NONCOMMUTATIVE DIFFERENTIAL CALCULUS
ON THE $\kappa$-MINKOWSKI SPACE

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ABSTRACT
Following the construction of the $\kappa$-Minkowski space from the bicrossproduct structure of the $\kappa$-Poincare group, we investigate possible differential calculi on this noncommutative space. We discuss then the action of the Lorentz quantum algebra and prove that there are no 4D bicovariant differential calculi, which are Lorentz covariant. We show, however, that there exist a five-dimensional differential calculus, which satisfies both requirements. We study also a toy example of 2D $\kappa$-Minkowski space and and we briefly discuss the main properties of its differential calculi.

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1 The $\kappa$-Poincare algebra

The $\kappa$-Poincare algebra has been introduced [1, 2] and studied extensively [3, 4, 5, 6, 7] as one of possible Hopf algebra deformations of the standard Poincare algebra. The momenta and the rotation generators remain unchanged and the deformation occurs only in the boost sector and the coproduct structure.

Recently, it was shown [8] that the $\kappa$-Poincare has a structure of a Hopf algebra extension of (classical) $U(so(1, 3))$ by the Hopf algebra of (deformed) translations $T$:

\[ \{ P_\mu, P_\nu \} = 0, \]
\[ \Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0, \]
\[ \Delta P_i = P_i \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes P_i. \]

The commutation relations of the Lorentz algebra generators, rotations $M_i$ and boosts $N_i$ are the standard ones:

\[ [M_i, M_j] = \epsilon_{ijk} M_k, \quad [M_i, N_j] = \epsilon_{ijk} N_k, \quad [N_i, N_j] = -\epsilon_{ijk} M_k, \]

whereas the cross relations and the coproduct structure of the Lorentz part are deformed:

\[ [P_0, M_i] = 0, \quad [P_i, M_j] = \epsilon_{ijk} P_k \]
\[ [P_0, N_i] = -P_i, \quad [P_i, N_j] = -\delta_{ij} \left( \frac{\kappa}{2} (1 - e^{-\frac{2P_0}{\kappa}}) + \frac{1}{2\kappa} P^2 \right) + \frac{1}{\kappa} P_i P_j, \]
\[ \Delta N_i = N_i \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes N_i + \epsilon_{ijk} P_j \otimes M_k, \quad \Delta M_i = M_i \otimes 1 + 1 \otimes M_i. \]

The classical Poincare algebra is obtained in the limit $\kappa \to \infty$.

2 $\kappa$-Minkowski space

Here we shall briefly outline the construction of the $\kappa$-Minkowski space ($\mathcal{M}_4^\kappa$) and the action of the $\kappa$-Poincare on its generators, as developed in [8].

As the $\kappa$ deformation of Minkowski space we take the dual Hopf algebra of the translation algebra $T$ and we denote its generators by $x_\mu$. From the relations (1-3) we immediately obtain:

\[ [x_i, x_j] = 0, \quad [x_i, x_0] = \frac{x_i}{\kappa}, \]
\[ \Delta x_\mu = x_\mu \otimes 1 + 1 \otimes x_\mu. \]

The canonical action of translations on our Minkowski space is:

\[ t \triangleright x = < x_{(1)}, t > x_{(2)}, \quad \forall x \in T^*, \quad t \in T, \]

where we use shorthand notation for $\Delta x = \sum x_{(1)} \otimes x_{(2)}$.

