Complete Weight Distribution and MacWilliams Identities for Asymmetric Quantum Codes

Chuangqiang Hu, Shudi Yang and Stephen S.-T. Yau

Abstract

In 1997, Shor and Laflamme defined the weight enumerators for quantum error-correcting codes and derived a MacWilliams identity. We extend their work by introducing our double weight enumerators and complete weight enumerators. The MacWilliams identities for these enumerators can be obtained similarly. With the help of MacWilliams identities, we obtain various bounds for asymmetric quantum codes.

Index Terms

MacWilliams identities, Asymmetric Quantum codes, Quantum Singleton bound.

I. INTRODUCTION

Quantum information theory is rapidly becoming a well-established discipline. It shares many of the concepts of classical information theory but involves new subtleties arising from the nature of quantum mechanics. Among the central concepts in common between classical and quantum information is that of error correction. Quantum error-correcting codes have been initially discovered by Shor [1] and Steane [2], [3] in 1995-1996 for the purpose of protecting quantum information from noise in computation or communication [4], [5], [6], [7], [8]. This discovery of [1] has revolutionized the field of quantum information and leads to a new research line. In [10], [11], [12] noiseless quantum codes were built using group theoretic methods [13], [14]. In [15], [5] quantum error correction was used to broader analyses of the physical principles. The authors in [16], [17], [18] gave various new constructions of quantum error-correcting codes.

It is well known that if further information about the error process is available, more efficient codes can be designed. Indeed, in many physical systems, the noise is likely to be unbalanced between amplitude (X-type) errors and phase (Z-type) errors. Recently a lot of attention has been put into designing codes for this situation and into studying their fault tolerance properties [19], [20], [21]. All these results use error models described by Kraus operators [22] that generalize Pauli operators.

In the classical theory the famous MacWilliams identity gives a relationship between the weight distributions of a code C and its dual code C⊥ without knowing specifically the codewords of C⊥ or anything else about its structure [23], [24]. The same technique was adapted to the quantum case by Shor and Laflamme [25] generalizing the classical case and they derived a MacWilliams identity. Rains [26] investigated the properties of quantum enumerators. In [27], Rains extended the work of [25] to general codes by introducing quantum shadow enumerators.

Several bounds are known for classical error-correcting codes. Delsarte [28] investigated the Singleton and Hamming bounds using linear programming approach. The first linear programming bound was generalized by Aaltonen [29] to the nonbinary case. See [30], [31], [32] for more information on available bounds for non-binary codes. Recently there has been intensive activity in the area of quantum codes. In particular, Knill and Laflamme [33] introduced the notion of the minimum distance of a quantum error-correcting code and showed that the error for entangled states is bounded linearly by the error for pure states. Shor and Laflamme [25] presented a linear-programming bound for quantum error-correcting codes. Cleve [34] demonstrated connections between quantum stabilizer codes and classical codes and gave upper bounds on the best asymptotic capacity. Rains [26], [27] showed that the minimum distance of a quantum code is determined by its enumerators. Ashikhmin and Litsyn [35] attained upper asymptotic bounds on the size of quantum codes. Aly [36] established asymmetric Singleton and Hamming bounds on asymmetric quantum and subsystem code parameters. Sarvepalli et al. [37] studied asymmetric quantum codes and derived upper bounds on the code parameters using linear programming. Wang et al. [38] extended the characterization of non-additive symmetric quantum codes given in [38], [17] to the asymmetric case and obtained an asymptotic bound from algebraic geometry codes.

It should be mentioned that there is another weight enumerator for a classical code that contain more detailed information about the codewords. Namely, the complete weight enumerator, which enumerates the codewords according to the number of alphabets of each kind contained in each codeword. MacWilliams [23], [24] also proved that there is an identity between the complete weight enumerators of C and its dual code C⊥. The complete weight enumerator and weight enumerator of...
We define the X and where asymmetric quantum codes. In Section III, we establish our main result on quantum MacWilliams identities by defining double case. Using the generalized MacWilliams identities, we will find new upper bounds on the minimum distance of asymmetric quantum codes. For and the first linear-programming-type bounds for asymmetric quantum codes. In Section V, we apply the key inequality to obtain Singleton-type, Hamming-type of the Krawtchouk polynomials and we prove the key inequality that allows us to get new upper bounds of the minimum weight enumerators and complete weight enumerators of quantum codes. In Section IV, we give a short survey of properties classical codes have been studied extensively, see [39], [40], [41], [42], [43], [44], [45] and the reference therein. However, to the best of our knowledge, there is no quantum analog complete weight enumerators as in classical coding theory. Therefore the purpose of the present paper is to introduce the notions of double weight enumerator and complete weight enumerator and then generalize the MacWilliams identities about complete weight enumerators from classical coding theory to the quantum case. Using the generalized MacWilliams identities, we will find new upper bounds on the minimum distance of asymmetric quantum codes.

Here is the plan of the rest of this paper. In Section II, we introduce some basic definitions and notations on symmetric and asymmetric quantum codes. In Section III we establish our main result on quantum MacWilliams identities by defining double weight enumerators and complete weight enumerators of quantum codes. In Section IV we give a short survey of properties of the Krawtchouk polynomials and we prove the key inequality that allows us to get new upper bounds of the minimum distance of an asymmetric quantum codes. In Section V we apply the key inequality to obtain Singleton-type, Hamming-type and the first linear-programming-type bounds for asymmetric quantum codes.

II. SYMMETRIC AND ASYMMETRIC QUANTUM CODES

We begin with some basic definitions and notations. Let C be a complex number field. We regard C^2 as a Hilbert space with orthonormal basis |0⟩ and |1⟩. Denote by (C^2)^⊗n = C^{2n} the n-th tensor of C^2. This space enables us to transmit n bits of information. The coordinate basis is given by

$$|j⟩ = |j_0⟩ ⊗ |j_1⟩ ⊗ ... ⊗ |j_{n-1}⟩,$$

for each $$j_r ∈ \{0, 1\}.$$ For two quantum states |u⟩ and |v⟩ in C^{2^n} with

$$|u⟩ = \sum_j u_j |j⟩, \quad |v⟩ = \sum_j v_j |j⟩,$$

the Hermitian inner product of |u⟩ and |v⟩ is defined by

$$⟨u|v⟩ = \sum_j \bar{u}_j v_j.$$

In the process of transmission over a channel the information can be altered by errors. There are several models of channels. Perhaps the most popular one is the completely depolarized channel, among which a vector v ∈ C^2 can be altered by one of the following error operators:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The error operators action on C^2 constitute a set

$$E := \{ e = σ_0 ⊗ ... ⊗ σ_{n-1} | σ_r ∈ \{ I, σ_x, σ_y, σ_z \} \}.$$ 

For e ∈ E, the number of non-identity matrices in expression of e is called the weight of e which is denoted by w_Q(e). Similarly, we denote by N_x(e), N_y(e) and N_z(e) the number of the matrices σ_x, σ_y and σ_z occurred in the expression of e respectively. It is clear that

$$w_Q(e) = N_x(e) + N_y(e) + N_z(e).$$

It is well known that each e ∈ E is the composition of two kinds of error operators, i.e. the bit flip and the phase flip. Precisely, fix an error operator e, there exist vectors a = (a_0 ... a_{n-1}) ∈ F_2^n and b = (b_0 ... b_{n-1}) ∈ F_2^n such that

$$e = i^{a·b} X(a) Z(b), \quad (II.1)$$

where

$$X(a) = ω_0 ⊗ ... ⊗ ω_{n-1}, \quad Z(b) = ω'_0 ⊗ ... ⊗ ω'_{n-1},$$

and

$$ω_i = \begin{cases} I, & \text{if } a_i = 0, \\ σ_x, & \text{if } a_i = 1, \end{cases} \quad ω'_i = \begin{cases} I, & \text{if } b_i = 0, \\ σ_z, & \text{if } b_i = 1. \end{cases}$$

