EFFECTIVE UPPER BOUND OF ANALYTIC TORSION UNDER ARAKELOV METRIC

CHANGWEI ZHOU

Abstract. Given a choice of metric on the Riemann surface, the regularized determinant of Laplacian (analytic torsion) is defined via the complex power of elliptic operators:
\[
det(\Delta) = \exp(-\zeta'(0))
\]
In this paper we gave an asymptotic effective estimate of analytic torsion under Arakelov metric. In particular, after taking the logarithm it is asymptotically upper bounded by \( g \) for \( g > 1 \).

The construction of a cohomology theory for arithmetic surfaces in Arakelov theory has long been an open problem. In particular, it is not known if \( h^1(X, L) \geq 0 \). We view this as an indirect piece of evidence that if such a cohomology theory exists, the \( h^1 \) term may be effectively estimated.

1. Introduction

Let \((X_\sigma, g)\) be a compact connected smooth Riemann surface without boundary (which we henceforth abbreviate as compact Riemann surface). The metric Laplacian is defined to be
\[
\Delta_g(f) = \frac{1}{\sqrt{\det(g)}} \partial_i(\sqrt{\det(g)} g^{ij} \partial_j f)
\]
The regularized determinant of the metric Laplacian (which we henceforth abbreviate as analytic torsion) is defined to be
\[
det(\Delta_g) = \exp(-\zeta'(0)), \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(e^{t\Delta} - P) dt
\]
where \( P \) is the projection operator onto the kernel of the Laplacian.

The present paper presents the following theorem:

Theorem 1.1. Let \( \pi : X \to B = \text{Spec}(O_K) \) be an integral, flat, projective scheme of dimension 2. Let \( \sigma : K \to \mathbb{C} \) be a fixed archimedean place of \( K \). Let \( \Delta_{Ar} \) denotes the metric Laplacian associated to Arakelov metric on \( X_\sigma \).

We have the following effective estimate of the analytic torsion under Arakelov metric: For \( g \) large enough (\( g > 10 \), for example):
\[
-\infty < \log(\det(\Delta_{Ar})) < g
\]

2010 Mathematics Subject Classification. 14G40, 58J52, 13D45.
In particular, for $g \geq 1$ the analytic torsion under Arakelov metric is always effectively upper bounded from above. The precise upper bound can be found in Appendix.

Our proof of this result heavily used previous results by Jorgenson and Kramer, Wentworth and Wilms. The main idea behind the paper is that with Richard Wentworth and Jay Jorgenson’s proof of the correct bosonization formula, we may write the scalar analytic torsion in terms of Faltings’ delta function and area of the compact Riemann surface under Arakelov metric. With Robert Wilms’s result, we may directly estimate Faltings’ delta function. Thus to estimate the scalar analytic torsion, it is enough to bound the area of the surface. This was done essentially by Jorgenson and Kramer in their paper [JK 2006]. This answers a question of Gillet and Soule in their paper [G-S 1998] in the setting of Arakelov metric.

We note that it is crucial that we restrict ourselves to metric Laplacian under Arakelov metric. As we mentioned earlier, the paper [Osgood-Phillips-Sarnak 1988] showed such estimate is already very subtle even for hyperbolic metric. For example, the precise extreme metric for $g \geq 2$ is unknown.

The organization of the paper is as follows: In section 2 we review the main background notation and previous work on this topic. In section 3-4 we present the proof and some ideas related to it, which is rather trivial. In Appendix section 5 we give an that lists effective estimates for arithmetic surfaces of small genus except $g = 1$. In Appendix section 6 we give a derivation for the case of elliptic curve.

I thank A.Borisov, J. Bost, C. Soule, R. Wentworth, R. Wilms, J. Kramer, J. Jorgenson, P. Andreae, X.Wang, M. Kierlanczyk for generously sharing their ideas during the research process. In particular based Prof.Kramer’s suggestion, I found the effective estimate.

In May 30th, 2019 I received a preprint [Jorgenson1992] from Prof. Jorgenson. It is clear from the preprint that some of the results from this paper may be well known to Prof. Jorgenson and Prof. Wentworth as early as 1990. Further, his paper went much deeper than I did at here.

Without the emotional support from friends like Jaiung Jun, Jingyu Zhao, Oleg Lazerev, Patrick Milano, Yiming Zhao and Zana Chan, the paper probably would never have been written down and I would be sweating on Leetcode instead.

For obvious reasons, I am indebted to Prof. Vasiu.