From the bicrossproduct structure of $\kappa$-Poincare we have the action of $U(so(1, 3))$ on translations $T$, which now, by duality, could be translated into action on the generators of the Minkowski space:

\[ M_i \triangleright x_j = \epsilon_{ijk} x_k, \quad M_i \triangleright x_0 = 0, \quad N_i \triangleright x_j = -\delta_{ij} x_0, \quad N_i \triangleright x_0 = -x_i, \]
which generalizes to the whole algebra by the covariance condition:

$$h \triangleright xy = (h_1 \triangleright x)(h_2 \triangleright y), \quad \forall h \in U(so(1, 3)), x, y \in T^*,$$

Finally, let us quote here another result of [8] that the lowest order Lorentz-invariant polynomial in $x_\mu$ is:

$$x_0^2 - \vec{x}^2 + \frac{3}{\kappa}x_0.$$

3 Differential calculus

In this section we shall present the main points of the construction of bicovariant differential calculi on the $\kappa$-Minkowski space. Let us remind that the (first order) bicovariant calculus on a Hopf algebra $A$ is defined by $(\Gamma, \Delta, \Delta', d)$, where $\Gamma, \Delta, \Delta'$ defines a bicovariant bimodule over $A$ (for details see [9]). In particular: $\Gamma : \Delta \to \Delta \otimes \Delta$, $\Delta : \Gamma \to \Delta \otimes \Gamma$, are such that:

$$\Delta(\omega a) = \Delta(\omega)\Delta(a),$$

$$\Gamma \Delta(a \omega) = \Gamma \Delta(a)\Delta(\omega),$$

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta,$$

$$(\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta.$$

In addition, the external derivative $d$ satisfies:

$$\Gamma d = (d \otimes \text{id})\Delta, \quad \Delta d = (\text{id} \otimes d)\Delta.$$

Now, we shall investigate the case of bicovariant differential calculi on the $\kappa$-Minkowski space. Let us begin with observation, that, due to the coproduct structure on $\mathcal{M}_4^k$ ([4]) and the bicovariance property (14) all $dx_\mu$ are simultaneously left and right-covariant:

$$\Delta(\omega a) = \Delta(\omega)\Delta(a),$$

$$\Gamma \Delta(a \omega) = \Gamma \Delta(a)\Delta(\omega),$$

therefore, from the general properties of the bicovariant bimodules (Theorem 2.1 [9]) we deduce that the commutator $[x_\mu, \chi_a]$ must have the following expansion:

$${[x_\mu, \chi_a] = \sum_{\mu, a, b} A_{\mu a}^b \chi_b.}$$

There are, of course, some consistency conditions for the above relations, which come from the mixed Jacobi identity:

$$[[x_\mu, x_\nu], \chi_a] + [[x_\nu, \chi_a], x_\mu] + [[\chi_a, x_\mu], x_\mu] = 0.$$
If we rewrite, for simplicity of notation, the commutation relations (3) on $\mathcal{M}_\kappa^4$, in a more general form:

$$[x_\mu, x_\nu] = B^{\rho}_{\mu\nu} x_\rho,$$

(19)

the relations (18) may be rewritten as:

$$A_{\nu c}^a A_{\rho b}^c - A_{\mu c}^a A_{\rho b}^c = B^{\rho}_{\mu\nu} A_{\rho b}^a,$$

(20)

which simply state that the map $\pi : x_\mu \to -A_\mu$, where $A_\mu$ is an $N \times N$ matrix, is a representation of the Lie algebra defined by the relations (3). Therefore the general theory of bicovariant bimodules on the $\kappa$-Minkowski space is linked with the finite dimensional representation theory of the Lie algebra generated by $x_\mu$. \(^1\)

Furthermore, due to (15) we obtain that $dx_\mu$ may be expressed as a linear combination of $\chi_a$:

$$dx_\mu = D_\mu^a \chi_a,$$

(21)

and if we impose the Leibnitz rule, by differentiating (19) we obtain another restriction:

$$D_\nu^b A_{\nu c}^a - D_\mu^b A_{\nu c}^a = B^{\rho}_{\mu\nu} D_\rho^a.$$

(22)

Both relations (20,22) are sufficient and necessary to determine a bicovariant differential calculus on the $\kappa$-Minkowski space. In what follows, we shall not attempt, however, to classify all possible bicovariant differential calculi, as we shall exploit other restrictions provided by rich structure of the $\kappa$-Poincare algebra.

