Probing Quantum Nonlinearities through Neutrino Oscillations

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Abstract

We investigate potential quantum nonlinear corrections to Dirac’s equation through its sub-leading effect on neutrino oscillation probabilities. Working in the plane-wave approximation and in the $\mu - \tau$ sector, we explore various classes of nonlinearities, with or without an accompanying Lorentz violation. The parameters in our models are first delimited by current experimental data before they are used to estimate corrections to oscillation probabilities. We find that only a small subset of the considered nonlinearities have the potential to be relevant at higher energies and thus possibly detectable in future experiments. A falsifiable prediction of our models is an energy dependent effective mass-squared, generically involving fractional powers of the energy.

1 Introduction

It appears that constant neutrino masses, though still not directly confirmed, are the simplest way of explaining of current data on neutrino oscillations [1]. Other possibilities, such as Lorentz violating dispersion relations [2], do not seem to be possible explanations of the leading order effects [3].

Neutrinos are a valuable probe of new physics because of their weak interactions and in this paper we will study how nonlinear modifications to the quantum mechanical propagation of a neutrino, that is a nonlinear Dirac
equation, affects neutrino oscillations. The propagating neutrino wavefunctions are assumed to follow

\[(i\gamma^\mu \partial_\mu - m + F) \psi = 0,\]

where \(F\) is a matrix in spinor space, depending on the wavefunction \(\psi\), its adjoint and their derivatives \([4]\). One may think of \(F\) as probing deviations from exact quantum-linearity, a possibility that had had some empirical tests in the non-relativistic regime placing bounds on the nonlinearity scale \([5, 6]\).

In \([7]\) it was suggested that quantum nonlinearities might be related to Lorentz violation, leading one to consider higher-energy tests. Although in this paper we initially adopt the more general possibility that \(F\) might be nonlinear but Lorentz invariant, we find that Lorentz invariant nonlinearities are not likely to be probed by neutrino oscillations. Furthermore, we leave open the possibility that \(F\) might only be an effective nonlinearity, summarising unknown microscopic physics, rather than a fundamental modification of quantum theory. Thus while in our construction of nonlinear Dirac equations in \([4]\) we required the equation to be invariant under the scaling \(\psi \rightarrow \lambda \psi\), as in the linear theory \([5]\), thus leading to nonpolynomial nonlinearities, here we consider also simpler polynomial nonlinearities which do not have that invariance.

We restrict our study to the \(\mu - \tau\) sector as this oscillation is more likely to be probed at higher energies in the near future compared to the \(e - \mu\) oscillation \([8]\). As in the standard formalism we take the weak eigenstates of the neutrinos in the \(\mu - \tau\) sector to be superpositions of the mass eigenstates,

\[\psi_\alpha(x) = \sum_i U^{*\alpha}_i \psi_i(x)\]

(2)

where \(\psi_\alpha(x)\) are the neutrino flavor eigenfunctions, \(\psi_i(x)\) are the “oscillating” eigenfunctions, and \(U\) is the Leptonic mixing matrix. For two-neutrino flavor oscillations the mixing matrix is

\[U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.\]

(3)

The mass eigenstates are assumed to be plane wave solutions of the nonlinear Dirac equation \([1]\) with modified dispersion relations,

\[\psi_i(x_1, t_i) = e^{-i(E_i t_i - \mathbf{p}_i \cdot \mathbf{x}_i)} u_i(E, \mathbf{p})\]

(4)

where \(u_i\) are spinors. Thus after travelling a distance \(L\) between source and detector, and averaging over the unobserved travel time, as in \([9]\), one obtains the usual flavour change probability formula

\[P_{\nu_\alpha \rightarrow \nu_\beta}(E, L) = \sin^2 2\theta \sin^2 \left(\frac{L \Delta p}{2}\right)\]

(5)
where $E$ is the beam energy, the momenta are taken in the direction of $L$ and $\Delta p = p_i - p_j$. Recall that in the $F = 0$ case, with small masses and large energies, one has $\Delta p = \Delta m^2 / 2E$ where $\Delta m^2 = m_i^2 - m_j^2$. Maximum oscillations occur when $L = L_0$, where

$$\frac{L_0 \Delta p}{2} = \frac{\pi}{2} \rightarrow L_0 = \frac{\pi \hbar}{\Delta p}. \quad (6)$$

