Theorem A. Let $M^d$ be closed, simply connected Spin-manifold that has at least one non-vanishing rational Pontryagin class and let $g \in \mathcal{RC}(M)$. Let $k \geq 1$ be such that $(d + k)$ is divisible by 4 and $k \leq \min(\frac{d-1}{3}, \frac{d-5}{2})$. Then map

$$
\pi_{k-1}(\text{Diff}(M, D)) \otimes \mathbb{Q} \longrightarrow \pi_{k-1}(\mathcal{RC}(M)) \otimes \mathbb{Q}
$$

induced by the orbit map $f \mapsto f^* g$ is nontrivial.

In particular: $\pi_{k-1}(\mathcal{RR}(M)) \otimes \mathbb{Q} \neq 0 \neq \pi_{k-1}(\mathcal{RS}(M)) \otimes \mathbb{Q}$, provided these spaces are non-empty.

Remark 1.1 (State of the art concerning $\mathcal{RS}(M)$). To the best of the author’s knowledge, the following is all that is known about the homotopy type of $\mathcal{RS}(M)$:

(i) Kreck–Stolz have shown in [KS93], that there exists manifolds $M$ such that $\mathcal{RS}(M)$ is not connected. Their result remains true fore the quotient $\mathcal{RS}(M)/\text{Diff}(M)$, so those components do not originate from the orbit map.

(ii) Crowley–Schick–Steimle have shown in [CSS18] that for every manifold $M$, the image of the orbit map $\pi_{k-1}(\text{Diff}(M)) \rightarrow \pi_{k-1}(\mathcal{RS}(M))$ contains $\mathbb{Z}/2$ as a subgroup provided $d + k \equiv 1, 2 \pmod{8}$ and $d \geq 6$. This extends earlier results of Hitchin [Hit74] and Crowley–Schick [CS13].

2010 Mathematics Subject Classification. 53C21, 55R40, 57R20, 57R22, 58D17, 58D05.

I am supported by the DFG (German Research Foundation) – 281869850 (RTG 2229).
(iii) Krannich–Kupers–Randal-Williams have proven the special case $M = \mathbb{H}P^2$, $k = 4$ of Theorem A in [KKRW20]. Their proof delivers an excellent blueprint for our generalisations. Like in loc.cit., we will construct an $M$-bundle $E \to S^k$ with a Spin-structure on the vertical tangent bundle and non-vanishing $\hat{A}$-genus. Since [KKRW20] is written rather densely, we chose to give a more detailed account of their argument in Section 2 before we go on to proving Theorem A in Section 3.

Remark 1.2.

(i) The bound on $k$ can be improved if $M$ is even-dimensional and $\ell$-connected. In this case Theorem A holds true for $k \leq \min(d - 4, 2\ell - 1)$ (cf. Remark 2.3).

(ii) As pointed out in [KKRW20] this answers a question of Schick [OWL, p. 30] and provides many examples for manifolds to which [BEW20, Theorem 2.1] is applicable.

According to [Zil14] there are the only known examples of positively manifolds curved in dimensions $4k + 3$ for $k \geq 2$ are spheres. Also, all 7-dimensional examples have finite fourth cohomology (cf. [Esc92; Goe14; GKS04]). Therefore, a positive answer to the following question would yield the first example of a manifold that admits infinitely many pairwise non-isotopic metrics of positive sectional curvature.

Question 1.3. Is there a positively curved manifold of dimension $4k + 3$, $k \geq 1$ with a non-vanishing rational Pontryagin class?

