BUFFEROVERFLOWS: JOINT LIMIT LAWS OF UNDERSHOOTS AND
OVERSHOOTS OF REFLECTED PROCESSES

ALEKSANDAR MIJATOVI ´ C AND MARTIJN PISTORIUS

ABSTRACT. Let \( \tau(x) \) be the epoch of first entry into the interval \((x, \infty)\), \(x > 0\), of the reflected process \( Y \) of a Lévy process \( X \), and define the overshoot \( Z(x) = Y(\tau(x)) - x \) and undershoot \( z(x) = x - Y(\tau(x)) \) of \( Y \) at the first-passage time over the level \( x \). In this paper we establish, separately under the Cramér and positive drift assumptions, the existence of the weak limit of \((z(x), Z(x))\) as \( x \) tends to infinity and provide explicit formulae for their joint CDFs in terms of the Lévy measure of \( X \) and the renewal measure of the dual of \( X \). We apply our results to analyse the behaviour of the classical M/G/1 queueing system at the buffer-overflow, both in a stable and unstable case.

1. Introduction

Consider a classical single server M/G/1 queueing system, consisting of a stream of jobs of sizes given by positive IID random variables \( U_1, U_2, \ldots \), arriving according to a standard Poisson process \( N \) with rate \( \lambda \), and a server that processes these jobs at unit speed. At a given time \( t \) the workload \( Y \) in the system is given by

\[
Y(t) = X(t) - X_*(t), \quad X_*(t) = \inf_{s \leq t} \{X(s), 0\},
\]

\[
X(t) = I(t) - O(t), \quad I(t) = u_0 + \sum_{n=1}^{N_t} U_n, \quad O(t) = t,
\]

where \( u_0 \geq 0 \) is the workload in the system at time 0, \( I(t) \) denotes the cumulative workload of all jobs that have arrived by time \( t \) and \( O(t) \) the cumulative capacity at time \( t \) (i.e. the amount of service that could have been provided if the server has never been idle up to time \( t \)). We refer to [1, 18] for background on queueing theory. In generalisations of the classical M/G/1 model it has been proposed to replace the compound Poisson process \( X \) in Eqn. (1.2) by a general Lévy process leading to the so-called Lévy-driven queues. In case the system in Eqns. (1.1)−(1.2) has a finite buffer of size \( x > 0 \) for the storage of the workload, two quantities of interest are the under- and overshoots of the workload \( Y \) at the first time \( \tau(x) \) of buffer-overflow (i.e. \( z(x) = x - Y(\tau(x)) \) and \( Z(x) = Y(\tau(x)) - x \) resp.), representing the level of the workload just before the buffer-overflow and the part of the job lost at \( \tau(x) \) (see e.g. [6, 13]).

Our main result (Theorem 2 below) states that if a Lévy process \( X \) satisfies the Cramér assumption and a non-lattice condition, the joint weak limit \((z(\infty), Z(\infty))\) of the under- and overshoot of \( Y \) at \( \tau(x) \) (as \( x \uparrow \infty \)) exists and is explicitly given by the following formula:

\[
P([z(\infty), Z(\infty)] \in du \otimes dv] = \frac{\gamma}{\phi(0)} \hat{V}_\gamma(u) \nu(dv + u) du, \quad \text{for } u, v > 0,
\]

2000 Mathematics Subject Classification. 60G51, 60F05, 60G17.

Key words and phrases. Reflected Lévy process, asymptotic undershoot and overshoot, Cramér condition, queueing.

MP’s research supported by NWO-STAR.
where \(\hat{\gamma}(u) = \int_{[0,u]} e^{\gamma(u-y)} \hat{V}(dy)\), \(\hat{V}(dy)\) is the renewal measure of the dual of \(X\), \(\phi\) the Laplace exponent of the ascending ladder-height process, \(\nu\) the Lévy measure of \(X\) and \(\gamma\) the Cramér coefficient (see Section 2 for precise definitions). Note that, unlike (1.3), analogous limit results for the Lévy process \(X\) at the moment of its first passage over the level \(x\) require conditioning on the event that the process reaches the level \(x\) in finites time (see e.g. Lemma 1(iii) below and [11, Thm. 7.1], [14, Thm. 4.2]).

In the case \(E[X(1)]\) is positive and finite and \(X(1)\) is non-lattice, we identify (in Theorem 1 below) the limiting joint law of the under- and overshoot of \(Y\) as:

\[
P[(z(\infty), Z(\infty)) \in du \otimes dv] = \frac{\hat{V}(u)}{\phi'(0)} \nu(dv + u) du, \quad \text{for } u, v > 0.
\]

In the classical queueing model given by (1.1)–(1.2), the random variable \(X(1)\) is non-lattice if the distribution \(F\) of the job-sizes \(U_i\) is. Our results yield the joint limit law explicitly in terms of the distribution \(F\) and the arrival rate \(\lambda\):

\[
P[(z(\infty), Z(\infty)) \in du \otimes dv] = \frac{\lambda}{\lambda m - 1}(1 - e^{-u}) F(dv + u) du, \quad u, v > 0,
\]

where \(m = \int_{(0,\infty)} x F(dx)\) is the mean of \(F\) and \(r^*\) denotes the largest (resp. smallest) root \(s\) in \(\mathbb{R}\) of the characteristic equation

\[
\lambda \int_{(0,\infty)} e^{sy} F(dy) - \lambda - s = 0,
\]

in the Cramér (resp. positive drift) case\(^1\). cf. Remarks (i) and (ii) after Theorem 2 in the next section.

While the formulæ in (1.3), (1.4) and (1.5) hold for any starting point \(Y_0 = u_0 \geq 0\), it is not hard to see that it suffices to establish those relations just for the value \(u_0 = 0\). In the Cramér case, the probability that the first time of buffer-overflow over the level \(x\) occurs before the end of the busy period (i.e. before the first time that \(Y\) reaches zero) tends to zero as \(x\) tends to infinity. Hence we may assume, by the strong Markov property of \(Y\) applied at the end time of the busy period, that we have \(Y_0 = 0\). In the positive drift case (i.e. when the queueing system is unstable), the probability of the complementary event that the buffer-overflow of size \(x\) occurs after the first visit of \(Y\) to the origin tends to zero (see Section 3.3.1). Hence, the proof of Theorem 1 yields that the joint limit law in (1.4) is equal to the asymptotic distribution, as \(x \uparrow \infty\), of the under- and overshoot of \(X\) (with \(X_0 = u_0\)) at the epoch of its first entrance into the set \((x, \infty)\). The latter limit law clearly does not depend on the starting level \(u_0\), due to the spatial homogeneity of the process \(X\).

