On Entropic Optimization and Path Integral Control

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Abstract
This article is motivated by the question whether it is possible to solve optimal control or dynamic optimization problems in a similar fashion to how static optimization problems can be addressed with Evolutionary Strategies. The latter maintain a sequence of Gaussian search distributions that converge to the optimum. For the moment, this question has been answered partially by a set of algorithms that are known as Path Integral Control methods. Those maintain a sequence of locally linear Gaussian feedback controllers or policies. So far Path Integral Control methods have been derived solely from the theory of Linearly Solvable Optimal Control, which, however beautiful, includes only a narrow subset of optimal control problems and has only limited application potential as a consequence. We aim to address this question within a far more general mathematical setting. Therefore, we first identify the framework of entropic inference as a suitable setting to synthesise stochastic search algorithms. Therewith we establish the formal framework of entropic optimization and provide a compelling justification for the inclusion of information-theoretic measures in stochastic optimization. From this theory follows a formal optimal search distribution sequence which converges monotonically to the Dirac delta distribution centered at the optimum. Then we demonstrate how this result can be used to derive Gaussian search distributions similar to existing Evolutionary Strategies. We then proceed to transfer these ideas from a static to a dynamic setting, therewith establishing the framework of Entropic Optimal Control which shares characteristics with Relative Entropy and Maximum Entropy Reinforcement Learning. From this theory we can construct a number of formal optimal path distribution sequences. Thence we derive the outlines of a generalised algorithmic framework complementing the set of existing Path Integral Control methods. The main ambition of the present theoretical inquiry is to reveal how all of these fields are related in a most exciting fashion. In future work we intend to study the numerical implications and practical applications.
1. Introduction

The first problem we are drawn to is that of static optimization where the goal is to find optimization variables \( x \in X \) that minimize the objective \( f : X \mapsto \mathbb{R} \), where \( X \subset \mathbb{R}^n \) represents the feasible subset. This is considered a standard problem, both in engineering as in applied sciences.

\[
\min_{x \in X} f(x) \tag{1}
\]

The second problem we address is that of dynamic optimization or optimal control, the paradigm overarching trajectory optimization methods, model based control methods and Reinforcement Learning. Here the goal is to find an input sequence \( \{a_0:T-1\} \) that minimizes cost, \( R : T \mapsto \mathbb{R} \), accumulated over the state-action trajectory \( \tau = \{s_1:T, a_0:T-1\} \). Function \( r : S \times A \mapsto \mathbb{R} \) represents the cost rate. The set \( T \) contains all trajectories \( \tau \) that agree with the dynamics of the system, i.e. \( s_{t+1} = f_t(s_t, a_t) \), and any constraints concerning feasible states and actions, \( s_t \in S_t \) and \( a_t \in A_t \). One may note that \( T \subset S_{1:T} \times A_{0:T-1} \).

\[
\min_{\tau \in T} R(\tau) = \sum_{t=0}^{T-1} r_t(s_t, a_t) + r_T(s_T) \tag{2}
\]

With the prefix dynamic, it is emphasized that the optimization variables are constrained by a causal structure which allows to break apart the problem into several subproblems that can be solved recursively. This problem property is also known as optimal substructure which can be exploited by dedicated solution algorithms. Correspondingly, problem (2) can also be represented by the recursive Bellman equation. Here \( V_t : X_t \mapsto \mathbb{R} \) represents the value function or optimal cost-to-go, i.e. the cost that is accumulated if we initialize the system in state \( s \) at time \( t \) and control it optimally until a final time \( T \)

\[
V_t(s) = \min_a r_t(s, a) + V_{t+1}(f_t(s, a)), \quad V_T(s) = r_T(s)
\]

In this article we are concerned with computational procedures, or so called search methods, that solve either problem (1) or (2). Search methods look for the optimum by querying the objective function for particular points of interest, iteratively exploring the feasible optimization space by exploiting the queried information. The way in which the objective provides such information, determines to a large extent whether the problem can be solved successfully.
Challenging conditions are set when algebraic models for the objective and feasible space are absent. Such problems are typically referred to as black-box optimization problems. A vast body of work exists on black-box optimization. Therefore, we will not engage into a comprehensive literature review and trust in that the reader is familiar with the common approaches. We are interested specifically in stochastic search methods. Stochastic search methods rely on randomness to probe the optimization space and maintain mechanisms that eventually channel that randomness towards prosperity.

Broadly speaking, any stochastic search method maintains a prior distribution of candidate solutions over the feasible space and generates a posterior distribution based on the prosperity of the individual candidates. Evolutionary Strategies (ESs) refer to a particular subclass of stochastic search algorithms tailored to problems of the form \( (1) \) \cite{1,2}. As opposed to population based algorithms \cite{3}, ES engage a parametric search distribution model, \( \pi(x|\theta): \mathcal{X} \rightarrow \mathbb{R}_{\geq 0} \). Every main iteration, \( g \), a sample population, \( \mathcal{X}_g = \{x^k_g\} \), is generated from a search distribution \( \pi(x|\theta_g) \) and the search distribution parameters, \( \theta_{g+1} \leftarrow \theta_g \), are updated based on the relative success of the individual samples. The objective function value \( f(x^k) \) is used as a discriminator between prosperous and poor behaviour of each individual, \( x^k \) \cite{4}. When iterated this concept spawns a sequence of distributions, \( \{\pi_g\} \). The update procedure is devised so that distribution sequence migrates gradually through the optimization space to converge towards the solution of \( (1) \). Although that limiting the sequence to a parametric distribution family may compromise the inherent expressiveness or elaborateness of the associated search, it also elevates the determination of update rules, from what are basically heuristics, to a more rigorous and theoretical body \cite{5,6}.

The most well-known members are the Covariance Matrix Evolutionary Adaptation Strategy (CMA-ES) \cite{1,5} and the Natural Evolutionary Strategy (NES) \cite{2}. Both CMA-ES as NES consider a multivariate Gaussian parametric distribution and provide appropriate update procedures for the mean, \( \mu_g \), and, covariance, \( \Sigma_g \). ESs are successful for such optimizing problems where gradient information is inaccessible or where traditional gradient based algorithms are prone to fall in local minima. Examples of application on medium to high scale problems for non-differential design optimization are \cite{7,8,9,10,11,12,13}. Such methods have also been picked up in Machine and Reinforcement Learning, for hyper-parameter \cite{14} and direct policy optimization \cite{15}.

\[
\pi_g(x) = \mathcal{N}(\theta_g) = \mathcal{N}(x|\mu_g, \Sigma_g)
\]
In the present work, we are driven by the question whether we can address problem (2), exploiting stochasticity as a natural means of exploration, in a similar fashion as how ESs address problem (1)? We might emphasize that problem (2) is fundamentally different from the static counterpart, in the sense that we can no longer probe the optimization space directly. We can only do so by applying stochastic policies to the system, the observe how the system evolves and draw conclusions from these system rollouts. Specifically, we are interested in locally linear Gaussian feedback policies of the following form and that we can apply to the system to inject the required stochasticity.

\[ \pi_{g,t}(a|s) = \mathcal{N}(\theta_{g,t}) = \mathcal{N}(a|a_{g,t} + K_{g,t}s, \Sigma_{g,t}) \]

In the previous decade a novel class of stochastic search algorithms was discovered that partially accommodate our question. The class is known as Path Integral Control (PIC). For now PIC always root back to the framework of Linearly Solvable Optimal Control (LSOC). LSOC is a narrow subset of the Stochastic Optimal Control (SOC) framework (see 2.2), characterised by a set of particularly peculiar properties (see 4.3) [16, 17, 18]. Kappen [19] was the first to demonstrate that in the specific setting of LSOC, the value function, \( V_t \), can be estimated from uncontrolled rollouts. Likewise a parametric policy can be estimated from the likelihood weighted rollouts statistics. Theodorou et al. then pioneered the idea to deliberately perturb deterministic policies therewith establishing the first dynamic stochastic search algorithms known as Path Integral Policy Improvement (PI²) [20, 21]. Stulp et al. [22, 23] first pointed out the structural similarities between PI² and CMA-ES and suggested the PI²-CMA algorithm. Their method deviates from the theory of LSOC and adapts the policy covariance in analogy with CMA-ES. This modification improved the convergence properties significantly yet, as said, ignored the theory. Other attempted generalisations include [24, 25].

There has been keen interest in such algorithms since stochastic search algorithms may display several advantages over gradient based algorithms [25, 22, 23]. So far it has been used in guided policy search to generate a set of prior optimal trajectories that were then used to fit a global policy [26] and is one of the two algorithms promoted by the Lyceum robot learning environment [27]. The use of PIC algorithms for real-time control applications was only recently considered as their execution is similar to Monte Carlo (MC) algorithms and were therefore thought to be too time critical to perform in real-time. However, with the rise of affordable GPUs and the ease
of parallelisation of MC based methods, it may become feasible in the near future to iterate dynamic stochastic search algorithms in real-time \cite{28, 29}. Nevertheless, practitioners of such methods have raised issues concerning the update of the covariance matrix \cite{22, 28, 29, 26}. It seems a mechanism is inherent to the existing framework that makes the covariance matrix vanish, compromising exploration. In other words, the search distribution collapses prematurely. Many authors suggested that the issue is limitedly understood and that it seems unlikely that it can be resolved with the theory at hand. It is clear that ESs and PIC methods exhibit obvious similarities and that PIC method may benefit from the rich body of work concerning ESs. However, since PIC are derived solely from the theory of LSOC, we argue that they are only limitedly understood and their similarity has been circumstantial.

Recently Williams et al. provide a novel derivation of the PI\textsuperscript{2} method from an information-theoretic background \cite{29}. Other authors have explored the relation between LSOC and information-theory \cite{29, 30, 31}, however these studies aimed for a physical connection and understanding.

In this paper, we venture on a different strategy. We aim to describe an overarching optimization principle that leads us to derive to ESs in the context of static optimization and PIC methods in the context of dynamic optimization. Hence we approach the problem from an algorithmic point of view, rather than searching for a deeper physical interpretation or understanding. To establish the overarching framework, we identify the principle of entropic inference as a suitable setting to synthesise stochastic search algorithms and derive an entropic optimization framework from it. This will also allow us to derive a generalised set of PIC methods which are no longer limited to the LSOC setting and therefore do not inherit any of its inherent limitations. Furthermore, the mutual theoretical background paves way for a knowledge transfer from ES to PIC. Finally, this viewpoint provides us with a unique opportunity to conclusively relate PIC to existing entropy-regularization paradigms in Reinforcement Learning \cite{32, 33, 34}.
2. Preliminaries

Our contributions navigate two seemingly disparate fields of research. In this preliminary section we provide key equations and concepts, establish notation and provide references for further reading.

2.1. Entropic inference

Inductive inference refers to the problem of how rational agents should update their state of knowledge, or so called belief, about some quantity when new information related to that process becomes available. Beliefs about any quantity $x \in X \subset \mathbb{R}^n$ are modelled as probability distributions over said space, for example $\pi$ or $\rho$. Inference procedures establish a computational framework to determine how new information can be integrated with the information held by any prior belief, say $\rho$, to consistently determine an informed posterior, say $\pi$. This model implies that the quantity $x$ has a unique value $x^*$ but that we are simply unable to pin it down exactly provided with the accessible information. In that sense the lack of certainty that is described is epistemological. In any case, inference procedures thus boil down to updating probabilities based on new information.

On the other hand, to date the concept of information is not well defined. Intuitively, data should somehow contain or convey information but clearly such a definition can not be wielded to spawn a rich mathematical framework. A more practical yet implicit definition of information is that what forces us to change our rational beliefs. Put differently, information is a constraint on the family of acceptable posteriors [35, 36]. Particularly, here we are interested in constraints of the following form where $g: \mathbb{R}^n \mapsto \mathbb{R}^m$ represents some measurement function and $\mu$ the expected measurement.

$$C_g(\mu) := \{\pi \in \Pi : \mathbb{E}_\pi[g_k] = \mu_k, \forall k\}$$

Any $\pi \in C_g(\mu)$ that satisfies the information constraint now qualifies as a potential posterior belief. The challenge thus reduces to identifying a unique posterior from among all those that could give rise to the observed measurements. The solution is to establish a ranking on the set $C_g(\mu)$ by determining a functional $F$ that associates a value to any $\pi \in \Pi$ relative to the prior $\rho$. The inference procedure is then determined uniquely.

$$\pi^* = \arg \min_{\pi \in C_g(\mu)} F[\pi, \rho]$$
Caticha et al. promotes a total of three criteria to determine such a functional uniquely, following an eliminative inductive procedure according to a principle they refer to as *minimal updating* [36]. The *principle of minimal updating* states that - assuming that any information held in the form of a prior belief is valuable - the functional $F$ should be chosen so that beliefs are updated only to the minimal extent required by new information represented by a constraint, and, gives rise to the four basic axioms promoted earlier by Jaynes [37, 38] and Johnson and Shore [39].