Acknowledgements. I would like to thank Jens Reinhold for comments on an earlier draft and Bernhard Hanke and Jost Eschenburg for valuable remarks.
Sing\(\bullet(X)\) is homotopy equivalent to \(X\) for any space \(X\) ([HAT, pp. 8]), we get an induced map

\[ B\text{Diff}(M) \rightarrow B\widetilde{\text{Diff}}(M). \]

Next, let \(\text{hAut}(M)\) denote the group-like\(^1\) topological monoid of (orientation preserving) homotopy equivalences of \(M\) with classifying space \(\text{BhAut}(M)\). Again, let \(\text{hAut}(M)\) be the realisation of the semisimplicial group of block homotopy equivalences defined analogously with \(\text{BhAut}(M)\) and \(\text{BhAut}(M)\) the corresponding classifying spaces. By [Dol63, Thm 6.1] \(\text{hAut}(M)\) and \(\widetilde{\text{hAut}}(M)\) are homotopy equivalent. Consider the following maps induced by inclusions:

\[ B\widetilde{\text{Diff}}(M) \rightarrow B\widetilde{\text{hAut}}(M) \simeq B\text{hAut}(M) \quad B\text{Diff}(M) \rightarrow B\text{hAut}(M) \]

and let \(\text{hAut}(M)/\text{Diff}(M)\) and \(\text{hAut}(M)/\text{Diff}(M)\) denote the respective homotopy fibres. Note that \(\text{hAut}(M)/\text{Diff}(M)\) classifies \(M\)-bundles that are homotopy equivalent to the trivial bundle through a homotopy that commutes with the projection of the bundle, i.e. fibre homotopy trivial \(M\)-bundles. We have the following comparison result which easily follows from [BL82, Corollary D].

**Lemma 2.2.** If \(k \leq \min\left(\frac{d-1}{3}, \frac{d-5}{2}\right)\) then the map

\[ \pi_k \left( \frac{\text{hAut}(M)}{\text{Diff}(M)} \right) \left[ \frac{1}{2} \right] \rightarrow \pi_k \left( \frac{\text{hAut}(M)}{\text{Diff}(M)} \right) \left[ \frac{1}{2} \right] \]

is (split-)surjective.

**Proof.** By [BL82, Corollary D] there exists a space \(S\) such that for \(k \leq \min\left(\frac{d-1}{3}, \frac{d-5}{2}\right)\) we have

\[ \pi_k \left( \frac{\text{hAut}(M)}{\text{Diff}(M)} \right) \left[ \frac{1}{2} \right] \cong \pi_{k-1}(\Omega \left( \frac{\text{hAut}(M)}{\text{Diff}(M)} \right) \left[ \frac{1}{2} \right]) \]

\[ \cong \pi_{k-1}(\Omega \left( \frac{\text{hAut}(M)}{\text{Diff}(M)} \right) \times \Omega S) \left[ \frac{1}{2} \right] \rightarrow \pi_k \left( \frac{\text{hAut}(M)}{\text{Diff}(M)} \right) \left[ \frac{1}{2} \right] \]

Therefore, an element of \(\pi_k(\text{hAut}(M)/\text{Diff}(M)) \otimes \mathbb{Q}\) yields an \(M\)-bundle \(E \rightarrow S^k\) that is fibre homotopy trivial, provided that the dimension of \(M\) is high enough. The advantage of working with \(\text{hAut}(M)/\text{Diff}(M)\) instead of \(\text{hAut}(M)/\text{Diff}(M)\) stems from the fact, that the former is accessible through surgery theory as we will review in the succeeding section.

**Remark 2.3.** Another approach to compare \(B\text{Diff}(M)\) and \(B\widetilde{\text{Diff}}(M)\) is by using Morlet’s lemma of disjunction as in [KKRW20, Lemma]. Let \(M^{2n}\) be even-dimensional and consider the following diagram of (homotopy) fibrations

\(^1\)A topological space \(X\) is called group-like if \(\pi_0(X)\) is a group
If $M$ is $\ell$-connected with $\ell \leq 2n - 4$, then the induced map on homotopy fibres is $(2\ell - 2)$-connected by Morlet’s lemma of disjunction (cf. [BLR75, Corollary 3.2 on page 29]). Now $\pi_k(B\Diff(D^{2n})) \cong \pi_0(\Diff(D^{2n+k-1}))$ is isomorphic to the finite group of exotic spheres in dimension $(2n + k)$ and $B\Diff(D^{2n})$ is rationally $2n - 5$-connected by [RW17, Theorem 4.1]. Therefore, $\Diff(D^{2n})/\Diff(M)$ is rationally $(2n - 5, 2\ell - 2)$-connected. This implies that the map

$$\pi_k(\Diff(M)) \otimes \mathbb{Q} \to \pi_k(B\Diff(M)) \otimes \mathbb{Q}$$

is surjective for $k \leq \min(2n - 4, 2\ell - 1)$.

