Chiral Determinant as an Overlap of Two Vacua

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Abstract

The effective action induced by chiral fermions can be written, formally, as an overlap of two states. These states are the Fock ground states of Hamiltonians for fermions in even dimensional space with opposite sign mass terms coupled to identical static vector potentials. A perturbative analysis of the overlap in the continuum framework produces the correct anomaly for Abelian gauge fields in two dimensions. When a lattice transfer matrix formalism is applied in the direction perpendicular to a domain wall on which chiral fermions live a lattice version of the overlap is obtained. The real part of the overlap is nonperturbatively defined and previous work indicates that the real part of the vacuum polarization tensor in four dimensions has the correct continuum limit for a chiral theory. The phase of the overlap represents the imaginary part of the chiral action and suffers from ambiguities.
1. Introduction

The existence of asymptotically free chiral gauge theories outside perturbation theory, with fermionic matter in an anomaly-free complex representation of the gauge group, has neither been established nor been disproved. Lattice techniques have not been successful so far [1], although they work quite well for vector gauge theories. On the one hand gauge invariant regularizations do not seem to distinguish between anomalous and anomaly-free theories and fail equally in both cases. On the other hand it is very difficult to give a final answer to the question of non-perturbative existence if the regularization breaks the gauge symmetry and fine tuning is necessary. Current approaches break the gauge symmetry at some stage and work under the assumption that a fully interacting continuum limit exists [2]. If indeed, theories made consistent perturbatively by delicate anomaly cancelations do not exist, it would mean that there are unremovable cutoff dependent terms in the theory; a situation somewhat similar to theories suffering from “triviality” ambiguities due to couplings that are not asymptotically free.

Recently, a new approach to the non-perturbative formulation of chiral gauge theories was developed. A simple way to understand it is to start from the observation that a chiral fermion is a zero eigenstate of the mass matrix $M$, while its partner of opposite chirality is a zero eigenstate of $M^\dagger$. If one makes $M$ infinite dimensional, one can choose it so that its index is unity [3,4]. In the continuum, an application of this approach that works to all orders in perturbation theory is achieved by using an infinite number of Pauli-Villars fields with heavy masses and alternating statistics [4,5]. In this paper, following up on the idea of Kaplan [3], we focus on a lattice realization of $M$ that avoids the doubling problem [6]. We extend our earlier work [4] to arrive at a particular representation of the chiral determinant produced by the integration over all the fermion fields. This representation has a general and suggestive structure and we propose it as a direction for more research.

Kaplan [3] was motivated by Callan and Harvey [7] and employed five dimensional gauge fields when targeting a four dimensional chiral theory. This approach encounters problems because the supersymmetry protecting the index of $M$ seems to break either due to strong fluctuations in the gauge fields in the fifth direction [8,9] or by boundary conditions [10]. We have the impression that [11] also speaks against fifth direction gauge fields. Suspecting problems one could encounter with these extra unwanted degrees of freedom, the gauge fields were kept strictly four dimensional and copied to all slices in
the fifth dimension in [4]. The extra fermion fields then look like regulators. Dropping the fifth dimensional character of the gauge fields was also suggested by Kaplan [12], but he introduced two Higgs like degrees of freedom in order to keep the fifth direction finite and to still have the gauge fields couple only to one fermion zero mode. This brings the model close to the Yukawa approach and this is a cause of worry [13].

We continue here to advocate keeping the dimension of $M$ strictly infinite. We aim at a well defined theory dealing directly with an infinite number of fermion fields without ever resorting to a limiting procedure. Alternative approaches seem to run into difficulties at too early a stage and for too indirect a reason for us to feel that they really carry a message about the existence of asymptotically free, anomaly free, chiral gauge theories.

Working with an $M$ of unit index forces us to deal with an infinity in the theory. The main idea in overcoming the infinity is based on the realization that the infinity that one encounters in the path integral formalism here is the same as the infinity one would encounter in a path integral of a quantum mechanical system with an infinite extent in time. But the latter infinity is just a way for the path integral to create an exact projection operator on the ground state of the system. In the problem at hand, namely the “wall” realization of chiral fermions, we have two transfer matrices describing the propagation on the two sides of the wall. An infinite extent on both sides of the wall exactly projects out the ground states of the two transfer matrices leaving us with an overlap of these two ground states as the effective gauge action induced by the fermions. The absolute value of the overlap is well defined and provides a gauge invariant regularized formula for the real part of the chiral action.

However, even ignoring possible degeneracies of the ground states, all one gets are projection operators on one dimensional subspaces, or “rays”. The phase of the overlap is left undetermined; in Minkowski space this phase turns into the parity violating part of the effective chiral action and as such is the most distinguishing feature of the theories we are after. In perturbation theory one usually computes the perturbed states together with their phases by fixing in some convenient way the necessarily present freedoms one encounters order by order. For example, in Brillouin–Wigner perturbation theory one simply requires the overlap of the perturbed space with the unperturbed one to be unity and fixes the norm (if one wishes) only at the end. We shall show that in a continuum version of the overlap, if one uses a specific choice to fix the ambiguity, say Brillouin–Wigner perturbation theory, the correct perturbative
anomaly is obtained in the simplest case of two dimensions and abelian gauge

group. Any other choice of fixing the ambiguity that differs from the above by
terms local in gauge fields will also reproduce the anomaly. Thus, the phase

ambiguity, at least in perturbation theory, seems to be controllable. If one

picks other conventions for the phase that differ by nonlocal terms from the

above, then gauge invariance may be restored (for example one could make the

phases constant on gauge orbits by construction) and one most likely ends up

with the wrong theory; another way to get a wrong theory is to simply define

the phase away, by making the overlap real. Unfortunately we can’t offer one

“good” nonperturbative choice that will likely produce the correct continuum

limit if one exists, and convince us beyond reasonable doubt that one doesn’t

if it fails. This is a subject for future research.

In section 2, starting from the path integral formalism on the lattice, we

extract the two transfer matrices and cast the effective action as a simple over-

lap between the two ground states. This representation can be easily extended
to the continuum. We show that, indeed, one can make a formal connection
between the overlap and a certain regularization of the chiral determinant.
This connection contains some indications for the conditions under which the

overlap would capture the phase of the determinant.

In section 3, we calculate the continuum version of the overlap in two
dimensions using perturbation theory. We focus on the imaginary part and

separate the pieces undetermined by perturbation theory from the unambigu-

ous ones. In other words, the computation is parameterized in terms of some
unknowns representing the ambiguity. Within this parameterization the un-

ambiguous piece has a non-local contribution and the anomaly resulting from
this term alone has the right form with “consistent” normalization. As long as
the ambiguous pieces are local in the gauge fields they won’t alter the result.
A discussion of consistent and covariant anomalies ends this section.

In the last section, we discuss various issues and propose future directions.
2. Transfer matrix and the overlap formula

In the “wall” realization, it is best to think of the infinite copies of the fermions, labeled by $s$, as living in an extra “dimension”. The fermionic action is [4]

$$S_F(\bar{\psi}, \psi, U) = \frac{1}{2} \sum_{n,s,\mu} \bar{\psi}_{n,s}(1 + \gamma_\mu)U_{n,\mu}\psi_{n+\mu,s} + \frac{1}{2} \sum_{n,s,\mu} \bar{\psi}_{n,s}(1 - \gamma_\mu)U^{\dagger}_{n-\mu,\mu}\psi_{n-\mu,s} + \frac{1}{2} \sum_{n,s} \bar{\psi}_{n,s}(1 + \gamma_5)\psi_{n,s+1} + \frac{1}{2} \sum_{n,s} \bar{\psi}_{n,s}(1 - \gamma_5)\psi_{n,s-1} - \sum_{n,s} [5 - m \text{ sign}(s + \frac{1}{2})] \bar{\psi}_{n,s}\psi_{n,s} \tag{2.1}$$

$\bar{\psi}_{n,s}$ and $\psi_{n,s}$ are Dirac spinors whose entries are elements of a Grassmann algebra. $n$ is a four component integer labeling sites on the lattice and $s$ labels the infinite copies of fermions. $n$ can run over a finite set of integers or an infinite set. When it runs over a finite set, which will be the case of practical interest, the boundary conditions on the fermions will be chosen to be periodic or anti-periodic. The first two lines in the equation (2.1) contain the usual gauge invariant coupling of the fermions to the gauge fields. The last three lines represent the mass matrix $M$ in the “wall” realization.

We would like to define the action, $S_{\text{eff}}(U)$, induced by the fermions via the usual path integral:

$$e^{S_{\text{eff}}(U)} = \int \prod_{n,s} d\bar{\psi}_{n,s}\psi_{n,s} e^{S_F(\bar{\psi}, \psi, U)}, \tag{2.2}$$

However, due to the infinite $s$-extent and the homogeneity of the gauge field, $S_{\text{eff}}(U)$ diverges and the divergence is $U$-dependent. † The action has a bulk contribution and an interface (at $s = 0$) contribution and the divergence is a

† If we allowed an $s$-dependence the divergence would be trivial for $U - 1$ of compact support.
bulk effect. A natural and simple procedure to take care of this bulk divergence is to define the interface effective action as

\[ S_i(U) = S_{\text{eff}}(U) - \frac{1}{2} \left[ S^+_{\text{eff}}(U) + S^-_{\text{eff}}(U) \right]. \]  

(2.3)

\( S^\pm_{\text{eff}}(U) \) is the effective action for the homogeneous mass term obtained by replacing the coefficient of the last term in (2.1) by \( 5 \mp m \) respectively.

(2.3) can be implemented order by order in perturbation theory by subtracting the contributions of diagrams with identical external legs before the sum over \( s \) is carried out. To obtain convergent sums on \( s \) the terms coming from positive and negative \( s \) in \( S^\pm_{\text{eff}}(U) \) have to be lumped together. There is some ambiguity in this procedure because one may elect to “shift” the \( s \)-sums for negative or positive values and be left after the cancelation with an additional finite part. This ambiguity should reflect the phase choice freedom discussed before but we have not traced the correspondence through.