The functional $F$ coincides with the relative entropy measure, stating that, subject to precisely stated prior data, amongst which both propositions that express testable information and prior beliefs, the posterior which best represents the current state of knowledge is the one with minimum relative entropy. In ordinary language, the principle can be said to express epistemic modesty or maximal ignorance. The posterior distribution then makes the least claim to being informed beyond the stated prior data, that is to say the one that admits the most ignorance beyond it.

The idea roots back at least to the maximum entropy principle first introduced by Jaynes, whom incorporated the work of Shannon [40] in information theory [1] and applied it to statistical inference problems. Kullback generalised the principle to minimum relative entropy updating [41]. Jaynes [42] and Tikochinsky et al. [43] soon provided compelling theoretical arguments for the entropy as the only consistent measure to express how uninformed one is about a specific inference problem, particularly the entropy concentration and consistency principles. The interpretation that we will wield here is that of a quantity measuring the amount of knowledge that is left to be specified so to determine some epistemological uncertain variable uniquely [2].

$$F[\pi, \rho] := E_\pi [\log \pi - \log \rho] := D[\pi \parallel \rho] = \int \pi(x) \log \frac{\pi(x)}{\rho(x)} \, dx$$

---

1 The foundation of information theory was provided by Shannon who introduced Boltzmann’s thermodynamic entropy as a measure defined over probability distributions that was somehow related to the information they carry. For a proper distinction, one refers to the information-theoretic measure with information entropy as dubbed by Shannon and von Neumann. The specific use of terminology initiated decades of debate about the true nature and meaning of information entropy, which has obscured the field of information theory ever since.

2 If not semantically correct, it is at least an interpretation that is consistent with the mathematics.
Finally, we end up solving the following problem

$$
\pi^* = \arg \min_{\pi \in C} \mathbb{D}[\pi \parallel \rho]
$$

which solution is stated by the following Lemma. For the proof we refer to Appendix A.

**Lemma 1.** The solution of the minimum relative entropy inference problem stated in (3) is given by (in vector notation)

$$
\pi^*(x) \propto \rho(x) \cdot e^{-\lambda^T g(x)}
$$

up to a normalization constant

$$
Z = \int \exp(-\lambda^T g(x)) dx
$$

2.2. Stochastic Optimal Control

Secondly, we will make helpful use of the finite horizon Stochastic Optimal Control (SOC) framework with known initial state $s_0$. This is a trivial stochastic generalisation of the deterministic optimal control problem which collapses onto the deterministic problem when all probabilities are replaced by the appropriate Dirac delta distributions [44].

$$
\min_{\pi_t} \mathbb{E}_{\omega}[R] = \int R(\tau) \omega_\pi(\tau) d\tau
$$

Here $\omega_\pi : T = S_{1:T} \times A_{0:T-1} \mapsto \mathbb{R}_{\geq 0}$ denotes the controlled path probability distribution. It is implied that we consider control problems which can be modelled as a Markov Decision Process with temporal transition probability $\omega_t : S_t \times A_t \times S_{t+1} \mapsto \mathbb{R}_{\geq 0}$ so that $\omega_t(s_{t+1}|s_t, a_t)$ denotes the probability of moving to state $s_{t+1}$ from state $s_t$ when applying action $a_t$ and stochastic policies $\pi_t : S \times A_t \mapsto \mathbb{R}_{\geq 0}$ where $\pi_t(s_t|a_t)$ denotes the probability of applying action $a_t$ when observing state $s_t$. Some dense notation is introduced to unload the notational burden in the remainder of this text.

$$
\omega_\pi(\tau) = \prod_{t=0}^{T-1} \omega_t(s_{t+1}|s_t, a_t) \pi_t(a_t|s_t) = \prod_t \omega_t \pi_t = \omega \cdot \pi
$$

The SOC problem also agrees with a recursive expression, i.e. has an optimal substructure, known as the stochastic Bellman equation, which, in its most general form, is expressed below. This is a well known result.

$$
V_t(s) = \min_{\pi_t} \int \pi_t(a|s) \left( r_t(s, a) + \int V_{t+1}(s') \omega_t(s'|s, a) ds' \right) da
$$
Here $V_t : S \mapsto \mathbb{R}$ denotes the stochastic value function or optimal cost-to-go. We might want to emphasize here that we assume the policy to be stochastic. However, unless there is the incentive to maintain a stochastic policy for the purpose of exploration, the expression minimizes for a deterministic action so that the expectation over the policy can be omitted \[45\]. This redundant form is however appealing for later comparison.

3. Entropic Optimization

Let us here elaborate the central idea of entropic optimization, which makes, as far as the authors are aware of, an original, and, compelling argument for the practice of entropy regularization in the context of optimization. We argue that the principle of entropic inference can be practised to serve the purpose of optimization. The first step in this, is to model any beliefs we might have about the solution of (1) with some prior distribution function, say $\rho$. Secondly, instead of supposing information in the form of the expected value of the objective $f$, here we only require that the expected value with respect to the posterior, $\pi$, is, some amount $\Delta > 0$, smaller than the expected value is with respect to the prior. In this fashion, we change our prior belief about the optimal solution but only to the minimal extent required to decrease the expectation taken over $f$ with some arbitrary value $\Delta$. Put differently, we obtain a posterior that makes least claim to being informed about the optimal solution beyond the stated lower limit on the expectation. This idea can be formalized accordingly

$$\min_{\pi \in \Pi} \mathbb{D}[\pi \| \rho] \quad \text{s.t.} \quad \mathbb{E}_{\pi}[f] + \Delta \leq \mathbb{E}_{\rho}[f]$$

(6)

Since the relative entropy minimizes for $\pi = \rho$, it follows immediately that the inequality tightens into an equality constraint. Nonetheless, we should be careful when we pick a value for $\Delta > 0$ that respects the bound $\Delta \leq \mathbb{E}_{\rho}[f] - f^\star$. This is however a practical concern that does not interfere with what we wish to accomplish here, which is to construct an entropic update procedure that we can practise to serve the purpose of optimization. To do so it suffices to solve the problem above for $\pi$.

**Lemma 2.** The solution of problem (6) is given by distribution $\pi$ where $\lambda > 0$

$$\pi(x) \propto \rho \cdot e^{-\lambda f(x)}$$

(7)
Proof. A Lagrangian can be constructed introducing the Lagrangian multipliers $\lambda > 0$ and $\eta$

$$L[\pi] = D[\pi \parallel \rho] + \lambda E_\pi[f] + \lambda \Delta - \lambda E_\rho[f] + \eta E_\pi - \eta$$

$$= \int (\log \pi(x) - \log \rho(x) + \lambda f(x) + \eta) \pi dx - \lambda \int f(x) \rho(x) dx + \lambda \Delta - \eta$$

According to the calculus of variations it must then hold that the variation of the functional $L$ should be zero for any function $\pi$. Accordingly, we find

$$\log \pi(x) - \log \rho(x) + \lambda f + 1 + \eta = 0$$

It follows that the posterior and the prior are proportional up to a normalization constant $\pi \propto \rho \cdot e^{-\lambda f}$. The normalization constant is associated to $\eta$.

The value of $\lambda$ can be determined exactly by substituting the expression $\pi$ back into the original problem and minimizing the so-called dual problem

$$G(\lambda) = \lambda \Delta - \lambda \int f(x) \rho(x) dx - \log \int e^{-\lambda f(x)} \rho(x) dx$$

so that $\nabla G = E_\pi[f] + \Delta - E_\rho[f]$. \hfill $\Box$

First of all, the structural result in (7) subjects to an elegant interpretation. The posterior distribution, $\pi$, is equal to the prior distribution, $\rho$, multiplied with a cost driven probability shift, $e^{-\lambda f}$, that makes rewarding regions more probable, resembling the concept of Bayesian inference. Clearly $p(f) = e^{-\lambda f}$ is a transformation that maps costs to probabilities. Indeed one may recognize the inverse log-likelihood transformation from probability to cost as it is often used in the context of Bayesian inference. Fig. 1 demonstrates the idea of the probability shift from the prior $\rho$ to the posterior $\pi$ according to the likelihood measure, $e^{-f}$.
Secondly, the exact value of $\lambda$ can be determined by solving the dual problem. Alternatively, we could also pick any $\lambda > 0$ without the risk of overshooting the constraint $\Delta \leq \mathbb{E}_\rho[f] - f^*$. Later we will show that when $\lambda \to \infty$, the expectation $\mathbb{E}_\pi[f]$ collapses on the exact solution $f^*$. In that respect, $\lambda$ simply reduces to a temperature like quantity that determines the amount of information that is added to $\rho$, where in the limit exactly so much information is admitted to precisely determine $x^*$ and $f^*$.

3.1. Theoretical search distribution sequence

Iterating the inference procedure suggested in (7), substituting the iterate posterior for the prior, generates an optimal search distribution sequence which is increasingly more informed about the optimum with each iteration. This sequence stipulates the fundamental backbone of the entropic optimization framework proposed here but is a well-known relation in the context of optimization and machine learning. However, as far the authors are aware of, it has never been derived from the theory of entropic inference, which make it possible to see it in a much more general light.

$$
\pi_{g+1}(x) \propto \pi_g(x) \cdot e^{-\lambda f(x)} \to \pi_g(x) = \pi_0(x) \cdot e^{-g\lambda f(x)}
$$

In practice we shall seek for algorithms that estimate a posterior distribution from samples taken from the prior where the update is governed by the equation above. In the following section we analyse some of its properties.

3.2. Properties

The sequence governed by equation (8) exhibits a number of interesting properties. First of all, the sequence collapses on the Dirac delta distribution centered at the global minimizer. Phrased in terms of entropic inference, in the limit it is completely informed about the optimizer.

**Theorem 1.** Assume that, without loss of generality, objective $f$ attains a unique global minima at the origin. Further define a sequence of search distributions, $\pi_g$, governed by equation (8). Then it holds that $\pi_g$ collapses on the Dirac delta distribution, i.e. $\delta(x)$, for $g \to \infty$, i.e. $\lim_{g \to \infty} \pi_g := \delta$.

**Proof.** Consider any $x^* \in \mathcal{X} : x^* \neq x^*$ and define $f^* = f(x^*) > f^*$, then there exists a set $\mathcal{X}_{f^* > f} \subset \mathcal{X}$ so that $\forall x \in \mathcal{X}_{f^* > f} : f^* > f(x)$ and a set $\mathcal{X}_{f^* \leq f} = \mathcal{X} / \mathcal{X}_{f^* > f}$ so that $\forall x \in \mathcal{X}_{f^* \leq f} : f^* \leq f(x)$. 


These definitions allow us to derive an upper bound for the value of $\pi^*$, specifically

$$ \pi_g^* = \frac{e^{-gf^*}}{\int_{X_{f^* > f}} e^{-gf} \, dx + \int_{X_{f^* \leq f}} e^{-gf} \, dx} $$

$$ = \frac{1}{\int_{X_{f^* > f}} e^{g(f^*-f)} \, dx + \int_{X_{f^* \leq f}} e^{g(f^*-f)} \, dx} $$

$$ \leq \frac{1}{\int_{X_{f^* > f}} e^{g(f^*-f)} \, dx} $$

Now since $\exp(f^* - f) > 1, \forall x \in X_{f^* > f}$ it follows that $\pi_g^*$ tends to 0 for $g \to \infty$. On the other hand if we choose $x^* = x^*$, one can easily verify that the denominator tends to 0 and thus $\pi_g^*$ tends to $\infty$ as $g \to \infty$. This limit behaviour agrees with that of the Dirac delta and the statement follows. 

Secondly, as implied by its implicit definition in (6), the sequence $E_{\pi_g}[f]$ converges monotonically to $f^*$.

**Corollary 1.** Consider the sequence of distributions, $\pi_g$, governed by equation (8), then it holds that $E_{\pi_g}[f]$ is a monotonically decreasing function of $g$. It follows that

$$ E_{\pi_{g+1}}[f] < E_{\pi_g}[f] $$

**Proof.** To prove that $E_{\pi_g}[f]$ is a monotonically decreasing function of $g$, we simply have to verify whether the derivative is strictly negative. Therefore, let us first express the expectation explicitly, introducing the normalizer

$$ E_{\pi_g}[f] = \frac{\int f e^{-gf} \rho \, dx}{\int e^{-gf} \rho \, dx} $$

Taking the derivative to $g$ yields

$$ \frac{d}{dg} E_{\pi_g}[f] = -\int f^2 e^{-gf} \rho \, dx + \int f e^{-gf} \rho \, dx \int f e^{-gf} \rho \, dx $$

$$ = -E_{\pi_g}[f^2] + E_{\pi_g}[f] = -\text{cov}_{\pi_g}[f] $$

Since the covariance is a strictly positive operator, expect for $\pi_\infty$, the statement follows. \qed
The latter theorem implies that if we construct a stochastic optimization algorithm that maintains a sequence of search distribution governed by (8), it follows that this algorithm will converge monotonically to the optimal solution. This sequence of search distributions is increasingly more informed about the solution and as a result its entropy content deteriorates at the same rate as the sequence converges to the minimum.

