On generalized Caputo fractional differential equations and inclusions with non-local generalized fractional integral boundary conditions

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Abstract
In this article, concerning nonlocal generalized fractional integral boundary conditions, we investigate the existence of solutions for new boundary value problems of generalized Caputo-type fractional differential equations and inclusions. In the case of equations, we make use of the Banach fixed point theorem and fixed point theorem due to O’Regan and the nonlinear alternative for contractive maps for inclusions. Examples are given to clarify our main results. Finally, we discuss some variants of the given problem.

Keywords
Fractional differential equations, generalized Caputo fractional derivative, Generalized Riemann-Liouville fractional integral, Non-local, Existence, Inclusions, Fixed point.

AMS Subject Classification
26A33, 34A08, 34A12, 34B10, 34A60.

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1. Introduction
The technique of dealing with fractional differential equations (FDEs) wasn’t much for young researchers until the generalized fractional derivatives defined by various fractional operators were exposed. Generalized fractional differential equations (GCFDEs) of Caputo type resulted in effective numerical solutions to differential equations. It is also well-known that the Caputo derivative plays an essential role in physical memory problems. Katugampola in [15] has also introduced a new fractional integral, which acts as a combined integral for the Riemann-Liouville and Hadamard integrals. Research work in this area has grown significantly worldwide due to extensive applications of FDEs in engineering and science. For examples and details see [1, 3, 6, 7, 9–13, 17, 21, 22, 27–31, 33] and the references cited therein. In recent years non-local boundary value problems (BVPs) for FDEs and inclusions have been studied by many researchers. Ntouyas et.al [23] has studied the existence of solutions with sum and integral boundary conditions for fractional differential inclusions. In [2] the authors analyzed the existence results for the fractional differential inclusion and the integrated boundary conditions of the form of type Erdelyi-Kober:

\[
D^q x(t) \in F(t, x(t)), \quad t \in [0, T],
\]

\[
x(0) = 0, \quad \alpha x(T) = \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i, \delta_i} x(\xi_i),
\]

where \(1 < q \leq 2\), \(D^q\) is the Riemann-Liouville fractional derivative (RLFD) of order \(q\), \(F : [0, 1] \times \mathbb{R} \rightarrow P(\mathbb{R})\) is a multivalued map, and \(P(\mathbb{R})\) is the family of all nonempty subsets of \(\mathbb{R}\), and \(\alpha, \beta_i \in \mathbb{R}, \xi_i \in (0, T), I_{t_i}^{\gamma_i, \delta_i}\) is the Erdelyi-Kober
fractional integral of order \(\delta > 0, \eta_i > 0, \gamma \in \mathbb{R} \) \(i = 1, 2, \ldots, m\) are given constants. In [4] the authors investigated the FDEs and inclusion with integral boundary conditions of the form of Erdelyi-Kober type:

\[
{}^cD^\rho x(t) = f(t, x(t)), \quad t \in [0, T],
\]

\[x(0) = g(x(t)), \quad x(T) = aI^\rho \delta x(\xi), \]

and

\[
{}^cD^\rho x(t) \in F(t, x(t)), \quad t \in [0, T],
\]

\[x(0) = g(x), \quad x(T) = aI^\rho \delta x(\xi), \]

where \(1 < q \leq 2, D^\rho\) is the Caputo fractional derivative (CFD) of order \(q\), \(f: \mathcal{J} \times \mathbb{R} \to \mathbb{R}\) and \(g: \mathcal{J} \times \mathbb{R} \to \mathbb{R}\) are given continuous functions, and \(F: [0, 1] \times \mathbb{R} \to \mathbb{P}(\mathbb{R})\) is a multi-valued map, and \(\mathbb{P}(\mathbb{R})\) is the family of all nonempty subsets of \(\mathbb{R}\). The general results for FDEs. Examples in section 4 are used to validate the solutions. Section 5 contains the main results for FDEs. Inclusions. The important observations of the results are made in Section 6.

\[\begin{align*}
\mathcal{L}_1^1(b, c) &\text{ refers to the measurable space of all Lebesgue functions } \phi \text{ on } (b, c) \text{ endowed with the norm:} \\
\|\phi\|_{\mathcal{L}_1^1} &= \int_c^b |\phi(u)| \, du < \infty.
\end{align*}\]

**Definition 2.1.** [16] The left and right-sided GRLFIs of order \(\rho > 0\) and \(p > 0\), of a function \(h \in T_\rho^p(b, c), V \rightarrow b < \tau < c \leq \infty\), is defined as

\[
(\rho \mathcal{D}_b^p, h)(\tau) = \rho^{1-p} \frac{1}{\Gamma(p)} \int_b^\tau \left(\frac{u^{1-p} h(u)}{(\tau - u)^{1-p}}\right)^{\rho^{1-p}} \, du,
\]

\[
(\rho \mathcal{D}_c^p, h)(\tau) = \rho^{1-p} \frac{1}{\Gamma(p)} \int_\tau^c \left(\frac{u^{1-p} h(u)}{(\tau - u)^{1-p}}\right)^{\rho^{1-p}} \, du.
\]

**Remark 2.2.** The above definition for GRLFIs reduce to the standard Riemann-Liouville fractional integrals for \(\rho \rightarrow 1\) (see [16]), and Hadamard fractional integrals \(p \rightarrow 0\) respectively (see [18]).

**Definition 2.3.** [14] For \(\rho \geq 0\) and \(h \in \mathcal{D}_c^\rho [b, c], \) the left and right-sided GCFDs of order \(\rho\) are described by

\[
(\rho \mathcal{D}_b^\rho, h)(\epsilon) = \rho^{\rho-1} \frac{\epsilon^{1-\rho}}{\rho} \int_b^\epsilon \left(\frac{h(u)}{\epsilon^{1-\rho}}\right)^{\rho-1} \, du,
\]

\[
(\rho \mathcal{D}_c^\rho, h)(\epsilon) = \rho^{\rho-1} \frac{\epsilon^{1-\rho}}{\rho} \int_\epsilon^c \left(\frac{h(u)}{\epsilon^{1-\rho}}\right)^{\rho-1} \, du.
\]

**Remark 2.4.** The above definitions for GCFDs reduces to the standard Caputo derivatives and Hadamard fractional derivatives (see for \(p \rightarrow 1\) (see [14])) and \(p \rightarrow 0\) respectively (see [14]).

