Simple Hamiltonian manifolds

JEAN-CLAUDE HAUSMANN AND TARA HOLM

A simple Hamiltonian manifold is a compact connected symplectic manifold equipped with a Hamiltonian action of a torus $T$ with moment map $\Phi : M \to t^*$, such that $M^T$ has exactly two connected components, denoted $M_0$ and $M_1$. We study the differential and symplectic geometry of simple Hamiltonian manifolds, including a large number of examples.

1. Introduction

Let $M$ be a compact connected symplectic manifold equipped with a Hamiltonian action of a torus $T = (S^1)^n$, and let $\Phi : M \to t^*$ denote the moment map. The celebrated Atiyah Guillemin–Sternberg convexity theorem states...
that the image of the moment map $\Phi$ is the convex hull of the image of the fixed points, $\Phi(M^T)$. This polytope is a single point, that is, the moment map is constant, if and only if the action is trivial. So long as the action is non-trivial, this polytope $\Phi(M)$ must have at least two extreme points. In this paper, we consider the simplest non-trivial case, when $M^T$ has exactly two components, and so $\Phi(M)$ is a 1-dimensional polytope.

**Definition 1.1.** A simple Hamiltonian manifold is a compact connected symplectic manifold equipped with a Hamiltonian action of a torus $T$ with moment map $\Phi : M \to \mathfrak{t}^*$, such that $M^T$ has exactly two connected components, denoted $M_0$ and $M_1$.

As noted above, a simple manifold has the minimum possible number of fixed components. We describe a simple Hamiltonian manifold by the triple $(M, M_0, M_1)$, and let $2m_i$ and $2m$ be the dimensions of $M_i$ and $M$, respectively, and set $2r_i = \text{codim } M_i = 2m - 2m_i$. As a consequence of some basic results in equivariant symplectic geometry, the torus action on a simple manifold necessarily factors into a trivial action and a residual effective circle action (Lemma 2.2). Thus, our results hold for torus actions, but generally require verification only for the residual circle action.

In what follows, we explore the geometry associated to simple Hamiltonian manifolds. We establish the basic topology of a simple Hamiltonian manifold, using the moment map as the key tool, in Section 2. This is where we discuss the residual circle action (Lemma 2.2). Then we turn to cohomology constraints on simple Hamiltonian manifolds in Section 3. The residual moment map is a Morse–Bott function on $M$, and so the cohomology of $M$ is determined from $M_0$ and $M_1$ (Proposition 3.1 and its Corollaries). This allows us to deduce relations among $m, m_0, m_1, r_0$ and $r_1$. This section also includes comments about how our work relates to several recent papers on this topic.

In Section 4, we study bundles over the $M_i$ and the gauge groups of these bundles, and prove our first main theorem giving necessary conditions for two simple Hamiltonian manifolds to be $T$-equivariantly diffeomorphic, Theorem 4.4. Next, in Section 5, we turn to the special case when $M_1$ has codimension 2 in $M$, and characterize $M$ in terms of $M_0$ (Theorem 5.4). In this special case, we must have that $M_1$ is diffeomorphic to $M_1$ (Corollary 5.5). In Section 6, we turn to the classification $M$ up to $T$-equivariant symplectomorphism, with a complete answer in the same special case $r_1 = 1$ (Theorems 6.2 and 6.3). In particular, when $r_0 = r_1 = 1$, then $M_0$
and $M_1$ must be $T$-equivariantly symplectomorphic (Corollary 6.4). Finally, the last section of the paper is devoted to examples of polygon spaces.

There has been a flurry of recent work on Hamiltonian $S^1$-manifolds that are in some sense minimal. Tolman introduces Betti number constraints in [14], and shows that only a finite number of cohomology rings can occur. These constraints are explored further in [11] when the fixed set has exactly two components, that is the manifold is a simple Hamiltonian manifold. The differential geometry of simple Hamiltonian manifolds with minimal Betti numbers is discussed in [12]; this work may be related to our results in Section 4. Another natural hypothesis is that the circle action be semifree, as is the case for weight simple Hamiltonian manifolds discussed below in Section 2. The implications of this hypothesis are developed further in [5, 15].

We now conclude this Introduction with a handful of examples of simple Hamiltonian manifolds.

**Example 1.2.** Let $M = \mathbb{C}P^n$ with a circle action given by

$$g \cdot [z_0 : \cdots : z_n] = [gz_0 : \cdots : g\bar{z}_k : z_{k+1} : \cdots : z_n],$$

for $g \in S^1$. This is a simple Hamiltonian manifold $(\mathbb{C}P^n, \mathbb{C}P^k, \mathbb{C}P^{n-k-1})$.

**Example 1.3.** A simple Hamiltonian manifold $M$ with $M^T$ discrete is diffeomorphic to $S^2$. In this case, the moment map is a Morse function with exactly two critical points, which implies that $M$ is homeomorphic to a sphere $S^n$. As $M$ is symplectic, it must be diffeomorphic to $S^2$.

**Example 1.4.** *The symplectic cut of a weight bundle.* We may use Lerman’s symplectic cuts [10] to produce a simple Hamiltonian manifold from a symplectic manifold equipped with a complex vector bundle. Let $M_0$ be a compact symplectic manifold and let $\nu_0: V \to M_0$ be a complex vector bundle of rank $k$. Viewing $S^1 \subset \mathbb{C}$ as the unit complex numbers, there is a natural $S^1$-action on this bundle, namely fiberwise complex multiplication. We assume that the total space $V$ is equipped with a symplectic form so that this $S^1$-action is Hamiltonian. The moment map $\phi: V \to \mathbb{R}$ has only 0 as a critical value. Let $M$ be the symplectic cut of $V$ at a regular value $\ell > 0$ of $\phi$. This gives a simple Hamiltonian manifold $(M, M_0, M_1)$ with $M_1$ the symplectic reduction of $V$ at $\ell$. The bundle projection descends to a map $M \to M_0$ with fiber $\mathbb{C}P^k$. Thus, $M = \mathbb{P}(\nu_0)$, the total space of the $\mathbb{C}P^k$-bundle associated to $\nu_0$ and $M_1 = \mathbb{P}(\nu_0)$, the total space of the $\mathbb{C}P^{k-1}$-bundle associated to $\nu_0$. The case $k = 1$ is described in [13, Example 5.10].
Example 1.5. Let $M = G_k(\mathbb{C}^r)$ be the Grassmannian manifold of complex $k$-planes in $\mathbb{C}^r$, endowed with a $U(r)$-invariant symplectic form. As a homogeneous space, $M \cong U(r)/(U(k) \times U(r-k))$. We may endow $M$ with a symplectic form by identifying it with the $U(r)$ coadjoint orbit of Hermitian $r \times r$ matrices with eigenvalues consisting of $k$ ones and $(r-k)$ zeros. The maximal torus $T$ of diagonal matrices in $U(r)$ acts in a Hamiltonian fashion on $M$, and we consider the last coordinate circle of this torus. Under the identifications we have made, this action has moment map

$$\Phi(A) = a_{r,r},$$

where $A$ is a symmetric matrix and $a_{r,r}$ its bottom right entry. Then $M$ is a simple Hamiltonian manifold with moment map image the interval $[0, 1]$. We identify

$$M_0 = \left\{ \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

where $B$ is a symmetric $(r-1) \times (r-1)$ matrix with eigenvalues consisting of $k$ ones and $(r-k-1)$ zeros. Thus, $M_0 \cong G_{k-1}(\mathbb{C}^{r-1})$. The second fixed component is

$$M_1 = \left\{ \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

where $B$ is a symmetric $(r-1) \times (r-1)$ matrix with eigenvalues consisting of $(k-1)$ ones and $(r-k)$ zeros; so $M_1 \cong G_k(\mathbb{C}^{r-1})$. The real locus (for complex conjugation) of this simple Hamiltonian manifold is discussed in [6, Example 5].

Example 1.6. If $M$ is a simple Hamiltonian manifold and $N$ is a connected compact symplectic manifold, then $M \times N$ is a simple Hamiltonian manifold, where $g \cdot (x, y) = (gx, y)$ for $g \in T$ and $(x, y) \in M \times N$.

Example 1.7. Grassmannian manifold $\tilde{G}_2(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. See figure 1 and its legend, describing moment polytopes for $\tilde{G}_2(\mathbb{R}^5)$ and $\tilde{G}_2(\mathbb{R}^7)$. These simple manifolds play an important role in [11, 12, 14].
Figure 1: In subfigure (a) there is the $T^2$-moment polytope for $\tilde{G}_2(\mathbb{R}^5)$, together with a projection for an $S^1$-moment map for which this manifold is a simple Hamiltonian manifold. Both $M_0$ and $M_1$ in this case are diffeomorphic to $\mathbb{P}^1$. Subfigure (b) shows the $T^3$-moment polytope for $\tilde{G}Sub_2(\mathbb{R}^7)$ and a projection for an $S^1$-moment map for which this manifold is a simple Hamiltonian manifold. Both $M_0$ and $M_1$ in this case are diffeomorphic to $\mathbb{P}^2$.

2. Preliminaries

Standard properties of moment maps, which may be found in [1], immediately imply the following:

**Lemma 2.1.** Let $(M, M_0, M_1)$ be a simple Hamiltonian manifold with moment map $\Phi : M \to t^*$. Then

(i) the moment polytope $\Delta = \Phi(M)$ is a line segment.

(ii) $\Phi$ is a Morse–Bott function onto $\Delta$ with exactly two critical values, namely the endpoints of $\Delta$.

We think of the circle $S^1$ as the complex numbers of norm 1. The Lie algebra $\text{Lie}(S^1)$ may then be identified as $i \mathbb{R}$, with basis vector $2\pi i$. We may use the dual basis to identify $\text{Lie}(S^1)^*$ with $\mathbb{R}$. The group of characters of $T$ is $\hat{T} = \text{Hom}(T, S^1)$, the set of smooth homomorphisms. This is isomorphic to the linear maps from $\mathbb{R} = \text{Lie}(S^1)^*$ to $t^*$ that send $\mathbb{Z}$ to the weight lattice. Taking the image of 1 identifies $\hat{T}$ with the weight lattice inside $t^*$. 
Lemma 2.2. Let \((M, M_0, M_1)\) be a simple Hamiltonian \(T\)-manifold with moment map \(\Phi: M \to \mathfrak{t}^*\). Then there is a unique character \(\chi \in \hat{T}\) such that

(i) the \(T\)-action \(\alpha: T \to \text{Diff}(M)\) is of the form \(\alpha = \bar{\alpha} \circ \chi\), where \(\bar{\alpha}: S^1 \to \text{Diff}(M)\) is an effective action making \((M, M_0, M_1)\) a simple Hamiltonian \(S^1\)-manifold. We call \(\bar{\alpha}\) the residual action.

(ii) The residual action \(\bar{\alpha}\) admits a moment map \(\bar{\Phi}: M \to \mathbb{R}\) such that \(\bar{\Phi}(M_0) = 0\) and \(\bar{\Phi}(M_1) > 0\).

Moreover, the above character \(\chi\), seen as an element of the weight lattice, is a positive multiple of \(\Phi(M_1) - \Phi(M_0)\).