### 4 The action of $\kappa$-Lorentz on the differential calculus.

Having discussed the structure of the possible bicovariant differential calculi on the $\kappa$-Minkowski space we shall now proceed to extend the action of the Lorentz algebra to the bimodule of one-forms.

We shall postulate that the action of the Lorentz algebra (11-12) extends to the differential algebra in a natural covariant way, i.e.:

$$h\triangleright(y dx) = (h_{(1)} \triangleright y)(d(h_{(2)} \triangleright x)),$$

$$h\triangleright(dx y) = (d(h_{(1)} \triangleright x))(h_{(2)} \triangleright y).$$

(23)

From the above definition and the action (11) we easily obtain the following identities:

$$N_{k\triangleright}[x_i, dx_j] = -\delta_{ki}[x_0, dx_j] - \delta_{kj}[x_i, dx_0] + \frac{1}{\kappa}(\delta_{kj}dx_i - \delta_{ij}dx_k),$$

(24)

$$N_{k\triangleright}[x_0, dx_i] = -[x_k, dx_i] - \delta_{ki}[x_0, dx_0] + \frac{1}{\kappa}\delta_{ki}dx_0,$$

(25)

$$N_{k\triangleright}[x_i, dx_0] = -[x_i, dx_k] - \delta_{ki}[x_0, dx_0],$$

(26)

\(^1\)Of course, this results are general and may be applied to construction and classification of bicovariant bimodules and bicovariant differential calculi on any universal enveloping algebra. In this paper, however, we restrict ourselves only to the case of $\kappa$ Minkowski space.

\(^2\)In fact, one may postulate as well that the action of translations (as discussed in 3) extends in the same way, however, as as $P_{\mu\triangleright}dx_\nu = 0$, due to (23) and the coproduct structure (3) one may see that every bicovariant differential calculus with relations (14) is automatically covariant in the above sense with respect to the action of translations alone. Therefore, we shall concentrate on the highly nontrivial Lorentz part of the action.

3
\[ N_k \triangleright [x_0, dx_0] = -[x_k, dx_0] - [x_0, dx_k] + \frac{1}{\kappa} dx_k, \]  
(27)

\[ M_k \triangleright [x_i, dx_j] = \epsilon_{kis} [x_s, dx_j] + \epsilon_{ksj} [x_i, dx_s], \]  
(28)

\[ M_k \triangleright [x_0, dx_i] = \epsilon_{kis} [x_0, dx_s], \]  
(29)

\[ M_k \triangleright [x_i, dx_0] = \epsilon_{kis} [x_s, dx_0] \]  
(30)

\[ M_k \triangleright [x_0, dx_0] = 0. \]  
(31)

From the above relations we can immediately see that if we postulate the \( \kappa \)-Lorentz covariance we can no longer have a commutative differential calculus on the subalgebra generated by \( x_i, i = 1, 2, 3 \), as we must have (at least) a non-vanishing term:

\[ [x_i, dx_j] = \delta_{ij} \frac{1}{\kappa} dx_0 + \ldots, \]  
(32)

which follows directly from the equation (24).

Now, if we consider only 4D bicovariant calculi, with the bimodule of one forms generated by \( dx_\mu \), taking into account the relations (17) (with \( \chi_\mu = dx_\mu \)) and plugging them into (24-31), we obtain system of linear equations for the coefficients \( A^\rho_{\mu \nu} \), which we can solve. It appears, that the solution is unique and gives us the following relations:

\[ [x_i, dx_j] = \delta_{ij} \frac{1}{\kappa} dx_0, \]  
(33)

\[ [x_0, dx_i] = 0, \]  
(34)

which, however, do not define a differential calculus as they fail to obey the condition (20)! Therefore we may conclude that there exist no bicovariant and \( \kappa \)-Lorentz covariant 4D differential calculus on the \( \kappa \)-Minkowski space. Before we proceed with further considerations of the 4D situation, let us study a much simpler model, of 2D \( \kappa \)-Poincare and \( \kappa \)-Minkowski space.

