Discrete uniqueness sets for functions with spectral gaps

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Abstract. It is well known that entire functions whose spectrum belongs to a fixed bounded set $S$ admit real uniformly discrete uniqueness sets. We show that the same is true for a much wider range of spaces of continuous functions. In particular, Sobolev spaces have this property whenever $S$ is a set of infinite measure having ‘periodic gaps’. The periodicity condition is crucial. For sets $S$ with randomly distributed gaps, we show that uniformly discrete sets $\Lambda$ satisfy a strong non-uniqueness property: every discrete function $c(\lambda) \in l^2(\Lambda)$ can be interpolated by an analytic $L^2$-function with spectrum in $S$.

Bibliography: 9 titles.

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§ 1. Introduction

Paley-Wiener space. We will use the standard form of the Fourier transform:

$$F(t) = \widehat{f}(t) := \int_{\mathbb{R}} e^{-2\pi itx} f(x) \, dx.$$

Given a measurable set $S \subseteq \mathbb{R}$, the Paley-Wiener space $PW_S$ consists of the inverse Fourier transforms of all square-integrable functions $F$ which vanish almost everywhere outside $S$. The set $S$ is called the spectrum of the space $PW_S$. Clearly, if the measure of $S$ is finite, then $F \in L^1(\mathbb{R})$, so every function $f \in PW_S$ is continuous.

Uniformly discrete sets. A set $\Lambda \subseteq \mathbb{R}$ is said to be uniformly discrete (u.d.) if $\delta(\Lambda) > 0$, where

$$\delta(\Lambda) := \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'|$$

(1)

(the infimal distance between distinct elements of $\Lambda$).

A set $\Lambda$ is said to have uniform density $D(\Lambda)$ if $\Lambda$ is regularly distributed in the following sense:

$$\text{Card}(\Lambda \cap (x, x + r)) = rD(\Lambda) + o(r) \quad \text{uniformly in } x \text{ as } r \to \infty.$$
Uniqueness problem. Let $M$ be a space of continuous functions on the real line $\mathbb{R}$. A set $\Lambda \subset \mathbb{R}$ is called a uniqueness set for $M$ if

$$f \in M \text{ and } f|_{\Lambda} = 0 \implies f = 0.$$ 

Otherwise, $\Lambda$ is called a nonuniqueness set for $M$.

We are interested in the following problem: \textit{which spaces of continuous functions on $\mathbb{R}$ admit uniformly discrete uniqueness sets?}

In this paper we consider this problem for spaces of functions whose spectrum belongs to a fixed set $S$. It is natural to distinguish between the following three cases: $S$ is a bounded set, $S$ is an unbounded set of finite measure, and $S$ is a set of infinite measure.

We focus on spaces of continuous functions whose spectrum lies in a set $S$ of infinite measure. In §§3–5 we establish that a wide range of spaces of such functions admit u.d. uniqueness sets, provided $S$ has periodic gaps. The periodicity condition is important. In particular, in §6, for sets $S$ with randomly distributed gaps we show that every u.d. set $\Lambda$ satisfies some strong nonuniqueness property.

We start with a short survey of the known results in the first two cases. A detailed discussion of these and related results can be found in [8]. For simplicity of presentation we focus on the one-dimensional case.

§ 2. Spectra of finite measure

Bounded spectra. The classical case is when $S = [a, b]$ is an interval. Then the elements of $PW_S$ are entire functions of exponential type. The distribution of zeros of such functions has been very thoroughly studied, see [4]. In particular, if the uniform density $D(\Lambda)$ exists, then the condition $D(\Lambda) > |S|$ is necessary while the condition $D(\Lambda) > |S|$ is sufficient for $\Lambda$ to be a uniqueness set for $PW_S$, where $|S|$ denotes the measure of $S$. This can be shown by standard complex variable techniques. A classical result of Beurling and Malliavin’s [2] states that the same is true for irregular sets $\Lambda$, provided uniform density is replaced by a certain exterior density (Beurling-Malliavin density).