The arguments outlined in the previous paragraph also imply that the limit distribution in (1.3) (and hence (1.5)) remains valid if \(Y\) is in its steady state, i.e. the workload process \(Y\) was started according to its stationary distribution, which exists since, under the Cramér assumption, \(Y\) is an ergodic strong Markov process and the corresponding queueing system is stable. Furthermore, it is worth noting that, in the Cramér case, the right-hand side of (1.3) (and hence that of (1.5)) is in fact also equal to the asymptotic distribution of the under- and overshoot conditional on the event that the buffer-overflow takes place in the busy period (this result follows directly by combining the proof of Theorem 2 below with the two-sided Cramér estimate for \(X\), see e.g. [17, Prop. 7]).

Various aspects of the law of the reflected process have been studied recently in a number of papers. The exact asymptotic decay of the distribution of the maximum of an excursion, under the Itô-exursion

\(^1\)Note that in the Cramér case it holds \(r^* > 0\), \(\lambda m < 1\) and in the (unstable) positive drift case we have \(r^* < 0\), \(\lambda m > 1\).
measure, was identified in \cite{3} under the Cramér condition. Also in the Cramér case, the joint asymptotic
distributions of the overshoot, the maximum and the current value of the reflected process were obtained in \cite{17}. In special cases a number of papers are devoted to the characterisation of the law of the reflected
process at the moment of buffer-overflow. For example, in the case of spectrally negative Lévy processes, the
joint Laplace-transform of the pair \((\tau(x), Y_{\tau(x)})\) was obtained in \cite{2}. A sex-tuple law extension of this
result, centred around the epoch of the first-passage of the reflected process, was given in \cite{16}.

The remainder of the paper is organised as follows: the main results are stated in Section \(\S 2\) and their
proofs are given in Section \(\S 3\). Appendix \(\text{A}\) contains the proof of Lemma \(\text{I}\) which plays an important role
in the proofs of Theorems \(\text{I}\) and \(\text{II}\) and is stated in Section \(\S 3.2\).

2. Joint limiting distributions

Let \(X\) be a Lévy process, that is, a stochastic process with independent and stationary increments and
càdlàg paths, with \(X(0) = 0\), and let \(Y = \{Y(t), t \geq 0\}\) be the reflected process of \(X\) at its infimum, i.e.

\(\begin{equation}
Y(t) = X(t) - \inf_{0 \leq s \leq t} X(s), \quad t \geq 0.
\end{equation}\)

The process \(Y\) crosses any positive level \(x\) in finite time almost surely, that is, the moment of first-passage
\(\tau(x) = \inf\{t \geq 0 : Y(t) \in (x, \infty)\}\)
is finite with probability 1, for any \(x > 0\). Denote by \(\Psi_x\) the joint (complementary) distribution function of the pair \((z(x), Z(x))\) of under- and overshoot of \(Y\),

\[
\Psi_x(u, v) = P(z(x) > u, Z(x) > v), \quad u, v \geq 0, \quad x > 0,
\]

where we defined \(z(x) = x - Y(\tau(x) -)\) and \(Z(x) = Y(\tau(x)) - x\).

Recall that the renewal function \(V : \mathbb{R}_+ \to \mathbb{R}_+\) of \(X\) is the unique non-decreasing right-continuous
function with the Laplace transform given by

\(\begin{equation}
\int_0^\infty \! e^{-sy} V(y) \, dy = (\theta \phi(\theta))^{-1}, \quad \text{where}
\end{equation}\)

\(\begin{equation}
\phi(s) = -\log E[e^{-sH(1)} I_{\{1 < L(\infty)\}}], \quad \text{for } s \geq 0,
\end{equation}\)

\(L\) denotes a local time of \(X\) at its running supremum \(X^*\), \(X^*(t) = \sup_{0 \leq s \leq t} X(s)\), with \(L(\infty) = \lim_{t \uparrow \infty} L(t)\), and \(H\) is the ascending ladder-height process of \(X\). The corresponding measure \(V(dy)\)
is the potential measure of \(H\), i.e. \(V(dy) = \int_0^\infty \! P(H(t) \in dy) \, dt\). Similarly \(\hat{L}, \hat{H}, \hat{\phi}\) and \(\hat{V}\) denote
the local time, the ladder process, its characteristic exponent and the renewal function of the dual process
\(\hat{X} \doteq -X\) respectively. We assume throughout the paper that \(L\) and \(\hat{L}\) are normalised in such a way that
\(-\log E[e^{i\theta X(1)}] = \phi(-i\theta) \phi(i\theta), \ \theta \in \mathbb{R}\), holds, and denote by \(\nu\) the Lévy measure of \(X\) and by 
\(\nu(a) = \nu((a, \infty)), a > 0\), its tail function. For the background on ladder processes and fluctuation theory
we refer to Bertoin \cite{4} Ch. VI).

The first limit result concerns the positive drift case:

**Theorem 1.** Let the law of \(X(1)\) be non-lattice and suppose \(E[|X(1)|] < \infty, E[X(1)] \in (0, \infty)\). Then \(\phi'(0) \in (0, \infty)\) and the limit \(\Psi_\infty(u, v) = \lim_{x \uparrow \infty} \Psi_x(u, v)\) exists and is given as follows:

\[
\Psi_\infty(u, v) = \frac{1}{\phi'(0)} \int_u^\infty \nu(v + z) \hat{V}(z) \, dz, \quad u, v \geq 0.
\]
In the negative drift case we will restrict ourselves to the classical Cramér setting.

**Assumption 1.** Suppose that the Cramér-assumption holds, i.e. there exists a $\gamma \in (0, \infty)$ such that $E[e^{\gamma X(1)}] = 1$, $X(1)$ is non-lattice with a finite mean and $E[|X(1)|e^{\gamma X(1)}] < \infty$.