In the last step, we have restored the $\hbar$'s and $c$'s. $L_0$ is the oscillation length, the path-length needed for a neutrino of flavour $\alpha$ to maximally oscillate to a neutrino of flavour $\beta$. Thus the oscillation length in the conventional approach is given by

$$L_0 = \frac{2\pi \hbar c E}{\Delta m^2 c^4}. \quad (7)$$

The expressions (5) and (6) are valid even in the nonlinear theory but with a modified $\Delta p$. Since we are adopting the plane wave approximation, the only difference from the conventional formalism will be modified dispersion relations in the expression for $\Delta p$.

We classify the types of $F$ in (1) into polynomial (P) or the non-polynomial (NP) forms studied in [4], then into Lorentz violating (LV) or Lorentz invariant (LI). In [4] we had studied the NP type of $F$'s in a double expansion in the degree of nonlinearity $n$ and number of derivatives $d$, for example

$$F = \left( \frac{\bar{\psi} \gamma^5 \psi}{\psi \psi} \right)^n \quad (8)$$

has $d = 0$ and degree $n$. In this paper, for simplicity we consider nonlinearities $F$ which consist of a single term with $n = \alpha$ and structure $F = (\ast)^\alpha$ with $(\ast)$ containing at most one derivative, so that $F$ has at most $\alpha$ derivatives. Furthermore we consider the general case of real $\alpha$, not necessarily an integer. While non-integral powers might not be surprising in an effective theory, interestingly the specific fraction $\alpha = 1/3$ appears when one demands conformal invariance of a simple nonlinear Dirac equation [10]. Also, as noted in [4], in the nonpolynomial case one can still preserve separability for general $\alpha$ through an appropriate construction.

Since $F$ is a matrix in spinor space, we will consider two special cases, $F \propto I$ and $F \propto \gamma^\mu$, representing corrections to the mass or kinetic terms of the usual Dirac equation. As the nonlinearity must be small on phenomenological grounds, we can compute the modified dispersion relations in perturbation theory using the plane wave solutions of the linear theory, see [4].

We describe the Lorentz violating cases using background gauge fields $A_\mu$, but only consider time-like fields $A_\mu \equiv (A_0, \vec{0})$ so that there are no direction dependent terms in our oscillation probabilities. The background fields will simultaneously control the magnitude of both the nonlinearity and the Lorentz violation. A more detailed discussion of Lorentz violating nonlinear Dirac equations is in [4].
The rest of the paper is structured as follows: In the next section we consider one example of a nonlinearity from the class NP-LV in detail and list results for other cases. In Section (3) we bound our parameters using current experimental data and then use our expressions for the modified oscillation probabilities to estimate corrections in future higher energy experiments. In Section (4) we discuss the other classes of nonlinearities while Section (5) concludes the paper.

The conventions we use are similar to those of the textbook [12] and Ref.[1]. We work in 3 + 1 dimensional flat spacetime with metric $\eta^{\mu\nu} = (1, -1, -1, -1)$, set $\hbar = 1 = c$ and restore them as and when needed.

2 The NP-LV class

2.1 $F_1$

An example of an $F$ of the type considered in [4] is given by

$$F_1 = \left( A_\mu \frac{\bar{\psi} \gamma^\mu \psi}{\bar{\psi} \psi} \right)^\alpha$$  \hspace{1cm} (9)

where $A_\mu$ is a real constant background field and $\alpha$ is any real number. Thus this is a NP-LV type of nonlinearity of degree $n = \alpha$, with $F$ proportional to $I$ and no derivatives, $d = 0$.

