PERIODICITY OF THE SPECTRUM OF A FINITE UNION OF INTERVALS

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Abstract. A set \( \Omega \), of Lebesgue measure 1, in the real line is called spectral if there is a set \( \Lambda \) of real numbers such that the exponential functions \( e_\lambda(x) = \exp(2\pi i \lambda x) \) form a complete orthonormal system on \( L^2(\Omega) \). Such a set \( \Lambda \) is called a spectrum of \( \Omega \). In this note we present a simplified proof of the fact that any spectrum \( \Lambda \) of a set \( \Omega \) which is finite union of intervals must be periodic. The original proof is due to Bose and Madan.

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1. Introduction and Statement of the Result

Let \( \Omega \subseteq \mathbb{R}^d \) be a bounded measurable set of Lebesgue measure 1. A set \( \Lambda \subseteq \mathbb{R}^d \) is called a spectrum of \( \Omega \) (and \( \Omega \) is said to be a spectral set) if the set of exponentials

\[
E(\Lambda) = \left\{ e_\lambda(x) = e^{2\pi i \lambda x} : \lambda \in \Lambda \right\}
\]

is a complete orthonormal set in \( L^2(\Omega) \). (The inner product in \( L^2(\Omega) \) is \( \langle f, g \rangle = \int_\Omega f \overline{g} \).)

It is easy to see (see, for instance, [5]) that the orthogonality of \( E(\Lambda) \) is equivalent to the packing condition

\[
\sum_{\lambda \in \Lambda} |\widehat{\chi_\Omega}|^2(x - \lambda) \leq 1, \quad \text{a.e. } (x),
\]

as well as to the condition

\[
\Lambda - \Lambda \subseteq \{0\} \cup \{\widehat{\chi_\Omega} = 0\}.
\]

The completeness of \( E(\Lambda) \) is in turn equivalent to the tiling condition

\[
\sum_{\lambda \in \Lambda} |\widehat{\chi_\Omega}|^2(x - \lambda) = 1, \quad \text{a.e. } (x).
\]

These equivalent conditions follow from the identity

\[
\langle e_\lambda, e_\mu \rangle = \int_\Omega e_\lambda \overline{e_\mu} = \widehat{\chi_\Omega}(\lambda - \mu)
\]

and from the completeness of all the exponentials in \( L^2(\Omega) \).

Example: If \( Q_d = (-1/2, 1/2)^d \) is the cube of unit volume in \( \mathbb{R}^d \) then \( \mathbb{Z}^d \) is a spectrum of \( Q_d \).

In the one dimensional case, which will concern us in this paper, condition (2) implies that the set \( \Lambda \) has gaps bounded below by a positive number, the smallest zero of \( \widehat{\chi_\Omega} \).

Research on spectral sets has been driven for many years by a conjecture of Fuglede [4] which stated that a set \( \Omega \) is spectral if and only if it is a translational tile. A set \( \Omega \) is a translational tile if we can translate copies of \( \Omega \) around and fill space without overlaps. More precisely there exists a set \( S \subseteq \mathbb{R}^d \) such that

\[
\sum_{s \in S} \chi_\Omega(x - s) = 1, \quad \text{a.e. } (x).
\]

This conjecture is now known to be false in both directions if \( d \geq 3 \) and both directions are still open in dimensions \( d = 1, 2 \).

In this paper we present a new proof of the periodicity of the spectrum, which is a considerable simplification of that in [1].

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Theorem 1 (Bose and Madan [1]). If $\Omega = \bigcup_{j=1}^{n}(a_j, b_j) \subseteq \mathbb{R}$ is a finite union of intervals of total length 1 and $\Lambda \subseteq \mathbb{R}$ is a spectrum of $\Omega$ then there exists a positive integer $T$ such that $\Lambda + T = \Lambda$.

This is the spectral analogue of a result [2, 3] which states that all translational tilings by a bounded measurable set (or by a compactly supported function) are necessarily periodic. The proof of Theorem 1 is given in the next section.

2. Proof of the periodicity of the spectrum

Let us observe first, as in [1], that the spectrum $\Lambda = \{\lambda_j : j \in \mathbb{Z}\}$, $\lambda_j < \lambda_{j+1}$, of any bounded set $\Omega \subseteq \mathbb{R}$ has “finite complexity”, in the sense that all gaps $\lambda_{j+1} - \lambda_j$ are drawn from the discrete set $(\widehat{\chi}_\Omega$ is analytic as $\chi_\Omega$ has bounded support) $\{\widehat{\chi}_\Omega = 0\}$. This implies that if we consider all intesections of $\Lambda$ with a sliding window of width $h$ $[\lambda, \lambda + h] \cap \Lambda$, (where $\lambda \in \Lambda$), then we only see finitely many different sets.

