Szegő projection and matrix Hilbert transform in Hermitean Clifford analysis

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Abstract The simultaneous null solutions of the two complex Hermitean Dirac operators are focused on in Hermitean Clifford analysis, where the matrix Hilbert transform was presented and proved to satisfy the analogous properties of the Hilbert transform in classical analysis and in orthogonal Clifford analysis. Under this setting we will introduce the Szegő projection operator for the Hardy space of Hermitean monogenic functions defined on a bounded subdomain of even dimensional Euclidean space, establish the Kerzman-Stein formula which closely connects the Szegő projection operator with the Hardy projection operator onto the Hardy space of Hermitean monogenic functions defined on a bounded subdomain of even dimensional Euclidean space, and get the Szegő projection operator in terms of the Hardy projection operator and its adjoint. Further we will give the algebraic and geometric characterizations for the matrix Hilbert transform to be unitary in Hermitean Clifford analysis.

Keywords: Hermitean Clifford analysis, Szegő projection, Matrix Hilbert transformation, Hardy space

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1. Introduction

The Hilbert transform in one dimensional space and its properties were mainly developed by Titchmarsh and Hardy though it is named after David Hilbert. This transform, which plays an important role in engineering science such as signal analysis, naturally appears when considering the boundary behavior of the Cauchy transform. The crucial formula connecting the boundary value of the Cauchy transform and Hilbert transform is the well-known Plemelj-Sokhotzki formula. The classical multidimensional analogue of the Hilbert transform is a tensorial one, studying the Riesz transforms for each of the Cartesian variables separately (see reference e.g. [1]). As opposed to these tensorial approaches, the orthogonal Clifford analysis (seen in references e.g. [2, 3, 4]) essentially provided a natural framework for generalizing a lot of classical results from complex analysis and harmonic analysis in the plane to the higher dimensional case. The central tool is the Cauchy transform which leads to the Plemelj-Sokhotzki type formula
when taking the boundary values. Also the properties of the corresponding singular operator was studied in virtue of the function theoretic methods (see reference e.g. [5] or elsewhere).

In [6], Kerzman and Stein proved the fundamental property of the Cauchy transform \( Cf \) of \( f \in L_2(\Sigma) \) where \( \Sigma \) is the smooth boundary of a bounded open domain \( D \) in the plane. They stated that the operator \( A = C - C^* \) (where \( C^* \) is the adjoint operator of the operator \( C \)) is a compact infinitely smooth operator on \( L_2(\Sigma) \) and the Szegö projection \( S \) and the Hardy projection \( C \) of \( L_2(\Sigma) \) onto the Hardy space \( H^2(\Sigma) \) are related by Kerzman-Stein formula (see references e.g. [6, 7, 8, 9]). Moreover, they showed that the disc is the only plane region on which the Hilbert transform \( H \) on \( L_2(\Sigma) \) is unitary. In [10, 11, 12], Bernstein, Calderbank, Delanghe and their collaborators generalized the Kerzman-Stein formula to the higher dimensional case. Furthermore Delanghe (seen in [12]) characterized the unitariness of the Hilbert transform under the setting of orthogonal Clifford analysis. More related results on the Szegö kernel and the Hilbert transform in orthogonal Clifford analysis can be also found in references e.g. [13, 14 – 16].

More recently, offering yet a refinement of the orthogonal case, Hermitean Clifford analysis in references e.g. [17 – 23] emerged as a new and successful branch of Clifford analysis. It focuses on the simultaneous null solutions of the two complex Hermitean Dirac operators, which are invariant under the action of the unitary group and were first studied in references e.g. [17 – 19]. The Cauchy integral formula for Hermitean monogenic functions defined in even dimensional Euclidean space taking values in the complex Clifford algebra \( \mathbb{C}_{2n} \) was constructed in the framework of circulant \((2 \times 2)\) matrix functions, and at the same time the intimate relationship with holomorphic function theory of several complex variables (see references e.g. [24, 25]) was established by Brackx, De Schepper, Sommen and so on (see [20]). The Hermitean Cauchy transform, which gave rise to the Hardy projection to be skew in Hermitean Clifford analysis, and the related decomposition problems of continuous functions were discussed in [21, 22]. The new Hilbert-like matrix operator was revealed by the non-tangential boundary limits of the Hermitean Cauchy transform and the analogues of characteristic properties of the matrix Hilbert transform in classical analysis and in orthogonal Clifford analysis were given in [23]. Much recent progress can be also seen elsewhere. Under this setting it is natural to think of the orthogonal Szegö projection. However up to the present, as far as we know, it has not been studied. In the underlying paper, based on [19 – 20, 23, 25, 12, 6, 14], we will first define an inner product on the space of square integral circulant \((2 \times 2)\) matrix functions defined on the boundary of a bounded subdomain in even dimensional Euclidean space, and introduce the Szegö projection operator to be orthogonal for the Hardy space of Hermitean monogenic functions defined on a bounded subdomain of even dimensional Euclidean space. Then we will establish the Kerzman-Stein formula which are closely related to the Szegö projection operator and the Hardy projection
operator onto the Hardy space of Hermitean monogenic functions defined on a bounded subdomain of even dimensional Euclidean space, and present the Szegö projection operator in explicit terms of the Hardy projection operator and its adjoint. Lastly we will give the algebraic and geometric characterizations for the matrix Hilbert transform to be unitary in Hermitean Clifford analysis.

The paper is organized as follows. In section 2, we recall some basic facts about Hermitean Clifford analysis which will be needed in the sequel. In section 3, we will introduce the Szegö projection operator to be orthogonal for the Hardy space of Hermitean monogenic functions defined on a bounded subdomain of even dimensional Euclidean space, establish the Kerzman-Stein formula which closely connects the Szegö projection operator with the Hardy projection operator onto the Hardy space of Hermitean monogenic functions defined on a bounded subdomain of even dimensional Euclidean space, and present the Szegö projection operator in explicit terms of the Hardy projection operator and its adjoint in Hermitean Clifford analysis. In the last section we will give the algebraic and geometric characterizations for the matrix Hilbert transform to be unitary in Hermitean Clifford analysis.

2. Preliminaries and notations

In this section we recall some basic facts about Clifford algebra and Hermitean Clifford analysis which will be needed in the sequel. More details can be also seen in the references e.g. [2, 4, 26, 27, 28−31] and [17−23, 25].

Let \{e_1, e_2, \cdots , e_m\} be an orthogonal basis of the Euclidean space \(\mathbb{R}^m\), let \(\mathbb{R}^m\) be endowed with a non-degenerate quadratic form of signature \((0,m)\) and let \(\mathbb{R}_{0,m}\) be the 2\(^n\)-dimensional real Clifford algebra constructed over \(\mathbb{R}^m\) with basis

\[\left\{ e_A : A = \{h_1, \cdots , h_r\} \in \mathcal{P}N, 1 \leq h_1 < h_2 \leq m \right\},\]

where \(N\) stands for the set \(\{1,2,\cdots ,m\}\) and \(\mathcal{P}N\) denotes for the family of all order-preserving subsets of \(N\). We denote \(e_0\) as \(e_0\) and \(e_A\) as \(e_{h_1\cdots h_r}\) for \(A = \{h_1, \cdots , h_r\} \in \mathcal{P}N\). The product in \(\mathbb{R}_{0,m}\) is defined by

\[\left\{ \begin{array}{l}
 e_{AB} = (-1)^{N(A\cap B)}(-1)^{P(A,B)}e_{AAB}, \quad \text{if } A, B \in \mathcal{P}N, \\
 \lambda \mu = \sum_{A,B \in \mathcal{P}N} \lambda_{AB}e_{AB}, \quad \text{if } \lambda = \sum_{A \in \mathcal{P}N} \lambda_{A}e_{A}, \mu = \sum_{B \in \mathcal{P}N} \mu_{B}e_{B},
\end{array} \right.\]

where \(N(A)\) is the cardinal number of the set \(A\), and \(P(A,B) = \sum_{j \in B} P(A,j)\), with \(P(A,j) = N\{i : i \in A, i > j\}\). It follows \(e_0\) is the identity element, now written as 1 and that in particular

\[\left\{ \begin{array}{l}
 e_i^2 = -1, \quad \text{if } i = 1, 2, \cdots , m, \\
e_i e_j + e_j e_i = 0, \quad \text{if } 1 \leq i < j \leq m, \\
e_{h_1} e_{h_2} \cdots e_{h_r} = e_{h_1 h_2 \cdots h_r}, \quad \text{if } 1 \leq h_1 < h_2 < \cdots < h_r \leq m.
\end{array} \right.\]
Thus the real Clifford algebra \( \mathbb{R}_{0,m} \) is a real linear, associative, but non-commutative algebra.