Our purpose here is to replace this perturbative prescription by a truly non-perturbative formula. The way to proceed is pretty obvious: Since we have exact homogeneity in the lattice version on both sides of the defect, there should be one transfer matrix describing propagation in the \( s \)-direction on one side of the defect and another transfer matrix for the other side of the defect. Imposing infiniteness in the two directions, projects out the ground states of the two transfer matrices. The resulting interface effective action should become an overlap between the two ground states. Each of these ground states is the solution to a Hamiltonian problem for fermions that live in an even dimensional Euclidean space and have nonzero mass. Once this overlap formula is obtained, one can forget about the “wall”, the additional \( s \)-dimension, and the bulk infinity. One can build up the argumentation for the overlap formula itself directly from first principles and thus circumvent the need to define the infinite \( s \)-extent by a limiting procedure. The purpose of this section is to construct an appropriate overlap formula for \( S_i(U) \).

2.1 Transfer matrix from the path integral

The basic problem we wish to solve is to find a way to replace the Grassmann path integral in equation (2.2) by matrix products. Since the number of \( s \)-slices is infinite, the number of matrices that are multiplied is infinite, and the full expression has only a formal meaning. However, the individual factors in the matrix product are perfectly well defined by the path integral. Our job is particularly simple because of several choices we already made in [4]: We
chose to work only with $1 \pm \gamma_\mu$ (corresponding to setting the variable usually denoted by $r$ in the Wilson fermion action to have absolute magnitude unity); therefore the slices $s$ and $s+1$ will be connected by a transfer matrix and we do not have to go two steps in $s$. We chose to have the higher dimensional mass term change only at the ends of one bond and this will result in a simple overlap formula without necessitating the insertion of an operator between the two states. Moreover, our transfer matrix is expected to come out positive definite because of reflection positivity in the bulk portions; only in this case will its action as a projector in an infinite product be an obvious property.

The way to get the transfer matrix is well known; we choose to follow Lüscher [14]. To start, the Grassmann action has to be written in such a manner that a natural split of the Grassmann integration variables between “holomorphic” and “antiholomorphic” sets is evident. One then proceeds to separate out from the action the inner product piece that connects different slices; this piece is kinematical. The left over “dynamical” part is then easily translated into an operator. The operator is the transfer matrix we are after. There is always some freedom of grouping together terms into individual factors in the product of operators; we fix this freedom by demanding to obtain a formula that only contains an overlap between states with no operator insertion in the middle.

Since we are only concentrating here on the $s$ dependence of the action we introduce a compact notation which hides irrelevant dependencies. We first write the four component Dirac spinors in terms of two component spinors as follows:

$$\bar{\psi}_{n,s} = (\bar{\chi}_{n,s}^L \quad \bar{\chi}_{n,s}^R); \quad \psi_{n,s} = \left(\chi_{n,s}^R \quad \chi_{n,s}^L\right). \tag{2.4}$$

The two-component spinors $\chi$ at a fixed $s$ are combined into a single vector. If $\bar{v}$ and $u$ are two such vectors, we define an inner product by

$$(\bar{v}, u) \equiv \sum_{n\alpha i} \bar{v}_{n\alpha i} u_{n\alpha i}. \tag{2.5}$$

The sum in the above equation is over the four dimensional space, $n$; over the two components of the spinor, $\alpha$; and over the color index, $i$; all at some fixed
The fermionic action in (2.1) can be written as

$$S_F(\bar{\chi}^R, \bar{\chi}^L, \chi^R, \chi^L, U) = -\sum_{s \geq 0} \left[ (\bar{\chi}^L_s, B^+ \chi^R_s) + (\bar{\chi}^R_s, B^+ \chi^L_s) \right]$$

$$-\sum_{s < 0} \left[ (\bar{\chi}^L_s, B^- \chi^R_s) + (\bar{\chi}^R_s, B^- \chi^L_s) \right]$$

$$+ \sum_s \left[ (\bar{\chi}^L_s, C \chi^L_s) - (\bar{\chi}^R_s, C^\dagger \chi^R_s) \right]$$

$$+ \sum_s \left[ \bar{\chi}^L_s \chi^R_{s+1} + \bar{\chi}^R_s \chi^L_{s+1} \right].$$

(2.6)

$B^\pm$ and $C$ are operators on the vector space defined above and they depend on the gauge fields. Explicitly,

$$B^\pm_{n\alpha i, m\beta j} = (5 \mp m) \delta_{nm} \delta_{\alpha\beta} \delta_{ij} - \frac{1}{2} \delta_{\alpha\beta} \sum_{\mu} \left[ \delta_{m,n+\mu} U^i_{n,\mu} + \delta_{n,m+\mu} U^j_{m,\mu} \right]$$

(2.7)

$$C_{n\alpha i, m\beta j} = \frac{1}{2} \sum_{\mu} \left[ \delta_{m,n+\mu} U^i_{n,\mu} - \delta_{n,m+\mu} U^j_{m,\mu} \right] \sigma_{\mu}^{\alpha\beta}$$

(2.8)

We have used the following representation for the $\gamma$ matrices:

$$\gamma_{\mu} = \begin{pmatrix} 0 & \sigma_{\mu} \\ \sigma_{\mu}^\dagger & 0 \end{pmatrix}; \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(2.9)

where $\sigma_0 = i$ and $\sigma_j; j = 1, 2, 3$ are the usual Pauli matrices.

The Grassmann path integral in (2.2) can now be converted into the second quantized operator form using fermion operators satisfying canonical anti commutation relations. We follow [14] closely and so we will be brief with some of the intermediate steps relegated to Appendix A. The result for the effective action, $S_{\text{eff}}(U)$, is

$$e^{S_{\text{eff}}(U)} = \lim_{s \to \infty} \left[ \frac{\det B^-}{\det B^+} \right]^{s+\frac{1}{2}} \left[ \frac{\det B^+}{\det B^-} \right]^{s+\frac{1}{2}}$$

$$\langle b | \hat{D}_- (\hat{T}_-)^{s-1} (\hat{T}_+)^{s-1} \hat{D}_+^\dagger | b+ \rangle$$

(2.10)

$$\hat{D}_\pm = e^{\hat{\alpha} \hat{Q}_\pm \hat{a}}$$

$$\hat{T}_\pm = e^{\hat{\alpha} \hat{H}_\pm \hat{a}}$$

(2.11)

(2.12)
\[ e^{Q_{\pm}} = \begin{pmatrix} \frac{1}{\sqrt{B_{\pm}}} & \frac{1}{\sqrt{B_{\pm}}} C \\ 0 & \frac{1}{\sqrt{B_{\pm}}} C \end{pmatrix} \] (2.13)

\[ e^{H_{\pm}} = \begin{pmatrix} \frac{1}{B_{\pm}} & \frac{1}{B_{\pm}} C \\ C^\dagger \frac{1}{B_{\pm}} & C^\dagger \frac{1}{B_{\pm}} C + B_{\pm} \end{pmatrix} \] (2.14)

Since \( B_{\pm} \) are both strictly positive (see Appendix A) all the above equations are well defined. \( |b_{\pm} > \) in (2.9) are boundary conditions at \( s = \pm \infty \) respectively. The choice of these boundary conditions is discussed in the next subsection. \( \hat{a}_{nAi} \) and \( \hat{a}_{nAi}^\dagger \) are fermion operators satisfying canonical anticommutation relations:

\[ \{ \hat{a}_{nAi}, \hat{a}_{mBj}^\dagger \} = \delta_{nm} \delta_{AB} \delta_{ij}; \quad \{ \hat{a}_{nAi}, \hat{a}_{mBj} \} = 0; \quad \{ \hat{a}_{nAi}^\dagger, \hat{a}_{mBj}^\dagger \} = 0. \] (2.15)

Here the index \( A \) and \( B \) runs over four spinor components. The two transfer matrix operators in (2.10) are hermitian due to the hermiticity of (2.14). One can compute the determinants of \( e^{H_{\pm}} \) and prove that the \( H_{\pm} \) are traceless. While this indicates the presence of both positive and negative eigenvalues the relation between the two sets of eigenvalues and eigenvectors is not as simple as it would be in the continuum.

2.2 Overlap formula and a phase ambiguity

We still need to supply the boundary conditions appearing in (2.10). Let us first absorb the operators \( \hat{D}_- \) and \( \hat{D}_+^\dagger \) into the boundary states and define new boundary states by

\[ |b'_{\pm} > \equiv \hat{D}_\pm^\dagger |b_{\pm} > . \] (2.16)

(2.10) is now given by

\[ e^{S_{\text{eff}}(U)} = \lim_{s \to \infty} | \det B^- |^{s + \frac{1}{2}} | \det B^+ |^{s + \frac{1}{2}} < b' - | (\hat{T}_-)^{s-1} (\hat{T}_+)^{s-1} | b' + > \] (2.17)

It is easy to see that the equations corresponding to (2.17) for \( S_{\text{eff}}^{\pm}(U) \) are

\[ e^{S_{\text{eff}}^{\pm}(U)} = \lim_{s \to \infty} | \det B^{\pm} |^{2s+1} < b' \pm | (\hat{T}_ \pm)^{2s-2} | b' \pm > \] (2.18)

\( |b'_{\pm} > \) are the modified (similar to (2.16)) boundary states for the homogeneous cases. The boundary states at \( s = \pm \infty \) are the same for the homogeneous cases. Further, owing to the hermiticity of \( B^{\pm} \) and \( H_{\pm} \) (c.f. (2.14))
the effective actions for the homogeneous cases are real. The result for the effective action at the interface in (2.3) is

\[
e_{s_i(U)} = \lim_{s \to \infty} \frac{< b' - |(\hat{T}_-)s^{-1}(\hat{T}_+)s^{-1}|b' + >}{\sqrt{< b' - |(\hat{T}_-)2s^{-2}|b' - > < b' + |(\hat{T}_+)2s^{-2}|b' + >}}
\]  

(2.19)

Note that the quantities inside the square root are positive. The limit \( s \to \infty \) projects out the ground states and yields

\[
e_{s_i(U)} = < b' - |0- > < 0 - |0+ > < 0 + |b' + > > < b' - |0- > | < b' + |0+ > |
\]  

(2.20)

\(|0\pm >\) are the ground states of \(\hat{T}_\pm\) respectively. In addition to \( < 0 - |0+ > \) (2.20) contains two complex factors of modulus unity; they must be there because, as written, eq. (2.20) seems independent of the phases of \(|0\pm >\). We may just as well admit that there is a dependence on the phases and that they are undetermined. Nothing changes if we make the replacement

\[
|b'\pm > \to |0\pm >.
\]  

(2.21)

We end up with the “overlap formula” for the effective action at one interface,

\[
e_{s_i(U)} = < 0 - |0+ > .
\]  

(2.22)

The real part of the above equation is easy to understand. If we imposed periodic boundary conditions in the \(s\)-direction for both the wall and the homogeneous cases the right hand side of (2.20) will be \(| < 0 - |0+ > |^2\). This is purely real and corresponds to the “wall”-“anti wall” case: One has a finite circular \(s\)-extent with a step up in the mass term followed by a step down halfway round the \(s\)-circle. The interface action is the sum from the two interfaces and has a “light” contribution from pairs of chiral fermions of opposite chirality.