**Corollary 2.** Consider the sequence of distributions, \( \pi_g \), governed by equation (8), then it holds that \( \mathbb{H}(\pi_g) \) is a monotonically decreasing function of \( g \). It follows that

\[
\mathbb{H}(\pi_{g+1}) < \mathbb{H}(\pi_g)
\]

Moreover, the sequence \( \mathbb{H}(\pi_g) \) converges with the exact same rate as \( \mathbb{E}_{\pi_g}[f] \).

**Proof.** The entropy of the distribution \( \pi_g \) is equal to

\[
\mathbb{H}(\pi_g) = \log \int e^{-gf}dx + g \int f \pi_g dx
\]

Taking the derivative to \( g \) yields

\[
\frac{d}{dg} \mathbb{H}(\pi_g) = -\frac{1}{\lambda} \mathbb{E}_{\pi_g}[f] + \frac{1}{\lambda} \mathbb{E}_{\pi_g}[f] + \frac{d}{dg} \mathbb{E}_{\pi_g}[f] = -\text{cov}_{\pi_g}[f] < 0
\]

where the second expression holds due to theorem (1) completing the proof.

Put differently, the latter theorem states that the entropy slowly evaporates. This is a crucial observation implying that the explorative incentive of the search distribution deteriorates for increasing \( g \). As stated in the introduction, it is our intention to use the theory developed here to derive stochastic search algorithms by projecting the theoretical distribution \( \pi_g \) on a parametric distribution model \( \pi(\cdot | \theta) \). Thence the parameters \( \theta_{g+1} \) are inferred from a sample population \( \mathcal{X}_g = \{x^k\} \) where \( x^k \sim \pi(\cdot | \theta_g) \). Considering that the information content associated to the population, \( \mathcal{X}_g \), is monotonically decreasing, it will become increasingly more likely that the correct parameters cannot be extracted properly from the statistical properties of the population and that therefore the distribution sequence may collapse prematurely if left uncompensated.
3.3. Generalised Entropic Optimization framework

To remedy the objection made in the previous section, we propose a generalised Entropic Optimization framework, stated in (9). First we conform to a more convenient representation, swapping the objective and the constraint in (6). Following Lagrangian duality these problems are equivalent up to the appropriate scaling. Secondly, we constraint the entropy of the search distribution from below to prevent it from collapsing on the Dirac delta distribution, that is $\mathbb{H}[\pi] > -\epsilon$. This constraint represents the desire to maintain an explorative incentive during the search. For an intuitive understanding of the problem, we substitute $-\mathbb{D}[\pi \parallel \mathcal{U}]$ for $\mathbb{H}[\pi]$ where $\mathcal{U} \propto 1$, revealing that we simultaneously bound the relative entropy between the uniform distribution and the prior belief. In conclusion, we address the problem

$$\min_{\pi \in \Pi} \mathbb{E}_\pi[f]$$

s.t. $\mathbb{D}[\pi \parallel \pi_g] \leq \Delta$

$$\mathbb{D}[\pi \parallel \mathcal{U}] \leq \epsilon$$

(9)

3.3.1. Information-geometric landscape

One could refer to these constraints as describing an information-geometric trust-region or landscape, see figure 2. The first constraint bounds the amount of information that we should add to transform our belief $\pi_g$ into the new belief $\pi_{g+1}$. Each time we repeat this procedure, using the old posterior as the new prior, however we are allowed to specify more information so that in the end $\pi$ will still collapse onto the Dirac delta distribution. The second constraint bounds the amount of information that is lacking to precisely specify the optimum and so we purposefully retain some level of uncertainty. Put differently, if we would repeat this procedure we deliberately retain $\pi$ from collapsing on the Dirac distribution and maintain an explorative incentive.
3.3.2. Entropic search distribution sequence

Again (9) poses a variational problem that we can solve correspondingly, yielding the following search probability distribution sequence. It follows that this formal problem amounts to an approximately similar update procedure apart from that the prior is exponentiated with a factor $0 < \lambda / (\lambda + \gamma) < 1$. This factor can be understood as a diffusion factor that keeps the distribution sequence from collapsing on the Dirac delta distribution.

**Lemma 3.** The solution of problem (9) is given by the distribution $\pi$ where $\lambda > 0, \gamma > 0$

$$\pi_{g+1}(x) \propto \pi_g \left( \frac{1}{\lambda + \gamma} \right)^{g} e^{-\frac{1}{\lambda + \gamma} f(x)} \rightarrow \pi_g(x) = \pi_0(x) \left( \frac{1}{\lambda + \gamma} \right)^g e^{-\left( \frac{1}{\lambda + \gamma} \right)^g \frac{1}{\gamma} f(x)} \quad (10)$$

**Proof.** We introduce Lagrangian multipliers $\lambda, \gamma > 0$ and $\eta$ and construct the Lagrangian

$$\mathcal{L}[\pi] = \mathbb{E}_{\pi} [f] + \lambda \mathbb{D} [\pi \| \pi_g] - \lambda \Delta + \gamma \epsilon - \gamma \mathbb{H} [\pi] + \eta \mathbb{E}_{\pi} [1] - \eta$$

$$= \int \left( f(x) + \lambda \log \frac{\pi(x)}{\pi_g(x)} + \gamma \log \pi(x) + \eta \right) \pi(x) dx - \lambda \Delta + \gamma \epsilon + \eta$$

Evaluating the variation $\delta \mathcal{L} = 0$ and solving for the distribution $\pi$, yields $\pi_{g+1} \propto \pi_g \left( \frac{1}{\lambda + \gamma} \right)^{g} e^{-\frac{1}{\lambda + \gamma} f(x)}$ where the proportionality can be made equal by normalizing the right hand side expression which would be equivalent to solving the normalization constraint to obtain an expression for $\eta$. The values of $\lambda$ and $\gamma$ depend on the values of $\Delta$ and $\epsilon$ and can be obtained by minimizing the appropriate dual function $G(\gamma, \lambda)$

$$G(\gamma, \lambda) = \gamma \epsilon - \lambda \Delta - (\lambda + \gamma) \log \int \rho(x) \left( \frac{1}{\lambda + \gamma} \right)^{\frac{1}{\gamma} f(x)} dx$$

which completes the proof. \hfill \Box

However in practice it is more convenient to address the Lagrangian problem as an auxiliary objective with added penalty functions administering the constraints and simply pick values for $\lambda$ and $\gamma$ instead of $\Delta$ and $\epsilon$.

Furthermore, it is easily verified that the sequence (10) does not collapse on the Dirac delta distribution and maintains an explorative incentive, even when it converges to the limit.

$$\lim_{g \to \infty} \pi_g(x) \propto \exp \left( -\frac{1}{\gamma} f(x) \right) \quad (11)$$
3.4. A basic algorithm

In order to illustrate the practical use of the entropic search distribution sequence, here we demonstrate how the formal theory can be leveraged to derive a practical search algorithm. To that end, we project the theoretical search distribution sequence $\pi_g$ onto a parametric distribution family $\pi_\theta$ and manipulate the resulting expression into an expectation over the prior belief $\pi_g$. As such we are able to establish a computable update procedure that infers parameters from an estimated expectation using samples taken from the prior. In particular we are interested in the Gaussian family, i.e. $\pi_\theta(x) = \mathcal{N}(x|\mu, \Sigma)$, which is commonly used in the context of evolutionary strategies [1, 2, 5]. As a projection operator we propose the relative entropy measure. Partially because it results into a problem that can be solved analytically. Otherwise one can think of it as minimizing the information that is lost by considering $\pi_\theta$ instead of $\pi_g$.

$$\min_{\mu, \Sigma} \mathbb{D}[\pi_{g+1} \parallel \pi_\theta] = \max_{\mu, \Sigma} \int \pi_{g+1}(x) \log \mathcal{N}(x|\mu, \Sigma) dx$$

$$\propto \max_{\mu, \Sigma} \mathbb{E}_{\pi_g} \left[ \pi_g(x)^{\frac{\lambda}{\lambda + \gamma}} e^{-\frac{1}{\lambda + \gamma} f(x)} \log \mathcal{N}(x|\mu, \Sigma) \right]$$ (12)

Then we substitute the empirical estimate $\sum_k h(x^k)$, $x^k \sim \pi_\theta$, for $\mathbb{E}_{\pi_g}[h]$ and solve for each parameter independently using a coordinate descent strategy, rendering both sub-problems concave (see Appendix B). We obtain the following update which readers, familiar with the class of Evolutionary Strategies, will recognize to be similar to those they are accustomed with.

$$\mu_{g+1} = \mu_g + \sum_k \frac{w_k^g}{\sum_k w_k^g} \delta x^k_g$$

$$\Sigma_{g+1} = \sum_k \frac{w_k^g}{\sum_k w_k^g} \delta x^k_g \delta x^k_g^\top$$

where $w_k^g = \pi_\theta(x^k)^{\frac{\gamma}{\lambda + \gamma}} e^{-\frac{1}{\lambda + \gamma} f(x^k)}$ and $\delta x^k_g = x^k - \mu_g$.

Finally, we note that the weights can also be expressed as illustrated below. This form will allow for an interesting comparison later on.

$$w_k^g \propto \exp \left( -\frac{1}{\lambda + \gamma} \left( f(x^k) - \frac{1}{2} \| \delta x^k_g \|_{\Sigma_g^{-1}}^2 \right) \right)$$

\footnote{Note that if it was not for the diffusion factor $\frac{\gamma}{\lambda + \gamma}$, the weights would simply equal the exponential transformed objective. We emphasize here that $f$ does not need to represent the physical objective but could be any rank preserving mapping implying these updates correspond with those of any other ES provided that the mapping is known.}
4. Entropic Optimal Control formulations

In this section, we generalise the entropic optimization framework to include the framework of optimal control.

Recall that we were motivated by the question whether we could solve the generic optimal control problems defined in (2) and (4) exploiting stochasticity as a natural means of exploration. We may now specify this desire, in that we wish to do so in a similar fashion as was illustrated in the static optimization setting. If we succeed in including the framework of optimal control in that of entropic optimization, we could potentially derive stochastic search algorithms by projecting a parametric trajectory distribution onto the theoretical optimal path probability distribution and infer a search policy from that. Such a policy could then be used to solve the underlying optimal control problem.

To answer this question, we shall directly apply the entropic optimization machinery on the Stochastic Optimal Control framework. Therefore we modify the entropic optimization problem (9) slightly, in that we only desire to constraint the entropy of the stochastic policy $\pi$, and not that of the path probability $\omega$. This wish can be facilitated by replacing the lower bound on the entropy $\mathbb{H}[\omega]$ with an upper bound on the relative entropy $\mathbb{D}[\omega_\pi,\omega_U]$ where $\omega_U = \prod_t \omega_t \cdot U_t = \omega \cdot U$ and $U \propto U_t \propto 1$.

We propose the following entropic optimal control problem

$$\begin{align*}
\min_{\pi_t} & \quad \mathbb{E}_{\omega_\pi}[R] \\
\text{s.t.} & \quad \mathbb{D}[\omega_\pi, \omega_{\pi_g}] \leq \Delta \\
& \quad \mathbb{D}[\omega_\pi, \omega_U] \leq \epsilon
\end{align*}$$

(13)

where $\omega_{\pi_g} = \omega \cdot \pi_g$.

4.1. Exact solution

There exists an exact solution to problem (13) when we solve for $\pi_t$.