Any Clifford number \( a \) in \( \mathbb{R}_{0,m} \) may thus be written as \( a = \sum_{N(A)=k} a_A e_A, a_A \in \mathbb{R} \), or still as \( a = \sum_{N(A)=k} [a]_k \), where \( [a]_k = \sum_{N(A)=k} e_A a_A \) is the so-called \( k \)-vector part of a \( (k = 0, 1, 2, \cdots, m) \). The Euclidean space \( \mathbb{R}^m \) is embedded in \( \mathbb{R}_{0,m} \) by identifying \( (x_1, x_2, \cdots, x_m) \) with the Clifford vector \( X \) given by

\[
X = \sum_{j=1}^m e_j x_j.
\]

The conjugation in \( \mathbb{R}_{0,m} \) is defined as follows:

\[
\bar{a} = \sum_A a_A \bar{e}_A, \bar{e}_A = (-1)^{k(k+1)/2} e_A, N(A) = k, a_A \in \mathbb{R}.
\]

and hence

\[
\overline{ab} = \overline{ba}, \text{for arbitrary } a, b \in \mathbb{R}_{0,m}.
\]

Note that the square of a vector \( X \) is scalar valued and equals the norm squared up to a minus sign \( X^2 = -\langle X, X \rangle = -|X|^2 \). The dual of the vector \( X \) is the vector valued first order differential operator

\[
\partial_X = \sum_{j=1}^m e_j \partial x_j
\]

is called Dirac operator. It is precisely this Dirac operator which underlies the notion of monogenicity of a function, a notion which is the higher dimensional counterpart or holomorphy in the complex plane. As the Dirac operator factorizes the Laplacian, \( \Delta_m = -\partial_X^2 \), monogenicity can be regarded as a refinement of harmonicity. We refer to this setting as the orthogonal case, since the fundamental group leaving the Dirac operator \( \partial_X \) invariant is the special orthogonal group \( SO(m; \mathbb{R}) \), which is doubly covered by the Spin(m) group of the Clifford algebra \( \mathbb{R}_{0,m} \). For this reason, the Dirac operator is called a rotation invariant operator.

When allowing for complex constants and moreover taking the dimension to be even, say \( m = 2n \), the same set of generators as above, \( \{e_1, e_2, \cdots, e_{2n}\} \), still satisfying the above defining relation, may in fact also produce the complex Clifford algebra \( \mathbb{C}_{2n} \). As \( \mathbb{C}_{2n} \) is the complexification of the real Clifford algebra \( \mathbb{R}_{0,2n} \), i.e. \( \mathbb{C}_{2n} = \mathbb{R}_{0,2n} \oplus i \mathbb{R}_{0,2n} \), any complex Clifford number \( \lambda \in \mathbb{C}_{2n} \) may be written as \( \lambda = a + ib, a, b \in \mathbb{R}_{0,2n} \), leading to the Hermitean conjugation \( \lambda^\dagger = (a + ib)^\dagger = \bar{a} - i\bar{b} \), where the bar denotes the usual Clifford conjugation in \( \mathbb{R}_{0,2n} \), i.e. the main anti-involution for which \( \bar{e}_j = -e_j, j = 1, 2, \cdots, 2n \). This Hermitean conjugation leads to a Hermitean inner product and its associated norm on \( \mathbb{C}_{2n} \) given by \( \langle \lambda, \mu \rangle = [\lambda^\dagger \mu]_0 \) and \( |\lambda| = \sqrt{\langle \lambda^\dagger \lambda \rangle}_0 = \left( \sum_A |\lambda_A|^2 \right)^{1/2} \). The
above framework will be referred to as the Hermitean Clifford analysis, as opposed to traditional orthogonal Clifford one. Hermitean Clifford analysis then focuses on simultaneous null solutions of two Hermitean Dirac operators $\partial Z$ and $\partial Z^\dagger$, introduced as follows.

One of the ways for introducing Hermitean Clifford analysis is by considering the complex Clifford algebra $\mathbb{C}_{2n}$ and a so-called complex structure on it, i.e. an $SO(2n, \mathbb{R})$-element $J$ for which $J^2 = -1$ (see e.g. [17 – 20]). More specifically, $J$ is chosen to act upon the generators $e_1, e_2, \ldots, e_{2n}$ of the Clifford algebra as

$$J[e_j] = -e_{n+j} \quad \text{and} \quad J[e_{n+j}] = e_j, \quad j = 1, 2, \ldots, n.$$ 

Let us recall that the main objects of the Hermitean setting are then conceptually obtained by considering the projection operators $\frac{1}{2}(1 \pm iJ)$ and letting them act on the corresponding protagonists of the orthogonal framework. First of all, the so-called Witt basis elements $\{f_j, f_j^\dagger | j = 1, 2, \ldots, n\}$ for the complex Clifford algebra $\mathbb{C}_{2n}$ are obtained through the action of $\frac{1}{2}(1 \pm iJ)$ on the orthogonal basis elements $e_j$:

$$f_j = \frac{1}{2}(1 + iJ)[e_j] = \frac{1}{2}(e_j - ie_{n+j}), \quad j = 1, 2, \ldots, n,$$

$$f_j^\dagger = -\frac{1}{2}(1 - iJ)[e_j] = -\frac{1}{2}(e_j + ie_{n+j}), \quad j = 1, 2, \ldots, n.$$

These Witt basis elements satisfy the Grassmann identities

$$f_j f_k + f_k f_j = f_j^\dagger f_k^\dagger + f_k^\dagger f_j^\dagger = 0, \quad j, k = 1, 2, \ldots, n,$$

and the duality identities

$$f_j f_k^\dagger + f_k^\dagger f_j = \delta_{jk}, \quad j, k = 1, 2, \ldots, n.$$

Next we identify a vector $\mathbf{X} = (X_1, X_2, \ldots, X_{2n}) = (x_1, x_2, \ldots, x_n, y_1, \ldots, y_n)$ in $\mathbb{R}^{2n}$ with the Clifford vector $\mathbf{X} = \sum_{j=1}^{n} (e_j x_j + e_{n+j} y_j)$ and we denote by $\mathbf{X}|$ the action of the complex structure $J$ on $\mathbf{X}$, i.e.

$$\mathbf{X}| = J[\mathbf{X}] = \sum_{j=1}^{n} (e_j y_j - e_{n+j} x_j).$$

Note that the vectors $\mathbf{X}$ and $\mathbf{X}|$ are orthogonal w.r.t. the standard Euclidean scalar product, which implies that the Clifford vectors $\mathbf{X}$ and $\mathbf{X}|$ are both anti-commutative. The Hermitean Clifford variables $Z$ and $Z^\dagger$ then arise through the action of the projection operators on the standard Clifford vector $\mathbf{X}$:

$$Z = \frac{1}{2}(1 + iJ)[\mathbf{X}] = \frac{1}{2}(\mathbf{X} + i\mathbf{X}|),$$

$$Z^\dagger = -\frac{1}{2}(1 - iJ)[\mathbf{X}] = -\frac{1}{2}(\mathbf{X} - i\mathbf{X}|).$$
They can be rewritten in terms of the Witt basis elements as

\[ Z = \sum_{j=1}^{n} f_j z_j, \] and \[ Z^\dagger = (Z)^\dagger = \sum_{j=1}^{n} f_j^\dagger z_j^\dagger, \]

where \( n \) complex variables \( z_j = x_j + iy_j \) have been introduced, with complex conjugates \( z_j^c = x_j - iy_j, j = 1, 2, \cdots, n \). Finally, the Hermitian Dirac operators \( \partial_Z \) and \( \partial_{Z^\dagger} \) are derived out of the orthogonal Dirac operator \( \partial_X \):

\[ \partial_{Z^\dagger} = \frac{1}{4} (1 + iJ) [\partial_X] = \frac{1}{4} (\partial_X + i\partial_X), \] and
\[ \partial_Z = -\frac{1}{4} (1 - iJ) [\partial_X] = -\frac{1}{4} (\partial_X - i\partial_X), \]

where we have introduced \( \partial_X = J[\partial_X] = \sum_{j=1}^{n} (e_j \partial_{y_j} - e_{n+j} \partial_{x_j}) \).