**Definition 2.5.** [14] Let \(\rho > 0, \) \(p > 0, \) \(n = [\rho] + 1, 0 < b < c \leq \infty. \) The operators

\[
(\rho \mathcal{D}_b^\rho, h)(\tau) = \frac{\rho^{\rho-n+1}}{(n-\rho)} \left(\tau^{1-\rho} \frac{d}{d\tau}\right)^n \int_b^\tau \left(\frac{h(u)}{(\tau - u)^{1-p}}\right)^{\rho^{1-p}} \, du,
\]

\[
(\rho \mathcal{D}_c^\rho, h)(\tau) = \frac{\rho^{\rho-n+1}}{(n-\rho)} \left(-\tau^{1-\rho} \frac{d}{d\tau}\right)^n \int_\tau^c \left(\frac{h(u)}{(\tau - u)^{1-p}}\right)^{\rho^{1-p}} \, du
\]

for \(\tau \in (b, c)\) are called the left and right sided generalized Riemann-Liouville fractional derivatives (GRLFDs) of fractional order \(\rho, \) respectively.

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**2. Preliminaries**

In section 2 we present some notations, fractional calculus definitions and preliminary results needed later on in our proof [14, 16, 18, 20, 26]. Let us define the space of complex-valued all Lebesgue measurable functions \(h: (b, c) \to \mathbb{R} \to \|h\|_{\mathcal{T}_\rho^p} < \infty, \) where \(a \in \mathbb{R}, \) \(1 \leq p \leq \infty\) and

\[
\|h\|_{\mathcal{T}_\rho^p} = \left( \int_b^c |v^p h(u)| \, du \right)^{\frac{1}{p}}, 1 \leq p \leq \infty.
\]

We emphasize that \(g(y)\) in (1.1) may be interpreted as

\[
g(y) = \sum_{j=1}^k c_j y(\tau_j), \quad \text{where} \quad \tau_1 < \tau_2 < \cdots < \tau_k \leq 1.
\]

Whatever is left of the paper is sorted out as follows. Section 2 shows the essential founding material identifying with our problem and has demonstrated an auxiliary lemma. Section 3 contains the main results for FDEs. Examples in section 4 are used to validate the solutions. Section 5 contains the main results for inclusions. The important observations of the results are made in Section 6.
Lemma 2.6. Let $\rho \geq 0$, $n = [\rho] + 1$ and $h \in AC^\rho_0[b,c]$, where $0 < b < c < \infty$. Then

(1) for $\rho \notin N$,
$$\rho \frac{\partial^\rho}{\partial x^\rho} h(x) = \frac{1}{\Gamma(n-\rho)} \int_b^x \frac{(v-x)^{n-\rho-1}}{\rho} (\delta^\rho h)(v) dv,$$
where $\delta^\rho h = (x^\rho - h(x)) / x^\rho$.

(2) for $\rho \in N$,
$$\rho \frac{\partial^\rho}{\partial x^\rho} h(x) = \frac{1}{\Gamma(n-\rho)} \int_b^x \frac{(v-x)^{n-\rho-1}}{\rho} (\delta^\rho h)(v) dv,$$
where $\delta^\rho h = (x^\rho - h(x)) / x^\rho$.

Lemma 2.7. Let $\rho \neq 0$, a function $h \in AC^\rho_0[b,c]$. Then
$$\rho \frac{\partial^\rho}{\partial x^\rho} h(x) = h(x) - \frac{1}{\Gamma(n-\rho)} \int_b^x \frac{(v-x)^{n-\rho-1}}{\rho} (\delta^\rho h)(v) dv,$$
where $\delta^\rho h = (x^\rho - h(x)) / x^\rho$.

In particular, if $0 < \rho \leq 1$, we have
$$\rho \frac{\partial^\rho}{\partial x^\rho} h(x) = h(x) - h(b),$$
$$\rho \frac{\partial^\rho}{\partial x^\rho} h(x) = h(x) - h(c).$$

Lemma 2.8. Let $\rho > 0$ and $\rho \in \mathbb{R}$. Then
$$\rho \frac{\partial^\rho}{\partial t^\rho} = \frac{1}{\Gamma(1+1)} \frac{\rho^p+1}{\rho^p+1} \rho^p.$$
The following hypotheses are necessary to prove the existence of theorems, we obtain the existence and uniqueness results.

**Theorem 2.15.** [25] Let \( \mathcal{W} \) be a Banach space, and \( \mathcal{U} \) a bounded neighborhood of 0 \( \in \mathcal{W} \). Let \( T_1 : \mathcal{W} \to \mathcal{E}_{cp,e}(\mathcal{W}) \) and \( T_2 : \mathcal{W} \to \mathcal{E}_{cp,e}(\mathcal{W}) \) be two multivalued operators satisfying (1) \( T_1 \) is contraction, and (2) \( T_2 \) is upper semi-continuous and compact.

Then, if \( \mathcal{M} = T_1 + T_2 \) either

(i) \( \mathcal{M} \) has a fixed point in \( \mathcal{W} \) or

(ii) there is a point \( v \in \partial \mathcal{U} \) and \( \beta \in (0, 1) \) with \( v = \beta \Theta(v) \) where \( \partial \mathcal{U} \) and \( \mathcal{U} \), respectively, represent the closure and boundary of \( \mathcal{U} \).

In this section, by using Banach and O’Regan’s fixed point theorems, we obtain the existence and uniqueness results.