**Theorem 2.** Consider the entropic optimal control problem defined in (13). The optimal temporal stochastic optimal policy is given by

$$\pi_{g+1,t}(a|s) = \pi_{g,t}(a|s) \frac{1}{\lambda + \gamma} \exp \left( -\frac{1}{\lambda + \gamma} \left( Q_{g+1,t}(s,a) - V_{g+1,t}(s) \right) \right)$$

where

$$Q_{g+1,t}(s,a) = r_t(s,a) + \int V_{g+1,t+1}(s')\omega_t(s'|s,a)ds'$$
and
\[ V_{g+1,t}(s) = (\lambda + \gamma) \log \int \pi_{g,t}(a|s)^{\frac{1}{\lambda+\gamma}} \exp \left( -\frac{1}{\lambda+\gamma} Q_{g+1,t}(s,a) \right) da \]

Proof. First let us address the corresponding Lagrangian, which we reorganise appropriately
\[
\min_{\pi} \mathbb{E}_{\omega_{\pi}} [R] + \lambda \mathbb{D} [\omega_{\pi} \parallel \omega_{\pi_0}] + \gamma \mathbb{D} [\omega_{\pi} \parallel \omega_{\tilde{\pi}}] \\
= \mathbb{E}_{\omega_{\pi}} [R] + \lambda \int \omega_{\pi} \log \frac{\omega_{\pi}}{\pi_{g}} \, d\tau + \gamma \int \omega_{\pi} \log \pi \, d\tau \\
= \int \omega_{\pi} \log e^{R} \, d\tau + \int \omega_{\pi} \log \frac{\pi^{\lambda+\gamma} e^{R}}{\pi_{0}^{\lambda+\gamma}} \, d\tau \\
= \int \prod_{\tau} \omega_{\pi} \, \log \frac{\prod_{\tau} \pi^{\lambda+\gamma} e^{R}}{\prod_{\tau} \pi_{0}^{\lambda+\gamma}} \, d\tau \\
= \int \prod_{\tau} \omega_{\tilde{\pi}} \, \log \frac{\prod_{\tau} \pi^{\lambda+\gamma} e^{R}}{\prod_{\tau} \pi_{0}^{\lambda+\gamma}} \, d\tau \\
\propto \int \prod_{\tau} \omega_{\tilde{\pi}} \, \log \frac{\prod_{\tau} \pi^{\lambda+\gamma} e^{R}}{\prod_{\tau} \pi_{0}^{\lambda+\gamma}} \, d\tau
\]

Now we will try to establish an expression that lends itself to a recursion. To that end, we reorganize the final expression as illustrated here
\[
\min_{\pi_0, T-1} \int \prod_{\tau} \omega_{\pi} \, \log \frac{\prod_{\tau} \pi^{\lambda+\gamma} e^{R}}{\prod_{\tau} \pi_{0}^{\lambda+\gamma}} \, d\tau \\
= \min_{\pi_0} \int \pi_0 \left( \log \frac{\pi_{0e}^{\lambda+\gamma} e^{R_0}}{\pi_{0}^{\lambda+\gamma}} + \int \omega_{0} \min_{\pi_{1:T-1}} \ldots ds_1 \right) \, da_0
\]

implying at what we refer to as the entropic Bellman equation
\[
\frac{1}{\lambda+\gamma} V_{g+1,t}(s) = \min_{\pi_t} \int \pi_t(a|s) \left( \frac{1}{\lambda+\gamma} r_t(s,a) + \log \frac{\pi_t}{\pi_{g,t}} \right. \\
\left. + \frac{1}{\lambda+\gamma} \int V_{g+1,t+1}(s') \omega_t(s'|s,a) ds' \right) \, da \quad (14)
\]

We note that this is the same recursive problem that we considered earlier in the context of stochastic optimal control, yet due to the presence of the log-penalty term, now there is the incentive to maintain a stochastic policy.
To complete the analogy with the formal theory of stochastic optimal control, one may also verify that problem (14) is formally equivalent to (5) extended with an entropic trust-region

\[ V_{g+1,t}(s) = \min_{\pi_t} \mathbb{E}_{\pi_t} [r_t(s,a) + \mathbb{E}_{\omega_t}[V_{g+1,t+1}(s')]] \]

s.t. \[ \mathbb{D} [\pi_t \| \pi_{g,t}] \leq \Delta(\lambda) \]
\[ \mathbb{D} [\pi_t \| \mathcal{U}_t] \leq \epsilon(\gamma) \]

Furthermore, remark that this is a variational problem in \( \pi_t \). It follows that

\[ \pi_{g+1,t}(a|s) \propto \pi_{g,t}(a|s) \frac{1}{\lambda+\gamma} e^{-\frac{1}{\lambda+\gamma} Q_{g+1,t}(s,a)} \]

where we defined

\[ Q_{g+1,t}(s,a) = r_t(s,a) + \int V_{g+1,t+1}(s')\omega_t(s'|s,a)ds' \]

Finally we might express the normalizer explicitly

\[ \pi_{g+1,t}(a|s) = \frac{\pi_{g,t}(a|s)}{\int \pi_{g,t}(a|s) \frac{1}{\lambda+\gamma} e^{-\frac{1}{\lambda+\gamma} Q_{g+1,t}(s,a)} da} \]

Substitution in the entropic Bellman equation then yields the identity

\[ \frac{1}{\lambda+\gamma} V_{g+1,t}(s) = \log \int \pi_{g,t}(a|s) \frac{1}{\lambda+\gamma} e^{-\frac{1}{\lambda+\gamma} Q_{g+1,t}(s,a)} da \]

completing the proof.

4.2. Naive optimal path probability

Here, we emphasize that (13) denotes a variational problem in \( \omega_t \) too, that we can solve correspondingly, yielding an explicit solution for \( \omega_{g+1} \) as well, in a similar fashion to the static entropic optimization problem. However, it is stressed that these two solutions are not equivalent. As a matter of fact, it is impossible to cast the solution of (13) into an explicit expression for the true optimal path probability sequence \( \omega_{g+1} \). Therefore we will refer to the explicit solution as the naive optimal path probability.

19
Corollary 3. Consider the entropic optimal control problem defined in (13). The naive optimal path probability distribution is given, with \( \lambda, \gamma > 0 \):

\[
\omega_{g+1} \propto \omega \cdot \pi_g^{\lambda/\gamma} \cdot e^{-\frac{1}{\lambda+\gamma} R} \rightarrow \omega_{g} = \omega \cdot \rho^{(\frac{1}{\lambda+\gamma})^g} \cdot \mathcal{U}(1-(\frac{1}{\lambda+\gamma})^g) \cdot e^{-\frac{1}{\lambda+\gamma} R}
\]

or equivalently

\[
\prod_{t=0}^{T-1} w_t \pi_{g,t} \propto e^{-\frac{1}{\lambda+\gamma} r_t} \prod_{t=0}^{T-1} w_t \pi_{g-1,t} e^{-\frac{1}{\lambda+\gamma} r_t}.
\]

Proof. The principle derivation is equivalent to the static version, yielding

\[
\omega_{g} \propto \omega_{g+1} \cdot \omega_{U} \cdot e^{-\frac{1}{\lambda+\gamma} R}
\]

which is easily reorganised in the form given. \( \square \)

Therefore we face the following problem. Since we cannot solve for the optimal path probability explicitly, it is also impossible to project it onto a parametric distribution sequence so to derive a computable stochastic search method. It is tempting to use the naive optimal path probability for this procedure regardless of it being incorrect.

Alternatively, we could revise the entropic optimal control problem in (13). In order to adapt the entropic optimal control problem into a problem that does solve into an explicit path probability that we can project accordingly, we will first review the framework of Linearly Solvable Optimal Control (LSOC) which is closely related to the EOC framework discussed so far.

4.3. Linearly Solvable Optimal Control

As we will illustrate here, the framework of EOC is closely related to the framework of Linearly Solvable Optimal Control. In essence, the LSOC framework refers to a specific subset of SOC problems that submit to an explicit solution [16, 17, 18]. This particular subset has been known for quite some time but was long deemed a mathematical peculiarity and no one attributed real application potential to it until a set of Path Integral Control methods were derived from it. See section 5.

In short the LSOC framework considers stochastic problems of the following form. For elaborate details we refer to the papers mentioned above.

\[
V_t(s) = \min_{\nu_t} l_t(s) + \lambda \mathbb{D}[\nu_t \parallel \sigma_t] + \mathbb{E}_{\nu_t}[V_{t+1}]
\]

where we defined the controlled transition probability \( \nu_t = \int \omega_t \pi_t d\alpha_t \) and the uncontrolled transition probability \( \sigma_t = \int \omega_t \delta d\alpha_t \). Let us here also introduce the associated expressions \( \nu = \prod_t \nu_t = \int \omega \pi d\alpha \) and \( \sigma = \prod_t \sigma_t = \int \omega \delta d\alpha \), representing the controlled and uncontrolled trajectory probabilities.
For proper comparison with the framework discussed so far, we cast the problem above into the following entropy constraint stochastic optimal control problem

\[
V_t(s) = \min_{\pi_t} \mathbb{E}_{\pi_t}[l_t(s) + \mathbb{E}_{\omega_t}[V_{t+1}(s')]] = l_t(s) + \mathbb{E}_{\nu_t}[V_{t+1}(s')]
\]

s.t. \(\mathbb{D}[\nu_t \| \sigma_t] \leq \Delta(\lambda)\)

We emphasize that only the state is penalized instead of the state-action couple, i.e. \(l_t(s)\) instead of \(r_t(s,a)\). The control problem therefore only accounts for the accumulated cost over the corresponding state trajectory. In this context the relative entropy constraint can be understood as a control penalization. Furthermore, since the the relative entropy between the free and controlled state transition probability is penalized rather then the entropy of the policy, there is no longer an incentive to maintain a stochastic policy. Put differently, the optimal policy \(\pi_t\) is a shifted Dirac delta distribution, i.e. \(\pi(a|s) = \delta(a - a^*(s))\). As a consequence, we can minimize the objective for \(\nu_t\) instead of \(\pi_t\).

Finally, iterating the recursion above yields the associated path probability optimization problem

\[
\min_{\nu} \mathbb{E}_{\nu}[L] + \lambda \mathbb{D}[\nu \| \sigma]
\]

This problem submits to a similar solution as the EOC framework.

**Theorem 3.** Consider the Linearly Solvable Optimal Control problem where \(L = \sum_t l_t\) and where \(l_t : S \mapsto \mathbb{R}\) and \(\lambda > 0\).

\[
\min_{\nu} \mathbb{E}_{\nu}[L] + \lambda \mathbb{D}[\nu \| \sigma]
\]  \hspace{1cm} (15)

The optimal state transition probability \(\nu_t^*\) is then given by

\[
\nu_t^*(s'|s) \propto \sigma_t(s'|s) \exp\left(-\frac{1}{\lambda}V_{t+1}(s')\right)
\]

where \(V_t\) is governed by the recursive expression

\[
V_t(s) = l_t(s) + \lambda \log \int \sigma_t(s'|s) \exp\left(-\frac{1}{\lambda}V_{t+1}(s')\right) ds'
\]

Equivalently the optimal path probability is given by

\[
\nu^*(\tau_s) \propto \sigma(\tau_s)e^{-\frac{1}{\lambda}L(\tau_s)}
\]

where \(\tau_s\) denotes a state trajectory \(\tau_s = \{s_0, \ldots, s_T\}\).
**Proof.** We retake from the Linearly Solvable Bellman equation

\[ V_t(s) = \min_{\nu_t} l_t(s) + \lambda \int \nu_t(s'|s) \log \frac{\nu_t(s'|s)}{\sigma_t(s'|s)} ds' + \int V_{t+1}(s') \nu_t(s'|s) ds' \]

so that the problem reduces to a variational problem in \( \nu_t \) which we can solve correspondingly (as was illustrated abundantly before)

\[ \nu_t^*(s'|s) \propto \sigma_t(s'|s) \exp \left( -\frac{1}{\lambda} V_{t+1}(s') \right) \]

Substitution in the recursive equation yields the following governing equation for the Value function

\[ V_t(s) = l_t(s) + \lambda \log \int \sigma_t(s'|s) \exp \left( -\frac{1}{\lambda} V_{t+1}(s') \right) ds' \]

Either by iterating the recursion or by solving the problem directly, it is easily verified that

\[ \nu^*(\tau_s) \propto \sigma(\tau_s) e^{-\frac{1}{\lambda} L(\tau_s)} \]

deriving the second statement.

It follows that, whether we solve for the optimal temporal state transition probability \( \nu_t^* \) or the optimal trajectory probability \( \nu^* \), each strategy would yield the same solution. Put differently, the optimal substructure is absent.

The technical reason for this is that, contrary to the setting with general EOC, here the value function is no longer integrated over the transition probability \( \omega_t \) in the expression for the optimal temporal stochastic policy \( \pi_{g+1,t} \). As a result, the exponential in the expression of the optimal transition probability \( \nu_t^* \) can directly act on the logarithm in the expression of the value function, so that the recursion can be evaluated explicitly. In conclusion, we can iterate the recursion and solve for the optimal path probability \( \nu^* \).

Although, now we do have access to an exact expression for the optimal trajectory probability, the conditions set by the Linearly Solvable Optimal Control framework are rather limiting and do not cover a broad range of optimal control problems. Furthermore the stochasticity of the problem is directly related to the penalization of the control and can not be exploited as a result (see section 5.2). Nevertheless, the insights wielded by its explicit solution inspire to formulate an entropic trajectory optimization problem to be introduced in the following section.
4.4. Entropic Trajectory Optimization

As explained earlier, in the general entropic case, there exists no explicit expression for the path probability distribution sequence. It is our desire to identify an entropic optimal control problem that does not depend on an entropic constraint to penalize the control effort, yet that solves for an explicit trajectory probability. To do so, clearly we must get rid of the stochastic policy and absorb it into the transition probability. Therefore, we propose to elevate the optimal control problem entirely to the state trajectory space and solve for an optimal controlled transition probability sequence $\nu_{g,t}$ instead. Therewith, we reformulate the general optimal control problem as a state trajectory optimization problem and consider the inputs to be implicit. One can think of it as latent variables that we shall infer from the optimal state trajectory probability later as is standard practice in many traditional or gradient based trajectory optimization schemes.

Equivalently, we assume there exists an inverse dynamics expression $h_t : S_t \times S_{t+1} \mapsto A_t$ so that $a_t = h_t(s_t, s_{t+1})$ if $s_{t+1} = f_t(s_t, a_t)$, which implies a trajectory cost rate function $c_t : S_t \times S_{t+1} \mapsto \mathbb{R}$ that accumulates in a trajectory cost $C = \sum_t c_t$. Specifically, we have that $c_t(s_t, s_{t+1}) := r_t(s_t, h_t(s_t, s_{t+1}))$.