Perturbing around the plane wave limit gives $F_2 = \left( \frac{A \cdot k}{m} \right)^\alpha$. The modified dispersion relation is then given by

$$k^2 = m^2 - 2m \left( \frac{A \cdot k}{m} \right)^\alpha + O(A^{2\alpha}) .$$  \hspace{1cm} (10)

We take the background field to be time-like. This is to avoid the directional dependence coming from the spatial part of the background field. Thus we have

$$E^2 - p^2 = m^2 - 2m \left( \frac{AE}{m} \right)^\alpha$$  \hspace{1cm} (11)

and for small masses

$$p \simeq E - \frac{m^2}{2E} + A^\alpha \left( \frac{E}{m} \right)^{\alpha-1} .$$  \hspace{1cm} (12)

The momentum difference is given by

$$\Delta p = \frac{\Delta m^2}{2E} - E^{\alpha-1} \Delta \left( \frac{A^\alpha}{m^{\alpha-1}} \right)$$  \hspace{1cm} (13)

where $\Delta \left( \frac{A^\alpha}{m^{\alpha-1}} \right) = \frac{A_0^\alpha}{m_0^{\alpha-1}} - \frac{A_f^\alpha}{m_f^{\alpha-1}}$. Note that we have indicated a possible species dependence in the background gauge field. The oscillation length is

$$L_0 = \frac{2\pi E}{\Delta m^2 - 2E^\alpha \Delta \left( \frac{A^\alpha}{m^{\alpha-1}} \right)}$$  \hspace{1cm} (14)
and may be written in the form

\[ L_0 = \frac{2\pi E}{\Delta m^2 (1 - X_1)} \] \hspace{1cm} (15)

### 2.2 Summary of Other NP-LV Cases

We list here the other types of \( F \) we study in this class.

- \( F \propto I, \ n = d = \alpha \).

\[ F_2 = \left[ iA_\mu \left( \frac{\partial^\mu \bar{\psi} - \left( \partial^\mu \bar{\psi} \right) \psi}{\bar{\psi} \psi} \right) \right]^\alpha \] \hspace{1cm} (16)

- \( F \propto \gamma^\mu, \ n = \alpha, \ d = 0 \)

\[ F_3 = A_\mu \gamma^\mu \left[ B_\nu \left( \bar{\psi} \gamma^\nu \psi \right) \right]^\alpha \] \hspace{1cm} (17)

- \( F \propto \gamma^\mu, \ n = d = \alpha \)

\[ F_4 = A_\mu \gamma^\mu \left[ iB_\nu \left( \bar{\psi} \partial^\nu \psi - \left( \partial^\nu \bar{\psi} \right) \psi \right) \right]^\alpha \] \hspace{1cm} (18)

All background fields are real, with only a nonzero time component. The dispersion relations are obtained by perturbing around plane waves as in the previous subsection to obtain the corresponding \( X \)'s in formula (15).

### 3 Empirical Bounds and Estimates

We have the modified oscillation lengths in the form

\[ L_0 = \frac{2\pi E}{\Delta m^2 (1 - X)} \] \hspace{1cm} (19)

where \( X \) is the leading order correction depending on the nonlinearity parameters and energy. Current neutrino oscillation data fit the conventional neutrino mass scenario quite well. Still, as there are the usual experimental uncertainties, we use those to estimate the size of \( X \). From an eyeball inspection of the data, we conservatively take \( X \) to be in the range 10% to 0.01% and will use this to constrain the range of values which \( \alpha \) can take for each type of nonlinearity considered. Following that, we will estimate the value of \( X \) in future higher energy experiments.
Since the background fields play the dual role of Lorentz violating and nonlinearity parameters, we rewrite them as follows:

For $F_1$: $A^\alpha \rightarrow \epsilon_1$ (20)
For $F_2$: $A^\alpha \rightarrow \epsilon_2$ (21)
For $F_3$: $AB^\alpha \rightarrow \epsilon_3$ (22)
For $F_4$: $AB^\alpha \rightarrow \epsilon_4$ (23)

Upon restoring the factors of $\hbar$’s and $c$’s, the length dimensions of the parameters is given by

\[
\frac{\epsilon_1}{\hbar} = L^{-1} \quad (24) \\
\frac{\epsilon_2}{\hbar} = L^{\alpha-1} \quad (25) \\
\frac{\epsilon_3}{\hbar} = L^{-1} \quad (26) \\
\frac{\epsilon_4}{\hbar} = L^{\alpha-1} \quad (27)
\]