In terms of the Witt basis elements, the Hermitian Dirac operators are expressed as

\[ \partial_Z = \sum_{j=1}^{n} f_j^{\dagger} \partial z_j, \] and \[ \partial_{Z^\dagger} = (\partial_Z)^\dagger = \sum_{j=1}^{n} f_j \partial z_j^c, \]

involving the classical Cauchy-Riemann operators \( \partial z_j = \frac{1}{2}(\partial x_j - i\partial y_j) \) and their complex conjugates \( \partial z_j^c = \frac{1}{2}(\partial x_j + i\partial y_j) \) in the complex \( z_j \)-planes, \( j = 1, 2, \cdots, n \). The Hermitean Dirac operators \( \partial_Z \) and \( \partial_{Z^\dagger} \) are invariant under the action of a realization, denoted \( \tilde{U}(n) \), of the unitary group in terms of the Clifford algebras (see e.g. [17, 19]). The group \( \tilde{U}(n) \subset \text{Spin}(2n) \) is given by

\[ \tilde{U}(n) = \left\{ s \in \text{Spin}(2n) | \exists \theta \geq 0 : sI = e^{-i\theta} I \right\} \]

its definition involving the self-adjoint primitive idempotent \( I = I_1 I_2 \cdots I_n \), with \( I_j = f_j f_j^\dagger = \frac{1}{2} (1 - ie_j e_{n+j}), j = 1, 2, \cdots, n \).

Finally observe for further use that the Hermitean vector variables and Dirac operators are isotropic, i.e.

\[ (Z)^2 = (Z^\dagger)^2 = 0 \] and \( (\partial_Z)^2 = (\partial_{Z^\dagger})^2 = 0 \).

Whence the Laplacian \( \Delta_{2n} = -\partial_Z^2 = -\partial_{Z^\dagger}^2 \) allows the decomposition

\[ \Delta_{2n} = 4(\partial_Z \partial_{Z^\dagger} + \partial_{Z^\dagger} \partial_Z) \]

and one also has that

\[ Z \partial_{Z^\dagger} + Z^\dagger \partial_Z = |Z|^2 = |Z^\dagger|^2 = |X|^2 = |X^\dagger|^2. \]
For further use, we introduce the Hermitean oriented surface elements $d\sigma_Z$ and $d\sigma_Z^\dagger$ as follows

$$\varepsilon(Z) = \frac{2}{w_{2n}} \frac{Z}{|Z|^{2n}}$$ and $$\varepsilon^\dagger(Z) = \frac{2}{w_{2n}} \frac{Z^\dagger}{|Z|^{2n}}.$$ 

$$d\sigma_Z = \sum_{j=1}^n f_j^\dagger \widehat{dz}_j$$ and $$d\sigma_Z^\dagger = \sum_{j=1}^n f_j \widehat{dz}_j^\dagger.$$ 

Explicitly,

$$d\sigma_Z = -\frac{1}{4} (-1)^{\frac{n(n+1)}{2}} (2i)^n \left( d\sigma_X - id\sigma_X^\dagger \right),$$

$$d\sigma_Z^\dagger = -\frac{1}{4} (-1)^{\frac{n(n+1)}{2}} (2i)^n \left( d\sigma_X + id\sigma_X^\dagger \right),$$

$$\varepsilon = -(E+iE|), \quad \varepsilon^\dagger = (E-iE|),$$

where $d\sigma_X$ denotes the vector valued oriented surface element and $d\sigma_X^\dagger = J[d\sigma_X^\dagger]$. They are explicitly given by means of the following differential forms of order $2n-1$

$$d\sigma_X = \sum_{j=1}^n \left( e_j (-1)^{j-1} \widehat{dx}_j + e_{n+j} (-1)^{n+j-1} \widehat{dy}_j \right),$$

$$d\sigma_X^\dagger = \sum_{j=1}^n \left( e_j (-1)^{n+j-1} \widehat{dy}_j + e_{n+j} (-1)^j \widehat{dx}_j \right),$$

with

$$\widehat{dx}_j = dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n,$$

$$\widehat{dy}_j = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_j \wedge dy_{j+1} \wedge \cdots \wedge dy_n.$$ 

We denote the outward pointing unit normal vector at $X$ on $\partial \Omega$ by $\nu(X)$, $dS(X)$ stands for the classical element on $\partial \Omega$, then

$$d\sigma_X = \nu(X)dS(X), \quad d\sigma_X^\dagger = \nu^\dagger(X)dS(X).$$

In this context the functions taking $\mathbb{C}_{2n}$-valued defined on an open region $\Omega$ of $\mathbb{R}^{2n}$ will be considered. The continuity, continuously differentiability, $L_p (1 < p < +\infty)$-integrable and so on of the function $f = \sum A f_A : \Omega(\subset \mathbb{R}^{2n}) \rightarrow \mathbb{C}_{2n}$ where $f_A : \Omega(\subset \mathbb{R}^{2n}) \rightarrow \mathbb{C}$, the space of which are denoted respectively by $C(\Omega, \mathbb{C}_{2n})$, $C^1(\Omega, \mathbb{C}_{2n})$, $L_p(\Omega, \mathbb{C}_{2n})$ and so on, are ascribed to each component $f_A$ which are respectively continuous, continuously differential, $L_p$-integrable and so on. A function $f(X)$ defined and differentiable in an open region $\Omega$ of $\mathbb{R}^{2n}$ with its boundary $\partial \Omega$ and taking values in $\mathbb{C}_{2n}$ is called (left) monogenic in $\Omega$ if $\partial_X f(X) = 0$. 

We introduce the particular circulant \((2 \times 2)\) matrices
\[
D_{(Z,Z^\dagger)} = \begin{pmatrix} \partial_Z \partial_{Z^\dagger} & \partial_{Z^\dagger} \partial_Z \end{pmatrix}, \quad (D_{(Z,Z^\dagger)})^\dagger = \begin{pmatrix} \partial_{Z^\dagger} \partial_Z & \partial_Z \partial_{Z^\dagger} \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} \varepsilon \varepsilon^\dagger \\ \varepsilon^\dagger \varepsilon \end{pmatrix}, \quad \delta = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix},
\]
then \(D_{(Z,Z^\dagger)} \mathcal{E} = \delta(Z)\), i.e. \(\mathcal{E}\) is the fundamental solution of \(D_{(Z,Z^\dagger)}\) (see e.g. [17, 18, 19, 20]).

In the same setting of circulant \((2 \times 2)\) matrices, we consider the functions \(L_1, L_2, L \in \mathcal{C}^1(\Omega, \mathbb{C}_n)\) and the corresponding circulant \((2 \times 2)\) matrix functions in the following
\[
L_2^1 = \begin{pmatrix} L_1 & L_2 \\ L_2 & L_1 \end{pmatrix} \text{ and } L_0 = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}.
\]

In the following context the operations of matrices such as addition and multiplication, and the operations between the complex numbers and the matrices respectively keep to the operation rules of the usual numerical matrices and of multiplication between the complex numbers and the usual numerical matrices.

