Three-nucleon force in chiral EFT with explicit $\Delta(1232)$ degrees of freedom: Longest-range contributions at fourth order

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We analyze the longest-range two-pion exchange contributions to the three-nucleon force at leading-loop order in the framework of heavy-baryon chiral effective field theory with explicit $\Delta(1232)$ degrees of freedom. All relevant low-energy constants which appear in the calculation are determined from pion-nucleon scattering. Comparing our results with the ones obtained in the $\Delta$-less theory at $N^4\text{LO}$, we find effects of the $\Delta$ isobar for this particular topology to be rather well represented in terms of resonance saturation of various low-energy constants in the $\Delta$-less approach.

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I. INTRODUCTION

Three-nucleon forces (3NF) and their impact on nuclear structure and reactions became an important frontier in nuclear physics, see Refs. [1–25] for a selection of recent studies along these lines and Refs. [26, 27] for review articles. Chiral effective field theory (EFT) provides a model-independent and systematically improvable theoretical framework to describe nuclear forces and low-energy nuclear structure and dynamics in harmony with the symmetries of QCD [28, 29]. Nucleon-nucleon (NN) scattering has been extensively studied in chiral EFT in the past two decades following the pioneering work by Weinberg [30] and Ordonez et al. [31]. In particular, NN potentials at next-to-next-to-next-to-leading order ($N^3\text{LO}$) in the chiral expansion are available since about one and a half decades [32, 33] and served as a basis for numerous \textit{ab initio} calculations of nuclear structure and reactions. Recently, accurate and precise chiral EFT potentials up to fifth order in the chiral expansion, i.e. $N^4\text{LO}$, have been developed [34–37]. In particular, the semilocal $N^4\text{LO}$ potentials of Ref. [37] provide a description of the 2013 Granada database of neutron-proton and proton-proton scattering data below $E_{\text{lab}} = 300$ MeV, which is comparable to or even better than that based on the available high-precision phenomenological potentials.

The chiral expansion of the 3NF at one-loop level, i.e. up to and including next-to-next-to-next-to-next-to-leading order ($N^4\text{LO}$) contributions, can be described in terms of six topologies depicted in Fig. 1. The first nonvanishing contributions emerge at next-to-next-to-leading order ($N^2\text{LO}$) from tree-level diagrams of type (a), (d) and (f) [38, 39]. The resulting 3NF at $N^2\text{LO}$ has been intensively explored in three- and four-nucleon scattering calculations as well as in nuclear structure calculations, see [1–11, 13, 17–24] for some recent examples and the review articles [26, 27] and references therein. The first corrections to the 3NF emerge at $N^3\text{LO}$ from all possible one-loop diagrams of type (a)-(e) in Fig. 1 constructed from the lowest-order vertices. The resulting parameter-free expressions have been worked out in Refs. [40, 41], see also Ref. [42]. An interesting feature of the $N^3\text{LO}$ 3NF contributions is their rather rich isospin-spin-momentum structure emerging primarily from the ring topology (c) in Fig. 1. This is in contrast with the quite restricted operator structure of the $N^2\text{LO}$ 3NF. Numerical implementation of the $N^3\text{LO}$ 3NF corrections requires their partial wave decomposition [15, 43] and a consistent implementation of the regulator. This work is currently in progress, see Refs. [44, 45, 46] for some preliminary results. We further emphasize that four-nucleon forces also start to contribute at $N^3\text{LO}$ and have been worked out in Refs. [45, 46]. Pioneering applications of the chiral four-nucleon forces to the $\alpha$-particle binding energy [47, 48] and neutron matter [11, 22, 49] indicate that their

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In this paper we, for the first time, extend the SSE for the nuclear forces to N^3LO and concentrate on the longest-range topologies, the N^3LO contributions to the long-range [50] and intermediate-range [51] 3NF corresponding to diagrams (a) and (b, c) in Fig. 1 respectively. The corresponding potentials at large distance emerge as parameter-free predictions as they are completely determined by the chiral symmetry of QCD and experimental information on pion-nucleon scattering needed to fix the relevant low-energy constants (LECs). More precisely, for the two-pion-exchange topology, the N^3LO 3NF contributions depend on some of the LECs c_3, d_{3,1} and e_{i,3} from the order-Q^2, Q^3 and Q^4 effective pion-nucleon Lagrangians, which have been extracted from the available πN partial wave analyses. The resulting longest-range 3NF was shown to converge reasonably fast [50]. The situation appears to be very different for the two-pion-one-pion (2π-1π) exchange and ring 3NF topologies corresponding to diagrams (b) and (c) in Fig. 1: the formally leading contributions emerging at N^3LO turn out to be rather small in magnitude while the first corrections at N^3LO are considerably larger [51]. The origin of such an unnatural convergence pattern can be understood if one assumes the intermediate Δ(1232) excitation as a dominant 3NF mechanism, which is well in line with various phenomenological studies [52–54]. In the standard formulation of chiral EFT based on pions and nucleons as the only explicit degrees of freedom and used, in particular, in Refs. [50, 51], all effects of the Δ (and heavier resonances as well as heavy mesons) are taken into account implicitly through (some of the) LECs starting from the subleading effective Lagrangian, i.e. c_3, d_{3,1}, e_{i,3}, ... In particular, the values of the LECs c_{3,4}, which contribute to the two-pion exchange 3NF at N^3LO, are known to receive large contributions from the Δ. Thus, for this longest-range 3NF topology, effects of the Δ are already, to a large extent, accounted for at the lowest order (N^2LO). The first corrections at N^3LO emerge from one-loop diagrams constructed from the leading-order pion-nucleon vertices, which are not affected by the Δ, and the corresponding potentials appear to be fairly small in magnitude. This explains the observed good convergence pattern of the chiral expansion for the two-pion exchange 3NF. On the other hand, for the intermediate-range topologies, the expansion starts at N^3LO while the first effects of the Δ appear at N^4LO and lead to large corrections. Moreover, since the N^4LO contributions to the 2π+1π and ring 3NFs are proportional to c_3, only effects due to single-delta excitations are implicitly taken into account at that order. This raises the question of whether the double- and triple-delta excitations, which in the standard Δ-less formulation of chiral EFT are taken into account at even higher orders, might lead to sizable 3NF contributions. While this question could, at least in principle, be clarified by extending the calculations to even higher orders in the chiral expansion, this would require calculation of two-loop diagrams and also dealing with a large number of new LECs which makes this strategy hardly feasible. Instead, we follow a different approach and use chiral EFT with explicit Δ degrees of freedom, which offers a more efficient way to resum the contributions due to intermediate Δ-excitations. To be specific, we employ a formulation in which the delta-nucleon mass splitting is treated on the same footing as the pion mass, which is known as the small-scale expansion (SSE) [55]. Following the pioneering calculations in Refs. [56, 57], we have already worked out the contributions of the Δ to the two- and three-nucleon forces up to N^2LO in the SSE [58] and also looked at isospin-breaking corrections to the NN potential [59]. These calculations confirmed a better convergence of the Δ-full EFT formulation compared to its standard, Δ-less version. Interestingly, for the 3NF, the only nonvanishing Δ contribution up to N^2LO is the two-pion exchange diagram with an intermediate Δ-resonance, commonly called the Fujita-Miyazawa force. This term is shifted in the Δ-full theory to next-to-leading order (NLO).

In this paper we, for the first time, extend the SSE for the nuclear forces to N^3LO and concentrate on the longest-range topologies, which contribute to the two-pion exchange 3NF at N^3LO, are known to receive large contributions from the Δ. Thus, for this longest-range 3NF topology, effects of the Δ are already, to a large extent, accounted for at the lowest order (N^2LO). The first corrections at N^3LO emerge from one-loop diagrams constructed from the leading-order pion-nucleon vertices, which are not affected by the Δ, and the corresponding potentials appear to be fairly small in magnitude. This explains the observed good convergence pattern of the chiral expansion for the two-pion exchange 3NF. On the other hand, for the intermediate-range topologies, the expansion starts at N^3LO while the first effects of the Δ appear at N^4LO and lead to large corrections. Moreover, since the N^4LO contributions to the 2π+1π and ring 3NFs are proportional to c_3, only effects due to single-delta excitations are implicitly taken into account at that order. This raises the question of whether the double- and triple-delta excitations, which in the standard Δ-less formulation of chiral EFT are taken into account at even higher orders, might lead to sizable 3NF contributions. While this question could, at least in principle, be clarified by extending the calculations to even higher orders in the chiral expansion, this would require calculation of two-loop diagrams and also dealing with a large number of new LECs which makes this strategy hardly feasible. Instead, we follow a different approach and use chiral EFT with explicit Δ degrees of freedom, which offers a more efficient way to resum the contributions due to intermediate Δ-excitations. To be specific, we employ a formulation in which the delta-nucleon mass splitting is treated on the same footing as the pion mass, which is known as the small-scale expansion (SSE) [55]. Following the pioneering calculations in Refs. [56, 57], we have already worked out the contributions of the Δ to the two- and three-nucleon forces up to N^2LO in the SSE [58] and also looked at isospin-breaking corrections to the NN potential [59]. These calculations confirmed a better convergence of the Δ-full EFT formulation compared to its standard, Δ-less version. Interestingly, for the 3NF, the only nonvanishing Δ contribution up to N^2LO is the two-pion exchange diagram with an intermediate Δ-resonance, commonly called the Fujita-Miyazawa force. This term is shifted in the Δ-full theory to next-to-leading order (NLO).
range contribution to the 3NF corresponding to diagram (a) in Fig. [1]. This topology is particularly challenging due to (i) the need to carry out a non-trivial renormalization program as will be explained later and (ii) the need to re-consider pion-nucleon scattering in order to determine the relevant LECs, see Ref. [50] where this program was carried out in the standard, ∆-less version of chiral EFT. We will also discuss in detail renormalization within the ∆-full framework and work out the ∆ contributions to the relevant low-energy constants in the effective Lagrangian. Although we do not expect to see large benefits from the explicit treatment of the ∆ for the $2\pi$-exchange 3NF, where the standard chiral expansion already shows a good convergence [40, 50], this calculation is a necessary prerequisite for analyzing the ∆ contributions to the more problematic intermediate-range diagrams. This work is in progress and will be reported in a separate publication.

Our paper is organized as follows. In section II, we describe the framework and specify all terms in the effective Lagrangian that are needed in the calculation. Renormalization of the lowest-order effective Lagrangian to leading loop order is carried out in section III. In section IV we provide analytic expressions for the contribution of the ∆ to the relevant LECs $c_i, \bar{d}_i$ and $\bar{e}_i$ and determine the numerical values of these LECs from pion-nucleon scattering. ∆ contributions to the $2\pi$-exchange 3NF at $N^{3}\text{LO}$ are worked out in section V. In particular, we provide here parameter-free expressions both in momentum and coordinate spaces. A comparison of our findings with the ones of Refs. [40, 50] is given in section VI. Finally, the main results of our work are briefly summarized in section VII. The Appendices contain the unitary transformations of the nuclear Hamiltonian and the delta-contributions up to $N^{3}\text{LO}$ to the $\pi N$ invariant amplitudes.

II. THE FRAMEWORK

In the following, we briefly describe the formalism we employ in our analysis, namely the heavy-baryon formulation of chiral EFT with explicit ∆(1232) degrees of freedom [55]. In this framework, the soft scales are given by small external momenta $Q$, pion mass $M_\pi$ and the delta-nucleon mass splitting $\Delta := m_\Delta - m_N$. The resulting expansion in powers of the small parameter $\epsilon$ defined as

$$\epsilon \in \left\{ \frac{Q}{\Lambda_\chi}, \frac{M_\pi}{\Lambda_\chi}, \frac{\Delta}{\Lambda_\chi} \right\}$$

(2.1)

with $\Lambda_\chi \sim 1$ GeV denoting the chiral symmetry breaking scale, is known in the literature as the SSE.

We begin with specifying the effective chiral Lagrangian for pions, nucleons and the ∆. It is well known that the free spin-$3/2$ Lagrangian is non-unique and can be written in the form

$$\mathcal{L}_{\Delta}^{\text{free}} = -\bar{\psi}_i A^{i \mu} \left[ (i\partial - m_\Delta) g_{\mu \nu} - \frac{1}{4} \gamma_\mu \gamma_\lambda (i\partial - m_\Delta) \gamma^\lambda \gamma^\nu \right] \xi_{3/2}^{ij} A^\nu \psi_j,$$

(2.2)

where the tensor

$$A^{\mu \nu} = g^{\mu \nu} + \frac{1}{2} A \gamma^\mu \gamma^\nu$$

(2.3)

parametrizes non-uniqueness in the description of a spin-$3/2$ theory in terms of a parameter $A$, which can be chosen arbitrarily subject to the restriction $A \neq -1/2$. Further, the quantity $\xi_{3/2}^{ij}$ is the isospin-$3/2$ projection operator given by

$$\xi_{3/2}^{ij} = \delta^{ij} - \frac{1}{3} \tau^i \tau^j,$$

(2.4)

where $\tau_i$ denote the isospin Pauli matrices. Physical observables do not depend on the choice of the parameter $A$ since the entire dependence on $A$ can be absorbed into a field redefinition of the delta field. In practical calculations, the choice of $A$ is a matter of convenience. In the covariant approach, one usually chooses $A = -1$, see e.g. [61, 65], since in this case the free Lagrangian takes the particularly simple form

$$\mathcal{L}_{\Delta}^{\text{free}} = \bar{\psi}_i \gamma^{\mu \alpha} (i\partial^\alpha - m_\Delta \gamma_\mu) \xi_{3/2}^{ij} \psi_j,$$

(2.5)

with

$$\gamma^{\mu \alpha} = \frac{1}{4} \{[\gamma_\mu, \gamma_\nu], \gamma_\alpha\}, \quad \gamma_\mu \gamma_\nu = \frac{1}{2} [\gamma_\mu, \gamma_\nu].$$

(2.6)
This form of the Lagrangian leads to a fairly compact and convenient expression for the free propagator of the delta field

\[ S^\mu\nu = \frac{\theta + m_\Delta}{p^2 - m_\Delta^2} \left( -g^\mu\nu + \frac{1}{3} \gamma^\mu \gamma^\nu + \frac{1}{3m_\Delta} (\gamma^\mu p^\nu - \gamma^\nu p^\mu) + \frac{2}{3m_\Delta^2} p^\mu p^\nu \right). \]  

(2.7)

For every interaction in the Lagrangian, one generally has a freedom to introduce an off-shell parameter. As a consequence, interaction terms depend, in addition to the point-transformation parameter \( A \), also on the off-shell parameters \( z_i \) via the tensor

\[ \tilde{O}_{\alpha\beta}^\mu \tilde{O}_{\alpha\beta}^\nu = g^\mu\nu + \left[ \frac{1}{2} \right] z_i (1 + 4z_i) A \gamma^\mu \gamma^\nu. \]

(2.8)

All terms proportional to the off-shell parameters are redundant meaning that their contributions to observables can be absorbed into a redefinition of the corresponding low-energy constants (LECs). A particular choice of the off-shell parameters in the calculations is, therefore, a matter of convention. For example, in the covariant calculation of Ref. [64] we have set \( A_{\text{Relativistic}} = -1 \) and all \( z_i \) of Ref. [64] we have set \( A_{\text{Relativistic}} = -1 \) and all \( z_i \) of Ref. [64] we have set \( A_{\text{Relativistic}} = -1 \) and all \( z_i \) of Ref. [64] we have set \( A_{\text{Relativistic}} = -1 \). In the present analysis, we employ the heavy-baryon 1/m-expansion worked out by Hemmert et al. [55], where the choice \( A_{\text{HB}} = 0 \) without specifying a particular value for the off-shell parameter \( z_0^{\text{HB}} \) of the leading-order pion-nucleon-delta coupling has been made. In order to be consistent with the convention used in the covariant calculation of Ref. [64], we have to set

\[ z_0^{\text{HB}} + \frac{1}{2} (1 + 4z_0^{\text{HB}}) A_{\text{HB}} = z_0^{\text{Relativistic}} + \frac{1}{2} (1 + 4z_0^{\text{Relativistic}}) A_{\text{Relativistic}} \Rightarrow z_0^{\text{HB}} = -\frac{1}{2}. \]

(2.9)

This choice will be used throughout this work.

