A remark on matrix product operator algebras, anyons and subfactors

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Abstract

We show that the mathematical structures in a recent work of Bultinck-Mariëna-Williamson-Şahinoğlu-Haegemana-Verstraete are the same as those of flat symmetric bi-unitary connections and the tube algebra in subfactor theory. More specifically, a system of flat symmetric bi-unitary connections arising from a subfactor with finite index and finite depth satisfies all their requirements for tensors and the tube algebra for such a subfactor and the anyon algebra for such tensors are isomorphic up to the normalization constants. Furthermore, the matrix product operator algebras arising from tensors corresponding to possibly non-flat symmetric bi-unitary connections are isomorphic to those arising from flat symmetric bi-unitary connections for subfactors.

1 Introduction

A recent work Bultinck-Mariëna-Williamson-Şahinoğlu-Haegemana-Verstraete [4] has caught much attention in the community of condensed matter physics in connection to 2-dimensional topological phases of matter. (See [12], [22], for example. They are also related to the Levin-Wen model [16].) As they themselves are aware and give a citation to [6], the mathematical structures in their work are similar to those in subfactor theory. In this short note, we present precise mathematical relations between the two settings. In fact, under some natural setting, we show that the machinery in [4] is mathematically the same as that of flat symmetric bi-unitary connections of Ocneanu [18], [7, Chapter 10]. We also discuss how to deal with non-flat symmetric bi-unitary connections which also naturally appear in this framework.

The Jones theory of subfactors [10] opened a vast new field in theory of operator algebras and his discovery of the Jones polynomial [11] initiated huge interest in quantum
invariants in 3-dimensional topology. New algebraic structures governing such mathematics are fusion categories and modular tensor categories. See [14] and references therein for operator algebraic approaches to these categories and connections to 2-dimensional conformal field theory. A modular tensor category is a useful tool to study anyons in 2-dimensional phases of matter. An anyon is a quasi-particle in 2-dimensional phases of matter and its exchanges follow braid group statistics. This viewpoint is also expected to be useful for mathematical understanding of topological quantum computations [21]. We would like to present how to use operator algebraic tools to study mathematical problems on anyons related to [4].

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2 Symmetric bi-unitary connections

In order to compare the framework of a recent paper Bultinck-Mariëna-Williamson-Şahinoğlu-Haegemana-Verstraete [4] with that in subfactor theory, we need to present machinery of bi-unitary connections [7, Chapter 11] in a little bit different setting from that in [7] so that we can establish a direct relation to that of tensors in [4]. In this paper, a subfactor always means a subfactor of the hyperfinite II_1 factor with finite Jones index and finite depth. We refer the reader to [7, Chapter 9] for basics of subfactor theory.

We introduce a notion of a symmetric bi-unitary connection. Let $V$ be a finite set. This is regarded as a common set of vertices of a family of finite oriented graphs. Let $G$ be a finite oriented graph whose vertices are in $V$ and $\Delta_G$ its adjacency matrix. That is, $(\Delta_G)_{vw}$ is equal to the number of edges of $G$ from $v$ to $w$. We assume that the matrix $\Delta_G$ is symmetric and for a sufficiently large $n$, all the entries of $\Delta_G^n$ are strictly positive. Let $H$ be another finite oriented graph whose vertices are again in $V$ and $\Delta_H$ be its adjacency matrix. We assume that for any $v \in V$, we have $w \in V$ with $(\Delta_H)_{vw} > 0$ and that for any $w \in V$, we have $v \in V$ with $(\Delta_H)_{vw} > 0$. We assume that each vertex $v \in V$ has a positive value $\mu(v)$. We further assume to have two positive numbers $\beta_G, \beta_H$ such that for each $v$, we have

$$
\beta_G \mu(v) = \sum_{w \in V} (\Delta_G)_{vw} \mu(w),
$$

$$
\beta_H \mu(v) = \sum_{w \in V} (\Delta_H)_{vw} \mu(w),
$$

$$
\beta_H \mu(v) = \sum_{w \in V} (\Delta_H)_{wv} \mu(w).
$$

That is, the vectors given by $\mu$ are the Perron-Frobenius eigenvectors for $\Delta_G, \Delta_H$ and the transpose of $\Delta_H$. 

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We consider a cell as in Fig. 1, where the edges $i, l$ are in $G$, the edges $j, k$ are in $H$, and we have $s(i) = s(k)$, $r(i) = s(j)$, $r(k) = s(l)$ and $r(j) = r(l)$. (Here $s$ and $r$ denote the source and range of an oriented edge, respectively.)

