Sample canonical correlation coefficients of high-dimensional random vectors with finite rank correlations

Zongming Ma *1 and Fan Yang †1

1Department of Statistics and Data Science, University of Pennsylvania

June 14, 2022

Abstract

Consider two random vectors \( \mathbf{x} = A \mathbf{z} + C_1^{1/2} \mathbf{y} \in \mathbb{R}^p \) and \( \tilde{\mathbf{y}} = B \mathbf{z} + C_2^{1/2} \mathbf{y} \in \mathbb{R}^q \), where \( \mathbf{x}, \mathbf{y}, \text{ and } \mathbf{z} \) are independent random vectors with i.i.d. entries of zero mean and unit variance, \( C_1 \) and \( C_2 \) are \( p \times p \) and \( q \times q \) deterministic population covariance matrices, and \( A \) and \( B \) are \( p \times r \) and \( q \times r \) deterministic factor loading matrices. With \( n \) independent observations of \( \mathbf{x} \) and \( \tilde{\mathbf{y}} \), we study the sample canonical correlations between them. Under the sharp fourth moment condition on the entries of \( \mathbf{x}, \mathbf{y}, \text{ and } \mathbf{z} \), we prove the BBP transition for the sample canonical correlation coefficients (CCCs). More precisely, if a population CCC is below a threshold, then the corresponding sample CCC converges to the right edge of the bulk eigenvalue spectrum of the sample canonical correlation matrix and satisfies the famous Tracy-Widom law; if a population CCC is above the threshold, then the corresponding sample CCC converges to an outlier that is detached from the bulk eigenvalue spectrum. We prove our results in full generality, in the sense that they also hold for near-degenerate population CCCs and population CCCs that are close to the threshold.

1 Introduction

Since the seminal work by Hotelling [29], the canonical correlation analysis (CCA) has been one of the most classical methods to study the correlations between two random vectors. Given two random vectors \( \mathbf{x} \in \mathbb{R}^p \) and \( \tilde{\mathbf{y}} \in \mathbb{R}^q \), CCA seeks two sequences of orthonormal vectors, such that the projections of \( \mathbf{x} \) and \( \tilde{\mathbf{y}} \) onto these vectors have maximized correlations, and the corresponding sequence of correlations are called canonical correlation coefficients (CCCs). More precisely, we first find a pair of unit vectors \( \mathbf{a}_1 \) and \( \mathbf{b}_1 \) that maximizes the correlation \( \rho(a, b) := \text{Corr}(\mathbf{a}^\top \tilde{\mathbf{x}}, \mathbf{b}^\top \mathbf{y}) \). Then, \( \rho_1 := \rho(\mathbf{a}_1, \mathbf{b}_1) \) is the first CCC and \( (\mathbf{a}_1^\top \tilde{\mathbf{x}}, \mathbf{b}_1^\top \mathbf{y}) \) is the first pair of canonical variables. Suppose we have obtained the first \( k \) CCCs, \( \rho_1, \ldots, \rho_k \), and the corresponding pairs of canonical variables. We then define inductively the \((k+1)\)-th CCC by seeking a pair of unit vectors \( (\mathbf{a}_{k+1}, \mathbf{b}_{k+1}) \) that maximizes \( \rho(a, b) \) subject to the constraint that \( (\mathbf{a}_k^\top \tilde{\mathbf{x}}, \mathbf{b}_k^\top \mathbf{y}) \) is uncorrelated with the first \( k \) pairs of canonical variables. Then, \( \rho_{k+1} := \rho(\mathbf{a}_{k+1}, \mathbf{b}_{k+1}) \) is the \((k+1)\)-th CCC.

There is a well-know representation of CCCs in terms of the eigenvalues of the population canonical correlation (PCC) matrix defined using the population covariance and cross-covariance matrices:

\[
\tilde{\Sigma}_{xx} := \text{Cov}(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}), \quad \tilde{\Sigma}_{yy} := \text{Cov}(\tilde{\mathbf{y}}, \tilde{\mathbf{y}}), \quad \tilde{\Sigma}_{xy} = \tilde{\Sigma}_{yx}^\top := \text{Cov}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}),
\]

*E-mail: zongming@wharton.upenn.edu
†E-mail: fyang75@wharton.upenn.edu
where for two random vectors \( u \) and \( v \), we define \( \text{Cov}(u, v) := \mathbb{E}[(u - \mathbb{E}u)(v - \mathbb{E}v)\top] \). It is known that \( \rho_i \) is the square root of the \( i \)-th largest eigenvalue, say \( t_i \), of the PCC matrix
\[
\tilde{\Sigma} := \tilde{\Sigma}_{xx}^{-1/2}\tilde{\Sigma}_{xy}\tilde{\Sigma}_{yy}^{-1/2}\tilde{\Sigma}_{yx}\tilde{\Sigma}_{xx}^{-1/2}.
\]
Suppose we observe \( n \) independent samples of \((\tilde{x}, \tilde{y})\). Then, we can study the population CCCs through their sample counterparts. More precisely, we form data matrices \( \tilde{X} \) and \( \tilde{Y} \) as
\[
\tilde{X} := n^{-1/2} \left( \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n \right), \quad \tilde{Y} := n^{-1/2} \left( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n \right),
\]
(1.1)
where \((\tilde{x}_i, \tilde{y}_i)\) are i.i.d. copies of \((\tilde{x}, \tilde{y})\) and \( n^{-1/2} \) is a convenient scaling, so that the sample covariance and cross-covariance matrices can be written concisely as
\[
\tilde{S}_{xx} := \frac{1}{n} \sum_{i=1}^{n} \tilde{x}_i\tilde{x}_i\top = \tilde{X}\tilde{X}\top, \quad \tilde{S}_{yy} := \frac{1}{n} \sum_{i=1}^{n} \tilde{y}_i\tilde{y}_i\top = \tilde{Y}\tilde{Y}\top, \quad \tilde{S}_{xy} = \tilde{S}_{yx} := \frac{1}{n} \sum_{i=1}^{n} \tilde{x}_i\tilde{y}_i\top = \tilde{X}\tilde{Y}\top.
\]
The squares of the sample CCCs, \( \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_{p\times q} \geq 0 \), are then defined as the eigenvalues of the sample canonical correlation (SCC) matrix
\[
\tilde{C}_{\tilde{X}\tilde{Y}} := \tilde{S}_{xx}^{-1/2}\tilde{S}_{xy}\tilde{S}_{yy}^{-1/2}\tilde{S}_{yx}\tilde{S}_{xx}^{-1/2}.
\]
If \( n \to \infty \) while \( p, q \) and \( r \) are fixed, the SCC matrix converges to the PCC matrix almost surely by the law of large numbers, and hence sample CCCs can be used as consistent estimators of population CCCs. However, many modern applications, such as statistical learning, wireless communications, medical imaging, financial economics and population genetics, are seeing a rapidly increasing demand in analyzing high-dimensional data, where \( p \) and \( q \) are comparable to \( n \) when \( n \) is large. In the high-dimensional setting, the behavior of the SCC matrix can deviate greatly from the PCC matrix due to the so-called “curse of dimensionality”.

There have been several works on the theoretical analysis of high-dimensional CCA. We mention some of them that are most related to this paper. First, we consider the null case where \( \tilde{x} \) and \( \tilde{y} \) are independent random vectors. When \( \tilde{x} \) and \( \til\) are independent Gaussian vectors, the eigenvalues of the SCC matrix have the same joint distribution as those of a double Wishart matrix [31]. In particular, the joint distribution of the eigenvalues of double Wishart matrices has been studied in the context of the Jacobi ensemble and F-type matrices [28, 31], where the largest few eigenvalues of the SCC matrix are shown to satisfy the Tracy-Widom law asymptotically. For generally distributed random vectors \( \tilde{x} \) and \( \tilde{y} \), the Tracy-Widom fluctuation of the largest eigenvalues of the SCC matrix is proved in [27] under the assumption that the entries of \( \tilde{x} \) and \( \tilde{y} \) have finite moments up to any order. The moment assumption is later relaxed to the finite fourth moment assumption in [47]. In the Gaussian case, it is shown in [12] that, almost surely, the empirical spectral distribution (ESD) of the SCC matrix converges weakly to a deterministic probability distribution (cf. [2,12]). In the general non-Gaussian case, both the convergence and the linear spectral statistics of the ESD of the SCC matrix have been proved [49, 50].

Next, we consider the case where \( \tilde{x} \) and \( \tilde{y} \) have finite rank correlations. If \( \tilde{x} \) and \( \tilde{y} \) are random Gaussian vectors, then the asymptotic distributions of sample CCCs have been derived when one of \( p \) and \( q \) is fixed as \( n \to \infty \) [24]. If \( p \) and \( q \) are both proportional to \( n \), the asymptotic distributions of sample CCCs have been established under the Gaussian assumption in [47]. Under certain sparsity assumptions, the theory of high-dimensional sparse CCA and it applications have been discussed in [25, 26]. In [30], the authors derived asymptotic null and non-null distributions of several test statistics for tests of redundancy in high-dimensional CCA. In [32], the authors studied the asymptotic behaviors of the likelihood ratio processes of CCA under the null hypothesis of no spikes and the alternative hypothesis of a single spike.
In this paper, we consider the following signal-plus-noise model for \( \tilde{x} \in \mathbb{R}^p \) and \( \tilde{y} \in \mathbb{R}^q \):

\[
\tilde{x} = Az + C_1^{1/2}x, \quad \tilde{y} = Bz + C_2^{1/2}y.
\]

Here, \( z \in \mathbb{R}^r \) is a rank-\( r \) signal vector with i.i.d. entries of mean zero and variance one and independent of \( x \) and \( y \), and \( A \) and \( B \) are \( p \times r \) and \( q \times r \) deterministic factor loading matrices, respectively. \( x \in \mathbb{R}^p \) and \( y \in \mathbb{R}^q \) are two independent noise vectors with i.i.d. entries of mean zero and variance one, and \( C_1 \) and \( C_2 \) are \( p \times p \) and \( q \times q \) deterministic population covariance matrices. Then, we can write the data matrices in \([1.1]\) as

\[
\tilde{X} := AZ + C_1^{1/2}X, \quad \tilde{Y} := BZ + C_2^{1/2}Y,
\]

where \( X, Y \) and \( Z \) are respectively \( p \times n, q \times n \) and \( r \times n \) matrices with i.i.d. entries of mean zero and variance \( n^{-1} \) and they are independent of each other. We consider the high-dimensional setting with a low-rank signal, that is, \( p/n \to c_1 \) and \( q/n \to c_2 \) as \( n \to \infty \) for some constants \( c_1 \in (0, 1) \) and \( c_2 \in (0, 1 - c_1) \), and \( r \) is a fixed integer that does not depend on \( n \).

For the model \([1.2]\), the PCC matrix is given by

\[
\tilde{\Sigma} = (C_1 + AA^T)^{-1/2}AB^T(C_2 + BB^T)^{-1}BA^T(C_1 + AA^T)^{-1/2},
\]

which is of rank at most \( r \). We order the nontrivial eigenvalues of \( \tilde{\Sigma} \) as \( t_1 \geq t_2 \geq \cdots \geq t_r \geq 0 \). Under the Gaussian assumption, that is, \( X, Y \) and \( Z \) are independent random matrices with i.i.d. Gaussian entries, Bao et al. \([7]\) proved that for any \( 1 \leq i \leq r \), \( \tilde{\lambda}_i \) exhibits very different behaviors depending on whether \( t_i \) is below or above the threshold \( t_c \), where

\[
t_c := \sqrt{\frac{c_1 c_2}{(1 - c_1)(1 - c_2)}}.
\]

More precisely, if \( t_i < t_c \), then the corresponding sample CCC \( \tilde{\lambda}_i \) sticks to the right edge \( \lambda_+ \) of the bulk eigenvalue spectrum (cf. \([2.13]\)) of the SCC matrix, and \( n^{2/3}(\tilde{\lambda}_i - \lambda_+) \) converges weakly to the type-1 Tracy-Widom distribution. On the other hand, if \( t_i > t_c \), then it gives rise to an outlier \( \tilde{\lambda}_i \) that lies around a fixed location \( \theta_i \in (\lambda_+, 1) \) determined by \( t_i, c_1 \) and \( c_2 \). Furthermore, \( n^{1/2}(\tilde{\lambda}_i - \theta_i) \) converges weakly to a centered Gaussian. Such an abrupt change of the behavior of \( \tilde{\lambda}_i \) when \( t_i \) crosses the threshold \( t_c \) is generally called a BBP transition, which dates back to the seminal work of Baik, Ben Arous and Pêché \([5]\) on spiked sample covariance matrices. The BBP transition has been observed in many random matrix ensembles with finite rank perturbations. Without attempting to be comprehensive, we mention the references \([13, 14, 21, 33, 34, 35]\) on deformed Wigner matrices, \([3, 5, 6, 12, 22, 30, 37]\) on spiked sample covariance matrices, \([17, 15, 48]\) on spiked separable covariance matrices, and \([8, 9, 10, 13]\) on several other types of deformed random matrix ensembles. In our setting, the SCC matrix \( C_{\tilde{X} \tilde{Y}} \) can be regarded as a finite rank perturbation of the SCC matrix in the null case with \( r = 0 \).

A natural question is whether the results in \([7]\) hold universally, that is, whether \( \tilde{\lambda}_i \) satisfies the same properties if we only assume certain moment conditions on the entries of \( X, Y \) and \( Z \). In fact, the proof in \([7]\) depends crucially on the rotational invariance of multivariate Gaussian distributions under orthogonal transforms, and it is hard (if possible) to be extended to the data matrices with generally distributed entries. In this paper, we answer the above question definitely, and show the universality of the results in \([7]\). Moreover, we highlight the following improvements over the previous results.

- Theorem \([2.14]\) shows that the following results hold assuming only a finite fourth moment condition (actually we require a slightly weaker condition \( [2.31] \)): for \( 1 \leq i \leq r \), \( n^{2/3}(\tilde{\lambda}_i - \lambda_+) \) converges weakly to the Tracy-Widom law if \( t_i < t_c \), while \( \tilde{\lambda}_i \to \theta_i \) in probability if \( t_i > t_c \).
We obtain quantitative versions of all the results under general moment assumptions: Theorem 2.4 provides almost sharp convergence rates for the sample CCCs; Theorem 2.11 provides an almost sharp eigenvalue sticking estimate, which shows that the eigenvalues of the SCC matrix stick to those of the null SCC matrix with \( r = 0 \).

Our results hold even when some \( t_i \)-s are close to the threshold \( t_c \) and when there are groups of near-degenerate \( t_i \)-s—both of these two cases are ruled out in the setting of \([7]\).

Instead of using the rotational invariance of multivariate Gaussian distributions, the proofs in this paper are based on a linearization method developed in \([47]\), which reduces the problem to the study of a \((p + q + 2n) \times (p + q + 2n)\) random matrix \( H \) that is linear in \( X \) and \( Y \) (cf. \([43]\)). Moreover, an optimal local law has been proved for the resolvent \( G := H^{-1} \) in \([47]\), which is the basis of all the proofs in this paper. Our approach is relatively more flexible and allows us to obtain precise convergence rates for the eigenvalues of the SCC matrix \( C_{\hat{X}_{\hat{Y}}} \).

Before concluding the introduction, we fix some notations that will be used frequently in the paper. For two quantities \( a_n \) and \( b_n \) depending on \( n \), we use \( a_n = O(b_n) \) to mean that \( |a_n| \leq C|b_n| \) for a constant \( C > 0 \), and use \( a_n = o(b_n) \) to mean that \( |a_n| \leq c_n|b_n| \) for a positive sequence of numbers \( c_n \downarrow 0 \) as \( n \to \infty \). We will use the notations \( a_n \lesssim b_n \) if \( a_n = O(b_n) \) and \( a_n \sim b_n \) if \( a_n = O(b_n) \) and \( b_n = O(a_n) \). For a matrix \( A \), we use \( \|A\| \) to denote its operator norm. For a vector \( v \), we use \( \|v\| \) to denote its Euclidean norm. In this paper, we will abbreviate an identity matrix as \( I \) or \( 1 \).

2 The model and main results

2.1 The model

We consider two independent families of data matrices \( X = (x_{ij}) \) and \( Y = (y_{ij}) \), which are of dimensions \( p \times n \) and \( q \times n \), respectively. We assume that the entries \( x_{ij} \), \( 1 \leq i \leq p \), \( 1 \leq j \leq n \), and \( y_{ij} \), \( 1 \leq i \leq q \), \( 1 \leq j \leq n \), are real independent random variables satisfying

\[
\mathbb{E}x_{ij} = \mathbb{E}y_{ij} = 0, \quad \mathbb{E}|x_{ij}|^2 = \mathbb{E}|y_{ij}|^2 = n^{-1}. \tag{2.1}
\]

To be more general, we do not assume that these random variables are identically distributed. We define the following data model with finite rank correlation:

\[
\hat{X} := C_1^{1/2}X + \tilde{A}Z, \quad \hat{Y} := C_2^{1/2}Y + \tilde{B}Z,
\]

where \( C_1 \) and \( C_2 \) are \( p \times p \) and \( q \times q \) deterministic positive definite symmetric covariance matrices, \( \tilde{A} \) and \( \tilde{B} \) are \( p \times r \) and \( q \times r \) deterministic matrices, and \( Z = (z_{ij}) \) is an \( r \times n \) random matrix which gives the nontrivial correlation between \( \hat{X} \) and \( \hat{Y} \). We assume that \( Z \) is independent of \( X \) and \( Y \), and the entries \( z_{ij}, 1 \leq i \leq r, 1 \leq j \leq n \), are independent random variables satisfying

\[
\mathbb{E}z_{ij} = 0, \quad \mathbb{E}|z_{ij}|^2 = n^{-1}. \tag{2.2}
\]

In this paper, we study the eigenvalues of the sample canonical correlation (SCC) matrix

\[
C_{\hat{X}_{\hat{Y}}} = (\hat{X}\hat{X}^\top)^{-1/2}\hat{X}\hat{Y}^\top(\hat{Y}\hat{Y}^\top)^{-1}\hat{Y}\hat{X}^\top(\hat{X}\hat{X}^\top)^{-1/2}.
\]

In particular, we are interested in the relations between the eigenvalues of \( C_{\hat{X}_{\hat{Y}}} \) and those of the population canonical correlation (PCC) matrix:

\[
\Sigma := \Sigma_{xx}^{-1/2}\Sigma_{xy}^{-1}\Sigma_{yy}^{-1}\Sigma_{yx}^{-1/2}, \quad \Sigma_{xx} := C_1 + \tilde{A}\tilde{A}^\top, \quad \Sigma_{yy} := C_2 + \tilde{B}\tilde{B}^\top, \quad \Sigma_{xy} = \Sigma_{yx} := \tilde{A}\tilde{B}^\top.
\]
It is well-known that the canonical correlation coefficients are the square roots of the eigenvalues of the PCC matrix. Note that $C_{XY}$ is similar to

$$
C_{XY}' := \hat{X}^\top (\hat{Y}^\top)^{-1} \hat{Y} \hat{X}^\top (\hat{X}^\top)^{-1}.
$$

Under the non-singular transformation $\hat{X} \to \tilde{X} := C_1^{-1/2} \hat{X}$ and $\hat{Y} \to \tilde{Y} := C_2^{-1/2} \hat{Y}$, we see that

$$
C_{\tilde{X}\tilde{Y}}' = C_1^{1/2} C_{XY}' C_1^{-1/2},
$$

which shows that $C_{\tilde{X}\tilde{Y}}$ and $C_{XY}$ have the same eigenvalues. A similar argument also shows that the eigenvalues of the PCC matrix are unchanged under the same non-singular transformation. Hence, without loss of generality, we only need to consider a simpler model

$$
\tilde{X} := X + AZ, \quad \tilde{Y} := Y + BZ, \quad \text{where} \quad A := C_1^{-1/2} \hat{A}, \quad B := C_2^{-1/2} \hat{B}.
$$

We assume that $A$ and $B$ have the following singular value decompositions:

$$
A = \sum_{i=1}^{r} a_i u_i^x (v_i^x)^\top, \quad B = \sum_{i=1}^{r} b_i u_i^y (v_i^y)^\top,
$$

where $\{a_i\}$ and $\{b_i\}$ are the singular values, $\{u_i^x\}$ and $\{v_i^x\}$ are the left singular vectors, and $\{v_i^y\}$ are the right singular vectors. We assume that for some constant $C > 0$,

$$
0 \leq a_r \leq \ldots \leq a_2 \leq a_1 \leq C, \quad 0 \leq b_r \leq \ldots \leq b_2 \leq b_1 \leq C.
$$

In this paper, we consider the high-dimensional setting, that is,

$$
c_1(n) := p/n \to \hat{c}_1 \in (0, 1), \quad c_2(n) := q/n \to \hat{c}_2 \in (0, 1 - \hat{c}_1).
$$

For simplicity of notations, we will always abbreviate $c_1(n) \equiv c_1$ and $c_2(n) \equiv c_2$ for the rest of the paper. Without loss of generality, we assume that $c_1 \geq c_2$.

We now summarize the main assumptions for future reference. For our purpose, we relax the assumptions (2.1) and (2.2) a little bit. The reader can refer to the explanation above Corollary 2.13 for the reason of this extension.

**Assumption 2.1.** Fix a small constant $\tau > 0$.

(i) $X = (x_{ij})$ and $Y = (y_{ij})$ are two real independent $p \times n$ and $q \times n$ random matrices. Their entries are independent random variables that satisfy the following moment conditions:

$$
\max_{i,j} |E x_{ij}| \leq n^{-2-\tau}, \quad \max_{i,j} |E y_{ij}| \leq n^{-2-\tau},
$$

$$
\max_{i,j} |E x_{ij}|^2 - n^{-1} \leq n^{-2-\tau}, \quad \max_{i,j} |E y_{ij}|^2 - n^{-1} \leq n^{-2-\tau}.
$$

(ii) $Z = (z_{ij})$ is a real $r \times n$ random matrix that is independent of $X$ and $Y$, and its entries are independent random variables that satisfy the following moment conditions:

$$
\max_{i,j} |E z_{ij}| \leq n^{-1-\tau}, \quad \max_{i,j} |E z_{ij}|^2 - n^{-1} \leq n^{-1-\tau}.
$$

(iii) We assume that

$$
r \leq \tau^{-1}, \quad \tau \leq c_2 \leq c_1, \quad c_1 + c_2 \leq 1 - \tau.
$$
(iv) We consider the data model in (2.3), where $A$ and $B$ satisfy (2.4) and (2.5).

In this paper, we study the SCC matrix

$$C_{XY} := (X^T X)^{-1/2} X^T (Y^T Y)^{-1} Y^T (X^T X)^{-1/2},$$

the null SCC matrix $C_{XY} := S_{xx}^{-1/2} S_{xy} S_{yx}^{-1} S_{yy} S_{yx}^{-1/2}$, where

$$S_{xx} := X X^T, \quad S_{yy} := Y Y^T, \quad S_{xy} := S_{yx} := Y^T X,$$

and the PCC matrix $\Sigma_{XY} := \Sigma_{xx}^{-1/2} \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx} \Sigma_{xx}^{-1/2}$, where

$$\Sigma_{xx} = I_p + A A^T, \quad \Sigma_{yy} = I_q + B B^T, \quad \Sigma_{xy} = \Sigma_{yx} = \Sigma_{xx}^{-1/2} \Sigma_{xy} \Sigma_{xx}^{-1/2}.$$

Moreover, we will also consider the following matrices:

$$C_{YX} := (Y^T Y)^{-1/2} Y^T (X^T X)^{-1} X^T (Y^T Y)^{-1/2},$$

$$C_{YX} := S_{yy}^{-1/2} S_{yx} S_{xx}^{-1} S_{xy} S_{yy}^{-1/2}, \quad \Sigma_{YX} := \Sigma_{xx}^{-1/2} \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx} \Sigma_{xx}^{-1/2}.$$

Finally, we define another null SCC matrix $C_{YX}^b$ as

$$C_{YX}^b := (S_{yy}^b)^{-1/2} S_{yx}^b S_{xx}^{-1} S_{xy}^b (S_{yy}^b)^{-1/2},$$

where $S_{yy}^b := Y Y^T$ and $S_{yx}^b := (S_{yx})^T := X Y^T$. The matrix $C_{YX}^b$ can be defined in the obvious way.

### 2.2 Preliminaries

We denote the eigenvalues of $C_{YX}$ by $\lambda_1 \geq \cdots \geq \lambda_q \geq 0$, while $C_{XY}$ shares the same eigenvalues with $C_{YX}$, except that it has $p - q$ more trivial zero eigenvalues $\lambda_{q+1} = \cdots = \lambda_p = 0$. We denote the ESD of $C_{YX}$ by

$$F_n(x) := \frac{1}{q} \sum_{i=1}^q \mathbf{1}_{\lambda_i \leq x}.$$  

It is known that, almost surely, $F_n$ converges weakly to a deterministic probability distribution $F(x)$ with density

$$f(x) = \frac{1}{2\pi c_2} \sqrt{\frac{(\lambda_+ - x)(x - \lambda_-)}{x(1 - x)}} \mathbf{1}_{\lambda_- \leq x \leq \lambda_+},$$

where

$$\lambda_{\pm} := \left( \frac{\sqrt{c_1(1 - c_2)} \pm \sqrt{c_2(1 - c_1)}}{x(1 - x)} \right)^2.$$  

For the model (2.3), we denote the eigenvalues of $C_{YX}$ by $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_q \geq 0$, while $C_{XY}$ has $p - q$ more trivial zero eigenvalues $\tilde{\lambda}_{q+1} = \cdots = \tilde{\lambda}_p = 0$. We denote the eigenvalues of $\Sigma_{XY}$ by

$$t_1 \geq \cdots \geq t_r \geq t_{r+1} = \cdots = t_q = 0.$$  

Suppose the entries of $X$, $Y$ and $Z$ are i.i.d. Gaussian. Then, it was proved in [4] that, if $t_i > t_c$ (recall (1.3)), $\tilde{\lambda}_i - \theta_i \to 0$ almost surely, where

$$\theta_i := t_i (1 - c_1 c_2) c_2 (1 - c_2 + c_2 t_i^{-1}).$$
if \( t_i \leq t_e, \) \( \tilde{X}_i - \lambda_+ \to 0 \) almost surely and \( n^{2/3}(\tilde{X}_i - \lambda_+) \) converges weakly to the Tracy-Widom law. Note that for \( t_i > t_e, \) we have \( \theta_i > \lambda_+, \) so \( \tilde{X}_i \) is an outlier that is detached from the support \([\lambda_-, \lambda_+]\) of the limiting distribution \( F(x). \) In Section 2.3, we will state our main results showing that the above BBP transition also holds without the Gaussian assumption. To state and explain the main results, we need to introduce more notations, assumptions and a preliminary result regarding the asymptotic behaviors of the eigenvalues of \( \mathcal{C}_{Y,X}. \)

In this paper, we will frequently use the following notion of stochastic domination. It was first introduced in [18] and subsequently used in many works on random matrix theory. It simplifies the presentation of the results and their proofs by systematizing statements of the form “\( \xi \) is bounded by \( \zeta \) with high probability up to a small power of \( n \).”