In the case of spectra $S$ consisting of several intervals or having more complicated structure, the uniqueness property of a set $\Lambda$ cannot be expressed in terms of its density: some ‘dense’ (relative to the measure of $S$) sets $\Lambda$ may be nonuniqueness sets for $PW_S$. For example, one can easily check that $\Lambda = \mathbb{Z}$ is a nonuniqueness set for $PW_S$, where $S = [0, \varepsilon] \cup [1, 1 + \varepsilon]$, $0 < \varepsilon < 1$.

On the other hand, some ‘sparse’ sets $\Lambda$ may be uniqueness sets for $PW_S$ with a ‘large’ spectrum $S$. This phenomenon was discovered by Landau [3], who proved that certain perturbations of $\mathbb{Z}$ produce uniqueness sets for $PW_S$ whenever $S$ is a finite union of intervals $[k + a, k + 1 - a]$, $k \in \mathbb{Z}$, $0 < a < 1/2$. The uniqueness sets $\Lambda$ constructed by Landau have a complicated structure.

A more general result was proved in [5].

\textbf{Theorem 1.} The set

$$\Lambda := \{n + 2^{-|n|}, n \in \mathbb{Z}\}$$

is a uniqueness set for $PW_S$ for every bounded set $S$ satisfying $|S| < 1$. 
This theorem remains true for bounded sets $S$ of arbitrarily large measure satisfying $|S_1| < 1$, where we let

$$S_a := (S + a \mathbb{Z}) \cap [0, a]$$

(2)
denote the ‘projection’ of $S$ onto $[0, a]$.

Moreover, the result is also true for unbounded sets of finite measure that display ‘moderate accumulation’ at infinity, see [5].

Using rescaling, we can formulate the corresponding result for any bounded set $S$ of fixed measure.

**Unbounded spectra of finite measure.** It was shown in [7] (see also [8], Lecture 10) that for every (bounded or unbounded) set $S$ of finite measure, the space $PW_S$ possesses a u.d. uniqueness set.

**Theorem 2.** For every set $S$ of finite measure, there exists a u.d. set $\Lambda$ satisfying $D(\Lambda) = |S|$ which is a uniqueness set for $PW_S$.

By the discussion above, the density condition $D(\Lambda) = |S|$ is optimal, since we cannot get a smaller density when $S$ is an interval.

§ 3. Sobolev spaces with periodic spectral gaps

Here we begin to study the general case when the spectrum $S$ is a set of infinite measure.

3.1. Periodic spectral gaps. We say that $S$ has periodic ‘strong’ gaps if there exists $a > 0$ such that

$$|\overline{S_a}| < a,$$

(3)

where $\overline{S_a}$ denotes the closure of $S_a$, and the set $S_a$ was defined in (2). Condition (3) means that there exists a nonempty interval $I \subset [0, a]$ such that

$$S \cap (I + a \mathbb{Z}) = \emptyset.$$

We say that $S$ has periodic ‘weak’ gaps if

$$|S_a| < a.$$  

(4)

Condition (4) means that there exists a set of positive measure $Q \subset [0, a]$ such that $S \cap (Q + a \mathbb{Z}) = \emptyset$.

Observe that *every set $S$ of finite measure has periodic weak gaps* since $|S_a| < a$ for every $a > |S|$.

3.2. Uniqueness sets for Sobolev spaces. Given a u.d. set $\Lambda$, it is obvious that there exists a nontrivial smooth function $f$ that vanishes on $\Lambda$. However, this is no longer so if the spectrum of $f$ has weak periodic gaps. Below, we will state the result for Sobolev spaces.

For every number $\alpha > 1/2$, we denote by $W^{(\alpha)}$ the Sobolev space of functions $f$ such that the Fourier transform $F = \hat{f}$ satisfies

$$\|F\|_{\alpha}^2 := \int_{\mathbb{R}} (1 + |t|^{2\alpha}) |F(t)|^2 \, dt < \infty.$$  

(5)
It is clear that the functions $F$ satisfying (5) belong to $L^1(\mathbb{R})$, and so $W^{(\alpha)}$ consists of continuous functions.