In the case of negative drift the limiting distribution is given as follows:

**Theorem 2.** Let Ass. [1] hold. Then $\phi(0) \in (0, \infty)$ and the limit $\Psi_\infty(u, v) \doteq \lim_{x \uparrow \infty} \Psi_x(u, v)$ exists and is given by

$$
\Psi_\infty(u, v) = \frac{\gamma}{\phi(0)} \int_u^\infty \varphi(v + z) \hat{V}_\gamma(z) dz, \quad u, v \geq 0,
$$

where we denote $\hat{V}_\gamma(z) \doteq \int_{[0,z]} e^{\gamma(z-y)} \hat{V}(dy)$.

**Remarks.**

(i) If $X$ is spectrally positive (i.e. $\nu((-\infty, 0)) = 0$) with $\psi(\theta) = \log E[e^{-\theta X(1)}]$ and satisfies Ass. [1] we have

$$
\hat{V}(y) = \frac{y}{\hat{\phi}(0)}, \quad \phi(0) = \frac{\psi'(0)}{\phi(0)}, \quad \hat{V}_\gamma(z) = (\gamma \hat{\phi}(0))^{-1}(e^{\gamma z} - 1),
$$

where $\hat{\phi}(0) > 0$, $\gamma$ is the largest root of $\psi(-\theta) = 0$ and the second equality follows from the Wiener-Hopf factorisation $-\psi(-\theta) = \phi(\theta) \hat{\phi}(-\theta)$ and equality $\hat{\phi}(0) = 0$. The limit distribution $\Psi_\infty$ is given by

$$
\Psi_\infty(u, v) = \frac{1}{\psi'(0)} \int_u^\infty (e^{\gamma z} - 1) \varphi(v + z) dz.
$$

(ii) If $X$ is spectrally positive with $E[X(1)] \in (0, \infty)$, we have the identities (see e.g. [4], p. 191)

$$
\phi'(0) = -\frac{\psi'(0)}{\Phi(0)}, \quad \hat{V}(y) = \frac{1 - e^{-\Phi(0)y}}{\Phi(0)},
$$

where $\psi(\theta) = \log E[e^{-\theta X(1)}]$ and $\Phi(0) = \hat{\phi}(0) > 0$ is the largest root of the equation $\psi(\theta) = 0$. The joint asymptotic distribution of under- and overshoot is in this case given explicitly by the formula

$$
\Psi_\infty(u, v) = -\frac{1}{\psi'(0)} \int_u^\infty \varphi(v + y)(1 - e^{-\Phi(0)y}) dy.
$$

(iii) In Corollary 2(ii) of [17], the marginal law of $Z(\infty)$ was identified and the following expression for the overshoot was given:

$$
(2.4) \quad P[Z(\infty) > v] = \frac{\gamma}{\phi(0)} e^{-\gamma v} \int_{v, \infty} \varphi(z) \varphi_H(z) dz, \quad v \geq 0,
$$

where $\varphi_H(a) \doteq \nu_H((a, \infty))$, $a > 0$, is the tail of the Lévy measure $\nu_H$ of $H$. Combining this with Theorem 2 we find that the equality $\Psi_\infty(0, v) = P[Z(\infty) > v]$ holds for all $v \geq 0$. Indeed,

$$
\frac{\phi(0)}{\gamma} \Psi_\infty(0, v) = \int_{[0,\infty)} \hat{V}(dy) \int_y^\infty e^{\gamma(z-y)} \varphi(v+z) dz = \int_{[0,\infty)} \hat{V}(dy) \int_0^\infty e^{\gamma z} \varphi(v+z+y) dz
$$

$$
= \int_0^\infty e^{\gamma z} \int_{[0,\infty)} \varphi(v+z+y) \hat{V}(dy) = \int_0^\infty e^{\gamma(z-y)} \int_{[0,\infty)} \varphi(z+y) \hat{V}(dy),
$$

\[ \text{If a non-trivial Lévy process } X \text{ satisfies Ass. [1] then } E[X(1)] \in (-\infty, 0) \text{ since } u \mapsto E[e^{uX(1)}] \text{ is strictly convex on } [0, \gamma]. \]
which is equal to \( \frac{\Phi(0)}{\gamma} \mathbb{P}[Z(\infty) > v] \) by (2.3) and Vigon’s identity (2.5) (established in [20]) and relating the tail \( \Phi_H \) of the Lévy measure \( \nu_H \) of \( H \) to the dual renewal function \( \hat{V} \) and the upper tail \( \Phi(a) = \nu((a, \infty)) \), \( a > 0 \), of the Lévy measure \( \nu \) of \( X \):

\[
(2.5) \quad \Phi_H(a) = \int_{[0, \infty)} \Phi(a+y) \hat{V}(dy).
\]

In (2.5) the local times \( L \) and \( \hat{L} \) are normalised so that \( -\log E[e^{\theta X(1)}] = \phi(-\theta) \hat{\phi}(\theta) \) holds for \( \theta \in [0, \gamma] \) (see e.g. [20] Thm. 2.1 and the remark that follows the theorem), as is assumed throughout this paper.

(iv) The assumption (in Theorems 1 and 2) that \( X(1) \) is non-lattice is satisfied if the Lévy measure \( \nu \) of \( X \) is non-lattice or if either the drift or the Gaussian coefficient of \( X \) are non-zero.

3. Proofs

3.1. Setting. We next describe the setting of the remainder of the paper, and refer to [1] Ch. I for further background on Lévy processes. Let \((\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t \geq 0}, P)\) be a filtered probability space that carries a Lévy process \( X \). The sample space \( \Omega \triangleq D([0, \infty)) \) is taken to be the Skorokhod space of real-valued functions that are right-continuous on \( \mathbb{R}_+ \) and have left-limits on \((0, \infty)\), \( \{\mathcal{F}(t)\}_{t \geq 0} \) denotes the completed filtration generated by \( X \), which is right-continuous, and \( \mathcal{F} \) is the completed sigma-algebra generated by \( \{X(t), t \geq 0\} \).

For any \( x \in \mathbb{R} \) denote by \( P_x \) the probability measure on \((\Omega, \mathcal{F})\) under which the shifted process \( X - x \) is a Lévy process and by \( E_x \) the expectation under \( P_x \). Throughout we identify \( P \equiv P_0 \) and \( E \equiv E_0 \) and let \( I_A \) denote the indicator of a set \( A \).