2. Review of background material

The main point for this section is to set notation and review work done by experts in the field. We first recall that Arakelov considered a metric on $X_\tau$ whose area is 1:
**Definition 2.1.** Let $X_\sigma$ be a Riemann surface of genus $g \geq 1$. Let $\alpha_j$ be an orthonormal basis of holomorphic 1-forms on $X_\sigma$ with scalar product

$$(\alpha, \beta) = \frac{i}{2} \int_X \alpha \wedge \overline{\beta}$$

Then the volume form given by

$$\mu_\sigma = \frac{i}{2g} \sum \alpha_i \wedge \overline{\alpha}_j$$

is called the **canonical metric**.

We now proceed to define the Arakelov metric ([Arakelov 1977]).

**Definition 2.2.** Let $G_\sigma : X_\sigma \times X_\sigma \to \mathbb{R}$ be Green function if it satisfies the following properties:

- $G_\sigma$ is $C^\infty$ on $X_\sigma \times X_\sigma$.
- $G_\sigma(P, Q)$ has a first order zero on the diagonal, and it is non-zero if $P \neq Q$.
- If $P \neq Q$, then we require
  $$\partial_P \overline{\partial_P} \log G_\sigma(P, Q) = \pi i \mu_\sigma(P)$$
- We require
  $$\int_X \log G_\sigma(P, z) \mu_\sigma(z) = 0$$

Let $\sigma$ be an archimdean place of $K$. We fix the embedding $\sigma$ and let $M = X_\sigma$ be a fixed Riemann surface. Then we have so called Poincare residue isomorphism:

$$\mathcal{O}_{M \times M}(-\Delta) \cong K_M$$

where $\Delta$ is the diagonal. Using Green functions, we may define a metric on the dual space $\mathcal{O}_{M \times M}(\Delta)$. Let $1_\Delta$ be its canonical section, we define the metric by

$$(\log |1_\Delta|)(P, Q) = \log G(P, Q)$$

**Definition 2.3.** The dual metric of above metric on $\mathcal{O}_{M \times M}(-\Delta)$ is the so called **Arakelov metric**. To be precise, we consider the diagonal morphism

$$\mu : M \to M \times M$$

and let $\mu^*$ be the pull-back metric on $\Omega^1(M)$. We want to mention the motivation of this construction (see [Arakelov 1977]): Classical adjunction formula suggests that

$$K_D = (K_X + D) |_D$$

This can be seen from the conormal exact sequence induced from the inclusion map $i : D \to X$:

$$0 \to i^* \mathcal{O}(-D) \to i^* K_X \to K_D \to 0$$

Arakelov wants the adjunction formula isomorphism to be an isometry for the case $D = P$:

$$\Omega_X \otimes \mathcal{O}_P \cong \Omega_P, \eta \otimes \frac{s}{f} \to s \frac{\partial \eta}{\partial f} |_{f=0}$$
Since the metric at $O_P$ is given by $G(P, Q)$, we have the Arakelov metric on $\Omega_X$ to be

$$|dz(P)|_{Ar} = \lim_{Q \to P} \frac{|z(Q) - z(P)|}{G(P, Q)}$$

where $\eta = dz$, $s = 1$, $f = z$. The Arakelov metric on $O_X$ is defined to be its dual metric:

$$\mu_{Ar} = \frac{|dz|}{\int |dz_{Ar}|}$$

In 1984, while working on an arithmetic analog of Riemann-Roch theorem for arithmetic surfaces, Faltings proved the arithmetic Noether formula [Faltings1984]: As before we begin with a curve $C$ over a number field $K$. After a finite field extension, we may assume $C$ has semi-stable reduction over $K$. Further we denote by $X$ the minimal regular model of $C$ over $B = \text{Spec}(O_K)$. We have:

$$12\deg \det \pi_* \omega_{X/B} = \omega_X/\omega_{X/B} + \sum_{v\in B} \delta_v \log(N(v)) + \sum_{\sigma: K \to C} \delta'(X_{\sigma})$$

By the geometric analogy, the interpretation of the formula is straightforward: The extra terms $\delta_v$ on the right hand side measures the degree of singularity of $X$. Thus it should ‘blow up’ on the boundary of the moduli space $M_g$. We note there is a discrepancy in normalization: The $\delta$-function we discuss at here, $\delta_{Fal}$ equals $\delta' + 4g \log(2\pi)$.

