Geometrical tools for embedding fields, submanifolds, and foliations

Antony J. Speranza*

Perimeter Institute for Theoretical Physics, 31 Caroline St. N, Waterloo, ON N2L 2Y5, Canada

Abstract

Embedding fields provide a way of coupling a background structure to a theory while preserving diffeomorphism-invariance. Examples of such background structures include embedded submanifolds, such as branes; boundaries of local subregions, such as the Ryu-Takayanagi surface in holography; and foliations, which appear in fluid dynamics and force-free electrodynamics. This work presents a systematic framework for computing geometric properties of these background structures in the embedding field description. An overview of the local geometric quantities associated with a foliation is given, including a review of the Gauss, Codazzi, and Ricci-Voss equations, which relate the geometry of the foliation to the ambient curvature. Generalizations of these equations for curvature in the nonintegrable normal directions are derived. Particular care is given to the question of which objects are well-defined for single submanifolds, and which depend on the structure of the foliation away from a submanifold. Variational formulas are provided for the geometric quantities, which involve contributions both from the variation of the embedding map as well as variations of the ambient metric. As an application of these variational formulas, a derivation is given of the Jacobi equation, describing perturbations of extremal area surfaces of arbitrary codimension. The embedding field formalism is also applied to the problem of classifying boundary conditions for general relativity in a finite subregion that lead to integrable Hamiltonians. The framework developed in this paper will provide a useful set of tools for future analyses of brane dynamics, fluid mechanics, and edge modes for finite subregions of diffeomorphism-invariant theories.

*asperanz@gmail.com
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1 Introduction

A number of physical systems can be described in terms of a background structure coupled to an otherwise generally covariant theory. One example is a local subregion in a gravitational theory, where the subregion boundary partially breaks the underlying diffeomorphism symmetry [1, 2]. Relativistic fluid dynamics provides another example, with the fluid rest frame determining the background structure [3, 4, 5, 6]. Branes coupled to gravity can similarly be viewed as a type of background structure [7], as can the preferred frame of Lorentz-violating gravity theories [8, 9, 10, 11].

In each of these examples, the background structure introduces additional degrees of freedom whose kinematics and dynamics require scrutiny. Embedding fields provide a convenient formalism with which to perform this analysis. An embedding field is a diffeomorphism $X$ from a reference space $M_0$ into the dynamical spacetime manifold $M$. It provides a means for coupling fields living on the different spaces by mapping them between the spaces via pullbacks. The background structure can be defined as a fixed object on $M_0$, and then coupled to the dynamical fields on $M$ using the $X$ mapping. Promoting $X$ to a dynamical object ensures that the coupling is covariant with respect to diffeomorphisms of $M$. This restores diffeomorphism symmetry to the theory, and the $X$ field encapsulates the additional degrees of freedom introduced by the background structure.

Fluid dynamics provides a canonical example of this procedure. The reference space $M_0$ is the so-called Lagrangian frame, and the fixed structure is a congruence of curves describing the flow of the fluid elements through time. The $X$ field maps this congruence into the spacetime manifold $M$, which in this context is also referred to as the Eulerian frame. $X$ contains the information of how the fluid flow distorts as it evolves through spacetime, and hence represents the fluid degrees of freedom. This framework for describing relativistic fluids has found applications in forming covariant Lagrangians for fluid mechanics [3, 5], describing the phase space for Einstein gravity coupled to a fluid [12, 4], and constructing effective field theories for fluid dynamics [6, 13, 14].

In other applications, the background structure is a single embedded submanifold, rather than a foliation. Strings and branes are examples of this type of object, and the embedding fields define the map from the worldvolume into the target space $M$ [7]. The boundary of a local subregion is another example of a single-submanifold background structure. Al-
though such a boundary should not define a dynamical object in the theory, being merely an arbitrary demarcation for a subregion in a larger space, describing the location of the boundary nevertheless introduces a background structure into the description. The additional degrees of freedom that inevitably appear must be eliminated during the process of gluing adjacent subregions to correctly reproduce the phase space of the larger region. This construction is particularly relevant when discussing entanglement entropy for a subregion in a gravitational theory, where the entangling surface serves as the subregion boundary. The additional degrees of freedom represented by the embedding fields contribute to the entropy of the subregion, and this edge mode entropy is believed to be an important contribution to the entropy associated to horizons and black holes [1]. A related construction by Isham and Kuchař in canonical general relativity extends the usual phase space by additional degrees of freedom representing the embedding of the Cauchy surface into spacetime [15, 16]. Introducing the embedding fields in this context produces a constraint algebra that represents \text{Diff}(M), as opposed to the field-dependent hypersurface deformation algebra that usually appears in the Hamiltonian analysis.

The entanglement wedge associated with a subregion of a holographic conformal field theory is another example where embedding fields are used to describe a single submanifold. The wedge is defined as the domain of dependence of a subregion bounded by a codimension-2 extremal surface in the bulk. This extremal surface is referred to as the RT surface, in honor of the Ryu-Takayanagi formula, which expresses the entanglement entropy of the boundary CFT in terms of the area of this bulk surface [17]. In many applications, one is interested in the response of the RT surface to perturbations of the bulk geometry or the boundary subregion. Examples include holographic proofs of the quantum null energy condition [18, 19, 20], discussions of bulk reconstruction from modular flow [21] and derivations of the Einstein equation from entanglement beyond linear order [22]. Embedding fields again can be applied in these situations, and give a way of cleanly separating perturbations induced by changing the geometry from those coming purely from varying the embedding into spacetime.

Embedding fields also find applications in situations where the background structure is a nondynamical metric. One such application is the parameterized scalar field [15], where the embedding fields impart diffeomorphism invariance on an ordinary scalar field theory. Another is nonlinear massive gravity, where a background metric appears in the construction of the mass interactions for the massive spin two field [23]. Embedding fields are sometimes used as Stueckelberg fields to restore diffeomorphism invariance into the theory, which aids in establishing the absence of ghosts [24].

Given the abundance of applications for embedding fields, it is desirable to have a systematic framework in which to perform calculations with them. The present work seeks to develop this framework, placing specific emphasis on maintaining manifest covariance in all expressions and derivations. The embedding field $X$ is viewed as a specific type of dynamical field to be included in the theory, namely, a diffeomorphism from the reference space $M_0$ into spacetime $M$. This means it has prescribed transformation properties under diffeomorphisms of $M$. Section 2 discusses these transformation properties of $X$, and describes generally how to couple $X$ to other dynamical fields in a diffeomorphism-invariant way. Often embedding fields are viewed as a collection of scalar fields representing the coordinates of the manifold,
which particularly when performing concrete calculations. In order to connect with this
description, section 2.1 discusses the coordinate expressions for the embedding fields.

Following this, section 3 develops the local geometry associated with foliations and sub-
manifolds. The fundamental object is the normal form to the submanifolds, $\nu$, which is
a differential form that annihilates all vectors tangent to the submanifolds. All geometric
quantities associated with the submanifold and foliation can be constructed in terms of $\nu$
and the metric $g_{ab}$. Section 3.1 performs these constructions explicitly, obtaining formulas
for the normal and tangential metrics as well as the extrinsic curvatures. The normal and
tangential covariant derivatives are introduced in section 3.2, and their associated curvatures
are related to the spacetime curvature via the Gauss, Ricci-Voss, and Codazzi identities in
3.3. In situations where one is only interested in a single submanifold, it is important to
know which quantities constructed for a foliation remain well-defined. Section 3.4 addresses
this question, where it is shown that tensors associated with the intrinsic geometry such as
the induced metric $h_{ab}$ and intrinsic curvature $\mathcal{R}^{a}_{bcd}$ are invariants of a single submanifold,
as are the extrinsic curvature tensor $K^{a}_{bc}$ and outer curvature $O^{a}_{bcd}$ (defined in (3.45)), but
notably the normal extrinsic curvature $L^{c}_{ab}$ (defined in (3.20)) is not invariant. The trans-
formation properties of $L^{c}_{ab}$ under a change in foliation are derived in that section, and its
relation to the existence of a normal coordinate system near the submanifold is discussed.

Section 4 then combines the embedding fields with the submanifold calculus of section
3 in order to derive variational formulas for the foliation geometry. Explicitly including
the embedding fields into the description allows one to separate out the variations induced
by changing the metric from variations coming solely from a change in embedding. As
a particular application, section 4.2.1 uses the variational formulas for the mean extrinsic
curvature $K^{a}_{a}$ to derive the Jacobi equation, which describes perturbations of an extremal
surface to a nearby extremal surface. This equation is of particular importance to holography
and RT surfaces, and the geometric description provided in this section will hopefully be
relevant to future holographic calculations.

The constructions of sections 3 and 4 are performed for foliations of arbitrary codimen-
sion. To connect with more common constructions, section 5 analyzes the specific cases of
hypersurfaces (codimension 1), congruences of curves (codimension $(d-1)$), and codimension-
2 surfaces. In each of these cases, certain simplifications occur, and connections are made
between the usual tensors defined for these foliations and the tensors associated with the
generic case. Following this, section 6 gives an application of the formalism by discussing the
boundary term than appears in the gravitational Hamiltonian for a local subregion. Section
7 concludes with a discussion of possible generalizations of the formalism to the case of null
submanifolds, and points to a number of future applications.

Finally, two appendices are included with additional geometric identities relevant to the
submanifold calculus of the paper. Appendix A generalizes the Gauss-Ricci-Voss-Codazzi
identities associated with the tangential covariant derivative $D_{a}$ by deriving analogous state-
ments for the normal covariant derivative $D_{a}$, as well as identities associated with a mixed
commutator between these two derivatives. Some of these identities have appeared be-
fore in the special case of foliations by one-dimensional curves [25, 26, 27, 28]; and general
treatments of arbitrary codimension foliations has appeared before in [29, Ch. V, Sec. 7],
although appendix A casts these identities in a modern light and derives additional relations from them. Appendix B gives numerous coordinate expressions for the geometric quantities associated with a foliation, which can be helpful in concrete computations. This appendix can be viewed as a generalization of the $3+1$ decomposition of spacetime to a $(d-p)+p$ decomposition for arbitrary codimension foliations.

The submanifold calculus developed in this work has been considered before by several authors, and it is worth pointing out the similarities and differences between this paper and previous work. The work of Carter [30, 7] provides much of the basis for the formalism developed here. A notable difference is that Carter was primarily concerned with single submanifolds, as are applicable to the study of branes, and hence employs a formalism that does not require an extension of the submanifold to a local foliation. The advantage of working with a local foliation is that it canonically determines a connection on the normal bundle to the surface through the normal extrinsic curvature tensor $L^{c}_{ab}$ (see section 3.2). As discussed in section 3.4, $L^{c}_{ab}$ is not an invariant tensor on the surface, since it depends on how the foliation is extended away from the surface. This is just the statement that there is no preferred connection on the normal bundle in the absence of a local foliation. In Carter’s work, the analog of the arbitrariness in the foliation is the choice of an orthonormal basis on the normal bundle, needed in order to define the twist pseudotensor. Section 3.4 shows that this twist pseudotensor coincides with the twist $P^{c}_{ab}$ of a foliation whose normal directions are geodesic. For this reason, the acceleration tensor $A^{c}_{ab}$ never appears in Carter’s formalism. Capovilla and Guven [31, 32] and Armas and Tarrió [33] similarly employ an orthonormal basis to describe the normal geometry, and compute many of the variational formulas appearing in section 4 of the present work. The main difference between the present paper and the above is favoring a choice of local foliation over a choice of orthonormal normal basis, since the goal is to have a formalism that applies both for single submanifolds and for foliations. This is more in line with related treatments of submanifold variations employed by Feng and Matzner [34], and Plebanski and Rozga [35]. A notable cosmetic difference between this paper and others is the treatment of the embedding field $X^{a}$ abstractly as a diffeomorphism, as opposed to using its coordinate expression $X^{a}_{\mu}$, discussed in section 2.1. This has some advantages in leading to clean variational expressions in terms of the variational vector field $\chi^{a}$ that appears in the pullback formula (2.3), and we stress that the abstract calculus provides concrete computational tools through repeated use of this formula. Finally, Engelhardt and Fischetti’s recent work [36], which appeared concurrently with the present one, presents a comprehensive, covariant treatment of submanifolds using distributional tensor fields, which do not require an extension to a local foliation. Their work therefore gives a complementary perspective on some of the results obtained in this paper.

1.1 Notation

Throughout this paper, Latin indices from the beginning of the alphabet, $a, b, c, \ldots$ are considered abstract indices [37, Section 2.4], while Greek indices $\mu, \nu, \ldots$ are used as coordinate indices on $M$. Since tensor fields on both $M_{0}$ and on $M$ appear in the calculations below, it is helpful to distinguish between them. This is done using boldface font, $g_{ab}, K_{bc}^{a}$, for
Figure 1: The embedding field $X$ is a map between a reference space $M_0$ and the spacetime manifold $M$. The metric $g_{ab}$ and other dynamical fields live on $M$, and can be mapped to pulled-back fields $X^* g_{ab}$ on the reference space using the embedding map.

tensors on the reference space $M_0$, and regular font, $g_{ab}$, $K^a_{bc}$ for tensors on $M$. The same letter is used for a bolded and unbolded tensor when the two are related by a pullback by the embedding field, so that in general, $\psi = X^* \psi$. Finally, many expressions involve antisymmetrization or symmetrization over multiple indices that are not adjacent. Rather than using brackets and parentheses, which become cumbersome in these cases, we instead underline indices to denote antisymmetrization, and overline indices for symmetrization, i.e. $T_{ab} \equiv T_{[ab]} = \frac{1}{2} (T_{ab} - T_{ba})$ and $T^{ab} \equiv T_{(ab)} = \frac{1}{2} (T_{ab} + T_{ba})$. The spacetime metric $g_{ab}$ is always assumed to have signature $(-,+,+,+,...)$. All tensors are presented as tensors on spacetime, and as such, the same types of letters are used to describe both normal and tangential indices, in line with the philosophy of [7]. This has a slight disadvantage of requiring the reader to remember which indices are normal and tangential (i.e. $K^a_{bc}$ is normal on $a$ and tangential on $b$ and $c$), but has the advantage of making clear how objects such as the spacetime covariant derivative $\nabla_a$ act on these tensors.

2 Overview of embedding fields

This section develops the basic tools for computing with embedding fields. The starting point is a collection of dynamical fields, collectively denoted $\phi$, which are tensor fields on a spacetime manifold $M$. For this paper, the sole dynamical field will be the metric $g_{ab}$, but the basic operations for coupling a generic field to the embedding fields is the same for fields of any type. The diffeomorphism group $\text{Diff}(M)$ acts on these fields via pullbacks, sending $\phi \mapsto Y^* \phi$ for $Y \in \text{Diff}(M)$. A theory in which all tensor fields needed to construct a covariant action are dynamical is diffeomorphism-invariant. An example of a non-invariant theory is a massless scalar, since its Lagrangian density $L = \eta^{ab} \nabla_a \phi \nabla_b \phi$ involves a fixed metric $\eta^{ab}$ which does not transform under diffeomorphisms. This metric therefore constitutes the background structure of the theory that prevents diffeomorphism-invariance. Similarly, a theory with an invariant Lagrangian defined in a finite subregion is still not fully diffeomorphism-invariant, since the normal form to the boundary of the subregion is again a fixed structure which does not transform.
Embedding fields provide a way of modifying theories with background structures to make them invariant. The embedding field $X$ is a diffeomorphism from a reference space $M_0$ into $M$ (Figure 1). This reference space can simply be viewed as a second copy of the spacetime manifold, although for calculations involving coordinate expressions (section 2.1), it is often useful to take it to be an open subset of $\mathbb{R}^d$. The embedding field transforms under diffeomorphisms of $M$ by the pullback by $Y^{-1}$,

$$X \mapsto (Y^{-1})^*X = Y^{-1} \circ X. \quad (2.1)$$

The natural way to couple the embedding field to the other dynamical fields is simply to form the pulled back fields, $\phi \equiv X^*\phi$. Throughout this work, bold font will indicate tensors that live on the reference space $M_0$, and are related to their non-bold counterparts through a pullback by $X$. The fields $\phi$ have the property of being invariant under diffeomorphisms of $M$, since

$$(Y^{-1} \circ X)^*Y^*\phi = X^*(Y^{-1})^*Y^*\phi = X^*\phi. \quad (2.2)$$

This invariance is intuitive from the perspective that $\phi$ live on a different manifold, $M_0$, and hence should not transform under diffeomorphisms of $M$.

A central goal of this work is to compute variations of objects constructed using $X$ and the dynamical fields. For this, the following formula is of central importance,

$$\delta X^*\phi = X^*(\delta \phi + \mathcal{L}_\chi \phi), \quad (2.3)$$

where $\chi^a$ is a vector field representing an arbitrary infinitesimal variation of the embedding map $X$ (see [1, 2] for derivations of this formula). The coordinate expression for $\chi^a$ is given in equation (2.14). A related formula involves mapping a field on $M_0$ to a field on $M$ with the inverse pullback $(X^{-1})^* \equiv X^*$, and reads

$$\delta X_*\phi = X_*\delta \phi - \mathcal{L}_\chi \phi. \quad (2.4)$$

Any expression involving $\delta$ should be taken to mean an arbitrary variation of that quantity. To denote a particular choice for the infinitesimal variation, we will write $I_{\hat{\Phi}}$ acting on the expression, where $\hat{\Phi}$ then contains the information of which variation is being considered. For example, we write $I_{\hat{\Phi}} \delta \phi = \Phi$ to denote the infinitesimal variation of the field $\phi$ given by the specific field configuration $\Phi$. This notation stems from viewing $\delta$ as the exterior derivative on the infinite-dimensional manifold of field configurations, in which case $I_{\hat{\Phi}}$ denotes contraction of a differential form with the vector field $\hat{\Phi}$ describing an infinitesimal change on this manifold [1, 2].

A particularly important class of variations are those induced by an infinitesimal diffeomorphism of $M$ generated by a vector field $\xi^a$. The contraction representing this transformation is denoted $I_{\hat{\xi}}$, so that $I_{\hat{\xi}} \delta \phi = \mathcal{L}_\xi \phi$, since the Lie derivative gives the transformation of a tensor field under an infinitesimal diffeomorphism. Note that since the pulled back fields $X^*\phi$ are diffeomorphism-invariant, they must satisfy $I_{\hat{\xi}} \delta X^*\phi = 0$, which implies via equation (2.3)

$$I_{\hat{\xi}} \chi^a = -\xi^a. \quad (2.5)$$
We can also consider diffeomorphisms of the reference space generated by a vector $\xi^a$, and this transformation on the fields will be represented by $I_\xi$. These diffeomorphisms of $M_0$ should not transform the dynamical fields which live on $\tilde{M}$, which implies the relations for this contraction

$$I_\xi \delta \phi = 0, \quad I_\xi \chi^a = X_* \xi^a = \xi^a.$$  \hfill (2.6)

Occasionally we will also be interested in second variations of quantities. In these cases, when $\delta$ acts on a tensor that already involves a variation, we take it to mean the antisymmetric combination of the two variations. This is consistent with the interpretation of $\delta$ as the exterior derivative on the space of field configurations. In particular, it implies $\delta \delta \phi = 0$. Also, any expression involving the product of two variational tensors will similarly always be assumed to be antisymmetrized in the variations, so that $\delta^2 \phi = 0$. With this interpretation, we note the useful formula for the variation of $\chi^a$, which does not vanish, but rather satisfies $\delta \chi^a = \chi^a \nabla_a X^\mu = \chi^a$.

\[
\delta \chi^a = \delta X_* \chi^a = X_* (\delta \chi^a - \mathcal{L}_\chi \chi^a) = -\frac{1}{2} [\chi, \chi]^a. \tag{2.7} \]

2.1 Coordinate expressions

While the abstract definitions of the $X$ field and the vector $\chi^a$ associated with its variation suffice for formal manipulations, it is also useful to have coordinate expressions for these objects when performing direct calculations. To obtain these, introduce a coordinate system on $M$, which is defined by a collection of functions $y^\mu : M \to \mathbb{R}$, $\mu = 0, \ldots, d-1$, mapping each point in $x \in M$ to its coordinate values $y^\mu(x)$.

The coordinate expression for the $X$ field is simply obtained by pulling the coordinate functions back by $X$. Taking $\mathbb{R}^d$ for the reference space $M_0$, this leads to a collection of $d$ scalar functions $X^\mu : \mathbb{R}^d \to \mathbb{R}$, defined by

$$X^\mu = X^* y^\mu = y^\mu \circ X. \tag{2.8}$$

The functions $X^\mu$ then define a coordinate system on $\mathbb{R}^d$. Promoting $X$ to a dynamical field in the theory therefore has the interpretation of giving dynamics to the coordinate system itself. A consequence of $X^\mu$ defining a coordinate system is that the gradients $\nabla_a X^\mu$ yield a basis for one-forms on $\mathbb{R}^d$. This in turn defines a basis $\partial^a_\mu$ for tangent vectors on $\mathbb{R}^d$ satisfying

$$\nabla_a X^\nu \partial^\nu_\mu = \delta^\nu_\mu \quad \text{(2.9)}$$

$$\nabla_a X^\mu \partial^b_\mu = \delta^b_a \quad \text{(2.10)}$$

$$[\partial^a_\mu, \partial^b_\nu] = 0. \quad \text{(2.11)}$$

These basis vectors are just the pullbacks of the coordinate basis vectors $\partial^a_\mu$ on $M$.

The variations of the functions $X^\mu$ again define a collection of $d$ functions $\delta X^\mu : \mathbb{R}^d \to \mathbb{R}$ interpretable as an infinitesimal change of coordinates. From the pullback formula (2.3) and the requirement that $\delta y^\mu = 0$, this variation is given by

$$\delta X^\mu = X^* \ell_\chi y^\mu = X^a \nabla_a X^\mu = X^\mu, \quad \text{(2.12)}$$

\footnote{Since the following analysis deals only with local quantities, it suffices to restrict attention to a single coordinate patch.}
so that $\delta X^\mu$ are simply the components of $X^a$ in the coordinate system on $\mathbb{R}^d$ defined by $X^\mu$,

$$\chi^a = \delta X^\mu \partial_a^\mu.$$  

(2.13)

This leads to the coordinate expression for $\chi^a$,

$$\chi^a = (X_* \delta X^\mu) \partial_a^\mu = (\delta X^\mu \circ X^{-1}) \partial_a^\mu,$$

(2.14)

which was presented in [1].