**Definition 2.2** (Stochastic domination and high probability event). (i) Let

\[
\xi = \left( \xi^{(n)}(u) : n \in \mathbb{N}, u \in U^{(n)} \right), \quad \zeta = \left( \zeta^{(n)}(u) : n \in \mathbb{N}, u \in U^{(n)} \right)
\]

be two families of nonnegative random variables, where \( U^{(n)} \) is an \( n \)-dependent parameter set. We say \( \xi \) is stochastically dominated by \( \zeta \), uniformly in \( u \), if for any small constant \( \varepsilon > 0 \) and large constant \( D > 0 \),

\[
\sup_{u \in U^{(n)}} \mathbb{P} \left[ \xi^{(n)}(u) > n^\varepsilon \zeta^{(n)}(u) \right] \leq n^{-D}
\]

for large enough \( n \geq n_0(\varepsilon, D) \), and we shall use the notation \( \xi < \zeta \). Throughout this paper, the stochastic domination will always be uniform in all parameters that are not explicitly fixed. If \( \xi \) is complex and we have \( |\xi| \leq \zeta \), then we will also write \( \xi < \zeta \) or \( \xi = O_<(\zeta) \).

(ii) We extend the definition of \( O_<(\cdot) \) to matrices in the sense of operator norm as follows. Let \( A \) be a family of matrices and \( \zeta \) be a family of nonnegative random variables. Then, \( A = O_<(\zeta) \) means that \( \|A\| < \zeta \).

(iii) We say an event \( \Xi \) holds with high probability if for any constant \( D > 0 \), \( \mathbb{P}(\Xi) \geq 1 - n^{-D} \) for large enough \( n \). Moreover, we say \( \Xi \) holds with high probability on an event \( \Omega \), if for any constant \( D > 0 \), \( \mathbb{P}(\Omega \cap \Xi) \leq n^{-D} \) for large enough \( n \).

The following lemma collects basic properties of stochastic domination \(<\), which will be used tacitly throughout this paper.

**Lemma 2.3** (Lemma 3.2 in [11]). Let \( \xi \) and \( \zeta \) be two families of nonnegative random variables, and let \( C > 0 \) be an arbitrary constant.

(i) Suppose that \( \xi(u,v) < \zeta(u,v) \) uniformly in \( u \in U \) and \( v \in V \). If \( |V| \leq n^C \), then \( \sum_{v \in V} \xi(u,v) < \sum_{v \in V} \zeta(u,v) \) uniformly in \( u \).

(ii) If \( \xi_1(u) < \zeta_1(u) \) and \( \xi_2(u) < \zeta_2(u) \) uniformly in \( u \in U \), then \( \xi_1(u) \xi_2(u) < \zeta_1(u) \zeta_2(u) \) uniformly in \( u \).

(iii) Suppose that \( \Psi(u) \geq n^{-C} \) is deterministic and \( \xi(u) \) satisfies \( \mathbb{E}|\xi(u)|^2 \leq n^C \) for all \( u \). If \( \xi(u) < \Psi(u) \) uniformly in \( u \), then we have \( \mathbb{E}\xi(u) < \Psi(u) \) uniformly in \( u \).

Now, we introduce a bounded support condition for random matrices considered in this paper.

**Definition 2.4** (Bounded support condition). We say a random matrix \( X \) satisfies the bounded support condition with \( \phi_n \) if

\[
\max_{i,j} |x_{ij}| < \phi_n. \tag{2.16}
\]

Whenever \( \phi_n \) holds, we say that \( X \) has support \( \phi_n \). In this paper, \( \phi_n \) is always a deterministic parameter satisfying that \( n^{-1/2} \leq \phi_n \leq n^{-c_\phi} \) for some small constant \( c_\phi > 0 \).
In this paper, we also consider the case where \(|t_i - t_c| = o(1)|i.e., the spike \(t_i\) is very close to the BBP transition threshold. Suppose that \(X\) and \(Y\) have bounded support \(\phi_n\) and \(Z\) has bounded support \(\psi_n\). Then, we make the following assumption.

**Assumption 2.5.** We assume that for some integer \(0 \leq r_+ \leq r\), the following statement holds:
\[
t_i - t_c \geq n^{-1/3} + \psi_n + \phi_n \quad \text{if and only if} \quad 1 \leq i \leq r_+.
\] (2.17)

The lower bound is chosen for definiteness, and it can be replaced with any \(n\)-dependent parameter that is of the same order.

**Remark 2.6.** There is some freedom in choosing the two parameters \(\phi_n\) and \(\psi_n\). In principle, one can choose any \(\phi_n, \psi_n \ll 1\) so that the following conditions hold:
\[
\max_{i,j} |x_{ij}| < \phi_n, \quad \max_{i,j} |y_{ij}| < \phi_n, \quad \max_{i,j} |z_{ij}| < \psi_n.
\]

However, since smaller \(\phi_n\) and \(\psi_n\) lead to weaker assumptions and stronger results, it is better to choose them as small as possible. In particular, under certain moment conditions on the entries of \(\phi\), \(Y\), and \(Z\) as in (2.26), the best choice of \(\phi_n\) and \(\psi_n\) is given in (2.27), which comes from a standard truncation argument.

We define the quantiles of the density (2.14), which give the classical locations of \(\lambda_i\)-s.

**Definition 2.7.** The classical location \(\gamma_j\) of the \(j\)-th eigenvalue is defined as
\[
\gamma_j := \sup_x \left\{ \int_x^{+\infty} f(t) dt > \frac{j-1}{q} \right\},
\] (2.18)
where \(f\) is defined in (2.14). Note that we have \(\gamma_1 = \lambda_+\) and \(\lambda_+ - \gamma_j \sim (j/n)^{2/3}\) for \(j > 1\).

The following eigenvalue rigidity and edge universality results for \(C_{YX}\) have been proved in [17].

**Lemma 2.8** (Theorem 2.5 of [17]). Suppose Assumption 2.7 (i) and (iii) hold. Suppose \(X\) and \(Y\) have bounded support \(\phi_n\) with \(n^{-1/2} \leq \phi_n \leq n^{-c_0}\) for some constant \(c_0 > 0\). Assume that
\[
\max_{i,j} E|x_{ij}|^3 \leq n^{-3/2}, \quad \max_{i,j} E|y_{ij}|^3 \leq n^{-3/2}, \quad \max_{i,j} E|x_{ij}|^4 \leq n^{-2}, \quad \max_{i,j} E|y_{ij}|^4 < n^{-2}.
\] (2.19)

Then, the eigenvalues of the null SCC matrix \(C_{YX}\) satisfy the following rigidity estimate: for any constant \(\delta > 0\) and all \(1 \leq i \leq (1 - \delta)q\),
\[
|\lambda_i - \gamma_i| < \delta^{-1/3} n^{-2/3}.
\] (2.20)

Moreover, we have that for any fixed \(k \in \mathbb{N}\),
\[
\lim_{n \to \infty} P \left( n^{2/3} \frac{\lambda_i - \lambda_+}{c_{TW}} \leq s_i \right)_{1 \leq i \leq k} = \lim_{n \to \infty} P^{GOE} \left( n^{2/3} (\lambda_i - 2) \leq s_i \right)_{1 \leq i \leq k}
\] (2.21)
for all \((s_1, s_2, \ldots, s_k) \in \mathbb{R}^k\), where
\[
c_{TW} := \left[ \frac{\lambda_+^2 (1 - \lambda_+)^2}{\sqrt{c_1 c_2 (1 - c_1) (1 - c_2)}} \right]^{1/3},
\]
and \(P^{GOE}\) stands for the law of GOE (Gaussian orthogonal ensemble), which is an \(n \times n\) symmetric matrix with independent (up to symmetry) Gaussian entries of mean zero and variance \(n^{-1}\).

Taking \(k = 1\) in (2.21), we obtain that \(n^{2/3} (\lambda_1 - \lambda_+)/c_{TW} \Rightarrow F_1\), where \(F_1\) is the famous type-1 Tracy-Widom distribution derived in [40] [41]. Moreover, the joint distribution of the largest \(k\) eigenvalues of GOE can be written in terms of the Airy kernel for any \(k\) [23]. Hence, (2.21) gives a complete description of the finite-dimensional correlation functions of the edge eigenvalues of \(C_{YX}\).
2.3 The main results

With the above preparations, we are now ready to state our main results on the eigenvalues of the SCC matrix $C_{XY}$. We define the following quantities, which characterize the distances from $t_r$-s to the BBP transition threshold:

$$
\Delta_i := |t_i - t_c|, \quad \alpha_+ := \min_{1 \leq i \leq r} |t_i - t_c|.
$$

(2.22)

We first describe the convergence of the outlier eigenvalues and the extreme non-outlier eigenvalues.

**Theorem 2.9.** Suppose Assumptions [2.1] and Assumption [2.5] hold. Suppose $X$ and $Y$ have bounded support $\phi_n$ and $Z$ has bounded support $\psi_n$ with $n^{-1/2} \leq \phi_n \leq n^{-c_\phi}$ and $n^{-1/2} \leq \psi_n \leq n^{-c_\psi}$ for some constants $c_\phi, c_\psi > 0$. Assume that (2.19) holds. Then, for any $1 \leq i \leq r_+$, we have that

$$
|\tilde{\lambda}_i - \theta_i| < (\psi_n + \phi_n)\Delta_i + n^{-1/2}\Delta_i^{1/2}.
$$

(2.23)

Let $\varpi \in \mathbb{N}$ be a fixed (large) integer. For any $r_+ + 1 \leq i \leq \varpi$ and small constant $\varepsilon > 0$, we have that

$$
-n^{-2/3+\varepsilon} < \tilde{\lambda}_i - \lambda_+ \leq n^2(\psi_n^2 + \phi_n^2 + n^{-2/3}) \quad \text{with high probability.}
$$

(2.24)

**Remark 2.10.** This theorem gives precise large deviation bounds on the locations of the outliers and largest few extreme non-outlier eigenvalues. Consider a small support case with $\phi_n + \psi_n \leq n^{-1/3}$ (which holds with probability $1-o(1)$ if we assume the existence of 12-th moment, see (2.27) below). Then, (2.23) and (2.24) show that the fluctuation of the $i$-th eigenvalue changes from the order $\Delta_i + n^{-1/2}\Delta_i^{1/2}$ to $n^{-2/3}$ when $\Delta_i$ crosses the scale $n^{-1/3}$. This implies the occurrence of the BBP transition.

For the non-outlier eigenvalues of $C_{XY}$, they stick to the corresponding eigenvalues of $C_{XY}$ as given by the following theorem.

**Theorem 2.11.** Suppose the assumptions of Theorem 2.9 hold. Assume that $\alpha_+ \geq n^{\varepsilon_0}(\psi_n + \phi_n)$ for some constant $\varepsilon_0 > 0$. Then, we have the eigenvalue sticking estimates:

$$
|\tilde{\lambda}_{i+r_+} - \lambda_i| < n^{-1}\alpha_+^{-1}
$$

(2.25)

for all $i \leq (1-\delta)q$, where $\delta > 0$ is any small constant.

**Remark 2.12.** This theorem establishes a large deviation bound on the non-outlier eigenvalues of $C_{XY}$ with respect to the eigenvalues of $C_{XY}$. Combining it with Lemma 2.8 above, we immediately obtain the asymptotic behaviors of the non-outlier eigenvalues of $C_{XY}$. In particular, when $\alpha_+ \gg n^{-1/3}$, the right-hand side of (2.25) is much smaller than $n^{-2/3}$. Together with (2.24) for $\lambda_i$, (2.25) implies that the largest non-outlier eigenvalue of $C_{XY}$ also converges to the Tracy-Widom law as long as $t_r$-s are away from the transition threshold $t_c$ at least by $\alpha_+ \gg n^{-1/3}$.

In many settings, people usually assume certain moment conditions on the entries of $X$, $Y$ and $Z$ instead of the bounded support condition. By using Markov’s inequality and a standard truncation argument, we can derive a bounded support condition from the moment assumptions. Then, with Theorems 2.9 and 2.11 we can easily obtain the following corollary. Since we did not assume the entries of $X$, $Y$ and $Z$ are identically distributed, the means and variances of the truncated entries may be different. This is why we have assumed the slightly more general mean and variance conditions (2.6)–(2.8).

**Corollary 2.13.** Assume that $X = (x_{ij})$, $Y = (y_{ij})$ and $Z = (z_{ij})$ are respectively $p \times n$, $q \times n$ and $r \times n$ matrices, whose entries are real independent random variables satisfying (2.1), (2.2) and

$$
\max_{i,j} \mathbb{E}[\sqrt{n}|x_{ij}|^b] \leq C, \quad \max_{i,j} \mathbb{E}[\sqrt{n}|y_{ij}|^b] \leq C, \quad \max_{i,j} \mathbb{E}[\sqrt{n}|z_{ij}|^b] \leq C,
$$

(2.26)
for some constants $a > 4$, $b > 2$ and $C > 1$. If Assumption 2.11 (iii)–(iv) and Assumption 2.3 hold with
\[ \phi_n = n^{-1/2+2/a}, \quad \psi_n = n^{-1/2+1/b}, \] (2.27)
then for any $1 \leq i \leq r_+$ and small constant $\varepsilon > 0$,
\[ \lim_{n \to \infty} \mathbb{P} \left( |\hat{\lambda}_i - \theta_i| \leq n^{\varepsilon} \left( (\psi_n + \phi_n) \Delta_i + n^{-1/2} \Delta_i^{1/2} \right) \right) = 1. \] (2.28)
Moreover, assume that the eigenvalues of $\Sigma_{XY}$ satisfy that
\[ \alpha_+ \geq n^0 (\psi_n + \phi_n) + n^{-1/3+\varepsilon_0} \] (2.29)
for a constant $\varepsilon_0 > 0$. Then, for any fixed $k \in \mathbb{N}$ and all $(s_1, s_2, \ldots, s_k) \in \mathbb{R}^k$,
\[ \lim_{n \to \infty} \mathbb{P} \left( \frac{n^{2/3} (\hat{\lambda}_i + r_+ - \lambda_i)}{ct_{FW}^2} \leq s_i \right)_{1 \leq i \leq k} = \lim_{n \to \infty} \mathbb{P}^{GOE} \left( \frac{n^{2/3} (\lambda_i - 2)}{\Delta_i} \leq s_i \right)_{1 \leq i \leq k}. \] (2.30)

If the entries of $X$, $Y$ and $Z$ are identically distributed, then we can obtain the following result under a weaker tail condition than (2.20). We believe this tail condition is sharp.

**Theorem 2.14.** Suppose Assumption 2.11 (iii)–(iv) and Assumption 2.3 hold. Assume that $x_{ij} = n^{-1/2} \hat{x}_{ij}$, $y_{ij} = n^{-1/2} \hat{y}_{ij}$ and $z_{ij} = n^{-1/2} \hat{z}_{ij}$, where $\{\hat{x}_{ij}\}$, $\{\hat{y}_{ij}\}$ and $\{\hat{z}_{ij}\}$ are three independent families of i.i.d. random variables with mean zero and variance one. Moreover, suppose the following tail condition holds:
\[ \lim_{t \to \infty} t^4 \left[ \mathbb{P} (|\hat{\lambda}_{11}| \geq t) + \mathbb{P} (|\hat{\gamma}_{11}| \geq t) \right] = 0. \] (2.31)
We assume that the eigenvalues of $\Sigma_{XY}$ converge as $n \to \infty$ with
\[ \lim_{n \to \infty} t_{r_+} > t_c > \lim_{n \to \infty} t_{r_+ + 1}. \] (2.32)
Then, both (2.30) and the following convergence in probability hold:
\[ \lim_{n \to \infty} \mathbb{P} \left( \hat{\lambda}_i - \theta_i \leq \varepsilon \right) = 1 \quad \text{for any constant } \varepsilon > 0. \] (2.33)

**Remark 2.15.** If $\hat{\lambda}_{11}$ and $\hat{\gamma}_{11}$ have finite fourth moments, then the tail condition (2.31) holds. Hence, (2.31) is strictly weaker than (2.20), and it gives a weaker result (2.33) without an explicit convergence rate for $\hat{\lambda}_i - \theta_i$. Note that Theorem 2.14 cannot be derived directly from Corollary 2.13 when $a = 4$ and $b = 2$, we have $\phi_n = \psi_n = 1$ in (2.24), and the result (2.25) becomes a trivial statement. We also remark that (2.32) means for large enough $n$, there are exactly $r_+$ outliers and $t_i$-s are all away from the BBP transition threshold by a small constant, i.e., $\alpha_+ \gtrsim 1$. This is consistent with Assumption 2.5 (up to a small constant) with $\phi_n = \psi_n = 1$.

Finally, we mention that for an outlier eigenvalue, $n^{1/2}(\hat{\lambda}_i - \theta_i)$ actually converges to a normal distribution, which has been proved in [7] for the Gaussian case and for well-separated outliers, i.e. every pair $t_i$ and $t_j$ are either exactly degenerate or separated from each other by a distance of order 1. The proof for the general distribution case with near-degenerate outliers is quite involved, and, considering the length of this paper, we include it into another paper [10].

The rest of this paper is organized as follows. In Section 3, we introduce the linearization method and collect some basic tools that will be used in the proof. Then, we will give the proof of Theorem 2.9 in Section 4. The proofs of Theorem 2.11 Corollary 2.13 and Theorem 2.14 will be presented in Sections A and C.
3 Linearization method and local laws

The self-adjoint linearization method has been proved to be useful in studying the local laws of random matrices of Gram type 1 2 15 16 35 44 45. We now introduce a generalization of this method, which was introduced in 47 to prove Lemma 2.8. For the discussion below, we assume that $X^T, Y^T, XX^T$ and $YY^T$ are all non-singular almost surely. (This is trivially true if, say, the entries of $X, Y$ and $Z$ have continuous densities.) Then, given $\lambda > 0$, it is an eigenvalue of $C_{XY}$ if and only if the following equation holds:

$$\det \left[ X^T (Y^T)^{-1} Y X^T - \lambda XX^T \right] = 0. \tag{3.1}$$

Using the Schur complement formula, we can check that equation (3.1) is equivalent to

$$\det \left( \begin{array}{cc} XX^T & \lambda^{-1/2} XY^T \\ \lambda^{-1/2} Y^T X^T & \lambda \end{array} \right) = 0.$$

By the Schur complement formula again, the above equation is equivalent to

$$\det \left[ \begin{array}{cc} 0 & \left( \begin{array}{cc} X & 0 \\ 0 & Y \end{array} \right) \\ \left( \begin{array}{cc} X^T & 0 \\ 0 & Y^T \end{array} \right) & \left( \begin{array}{cc} \lambda I_n & \lambda^{-1/2} I_n \\ \lambda^{-1/2} I_n & \lambda I_n \end{array} \right)^{-1} \end{array} \right] = 0 \quad \text{if} \quad \lambda \notin \{0, 1\}. \tag{3.2}$$

Inspired by equation (3.2), we define the following $(p + q + 2n) \times (p + q + 2n)$ symmetric block matrix

$$H(\lambda) := \left( \begin{array}{cc} 0 & \left( \begin{array}{cc} X & 0 \\ 0 & Y \end{array} \right) \\ \left( \begin{array}{cc} X^T & 0 \\ 0 & Y^T \end{array} \right) & \left( \begin{array}{cc} \lambda I_n & \lambda^{-1/2} I_n \\ \lambda^{-1/2} I_n & \lambda I_n \end{array} \right)^{-1} \end{array} \right). \tag{3.3}$$

In general, we can extend the argument $\lambda$ to $z \in \mathbb{C}_+: = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \}$ and call it $H(z)$, where we take $z^{1/2}$ to be the branch with positive imaginary part. Then, using (2.3) and (2.4), we can write equation (3.2) as

$$\det \left[ H(\lambda) + \begin{pmatrix} U & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & E^T \end{pmatrix} \right] = 0, \tag{3.4}$$

where $D$ is a $2r \times 2r$ matrix with

$$D := \left( \begin{array}{cc} \Sigma_a & 0 \\ 0 & \Sigma_b \end{array} \right), \quad \Sigma_a := \text{diag} (a_1, \cdots, a_r), \quad \Sigma_b := \text{diag} (b_1, \cdots, b_r), \tag{3.5}$$

and $U$ and $E$ are $(p + q) \times 2r$ and $2n \times 2r$ matrices, respectively:

$$U := \left( \begin{array}{c} u_1^a, \cdots, u_r^a \\ 0 \end{array} \right), \quad E := \left( \begin{array}{c} (Z^T v_1^a, \cdots, Z^T v_r^a) \\ 0 \end{array} \right). \tag{3.6}$$

If $\lambda$ is not an eigenvalue of $C_{XY}$, then $H(\lambda)$ is non-singular by the Schur complement formula and (3.4) is equivalent to

$$\det \left[ 1 + \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \begin{pmatrix} U^T & 0 \\ 0 & E^T \end{pmatrix} \frac{1}{H(\lambda)} \begin{pmatrix} U & 0 \\ 0 & E \end{pmatrix} \right] = 0, \tag{3.7}$$
where we used the identity \( \det(1 + M_1M_2) = \det(1 + M_2M_1) \) for any matrices \( M_1 \) and \( M_2 \) of conformable dimensions. Inspired by the above discussion, we define the resolvent (or Green’s function)
\[
G(z) := [H(z)]^{-1}, \quad z \in \mathbb{C} \cup \mathbb{R},
\]
whenever the inverse exists. Note that although \( H(\lambda) \) is not well-defined for \( \lambda = 1 \), we can still define \( G(1) = \lim_{z \to 1} G(z) \) using the Schur complement, see (3.14) and (3.15) below. In order to study the eigenvalues of \( \mathcal{C}_{XY} \), we need to obtain some estimates on the \( 4r \times 4r \) matrix
\[
\begin{pmatrix}
U^\top & 0 \\
0 & E^\top
\end{pmatrix} G(\lambda) \begin{pmatrix}
U & 0 \\
0 & E
\end{pmatrix}.
\]
This is provided by the anisotropic local law of \( G(z) \), which is one of the main results in [17]. We will state it in Theorem 3.7 below.

For the proof of Theorem 2.11 we will also use a different representation of (3.7): if \( \lambda \) is not an eigenvalue of \( \mathcal{C}_{XY} \), then \( \lambda \) is an eigenvalue of \( \mathcal{C}_{XY} \) if and only if
\[
\det \left[ 1 + \begin{pmatrix}
0 & \mathcal{D}_a \\
\mathcal{D}_a & 0
\end{pmatrix} \begin{pmatrix}
U_a^\top & 0 \\
0 & E_a
\end{pmatrix} G^b(\lambda) \begin{pmatrix}
U_a & 0 \\
0 & E_a
\end{pmatrix} \right] = 0,
\]
where
\[
G^b(\lambda) := [H^b(\lambda)]^{-1}, \quad H^b(\lambda) := \begin{bmatrix}
0 & \begin{pmatrix}
X & 0 \\
0 & Y
\end{pmatrix} \\
\begin{pmatrix}
X^\top & 0 \\
0 & Y^\top
\end{pmatrix} & \left( \begin{pmatrix}
zI_n & 0 \\
0 & z^{1/2}I_n
\end{pmatrix} \right)^{-1}
\end{bmatrix},
\]
\[
\mathcal{D}_a := \begin{pmatrix}
\Sigma_a & 0 \\
0 & 0
\end{pmatrix}, \quad U_a := \begin{pmatrix}
u_1^a & \cdots & u_n^a
\end{pmatrix}, \quad E_a := \begin{pmatrix}
v_1^a & \cdots & v_n^a
\end{pmatrix}.
\]

For simplicity of notations, we introduce the following index sets for linearized matrices.

**Definition 3.1 (Index sets).** We define the index sets
\[
\mathcal{I}_1 := [1, p], \quad \mathcal{I}_2 := [p + 1, p + q], \quad \mathcal{I}_3 := [p + q + 1, p + q + n], \quad \mathcal{I}_4 := [p + q + n + 1, p + q + 2n].
\]
We will consistently use latin letters \( i, j \in \mathcal{I}_1 \cup \mathcal{I}_2 \) and greek letters \( \mu, \nu \in \mathcal{I}_3 \cup \mathcal{I}_4 \). Moreover, we will use notations \( a, b \in \mathcal{I} := \cup_{i=1}^4 \mathcal{I}_i \).

Next, we define several other types of resolvents that will be used in the proof.

