Conformal Submersion and Statistical Manifolds

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Abstract In this paper, we prove a necessary and sufficient condition for tangent bundle $TM$ to become a statistical manifold with respect to Sasaki lift metric and complete lift connection. Also, we study statistical structure on the manifold $B$ induced by affine submersion with horizontal distribution. A necessary and sufficient condition for a submersed statistical manifold to be dually flat is given. We introduced the conformal submersion with horizontal distribution which is a generalization of affine submersion with horizontal distribution and generalized the results of Abe and Hasegawa.

Keywords Statistical Manifold · Conformal Submersion · Tangent bundle

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1 Introduction

Information geometry has emerged from studies of invariant geometric structures involved in statistical inference. It defines a Riemannian metric together with dually coupled affine connections in a manifold of probability distributions [2]. Statistical manifold was originally introduced by S.L Lauritzen [4], later Kurose [3] reformulated this from the viewpoint of affine differential geometry. The notion of submersion between statistical manifolds was first introduced by Abe and Haswgawa [1]. They studied Riemannian submersion and affine submersion with horizontal distribution. They obtained a necessary and sufficient condition for $(M, \nabla, g_m)$ to become a statistical manifold with respect to affine submersion with horizontal distribution. Statistical structure on tangent bundle was studied by Mastuzoe and Inoguchi [6]. They obtained
necessary and sufficient condition for the tangent bundle $TM$ to become a statistical manifold with respect to Sasaki lift metric and horizontal lift connection.

In this paper we prove a necessary and sufficient condition for tangent bundle $TM$ to become a statistical manifold with respect to Sasaki lift metric and complete lift connection. In section 2, we obtained a statistical structure on ambient manifold $B$ induced by affine submersion $\pi: M \to B$ with horizontal distribution $\mathcal{H}(M) = V^\perp(M)$. Also obtained a necessary and sufficient condition for submersed statistical manifold to be dually flat. In section 3, The necessary and sufficient condition for $(TM, h^s, \nabla^c)$ to become a statistical manifold, where $h^s$ is the Sasaki lift metric and $\nabla^c$ is the complete lift of affine connection $\nabla$ on $M$ is given. In section 4, we introduced the conformal submersion with horizontal distribution which is a generalization of affine submersion with horizontal distribution and generalized the results of Abe and Hasegawa [1].

2 Statistical Manifold and Riemannian Submersion

A pseudo - Riemannian Manifold $(M, g)$ with a torsion free affine connection $\nabla$ is called a statistical manifold if $\nabla g$ is symmetric. For a statistical manifold $(M, \nabla, g)$ the dual connection $\nabla$ is defined by

$$X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for vector fields $X, Y$ and $Z$ in $\mathcal{X}(M)$. If $(\nabla, g)$ is a statistical structure on $M$ so is $(\nabla, g)$. Then $(M, \nabla, g)$ become a statistical manifold called the dual statistical manifold of $(M, \nabla, g)$. Let $R^\nabla$ and $R^{\nabla}$ be the curvature tensors of $\nabla$ and $\nabla$, respectively. It follows from (1) that

$$g(R^\nabla(X, Y) Z, W) = -g(Z, R^{\nabla}(X, Y)W)$$

for $X, Y, Z$ and $W$ in $\mathcal{X}(M)$. We say $(M, \nabla, \nabla, g)$ has constant curvature $k$ if

$$R^\nabla(X, Y) Z = k\{g(Y, Z)X - g(X, Z)Y\}.$$  

A statistical manifold with curvature zero is called a flat statistical manifold and in that case $(M, \nabla, \nabla, g)$ is called a dually flat statistical manifold.

2.1 Riemannian Submersion

Let $M$ be an $n$ dimensional manifold and $B$ be an $m$ dimensional manifold ($n > m$). An onto map $\pi: M \to B$ is called a submersion if $\pi_p: T_pM \to T_{\pi(p)}B$ is onto for all $p \in M$. For a submersion $\pi: M \to B$, $\pi^{-1}(b)$ is a submanifold of $M$ of dimension $n - m$ for each $b \in B$. We call these submanifolds $\pi^{-1}(b)$ as fibers. Set $\mathcal{V}(M)_x = Ker(\pi_*x)$ for each $x \in M$. 

Definition 1 A submersion $\pi : M \to B$ is called submersion with horizontal distribution if there is a smooth distribution $x \mapsto \mathcal{H}(M)_x$ such that

$$T_x M = \mathcal{V}(M)_x \bigoplus \mathcal{H}(M)_x. \quad (4)$$

We call $\mathcal{V}(M)_x$ ( $\mathcal{H}(M)_x$) the vertical (horizontal) subspace of $T_x M$. $\mathcal{H}$ and $\mathcal{V}$ denote the projection of the tangent space of $M$ onto the horizontal and vertical subspaces, respectively.

