Quantum Hall States of Gluons in Quark Matter

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Savvidy vacuum of SU(2) gauge theory is known to be a color ferromagnetic state, but to be unstable. We show that unstable modes in the vacuum condense to form a stable quantum Hall state. As a result, the ferromagnetic state involving the quantum Hall state is realized as a stable ground state. We discuss that such a state arises in dense quark matter.

§1. introduction

Quark matter is known or expected to have several phases, hadronic phase, quark gluon plasma phase and color superconducting phase.1) Among the phases only hadronic phase is observed. When the temperature is sufficiently large, the matter is expected to form the quark gluon plasma, which is accessible in the present experiments. Although the color superconducting phase is very intriguing, present experiments could not produce such a phase because large chemical potential like 1 GeV is needed for the realization of the phase.

We have recently discussed 2) a possible existence of the stable color ferromagnetic states of dense quark matter. The ferromagnetic state is caused by the condensation of the color magnetic field, not by the alignment of the quark’s magnetic moments. The states are realized between the hadronic state and the color superconducting state when the chemical potential is varied. Thus, the phase is expected to be observed in the very near future. The ferromagnetic states possess a spontaneously generated color magnetic field in maximal Abelian sub-algebra and also involve a quantum Hall state of off diagonal gluons. The gluons are ones known as unstable modes6) in the color magnetic field $B$. They occupy the lowest Landau level with their spins pointed to the magnetic field and with their energies being imaginary. We have shown 2) recently that the formation of the gluon’s QHS makes the ferromagnetic state be stabilized.

In this paper we review our works along with quantum Hall state (QHS)4) of electrons in semiconductor and Chern-Simons gauge theory5) for describing the QHS. Applying the theory to the unstable gluons, we show that the gluons form the QHS and the instability disappears in the state. We discuss mainly SU(2) gauge theory with massless quarks of the two flavours, but make brief comments on the case of SU(3).
As is well known, \(^3, ^6\) the one loop effective potential \(V\) of the constant color magnetic field in the SU\((n_c)\) gauge theory with massless quarks of \(n_f\) flavours is given by \(V_{\text{eff}} = \frac{11N}{96\pi^2}g^2B^2 \left( \log (gB/A^2) - \frac{1}{2} \right) - \frac{i}{8\pi}g^2B^2\), with an appropriate renormalization \(^7\) of the gauge coupling \(g\), where \(N = n_c - 2n_f/11\). Since we consider SU\((2)\) gauge theory with massless quarks of two flavours, \(N = 18/11\). The potential implies the spontaneous generation of a color magnetic field. The state is called as Savvidy vacuum. The directions of the magnetic field in real space can be arbitrarily chosen and the direction in color space can be taken in general such as \(B\) is in the maximal Abelian sub-algebra. In any case of their choices the spontaneous generation of the magnetic field breaks the spatial rotational symmetry and the SU\((2)\) gauge symmetry (SU\((3)\) gauge symmetry) into the gauge symmetry of U\((1)\) (U\((1)\times U(1))\). This fact has made invalid the adoption of the state as a true vacuum of the gauge \(U(1)\).

It apparently seems that the spontaneous generation of the color magnetic field, namely, the realization of a ferromagnetic state occurs. But it is not so simple since the imaginary part in \(V\) \((eB)\) is present when \(eB \neq 0\). It means that the state (Savvidy vacuum) with the magnetic field is unstable as well as a perturbative vacuum state with \(eB = 0\). Actually, the unstable modes of gluons are present in the magnetic field (in the ferromagnetic state). Thus, this naive ferromagnetic state is unstable. \(^7\) The modes are expected to make a stable state. What kind of the stable state is formed of the unstable gluons? We have shown \(^2\) that the state is a QHS of the gluons with the color magnetic field. In order to explain it in the simple case of SU\((2)\) gauge theory, we decompose the gluon’s Lagrangian with the use of the variables, “electromagnetic field” \(A_\mu = A^3_\mu\), and “charged vector field” \(\Phi_\mu = (A^1_\mu + iA^2_\mu)/\sqrt{2}\) where indices 1 \(\sim\) 3 denote color components,

\[
L = -\frac{1}{4} \tilde{F}^{\mu\nu}_{\mu\nu} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2} |D_\mu \Phi_\nu - D_\nu \Phi_\mu|^2 + i e (\partial_\mu A_\nu - \partial_\nu A_\mu) \Phi_\nu^i \Phi_\mu^j + \frac{e^2}{4} (\Phi_\nu^i \Phi_\nu^j - \Phi_\mu^i \Phi_\mu^j)^2
\]