The effective heavy-baryon Lagrangians which contribute to the nuclear forces up to N3LO are given by

\[ \mathcal{L}_{\text{SSE}} = \mathcal{L}_{\pi\pi}^{(2)} + \mathcal{L}_{\pi\pi}^{(4)} + \mathcal{L}_{\pi\pi}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{\pi N}^{(3)} + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{\pi N}^{(2)} + \delta \mathcal{L}_{\pi N}^{(2)}, \]

(2.10)

where the subscripts refer to the small-scale dimension. Notice that the last term denotes the contribution to the pion-nucleon effective Lagrangian induced by the non-propagating spin-1/2 components of the Rarita-Schwinger field for the delta. The relevant terms in the pion Lagrangians have the form [69]

\[ \mathcal{L}_{\pi\pi}^{(2)} = \frac{1}{2} \left( \partial_\mu \pi \cdot \partial_\nu \pi - M^2 \pi \cdot \pi \right) + \frac{M^2}{8F^2} (8\alpha - 1)(\pi \cdot \pi)^2 + \frac{1}{2F^2} (1 - 4\alpha) (\pi \cdot \partial_\mu \pi)(\pi \cdot \partial_\nu \pi) \]

\[ - \frac{\alpha}{F^2} \pi \cdot \pi \partial_\mu \pi \cdot \partial_\nu \pi, \]

(2.11)

\[ \mathcal{L}_{\pi\pi}^{(4)} = - \frac{l_3}{F^2} M^4 \pi \cdot \pi + \frac{l_4}{F^2} M^2 (\partial_\mu \pi \cdot \partial_\nu \pi - M^2 \pi \cdot \pi), \]

(2.12)

where \( \pi, M \) and \( F \) refer to the pion fields in the chiral limit, the pion mass to leading order in quark masses and the pion decay coupling in the chiral limit, while \( l_3 \) and \( l_4 \) are further LECs. Here and in what follows, \( \hat{X} \) indicates that the quantity \( X \) is taken in the chiral limit. Further, the parameter \( \alpha \) reflects the freedom in the choice of a particular parametrization for the pion field. All physical quantities are, of course, independent of this parameter. We do not give here explicitly \( \mathcal{L}_{\pi N}^{(1)}, \mathcal{L}_{\pi N}^{(2)}, \mathcal{L}_{\pi N}^{(3)} \) as all relevant terms are listed in Ref. [60] where, in order to be consistent with our notation, the LECs \( c_i \) and \( d_i \) should be replaced by \( \tilde{c}_i \) and \( \tilde{d}_i \). The remaining Lagrangians in Eq. (2.10) are given by [50]

\[ \mathcal{L}_{\pi N}^{(1)} = - \frac{\tilde{h}_A}{F} \tilde{N}_i^\dagger \tilde{T}_\mu^i \partial^\mu \tilde{\pi} + h.c., \]

(2.13)

\[ \mathcal{L}_{\pi N}^{(2)} = i \frac{I}{F} (h_3 + h_6) \tilde{N}_i^\dagger \tilde{T}_\mu^i \partial^\mu \partial \cdot v \tilde{\pi} + i \frac{\tilde{h}_A}{F} \tilde{N}_i^\dagger \partial^\mu \tilde{T}_\mu^i \partial \cdot v \pi^i + h.c., \]

\[ \mathcal{L}_{\pi N}^{(3)} = \frac{2}{F} (2h_7 - h_8 - 2h_9 - 2h_{10}) M^2 \tilde{N}_i^\dagger \tilde{T}_\mu^i \partial^\mu \tilde{\pi}^i - \frac{1}{F} (h_{12} + h_{13}) \tilde{N}_i^\dagger \tilde{T}_\mu^i (\partial \cdot v)^2 \partial^\mu \tilde{\pi}^i + h.c., \]

\[ \mathcal{L}_{\pi N}^{(1)} = - \tilde{T}_\mu^i \left( i \partial \cdot v - \Delta - \frac{\tilde{g}_1}{F} \tau \cdot (\partial_\nu \pi) S^\alpha \right) \tilde{T}_\nu^j g^\mu\nu \delta_{ij}, \]

\[ \mathcal{L}_{\pi N}^{(2)} = -4M^2 c_1^i \tilde{T}_\mu^i \tilde{T}_\nu^j g^\mu\nu \delta_{ij} + \frac{1}{2m_\Delta^2} \tilde{T}_\nu^j (\partial^2 - (\partial \cdot v)^2) \tilde{T}_\nu^j g^\mu\nu \delta_{ij}, \]
\[ \delta \mathcal{L}^{(2)}_{\pi N} = \frac{\hat{h}_A^2}{18 F^2 m} \hat{N}_v^i \left( i \tau \cdot (\partial_\mu \hat{\pi} \times \partial_\mu \hat{\pi}) - 2 \partial_\mu \hat{\pi} \cdot \partial_\mu \hat{\pi} \right) \times \left( 4 \left( 1 + 8 z_0^{\text{HB}} + 12 (z_0^{\text{HB}})^2 \right) S'^\mu S'^\nu + \left( 5 - 8 z_0^{\text{HB}} - 4 (z_0^{\text{HB}})^2 \right) v'^\mu v'^\nu \right) \hat{N}_v, \]

where \( N_v \) and \( T_\mu^i \) denote the large components of the nucleon and delta field, respectively, \( v \) is the four-velocity and \( m \) is the nucleon mass in the chiral limit. For the sake of compactness, we do not show the velocity index explicitly in the case of the delta fields \( T_\mu^i \). The quantity \( h_A \) denotes the \( \pi N \Delta \) axial coupling, \( b_i, h_i \) and \( c_i^{\Delta} \) are further LECs and the covariant spin operator is defined via

\[
S_\mu = \frac{1}{2} i \gamma_5 \sigma_{\mu\nu} v^\nu, \quad \sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu].
\]

Last but not least, we emphasize that we adopt in the present work the convention for the pion-nucleon LECs which maintains an explicit decoupling of the delta. To be specific, the results for a given amplitude or nuclear potential \( \mathcal{M} \) have the form

\[
\mathcal{M} = \mathcal{M}_\Delta + \mathcal{M}_\Delta,
\]

where \( \mathcal{M}_\Delta \) denotes the contribution associated with the delta degrees of freedom while \( \mathcal{M}_\Delta \) is the purely nucleonic part. As guaranteed by the decoupling theorem \cite{10}, all effects of the delta isobar at low energy can be accounted for in an implicit way, i.e. through its contributions to the effective pion-nucleon Lagrangian. Expanding the delta contribution \( \mathcal{M}_\Delta \) around \( \Delta \to \infty \), one generally finds terms with both positive and negative powers of the \( \Delta \). While the latter ones can be identified with the \( \Delta \)-resonance saturation of the pion-nucleon LECs, see section \cite{13} for more details, terms with positive powers of the delta-nucleon mass splitting can, as a matter of convention, be eliminated by an appropriate redefinition of the pion-nucleon LECs. This is the convention we adopt in our analysis. It guarantees, that no positive powers of the \( \Delta \) appear in the finite expressions for all physical quantities and the \( \Delta (1232) \) contributions decouple (vanish) in the large-\( \Delta \) limit. Stated differently, this convention ensures that our results actually correspond to a partial resummation of the \( \Delta \)-resonance contributions to the pion-nucleon LECs within the \( \Delta \)-less formulation.

\section{Renormalization of the Effective Lagrangian to Leading Loop Order}

We now discuss in detail renormalization of the lowest-order effective chiral Lagrangian at the one-loop level which is achieved by expressing all quantities in terms of renormalized parameters rather than their chiral limit values. We do not consider here renormalization in the pionic sector as it is extensively discussed in the literature and concentrate entirely on the nucleon and delta sectors. We begin with introducing the renormalized fields and coupling constants via the relations

\[
\hat{N}_v = \sqrt{Z_N} N_v, \quad \hat{T}_\mu^i = \sqrt{Z_\Delta} T_\mu^i, \quad \hat{\pi}^i = \sqrt{Z_\pi} \pi^i, \quad M = M_N + \delta M, \quad m = m_N + \delta m, \quad \hat{\Delta} = \Delta + \delta \Delta, \quad Z_\pi = 1 + \delta Z_\pi, \quad Z_N = 1 + \delta Z_N, \quad Z_\Delta = 1 + \delta Z_\Delta, \quad F = F_\pi + \delta F, \quad \hat{g}_A = g_A + \delta g_A, \quad \hat{h}_A = h_A + \delta h_A, \quad \hat{g}_1 = g_1 + \delta g_1.
\]

and determine the shifts \( \delta X \) with \( X \in \{ M, F, m, \Delta, Z_N, Z_\Delta, g_A, h_A, g_1 \} \) order by order in the small scale expansion. Notice that in this formulation, the heavy baryon expansion corresponds to a \( 1/m_N \)-expansion, where \( m_N \) is now the physical nucleon mass and not the nucleon mass in the chiral limit, see \cite{11} for more details. We further emphasize that \( T_\mu^i \) does not correspond to an interpolating field of an asymptotic state so that its renormalization prescription is conventional. Even if \( Z_\Delta \) is a complex number, the replacement \( \hat{T}_\mu^i = \sqrt{Z_\Delta} T_\mu^i \) in the Lagrangian does not lead to a violation of unitarity in the kinematical region we are interested in simply because there are no external delta lines. Indeed, the complex renormalization factor \( Z_\Delta \), which shows up in the delta propagator, is compensated by vertices to which this delta propagator is attached so the amplitude does not depend on \( Z_\Delta \). This argument does not rely on whether the renormalization factor is a real or complex number. The main motivation for us to make the replacement \( T_\mu^i = \sqrt{Z_\Delta} T_\mu^i \) is to ensure that we can treat delta fields in the same manner as stable particles. This procedure is a matter of convention and does not affect the final result for the amplitudes. In the following, we discuss in detail renormalization of the various quantities in Eq. \ref{3.17}. 

\[ \frac{\hat{h}_A^2}{18 F^2 m} \hat{N}_v^i \left( i \tau \cdot (\partial_\mu \hat{\pi} \times \partial_\mu \hat{\pi}) - 2 \partial_\mu \hat{\pi} \cdot \partial_\mu \hat{\pi} \right) \times \left( 4 \left( 1 + 8 z_0^{\text{HB}} + 12 (z_0^{\text{HB}})^2 \right) S'^\mu S'^\nu + \left( 5 - 8 z_0^{\text{HB}} - 4 (z_0^{\text{HB}})^2 \right) v'^\mu v'^\nu \right) \hat{N}_v, \]
Nucleon mass and field renormalization

To study nucleon-mass and field-renormalization up to order $\epsilon^3$ one needs to calculate the self-energy diagrams shown in Fig. 2. The full nucleon propagator in the rest-frame of the nucleon can be parametrized via

$$D_N(p \cdot v) = \frac{1}{p \cdot v - \Sigma_N(p \cdot v) + i\epsilon}, \quad (3.18)$$

where $\Sigma_N(p \cdot v)$ denotes the nucleon self-energy. In the vicinity of $p \cdot v = 0$, the propagator of the renormalized physical nucleon fields has a simpler form

$$D_N(p \cdot v) = \frac{1}{p \cdot v + i\epsilon} + O((p \cdot v)^0). \quad (3.19)$$

Making the Taylor expansion

$$p \cdot v - \Sigma_N(p \cdot v) = -\Sigma_N(0) + (1 - \Sigma_N'(0))p \cdot v + O((p \cdot v)^2), \quad (3.20)$$

we obtain renormalization conditions for the nucleon mass and the $Z$-factor $Z_N$:

$$\Sigma_N(0) = 0 \quad \text{and} \quad \Sigma_N'(0) = 0. \quad (3.21)$$

The contribution of the first diagram in Fig. 2 to the nucleon self-energy is given by

$$\Sigma^{\text{tree}}_N(p \cdot v) = \delta m - 4\epsilon_1 M_\pi^2 - \delta Z_N p \cdot v. \quad (3.22)$$

The contribution of the nucleonic one-loop diagram, see second graph in Fig. 2 to the self-energy at the order we are working is given by

$$\Sigma^{\text{loop}}_N(p \cdot v) = \frac{3g_A^2}{4F_\pi^2} p \cdot v (d : 0) + \frac{3g_A^2}{4F_\pi^2} (M_\pi^2 - (p \cdot v)^2) I(d : 0; (p, 0)), \quad (3.23)$$

while the delta-loop contribution emerging from the last diagram in Fig. 2 is given by

$$\Sigma^{\text{loop}}_N(p \cdot v) = -\frac{2(d - 2)h_A^2}{(d - 1) F_\pi^2} (\Delta - p \cdot v) I(d : 0) + \frac{2(d - 2)h_A^2}{(d - 1) F_\pi^2} (M_\pi^2 - \Delta^2 + 2\Delta p \cdot v - (p \cdot v)^2) I(d : 0; (p, \Delta)). \quad (3.24)$$

Here, scalar master integrals in $d$ dimensions are defined according to:

$$I(d : p_1, \ldots, p_n) = \frac{1}{\Gamma(d - 4 - i)} \int \frac{d^dl}{(2\pi)^d} \frac{1}{(l + p_1)^2 - M_\pi^2 + i\epsilon} \cdots \frac{1}{(l + p_n)^2 - M_\pi^2 + i\epsilon}, \quad (3.25)$$

$$I(d : p_1, \ldots, p_n; (p, \delta)) = \mu^{4-d} \frac{1}{\Gamma(d - 4 - i)} \int \frac{d^dl}{(2\pi)^d} \frac{1}{(l + p_1)^2 - M_\pi^2 + i\epsilon} \cdots \frac{1}{(l + p_n)^2 - M_\pi^2 + i\epsilon} \frac{1}{(l + p)^2 - \delta + i\epsilon}. \quad (3.26)$$

Using the renormalization conditions in Eq. 3.21 we obtain the following expressions at order $\epsilon^3$ in four dimensions:

$$\delta m = 4\epsilon_1 M_\pi^2 + \frac{3g_A^2 M_\pi^2}{32\pi^2 F_\pi^2} + \frac{h_A^2}{36\pi^2 F_\pi^2} (2\Delta^2 - 3M_\pi^2) + \frac{8h_A^2}{3F_\pi^2} (2\Delta^2 - 3M_\pi^2) \lambda_\pi + \frac{4h_A^2}{F_\pi^2} (\Delta^2 - M_\pi^2) \bar{J}_0(-\Delta),$$

$$\delta Z_N = -\frac{3g_A^2 M_\pi^2}{32\pi^2 F_\pi^2} + \frac{h_A^2}{4\pi^2 F_\pi^2} (2\Delta^2 - M_\pi^2) + \frac{1}{2F_\pi^2} (16h_A^2 (2\Delta^2 - M_\pi^2) - 9g_A^2 M_\pi^2) \lambda_\pi + \frac{4h_A^2}{F_\pi^2} \bar{J}_0(-\Delta), \quad (3.26)$$

where the quantities $\lambda_\pi$ and $\bar{J}_0$ are defined in appendix B.
vanishing Feynman diagrams which contribute to $G_A(0)$ up to order $\varepsilon^3$. Wavy lines represent external axial sources. For the remaining notation see Fig. 2.

• **Renormalization of the nucleon axial coupling**

To renormalize the axial-vector coupling constant $\tilde{g}_A$, we consider the axial-vector form factor of the nucleon as shown in Fig. 3. In the Breit frame ($q_0=0$), the matrix element can be parametrized via (3.28)

$$M(p', p, q) = -\tau^j \varepsilon_A^j \cdot \hat{\sigma} \frac{E}{2m_N} G_A(q^2) + \ldots,$$

(3.27)

where the ellipses refer to terms which are of no relevance for renormalization of $\tilde{g}_A$. In the above expression, $E$ denotes the energy of the incoming nucleon (which in the Breit-frame is also equal to the energy of the outgoing nucleon) while $\varepsilon_A^j$ is the polarization vector of the $j$-th component of an isotriplet external axial field. The physical value of the nucleon axial coupling $g_A$ is defined as

$$g_A = G_A(0).$$

(3.28)

Up to order $\varepsilon^3$, the contributions to the axial form factor $G_A(0)$ emerge from the tree-level diagrams

$$G_A^{tree}(0) = g_A + \delta g_A + g_A \delta Z_N + 4d_{16} M^2_p,$$

(3.29)

one-loop diagrams without delta excitations

$$G_A^{loop, \pi N}(0) = -\frac{g_A}{4F^2_\pi} (g_A^2 (d - 3) - 4) I(d:0)$$

(3.30)

and one-loop diagrams with intermediate delta excitations, see the second raw of the right panel of Fig. 3.

$$G_A^{loop, \pi N \Delta}(0) = -\frac{2(d-2)h_A^2}{9(d-1)^2 F^2_\pi} (24g_A + 5(d^2 - 2d - 3)g_1) I(d:0) + \frac{16(d-2)g_A h_A^2 M^2_\pi}{3(d-1)^2 F^2_\pi \Delta} I(d:0; (0,0))$$

$$+ \frac{2(d-2)h_A^2}{9(d-1)^2 F^2_\pi \Delta} (24g_A(M^2_\pi - \Delta^2) - 5(d^2 - 2d - 3)g_1 \Delta^2) I(d:0; (0, \Delta)).$$

(3.31)

Using the renormalization condition in Eq. (3.28) we obtain in four dimensions

$$\delta g_A = -\delta Z_N g_A - 4d_{16} M^2_p - \frac{M^2_\pi}{7776 \pi^2 F^2_\pi} (243g_A^3 - 576g_A h_A^2 + 1240h_A^2 g_1) - \frac{h_A^2 \Delta^2}{486 \pi^2 F^2_\pi} (24g_A - 155g_1)$$

$$- \frac{4g_A h_A^2 M^2_\pi}{27 \pi F^2_\pi \Delta} \left( \frac{16h_A^2 \Delta^2}{81 F^2_\pi} (24g_A + 25g_1) - \frac{M^2_\pi}{162 F^2_\pi} (81g_A^3 + 36g_A (32h_A^2 - 9) + 400h_A^2 g_1) \right) \lambda_\pi$$

$$+ \left( \frac{4h_A^2 \Delta^2}{81 F^2_\pi} (24g_A + 25g_1) - \frac{32g_A h_A^2 M^2_\pi}{27 \pi F^2_\pi \Delta} \right) \bar{J}_0(-\Delta).$$

(3.32)

• **Delta mass and field renormalization**

To study the delta mass and field renormalization one needs to calculate the corresponding nonvanishing self-energy diagrams shown in Fig. 4. In general, the self-energy of the $\Delta$-resonance in the rest-frame can be parametrized via

$$\Sigma_{\Delta}(p \cdot v)^{ij}_{\mu \nu} = P^{3/2} \varepsilon_{i=3/2} \Sigma_{\Delta}(p \cdot v),$$

(3.33)
where the spin- and isospin-3/2 projector operators are defined by
\[ P_{3/2}^{\mu\nu} = g_{\mu\nu} - v_\mu v_\nu + \frac{4}{3} S_\mu S_\nu \quad \text{and} \quad \xi^{ij}_{I=3/2} = \delta^{ij} - \frac{1}{3} \tau^i \tau^j, \] (3.34)
respectively. The contribution of the tree-level diagram, see the first graph in Fig. 4 to the delta self-energy is given by
\[ \Sigma_{\text{tree}}^{\Delta}(p \cdot v) = -4c_\Delta M^2. \] (3.35)

The contributions of the two one-loop diagrams with the πN and πΔ cuts in d space-time dimensions have the form
\[ \Sigma_{\Delta, \text{loop}, \pi N}(p \cdot v) = \frac{h_\Delta^2 (d-1) F_\pi^2}{12 (d-1)^2 F_\pi^2} \left[ (p \cdot v - \Delta) I(d : 0) + (M_\pi^2 - (p \cdot v - \Delta)^2) I(d : 0; (p, \Delta)) \right], \] (3.36)
\[ \Sigma_{\Delta, \text{loop}, \pi \Delta}(p \cdot v) = 5 \left( d^2 - 2d - 3 \right) g_\Delta^2 \left[ (p \cdot v - \Delta) I(d : 0) + (M_\pi^2 - (p \cdot v - \Delta)^2) I(d : 0; (p, \Delta)) \right]. \]

The full ∆-propagator in the rest-frame of the ∆-resonance can be written as
\[ D_{\Delta}(p \cdot v)^{ij}_{\mu\nu} = -D_{\Delta}(p \cdot v) P_{3/2}^{ij} \xi^{ij}_{I=3/2}, \quad \text{with} \quad D_{\Delta}(p \cdot v) = \frac{i}{p \cdot v - \Delta - \Sigma_{\Delta}(p \cdot v)}. \] (3.37)

In the vicinity of the pole, the full delta-propagator has a simpler structure, namely
\[ D_{\Delta}(p \cdot v) \approx \frac{i}{p \cdot v - \Delta + i \Gamma_{\Delta}/2}. \] (3.38)

Here, Δ and Γ_{\Delta} denote the (pole-position) mass and width of the delta resonance, respectively. Expanding the full propagator around the pole one extracts the mass, width and the complex Z_{\Delta}-factor:
\[ p \cdot v - \Delta - \Sigma_{\Delta}(p \cdot v) = \Delta - i \frac{\Gamma_{\Delta}}{2} - \Delta - \Sigma_{\Delta} \left( \Delta - i \frac{\Gamma_{\Delta}}{2} \right) + \left( 1 - \Sigma'_{\Delta} \left( \Delta - i \frac{\Gamma_{\Delta}}{2} \right) \right) (p \cdot v - \Delta + i \frac{\Gamma_{\Delta}}{2}) + \mathcal{O} \left( (p \cdot v - \Delta + i \frac{\Gamma_{\Delta}}{2})^2 \right). \] (3.39)