Note 1 Let $\pi : M \to B$ be a submersion with horizontal distribution $\mathcal{H}(M)$. Then $\pi_* |_{\mathcal{H}(M)_p} : \mathcal{H}(M)_p \to T_{\pi(p)} B$ is an isomorphism for each $p \in M$.

Definition 2 Let $(M, g_m)$ be a Riemannian manifold of dimension $n$, $(B, g_B)$ be a Riemannian manifold of dimension $m$ ($n > m$). A submersion $\pi : M \to B$ is called a Riemannian submersion if all fibers are Riemannian submanifold of $M$ and $\pi_*$ preserves the length of horizontal vectors.

Note 2 A vector field $E$ on $M$ is said to be projectable if there exist a vector field $E_\pi$ on $B$ such that $\pi_*(E_p) = E_{\pi(p)}$ for each $p \in M$, that is $E$ and $E_\pi$ are $\pi$-related. A vector field $X$ on $M$ is said to be basic if it is projectable and horizontal. Every vector field $X$ on $B$ has a unique smooth horizontal lift, denoted by $\tilde{X}$, to $M$.

Definition 3 Let $\nabla$ and $\nabla^*$ be affine connections on $M$ and $B$ respectively. $\pi : (M, \nabla) \to (B, \nabla^*)$ is said to be an affine submersion with horizontal distribution if $\pi : M \to B$ is a submersion with horizontal distribution and satisfies $\mathcal{H}(\nabla^* \tilde{Y}) = (\nabla^* Y)$ for all vector fields $X$ and $Y$ on $B$.

Proposition 1 Let $\pi : M \to B$ be a submersion with horizontal distribution and $\nabla$ be an affine connection on $M$. If $\mathcal{H}(\nabla^* \tilde{Y})$ is projectable for all vector fields $X$ and $Y$ on $B$, then there exists a unique affine connection $\nabla^*_X Y = \pi_* (\nabla^*_X \tilde{Y})$ on $B$ such that $\pi : (M, \nabla) \to (B, \nabla^*)$ is an affine submersion with horizontal distribution.

Proof Let $X$ be a vector field on $B$ and $f$ be a smooth real valued function on $B$, then $(f X) = (f \circ \pi) \tilde{X}$. Then

$$\nabla^*_X f Y = \pi_* (\nabla^*_X (f \tilde{Y}))$$

$$= \pi_* (\nabla^*_X (f \circ \pi) Y)$$

$$= \pi_* (\tilde{X} (f \circ \pi) \tilde{Y}) + \pi_* ((f \circ \pi) \nabla^*_X \tilde{Y})$$

$$= X (f) Y + f \nabla^*_X Y$$

Similarly we can prove the other conditions of affine connection.

Note 3 A connection $\nabla^0 \nabla$ on the subbundle $\mathcal{V}(M)$ is defined by $(\nabla^0 \nabla)_E V = \nabla(\nabla_E V)$ for any vertical vector field $V$ and any vector field $E$ on $M$. For each $b \in B$, $\nabla^0 \nabla$ induces a unique connection $\nabla^b$ on the fiber $\pi^{-1}(b)$. $\nabla^b$ we simply denoted as $\nabla$. The torsion of $\nabla$ is denoted by $\text{Tor}(\nabla)$. In [1] Abe and Hasegawa proved that if $\nabla$ is torsion free, then $\nabla$ and $\nabla^*$ are also torsion free.
Definition 4 Let $M$ be an $n$-dimensional manifold with affine connection $\nabla$. Then the fundamental tensors $T$ and $A$ are defined as

$$T_E F = \mathcal{H}(\nabla_{VE} VF) + \mathcal{V}(\nabla_{VE} HF)$$

(5)

$$A_E F = \mathcal{V}(\nabla_{HE} HF) + \mathcal{H}(\nabla_{HE} VF)$$

(6)

for arbitrary vector fields $E$ and $F$ on $M$.

Note that $T_E$ and $A_E$ reverses the horizontal and vertical subspaces and $T_E = T_{VE}, A_E = A_{HE}$.

The inclusion map $(\pi^{-1}(b), \tilde{\nabla}^b) \rightarrow (M, \nabla)$ is an affine immersion in the sense of [7]. The following equations are corresponding to the Gauss and Weingarten formulae. Let $X$ and $Y$ be horizontal vector fields, and $V$ and $W$ be vertical vector fields. Then

$$\nabla_{V} W = T_{V} W + \tilde{\nabla}_{V} W$$

$$\nabla_{V} X = \mathcal{H}(\nabla_{V} X) + T_{V} X$$

$$\nabla_{X} V = \mathcal{V}(\nabla_{X} V) + A_{X} V$$

$$\nabla_{X} Y = \mathcal{H}(\nabla_{X} Y) + A_{X} Y$$

Let $\pi : M \rightarrow B$ be a Riemannian submersion with horizontal distribution if $A$ is parallel then $A$ vanishes, similarly for $T$.