We denote the subspace of $W^{(\alpha)}$ of functions $f$ with spectrum in $S$ by $W^{(\alpha)}_S$; this means that $F = 0$ almost everywhere outside $S$.

**Theorem 3.** Suppose a set $S$ satisfies $|S_a| < a$, for some $a > 0$. Then there exists a u.d. set $\Lambda$ of density $D(\Lambda) = a$ that is a uniqueness set for $W^{(\alpha)}_S$.

### 3.3. Decomposition of $\mathbb{Z}$

**Lemma 1.** Let $A \subset [0,1], |A| < 1$. Then there exist pairwise disjoint sets $Z_j \subset \mathbb{Z}$, $j \in \mathbb{N}$, such that every exponential system

$$\{e^{-i2\pi nt}, n \in Z_j\} \tag{6}$$

is complete in $L^2(A)$.

**Proof.** 1. First, we remark that the for every natural number $N$ the exponential family

$$\{e^{-i2\pi nt}, |n| > N\} \tag{7}$$

is complete in $L^2(A)$. Indeed, assume that there exists a nontrivial function $F \in L^2(A)$ orthogonal to the system (7). Extend $F$ by zero to $[0,1] \setminus A$. Since the trigonometric system forms an orthonormal basis in $L^2(0,1)$, we conclude that $F$ is a trigonometric polynomial:

$$F(t) = \sum_{|n| \leq N} c_ne^{-i2\pi nt}.$$  

Clearly, $F$ cannot vanish on the set of positive measure $[0,1] \setminus A$, which is a contradiction.

2. Fix a sequence $\varepsilon_k, k \in \mathbb{N}$, satisfying $\varepsilon_k \to 0$ as $k \to \infty$. We will now construct a sequence of disjoint finite symmetric sets $\Gamma_k \subset \mathbb{Z}$, $k \in \mathbb{N}$, with the following property: for every $m$, $|m| \leq k$, there exists a trigonometric polynomial $P_{k,m}$ whose frequencies belong to $\Gamma_k$; such that

$$\|e^{i2\pi mt} - P_{k,m}(t)\|_{L^2(A)} < \varepsilon_k. \tag{8}$$

Set $\Gamma_1 := \{-1,0,1\}$. Clearly, (8) holds with $m = 0,-1,1$. Then we set

$$\Gamma_k := \{n: n_{k-1} < |n| \leq n_k\},$$

where $n_1 = 1$, and choose $n_j$, $j > 1$, inductively as follows. Since the system (7) is complete, there exists $n_2$ so large that for every $|m| \leq 2$ there exists a polynomial $P_{2,m}$ satisfying (8) with $k = 2$ and whose frequencies belong to the set $\Gamma_2$, and so on. At the $k$th step we choose an integer $n_k$ so large that for every $|m| \leq k$ there exists a polynomial $P_{k,m}$ satisfying (8) and whose frequencies belong to the set $\Gamma_k$.

3. Now we take disjoint infinite subsets $\Delta_j \subset \mathbb{N}$ and set

$$Z_j := \bigcup_{k \in \Delta_j} \Gamma_k.$$  

It follows from the construction above that every exponential system (6) is complete in $L^2(A)$. 


3.4. Periodization and the Fourier transform. For an integrable function $H$ on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, we denote the Fourier coefficients of $H$ by

$$c_n(H) := \int_{\mathbb{T}} H(t)e^{2\pi i nt} dt, \quad n \in \mathbb{Z}.$$ 

Given $F \in L^1(\mathbb{R})$, consider its ‘periodization’

$$H(t) := \sum_{k \in \mathbb{Z}} F(t + k), \quad t \in [0, 1).$$

Clearly, $H$ is defined almost everywhere and belongs to $L^1(\mathbb{T})$. Direct calculation shows that $c_n(H) = f(n)$, where $f$ is the inverse Fourier transform of $F$.

