COHERENT STATE QUANTIZATION OF QUATERNIONS

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Abstract. Parallel to the quantization of the complex plane, using the canonical coherent states of a right quaternionic Hilbert space, quaternion field of quaternionic quantum mechanics is quantized. Associated upper symbols, lower symbols and related quantities are analysed. Quaternionic version of the harmonic oscillator and Weyl-Heisenberg algebra are also obtained.

1. Introduction

Quantization is commonly understood as the transition from classical to quantum mechanics. One may also say, to a certain extent, quantization relates to a larger discipline than just restricting to specific domains of physics. In physics, the quantization is a procedure that associates with an algebra $A_{cl}$ of classical observables an algebra $A_{q}$ of quantum observables. The algebra $A_{cl}$ is usually realized as a commutative Poisson algebra of derivable functions on a symplectic (or phase) space $X$. The algebra $A_{q}$ is, however, non commutative in general and the quantization procedure must provide a correspondence $A_{cl} \mapsto A_{q} : f \mapsto A_{f}$. Most physical quantum theories may be obtained as the result of a canonical quantization procedure. However, among the various quantization procedures available in the literature, the coherent state quantization (CS quantization) appear quite arbitrary because the only structure that a space $X$ must possess is a measure. Once a family of CS or frame labeled by a measure space $X$ is given one can quantize the measure space $X$. Various quantization schemes and their advantages and drawbacks are discussed in detail, for example, in [3, 6, 12, 2].

Due to the non commutativity of quaternions, quaternionic Hilbert spaces are formed by right or left multiplication of vectors by quaternionic scalars; the two different conventions give isomorphic versions of the theory. Quaternions can always be represented, through symplectic component functions, as a pair of complex numbers and thereby quaternions possess a symplectic structure. However, the quaternionic quantum mechanics is inequivalent to complex quantum mechanics. In analogy with the complex quantum mechanics, states of quaternionic quantum mechanics are described by vectors of a separable quaternionic Hilbert space and observables in quaternionic quantum mechanics are represented by quaternion linear and self-adjoint operators [1].

The CS quantization in the complex quantum mechanics is a well-known and well-studied problem. Using the method of CS quantization, various phase spaces such as complex field, complex unit disc, circle in complex plane, and cylindrical phase spaces, to name a few, have been quantized [2, 4, 6, 7, 5]. However, quantization of the quaternion field

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has not been studied yet. In this regard, parallel to the (CS) quantization of complex field, in this note, we present CS quantization of the quaternion field using CS of a right quaternionic Hilbert space, compute upper and lower symbols, and study the matrix elements. The quaternionic version of the harmonic oscillator and Weyl-Heisenberg algebra are also obtained. Since the properties of operators from complex Hilbert spaces do not directly translate to the operators on quaternionic Hilbert spaces \cite{13}, we shall investigate quaternionic operator properties associated with the quantization as needed.

2. Mathematical preliminaries

In order to make the paper self-contained, we recall a few facts about quaternions which may not be well-known. In particular, we revisit the $2 \times 2$ complex matrix representations of quaternions, quaternionic Hilbert spaces as needed here. For details we refer the reader to \cite{1} \cite{8} \cite{17} \cite{18}.

2.1. Quaternions. Let $H$ denote the field of quaternions. Its elements are of the form $q = x_0 + x_1 i + x_2 j + x_3 k$ where $x_0, x_1, x_2$ and $x_3$ are real numbers, and $i, j, k$ are imaginary units such that $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i$ and $ki = -ik = j$. The quaternionic conjugate of $q$ is defined to be $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$. We shall find it convenient to use the representation of quaternions by $2 \times 2$ complex matrices:

\begin{equation}
q = x_0 \sigma_0 + i x_1 \sigma_1 + j x_2 \sigma_2 + k x_3 \sigma_3,
\end{equation}

with $x_0 \in \mathbb{R}, \quad \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad \sigma_0 = I_2, \quad \sigma_1, \sigma_2, \sigma_3$ the usual Pauli matrices. The quaternionic imaginary units are identified as, $i = \sqrt{-1} \sigma_1, \quad j = \sqrt{-1} \sigma_2, \quad k = \sqrt{-1} \sigma_3$. Thus,

\begin{equation}
q = \begin{pmatrix}
x_0 + i x_3 & -x_2 + i x_1 \\
x_2 + i x_1 & x_0 - i x_3
\end{pmatrix}
\end{equation}

and $\overline{q} = q^\dagger$ (matrix adjoint). Introducing the polar coordinates:

\begin{align*}
x_0 &= r \cos \theta, \\
x_1 &= r \sin \theta \sin \phi \cos \psi, \\
x_2 &= r \sin \theta \sin \phi \sin \psi, \\
x_3 &= r \sin \theta \cos \phi,
\end{align*}

where $(r, \phi, \theta, \psi) \in [0, \infty) \times [0, \pi] \times [0, 2\pi]^2$, we may write

\begin{equation}
q = A(r) e^{i \theta \sigma(\vec{n})},
\end{equation}

where

\begin{equation}
A(r) = r \sigma_0
\end{equation}

and

\begin{equation}
\sigma(\vec{n}) = \begin{pmatrix}
\cos \phi & \sin \phi e^{i \psi} \\
\sin \phi e^{-i \psi} & -\cos \phi
\end{pmatrix}.
\end{equation}

The matrices $A(r)$ and $\sigma(\vec{n})$ satisfy the conditions,

\begin{equation}
A(r) = A(r)^\dagger, \quad \sigma(\vec{n})^2 = \sigma_0, \quad \sigma(\vec{n})^\dagger = \sigma(\vec{n})
\end{equation}

and $[A(r), \sigma(\vec{n})] = 0$. Note that a real norm on $H$ is defined by

\[|q|^2 := \overline{q}q = r^2 \sigma_0 = (x_0^2 + x_1^2 + x_2^2 + x_3^2).\]
A typical measure on $H$ may take the form

$$d\varsigma(r, \theta, \phi, \psi) = d\tau(r) d\theta d\Omega(\phi, \psi)$$

with $d\Omega(\phi, \psi) = \frac{1}{4\pi} \sin \phi \, d\phi \, d\psi$. Note also that for $p, q \in H$, we have $pq = qp$, $pq \neq qp$, $q\overline{q} = \overline{q} q$, and real numbers commute with quaternions. In defining the position and momentum operators, we shall also need the sliced version of quaternions. We borrow the materials as needed here from [9]. Let

$$S = \{ q = x_1 i + x_2 j + x_3 k \mid x_1, x_2, x_3 \in \mathbb{R}, x_1^2 + x_2^2 + x_3^2 = 1 \}$$

we call it a quaternion sphere.