Because $\partial_a^\mu$ depends on $X^\mu$, the antisymmetric variation of the vector field $\chi^a$ does not vanish; rather, it is given by

$$\delta \chi^a = -\delta X^\mu \delta (\partial_a^\mu)$$

$$= -\delta X^\mu \partial_a^\mu (\nabla_b X^\nu) \delta (\partial_b^\nu)$$

$$= \chi^b \nabla_b (\delta X^\nu) \partial_a^\nu$$

$$= \chi^b (\nabla_b \chi^a - \delta X^\nu \nabla_b \partial_a^\nu)$$

$$= \frac{1}{2} [\chi, \chi]^a.$$  

(2.15)

The second line follows after applying equation (2.9), while the third line uses the identity $\nabla_a (\delta X^\mu) \partial_a^\mu = -\nabla_a X^\mu \delta (\partial_a^\mu)$, obtained by acting with $\delta$ on equation (2.10). Finally, the fifth line uses that $\chi^b \nabla_b \chi^a = \frac{1}{2} [\chi, \chi]^a$ due to the implied antisymmetrization between the two $\chi^a$’s in this expression, and also that $\nabla_b$ can be taken to be the derivative operator associated with the $X^\mu$ coordinate system, which annihilates the vectors $\partial_a^\mu$. From this, we can also compute the variation of the vector field $\chi^a$ defined on $M$,

$$\delta \chi^a = \delta X_* \chi^a = X_* \delta \chi^a - \mathcal{L}_\chi X_* \chi^a = \frac{1}{2} [\chi, \chi]^a - [\chi, \chi]^a = -\frac{1}{2} [\chi, \chi]^a,$$

(2.16)

giving a derivation of equation (2.7).

Finally, although this paper refrains from employing explicit coordinate expressions for the pullback of the metric $g_{\alpha\beta} = X^* g_{ab}$, it can be useful to have them on hand. Letting $g_{\mu\nu} = g_{ab} \partial_a^\mu \partial_b^\nu$ denote the components of the metric in the $y^\mu$ coordinate system, and $g_{\alpha\beta}$ the components of $g_{ab}$ in a coordinate system $x^\alpha$ on the reference space, they are related via

$$g_{\alpha\beta}(x) = \frac{\partial X^\mu}{\partial x^\alpha} \frac{\partial X^\nu}{\partial x^\beta} g_{\mu\nu}(X(x)).$$  

(2.17)

### 3 Foliations by submanifolds

The main application for the embedding fields in this paper is in describing embedded submanifolds and foliations. This section discusses the basic construction of a foliation, and

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2With any other derivative operator, one finds that the connection coefficients drop out of this expression, so this choice is not restrictive.
Figure 2: The fixed foliation on $M_0$ is specified through the normal form $\nu$ by requiring all tangent vectors to the foliation annihilate it. $X$ maps $\nu$ to the form $\nu = X.\nu$, which defines a foliation in $M$. The $M$ foliation can be varied by changing the embedding map $X$.

derives the local geometric data associated with it. The idea is to fix a specific submanifold or foliation in the reference space $M_0$, and then use $X$ to map it into spacetime. Variations of geometric quantities under a change in the foliation are then parameterized by variations of $X$. In addition, this setup cleanly separates the effects of varying the spacetime metric from those coming from varying the embedding.

A common way to define a submanifold of codimension $p$ on the reference space is to give $p$ functions $F^A(x), A = 1, \ldots, p$, which simultaneously vanish on the submanifold. These functions also define a foliation via their simultaneous level sets. One disadvantage of this description is that it is highly redundant: any reparameterization of the form $G^A(x) = G^A(F^B(x))$ defines an equivalent foliation. A way to avoid this redundancy is to instead work with a differential $p$-form $\nu$, which vanishes when restricted to the submanifolds, i.e. the tangent vectors $v^a$ to the submanifolds are precisely the ones which annihilate the $p$-form, $i_v \nu = 0$. One can express $\nu$ in terms of the functions $F^A$ simply via

$$\nu = dF^1 \wedge \ldots \wedge dF^p$$

Under the reparameterization $G^A = G^A(F^B)$, $\nu$ changes by an overall rescaling from the Jacobian. Hence, $\nu$ and $f(x)\nu$ define equivalent foliations for any positive function $f(x)$. Working directly with $\nu$ reduces the redundancy of the description from arbitrary reparameterizations of $F^A$ to a single overall rescaling ambiguity in $\nu$.

Going forward, we will forget entirely about the functions $F^A$ and instead work only with $\nu$. It is important to note at this point that an arbitrary $p$-form is not suitable for defining the submanifolds; rather, it must satisfy two additional conditions. The first condition is

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3By assuming that these functions are defined everywhere in a neighborhood of the submanifold, we are implicitly imposing that the normal bundle of the submanifold is trivial. This is not overly restrictive for many applications; however, for the most general class of submanifolds and foliations, the existence of such functions can only be established locally. A simple example where global existence fails is a circle embedded in a Mobius strip, since the normal form would return to its negative after going around the circle.
that the rank of $\nu$ must equal $p$, which means there exists a set of one-forms $e^A$ with $A$ ranging from 1 to $p$, in terms of which $\nu$ can be expressed as

$$\nu = \nu_{A_1\ldots A_p} e^{A_1} \wedge \ldots \wedge e^{A_p}. \quad (3.2)$$

A generic $p$-form requires more than $p$ basis one-forms to express it, so this requirement is that $\nu$ has the minimal possible rank, i.e., it is a simple form. This restriction is an algebraic condition on $\nu$ that must hold pointwise. The reason for it is to ensure the dimension of the submanifold is precisely $(d - p)$. If $\nu$ had rank greater than $p$, fewer vectors would annihilate it, and it would therefore define lower dimensional submanifolds. The second condition is an integrability condition that ensures that the vectors annihilating $\nu$ are tangent to submanifolds. In terms of the vector fields, this means that $[\nu, w]^a$ must annihilate $\nu$ whenever $v^a$ and $w^a$ do. The condition this imposes on $\nu$ is

$$d\nu = \rho \wedge \nu \quad (3.3)$$

for some one-form $\rho$ [38, Section IV.C]. Occasionally, the stronger condition $d\nu = 0$ is imposed, especially when $\nu$ has additional physical meaning, such as a conserved current. An example of this is ideal fluid mechanics, where $\nu$ can represent a conserved particle number or entropy current [5].

Once $\nu$ satisfying the simplicity and integrability conditions has been chosen, it defines a fixed family of submanifolds in the reference space. It can be mapped to spacetime using the embedding fields,

$$\nu = X_\ast \nu, \quad (3.4)$$

and $\nu$ defines a foliation in spacetime, since the pushforward preserves the conditions imposed on $\nu$ (see Figure 2). Finally, we can require that $\nu$ is held fixed under all variations of dynamical fields, $\delta \nu = 0$, which fixes the variation of $\nu$ to be that induced by the variation of the embedding fields $X$,

$$\delta \nu = -\mathcal{L}_X \nu. \quad (3.5)$$

using equation (2.4). Note that this formula ensures that $\nu$ transforms as an ordinary tensor field under spacetime diffeomorphisms, meaning that

$$I_\xi \delta \nu = \mathcal{L}_\xi \nu, \quad (3.6)$$

by equation (2.5), which guarantees that any tensor constructed from $\nu$ and the dynamical fields will also have covariant transformation properties. Variations of objects constructed from $\nu$ are treated in detail in section 4.

### 3.1 Intrinsic and extrinsic geometry

In some applications, such as descriptions of fluids, the unnormalized form $\nu$ is treated as a physical quantity that can be used to build actions and describe dynamics [5, 6]. However, geometric invariants of the foliation should not involve $\nu$ directly, since it contains additional information beyond the embedding, due to the rescaling ambiguity. Instead, the appropriate
object to work with is the normalized unit normal. Normalization of \( \nu \) requires a metric, which is obtained from the dynamical metric on spacetime by an \( X^* \) pullback,

\[
g_{ab} = X^* g_{ab}. \tag{3.7}\]

Alternatively, we can map the normal form to spacetime, giving \( \nu = X_* \nu \), which then defines the foliation on \( M \). The various quantities constructed below from \( \nu \) are mapped to their counterparts on the reference space \( M_0 \) via \( X^* \). Hence, the formulas derived in this section are valid for both bolded and unbolded tensors.

The unit normal \( p \)-form is defined as

\[
n_{a_1 \ldots a_p} = N \nu_{a_1 \ldots a_p}, \tag{3.8}\]

which must satisfy the normalization condition

\[
n_{a_1 \ldots a_p} n^{a_1 \ldots a_p} = \varepsilon p!, \tag{3.9}\]

where \( \varepsilon = \pm 1 \) depending on whether the foliation is spacelike \((-1)\) or timelike \((+1)\).\(^4\) \( N \) must therefore be given by

\[
N = \sqrt{p! \left( \varepsilon g^{a_1 b_1} \ldots g^{a_p b_p} \nu_{a_1 \ldots a_p} \nu_{b_1 \ldots b_p} \right)^{-1/2}}. \tag{3.10}\]

The normal metric \( s_{ab} \) is constructed from the normal form as

\[
s_{ab} = \frac{\varepsilon}{(p-1)!} n_{a_1 \ldots a_p} n_{b_1 \ldots b_p} g^{a_1 b_1} \ldots g^{a_p b_p}. \tag{3.11}\]

The normalization of this tensor comes from requiring that \( s_{ab} \) be a normal projector. One can verify using (3.11) that if \( \alpha_a \) is a normal 1-form, then

\[
s^{a} b \alpha_a = \varepsilon (-1)^{p-1} (\star \star \alpha)_b = \alpha_b, \tag{3.12}\]

where \( \star \) is the Hodge dual on the normal space. This is the point where the simplicity of \( \nu \) is used: the normalization condition (3.9) and the projector requirement \( s^{a} b s_{b} c = s_{c} c \) cannot both hold unless \( \nu \) is simple. More generally, one can form the projector onto the space of normal \( q \)-forms by contracting \( (p-q) \) indices of a pair of unit normals,

\[
s^{a_1 \ldots a_q} b_1 \ldots b_q = s^{a_1} b_1 \ldots s^{a_q} b_q = \frac{\varepsilon}{q! (p-q)!} n^{a_1 \ldots a_q e_{q+1} \ldots e_p} n_{b_1 \ldots b_q e_{q+1} \ldots e_p}, \tag{3.13}\]

and its status as a projector also follows by expressing its action on a \( q \)-form in terms of Hodge duals.

The tangential metric \( h_{ab} \) is then constructed via

\[
h_{ab} = g_{ab} - s_{ab}, \tag{3.14}\]

\(^4\) Null surfaces are excluded from this analysis. The inability to normalize the normal form is one of several reasons the geometric quantities considered in this section cannot be defined for a null surface. A lengthier discussion of how one might adapt the formalism to null surfaces is given in section 7.1.
and the tangential projector \( h^a_b \) is obtained by simply raising the index with the metric. When pulled back to the submanifolds, \( h_{ab} \) defines the induced metric that determines the intrinsic geometry. The volume forms \( \mu_{b_1 \ldots b_{d-p}} \) for the surfaces are obtained by contracting the spacetime volume form \( \epsilon_{a_1 \ldots a_d} \) with the unit normal form, giving the equation

\[
\mu_{b_1 \ldots b_{d-p}} = -\frac{\varepsilon}{p!} n^{a_1 \ldots a_p} \epsilon_{a_1 \ldots a_p b_1 \ldots b_{d-p}}.
\]  

(3.15)

This ensures that the normal form and induced volume form combine into the full spacetime volume form through the equation

\[
\epsilon = -n \wedge \mu.
\]  

(3.16)

Additional geometric information is contained in the covariant derivative of \( s^a_b \). It is straightforward to see that \( \nabla_a s^b_c \) vanishes when \( b \) and \( c \) are both projected in the normal direction, or both tangentially. Of the remaining components, the ones with \( a \) and \( c \) tangent (or equivalently, \( a \) and \( b \)) encode the extrinsic geometry of the embedding through the extrinsic curvature tensor, defined as

\[
K^a_{bc} = h^d_b h^e_c \nabla_d s^a_e = -h^d_b h^e_c \nabla_d h^a_e,
\]  

(3.17)

which is manifestly tangential on \( b \) and \( c \) and normal on \( a \). An important property of the extrinsic curvature tensor \( K^a_{bc} \) is that it is symmetric on its tangential indices, \( b \) and \( c \), which follows from the integrability condition (3.3) for the normal form \( \nu \). Integrability implies that \( \nabla_a n_{c_2 \ldots c_p} = \tilde{\rho}_a n_{c_2 \ldots c_p} \) for some one form \( \tilde{\rho}_a \), and so in particular, \( d n \) has at most one tangential component. We can then write out the condition that \( d n \) vanishes when projected tangentially on two indices to derive

\[
0 = (p + 1) h^a_b h^e_d \nabla_d n_{c_2 \ldots c_p} - h^a_b h^e_d \nabla_d h^a_e,
\]  

(3.18)

From this, the symmetry of \( K^a_{bc} \) follows straightforwardly,

\[
K^a_{bc} = \frac{\varepsilon}{(p-1)!} h^d_b h^e_c n^{a c_2 \ldots c_p} \nabla_d n_{c_2 \ldots c_p} = \frac{\varepsilon}{(p-1)!} h^d_b h^e_c h^a_{c_2 \ldots c_p} \nabla_d n_{c_2 \ldots c_p} = K^a_{cb}
\]  

(3.19)

The components of \( \nabla_a s^b_c \) with \( a \) and \( c \) normal (equivalently, \( a \) and \( b \)) encode the extrinsic geometry of the normal planes to the surface through the normal extrinsic curvature tensor,

\[
L^a_{bc} = -s^d_b s^e_c \nabla_d s^a_e = s^d_b s^e_c \nabla_d h^a_e,
\]  

(3.20)

which is manifestly normal on \( b \) and \( c \) and consequently tangential on \( a \). Unlike \( K^a_{bc} \), the normal extrinsic curvature \( L^a_{bc} \) is not symmetric on its lower indices, since in general the

\[\text{The minus depends on the choice of orientation for both spacetime and the submanifolds. The choice made here is most convenient for spacelike submanifolds with a future-pointing timelike normal.}\]
normal planes are not integrable. Hence, \( L^a_{bc} \) further decomposes into its antisymmetric and symmetric parts

\[
F^a_{bc} = -2L^a_{bc} \tag{3.21}
\]
\[
A^a_{bc} = L^a_{bc}. \tag{3.22}
\]

\( A^a_{bc} \) is called the generalized acceleration tensor, because its contraction with a normal vector \( W^b \) gives

\[
W^b W^c A^a_{bc} = -h^a_c W^b \nabla_b W^c. \tag{3.23}
\]

The second expression is the tangential component of the acceleration of a flow along \( W^a \). Since a symmetric tensor is entirely determined by repeated contractions of this form, knowledge of the accelerations in all possible normal directions completely determines \( A^a_{bc} \).

The twist tensor \( F^a_{bc} \) measures the non-integrability of the normal planes. This follows from viewing the foliation as a fiber bundle, with the leaves of the foliation comprising the fibers. The tangential directions to the leaves coincide with the vertical directions of the fiber bundle, and hence the tangential projector \( h^a_b \) defines a connection on the fiber bundle. Its curvature is given by the Frölicher-Nijenhuis bracket of \( h^a_b \) with itself, and measures the obstruction to integrability of the horizontal (i.e. normal) planes. This bracket can be related to \( F^a_{bc} \), by noting first that \([h, h] = [\delta - s, \delta - s] = [s, s]\) since the identity \( \delta^a_b \) has vanishing bracket with everything, and then computing (see [39, Section II.8])

\[
\frac{1}{2} [h, h]^a_{bc} = \frac{1}{2} [s, s]^a_{bc} = 2 \left( s^d_{\xi} \nabla_d s^a_{\xi} - s^a_d \nabla_b s^d_{\xi} \right) \tag{3.24}
\]

Projecting \( a \) onto the normal direction is seen to give something proportional to \( K^a_{bc} \) which vanishes, hence the bracket must be tangential on \( a \), giving

\[
\frac{1}{2} [h, h]^a_{bc} = 2 h^a_c s^d_{\xi} s^f_{\xi} \nabla_d s^e_{\xi} = F^a_{bc}. \tag{3.25}
\]

Another way to see that \( F^a_{bc} \) measures normal integrability obstruction is to contract with two normal vector \( W^b \) and \( U^c \), which gives

\[
W^b U^c F^a_{bc} = h^a_c [W, U]^c. \tag{3.26}
\]

Hence when \( F^a_{bc} \) is nonvanishing, the commutator of normal vectors fails to remain normal, signaling a lack of integrability.

A final note on the normal extrinsic curvature \( L^a_{bc} \) is that although it is a globally well-defined tensor on the surface, it can become singular if the foliation crosses itself at caustics. Such behavior is inevitable unless the normal bundle is topologically trivial, and we will comment below on the effects of this singular behavior when it occurs.

### 3.2 Covariant derivative operators

Although the induced metric \( h_{ab} \) is defined as a spacetime tensor, restricting its action to vectors tangent to the submanifolds of the foliation naturally defines a metric on these
submanifolds. It therefore is fruitful to discuss the covariant derivative operator compatible with this induced metric. This tangential covariant derivative operator is denoted $D_a$, and its action on a tangential vector $V^b$ is the tangential projection of the spacetime covariant derivative of $V^b$,

$$D_a V^b = h^c_a h^b_d \nabla_c V^d = h^c_a \nabla_a V^b + K^b_a V^d,$$

(3.27)

where the second equality follows from applying equation (3.17). The action of $D_a$ on covectors and multi-index tensors is defined similarly by projecting the covariant derivative of the tensor tangentially on all indices. A consequence of this definition is that $h^a_b$ is annihilated by $D_a$, since

$$D_a h^a_b = h^d_a h^c_b h^e_f \nabla_d h_{ef} = h^d_a (h^c_b \nabla_d h_{ec} - h^f_b \nabla_d h_{fc}) = 0.$$

(3.28)

Hence, $D_a$ is the unique covariant derivative compatible with the induced metric on the submanifolds. This fact leads to the usual coordinate expressions for the connection coefficients of $D_a$ in terms of derivatives of $h^a_b$, written in detail in appendix B.2.

It is also interesting to consider derivatives of normal vectors along the tangential directions of the submanifold. The covariant derivative associated with vectors in the normal bundle will also be denoted by $D_a$, and its action on a normal vector $W^b$ is obtained by a different projection of the spacetime covariant derivative, namely,

$$D_a W^b = h^c_a s^b_d \nabla_c W^d = h^c_a \nabla_c W^b - K^b_a W^d,$$

(3.29)

where again the second equality follows from equation (3.17). This definition guarantees that $D_a W^b$ remains normal on the $b$ index. $D_a$ can be extended to act on tensors with multiple normal indices by projecting all indices in the normal direction after acting with $h^b_a \nabla_b$. Hence, using an argument similar to (3.28), one finds that $D_a s_{bc} = 0$. Appendix B.2 derives the coordinate expressions for the normal connection coefficients, and it is here that one finds that $L^c_{ab}$ is associated with these coefficients. This follows from the equation

$$D_a W^b = s^b_d h^e_c \partial_d W^c + L^b_{ac} W^c$$

(3.30)

where $\partial_c$ is a coordinate derivative for a coordinate system compatible with the foliation, (see appendix B).

The appearance of $L^b_{ac}$ as connection coefficients suggests a modified connection on the normal bundle, $\tilde{D}_b$, which acts on normal vectors and covectors as

$$\tilde{D}_b W^b = D_a W^b - L^b_{ac} W^c; \quad \tilde{D}_b W^b = D_a W^b + L^b_{ac} W^c.$$

(3.31)

This connection simply subtracts off the contribution from $L^b_{ac}$ in (3.30), and so acts like a coordinate derivative on the components of the normal vectors. It appears in a number of variational formulas in section 4. $\tilde{D}_b$ annihilates the tangential metric $h_{ab}$ because it agrees with $D_b$ acting on tangential tensors; however, it is not compatible with the normal metric $s_{ab}$,

$$\tilde{D}_a s_{bc} = 2A_{abc}.$$

(3.32)

This equation is directly related to the coordinate expression for the tensor $A_{abc}$ derived in appendix B.3. If $L^b_{ac}$ has singularities due to caustics in the foliation, the modified
connection $\tilde{D}_a$ is only well-defined away from these singular points. In particular, $\tilde{D}_a$ can be globally defined only for certain normal bundle topologies, including, but not limited to, trivial normal bundles. This is in contrast to $D_a$, which is well-defined for any normal bundle topology.

