Cosmology of a covariant Galileon field

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We study the cosmology of a covariant scalar field respecting a Galilean symmetry in flat space-time. We show the existence of a tracker solution that finally approaches a de Sitter fixed point responsible for cosmic acceleration today. The viable region of model parameters is clarified by deriving conditions under which ghosts and Laplacian instabilities of scalar and tensor perturbations are absent. The field equation of state exhibits a peculiar phantom-like behavior along the tracker, which allows a possibility to observationally distinguish the Galileon gravity from the ΛCDM model.

The problem of dark energy responsible for cosmic acceleration today has motivated the idea that the gravitational law may be modified from General Relativity (GR) on large scales (see Refs. [1] for reviews). On the other hand, one needs to recover Newton gravity at short distances for the compatibility with solar-system experiments. Besides the chameleon mechanism [2] based on the density-dependent matter coupling with a scalar field (used also in f(R) theories [3,4]), there is another way to recover GR at short distances: the Vainshtein mechanism [3] based on non-linear field self-interactions such as $\Box \phi (\Box \phi)^2$, where $(\Box \phi)^2 = \partial^\mu \phi \partial^\mu \phi$. This non-linear effect has been employed for the brane-bending mode of the self-accelerating branch in the Dvali-Gabadadze-Porrati (DGP) braneworld [4], but the DGP model is unfortunately plagued by a ghost problem [5].

In order to avoid the appearance of ghosts, it is important to keep the field equations up to second-order in time-derivatives. A scalar field $\phi$ called “Galileon” [5], whose action is invariant under the Galilean symmetry $\partial_\mu \phi \rightarrow \partial_\mu \phi + b_\mu$ in flat space-time, allows five field Lagrangians that give rise to derivatives up to second-order (see Refs. [6,7] for related works). If the analysis in [6] is extended to the curved space-time, one needs to introduce couplings between the field $\phi$ and the curvature tensors for constructing the Lagrangians free from higher-order derivatives in the equations of motion [6].

The five covariant Lagrangians that respect the Galilean symmetry in flat space-time are given by

$$\begin{align*}
L_1 &= M^3 \phi, \\
L_2 &= (\Box \phi)^2, \\
L_3 &= (\Box \phi)(\Box \phi)^2/M^3, \\
L_4 &= (\Box \phi)^2 [2(\Box \phi)^2 - 2\phi_{\mu\nu}\phi_{\mu\nu} - R(\Box \phi)^2/2] /M^6, \\
L_5 &= (\Box \phi)^2((\Box \phi)^3 - 3(\Box \phi)\phi_{\mu\nu}\phi_{\mu\nu} + 6\phi_{\mu\nu}\phi_{\nu\rho}\phi_{\rho\mu} - 6\phi_{\mu\nu}\phi_{\nu\rho}\phi_{\rho\mu}G_{\nu\rho})/M^9,
\end{align*}$$

where a semicolon represents a covariant derivative, $M$ is a constant having a dimension of mass, and $G_{\nu\rho}$ is the Einstein tensor. In this Letter we study the cosmology based on the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{pl}^2}{2} R + \sum_{i=1}^{5} c_i L_i \right] + \int d^4x L_M, \tag{2}$$

where $M_{pl}$ is the reduced Planck mass and $c_i$ are constants. For the matter Lagrangian $L_M$ we take into account perfect fluids of radiation (density $\rho_r$) and non-relativistic matter (density $\rho_m$). Although the cosmological dynamics up to $L_4$ were discussed in Ref. [11], we will show that inclusion of $L_5$ is crucially important to determine the full Galileon dynamics. Moreover the viable parameter space will be clarified for such full theory.

In the flat Friedmann-Lemaître-Robertson-Walker (FLRW) Universe with a scale factor $a(t)$, the variation of the action (2) leads to the following equations of motion

$$\begin{align*}
3M_{pl}^2 H^2 &= \rho_{DE} + \rho_m + \rho_r, \tag{3} \\
3M_{pl}^2 H^2 + 2M_{pl}^2 \dot{H} &= -P_{DE} - \rho_r/3, \tag{4}
\end{align*}$$

where $H = \dot{a}/a$ is the Hubble parameter (a dot represents a derivative with respect to cosmic time $t$), and