We define the X-weight w_X(e) and the Z-weight w_Z(e) to be the Hamming weights of a and b of Equation (II.1) respectively. In fact, they alternatively can be defined as

$$w_X(e) = N_x(e) + N_y(e), \quad w_Z(e) = N_y(e) + N_z(e).$$
In the following section we want to investigate some natural partitions of the set \( E \), so we define
\[
E[i, j, k] := \{ e \in E \mid N_x(e) = i, N_y(e) = j, N_z(e) = k \},
\]
\[
E[i, j] := \{ e \in E \mid w_X(e) = i, w_Y(e) = j \},
\]
\[
E[i] := \{ e \in E \mid w_Q(e) = i \}.
\]

**Definition 1.** A quantum code of length \( n \) is a linear subspace of \( \mathbb{C}^{2^n} \) with dimension \( K \geq 1 \). Such a quantum code can be denoted as \((n, K)\) code or \([n, k]\) code, where \( k = \log K \).

**Remark 2.** Here and thereafter, the logarithms are base 2.

**Definition 3.** Let \( Q \) be a quantum code. An error \( e \) in \( E \) is called detectable if
\[
\langle v | e | w \rangle = 0
\]
for all orthogonal vectors \( v \) and \( w \) from the code \( Q \).

Denote by \( P \) the orthogonal projection from \( \mathbb{C}^{2^n} \) onto a quantum code \( Q \). We have an alternative definition for detectable errors. It is deduced from [35] that \( e \) is detectable if and only if
\[
P e P = \lambda_e P
\]
for a constant \( \lambda_e \) depending on \( e \).

**Definition 4.** Let \( Q \) be a quantum code with parameters \((n, K)\). The minimum distance of \( Q \) is the maximum integer \( d \) such that any error \( e \in E[i] \) with \( i < d \) is detectable. Such a quantum code is called a symmetric quantum code with parameters \((n, K, d)\) or \([n, k, d]\).

The classical Singleton bound can be extended to quantum codes.

**Theorem 5.** Let \( Q \) be a quantum code with parameters \([n, k, d]\). We have
\[
n \geq k + 2d - 2.
\]

In [36, 18], the authors have proved the Singleton bound for stabilizer asymmetric quantum codes. That is
\[
n \geq k + d_x + d_z - 2.
\]

However, we can not find the proof of Singleton bound for general quantum codes.

### III. Weight Distributions and Enumerators

The weight distribution for classical codes can be generalized to the case of quantum codes. According to [25], the weight distribution for quantum codes is defined by the following two sequences of numbers
\[
B_i = \frac{1}{K^2} \sum_{e \in E[i]} \text{Tr}^2(eP),
\]
\[
B_i^\perp = \frac{1}{K} \sum_{e \in E[i]} \text{Tr}(eP eP).
\]

Moreover, the corresponding weight enumerator is defined to be the following two bivariate polynomials
\[
B(X, Y) := \sum_{i=0}^{n} B_i X^{n-i} Y^i,
\]
\[
B^\perp(X, Y) := \sum_{i=0}^{n} B_i^\perp X^{n-i} Y^i.
\]

In a similar manner, we introduce the double weight distribution
\[
C_{i,j} = \frac{1}{K^2} \sum_{e \in E[i,j]} \text{Tr}^2(eP),
\]
\[
C_{i,j}^\perp = \frac{1}{K} \sum_{e \in E[i,j]} \text{Tr}(eP eP),
\]
and the complete weight distribution

\[ D_{i,j,k} = \frac{1}{K^2} \sum_{e \in E[i,j,k]} \text{Tr}^2(eP), \]

\[ D^-_{i,j,k} = \frac{1}{K} \sum_{e \in E[i,j,k]} \text{Tr}(ePeP). \]

Then the double weight enumerator and the complete weight enumerator are defined by

\[ C(X, Y, Z, W) := \sum_{i,j=0}^n C_{i,j} X^{n-i} Y^i Z^{n-j} W^j, \]

\[ C^\perp(X, Y, Z, W) := \sum_{i,j=0}^n C^\perp_{i,j} X^{n-i} Y^i Z^{n-j} W^j, \]

\[ D(X, Y, Z, W) := \sum_{i+j+k \leq n} D_{i,j,k} X^i Y^j Z^k W^{n-i-j-k}, \]

\[ D^\perp(X, Y, Z, W) := \sum_{i+j+k \leq n} D^\perp_{i,j,k} X^i Y^j Z^k W^{n-i-j-k}. \]

These enumerators are related by the following theorem.

**Theorem 6.** Let \( Q \) be a quantum code with enumerators \( B, B^\perp, C, C^\perp, D \) and \( D^\perp \). Then the following four identities hold:

\[ B(X, Y) = D(Y, Y, Y, X), \quad (\text{III.1}) \]

\[ B^\perp(X, Y) = D^\perp(Y, Y, Y, X), \quad (\text{III.2}) \]

\[ C(X, Y, Z, W) = D(YZ, YW, XW, XZ), \quad (\text{III.3}) \]

\[ C^\perp(X, Y, Z, W) = D^\perp(YZ, YW, XW, XZ). \quad (\text{III.4}) \]

**Proof.** We shall investigate the explicit expressions for these enumerators. Recall that the coordinate basis for \( \mathbb{C}^{2^n} \) is given by

\[ |j\rangle = |j_0\rangle \otimes \ldots \otimes |j_{n-1}\rangle, \]

where \( 0 \leq j < 2^n \) and \( j = j_0 + j_1 2 + \ldots + j_{n-1} 2^{n-1} \). Denote by \( e_{i,j}, p_{i,j} \) the entries of \( e \) and \( P \) respectively with respect to our coordinate basis. For \( e = \sigma_0 \otimes \ldots \otimes \sigma_{n-1} \in E \), the identity \( e|i\rangle = \sum_j e_{i,j} |j\rangle \) implies

\[ e_{i,j} = (\sigma_0)_{i_0,j_0} (\sigma_1)_{i_1,j_1} \ldots (\sigma_{n-1})_{i_{n-1},j_{n-1}}. \]