Renormalization of the delta mass and width is determined from the condition
\[ \Delta - i \frac{\Gamma_{\Delta}}{2} - \Delta - \Sigma_{\Delta} \left( \Delta - i \frac{\Gamma_{\Delta}}{2} \right) = 0. \] (3.40)

From the real part of this condition we deduce the delta-mass renormalization as
\[ \Delta - \Delta - \Re \Sigma_{\Delta} \left( \Delta - i \frac{\Gamma_{\Delta}}{2} \right) = 0, \] (3.41)
while the imaginary part of this condition yields the following result for the width:
\[ \frac{\Gamma_{\Delta}}{2} + \Im \Sigma_{\Delta} \left( \Delta - i \frac{\Gamma_{\Delta}}{2} \right) = 0. \] (3.42)

The complex-valued Z_{\Delta}-factor is determined by the relation
\[ \Sigma'_{\Delta} \left( \Delta - i \frac{\Gamma_{\Delta}}{2} \right) = 0. \] (3.43)
At the one-loop level we can replace the relations (3.41), (3.42) and (3.43) by

\[ \Delta - \tilde{\Delta} - \text{Re} \Sigma_\Delta(\Delta) = 0, \quad \frac{\Gamma_\Delta}{2} + \text{Im} \Sigma_\Delta(\Delta) = 0 \quad \text{and} \quad \Sigma'_\Delta(\Delta) = 0, \]  

(3.44)

One immediately sees that the above relations coincide with the Breit-Wigner conditions. The pole conditions in Eqs. (3.41), (3.42) and (3.43) and the Breit-Wigner conditions start to differ from each other at the two-loop level which is beyond the order we are working at. From the conditions in Eq. (3.44) we finally obtain

\[ \delta \Delta = 4e_\Delta^2 M^2_\pi + \frac{2h_\Delta^2}{3F^2_\pi} (3M^2_\pi - 2\Delta^2) \lambda_\pi + \frac{25g^2_1 M^2_\pi}{864\pi^2 F^2_\pi} + \frac{h^2_\Delta(2\Delta^2 - 3M^2_\pi)}{72\pi^2 F^2_\pi} - \frac{h^2_\Delta(M^2_\pi - \Delta^2)}{F^2_\pi} \text{Re} \tilde{J}_0(\Delta), \]

\[ \delta Z_\Delta = - \frac{1}{18F^2_\pi} (36h_\Delta^2(M^2_\pi - 2\Delta^2) + 25g^2_1 M^2_\pi \lambda_\pi - 65g^2_1 M^2_\pi) \lambda_\pi - \frac{h^2_\Delta M^2_\pi}{F^2_\pi} \text{Re} \tilde{J}_0(\Delta). \]  

(3.45)

- **Renormalization of the \( \pi N \Delta \) axial coupling**

To renormalize the LEC \( h_A \) we consider the axial-vector nucleon-delta transition form factor, see Fig. 5 in the rest-frame of the delta:

\[ M(p_\Delta, p_N, q)^i_\mu = F^{3/2}_\Delta \xi^{ij}_{I = 3/2} \epsilon_A(q)_\nu^j H_A(p_\Delta \cdot v, q^2, p \cdot v) + \ldots, \]  

(3.46)

where the ellipses refer to other terms which are not relevant for renormalization of the \( \pi N \Delta \) axial coupling constant. We analytically continue the form factor \( H_A \) and choose the renormalization point to be

\[ h_A = \text{Re} H_A \left( \Delta - i \frac{\Gamma_\Delta}{2}, \left( \Delta - i \frac{\Gamma_\Delta}{2} \right)^2, 0 \right), \]  

(3.47)

which, in the one-loop approximation, becomes

\[ h_A = \text{Re} H_A (\Delta, \Delta^2, 0) \ldots \]  

(3.48)

Up to order \( e^3 \), the quantity \( H_A(\Delta, \Delta^2, 0) \) receives contributions from the tree-level diagram (first graph in the right panel of Fig. 5)

\[ H_A^{\text{tree}}(\Delta, \Delta^2, 0) = h_A + \delta h_A - (b_2 + b_7) \Delta + \frac{h_\Delta^2}{2}(\delta Z_N + \delta Z_\Delta) + 2(h_8 + 2(h_9 + h_{10}))M^2_\pi, \]  

(3.49)

one-loop diagrams without delta excitations (the remaining two diagrams in the upper raw of the right panel of Fig. 5)

\[ H_A^{\text{loop,} \pi N}(\Delta, \Delta^2, 0) = \frac{h_A(d - 1 - g^2_\lambda)}{(d - 1)F^2_\pi} I(d : 0) + \frac{g^2_1 h_A M^2_\pi}{(d - 1)F^2_\pi} I(d : 0; (0, 0)) \]

\[ - \frac{g^2_1 h_A (M^2_\pi - \Delta^2)}{(d - 1)F^2_\pi} I(d : 0; (\Delta, 0)) \]  

(3.50)
and one-loop graphs with pions, nucleons and delta degrees of freedom (diagrams in the second row of the right panel of Fig. 5).

\[
H_A^{\text{loop}, \pi N \Delta} (\Delta, \Delta^2, 0) = - \frac{(d - 3) h_A}{36(d - 1)^3 F_\pi^2} (12(d - 1) h_A^2 + 5(d + 1) g_l (3(d - 1)^2 g_A + 4 g_l)) I(d : 0) \\
- \frac{5(d^2 - 2d - 3) h_A g_l^2 M_\pi^2}{9(d - 1)^3 F_\pi^2 \Delta} I(d : 0; (0, 0)) - \frac{(d - 3) h_A^2 (M_\pi^2 - \Delta^2)}{6(d - 1)^2 F_\pi^2 \Delta} I(d : 0; (\Delta, 0)) \\
+ \frac{(d - 3) h_A (M_\pi^2 - \Delta^2)}{18(d - 1)^2 F_\pi^2 \Delta} (10(d + 1) g_l^2 + 3(d - 1) h_A^2) I(d : 0; (0, \Delta)). \tag{3.51}
\]

Substituting these expressions into the renormalization condition given in Eq. (3.48), we obtain the following order-\(\epsilon^3\) expression for \(\delta h_A\) in four dimensions:

\[
\delta h_A = - \frac{h_A}{2} (\delta Z_N + \text{Re} \, \delta Z_\Delta) + \Delta (b_2 + b_7) - 2 (h_8 + 2(h_9 + h_{10})) M_\pi^2 + (3h_A^2 + 5g_l^2 - 27g_A^2) \frac{h_A \Delta^2}{972\pi^2 F_\pi^2} \\
- \frac{h_AM_\pi^2}{2592\pi^2 F_\pi^2} (12h_A^2 - 108g_A^2 + 20g_l^2 + 195g_A g_l) + (81g_A^2 - 25g_l^2) h_A \frac{M_\pi^3}{1944\pi F_\pi^2 \Delta} \\
+ h_A (81g_A^2 + 9h_A^2 + 25g_l^2) \frac{4\Delta^2}{243 F_\pi^2} - h_A (100g_l^2 + 225g_A g_l + 36(h_A^2 + 9g_A^2 - 1)) \frac{M_\pi^2}{162 F_\pi^2} \lambda_\pi \\
- (9h_A^2 + 50g_l^2) \frac{h_A (M_\pi^2 - \Delta^2)}{486 F_\pi^2 \Delta} J_0 (-\Delta) + (h_A^2 + 18g_A^2) \frac{h_A (M_\pi^2 - \Delta^2)}{54 F_\pi^2 \Delta} \text{Re} J_0 (\Delta). \tag{3.52}
\]

**IV. DETERMINATION OF THE LECs FROM \(\pi N\) SCATTERING**

Given that the LECs in the effective Lagrangian with and without explicit delta degrees of freedom have a different meaning, we cannot use the values of the various LECs from our earlier work [50] based on the \(\Delta\)-less formulation and have to redo the analysis of the pion-nucleon system utilizing the small-scale expansion. Specifically, we need to calculate the \(\pi N\) scattering amplitude up to order \(\epsilon^3\).

Before discussing renormalization of the \(\pi N\) amplitude in the explicit decoupling scheme as explained in section [II], we first perform the following shifts in the LECs in order to get rid of redundant terms:

\[
h_A \rightarrow h_A - \Delta (b_2 + b_3 + b_6 + b_7) + \Delta^2 (h_{12} + h_{13}) + 4M_\pi^2 h_7, \\
\tilde{c}_2 \rightarrow \tilde{c}_2 + \frac{4(d - 2)}{3(d - 1)} h_A (b_3 + b_6) - \frac{2(d - 2)}{3(d - 1)} \Delta (b_3 + b_6)^2 - \frac{4(d - 2)}{3(d - 1)} \Delta h_A (h_{12} + h_{13}), \\
\tilde{c}_3 \rightarrow \tilde{c}_3 - \frac{4(d - 2)}{3(d - 1)} h_A (b_3 + b_6) + \frac{2(d - 2)}{3(d - 1)} \Delta (b_3 + b_6)^2 + \frac{4(d - 2)}{3(d - 1)} \Delta h_A (h_{12} + h_{13}), \\
\tilde{c}_4 \rightarrow \tilde{c}_4 + \frac{4}{3(d - 1)} h_A (b_3 + b_6) - \frac{2}{3(d - 1)} \Delta (b_3 + b_6)^2 - \frac{4}{3(d - 1)} \Delta h_A (h_{12} + h_{13}), \\
\tilde{d}_1 + \tilde{d}_2 \rightarrow \tilde{d}_1 + \tilde{d}_2 + \frac{d - 2}{6(d - 1)} (b_3 + b_6)^2 - \frac{d - 2}{3(d - 1)} h_A (h_{12} + h_{13}), \\
\tilde{d}_3 \rightarrow \tilde{d}_3 - \frac{d - 2}{6(d - 1)} (b_3 + b_6)^2 + \frac{d - 2}{3(d - 1)} h_A (h_{12} + h_{13}), \\
\tilde{d}_{14} - \tilde{d}_{15} \rightarrow \tilde{d}_{14} - \tilde{d}_{15} - \frac{2}{3(d - 1)} (b_3 + b_6)^2 + \frac{4}{3(d - 1)} h_A (h_{12} + h_{13}). \tag{4.53}
\]

Notice that these replacements are performed in the amplitude written in \(d\) dimensions. After this shift the amplitude does not depend on the LECs \(b_3 + b_6, b_2 + b_7, h_{12} + h_{13}\) and \(h_7\) anymore.

Let us now discuss renormalization of the pion-nucleon amplitude. All divergences which remain after expressing the amplitude in terms of physical quantities as discussed in the previous section are absorbed into redefinition of the LECs \(c_i\) and \(d_i\) entering the order-\(Q^2\) and \(Q^3\) effective pion-nucleon Lagrangians. While the LECs \(c_i\) are finite in the
\(\Delta\)-less framework provided one uses dimensional regularization with the \(\overline{\text{MS}}\)-scheme, this does not hold true anymore in the \(\Delta\)-full theory due to the appearance of ultraviolet divergencies \(\propto \Delta\) at \(\epsilon^3\) and higher powers of \(\Delta\) at orders beyond \(\epsilon^3\). We parametrize the bare LECs \(\bar{c}_i\) and \(\bar{d}_i\) via

\[
\bar{c}_i = c_i + \Delta \left( \frac{\beta_i^c}{F_\pi^2} \lambda_\pi + \frac{1}{4\pi F_\pi^2} \epsilon_i^c \right), \quad \bar{d}_i = \frac{\beta_i^d}{F_\pi^2} \lambda_\pi + d_i + \frac{1}{4\pi F_\pi^2} \epsilon_i^d, \tag{4.54}
\]

where the various \(\beta\)-functions relevant for pion-nucleon scattering are given by

\[
\begin{align*}
\beta_1^c &= 2 h_A^2, \\
\beta_2^c &= \frac{80}{2187} h_A^2 (9 g_A - 5 g_1)^2, \\
\beta_3^c &= \frac{16}{2187} h_A^2 \left( 729 + 5 (9 g_A - 5 g_1)^2 \right), \\
\beta_4^c &= -\frac{2}{2187} h_A^2 (972 + 2349 g_A^2 + 1152 h_A^2 - 2250 g_A g_1 + 125 g_1^2), \\
\beta_{14}^d &= \frac{1}{6} g_A^4, \\
\beta_{15}^d &= \beta_{18}^d = 0, \\
\beta_{14}^{d,\Delta} + \beta_{2}^{d,\Delta} + \beta_{3}^{d,\Delta} &= \frac{10}{27} h_A^2, \\
\beta_3^{d,\Delta} &= \frac{h_A^2}{2187} (125 g_1^2 + 288 h_A^2 - 243 g_A^2 - 450 g_A g_1), \\
\beta_5^{d,\Delta} &= -\frac{5}{27} h_A^2, \\
\beta_{18}^{d,\Delta} &= 0, \\
\beta_{14}^{d,\Delta} - \beta_{15}^{d,\Delta} &= \frac{2 h_A^2}{2187} (288 h_A^2 - 243 g_A^2 - 450 g_A g_1 + 125 g_1^2). \tag{4.55}
\end{align*}
\]

This particular form of the \(\beta\)-functions guarantees that the amplitude remains finite in the \(d \to 4\) limit. We use here the notation, in which the divergencies associated with loop diagrams without delta excitations (with delta excitations) are cancelled by terms \(\propto \beta_{14}^{d,N} (\propto \beta_{18}^{d,\Delta} \text{ and } \propto \beta_{15}^{d,\Delta})\). Furthermore, in order to maintain the explicit decoupling scheme, we have introduced additional finite dimensionless shifts \(c_{\Delta}^c\) and \(d_{\Delta}^d\). The explicit decoupling scheme is defined by the requirement that all observables calculated in the SSE include only nucleonic contributions after taking the \(\Delta \to \infty\) limit. In this limit all contributions emerging from the intermediate delta excitations have to vanish (in the explicit decoupling scheme) so that the delta isobar explicitly decouples from the theory. In order to satisfy the explicit decoupling, the values of the LECs \(c_{\Delta}^c\) and \(d_{\Delta}^d\) have to be chosen as

\[
\begin{align*}
c_{\Delta}^c &= 2 h_A^2 \log \left( \frac{2 \Delta}{\mu} \right), \\
c_{\Delta}^2 &= -\frac{2 h_A^2}{6561} (6399 g_A^2 - 8910 g_A g_1 + 3575 g_1^2) - \frac{80 h_A^2}{2187} (9 g_A - 5 g_1)^2 \log \left( \frac{2 \Delta}{\mu} \right), \\
c_{\Delta}^3 &= \frac{2 h_A^2}{6561} (6399 g_A^2 - 8910 g_A g_1 + 3575 g_1^2) + \frac{16 h_A^2}{2187} (729 + 5 (9 g_A - 5 g_1)^2) \log \left( \frac{2 \Delta}{\mu} \right), \\
c_{\Delta}^4 &= \frac{h_A^2}{6561} (4860 - 35559 g_A^2 + 1728 h_A^2 + 28350 g_A g_1 - 4775 g_1^2) \\
&\quad - \frac{2 h_A^2}{2187} (972 + 2349 g_A^2 + 1152 h_A^2 - 2250 g_A g_1 + 125 g_1^2) \log \left( \frac{2 \Delta}{\mu} \right). \tag{4.56}
\end{align*}
\]
Here, \( \omega, t \) and \( \bar{\omega}, \bar{t} \) are the pion cms energy, \( E_1 = E_2 = E = (q^2 + m^2)^{1/2} \) the nucleon energy and \( \bar{q}_1^2 = \bar{q}_2^2 \equiv \bar{q}^2 = ((s - M_+^2 - m^2)^2 - 4m^2M_+^2)/(4s) \). Further, \( t = (q_1 - q_2)^2 \) is the invariant momentum transfer squared while \( s \) denotes the total cms energy squared. The quantities \( q^\pm(\omega, t) \) (\( h^\pm(\omega, t) \)) refer to the isoscalar and isovector non-spin-flip (spin-flip) amplitudes and can be calculated in chiral perturbation theory. The contributions to the amplitudes which do not involve intermediate delta excitations up to order \( Q^4 \) (i.e. subleading one-loop order) are given in Ref. [50]. In Appendix \[ \boxed{3} \] we give explicitly the delta-isobar contributions up to order \( \epsilon^3 \). In a complete analogy to the \( \Delta \)-less calculation reported in Ref. [50], the phase shifts are obtained from the partial-wave amplitudes in the isospin basis \( f_{1\pm}(s) \) by means of the following unitarization prescription

\[
\delta^I_{1\pm}(s) = \arctan \left( \frac{q^2}{\Re f^I_{1\pm}(s)} \right) .
\] (4.60)

Determination of the LECs is carried out using exactly the same procedure as in our \( \Delta \)-less calculations [50]. While the \( \pi N \) scattering amplitude is worked out here only to order \( \epsilon^3 \), we decided to include also the order-\( Q^4 \) terms
obtained within the Δ-less theory when fitting the phase shifts in order to facilitate a direct comparison with the results of Ref. \[50\]. This way we make sure that the differences between the values of the LECs obtained in the two analyses are solely due to the explicit treatment of the delta degrees of freedom. The impact of the \(Q^4\) terms on the 3NF will be discussed in section VI.

For the pion-nucleon contributions to the scattering amplitude we proceed in exactly the same way as in Ref. \[50\]. We remind the reader that certain LECs \(\bar{c}_i\) from \(\mathcal{L}_{\pi NN}^{(4)}\) enter the amplitude only in linear combinations with the LECs \(c_i\) and, therefore, cannot be determined from \(\pi N\) scattering data. Following Refs. \[73\] and \[50\], these \(\bar{c}_i\)-contributions are absorbed into redefinition of the \(c_i\)’s by setting

\[
e_{22} - 4e_{38} - \frac{l_3c_1}{F_\pi^2} = 0, \quad e_{20} + e_{35} = 0, \quad 2e_{19} - e_{22} - e_{36} + 2\frac{l_3c_1}{F_\pi^2} = 0, \quad 2e_{21} - e_{37} = 0,
\]

without loss of generality. The LEC \(d_{18}\) from \(\mathcal{L}_{\pi NN}^{(4)}\) can be fixed by means of the Goldberger-Treiman discrepancy

\[
g_{\pi NN} = \frac{g_A m_N}{F_\pi} \left(1 - \frac{2M_\pi^2 d_{18}}{g_A}\right),
\]

where for \(g_{\pi NN}\) we adopt the value from Ref. \[74\] of \(\frac{g_{\pi NN}^2}{(4\pi)} \approx 13.54\), which also agrees with the determination in Ref. \[75\] based on the Goldberger-Miyazawa-Oehme sum rule and utilizing the most accurate available data on the pion-nucleon scattering lengths. We set \(d_{10} = 0\) and use the effective, larger value for \(g_A\) of

\[
g_A = \frac{F_\pi g_{\pi NN}}{m_N} \approx 1.285
\]

in all expressions. This is a legitimate procedure at the order we are working. This leaves us with 13 independent (linear combinations of the) low energy constants in the nucleonic contributions to the scattering amplitude which have to be fixed from a fit to the data, namely \(c_{1,2,3,4}, d_{1}+d_{2}, d_3, d_4-d_{15}, \) and \(\bar{c}_{14,15,16,17,18}\), see Ref. \[50\] for more details and explicit expressions. We also use the same values for the pion mass and decay constant as in that reference, namely \(M_\pi = 138.03\) MeV and \(F_\pi = 92.4\) MeV.