The action of $D_a$ on tensors with indices of both normal and tangential type is given by first acting with $h^b_a \nabla_b$, and then projecting tangentially all indices that were originally tangential, and projecting normally all indices that were originally normal. For example, the extrinsic curvature tensor $K^a_{bc}$ is normal on $a$ and tangential on $b$ and $c$, so its tangential covariant derivative is

$$D_a K^b_{cd} = h^m_a s^n_b h^p_c h^q_d \nabla_m K^n_{pq}. \quad (3.33)$$

For a tensor with indices that are not definitely tangential or normal, the action of $D_a$ is defined by first decomposing the tensor into tangential and normal pieces, and then acting according to the above definitions. A straightforward application of this rule shows that the spacetime metric $g_{ab}$ is annihilated by $D_a$ since

$$D_a g_{bc} = D_a (h_{bc} + s_{bc}) = D_a h_{bc} + D_a s_{bc} = 0. \quad (3.34)$$

When working with foliations of submanifolds, it is important to also consider derivatives in the normal direction. Analogous to the tangential covariant derivative, we define the normal covariant derivative $\overline{\nabla}_a$ acting on a normal vector $W^b$ by

$$\overline{\nabla}_a W^b = s^c_a s^d_b \nabla_c W^d = s^c_a \nabla_c W^b + L^b_{ad} W^d, \quad (3.35)$$

and on a tangential vector $V^b$ by

$$\overline{\nabla}_a V^b = s^c_a h^b_d \nabla_c V^d = s^c_a \nabla_c V^b - L^b_{da} V^d. \quad (3.36)$$

Its action on multiple indices of mixed tangential and normal type are defined completely analogously to the tangential covariant derivative $D_a$. The normal covariant derivative $D_a$ has many properties analogous $D_a$, including annihilating the normal and tangential metric, $D_a h_{bc} = D_a s_{bc} = 0$. A notable difference arises from the fact that the tangent planes are not necessarily integrable, which means that there are no submanifolds on which $D_a$ restricts to a genuine affine connection. This means that $D_a$ behaves somewhat like a connection with torsion. This can be quantified by computing the commutator $[D_a, D_b]$ acting on a scalar function,

$$(D_a D_b - D_b D_a) f = 2s^e_a s^d_b \nabla_c (s^c_d \nabla_e f) = F^e_{ab} D_e f. \quad (3.37)$$

Here, the negative twist tensor $-F^e_{ab}$ is acting like a torsion for the connection $D_a$. However, it differs from a usual torsion tensor in that the derivative appearing on the right hand side of (3.37) is the tangential derivative $D_e$, rather than $D_e$. This means the torsion points in a tangential direction rather than a normal direction. The tensor $-F^e_{ab}$ is sometimes called the deficiency of the connection $D_a$, to distinguish it from genuine torsion [40, 28]. Additional discussion of this interpretation of $-F^e_{ab}$ as a type of torsion for $D_a$ is provided in appendix A.2.
There is also a modification of $D_b$ acting on tangent vectors that appears in many of the variational formulas. This modified connection $\tilde{D}_b$ acts on tangential vectors as
\begin{equation}
\tilde{D}_a V^b = D_a V^b - K_{ac}^b v^c; \quad \tilde{D}_a V_b = D_a V_b + K_{ab}^c v_c.
\end{equation}
Similar to $D_a$, this modified connection is compatible with the normal metric $s_{ab}$, but not the tangential metric,
\begin{equation}
\tilde{D}_a h_{bc} = 2K_{abc}.
\end{equation}

### 3.3 Gauss, Codazzi, and Ricci-Voss identities

This section describes the curvature tensors associated with the tangential covariant derivative $D_a$ and their relationships to the spacetime curvature tensors. The derivation of these equations as well as their generalizations to the normal connection $\tilde{D}_a$ are given in appendix A. Since the tangential connection $D_a$ is compatible with the induced metric on the submanifolds, its curvature gives information about their intrinsic geometry. This intrinsic curvature $R^a_{bcd}$ tensor is defined through the commutator of two covariant derivatives acting on a tangential vector $V^a$ according to the equation
\begin{equation}
(D_c D_d - D_d D_c)V^a = R^a_{bcd} V^b.
\end{equation}
Its relationship to the ambient spacetime curvature and extrinsic geometry of the surface is encoded in the Gauss equation, which reads
\begin{equation}
R_{abcd} = h^m_a h^n_b h^p_c h^q_d R_{mnpq} + K^e_{ac} K_{ebd} - K^e_{ad} K_{ebc},
\end{equation}
showing that the tangential components of the spacetime curvature tensor and the extrinsic curvature determine the intrinsic curvature of the submanifolds.

$D_a$ also defines a connection on the normal bundle, and there is an outer curvature tensor $O^a_{bcd}$ associated with this connection. It is defined by the commutator $[D_c, D_d]$ acting on a normal vector $W^a$ via
\begin{equation}
(D_c D_d - D_d D_c)W^a = O^a_{bcd} W^d.
\end{equation}
It also satisfies an equation relating it to the spacetime curvature and extrinsic curvature known as the Ricci-Voss equation,
\begin{equation}
O_{abcd} = s^m_a s^n_b h^p_c h^q_d R_{mnpq} + K^e_b d K_{ace} - K^e_b c K_{ade}.
\end{equation}
Since $O_{abcd}$ is normal and antisymmetric in its first two indices, and tangential and antisymmetric in its second two indices, it is trivially traceless on all indices. One can also straightforwardly see that the traces on the right hand side of the equation drop out as well, so it can instead be expressed in terms of the spacetime Weyl tensor $C_{abcd}$ and the traceless extrinsic curvature $\tilde{K}_{abc} = K_{abc} - \frac{1}{d-2} K_{a}^{bc} h_{bc}$ [30],
\begin{equation}
O_{abcd} = s^m_a s^n_b h^p_c h^q_d C_{mpq} + \tilde{K}^e_b d \tilde{K}_{ace} - \tilde{K}^e_b c \tilde{K}_{ade}.
\end{equation}
From equation (3.30), we know that the tensor $L_{ab}^c$ fulfills the role of the normal bundle connection coefficients, and hence there should also be an expression for $O_{abcd}$ in terms of $L_{ab}^c$. This alternative equation for the outer curvature is derived in appendix A.3, and reads

$$O_{abcd} = D_c L_{dba} - D_d L_{cba} + L_{eb}^e L_{dea} - L_{eb}^e L_{cea}.$$  

(3.45)

One can further decompose this expression by separating $L_{abc}$ into its symmetric and antisymmetric parts according to equations (3.21) and (3.22), which leads to the final expression for the outer curvature,

$$O_{abcd} = D_c F_{dab} + \frac{1}{2} F_{eb}^e F_{dea} + 2 \tilde{A}_{cb}^e \tilde{A}_{dea},$$

(3.46)

where only the traceless part of the acceleration tensor $\tilde{A}_{cea} = A_{cea} - \frac{1}{p} A_c s_{ea}$ appears.

The modified connection $\tilde{D}_a$ introduced in (3.31) in principle also has an associated outer curvature, but appendix A.4 demonstrates that this curvature vanishes. This is consistent with the interpretation of $\tilde{D}_a$ treating normal vectors as a collection of scalars, and hence acting as a tangential partial derivative. Note that for this flat connection to be globally defined, the normal bundle must satisfy certain topological restrictions. In particular, it requires any topological invariants constructed from the outer curvature, such as the Euler number, to vanish.

Finally, there is an identity associated with the requirement that $(D_c D_d - D_d D_c)V^a$ is tangential on $a$ for $V^a$ tangential, which according to the definition of $D_c$ is trivially true. However, expressing this condition in terms of the spacetime curvature leads to the Codazzi equation,

$$h^n_a h^n_b h^p_c s^q_d R_{mpq} = D_a K_{dbc} - D_b K_{dac}.$$  

(3.47)

This same equation arises from requiring that $(D_c D_d - D_d D_c)W^a$ is normal on $a$ for $W^a$ normal.

There are similar curvature quantities and identities associated with the normal covariant derivative $D_a$, which are discussed in more detail in appendix A.2.

### 3.4 Invariant tensors of a submanifold

For some applications, one is interested only in the properties of a single submanifold of a foliation. In these cases, it is important to know which quantities are independent of how the foliation is extended away from the submanifold. These quantities will be called invariant tensors of the submanifold. This section argues that the invariant tensors consist of the tangential metric $h_{ab}$, the normal metric $s_{ab}$, the tangential extrinsic curvature $K_{abc}^a$, and any quantities constructed from tangential covariant derivatives $D_a$ of invariant tensors, which include the intrinsic curvature $\mathcal{R}^{abc}_d$ and outer curvature $O^{abc}_d$. Of course, tensors that are defined without reference to the embedding, such as the spacetime Riemann tensor $R_{abcd}$, are also invariant. Notably, the normal extrinsic curvature $L_{ab}^c$ is not an invariant tensor, and its transformations under a change in the foliation away from the submanifold is
derived below. Although it is always possible to set $L_{ab}^c$ to zero at a point, the fact that the invariant outer curvature is constructed from $L_{ab}^c$ means that it cannot vanish everywhere, as is typical of connection coefficients in the presence of curvature. Nevertheless, there always exists a choice of foliation such that the symmetric piece $A_{ab}^c$ vanishes, and this coincides with extending the foliation away from the submanifold by flowing radially along normal geodesics. With this choice of foliation, there is still additional freedom to adjust the antisymmetric part $F_{ab}^c$, which can be used to impose gauge conditions on this tensor.

The analysis of invariant tensors begins by noting that the unit normal $n$ is invariant since it depends algebraically on the normal form $\nu$, and normalization ensures it does not change when $\nu$ is rescaled. Any tensors constructed algebraically from $n$ are then also invariant, and these include the normal metric $s_{ab}$, tangential metric $h_{ab}$, and induced volume form $\mu$.

Next we show that the tangential covariant derivative $D_a$ acting on an invariant tensor produces another invariant tensor. This follows immediately from the fact that if two tensors $T_{ab}^{\cdots b...}$ and $U_{ab}^{\cdots b...}$ agree on a submanifold $\Sigma$, then the gradient of their difference is tensorial and normal to the surface. This is because $\nabla_e(T_{ab}^{\cdots b...} - U_{ab}^{\cdots b...}) = \partial_e(T_{ab}^{\cdots b...} - U_{ab}^{\cdots b...})$ since the contributions for connection coefficients all involve $T_{ab}^{\cdots b...} - U_{ab}^{\cdots b...}$ undifferentiated at the surface, which vanishes. Furthermore, contracting with a tangent vector on $e$ also gives a vanishing result since $T_{ab}^{\cdots b...} - U_{ab}^{\cdots b...}$ vanishes everywhere on the surface. Now if $T_{ab}^{\cdots b...}$ and $U_{ab}^{\cdots b...}$ are taken to be invariant tensors for different foliations that agree at $\Sigma$, their difference must vanish on $\Sigma$, and

$$D_e T_{ab}^{\cdots b...} - D_e U_{ab}^{\cdots b...} = h^d e \nabla_d(T_{ab}^{\cdots b...} - U_{ab}^{\cdots b...}) = 0. \quad (3.48)$$

Hence, $D_e T_{ab}^{\cdots b...}$ defines an invariant tensor. This also implies that the curvatures $\mathcal{R}_{abcd}$ and $\mathcal{O}_{abcd}$ are invariant, since they can be expressed in terms of $D_a$ acting on invariant tensors.

The invariance of the extrinsic curvature $K_{ab}^c$ follows from a similar argument. Note that $s_{ab}^e$ is invariant, and let $\bar{s}_{ab}^e$ denote the normal projector for a different foliation that agrees at $\Sigma$. Then $\nabla_e(s_{ab}^e - \bar{s}_{ab}^e)$ is normal on $e$, and so

$$K_{ab}^c - \bar{K}_{ab}^c = h_{p \bar{p}} h^d \nabla_d(s_{ab}^e - \bar{s}_{ab}^e) = 0. \quad (3.49)$$

We now turn to the transformation properties of $L_{ab}^c$ under a change in foliation. First, consider the expression for the covariant derivative of the unit normal, $\nabla_e n_{a_1...a_p}$. This has no component that is normal on all the $a_i$ indices, since this would be proportional to $n_{a_1...a_p}$ due to antisymmetry, but $n_{a_1...a_p} \nabla_e n_{a_1...a_p} = \frac{1}{2} \nabla_e (n_{a_1...a_p} n_{a_1...a_p}) = 0$. Hence, $\nabla_e n_{a_1...a_p}$ is tangential on at least one $a_i$ index, and by projecting tangentially we derive that

$$\nabla_e n_{a_1...a_p} = p \left( K_{e a_1}^{b a_2} - L_{a_1}^{b a_2} \right) n_{b a_2...a_p} \quad (3.50)$$

Now take $\bar{n}$ to be the unit normal of a different foliation that agrees with $n$ at $\Sigma$. From the above formula, we find that their normal gradients differ according to

$$\nabla_e (\bar{n}_{a_1...a_p} - n_{a_1...a_p}) = p l_{a_1}^{b a_2...a_p} n_{b a_2...a_p} \quad (3.51)$$

with

$$l_{a_1}^{b a_2...a_p} = L_{a_1}^{b a_2...a_p} - \bar{L}_{a_1}^{b a_2...a_p}. \quad (3.52)$$
Hence, the tensor $L_{ae}^b$ shifts under a change in the foliation by $-l_{ae}^b$, which characterizes the first order change in the unit normal when moving off of $\Sigma$ by equation (3.51).

The transformation of $L_{ae}^b$ in (3.52) suggests that it could be set to zero by choosing $l_{ae}^b$; however, not all tensors $l_{ae}^b$ define a valid change in the foliation. The new unit normal $\bar{n}$ must remain normalized according to (3.9) and satisfy the simplicity and integrability constraints arising from (3.2) and (3.3). After ensuring $\bar{n}$ is normalized, the latter two conditions are equivalent to

$$\bar{s}_a^b \bar{s}_c^b = \bar{s}_c^a, \quad (3.53)$$
$$\bar{s}_d^a [\bar{s}, \bar{s}]_{bc}^d = 0, \quad (3.54)$$

where the second condition involves the Frölicher-Nijenhuis bracket (see equation (3.24)). Taking gradients of these equations with $s_a^b$ expressed in terms of $\bar{n}$ leads to additional constraints on the gradients of $(\bar{n} - n)$. In particular, the gradient of (3.54) gives a differential restriction on $l_{ae}^b$. A quicker way to derive this restriction is to note that the outer curvature $\mathcal{O}_{abcd}$ is an invariant tensor, and hence should be the same whether computed using $L_{ab}^c$ or $\tilde{L}_{ab}^c$. Using equation (3.45), this implies that

$$D_\xi l_{dea} + L_{\xi b}^e l_{dea} + l_{\xi b}^e L_{dea} - l_{\xi b}^e l_{dea} = 0, \quad (3.55)$$

which can be simplified by using the modified connection $\tilde{D}_a$ (3.31),

$$\tilde{D}_\xi l_{dea} + 2l_{\xi b}^e A_{dea} - l_{\xi b}^e l_{dea} = 0 \quad (3.56)$$

or equivalently

$$\tilde{D}_\xi l_{gb}^a - l_{\xi b}^e l_{dea} = 0. \quad (3.57)$$

(3.57) looks like a condition of vanishing curvature for a connection, and so it is easy to write down solutions. Choose any tensor $m_a^e$ with normal indices and require that it be invertible in the sense that $(m^{-1})_b^e m_a^e = s_b^a$ for some tensor $(m^{-1})_b^e$. Then

$$l_{db}^a = -(m^{-1})_b^e \tilde{D}_d m_a^e \quad (3.58)$$

solves (3.57), which is seen using the identity $\tilde{D}_c (m^{-1})_b^e = -(m^{-1})_b^f \tilde{D}_c m_f^g (m^{-1})_g^e$, and the fact that the outer curvature of $\tilde{D}_c$ vanishes, as shown in appendix A.4. A particularly useful class of solutions are those in which $l_{dea} = A_{dea}$, since this transforms to a new foliation in which $A_{dea} = 0$ on $\Sigma$. Expressing this condition in terms of $m_a^e$ gives

$$2A_{dea} = \tilde{D}_ds_{ba} = -(m^{-1})_b^e s_{ae} \tilde{D}_d m_a^e - (m^{-1})_a^e s_{be} \tilde{D}_d m_a^e = \tilde{D}_d (m_b^e m_a^e s_{ec}) \quad (3.59)$$

Hence, the matrix $m_b^e$ must transform $s_{ec}$ into a metric $\eta_{ab} = m_a^e m_b^e s_{ec}$ that is compatible with the modified connection $\tilde{D}_d$. Since all metrics of the same signature are related by some $GL(p)$ transformation, such a matrix can always be found pointwise, but there can be topological obstructions to choosing it smoothly globally. Hence, $\tilde{A}_{dea} = 0$ can be set to zero.
everywhere on the surface only for special topologies, although for many applications the topology is sufficiently trivial that the acceleration can be set to zero globally.

The tensor \( m^b_a \) is not unique; the additional freedom in choosing it consists of transformations that leave \( \eta_{ab} \) invariant. These comprise an orthogonal group, \( SO(p-1,1) \) for \( \varepsilon = -1 \) or \( SO(p) \) for \( \varepsilon = +1 \), and they can be used to change \( F^a_{bc} \) while leaving \( A^a_{bc} \) invariant. Since \( F^a_{bc} \) is antisymmetric on its normal indices \( b \) and \( c \), it can be viewed on \( \Sigma \) as a vector valued in the Lie algebra of the appropriate orthogonal group. On the other hand, \( l^c_{ab} \) in (3.58) is a gradient of a group element, so can be used to cancel a scalar degree of freedom from \( F^a_{bc} \).

For example, one might use this remaining freedom to impose an axial-like gauge \( V^a F^a_{bc} = 0 \) for some fixed tangential covector \( V_a \), or a Coulomb-like gauge \( \tilde{D}^a F^a_{bc} = 0 \).

A tensor \( m^b_a \) that sets the acceleration to zero is closely related to a vielbein for the normal space. The inverse vielbein is \( m^A_a \), obtained from \( m^b_a \) by expressing its lower index in a coordinate basis. It satisfies \( \eta^{AB} m^A_a m^b_B = s^{ab} \), where \( \eta^{AB} \) are the components of the inverse metric compatible with the flat normal bundle connection \( \tilde{D}_a \), which can be taken to be constant. One can further show that the modified normal extrinsic curvature can be expressed in terms of \( m^b_a \) as

\[
\bar{L}^c_{ab} = (m^{-1})_b^A D_a m^c_A , \tag{3.60}
\]

which only involves the twist term \(-2F^c_{ab}\) when \( m^c_A \) defines an inverse vielbein. The use of vielbeins is common in Carter’s treatment of embedded submanifolds \([30, 7]\), where an analogous formula to (3.60) is used to define the twist tensor. The above discussion shows that the acceleration tensor will never appear when vielbeins are employed, which explains why such an object is not discussed in Carter’s work. Allowing for an acceleration tensor represents a more general choice of parameterization of the normal bundle, and is indispensable when considering a foliation as opposed to single submanifold. This acceleration tensor can also encode topological information about the normal bundle, since it must be nonvanishing somewhere when there is a topological obstruction to choosing \( m^b_a \) smoothly. In particular, topological invariants constructed the outer curvature \( O^a_{bcd} \), such as the normal bundle Euler number, can be expressed as global integrals in terms of the original \( F^c_{ab} \) and \( A^c_{ab} \), but must be generally constructed in local patches when working with an orthonormal normal basis and \( \tilde{F}^c_{ab} \).

The coordinate expression (B.26) for \( A^a_{bcd} \) shows that choosing the foliation away from \( \Sigma \) to set \( A_{abc} \) to zero is equivalent to setting the components \( s_{AB} \) of the normal metric to constant values on \( \Sigma \). Furthermore, the tensor \( A^a_{bcd} \) measures the acceleration of flows normal to \( \Sigma \) due to equation (3.23). Hence, a foliation in which neighboring surfaces are reached by following a normal geodesic necessarily will have \( A_{abc} = 0 \) on \( \Sigma \). These foliations therefore define a class of normal coordinates adapted to the surface. In such a coordinate system, one would like to define the transverse basis vectors to be tangent to a family of normal geodesics; however, this is generally not possible when \( F^a_{bc} \) is nonzero [30, 7]. This is because \( F^a_{bc} \) measures the tangential component of the commutator of normal vectors according to equation (3.26). Since coordinate basis vectors must commute, generically one cannot choose the coordinate basis vectors to be normal to the surface. At best, the freedom to shift \( F^a_{bc} \) can be used to set it to zero at a point, and the basis vectors can be chosen to be normal there. Away from the point where \( F^a_{bc} \) vanishes, some other prescription must be
given to determine the transverse basis vectors, and the local shift freedom in $F^a_{bc}$ will lead to different available choices in this prescription. Hence, unless $F^a_{bc}$ vanishes everywhere, which is only possible if the outer curvature $O_{abcd}$ vanishes, there is no unique set of normal coordinates.

4 Variational formulas

A central motivation for working with the embedding field $X$ comes from considering variations of geometric quantities of surfaces. Variations of the spacetime metric $g_{ab}$ and the embedding map both lead to changes in the geometry, and the $X$ field allows one to cleanly separate the contributions coming from each type of variation. This section works out a number of formulas for these variations.

The key property that determines the variations is that the normal form $\nu$ is fixed in the reference space, $\delta \nu = 0$. As mentioned in equation (3.5), this sets the variation of the normal form on spacetime to be $\delta \nu = -\mathcal{L}_X \nu$. It receives no contribution from the variation of the metric, which is expected since the foliation is defined without reference to a metric. $\nu$ transforms as a tensor under spacetime diffeomorphisms by equation (3.6), and this ensures any object built from $\nu$ and spacetime tensor fields will also transform covariantly. On the other hand, $\nu$ is not covariant under diffeomorphisms of the reference space, since $I^\xi \delta \nu = 0$, which generally differs from $\mathcal{L}_\xi \nu$. This means the geometric quantities constructed from $\nu$ will not transform as tensors on the reference space, so some care must be taken when computing their variations under a change in the embedding $X$ induced by a reference space diffeomorphism. Nevertheless, it will be shown below that the unit normal $n$ is covariant with respect to foliation-preserving transformations of the references space, generated by vectors with a covariantly constant normal component with respect to $\tilde{D}_a$ defined in equation (3.31). These include, in particular, purely tangential vectors. Since all other geometric tensors are constructed from $n$ and $g_{ab}$, their variation under a tangential diffeomorphism will always be given by a Lie derivative. The variations with respect to normal diffeomorphisms are generally more involved, and require a careful analysis to determine.

Before doing any calculations, there are a few general rules that help simplify the computations. First, since the restriction $\delta \nu = 0$ is imposed in the reference space, it often is easier to compute variations of quantities $\phi$ in the reference space first, and from this infer the variation of the corresponding spacetime quantity using $\delta \phi = X_\ast \delta \phi - \mathcal{L}_X \phi$. Second, variations of tensors with mixed tangential and normal indices tend to be simplest if all normal indices are taken to be covariant (downstairs) and all tangential indices taken to be contravariant (upstairs). This is related to the fact that being tangential for a contravariant index depends only on $\nu$ and not the metric, and so the variation will remain tangential. Being normal for covariant indices is also metric-independent, so these also remain normal when varied. The variations of tensors with different index placements can then be straightforwardly computed using $\delta \phi^a = \delta g_{ab} \phi^b + g_{ab} \delta \phi^b$ and $\delta \psi^a = -\delta g_{ab} g^{ca} \psi^b + g^{ab} \delta \psi_b$. 