The ambiguity in (2.22) arising from the phases affects only the imaginary part of \(S_i(U)\). Gauge invariance can be broken only by these phases. Indeed, it was understood for quite a while that the definition of the real part of the action of chiral fermions ought to be a trivial matter, it is the imaginary part that is special to the chiral character reflecting the basic absence of a parity operator in Minkowski space. Nevertheless, without exception as far as we know, all older lattice approaches fail even for the real part. Thus, we feel
that some progress has been made. The real part is defined nonperturbatively and it is represented by the same object that is supposed to also produce the imaginary part. It should be stressed that even if we believe that some chiral gauge theories exist, we still should expect some difficulty in defining the phase of the chiral determinant in a gauge invariant approach as long as the construction itself is insensitive to anomalies.

To make some progress in understanding the phase problem we extend the overlap formula to the continuum and compute it in perturbation theory. In perturbation theory, one should be able to separate the ambiguous and the unambiguous contributions order by order. In section 3, we attack the simple problem of perturbation theory in two dimensions and show that the overlap formula indeed produces the correct anomaly.

2.3 First quantized form of the overlap formula

In the last subsection we have reduced the computation of the effective action to a simple overlap between the ground states of the transfer matrix operators, \( \hat{T}_+ \) and \( \hat{T}_- \). These are two positive and hermitian operators. The ground states of \( \hat{T}_\pm \) are obtained by filling all the states corresponding to the negative eigenvalues of the traceless operators \( \hat{H}_\pm \) respectively. Let \( R \) and \( L \) diagonalize \( \hat{H}_+ \) and \( \hat{H}_- \) respectively:

\[
R_{\alpha,n,A_i} \hat{H}_+ n_{n,A_i, m_{B_j}} R_{m_{B_j}, \beta}^\dagger = \lambda_+^\alpha \delta_{\alpha \beta} \tag{2.23}
\]

\[
L_{\alpha,n,A_i} \hat{H}_- n_{n,A_i, m_{B_j}} L_{m_{B_j}, \beta}^\dagger = \lambda_-^\alpha \delta_{\alpha \beta} \tag{2.24}
\]

\( \lambda_+^\alpha \) are the eigenvalues of \( \hat{H}_+ \). Repeated indices are summed over with the exception of \( \alpha \) and \( \beta \). We assume that the diagonalizing matrices are so chosen that the eigenvalues are arranged in an increasing order. Let us assume that there are \( N \) eigenvalues that are negative for both \( \hat{H}_\pm \). If the numbers are different then the overlap will be zero. The two ground states are

\[
|0+\rangle = R_{1,n_1,A_1}^* \cdots R_{N,n_N,A_N}^* \hat{a}^\dagger_{n_1,A_1 i_1} \cdots \hat{a}^\dagger_{n_N,A_N i_N} |0\rangle \tag{2.25}
\]

\[
|0-\rangle = L_{1,n_1,A_1}^* \cdots L_{N,n_N,A_N}^* \hat{a}^\dagger_{n_1,A_1 i_1} \cdots \hat{a}^\dagger_{n_N,A_N i_N} |0\rangle \tag{2.26}
\]

\( |0\rangle \) is the vacuum state annihilated by all \( \hat{a}_{n,A_i} \). Let \( O \) be the restriction of the matrix \( LR^\dagger \) to the negative eigenvalues of both \( \hat{H}_\pm \). Then we show in Appendix B that

\[
<0-|0+\rangle = \det O \tag{2.27}
\]
2.4 The free field case

It is instructive to see the explicit structure of the transfer matrices in the absence of gauge fields. This would be the first step in setting up perturbation theory.

Going to Fourier space, for which we maintain a discrete notation even if the four dimensional volume is infinite, we get the following forms in a plane wave basis:

\[
[eH^\pm]_{\mu,\nu}^0 = \delta_{\mu\nu}\delta_{ij}
\left(\begin{array}{cc}
\frac{1}{1+m+\frac{1}{2}\bar{p}^2} & \frac{i\sigma\cdot\bar{p}}{1+m+\frac{1}{2}\bar{p}^2} \\
\frac{-i\sigma^\dagger\cdot\bar{p}}{1+m+\frac{1}{2}\bar{p}^2} & \frac{\bar{p}^2}{1+m+\frac{1}{2}\bar{p}^2} + 1 \mp m + \frac{1}{2}\bar{p}^2
\end{array}\right)
\]

(2.28)

In the above equation only the spinor structure is non-diagonal. As usual we use \(\bar{p}_\mu = \sin(p_\mu)\) and \(\hat{p}_\mu = 2\sin\frac{p_\mu}{2}\). The eigenvalues \(\lambda\) of the matrices are the roots of the equation

\[
\lambda + \frac{1}{\lambda} = \frac{1 + \bar{p}^2}{1 \mp m + \frac{1}{2}\bar{p}^2} + 1 \mp m + \frac{1}{2}\bar{p}^2
\]

(2.29)

and are easily recognized as appearing in the free propagator obtained in [4]. The complete free propagator can be obtained from the explicit form of the free transfer matrices. Generically, one of the roots is larger than one and the other is its inverse.

Let us now consider the eigenvectors corresponding to the above eigenvalues. Following [4] we denote \(a_\pm = 1 \mp m + \frac{1}{2}\bar{p}^2\). Let the eigenvector(s) corresponding to \(\lambda\) be \(\psi^\pm_\lambda\) with degeneracy indices suppressed. Write

\[
\psi^\pm_\lambda = \begin{pmatrix} u^\pm \\ v^\pm \end{pmatrix}
\]

(2.30)

with

\[
\begin{align*}
\sigma\cdot\bar{p}u^\pm = (a_\pm - 1)u^\pm \\
\sigma^\dagger\cdot\bar{p}u^\pm = (a_\pm - 1)u^\pm
\end{align*}
\]

(2.31)

We are interested in the eigenvectors corresponding to \(\lambda < 1\). For \(a_-\) we have \(a_-/\lambda > 1\) for all momenta and we can use the second equation, expressing \(v\) in terms of \(u\), globally over momentum space. For \(a_+\) however a global choice is impossible. In selecting the definition of the eigenvector we must check for all locations where the right hand side has a vanishing prefactor. The identity \((a_+/\lambda - 1)(1 - \lambda a_+) = \bar{p}^2\) shows that all the trouble spots are at the places where massless fermions would appear in the naïve fermion action. Both
factors in the identity cannot vanish simultaneously. It is easy to check now that the zero momentum point corresponds to the vanishing of \( a_+ / \lambda - 1 \) while all the other points to the vanishing of \( 1 - \lambda a_+ \). Hence, in a small region around the origin we have to define the eigenvector by the first equation while outside that region we should choose the other. The singling out of the positive side is reminiscent of the fact that only this side carries a Chern-Simons current in the \( s \)-direction in the odd dimensional formulation [15].

As we will see in section 3, the situation in the continuum will be slightly different. In the continuum equations similar to (2.31) we will have to pick one equation on the positive side and the other equation on the negative side for all momenta. We should observe, in view of the above, that in finite volume formulations on the lattice, one probably should ascertain that the definition chosen for the phase is such that it be compatible with a smooth infinite volume limit.

2.5 Chiral determinant and the overlap

We arrived at (2.22) by a rather circuitous route: The first step is in the paper of Callan and Harvey [7] where the mass defect is introduced as a smooth background in a five dimensional theory. Homogeneity in the fifth direction holds only asymptotically. The second step is taken in Kaplan’s work [3] where the defect is put on the lattice and becomes strictly localized in the fifth direction. The third step was taken in our previous paper [4] where any five dimensional aspects of the gauge field were abolished and the lattice action was chosen so as to have a good transfer matrix in the fifth direction. Also, the defect was chosen to have the shortest possible extent of just one bond. Clearly, if (2.22) is right it shouldn’t depend on such an indirect derivation and the simplicity of the expression suggests that it has wider validity, beyond the lattice.

Our goal is to provide a simple and direct, albeit formal, argument for the validity of the overlap formula. Let the chiral Dirac operator in four Euclidean dimensions be denoted by \( X \). The group of Euclidean space rotations, \( O(4) \), is represented inequivalently in the domain and image of \( X \). Therefore, the usual definition of the determinant needs an extension. However, both spaces connected by \( X \) are Hilbert so the operators \( XX^\dagger \) and \( X^\dagger X \) are well defined and have usual determinants. It is only the phase of \( \det(X) \) that really needs to be defined [16]. Since we are only interested in the gauge dependence of this phase it is sufficient to be able to define \( \det(X_0^{-1}X) \) for some fixed \( X_0 \) and arbitrary \( X \). The latter determinant is of the usual type and will need
regularization; usually this involves some related eigenvalue problem and this would have been difficult to formulate for $X$ itself.

Ignoring the infinity of the dimensions of the spaces we define $\det(X)$ as follows: Let the space on which $X$ acts be denoted by $V_L$ and the space to which $V_L$ is mapped be $V_R$. Let $\{v_L^{(i)}\}$ and $\{v_R^{(i)}\}$ be orthonormal bases of $V_{L,R}$ respectively. We now define $\det(X) = \det_{ij} < v_R^{(i)}, X v_L^{(j)} >$. If $Y : V_L \to V_L$ then $\det(X Y) = \det(X) \det(Y)$, and replacing $X$ by $X_0$ and $Y$ by $X_0^{-1} X$ we see that ratios will come out correctly.