If $\Omega = \bigcup_{j=1}^{n}(a_j, b_j) \subseteq \mathbb{R}$ it follows by a simple calculation that

$$
\widehat{\chi}_\Omega(x) = \frac{1}{2\pi i} \sum_{j=1}^{n} \left( e^{-2\pi i a_j x} - e^{-2\pi i b_j x} \right).
$$

The important ingredient of the approach in [1] that we keep in our approach is the view of the spectrum as a linear space via the map $\phi = \phi_\Omega : \mathbb{R} \to \mathbb{C}^{2n}$ given by

$$
x \to \left( e^{-2\pi i a_1 x}, \ldots, e^{-2\pi i a_n x}, e^{-2\pi i b_1 x}, \ldots, e^{-2\pi i b_n x} \right).
$$

Define the bilinear form $A$ on $\mathbb{C}^{2n}$ by (writing $z = (z_1, z_2)$, $z_1, z_2 \in \mathbb{C}^n$)

$$
A(z, w) = \langle z_1, w_1 \rangle - \langle z_2, w_2 \rangle,
$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{C}^n$. Using (6) we see that if $\lambda \neq \mu$ then $e_\lambda \perp e_\mu$ if and only if $A(\phi(\lambda), \phi(\mu)) = 0$.

Write

$$
V(\Lambda) = \text{span} \phi(\Lambda)
$$

for the subspace of $\mathbb{C}^{2n}$ generated by the set $\phi(\Lambda) = \{\phi(\lambda) : \lambda \in \Lambda\}$.

Suppose now that $B = \{b_1, \ldots, b_m\} \subseteq \Lambda$ is a generating set, i.e., that $V(\Lambda) = \text{span} \phi(B)$. It follows that $x \in \Lambda$ if and only if $A(\phi(x), \phi(b_j)) = 0$ for $j = 1, 2, \ldots, m$. Indeed, if the latter condition is true it follows by linearity that $A(\phi(x), \phi(\mu)) = 0$ for all $\mu \in \Lambda$ and hence that $e_x \perp e_\mu$, $\mu \in \Lambda \setminus \{x\}$. This implies that $x \in \Lambda$, otherwise $E(\Lambda)$ would not be a complete set of exponentials for $L^2(\Omega)$. As remarked in [1] this means that $\Lambda$ is determined by any such generating set $B$.

Lemma 1. Let $\Omega$ be a finite union of intervals. If $A \subseteq \mathbb{R}$ is a set of positive minimum gap $\delta$ then for $R > 0$ we have

$$
\sum_{\substack{\lambda \in \Lambda \cap A \setminus \{a\} \text{ for } |a| > R}} |\widehat{\chi}_\Omega|^2(a) \leq C/R,
$$

for some constant $C > 0$ that may depend on $\Omega$ and $\delta$ only.

Proof. This is immediate from the fact that $|\widehat{\chi}_\Omega|^2(y) \leq C/|y|^2$ (see [1]). \hfill \Box

Lemma 2. There is a finite $T > 0$ such that for all $x \in \mathbb{R}$ the set $\Lambda \cap (x, x + T)$ is a generating set.

Proof. Suppose not, so that there is a sequence $m_k \in \Lambda$, $k = 1, 2, \ldots$, such that

$$
\dim \text{span} (\phi(\Lambda \cap (m_k - k, m_k + k))) < \dim \text{span} \phi(\Lambda).
$$

Consider the sequence of finite sets

$$
M_k = [\Lambda \cap (m_k - k, m_k + k)] - m_k,
$$

i.e., the sets $\Lambda \cap (m_k - k, m_k + k)$ translated so that they are centered at 0 (therefore they all contain 0). Observe that in any given interval $(-t, t)$ the sets $M_k$ may only take finitely many forms.

For $n = 1, 2, 3, \ldots$ in turn we look at the infinite sequence

$$
M_k \cap (-n, n), \quad k = 1, 2, \ldots.
$$
There is an infinite sequence of $k$’s such that all sets $M_k \cap (-n, n)$ are the same. Keep only these indices and define $L_n$ to be this common set. In this way we define an increasing infinite sequence of sets $L_n$, $L_n \subseteq L_{n+1}$, each of which contains 0 and is of the form

$$L_n = \Lambda \cap (c_n - n, c_n + n) - c_n,$$

for some $c_n \in \Lambda$.

Let $L = \bigcup_{n=1}^{\infty} L_n$. Since each finite part of $L$ is a translate of a part of $\Lambda$ it follows that the elements of $E(L)$ are orthogonal. We now show that $E(L)$ is also complete and is thus also a spectrum of $\Omega$.