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4.1 Intrinsic geometry

We begin with the unit normal \( n \) defined in equation (3.8). Its variation comes entirely from the variation of the norm \( N \) from equation (3.10), which is calculated as follows

\[
\delta N = -\frac{1}{2} \sqrt{p!} (\varepsilon \nu_{e_1} \cdots \nu_{e_p} \nu^{e_1} \cdots \nu^{e_p})^{-3/2} \varepsilon p \delta g^{a_1b_1} \nu_{a_1} \nu_{a_2} \cdots \nu_{a_p} n^{a_2} \cdots a_p \\
= \frac{1}{2} N^3 \varepsilon p \delta g_{cd} \nu^{c_{a_2}} \cdots a_p \nu^{d_{a_2}} \cdots a_p \\
= \frac{1}{2} N s^{cd} \delta g_{cd},
\]

(4.1)

where the definition (3.11) was used. This immediately leads to the formula for the variation of the unit normal,

\[
\delta n = \frac{1}{2} s^{cd} \delta g_{cd} n.
\]

(4.2)

It is interesting to single out the contribution to this variation coming purely from a change in the embedding, encapsulated by the terms involving \( \chi^a \). They appear implicitly in (4.2) through \( \delta g_{cd} \) since \( \delta g_{cd} = \chi^a \delta g_{cd} + \chi^a \delta g_{cd} \). The terms involving \( \chi^a \) in \( \delta n \) will be denoted \( L_{\chi} n \equiv I_{\chi} \delta n \), employing the notation of section 2 for this variation. It is helpful to decompose \( \chi^a = \tau^a + \sigma^a \) in to its tangential \( \tau^a \) and normal \( \sigma^a \) pieces,

\[
\tau^a = h^a_b \chi^b, \quad \sigma^a = s^a_b \chi^b.
\]

(4.3)

Then (4.2) leads immediately to

\[
L_{\chi} n = s^{cd} (\nabla_c \chi_d) n = (D_c \sigma^c + \tau^c A_c) n.
\]

(4.4)

It is helpful to compare this expression to \( L_{\chi} n \). The \( \tau^a \) contribution remains proportional to \( n \), since \( L_{\tau} n = i_{\tau} d n = \tau^c (D_c \log N + \rho_c) n \) with \( \rho_c \) defined by the integrability condition (3.3) for \( \nu \). By expanding out the Lie derivative and contracting with \( \frac{\varepsilon}{p!} n^{b_1} \cdots b_p \), a different expression for the \( n \) coefficient is derived,

\[
\frac{\varepsilon}{p!} n^{b_1} \cdots b_p \left( \tau^c \nabla_e n_{b_1} \cdots b_p + p(\nabla_b \tau^e) n_{e b_2} \cdots b_p \right) = s^b_1 \nabla_b \tau^e = \tau^c A_c,
\]

(4.5)

which matches the term appearing in (4.4). Incidentally, it also gives a relation for \( \rho_c \),

\[
\rho_c = A_c - D_c \log N.
\]

(4.6)

For the \( \sigma^a \) contribution, there will be a piece proportional to \( n \), and a term with one tangential index. The purely normal term is isolated as before by contracting with \( \frac{\varepsilon}{p!} n^{b_1} \cdots b_p \), which is seen to give \( D_b \sigma^b \). The term with a tangential index is

\[
\frac{\varepsilon}{(p-1)!} n^{c_{b_2} \cdots b_p} h^b_d \left( \sigma^e \nabla_{e} n_{b_2} \cdots b_p + p(\nabla_b \sigma^e) n_{e b_2} \cdots b_p \right) = D_d \sigma^e - L_{cd} \sigma^e = \tilde{D}_d \sigma^e.
\]

(4.7)

Hence the normal Lie derivative is

\[
L_{\sigma} n_{b_1} \cdots b_p = (D_e \sigma^e) n_{b_1} \cdots b_p + p(\tilde{D}_{b_1} \sigma^e) n_{b_2} \cdots b_p,
\]

(4.8)
and only matches (4.4) if $\tilde{D}_b \sigma^c = 0$, which is the condition for the diffeomorphism to preserve the foliation. \(^6\)

Next, applying the definition (3.11) for the normal metric, its variation is

$$\delta s_{ab} = \frac{\varepsilon}{(p-1)!}(s^{cd}\delta g_{cd}n_{ae_2...e_p}n_{b}^{e_2...e_p} - (p-1)\delta g_{cd}n_{a}^{ec_3...e_p}n_{b}^{d\ldots e_p})$$

$$= s^{cd}\delta g_{cd}s_{ab} - \frac{\varepsilon}{(p-1)!}(p-1)(s_{ab}s^{cd} - s_{d}\delta g_{cd})$$

$$= s^{d}a\delta s_{b}^c\delta g_{cd}. \tag{4.9}$$

where the second line applied the identity (3.13) with $q = 2$. Hence the variation of the normal metric is simply the normal projection of the full metric variation. The inverse metric variation is a bit more complicated because it has contravariant normal indices,

$$\delta s^{ab} = \delta g_{cd}(s^{ca}s^{db} - s^{ca}g^{bd} - g^{ca}s^{db}) = -\delta g_{cd}(s^{ca}s^{db} + s^{ca}h^{db} + h^{ca}s^{db}) \tag{4.10}$$

which involves both normal-normal and normal-tangential components of the metric variation.

The normal and tangential projector variations are closely related since $\delta s^a_{\ b} = \delta (\delta^a_{\ b} - h^a_{\ b}) = -\delta h^a_{\ b}$. Explicitly, this variation is

$$\delta s^a_{\ b} = \delta g_{cd}(-g^{ac}s^{db} + s^{ac}s^{db}) = -h^{ac}s^{db}\delta g_{cd} \tag{4.11}$$

The inverse tangential metric also has a simple variation, given by

$$\delta h^{ab} = \delta g^{ab} - \delta(g^{ac}g^{bd}s_{cd})$$

$$= \delta g_{cd}(-g^{ac}g^{bd} + g^{ac}s^{db} + s^{ac}g^{bd} - s^{ac}s^{db})$$

$$= -h^{ac}h^{bd}\delta g_{cd}. \tag{4.12}$$

This involves only the tangential components of the metric variation. For the tangential metric with covariant indices, there are additional contributions coming from normal-tangential components of $\delta g_{ab}$, similar to how they arose in (4.10),

$$\delta h_{ab} = \delta g_{cd}(h^c_{\ a}h^d_{\ b} + h^c_{\ b}h^d_{\ a} + s^c_{\ a}h^d_{\ b}). \tag{4.13}$$

Equations (4.9), (4.11), and (4.12) cover all of the independent components of the metric variation, and determines its decomposition into normal and tangential pieces to be

$$\delta g_{de} = \delta s_{de} + h^{dm}\delta h^m_e + h^e_n\delta h^n_d - h^{dm}h^{en}\delta h^{mn}. \tag{4.14}$$

\(^6\)As an aside, it is also interesting to consider when the Lie derivative of the unnormalized form $\nu$ agrees with its variation under a change in embedding, which simply vanishes $L_\chi \nu = 0$. For tangential vectors, $\mathcal{L}_\tau \nu = \tau^\gamma p_\gamma \nu$. This vanishes automatically if $d\nu = 0$, which is relevant in applications where $\nu$ represents a conserved quantity, such as the entropy current in fluid dynamics. The normal Lie derivative is $\mathcal{L}_\nu \nu_{a...p} = \nu_{c}(\frac{\partial}{\partial x^c})n_{a...p} + p(\nu_{c}\sigma^a)\nu_{c...p}$, so for this to vanish, $\nu$ must both be foliation-preserving and satisfy $D_e (\sigma^a) = 0$, which is interpreted as a volume-preserving condition in the normal directions. Volume-preserving diffeomorphism symmetry is often used as an organizing principle in treatments of finite-temperature fluid mechanics [6, 13, 14].
We also need the variation of the induced volume form \( \mu \). From its definition (3.15) we have (suppressing the \(( d - p)\) tangential indices)

\[
\delta \mu = -\frac{\varepsilon}{p!} (\delta n_{a_1...a_p} \epsilon^{a_1...a_p} \mu - p \delta g_{cd} n^{c a_2...a_p} \epsilon_{a_2...a_p} + n^{a_1...a_p} \delta \epsilon_{a_1...a_p}) \\
= \delta g_{cd} \left( \frac{1}{2} - \frac{\varepsilon}{(p-1)!} \epsilon(p-1)! + \frac{1}{2} \right) + \frac{\varepsilon}{(p-1)!} n^{c a_2...a_p} h^{d e} \epsilon_{a_2...a_p} \\
= \delta g_{cd} \left( \frac{1}{2} h^{c d} \mu + \frac{\varepsilon}{(p-1)!} n^{c a_2...a_p} h^{d e} \epsilon_{a_2...a_p} \right) \\
= \delta g_{cd} \left( \frac{1}{2} h^{c d} \mu + \frac{\varepsilon}{(p-1)!} n^{c a_2...a_p} h^{d e} \epsilon_{a_2...a_p} \right) \quad (4.15)
\]

The second term can be simplified somewhat by recalling that \( \epsilon = -n \wedge \mu \) and employing a useful formula for the wedge product,

\[
(\alpha \wedge \beta)_{a_1...a_p b_1...b_q} = \sum_{n=0}^{\min(p,q)} (-1)^n \binom{p}{n} (q - n)! \alpha_{a_1...a_n b_{n+1}...b_q} \beta_{b_1...b_n a_{n+1}...a_q} \quad (4.16)
\]

where the antisymmetrization occurs separately for the single-underlined and double-underlined indices. Applying this to the second term in (4.15) gives

\[
\delta \mu_{b_1...b_{d-p}} = \delta g_{cd} \left( \frac{1}{2} h^{c d} \mu_{b_1...b_{d-p}} + (d - p) s^{c d} \mu_{b_1...b_{d-p}} \right) \quad (4.17)
\]

The first term gives the expected expression for the variation of the intrinsic volume form on the surfaces, and is the only term that remains when \( \delta \mu \) is restricted to the surface. The second term tracks the component that must be added to the original volume form after the variation to keep it tangential, since tangential covectors do not remain tangential when the metric is varied.

### 4.2 Tangential extrinsic curvature

Next we turn to the variational formulas for the extrinsic curvatures. These feature prominently when considering perturbations to extremal surfaces and lead to a derivation of the Jacobi equation for such surfaces. We begin with the tangential extrinsic curvature tensor defined by (3.17). As usual, the variation is simplest with the tangential indices up and normal index down, and applying (4.9), (4.11), and (4.12) leads to

\[
\delta K^a_{bc} = \delta (s^{a d} h^{b e} h^{c f} \nabla_e s^d_f) \\
= \delta g_{de} (s^d_a K^{ebc} - h^{bd} K^a_{bc} - h^{cd} K^a_{bc}) - \delta \Gamma^f_{de} s^f_a h^{bd} h^{ce} \quad (4.18)
\]

where the variation of the Christoffel symbol is

\[
\delta \Gamma^f_{de} = \frac{1}{2} g^{fm} (\nabla_d \delta g_{me} + \nabla_e \delta g_{md} - \nabla_m \delta g_{de}) \quad (4.19)
\]

The contribution of \( \chi^a \) to this expression is of particular interest. Since \( K^a_{bc} \) is constructed from \( n \) and \( g_{ab} \), the contributions from \( \tau^a = h^a_b \chi^b \) will be simply given by \( \pounds \chi^a \). Hence,
we focus on the normal contribution involving $\sigma^a = s^a_\chi^b$, denoted $L_\sigma K_a^{bc}$. First, consider the $\delta \Gamma^f_{de}$ term of (4.18),

$$-(L_\sigma \Gamma^f_{de}) s_{af} h^{bd} h^{ce} = -h^{bd} h^{ce} s_{af} (\nabla_d \nabla_f \sigma^e + \nabla_i \nabla_f \sigma^e - \nabla_f \nabla^e \sigma^e) = h^{bd} h^{ce} s_{af} \left( R^m_{ef} \sigma_m - \nabla_d \nabla_f \sigma_f \right).$$

(4.20)

The second term here can be expanded by employing the identity (A.5) derived in appendix A.1 to give

$$-h^{bd} h^{ce} s_{af} \nabla_d \nabla_f \sigma_f = -D^\nu D^\nu \sigma_a - K^{ebc} D_e \sigma_a + K_a \tilde{D}^e K_a^\nu \sigma_d$$

(4.21)

The remaining terms in (4.18) involving $\delta g_{de}$ are straightforward to evaluate, giving

$$2 \nabla_d \sigma_f (s^d_a K^{ebc} - h^{bd} K_a^{ce} - h^{cd} K_a^{be}) = K^{dbce} (D_d \sigma_a + D_a \sigma_d) - 4 K^d \tilde{D}^e K_a^\nu \sigma_d$$

(4.22)

Combining (4.21) and (4.22), we arrive at the final expression for the change in $K_a^{bc}$ under a change in the embedding,

$$L_\sigma K_a^{bc} = K_a \tilde{D}^e K_a^\nu \sigma_d + \left( h^{bm} h^{ce} s_{af} R^d_{ef} - 3 K^d \tilde{D}^e K_a^\nu \sigma_d \right) \sigma_d$$

(4.23)

The full extrinsic curvature variation (4.18) also receives contributions from the spacetime metric variation in the form $X^* \delta g_{de}$. The expression involving these terms can be simplified by decomposing the metric variation into its normal and tangential components as in (4.14) and computing each contribution separately. The first thing to notice is that the terms in (4.18) involving $\delta s_{de}$ drop out,

$$\delta s_{de} s^d_a K^{ebc} - h^{bd} h^{ce} s_{af} \nabla_d \delta s_{ef} = 0.$$ 

(4.24)

The terms coming from $\delta h^{ce}_a$ are

$$- \frac{1}{2} \left( D^b \delta h^{ce} + D^c \delta h^{be} + L^c_a \delta h^{be} + L^b_a \delta h^{ce} \right) = -\tilde{D}^\nu \delta h^c_a.$$ 

(4.25)

Finally, the remaining pieces from $\delta h^{bc}$ are

$$\delta h^{bc} K_a^{ce} + \delta h^{ec} K_a^{bc} - \frac{1}{2} \left( K_a^{be} \delta h^{ce} + K_a^{ce} \delta h^{be} + D_a \delta h^{bc} \right) = -\frac{1}{2} \tilde{D}^\nu \delta h^{bc}.$$ 

(4.26)

Of course, these expressions are simply a rewriting of (4.18), hence still contain contributions from $X^a$; however, it is straightforward to separate off the spacetime metric variation. Denoting this contribution $X^* \kappa_a^{bc}$, we have

$$\kappa_a^{bc} = \frac{1}{2} \tilde{D}_a (h^{bd} h^{ce} \delta g_{de}) - \tilde{D}^e (h^{de} s_{af} \delta g_{de}).$$ 

(4.27)

Thus, we have the general formula that $\delta K_a^{bc} = X^* \kappa_a^{bc} + L_\sigma K_a^{bc} + \tilde{L}_\tau K_a^{bc}$, with the individual contributions given by (4.23) and (4.27).
The formula for $\delta K_a$ then follows from those for $\delta K_a^\ bc$ since $\delta K_a = h_{bc} \delta K_a^\ bc + K_a^\ bc \delta h_{bc}$. Applying (4.18) then gives
\[
\delta K_a = \delta g_{bc} K_a^\ bc - \delta \Gamma_{bc}^\ d s_{ad} h_{bc}.
\] (4.28)
The contribution involving $\sigma^a$ follows in a similar manner from (4.23) giving
\[
\hat{L}_\sigma K_a = K_d D_a \sigma^d - D_b D^b \sigma_a + (s^e_a h_{bc}^\ dr_{bc} - K_{abc}^\ d K_a^\ bc) \sigma_d,
\] (4.29)
and again the $L_\tau K_a$ terms are simply $\hat{L}_\tau K_a$. The remaining parts of $\delta K_a$ involving the spacetime metric are denoted $X^* \kappa_a$, and since $\kappa_a = h_{bc} \kappa_a^\ bc + K_a^\ bc \delta g_{bc}$, it is expressed using (4.27) as
\[
\kappa_a = \frac{1}{2} D_a (h_{bc} \delta g_{bc}) - \tilde{D}_b (h_{bd} s^e_a \delta g_{bc}).
\] (4.30)
The full variation of $K_a$ is then given by the sum $\delta K_a = X^* \kappa_a + \hat{L}_\sigma \kappa_a + \hat{L}_\tau K_a$ involving (4.29), (4.30), and the tangential piece.

### 4.2.1 Jacobi equation

An immediate application of the variational formula (4.29) for $K_a$ is a derivation of the Jacobi equation, which describes deformations of extremal surfaces which leave them extremal. These surfaces are described by an embedding map $X$ that is a stationary point of the volume functional,
\[
V = \int_\Sigma \mu,
\] (4.31)
where $\Sigma$ is a fixed $(d - p)$-dimensional surface in the reference space with normal form $\nu$. Using (4.17) for $\delta \mu$, the variation of this functional is
\[
\delta V = \int_\Sigma \delta \mu = \int_\Sigma h^{cd} \delta g_{cd} \mu = \int_\Sigma \sigma^e K_e \mu + \int_{\partial \Sigma} \tau^e \mu_e + \frac{1}{2} \int_\Sigma X^* (h_{cd} \delta g_{cd} \mu)
\] (4.32)
where in the last expression the variation has been split into a contribution from the spacetime metric variation $\delta g_{cd}$, and the contributions from the normal and tangential components of $X^a$. Hence, for fixed spacetime metric, the embedding must satisfy $K_e = 0$ to be a stationary point of $V$.

The Jacobi equation involves perturbations to nearby stationary surfaces, so it comes from demanding that $K_e$ remains 0 after varying the embedding. For a specific choice of the deformation described by the vector $\xi^a$, this is just the statement that $L_\xi K_e = 0$. The tangential component $\xi^a = h^{a} \xi^b$ does not contribute to this equation since it is just given by $L_\xi K_e$, which vanishes since $K_e = 0$ everywhere on the surface. The normal contribution $\eta^a = s^a \beta \eta ^b$ is nontrivial, and equation (4.29) shows that it must satisfy
\[
D_b D^b \eta^a + (K_{bc}^\ d K_a^\ bc - s^ea h_{bc} R_{d b c e}) \eta^d = 0,
\] (4.33)
\footnote{Additionally, boundary conditions should be imposed on the tangential component so that $\tau^e \mu_e |_{\partial \Sigma} = 0$.}
which is the Jacobi equation for the vector field $\eta^a$. Note that since $K_a = 0$ on the surface, the extrinsic curvatures appearing in this equation can be taken to be traceless. This equation was first derived for embedded surfaces of arbitrary codimension in [41].

It is interesting to express this equation using the modified normal bundle connection from equation (3.31), which gives

$$\tilde{D}_b \tilde{D}^b \eta^a + 2L^a_{cde} \tilde{D}_b \eta^e + \left( D_b L^a_d + L_{bcde} L^{bae} + K^a_{bc} K^e_{de} - s^{ae} h^{bc} R_{dbee} \right) \eta^d = 0. \quad (4.34)$$

After applying the identity (A.41), this simplifies to

$$\tilde{D}_b \tilde{D}^b \eta^a + 2L^a_{cde} \tilde{D}_b \eta^e - \tilde{D}_d K^a \eta^d = 0. \quad (4.35)$$

An advantage of using the modified connection is that $\tilde{D}_b \tilde{D}^b$ acts like a scalar Laplacian as opposed to a vector one. This is because $\tilde{D}_b$ annihilates the normal basis vectors $w_A^a$, defined in appendix B, equation (B.2), so by expressing the deformation vector in terms of its components $\eta^a = \eta^A w_A^a$, this equation becomes

$$D_i D^i \eta^A + 2L^i_{AB} \partial_i \eta^B - (D_B K^A) \eta^B = 0, \quad (4.36)$$

with $D_i D^i$ the scalar Laplacian, and $\eta^A$ are now viewed as a collection of scalar functions. The drawback of (4.35) is that $\tilde{D}_d K^a$ appears, which involves derivatives of $K^a$ away from the surface. These derivatives were not specified by demanding that $K^a$ vanish on the surface, so it seems the only way to make sense of this object is to replace it with the equivalent expression in (4.34).

### 4.3 Normal extrinsic curvature

The last set of variational formulas we consider are for $L^a_{bc}$. Using its definition (3.20) and applying (4.11), we find

$$\delta L^a_{bc} = \delta (h^a_f s^d_b s^e_c \nabla_d h^e_f) = \delta g_{de} s^d_b K^e_c + h^a_m s^d_b s^e_c (-\delta \Gamma^m_{de} + \nabla_d (h^{mf} s^n_e \delta g_{mn})) = \delta g_{de} (s^d_b K^e_c + s^e_c L^a_{be} - h^a_d L^e_{de}) + h^{ae} s^d_b s^e_c \delta \Gamma^f_{de}. \quad (4.37)$$

This can be simplified by decomposing the metric variation into its normal and tangential components according to (4.14). First the $\delta s_{de}$ terms are

$$\delta s_{ce} L^a_{be} + \frac{1}{2} (D^a \delta s_{bc} - L^a_{b e} \delta s_{ce} + L^a_{e c} \delta s_{be}) = \frac{1}{2} \tilde{D}^a \delta s_{be}. \quad (4.38)$$

Next, the purely tangential contribution involving $\delta h^{mn}$ is

$$\delta h^{ae} L_{ebc} + \frac{1}{2} (-L_{ebc} \delta h^{ea} + L_{ecb} \delta h^{ea}) = \delta h^{ae} A_{ebc}. \quad (4.39)$$
Lastly, the normal-tangential terms involving $\delta h^m_e$ are

$$\delta h^m_e K_{ce} + \frac{1}{2} (D_b \delta h^a_c - D_c \delta h^a_b - K_{b c} \delta h^e_c - K_{c e} \delta h^b_b) = \tilde{D}_e \delta h^a_e$$  \hspace{1cm} (4.40)

The expressions in (4.38) and (4.39) are manifestly symmetric on $b$ and $c$, and hence contribute only to $\delta A_{bc}$, while (4.40) is antisymmetric in $b$ and $c$ and hence contributes to only $\delta F_{bc}$. This then leads to expressions for the contribution of $\sigma^a$ to the variations

$$L_{\sigma} A_{bc} = D^a \sigma_{\sigma} - 2 A_{abc} K^{de} a \sigma_d$$  \hspace{1cm} (4.41)

$$L_{\sigma} F_{bc} = 2 \tilde{D}_e (D^a \sigma_e - L^a_{bc} \sigma_e)$$  \hspace{1cm} (4.42)

The effects of a pure metric variation also can be read off from (4.38), (4.39), and (4.40). These are captured by the spacetime variational tensors

$$\alpha_{bc} = \frac{1}{2} \tilde{D} (s^d_b s^e_c \delta g_{de}) - A_{bc} h^a \delta g_{de}$$  \hspace{1cm} (4.43)

$$\Gamma_{bc} = 2 \tilde{D}_e (h^a \delta g_{de})$$  \hspace{1cm} (4.44)

$$\lambda_{bc} = \alpha_{bc} - \frac{1}{2} \Gamma_{bc}$$  \hspace{1cm} (4.45)

with $X^* \alpha_{bc}$ giving the pure metric variation in $\delta A_{bc}$, $X^* \Gamma_{bc}$ the contribution in $\delta F_{bc}$, and $X^* \lambda_{bc}$ the contribution in $\delta L_{bc}$. As before, the full variation satisfies $\delta A_{bc} = X^* \alpha_{bc} + L_{\sigma} A_{bc} + \tau A_{bc}$, and similarly for $\delta F_{bc}$ and $\delta L_{bc}$.