**Definition 3.2 (Resolvents).** We denote the \((\mathcal{I}_1 \cup \mathcal{I}_2) \times (\mathcal{I}_1 \cup \mathcal{I}_2)\) block of \( G(z) \) by \( \mathcal{G}_{L}(z) \), the \((\mathcal{I}_1 \cup \mathcal{I}_2) \times (\mathcal{I}_3 \cup \mathcal{I}_4)\) block by \( \mathcal{G}_{LR}(z) \), the \((\mathcal{I}_3 \cup \mathcal{I}_4) \times (\mathcal{I}_1 \cup \mathcal{I}_2)\) block by \( \mathcal{G}_{RL}(z) \), and the \((\mathcal{I}_3 \cup \mathcal{I}_4) \times (\mathcal{I}_3 \cup \mathcal{I}_4)\) block by \( \mathcal{G}_{R}(z) \). We denote the \( \mathcal{I}_\alpha \times \mathcal{I}_\alpha \) block of \( G(z) \) by \( \mathcal{G}_\alpha(z) \) for \( \alpha = 1, 2, 3, 4 \). Then, we define the partial traces
\[
m_\alpha(z) := \frac{1}{n} \text{Tr} \mathcal{G}_\alpha(z) = \frac{1}{n} \sum_{a \in I_\alpha} G_{aa}(z), \quad \alpha = 1, 2, 3, 4.
\]
Recalling the notations in (2.10), we define \( \mathcal{H} := S_{xx}^{-1/2} S_{xy} S_{yy}^{-1/2} \) and
\[
R_1(z) := (\mathcal{H}\mathcal{H}^\top - z)^{-1}, \quad R_2(z) := (\mathcal{H}^\top \mathcal{H} - z)^{-1}, \quad m(z) := q^{-1} \text{Tr} R_2(z).
\]
(3.11)
Note that we have $R_1\mathcal{H} = \mathcal{H}R_2$, $\mathcal{H}^T R_1 = R_2\mathcal{H}^T$, and
\[
\text{Tr } R_1 = \text{Tr } R_2 - \frac{p-q}{z} = q m(z) - \frac{p-q}{z}, \tag{3.12}
\]
since $C_{XY} = \mathcal{H}\mathcal{H}^T$ has $p-q$ more zero eigenvalues than $C_{YX} = \mathcal{H}^T\mathcal{H}$. Moreover, we define
\[
R(z) := \begin{pmatrix} -\frac{z}{z^{-1/2}\mathcal{H}^T} & -z^{1/2}\mathcal{H}^{-1} \\ -z^{-1/2}\mathcal{H}^T & -z \end{pmatrix}.
\]
Finally, we can define $G^b_L(z)$, $G^b_R(z)$, $m^b(z)$, $\mathcal{H}^b$, $R^b$ etc. in the obvious way by replacing $Y$ with $Y$.

Using the Schur complement formula, we can check that
\[
R(z) := \left( \frac{R_1}{-z^{-1/2}\mathcal{H}^T R_1} \right) \left( \begin{array}{cc} -z^{-1/2}\mathcal{H}^T R_2 & z \end{array} \right).
\]
Let $\mathcal{H} = \sum_{k=1}^q \sqrt{\lambda_k} \xi_k \xi_k^T$ be a singular value decomposition of $\mathcal{H}$, where $\sqrt{\lambda_1} \geq \cdots \geq \sqrt{\lambda_q} \geq 0 = \sqrt{\lambda_{q+1}} = \cdots = \sqrt{\lambda_p}$ are the singular values, $\{\xi_k\}_{k=1}^q$ are the left-singular vectors, and $\{\zeta_k\}_{k=1}^q$ are the right-singular vectors. Then, we have the following eigendecomposition of $R(z)$:
\[
R(z) = \sum_{k=1}^q \frac{1}{\lambda_k - z} \begin{pmatrix} \xi_k \xi_k^T & -z^{-1/2}\sqrt{\lambda_k} \xi_k \zeta_k^T \\ -z^{-1/2}\sqrt{\lambda_k} \xi_k \zeta_k^T & \zeta_k \zeta_k^T \end{pmatrix} - \frac{1}{z} \begin{pmatrix} \sum_{k=q+1}^p \xi_k \xi_k^T & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.13}
\]
On the other hand, applying the Schur complement formula to $G(z)$, we get that
\[
G_L = \begin{pmatrix} S_{xx}^{-1/2} & 0 \\ 0 & S_{yy}^{-1/2} \end{pmatrix} R(z) \begin{pmatrix} S_{xx}^{-1/2} & 0 \\ 0 & S_{yy}^{-1/2} \end{pmatrix}.
\tag{3.14}
\]
Moreover, the other blocks take the forms
\[
G_R = \begin{pmatrix} z I_n & z^{1/2} I_n \\ z^{1/2} I_n & z I_n \end{pmatrix} + \begin{pmatrix} z I_n & z^{1/2} I_n \\ z^{1/2} I_n & z I_n \end{pmatrix} \begin{pmatrix} X^T & 0 \\ 0 & Y^T \end{pmatrix} G_L \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} z I_n & z^{1/2} I_n \\ z^{1/2} I_n & z I_n \end{pmatrix}, \tag{3.15}
\]
\[
G_{LR}(z) = -G_L(z) \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} z I_n & z^{1/2} I_n \\ z^{1/2} I_n & z I_n \end{pmatrix}, \quad G_{RL}(z) = G_{LR}(z)^T. \tag{3.16}
\]
Expanding the product in (3.15) using (3.13) and calculating partial traces, we can check that
\[
m_3(z) = z + \frac{1}{n} (-2 z p - z^2 \text{Tr } R_1 + z \text{Tr } R_2) = c_2 z (1 - z) m(z) + (1 - c_1 - c_2) z, \tag{3.17}
\]
\[
m_4(z) = z + \frac{1}{n} (-2 z q - z^2 \text{Tr } R_2 + z \text{Tr } R_1) = c_2 z (1 - z) m(z) - (c_1 - c_2) + (1 - 2 c_2) z, \tag{3.18}
\]
where we also used (3.12) in the derivations. In particular, we have the identity
\[
m_3(z) - m_4(z) = (1 - z) (c_1 - c_2). \tag{3.19}
\]
We remark that all the above identities also hold for $G^b_L(z)$, $G^b_R(z)$, $m^b(z)$ etc. with some obvious changes of notations.

Since $S_{xx}$ and $S_{yy}$ are standard sample covariance matrices, it is well-known that their eigenvalues are all inside the supports of Marchenko-Pastur laws, $[(1 - \sqrt{\rho}^2), (1 + \sqrt{\rho}^2)]$ and $[(1 - \sqrt{\rho}^2), (1 + \sqrt{\rho}^2)]$ etc. with probability $1 - o(1)$ \footnote{In our proof, we will need some slightly stronger estimates on the extreme eigenvalues of $S_{xx}$ and $S_{yy}$, denoted by $\lambda_1(S_{xx}) \geq \lambda_p(S_{xx})$ and $\lambda_1(S_{yy}) \geq \lambda_q(S_{yy})$, which are given by the following lemma.}.
Lemma 3.3. Suppose Assumption 2.1 holds. Suppose $X$ and $Y$ have bounded support $\phi_n$ and $Z$ has bounded support $\psi_n$ with $n^{-1/2} \leq \phi_n \leq n^{-c_\phi}$ and $n^{-1/2} \leq \psi_n \leq n^{-c_\psi}$ for some constants $c_\phi, c_\psi > 0$. Then, for any constant $\varepsilon > 0$, we have that with high probability,

\begin{align}
(1 - \sqrt{c_1})^2 - \varepsilon &\leq \lambda_p(S_{xx}) \leq \lambda_1(S_{xx}) \leq (1 + \sqrt{c_1})^2 + \varepsilon, \quad (3.20) \\
(1 - \sqrt{c_2})^2 - \varepsilon &\leq \lambda_q(S_{yy}) \leq \lambda_1(S_{yy}) \leq (1 + \sqrt{c_2})^2 + \varepsilon. \quad (3.21)
\end{align}

Moreover, there exists a constant $c > 0$ such that with high probability,

$$c \leq \lambda_q(S_{yy}) \leq \lambda_1(S_{yy}) \leq c^{-1}, \quad (3.22)$$

where $\lambda_1(S_{yy})$ and $\lambda_q(S_{yy})$ are respectively the largest and smallest eigenvalues of $S_{yy}$.

Proof. The estimates (3.20) and (3.21) have been proved in Lemma 3.3 of [47]. To get (3.22), we write

$$S_{yy} = (I_q, B) WW^\top \begin{pmatrix} I_q \\ B^\top \end{pmatrix}, \quad W := \begin{pmatrix} Y \\ Z \end{pmatrix}.$$ 

Since $r/n \to 0$, the estimate (3.21) applied to $WW^\top$ gives that with high probability,

$$\lambda_1(WW^\top) \leq (1 + \sqrt{c_2})^2 + \varepsilon.$$ 

Then, using that for any unit vector $v \in \mathbb{R}^q$, $\|v\| \sim \|u\|$ for $u := \begin{pmatrix} I_q \\ B^\top \end{pmatrix} v$, we conclude (3.22). \hfill \Box

Let $m_{\alpha c}$ be the asymptotic limits of $m_{\alpha}$ for $\alpha = 1, 2, 3, 4$. In [47], we have obtained that

\begin{align}
m_{1c}(z) &= \frac{-z + c_1 + c_2 + \sqrt{(z - \lambda_-)(z - \lambda_+)}}{2(1 - c_1)z(1 - z)} \cdot \frac{c_1}{(1 - c_1)z}, \quad (3.23) \\
m_{2c}(z) &= \frac{-z + c_1 + c_2 + \sqrt{(z - \lambda_-)(z - \lambda_+)}}{2(1 - c_2)z(1 - z)} \cdot \frac{c_2}{(1 - c_2)z}, \quad (3.24) \\
m_{3c}(z) &= \frac{1}{2} \left[ (1 - 2c_1)z + c_1 - c_2 + \sqrt{(z - \lambda_-)(z - \lambda_+)} \right], \quad (3.25) \\
m_{4c}(z) &= \frac{1}{2} \left[ (1 - 2c_2)z + c_2 - c_1 + \sqrt{(z - \lambda_-)(z - \lambda_+)} \right], \quad (3.26)
\end{align}

where $\lambda_\pm$ are defined in (2.13). It is easy to see that when $z \to 1$, both $m_{1c}(z)$ and $m_{2c}(z)$ have finite limits, and without loss of generality, we still denote them by $m_{1c}(1)$ and $m_{2c}(1)$. With (3.17), we can easily obtain the asymptotic limit of $m(z)$ as

$$m_{\alpha}(z) = \frac{m_{3c}(z) + (c_1 + c_2 - 1)z}{c_2z(1 - z)} = \frac{1 - c_2}{c_2} m_{2c}(z). \quad (3.27)$$

Through direct calculations, we can check that $m_{\alpha c}$s satisfy the following equations:

$$m_{1c} = -\frac{c_1}{m_{3c}}, \quad m_{2c} = -\frac{c_2}{m_{4c}}, \quad m_{3c}(z) - m_{4c}(z) = (1 - z)(c_1 - c_2). \quad (3.28)$$

Finally, we introduce the function

$$h(z) = \frac{z^{-1/2}m_{3c}(z)}{1 + (1 - z)m_{2c}(z)} = \frac{z^{-1/2}m_{4c}(z)}{1 + (1 - z)m_{1c}(z)} = z^{-1/2} \frac{2}{2} \left[ -z + (2 - c_1 - c_2) + \sqrt{(z - \lambda_-)(z - \lambda_+)} \right]. \quad (3.29)$$
Now, with the functions \( m_{\alpha c} \) and \( h \), we can define the matrix limit of \( G(z) \) as

\[
\Pi(z) := \begin{pmatrix}
\left( c_1^{-1} m_{1c}(z) I_p \right) & 0 & 0 \\
0 & \left( c_2^{-1} m_{2c}(z) I_q \right) & 0 \\
0 & 0 & \left( \frac{m_{3c}(z) I_n}{h(z) I_n} \right)
\end{pmatrix}.
\] (3.30)

Given \( z = E + i \eta \), we define its distance (along the real axis) to the two edges as

\[
\kappa \equiv \kappa_E := \min \{|E - \lambda_-|, |E - \lambda_+|\}.
\] (3.31)

We have the following lemma, which can be proved through direct calculations using (3.30)–(3.36).

**Lemma 3.4.** If (3.30) holds, then the following estimates hold for any constants \( c, C > 0 \).

1. For \( z \in \mathbb{C}_+ \cap \{z : c \leq |z| \leq C\} \), we have

\[
|m_c(z)| \sim 1, \quad 0 \leq \text{Im} m_c(z) \sim \begin{cases} \eta/\sqrt{\kappa + \eta}, & E \notin [\lambda_-, \lambda_+] \\ \sqrt{\kappa + \eta}, & E \in [\lambda_-, \lambda_+] \end{cases}.
\] (3.32)

2. For \( z, z_1, z_2 \in \mathbb{C}_+ \cap \{z : c \leq |z| \leq C\} \cap \{\text{Re} z > \lambda_+\} \), we have

\[
|m_c(z) - m_c(\lambda_+)| \sim |z - \lambda_+|^{1/2}, \quad |m'_c(z)| \sim |z - \lambda_+|^{-1/2},
\] (3.33)

\[
|m_c(z_1) - m_c(z_2)| \sim \frac{|z_1 - z_2|}{\max_{i=1,2} |z_i - \lambda_+|^{1/2}}.
\] (3.34)

The above estimates also hold for \( m_{\alpha c} \), \( \alpha = 1, 2, 3, 4 \). Finally, \( h(z) \) also satisfies (3.33), (3.34) and the first estimate in (3.32).

For simplicity of notations, we introduce the following notion of generalized entries.

**Definition 3.5 (Generalized entries).** Given \( \mathbf{v}, \mathbf{w} \in \mathbb{C}^I \), \( a \in \mathcal{I} \) and \( \mathcal{I} \times \mathcal{I} \) matrix \( A \), we denote

\[
A_{\mathbf{v} \mathbf{w}} := \langle \mathbf{v}, A \mathbf{w} \rangle, \quad A_{\mathbf{v} a} := \langle \mathbf{v}, A e_a \rangle, \quad A_{a \mathbf{w}} := \langle e_a, A \mathbf{w} \rangle,
\] (3.35)

where \( e_a \) is the standard unit vector along the \( a \)-th coordinate axis, and the inner product is defined as \( \langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{v}^* \mathbf{w} \) with \( \mathbf{v}^* \) being the conjugate transpose of \( \mathbf{v} \). Given a vector \( \mathbf{v} \in \mathbb{C}^{I_1} \), \( \alpha = 1, 2, 3, 4 \), we always identify it with its natural embedding in \( \mathbb{C}^I \). For example, we shall identify \( \mathbf{v} \in \mathbb{C}^{I_1} \) with a vector \( \mathbf{v}' \in \mathbb{C}^I \) with \( \mathbf{v}'(i) = \mathbf{v}(i) \) for \( i \in I_1 \) and \( \mathbf{v}'(i) = 0 \) for \( i \notin I_1 \).

We define the following spectral domains for the local laws of \( G(z) \).

**Definition 3.6 (Spectral domains).** For any constant \( \varepsilon > 0 \), we define the following two domains:

\[
S(\varepsilon) := \{ z = E + i \eta : \varepsilon \leq E \leq 2, n^{-1+\varepsilon} \leq \eta \leq \varepsilon^{-1} \},
\] (3.36)

\[
S_{\text{out}}(\varepsilon) := S(\varepsilon) \cap \{ z = E + i \eta : E \notin [\lambda_-, \lambda_+], n \eta \sqrt{\kappa + \eta} \geq n^2 \}.
\] (3.37)

Correspondingly, we define the following two domains that are away from \( z = 1 \): for any fixed \( \bar{\varepsilon} > 0 \),

\[
\bar{S}(\varepsilon, \bar{\varepsilon}) := \{ z = E + i \eta : \varepsilon \leq E \leq 1 - \bar{\varepsilon}, n^{-1+\varepsilon} \leq \eta \leq \varepsilon^{-1} \}, \quad \bar{S}_{\text{out}}(\varepsilon, \bar{\varepsilon}) := \bar{S}(\varepsilon, \bar{\varepsilon}) \cap S_{\text{out}}(\varepsilon).
\]
Lemma 3.9 of [47]

Lemma 3.8

the rigidity estimate (2.20) together give the following delocalization of eigenvectors.

\[ u \]

All the above estimates are uniform in the spectral parameter

Theorem 3.9 (Theorem 2.13 and Theorem 2.14 of [47]). Suppose the assumptions of Lemma 2.8 hold. Then, for any fixed \( \varepsilon, \delta > 0 \), the following estimates hold.

(1) **Anisotropic local law**: For any \( z \in S(\varepsilon) \) and deterministic unit vectors \( u, v \in \mathbb{C}^\mathcal{T} \), we have

\[
|G_{uv}(z) - \Pi_{uv}(z)| < \phi_n + \Psi(z).
\]

(3.39)

(2) **Averaged local law**: For any \( z \in \tilde{S}(\varepsilon, \tilde{\varepsilon}) \), we have

\[
|m(z) - m_c(z)| < (n\eta)^{-1}.
\]

Moreover, outside of the spectrum, we have a stronger estimate for any \( z \in \tilde{S}_{out}(\varepsilon, \tilde{\varepsilon}) \):

\[
|m(z) - m_c(z)| < \frac{1}{n(\kappa + \eta)} + \frac{1}{(n\eta)^2\sqrt{\kappa + \eta}}.
\]

The estimates (3.40) and (3.41) also hold for \( m_{\alpha}(z) - m_{\alpha c}(z) \), \( \alpha = 1, 2, 3, 4 \).

All the above estimates are uniform in the spectral parameter \( z \).

The averaged local law implies the rigidity of eigenvalues in (2.20). The anisotropic local law (3.39) and the rigidity estimate (2.20) together give the following delocalization of eigenvectors.

**Lemma 3.8** (Lemma 3.9 of [47]). Suppose (3.39) and (2.20) hold. Then, for any small constant \( \delta > 0 \) and deterministic unit vectors \( u_\alpha \in \mathbb{C}^{\mathcal{T}_\alpha} \), \( \alpha = 1, 2, 3, 4 \), the following estimates hold:

\[
\max_{1 \leq k \leq (1 - \delta)q} \left\{ \left| \langle u_1, S_{xx}^{-1/2} \xi_k \rangle \right|^2 + \left| \langle u_2, S_{yy}^{-1/2} \zeta_k \rangle \right|^2 \right\} < n^{-1},
\]

(3.42)

\[
\max_{1 \leq k \leq (1 - \delta)q} \left\{ \left| \langle u_3, X^\top S_{xx}^{-1/2} \xi_k \rangle \right|^2 + \left| \langle u_4, Y^\top S_{yy}^{-1/2} \zeta_k \rangle \right|^2 \right\} < n^{-1}.
\]

(3.43)

Away from the support \( [\lambda_-, \lambda_+] \), the anisotropic local law can be strengthened as follows.

**Theorem 3.9** (Anisotropic local law outside the bulk spectrum). Suppose the assumptions of Lemma 2.8 hold. Fix any constant \( \varepsilon > 0 \). For any

\[
z \in D_{out}(\varepsilon) := \left\{ z = E + i\eta : \lambda_+ + n^{-2/3 + \varepsilon} \leq E \leq 2, 0 \leq \eta \leq 1 \right\},
\]

(3.44)

and deterministic unit vectors \( u, v \in \mathbb{C}^\mathcal{T} \), the following anisotropic local law holds:

\[
|G_{uv}(z) - \Pi_{uv}(z)| < \phi_n + \sqrt{\frac{\text{Im} \ m_c(z)}{n\eta}} = \phi_n + n^{-1/2}(\kappa + \eta)^{-1/4}.
\]

(3.45)
Proof. The second step of (3.49) follows from (3.32). Using (3.39) and \( \kappa \geq n^{-2/3+\varepsilon} \), we can get that (3.45) holds for \( z \in S(\varepsilon) \cap D_{\text{out}}(\varepsilon) \) with \( \eta \geq \eta_0 := n^{-1/2} \kappa^{-1/4} \). Hence, it remains to prove that for \( z \in D_{\text{out}}(\varepsilon) \) with \( 0 \leq \eta \leq \eta_0 \), we have

\[
|G_{\text{vv}}(X, z) - \Pi_{\text{vv}}(z)| < \phi_{\eta_0} + n^{-1/2} \kappa^{-1/4},
\]

(3.46)

for any deterministic unit vector \( v \in \mathbb{C}^I \). Note that (3.46) implies (3.45) by the polarization identity

\[
\langle u, Mv \rangle = \frac{1}{4} \langle (u + v), M(u + v) \rangle - \frac{1}{4} \langle (u - v), M(u - v) \rangle
\]

+ \frac{i}{4} \langle (iu + v), M(iu + v) \rangle - \frac{i}{4} \langle (iu - v), M(iu - v) \rangle

for any \( I \times I \) matrix \( M \). Now, fix any \( z = E + i\eta \in D_{\text{out}}(\varepsilon) \) with \( \eta \leq \eta_0 \). We denote \( z_0 := E + i\eta_0 \). Since (3.46) holds at \( z_0 \), it suffices to prove the following estimates:

\[
\Pi_{\text{vv}}(z) - \Pi_{\text{vv}}(z_0) < n^{-1/2} \kappa^{-1/4},
\]

(3.47)

\[
G_{\text{vv}}(z) - G_{\text{vv}}(z_0) < n^{-1/2} \kappa^{-1/4}.
\]

(3.48)

The estimate (3.47) follows immediately from (3.34). It remains to show (3.48).

We write \( v = (v_1^*, v_2^*)^T \), where \( v_\alpha \in \mathbb{C}^{\mathbb{Z}_\alpha} \), \( \alpha = 1, 2, 3, 4 \). We claim that

\[
\left( v_1^*, v_2^* \right) \left[ G_L(z) - G_L(z_0) \right] \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} < n^{-1/2} \kappa^{-1/4}.
\]

(3.49)

For simplicity of notations, in the following proof, we will always identify \( v_\alpha \), \( \alpha = 1, 2, 3, 4 \), with their natural embeddings in \( \mathbb{C}^I \) (recall Definition 3.3). Using (3.33) and (3.14), and recalling that with high probability \( E - \lambda_k \gtrsim 1 \) for \( k \geq (1 - \delta)q \) by the rigidity estimate (2.20), we obtain that

\[
|\langle v_1, (G(z) - G(z_0)) v_1 \rangle| < \sum_{k \leq 1 - \delta \eta} \eta_0 |(E - \lambda_k)^2 + \eta_0^2|^{1/2} \left[ (E - \lambda_k)^2 + \eta_0^2 \right]^{1/2}
\]

(3.50)

+ \eta_0 \sum_{k > 1 - \delta \eta} |\langle v_1, S^{-1/2}_{xx} \zeta_k \rangle|^2.

By (2.20), we have that for any \( k \geq 1 \), \( E - \lambda_k \gtrsim \kappa \gtrsim \eta_0 \) with high probability. Then, using (3.42) and (3.20), we can bound (3.50) by

\[
|\langle v_1, (G(z) - G(z_0)) v_1 \rangle| < \eta_0 + \frac{1}{n\kappa} \sum_{k=1}^{q} \eta_0 |(E - \lambda_k)^2 + \eta_0^2| = \eta_0 + \text{Im} m(z_0)
\]

\[
< \eta_0 + \frac{1}{(n\eta_0)^2} + \frac{1}{(n\eta_0)^2} \sqrt{\kappa} + \text{Im} m_c(z_0)
\]

\[
\lesssim \frac{1}{n\kappa} + \frac{1}{(n\eta_0)^2} \sqrt{\kappa} + \frac{\eta_0}{\sqrt{\kappa} + \eta_0} \lesssim n^{-1/2} \kappa^{-1/4},
\]

where we used the spectral decomposition for \( m(z) \) in the second step, (3.41) in the third step, and (3.32) in the fourth step. Similarly, we have

\[
|\langle v_1, (G(z) - G(z_0)) v_2 \rangle| < 1 - \sum_{z_0} |\langle v_1, G(z_0) v_2 \rangle| + \sum_{k=1}^{q} \eta_0 |\langle v_1, S^{-1/2}_{xx} \zeta_k \rangle| |\langle v_2, S^{-1/2}_{yy} \zeta_k \rangle|
\]

\[
|\lambda_k - z||\lambda_k - z_0|
\]

< \eta_0 + \text{Im} m(z_0) < n^{-1/2} \kappa^{-1/4}.

17
Similar arguments also apply to \( \langle v_2, (G(z) - G(z_0)) v_1 \rangle \) and \( |\langle v_2, (G(z) - G(z_0)) v_2 \rangle| \). Hence we conclude (3.49). Finally, using (3.49), (3.15), (3.16) and Lemma 3.8 we can get (3.48). We omit the details.

The second moment of \( \langle u, (G(z) - \Pi(z))v \rangle \) in fact satisfies a stronger bound. It will be used in the proof of Theorem 2.14.

**Lemma 3.10.** Suppose the assumptions of Lemma 2.8 hold. Fix any constant \( \varepsilon > 0 \). For any deterministic unit vectors \( u, v \in \mathbb{C}^n \), we have that uniformly in \( z \in S(\varepsilon) \) (recall (3.46)),

\[
E |G_{uv}(z) - \Pi_{uv}(z)|^2 < \Psi^2(z),
\]

and uniformly in \( z \in D_{\text{out}}(\varepsilon) \) (recall (3.44)),

\[
E |G_{uv}(z) - \Pi_{uv}(z)|^2 < \frac{1}{n^\sqrt{\kappa + \eta}}.
\]

**Proof.** The estimate (3.51) has been proved in Lemma 3.10 of [47]. The estimate (3.52) can be proved using almost the same argument, where the only difference is that we replace the local law (3.39) with the stronger one (3.45) in the proof. We omit the details.