We now turn to the overlap and show that it is formally related to the determinant of an operator that has the appearance of an operator regularized version of $X$. The relationship is formal because we work with operators in infinite spaces and we do not worry about the finiteness of the expressions we are writing down. The final expression admits a lattice regularization and probably many others. The lattice regularized version is equation (2.22).

Define a Hermitian traceless operator $H$ in the space $V = V_R \oplus V_L$

$$H = \begin{pmatrix} m & X^\dagger \\ X & -m \end{pmatrix} \tag{2.32}$$

$H$ is the single particle Hamiltonian of a five dimensional Minkowski massive Dirac system with real space identified with Euclidean four space. Following (2.30) and (2.31) we write:

$$H \psi_\lambda = \lambda \psi_\lambda; \quad \psi_\lambda = \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix} \tag{2.33}$$

$$X^\dagger v_\lambda = (\lambda - m) u_\lambda; \quad X u_\lambda = (\lambda + m) v_\lambda \tag{2.34}$$

To get the overlap we do not need the exact eigenvectors: We only need two sets of linearly independent vectors spanning the $\lambda > 0$ subspaces of $V$ for $m > 0$ and for $m < 0$. This is easily achieved by observing that the $\lambda$ terms in the above equation can be replaced by $X$ dependent operators via:

$$X X^\dagger v_\lambda = (\lambda^2 - m^2) v_\lambda; \quad X^\dagger X u_\lambda = (\lambda^2 - m^2) u_\lambda \tag{2.35}$$

For $m = |m|$ let $u^{(i)}$ be a basis of $V_L$; then the subspace of interest in $V$ is spanned by

$$\psi_i^+ = N_i^+ \frac{u^{(i)}}{\sqrt{X X^\dagger + m^2 + |m|}} \tag{2.36}$$
Similarly, for \( m = -|m| \), let \( v^{(i)} \) be a basis of \( V_R \); we are now interested in the span of

\[
\psi_i^- = N_i^- \left( \frac{1}{\sqrt{XX^\dagger + m^2 + |m|}} v^{(i)} \right)
\]  

(2.37)

The overlap is given by

\[
\det_{ij} \left[ 2N_i^- N_j^+ < v^{(i)}, \frac{1}{\sqrt{XX^\dagger + m^2 + |m|}} X u^{(j)} > \right]
\]  

(2.38)

This expression can be viewed as a (partially) regularized version of \( \det(X) \) as defined above (for large eigenvalues of \( XX^\dagger \) both the normalization factors tend to \( \frac{1}{\sqrt{2}} \)). (2.38) will have the same phase as \( \det(X) \) for finite square matrices, if the bases \( u^{(i)} \) & \( v^{(i)} \) were chosen to be independent of \( X \); this is quite natural since these bases can be chosen before \( X \) is given. Also, note that the choice of a different pair of equations from (2.34) and (2.35) was instrumental and was justified by demanding that the expressions (2.36) and (2.37) be valid even when \( X \) or \( X^\dagger \) have zero modes. This is reminiscent of (2.31) and the discussion thereafter.

The regularization of the second quantized system built from \( H \) can be carried out gauge invariantly using the gauge invariant single particle energies. On the lattice one loses the exact match between the two cases \( m = \pm |m| \). As a matter of fact, for an arbitrary background gauge field there is no guarantee that there will be equal numbers of single particle states in both states making up the overlap and therefore the latter can easily vanish. It is in this way that we expect the lattice to correctly reproduce instanton effects. Note that the two transfer matrices provide, via the dimensionalities of the appropriate subspaces a new definition of the topological charge associated with a lattice configuration of gauge fields. This definition is manifestly gauge invariant and naturally integer valued. More precisely, the topological charge should be identified with the signed number of level crossings through eigenvalue unity when the mass term is changed from positive to negative. Explicitly,

\[
n_{\text{top}} = \frac{1}{2} Tr \left( \frac{H_+}{\sqrt{H_+^2}} - \frac{H_-}{\sqrt{H_-^2}} \right)
\]  

(2.39)

As expected with any lattice definition, there are “singular” gauge field configurations for which the topological charge is not defined; this permits the
integer to change when the background gauge fields are smoothly deformed. One can smoothly deform gauge fields from one topological class to another because the lattice has wiped out the manifold structure of spacetime and the space of gauge transformations is no longer disconnected. In this case the “singular gauge fields” are the ones for which $H_\pm$ have zero eigenvalues. This can’t happen for gauge fields sufficiently close to the identity in view of our study of the free case. Note that in a perturbative expansion around the free case the finite radius of convergence of the series will be determined by the first encountered “singular” gauge configuration.

It should be no surprise that we unintentionally obtained a lattice definition of $n_{\text{top}}$: the dynamics of chiral fermions has a universal sensitivity to the topological charge of the gauge background and any respectable approach to the regularization of chiral gauge theories must have some definition of $n_{\text{top}}$ hidden in it [17].

3. Two dimensional anomaly from the continuum overlap formula

The formal continuum limits of $H_\pm$ in (2.14) correspond to Hamiltonians of massive Dirac fields coupled to a gauge field with the masses being $\pm m$ respectively †. To obtain this, one uses the standard representation for $U_{n,\mu}$ and expands the right-hand side of (2.14) in powers of the lattice spacing, $a$. The leading order term gives the continuum Hamiltonians. The lattice overlap is replaced by the overlap between the ground states of these two Dirac Hamiltonians. This introduces a significant simplification in the algebra. In this section, starting from the Hamiltonians in two dimensions and using Schrödinger perturbation theory, we show that the overlap produces the correct consistent anomaly for the case of Abelian gauge field in the limit where $m \to \infty$. In this limit the other modes become infinitely heavy and we expect to be left with only the interesting piece, namely the zero mode attached to the abrupt mass defect.

† In this section $m \equiv |m|$.
3.1 Problem definition

The continuum limits of \( H_{\pm} \), in momentum space, are

\[
-H_{\pm}(p,q) = H_{0}^{\pm}(p)\delta_{p,q} + \sum_{k} z_{k} \tau \delta_{q,p+k} + \sum_{k} z_{k}^{*} \tau \delta_{p,q+k}
\]

(3.1)

\[
H_{0}^{\pm}(p) = \begin{pmatrix}
\mp m & p_{1} - ip_{2} \\
p_{1} + ip_{2} & \pm m
\end{pmatrix}
\]

(3.2)

\[
\tau = \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\]

(3.3)

\[
z_{k} = \int d^{2}x e^{ikx}[A_{1}(x) + iA_{2}(x)]
\]

(3.4)

The above Hamiltonians are the Hamiltonians associated with the continuum limit of the action, (2.1), with the coefficient of the last term being \((5 \mp m)\). The \( s \)-direction plays the role of Euclidean time. The two Dirac matrices are the Pauli matrices \( \sigma_{1} \) and \( \sigma_{2} \). \((1 \pm \sigma_{3})\) projects onto right and left chiral states. In the above equations, \( p,q \) and \( k \) are two-component momenta. \( \sum_{k} \) is a notation for the momentum integral \( \frac{1}{(2\pi)^{2}} \int d^{2}k \). \( A_{1}(x) \) and \( A_{2}(x) \) are the two components of the Abelian gauge field.

The unperturbed Hamiltonians are diagonal in momentum space. For each momentum, the unperturbed Hamiltonians have two eigenstates one with positive energy and one with negative energy. We denote these two unperturbed states by \( \psi_{p}^{\pm} \) (positive energy) and \( \chi_{p}^{\pm} \) (negative energy). The states are labeled by their momentum. Let

\[
-H_{\pm}\Psi_{p}^{\pm} = \Lambda_{p}^{\pm}\Psi_{p}^{\pm}, \quad \Lambda_{p}^{\pm} > 0
\]

(3.5)

be the set of eigenvectors of the full Hamiltonians corresponding to positive eigenvalues. The unperturbed limit of \( \Psi_{p}^{\pm} \) is \( \psi_{p}^{\pm} \), hence the labeling of the states by \( p \). The overlap matrix is defined as

\[
O_{pq} = \Psi_{p}^{-1}\Psi_{q}^{+}
\]

(3.6)

and the effective action at the interface is given by

\[
e^{S_{i}(A)} = \det O
\]

(3.7)
This is the continuum limit of (2.22). We now proceed to compute $S_t(A)$ in perturbation theory. We assume that there is no zero momentum term in the gauge field; i.e., $z_0 = 0$. If $z_0 \neq 0$, it can be combined with the unperturbed Hamiltonian and will amount to a shift in the momenta.

3.2 Eigenvectors and eigenvalues of $H_0^\pm$

$H_0^\pm$ have the same set of eigenvalues with the positive and negative eigenvalues occurring in pairs.

$$
H_0^\pm \psi_p^\pm = \lambda_p \psi_p^\pm; \quad \lambda_p > 0
$$

(3.8)

The eigenvectors are labeled by a momentum index and are given by

$$
\psi_p^+(q) = \frac{1}{N(p)} \left( \frac{p_1 - ip_2}{\lambda_p + m} \right) \delta_{pq}; \quad \psi_p^-(q) = \frac{1}{N(p)} \left( \frac{\lambda_p + m}{p_1 + ip_2} \right) \delta_{pq}
$$

(3.9)

$$
\chi_p^+(q) = \frac{1}{N(p)} \left( \frac{-(\lambda_p + m)}{p_1 + ip_2} \right) \delta_{pq}; \quad \chi_p^-(q) = \frac{1}{N(p)} \left( \frac{-(p_1 - ip_2)}{\lambda_p + m} \right) \delta_{pq};
$$

(3.10)

$$
\lambda_p = \sqrt{|p|^2 + m^2} > 0; \quad N(p) = \sqrt{2\lambda_p(\lambda_p + m)}
$$

(3.11)

The eigenvectors are globally smooth in the momentum index $p$. Both sets, $\{\psi_p^+, \chi_p^+\}$ and $\{\psi_p^-, \chi_p^-\}$, are orthonormal and complete.

$$
\sum_k [\psi_p^+(k)]^\dagger \psi_q^-(k) = \delta_{pq}
$$

$$
\sum_k [\chi_p^+(k)]^\dagger \chi_q^-(k) = \delta_{pq}
$$

(3.12)

$$
\sum_k [\psi_p^+(k)]^\dagger \chi_q^+(k) = 0
$$

The set $\{\psi_p^-, \chi_p^-\}$ is related to $\{\psi_p^+, \chi_p^+\}$ by

$$
\psi_p^- = \Sigma_3 \chi_p^+; \quad \chi_p^- = \Sigma_3 \psi_p^+
$$

$$
\Sigma_3(q,k) = \sigma_3 \delta_{qk}
$$

(3.13)

where $\sigma_3$ is the third Pauli matrix. The overlap matrices between the eigenstates of $H_0^+$ and $H_0^-$ are

$$
[\psi_p^+]^\dagger \psi_q^+ = [\chi_p^+]^\dagger \chi_q^- = \frac{p_1 - ip_2}{\lambda_p} \delta_{pq} \equiv P_{pq}
$$

(3.14)
\[ [\chi_p^-]^{\dagger}\psi_q^+ = -[\psi_p^-]^{\dagger}\chi_q^+ = \frac{m}{\lambda_p}\delta_{pq} \equiv Q_{pq} \]  

(3.15)

The above equations follow from (3.9) and (3.10).