Remark 2.3. The character \(\chi\) of Part (ii) of Lemma 2.2 is the associated character to the simple Hamiltonian manifold \((M, M_0, M_1)\). The moment map \(\bar{\Phi}: M \to \mathbb{R}\) is called the residual moment map. This lemma reduces the classification of simple Hamiltonian \(T\)-manifolds to the case of simple Hamiltonian \(S^1\)-manifolds, for effective circle actions.

Proof. As the moment polytope is 1-dimensional, \(\bar{T} = T / \ker \alpha\) is a 1-dimensional torus (see e.g. [1, Section III.2.b]). Choosing an identification of \(\bar{T}\) with \(S^1\) gives a character \(\chi\) and a residual action \(\bar{\alpha}\) with moment map \(\bar{\Phi}: M \to \mathbb{R}\) (with \(\bar{\Phi}(M_0) = 0\)). We denote by \(2m_i\) and \(2m\) the dimensions of \(M_i\) and \(M\) and we set \(2r_i = \text{codim} M_i\). As \(\alpha = \bar{\alpha} \circ \chi\), this implies that \(\chi \circ \bar{\Phi}\) is a moment map for \(\alpha\), proving the last statement. The uniqueness statement (ii) follows from the fact that the two identifications of \(\bar{T}\) with \(S^1\) differ by the sign of \(\bar{\Phi}\). \(\square\)

Let \((M, M_0, M_1)\) be a simple Hamiltonian \(T\)-manifold. Recall that \(M\) always admits a \(T\)-invariant almost complex structure \(J\) that is \(\omega\)-compatible: \(J\) is an isometry for \(\omega\) and \(\omega(v, Jv) > 0\) for all non-zero tangent vectors \(v\) to \(M\) (see, for example, [13, Section 2.5] or [2, Part V]). Then \(\langle v, w \rangle = \omega(v, Jw)\) defines a Riemannian metric on \(M\) and \(\langle \cdot, \cdot \rangle + i\omega(\cdot, \cdot)\) is a \(T\)-invariant Hermitian metric. The space of \(T\)-invariant \(\omega\)-compatible almost complex structures on \(M\) is denoted by \(\mathcal{J}(M, \omega)\) and is contractible (see [13, Proposition 4.1 and 2.49] or [2, Proposition 13.1]). Therefore, choosing \(J \in \mathcal{J}(M, \omega)\) endows the tangent bundle \(TM\) with a \(U(r)\)-structure whose isomorphism class is well-defined. As the \(M_i\) are symplectic submanifolds, the normal bundles \(\nu_i = TM|_{M_i}/TM_i\) are also Hermitian bundles, with structure group \(U(r_i)\), and these structures are well-defined up to isomorphism. Observe that \(\nu_i\) is isomorphic to the orthogonal complement to \(TM_i\) in \(TM|_{M_i}\) with respect to the Riemannian metric associated to \(J\).
The $U(r_i)$ structure on $\nu_i$ is $T$-invariant, so the bundle $\nu_i$ decomposes into a Whitney sum of $T$-weight bundles. It follows from Lemmas 2.1 and 2.2 that the weights which occur are multiples of $\chi$.

**Definition 2.4.** If $\nu_0$ (or, equivalently, $\nu_1$) is itself a weight bundle, we call $M$ a **weight simple Hamiltonian manifold**.

For instance, $M$ is a weight simple Hamiltonian manifold when $\text{codim } M_0 = 2$ or $\text{codim } M_1 = 2$. The Grassmannian manifold $\tilde{G}_2(\mathbb{R}^{m+2})$ of Example 1.1.6 is not a weight simple manifold. Observe that $M$ is a weight simple Hamiltonian manifold if and only if the residual action is semi-free. By [11, Proposition 8.1], a simple Hamiltonian manifold $(M, M_0, M_1)$ with $m = m_0 + m_1 + 1$ is a weight simple Hamiltonian manifold unless $\dim M_0 = \dim M_1$.

**Remark 2.5.** In the above discussion, the Hermitian bundle $\nu_i$ is the underlying bundle of a Hermitian bundle $\hat{\nu}_i$ endowed with a $T$-action. We do not distinguish these two notions because in the case of interest for us, where $(M, M_0, M_1)$ is a weight simple manifold, $\hat{\nu}_i$ is determined by $\nu_i$. Indeed, $T$ acts on $\nu_0$ via the character $\chi: T \to S^1$ composed with complex multiplication on the fibers. The same holds for $\nu_1$, replacing $\chi$ by $\chi^{-1}$.

Let $(M, M_0, M_1)$ be a simple Hamiltonian $T$-manifold with residual moment map $\bar{\Phi}: M \to \mathbb{R}$. Let $\ell > 0$ defined by $\{\ell\} = \bar{\Phi}(M_1)$. Define

$$V_0 = \bar{\Phi}^{-1}([0, \ell/2]) \quad \text{and} \quad V_1 = \bar{\Phi}^{-1}([\ell/2, \ell]).$$

**Lemma 2.6.** For $i = 0$ and $1$, the subspace $V_i$ of (2.1) is a $T$-invariant (closed) tubular neighborhood of $M_i$ in $M$.

**Proof.** We prove this for the case $i = 0$, and mention the necessary adaptations to complete the case $i = 1$. The proof introduces techniques which are useful in subsequent sections (see Remark 2.7 for the idea of a more direct argument). Passing to the residual action, we suppose that $T = S^1$.

Choose an $S^1$-invariant almost complex structure $J$ on $M$. This makes $\nu_0$ an $S^1$-equivariant Hermitian bundle with structure group $U(r_0)$. We denote by $E(\nu_0)$ its total space and by $p: E(\nu_0) \to M_0$ the bundle projection. Denote by $S(\nu_0) \subset E(\nu_0)$ the associated unit sphere bundle. For $\varepsilon > 0$, let $D_\varepsilon(\nu_0) \subset E(\nu_0)$ the disk bundle formed by the elements of $E(\nu_0)$ of norm $\leq \varepsilon$. An element of $D_\varepsilon(\nu_0)$ may be written under the form $rz$, with $z \in S(\nu_0)$ and $r \in [0, \varepsilon]$, with the identification $0z = p(z)$.
As \( \nu_0 \) is a Hermitian bundle, each fiber of \( E(\nu_0) \) carries a symplectic form, isomorphic to the standard form on \( \mathbb{C}^r \) via a trivialization. The orthogonal sum with the symplectic form on \( M_0 \) provides a symplectic form \( \omega^0 \) on \( E(\nu_0) \). The same construction works for the almost complex structure and the Riemannian metric, so there is a compatible triple \( (\omega^0, J^0, \langle \cdot, \cdot \rangle^0) \) over \( E(\nu_0) \), extending the given one over \( M_0 \).

Let \( b: D_\varepsilon(\nu_0) \rightarrow M \) be the \( S^1 \)-equivariant tubular neighborhood embedding given by the exponential with respect to the Riemannian metric \( \langle \cdot, \cdot \rangle \), for \( \varepsilon > 0 \) small enough. The two symplectic forms \( \omega^0 \) and \( b^* \omega \) coincide on \( M_0 \). By [13, Lemma 3.14], there is a tubular neighborhood embedding \( h: D_\varepsilon'(\nu_0) \rightarrow D_\varepsilon(\nu_0) \) such that \( h^* b^* \omega = \omega^0 \). Based on Moser’s argument, the construction of \( h \) can be made \( S^1 \)-invariant (see, e.g. [1, Remark II.1.13]). Thus, replacing \( b \) with \( b \circ h \) and \( \varepsilon \) with \( \varepsilon' \) if necessary, we may assume that \( b^* \omega = \omega^0 \). Pushing the triple \( (\omega^0, J^0, \langle \cdot, \cdot \rangle^0) \) down to \( M \) via \( b \), we get a compatible triple \( (\omega, J^0, \langle \cdot, \cdot \rangle^0) \) near \( M_0 \).

Choose a smooth function \( \delta_0: [0, \ell] \rightarrow [0, 1] \) which is equal to 0 near 0 and so that the support of \( (1 - \delta_0) \circ \Phi \) is contained in the interior of \( b(D_\varepsilon(\nu_0)) \). Recall that the space \( J(b(D_\varepsilon(\nu_0)), \omega) \) of \( S^1 \)-invariant \( \omega \)-compatible almost complex structures on \( b(D_\varepsilon(\nu_0)) \) is contractible. The standard proof of this, for example in [13, Propositions 4.1 and 2.49], actually provides a path \( J^s (s \in [0, 1]) \) from \( J^0 \) to \( J^1 = J \). The formula

\[
J'_x = J_{x}^{\delta_0 \circ \Phi(x)} \in \text{Aut}_R T_x M
\]

makes sense for all \( x \in M \) and provides a \( \omega \)-compatible almost complex structure on \( M \). We say that \( J' \) is obtained by straightening \( J \) around \( M_0 \), using the straightening function \( \delta_0 \). The almost complex structure \( J' \) determines a Riemannian metric \( \langle \cdot, \cdot \rangle' \) on \( M \), and hence we have an \( S^1 \)-invariant compatible triple \( (\omega, J', \langle \cdot, \cdot \rangle') \) on \( M \).

Let us consider the gradient vector field \( \text{Grad} \Phi \) for the metric \( \langle \cdot, \cdot \rangle' \). This vector field depends only on \( J' \), since \( \text{grad} \Phi = J' X \), where \( X \) is the fundamental vector field of the Hamiltonian residual circle action. A \( J' \)-gradient line is the closure of a trajectory of \( \text{Grad} \Phi \).

Suppose that \( M \) is a weight simple manifold. We claim that for each vector \( z \in S(\nu_0) \), there is a unique \( J' \)-gradient line \( \Gamma_z \) that is tangent to \( z \) and that hits \( M_0 \) at a point \( p(z) \). This process parameterizes the gradient lines by \( S(\nu_0) \). To see this, we transport ourselves into \( D_\varepsilon(\nu_0) \) via \( b \). If \( M \) is a weight manifold, the restriction of the moment map \( \Phi \circ b \) on each fiber is just the norm square, whose level surfaces of \( \Phi \circ b \) are round spheres and the \( J' \)-gradient lines are the radial lines to the zero sections. Checking this also
makes it clear that the equation

\[ (2.2) \quad \beta_0(rz) = \Gamma_z \cap \Phi^{-1} \left( \frac{\ell r^2}{2} \right) \]

defines a map \( \beta_0: D_1(\nu_0) \to M \) which is an \( S^1 \)-equivariant smooth embedding with image \( V_0 \). This completes the proof of Lemma 2.6 for \( i = 0 \) when \( M \) is a weight simple manifold. The case \( i = 1 \) is analogous. We reverse the orientation of the gradient lines, and for \( rz \in [0, \sqrt{\ell}] \times D_1 \), we define \( \beta_1(rz) \) to be the point \( y \in \Gamma_z \) such that \( \Phi(y) = \frac{\ell - r^2}{2} \).

