A CHARACTERIZATION OF ERDŐS SPACE FACTORS

BY

DAVID S. LIPHAM

Department of Mathematics, Auburn University at Montgomery
Montgomery, AL 36117, USA
e-mail: dlipham@aum.edu, dsl0003@auburn.edu

ABSTRACT

We present a new characterization of Erdős space factors, and show that the Erdős space $E$ is unstable in the class of almost zero-dimensional $F_{\sigma\delta}$-spaces. The latter answers a question by Dijkstra and van Mill.

1. Introduction

All spaces under consideration are non-empty, separable and metrizable.

An element $X$ of a class of topological spaces is called the stable space for that class if for every space $Y$ in the class we have that $X \times Y$ is homeomorphic to $X$. Examples of stable spaces include the Cantor set $2^\omega$ for the class of zero-dimensional compacta, the irrationals $\mathbb{P}$ for the complete zero-dimensional spaces, and $\mathbb{Q}^\omega$ (the countably infinite power of the rationals) for the zero-dimensional $F_{\sigma\delta}$-spaces. The last of these results was proved by van Engelen in the 1980’s [14]. This paper adds to the more recent study of stability in almost zero-dimensional spaces [6, 4, 5].

We say that a subset $A$ of a space $X$ is a C-set in $X$ if $A$ can be written as an intersection of clopen subsets of $X$. A space $X$ is almost zero-dimensional if every point $x \in X$ has a neighborhood basis consisting of C-sets of $X$. Almost zero-dimensional spaces of positive dimension include

- Erdős space $E = \{x \in \ell^2 : x_i \in \mathbb{Q} \text{ for each } i < \omega\}$ and
- complete Erdős space $E_c = \{x \in \ell^2 : x_i \in \mathbb{P} \text{ for each } i < \omega\}$,

where $\ell^2$ is the Hilbert space of square summable sequences of real numbers [7].

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Dijkstra, van Mill and Steprāns proved that $\mathcal{E}_c$ is unstable because it is not homeomorphic to its $\omega$-power $\mathcal{E}_c^\omega$ [6]. On the other hand, $\mathcal{E}_c^\omega$ is stable in the class of complete almost zero-dimensional spaces [4] and is therefore the almost zero-dimensional analogue of $\mathbb{P}$. In their 2010 paper [5], Dijkstra and van Mill proved $\mathcal{E} \simeq \mathcal{E}_c^\omega$ [5, Corollary 9.4] and asked whether $\mathcal{E}$ is stable in the class of almost zero-dimensional $F_{\sigma\delta}$-spaces.

**Question 1** ([5, Question 9.7]): Is every almost zero-dimensional $F_{\sigma\delta}$-space an Erdős space factor?

A space $X$ is called an **Erdős space factor** if $X \times \mathcal{E}$ is homeomorphic to $\mathcal{E}$; see [5, Theorem 9.2] for a list of equivalent formulations.

In this paper we answer Question 1 in the negative. Our main counterexample is the space

$$\mathcal{X} := \{ x \in \mathcal{E}_c : x_0 \in \mathbb{Q} + \sqrt{2} \text{ or } \|x\| \in \mathbb{Q} \},$$

where $\|x\|$ stands for the $\ell^2$-norm of $x$. We will see that similar examples emerge in complex dynamics. Finally, we will provide an intrinsic characterization of Erdős space factors, along with some applications.

### 2. Instability of $\mathcal{E}$

Observe that $\mathcal{X}$ is an $F_\sigma$-subset of $\mathcal{E}_c$, and is therefore an almost zero-dimensional $F_{\sigma\delta}$-space. In order to show that $\mathcal{X}$ is not an Erdős space factor, we will require the following basic fact about the topology of $\ell^2$.

**Lemma 1**: $\ell^2$ has a basis of neighborhoods of the form

$$\{ x \in U : \|x\| \leq q \}$$

where $U$ is open in $\mathbb{R}^\omega$ and $q \in \mathbb{Q}$.