The contributions to the amplitude associated with the delta excitations and given in appendix B involve further LECs, namely \(h_A\) and \(g_1\). For the \(\pi N\Delta\) axial vector constant we adopt the value of \(h_A = 1.34\) which is fixed from the width of the delta-resonance and also agrees well with the large-\(N_c\) prediction \[76,77\]. Notice that similarly to the convention adopted for \(g_A\), the \(\pi N\Delta\) Goldberger-Treiman discrepancy is implicitly taken into account by using the above value of the LEC \(h_A\). Our fits to \(\pi N\) data turn out to be fairly insensitive to a particular value of \(g_1\). For this reason we decided to fix it to its large-\(N_c\) value \(g_1 = 9/5 g_A \approx 2.31\) \[76,78\]. We, therefore, have to finally fix exactly the same combinations of the low-energy constants as in the Δ-less theory.

As in Ref. \[50\] we performed a combined fit for all \(s-, p-,\) and \(d-\)waves. We remind the reader that it is crucial to include \(d\)-waves in the fit as they impose severe constraints on some of the \(e_i\) constants, especially on \(e_{14}\) and \(\bar{e}_{17}\), which also enter the \(N^4LO\) expressions for the three-body force. The results of the fits using the partial wave analysis (PWA) by the George-Washington University group (GW) \[79\] and the Karlsruhe-Helsinki group (KH) \[80\] are presented in Figs. 6 and 7 respectively. In these figures we show the full, i.e. order-\(e^3 + Q^4\) results (solid curves) as well as the phase shifts calculated up to order \(e^3\) without \(Q^4\) terms (dashed curves) and \(e^2\) (dashed-dotted curves) using the same parameters (from the order-\(e^3 + Q^4\) fit) in all curves. We fit the data points from threshold up to \(p_{\text{Lab}} = 150\) MeV/c, and obtain a description of the phase shifts similar to the Δ-less case. Naturally, the description of the \(P_{33}\) partial wave (delta s-channel) is significantly improved.

Notice that more sophisticated studies of pion-nucleon scattering employing a covariant formulation of baryon chiral effective field theory with and without explicit \(\Delta(1232)\) degrees of freedom have been carried out recently, see Refs. \[78,81,84\]. Also, more reliable ways to extract the low-energy constants from the \(\pi N\) reaction and to estimate their uncertainties have been explored as compared to the ones employed in our analysis. Those include, in particular, analytic extrapolations of the scattering amplitude into the subthreshold region using the solutions to the Roy-Steiner formalism.

---

1 The results for the 3NF are expected to be much less sensitive to the precise value of \(h_A\) than to the value of \(g_A\). This is because the changes in \(h_A\) can, to some extent, be compensated by the changes in the LECs from the subleading and higher-order effective Lagrangian.
FIG. 6: Results of the fit for $\pi N$ $s$, $p$ and $d$-wave phase shifts using the GW partial wave analysis of Ref. [79]. The solid curves correspond to the $\epsilon^3 + Q^4$ results, the dashed curves to the order-$\epsilon^3$ results, and the dashed-dotted curves to the order-$\epsilon^2$ calculation.

equation and a direct determination of the LECs from the available $\pi N$ scattering data in the physical region instead of using partial-wave analyses, see Refs. [78 82 83 85–87]. Future studies of nuclear forces and few-nucleon systems should, obviously, employ the most reliable available values of the $\pi N$ LECs such as e.g. the ones from Refs. [78 84 85]. In this paper we, however, focus mainly on the $\Delta$ contributions to the 3NF. To facilitate a comparison between the $\Delta$-full and $\Delta$-less calculation of Refs. [50, 51] and to allow for an unambiguous interpretation of our results in terms of resonance saturation, we follow here the same procedure for the determination of various LECs as adopted in Ref. [50].

We finally turn to the discussion of the extracted parameters. The obtained values of the low energy constants are collected in Table I. We also looked at the statistical errors of the fitted parameters in order to see qualitatively which low-energy constants (or their linear combinations) are well constrained by the data and which of them are poorly

|          | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $d_1 + d_2$ | $d_3$ | $d_5$ | $d_{14} - d_{15}$ | $\bar{e}_{14}$ | $\bar{e}_{15}$ | $\bar{e}_{16}$ | $\bar{e}_{17}$ | $\bar{e}_{18}$ |
|----------|-------|-------|-------|-------|------------|-------|-------|------------------|----------------|----------------|----------------|----------------|----------------|
| fit to the GW PWA [79] | $-1.32$ | $0.39$ | $-2.68$ | $1.86$ | $1.46$ | $-1.01$ | $-0.10$ | $-2.16$ | $0.06$ | $-2.47$ | $-0.05$ | $-0.56$ | $0.54$ |
| statistical error | $0.45$ | $1.34$ | $0.16$ | $0.07$ | $0.17$ | $0.31$ | $0.19$ | $0.33$ | $0.03$ | $0.07$ | $0.80$ | $0.38$ | $4.66$ |
| fit to the KH PWA [80] | $-0.85$ | $0.45$ | $-1.91$ | $1.49$ | $2.07$ | $-2.45$ | $0.66$ | $-3.86$ | $-0.12$ | $-7.05$ | $3.39$ | $-0.38$ | $2.85$ |
| statistical error | $0.50$ | $1.47$ | $0.18$ | $0.10$ | $0.19$ | $0.33$ | $0.20$ | $0.36$ | $0.03$ | $0.08$ | $0.90$ | $0.48$ | $5.12$ |

TABLE I: Low-energy constants obtained from a fit to the empirical $s$, $p$- and $d$-wave pion-nucleon phase shifts using partial wave analysis of Ref. [79] and of Ref. [80]. Values of the LECs are given in GeV$^{-1}$, GeV$^{-2}$ and GeV$^{-3}$ for the $c_i$, $d_i$ and $\bar{e}_i$, respectively.
determined. Similarly to the strategy utilized in Ref. [SS] we assigned the same relative error to each data point from the partial wave analyses equal to 5%. This ansatz is somewhat arbitrary but seems reasonable for an estimate of the relative uncertainties of different low-energy constants. Notice further that the statistical errors are calculated in the linearized approximation, i.e. the covariance matrix is taken to be the inverse of the Hessian matrix of the $\chi^2$ function at its minimum. Such an approximation is sufficient for a qualitative analysis that we are going to perform. The resulting statistical uncertainties for all low energy constants are listed in Table I and appear to be almost the same for both the KH and GW analyses. Moreover, they change very little when the fit is performed in the $\Delta$-less case as in ref. [54]. One can see that the low energy constants $c_2$, $\bar{e}_{15}$ and $\bar{e}_{16}$ have the largest errors ($0.8 - 5.1$) in the corresponding natural units (GeV$^{-1}$, GeV$^{-2}$ and GeV$^{-3}$ for the $c_i$, $\bar{d}_i$ and $\bar{e}_i$, respectively), indicating that these parameters are not well determined in the fit. On the other hand, $\bar{e}_{17}$, $\bar{e}_{14}$, which are the only LECs $\bar{e}_i$ contributing to the 3NF at order $Q^5$, and the LEC $c_4$ are strongly constrained by the data.

Another important quantity is a correlation between pairs of parameters. The largest in magnitude values for the correlation coefficients are obtained for the pairs $c_1 - c_2$ (0.99), $c_1 - \bar{e}_{16}$ ($-0.98$), $c_2 - \bar{e}_{16}$ ($-0.99$), $(\bar{d}_{1} + \bar{d}_{2}) - (\bar{d}_{14} - \bar{d}_{15})$ ($-0.97$). In order to get a more detailed information on the correlation among various parameters we have computed eigenvalues of the covariance matrix. The square roots of their numerical values in natural units are $5.41, 0.59, 0.45, 0.35, 0.29, 0.12, 0.05, 0.03, 0.02, 0.02, 0.01, 0.01$ for the KH analysis and $4.92, 0.51, 0.37, 0.32, 0.27, 0.11, 0.04, 0.03, 0.02, 0.02, 0.01, 0.01, 0.01$ for the GW analysis. One can see that the first eigenvalue is at least two orders of magnitude larger than any of the other eigenvalues. This indicates that fixing certain linear combination of parameters results in very slow changes in the $\chi^2$ even if the individual values of the LECs entering this linear combination change significantly. This combination is the corresponding eigenvector and is approximately equal to $-0.1c_1 - 0.3c_2 - 0.1\bar{e}_{15} + 0.9\bar{e}_{16}$ (the other constants enter with much smaller coefficients). The coefficients are given in natural units. We indeed observe that these four parameters are strongly correlated and

![Fig. 7](image-url)
one can obtain fits comparable with the best one with those parameters being significantly shifted. The appearance of such a strong correlation among the parameters reflects the fact that one cannot fully resolve the energy dependence of the amplitude with a good accuracy in the low-energy regime. This interpretation is confirmed by performing a fit to higher energy, namely \( p_{\text{lab}} = 200 \text{ MeV/c} \), see Table II. Comparing the second and the third rows of this table, one observes a sizable reduction in the statistical errors and the correlations (including the ones among \( c_1, c_2, \bar{c}_1, \bar{c}_6 \)) do become significantly smaller. Unfortunately, the purely perturbative approach cannot be expected to be applicable at such energies as the phase shifts become quite large.

Finally, it is interesting to compare the values of the LECs with the ones obtained in the \( \Delta \)-less approach. As already pointed out before, one expects to find more natural values of the LECs in the \( \Delta \)-full theory. This is indeed the case as one can see from Table III, where such a comparison is carried out for the KH fits. The situation for the GW fits is similar, see Table II of [50]. Comparing the second and the third rows of this table, one observes a sizable reduction in magnitude for most of the LECs when the delta is included as an explicit degree of freedom. This raises the question of whether these differences can be understood analytically. In the following, we address this question by isolating the contributions of the delta to the various LECs. To this aim, we make a 1/\( \Delta \)-expansion of the delta-resonance contributions to the \( \pi N \) amplitude and match the expanded expressions to the amplitude obtained in the \( \Delta \)-less theory up to order \( Q^4 \). We decompose the various renormalized LECs into \( \Delta \)-less (\( \Delta \)) and delta contributions (\( \Delta \)) via

\[
\tilde{c}_i = c_i(\Delta) + c_i(\Delta), \quad \tilde{d}_i = d_i(\Delta) + d_i(\Delta), \quad \tilde{e}_i = e_i(\Delta) + e_i(\Delta),
\]

(4.64)

Expanding the \( \epsilon^1 \)-result up to order 1/\( \Delta \) we recover the well-known results for the \( c_i \)'s [89]

\[
c_1(\Delta) = 0, \quad c_2(\Delta) = \frac{4 h_A^2}{9 \Delta}, \quad c_3(\Delta) = -\frac{4 h_A^2}{9 \Delta}, \quad c_4(\Delta) = \frac{2 h_A^2}{9 \Delta}.
\]

(4.65)

From the 1/\( \Delta^2 \) terms of the order-\( \epsilon^1 \) \( \pi N \)-amplitude, we obtain the delta contributions to the LECs \( \tilde{d}_i \) given by

\[
d_1(\Delta) + d_2(\Delta) = \frac{h_A^2}{9 \Delta^2}, \quad d_3(\Delta) = -\frac{h_A^2}{9 \Delta^2}, \quad d_4(\Delta) - d_5(\Delta) = -\frac{2 h_A^2}{9 \Delta^2}.
\]

(4.66)

In principle, one could also expect 1/\( \Delta \) contributions from the order-\( \epsilon^2 \) \( \pi N \)-amplitude. However, all such terms turn out to contribute to renormalization of \( h_A \) and do not lead to resonance saturation of \( d_i \). One observes from table II that the delta contributions explain at least a half of the size of the LECs \( d_1 + d_2, d_3 \) and \( d_4 - d_5 \), which appear to be unnaturally large in the \( \Delta \)-less approach, see also Ref. [78] for similar conclusions.

To explore delta-resonance saturation of the LECs \( e_i \) from \( \mathcal{L}^{(4)}_{\pi N} \) which enter the order-\( Q^4 \) pion-nucleon amplitude, we need to analyze the following terms:

\[
\begin{array}{cccccccccccc}
\text{c}_1 & \text{c}_2 & \text{c}_3 & \text{c}_4 & \text{c}_5 & \text{c}_6 & \text{c}_7 & \text{c}_8 & \text{c}_9 & \text{c}_{10} & \text{c}_{11} & \text{c}_{12} \\
\text{d}_1 & \text{d}_2 & \text{d}_3 & \text{d}_4 & \text{d}_5 & \text{d}_6 & \text{d}_7 & \text{d}_8 & \text{d}_9 & \text{d}_{10} & \text{d}_{11} & \text{d}_{12} \\
\text{e}_1 & \text{e}_2 & \text{e}_3 & \text{e}_4 & \text{e}_5 & \text{e}_6 & \text{e}_7 & \text{e}_8 & \text{e}_9 & \text{e}_{10} & \text{e}_{11} & \text{e}_{12} \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\text{Q}^4, \text{KH PWA [80]} & -0.75 & 3.49 & -4.77 & 3.34 & 6.21 & -6.83 & 0.78 & -12.02 & 1.52 & -10.41 & 6.08 & -0.37 & 3.26 \\
\text{c}^3 + \text{Q}^4, \text{KH PWA [80]} & -0.85 & 0.45 & -1.91 & 1.49 & 2.07 & -2.45 & 0.66 & -3.86 & -0.12 & -7.05 & 3.39 & -0.38 & 2.85 \\
\text{\Delta}-\text{contribution} & 0 & 2.81 & -2.81 & 1.40 & 2.39 & -2.39 & 0 & -4.77 & 1.87 & -4.15 & 4.15 & -0.17 & 1.32 \\
\end{array}
\]
• $1/{\Delta}^3$-contributions from $\epsilon^1$-amplitude
• $1/{\Delta}^2$-contributions from $\epsilon^2$-amplitude (these terms vanish after renormalization of $h_A$),
• $1/{\Delta}$-contributions from $\epsilon^3$-amplitude.

The complete contribution of the delta to these LECs is given by a sum of these terms and has the form:

\[
\begin{align*}
\bar{e}_{14}(\Delta) &= \frac{h_A^2}{864 F_\pi^2 \pi^2 \Delta} \left(7 + 10 \log \left(\frac{2 \Delta}{M_\pi}\right)\right), \\
\bar{e}_{15}(\Delta) &= -\frac{h_A^2}{18 \Delta^3} \frac{h_A^3}{839808 F_\pi^2 \pi^2 \Delta} (3969 g_A^2 - 4050 g_A g_1 + 1225 g_1^2), \\
\bar{e}_{16}(\Delta) &= \frac{h_A^2}{18 \Delta^3} \frac{h_A^3}{839808 F_\pi^2 \pi^2 \Delta} (3969 g_A^2 - 4050 g_A g_1 + 1225 g_1^2), \\
\bar{e}_{17}(\Delta) &= -\frac{h_A^2}{1728 F_\pi^2 \pi^2 \Delta} (1 + 2 \log \left(\frac{2 \Delta}{M_\pi}\right)), \\
\bar{e}_{18}(\Delta) &= \frac{h_A^2}{36 \Delta^3} \frac{h_A^3}{839808 F_\pi^2 \pi^2 \Delta} (2025 g_A^2 + 3456 h_A^2 - 450 g_A g_1 + 425 g_1^2) \\
&\quad - \frac{h_A^2 g_A^2}{108 F_\pi^2 \pi^2 \Delta} \log \left(\frac{2 \Delta}{M_\pi}\right), \\
\bar{e}_{19}(\Delta) - \frac{1}{2} \bar{e}_{36}(\Delta) - 2 \bar{e}_{38}(\Delta) &= -\frac{h_A^3}{93312 F_\pi^2 \pi^2 \Delta} (351 + 1296 g_A^2 + 400 g_1^2) \\
&\quad + \frac{h_A^2}{5184 F_\pi^2 \pi^2 \Delta} (-33 + 81 g_A^2 - 50 g_A g_1 + 25 g_1^2) \log \left(\frac{2 \Delta}{M_\pi}\right), \\
\bar{e}_{20}(\Delta) + \bar{e}_{35}(\Delta) &= \frac{h_A^2}{5832 F_\pi^2 \pi^2 \Delta} (81 g_A^2 + 25 g_1^2) - \frac{h_A^2}{5184 F_\pi^2 \pi^2 \Delta} (81 g_A^2 - 50 g_A g_1 + 25 g_1^2) \log \left(\frac{2 \Delta}{M_\pi}\right), \\
\bar{e}_{21}(\Delta) - \frac{1}{2} \bar{e}_{37}(\Delta) &= -\frac{h_A^2}{62208 F_\pi^2 \pi^2 \Delta} (72 - 999 g_A^2 + 384 h_A^2 + 750 g_A g_1 - 175 g_1^2) \\
&\quad + \frac{h_A^2}{10368 F_\pi^2 \pi^2 \Delta} (-24 + 135 g_A^2 + 50 g_A g_1 - 25 g_1^2) \log \left(\frac{2 \Delta}{M_\pi}\right), \\
\bar{e}_{22}(\Delta) - 4 \bar{e}_{38}(\Delta) &= -\frac{h_A^2}{72 F_\pi^2 \pi^2 \Delta} \left(1 + \log \left(\frac{2 \Delta}{M_\pi}\right)\right). 
\end{align*}
\]

(4.67)

The appearance of logarithms of the physical pion mass in the above expressions is due to our choice of the renormalization scale $\mu = M_\pi$ in the definitions of $c_i$, $d_i$, and $e_i$. The LECs $\bar{e}_{15}(\Delta)$, $\bar{e}_{16}(\Delta)$ and $\bar{e}_{18}(\Delta)$ receive $1/{\Delta}^2$-contributions from the order-$\epsilon^1$ $\pi N$-amplitude. Numerically, these terms dominate over the loop contributions (as one would expect from naive dimensional analysis) and explain a half of the size of the LECs $\bar{e}_{15}$, $\bar{e}_{16}$ and $\bar{e}_{18}$ which appear to be unnaturally large in the $\Delta$-less theory, see Table III. It is comforting to see that the delta contributions to the LECs $c_i$, $d_i$ and $e_i$ given in the above expressions, whose numerical values are listed in table III, are in a very good agreement with the differences between the $\Delta$-less and $\Delta$-full fits. Clearly, one should not expect this agreement to be perfect since the delta contributions to the amplitude involve terms beyond the order-$Q^4$ $\Delta$-less result. Our findings, however, indicate that these resummed contributions are likely to be small and the most important terms are well represented by the $\Delta$-resonance contributions to the LECs $c_i$, $d_i$ and $e_i$.