We consider an assignment map $a$ of a complex number to each cell and denote its value as in Fig. 2. We sometimes draw a picture of a cell where at least one of the requirements $s(i) = s(k)$, $r(i) = s(j)$, $r(k) = s(l)$ and $r(j) = r(l)$ fail. In such a case, the picture in Fig. 2 represents a zero value.

Consider a matrix $a$ defined as in Fig. 3, where each row is labeled with a pair $(i, j)$ with $r(i) = s(j)$ and each column is labeled with a pair $(k, l)$ with $r(k) = s(l)$. We use the same symbol $a$ for this matrix as the assignment map. (Note that if we have $s(i) \neq s(k)$ or $r(j) \neq r(l)$, then the entry is 0 by our convention.) Then our first requirement is that this matrix $a$ is unitary.

Next consider a matrix $\bar{a}$ defined as in Fig. 4, where $\tilde{j}$ and $\tilde{k}$ denote the edges $j$ and $k$ with reversed orientations and the bar above the square on the right hand side means the complex conjugate. Note that each row is labeled with a pair $(l, \tilde{j})$ with $r(j) = r(l)$ and each column is labeled with a pair $(\tilde{k}, i)$ with $s(i) = s(k)$. Then our second requirement is that the matrix $\bar{a}$ is also unitary. The two unitarity requirements together are called bi-unitarity.

We further assume that we have the identity as in Fig. 5. This is called the symmetric property of $a$.

If all of the above are satisfied, we say that $a$ is a symmetric bi-unitary connection on $G$.
\[ \bar{a}_{(i,j),(k,l)} = \sqrt{\frac{\mu(s(i))\mu(r(j))}{\mu(r(i))\mu(r(k))}} \]

Figure 4: The second matrix

\[ j \begin{array}{c} \bar{i} \\ a \\ \bar{j} \end{array} k = \sqrt{\frac{\mu(s(i))\mu(r(j))}{\mu(r(i))\mu(r(k))}} \]

Figure 5: The symmetric property of \( a \)

Let \( a \) be a symmetric bi-unitary connection on \( G, H \) as above. Let \( U = (U_{j'j}) \) be a unitary matrix where \( j, j' \) are edges of the graph \( H \). We assume that \( U_{j'j} = 0 \) if \( s(j) \neq s(j') \) or \( r(j) \neq r(j') \). For such \( a, U \), we set a new symmetric bi-unitary connection \( \tilde{a} \) as in Fig. 6. In this case, we say the symmetric bi-unitary connections \( \tilde{a} \) and \( a \) are equivalent.

Let \( a \) be a symmetric bi-unitary connection on \( G, H_1 \) and \( b \) be a symmetric bi-unitary connection on \( G, H_2 \). We define a new graph \( H_3 \) simply by adding the edges of \( H_1 \) and \( H_2 \). That is, the adjacency matrix is given by \( \Delta_{H_3} = \Delta_{H_1} + \Delta_{H_2} \). We can define a new symmetric bi-unitary connection \( c \) by setting a matrix \( c_{(i,j),(k,l)} \) as follows.

\[
c_{(i,j),(k,l)} = \begin{cases} 
a_{(i,j),(k,l)}, & \text{if } j, k \in H_1, \\
b_{(i,j),(k,l)}, & \text{if } j, k \in H_2, \\
0, & \text{otherwise}. \end{cases}
\]

Note that we have \( \beta_{H_3} = \beta_{H_1} + \beta_{H_2} \). We call \( c \) a sum of \( a \) and \( b \), and write \( c = a + b \).

We say a symmetric bi-unitary connection is irreducible if it is not equivalent to a sum of two symmetric bi-unitary connections.

Let \( a \) be a symmetric bi-unitary connection on \( G, H_1 \) and \( b \) be a symmetric bi-unitary connection on \( G, H_2 \). We define a new graph \( H_3 \) simply by concatenating the graphs \( H_1 \) and \( H_2 \) vertically. That is, the adjacency matrix is given by \( \Delta_{H_3} = \Delta_{H_1}\Delta_{H_2} \). We define a new symmetric bi-unitary connection \( ab \) on \( G, H_3 \) as in Figures 7 and 8.

\[ k \begin{array}{c} \bar{i} \\ a \\ \bar{j} \end{array} j = \sum_{j',k'} \tilde{U}_{k'k} k' \begin{array}{c} \bar{i} \\ a \\ \bar{j'} \end{array} U_{j'j} \]

Figure 6: Equivalence of symmetric bi-unitary connections

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For a symmetric bi-unitary connection $a$, we also define a new one $\bar{a}$ as in Fig. 9. This is called the dual symmetric bi-unitary connection of $a$.