2.2 Riemannian Curvature and Fundamental equations

Let $(M, g_m)$ be a Riemannian manifold of dimension $n$. Let $R$ denote the Riemannian curvature of $(M, g_m)$ with respect to the affine connection $\nabla$. For any $E, F, G, H \in \mathcal{X}(M)$ we put $R(E, F, G, H) = g_m(R(G, H, F), E)$, where $R(G, H, F) = (\nabla_G, \nabla_H) - \nabla_{[G, H]} F$.

Definition 5 Let $\pi : (M, g_m) \rightarrow (B, g_b)$ be a Riemannian submersion with horizontal distribution. $R$ be the Riemannian curvature of $(M, g_m)$ with respect to $\nabla$ and $R^*$ be the Riemannian curvature of $(B, g_b)$ with respect to $\nabla^*$. Define a $(1, 3)$- tensor field $\hat{R}$ for the horizontal vector fields of $M$ as, $\pi_* (\hat{R}(X, Y) Z) = R^*(\pi_* X, \pi_* Y, \pi_* Z)$

The fundamental equations are

1. $R(U, V, F, W) = \hat{R}(U, V, F, W) + g_m(T_{U} W, T_{V} F) - g_m(T_{V} W, T_{U} F)$ (Gauss)

2. $R(U, V, W, X) = g_m((\nabla_{U} T)(U, W), X) - g_m(\nabla_{V} T)(V, W), X)$ (Codazzi)

where $\hat{R}$ denotes the Riemannian curvature of any fiber $(\pi^{-1}(x), \tilde{g}_x)$. $U, V, W, F$ are vertical vector fields on $M$ and $X$ is a horizontal vector field on $M$. Also we have the following fundamental equations.

$$R(X, Y, Z, V) = g((\nabla_{Z} A)(X, Y), V) + g(A_{X} Y, T_{V} Z)$$

$$-g(A_{V} Z, T_{X} W) - g(A_{Z} X, T_{V} Y)$$

(7)
\[ R(X, Y, Z, V) = R'(X, Y, Z, H) - 2g(A_XY, A_ZH) \]

\[ + g(A_YZ, A_XH) \]

\[ R(X, Y, V, W) = g((\nabla_V A)(X, Y), W) - g((\nabla_W A)(X, Y), V) \]

\[ + g(A_XV, A_YW) - g(A_XW, A_YV) - g(T_{YX}, T_{YW}) \]

\[ + g(T_{WX}, T_{VY}) \]

\[ R(X, Y, V, W) = g((\nabla_X T)(V, W), Y) - g((\nabla_V A)(X, Y), W) \]

\[ + g(A_XV, A_YW) - g(T_{YX}, T_{YW}) \]

for any horizontal vector fields \( X, Y, Z, H \) and vertical vector fields \( V, W \).

2.3 Statistical manifold and Submersion

Let \((M, g_m, \nabla)\) be an \(n\)-dimensional Riemannian manifold with affine connection \(\nabla\) and \((B, g_b, \nabla^*)\) be an \(m\)-dimensional Riemannian manifold with affine connection \(\nabla^*\). \(\nabla\) and \(\nabla^*\) be the dual connection of \(\nabla\) and \(\nabla^*\) respectively. In [1] Abe and Hasegawa proved

**Proposition 2** Assume that \(\pi : (M, g_m) \rightarrow (B, g_b)\) is a Riemannian submersion. Then \(\pi : (M, \nabla) \rightarrow (B, \nabla^*)\) is an affine submersion with horizontal distribution \(\mathcal{H}(M) = V^\perp(M)\) if and only if \(\pi : (M, \nabla) \rightarrow (B, \nabla^*)\) is an affine submersion with horizontal distribution.

Set \(S_E F = \nabla_E F - \nabla_F E\) for any vector fields \(E\) and \(F\) on \(M\).

**Theorem 1** Assume that \(\text{Tor}(\nabla) = 0\), \(\pi : (M, g_m) \rightarrow (B, g_b)\) is a Riemannian submersion and \(\pi : (M, \nabla) \rightarrow (B, \nabla^*)\) is an affine submersion with horizontal distribution \(\mathcal{H}(M) = V^\perp(M)\). Then \((M, \nabla, g_m)\) is a statistical manifold if and only if

1. \(\mathcal{H}(S_Y X) = A_X V - \overline{A_X V}\)
2. \(V(S_Y V) = T_Y X - \overline{T_Y X}\)
3. \((\pi^{-1}(b), \nabla^*_b, g^b_m)\) is a statistical manifold for each \(b \in B\).
4. \((B, \nabla^*, g_b)\) is a statistical manifold.

Assume that \(\text{Tor}(\nabla) = 0\), \(\pi : (M, g_m) \rightarrow (B, g_b)\) is a Riemannian submersion and \(\pi : (M, \nabla) \rightarrow (B, \nabla^*)\) is an affine submersion with horizontal distribution, from the fundamental equations we have

**Theorem 2** If the fundamental tensors \(T\) and \(A\) are parallel, then \((M, \nabla, \nabla^*, g_m)\) is dually flat if and only if \((\pi^{-1}(b), \nabla^*_b, \overline{\nabla^*_b}, \overline{g^b_m})\) and \((B, \nabla^*, \nabla^*, g_b)\) are dually flat.