**Definition 2.1.** Suppose that \(L_2^1(L_0) \in \mathcal{C}^1(\Omega, \mathbb{C}_n)\) which means that each entry of \(L_2^1(L_0)\) belongs to \(\mathcal{C}^1(\Omega, \mathbb{C}_n)\). \(L_2^1(L_0)\) is called as (left) H-monogenic if and only if it satisfies the following system
\[
D_{(Z,Z^\dagger)}L_2^1 = 0 \left(D_{(Z,Z^\dagger)}L_0 = 0\right),
\]
where \(0\) denotes the \((2 \times 2)\) matrix with zero entries. Similarly, it is obvious in the following that \(L_2^1(L_0) \in \mathcal{C}(\partial \Omega, \mathbb{C}_n), \mathbb{H}^p(\partial \Omega, \mathbb{C}_n), L_p(\partial \Omega, \mathbb{C}_2n)(1 < p < +\infty)\) and so on which mean each entry of \(L_2^1(L_0)\) belongs to \(\mathcal{C}(\partial \Omega, \mathbb{C}_2n), \mathbb{H}^p(\partial \Omega, \mathbb{C}_2n), L_p(\partial \Omega, \mathbb{C}_2n)\) and so on.

In the following we introduce
\[
V = \frac{1}{2}(Y + iY^\dagger), V^\dagger = -\frac{1}{2}(Y - iY^\dagger),
\]
\[
dV_{(Z,Z^\dagger)} = (dz_1 \wedge dz_2^\dagger) \wedge (dz_2 \wedge dz_2^\dagger) \ldots \wedge (dz_n \wedge dz_n^\dagger),
\]
where \(dV_{(Z,Z^\dagger)}\) denote the Hermitean volume element.

For the functions \(L_i \in L_p(\partial \Omega, \mathbb{C}_2n)(1 < p < +\infty, i = 1, 2)\), we define the orthogonal Cauchy type integrals as follows
\[
\mathcal{C}[L_i](\mathcal{Y}) \triangleq (C_{\partial \Omega} L_i)(\mathcal{Y}) = \int_{\partial \Omega} E(X - X) \sigma_X L_i(X), \mathcal{Y} \notin \partial \Omega,
\]
\[
\mathcal{C}[|L_i](\mathcal{Y}) \triangleq (C_{|\partial \Omega} L_i)(\mathcal{Y}) = \int_{\partial \Omega} E(|X - X) \sigma_X L_i(X), \mathcal{Y} \notin \partial \Omega,
\]
which are well-defined (see references e.g. [4, 15]), where
\[
E(X) = \frac{1}{w_{2n}} \frac{X}{|X|^{2n}}, \quad E(|X) = \frac{1}{w_{2n}} \frac{|X|}{|X|^{2n}},
\]
and $d\sigma_X, d\sigma_Y$ as above. Then for $Y \notin \partial \Omega$,
\[
\partial \Sigma C[L_i](Y) = 0, \partial \Sigma C[[L_i](Y) = 0 (i = 1, 2).
\]

For the functions $L_1^2, L_0 \in L_p(\partial \Omega, \mathbb{C}_{2n})$, the Hermitean Cauchy type integrals are defined by
\[
(2.1) \quad [CL_2^2](Y) = \int_{\partial \Omega} \mathcal{E}(Z - V) d\Sigma(Z, Z^1)^2 L_2(X), Y \notin \partial \Omega,
\]
\[
(2.2) \quad [CL_0^0](Y) = \int_{\partial \Omega} \mathcal{E}(Z - V) d\Sigma(Z, Z^1)^2 L_0(X), Y \notin \partial \Omega,
\]
where
\[
d\Sigma(Z, Z^1) = \begin{pmatrix} d\sigma_Z & -d\sigma_{Z^1} \\ -d\sigma_{Z^1} & d\sigma_Z \end{pmatrix}
\]
with $d\sigma_Z$ and $d\sigma_{Z^1}$ as above.

In the following we introduce the vector space
\[
\mathcal{L}_2(\partial \Omega) = \left\{ \mathcal{L}_2 = \begin{pmatrix} L_1 & L_2 \\ L_2 & L_1 \end{pmatrix} \big| L_1, L_2 \in \mathcal{L}_2(\partial \Omega, \mathbb{C}_{2n}) \right\},
\]
on which, inspired by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}_2}$ on $\mathcal{L}_2(\partial \Omega, \mathbb{C}_{2n})$ given by
\[
\langle L_1, L_2 \rangle_{\mathcal{L}_2} = \left[ \int_{\partial \Omega} L_1^*(X) L_2(X) dS_X \right]_0,
\]
where $[\cdot]_0$ denotes the scale part of any $\cdot$ in $\mathbb{C}_{2n}$. We introduce the following bilinear form
\[
\langle \cdot, \cdot \rangle_{\mathcal{L}_2} : \mathcal{L}_2(\partial \Omega) \times \mathcal{L}_2(\partial \Omega) \rightarrow \mathbb{C},
\]
\[
\left\langle \begin{pmatrix} L_1 & L_2 \\ L_2 & L_1 \end{pmatrix}, \begin{pmatrix} K_1 & K_2 \\ K_2 & K_1 \end{pmatrix} \right\rangle_{\mathcal{L}_2} \mapsto \langle L_1, K_1 \rangle_{\mathcal{L}_2} + \langle L_2, K_2 \rangle_{\mathcal{L}_2}.
\]
Then by directly calculating, for arbitrary $\mathcal{L}_2^1, \mathcal{K}_2^1, \mathcal{G}_2^1 \in \mathcal{L}_2(\partial \Omega, \mathbb{C}_{2n})$ where $\mathcal{G}_2^1$ is defined similarly to $\mathcal{L}_2^1$ as above and arbitrary $\lambda \in \mathbb{C}$, we can check
\[
(i) \quad \langle \mathcal{L}_2^1, \lambda \mathcal{K}_2^1 + \mathcal{G}_2^1 \rangle_{\mathcal{L}_2} = \lambda \langle \mathcal{L}_2^1, \mathcal{K}_2^1 \rangle_{\mathcal{L}_2} + \langle \mathcal{L}_2^1, \mathcal{G}_2^1 \rangle_{\mathcal{L}_2},
\]
\[
(ii) \quad \left( \langle \mathcal{L}_2^1, \mathcal{K}_2^1 \rangle_{\mathcal{L}_2} \right)^\dagger = \langle \mathcal{K}_2^1, \mathcal{L}_2^1 \rangle_{\mathcal{L}_2},
\]
\[
(iii) \quad \langle \mathcal{L}_2^1, \mathcal{L}_2^1 \rangle_{\mathcal{L}_2} \geq 0 \text{ and } \langle \mathcal{L}_2^1, \mathcal{L}_2^1 \rangle_{\mathcal{L}_2} = 0 \text{ if and only if } \mathcal{L}_2^1 = 0,
\]
where the operator $(\cdot)^\dagger$ as above and $0$ denotes the $(2 \times 2)$ matrix with zero entries. Therefore $\langle \cdot, \cdot \rangle_{\mathcal{L}_2}$ is an inner product, which derives the norm on $\mathcal{L}_2(\partial \Omega)$ by
\[
\| \mathcal{L}_2^1 \| = \sqrt{\langle L_1, L_1 \rangle_{\mathcal{L}_2} + \langle L_2, L_2 \rangle_{\mathcal{L}_2}}.
\]
Hence $\left( \mathcal{L}_2(\partial \Omega), \| \cdot \| \right)$ is the Hilbert space which is different from the space of $\mathcal{L}_2(\partial \Omega)$ in references e.g. [23, 25]. Under this setting, we have the following
Lemma without proof which was also stated in [23, 25] in the sense of different topology. For convenience without confusion and ambiguity, \( \mathcal{L}_2(\partial \Omega), \| \cdot \| \) still denotes \( \mathcal{L}_2(\partial \Omega) \) in the following context.

