On $\sigma$-semipermutable subgroups of finite groups*

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Abstract

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$, $G$ a finite group and $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$. A set $\mathcal{H}$ of subgroups of $G$ is said to be a complete Hall $\sigma$-set of $G$ if every member $\neq 1$ of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$ for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $\sigma_i \in \sigma(G)$. A subgroup $H$ of $G$ is said to be: $\sigma$-semipermutable in $G$ with respect to $\mathcal{H}$ if $HH_i^x = H_i^xH$ for all $x \in G$ and all $H_i \in \mathcal{H}$ such that $(|H|, |H_i|) = 1$; $\sigma$-semipermutable in $G$ if $H$ is $\sigma$-semipermutable in $G$ with respect to some complete Hall $\sigma$-set of $G$.

We study the structure of $G$ being based on the assumption that some subgroups of $G$ are $\sigma$-semipermutable in $G$.

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $p \in \pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If $n$ is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing $n$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of $G$.

In what follows, $\sigma = \{\sigma_i | i \in I \subseteq \mathbb{N}\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. Let $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$.

In the mathematical practice, we often deal with the following two special partitions of $\mathbb{P}$: $\sigma = \{\{2\}, \{3\}, \ldots\}$ and $\sigma = \{\pi, \pi'\}$ (in particular, $\sigma = \{\{p\}, \{p\}'\}$, where $p$ is a prime).

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A set $\mathcal{H}$ of subgroups of $G$ is a complete Hall $\sigma$-set of $G$ \cite{1} \cite{2} if every member $\neq 1$ of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$ for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exact one Hall $\sigma_i$-subgroup of $G$ for every $\sigma_i \in \sigma(G)$.

Subgroups $A$ and $B$ of $G$ are called permutable if $AB = BA$. In this case they also say that $A$ permutes with $B$.

**Definition 1.1.** Suppose that $G$ possesses a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$. A subgroup $H$ of $G$ is said to be: $\sigma$-semipermutable in $G$ with respect to $\mathcal{H}$ if $HH_i^x = H_i^xH$ for all $x \in G$ and all $i$ such that $(|H|, |H_i|) = 1; \sigma$-semipermutable in $G$ if $H$ is $\sigma$-semipermutable in $G$ with respect to some complete Hall $\sigma$-set of $G$.

Many known results deal with two special cases of the $\sigma$-semipermutability condition: when $\sigma = \{2\}, \{3\}, \ldots$ and $\sigma = \{\pi, \pi'\}$.

Consider some typical examples.

**Example 1.2.** A subgroup $H$ of $G$ is said to be $S$-permutable in $G$ if $H$ permutes with all Sylow subgroups $P$ of $G$ satisfying $(|H|, |P|) = 1$. Thus $H$ is $S$-permutable in $G$ if and only if it is $\sigma$-permutable in $G$ where $\sigma = \{2\}, \{3\}, \ldots$.

The $S$-permutable condition can be found in many known results (see for example Section 3 in \cite{8}, VI, Chapter 3 in \cite{4} and also the recent papers \cite{5, 6, 7}).

Before continuing, let’s make the following remark.

**Remarks 1.3.** Let $G = AB$ by a product of subgroups $A$ and $B$ and $K \leq B$. Suppose that $A$ permutes with $K^b$ for all $b \in B$. Then:

(i) For any $x = ab$, where $a \in A$ and $b \in B$, we have $AK^x = Aa(K^b)a^{-1} = a(K^b)a^{-1}A = K^xA$ and hence $A$ permutes with all conjugates of $K$.

(ii) $A^xK = KA^x$ for all $x \in G$. Indeed, $(A^xK)^{x^{-1}} = AK^{x^{-1}} = K^{x^{-1}}A$ by Part (i), so $(AK^{x^{-1}})^x = A^xK = KA^x$.

**Example 1.4.** A subgroup $H$ of $G$ is said to be $SS$-quasinormal if $G$ has a subgroup $T$ such that $HT = G$ and $H$ permutes with all Sylow subgroups of $T$. If $P$ is a Sylow subgroup of $T$ satisfying $(|H|, |P|) = 1$, then $P$ is a Sylow subgroup of $G$ and so $H$ is $\sigma$-permutable in $G$, where $\sigma = \{2\}, \{3\}, \ldots$, by Example 1.2 and Remark 1.3(i). Various applications of $SS$-quasinormal subgroups can be found in \cite{8, 9, 10} and in many other papers.

**Example 1.5.** In \cite{11}, Huppert proved that if a Sylow $p$-subgroup $P$ of $G$ of order $|P| > p$ has a complement $T$ in $G$ and $T$ permutes with all maximal subgroups of $P$, then $G$ is $p$-soluble. In view of Remark 1.3 the condition "$T$ permutes with all maximal subgroups of $P$" is equivalent to the condition "all maximal subgroups of $P$ are $\sigma$-permutable in $G$ with respect to $\{P, T\}$", where $\sigma = \{\{p\}, \{p\}'\}$. The result of Huppert was developed in the papers \cite{12, 13}, where instead of maximal subgroups we considered the subgroups of $P$ of fixed order $p^k$.

Further, the results in \cite{11} \cite{12} \cite{13} were generalized in \cite{14} \cite{15}, where instead of a Sylow $p$-subgroup of $G$ was considered a Hall subgroup of $G$ (see Section 4 below).
Finally, note that all the above-mentioned results deal with two special cases: a "binary" case, when $\sigma = \{\pi, n'\}$, and an "n-ary" case, when $\sigma = \{\{2\}, \{3\}, \ldots\}$.

In this paper, we consider the $\sigma$-semipermutability condition for arbitrary partition $\sigma$ of $\mathbb{P}$.

In fact, our main results are the following two observations.

**Theorem A.** Let $P$ be a Sylow $p$-subgroup of $G$. Suppose that $G$ has a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $H_1$ is $p$-supersoluble of order divisible by $p$. Suppose also that there is a natural number $k$ such that $p^k < |P|$ and every subgroup of $P$ of order $p^k$ and every cyclic subgroup of $P$ of order 4 (if $p^k = 2$ and $P$ is non-abelian) are $\sigma$-semipermutable in $G$ with respect to $\mathcal{H}$. Then $G$ is $p$-supersoluble.