3.2. Undershoots and overshoots of \( X \). An important step in the proof of Theorems 1 and 2 consists in the identification of the limiting joint distribution of the under- and overshoot of \( X \), given in Lemma 1 below. Let \( T(x) \triangleq \inf \{t \geq 0 : X(t) > x\} \) denote the first-passage time of \( X \) over the level \( x \). On the set \( \{T(x) < \infty\} \), the overshoot \( K(x) \) (resp. undershoot \( k(x) \)) of the process \( X \) at the level \( x \) is the distance between \( x \) and the positions of \( X \) at (resp. just before) the moment \( T(x) \):

\[
(3.1) \quad k(x) \triangleq x - X(T(x)-), \quad K(x) \triangleq X(T(x)) - x.
\]

For any \( x > 0 \) the joint (complementary) distribution of the pair \((k(x), K(x))\) is denoted by \( \Phi_x \), viz.

\[
\Phi_x(u, v) \triangleq P(k(x) > u, K(x) > v, T(x) < \infty), \quad u, v \geq 0.
\]

In the case \( P(T(x) < \infty) < 1 \), or equivalently, when \( X \) tends to \(-\infty \) (as is the case under As. 1), the distribution function \( \Phi_x \) is defective for any \( x > 0 \). Assuming \( P(T(x) < \infty) > 0 \) (as is the case under As. 1), consider the conditioned distribution \( \Phi_x^\# \) defined as follows:

\[
\Phi_x^\#(u, v) \triangleq P(k(x) > u, K(x) > v|T(x) < \infty).\]

Lemma 1. (i) Recall that \( \phi \) is defined in (2.3). Then, if \( X(1) \) is integrable with \( E[X(1)] \in (0, \infty) \), it holds \( \phi'(0) = E[H(1)] \in (0, \infty) \).

(ii) Suppose \( E[|X(1)|] < \infty \) and \( E[X(1)] \in (0, \infty) \) and let the law of \( X(1) \) be non-lattice. Then the limit \( \Phi_\infty(u, v) \triangleq \lim_{x \to \infty} \Phi_x(u, v) \) exists and is given as follows:

\[
(3.2) \quad \Phi_\infty(u, v) = \frac{1}{\phi'(0)} \int_u^\infty \Phi(v + z) \hat{V}(z)dz, \quad u, v \geq 0.
\]
(iii) Assume that As. 1 is satisfied. Then $\Phi^\#_{\infty}(u,v) = \lim_{x \to \infty} \Phi^\#_x(u,v)$ exists and is equal to
\begin{equation}
\Phi^\#_{\infty}(u,v) = \frac{\gamma}{\phi(0)} \int_{u}^{\infty} \nu(v + z) \hat{V}(z) dz, \quad u, v \geq 0,
\end{equation}
where $\hat{V}(z) = \int_{[0,z]} e^{\gamma(z-y)} \hat{V}(dy)$ is as defined in Theorem 2.

Remark. The marginal asymptotic distributions of the overshoot and undershoot of $X$ over a positive level under As. 1 (cf part (iii) of the lemma) were derived in [12, Thm. 4.1]. A direct proof of the existence of the joint limit law $(k(\infty), K(\infty))$ and its explicit description in Lemma 1(iii) is given in the appendix.

3.3. Proofs of Theorems 1 and 2. In this section we establish our main results.

3.3.1. Proof of Theorem 1. Fix $0 < M < x$ and $u, v \geq 0$. Define $\hat{T}(y) = \inf\{t \geq 0 : X(t) < -y\}$ for any $y \geq 0$. The strong Markov property of $Y$ implies the following:
\begin{align*}
\Psi_x(u,v) &= E[\mathbf{1}_{\{Y(\tau(M)) < x\}} P[A_{u,v}(x), \tau(x) < \tau_0|Y(\tau(M))]] \\
&+ P[Y(\tau(M)) \geq x, A_{u,v}(x)] + E[P[A_{u,v}(x), \tau(x) > \tau_0|Y(\tau(M))]],
\end{align*}
where $A_{u,v}(x) \equiv \{z(x) > u, Z(x) > v\}$ and $\tau_0 = \inf\{t \geq 0 : Y(t) = 0\}$. Since, for any $y > 0$, the processes $\{Y(t), Y(0) = y, t \leq \tau_0\}$ and $\{X(t), X(0) = y, t \leq \hat{T}(0)\}$ are equal in law, for any $z \in [M, x)$ we have:
\begin{align*}
P[A_{u,v}(x), \tau(x) < \tau_0|Y(\tau(M)) &= \Phi_{x-z}(u,v) - P_z[B_{u,v}(x), \hat{T}(0) < T(x)], \\
P[A_{u,v}(x), \tau(x) > \tau_0|Y(\tau(M)) &= P_z[\hat{T}(0) < T(x)] - P_z[\hat{T}(0) < \infty] \leq P_M[\hat{T}(0) < \infty],
\end{align*}
where we denote $P_z[.] = P[\cdot|X_0 = z]$ and $B_{u,v}(x) = \{b(x) > u, K(x) > v, T(x) < \infty\}$ (see (3.1)). Since $\{Y(\tau(M)) \geq x, A_{u,v}(x)\} \subset \{\tau(M) = \tau(x)\}$ and $P[\hat{T}(M) < \infty] = P_M[\hat{T}(0) < \infty]$, the inequalities above yield the following estimate:
\begin{align*}
|\Psi_x(u,v) - \Phi_x(u,v)| &\leq 2P[\hat{T}(M) < \infty] + P[\tau(M) = \tau(x)] \\
&+ E[\mathbf{1}_{\{Y(\tau(M)) < x\}} \Phi_{x-Y(\tau(M))}(u,v)] - \Phi_x(u,v).
\end{align*}
Lemma 2(ii) implies $\mathbf{1}_{\{Y(\tau(M)) < x\}} \Phi_{x-Y(\tau(M))}(u,v) \to \Phi_{\infty}(u,v)$ $P$-a.s. as $x \to \infty$. Also $\mathbf{1}_{\{\tau(M) = \tau(x)\}} \to 0$ $P$-a.s. as $x \to \infty$ and the dominated convergence theorem yields the following estimate for any $M > 0$:
\begin{equation}
\limsup_{x \to \infty} |\Psi_x(u,v) - \Phi_x(u,v)| \leq 2P[\hat{T}(M) < \infty].
\end{equation}
Since $E[X(1)] > 0$, $X$ drifts to $+\infty$. Hence $P[\hat{T}(M) < \infty] \to 0$ as $M \to \infty$ and the theorem follows. \hfill \Box

3.3.2. Proof of Theorem 2. In this section we assume throughout that As. 1 is satisfied. The proof is based on Itô-exursion theory. Refer to [4, Chs O, IV] for a treatment of Itô-exursion theory for Lévy processes and for further references.