Finally, when \( M \) is not a weight manifold, the level surfaces of \( \Phi \circ b \) are ellipsoids and the above process does not work: it requires that the Hessian of \( \Phi \circ b \) be proportional to the metric \( \langle \cdot, \cdot \rangle^0 \). To get around this difficulty, we precompose \( b \) with an automorphism of \( \nu_0 \) which transforms the ellipsoids into round spheres. We use this new tubular neighborhood \( b'': D_{\varepsilon''} \to M \) to transport the metric \( \langle \cdot, \cdot \rangle^0 \) on a neighborhood of \( M_0 \) in \( M \), providing a Riemannian metric \( \langle \cdot, \cdot \rangle'' \) on this neighborhood. This metric may be mixed with \( \langle \cdot, \cdot \rangle \) using a function like \( \delta \) to obtain an \( S^1 \)-invariant Riemannian metric \( \langle \cdot, \cdot \rangle^- \) on \( M \). Then Equation (2.2) together with the metric \( \langle \cdot, \cdot \rangle^- \) provides an \( S^1 \)-invariant smooth tubular neighborhood embedding with image \( V_0 \). Note that the metric \( \langle \cdot, \cdot \rangle^- \) is no longer compatible with the symplectic form, but this is not necessary for the proof of Lemma 2.6.

Remark 2.7. The above proof of Lemma 2.6 was designated to introduce techniques useful in subsequent sections. For a more direct proof, recall that the Morse Lemma provides an embedding \( \psi: D_1(\nu_0) \to M \) with image a tubular neighborhood \( \mathcal{D} \) of \( M_0 \), such that each gradient line of \( \Phi \) intersects the boundary of \( \mathcal{D} \) transversally in one point. This enables us to construct a diffeomorphism \( \beta_0: D_1(\nu_0) \to V_0 \) as in (2.2). Thus, \( V_0 \) is a tubular neighborhood of \( M_0 \) (note that \( V_0 \) is \( T^1 \)-invariant by definition).

3. Cohomology constraints

In this paper, \( H^* \cdot \cdot \cdot \) denotes the cohomology ring of a space with rational coefficients. Recall that, for \( (M, M_0, M_1) \) a simple Hamiltonian manifold, \( 2m_i \) and \( 2m \) are the dimensions of \( M_i \) and \( M \), respectively, and that \( 2r_i = \text{codim} M_i \).
Proposition 3.1. Let \((M, M_0, M_1)\) be a simple Hamiltonian manifold. Then for \(i, j \in \{0, 1\}\) and \(i \neq j\), there are short exact sequences

\[
\begin{align*}
0 & \to H^{*-2r_j}(M_j) \to H^*(M) \to H^*(M_i) \to 0 \\
0 & \to H_{2m-*}(M_j) \to H^*(M) \to H^*(M_i) \to 0,
\end{align*}
\]

where the right hand homomorphisms are induced by inclusion.

Remark 3.2. This is related to the results in [8, Section 3]. Here we do not need to assume that the cohomology of \(M_0\) and \(M_1\) is concentrated in even degrees because the moment map provides a perfect Morse–Bott function that allows us to deduce the result.

Proof. Let \(V_j\) be the tubular neighborhood near \(M_j\) given by Lemma 2.6 that satisfies \(V_i = M - \text{int} V_j\). We first note that the cohomology exact sequence of the pair \((M, V_i)\) splits into short exact sequences

\[
0 \to H^*(M, V_i) \to H^*(M) \to H^*(V_i) \to 0.
\]

This is related to the fact that the residual moment map is a perfect Morse–Bott function. A proof of (3.3) for the \(T\)-equivariant cohomology is given in [15, Proposition 2.1]. Exactness of (3.3) then follows because \(M_i\) is \(T\)-fixed, so the map \(H^*_T(M_i) \to H^*(M_i)\) is onto. By excision of \(\text{int} V_i\) and the Thom isomorphism,

\[
H^*(M, V_i) \approx H^*(V_j, \partial V_j) \approx H^{*-2r_j}(M_j).
\]

Then (3.3) and (3.4) give exactness of Sequence (3.1).

Next, Poincaré duality for \(V_j\) implies that

\[
H^*(M, V_i) \approx H^*(V_j, \partial V_j) \approx H_{2m-*}(V_j) \approx H_{2m-*}(M_j).
\]

Thus (3.3) and (3.5) imply exactness of Sequence (3.2). \(\square\)
Let $P, P_i \in \mathbb{Z}[t]$ be the Poincaré polynomials of $M$ and $M_i$.

**Corollary 3.3.** The Poincaré polynomial $P_0$ together with $r_0$ and $r_1$ determine both $P_1$ and $P$ by the following equations:

\[
\begin{cases}
(1 - t^{2r_1})P_1 = (1 - t^{2r_0})P_0 \\
(1 - t^{2r_1})P = (1 - t^{2(r_0 + r_1)})P_0
\end{cases}
\]

**Proof.** Sequences (3.1) for $i = 0$ and $i = 1$ immediately give the following equations:

\[
\begin{cases}
P = P_0 + t^{2r_1}P_1 \\
P = t^{2r_0}P_0 + P_1
\end{cases}
\]

from which we may deduce the equations of Corollary 3.3. Note that Equations (3.7) are just the Morse–Bott equalities for the residual moment map and its opposite. □

**Corollary 3.4.** Let $(M, M_0, M_1)$ be a simple Hamiltonian manifold with $r_1 = 2$. Then there are additive isomorphisms

\[
\begin{align*}
H^*(M_1) &\approx_{\text{add}} H^*(M_0) \otimes H^*(\mathbb{C}P^{r_0-1}) \\
H^*(M) &\approx_{\text{add}} H^*(M_0) \otimes H^*(\mathbb{C}P^{r_0})
\end{align*}
\]

**Proof.** Suppose that $M$ is obtained by a symplectic cut of the trivial bundle $M_0 \times \mathbb{C}^{r_0}$. Then $M_1 = M_0 \times \mathbb{C}P^{r_0-1}$ and $M = M_0 \times \mathbb{C}P^{r_0}$, which proves the lemma in this case. The general case follows from Corollary 3.3. □

**Remark 3.5.** It is not true that $P_0$ together with $r_1$ determines the cohomology ring $H^*(M)$. For instance, for the symplectic cut of a weight bundle $\nu_0$ over $M_0$ given in Example 1.4, the ring structure on $H^*(M)$ depends on the bundle $\nu_0$. For $M_0 = S^2$ and $r_0 = 1$, $M$ is diffeomorphic to $S^2 \times S^2$ if $c_1(\nu_0)$ is even and to $\mathbb{C}P^2 \# \mathbb{C}D^2$ if $c_1(\nu_0)$ is odd.

The first equation in (3.6) immediately implies the following corollary.

**Corollary 3.6.** If $r_0 = r_1$, the Poincaré polynomials of $M_0$ and $M_1$ are identical: $P_0 = P_1$.

The following proposition appears as a special case of the first centered equation in [11]. In the case of a simple Hamiltonian manifold, their inequality is precisely this one.
Proposition 3.7. Let $(M, M_0, M_1)$ be a simple Hamiltonian manifold. Then

$$m \leq m_0 + m_1 + 1.$$ 

Proof. Suppose that $2r_1 > 2m_0 + 2$. The first equation of (3.7) then implies that $H^{2m_0+2}(M) = 0$, which is impossible as $M$ is a compact symplectic manifold of dimension $\geq 2m_0 + 2$. Hence, $2r_1 \leq 2m_0 + 2$, which implies that $2m \leq 2m_0 + 2m_1 + 2$. \(\square\)

Lemma 3.8. Let $(M, M_0, M_1)$ be a simple Hamiltonian manifold. Then

$$H^1(M_0) \approx H^1(M) \approx H^1(M_1),$$

these isomorphisms being induced by the inclusions $M_i \subset M$.

Proof. As $m_i \geq 1$, the abstract isomorphisms come from Equations (3.7). By Proposition 3.1, inclusions $M_i \subset M$ induce surjective homomorphisms, which are then isomorphisms. \(\square\)

Proposition 3.9. For a simple Hamiltonian manifold $(M, M_0, M_1)$, the following conditions are equivalent.

(a) $H^{\text{odd}}(M_0) = 0$.
(b) $H^{\text{odd}}(M_1) = 0$.
(c) $H^{\text{odd}}(M) = 0$.

Proof. By the first equation of (3.6), Conditions (a) and (b) are equivalent. By Equation (3.7), (c) is equivalent to (a) and (b) together. \(\square\)

Example 3.10. Suppose that $M$ has the cohomology of $CP^n$. Then $M_0$ and $M_1$ have the cohomology ring of a complex projective space. Indeed, their cohomology groups vanish in odd degree by Proposition 3.9. Also, their Betti numbers are $\leq 1$ by Proposition 3.1 and they are symplectic manifolds. The first equation of (3.7) implies that $m_0 + m_1 + 1 = m$, as in Example 1.2 (For $M_1 = pt$, this is a result of [6, Theorem 1]).

Remark 3.11. The extreme case in Proposition 3.7, i.e., $m = m_0 + m_1 + 1$, is studied in [11, 12, 14]. Much stronger restrictions than what we prove in this section hold in that special case. In that context, the ring $H^*(M; \mathbb{Z})$ must be isomorphic either to $H^*(CP^n)$ or to $H^*(\tilde{G}_2(\mathbb{R}_m^2))$, and $M$ is necessarily
simply connected. Moreover, $M_1$ and $M_2$ each have the homotopy type of a complex projective space.

4. Diffeomorphism invariants

Let $M_0^a$ and $M_1^a$ be fixed compact smooth manifolds (the exponent $a$ stands for abstract). We also fix two Hermitian vector bundles $\nu_i^a: E_i \to M_i^a$ of complex rank $r_i$. The isomorphism class $[\nu_i^a]$ of the abstract normal bundle may be considered as an element of $[M_i^a, BU(r_i)]$; we write

$$[\nu^a] = ([\nu_0^a], [\nu_1^a]) \in [M_0^a, BU(r_0)] \times [M_1^a, BU(r_1)].$$

**Definition 4.1.** A $[\nu^a]$-simple Hamiltonian $T$-manifold consists of a weight simple Hamiltonian $T$-manifold $(M, M_0, M_1)$ together with diffeomorphisms $\alpha_i: M_i^a \approx \to M_i$ for $i = 0, 1$, such that $\alpha_i^* [\nu_i] = [\nu_i^a]$. Here, $\nu_i = TM_i|_{M_i}/TM_i$ is called the concrete normal bundle to $M_i$ in $M$. It can be endowed with a $U(r_i)$-structure group via the choice of an almost complex structure $J \in J(M, \omega)$.

The isomorphism class $[\nu_i]$ is well-defined (see the Discussion before Remark 2.5). Two such objects $((M, M_0, M_1), \alpha_i)$ and $((M', M'_0, M'_1), \alpha'_i)$ are considered equivalent if there is a $T$-equivariant symplectomorphism $h: M \to M'$ such that $h \circ \alpha_i = \alpha'_i$. The set of equivalence classes of $[\nu^a]$-simple Hamiltonian $T$-manifolds is denoted $\mathcal{H}(\nu^a)$.

The first invariant associated to a class $\mathcal{M} \in \mathcal{H}(\nu^a)$ is the character $\chi(\mathcal{M}) \in \hat{T}$ defined in Lemma 2.2. Note that, since $M \neq M^T$, the map $\chi: T \to S^1$ is surjective. As we are dealing with weight manifolds, the residual action is semi-free, with residual moment map: $\Phi: M \to [0, \ell]$, that sends $M_0$ to 0. The number $\ell = \ell(\mathcal{M}) > 0$ is another invariant of the class $\mathcal{M} \in \mathcal{H}(\nu^a)$, called the $T$-size of $\mathcal{M}$.