### 3.1 Current Experiments

From current data [13], we may take the size of the small dimensionless LV parameters to be

\[ f \sim 10^{-27} \quad (28) \]

and if the parameters are neutrino species dependent then we assume $\Delta f \sim f$. The relevant neutrino data are taken to be

\[ \Delta m^2 = 2.5 \times 10^{-3} eV^2 \quad (29) \]
\[ E = 100 GeV \quad (30) \]

that is we use the mean beam energy $E = 100 GeV$. We assume that the mass of neutrinos is of the same order as $\sqrt{\Delta m^2}$. In the same, “order of magnitude” spirit, we estimate the expression $\Delta(m\epsilon)$ by $(\Delta m) (\epsilon)$. The nonlinear parameter itself may be written, taking $F_2$ as an example,

\[ \epsilon_2 = \lambda_2^{-1} f \quad (31) \]

where $\lambda_2$ is the nonlinearity length scale. As linear quantum mechanics works well, we will require the nonlinearity length scale to be much smaller than the Compton wavelength of the neutrino,

\[ \lambda_c = \frac{\hbar}{mc} \quad (32) \]

but larger than the Planck length $\lambda_p$ so that we may continue to work in the flat space approximation (however note that the quantum nonlinearity
might plausibly already include, in an effective way, the effects of gravity). Actually, as we shall see below, in conjunction with \(10^{-4} \ll X \ll 10^{-1}\) often only one of the constraints \(\lambda \ll \lambda_c\) or \(\lambda P \ll \lambda\) is actually useful, the other being automatically satisfied.

As the actual mass of neutrinos may be 1 or 2 orders of magnitude larger than what we have assumed above, we compensate for this possibility and the other approximations above by taking

\[
X \text{ of order } 10^{-1} \text{ to } 10^{-4} \quad (33)
\]

\[
\frac{\lambda}{\lambda_c} \text{ of order } 10^{-1} \text{ to } 10^{-4} \quad (34)
\]

\[
\frac{\lambda}{\lambda_p} \text{ of order } 10^4 \text{ to } 10^4 \quad (35)
\]

The various expressions of \(X\)'s upon restoring the \(h\)'s and \(c\)'s, and using the above assumptions, are given by

\[
X_1 = \frac{2hc\Delta f}{\lambda_2 (\Delta m^2 c^4)^{(\alpha+1)/2}} E^\alpha \quad (36)
\]

\[
X_2 = \frac{2^{\alpha+1} \lambda_3^{\alpha-1} \Delta f}{(hc)^{\alpha-1} (\Delta m^2 c^4)^{1/2}} E^\alpha \quad (37)
\]

\[
X_3 = \frac{2hc\Delta f}{\lambda_4 (\Delta m^2 c^4)^{(\alpha+2)/2}} E^{\alpha+1} \quad (38)
\]

\[
X_4 = \frac{2^{\alpha+1} \lambda_5^{\alpha-1} \Delta f}{(hc)^{\alpha-1} \Delta m^2 c^4} E^{\alpha+1} \quad (39)
\]

Our procedure is as follows: Since \(X\) depends on the characteristic length, \(\lambda\), we invert the relationship to plot \(\lambda\) as a function of \(\alpha\) using the values for \(f, m, E\) mentioned above and for the chosen range of \(X\) values. From these plots, we can determine the values of \(\alpha\)'s for which the nonlinearity scale \(\lambda\) lies within the required range. The plots of \(\lambda\) versus \(\alpha\) for \(X_1\) and \(X_2\) are shown in Figure 1 and Figure 2 respectively. From those figures, and similar ones for the other \(X\)'s, we obtain the corresponding range of values for the \(\alpha\)'s:

For \(F_1\) \(1.6 \leq \alpha_1 \leq 2.1\) \(\quad (40)\)

For \(F_2\) \(1.8 \leq \alpha_2 \leq 2.4\) or \(0.2 \leq \alpha_2 \leq 0.4\) \(\quad (41)\)

For \(F_3\) \(0.6 \leq \alpha_3 \leq 1.1\) \(\quad (42)\)

For \(F_4\) \(0.9 \leq \alpha_4 \leq 1.1\) \(\quad (43)\)

Since we expect the nonlinear effects to be small, the \(\alpha\)'s are more likely to be observed near the lower bound in \(X\). These \(\alpha\)'s are indicated in boldface in the above equations. Note that the bounds depend on the choice of \(X\) and \(\lambda_c, \lambda_p\) and so must be updated as more accurate data becomes available.
3.2 Future Experiments

We now reverse the argument. We plot $X$ versus $E$ for the range of $\alpha$ values determined in the previous section and for the allowed regions of $\lambda$ as indicated in Eqns.(34-35). From these plots, we can estimate the values of $X$’s expected in future experiments where higher energies will be available [8]. The plots are shown in Figures 3 to 5 for some cases and parameter values. The general trend of $X(E)$ can be seen from equations (36-39). Of course since we evaluated the $X$’s perturbatively, the expressions and plots are valid only for small $X$, while for larger $X$ the indicated trend is only qualitative.