At the end of this section, we recall relations between symmetric bi-unitary connections and bimodules over $\text{II}_1$ factors. (See [7, Chapter 9] for general theory of bimodules.) We fix a vertex $*$ in $V$. Using the graph $G$, we obtain a hyperfinite $\text{II}_1$ factor $M$ with the string algebra construction as in [7, Section 11.3]. From a symmetric bi-unitary connection $a$, we obtain an $M$-$M$ bimodule $H_a$ with the open string bimodule construction as in [20], [11 Section 3]. Note that here we have $\dim_M H_a < \infty$ and $\dim(H_a)_M < \infty$. We recall the following result in [11 Section 3]. (Note that here we use only symmetric bi-unitary connections here.)

**Theorem 2.1** For symmetric bi-unitary connections $a, b$, we have $H_{a+b} \cong H_a \oplus H_b$, $H_{ab} \cong H_a \otimes_M H_b$ and $\bar{H}_a \cong H_{\bar{a}}$. These are $M$-$M$ bimodule isomorphisms.

We have $H_a \cong H_b$ if and only if $a$ and $b$ are equivalent. We also have $a$ is irreducible if and only if $H_a$ is an irreducible $M$-$M$ bimodule.

Let $a$ be a symmetric bi-unitary connection on $G, H_1$ and $b$ be a symmetric bi-unitary connection on $G, H_2$. By the results in [11 Section 3], we can describe the bimodule homomorphism space $\text{Hom}(H_a, H_b)$ in terms of symmetric bi-unitary connections.

$$\tilde{a} \circ \mu(s(i)) \mu(r(j)) = \frac{\mu(s(i)) \mu(r(j))}{\mu(r(i)) \mu(r(k))} \tilde{a}$$

Figure 9: The new connection $\bar{a}$
We next introduce a family of flat bi-unitary connections arising from a subfactor. Let $N \subset M$ be a subfactor (of the hyperfinite $\text{II}_1$ factor with finite Jones index and finite depth) as in \cite[Chapter 9]{ref1}. We do not assume the irreducibility $N' \cap M = \mathbb{C}$ here. We set $N = M_{-1}, M = M_0$ and apply the Jones tower/tunnel construction \cite[Section 9.3]{ref1} to get

$$
\cdots \subset M_{-2} \subset M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset \cdots
$$

Each algebra $M_j$ is again a type $\text{II}_1$ factor. We then consider a double sequence $A_{jk} = M'_{-k} \cap M_j$ of finite dimensional $C^*$-algebras for $j, k \geq 0$. They form commuting squares as in \cite[Chapter 9]{ref1}. We label minimal direct summands of $M'_{-k} \cap M_{2j+1}$ with labels $a, b, c, \ldots$. The labels of those for $M'_{-k} \cap M_{2j+1}$ are naturally identified with those for $M'_{-k} \cap M_{2j+3}$. Fix a label $a$ and choose a minimal projection $p$ from the summand of $M'_{-1} \cap M_{2j+1}$ labeled with $a$ and consider the sequence of commuting square

$$
(M'_{-1} \cap M_{-1})p \subset (M'_{-3} \cap M_{-1})p \subset (M'_{-5} \cap M_{-1})p \subset (M'_{-7} \cap M_{-1})p \subset \cdots
$$

$$
p(M'_{-1} \cap M_{2j+1})p \subset p(M'_{-3} \cap M_{2j+1})p \subset p(M'_{-5} \cap M_{2j+1})p \subset p(M'_{-7} \cap M_{2j+1})p \subset \cdots
$$

We denote the symmetric bi-unitary connection which gives these commuting squares by $a$. (Since we use only $M_{2j+1}$, this we have the symmetric property.) The symmetric bi-unitary connection $a$ is well-defined up to equivalence regardless of the choices of $j$ and $p$. Furthermore, this satisfies flatness as in Fig. 10. (See \cite{ref2}, \cite{ref3}, \cite[Chapter 10]{ref1}. The name “flatness” comes from analogy to a flat connection in differential geometry. A graph is regarded as a discrete analogue of a manifold.)