Note that $\nu_i^a$ and the character $\chi$ determine unique $T$-equivariant weight bundles, as discussed in Remark 2.5. Thus, $\nu_i^a$ is $T$-equivariantly isomorphic to the concrete normal bundle $\nu_i$ of a representative of $\mathcal{H}(\nu^a)$. Associated to the abstract normal Hermitian bundle $\nu_i^a$, we have the following.

**Definition 4.2.** For the bundle $\nu_i^a$, denote the total space $E_i$ with its bundle projection $p_i: E_i \to M_i^a$. This has associated bundles and structure groups:
4.2.1 the abstract sphere bundle $S_i \to M_i^a$ (fiber $S^{2r_i-1}$), where

$$S_i = \{ z \in E_i \mid |z| = 1 \}.$$  

4.2.2 The abstract disk bundle $D_i \to M_i^a$ (fiber the unit disk in $\mathbb{C}^{r_i}$), where $D_i = \{ z \in E_i \mid |z| \leq 1 \}$. We also consider the disk bundle $D_{i,\varepsilon} = \{ z \in E_i \mid |z| \leq \varepsilon \}$.

4.2.3 The abstract projective bundle $P_i \to M_i^a$ (fiber $\mathbb{C}P^{2r_i-1}$), where $P_i = S_i/S^1$. The projection $\eta_i : S_i \to P_i$ is a principal $S^1$-bundle with Euler class $e(\eta_i) \in H^2(P_i; \mathbb{Z})$.

4.2.4 The extended gauge group $\hat{G}(\nu_i^a)$, defined by pairs of isomorphisms that fit into commutative diagrams

$$\begin{array}{ccc}
E_i & \xrightarrow{g} & E_i \\
\downarrow & & \downarrow \\
M_i^a & \xrightarrow{\bar{g}} & M_i^a
\end{array},$$

where $g$ is smooth and its restriction to each fiber is an isometry. Those isomorphisms with $\bar{g} = \text{id}$ form the usual gauge group $G(\nu_i^a)$. There is thus an exact sequence

$$(4.1) \quad 1 \to G(\nu_i^a) \to \hat{G}(\nu_i^a) \to \text{Diff} (M_i^a, [\nu_i^a]) \to 1,$$

where $\text{Diff} (M_i^a, [\nu_i^a])$ denotes the group of diffeomorphisms $h : M_i^a \to M_i^a$ that satisfy $h^* [\nu_i^a] = [\nu_i^a]$. The group $\hat{G}(\nu_i^a)$ acts naturally on each of the above-associated bundles.

4.2.5 The extended gauge group $\hat{G}(\eta_i)$, defined by pairs of isomorphisms that fit into commutative diagrams

$$\begin{array}{ccc}
S_i & \xrightarrow{g} & S_i \\
\downarrow & & \downarrow \\
P_i & \xrightarrow{\bar{g}} & P_i
\end{array},$$

such that $g$ is smooth and $S^1$-equivariant. Those isomorphisms with $\bar{g} = \text{id}$ form the usual gauge group $G(\eta_i)$.

The $T$-action on $\nu_i^a$ induces a $T$-action on all the abstract sphere and disk bundles which commutes with the actions of the extended gauge groups.
Let \((M, M_0, M_1), \alpha_i\) represent an element of \(H([\nu^a])\). Choose a compatible almost complex structure \(J\) on \(M\), and consider its associated Riemannian metric. As discussed above, this endows the concrete normal bundle \(\nu_i = TM|_{M_i}/TM_i\) with a \(T\)-invariant Hermitian structure, making it isometric to the orthogonal complement of \(TM_i\) in \(TM\). Choose Hermitian vector bundle isomorphisms \(\gamma_i: E_i \rightarrow E(\nu_i)\) covering \(\alpha_i\). These induce isomorphisms on the associated bundles: \(\gamma_i: S_1 \rightarrow S(\nu_i)\) and so forth. We also get a tubular neighborhood embedding \(b: D_{i, \varepsilon} \rightarrow M\) of \(M_i\) in \(M\). We now proceed as in the proof of Lemma 2.6. We may use the embedding \(b\) to straighten the Riemannian metric around \(M_i\), using straightening functions \(\delta_i: [0, \ell] \rightarrow [0, 1]\). The gradient lines for the moment map \(\Phi\) and the straightened metric provide a \(T\)-equivariant smooth embedding \(\beta_0: D_0 \rightarrow M\) by

\[
\beta_0(rz) = \Gamma_{\gamma_0}(z) \cap \Phi^{-1} \left( \frac{\ell r^2}{2} \right),
\]

where \(\Gamma_{\gamma_0}(z)\) is the unique gradient line starting from \(p_0(z)\) in the direction of \(\gamma_0(z)\). The \(T\)-equivariant embedding \(\beta_1: D_1 \rightarrow M\) is defined symmetrically. The image of \(\beta_0\) and \(\beta_1\) are the \(T\)-invariant tubular neighborhoods \(V_0 = \phi^{-1}([0, \ell/2])\) and \(V_1 = \phi^{-1}([\ell/2, \ell])\).

The map

\[
(4.2) \quad \psi = \beta_0^{-1} \circ \beta_1: S_1 \rightarrow S_0
\]

is a diffeomorphism which anti-commutes with the \(S^1\)-action. Let \(E(\nu^a)\) be the space of such diffeomorphisms \(\psi: S_1 \rightarrow S_0\). Observe that \(\psi\) descends to a diffeomorphism \(\tilde{\psi}: \mathbb{P}_1 \rightarrow \mathbb{P}_0\). By pre-composition, the extended gauge group \(\hat{G}(\nu^a_1)\) acts on the right on \(E(\nu^a)\) and, by post-composition, \(\hat{G}(\nu^a_0)\) acts on the left on \(E(\nu^a)\). These two actions commute and descend to the isotopy classes, giving actions of \(\pi_0(\hat{G}(\nu^a_1))\) and \(\pi_0(\hat{G}(\nu^a_0))\) on \(\pi_0(E(\nu^a))\). We can restrict these actions to the usual gauge groups. Define the set \(E([\nu^a])\) by

\[
(4.3) \quad E([\nu^a]) = \pi_0(\hat{G}(\nu^a_0)) \backslash \pi_0(E(\nu^a))/(\pi_0(\hat{G}(\nu^a_1))).
\]

The notation \(E([\nu^a])\) makes sense because the above double coset depends only on \([\nu^a]\). More precisely, let \(\nu' = (\nu'_0, \nu'_1)\) and \(\nu'' = (\nu''_0, \nu''_1)\) be two representatives of \([\nu^a]\). Choosing principal bundle isomorphisms \(\kappa_i: E(\nu'_i) \rightarrow E(\nu''_i)\) produces a bijection \(\kappa\) between the double quotient (4.3) for \(\nu'\) and \(\nu''\). Since we have divided out by the action of the gauge groups, the bijection \(\kappa\) does not depend on the choice of the \(\kappa_i\)’s.
Lemma 4.3. The above construction provides a well-defined map

\[ \Psi : \mathcal{H}(\nu^a) \to \mathcal{E}(\nu^a). \]

Proof. Let \((M, M_0, M_1, \alpha_i)\) represent a class \(M \in \mathcal{H}(\nu^a)\). The definition of the diffeomorphism \(\psi\) of (4.2) involves three choices:

(a) The compatible almost complex structure \(J\) on \(M\);
(b) The \(U(r_i)\)-isomorphism \(\gamma_i : E_i \to E(\nu_i)\); and
(c) The straightening functions \(\delta_i : [0, \ell] \to [0, 1]\).

Once the choices (a) and (b) have been made, the straightening functions \(\delta_0\) and \(\delta_1\) belong to convex spaces, so their choice does not change \(\psi\) in \(\pi_0(\mathcal{E}(\nu))\). If we choose instead \(\tilde{\gamma}_i : E_i \to E(\nu_i)\) for (b), then \(\tilde{\gamma}_i = g_i \circ \gamma_i\) with \(g_i \in \mathcal{G}(\nu_i)\). Hence, \(\tilde{\psi} = g_0 \circ \psi \circ g_1\), proving that \(\tilde{\psi}\) and \(\psi\) represent the same class in \(\mathcal{E}(\nu^a)\). Finally, the choice of (a) does not change the class since compatible almost complex structures on \(M\) form a contractible space.

Now, let \(((\bar{M}, \bar{M}_0, \bar{M}_1), \bar{\alpha}_i)\) be another representative of \(\mathcal{M}\). Let \(h : M \to \bar{M}\) be a \(T\)-equivariant symplectomorphism realizing the equivalence. Choose a compatible almost complex structure \(\bar{J}\) on \(\bar{M}\). Then \(J = Th^{-1} \circ J \circ Th\) is a compatible almost complex structure on \(M\), which may be used, together with the above Hermitian bundle isomorphisms \(\gamma_i\) to get a representative \(\tilde{\psi}\) of \(\bar{\Psi}(\mathcal{M})\). The construction is transported via \(h\) to \(\bar{M}\), using \(J\), setting \(\tilde{\gamma}_i = Th \circ \gamma_i\), and using the same straightening function. We thus get embeddings \(\bar{\beta}_i = h \circ \beta_i : \mathbb{D}_i \to M'\) which can be used to define \(\bar{\psi} : S_1 \to S_0\), which then satisfies

\[ \bar{\psi} = \bar{\beta}_0^{-1} \circ \bar{\beta}_1 = \beta_0^{-1} \circ h^{-1} \circ h \circ \beta_1 = \psi. \]

\[ \square \]

Let us consider the following quotients of the set \(\mathcal{E}(\nu^a)\):

\[ (4.4) \quad \mathcal{E}^1(\nu^a) = \pi_0(\mathcal{G}(\nu_0^a) \backslash \pi_0(\mathcal{E}(\nu^a)) / \pi_0(\hat{\mathcal{G}}(\nu_1^a)) \]

and

\[ (4.5) \quad \mathcal{E}^{01}(\nu^a) = \pi_0(\hat{\mathcal{G}}(\nu_0^a) \backslash \pi_0(\mathcal{E}(\nu^a)) / \pi_0(\hat{\mathcal{G}}(\nu_1^a)) \]

The compositions of the map \(\Psi : \mathcal{H}(\nu^a) \to \mathcal{E}(\nu^a)\) with the projections onto \(\mathcal{E}^1(\nu^a)\) and \(\mathcal{E}^{01}(\nu^a)\) are denoted by \(\Psi^1\) and \(\Psi^{01}\).
Theorem 4.4. Let \((M, M_0, M_1), \alpha_i\) and \((M', M'_0, M'_1), \alpha'_i\) represent classes \(\mathcal{M}\) and \(\mathcal{M}'\) in \(\mathcal{H}(\mathcal{H})\), with \(\chi(\mathcal{M}) = \chi(\mathcal{M}')\). Denote the reduced moment maps by \(\Phi: M \rightarrow [0, \ell]\) and \(\Phi': M \rightarrow [0, \ell']\), where \(\ell\) and \(\ell'\) are the \(T\)-sizes of \(\mathcal{M}\) and \(\mathcal{M}'\).