**Proposition 2.1.** [9] For any non-real quaternion $q \in H \setminus \mathbb{R}$, there exist, and are unique, $x, y \in \mathbb{R}$ with $y > 0$, and $I \in S$ such that $q = x + yI$.

**Definition 2.2.** (Slice [9]) For every quaternion $I \in S$, the complex line $L_I = \mathbb{R} + I\mathbb{R}$ passing through the origin, and containing 1 and $I$, is called a quaternion slice.

From the definition we can see that

$$H = \bigcup_{I \in S} L_I \quad \text{and} \quad \bigcap_{L_I} = \mathbb{R}$$

One can also easily see that $L_I \subset H$ is commutative, while, elements from two different quaternion slices, $L_I$ and $L_J$ (for $I, J \in S$ with $I \neq J$) do not necessarily commute.

**2.2. Quaternionic Hilbert spaces.** In this subsection we define left and right quaternionic Hilbert spaces. For details we refer the reader to [11]. We also define the Hilbert space of square integrable functions on quaternions based on [17].

**2.2.1. Right Quaternionic Hilbert Space.** Let $V^R_H$ be a linear vector space under right multiplication by quaternionic scalars (again $H$ standing for the field of quaternions). For $f, g, h \in V^R_H$ and $q \in H$, the inner product

$$\langle \cdot | \cdot \rangle : V^R_H \times V^R_H \rightarrow H$$

satisfies the following properties

(i) $\overline{\langle f | g \rangle} = \langle g | f \rangle$
(ii) $\| f \|^2 = \langle f | f \rangle > 0$ unless $f = 0$, a real norm
(iii) $\langle f | g + h \rangle = \langle f | g \rangle + \langle f | h \rangle$
(iv) $\langle f | g q \rangle = \langle f | g \rangle q$
(v) $\langle f q | g \rangle = \overline{\langle f | g \rangle}$

where $\overline{q}$ stands for the quaternionic conjugate. We assume that the space $V^R_H$ is complete under the norm given above. Then, together with $\langle \cdot | \cdot \rangle$ this defines a right quaternionic Hilbert space, which we shall assume to be separable. Quaternionic Hilbert spaces share most of the standard properties of complex Hilbert spaces. In particular, the Cauchy-Schwarz inequality holds on quaternionic Hilbert spaces as well as the Riesz representation theorem for their duals. Thus, the Dirac bra-ket notation can be adapted to quaternionic Hilbert spaces:

$$| f q \rangle = | f \rangle q, \quad \langle f q | = \overline{q}(f |,$$
for a right quaternionic Hilbert space, with \( |f\rangle \) denoting the vector \( f \) and \( \langle f| \) its dual vector. Let \( O_R \) be an operator on a right quaternionic Hilbert space. The scalar multiple of \( O_R \) should be written as \( qO_R \) and the action must take the form

\[
(qO_R) | f \rangle = (O_R | f \rangle)q.
\]

The adjoint \( O_R^\dagger \) of \( O_R \) is defined as

\[
\langle g | O_R f \rangle = \langle O_R^\dagger g | f \rangle; \quad \text{for all } f, g \in V_R^n.
\]

2.2.2. **Left Quaternionic Hilbert Space.** Let \( V_L^H \) be a linear vector space under left multiplication by quaternionic scalars. For \( f, g, h \in V_L^H \) and \( q \in H \), the inner product \( \langle \cdot | \cdot \rangle : V_L^H \times V_L^H \to H \) satisfies the following properties

(i) \( \langle f | g \rangle = \langle g | f \rangle \)

(ii) \( \| f \|^2 = \langle f | f \rangle > 0 \) unless \( f = 0 \), a real norm

(iii) \( \langle f | g + h \rangle = \langle f | g \rangle + \langle f | h \rangle \)

(iv) \( \langle qf | g \rangle = q\langle f | g \rangle \)

(v) \( \langle f | qg \rangle = \langle f | g \rangle q \)

Again, we shall assume that the space \( V_L^H \) together with \( \langle \cdot | \cdot \rangle \) is a separable Hilbert space. Also,

\[
| qf \rangle = | f \rangle q, \quad (qf | = q\langle f | .
\]

Note that, because of our convention for inner products, for a left quaternionic Hilbert space, the bra vector \( \langle f | \) is to be identified with the vector itself, while the ket vector \( | f \rangle \) is to be identified with its dual. Note also that there is a natural left multiplication by quaternionic scalars on the dual of a right quaternionic Hilbert space and a similar right multiplication on the dual of a left quaternionic Hilbert space.