2.2. Surgery theory. Let $X$ be a simply connected manifold with boundary $\partial X$. The structure set $\mathcal{S}(X, \partial X)$ (sometimes written as $\mathcal{S}_0(X)$) is defined to be the set of equivalence classes of tuples $(W, \partial W, f)$ where $W$ is a manifold with boundary $\partial W$ and $f$ is a homotopy equivalence that restricts to a diffeomorphism on the boundary. Two such tuples $(W_0, \partial W_0, f_0)$ and $(W_1, \partial W_1, f_1)$ are equivalent, if there exists a diffeomorphism $\alpha: W_0 \to W_1$ such that $f_0 = f_1 \circ \alpha$.

It is a consequence of the $h$-cobordism theorem that we have the following isomorphism ([BM13, Section 3.2, pp.33])

$$\pi_k \left( \frac{h\Aut(M)}{\Diff(M)} \right) \cong \mathcal{S}_0(D^k \times M).$$

The main result of surgery theory is that the structure set $\mathcal{S}_0(D^k \times M)$ fits into an exact sequence of sets known as the surgery exact sequence (cf. [CLM, Theorem 10.21 and Remark 10.22]):

$$L_{k+d+1}(\mathbb{Z}) \to \mathcal{S}_0(D^k \times M) \to \mathcal{N}_0(D^k \times M) \to L_{k+d}(\mathbb{Z}).$$

Here, $\mathcal{N}_0(D^k \times M)$ is the set of normal invariants which is given by equivalence classes of tuples $(W, f, \hat{f}, \xi)$, where $W$ is a $d + k$-dimensional manifold with (stable) normal bundle $\nu_W$, $\xi$ is a stable vector bundle over $D^k \times M$ and $f: W \to D^k \times M$ is a map of degree 1 covered by a bundle map $\hat{f}: \nu_W \to \nu_{D^k \times M} \oplus \xi$ such that $(f, \hat{f})$ restricts to the identity on the boundary and the equivalence relation is given by cobordism.

---

2Since we assume $X$ to be simply connected, every homotopy equivalence is simple and we do not need to require this in the definition.
Since we only consider simply connected manifolds, the relevant $L$-groups are 4-periodic and given by (cf. [CLM, Theorem 7.96])

\[
L_n(\mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & \text{if } n \equiv 0 \pmod{4} \\
\mathbb{Z}/2 & \text{if } n \equiv 2 \pmod{4} \\
0 & \text{otherwise}
\end{cases}
\]

and the map $\sigma$ in the surgery exact sequence (1) is the so-called surgery obstruction map, which in degrees $d+k \equiv 0 \pmod{4}$ for simply connected $M$ is given by

\[
\sigma(W, f, \hat{f}, \xi) = \frac{1}{8} \left( \text{sign}(W \cup (D^k \times M)) - \text{sign}(S^k \times M) \right) = \frac{1}{8} \text{sign}(W')
\]

where sign denotes the signature (cf. [CLM, Lemma 7.170, Exercise 7.188]). The signature of $W'$ can be computed via Hirzebruch’s signature theorem, which constructs a power series

\[
\mathcal{L}(x_1, x_2, \ldots) = 1 + s_1 x_1 + s_2 x_1 + \cdots + s_i x_i + \cdots + s_i j x_i \cdot x_j + \cdots
\]

such that $\text{sign}(W') = (\mathcal{L}(p_1(TW'), p_2(TW'), \ldots), [W'])$. Here $p_i(TW')$ are the Pontryagin classes of $W'$. Note that $f, \hat{f},$ and $\xi$ can be extended trivially to $W'$. Since the map $\hat{f}$ is of degree one, evaluating the Pontryagin classes of $W'$ against $[W']$ yields the same result as evaluating the Pontryagin classes of $-\xi \oplus T(S^k \times W)$, where $-\xi$ denotes the (stable) orthogonal complement to $\xi$.