### 4 Proof of Theorem 2.9

In this section, we prove Theorem 2.9 using the local laws, Theorems 3.7 and 3.9, and the eigenvalue rigidity (2.20). During the proof, in order to avoid some non-generic events, we assume that the entries \( x_{ij}, y_{ij} \) and \( z_{ij} \) have continuous densities. (4.1)

It can be achieved by adding a small perturbation to \( X, Y \) and \( Z \). For example, we can add to each matrix a small Gaussian matrix:

\[
X \rightarrow X + \delta e^{-n} X_G, \quad Y \rightarrow Y + \delta e^{-n} Y_G, \quad Z \rightarrow Z + \delta e^{-n} Z_G.
\]

These Gaussian components are negligible for our results and can be easily removed by taking \( \delta \to 0 \). Under (4.1), the matrices \( XX^T, YY^T, XX^T \) and \( YY^T \) are all non-singular almost surely. Moreover, almost surely, \( \lambda = 1 \) is not in the spectrum of \( C_{XY} \) or \( C_{YX} \). By (3.7), \( 0 < \lambda < 1 \) is an eigenvalue of \( C_{XY} \) if and only if

\[
\det \left[ 1 + \left( \begin{array}{cc} 0 & \mathcal{D} \\ \mathcal{D} & 0 \end{array} \right) \left( \begin{array}{cc} U^T & 0 \\ 0 & \mathbf{E}^T \end{array} \right) G(\lambda) \left( \begin{array}{cc} U & 0 \\ 0 & \mathbf{E} \end{array} \right) \right] = 0.
\]

Using a standard large deviation estimate (e.g., Lemma 3.8 of [19]), we can derive the following approximate isometry condition for \( Z \):

\[
\|Z Z^T - I_r\| < \psi_n.
\]

Now, for any \( \lambda \in D_{\text{out}}(\varepsilon) \), using Theorem 3.9 and (4.3), we can write (4.2) as

\[
0 = \det \left[ 1 + \left( \begin{array}{cc} 0 & \mathcal{D} \\ \mathcal{D} & 0 \end{array} \right) (\Pi_r(\lambda) + \mathcal{E}_r) \right]
\]

\[
= \det \left[ \begin{array}{cc} I_{2r} & \mathcal{D} \left( \begin{array}{cc} m_{3c}(\lambda) I_r & h(\lambda) M_r \\ h(\lambda) M_r^T & m_{4c}(\lambda) I_r \end{array} \right) \\ \mathcal{D} \left( \begin{array}{cc} c_1^{-1} m_{1c}(\lambda) I_r & 0 \\ 0 & c_2^{-1} m_{2c}(\lambda) I_r \end{array} \right) & I_{2r} \end{array} \right] + \left( \begin{array}{cc} 0 & \mathcal{D} \\ \mathcal{D} & \mathcal{E}_r \end{array} \right).
\]
Here, \( \mathcal{E}_{4r} \) is a \( 4r \times 4r \) random matrix satisfying
\[
\| \mathcal{E}_{4r} \| < \psi_n + \phi_n + n^{-1/2} \kappa_\lambda^{-1/4}, \quad \text{with} \quad \kappa_\lambda := \min \{|\lambda - \lambda_-|, |\lambda - \lambda_+|\},
\]
and \( \mathcal{M}_r \) is an \( r \times r \) orthogonal matrix with entries
\[
(\mathcal{M}_r)_{ij} := (\psi_i^a)^\top \psi_j^b, \quad 1 \leq i, j \leq r,
\]
and \( \Pi_r(\lambda) \) is defined as
\[
\Pi_r(\lambda) := \begin{pmatrix}
(c_{1}^{-1}m_{1c}(\lambda)I_r & 0 & 0 \\
0 & c_2^{-1}m_{2c}(\lambda)I_r & 0 \\
(\nu^a I_r) & h(\lambda)M_r & h(\lambda)M_r^\top
\end{pmatrix}.
\]
Applying the Schur complement formula and using (3.28), we obtain that (4.4) is equivalent to
\[
\Pi_r(\lambda) := \begin{pmatrix}
(c_{1}^{-1}m_{1c}(\lambda)I_r & 0 & 0 \\
0 & c_2^{-1}m_{2c}(\lambda)I_r & 0 \\
(\nu^a I_r) & h(\lambda)M_r & h(\lambda)M_r^\top
\end{pmatrix}.
\]
Next, we show that if \( \mathcal{E}_r \) also satisfies the same bound as in (4.5), then solving equation (4.7) gives the classical locations \( \theta_i \) defined in (2.14). After a change of basis, (4.5) reduces to
\[
\det \left( \frac{m_{3c}(\lambda)m_{4c}(\lambda)}{h^2(\lambda)}I_r - \frac{\Sigma_a}{(I_r + \Sigma_a^2)^{1/2}}M_r - \frac{\Sigma_b}{(I_r + \Sigma_b^2)^{1/2}}M_r^\top \right) = 0,
\]
where \( \mathcal{E}_r \) also satisfies the same bound as in (4.5).

Next, we show that if \( \mathcal{E}_r = 0 \), then solving equation (4.7) gives the classical locations \( \theta_i \) defined in (2.14). Using (3.20), (3.20), and (3.29), we can calculate that
\[
f_c(z) := \frac{m_{3c}(z)m_{4c}(z)}{h^2(z)} = z[1 + (1 - z)m_{1c}(z)][1 + (1 - z)m_{2c}(z)]
\]
\[
= z - (c_1 + c_2 - 2c_1c_2) + \sqrt{(z - \lambda_+)(z - \lambda_+)}
\]
\[
2(1 - c_1)(1 - c_2).
\]
We can find the inverse function of \( f_c(z) \) for \( z \notin [\lambda_, \lambda_+] \) as
\[
g_c(\xi) := \xi(1 - c_1 + c_1\xi^{-1})(1 - c_2 + c_2\xi^{-1}).
\]
Note that \( f_c(\lambda) \) is monotonically increasing in \( \lambda \) for \( \lambda > \lambda_+ \), so the function \( f_c(\lambda) - t_i = 0 \) has a solution in \( (\lambda_+, \infty) \) if and only if (recall (1.33))
\[
f_c(\lambda_+) < t_i \iff t_c < t_i.
\]
If (1.3) holds, the classical location of the outlier corresponding to \( t_i \) is \( \theta_i = g_c(t_i) \), which gives (2.15).

With direct calculations, one can verify the following simple estimates on \( f_c \) and \( g_c \).
Lemma 4.1. Fix a large constant $C > 0$. Let $z, z_1, z_2 \in \mathbb{D} := \{ z \in \mathbb{C} : \lambda_+ < \text{Re} \ z < C, 0 < \text{Im} \ z \leq C \}$ and $\xi, \xi_1, \xi_2 \in f_c(\mathbb{D})$. The following estimates hold:

$$
|f_c(z) - f_c(\lambda_+)| \sim |z - \lambda_+|^{1/2}, \quad |f_c(\xi)| \sim |\xi - \lambda_+|^{-1/2}, \quad \text{(4.9)}
$$

$$
|g_c(\xi) - \lambda_+| \sim |\xi - t_c|^2, \quad |g_c'(\xi)| \sim |\xi - t_c|, \quad \text{(4.10)}
$$

$$
|f_c(z_1) - f_c(z_2)| \sim \frac{|z_1 - z_2|}{\max_{i=1,2} |z_i - \lambda_+|^{1/2}}, \quad |g_c(\xi_1) - g_c(\xi_2)| \sim |\xi_1 - \xi_2| \cdot \max_{i=1,2} |\xi_i - t_c|. \quad \text{(4.11)}
$$

The estimate (4.9) also holds for $z$ with $\lambda_- + \epsilon \leq \text{Re} \ z \leq \lambda_+$ and $0 < \text{Im} \ z \leq \epsilon^{-1}$ for any small constant $\epsilon > 0$.

For the proof of Theorem 2.9 we record the following eigenvalue interlacing result:

$$
\tilde{\lambda}_i \in [\lambda_{i+2r}, \lambda_{i-2r}], \quad \text{(4.12)}
$$

where we adopt the convention that $\lambda_i = 1$ if $i < 1$ and $\lambda_i = 0$ if $i > q$. For the reader’s convenience, we briefly describe why (4.12) holds. We first consider a 1-dimensional perturbation:

$$
X_1 := X + u_1 v_1^T, \quad u_1 \in \mathbb{R}^p, \quad v_1 \in \mathbb{R}^n.
$$

Then, it is easy to see that $P_X := X^T (X X^T)^{-1} X$ is a projection onto the subspace $W$ spanned by the rows of $X$. Similarly, $P_{X_1} := (X_1 X_1^T)^{-1} X_1$ is a projection onto the subspace $W_1$ spanned by the rows of $X_1$. Moreover, $W$ and $W_1$ differ at most by a 1-dimensional subspace. Hence, by Cauchy interlacing, we have

$$
\lambda_i(P_{X_1} P_Y P_{X_1}) \in [\lambda_{i+1}(P_X P_Y P_X), \lambda_{i-1}(P_X P_Y P_X)], \quad \text{where} \quad P_Y := Y^T \frac{1}{Y Y^T} Y.
$$

Notice that $P_X P_Y P_X$ (resp. $P_{X_1} P_Y P_{X_1}$) has the same nonzero eigenvalues as $C_{XY}$ (resp. $C_{X_1 Y}$): if $u$ is an eigenvector of $C_{XY}$ with eigenvalue $\lambda$, then $X^T (X X^T)^{-1} X u$ is an eigenvector of $P_X P_Y P_X$ with the same eigenvalue. Thus, we get

$$
\lambda_i(C_{X_1 Y}) \in \lambda_i+1(C_{XY}), \lambda_i-1(C_{XY})].
$$

Repeating this estimate $r$ times for the rank-$r$ perturbation $X$, we get

$$
\lambda_i(C_{X r Y}) \in \lambda_i+r(C_{XY}), \lambda_i-r(C_{XY})],
$$

where $C_{X r Y}$ is defined by replacing $X$ with $X$ in $C_{XY}$. Obviously, the same argument works for the rank-$r$ perturbation of $Y$, which leads to (4.12).

With (4.7) and (4.12), the rest of the proof for Theorem 2.9 is similar to those in [1,2 Section 4] and [3, Section 6], but these references have only considered cases with small support $\phi_n < n^{-1/2}$. We need to adapt their proofs to our setting with larger $\phi_n$ and $\psi_n$.

Proof of Theorem 2.9. For simplicity of presentation, in this proof we abbreviate $\phi_n + \psi_n$ as $\phi_n$ because these two factors always appear together. By Theorems 6.7 and 6.9 and equations (2.20) and (4.13), for any fixed $\epsilon > 0$, we can choose a high-probability event $\Xi$ on which the following estimates hold:

$$
\left\| \left( \begin{array}{cc} U^T & 0 \\ 0 & E^T \end{array} \right) G(z) \left( \begin{array}{cc} U & 0 \\ 0 & E \end{array} \right) - \Pi_r(z) \right\| \leq n^{\epsilon/2} \left( \phi_n + \Psi(z) \right), \quad \text{for} \ z \in S(\epsilon); \quad \text{(4.13)}
$$

$$
\left\| \left( \begin{array}{cc} U^T & 0 \\ 0 & E^T \end{array} \right) G(z) \left( \begin{array}{cc} U & 0 \\ 0 & E \end{array} \right) - \Pi_r(z) \right\| \leq n^{\epsilon/2} \left( \phi_n + n^{-1/2} \kappa^{-1/4} \right), \quad \text{for} \ z \in D_{\text{out}}(\epsilon); \quad \text{(4.14)}
$$

20
for a fixed large integer \( \varpi \in \mathbb{N} \),
\[
|\lambda_i - \lambda_+| \leq n^{-2/3+\varepsilon}, \quad \text{for } 1 \leq i \leq \varpi + 2r.
\]

(4.15)

We remark that the randomness of \( X \) and \( Y \) only comes into play to ensure that \( \Xi \) holds with high probability. The rest of the proof will be entirely deterministic once restricted to \( \Xi \). In the following proof, we assume that \( \varepsilon \) is a sufficiently small constant.

We now define the index sets
\[
\mathcal{O}_\varepsilon := \left\{ i : t_i - t_\varepsilon \geq n^\varepsilon \phi_n + n^{-1/3+\varepsilon} \right\}.
\]

(4.16)

Since the constant \( \varepsilon \) is arbitrary, in order to prove (2.23) and (2.24), it suffices to show that there exists a constant \( C > 0 \) such that
\[
1(\Xi) \left| \tilde{\lambda}_i - \theta_i \right| \leq C n^{2\varepsilon} \left( \phi_n \Delta_i + n^{-1/2} \Delta_i^{1/2} \right)
\]
for all \( i \in \mathcal{O}_{4\varepsilon} \), and
\[
-n^{-2/3+\varepsilon} \leq 1(\Xi) \left( \tilde{\lambda}_i - \lambda_+ \right) \leq C n^{2\varepsilon} \phi_n^2 + C n^{-2/3+12\varepsilon}
\]
for all \( i \in \{1, \ldots, \varpi\} \setminus \mathcal{O}_{4\varepsilon} \). For the rest of the proof, we assume that \( \Xi \) holds.

**Step 1:** Our first step is to prove that on \( \Xi \), there are no eigenvalues outside the neighborhoods of \( \theta_i \)'s. For \( 1 \leq i \leq r_+ \), we define the permissible intervals
\[
I_i \equiv I_i(t) := \left[ \theta_i - n^\varepsilon \left( \phi_n \Delta_i + n^{-1/2} \Delta_i^{1/2} \right), \theta_i + n^\varepsilon \left( \phi_n \Delta_i + n^{-1/2} \Delta_i^{1/2} \right) \right],
\]
where \( t \) denotes the vector \( t := (t_1, t_2, \ldots, t_r) \). We then define
\[
I \equiv I(t) := I_0 \cup \bigcup_{i \in \mathcal{O}_\varepsilon} I_i(t), \quad I_0 := \left[ 0, \lambda_+ + n^{2\varepsilon} \phi_n^2 + n^{-2/3+3\varepsilon} \right].
\]

(4.20)

We claim the following result.

**Lemma 4.2.** The complement of \( I(t) \) contains no eigenvalues of \( C_{XY} \).

Proof. The main idea is similar to the ones for [33, Proposition 6.5] and [17, Lemma S.4.2]. It suffices to show that for any \( 1 \leq i \leq r \), if \( x \notin I(t) \), then
\[
|f_c(x) - t_i| \geq c \left( n^\varepsilon \phi_n + n^{-1/2+\varepsilon} \kappa_x^{-1/4} \right)
\]
for some constant \( c > 0 \). Thus, (4.17) cannot hold on \( \Xi \) by (4.14).

For \( x \notin I_0 \), with (4.17), we get that
\[
f_c(x) - t_c = f_c(x) - f_c(\lambda_+) \geq c \kappa_x^{1/2} \geq c' \left( n^\varepsilon \phi_n + n^{-1/2+\varepsilon} \kappa_x^{-1/4} \right),
\]
for some constants \( c, c' > 0 \). This concludes (4.21) for \( i \geq r_+ \) by using \( t_i \leq t_c + n^{-1/3} + \phi_n \).

For the case \( 1 \leq i \leq r_+ \), we take any \( x \notin I_0 \cup I_i(t) \). We first assume that there exists a constant \( \tilde{c} > 0 \) such that \( \theta_i \notin [x - \tilde{c} \kappa_x, x + \tilde{c} \kappa_x] \). Since \( f_c \) is monotonically increasing on \( (\lambda_+, +\infty) \), we have that
\[
|f_c(x) - t_i| = |f_c(x) - f_c(\theta_i)| \geq |f_c(x) - f_c(x \pm \tilde{c} \kappa_x)| \geq c \kappa_x^{1/2} \geq c' \left( n^\varepsilon \phi_n + n^{-1/2+\varepsilon} \kappa_x^{-1/4} \right).
\]

(4.21)
for some constants $c, c' > 0$, where we used (4.11) in the third step. On the other hand, suppose $\theta_i \in [x - \tilde{c}\kappa_x, x + \tilde{c}\kappa_x]$, in which case we have that $\theta_i - \lambda_x \sim \kappa_x$. By (4.10), we have $\kappa_x \sim \theta_i - \lambda_+ \sim \Delta_i^2$. Then, using (4.11) and the definition of $I_i(t)$, we get that for $x \notin I_i(t)$,

$$|f_c(x) - t_i| = |f_c(x) - f_c(\theta_i)| \geq c\Delta_i^{-1} \left(n^\xi\phi_n \Delta_i + n^{-1/2+\varepsilon}\Delta_i^{1/2}\right) \geq c' \left(n^\xi\phi_n + n^{-1/2+\varepsilon}\kappa_x^{-1/4}\right),$$

for some constants $c, c' > 0$. This concludes (4.21) and hence Lemma 4.2.

**Step 2:** Before giving the general proof, for heuristics, we consider an easy case where the $t_i$'s are independent of $n$ and satisfy that

$$t_1 > t_2 > \cdots > t_{r_+} > \lambda_+.$$  

(4.22)

We claim that each $I_i(t)$, $1 \leq i \leq r_+$, contains precisely one eigenvalue of $C_{X,Y}$. Fix any $1 \leq i \leq r_+$, we choose a small $n$-independent positively oriented closed contour $\Gamma \subset \mathbb{C}/[0, \lambda_+]$ that encloses $\theta_i$ but no other points of the set $\{\theta_i : 1 \leq i \leq n\}$. Define two functions

$$f_1(z) := \det (f_c(z)I_r - \text{diag} (t_1, \cdots, t_r)),
\quad f_2(z) := \det (f_c(z)I_r - \text{diag} (t_1, \cdots, t_r) + \mathcal{E}'(z)),$$

(4.23)

where $\mathcal{E}'$ is defined in (4.17). The functions $f_1, f_2$ are holomorphic on and inside $\Gamma$ when $n$ is sufficiently large, because $\Gamma$ does not enclose any pole of $G(z)$ by (4.15). Moreover, by the construction of $\Gamma$, the function $f_1$ has precisely one zero inside $\Gamma$ at $\theta_i$. By (4.11), we have

$$\min_{z \in \Gamma} |f_1(z)| \geq 1, \quad \max_{z \in \Gamma} |f_1(z) - f_2(z)| = o(1).$$

The claim then follows from Rouché’s theorem.

**Step 3:** In order to extend the argument in Step 2 to an arbitrary $n$-dependent configuration $t$, we need to deal with the case where some of the intervals $I_i$ and $I_j$, $i \neq j$, have non-empty overlaps. For any constant $\varepsilon > 0$, we denote $r_\varepsilon := |\mathcal{O}_\varepsilon|$. In this step, we prove the following claim for the first $r_{2\varepsilon}$ eigenvalues.

**Claim 4.3.** On event $\Xi$, the estimate (4.17) holds for $i \in \mathcal{O}_{3\varepsilon}$.

**Proof.** Let $\mathcal{B}$ denote the finest partition of $\{1, \cdots, r_+\}$ in the sense that $i$ and $j$ belong to the same block of $\mathcal{B}$ whenever $I_i \cap I_j \neq \varnothing$. We now fix any $1 \leq i \leq r_{2\varepsilon}$ and denote by $B_i$ the block of $\mathcal{B}$ that contains $i$. Our first task is to estimate $\theta_{j-1} - \theta_j$ for $j, j - 1 \in B_i$. We claim that there exists a constant $C_1 > 0$ such that

$$\theta_{j-1} - \theta_j \leq C_1 \left(n^\xi\phi_n \Delta_j + n^{-1/2+\varepsilon}\Delta_j^{1/2}\right), \quad \text{if } j \in B_i \text{ and } j - 1 \in B_i.$$  

(4.24)

First, we assume that $j \in \mathcal{O}_{3\varepsilon}$. We pick any $x \in I_j \cap I_{j-1}$ such that $\theta_j \leq x \leq \theta_{j-1}$. Then, using (4.14) and (4.11), we obtain that

$$|f_c(x) - t_j| = |f_c(x) - f_c(\theta_j)| \leq C \left(n^\xi\phi_n + n^{-1/2+\varepsilon}\Delta_j^{-1/2}\right) \ll \Delta_j,$$

since $\Delta_j \geq n^3\phi_n + n^{-1/3+3\varepsilon}$ for $j \in \mathcal{O}_{3\varepsilon}$. Thus, we get that $|f_c(x) - t_c| = (1 + o(1))\Delta_j$. Similarly, we can show that $|f_c(x) - t_c| = (1 + o(1))\Delta_{j-1}$. This gives (4.24) due to the choice of $x$ and the definition of $I_i$ and $I_{j-1}$. In addition, we also get that

$$\Delta_j = (1 + o(1))\Delta_{j-1}, \quad \text{if } j \in B_i \text{ and } j - 1 \in B_i.$$  

(4.25)
It remains to verify that \( j \in \mathcal{O}_{3\epsilon} \) for all \( j \in B_i \). Let \( j_0 \) be the smallest integer such that \( \theta_{j_0} \notin B_i \). Since \( |B_i| \leq r \), by (4.24) we have that
\[
\theta_{j_0 - 1} > \theta_i - C \left( n^\epsilon \phi_n \Delta_i + n^{-1/2+\epsilon} \Delta_i^{1/2} \right)
\]
for some constant \( C > 0 \). Then, using \( i \in \mathcal{O}_{4\epsilon}, j_0 \notin \mathcal{O}_{3\epsilon} \) and (4.10), we can check that
\[
\theta_{j_0 - 1} - \theta_{j_0} > \left( n^\epsilon \phi_n \Delta_{j_0 - 1} + n^{-1/2+\epsilon} \Delta_{j_0 - 1}^{1/2} \right) + \left( n^\epsilon \phi_n \Delta_{j_0} + n^{-1/2+\epsilon} \Delta_{j_0}^{1/2} \right),
\]
which contradicts the definition of \( B_i \). This concludes Claim 4.3 by (4.26).

Now, with (4.24), (4.25) and (4.26), we have that
\[
d_i := \text{diam} \left( \bigcup_{j \in B_i} I_j \right) \leq C_{\frac{r}{r}} \left( n^\epsilon \phi_n \Delta_i + n^{-1/2+\epsilon} \Delta_i^{1/2} \right),
\]
for some constant \( C_r > 0 \) depending on \( r \) and \( C_1 \) only. On the other hand, by (4.10) we have that
\[
\theta_i - \lambda_+ - d_i \geq c \Delta_i^2 - C \left( n^\epsilon \phi_n \Delta_i + n^{-1/2+\epsilon} \Delta_i^{1/2} \right) \geq n^{2\epsilon} \phi_n^2 + n^{-2/3+3\epsilon},
\]
where we used \( \Delta_i \geq n^{4\epsilon} \phi_n + n^{-1/3+4\epsilon} \) for \( i \in \mathcal{O}_{4\epsilon} \) in the second step. Hence, there is a gap between the right edge of \( I_0 \) and the left edge of \( \bigcup_{j \in B_i} I_j \).

Let \( x_i \) and \( y_i \) be the left and right end points of the interval \( \bigcup_{j \in B_i} I_j \). Then, we pick the contour
\[
\Gamma_i := \{ z = x_i + iy : -d_i \leq y \leq d_i \} \cup \{ z = y_i + iy : -d_i \leq y \leq d_i \} \cup \{ z = E \pm id_i : x_i \leq E \leq y_i \},
\]
which lies in the half plane on the right of \( I_0 \), and only includes \( \theta_i \)'s with \( j \in B_i \) but no other points of the set \( \{ \theta_i : 1 \leq i \leq r_+ \} \). We again consider the functions \( f_1 \) and \( f_2 \) in (4.23). We know that \( f_1(z) \) has exactly \( |B_i| \) eigenvalues at \( \theta_j, j \in B_i \). Moreover, with the arguments in Lemma 4.2 one can show that
\[
\|E(z)\| = o(1) \quad \text{for} \quad z \in \Gamma_i, \quad \text{where} \quad E(z) := \left[ f_c(z) \Gamma - \text{diag}(t_1, \ldots, t_r) \right]^{-1} E'(z).
\]
Thus, we have
\[
|f_2(z) - f_1(z)| = |f_1(z)| |\text{det} \left( 1 + E(z) \right) - 1| < |f_1(z)| \quad \text{for} \quad z \in \Gamma_i.
\]
By Rouché’s theorem, \( f_2(z) \) has exactly \( |B_i| \) eigenvalues in \( \bigcup_{j \in B_i} I_j \). Together with Lemma 4.2 and a simple eigenvalue counting argument, we get that \( \tilde{\lambda}_i \in \bigcup_{j \in B_i} I_j \), and hence
\[
|\tilde{\lambda}_i - \theta_i| \leq d_i, \quad i \in \mathcal{O}_{3\epsilon}.
\]
This concludes Claim 4.3 by (4.26).

**Step 4:** Finally, we consider the eigenvalues \( \tilde{\lambda}_i \) with \( i \notin \mathcal{O}_{3\epsilon} \). First, by (4.15) and (4.12), we have that
\[
\tilde{\lambda}_i \geq \lambda_+ - n^{-2/3+\epsilon}, \quad i \leq \omega,
\]
which verifies the lower bound in (4.18). For the upper bound, we consider the intervals in (4.19) and
\[
\hat{I}_0 := \left[ 0, \lambda_+ + \tilde{C}_1 \left( n^{8\epsilon} \phi_n^2 + n^{-2/3+12\epsilon} \right) \right],
\]
where \( \tilde{C}_1 \) is a constant.
for a constant $\tilde{C}_1 > 0$. Then, we define a partition $\mathcal{B}$ as in Step 3, where $B_0$ is the block of $\mathcal{B}$ that contains $i$. With the same arguments as in the proof of Claim 4.3, we can prove that

$$\hat{I}_0 \cup \left( \bigcup_{j \in B_0} I_j \right) \subset \left[ 0, \lambda_\star + C_2 \left( n^{8\epsilon} \phi_n^2 + n^{-2/3 + 12\epsilon} \right) \right]$$

(4.28)

for a large enough constant $C_2 > 0$. Moreover, for any $j \notin B_0$, we have that $j \in \mathcal{O}_{4\epsilon}$ by (4.20) as long as $\tilde{C}_1$ is chosen large enough. Thus, using Lemma 4.2, the result of Step 3 and a simple eigenvalue counting argument, we get that

$$\hat{\lambda}_i \in \hat{I}_0 \cup \left( \bigcup_{j \in B_1} I_j \right), \quad i \notin \mathcal{O}_{4\epsilon}.$$ 

This concludes the upper bound in (4.28) by (4.28), and hence completes the proof of Theorem 2.9. \hfill \Box

### A Proof of Theorem 2.11

To conclude Theorem 2.11, we claim that it suffices to prove the following eigenvalue sticking estimate:

$$|\hat{\lambda}_i + r_+ - \lambda^b_i| < n^{-1} \alpha^{-1}_\star,$$

(A.1)

where $\lambda^b_1 \geq \lambda^b_2 \geq \cdots \geq \lambda^b_q$ denote the eigenvalues of $\mathcal{C}_{XY}$. In fact, this estimate shows that the non-outlier eigenvalues of $\mathcal{C}_{XY}$ with non-trivial $A$ and $B$ stick to those of $\mathcal{C}^b_{XY}$ with $A = 0$ and the same $B$. On the other hand, notice that $\mathcal{C}^b_{XY}$ has no outlier eigenvalues, because its PCC matrix is zero. Hence, as a special case of (A.1), we also know that $\lambda^b_i$ stick to the eigenvalues $\lambda_i$ of $\mathcal{C}_{XY}$ with $A = 0$ and $B = 0$. Together with (A.1), it gives the eigenvalue sticking estimate (2.25).