3.3 The overlap formula in perturbation theory

In perturbation theory, the anomaly in two dimensions should show up at second order. To second order the eigenvectors corresponding to the positive eigenvalues will have the following form:

\[
\begin{align*}
\Psi^+_p &= \psi^+_p + \psi^+_q A_{qp} + \chi^+_q B_{qp} + \psi^+_q W_{qp} + \chi^+_q X_{qp} \\
\Psi^-_p &= \psi^-_p + \psi^-_q C_{qp} + \chi^-_q D_{qp} + \psi^-_q Y_{qp} + \chi^-_q Z_{qp}
\end{align*}
\]

(3.16)

\(A, B, C\) and \(D\) are coefficients linear in the gauge field and \(W, X, Y\) and \(Z\) are coefficients quadratic in the gauge field. Summation over repeated indices is implied. The overlap matrix, (3.6), to second order in the gauge field is

\[
O = P + PA - QB + C^\dagger P + D^\dagger Q + PW - QX + Y^\dagger P \\
+ Z^\dagger Q + C^\dagger PA - C^\dagger QB + D^\dagger QA + D^\dagger P^\dagger B
\]

(3.17)

In deriving the above equation from (3.16), we have used (3.12), (3.14) and (3.15). The effective action, \(S_i(A)\), to second order in the gauge field is

\[
S_i(A) - S_i(0) = \ln \det P^{-1}O \\
= \text{Tr} \ln P^{-1}O \\
= T_{11} + T_{12} + T_{21} + T_{22} + T_{23} + T_{24}
\]

(3.18)

\[T_{11} = \text{Tr}(A + C^\dagger)\]
\[T_{12} = \text{Tr}P^{-1}Q(D^\dagger - B)\]
\[T_{21} = \text{Tr}(W + Y^\dagger)\]
\[T_{22} = -\frac{1}{2} \text{Tr}(A^2 + C^\dagger)^2\]
\[T_{23} = -\frac{1}{2} \text{Tr}(P^{-1}QB)^2 - \frac{1}{2} \text{Tr}(P^{-1}QD^\dagger)^2 + \text{Tr}P^{-1}D^\dagger P^{-1}Q^2 B \\
+ \text{Tr}P^{-1}D^\dagger P^\dagger B + \text{Tr}P^{-1}QBA - \text{Tr}P^{-1}QC^\dagger D^\dagger\]
\[T_{24} = \text{Tr}P^{-1}Q(Z^\dagger - X)\]

(3.19)

\(T_{11}\) and \(T_{22}\) are linear in the gauge field and the rest are quadratic in the gauge field. We have used the fact that \(P\) and \(Q\) are diagonal (c.f.(3.14) and (3.15)).
3.4 First and second order perturbation theory

The perturbing term is the same for both the Hamiltonians, $\mathbf{H}_\pm$, (c.f.(3.1)):

$$H_1(p, q) = \sum_k z_k \tau \delta_{q, p+k} + \sum_k z_k^* \tau^\dagger \delta_{p, q+k}. \quad (3.20)$$

The eigenvalues, $\Lambda_\pm^\pm$, in (3.5) have a perturbation expansion of the form

$$\Lambda_\pm^\pm = \lambda_p + \lambda_\pm^{(1)} + \lambda_\pm^{(2)} \quad (3.21)$$

$\lambda_p^{(1)}$ is linear in gauge field and $\lambda_p^{(2)}$ is quadratic in gauge field. Standard Schrödinger perturbation theory gives

$$\lambda_p^{(1)} = \psi_p^{\dagger} H_1 \psi_p^\pm \quad (3.22)$$

$$A_{pq} = \frac{\psi_p^{\dagger} H_1 \psi_q^+}{\lambda_q - \lambda_p}; \quad p \neq q \quad (3.23)$$

$$B_{pq} = \frac{\lambda_p^{\dagger} H_1 \psi_p^+}{\lambda_q + \lambda_p} \quad (3.24)$$

$$C_{pq} = \frac{\psi_p^{\dagger} H_1 \psi_q^-}{\lambda_q - \lambda_p}; \quad p \neq q \quad (3.25)$$

$$D_{pq} = \frac{\lambda_p^{\dagger} H_1 \psi_q^-}{\lambda_q + \lambda_p} \quad (3.26)$$

The diagonal terms, $A_{pp}$ and $C_{pp}$, are not determined above. Imposing orthonormality of the eigenvectors,

$$\psi_p^{\dagger} \psi_q^\pm = \delta_{pq}, \quad (3.27)$$

to first order, yields

$$A_{pp} + A_{pp}^* = 0; \quad C_{pp} + C_{pp}^* = 0; \quad \forall p. \quad (3.28)$$

The imaginary parts of $A_{pp}$ and $C_{pp}$ remain undetermined and this arbitrariness is the phase ambiguity in the eigenvectors and the overlap. One choice to fix this ambiguity is to impose the condition that the overlap between the true
eigenstate and the unperturbed one, namely $\Psi_p^\pm \psi_p^\pm$, be real. This condition corresponds to Brillouin–Wigner perturbation theory.

Since $H_1$ is hermitian (c.f.(3.20)) it follows from (3.23) and (3.25) that

$$A^\dagger = -A; \quad C^\dagger = -C. \quad (3.29)$$

This is true for any choice of fixing the ambiguity. $\Sigma_3$ defined in (3.13) has the property

$$\Sigma_3^\dagger H_1 \Sigma_3 = -H_1 \quad (3.30)$$

This along with the hermitian nature of $H_1$ relates $B$ and $D$ in (3.24) and (3.26) by

$$D = -B^\dagger. \quad (3.31)$$

Second order perturbation theory is needed to evaluate $T_{21}$ and $T_{24}$. Owing to the diagonal nature of $P$ and $Q$ (c.f.(3.14) and (3.15)), we only need the diagonal terms of $W,X,Y$ and $Z$. The diagonal terms of $W$ and $Y$ are not determined by the eigenvector condition. Enforcing the orthonormality, (3.27), to second order, yields

$$W_{pp} + W_{pp}^* = -\sum_q \left[ A_{pq}^\dagger A_{qp} + B_{pq}^\dagger B_{qp} \right]$$

$$Y_{pp} + Y_{pp}^* = -\sum_q \left[ C_{pq}^\dagger C_{qp} + D_{pq}^\dagger D_{qp} \right] \quad (3.32)$$

As with $A$ and $C$, the imaginary parts of $W_{pp}$ and $Y_{pp}$ remain undetermined. Brillouin–Wigner perturbation theory sets all of them to zero:

$$A_{pp} = C_{pp} = W_{pp} = Y_{pp} = 0. \quad (3.33)$$

For $X$ and $Z$, second order perturbation theory gives

$$2\lambda_p X_{pp} = \sum_{q \neq p} \left[ (\lambda_p + \lambda_q) B_{pq} A_{qp} + (\lambda_p - \lambda_q) C_{pq} B_{qp} \right] + \left[ 2\lambda_p A_{pp} - \lambda_p^{(1)} - \lambda_p^{(1)} \right] B_{pp} \quad (3.34)$$

$$2\lambda_p Z_{pp} = \sum_{q \neq p} \left[ (\lambda_p + \lambda_q) D_{pq} C_{qp} + (\lambda_p - \lambda_q) A_{pq} D_{qp} \right] + \left[ 2\lambda_p C_{pp} - \lambda_p^{(1)} - \lambda_p^{(1)} \right] D_{pp} \quad (3.35)$$
3.5 Imaginary part of the effective action

Using the results of the previous subsection, we now compute the imaginary part of the effective action given by (3.18). We define

\[ I_1 = \frac{1}{2}(T_{11} - T_{11}^*); \quad I_2 = \frac{1}{2}(T_{21} - T_{21}^*). \]  

(3.36)

Because they only involve the imaginary parts of \( A_{pp}, C_{pp}, W_{pp} \) and \( Y_{pp} \), they are both completely arbitrary.

From (3.9), (3.10), (3.20), (3.24) and (3.26) it is clear that

\[ B_{pp} = D_{pp} = 0 \]  

(3.37)

if \( z_0 = 0 \). Therefore

\[ T_{12} = 0 \]

in (3.19). Because of (3.29), \( T_{22} \) in (3.19) is real and does not contribute to the imaginary part of \( S_i(A) \).