### 3. Main Results : Single-valued case

For computational purposes, we represent:

\[
\Theta_1 = \frac{T^\rho}{\rho^\theta \Gamma(\rho + 1)} + \frac{T^\rho}{\rho^\sigma} \left[ \frac{\sigma^\rho (\rho + \zeta)}{\rho^\rho + \zeta \Gamma(\rho + \zeta + 1)} \right], \quad (3.1)
\]

and

\[
\Theta_2 = \frac{T^\rho}{\rho^\sigma}. \quad (3.2)
\]

The following hypotheses are necessary to prove the existence and uniqueness results. Let \( h : \mathcal{J} \times \mathbb{R} \to \mathbb{R} \) and \( g : \mathcal{G} \times \mathcal{J} \times \mathbb{R} \to \mathbb{R} \) be continuous functions.

\((\mathcal{J}_2)\) A constant \( \mathcal{K} > 0 \) exists such that \( |h(t, y, z)| \leq \mathcal{K}|y - z| \), for each \( t \in \mathcal{J} \) and \( y, z \in \mathcal{G} \).

\((\mathcal{J}_2)\) A constant \( k > 0 \) exists such that \( |g(y) - g(z)| \leq k|y - z|, k < \frac{1}{\Theta_2} \), for each \( y, z \in \mathcal{G} \).

\((\mathcal{J}_3)\) \( g(0) = 0 \).

\((\mathcal{J}_4)\) A nonnegative function \( \theta \in \mathcal{C}(\mathcal{J}, (0, \infty)) \) and a non-decreasing function \( \xi : [0, \infty) \to (0, \infty) \) satisfy \( h(t, \xi(|y|)) \leq \theta(t) \xi(|y|) \) for any \( (t, y) \in \mathcal{J} \times \mathcal{G} \).

\((\mathcal{J}_5)\) \[ \sup_{v \geq 0} \frac{\Theta_1 \| \theta \|_{L^1(v)}}{\xi(v)} \geq 1 - \Theta_2 k. \]

\((\mathcal{J}_6)\) \[ H : \mathcal{J} \times \mathcal{J} \to \mathcal{E}_{cp,e}(\mathbb{R}) \] is \( L^1 \)-Caratheodory multivalued map.\(\mathcal{G} \)

\((\mathcal{J}_7)\) A continuous nondecreasing function \( \xi : [0, \infty) \to (0, \infty) \) and a function \( \psi \in \mathcal{C}(\mathcal{J}, (0, \infty)) \) satisfy \( H(t, \phi) \|_{\mathcal{E}} = \sup \{ |z| : z \in H(t, y) \} \leq \theta(t) \xi(|y|) \) for each \( (t, y) \in \mathcal{J} \times \mathcal{G} \).

\((\mathcal{J}_8)\) A number \( \alpha > 0 \) exists such that

\[
\frac{(1 - \Theta_2 k) \alpha}{\Theta_1 \| \theta \|_{L^1(\alpha)}} > 1, \quad (3.3)
\]

where \( \Theta_1 \) and \( \Theta_2 \) are given by (3.1) and (3.2) respectively.

**Theorem 3.1.** Suppose that \((\mathcal{J}_1)\) and \((\mathcal{J}_2)\) holds. If

\[
\eta := \mathcal{K} \Theta_1 + k \Theta_2 < 1, \quad (3.4)
\]

where \( \Theta_1, \Theta_2 \), are defined by (3.1) and (3.2). Then the BVP (1.1) has a unique solution on \( \mathcal{J} \).

**Proof.** For \( y, z \in \mathcal{G} \) and each \( \tau \in \mathcal{J} \), the operator \( \Psi : \mathcal{G} \to \mathcal{G} \) is contraction, and \( (\mathcal{J}_3), (\mathcal{J}_4), (\mathcal{J}_5) \) holds. If \( \Psi_1 \) and \( \Psi_2 \), we obtain

\[
\| \Psi_1(y) - \Psi_2(z) \| \leq \max_{\tau \in \mathcal{J}} \left\{ \frac{\rho^1 - \rho}{\Gamma(\rho)} \int_0^\tau \frac{u^{\rho-1}}{(\rho - u)^{1-\rho}} \right\} \| h(\tau, y) - h(\tau, z) \| d\tau + \frac{\tau^\rho}{\rho \sigma} \left[ \frac{\sigma^\rho (\rho + \zeta)}{\rho^\rho + \zeta \Gamma(\rho + \zeta + 1)} \right] \int_0^\tau (\rho - u)^{1-\rho} |h(\tau, y) - h(\tau, z)| du \]

\[
+ \frac{\tau^\rho}{\rho \sigma} |g(y) - g(z)| \right\} \]

\[
\leq \max_{\tau \in \mathcal{J}} \left\{ \mathcal{K} \| y - z \| \left( \frac{\rho^1 - \rho}{\Gamma(\rho)} \int_0^\tau \frac{u^{\rho-1}}{(\rho - u)^{1-\rho}} du \right) + \frac{\tau^\rho}{\rho \sigma} \left[ \frac{\sigma^\rho (\rho + \zeta)}{\rho^\rho + \zeta \Gamma(\rho + \zeta + 1)} \right] \int_0^\tau (\rho - u)^{1-\rho} |h(\tau, y) - h(\tau, z)| du \right\}
\]

\[
+ \frac{\tau^\rho}{\rho \sigma} |g(y) - g(z)| \right\} \leq (\mathcal{K} \Theta_1 + k \Theta_2) \| y - z \|.
\]

Therefore,

\[
\| \Psi_1(y) - \Psi_2(z) \| \leq (\mathcal{K} \Theta_1 + k \Theta_2) \| y - z \|
\]

This follows from the statement (3.4) that \( \Psi \) is a contraction in itself from Banach space \( \mathcal{G} \). As a consequence, operator \( \Psi \) has a fixed point by Banach’s fixed point theorem, which corresponds to the unique solution to the problem (1.1).

**Theorem 3.2.** Suppose that \((\mathcal{J}_2), (\mathcal{J}_3), (\mathcal{J}_4), (\mathcal{J}_5), (\mathcal{J}_6)\) holds. Then the BVP (1.1) has at least one solution on \( \mathcal{J} \).