Denote by $\epsilon = \{\epsilon(t), t \geq 0\}$ the excursion process of $Y$ away from zero. Since, under As. 1 $Y$ is a recurrent strong Markov process under $P$, Itô’s characterisation yields that $\epsilon$ is a Poisson point process under $P$. Its intensity measure under $P$ will be denoted by $\nu$. Let $\zeta(\epsilon)$ denote the lifetime of a generic excursion $\epsilon$ and let $\rho(x, \epsilon)$ denote the first time that the excursion $\epsilon$ enters $(x, \infty)$, viz.
\begin{equation}
\rho(x, \epsilon) = \inf\{t \geq 0 : \epsilon(t) > x\}.
\end{equation}
In the sequel we will drop the dependence of $\rho(x, \varepsilon)$ and $\zeta(\varepsilon)$ on $\varepsilon$, and write $\zeta$ and $\rho(x)$, respectively.

Theorem 2 follows directly by combining Lemmas 2 and 3 below.

Lemma 2. For any $u, v \geq 0$ and $x > 0$ the following holds true:

\[
P(z(x) > u, Z(x) > v) = n(E_{u,v}(x)|\rho(x) < \zeta) = \frac{n(E_{u,v}(x), \rho(x) < \zeta)}{n(\rho(x) < \zeta)},
\]

where $E_{u,v}(x) = \{\nu - \varepsilon(e(\nu)\varepsilon), \varepsilon(\rho(x)) - x > v, \rho(x) < \zeta\}$.

Proof of Lemma 2. By [4, Ch. O, Prop. 2], for sets $A, B$ with $n(A) \in (0, \infty)$, we have $P(\varepsilon(T_A) \in B) = n(B|A) = n(A \cap B)/n(A)$ where $T_A = \inf\{t \geq 0 : \varepsilon(t) \in A\}$. The lemma now follows by noting that the left-hand side of (3.5) is the probability that the first excursion in $A = \{\rho(x) < \zeta\}$ is in $B = E_{u,v}(x)$.

Lemma 3. Let $u, v \geq 0$ and recall $\hat{V}_\gamma(z) = \int_{[0,1]} e^{\gamma(z-y)} \hat{V}(dy)$. The following holds true:

\[
\lim_{x \to \infty} n(E_{u,v}(x)|\rho(x) < \zeta) = \frac{\gamma}{\phi(0)} \int_u^\infty \varphi(v + z) \hat{V}_\gamma(z) dz.
\]

Remarks. (i) The proof of Lemma 3 uses the following facts, which hold by [5] and [8], respectively, if 0 is regular for $(0, \infty)$ under the law of $X$ and As. 1 is satisfied:

\[
P(T(x) < \infty) \sim C_\gamma e^{-\gamma x} \quad \text{as} \quad x \to \infty, \quad \text{where} \quad C_\gamma = \frac{\phi(0)}{\gamma \phi'(0)}.
\]

The following holds true:

\[
n(\rho(x) < \zeta) \sim C_\gamma \hat{\phi}(\gamma) e^{-\gamma x} \quad \text{as} \quad x \to \infty.
\]

Here and throughout the paper we write $f(x) \sim g(x)$ as $x \to \infty$ if $\lim_{x \to \infty} f(x)/g(x) = 1$.

(ii) A further ingredient of the proof of Lemma 3 are the following asymptotic identities, established in [17, Lemma 9]:

\[
n(e^{\gamma(x)}(\rho(x))) \mathbf{1}_{(\rho(x) < \zeta)} \sim \hat{\phi}(\gamma) \quad \text{as} \quad x \to \infty,
\]

\[
e^{\gamma x} n(\varepsilon(\rho(x)) > x, \rho(x) < \zeta) = O(1) \quad \text{as} \quad x \to \infty, \text{for any} \quad z > 0.
\]

(iii) The key observation in [5] is that $V(dz)$ is a renewal measure corresponding to the distribution $P[H(\Theta) \in dz]$, where $\Theta$ is an exponential random variable with $E[\Theta] = 1$, independent of $X$ (and hence of $H$). Estimate (3.6) then follows from the classical renewal theorem for non-lattice random walks with the step-size distribution $H(\Theta)$, which needs to be non-lattice for the theorem to be applicable (see the conclusion of the proof of the Theorem in [5] for this argument and [10, p. 363] for the statement of the renewal theorem). The assumption in [5], which ensures this, stipulates that 0 is regular for $(0, \infty)$ under the law of $X$. Note that this assumption also implies the non-lattice condition As. 1.

Furthermore, if $X(1)$ is non-lattice, so is $H(\Theta)$ (indeed, if $H(\Theta)$ were lattice, Theorem 30.10 in [10] would yield that $H$ is a compound Poisson process, necessarily with a Lévy measure that has lattice support, hence implying that $X$ itself is a compound Poisson process with a Lévy measure that has lattice support). Since the argument in [5] only requires $H(\Theta)$ to be non-lattice, the estimate in (3.6) remains valid under As. 1.