**Definition 6** Let \(\pi : (M, \nabla) \rightarrow (B, \nabla')\) be an affine submersion with horizontal distribution \(V^\perp(M)\) and \(g\) be a Riemannian metric on \(M\), \(\mathcal{H}(\nabla X)\) is projectable. Define the induced metric \(\tilde{g}\) and the induced connection \(\nabla'\) on \(B\) as
\[ \tilde{g}(X, Y) = g(\tilde{X}, \tilde{Y}) \]  
\[ \nabla^\pi_X Y = \pi_*(\nabla_X \tilde{Y}) \]

where \( X, Y \) are vector field on \( B \).

Now we give a condition for \((B, \nabla^*, \tilde{g})\) to be a statistical manifold.

**Proposition 3** Let \((M, \nabla, g)\) be a statistical manifold and \( \pi : M \rightarrow B \) be an affine submersion with \( \mathcal{H}(M) = \mathcal{V}^\pi(M) \). If \( \mathcal{H}(\nabla_X \tilde{Y}) \) is projectable, then \((B, \nabla^*, \tilde{g})\) is also a statistical manifold.

**Proof** Let \( X, Y, Z \) be arbitrary vector fields on \( B \), we have

\[ (\nabla^\pi_X \tilde{g})(Y, Z) = Xg(\tilde{Y}, \tilde{Z}) - \tilde{g}(\nabla^\pi_X Y, Z) - \tilde{g}(Y, \nabla^\pi_X Z) \]
\[ = \tilde{X}g(\tilde{Y}, \tilde{Z}) - \tilde{g}(\nabla^\pi_X \tilde{Y}, \tilde{Z}) - \tilde{g}(\tilde{Y}, \nabla^\pi_X \tilde{Z}) \]
\[ = (\nabla^\pi_X \tilde{g})(\tilde{Y}, \tilde{Z}) \]

Since \((M, \nabla, g)\) is a statistical manifold, \((B, \nabla^*, \tilde{g})\) is also a statistical manifold.

### 3 Statistical Structures on Tangent Bundles

Let \( M \) be an \( n \)-dimensional manifold. \( TM = \bigcup_{p \in M} T_p M \) denote the tangent bundle on \( M \). Let \( \pi : TM \rightarrow M \) be the natural projection defined by \( \tilde{X}_p \in T_p M \rightarrow p \in M \). Let \((U, x^1, ..., x^n)\) be a local coordinate system on \( M \). Denote the induced co-ordinate system on \( \pi^{-1}(U) \) by \((x^1, ..., x^n; u^1, ..., u^n)\). Let \((x; u)\) be a point on \( TM \), denote the kernel of \( \pi_{v(x; u)}\) by \( V_{(x; u)} \) called the vertical subspace of \( T_{(x; u)}(TM) \) at \((x; u)\).

**Remark 1** The vertical subspace \( V_{(x; u)} \) is spanned by \( \{ \frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, ..., \frac{\partial}{\partial u^n} \} \). The two linear spaces \( T_x M \) and \( V_{(x; u)} \) have same dimension, so there is a canonical linear isomorphism \( V : T_x M \rightarrow V_{(x; u)} \) called the vertical lift.

**Definition 7** Let \( f : M \rightarrow R \) be a smooth function on \( M \) and \( \pi : TM \rightarrow M \) be the natural projection. The vertical lift of \( f \) is denoted by \( f^v \) and defined as \( f^v = f \circ \pi \). For a vector field \( X = X^i \frac{\partial}{\partial x^i} \) on \( M \) the vertical lift is denoted by \( X^v \) and defined as \( X^v = (X^i)^v \frac{\partial}{\partial u^i} \).

**Note 4** By direct calculation we can see that for any vector fields \( X, Y \) on \( M \) \([X^v, Y^v] = 0\).

**Definition 8** Let \( f : M \rightarrow R \) be a smooth map, the complete lift \( f^c \) of \( f \) on \( TM \) is defined as \( f^c = df = u^i \frac{\partial f}{\partial u^i} \). The complete lift of vector field \( X \) on \( M \) is defined as the one \( X^c \) on \( TM \) which characterized by the formula \( X^c(f^c) = (X^v)^c \) for all \( f \in C^\infty(M) \). In local co-ordinate, the complete lift \( X^c \) of \( X = X^i \frac{\partial}{\partial x^i} \) has the local expression

\[ X^c = (X^i)^c \frac{\partial}{\partial x^i} + u^j \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial u^j} \]
The complete lift to \(1\)–from \(\omega\) is defined as

\[ \omega^c(X) = (\omega(X))^c. \]

More generally the complete lift to full tensor algebra \(T(M)\) is given by the rule

\[ (P \otimes Q)^c = P^c \otimes Q^v + P^v \otimes Q^c \]

for any tensor fields \(P\) and \(Q\) on \(M\). Let \(\nabla\) be a linear connection on \(M\), then the complete lift \(\nabla^c\) on \(TM\) is defined as \(\nabla^c_{\nabla X}Y^c = (\nabla_X Y)^c\) for every \(X, Y \in \mathfrak{X}(M)\).