**Proof.** Consider the operator \( \Psi : \mathcal{G} \to \mathcal{G} \) defined by (2.5). We break down the \( \Psi \) into two operators,

\[
(\Psi_1)(\tau) = (\Psi_1)(\tau) + (\Psi_2)(\tau), \quad \tau \in \mathcal{J}, \quad (3.5)
\]

where

\[
(\Psi_1)(\tau) = \frac{\rho^1 - \rho}{\Gamma(\rho)} \int_0^\tau \frac{u^{\rho-1}}{(\rho - u)^{1-\rho}} h(\tau, y(\tau)) du
\]

\[
- \frac{\tau^\rho}{\rho \sigma} \left[ \frac{\sigma^\rho (\rho + \zeta)}{\rho^\rho + \zeta \Gamma(\rho + \zeta + 1)} \right] \int_0^\tau (\rho - u)^{1-\rho} |h(\tau, y(\tau)) - h(\tau, z(\tau))| du, \quad \tau \in \mathcal{J}, \quad (3.6)
\]

and

\[
(\Psi_2)(\tau) = \frac{\tau^\rho}{\rho \sigma} g(y), \quad \tau \in \mathcal{J}. \quad (3.7)
\]

Let \( \mathcal{D}_v = \{ y \in \mathcal{G} : \| y \| < v \} \). From the assumption of (3.5), a number \( v_0 > 0 \) exists

\[
\frac{v_0}{\Theta_1 \| \theta \|_{L^1(v_0)}} > \frac{1}{1 - \Theta_2 k}. \quad (3.8)
\]
We will continue to prove that operators $\Psi_1, \Psi_2$ meet all Theorem 2.14 requirements.

**Step 1.** The set $\Psi(\mathcal{D}_{\mathcal{V}_0})$ is bounded. For any $y \in \mathcal{D}_{\mathcal{V}_0}$, we procure

$$\|\Psi_1y\| \leq \frac{\rho^{1-\rho}}{\Gamma(\rho)} \int_0^\tau \frac{u^{\rho-1}}{(\tau^\rho - u^{\rho})^{1-\rho}} |h(u, y(u))| du$$

$$+ T^\rho \left( \frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho + \varsigma)} \int_0^\varphi \frac{v^{\rho-1}}{(\rho^{\rho - v^{\rho}})^{1-(\rho+\varsigma)}} |h(v, y(v))| du \right)^{\frac{1}{\rho+\varsigma}}$$

$$\times |h(u, y(u))| du$$

$$\leq \|\vartheta\| \xi(\mathcal{V}_0) \left\{ \frac{T^\rho}{\rho^\rho(\rho + 1)} \right\}$$

$$+ T^\rho \left( \frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho + \varsigma)} \int_0^\varphi \frac{v^{\rho-1}}{(\rho^{\rho - v^{\rho}})^{1-(\rho+\varsigma)}} |h(v, y(v))| du \right)^{\frac{1}{\rho+\varsigma}}$$

$$\leq \frac{\|\vartheta\| \xi(\mathcal{V}_0)}{\rho^\rho(\rho + 1)}$$

It demonstrates that $\Psi_1(\mathcal{D}_{\mathcal{V}_0})$ is uniformly bounded. The assumptions of $(\mathcal{J}_2)$ and $(\mathcal{J}_3)$ imply that

$$\|\Psi_2y\| \leq \frac{T^\rho}{\rho^\rho} k_{\mathcal{V}_0},$$

for any $y \in \mathcal{D}_{\mathcal{V}_0}$. Thus, the set $\Psi(\mathcal{D}_{\mathcal{V}_0})$ is bounded.

**Step 2.** Continuous and completely continuous operator $P$. Step 1 means that $\Psi(\mathcal{D}_{\mathcal{V}_0})$ is uniformly bounded. Furthermore, for any $\tau_1, \tau_2 \in \mathcal{J}$, we have

$$\|\Psi_1(y)(\tau_2) - (\Psi_1(y))(\tau_1)\| \leq \frac{\rho^{1-\rho}}{\Gamma(\rho)} \left\{ \frac{T^\rho}{\rho^\rho(\rho + 1)} \right\}$$

$$\times |h(u, y(u))| du$$

$$\leq \|\vartheta\| \xi(\mathcal{V}_0) \left\{ \frac{T^\rho}{\rho^\rho(\rho + 1)} \right\}$$

$$\times |h(u, y(u))| du$$

$$\leq \frac{\|\vartheta\| \xi(\mathcal{V}_0)}{\rho^\rho(\rho + 1)}$$

which is independent of $y$ and tends to zero as $\tau_2 - \tau_1 \to 0$. Thus, $\Psi_1$ is equicontinuous. Hence, by the Arzela-Ascoli Theorem, $\Psi(\mathcal{D}_{\mathcal{V}_0})$ is relatively compact. Now let $y_m, y \in \mathcal{D}_{\mathcal{V}_0}$ with $|y_m - y| \to 0$. Then the limit $|y_m(\tau) - y(\tau)| \to 0$ is uniformly valid on $\mathcal{J}$. It follows that $|h(\tau, y_m(\tau)) - h(\tau, y(\tau))| \to 0$ is uniformly valid on $\mathcal{J}$ from the uniform continuity of $h(\tau, y)$ on the compact set $\mathcal{J} \times [-v_0, v_0]$. Then

$$\|\Psi_1y_m - \Psi_1y\| \to 0 \text{ as } m \to \infty$$

that proves $\Psi_1$ is continuity. The operator $\Psi_1$ is continuous and completely continuous.

**Step 3.** The operator $\Psi_2 : \mathcal{D}_{\mathcal{V}_0} \to \mathcal{H}$ is contractive. Observe that

$$|\langle \Psi_2\rangle(y) - \langle \Psi_2\rangle(z)\rangle| = \frac{T^\rho}{\rho^\rho} \|g(y) - g(z)\|$$

$$\leq \frac{T^\rho}{\rho^\rho} k \|y - z\| = \lambda \|y - z\|,$$

with $\lambda = \Theta_2 k < 1$ by the assumptions of $(\mathcal{J}_2)$. Hence $\Psi_2$ is contractive.