Similarly, for the periodization $H_v$ of the function $F_v(t) := e^{2\pi ivt}F(t)$ we have

$$c_n(H_v) = f(n + v), \quad n \in \mathbb{Z}. \quad (9)$$

It is easy to check that the periodization of $F \in L^1 \cap L^2(\mathbb{R})$ does not always belong to $L^2(\mathbb{T})$. However, it does when $f$ belongs to the Sobolev space.

**Lemma 2.** Assume that $F$ satisfies $\|F\|_\alpha < \infty$. Then

$$\int_0^1 |H(t)|^2 dt < \infty.$$ 

**Proof.** Now,

$$|H(t)|^2 = \left| \sum_{k \in \mathbb{Z}} F(t + k) \right|^2 \leq \sum_{k \in \mathbb{Z}} \frac{1}{(1 + |k|^\alpha)^2} \sum_{k \in \mathbb{Z}} |F(t + k)|^2 (1 + |k|)$$

and the lemma follows easily from the definition of $\|F\|_\alpha$ in (5).

3.5. Proof of Theorem 3. After rescaling we can assume that $a = 1$.

Fix a sequence $\{\alpha_j\}$ dense in $[0, 1]$. We can now choose sets $Z_j$ in Lemma 1 so that the set

$$\Lambda := \bigcup_{j=1}^{\infty} (Z_j + \alpha_j) \quad (10)$$

is u.d. and $D(\Lambda) = 1$.

By Lemma 1 with $A = S_1$, each exponential system (6) is complete in $L^2(S_1)$. This means that each $Z_j$ is a uniqueness set for the space $PW_{S_1}$.

Now we will prove that $\Lambda$ is a uniqueness set for the space $W_{S_1}^{(\alpha)}$. We have to show that every function $f \in W_{S_1}^{(\alpha)}$ satisfying

$$f|_{\Lambda} = 0 \quad (11)$$

vanishes identically on $\mathbb{R}$. 
For $j \in \mathbb{N}$ we consider the function

$$F_j(t) := e^{2\pi \alpha_j t} F(t)$$

and its periodization $H_j$. Recall that $F$ vanishes almost everywhere outside $S$. Since $S \subset S_1 + \mathbb{Z}$, we have

$$H_j = 0 \quad \text{almost everywhere on } \mathbb{T} \setminus S_1.$$ 

Also, by Lemma 2, $H_j \in L^2(\mathbb{T})$.

By (9)–(11),

$$c_n(H_j) = f(\alpha_j + n) = 0, \quad n \in Z_j.$$ 

Since the system (6) is complete in $L^2(S_1)$, we have $H_j = 0$ almost everywhere. By (9) this means that

$$f(n + \alpha_j) = 0, \quad n \in \mathbb{Z}.$$ 

Since this equality is true for all $j$, $f$ is continuous and the sequence $\{\alpha_j\}$ is dense in $[0, 1]$, we conclude that $f = 0$.

§ 4. Uniqueness sets for rapidly decreasing functions

Theorem 3 shows that classes of smooth functions $f$ having periodic weak spectral gaps admit u.d. uniqueness sets. In this section we show that a similar result holds for rapidly decreasing functions.

We denote the space of continuous functions $f$ satisfying

$$\sup_{x \in \mathbb{R}} (1 + x^2)|f(x)| < \infty$$

by $Y$, and the subspace of $Y$ of functions $f$ such that the function $F = \hat{f}$ vanishes outside $S$ by $Y_S$.

Theorem 4. Suppose a set $S$ satisfies $|S_a| < a$, for some $a > 0$. Then there exists a u.d. set $\Lambda$ of density $D(\Lambda) = a$ that is a uniqueness set for $Y_S$.

Condition (12) in the definition of $Y$ can be relaxed somewhat. However, the result is no longer true if no decay condition is imposed; see Theorem 6 below.

4.1. Proof of Theorem 4. The proof follows the same idea as the proof of Theorem 3. However, since $F$ need not be integrable, the periodization of $F$ cannot be defined pointwise.

We will use the following corollary of the classical Poisson summation formula.