**Theorem B.** Let $X \leq E$ be normal subgroups of $G$. Suppose that $G$ has a complete Hall $\sigma$-set $\mathcal{H}$ such that every member of $\mathcal{H}$ is supersoluble. Suppose also that for every non-cyclic Sylow subgroup $P$ of $X$ there is a natural number $k = k(P)$ such that $p^k < |P|$ and every subgroup of $P$ of order $p^k$ and every cyclic subgroup of $P$ of order 4 (if $p^k = 2$ and $P$ is non-abelian) are $\sigma$-semipermutable in $G$ with respect to $\mathcal{H}$. If $X = E$ or $X = F^*(E)$, then every chief factor of $G$ below $E$ is cyclic.

In this theorem $F^*(E)$ denotes the generalized Fitting subgroup of $E$, that is, the product of all normal quasinilpotent subgroups of $E$.

We prove Theorems A and B in Section 3. In Section 4 we discuss some applications of these two results.

All unexplained notation and terminology are standard. The reader is referred to [16], [17], [18] or [4] if necessary.

## 2 Base lemmas

Suppose that $G$ has a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$. For any subgroup $H$ of $G$ we write $H \cap \mathcal{H}$ to denote the set $\{H \cap H_1, \ldots, H \cap H_t\}$. If $H \cap \mathcal{H}$ is a complete Hall $\sigma$-set of $H$, then we say that $\mathcal{H}$ reduces into $H$.

**Lemma 2.1.** Suppose that $G$ has a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that a subgroup $H$ of $G$ is $\sigma$-semipermutable with respect to $\mathcal{H}$. Let $R$ be a normal subgroup of $G$ and $H \leq L \leq G$. Then:

1. $\mathcal{H}_0 = \{H_1 R/R, \ldots, H_t R/R\}$ is a complete Hall $\sigma$-set of $G/R$. Moreover, if for every prime $p$ dividing $|H|$ and for a Sylow $p$-subgroup $H_p$ of $H$ we have $H_p \not\leq R$, then $H R/R$ is $\sigma$-semipermutable in $G/N$ with respect to $\mathcal{H}_0$.

2. If $\mathcal{H}$ reduces into $L$, then $H$ is $\sigma$-semipermutable in $L$ with respect to $L \cap \mathcal{H}$. In particular, if $L$ is normal in $G$, then $H$ is $\sigma$-semipermutable in $L$ with respect to $L \cap \mathcal{H}$.

3. If $L \leq H_i$, for some $i$, then $\mathcal{H}$ reduces into $LR$.

4. If $H \leq H_i$, for some $i$, then $H$ is $\sigma$-semipermutable in $HR$.

5. If $H$ is a $p$-group, where $p \in \pi(H_i) \subseteq \sigma_i$ and $R$ is a $\sigma_i$-group, then $|G : N_G(H \cap R)|$ is a
\[ \sigma_i \text{-number.} \]

**Proof.** Without loss of the generality we can assume that \( H_i \) is a \( \sigma_i \)-group for all \( i = 1, \ldots, t \).

(1) It is clear that \( \mathcal{H}_0 = \{H_1R/R, \ldots, H_tR/R\} \) is a complete Hall \( \sigma \)-set of \( G/R \). Let \( i \in \{1, \ldots, t\} \) such that \( (|HR/R|, |H_iR/R|) = 1 \). Let \( p \in \pi(H) \) and \( H_p \) a Sylow \( p \)-subgroup of \( H \). Assume that \( p \) divides \( |H_i| \). Then \( H_i \) contains a Sylow \( p \)-subgroup of \( G \) since it is a Hall subgroup of \( G \) and so \( H_p \leq R \), contrary to the hypothesis. Hence \( (|H|, |H_i|) = 1 \). By hypothesis, \( HH_i^x = H_i^xH \) for all \( x \in G \). Then

\[
(HR/R)(H_iR/R)^xR = HH_i^xR/R
= H_i^xHR/R = (H_iR/R)^x(RR/R),
\]

so \( HR/R \) is \( \sigma \)-semipermutable in \( G/R \) with respect to \( \mathcal{H}_0 \).

(2) Let \( L_i = H_i \cap L \) for all \( i = 1, \ldots, t \) and \( \mathcal{L} = \{L_1, \ldots, L_t\} \). By hypothesis, \( \mathcal{L} \) is a complete \( \sigma \)-Hall set of \( L \). Let \( i \in \{1, \ldots, t\} \) such that \( (|H|, |L_i|) = 1 \) and let \( a \in L \). Then \( (|H|, |H_i|) = 1 \). Hence, by hypothesis, \( HH_i^a = H_i^aH \) for all \( a \in L \), so \( L \cap H \cap L_i^a = H \cap (L \cap H \cap L_i^a) = H \cap L_i^a = H_i^aH \). This shows that \( H \) is \( \sigma \)-semipermutable in \( L \) with respect to \( L \cap \mathcal{H} \).

(3) Since \( H_i \cap R \) is a Hall \( \sigma_i \)-subgroup of \( R \) and \( H_i \cap LR = L(H_i \cap R) \), we have \( LR : H_i \cap LR \mid R : H_i \cap R \). Hence \( H_i \cap LR \) is a Hall \( \sigma_i \)-subgroup of \( LR \). It is clear also that \( H_j \cap LR = H_j \cap R \) is a Hall \( \sigma_j \)-subgroup of \( LR \) for all \( j \neq i \). Hence \( \mathcal{H} \) reduces into \( LR \).

(4) This follows from Parts (2) and (3).

(5) For any \( j \neq i \), \( H_jH = HH_j \) is a subgroup of \( G \) and \( HH_j \cap R = (H \cap R)(H_j \cap R) = H \cap R \), so \( H_j \leq N_G(H \cap R) \). Hence \( |G : N_G(H \cap R)| \) is a \( \sigma_i \)-number.

**Lemma 2.2** (See Kegel [19]). Let \( A \) and \( B \) be subgroups of \( G \) such that \( G \neq AB \) and \( AB^x = B^x A \), for all \( x \in G \). Then \( G \) has a proper normal subgroup \( N \) such that either \( A \leq N \) or \( B \leq N \).