Thus the estimate in (3.6) holds in our setting. Likewise, the argument in [8] relies solely on the fact that $V(dz)$ is a renewal measure of a non-lattice law and therefore estimate (3.7) also holds under As. 1.
Proof of Lemma 3: Fix $M > 0$ and pick $u, v \geq 0$. The proof starts from the elementary observation that relates the following two conditional $n$-measures:

\begin{equation}
(3.10) \quad n(E_{u,v}(x) | \rho(x) < \zeta) = n(E_{u,v}(x) | \rho(M) < \zeta) \cdot \frac{n(\rho(M) < \zeta)}{n(\rho(x) < \zeta)}, \quad x > M.
\end{equation}

Recall that the coordinate process under the probability measure $n(\cdot | \rho(M) < \zeta)$ has the same law as the first excursion of $Y$ away from zero with height larger than $M$. The strong Markov property under $n(\cdot | \rho(M) < \zeta)$ implies that $\varepsilon \circ \theta_{\rho(M)}$ has the same law under $n(\cdot | \rho(M) < \zeta)$ as the coordinate process of $X$ under $P$, with entrance law $\mu_M(dy) = n(\varepsilon(\rho(M))) \in dy | \rho(M) < \zeta)$, that is killed upon its first entrance into $(-\infty, 0)$. Recall $\hat{T}(y) = \inf\{t \geq 0 : X(t) < -y\}$, for $y \geq 0$, and note that for every $x > M$ we have:

\begin{equation}
(3.11) \quad n(E_{u,v}(x) | \rho(M) < \zeta) = \int_{[M,x]} P_z[B_{u,v}(x)] \mu_M(dz) + O_{u,v}(x),
\end{equation}

where $B_{u,v}(x) = \{k(x) > u, K(x) > v, T(x) < \infty\}$ and $O_{u,v}(x)$ is given by the following expression:

\[ O_{u,v}(x) = n(E_{u,v}(x), \varepsilon(\rho(M)) > x | \rho(M) < \zeta) - \int_{[M,x]} P_z[B_{u,v}(x), \hat{T}(0) < T(x) < \infty] \mu_M(dz). \]

Note that $P_z[B_{u,v}(x), \hat{T}(0) < T(x) < \infty] \leq P_z[\hat{T}(0) < \hat{T}(x-z) < \infty] \sim C \gamma e^{-\gamma z} E[e^{\gamma(X(\hat{T}(z)+z))}]$ as $x \to \infty$.

The following facts hold: $X(\hat{T}(z)) + z \leq 0$, the measure $\mu_M(dy)$ is concentrated on $[M, \infty)$ with $\mu_M([M, \infty)) = 1$ for any $M > 0$ and equality (3.9) is satisfied. Hence, for a fixed $M > 0$, we have

\begin{equation}
(3.12) \quad -C \gamma e^{-\gamma x} \leq O_{u,v}(x) \leq e^{-\gamma x} o(1) \quad \text{as} \ x \to \infty.
\end{equation}

By Lemma 1 (iii) we have $\lim_{x \to \infty} P_z[k(x) > u, K(x) > v | T(x) < \infty] = \Phi^\#(u, v)$ for any fixed $z \geq 0$. Equality (5.6) implies $P_z[T(x) < \infty] = \Phi^\#(u, v) = e^{-\gamma z} C \gamma e^{-\gamma z} (1 + r(x-z))$ as $x \to \infty$ for any $z \geq 0$, where $r : \mathbb{R}_+ \to \mathbb{R}$ is bounded and measurable with $\lim_{x' \to \infty} r(x') = 0$. By (3.8), $z \to e^{\gamma z}, z \in [M, \infty)$, is in $L^1(\mu_M)$ for all large $M$. The dominated convergence theorem and (3.7) therefore imply:

\begin{equation}
(3.13) \quad \lim_{x \to \infty} \int_{[M,x]} \frac{P_z[B_{u,v}(x)]}{\mu_M(dz)} \mu_M(dz) = \lim_{x \to \infty} \int_{[M,x]} \frac{P_z[B_{u,v}(x)]}{\mu_M(dz)} \mu_M(dz).
\end{equation}

Recall that $E[X(1)] < 0$ by As. 1 and hence $\hat{\phi}(\gamma) > 0$ since $\hat{H}$ is a non-trivial subordinator. Hence (3.7), (3.10), (3.11), (3.12) and (3.13) imply the following inequalities for any fixed $M > 0$:

\[-\hat{\phi}(\gamma)^{-1} n(\rho(M) < \zeta) \leq \liminf_{x \to \infty} n(E_{u,v}(x) | \rho(x) < \zeta) - \Phi^\#(u, v) \cdot n(e^{\gamma z} \rho(M)) \mathbb{I}_{\rho(M) < \zeta} \cdot \hat{\phi}(\gamma)^{-1} \leq \limsup_{x \to \infty} n(E_{u,v}(x) | \rho(x) < \zeta) - \Phi^\#(u, v) \cdot n(e^{\gamma z} \rho(M)) \mathbb{I}_{\rho(M) < \zeta} \cdot \hat{\phi}(\gamma)^{-1} \leq 0.\]

Since these inequalities hold for all large $M > 0$, in the limit as $M \to \infty$ equation (3.8) implies $\lim_{x \to \infty} n(E_{u,v}(x) | \rho(x) < \zeta) = \Phi^\#(u, v)$. This, together with Lemma 1 (iii), concludes the proof. □
Appendix A. Proof of Lemma 1

Proof. (i) $E[X(1)] > 0$ implies $X(t) \to \infty$ P-a.s. as $t \uparrow \infty$. Hence $E[H(1)] \in (0, \infty]$ and $E[\hat{L}(\infty)] = 1/\hat{\phi}(0) < \infty$. Since $E[X(1)] < \infty$, we have $\int_{[1,\infty)} y\nu(dy) < \infty$. By definition we have $\hat{V}(\infty) = \lim_{y \to \infty} \hat{V}(y) = E[\hat{L}(\infty)]$. Fabuini’s theorem, the estimate $\int_{[1,\infty)} z\nu(y + dz) \leq \int_{[1,\infty)} (z + y)\nu(y + dz) \leq \int_{[1,\infty)} x\nu(dx) < \infty$ for any $y \geq 0$ and (2.5) imply

$$\int_{[1,\infty)} y\nu_H(dy) = \int_{[0,\infty)} \hat{V}(dy) \int_{[1,\infty)} z\nu(y + dz) \leq \hat{V}(\infty) \int_{[1,\infty)} x\nu(dx) < \infty,$$

and part (i) of the lemma follows.