In 1987, Deligne proved a version of Riemann-Roch theorem using analytic torsion. In 1988 Bost, Gillet and Soule refined his result. They found that

$$\delta_{Fal}(X) = -6 D_{Ar}(X) + a(g_X),$$

where

$$D_{Ar}(X) = \log\left(\frac{\det(\Delta)_{Ar}}{\text{Area}_{Ar}(X)}\right),$$

and

$$a(g) = -8g \log(2\pi) + (1 - g)(-24\zeta'_Q(-1) + 1 - 6\log(2\pi) - 2\log(2))$$

It is thus clear that $\delta_{Fal}(X)$’s growth only depend on the analytic torsion and the area of $X$ under Arakelov metric. The original computation of $a(g)$ in Soule’s paper [Soule 1988] turned out to be incorrect, most notably an sign error $(g - 1)$ instead of $1 - g$. Later in 2008, Richard Wentworth re-derived the correct formula rigorously in his paper [Wentworth 2009]. We wish to mention there is an independent proof by Jay Jorgenson [Jorgenson1992].

In [JK 2001] and [JK 2006], Jorgenson and Kramer systematically investigated the area of a compact Riemann surface under Arakelov metric. We note that everything below is only valid for $g > 1$. Their result, presented in compact form (page 7, Corollary 3.3), is that

$$\log\left(\frac{\mu_{Ar}(z)}{\mu_{hyp}(z)}\right) \leq 4\pi(1 - \frac{1}{g}) \int_1^\infty K_H(t, 0) dt - \frac{c_{set}}{g(g - 1)} + \frac{1}{g - 1} - \log(4)$$

where $\mu_{hyp}(z)$ denotes the hyperbolic metric and $\mu_{Ar}(z)$ denotes the Arakelov metric. Here the constant $K_H(t, 0)$ is the hyperbolic heat kernel.
In 2016, Robert Wilms proved in his paper ([Wilms 2017]) that
\[ \delta_{Fal}(X) > -2g \log(2\pi^4) \]
for all \( g \). We refer the reader to his thesis for a detailed exposition of his paper.

3. A HEURISTIC INTERPRETATION OF THE RESULT

Before we elaborate the proof, we want to offer a few comments why the theorem 'should be expected' other than that it fits into arithmetic intersection theory framework established by Faltings and Gillet-Soule. In fact, Jorgenson and Kramer's earlier result in [JK 2006-2] indicates that the area of the surfaces shrinks to 0 near the boundary of \( M_g \). Thus an easy argument using the compactness of \( M_g \) showed the upper bound exists and only dependent on \( g \) for \( g > 1 \). The real surprise is that the upper bound is only linear. This feels rather mysterious at first sight.

Here we note that using gluing formula of determinant of elliptic operators we have
\[
\frac{\det_{Ar}(X)}{\text{Area}_{Ar}(X)} = \det(X_+, Ar) \det(X_-, Ar) \frac{\det(N_{l, \gamma, h})}{\text{length}_{Ar}(\gamma)}
\]
The notation is borrowed from Wentworth's paper: \( \gamma \) is a simple separating curve on \( X \) that separates \( X \) into two parts, \( X_+ \) and \( X_- \). The determinant on the right hand side is evaluated in terms of Laplacian with Dirichlet boundary condition on \( \gamma \). The symbol \( N_{l, \gamma} \) denotes the Newmann jumping operator.

An 'obvious idea' now is to take logarithm, and use induction on \( g \) to control the growth of each term. However, the Arakelov metric inherited this way for the surfaces with boundary of genus \( g - 1 \) is not the same Arakelov metric for closed surface \( g \). So far more involved analysis similar to Wentworth’s work is necessary. We conjecture that the term involving the Newmann jumping operator is bounded from above. Heuristically, the determinant 'factorizes' and as \( g \) grows the extra contribution should be at most linear.

4. PROOF OF MAIN THEOREM

The associated integral may be explicitly bounded by
\[
\int_1^{\infty} K_H(t, 0) dt \leq \int_1^{\infty} \frac{e^{-t/4}}{4\pi t} dt \leq 0.0832 < 0.1
\]
Thus for \( g > 1 \), the constant term in the above sum may be evaluated to be bounded above by \( \frac{1}{g-1} - 0.33 \leq 0.67 \leq 1 \). The only term left to estimate is the \( c_{sel} \) for hyperbolic surfaces, which Jorgenson and Kramer estimated in [JK 2001] to be (page 13, remark 3.5):
\[
c_{sel} \geq -4\log(1366(g - 1))
\]
As a result we have
\[
\log \left( \frac{\mu_{Ar}(z)}{\mu_{hyp}(z)} \right) \leq 1 + \frac{4\log(1366(g - 1))}{g(g - 1)}
\]
Exponentiate both sides and integrate, we have
\[ \text{Area}(X_\sigma) < e \ast 4\pi(g-1) \ast (1366(g-1))^\frac{4}{\pi(g-1)} < 36(g-1)(1366(g-1))^\frac{4}{\pi(g-1)} \]
where we used the fact for hyperbolic surfaces the area is given by \( 4\pi(g-1) \) (Gauss-Bonnet).