4 Other Classes of Nonlinearities

4.1 P-LV

Polynomial type of nonlinearities lead to a non-separable Dirac equation. For example from the NP-LV, $F \propto I$ cases considered before one may remove the denominators to get corresponding P-LV type of nonlinearities. But now one sees that the modified dispersion relation will depend on the normalisation chosen for the wavefunctions. If one chooses the usual plane wave normalisation such that $\bar{\psi}\psi = 1$ then the results for the P-LV cases mentioned above would be the same as for the NP class.

So to obtain new results let us explore the popular normalisation $\psi^\dagger\psi = 1$ which makes $\bar{\psi}\psi = m/E$. But this energy factor from the normalisation will cancel that from the nonlinearity in the two $F \propto I$ P-LV cases obtained in the previous paragraph, thus making the modified dispersion relation energy independent. Therefore only the $F \propto \gamma^\mu$ cases are relevant. We label these as

$$F_5 = A\mu\gamma^\mu(\bar{\psi}\psi)^\alpha$$

$$F_6 = A\mu\gamma^\mu\left[iB\nu\left(\bar{\psi}\partial^\nu\psi - \left(\partial^\nu\bar{\psi}\right)\psi\right)\right]^\alpha$$

(44)

(45)

Note that while $F_6$ is just the numerator of $F_4$, $F_5$ is not the numerator of $F_3$ as now we have a simpler way to generate a P-LV $n = \alpha, d = 0$ nonlinearity. The corresponding $X$’s are

$$X_5 = \frac{2\hbar c}{\lambda_5} (\Delta m^2 c^4)^{(\alpha-2)/2} \frac{\Delta f}{E^{1-\alpha}}$$

$$X_6 = \frac{2^{\alpha+1}\lambda_6^{\alpha-1}}{(\hbar c)^{\alpha-1}} \frac{(\Delta m^2 c^4)^{(\alpha-2)/2} \Delta f}{E}$$

(46)

(47)

and the range of $\alpha$ values are found to be

$$0 < \alpha_5 \leq 0.9$$

$$0.5 \leq \alpha_6 \leq 0.6$$

(48)

(49)
Again the α’s in boldface are near the more likely values.

4.2 NP-LI

If the nonlinearity is Lorentz invariant, the modified dispersion relations remain covariant \( E^2 = p^2 + M^2 \) but with an effective mass \( M \) that depends on the nonlinearity parameters [4]. If we take these nonlinearity parameters to be non-universal, meaning that the different neutrinos get different effective mass corrections, then the oscillation probabilities are modified. However as the NP type of nonlinearities are invariant to the normalisation of \( \psi \), the modification is energy independent and thus not relevant for high-energy tests.

4.3 P-LI

Choosing the same energy dependent normalisation as in the P-LV cases discussed above, and noting the discussion in the previous subsection, one sees that in P-LI cases such as \((\bar{\psi}\psi)^\alpha\) the \( X \) decreases with increasing energy and thus becomes irrelevant at high energies. (We have looked at a few simple cases of \( F \) in the class P-LI and they show a similar trend).

5 Discussion

The way we have parametrized our corrections, the modification to the conventional neutrino oscillation probabilities may be described in terms of an effective energy dependent mass-squared difference,

\[
\Delta m^2_{\text{eff}}(E) = (1 - X)\Delta m^2
\]

(50)

Hopefully such an effect may be detectable with higher statistics and energies in future experiments. Note that our \( X \)'s are positive because we took the constant in front of the \( F \)'s in (1) to be positive and absorbed it into the nonlinearity parameter. More generally then, the right-hand-side of (50) should read \((1 \pm X)\Delta m^2\) so that the effective mass might increase or decrease with energy.