(ii) The compensation formula applied to the Poisson point process \{\Delta X(t), t \geq 0\} (here $X(0-) = 0$ and $\Delta X(t) = X(t) - X(t-) \geq 0$ for $t \geq 0$) and the form of the resolvent of $X$ killed upon entering $(x, \infty)$ (see [4, p.176]) imply the following identity (recall $\mathcal{P}(a) = \nu((a, \infty))$, $a > 0$, is the tail of the Lévy measure $\nu$):

$$\Phi_x(u, v) = E \sum_{t>0} \mathbf{I}_{\{x^+(t) < x, x - X(t-) > u, X(t-) + \Delta X(t) - x > v\}}$$

$$= E \int_0^\infty \overline{\nu}(x + X(t)) \mathbf{I}_{\{x - X(t-) > u, x^+(t) < x\}} dt = E \int_0^\infty \overline{\nu}(x + X(t)) \mathbf{I}_{\{x - X(t+) > u\}} dt$$

$$= \int_{[0,x]} \overline{\mathcal{F}}(x - z)V(dz),$$

where $\overline{\mathcal{F}}$ is given by the expression

$$\overline{\mathcal{F}}(z) = \int_{[0,\infty)} \overline{\nu}(v + z + y) \mathbf{I}_{\{z + y > u\}} \hat{V}(dy), \quad \text{for any } z \geq 0,$$

and the function $V$ is defined in (2.2) (an argument based on the quintuple law from [7, Thm. 3] can also be applied to establish (A.1)). The function $z \to \overline{\nu}(v + z)$ is integrable on $(0, \infty)$ by the assumption that $E[|X(1)|] < \infty$ and the inequality $\hat{V}(\infty) < \infty$ holds (see e.g. proof of Lemma 1(i) above). Hence the inequalities $0 \leq \overline{\mathcal{F}}(z) \leq \overline{\nu}(z + v) \hat{V}(\infty)$ hold for all $z \geq 0$, making the function $\overline{\mathcal{F}}$ directly Riemann integrable as defined in [10, Definition on p. 362].

Let $\Theta$ be independent of $H$ and exponentially distributed with $E[\Theta] = 1$. The law $P[H(\Theta) \in dz]$ has the mean equal to $\phi'(0)$. By Remark (iii) following Lemma 3 the renewal theorem in [10, Thm. on p. 363] and (A.1) imply that $\Phi_\infty(u, v) = \lim_{x \to \infty} \Phi_x(u, v)$ exists and is equal to

$$\Phi_\infty(u, v) = \frac{1}{\phi'(0)} \int_{[0,\infty)} \overline{\mathcal{F}}(z)dz.$$

The definition of $\overline{\mathcal{F}}$ in (A.2) and several applications of Fubini’s theorem yield the following:

$$\int_{[0,\infty)} \overline{\mathcal{F}}(z)dz = \int_{[0,\infty)} dz \int_{[u - z, \infty)} \overline{\nu}(v + z + y) \hat{V}(dy) + \int_{[u, \infty)} dz \int_{[0,\infty)} \overline{\nu}(v + z + y) \hat{V}(dy)$$

$$= \int_{[0,\infty)} \hat{V}(dy) \int_{[u - y, \infty)} \overline{\nu}(v + z + y)dz + \int_{[0,\infty)} \hat{V}(dy) \int_{[u, \infty)} \overline{\nu}(v + z + y)dz$$

$$= \int_{[0,\infty)} \hat{V}(dy) \int_{[u - y, \infty)} \overline{\nu}(v + z + y)dz,$$
where as usual \((u - y)^+ = \max\{u - y, 0\}\). The equality in (A.3) and further applications of the Fubini
theorem imply part (ii) of the lemma:

\[
\int_{[0,\infty)} \mathcal{F}(z) dz = \int_{[0,u]} \hat{V}(dy) \int_{[u-y,\infty)} \mathcal{P}(v + z + y) dz + \int_{(u,\infty)} \hat{V}(dy) \int_{[0,\infty)} \mathcal{P}(v + z) dz
\]

\[
= \int_{[0,u]} \hat{V}(dy) \int_{[u,\infty)} \mathcal{P}(v + z) dz + \int_{(u,\infty)} \hat{V}(dy) \int_{(u,\infty)} \mathcal{P}(v + z) dz
\]

\[
= \hat{V}(u) \int_{[u,\infty)} \mathcal{P}(v + z) dz + \int_{[u,\infty)} \hat{V}(dy) \int_{[u,\infty)} \mathcal{P}(v + z) dz
\]

\[
= \hat{V}(u) \int_{[u,\infty)} \mathcal{P}(v + z) dz + \int_{[u,\infty)} (\hat{V}(z) - \hat{V}(u)) \mathcal{P}(v + z) dz
\]

\[
= \int_{[u,\infty)} \hat{V}(z) \mathcal{P}(v + z) dz.
\]

(iii) Let \(P^{(\gamma)}\) be the Cramér measure on \((\Omega, \mathcal{F})\), the restriction of which to \(\mathcal{F}(t)\) is defined by \(P^{(\gamma)}(A) = E[e^{\gamma X(t)} 1_A]\) for any \(A \in \mathcal{F}(t), t \in \mathbb{R}_+\). Under \(P^{(\gamma)}\) it holds \(E^{(\gamma)}(|X(1)|) = E(|X(1)|e^{\gamma X(1)}) < \infty\) and \(E^{(\gamma)}(X(1)) > 0\) and hence \(P^{(\gamma)}(T(x) < \infty) = 1\). Define \(\Phi^{(\gamma)}(u, v) = P^{(\gamma)}(k(x) > u, K(x) > v, T(x) < \infty)\) for any \(u, v \geq 0\). Changing the measure yields

\[
\Phi^{(\gamma)}(u, v) = e^{-\gamma x} E^{(\gamma)}[e^{-\gamma K(x)} 1_{\{k(x) > u, K(x) > v, T(x) < \infty\}}] = e^{-\gamma x} \int_{(v,\infty)} e^{-\gamma w} \Phi^{(\gamma)}(u, dw).
\]

By part (ii) of the lemma, the limit \(\Phi^{(\gamma)}(u, v) \to \Phi^{(\gamma)}(u, v)\), as \(x \to \infty\), exists for all \(u, v \geq 0\). Assume first \(\Phi^{(\gamma)}(u, v) > 0\) and note that the probability measures \(1_{\{w > v\}} \Phi^{(\gamma)}(u, dw)/\Phi^{(\gamma)}(u, v)\) on \(\mathbb{R}\) converge weakly to the probability measure \(1_{\{w > v\}} \Phi^{(\gamma)}(u, dw)/\Phi^{(\gamma)}(u, v)\) as \(x \to \infty\). Hence [9, Thm.3.9.1(vi)] applied to the bounded function \(w \mapsto 1_{\{w > v\}} e^{-\gamma w}\), the Cramér’s asymptotics in (3.6) and Lemma 1 (ii) imply the following equalities

\[
(A.4) \quad \lim_{x \to \infty} \Phi^{(\gamma)}(u, v) = C^{-1} \int_{(v,\infty)} e^{-\gamma w} \Phi^{(\gamma)}(u, dw) = \frac{C^{-1}}{\phi'(0)} \int_{(v,\infty)} e^{-\gamma w} \nu(y + dw) \hat{V}(y) dy.
\]

In the case \(\Phi^{(\gamma)}(u, v) = 0\) we note \(\int_{(v,\infty)} e^{-\gamma w} \Phi^{(\gamma)}(u, dw) \leq \Phi^{(\gamma)}(u, v)\). Hence by (3.6) and Lemma 1 (ii) we find \(\lim_{x \to \infty} \Phi^{(\gamma)}(u, v) = \lim_{x \to \infty} \frac{e^{-\gamma x}}{\mathcal{P}(x)} \int_{(v,\infty)} e^{-\gamma w} \Phi^{(\gamma)}(u, dw) = 0\). Therefore the first equality in (A.4) holds also in the case \(\Phi^{(\gamma)}(u, v) = 0\).

The Wiener-Hopf factorisation [4] p. 166, Eqn. (4)] implies \(\phi(\theta) = \phi(\theta - \gamma)\) and \(\hat{\phi}(\theta + \gamma)\) for all \(\theta \geq 0\). The elementary equality \(\nu(y) = e^{\gamma y}\) and the form of \(C^{-1}\) given in (3.6) therefore yield

\[
\lim_{x \to \infty} \Phi^{(\gamma)}(u, v) = C^{-1} \int_{(v,\infty)} \hat{V}(y) dy.
\]

The Laplace transform of \(\hat{V}(\gamma)\) is given by \([\phi(\hat{\phi}(\theta + \gamma))]^{-1} = [\hat{\phi}(\theta + \gamma)]^{-1}\) (cf. (22)). It follows that the Laplace transforms of the function \(y \mapsto e^{\gamma y} \hat{V}(\gamma)(y)\) and the convolution \(y \mapsto \hat{V}(\gamma)(y) = \int_{[0,y]} e^{\gamma(y-z)} \hat{V}(dz)\) are both equal to \([\theta - \gamma \hat{\phi}(\theta)]^{-1}\) (recall that \(\int_{(0,\infty)} e^{-\gamma z} \hat{V}(dz) = 1/\hat{\phi}(\theta)\)). Hence the two functions can only differ on a set with at most countably many points, which has Lebesgue measure zero. Therefore the formula in (3.3) and the lemma follow.
References

[1] Asmussen, S (2003). Applied probability and queues, volume 51 of Applications of Mathematics (New York). Springer-Verlag, New York, second edition. Stochastic Modelling and Applied Probability.

[2] Avram, F., Kyprianou, A.E. and Pistorius, M.R. (2004). Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. Ann. Appl. Probab. 14:215–238.

[3] Baardoux, E. (2009). Some excursion calculations for reflected Lévy processes. ALEA, 6:149-162.

[4] Bertoin, J. (1996). Lévy processes, volume 121 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge.

[5] Bertoin, J. and Doney, R. A. (1994). Cramér’s estimate for Lévy processes. Statist. Probab. Lett., 21:363–365.

[6] Chydzinski, A. (2007). Time to reach buffer capacity in a BMAP queue. Stochastic Models, 23:195–209.

[7] Doney, R. A., and Kyprianou, A.E. (2006). Overshoots and undershoots of Lévy processes. Ann. Appl. Prob. 16, 91–106.

[8] Doney, R. A. and Maller, R. A. (2005). Cramér’s estimate for a reflected Lévy process. Ann. Appl. Probab., 15:1445–1450.

[9] Durrett, R. (2010). Probability: Theory and Examples. Cambridge University Press, Cambridge, fourth edition.

[10] Feller, W. (1971). An introduction to probability theory and its applications Vol. II. 2nd edition. John Wiley & Sons Inc., New York.

[11] Griffin, P.S. and Maller, R.A. (2012). Path decomposition of ruinous behaviour for a general Lévy insurance risk process. Ann. Appl. Probab., 22(4):1411–1449.

[12] Griffin, P.S., Maller, R. A., and van Schaik, K. (2011). Asymptotic distributions of the overshoot and undershoots for the Lévy insurance risk process in the Cramér and convolution equivalent case. arXiv:1106.3292.

[13] Kempa, W.M. (2012). On the distribution of the time to buffer overflow in a queueing system with a general-type input stream. Telecommunications and Signal Processing (TSP), 2012 35th International Conference Proceedings, 207–211, 3-4 July 2012, doi: 10.1109/TSP.2012.6256283

[14] Klüppelberg, C., Kyprianou, A. E. and Maller, R. A. (2004). Ruin probabilities and overshoots for general Lévy insurance risk processes. Ann. Appl. Probab., 14:1766–1801

[15] Kyprianou, A.E. (2006). First passage of reflected strictly stable processes. ALEA, 2:119–123.

[16] Mijatović, A. and Pistorius, M.R. (2012). On the drawdown of completely asymmetric Lévy processes Stoch. Proc. Appl. 122:3812–3836.

[17] Mijatović, A. and Pistorius M.R. (2013). Joint asymptotic distribution of certain path functionals of the reflected process. Submitted. arXiv:1306.6746

[18] Prabhhu, N. U. (1998) Stochastic storage processes, volume 15 of Applications of Mathematics (New York), Springer Verlag, New York, second edition.

[19] Sato, K. (1999). Lévy processes and infinitely divisible distributions, Cambridge University Press, Cambridge.

[20] Vigné, V. (2002). Votre Lévy rampe-t-il? J. London Math. Soc. 65:243–256.

Department of Mathematics, Imperial College London
E-mail address: a.mijatovic@imperial.ac.uk

Department of Mathematics, Imperial College London
E-mail address: m.pistorius@imperial.ac.uk