**Lemma 2.3.** Let \( P \) be a Sylow \( p \)-subgroup of \( G \) and \( \mathcal{H} = \{H_1, \ldots, H_t\} \) a complete Hall \( \sigma \)-set of \( G \) such that \( p \in \pi(H_1) \). Suppose that for any \( x \in G \), \( P^xH_i \) is a \( p \)-soluble subgroup of \( G \) for all \( i = 2, \ldots, t \). Then \( G \) is \( p \)-soluble.

**Proof.** Assume that this is false and let \( G \) be a counterexample of minimal order. First note that the hypothesis holds for every normal subgroup \( R \) of \( G \). Therefore every proper normal subgroup of \( G \) is \( p \)-soluble by the choice of \( G \). Moreover, the choice of \( G \) and the hypothesis imply that \( PH_i \neq G \) for all \( i = 2, \ldots, t \). By Lemma 2.2, we have either \( P^G \neq G \) or \( (H_2)^G \neq G \). Hence \( G \) has a proper non-identity normal subgroup \( R \). But then \( R \) is \( p \)-soluble. On the other hand, the hypothesis holds for \( G/R \), so \( G/R \) is also \( p \)-soluble by the choice of \( G \). This implies that \( G \) is \( p \)-soluble.

A group \( G \) is said to be **strictly \( p \)-closed** [20, p.5] whenever \( G_p \), a Sylow \( p \)-subgroup of \( G \), is normal in \( G \) with \( G/G_p \), abelian of exponent dividing \( p - 1 \). A normal subgroup \( H \) of \( G \) is called **hypercyclically embedded** in \( G \) if every chief factor of \( G \) below \( H \) is cyclic.

**Lemma 2.4** A normal \( p \)-subgroup \( P \) of \( G \) is hypercyclically embedded in \( G \) if and only if \( G/C_G(P) \) is strictly \( p \)-closed.

**Proof.** If \( P \) is hypercyclically embedded in \( G \), then for any chief factor \( H/K \) of \( G \) below \( P \),
\( G/C_G(H/K) \) is abelian of exponent dividing \( p - 1 \). Hence \( G/C \), where \( C \) the intersection the centralizers of all such factors, is also an abelian group of exponent dividing \( p - 1 \). On the other hand, \( C/C_G(P) \) is a \( p \)-group by [21] Ch.5, Corollary 3.3]. Hence \( G/C_G(P) \) is strictly \( p \)-closed.

Now assume that \( G/C_G(P) \) is strictly \( p \)-closed and let \( H/K \) be any chief factor below \( P \). Since \( C_G(P) \leq C_G(H/K) \), \( G/C_G(H/K) \) is strictly \( p \)-closed. But since \( O_\nu(G/C_G(H/K)) = 1 \) [16] Ch.A, Lemma 13.6], \( G/C_G(H/K) \) is abelian of exponent dividing \( p - 1 \). It follows from [20] Ch.1, Theorem 1.4] that \( |H/K| = p \). Thus \( P \) is hypercyclically embedded in \( G \).

Let \( P \) be a \( p \)-group. If \( P \) is not a non-abelian 2-group, then we use \( \Omega(P) \) to denote the subgroup \( \Omega_1(P) \). Otherwise, \( \Omega(P) = \Omega_2(P) \).

**Lemma 2.5** (See [22] Lemma 12]). Let \( P \) be a normal \( p \)-subgroup of \( G \) and \( D = \Omega(C) \), where \( C \) is a Thompson critical subgroup of \( P \). If either \( P/\Phi(P) \) is hypercyclically embedded in \( G/\Phi(P) \) or \( D \) is hypercyclically embedded in \( G \), then \( P \) is also hypercyclically embedded in \( G \).

**Lemma 2.6.** Let \( C \) be a Thompson critical subgroup of a \( p \)-group \( P \). Then the group \( D = \Omega(C) \) is of exponent \( p \) if \( p \) is odd prime or exponent 4 if \( P \) is non-abelian 2-group. Moreover, every non-trivial \( p' \)-automorphism of \( P \) induces a non-trivial automorphism of \( D \).

**Proof.** The first assertion follows from [21] Ch. 5, Theorem 3.11] and [22] Lemma 2.11]. The second one directly follows from [21] Ch. 5, Theorem 3.11].

**Lemma 2.7.** Let \( E \) be a normal subgroup of \( G \) and \( P \) a Sylow \( p \)-subgroup of \( E \) such that \( (p - 1, |G|) = 1 \). If either \( P \) is cyclic or \( G \) is \( p \)-supersoluble, then \( E \) is \( p \)-nilpotent and \( E/O_{p'}(E) \leq Z_{\infty}(G/O_{p'}(E)) \).

**Proof.** First note that in view of [18] Ch.IV, Theorem 5.4] and the condition \( (p - 1, |G|) = 1 \), \( E \) is \( p \)-nilpotent. Let \( H/K \) be any chief factor of \( G \) such that \( O_{p'}(E) \leq K < H \leq E \).

Thus \( E/O_{p'}(E) \leq Z_{\infty}(G/O_{p'}(E)) \).

The following lemma is well-known (see, for example, [18] Lemma 2.1.6]).

**Lemma 2.8.** If \( G \) is \( p \)-supersoluble and \( O_{p'}(G) = 1 \), then \( p \) is the largest prime dividing \( |G| \), \( G \) is supersoluble and \( F(G) = O_p(G) \) is a Sylow \( p \)-subgroup of \( G \).

**Lemma 2.9** (See [23]). Let \( H, K \) and \( N \) be pairwise permutable subgroups of \( G \) and \( H \) is a Hall subgroup of \( G \), then \( N \cap HK = (N \cap H)(N \cap K) \).

The following fact is also well-known (see for example [1] Ch.1, Lemma 5.35(6)]).

**Lemma 2.10** If \( H \) is a subnormal \( \pi \)-subgroup of \( G \), then \( H \leq O_\pi(G) \).

**Lemma 2.11** (See [24] Theorem C]). Let \( E \) be a normal subgroup of \( G \). If \( F^*(E) \) is hypercyclically embedded in \( G \), then \( E \) is hypercyclically embedded in \( G \).