According to the definition of \( D \), we get

\[ D(X, Y, Z, W) = \sum_{i,j,k} X^i Y^j Z^k W^{n-i-j-k} D_{i,j,k} \]

\[ = \frac{1}{K^2} \sum_{i,j,k} X^i Y^j Z^k W^{n-i-j-k} \sum_{e \in E[i,j,k]} \sum_{r,s,t,u} e_{r,s} p_{s,r} e_{t,u} p_{u,t} \]

\[ = \frac{1}{K^2} \sum_{r,s,t,u} p_{s,r} p_{u,t} \sum_{e \in E} X^{N_s(e)} Y^{N_r(e)} Z^{N_t(e)} W^{n-w_Q(e)} e_{r,s} e_{t,u} \]

\[ = \frac{1}{K^2} \sum_{r,s,t,u} p_{s,r} p_{u,t} \prod_{\lambda=0}^{n-1} d_\lambda(X, Y, Z, W), \]

where

\[ d_\lambda(X, Y, Z, W) = (\sigma_x)_{r_\lambda,s_\lambda} (\sigma_x)_{t_\lambda,u_\lambda} X + (\sigma_y)_{r_\lambda,s_\lambda} (\sigma_y)_{t_\lambda,u_\lambda} Y + (\sigma_z)_{r_\lambda,s_\lambda} (\sigma_z)_{t_\lambda,u_\lambda} Z + (I)_{r_\lambda,s_\lambda} (I)_{t_\lambda,u_\lambda} W. \]
By the same method, we have
\[ C(X,Y,Z,W) = \sum_{i,j} X^{n-i} Y^i Z^{n-j} W^j C_{i,j} \]
\[ = \frac{1}{K^2} \sum_{i,j} X^{n-i} Y^i Z^{n-j} W^j \sum_{e \in E[i]} \sum_{r,s,t,u} e_{r,s} p_{r,u} e_{t,u} p_{t,u} \]
\[ = \frac{1}{K^2} \sum_{r,s,t,u} p_{r,u} p_{t,u} \sum_{e \in E[i]} X^{n-wx(e)} Y^{wz(e)} Z^{n-wz(e)} W^{wz(e)} e_{r,s} e_{t,u} \]
\[ = \frac{1}{K^2} \sum_{r,s,t,u} p_{r,u} p_{t,u} \prod_{\lambda=0}^{n-1} c_\lambda(X,Y,Z,W), \]
where
\[ c_\lambda(X,Y,Z,W) = (\sigma_x)_{r,s} (\sigma_x)_{t,u} YZ + (\sigma_y)_{r,s} (\sigma_y)_{t,u} YW + (\sigma_z)_{r,s} (\sigma_z)_{t,u} XW + (I)_{r,s} (I)_{t,u} XZ. \]

Moreover, we obtain
\[ B(X,Y) = \sum_{i,j} X^{n-i} Y^i B_{i,j} \]
\[ = \frac{1}{K^2} \sum_{i,j} X^{n-i} Y^i \sum_{e \in E[i]} \sum_{r,s,t,u} e_{r,s} p_{r,u} e_{t,u} p_{t,u} \]
\[ = \frac{1}{K^2} \sum_{r,s,t,u} p_{r,u} p_{t,u} \sum_{e \in E[i]} X^{n-wQ(e)} Y^{wQ(e)} e_{r,s} e_{t,u} \]
\[ = \frac{1}{K^2} \sum_{r,s,t,u} p_{r,u} p_{t,u} \prod_{\lambda=0}^{n-1} b_\lambda(X,Y), \]
where
\[ b_\lambda(X,Y) = (\sigma_x)_{r,s} (\sigma_x)_{t,u} Y + (\sigma_y)_{r,s} (\sigma_y)_{t,u} Y + (\sigma_z)_{r,s} (\sigma_z)_{t,u} Y + (I)_{r,s} (I)_{t,u} X. \]

So Equations (III.1) and (III.3) are deduced from
\[ d_\lambda(YZ, YW, XW, XZ) = c_\lambda(X,Y,Z,W), \]
and
\[ d_\lambda(Y, Y, X, X) = b_\lambda(X, Y). \]

Applying the same argument, Equations (III.2) and (III.4) follow immediately.

The classical MacWilliams identity provides a relationship between classical linear codes and their dual codes. It is interesting to see that the MacWilliams identity also holds for quantum weight enumerators [25]. That is
\[ B(X,Y) = \frac{1}{K} B^\perp \left( \frac{X+3Y}{2}, \frac{X-Y}{2} \right). \]  

Our main result is to show that quaternary MacWilliams identities for the double weight enumerator and complete weight enumerator hold similarly.

**Theorem 7** (quaternary MacWilliams identities). *Let Q be an (n, K) quantum code. With the notation introduced above, we have*
\[ C(X,Y,Z,W) = \frac{1}{K} C^\perp (Z + W, Z - W, \frac{X+Y}{2}, \frac{X-Y}{2}), \]  
\[ D(X,Y,Z,W) = \frac{1}{K} D^\perp \left( \frac{X-Y - Z + W}{2}, \frac{-X+Y - Z + W}{2}, \frac{-X-Y + Z + W}{2}, \frac{X+Y + Z + W}{2} \right). \]

**Proof.** We maintain all notations in the proof of the previous theorem. Using the same method, one can show that
\[ D^\perp (X,Y,Z,W) = \frac{1}{K} \sum_{r,s,t,u} p_{r,u} p_{t,u} \prod_{\lambda=0}^{n-1} d_\lambda(X,Y,Z,W), \]
where
\[ d_\lambda(X,Y,Z,W) = (\sigma_x)_{r,s} (\sigma_x)_{t,u} X + (\sigma_y)_{r,s} (\sigma_y)_{t,u} Y + (\sigma_z)_{r,s} (\sigma_z)_{t,u} Z + (I)_{r,s} (I)_{t,u} W. \]
We can check directly that for arbitrary \(r, s, t, u \in \{0, 1\},\)
\[
d_\lambda(X, Y, Z, W) = d_\lambda^+(\frac{X - Y - Z + W}{2}, \frac{-X + Y - Z + W}{2}, \frac{-X - Y + Z + W}{2}, \frac{X + Y + Z + W}{2}).
\]
This implies Equation (III.7). Using this equation and the relationship between \(C\) and \(D,\) we get
\[
C(X, Y, Z, W) = D(YZ, YW, XW, XZ)
\]
\[
= \frac{1}{K} D^+(\frac{(X + Y)(Z - W)}{2}, \frac{(X - Y)(Z - W)}{2}, \frac{(X - Y)(Z + W)}{2}, \frac{(X + Y)(Z + W)}{2})
\]
\[
= \frac{1}{K} C^+(\frac{Z + W, Z - W}{2}, \frac{X + Y, X - Y}{2}),
\]
where in the last step we use Equation (III.4). This completes the proof of the theorem.

The following theorem generalize Theorem 3 in [35] to the case of double weight distribution.

**Theorem 8.** Let \(((Q, K, d_z/d_x))\) be an asymmetric quantum code with double weight distribution \(C_{i,j}^+\) and \(C_{i,j}^-\). Then

1. \(C_{i,j}^+ \geq C_{i,j}^- \geq 0\) for \(0 \leq i, j \leq n,\) and \(C_{0,0} = C_{0,0}^+ = 1.\)
2. If \(t_x, t_z\) are the two largest integers such that \(C_{i,j}^- = C_{i,j}^+\) for \(i < t_x\) and \(j < t_z,\) then \(d_x = t_x\) and \(d_z = t_z.\)

**Proof.** The proof is similar to that of Theorem 3 in [35] and so is omitted here.

**Remark 9.** Suppose that \(Q\) is an additive quantum code of length \(n\) constructed from a classical linear code \(C\) over \(\mathbb{F}_4,\) i.e., \(\mathbb{F}_4 = \{\alpha, \alpha^2, \alpha^3 = \alpha + \alpha^2, 0\}.\) Write
\[
c = (c_1, c_2, \cdots, c_n) = (a_1\alpha + b_1\alpha^2, a_2\alpha + b_2\alpha^2, \cdots, a_n\alpha + b_n\alpha^2) \in \mathbb{F}_4^n,
\]
where \((a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n) \in \mathbb{F}_2^{2n}.\) Then the double weight distribution of \(Q\) is deduced from \(C\) and its symplectic dual \(C_{i,h}^+,\) i.e.,
\[
C_{i,j} = \# \{c \in C : \text{comp}(c) = i, \sum_{s=1}^n a_s = i, \sum_{s=1}^n b_s = j \},
\]
\[
C_{i,j}^+ = \# \{c \in C_{i,h}^+ : \text{comp}(c) = i, \sum_{s=1}^n a_s = i, \sum_{s=1}^n b_s = j \}.
\]

