. Since the origin is chosen at a trough, it is reasonable to use the subspace of $C[-\pi, \pi]$ consisting of odd functions vanishing at zero and $\pi$. The closed convex cone $K$ in this subspace consists of functions such that (i) $f(t) \geq 0$ for $t \in [0, \pi]$; (ii) $f(t)/\sin(t/2)$ is nonincreasing on $[0, \pi]$; (iii) $f(t) \leq f(s)$ for all $t \in [\pi/2, \pi]$ and $s \in [\pi - t, t]$. The crucial point is that the operator $B$ maps both the cone of nonnegative functions on $[0, \pi]$ and $K$ into themselves. We are now in a position to formulate Keady and Norbury’s theorem.

Equation (15) has an unbounded continuum $C$ of solutions in $[0, \infty) \times K$ and $(\mu, 0)$ belongs to $C$ if and only if $\mu = 3$. Furthermore, each $(\mu, \Phi) \in C$ has the following properties:

(i) $\Phi$ vanishes identically for $\mu \leq 3$,
(ii) $0 < \Phi < \pi/3$ and $\Phi(\theta)/\theta$ is nonincreasing on $(0, \pi)$,
(iii) $\Phi'(\theta) \leq 0$ for $\theta \in [\pi/2, \pi]$.

An interesting open question is whether there are solutions of (15) which do not belong to $C$, but lie in a wider cone, for example, the cone defined only by conditions (i) and (ii). It is also not clear whether $C$ is a curve; numerical computations support this conjecture (see Craig and Nicholls [14], in particular Figure 1), but there is no analytical proof at the time of writing.

There are actually infinitely many continua of solutions of equation (15). Indeed, a change of variables shows that each set

$$\{(n\mu, \Phi(n\theta)) : (\mu, \Phi(\theta)) \in C\}, \quad n = 1, 2, \ldots,$$
is also a continuum of solutions bifurcating from \((3n, 0)\); each element of this continuum yields a wave of minimal period \(\lambda/n\).

To the best of author’s knowledge there are no analogous existence results for Eq. (30) describing waves on water of finite depth. However, another integral equation for this case was investigated by Norbury [55]. This equation arises after a hodograph transform; the unknown is the inclination of the flow velocity as a function of the potential along the free surface. The corresponding nonlinear operator is positive and completely continuous on the space of continuous functions, and so the same approach yields an unbounded continuum of solutions; moreover, it includes those found in [34].

**Properties of solutions belonging to \(C\).** Suppose that \((\mu, \Phi_\mu) \in C\) with \(\mu > 3\). The solution \(\Phi_\mu\), which we suppose does not vanish identically, has several additional interesting properties. McLeod [41] demonstrated that \(\|\Phi_\mu\|_\infty > \pi/6\) for sufficiently large values of \(\mu\). Subsequently, Amick [1] obtained an upper bound for this norm; combining its slightly improved version (see [72], p. 36) and McLeod’s result, one finds that

\[
\frac{\pi}{6} < \sup\{\|\Phi\| : (\mu, \Phi) \in C\} < 0.5434 \approx (1.0378)\pi/6 \approx 31.13^\circ.
\]  

(32)

It should be mentioned that this upper bound is very close to the value 0.530 \((\approx 30.37^\circ)\) calculated numerically in [39]. In the paper [1], Amick also showed that that \(\|\Phi\| < 0.544\) for solutions of (30) for periodic waves on water of finite depth; letting the wavelength \(\lambda\) tend to infinity extends this bound to solitary waves.

Clearly, one can reformulate (15) and (19) as

\[
\Phi(\theta) = \int_{-\pi}^{\pi} \frac{\sin \Phi(\tau)}{\nu + \int_{0}^{\tau} \sin \Phi(\zeta) \, d\zeta} K(\theta, \tau) \, d\tau, \quad \theta \in [-\pi, \pi],
\]  

(33)

where \(\nu = \mu^{-1}\), but one can also consider this equation with \(\nu = 0\) (see below). Important results concerning solutions \(\Phi\) of (33) with \(\mu > 3\) were obtained by Amick and Toland; see Appendix in their paper [5]. They showed that \(\Phi\) is a real-analytic function (which in fact is a corollary of Lewy’s theorem [38]), and recovered the fluid domain and velocity field from \(\Phi\) in the following theorem.