From (3.14) and (3.15) it follows that

\[ P^\dagger = P^{-1}(1 - Q^2). \]  

(3.38)

This, along with (3.29) and (3.31), reduces \( T_{23} \) in (3.19) to

\[ T_{23} = -\text{Tr}(P^{-1}QB)^2 - \text{Tr}(P^{-1}B)^2 + \text{Tr}P^{-1}Q(BA - CB) \]  

(3.39)

Using (3.29), (3.31) and (3.37), (3.34) and (3.35) yield

\[ Z_{pp}^* - X_{pp} = \sum_q \frac{\lambda_q}{\lambda_p}(C_{pq}B_{qp} - B_{pq}A_{qp}) \]  

(3.40)

Using (3.14), (3.15) and (3.40), \( T_{24} \) in (3.19) is given by

\[ T_{24} = \sum_{pq} \frac{m}{p_1 - i\lambda_p}(C_{pq}B_{qp} - B_{pq}A_{qp}) \]  

(3.41)

To proceed further with \( T_{23} \) and \( T_{24} \) we need the explicit expressions for \( A, B \) and \( C \). The defining equations for \( A, B \) and \( C \) are (3.23), (3.24) and
(3.25). Using the eigenvectors given by (3.9) and (3.10), and $H_1$ given by (3.3), (3.4) and (3.20), we get

$$A_{pq} = z_{q-p} a(p,q) - z_{p-q}^* a^*(q,p)$$  
(3.42)

$$B_{pq} = z_{q-p} b(p,q) - z_{p-q}^* d(p,q)$$  
(3.43)

$$C_{pq} = -z_{q-p} a(q,p) + z_{p-q}^* a^*(p,q)$$  
(3.44)

$$a(p,q) = (\lambda_p + m)(q_1 - iq_2)$$  
(3.45)

$$b(p,q) = (p_1 - ip_2)(q_1 - iq_2)$$  
(3.46)

$$d(p,q) = (\lambda_p + m)(\lambda_q + m)$$  
(3.47)

Using the above equations for $A, B$ and $C$ along with (3.14) and (3.15), the imaginary part of $T_{23}$ in (3.39) and the imaginary part of $T_{24}$ in (3.41) give

$$I_3 = \frac{1}{2}(T_{23} + T_{24} - T_{23}^* - T_{24}^*)$$  
(3.48)

$$k(p) = \sum_p c(p, q, q) \frac{1}{[p_1 + q_1 + i(p_2 + q_2)][q_1 + iq_2]}$$  
(3.49)

$$c(p, q) = \frac{m(\lambda_p \lambda_q + \lambda_p^2 + \lambda_q^2 - m^2)}{4\lambda_p \lambda_q (\lambda_p + \lambda_q)}$$  
(3.50)

The above two equations have a semblance to the “bubble” diagram in ordinary Feynman diagram language. $p$ is the external momenta. The second factor in (3.49) is the product of the two chiral fermion propagators occurring in the bubble. $c(p, q)$ is a regulator for high momenta introduced by the overlap definition of the effective action.

Finally,

$$\text{Im}[S_i(A) - S_i(0)] = I_1 + I_2 + I_3$$  
(3.51)

The first two terms are ambiguous and the last term is completely determined. It is worthwhile noting here that the condition $z_0 = 0$ is not needed to remove all arbitrariness in $I_3$. This is because the arbitrariness in the last terms in (3.34) and (3.35) is exactly cancelled by the arbitrariness in the last term in (3.39).
3.6 Anomaly

We now want to check if (3.51) produces the correct anomaly. The anomaly arises due to non-local contributions to $I_3$. The arbitrariness in $I_1$ and $I_2$ arising from choices for the diagonal terms of $A, C, W$ and $Y$ should be local and cannot affect the anomaly.

We start by showing that $I_3$ in (3.48) has a non-local piece. Note that $k(p)$ defined in (3.49) is finite. It can be rewritten as

$$ k(p) = \frac{1}{p_1 + ip_2} \sum_q \frac{1}{q_1 + iq_2} [c(q + p, q) - c(q - p, q)]. \quad (3.52) $$

We now expand $[c(q + p, q) - c(q - p, q)]$ in powers of $p$. From (3.50), we find

$$ c(q + p, q) - c(q - p, q) = -\frac{3m|q|^2}{8\lambda_q^5} (q_1 p_1 + q_2 p_2) + O(p^2) \quad (3.53) $$

Using (3.53) in (3.52) gives

$$ k(p) = -\frac{p_1 - ip_2}{p_1 + ip_2} \sum_q \frac{3m|q|^2}{16\lambda_q^5} + O(|p|/m) \quad (3.54) $$

The integral in the above equation is finite. When $m \to \infty$, we get the following non-local expression for $I_3$ in (3.48):

$$ I_3 = \frac{1}{16\pi} \sum_p \left[ \frac{p_1 + ip_2}{p_1 - ip_2} z_p^{*} z^{*}_{-p} - \frac{p_1 - ip_2}{p_1 + ip_2} z_p^{*} z_{-p} \right] \quad (3.55) $$

Note that only in the limit $m \to \infty$ all the additional fields become infinitely massive and we are left with one chiral fermion.

Now we focus on $I_3$ to extract the anomaly [18]. The anomaly is defined as

$$ A(x) = \sum_\mu \partial_\mu \frac{\delta I_3}{\delta A_\mu(x)} \quad (3.56) $$

The momentum space equivalent of the above equation using (3.4) is

$$ iA_p = \left[ \frac{\delta I_3}{\delta z_{-p}} + p^* \frac{\delta I_3}{\delta z_p^{*}} \right] $$

$$ = \frac{1}{8\pi} \left[ (p_1 + ip_2) z_p^{*} z_{-p} - (p_1 - ip_2) z_p^{*} \right] \quad (3.57) $$
We have used (3.55) in deriving the last line above. From (3.4) we have

\[ z_p = A_{1,p} + iA_{2,p}; \quad z^*_{-p} = A_{1,p} - iA_{2,p}. \]  

(3.58)

Using (3.58), the anomaly, (3.57), becomes

\[ A_p = \frac{1}{4\pi} (p_2 A_{1,p} - p_1 A_{2,p}) \]  

(3.59)

Since the field strength in two dimensions is

\[ F_{12}(p) = i(p_2 A_{1,p} - p_1 A_{2,p}) \]  

(3.60)

the anomaly, (3.59), can be rewritten as

\[ A_p = -\frac{i}{4\pi} F_{12}(p) \]  

(3.61)

The normalization of the right hand side tells us that, although we computed only the abelian anomaly, we still can conclude that it is in the consistent rather than the covariant form \( ^{\dagger}\).

3.7 Consistent versus covariant anomalies

Once we have an explicit formula for the effective action induced by chiral fermions the currents defined by varying this action with respect to the external gauge fields will, by construction, have anomalies in the “consistent” form at the expense of covariance under gauge transformations [19]. However, it is known that in the case that the defect is string–like rather than wall–like (i.e. the codimension of the defect is two rather than one) that the anomaly one obtains from the Callan-Harvey arguments [7] has the covariant form. This was emphasized by Naculich [19]. It also is true of the wall case. Thus, in our case, the Callan-Harvey analysis cannot fully account for the anomaly, in particular the result obtained in eq. (3.61) above. This is true even allowing for the additional changes induced by the lattice regularization of the five dimensional calculation of the induced Chern-Simons term, namely its appearance on only one side of the defect [15].

For simplicity, we shall again restrict ourselves to two dimensions. To see the difference between consistent and covariant forms and not have to rely on

\( ^{\dagger}\) In the covariant case the factor of \( \frac{1}{4\pi} \) would be replaced by \( \frac{1}{2\pi} \).
normalizations we have chosen to work here with the non-abelian case. Let us start by explaining how Callan and Harvey end up with the covariant form of the anomaly. We use a compact differential form notation that has proven useful in studying anomalies in arbitrary dimensions and the relations between them [20].

Let $\mathcal{M}$ be a three dimensional manifold with boundary. $A$ is a three dimensional vector potential written as a Lie Algebra valued one form. $F$ is the associated curvature $F = dA + A^2$. $tr$ denotes trace over representation indices. The three dimensional Chern-Simons three–form is given by

$$\Omega = tr[AdA + \frac{2}{3}A^3] \quad (3.62)$$

We choose the Chern-Simons action as $S_{CS} = -\frac{i}{4\pi} \int_{\mathcal{M}} \Omega$ with no boundary terms added. Under an arbitrary variation $\delta$ we have

$$\delta S_{CS} = -\frac{i}{2\pi} \int_{\mathcal{M}} tr(\delta AF) - \frac{i}{4\pi} \int_{\partial \mathcal{M}} tr(\delta AA) \quad (3.63)$$

Restricting the variation to a gauge variation $\delta A = \delta h \equiv dh + [A, h]$ the first integrand becomes the $d$ of something and we obtain

$$\delta h S_{CS} = -\frac{i}{4\pi} \int_{\partial \mathcal{M}} tr(hdA) \quad (3.64)$$

The boundary of the manifold is two dimensional and restricting the gauge fields to it we obtain the two dimensional anomaly. Clearly $dA$ does not transform covariantly because the term $A^2$ is missing.

Suppose we have the Callan-Harvey wall set-up and $x_3 \equiv s$ is the third direction. We view the system as three dimensional and define the three dimensional currents (more precisely, their Hodge duals) $\mathcal{J} = \frac{\delta S_{CS}}{\delta A}$. Far away from the defect, at $x_3 \to \pm \infty$, the integration of the massive fermions is presumed to induce $\pm \frac{1}{2} S_{CS}$ terms. † Note however that the computations leading to this assumption are most naturally defined for a homogeneous system on a boundary free manifold in the limit it becomes infinite [21]. While we cannot write explicit formulae for $\mathcal{J}$ everywhere, we do know the currents in the asymptotic regime. Specialize now to a background which is two dimensional

† The reader should not worry about overall normalizations, because we only want to trace the difference between the two forms of the anomaly.
and $s$ independent. We then have, as $s \to \infty$, $\mathcal{J}_3 = -\frac{i}{4\pi} F_{12}$ and, as $s \to -\infty$, $\mathcal{J}_3 = \frac{i}{4\pi} F_{12}$; the 1 and 2 components of $\mathcal{J}$ vanish in the asymptotic regions because of our choice of background.

Because of gauge invariance and absence of three dimensional anomalies

$$D_\mu J_\mu = 0$$

(3.65)

everywhere. $D_\mu$ is the covariant derivative in the $\mu^{th}$ direction. Integrating the equation over $s$ from $-\infty$ to $\infty$ we get

$$D_i J_i = \frac{i}{2\pi} F_{12}$$

(3.66)

In (3.65) $\mu = 1, 2, 3$ and in (3.66) $i = 1, 2$ and $J_i = \int_s \mathcal{J}_i(x_1 x_2 x_3 = s)$. We have obviously obtained the covariant form of the anomaly $\dagger$. This means that the currents $J_i$ can’t be written as the variation of something; this is a bit strange because the $\mathcal{J}$ were defined as a variation. The problem was introduced when we evaluated $\mathcal{J}$: we tacitly assumed that the variation $\delta A$ vanishes at infinity (the boundary of our manifold) and therefore ignored the boundary term in (3.63). This procedure is correct for the three dimensional current in the bulk but wrong at infinity. However, the justification for the Chern-Simons form of the induced action holds only in bulk. To get the consistent form of the anomaly we have seen above that one needs to know that the Chern-Simons form of the induced actions holds also on the boundary of the manifold. We do not know this to be true but we could easily accept that a carefully chosen boundary condition at $s = \pm \infty$ will make this happen. We must assume that this choice of boundary conditions has been carried out if we want $S_i(U)$ to behave properly.