Last but not least, we emphasize that the (linear combinations of the) LECs $\bar{e}_{19, 20, 21, 22, 35, 36, 37, 38}$ and $c_1$ absorbed into redefinition of $c_i$’s, see Eq. (4.61), do receive contributions due to the delta resonance, see Eq. (4.67). Assuming that these LECs are saturated by the delta, we may estimate the shifts in the $c_i$ induced by absorbing these order-$Q^4$ contributions via

\[
\begin{align*}
c_1 &\to c_1 + 2 M_\pi^2 \left(\bar{e}_{22} - 4 \bar{e}_{38} + \frac{t_3 c_1}{F_\pi^2}\right), \\
c_2 &\to c_2 - 8 M_\pi^2 (\bar{e}_{20} + \bar{e}_{35}), \\
c_3 &\to c_3 - 4 M_\pi^2 (2 \bar{e}_{19} - \bar{e}_{22} - \bar{e}_{36}),
\end{align*}
\]
\[
c_4 \rightarrow c_4 - 4M^2_\pi (2\bar{e}_{21} - \bar{e}_{37}).
\] (4.68)

Since the delta-resonance contributions to the induced shifts of \(c_i\)'s start with the loop corrections and do not have any \(1/\Delta^3\) contribution from the order-\(\epsilon^1\) terms, the induced shifts appear to be rather small:

\[
2M^2_\pi \left(\bar{e}_{22}(\Delta) - 4\bar{e}_{38}(\Delta) + \frac{\bar{t}_3 c_1(\Delta)}{F^2_\pi}\right) = -0.10 \text{ GeV}^{-1},
\]
\[
-8M^2_\pi (\bar{e}_{20}(\Delta) + \bar{e}_{35}(\Delta)) = -0.14 \text{ GeV}^{-1},
\]
\[
-4M^2_\pi (2\bar{e}_{19}(\Delta) - \bar{e}_{22}(\Delta) - \bar{e}_{36}(\Delta)) = 0.10 \text{ GeV}^{-1},
\]
\[
-4M^2_\pi (2\bar{e}_{21}(\Delta) - \bar{e}_{37}(\Delta)) = -0.26 \text{ GeV}^{-1}.
\] (4.69)

V. \(\Delta(1232)\) Contributions to the Two-Pion Exchange 3NF

After these preparations, we are now in the position to discuss the contributions to the two-pion exchange 3NF emerging from the intermediate \(\Delta\)-excitations up to the leading one-loop order (i.e. \(N^3\text{LO}\)).

In the isospin and static limits, the general structure of the two-pion exchange 3NF in momentum space has the following form (modulo terms of a shorter range, see Ref. [50] for more details):

\[
V_{2\pi} = \frac{\bar{\sigma}_1 \cdot \bar{q}_1 \bar{\sigma}_3 \cdot \bar{q}_3 (\bar{\tau}_1 \cdot \tau_3 A(q_2) + \tau_1 \times \tau_3 \cdot \bar{\tau}_2 \bar{q}_1 \times \bar{q}_3 \cdot \bar{\sigma}_2 B(q_2))}{[q_1^2 + M^2_i][q_3^2 + M^2_f]},
\] (5.70)

where \(\bar{\sigma}_i (\tau_i)\) denote the Pauli spin (isospin) matrices for the nucleon \(i\) while \(\bar{q}_i\) is the momentum transfer, \(\bar{q}_i = \bar{p}_i' - \bar{p}_i\), with \(\bar{p}_i'\) and \(\bar{p}_i\) being the final and initial momenta of the nucleon \(i\). Here and in what follows, we use the notation: \(q_i \equiv |\bar{q}_i|\). Unless stated otherwise, the expressions for the 3NF are given for a particular choice of the nucleon labels. The complete result can then be obtained by taking into account all possible permutations of the nucleons,

\[
V_{3N}^{\text{full}} = V_{3N} + 5 \text{ permutations}.
\] (5.71)

The quantities \(A(q_2)\) and \(B(q_2)\) in Eq. (5.70) are scalar functions of the momentum transfer \(q_2\) of the second nucleon whose explicit form is derived within the chiral expansion. In the \(\Delta\)-less framework, this expansion starts at \(N^2\text{LO}\) which corresponds to the order \(Q^3\). The explicit expressions for \(A(q_2)\) and \(B(q_2)\) up to \(N^4\text{LO}\), i.e. up to order \(Q^5\), can be found in Ref. [50]. In the \(\Delta\)-full framework, the leading contributions are shifted from \(N^2\text{LO}\) to NLO, i.e. to order \(\epsilon^2\). These leading delta contributions have the form

\[
A_{\Delta}^{(2)}(q_2) = -\frac{g_A^2 h_A^2}{18 \Delta F^2_\pi} (2M^2_\pi + q_2^2),
\]
\[
B_{\Delta}^{(2)}(q_2) = \frac{g_A^2 h_A^2}{36 \Delta F^2_\pi},
\] (5.72)

and are known to provide the dominant long-range mechanism of the 3NF [52]. There are no contributions of the delta to \(A(q_2)\) and \(B(q_2)\) at \(N^3\text{LO}\) [59], i.e. at order \(\epsilon^3\), except for the shift of the LEC \(h_A\) as discussed in section IV see Eq. (4.53). At \(N^3\text{LO}\) \((\epsilon^4)\) one has to take into account the contributions emerging from the diagrams shown in Fig. 8. These graphs are analogous to the \(\Delta\)-less ones shown in Fig. 2 of Ref. [10] but involve at least one intermediate \(\Delta\) excitation. Notice that in contrast to that work, we do not show in Fig. 8 certain diagrams which yield vanishing results for the sake of compactness. This concerns, for example, one-loop graphs leading to integrals involving an odd power of the loop momentum to be integrated over.

The last three diagrams in Fig. 8 contribute to renormalization of the pion field and the lowest-order pion-nucleon and pion-nucleon-delta vertices and also give rise to the corresponding Goldberger-Treiman discrepancy relations. These contributions are automatically taken into account by expressing the 3NF in terms of physical quantities and using the effective values of the LECs \(g_A\) and \(h_A\) which account for the Goldberger-Treiman discrepancy, see the discussion in the previous section. We are, therefore, left with one-loop diagrams constructed out of the lowest-order and tree-order graphs which involve a single insertion of the \(1/m_N\)-vertices which give rise to the leading relativistic corrections. We remind the reader that the power counting scheme used to derive the nuclear forces and currents in
FIG. 8: Two-pion exchange 3N diagrams involving intermediate delta excitations at $N^3$LO. Solid dots and filled rectangles denote vertices from $\mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(1)} \Delta + \mathcal{L}_{\pi N}^{(3)} \Delta$, respectively. Open rectangles refer to $1/m$-vertices from $\mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{\pi N}^{(2)} \Delta$. Diagrams which result from the interchange of the nucleon lines and/or application of the time reversal operation are not shown. Also not shown are diagrams which lead to vanishing contributions to the 3NF. For the remaining notation see Fig. 2.

Refs. [32, 35, 37, 40, 41, 45, 46, 50, 51, 90] and [91–93] makes the assignment $Q/m_N \sim Q^2/\Lambda^2$ for the nucleon mass, see Ref. [94] for more details. This implies that $1/m_N$-corrections to the nuclear forces and currents are shifted to higher orders compared to the corresponding static contributions. In particular, the leading relativistic corrections appear at the same order with the leading one-loop diagrams.

It is important to keep in mind that, in order to derive the genuine 3NF contributions, one needs to separate the

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2 Notice that the same power counting is employed to determine the LECs from pion-nucleon scattering in Ref. [33] and in this work.
irreducible parts in the corresponding amplitudes in order to avoid double counting when iterating the potentials in the scattering equation. While this can be achieved in different ways, see Ref. [95] for more details, we employ here the method of unitary transformation which was first applied in the context of chiral EFT in Ref. [96]. A comprehensive discussion of this approach can be found in Ref. [97]. The same method was used in our earlier work on the derivation of the three- [40] [41] [50] [51] and four-nucleon forces [45] [46] and electroweak nuclear current operators [91] [92]. We remind the reader that in this approach one first applies the canonical formalism to the effective chiral Lagrangian expressed in terms of renormalized fields to derive the Hamilton density in the pion-nucleon sector. In the second step, one decouples the purely nucleonic subspace of the Fock space from the rest via a suitably chosen unitary transformation. The determination of the unitary operator and the resulting nuclear potentials is carried out perturbatively within the EFT approach. Clearly, there is always certain ambiguity in the choice of the unitary operator. However, as was found in Refs. [45] [46], most of the choices of the unitary operator lead to nuclear potentials which cannot be renormalized, i.e. the corresponding matrix elements involve ultraviolet-divergent integrals even after expressing all LECs in terms of their physical values. While this is, of course, not a fundamental problem since nuclear potentials do not correspond to observable quantities, it is desirable to have a well-defined and finite nuclear Hamiltonian. Enforcing renormalizability at the level of the Hamiltonian strongly restricts the unitary ambiguity mentioned above.

In particular, the renormalizability requirement was found to lead to an unambiguous result for the static parts of the three- [40] [41] [50] [51] and four-nucleon potentials [45] [46], while the leading relativistic corrections still depend on two arbitrary constants, which parametrize the unitary ambiguity at this order in the chiral expansion [41]. Explicit expressions for the nuclear Hamiltonian in the operator form after fixing the unitary ambiguity up to NLO in the \( \Delta \)-less approach can be found in Refs. [41] [46] [50].

To employ the method of unitary transformation within the small scale expansion one can follow the lines of Ref. [46]. The crucial difference is that one now needs to decouple not only pions but also the delta degrees of freedom. As discussed in that work, it is convenient to start with the minimal parametrization of the unitary operator using the ansatz proposed by Okubo [98], see Eq. (2.12) of Ref. [46]. Using this parametrization, the unitary operator can be calculated via a perturbative solution of the decoupling equation (2.13) of Ref. [46] within the small-scale expansion. The resulting rather lengthy expressions for the delta contributions to the nuclear force in the operator form are not listed in this work but can be made available as a Mathematica notebook upon request from the authors. Notice further that the resulting nuclear Hamiltonian is defined unambiguously within this ansatz but is not renormalizable as explained before. Following the lines of Ref. [46], we exploit the unitary ambiguity to ensure renormalizability of the nuclear potentials. This is achieved by applying all possible additional unitary transformations acting on the nucleonic subspace of the Fock space which can be constructed at a given order in the SSE, see Ref. [46] for more details. The corresponding transformation angles are to be chosen in such a way that the resulting Hamiltonian is finite. In the \( \Delta \)-less approach, we had to introduce six such additional unitary transformations (plus two more transformations involving \( 1/m_N \)-corrections) whose generators are given in Ref. [46] (Ref. [92]). The inclusion of the delta excitations in the intermediate states allows for much more flexibility in the construction of the additional unitary transformations. In particular, we were able to write 50 antihermitian generators \( S_\Delta \) which are listed in Eq. (A.4). The corresponding unitary transformations generate additional contributions to the nuclear Hamiltonian which depend on 50 real parameters \( \alpha_\Delta \). In order to derive nuclear potentials, one has to evaluate the corresponding matrix elements of the nuclear Hamilton operator written in second-quantized form. Calculating the 3NF contributions, expressing them in terms of physical parameter and requiring that there are no ultraviolet divergencies lead to constraints on \( \alpha_\Delta \), which are given in Eq. (A.5). In particular, we find that 23 specific linear combinations of the \( \alpha_\Delta \)'s have to vanish. While these constraints obviously do not allow for a unique determination of these parameters, we find that they lead to an unambiguous result for the 3NF, which does not depend on any of the undetermined linear combinations of \( \alpha_\Delta \)'s.

We now turn to the results for the 3NF and consider first the static terms. We obtain the following contributions of the delta-isobar to the functions \( A_\Delta(q_2) \) and \( B_\Delta(q_2) \) at leading one-loop order:

\[
A_\Delta^{(4)}(q_2) = -\frac{g_A^2 h_A^2}{13996\pi^2 \Delta^3 F_0^2} \left( 81 g_A^2 (40 \Delta^4 + 34 M_\pi^4 - \pi(\Delta^2 M_\pi^2 - 13 \Delta^2 M_\pi^2)) (2M_\pi^2 + q_2^2) \right) - 450 g_A g_1 (8 \Delta^4 + 2 M_\pi^4 - \pi(\Delta^2 M_\pi^2 - 5 \Delta^2 M_\pi^2 - 13 \Delta^2 M_\pi^2)) (2M_\pi^2 + q_2^2) + 36 \Delta (20 \pi h_A^2 M_\pi^2 (2M_\pi^2 + q_2^2)) - 27 (2 \Delta^2 M_\pi^2 - \Delta^3) (M_\pi^2 + 2 q_2^2) + 25 q_2^2 (40 \Delta^4 + 34 M_\pi^4 - 17 \pi(\Delta^2 M_\pi^2 - 13 \Delta^2 M_\pi^2)) (2M_\pi^2 + q_2^2) \right)
\] 
\[
+ \frac{g_A^2 h_A^2}{144\pi^2 F_0^2} \Delta D(q_2) (M_\pi^2 + 2 q_2^2) (-2 \Delta^2 + 2 M_\pi^2 + q_2^2) - \frac{g_A^2 h_A^2}{144\pi^2 F_0^2} \Delta L(q_2) (M_\pi^2 + 2 q_2^2)
\] 
\[
+ \frac{g_A^2 h_A^2}{13996\pi^2 \Delta^3 F_0^2} H(0) (81 g_A^2 (40 \Delta^4 + 34 M_\pi^4 - 47 \Delta^2 M_\pi^2)) (2M_\pi^2 + q_2^2)
\]
where we use the notation for the various loop functions introduced in Ref. [57] which, in the case of dimensional regularization, reads:

\[
L(q) = \sqrt{q^2 + 4M_x^2} \ln \frac{\sqrt{q^2 + 4M_x^2} + q}{2M_x},
\]

\[
D(q) = \frac{1}{\Delta} \int_{2M_x}^{\infty} \frac{d\mu}{q^2 + \mu^2} \arctan \frac{\sqrt{\mu^2 - 4M_x^2}}{2\Delta},
\]

\[
H(q) = \frac{4M_x^2 + 2q^2 - 4\Delta^2}{4M_x^2 + q^2 - 4\Delta^2} \left[ L(q) - L \left( \frac{2\Delta}{M_x} \right) \right].
\]

The expressions for the loop functions resulting in the framework of spectral function regularization introduced in Ref. [59] can be found in [58]. $1/\Delta$-expansion of the loop functions is given by

\[
D(q) = \frac{-L(q) + \log \left( \frac{2\Delta M_x^2}{3M_x} \right) + 1}{2\Delta^2} + \frac{-3L(q) (4M_x^2 + q^2) + 3 (6M_x^2 + q^2) \log \left( \frac{2\Delta M_x^2}{3M_x} \right) + 3M_x^2 + q^2}{72\Delta^4} + \frac{10L(q) (4M_x^2 + q^2)^2 - 15M_x^4 + 10M_x^2 q^2 + 10 (30M_x^4 + 10M_x^2 q^2 + q^4) \log \left( \frac{2\Delta M_x^2}{3M_x} \right) + 2q^4}{1600\Delta^6} + O(1/\Delta^8),
\]

\[
H(0) = 1 - \log \left( \frac{2\Delta}{M_x} \right) + \frac{2M_x^2 - 4M_x^2 \log \left( \frac{2\Delta}{M_x} \right)}{8\Delta^2} + \frac{7M_x^4 - 12M_x^4 \log \left( \frac{2\Delta}{M_x} \right)}{32\Delta^4} + \frac{74M_x^6 - 120M_x^6 \log \left( \frac{2\Delta}{M_x} \right)}{384\Delta^6} + O(1/\Delta^8).
\]

These expressions indicate that the log terms in $A^{(4)}_{\Delta}(q_2)$ and $B^{(4)}_{\Delta}(q_2)$ are essential for vanishing of $A^{(4)}_{\Delta}(q_2)$ and $B^{(4)}_{\Delta}(q_2)$ in the $\Delta \to \infty$ limit as required by the decoupling theorem. Notice further that the two-pion exchange diagrams shown in Fig. 8 also induce shorter-range contributions which will be discussed in a separate publication.