Lemma 3. Assume that a continuous function $f$ satisfies (12) and $\hat{f}(t) = 0$, $t \in Q + \mathbb{Z}$, for some set $Q \subset [0, 1]$, $|Q| > 0$. Then for every $x \in [0, 1]$

$$\sum_{n \in \mathbb{Z}} f(x + n)e^{-i2\pi nt} = 0, \quad t \in Q.$$ (13)
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Proof. If $F = \hat{f}$ is also rapidly decreasing, then the result follows directly from the Poisson summation formula.

Otherwise, take any smooth function $h$ vanishing outside $(-1, 1)$ and satisfying $\int h = 1$. Set

$$h_\varepsilon(x) := \frac{h(x/\varepsilon)}{\varepsilon}.$$

Fix $x \in [0, 1)$ and $t \in Q$ and apply the Poisson formula to the convolution $f * h_\varepsilon$:

$$\sum_{n \in \mathbb{Z}} (f * h_\varepsilon)(x + n)e^{-i2\pi nt} = \sum_{n \in \mathbb{Z}} (\hat{f} \cdot \hat{h}_\varepsilon)(t + n)e^{-i2\pi nx} = 0.$$

We will show that

$$\sum_{n \in \mathbb{Z}} (f * h_\varepsilon)(x + n)e^{-i2\pi nt} \to \sum_{n \in \mathbb{Z}} f(x + n)e^{-i2\pi nt}$$

as $\varepsilon \to 0$, which proves the lemma.

So we fix $\delta > 0$ and decompose the left-hand side into two sums: $\sum_{|n| < N} + \sum_{|n| \geq N}$. In view of (12) we can choose $N = N(\delta)$ so that for every $x, t \in [0, 1]$ and $0 < \varepsilon < 1$ the modulus of the second summand is less than $\delta$. Clearly, each term of the first summand tends to $f(x + n)e^{-i2\pi nt}$ as $\varepsilon \to 0$, due to the continuity of $f$.

Now, we can finish the proof of Theorem 4. After rescaling, we can assume that $a = 1$, so that $|S_1| < 1$.

Following the proof of Theorem 3 we can find pairwise disjoint sets $Z_j \subset \mathbb{Z}$, $j \in \mathbb{N}$, such that for every $j$ the system

$$E(Z_j) := \{e^{-2\pi ik}, k \in Z_j\}$$

is complete in $L^2(S_1)$.

Set

$$\Lambda := \bigcup_{j \in \mathbb{N}} (Z_j + \alpha_j),$$

where $\{\alpha_l, l \in \mathbb{N}\}$ is dense in $(0, 1)$. It remains to check that $\Lambda$ is a uniqueness set for $Y_S$.

Assume that $f|\Lambda = 0$ for some $f \in Y_S$, that is,

$$f|Z_j + \alpha_j = 0, \quad j = 1, 2, \ldots.$$

Fix $j$ and consider a 1-periodic function

$$g_j(x) := \sum_{n \in \mathbb{Z}} f(n + \alpha_j)e^{-i2\pi nx}.$$

Clearly, $g \in L^2(0, 1)$ and $g$ is orthogonal to all the exponential functions in $E(Z_j)$. On the other hand, by Lemma 3,

$$g_j(x) = 0, \quad t \in Q := [0, 1] \setminus S_1.$$

As $E(Z_j)$ is complete in $L^2(S_1)$, thus $g_j = 0$ almost everywhere. Hence

$$f(n + \alpha_j) = 0, \quad n \in \mathbb{Z}.$$

This is true for every $j$. Recalling that $\{\alpha_j\}$ is dense on $[0, 1]$ and $f \in C(\mathbb{R})$, we conclude that $f = 0$ on $\mathbb{R}$.
§ 5. Distributions with periodic spectral gaps

5.1. Strong gaps. If \( S \) has periodic strong gaps, then Theorem 4 can be extended to wider function spaces.

Denote the space of continuous functions that have at most polynomial growth on \( \mathbb{R} \) by \( X \). Every element \( f \in X \) is a Schwartz distribution. Its spectrum is the minimal closed set \( S \) such that

\[
\int_{\mathbb{R}} f(t) \varphi(t) \, dt = 0
\]

for every test function \( \varphi \) satisfying \( \hat{\varphi} = 0 \) in a neighbourhood of \( S \).