Finally, according to the discussion of section 3.2, $L_{ab}^c$ has an interpretation as the connection coefficients of the normal bundle connection, and hence one might expect its variation to be an invariant tensor of the surface, even though $L_{ab}^c$ is not. This is the case, provided that its first index is projected tangentially and last index normally. To see this, note that the expression (4.37) leads straightforwardly to

$$\delta L_{ab}^c = \delta g_{de} (s^d_b K_{ab}^e + h^d L_{ab}^e + s^d_a L_{ab}^e) + h^e_a s^d_b s^e_f \delta \Gamma_{de}$$  \hspace{1cm} (4.46)

and the projection gives

$$h^e_m s^m_a \delta L_{ab}^c = \delta g_{de} s^d_b K_{ab}^e + h^e_m s^d_b s^e_f \delta \Gamma_{de}$$  \hspace{1cm} (4.47)

Since $\delta \Gamma_{de}$ is expressed solely in terms of the spacetime metric variation and its covariant derivative, it is an invariant tensor of the surface by the arguments of section 3.4. $K_{ab}^e$, $s^a_b$, $h^a_b$, are also all invariant, and so every term in (4.47) is invariant.

## 5 Special Cases

This section applies the formalism developed above to special choices for the codimension of the surfaces. The codimension-1 hypersurface is presented first, since this case is likely the most familiar and allows an easy comparison between the geometric quantities defined above.
and the usual quantities associated with a hypersurface. Next, one-dimensional submanifolds are treated, which is the case of a congruence of curves, and again comparisons are made between the constructions in this work and the expansion, shear, twist, and acceleration usually associated with such a congruence. Finally, we treat the case of codimension-2, which exhibits some special features over the generic case.

5.1 Codimension 1

The normal form for a codimension-1 hypersurface is simply a one-form \( \nu_a \), and the unit normal is given by

\[
 n_a = N \nu_a. \tag{5.1}
\]

We will take the hypersurfaces to be spacelike, so that \( \varepsilon = -1 \). A common situation where a foliation by hypersurfaces is employed is the 3 + 1 split used in the canonical analysis of general relativity. There, the normal is \( \nu = -dT \), with \( T \) the time function labelling the hypersurfaces, and \( N \) is the lapse. The normal and tangential metrics are

\[
 s_{ab} = -n_a n_b, \quad h_{ab} = g_{ab} + n_a n_b. \tag{5.2}
\]

Since the normal space is one dimensional, the unit normal \( n_a \) provides a natural vielbein for this space, as the equation for \( s_{ab} \) shows.

The one dimensional normal space also means that the extrinsic curvature tensor \( K^a_{bc} \) is entirely determined by its contraction with \( n_a \) on its normal index. This is seen explicitly by applying its definition (3.17),

\[
 K^a_{bc} = -h^d_b h^e_c \nabla_d (n^a n_e) = -n^a h^d_b h^e_c \nabla_d n_e = -n^a K_{bc}, \tag{5.3}
\]

where the last equality involves the usual extrinsic curvature tensor of the hypersurface,

\[
 K_{bc} = n_a K^a_{bc} = h^d_b h^e_c \nabla_d n_e. \tag{5.4}
\]

For the normal extrinsic curvature tensor \( L^a_{bc} \), it is completely determined by its contraction with \( n_b \) on both of its normal indices. It cannot have an antisymmetric piece, so \( F^a_{bc} \) vanishes identically, and the symmetric piece is pure trace, \( A^a_{bc} = A^a s_{bc} \). The tangential vector \( A^a \) can be shown to be the acceleration of \( n^a \),

\[
 A^a = -n^b n^c A^a_{bc} = -n^b n^c \nabla_b (n^a n_c) = n^b \nabla_b n^a. \tag{5.5}
\]

The outer curvature tensor \( O_{abcd} \) from (3.45) vanishes identically since it is normal and antisymmetric on \( a \) and \( b \), and consequently all terms in the Ricci-Voss equation (3.43) vanish separately. Hence, this equation has no content when considering hypersurfaces.

According to the discussion of section 3.4, if one is free to choose how to extend the foliation away from an initial hypersurface \( \Sigma \), there exists a choice that sets \( A^a = 0 \). From equation (5.5), this choice simply corresponds to extending the surfaces along geodesics in the normal direction. This choice defines a Gaussian normal coordinate system, in which the
lapse $N$ is constant, and can therefore be set to 1, and the shift vector $N^i = N^i_T$, discussed in appendix B, vanishes. Applying the formulas in section B.5, we see that the Christoffel symbols with two or three $T$ components, $\Gamma^T_{Ti}$, $\Gamma^T_{TT}$ and $\Gamma^T_{TT}$, all vanish in this coordinate system.

The Jacobi equation for perturbations between maximal volume slices takes a simple form due to the one-dimensionality of the normal space. The normal deformation vector field must be proportional to the normal vector, so by writing $\eta^a = H n^a$, the Jacobi equation (4.33) becomes a scalar equation for the function $H$. Recalling that $D_a n_b = 0$, the Jacobi equation reduces to

$$D_a D^a H - (K_{ab} K^{ab} + n^a n^b R_{ab}) H = 0.$$ (5.6)

### 5.2 Codimension $(d - 1)$

A congruence of curves is a foliation by one-dimensional submanifolds, and this gives another special case where a number of simplifications occur. This case is relevant to many applications on global properties of spacetime [42], and is also important for relativistic fluid mechanics, where the curves define the world lines of the fluid elements. We take the curves to be timelike, so $\varepsilon = +1$. The normal metric $s_{ab}$ is given by (3.11), and has rank $d - 1$. On the other hand, the tangential metric is rank 1, and hence is given by

$$h_{ab} = -u_a u_b,$$ (5.7)

where $u^a$ is the unit tangent vector to the curves.

Since the tangential space is one-dimensional, $K^a_{bc}$ is totally determined by its contraction with $u^b$ on both its tangential indices. This means it must be pure trace, $K^a_{bc} = K^a h_{bc}$, and the normal vector $K^a$ is simply the acceleration of the curves,

$$K^a = -u^b u^c K^a_{bc} = u^b u^c \nabla_b h^a_c = u^b \nabla_b u^a.$$ (5.8)

The normal extrinsic curvature $L^a_{bc}$ contains the remaining geometric information about the congruence. It is totally determined by its contraction with $u_a$ on its tangential index, and takes the form $L^a_{bc} = -u^a L_{bc}$, where

$$L_{bc} = -u_a s^d s^e \nabla_d (u^a u_e) = s^d s^e \nabla_d u_e.$$ (5.9)

The twist and acceleration tensors defined in (3.21) and (3.22) are similarly determined by their contraction with $u_a$, which determine the tensors $F_{bc} = u_a F^a_{bc} = -2 L_{bc}$ and $A_{bc} = u_a A^a_{bc} = L_{bc}$. These are then related to the usual twist $\omega_{bc}$, expansion $\theta$, and shear $\sigma_{bc}$ of the congruence by

$$\omega_{bc} = -\frac{1}{2} F_{bc},$$ (5.10)

$$\theta = s_{bc} A_{bc},$$ (5.11)

$$\sigma_{bc} = \tilde{A}_{bc} = A_{bc} - \frac{1}{d-1} \theta s_{bc}.$$ (5.12)

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The outer curvature $O_{abcd}$ of (3.45) again vanishes, since it is antisymmetric and tangential on $c$ and $d$, and again all terms in the Ricci-Voss equation (3.43) are identically zero. Similarly, the intrinsic curvature of the curves also vanishes since they are one-dimensional, and the Gauss equation (3.41) is also trivial. However, as discussed in appendix A.2, there is an analog of the Gauss relation for the normal connection $D_a$, and gives an identity (A.12) that relates the normal curvature tensor $C_{abcd}$ defined by equation (A.10) to the spacetime Riemann tensor and $L_{bc}$ [27, 28].

In the situation where only a single curve is determined and the foliation away from the curve is freely specifiable, the discussion of section 3.4 shows that $A_{bc}$ can be chosen to vanish. Furthermore, the additional freedom in changing the foliation after $A_{bc}$ has been set to zero is enough to also set $F_{bc}$ to zero. This is just the axial gauge choice $u_a F_a^{bc} = 0$ discussed in section 3.4. Hence, when the congruence is extended according to this prescription, the expansion, shear, and twist all vanish along the central curve. Since $F_{bc}$ was set to zero, one can pick a basis of normal vectors which are all commuting, and hence can serve as coordinate basis vectors. This then leads to the notion of Fermi normal coordinates along the curve. In terms of the coordinate expressions discussed in appendix B, these coordinates set the normal metric $s_{AB}$ along the curve to a constant, and set its first transverse derivative to zero; additionally, they set the shift $N^0_A$ and its first transverse derivative to zero (here 0 denotes the coordinate along the curve). Applying the formulas in section B.5, the only nonvanishing Christoffel symbols along the curve in these coordinates are $\Gamma^A_{00}$ and $\Gamma^0_{A0}$, and if the curve is a geodesic, these vanish as well.

The covariant derivative $D_a$ is also determined by its contraction with $u^a$, and this contraction $D_F = u^a D_a$ is referred to as the Fermi derivative along the curve [42]. From the fact that $D_F u^a = 0$, the Jacobi equation (4.33) for a normal vector $\eta^a$ that defines a variation to a nearby geodesic reads

$$D_F^2 \eta^a + \mathbf{s}^{\epsilon a} u^b u^c R_{dbce} \eta^d = 0,$$

which is also known as the equation of geodesic deviation.

### 5.3 Codimension 2

For codimension-2 submanifolds, there are certain simplifications that occur in the expressions for the twist tensor $F^a_{bc}$ and the outer curvature tensor $O_{abcd}$. This is because they contain normal, antisymmetric pairs of indices, which, since the normal space is two-dimensional, can be simplified by contracting with the unit normal $n^{ab}$. Hence, the twist tensor can reduced to a single tangential vector,

$$F^a = \frac{1}{2} n^{bc} F^a_{bc}, \quad F^a_{bc} = \epsilon F^a r_{bc},$$

and similarly the outer curvature reduces to a tangential 2-form,

$$O_{cd} = \frac{1}{2} n^{ab} O_{abcd}, \quad O_{abcd} = \epsilon n_{ab} O_{cd}.$$
This then leads to a simplification in the expression of $O_{cd}$ in terms of $F_a$ and $A^a_{bc}$, which follows from (3.46),

$$O_{cd} = D_c F_d + n^{ab} \tilde{A}_{zb}^e \tilde{A}_{dca}, \quad (5.16)$$

which no longer involves a term quadratic in $F^a_{bc}$. Further, when the extension of the foliation away from the initial surface is allowed to be chosen to set $A_{ab}^c$ to zero, the outer curvature restricted to the surface is an exact form,

$$O_{cd} = D_c F_d. \quad (5.17)$$

This reflects the fact that the orthogonal group acting on the normal bundle is abelian for codimension-2 surface. One can then form topological invariants of the surface by wedging $O_{cd}$ together to form a top form (when the submanifold is even dimensional), and then integrating it over the surface [30]. This will result in the Euler number of the normal bundle. Note that if a global choice of tensor $m^b_a$ can be found that transforms $A_{ab}^c$ to zero everywhere, as discussed in 3.4, equation (5.17) says that the outer curvature is globally exact on the surface. This would imply that the Euler number vanishes, being the integral of an exact form. When $A_{ab}^c$ cannot be set to zero everywhere and equation (5.16) is used to compute the Euler number, all terms involving $F_d$ are exact and drop out of the integral. Hence, the acceleration tensor completely determines the Euler number in this case, and, conversely, the Euler number represents an obstruction to setting $A_{ab}^c$ to zero everywhere on the surface.

6 Boundary term in gravitational Hamiltonian

One application of this formalism is in analyzing boundary terms of Hamiltonians that arise in the covariant canonical analysis of general relativity. Given a finite subregion defined by a hypersurface $\Sigma$ with boundary $\partial \Sigma$, one can form a symplectic form $\Omega$ associated with the subregion by integrating the symplectic current $(d-1)$-form $\omega$ over the surface,

$$\Omega = \int_{\Sigma} \omega. \quad (6.1)$$

The symplectic current is a 2-form on field space, so that $\delta g_{ab}$ appears quadratically, and it is constructed from the field space exterior derivative of a symplectic potential $\theta$, a spacetime $(d-1)$-form. This potential arises as the boundary term in the variation of the Einstein-Hilbert Lagrangian for general relativity, $L = \frac{1}{16\pi G} \epsilon R$, through the equation

$$\delta L = E^{ab} \delta g_{ab} + d \theta, \quad (6.2)$$

where $E^{ab}$ are the field equations, and the expression for $\theta$ is [43]

$$\theta = 2 \epsilon_a E^{abcd} \nabla_d \delta g_{bc}, \quad E^{abcd} = \frac{1}{32\pi G} (g^{ac} g^{bd} - g^{ad} g^{bc}). \quad (6.3)$$
Time evolution along the flow of a vector field $\xi^a$ should be generated by a Hamiltonian $H_\xi$ on the phase space. This Hamiltonian is required to satisfy

$$\delta H_\xi = \int_{\partial \Sigma} (\delta Q_\xi - i_\xi \theta),$$

(6.4)

where $Q_\xi$ is the Noether charge \[43\], and $i_\xi$ denotes contraction of the vector $\xi^a$ into a differential form. This formula assumes that the vector field $\xi^a$ is independent of the dynamical fields, $\delta \xi^a = 0$, and we also assume that the embedding $X$ is fixed, so there are no contributions from $\chi^a$ (relaxing these constraints will be discussed later). In general, if $\xi^a$ has a transverse component to $\partial \Sigma$, this equation has no solutions since $\theta$ is not a total variation \[44\]. However, one can look for boundary conditions to impose on the fields so that $i_\xi \theta = \delta B_\xi$ for some $B_\xi$. In this case the Hamiltonian is equal to $\int_{\partial \Sigma} (Q_\xi - B_\xi)$, up to a constant.

To classify the possible boundary conditions, it is helpful to decompose $i_\xi \theta |_{\partial \Sigma}$ in terms of the geometric quantities associated with the surface $\partial \Sigma$. First, note that only the normal component of $\xi^a$ is relevant, since any tangential component contracting with $\theta$ will lead to a form that vanishes when restricted to $\partial \Sigma$. Then applying $\epsilon = -n \wedge \mu$ to the formula (6.3) gives

$$i_\xi \theta |_{\partial \Sigma} = -\frac{\mu}{16\pi G} \xi^c n_{ae} (s^{ae} g^{bd} - s^{ad} g^{be}) \nabla_d \delta g_{bc}.$$  

(6.5)

It is useful now to use the decomposition of the metric variation (4.14), which can be applied to $\delta g_{ab}$ as opposed to the pullback $\delta g_{ab}$ because the embedding is held fixed. First, the terms involving $\delta s_{ab}$ are

$$n^c_e (s^{bd}(D_d \delta s_{bc} - D_c \delta s_{bd}) + K^b \delta s_{bc}) = n^{cd} D_d \delta s_{ec} + n^c_e K^b \delta s_{bc}.$$  

(6.6)

Next, the contribution involving $\nabla_d (h_{mb} \delta h^m_{bc} + h_{mc} \delta h^m_{b})$ is computed noting that the contraction with $g^{bc}$ gives zero since $\delta h^m_{bc}$ is normal on its lower index. The remaining terms involving $\delta h^m_{bc}$ are

$$n^c_e (A_m \delta h^m_{be} + L^b_{me} \delta h^m_{b} + D_d \delta h^c_{de}).$$  

(6.7)

Finally, the $\delta h^m_{mn}$ terms are

$$n^c_e (K_{cmn} \delta h^m_{mn} + D_c (h_{mn} \delta h^m_{b})) = n^c_e (K_{cbd} \delta h^{bd} - 2 \delta K_c - 2 D_b \delta h^b_{c} - 2 L^b_{mc} \delta h^m_{b}).$$

(6.8)

where the second expression employed the formula (4.27) for the variation of $K_c$. Combining these expressions in (6.5) leads to

$$i_\xi \theta |_{\partial \Sigma} = \frac{\mu}{16\pi G} \xi^c n^c_e \left[ \delta K_c + s_{mc} h^{bd} \delta K^m_{bd} + D_b \delta h^c_{d} - s^{bd}(D_d \delta s_{bc} - D_c \delta s_{bd}) + 2 L^a_{dc} \delta h^d_{a} - A_b \delta h^b_{c} - L^d_{bc} \delta h^b_{d} \right].$$

(6.9)

Each term on the first line of this expression consists of tensors that are invariant with respect to changes in the foliation away from $\partial \Sigma$, according to the discussion in section 3.4. The terms on the second line are not individually invariant, due to the appearance of normal covariant derivatives $D_b$ and normal extrinsic curvatures $L_{dc}$; however, together they form an invariant object, since

$$-s^{bd}(D_d \delta s_{bc} - D_c \delta s_{bd}) + 2 L^a_{dc} \delta h^d_{a} - A_b \delta h^b_{c} - L^d_{bc} \delta h^b_{d} = -s^{bd}(\nabla_d \delta g_{bc} - \nabla_c \delta g_{bd}).$$

(6.10)

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The expression (6.9) can be organized into a somewhat simpler form by introducing $\delta J^{a}_{bc}$, the variation of the connection coefficients for $D_a$. $J^{a}_{bc}$ is defined in appendix B.2 through equation (B.18), and, in a coordinate system adapted to the foliation, it is given by the normal projection of the Christoffel symbols, as in equation (B.36). Its variation satisfies

$$s^d \delta J^{a}_{bc} = -\delta h^e_b L_{ec}^d - \delta h^e_c L_{eb}^d + s^d s^f \delta \Gamma^{a}_{ef}, \quad (6.11)$$

where the relation (B.40) was used. One can then calculate that

$$n^b_a \delta J^{a}_{be} = n^c_e \left[ - s^{bd} (D_d \delta s_{bc} - D_c \delta s_{bd}) + 2 A_{dc}^a \delta h^d_a - 2 A_b \delta h^b_d \right] \quad (6.12)$$

so that (6.9) becomes

$$i_\xi \theta \big|_{\partial \Sigma} = \frac{\mu}{16 \pi G} \xi^e \left[ n^c_e \left( \delta K^c_e + s_{mc} h^{bd} \delta K^m_{bd} + D_b \delta h^c_e + A_d \delta h^d_e - L_b^d \delta h^b_d \right) + n^b_a \delta J^{a}_{be} + s^d e \delta F_d \right] \quad (6.13)$$

Since $\theta$ is defined as the boundary term obtained when varying the Lagrangian, it is ambiguous by the addition of an exact form, $\theta \to \theta + d\beta$. The presence of the divergence term $D_b \delta h^b_e$ in (6.9) suggest that some terms may be canceled by the appropriate choice of $\beta$. This is indeed the case, and coincides with the natural choice for this ambiguity suggested in [2]:

$$\beta_{e_2 ... e_{d-1}} = \frac{1}{16 \pi G} \epsilon^{b e_2 ... e_{d-1}} \delta h^b_c \quad (6.14)$$

gives

$$(d\beta)_{e_1 ... e_{d-1}} = \frac{1}{16 \pi G} \left[ \epsilon^{b e_2 ... e_{d-1}} \nabla_{e_1} \delta h^b_c - (d - 2) \epsilon^{e_1 e_2 ... e_{d-1}} \nabla_{e_2} \delta h^b_c \right]. \quad (6.15)$$

Contracting with the normal component of $\xi^e$ and restricting the form to $\partial \Sigma$ forces the indices $e_2, \ldots, e_{d-1}$ to be tangential. In the second term above, the only nonzero contribution then comes from $c$ normal and $b$ tangential in $\epsilon^{e_1 e_2 ... e_{d-1}}$, and the expression simplifies to

$$i_\xi d\beta \big|_{\partial \Sigma} = \frac{\mu}{16 \pi G} \xi^e \left[ - n^e_c D_b \delta h^b_c - n^b_e L_{deb} \delta h^d_c \right]
= \frac{\mu}{16 \pi G} \xi^e \left[ - D_b \delta h^b_c - A_d \delta h^d_e + L_d^b \delta h^b_e \right]. \quad (6.16)$$

Adding this term to (6.13) then gives

$$i_\xi (\theta + d\beta) \big|_{\partial \Sigma} = \frac{\mu}{16 \pi G} \xi^e \left[ n^c_e \left( \delta K^c_e + s_{mc} h^{bd} \delta K^m_{bd} \right) + n^b_a \delta J^{a}_{be} + s^d e \delta F_d \right] \quad (6.17)$$

This gives a fairy simple expression for the additional boundary contribution to the Hamiltonian associated with the flow along $\xi^e$; however, a word of caution is in order. Although $\beta$ defined in (6.14) is an invariant form on the surface, its spacetime exterior derivative is not, since it involves normal derivatives. The expression (6.16) is similarly not invariant, which is easily seen due to the explicit appearance of $L_{deb}$ in the first line. Since the original boundary term (6.9) is invariant, the modified one (6.17) necessarily is not. Although simpler to analyze than (6.9), one should keep in mind that quantities in equation (6.17) depend on how the foliation is extended away from $\partial \Sigma$. Other choices for
the ambiguity term $\beta$ generally share this feature of breaking refoliation-invariance of $i_\xi \theta$, unless $\beta$ is constructed solely from spacetime-covariant fields, such as $g_{ab}$ and $\delta g_{ab}$, as opposed to tensors associated with the surface, such as $\delta h^a$. For pure general relativity, there are no such choices for $\beta$ without using derivatives of $g_{ab}$ and $\delta g_{ab}$, hence the expression (6.9) is the unique refoliation-invariant choice where all terms involve only one derivative of the metric or its variation.