**Let \(\Phi\) be a solution of (33) with \(\mu > 3\) and suppose that \(\lambda, c\) are positive real numbers such that**

\[
\left[ 3g\lambda \frac{\sqrt{2\pi c^2}}{2\pi c^2} \right]^{1/3} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos \Phi(\tau) \, d\tau \int_{-\pi}^{\pi} \left[ \mu^{-1} + \int_{0}^{\tau} \sin \Phi(\zeta) \, d\zeta \right]^{1/3}.
\]  

(34)

There exists a fluid domain \(D_z\) with width \(\lambda\), corresponding to a single wave whose profile is characterised locally by the angle \(\Phi\), and a periodic complex velocity potential defined on \(\overline{D_z}\) which satisfies the free-surface boundary conditions on the upper part of \(\partial D_z\). The real number \(c\) satisfies the asymptotic condition (1) as \(y \to -\infty\).

It is worth mentioning that a semi-explicit expression is obtained for the function \(\eta(x)\) describing the upper boundary of \(D_z\). It is rather complicated and involves two integrals, each of which has a variable limit of integration; one of these integrals has the same integrand as (34).
We conclude this section with the result of Kobayashi [30], which gives a partial answer to Toland’s conjecture that \( C \) is a curve (see the end of his paper [73]). Kobayashi established a uniqueness theorem for solutions of equation (33) for \( \mu \in (3, 170) \) using two different approaches and three subintervals. For \( \mu \in (3, 3.009] \) the result was proved analytically. It was demonstrated that uniqueness holds if the maximum of a certain rather complicated function is strictly less than one, and this inequality is verified by tedious estimates. The two remaining segments—[3.009, 3.3] and [3.3, 170]—were examined with the help of a numerical verification method. In the former case one constructs a function whose maximum is strictly less than unity to use in the contraction mapping principle; the inequality is checked numerically with controlled rounding error. The latter case is treated in a similar, but slightly different, fashion.

3.2 Nekrasov’s Equation for Waves of Extreme Form on Deep Water

Let us consider equation (33) with \( \nu = 0 \), which describes a wave with stagnation points at its crests (\( \nu = (2\pi/3) q_0^3/(g\xi) \)) according to (16), where \( q_0 \) is the velocity at a crest). The existence of an odd solution of

\[
\Phi^*(\theta) = \int_{-\pi}^{\pi} \frac{\sin \Phi^*(\tau)}{\int_{0}^{\tau} \sin \Phi^*(\xi) \, d\xi} K(\theta, \tau) \, d\tau, \quad \theta \in [-\pi, \pi],
\]

was proved by Toland [72], who based his approach on Keady and Norbury’s result concerning the existence of solutions of (33) with nonzero \( \nu \). Their method cannot be applied directly because the operator in (35) is not completely continuous. Toland therefore considered a sequence \( \{(\nu_k, \Phi_k)\} \) of solutions to (33) for which \( \nu_k \to 0 \). Since the absolute values of \( \Phi_k \) are bounded, this sequence converges weakly in \( L^2 \) to a nontrivial \( \Phi^* \), and Toland demonstrated that \( \{\Phi_k\} \) in fact converges strongly in \( L^2 \), thus proving that \( \Phi^* \) is a solution of (35). In his paper \( \Phi^* \) was also shown to be continuous on \( [-\pi, \pi] \), except for point of discontinuity at zero, whose nature was not resolved in [72].

This discontinuity was the topic of a conjecture made by Stokes in his “Considerations relative to the greatest height of oscillatory irrotational waves which can be propagated without change of form”—an appendix to [67] (see [68], pp. 225–228). Since the origin is chosen at a trough and the wavelength is \( 2\pi \), the conjecture can be formulated as

\[
\lim_{\theta \uparrow \pi} \Phi^*(\theta) = \pi/6.
\]

Stokes gave a formal asymptotic argument near a stagnation point in support of his conjecture and added

“[…] whether in the limiting form the inclination of the wave to the horizon continually increases from the trough to the summit […] is a question which I cannot certainly decide, though I feel little doubt that .. [convexity].. represents the truth.”
The assertion that the free surface profile \( \eta \) is convex between its successive maxima is nowadays referred to as Stokes’s second conjecture about waves of extreme form. It is now known that there indeed exists an extreme wave having this property (see [57]). The idea of the proof is to study a parametrized family of integral equations, each associated with a free-boundary problem. When the parameter is equal to 1 the equation has a unique solution with an associated convex free boundary, and a continuation argument shows that the existence and convexity of this solution persists as the parameter varies from 1 to 1/3; the equation with parameter value 1/3 is precisely (35). This approach does however leave the question of whether every extreme wave is convex between its successive maxima unanswered.