**Remark 2** In [6] Matsuzoe and Inoguchi has proved that if \((M, \nabla, h)\) is a statistical manifold, then \((TM, h^c, \nabla^c)\) is a statistical manifold. Moreover the conjugate connection of \(\nabla^c\) is \((\nabla^c)^c = (\nabla)^c\).

### 3.1 Horizontal lift on Tangent bundle

Let \(M\) be a smooth \(n\)–dimensional manifold, \(\nabla\) be a linear connection on \(M\), let \(V_{(x;u)} = \ker \pi_{(x;u)}\) be the vertical subspace of \(T_{(x;u)}(TM)\) at \((x; u)\). Vertical subspaces \(V_{(x;u)}\) defines a smooth distribution \(V\) on \(TM\) called the vertical distribution, also there exists a smooth distribution \(x \mapsto \mathcal{H}(TM)_x\) depending on linear connection \(\nabla\) such that

\[ T_{(x;u)}(TM) = \mathcal{H}(TM)_x \oplus V_{(x;u)}. \]

This distribution is denoted by \(\mathcal{H}_\nabla\), it is the called horizontal distribution. Let \(X\) be a vector field on \(M\), then the horizontal lift of \(X\) on \(TM\) is the unique vector field \(X^H\) on \(TM\) such that \(\pi_*(X^H_{(x;u)}) = X_{\pi((x;u))}\) for all \((x; u) \in TM\).

In local co-ordinates if \(X = X^i \frac{\partial}{\partial x^i}\), then

\[ X^H = X^i \frac{\partial}{\partial x^i} - X^i u^k \Gamma^i_{jk} \frac{\partial}{\partial u_j}. \]

Here \(\Gamma^i_{jk}\) is the connection coefficient of \(\nabla\).

Let \(h\) be a semi-Riemannian metric on \(M\), then the horizontal lift of \(h\) is defined as \(h^H(X^H, Y^H) = h^H(X^v, Y^v) = 0\) and \(h^H(X^H, Y^v) = h(X, Y)\), for \(X, Y \in \mathfrak{X}(M)\). The horizontal lift of linear connection \(\nabla\) on \(M\) is defined as \(\nabla^H_{X^H}Y^v = 0\), \(\nabla^H_{X^H}Y^v = 0\), \(\nabla^H_{X^H}Y^v = (\nabla_X Y)^v\), \(\nabla^H_{X^H}Y^v = (\nabla_X Y)^H\), for \(X, Y \in \mathfrak{X}(M)\).

**Remark 3** Even if \(\nabla\) is torsion free, its horizontal lift \(\nabla^H\) has non-trivial torsion.

**Definition 9** (Sasaki lift metric) Let \(h\) be a semi-Riemannian metric on \((M, \nabla)\). We define a semi-Riemannian metric \(h^s\) on \(TM\) as, \(h^s_{(x;u)}(X^H, Y^H) = h^s(X, Y)\), \(h^s_{(x;u)}(X^H, Y^v) = 0\), \(h^s_{(x;u)}(X^v, Y^v) = h^s(X, Y)\). The metric \(h^s\) is called the Sasaki lift metric.
Remark 4 In [6] Matsuzoe and Inoguchi proved that if \((M, h, \nabla)\) is a statistical manifold, then \((TM, h^s, \nabla^H)\) or \((TM, h^H, \nabla^H)\) is a statistical manifold if and only if \(\nabla h = 0\). Also they obtained for a statistical manifold \((M, \nabla, h)\), \((TM, h^s, C^H)\) and \((TM, h^H, C^H)\) are statistical manifolds, where \(C^H\) is the horizontal lift of the cubic form \(C = \nabla h\).

Remark 5 In [9] Yano and Ishihara introduced \(\gamma\) operator for defining horizontal lift from complete lift. Let \(X\) be a vector field on \(M\), with local expression \(X = X^i \frac{\partial}{\partial x^i}\), \(X = X^i \frac{\partial}{\partial u^i} + X^k \Gamma^i_{jk}\). Define \(\gamma(\nabla X) = u^j X^i \frac{\partial}{\partial u^i}\) with respect to the induced co-ordinate \((x^1, ..., x^n; u^1, ..., u^n)\). Then we can see that \(X^H = X^c - \gamma(\nabla X)\), note that \(\gamma(\nabla X)\) is the vertical part of \(X^c\).

Consider the submersion \(\pi : TM \rightarrow M\). Let \(\nabla\) be an affine connection on \(M\). Then there is a horizontal distribution \(\mathcal{H}_{\nabla}\) such that

\[ T(x; u)(TM) = \mathcal{H}_{(x; u)}(TM) + \nu \]

for every \((x; u) \in TM\).