**Lemma 2.1.** Suppose that \( \Omega \) is an open bounded subset of \( \mathbb{R}^{2n} \) with smooth boundary \( \partial \Omega \). The functions \( [\mathcal{CL}_2^1](X) \) and \( [\mathcal{CL}_0^1](X) \) are defined similarly to \( [\mathcal{CL}_2^1](X) \) and \( [\mathcal{CL}_0^1](X) \) as above. If the functions \( \mathcal{L}_2^1(X), \mathcal{L}_0^1(X) \in \mathbf{L}_p(\partial \Omega, C_{2n}) \), \( (1 < p < +\infty) \), then for arbitrary \( T \in \partial \Omega \),

(i) for arbitrary \( X \in \mathbb{R}^{2n} \setminus \partial \Omega, D_{(Z, \bar{Z})} \mathcal{L}_2^1(X) = 0 \), \( D_{(Z, \bar{Z})} \mathcal{L}_0^1(X) = 0 \),

i.e. \( \mathcal{L}_2^1(X), \mathcal{L}_0^1(X) \) are both \( H \)-monogenic,

(ii) \( [\mathcal{CL}_2^1](T) \triangleq \lim_{\Omega \ni \tilde{X} \to T} [\mathcal{CL}_2^1](X) = (-1)^{(n+1)/2} \left( \pm \mathcal{L}_2^1(T) + [H \mathcal{L}_2^1](T) \right) \),

\( [\mathcal{CL}_0^1](T) \triangleq \lim_{\Omega \ni \tilde{X} \to T} [\mathcal{CL}_0^1](X) = (-1)^{(n+1)/2} \left( \pm \mathcal{L}_0^1(T) + [H \mathcal{L}_0^1](T) \right) \),

(iii) \( [\mathcal{CL}_2^1](T) \in \mathbf{L}_p(\partial \Omega, C_{2n}) \left( [\mathcal{CL}_0^1](T) \in \mathbf{L}_p(\partial \Omega, C_{2n}) \right) \),

where the limits of (ii) mean the the non-tangential limits and it is the same in this context,

\[
H = \frac{1}{2} \left( \begin{array}{cc} H + H & -H + H \\ -H + H & H + H \end{array} \right),
\]

and

\[
[Hf](T) = \text{p.v.} \int_{\partial \Omega} E(Y - T) d\sigma_Y f(Y), T \in \partial \Omega,
\]

\[
[Hf](T) = \text{p.v.} \int_{\partial \Omega} E(|Y - T|) d\sigma_Y f(Y), T \in \partial \Omega,
\]

which are both Cauchy principle value integrals in the sense of \( \mathbf{L}_p(1 < p < +\infty) \).

When the variables are omitted without confusion and ambiguity, for convenience \( [Hf](T), [Hf](T) \) are for short of \( Hf, Hf \) respectively and it is similar in the following context.

### 3. Szegö projection

In this section, we will introduce the Szegö projection operator for the Hardy space of Hermitean monogenic functions defined on a bounded subdomain of even dimensional Euclidean space, establish the Kerzman-Stein formula which is closely related to the Szegö projection and the Hardy projection for the Hardy space of Hermitean monogenic functions defined on a bounded subdomain of even dimensional Euclidean space, and get the Szegö projection operator in explicit terms of the Hardy projection operator and its adjoint in the setting of Hermitean Clifford analysis.
In what follows we will consider the Hardy space
\[ \mathbb{H}^2(\Omega) = \left\{ \mathcal{L}_2^1 : \Omega \to (\mathbb{C}_n)^{2 \times 2} \mid D_{(Z,Z')} \mathcal{L}_2^1 = 0 \text{ and } L_1|_{\partial \Omega}, L_2|_{\partial \Omega} \in \mathcal{L}_2(\partial \Omega, \mathbb{C}_{2n}) \right\} \]
and \( \mathbb{H}^2(\partial \Omega) \) denotes the \( \mathcal{L}_2(\partial \Omega) \)-closure of the set of boundary values of elements of \( \mathbb{H}^2(\Omega) \). Then associating Lemma 2.1, the Hermitean Cauchy transform \( C \) maps \( \mathcal{L}_2(\partial \Omega, \mathbb{C}_{2n}) \) onto \( \mathbb{H}^2(\partial \Omega) \) for arbitrary \( \mathcal{L}_2^1 \in \mathcal{L}_2(\partial \Omega, \mathbb{C}_{2n}) \), which is skew and so-called the Hardy projection.

By the same argument in [23], associating the definition of the above \( \mathbb{C} \)-valued inner product on \( \mathcal{L}_2(\partial \Omega) \), we have the following Lemma which is only stated without proof.

**Lemma 3.1** Suppose that \( \mathcal{L}_2(\partial \Omega), \nu \) and \( \mathbb{H}^2(\partial \Omega) \) as above. Then

1. \( H^2 = I \),
2. \( H^* = \nu H \nu \),
3. for arbitrary \( \mathcal{L}_2^1 \in \mathcal{L}_2(\partial \Omega), \mathcal{H} \mathcal{L}_2^1 = \mathcal{L}_2^1 \) if and only if \( \mathcal{L}_2^1 \in \mathbb{H}^2(\partial \Omega) \),
4. \( \mathcal{L}_2^1 \in \mathcal{L}_2(\partial \Omega) = \mathbb{H}^2(\partial \Omega) \oplus \nu \mathbb{H}^2(\partial \Omega) \) (w.r.t. \( \langle \cdot, \cdot \rangle_{\mathcal{L}_2} \)),

where
\[ \nu = \frac{1}{2} \left( \nu + \nu | - \nu + \nu \right). \]

**Remark 3.1** The same results in Lemma 3.1 were gotten in [23] with respect to \((\mathbb{C}_n)^{2 \times 2}\)-valued inner product which is different from the above \( \mathbb{C} \)-valued inner product on \( \mathcal{L}_2(\partial \Omega) \).

The orthogonal projection operator \( S \) of \( \mathcal{L}_2(\partial \Omega) \) onto \( \mathbb{H}^2(\partial \Omega) \), which is so-called the Szegö projection operator, may be Hermitean monogenically extended to \( \mathbb{H}^2(\Omega) \) by
\[ \mathcal{S}(\mathcal{L}_2^1(X)) = \int_{\partial \Omega} \mathcal{S}_X(Y) \mathcal{L}_2^1(Y) dS_Y, \]
where \( \mathcal{S}_X(Y) \) is so-called the Szegö kernel.
That is, for arbitrary \( X \in \Omega \),
\[ \mathcal{S}(\mathcal{L}_2^1(X)) = \int_{\partial \Omega} \mathcal{S}_X(Y) \mathcal{L}_2^1(Y) dS_Y = \mathcal{L}_2^1(X). \]

**Remark 3.2** Particularly when \( \Omega = B(1) \) the unit ball centered at 0 of \( \mathbb{R}^{2n} \), \( \partial \Omega = S^{2n-1} \) the unit sphere of \( \mathbb{R}^{2n} \) and \( \nu(W) = W, \nu(W) = W \) for arbitrary \( W \in S^{2n-1} \). Then
\[ \mathcal{L}_2(S^{2n-1}) = \mathbb{H}^2(S^{2n-1}) \oplus \nu|_{S^{2n-1}} \mathbb{H}^2(S^{2n-1}), \]
where
\[ \nu|_{S^{2n-1}} = \frac{1}{2} \left( \frac{W + W^2}{W + W^2} - \frac{W + W}{W + W^2} \right). \]
Given the boundary data $\mathcal{L}_1^1 \in \mathbf{L}_2(S^{2n-1}, \mathbb{C}_{2n})$, find the function $\mathcal{K}_2^1$ such that

\[
\begin{cases}
\Delta \mathcal{K}_2^1(X) = 0, X \in B(1), \\
\mathcal{K}_2^1(X) = \mathcal{L}_2^1(X), X \in S^{2n-1},
\end{cases}
\Rightarrow
\begin{cases}
\Delta_2n \mathcal{K}_1(X) = 0, X \in B(1), \\
\mathcal{K}_1(W) = L_1(W), W \in S^{2n-1}, \\
\Delta_2n \mathcal{K}_2(X) = 0, X \in B(1), \\
\mathcal{K}_2(W) = L_2(W), W \in S^{2n-1},
\end{cases}
\]

where $\mathcal{L}_2^1$ as above, $\mathcal{K}_2^1 = \begin{pmatrix} K_1 & K_2 \\ K_2 & K_1 \end{pmatrix}$ is defined similarly to $\mathcal{L}_2^1$ and $\Delta = \begin{pmatrix} \Delta_2n & 0 \\ 0 & \Delta_2n \end{pmatrix}$.