From here, one could classify the possible boundary conditions that allow (6.9) or (6.17) to be written as a total variation. We will not attempt this general analysis here, although this problem has been partially considered before. In particular, [45] found the necessary boundary conditions under the assumption of fixed normal metric $\delta s_{ab} = 0$ and a partial fixing of the tangential projector variation $\delta h^a$. These boundary conditions turn out to be quite stringent: they require either a fixed volume form $\delta \mu = 0$ or $K_c = 0$, and also fixing the traceless extrinsic curvature $\delta \tilde{K}^a = 0$ or fixing the conformal class of the induced metric $\delta h_{ab} - \frac{1}{d-2} h_{ab} h^{cd} \delta h_{cd} = 0$. Similar classifications of boundary conditions have appeared in [46].

For a finite subregion, it can appear overly restrictive to try to demand that this Hamiltonian be integrable for a vector field that generates a diffeomorphism transverse to the surface. This is because one would expect symplectic flux to leak out if the surface is moved to a different location. However, another application for the above analysis is in the considerations of diffeomorphism edge modes, which characterize the gauge degrees of freedom that become physical in the presence of the fixed surface $\partial \Sigma$ [1]. In order to build up a larger phase space by assembling phase spaces associated with subregions, these edge modes are necessary in order to implement gauge constraints in the larger space through a symplectic reduction procedure. This procedure requires a symmetry algebra to act on the edge modes as Hamiltonian transformations in the local phase space, and hence it is important that integrable Hamiltonians can be found for these transformations, including the diffeomorphisms transverse to $\partial \Sigma$.

The edge mode degrees of freedom are contained in the embedding fields $X$, and the Hamiltonian for a diffeomorphism in the reference space must satisfy

$$\delta H_\xi = \int_{\partial \Sigma} (\delta Q_\xi - i_\xi \theta) \tag{6.18}$$

where $\theta$ is now a function of the pulled back metric variation $\delta g_{ab}$. This case now reduces to the same analysis as before, and the conclusion that possibly overly strong boundary conditions are necessary. However, one can generalize the allowed symmetry transformations by letting the generator $\xi^a$ depend on the dynamical fields, so that $\delta \xi^a \neq 0$. For such a field-dependent generator, the Hamiltonian variation is instead

$$\delta H_\xi = \int_{\partial \Sigma} (\delta Q_\xi - Q_\delta \xi - i_\xi \theta), \tag{6.19}$$

and the extra freedom in $\delta \xi^a$ makes it plausible that this equation has solutions without overly restrictive boundary conditions on the fields. The algebra satisfied by these field-dependent generators is modified from the Lie bracket of vector fields on spacetime to the
Lie bracket of the associated vector fields on field space, given in [47, 48, 49]. Additionally, when the algebra is represented with Poisson brackets of the Hamiltonians $H_\xi$, it can acquire central extensions. Examples of field-dependent generators leading to central charges include the Brown-Henneaux analysis of asymptotically AdS$_3$ gravity [50], Carlip’s work on near horizon symmetry algebras of black holes [51, 52], and the Barnich-Troessart analysis of the extended BMS algebras of asymptotically flat space [47, 53]. We leave further analysis of such field-dependent generators and their algebras to future work.

7 Discussion

We conclude with a discussion of possible generalizations of the above constructions with embedding fields and foliations, and point to additional applications for this formalism.

7.1 Null surfaces

A notable deficiency in the formalism developed in this work is that it cannot be applied unmodified to null surfaces. This is unfortunate, since numerous recent works on asymptotic symmetries [54, 47, 53, 55], actions for local subregions [56, 57, 58], and degrees of freedom on null Cauchy surfaces [59, 60, 61], among many others, all utilize null surfaces or null foliations. There are a number of obstacles in trying to adapt the submanifold calculus of this paper to null surfaces. First, although a null surface can still be defined by a normal form $\nu$, it is no longer possible to form a unit normal as in equation (3.8), since nullness means that $\nu$ has norm zero. One can still work with $\nu$ directly as an unnormalized normal form, but it should not be considered an invariant tensor of the foliation, since rescaling $\nu$ defines the same foliation.

The ability to normalize $\nu$ led to expressions for normal and tangential projectors, $s^a_b$ and $h^a_{\ b}$, which defined canonical decompositions of all vectors and covectors into normal and tangential parts. With a null surface $\Sigma$, no such projectors are available, and the decompositions of the tangent and cotangent spaces are more complicated. For vectors, the subspace of tangential vectors, $T\Sigma$, defined as all vectors which annihilate $\nu$ upon contraction, is well-defined and independent of the metric. Since $\Sigma$ is a null surface, there is a one-dimensional subspace in $T\Sigma$ consisting of all null vectors tangent to $\Sigma$. Call this the lightlike subspace $L\Sigma$, and let $k^a$ denote a generic null vector in this space. The metric can be used to define the orthogonal complement $L\Sigma^\perp$, the subspace of all vectors with zero inner product with $k^a$. These spaces are related to each other by the following inclusions

$$L\Sigma \subseteq T\Sigma \subseteq L\Sigma^\perp \quad (7.1)$$

The second inclusion reflects the fact that $k_a$ is a normal one-form, so that $\nu = k \wedge \nu$ for some $(p-1)$-form $\nu$, and hence all tangential vectors must annihilate $k_a$. When dealing with a congruence of null curves ($p = d - 1$), the first inclusion is an equality, whereas for a null hypersurface ($p = 1$), the second inclusion is an equality.
\(L\Sigma^\perp\) is the largest subspace in this discussion, and is always \((d-1)\)-dimensional. Hence, there are still vectors that do not lie in any of these subspaces, and there is no canonical subspace to associate with the vectors transverse to \(L\Sigma^\perp\). Instead, one must work with quotients of vector spaces. Two quotient spaces that are particularly relevant are the transverse space \(TR = TM/T\Sigma\), where two vectors are equivalent if they differ by a tangential vector, and the spatial space \(S = T\Sigma/L\Sigma\), consisting of tangential vectors to \(\Sigma\) defined modulo arbitrary multiples of \(k^a\).

Similar subtleties exist for the decomposition of the cotangent space \(T^*M\). There is a natural one-dimensional lightlike subspace \(L^*\Sigma\) generated by \(k_a\), which is contained in the normal subspace \(N^*\Sigma\) defined by \(\nu\). Finally, there is \((d-1)\)-dimensional horizontal subspace \(H^*\) consisting of all covectors that annihilate \(k^a\) (which implies \(H^* = L^*\Sigma^\perp\)), and these subspaces satisfy the inclusions

\[
L^*\Sigma \subseteq N^*\Sigma \subseteq H^*.
\] (7.2)

Equality occurs in the first inclusion for the case of a null hypersurface, and in the second in the case of a null congruence of curves. Again, one must form quotients when performing general decompositions of covectors. Important quotients are the tangential space \(T^*\Sigma = T^*M/N^*\Sigma\), consisting of covectors defined modulo normal covectors which should serve as the cotangent bundle for the null surface, and the transverse space \(H^*/L^*\Sigma\), which is used in discussions of geodesic deviation for null congruences.

Quotient spaces complicate the description of geometric quantities on the surface, since tensors defined in the quotient space do not give rise to well-defined tensors on \(M\). Instead, there are conditions a spacetime tensor must satisfy in order for it to define an unambiguous tensor on the quotient space of interest [37, pg. 222].

Another issue related to the lack of a projector is that not all of the subspaces or quotient spaces come equipped with a metric. For example, the spacetime metric \(g_{ab}\) does not induce a metric on the subspace \(L\Sigma\) generated by the null vector \(k^a\) on \(\Sigma\), since the only possible inner product vanishes, \(k^a k^b g_{ab} = 0\). There is a metric induced on \(T\Sigma\), but it is degenerate, which means, for one, that a metric-compatible connection on \(T\Sigma\) is not uniquely defined, and in fact a torsion-free one does not exist in general. Without such an object, it is more difficult to discuss intrinsic and extrinsic curvatures for the null surface and to formulate analogs of the Gauss, Ricci-Voss, and Codazzi identities.

In addition to these obstructions to standard geometric analyses of null surfaces, there are also problems related to the constraint nullness puts on metric variations. For timelike or spacelike surfaces, one is free to vary the spacetime metric and the embedding fields independently, since the norm of \(\nu\) is not fixed, but merely required to be positive or negative. On the other hand, null surfaces require the norm of \(\nu\) to be zero, which, after imposing that \(\nu\) does not vary, necessarily restricts the allowed variations of the metric. In the case of a hypersurface, this condition is \(\delta g^{ab} \nu_a \nu_b = X^* (\delta g^{ab} + \mathcal{L}_\chi g^{ab}) \nu_a \nu_b = 0\), which relates some components of the metric variation to the variation of the embedding described by \(\chi^a\). Although one could choose to work only with embedding maps \(X\) and metrics \(g_{ab}\) that maintain nullness of \(\nu\), it is somewhat contrary to the perspective of this work where the two objects are meant to be varied independently.
Imposing nullness by hand can also lead to more severe restrictions on the class of metrics under consideration. For example, null hypersurfaces are generated by geodesics, and therefore generically develop caustics and crossovers. These would be reflected in a singularity in either the embedding or the normal form $\nu$. By not allowing for such singularities, one is making a strong assumption about the metric, in that it admits a complete, caustic-free, null hypersurface.

Clearly a different formalism is required in order to discuss the geometry and dynamics of null surfaces and foliations. This section concludes by offering a number of suggestions on how the various issues outlined above might be addressed, and leaves full analysis of these possibilities to future work.

A common way of dealing with the necessary appearance of quotient spaces is to introduce additional structures that allow these quotients to be canonically identified with subspaces of the spacetime tangent and cotangent bundles. One way of doing this for the null surface is to choose an arbitrary slicing of the null surface by spatial sections. Doing so essentially converts the problem to the analysis of a foliation by spatial submanifolds with codimension one higher than the null surface. One can then define the intrinsic and extrinsic geometry of this spatial foliation, and many quantities computed in this way can be argued to produce well-defined tensors on the quotient spaces. For many applications, this construction is sufficient, although the issue of caustics and the assumed regularity of the null surface is still present.

A related approach is to define a Carroll structure on the null surface [62]. This is simply a choice of preferred spatial subspace to identify with the spatial quotient space by way of an Ehresmann connection [63]. The Carroll structure is more general than the choice of spatial slicing since the spatial subspaces are not required to be integrable. A related construction parameterizes the transverse space to $\Sigma$ by choosing an auxiliary null vector $\ell^a$ that satisfies $\ell^a k_a = -1$, in terms of which the metric can be fully decomposed (see e.g. [64]). An interesting question in these approaches is whether these additional structures can be used to fix a unique connection $D_a$ on $\Sigma$. Note that it is generically not possible to require that $D_a$ be compatible with the degenerate induced metric on the surface. It seems possible that once a preferred $k^a$ and additional Carroll or transverse structure has been chosen, a connection could be fixed by imposing compatibility conditions with these structures (see, e.g. [65]).

The issue of nullness overconstraining the metric variations could possibly be addressed by only imposing that $\nu$ is null in the background, and not requiring that variations preserve nullness. This approach was advocated for recently in [66], which provided a physical interpretation for metric variations that do not preserve nullness of the surface. Note there are still challenges in this case related to quotient spaces and the inability to construct projectors. Furthermore, variations of the metric could now produce a surface whose induced metric changes signature, which can cause the intrinsic geometry to look singular. Such problems would have to be addressed when pursuing this construction, perhaps along the lines of [67].

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*I thank Rafael Sorkin for pointing this out.*
Finally, a possible way forward with the problem of caustics and crossovers would be to consider embeddings into the cotangent bundle $T^*M$ or a tensor product thereof, rather than into the manifold $M$ [68]. The image of the embedding specifies both the location of the surface and its normal form. Hence, surfaces with caustics or crossovers can look singular when embedded in $M$, but are perfectly smooth as surfaces in $T^*M$. There are drawbacks in this approach in that it allows for immersed surfaces in $M$ as opposed to embeddings, since the surfaces can now intersect themselves. While this raises additional complications, it seems to be a promising way forward in analyzing the geometry of generic null surfaces.

7.2 Hamiltonians for field-dependent generators and asymptotic symmetries

Section 6 examined boundary terms that appear when defining a Hamiltonian in general relativity for transformations that move the boundary of a finite subregion. Although the conditions for which this Hamiltonian is integrable were not rigorously analyzed, it was clear that the necessary conditions are likely overconstraining in that most metric components must be held fixed. An alternative was suggested that allows for the vector field generating the flow to be field-dependent, $\delta \xi^a \neq 0$, to allow for more freedom in imposing integrability of the Hamiltonian. Many analyses of asymptotic symmetries employ such field-dependent generators, since they are necessary when attempting to preserve a privileged asymptotic structure. Note that field-dependent generators can lead to algebras that differ from the Lie algebra of the vector fields on $M$. This occurs for two reasons. First, the Lie bracket is explicitly modified due to the field dependence [47]. Second, representing the algebra as a Poisson bracket of Hamiltonians can give rise to central extensions, as found in, e.g., [50, 51, 52, 47, 53]. Together, these modifications of the algebra can have important implications when one considers quantizations of the theory in question. The case of asymptotic symmetries of AdS$_3$ gravity is a renowned example, where the central charge of the asymptotic Virasoro algebra is related to the entropy of black holes in the theory [69].

7.3 Edge modes and entanglement entropy

Another reason to consider field-dependent generators comes from the application of embedding fields to the problem of entanglement entropy in a gravitational theory. The $X$ field encapsulates the extra edge mode degrees of freedom that must be incorporated into a finite subregion in order to properly implement gauge-invariance when gluing to an adjacent subregion [1]. Variations of the embedding transverse to the entangling surface are among the degrees of freedom encoded in $X$. In order to analyze the contribution of the edge modes to the entanglement entropy, one must have a handle on their quantization, for which their symmetry algebra plays an important role. It may be necessary to include the surface deformations in the symmetry algebra to account for the full diffeomorphism invariance, and such a symmetry algebra will likely involve field-dependent generators to avoid overconstraining boundary conditions on the metric.
7.4 Boundary terms in gravitational actions

An application of the general geometric framework developed in this paper is to the problem of finding boundary terms in the action of higher curvature gravitational theories. These boundary terms are added to ensure that the action is stationary only for field variations that satisfy the appropriate boundary conditions. For general relativity, the Gibbons-Hawking boundary term is added on spacelike or timelike codimension-1 boundaries to implement a Dirichlet condition on the induced metric at the boundary [70]. Additional contributions must be added when the region has null components or corners of higher codimension, as were recently analyzed in detail in [57, 58]. For higher curvature theories, less is known about the necessary boundary terms, although see [71, 72, 73, 74] for some results in this area. One reason these theories are more complicated is that the presence of higher derivatives in the field equations entails additional boundary conditions on derivatives of the fields in the variation principal. Nevertheless, a general analysis of boundary terms and boundary conditions in these theories is lacking at present, and the systematic decompositions described in this work may lend themselves to addressing this problem in the future.

7.5 Perturbations of RT surfaces

RT surfaces are important objects in holography since they bound subregions in the bulk that are dual to corresponding subregions in the boundary [75]. Perturbations of these surfaces in response to a change in the boundary subregion or the bulk dynamical fields are of interest, since these can be used in perturbative calculations of entanglement entropy [76, 77], proofs of CFT energy conditions [78, 18, 19, 20], and considerations of bulk reconstruction [21, 79]. The embedding fields provide a covariant description of these perturbations, and perturbations to nearby extremal surfaces are controlled by solutions to the Jacobi equation, discussed in section 4.2.1 (see [80, 81] for a perturbative analyses of solutions to this equation). Recently, [36] used a similar submanifold formalism to the one developed in this paper to recast and extend some of these holographic proofs involving RT surfaces in a covariant language. Further application of the covariant formalism may lead to deeper understanding of the mechanisms at play in these holographic constructions, and suggest possible generalizations.

7.6 Magnetohydrodynamics and force-free electrodynamics

A final application of the embedding field formalism is to the theory of relativistic magnetohydrodynamics (MHD) and force-free electrodynamics (FFE). This application is similar to the use of embedding fields in fluid dynamics. It differs, however, in that the foliation is by two-dimensional timelike manifolds, as opposed to the one-dimensional flow lines. MHD and FFE are thus theories of a string fluid.

This foliation arises since the abundance of free charges tends to short out any electric fields in the rest frame of the fluid, implying that the electromagnetic field tensor satisfies $F_{ab}u^b = 0$, where $u^b$ is the fluid velocity. Since $u^a$ is a zero eigenvector of $F_{ab}$, the field
strength must have rank 2, and is also closed, \( dF = 0 \). Hence, \( F_{ab} \) is normal to a foliation of two-dimensional manifolds according to the discussion of 3, which comprise the strings of the fluid description. In fact, the entire theory can be recast as the theory of this foliation using embedding fields and a fixed two-form in a reference space \( M_0 \) [82]. An intriguing relation geometrical relation holds for the electromagnetic stress tensor, which for a degenerate field takes the form [82]

\[
T_{ab}^{\text{EM}} = \frac{1}{4} F^{cd} F_{cd} \left( s_{ab} - h_{ab} \right),
\]

being proportional to the difference of the normal and tangential metrics of the foliation. It could be useful to express other geometrical tensors associated with the foliation in terms of the electromagnetic field strength, which could lead to deeper intuition for properties of MHD and FFE solutions [83].

The string fluid is also the starting point for the effect field theory of MHD and FFE, where the foliation has an interpretation in terms of the generalized global symmetries [84]. The one-form symmetry of electromagnetism has a conserved charge that is integrated over codimension-2 surfaces, and this flux counts the number of strings passing through the surface, which are the fundamental charged particles for the one-form symmetry [85]. This is analogous to the viewpoint that the charge of an ordinary symmetry is the integral over a spacelike hypersurface, and the flux counts the number of charged particle worldlines piercing the surface. The effective field theory for the one-form symmetry of electromagnetism should arise from the most general action consistent with this symmetry, and the string fluid and embedding fields provide an efficient way of writing terms that are manifestly symmetric [85, 86, 87].

Force-free electrodynamics is known to be dynamically well-posed if the field strength is magnetically dominated, \( B^2 > E^2 \), or equivalently, \( F_{ab} F^{ab} > 0 \). There is considerable interest in the question of how and when this condition breaks down, signaling the onset of an instability which can have observable astrophysical consequences. At saturation, \( B^2 = E^2 \), the foliation becomes null, implying that a singularity develops in its intrinsic geometry. Additionally, magnetic reconnection events are associated with crossing field lines, which can be interpreted as the development of a caustic from the foliation perspective. In order to better characterize how these breakdowns occur, it would be helpful to develop singularity and focusing theorems for the intrinsic and extrinsic geometry of the field sheets. A possible useful tool would be a generalization of the Raychaudhuri equation that applies to the two dimensional foliation. The starting point for such a generalization would be equation (A.31), which relates the tangential derivative of \( L_{cd} \) to other quantities on the surface. Some initial analysis of this expression, interpreted as a generalized Raychaudhuri equation, has appeared in [32]. Further analysis of the force-free equations using the geometric tools developed in this work could lead interesting and useful results.
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A Curvature identities for $D_a$, $\mathcal{D}_a$

As discussed in section 3.3, there are curvature tensors $\mathcal{R}_{abcd}$ and $\mathcal{O}_{abcd}$ associated with the commutator of two tangential covariant derivatives $[D_c, D_d]$. These are related to the space-time Riemann curvature and extrinsic curvature tensors through the Gauss and Ricci-Voss equations. Additionally, the Codazzi equation arises from the requirement that $[D_c, D_d]V^a$ remain tangential on a if $V^a$ is itself tangential. These three equations relate certain components of $\mathcal{R}_{abcd}$ to intrinsic and extrinsic geometrical quantities at the surface. The specific components of $\mathcal{R}_{abcd}$ covered by these identities are those with all indices tangential, those with three indices tangential and one normal, and those with the first two indices normal and second two tangential (or equivalently, second two normal and first two tangential).

A natural generalization of these equations comes from the commutator of two normal covariant derivatives $[\mathcal{D}_a, \mathcal{D}_b]$. This case is slightly more subtle due to the presence of torsion-like contributions; however, one can find a tensorial expression that can be called the curvature of the normal connection, along with the analog of the outer curvature. These are then related to the purely normal components of $\mathcal{R}_{abcd}$ and its tangent-tangent-normal-normal components. An analog of the Codazzi equation gives a relation between $L_{abc}$ and the components of $\mathcal{R}_{abcd}$ with three normal and one tangential index.

The remaining components of $\mathcal{R}_{abcd}$ that are not constrained by these equations are normal on $a$ and $c$ and tangential on $b$ and $d$. The desired relation on these components is obtained by considering the commutator of one tangential and one normal covariant derivative, $[D_c, D_d]$. Mixed curvature tensors associated with this commutator can be constructed, which are related to $K_{abc}$ and $L_{abc}$. Finally, by examining the Codazzi identity for this commutator, a number of additional relations between the various curvature quantities can be derived, one of which relates the remaining components of $\mathcal{R}_{abcd}$ to $K_{abc}$ and $L_{abc}$.