Separable quaternionic Hilbert spaces admit countable orthonormal bases. Let \( V_R^H \) be a right quaternionic Hilbert space and let \( \{e_\nu\}_{\nu=0}^N \) (\( N \) could be finite or infinite) be an orthonormal basis for it. Then, \( \langle e_\nu | e_\mu \rangle = \delta_{\nu\mu} \) and any vector \( f \in V_R^H \) has the expansion \( f = \sum_\nu e_\nu f_\nu \), with \( f_\nu = \langle e_\nu | f \rangle \in H \). Using such a basis, it is possible to introduce a multiplication from the left on \( V_R^H \) by elements of \( H \). Indeed, for \( f \in V_R^H \) and \( q \in H \) we define,

\[
qf = \sum_\nu e_\nu(qf_\nu).
\]

Further, \( \langle qf | g \rangle = \langle f | qg \rangle \) (see [16]). The field of quaternions \( H \) itself can be turned into a left quaternionic Hilbert space by defining the inner product \( \langle q | q' \rangle = qq'^\dagger = qq'^\dagger \) or into a right quaternionic Hilbert space with \( \langle q | q' \rangle = q'^\dagger q = qq'^\dagger \). Further note that, due to the non-commutativity of quaternions the sum

\[
\sum_{m=0}^\infty \frac{pq)^m}{m!} \neq \sum_{m=0}^\infty \frac{(pq)^m}{m!},
\]
thereby it cannot be written as \( \exp(pq) \). However, in any Hilbert space the norm convergence implies the convergence of the series and
\[
\sum_{m=0}^{\infty} \left| \frac{p^m q^m}{m!} \right| \leq \sum_{m=0}^{\infty} \frac{|p|^m |q|^m}{m!} = \sum_{m=0}^{\infty} \frac{(|p||q|)^m}{m!} = e^{|p||q|}.
\]
Thus the series (2.13) converges and, where ever needed, we call it \( E(p, q) \).

2.2.3. Quaternionic Hilbert Spaces of Square Integrable Functions. Let \( (X, \mu) \) be a measure space and \( H \) the field of quaternions, then
\[
\{ f : X \to H \mid \int_X |f(x)|^2 d\mu(x) < \infty \}
\]
is a right quaternionic Hilbert space which is denoted by \( L^2_H(X, \mu) \), with the (right) scalar product
\[
\langle f \mid g \rangle = \int_X \overline{f(x)} g(x) d\mu(x),
\]
where \( \overline{f(x)} \) is the quaternionic conjugate of \( f(x) \), and (right) scalar multiplication \( fa, a \in H \), with \( (fa)(q) = f(q)a \) (see [17] for details). Similarly, one could define a left quaternionic Hilbert space of square integrable functions.

3. Coherent states on right quaternion Hilbert spaces

The main content of this section is extracted from [15] as needed here. For an enhanced explanation we refer the reader to [15]. In [15] the authors have defined coherent states on \( V^R_H \) and \( V^L_H \), and also established the normalization and resolution of the identities for each of them. We briefly revisit the coherent states of \( V^R_H \) and the normalization and resolution of the identity. Let \( \{ |f_m\rangle \}_{m=0}^{\infty} \) be an orthonormal basis of \( V^R_H \). For \( q \in V^R_H \), the coherent states are defined as vectors in \( V^R_H \) in the form of
\[
|q\rangle = N(|q|)^{-\frac{1}{2}} \sum_{m=0}^{\infty} |f_m\rangle \frac{q^m}{\sqrt{\rho(m)}},
\]
where \( N(|q|) \) is the normalization factor and \( \{ \rho(m) \}_{m=0}^{\infty} \) is a positive sequence of real numbers. Using conditions (2.7), we can determine the normalization factor \( N(|q|) \), and the resolution of the identity. In order for the norm of \( |q\rangle \) to be finite, we must have
\[
\langle q \mid q \rangle = N(|q|)^{-1} \sum_{m=0}^{\infty} \frac{\rho^{2m}(m)}{\rho(m)} \quad < \infty.
\]
Therefore, if the positive sequence \( \{ \rho(m) \}_{m=0}^{\infty} \) of real numbers converges to \( \ell > 0 \), then we are required to restrict the domain into
\[
D = [0, \sqrt{\ell}] \times [0, \pi] \times [0, 2\pi)^2
\]
so that the convergence of the above series is guaranteed. The typical measure (2.7) is an appropriate one on the domain $D$ too. By requiring $\langle q | q \rangle = 1$, the normalization factor is obtained as

$$N(|q|) = \sum_{m=0}^{\infty} \frac{r^{2m}}{\rho(m)}.$$  

Using the measure $d\varsigma(r, \theta, \phi, \psi)$ one can obtain the following operator valued integral on the domain $D$ of (3.3):

$$\int_{D} |q\rangle\langle q | d\varsigma(r, \theta, \phi, \psi) = \sum_{m=0}^{\infty} \frac{2\pi}{\rho(m)} \int_{0}^{\sqrt{\ell_0}} \frac{r^{2m}}{N(|q|)} |f_m\rangle\langle f_m| d\tau(r),$$

and in obtaining it we have used the identity

$$\int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{2\pi} e^{i(m-l)\theta} \sin \phi d\phi d\theta d\varphi = 2\pi \delta_{ml} \mathbb{I}_2,$$

where $\delta_{ml}$ is the Kronecker’s delta. The resolution of the identity,

$$\int_{D} |q\rangle\langle q | d\varsigma(r, \theta, \phi, \psi) = I_{VR^H},$$

where $I_{VR^H}$ is the identity operator on $V_{RH}$, is obtained if there is a measure to satisfy the moment problem,

$$\int_{0}^{\sqrt{\ell_0}} r^{2m} \lambda(r) dr \mathbb{I}_2 = N(|q|) I_{2}.$$  

If the measure $d\tau(r)$ is chosen such that

$$d\tau(r) = \frac{N(|q|)}{2\pi} \lambda(r) dr,$$

then there exists an auxiliary density $\lambda(r)$ to solve (3.8), that is, we get

$$\int_{0}^{\sqrt{\ell_0}} r^{2m} \lambda(r) dr \mathbb{I}_2 = \rho(m) \mathbb{I}_2.$$

Particularly, if $\rho(m) = m!$, then the normalization factor $N(|q|) = e^{|q|^2}$ and $\ell = \infty$. The resolution of the identity can be established for (3.1) with $\lambda(r) = 2re^{-r^2}$. In this case $D = H$ and the CS are called quaternionic canonical coherent states. For the purpose of quantizing the quaternions we shall use these canonical set of CS.