Among the various types of nonlinearities we have studied, we find that only six have the potential to be detected in future higher energy experiments through their increasing energy dependent effect on the neutrino oscillation probabilities. In the two \( F \propto I \) cases, \( F_1, F_2 \), each of the discrete symmetries is preserved while in the remaining \( F \propto \gamma^\mu \) cases the nonlinearities are \( C \) and \( CPT \) odd. Thus the discrete symmetries might be one way of partly discriminating among the possibilities. The mildest energy dependence is seen for \( F_5, F_6 \) but note that in those cases the nonlinearity is dependent on the energy dependent normalisation.
Since we worked in the plane wave approximation, the above-mentioned effects only probe modified dispersion relations rather than the nonlinearity itself. However there are various ways of distinguishing our results from other proposals in the literature. Firstly, we found that in modeling the nonlinearity by a single term $F \sim (\ast)^{\alpha}$ in the evolution equation, $\alpha$ turns out to have generically noninteger values so that

$$\frac{\Delta m^2(E) - \Delta m^2(0)}{\Delta m^2(0)} \propto \pm E^{\beta}$$

for some positive and typically fractional $\beta$. While even this can be obtained simply from a modified dispersion relation, [14, 15], independent of a quantum nonlinearity, the availability of a quantum evolution equation in our approach will enable a more refined analysis of the phenomena when sufficient data become available. We also emphasize that our nonlinearity simultaneously violates Lorentz invariance and that is an additional distinguishing feature.

Further work in this direction would involve going beyond the plane wave approximation, leading to genuine nonlinear effects and perhaps leading to an understanding of the mixing angles too [7]. Also, a subleading directional dependence of the oscillation probability can be examined by choosing space-like background fields. Finally, one should explore if current oscillation data can be fit using purely energy dependent effective neutrino masses as suggested, for example, in [7].

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6 Figures

Figure 1: This is a plot of $\lambda$ vs $\alpha$ for $X_1$. The vertical axis, plotted in log scale, has units of metre while the horizontal axis is dimensionless. The solid and the dashed lines represent $X = 10^{-1}$ and $X = 10^{-4}$ respectively. The horizontal lines are the bounds $10^{-1}\lambda_c$ and $10^{-4}\lambda_c$.

Figure 2: This is a plot of $\lambda$ vs $\alpha$ for $X_2$. The vertical axis, plotted in log scale, has units of metre while the horizontal axis is dimensionless. The full and dashed lines represent $X = 10^{-1}$ and $X = 10^{-4}$ respectively. The upper two horizontal lines are the bounds $10^{-1}\lambda_c$ and $10^{-4}\lambda_c$ while the lower two horizontal lines are the bounds $10\lambda_p$ and $10^4\lambda_p$.

Figure 3: This is the log-log plot of $X_1$ vs energy. The full and dashed lines have $\alpha$ values of 1.6 and 2.1 respectively. Here we have assumed $\lambda_1 = 0.1\lambda_c$. Note that $X_3(E)$ has an identical plot to $X_1(E)$ after the following redefinition of $\alpha$: The full and dashed lines have $\alpha$ values of 0.6 and 1.1 respectively. Again we assume $\lambda_3 = 0.1\lambda_c$.

Figure 4: This is the log-log plot of $X_2$ vs energy. The full and dashed lines have $\alpha$ values of 1.8 and 2.4 respectively, where we have assumed $\lambda_2 = 10^{-4}\lambda_c$. The dot line has $\alpha = 0.2$ where we have assumed $\lambda_2 = 10\lambda_p$. Note that for $\alpha = 0.4$, the energy required for $X_2$ to be significant (say $X = 0.2$) is much higher than the rest ($\sim 10^{11} GeV$). Thus it is not plotted.

Figure 5: This is the log-log plot of $X_4$ vs energy. The full and dashed lines have $\alpha$ values of 0.9 and 1.1 respectively. Here we have assumed $\lambda_4 = 10^4\lambda_p$. Note that if we choose $\lambda_4 = 10^{-4}\lambda_c$, the results will still be of a similar order of magnitude: The reason is that the exponent for $\lambda_4$ is close to zero.
Figure 1:

Figure 2:
Figure 5: