Optimal Resistor Networks

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Abstract

Given a graph on \( n \) vertices with \( m \) edges, each of unit resistance, how small can the average resistance between pairs of vertices be? There are two very plausible extremal constructions – graphs like a star, and graphs which are close to regular – with the transition between them occurring when the average degree is 3. However, in this paper we show that there are significantly better constructions for a range of average degree including average degree near 3.

A key idea is to link this question to a analogous question about rooted graphs – namely ‘which rooted graph minimises the average resistance to the root?’ The rooted case is much simpler to analyse that the unrooted, and the one of the main results of this paper is that the two cases are asymptotically equivalent.

1 Introduction

In this paper we shall be considering resistor networks. For our purposes a resistor network is a graph \( G \) (possibly containing multiple edges) where we view each edge as having unit resistance. We refer any reader unfamiliar with the theory of electrical networks, to Snell and Doyle [8] or Chapters 2 and 9 of Bollobás [4].

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1 Alternatively the arXiv version of this paper contains an appendix summarising all the facts we need.
$V = [n]$ and edge set $E$, and for any such graph $G$, we define the average resistance to be

$$A(G) = \frac{1}{\binom{n}{2}} \sum_{x < y}^{x, y \in V} R_{xy},$$

where $R_{xy}$ denotes the effective resistance between $x$ and $y$ in $G$. We would like to know which graphs minimise this quantity for a given number $n$ of vertices and $m$ of edges. Trivially, if $G$ is not connected then $A(G) = \infty$, so we shall always assume $m \geq n - 1$.

Before going any further we give two examples. If $m = n$ then, since the optimal graph must be connected, it is a tree, and in a tree the resistance between any two points is just the distance between those points. Obviously a star minimises the average distance between points. It has $\binom{n-1}{2}$ pairs at distance 2, and $n-1$ pairs at distance 1, giving an average of

$$\frac{2\binom{n-1}{2} + (n-1)}{\binom{n}{2}} = \frac{2((n-1)(n-2) + n-1)}{n(n-1)} = 2 - 2/n.$$

At the other extreme consider the complete graph $K_n$. Obviously all pairs of vertices have the same resistance and this is given by the edge joining the two vertices, and the $n-2$ paths of length 2 joining them, all combined in parallel. This gives resistance $2/n$.

In the next section we describe two very plausible optimal graphs: one is roughly ‘star-like’ (i.e., contains a vertex of very high degree), and the other is ‘regular-like’ (i.e., all vertices look very much the same – the complete graph is an extreme example of this case). It is easy to verify that for $m < 3n/2$ the star-like constructions are better than the regular-like constructions, whereas the reverse is true when $m > 3n/2$, with both types of graph giving $A(G) = 4/3$ when $m = 3n/2$. Thus it seems natural to expect that the optimal graph starts out star-like and switches to regular (or regular-like) when $m = 3n/2$. However, contrary to our expectations, this turns out not to be the case: there is a significantly better construction for $m = 3n/2$.

**Theorem 1.** For all sufficiently large $n$ there is a graph $G$ with $n$ vertices and at most $3n/2$ edges and $A(G) < 1.2908$.

Describing this construction and proving an analogue of Theorem 1 for a range of $m$ is one of the main aims of this paper.
One key idea of the proof is to consider a related model where we insist that all current go ‘via’ a special designated ‘root’ vertex. More precisely, we consider graphs with a root $\rho$ and let $B(G)$ be the average resistance of a vertex to the root.

Since the triangle inequality holds for resistances, for any $x$ and $y$ we have $R_{xy} \leq R_{xp} + R_{py}$ and, hence, that $A(G) \leq 2B(G)$. It follows that graphs giving upper bounds on $B(G)$ automatically give upper bounds for $A(G)$. Our second main result shows that, by slightly modifying the graph and allowing a small error term, we can obtain the reverse inequality.

**Theorem 2.** Suppose that $G$ is a graph and that $s \in \mathbb{N}$ is given. Then there is a graph $G'$ formed by adding a root vertex to $G$ joined to $G$ by $s$ edges with

$$B(G') \leq \frac{1}{2}A(G) + \frac{1}{2s}A(G) + \frac{1}{s}.$$ 

The star shows that any optimal graph has $A(G) \leq 2$, so Theorem 2 shows that, for any such graph $G$, there is a graph $G'$ such that $B(G') \leq A(G)/2 + 2/s$. Letting $s$ tend to infinity with $n$ shows that the two models are equivalent asymptotically. Thus, it suffices to prove all our bounds for the rooted model. This is useful as the rooted model is often much simpler to analyse.

**Motivation**

In addition to being mathematically natural, the question of minimising $A(G)$ arises in statistics [1, 14]. Indeed, suppose that $G$ is the concurrence graph of an experimental design. Then $R_{xy}$ is proportional to the variance of the best linear unbiased estimator for the difference between treatments $x$ and $y$. It follows that the graphs for which $A(G)$ is minimised correspond to designs for which the average of these variances is as small as possible. See the surveys [2, 3] for more details. We note that whilst our primary interest in the rooted model is to allow us to prove results about the general model, it does, in fact, have a natural interpretation in this application. A statistician would call the rooted model a ‘design with a control.’

The parameter $A(G)$ has also been studied in the context of mathematical chemistry [11] and network robustness [9]. In these areas it is sometimes referred to as the Kirchhoff Index of a graph.
Finally, resistance plays an important role in the theory of random walks on graphs. This connection was first shown in Snell and Doyle [8], and has been further developed by many authors including Chandra et al. [6], Coppersmith, Feige and Shearer [7], and Tetali [13].

Layout of the Paper

We start by formalising both the unrooted and rooted models and introduce some notation. Then in Section 3 we use some basic ideas from electrical network theory to give some simple general bounds. These bounds justify our choice of $m$ linear in $n$ as being the most interesting case of the problem.

In Section 4 we start the main paper by proving some elementary results about the rooted average resistance and, in Section 5, we prove that the rooted and unrooted models have the same asymptotic behaviour.

Having concluded the necessary preparation we prove our upper bounds including Theorem 1 and a proof of a bound given by a random regular graph heuristic in Section 6; and improve our lower bounds in Section 7.

In Section 8 we show how the structure changes as $m$ increases – in broad terms, from graphs which are like the star to graphs which are closer to a regular graph. In Section 9 we discuss the rooted model with the extra requirement that the root must be joined to all vertices. This model is sometimes called the Queen-Bee model by statisticians (see e.g., [3]). We show that, for any $n \leq m \leq 3n/2$, the best Queen-Bee network is exactly a star of triangles and leaves. However, we will see that no Queen-Bee network is even asymptotically optimal unless $m = (1 + o(1))n$.

We conclude with some open problems.

2 Definitions and Notation

Recall from the introduction that we define

$$A(G) = \frac{1}{\binom{n}{2}} \sum_{x,y \in V, x < y} R_{xy},$$

to be the average resistance between pairs of vertices. However, in most of our calculations it turns out to be easier to consider the average over all pairs, rather than all distinct pairs – as the resistance of a vertex to itself is
zero, this does not change the sum, but reduces the average slightly as we are averaging over more terms. Hence, we define

\[ A'(G) = \frac{1}{n^2} \sum_{x,y \in V(G)} R_{xy}. \]

Since \( R_{xx} = 0 \) for all \( x \) we have

\[ A(G) = \frac{n^2}{n(n-1)} A'(G) = \frac{n}{n-1} A'(G) \]

so, in particular, the asymptotic behaviour is the same for \( A \) and \( A' \).

For the rooted model we define

\[ B(G) = \frac{1}{|V(G)|-1} \sum_{x \in V(G)} R_{\rho x} \]

to be the average resistance to the root vertex \( \rho \).

We are interested in minimising these two quantities for a given size of graph. As we shall see in the next section, it is simpler to parameterise by the average degree of the graph rather than the number of edges. Thus define

\[ a_n(\alpha) = \min \{ A'(G) : |V(G)| = n, \ e(G) \leq \frac{\alpha n}{2} \} \]

and

\[ b_n(\alpha) = \min \{ B(G) : |V(G)| = n+1, \ e(G) \leq \frac{\alpha n}{2} \}. \]

Requiring \( |V(G)| = n+1 \) in the second definition makes the calculations a little cleaner: it means that \( n \) is the number of non-root vertices. With a slight abuse of terminology we will refer to \( \alpha \) as the average degree in both cases.

As most of our work is with the rooted model it is convenient to define some extra notation. The total effective resistance \( R_{\text{tot}} = \sum_{x \in V(G)} R_{\rho x} \). In some cases we will want to refer to the graphs occurring in the definition of \( b_n(\alpha) \). Thus we define

\[ \mathcal{B}(n, \alpha) = \{ G : |V(G)| = n+1, \ e(G) \leq \frac{\alpha n}{2} \}. \]

Many of our results will be asymptotic and we define

\[ a(\alpha) = \lim_{n \to \infty} a_n(\alpha) \quad \text{and} \quad b(\alpha) = \lim_{n \to \infty} b_n(\alpha). \]

We prove that these limits exist in the next two sections.
background and elementary remarks

In this section we discuss some easy cases of our problem, and give some simple general bounds. The results in this section are not original (see [14] or [1] for example) and are given to place our main result (Theorem 1) into context.

As we remarked in the introduction, if $G$ is not connected then $A(G)$ is infinite. Thus, the first non-trivial case is $m = n - 1$ and then any $G$ minimising $A(G)$ must be a tree. Since a tree contains a unique path between any two points, the resistance between any two vertices is just the distance between them. This implies that a star is the unique (up to isomorphism) graph minimising $A(G)$. In this case, the average resistance is $2 - 2/n$.

In the next case $n = m$ the situation is a little more complicated. As above $G$ must be connected so it must be unicyclic. It is easy to check that the graph minimising $A(G)$ must consist of a cycle with (possibly) some leaves attached to one vertex of the cycle. Then it is a simple calculation to find the optimal length for the cycle: it turns out that for $n \leq 13$ a 4-cycle is optimal whereas for $n \geq 13$ a 3-cycle is optimal (they are both optimal for $n = 13$); see [1] for details.

The fact that the optimal graph switches as $n$ increases leads us to our first choice: in this paper we will almost always be looking at the behaviour for ‘large $n$’. Thus, we suppose that $n$ is large, and that $G$ has average degree $\alpha = \alpha(n)$ so $m = n\alpha/2$. Although we have allowed $\alpha$ to vary with $n$, as we shall see shortly, the case where $\alpha$ is constant is the most interesting.

If $\alpha$ is an even integer then one possible graph is a star where each vertex is joined to the centre of the star by $\alpha/2$ edges. This gives a graph with average degree approximately $\alpha$ and average resistance approximately $4/\alpha$.

What about lower bounds? The first trivial bound we might consider is that, for any $x \neq y$, $R_{xy} \geq 1/d_x$ where $d_x$ denotes the degree of $x$. Thus, by convexity,

$$A(G) \geq \frac{1}{n} \sum_{x \in V} \frac{1}{d_x} \geq \left(\frac{1}{n} \sum_x d_x\right) = \frac{1}{\alpha};$$

i.e., $A(G)$ is at least the reciprocal of the average degree. Combining this lower bound with the upper bound given by the multi-edge-star construction above we see that, as the average degree $\alpha$ tends to infinity, the average
resistance for an extremal graph is $\Theta(1/\alpha)$.

These upper and lower bounds are approximately a factor of four apart; it is easy, however, to improve the lower bound in (1). For all pairs $x$, $y$ which are not neighbours

$$R_{xy} \geq \frac{1}{d_x} + \frac{1}{d_y}.$$ 

Thus, since $d_x \geq 1$ for all $x$,

$$A(G) \geq \frac{1}{n(n-1)} \sum_{x,y \neq \Gamma(x)} \left( \frac{1}{d_x} + \frac{1}{d_y} \right) \geq \frac{1}{n(n-1)} \sum_{x,y \neq y} \left( \frac{1}{d_x} + \frac{1}{d_y} \right) - \frac{4m}{n(n-1)} \geq \frac{2}{\left(\frac{1}{n} \sum_x d_x\right)} - O\left(\frac{m}{n(n-1)}\right) = \frac{2}{\alpha} - O\left(\frac{\alpha}{n}\right).$$

In other words, provided the average degree $\alpha$ is $o(n)$ the average resistance cannot be (much) less than $2/\alpha$.

Another approach for a lower bound is to prove a lower bound for the ‘continuous’ version of the model where non-integral resistances are allowed: that is $G$ is a complete graph with arbitrary resistances on the edges. The natural analogue of the bound on the number of edges is to bound the total conductance, that is $\sum_{uv \in E(G)} 1/r_{uv} \leq m$, where $r_{uv}$ denotes the resistance of the edge $uv$.

It is well known that, given two networks $G_1$ and $G_2$ with total conductance at most $m$, the network $H = \frac{1}{2}(G_1 + G_2)$ (i.e. with the conductance of an edge the average of the conductance in the two networks) has $A(H) \leq \frac{1}{2} (A(G_1) + A(G_2))$. This implies that the network with all conductances equal is an optimal network. In this network each edge has conductance $m/\binom{n}{2}$, and it easy to check that average resistance is $(n-1)/m$, which is approximately $2/\alpha$.

Having found some simple lower bounds let us give some more constructions; we concentrate on the case of small $\alpha$. The above averaging argument indicates that, in a sense, the star construction is the worst possible: we
have ‘averaged’ as little as possible. Thus let us look at the opposite extreme: a very uniform graph. Let $G$ be an $\alpha$-regular graph. Suppose that two vertices $x$ and $y$ are far apart and that around each vertex the graph locally looks like a tree – which would be the case for most pairs of vertices if we chose the $\alpha$-regular graph at random – then the resistance between these two vertices would be about twice the ‘resistance of a vertex to infinity’ along an $\alpha$-regular tree. (Resistance is well defined in infinite graphs see [8]; alternatively just consider the limit for finite $\alpha$-regular trees of depth tending to infinity.) By identifying vertices at the same depth in the tree (since they will all have the same potential in the flow from the vertex to infinity) it is easy to see that the resistance in an $\alpha$-regular tree is

$$\sum_{k=0}^{\infty} \frac{1}{\alpha(\alpha - 1)^k} = \frac{\alpha - 1}{\alpha(\alpha - 2)}.$$  

This suggests that the average resistance between vertices of a random $\alpha$-regular graph is $2(\frac{\alpha - 1}{\alpha(\alpha - 2)})$ – of course to prove this we would have to deal with the cases where the graph is not locally a tree.

If we compare this with the lower bound of $2/\alpha$ we proved above, we see that if $\alpha \to \infty$ then our upper and lower bounds are both $\sim 2/\alpha$. (We emphasise, though, that the upper bound as we have given it here is just a heuristic; we discuss this more in Section 6.) However, for $\alpha$ a fixed constant the upper and lower bounds do differ significantly.

Let us compare this new upper bound with the upper bound from the ‘star construction’. We see that the regular construction is better than the star construction for $\alpha \geq 4$. For $\alpha = 3$ this heuristic gives average resistance $4/3$ which is the same as the bound for the star when $\alpha = 3$. The bound for the star was only valid for $\alpha$ an even integer; however there is another star-like construction: a star of triangles. In this construction one vertex is joined to all others, and a matching is added between all the non-centre vertices (if $n$ is even there is one vertex left over, which we can ignore as we are looking at the asymptotics). This does have average resistance asymptotically $4/3$. Thus, for $\alpha = 3$ the ‘star-like’ construction and the random regular heuristic give the same average resistance.

Finally, we note that the observations we have given so far naturally motivate the rooted model we defined in the introduction. Indeed, let us consider the case $m = n$ again: we saw above that, depending on $n$, the
optimal graph was either a 4-cycle with leaves, or a 3-cycle with leaves. The reason that the optimal graph switches is that, for small \( n \), the resistances between pairs of vertices in the cycle form a significant part of the average, whereas for large \( n \) the average is dominated by leaf vertex to leaf vertex resistance (always 2) followed by the contribution from leaf vertices to cycle vertices. The rooted model naturally avoids this change in behaviour as it insists that all current go to the root vertex; so in the example above the optimal rooted graph is always a 3-cycle with leaves.

4 Basic Properties of the Rooted Model

In this section we work entirely with the rooted model; in the next section we show that the rooted model and the unrooted model behave the same asymptotically; so most of the results proved in this section also apply to the unrooted model. It is worth remarking, however, that we do not have direct proofs of most of the analogues of these results for the unrooted model: the only proof we know is via the results of this section and the equivalence between the two models given in the next section.

We first show that the limit in the definition of the function \( b \) exists.

**Lemma 3.** For any \( \alpha \geq 2 \), the limit \( b(\alpha) = \lim_{n \to \infty} b_n(\alpha) \) exists and is equal to \( \inf_n b_n(\alpha) \).

**Proof.** Given \( \alpha \geq 2 \) let \( b = \inf_n b_n(\alpha) \). Given \( \varepsilon > 0 \) choose \( n \) so that \( b_n(\alpha) < b + \varepsilon \) and let \( G \in \mathcal{B}(n, \alpha) \) be a rooted graph with \( B(G) = b_n(\alpha) \).

Any \( N \) can be written as \( N = kn + l \) with \( 0 \leq l < n \). Consider the graph \( H \) on \( N+1 \) vertices comprising \( k \) copies of \( G \) with a common root vertex, but otherwise disjoint, together with \( l \) leaves adjacent to the root. This graph has \( e(H) = ke(G) + l \leq \frac{kn \alpha}{2} + l \leq \frac{\alpha N}{2} \) and so \( H \in \mathcal{B}(N, \alpha) \). The average rooted resistance is \( B(H) = \frac{1}{N} + (1 - \frac{1}{N}) B(G) < \frac{n}{N} + b + \varepsilon \). Therefore, for any \( N > \frac{n}{\varepsilon} \) we have \( B(H) < b + 2\varepsilon \). Thus, \( \limsup_{N \to \infty} b_N(\alpha) < b + 2\varepsilon \) and the result follows.

**Lemma 4.** The function \( b \) is convex and continuous on \([2, \infty)\).

**Proof.** Suppose that \( 2 \leq \alpha \leq \beta \), that \( t \in [0, 1] \) and that \( \gamma = ta + (1 - t)\beta \). We must show that \( b(\gamma) \leq tb(\alpha) + (1 - t)b(\beta) \).
Given \( \varepsilon > 0 \) take \( n \) large enough that \( b_n(\alpha) < b(\alpha) + \varepsilon \) and \( b_n(\beta) < b(\beta) + \varepsilon \). Let \( G_1 \in \mathcal{B}(n, \alpha), G_2 \in \mathcal{B}(n, \beta) \) be rooted graphs with \( B(G_1) = b_n(\alpha) \) and \( B(G_2) = b_n(\beta) \).

For any integer \( k \) let \( N_k = kn \) and let \( H \) be the graph comprising \( \lceil tk \rceil \) copies of \( G_1 \) and \( \lfloor (1 - t)k \rfloor \) copies of \( G_2 \) with a common root vertex but otherwise disjoint. The graph \( H \) has \( (\lceil tk \rceil + \lfloor (1 - t)k \rfloor)n + 1 = N_k + 1 \) vertices and \( e(H) \leq \gamma N_k^2 \); i.e., \( H \in \mathcal{B}(N_k, \gamma) \). Also \( B(H) = \frac{\lceil tk \rceil}{k} B(G_1) + \frac{\lfloor (1 - t)k \rfloor}{k} B(G_2) < (t + \varepsilon)(b(\alpha) + \varepsilon) + (1 - t)(b(\beta) + \varepsilon) \) provided that \( k \) is large. This expression tends to \( tb(\alpha) + (1 - t)b(\beta) \) as \( \varepsilon \to 0 \) and so \( b(\gamma) \leq tb(\alpha) + (1 - t)b(\beta) \) as required.

This shows that \( b \) is convex on \([2, \infty)\) which implies it is continuous on \((2, \infty)\). To prove that \( b \) is continuous at \( 2 \) we need to use the lower bound of \( 1/(\alpha - 1) \) for \( b(\alpha) \) that we prove later in Theorem 13. Since this bound is continuous and takes the value 1 when \( \alpha = 2 \) this completes the proof. \( \square \)

**Remark.** This result is only true in the limit: that is \( b \) is convex but, for fixed \( n \) the function \( b_n \) is not. Indeed, we have seen that \( b_n(n - 1) = 1 \) and \( b_n(n) = 1 - \frac{2}{3n} \). On the other hand, by Theorem 1 combined with the equivalence between the rooted and unrooted models (Theorem 6, see later), we see that \( b_n\left( \frac{3n}{2} \right) \leq 0.65 < \frac{2}{3} \) for all sufficiently large \( n \).

Although we are interested in minimising the average resistance, it will sometimes be useful to have a bound on the maximum resistance in an optimal graph. The following simple lemma provides this.

**Lemma 5.** If \( G \in \mathcal{B}(n, \alpha) \) and \( R_{x, \rho} > 1 \) for some \( x \in V(G) \) then there is a graph \( H \in \mathcal{B}(n, \alpha) \) with \( B(H) < B(G) \).

**Proof.** Suppose that \( R_{x, \rho} > 1 \). Contracting an edge incident with \( x \) gives a graph \( G' \) with \( n - 1 \) non-root vertices and at most \( e(G) - 1 \) edges (exactly \( e(G) - 1 \) unless the contracted edge was one of a set of parallel edges). We have not increased the resistance between any of the remaining vertices and the root so \( R_{\text{tot}}(G') < R_{\text{tot}}(G) - 1 \). Now adding a leaf to the root gives a graph with \( n \) non-root vertices and at most \( e(G) \) edges. The new vertex has resistance 1 to the root and so \( R_{\text{tot}}(H) \leq R_{\text{tot}}(G') + 1 < R_{\text{tot}}(G) \). Since \( H \)
has the same number of vertices as $G$ and at most as many edges this shows that $B(H) < B(G)$ as claimed.

\[ \square \]

5 Equivalence of Models

In this section we prove Theorem 2 showing that the rooted and unrooted models behave the same asymptotically. As a corollary we deduce that many of the results of the previous section also hold in the unrooted case.

Definition. For any graph $G$ and multiset $S$ of vertices let $G_S$ be the graph formed from $G$ by adding a new vertex $\rho$ joined to each vertex of $S$ (by multiple edges if the vertex occurs multiple times). We call the new vertex the root and the vertices in $S$ the sinks.

Proof of Theorem 2. In fact we show that there is a multiset $S$ of vertices with $|S| = n$ such that the graph $G_S$ satisfies

\[ B(G_S) \leq \frac{1}{2} A'(G) + \frac{1}{2s} A'(G) + \frac{1}{s}. \]

Since $A'(G) = \frac{n-1}{n} A(G)$ this implies the theorem with $G' = G_S$.

We do our calculations in $G$. Fix one vertex $v \in G$ and let $I_i(e)$ be the current in edge $e$ when a current of size 1 flows into the network at $i$ and out from the network at $v$. By the principle of superposition the current in the edge $e$ when a current of size 1 flows in at $i$ and out at $j$ is $I_i(e) - I_j(e)$.

The total power of the flow when a current of size 1 enters at $i$ and leaves at $j$ equals the resistance $R_{ij}$. Thus

\[ R_{ij} = \sum_e (I_j(e) - I_i(e))^2 \]

and, hence, the average resistance over all pairs is

\[ \frac{1}{n^2} \sum_{i,j} R_{ij} = \frac{1}{n^2} \sum_{i,j} \sum_e (I_j(e) - I_i(e))^2 = \sum_e \frac{1}{n^2} \sum_{i,j} (I_j(e) - I_i(e))^2. \]  \( \text{(2)} \)

For each $e$ let $X(e)$ be the random variable obtained by picking a vertex $i$ uniformly at random and setting $X(e) = I_i(e)$ and let $X'(e)$ be an
independent copy of $X(e)$. Then equation (2) becomes

$$A'(G) = \frac{1}{n^2} \sum_{i,j} R_{ij} = \sum_e \text{Var}(X(e) - X'(e))$$

since $E(X(e) - X'(e)) = 0$

$$= \sum_e \text{Var}(X(e)) + \text{Var}(X'(e))$$

since $X$ and $X'$ are independent

$$= 2 \sum_e \text{Var}(X(e)).$$

Next we bound $B(G_S)$. Suppose that $S$ is a fixed multiset of $s$ vertices. One flow of size 1 entering at $i$ and leaving at $\rho$ is given by sending a current of $1/s$ to each of the sinks and on to the root: that is the flow given by $\frac{1}{s} \sum_{j \in S} (I_i - I_j)$, together with a current of size $1/s$ in each of the edges to the root. We know that the actual current flow is the one minimising the power, so the actual power is at most the power in this flow. Thus

$$R_{i\rho} \leq \sum_e \left( \frac{1}{s} \sum_{j \in S} (I_i(e) - I_j(e)) \right)^2 + s \cdot \frac{1}{s^2}.$$

Averaging this over all $i$ gives

$$B(G_S) \leq \frac{1}{n} \sum_{i \in V} \sum_e \left( \frac{1}{s} \sum_{j \in S} (I_i(e) - I_j(e)) \right)^2 + \frac{1}{s}$$

$$\leq \frac{1}{n} \sum_{i \in V} \sum_e \left( I_i(e) - \frac{1}{s} \sum_{j \in S} I_j(e) \right)^2 + \frac{1}{s}$$

$$= E_X \left( \sum_e \left( X(e) - \frac{1}{s} \sum_{j \in S} I_j(e) \right)^2 \right) + \frac{1}{s}$$

$$= \sum_e E_X \left( \left( X(e) - \frac{1}{s} \sum_{j \in S} I_j(e) \right)^2 \right) + \frac{1}{s}$$

Now, suppose that we pick $S$ uniformly at random from all multisets of $s$ vertices (i.e., a random $s$-tuple of vertices), and independently of $X$ and
Obviously

\[ E_S \left( \frac{1}{s} \sum_{j \in S} I_j(e) \right) = E_X(X(e)) \quad \text{so} \quad E_{X,S} \left( X(e) - \frac{1}{s} \sum_{j \in S} I_j(e) \right) = 0 \]

Hence, we get

\[
E_S(B(G_S)) \leq \sum_e E_{X,S} \left( \left( X(e) - \frac{1}{s} \sum_{j \in S} I_j(e) \right)^2 \right) + \frac{1}{s}
\]
\[
= \sum_e \text{Var}_{X,S} \left( X(e) - \frac{1}{s} \sum_{j \in S} I_j(e) \right) + \frac{1}{s}
\]
\[
= \sum_e \text{Var}_X(X(e)) + \sum_e \text{Var}_S \left( \frac{1}{s} \sum_{j \in S} I_j(e) \right) + \frac{1}{s}
\]
\[
= \sum_e \text{Var}_X(X(e)) + \frac{1}{s} \sum_e \text{Var}_X(X(e)) + \frac{1}{s}
\]
\[
= \frac{A'(G)}{2} + \frac{A'(G)}{2s} + \frac{1}{s}.
\]

Thus the result is true ‘on average’ so there must be a set \( S \) for which it holds.

We remark that we could insist that \( S \) is actually a set (rather than a multiset). Indeed, standard results for the variance of a sum with, and without, replacement show that

\[ \text{Var}_S \left( \frac{1}{s} \sum_{j \in S} I_j(e) \right) \]

is smaller for sets than multisets, and using this in the estimation of \( E_S(B(G_S)) \) implies this slightly stronger result.

**Theorem 6.** For any \( \alpha \) we have \( b(\alpha) = a(\alpha)/2 \).

**Proof.** First observe that \( a(\alpha) \leq 2b(\alpha) \). Indeed, for any rooted graph \( G \) the triangle inequality for resistances gives \( R_{xy} \leq R_{xp} + R_{py} \) which implies \( A(G) \leq 2B(G) \).
For the reverse inequality suppose that \( G_n \) is a sequence of (unrooted) graphs with \( n \to \infty \), \( e(G_n) \leq \alpha n/2 \) and \( A(G_n) \to a(\alpha) \). Let \( s = \log n \) and form graphs \( H_n \) as given by Theorem 2. The graph \( H_n \) has \( n + 1 \) vertices, at most \( \alpha n + \log n \) edges, and resistance to the root at most \( A'(G)/2 + O(1/\log n) \). Thus

\[
\lim_{x \to \alpha^+} b(x) \leq a(\alpha)/2.
\]

Since \( b \) is continuous (Lemma 4) the result follows.

Theorem 6 and Lemmas 3 and 4 combine to prove

**Corollary 7.** The function \( a(\alpha) \) exists and is convex and continuous on \([2, \infty)\).

As we remarked previously we do not know of any direct proof of the convexity of \( a \) without going via the equivalence with \( b \).