In virtue of (iv) in Lemma 3.1, we have

$\mathcal{L}_2^1 = \mathcal{G}_2^1 + \nu \mathcal{H}_2^1,$

where $\mathcal{G}_2^1, \mathcal{H}_2^1 \in \mathbb{H}^2(S^{2n-1})$ are defined similarly to $\mathcal{L}_2^1$.

Then the above Dirichlet problem exists the unique solution. Moreover the solution is formulated in the following form

$\mathcal{K}_2^1(X) = \tilde{\mathcal{G}}_2^1 + \mathcal{X}\tilde{\mathcal{H}}_2^1, X \in B(1),$

where $\tilde{\mathcal{G}}_2^1, \tilde{\mathcal{H}}_2^1 \in \mathbb{H}^2(B(1))$ are Hermitean monogenic extension of $\mathcal{G}_2^1, \mathcal{H}_2^1$, respectively (i.e. $\mathcal{G}_2^1, \mathcal{H}_2^1$ are the non-tangential boundary value limits of $\tilde{\mathcal{G}}_2^1, \tilde{\mathcal{H}}_2^1$)

and $\mathcal{X} = \begin{pmatrix} X+X| & -X+X| \\ -X+X| & X+X| \end{pmatrix}$.

In what follows, we introduce the matrix Kerzman-Stein operator on $\mathcal{L}_2(\partial \Omega)$ by

$\mathcal{A} = \frac{1}{2} \begin{pmatrix} A + A| & -A + A| \\ -A + A| & A + A| \end{pmatrix}$,

where $A = \mathcal{C} - \mathcal{C}^*$ and $A| = \mathcal{C}| - \mathcal{C}^*|$ are both well-defined, with $\mathcal{C}^*$ and $\mathcal{C}|$ respectively denoting the adjoint operators of $\mathcal{C}$ and $\mathcal{C}|$ on the Hilbert space of $\mathcal{L}_2(\partial \Omega, \mathbb{C}_{2n})$ given by

$\mathcal{C}^* = \frac{1}{2}(1 + \nu \mathcal{H} \nu) : \mathbf{L}_2(\partial \Omega, \mathbb{C}_{2n}) \to H^2(\partial \Omega),$

$\mathcal{C}|^* = \frac{1}{2}(1 + \nu| \mathcal{H} |\nu|) : \mathbf{L}_2(\partial \Omega, \mathbb{C}_{2n}) \to H^2(\partial \Omega),$

with

$H^2(\partial \Omega) = \left\{ L_1 : \Omega \to \mathbb{C}_{2n} | \partial_X L_1 = 0 \text{ and } L_1|_{\partial \Omega} \in \mathbf{L}_2(\partial \Omega, \mathbb{C}_{2n}) \right\}$

and $\nu, \nu|, \mathcal{H}$ as above and $1$ being the identity operator. More detail can be seen in [10, 12, 11].

Applying Lemma 3.1, we directly get
Lemma 3.2 Suppose that $A$ and $A|_-$ as above. Then

$$A = C - C^* = H - H^*, \ i.e. \ A = H - \nu H_L,$$

where $H^*$ as above and $C^* = \frac{1}{2}(I + H^*)$ mean the adjoint operators of $H$ and $C$ on $L_2(\partial \Omega)$.

Theorem 3.1 Suppose that $S$ and $C$ as above. Then

$$S(I + A) = C,$$

where $I$ denote $(2 \times 2)$ identity matrix operator.

Proof Since the operator $S$ is orthogonal projection operator on the Hilbert space $L_2(\partial \Omega)$, applying the property of the orthogonal operator on the Hilbert space (see reference e.g. [32]), then $S^*$ is well-defined, where $S^*$ means the adjoint operator of $S$. Moreover, $S$ is the self-adjoint operator on $L_2(\partial \Omega)$, that is, $S = S^*$.

Then as operators from $L_2(\partial \Omega)$ to $H^2(\partial \Omega)$,

$$SC = C \text{ and } CS = S.$$

Applying the property of the adjoint operator on the Hilbert space of $L_2(\partial \Omega)$ (see reference e.g. [32]), $(SC)^*$ is well-defined and $(SC)^* = C^*S^*$, where $C^*$ means the adjoint operator on $L_2(\partial \Omega)$. Taking the adjoint operators with respect to $\langle . , . \rangle_{L_2}$, we have

$$C^*S = (SC)^* = C^* \text{ and } SC^* = (CS)^* = S.$$

Hence

$$S = SC - SC^* = S - C.$$

Therefore

$$S(I + A) = C,$$

where $I$ denotes $(2 \times 2)$ identity matrix operator.

The proof of the result completes.

Remark 3.3 Theorem 3.1 characterizes the relation between Hermitian Hardy projection operator and Szegö projection operator, which is the generalization of classical Kerzman-Stein formula in the setting of Hermitian Clifford analysis.

We define the matrix operator as follows

$$B = \frac{1}{2} \left( \begin{array}{cc} (1 + A)^{-1} + (1 + A|^1)^{-1} & -(1 + A)^{-1} + (1 + A|^1)^{-1} \\ -(1 + A)^{-1} + (1 + A|^1)^{-1} & (1 + A)^{-1} + (1 + A|^1)^{-1} \end{array} \right),$$

where $1$ denotes the identity operator on $L_2(\partial \Omega, \mathbb{C}_{2n})$.

Applying Lemma 4.5 in [10], the operator $1 + A$ and $1 + A|$ are invertible on $L_2(\partial \Omega, \mathbb{C}_{2n})$ (also see references [11, 32] or elsewhere), the matrix operator $B$ is well defined on $L_2(\partial \Omega)$. 
Theorem 3.2 Suppose that $S$ and $C$ as above. Then the Szegö projection operator is explicitly formulated by

\begin{equation}
S = C(I + A)^{-1}.
\end{equation}

where I denote $(2 \times 2)$ identity matrix operator.

Proof Since the matrix operators $I + A$ and $B$ as above, calculating directly, we get

\begin{equation}
I = \begin{pmatrix}
2 + A + A| & -A + A| \\
-A + A| & 2 + A + A|
\end{pmatrix}
\end{equation}

where

\begin{equation}
M = \begin{pmatrix}
(1 + A)^{-1} + (1 + A|)^{-1} & -(1 + A)^{-1} + (1 + A|)^{-1} \\
-(1 + A)^{-1} + (1 + A|)^{-1} & (1 + A)^{-1} + (1 + A|)^{-1}
\end{pmatrix}.
\end{equation}

Then

\begin{equation}
(I + A)B = I,
\end{equation}

i.e. the matrix operator $I + A$ is invertible and its inverse is given by

\begin{equation}
(I + A)^{-1} = B.
\end{equation}

So it follows that

\begin{equation}
S = C(I + A)^{-1}.
\end{equation}

4. Characterization of matrix Hilbert transform

In this section, we will give the algebraic and geometric characterizations for the matrix Hilbert transform to be unitary in Hermitean Clifford analysis, which is analogous to the corresponding characterization of the Hilbert transform in classical analysis and orthogonal Clifford analysis.

In the sequel we introduce the following functions

\begin{equation}
\alpha(X) = \frac{1}{2}(1 + iv(X)), \quad \beta(X) = \frac{1}{2}(1 - iv(X)), X \in \mathbb{R}^{2n},
\end{equation}

\begin{equation}
\alpha| (X) = \frac{1}{2}(1 + iv|(X)), \quad \beta|(X) = \frac{1}{2}(1 - iv|(X)), X \in \mathbb{R}^{2n}.
\end{equation}

By directly calculating, it is easy to obtain the Lemma as follows.