$$\begin{align*}
\rho_{DE} &= -c_1 M^3 \dot{\phi}/2 - c_2 \dot{\phi}^2/2 + 3c_3 H \ddot{\phi}/M^3 \\
&\quad -45c_4 H^2 \dddot{\phi}/(2M^6) + 21c_5 H^3 \dddot{\phi}/M^9, \tag{5} \\
P_{DE} &= c_1 M^3 \ddot{\phi}/2 - c_2 \ddot{\phi}^2/2 - 3c_3 \dddot{\phi}/M^3 \\
&\quad + 3c_4 \dddot{\phi}[8H \dddot{\phi} + (3H^2 + 2\dot{H})\ddot{\phi}]/(2M^6) \\
&\quad - 3c_5 H \dddot{\phi}[5H \dddot{\phi} + 2(H^2 + \dot{H})\ddot{\phi}]/M^9. \tag{6}
\end{align*}$$

The matter fluids obey the continuity equations $\dot{\rho}_m + 3H \rho_m = 0$ and $\dot{\rho}_r + 4H \rho_r = 0$. From Eqs. (3) and (4) the dark component also satisfies $P_{DE} + 3H (\rho_{DE} + P_{DE}) = 0$. We define the dark energy equation of state $w_{DE}$ and the effective equation of state $w_{eff}$ as $w_{DE} = P_{DE}/\rho_{DE}$ and $w_{eff} = -1 - 2\dot{H}/(3H^2)$. The latter is known by the background expansion history of the Universe.

Since we are interested in the case where the late-time cosmic acceleration is driven by field kinetic terms without a potential, we set $c_1 = 0$ in the following discussion. When $c_1 = 0$ and $c_2 \neq 0$ the only solution to Eqs. (3) and (4) in the Minkowski background ($H = 0$) without matter corresponds to $\ddot{\phi} = 0$. We introduce the following quantities useful to describe the cosmological dynamics

$$r_1 = \phi_{4S}H_{4S}/(\dot{\phi}H), \quad r_2 = (\dddot{\phi}/\phi_{4S})^2/r_1, \tag{7}$$

where $\phi_{4S}$ and $H_{4S} \approx 10^{-60} M_{pl}$ are the field velocity and the Hubble parameter at the de Sitter (dS) solution, respectively. The mass $M$ is related to $H_{4S}$ via $M^3 = M_{pl}H_{4S}^2$. At the dS point one has $r_1 = 1$ and $r_2 = 1$. 

\* \* \*
Equation (8) can be written as $\Omega_m + \Omega_r + \Omega_{DE} = 1$, where $\Omega_m = \rho_m / (3M^2 \rho^2 H^2)$, $\Omega_r = \rho_r / (3M^2 \rho^2 H^2)$, and

$$\Omega_{DE} = \frac{1}{6} c_2 x_{ds}^{-1} r_2^3 + c_3 x_{ds}^{-1} r_2^3 - \frac{15}{2} c_4 x_{ds} r_2^3 + 7 c_5 x_{ds}^3 r_2^2,$$

where $x_{ds} = \phi_{ds} / (H_{ds} M_p^4)$. Since $\Omega_{DE} = 1$ at the dS point, Eq. (8) gives a relation between the terms $c_2 x_{ds}^2$, $c_3 x_{ds}^2$, $\alpha = \frac{c_2 x_{ds}^2}{\Omega_r}$, $\beta = \frac{c_2 x_{ds}^3}{\Omega_r}$ $\phi_{ds}$ and $x_{ds}$, where $\gamma$ is a real number. Therefore rescaled choices of $c_i$ will lead to the same dynamics (as they have the same $\alpha$ and $\beta$ for both the background and the linear perturbation, which implies that redefining the coefficients $c_i$ in terms of $\alpha$ and $\beta$ is convenient.

The autonomous equations for the variables $r_1, r_2, \Omega_r$ follow from Eqs. (8), (9), and fluid equations. One can show that there is an equilibrium point characterized by

$$r_1 = 1, \quad \text{i.e. } \dot{\phi}H = \text{constant},$$

at which the variables $r_2$ and $\Omega_r$ satisfy

$$r_2' = \frac{2 r_2 (3 - 3r_2 + \Omega_r)}{1 + r_2}, \quad \Omega_r' = \frac{\Omega_r (\Omega_r - 1 - 7r_2)}{1 + r_2},$$

where a prime represents a derivative with respect to $N = \ln a$. This result is interesting because it shows the universality of the equations of motion without any dependence on $\alpha$ and $\beta$. Along the solution (11), the field velocity evolves as $\dot{\phi} \propto t$ during radiation and matter eras ($H \propto 1/t$). There is also a simple relation $\Omega_{DE} = r_2$ along the solution $r_1 = 1$.