6 Constructions

6.1 Stars of Edges and Triangles

We start with a simple construction for \( \alpha \) between 2 and 3.

**Lemma 8.** For \( \alpha \in [2, 3] \) we have \( b(\alpha) \leq \frac{5-\alpha}{3} \).

**Proof.** For \( \alpha = 3 \) and \( n \) even, let \( G \) be a star of triangles: that is a collection of triangles each containing the root but otherwise disjoint. It is easy to check that the resistance of any vertex to the root is 2/3 and hence \( b_n(3) \leq 2/3 \). Since \( b_n(2) \leq 1 \) (the star) the convexity of \( b \) (Lemma 4) gives the result.

Indeed, the proof of convexity gives an explicit construction for all \( \alpha \in [2, 3] \) – namely the graph with \( n \) vertices together with a root vertex, consisting of \((\alpha - 2)n/2\) triangles and \((3 - \alpha)n\) edges intersecting in a common root.
6.2 Mixed Constructions

For \( \alpha \geq 3 \) we have the bound \( b(\alpha) \leq \frac{(\alpha-1)}{\alpha(\alpha-2)} \) coming from the random \( \alpha \)-regular graph (currently this is just a heuristic bound but it will be made rigorous shortly). Clearly this only holds for integer \( \alpha \). However, we can use the fact that the graph of \( b(\alpha) \) against \( \alpha \) is convex (Lemma 4) to combine constructions. For instance, the graph of the convex hull of the points \((2,1)\) and the points \( \{(\alpha, \frac{(\alpha-1)}{\alpha(\alpha-2)}): \alpha = 3, 4, \ldots\} \) is an upper bound for \( b(\alpha) \), achieved by adding leaves to the root of a random \( \alpha \)-regular graph. Unfortunately, this does not give a better bound than Lemma 8. That is, we cannot improve the bound \( b(3) \leq 2/3 \) by this process.

However, the convex hull of \((2,1)\) and the curve \( \{(\alpha, \frac{(\alpha-1)}{\alpha(\alpha-2)}): \alpha \in [3, \infty)\} \) does give a better bound. Of course, random regular graphs only exist for integer \( \alpha \). However, if we could achieve something close to the random regular heuristic at non-integer \( \alpha \) we would expect to be able to give an improved construction at, for instance, \( \alpha = 3 \). Perhaps surprisingly, this can be done.

To achieve this we first prove a bound on the average resistance based on the ‘local’ resistance from a vertex out into the graph (recall that this was the basis of the heuristic argument in the Introduction). This is Theorem 10 in the next section. The setting for this is a slightly modified model which we will call a random \( p \)-rooted graph. This result will allow us to prove bounds close to the analogue of the random regular heuristic bound for our constructions with non-integer average degree. Specifically, we will show that there is a graph \( G \) with \( 3 < \alpha < 4 \) and \( B(G) \) (calculated via our ‘local’ resistance result) only a little larger than \( \frac{(\alpha-1)}{\alpha(\alpha-2)} \).

It is worth observing that if we were only interested in proving the heuristic bound for random regular graphs then a shorter argument using eigenvalue methods could be used. The distribution of eigenvalues of the adjacency matrix of a random \( \alpha \)-regular graph was determined by McKay [12]. An \( \alpha \)-regular graph with adjacency matrix \( M \) has Laplacian matrix \( \alpha I_n - M \) and so McKay’s result also gives the distribution of Laplacian eigenvalues from which \( A(G) \) can be calculated. It follows that if \( G \) is a random \( \alpha \)-regular graph on \( n \) vertices then with high probability \( A(G) = \frac{2(\alpha-1)}{\alpha(\alpha-2)} + o(1) \) (we thank László Lovász and Bojan Mohar for pointing out this eigenvalue argument). However, it is important for our application that the bound
holds for non-regular graphs for which we need the $p$-rooted model and the full strength of Theorem 10.

### 6.3 Bounds Using Local Resistance

We start by defining the graph construction we will use. The construction is based on a random rooting similar to that used in Theorem 2. We will prove an upper bound on the average resistance of graphs constructed by this random rooting which shows, for instance, that applying it to any $\alpha$-regular graph $G$ with suitably large girth gives a graph $G'$ with $A(G') = \frac{2(\alpha-1)}{\alpha(\alpha-2)} + o(1)$.

**Definition.** Suppose that $G$ is a graph and $0 < p < 1$. Define $\hat{G}$ the random $p$-rooted graph formed from $G$ to be the following (random) rooted graph. Form a random set $S$, the sinks, by including each vertex independently with probability $p$. Add a root vertex $\rho$, and for each vertex $x \in S$ add $d_G(x) - 1$ parallel edges from $x$ to $\rho$.

First we prove a lemma about the resistance of this construction when applied to a small tree. Since we will be considering several different trees it is convenient to make the following definition.

**Definition.** Suppose that $T$ is a tree and $x \in V(T)$. Then $R(x, T)$, the resistance of the tree from $x$, is the resistance between $x$ and the leaves of the tree identified to a single vertex.

For a tree $T$ we would like to bound the expected resistance of $x$ to the root in $\hat{T}$. However, there is a positive chance that there are no edges between the root and $T$ which would give an infinite resistance to the root. Instead, we show that the resistance in $\hat{T}$ is ‘unlikely to be large’.

**Lemma 9.** Let $0 < \varepsilon < 1$ be fixed. Suppose that $T$ is a tree, $x \in V(T)$, and the depth of $T$ from $x$ is $\ell$. Let $\hat{T}$ be the random $\varepsilon$-rooted graph formed from $T$. Then,

$$\mathbb{P} \left( R_{x\rho}(\hat{T}) > (1 + \varepsilon)R(x, T) \right) \leq \frac{4(1 - \varepsilon)^\ell}{\varepsilon}$$

**Proof.** Let $D_\ell$ denote all the vertices of $T$ at distance exactly $\ell$ from $x$, and let $N = T \setminus D_\ell$. We consider the paths from $x$ to $D_\ell$. We define the $T$-current in a path to be the current in the final edge when a current of size
1 flows from $x$ to $D_\ell$. This means that the current in any edge is the sum of the currents in all the paths through that edge.

Let $S$ be the set of sinks. We say a path from $x$ to $D_\ell$ in $T$ is *good* if it meets a vertex of $S \cap N$ and *bad* otherwise. Let $T_S$ denote the subgraph of $T$ formed by taking the union of all good paths. Further, let $\delta$ be the sum of the currents flowing in the bad paths in the $T$-flow.

The proof consists of three steps. First, we show that the resistance $R_{x\rho}(\bar{T}) \leq R(x, T_S)$. Secondly, we show that $R(x, T_S) \leq (1 - \delta)^{-2} R(x, T)$. Finally we put these results together to prove the bound in the statement of the Lemma.

**Step 1:** $R_{x\rho}(\bar{T}) \leq R(x, T_S)$

Let $N_S$ be the vertices on the good paths of $T$ up to and including the first vertex in $S$ on the path.

Let $T'$ be the subgraph of $\bar{T}$ induced by $N_S \cup \{\rho\}$. Since $T'$ is a subgraph of $\bar{T}$, $R_{x\rho}(\bar{T}) \leq R_{x\rho}(T')$.

Now consider the graph $T'_S$ obtained from $T_S$ by identifying all vertices after the first vertex in $S$ (which necessarily includes all of $T_S \cap D_\ell$) together to one vertex $u$. It is easy to see that $T'$ is isomorphic to $T'_S$. (This is why we chose to add $d(x) - 1$ edges to each vertex $x$ in $S$). Since it is formed by identifying vertices in $T_S$ we see that $R(x, T_S) \geq R_{xu}(T'_S) = R_{x\rho}(T')$.

Combining this with the previous inequality completes this step.

**Step 2:** $R(x, T_S) \leq (1 - \delta)^{-2} R(x, T)$

We construct a current flow in $T_S$ by restricting the flow in $T$ to $T_S$: that is we just delete all the flow that exits from a vertex of $D_\ell \cap (T \setminus T_S)$.

Obviously the current in an edge is no greater than in the original $T$-flow. The power of the original flow in $T$ is exactly the resistance $R(x, T)$ and hence the power of the flow in $T_S$ is at most $R(x, T)$.

The total current of this flow in $T_S$ is only $1 - \delta$. However, if we multiply all currents of this flow by $(1 - \delta)^{-1}$ then we have a unit current from $x$ to $D_\ell$. The power dissipated in each edge has gone up by a factor of $(1 - \delta)^{-2}$ and, therefore, the power of the new flow is at most $(1 - \delta)^{-2} R(x, T)$. Since the resistance $R(x, T_S)$ is the minimum power over all unit flows this step is complete.
**Step 3**

The probability that a particular path is bad is $(1 - \varepsilon)^\ell$. Thus, $E(\delta) = (1 - \varepsilon)^\ell$. Thus,

$$\Pr \left( R_{x\rho}(\hat{T}) > (1 + \varepsilon)R(x, T) \right) \leq \Pr \left( \delta > 1 - \frac{1}{\sqrt{1 + \varepsilon}} \right)$$

by Steps 1 and 2

$$\leq \Pr \left( \delta > \frac{\varepsilon}{4} \right)$$

since $\varepsilon < 1$

$$\leq \frac{\mathbb{E}(\delta)}{\varepsilon/4}$$

Markov’s Inequality

$$\leq \frac{4(1 - \varepsilon)^\ell}{\varepsilon}$$

as required.