### 3 Proofs of Theorems A and B

Theorem A is a corollary of the following two general results.
**Theorem 3.1.** Let $E$ be a $p$-soluble normal subgroup of $G$ and $P$ a Sylow $p$-subgroup of $E$. Suppose that $G$ has a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $H_1$ is $p$-supersoluble of order divisible by $p$. Suppose also that there is a natural number $k$ such that $p^k < |P|$ and every subgroup of $P$ of order $p^k$ and every cyclic subgroup of $P$ of order 4 (if $p^k = 2$ and $P$ is non-abelian) are $\sigma$-semipermutable in $G$ with respect to $\mathcal{H}$. Then $E/O_{p'}(E)$ is hypercyclically embedded in $G/O_{p'}(E)$.

**Theorem 3.2.** Let $P$ be a Sylow $p$-subgroup of $G$. Suppose that $G$ has a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $H_1$ is $p$-supersoluble of order divisible by $p$. Suppose also that there is a natural number $k$ such that $p^k < |P|$ and every subgroup of $P$ of order $p^k$ and every cyclic subgroup of $P$ of order 4 (if $p^k = 2$ and $P$ is non-abelian) are $\sigma$-semipermutable in $G$ with respect to $\mathcal{H}$. Then $G$ is $p$-soluble.

**Proof of Theorem 3.1.** Assume that this theorem is false and let $G$ be a counterexample with $|G| + |E|$ minimal. Let $|P| = p^n$. Then:

(1) $O_{p'}(N) = 1$ for every subgroup $N$ of $E$. Hence $O_p(G) \neq 1$.

Suppose that for some subgroup $N$ of $G$ contained in $E$ we have $O_{p'}(N) \neq 1$. Then $O_{p'}(N)$ is normal in $G$ and so $O_{p'}(N) \leq O_p(G)$ by Lemma 2.10. On the other hand, by Lemma 2.1(1), the hypothesis holds for $(G/(E \cap O_{p'}(G)), E/(E \cap O_{p'}(G))) = (G/O_{p'}(E), E/O_{p'}(E))$. Hence $E/O_{p'}(E)$ is hypercyclically embedded in $G/O_{p'}(E)$ by the choice of $G$, a contradiction. Thus we have (1).

(2) Let $U = O_p(E)$. Then $U$ is not hypercyclically embedded in $G$.

Assume that $U$ is hypercyclically embedded in $G$. Since $E$ is $p$-soluble by hypothesis and $O_{p'}(E) = 1$ by Claim (1), $U \neq 1$ and $C_E(U) \leq U$ by the Hall-Higman lemma [3 Ch.VI, Lemma 6.5]. But since $U$ is hypercyclically embedded in $G$, $G/C_G(U)$ is strictly $p$-closed by Lemma 2.4 and so $G/C_G(U)$ is supersoluble by [20 Ch.1, Theorem 1.9]. Now in view of the $G$-isomorphism $E_C(U)/C_G(U) \simeq E/E \cap C_G(U)$, we conclude that $E$ is hypercyclically embedded in $G$, a contradiction.

(3) $k > 1$.

Assume that $k = 1$. We show that in this case $U$ is hypercyclically embedded in $G$. Assume that this is false. Let $U/R$ be a chief factor of $G$. Then by the choice of $G$ we have $R$ is hypercyclically embedded in $G$, so for any normal subgroup $V$ of $G$ such that $V < U$ we have $V \leq R$ and $U/R$ is not cyclic. Let $B$ be a Thompson critical subgroup of $U$ and $\Omega = \Omega(B)$. We claim that $\Omega = U$. Indeed, if $\Omega < U$, then $\Omega \leq R$ and so $\Omega$ is hypercyclically embedded in $G$. Hence $U$ is hypercyclically embedded in by Lemma 2.5, a contradiction. Thus $\Omega = U$. Since $U \leq H_1$ and $H_1$ is $p$-supersoluble by hypothesis, there is a subgroup $L/R \leq U/R$ of order $p$ such that $L/R$ is normal in $H_1/R$. Let $x \in L \setminus R$ and $H = \langle x \rangle$. Since $\Omega = U$ and $L \leq U$, $|H|$ is either prime or 4. Then, by hypothesis, $H$ is $\sigma$-semipermutable in $G$ with respect to $\mathcal{H}$. Hence $HR/R$ is $\sigma$-semipermutable in $G/R$ with respect to $\{H_1R/R, \ldots, H_tR/R\}$ by Lemma 2.1(1). Then, by Lemma 2.1(5), $|G/R : N_{G/R}(HR/R)| = |G/R : N_{G/R}(L/R)|$ is a $\pi(H_1)$-number. It follows that $L/R$ is normal in $G/R$, and so $U/R = L/R$ is cyclic, a contradiction. This shows that $U$ is hypercyclically embedded in $G$, contrary to Claim (2). Hence we have (3).
(4) \(|N| \leq p^k\) for any minimal normal subgroup \(N\) of \(G\) contained in \(P\).

Indeed, suppose that \(|N| > p^k\). Then there exists a non-identity proper subgroup \(H\) of \(N\) such that \(H\) is normal in \(H_1\) and \(H\) is \(\sigma\)-semipermutable in \(G\) with respect to \(\mathcal{H}\). But then \(H\) is normal in \(G\) by Lemma 2.1(5), which contradicts the minimality of \(N\).

(5) If \(P\) is a non-abelian 2-group, then \(k > 2\).

Assume that \(k = 2\). We shall show that in this case every subgroup \(H\) of \(P\) of order 2 is \(\sigma\)-semipermutable in \(G\) with respect to \(\mathcal{H}\). This means that \(k = 1\) is possible, which will contradicts Claim (3).

First show that for any subgroup \(V = A \times B \leq P\) where \(|A| = 2 = |B|\), if both \(V\) and \(A\) are \(\sigma\)-semipermutable in \(G\) with respect to \(\mathcal{H}\), then \(B\) is \(\sigma\)-semipermutable in \(G\) with respect to \(\mathcal{H}\). Indeed, let \(i > 1\) and \(x \in G\). Then \(AH_i^x\) and \(VH_i^x\) are subgroups of \(G\) and \(|VH_i^x : AH_i^x| = 2\). Hence \(VH_i^x\) is 2-nilpotent, so \(H_i^xB = H_i^2B\) since \(H_i^2\) is normal in \(H_i^2V\). Similarly, if \(V = \langle a \rangle\) is a cyclic subgroup of order 4, then \(\langle a^2 \rangle\) is \(\sigma\)-semipermutable in \(G\) with respect to \(\mathcal{H}\).