We have three fixed points: (a) $(r_1, r_2, \Omega_r) = (1, 0, 0)$ [radiation], (b) $(r_1, r_2, \Omega_r) = (1, 0, 0)$ [matter], (c) $(r_1, r_2, \Omega_r) = (1, 0, 1)$ [dS]. The stability of these points can be analyzed by considering linear perturbations $\delta r_1, \delta r_2, \delta \Omega_r$ about them. The perturbation $\delta r_1$ satisfies

$$\delta r_1' = -\frac{9 + \Omega_r + 3r_2}{2} \delta r_1,$$

which shows that, in the regime $0 \leq r_2 \leq 1$ and $\Omega_r \geq 0$, the solution is stable in the direction of $r_1$. Since the dS point is stable in the other two directions, the solutions finally approach it. The points (a) and (b) are saddle because they are unstable in the direction $r_1$.

Along the solution (11), we have $\rho_{DE} = 3M^6 / H^2$, $P_{DE} = -3M^6 (2 + w_{eff}) / H^2$, and

$$w_{DE} = -2 - w_{eff} = -\frac{\Omega_r + 6}{3 (r_2 + 1)}, \quad w_{eff} = \frac{\Omega_r - 6r_2}{3 (r_2 + 1)}.$$

From the radiation era to the dS epoch the effective equation of state evolves as $w_{eff} = 1 / 3 \to 0 \to -1$, whereas the dark energy equation of state exhibits a peculiar evolution: $w_{DE} = -7 / 3 \to -2 \to -1$.

The evolution of dark energy is different depending on the initial conditions of $(r_1, r_2, \Omega_r)$. If they are chosen to be close to the fixed point (a) at the onset of the radiation era, then the solutions follow the sequence (a) $\to$ (b) $\to$ (c). If $r_2 \ll 1$ initially, the dominant contribution to $\Omega_{DE}$ comes from the term $\mathcal{L}_3$, i.e. $\Omega_{DE} \simeq 7 \beta r_2$. In this case the solutions approach $r_1 = 1$ at late times with the increase of $r_1$. For the initial conditions with $r_1 \gg 1$ the term $\mathcal{L}_2$ gives the dominant contribution to $\Omega_{DE}$, but this case is not viable because the field kinetic energy decreases rapidly as in quintessence without a potential. Numerical simulations show that if $r_2 \simeq 2$ initially the solutions approach $r_1 = 1$, but in the opposite case the Universe finally reaches the matter-dominated epoch.

Let us find the allowed parameter space in terms of $(\alpha, \beta)$ by deriving the conditions for the avoidance of ghosts and instabilities of scalar and tensor perturbations. Using the Faddeev-Jackiw method [12], the action (2) can be expanded at second-order in perturbations. Following the similar procedure to that given in Refs. [13], the no-ghost condition for the scalar sector of the action (2) is given by

$$Q_S \equiv -s / (1 + \mu_3^2) > 0,$$

where $s = 6 (1 + \mu_1) (1 + \mu_2 + 1 \mu_3 - \mu_3^2)$, and

$$\mu_1 = 3 \alpha_1 r_2 / 2 - 3 \beta r_2,$$

$$\mu_2 = (3 \alpha - 4 \beta + 2 r_2^7 r_2') / 2 - 2 (9 \alpha - 9 \beta + 2) r_2^3 r_2' + 45 \alpha_1 r_2 / 2 - 28 \beta r_2,$$

$$\mu_3 = -(9 \alpha - 9 \beta + 2) r_2^7 r_2' / 2 + 15 \alpha_1 r_2 / 2 - 21 \beta r_2 / 2.$$ (17)