We are ready to prove the main theorem of this section which bounds the average resistance (rooted or unrooted) based on the ‘local’ resistance from a point out into the general graph. Since we can easily calculate this local resistance we can use it to construct graphs in which we can bound the average resistance.

**Theorem 10.** Suppose that $(G_n)_{n=1}^\infty$ is a sequence of graphs with the following properties. $G_n$ has $n$ vertices, average degree at most $\alpha$, and girth at least $2\ell + 2$ with $\ell = \ell(n)$ tending to infinity with $n$.

For a vertex $x$, let $T_x$ be the subgraph of $G_n$ formed from all vertices at distance at most $\ell$ from $x$ (which is necessarily a tree by the girth condition).

Then, for any $\varepsilon > 0$ and all sufficiently large $n$, there are rooted graphs $G'_n$ formed from $G_n$ with average degree at most $\alpha + \varepsilon$ such that, for all $x \in V(G)$,

$$R_{x\rho}(G'_n) \leq (1 + \varepsilon)R(x, T_x).$$

In particular, the rooted resistance $B(G'_n)$ satisfies

$$B(G'_n) \leq \frac{1 + \varepsilon}{|G_n|} \sum_{x \in G_n} R(x, T_x).$$

**Proof.** We work with each $G_n$ in turn so, for notational simplicity, fix $G = G_n$. Let $p = \varepsilon/8\alpha$ and $\hat{G}$ be the random $p$-rooted graph formed from $G$ and let $S$ be the sinks of this graph. The expected number of edges we add is
\[ p \sum_x d(x) = pcn = \varepsilon n/8. \] Hence, with probability at least one half we do not add more than \( \varepsilon n/4 \) edges.

Now, for each vertex \( x \), the graph \( \hat{T}_x \) is a subgraph of \( \hat{G} \) so \( R_{x\rho}(\hat{G}) \leq R_{x\rho}(\hat{T}_x) \). By Lemma 9, we have

\[
P \left( R_{x\rho}(\hat{G}) > (1 + \varepsilon)R(x, T_x) \right) \leq \frac{4(1 - \varepsilon)^{\ell}}{\varepsilon} = o(1).
\]

Let \( S' = \{ x \in G : R_{x\rho}(\hat{G}) > (1 + \varepsilon)R(x, T_x) \} \) be the set of vertices that do not satisfy this and let \( D = \sum_{x \in S'} d_G(x) \). Then

\[
\mathbb{E}(D) = \sum_{x \in V} \mathbb{P}(x \in S')d_G(x) \leq \frac{4(1 - \varepsilon)^{\ell}}{\varepsilon} \sum_{x \in V} d_G(x) = o(n)
\]

so, by Markov’s inequality, \( \mathbb{P}(D > \varepsilon n/4) = o(1) \).

Combining these two bounds we see that with positive probability both the number of edges from \( S \) to the root is at most \( \varepsilon n/4 \) and \( D \leq \varepsilon n/4 \). Fix \( \hat{G} \) to be a graph satisfying both of these properties. Form \( G' \) from \( \hat{G} \) by adding \( d_G(x) \) edges between each vertex \( x \) of \( S' \) and the root.

We claim all vertices \( x \) in \( G' \) satisfy

\[ R_{x\rho}(G') \leq (1 + \varepsilon)R(x, T_x). \]

Indeed vertices not in \( S' \) trivially satisfy this condition and, since we joined each vertex \( x \) in \( S' \) to the root with \( d(x) \) edges, \( R_{x\rho}(G') \leq 1/d(x) \leq R(x, T_x) \) for all vertices in \( S' \).

Finally we just need to check the average degree of \( G' \). There are at most \( \varepsilon n/4 \) edges from \( S \) to the root and we added \( D \leq \varepsilon n/4 \) edges from \( S' \) to the root. Hence we have added at most \( \varepsilon n/2 \) edges in total which increases the average degree by most \( \varepsilon \): i.e. the average degree of \( G' \) is at most \( \alpha + \varepsilon \) as claimed.

To illustrate Theorem 10, we prove the bound given by the heuristic for random regular graphs. Essentially we just need to observe that for any integer \( \alpha \geq 3 \) there exist \( \alpha \)-regular graphs with girth tending to infinity. Indeed a random \( \alpha \)-regular graph has girth at least \( \ell \) with positive probability; see Bollobás [5] for details. Alternatively, for some explicit constructions with much stronger bounds on the girth see Lubotzky, Phillips and Sarnak [10].
Corollary 11. For $\alpha \in \{3, 4, \ldots\}$ we have that $b(\alpha) \leq \frac{\alpha - 1}{\alpha(a - 2)}$ and, thus, $a(\alpha) \leq 2\frac{(\alpha - 1)}{\alpha(a - 2)}$.

Proof. Fix $\varepsilon > 0$ and let $G_n$ be an $\alpha$-regular graph of girth tending to infinity with $n$, and form the graph $G'$ as in Theorem 10. In a regular graph all the trees $T_x$ are the same and have resistance less than the corresponding infinite tree: i.e., $R(T_x) \leq \frac{\alpha - 1}{\alpha(a - 2)}$. Thus $G'$ has average degree at most $\alpha + \varepsilon$ and average resistance at most $\frac{\alpha - 1}{\alpha(a - 2)} + o(1)$.

Hence $b(\alpha + \varepsilon) \leq \frac{\alpha - 1}{\alpha(a - 2)}$ for all $\varepsilon > 0$ and the continuity of $b$ (Lemma 4) shows $b(\alpha) \leq \frac{\alpha - 1}{\alpha(a - 2)}$. This, together with Theorem 6, implies the bound for $a$.

As we noted earlier, this bound can also be proved more directly by eigenvalue arguments. In the next section we will prove Theorem 1 using a similar application of Theorem 10 to non-regular graphs where eigenvalue methods are not enough.

6.4 Proof of Theorem 1

Recall that our aim is to construct a graph $G$ with $3 < \alpha < 4$ and $B(G)$ only a little larger than $\frac{(\alpha - 1)}{\alpha(a - 2)}$.

For non-integer values of $\alpha$ we certainly have (using Lemma 4 again) that the convex hull of the points $\{(\alpha, \frac{(\alpha - 1)}{\alpha(a - 2)}) : \alpha = 3, 4, \ldots\}$ is an upper bound for $r(\alpha)$. This corresponds to taking a union of a rooted random $\lfloor \alpha \rfloor$-regular graph and a rooted random $\lceil \alpha \rceil$-regular graph which intersect only in the root vertex. However, as we shall see, if we take a high girth graph of average degree $\alpha$ in which all vertices have degree $\lfloor \alpha \rfloor$ or $\lceil \alpha \rceil$, then the average rooted resistance of a rooting of this graph is slightly lower. Hence, it is plausible that by taking such a rooting and adding some leaves (i.e., taking the convex hull in $b$) we would get an improved bound for 3 and, indeed, this is the case.

By taking care that the vertices of different degrees are suitably distributed in the graph we can do even better. A simple example of such a construction is a high girth bipartite graph with bipartition into a part containing $\frac{3n}{7}$ vertices of degree 4 and a part containing $\frac{4n}{7}$ vertices of degree 3. This graph has average degree $\frac{24}{7}$ and has 2 kinds of trees $T_x$ depending on whether $x$ has degree 3 or degree 4. Calculating the resistance of these trees.
and applying Theorem 10 yields a bound of $B(G) = \frac{209}{420} + o(1) \approx 0.4976$. We omit the details since the next example is better for our constructions and involves very similar calculations.

Let $G_0$ be a high girth 4-regular bipartite graph. It is clear that such graphs exist since, for example, a random 4-regular bipartite graph has a positive chance of having girth greater than any fixed size. This is very similar to the results of random regular graphs discussed above: see [15] for full details.

Then form $G$ by replacing each vertex in one partition by two vertices joined by an edge, with the four neighbours being shared two to each new vertex. Then $G$ has average degree $\frac{10}{3}$. We calculate the ‘resistance to infinity’ in the related tree: that is in the tree obtained from a 4-regular tree by replacing all vertices at even distance from some designated vertex with two vertices of degree 3. Let $x$ be the resistance to infinity in this tree from a degree 3 vertex along one of the edges to a degree 4 vertex, and let $y$ be the resistance to infinity from a degree 4 vertex along any one of its edges. Then, by a simple application of the series and parallel laws, we have

$$x = 1 + \frac{y}{3} \quad \text{and} \quad y = 1 + \frac{1}{x + \frac{1}{1+x/2}}.$$ 

Solving this gives $x = \frac{1+\sqrt{5}}{2}$ and $y = -\frac{3+3\sqrt{5}}{4}$, which corresponds to a resistance to infinity from a degree 3 vertex of $1/(2/x + 1/(1 + x/2)) = \frac{\sqrt{5}}{4}$ and from a degree 4 vertex of $y/4 = -\frac{3+3\sqrt{5}}{8}$. Thus, the average resistance to infinity is $-\frac{3+7\sqrt{5}}{24} \approx 0.527186$.

By applying Theorem 10, we can construct a rooted graph with average degree $\frac{10}{3} + o(1)$ and average rooted resistance $0.527186 > -\frac{3+7\sqrt{5}}{24}$. We have proved the following theorem which includes Theorem 1 as a special case.

**Theorem 12.** For $\alpha \in [2, 10/3]$ we have that $b(\alpha) \leq 1 - 0.3546(\alpha - 2)$. In particular $b(3) \leq 0.6454$ and $a(3) \leq 1.2908$.