Since \(P\) is a non-abelian 2-group, \(P\) has a cyclic subgroup \(H = \langle a \rangle\) of order 4. Then \(H\) is \(\sigma\)-semipermutable in \(G\) with respect to \(\mathcal{H}\) by hypothesis, so \(A = \langle a^2 \rangle\) is also \(\sigma\)-semipermutable in \(G\) with respect to \(\mathcal{H}\). Then every subgroup \(B\) of \(Z(P)\) of order 2 is \(\sigma\)-semipermutable in \(G\) with respect to \(\mathcal{H}\), and so every subgroup \(P\) of order 2 is \(\sigma\)-semipermutable in \(G\) with respect to \(\mathcal{H}\).

(6) If \(N\) is a minimal normal subgroup of \(G\) contained in \(P\), then \((E/N)/O'_p(E/N)\) is hypercyclically embedded in \((G/N)/O'_p(E/N)\).

It is enough to show that the hypothesis holds for \(G/N\). Since \(E/N\) is \(p\)-soluble, we can assume that \(|P/N| > p\).

If either \(p > 2\) and \(|N| < p^k\) or \(p = 2\) and \(|N| < 2^{k-1}\), then it is clear by Lemma 2.1(1). Now let either \(p > 2\) and \(|N| = p^k\) or \(p = 2\) and \(|N| \in \{2^{k-1}, 2^{k-1}\}\).

In view of Claim (3), \(k > 1\). Suppose that \(|N| = p^k\). Then \(N\) is non-cyclic and so every subgroup of \(G\) containing \(N\) is not cyclic. Let \(N \leq K \leq P\), where \(|K : N| = p\). Since \(K\) is non-cyclic, it has a maximal subgroup \(L \neq N\). Consider \(LN/N\). Since \(L\) is \(\sigma\)-semipermutable in \(G\) with respect to \(\mathcal{H}\), \(LN/N\) is also \(\sigma\)-semipermutable in \(G/N\) with respect to \(\{H_1R/R, \ldots, H_2R/R\}\) by Lemma 2.1(1). Therefore, if \(P/N\) is abelian, the hypothesis is true for \((G/N, P/N)\). Next suppose that \(P/N\) is a non-abelian 2-group.

Then \(P\) is non-abelian and so \(k > 2\) by Claim (5). Since \(|P/N| > 2, n-k \geq 2\). We may, therefore, let \(N \leq K \leq V \leq P\) such that \(|V : N| = 4, V/N\) is cyclic and \(|V : K| = 2\). Since \(V/N\) is not elementary, \(N \notin \Phi(V)\). Hence for some maximal subgroup \(K_1\) of \(V\) we have \(V = K_1N\). Suppose that \(K_1\) is cyclic. Then \(|K_1 \cap N| = 2\) and \(2 = |V : K_1| = |K_1N : K_1| = |N : K_1 \cap N|\). This implies that \(|N| = 4\). But then \(k = 2\), a contradiction. Hence \(K_1\) is not cyclic. Let \(S\) and \(R\) be two different maximal subgroups of \(K_1\). Then \(K_1 = SR\). If \(SN \leq K\) and \(RN \leq K\), then \(K_1 = SR \leq K\), which contradicts the choice of \(K_1\). Now since \(N/N < K/N < V/N\) where \(K/N\) is a maximal subgroup of \(V/N\), we have that \(V/N = K_1N/N = SRN/N = (SN/N)(RN/N)\). But since \(V/N\) is cyclic, eight \(V/N = SN/N\) or \(V/N = RN/N\). Without loss of generality, we may assume that \(NS = V\). Since
$S$ is a maximal subgroup of $K_1$ and $K_1$ is a maximal subgroup of $V$, $|S| = |N| = p^k$. Then $S$ is $\sigma$-semipermutable in $G$ with respect to $\mathfrak{H}$. Hence by Lemma 2.1(1), $V/N$ is $\sigma$-semipermutable in $G/N$ with respect to $\{H_1R/R, \ldots, H_tR/R\}$. This shows that the hypothesis is true for $(G/N, P/N)$.

Now suppose that $2^{k-1} = |N|$. If $|N| > 2$, then $N$ is not cyclic and as above one can show that every subgroup $\bar{H}$ of $P/N$ with order 2 and every cyclic subgroup of $P/N$ of order 4 (if $P/N$ is a non-abelian 2-group) is $\sigma$-semipermutable in $G/N$ with respect to $\{H_1R/R, \ldots, H_tR/R\}$. Finally, if $|N| = 2$ and $P/N$ is non-abelian, then $P$ is non-abelian and $k = 2$, which contradicts Claim (5). Thus (6) holds.

(7) $\Phi(U) = 1$.

Assume that for some minimal normal subgroup $N$ of $G$ we have $N \leq \Phi(U)$. Then, by Claim (6), every chief factor of $G/N$ between $O_{p'}(E/N)$ and $E/N$ is cyclic. Note that if $V/N = O_{p'}(E/N) \neq 1$ and $W$ is a $p$-complement in $V$, then by the Frattini argument, $G = VN_G(W) = NWN_G(W) = N_G(W)$ since $N \leq \Phi(O_{p'}(E)) \leq \Phi(G)$. Hence $W = 1$ by Claim (1). Therefore every chief factor of $G$ between $E$ and $N$ is cyclic. Now applying Lemma 2.5, we deduce that $E$ is hypercyclically embedded in $G$, a contradiction. Hence we have (7).

**Final contradiction.** In view of Claims (2) and (7), $U$ is an elementary group and for some minimal normal subgroup $N$ of $G$ contained in $U$ we have $|N| > p$. Let $S$ be a complement of $N$ in $U$. Since $N \leq H_1$ and $|N| \leq p^k$ by (4), there are a maximal subgroup $V$ of $N$ and a subgroup $W$ of $S$ such that $V$ is normal in $H_1$ and $|VW| = p^k$. Then $VW$ is $\sigma$-semipermutable in $G$ with respect to $\mathfrak{H}$ by hypothesis, so $V = VW \cap N$ is normal in $G$ by Lemma 2.1(5). Thus $V = 1$, and so $|N| = p$. This final contradiction completes the proof of the result.