Lemma 4.1 Suppose that $\alpha(X), \alpha|(X)$ and $\beta(X), \beta|(X)$ as above. Then

\begin{equation}
(i) \alpha^2(X) = \alpha(X), \beta^2(X) = \beta(X),
\end{equation}

\begin{equation}
(ii) \alpha(X)\beta(X) = 0, \alpha|(X)\beta|(X) = 0,
\end{equation}

\begin{equation}
(iii) \alpha|(X) = \alpha^\dagger(X), \beta|(X) = \beta^\dagger(X),
\end{equation}

\begin{equation}
(iv) \alpha(X) + \beta(X) = 1.
\end{equation}
Related results can be also found in [12, 16, 25] or monographs on Fourier analysis elsewhere. In what follows we introduce matrix functions

\[
\alpha = \frac{1}{2} \begin{pmatrix} \alpha + \alpha| - \alpha + \alpha| \\ -\alpha + \alpha| \alpha + \alpha| \end{pmatrix}, \quad \beta = \frac{1}{2} \begin{pmatrix} \beta + \beta| - \beta + \beta| \\ -\beta + \beta| \beta + \beta| \end{pmatrix},
\]

where \( \alpha, \beta, \alpha|, \beta| \) are for short of \( \alpha(\mathbf{X}), \beta(\mathbf{X}), \alpha|(\mathbf{X}), \beta|(\mathbf{X}) \). In the following context when without confusion and ambiguity, the independent variable of considered functions are omitted.

Making use of the above Lemma 4.1 and directly calculating of the matrix functions, we get the following Lemma.

**Lemma 4.2** Suppose that \( \alpha \) and \( \beta \) as above. Then

\[
(i) \quad \alpha \beta = 0,
(ii) \quad \alpha = \alpha^\dagger, \beta = \beta^\dagger,
(iii) \quad \alpha + \beta = 1, \nu^2 = -1,
(iv) \quad \alpha^2 = \alpha, \beta^2 = \beta,
\]

where \( 0, 1 \) denote \( (2 \times 2) \) zero matrix and identity matrix respectively.

Associating Lemmas 4.2, 4.1, we directly get the algebraic decomposition of \( \mathcal{L}_2(\partial \Omega) \) as follows.

**Corollary 4.1** For arbitrary \( \mathcal{L}_1^1, \mathcal{K}_1^1 \in \mathcal{L}_2(\partial \Omega) \),

\[
(i) \quad \langle \alpha \mathcal{L}_1^1, \beta \mathcal{K}_1^1 \rangle_{\mathcal{L}_2} = 0,
(ii) \quad \mathcal{L}_2(\partial \Omega) = \alpha \mathcal{L}_2(\partial \Omega) \oplus \beta \mathcal{L}_2(\partial \Omega) \text{ (w.r.t. } \langle \cdot, \cdot \rangle_{\mathcal{L}_2}).
\]

**Remark 4.1** The above Corollary 4.1 gives the algebraic decomposition of \( \mathcal{L}_2(\partial \Omega) \). The analogous results can be found in [25], based on which the unique solution to the classical Dirichlet problem on the unit ball in Hermitian Clifford analysis is explicitly expressed.
**Theorem 4.1** Suppose that \( \Omega \) be a bounded open domain of \( \mathbb{R}^{2n} \) with smooth boundary \( \partial \Omega \). Let \( \mathcal{L}_2^1 \) and \( 0 \) be as above. Then the following are equivalent

(i) \( \alpha [H\alpha \mathcal{L}_2^1] = 0, \beta [H\beta \mathcal{L}_2^1] = 0 \) for arbitrary \( \mathcal{L}_2^1 \in \mathcal{L}_2(\partial \Omega) \),

(ii) \( H\alpha \mathcal{L}_2^1 = \beta \mathcal{L}_2^1 \) for arbitrary \( \mathcal{L}_2^1 \in \mathbb{H}^2(\partial \Omega) \),

(iii) \( H\beta \mathcal{L}_2^1 = \alpha \mathcal{L}_2^1 \) for arbitrary \( \mathcal{L}_2^1 \in \mathbb{H}^2(\partial \Omega) \),

(iv) \( H\nu \mathcal{L}_2^1 = -\nu \mathcal{L}_2^1 \) for arbitrary \( \mathcal{L}_2^1 \in \mathbb{H}^2(\partial \Omega) \),

(v) \( H \) is unitary, i.e. \( HH^* = H^*H = I \) with \( I \) being identity matrix operator,

(vi) \( A = 0 \),

(vii) \( \Omega \) is a ball,

(viii) \( S_X(Y) = C_X(Y) \),

with \( S_X(Y), C_X(Y) \) denoting the Szegö kernel and the Cauchy kernel respectively, i.e. the Szegö kernel and the Cauchy kernel coincide.

**Proof** "(i) \( \Rightarrow (ii) \). For arbitrary \( \mathcal{L}_2^1 \in \mathbb{H}^2(\partial \Omega) \), by (iii) in Lemma 3.1, we have

\[
H\mathcal{L}_2^1 = \mathcal{L}_2^1, \quad \text{i.e.} \quad \beta H\mathcal{L}_2^1 = \beta \mathcal{L}_2^1 \quad \text{and} \quad \alpha H\mathcal{L}_2^1 = \alpha \mathcal{L}_2^1. \tag{4.6}
\]

Associating the condition (i) and (iii) in Lemma 4.2, we get

\[
\beta H\mathcal{L}_2^1 - \beta [H\beta \mathcal{L}_2^1] = \beta \mathcal{L}_2^1, \tag{4.7}
\]

Making use of the condition \( \alpha [H\alpha \mathcal{L}_2^1] = 0 \), in term of (iii) in Lemma 4.2, one gets

\[
\beta [H\alpha \mathcal{L}_2^1] = \beta \mathcal{L}_2^1. \tag{4.8}
\]

"(ii) \( \Rightarrow (i) \). For arbitrary \( \mathcal{L}_2^1 \in \mathcal{L}_2(\partial \Omega) \), using (iv) in Lemma 3.1, we have

\[
\mathcal{L}_2^1 = G_2^1 + \nu \mathcal{H}_2^1, \tag{4.9}
\]

where \( G_2^1, \mathcal{H}_2^1 \in \mathbb{H}^2(\partial \Omega) \) are defined similarly to \( \mathcal{L}_2^1 \). Therefore

\[
H\alpha \mathcal{L}_2^1 = H\alpha G_2^1 + H\alpha \nu \mathcal{H}_2^1. \]

The condition (ii) acts on \( G_2^1 \in \mathbb{H}^2(\partial \Omega) \), associating (i) in Lemma 4.2, we get

\[
\alpha H\alpha \mathcal{L}_2^1 = \alpha H\alpha G_2^1 + \alpha H\alpha \nu \mathcal{H}_2^1 = \alpha \beta G_2^1 + \alpha H\alpha \nu \mathcal{H}_2^1 = \alpha H\alpha \nu \mathcal{H}_2^1.
\]

i.e.

\[
i\alpha H\alpha \mathcal{L}_2^1 = \alpha H\alpha \nu \mathcal{H}_2^1.
\]

Applying the condition (ii), we have

\[
0 = \alpha \beta \mathcal{H}_2^1 = \alpha H\alpha \mathcal{H}_2^1.
\]