**Proof.** This follows instantly from convexity, $b(2) = 1$ and the above construction showing that $b(10/3) \leq 0.527186$. \qed
7 Lower Bounds

In the introduction we proved that

\[ A(G) \geq \frac{2}{(\frac{1}{n}\sum d_x) + O(m/n^2)} \]

which showed that \( a(\alpha) \geq 2/\alpha \) or, equivalently, that \( b(\alpha) \geq 1/\alpha \). The rough idea was to bound the resistance from a vertex to the rest of the graph by the resistance to its neighbourhood. We improve this bound by considering the resistance from a vertex to its two-step neighbourhood. It is more convenient to work in the rooted case.

**Theorem 13.** For all \( \alpha \geq 2 \) we have \( b(\alpha) \geq 1/(\alpha - 1) \).

**Proof.** Let \( G \) be an optimal graph and let \( G_x \) be the graph formed from \( G \) by identifying all vertices not in \( x \cup \Gamma(x) \) (i.e., all vertices at distance at least two from \( x \)) to the root vertex. This decreases the resistance from \( x \) to the root. We call the resistance from \( x \) to the root in \( G_x \) the two-step resistance of \( x \) and similarly for the two-step conductance.

Let \( d_x \) be the degree of a vertex and let \( e_x \) be the number of those edges which join it to the root. Thus

\[ \sum_{x \neq \rho} (d_x + e_x) = \alpha n \]

Let \( \Gamma' \) denote the neighbourhood of \( x \) viewed as a multiset and excluding the root vertex. Thus \( |\Gamma'(x)| = d_x - e_x \). Note, that for any (non-root) vertex \( x \) and vertex \( y \in \Gamma'(x) \) we have \( d_y \geq 2 \): indeed, otherwise \( y \) would be a leaf vertex not joined to the root contradicting the assumed optimality of \( G \).

Having set up the notation observe that the two-step conductance from a vertex \( x \) is at most

\[ \sum_{y \in \Gamma'(x)} \left( 1 + \frac{1}{d_y - 1} \right)^{-1} + e_x. \]

To complete the proof we sum this bound over all vertices and simplify. To reduce the notation all vertex sums are over all vertices except the root. The
The sum of the two-step conductances is at most

\[
\sum_x \left( \sum_{y \in \Gamma'(x)} \left( 1 + \frac{1}{d_y - 1} \right)^{-1} + e_x \right) = \sum_x \sum_{y \in \Gamma'(x)} \frac{d_y - 1}{d_y} + \sum_x e_x \\
= \sum_x \sum_{y \in \Gamma'(x)} 1 - \sum_x \sum_{y \in \Gamma'(x)} \frac{1}{d_y} + \sum_x e_x \\
= \sum_x d_x - \sum_y \sum_{x \in \Gamma'(y)} \frac{1}{d_y} \\
= \sum_x d_x - \sum_y \frac{d_y - e_y}{d_y} \\
= \sum_x (d_x + e_x) - \sum_y \frac{d_y - e_y}{d_y} - \sum_y e_y \\
= \sum_x (d_x + e_x) - \sum_y 1 + \sum_y \frac{e_y}{d_y} - \sum_y e_y \\
\leq \alpha n - n = (\alpha - 1)n.
\]

Hence the average two-step conductance is at most \(\alpha - 1\) and, thus, the average two-step resistance is at least \(1/(\alpha - 1)\).

We might expect that, with a bit more ingenuity and algebraic manipulation, we could extend the above to 3 and more steps. This does not appear to be easy and, indeed, the natural analogue of Theorem 13 for five or more steps would contradict Theorem 12, so cannot possibly work.

We see that the lower bound for \(\alpha = 2\) now matches the star construction. Recall from Lemma 4 that we needed this result to prove the continuity of \(b\) (and thus \(a\)) at 2.

8 The Structure of Optimal Graphs

In this section we prove some results describing what optimal graphs look like. One particular aim is to show that, as \(\alpha\) increases, the graph changes from ‘star-like’ to ‘regular-like’.

First we show that for large \(\alpha\) no optimal graph, in either the rooted or unrooted case, has a significant number of leaves. This is a simple application of the convexity of \(b\).
Lemma 14. Suppose that $\alpha \geq 3.83$ and that $G$ is any graph with average degree $\alpha$ and $\gamma n$ leaves. Then $A(G) - a(\alpha) \geq \gamma/100 + o(1)$. In particular the optimal graph has $o(n)$ leaves.

Proof. We start by proving a similar result for the rooted case. Trivially, $b(2) = 1$, and, by Theorem 12, $b(10/3) \leq 0.527186$. Also, by Theorem 13, $b(\alpha) > 1/(\alpha - 1)$. Let $l = l(x)$ be the function giving the line through the points $(10/3, 0.527186)$ and $(\alpha, b(\alpha))$, and $l_0$ the line through $(10/3, 0.527186)$ and $(3.83, 1/2.83)$. Since $b$ is convex and $\alpha \geq 3.83 > 10/3$ we see that for all $x \geq \alpha$ we have $b(x) \geq l(x)$. Note that, $l_0(2) \leq 0.995$ so, again by convexity, $l(2) \leq 0.995$.

Split $G$ into two subgraphs disjoint except for the root: the leaves $L$, and the rest of the graph $H$. The average degree of $G$ satisfies

$$\alpha(G) = \gamma \alpha(L) + (1 - \gamma) \alpha(H) = 2\gamma + (1 - \gamma) \alpha(H),$$

and, in particular, $\alpha(H) \geq \alpha(G)$.

Turning to resistances, we have

$$B(G) = \gamma B(L) + (1 - \gamma) B(H) \geq \gamma + (1 - \gamma) b(\alpha(H)) \geq \gamma l(2) + (1 - \gamma) l(\alpha(H)) + (1 - 0.995) \gamma = l(\alpha(G)) + \gamma/200 = b(\alpha(G)) + \gamma/200.$$

Finally, we lift this result back to the unrooted case. Let $G'$ be a rooting of the graph $G$ with $2B(G') \leq A(G) + o(1)$ and $\alpha(G') \leq \alpha(G) + o(1)$. Since the rooting process only added $o(n)$ edges to $G$ we see that $G'$ has $\gamma n - o(n)$ leaves.

Thus

$$A(G) \geq 2B(G') + o(1) \geq 2b(\alpha(G')) + \gamma/100 + o(1) \geq 2b(\alpha(G)) + \gamma/100 + o(1) = a(\alpha(G)) + \gamma/100 + o(1)$$

as required. \qed
We remark that, if we just used the simple bound \( b(3) \leq 2/3 \) given by a star of triangles, then we would prove a similar result for the slightly weaker case when \( \alpha \geq 4 \).

Next we show that, if \( \alpha \) is not much greater than 2, then the optimal graph must have leaves.

**Theorem 15.** Let \( G \) be a graph on \( n \) vertices with average degree \( \alpha = 2\frac{1}{4} - \varepsilon \), for some \( \varepsilon > 0 \), and \( \gamma n \) leaves. Then \( A(G) - a(\alpha) \geq \frac{2}{25}(\varepsilon - \gamma/4) + o(1) \). In particular, the optimal such graph must contain at least \( 4\varepsilon n - o(n) \) leaves.

**Remark.** The bound of \( \alpha < 2\frac{1}{4} \) can be improved significantly but the important point for our purposes is that the bound is strictly greater than 2. We discuss the limits of the basic technique after the proof.

Roughly, our aim is to show that there is a vertex \( x \) with resistance to the root at least 1; then we can contract an edge incident to \( x \) reducing the total resistance by 1 and add a leaf.

We will be considering a number of modified graphs. Let \( G \) be the original graph and suppose that \( x \) is a degree-2 vertex with neighbours \( y_1 \) and \( y_2 \). Let \( G \setminus x \) denote the graph with vertex \( x \) deleted; \( G/x \) denote the graph with the edge \( y_1x \) contracted (i.e., \( G \setminus x \) with the edge \( y_1y_2 \) added). Finally, let \( G^+/x \) denote \( G/x \) with an additional leaf added to the root so, in particular, \( G^+/x \) has the same number of vertices and edges as \( G \).

**Lemma 16.** Let \( G \) be a rooted graph with no leaves and average degree less than 9/4. Then there exists a vertex \( x \) with \( R_{\text{tot}}(G) - R_{\text{tot}}(G^+/x) \geq \frac{1}{100} \).

**Proof.** We claim that there is a degree-2 vertex which has both neighbours also degree-2. Indeed, suppose not. Let \( S \) be the set of vertices of degree 2. Since \( G \) has no leaves, the sum of the degrees is at least \( 2|S| + 3|V \setminus S| \leq \frac{9}{4} n \), so \( |S| \geq 3n/4 \). Let \( t \) be the number of edges from \( S \) to \( V \setminus S \). Since no vertex in \( S \) has both neighbours in \( S \) we have \( t \geq |S| \). Thus, by considering the degrees in \( G[S] \) we see that \( 2E(S) = 2|S| - t \). The total number of edges in \( G \) is at least

\[
t + |E(S)| = t/2 + |S| \geq \frac{3}{2}|S| \geq \frac{9}{8} n
\]

which contradicts the average degree being less than 9/4.

Let \( x \) be such a vertex, and \( y_1, y_2 \) its two neighbours, and \( z_1, z_2 \) their two neighbours (we do not exclude \( z_1 = z_2 \)). We immediately have \( R_{zp} \geq 1 \) but
we need something slightly stronger. If either of \( z_1, z_2 \) has resistance to the root in \( G \setminus \{ x, y_1, y_2 \} \) at least \( \frac{1}{24} \), then \( R_{x^\rho}(G) \geq \frac{98}{97} \). Otherwise we consider the contribution to \( R_{\text{tot}} \) of \( y_1 \). We have \( R_{y_1^\rho}(G) \geq 3/4 \) and \( R_{y_1^\rho}(G/x) \leq \frac{1}{24} + \frac{3}{4} \). In either case we get that \( R_{\text{tot}}(G) - R_{\text{tot}}(G^+/x) \geq 1/100 \) giving the result.