5 2D \( \kappa \)-Minkowski space and the differential calculi

The two-dimensional \( \kappa \)-Poincare algebra is defined in a similar way as the 4D one (see [10] for details.) For simplicity of notation let us denote the momentum generators as \( E \) and \( P \) and the boost operator by \( N \). Then the commutation relations and the coproduct structure are as follows:

\[ [P, E] = 0 \quad [N, E] = P, \quad [N, P] = -\frac{\kappa}{2} (1 - e^{-\frac{2\kappa}{E}}) + \frac{1}{2\kappa} P^2, \]  
(35)

\[ \Delta P = P \otimes 1 + e^{-\frac{E}{\kappa}} \otimes P, \quad \Delta E = E \otimes 1 + 1 \otimes E, \quad \Delta N = N \otimes 1 + e^{-\frac{E}{\kappa}} \otimes N. \]  
(36)

The 2D \( \kappa \)-Minkowski could be defined like in the 4D case as a Hopf dual to the algebra of momenta. If we call its generators \( x, t \), we shall have:

\[ [x, t] = \frac{1}{\kappa} x, \quad \Delta x = x \otimes 1 + 1 \otimes x, \quad \Delta t = t \otimes 1 + 1 \otimes t. \]  
(37)

The bicrossproduct construction could be also repeated in this case, and the action of the boost \( N \), which follows from it is:

\[ N \triangleright t = -x, \quad N \triangleright x = -t. \]  
(38)

which extends onto the whole of the algebra according to the covariance prescription [12].
5.1 2D differential calculus

Using the same arguments as in the 4D situation, we shall postulate both bicovariance and Lorentz covariance of the differential calculus. First let us present the relations following from the latter requirement:

\[ N \triangleright [t, dt] = -[x, dx] - [t, dx] + \frac{1}{\kappa} dt, \]
\[ N \triangleright [x, dt] = -[t, dt] - [x, dx], \]
\[ N \triangleright [t, dx] = -[t, dt] - [x, dx] + \frac{1}{\kappa} dx, \]
\[ N \triangleright [x, dx] = -[t, dx] - [x, dt]. \]

Again, if we look for two-dimensional bicovariant calculi, which satisfy the above covariance property, we may solve the corresponding system of linear equations, obtaining:

\[ [x, dx] = \frac{1}{\kappa} dt, \quad [x, dt] = \frac{1}{\kappa} dx, \quad [t, dx] = 0, \quad [t, dt] = 0, \]

which, though being the solution to (39-42), does not give a differential calculus, since it fails to obey (20). This result is hardly surprising, as we already know that this was the case in four dimensions. Therefore we must look for other possibilities, the simplest of which, is the higher-dimensional calculus.

5.2 3D differential calculus

Let us assume that the bimodule of one-forms is generated by left-invariant forms \( dx, dt \) and \( \phi \). Furthermore, we shall assume that \( N \triangleright \phi = 0 \). From the general theory we already know (17) the most general form of the commutators. We may always choose \( \phi \) in such a way that \( [x, \phi] = \alpha dx \).

Then, by applying \( N \) to both sides, we verify that the only consistent case of Lorentz covariant calculus is:

\[ [x, \phi] = dx, \quad [t, \phi] = dt + \phi. \]

while the rest of the relations are still unknown:

\[ [x_\mu, \chi_a] = A^b_{\mu a} \chi_b, \quad t = x_0, \quad x = x_1; \quad \chi_0 = dt, \quad \chi_1 = dx, \quad \chi_2 = \phi. \]

Now, solving the system of linear equations (39-42) for the coefficients \( A^b_{\mu a} \) restricts the number of free parameters from 12 to 2! If we denote the free parameters as \( a, b \), the matrices \( A_t, A_x \) introduced earlier to define the commutation relations are:

\[ A_t = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 1 & \beta & 0 \end{pmatrix}, \quad A_x = \begin{pmatrix} 0 & \frac{1}{\kappa} & -b \\ \frac{1}{\kappa} & 0 & -a \\ 0 & 1 & 0 \end{pmatrix}. \]

Now, if we impose both consistency conditions (20,22) it appears that they would be satisfied only and only if \( a = \frac{1}{\kappa} \) and \( b = 0 \).