4. Coherent state quantization: General scheme

Let $(X, \mu)$ be a measure space and $L^2(X, \mu)$ be given by

$$\left\{ f : X \to \mathbb{C} \mid \int_{X} |f(x)|^2 d\mu(x) < \infty \right\}.$$  

The Berezin-Toeplitz or anti-Wick or coherent state quantization, as used by various authors in the literature, associates a classical observable that is a function $f(x)$ on $X$ to an operator valued integral. We continue with the general procedure described in [6] and applied, for example, in [5, 4].

Choose a countable orthonormal basis

$$\mathcal{O} = \{ \phi_n \mid n = 0, 1, 2 \ldots \}$$
in $L^2(X, \mu)$, that is
\begin{equation}
\langle \phi_n | \phi_m \rangle = \int_X \phi_n(x) \phi_m(x) d\mu(x) = \delta_{mn},
\end{equation}
and assume that
\begin{equation}
0 < \sum_{n=0}^{\infty} |\phi_n(x)|^2 := N(x) < \infty \quad a.e.
\end{equation}
holds. Let $\mathcal{H}$ be a separable complex Hilbert space with orthonormal basis $\{|e_n| \mid n = 0, 1, 2 \cdots \}$ in 1-1 correspondence with $O$. In particular $\mathcal{H}$ can be taken as $\mathcal{H} = \text{span}O$ in $L^2(X, \mu)$. Then the family $\mathcal{F}_H = \{|x| \mid x \in X\}$ with
\begin{equation}
|x| = N(x)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \phi_n(x)|e_n\rangle \in \mathcal{H}
\end{equation}
forms a set of coherent states (CS). From (4.1) and (4.2) we have
\begin{equation}
\langle x | x \rangle = 1
\end{equation}
and
\begin{equation}
\int_X N(x)|x\rangle\langle x|d\mu(x) = I_H,
\end{equation}
where $I_H$ is the identity operator on $H$. We call the set $\mathcal{F}_H$ a set of CS only for satisfying the normalization and a resolution of the identity. Equation (4.5) allows us to implement CS or frame quantization of the set of parameters $X$ by associating a function $X \ni x \mapsto f(x)$ that satisfies appropriate conditions the following operator in $H$
\begin{equation}
f(x) \mapsto A_f = \int_X N(x)f(x)|x\rangle\langle x|d\mu(x).
\end{equation}
The matrix elements of $A_f$ with respect to the basis $\{|e_n\rangle\}$ are give by
\begin{equation}
(A_f)_{mn} = \langle e_m | A_f | e_n \rangle = \int_X f(x) \phi_n(x) \phi_m(x) d\mu(x).
\end{equation}
The operator $A_f$ is
(a) symmetric if $f(x)$ is real valued.
(b) bounded if $f(x)$ is bounded.
(c) self-adjoint if $f(x)$ is real semi-bounded (through Friedrich’s extension).
In order to view the upper symbol $f$ of $A_f$ as a quantizable object (with respect to the family $\mathcal{F}_H$) a reasonable requirement is that the so-called lower symbol of $A_f$ defined as
\begin{equation}
\hat{f}(x) = \langle x | A_f | x \rangle = \int_X N(x')f(x')|x\rangle\langle x'|d\mu(x')
\end{equation}
be a smooth function on $X$ with respect to some topology assigned to the set $X$. Associating to the classical observable $f(x)$ the mean value $\langle x | A_f | x \rangle$ one can also get the so-called Berezin transform $B[f]$ with $B[f](x) = \langle x | A_f | x \rangle$, for example, see [11] for details.
5. Quantization of the quaternions

In this section we shall adapt the general procedure outlined in the above section to quaternions. Since \((H, d\varsigma(r, \theta, \phi, \psi))\) is a measure space, the set
\[
\left\{ f : H \to H \mid \int_H |f(x)|^2 d\varsigma(r, \theta, \phi, \psi) < \infty \right\}
\]
is the space of right quaternionic square integrable functions and is denoted by \(L^2_H(H, d\varsigma(r, \theta, \phi, \psi))\).

Define the sequence of functions \(\{\phi_n\}_{n=0}^{\infty}\) such that
\[
\phi_n : H \to H \quad \text{by} \quad (5.1) \quad \phi_n(q) = \frac{q^n}{\sqrt{n!}}, \quad \text{for all} \quad q \in H.
\]

Then \(\phi_n \in L^2_H(H, d\varsigma(r, \theta, \phi, \psi))\), for all \(n = 0, 1, 2 \cdots\) and from (3.6) \(\langle \phi_m \mid \phi_n \rangle = \delta_{mn}\) (see [15]). That is,
\[
\mathcal{O} = \{ \phi_n \mid n = 0, 1, 2 \cdots \}
\]
is an orthonormal set in \(L^2_D(D, d\varsigma(r, \theta, \phi, \psi))\). The right quaternionic span of \(\mathcal{O}\) is the space of anti-right-regular functions [14] (the counter part of complex anti-holomorphic functions). Let \(\mathfrak{H}\) be a separable right quaternionic Hilbert space with an orthonormal basis
\[
\mathcal{E} = \{ |e_n\rangle \mid n = 0, 1, 2 \cdots \}
\]
which is in \(1 \sim 1\) correspondence with \(\mathcal{O}\). Then the coherent states (3.1) become
\[
(5.2) \quad |\gamma_q\rangle = e^{i|q|^{-1}} \sum_{m=0}^{\infty} |e_m\rangle \overline{\phi_m}.
\]

Using the set of CS (5.2) we shall establish the coherent state quantization on \(\mathfrak{H}\) by associating a function
\[
H \ni q \mapsto f(q, \overline{q}).
\]