Let us now first rewrite the stochastic Bellman equation (5) as follows

$$V_t(s) = \min_{\pi_t} \int \int (r_t(s, a) + V_{t+1}(s')) \omega_t(s', a) \pi_t(a|s) da ds'$$

and then substitute the expressions above to yield

$$V_t(s) = \min_{\nu_t} \int (c_t(s, s') + V_{t+1}(s')) \nu_t(s'|s) ds'$$

The former problem then suggests an entropic Bellman equation of the form

$$V_{g+1,t}(s) = \min_{\nu_t} \int \left( c_t(s, s') + V_{g+1,t+1}(s') 
+ \lambda \log \frac{\nu_t(s'|s)}{\nu_{g,t}(s'|s)} + \gamma \log \frac{\nu_t(s'|s)}{U(s'|s)} \right) \nu_t(s'|s) ds'$$

which can be iterated and is found to correspond with the following Entropic Trajectory Optimization (ETO) problem

$$\min_{\nu_t} \mathbb{E}_\nu[C]$$

s.t. $\mathbb{D}[\nu \parallel \nu_g] \leq \Delta$

$\mathbb{D}[\nu \parallel U] \leq \epsilon$ (17)

23
This problem admits a similar solution as for the LSOC setting yet is far more general and exhibits superior application potential as a result.

**Theorem 4.** Consider the entropic trajectory optimization problem defined in (16). The optimal state transition probability $\nu_{g+1,t}$ is then given by

$$\nu_{g+1,t}(s'|s) \propto \nu_{g,t}(s'|s) \frac{1}{e^{\frac{1}{2\lambda+\gamma}} e^{-\frac{1}{2\lambda+\gamma}C_t(s,s') - \frac{1}{\lambda+\gamma} V_{g+1}(s')}}$$

where

$$\frac{1}{\lambda+\gamma} V_{g+1,t}(s) = \log \int \nu_{g,t}(s'|s) \frac{1}{e^{\frac{1}{2\lambda+\gamma}}} \exp \left( -\frac{1}{2\lambda+\gamma} c_t(s,s') - \frac{1}{\lambda+\gamma} V_{g+1,t+1}(s') \right) ds'$$

Equivalently the optimal trajectory path probability is given by

$$\nu_{g+1}(\tau_s) \propto \nu_{g}(\tau_s) \frac{1}{e^{\frac{1}{2\lambda+\gamma} C(\tau_s)}}$$

**Proof.** The equation in (16) determined a variational problem in $\nu_t$ which we can solve correspondingly

$$\nu_{g+1,t}(s'|s) \propto \nu_{g,t}(s'|s) \frac{1}{e^{\frac{1}{2\lambda+\gamma}}} e^{-\frac{1}{2\lambda+\gamma}C_t(s,s') - \frac{1}{\lambda+\gamma} V_{g+1}(s')}$$

and that we can substitute back into (16) to yield the recursive equation

$$\frac{1}{\lambda+\gamma} V_{g+1,t}(s) = \log \int \nu_{g,t}(s'|s) \frac{1}{e^{\frac{1}{2\lambda+\gamma}}} \exp \left( -\frac{1}{2\lambda+\gamma} c_t(s,s') - \frac{1}{\lambda+\gamma} V_{g+1,t+1}(s') \right) ds'$$

Either by iterating the recursion or by solving problem (17) directly, it is then easily verified that

$$\nu_{g+1}(\tau_s) \propto \nu_{g}(\tau_s) \frac{1}{e^{\frac{1}{2\lambda+\gamma} C(\tau_s)}}$$

proving the second statement.

It follows that now we dispose of an explicit optimal trajectory probability sequence whilst only introducing mild additional assumptions on the generic entropic optimal control framework.

Furthermore, it is easily verified that sequence converges to

$$\lim_{g \to \infty} \nu_{g}(\tau_s) \propto \exp \left( -\frac{1}{\gamma} C(\tau_s) \right)$$

In the following section we will discuss its practical use in terms of a set of generalised Path Integral Control methods.
5. Path Integral Control

The framework of Linearly Solvable Optimal Control has been practised in the past to derive a class of so called Path Integral Control methods [19, 23, 24, 29, 46, 27, 20, 21, 25, 47]. The idea in the context of Path Integral Control (PIC) methods is to deploy a search distribution sequence to solve dynamic optimization or optimal control problems, in a similar fashion as one deploys Evolutionary Strategies to solve static optimization problems. Such a method could be practised in complex simulation environments where traditional gradient based trajectory optimizers fall short, or, to derive reinforcement learning algorithms. Clearly the Path Integral Control framework answers the question we posed earlier, however since it derives from the LSOC framework, it inherits the associated limitations. In this section we derive and discuss a generalised set of PIC methods from the ETO context.

5.1. PIC schemes

We might emphasize that the present dynamic context is fundamentally different from its static counterpart, in the sense that we can no longer probe the optimization space directly, as was the case for static optimization problems, but must do so by applying stochastic policies to the system and by observing the corresponding state evolution. Therefore our aim is to obtain a parametrized policy distribution which we can apply to the system to generate a set of sample paths, or so called rollouts. Updated policy parameters are inferred from these rollouts and in this fashion the optimal policy is derived. This constructs an iterative update procedure that ultimately collapses onto the formal solution. As it will turn out, the evaluation of these update procedures will depend on calculating path integrals over the sampled paths, hence explaining the terminology.

First we elaborate how one may derive a PIC method from the specific conditions set by the LSOC framework. Afterwards we cast these ideas in the general setting of ETO.

5.1.1. LSOC based

In the classic derivation, the theoretical state trajectory distribution, \( \nu^* \), is projected onto a parametrized trajectory probability distribution, \( \nu_\theta \). Originally, to extract an expression for the policy, as in the setting of LSOC, there exist no exact expression for it. In later work, it was pointed out that the policy could be extracted from observing the uncontrolled system and hence the framework lend itself to construct a stochastic search method.
5.1.1.1. Policy parametrization. To that end, the controlled state transition probability $\nu_t$ is parametrised in such a fashion that we can derive the policy. Since \( [15] \) is essentially a conventional stochastic optimal control problem, implying that its solution is a deterministic policy, it is trivial to choose a parametrized deterministic policy, e.g. $a_{\theta_t} = a_t$, so that

$$\nu_t(s'|s; \theta_t) = \omega_t(s'|s, a_{\theta_t})$$

Equivalently $\nu_{\theta} = \prod_t \nu_t(s'|s; \theta_t)$.

Alternatively, we can lift these ideas from a stochastic context and practice them in the context of the deterministic optimal control problem \([2]\). That is, one deliberately introduces stochasticity as an instrument to facilitate a search method. This coincides with the earlier framework, only the state transition probability is now defined by the following expression

$$\nu_t(s'|s; \theta_t) = \int \delta (s' - f_t(s, a)) \pi(a|s; \theta_t)da$$

where we are specifically interested in locally linear Gaussian feedback policies

$$\pi(a|s; \theta_{g,t}) = \mathcal{N}(a|a_{g,t} + K_{g,t}s, \Sigma_t)$$

At this point one might be tempted to identify $\Sigma_t$ as a policy parameter apart from $a_{g,t}$ and $K_{g,t}$. However, provided that the control effort was penalized using the relative entropy, in fact, $\Sigma_t$, can not be chosen arbitrarily given that its value will determine the control penalization (see sec. 5.2).

5.1.1.2. Path Integrals. For the projection operator we choose the relative entropy again. Therewith we minimize the information required to cast $\nu_{\theta}$ into $\nu^*$.

$$\min_{\theta} \mathbb{D} [\nu^* \| \nu_{\theta}] \propto \max_{\theta} \int \sigma \cdot e^{-\frac{1}{2}L} \log \nu_{\theta} ds$$

$$= \max_{\theta} \mathbb{E}_{\sigma} \left[ e^{-\frac{1}{2}L} \log \nu_{\theta} \right]$$

One may observe that the $\log \nu_{\theta}$ can be rewritten as a sum over the $\log \nu_{\theta_t}$, which suggests that we can address the optimization problem for each $\theta_t$ separately, significantly simplifying our calculations.

$$\max_{\theta_t} \mathbb{E}_{\sigma} \left[ e^{-\frac{1}{2}L} \log \nu_{\theta_t} \right]$$

(18)
To complete the algorithm, the expectation operator is estimated taking the sample average over uncontrolled sample paths. In practice, one generates a set of state trajectory rollouts $T_g = \{\tau^k_s\} = \{s^k_{0:T}\}$ and estimates (18) as

$$E_\sigma \left[ e^{-\frac{1}{\lambda} L \log \nu_{\theta_t}} \right] \approx \sum_k w^k \sum_i w^k \log \nu_{\theta_t}$$

where the weights depend on the path integral taken over the corresponding sample

$$w^k = \exp \left( -\frac{1}{\lambda} L(s^k) \right) = \exp \left( -\frac{1}{\lambda} \sum_t l_t(s^k_t) \right)$$

Solving this optimization problem would successfully extract the corresponding optimal policy from the optimal path probability. In practice however, we can not solve the problem for just any system, which will force us to invoke secondary assumptions on the governing dynamics. Secondly, exploration of the probability space spanned over $T$ is left completely to the intrinsic space covering properties of the free path probability, $\sigma$. The prior objection is addressed later, the second is addressed first.

5.1.1.3. Policy Improvement. The former procedure is not iterative, and it may take an excessive amount of samples to pin down the policy exactly. To accommodate this shortcoming, we could enforce an iterative procedure by engaging the concept of importance sampling using the old estimate, $\nu_g$, to spawn a set of directed rollouts. In this way we initiate a sequence of policy parameters governed by the following update. A similar procedure has been proposed by Williams [29] and Drews [46] and can be considered state-of-art in PIC. The trajectory probability ratio is easily accounted for by adapting the weights to $w^k_g = \sigma(\tau^k_s) \cdot \nu^{-1}_g(\tau^k_s) \cdot \exp(-\frac{1}{\lambda} C(\tau^k_s))$.

$$\theta_{g+1,t} = \max_{\theta_t} \mathbb{E}_{\nu_{\theta_t}} \left[ \frac{\sigma}{\nu_{\theta_g}} \cdot e^{-\frac{1}{\lambda} L \log \nu_{\theta_t}} \right]$$

(19)

5.1.1.4. Limitations and attempted generalisations. The limitations of described strategy should be clear. The iterative algorithm is elementary and a recursive relation is obtained only by virtue of a trick rather then it is inherent to the framework. Furthermore, as a result of described proportionality between the perturbation noise and penalization of control authority (see sec. 5.2 for additional details), it is simply prohibited to tune the search policy' s covariance in the pursuit of superior convergence properties. For the same reason, the solution retrieved with this method remains subject to the stochastic optimal control problem underlying LSOC.
Therefore, the strategy is not applicable for deterministic optimal control problems nor problems characterised by a cost rate function that is not quadratic in the control effort (sec. 5.2). A number of extensions have been suggested, the most noteworthy are PI\textsuperscript{2} with Covariance Matrix Adaptation (PI\textsuperscript{2}-CMA) by Stulp et al. \cite{22, 26} and PI\textsuperscript{2} with Differential Dynamic Programming (PI\textsuperscript{2}-DDP) \cite{25, 24}. However all fall short of rigorousness.

5.1.2. ETO based PIC

To overcome the limitations corresponding PIC class derived from the theoretical setting of LSOC, we aim to set up a similar procedure in the context of ETO. We emphasize that in the context of ETO, we deliberately orchestrate a stochastic search method and so, inherently, the framework boasts an iterative and stochastic policy.

In this setting, we project the trajectory probability distribution sequence, $\nu_g$, on the parametric trajectory probability, $\nu_\theta$. The former is taken from the ETO framework. The latter is again obtained by modelling the optimal stochastic policy sequence, $\pi_{g,t}$, as a sequence of locally linear Gaussian feedback policies. Finally, we retake the projection problem that was proposed earlier and manipulate the expression into an expectation over $\nu_g$.

$$
\min_\theta \mathbb{D} [\nu_{g+1} \| \nu_g] \propto \max_\theta \int \nu_g^{\lambda + \gamma} \cdot e^{-\frac{1}{\lambda + \gamma} C} \log \nu_\theta ds
$$

$$
= \max_\theta \sum_t \mathbb{E}_{\nu_g} \left[ \nu_g^{\lambda + \gamma} \cdot e^{-\frac{1}{\lambda + \gamma} C} \log \nu_{\theta_t} \right]
$$

Again the expectation can be approximated by approximating the expectation by the sample average

$$
\max_\theta \mathbb{E}_{\nu_g} \left[ \nu_g^{\lambda + \gamma} \cdot e^{-\frac{1}{\lambda + \gamma} C} \log \nu_{\theta_t} \right] \approx \sum_k \frac{w^k_g}{\sum_k w^k_g} \log \nu_{\theta_t} \tag{20}
$$

where $w^k_g = \nu_\theta^{\lambda + \gamma} (\tau^k_s) \cdot \exp(-\frac{1}{\lambda + \gamma} C(\tau^k_s))$ representing the empirical path integrals.