**Lemma 17.** Fix \( n < m \) with \( 2m < 9n/4 \), and let \( \ell = 9n - 8m \). Suppose that \( G \) is a rooted graph with \( n \) non-root vertices, \( m \) edges, and \( k \leq \ell \) leaves. Then there exists a rooted graph \( G' \) also with \( n \) vertices and \( m \) edges and \( R_{\text{tot}}(G') \leq R_{\text{tot}}(G) - \frac{1}{100}(\ell - k) \).

**Proof.** We prove this by induction on \( \ell - k \). It is trivial if \( \ell - k = 0 \). Thus suppose that \( \ell - k \geq 1 \) and that the result holds for any graph with smaller \( \ell' \).

Let \( L \) be the set of vertices of degree one (i.e., the leaves), and let \( G_0 = G[V \setminus L] \). Then \( G_0 \) has \( n - k \) non-root vertices and \( m - k \) edges. Thus \( G_0 \) has average degree

\[
\frac{2(m - k)}{n - k} < \frac{2(\ell - \ell)}{n - \ell} = \frac{18(m - n)}{8(m - n)} = \frac{9}{4}.
\]

Thus, by Lemma 16, there exists \( x' \) with \( R_{\text{tot}}(G_0) - R_{\text{tot}}(G_0^+/x') \geq \frac{1}{100} \). Trivially, this implies that \( R_{\text{tot}}(G) - R_{\text{tot}}(G^+/x') \geq \frac{1}{100} \).

Now, \( G^+/x' \) has \( n \) vertices and \( m \) edges, and \( k + 1 \) leaves. Hence, by induction, there exists \( G' \) with \( n \) vertices and \( m \) edges and

\[
R_{\text{tot}}(G') \leq R_{\text{tot}}(G^+/x') - \frac{1}{100}(\ell - (k + 1)) \leq R_{\text{tot}}(G) - \frac{1}{100}(\ell - k),
\]

as claimed. \( \square \)

**Proof of Theorem 15.** Suppose that \( G \) is as in the theorem. By Theorem 10 we let \( G' \) be a rooting of \( G \) with \( B(G') < A(G)/2 + o(1) \) and \( m' = (1 + o(1))m \) edges. Observe that \( G' \) has at most as many leaves as \( G \), so \( G' \) has at most \( \gamma n \) leaves. Thus, by Lemma 17, there exists \( G'' \) with the same number of vertices and edges as \( G' \) and

\[
R_{\text{tot}}(G') - R_{\text{tot}}(G'') \geq \frac{1}{100} (9n - 8m' - \gamma n)
\]

\[26\]
Therefore

\[
B(G') - B(G'') = \frac{R_{\text{tot}}(G') - R_{\text{tot}}(G'')}{n} \\
\geq \frac{9n - 8m' - \gamma n}{100n} \\
= \frac{9 - 4\alpha - \gamma}{100} + o(1) \\
\geq \frac{4\varepsilon - \gamma}{100} + o(1).
\]

Finally, \(B(G'') \geq a(2m'/n)/2 = a(\alpha)/2 + o(1)\). Putting this all together we have

\[
A(G) \geq 2B(G') + o(1) \\
\geq 2B(G'') + \frac{4\varepsilon - \gamma}{50} + o(1) \\
\geq a(\alpha) + \frac{2}{25}(\varepsilon - \gamma/4) + o(1).
\]

As mentioned above, with substantially more effort, the bound could be improved significantly: to a little beyond \(2\frac{1}{2}\). However, the key technique we used was to find a vertex such that contracting one of its edges and adding a leaf reduces the total resistance. A star of triangles shows that there may be no such vertex when \(\alpha \geq 3\). Thus improving the bound beyond this would require new ideas.

We would like to say that there is phase transition where the graph changes from star-like to regular-like. We have seen that when \(\alpha > 3.83\) the optimal graph cannot contain a positive proportion of leaves, and when \(\alpha < 9/4\) the optimal graph must contain a positive proportion of leaves. It seems likely that there is a threshold when the optimal graph changes; i.e., that there is a phase transition. But we are not able to show this and there could be a region where there are optimal graphs with, and without, a positive proportion of leaves.

9 Queen-Bee Model

A naive approach to the rooted problem would be to insist that we join the root to every other vertex. This uses \(n\) edges (since we have defined the
rooted model to have \( n \) non-root vertices). As well as being a natural sub-class of rooted graphs, it also has a natural interpretation in the statistical application where it corresponds to the situation that every treatment is being compared with one control treatment. Following Bailey and Cameron [3], we call such graphs Queen-Bee networks. Note we do allow multiple edges between the root and non-root vertices; we insist only that there is at least one edge between the root and each non-root vertex.

In this section we show that the optimal Queen-Bee network for \( \alpha \in [2, 3] \) is the star of triangles and leaves with the correct number of edges, but that a Queen-Bee network is not optimal (in the space of all rooted graphs) for any \( \alpha > 2 \).

We prove our bound by moving into the space of networks where each edge, including the edges from the root vertex, can have any non-negative real conductance and, as in the introduction, the total conductance is bounded by \( m \). We say a configuration is legal if the non-root vertices can be partitioned into ‘components’ satisfying

1. edges between distinct components have zero conductance
2. each component of size \( s \) has a total conductance internally an integer at least \( s - 1 \), and total conductance to the root an integer at least \( s \).

**Lemma 18.** Suppose that \( C \) is a component of a legal configuration on \( s \) vertices with sum of the conductances of the edges \( m = 2s - 1 + t \). Then the total effective resistance to the root is at least \( \frac{s+2}{3} - \frac{2t}{3} = s - \frac{2(m-s)}{3} \). Moreover this bound is only obtained if \( m = 2s - 1 \).

**Proof.** If \( t = 0 \) (the minimum allowed in a legal configuration) then by the usual averaging argument the conductances on all edges inside a component are equal, and the conductances to the root from a component are all equal. Thus, by the definition of legality we know that each edge to the root has conductance 1, and each edge in the component has conductance \( 2/s \). Easy calculation shows that the resistance to the root is \( \frac{s+2}{3} \) and the bound is tight in this case.

Now suppose that \( t > 0 \). If \( s = 1 \) then the resistance to the root is \( \frac{1}{t+1} \) which is at least \( 1 - \frac{2t}{3} \) for all positive integers. Hence the only remaining case is \( t > 0 \) and \( s > 1 \).
We do not know (we could calculate but it is not informative) whether to put the extra conductance in the component or to the root. However, we just do both: we show that the bound still holds if the total conductance in the component is $s - 1 + t$ and the total conductance to the root is $s + t$. Indeed, if we take the example above on $s$ vertices for $t = 0$ and multiply each conductance by $1 + t/(s - 1)$ then we get a network with conductances at least this large and the resistance to the root goes down by

$$\frac{s + 2}{3} \left(1 - \frac{1}{1 + \frac{t}{s - 1}}\right) = \frac{s + 2}{3} \left(1 - \frac{s - 1}{s - 1 + t}\right) = \frac{s + 2}{3} \frac{t}{s - 1 + t} \leq \frac{2}{3}$$

with equality only if $s = 2$ and $t = 1$. It is easy to verify that the bound in this case is strict too. \hfill \Box

**Theorem 19.** Suppose that $G$ is a Queen-Bee network with $n$ vertices and average degree $\alpha$. Then the average resistance to the root $B(G)$ is at least $\frac{5 - \alpha}{3}$. Moreover, for $\alpha \in [2, 3]$ this is attainable.

**Proof.** Let $m = \alpha n/2$. Split the graph (without the root) into components. Each component is necessarily a legal component in the above definition. We write $n_i$ for the sizes, and $m_i$ for the weights, of the components. By Lemma 18 the total resistance to the root is at least

$$\sum_i \left(n_i - 2(m_i - n_i) \right) = n - \frac{2m - 2n}{3} = \frac{n(5 - \alpha)}{3}$$

and, thus, the average resistance to the root is at least $\frac{5 - \alpha}{3}$.

Finally, to see that this is attainable when $\alpha \in [2, 3]$ consider a star of leaves and triangles with the appropriate number of edges. The bound in Lemma 18 is attained for both leaves and triangles (components with $t = 0$ and either $r = 1$ or $r = 2$) and hence the above bound is attained. \hfill \Box

Next we improve this bound for large $\alpha$.

**Lemma 20.** Suppose that $G$ is a Queen-Bee network with average degree $\alpha$. Then

$$B(G) \geq \begin{cases} \frac{5 - \alpha}{3} & \text{if } \alpha \leq 3\frac{1}{2} \\ \frac{1}{\alpha - 3/2} & \text{if } \alpha \geq 3\frac{1}{2} \end{cases}$$
Proof. Let \( f = f(\alpha) \) be the claimed lower bound (i.e., the right hand side above) and note that \( f \) is convex.

We start by showing that, if \( G \) has no leaves, then \( B(G) \geq 1/(\alpha - 3/2) \). This is actually an easy consequence of the proof of the two-step lower bound (Theorem 13). The penultimate line of that proof showed that the sum of the two-step conductances is at most

\[
\sum_x (d_x + e_x) - \sum_y 1 + \sum_y \frac{e_y}{d_y} - \sum_y e_y \tag{3}
\]

In that proof we just used that the sum of the final pair of terms,

\[
\sum_y \frac{e_y}{d_y} - \sum_y e_y
\]

is negative. However, for a Queen-Bee network we know that \( e_y \geq 1 \) for all \( y \). Since \( G \) has no leaves, \( d_y \geq 2 \) and we see that \( e_y - e_y/d_y \geq 1/2 \) for all vertices \( y \). Substituting this improved bound into equation (3) we get

\[
\sum_x (d_x + e_x) - \sum_y 1 + \sum_y \frac{e_y}{d_y} - \sum_y e_y \leq \alpha n - n - n/2 = (\alpha - 3/2)n
\]

and thus the average resistance to the root in \( G \) is at least \( \frac{1}{\alpha - 3/2} \).