For a vector \(c = (c_1, c_2, \cdots, c_n) \in \mathbb{F}_4^n,\) the composition of \(c,\) denoted by \(\text{comp}(c),\) is defined as \(\text{comp}(c) = (k_1, k_2, k_3, n - k_1 - k_2 - k_3),\) where \(k_j (j \neq 0)\) is the number of components \(c_s\) of \(c\) that are equal to \(\alpha^j.\) Then the complete weight distribution of \(Q\) is also deduced from \(C\) and its symplectic dual \(C_{i,h}^+,\) i.e.,
\[
D_{i,j,k} = \# \{c \in C : \text{comp}(c) = (i, j, k, n - i - j - k) \},
\]
\[
D_{i,j,k}^+ = \# \{c \in C_{i,h}^+ : \text{comp}(c) = (i, j, k, n - i - j - k) \}.
\]

Here we provide a concrete example to illustrate our main results.

**Example 10.** Let \((q, m) = (4, 2).\) A \([5, 3, 3]\) Hamming code \(H_2\) over \(\mathbb{F}_4\) of length \(n = (q^m - 1)/(q - 1) = 5\) has check matrix
\[
\begin{bmatrix}
1 & 0 & 1 & \alpha^2 & \alpha^2 \\
0 & 1 & \alpha^2 & \alpha^2 & 1
\end{bmatrix},
\]
where \(\alpha\) is a fixed primitive element of \(\mathbb{F}_4.\) Its dual code \(H_2^+\) is a \([5, 2, 4]\) linear code over \(\mathbb{F}_4.\) The code \(H_2^+\) induces an additive quantum code \(Q\) with parameters \([[n, n - 2m, 3]] = [[5, 1, 3]].\) The weight enumerator of \(Q\) is computed from \(C := H_2^+\) and its symplectic dual \(C_{i,h}^- := H_2,\) namely,
\[
B(X, Y) = X^5 + 15XY^4,
\]
\[
B^+(X, Y) = X^5 + 30X^2Y^3 + 15XY^4 + 18Y^5.
\]
One verifies that the MacWilliams identity (III.5) holds for $B$ and $B^\perp$. The double weight enumerator of $Q$ is as follows

$$C(X, Y, Z, W) = X^5Z^5 + 5X^3Y^2Z^3W^2 + 5X^3Y^2ZW^4 + 5XY^4Z^3W^2,$$

$$C^\perp(X, Y, Z, W) = X^5Z^5 + X^5W^5 + 5X^4YZ^2W^2 + 5X^4Y^2Z^2W^3 + 5X^3Y^2Z^2W + 5X^3Y^2Z^3W^2
\quad + 5X^2Y^2Z^2W^2 + 5X^2Y^2Z^3W^3 + 5X^2Y^3Z^2W^3 + 5X^2Y^3Z^3W^3.
\quad + 5X^2Y^3ZW^4 + 5XY^4Z^2W^3 + 5XY^4Z^3W + 5Y^5Z^5 + 5Y^5W^5.$$

The complete weight enumerator of $Q$ is given below

$$D(X, Y, Z, W) = W^5 + 5WX^2Y^2 + 5WX^2Z^2 + 5WY^2Z^2,$$

$$D^\perp(X, Y, Z, W) = W^5 + X^5 + Y^5 + Z^5 + 5WX^2Y^2 + 5W^2X^2Y + 5W^2X^2Z + 5W^2XY^2 + 5W^2XZ^2 + 5W^2Y^2Z
\quad + 5W^2Y^2Z^2 + 5WX^2Y^2 + 5WX^2Z^2 + 5WY^2Z^2 + 5X^2Y^2Z + 5X^2Y^2Z^2 + 5XY^2Z^2 + 5XY^2Z.$$

The above experimental results by Magma are consistent with the conclusions of Theorems 6 and 7.

IV. KRAWTCHOUK POLYNOMIALS AND THE KEY INEQUALITY

In this section, we introduce Krawtchouk polynomials and summarize their properties. This allows us to obtain the close relationship between $C_{i,j}$ and $C^\perp_{i,j}$ of an asymmetric quantum code. Then we propose the key inequality which enables us to reduce the problem of upper-bounding the size of asymmetric quantum codes to a problem of finding polynomials possessing special properties.

Fix an integer $n$. For $0 \leq i \leq n$, the polynomial

$$P_i(x) = \sum_{j=0}^{i} (-1)^j \binom{x}{j} \binom{n-x}{i-j}$$

is called the $i$-th Krawtchouk polynomial. The first few polynomials are

$$P_0(x) = 1, P_1(x) = n - 2x, P_2(x) = 2x^2 - 2nx + \binom{n}{2}, \ldots$$

These polynomials have the generating function

$$(X + Y)^{n-r}(X - Y)^r = \sum_{i=0}^{n} P_i(r)X^{n-i}Y^i.$$

Now we recall several important properties of the Krawtchouk polynomials, see [24] for more information. The Krawtchouk polynomials satisfy the reciprocity formula

$$\binom{n}{i} P_s(i) = \binom{n}{s} P_i(s). \quad (IV.1)$$

They also have the following property

$$\sum_{i=0}^{n} \binom{n-i}{n-j} P_i(x) = 2^i \binom{n-x}{j}. \quad (IV.2)$$

The Krawtchouk polynomials are orthogonal to each other, i.e.,

$$\sum_{i=0}^{n} P_r(i)P_i(s) = 2^n \delta_{r,s}. \quad (IV.2)$$

Many important facts follow from this orthogonality. For example, there is a three-term recurrence:

$$(i + 1)P_{i+1}(x) = (n - 2x)P_i(x) - (n - i + 1)P_{i-1}(x). \quad (IV.3)$$

The Christoffel-Darboux formula (see Corollary 3.5 of [32]) also holds:

$$P_{i+1}(x)P_i(a) - P_t(x)P_{i+1}(a) = \frac{2(a-x)}{t+1} \binom{n}{t} \sum_{i=0}^{t} P_i(x)P_i(a) \binom{n}{i}. \quad (IV.4)$$

Using (IV.1) and (IV.3), the ratio $P_t(x+1)/P_t(x)$ is given by McEliece et al. [46]

$$\frac{P_t(x+1)}{P_t(x)} = \frac{n - 2t + \sqrt{(n-2t)^2 - 4j(n-x)}}{2(n-x)} \left(1 + O\left(\frac{1}{n}\right)\right). \quad (IV.5)$$
We will also need a result on asymptotic behavior of the smallest root \( r_t \) of \( P_t(x) \). For \( t \) growing linearly in \( n \) and \( \tau = t/n \) (see, e.g., Eq A.20 of [46])

\[
\gamma_t = \frac{r_t}{n} = \frac{1}{2} - \sqrt{\tau(1-\tau)} + o(1).
\] (IV.6)

Remember that \( o(1) \) tends to 0 as \( n \) grows.

The following theorem shows that the Krawtchouk polynomials have close relation to double weight distribution \( C_{i,j} \) and \( C^\perp_{i,j} \) of a quantum code.