Now let us define the operator on \(\mathfrak{H}\) by
\[
(5.3) \quad f(q, \overline{q}) \mapsto A_f
\]
where \(A_f\) is given by the operator valued integral
\[
(5.4) \quad A_f = \int_H |\gamma_q\rangle f(q, \overline{q}) \langle \gamma_q | \: d\varsigma(r, \theta, \phi, \psi).
\]

Remark 5.1. The operator \(A_f\) is formed by the vector \(|\gamma_q\rangle f(q, \overline{q})\), which is the right scalar multiple of the vector \(|\gamma_q\rangle\) by the scalar \(f(q, \overline{q})\), and the dual vector \(\langle \gamma_q |\). Instead if one takes
\[
(5.5) \quad A_f = \int_H f(q, \overline{q}) |\gamma_q\rangle \langle \gamma_q | \: d\varsigma(r, \theta, \phi, \psi),
\]
then it is formed by \(f(q, \overline{q}) |\gamma_q\rangle\) (a left scalar multiple of a right Hilbert space vector) and the dual vector \(\langle \gamma_q |\), which is unconventional. Further, due to the noncommutativity of quaternions, the \(A_f\) in the form (5.5) shall cause severe technical problems in the follow up computations.
Now

\[ A_f = \int_H |\gamma_q f(q, \overline{q})| \langle \gamma_q | d\zeta(r, \theta, \phi, \psi) \]

\[ = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{|e_m| J_{m,l}(e_l | \sqrt{m! l!}}, \]

where the integral \( J_{m,l} \) is given by

\[ J_{m,l}(q) = \frac{a^m f(q, \overline{q} / |e_l^2 \rangle d\zeta(r, \theta, \phi, \psi).} \]

By direct calculation we have that if \( f(q, \overline{q}) = q \), then

\[ A_q = \sum_{m=0}^{\infty} \sqrt{(m + 1)} |e_m| \langle e_{m+1} | \]

and if \( f(q, \overline{q}) = \overline{q} \), then

\[ A_{\overline{q}} = \sum_{m=0}^{\infty} \sqrt{(m + 1)} |e_{m+1} \langle e_m |. \]

Moreover if \( f(q, \overline{q}) = 1 \), then \( A_1 = I_H \). It should be mentioned that, since the operator \( A_f \) is a quaternionic operator, the usual properties of its complex counterpart may not hold. In this regard, each property used must be validated. First let \( |f\rangle, |g\rangle \in H \). Since \( H \) is a right Hilbert space, there are scalars \( \{\alpha_i\}, \{\beta_j\} \) in \( H \) such that

\[ |f\rangle = \sum_{l=0}^{\infty} |e_l\rangle \alpha_l \quad \text{and} \quad |g\rangle = \sum_{j=0}^{\infty} |e_j\rangle \beta_j. \]

With these it can be seen that

\[ \langle A_{\overline{q}} g | f \rangle = \langle g | A_q f \rangle = \sum_{m=0}^{\infty} \overline{\beta_m} \alpha_{m+1} \sqrt{m+1}. \]

That is,

\[ \langle A_{\overline{q}} g | f \rangle = \langle g | A_q f \rangle; \quad \text{for all} \quad |f\rangle, |g\rangle \in H. \]

Hence \( A_{\overline{q}} \) is the adjoint of \( A_q \). Now \( A_f \) is an operator from \( H \) to \( H \), and if \( H = \overline{\text{span}} \) (right linear span over \( H \)), then it is a subspace of \( L^2_H(\sigma(r, \theta, \phi, \psi)) \). That is,

\[ A_f : H \rightarrow \overline{\text{span}} \quad \text{by} \quad A_f(u) = A_f | u \rangle, \]

for all \( u \in H \). Hence, \( A_f(u) \) will be determined by this integral

\[ \int_H |\gamma_q f(q, \overline{q})| \langle \gamma_q | u \rangle d\zeta(r, \theta, \phi, \psi). \]

Moreover, for each \( u \in H \), \( A_f | u \rangle \in H \). It can also be considered as a function

\[ A_f : H \times H \rightarrow H \quad \text{by} \quad A_f(u, v) = \langle u | A_f | v \rangle, \]

for all \( (u, v) \in H \times H \). Thereby, \( A_f(u, v) \) will be determined by the quaternion valued integral

\[ \int_H \langle u | \gamma_q f(q, \overline{q})| v \rangle d\zeta(r, \theta, \phi, \psi). \]
Since $|\gamma_q\rangle$ is a column vector and $\langle\gamma_q|$ is a row vector, we can see that the operator $A_f$ is a matrix and the matrix elements with respect to the basis $\{|e_n\rangle\}$ are given by

$$(A_f)_{mn} = \langle e_m | A_f | e_n \rangle.$$ 

That is, $(A_f)_{mn}$ is determined by the integral

$$\int_{H} \langle e_m | \gamma_q \rangle f(q, \bar{q}) \langle\gamma_q| e_n \rangle d\varsigma(r, \theta, \phi, \psi).$$

We have

$$\langle e_m | \gamma_q \rangle = N(|q\rangle)^{-\frac{1}{2}} \phi_m(q)$$

and

$$\langle\gamma_q| e_n \rangle = \langle e_n | \gamma_q \rangle = N(|q\rangle)^{-\frac{1}{2}} \phi_n(q).$$

Therefore

$$(A_f)_{mn} = \int_{H} \mathcal{N}(|q\rangle)^{-1}\phi_m(q)f(q, \bar{q})\phi_n(q) d\varsigma(r, \theta, \phi, \psi).$$

Further, it can easily be seen that

$$(A_q)_{k,l} = \langle e_k | A_q | e_l \rangle = \begin{cases} \sqrt{k+1} & \text{if} \quad l = k + 1 \\ 0 & \text{if} \quad l \neq k + 1, \end{cases}$$

and

$$(A_{\bar{q}})_{k,l} = \langle e_k | A_{\bar{q}} | e_l \rangle = \begin{cases} \sqrt{k} & \text{if} \quad l = k - 1 \\ 0 & \text{if} \quad l \neq k - 1. \end{cases}$$

Let us realize the operator $A_f$ as annihilation and creation operators. From (5.6) and (5.7) we have $A_q |e_0\rangle = 0$, $A_q |e_m\rangle = \sqrt{m} |e_{m-1}\rangle$; $m = 1, 2, \ldots$

and

$$A_{\bar{q}} |e_m\rangle = \sqrt{m + 1} |e_{m+1}\rangle; \quad m = 0, 1, 2, \ldots$$

That is $A_q, A_{\bar{q}}$ are annihilation and creation operators respectively. Moreover, one can easily see that $A_q |\gamma_q\rangle = |\gamma_q\rangle q$, which is in complete analogy with the action of the annihilation operator of the ordinary harmonic oscillator and the result obtained in [15].