(1.1)

with \(D_\mu = \partial_\mu + ieA_\mu\), where we have omitted a gauge term \(D_\mu \Phi_\mu = 0\). We can derive using the Lagrangian that the energy \(E\) of the charged vector field \(\Phi_\mu \propto e^{ie\mu}\) in the magnetic field, \(A_\mu = A^B_\mu\), is given by \(E^2 = k_3^2 + 2eB(n + 1/2) \pm 2eB\) with a gauge choice, \(A^B_{\mu j} = (0, x_2B, 0)\) and \((\partial_\mu + ieA^B_\mu)\Phi_\mu = 0\), where we have taken the spatial direction of \(\vec{B}\) being along \(x_3\) axis. \(\pm 2eB\) (the integer \(n \geq 0\)) denote contributions of spin components of \(\Phi_\mu\) (Landau levels) and \(k_3\) denotes momentum in the direction parallel to the magnetic field. (Similar decomposition of gluons in SU\((3)\) gauge theory is possible. Taking Abelian gauge field as \(A_\mu = A^3_\mu\) \(\cos \theta + A^8_\mu \sin \theta\), off diagonal components are given by \(\Phi^a_\mu = (A^1_\mu + iA^2_\mu)/\sqrt{2}, \Phi^b_\mu = (A^4_\mu + iA^5_\mu)/\sqrt{2}\), and \(\Phi^c_\mu = (A^6_\mu - iA^7_\mu)/\sqrt{2}\), where the direction, \(\theta\), of the color magnetic field in color space \((\lambda_3, \lambda_8)\) is determined dynamically.)

Obviously, the modes with \(E^2(n = 0) < 0\) are unstable modes occupying the...
lowest Landau level and with spin parallel to $\vec{B}$. Among them, the modes with $k_3 = 0$ are the most unstable ones, which means that they have the largest negative value of $E^2(k_3 = 0)$. Thus, they are expected to form a stable state as in a simple model of a complex scalar field $\phi$ with a double well potential, $-m^2|\phi|^2 + \lambda|\phi|^4/2$. (The unstable mode $\phi(k = 0)$, not $\phi(k \neq 0)$ around the state $\langle \phi \rangle = 0$ form the stable ground state $\langle \phi \rangle = \sqrt{m^2/\lambda}$.) Since they have no $x_3$ dependence, they are two dimensional objects occupying the lowest Landau level. The situation is quite similar to the one of the two dimensional electrons forming QHSs. The only difference is that in the gauge theory gluons are bosons, while electrons are fermions.

In order to find the stable state in the gauge theory, we extract only the most unstable modes from the Lagrangian equation (1.1), ignoring the other modes coupled with them and obtain two dimensional Lagrangian,

$$L_{\text{unstable}} = |(i\partial_\nu - eA^B_\nu)\phi_u|^2 + 2eB|\phi_u|^2 - \frac{\lambda}{2}|\phi_u|^4;$$  

with $\lambda = e^2/l$, where the field $\phi_u = (\Phi_1 - i\Phi_2)\sqrt{1/2}$ denotes the unstable modes in the lowest Landau level. $l$ denotes the coherent length of the magnetic field, namely, its extension in the direction of the field. The index $\nu$ runs from 0 to 2. We note that the field $\phi_u$ has a color charge associated with $\lambda_3$ of SU(2) algebra. This color charge is only a conserved quantity when the spontaneous generation of the color magnetic field $\propto \lambda_3$ occurs in the SU(2) gauge theory. In the case of SU(3) gauge theory, the potential term of the unstable gluons is a little bit complicated (see our paper$^2$) due to the presence of three types of the unstable modes.