**Proof.** Let $i : \ell^2 \hookrightarrow \mathbb{R}^\omega$ be the identity map. By the Lemma in [12],

$$h(x) = \langle i(x), \|x\| \rangle$$

defines a homeomorphism from $\ell^2$ onto the subspace $H = h(\ell^2)$ of $\mathbb{R}^\omega \times [0, \infty)$. Basic neighborhoods of $h(x)$ in $H$ are of the form

$$V = (U \times ([\|x\| - \varepsilon, \|x\| + \varepsilon]) \cap H, \quad U \text{ being open in } \mathbb{R}^\omega.$$

If $U$ is a small enough neighborhood of $i(x)$, then $\|y\| > \|x\| - \varepsilon$ for all $y \in U$. Given such $U$ and $q \in \mathbb{Q} \cap (\|x\|, \|x\| + \varepsilon)$, we get that $(U \times [0, q]) \cap H \subset V$ is a neighborhood of $h(x)$ in $H$. Since

$$(U \times [0, q]) \cap H = h(\{y \in U : \|y\| \leq q\}),$$

the proof is complete. 

We say that a space $X$ is **C-Baire** if no neighborhood in $X$ can be covered by countably many nowhere dense C-sets of $X$.

**Lemma 2:** $\mathcal{X}$ is C-Baire.

**Proof.** Let $A$ be a neighborhood in $\mathcal{X}$, and let $\{A_n : n < \omega\}$ be any countable C-set covering of $A$. By Lemma 1 we may assume that $A = \{x \in C : \|x\| \leq q\} \cap \mathcal{X}$ where $C$ is clopen in $\mathbb{P}^\omega$ and $q \in \mathbb{Q}$. Then $\{x \in C : \|x\| = q\}$ is complete and non-empty, and its topology as a subspace of $\mathcal{X}$ is the same as the topology it inherits from $\mathbb{P}^\omega$. So by Baire’s theorem [11, Theorem 48.2] there is a clopen set $B \subset C$ and $n < \omega$ such that

$$\emptyset \neq \{x \in B : \|x\| = q\} \subset A_n.$$ 

Note that $W := \{x \in B : \|x\| < q\} \cap \mathcal{X}$ is a non-empty open subset of $\mathcal{X}$. We claim that $W \subset A_n$. Aiming at a contradiction, suppose that there exists $y \in W \setminus A_n$. Then since $A_n$ is a C-set in $\mathcal{X}$, there exists a clopen subset $O$ of $\mathcal{X}$ containing $A_n$ and missing $y$. Then $W \setminus O$ is a norm-bounded non-empty clopen subset of $\mathcal{X}$. This is a contradiction because

$$\mathcal{E}' := \{x \in \ell^2 : x_n \in \mathbb{Q} + \sqrt{2} \text{ for all } n < \omega\}$$

is a dense subset of $\mathcal{X}$, and every non-empty clopen subset of $\mathcal{E}'$ is unbounded in the $\ell^2$-norm [7]. Therefore $W \subset A_n$ and $A_n$ has non-empty interior in $\mathcal{X}$. 

**Theorem 3:** $\mathcal{X}$ is not an Erdő space factor.

**Proof.** Suppose for a contradiction that $\mathcal{X}$ is a factor of the Erdő space $\mathcal{E}$. Then $\mathcal{X}$ is homeomorphic to a closed subspace $\mathcal{Y}$ of $\mathcal{E}$. For every $q \in \mathbb{Q}$ the set $\{x \in \mathcal{E}_c : x_0 = q + \sqrt{2} \text{ or } \|x\| = q\}$ is nowhere dense in $\mathcal{X}$, so we can write

$$\mathcal{Y} = \bigcup\{Y_n : n < \omega\}$$

where each $Y_n$ is nowhere dense in $\mathcal{Y}$. 