(a) If we have \(\Psi(\mathcal{M}) = \Psi(\mathcal{M}')\), then there is a \(T\)-equivariant diffeomorphism \(h: M \rightarrow M'\) satisfying

\[
\Phi' \circ h = \frac{\ell'}{\ell} \Phi
\]

and such that \(h \circ \alpha_i = \alpha'_i\) for \(i = 0, 1\).

(b) If we have \(\Psi^1(\mathcal{M}) = \Psi^1(\mathcal{M}')\), then there is a \(T\)-equivariant diffeomorphism \(h: M \rightarrow M'\) satisfying (4.6) and such that \(h \circ \alpha_0 = \alpha'_0\).

(c) If \(\Psi^{01}(\mathcal{M}) = \Psi^{01}(\mathcal{M}')\), then there is a \(T\)-equivariant diffeomorphism \(h: M \rightarrow M'\) satisfying (4.6).

Equation (4.6) means that \(\Phi' \circ h = \sigma \circ \Phi\), where \(\sigma\) is an affine isomorphism of \(t^*\) of ratio \(\ell'/\ell\).

Proof. For Part (a), choose \((J, \gamma_i)\) and \((J', \gamma'_i)\) as above, getting \(T\)-equivariant embeddings \(\beta_i\) and \(\beta'_i\) and \(\psi, \psi' \in \mathcal{E}(\nu)\). The condition \(\Psi(\mathcal{M}) = \Psi(\mathcal{M}')\) implies an equation in \(\pi_0(\mathcal{E}(\nu))\) of the form \([\psi'] = g_1[\psi]g_0\) with \(g_i \in \mathcal{G}_i\). Changing \(\gamma_i\) into \(\gamma_i \circ g_i^{-1}\), we get that \([\psi] = [\psi']\) in \(\pi_0(\mathcal{E}(\nu))\). Now, the embeddings \(\beta_i\) produce a \(T\)-equivariant diffeomorphism \(N_\psi = \mathbb{D}_0 \cup_\psi \mathbb{D}_1 \xrightarrow{q} M\) extending \(\alpha_0\) and \(\alpha_1\). In the same way, the embeddings \(\beta'_i\) produce a \(T\)-equivariant diffeomorphism

\[
N_{\psi'} = \mathbb{D}_0 \cup_{\psi'} \mathbb{D}_1 \xrightarrow{q'} M'
\]

extending \(\alpha'_0\) and \(\alpha'_1\). As \([\psi'] = [\psi]\), there is a smooth \(T\)-equivariant isotopy

\[
b: S_0 \times [1/2, 1] \rightarrow S_0 \times [1/2, 1],
\]

preserving the projection onto \([1/2, 1]\), such that \(b(z, t) = (z, t)\) for \(t\) near \(1/2\), and \(b(z, t) = (\psi' \circ \psi^{-1}(z), t)\) for \(t\) near \(1\). This isotopy extends, by the identity near the null-section, to a \(T\)-equivariant diffeomorphism \(b: \mathbb{D}_0 \rightarrow \mathbb{D}_0\). Now, \(b\) together with the identity on \(\mathbb{D}_1\) gives a \(T\)-equivariant diffeomorphism \(B: N_\psi \xrightarrow{\approx} N_{\psi'}\). Finally, observe that the level sets of the maps \(\Psi \circ q\) and \(\Psi \circ q'\) are the manifolds \(|z| = \text{constant}\) in \(\mathbb{D}_i\). These level sets are preserved by the diffeomorphism \(B\). By the definition of the embeddings \(\beta_i\) and \(\beta'_i\),
this proves Equation (4.6) and completes the proof of (a). Parts (b) and (c) are proven in the same way, but the elements $g_i$ that occur in the above argument are now in $\hat{G}_i$ instead of $G_i$. □

In order to get applications of Theorem 4.4, we now provide a different description of $E([\nu])$ and its quotients. Choose an element $h \in E([\nu^a])$, if $E([\nu^a])$ is non-empty. Then any $\tilde{h} \in E([\nu^a])$ is of the form $\tilde{h} = h \circ h^{-1} \circ \hat{g}$ and $h^{-1} \circ \hat{g} \in \hat{G}(\eta_1)$. Hence, the map $g \mapsto h \circ \hat{g}$ provides a bijection from $\hat{G}(\eta_1)$ onto $E([\nu])$. Now, there is an injection $\hat{G}(\eta_1) \hookrightarrow E([\nu^a])$ given by $\gamma \mapsto h \circ \gamma \circ h$. Composed with the above bijection $\hat{G}(\eta_1) \approx E([\nu^a])$ gives an injective homomorphism

$$\hat{G}(\nu^a_0) \rightarrow \hat{G}(\eta_1)$$

defined by $\gamma \mapsto h \circ \gamma \circ h^{-1}$. We have proven the following proposition.

**Proposition 4.5.** If $E([\nu^a])$ is not empty, the choice of $h \in E(\nu^a)$ provides bijections

$$E([\nu^a]) \approx \pi_0(\hat{G}(\nu^a_0)) \backslash \pi_0(\hat{G}(\eta_1)) / \pi_0(\hat{G}(\nu^a_1)),$$

$$E^1([\nu^a]) \approx \pi_0(\hat{G}(\nu^a_0)) \backslash \pi_0(\hat{G}(\eta_1)) / \pi_0(\hat{G}(\nu^a_1))$$

and

$$E^{01}([\nu^a]) \approx \pi_0(\hat{G}(\nu^a_0)) \backslash \pi_0(\hat{G}(\eta_1)) / \pi_0(\hat{G}(\nu^a_1)),$$

where the inclusion $\hat{G}(\nu^a_0) \hookrightarrow \hat{G}(\eta_1)$ is given by $\gamma \mapsto h^{-1} \circ \gamma \circ h$.

5. The case $r_1 = 1$

The results of this section follow from the following proposition.

**Proposition 5.1.** Let $M^a_i$ be compact smooth manifolds for $i = 0, 1$. Let

$$[\nu^a] = ([\nu^a_0], [\nu^a_1]) \in [M^a_0, BU(r_0)] \times [M^a_1, BU(r_1)].$$

Suppose that $r_1 = 1$. Then $E^1([\nu^a])$ is either empty or contains a single element.
Proof. If $\mathcal{E}^1([\nu^a])$ is not empty, then it is, by Proposition 4.5, in bijection with
\[ \pi_0(\mathcal{G}(\nu^a_0)) \setminus \pi_0(\hat{\mathcal{G}}(\eta_1)) / \pi_0(\hat{\mathcal{G}}(\nu^a_1)). \]
As $r_1 = 1$, $\nu^a_1$ is isomorphic to the complex line bundle associated to $\eta_1$. Hence, $\hat{\mathcal{G}}(\nu^a_1) = \hat{\mathcal{G}}(\eta_1)$ which implies that $\mathcal{E}^1([\nu])$ consists of a single element. \qed

We now provide a criterion to determine, in Proposition 5.1, whether $\mathcal{E}^1([\nu^a])$ is non-empty. Let $\eta_i : S_i \to P_i$ be the $S^1$-principal bundle associated to $\nu^a_i$. Let $L_i \to P_i$ be the Hermitian line bundle associated to $\eta_i$. Let $L_i^- \to P_i$ be the conjugate line bundle, and denote its isomorphism class by $[\eta_i^-]$.

**Proposition 5.2.** Let $M_i^a$ and $[\nu^a]$ as in Proposition 5.1. The set $\mathcal{E}^1([\nu^a])$ is non-empty if and only if there exists a diffeomorphism $\kappa : M_i^a \to P_0$ such that $\kappa^*[\eta_0^-] = [\nu^a_1]$.

**Proof.** The diffeomorphism $\kappa$ would be covered by a diffeomorphism $\tilde{\kappa} : S_1 \to S_0$ which anti-commutes with the $S^1$-action. Such a $\tilde{\kappa}$ defines a class in $\mathcal{E}^1([\nu^a])$.

Conversely, a class in $\mathcal{E}^1([\nu^a])$ is represented by a diffeomorphism $h : S_1 \to S_0$ which anti-commutes with the $S^1$-action. This descends to $\tilde{h} : P_1 \to P_0$ satisfying $\tilde{h}^*[\eta_0^-] = [\eta_1]$. As $r_1 = 1$, there is a bundle isomorphism between $\mathbb{E}_1$ and $L(\eta_1)$ (over the identity of $M_i^a$). Hence, $\kappa = \tilde{h}$ is the desired diffeomorphism. \qed

We now describe in details a basic example.

**Example 5.3.** Let $N$ be a compact symplectic manifold. Let $\xi : E \to N$ be a Hermitian vector bundle of complex rank $r$. Each fiber of $\xi$ is equipped with a symplectic form coming from the standard symplectic form on $\mathbb{C}^r$ via a trivialization. Then the symplectic form on $N$ as well as those on the fibers of $\xi$ are the restriction of a unique symplectic form $\omega$ on $E$. The action of $S^1$ by complex multiplication is Hamiltonian, with moment map $\Phi(z) = \frac{1}{2}||z||^2$. Any $\ell > 0$ is a regular value, so we may take the symplectic cut $P_\ell(\xi)$ of $E$ at $\ell$. We thus get a simple $S^1$-Hamiltonian manifold $(P_\ell(\xi), N, P_\ell(\xi))$, where $P_\ell(\xi)$ is the symplectic reduction of $E$ at $\ell$. Using a non-trivial character $\chi : T \to S^1$, we thus get a weight simple $T$-Hamiltonian manifold with residual moment map $\tilde{\phi}$. We denote this simple Hamiltonian manifold by $C_\chi(N, \xi, \ell)$. 

---

**Simple Hamiltonian manifolds 407**

---
Let us define abstract manifolds and normal bundles for \( C_\chi(N, \xi, \ell) \). We can take \( M_0^a = N, \alpha_0 = \text{id} \) and \( \nu_0^a = \xi \). Then there is a canonical diffeomorphism \( \alpha_1 : \mathbb{P}_0 \xrightarrow{\cong} P_\ell(\xi) \) obtained by following the real vector lines in \( E_0 = E \). Hence, together with \( \alpha_0 \) and \( \alpha_1 \), \( C_\chi(N, \xi, \ell) \) is a \((N, \mathbb{P}_0)\)-simple Hamiltonian \( T \)-manifold. As seen in Proposition 5.2, \([\nu_1] = [\eta_0^-]\).

The \( T \)-embedding \( \beta_0 \) is induced by the embedding \( \tilde{\beta}_0 : \mathbb{D}_0 \rightarrow \mathbb{E}_0 \) defined by \( \tilde{\beta}_0(sz) = r\sqrt{\ell} z \). Using the identification \( S_1 = S_0^- \), the elements of \( \mathbb{D}_1 \) may be written under the form \( rz \) with \( r \in [0, 1] \) and \( z \in S_0, \) with the identification \( 0z = 0z' = p(z) = p(z') \) when the projection of \( z \) and \( z' \) onto \( \mathbb{P}_0 \) coincide. The \( T \)-embedding \( \beta_1 \) is then induced by the \( T \)-map \( \tilde{\beta}_1 : \mathbb{D}_0^- \rightarrow \mathbb{E}_0 \) defined by \( \tilde{\beta}_1(rz) = r(\sqrt{\ell} - \sqrt{2\ell}) \bar{z} \). Hence,

\[
\Psi(P) = [\text{id}]
\]

(the identity from \( S_0 \) to \( S_0^- = S_1 \) anti-commuting with the \( S^1 \)-multiplication, as expected).