Now a direct calculation shows that

$$A_q A_{\bar{q}} = \sum_{m=0}^{\infty} (m + 1) |e_m\rangle \langle e_m|$$

and

$$A_{\bar{q}} A_q = \sum_{m=0}^{\infty} (m + 1) |e_{m+1}\rangle \langle e_{m+1}|.$$ 

Thereby the commutator of $A_q, A_{\bar{q}}$ takes the form

$$[A_q, A_{\bar{q}}] = A_q A_{\bar{q}} - A_{\bar{q}} A_q = \sum_{m=0}^{\infty} |e_m\rangle \langle e_m| = \mathbb{I}_0.$$

5.1. Number, position and momentum operators and Hamiltonian. Let \( N = A_{\overline{q}}A_q \), then we have

\[
N \mid e_k \rangle = A_{\overline{q}}A_q \mid e_k \rangle = \sum_{m=0}^{\infty} \mid e_{m+1} \rangle \langle e_{m+1} \mid e_k \rangle (m + 1) = \mid e_k \rangle k.
\]

Thereby \( N \) acts as the number operator and the Hilbert space \( \mathcal{H} \) is the quaternionic Fock space. As an analog of the usual harmonic oscillator Hamiltonian, if we take \( H = N + \mathbb{I}_{\mathcal{H}} \), then \( H \mid e_n \rangle = \mid e_n \rangle(n + 1) \), which is a Hamiltonian in the right quaternionic Hilbert space \( \mathcal{H} \) with spectrum \((n + 1)\) and eigenvector \( \mid e_n \rangle \).

**Remark 5.2.** In the complex quantum mechanics, for the canonical CS \( \mid z \rangle \), \( z \in \mathbb{C} \), the lower symbol or the expectation value of the number operator, \( \langle z \mid N \mid z \rangle \), is precisely \( |z|^2 \).

The position and momentum coordinates are \( q = \frac{1}{\sqrt{2}}(z + \overline{z}) \) and \( p = \frac{1}{\sqrt{2}}(z - \overline{z}) \) and by linearity one infers that the position and momentum operators as \( Q = \frac{1}{\sqrt{2}}(A_z + A_{\overline{z}}) \) and \( P = \frac{-i}{\sqrt{2}}(A_z - A_{\overline{z}}) \). The CS quantized classical harmonic oscillator, \( H = \frac{1}{2}(q^2 + p^2) \) is \( A_H = A_{|q|^2} = N + I_{\mathcal{H}} \), where \( I_{\mathcal{H}} \) is the identity operator of the Fock space. The operators \( Q \) and \( P \) satisfy the commutation rule \([Q, P] = iI_{\mathcal{H}}\) and are self-adjoint. If one simply takes the canonical quantization of the classical Hamiltonian it becomes \( \hat{H} = \frac{1}{2}(\hat{Q}^2 + \hat{P}^2) = N + \frac{1}{2}I \). For details we refer the reader to [5, 6].

In the case of quaternions we have three imaginary units, \( i, j \) and \( k \), and if one try to duplicate the position and momentum coordinates with one of \( i, j \) or \( k \), that is, if we take

\[
q = \frac{1}{\sqrt{2}}(q + \overline{q}) \quad \text{and} \quad p = \frac{-i}{\sqrt{2}}(q - \overline{q}),
\]

then a simple calculation shows that \( H = \frac{1}{2}(q^2 + p^2) \neq |q|^2 \). However, the lower symbol of \( N \) is \( \langle \gamma_q \mid N \mid \gamma_q \rangle = |q|^2 \) and through a rather lengthy calculation we can see that \( A_{|q|^2} = N + I_{\mathcal{H}} \). The best way to avoid the difficulty in defining the position and momentum coordinates is to consider quaternion slices. In this regard, let \( q \in H \), then there exists \( I \in \mathbb{S} \) such that \( q = x + Iy \) for some \( x, y \in \mathbb{R} \). Now note that \( qI = (x + Iy)I = I(x + Iy) = Iq \), similarly \( \overline{q}I = (x - Iy)I = I(x - Iy) = I\overline{q} \). That is, the commutativity holds among \( I, q \) and \( \overline{q} \). Let us define the position and momentum coordinates by

\[
q = \frac{1}{\sqrt{2}}(q + \overline{q}) \quad \text{and} \quad p = \frac{-i}{\sqrt{2}}(q - \overline{q}),
\]

then, with the aid of the commutativity among \( I, q \) and \( \overline{q} \), the Hamiltonian can be calculated as

\[
H = \frac{1}{2}(q^2 + p^2) = |q|^2.
\]