Let us turn to conjecture (36), which was proved in 1982, independently in [3] and [56]. The authors of [3] discuss the key difficulties of their proof in an appendix (see also the slightly amended version in [73]). In particular, it is noted that the operator in (35) lacks some properties needed for the application of the fixed-point theorems used when \( \nu \neq 0 \). To overcome these difficulties the approximate integral equation

\[
\phi(\theta) = \frac{1}{3\pi} \int_0^\infty \frac{\sin \phi(\tau)}{\sin \phi(\xi)} \log \left| \frac{\theta + \tau}{\theta - \tau} \right| d\tau, \quad \theta \in (0, \infty),
\]

which has the explicit solution \( \phi \equiv \pi/6 \), is introduced. It is proved that this constant solution is the only pointwise bounded solution of (37) such that

\[
\inf_{\theta \in (0, \infty)} \phi(\theta) > 0
\]

and

\[
\sup_{\theta \in (0, \infty)} \phi(\theta) \leq \pi/3;
\]

moreover this property of \( \phi \) implies (36) for any odd solution \( \Phi^* \) of (35) satisfying

\[
0 \leq \Phi^*(\theta) < \pi/3 \quad \text{for} \quad \theta \in (-\pi, \pi)
\]

and

\[
\lim \inf_{\theta \uparrow \pi} \Phi^*(\theta) > 0;
\]

these properties of \( \Phi^* \) were of course established in [5].

The asymptotic formula

\[
\Phi^*(\theta) = \frac{\pi}{6} + C_1(\pi - \theta)^{\beta_1} + C_2(\pi - \theta)^{2\beta_1} + O\left( (\pi - \theta)^{3\beta_1} \right) \quad \text{as} \quad \theta \uparrow \pi,
\]

refines Stokes’s conjecture (36); here \( \beta_1 \approx 0.802679 \) is the so-called Grant number—the smallest positive root of \( \sqrt{3}(1 + \beta) = \tan(\pi\beta/2) \). The formula was established
rigorously in [2] and [42]; the inequalities $C_1 < 0$ and $C_2 > 0$ were proved in respectively [42] and [2].

It is worth mentioning two related results at this point. Fraenkel [19] developed a constructive approach to a variant of the limiting Nekrasov equation describing extreme waves. Along with the existence proof, he provided a rather accurate approximation of its solution. However, his “results depend on the numerical evaluation and numerical integration of functions defined by explicit formulae. […] Therefore, purists may believe that the theorems in the paper have not been proved.” Varvaruca [76] considered a version of Eq. (35) generalised by introducing a parameter into its right-hand side (as in [57]). He derived an equivalent form of his equation involving the $2\pi$-periodic Hilbert transform which allowed him to simplify the proof of Stokes’s (generalised) first conjecture.

We conclude this section with the another result by Kobayashi [31], namely a computer-assisted proof that equation (35) has a unique odd solution. Combining this result with that of Plotnikov and Toland [57], he concluded that every extreme wave is convex between its successive maxima, thus settling Stokes’s second conjecture about these waves. Kobayashi obtained his result in the same way as in his first paper [30]. He proved that solutions to (35) are unique if the supremum of a certain rather complicated function is strictly less than 1, and computed the supremum numerically with controlled rounding error guaranteeing that the approximate result 0.99290443370699 is indeed less than 1.

4 Concluding Remarks

The tale of Nekrasov’s integral equations would not be complete without mentioning further developments during the past forty years, beginning with the seminal papers by Amick and Toland ([4] and [5]) in 1981.

Using a procedure resembling that applied by Nekrasov, an integral equation analogous to (19) for solitary waves was derived in [4], namely

$$\hat{\Phi}(\theta) = \mu \int_{-\pi/2}^{\pi/2} \frac{\sec \tau \sin \hat{\Phi}(\tau)}{1 + \mu \int_0^\pi \sec \zeta \sin \Phi(\zeta) d\zeta} \hat{K}(\theta, \tau) d\tau, \ \theta \in (-\pi/2, \pi/2).$$

(38)

Here $\hat{\Phi}$ denotes the angle between the real axis in the $z$-plane and the negative velocity vector at the free surface (parametrised in a special way because it has infinite extent), while the kernel

$$\hat{K}(\theta, \tau) = \frac{1}{3\pi} \sum_{k=1}^{\infty} \sin 2k\theta \sin 2k\tau \frac{\sin 2k\tau}{k}$$

resembles that in (19). The parameter in the equation is expressed in terms of characteristics of the flow by the formula

$$\mu = \frac{6ghc}{\pi q_0^3}.$$
This formula is analogous to (16), but of course involves the depth $h$ of the flow at infinity.