In order to further analyse $\mathcal{N}_0(D^k \times M)$, let us define $G(n) = \{f : S^{n-1} \rightarrow S^n \rightarrow M \text{ homotopy equivalence} \}$ and $BG := \text{colim}_{n \rightarrow \infty} BG(n)$. Note, that the index shift stems from the fact that one wants to have an inclusion $O(n) \subset G(n)$ of the orthogonal group. Analogously, let $BO := \text{colim}_{n \rightarrow \infty} BO(n)$. Note that $BG$ is the classifying space for stable spherical fibrations whereas $BO$ is the classifying space for stable vector bundles. The inclusion $O(n) \hookrightarrow G(n)$ induces a map $BO \rightarrow BG$ and we denote the homotopy fibre by $G/O$. By [CLM, Remark 10.28] there is an identification

\[
\mathcal{N}_0(D^k \times M) \cong [S^k \wedge M_+, G/O]_*.
\]

Here $M_+$ is $M$ with a disjoint base point and $\wedge$ denotes the smash product of pointed spaces given by $(X, x) \wedge (Y, y) := (X \times Y) / (X \times \{y\} \cup \{x\} \times Y)$.

The functor $S^k \wedge (-)_+$ is adjoint to the $k$-fold loop space functor $\Omega^k(-)$ and so we get $[S^k \wedge M_+, G/O]_* \cong [M, \Omega^k G/O]$. Now $\Omega^{k+1}BG$ is the homotopy fibre of the map $\Omega^k G/O \rightarrow \Omega^k BO$. By obstruction theory (cf. [Hat02, p. 418]) the obstructions to the lifting problem

\[
\Omega^k G/O \xrightarrow{\text{map}} \Omega^k BO
\]

live in the groups $H^{i+1}(M; \pi_i(\Omega^{k+1}BG)) \cong H^{i+1}(M; \pi_{k+i+1}(BG))$. The homotopy groups of $\pi_k(BG)$ are isomorphic to the shifted stable homotopy groups of spheres $\pi^s_k$ by [CLM, p. 135]. By Serre’s finiteness theorem,
these groups are finite for \( k \geq 2 \) and hence all obstruction groups vanish rationally, since we assumed that \( k \geq 1 \). Since maps \( s(M, \Omega^k BO) \) is an \( H \)-space, we see that for every (pointed) map \( f : M \to \Omega^k BO \), some multiple of \( f \) can be lifted to \( \Omega^k G / \mathbb{O} \). Therefore it suffices for us to specify an element in

\[
[S^k \wedge M_+, BO]_s = \widetilde{KO}^0(S^k \wedge M_+)
\]

in order to get a normal invariant. Next, consider the isomorphism given by the Pontryagin character:

\[
\text{ph}(\omega) := \text{ch}(\omega \otimes \mathbb{C}) : \widetilde{KO}^0(S^k \wedge M_+) \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_{i \geq 0} \tilde{H}^{4i}(S^k \wedge M_+; \mathbb{Q})
\]

\[
\cong u_k \cdot \bigoplus_{i \geq 0} H^{4i-k}(M; \mathbb{Q})
\]

for \( u_k \) the cohomological fundamental class in \( H^k(S^k) \). The \( i \)-th component of the Pontryagin character is given by

\[
\text{ph}_i(\xi) = \text{ch}_{2i}(\xi \otimes \mathbb{C}) = \frac{1}{(2i)!} \left( (-2i)c_{2i}(\xi) + f(c_1(\xi), \ldots, c_{2i-1}(\xi)) \right)
\]