Recall that on a right quaternionic Hilbert space operators are multiplied on the left by quaternion scalars. From the position and momentum coordinates, using linearity, we get the position operator, \( Q \), and the momentum operator, \( P \), as

\[
Q = \frac{1}{\sqrt{2}}(A_q + A_{\overline{q}}) \quad \text{and} \quad P = \frac{-i}{\sqrt{2}}(A_q - A_{\overline{q}}).
\]
Since \((A_\mathbf{q})^\dagger = A_\mathbf{q}\) and \((-I)^\dagger = I\), the operators \(P\) and \(Q\) are self-adjoint. Using the fact \((2.10)\) we can see that \(A_\mathbf{q}(IA_\mathbf{q}) = IA_\mathbf{q}A_\mathbf{q}\). With the aid of this we get
\[
Q \cdot P = \left[\frac{(A_\mathbf{q} + A_\mathbf{\bar{q}})}{\sqrt{2}}\right] \left[ -I \left(\frac{A_\mathbf{q} - A_\mathbf{\bar{q}}}{\sqrt{2}}\right) \right] = -\frac{1}{2}I \left[ A_\mathbf{q}^2 + A_\mathbf{\bar{q}}A_\mathbf{q} - A_\mathbf{q}A_\mathbf{\bar{q}} - A_\mathbf{\bar{q}}^2 \right]
\]
and
\[
P \cdot Q = \left[ -I \left(\frac{A_\mathbf{q} - A_\mathbf{\bar{q}}}{\sqrt{2}}\right) \right] \left[ \frac{(A_\mathbf{q} + A_\mathbf{\bar{q}})}{\sqrt{2}} \right] = -\frac{1}{2}I \left[ A_\mathbf{q}^2 - A_\mathbf{\bar{q}}A_\mathbf{q} + A_\mathbf{q}A_\mathbf{\bar{q}} - A_\mathbf{\bar{q}}^2 \right].
\]

Thereby we have the commutator
\[
[Q, P] = Q \cdot P - P \cdot Q = I \left[ A_\mathbf{q}, A_\mathbf{\bar{q}} \right] = I \mathbb{I}_\delta.
\]

We also have
\[
Q^2 = \frac{1}{2} \left[ A_\mathbf{q}^2 + A_\mathbf{\bar{q}}A_\mathbf{q} + A_\mathbf{q}A_\mathbf{\bar{q}} + A_\mathbf{\bar{q}}^2 \right] \quad \text{and} \quad P^2 = -\frac{1}{2} \left[ A_\mathbf{q}^2 - A_\mathbf{\bar{q}}A_\mathbf{q} - A_\mathbf{q}A_\mathbf{\bar{q}} - A_\mathbf{\bar{q}}^2 \right]
\]

Hence
\[
H = \frac{Q^2 + P^2}{2} = \frac{1}{2} \left[ A_\mathbf{q}A_\mathbf{q} + A_\mathbf{q}A_\mathbf{\bar{q}} \right] = A_\mathbf{q}A_\mathbf{q} + \frac{1}{2} \left[ A_\mathbf{q}A_\mathbf{\bar{q}} - A_\mathbf{\bar{q}}A_\mathbf{q} \right] = N + \frac{1}{2} \mathbb{I}_\delta,
\]
which is in complete analogy with the complex case.

**5.2. Oscillator algebra.** As before a simple calculation shows that
\[
NA_\mathbf{q} = \sum_{m=0}^{\infty} m \sqrt{m+1} | e_m \rangle \langle e_{m+1} |
\]
\[
A_\mathbf{q}N = \sum_{m=0}^{\infty} (m+1) \sqrt{m+1} | e_m \rangle \langle e_{m+1} |
\]
\[
NA_\mathbf{\bar{q}} = \sum_{m=0}^{\infty} (m+1) \sqrt{m+1} | e_{m+1} \rangle \langle e_m |
\]
\[
A_\mathbf{\bar{q}}N = \sum_{m=0}^{\infty} m \sqrt{m+1} | e_{m+1} \rangle \langle e_m |
\]

Thereby we get
\[
[N, A_\mathbf{q}] = NA_\mathbf{q} - A_\mathbf{q}N = -A_\mathbf{q}, \quad [N, A_\mathbf{\bar{q}}] = NA_\mathbf{\bar{q}} - A_\mathbf{\bar{q}}N = A_\mathbf{\bar{q}}.
\]
and we already have \([A_\mathbf{q}, A_\mathbf{\bar{q}}] = A_1 = \mathbb{I}_\delta\). That is, the right quaternionic operators \(A_1, A_\mathbf{q}, A_\mathbf{\bar{q}}\) and \(N\) satisfy the Weyl-Heisenberg commutation relations, and therefore these operators together with their commutators form the quaternionic version of the Weyl-Heisenberg algebra.

**5.3. Upper symbols as differential operators.** Recall the sets \(\mathcal{E} = \{ |e_m \rangle \mid m = 0, 1, 2, \cdots \}\) and \(\mathcal{O} = \{ \phi_m \mid m = 0, 1, 2, \cdots \}\), where \(\phi_m\) is as in \((5.1)\). Let \(\mathcal{F} = \{ \overline{\phi_m} \mid m = 0, 1, 2, \cdots \}\). The sets \(\mathcal{E}, \mathcal{O}\) and \(\mathcal{F}\) are in 1-1 correspondence to each other and \(\mathcal{E}\) is the basis of \(\mathcal{F}\). If one replaces the Hilbert space \(\mathcal{H}\) by the Hilbert space of right regular functions \(\mathcal{H}_{\text{reg}} = \text{span}\mathcal{F}\) or by the Hilbert space of right anti-regular functions \(\mathcal{H}_{\text{a-reg}} = \text{span}\mathcal{O}\),
where span means right linear span over $H$, then the actions of the operators $A_q$ and $A_{\overline{q}}$ on the basis sets $\mathcal{O}$ and $\mathcal{F}$ take the form

\begin{align*}
A_q \phi_m &= \sqrt{m+1} \phi_{m+1}, \\
A_{\overline{q}} \phi_m &= \sqrt{m+1} \phi_{m+1}, \\
A_q \overline{\phi}_m &= \sqrt{m} \overline{\phi}_{m-1}, \\
A_{\overline{q}} \overline{\phi}_m &= \sqrt{m+1} \overline{\phi}_{m+1}.
\end{align*}

Therefore, on the space $\mathcal{H}_{\alpha-reg}$ the operator $A_q = \frac{\partial}{\partial q}$ and $A_{\overline{q}}$ is multiplication by $q$, and similarly, on the space $\mathcal{H}_{reg}$ the operator $A_q = \frac{\partial}{\partial q}$ and $A_{\overline{q}}$ is multiplication by $q$, where the quaternionic derivatives are right Cullen derivatives (again in complete analogy with the complex case). For details on Cullen derivatives we refer the reader to [9, 10] and for application of Cullen derivatives on quaternionic CS we refer to [14]. Also note that the spaces $\mathcal{H}_{reg}$ and $\mathcal{H}_{\alpha-reg}$ are the quaternionic analogue of the Bargmann space of analytic functions and the Bargmann space of anti-analytic functions respectively [14].