In the proof of (A.1), we will need to use the following eigenvalue rigidity estimate for $\lambda^b_i$. When $B = 0$, it reduces to (2.20) in Lemma 2.8.

**Lemma A.1.** Suppose the assumptions of Theorem 2.2 hold. Then, we have the following eigenvalue rigidity estimate: for any constant $\delta > 0$ and all $1 \leq i \leq (1 - \delta)q$,

$$|\lambda^b_i - \gamma_i| < i^{-1/3} n^{-2/3}.$$  

(A.2)

Another tool for the proof of (A.1) is the anisotropic local law for $G^b(z)$, which can be derived easily from the local law, Theorem 3.7, for $G(z)$ by using the approximate isometry condition (4.28) and the following Woodbury matrix identity: for $\mathcal{A}, \mathcal{S}, \mathcal{B}, \mathcal{T}$ of conformable dimensions,

$$(\mathcal{A} + \mathcal{S} \mathcal{B} \mathcal{T})^{-1} = \mathcal{A}^{-1} - \mathcal{A}^{-1} \mathcal{S} (\mathcal{B}^{-1} + \mathcal{T} \mathcal{A}^{-1} \mathcal{S})^{-1} \mathcal{T} \mathcal{A}^{-1}.$$  

(A.3)

We define

$$\Pi^b(z) := \Pi(z) - \Pi(z) \begin{pmatrix} \mathbf{U}_b & 0 \\ 0 & \mathbf{E}_b \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_2 m^{-1}_2 (z) \Sigma_b \mathcal{M}_b & 0 & 0 \\ 0 & 0 & \mathcal{M}_b & 0 \\ 0 & \Sigma_b \mathcal{M}_b & 0 & \Sigma_b \mathcal{M}_b \end{pmatrix} \begin{pmatrix} \mathbf{U}_b^\top & 0 \\ 0 & \mathbf{E}_b^\top \end{pmatrix} \Pi(z),$$

where

$$\mathcal{M}_b := \frac{\Sigma_b}{1 + \Sigma_b^2}, \quad \mathbf{U}_b := \begin{pmatrix} 0 \\ 0 \\ (\mathbf{u}_1^b, \cdots, \mathbf{u}_q^b) \end{pmatrix}, \quad \mathbf{E}_b := \begin{pmatrix} 0 \\ 0 \\ (Z^\top \mathbf{v}_1^b, \cdots, Z^\top \mathbf{v}_q^b) \end{pmatrix}.$$  

(A.4)
Lemma A.2 (Anisotropic local law for $G^b$). Suppose the assumptions of Theorem 2.7 hold. Fix any constant $\varepsilon > 0$ and unit vectors $u, v \in \mathbb{C}^2$ that are independent of $X$ and $Y$. We have that uniformly for $z \in S(\varepsilon)$ (recall (3.39)),

$$\left| G^b_{uv}(z) - \Pi^b_{uv}(z) \right| < \psi_n + \phi_n + \Psi(z),$$

and uniformly for $z \in D_{\text{out}}(\varepsilon)$ (recall (3.44)),

$$\left| G^b_{uv}(z) - \Pi^b_{uv}(z) \right| < \psi_n + \phi_n + n^{-1/2}(\kappa + \eta)^{-1/4}.$$

Moreover, (A.5) and (A.2) together imply that for any constant $\delta > 0$,

$$\max_{1 \leq k \leq (1-\delta)q} \left\{ \left| \langle u_1, S_{xx}^{-1/2} \xi_k^b \rangle \right|^2 + \left| \langle u_2, (S_{yy}^b)^{-1/2} \xi_k^b \rangle \right|^2 \right\} < n^{-1},$$

(A.7)

$$\max_{1 \leq k \leq (1-\delta)q} \left\{ \left| \langle u_3, X^\top S_{xx}^{-1/2} \xi_k^b \rangle \right|^2 + \left| \langle u_4, Y^\top (S_{yy}^b)^{-1/2} \xi_k^b \rangle \right|^2 \right\} < n^{-1},$$

(A.8)

where $\{\xi_k^b\}_{k=1}^p$ are the left and right singular vectors of $H^b$ (recall Definition 3.2), respectively, and $u_1, v_2 \in \mathbb{C}_{\Sigma^b}$ are unit vectors independent of $X$ and $Y$.

Proof. Using (A.3), we can write $G^b(z)$ in (3.10) as

$$G^b = G - G \begin{pmatrix} U_b^0 & 0 \\ 0 & E_b \end{pmatrix} \begin{pmatrix} 0 & D_b^{-1} \\ D_b^{-1} & 0 \end{pmatrix} \begin{pmatrix} U_b^0 & 0 \\ 0 & E_b \end{pmatrix}^{-1} \begin{pmatrix} U_b^0 & 0 \\ 0 & E_b \end{pmatrix} G,$$

(A.9)

where $D_b := \begin{pmatrix} 0 & 0 \\ 0 & \Sigma_b \end{pmatrix}$. Since $D_b^{-1}$ is not well-defined, the above expression should be understood as

$$\begin{pmatrix} 0 & D_b^{-1} \\ D_b^{-1} & 0 \end{pmatrix} = 1 + \begin{pmatrix} 0 & D_b \\ D_b & 0 \end{pmatrix} \begin{pmatrix} U_b^0 & 0 \\ 0 & E_b \end{pmatrix}^{-1} \begin{pmatrix} U_b^0 & 0 \\ 0 & E_b \end{pmatrix} \begin{pmatrix} 0 & D_b \\ D_b & 0 \end{pmatrix}.$$

Combining (A.9) with Theorem 3.7, Theorem 3.9 and (4.43), we can conclude (A.5) and (A.6). The estimates (A.7) and (A.8) follow from (A.5) and (A.2) as in the proof of Lemma 3.8 where the details can be found in the proof of Lemma 3.9 in [27].

As in Section 4 from equation (3.9), we can derive a similar equation as (4.7). More precisely, suppose $\lambda$ is not an eigenvalue of $C_{X\lambda}$ and the following local law holds for $G^b(\lambda)$:

$$\left( U_a^0 \\ E_a^0 \right) \begin{pmatrix} U_a^0 & 0 \\ 0 & E_a \end{pmatrix} G^b(\lambda) \begin{pmatrix} U_a^0 & 0 \\ 0 & E_a \end{pmatrix}^{-1} - \Pi^b(\lambda) = O(\Phi_n) \quad \text{with high probability},$$

where $\Phi_n$ is a deterministic parameter satisfying $0 < \Phi_n \leq n^{-c}$ for a small constant $c > 0$, and

$$\Pi^b(\lambda) := \begin{bmatrix} (c^{-1} m_{1c}(\lambda)) I_r & 0 \\ 0 & (m_{3c}(\lambda)) I_r \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k^2}{m_{1c}(\lambda)} M_r \frac{\Sigma^2}{4 + \Sigma^2} M_r^\top \end{bmatrix}.$$
Then, \( \lambda \) is an eigenvalue of \( C_{XY} \) if and only if
\[
\det \left( f_c(\lambda) I_r - \text{diag}(t_1, \ldots, t_r) + \mathcal{E}_r(\lambda) \right) = 0,
\]
where \( \mathcal{E}_r \) is an error term satisfying \( \|\mathcal{E}_r\| \leq n^\varepsilon \psi_n + \Phi_n \) with high probability for any small constant \( \varepsilon > 0 \). Moreover, similar to (4.12), we have the following eigenvalue interlacing,

\[
\bar{\lambda}_i \in [\lambda_i^b, \lambda_i^b],
\]
where we adopt the convention that \( \lambda_i^b = 1 \) if \( i < 1 \) and \( \lambda_i^b = 0 \) if \( i > q \). This is the main reason why we need to prove (A.1) instead of proving (2.25) directly: our proof of (A.1) will use crucially the rank-\( r \) interlacing in (A.12), while the rank-2r interlacing in (4.12) is not strong enough to yield (2.25) directly.

Proof of Theorem 2.11: For simplicity, in the following proof, we abbreviate \( \phi_n + \psi_n \) as \( \phi_n \). As a byproduct of the proof of Lemma A.1 in Section D, we obtain an averaged local law for \( m^b(x) \) in equation (D.3) below. By (D.3), (A.2), Theorem 2.9, Lemma 3.3, (4.3) and Lemma A.2, for any small constants \( \varepsilon, \tilde{\varepsilon}, \delta > 0 \) and fixed integer \( \omega \in \mathbb{N} \), we can choose a high-probability event \( \Xi \) on which the following estimates hold:

\[
|n^{b}(z) - m_c(z)| \leq \frac{n^{\varepsilon/4}}{n^\eta}, \quad \text{for } z \in S(\varepsilon, \tilde{\varepsilon});
\]

(A.12)

\[
|\lambda_i^b - \lambda_+| \leq n^{-2/3 + \varepsilon/2}, \quad \text{for } 1 \leq i \leq \omega;
\]

(A.13)

\[
|\lambda_i^b - \gamma_i| \leq i^{-1/3} n^{-2/3 + \varepsilon/2}, \quad \text{for } 1 \leq i \leq (1 - \delta)q;
\]

(A.14)

\[
|\lambda_i - \theta_i| \leq n^\varepsilon \phi_n \Delta_i + n^{-1/2 + \varepsilon} \Delta_i^{1/2}, \quad \text{for } 1 \leq i \leq r_+;
\]

(A.15)

\[
- n^{-2/3 + \varepsilon/2} \leq \lambda_i - \lambda_+ \leq n^{-2/3 + \varepsilon/2}, \quad \text{for } r_+ + 1 \leq i \leq \omega;
\]

(A.16)

\[
c_0 \leq \min \{ \lambda_p(S_{xx}), \lambda_q(S_{yy}) \} \leq \max \{ \lambda_1(S_{xx}), \lambda_1(S_{yy}) \} \leq c_0^{-1};
\]

(A.17)

\[
\| ZZ^T - I_r \| \leq n^{-\delta/4} \phi_n;
\]

(A.18)

\[
\left\| \begin{pmatrix} U_a^T & 0 \\ 0 & E_a \end{pmatrix} G^b(z) \begin{pmatrix} U_a & 0 \\ 0 & E_a \end{pmatrix} - \Pi^b_r(z) \right\| \leq n^{\varepsilon/2} (\phi_n + \Psi(z)), \quad \text{for } z \in S(\varepsilon);
\]

(A.19)

\[
\max_{1 \leq k \leq (1 - \delta)q} \left\{ \left| \langle u_1, S_{xx}^{-1/2} \xi_k^b \rangle \right|^2 + \left| \langle u_2, (S_{yy}^b)^{-1/2} \xi_k^b \rangle \right|^2 \right\} \leq n^{-1 + \varepsilon/20};
\]

(A.20)

\[
\max_{1 \leq k \leq (1 - \delta)q} \left\{ \left| \langle u_3, X^T S_{xx}^{-1/2} \xi_k^b \rangle \right|^2 + \left| \langle u_4, Y^T (S_{yy}^b)^{-1/2} \xi_k^b \rangle \right|^2 \right\} \leq n^{-1 + \varepsilon/20}.
\]

(A.22)

Here, \( c_0 \) is a small enough constant, and the vectors \( u_\alpha \in \mathbb{C}^{2m}, \alpha = 1, 2, 3, 4 \), belong to a set of vectors that is independent of \( X \) and \( Y \), has cardinality \( n^{O(1)} \), and includes all the unit vectors that will be used in the proof. Again, the randomness of \( X, Y \) and \( Z \) only comes into play to ensure that \( \Xi \) holds with high probability, and the rest of the proof will be entirely deterministic on the event \( \Xi \).

Step 1: As in the proof of Theorem 2.9 we first find a permissible region. For any \( i \), we define the set

\[
\Omega_i := \left\{ x \in [\lambda_{i+r_+}, \lambda_+ + n^{2\varepsilon} \phi_n^2 + n^{-2/3 + 2\varepsilon}] : \text{dist} \left( x, \text{Spec}(C_{XY}^b) \right) > n^{-1 + \varepsilon} \alpha_+^{-1} \right\},
\]

(A.23)

where \( \text{Spec}(C_{XY}^b) \) stands for the eigenvalue spectrum of \( C_{XY}^b \).
Lemma A.3. There exists a constant $C_1 > 0$ such that for $\alpha_+ \geq C_1 (n^{7} \phi_n + n^{-1/3 + \varepsilon})$ and $i \leq n^{-1/2} \alpha_+^3$, the set $\Omega_i$ contains no eigenvalue of $\mathcal{C}_{XY}$.

Proof. In the proof, we always use the following spectral parameter

$$z_x = x + i \eta_x, \quad \text{with} \quad \eta_x := n^{-1+\varepsilon} \alpha_+^{-1}. \tag{A.24}$$

Suppose $x \in \Omega_i$. We first claim that for any deterministic unit vectors $u, v \in \Gamma$,

$$|G^b_{u,v}(z_x) - G^b_{u,v}(x)| \leq C n^{\varepsilon/20} \text{Im } m^b(z_x) + C n^{\varepsilon/20} \eta_x, \quad x \in \Omega_i. \tag{A.25}$$

We use a similar argument as in the proof of Theorem 3.9. To illustrate the idea, for deterministic unit vectors

$$v = (v_1^T, v_2^T, v_3^T, v_4^T)^T, \quad u = (u_1^T, u_2^T, u_3^T, u_4^T)^T$$

with $u, v \in \mathbb{C}^{\mathcal{I}_n}$, we calculate $G^b_{u_1, v_1}(z_x) - G^b_{u_1, v_1}(x)$ as an example. As in (3.50), we have

$$|G^b_{u_1, v_1}(z_x) - G^b_{u_1, v_1}(x)| \leq \sum_{k \leq (1-\delta)q} \eta_x \langle v_1, S_{xx}^{-1/2} \xi_k \rangle \langle u_1, S_{xx}^{-1/2} \xi_k \rangle + \eta_x \sum_{k > (1-\delta)q} \langle v_1, S_{xx}^{-1/2} \xi_k \rangle \langle u_1, S_{xx}^{-1/2} \xi_k \rangle \leq n^{-1+\varepsilon/20} \left( \sum_{k=1}^q \frac{\eta_x}{(\lambda_k - x)^2 + \eta_x^2} \right) + \eta_x \leq n^{\varepsilon/20} \text{Im } m^b(z_x) + \eta_x,$$

where in the second step we used (A.17), (A.21) and $|\lambda_k - x| \geq \eta_x$ for $x \in \Omega_i$, and in the last step we used the spectral decomposition of $m^b(z_x)$. The proofs for the rest of the cases $G^b_{u_a, v_2}(z_x) - G^b_{u_a, v_2}(x)$, $\alpha, \beta = 1, 2, 3, 4$, are similar, so we omit the details.

Recall that $x \in \Omega_i$ is an eigenvalue of $\mathcal{C}_{X,Y}$ if and only if (A.10) holds, where $\mathcal{E}_r$ satisfies the following bound by (A.25), (A.12) and (A.19):

$$\|\mathcal{E}_r(x)\| \leq C \left( n^{\varepsilon/20} \text{Im } m_c(z_x) + n^{\varepsilon/20} \eta_x + n^{\varepsilon/2} \phi_n + n^{\varepsilon/2} \Psi(z_x) + \frac{n^{3\varepsilon/10}}{n \eta_x} \right)$$

for some constant $C > 0$. With (3.32) and the definition of $\Psi(z_x)$ in (3.38), we can further bound that

$$\|\mathcal{E}_r(x)\| \leq C' \left( n^{\varepsilon/2} \phi_n + n^{\varepsilon/2} \text{Im } m_c(z_x) + \frac{n^{\varepsilon/2}}{n \eta_x} \right)$$

for some constant $C' > 0$. Now, to prove Lemma A.3 it suffices to show that for any $1 \leq j \leq r$,

$$|f_c(x) - t_j| > C' \left( n^{\varepsilon/2} \phi_n + n^{\varepsilon/2} \text{Im } m_c(z_x) + \frac{n^{\varepsilon/2}}{n \eta_x} \right), \quad x \in \Omega_i. \tag{A.27}$$

Since $i \leq n^{1-2\varepsilon} \alpha_+^3$, by (A.14) we have that for $x \in \Omega_i$,

$$- \left( n^{2\varepsilon} \phi_n^2 + n^{-2/3 + 2\varepsilon} \right) \leq \lambda_+ - x \leq (i/n)^{2/3} + i^{-1/3} n^{-2/3 + \varepsilon/2} \leq n^{-4\varepsilon/3} \alpha_+^2, \tag{A.28}$$

where we used $\gamma_i \sim (i/n)^{2/3}$ and $\alpha_+ \geq C_1 n^{-1/3 + \varepsilon}$. Then, by (4.3), we have

$$|f_c(x) - t_j| = |f_c(x) - f_c(\lambda_+)| \leq C n^{-2\varepsilon/3} \alpha_+, \quad x \in \Omega_i \cap \{ x : x \leq \lambda_+ \},$$

27
\[ |f_c(x) - t_c| = |f_c(x) - f_c(\lambda_+)| \leq C \left( n^{\varepsilon} \phi_n + n^{-1/3 + \varepsilon} \right), \quad x \in \Omega_1 \cap \{ x : x > \lambda_+ \}, \]

for a constant \( C > 0 \) that does not depend on \( C_1 \). Hence, as long as \( C_1 \) is large enough, we have

\[ |f_c(x) - t_c| \leq \frac{1}{4} \alpha_+ \quad \Rightarrow \quad |f_c(x) - t_j| \geq \frac{3}{4} \alpha_+, \quad (A.29) \]

where we used the definition of \( \alpha_+ \) in (2.22). On the other hand, with (3.32), (A.24) and (A.28), we can verify that

\[ C' \left( n^{\varepsilon/2} \phi_n + n^{\varepsilon/2} \text{Im} m_c(z_x) + \frac{n^{\varepsilon/2}}{m_n} \right) \leq C'' \left( n^{\varepsilon/2} \phi_n + n^{\varepsilon/2} \sqrt{\kappa_x + \eta_x} + n^{-\varepsilon/2} \alpha_+ \right) \ll \alpha_+ \]

for \( x \in \Omega_1 \cap \{ x : x \leq \lambda_+ \} \), and

\[ C' \left( n^{\varepsilon/2} \phi_n + n^{\varepsilon/2} \text{Im} m_c(z_x) + \frac{n^{\varepsilon/2}}{m_n} \right) \leq C'' \left( n^{\varepsilon/2} \phi_n + n^{\varepsilon/2} \eta_x \sqrt{\kappa_x + \eta_x} + n^{-\varepsilon/2} \alpha_+ \right) \ll \alpha_+ \]

for \( x \in \Omega_1 \cap \{ x : x > \lambda_+ \} \). Together with (A.29), these two estimates imply that (A.27) holds. This concludes the proof. \( \Box \)

**Step 2:** In this step, we perform a counting argument for a special case as in the following lemma. We postpone its proof until we finish the proof of Theorem 2.11.

**Lemma A.4.** Given \( 0 \leq r_+ \leq r \), we choose a matrix \( A = A(0) \) of rank \( r_+ \) such that the eigenvalues configuration \( t \equiv t(0) := (t_1, t_2, \cdots, t_r) \) of the PCC matrix satisfies that

\[ (t_{r_+} - t_c) \wedge \min_{1 \leq i \leq r_+ - 1} (t_i - t_{i+1}) \geq 1, \quad t_{r+1} = \cdots = t_r = 0. \quad (A.30) \]

Then, for \( i \leq n^{1-4\varepsilon} \alpha_+^3(0) \), we have

\[ |\tilde{\lambda}_{i+r_+} - \lambda_i^b| \leq n^{-1+2\varepsilon} \alpha_+^{-1}(0), \quad (A.31) \]

where \( \alpha_+(0) \) is defined as in (2.22) for \( t(0) \). (The meaning of the argument 0 will be clear in Step 3.)

**Step 3:** In this step, we employ a continuity argument as in [33, Section 6.5] and [17, Section S.4.2]. We choose a continuous (n-dependent) path \( A(s) \) for \( 0 \leq s \leq 1 \), such that \( A(1) = A \) is the matrix in Theorem 2.11 and \( A(0) \) gives an eigenvalue configuration \( t(0) \) satisfying (A.30). Correspondingly, we have a continuous path of the configuration \( t(s) \) and the sample eigenvalues \( \{ \lambda_i(s) \}_{i=1}^n \). We can choose \( A(s) \) such that

\[ \inf_{s \in [0, 1]} \alpha_+(s) \geq \alpha_+ \equiv \alpha_+(1), \]

where \( \alpha_+(s) \) is defined as in (2.22) for the eigenvalue configuration \( t(s) \).

In this step we consider the case where \( \alpha_+ \geq C_1(n^{\varepsilon} \phi_n + n^{-1/3 + \varepsilon}) \) and \( i \leq n^{1-4\varepsilon} \alpha_+^3. \) Without loss of generality, we rename \( \alpha_+ := \inf_{s \in [0, 1]} \alpha_+(s) \). Define

\[ \tilde{\Gamma}_0 := \left\{ x \in [0, \lambda_+ + n^{2\varepsilon} \phi_n^2 + n^{-2/3+2\varepsilon}] : \text{dist} \ (x, \text{Spec}(C_{\chi}^b)) \leq n^{-1+\varepsilon} \alpha_+^{-1} \right\}. \]

Note that \( \tilde{\Gamma}_0 \) is a union of connected intervals. Due to the interlacing (A.11), we have

\[ \lambda_{i+r}^b \leq \tilde{\lambda}_i(s) \leq \lambda_{i-r}^b, \quad s \in [0, 1]. \quad (A.32) \]
By Lemma A.3 and Lemma A.4, we know
\[ |\tilde{\lambda}_{i+r_+}(0) - \lambda_b^i| \leq n^{-1+2\varepsilon} \alpha_+^{-1}, \]
and
\[ \text{dist} \left( \tilde{\lambda}_{i+r_+}(s), \text{Spec}(C_b^b) \right) \leq n^{-1+\varepsilon} \alpha_+^{-1}, \quad s \in [0, 1]. \] (A.33)

In addition, by continuity of eigenvalues with respect to \( s \), we know that \( \tilde{\lambda}_{i+r_+}(s) \) is in the same connected component of \( \tilde{I}_0 \) as \( \tilde{\lambda}_{i+r_+}(0) \). For any \( i \), let \( B_i \) be the set of \( j \) such that \( \lambda_b^i \) and \( \lambda_b^j \) are in the same connected component of \( \tilde{I}_0 \). Then, we conclude that
\[ \tilde{\lambda}_{i+r_+}(s) \in \bigcup_{j \in B_i \cap [i+r_+-1]} [\lambda_b^i - n^{-1+2\varepsilon} \alpha_+^{-1}, \lambda_b^j + n^{-1+2\varepsilon} \alpha_+^{-1}], \]
which gives that
\[ |\tilde{\lambda}_{i+r_+}(s) - \lambda_b^i| \leq 2rn^{-1+2\varepsilon} \alpha_+^{-1}, \quad s \in [0, 1]. \] (A.34)

**Step 4:** Finally, we consider the case where \( \alpha_+ < C_1(n^r \phi_n + n^{-1/3+\varepsilon}) \) or \( i > n^{-4\varepsilon} \alpha_+^3 \). Suppose first that \( \alpha_+ < C_1(n^r \phi_n + n^{-1/3+\varepsilon}) \). Then, by the assumption of Theorem 2.11 if \( \varepsilon \) is small enough such that \( \varepsilon < \varepsilon_0 \), we must have
\[ \phi_n \leq n^{-1/3}, \quad \alpha_+ \leq n^{-1/3+\varepsilon}. \] (A.35)

Now, using (A.35), (A.11), (A.14) and (A.16), we find that
\[ |\tilde{\lambda}_{i+r_+} - \lambda_b^i| \leq n^{-2/3+\varepsilon} \leq n^{-1+2\varepsilon} \alpha_+^{-1}. \]

On the other hand, suppose \( i > n^{-4\varepsilon} \alpha_+^3 \). If \( i \leq r \), then we have \( \alpha_+ \leq n^{-1/3+4\varepsilon/3} \), and with the same argument as above, we get
\[ |\tilde{\lambda}_{i+r_+} - \lambda_b^i| \leq n^{-2/3+\varepsilon} \leq n^{-1+3\varepsilon} \alpha_+^{-1}. \]

Otherwise, using (A.11) and (A.14), we get
\[ |\tilde{\lambda}_{i+r_+} - \lambda_b^i| \leq i^{-1/3}n^{-2/3+\varepsilon/2} \leq n^{-1+2\varepsilon} \alpha_+^{-1}. \]

Combining the above three estimates with (A.34), we conclude (A.1), since \( \varepsilon > 0 \) can be arbitrarily small. \( \Box \)

For the proof of Lemma A.1 we shall use an argument that extends the one in the proof of Proposition 6.8 in [33]. However, the proof in [7] Section 7 may also work, where the authors proved essentially the same result but only for \( i \leq \bar{\varepsilon} \) with \( \bar{\varepsilon} \) being a fixed integer.

**Proof of Lemma A.4** Note that in this lemma, we have \( \alpha_+ \equiv \alpha_+(0) \sim 1 \). In the first step, we group together the eigenvalues \( \lambda_i \) that are close to each other. More precisely, let \( \mathcal{B} = \{ B_k \} \) be the finest partition of \( \{1, \cdots, q\} \) such that \( i < j \) belong to the same block of \( \mathcal{B} \) if
\[ |\lambda_b^i - \lambda_b^j| \leq n^{-1+7\varepsilon/6} \alpha_+^{-1}. \]

Note that each block \( B_k \) of \( \mathcal{B} \) consists of a sequence of consecutive integers. We order the blocks in the descending order, that is, if \( k < l \) then \( \lambda_b^k > \lambda_b^l \) for all \( i_k \in B_k \) and \( i_l \in B_l \).

We first derive a bound on the sizes of the blocks. We define \( k^* \) such that \( n_0 := \lfloor n^{-4\varepsilon} \alpha_+^3 \rfloor \in B_{k^*} \). For any \( k \leq k^* \), we take \( i < j \) such that \( i \) and \( j \) both belong to the block \( B_k \). Then, by (A.11) and (A.14), we have that for some constants \( c, C > 0 \),
\[ c \left( (j/n)^{2/3} - (i/n)^{2/3} \right) - C_i^{-1/3}n^{-2/3+\varepsilon/2} \leq \lambda_b^i - \lambda_b^j \leq C(j-i)n^{-1+7\varepsilon/6} \alpha_+^{-1}. \]
Now, using \(j^{2/3} - i^{2/3} \geq j^{-1/3}(j - i)\), we obtain that
\[
\left(j^{-1/3} - Cn^{-1/3+\varepsilon/6}\alpha_n^{-1}\right)(j - i) \leq Ci^{-1/3}n^{\varepsilon/2}.
\]

From this estimate, we conclude that if \(i\) and \(j\) satisfy
\[
1 \leq i \leq j \leq n^{1-15\varepsilon/4},
\]
then we have
\[
\alpha = C(j/i)^{1/3}n^{\varepsilon/2}.
\]

Now, we claim that
\[
j - i \leq C(j/i)^{1/3}n^{\varepsilon/2}.
\]

and for any given \(i_k \in B_k\),
\[
|\lambda_i^b - \gamma_{i_k}| \leq i^{-1/3}n^{-2/3+\varepsilon}
\]
for all \(i \in B_k\).

To prove (A.38) and (A.39), we denote \(\alpha_k := \max_{i \in B_k} \gamma_{i_k}\) and \(\beta_k := \min_{i \in B_k} \gamma_{i_k}\). If \(i \in B_k\) satisfies \(i \geq \alpha_k/2\), then (A.37) gives that \(\alpha_k - i \leq Cn^{\varepsilon/2}\), with which we obtain that
\[
|\gamma_i - \gamma_{\alpha_k}| \leq C\lambda_i^{-1/3}n^{-2/3+\varepsilon/2}.
\]

On the other hand, if \(i \in B_k\) satisfies \(i \leq \alpha_k/2\), then (A.37) gives that \(\alpha_k - i \leq Cn^{\varepsilon/4}\). Thus,
\[
|\gamma_i - \gamma_{\alpha_k}| \leq Cn^{-2/3+\varepsilon/2} \leq C\lambda_i^{-1/3}n^{-2/3+\varepsilon/4}.
\]

Combining the above two estimates with (A.14), we obtain that
\[
|\lambda_i^b - \gamma_{i_k}| \leq |\lambda_i^b - \gamma_i| + |\gamma_i - \gamma_{\alpha_k}| + |\gamma_{\alpha_k} - \gamma_{i_k}| \leq C\lambda_i^{-1/3}n^{-2/3+3\varepsilon/4} \leq i^{-1/3}n^{-2/3+\varepsilon}.
\]

From the above proof, we see that (A.38) and (A.39) as long as (A.30) holds. We still need to prove (A.36) for \(i, j \in B_k\) with \(k \leq k^*\). In fact, if there is \(j \in B_k\) such that \(j \geq n^{1-15\varepsilon/4}\), then we can find \(j' \in B_k\) such that \(n^{\varepsilon} \leq j' - n_0 \leq 2n^{\varepsilon}\), which contradicts (A.37) and (A.38).

We are now ready to complete the proof. For any \(1 \leq k \leq k^*\), we denote
\[
ak := \min_{i \in B_k} \lambda_i = \lambda_{\alpha_k}, \quad bk := \max_{i \in B_k} \lambda_i = \lambda_{\beta_k}.
\]

We introduce a continuous path as
\[
x_{s}^{k} := (1 - s)(ak - \delta_n/3) + s(bk + \delta_n/3), \quad s \in [0, 1],
\]
where \(\delta_n := n^{-1+7\varepsilon/6}\alpha_n^{-1}\). The interval \([x_{0}^{k}, x_{1}^{k}]\) contains precisely the eigenvalues of \(C_{XY}^k\) that are in \(B_k\), and the endpoints \(x_{0}^{k}\) and \(x_{1}^{k}\) are at distances at least \(\delta_n/3\) from any eigenvalue of \(C_{XY}^k\). Then, we have the following proposition. We postpone its proof until we finish the proof of Lemma A.4.

**Proposition A.5.** Almost surely, there are at least \(|B_k|\) eigenvalues of \(C_{XY}\) in \([x_{0}^{k}, x_{1}^{k}]\).

Here, “almost surely” in the statement is due to the assumption (A.11): in the proof we discard a measure zero non-generic event. We postpone the proof of Proposition A.5 until we complete the proof of Lemma A.4.

We now use a standard interlacing argument to show that \(C_{XY}\) has at most \(|B_k|\) eigenvalues in \([x_{0}^{k}, x_{1}^{k}]\). By (A.11), there are at most \(|B_1| + r_+\) eigenvalues of \(C_{XY}\) in \([x_{0}^{k}, \infty)\) (recall that the rank of \(A(0)\) is \(r_+\)). Moreover, with the argument in Section 4, we can prove that (A.15) holds in the case \(A = A(0)\), i.e. there
are exactly \( r_k \) outliers. Then, together with Proposition \( \ref{prop:approximation} \), it gives that there are exactly \(|B_1|\) eigenvalues of \( C_{X,Y} \) in \([x_0, x_1]\). Repeating this argument, we can show that \( C_{X,Y} \) has exactly \(|B_k|\) eigenvalues in \([x_k, x_{k+1}]\) for all \( k = 2, \ldots, k^* \). Moreover, using \( \ref{lem:spectral-decomposition} \), we find that for any \( i \in B_k \),

\[
\sup \left\{ |x - \lambda_i^b| : \alpha \in [x_0^k, x_1^k] \right\} \leq C n^{3\varepsilon/4} \left( n^{-1+7\varepsilon/6} \alpha_+^{-1} \right) \leq n^{-1+2\varepsilon/3} \alpha_+^{-1},
\]

which concludes Lemma \( \ref{lem:spectral-decomposition} \).

Finally, we give the proof of Proposition \( \ref{prop:approximation} \).

**Proof of Proposition \( \ref{prop:approximation} \).** For the spectral decomposition of \( R^b(z) \) (which takes a similar form as \( \ref{eq:approximation} \)), we define

\[
P_{B_k} R^b(z) := \sum_{l \in B_k} \frac{1}{\lambda_l^b - z} \begin{pmatrix}
\xi_l^b (c_l^b) \top & -z^{-1/2} (\lambda_l^b)^{1/2} c_l^b (c_l^b) \top \\
-z^{-1/2} (\lambda_l^b)^{1/2} c_l^b (\xi_l^b) \top & (\xi_l^b) \top
\end{pmatrix},
\]

(A.42)

and \( P_{B_k} R^b(z) := R^b(z) - P_{B_k} R^b(z) \). We define \( P_{B_k} G^b \) by replacing \( R \) and \( Y \) with \( P_{B_k} R \) and \( Y \) in \( \ref{eq:approximation} \), \( \ref{eq:approximation2} \) and \( \ref{eq:approximation3} \), that is,

\[
P_{B_k} G^b := \left( P_{B_k} G^b_{L} P_{B_k} G^b_{LR} P_{B_k} G^b_{R} \right),
\]

where

\[
P_{B_k} G^b_L := \begin{pmatrix}
(X X \top)^{-1/2} & 0 \\
0 & (Y Y \top)^{-1/2}
\end{pmatrix} P_{B_k} R^b \begin{pmatrix}
(X X \top)^{-1/2} & 0 \\
0 & (Y Y \top)^{-1/2}
\end{pmatrix},
\]

(A.43)

\[
P_{B_k} G^b_R := \begin{pmatrix}
\frac{z I_n}{z^2 I_n} & \frac{z^2}{z^2} I_n \\
\frac{z^2}{z^2} I_n & \frac{z^2}{z^2} I_n
\end{pmatrix} + \begin{pmatrix}
\frac{z I_n}{z^2 I_n} & \frac{z^2}{z^2} I_n \\
\frac{z^2}{z^2} I_n & \frac{z^2}{z^2} I_n
\end{pmatrix} \begin{pmatrix} X \top & 0 \end{pmatrix} P_{B_k} G^b_L \begin{pmatrix} X \top & 0 \end{pmatrix}
\frac{z I_n}{z^2 I_n} \frac{z^2}{z^2} I_n
\frac{z^2}{z^2} I_n \frac{z^2}{z^2} I_n,
\]

(A.44)

\[
P_{B_k} G^b_{LR}(z) := -P_{B_k} G^b_L(z) \begin{pmatrix} X \top & 0 \end{pmatrix} \begin{pmatrix} z^2 I_n & \frac{z^2}{z^2} I_n \frac{z^2}{z^2} I_n \frac{z^2}{z^2} I_n, \quad P_{B_k} G^b_{RL}(z) := P_{B_k} G^b_R(z) \top.
\]

(A.45)

Then, we define \( P_{B_k} G^b(z) := G^b(z) - P_{B_k} G^b(z) \). Given any \( x \in [x_0^k, x_1^k] \), we denote \( z_x := x + i \eta_x \) with \( \eta_x := n^{-1+7\varepsilon/6} \alpha_+^{-1} \). We claim that

\[
\left| \begin{pmatrix} U_a^\top & 0 \\
0 & E_a^\top
\end{pmatrix} \left[ P_{B_k} G^b(z_x) - P_{B_k} G^b(z) \right] \begin{pmatrix} U_a^\top & 0 \\
0 & E_a^\top
\end{pmatrix} \right| \leq n^{\varepsilon/20} \Im m^b(z_x) + n^{\varepsilon/20} \eta_x.
\]

(A.46)

The proof is very similar to that for \( \ref{eq:approximation-error} \). For example, for deterministic unit vectors \( v \) and \( u \) in \( \ref{eq:approximation-error} \), using \( \ref{eq:approximation-error-bound} \), \( \ref{eq:approximation-error-bound2} \) and \( \ref{eq:approximation-error-bound3} \), we get

\[
\left| P_{B_k} G^b_{u,v}(z_x) - P_{B_k} G^b_{u,v}(x) \right| \\
\leq \sum_{l \notin B_k, l \leq (1-\delta) q} \frac{\eta_x |\langle v_1, S_x^{-1/2} \xi_l^b \rangle| |\langle u_1, S_x^{-1/2} \xi_l^b \rangle| + \eta_x \sum_{l > (1-\delta) q} |\langle v_1, S_x^{-1/2} \xi_l^b \rangle| |\langle u_1, S_x^{-1/2} \xi_l^b \rangle|}
\leq n^{-1+\varepsilon/20} \left( (\lambda_l^b - x)^2 + \eta_x^2 \right) + \eta_x \leq n^{\varepsilon/20} \Im m^b(z_x) + \eta_x,
\]

where in the second step we used \( |\lambda_l^b - x| \geq \eta_x \) for \( l \notin B_k \). The proofs for the rest of the cases \( P_{B_k} G^b_{u,v}(z_x) - P_{B_k} G^b_{u,v}(x) \), \( \alpha, \beta = 1, 2, 3, 4 \), are similar, so we omit the details.

Next, we claim that

\[
|P_{B_k} G^b_{u,v}(z_x)| + |P_{B_k} G^b_{u,v}(x_0^k)| \leq n^{-\varepsilon/3}.
\]

(A.47)
For example, for the $z_x$ term, we have
\[
|P_{B_k}G_{u^b_{\alpha},v_\beta}(z_x)| = \left| \sum_{l \in B_k} \frac{(u_1, S_{XX}^{1/2} - \xi_l^b)(\xi_l^b S_{XX}^{1/2}, v_\beta)}{\lambda_l^b - z_x} \right| \lesssim C n^{3/4} \eta_x^{-1} n^{-1+3/20} \ll n^{-3/4},
\]
where we used (A.21) and (A.38) in the second step. The proofs for the rest of the cases $P_{B_k}G_{u^b_{\alpha},v_\beta}(z_x)$, $\alpha, \beta = 1, 2, 3, 4$, are similar. For $z = x_0^b$, the proof is the same except that we need to use $|\lambda_l^b - x_0^b| \lesssim n^{-1+7\varepsilon/6} \alpha_x^{-1}$ for $l \in B_k$.

Now, we remove the zero singular values of $A$ and redefine that
\[
\Sigma_a := \text{diag}(a_1, \ldots, a_{\varepsilon_0}), \quad U_a = (u_1^a, \ldots, u_{\varepsilon_0}^a), \quad E_a = (Z^\top v_1^a, \ldots, Z^\top v_{\varepsilon_0}^a).
\]

Inspired by (3.9), for $x \notin \text{spec}(C^b_{\chi_y})$, we define
\[
\mathcal{M}(x) := \left( \begin{array}{cc} 0 & \Sigma_a^{-1} \\ \Sigma_a & 0 \end{array} \right) + \left( \begin{array}{cc} U_a^\top & 0 \\ 0 & E_a^\top \end{array} \right) \left( \begin{array}{cc} G_l^b(x) & G_{13}^b(x) \\ G_{21}^b(x) & G_{33}^b(x) \end{array} \right) \left( \begin{array}{cc} U_a & 0 \\ 0 & E_a \end{array} \right),
\]
where we recall that $G_l^b$ is the $L_\alpha \times L_\beta$ block of $G^b$ (cf. Definition (3.2)), and we have used $G_{l,\beta}^b$ to denote the $L_\alpha \times L_\beta$ block of $G^b$. We know that almost surely, $x \in \text{Spec}(C_{\chi_y}) \setminus \text{Spec}(C^b_{\chi_y})$ if and only if $\mathcal{M}(x)$ is singular. To simplify notations, we denote
\[
[G^b(x)]_{1,3} := \left( \begin{array}{c} G_{l1}^b(x) \\ G_{21}^b(x) \\ G_{31}^b(x) \end{array} \right).
\]

Now, using (A.12), (A.19), (A.46) and (A.47), we obtain that
\[
\begin{align*}
\mathcal{M}(x) &= \left( \begin{array}{cc} 0 & \Sigma_a^{-1} \\ \Sigma_a & 0 \end{array} \right) + \left( \begin{array}{cc} U_a^\top & 0 \\ 0 & E_a^\top \end{array} \right) [P_{B_k}G^b(x)]_{1,3} \left( \begin{array}{cc} U_a & 0 \\ 0 & E_a \end{array} \right) \\
&+ \left( \begin{array}{cc} U_a^\top & 0 \\ 0 & E_a^\top \end{array} \right) [P_{B_k}G^b(x) - C^b(z_x)]_{1,3} \left( \begin{array}{cc} U_a & 0 \\ 0 & E_a \end{array} \right) \\
&= \left( \begin{array}{cc} 0 & \Sigma_a^{-1} \\ \Sigma_a & 0 \end{array} \right) + \left( \begin{array}{cc} U_a^\top & 0 \\ 0 & E_a^\top \end{array} \right) [P_{B_k}G^b(x)]_{1,3} \left( \begin{array}{cc} U_a & 0 \\ 0 & E_a \end{array} \right) + [\Pi_{l}^b(z_x)]_{1,3} + R_0(x) \\
&= \left( \begin{array}{cc} 0 & \Sigma_a^{-1} \\ \Sigma_a & 0 \end{array} \right) + \left( \begin{array}{cc} U_a^\top & 0 \\ 0 & E_a^\top \end{array} \right) [P_{B_k}G^b(x)]_{1,3} \left( \begin{array}{cc} U_a & 0 \\ 0 & E_a \end{array} \right) + [\Pi_{l}^b(\lambda_+)]_{1,3} + R(x),
\end{align*}
\]
where
\[
[\Pi_{l}^b(\lambda_+)]_{1,3} := \left( c_{l1}^{-1} m_{l_1}(z_x) I_r - \frac{\kappa_x(z_0)}{m_{l_3}(z_x)} M_r \right) \Sigma_{l_3}^2 \left( I_r + M_r \right),
\]
and $R_0$ and $R_1$ are two matrices satisfying that
\[
\|R_0(x)\| = O\left(n^{\varepsilon/20} \eta_x + n^{\varepsilon/20} \text{Im} m_{l_3}(z_x) + n^{\varepsilon/2} \Psi(z_x) + n^{\varepsilon/2} \phi_n + n^{-\varepsilon/3}\right) = O\left(n^{-\varepsilon/3}\right),
\]
\[
\|R(x)\| = \|R_0(x)\| + O\left(\sqrt{\kappa_x} + \eta_x\right) = O\left(n^{-\varepsilon/3}\right).
\]
In bounding the $\|R_0(x)\|$ and $\|R(x)\|$, we also used Lemma 3.21 and 3.38, and that
\[
\kappa_x \leq \max \{ |\lambda_+ - x_0^b|, |\lambda_- - x_0^b| \} \lesssim (n^{1-15\varepsilon/4}/n)^{2/3} + n^{-2/3+\varepsilon} + n^{-1+7\varepsilon/6} \alpha_x^{-1} \ll n^{-\varepsilon/3},
\]
where in the second step we used (A.36) and (A.39) and the definition (A.41). Moreover, $R(x)$ is real symmetric (because all the other terms in the line (A.48) are real symmetric) and continuous in $x$ on the extended real line $\mathbb{R}$.
The rest of the proof follows from a continuity argument, which is exactly the same as the one in \[33\] Section 6.4. Instead of writing down all the details, we shall give an almost rigorous argument to show how equation (A.43) implies Proposition (A.3).

First, we claim that \(M(x)\) has some negative singular values when \(x = x_0^k\). By (A.47), equation (A.48) gives that

\[
M(x_0^k) = \left( \frac{\Sigma_0}{\Sigma_{a-1}} \right)_{1,3} + O(n^{1/3}).
\]

Let \(v_i\) be an eigenvector of

\[
\frac{\Sigma_0}{(1 + \Sigma_{a-1}^2)^{1/2}} M_a \Sigma_0^2 \frac{\Sigma_0}{(1 + \Sigma_{a-1}^2)^{1/2}}
\]

with eigenvalue \(t_i\). Then, for \(u_i = \left( \frac{m_3 c(\lambda_+)(1 + \Sigma_{a-1}^2)^{-1/2} v_i}{\Sigma_0(1 + \Sigma_{a-1}^2)^{-1/2}} \right)\), we can verify that

\[
u_i^T M(x_0^k) u_i = \frac{h^2(\lambda_+)}{m_4 c(\lambda_+)} (f_c(\lambda_+) - t_i) \| v_i \|^2 + O(n^{-1/3}) \| v_i \|^2 < 0,
\]

where we used \(m_4 c(\lambda_+)^2 > 0\), \(t_i > t_c = f_c(\lambda_+)\) and \(t_i - t_c \sim 1\) for \(1 \leq i \leq r_+\).

Next, we claim that for \(l \in B_k\), almost surely, \(M(x)\) is positive definite when \(x \uparrow \lambda_l^+\) and negative definite when \(x \downarrow \lambda_l^-\). To see why, we pick any unit vector \(v = (v_1^T, v_2^T)^T\), \(v_1, v_2 \in \mathbb{R}^{r_+}\), and denote \(\tilde{v} = (v_1^T, 0, v_2^T, 0)^T\). Then,

\[
v^T M(x) v = O(1) + \tilde{v}^T \left( U_a \begin{bmatrix} 0 & 0 \\ 0 & E_a \end{bmatrix} P_{B_k} \begin{bmatrix} G^b_1(x) \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & E_a \end{bmatrix} \right) \tilde{v}
\]

\[
= O(1) + \tilde{w}^T \begin{bmatrix} P_{B_k} G^b_1(x) \\ -P_{B_k} G^b_2(x) \\ P_{B_k} G^b_1(x) \end{bmatrix} \tilde{w} = O(1) + w^T P_{B_k} R^b(x) w,
\]

where in the second step we used (A.44) and (A.45) with

\[
\tilde{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := \begin{bmatrix} I_{1+p+q} \\ X \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & x^{1/2} I_n \\ x^{1/2} I_n & x I_n \end{bmatrix} \begin{bmatrix} U_a \\ 0 \end{bmatrix} \tilde{v}, \quad w_1, w_2 \in \mathbb{R}^{p+q},
\]

and in the third step we used (A.43) with

\[
w := \begin{bmatrix} S_{xx}^{-1/2} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ (S_{yy}^b)^{-1/2} \end{bmatrix} (w_1 - w_2).
\]

Using the spectral decomposition (A.42), we can write

\[
P_{B_k} R^b(x) = \frac{1}{2} \sum_{l \in B_k} \frac{x^{-1/2}}{(\lambda_l^b)^{1/2} - x^{1/2}} \begin{bmatrix} -\xi_l^b & \xi_l^b \end{bmatrix} \begin{bmatrix} -\xi_l^b \\ \xi_l^b \end{bmatrix}^T - \frac{x^{-1/2}}{(\lambda_l^b)^{1/2} + x^{1/2}} \begin{bmatrix} \xi_l^b \\ -\xi_l^b \end{bmatrix} \begin{bmatrix} \xi_l^b \\ -\xi_l^b \end{bmatrix}^T.
\]

In particular, it has poles at \(x = \lambda_l^b\) for \(l \in B_k\). Combining (A.49) and (A.50), we conclude the claim.

With the above two claims and a simple continuity argument, we see that there exists \(x \in (\lambda_0^k, \lambda_{\alpha_k}^k)\) (recall (A.10)) such that \(M(x)\) is singular, and for any \(l, l - 1 \in B_k\), there exists \(x \in (\lambda_l^k, \lambda_{l-1}^k)\) such that \(M(x)\) is singular. This gives at least \(|B_k|\) eigenvalues of \(C_{XY}\) inside \([x_0^k, x_1^k]\) and hence completes the proof. Writing down a rigorous continuity argument involves discussions on some non-generic measure zero events, and we refer the reader to \[33\] Section 6.4 for more details. \(\square\)
B Proof of Corollary 2.13

For \( \phi_n \) and \( \psi_n \) in (2.27), we define the truncated matrices \( \tilde{X}, \tilde{Y} \) and \( \tilde{Z} \) with entries

\[
\tilde{x}_{ij} := x_{ij} \mathbf{1}_{|x_{ij}| \leq \phi_n n^z}, \quad \tilde{y}_{ij} := y_{ij} \mathbf{1}_{|y_{ij}| \leq \phi_n n^z}, \quad \tilde{z}_{ij} := z_{ij} \mathbf{1}_{|z_{ij}| \leq \psi_n n^z},
\]

for a sufficiently small constant \( \varepsilon > 0 \). Combining the moment conditions in (2.26) with Markov’s inequality, we obtain that

\[
P(\tilde{X} \neq X, \tilde{Y} \neq Y, \tilde{Z} \neq Z) = O\left(n^{-\alpha} + n^{-b\varepsilon}\right)
\]

for a sufficiently small constant \( \varepsilon > 0 \). Combining the moment conditions in (2.26) with Markov’s inequality, we obtain that

\[
P(\tilde{X} \neq X, \tilde{Y} \neq Y, \tilde{Z} \neq Z) = O\left(n^{-\alpha} + n^{-b\varepsilon}\right)
\]

by a simple union bound. Using (2.26) and integration by parts, we can also check that

\[
\mathbb{E}|x_{ij}| \mathbf{1}_{|x_{ij}| > \phi_n n^z} \leq n^{-2-\varepsilon}, \quad \mathbb{E}|x_{ij}|^2 \mathbf{1}_{|x_{ij}| > \phi_n n^z} \leq n^{-2-\varepsilon}.
\]

For example, for the first estimate in (B.2), we have that

\[
\mathbb{E}|\tilde{x}_{ij}| = \int_0^{\phi_n n^z} \mathbb{P}(|x_{ij}| > \phi_n n^z) \, ds + \int_{\phi_n n^z}^\infty \mathbb{P}(|x_{ij}| > s) \, ds
\]

\[
\lesssim \int_0^{\phi_n n^z} (n^{1/2+\varepsilon} \phi_n)^{-a} \, ds + \int_{\phi_n n^z}^\infty (\sqrt{n}s)^{-a} \, ds \leq n^{-\frac{1}{2} - 2\varepsilon - \frac{1}{a} - (a-1)\varepsilon} \leq n^{-2-\varepsilon},
\]

where in the third step we used (2.26) and Markov’s inequality, and in the last step we used \( a > 4 \). The second estimate of (B.2) can be proved in a similar way. Note that (B.2) implies

\[
|\mathbb{E}\tilde{x}_{ij}| \leq n^{-2-\varepsilon}, \quad \mathbb{E}|\tilde{x}_{ij}|^2 = n^{-1} + O(n^{-2-\varepsilon}).
\]

Moreover, we trivially have

\[
\mathbb{E}|\tilde{x}_{ij}|^3 \leq \mathbb{E}|x_{ij}|^3 = O(n^{-3/2}), \quad \mathbb{E}|\tilde{x}_{ij}|^4 \leq \mathbb{E}|x_{ij}|^4 = O(n^{-2}).
\]

Similar estimates also hold for the entries of \( \tilde{Y} \). Hence, \( \tilde{X} \) and \( \tilde{Y} \) are random matrices satisfying Assumption 2.1 (i) and condition 2.19. For \( \tilde{Z} \), using (2.26) and a similar argument, we can check that

\[
|\mathbb{E}\tilde{z}_{ij}| \leq n^{-1-\varepsilon}, \quad \mathbb{E}|\tilde{z}_{ij}|^2 = n^{-1} + O(n^{-1-(b-2)\varepsilon}).
\]

Hence, \( Z \) is a random matrix satisfying Assumption 2.1 (ii). Now, combining (B.1) with Theorem 2.9, we conclude (2.28). Combining (B.1) with Theorem 2.11, we obtain that

\[
|\tilde{x}_{r+i} - \lambda_i^b| < n^{-1}\alpha_i^{-1} \leq n^{-2/3-\varepsilon_0}, \quad 1 \leq i \leq k,
\]

for \( \alpha_i \) satisfying (2.29). Together with Lemma 2.8 it concludes (2.30).

C Proof of Theorem 2.14

For the proof of Theorem 2.14, we adopt a similar argument as that for Theorem 2.7 in [47], that is, we decompose \((X, Y, Z)\) (in distribution) into well-behaved random matrices \((X^{*}, Y^{*}, Z^{*})\) with bounded support \( n^{-\varepsilon} \) plus a perturbation matrix. However, our setting here is more complicated. We now define the precise
defined through $\Omega$ for any event $\Omega$ and its rank is at most $C$.2

Remark a negligible deterministic matrix, see (C.2) below. The matrix $X$ can be decomposed of $X$. First, we introduce a cutoff on the matrix entries of $X$ at the level $n^{-\varepsilon}$ for a sufficiently small constant $\varepsilon > 0$:

$$\alpha_n^{(1)} := \mathbb{P}\left( |\hat{z}_{11}| > n^{1/2 - \varepsilon} \right), \quad \beta_n^{(1)} := \mathbb{E}\left[ 1 \left( |\hat{z}_{11}| > n^{1/2 - \varepsilon} \right) \right].$$

Using (2.31), we can check with integration by parts that for any small constant $\delta > 0$,

$$\alpha_n^{(1)} \leq \delta n^{-2 + 4\varepsilon}, \quad |\beta_n^{(1)}| \leq \delta n^{-3/2 + 3\varepsilon}. \quad \text{(C.1)}$$

Now, we define independent random variables $\hat{z}_{ij}^+, \hat{z}_{ij}^-, c_{ij}^{(1)}, 1 \leq i \leq p, 1 \leq j \leq n$, as follows.

**Definition C.1.** We define $\hat{z}_{ij}^+$ as a random variable that has law $\rho_n^{(1)}$ defined through

$$\rho_n^{(1)}(\Omega) = \frac{1}{1 - \alpha_n^{(1)}} \int \mathbf{1} \left( x + \frac{\beta_n^{(1)}}{1 - \alpha_n^{(1)}} \in \Omega \right) \mathbf{1} \left( |x| \leq n^{1/2 - \varepsilon} \right) \rho^{(1)}(dx)$$

for any event $\Omega$, where $\rho^{(1)}(dx)$ is the law of $\hat{z}_{ij}$. We define $\hat{z}_{ij}^-$ as a random variable that has law $\rho_n^{(1)}$ defined through

$$\rho_n^{(1)}(\Omega) = \frac{1}{1 - \alpha_n^{(1)}} \int \mathbf{1} \left( x + \frac{\beta_n^{(1)}}{1 - \alpha_n^{(1)}} \in \Omega \right) \mathbf{1} \left( |x| > n^{1/2 - \varepsilon} \right) \rho^{(1)}(dx)$$

for any event $\Omega$. We define $c_{ij}^{(1)}$ as a Bernoulli $0$-$1$ random variable with

$$\mathbb{P}(c_{ij}^{(1)} = 1) = \alpha_n^{(1)}, \quad \mathbb{P}(c_{ij}^{(1)} = 0) = 1 - \alpha_n^{(1)}.$$

Finally, let $X^s$, $X^l$ and $X^c$ be independent random matrices with entries

$$x_{ij}^s = n^{-1/2} \hat{z}_{ij}^+, \quad x_{ij}^l = n^{-1/2} \hat{z}_{ij}^-, \quad x_{ij}^c = c_{ij}^{(1)}.$$

**Remark C.2.** With the above definition, $X$ can be decomposed as $X^s + (X^l - X^s) \circ X^c$ in distribution up to a negligible deterministic matrix, see (C.2) below. The matrix $X^c$ gives the locations of the nonzero entries and its rank is at most $n^5$ with high probability, see (C.9) below. The matrix $X^l$ contains the large entries above the cutoff, but the tail condition (2.31) guarantees that the sizes of these entries are of order $o(1)$ in probability, see (C.13). Hence, the perturbation is of low rank and has small signal strengths. We expect that, as in the famous BBP transition [5], the effect of this weak perturbation on the largest few eigenvalues is negligible.

In Definition C.1 $\rho_n^{(1)}$ and $\rho_n^{(1)}$ are defined in a way such that $\hat{z}_{ij}^+$ and $\hat{z}_{ij}^-$ are both centered random variables. We can easily check that

$$x_{ij} \overset{d}{=} x_{ij}^s \left( 1 - x_{ij}^c \right) + x_{ij}^l x_{ij}^c - \frac{1}{\sqrt{n}} \frac{\beta_n^{(1)}}{1 - \alpha_n^{(1)}}, \quad \text{(C.2)}$$

where $\overset{d}{=}$ means “equal in distribution”. Similarly, we decompose $Y$ as

$$y_{ij} \overset{d}{=} y_{ij}^s \left( 1 - y_{ij}^c \right) + y_{ij}^l y_{ij}^c - \frac{1}{\sqrt{n}} \frac{\beta_n^{(2)}}{1 - \alpha_n^{(2)}}, \quad \text{(C.3)}$$
where the entries $y_{ij}^{(e)}$, $y_{ij}^{(t)}$, and $y_{ij}^{(c)}$ of the independent random matrices $Y^e$, $Y^t$ and $Y^c$ are defined in similar ways using

$$
\alpha_n^{(2)} := \mathbb{P}\left( |\hat{y}_{11}| > n^{1/2-\epsilon}\right), \quad \beta_n^{(2)} := \mathbb{E}\left[ 1\left( |\hat{y}_{11}| > n^{1/2-\epsilon}\right)\right].
$$

Notice that the deterministic matrix $M_1$ with entries

$$(M_1)_{ij} = -\frac{1}{\sqrt{n}} \frac{\beta_n^{(1)}}{1 - \alpha_n^{(1)}}, \quad 1 \leq i \leq p, \ 1 \leq j \leq n,$$

has operator norm $O(n^{-1+3\epsilon})$, which, by Weyl’s inequality, perturbs the singular values of $X$ at most by $O(n^{-1+3\epsilon})$. Similarly, we will also omit the constant term in $O(\epsilon)$. Finally, we decompose $Z$ as $Z = Z^s + Z^t$, where

$$
\beta_n^{(3)} := \mathbb{E}\left[ 1\left( |Z_{ij}| > n^{-\epsilon}\right)Z_{ij}\right].
$$

Using (2.2) and integration by parts, one can check that

$$
\beta_n^{(3)} = O(n^{-1+\epsilon}).
$$

Using (2.2) and integration by parts, one can check that

$$
\beta_n^{(3)} = O(n^{-1+\epsilon}).
$$

The deterministic vector $(\beta_n^{(3)}, \ldots, \beta_n^{(3)})^T \in \mathbb{R}^n$ has Euclidean norm $O(n^{-1+2\epsilon})$ and is also negligible for the following proof. Hence, for simplicity of the proof, we will also omit it.

With (2.3) and integration by parts, we can obtain that

$$
\mathbb{E}[\hat{Z}_{11}^2 = 0, \mathbb{E}|\hat{Z}_{11}|^2 = 1 - O(n^{-1+2\epsilon}), \mathbb{E}|\hat{Z}_{11}|^3 = O(1), \mathbb{E}|\hat{Z}_{11}|^4 = O(\log n). (C.4)
$$

Similar estimates hold for $\hat{Z}_{11}^t$. Hence, $X_1 := (\mathbb{E}|\hat{Z}_{11}^s|^2)^{-1/2}X^s$ and $Y_1 := (\mathbb{E}|\hat{Z}_{11}^t|^2)^{-1/2}Y^t$ are random matrices that satisfy the assumptions for $X$ and $Y$ in Lemma A.1, Theorem 2.9 and Theorem 2.11 with $\phi_n = O(n^{-\epsilon})$. Moreover, the small errors $O(n^{-1+2\epsilon})$ in $\mathbb{E}|\hat{Z}_{11}^s|^2$ and $\mathbb{E}|\hat{Z}_{11}^t|^2$ are negligible for our purpose. For $Z$, using $\lim_{n \to \infty} \mathbb{E}[|\hat{Z}_{11}|^2 1(|\hat{Z}_{11}| > t)] = 0$, we get that

$$
\mathbb{E}|\hat{Z}_{11}^s|^2 = 1 - o(1), \quad \mathbb{E}|\hat{Z}_{11}^t|^2 = o(1),
$$

where we have used notations $\hat{Z}_{11}^s := \sqrt{n}Z_{11}^s$ and $\hat{Z}_{11}^t := \sqrt{n}Z_{11}^t$. Then, $Z_1 := (\mathbb{E}|\hat{Z}_{11}^s|^2)^{-1/2}Z^s$ satisfies the assumptions for $Z$ in Lemma A.1, Theorem 2.9 and Theorem 2.11 with $\psi_n = O(n^{-\epsilon})$. Note that the scaling of $Z^s$ by $(\mathbb{E}|\hat{Z}_{11}^s|^2)^{-1/2}$ amounts to a rescaling of $A$ and $B$ by $(\mathbb{E}|\hat{Z}_{11}^s|^2)^{1/2}$, i.e.,

$$
A \to A_1 := (\mathbb{E}|\hat{Z}_{11}^s|^2)^{1/2}A, \quad B \to B_1 := (\mathbb{E}|\hat{Z}_{11}^s|^2)^{1/2}B,
$$

so that $A_1Z_1 = AZ^s$ and $B_1Z_1 = BZ^s$. In particular, we have that

$$
\text{the } t_i's \text{ in (2.14) are only perturbed by an amount of } o(1). \quad (C.5)
$$

Denote by $C_{XY}^s$ and $C_{XY}^t$ the SCC matrices obtained by replacing $(X, Y, Z)$ with $(X^s, Y^s, Z^s)$ in the corresponding definitions. Let $\lambda_i^s$ and $\lambda_i^t$ be their eigenvalues. Then, by Theorem 2.9 and (C.5), for any $1 \leq i \leq r_+$, we have that

$$
|\hat{\lambda}_i^s - \theta_i| = o(1) \text{ with high probability}, \quad (C.6)
$$

and by Lemma 2.8 we have that for all $s_1 \in \mathbb{R}$,

$$
\lim_{n \to \infty} \mathbb{P}\left( n^{2/3} \frac{\lambda_{r+1}}{c_TW} \leq s_1 \right) = \lim_{n \to \infty} \mathbb{P}^{GOE}\left( n^{2/3} (\lambda_{r+1} - 2) \leq s_1 \right). \quad (C.7)
$$
Moreover, applying Theorem 2.11 gives \( |\tilde{\lambda}_{1+r_+} - \lambda|^2 < n^{-1} \). Combining it with (C.7), we obtain that

\[
\lim_{n \to \infty} \mathbb{P} \left( n^{2/3} \frac{\tilde{\lambda}_{1+r_+} - \lambda_r}{c_{TW}} \leq s_1 \right) = \lim_{n \to \infty} \mathbb{P}^{GOE} ( n^{2/3} (\lambda_1 - 2) \leq s_1 ) . \tag{C.8}
\]

(For simplicity, we only consider the largest non-outlier eigenvalue. The extension to the case with multiple non-outlier eigenvalues is straightforward.) We write the right-hand sides of (C.2) and (C.3) as

\[
x^c_{ij} (1 - x^c_{ij}) + x^c_{ij} x^c_{ij} = x^c_{ij} + \Delta^{(1)} x^c_{ij}, \quad \Delta^{(1)} := x^c_{ij} - x^c_{ij},
\]

\[
y^c_{ij} (1 - y^c_{ij}) + y^c_{ij} y^c_{ij} = y^c_{ij} + \Delta^{(2)} y^c_{ij}, \quad \Delta^{(2)} := y^c_{ij} - y^c_{ij}.
\]

We define matrices

\[
\mathcal{E}^{(1)} := \{ (\Delta^{(1)} x^c_{ij} : 1 \leq i < p, 1 \leq j \leq n) \}, \quad \mathcal{E}^{(2)} := \{ (\Delta^{(2)} y^c_{ij} : 1 \leq i < q, 1 \leq j \leq n) \}.
\]

It suffices to show that the effect of \( \mathcal{E}^{(1)} \), \( \mathcal{E}^{(2)} \) and \( Z' \) on \( \tilde{\lambda}_i \), \( 1 \leq i \leq r_+ \) and \( \tilde{\lambda}_{r_+ + 1} \) is negligible.

Define the event

\[
\mathcal{A} := \{ \#(i, j) : x^c_{ij} = 1 \leq n^{5\epsilon} \} \cap \{ x^c_{ij} = x^c_{kl} = 1 \Rightarrow \{ i, j \} \text{ or } \{ i, j \} \cap \{ k, l \} = \emptyset \}.
\]

By a Chernoff bound, we get that

\[
\mathbb{P} \left( \{ \#(i, j) : x^c_{ij} = 1 \leq n^{5\epsilon} \} \right) \geq 1 - \exp(-n^\epsilon). \tag{C.9}
\]

If the number \( n_0 \) of the nonzero elements in \( X^c \) satisfies \( n_0 \leq n^{5\epsilon} \), then we can check that

\[
\mathbb{P} (\exists i < j \neq l \text{ or } i \neq k, j = l \text{ so that } x^c_{ij} = x^c_{kl} = 1 \Rightarrow \{ i, j \} \neq \{ k, l \} \cup \{ i, j \} \cap \{ k, l \} = \emptyset ) = O(n_0^2 / n). \tag{C.10}
\]

Combining the estimates (C.9) and (C.10), we get that

\[
\mathbb{P}(\mathcal{A}) \geq 1 - O(n^{-1+10\epsilon}). \tag{C.11}
\]

Similarly, for the event

\[
\mathcal{B} := \{ \#(i, j) : y^c_{ij} = 1 \leq n^{5\epsilon} \} \cap \{ y^c_{ij} = y^c_{kl} = 1 \Rightarrow \{ i, j \} \neq \{ k, l \} \cup \{ i, j \} \cap \{ k, l \} = \emptyset \},
\]

we have

\[
\mathbb{P}(\mathcal{B}) \geq 1 - O(n^{-1+10\epsilon}). \tag{C.12}
\]

On the other hand, using condition (2.31) and Markov’s inequality, we get

\[
\mathbb{P} \left( |\mathcal{E}^{(1)}_{ij} | \geq \omega \right) + \mathbb{P} \left( |\mathcal{E}^{(2)}_{ij} | \geq \omega \right) \leq \mathbb{P} \left( |\tilde{x}_{ij} | \geq \frac{\omega}{2} n^{1/2} \right) + \mathbb{P} \left( |\tilde{y}_{ij} | \geq \frac{\omega}{2} n^{1/2} \right) = o(n^{-2}),
\]

for any fixed constant \( \omega > 0 \). With a simple union bound, we get

\[
\mathbb{P} \left( \max_{ij} |\mathcal{E}^{(1)}_{ij} | \geq \omega \right) + \mathbb{P} \left( \max_{ij} |\mathcal{E}^{(2)}_{ij} | \geq \omega \right) = o(1). \tag{C.13}
\]

Define the event

\[
\mathcal{E}_1 := \left\{ \max_{ij} |\mathcal{E}^{(1)}_{ij} | \leq \omega \right\} \cap \left\{ \max_{ij} |\mathcal{E}^{(2)}_{ij} | \leq \omega \right\}.
\]
Combining (C.11), (C.12) and (C.13), we get
\[ \mathbb{P}(\mathcal{F} \cap \mathcal{B} \cap \mathcal{C}_1) = 1 - o(1). \]  
(C.14)

We also define the event
\[ \mathcal{C}_2 := \left\{ \|(Z^s)^\top Z^s - I_r\| \leq w, \|(Z^t)^\top Z^t\| \leq w^2, \|(Z^s)^\top Z^t\| \leq w \right\}. \]  
(C.15)

By the law of large numbers, we have \( \mathbb{P}(\mathcal{C}_2) = 1 - o(1) \).

Recalling (3.2), we only need to study the zeros of \( \det[\tilde{H}_1(\lambda)] \) on event \( \mathcal{F} \cap \mathcal{B} \cap \mathcal{C}_1 \cap \mathcal{C}_2 \). Here, we define \( \tilde{H}_i(\lambda), t \in [0, 1] \), as
\[ \tilde{H}_i(\lambda) := \tilde{H}^*(\lambda) + t \left[ \begin{array}{c c c}
0 & (\mathcal{E}^{(1)} + AZ^t) & 0 \\
(\mathcal{E}^{(2)} + BZ^t)^\top & 0 & \mathcal{E}^{(2)} + BZ^t \\
0 & (\mathcal{E}^{(2)} + BZ^t)^\top & 0
\end{array} \right], \]
where
\[ \tilde{H}^*(\lambda) := H^*(\lambda) + \left[ \begin{array}{c c c}
0 & (AZ^s)^\top & 0 \\
(BZ^s)^\top & 0 & BZ^s \\
0 & (BZ^s)^\top & 0
\end{array} \right], \]
with
\[ H^*(\lambda) := \left[ \begin{array}{c c c}
0 & \left( X^s \begin{array}{c}
0 \\
0
\end{array} \end{array} \right) & \left( X^s \begin{array}{c}
0 \\
0
\end{array} \right) \\
0 & \left( Y^s \begin{array}{c}
0 \\
0
\end{array} \right) & \left( Y^s \begin{array}{c}
0 \\
0
\end{array} \right) \\
0 & \left( Y^s \begin{array}{c}
\Lambda^{-1/2} I_n \\
\Lambda^{-1/2} I_n
\end{array} \right) & \left( \Lambda^{-1/2} I_n \right)^{-1}
\end{array} \right]. \]

We would like to extend (C.6) and (C.8) at \( t = 0 \) all the way to \( t = 1 \) using a continuity argument. Correspondingly, for any \( t \in [0, 1] \), we define the PCC matrix \( C_{\mathcal{X}\mathcal{Y}}(t) \) for \( \mathcal{X}(t) := X^s + t\mathcal{E}^{(1)} + A(Z^s + tZ^t) \) and \( \mathcal{Y}(t) := Y^s + t\mathcal{E}^{(2)} + B(Z^s + tZ^t) \), and we denote its eigenvalues by \( \tilde{\lambda}_i(t) \). Note that \( \tilde{\lambda}_i = \tilde{\lambda}_i(1) \) are the eigenvalues we are interested in, and the eigenvalues \( \tilde{\lambda}_i(0) \) satisfy (C.6) and (C.8). Moreover, \( \tilde{\lambda}_i(t) \) is continuous with respect to \( t \) on the extended real line \( \mathbb{R} \).

**Proof of (2.33).** For any \( 1 \leq i \leq r_+ \), we pick a sufficiently small constant \( \delta > 0 \) such that the following properties hold for large enough \( n \): (i) the interval \( J_i := [\theta_i - \delta, \theta_i + \delta] \) only contains \( \theta_j \)'s that converge to the same limit as \( \theta_i \) when \( n \to \infty \), (ii) \( J_i \) is away from all the other \( \theta_j \)'s at least by \( \delta \), and (iii) \( J_i \) is away from \( \lambda_+ \) at least by \( \delta \). By (C.6), we know \( \tilde{\lambda}_i(0) \in J_i \) with high probability. Now, for \( \mu := \theta_i \pm \delta \), we claim that
\[ \mathbb{P} \left( \det \tilde{H}_i(\mu) \neq 0 \text{ for all } 0 \leq t \leq 1 \right) = 1 - o(1). \]  
(C.16)

If (C.16) holds, then \( \mu \) is not an eigenvalue of \( C_{\mathcal{X}\mathcal{Y}}(t) \) for all \( t \in [0, 1] \) with probability \( 1 - o(1) \). By continuity of \( \tilde{\lambda}_i(t) \) with respect to \( t \), we have \( \tilde{\lambda}_i = \tilde{\lambda}_i(1) \in J_i \) with probability \( 1 - o(1) \), that is,
\[ \mathbb{P}(|\tilde{\lambda}_i - \theta_i| \leq \delta) = 1 - o(1). \]
This concludes (2.33) since \( \delta \) can be arbitrarily small.

For the proof of (C.16), we will condition on \( \mathcal{F} \cap \mathcal{B} \) and the event \( \mathcal{C}_{n_x, n_y} \) that \( X^c \) and \( Y^c \) have \( n_x \) and \( n_y \) nonzero entries with \( \max\{n_x, n_y\} \leq n^{5/8} \). Moreover, we assume that the indices of the \( n_x \) nonzero entries of \( X^c \) are \( (\sigma_x(1), \pi_x(1)), \ldots, (\sigma_x(n_x), \pi_x(n_x)) \), and the indices of the \( n_y \) nonzero entries of \( Y^c \) are
Here, we used $\sigma_x : \{1, \ldots, n_x\} \to \{1, \ldots, p\}$, $\pi_x : \{1, \ldots, n_x\} \to \{1, \ldots, n\}$, $\sigma_y : \{1, \ldots, n_y\} \to \{1, \ldots, q\}$ and $\pi_y : \{1, \ldots, n_y\} \to \{1, \ldots, n\}$ are all injective functions. Then, we can rewrite that

$\tilde{H}_t(\mu) = H^*(\mu) + O_t \begin{bmatrix} 0 & 0 & (D & 0 & tD_c) \end{bmatrix} O_t^T, \quad O_t := \begin{bmatrix} (U, F_1) & 0 \end{bmatrix}$,

where $D$ and $U$ have been defined in (3.5) and (3.6); $D_c := \begin{pmatrix} \Sigma_c^{(1)} & 0 \\ 0 & \Sigma_c^{(2)} \end{pmatrix}$ with

$\Sigma_c^{(1)} := \text{diag}(E_{\sigma_y(1)}^{(1)} \pi_y \pi_{y(a)}, \ldots, E_{\sigma_y(n_y)}^{(1)} \pi_y \pi_{y(n_y)}), \quad \Sigma_c^{(2)} := \text{diag}(E_{\sigma_y(1)}^{(2)} \pi_y \pi_{y(a)}, \ldots, E_{\sigma_y(n_y)}^{(2)} \pi_y \pi_{y(n_y)}):$

$E_t := \begin{bmatrix} (Z_t^T v_1^a, \ldots, Z_t^T v_r^a) & 0 \\ 0 & (Z_t^T v_1^b, \ldots, Z_t^T v_r^b) \end{bmatrix}$, with $Z_t := Z^* + tZ'$;

$F_1 := \begin{bmatrix} (e_{\sigma_x(1)}^{(p)} \pi_{x(a)}, \ldots, e_{\sigma_x(n_x)}^{(p)} \pi_{x(a)}) & 0 \\ 0 & (e_{\sigma_x(1)}^{(q)} \pi_{x(a)}, \ldots, e_{\sigma_x(n_x)}^{(q)} \pi_{x(a)}) \end{bmatrix}$;

$F_2 := \begin{bmatrix} (e_{\pi_x(1)}^{(n)} \pi_{x(a)}, \ldots, e_{\pi_x(n_x)}^{(n)} \pi_{x(a)}) & 0 \\ 0 & (e_{\pi_y(1)}^{(n)} \pi_{y(a)}, \ldots, e_{\pi_y(n_y)}^{(n)} \pi_{y(a)}) \end{bmatrix}$.

Here, we used $e_i^{(l)}$ to denote the standard unit vector along $i$-th coordinate in $\mathbb{R}^l$.

Applying the identity $\det(1 + AB) = \det(1 + BA)$, we obtain that

$\det \tilde{H}_t(\mu) = \det [G^*(\mu)] \cdot \det \left[ 1 + \tilde{F}_t(\mu) + \mathcal{E}_t(\mu) \right], \quad (C.17)$

where

$\tilde{F}_t(\mu) := \begin{bmatrix} 0 & 0 & (D & 0 & tD_c) \end{bmatrix} O_t^T \Pi(\mu) O_t,$

$\mathcal{E}_t(\mu) := \begin{bmatrix} 0 & 0 & (D & 0 & tD_c) \end{bmatrix} O_t^T [G^*(\mu) - \Pi(\mu)] O_t.$

Since $O_t$ is deterministic conditioning on $Z$, by Lemma 3.10, we have that (recall $C.15$)

$\mathbb{E} \left[ |O_t^T (G^*(\mu) - \Pi(\mu)) O_t|_{ij}^2 \right] E_{n_x n_y, Z, \mathcal{E}_2} < n^{-1}, \quad 1 \leq i, j \leq 2r + n_x + n_y.$

Applying Markov's inequality to this estimate and using a simple union bound, we get that

$\max_{1 \leq i, j \leq 2r + n_x + n_y} \left| O_t^T (G^*(\mu) - \Pi(\mu)) O_t \right|_{ij} \leq n^{-1/4} \quad (C.18)$
with probability $1 - O(n^{-1/2 + 11\varepsilon})$ conditioning on $C_{n, n_0}$, $Z$ and $C_2$. Next, we claim that on $C_1 \cap C_2$,
\[
\sup_{0 \leq t \leq 1} \left\| \tilde{F}_t(\mu) - \tilde{F}_0(\mu) \right\| \leq C\omega, \tag{C.19}
\]
for some constant $C > 0$ that does not depend on $\omega$. In fact, expanding $\tilde{F}_t(\mu)$ and using $\|\Pi(\mu)\| = O(1)$, $\|t\Sigma^{(1)}\| \leq \omega$, $\|t\Sigma^{(2)}\| \leq \omega$ and $\|E_t - E_0\| = O(\omega)$ on $C_1 \cap C_2$, we can easily obtain (C.19). Then, combining (C.18) and (C.19), we get that on the event $\mathcal{A} \cap \mathcal{B} \cap C_1 \cap C_2$,
\[
\det \left( 1 + \tilde{F}_t(\mu) + \mathcal{E}_t(\mu) \right) = \det \left( 1 + \tilde{F}_0(\mu) + O(\omega) \right) \quad \text{for all} \quad t \in [0, 1], \tag{C.20}
\]
with probability $1 - o(1)$. When $t = 0$, the discussion at the beginning of Section 4 (i.e. the argument leading to (4.7)) gives that at $\mu = \theta_1 \pm \delta$, $\|(1 + \tilde{F}_0(\mu))^{-1}\| \leq C_\delta$ with high probability for some constant $C_\delta > 0$. Thus, by (C.20), as long as $\omega$ is sufficiently small, we have that with probability $1 - o(1)$, $\det(1 + \tilde{F}_t(\mu) + \mathcal{E}_t(\mu)) \neq 0$ for all $t \in [0, 1]$. This concludes (C.16), which further gives (2.33).

**Proof of (2.30) for Theorem 2.14.** Similar to (C.16), we claim that
\[
P \left( \det \tilde{H}_t(\mu) \neq 0 \quad \text{for all} \quad 0 \leq t \leq 1 \right) = 1 - o(1), \tag{C.21}
\]
for $\mu := \lambda_1(0) \pm n^{-3/4} = \lambda^*_1 \pm n^{-3/4}$. At $t = 0$, Theorem 2.11 gives that
\[
\tilde{\lambda}_{1+r_+}(0) \in [\lambda^*_1 - n^{-3/4}, \lambda^*_1 + n^{-3/4}] \quad \text{with high probability.}
\]
If (C.21) holds, then due to the continuity of $\tilde{\lambda}_{1+r_+}(t)$ with respect to $t$, we have that
\[
\tilde{\lambda}_{1+r_+} \equiv \tilde{\lambda}_{1+r_+}(1) \in [\lambda^*_1 - n^{-3/4}, \lambda^*_1 + n^{-3/4}] \quad \text{with probability} \quad 1 - o(1),
\]
which concludes (2.30) for $k = 1$ together with (C.7).

In the following proof, we choose $z = \lambda_+ + in^{-2/3}$. As in (C.17), we need to study
\[
\det \left( 1 + \tilde{F}_t(z) + \mathcal{E}_t(z) + \begin{bmatrix}
0 & D \\
D & -tD_e
\end{bmatrix} \right) \left[ G^s(\mu) - G^s(z) \right] O_t,
\]
where we used the simple identity
\[
O_t^T G^s(\mu) O_t = O_t^T \left[ G^s(\mu) - G^s(z) \right] O_t + O_t^T G^s(z) O_t.
\]
Repeating the proof below (C.17), we can show that with probability $1 - o(1)$,
\[
1 + \tilde{F}_t(z) + \mathcal{E}_t(z) = 1 + \tilde{F}_0(z) + O(\omega) \quad \text{for all} \quad t \in [0, 1], \tag{C.22}
\]
and $\|(1 + \tilde{F}_0(z))^{-1}\| \leq C$ with high probability for some constant $C > 0$ that is independent of $\omega$. Moreover, we have that
\[
\|O_t^T \left[ G^s(\mu) - G^s(z) \right] O_t\| \leq n^{-1/6} \quad \text{with probability} \quad 1 - o(1), \tag{C.23}
\]
which is proved as (5.16) in [47]. Combining (C.22) and (C.23), we get that with probability $1 - o(1)$,
\[
\det \left( 1 + \tilde{F}_t(\mu) + \mathcal{E}_t(\mu) \right) = \det \left( 1 + \tilde{F}_0(z) + O(\omega) \right) \neq 0 \quad \text{for all} \quad t \in [0, 1],
\]
as long as $\omega$ is sufficiently small. This concludes (C.21), which completes the proof of (2.30) for the $k = 1$ case. It is easy to extend the above proof to the $k > 1$ case, and we omit the details. \[\square\]
D Proof of Lemma A.1

Finally, in this section, we present the proof of Lemma A.1. It has been proved in [47] for the $B = 0$ case, and we need to show that adding the BZ term to $Y$ does not affect the results. We remark that, since (A.2) has been used in the proof of (A.1), we cannot apply Theorem 2.11 and (2.20) to conclude Lemma A.1. A separate argument is needed. We first prove an averaged local law for $G^b(z)$ as in (3.40) and (3.41), using the following resolvent estimates.

Lemma D.1 (Lemma 3.8 of [47]). For any deterministic unit vectors $v_\beta \in \mathbb{C}^d$, $\beta = 3, 4$, we have that

$$\sum_{a \in I} |G_{av_\beta}|^2 < 1 + \frac{|\text{Im}(UG_Rv_\beta v_\beta^*)|}{\eta}, \quad \sum_{a \in I} |G_{v_\beta a}|^2 < 1 + \frac{|\text{Im}(G_RU^*)v_\beta v_\beta^*)|}{\eta},$$

where

$$U := z^{1/2} \left( \begin{array}{cc} I_n & z^{1/2} I_n \\ z^{1/2} I_n & z I_n \end{array} \right)^{-1}.$$

We calculate $m^b_3(z) = n^{-1} \sum_{\mu \in \mathcal{I}_3} G_{\mu \mu}^b(z)$ using (A.9). By the anisotropic local law (A.3), we have that with high probability,

$$\left[ 1 + \left( \begin{array}{cc} D_b & U_b^T \\ 0 & E_b \end{array} \right) G(z) \left( \begin{array}{cc} U_b & 0 \\ 0 & E_b \end{array} \right) \right]^{-1} \left( \begin{array}{cc} 0 & D_b \\ D_b & 0 \end{array} \right) = O(1).$$

Hence, using (A.9), we obtain that

$$|m^b_3(z) - m_3(z)| < \frac{1}{n} \max_{1 \leq k \leq r} \sum_{\mu \in \mathcal{I}_3} \left( |G_{\mu \mu}^b(z)|^2 + |G_{\mu \psi_k^b}(z)|^2 \right),$$

where we have abbreviated that $\psi_k^b := \frac{z^2}{2} \Phi_k^b$. Note that $\psi_k^b$ are approximately orthonormal vectors by (A.8). Then, using (D.1), we obtain that for $z \in \tilde{S}(\varepsilon, \zeta)$,

$$|m^b_3(z) - m_3(z)| < \frac{1}{n} \max_{1 \leq k \leq r} \left( |\text{Im}(UG_Ru_k^b u_k^*)| + |\text{Im}(UG_R \psi_k^b \psi_k^*)| \right),$$

$$< \frac{1}{n} \max_{1 \leq k \leq r} \frac{\eta + \text{Im} m_e(z) + \Psi(z) + \psi_n + \phi_n}{\eta} \lesssim \frac{\psi_n + \phi_n}{\eta},$$

where in the second step we used the local law (A.5) and that

$$\left| \text{Im}(UG_R(z))u_k^b u_k^* \right| + \left| \text{Im}(UG_R(z))\psi_k^b \psi_k^* \right| \lesssim \text{Im} m_e(z) + \eta.$$

Here, $\Pi^b_3(z)$ denotes the $(\mathcal{I}_3 \cup \mathcal{I}_4) \times (\mathcal{I}_3 \cup \mathcal{I}_4)$ block of $\Pi^b$. Combining (D.2) with the averaged local laws (3.40)–(3.41) for $m_3(z)$ and equation (3.17) for $m^b_3(z)$ and $m^b(z)$, we obtain the following local laws: for any fixed $\varepsilon, \zeta > 0$,

$$|m^b(z) - m_3(z)| < (n\eta)^{-1}$$

uniformly in $z \in \tilde{S}(\varepsilon, \zeta)$, and

$$|m^b(z) - m_e(z)| < \frac{\psi_n + \phi_n}{n\eta} + \frac{1}{n(n + \kappa + \eta)} + \frac{1}{(n\eta)^2 \sqrt{\kappa + \eta}}$$

uniformly in $z \in \tilde{S}_{out}(\varepsilon, \zeta)$. 

41
Definition D.2 (Regularized resolvents). For \( z = E + i\eta \in \mathbb{C}_+ \), we define the regularized resolvent \( \hat{G}(z) \) as
\[
\hat{G}(z) := \left[ H(z) - zn^{-10} \begin{pmatrix} I_{p+q} & 0 \\ 0 & 0 \end{pmatrix} \right]^{-1}.
\]
Moreover, we define
\[
\hat{H} := \hat{S}_{xx}^{-1/2} S_{xy} \hat{S}_{yy}^{-1/2}, \quad \hat{S}_{xx} := S_{xx} + n^{-10}, \quad \hat{S}_{yy} := S_{yy} + n^{-10}.
\]
The resolvents \( \hat{R}(z), \hat{G}^b(z) \) and \( \hat{R}^b(z) \) etc. can be defined in the obvious way as in Definition D.3.

With the Schur complement formula, we can obtain similar expressions for \( \hat{G}_L, \hat{G}_R \) and \( \hat{G}_{LR} \) as in (3.13) – (3.16). The main reason for introducing regularized resolvents is that they satisfy the following deterministic bounds: for some constant \( C > 0 \),
\[
\| \hat{G}(z) \| \leq \frac{Cn^{10}}{\eta}, \quad \| \hat{G}^b(z) \| \leq \frac{Cn^{10}}{\eta}.
\]
This estimate has been proved in Lemma 3.6 of [47]. With a standard perturbation argument, we can control the difference between \( \hat{G}(z) \) and \( G(z) \) as in the following claim.

Claim D.3. Suppose there exists a high probability event \( \Xi \) on which \( \|G(z)\|_{\text{max}} = O(1) \) for \( z \) in some subset, where \( \|G\|_{\text{max}} := \max_{i,j} |G_{ij}| \) denotes the max-norm. Then, we have that
\[
\|G(z) - \hat{G}(z)\|_{\text{max}} \leq n^{-8} \quad \text{on} \quad \Xi.
\]
The same bound holds for \( \|G^b(z) - \hat{G}^b(z)\|_{\text{max}} \) on the events \( \{\|G^b(z)\|_{\text{max}} = O(1)\} \) and \( \{\|\hat{G}^b(z)\|_{\text{max}} = O(1)\} \).

Proof. For \( t \in [0, 1] \), we define
\[
G_t(z) := \left[ H(z) - tzn^{-10} \begin{pmatrix} I_{p+q} & 0 \\ 0 & 0 \end{pmatrix} \right]^{-1}, \quad \text{with} \quad G_0(z) = G(z), \quad G_1(z) = \hat{G}(z).
\]
Taking derivatives with respect to \( t \), we get that
\[
\partial_t G_t(z) = zn^{-10}G_t(z) \begin{pmatrix} I_{p+q} & 0 \\ 0 & 0 \end{pmatrix} G_t(z).
\]
Thus, applying Gronwall’s inequality to
\[
\|G_t(z)\|_{\text{max}} \leq \|G(z)\|_{\text{max}} + Cn^{-9} \int_0^t \|G_s(z)\|_{\text{max}}^2 ds,
\]
we obtain that \( \|G_t(z)\|_{\text{max}} \leq C \) for all \( 0 \leq t \leq 1 \) on \( \Xi \). Then, using (D.7) again, we get (D.6). \( \square \)

Note that the bound (D.6) is purely deterministic on \( \Xi \), so we do not lose any probability here. Moreover, such a small error \( n^{-8} \) will not affect any of our results.

Proof of Lemma A.1. With the same argument as those for [19] Theorems 2.12 and 2.13, [20] Theorem 2.2 and [39] Theorem 3.3, from the averaged local law [D.3] we can derive that for any small constants \( \delta, \varepsilon > 0 \), (A.2) holds for all \( n^{2} \leq i \leq (1 - \delta)q \). To conclude (A.2) for the first \( n^{2} \) eigenvalues, we still need to prove an upper bound on them. More precisely, it suffices to show that for any small constant \( \varepsilon > 0 \),
\[
\lambda_1^b \leq \lambda_+ + n^{-2/3+\varepsilon}, \quad \text{w.h.p.}
\]
Combining this estimate with the rigidity estimate for \( \lambda_{n,i}^b \), we can conclude that (A.2) holds all \( 1 \leq i < (1 - \delta)q \) since \( \varepsilon \) can be arbitrarily small.

First, using the local law (D.1), we can obtain that for any small constants \( c, \varepsilon > 0 \),

\[
\# \{ i : \lambda_i^b \in [\lambda_+ + n^{-2/3 + \varepsilon}, 1 - c] \} = 0, \quad \text{w.h.p.} \tag{D.9}
\]

The proof is standard and similar to the one for (4.7) of \[47\], so we omit the details. It remains to prove that for a sufficiently small constant \( c > 0 \),

\[
\# \{ i : \lambda_i^b \in [1 - c, 1] \} = 0, \quad \text{w.h.p.} \tag{D.10}
\]

We define a continuous path of interpolated random matrices between \( Y \) and \( Y + BZ \) as

\[
Y_t := Y + tBZ, \quad t \in [0, 1].
\]

By replacing \( Y \) with \( Y_t \) in (3.10) and Definition D.2 we can define \( H_t^b(z), G_t^b(z), \tilde{H}_t^b(z) \) and \( \tilde{G}_t^b(z) \) correspondingly. First, we claim the following result.

**Claim D.4.** With high probability, we have that

\[
\| G_t^b(1 - c) \|_{\text{max}} < \infty \quad \text{for all} \quad t \in [0, 1]. \tag{D.11}
\]

We postpone the proof of this claim until we complete the proof of (A.2). Let \( \lambda_i^b(t) \geq \lambda_2^b(t) \geq \cdots \geq \lambda_q^b(t) \) be the eigenvalues of \( C_{Y_t} \). For any \( 1 \leq i \leq q \), \( \lambda_i^b(t) : [0, 1] \to \mathbb{R} \) is a continuous function with respect to \( t \) on the extended real line \( \mathbb{R} \). By (2.20), the eigenvalues \( \lambda_i^b(0) \) of \( C_{X^Y} \) are all inside \( [0, \lambda_+ + n^{-2/3 + \varepsilon}] \) with high probability. If (D.11) holds, then we have that

\[
m_i^b(1 - c) = \frac{1}{q} \sum_{i=1}^{q} \frac{1}{\lambda_i^b(t) - (1 - c)} \quad \text{is finite for all} \quad t \in [0, 1].
\]

It means that the eigenvalue \( \lambda_i^b(t) \) does not cross the point \( E = 1 - c \) for all \( t \in [0, 1] \). Thus, we conclude (D.10), which further concludes (D.8) together with (D.9).

Finally, we give the proof of Claim D.4.

**Proof of Claim D.4.** Take a discrete net of \( t, t_k = kn^{-50}, 0 \leq k \leq n^{50} \). First, we claim that there exists a high probability event \( \Xi_1 \), so that

\[
\mathbf{1}(\Xi_1) \max_{0 \leq k \leq n^{50}} \| \tilde{G}_{t_k}^b(E + in^{-10}) \|_{\text{max}} \leq C \quad \text{for} \quad E := 1 - c, \tag{D.12}
\]

for some large constant \( C > 0 \). In fact, notice that \( Y_t \) also satisfies the assumptions for \( Y \) in Lemma A.1. Hence, using (D.9), we obtain that for any \( t_k \), the eigenvalues \( \lambda_i^b(t_k) \) are inside \( [0, \lambda_+ + n^{-2/3 + \varepsilon}] \cup [1 - c/2, 1] \) with high probability. By taking a union bound, we get that

\[
\min_{0 \leq k \leq n^{50}} \min_{1 \leq i \leq q} | E - \lambda_i^b(t_k) | \geq 1 \quad \text{w.h.p.} \tag{D.13}
\]

Applying the spectral decomposition (3.13) to \( R_z \), we obtain from (D.13) that

\[
\max_{0 \leq k \leq n^{50}} | R_{t_k}^b(z) | \leq C \quad \text{for} \quad z = E + in^{-10}.
\]

43
Combining this bound with (3.14)–(3.16) and using Lemma 3.3, we get that
\[
\max_{0 \leq k \leq n^{50}} \| G_{tk}^b(z) \| \leq C, \quad \text{w.h.p.}
\]
Next, applying Claim D.3, we get (D.12) for \( G^b \).

Now, given (D.12), using the deterministic bound (D.5) for \( G^b \), we get that on \( \Xi_1 \),
\[
\left\| \hat{G}_t^b(E + in^{-10}) - \hat{G}_{tk}^b(E + in^{-10}) \right\|_{\max} \leq n^{-50} \left\| \hat{G}_t^b(E + in^{-10}) \right\| \cdot \left\| \hat{G}_{tk}^b(E + in^{-10}) \right\| \\
\leq n^{-50} \cdot (n^{-20})^2 \cdot \| Z \| \leq n^{-10} \cdot \| Z \|,
\]
for any \( t_{k-1} \leq t \leq t_k \). By the bounded support condition of \( Z \), we have that \( \| Z \| = O(\sqrt{n}) \) on a high probability event \( \Xi_2 \). Thus, on the high probability event \( \Xi_1 \cap \Xi_2 \),
\[
\left\| \hat{G}_t^b(E + in^{-10}) - \hat{G}_{tk}^b(E + in^{-10}) \right\|_{\max} \leq n^{-50} \cdot (n^{-20})^2 \cdot \sqrt{n} \leq n^{-9},
\]
which gives that
\[
1(\Xi_1 \cap \Xi_2) \max_{0 \leq t \leq 1} \left\| \hat{G}_t^b(E + in^{-10}) \right\|_{\max} \leq C.
\]
Finally, using the same perturbation argument as in the proof of Claim D.3 we can remove both the \( in^{-10} \) and the regularization in \( \hat{G} \), which gives (D.11) on \( \Xi_1 \cap \Xi_2 \).

**Acknowledgments**

We would like to thank the editor, the associated editor and an anonymous referee for their helpful comments, which have resulted in a significant improvement of the paper. The second author is supported in part by the Wharton Dean’s Fund for Postdoctoral Research.

**References**

[1] J. Alt. Singularities of the density of states of random Gram matrices. *Electron. Commun. Probab.*, 22:13 pp., 2017.

[2] J. Alt, L. Erdős, and T. Krüger. Local law for random Gram matrices. *Electron. J. Probab.*, 22:41 pp., 2017.

[3] Z. Bai and J. Yao. Central limit theorems for eigenvalues in a spiked population model. *Ann. Inst. H. Poincaré Probab. Statist.*, 44(3):447–474, 2008.

[4] Z. D. Bai and J. W. Silverstein. No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices. *Ann. Probab.*, 26(1):316–345, 1998.

[5] J. Baik, G. Ben Arous, and S. Péché. Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. *Ann. Probab.*, 33(5):1643–1697, 2005.

[6] J. Baik and J. W. Silverstein. Eigenvalues of large sample covariance matrices of spiked population models. *J. Multivariate Ana.*, 97(6):1382 – 1408, 2006.

[7] Z. Bao, J. Hu, G. Pan, and W. Zhou. Canonical correlation coefficients of high-dimensional Gaussian vectors: Finite rank case. *Ann. Statist.*, 47(1):612–640, 2019.
[8] S. T. Belinschi, H. Bercovici, M. Capitaine, and M. Février. Outliers in the spectrum of large deformed unitarily invariant models. *Ann. Probab.*, 45(6A):3571–3625, 2017.

[9] F. Benaych-Georges, A. Guionnet, and M. Maida. Fluctuations of the extreme eigenvalues of finite rank deformations of random matrices. *Electron. J. Probab.*, 16:621–1662, 2011.

[10] F. Benaych-Georges and R. R. Nadakuditi. The eigenvalues and eigenvectors of finite, low rank perturbations of large random matrices. *Adv. Math.*, 227(1):494 – 521, 2011.

[11] A. Bloemendal, L. Erdős, A. Knowles, H.-T. Yau, and J. Yin. Isotropic local laws for sample covariance and generalized Wigner matrices. *Electron. J. Probab.*, 19(33):1–53, 2014.

[12] A. Bloemendal, A. Knowles, H.-T. Yau, and J. Yin. On the principal components of sample covariance matrices. *Probab. Theory Relat. Fields*, 164(1):459–552, 2016.

[13] M. Capitaine, C. Donati-Martin, and D. Féral. The largest eigenvalues of finite rank deformation of large Wigner matrices: Convergence and nonuniversality of the fluctuations. *Ann. Probab.*, 37(1):1–47, 2009.

[14] M. Capitaine, C. Donati-Martin, and D. Féral. Central limit theorems for eigenvalues of deformations of Wigner matrices. *Ann. Inst. H. Poincaré Probab. Statist.*, 48(1):107–133, 2012.

[15] X. Ding and F. Yang. Edge statistics of large dimensional deformed rectangular matrices. *arXiv:2009.00389*, 2020.

[16] X. Ding and F. Yang. Tracy-widom distribution for the edge eigenvalues of Gram type random matrices. *arXiv:2008.04766*, 2020.

[17] X. Ding and F. Yang. Spiked separable covariance matrices and principal components. *Ann. Statist.*, 49(2):1113 – 1138, 2021.

[18] L. Erdős, A. Knowles, and H.-T. Yau. Averaging fluctuations in resolvents of random band matrices. *Ann. Henri Poincaré*, 14:1837–1926, 2013.

[19] L. Erdős, A. Knowles, H.-T. Yau, and J. Yin. Spectral statistics of Erdős-Rényi graphs I: Local semicircle law. *Ann. Probab.*, 41(3B):2279–2375, 2013.

[20] L. Erdős, H.-T. Yau, and J. Yin. Rigidity of eigenvalues of generalized Wigner matrices. *Adv. Math.*, 229:1435 – 1515, 2012.

[21] D. Féral and S. Péché. The largest eigenvalue of rank one deformation of large Wigner matrices. *Comm. Math. Phys.*, 272(1):185–228, 2007.

[22] D. Féral and S. Péché. The largest eigenvalues of sample covariance matrices for a spiked population: Diagonal case. *J. Math. Phys.*, 50(7):073302, 2009.

[23] P. Forrester. The spectrum edge of random matrix ensembles. *Nucl. Phys. B*, 402(3):709 – 728, 1993.

[24] Y. Fujikoshi. High-dimensional asymptotic distributions of characteristic roots in multivariate linear models and canonical correlation analysis. *Hiroshima Math. J.*, 47(3):249–271, 2017.

[25] C. Gao, Z. Ma, Z. Ren, and H. H. Zhou. Minimax estimation in sparse canonical correlation analysis. *Ann. Statist.*, 43(5):2168–2197, 2015.
[26] C. Gao, Z. Ma, and H. H. Zhou. Sparse CCA: Adaptive estimation and computational barriers. *Ann. Statist.*, 45(5):2074–2101, 2017.

[27] X. Han, G. Pan, and Q. Yang. A unified matrix model including both CCA and F matrices in multivariate analysis: The largest eigenvalue and its applications. *Bernoulli*, 24(4B):3447–3468, 2018.

[28] X. Han, G. Pan, and B. Zhang. The Tracy-Widom law for the largest eigenvalue of F type matrices. *Ann. Statist.*, 44(4):1564–1592, 2016.

[29] H. Hotelling. Relations between two sets of variates. *Biometrika*, 28(3-4):321–377, 1936.

[30] I. M. Johnstone. On the distribution of the largest eigenvalue in principal components analysis. *Ann. Statist.*, 29:295–327, 2001.

[31] I. M. Johnstone. Multivariate analysis and Jacobi ensembles: Largest eigenvalue, Tracy-Widom limits and rates of convergence. *Ann. Statist.*, 36(6):2638–2716, 2008.

[32] I. M. Johnstone and A. Onatski. Testing in high-dimensional spiked models. *Ann. Statist.*, 48(3):1231–1254, 2020.

[33] A. Knowles and J. Yin. The isotropic semicircle law and deformation of Wigner matrices. *Comm. Pure Appl. Math.*, 66:1663–1749, 2013.

[34] A. Knowles and J. Yin. The outliers of a deformed Wigner matrix. *Ann. Probab.*, 42(5):1980–2031, 2014.

[35] A. Knowles and J. Yin. Anisotropic local laws for random matrices. *Probab. Theory Relat. Fields*, 169(1):257–352, 2017.

[36] R. Oda, H. Yanagihara, and Y. Fujikoshi. Asymptotic null and non-null distributions of test statistics for redundancy in high-dimensional canonical correlation analysis. *Random Matrices: Theory and Applications*, 08(01):1950001, 2019.

[37] D. Paul. Asymptotics of sample eigenstructure for a large dimensional spiked covariance model. *Statist. Sinica*, 17(4):1617–1642, 2007.

[38] S. Péché. The largest eigenvalue of small rank perturbations of Hermitian random matrices. *Probab. Theory Relat. Fields*, 134(1):174–174, 2006.

[39] N. S. Pillai and J. Yin. Universality of covariance matrices. *Ann. Appl. Probab.*, 24:935–1001, 2014.

[40] C. A. Tracy and H. Widom. Level-spacing distributions and the airy kernel. *Comm. Math. Phys.*, 159:151–174, 1994.

[41] C. A. Tracy and H. Widom. On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys.*, 177:727–754, 1996.

[42] K. W. Wachter. The limiting empirical measure of multiple discriminant ratios. *Ann. Statist.*, 8(5):937–957, 1980.

[43] Q. Wang and J. Yao. Extreme eigenvalues of large-dimensional spiked Fisher matrices with application. *Ann. Statist.*, 45(1):415–460, 2017.

[44] H. Xi, F. Yang, and J. Yin. Local circular law for the product of a deterministic matrix with a random matrix. *Electron. J. Probab.*, 22:77 pp., 2017.
[45] F. Yang. Edge universality of separable covariance matrices. *Electron. J. Probab.*, 24:57 pp., 2019.

[46] F. Yang. Limiting distribution of the sample canonical correlation coefficients of high-dimensional random vectors. *arXiv:2103.08014*, 2021.

[47] F. Yang. Sample canonical correlation coefficients of high-dimensional random vectors: Local law and Tracy-Widom limit. *Random Matrices: Theory and Applications*, 11(01):2250007, 2022.

[48] F. Yang, S. Liu, E. Dobriban, and D. P. Woodruff. How to reduce dimension with PCA and random projections? *IEEE Trans. Inf. Theory*, 67(12):8154–8189, 2021.

[49] Y. Yang and G. Pan. The convergence of the empirical distribution of canonical correlation coefficients. *Electron. J. Probab.*, 17:13 pp., 2012.

[50] Y. Yang and G. Pan. Independence test for high dimensional data based on regularized canonical correlation coefficients. *Ann. Statist.*, 43(2):467–500, 04 2015.