**Theorem 11.** Let \( ([Q, K, d_z/d_x]) \) be an asymmetric quantum code with double weight distribution \( C_{i,j} \) and \( C^\perp_{i,j} \). Then

\[
C_{i,j} = \frac{1}{2^n K} \sum_{r,s=0}^{n} P_i(s)P_j(r)C^\perp_{r,s},
\] (IV.7)

\[
C^\perp_{r,s} = \frac{K}{2^n} \sum_{i,j=0}^{n} P_i(j)P_s(i)C_{i,j}.
\] (IV.8)

**Proof.** We have from Theorem 7 that

\[
\sum_{i,j=0}^{n} C_{i,j} X^{n-i}Y^{j}Z^{n-j}W^{j} = \frac{1}{2^nK} \sum_{r,s=0}^{n} C^\perp_{r,s}(Z + W)^{n-r}(Z - W)^{r}(X + Y)^{n-s}(X - Y)^{s}
\]

\[
= \frac{1}{2^nK} \sum_{r,s=0}^{n} C^\perp_{r,s} \sum_{i,j=0}^{n} P_i(s)P_j(r)X^{n-i}Y^{j}Z^{n-j}W^{j}.
\]

\[
= \frac{1}{2^nK} \sum_{i,j=0}^{n} \sum_{r,s=0}^{n} C^\perp_{r,s} P_i(s)P_j(r)X^{n-i}Y^{j}Z^{n-j}W^{j}.
\]

This completes the proof of (IV.7). Taking into account that

\[
C^\perp(X, Y, Z, W) = K \cdot C(Z + W, Z - W, X + Y, X - Y)
\]

from (III.6), the identity (IV.8) follows immediately.

The key inequality is given below, which will be needed in the sequel.

**Lemma 12.** Let \( Q \) be an \( ([n, K, d_z/d_x]) \) quantum code. Assume that the polynomial \( f(x, y) = \sum_{i,j=0}^{n} \alpha_{i,j} P_i(y)P_j(x) \) satisfies the following conditions

1) \( \alpha_{i,j} \geq 0 \) for \( 0 \leq i, j \leq n \),
2) \( f(r, s) > 0 \) for \( 0 \leq r < d_x \) and \( 0 \leq s < d_z \),
3) \( f(r, s) \leq 0 \) for \( r \geq d_x \) or \( s \geq d_z \).

Then

\[
K \leq \frac{1}{2^n} \max_{0 \leq i < d_x, 0 \leq j < d_z} \frac{f(i,j)}{\alpha_{i,j}}.
\]

**Proof.** It follows from Theorems 8 and 11 that

\[
2^n K \sum_{i=0}^{d_x-1} \sum_{j=0}^{d_z-1} \alpha_{i,j} C_{i,j} \leq 2^n K \sum_{i,j=0}^{n} \alpha_{i,j} C_{i,j}
\]

\[
= \sum_{i,j=0}^{n} \alpha_{i,j} \sum_{r,s=0}^{n} P_i(s)P_j(r)C^\perp_{r,s}
\]

\[
= \sum_{r,s=0}^{n} f(r, s)C^\perp_{r,s}
\]

\[
\leq \sum_{r=0}^{d_x-1} \sum_{s=0}^{d_x-1} f(r, s)C^\perp_{r,s}
\]

\[
= \sum_{r=0}^{d_x-1} \sum_{s=0}^{d_z-1} f(r, s)C_{r,s}.
\]
Thus we have
\[
2^n K \leq \sum_{i=0}^{d_z-1} \sum_{j=0}^{d_z-1} f(i, j) C_{i,j} \leq \max_{0 \leq i < d_z, 0 \leq j < d_z} f(i, j),
\]
completing the proof of this lemma. 

The following lemma is a consequence of Lemma 12.

Lemma 13. Let \( Q \) be an \((n, K, d_x/d_z)\) quantum code. Define \( f(x, y) = f_1(x)f_2(y) \) where \( f_1(x) = \sum_{j=0}^{n} \beta_j P_j(x) \) and \( f_2(y) = \sum_{i=0}^{n} \alpha_i P_i(y) \). Assume that the polynomial \( f(x, y) \) satisfies the following conditions
1) For even \( i, \alpha_i > 0, \beta_i > 0 \). For odd \( i, \alpha_i = \beta_i = 0 \).
2) \( f(r, s) \geq 0 \) for \( 0 \leq r < d_x \) and \( 0 \leq s < d_z \). For even \( r, f_1(r) > 0, f_2(r) > 0 \). For odd \( r, f_1(r) = f_2(r) = 0 \).

Then
\[
K \leq \frac{1}{2} \max_{0 \leq i < d_z} f_1(i) \max_{0 \leq j < d_z} f_2(j).
\]

Lemma 14. Let \( A(x) = 2^{n-d+1} \prod_{r=d}^{n-1} \left( 1 - \frac{x}{r} \right) \). Then
\[
A(x) = \sum_{i=0}^{n} \alpha_i P_i(x)
\]
where
\[
\alpha_i = \alpha_i(d) = \frac{(n-i)}{(d-1)} / \left( \frac{n}{n-d+1} \right).
\]

Proof. By definition, we have
\[
A(x) = 2^{n-d+1} \prod_{r=d}^{n-1} \left( 1 - \frac{x}{r} \right) = 2^{n-d+1} \left( \frac{n-x}{n-d+1} \right) / \left( \frac{n}{n-d+1} \right).
\]
It is known from the book [24] (Exercise 41) that
\[
\sum_{r=0}^{n} \left( \frac{n-r}{n-d+1} \right) P_r(i) = 2^{d-1} \left( \frac{n-i}{d-1} \right),
\]
This implies that
\[
\alpha_i = 2^{-n} \sum_{r=0}^{n} A(r) P_r(i) = \left( \frac{n-i}{d-1} \right) / \left( \frac{n}{n-d+1} \right),
\]
completing the proof of this lemma. 

V. UPPER BOUNDS

In this section, we extend the work of [35] and derive asymptotic upper bounds on the size of an arbitrary asymmetric quantum code of given length and minimum distance. Precisely, the Singleton bound, the Hamming bound and the first linear programming bound are determined utilizing the key inequality presented in Section IV.

A. A Singleton-type bound

Theorem 15 (Quantum Singleton Bound). Let \( Q \) be an \([n, k, d_x/d_z]\) quantum code. Then \( n \geq k + d_x + d_z - 2 \).

Proof. Set \( \alpha_{i,j} = \alpha_i(d_z) \alpha_j(d_z) \geq 0 \) for \( 0 \leq i, j \leq n \), where \( \alpha_i(d) \) is defined in (IV.9). Let
\[
f(x, y) = \sum_{i,j=0}^{n} \alpha_{i,j} P_i(x) P_j(y)
\]
\[
= \sum_{i,j=0}^{n} \alpha_i(d_z) \alpha_j(d_z) P_i(x) P_j(y)
\]
\[
= 2^{2n-d_x-d_z+2} \prod_{r=d_x}^{n} \left( 1 - \frac{x}{r} \right) \prod_{s=d_z}^{n} \left( 1 - \frac{y}{s} \right).
\]
One may check that this polynomial verifies all conditions of Lemma 12. So
\[
K \leq \frac{1}{2^n} \max_{0 \leq i < d_x, 0 \leq j < d_z} \frac{f(i, j)}{\alpha_{i,j}}
\]
\[
= 2^{n-d_x-d_z+2} \max_{0 \leq i < d_x} g(i; d_x) \max_{0 \leq j < d_z} g(j; d_z),
\]
where
\[
g(i; d) = \binom{n-i}{n-d+1}/\binom{n-i}{d-1}.
\]
For \(d \leq \frac{n}{2} + 1\), we have \(g(i; d)/g(i+1; d) \geq 1\). Therefore, we obtain
\[
K \leq 2^{n-d_x-d_z+2} g(0, d_x)g(0, d_z) = 2^{n-d_x-d_z+2}.
\]
Since \(K = 2^k\), we find \(n \geq k + d_x + d_z - 2\).