For each \( n < \omega \) and \( q_0, \ldots, q_n \in \mathbb{Q} \) define
\[
F_{q_0, \ldots, q_n} = \{ x \in \ell^2 : x_i = q_i \text{ for all } i \leq n \}.
\]

Note that \( F_{q_0, \ldots, q_n} \cap \mathcal{E} \) is a C-set in \( \mathcal{E} \). We shall inductively choose rationals \( q_0, q_1, \ldots \) and non-empty \( \mathcal{Y} \)-open sets \( W_0 \supset W_1 \supset \cdots \) such that for every \( n < \omega \):
- \( W_n \subset F_{q_0, \ldots, q_n} \),
- \( \text{diam}(W_n) < \frac{1}{n} \), and
- \( \overline{W_n} \cap Y_n = \emptyset \) (the closure being taken in \( \ell^2 \)).

Well, by Lemma 2 there exists \( q_0 \in \mathbb{Q} \) such that \( F_{q_0} \cap \mathcal{Y} \) has non-empty interior in \( \mathcal{Y} \). So there is a non-empty \( \mathcal{Y} \)-open set \( W_0 \subset F_{q_0} \) such that
\[
\text{diam}(W_0) < \infty \quad \text{and} \quad \overline{W_0} \cap Y_0 = \emptyset.
\]

Now suppose that \( q_0, \ldots, q_n \) and \( W_n \) have already been defined. Let \( A \subset W_n \) be a C-set neighborhood in \( \mathcal{Y} \), and observe that
\[
\{ F_{q_0, \ldots, q_n, q} \cap A : q \in \mathbb{Q} \}
\]
is a countable covering of \( A \) by C-sets of \( \mathcal{Y} \). By Lemma 2 there exists \( q_{n+1} \in \mathbb{Q} \) such that \( F_{q_0, \ldots, q_n, q_{n+1}} \cap A \) has non-empty interior in \( \mathcal{Y} \). So there is a non-empty \( \mathcal{Y} \)-open set \( W_{n+1} \subset F_{q_0, \ldots, q_n, q_{n+1}} \cap A \) with
\[
\text{diam}(W_{n+1}) < \frac{1}{n+1} \quad \text{and} \quad \overline{W_{n+1}} \cap Y_{n+1} = \emptyset.
\]

By completeness of \( \ell^2 \) there exists
\[
y \in \bigcap_{n < \omega} \overline{W_n}.
\]

Then \( y \in F_{q_0, \ldots, q_n} \) for every \( n < \omega \) and thus \( y \in \mathcal{E} \). Since \( \mathcal{Y} \) is closed in \( \mathcal{E} \), we get \( y \in \mathcal{Y} \). This is impossible because \( y \notin \bigcup \{ Y_n : n < \omega \} \).

Remark 1: van Mill [15] proved that \( \mathbb{Q} \times \mathbb{P} \) is the unique zero-dimensional space which is strongly \( \sigma \)-complete, nowhere \( \sigma \)-compact, and nowhere complete. There is no such classification of almost zero-dimensional spaces, as \( \mathcal{X} \) is strongly \( \sigma \)-complete, nowhere \( \sigma \)-compact, nowhere complete, and is not homeomorphic to any \( \mathbb{Q} \)-product (by Lemma 2).
Remark 2: An example similar to $X$ is generated by the complex exponential $f(z) = \exp(z) - 1$. Indeed, the Julia set $J(f)$ is a Cantor bouquet of rays in the complex plane [1], and has a natural endpoint set $E(f)$ which is homeomorphic to $\mathcal{E}_c$ [8]. For each $n \in \mathbb{N}$ let

$$f^n = f \circ f \circ \cdots \circ f$$

$n$ times
denote the $n$-fold composition of $f$, and consider the set

$$\hat{E}(f) := \{z \in E(f) : \overline{\{f^n(z) : n \in \mathbb{N}\}} \neq J(f)\}$$

consisting of all endpoints whose forward orbits are not dense in the Julia set. By [2, Lemma 1], $\hat{E}(f)$ is a first category $F_\sigma$-subset of $E(f)$. Apparently,

$$\hat{E}(f) := \{z \in E(f) : |f^n(z)| \to \infty\} \subset \hat{E}(f),$$

and $\hat{E}(f)$ is dense in $E(f)$ by Montel’s theorem. In [9] we proved that $\hat{E}(f)$ is C-Baire (see [9, Remark 6.2]). It follows that $\hat{E}(f)$ is C-Baire. The proof of Theorem 3 shows that a first category C-Baire space cannot be an Erdős space factor. Therefore $\hat{E}(f)$ is not an Erdős space factor.