**Theorem 5.4.** Let \((M, M_0, M_1)\) be a simple Hamiltonian \( T \)-manifold with \( T \)-size \( \ell \), and associated character \( \chi \). Suppose that \( r_1 = 1 \). Then there exits a \( T \)-equivariant diffeomorphism

\[
F : C_\chi(M_0, \nu_0, \ell) \xrightarrow{\cong} M
\]

commuting with the residual moment maps and such that \( F|_{M_0} = \text{id} \).

**Proof.** As \( r_1 = 1 \), we know \( M \) is a weight simple Hamiltonian manifold. Define \( M_0^a = M_0 \) and set \( \alpha_0 = \text{id} \). Fix an almost complex structure on \( M \) compatible with the symplectic form and let \( \nu_0^a \) be the orthogonal complement of \( TM_0 \) in \( TM \) for the associated metric. By Proposition 5.2, there exists a diffeomorphism \( \alpha_1 : \mathbb{P}_0 \rightarrow M_1 \) such that \( \alpha_1^* [\nu_0^a] = [\eta_0^-] \). Hence, \(((M, M_0, M_1), \alpha_1)\) represents a class in \( \mathcal{H}([\nu]) \) for \( [\nu] = ([\nu_0^a], [\eta_0^-]) \). So does the simple Hamiltonian manifold \( C_\chi(M_0, \nu_0, \ell) \) of Example 5.3, with its own \( \alpha_i \)'s. By Theorem 4.4 and Proposition 5.1, this completes the proof of Theorem 5.4. \( \square \)

Theorem 5.4 implies that \( M_1 \) is diffeomorphic to \( \mathbb{P}_0 \). If, in addition \( r_0 = 1 \), then \( \mathbb{P}_0 \) is diffeomorphic to \( M_0 \) and we have the following corollary, also found in \([4, \text{Lemma 3.2}]\).

**Corollary 5.5.** Let \((M, M_0, M_1)\) be a simple Hamiltonian manifold with \( r_0 = r_1 = 1 \). Then \( M_1 \) is diffeomorphic to \( M_0 \).
6. Classification up to $T$-equivariant symplectomorphism

The philosophy of this section is slightly different from that in Section 4. We fix a single compact smooth manifold $M_0^a$ and a Hermitian vector bundle $\nu_0^a: E_0 \to M_0^a$ of complex rank $r_0$, whose isomorphism class is denoted by $[\nu_0^a] \in [M_0^a, BU(r_0)]$. The associated bundles $S_0 \to M_0$ and so forth, as well as $\eta_0$, are defined as in Section 4.

**Definition 6.1.** A $[\nu_0^a]$-simple Hamiltonian $T$-manifold consists of a weight simple Hamiltonian $T$-manifold $(M, M_0, M_1)$ together with a diffeomorphism $\alpha_0: M_0^a \to M_0$ such that $\alpha_0^* [\nu_0] = [\nu_0^a]$.

Here, $\nu_0$ is the concrete normal bundle to $M_0$ in $M$, represented by the orthogonal complement of $TM_0$ in $TM$ for the Riemannian metric associated to a $T$-invariant almost complex structure on $M$ compatible with the symplectic form. In particular, $\omega^a = \alpha_0^* \omega_0$ is a symplectic form on $M_0^a$. Two such objects $((M, M_0, M_1), \alpha_0)$ and $((M', M'_0, M'_1), \alpha'_0)$ are considered as equivalent if there is a $T$-equivariant symplectomorphism $h: M \to M'$ such that $h \circ \alpha_0 = \alpha'_0$. The following are invariants of an equivalence class:

- The associated character $\chi$ and the residual action, which is semi-free, since we are in the case of weight simple Hamiltonian manifolds;
- The $T$-size $\ell > 0$;
- The symplectic form $\omega_0^a$ on $M_0^a$; and
- The codimensions $r_0$ and $r_1$.

Fixing $[\nu_0^a]$, $\omega_0^a$, $\ell$ and $r_1$, we get a set of equivalence classes denoted by

$$S^0([\nu_0^a], \omega_0^a, r_1, \ell).$$

We are especially interested in the case $r_1 = 1$. By Theorem 5.4, elements of $S^0([\nu_0], \omega_0^a, 1, \ell)$ are in bijection with classes of symplectic forms on $\mathcal{C}_\chi(M_0^a, \nu_0^a, \ell)$ coinciding with $\omega_0^a$ on $M_0^a$ and for which the $T$-action is Hamiltonian. Two such forms $\omega$ and $\omega'$ are equivalent if there is a self-diffeomorphism $F$ of $\mathcal{C}_\chi(M_0^a, \nu_0, \ell)$, commuting with the reduced moment maps, such that $F^* \omega = \omega'$ and $F|_{M_0^a} = \text{id.}$
Let $\Omega^{\text{sym}}(M^a_0)$ be the space of symplectic forms on $M^a_0$, with the topology induced by the $C^\infty$-topology in $\Omega^2(M^a_0)$. Define
\[
\mathcal{D}((M^a_0, \omega^a_0), [\nu^a_0], \ell) = \left\{ \omega: [0, \ell] \to \Omega^{\text{sym}}(M^a_0) \mid \omega(0) = \omega^a_0 \text{ and } [\omega(\lambda)] = [\omega^a_0] + \lambda e(\eta_0) \right\},
\]
where the last equation holds in de Rham cohomology $H^2_d(M^a_0)$.

**Theorem 6.2.** Suppose that $r_1 = 1$. Then there exists a bijection
\[
\Theta: \mathcal{S}^0([\nu^a_0], \omega^a_0, 1, \ell) \xrightarrow{\approx} \pi_0(\mathcal{D}((M^a_0, \omega^a_0), [\nu^a_0], \ell)).
\]

**Proof.** Let $M = \mathcal{C}_\chi(M^a_0, \nu_0, \ell)$. As noted above, a class of $a \in \mathcal{S}^0([\nu^a_0], \omega^a_0, 1, \ell)$ is represented by a symplectic form $\omega$ on $M$. Observe that there is a diffeomorphism from $M/S^1$ to $[0, \ell] \times M^a_0$. The first component is given by the residual moment map and the second one is induced by the projection $\mathbb{E}_0 \to M^a_0$. Each slice $\{\lambda\} \times M_0$ is then endowed with a symplectic form $\omega(\lambda)$ given by the symplectic reduction of $\mathbb{E}_0$ at $\lambda$. This provides a map $\omega: [0, \ell] \to \Omega^{\text{sym}}(M^a_0)$ with $\omega(0) = \omega^a_0$. The equation $[\omega(\lambda)] = [\omega^a_0] + \lambda e(\eta_0)$ holds in $H^2(M^a_0)$ by the Duistermaat–Heckman theorem. Hence, $\omega()$ defines a class in $\mathcal{D}((M^a_0, \omega^a_0), [\nu^a_0], \ell)$ which we define to be $\Theta(a)$.

To see that $\Theta$ is well-defined, suppose that $\omega'$ is a symplectic form on $M$ equivalent to $\omega$. Let $F$ be a self-diffeomorphism of $M$ realizing the equivalence, so $F^*\omega() = \omega'()$. The map $F$ descends to a self-diffeomorphism $\tilde{F}$ of $[0, \ell] \times M^a_0$ commuting with the projection onto $[0, \ell]$. Hence, $\tilde{F}$ is of the form $\tilde{F}(\lambda, x) = (\lambda, \tilde{F}_\lambda(x))$ where $\tilde{F}_\lambda$ is a self-diffeomorphism of $M^a_0$ such that $\tilde{F}_\lambda^*\omega(\lambda) = \omega'(\lambda)$ and $\tilde{F}_0 = \text{id}$. For $t \in [0, 1]$, let $\omega_t: [0, \ell] \to \Omega^{\text{sym}}(M^a_0)$ be defined by $\omega_t(\lambda) = \tilde{F}_\lambda^*\omega$. The map $t \mapsto \omega_t()$ is a path in $\Omega^{\text{sym}}(M^a_0)$ from $\omega()$ to $\omega'()$. This shows that the two forms are cohomologous and so $\Theta$ is well-defined.

Let us now prove that $\Theta$ is surjective. Let $\omega()$ represent a class in $\mathcal{D}((M^a_0, \omega^a_0), [\nu^a_0], \ell))$. Let $S^0_0 \to M^a_0$ be the $S^1$-bundle associated to $\nu^a_0$. Using the normal form for reduced spaces [2, Section 30.3], we can extend the map $\omega()$ to a smooth map $\omega: [-\varepsilon, 1 + \varepsilon] \to \Omega^{\text{sym}}(M^a_0)$. For such map there is a symplectic form $\tilde{\omega}$ on $S^0_0 \times [-\varepsilon, 1 + \varepsilon]$ such that the $S^1$-action is Hamiltonian with moment map the projection onto $[-\varepsilon, 1 + \varepsilon]$, as shown in [13, Proposition 5.8]. Performing symplectic cuts at 0 and 1 provides a simple Hamiltonian manifolds $(N; M^a_0, N_1)$ defining a class $a \in \mathcal{S}^0([\nu^a_0], \omega^a_0, 1, \ell)$ and using $\alpha = \text{id}$ so that $\Theta(a) = [\omega()]$. 
To prove the injectivity of $\Theta$, suppose $a, a' \in \mathcal{S}^0([\nu_0^a], \omega_0^a, 1, \ell)$ are represented by symplectic forms $\tilde{\omega}$ and $\tilde{\omega}'$ on $M = C_\chi(M_0, \nu_0, \ell)$. These give rise to $\omega()$ and $\omega(')$ in $\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)$ representing $\Theta(a)$ and $\Theta(a')$. If $\Theta(a) = \Theta(a')$, there exists a path $\omega_t() \in \mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)$ joining $\omega()$ to $\omega(')$. Because of the cohomology constraint in the definition of $\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)$, the cohomology class of $\omega_t(\lambda)$ is independent of $t$. By Moser’s theorem [2, Theorem 7.3], there exists an isotopy $\rho_t: M_0^a \times [0, \ell] \rightarrow M_0^a \times [0, \ell]$, with $\rho_0 = \text{id}$, such that $\omega_t() = \rho_t^* \omega()$. This isotopy may be covered by an isotopy $\tilde{\rho}_t: M \rightarrow M$ with $\tilde{\rho}_0 = \text{id}$. Let $\tilde{\omega}_t = \tilde{\rho}_t^* \tilde{\omega}$. By [13, Proposition 5.8], we may deduce that $\tilde{\omega}_1 = \tilde{\omega}'$. This proves that $a = a'$, completing the proof.

Theorem 6.2 reduces the identification of $\mathcal{S}^0([\nu_0^a], \omega_0^a, 1, \ell)$ to computing $\pi_0(\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell))$. We only have results when the latter is reduced to one element.

**Theorem 6.3.** Suppose that $r_1 = 1$. Then $\pi_0(\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)) = \ast$ if $[\omega_0^a]$ and $e(\nu_0^a)$ are linearly dependent in the vector space $H^2_{dr}(M_0^a)$ of de Rham cohomology.