Now we show that the submersion \(\pi\) of \(TM\) into \(M\) with complete lift of affine connection is an affine submersion with horizontal distribution.

**Proposition 4** The submersion \(\pi : (TM, \nabla^c) \rightarrow (M, \nabla)\) is an affine submersion with horizontal distribution.

**Proof** We need to show that

\[ \mathcal{H}(\nabla^c_{X^H} Y^H) = (\nabla X Y)^H \]

Consider, \(X^H = X^c - \gamma(\nabla X)\), then

\[
\nabla^c_{X^H} Y^H = \nabla^c_{X^c - \gamma(\nabla X)} Y^c - \gamma(\nabla Y) \\
= \nabla^c_{X^c} Y^c - \nabla^c_{X^c - \gamma(\nabla X)} \gamma(\nabla Y) \\
= \nabla^c_{X^c} Y^c - \nabla^c_{\gamma(\nabla X)} Y^c - \nabla^c_{X^c} \gamma(\nabla Y) + \nabla^c_{\gamma(\nabla X)} \gamma(\nabla Y)
\]

Using \(\nabla^c_{X^c} Y^v = 0\) \([6]\) we have

\[ \nabla^c_{X^H} Y^H = (\nabla X Y)^c - [\nabla^c_{\gamma(\nabla X)} Y^c + \nabla^c_{X^c} \gamma(\nabla Y)] \] (13)

By definition

\[ (\nabla X Y)^c = (\nabla X Y)^H + \gamma(\nabla \nabla X Y) \] (14)

From (13) and (14)

\[ \mathcal{H}(\nabla^c_{X^H} Y^H) = (\nabla X Y)^H. \]

Hence the submersion \(\pi : (TM, \nabla^c) \rightarrow (M, \nabla)\) is an affine submersion with horizontal distribution.
Proposition 5 The submersion \( \pi : (TM, h^s) \rightarrow (M, h) \) is a semi-Riemannian submersion.

Proof Clearly \( \pi^{-1}(p) = T_pM \) for all \( p \in M \) is a semi-Riemannian submanifold of \( TM \) also by definition of \( h^s \) we have
\[
h^s(X^H, Y^H) = h(X, Y)
\]
Hence \( \pi \) is a semi-Riemannian submersion.

Since \( \pi : (TM, h^s) \rightarrow (M, h) \) is a semi-Riemannian submersion and \( \pi : (TM, V^c) \rightarrow (M, V) \) is an affine submersion with horizontal distribution, we have the following result similar to theorem(1)

Theorem 3 \((TM, h^s, V^c)\) is a statistical manifold if and only if
1. \( H(SVX) = AXV - \overrightarrow{AXV} \)
2. \( V(SXV) = TVX - \overrightarrow{TVX} \)
3. \((T_pM, \nabla, h^s)\) is a statistical manifold for each \( p \in M \).
4. \((M, \nabla, h)\) is a statistical manifold.

Note that, since \( h^s(X^H, Y^V) = 0 \), we can take \( H(\nabla M) = V(M)^\perp \).

4 Conformal Submersion and Statistical Manifolds

Definition 10 Let \((M, g_m)\) and \((B, g_b)\) be semi-Riemannian manifolds. A submersion \( \pi : (M, g_m) \rightarrow (B, g_b) \) is called a conformal submersion if there exists a \( \phi \in C^\infty(M) \) such that
\[
g_m(X, Y) = e^{2\phi} g_b(\pi_*X, \pi_*Y). \tag{15}
\]

Proposition 6 Let \( \pi : (M, g_m) \rightarrow (B, g_b) \) be a conformal submersion, \( \nabla \) be the Levi-Civita connection on \( M \) and \( \nabla^* \) be the Levi-Civita connection on \( B \). Then
\[
g_b(\pi_*X Y, Z) = g_b(\nabla X Y, Z) - d\phi(\nabla X Y, Z) + d\phi(\nabla Y Z, X) + d\phi(\nabla Z X, Y)
\]
where \( X, Y, Z \in \mathfrak{X}(B) \) and \( \tilde{X}, \tilde{Y}, \tilde{Z} \) denote its unique horizontal lift on \( M \).