The additional piece of current that would turn the above defined $J_i$ into a variation of an action resides at the boundaries at $s = \pm \infty$. It is easy to see that it is given by $\Delta J_i = \frac{i}{4\pi} \epsilon_{ij} A_j$ and that the new current $J_i + \Delta J_i$ has the consistent anomaly. In summary, to get the right form for the anomaly one must ensure that the boundary conditions at infinity are as undistruptive as possible. This is reminiscent of the boundary conditions needed in defining the APS index theorem for manifolds with boundary [22].

We now proceed to propose an overlap formula for the covariant currents. The formula has some geometric appeal. In computing the anomaly of these

$\dagger$ The nonabelian field strength appears rather than $\epsilon_{ij} \partial_j A_i$. The prefactor is $\frac{1}{2\pi}$ rather than $\frac{1}{4\pi}$.
currents we shall realize that the difference between them and the previously s-integrated $J_i$’s is due to higher energy excitations in the massive fermion systems. Naïvely one would have ignored such excitations because of suppression by large energy denominators, but, they are sufficiently numerous to make a difference. This formula might be particular useful when defining currents associated with global anomalous symmetries† in an anomaly–free gauge theory [24].

Our main purpose is to identify the terms in the variation of the overlap that produce the $\Delta J_i$ contribution. Let $A_{\mu}^{\text{ext}}(x)$ be an external vector potential coupled to the two Hamiltonians that would correspond to a symmetry whose current we wish to calculate. Make the external vector potential infinitesimal; a current with the consistent form of the anomaly would emerge if we evaluated $\delta(<0^+|\psi>|0^+>)$. The variation of the states $|0\pm>$ contains a piece proportional to the state itself and additional contributions from excited states. If we neglect the excited states we obtain for the candidate covariant current an expression made out of two contributions, one from each side of the overlap: $<0^+|\delta 0^+> + <\delta 0^-|0^->$. More explicitly, we have:

$$J_{\mu}^{\text{cov}}(x) = <0 +|\frac{\partial}{\partial A_{\mu}^{\text{ext}}(x)}|0^+> - <0 - |\frac{\partial}{\partial A_{\mu}^{\text{ext}}(x)}|0^-> \quad (3.67)$$

Each one of the terms in the above equation has a geometrical interpretation: Remember that the phases of $|0\pm>$ are ambiguous to some extent. The related $U(1)$ gauge symmetry over the space of vector potentials has a naturally associated connection [25], and the current is written as the difference between these two connections. Under a $U(1)$ gauge transformation the phases of the states change but this should leave the anomaly unchanged as long as the gauge change is local. So the anomaly is a $U(1)$ gauge invariant (under a class of gauge functions restricted by locality) and should reside in the curvature of the connection. Of course, had we insisted that the current be a total derivative there would have been no curvature. It so happens that all the covariant anomaly is contained in the curvature. We really need only the difference between the two curvatures, denoted by $\mathcal{F}$:

$$\mathcal{F}_{\mu,x,\nu,y} = \frac{\partial J_{\mu}^{\text{cov}}(x)}{\partial A_{\nu}^{\text{ext}}(y)} - \frac{\partial J_{\nu}^{\text{cov}}(y)}{\partial A_{\mu}^{\text{ext}}(x)} \quad (3.68)$$

† Recent studies of a global anomaly in a theory regularized by infinitely many Pauli-Villars fields have given encouraging results [23].
When $F$ is integrated over a two dimensional disk embedded in the space of vector potentials it gives the difference between the Berry phases \[26\) accumulated when each one of the vacua is adiabatically moved around the circumference of the disk. The unremovability of these phases is related to the presence of degeneracy submanifolds in the space of vector potentials.

Let us sketch the perturbative evaluation of $F$ in an abelian two dimensional situation. The calculation has been first carried out by Niemi et. al. \[27\]. Since we need the currents only to first order in the gauge fields $F$ is needed only to zeroth order. Using intermediate states and going to first quantized formalism we have

$$F^{\pm}_{\mu x, \nu y} = F^{+}_{\mu x, \nu y} - F^{-}_{\mu x, \nu y}$$  \hspace{1cm} (3.69)

$$F^{\pm}_{\mu x, \nu y} = - \sum_{p,q} \frac{1}{(\lambda_p + \lambda_q)^2} \left\{ \left[ \langle \psi_p^\pm | \frac{\partial H^\pm}{\partial A^{\text{ext}}_\mu (x)} | \chi_q^\pm \rangle < \chi_q^\pm | \frac{\partial H^\pm}{\partial A^{\text{ext}}_\nu (y)} | \psi_p^\pm \rangle > 

- \langle \psi_p^\pm | \frac{\partial H^\pm}{\partial A^{\text{ext}}_\nu (y)} | \chi_q^\pm \rangle < \chi_q^\pm | \frac{\partial H^\pm}{\partial A^{\text{ext}}_\mu (x)} | \psi_p^\pm \rangle > \right]\right.$$

$$- [\mu x \leftrightarrow \nu y] \right\} \right. \hspace{1cm} (3.70)$$

Here we have used the notations from section (3.1). Plugging in explicit formulae one gets after some algebra:

$$F^{\pm}_{\mu x, \nu y} = 4 \epsilon_{\mu \nu} \sum_{k,k'} \frac{e^{i(k'-k) \cdot (x-y)}}{\lambda_k \lambda_{k'} (\lambda_k + \lambda_{k'})}$$  \hspace{1cm} (3.71)

The structure is reminiscent of equation (3.50). The following identity converts the momentum integral to a three dimensional integral associated with a bubble diagram:

$$\int \frac{d^2 p}{(2\pi)^2} \frac{1}{\lambda_{p+q/2} \lambda_{p-q/2} (\lambda_{p+q/2} + \lambda_{p-q/2})} = 2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{(p_0^2 + (p + q/2)^2 + m^2)(p_0^2 + (p - q/2)^2 + m^2)}$$  \hspace{1cm} (3.72)

In the limit of large mass one is left with

$$F^{\pm}_{\mu x, \nu y} = \frac{i}{2\pi} \delta(x - y) \epsilon_{\mu \nu}$$  \hspace{1cm} (3.73)
From $F_{\mu x,\nu y}$ one can construct the current $J_{\mu}^{\text{cov}}$ to linear order in $A_{\mu}(x)$ and then obtain the anomaly. The answer is just twice as large as the one obtained in eq. (3.61). This factor indicates that now we have obtained the covariant anomaly. Moreover, the Berry phase computation is known to reproduce the results for other calculations of the induced effective actions in odd-dimensional space time. These calculations, as stressed above, are made in the boundary free case and hence are only concerned with the bulk effects that we already showed give the covariant rather than the consistent anomaly.

As for the nonconservation of the charge associated with $J_{\mu}^{\text{cov}}$ in the global case we note that it is not related to extra light particles anywhere in the system; rather it has to do with some global degrees of freedom measuring the sensitivity of the phases associated with the two Fock vacua entering the overlap.

4. Summary and outlook

In this paper, inspired by the “wall” realization of chiral fermions due to Kaplan [3], we arrived at a connection between the chiral fermion determinant and the overlap of two complex vectors. The two vectors are the ground states for the two transfer matrices describing the propagation on the two sides of the defect in the “wall” realization. This relation can be extended to the continuum. We analyzed the two dimensional continuum overlap in perturbation theory. Since the ground states have a phase ambiguity there is a corresponding ambiguity in the imaginary part of the effective action. We can fix the ambiguity by doing Brillouin-Wigner perturbation theory, and disallowing choices that differ by non-local phase redefinitions the expected form of the anomaly is obtained. Brillouin–Wigner perturbation theory is simply the condition that the overlap between the ground state at some non-zero gauge field and the ground state for the free Hamiltonian is real. This is a condition that can be easily imposed in a numerical computation.

The above result indicates that if the gauge fields are perturbative (i.e. sufficiently close to zero) this condition will produce the correct anomaly also on the lattice. An explicit check of this is still needed. If this works out it would be necessary to see what happens for large gauge fields.

Another phase choice, that has the same effect in perturbation theory to the order we went, is to use adiabatic deformations. Here we choose a certain
interpolation that connects the given gauge configuration to zero field and
demand that the change in the eigenvectors of the Hamiltonian at any point
on this interpolating curve is orthogonal to the eigenvectors at that point. It
is possible that this choice is more attractive for numerical calculations. In
the same vein, one might consider the fact that we are only interested in the
relative phase of the two states contributing to the overlap insufficiently rep-
resented in the above alternatives. One may try to use adiabatic deformation
of the mass term itself from positive to negative values as a way to interlock
the two phases. This procedure will encounter obstacles when the background
gauge becomes “singular” for any of the intermediate mass values. It is quite
likely that these obstacles are the heart of the matter and their presence is a
good way to single out from the phase difference the part that is physically
significant.

The numerical simulations will have to evaluate the overlap for a given
background gauge configuration. The real part will be used for the update
and the imaginary part will have to be combined with the observable being
measured. All the eigenvectors corresponding to the negative eigenvalues of
\( H_\pm \) will be found by some iterative procedure. In an updating procedure only
a single link is changed at one time and hence the matrix in (2.14) will only
change slightly; this would make it easier to find the new eigenvectors given
the old ones. It should be noted that, to evaluate the overlap we do not need
explicitly all the negative eigenvectors. What we really need are two bases in
the spaces \( \text{Im} P_\pm \) where
\[
P_\pm = -\frac{1}{2} \left( \frac{H_\pm}{\sqrt{H_\pm^2 + 1}} - 1 \right)
\]
are projector operators on
the filled states in the Dirac seas.