B. A Hamming-type bound

Let \(\phi = \left\lfloor \frac{d_x-1}{2} \right\rfloor\) and \(\theta = \left\lfloor \frac{d_z-1}{2} \right\rfloor\). Define \(\alpha_i(d_z) = (P_b(i))^2\), \(\beta_j(d_x) = (P_{\phi}(j))^2\) and
\[
f(x, y) = \sum_{i, j=0}^{n} \alpha_i(d_z)\beta_j(d_x)P_i(y)P_j(x).
\]

**Lemma 16** ([46], Eq A.19). Any product \(P_i(x)P_j(x)\) can be expressed as a linear combination of the \(P_k(x)\) as follows:
\[
P_i(x)P_j(x) = \sum_{k=0}^{n} \binom{n-k}{(i+j-k)/2} \binom{k}{(i-j+k)/2} P_k(x),
\]
where a binomial coefficient with fractional or negative lower index is to be interpreted as zero.

Using the lemma, we get
\[
\alpha_i(d_z) = \sum_{k=0}^{n} \binom{n-k}{\theta - k/2} \binom{k}{k/2} P_k(i)
\]
and
\[
\beta_j(d_x) = \sum_{k=0}^{n} \binom{n-k}{\phi - k/2} \binom{k}{k/2} P_k(j).
\]

It then follows that
\[
\sum_{i=0}^{n} \alpha_i(d_z)P_i(y) = \sum_{i=0}^{n} \sum_{k=0}^{n} \binom{n-k}{\theta - k/2} \binom{k}{k/2} P_k(i)P_i(y)
\]
\[
= \sum_{k=0}^{n} \binom{n-k}{\theta - k/2} \binom{k}{k/2} \sum_{i=0}^{n} P_k(i)P_i(y)
\]
\[
= 2^n \binom{n-y}{\theta - y/2} \binom{y}{y/2},
\]
where in the last step we use \((V.2)\). Similarly we have
\[
\sum_{j=0}^{n} \beta_j(d_x)P_j(x) = 2^n \binom{n-x}{\phi - x/2} \binom{x}{x/2},
\]
Hence
\[
f(x, y) = 2^{2n} \binom{n-x}{\phi - x/2} \binom{x}{x/2} \binom{n-y}{\theta - y/2} \binom{y}{y/2}.
\]

For later use, we define the binary entropy function \(H(x) = -x \log x - (1-x) \log (1-x)\) for \(0 \leq x \leq 1\). Taking into account that
\[
\frac{1}{n} \log \binom{n}{k} = H\left(\frac{k}{n}\right) + O\left(\frac{1}{n}\right)
\]
\[(V.2)\]
and denoting $\xi = x/n$, $\eta = y/n$, $\tau = \phi/n$ and $\sigma = \theta/n$, we get
\[
\frac{1}{n} \log \left[ \binom{n-x}{\frac{\phi-x}{2}} \frac{x}{x/2} \binom{n-y}{\frac{\theta-y}{2}} \frac{y}{y/2} \right] = \frac{1}{n} \log \left( \frac{n-x}{\phi-x/2} \right) + \frac{1}{n} \log \left( \frac{x}{x/2} \right) + \frac{1}{n} \log \left( \frac{n-y}{\theta-y/2} \right) + \frac{1}{n} \log \left( \frac{y}{y/2} \right) = (1-\xi) H(\frac{\tau-\xi/2}{1-\xi}) + \xi H(\frac{1}{2}) + (1-\eta) H\left( \frac{\sigma-\eta/2}{1-\eta} \right) + \eta H\left( \frac{1}{2} \right) + O\left(\frac{1}{n}\right)
\]
This yields
\[
\frac{1}{n} \log f(x,y) = 2 + \xi + \eta + (1-\xi) H\left( \frac{\tau-\xi/2}{1-\xi} \right) + (1-\eta) H\left( \frac{\sigma-\eta/2}{1-\eta} \right) + O\left(\frac{1}{n}\right).
\]
To derive an estimate for $\alpha(x,y) = \alpha(y)\alpha_x(d_z)$, we need bounds on values of Krawtchouk polynomials. Recall that by (IV.6)
\[
\gamma_\phi = \frac{r_\phi}{n} = \frac{1}{2} - \sqrt{\tau(1-\tau)} + o(1),
\]
and
\[
\gamma_\theta = \frac{r_\theta}{n} = \frac{1}{2} - \sqrt{\sigma(1-\sigma)} + o(1).
\]
We also recall the following equations, see [47]:
\[
\frac{1}{n} \log P_\phi(x) = H(\tau) + \int_0^\xi \log \left( \frac{1-2\tau + \sqrt{(1-2\tau)^2 - 4z(1-z)}}{2(1-z)} \right) dz + O\left(\frac{1}{n}\right),
\]
for $\xi < \gamma_\phi$ and
\[
\frac{1}{n} \log P_\theta(y) = H(\sigma) + \int_0^\eta \log \left( \frac{1-2\sigma + \sqrt{(1-2\sigma)^2 - 4z(1-z)}}{2(1-z)} \right) dz + O\left(\frac{1}{n}\right),
\]
for $\eta < \gamma_\theta$. Hence we obtain
\[
\frac{1}{n} \log \alpha_{x,y} = \frac{1}{n} \log \left( \left( P_\phi(x) \right)^2 (P_\phi(y))^2 \right) = 2H(\tau) + 2 \int_0^\xi \log \left( \frac{1-2\tau + \sqrt{(1-2\tau)^2 - 4z(1-z)}}{2(1-z)} \right) dz + 2H(\sigma) + 2 \int_0^\eta \log \left( \frac{1-2\sigma + \sqrt{(1-2\sigma)^2 - 4z(1-z)}}{2(1-z)} \right) dz + O\left(\frac{1}{n}\right).
\]
Now we are in a position to give the Hamming type bound.

**Theorem 17** (Hamming-type Bound). Let $\tau = \lfloor \frac{d_z-1}{2} \rfloor /n$ and $\sigma = \lfloor \frac{d_z-1}{2} \rfloor /n$. Define
\[
\Omega_\tau(\xi) := \xi + (1-\xi) H\left( \frac{\tau-\xi/2}{1-\xi} \right) - 2H(\tau) - 2 \int_0^\xi \log \left( \frac{1-2\tau + \sqrt{(1-2\tau)^2 - 4z(1-z)}}{2(1-z)} \right) dz.
\]
Suppose that $2\tau < \gamma_\phi$ and $2\sigma < \gamma_\theta$ where $\gamma_\phi$ and $\gamma_\theta$ are given in (V.3) and (V.5), respectively. Then for an $((n, K, d_z/d_x))$ quantum code we have
\[
\frac{\log K}{n} \leq 1 + \max_{0 \leq \xi \leq 2\tau} \Omega_\xi(\xi) + \max_{0 \leq \eta \leq 2\sigma} \Omega_\eta(\eta) + o(1).
\]