By combing Wilms’ result and Wentworth’s result mentioned earlier, we have
\[ -6D_{A_r}(X) + a(g_X) > -2\log(2\pi^4) \]
where
\[ D_{A_r}(X) = \log(\frac{\det(\Delta_{A_r})}{\text{Area}_{A_r}(X)}), \]
and
\[ a(g) = -8g\log(2\pi) + (1-g)(-24\zeta'_Q(-1) + 1 - 6\log(2\pi) - 2\log(2)) \]
To ease notation we ignore the \( A_r \) in the subscript. Rearranging and clearing off the terms, we have
\[ \log(\det(\Delta)) < \frac{\log(2\pi^4)}{3}g + \frac{1}{6}a(g) + \log(\text{Area}(X)) \]
By previous computation we have
\[ \log(\text{Area}(X)) < \log[36] + \log[g-1] + \frac{4}{g(g-1)}\log(1366(g-1)) \sim O(\log(g)) \]
We now focus on the leading term of size \( O(g) \). Combining with the first term we have an effective asymptotic upper bound to be
\[ \left( \frac{\log(2\pi^4)}{3} - 8/6 \ast \log[2\pi] - \frac{1}{6}(-24\zeta'_Q(-1) + 1 - 6\log(2\pi) - 2\log(2)) \right)g \]
The constants can be explicitly evaluated to be
\[ \approx 1.7573 - 2.4505 + 2.07 - \frac{1}{6} + 4\zeta'_Q(-1) < 1.21 - 0.67 = 0.56 < 1 \]
As a result we have the final effective estimate to be (valid for all \( g > 1 \))
\[ -\infty < \log(\det(\Delta)) < 0.56g + E(g) \]
where \( E(g) \leq 2\log(g) < 0.44g \) for \( g \) large enough. As a result the whole term is asymptotically bounded by \( g \). We have the following effective formula for \( E(g) \):
\[ E(g) = \log[36] + \log[g-1] + \frac{4}{g(g-1)}\log(1366(g-1)) + \frac{1}{6}(-24\zeta'_Q(-1) + 1 - 6\log(2\pi) - 2\log(2)) \]
and a more refined estimate can be given by
\[ E(g) = \frac{1}{g-1} + \log(g-1) + \frac{4}{g(g-1)}\log(1366(g-1)) + \frac{1}{6}(-24\zeta'_Q(-1) + 1 - 6\log(2\pi) - 2\log(2)) + 2.1890125 \]
In particular arithmetic computation showed that for \( g > 10 \), we have \( E(g) < 0.44g \). Thus for the above estimate, it suffice to let \( g > 10 \).

**Corollary 4.1.** For a compact Riemann surface of genus \( g \geq 1 \), the difference of Faltings metric and Quillen metric’s logarithm has an asymptotic lower bound by a constant. The constant only depends on \( g \).
Proof. By Jorgenson’s result in his preprint (page 40) we have
\[ h_F(L) - h_Q(L) = \log(\text{Area}) - \log(\det(\Delta)) - 2\log(2\pi) \]
where \( h_F(L), h_Q(L) \) denotes the logarithm of Faltings and Quillen metric respectively. Re-write it we have
\[
h_F(L) - h_Q(L) \geq -Dr_{Ar}(X) - 2\log(2\pi)
= -2\log(2\pi) - \frac{\alpha(g)}{6} - \frac{g}{3} \log(2\pi^4)
= -2\log(2\pi) - \frac{-8g\log(2\pi) + (1 - g)(-24\zeta'_Q(-1) + 1 - 6\log(2\pi) - 2\log(2))}{6} - \frac{g}{3} \log(2\pi^4)
= (\frac{4}{3} \log(2\pi) - \log(2\pi^4))/3 + 4\zeta'_Q(-1) - \frac{1}{6} + \log(2\pi) + \frac{1}{3} \log(2))g + C\]
where \( C \) is the effective constant
\[
C = -\log(2\pi) + 4\zeta'_Q(-1) - \frac{1}{6} + \frac{1}{3} \log(2) \approx -3.6113717392987086 - 0.661685 \approx -4.273
\]
And the coefficient for \( g \) is bounded below by
\[
(\frac{4}{3} \log(2\pi) - \log(2\pi^4))/3 + 4\zeta'_Q(-1) - \frac{1}{6} + \log(2\pi) + \frac{1}{3} \log(2)) \approx 1.93372164089272 \geq 1.934
\]
As a result the difference can be estimated to be
\[
h_F(L) - h_Q(L) \geq 1.934g - 4.273 > -2.334
\]
effective for all \( g \geq 1 \).

Remark 4.2. We note that obtaining an effective upper bound on \( M_g \) would be much more difficult as we would have to study the degeneration of \( \delta_X \) in \( M_g \). For recent work on this we cite [Robin de Jong 2017].

Remark 4.3. By Elkies’ result, the Faltings metric is ‘much larger’ than the \( L^2 \) metric when \( \text{deg}(L) \) is large enough. It would be interesting to know if semi-positive condition of the curvature under Arakelov metric induces an effective estimate of analytic torsion.

5. Appendix: Effective bounds for small genus

For the convenience of the reader we list some of the upper bounds we obtained for small \( g \). We note that the case for \( g = 0 \) was slightly misleading as Arakelov metric in this case is not exactly the round metric of the unit sphere (it is a multiple of Fubini-Study metric), which caused the error in Soulé’s Bourbaki paper. We did the numerical computation using Wolfram Alpha:

- \( g = 0 \): Computational result by [Jorgenson1992] is
  \[
  \det(\Delta_g) = \exp(-4\zeta'(-1) + 7/6 - 4/3 \log(2)) \approx 2.46984
  \]

- \( g = 1 \): After identifying \( X_g = \langle 1, \tau \rangle \) with \( \tau = \langle x, iy \rangle \) we have:
  \[
  \det(\Delta_g) \leq 2\pi y^2 e^{-\frac{2}{3}y^2 + \frac{1}{3}x^2}
  \]
  (For an elementary derivation, see section 6)
6. Appendix: The case for elliptic curve

We identify the elliptic curve to be generated by the lattice \( \langle 1, \tau \rangle \):

**Lemma 6.1.** Let \( \tau = x + iy \). Then we have the Arakelov metric associated to the elliptic curve to be

\[
|dz|_{Ar} = \frac{\sqrt{y}}{2\pi} \| \eta(\tau) \|^{-2} = \frac{\sqrt{y}}{2\pi} \cdot \frac{1}{\sqrt{y} \eta(\tau)^2} = \frac{1}{2\pi \eta(\tau)^2}
\]

**Proof.** This was proved by Faltings in [Faltings1984]. \( \square \)

**Lemma 6.2.** The Arakelov metric over functions is given by

\[
\mu_{Ar} = \pi |\eta(\tau)|^2 dz \wedge \overline{dz}
\]

**Proof.** Since the metric on \( \mathcal{O}_X \) is the dual metric of the metric on \( K_X \), we have

\[
\mu_{Ar} = \frac{|dz|}{|dz|_{Ar}} \frac{i}{2} dz \wedge \overline{dz} = \pi i |\eta(\tau)|^2 dz \wedge \overline{dz}
\]

**Lemma 6.3.** The Arakelov area associated to the Arakelov metric is

\[
\text{Area}_{Ar} = 2\pi y|\eta(\tau)|^2
\]

**Proof.** We know that the area associated to the fundamental domain is

\[
\frac{i}{2} \int_D dz \wedge \overline{dz}
\]

As a result, we have

\[
\text{Area}_D = \int_0^{x+iy} \int_0^1 |\mu|_{Ar} = 2\pi y|\eta(\tau)|^2
\]

**Lemma 6.4.** If we normalize the unit area to be 1, such that \( E \cong \mathbb{R}^2/L \), where \( L = \mu(1, z) \), then the regularized determinant associated to the flat metric for a lattice of unit area is

\[
\det(\Delta) = y|\eta(z)|^4
\]
Proof. The computation here was done by Osgood, Philips and Sarnark. They pointed out that for flat metric with area 1 on the complex plane, we have ([Osgood-Phillips-Sarnak 1988], page 204):

$$\log(\det(\Delta)) = \log(y|\eta(z)|^4)$$

□

Lemma 6.5. The regularized determinant associated to the Arakelov metric is

$$\log(\det(\Delta_{Ar})) = \log(2\pi) + 2\log(y) + 6\log(|\eta(\tau)|)$$

Proof. By Lemma 6.2, the Arakelov metric is a scaled version of the flat metric on \((1, \tau)\). By Lemma 6.3, the scaling factor versus unit area metric is

$$\gamma = \sqrt{2\pi y|\eta(\tau)|^2}$$

By Lemma 6.4, the regularized determinant associated to the lattice with unit area flat metric is

$$\log(\det(\text{Unit})) = \log(2\pi y|\eta(\tau)|^2)$$

We now cite a result in ([Osgood-Phillips-Sarnak 1988], page 156): For elliptic curve, under a scaling factor of \(\gamma^2 : g(x, x) \rightarrow \gamma^2 g(x, x)\) of the metric \(g\), we have

$$\log(\det(\gamma^2 g)) = 2\log(\gamma) + \log(\det(g))$$

Indeed this can be proved directly: We have (note \(Z(0) = -1\):

$$Z_L(s) = \sum_{l \in L^*} \frac{1}{(2\pi |l|)^{2s}} \rightarrow Z_{\gamma L} = \gamma^{2s} \sum_{l \in L^*} \frac{1}{(2\pi |l|)^{2s}}$$

Combining everything together we have

$$\log(\det(\Delta_{Ar})) = \log(2\pi y|\eta(\tau)|^2) + \log(y|\eta(\tau)|^4) = \log(2\pi) + 2\log(y) + 6\log(|\eta(\tau)|)$$

□

Corollary 6.6. We have

$$\log(y^2|\eta(\tau)|^6) = \log(y^2 e^{-\frac{\pi y}{2}} \prod_{n=1}^{\infty} |1 - e^{2\pi in\tau}|^6)$$

to be bounded above by

$$2\log(y) - \frac{\pi y}{2} + \frac{6}{\pi y}$$

Proof. We have the following elementary inequality (I learned this by reading Rafael von Kanel):

$$\log\left(\prod_{n=1}^{\infty} (1 - q^n)\right) \leq \frac{|q|}{(1 - |q|)} , q = e^{2\pi i\tau}$$

As the result the above term is bounded above by

$$2\log(y) - \frac{\pi y}{2} + \frac{6}{1 - e^{2\pi y}} = 2\log(y) - \frac{\pi y}{2} + \frac{6}{e^{2\pi y} - 1} \leq 2\log(y) - \frac{\pi y}{2} + \frac{3}{\pi y}$$

where we recall \(q = e^{2\pi i\tau} = e^{2\pi ix} \cdot e^{-2\pi y}\), so we have \(|q| = e^{-2\pi y}\).

□
[Aitken 1995] Wayne Aitken, An arithmetic Riemann-Roch theorem for singular arithmetic surfaces, Memoir of AMS, 1995

[ACG 2010] Enrico Arbarello, Maurizio Cornalba, Phillip A. Griffiths, Geometry of algebraic curves II, 2010, Grundlehren der mathematischen Wissenschaften

[Arakelov 1977] S J Arakelov, Intersection theory of divisors on an arithmetic surface, (1974) Math. USSR Izv. 8 1167

[Berman] Robert J Berman, Analytic torsion, vortices and positive Ricci curvature, https://arxiv.org/pdf/1006.2988.pdf

[Bismut-Vasserot 1989] The asymptotics of the Ray-Singer analytic torsion associated with high powers of a positive line bundle, Comm. Math. Phys. Volume 125, Number 2 (1989), 355-367.

[Bost 1986-1] J.B.Bost, Conformal And Holomorphic Anomalies On Riemann Surfaces And Determinant Line Bundles, Marseille 1986, Proceedings, Mathematical Physics* 768-775. Presented at Conference: C86-07-16.1, p.768-775 Print-86-1323 (Ecole Normale)

[Bost 1986-2] J.B.Bost, Luis Alvarez-Gaum, Jean-Benot Bost, Gregory Moore, Philip Nelson, Cumrun Vafa Bosonization in arbitrary genus, Physics Letters B

[Bost 1987] Luis Alvarez-Gaum, Jean-Benot Bost, Gregory Moore, Philip Nelson, Cumrun Vafa, Bosonization on higher genus Riemann surfaces, Comm. Math. Phys. Volume 112, Number 3 (1987), 503-552.

[Bost2015] J.B.Bost, Theta invariants of euclidean lattices and infinite-dimensional hermitian vector bundles over arithmetic curves, https://arxiv.org/abs/1512.08946

[Bost 2017] J. B. Bost, Question and answer session, Gomtrie d’Arakelov et applications diophantienes 2017.

[Borisov 2003] Alexander Borisov, Convolution structure and arithmetic cohomology, Compositio Mathematica (2003) no. 136, page 237-254.

[Chinburg 1984] Ted Chinburg, An introduction to Arakelov intersection theory, Arithmetic Geometry, G. Cornell and J.H. Silverman eds, Spring-Verlag, Berlin Heidelberg New York (1986), p. 289-307.

[Chinburg 2015] T.Chinburg, G.Pappas, M.J.Taylor, Higher adèles and non-abelian Riemann-Roch, Advances in Mathematics, Volume 281, 20 August 2015, Pages 928-1024

[Cipriani-Ginkel] Alessandra Cipriani and Bart Van Ginkel, The discrete gaussian free field on a compact manifold, https://arxiv.org/pdf/1809.03382.pdf

[Dedekind] Erlunterungen zu den vorstehenden Fragmenten von R. Dedekind. Die Entstehungszeit (September 1852)

[Deligne 1987] P. Deligne, Le dterminant de la cohomologie, Contemporary mathematics, Volume 67.

[Dolce] Paolo Dolce, Adelic geometry on arithmetic surfaces I: adelic and adelic interpretation of the Deligne pairing, https://arxiv.org/abs/1812.10834

[Robin de Jong 2004] Robin de Jong, Explicit Arakelov Theory, https://wiki.epfl.ch/waldspurger/documents/dejong-thesis.pdf

[Robin de Jong 2005] Robin de Jong, On the Arakelov theory of elliptic curves, l’Ens. Math. 51 (2005), 179–201, https://arxiv.org/abs/math/0312359

[Robin de Jong 2017] Robin de Jong, Faltings invariant and semi-stable degeneration, https://arxiv.org/pdf/1511.06567.pdf

[Grothendieck 1957] A. Grothendieck, Sur quelques points d’algbre homologique, Thoku Mathematical Journal

[Hartshorne 1977] Robin Hartshorne, Algebraic Geometry, Graduate Text in Mathematics, 52

[Faltings1984] Gerd Faltings, Calculus on arithmetic surfaces, Annals of Mathematics, (1984), page 387-424

[Gelfand 1964] I.M.Gelfand and N.Ya.Yilenkin, Applications of harmonic analysis, Academy Press, 1964

[G-S 1991] H. Gillet, C. Soule, On the number of lattice points in convex symmetric bodies and their duals, Israel Journal Of Mathematics, Vol. 74, Nos. 2-3, 1991

[G-S 1998] H. Gillet, C. Soule, Upper Bounds for Regularized Determinants, Communications in Mathematical Physics, December 1998, Volume 199, Issue 1, pp 99115
[G-S 2000] Van der Geer, G. & Schoof, Effectivity of Arakelov Divisors and the Theta Divisor of a Number Field, Selecta Mathematica, New ser. (2000) 6: 377

[Harijac 2017] Paul Harijac, Private communication via email.

[Jorgenson 1990] Jay, Jorgenson, Asymptotic behavior of Faltings’ delta function, Duke Mathematical Journal Volume 51, Number 1 (August 1990), 221-254.

[Jorgenson 1991] Jay, Jorgenson, Analytic torsion for line bundles on Riemann surfaces, Duke Mathematical Journal Volume 62, Number 3 (April 1991), 527-549.

[Jorgenson 1992] Jay, Jorgenson, Degenerating hyperbolic Riemann surfaces and an evaluation of the constant in Deligne’s arithmetic Riemann-Roch theorem, Future preprint

[JK 2001] J. Jorgenson, J. Kramer: Bounds for special values of Selberg zeta functions of Riemann surfaces, J. reine angew. Math. 541 (2001), 1-28.

[JK 2006] Jorgenson, Kramer, Expressing Arakelov invariants using hyperbolic heat kernels, In: J. Jorgenson and L. Walling (eds.), The Ubiquitous Heat Kernel. Contemp. Math. 398, 295-309. Amer. Math. Soc., Providence, RI, 2006.

[JK 2006-2] Jorgenson, Kramer, Non-completeness of the Arakelov-induced metric on moduli space of curves, Manuscripta Math. 119 (2006), 453-463.

[JK 2006-3] Jorgenson, Kramer. Bounds on Canonical Green’s Functions. Compositio Mathematica 142, no. 3 (2006): 679-700. doi:10.1112/S0010437X05001990.

