A Note on General Covariant Stability Theory

M.I. Wanas\textsuperscript{1} ; M.A. Bakry \textsuperscript{2}

Abstract
In the present work we suggest a general covariant theory which can be used to study the stability of any physical system treated geometrically. Stability conditions are connected to the magnitude of the deviation vector. This theory is a modification of an earlier joint work, by the same authors, concerning stability. A comparison between the present work and the earlier one is given. The suggested theory can be used to study the stability of planetary orbits, astrophysical configurations and cosmological models.

1 Introduction
In a previous paper \cite{1} the authors have suggested the use of geodesic deviation equations to study stability of gravitating systems. In that paper, they have generalized the classical perturbation scheme, usually used to deal with such problem. We have suggested the use of components of the deviation vector, representing the solution of the equation of geodesic deviation,

\[
\frac{d^2 \xi^\alpha}{ds^2} + 2 \left\{ \begin{array}{c} \alpha \\ \beta \gamma \end{array} \right\} \frac{d \xi^\beta}{ds} U^\gamma + \left\{ \begin{array}{c} \alpha \\ \beta \gamma \end{array} \right\}_\lambda U^\beta U^\gamma \frac{d \xi^\lambda}{ds} = 0
\]  

where $\xi^\alpha$ is the deviation vector, $U^\beta$ is the unit tangent to the geodesic, $\left\{ \begin{array}{c} \alpha \\ \beta \gamma \end{array} \right\}$ is the Christoffel symbol of the second kind and $(s)$ is an invariant parameter. Now, $\xi^\alpha(s)$ is the solution of equation (1) in the interval $[a, b]$ in which the functions $\xi^\alpha(s)$ behave monotonically. This vector reflects the reaction of the system under perturbation. The quantities, that have been suggested in \cite{1}, to be used as sensors for stability of the system, are

\[
q^\alpha \overset{\text{def}}{=} \lim_{s \to b} \xi^\alpha(t).
\]  

The criterion suggested is that, if $q^\alpha \to \infty$, the system would be unstable, otherwise it would be stable. This criteria has been used to study stability of a number of cosmological models. Applications in cosmological models, using this criterion, is somewhat easy since most of these models depend on one function, the scale factor. Further applications show the non-covariance, of the scheme, under coordinate transformations.

It appears that if stability conditions are obtained depending on the quantities (2), these conditions would not be, in general, covariant. This is because the components of the deviation vector depend on the coordinates system used. In other words, the stability conditions obtained would be coordinate dependent. We are going to call the scheme suggested in \cite{1} the ”Coordinate Dependent Scheme”, (CDS).

The aim of the present note is to modify the quantity (2) in order to get covariant stability conditions.

\textsuperscript{1}Astronomy Department, Faculty of Science, Cairo University, Giza, Egypt.
E-mail: wanasa@frcu.eun.eg

\textsuperscript{2}Mathematics Department, Faculty of Education, Ain Shams University, Cairo, Egypt.
2 Covariant Stability Conditions

To get covariant results, independent of the coordinate system used, one has to replace the contravariant components of the deviation vector used in (2) by its magnitude, then we examine the limit
\[ q \equiv \lim_{s \to b} (\xi^\alpha \xi_\alpha)^{\frac{1}{2}}. \] (3)

Now, if \( q \to \infty \), then the system is unstable. Otherwise, it would be stable.

To summarize how to apply the covariant scheme suggested, one has to follow the following steps:
1. Having a well defined problem, we solve the field equations controlling this problem to know the type of geometry associated with the system under consideration (the metric).
2. Knowing the metric of space-time, we solve the geodesic equation to get the unit tangent vector \( U^\alpha \).
3. Using the information, obtained in the above two steps, substituting in the geodesic deviation equation (1) and solving it, we get the deviation vector \( \xi^\alpha \).
4. Evaluating the scalar \( \xi^\mu \xi_\mu \) and examining its limit as given by (3), one can answer stability question.
   If \( q \to \infty \), the system will be unstable. Otherwise, it will be stable.
5. A strong stability condition can be achieved if,
\[ \lim_{t \to \infty} (\xi^\alpha \xi_\alpha)^{\frac{1}{2}} = 0. \] (3)

We are going to call this scheme "The Coordinate Independent Scheme", (CIS).

3 Discussion

If we use the scheme suggested in the present work CIS and apply it to some of the world models examined in the previous work [1] we get the results that are summarized and compared, to those obtained using the CDS, in Table 1. In this table, the cosmological models treated are classified as follows. The first set of models represents world models constructed using "General Relativity" (GR). In the second set, we examine a world model depending on "Miln Kinematical Relativity" (KR) and another one constructed using "Brans-Dicke Theory" (BD). The third set contains models resulting from "Møller’s Tetrad Theory of Gravitation" (MTT)[2]. The last set contains models obtained using the "Generalized Field Theory" (GFT) [3]. The sample, in Table 1, is chosen in such a way that it represents models depending on different geometric field theories. It is clear from the following table that the use of the covariant scheme, suggested in the present work, gives results, some of which are different from those obtained in the previous work.
Table 1: Stability of Some World Models Using CDS and CIS.

| Theory | Model                  | CDS       | CIS       |
|--------|------------------------|-----------|-----------|
| GR     | Einstein [4]           | Unstable  | Unstable  |
|        | De Sitter [4]          | Stable    | Unstable  |
|        | Einstein-De Sitter [4] | Stable    | Unstable  |
|        | Radiation [5]          | Stable    | Unstable  |
| KR     | Miln [4]               | Stable    | Stable    |
| BD     | Brans-Dick [6]         | Stable    | Unstable  |
| MTT    | $D < 0$ [7]            | Stable    | Unstable  |
|        | $D > 0$ [7]            | Conditional | Unstable  |
| GFT    | $k = -1$ [8]           | Stable    | Stable    |
|        | $k = 0$ [8]            | Unstable  | Unstable  |

The similar results obtained, using the suggested scheme and the previous one [1], are just coincidence. It is obvious that changing the coordinate system used to construct a world model will not affect the results of the last column of Table 1, while it may change those given in the third column.

The scheme suggested in the present work has been successfully used to study stability of non-singular black holes [9]. Further details will be published elsewhere.

References

[1] Wanas, M.I. and Bakry, M.A. (1995) Astrophys. Space Sci. 228, 239.
[2] Møller, C. (1978) Mat. Fys. Skr. Dan. Vid. selk. 39,13, 1.
[3] Mikhail, F.I. and Wanas, M.I. (1977) Proc. Roy. Soc. Lond. A 356, 471.
[4] McVittie, G.C. (1961) ”Facts and Theory of Cosmology”, Eyre & spittwoode, London.
[5] Sciama, D.W. (1971) ”Modern Cosmology”, Cambridge, London.
[6] Wienberg, S. (1972)”Gravitation and Cosmology” John Wily & Sons.
[7] Saez, D. and de-Juan , T. (1984) Gen.Rel Grav. 16, 5.
[8] Wanas, M.I. (1989) Astrophys. Space Sci. 154, 165.
[9] Nashed, G.G.L. (2003) Chaos, Solitons and Fractals, 15, 841.