For this it suffices to show that $F(x) := \sum_{\ell \in L} |\hat{\chi}_{\Omega}\ell|^2(x - \ell) = 1$ for almost every $x \in \mathbb{R}$. Assume for simplicity that $x \geq 0$. We have for $t > 2x$

$$1 \geq F(x) \quad \text{(from } (11), \text{ since } E(L) \text{ is an orthogonal set)}$$

$$\geq \sum_{\ell \in (-t,t) \cap L} |\hat{\chi}_{\Omega}|^2(x - \ell)$$

$$= \sum_{\ell \in (-t,t) \cap L_n} |\hat{\chi}_{\Omega}|^2(x - \ell) \quad \text{(for some } n = n(t) > t)$$

$$= \sum_{\ell \in \Lambda - c_n, |\ell| < t} |\hat{\chi}_{\Omega}|^2(x - \ell)$$

$$= 1 - \sum_{\ell \in \Lambda - c_n, |\ell| \geq t} |\hat{\chi}_{\Omega}|^2(x - \ell) \quad \text{(by } (3), \text{ since } \Lambda \text{ is a spectrum)}$$

$$\geq 1 - \sum_{\ell \in \Lambda - c_n, |x - \ell| \geq t/2} |\hat{\chi}_{\Omega}|^2(x - \ell) \quad \text{(as } |\ell| \geq t > 2x \text{ implies } |x - \ell| \geq t/2)$$

$$= 1 - \sum_{a \in x - \Lambda + c_n, |a| \geq t/2} |\hat{\chi}_{\Omega}|^2(a) \quad \text{(with } a = x - \ell)$$

$$\geq 1 - \frac{C}{t} \quad \text{(from Lemma } 1 \text{ applied to the set } x - \Lambda + c_n).$$

Letting $t \to \infty$ we obtain that $F(x) = 1$ for all $x \in \mathbb{R}$. (Notice that the constant $C$ that appears above does not depend on $n$.)

Since every finite subset of $L$ is contained in some $L_n$ it follows that

$$\dim \text{ span } \phi(L) < \dim \text{ span } \phi(\Lambda). \quad (7)$$

To derive a contradiction let the finite set $\Lambda' \subseteq \Lambda$ be such that $\phi(\Lambda')$ is a basis of $\text{span } \phi(\Lambda)$ and also let the finite set $L' \subseteq L$ be such that $\phi(L')$ is a basis of $\text{span } \phi(L)$. Some translate $s + L'$ of the finite set $L'$ is contained in $\Lambda$, hence

$$A(\phi(s + \ell'), \phi(\lambda')) = 0, \quad \text{(for all } \ell' \in L' \text{ and } \lambda' \in \Lambda'),$$

which implies

$$A(\phi(\ell'), \phi(-s + \lambda')) = 0, \quad \text{(for all } \ell' \in L' \text{ and } \lambda' \in \Lambda'),$$

and this means that $-s + \Lambda' \subseteq L$ and therefore that

$$\dim \text{ span } \phi(L) \geq \dim \text{ span } \phi(-s + \Lambda') = \dim \text{ span } \phi(\Lambda') = \dim \text{ span } \phi(\Lambda),$$

in contradiction with $(7)$. We have used the easy fact that $\dim \text{ span } \phi(A + x) = \dim \text{ span } \phi(A)$ for any $x \in \mathbb{R}, A \subseteq \mathbb{R}$. \hfill \Box

**Completion of the proof:** The set $\Lambda$ is periodic.

Let $T$ be as in Lemma 2 and consider all subsets of $\Lambda$ of the form

$$B_\lambda = \Lambda \cap [\lambda, \lambda + T], \quad \lambda \in \Lambda.$$  

It follows from Lemma 2 that $B_\lambda$ is a generating set for each $\lambda$. But there are only finitely many different forms the set $B_\lambda - \lambda$ can take, hence there are $\lambda_1, \lambda_2 \in \Lambda$, $\lambda_1 > \lambda_2$, such that

$$B_{\lambda_1} - \lambda_1 = B_{\lambda_2} - \lambda_2,$$

or

$$B_{\lambda_1} = B_{\lambda_2} + \lambda_1 - \lambda_2.$$
Since $B_{\lambda_1}$ and $B_{\lambda_2}$ are both generating sets for $\phi(\Lambda)$ it follows that
\[ x \in \Lambda \iff e_x \perp e_y \ (y \in B_{\lambda_2}) \]
\[ \iff e_{x+\lambda_1-\lambda_2} \perp e_y \ (y \in B_{\lambda_1}) \]
\[ \iff x + (\lambda_1 - \lambda_2) \in \Lambda. \]

In other words, $T = \lambda_1 - \lambda_2$ is a period of $\Lambda$.

Let us also remark that any period of $\Lambda$ must be an integer. This is a consequence of the fact that $\Lambda$ has density 1: if $T$ is a period of $\Lambda$ this implies that there are exactly $T$ elements of $\Lambda$ in each interval $[x, x+T)$ hence $T$ is an integer.

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