Carrying out the above program leads to six curvature tensors associated with the normal and tangential covariant derivatives, which are related in various ways to $\mathcal{R}_{abcd}$, $K_{abc}$ and $L_{abc}$. In addition, three Codazzi identities lead to further relationships between $\mathcal{R}_{abcd}$, $K_{abc}$ and $L_{abc}$. This appendix systematically derives these identities and works out some of their consequences. An early treatment of many of the concepts presented in this appendix was given by Schouten in [29, Ch. V, Sec. 7].
A.1 Tangential curvatures

The intrinsic tangential curvature $R_{abcd}^a$ is defined by $[D_c, D_d]$ acting on a tangential vector $V^a$ by the equation

$$\left(D_c D_d - D_d D_c \right) V^a = R_{abcd}^a V^b.$$  \hfill (A.1)$$

The Gauss equation for this curvature can be derived by simply writing out the definition for the tangential covariant derivative,

$$D_c D_d V^a = h_p^a h_q^d h_m^a \nabla_p(D_q V^m)$$

$$= h_p^a h_q^d h_m^a \nabla_p(h^e_q \nabla_e V^m + K^m_{qb} V^b)$$

$$= -K^e_{cd} D_e V^a + h_p^a h_q^d h_m^a \nabla_p \nabla_q V^m + K^a_{mc} K^m_{db} V^b.$$  \hfill (A.2)$$

Subtracting $D_d D_a V^a$ from this expression and recalling that $K^e_{cd} = 0$ and that $(\nabla_p \nabla_q - \nabla_q \nabla_p) V^m = R^m_{npq} V^n$, produces the Gauss equation,

$$R_{abcd} = h_m^a h_n^b h_p^c h_q^d R_{mnpq} + K^a_{eca} K^e_{db} - K^a_{cda} K^e_{cb}.$$  \hfill (A.3)$$

The tangential outer curvature $O_{abcd}^a$ is defined by $[D_c, D_d]$ acting on a normal vector $W^a$ through the equation

$$\left(D_c D_d - D_d D_c \right) W^a = O_{abcd}^a W^b.$$  \hfill (A.4)$$

To derive the Ricci-Voss equation, we explicitly compute the action of the two derivatives acting on $W^a$,

$$D_c D_d W^a = h_p^a h_q^d h_m^a \nabla_p(D_q W^m)$$

$$= h_p^a h_q^d h_m^a \nabla_p(h^e_q \nabla_e W^m - K^m_{qb} W^b)$$

$$= -K^e_{cd} D_e W^a + h_p^a h_q^d h_m^a \nabla_p \nabla_q W^m + K^a_{cm} K^m_{bd} W^b.$$  \hfill (A.5)$$

Subtracting the expression with $c$ and $d$ reversed yields the Ricci-Voss equation,

$$O_{abcd} = s^m_a s^n_b h^p_c h^q_d R_{mnpq} + K^a_{ace} K^e_{db} - K^a_{ade} K^e_{bc}.$$  \hfill (A.6)$$

Finally, the Codazzi equation arises by requiring that $[D_a, D_b]V^q$ is tangential on the $q$ index. This leads to

$$0 = s^d_q D_a D_b V^q = s^d_q h^m_a h^q_b h^e_c \nabla_m(D_n V^e)$$

$$= s^d_q h^m_a h^q_b \nabla_m(h^p_n \nabla_p V^q + K^q_{nc} V^c) + K^d_{ae} D_b V^e$$

$$= -K^p_{ab} L^d_{ap} V^q + s^d_q h^m_a h^q_b \nabla_m \nabla_q V^p + D_a K^d_{be} V^c + K^d_{be} D_a V^e + K^d_{ae} D_b V^e.$$  \hfill (A.7)$$

Subtracting the expression with $a$ and $b$ reversed and rearranging terms yields the Codazzi equation

$$h^m_a h^q_b h^p_c s^q_d R_{mnpq} = D_a K^d_{bec} - D_b K^d_{dac}.$$  \hfill (A.8)$$

Note that the identity obtained from requiring that $[D_a, D_b] V^q$ is normal on $q$ for $W^q$ normal is equivalent to equation (A.8).
A.2 Normal curvatures

Next we look to define a tensorial quantity to associate with the curvature of the normal connection $D_a$. This case is more subtle than the tangential curvature since, as discussed around equation (3.37), the twist tensor $F_{ab}$ imbues the connection $D_a$ with a property similar to torsion. This causes the commutator $[D_c, D_d]W^a$ that to not be tensorial, and an additional term of the form $F_{cd}D_eW^a$ must be subtracted in order to define a tensor. We take this modified commutator as the definition of the curvature of the normal connection, and in the process derive the analogue of the Gauss equation for this curvature. Taking $W^a$ to be normal, we start by computing

$$D_cD_dW^a = s^p e^q s^r m \nabla_p (D_q W^m)$$

$$= s^p e^q s^r m \nabla_p (s^e_q \nabla_e W^m + L_{mq}^{m} W^b)$$

$$= -L_c^{e} D_e W^a + s^p e^q s^r m \nabla_p \nabla_q W^m + L_{me}^{a} L_{m}^{d} W^b$$

(A.9)

When we subtract $D_d D_c W^a$ from this expression, the right hand side still contains the term $F_{cd} D_e W^a$, which prevents the commutator from being tensorial. However, we can simply move this term involving a derivative of $W^a$ to the left hand side, and take the definition of the normal curvature $\mathcal{C}_{bcd}^a$ to be\footnote{The letter $\mathcal{C}$ is employed for this curvature tensor in honor of Cattaneo-Gasparini, who appears to have been the first to consider this quantity in the special case of a foliation by one-dimensional curves [25, 26].}

$$(D_cD_d - D_d D_c - F_{cd} D_e)W^a = \mathcal{C}_{bcd}^a W^b$$

(A.10)

This equation for the curvature further justifies the interpretation of $-F_{cd}$ as a type of torsion for the connection $D_a$. For comparison, the definition of the curvature $\bar{R}_{bcd}^a$ for an affine connection $\tilde{\nabla}_a$ with torsion $\bar{T}_{ab}$ is

$$(\tilde{\nabla}_e \tilde{\nabla}_d - \tilde{\nabla}_d \tilde{\nabla}_e + T_{cd} \tilde{\nabla}_e) V^a = \bar{R}_{bcd}^a V^b.$$  

(A.11)

Combining equation (A.9) with the definition (A.10) leads to the Gauss equation for the normal curvature,

$$\mathcal{C}_{abcd} = s^m a s^n b s^p e s^q d R_{mnpq} + L_{ec}^{a} L_{d}^{e} - L_{ea}^{d} L_{cb}^{e} \quad (A.12)$$

This expression shows that the normal curvature enjoys the symmetries $\mathcal{C}_{abcd} = -\mathcal{C}_{bacd} = -\mathcal{C}_{abdc};$ however, it does not satisfy either of the remaining symmetries associated with a torsionless curvature tensor, since $\mathcal{C}_{abcd} \neq \mathcal{C}_{cdab}$ and $\mathcal{C}_{abcd} \neq 0$.

We can also define a normal outer curvature tensor $\mathcal{P}_{bcd}^a$ by acting with the derivative operator in (A.10) on a tangential vector $V^a$,

$$(D_c D_d - D_d D_c - F_{cd} D_e) V^a = \mathcal{P}_{bcd}^a V^b$$

(A.13)

Explicitly expanding out the derivatives, we find

$$D_c D_d V^a = s^p e^q s^r m \nabla_p (D_q V^m)$$

$$= s^p e^q s^r m \nabla_p (s^e_q \nabla_e V^m - L_{mq}^{m} V^b)$$

$$= -L_c^{e} D_e V^a + s^p e^q s^r m \nabla_p \nabla_q V^m + L_{me}^{a} L_{m}^{d} V^b$$

(A.14)
Using this to form the combination of derivatives in (A.13), we derive the Ricci-Voss equation for the normal outer curvature,

$$
\mathcal{P}_{abcd} = h^m_a h^n_b s^p_c s^q_d R_{mpnq} + L_{ace} L_{bd}^e - L_{ade} L_{bc}^e
$$  \hspace{1cm} (A.15)

$$
= \mathcal{O}_{cdab} - K_{cae} K_{db}^e + K_{cbe} K_{da}^e + L_{ace} L_{bd}^e - L_{ade} L_{bc}^e,
$$  \hspace{1cm} (A.16)

where the second expression was obtained by applying the tangential Ricci-Voss equation, (A.6). Hence we see that the outer curvature for $D_a$ is expressible in terms of the outer curvature for $D_a$ and $K_{abc}$ and $L_{abc}$.

Finally, we can derive a Codazzi identity for the normal connection by requiring that $[D_a, D_b] W^q$ is normal on $q$. We have

$$
0 = h^d_a D_a D_b W^q = h^d_q s^m_a s^b_s s^e_q \nabla_m (D_n W^e)
\nonumber
$$

$$
= h^d_q s^m_a s^n_b s^q_e \nabla_m (s^a_n \nabla_p W^q + L^a_{nc} W^c) + L^d_{ae} D_b W^e
\nonumber
$$

$$
= -L^p_{ab} K_{qp}^d W^q + h^d_q s^m_a s^n_b \nabla_m \nabla_n W^q + D_a L^d_{bc} W^c + L^d_{bc} D_a W^c + L^d_{ae} D_b W^e.  \hspace{1cm} (A.17)
$$

Subtracting the expression with $a$ and $b$ reversed leads to the normal Codazzi identity,

$$
s^m_a s^n_b s^p_c h^q_d R_{mpnq} = D_a L_{dbc} - D_b L_{dac} + F_{a e} K_{ced}.  \hspace{1cm} (A.18)
$$

### A.3 Mixed curvatures

Additional relations arise from considering the mixed commutator of the two derivatives, $[D_c, D_d] V^a$. Taking $V^a$ tangent, we compute

$$
D_c D_d V^a = s^p_c h^q_d h^a_m \nabla_p (D_q V^m)
\nonumber
$$

$$
= s^p_c h^q_d h^a_m \nabla_p (h^e_q \nabla_e V^m + K^m_{qp} V^b)
\nonumber
$$

$$
= L^e_{dc} D_e V^a + s^p_c h^q_d h^a_m \nabla_p \nabla_q V^m - L^a_{cm} K^m_{db} V^b.  \hspace{1cm} (A.19)
$$

We also need to separately calculate the expression with the derivatives in the opposite order,

$$
D_d D_c V^a = h^q_d s^p_c h^a_m \nabla_q (D_p V^m)
\nonumber
$$

$$
= h^q_d s^p_c h^a_m \nabla_q (s^e_p \nabla_e V^m - L^m_{qp} V^b)
\nonumber
$$

$$
= K^e_{cd} D_e V^a + h^q_d s^p_c h^a_m \nabla_q V^m - K^a_{md} L^m_{bc} V^b.  \hspace{1cm} (A.20)
$$

By subtracting these two expressions, we can form a tensor by considering the following combination of derivatives of $V^a$,

$$
(D_c D_d - D_d D_c + K^e_{cd} D_e - L^e_{dc} D_e)V^a = \mathcal{M}^a_{bcd} V^b.  \hspace{1cm} (A.21)
$$

The tangential mixed curvature tensor $\mathcal{M}_{abcd}$ then satisfies the identity

$$
\mathcal{M}_{abcd} = h^m_a h^n_b s^p_c h^q_d R_{mpnq} + L_{bce} K^e_{da} - L_{ace} K^e_{db}
\nonumber
$$

$$
= D_b K_{cad} - D_a K_{cbd} + L_{bce} K^e_{da} - L_{ace} K^e_{db},  \hspace{1cm} (A.22)
$$

$$
\mathcal{M}_{abcd} = h^m_a h^n_b s^p_c h^q_d R_{mpnq} + L_{bce} K^e_{da} - L_{ace} K^e_{db}
\nonumber
$$

$$
= D_b K_{cad} - D_a K_{cbd} + L_{bce} K^e_{da} - L_{ace} K^e_{db},  \hspace{1cm} (A.23)
$$

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where the second line is obtained by employing the tangential Codazzi identity (A.8). Note that the mixed curvature possesses the following symmetries: \( M_{abcd} = -M_{bacd} \), and \( M_{abcd} = 0 \).

We can also form a normal mixed curvature tensor \( \mathcal{N}_{abcd} \) through a definition similar to (A.21), except acting on a normal vector \( W^a \),

\[
(D_c D_a - D_a D_c + K_{cd}^e D_e - L_{dc}^e D_e)W^a = N_{abcd}^a W^b. \tag{A.24}
\]

Again expanding out the derivatives that appear in this expression, we find

\[
D_c D_d W^a = s^p_c h^q_d s^a_m \nabla_p (D_q W^m)
= s^p_c h^q_d s^a_m \nabla_p (h^e_q \nabla_e W^m - K_{bq}^m W^b)
= L_{dc}^e D_e W^a + s^p_c h^q_d s^a_m \nabla_p \nabla_q W^m - L_{mc}^a K_{bd}^m W^b. \tag{A.25}
\]

\[
D_d D_c W^a = h^q_d s^p_c s^a_m \nabla_q (D_p W^m)
= h^q_d s^p_c s^a_m \nabla_q (s^e_p \nabla_e W^m + L_{pq}^m W^b)
= K_{cd}^e D_e W^a + h^q_d s^p_c s^a_m \nabla_q W^m - K_{dm}^a L_{eb}^m W^b. \tag{A.26}
\]

These expressions then lead to the identity satisfied by the normal mixed curvature,

\[
\mathcal{N}_{abcd} = s^m_a s^m_b s^p_c h^q_d R_{mpnq} + K_{ade}^e L_{cb}^e - K_{bde}^e L_{ca}^e
= D_a L_{dbc} - D_b L_{dac} + F_{ab}^e K_{ced} + K_{ade}^e L_{cb}^e - K_{bde}^e L_{ca}^e. \tag{A.27}
\]

where the normal Codazzi identity (A.18) was used to obtain the second line. The only symmetry that the normal mixed curvature possesses is \( \mathcal{N}_{abcd} = -\mathcal{N}_{bacd} \), so in general \( \mathcal{N}_{abcd} \neq 0 \).

Finally, we can examine the mixed Codazzi identity, which will lead to a number of useful relations between various curvature quantities. These arise from requiring \([D_a, D_b] V^q\) is tangential on \( q \). Computing in a similar manner as before, we find

\[
0 = s^q_d D_a D_b V^q = s^q_d s^m_a h^p_n h^q_e \nabla_m (D_n V^e)
= s^q_d s^m_a h^p_n \nabla_m (h^p_n \nabla_p V^q + K_{nc}^q V^c) - L_{ea}^d D_b V^e
= L_{ba}^p L_{qp}^d V^d + s^q_d s^m_a h^p_n \nabla_m \nabla_n V^q + D_a K_{be}^d V^e + K_{bc}^d D_a V^e - L_{ea}^d D_b V^e. \tag{A.29}
\]

\[
0 = s^q_d D_b D_a V^q = s^q_d h^p_n s^m_a h^q_e \nabla_n (D_m V^e)
= s^q_d h^p_n s^m_a \nabla_n (s^p_m \nabla_p V^q - L_{cm}^q V^c) + K_{be}^d D_a V^e
= -K_{ab}^p K_{pq}^d V^q + s^q_d h^p_n s^m_a \nabla_n \nabla_m V^q - D_b L_{ca}^d V^e - L_{ea}^d D_b V^e + K_{be}^d D_a V^e. \tag{A.30}
\]

Subtracting these two expressions produces the mixed Codazzi identity,

\[
s^m_a h^p_n s^q_d h^c_e R_{qpnm} = -D_b L_{cde} - L_{ba}^e L_{ced} - D_a K_{dbc} - K_{ab}^e K_{dec}. \tag{A.31}
\]

Note that this relation appeared previously in [32], equation 3.1, where it was presented as a generalization of the Raychaudhuri equation for foliations with arbitrary dimension for the leaves. This interpretation takes the submanifolds to be timelike, and views (A.31) as an
evolution equation for $L_{cad}$. In the case of one-dimensional manifolds, the resulting equation captures the evolution of the expansion, shear, and twist, using the identifications between these quantities and $L_{abc}$ described in section 5.2.

(A.31) can be unpackaged somewhat by symmetrizing or antisymmetrizing on various indices. First, consider the antisymmetrization on $b$ and $c$, which leads to

$$s^q_d s^m_a h^n_b h^p_c R_{qmpn} = D_c L_{bad} - D_b L_{cad} + L_{ca} e L_{bed} - L_{ba} e L_{ced} + K_{ac} e K_{de} - K_{ab} e K_{dec}. \quad (A.32)$$

The Riemann tensor component in this equation is the same as the one appearing in the Ricci-Voss equation (A.6), and so we can substitute in the outer curvature tensor to this equation. This leads to an alternative expression for $O_{abcd}$,

$$O_{abcd} = D_c L_{dba} - D_d L_{cba} + L_{cb} e L_{dea} - L_{db} e L_{cea}. \quad (A.33)$$

Similarly, considering the antisymmetric part of (A.31) on $a$ and $d$ and comparing to the normal Ricci-Voss equation for $P_{abcd}$ (A.15) gives an alternative expression for this tensor,

$$P_{abcd} = D_c K_{dab} - D_d K_{cab} + K_{cb} e K_{dea} - K_{db} e K_{cea} + D_b F_{adc} + L_{db} e F_{aec} - L_{bc} e F_{aed}. \quad (A.34)$$

The symmetric part of (A.31) gives a relation on the remaining components of the space-time Riemann tensor that have not appeared in any of the previous identities. Two identities are thus obtained by symmetrizing on either $b$ and $c$, or on $a$ and $d$, and these read

$$s^m_a h^n_b h^p_c s^q_d R_{qmpn} = D_a K_{dbc} + K_{ab} e K_{dec} + D_b L_{rad} + L_{ta} e L_{rde} \quad (A.35)$$

$$s^m_a h^n_b h^p_c s^q_d R_{qmpn} = D_b A_{cad} + L_{ta} e L_{rde} + D_{tb} K_{abc} + K_{tb} e K_{dec}. \quad (A.36)$$

After symmetrizing on one pair of indices as in the above expressions, the Riemann tensors appearing are automatically symmetric in their second pair of indices, since $R_{qmpn} - R_{mpqn} = 0$. Hence, antisymmetrizing the above expressions on the remaining index pair will lead to differential identities involving only the $K_{abc}$ and $L_{abc}$ tensors. These are

$$D_a K_{dbc} - D_d K_{abc} = D_b F_{rad} + L_{ta} e L_{rde} - L_{ta} e L_{rde} \quad (A.37)$$

$$D_b A_{cad} - D_c A_{bad} = L_{ta} e L_{rde} - L_{ta} e L_{rde}. \quad (A.38)$$

which simplify using the modified connection $\tilde{D}_a$ from (3.31) to

$$D_a K_{dbc} - D_d K_{abc} = \tilde{D}_b F_{rad} \quad (A.39)$$

$$\tilde{D}_b A_{cad} - \tilde{D}_c A_{bad} = 0. \quad (A.40)$$

It is also useful to consider the traces of the above identities, since, for example, the trace of (A.35) on $b$ and $c$ contains an expression that appears in the Jacobi equation for perturbations of extremal surfaces. Hence, the traced identities read

$$s^m_a h^n_b h^p_c s^q_d R_{qmpn} = D_a K_{d} + K_{a} b c K_{dcb} + D_b L_{ad} + L_{b} e L_{bca} \quad (A.41)$$

$$s^m_a h^n_b h^p_c s^q_d R_{qmpn} = D_b A_{c} + L_{b} c e L_{cea} + D_{a} K_{bc} a + K_{a} b c K_{aec}. \quad (A.42)$$

$$D_a K_{d} - D_d K_{a} = D_b F_{ad} + L_{b} c e L_{bca} - L_{b} c e L_{bca} \quad (A.43)$$

$$D_b A_{c} - D_c A_{b} = 0. \quad (A.44)$$
Finally, equations (A.37) and (A.38) can be used in the expressions (A.33) and (A.34) for the outer curvature tensors to further simplify their expressions and make manifest some of the index symmetries. The resulting equations are

\begin{align}
\mathcal{O}_{abcd} &= D_c F_{dab} + \frac{1}{2} F_{ce}^a F_{dab} + 2 A_{ce}^a A_{dab} \\
\mathcal{P}_{abcd} &= D_a F_{bcd} + L_{ac}^e F_{bed} - L_{ad}^e F_{bec} + 2 K_{ce}^e K_{de}^a
\end{align}

(A.45)

(A.46)

(A.47)

\begin{align}
\mathcal{O}_{abcd} &= D_c F_{dab} + \frac{1}{2} F_{ce}^a F_{dab} + 2 A_{ce}^a A_{dab} \\
\mathcal{P}_{abcd} &= D_a F_{bcd} + L_{ac}^e F_{bed} - L_{ad}^e F_{bec} + 2 K_{ce}^e K_{de}^a
\end{align}

(A.48)

(A.49)

(A.50)

(A.51)

(A.52)

(A.53)

(A.54)

(A.55)

\textbf{B. Modified outer curvatures}

The modified connections \( \tilde{D}_a \) and \( \tilde{D}_a \) defined in (3.31) and (3.38) also have curvature identities associated with them. We can define the outer curvatures associated with these connections in analogy with equations (A.4) and (A.13),

\begin{align}
(\tilde{D}_c \tilde{D}_d - \tilde{D}_d \tilde{D}_c) W^a &= \tilde{\mathcal{O}}_{bcd}^a W^b \\
(\tilde{D}_c \tilde{D}_d - \tilde{D}_d \tilde{D}_c - F_{ce}^a D_e) V^a &= \tilde{\mathcal{P}}_{bcd}^a V^b.
\end{align}

(A.48)

(A.49)

\begin{align}
\tilde{D}_c \tilde{D}_d W^a &= \tilde{D}_c (D_d W^a - L_{db}^a W^b) \\
&= D_c D_d W^a - L_{cb}^a D_d W^b - D_e L_{db}^a W^b - L_{db}^a D_c W^b + L_{ce}^a L_{db}^e W^b.
\end{align}

(A.50)

(A.51)

Antisymmetrization on \( c \) and \( d \) then leads to

\begin{align}
\tilde{\mathcal{O}}_{bcd}^a &= \mathcal{O}_{bcd}^a - 2 \left( D_c L_{db}^a + L_{de}^a L_{cb}^e \right) = 0
\end{align}

(A.52)

by equation (A.33). Hence, \( \tilde{D}_a \) is a flat connection on the normal bundle.

\begin{align}
\tilde{D}_c \tilde{D}_d V^a &= \tilde{D}_c (D_d V^a - K_{db}^a V^b) \\
&= D_c D_d V^a - K_{cb}^a D_d V^b - D_e K_{db}^a V^b - K_{de}^a D_c V^b + K_{ce}^a K_{db}^e V^b.
\end{align}

(A.53)

(A.54)

Forming the combination of derivatives appearing in (A.49) then gives the relation

\begin{align}
\tilde{\mathcal{P}}_{bcd}^a &= \mathcal{P}_{bcd}^a - 2 \left( D_c K_{db}^a + K_{de}^a K_{cb}^e \right) = \tilde{D}_b F_{dc}^a
\end{align}

(A.55)

using (A.34).