\[
= \frac{(-1)^{i+1}}{(2i-1)!} p_i(\xi)
\]

where \( f(c_1(\xi), \ldots, c_{2i-1}(\xi)) \) is a polynomial in Chern classes of \( \xi \) homogenous of degree \( 2i \) which vanishes since all products in \( \tilde{H}^*(S^k \wedge M_+; \mathbb{Q}) \) are trivial. Hence, for any collection \( (x_i) \in H^{4i-k}(M; \mathbb{Q}) \) and \( (A_i) \in \mathbb{Q} \) there exists a \( \lambda \in \mathbb{Z} \setminus \{0\} \) and a normal invariant \( \langle W, f, \tilde{f}, \xi \rangle \in \mathcal{N}_\mathcal{Q}(D^k \times M) \) such that

\[
p_i(\xi') = (-1)^{i+1}(2i-1)!\lambda A_i \cdot u_k \cdot x_i,
\]

where \( u_k \) denotes the cohomological fundamental class of \( S^k \). This allows us to construct a normal invariant such that the underlying stable vector bundle has prescribed Pontryagin classes, which we will do in the succeeding section.

3. Proof of Main Theorem

3.1. Prescribing Pontryagin classes. Let \( M^d \) be as in Theorem A and let \( m := \frac{d+1}{2} \). Let \( j := \min\{i \geq 1 : p_i(TM) \neq 0 \in H^{4i}(M; \mathbb{Q})\} \in \{1, \ldots, \lfloor \frac{d}{2} \rfloor \} \), where \( p_i(TM) \) denotes the \( i \)-th Pontryagin class of \( M \).

**Lemma 3.1.** There exists a normal invariant \( \eta \in \mathcal{N}_\mathcal{Q}(D^k \times M) \) with underlying stable vector bundle \( \xi \to D^k \times M \) with the following property: For \( \xi' \) the extension of \( \xi \) by the trivial bundle to \( S^k \times M \), we have

\[
\langle p_j(TM \oplus -\xi'), p_{m-j}(TM \oplus -\xi'), [S^k \times M] \rangle \neq 0 \neq \langle p_j(TM \oplus -\xi'), [S^k \times M] \rangle
\]

are the only non-vanishing elementary Pontryagin numbers of \( TM \oplus -\xi \), and \( \sigma(\eta) = 0 \).

**Proof.** Let \( u_M \in H^{4m-k}(M; \mathbb{Q}) \) denote the cohomological fundamental class of \( M \). Since the cup product induces a perfect pairing

\[
H^{4j}(M; \mathbb{Q}) \times H^{4(m-j)-k}(M; \mathbb{Q}) \to \mathbb{Q},
\]

there exists a class \( x \in H^{4(m-j)-k}(M; \mathbb{Q}) \) such that \( x \cdot p_j(TM) = u_M \). By the discussion in section 2 for every \( A \in \mathbb{Q} \) there exists a \( \lambda \in \mathbb{Z} \setminus \{0\} \) and a normal
invariant $\eta = (W, f, \hat{f}, \xi)$ such that the (extended) stable vector bundle $\xi'$ has only 2 higher non-vanishing rational Pontryagin classes, namely:

$$p_0(\xi') = 1$$
$$p_{m-j}(\xi') = -(-1)^{m-j+1}(2m - 2j)! \cdot \lambda \cdot u_k \cdot x$$
$$p_m(\xi') = -(-1)^{m+1}(2m)! \lambda A \cdot u_k \cdot u_M$$