The linear dependence condition is automatically fulfilled when $H^2_{dr}(M_0^a) \approx \mathbb{R}$, as when $M_0^a$ is a complex Grassmannian or $\tilde{G}_2(\mathbb{R}^{m+2})$ of Example 1.1.6.

**Proof.** Let $\omega: [0, \ell] \rightarrow \Omega_{\text{sym}}(M_0^a)$ represent an element of $\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)$. As $[\omega_0^a] \neq 0$, our hypothesis of linear dependence implies that there is a unique $s \in \mathbb{R}$ such that $e(\nu_0^a) = s [\omega_0^a]$. Hence,

$$[\omega(\lambda)] = [\omega_0^a] + \lambda e(\nu_0^a) = (1 + \lambda s)[\omega_0^a].$$

As $[\omega(\lambda)] \neq 0$, we know that $(1 + \lambda s) > 0$. The symplectic form $(1 + \lambda s)^{-1} \omega(\lambda)$ thus satisfies $[(1 + \lambda s)^{-1} \omega(\lambda)] = [\omega_0^a]$. By Moser’s theorem [2, Theorem 7.3], there exists an isotopy

$$\rho_\lambda: M_0^a \rightarrow M_0^a$$

with $\rho_0 = \text{id}$, such that $\omega(\lambda) = \rho_\lambda^* \omega_0^a$. Hence, the formula

$$\omega_t(\lambda) = (1 + \lambda s)\rho_{t\lambda}^* \omega_0^a \quad (t \in [0, 1])$$

defines a path in $\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell)$ joining $\omega$ to $(1 + \lambda s)\omega_0^a$. This shows that $\pi_0(\mathcal{D}((M_0^a, \omega_0^a), [\nu_0^a], \ell))$ has only one element.

Using Theorem 6.2, Theorem 6.3 and its proof have the following corollary.

---

Simple Hamiltonian manifolds 411
Corollary 6.4. Let \((M, M_0, M_1)\) be a \([\nu^0_a]\)-simple Hamiltonian \(T\)-manifold with \(T\)-size \(\ell\) and associated character \(\chi\). Suppose that \(r_0 = r_1 = 1\) and that \(e(\nu^0_a) = s [\omega^0_a]\) for some \(s \in \mathbb{R}\). Then there exits a \(T\)-equivariant symplectomorphism \(\alpha : C^\chi(\nu^0_a) \approx M\) such that \(\alpha |_{M_0} = \alpha_0\). Moreover \((M_1, \omega_1)\) is symplectomorphic to \((M_0^a, (1 + s\ell) \omega^0_a)\).

As a corollary below, we may reproduce Delzant’s result [3, Theorem 1.2] in a slightly more precise way, with essentially the same proof rephrased in our framework. For the diagonal action of \(S^1\) on \((\mathbb{C}P^m)^{m+1}\), with moment map \(\tilde{\Phi}(z) = \frac{1}{2} |z|^2\), denote by \((\mathbb{C}P^m)_\ell\) the symplectic reduction at \(\ell\):

\[
(\mathbb{C}P^m)_\ell = \mathbb{C}^{m+1} \sslash S^1.
\]

We also consider the symplectic cut \((\mathbb{C}P^m)_\ell\) of \(\mathbb{C}^{m+1}\) at \(\ell\), equipped with the induced \(S^1\)-action and induced moment map \(\hat{\Phi} : (\mathbb{C}P^m)_\ell \to [0, \ell]\). Observe that \((\mathbb{C}P^m)_\ell\) is symplectomorphic to \((\mathbb{C}P^{m+1})_\ell\). Indeed, as the symplectic forms vary linearly in \(\ell\), it is enough to prove this for \(\ell = 1\). But \((\mathbb{C}P^m)_1\) and \((\mathbb{C}P^{m+1})_1\) are both toric manifolds admitting as moment polytope an \((m+1)\)-simplex intersecting the weight lattice at its vertices.

Corollary 6.5. Let \((M^2m, M_0, M_1)\) be a simple Hamiltonian \(S^1\)-manifold of \(S^1\)-size \(\ell\), with \(M_0\) a single point. Then

1. \(M\) is \(S^1\)-equivariantly symplectomorphic to \((\mathbb{C}P^m)_\ell\), endowed with a standard \(S^1\)-action (multiplication on a single coordinate).

2. \(M_1\) is symplectomorphic to \((\mathbb{C}P^{m-1})_\ell\).

Proof. Let \((W, W_0, W_1) = (\mathbb{C}P^m)_\ell, pt, (\mathbb{C}P^m)_\ell\) and, for \(A \subset \mathbb{R}\), let \(X_A = \{z \in \mathbb{C} | |z| \in A\}\). Let \(0 < \varepsilon < \varepsilon' < \ell\). Performing a symplectic cut to \(W\) at \(\varepsilon\) gives rise to two simple manifolds, the “lower” one \((W_-, pt, V_\varepsilon)\) and the “upper” one \((W_+, V_\varepsilon, W_1)\), together with symplectic \(S^1\)-equivariant embeddings \(h_- : X_{(0, \varepsilon)} \to W_-\) and \(h_+ : X_{\varepsilon, \varepsilon'} \to W_+\). From these, one can recover \(W\). The quotient map \(p : X_{[0, \ell]}\) induces to an \(S^1\)-equivariant symplectomorphism

\[
W \approx (W_-- V_\varepsilon) \cup_{h_-} X_{(0, \varepsilon')} \cup_{h_+} (W_+ - V_\varepsilon).
\]

In the same way, performing a symplectic cut of \(M\) at \(\varepsilon\) gives rise to two simple manifolds \((M_-, pt, N_\varepsilon)\) and \((M_+, N_\varepsilon, M_1)\). If \(\Phi : M \to [0, \ell]\) denotes the
moment map, we get $S^1$-symplectomorphisms $\alpha_-: \Phi^{-1}(0, \varepsilon) \to M_- - N_\varepsilon$ and $\alpha_+: \Phi^{-1}(\varepsilon, \ell) \to M_+ - N_\varepsilon$. By the local forms around a fixed point, there is an $S^1$-equivariant symplectomorphism $q: U \to U'$ between neighborhoods $U$ and $U'$ of $W_0$ and $M_0$, respectively. Choose $\varepsilon'$ small enough so that $p(X_{(0,\varepsilon')}) \subset U$. We thus get two symplectic $S^1$-equivariant embeddings

$$g_- = \alpha_- \circ q \circ p: X_{(0,\varepsilon)} \to W_- \quad \text{and} \quad g_+ = \alpha_+ \circ q \circ p: X_{\varepsilon,\varepsilon'} \to W_+,$$

and an $S^1$-equivariant symplectomorphism

$$(6.2) \quad M \approx (M_- - N_\varepsilon) \cup g_- X_{(0,\varepsilon')} \cup g_+ (M_+ - N_\varepsilon).$$

The symplectomorphism $q$ induces an $S^1$-equivariant symplectomorphism $q_-: W_- \to M_-$ such that $q_- \circ h_- = g_-$, and also a symplectomorphism $q_+: V_\varepsilon \to N_\varepsilon$.

In order to get an $S^1$-equivariant symplectomorphic from $W$ to $M$ it is then enough, given (6.1) and (6.2), to construct an $S^1$-equivariant symplectomorphism $q_+: W_+ \to M_+$ such that $q_+ \circ h_+ = g_+$. The problem may be reformulated as follows. Let $Y$ be the upper manifold of the symplectic cut of $X_{(0,\varepsilon')}$ at $\varepsilon$. The embedding $h_+$ extends to a symplectic $S^1$-equivariant embedding $\hat{h}_+: Y \to W_+$ onto a tubular neighborhood of $V_\varepsilon$ in $W_+$; $h_+$ and $\hat{h}_+$ determine each other. In the same way, $g_+$ extends to a symplectic $S^1$-equivariant embedding $\hat{g}_+: Y \to M_+$ onto a tubular neighborhood of $N_\varepsilon$ in $M_+$; $g_+$ and $\hat{g}_+$ determine each other. We are then looking for an $S^1$-equivariant symplectomorphism $q_+: W_+ \to M_+$ such that $q_+ \circ \hat{h}_+ = \hat{g}_+$. As $\hat{g}_+$ coincides with $q_+$ on $V_\varepsilon$, it actually suffices to construct an $S^1$-equivariant symplectomorphism $q_+: W_+ \to M_+$ extending $q_\varepsilon$. Indeed, by the uniqueness of $S^1$-invariant tubular neighborhood of $N_\varepsilon$ up to symplectomorphism, it will be possible, taking $\varepsilon'$ smaller if necessary, to modify $q_+\varepsilon$ by an isotopy so that the condition $q_+ \varepsilon \circ \hat{h}_+ = \hat{g}_+$ remains true.

By construction, $W_+$ is identified with $\mathcal{C}_\varepsilon((CP^{n-1}, \eta, \ell - \varepsilon))$, where $\eta$ is the Hopf bundle. The simple manifold $(M_+, N_\varepsilon, M_1)$ has $S^1$-size $\ell - \varepsilon$ and the existence of the diffeomorphism $\hat{g}_+$ implies that $g_\varepsilon^*\nu(N_\varepsilon) = \eta$. By Corollary 6.4, $g_\varepsilon$ extends to an $S^1$-equivariant symplectomorphism $q_+: W_+ \to M_+$ as required. □

7. Examples of polygon spaces

This section provides examples using polygon spaces. We recall below some minimal theory to state the results. For more developments, classification and references; see e.g. [7, 8].

---

Simple Hamiltonian manifolds 413
Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_r^n$, where $\mathbb{R}_r^n := \{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \mid 0 < \alpha_1 \leq \cdots \leq \alpha_n \}$. Let $S^2_{\alpha_i}$ denote the sphere in $\mathbb{R}^3$ with radius $\alpha_i$. We identify $\mathbb{R}^3$ with $so(3)^*$ so that the Lie–Kirillov–Kostant–Souriau symplectic structure gives $S^2_{\alpha_i}$ the symplectic volume $2\alpha_i$.

**Definition 7.1.** The polygon space $N_\alpha$ is the symplectic reduction at 0

$$N_\alpha = \left( \prod_{i=1}^m S^2_{\alpha_i} \right) \big/_{0} SO_3$$

for the the diagonal co-adjoint action of $SO(3)$.

The moment map for the co-adjoint action on the product of spheres maps $\rho \mapsto \sum \rho_i$, so we get

(7.1) $$N_\alpha = \left\{ \rho = (\rho_1, \ldots, \rho_m) \in (\mathbb{R}^3)^m \mid \forall i, |\rho_i| = \alpha_i \text{ and } \sum_{i=1}^m \rho_i = 0 \right\} /_{SO_3}$$

as the moduli space of spatial configurations of a polygon with length-side vector $\alpha$. Note that $N_\alpha$ is denoted by Pol $(\alpha)$ in [8] and by $N_{3\alpha}^n(\alpha)$ in [7]). The origin is a regular value for the moment map if and only if there is no aligned configuration, that is the equation

$$\sum_{i=1}^n \epsilon_i \alpha_i = 0$$

has no solution with $\epsilon_i = \pm 1$. Such length vectors $\alpha$ are called **generic**.