It is well known that anomalies very often can be given a topological
interpretation [28]; this topology has to do with smooth space or space–time
and is lost on the lattice. To make sure that the lattice has not completely
wiped out the remnants of the topological effects it would be necessary to
perform calculations on a sequence of lattices and verify that quantities are
behaving smoothly. The nonperturbative \( SU(2) \) anomaly [29] will also have
to be shown to hold on the lattice.

An important point to note is that the overlap can be zero. This will hap-
pen whenever the number of states with eigenvalues less than one for the two
transfer matrices are different. In order to study this it would be interesting
to study the overlap near instanton configurations.

It should be clear to the reader that we do not claim to have here a com-
plete solution to the problem of regularizing nonperturbatively chiral gauge
theories. Nevertheless we feel that progress has been made irrespectively of whether chiral gauge theories will turn out to ultimately exist or not. The flavor of the overlap formulation seems right; for the first time we see a lattice framework that makes contact with the important developments in our understanding of anomalies that took place in the mid eighties and incorporates them. If one wishes to phrase the problem in the mnemonic advocated by Fujikawa [30] the infiniteness of the number of fermion fields has made the Grassmann “measure” ill defined and hence not necessarily gauge invariant (despite appearances). If one likes more the approach of Alvarez–Gaume and Della Pietra [31] we have a clear separation between the imaginary and real parts of the chiral action. The quantities we are dealing with are reminiscent of the \( \eta \) invariant used by these authors. Instanton effects have a clear place and the partition function can easily vanish. Global anomalies seem also to fit in.

We would like to suggest that the right line of approach is not to try to jump prematurely to a conclusion whether a perturbatively consistent chiral gauge theory can exist but rather try to build a framework which, where it to ultimately fail, would provide a strong indication that indeed chiral theories cannot be freed of all cutoff dependence. Of course, much more work will be needed to see if the overlap is a useful approach to the problem of non-perturbatively defining anomaly free chiral gauge theories.

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Appendix A

In this appendix we provide some intermediate steps in going from (2.6) to (2.10).

We start by noting that, if $0 < m < 1$, $B^{\pm}$ defined in (2.7) is strictly positive for any gauge configuration. To see this, first note that $B^\mu$ defined by

$$B^\mu_{\alpha i, m\beta j} = \delta_{\alpha \beta} \delta_{m,n} + \hat{U}^{ij}_{n,\mu}$$

is unitary. Therefore,

$$||B^\mu|| \leq 1.$$  \hspace{1cm} \text{(A.2)}

This shows that the norm of the second term in (2.7) is less than or equal to 4. The positivity of $B^{\pm}$ now follows.

We now define the following change of Grassmann variables to make the first two lines in (2.6) into a simple inner product. For $s \geq 0$, let

$$c_s = \sqrt{B^{+}} \chi^R_s; \quad c^*_s = \sqrt{B^{+}} \chi^L_s; \quad d^*_s = \sqrt{B^{+}} \chi^L_s; \quad d_s = -\sqrt{B^{+}} \chi^R_s.$$

\hspace{1cm} \text{(A.3)}

For $s < 0$, let

$$c_s = \sqrt{B^{-}} \chi^R_s; \quad c^*_s = \sqrt{B^{-}} \chi^L_s; \quad d^*_s = \sqrt{B^{-}} \chi^L_s; \quad d_s = -\sqrt{B^{-}} \chi^R_s.$$  \hspace{1cm} \text{(A.4)}

The superscript $t$ denotes the transpose operation. Starting from (2.6) and using (A.3) and (A.4), the effective action in (2.2) becomes

$$e^{S_{\text{eff}}(U)} = \prod_{s \geq 0} [\det B^{+}]^2 \prod_{s < 0} [\det B^{-}]^2 \int \prod_s [dc_s^*] [dc_s] [dd_s^*] [dd_s] e^{-(c_s^*, c_s) - (d_s^*, d_s)}$$

$$\prod_{s \leq -2} R_-(c_s^*, d_s^*) T_-(c_s^*, d_s^*; c_{s+1}, d_{s+1}) R_+^\dagger(c_{s+1}, d_{s+1})$$

$$\prod_{s \geq 0} R_+ (c_s^*, d_s^*) T_+(c_s^*, d_s^*; c_{s+1}, d_{s+1}) R_+^\dagger(c_{s+1}, d_{s+1})$$

$$R_\pm(c_s^*, d_s^*) = \exp\left\{ \left( c_s^*, \frac{1}{\sqrt{B^{\pm}}} C \frac{1}{\sqrt{B^{\pm}}} d_s^* \right) \right\}$$  \hspace{1cm} \text{(A.6)}

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\[ R^\pm(c_s, d_s) = \exp \left\{ \left( d_s, \frac{1}{\sqrt{B^\pm}} C^\dagger \frac{1}{\sqrt{B^\pm}} c_s \right) \right\} \]  
(A.7)

\[ T_\pm(c_s^*, d_s^*; c_{s+1}, d_{s+1}) = \exp \left\{ \left( c_s^*, \frac{1}{B^\pm} c_{s+1} \right) + \left( d_s^*, \frac{1}{B^\pm} d_{s+1} \right) \right\} \]  
(A.8)

\[ T_0(c_{s-1}^*, d_{s-1}^*; c_0, d_0) = \exp \left\{ \left( c_{s-1}^*, \frac{1}{\sqrt{B^+} \sqrt{B^-}} c_0 \right) + \left( d_{s-1}^*, \frac{1}{\sqrt{B^+} \sqrt{B^-}} d_0 \right) \right\} \]  
(A.9)

At this point we can use the formulas in the appendix of \cite{14} to convert (A.5) into operator notation. The Grassmann variables \( c^*, d^*, c, d \) go into fermion operators \( \hat{c}^\dagger, \hat{d}^\dagger, \hat{c}, \hat{d} \) respectively. These fermion operators satisfy the following canonical anticommutation relations:

\[ \{ \hat{c}_{n\alpha i}, \hat{c}_{m\beta j}^\dagger \} = \delta_{nm} \delta_{\alpha \beta} \delta_{ij}; \quad \{ \hat{d}_{n\alpha i}, \hat{d}_{m\beta j}^\dagger \} = \delta_{nm} \delta_{\alpha \beta} \delta_{ij}. \]  
(A.10)

All other anticommutators are zero. The operator equations associated with (A.6)–(A.9) are

\[ \hat{R}_\pm = \exp \left\{ \left( \hat{c}^\dagger, \frac{1}{\sqrt{B^\pm}} C \frac{1}{\sqrt{B^\pm}} \hat{d} \right) \right\} \]  
(A.11)

\[ \hat{R}_\pm^\dagger = \exp \left\{ \left( \hat{d}, \frac{1}{\sqrt{B^\pm}} C^\dagger \frac{1}{\sqrt{B^\pm}} \hat{c} \right) \right\} \]  
(A.12)

\[ \hat{T}_\pm = \exp - \left\{ \left( \hat{c}^\dagger, \log B^\pm \hat{c} \right) + \left( \hat{d}^\dagger, \log B^\pm \hat{d} \right) \right\} \]  
(A.13)

\[ \hat{T}_0 = \exp - \left\{ \left( \hat{c}^\dagger, \frac{1}{2} \log (B^+ B^-) \hat{c} \right) + \left( \hat{d}^\dagger, \frac{1}{2} \log (B^- B^+) \hat{d} \right) \right\} \]  
(A.14)

The operator equation for the effective action in (A.5) is

\[ e^{S_{eff}(U)} = \prod_{s \geq 0} [\det B^+]^2 \prod_{s < 0} [\det B^-]^2 \] 

\[ < b - | \left\{ \prod_{s \leq -2} \left[ \hat{R}_- \hat{T}_- \hat{R}_-^\dagger \right] \left[ \hat{R}_- \hat{T}_0 \hat{R}_-^\dagger \right] \prod_{s \geq 0} \left[ \hat{R}_+ \hat{T}_+ \hat{R}_+^\dagger \right] \right\} | b^+ > \]  
(A.15)

\[ < b - > \text{ and } | b^+ > \text{ are the boundary conditions at } s = -\infty \text{ and } s = \infty \text{ respectively expressed as states in the operator formalism. To obtain (2.10) all that needs to be done now is to define the fermion operators } \hat{a}^\dagger \text{ and } \hat{a} \text{ as } \]  
\[ \hat{a}^\dagger \equiv \begin{pmatrix} \hat{c}^\dagger \\ \hat{d}^\dagger \end{pmatrix}; \quad \hat{a} \equiv \begin{pmatrix} \hat{c} \\ \hat{d} \end{pmatrix} \]  
(A.16)
These fermion operators also satisfy canonical anti commutation relations (equation (2.15)) and in terms of (A.16), (A.15) becomes (2.10). A useful formula for executing this last step is

\[ e^\hat{a}K \hat{a} = e^\hat{a}K_1 \hat{a} e^\hat{a}K_2 \hat{a}, \]

where,

\[ e^K = e^{K_1} e^{K_2}, \]

for any two matrices \( K_1 \) and \( K_2 \).

**Appendix B**

In this appendix we present the derivation of (2.27) from (2.25) and (2.26). Let us combine all the three indices \( nAi \) into one big index \( I \). From (2.25) and (2.26), the overlap between the two ground states is

\[
<0^-|0^+> = \sum_{I_1,\ldots,I_N} L_{1,I_1} \cdots L_{N,J_N} R_{I_1,1}^\dagger \cdots R_{I_N,N}^\dagger \]

\[<0^+|\hat{a}_{I_N} \cdots \hat{a}_{I_1} \hat{a}_{J_1}^\dagger \cdots \hat{a}_{J_N}^\dagger|0^+>\]  

For a non-zero contribution to the above sum, the set \( \{J_1, \ldots, J_N\} \) has to be a permutation of \( \{I_1, \ldots, I_N\} \). Further all the \( I_k \)'s have to different. This yields

\[<0^-|0^+> = \sum_\pi \text{sign}(\pi) \sum_{I_1 \neq I_2 \ldots \neq I_N} L_{\pi_1 I_1} R_{I_1,1}^\dagger \cdots L_{\pi_N I_N} R_{I_N,N}^\dagger \]

\[\pi \text{ denotes a permutation of the set } 1, \ldots, N. \] The restriction that all the \( I_k \)'s are different in the above sum can be removed, because if they are included, the sum over \( \pi \) will, in any case, render them zero. The result then is (2.27).
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