The static contributions discussed above depend only on the momentum transfers $q_i$ and are, therefore, local. It is thus natural to switch to the coordinate space representation of these 3NF terms. The Fourier transform of a local
potential is given by

$$\tilde{V}_{3N}(\vec{r}_{12}, \vec{r}_{32}) = \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} e^{i\vec{q}_1 \cdot \vec{r}_{12}} e^{i\vec{q}_3 \cdot \vec{r}_{32}} V_{3N}(\vec{q}_1, \vec{q}_3), \quad (5.76)$$

where $\vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j$ is the distance between the nucleons $i$ and $j$. For the two-pion-exchange contribution, we obtain from Eq. (5.70)

$$\tilde{V}_{2\pi}(\vec{r}_{12}, \vec{r}_{32}) = -\vec{\sigma}_1 \cdot \vec{\nabla}_{12} \vec{\sigma}_3 \cdot \vec{\nabla}_{32} \left( \begin{bmatrix} \tau_1 \cdot \tau_3 \cdot \tilde{A}(\vec{r}_{12}, \vec{r}_{32}) - \tau_1 \times \tau_3 \cdot \vec{\nabla}_{12} \times \vec{\nabla}_{32} \cdot \tilde{B}(\vec{r}_{12}, \vec{r}_{32}) \end{bmatrix} \right). \quad (5.77)$$

Here and in what follows, the differential operators $\vec{\nabla}_{ij}$ are defined in terms of dimensionless variables $\vec{x}_{ij} = \vec{r}_{ij} M_\pi$ while the functions $\tilde{A}$ and $\tilde{B}$ are defined via

$$\tilde{A}(\vec{r}_{12}, \vec{r}_{32}) = \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} e^{i\vec{q}_1 \cdot \vec{r}_{12}} e^{i\vec{q}_3 \cdot \vec{r}_{32}} \frac{1}{q_1^4 + M_\pi^2 q_3^2 + M_\pi^2} A(q_2),$$

$$\tilde{B}(\vec{r}_{12}, \vec{r}_{32}) = \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} e^{i\vec{q}_1 \cdot \vec{r}_{12}} e^{i\vec{q}_3 \cdot \vec{r}_{32}} \frac{1}{q_1^4 + M_\pi^2 q_3^2 + M_\pi^2} B(q_2). \quad (5.78)$$

To perform the integrations, we employ the spectral-function representation of the functions $A$ and $B$. The only non-polynomial in $q_2$ terms in Eq. (5.73) emerge from the scalar loop functions $L(q_2)$ and $D(q_2)$. Their spectral-function representation is given by

$$L(q_2) = 1 + q_2^2 \int_{M_N}^{\infty} d\mu \rho_L(\mu) = \frac{1}{\Delta} \arctan \left( \frac{\sqrt{\mu^2 - 4M_\pi^2}}{2\mu} \right), \quad (5.79)$$

$$D(q_2) = \int_{M_N}^{\infty} d\mu \rho_D(\mu) = \frac{1}{\Delta} \arctan \left( \frac{\sqrt{\mu^2 - 4M_\pi^2}}{2\mu} \right).$$

Therefore, the Fourier transform of the 3NF terms involving these functions can be written as

$$\int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} e^{i\vec{q}_1 \cdot \vec{r}_{12}} e^{i\vec{q}_3 \cdot \vec{r}_{32}} \frac{1}{q_1^4 + M_\pi^2 q_3^2 + M_\pi^2} L(q_2) = \frac{M_\pi^2}{(4\pi)^2} U_1(x_{12}) U_1(x_{32})$$

$$- M_\pi^2 \left( \vec{\nabla}_{12} + \vec{\nabla}_{32} \right)^2 \int d^3x U_1(|\vec{x}_{12} + \vec{x}|) U_1(|\vec{x}_{32} + \vec{x}|) V_1(x),$$

$$\int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} e^{i\vec{q}_1 \cdot \vec{r}_{12}} e^{i\vec{q}_3 \cdot \vec{r}_{32}} \frac{1}{q_1^4 + M_\pi^2 q_3^2 + M_\pi^2} D(q_2) = \frac{1}{(4\pi)^3} \int d^3x U_1(|\vec{x}_{12} + \vec{x}|) U_1(|\vec{x}_{32} + \vec{x}|) Q_1(x), \quad (5.80)$$

where the profile functions are given by

$$U_1(x) = \frac{4\pi}{M_\pi} \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{x}/M_\pi} = \frac{e^{-x}}{x},$$

$$V_1(x) = \int 2M_\pi \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{x}/M_\pi} \int_{2M_\pi}^{\infty} d\mu \frac{1}{\mu^2 + \mu^2} \sqrt{\mu^2 - 4M_\pi^2} = \frac{4\pi}{x} \int_2^{\infty} d\mu \frac{e^{-x}\mu}{\mu^2 - \mu^2 - 4},$$

$$Q_1(x) = \int \frac{d\mu}{x} M_\pi \frac{e^{-x\mu}}{\Delta} \arctan \left( \frac{M_\pi \sqrt{\mu^2 - 4}}{2\mu} \right). \quad (5.81)$$

We are interested here only in the long-range terms and, therefore, restrict ourselves to the case $x_{ij} \neq 0$. All terms involving positive powers of momenta $\vec{q}_1$ and $\vec{q}_3$ can be expressed through gradients $-iM_\pi \vec{\nabla}_{12}$ and $-iM_\pi \vec{\nabla}_{32}$ which can be taken out of the integrals.

For the NLO delta contributions to 3NF, we obtain the following coordinate-space expressions:

$$\tilde{A}_D^{(2)}(\vec{r}_{12}, \vec{r}_{32}) = \frac{g_\pi^2 h_A^2}{288\pi^2 \Delta F_{A}^2} M_\pi^6 \left( (\vec{\nabla}_{12} + \vec{\nabla}_{32})^2 - 2 \right) U_1(x_{12}) U_1(x_{32}),$$

$$\tilde{B}_D^{(2)}(\vec{r}_{12}, \vec{r}_{32}) = \frac{g_\pi^2 h_A^2}{576\pi^2 \Delta F_{A}^2} M_\pi^6 U_1(x_{12}) U_1(x_{32}). \quad (5.82)$$
\[ \bar{A}^{(4)}(\vec{r}_1, \vec{r}_2) = \frac{g_A^2 h_A^2}{22394848\pi^3 F_R^6} M^6 \left( 243\Delta^4 \left( 2 \left( \vec{\nabla}_{12} + \vec{\nabla}_{32} \right)^2 - 1 \right) \right) \times \left( 2\Delta^2 + M^2 \left( \left( \vec{\nabla}_{12} + \vec{\nabla}_{32} \right)^2 - 2 \right) \right) Q_1(\vec{x}_{12}, \vec{x}_{32}) + \pi M^2 \left( -25g_1^2 \left( \left( \vec{\nabla}_{12} + \vec{\nabla}_{32} \right)^2 - 2 \right) \right) (40\Delta^4(H(0) - 1) + 34(H(0) - 1)M^4) + \Delta^2(13 - 47H(0))M^2 + 17\pi M^2) + 1944\Delta^2(H(0) - 1) \left( 2 \left( \vec{\nabla}_{12} + \vec{\nabla}_{32} \right)^2 - 1 \right) (M^2 - \Delta^2) + 4\Delta^4 \left( 729 \left( 1 - 2 \left( \vec{\nabla}_{12} + \vec{\nabla}_{32} \right)^2 \right) - 250g_1^2 \left( \left( \vec{\nabla}_{12} + \vec{\nabla}_{32} \right)^2 - 2 \right) \right) \log \left( \frac{2\Delta}{M} \right) U_1(x_{12})U_1(x_{32}) \] 

\[ \times \frac{g_A^2 h_A^2}{9216\pi^5 F_R^6} \Delta M^6 \left( \vec{\nabla}_{12} + \vec{\nabla}_{32} \right)^2 \left( 1 - 2 \left( \vec{\nabla}_{12} + \vec{\nabla}_{32} \right)^2 \right) V_1(\vec{x}_{12}, \vec{x}_{32}) - \frac{g_A^2 h_A^2}{27648\pi^5 \Delta^3 F_R^6} M^6 \left( \left( \vec{\nabla}_{12} + \vec{\nabla}_{32} \right)^2 - 2 \right) U_1(x_{12})U_1(x_{32}) \left( 40\Delta^4(H(0) - 1) + 34(H(0) - 1)M^4 \right) \] 

\[ + \Delta^2(13 - 47H(0))M^2 + \pi M^2 + 40\Delta^4 \log \left( \frac{2\Delta}{M} \right) \] 

\[ + \frac{25g_1^2 h_A^2 g_1 M^6}{124416\pi^4 \Delta^3 F_R^6} \left( \vec{\nabla}_{12} + \vec{\nabla}_{32} \right)^2 \left( 1 - 2 \left( \vec{\nabla}_{12} + \vec{\nabla}_{32} \right)^2 \right) U_1(x_{12})U_1(x_{32}) \left( 8\Delta^4(H(0) - 1) + 2(H(0) - 1)M^4 \right) \] 

\[ + \Delta^2(5 - 7H(0))M^2 + \pi M^2 + 8\Delta^4 \log \left( \frac{2\Delta}{M} \right) \] 

\[ + \frac{5g_1^2 h_A^2}{15552\pi^3 \Delta^3 F_R^6} M^6 U_1(x_{12})U_1(x_{32}) \left( 5\Delta^2(H(0) - 1) + 34(H(0) - 1)M^4 \right) \] 

\[ + 2\Delta^2(8H(0) - 25)M^2 + \pi M^2 \right) - \frac{29g_1^2 h_A^2}{27648\pi^4 \Delta^3 F_R^6} \Delta M^6 U_1(x_{12})U_1(x_{32}) \log \left( \frac{2\Delta}{M} \right) \] 

\[ + \frac{25g_1^2 h_A^2 g_1 M^6 U_1(x_{12})U_1(x_{32})}{248832\pi^4 \Delta^3 F_R^6} \left( 10\Delta^4(H(0) - 1) + 10(H(0) - 1)M^4 \right) \] 

\[ - 2\Delta^2(4H(0) + 1)M^2 + 5\pi M^2 \right) + \frac{12g_1^2 h_A^2 g_1}{124416\pi^4 \Delta^3 F_R^6} M^6 U_1(x_{12})U_1(x_{32}) \log \left( \frac{2\Delta}{M} \right) \] 

\[ + \frac{g_A^2 h_A^2}{31104\pi^4 \Delta^3 F_R^6} M^6 U_1(x_{12})U_1(x_{32}) \left( -16\Delta^4(H(0) - 1) + 8(H(0) - 1)M^4 \right) \] 

\[ + 8\Delta^2(H(0) - 2)M^2 + 9\pi M^2 \right) - \frac{29g_1^2 h_A^2}{4478976\pi^4 \Delta^3 F_R^6} g_1^2 M^6 U_1(x_{12})U_1(x_{32}) \left( 10\Delta^4(H(0) - 1) \right) \] 

\[ + 34(H(0) - 1)M^4 + 2\Delta^2(16H(0) + 1)M^4 + 17\pi \Delta M^3 \right) \] 

\[ + \frac{g_A^2 h_A^2}{4608\pi^4 \Delta^3 F_R^6} M^6 \left( M^2 - \Delta^2 \right) U_1(x_{12})U_1(x_{32}) \left( -\frac{g_A^2 h_A^2}{1944\pi^4 \Delta^3 F_R^6} \Delta M^6 U_1(x_{12})U_1(x_{32}) \log \left( \frac{2\Delta}{M} \right) \right) \] 

\[ - \frac{12g_1^2 h_A^2 g_1^2}{22394848\pi^4 \Delta^3 F_R^6} M^6 U_1(x_{12})U_1(x_{32}) \log \left( \frac{2\Delta}{M} \right) - \frac{g_A^2 h_A^2}{2304\pi^4 \Delta^3 F_R^6} \Delta M^6 U_1(x_{12})U_1(x_{32}) \log \left( \frac{2\Delta}{M} \right), \] 5.83)

where the scalar integrals \( Q_1(\vec{x}_{12}, \vec{x}_{32}) \) and \( V_1(\vec{x}_{12}, \vec{x}_{32}) \) are defined as

\[ Q_1(\vec{x}_{12}, \vec{x}_{32}) = \int d^3 x U_1(|\vec{x}_{12} + \vec{x}|)U_1(|\vec{x}_{32} + \vec{x}|)Q_1(x), \]
\[ V_1(\vec{x}_{12}, \vec{x}_{32}) = \int d^3x \ U_1(|\vec{x}_{12} + \vec{x}|)U_1(|\vec{x}_{32} + \vec{x}|)V_1(x). \quad (5.84) \]

At N$^3$LO one also has to take into account relativistic corrections. Nucleonic contributions are already discussed in Ref. [50]. Here we give only the corresponding $\Delta$-contributions from the diagrams (36 - 39) of Fig. 5 proportional to $1/m_N$:

\[
V_{2\pi,1/m_N} = \frac{\hat{q}_1 \cdot \vec{\tau}_1 \hat{q}_3 \cdot \vec{\tau}_3}{\hat{q}_1^2 + M_N^2} \frac{g_A^2 h_A^2}{72 \Delta^2 F^4_\pi m_N} \left\{ \vec{\tau}_1 \cdot \vec{\tau}_3 \left[ -4(\hat{q}_1 \cdot \hat{q}_3) \right] + \hat{q}_1 \cdot \hat{q}_3 \times \vec{\sigma}_2 \right\}.
\]

At this stage several comments are in order.

- There are no contributions from $1/m_N$-corrections to pion-nucleon and pion-nucleon-$\Delta$ vertices since both of them are proportional to zeroth components of momenta and for this reason vanish in the kinematics relevant for nuclear forces. This argument is, however, only applicable to irreducible topologies since the corresponding 3NF contributions can be calculated using Feynman rules. All diagrams with intermediate delta excitations involving $1/m_N$-corrections to pion-nucleon or pion-nucleon-$\Delta$ vertices are indeed irreducible. The situation is different in the case of nucleonic contributions, where we used the unitary transformation technique to extract the corresponding irreducible pieces, see [41] for more details.

- Since the Fujita-Miyazawa 3NF corresponds to an irreducible diagram, we can not construct additional unitary transformations which would affect relativistic corrections to it. For this reason, the expression in Eq. (5.85) is unambiguous. This is, again, different for nucleonic contributions, where additional unitary transformations can be employed and the corresponding relativistic corrections do depend on arbitrary parameters $\beta_8$ and $\beta_9$ [41].

- Finally, we emphasize that Eq. (5.85) is consistent with the resonance saturation of the nucleonic N$^3$LO two-pion-exchange tree-level diagram with one of the vertices taken from the order-$Q^3 \pi N$-Lagrangian proportional to $d_4$. Indeed if we replace the $d_4$-constants of this diagram with their resonance saturation values given in Eq. (4.66), we reproduce the result of Eq. (5.85).

In coordinate space, the corresponding relativistic corrections are given by

\[
\hat{V}_{2\pi,1/m_N} = \frac{g_A^2 h_A^2 M_\pi^2}{1152 \pi^2 \Delta^2 F^4_\pi m_N} \hat{V}_{12} \cdot \hat{\sigma}_1 \hat{V}_{32} \cdot \hat{\sigma}_3 \left( \hat{V}_{12} \cdot \hat{V}_{32} \vec{\tau}_1 \cdot \vec{\tau}_2 \cdot \vec{\tau}_3 (-2 \hat{k}_1 \cdot \hat{V}_{12} + \hat{k}_1 \cdot \hat{V}_{32} - \hat{k}_3 \cdot \hat{V}_{12} + \hat{k}_3 \cdot \hat{V}_{32}) \right) \\
+ \frac{2 \hat{k}_3 \cdot \hat{V}_{32} + M_\pi \hat{V}_{12} \cdot \hat{V}_{32} + \vec{\sigma}_2}{m_N} \hat{k}_1 \cdot \vec{\sigma}_1 \hat{k}_3 \cdot \vec{\sigma}_3 \hat{k}_1 \cdot \hat{V}_{12} + \hat{k}_1 \cdot \hat{V}_{32} + \hat{k}_3 \cdot \hat{V}_{12} - 2 \hat{k}_3 \cdot \hat{V}_{32} \hat{V}_{12} \cdot \hat{V}_{32} + \vec{\sigma}_2) \\
+ \frac{4 M_\pi (\hat{V}_{12} \cdot \hat{V}_{32})^2 \vec{\tau}_1 \cdot \vec{\tau}_3}{\Delta^2} U_1(x_{12}) U_1(x_{32}).
\]

VI. DISCUSSION

Having constructed explicitly the two-pion-exchange $\Delta$-full three-nucleon force and having determined all the relevant low-energy constants, we are now in the position to analyze the convergence of chiral expansion for the long-range part of the 3NF. In Fig. 5 we show the results for the functions $A(q_2)$ and $B(q_2)$ for small values of the momentum transfer $q_2$, $q_2 < 300$ MeV at various orders in the small scale expansion. In addition to the $\epsilon^2$, $\epsilon^3$ and $\epsilon^4$ results we show also the results, where the purely nucleonic contributions of order $Q^3$ are added to the $\epsilon^4$ result in order to compare it with the $\Delta$-less N$^3$LO calculation from Ref. [50] (double-dashed dotted lines in Fig. 5). One should, however, keep in mind that this is not a complete $\epsilon^5$ result. We use here at all orders the low-energy constants $c_i$, $d_i$ and $\tilde{c}_i$ determined from the order-$\epsilon^3 + Q^3$ fit to the KH and GW partial wave analyses as described in section IV and listed in Table III. We also adopt the same conventions regarding the LECs as in the case of pion-nucleon scattering,
see Eqs. (4.61) and (4.63). Notice that although some of the $\bar{e}_i$-constants $(\bar{e}_{15,16,18})$ are rather sensitive to a particular choice of the partial wave analysis in pion-nucleon scattering, see Table [1], the functions $\mathcal{A}$ and $\mathcal{B}$ depend only on the LECs $\bar{e}_{14,17}$, which are quite stable.

One observes a fairly slow convergence for the functions $\mathcal{A}(q_2)$ and $\mathcal{B}(q_2)$ when going from order $\epsilon^2$ to $\epsilon^4$. On the other hand, the difference between the results at orders $\epsilon^4$ and $\epsilon^4 + Q^5$ is small for the function $\mathcal{B}(q_2)$ and almost negligible for the function $\mathcal{A}(q_2)$, which may indicate that convergence is reached at this order. Making a more definite statement about the convergence would, however, require performing a complete $\epsilon^5$ derivation of the 3NF.

It is also comforting to see that the results at order $\epsilon^4 + Q^5$ are very close to the $\Delta$-less calculation at order $Q^5$. This indicates that the contributions of the $\Delta$-isobar to the two-pion exchange 3NF topology can be well represented in terms of resonance saturation of the LECs $c_1$, $\bar{d}_i$ and $\bar{e}_i$ at N$^4$LO in the $\Delta$-less approach. This also indicates that nucleonic terms at order $Q^6$ and higher saturated by the double and triple delta excitations are small.

Another instructive way to quantitatively analyze the obtained three-nucleon forces is to look at the structure functions $\mathcal{F}_i(r_{12}, r_{23}, r_{31})$ for the equilateral triangle configuration of the nucleons given by the condition $r_{12} = r_{23} = r_{31} = r$. Structure functions $\mathcal{F}_i(r_{12}, r_{23}, r_{31})$ are the coefficients in the expansion of a general local three-nucleon force in the basis of 20 operators $\tilde{G}_i$ (for their explicit form see Ref. [100]):

$$V_{\Delta N}^{3N} = \sum_{i=1}^{20} \tilde{G}_i \mathcal{F}_i(r_{12}, r_{23}, r_{31}) + 5 \text{ permutations.} \quad (6.87)$$

Only 8 out of 20 structure functions do not vanish for the two-pion-exchange topology, namely $\mathcal{F}_4$, $\mathcal{F}_6$, $\mathcal{F}_{15}$, $\mathcal{F}_{16}$, $\mathcal{F}_{17}$, $\mathcal{F}_{18}$, $\mathcal{F}_{19}$ and $\mathcal{F}_{20}$. Our results for these structure functions are visualized in Figs. [10] [11] [12].
We first comment on the convergence pattern of the chiral expansion for the $\Delta$-less ($\Delta$-full) scheme, see Figs. 10 and 11 by looking at the tree-level results and at the results at orders $Q^4$ ($\epsilon^4$) and $Q^5$ ($\epsilon^4 + Q^5$). We observe a better convergence of the $\Delta$-full approach, which is reflected in significantly smaller bands on the right panels of Figs. 10 and 11 which indicate the size of the purely nucleonic contributions at order $Q^5$. This means that the large loop contributions at order $Q^5$ in the $\Delta$-less theory reported in Refs. [50, 110] are, to a considerable extent, saturated by the lower-order ($\epsilon^4$) contributions in the $\Delta$-full scheme. As the distance increases to $r \sim 2.5 - 3.0$ fm, the results at all orders get closer together for both the $\Delta$-less and $\Delta$-full approaches fully in line with the general expectation that the chiral expansion converges most rapidly at large distances.