3By rescaling \( \phi \) we may set \( \alpha = 1 \) unless it is 0, the latter case, however, does not give any solutions.

4As the calculations are straightforward, we shall not present them here.
Therefore, there exist only one 3D bicovariant and Lorentz covariant differential calculus on the 2D \( \kappa \)-Minkowski space, with the following commutation relations:

\[
\begin{align*}
[x, dx] &= \frac{1}{\kappa}(dt - \frac{1}{\kappa} \phi), \quad [x, dt] = \frac{1}{\kappa} dx, \quad [x, \phi] = dx, \\
[t, dx] &= 0, \quad [t, dt] = \frac{1}{\kappa} \phi, \quad [t, \phi] = dt.
\end{align*}
\] (46)

Before we turn back to the four-dimensional situation let us comment briefly on the obtained result. First of all, let us rewrite slightly the relations (46) introducing an one-form \( \psi = dt - \frac{1}{\kappa} \phi \):

\[
\begin{align*}
[x, dx] &= \frac{1}{\kappa} \psi, \quad [x, \phi] = dx, \quad [x, \psi] = 0, \\
[t, dx] &= 0, \quad [t, \psi] = -\frac{1}{\kappa} \psi, \quad [t, dt] = \frac{1}{\kappa^2} \phi.
\end{align*}
\] (47)

Though it is merely a simply change of basis, we find it convenient to present the rules of differentiation for elements of algebra constructed of \( x \) and \( t \) alone and interpreted as usual functions (polynomials) on the real line. In the case of \( x \) we have:

\[
df(x) = dx \partial_x f(x) + \psi \frac{1}{2\kappa} \partial_{xx} f(x),
\] (48)

and since the commutation relations between \( x, dx, \psi \) are closed, this defines a differential submodule of our bigger one. Let us point out here that this specific \( \kappa \)-deformed calculus on the real line obtained as a restriction of the bigger structure is equivalent to the example of higher-derivatives calculus discussed elsewhere [11].

For the \( t \) variable the differential structure is, however, much different. It is convenient to use the forms \( \psi \) and \( \tilde{\psi} = dt + \frac{1}{\kappa} \phi \), which have much simpler commutation relations with \( t \):

\[
[t, \psi] = -\frac{1}{\kappa} \psi, \quad [t, \tilde{\psi}] = \frac{1}{\kappa} \tilde{\psi}.
\] (49)

Now it appears that:

\[
df(t) = \tilde{\psi} (1 - e^{\frac{1}{\kappa} \phi}) f(t) + \psi (1 - e^{-\frac{1}{\kappa} \phi}) f(t).
\] (50)

This differential calculus is a system is far more complicated than the one considered earlier, as it involves the partial derivatives of all orders.

Finally, let us consider the action of the external derivative \( d \) on an arbitrary function \( f(t, x) \), with the normal ordering : \( f : \), which denotes that all powers of \( t \) are shifted to the left:

\[
df = dx \partial_x : f : + \psi (1 - e^{-\frac{1}{\kappa} \phi}) \frac{1}{2\kappa} e^{-\frac{1}{\kappa} \phi} \partial_{xx} : f : + \tilde{\psi} (1 - e^{\frac{1}{\kappa} \phi}) : f :,
\] (51)

which is a highly complicated expression.