It is interesting to note that these procedures generate a very similar framework to that proposed in the context of Linearly Solvable Optimal Control apart from the precise expression used to evaluate the weights $w^k_g$. In particular the ratio, $\sigma \nu_g^{-1}$, in LSOC setting, that is replaced by the ratio, $\nu_g^{\lambda + \gamma} = \nu_g^{\lambda + \gamma} \nu_g^{-1}$, in the context of ETO.

28
5.1.3. EOC based PIC

Finally, we apply the same procedure for the naive optimal path probability distribution sequence stemming from the EOC framework, as a means of comparison and to gain additional insight in the relation between LSOC, ETO and EOC. We might want to emphasize again that the naive optimal path probability does not correspond with the actual optimal path probability. This produces a parameter estimation problem of the following form, with $w^k_g = \pi^\gamma \lambda \gamma \theta^g \left( \tau^k_s \right) e^{-1 \lambda + \gamma R^g \left( \tau^k_s \right)}$, which we leave to the reader to verify.

$$\max_{\theta_t} \mathbb{E}_{\omega^g} \left[ \pi^\gamma \lambda \gamma \theta^g \left( \tau^k_s \right) e^{-1 \lambda + \gamma R^g \left( \tau^k_s \right)} \right] \approx \sum_k w^k_g \sum_k w^k_g \log \pi_{\theta_t}$$ (21)

5.2. Explicit algorithms for control affine dynamics

In conclusion we will now practice these techniques for the particular case where the system is governed by deterministic control affine dynamics, i.e. $s_{t+1} = f_t + G_t a_t$, where $f_t$ and $G_t$ may both depend on $s$. This assumption has little practical effect since many systems comply to this system model. The corresponding transition probability is therefore given by

$$\nu_t(s'|s) = \int \delta(s' - f_t - G_t a_t) \mathcal{N}(a|a_t(s), \Sigma_t) da = \mathcal{N}(s'|f_t + G_t a_t(s), G_t \Sigma_t G_t^\top)$$

5.2.1. LSOC based algorithm

First we study the problem in the LSOC context. Recall that the control was penalized here implicitly through the relative entropy using the controlled and uncontrolled state transition probability. Provided with the system model above, we can show that this coincides with a quadratic control penalty where the control authority is inversely proportional to the stochasticity that is injected into the system, where $R_t = G_t^\top (G_t \Sigma_t G_t^\top)^{-1} G_t$.

As a matter of fact, $\Sigma_t$, can not be chosen arbitrarily but governs the proportionality between control effort and control noise. From a control engineering perspective this appears to be an acceptable condition for the larger is the noise amplitude, the larger any administrable control authority can be. Furthermore the parameter $\lambda$ can be tuned to obtain arbitrary control settings. However, seen from an algorithmic point of view, this conditions limits our choice for the policy search distribution nonetheless.
All the same, upon substituting these expressions into (19), we obtain an optimization problem that we can solve for \( a_{g,t} \) (here we consider a locally linear Gaussian policy without feedback), providing us with the following update procedure (we refer to Appendix \( \text{C} \) for a proper derivation)

\[
a_{g+1,t} = a_{g,t} + \sum_k w^k_g \delta a^k_t
\]

where \( \delta a^k_t = a^k_t - a_{g,t} \sim \mathcal{N}(0, \Sigma_t) \) and

\[
w^k_g = \frac{\sigma(\tau^k_s)}{\nu_{\theta_g}(\tau^k_s)} \exp \left( -\frac{1}{\lambda} L(\tau^k_s) \right)
\]

The probabilities \( \sigma(\tau^k_s) \) and \( \nu_{\theta_g}(\tau^k_s) \) can be evaluated respectively as

\[
\sigma(\tau^k_s) \propto \prod_t \exp \left( -\frac{1}{2} \| a_{g,t} + \delta a^k_t \|^2_{R^k_t} \right)
\]

\[

\nu_{\theta_g}(\tau^k_s) \propto \prod_t \exp \left( -\frac{1}{2} \| \delta a^k_t \|^2_{R^k_t} \right)
\]

so that the weights can be rewritten as \( w^k_g \propto \exp(-\frac{1}{\lambda} P_{\text{LSOC}}(\tau^k_s)) \) where

\[
P_{\text{LSOC}}(\tau^k_s) = l_T(s^k_T) + \sum_{t=0}^{T-1} l_t(s^k_t) + \lambda \| a^k_t \|_{R^k_t}^2 - \lambda \| \delta a^k_t \|_{R^k_t}^2
\]

In conclusion we note that this strategy is often referred to as Path Integral Policy Improvement (PI\(^2\)) or Model-based Predictive Control with Path Integrals (MPPI) ([28, 29, 46, 26]), depending whether it is applied for trajectory optimization or real-time control.

5.2.2. ETO based algorithm

In this section we will generalise the policy to locally linear Gaussian feedback policies of the form \( \mathcal{N}(a|a_{g,t} + K_{g,t}s, \Sigma_{g,t}) \). In the ETO setting, the proportionality between the control penalization and the injected noise is lifted and therefore we gain access to the full parametrization of the policies, namely \( \theta_{g,t} = \{a_{g,t}, K_{g,t}, \Sigma_{g,t}\} \). This procedure then yields the following elaborate updates (we refer to Appendix \( \text{C} \) for a proper derivation)

\[
a_{g+1,t} = \hat{a}_{g,t} + \Delta \hat{\mu}_{a,g,t} - \hat{\Sigma}_{as,g,t} \hat{\Sigma}_{ss,g,t}^{-1} (\hat{s}_{g,t} + \Delta \hat{\mu}_{s,g,t})
\]

\[
K_{g+1,t} = \hat{\Sigma}_{as,g,t} \hat{\Sigma}_{ss,g,t}^{-1}
\]

\[
\Sigma_{g+1,t} = \hat{\Sigma}_{aa,g,t} - \hat{\Sigma}_{as,g,t} \hat{\Sigma}_{ss,g,t} \hat{\Sigma}_{sa,g,t}
\]

(22)
with

\[ \Delta \hat{\mu}_{a,g,t} = \langle \langle \Delta a^k_t \rangle \rangle \]
\[ \Delta \hat{\mu}_{s,g,t} = \langle \langle \Delta s^k_t \rangle \rangle \]
\[ \hat{\Sigma}_{as,g,t} = \langle \langle \Delta a^k_t \Delta s^{k,T}_t \rangle \rangle \]
\[ \hat{\Sigma}_{ss,g,t} = \langle \langle \Delta s^k_t \Delta s^{k,T}_t \rangle \rangle \]

where \( \Delta s^k_t = s^k_t - \hat{s}_{g,t} \) with \( \hat{s}_{g,t} = \langle s^k_t \rangle \) and \( \Delta a^k_t = a^k_t - \hat{a}_{g,t} \) with \( \hat{a}_{g,t} = \langle a^k_t \rangle = a_{g,t} + K_{g,t} \hat{s}_{g,t} \) so that \( \Delta a^k_t = \delta a^k_t + K_{g,t} \Delta s^k_t \).

Notation \( \langle \langle \cdot \rangle \rangle \) is shorthand for the likelihood weighted average \( \sum_k w^k_{g} \sum_k w^k_g \) where the weights are given by

\[ w^k_g = \nu_{\theta^g}(\tau^k_s) \frac{1}{\lambda + \gamma} \exp \left( -\frac{1}{\lambda + \gamma} C(\tau^k_s) \right) \]

The probability \( \nu_{\theta^g}(\tau^k_s) \) can be evaluated as before so that the weights can be rewritten as \( w^k_g \propto \exp \left( -\frac{1}{\lambda + \gamma} P_{ETO}(\tau^k_s) \right) \) where

\[ P_{ETO}(\tau^k_s) = r_T(s^k_T) + \sum_{t=0}^{T-1} r_t(s^k_t, a^k_t) - \gamma \frac{1}{2} \| \delta a^k_t \|^2_{R^k_t} \]

Note that here we have assumed the existence of an inverse dynamic functions so that we could substitute \( a^k_t \) for \( h(s^k_t, s^{k+1}_t) \) so that we obtain an expression if function of \( r_t \) instead of \( c_t \). It is interesting to note that the weights depend on the accumulated cost over the corresponding trajectory and that a cost is subtracted to stimulate exploration, which proportional to the likelihood of the corresponding control sequence.

### 5.2.3. EOC based algorithm

Finally, we can repeat the procedure for the general EOC setting. It is easily verified that we obtain the exact same updates as in the ETO scheme apart from the likelihood related penalty term which is directly related to the covariance matrix of the locally linear Gaussian feedback policy. However, we note that if there exist an inverse dynamics function, implying that matrices \( G^k_t \) are invertible, both schemes are equivalent. This illustrates the error introduced by using the naive optimal path distribution.

\[ P_{EOC}(\tau^k_s) = r_T(s^k_T) + \sum_{t=0}^{T-1} r_t(s^k_t, a^k_t) - \gamma \frac{1}{2} \| \delta a^k_t \|^2_{\Sigma^{-1}_{g,t}} \]
5.3. Discussion

As was made clear throughout the rest of the paper, it is not our intention to provide a numerical analysis or study, yet. Here we are merely concerned with the relation between all the subjects that we touched upon. In conclusion, we will discuss therefore a number of observations that are of interest to fully grasp the relation between the entropic optimization framework, Evolutionary Strategies, the entropic optimal control framework, Path Integral Control methods and Reinforcement Learning.

5.3.1. Related work in Evolutionary Strategies and optimization

As far as we are aware of we provide an original derivation of a class of evolutionary strategies based on the framework of entropic inference. A similar framework that makes use of information-geometric constraints was addressed by [48], however they did not link the subject to entropic inference. The use of information-geometric trust-regions in the context of optimization was also discussed by Ollivier et al. [6]. The authors did however approach the problem within a continuous setting and formulated a continuous Ricci flow rather then a sequence of distributions. Finally, Luo described a similar distribution (the minima distribution) from the sole condition that the distribution sequence should converge to the dirac delta distribution [49].

5.3.2. Related work in Reinforcement Learning

The work of Toussaint et al. proposed a stochastic optimal control procedure based on approximate inference promoting similar expressions as those presented in this work [50, 51]. The principal idea of Toussaint is to establish a Bayesian inference procedure, boosting a prior trajectory probability distribution $\omega_\pi_g$ and a likelihood function proportional to $e^{-\frac{1}{\lambda} R}$ implying that $\omega_{\pi_{g+1}} \propto e^{-\frac{1}{\lambda} R}$. In order to extract the corresponding policy from the inferred trajectory distribution $\omega_{\pi_{g+1}}$ they propose to solve a projection problem of the form $\min_\pi D[\omega_\pi \| \omega_{\pi_{g+1}}]$ which, in retrospect is equivalent to $\min_\pi E_{\omega_\pi}[R] + \lambda D[\omega_\pi \| \omega_{\pi_{g}}]$. Contrary to our work, Toussaint et al. do not provide a principle justification. Instead, the likelihood expression and projection problem are simply postulated and shown to generate a consistent inference procedure. Their framework thus boils down to the entropic optimal control framework for the specific case where the relative entropy constraint with respect to the uniform distribution is omitted. Furthermore they do not explore the idea in the context of stochastic search algorithms but only in the context of Reinforcement Learning.
In that context we also add that the use of a relative entropy constraint has been proposed by Peters et al. [33, 52, 53] who noted that natural policy gradients, whom premultiply the descent direction with the inverse Fisher information matrix, outperform vanilla gradients, and, combined that observation with the fact that the Fisher information matrix is equal to the Hessian of the relative entropy. From this insight Peters et al. derived the framework of Relative Entropy Policy Search (REPS). Similar information geometric constraints also emerged in the work of Schulman et al. [54].

Recently authors within the Reinforcement Learning community founded the frameworks of Entropy-Regularized Markov Decision Processes [34] and Maximum Entropy Reinforcement Learning, giving rise to the Soft Actor-Critic method [55, 55, 32]. Levine et al. and Neu et al. specifically search for stochastic policies rather then deterministic policies to promote exploration. In the line of work of Toussaint, Levine et al. suggest general energy-based policies of the form \( \pi \propto \exp(-\mathcal{E}(s, a)) \) where \( \mathcal{E} \) is an energy function and proceed to propose the action-value function \( Q \) to represent that energy function. This choice roots back to the soft versions of the value and action-value functions first proposed by Ziebart et al. in the context of Inverse Reinforcement Learning [56]. We might note that Levine et al. nor Neu et al. consider any iterative procedures but simply extend the stochastic optimal control problem with an entropy penalty term to promote exploration. Neither do they draw an exact connection with the inference procedure of Toussaint et al., the relative entropy framework work of Peters et al. and the foundational work of Ziebart on the principle of Maximum Causal Entropy [57].

5.3.3. Derivation from likelihood weighted features

Here we attempt to wield some insight in the assumptions that are made implicitly by the derivations in section 5.2. Recall that we obtained update procedures by projecting the theoretical path distribution, \( \omega_{\pi g} \), on a parametrized distribution \( \omega_{\pi g} \).