It is also instructive to compare with each other the results within the $\Delta$-less and $\Delta$-full approaches at the highest considered orders $Q^{5}$ and $\epsilon^4 + Q^5$, respectively. As shown in Fig. 12, the $\Delta$-isobar contributions to the structure functions $F_6$, $F_{16}$, $F_{18}$, $F_{19}$ and $F_{20}$ are almost completely given by the resonance saturation of the corresponding LECs. Indeed, the bands indicating the difference between the order-$Q^5$ $\Delta$-less and order-($\epsilon^4 + Q^5$) $\Delta$-full results are almost invisible in those cases even at relatively short distances of $r \sim 1.0 - 1.5$ fm. For the functions $F_{15}$ and $F_{17}$ the saturation at this chiral order explains only a part of the $\Delta$-contributions. For the $F_{11}$-function, the $\Delta$-less and $\Delta$-full results turn out to be of a different sign at short distances. Notice, however, that the corresponding structure function is rather small in magnitude as compared to other ones. For larger distances of $r \sim 2.5 - 3.0$ fm, the saturation pattern improves and holds true for all structure functions. This means that $\Delta$-resonance saturation of the $\Delta$-full approach, which is reflected in significantly smaller bands on the right panels of Figs. 10 and 11, which indicate the size of the purely nucleonic contributions at order $Q^4$.

VII. SUMMARY AND CONCLUSIONS

In this paper, we have analyzed the longest-range contribution to the three-nucleon force at N$^3$LO utilizing the heavy-baryon formulation of chiral EFT with pions, nucleons and deltas as the only explicit degrees of freedom. The pertinent results of our study can be summarized as follows.

- We worked out in detail renormalization of the lowest-order effective chiral Lagrangian at the one-loop level.
- Employing renormalization conditions which maintain the explicit decoupling of the $\Delta$ isobar, we derived the delta contributions to those LECs $c_i$, $\bar{d}_i$ and $\bar{e}_i$ from the effective Lagrangians $\mathcal{L}_{\pi N}^{(2)}$, $\mathcal{L}_{\pi N}^{(3)}$ and $\mathcal{L}_{\pi N}^{(4)}$ which contribute to pion-nucleon scattering at the considered order.
- In order to determine the LECs $c_i$, $\bar{d}_i$ and $\bar{e}_i$ contributing to the $2\pi$-exchange 3NF, we re-analyzed pion-nucleon scattering at order $\epsilon^4 + Q^4$ employing the same power counting scheme as in the derivation of the nuclear forces and using the same fitting protocol as in the $\Delta$-less analysis of Ref. 104. We used the available partial wave analyses of the pion-nucleon scattering data to determine all relevant LECs. The resulting values turn out to be rather stable and consistent with our $\Delta$-less analysis reported in Ref. 100.
- We worked out the N$^3$LO delta contributions to the $2\pi$-exchange 3NF. The unitary ambiguity of the Hamilton operator is parametrized by 50 additional unitary transformations. After imposing the renormalizability constraint, i.e. the requirement that the resulting 3NF matrix elements are finite, the expressions for the 3NF appear to be defined unambiguously. These findings pave the way for the derivation of the remaining 3NF contributions at the same order, which are not considered in this paper.
- The obtained results for the $2\pi$-exchange 3NF at N$^3$LO of the SSE are in a good agreement with the N$^3$LO calculations of Ref. 100 within the $\Delta$-less approach. The agreement becomes even better when adding the nucleonic contributions at order N$^3$LO to the expressions at N$^4$LO of the SSE. This indicates that the effects of the $\Delta$ isobar for this particular topology are well represented by resonance saturation of the LECs $c_i$, $\bar{d}_i$ and $\bar{e}_i$ at N$^4$LO in the $\Delta$-less approach.

The presented calculations should be extended to the intermediate-range topologies, where we do expect significant contributions of the $\Delta$ to be still missing in the N$^4$LO analysis of Ref. 111 within the $\Delta$-less framework, as well as to short-range contributions. Work along these lines is in progress.
FIG. 10: (Color online) Profile functions $F_4(r)$, $F_6(r)$, $F_{15}(r)$, $F_{16}(r)$ in units of MeV generated by the two-pion exchange 3NF topology in the $\Delta$-less approach of Ref. [30] (left panel) and in the $\Delta$-full approach of the current work (right panel). The dash-dotted, dashed and solid lines are the results of the calculation at order $Q^3(\epsilon^3)$, $Q^4(\epsilon^4)$ and $Q^5(\epsilon^4 + Q^2)$, respectively. The bands indicate the purely nucleonic contribution at order $Q^3$. 
FIG. 11: (Color online) Profile functions $F_17(r), F_{18}(r), F_{19}(r), F_{20}(r)$ in units of MeV generated by the two-pion exchange 3NF topology in the $\Delta$-less approach of Ref. [50] (left panel) and in the $\Delta$-full approach of the current work(right panel). The dash-dotted, dashed and solid lines are the results of the calculation at order $Q^3(\epsilon^3), Q^4(\epsilon^4)$ and $Q^5(\epsilon^4 + Q^5)$, respectively. The bands indicate the purely nucleonic contribution at order $Q^3$. 
We are grateful to Ulf-G. Meißner for sharing his insights into the considered topics and useful discussions. This work was supported by BMBF (contract No. 05P2015 - NUSTAR R&D) and by DFG through funds provided to the Sino-German CRC 110 “Symmetries and the Emergence of Structure in QCD” (Grant No. TRR110).
Appendix A: Unitary ambiguity of the 3NF and constraints imposed by the renormalizability requirement

To derive the effective potential we use the method of unitary transformation. A detailed discussion of this approach including the explicit form of the unitary operator at low orders in the chiral expansion can be found in Ref. [46]. As explained in section V, this method can be straightforwardly extended to carry out calculations within the $\Delta$-full chiral EFT approach. In this appendix we discuss the restrictions on the choice of the unitary transformation imposed by the condition that the resulting nuclear Hamiltonian is renormalizable.

We first specify our notation. The indices in the interaction vertices $H_{a,b,c,d,e}^{(\kappa)}$ have the following meaning:

\[
\begin{align*}
    a &= \text{Number of pion fields} \\
    b &= \text{Number of outgoing nucleons} \\
    c &= \text{Number of outgoing deltas} \\
    d &= \text{Number of incoming nucleons} \\
    e &= \text{Number of incoming deltas} \\
    \kappa &= d + \frac{3}{2}(b + c + d + e) + a - 4,
\end{align*}
\]

where $d$ is the number of derivatives at a given vertex. We also introduce the projection operators $\eta$ and $\lambda$ onto the purely nucleonic and the remaining parts of the Fock space, respectively. These operators satisfy the usual relations $\eta^2 = \eta$, $\lambda^2 = \lambda$, $\eta\lambda = \lambda\eta = 0$ and $\lambda + \eta = 1$. We also need to differentiate the states from the $\lambda$-subspace by introducing the operators $\lambda^{a,b}$, where $a$ and $b$ refer to the number of pions and deltas in the corresponding intermediate state, respectively. The total energy of the pions and deltas in the corresponding state will be denoted by $E_{\pi\Delta} = O(\epsilon)$.

As pointed out in section V renormalizability of the nuclear Hamiltonian is achieved by performing all possible $\eta$-space unitary transformations after decoupling of pions and deltas by means of the (minimal) Okubo-type unitary transformation. Such additional unitary operators have a general form

\[
U = e^S,
\]

where $S$ is an antihermitian operator ($S^\dagger = -S$) acting on the $\eta$-space. We parametrize the operator $S$ as

\[
S = \sum_i \alpha_i S_i + \sum_i \alpha_i^\Delta S_i^\Delta,
\]

where $\alpha_i$ and $\alpha_i^\Delta$ are real numbers. The operators $S_i$ include only nucleon degrees of freedom and have already been discussed earlier [46, 50]. We now give 50 operators $S_i^\Delta$ which include delta contribution:

\[
\begin{align*}
S_1^\Delta &= \eta H_{1,1,0,1,0}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{0,1}^1}{E_{\pi\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \\
S_2^\Delta &= \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} H_{1,1,0,0,1}^{(1)} \frac{\lambda_{0,1}^1}{E_{\pi\Delta}} H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \\
S_3^\Delta &= \eta H_{1,1,0,1,0}^{(1)} \frac{\lambda_{1,0}^1}{E_{\pi\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{0,1}^1}{E_{\pi\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{1,0}^1}{E_{\pi\Delta}} H_{1,1,0,1,0}^{(1)} \eta - hc, \\
S_4^\Delta &= \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,0}^1}{E_{\pi\Delta}} H_{1,1,0,0,1}^{(1)} \frac{\lambda_{0,1}^1}{E_{\pi\Delta}} H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,0}^1}{E_{\pi\Delta}} H_{1,1,0,1,0}^{(1)} \eta - hc, \\
S_5^\Delta &= \eta H_{1,1,0,1,0}^{(1)} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} \frac{\lambda_{0,1}^1}{E_{\pi\Delta}} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} \eta - hc, \\
S_6^\Delta &= \eta H_{1,1,0,0,1}^{(1)} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} \frac{\lambda_{0,1}^1}{E_{\pi\Delta}} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} \eta - hc, \\
S_7^\Delta &= \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{0,1}^1}{E_{\pi\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} H_{1,1,0,1,0}^{(1)} \eta - hc, \\
S_8^\Delta &= \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{0,1}^1}{E_{\pi\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi\Delta}} H_{1,1,0,1,0}^{(1)} \eta - hc,
\end{align*}
\]
\[ S_{9}^{\Delta} = \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^{1,1}}{E_{\pi}^{\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{2,1}^{2,1}}{E_{\pi}^{\Delta}} H_{1,0,1,1,0}^{(1)} \frac{\lambda_{1,0}^{1,0}}{E_{\pi}^{\Delta}} H_{1,1,0,1,0}^{(1)} \eta - \hbar c, \]

\[ S_{10}^{\Delta} = \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^{1,1}}{E_{\pi}^{\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{2,0}^{2,0}}{E_{\pi}^{\Delta}} H_{1,0,1,1,0}^{(1)} \frac{\lambda_{1,0}^{1,0}}{E_{\pi}^{\Delta}} H_{1,1,0,1,0}^{(1)} \eta - \hbar c, \]

\[ S_{11}^{\Delta} = \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^{1,1}}{E_{\pi}^{\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{2,0}^{2,0}}{E_{\pi}^{\Delta}} H_{1,0,1,1,0}^{(1)} \frac{\lambda_{1,0}^{1,0}}{E_{\pi}^{\Delta}} H_{1,1,0,1,0}^{(1)} \eta - \hbar c, \]

\[ S_{12}^{\Delta} = \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^{1,1}}{E_{\pi}^{\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{2,0}^{2,0}}{E_{\pi}^{\Delta}} H_{1,0,1,1,0}^{(1)} \frac{\lambda_{1,0}^{1,0}}{E_{\pi}^{\Delta}} H_{1,1,0,1,0}^{(1)} \eta - \hbar c, \]

\[ S_{13}^{\Delta} = \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^{1,1}}{E_{\pi}^{\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{2,0}^{2,0}}{E_{\pi}^{\Delta}} H_{1,0,1,1,0}^{(1)} \frac{\lambda_{1,0}^{1,0}}{E_{\pi}^{\Delta}} H_{1,1,0,1,0}^{(1)} \eta - \hbar c, \]

\[ S_{14}^{\Delta} = \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^{1,1}}{E_{\pi}^{\Delta}} H_{1,1,0,1,0}^{(1)} \frac{\lambda_{2,0}^{2,0}}{E_{\pi}^{\Delta}} H_{1,0,1,1,0}^{(1)} \frac{\lambda_{1,0}^{1,0}}{E_{\pi}^{\Delta}} H_{1,1,0,1,0}^{(1)} \eta - \hbar c, \]
\[ S_3^\Delta = \eta H_{2,1,0,1,0}^{(2)} \frac{\lambda_{1,1}^0}{E_{\pi \Delta}} H_{1,0,1,0,1}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{34}^\Delta = \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^0}{E_{\pi \Delta}} H_{2,1,0,1,0}^{(2)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{35}^\Delta = \eta H_{2,1,0,0,1}^{(2)} \frac{\lambda_{2,1}^0}{E_{\pi \Delta}} H_{1,0,1,0,1}^{(1)} \frac{\lambda_{1,0}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{36}^\Delta = \eta H_{2,1,0,0,1}^{(2)} \frac{\lambda_{2,1}^0}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \frac{\lambda_{1,0}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{37}^\Delta = \eta H_{2,1,0,0,1}^{(2)} \frac{\lambda_{2,1}^0}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{38}^\Delta = \eta H_{2,1,0,0,1}^{(2)} \frac{\lambda_{2,1}^0}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{39}^\Delta = \eta H_{2,1,0,0,1}^{(2)} \frac{\lambda_{2,1}^0}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{40}^\Delta = \eta H_{2,1,0,0,1}^{(2)} \frac{\lambda_{2,1}^0}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{41}^\Delta = \eta H_{1,1,0,1,0}^{(1)} \frac{\lambda_{1,0}^1}{E_{\pi \Delta}} H_{2,1,0,0,1}^{(2)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{42}^\Delta = \eta H_{1,1,0,1,0}^{(1)} \frac{\lambda_{1,0}^1}{E_{\pi \Delta}} H_{2,1,0,0,1}^{(2)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{43}^\Delta = \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{2,0,1,0,1}^{(2)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{44}^\Delta = \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{2,0,1,0,1}^{(2)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{45}^\Delta = \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{2,0,1,0,1}^{(2)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{46}^\Delta = \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{2,0,1,0,1}^{(2)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{47}^\Delta = \eta H_{2,0,0,0,2}^{(2)} \frac{\lambda_{0,2}^1}{E_{\pi \Delta}} H_{1,0,0,1,0}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{48}^\Delta = \eta H_{2,0,0,0,2}^{(2)} \frac{\lambda_{0,2}^1}{E_{\pi \Delta}} H_{1,0,0,1,0}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{49}^\Delta = \eta H_{2,0,0,0,2}^{(2)} \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} \frac{1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc, \]

\[ S_{50}^\Delta = \eta H_{1,1,0,0,1}^{(1)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{2,0,1,1,1}^{(2)} \frac{\lambda_{1,1}^1}{E_{\pi \Delta}} H_{1,0,1,1,0}^{(1)} \eta - hc. \] 

The requirement that the delta contributions to the 3NF are renormalizable leads to the following constraints on the coefficients \( \alpha_i^\Delta \):

\[ \alpha_7^\Delta - \alpha_{13}^\Delta = 0, \]

\[ -\alpha_{13}^\Delta + \alpha_8^\Delta + \frac{1}{2} = 0, \]

\[ \alpha_9^\Delta = 0, \]

\[ \alpha_{10}^\Delta - \alpha_{13}^\Delta = 0, \]

\[ \alpha_{13}^\Delta + 2\alpha_{29}^\Delta - \frac{1}{4} = 0, \]

\[ \alpha_{13}^\Delta - \alpha_{25}^\Delta = 0, \]

\[ \alpha_{21}^\Delta - \alpha_{26}^\Delta = 0. \]
\[ \begin{align*}
\alpha_1^\Delta - \alpha_2^\Delta &= 0, \\
\alpha_3^\Delta - \alpha_4^\Delta &= 0, \\
\alpha_5^\Delta - \alpha_2^\Delta &= 0, \\
\alpha_6^\Delta - \alpha_3^\Delta &= 0, \\
\alpha_7^\Delta - \alpha_8^\Delta &= 0, \\
\alpha_9^\Delta - \alpha_1^\Delta &= 0, \\
\alpha_{11}^\Delta - 2\alpha_{29}^\Delta + \alpha_5^\Delta + \frac{1}{4} &= 0, \\
-2\alpha_{29}^\Delta + \alpha_3^\Delta + \frac{1}{4} &= 0, \\
-2\alpha_{29}^\Delta + \alpha_4^\Delta + \frac{1}{4} &= 0, \\
\alpha_5^\Delta - \alpha_{11}^\Delta &= 0, \\
-\alpha_{11}^\Delta - 2\alpha_{29}^\Delta + \alpha_6^\Delta - \frac{1}{4} &= 0, \\
\alpha_{12}^\Delta + 4\alpha_{29}^\Delta &= 0, \\
\alpha_{25}^\Delta - \alpha_{37}^\Delta &= 0, \\
\alpha_{36}^\Delta - \alpha_{35}^\Delta + \alpha_3^\Delta + \frac{1}{2} &= 0, \\
\alpha_{37}^\Delta - \alpha_{35}^\Delta + \alpha_4^\Delta - \frac{1}{2} &= 0, \\
\alpha_{30}^\Delta - \frac{1}{2} &= 0.
\end{align*} \]

(A.5)

Appendix B: Delta-isobar contributions to the invariant \( \pi N \) amplitudes \( g^\pm(\omega, t) \) and \( h^\pm(\omega, t) \)

In this Appendix we present the explicit expressions for the invariant amplitudes \( g^\pm(\omega, t) \) and \( h^\pm(\omega, t) \) which parametrize the pion-nucleon scattering matrix at first three orders in \( \epsilon \) expansion (for an earlier calculation see Ref. [101]). We give only contributions due to intermediate delta excitations, which have to be added to the nucleonic terms calculated within the \( \Delta \)-less theory and listed in Ref. [50].