**Proof of Theorem 3.2.** Assume that this theorem is false and let $G$ be a counterexample of minimal order. Without loss of generality we can assume that $P \leq H_1$ and $H_i$ is a $\sigma_i$-group for all $i = 1, \ldots, t$. Let $|P| = p^n$ and $V$ be a normal subgroup of $G$ such that $G/V$ is a simple group.

1. $O_{p'}(N) = 1$ for any subnormal subgroup $N$ of $G$ (See Claim (1) in the proof of Theorem 3.1).

2. $P \not\leq N$ for any proper normal subgroup $N$ of $G$ (In view of Lemma 2.1(4), this follows from the choice of $G$).

3. If the hypothesis holds for $V$, then $G/V$ is non-abelian, $O_{p}(V)$ is a Sylow $p$-subgroup of $V$ and $O_{p}(V)$ is hypercyclically embedded in $G$.

The choice of $G$ implies that $V$ is $p$-soluble. Hence $V$ is $p$-supersoluble by Theorem A. Since $O_{p'}(V) = 1$ by Claim (1), $V$ is supersoluble and $O_{p}(V)$ is a Sylow $p$-subgroup of $V$ by Lemma 2.8.

It is clear that $O_{p}(V)$ is normal in $G$, so $O_{p}(V)$ is hypercyclically embedded in $G$ by Theorem 3.1.

4. $k > 1$.

Assume that $k = 1$. Then:

(a) For a Sylow $p$-subgroup $V_p$ of $V$ we have $V_p \not\leq Z_{\infty}(G)$.

Indeed, assume that $V_p \leq Z_{\infty}(G)$. By [3], Ch. IV, Theorem 5.4, $G$ has a $p$-closed Schmidt subgroup $A$ and $A = A_p \times A_q$, where the Sylow subgroup $A_p$ of $A$ is of exponent $p$ or exponent
4 (if \( p = 2 \) and \( A_2 \) is non-abelian), and if \( \Phi = \Phi(A_p) \), then \( A_p/\Phi \) is a non-central chief factor of \( A \). Without loss of generality, we may assume that \( A_p \leq P \). Then \( V_p \cap A \leq Z_\infty(A) \cap A_p \leq \Phi \) and so there exists a subgroup \( H \) of \( A_p \) such that \( H \not\leq V \) and \( H \) is a cyclic group of order \( p \) or of order 4 (if \( p = 2 \) and \( A_2 \) is non-abelian). By hypothesis, \( H \) is \( \sigma \)-semipermutable in \( G \), so \( HV/V \) is \( \sigma \)-semipermutable subgroup of \( G/V \) by Lemma 2.1(1). Note that \( G \neq HH_2 \) (In fact, if \( |H| = p \), it is clear since \( |P| > p \). If \( HH_2 = G \) and \( H \) is a cyclic group of order 4, then \( G \) is \( p \)-soluble, contrary to the choice of \( G \)). Hence \( G/V \) is not simple by Lemma 2.2, a contradiction. Hence we have (a).

(b) If \( |V_p| = p \), then \( V \) is not \( p \)-soluble, and so \( H_1V = G \).

Indeed, if \( V \) is \( p \)-soluble, then \( V_p \) is normal in \( G \) by Claim (1). Hence \( V_p \) and \( C_G(V_p) \) are normal in \( G \). Claim (a) implies that \( P \leq C_G(V_p) < G \), which contradicts Claim (2). Therefore \( V \) is not \( p \)-soluble. But since the hypothesis holds for \( H_1V \) by Lemma 2.1(2)(3), the choice of \( G \) implies that \( H_1V = G \).

(c) \( |V_p| \neq p \). Hence the hypothesis holds for \( V \) by Lemma 2.1(2) and \( |P| > p^2 \).

Assume that \( |V_p| = p \). If \( V_p = V \cap P \leq \Phi(P) \), then \( V \) is \( p \)-nilpotent by the Tate theorem [3] Ch. IV, Theorem 4.7], contrary to (1). Hence \( V_p \) has a complement \( W \) in \( P \). Let \( L \) be a subgroup of order \( p \) in \( W \). Assume that \( L < W \). Then the hypothesis holds for \( VW \) by Lemma 2.1(2)(3), so \( VW \) is \( p \)-soluble, contrary to Claim (b). Therefore \( |W| = p \), so \( |P| = p^2 \) and \( P = V_pW \) is not cyclic.

Let \( E = (H_2 \cdots H_i)^G \). Then in view of Claim (b), we can assume, without loss of generality, that \( E \leq V \). We show that there is a subgroup \( W_0 \) of \( P \) order \( p \) such that \( W_0 \ntriangleleft V \) and \( W_0 \ntriangleleft C_G(E) \). Indeed, suppose that \( W \leq C_G(E) \). Note that \( C_G(E) \neq G \) by Claim (1). Hence \( V_p \ntriangleleft C_G(E) \) by Claim (2). It follows Claim (1) that \( C_G(E) \cap V = 1 \). Consequently \( G = C_G(E) \times V \). Let \( W = \langle a \rangle \), \( V_p = \langle b \rangle \) and \( W_0 = \langle ab \rangle \). Then \( W_0 \cap C_G(E) = 1 = W \cap V \).

Now let \( i > 1 \). Then \( W_0 H_i^x = H_i^x W_0 \) for all \( x \in G \) by hypothesis. Let \( L = H_i^{W_0} \cap W_0 H_i \). Then \( L \) is a subnormal subgroup of \( G \) by [25, Theorem 7.2.5]. Suppose that \( L \neq 1 \) and let \( L_0 \) be a minimal subnormal subgroup of \( G \) contained in \( L \). Then \( S = L_0 \cap W_0 \) is a Sylow \( p \)-subgroup of \( L_0 \) since \( L \leq W_0 H_i \). Moreover, in view of Claim (1) and Lemma 2.10, \( S \neq 1 \), and so \( W_0 = S \). If \( L_0 \) is abelian, then \( S = W_0 \leq O_p(G) \), where \( O_p(G) < P \) by Claim (2). Hence \( W_0 = O_p(G) \not\leq V \). Consequently \( W_0 \leq C_G(V) \leq C_G(E) \). This contradiction shows that \( L_0 \) is non-abelian. But then \( L_0 = L_0^G \) is a minimal normal subgroup of \( G \) by Claim (2) since \( |P| = p^2 \), which again implies that \( W_0 \leq C_G(E) \). This contradiction shows that \( L = 1 \). Therefore for every \( x \in G \) and every \( i > 1 \) we have \( (H_i^x)^{W_0} \cap W_0^{H_i} = 1 \), and so

\[
[W_0, H_i^x] \leq [(H_i^x)^{W_0}, W_0^{H_i}] = 1.
\]

Therefore \( W_0 \leq C_G(E) \), a contradiction. Hence we have (c).