Since we have the finite depth assumption, we have finitely many labels $a, b, c, \ldots$ for irreducible flat symmetric bi-unitary connections arising from $N \subset M$. By Theorem 2.1 and the remark after that, we have a fusion category whose objects are labeled with $a, b, c, \ldots$ and this fusion category is equivalent to that of $N$-$N$ bimodules arising from the subfactor $N \subset M$. (See \cite{ref4} for general theory of fusion categories and its relation to
operator algebras.) Each vertex $v$ in $V$ corresponds to an irreducible $N$-$N$ bimodule $H_v$ arising from the subfactor $N \subset M$ and $\mu(v)$ is given by the (quantum) dimension of $H_v$. One advantage of using flat symmetric bi-unitary connections is that all information can be encoded into a single matrix (by making a direct sum of matrices) while we have to with infinite dimensional Hilbert spaces in the bimodule approach to fusion categories (or endomorphisms of an infinite dimensional von Neumann algebras in the sector approach [17]).

Note that the flat symmetric bi-unitary connection labeled with $a$ is given by the quantum $6j$-symbols for the $N$-$N$ bimodules arising from the subfactor $N \subset M$ where two of the six bimodules are labeled with $a$ and given by $NM$. (See [7, Chapter 12] for quantum $6j$-symbols arising from subfactors. The flatness condition roughly corresponds to the pentagon relations of quantum $6j$-symbols as explained in [7, Chapter 12].)

We now relate the above definitions to the setting of [4]. We define a tensor $a$ with four legs as in Fig. 11. (We use the same label for a tensor and a flat symmetric bi-unitary connection.) Recall that we keep the convention that if $s(i) \neq s(k), r(i) \neq s(j), r(k) \neq s(l)$ or $r(j) \neq r(l)$, we have the value 0. Note that if the fusion category is realized with a finite group with a 3-cocycle, then all the normalizing constants in Fig. 11 are 1.

It is easy to see that the tensor $ab$ corresponding to the product of the two flat symmetric bi-unitary connections $a$ and $b$ is given by the vertical concatenation and contraction of the tensors $a$ and $b$, and we have Fig. 12 for the tensor $\overline{a}$ corresponding the dual flat symmetric bi-unitary connection. (Note that the latter holds due to our convention of the normalizing constants as in Fig. 11.)

Then the zipper condition, which is assumed in [4, (2)], now holds for our setting as in Fig. 13. This is due to the irreducible decomposition of the product of two flat symmetric bi-unitary connections.

The advantage of using flat symmetric bi-unitary connections is that we get (a part of) the irreducible decomposition rules canonically from the (dual) principal graphs. The zipper condition holds for general symmetric bi-unitary connections even without flatness, but then it is very difficult to see the irreducible decomposition rules and there is a
possibility that we need infinitely many tensors and do not have a fusion category, which happens when the subfactor has an infinite depth.

Next we introduce a matrix product state. We concatenate $L$ tensors with 3 legs and use the value of the contraction for a state labeled with $i_1, i_2, \ldots, i_L$ where these indices label tensors. See Fig. 14 where $L = 4$. A matrix product state was first introduced in [8] and has been important in recent studies of gapped Hamiltonians.

In a similar way, we introduce a matrix product operator by using $L$ tensors with 4 legs as in Fig. 15, where $L$ is again 4. When we use the tensor labeled with $a$ arising from a flat symmetric bi-unitary connection, we denote the resulting matrix product operator by $O_a^L$.

The product of two matrix product operators $O_a^L, O_b^L$ is given as in Fig. 16.

By the zipper condition in Fig. 13, the product $O_a^L O_b^L$ decomposes as $\sum_c N_{ab}^c O_c^L$, where the coefficients $N_{ab}^c$’s are given in the decomposition of the product $ab$ of flat symmetric bi-unitary connections into $\sum_c N_{ab}^c$. We also see that $O_a^L$’s give a fusion category where the dual object of $O_a^L$ is $O_a^{\bar{L}}$. Note that this fusion category is determined uniquely up to equivalence regardless of $L$. This is clear from the zipper condition, Fig. 13. From a subfactor theory viewpoint, this is because horizontal basic construction of a commuting square does not change a resulting subfactor. Also, we have $P_L = \sum_a w_a O_a^L$ as in [4] (13),

$$
\sum_{i_1, i_2, i_3, i_4} \langle i_1 i_2 i_3 i_4 | \langle i_1 i_2 i_3 i_4 | \rho_{a_{i_1} a_{i_2} a_{i_3} a_{i_4}} \rangle | i_1 i_2 i_3 i_4 \rangle
$$

Figure 13: The zipper condition

Figure 14: A matrix product state

Figure 15: A matrix product operator
where $w_a = d_a / w$, $w = \sum_a d_a^2$, and $d_a$ is the Perron-Frobenius dimension of $a$.

We next consider Ocneanu’s tube algebra [6], [7, Section 12.6], [9], [15]. We recall the definition following [9].