Contributions at order \( \epsilon^1 \):

\[ \begin{align*}
ge^+ &= \frac{4\Delta h_A^2 \left(-2M_\pi^2 + t + 2\omega^2\right)}{9F^2_\pi(\Delta - \omega)(\Delta + \omega)}, \\
ge^- &= -\frac{2h_A^2 \omega \left(-2M_\pi^2 + t + 2\omega^2\right)}{9F^2_\pi(\Delta - \omega)(\Delta + \omega)}, \\
h^+ &= \frac{4h_A^2 \omega}{9F^2_\pi(\omega^2 - \Delta^2)}, \\
h^- &= \frac{2\Delta h_A^2}{9F^2_\pi(\Delta^2 - \omega^2)}. \end{align*} \]  

(B.1)

Contributions at order \( \epsilon^2 \):

\[ g^+ = 0, \quad g^- = 0, \quad h^+ = 0, \quad h^- = 0. \]  

(B.2)

Contributions at order \( \epsilon^3 \):

\[ g^\pm = g^\pm_{SL} + g^\pm_{1/m_N}, \quad h^\pm = h^\pm_{SL} + h^\pm_{1/m_N}. \]  

(B.3)
with static limit contributions given by

\[ g_{SL}^+ = \frac{h_{A}^2}{486 F_\pi^2 \omega^2} \bar{J}_0(-\Delta) \left( (81 g_A^2 - 50 g_{A\gamma} + 25 g_{1\gamma}) (M_{\pi}^2 - \Delta^2) \right) \left( -2 M_z^2 + t + 2 \omega^2 \right) - 108 \omega^2 (M_z^2 - 2 t) \]

\[ + \frac{h_{A}^2}{4374 F_\pi^2 \omega^2 (\Delta - \omega)^2} \bar{J}_0(\omega - \Delta) (M_{\pi}^2 - (\Delta - \omega)^2) \left( 2 M_z^2 - t - 2 \omega^2 \right) (9 \Delta^2 (81 g_A^2 - 50 g_{A\gamma} + 25 g_{1\gamma})) \]

\[ - 10 \omega (\Delta - \omega) (9 g_A - 5 g_{1\gamma})^2 + \frac{h_{A}^2}{36 \pi^2 F_\pi^2} D (\sqrt{-t}) \Delta (M_z^2 - 2 t) \left( -2 \Delta^2 + 2 M_z^2 - t \right) \]

\[ + \frac{8 h_{A}^2}{243 F_\pi^2 (\Delta^2 - \omega^2)^2} \bar{J}_0 (\omega) \left( 2 M_z^2 + t + 2 \omega^2 \right) (9 g_A^2 (\Delta^2 - \omega^2) + h_{A}^2 (5 \Delta^2 + 4 \omega^2)) \]

\[ + \frac{h_{A}^2}{34992 \pi^2 F_\pi^2 (\Delta^2 - \omega^2)^2} \left( (2 M_z^2 - t) \left( (81 g_A^2 - 50 g_{A\gamma} + 25 g_{1\gamma}) \left( 9 \Delta (\Delta^2 - \omega^2)^2 \right) \right) \right) \]

\[ + \pi M_{\pi}^2 (17 \Delta^2 + \omega^2) + 9 \Delta M_{\pi}^2 (\omega^2 - \Delta^2) + 400 \pi g_{A\gamma} M_{\pi}^2 (\Delta^2 - \omega^2) - 486 \Delta (\Delta^2 - \omega^2)^2 (M_z^2 - 2 t) \]

\[ + \frac{\Delta h_{A}^2 \log \left( \frac{\Delta}{\pi} \right)}{8748 \pi^2 F_\pi^2} \left( 10 (9 g_A - 5 g_{1\gamma}) (2 M_z^2 - t - 2 \omega^2) + 729 (M_z^2 - 2 t) \right) \]

\[ + \frac{2 \Delta h_{A}^2 \bar{I}_{20} (\omega) (M_z^2 - 2 t)}{9 F_\pi^2} \]

\[ + (\omega \to -\omega), \quad (B.4) \]

\[ g_{SL}^- = \frac{h_{A}^2}{69984 F_\pi^2 \omega^2 \Delta \omega (\Delta^2 - \omega^2)^2} \left( (2 M_z^2 - 2 \omega^2 - t) (81 (2 \pi (12 \Delta^4 - 35 \omega^2 \Delta^2 + 20 \omega^4)) M_{\pi}^4 \right) \]

\[ - 3 (6 \Delta^3 - 13 \omega^2 \Delta^3 + 7 \omega^4 \Delta) M_{\pi}^2 + 9 \Delta (\Delta^2 - \omega^2)^2 (4 \Delta^2 + 3 \omega^2) \right) g_A^2 \]

\[ + 150 g_1 \Delta \left( 6 \pi \Delta \omega^2 M_{\pi}^2 - 3 (2 \Delta^4 - 3 \omega^2 \Delta^2 + \omega^4) M_{\pi}^2 + (\Delta^2 - \omega^2)^2 (12 \Delta^2 + 13 \omega^2) \right) g_A \]

\[ + 25 g_{1\gamma}^2 (-23 \Delta \omega^6 + (8 \pi M_{\pi}^2 - 3 \Delta M_{\pi}^2 + 34 \Delta^5) \omega^4 + \Delta^2 (-26 \pi M_{\pi}^2 - 3 \Delta M_{\pi}^2 + \Delta^5) \omega^2 + 6 \Delta^5 (M_z^2 - 2 \Delta^2))) \]

\[ - 18 \Delta \omega^2 (\Delta^2 - \omega^2) \left( 4 h_{A}^2 (2 M_z^2 - 2 \omega^2 - t) \left( 3 M_z^2 - 4 \Delta^2 - 2 \omega^2 \right) - 3 (24 M_z^2 - 72 \Delta^2 - 19 t) \left( \Delta^2 - \omega^2 \right) \right) \]

\[ + \frac{h_{A}^2}{18 F_\pi^2 \omega^2} \left( 2 M_z^2 + 2 \Delta^2 + t \right) \omega \bar{J}_0(\omega) \left( M_z^2 \right) + \frac{8 (M_z^2 - 12 \Delta^2 - 5 t) \omega \bar{J}_{20}(\omega)}{27 F_\pi^2} h_{A}^2 \]

\[ + \frac{h_{A}^2}{4374 F_\pi^2 \omega^2 (\Delta^2 - \omega^2)^2} \left( (2 M_z^2 - 2 \omega^2 - t) \left( 36 h_{A}^2 (4 \Delta^4 - 10 \omega^2 \Delta^2 + M_{\pi}^2 (5 \Delta^2 + \omega^2)) \omega^2 \right) \right) \]

\[ + (\Delta^2 - \omega^2) \left( 81 (19 \Delta^4 + 15 \omega^2 \Delta^2 + M_{\pi}^2 (12 \omega^2 - 8 \Delta^2)) g_A^2 \right) + 1350 g_1 \Delta^5 (\Delta^2 - \omega^2) g_A \]

\[ + 25 g_{1\gamma}^2 (4 M_{\pi}^2 - 13 \Delta^2) (\Delta^2 - \omega^2)) \right) - 972 \Delta^2 \omega^2 \left( \Delta^2 - \omega^2 \right) \bar{J}_0(-\Delta) \]

\[ - 2 \left( 2 M_z^2 - 2 \omega^2 - t \right) (\omega^2 - M_z^2) \left( 2 h_{A}^2 \omega (\Delta^2 - 12 \omega \Delta + 2 \omega^2) - 9 g_A^2 (3 \Delta - \omega) (\Delta^2 - \omega^2) \right) \bar{J}_0(\omega) h_{A}^2 \]

\[ + \frac{243 F_\pi^2 \omega (\Delta^2 - \omega^2)^2}{8748 F_\pi^2 \omega^2 (\Delta - \omega)^2} \left( 6 (243 g_A^2 - 150 g_{A\gamma} + 25 g_{1\gamma}) \Delta^2 \right) - 5 (9 g_A - 5 g_{1\gamma})^2 (4 \Delta - \omega) \right) \]

\[ \times (2 M_z^2 - 2 \omega^2 - t) \bar{J}_0(\omega - \Delta) \]

\[ \omega \left( (243 g_A^2 + 450 g_{A\gamma} - 288 h_{A}^2 - 125 g_{1\gamma}) (2 M_z^2 - 2 \omega^2 - t) - 810 t \right) \log \left( \frac{\Delta}{\pi} \right) h_{A}^2 \]

\[ + \frac{243 F_\pi^2 \omega (\Delta^2 - \omega^2)^2}{34992 F_\pi^2 \omega^2} \left( \left( \Delta^2 - \omega^2 \right) \right) \left( (81 g_A^2 - 3 \Delta g_A + 25 g_{1\gamma}) \right) \]

\[ + \frac{h_{A}^2}{2187 F_\pi^2 \omega^2 (\Delta^2 - \omega^2)^2} \bar{J}_0(-\Delta) \right) (\Delta^2 - \omega^2) \left( (81 g_A^2 - 3 \Delta^2 (5 \Delta^2 + 4 M_z^2) + 8 \Delta^2 M_z^2) \right) \]

\[ + (\omega \to -\omega), \quad (B.5) \]

\[ h_{SL}^+ = \frac{h_{A}^2}{4374 F_\pi^2 \omega^2 (\Delta - \omega)^2} \bar{J}_0(\omega - \Delta) \left( M_z^2 - (\Delta - \omega)^2 \right) \left( 2 M_z^2 - t - 2 \omega^2 \right) (243 g_A^2 - 150 g_{A\gamma} + 25 g_{1\gamma}) \]

\[ + (5 \omega^2 - 20 \Delta \omega)(9 g_A - 5 g_{1\gamma})^2 \]

\[ + \frac{h_{A}^2}{2187 F_\pi^2 \omega^2 (\Delta^2 - \omega^2)^2} \bar{J}_0(-\Delta) \right) \left( (\Delta^2 - \omega^2) \right) \left( (81 g_A^2 - 3 \Delta^2 (5 \Delta^2 + 4 M_z^2) + 8 \Delta^2 M_z^2) \right) \]
\[
\begin{align*}
&- 1350\Delta^2 g_{A g_1} (\Delta^2 - \omega^2) - 25g_{1}^2 (\Delta^2 - \omega^2) (4M_{\pi}^2 - 13\Delta^2)) \\
&- 36h_{A}^2 \omega^2 (4\Delta^4 - 10\Delta^2 \omega^2 + M_{\pi}^2 (5\Delta^2 + \omega^2))) \\
&- \frac{4h_{A}^2 J_0(\omega) (M_{\pi}^2 - \omega^2) (2h_{A}^2 \omega (\Delta^2 - 12\Delta \omega + 2\omega^2) - 9g_{A}^2 (\Delta - \omega)(3\Delta - \omega)(\Delta + \omega))}{243F_{\pi}^4 \omega (\Delta^2 - \omega^2)^2} \\
&+ \frac{h_{A}^2}{34992\pi^2 F_{\pi}^4 \omega (\Delta^2 - \omega^2)^2} ((\Delta^2 - \omega^2) (\Delta (\Delta^2 - \omega^2)) (729g_{A}^2 (4\Delta^2 + 3\omega^2) - 150g_{A} g_{1} (12\Delta^2 + 13\omega^2) \\
&+ 25g_{1}^2 (12\Delta^2 + 23\omega^2)) - 72\Delta h_{A}^2 \omega^2 (4\Delta^2 - 3M_{\pi}^2 + 2\omega^2)) + 81g_{A}^2 (2\pi M_{\pi}^2 (12\Delta^4 - 35\Delta^2 \omega^2 + 20\omega^4) \\
&- 3M_{\pi}^2 (6\Delta^5 - 13\Delta^3 \omega^2 + 7\Delta \omega^4)) - 150\Delta g_{A} g_{1} (6\pi M_{\pi}^2 \omega^2 - 3M_{\pi}^2 (2\Delta^4 - 3\Delta^2 \omega^2 + \omega^4)) \\
&+ 25g_{1}^2 (\pi M_{\pi}^2 (26\Delta^2 \omega^2 - 8\omega^4) + 32M_{\pi}^2 (-2\Delta^4 + \Delta^2 \omega^2 + \omega^4)) \\
&+ \frac{h_{A}^2 \omega (-243g_{A}^2 - 450g_{A} g_{1} + 288h_{A}^2 + 125g_{1}^2) \log \left(\frac{2\Delta}{M_{\pi}}\right)}{17496\pi^2 F_{\pi}^4} \\
&+ (\omega \rightarrow -\omega),
\end{align*}
\]

\[
\hbar_{SL} = \frac{\Delta h_{A}^2 D (\sqrt{-t}) (4\Delta^2 - 4M_{\pi}^2 + t)}{144\pi^2 F_{\pi}^4} - \frac{\Delta h_{A}^2 f_{20}(t)}{9F_{\pi}^4} \\
+ \frac{h_{A}^2}{4374F_{\pi}^4 \omega (\Delta^2 - \omega^2)^2} J_0(-\Delta) \left(10 (\Delta^2 - \omega^2)^2 (81g_{A}^2 - 90g_{A} g_{1} + 5g_{1}^2) (M_{\pi}^2 - \Delta^2) \\
+ 648\Delta^2 g_{A}^2 (\Delta^2 - \omega^2) (M_{\pi}^2 - \Delta^2) + 9\omega^2 (27 (\Delta^2 - \omega^2)^2 + 32h_{A}^2 (2\Delta^2 (\Delta^2 - \omega^2) + M_{\pi}^2 (\Delta^2 + \omega^2))) \right) \\
- \frac{4h_{A}^2 J_0(\omega) (M_{\pi}^2 - \omega^2) (h_{A}^2 \omega (9\Delta^2 - 8\Delta \omega + 8\omega^2) - 9g_{A}^2 (\Delta - \omega)) (\Delta^2 - \omega^2))}{243F_{\pi}^4 \omega (\Delta^2 - \omega^2)^2} \\
+ \frac{h_{A}^2}{69984\pi^2 F_{\pi}^4 (\Delta^2 - \omega^2)^2} \left(\Delta (\Delta^2 - \omega^2) (M_{\pi}^2 (81g_{A}^2 - 450g_{A} g_{1} - 576h_{A}^2 + 225g_{1}^2) + 1152\Delta^2 h_{A}^2) \\
+ \Delta (\Delta^2 - \omega^2) (-7047g_{A}^2 + 3150g_{A} g_{1} + 25g_{1}^2) + \pi M_{\pi}^2 (5g_{1} - 9g_{A}) (\Delta^2 (297g_{A} - 85g_{1}) - \omega^2 (279g_{A} + 5g_{1})) \right) \\
- \frac{\Delta h_{A}^2 (2349g_{A}^2 - 2250g_{A} g_{1} + 1152h_{A}^2 + 125g_{1}^2 + 972) \log \left(\frac{2\Delta}{M_{\pi}}\right)}{3492\pi^2 F_{\pi}^4} \\
- \frac{h_{A}^2 (9g_{A} - 5g_{1}) J_0(\omega - \Delta) (M_{\pi}^2 - (\Delta - \omega)^2) (-360\Delta \omega g_{A} + 4\Delta^2 (8g_{A} - 5g_{1}) + 5\omega^2 (9g_{A} - 5g_{1}))}{8748F_{\pi}^4 \omega (\Delta^2 - \omega^2)} \\
+ (\omega \rightarrow -\omega),
\]

and \(1/m_N\) corrections given by

\[
\begin{align*}
g_{1/m_N}^{+} &= - \frac{h_{A}^2}{9F_{\pi}^4 m_N (\Delta^2 - \omega^2)^2} \left\{ 16\Delta^4 \omega^2 - 4\Delta^3 \omega (-4M_{\pi}^4 + t + 4\omega^2) \\
&+ \Delta^2 (4M_{\pi}^4 - 4M_{\pi}^2 (t + 6\omega^2) + t^2 + 8t \omega^2 - 12\omega^4) - 2\Delta \omega (8M_{\pi}^4 - 2M_{\pi}^2 (3t + 4\omega^2) + t (t + 4\omega^2)) \\
&+ \omega^2 (4M_{\pi}^4 - 4M_{\pi}^2 (t - 2\omega^2) + t^2 + 4\omega^4) \right\} + \frac{16h_{A}^2 \omega^2}{9F_{\pi}^4 m_N}, \end{align*}
\]

\[
\begin{align*}
g_{1/m_N}^{-} &= - \frac{h_{A}^2}{18F_{\pi}^4 m_N (\Delta^2 - \omega^2)^2} \left\{ - 4\Delta^3 \omega (-4M_{\pi}^2 t + t + 4\omega^2) \\
&+ \Delta^2 (8M_{\pi}^4 - 2M_{\pi}^2 (3t + 16\omega^2) + t^2 + 10t \omega^2 + 24\omega^4) \\
&- 2\Delta \omega (4M_{\pi}^4 - 4M_{\pi}^2 t + t^2 + 2t \omega^2 - 4\omega^4) + \omega^2 (8M_{\pi}^4 - 6M_{\pi}^2 t + t^2 + 2t \omega^2 - 8\omega^4) \right\} \end{align*}
\]

\[
\begin{align*}
&- \frac{h_{A}^2}{9F_{\pi}^4 m_N \Delta} \omega (t + 8\omega^2 - M_{\pi}^2)), \\
&h_{1/m_N}^{+} = \frac{4h_{A}^2 [4\Delta^3 \omega + \Delta^2 (4M_{\pi}^2 t - t - 8\omega^2) + 2\Delta \omega (t - 2M_{\pi}^2) + \omega^2 (4M_{\pi}^2 - t)]}{9F_{\pi}^4 m_N (\Delta^2 - \omega^2)^2} - \frac{4h_{A}^2 \omega}{9F_{\pi}^4 m_N \Delta},
\end{align*}
\]
\[ h_{1/m_N}^- = \frac{h_A^2 \left( 4\Delta^3\omega + \Delta^2 \left( 2M^2_{\pi} - t - 6\omega^2 \right) + 2\Delta\omega \left( -4M^2_{\pi} + t + 2\omega^2 \right) + \omega^2 \left( 2M^2_{\pi} - t + 2\omega^2 \right) \right)}{18F^2_{\pi}m_N(\Delta^2 - \omega^2)^2}. \]  

(B.8)

The loop functions are defined via:

\[ 2M^2_{\pi} \lambda_n = \mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{l^2 - M^2_{\pi} + i\epsilon}{l^2 - \mu^2} \]

\[ J_0(\omega) = \begin{cases} \frac{\omega}{8\pi^2} + \frac{\sqrt{\omega^2 - M^2_{\pi}}}{4\pi} \text{arccosh} \left( -\frac{\omega}{M_{\pi}} \right) & \text{for } \omega < -M_{\pi} \\ \frac{\omega}{8\pi^2} - \frac{\sqrt{\omega^2 - M^2_{\pi}}}{4\pi} \left( \text{arccosh} \left( \frac{\omega}{M_{\pi}} \right) - i\pi \right) & \text{for } \omega > M_{\pi} \\ \frac{\omega}{8\pi^2} - \frac{\sqrt{M^2_{\pi} - \omega^2}}{4\pi} \text{arccos} \left( -\frac{\omega}{M_{\pi}} \right) & \text{for } |\omega| < M_{\pi} \end{cases} \]

\[ D(x) = \frac{1}{\Delta} \int_{2M_{\pi}}^\infty \frac{d\mu}{\mu^2 + x^2} \text{arctan} \left( \frac{\sqrt{\mu^2 - 4M^2_{\pi}}}{2\Delta} \right) \]

\[ \tilde{I}_{20}(t) = \frac{1}{16\pi^2} - \sqrt{\frac{1 - 4M^2_{\pi}/t}{16\pi^2}} \log \sqrt{\frac{1 - 4M^2_{\pi}/t + 1}{1 - 4M^2_{\pi}/t - 1}}. \]  

(B.9)