Proof Consider,
\[
2g_m(\nabla X Y, Z) = \tilde{X} g_m(\tilde{Y}, \tilde{Z}) + \tilde{Y} g_m(\tilde{Z}, \tilde{X}) - \tilde{Z} g_m(\tilde{X}, \tilde{Y}) - g_m(\tilde{X}, [\tilde{Y}, \tilde{Z}])
\]
\[
+ g_m(\tilde{Y}, [\tilde{Z}, \tilde{X}]) + g_m(\tilde{Z}, [\tilde{X}, \tilde{Y}]) \tag{16}
\]
Now consider
\[
\tilde{X} g_m(\tilde{Y}, \tilde{Z}) = \tilde{X} (e^{2\phi} g_b(Y, Z))
\]
\[
= \tilde{X} (e^{2\phi} g_b(Y, Z) + e^{2\phi} \tilde{X} g_b(Y, Z))
\]
\[
= 2e^{2\phi} d\phi(\tilde{X}) g_b(Y, Z) + e^{2\phi} X g_b(Y, Z)
\]
Similarly we have,
\[ \tilde{Y}_m(\tilde{X}, \tilde{Z}) = 2e^{2\phi}d\phi(\tilde{Y})g_b(X, Z) + e^{2\phi}Yg_b(X, Z) \]
\[ \tilde{Z}_m(\tilde{X}, \tilde{Y}) = 2e^{2\phi}d\phi(\tilde{Z})g_b(X, Y) + e^{2\phi}Zg_b(X, Y) \]
Also we have
\[ g_m(\tilde{X}, \tilde{Y}, \tilde{Z}) = e^{2\phi}g_b(X, Y, Z) \]
Then from equation (13) and above equations we get
\[ 2g_m(\nabla^*_X Y, Z) = 2d\phi(\tilde{X})e^{2\phi}g_b(Y, Z) + 2d\phi(\tilde{Y})e^{2\phi}g_b(X, Z) \]
This implies
\[ g_b(\pi_*(\nabla^*_X Y, Z) - d\phi(\tilde{Z})g_b(X, Y) \]
\[ + \{d\phi(\tilde{X})g_b(Y, Z) + d\phi(\tilde{Y})g_b(Z, X) \} \]
We generalize the concept of affine submersion with horizontal distribution as follows.

**Definition 11** \( \pi : (M, \nabla) \longrightarrow (B, \nabla^*) \) is called conformal submersion with horizontal distribution if \( \pi : M \longrightarrow B \) be submersion with horizontal distribution and satisfies
\[ g_b(\pi_*(\nabla_X Y, Z) - d\phi(\tilde{Z})g_b(X, Y) \]
\[ + \{d\phi(\tilde{X})g_b(Y, Z) + d\phi(\tilde{Y})g_b(Z, X) \} \]
for some \( \phi \in C^\infty(M) \) and for all \( X, Y, Z \in \mathcal{X}(B) \).

**Note 5** If \( \phi \) is constant it turns out to be an affine submersion with horizontal distribution.

In the conformal submersion case we have

**Proposition 7** Let \( \pi : (M, g_m) \longrightarrow (B, g_b) \) be a conformal submersion. Then \( \pi : (M, \nabla) \longrightarrow (B, \nabla^*) \) is a conformal submersion with horizontal distribution \( \mathcal{H}(M) \) if and only if \( \pi : (M, \nabla) \longrightarrow (B, \nabla^*) \) is a conformal submersion with same horizontal distribution.

**Proof** Consider,
\[ \tilde{X}_m(\tilde{Y}, \tilde{Z}) = 2e^{2\phi}d\phi(\tilde{X})g_b(Y, Z) + e^{2\phi}Xg_b(Y, Z) \]
\[ = 2e^{2\phi}d\phi(\tilde{X})g_b(Y, Z) + e^{2\phi}\{g_b(\nabla^*_X Y, Z) + g_b(Y, \nabla^*_X Z) \} \]
Now consider
\[
\tilde{X}g_m(\tilde{Y}, \tilde{Z}) = g_m(\nabla_{\tilde{X}}\tilde{Y}, \tilde{Z}) + g_m(\tilde{Y}, \nabla_{\tilde{X}}\tilde{Z})
= e^{2\phi}g_b(\pi_*(\nabla_{\tilde{X}}\tilde{Y}), Z) + e^{2\phi}g_b(Y, \pi_*(\nabla_{\tilde{X}}\tilde{Z}))
\] (17)
Since,
\[
g_b(\pi_*(\nabla_{\tilde{X}}\tilde{Y}), Z) = g_b(\nabla_{\tilde{X}}Y, Z) - d\phi(\tilde{Z})g_b(X, Y)
+ \{d\phi(\tilde{X})g_b(Y, Z) + d\phi(\tilde{Y})g_b(Z, X)\}
\] (18)
from (17) and (18) we get
\[
g_b(\pi_*(\nabla_{\tilde{X}}\tilde{Y}), Z) = g_b(\nabla_{\tilde{X}}Y, Z) - d\phi(\tilde{Z})g_b(X, Y)
+ \{d\phi(\tilde{X})g_b(Y, Z) + d\phi(\tilde{Y})g_b(Z, X)\}
\]
Hence, \(\pi : (M, \nabla) \rightarrow (B, \nabla^*)\) is a conformal submersion with horizontal distribution.
Converse is follows by Interchanging \(\nabla, \nabla^*\) with \(\nabla, \nabla^*\) in the above part.