By Theorem 19 we see that \( B(G) \geq 5 - \frac{\alpha}{3} \) so combining these we have \( B(G) \geq f(\alpha) \).

So far we have only considered the case when \( G \) has no leaves, and we turn now to the general case. First observe that we may assume that all leaves are joined to the root, since we can move them to the root without increasing the resistance.

Given any such Queen-Bee network \( G \), we write it as the union of a Queen-Bee network \( G_1 \) and a star \( G_2 \) (i.e., the union of some leaves) sharing a common root but otherwise disjoint. Let \( \alpha_1 \) be the average degree of \( G_1 \) and \( \alpha_2 = 2 \) the average degree of \( G_2 \). Since, by the above, \( B(G_1) \geq f(\alpha_1) \), and, trivially, \( B(G_2) = 1 = f(\alpha_2) \), the convexity of \( f \) implies that \( B(G) \geq f(\alpha) \).

These bounds allow us to show that, with the exception of \( \alpha = 2 \) (essentially the star), no Queen-Bee network is optimal.

**Theorem 21.** No Queen-Bee network with average degree strictly greater than 2 is optimal.
Proof. We just need to show that our (non-Queen-Bee) constructions give average resistance strictly less than the lower bound proved in Lemma 20.

We have given constructions of average degree $\alpha$ and resistance at most $r$ for any point in the convex hull of the star ($\alpha = 2, r = 1$), the random $10/3$ regular construction used for Theorem 12 ($\alpha = \frac{10}{3}, r = 0.528$), and the regular construction (Corollary 11) for $\alpha \geq 4$ with $\alpha \in \mathbb{N}$ (which gives $r = \frac{\alpha}{\alpha(\alpha-2)}$). An easy but lengthy calculation (given in the appendix) shows that these constructions have average resistance less than $\frac{1}{\alpha - 3/2}$ for all $\alpha \geq 3$ and less than $(5 - \alpha)/3$ for $2 < \alpha \leq 3^{1/2}$.

10 Open Problems

Our main open question is to find the function $a$. To state our conjecture we want to define a function $f$ that is the maximal convex function which is less than the value of $b$ given by the star, and less than the random regular heuristic for $b$. More precisely, let $f: [2, \infty) \to \mathbb{R}$ be the maximal convex function such that $f(2) \leq 1$ and $f(x) \leq \frac{x - 1}{x(x-2)}$ for all $x \in (2, \infty)$. (Since the supremum of a family of convex functions is also convex it is clear that this function exists.)

**Conjecture 1.** Let $f$ be defined as above. Then $a(x) \geq 2f(x)$ for all $x$ (equivalently $b(x) \geq f(x)$).

In particular, since we have the corresponding upper bound for integer $\alpha$ at least four, this would imply that $a(\alpha) = \frac{\alpha - 1}{\alpha(\alpha-2)}$ for all such $\alpha$.

Our next two conjectures concern the existence of a phase transition between star-like and regular-like behaviour as discussed in Section 8.

**Conjecture 2.** There is a threshold $\alpha_0$ on the average degree below which all optimal graphs have a positive proportion of leaves, and above which all optimal graphs have $o(n)$ leaves.

In fact we would expect this also to be true for graphs which are not optimal but are sufficiently close to optimal.

Of course the existence of leaves is not the only way a graph could fail to be ‘regular-like’ – it could have vertices of high degree.

**Conjecture 3.** There is a threshold $\alpha_1$ on the average degree below which the optimal graph has a vertex with degree $\Omega(n)$, and above which all vertices have degree $o(n)$.
If these conjectures are true then, obviously, \( \alpha_1 \geq \alpha_0 \), but perhaps they are, in fact, equal. This would imply that removing the leaves from any optimal the graph would yield a graph with maximum degree \( o(n) \) (since removing the leaves must yield a graph of average degree greater than \( \alpha_0 \)).

Theorem 21 showed that it was not optimal, for any \( \alpha > 2 \), for a vertex to have degree \( n \). The same argument shows that if the root of \( G \) has \( \eta \) non-leaf neighbours then \( B(G) \geq \frac{1}{\alpha - 1 - \eta/2} \). Comparison with our upper bound of \( \frac{\alpha - 1}{\alpha(\alpha - 2)} \) then shows that, for \( \alpha \in \mathbb{N} \), \( \eta \) is at most \( \frac{2}{\alpha - 1} \). Since Lemma 14 shows that for \( \alpha \geq 4 \) the optimal graph has \( o(n) \) leaves, we see that the root has degree at most \( 2n/(\alpha - 1) \). Since we can short any other vertex to the root (removing one vertex) this shows that the maximum degree is at most \( 2n/(\alpha - 1) \) and, in particular, that the maximum degree (as a proportion of all vertices) tends to zero as \( \alpha \to \infty \).

We also have an open question, motivated by the statistical applications which is, perhaps, less natural from a pure mathematical perspective. It describes the situation when treatments are compared in blocks of size \( r \) for some \( r > 2 \); in comparison the model we have been considering is the case when the blocks have size 2.

**Question 4.** Fix \( n \), \( m \) and \( r \). Given any collection \( E_1, E_2, \ldots, E_m \) of the \( r \)-sets of \( [n] \) we can form a graph \( G \) on \( [n] \) with edge (multi)-set \( \bigcup_{i=1}^{m} E_i^{(2)} \) (i.e., the union of the cliques on each \( E_i \)). Which such collection minimises \( A(G) \)?

Finally, the corresponding question for maximum rather than average resistance is also very natural:

**Question 5.** Which graphs \( G \) with \( |V(G)| = n \) and \( e(G) \leq \frac{an}{2} \) minimise \( \max_{x,y \in V(G)} R_{xy} \).

The maximum seems to behave very differently from the average. Indeed, in the rooted version of the problem, the maximum is 1 for all \( \alpha \in [2, 9/4] \) (since the graph either has a leaf, or a vertex of degree 2 with both neighbours having degree 2, as in the proof of Lemma 16) whereas the star of triangles shows the maximum at \( \alpha = 3 \) is \( 2/3 \). In particular, unlike the average, the maximum is not a convex function.
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Appendix: Proof of Theorem 21

We prove the bound claimed at the end of the proof of Theorem 21. This is elementary but tedious, and the calculations given were done with the aid of a computer algebra package.

Define the function \( l(x) = \frac{x-1}{x(x-2)} \) and \( \Delta l(x) = l(x+1) - l(x) \) (which will be negative). Then, for any integer \( t \), the line through the points \((t, l(t))\) and \((t+1, l(t+1))\) has equation

\[
y = y(x) = l(t) + (x - t)\Delta l(t).
\]

This is the equation of our upper bound construction (the convex combination of \( t \)-regular and \( (t+1) \)-regular graphs) for \( x \) between \( t \) and \( t+1 \). We need to show that, in this range, this function is less than the Queen-Bee lower bound of \( 1/(x - 3/2) \). In fact we show, that this function is less than \( 1/(x - 3/2) \) for all \( x > 3/2 \). Since \( 1/(x - 3/2) \) is continuous and greater than \( y(x) \) for \( x \) near \( 3/2 \) it suffices to show that the equation \( 1/(x - 3/2) = y(x) \) has no solutions. This equation rearranges to

\[
\Delta l(t)(x - t)(x - 3/2) + (x - 3/2)l(t) - 1 = 0
\]
which expands to
\[\Delta l(t)x^2 + (l(t) - dl(t)(t + 3/2))x + \frac{3t\Delta l(t)}{2} - \frac{3l(t)}{2} - 1 = 0.\]
Finding the discriminant of the quadratic we get
\[-\frac{8 t^5 - 41 t^4 + 66 t^3 - 71 t^2 + 38 t - 1}{4 (t + 1)^2 (t - 1)^2 t^2 (t - 2)^2}.\]
Writing \( t = 4 + s \) this becomes
\[-\frac{8 s^5 + 119 s^4 + 690 s^3 + 1905 s^2 + 2382 s + 935}{4 (s + 5)^2 (s + 3)^2 (s + 4)^2 (s + 2)^2}\]
which is obviously negative for all \( s \geq 0 \), i.e., for all \( t \geq 4 \).

The next case is when \( \alpha \in [3\frac{1}{2}, 4] \). We need to use the convex combination of our construction for average degree 10/3 and the four regular construction. For all \( x \leq 4 \) this lies below the line with equation
\[y(x) = (x - 10/3)^{3/8} - 0.528 + 0.528.\]
As above we look for solutions to \( y(x) = 1/(x - 3/2) \). This equation rearranges to
\[-0.2295x^2 + 1.63725x - 2.9395 = 0\]
which has discriminant less than 0. Thus there are no roots, and this case is complete.

Next we need to consider the comparison to the other lower bound for Queen-Bee networks: namely \((5 - \alpha)/3\). For \( \alpha < 10/3 \) this is trivially above the line through the star (2, 1), and our construction for 10/3 (the point \((10/3, 0.528))\).

Finally we have to compare the bound of \((5 - x)/3\) and the convex combination of our construction for 10/3 and the regular graph for \( x \leq 3\frac{1}{2} \).
Writing \( y(x) \) for this bound we get
\[\frac{5 - x}{3} - y(x) = 0.373666667 - 0.1038333333x\]
which is zero at approximately 3.59, so positive before that. This completes the proof.