**Proof.** It can be easily verified that the polynomial $f(x,y)$ of (V.1) satisfies all the conditions of Lemma 13. So we get from (V.3), (V.7) and Lemma 13 that
\[
\frac{\log K}{n} \leq -1 + \max_{0 \leq \xi \leq 2\tau} \left\{ \frac{1}{n} \log f(x,y) - \frac{1}{n} \log \alpha_{x,y} \right\} = -1 + \max_{0 \leq \xi \leq 2\tau} \left\{ \frac{1}{n} \log f(x,y) - \frac{1}{n} \log \alpha_{x,y} \right\} = 1 + \max_{0 \leq \xi \leq 2\tau} \left\{ \xi + (1-\xi) H\left( \frac{\tau-\xi/2}{1-\xi} \right) - 2H(\tau) - 2 \int_0^\xi \log \left( \frac{1-2\tau + \sqrt{(1-2\tau)^2 - 4z(1-z)}}{2(1-z)} \right) dz \right\} = 1 + \max_{0 \leq \eta \leq 2\sigma} \left\{ \eta + (1-\eta) H\left( \frac{\sigma-\eta/2}{1-\eta} \right) - 2H(\sigma) - 2 \int_0^\eta \log \left( \frac{1-2\sigma + \sqrt{(1-2\sigma)^2 - 4z(1-z)}}{2(1-z)} \right) dz \right\} + o(1),
\]
where in the third step we use the fact that \( \xi = x/n \leq 2\phi/n = 2\tau < \gamma \) and \( \eta = y/n \leq 2\theta/n = 2\sigma < \gamma \). This completes the proof. \( \square \)

Matlab shows that the function in Theorem 17 achieves its maximum at \( \xi = 0 \) and \( \eta = 0 \) for any \( 2\tau < \gamma \) and \( 2\sigma < \gamma \). A straightforward calculation gives that

\[
h(x) = 2x + \sqrt{x(1-x)} - 1/2
\]

is a monotone increasing function in \( x \) if \( 0 \leq x \leq 1/2 \).

Denote \( \delta_x = dx/n \) and \( \delta_z = dz/n \). Assume that \( 0 \leq \delta_x \leq 1/5 \) and \( 0 \leq \delta_z \leq 1/5 \). Then we have

\[
\tau = \left[ \frac{d-1}{2} \right] /n < d/2n = \delta_x/2.
\]

Let \( \Delta = h'(1/10)(\tau - 1/10) \). Then we have

\[
\frac{h(1/10) - h(\tau)}{1/10 - \tau} = h'(\tau) > h'(1/10),
\]

where \( \tau < \vartheta < 1/10 \). Since \( h(1/10) = 0 \), then

\[
h(\tau) < h'(1/10)(\tau - 1/10) = \Delta.
\]

It follows that \( 2\tau < 1/2 - \sqrt{\tau(1-\tau)} + \Delta \leq \gamma \) by noting that \( \Delta < o(1) \) for \( n \) sufficiently large. Therefore we conclude that if \( 0 \leq \delta_x \leq 1/5 \) and \( 0 \leq \delta_z \leq 1/5 \) then \( 2\tau < \gamma \) and \( 2\sigma < \gamma \). This means the conventional Hamming bound is valid when \( 0 \leq \delta_x \leq 1/5 \) and \( 0 \leq \delta_z \leq 1/5 \).

Corollary 18. The conventional Hamming bound is valid for quantum codes, i.e., if \( Q \) is an \( ((n,K,d_x/d_z)) \) quantum code then

\[
\log_2 \frac{K}{n} \leq 1 - H(\frac{\delta_x}{2}) - H(\frac{\delta_z}{2}) + o(1),
\]

where \( \delta_x = d_x/n \) and \( \delta_z = d_z/n \) stand for the relative distances of the code, \( 0 \leq \delta_x \leq 1/5 \) and \( 0 \leq \delta_z \leq 1/5 \).

Proof. Taking \( \xi = 0 \) and \( \eta = 0 \) in Theorem 17 yields that

\[
\log_2 \frac{K}{n} \leq 1 - H(\tau) - H(\sigma) + o(1)
\]

\[
= 1 - H(\frac{\delta_x}{2}) - H(\frac{\delta_z}{2}) + o(1),
\]

where in the last step we use the fact \( H(\tau) = H(\delta_x/2) + o(1) \) and \( H(\sigma) = H(\delta_z/2) + o(1) \). \( \square \)

C. The First Linear Programming Bound

To get the first linear programming bound for an \( ((n,K,d_x/d_z)) \) code, one has to choose integers \( s \) and \( t \) such that

\[
\frac{t}{n} = \frac{1}{2} - \sqrt{\delta_x (1-\delta_x)} + o(1),
\]

\[
\frac{s}{n} = \frac{1}{2} - \sqrt{\delta_z (1-\delta_z)} + o(1),
\]

where \( \delta_x = d_x/n \), \( \delta_z = d_z/n \). Then we choose integers \( a \) and \( b \) such that \( r_t+1 < a < r_t \), \( r_s+1 < b < r_s \), \( P_t(a)/P_{t+1}(a) = -1 \) and \( P_s(b)/P_{s+1}(b) = -1 \). Define

\[
f(x,y) = F(x)G(y),\quad (V.8)
\]

where

\[
F(x) = \frac{1}{a-x} \{ P_{t+1}(x)P_t(a) - P_t(x)P_{t+1}(a) \}^2,
\]

\[
G(y) = \frac{1}{b-y} \{ P_{s+1}(y)P_s(b) - P_s(y)P_{s+1}(b) \}^2.
\]

This polynomial \( (V.8) \) will yield the first linear programming bound for classical codes over \( \mathbb{F}_4 \). By the Christoffel-Darboux formula \( (IV.4) \)

\[
F(x) = \frac{2}{t+1} \binom{n}{t} \{ P_{t+1}(x)P_t(a) - P_t(x)P_{t+1}(a) \} \sum_{i=0}^{t} \frac{P_i(a)}{\binom{n}{i}} \{ P_{i+1}(x)P_i + P_i(x)P_{i+1} \},
\]

and

\[
= \frac{2}{t+1} \binom{n}{t} P_t(a) \sum_{i=0}^{t} \frac{P_i(a)}{\binom{n}{i}} \{ P_{i+1}(x)P_i + P_i(x)P_{i+1} \}.
\]
It follows from Lemma [16] that

\[ F(x) = \frac{2}{t+1} \left( \begin{array}{c} n \\ t \end{array} \right) P_t(a) \sum_{i=0}^{t} \frac{P_i(a)}{\binom{n}{i}} \times \left\{ \sum_{j=0}^{n} P_j(x) \left( \frac{n-j}{t+1-i-j/2} \right) \left( \frac{j}{t+1-i+j/2} \right) + \sum_{j=0}^{n} P_j(x) \left( \frac{n-j}{t+i-j/2} \right) \left( \frac{j}{t+i-j/2} \right) \right\} \]

\[ = \sum_{j=0}^{n} P_j(x) \left( \frac{2}{t+1} \left( \begin{array}{c} n \\ t \end{array} \right) P_t(a) \sum_{i=0}^{t} \frac{P_i(a)}{\binom{n}{i}} \right) \times \left\{ \left( \frac{n-j}{t+1+i-j/2} \right) \left( \frac{j}{t+1-i+j/2} \right) + \left( \frac{n-j}{t+i-j/2} \right) \left( \frac{j}{t-i+j/2} \right) \right\} \]

where we denote

\[ F_j = \frac{2}{t+1} \left( \begin{array}{c} n \\ t \end{array} \right) P_t(a) \sum_{i=0}^{t} \frac{P_i(a)}{\binom{n}{i}} \times \left\{ \left( \frac{n-j}{t+1+i-j/2} \right) \left( \frac{j}{t+1-i+j/2} \right) + \left( \frac{n-j}{t+i-j/2} \right) \left( \frac{j}{t-i+j/2} \right) \right\} \].