To that end, let us now model the entire joint path probability distribution as a Gaussian

\[
\omega_{\pi g} \approx \mathcal{N}(\tau | \mu_{\tau g}, \Sigma_{\tau g})
\]

In contrast to the projection strategy used earlier, here we propose to estimate the updated mean and covariance directly from the likelihood weighted sample path statistics each new generation. This procedure is equivalent to extracting distribution features from the set of sample paths and match them with those of a parametric distribution, in this case a Gaussian.
The temporal state-action distribution, \( \omega_{\tau,g,t} \approx \mathcal{N}(\tau_t|\mu_{\tau,g,t}, \Sigma_{\tau,t,g}) \), can then be obtained by marginalizing the joint trajectory probability distribution. Therefore we can write the Gaussian parameter update procedure as

\[
\hat{\mu}_{\tau,g,t+1} = \mu_{\tau,g,t} + \left\langle \Delta \tau_t \right\rangle = \hat{\tau}_{g,t} + \left\langle \Delta \tau_t \right\rangle
\]

\[
\hat{\Sigma}_{\tau,t,g+1} = \left\langle \Delta \tau_t \Delta \tau_t^\top \right\rangle
\]

where the set of sample paths is supposedly spawned from the distribution \( \mathcal{N}(\tau_t|\mu_{\tau,g}, \Sigma_{\tau,t,g}) \).

Further note that we can decompose \( \mu_{\tau,g,t} \) and \( \Sigma_{\tau,t,g} \) as

\[
\mu_{\tau,g,t} = \left( \begin{array}{c}
\mu_{s,g,t} \\
\mu_{a,g,t}
\end{array} \right)
\]

\[
\Sigma_{\tau,t,g} = \left( \begin{array}{cc}
\Sigma_{ss,g,t} & \Sigma_{sa,g,t} \\
\Sigma_{as,g,t} & \Sigma_{aa,g,t}
\end{array} \right)
\]

Clearly we can not spawn sample paths from the distribution \( \mathcal{N}(\tau_t|\mu_{\tau,g}, \Sigma_{\tau,t,g}) \) directly. Therefore we extract a search policy distribution from the optimal state-action distribution by calculating the conditional probability distribution of the action as a function of the state from the temporal joint probability distribution \( \mathcal{N}(\tau_t|\mu_{\tau,g,t}, \Sigma_{\tau,t,g}) \), that is

\[
\mathcal{N}(a_t|\mu_{a|s,g,t}, \Sigma_{a|s,g,t})
\]

where (see for example \[58\])

\[
\hat{\mu}_{a|s,g+1,t} = \hat{\mu}_{a,g+1,t} - \hat{\Sigma}_{as,g+1,t}\hat{\Sigma}_{ss,g+1,t}^{-1}\hat{\mu}_{s,g+1,t} + \hat{\Sigma}_{as,g+1,t}\hat{\Sigma}_{ss,g+1,t}^{-1}s_t
\]

\[
\hat{\Sigma}_{a|s,g+1,t} = \hat{\Sigma}_{aa,g+1,t} - \hat{\Sigma}_{as,g+1,t}\hat{\Sigma}_{ss,g+1,t}^{-1}\hat{\Sigma}_{sa,g+1,t}
\]

It can be verified that we retrieve identical update procedures as derived from the projection procedure in \[22\].

5.3.4. Analogies with Differential Dynamic Programming

Finally, let us postulate the following proportionalities and assume they are true. Here \( Q \) represents the local \( Q \) function and \( \Delta \tau_t = \tau_t^k - \langle \tau_t^k \rangle \). Note that we make use of subscripts to denote partial derivatives for the occasion.

Subscript \( t \) is omitted for notational convenience.

\[
\Delta \hat{\mu}_{\tau,g} = \left\langle \Delta \tau_t^k \right\rangle \propto -Q_{\tau,t,g}^{-1}Q_{\tau,g} = -Q_{\tau,t,g}^{-1}Q_{s,g}
\]

\[
\hat{\Sigma}_{\tau,t,g+1} = \left\langle \Delta \tau_t^k \Delta \tau_t^{k,\top} \right\rangle \propto Q_{\tau,t,g}^{-1} = \left( \begin{array}{cc}
Q_{ss,g} & Q_{sa,g} \\
Q_{as,g} & Q_{aa,g}
\end{array} \right)^{-1}
\]

34
Essentially we assume that the likelihood weighted features extract local gradient information from the rollouts statistics. According to the block matrix inversion lemma it should therefore hold that (see for example [58])

\[ \hat{\Sigma}_{aa,g}^{-1} + \hat{\Sigma}_{sa,g}^{-1} \propto Q_{aa,g}^{-1} Q_{sa,g} \]

\[ \hat{\Sigma}_{ss,g}^{-1} + \hat{\Sigma}_{sa,g}^{-1} \propto Q_{ss,g}^{-1} Q_{sa,g} \]

\[ \hat{\Sigma}_{aa,g}^{-1} \propto -Q_{ss,g}^{-1} Q_{sa,g} \]

Now also consider that the Differential Dynamic Programming trajectory optimization algorithm (which is a state-of-the-art gradients based algorithm, see appendix Appendix D) proposes following action update procedure

\[ \Delta a_g = k_g + K_g (s - s_g) \]

where

\[ k_g = -Q_{aa,g}^{-1} Q_{a,g} \]
\[ K_g = -Q_{aa,g}^{-1} Q_{as,g} \]

The postulated proportionalities can be reconsidered carefully to yield

\[ k_g = -Q_{aa,g}^{-1} Q_{a,g} \propto \Delta \hat{\mu}_{a,g} - \hat{\Sigma}_{as,g+1} \hat{\Sigma}_{ss,g}^{-1} \Delta \hat{\mu}_{s,g} \]
\[ K_g = Q_{aa,g}^{-1} Q_{as,g} \propto \hat{\Sigma}_{as,g+1} \hat{\Sigma}_{ss,g}^{-1} \]
\[ Q_{aa,g}^{-1} \propto \hat{\Sigma}_{as,g+1} - \hat{\Sigma}_{as,g+1} \hat{\Sigma}_{ss,g+1} \hat{\Sigma}_{sa,g+1} \]

Therewith it is revealed that [22] supports a similar update procedure as if we would obtain by substituting alleged extracted gradient information in equation [23]. We emphasize that this analogy is entirely based on the assumption that the postulated proportionalities hold true and was the starting point in the contributions of [25, 24]. We shall however not pursue to justify or motivate this equivalence here.

Moreover, it is suggested that the feedback gain matrices \( K_{g,t} \) are determined such that the next closed-loop sample statistic will exhibit a beneficial correlation. We might remark here that the likelihood weighted rollouts statistics will exhibit correlation between the state and action trajectories, even when the rollouts are generated using open-loop dynamics, i.e. by setting the feedback gain matrices, \( \{K_{g,t}\} \), equal to zero.
That is because, although the rollout set may not be correlated, the likelihood weights, $w^k$, will prioritize correlated state-action trajectories with high rewards. As a result the likelihood weighted state-action rollout statistics will be correlated anyhow. The presence of this mechanism implies that the PIC procedures could operate without using a feedback procedure which might inject an abundance of stochasticity in the sample paths deteriorating the actual performance. There is however no firm theoretical ground to support such a hypothesis and the matter should be clarified numerically.

6. Conclusion

In this article we established the theoretical foundation necessary to support a rigorous derivation of the class of PIC methods from the generic framework of entropic optimization and provide blueprints of associated algorithms. Therewith we answer the question posed in the introduction of the article. How can we address optimal control problems leaning on stochasticity as a natural means of exploration in a similar fashion as how the class of ESs exploit stochasticity in the context of static optimization? This question was motivated primarily by an unsatisfactory understanding and generality of the class of PIC methods that can be derived from the theoretical conditions known as Linearly Solvable Optimal Control, despite the obvious similarities with ESs. Moreover, since the referred class of dynamic search methods supports interesting applications ranging from guided policy search to robust model based predictive control. With respect to this background, we identify our three main contributions.

Firstly, we give an original and compelling argument for the use of information-geometric measures in the context of stochastic search algorithms based on the principle of entropic inference, therewith synthesising a number of ideas which were so far considered independent in the machine learning, reinforcement learning and optimization research communities. The main idea is to maintain a belief function over the solution space that is least committed to any assumption about the distribution, apart from the requirement that the expectation over the objective should decrease monotonically between updates. We build on the entropic inference procedure to construct an iterative stochastic search procedure and introduce an entropy regularization constraint to maintain an explorative incentive, even in the limit. The resulting Entropic Optimization framework serves as an overarching paradigm to derive stochastic search algorithms.
Secondly, we introduce and discuss a number of entropy regularized problem formulations tailored to dynamic optimization or optimal control problems, therewith establishing the formal concepts of Entropic Optimal Control and Entropic Trajectory Optimization. The explicit solution of each is discussed in light of its possible practice in the context of Path Integral Control where the existence of an associated explicit optimal path distributions sequence is essential. The Entropic Trajectory Optimization framework is then identified as the unique formulation that solves into an optimal trajectory distributions sequence, combining characteristics of both the general Entropic Optimal Control and Linearly Solvable Optimal Control frameworks and is therefore suited as a starting point to derive PIC methods.

Finally, we derive the blueprint of a number of algorithms that correspond with the known class of Path Integral Control methods from the overarching framework of Entropic Optimization. Provided that now there is no longer a formal difference between the class of Evolutionary Strategies, tailored to static optimization problems, and that of Path Integral Control methods, tailored to dynamic optimization or optimal control problems, we anticipate that the latter will admit additional improvements by exploiting this equivalence. Such investigation might be an interesting starting point for future research including numerical results benefiting described equivalences and proposed algorithms.
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Appendix A. Proof of Lemma

Proof. A Lagrangian can be constructed introducing multipliers \( \lambda \) and \( \eta \)

\[
\mathcal{L}[\pi] = \mathbb{D}[\pi || \rho] + \sum_k \lambda_k \mathbb{E}_\pi[g_k] - \lambda_k \mu_k + \eta \mathbb{E}_\pi[1] - \eta \\
= \int \left( \log \pi - \log \rho + \sum_k \lambda_k g_k + \eta \right) \pi dx - \sum_k \lambda_k \mu_k - \eta
\]

According to the calculus of variations it must then hold that the variation of the functional \( \mathcal{L} \) should be zero for any function \( \pi \). Accordingly

\[
\log \pi - \log \rho + \sum_k \lambda_k g_k + 1 + \eta = 0
\]

It follows that the posterior and the prior are proportional up to a normalization constant \( \pi \propto \rho \cdot e^{-\sum_k \lambda_k g_k} \) or in vector notation \( \pi \propto \rho \cdot e^{\lambda \top g} \). The normalization constant is associated to \( \eta + 1 \) and is found by enforcing the distribution normalization constraint. The value of \( \lambda \) can be determined exactly by substituting the expression \( \pi \) back into the original problem and minimizing the so-called dual problem

\[
\mathcal{G}(\lambda) = \log \int \exp \left( -\lambda \top g \right) dx + \lambda \top \mu
\]

so that \( \nabla_\lambda \mathcal{G} = \mu - \mathbb{E}_\pi[g] \). \( \square \)

Appendix B. Derivation of algorithm

We restate the optimization problem in (12).

\[
\max_{\mu, \Sigma} L(\mu, \Sigma) = \mathbb{E}_{\pi_g} \left[ \pi_g(x) \frac{\gamma}{\lambda + \gamma} e^{-\frac{1}{\lambda + \gamma} f(x)} \log \mathcal{N}_\theta(x) \right] \tag{B.1}
\]

This problem can be solved explicitly as such. However, we propose solving it for each parameter independently using a coordinate descent strategy, substituting the previous value of the respective other in the objective. This strategy renders each independent problem concave [5].

\[
\nabla_\mu L|_{\mu, \Sigma} = -\Sigma^{-1} \mathbb{E}_{\pi_g} \left[ w_g(x) (\delta x_g - \mu + \mu_g) \right] \\
\nabla_\Sigma L|_{\mu_g, \Sigma} = \mathbb{E}_{\pi_g} \left[ w_g(x) \Sigma^{-1} \delta x_g \delta x_g \top \Sigma^{-1} \right] - \Sigma^{-1}
\]

where \( w_g(x) = \pi_{\theta_g}(x) \frac{\gamma}{\lambda + \gamma} e^{-\frac{1}{\lambda + \gamma} f(x)} \) and \( \delta x_g = x - \mu_g \).
We can solve these equations to provide independent update procedures for the distribution parameters. Finally approximating the expectation using an empirical estimate will provide the earlier update procedures.

\[
\begin{align*}
\mu_{g+1} &= \mu_g + \mathbb{E}_{N_g}[w_g \delta x_g] \\
\Sigma_{g+1} &= \mathbb{E}_{N_g}[w_g \delta x_g \delta x_g^\top]
\end{align*}
\]

**Appendix C. Derivation of algorithms 5.2**

Here we address the optimization problems derived in section 5. In particular, we refer to the problem in (19), (20) and (21) which are roughly equivalent from an arithmetic perspective. In addition, we address the specific context where the system dynamics are control affine and where we use a locally linear Gaussian feedback policy. In general we then obtain an expression of the following form nd where \( w_g, x, \mu \text{ and } \Lambda \) are problem specific.

\[
\min_{\mu, K, \Sigma} L(a, K, \Sigma) = \sum_k \sum_{w_g} \log N(x_k|\mu(s_k), \Lambda^k) = \langle \langle \log N(x_k|\mu(s_k), \Lambda^k) \rangle \rangle
\]

where we introduced notation \( \langle \langle \cdot \rangle \rangle \) to denote the likelihood weighted average in addition to notation \( \langle \cdot \rangle \) for the actual mean.