Further, we may construct higher-order forms, and it appears that the following set of rules defines the multiplication of one-forms:

\[
\begin{align*}
dx \bullet dt &= -dt \bullet dx, \quad dt \bullet \phi = -\phi \bullet dt, \quad dx \bullet \phi = -\phi \bullet dx, \\
dx \bullet dx &= -dt \bullet dt, \quad d\phi = \kappa^2 (dt \bullet dt - dx \bullet dx).
\end{align*}
\] (52)

This is the weakest set of constraints on the higher-order calculus, we may as well consider the quotient of the above with \( dx \bullet dx = 0 \), thus enforcing \( dt \bullet dt = 0 \) and \( d\phi = 0 \).
6 5D Lorentz covariant, bicovariant differential calculus on 4D $\kappa$-Minkowski space

Following the example of the last section with the toy model of 2D bicovariant calculus, we shall attempt to construct a corresponding structure in the four-dimensional situation.

Let us take the bimodule of one-forms generated by $dx_\mu$, $\mu = 0, \ldots, 3$ and an additional one-form $\phi$. Moreover, motivated by the 2D example, we shall assume that the form $\phi$ is Lorentz invariant:

$$N_i \triangleright \phi = 0, \quad M_i \triangleright \phi = 0,$$

and the commutation relations between the coordinates $x_\mu$ and all generating one-forms $\phi$ are the following:

$$[x_\mu, \phi] = dx_\mu, \quad [x_0, dx_0] = \frac{1}{\kappa} \phi,$$

$$[x_i, x_j] = \delta_{ij} \frac{1}{\kappa} (dx_0 - \frac{1}{\kappa} \phi), \quad [x_0, dx_i] = 0, \quad [x_i, dx_0] = \frac{1}{\kappa} dx_i.$$ (54)

It could be easily checked that these relations satisfy the Lorentz covariance conditions (24-31), and both remaining conditions (20) and (22) and therefore they define a five-dimensional bicovariant and Lorentz covariant differential calculus on the $\kappa$-Minkowski space.

The differential calculus presented above has the same structure as its two-dimensional analogue and one can repeat all steps and calculate the explicit expression for $d$ and products of higher order form. We shall not do it here, as the results are exactly the same is in the 2D case, though, of course, a single space variable $x$ should now be replaced by a triple $x_1, x_2, x_3$.

We have demonstrated here the existence of such calculus, however, we cannot yet claim that it is unique, though it seems to be a reasonable hypothesis, and we shall address in future work.

7 Conclusions

The $\kappa$ deformation of the Poincare algebra is a good example and a testing ground for possible deformation of physical theories. The construction of the $\kappa$-Minkowski space enables us to use the tools noncommutative geometry to construct $\kappa$ deformations of field theory. The differential calculus, being the most important tool, is therefore a crucial point of these efforts. As we have learned from the studies of quantum groups, the requirement of bicovariance, though a strong one, seems to be at least an elegant way of choosing a particular differential structure. Of course, there may by many bicovariant differential calculi, also in our case of the $\kappa$-Minkowski space. Therefore, one may look for further constraints, which in our case are provided naturally by the action of the Lorentz algebra. The requirement of Lorentz covariance (or rather full $\kappa$-Poincare covariance, since, as we have already mentioned every bicovariant calculus is automatically covariant with respect to the action of translations) covariance seems to be a reasonable choice.

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5Let us mention here the simplest (and natural) example [12]: $[x_0, dx_i] = -\frac{1}{\kappa} dx_i$, and other commutators vanishing.
though its consequences are much more significant than one would suspect. The fact that there are no differential calculi of the same dimension as the underlying space (which we have shown for \( D = 2 \) and \( D = 4 \)) is very interesting, though hardly surprising in the \( q \)- or \( \kappa \)-deformed world. Let us stress, however, the difference: here it is not only the bicovariance, which enforces it, but some more requirements coming from some 'external' symmetries.

The higher-dimensional differential calculus, which we have constructed in both cases is therefore the most reasonable candidate as a tool for constructing models of \( \kappa \)-deformed field theory. This, as well as some detailed studies of its properties, is an interesting topic, which we shall investigate in future. What physical consequences this calculus may have and how would contribute to verification of \( \kappa \)-deformed physics as a feasible theory is also an open problem.

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