\textbf{B. Coordinate expressions}

It is often useful when performing computations to have coordinate expressions for the various curvature quantities defined for a foliation. In this appendix we will derive the relevant
quantities for a foliation-adapted coordinate system. This adapted coordinate system splits the spacetime coordinates \( y^\mu \) into \( p \) normal coordinates \( y^A \), \( A = 0, \ldots, p - 1 \), and \( (d - p) \) tangential coordinates \( y^i \), \( i = p, \ldots, d - 1 \), where the normal coordinates are required to be constant on each surface of the foliation, i.e. \( \nabla_v y^A \) are normal forms. This means the coordinate basis vectors \( \partial_\mu \) are tangential, and so \( y^i \) define an intrinsic coordinate system on surface. The remaining coordinate basis vectors \( \partial_A \) are transverse to the surface, but in general are not normal, since determining if a vector is normal to the surface requires a metric. Once a metric has been specified, we can determine the tangential piece of \( \partial_A \) and subtract it off to form a normal vector \( v_A = \partial_A + N_A \partial_\mu \). This equation determines the shift vectors \( N_A = N_A \partial_\mu \) as simply the tangential vector that must be added to \( \partial_A \) to produce a normal vector. It is then straightforward to see that although \( \nabla_v y^i \) are not tangential forms to the surface, the normal piece is also determined by the shift, so instead the covectors \( w^i = \nabla_v y^i - N_A \nabla_v y^A \) are tangential. This leads us to define a basis \( v_\mu^a \) for the tangent space,

\[
v_\mu^a = \begin{cases} 
\partial_\mu^a + N_\mu^i \partial_i^a & \text{for } \mu = 0, \ldots, p - 1 \\
\partial_\mu^a & \text{for } \mu = p, \ldots, d - 1
\end{cases}
\]  

so that \( v_\mu^a \) are a basis for the normal subspace while \( v_\mu^a \) are a basis for the tangential subspace. Similarly, a basis \( w_\mu^a \) for the cotangent space is given by

\[
w_\mu^a = \begin{cases} 
\nabla_\mu y^a & \text{for } \mu = 0, \ldots, p - 1 \\
\nabla_\mu y^a - N_\mu^i \nabla_\mu y^A & \text{for } \mu = p, \ldots, d - 1
\end{cases}
\]  

with \( w_\mu^A \) a basis for the normal subspace and \( w_\mu^i \) a basis for the tangential subspace.

### B.1 Metrics

The coordinate expressions for the normal and tangential metrics can then be expressed in terms of these bases,

\[
s_{ab} = s_{AB} w_a^A w_b^B = s_{AB} \nabla_v y^A \nabla_v y^B 
\]

\[
h_{ab} = h_{ij} w_a^i w_b^j = h_{ij} (\nabla_v y^i - N^i_A \nabla_v y^A) (\nabla_v y^j - N^j_B \nabla_v y^B),
\]

where \( s_{AB} \) and \( h_{ij} \) are the components of the normal and tangential metric. Since the spacetime metric is given by the sum \( g_{ab} = s_{ab} + h_{ab} \), (B.3) and (B.4) lead to the following expressions for the line element,

\[
ds^2 = s_{AB} dy^A dy^B + h_{ij} (dy^i - N^i_A dy^A) (dy^j - N^j_B dy^B) = (s_{AB} + h_{ij} N^i_A N^j_B) dy^A dy^B - 2h_{ij} N^i_A dy^A dy^i + h_{ij} dy^i dy^j
\]

The mixed-index projectors can be constructed directly from the basis vectors and covectors, and depend only on the shift vectors,

\[
s_b^a = v_A^a w_b^A = (\partial_\mu^a + N_\mu^i \partial_i^a) \nabla_v y^A
\]

\[
h_b^a = v_i^a w_b^i = (\nabla_v y^i - N^i_A \nabla_v y^A) \partial_i^a.
\]

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The inverse metrics are constructed in a similar manner,

\[ s^{ab} = s^{AB} v^a_A v^b_B = s^{AB} (\partial^a_A + N^a_i \partial^i_A)(\partial^b_B + N^i_j \partial^j_B) \]  
(B.9)

\[ h^{ab} = h^{ij} a^a_i b^b_j = h^{ij} \partial^a_i \partial^b_j. \]  
(B.10)

Here, \( s^{AB} \) must be the the matrix inverse of \( s_{BC} \), which follows from the requirement that \( s^{ab} s_{bc} = s^a_c \), using the expressions (B.3) and (B.7). Unsurprisingly, \( h^{ij} \) must be the the matrix inverse of \( h_{jk} \) by an analogous argument. The decomposition of the line element in (B.6) shows that for a fixed foliation, the metric variation can be parameterized by variations of \( s_{AB}, N^i_B \) and \( h^{ij} \). These match onto the respective terms in the covariant decomposition of \( \delta g_{ab} \) given in (4.14).

The expression for the unit normal form is\(^{10}\)

\[ n_{a_1...a_p} = -\frac{\sqrt{\varepsilon s}}{p!} \varepsilon_{A_1...A_p} w_{a_1}^{A_1} \ldots w_{a_p}^{A_p} = -\sqrt{\varepsilon s} (dy^0 \wedge \ldots \wedge dy^{p-1})_{a_1...a_p}, \]  
(B.11)

where \( s = \det(s_{AB}) \), and \( \varepsilon_{A_1...A_p} \) is the antisymmetric symbol with \( \varepsilon_{01...(p-1)} = 1 \). The prefactor \( \sqrt{\varepsilon s} \) can be derived by applying the normalization condition (3.9) to \( n_{a_1...a_p} \). The induced volume form \( \mu_{a_1...a_{d-p}} \) on the surfaces has a similar expression,

\[ \mu_{a_1...a_{d-p}} = \frac{\sqrt{-\varepsilon h}}{(d-p)!} \varepsilon_i ... i_{d-p} w_{a_1}^{i_1} \ldots w_{a_{d-p}}^{i_{d-p}} \]  
(B.12)

\[ = \sqrt{-\varepsilon h} [(dy^p - N^p_A dy^A) \wedge \ldots \wedge (dy^{d-1} - N^{d-1}_B dy^B)]_{a_1...a_{d-p}} \]  
(B.13)

where again \( h = \det(h_{ij}) \). Finally, since the spacetime volume form \( \varepsilon \) is related to the unit normal form and induced volume form by \( \varepsilon = -n \wedge \mu \), and since it also has the coordinate expression \( \varepsilon = \sqrt{-g} dy^0 \wedge \ldots \wedge dy^{d-1} \) with \( g = \det g_{\mu\nu} \), from (B.11) and (B.13) we find the relation

\[ \sqrt{-g} = \sqrt{\varepsilon s} \sqrt{-\varepsilon h}. \]  
(B.14)

### B.2 Connection coefficients

The next objects to consider are the connection coefficients for the tangential and normal covariant derivatives, \( D_a \) and \( D_\alpha \). Beginning with \( D_a \) acting on a tangent vector \( V^a = V^j \partial^a_j \), the connection coefficients \( \gamma_{bc}^a \) are defined by the equation

\[ D_b V^a = h^a_d h^c_b \partial_c V^d + \gamma_{bc}^a V^d = (\partial_d V^j + \gamma_{dk}^i V^k) v^a_d v^i_j \]  
(B.15)

where \( \partial_c \) is the coordinate derivative operator associated with the surface-adapted coordinate system \( y^\mu \). Its action extends to covectors and multi-index vectors in the usual way. From the relation that \( D_a h_{bc} = 0 \), we conclude that \( D_a \) is the unique metric-compatible, torsion-free

---

\(^{10}\)The minus sign is conventional when the normal is timelike, and reflects the fact that \(-\nabla^a y^0\) is a future pointing timelike vector, when \( y^0 \) is a timelike coordinate. It can be omitted for timelike foliations with a spacelike normal.
connection on the surface, and hence the connection coefficients have the usual expression in terms of derivatives of the intrinsic metric components,
\[ \gamma^i_{jk} = \frac{1}{2} h^d (\partial_j h_{ik} + \partial_k h_{ij} - \partial_l h_{jk}). \] (B.16)

For \( D_b \) acting on a normal vector \( W^a = W^A v^a_A \), the connection coefficients can be obtained by writing
\[ D_b W^a = h^d b s^a_c \partial_d W^c + L_{de} W^e = (\partial_i W^A + L_i B^A W^B) v^a_A w^i_B, \] (B.17)
where the second equality employs the relation (B.40). This shows that the tensor \( L_{de} \) gives the connection coefficients for \( D_b \) acting on the normal bundle.

The normal connection coefficients \( \gamma^a_{bc} \) can be defined by the action on a normal covector \( W^c = W_A \nabla_c y^A \),
\[ D_b W^c = s^d_b s^a_c \partial_d W^a - \gamma^c_{bc} W^a = (\partial_B W^A + N^i_B \partial_i W^A - \gamma^A_{BC} W^A) w^B_a w^C_b \] (B.18)
which involves both transverse \( \partial_A \) and tangential \( \partial_i \) derivatives, due to the form of the normal projector from (B.7). It is convenient to define a modified transverse derivative
\[ \partial^*_A = \partial_A + N^i_A \partial_i, \] (B.19)
which will appear often in the following formulas. The coordinate expression for \( \gamma^A_{BC} \) again follows from the fact that \( D_a s_{bc} = 0 \). However, since the derivatives appearing in this expression are \( \partial^*_A \) rather than \( \partial_A \), the formula for the connection coefficients will be the usual expression except with derivatives of \( s_{AB} \) with respect to \( \partial^*_C \) instead of \( \partial_C \),
\[ \gamma^A_{BC} = \frac{1}{2} s^{AD} (\partial^*_B s_{DC} + \partial^*_C s_{DB} - \partial^*_D s_{BC}). \] (B.20)

The connection coefficients for \( D_b \) acting on a tangent vector \( V^a = V^i v^a_i \) can be obtained applying (B.39),
\[ D_b V^a = s^d_b h^c_e (\partial_d V^c + \Gamma^c_{de}) V^e = (\partial_A V^i + (K^i_{Aj} - \partial_j N^i_A)) v^a_A w^i_B, \] (B.21)
so that \( K^i_{Aj} - \partial_j N^i_A \) are the connection coefficients.

### B.3 Extrinsic curvatures

The extrinsic curvatures are the next items to consider. To derive the coordinate expression for the tangential extrinsic curvature, it is useful to start with the interpretation of \( K^a_{bc} \) as measuring the change in the induced metric on the surface under a flow in the normal direction. Specifically, the projected Lie derivative of \( h_{ab} \) along a normal vector \( W^c \) satisfies
\[ \frac{1}{2} h^c_e h^d_b \mathcal{L}_W h_{cd} = \frac{1}{2} h^c_e h^d_b (W^e \nabla_c h_{ab} + \nabla_a W^e h_{eb} + \nabla_b W^e h_{ae}) = W^e K^i_{eab}. \] (B.22)
This formula can then be converted straightforwardly to a coordinate expression by writing

\[ K^A_{ij} = u^A_d s^d a K_{abc} v^b_i v^c_j = s^{AB} v^a_B K_{abc} v^b_i v^c_j = \frac{1}{2} s^{AB} (L_{\partial_B} h_{bc} + N^d_B \partial_k h_{bc}) v^b_i v^c_j. \tag{B.23} \]

The first term \( L_{\partial_B} h_{bc} \) will just produce the partial derivative of the components \( h_{ij} \) of the induced metric. For the second term, note that \( N^d_B \partial_k \) is a tangent vector, and hence we can evaluate this Lie derivative using the metric-compatible derivative \( D_a \). This results in

\[ K^A_{ij} = \frac{1}{2} s^{AB} (\partial_B h_{ij} + \tilde{D}_j N_{jB} + \tilde{D}_j N_{iB}), \tag{B.24} \]

where the modified normal connection of \((3.31)\) is used, given by \( \tilde{D}_j N_{jB} = \partial_j N_{jB} - \gamma_{ij}^k N_{kB} \) and \( N_{jB} = h_{jt} N^t_B \). In particular, the tilde means to treat \( N_{jB} \) as a collection of covectors indexed by \( B \), as opposed to a two-index tensor with one tangential index \( j \) and one normal index \( B \).

A slight modification of this argument leads to the coordinate expression for \( A^i_{AB} \). It is straightforward to see that this object measures the change in the normal metric \( s_{ab} \) when flowing in a tangential direction, since it satisfies the analog of equation \((B.22)\) for a tangent vector \( V^c \),

\[ \frac{1}{2} s^c a s^d b L_V s_{cd} = V^e A_{eab}. \tag{B.25} \]

Performing the analogous calculation to equation \((B.23)\) leads to

\[ A^i_{AB} = \frac{1}{2} \tilde{g}^{ij} \partial_j s_{AB} \tag{B.26} \]

which involves no shift vectors since the tangential basis vectors \( v_i^a \) are defined independent of \( N^i_A \).

The coordinate expression for the twist tensor \( F^{ai}_{bc} \) can be obtained using the expression \((B.7)\) for the normal projector and the direct definition of the twist \((3.21)\). This leads to

\begin{align*}
F^{ai}_{AB} &= w^a_i v^b_A v^c_B F^{ab}_{bc} = 2w^a_i v^b_A v^c_B \nabla_b s^a_c \\
&= 2w^a_i v^b_A v^c_B [(\partial_2 N^j_C) \partial_j s_{BC} + \Gamma^a_{2d} s_{cd} - \Gamma^d_{bc} s_{da}] \\
&= \partial_{sA} N^i_B - \partial_{sB} N^j_A. \tag{B.27}
\end{align*}

Note this is consistent with with equation \((3.62)\) of reference \([1]\) for the twist tensor.

**B.4 Intrinsic curvatures**

The coordinate expressions for the curvature tensors are derived directly from their defining formulas in terms of commutators of derivative operators, which were presented in appendix \(A\). Equation \((A.1)\) defines the intrinsic curvature \( R^{ai}_{bcd} \) of the tangential connection \( D_a \). To compute its coordinate expression, we apply the formula \((B.15)\) for the coordinate expression of \( D_b V^c \), along with its generalization to covectors and multi-index tensors. This leads to

\begin{align*}
v^a_i v^b_j D_a D_b V^c &= v^a_i v^b_j D_a \left[ (\partial_m V^i + \gamma^i_{mj} V^j) v^c_i v^m_j \right] \\
&= [\partial_k \partial_l V^i + \partial_k \gamma^i_{lj} V^j + \partial_l \gamma^i_{kj} V^j + \gamma^i_{km} (\partial_l V^m + \gamma^m_{lj} V^j) - \gamma^m_{kl} (\partial_m V^i + \gamma^i_{mj} V^j)] v^c_i. \tag{B.28}
\end{align*}
Subtracting the expression with the derivative order reversed retains only the antisymmetric piece on \( k \) and \( l \), and this leads to the expected formula for the intrinsic curvature components,
\[
\mathcal{R}^i_{jk} = \partial_k \gamma^i_j - \partial_l \gamma^i_k + \gamma^i_m \gamma^m_l - \gamma^m_k \gamma^m_j. \tag{B.29}
\]

For the intrinsic normal curvature \( C^a_{bcd} \), we utilize the definition (A.10) along with the coordinate expression \( D_b W^c = (\partial_s D^A W^A + \mathcal{J}^A_{DB} W^B)w^D_b v^e_A \) and its generalization to covectors and tensors. This gives
\[
v^c_a v^d_b D_a D_b W^c = v^c_a v^d_b D_a \left[ (\partial_s W^A + \mathcal{J}^A_{EB} W^B)w^E_b v^e_A \right]
= \left[ \partial_s \partial_s W^A + \partial_s \mathcal{J}^A_{DB} W^B + \mathcal{J}^A_{DB} \partial_s W^B \right.
+ \mathcal{J}^A_{CE} (\partial_s W^E + \mathcal{J}^E_{DB} W^B) - \mathcal{J}^A_{CD} (\partial_s W^A + \mathcal{J}^A_{EB} W^B) \right]
\tag{B.30}
\]

This time when subtracting the expression with the derivatives reversed, the term involving two derivatives of \( W^A \) is nonzero due to the shift vectors present in the starred derivatives. Instead, we find using (B.27)
\[
(\partial_s \partial_s D - \partial_s D \partial_s) W^A = F^i_{CD} \partial_i W^A = F^i_{CD} (D_i W^A - L_i B^A W^B). \tag{B.31}
\]

The first term here is exactly the piece that must be subtracted from the \([D_a, D_b]\) commutator to form the curvature according to (A.10). Adding the second term to the antisymmetric pieces in (B.30) on \( C \) and \( D \) then yields the coordinate expression for the normal curvature,
\[
C^A_{BCD} = \partial_s \mathcal{J}^A_{DB} - \partial_s \mathcal{J}^A_{CB} + \mathcal{J}^A_{CE} \mathcal{J}^E_{DB} - \mathcal{J}^A_{DE} \mathcal{J}^E_{CB} - F^i_{CD} L_i B^A, \tag{B.32}
\]
and the components \( L_i B^A \) come from (B.26) and (B.27) using \( L_m^b a = A_m^b a - \frac{1}{2} F_m^b a \). This coordinate expression for the normal curvature has appeared previously in the special case of a one-dimensional foliation in \([25, 27, 28]\).

For the remaining curvature tensors, the formulas derived in appendix A.3 lead directly to the coordinate expressions. For the outer curvature tensors \( O_{ABkl} \) and \( P_{ijCD} \) one simply uses (A.45) and (A.46) along with the expressions (B.24), (B.26) and (B.27) for \( K^A_{ij} \), \( A^i_{AB} \) and \( F^i_{AB} \). Similarly, equations (A.23) and (A.28) lead to coordinate expressions for the mixed curvatures \( M_{abcd} \) and \( N_{abcd} \).

### B.5 Spacetime Christoffel symbols

The final set of formulas will relate the spacetime Christoffel symbols for the metric defined by (B.6) to the intrinsic connection coefficients \( \gamma^i_{jk} \), \( \mathcal{J}^A_{BC} \), extrinsic curvatures \( K^A_{ij} \), \( L^i_{AB} \), and shift vectors \( N^i_B \). First, note that for the tangential covariant derivative,
\[
D_a V^b = h^c_a h^b_d (\partial_c V^d + \Gamma^d_{ce} V^e) = h^c_a h^b_d (\partial_c V^d + \gamma^d_{ce} V^e), \tag{B.33}
\]
leading to the conclusion
\[
\gamma^a_{bc} = h^a_d h^c_e h^f_d \Gamma^d_{ef}. \tag{B.34}
\]
A similar argument applies to the normal connection,

$$ D_a W^b = s^c a s^d b (\partial_c W^d + \Gamma_{ce}^d W^e) = s^c a s^d b (\partial_c W^d + \sum_{ce} V^e), $$

which implies

$$ \sum_{bc} = s^a e s^b s^c e \Gamma_{ef}^d. $$

The remaining components of the Christoffel symbols can be arrived at by examining the covariant derivative of \( s^a b \). From its coordinate expression (B.7), we find

$$ \nabla_b s^a _c = (\nabla_b N^i_A) \partial^a_i \nabla_c y^A + \Gamma_{bc}^a s^d_e - \Gamma_{bc}^a s^d_a. $$

and projecting onto different components yields the following expressions,

$$ s^d e h^e b h^f c \Gamma_{ef}^d = -K_{bc}^a $$
$$ h^d e h^e b s^f c \Gamma_{ef}^d = K_{cb}^a - (h^e b \nabla_e N^i_A) \partial^a_i \nabla_c y^A $$
$$ s^d e h^e b h^f c \Gamma_{ef}^d = L_{cb}^a $$
$$ h^d e h^e b s^f c \Gamma_{ef}^d = -(s^b e \nabla_e N^i_A) \partial^a_i \nabla_c y^A - L_{bc}^a $$

These four relations along with (B.34) and (B.36) lead to coordinate expressions for all components of the Christoffel symbols:

$$ \Gamma_{jk}^i = \gamma_{jk}^i - N^i_A K_{jk}^A $$
$$ \Gamma_{ij}^A = -K_{ij}^A $$
$$ \Gamma_{Aj}^i = K_{Aj}^i - D_j N^i_A + N^k_A N^i_B K_{kj}^B $$
$$ \Gamma_{Bi}^A = L_{iB}^A + N^i_B K_{ji}^A $$
$$ \Gamma_{AB}^i = -A_{AB}^i - D_{\alpha A} N^i_B - N^j_A K_{Bj}^i + 2 N^j_A D_j N^i_B - \gamma_{jk}^i N^j_A N^k_B - N^j_A N^k_B N^i_B K_{jk}^C $$
$$ \Gamma_{BC}^A = F_{BC}^A - N^j_B L_{jC}^A - N^i_C L_{jB}^A - N^j_B N^i_C K_{ij}^C. $$

The explicit expressions for the covariant derivatives of the shift vectors appearing in (B.44) and (B.46) are

$$ D_j N^i_A = \partial_j N^i_A + \gamma_{jk}^i N^k_B - L^B_{jA} N^k_B $$
$$ D_A N^i_B = \partial_A N^i_B - \sum_{AB} N^i_B + K_{Aj}^i N^j_B - (\partial_j N^i_A) N^j_B $$
$$ D_{\alpha A} N^i_B = \partial_{\alpha} N^i_B - \sum_{AB} N^i_B + K_{Aj}^i N^j_B. $$

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