Since $j < m$ and $p_i(TM) = 0$ for all $0 < i < j$, we have\footnote{Since we are only interested in rational Pontryagin classes we have $p(V \oplus W) = p(V) \cdot p(W)$.}

$$p_n(TM \oplus -\xi') = \sum_{i=0}^{n} p_i(TM) \cdot p_{n-i}(-\xi')$$

$$= \begin{cases} p_m(TM) + p_j(TM) \cdot p_{m-j}(-\xi') + p_m(-\xi') & \text{if } n = m \\ p_{m-j}(TM) + p_m(-\xi') & \text{if } n = m - j \\ p_n(TM) & \text{otherwise} \end{cases}$$

$$\langle p_j(TM \oplus -\xi') \cdot p_{m-j}(TM \oplus -\xi'), [S^k \times M] \rangle$$

$$= \langle p_j(TM) \cdot (p_{m-j}(TM) + p_m(-\xi')), [S^k \times M] \rangle$$

$$= \langle p_j(TM) \cdot (-1)^{m-j+1}(2m - 2j)! \lambda \cdot x \cdot u_k, [S^k \times M] \rangle$$

$$= (-1)^{m-j+1}(2m - 2j)! \lambda =: b \neq 0$$

We will choose $A$ later. Note that every non-vanishing elementary Pontryagin number of $TM \oplus -\xi$ must contain a Pontryagin class of $\xi$. Otherwise it would be a Pontryagin number of $TM$ of total degree $d + k$ which would evaluate trivially against $[S^k \times M]$. Therefore, any non-vanishing elementary Pontryagin-number of $TM \oplus -\xi$ must either contain $p_{m-j}$ or $p_m$ and since $p_i(TM) = 0$ for all $0 < i < j$, the above are the only possibly non-vanishing ones. It remains to compute the surgery obstruction using Hirzebruch’s signature theorem:

$$\sigma(\eta) = \text{sign}(W') = \langle \mathcal{L}(W'), [W'] \rangle = \langle \mathcal{L}(W'), f_*[S^k \times M] \rangle$$

$$= \langle \mathcal{L}(S^k) \cdot \mathcal{L}(TM \oplus \xi), [S^k \times M] \rangle$$

$$= \langle s_j \cdot p_j(TM \oplus \xi)p_{m-j}(TM \oplus \xi) + s_m p_m(TM), [S^k \times M] \rangle$$

$$= (s_j \cdot m - s_m) b + s_m \cdot c \cdot A.$$
By the discussion in Section 2, there exists a bundle $E \to S^k$ with the same two non-vanishing elementary Pontryagin numbers and by [FR21, Lemma 2.5] the $\hat{A}$-genus of $E$ does not vanish. Also note that $E$ is fibre homotopy equivalent to the trivial bundle.

**Lemma 3.2.** If $j := \min\{i \geq 1: p_i(TM) \neq 0\}$ is smaller than $d/4$, then there exists a bundle $E \to S^k$ as above that has a cross-section with trivial normal bundle.

**Proof.** Let $\text{triv} : S^k \to S^k \times M$ be the trivial section. Since the bundle $E$ constructed in Lemma 3.1 is fibre homotopy equivalent to the trivial bundle via $f : S^k \times M \simeq E$ we get a section $s := f \circ \text{triv} : S^k \rightarrow E$. We have

$$s^* p_n(TE) = \text{triv}^* \left( \sum_{i=0}^n p_i(TM) \cdot p_{n-i}(-\xi) \right) = \sum_{i=0}^n \text{triv}^* p_i(TM) \cdot \text{triv}^* p_{n-i}(-\xi) = \text{triv}^* p_n(-\xi)$$

Recall, that the only non-vanishing Pontryagin classes of $\xi$ are $p_{m-j}$ and $p_m$ and let $\nu_s$ denote the normal bundle of $s$. Since the rank of this bundle is bigger than $k$, the bundle $\nu_s$ is stable in the sense that it is classified by an element in

$$\pi_k(BO) = KO^{-k}(pt) \cong \begin{cases} 
\mathbb{Z} & \text{for } k \equiv 0 \ (4) \\
\mathbb{Z}/2 & \text{for } k \equiv 1, 2 \ (8) \\
0 & \text{otherwise}
\end{cases}$$