The escaping endpoint set $\hat{E}(f)$ is another almost zero-dimensional $F_{\sigma\delta}$-space which is not a factor of $\mathcal{E}$. The space $\hat{E}(f)$ differs from the previous examples because it is not $\sigma$-complete [10].

3. Characterizing $\mathcal{E}$-factors

A tree $T$ over a set $A$ is a subset of $A^{<\omega}$ that is closed under initial segments, i.e., if $\beta \in T$ and $\alpha \prec \beta$ then $\alpha \in T$. An element $\lambda \in A^{\omega}$ is an infinite branch of $T$ provided $\lambda \upharpoonright n \in T$ for every $n < \omega$. We let $[T]$ denote the set of all infinite branches of $T$. If $\alpha, \beta \in T$ are such that $\alpha \prec \beta$ and $\dom(\beta) = \dom(\alpha) + 1$, then we say that $\beta$ is an immediate successor of $\alpha$ and $\text{succ}(\alpha)$ denotes the set of immediate successors of $\alpha$ in $T$.

A system $(X_\alpha)_{\alpha \in T}$ is a Sierpiński stratification of a space $X$ if:

1. $T$ is a non-empty tree over a countable set,
2. each $X_\alpha$ is a closed subset of $X$,
3. $X_\varnothing = X$ and $X_\alpha = \bigcup\{X_\beta : \beta \in \text{succ}(\alpha)\}$ for each $\alpha \in T$, and
4. if $\lambda \in [T]$ then the sequence $X_{\lambda|0}, X_{\lambda|1}, \ldots$ converges to a point in $X$. 

Sierpiński proved that a space $X$ is absolutely $F_{\sigma\delta}$ if and only if $X$ has a Sierpiński stratification [13, Théorème]. In zero-dimensional spaces closed sets are the same as C-sets, so the stability of $\mathbb{Q}^\omega$ amongst zero-dimensional $F_{\sigma\delta}$-spaces can be stated:

A zero-dimensional space $X$ is a factor of $\mathbb{Q}^\omega$ if and only if $X$ has a Sierpiński stratification of C-sets.

Factors of $\mathcal{E}$ have a similar description.

**Theorem 4:** An almost zero-dimensional space $X$ is an Erdős space factor if and only if $X$ has a Sierpiński stratification of C-sets.

**Proof.** Suppose that $X$ is an Erdős space factor. Since $\mathcal{E} \simeq \mathbb{Q}^\omega \times \mathcal{E}_c$ [5, Proposition 9.1], we may assume that $X$ is a closed subset of $\mathbb{Q}^\omega \times \mathcal{E}_c$. Let $(A_\alpha)_{\alpha \in \mathbb{Q}^\omega}$ be the obvious Sierpiński stratification of $\mathbb{Q}^\omega$. Let $d$ be a complete metric for $\mathcal{E}_c$. For each non-empty $\beta \in \omega^{<\omega}$ define $B_\beta = \bigcap\{B_\beta(n) : n < \text{dom}(\beta)\}$. Then by completeness of $(\mathcal{E}_c, d)$ we have that $(B_\beta)_{\beta \in \omega^{<\omega}}$ is a Sierpiński stratification of $\mathcal{E}_c$. If $\alpha \in \mathbb{Q}^{<\omega}$, $\beta \in \omega^{<\omega}$, and $n = \text{dom}(\alpha) = \text{dom}(\beta)$, then let $\alpha * \beta = \langle \langle \alpha(0), \beta(0) \rangle, \ldots, \langle \alpha(n - 1), \beta(n - 1) \rangle \rangle$.