**Step 4.** Furthermore, it will be shown that the case (ii) in Theorem 2.14 does not occur. For this, we presume that case (ii) is true. Then, we have that $\exists \xi \in (0, 1)$ and $y \in \mathcal{D}_{\mathcal{V}_0} \ni y = \xi \Psi_y$. So, we have $\|y\| = v_0$ and for $\tau \in \mathcal{J}$,

$$y(\tau) = \xi \left\{ \frac{T^\rho}{\rho^\rho(\rho + 1)} \int_0^\tau \frac{u^{\rho-1}}{(\tau^\rho - u^{\rho})^{1-\rho}} h(u, y(u)) du + \frac{T^\rho}{\rho^\rho} \int_0^\varphi \frac{v^{\rho-1}}{(\rho^{\rho - v^{\rho}})^{1-(\rho+\varsigma)}} h(v, y(v)) du \right\}.$$
operator $\Psi$ has at least one fixed point $y \in \mathcal{D}_0$, which is the solution of the BVP (1.1).

**Remark 3.3.** Setting $\rho \to 1$, in the problem (1.1), then the problem reduces to

$$
\varphi \mathcal{D}_\rho^\gamma y(t) = h(t,y(t)), \quad t \in \mathcal{J},
$$

$$
y(0) = 0, \quad \mathcal{J}_\sigma^\alpha y(\varphi) = g(y), \quad 1 < \rho \leq 2, 0 < \varsigma < 1, \quad \varphi \in (0,T).
$$

(3.9)

In this case the values of $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are found to be

$$
\hat{\Theta}_1 = \frac{T}{\Gamma(\varsigma + 1)} + \frac{T}{\Gamma(\rho + \varsigma + 1)} ,
$$

(3.10)

$$
\hat{\Theta}_2 = \frac{T}{\Gamma(\sigma)}.
$$

(3.11)

and The operator form (2.5) changes

$$
\hat{\mathcal{P}}(y)(\tau) = \int_0^\tau \left( \frac{(\tau - \nu)^{\rho - 1}}{\Gamma(\rho)} h(u,y(u))\right) d\nu + \frac{\tau}{\sigma} \int_0^\tau g(y) - \frac{\sigma}{\Gamma(\rho + 1)} h(u,y(u)) d\nu , \quad \tau \in \mathcal{J},
$$

where $\hat{\mathcal{P}} = \frac{\sigma \rho^\alpha + 1}{\Gamma(2 + \varsigma)} \neq 0$.

**Corollary 3.4.** Suppose that $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ holds. Then the BVP (3.9), provided that

$$
\eta_1 := K\hat{\Theta}_1 + k\hat{\Theta}_2 < 1,
$$

where $\hat{\Theta}_1, \hat{\Theta}_2$ is described by (3.10) and (3.11) respectively, has a unique solution on $J$.

**Corollary 3.5.** Suppose that $(\mathcal{P}_2)$, $(\mathcal{P}_3), (\mathcal{P}_4)$ and $(\mathcal{P}_5)$ holds. Then the BVP (3.9), provided that

$$
\frac{v_0}{\Theta_1(\|\mathcal{P}\|v_0)} > \frac{1}{1 - k\hat{\Theta}_2},
$$

has at least one solution on $\mathcal{J}$.

Examples are given in this section to illustrate the feasibility of the results obtained.

### 4. Examples

**Example 4.1.** Consider the following BVP

$$
\frac{\partial}{\partial \tau}^\gamma y(\tau) = 1 + \left( \frac{|y|}{|y| + 1} \right) \frac{e^{-\tau}}{4(\tau + \sqrt{4})^2} ,
$$

$$
y(0) = 0, \quad \frac{1}{10} y(\frac{3}{2}) \leq \frac{1}{3} \frac{1}{\tau^2} y(\frac{1}{2}), \quad \tau \in [0,2].
$$

(4.1)

Here $\rho = \frac{3}{2}, \quad \varsigma = \frac{3}{2}, \quad \sigma = 1, \quad T = 2, \quad \rho = \frac{1}{2}, \quad \varphi = \frac{3}{4}, \quad k = \frac{1}{10}, \quad K = \frac{1}{10}, \quad g(y) = \frac{1}{10} y(\frac{3}{2})$ and $h(\tau,y) = 1 + \left( \frac{|y|}{|y| + 1} \right)$.

$$
e^{-\tau^2} \frac{4(\tau + \sqrt{4})^2}{4(\tau + \sqrt{4})^2} . \quad \text{We can acquire values using specified information: } \sigma \approx 3.991024198349486, \quad \Theta_1 \approx 6.006235934230121,
$$

$$
\Theta_2 \approx 0.708697062251816, \quad \eta \approx 0.4462594521145642, \quad \text{and}
$$

$$
|h(\tau,y) - h(\tau,z)| \leq \frac{1}{16} \frac{|y|}{|y| + 1} - \frac{|z|}{|z| + 1} \leq \frac{1}{16} |y - z|.
$$

With $K\Theta_1 + k\Theta_2 < 1 \approx 0.4462594521145642$, The Theorem 3.1 presumptions are fulfilled. The BVP (4.1) has a unique solution for $[0,2]$.

**Example 4.2.** Consider the following BVP

$$
\frac{1}{2} \mathcal{D}_\rho^\gamma y(\tau) = \left( \frac{|y|}{2 + |y|} \right) \frac{\tau}{200} , \quad \tau \in [0,2],
$$

(4.2)

Example 4.1 boundary conditions augmented. Here $\rho = \frac{3}{2}, \quad \varsigma = \frac{3}{2}, \quad \sigma = 1, \quad T = 2, \quad \rho = \frac{1}{2}, \quad \varphi = \frac{3}{4}, \quad k = \frac{1}{10}, \quad g(y) = \frac{1}{2} y(\frac{3}{2})$ and $h(\tau,y) = \left( \frac{|y| + \frac{1}{2}|y|}{2 + |y|} \right) \frac{\tau}{200}$. Using the specified information, we can acquire values: $\sigma \approx 4.038195868699365, \quad \Theta_1 \approx 6.001791053992137,
$$

$$\Theta_2 \approx 0.700418539414, \quad |h(\tau,y)| \leq \left| \left( \frac{|y| + \frac{1}{2}|y|}{2 + |y|} \right) \frac{\tau}{200} \right| \leq \left( 1 + 3|y| + 3|y| \right) \frac{\tau}{200}$$. We choose $\vartheta(\tau) = \frac{\tau}{200}$ and $\xi(\tau) = 1 + |y| + |\tau|^2, \quad |\tau|^2 \leq 3|y| + 3|y|$. and we find that $\sup_{\tau \in (0,\infty)} \Theta_1\|\mathcal{P}\|v(\tau) = \frac{3.33233860}{274471} > 1.0753172110360703 \approx \frac{1}{1 - \Theta_2 k}$. The Theorem 3.2 presumptions are fulfilled. The BVP (4.2) has at least one solution for $[0,2]$.