Since we are only interested in the problem rationally, it suffices to consider the case $k \equiv 0 \ (4)$. It follows, that $\nu_s$ is trivial if $p_{k/4}(\nu_s) = 0$ and as $p(S^k) = 1$, the Pontryagin class $p_{k/4}$ of $\nu_s$ satisfies

$$p_{k/4}(\nu_s) = p_{k/4}(s^*TE) = s^* p_{k/4}(TE) = \text{triv}^* p_{k/4}(\xi) = 0$$

since by our assumption $k/4 < \frac{d+k}{4} - j = m - j$ and $p_{m-j}$ and $p_m$ are the only Pontryagin classes of $\xi$. \[\square\]

**Remark 3.3.** If $d \neq 0 \ (4)$, the requirement from the lemma is automatically full-filled. If $d \equiv 0 \ (4)$ and $j = d/4$, then $M$ has only one non-vanishing Pontryagin number, namely $\langle p_{d/4}(TM) , [M] \rangle$. Since all coefficients in the $\hat{A}$-polynomial are nonzero by [BB18], we have $\hat{A}(M) = a \cdot \langle p_{d/4}(TM) , [M] \rangle \neq 0$ for some $a \in \mathbb{Z} \setminus \{0\}$. If additionally $M$ admits a Spin-structure, then by the Lichnerowicz-formula and the Atiyah–Singer index theorem [AS63; Lic63], $M$ does not support a metric of positive scalar curvature. Hence, for a Spin-manifold of positive scalar curvature, we have $j < d/4$ and Lemma 3.2 applies.

From the discussion in the preceding section we get:

**Proposition 3.4.** Let $k \geq 1$ and let $M$ be an oriented, simply connected manifold of dimension $d \geq \max(3k + 1, 2k + 5)$ that has at least one non-vanishing Pontryagin class. If $d + k \equiv 0 \ (4)$, then there exists a smooth, oriented $M$-bundle $E \to S^k$ that is fibre homotopy equivalent to the trivial bundle and satisfies $\hat{A}(E) \neq 0$. If $M$ admits a Spin-structure and a metric of
positive scalar curvature, then the bundle admits a cross-section with trivial normal bundle.

Remark 3.5. (i) This recovers [HSS14, Theorem 1.4] and provides an upgrade: the result in loc.cit. is “based on abstract existence results [and] does not yield an explicit description of the diffeomorphism type of the fibre manifold” [HSS14, p. 337]. In contrast, our result states, that it is correct for generic manifolds.

(ii) By [HSS14, Proposition 1.9] and [Wie19, Lemma 2.3] a bundle \( M \rightarrow E \rightarrow S^k \) is rationally nullcobordant, if all rational Pontryagin classes vanish or if \( \dim(M) < \frac{k}{2} \). This shows that both assumptions on \( M \) from Proposition 3.4 (and hence from Theorem A) are actually necessary, even though the dimension bound is not be optimal (cf. Remark 2.3).

Recall that an oriented manifold \( M \) is called \( \hat{A} \)-multiplicative fibre in degree \( k \) if for every oriented \( M \)-bundle \( E \rightarrow S^k \) we have \( \hat{A}(E) = 0 \) (cf. [HSS14, Definition 1.8]). From Proposition 3.4 and Remark 3.5 (ii) we deduce the following corollary.

**Corollary 3.6.** A manifold \( M \) of dimension \( d \geq \max(3k + 1, 2k + 5) \) is an \( \hat{A} \)-multiplicative fibre in degree \( k \) if and only if all its rational Pontryagin classes vanish.