[K-L 2015] K. Sugahara, L. Weng, Arithmetic Cohomology Groups, https://arxiv.org/abs/1507.06074

[Menares 2011] Ricardo Menares, Correspondences in Arakelov geometry and applications to the case of Hecke operators on modular curves.

[Moriwaki 2012] Atsushi Moriwaki, Zariski Decompositions on Arithmetic Surfaces, Publ. RIMS Kyoto Univ. 48 (2012), 799-898

[Neukirch 1997] Jrgen Neukirch, Algebraic Number Theory, 1999, Springer

[Osgood-Phillips-Sarnak 1988] Osgood-Phillips-Sarnak, Extremals of determinant of Laplacians, Journal of Functional Analysis, Volume 80, Issue 1, September 1988, Pages 148-211

[Quillen 1984] Daniel Quillen, Determinants of Cauchy-Riemann operators over a Riemann surface, Functional Analysis and Its Applications

[Ray-Singer 1971] Ray, Daniel B.; Singer, Isadore M, R-torsion and the Laplacian on Riemannian manifolds, Advances in Mathematics, 7 (2): 145-210

[Ray-Singer 1973] Ray, Daniel B.; Singer, Isadore M, Analytic torsion for complex manifolds, Annals of Mathematics, 2, 98 (1): 154-177

[Schwarz 1979] Schwarz, A. S. The partition function of a degenerate functional. Comm. Math. Phys. 67 (1979), no. 1, 1-16.
[Seeley 1967] R.T. Seeley, *Complex powers of an elliptic operator*, Proc. Symp. Pure Math. 10 (1967) 288-307

[Sheffield 2003] Scott Sheffield, *Gaussian free field for mathematicians*. https://arxiv.org/abs/math/0312099

[Smit 1988] Dirk-Jan Smit, *String theory and algebraic geometry of moduli spaces* Communications in Mathematical Physics, December 1988, Volume 114, Issue 4, pp 645-685

[Soule 2017] Christopher Soule, *arithmetic intersection*, Gomtrie d’Arakelov et applications diophantiennes 2017, To be published in Proceedings

[Soule 2017] Christopher Soule, *arithmetic intersection*, Gomtrie d’Arakelov et applications diophantiennes 2017, Private communication

[Soule 1988] Christopher Soule, *Geometrie d’Arakelov des surfaces arithmetiques*, Seminaire N. Bourbaki, no. 327-343

[Soule 1992] Christopher Soule, written with D. Abramovich, J-F. Burnol, J. Kramer, *Lectures on Arakelov geometry*, Cambridge University Press, (1992)

[Wentworth 1990] Richard Wentworth, *Asymptotics of determinants from functional integration*, J. Math. Phys. 32 (7) (1991), 1767-1773

[Wentworth 2009] Richard Wentworth, *Precise constants in bosonization formulas on Riemann surfaces I*, Commun. Math. Phys. 282 (2) (2008)

[Wilms 2016] Robert Wilms, *New explicit formulas for Faltings delta-invariant*, Invent. math. (2017) 209: 481. https://doi.org/10.1007/s00222-016-0713-1, https://link.springer.com/article/10.1007/s00222-016-0713-1

[Wilms 2017] Robert Wilms, *Mathoverflow Forum*, https://mathoverflow.net/questions/255722/why-are-green-functions-involved-in-intersection-theory

[Vardi 1988] Ilan Vardi, *Determinants of Laplacians and Multiple Gamma Functions*, SIAM Journal on Mathematical Analysis, 1988, Vol. 19, No. 2 : pp. 493-507

[Voisin 2002] Claire Voisin, *Hodge theory and complex algebraic geometry I, II*, Cambridge University Press, New York, 2002, ix+322 pp., 65, ISBN 0-521-80260-1 (Vol. I); 2003, ix+351 pp., 65, ISBN 0-521-80283-0 (Vol. II)

[Yuan 2013] Xinyi Yuan and Tong, *Effective bound of linear series on arithmetic surfaces*, Zhang, Duke Math. J. Volume 162, Number 10 (2013), 1723-1770.

[Yuan 2012] Xinyi Yuan, *Big Line Bundles Over Arithmetic Varieties*, Inventiones Mathematicae, 173 (2008), no. 3, 603-649

[Zha 1999] Yuhan Zha, *A general arithmetic Riemann-Roch theorem*, Thesis, University of Chicago, Dept. of Mathematics, June 1998

[Zhang 1990] Zhang Shouwu, *Positive Line Bundles on arithmetic Surfaces*, Annals of Mathematics, 1992