In this section, we obtain existence results for the BVP (1.2) by using the nonlinear alternative for contractive maps.

### 5. Main Results : Multi-valued case

**Theorem 5.1.** Suppose that $(\mathcal{P}_2), (\mathcal{P}_6), (\mathcal{P}_7)$ and $(\mathcal{P}_8)$ holds. Then the BVP (1.2) has at least one solution on $\mathcal{J}$.

**Proof.** In order to convert the problem (1.2) into a fixed point question, the operator $\Omega : \mathcal{H} \to \mathcal{E}(\mathcal{H})$ specified by

$$
\Omega(y) = \left\{ \begin{array}{ll}
q \in \mathcal{H} : \\
q = \rho^1 \rho^{-1} + \rho \rho^{-1} h(u) d\nu
\end{array} \right\}
$$

(4.3)

is considered. For $h \in \mathcal{P}_H$. Next, we intimate two operators: $\Omega_1 : \mathcal{H} \to \mathcal{H}$ by

$$
\Omega_1 y(\tau) = \frac{\tau^p}{\rho^p} q(y),
$$

(5.1)
and the multivalued operator $\Omega_2 : \mathcal{H} \to \mathcal{E}(\mathcal{H})$ by

$$\Omega_2(y) = \begin{cases} q \in \mathcal{H} : & q(\tau) = \frac{\theta^{1-\rho}}{\Gamma(\rho)} \int_0^\tau \frac{v^{\rho-1}}{(\tau - v)^{1-\rho}} h(v) dv \\ - \frac{\tau^\rho}{\rho \sigma} \left( \frac{\theta^{1-\rho}}{\Gamma(\rho + \frac{\theta}{\sigma})} \right) \int_0^\frac{\theta}{\sigma} \frac{v^{\rho-1}(\tau - v)^{1-\rho}}{(\phi^\rho - v^{\rho})^{1-(\rho + \sigma)}} dv \end{cases}.$$

Observe that $\Omega = \Omega_1 + \Omega_2$. We define the operators $\Omega_1$ and $\Omega_2$ that follow all of Theorem 2.15 assumptions on $\mathcal{F}$. For that, consider the operators $\Omega_1, \Omega_2 : \mathcal{B}_v = \mathcal{E}_{\mathcal{F},v}(\mathcal{H}),$ where $\mathcal{B}_v = \{ y \in \mathcal{H} : \|y\| \leq v \}$ is a bounded set in $\mathcal{H}$. Next, we prove that $\Omega_2$ is compact valued on $\mathcal{B}_v$. Considering that operator $\Omega_2$ is the $\mathcal{G} \circ \Psi_H$ composition where $\mathcal{G}$ is the linear continuous operator $\mathcal{L}^1(\mathcal{F}, \mathcal{R})$ into $\mathcal{H}$, as described by

$$\mathcal{G}(\theta)(\tau) = \frac{\theta^{1-\rho}}{\Gamma(\rho)} \int_0^\tau \frac{v^{\rho-1}}{(\tau - v)^{1-\rho}} \theta(v) dv \\ - \frac{\tau^\rho}{\rho \sigma} \left( \frac{\theta^{1-\rho}}{\Gamma(\rho + \frac{\theta}{\sigma})} \right) \int_0^\frac{\theta}{\sigma} \frac{v^{\rho-1}(\tau - v)^{1-\rho}}{(\phi^\rho - v^{\rho})^{1-(\rho + \sigma)}} dv.$$

Suppose $y \in \mathcal{B}_v$ is arbitrary and allow $\{ \theta_m \}$ to be a sequence in $\Psi_{H,v}$. In this case, we have $\theta_m(\tau) \in H(\tau, y(\tau))$ for almost all $\tau \in \mathcal{J}$, by definition of $\Psi_{H,v}$. Because $H(\tau, y(\tau))$ is compact for all $\tau \in \mathcal{J}$, it has a convergent subsequence of $\{ \theta_m(\tau) \}$, which converges for almost all $\tau \in \mathcal{J}$ to some $\theta(\tau) \in \Psi_{H,v}$. First, $\mathcal{G}$ is continuous, so $\mathcal{G}(\theta_m)(\tau) \to \mathcal{G}(\theta)(\tau)$ point-wise on $\mathcal{J}$. To prove that the convergence is uniform, we retain to demonstrate that the equiconvergent sequence is $\{ \mathcal{G}(\theta_m) \}$. Let $\tau_1, \tau_2 \in \mathcal{J}$ with $\tau_1 < \tau_2$. Then, we find that $\| \mathcal{G}(\theta_m)(\tau_2) - \mathcal{G}(\theta_m)(\tau_1) \|$

$$\leq \frac{\theta^{1-\rho}}{\Gamma(\rho)} \int_0^{\tau_2} \frac{v^{\rho-1}}{(\tau_2 - v)^{1-\rho}} \theta_m(v) dv \\ - \frac{\tau_2^{\rho}}{\rho \sigma} \left( \frac{\theta^{1-\rho}}{\Gamma(\rho + \frac{\theta}{\sigma})} \right) \int_0^\frac{\theta}{\sigma} \frac{v^{\rho-1}(\tau_2 - v)^{1-\rho}}{(\phi^\rho - v^{\rho})^{1-(\rho + \sigma)}} dv.$$

Then, for $\tau \in \mathcal{J}$, we have

$$q(\tau) = \frac{\theta^{1-\rho}}{\Gamma(\rho)} \int_0^\tau \frac{v^{\rho-1}}{(\tau - v)^{1-\rho}} h(v) dv \\ - \frac{\tau^\rho}{\rho \sigma} \left( \frac{\theta^{1-\rho}}{\Gamma(\rho + \frac{\theta}{\sigma})} \right) \int_0^\frac{\theta}{\sigma} \frac{v^{\rho-1}(\tau - v)^{1-\rho}}{(\phi^\rho - v^{\rho})^{1-(\rho + \sigma)}} dv.$$

The inequality above tends to be zero like $\tau_2 \to \tau_1$. Therefore, the sequence $\{ \mathcal{G}(\theta_m) \}$ is equicontinuous and we obtain a uniform convergent subsequence using the Arzela-Ascoli theorem. So, there is a subsequence of $\{ \theta_m \}$, $\mathcal{G}(\theta_m) \to \mathcal{G}(\theta)$. Considering $\mathcal{G}(\theta) \in \mathcal{E}(\Psi_{H,v})$. Consequently, $\Omega_2(y) = \mathcal{G}(\Psi_{H,v})$ is compact $\forall y \in \mathcal{B}_v$. So $\Omega_2(y)$ is compact. Now, we depict that $\Omega_2(y)$ is convex $\forall y \in \mathcal{H}$. Let $v_1, v_2 \in \Omega_2(y)$. We nominate $h_1, h_2 \in \Psi_{H,v} \cap \mathcal{F}$. Then, we have

$$v_j(\tau) = \frac{\theta^{1-\rho}}{\Gamma(\rho)} \int_0^\tau \frac{v^{\rho-1}}{(\tau - v)^{1-\rho}} h_j(v) dv \\ - \frac{\tau^\rho}{\rho \sigma} \left( \frac{\theta^{1-\rho}}{\Gamma(\rho + \frac{\theta}{\sigma})} \right) \int_0^\frac{\theta}{\sigma} \frac{v^{\rho-1} h_j(v) dv}{(\phi^\rho - v^{\rho})^{1-(\rho + \sigma)}}.$$

Since $H$ has convex values, so $\mathcal{G}_{H,v}$ is convex and $\mathcal{G}(h_1(v) + (1 - \gamma) h_2(v)) \in \mathcal{G}_{H,v}$. Thus, $\gamma_1 + (1 - \gamma) v_2 \in \Omega_2(y)$. Hence, $\Omega_2$ is convex-valued. Undoubtedly, $\Omega_1$ is compact and convex-valued. The remaining facts contains multiple phases and statements.

Step 1. We demonstrate that $\Omega_1$ is a contraction on $\mathcal{H}$. This is a result of $(\mathcal{P}_2)$, and the evidence in Step 2 of Theorem 3.2 is similar to that of Operator $\Psi_2$.

Step 2. We will proceed to demonstrate that the operator $\Omega_2$ is compact and upper semicontinuous. This is shown in number of claims.

Claim I: $\Omega_2$ maps bounded sets into bounded sets in $\mathcal{H}$. To see this, let $\mathcal{B}_v = \{ y \in \mathcal{H} : \|y\| \leq v \}$ be a bounded set in $\mathcal{H}$. Then, for each $q \in \Omega_2(y), y \in \mathcal{B}_v$, $\exists h \in \Psi_{H,v} \cap \mathcal{F}$

$$q(\tau) = \frac{\theta^{1-\rho}}{\Gamma(\rho)} \int_0^\tau \frac{v^{\rho-1}}{(\tau - v)^{1-\rho}} h(v) dv \\ - \frac{\tau^\rho}{\rho \sigma} \left( \frac{\theta^{1-\rho}}{\Gamma(\rho + \frac{\theta}{\sigma})} \right) \int_0^\frac{\theta}{\sigma} \frac{v^{\rho-1} h(v) dv}{(\phi^\rho - v^{\rho})^{1-(\rho + \sigma)}}.$$
Thus, $\|q\| \leq \xi(\|v\|)\Theta_1\|\vartheta\|$.

**Claim II:** Next, we show that $\Omega_2$ maps bounded sets into equicontinuous sets. Let $\tau_1, \tau_2 \in J$, and $y \in \mathfrak{B}_v$. For each $q \in \Omega_2(y)$, we get

$$|q(\tau_2) - q(\tau_1)| \leq \frac{\rho^{1 - \rho}}{\Gamma(\rho)} \int_{\tau_1}^{\tau_2} \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho)} - \frac{\tau^\rho}{\rho\sigma} \right) \left[ \frac{\|\vartheta\|}{\rho\sigma} \int_0^\varphi \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho + \varphi)} \right) \left( h(v)dv \right) \right]$$

The right side of the above inequality obviously tends to be zero, independent of $y \in \mathfrak{B}_v$ as $\tau_2 - \tau_1 \to 0$. Therefore, the Arzelà-Ascoli theorem that $\Omega_2 : \mathfrak{H} \to \varepsilon_{\beta, c}(\mathfrak{H})$, is completely continuous. By Lemma 2.12, $\Omega_2$ is semi-continuous if we prove it has a closed graph since $\Omega_2$ is already shown to be absolutely continuous. We set it out in the next paragraph.