The condition for the avoidance of Laplacian instabilities associated with the scalar field propagation speed is

$$c_S^2 = \{(1 + \mu_1)^2 (2 \mu_3 - (1 + \mu_3)) (5 + 3 w_{eff}) + 3 \Omega_m + 4 \Omega_r\}
-4 \mu_1 (1 + \mu_1) (1 + \mu_2) + 2 (1 + \mu_3)^2 (1 + \mu_4) / s > 0,$$

where

$$\mu_4 = -\alpha_1 r_2 / 2 + 3 \beta r_2 (3 + 3 w_{eff} - 3 r_2') / (r_1 - r_2') / 2.$$ (19)

Similar calculations for the tensor perturbation lead to

$$Q_T \equiv 3 \alpha_1 r_2 / 2 - 3 \beta r_2 / 2 + 1 / 2 > 0,$$

$$c_T^2 = \frac{2 r_1 (2 - \alpha_1 r_2) - 3 \beta (r_2^2 + \alpha_2^2)}{2 r_1 (2 + 3 \alpha_1 r_2 - 3 \beta r_2)} > 0.$$ (21)

We consider the following three different regimes.

- (i) $r_1 \ll 1$, $r_2 \ll 1$

This characterizes the early cosmological epoch in which the term $\mathcal{L}_5$ dominates the dynamics of the field. For the scalar modes we have $Q_S \simeq 60 \beta r_2^2$ and $c_S^2 \simeq (1 + \Omega_r) / 40$. The sign change of $r_2$ implies the appearance of ghosts. For the initial conditions
with $r_2 > 0$, it is required that $\beta > 0$. The Laplacian instabilities of the scalar modes can be avoided because $c_S^2 \simeq 1/20$ and $c_T^2 \simeq 1/10$ during radiation and matter eras, respectively. Since $Q_T \simeq 1/2$ and $c_T^2 \simeq 1 + 3\beta r_2 (5 - 3\Omega_r)/8 \simeq 1$, the tensor modes do not provide additional constraints. We also have

$$w_{DE} \simeq - (1 + \Omega_r)/8, \quad w_{\text{eff}} \simeq \Omega_r/3,$$  \hspace{1cm} (22)

which is valid for $\Omega_r \gg \{r_1, r_2\}$.

- (ii) $r_1 = 1$, $r_2 \ll 1$

This corresponds to the equilibrium point (10) during radiation or matter domination. The conditions (13) and (15) reduce to

$$Q_S \simeq 3(2 - 3\alpha + 6\beta) r_2 > 0,$$  \hspace{1cm} (23)

$$c_S^2 \simeq \frac{8 + 10\alpha - 9\beta + \Omega_r(2 + 3\alpha - 3\beta)}{3(2 - 3\alpha + 6\beta)} > 0.$$  \hspace{1cm} (24)

For the branch $r_2 > 0$ the first condition reduces to $2 - 3\alpha + 6\beta > 0$. For the tensor modes, we have $c_T^2 \simeq 1 - r_2(4\alpha + 3\beta + 3\Omega_r)/2 \simeq 1$ and $Q_T > 0$.

- (iii) $r_1 = 1$, $r_2 = 1$

This corresponds to the de Sitter point, at which the conditions (13), (20), (15), and (24) are given by

$$Q_S = \frac{4 - 9(\alpha - 2\beta)^2}{3(\alpha - 2\beta)^2} > 0,$$  \hspace{1cm} (25)

$$Q_T = \frac{(2 + 3\alpha - 6\beta)/4}{4} > 0,$$  \hspace{1cm} (26)

$$c_S^2 = \frac{(\alpha - 2\beta)(4 + 15\alpha^2 - 48\alpha\beta + 36\beta^2)}{2[4 - 9(\alpha - 2\beta)^2]} > 0,$$  \hspace{1cm} (27)

$$c_T^2 = \frac{2 - \alpha}{2 + 3\alpha - 6\beta} > 0.$$  \hspace{1cm} (28)

If $\beta > 0$, $c_T^2$ can have a minimum during the transition from the regime $r_2 \ll 1$ to $r_2 \simeq 1$. This value tends to decrease as $\beta$ approaches 1. Imposing that $c_T^2 > 0$ at the minimum, we obtain $\alpha < 12\sqrt{3} - 9\beta - 2$. In Fig. 1 we illustrate the region in which this condition as well as the conditions (23–24) are satisfied for $r_2 > 0$. Numerical simulations confirm that for the parameters inside the shaded region in Fig. 1 the no-ghost and stability conditions are not violated even in the intermediate cosmological epoch.