In order to see explicitly the QHS of the field, we introduce Chern-Simons gauge field to make composite gluons; bosons attached with the Chern-Simons flux. Then, a relevant Lagrangian is given by

$$L_a = |(i\partial_\nu - eA^B_\nu + a_\nu)\phi_a|^2 + 2eB|\phi_a|^2 - \frac{\lambda}{2}|\phi_a|^4 - \frac{\epsilon^{\mu\nu\lambda}}{4\alpha}a_\mu\partial_\nu a_\lambda;$$

where $\alpha$ called as statistical factor should be taken as $\alpha = 2\pi \times \text{integer}$ to keep the equivalence of the system described by $L_a$ to that of $L_{\text{unstable}}$. The field $\phi_a$ represents so-called composite gluons attached with the Chern-Simons flux $a_i$.

The equivalence between $L_{\text{unstable}}$ and $L_a$ has been shown$^8$ in an operator formalism although the equivalence had been known in the path integral formalism using the world lines of the $\phi_a$ particles. (In the formalism the last term in Eq. (1.3) produces a phase, $e^{i\lambda}/\pi$, in wavefunctions when trajectories of two particles are interchanged.)

In order to explain why we introduce the Chern-Simons gauge field for deriving QHSs of the gluons, we review briefly the theory of electron’s QHSs in semiconductors.

QHS of electrons has been observed$^7$ in 1980 by von Klitzing. He has observed quantized Hall conductivities $\sigma_{xy}$ with the unit of the fundamental constant $e^2/2\pi\hbar$ in a two dimensional quantum well fabricated of a semiconductor. The observation indicated the existence of a specific state of two dimensional electrons in the well under the strong magnetic field, $B$ perpendicular to the well. The original QHS
was called as integer quantum Hall state since the state is observed at filling factor being integer; the filling factor is defined as \( \rho/(eB/2\pi) \) (\( \rho \) is two dimensional number density of electrons). States of the two dimensional electrons in the magnetic field are specified by Landau levels, each of which has a large number of degenerate states; the degeneracy per unit area is given by \( eB/2\pi \). Thus, the filling factor means a fraction of electron occupation in a Landau level. For example, the filling factor \( \nu = 1/3 \) implies that electrons occupy one of third in the lowest Landau level. The integer filling factor implies that some of Landau levels are occupied completely. The integer QHS is understood as a localization property of each two dimensional electron. Many body effects are not important.

On the other hand, fractional quantum Hall effects were observed\(^{10}\) in 1982 by Tsui at the filling factor being fractional numbers, e.g. 1/3, 2/3, where electrons occupy a fraction in the lowest Landau level. The QHSs have been understood to be caused by many body effects of electrons, just like superconductivity. Laughlin\(^{11}\) proposed a wavefunction for this QHS, called as Laughlin wave function. Numerical simulations shows that the groundstates of the electrons at the fractional filling factors are well described by the Laughlin wavefunctions. It is difficult task to derive analytically the Laughlin wave functions. But one of interesting approaches is to use the Chern-Simons gauge theory. This is a theory of composite electrons describing real electrons in two dimensional spaces with the use of the boson field \( \phi_e \). The theory is given by,\(^{12,5}\)

\[
L_{\text{QHS}} = \phi_e^\dagger(i\partial_0 - a_0)\phi_e + c.c. - \frac{1}{2m_e}|(i\partial_i - eA_i^B + a_i)\phi_e|^2 - V_{\text{Coulomb}} - \frac{1}{4\alpha}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda,
\]

where \( m_e \) denotes mass of electrons and \( V_{\text{Coulomb}} = \int d^2x d^2y (\phi_e(x) - \rho) \frac{1}{2|x-y|} (\phi_e(y) - \rho) \) describes the Coulomb interaction between electrons. The term \( A_i^B \) describes the external magnetic field imposed for the realization of QHS. The statistical factor of \( \alpha \) should be taken as \( \pi \times \text{odd integer} \) for the boson field \( \phi_e \) describing real electrons.

The boson field \( \phi_e \) describes composite electrons; boson \( \phi_e \) attached with flux of \( a_i \). That is, particles with Fermi statistics can be described in two dimensional space by bosonic particles attached with a fictitious flux \( 2\alpha \) of Chern-Simons gauge field \( a_i \). Owing to this flux, the exchange of the bosonic particles induces a phase \( e^{i\alpha} \) in their wavefunction. Thus, with the choice of \( \alpha = \pi \times \text{odd integer} \), the wavefunctions describe particles with Fermi statistics. This situation is described mathematically by \( L_{\text{QHE}} \).