3.2. Spin-structures and positive (scalar) curvature. Let \( M \) be Spin and let \( B\text{Diff}^{\text{Spin}}(M) \) be the classifying space for \( M \)-bundles with a Spin-structure on the vertical tangent bundle\(^4\). By [Ebe06, Lemma 3.3.6] the homotopy fibre of the forgetful map \( B\text{Diff}^{\text{Spin}}(M) \rightarrow B\text{Diff}(M) \) is a \( K(\mathbb{Z}/2,1) \) if \( M \) is simply connected. Therefore the induced map

\[
\pi_n(B\text{Diff}^{\text{Spin}}(M)) \otimes \mathbb{Q} \rightarrow \pi_n(B\text{Diff}(M)) \otimes \mathbb{Q}
\]

is an isomorphism and we may assume without loss of generality that the bundles from Section 3 carry a Spin-structure on the vertical tangent bundle and hence on the total space, provided that \( M \) admits one.

It is a well known consequence of the Atiyah–Singer Index theorem and the Lichnerowicz formula that Spin-manifolds with non-vanishing \( \hat{A} \)-genus do not admit a metric of positive scalar curvature [AS63; Lic63]. Theorem A then follows from another standard argument that goes back to Hitchin [Hit74] (see [HSS14, Remark 1.5] or [FR21, Proposition 3.7]) from Proposition 3.4. Together with [Fre19a, Theorem A] (see also [Fre19b, Corollary E]) we also derive the following result.

**Corollary 3.7.** Let \( M \) be a simply connected Spin-manifold of dimension at least 6 that admits a metric of positive scalar curvature. Then the action

\[
\pi_0(\text{Diff}(M)) \rightarrow \pi_0\text{hAut}(\mathcal{R}_{\text{scal}>0}(M))
\]

factors through a finite group if and only if \( d \not\equiv 3(4) \) or \( d \equiv 3(4) \) and all Pontryagin classes of \( M \) vanish.

\(^4\)A model for \( B\text{Diff}^{\text{Spin}}(M) \) is given by

\[
B\text{Diff}^{\text{Spin}}(M) := \{(N, \tilde{\ell}_n), M \cong N \subset \mathbb{R}^\infty, \tilde{\ell}_N \in \text{Bun}(TN, \theta'U_d)\}
\]

for \( \theta : B\text{Spin}(d) \rightarrow B\text{SO}(d) \) the 2-connected cover, \( U_d \rightarrow B\text{SO}(d) \) the universal oriented vector bundle and \( \text{Bun}(\_\_\_) \) the space of bundle maps.
Theorem A also recovers [HSS14, Theorem 1.1 a]):

Corollary 3.8. Let $k \geq 1$ and let $N$ be a Spin-manifold with $\dim(N) = d \geq \max(3k + 1, 2k + 5)$ and $d + k \equiv 0 \pmod{4}$. Then $\pi_{k-1}(\mathcal{R}_{\text{scal}} > 0(N))$ contains an element of infinite order.

Proof. Let $K$ be a $K3$-surface. Then for $n := d - 4 \geq 2$, the manifold $K \times S^n$ satisfies the hypothesis of Theorem A and there is a $K \times S^n$-bundle $E \to S^k$ that has non-vanishing $\mathcal{A}$-genus and admits a cross section with trivial normal bundle. If $N$ is an arbitrary Spin-manifold of dimension $d \geq 6$, then gluing in the trivial $N \setminus D^d$-bundle along this cross section yields a $N \# (K \times S^{d-4})$-bundle over $S^k$ with non-vanishing $\mathcal{A}$-genus. Hence the group $\pi_{k-1}(\mathcal{R}_{\text{scal}} > 0(N \# (K \times S^{d-4})))$ contains an element of infinite order. Since $N$ is cobordant to $N \# (K \times S^{d-4})$ in $\Omega^d_{\text{Spin}}(B\pi_1(N))$, the corresponding spaces of positive scalar curvature metrics are homotopy equivalent.

□

Remark 3.9. A more general result without any dimension restriction has been proven by Botvinnik–Ebert–Randal-Williams [BERW17]. The methods from loc.cit. are however not constructive and do not give a way to decide if the obtained elements arise from the orbit of the action $\text{Diff}(M) \rtimes \mathcal{R}_{\text{scal}} > 0(M)$.

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