### 5.4. Overlap of the CS and lower symbols

The overlap of two CS is required to compute the lower symbols. Let $q, p \in H$, then the overlap of the CS becomes

$$\langle \gamma_q \mid \gamma_p \rangle = \frac{1}{\sqrt{\mathcal{N}(\|q\|)\mathcal{N}(\|p\|)}} \sum_{m=0}^{\infty} \frac{1}{m!} q^m \overline{p}^m.$$

That is,

$$\langle \gamma_q \mid \gamma_p \rangle = e^{-\frac{(\|q\|^2+\|p\|^2)}{2}} E(q, \overline{p}).$$

Now the lower symbol of $A_f$ can be computed as

$$\tilde{f} = \langle \gamma_p \mid A_f \mid \gamma_p \rangle = \int_H \langle \gamma_p \mid f(q, \overline{q}) \rangle \langle \gamma_q \mid \gamma_p \rangle d\varsigma(r, \theta, \phi, \psi) = \int_H \Phi(p, q) d\varsigma(r, \theta, \phi, \psi);$$

where

$$\Phi(p, q) = e^{-\frac{(\|q\|^2+\|p\|^2)}{2}} E(p, \overline{q}) f(q, \overline{q}) E(q, \overline{p}).$$

### 6. Conclusion

Using the general scheme of CS quantization the quaternion field is quantized. As expected, the operators $A_q$ and $A_{\overline{q}}$ act as annihilation and creation operators respectively. The matrix representations of these operators are also similar to the complex case. Further, using the so-called Cullen derivatives we have written these operators as differential operators. In complete analogy with the complex harmonic oscillator, the quaternionic version of the harmonic oscillator and Weyl-Heisenberg algebras are obtained. In conclusion, even though the noncommutativity of quaternions caused technical difficulties, most part of the quantization procedure of quaternions followed its complex counterpart. As the quantization procedure play an important role in complex quantum mechanics, the quantization presented in this manuscript can also play a vital role in the quaternionic quantum mechanics.
References

[1] Adler, S.L., *Quaternionic quantum mechanics and Quantum fields*, Oxford University Press, New York, 1995.

[2] Ali, S.T., Antoine, J-P. and Gazeau, J-P., *Coherent States, Wavelets and Their Generalizations*, Springer, New York, 2000.

[3] Ali, S.T., Englis, M., *Quantization method: a guide for Physicist and analysts*, Rev. Math. Phys., 17 (2005), 391-490.

[4] Aremua, I., Gazeau, J-P., Hounkonnou, M.N., *Action-angle coherent states for quantum systems with cylindrical phase space*, J. Phys. A. 45 (2012), 335302 (16pp).

[5] Cotfas, N., Gazeau, J-P., and Gorska, K., *Complex and real Hermite polynomials and related quantizations*, J. Phys.A: Math. Theor. 43 (2010), 305304.

[6] Gazeau, J-P., *Coherent states in quantum physics*, Wiley-VCH, Berlin (2009).

[7] Gazeau, J-P. and Szafraniec, F.H., *Holomorphic Hermite polynomials and non-commutative plane*, J. Phys. A: Math. Theor. 44 (2011), 495201.

[8] Ghiloni, R., Moretti, W. and Perotti, A., *Continuous slice functional calculus in quaternionic Hilbert spaces*, Rev. Math. Phys. 25 (2013), 1350006 (83 pages).

[9] Gentili, G. and Struppa, D.C., *A new theory of regular functions of a quaternionic variable*, Adv. Math. 216 (2007), 279-301.

[10] Gentili, G. and Stoppato, C., *Power series and analyticity over the quaternions*, Math. Ann. 352 (2012), 113-131.

[11] Mouayn, Z., *Coherent states quantization for generalized Bargmann spaces with formulae for their attached Berezin transform in terms of the Laplacian on C^n*, J. Fourier. Anal. Appl. 18, (2012), 609-625.

[12] Perelomov, A., *Generalized coherent states and their applications*, Springer-Verlag, Berlin (1986).

[13] Riccardo Ghiloni, Valter Moretti and Alessandro Perotti *Continuous slice functional calculus in quaternionic Hilbert space*, arXiv:1207.0666v2 [math.FA].

[14] Thirulogasanthar, K., Twareque Ali, S., *Regular subspaces of a quaternionic Hilbert space from quaternionic Hermite polynomials and associated coherent states*, J. Math. Phys., 54 (2013), 013506 (19pp).

[15] Thirulogasanthar, K., Honouvo, G. and Krzyzak, A., *Coherent states and Hermite polynomials on Quaternionic Hilbert spaces*, J. Phys.A: Math. Theor. 43 (2010), 385205.

[16] Torgasev, A., *Dual space of a quaternion Hilbert space*, Ser. Math. Inform. 14 (1999), 71-77.

[17] Viswanath, K., *Normal operators on quaternionic Hilbert spaces*, Trans. Am. Math. Soc 162 (1971), 337350.

[18] Zhang, F., *Quaternions and Matrices of Quaternions*, Linear Algebra and its Applications, 251 (1997), 21-57.

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