When $\alpha_i \neq \alpha_j$ for some $i, j$, then $\Phi_{i,j}(\rho) = |\rho_i + \rho_j|$ defines a smooth function $\Phi_{i,j}: N_\alpha \to \mathbb{R}$. This is the moment map of a Hamiltonian $S^1$-action on $N_\alpha$, a particular case of a **bending flow** [9]. It acts on $\rho$ by rotating $\rho_i$ and $\rho_j$ at constant speed around the axis $\rho_i + \rho_j$. The critical points for $\Phi_{i,j}$ are those configurations $\rho$ for which $\{ \rho_k \mid k \neq i, j \}$ generate a one-dimensional space.

If $\alpha \in \mathbb{R}_r^n$ satisfies the inequalities

(7.2) $$\alpha_n < \sum_{i<n} \alpha_i \text{ and } \alpha_n + \alpha_1 > \sum_{i=2}^{n-1} \alpha_i,$$

then $N_\alpha$ is known to be diffeomorphic to $\mathbb{C}P^{n-3}$, as shown in [7, Example 2.6]. Using Corollary 6.5, we get a precise symplectic description.
**Proposition 7.2.** Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \) \((n \geq 4)\) satisfying (7.2). Then \( N_\alpha \) is symplectomorphic to \((\mathbb{C}P^{n-3})_\ell\) for \( \ell = \alpha_1 + \cdots + \alpha_{n-1} - \alpha_n. \)

**Proof.** Since \( n \geq 4, \) the second equation in (7.2) implies that
\[
\alpha_n - \alpha_{n-1} > \alpha_2 + \cdots + \alpha_{n-2} - \alpha_1 > 0.
\]
Hence, the bending flow \( \Phi = \Phi_{n,n-1} \) is defined, with image
\[
I = [\alpha_n - \alpha_{n-1}, \alpha_1 + \cdots + \alpha_{n-2}],
\]
an interval of length \( \ell = \alpha_1 + \cdots + \alpha_{n-1} - \alpha_n. \) The fact that \( \alpha \in \mathbb{R}^n \) together with the second inequality of (7.3) imply that there are no critical points for \( \Phi \) in the interior of \( I. \) Hence, \( \Phi \) makes \( N_\alpha \) a simple Hamiltonian manifold with \( S^1 \)-size equal to \( \ell. \) The manifold \( \Phi^{-1}(\alpha_1 + \cdots + \alpha_{n-2}) \) is equal to a point. Proposition 7.2 then follows from Corollary 6.5 (exchanging the role of \( M_0 \) and \( M_1). \)

We now study the operation of adding a tiny edge to a polygon. Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \) be generic. If \( \varepsilon > 0 \) is small enough, then, for all integer \( j \in \{1, \ldots, n\}, \) the \( n \)-tuple
\[
\alpha(j, \delta) = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j + \delta, \alpha_{j+1}, \ldots, \alpha_n)
\]
belongs to \( \mathbb{R}^n \) and is generic when \( |\delta| \leq \varepsilon. \) We say that \( \varepsilon \) is \( \alpha \)-tiny. The manifolds \( N_{\alpha(j, \delta)} \) are then canonically diffeomorphic to \( N_\alpha, \) see [7, Lemma 1.2 and its proof].

We shall now describe the symplectic manifold \( N_{\alpha^*} \) where
\[
\alpha^* = (\varepsilon, \alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n+1}
\]
and \( \varepsilon \) is \( \alpha \)-tiny. For convenience, we will now index the coordinates by 0 to \( n. \) We check that the bending flow
\[
\Phi_{j,0} : N_{\alpha^*} \rightarrow I_j = [\alpha_j - \varepsilon, \alpha_j + \varepsilon]
\]
is well-defined and makes \( N_{\alpha^*} \) a simple Hamiltonian \( S^1 \)-manifold of \( S^1 \)-size equal to \( 2\varepsilon, \) with \( M_0 = N_{\alpha(j, -\varepsilon)} \) and \( M_1 = N_{\alpha(j, \varepsilon)}. \)

For \( i = 0, \ldots, n, \) consider the space \( E_i \) of configurations \( \rho \) as in (7.1) such that \( \rho_i = (0, 0, \alpha_i). \) This is the total space of a principal \( S^1 \)-bundle \( \xi \) over \( N_{\alpha^*}, \) or over \( N_{\alpha(i, -\varepsilon)} \) if \( 1 \leq i \leq n. \) We also denote by \( \xi_i \) its associated complex line bundle.
Proposition 7.3. Let $\alpha \in \mathbb{R}^n$ and let $\varepsilon > 0$ be tiny for $\alpha$. Then for each $1 \leq j \leq n$, the bending flow $\Phi_{j,0}$ makes the manifold $\mathcal{N}_{\alpha^*}$ $S^1$-equivariantly symplectomorphic to $C_{\text{id}}(\mathcal{N}_{\alpha(j,\varepsilon)}, \xi_j, 2\varepsilon)$.

For two descriptions of $\mathcal{N}_{\alpha^*}$ as a smooth manifold, see [7, Proposition 2.2].

Proof. Choose an $S^1$-invariant almost complex structure on $\mathcal{N}_{\alpha^*}$ compatible with the symplectic form and let $\nu$ be the normal bundle to $\mathcal{N}_{\alpha(j,\varepsilon)}$ in $\mathcal{N}_{\alpha^*}$. As $r_0 = r_1 = 1$, Corollary 6.4 implies that there is an $S^1$-equivariant symplectomorphism from $C_{\text{id}}(\mathcal{N}_{\alpha(j,\varepsilon)}, \nu, 2\varepsilon)$ to $\mathcal{N}_{\alpha^*}$. We have to identify $\nu$ with $\xi_j$.

The symplectic reduction of $\mathcal{N}_{\alpha^*}$ at $\lambda \in I_j$ is $\mathcal{N}_{\alpha(j,\lambda)}$. Identifying the latter with $\mathcal{N}_\alpha$ gives a symplectic form $\omega_\lambda \in \Omega^2(\mathcal{N}_\alpha)$ which, by the Duistermaat–Heckman theorem satisfies the equation
\[
\omega(\lambda) = [\omega_0^q] + \lambda e(\nu)
\]
in $H^2_{\text{dr}}(\mathcal{N}_\alpha)$. Hence,
\[
(7.5) \quad e(\nu) = \frac{d}{\lambda}[\omega(\lambda)].
\]
But, by [8, Remark 7.5.d],
\[
(7.6) \quad [\omega(0)] = \sum_{i=1}^n \alpha_i e(\xi_i).
\]
By (7.5) and (7.5), we deduce that $\nu$ is isomorphic to $\xi_j$. \hfill \Box

Proposition 7.4. Let $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1}$ satisfying
\[
(7.7) \quad \alpha_n + \alpha_0 < \sum_{i<n} \alpha_i \quad \text{and} \quad \alpha_n + \alpha_1 > \sum_{i=2}^{n-1} \alpha_i,
\]
let $\ell = \alpha_0 + \cdots + \alpha_{n-1} - \alpha_n$. Then $\mathcal{N}_\alpha$ is symplectomorphic to a symplectic cut of $(\mathbb{C}P^{m-2})_\ell$ so that the symplectic slice has size $\ell - 2\alpha_0$.

In particular, $\mathcal{N}_\alpha$ is diffeomorphic to $\mathbb{C}P^{m-2} \# \overline{\mathbb{C}P^{m-2}}$. For a generalization of this fact, see [7, Example 2.12].

Proof. We note that $\alpha = \beta^\alpha_0$ in the sense of (7.4), where $\beta = (\alpha_1, \ldots, \alpha_n)$ satisfies (7.2). We use Proposition 7.3 and its notations, with the bending flow $\Phi_{n,0}$. Hence, $\mathcal{N}_\alpha$ is symplectomorphic to $C_{\text{id}}(\mathcal{N}_{\alpha(n,-\alpha_0)}, \xi_n, 2\alpha_0)$. Using (7.5) and [8, Proposition 7.3], we deduce that $e(\xi_n) = -1$. 
Thus, $\Phi_{n,0}$ makes $\mathcal{N}_\alpha$ a simple Hamiltonian manifold $(\mathcal{N}_\alpha, M_0, M_1)$ with $M_0 = (\mathbb{CP}^{n-3})_\ell$ and $M_1 = (\mathbb{CP}^{n-3})_{\ell-2\alpha_0}$, using Proposition 7.2 to identify $M_0$. 

Acknowledgments

The first author (J.-C. H) had several fruitful conversations with Y. Karshon and S. Tolman at the conference on moment maps organized in the Bernoulli Center (EPFL) in August 2008. The second author (T. H) was supported in part by NSF Grant DMS-0835507. She would like to thank Y. Karshon, D. McDuff and S. Tolman for useful conversations. Both authors would like to thank the anonymous referees for helpful comments.

References

[1] M. Audin, *The topology of torus actions on symplectic manifolds*, Birkhäuser, 2nd edn., 2004.
[2] A. Cannas Da Silva, *Lectures on Symplectic Geometry* Springer, Lecture Notes 1764, 2001.
[3] T. Delzant, *Hamiltoniens périodiques et images convexes du moment*, Bull. Soc. Math. de France 116 (1988), 315–339.
[4] V. Guillemin and T. Holm, *GKM theory for torus actions with nonisolated fixed points*, Int. Math. Res. Not. 40 (2004), 2105–2124.
[5] E. Gonzalez, *Classifying semi-free Hamiltonian $S^1$-manifolds*, Int. Math. Res. Notices (2010) doi: 10.1093/imrn/rnq076. First published online April 25, 2010.
[6] D. Haibao and E. Rees, *Functions whose critical set consists of two connected manifolds*, Boletín de la Soc. Matem. Mexicana 37 (1992), 139–149.
[7] J.-C. Hausmann, *Geometric descriptions of polygon and chain spaces*, Topology and Robotics, Contemp. Math., 438, Amer. Math. Soc., Providence, RI (2007), 47–57.
[8] J.-C. Hausmann and A. Knutson, *The cohomology rings of polygon spaces*, Ann. Inst. Fourier 48 (1998), 281–321.
[9] M. Kapovich and J. Millson, *The symplectic geometry of polygons in Euclidean space*, J. Differential Geom. 44 (1996), 479–513.
[10] E. Lerman, *Symplectic cuts*, Math. Res. Lett. 2 (1995), 247–258.

[11] H. Li and S. Tolman, *Hamiltonian circle actions with minimal fixed sets*, arXiv:0905.4049.

[12] H. Li, M. Olbermann and D. Stanley, *One-connectivity and finiteness of Hamiltonian circle actions with minimal fixed sets*, arXiv:1010.2505.

[13] D. McDuff and D. Salamon, *Introduction to symplectic topology*. 2nd edn., Oxford University Press, 1998.

[14] S. Tolman, *On a symplectic generalization of Petrie’s conjecture*, Trans. AMS 362 (2010), 3963–3996.

[15] S. Tolman and J. Weitsman, *On the cohomology rings of Hamiltonian T-spaces*, Northern California Symplectic Geometry Seminar, Amer. Math. Soc. Transl. Ser. 2 196 (1999), 251–258.

Mathématiques-Université
BP. 64
CH-1211 Genève 4, Switzerland
E-mail address: Jean-Claude.Hausmann@unige.ch

Department of Mathematics
Cornell University
Ithaca, NY 14853-4201
USA
E-mail address: tsh@math.cornell.edu

Received July 8, 2011