Hence

\[
i\alpha H\alpha \mathcal{L}_2^1 = \alpha H\alpha (1 + i\nu) \mathcal{H}_2^1 = \alpha H\alpha \mathcal{H}_2^1 = \alpha H\alpha \mathcal{H}_2^1.
\]
By the condition (ii), we get
\[ \alpha \mathcal{H} \mathcal{L}_2^1 = \alpha \mathcal{H} \mathcal{H}_2^1 = \alpha \beta \mathcal{H}_2^1 = 0. \]
“(ii) ⇒ (iii)”. For arbitrary \( \mathcal{L}_2^1 \in \mathbb{H}^2(\partial \Omega) \), applying the term (4.6) and the condition (ii), we have
\[ \mathcal{H} \searrow \mathbf{L}_2^1 = \mathcal{H} \mathcal{L}_2^1 - \mathcal{H} \alpha \mathcal{L}_2^1 = \mathcal{L}_2^1 - \beta \mathcal{L}_2^1 = \alpha \mathcal{L}_2^1. \]
“(iii) ⇒ (ii)”. It is similar to the procedure of (iii) ⇒ (ii).
“(iii) ⇒ (iv)”. For arbitrary \( \mathcal{L}_2^1 \in \mathbb{H}^2(\partial \Omega) \), by the condition (iii) and the term (4.6), we get
\[ -i \mathcal{H} \nu \mathcal{L}_2^1 = 2 \mathcal{H} \alpha \mathcal{L}_2^1 - \mathcal{H} \mathcal{L}_2^1 = 2 \alpha \mathcal{L}_2^1 - \mathcal{L}_2^1 = i \nu \mathcal{L}_2^1. \]
Hence
\[ \mathcal{H} \nu \mathcal{L}_2^1 = -i \nu \mathcal{L}_2^1. \]
“(iv) ⇒ (iii)”. Since the procedure of (iii) ⇒ (iv), the result of (iii) follows.
“(iv) ⇒ (v)”. For arbitrary \( \mathcal{L}_2^1 \in \mathbb{L}_2(\partial \Omega) \), using (iv) in Lemma 3.1 and the term (4.9), we have
\[ \mathcal{H}^* \mathcal{L}_2^1 = \mathcal{H}_2 \mathcal{L}_2^1 + \mathcal{H} \nu \mathcal{H}^2 \mathcal{L}_2^1. \]
Making use of the condition (iv), we get
\[ \mathcal{H}^* \mathcal{L}_2^1 = -\mathcal{H} \nu \mathcal{L}_2^1 - \mathcal{H} \nu \mathcal{H} \mathcal{L}_2^1 = \mathcal{L}_2^1 - \mathcal{H} \nu \mathcal{H}_2^1 = \mathcal{L}_2^1. \]
Therefore for arbitrary \( \mathcal{L}_2^1 \in \mathbb{L}_2(\partial \Omega) \),
\[ \mathcal{H}^* \mathcal{L}_2^1 = \mathcal{L}_2^1. \]
i.e. \( \mathcal{H}^* = \mathbf{I} \),
where \( \mathbf{I} \) denotes the \((2 \times 2)\) identity matrix operator.
“(v) ⇒ (iv)”. Since \( \mathcal{H} \) is unitary, then \( \mathbf{I} = \mathcal{H}^* \mathcal{H} \). Associating (ii) in Lemma 3.1, for arbitrary \( \mathcal{L}_2^1 \in \mathbb{H}^2(\partial \Omega) \), we have
\[ \mathcal{L}_2^1 = \mathcal{H}^* \mathcal{H} \mathcal{L}_2^1 = \mathcal{H} \nu \mathcal{H} \mathcal{L}_2^1 = \nu \mathcal{H} \mathcal{L}_2^1. \]
Hence for arbitrary \( \mathcal{L}_2^1 \in \mathbb{H}^2(\partial \Omega) \),
\[ \mathcal{H} \nu \mathcal{L}_2^1 = -\nu \mathcal{L}_2^1. \]
“(v) ⇒ (vi)”. From the condition (v), \( \mathbf{I} = \mathcal{H}^* \mathcal{H} \). Applying (i) in Lemma 3.1, we have
\[ \mathcal{H} = \mathcal{H}^* \mathcal{H}^2 = \mathcal{H}^*. \]

By Lemma 3.2, the result of (vi) establishes.
“(vi) ⇒ (v)”. Applying Lemma 3.2 and Lemma 3.1, it is easy to get the result.
“(vi) ⇒ (vii)”. From the condition (vi), \( \mathcal{A} = 0 \). Hence \( \mathcal{A} = \mathcal{H} - \nu \mathcal{H} \nu = 0 \).
Since
\[ \mathcal{H} = \frac{1}{2} \left( \begin{array}{c} \mathcal{H} + \mathcal{H}^* \\ \mathcal{H} + \mathcal{H}^* \end{array} \right) \]
and
\[ \nu = \frac{1}{2} \left( \begin{array}{c} \nu + \nu^* \\ -\nu + \nu^* \end{array} \right), \]

ByLemma 3.2, the result of (vi) establishes.
“(vi) ⇒ (v)”. Applying Lemma 3.2 and Lemma 3.1, it is easy to get the result.
“(vi) ⇒ (vii)”. From the condition (vi), \( \mathcal{A} = 0 \). Hence \( \mathcal{A} = \mathcal{H} - \nu \mathcal{H} \nu = 0 \).
Since
\[ \mathcal{H} = \frac{1}{2} \left( \begin{array}{c} \mathcal{H} + \mathcal{H}^* \\ \mathcal{H} + \mathcal{H}^* \end{array} \right) \]
and
\[ \nu = \frac{1}{2} \left( \begin{array}{c} \nu + \nu^* \\ -\nu + \nu^* \end{array} \right), \]
then
\[ \nu H\nu = \frac{1}{8} \left( \nu + \nu \mid - \nu + \nu \right) \left( H + H\mid - H + H\mid \right) \left( \nu + \nu \mid - \nu + \nu \right). \]

Therefore
\[ \frac{1}{2} \left( H + H\mid - H + H\mid \right) = H = \nu H\nu = \frac{1}{2} \left( \nu H\nu + \nu H\nu \mid - \nu H\nu + \nu H\nu \right). \]

Hence
\[ H = \nu H\nu. \text{ i.e. } \nu_Y (Y - T) + (Y - T) \nu_T = 0, \]
\[ \langle Y - T, \nu_T \rangle + \langle Y - T, \nu_Y \rangle = 0, \text{ i.e. } \langle Y - T, \nu_T + \nu_Y \rangle = 0, \]
where \( Y, T \in \partial \Omega \) with \( Y \neq T \) and \( \nu_Y, \nu_T \) denote outward pointing unit vectors at \( Y, T \in \partial \Omega \) respectively (also see reference e.g. [16] or elsewhere). The result (vi) follows.

“(vii) \Rightarrow (vi)”. Since \( \Omega \) is a ball and the procedure of proof in “(vi) \Rightarrow (vii)”, the \( H = \nu H\nu \) and \( \nu = \nu H\nu \). So the result (vi) holds.

“(vii) \Rightarrow (viii)”. Since \( \Omega \) is a ball (i.e. \( \mathbb{A} = 0 \)), by Theorem 3.2, \( S = C \). That is, the Szegö projection and the Hardy projection coincide.

“(viii) \Rightarrow (vii)”. If the Szegö kernel and the Cauchy kernel coincide, \( S = C \). As the Szegö projection \( S \) is orthogonal on the Hilbert space of \( \mathcal{L}_2(\partial \Omega) \), then \( S = S^* \).

Hence \( C = S = S^* = C^* \), where \( S^*, C^* \) denote the adjoint operators of \( S, C \) on \( \mathcal{L}_2(\partial \Omega) \). Then \( \mathbb{A} = C = C^* = 0 \) respectively. That is, (vii) establishes.

The proof of Theorem 4.1 is complete.

Remark 4.2 “(vii) \Rightarrow (vi)” can be proved in virtue of direct calculation (see reference [25]), which leads to the solutions to half Dirichlet problems in the setting of Hermitian Clifford analysis.

Remark 4.3 The above theorem 4.1 implies that the matrix Hilbert transform \( H \) is unitary if and only if the bounded open subdomain \( \Omega \) of \( \mathbb{R}^{2n} \) is a ball. By Lemma 2.1, the matrix Hilbert transform \( H \) is unitary if and only if the Hardy projection operator \( C \) is self-adjoint.

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