The logarithm can be expressed as

\[
\log N(x|\mu, \Lambda) \propto -\log |\Lambda| - \text{tr} \left( \Lambda^{-1}(x - \mu)(x - \mu)^\top \right) + c
\]

where \( c \) is some constant.

For problems (19) and (20), we have that

\[
\begin{align*}
x^k &= s_{k+1}^k = f_t^k + G_t^k a_{g,t} + G_t^k K_{g,t} s_t^k + G_t^k \delta a_t^k \\
\mu^k &= f_t^k + G_t^k a + G_t^k K s_t^k \\
\Lambda^k &= G_t^K \Sigma G_t^{k,\top}
\end{align*}
\]

whilst for problem (21), we have that

\[
\begin{align*}
x^k &= a_t^k = a_{g,t} + K_{g,t} s_t^k + \delta a_t^k \\
\mu^k &= a + K s_t^k \\
\Lambda^k &= \Sigma
\end{align*}
\]

It follows that

\[
x - \mu = G_t^k (a_{g,t} - a + (K_{g,t} - K)s_t^k + \delta a_t^k)
\]
be it
\[ x - \mu = a_{g,t} - a + (K_{g,t} - K)s^k_t + \delta a^k_t \]

Regardless of the procedure that is used, we can express the first order optimality conditions, where the proportionality includes matrix multiplications with positive definite matrices.

\[ \nabla_a L \propto \langle \langle a_{g,t} - a + (K_{g,t} - K)s^k_t + \delta a^k_t \rangle \rangle = 0 \]
\[ \nabla_K L \propto \langle \langle (a_{g,t} - a + (K_{g,t} - K)s^k_t + \delta a^k_t)^T s^k_t \rangle \rangle = 0 \]
\[ \nabla_{\Sigma} L \propto \Sigma^{-1} - \langle \langle \Sigma^{-1}(x^k - \mu^k)(x^k - \mu^k)^T \Sigma^{-1} \rangle \rangle = 0 \]

These equations can be solved to yield expressions for \( a^{g+1}_{g+1,t} \), \( K^{g+1}_{g+1,t} \) and \( \Sigma^{g+1}_{g+1,t} \):

\[ a_{g+1,t} = a_{g,t} + \Delta \hat{\mu}_{a,g,t} - \hat{\Sigma}_{as,g,t} \hat{\Sigma}_{ss,g,t}^{-1} (\hat{s}_{g,t} + \Delta \hat{\mu}_{s,g,t}) \]
\[ K_{g+1,t} = \hat{\Sigma}_{as,g,t} \hat{\Sigma}_{ss,g,t}^{-1} \]
\[ \Sigma_{g+1,t} = \hat{\Sigma}_{aa,g,t} - \hat{\Sigma}_{as,g,t} \hat{\Sigma}_{ss,g,t}^{-1} \hat{\Sigma}_{sa,g,t} \]
\[ \Delta \hat{\mu}_{a,g,t} = \langle \langle \Delta a^k_t \rangle \rangle \]
\[ \Delta \hat{\mu}_{s,g,t} = \langle \langle \Delta s^k_t \rangle \rangle \]
\[ \hat{\Sigma}_{as,g,t} = \langle \langle \Delta a^k_t \Delta s^{k,T}_t \rangle \rangle \]
\[ \hat{\Sigma}_{ss,g,t} = \langle \langle \Delta s^k_t \Delta s^{k,T}_t \rangle \rangle \]

where \( \Delta s^k_t = s^k_t - \hat{s}_{g,t} \) with \( \hat{s}_{g,t} = \langle s^k_t \rangle \) and \( \Delta a^k_t = a^k_t - \hat{a}_{g,t} \) with \( \hat{a}_{g,t} = \langle a^k_t \rangle = a_{g,t} + K_{g,t} \hat{a}_{g,t} \) so that \( \Delta a^k_t = \delta a^k_t + K_{g,t} \Delta s^k_t \).

This concludes the derivation.

Appendix D. Traditional trajectory optimization algorithms

In this appendix we review two traditional Newton-type (i.e. gradient based) trajectory optimization algorithms tailored to deterministic optimal control problems of the form \( (2) \). The algorithms that we will discuss are the Direct Single Shooting (DSS) method and the Differential Dynamic Programming (DDP) method. These methods are referred to as trajectory optimization methods since they iterate state and action trajectories, \{s_{g,t}\} and \{a_{g,t}\}. Each generation the trajectories are updated based on gradient information about the dynamics and cost, that has been collected along the current state-action trajectory, \( \tau_g = \{s_{g,0:T}, a_{g,0:T-1}\} \). The difference between DSS and DDP is in the update mechanism.
The DSS updates the state and action trajectories separately, using a feedforward law for the action, \(a_{g+1,t} \leftarrow a_{g,t} + k_{g,t}\), and, then forward integrating the system to calculate the corresponding state trajectory, \(s_{g+1,t+1} \leftarrow f_t(s_{g+1,t}, a_{g+1,t})\). The DSS method is a historically relevant method, yet it is outdated nowadays, for it does not perform well on the unstable systems, as a result of the open-loop system integration.

The DDP algorithm updates the state and action trajectories simultaneously using a feedforward plus a closed-loop state-feedback update procedure during the forward system integration: \(a_{g+1,t} \leftarrow a_{g,t} + k_{g,t} + K_{g,t}(s_{g+1,t} - s_{g,t})\).

As a result the DSS and DDP method exhibit first- and second-order convergence properties respectively. Moreover, it can be show that the DSS method is a limit case of the DDP method when the closed-loop state-feedback mechanism is neglected. Therefore we can concentrate on the DDP method in the remainder of this discussion.

**Appendix D.1. Derivation of Differential Dynamic Programming**

For clarity let us restate the unconstrained, deterministic and discrete-time optimal control problem:

\[
\min_{\tau} R(\tau) = r_T(s_T) + \sum_{t=0}^{N-1} r_t(s_t, a_t)
\]

s.t. \(s_0 = s(0)\)

\(s_{t+1} = f_t(s_t, a_t)\)

where the dynamics are governed by a discrete time nonlinear state-space equation and \(s_0\) denotes a fixed or given initial state, equal to the current state measurement in real-time applications.

Let us now also reconsider the deterministic Bellman equation, which establishes a nested definition of the time dependent value function, \(V_t\). We drop superscripts \(t\). An accent is used to indicate that the affected quantity is assessed at the next time instant.

\[
V(s) = \min_a [r(s, a) + V'(s')]
\]

Let us further define the function \(Q\) as the argument of the latter optimization problem:

\[
Q(s, a) = r(s, a) + V'(f(s, a))
\]

and consider the second-order expansion coefficients stated below.

47
\[ Q_\tau = r_\tau + f_\tau V'_s \]
\[ Q_{\tau\tau} = r_{\tau\tau} + f_\tau V'_{ss} f_\tau^T + \langle V'_s, f_{\tau\tau} \rangle \]

Here subscript \( \tau \) and \( \tau\tau \) denote the first and second order partial derivatives to the state-action trajectory instant \((s,a)\), respectively. For example \( \nabla_\tau Q = Q_\tau \). Furthermore, \( \langle V'_s, f_{\tau\tau} \rangle \) is defined as the tensor product between the vector \( V'_s \) and the three dimensional tensor \( f_{\tau\tau} \).

The second order difference of \( Q \) can be approximate as such

\[
\Delta Q \approx Q_\tau^T \Delta \tau + \frac{1}{2} \Delta \tau^T Q_{\tau\tau} \Delta \tau
\]

\[
= \frac{1}{2} \begin{pmatrix} 1 \\ \Delta s \end{pmatrix}^T \begin{pmatrix} 0 & Q_\tau^T \\ Q_\tau & Q_{\tau\tau} \end{pmatrix} \begin{pmatrix} 1 \\ \Delta a \end{pmatrix}
\]

\[
\approx \frac{1}{2} \begin{pmatrix} 1 \\ \Delta s \\ \Delta a \end{pmatrix}^T \begin{pmatrix} 0 & Q_s^T & Q_a^T \\ Q_s & Q_{ss} & Q_{sa} \\ Q_a & Q_{as} & Q_{aa} \end{pmatrix} \begin{pmatrix} 1 \\ \Delta s \\ \Delta a \end{pmatrix}
\]

We can solve the second-order approximation of (D.1) accordingly to the action difference \( \Delta a \)

\[
\Delta a = \arg \min_{\Delta a} \Delta Q(\Delta s, \Delta a)
\]

\[
= -Q_{aa}^{-1} Q_a - Q_{aa}^{-1} Q_{as} \Delta s
\]

\[
= k + K \Delta s
\]

where \( k \) and \( K \) are defined as

\[
k = -Q_{aa}^{-1} Q_a
\]

\[
K = -Q_{aa}^{-1} Q_{as}
\]

Substituting this result back into \( \Delta Q \) provides an expression for \( \Delta V \).

Here the equality holds since \( \Delta a^* \) is optimal.

\[
\Delta V(\Delta s) = \Delta Q(\Delta s, \Delta a^*)
\]

\[
= \frac{1}{2} \begin{pmatrix} 1 \\ \Delta s \end{pmatrix}^T \begin{pmatrix} -Q_a^T Q_{aa}^{-1} Q_a & Q_s^T - Q_a^T Q_{aa}^{-1} Q_{as} \\ Q_{ss} - Q_{sa} Q_{aa}^{-1} Q_a & Q_{ss} - Q_{sa} Q_{aa}^{-1} Q_{as} \end{pmatrix} \begin{pmatrix} 1 \\ \Delta s \end{pmatrix}
\]

It follows that

\[
V_s = Q_s - Q_{sa} Q_{aa}^{-1} Q_a
\]

\[
V_{ss} = Q_{ss} - Q_{sa} Q_{aa}^{-1} Q_{as}
\]
In a forward pass, the DDP algorithm propagates the system according to

\[
\begin{align*}
    s_{g+1,0} &= s_0 \\
    a_{g+1,t} &= a_{g,t} + k_{g,t} + K_{g,t} (s_{g+1,t} - s_{g,t}) \\
    s_{g+1,t+1} &= f_t(s_{g+1,t}, a_{g+1,t})
\end{align*}
\]

collecting the derivative information \(\{r_{\tau,g+1,t}, r_{\tau\tau,g+1,t}, f_{\tau,g+1,t}, f_{\tau\tau,g+1,t}\}\) along the way.

When this forward pass has been completed, in a backward pass, the DDP algorithm back-propagates the value and policy approximations according to

\[
\begin{align*}
    Q_{\tau,g+1,t} &= r_{\tau,g+1,t} + f_{\tau,g+1,t+1} V_{s,g+1,t+1} \\
    Q_{\tau\tau,g+1,t} &= r_{\tau\tau,g+1,t} + f_{\tau,g+1,t+1} V_{ss,g+1,t+1} f_{\tau,g+1,t+1}^T + \langle V_{s,g+1,t+1}, f_{\tau\tau,g+1,t+1} \rangle \\
    V_{s,g+1,t} &= Q_{s,g+1,t} - Q_{s,a,g+1,t} Q_{a,a,g+1,t}^{-1} Q_{a,g+1,t} \\
    V_{ss,g+1,t} &= Q_{ss,g+1,t} - Q_{sa,g+1,t} Q_{a,a,g+1,t}^{-1} Q_{ss,g+1,t} \\
    k_{g+1,t} &= -Q_{a,a,g+1,t}^{-1} Q_{a,g+1,t}^{-1} \\
    K_{g+1,t} &= -Q_{a,a,g+1,t}^{-1} Q_{ss,g+1,t}
\end{align*}
\]

with initial values

\[
\begin{align*}
    V_{s,g+1,N} &= r_{s,g+1,N} \\
    V_{ss,g+1,N} &= r_{ss,g+1,N}
\end{align*}
\]

Iterating the forward and backward passes constitute the DDP algorithm.

Also note that each generation a locally linear feedback policy is generated that can be used directly to control the physical system. It follows that DDP provides both the optimal reference trajectory as well as the optimal tracking feedback.

**Appendix D.2. Direct Single Shooting**

As noted, the DSS method can be obtained as a limit case of the DDP method, setting the feedback gain \(K_{g+1,t}\) to zero. Otherwise the update mechanism is equivalent to that of DDP.

\[
\begin{align*}
    s_{g+1,0} &= s_0 \\
    a_{g+1,t} &= a_{g,t} + k_{g,t} \\
    s_{g+1,t+1} &= f_t(s_{g+1,t}, a_{g+1,t})
\end{align*}
\]