Note that $T := \{\alpha * \beta : \alpha \in \mathbb{Q}^{<\omega}, \beta \in \omega^{<\omega}, \text{ and } \text{dom}(\alpha) = \text{dom}(\beta)\}$ is a tree over $\mathbb{Q} \times \omega$, $(A_\alpha \times B_\beta)_{\alpha, \beta \in T}$ is a Sierpiński stratification of $\mathbb{Q}^\omega \times \mathcal{E}_c$, and each $A_\alpha \times B_\beta$ is a C-set in $\mathbb{Q}^\omega \times \mathcal{E}_c$. Then $((A_\alpha \times B_\beta) \cap X)_{\alpha, \beta \in T}$ is a Sierpiński stratification of $X$ consisting of C-sets in $X$.

Conversely, suppose that $(A_\alpha)_{\alpha \in T}$ is a Sierpiński stratification of $X$ where every $A_\alpha$ is a C-set in $X$. For each $\alpha \in T$ write $A_\alpha = \bigcap\{C_\alpha^n : n < \omega\}$ where each $C_\alpha^n$ is clopen in $X$. Let $\{B_i : i < \omega\}$ be a neighborhood basis of C-sets for $X$, and for each $i < \omega$ write $B_i = \bigcap\{D_{ij} : j < \omega\}$ where each $D_{ij}$ is clopen in $X$. The topology $\mathcal{T}$ that is generated by the sub-basis

$$\{C_\alpha^n \setminus X, C_\alpha^n : \alpha \in T, n < \omega\} \cup \{D_{ij} \setminus X : i,j < \omega\}$$

is easily seen to be a second countable, regular, zero-dimensional topology on $X$. Clearly each $A_\alpha$ is $\mathcal{T}$-closed, and moreover $(A_\alpha)_{\alpha \in T}$ is a Sierpiński stratification of $(X, \mathcal{T})$. By Sierpiński’s theorem $(X, \mathcal{T})$ is an $F_{\sigma\delta}$-space. In addition, $\mathcal{T}$ is coarser than the original topology of $X$ and every $B_i$ is $\mathcal{T}$-closed. By condition (7) in [5, Theorem 9.2], $X$ is an Erdős space factor. \qed
We end with a few applications of Theorem 4.

**Corollary 5:** If $X$ is an almost zero-dimensional space which is the union of countably many C-set Erdős space factors, then $X$ is an Erdős space factor.

**Proof.** Suppose that $X$ is almost zero-dimensional and $X = \bigcup\{A_n : n < \omega\}$ where each $A_n$ is both an Erdős space factor and a C-set in $X$. By Theorem 4, for each $n < \omega$ there is a Sierpiński stratification $(B^n_\alpha)_{\alpha \in T_n}$ of $A_n$ such that each $B^n_\alpha$ is a C-set in $A_n$. Define a tree

$$T = \bigcup_{n<\omega} \{\langle n \rangle \dashv \alpha : \alpha \in T_n\}.$$ 

Put $X_\emptyset = X$ and $X_{\langle n \rangle \dashv \alpha} = B^n_\alpha$ for each $n < \omega$ and $\alpha \in T_n$. By [5, Corollary 4.20] each $B^n_\alpha$ is a C-set in $X$. Thus $(X_\alpha)_{\alpha \in T}$ is a Sierpiński stratification of $X$ consisting of C-sets in $X$. By Theorem 4, $X$ is an Erdős space factor.

**Remark 3:** Combining Corollary 5 with [16, Proposition 2.2 & Corollary 5.2] will show that if $X$ is an Erdős space factor then so is the Vietoris hyperspace $\mathcal{F}(X)$ consisting of all non-empty finite subsets of $X$.

By Corollary 5 and [5, Corollary 9.3] we get:

**Corollary 6:** If $X$ is almost zero-dimensional space that is a countable union of complete C-sets, then $X$ is an Erdős space factor.

**Remark 4:** Corollary 6 applies to Dijkstra’s main example $T(\tilde{E}, E')$ in [3].

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