**Definition 3.1** Let $\mathcal{C}$ be a unitary fusion category. We set the tube algebra $\text{Tube}(\mathcal{C})$ to be

$$\bigoplus_{\lambda, \nu \in \text{Irr}(\mathcal{C}), \mu \in \text{Irr}(\mathcal{C})} \text{Hom}(\lambda \mu, \mu \nu)$$

as a complex linear space. We define its algebra structure and $*$-structure by the formulas as in [9, page 134].

We then see that under the identification of flat symmetric bi-unitary connections and tensors as in Fig. 11, the tube algebra arising from the unitary fusion category of the flat symmetric bi-unitary connections and the anyon algebra defined for the tensors as in [4, page 199] are isomorphic. We thus have the following result.

**Theorem 3.2** A family of tensors arising from the bi-unitary flat connections as above satisfy all the requirements in Bultinck-Mariëna-Williamson-Şahinoğlu-Haegeman-Verstraete [4]. The resulting anyon algebra and the tube algebra for the subfactor are isomorphic. The tensor categories arising from these two algebras are the equivalent, hence both are modular tensor categories and the Verlinde formula holds for the former.

That is, the new tensor category arising from the anyon algebra in [4] is indeed the Drinfel’d center of the original unitary fusion category. They define the fusion rules of the new tensor category in [4], but they say “The multiplicities — the specific values of $N_{ij}^k$ — are in general harder to obtain directly since they arise from the number of linearly independent ways the MPO strings emanating from the idempotents can be connected on the virtual level.” So it is not clear whether the structure constants $N_{ij}^k$ are well-defined or not in [4], but these are well-defined by [7], [9], [15]. In [4], they also say “One could of course also just calculate the fusion multiplicities from the $S$ matrix using the Verlinde formula,” but it is not clear why the Verlinde formula applies to this setting. Now our identification clearly shows that the Verlinde formula applies as in [7 Section 12.7], [9], [15].
4 Non-flat connections and their flat parts

We have seen a family of flat symmetric bi-unitary connections arising from a subfactor produces tensors satisfying the requirement of [4], but more general tensors also satisfy that. We now look into this matter.

Consider the Dynkin diagram $E_7$. It has a bi-unitary connection and this is not flat as in [5], [7, Chapter 11]. Though this bi-unitary connection is not symmetric, it is easy to obtain a symmetric one through horizontal and vertical basic constructions. Starting with this symmetric bi-unitary connection, we get a family of irreducible symmetric bi-unitary connections. (The set $V$ consists of the four even vertices of $E_7$.) This situation was considered by Ocneanu [19]. Since the principal graph of the subfactor arising from the bi-unitary connection on $E_7$ is $D_{10}$ as in [5], we obtain 6 irreducible symmetric bi-unitary connections in this way. (The number 6 is give by the even vertices of $D_{10}$.) Actually, one can obtain a larger fusion category of bi-unitary connections as in Ocneanu [19] for $E_7$. He has 17 irreducible objects there, but we now deal with only symmetric bi-unitary connections, so we have 10 irreducible objects corresponding to the even vertices. (Again, it is easy to have “symmetric” ones using the horizontal basic construction. See Fig. 42 in [3] from a viewpoint of $\alpha$-induction [2].) In this case, we have an example of a fusion unitary category of tensors which do not come from a family of flat connections. It is not clear what kind of mathematical assumptions we need for the setting of [4], but the case we are interested in most is when we have a unitary fusion category as in [4, Appendix A]. It is well-known that any unitary fusion category is realized with a family of flat connections from a (possibly reducible) subfactor as in Section 2. (See [7, Section 12.5], for example.) So we have the following theorem.

**Theorem 4.1** Suppose tensors satisfying Bultinck-Mariëna-Williamson-Şahinoğlu-Haegeman-Verstraete [4] give a unitary fusion category through the matrix product operator algebra. Then there exists a subfactor that produces the fusion category through a family of flat symmetric bi-unitary connections, and the tube algebra arising from such connections and the anyon algebra arising from the tensors are isomorphic.

Note that a problem of finding (a family of) flat symmetric bi-unitary connections for a unitary fusion category given by a family of non-flat symmetric bi-unitary connections is known as that of computing the “flat part” of a non-flat symmetric bi-unitary connection.

That is, if we have a finite family of non-flat symmetric bi-unitary connections closed under the product of such connections, computing the flat part gives a new family of flat symmetric bi-unitary connections. The zipper condition holds both for non-flat and flat symmetric bi-unitary connections, but the latter give a canonical form of tensors and are more appropriate for actual computations of tensors.

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