Final contradiction for (4). Let \( C = C_G(V_p) \). By Claims (3) and (c), \( V_p \) is normal in \( G \) and it is hypercyclically embedded in \( G \). Hence \( G/C \) is strictly \( p \)-closed by Lemma 2.4. If \( V_p \ntriangleleft Z(G) \), then there is a normal maximal subgroup \( M \) of \( G \) such that \( C \leq M \). But since \( |P| > p^2 \), the hypothesis holds for \( M \), so \( M \) is \( p \)-soluble and so \( G \) does. This contradiction shows that \( V_p \leq Z(G) \), which contradicts Claim (a). Hence we have (4).
(5) $|N| \leq p^k$ for any minimal normal subgroup $N$ of $G$ contained in $P$ (See Claim (4) in the proof of Theorem 3.1).

(6) $k = n - 1$.

Assume that $k < n - 1$. Then $VP \neq G$. Indeed, if $VP = G$, then $|G : V| = p$ and the hypothesis holds for $V$. Hence $V$ is $p$-soluble by the choice of $G$ and so $G$ is $p$-soluble, a contradiction. By Lemma 2.1(4) the hypothesis holds for $VP$, so $VP$ is $p$-soluble by the choice of $G$ since $VP \neq G$. Therefore $V$ is $p$-soluble, so $O_p(V) \neq 1$ by Claim (1). Let $N$ be a minimal normal subgroup of $G$ contained in $O_p(V)$. It is clear that $N \neq P$. Since $k < n - 1$, $|P : N| > p$ by Claim (5). Now repeating some arguments in Claim (6) of the proof of Theorem A one can show that the hypothesis holds for $G/N$, so $G/N$ is $p$-soluble by the choice of $G$. But then $G$ is $p$-soluble, a contradiction. Hence we have (6).

(7) If $O_p(G) \neq 1$, then $P$ is not cyclic.

Suppose that $P$ is cyclic. Let $L$ be a minimal normal subgroup of $G$ contained in $O_p(G) \leq P$. Assume that $C_G(L) = G$. Then $L \leq Z(G)$. Let $N = N_G(P)$. If $P \leq Z(N)$, then $G$ is $p$-nilpotent by Burnside’s theorem [3, Ch. IV, Theorem 2.6], a contradiction. Hence $N \neq C_G(P)$. Let $x \in N \setminus C_G(P)$ with $(|x|, |P|) = 1$ and $K = P \times \langle x \rangle$. By [3, Ch. III, Theorem 13.4], $P = [K, P] \times (P \cap Z(K))$. Since $L \leq P \cap Z(K)$ and $P$ is cyclic, it follows that $P = P \cap Z(K)$ and so $x \in C_K(P)$. This contradiction shows that $C_G(L) \neq G$.

Since $P$ is cyclic, $|L| = p$. Hence $G/C_G(L)$ is a cyclic group of order dividing $p - 1$. But then $P \leq C_G(L)$, so $C_G(L)$ is $p$-soluble by the choice of $G$. Hence $G$ is $p$-soluble. This contradiction shows that we have (7).

(8) $G \neq PH_i$ for any $i > 1$.

Without lose of generality, assume that $G = PH_2$. Let $V_1, \ldots, V_r$ be the set of all maximal subgroups of $P$ and $D_i = V_i^G$. Then $D_i = V_i^{PH_2} = V_i^{H_2} \leq V_i H_2 = H_2 V_i$ by Claim (6).

Suppose that for some $i$, say $i = 1$, we have $D_1 P < G$. Then $D_1 P$ is $p$-soluble by the choice of $G$. Hence $O_p(G) \neq 1$. By Claim (7), $P$ is not cyclic. Moreover, for any $i > 1$, we have that $G = P^G = D_1 D_i$. Hence for all such $i > 1$, we have that $D_i P = G$ and so $D_i = V_i H_2$. It is also clear that $V_2 \cap \cdots \cap V_r = \Phi(P)$. Let $E = V_2 H_2 \cap \cdots \cap V_r H_2$. Then

$$P \cap E = (P \cap V_2 H_2) \cap \cdots \cap (P \cap V_r H_2) = V_2 (P \cap H_2) \cap \cdots \cap V_r (P \cap H_2) = V_2 \cap \cdots \cap V_r = \Phi(P).$$

Hence $E$ is $p$-nilpotent by the Tate theorem [3, Ch. IV, Theorem 4.7]. It follows that $1 < H_2 \leq O_{p'}(G)$, contrary to Claim (1). Hence we have (8).

(9) $P^G = G$, so $P \nleq H_i^G < G$ for all $i > 1$.

First note that $P^G = G$ by Claim (2) and $PH_i \neq G$ by Claim (8). If $P$ is not cyclic, then $PH^x = PH_x$ for all $x \in G$. Hence $H_i^G < G$ by Lemma 2.2. Now assume that $P$ is cyclic and $V$ be a maximal subgroup of $P$. Lemma 2.2 implies that either $V^G < G$ or $H_i^G < G$. But if $V^G < G$, then $P \nleq V^G$ and so $V^G \cap P \leq \Phi(P)$. Thus $V^G$ is $p$-nilpotent by the Tate theorem [3, IV, 4.7], which implies that $V^G = V$, contrary to Claim (7). Hence $H_i^G < G$. 