**Claim III:** $\Omega_2$ has a closed graph. Let $y_m \to y_*$ and $q_m \in \Omega_2(y_m)$ and $q_m \to q_*$. Then, we need to show that $q_* \in \Omega_2(y_*)$ associated with $q_m \in \Omega_2(y_m)$, $\exists h_m \in \mathcal{H}_{\beta, y_m}$ for each $\tau \in J$,

$$q_m(\tau) = \frac{\rho^{1 - \rho}}{\Gamma(\rho)} \int_0^\tau \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho)} - \frac{\tau^\rho}{\rho\sigma} \right) \left[ \frac{\|\vartheta\|}{\rho\sigma} \int_0^\varphi \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho + \varphi)} \right) \left( h_m(v)dv \right) \right]$$

And it is enough to show $\exists h_* \in \mathcal{H}_{\beta, y_*}$, $\forall$ for each $\tau \in J$,

$$q_*(\tau) = \frac{\rho^{1 - \rho}}{\Gamma(\rho)} \int_0^\tau \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho)} - \frac{\tau^\rho}{\rho\sigma} \right) \left[ \frac{\|\vartheta\|}{\rho\sigma} \int_0^\varphi \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho + \varphi)} \right) \left( h(v)dv \right) \right]$$

Let us consider the linear operator $\Omega : \mathcal{L}^1(J, \mathbb{R}) \to \mathfrak{H}$ defined by

$$h \to \Omega(h)(\tau) = \frac{\rho^{1 - \rho}}{\Gamma(\rho)} \int_0^\tau \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho)} - \frac{\tau^\rho}{\rho\sigma} \right) \left[ \frac{\|\vartheta\|}{\rho\sigma} \int_0^\varphi \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho + \varphi)} \right) \left( h(v)dv \right) \right]$$

Observe that

$$\|q_m(\tau) - q_*(\tau)\| = \left\| \frac{\rho^{1 - \rho}}{\Gamma(\rho)} \int_0^\tau \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho)} - \frac{\tau^\rho}{\rho\sigma} \right) \left[ \frac{\|\vartheta\|}{\rho\sigma} \int_0^\varphi \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho + \varphi)} \right) \left( h_m(v)dv \right) \right] \right\|$$

for some $h_* \in \mathcal{H}_{\beta, y_*}$. Consequently $\Omega_2$ has a closed graph. $\Omega_2$ is upper semi-continuous. Operators $\Omega_1$ and $\Omega_2$ satisfy all the criteria of the theorem and its implementation, thus results, in either case, (i) or case (ii). We define the case (ii), as not probable. If $y \in \mu \mathcal{O}_1(\mu) + \mu \mathcal{O}_2(\mu)$ for $\mu \in (0, 1)$, then $\exists \overline{h} \in \mathcal{H}_{\beta, \overline{y}}$

$$y(\tau) = \mu \left\{ \frac{\rho^{1 - \rho}}{\Gamma(\rho)} \int_0^\tau \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho)} - \frac{\tau^\rho}{\rho\sigma} \right) \left[ \frac{\|\vartheta\|}{\rho\sigma} \int_0^\varphi \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho + \varphi)} \right) \left( h(v)dv \right) \right] \right\}$$

for $\tau \in J$. Consequently, we have

$$|y(\tau)| \leq \bar{\xi}(\|y\|)\|\vartheta\| \left\| \frac{\rho^{1 - \rho}}{\Gamma(\rho)} \int_0^\tau \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho)} - \frac{\tau^\rho}{\rho\sigma} \right) \left[ \frac{\|\vartheta\|}{\rho\sigma} \int_0^\varphi \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho + \varphi)} \right) \left( h(v)dv \right) \right] \right\|$$

which, on taking supremum over $\tau \in J$, yields

$$\|y\| \leq \bar{\xi}(\|y\|)\|\vartheta\| \left\| \frac{\rho^{1 - \rho}}{\Gamma(\rho)} \int_0^\tau \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho)} - \frac{\tau^\rho}{\rho\sigma} \right) \left[ \frac{\|\vartheta\|}{\rho\sigma} \int_0^\varphi \left( \frac{\rho^{1 - \rho}}{\Gamma(\rho + \varphi)} \right) \left( h(v)dv \right) \right] \right\|$$

If case (ii) of Theorem 2.15 holds, then $\exists \mu \in (0, 1)$ and $y \in \partial \mathcal{B}_\alpha$ with $y = \mu \mathcal{O}(y)$. Then, $y$ is a solution of (3.5) with $\|y\| = \alpha$. Now, the last inequality implies that

$$\alpha(1 - k \Theta_1) \leq 1,$$
which contradicts (3.3). Hence $\Omega$ has a fixed point in $\mathcal{F}$ by Theorem 2.15, and hence the problem (1.2) has a solution.

The problem comparison results (4.1) and (3.9) are presented in this section.

### 6. Discussion

Here, two cases, where been discussed, and also $\rho = 0.5$ happens to be case 1, case 2 is for $\rho = 1$.

Rest of the values are kept in common for problems (4.1) and (3.9).

Problem (4.1) signifies the generalized Caputo case and problem (3.9) delineates the Caputo case.

The assumption value of the uniqueness of solutions for the problem (4.1) is $\eta$ and is illustrated in Figure 1.

Likewise, $\eta_1$ depicts the assumption value for the problem (3.9) and is represented in Figure 2.

Here we demonstrate the comparison results of assumption values of the problem (4.1), and (3.9) is represented in Figure 3.

From the above-said figure, we justify the values of $\eta, \eta_1$ by showing the influence of $\rho$ for its differing values on the characteristics of a fractional derivative. It is evident from the figure that when $\rho > 0$, we can get positive solutions under the assumptions of Theorem 3.1. According to Figure 1 and Figure 2, the behavior of the fractional derivative concerning $\rho$ leads to a new path regarding control applications.

We came to realize that in favor of the results stated at the top, the problem of GCFDEs with non-local GRLFI boundary condition holds good for existing conditions. Therefore, the reader can build the problem with ample ideas with certain consistent estimates of the problem parameters.

We enrolled below a few special cases.

- If $\rho \to 1$, after that we acquire the solution for the problem of Caputo type FDEs with non-local Riemann-Liouville fractional integral boundary conditions.

- If $\rho \to 0$, in that case, we come to have the results (by noting that

  $\lim_{\rho \to 0} \left( \frac{\tau^\rho - b^\rho}{\rho} \right) = \ln \left( \frac{\tau}{b} \right)$
  $\lim_{\rho \to 0} \left( \frac{c^\rho - \tau^\rho}{\rho} \right) = \ln \left( \frac{c}{\tau} \right)$

for the problem of Caputo-Hadamard type FDEs with non-local Hadamard fractional integral boundary conditions. Make note that the generalized fractional derivative and integral in the problem reduces to the Caputo-Hadamard fractional derivative and Hadamard integral in the limit $\rho \to 0$ with the help of L’Hospital’s rule.

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