**Lemma 1** Let \(\pi : (M, g_m) \rightarrow (B, g_b)\) be a conformal submersion and \(\pi : (M, \nabla) \rightarrow (B, \nabla^*)\) be a conformal submersion with horizontal distribution \(V(M)\), then for horizontal vectors \(X, Y\) and vertical vectors \(U, V, W\)

\[
(\nabla_{\tilde{X}}g_m)(\tilde{X}_1, \tilde{X}_2) = e^{2\phi}(\nabla^*_{\tilde{X}}g_b)(X_1, X_2)
\] (19)
\[
(\nabla_{\tilde{V}}g_m)(\tilde{X}, Y) = g_m(S_\tilde{V}X, Y)
\] (20)
\[
(\nabla_{\tilde{X}}g_m)(\tilde{V}, Y) = g_m(AXV, Y) + g_m(\tilde{A}_VX, Y)
\] (21)
\[
(\nabla_{\tilde{X}}g_m)(\tilde{V}, W) = g_m(S_\tilde{X}V, W)
\] (22)
\[
(\nabla_{\tilde{V}}g_m)(\tilde{X}, W) = g_m(T_\tilde{V}X, W) + g_m(\tilde{T}_VX, W)
\] (23)
\[
(\nabla_{\tilde{U}}g_m)(\tilde{V}, W) = (\tilde{V}_U\tilde{g}_m)(\tilde{V}, W)
\] (24)
where \(\tilde{X}_i\) are the horizontal lift of vector fields \(X_i\) on \(B\), \(\tilde{g}\) is the induced metric on fibers and \(S_\tilde{V}X = \nabla_{\tilde{V}}X - \tilde{\nabla}_VX\).

**Proof** Consider

\[
(\nabla_{\tilde{X}}g_m)(\tilde{X}_1, \tilde{X}_2) = \tilde{X}g_m(\tilde{X}_1, \tilde{X}_2) - g_m(\nabla_{\tilde{X}}\tilde{X}_1, \tilde{X}_2) - g_m(\tilde{X}_1, \nabla_{\tilde{X}}\tilde{X}_2)
= \tilde{X}g_b(X_1, X_2) - e^{2\phi}g_b(\pi_*(\nabla_{\tilde{X}}\tilde{X}_1), X_2)
- e^{2\phi}g_b(X_1, \pi_*(\nabla_{\tilde{X}}\tilde{X}_2))
- 2e^{2\phi}d\phi(\tilde{X})g_b(X_1, X_2) + e^{2\phi}g_b(X_1, X_2)
- e^{2\phi}g_b(X_1, \pi_*(\nabla_{\tilde{X}}\tilde{X}_2))
\]
Since,
\[
g_b(\pi_*(\nabla_{\tilde{X}}\tilde{X}_1), X_j) = g_b(\nabla^*_{\tilde{X}}X_i, X_j) - d\phi(\tilde{X}_j)g_b(X, X_i)
+ \{d\phi(\tilde{X})g_b(X_i, X_j) + d\phi(\tilde{X}_i)g_b(X_j, X)\}
\]
where \(i, j = 1, 2\) and \(i \neq j\). We get
\[
(\nabla_{\tilde{X}}g_m)(\tilde{X}_1, \tilde{X}_2) = e^{2\phi}(\nabla^*_{\tilde{X}}g_b)(X_1, X_2)
\]
Similarly we can prove other equations.
In the conformal submersion case also we obtain a result similar to theorem(1)

**Theorem 4** Assume that \( \text{Tor}(\nabla) = 0, \pi : (M, g_m) \to (B, g_b) \) is a conformal submersion and \( \pi : (M, \nabla) \to (B, \nabla^*) \) is a conformal submersion with horizontal distribution \( H(M) = \nabla^\perp(M) \). Then \((M, \nabla, g_m)\) is a statistical manifold if and only if

1. \( H(S_X V) = A_X V - \overline{A}_X V \)
2. \( V(S_X V) = T_Y X - \overline{T}_Y X \)
3. \((\pi^{-1}(b), \hat{\nabla}^b, \hat{g}^b_{m})\) is a statistical manifold for each \( b \in B \).
4. \((B, \nabla^*, g_b)\) is a statistical manifold.

**Proof** Suppose \((M, \nabla, g_m)\) is a statistical manifold, then \( \nabla g \) is symmetric. From (20) and (21) of above lemma we get

\[ H(S_X V) = A_X V - \overline{A}_X V \]

From (22) and (23) of above lemma we get

\[ V(S_X V) = T_Y X - \overline{T}_Y X \]

from (24) of above lemma \( \hat{\nabla}^b g^b \) is symmetric, so \((\pi^{-1}(b), \hat{\nabla}^b, \hat{g}^b_{m})\) is a statistical manifold. Also from (19) of above lemma \((B, \nabla^*, g_b)\) is a statistical manifold. Conversely, if all the four conditions hold then from the above lemma \( \nabla g_m \) is symmetric, so \((M, \nabla, g_m)\) is a statistical manifold.

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