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Final contradiction. Claim (8) implies that \( PH_i \neq G \) for all \( i = 2, \ldots, t \). Hence in view of Claim (9), \( H^G_2 < G \). Assume that \( P = KL \), where \( K \) and \( L \) are different maximal subgroups of \( P \). Then the hypothesis and claim (6) imply that \( PH_i = KLH_i = H_iKL = H_iP \) for all \( i \). On the other hand, the hypothesis holds for \( PH_i \), so \( PH_i \) is \( p \)-soluble by the choice of \( G \). Now Lemma 2.3 implies that \( G \) is \( p \)-soluble. This contradiction shows that \( P \) is cyclic. But \( P \not\leq H^G_2 \) by Claim (9), so \( H^G_2 \cap P \leq \Phi(P) \). Therefore \( H^G_2 \) is \( p \)-nilpotent by the Tate theorem [3, Ch.IV, 4.7]. It follows from Claim (1) that \( H^G_2 \) is a \( p \)-subgroup. This final contradiction completes the proof.

Proof of Theorem B. Assume that this theorem is false and let \( G \) be a counterexample with \( |G| + |E| \) minimal.

First suppose \( X = E \). Let \( p \) be the smallest prime dividing \( |E| \) and \( P \) a Sylow \( p \)-subgroup of \( E \). Then \( E \) is \( p \)-nilpotent. Indeed, if \( |P| = p \), it follows directly from Lemma 2.7. If \( |P| > p \), then \( E \) is \( p \)-supersoluble by Theorems 3.1 and 3.2, so \( E \) is \( p \)-nilpotent again by Lemma 2.7. Let \( V = O^p_p(E) \). Since \( V \) is characteristic in \( E \), it is normal in \( G \) and the hypothesis holds for \( (G,V) \) and \( (G/V,E/V) \) by Lemma 2.1(1)(4).

The choice of \( G \) and Theorem 3.1 implies that \( P \neq E \). Hence \( V \neq 1 \), so \( E/V \) is hypercyclically embedded in \( G/V \) by the choice of \( (G,E) \). It is also clear that \( V \) is hypercyclically embedded in \( G \). Hence \( E \) is hypercyclically embedded in \( G \) by the Jordan-Hölder theorem for the chief series, a contradiction. Therefore in the case, when \( X = E \), the theorem is true. Finally, if \( X = F^*(E) \), then the assertion follows from Lemma 2.11. The result is proved.

4 Applications

Theorems A, B, Theorems 3.1 and 3.2 cover many known results. Hear we list some of them.

Corollary 4.1 (Gaschütz and N. Ito [3, Ch. IV, Theorem 5.7]). If every minimal subgroup of \( G \) is normal in \( G \), then \( G \) is soluble and \( G' \) has a normal Sylow 2-subgroup with nilpotent factor group.

Proof. This follows from the fact that \( G \) is \( p \)-supersoluble for all odd prime \( p \) dividing \( |G| \) by Theorem A.

Corollary 4.2 (Buckley [26]). If every minimal subgroup of a group \( G \) of odd order is normal in \( G \), then \( G \) supersoluble.

In view of Example 1.5 we get from Theorem 3.2 the following results.

Corollary 4.3 (Huppert [11]). Suppose that for a Sylow \( p \)-subgroup \( P \) of \( G \) we have \( |P| > p \). Assume that \( G \) has a \( p \)-complement \( E \) such that \( E \) permutes with all maximal subgroups of \( P \). Then \( G \) is \( p \)-soluble.

Corollary 4.4 (Sergienko [12], Borovik [13]) Suppose that for a Sylow \( p \)-subgroup \( P \) of \( G \) we have \( |P| > p \). Assume that \( G \) has a \( p \)-complement \( E \) and there is a natural number \( k \) such that \( p^k < |P| \) and every subgroup of \( P \) of order \( p^k \) permutes with \( E \). Suppose also that in the case when \( p = 2 \) the Sylow 2-subgroups of \( G \) are abelian. Then \( G \) is \( p \)-supersoluble.

Corollary 4.5 (Guo, Shum and Skiba [14]). Suppose that \( G = AT \), where \( A \) is a Hall \( \pi \)-subgroup
of $G$ and $T$ a nilpotent supplement of $A$ in $G$. Suppose that $A$ permutes with all subgroups of $T$. Then $G$ is $p$-supersoluble for each prime $p \not\in \pi$ such that $|T_p| > p$ for the Sylow $p$-subgroup $T_p$ of $T$.

**Proof.** Let $E$ be the Hall $\pi'$-subgroup of $T$. Then every subgroup $H$ of $E$ permutes with $A^x$ for all $x \in G$ by Remark 1.3. Hence $H$ is $\sigma$-semipermutable in $G$ with respect to $\{A, E\}$, so $G$ is $p$-supersoluble by Theorem A.

**Corollary 4.6** (Guo, Shum and Skiba [15]). Suppose that $G = AT$, where $A$ is a Hall $\pi$-subgroup of $G$ and $T$ a minimal nilpotent supplement of $A$ in $G$. Suppose that $A$ permutes with all maximal subgroups of any Hall subgroup of $T$. Then $G$ is $p$-supersoluble for each prime $p \not\in \pi$ such that $|T_p| > p$ for the Sylow $p$-subgroup $T_p$ of $T$.

In view of Example 1.4 we get from Theorem A the following

**Corollary 4.7** (Wei, Guo [10]). Let $p$ be the smallest prime dividing $|G|$ and $P$ be a Sylow $p$-subgroup of $G$. If there a subgroup $D$ of $P$ with $1 < |D| < |P|$ such that every subgroup $H$ of $P$ with order $|D|$ or order $2|D|$ (if $|D| = 2$) is SS-quasinormal in $G$, then $G$ is $p$-nilpotent.

From Example 1.4 and Theorem B we get the following three results.

**Corollary 4.8** (Li, Shen and Liu [8]). Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and $E$ a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that for every maximal subgroup of every non-cyclic Sylow subgroup of $E$ is SS-quasinormal in $G$. Then $G \in \mathcal{F}$.

**Corollary 4.9** (Li, Shen and Kong [9]). Let $E$ a normal subgroup of $G$ such that $G/E$ is supersoluble. Suppose that for every maximal subgroup of every Sylow subgroup of $F^*(E)$ is SS-quasinormal in $G$. Then $G$ is supersoluble.

**Corollary 4.10** (Li, Shen and Kong [9]). Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and $E$ a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that for every maximal subgroup of every Sylow subgroup of $F^*(E)$ is SS-quasinormal in $G$. Then $G \in \mathcal{F}$.

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