Taking \( j = x \) and estimating \( F_x \) by the term with \( i = t \), we obtain

\[ F_x \geq \frac{2}{t+1} \left( \begin{array}{c} n \\ t \end{array} \right) P_t(a)^2 \left( \frac{n-x}{t-x/2} \right) \left( \frac{x}{x/2} \right) \]

\[ = \frac{2}{t+1} P_t(a)^2 \left( \frac{n-x}{t-x/2} \right) \left( \frac{x}{x/2} \right). \]

Denote \( \xi = x/n, \tau = t/n \). Then, similarly to the derivation of the Hamming bound, we have

\[ \frac{1}{n} \log \left\{ \left( \frac{n-x}{t-x/2} \right) \left( \frac{x}{x/2} \right) \right\} = (1-\xi)H\left( \frac{\tau-\xi/2}{1-\xi} \right) + \xi + O\left( \frac{1}{n} \right). \quad (V.9) \]

Then, using (IV.1), we get

\[ \frac{F(x)}{F_x} \leq \frac{(t+1)P_t(a)^2 \left\{ P_{t+1}(x) + P_t(x) \right\}^2}{2(a-x)P_t(a)^2 \left( \frac{n-x}{t-x/2} \right) \left( \frac{x}{x/2} \right)} \]

\[ = \frac{(t+1) \left\{ \left( \begin{array}{c} n \\ t+1 \end{array} \right) P_{t+1}(t+1) + \left( \begin{array}{c} n \\ t \end{array} \right) P_t(t) \right\}^2}{2(a-x) \left( \frac{n-x}{t-x/2} \right) \left( \frac{x}{x/2} \right)} \]

\[ = \frac{(t+1) \left( \begin{array}{c} n \\ t \end{array} \right)^2 \left\{ \frac{n-t}{t+1} P_{t+1}(t+1) + P_t(t) \right\}^2}{2(a-x) \left( \frac{n-x}{t-x/2} \right) \left( \frac{x}{x/2} \right)}. \]

It then follows from (IV.1) and (IV.5) that

\[ \frac{F(x)}{F_x} \leq \frac{P_t(a)^2 \left( \frac{n-2x + \sqrt{(n-2x)^2 - 4t(n-l)}}{2} + t+1 \right)^2}{2(a-x)(t+1) \left( \frac{n-x}{t-x/2} \right) \left( \frac{x}{x/2} \right)}. \]

Taking logarithm on both sides and dividing by \( n \), we obtain from (V.2), (V.6) and (V.9) that

\[ \frac{1}{n} \log \frac{F(x)}{F_x} \leq 2H(\tau) + 2 \int_{0}^{\xi} \log \left( \frac{1-2\tau + \sqrt{(1-2\tau)^2 - 4z(1-z)}}{2(1-z)} \right) dz - (1-\xi)H\left( \frac{\tau-\xi/2}{1-\xi} \right) - \xi + O\left( \frac{1}{n} \right), \quad (V.10) \]

for \( \xi < \gamma_\varphi \), where \( \gamma_\varphi \) is given in (V.4). By a similar argument as above, we have

\[ G(y) = \sum_{i=0}^{n} P_i(y)G_i, \]
Theorem 19. Let $\eta < \gamma$ and consequently we obtain

$$\frac{1}{n} \log_2 \frac{G(y)}{G_y} \leq 2H(\sigma) + 2 \int_0^\tau \log \left( \frac{1 - 2\sigma + \sqrt{(1 - 2\sigma)^2 - 4\sigma(1 - z)}}{2(1 - z)} \right) dz - (1 - \eta)H\left( \frac{\sigma - \eta/2}{1 - \eta} \right) - \eta + O\left( \frac{1}{n} \right),$$

(V.11)

for $\eta < \gamma$, where $\sigma = s/n$, $\eta = y/n$ and $\gamma$ is given in (V.3).

With the above preparation, we can get the following theorem.

Theorem 19. Let $\delta_x = d_x/n$, $\delta_z = d_z/n$, $\tau = \frac{1}{2} - \sqrt{\delta_x(1 - \delta_x)}$ and $\sigma = \frac{1}{2} - \sqrt{\delta_z(1 - \delta_z)}$. Let

$$\Gamma_\tau(\xi) := 2H(\tau) + 2 \int_0^\xi \log \left( \frac{1 - 2\tau + \sqrt{(1 - 2\tau)^2 - 4\tau(1 - z)}}{2(1 - z)} \right) dz - (1 - \xi)H\left( \frac{\tau - \xi/2}{1 - \xi} \right) - \xi.$$

Suppose that $\delta_x < \gamma$ and $\delta_z < \gamma$. Then for an $(n, K, d_z/d_x)$ code, we have

$$\log_2 K \leq -1 + \max_{0 \leq \xi < \delta_x} \Gamma_\tau(\xi) + \max_{0 < \eta < \delta_z} \Gamma_\sigma(\eta) + o(1).$$

Proof. One verifies that the polynomial $f(x, y)$ of (V.8) satisfies all the conditions of Lemma [2]. Therefore applying Lemma [12], (V.10) and (V.11) gives that

$$\log_2 K \leq -1 + \max_{0 \leq \xi < \delta_x} \left\{ \frac{1}{n} \log_2 \frac{F(x)}{F_x} + \frac{1}{n} \log_2 \frac{G(y)}{G_y} \right\} - 1 + O\left( \frac{1}{n} \right)$$

$$= -1 + \max_{0 \leq \xi < \delta_x} \left\{ 2H(\tau) + 2 \int_0^\xi \log \left( \frac{1 - 2\tau + \sqrt{(1 - 2\tau)^2 - 4\tau(1 - z)}}{2(1 - z)} \right) dz - (1 - \xi)H\left( \frac{\tau - \xi/2}{1 - \xi} \right) - \xi \right\}$$

$$+ \max_{0 < \eta < \delta_z} \left\{ 2H(\sigma) + 2 \int_0^\eta \log \left( \frac{1 - 2\sigma + \sqrt{(1 - 2\sigma)^2 - 4\sigma(1 - z)}}{2(1 - z)} \right) dz - (1 - \eta)H\left( \frac{\sigma - \eta/2}{1 - \eta} \right) - \eta \right\} + o(1),$$

where in the second step we take $\xi = x/n$ and $\eta = y/n$. This completes the proof. $\square$

Computations with Matlab show that this function achieves its minimum at $\xi = 0$ and $\eta = 0$ for any $\delta_x \leq 0.1865$ and $\delta_z \leq 0.1865$.

Corollary 20 (The First Linear Programming Bound). If $0 \leq \delta_x \leq 0.1865$ and $0 \leq \delta_z \leq 0.1865$ then the conventional linear programming bound is valid for quantum codes, i.e., if $Q$ is an $(n, K, d_z/d_x)$ quantum code then

$$\log_2 K \leq H\left( \frac{1}{2} - \sqrt{\delta_x(1 - \delta_x)} \right) + H\left( \frac{1}{2} - \sqrt{\delta_z(1 - \delta_z)} \right) - 1 + o(1).$$

A straightforward computation gives that when $\delta_x = \delta_z = 0.1865$, $\log_2 K/n \approx 0.0028$. So for all $0.0028 \leq \log_2 K/n \leq 1$, the conventional first linear programming bound is valid.

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