We should note that if the Chern-Simons gauge fields \( a_\mu \) are absent and \( \phi_e \) obeys Fermi statistics in \( L_{\text{QHS}}(a_\mu = 0) \), \( L_{\text{QHS}}(a_\mu = 0) \) describes ordinary electrons in two dimensions.

From the above Lagrangian, we can show that the QHS is described by the condensed state of the bosonised electrons \( \langle \phi_e \rangle \neq 0 \). The state can be obtained only under the condition that \( eA_i^B = a_i \). Namely, the magnetic field \( eA_i^B \) is cancelled by the Chern-Simons gauge field \( a_i \); the field is given by the density \( \rho \) of electrons \( \phi_e^\dagger\phi_e \) such as \( \phi_e^\dagger\phi_e = \epsilon_{ij}\partial_i a_j / 2\alpha \), an equation derived by taking variational derivative of \( L_{\text{QHS}} \) in \( a_0 \). Hence, the QHS is obtained when the filling factor \( \nu = 2\pi\rho/eB \) is given.
by $\pi/\alpha = 1/3, 1/5$, etc. for $\pi/\alpha = 3\pi, 5\pi$, etc. The Laughlin wavefunctions can be derived\(^{13}\) approximated from these condensed states.

Up to now, we have reviewed the Chern-Simons gauge theory of the QHSs of ordinary electrons. Let us proceed to explain that the unstable gluons occupying the lowest Landau level under the color magnetic field form a QHS as a possible stable state. Lagrangian in Eq. (1.3) describes the dynamics of the unstable gluons occupying the lowest Landau level. The derivation of the gluon’s QHS is straightforward. Namely, from equations of motion we find spatially uniform solutions such as $\langle \phi_a \rangle \neq 0$. Such solutions are possible only when the cancellation between the magnetic field and the Chern-Simons gauge field arises;

$$eA_i = a_i.$$  

The solutions are given by solving the following equations,

$$2a_0 |\phi_a|^2 = eB/2\alpha \quad \text{and} \quad a_0^2 + 2eB = \lambda |\phi_a|^2. \quad (1.5)$$

The first equation implies the filling factor $2\pi \rho_c/eB$ being given by $\pi/\alpha$ where $\rho_c = 2a_0 |\phi_a|^2$ denotes the color charge density associated with $\lambda_3$. Thus, in the QHS of the gluons the factor takes values such as $1/2, 1/4, \cdots$ because the gluons are bosons. Solving the above equations, we find $a_0$ and $\phi_a$ in terms of $eB$ and $\lambda$.

Using the ground state solution of QHS we can show that the state is stable against for small fluctuations $\delta \phi_a$ and $\delta a_i$. Furthermore, we can show that the state is also stable against for the formation of the vortex solitons which correspond to Laughlin’s quasiparticles in ordinary QHS. Namely, the energy of the solitons is positive. The fact is not necessarily obvious in this system. In this way, the Savvidy vacuum (color ferromagnetic state) is stabilized by the formation of quantum Hall states of the unstable gluons.

In the case of $\text{SU}(3)$ gauge theory, several types of QHSs may arise owing to the presence of three different unstable gluons. In particular, it is interesting that Higgs mechanism works for the stabilization of unstable gluons in the QHSs. For example, only one of unstable gluons $\phi^a$ condenses to form a QHS where the other two gluons gain large positive masses due to the condensation. As a result, the two gluons are stabilized as well as $\phi^a$ which forms the QHS. Very rich structures might arise in $\text{SU}(3)$ gauge theory. We have not yet fully understood these points.

We should mention that the formation of the QHS requires color charge supply from outside of the gluon sector. Namely, the condensate of $\phi_a$ carries the color charges associated with $\lambda_3$. Realistically, the state is formed in dense quark matter. Namely, the color charges of the quarks are transmitted to the gluons, which condensate to form QHSs. Thus, if the number density of the quarks is not sufficient for the formation, such a stable color ferromagnetic state does not arise. This gives a lower limit of the chemical potential of the matter for the realization of the ferromagnetic state. The detail should be referred to our paper.\(^2\) Anyway, original unstable Savvidy vacuum (color ferromagnetic state) is stabilized by the formation of QHSs of unstable gluons occupying the lowest Landau level just as electrons in quantum wells of semiconductors. The formation of the QHS is realized in the dense quark matters which supply color charges to the condensate of the gluons.
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