In Fig. 2 we plot the variation of $w_{DE}$ and $w_{\text{eff}}$ versus the redshift $z$ for several different model parameters and initial conditions. In the case (A) the initial condition is chosen to be $r_1 = 1$, so that $w_{DE}$ and $w_{\text{eff}}$ evolve according to Eq. (13) with the variation of $r_2$ and $\Omega_r$. While the evolution of $w_{\text{eff}}$ is similar to that in the ΛCDM model, the dark energy equation of state evolves from the regime $w_{DE} < -1$ to the de Sitter attractor with $w_{DE} = -1$. The cases (B) and (B') in Fig. 2 correspond to the initial conditions in the regime (i). As estimated by Eq. (22), $w_{DE}$ starts to evolve from $-1/4$ and reaches the value $-1/8$ during the matter era. The evolution of $w_{DE}$ is different depending on the epoch at which $r_1$ grows to the order of 1. In the case (B) the solutions reach the regime $r_1 \sim 1$ only recently, whereas in the case (B') the approach of this regime occurs much earlier. The equilibrium point (10) can be regarded as a tracker that attracts solutions with different initial conditions to a common trajectory. Before approaching the tracker, the solutions cross the boundary $w_{DE} = -1$ without any unstable behavior of perturbations. Note that there are no significant differences for the variation of $w_{\text{eff}}$ between the cases (A) and (B) [also (B')].

In Fig. 3 we show the evolution of $c_S^2$ and $c_T^2$ for the
same model parameters and initial conditions as in Fig. 4. In the case (A) the scalar propagation speed remains sub-luminal, as estimated by Eqs. (24) and (27). For \( \alpha = -1.4 \) and \( \beta = -0.8 \), Eq. (28) shows that at the dS point the tensor mode becomes super-luminal. However, both the scalar and tensor modes can be sub-luminal at the dS point, as in the case (B) of Fig. 3 (\( \alpha = 0.1, \beta = 0.049 \)).

For the initial conditions starting from the regime (i) we require \( \beta > 0 \) to avoid ghosts. Under the conditions (24), (26), and (27) with \( \beta > 0 \), one can show that \( c_S^2 \) in Eq. (24) grows to the order of 1. If the solutions approach the tracker in the regime (ii) long before the dS epoch, there is a period in which \( c_T^2 \) exceeds 1. This super-luminal propagation can be avoided if \( r_1 \) grows to the order of unity only recently. The case (B) in Fig. 3 corresponds to such an example for which \( c_T^2 \) has a peak smaller than 1 after the matter era. In this case the tensor mode is slightly super-luminal in the regime (i). In general there is a period in which the propagation speed of either scalar or tensor modes exceeds 1. However, this does not necessarily imply the inconsistency of theory because of the possibility for the absence of closed causal curves [11].

In summary we have studied the cosmology for the full Galileon action (2) and derived all conditions for the consistency of such theory. We have shown that, under the conditions [12], there exist stable dS solutions responsible for dark energy. In spite of the complexities of Galileon Lagrangians, the conditions for the avoidance of ghosts and Laplacian instabilities constrain the allowed parameter space in terms of the variables \( \alpha \) and \( \beta \) in a simple way. While the evolution of \( w_{DE} \), \( c_S^2 \) and \( c_T^2 \) is different depending on the model parameters and the initial conditions of \( r_1 \), we have derived convenient analytic formulas to evaluate those quantities in three distinct regimes.

There are several interesting applications and generalizations of Galileon gravity. First, the study of cosmological perturbations may provide some signatures for the modification of gravity from GR. The last term of \( \mathcal{L}_4 \) in Eq. (1), for example, gives rise to a correction of the order \( \alpha r_1 r_2 \) to the bare gravitational constant. This affects the effective gravitational coupling \( G_{eff} \) that appears in the equation of matter perturbations. Also it will be possible to constrain the Galileon models from the time variation of \( G_{eff} \). Second, the study of spherically symmetric solutions in both weak and strong gravitational backgrounds can allow us to understand how the Vainshtein mechanism works in general. Third, it is possible to extend the Galileon Lagrangian (1) to the theory in which the field \( \phi \) is replaced by a general function \( f(\phi) \). We expect that such analyses will provide us deep insight on the possible modification of gravity and that it will shed new light on the nature of dark energy.

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