Unfortunately global bifurcation theory is not directly applicable directly to equation (38) because $\sec \tau$ is singular at $\tau = \pm \pi/2$. To overcome this difficulty Amick and Toland considered the sequence

$$
\hat{\Phi}_n(\theta/2) = 2\mu_n \int_0^\pi \frac{f_n(\tau) \sin(J \hat{\Phi}_n(\tau/2))}{1 + \mu_n \int_0^\theta f_n(\xi) \sin(J \hat{\Phi}_n(\xi/2)) \, d\xi} K(\theta, \tau) \, d\tau, \quad \theta \in (0, \pi),
$$

(39)
of Nekrasov-type equations. Here $K$ is the kernel from (19), $J : \mathbb{R} \to \mathbb{R}$ is the continuous function

$$
Ja = \begin{cases}
  a & \text{if } |a| \leq \pi, \\
  \pi & \text{if } a \geq \pi, \\
  -\pi & \text{if } a \leq -\pi,
\end{cases}
$$

and the sequence of functions $\{f_n\}$ is defined by

$$
f_n(\tau) = \begin{cases}
  2^{-1} \sec(\tau/2) & \text{if } |\tau| \leq \pi - n^{-1}, \\
  2^{-1} \sec([\pi - n^{-1}]/2) & \text{if } \pi - n^{-1} \leq |\tau| \leq \pi.
\end{cases}
$$

Equation (39) is tractable in the same way as (19) (see Sect. 3.1): for every integer $n \geq 1$ there exists an unbounded, closed and connected set $C_n$ of solutions. Amick and Toland demonstrated that the sets $C_n$ converge in a certain sense to an unbounded, closed and connected set whose elements $(\mu, \hat{\Phi})$ are solutions of (38); moreover, there is a solution for every $\mu \in [6/\pi, \infty)$. Properties of these solutions are investigated in detail and the existence of a solitary wave of extreme form is also established.

Amick and Toland developed their technique further in [5]. In this paper they give a detailed discussion of steady water waves on water of finite depth, present a new integral equation for the periodic problem and adapt the methods of [4] to show the convergence of periodic waves to solitary waves in the long-wave limit. Furthermore, they show how the classical formulation due to Nekrasov yields, via the maximum principle, new results about properties of periodic waves.

An integral equation alternative to Nekrasov’s which describes all steady waves (periodic and solitary) on water of finite depth was proposed by Byatt-Smith [8] in 1970; unfortunately, no existence theory (even local) is available for his equation. It was Konstantin Ivanovich Babenko (1919–1987), who invented a real alternative to Nekrasov’s equation. He proved a local existence theorem for his integro-differential equation describing steady periodic waves on deep water; see the brief notes [6] and [7] published in 1987. A detailed analysis of Babenko’s quasilinear equation is given in [10] and [11] on the basis of properties of the $2\pi$-periodic Hilbert transform involved in the equation and its variational structure; indeed, it is the Euler–Lagrange equation of a simple functional introduced in [10]. A relation between solutions of Babenko’s and Nekrasov’s equations established in [11], p. 252, demonstrates that there is an integro-differential form of the latter equation equivalent to (19). This form
involves the Hilbert transform and differentiation instead of the integral operator with the kernel $K$. In [64], Section 3.6, this new form of Nekrasov’s equation was extended to general Bernoulli free-boundary problems. Conversely, Nekrasov’s equation has also been used for obtaining some properties of solutions to Babenko’s equation; see [12], Theorem 10.4.4.

It was recently demonstrated that the approach used in [6] yields an equation of the same form as Babenko’s for water of finite depth (see [35]), the difference being merely that the $2\pi$-periodic Hilbert transform is now perturbed by a compact operator, so that the existence of local solution branches follows analogously to the deep-water case. Moreover, this equation (and a modification proposed in [17]) are convenient for numerical computation of global bifurcation diagrams, in particular, for (possibly multiple) secondary bifurcations (see [17], Figure 4).

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**Data availability statement** This article contains two quotations and three figures, all of which are in the public domain. The quotation from *A Tale of Two Cities* by Charles Dickens is available at https://en.wikipedia.org/wiki/A_Tale_of_Two_Cities. The second quotation is from Stokes’s Appendix to [67] published in the book [68]; it is available at https://archive.org/details/mathphyspapers01stokrich/page/n9/mode/2up. Figure 1 is available at http://ispu.ru/files/imagecache/640x480/cck-images/Prepodavateli_1922.jpg, while Figures 2 and 3 are scans from the books [53] and [52], which were published by the Soviet Academy of Sciences during the period when there was no copyright law in the USSR.

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