Given a closed set \( S \), we denote the subspace of \( X \) consisting of functions with spectrum in \( S \) by \( X_S \).

Without loss of generality, we can assume that the spectral gaps contain \((-\delta, 0) + \mathbb{Z}\).

**Theorem 5.** There exist a u.d. set \( \Lambda \), \( D(\Lambda) = 1 \), that is a uniqueness set for \( X_S \), where \( S = [0, 1 - \delta] + \mathbb{Z} \), for every \( \delta, 0 < \delta < 1 \).

**Proof.** Consider \( Z_j \) as in Lemma 1 (this can be done independent of \( \delta \)). Choose \( \Lambda \) as in the proof of Theorem 4. Given \( f \in X_S \), consider the function \( g := f \cdot \varphi \), where \( \hat{\varphi} \) is a Schwartz function supported on \([0, \delta/2] \). It is easy to see that \( g \) satisfies the assumptions of Theorem 4 with \( a = 1 \) and \( S = [0, 1 - \delta/2] \). If \( f|\Lambda = 0 \), then the same is true for \( g \). So Theorem 4 implies that \( f = 0 \).

5.2. Weak gaps. Here we show that Theorem 5 is no longer true for weak spectral gaps. This is a direct corollary of Theorem 6 below (see also the comment after Lemma 5 in §6.1).

We need the following definition.

**Definition 1.** Given a closed set \( S \), the Bernstein space \( B_S \) is the set of continuous bounded functions \( f \) on \( \mathbb{R} \) whose spectrum (in the sense of distributions) lies in \( S \).

**Theorem 6** (see [6]). There exists a closed set \( S \) of Lebesque measure zero such that every bounded function \( c(\lambda) \) defined on a u.d. set \( \Lambda \) can be interpolated by a function \( f \in B_S \).

It is obvious that every set of measure zero has weak periodic gaps with an arbitrary period \( a \). However, if \( S \) is the set in Theorem 6, then no u.d. set \( \Lambda \) is a uniqueness set for \( B_S \).

A few words about the proof of Theorem 6. It is based on Menshov’s classical result of 1916 (see [1]): there exists a probability measure \( \mu \) on \( \mathbb{R} \) supported by a compact set \( K \) of measure zero, such that its Fourier transform

\[
\hat{\mu}(x) = \int_K e^{-2\pi i tx} \, d\mu(t)
\]

vanishes at infinity.
Here is a short sketch of the proof of Theorem 6 (for the details see [8], Lecture 10).

Proof. 1. Using Menshov’s result, given a positive $\delta$, after rescaling we can get a probability measure $\mu_\delta$ supported by a compact $K$ of Lebesgue measure zero, such that

\[ \hat{\mu}_\delta(0) = 1 \quad \text{and} \quad |\hat{\mu}_\delta(x)| < \delta, \quad |x| > \delta. \]

2. Using this we can construct a family of compact sets $K_j$ of measure zero which goes to infinity and functions $g_j \in B_{K_j}$, $j \in \mathbb{N}$, satisfying

\[ \|g_j\|_\infty = g_j(0) = 1, \quad \|g_j(t)\| < e^{-j}, \quad |t| > e^{-j}. \]

3. Set

\[ S := \bigcup_{j=1}^{\infty} K_j. \]

It is a closed (noncompact) set of measure zero.

Fix a u.d. set $\Lambda$. Using appropriate translates of the functions $g_j$ we can define a function $f_\lambda \in B_S$ satisfying

\[ \|f_\lambda\|_\infty = f_\lambda(\lambda) = 1, \quad \lambda \in \Lambda, \]

and such that $f_\lambda$ is so small outside a small neighbourhood of $\lambda$ that we have

\[ \sum_{\lambda' \in \Lambda, \lambda' \neq \lambda} |f_{\lambda'}(\lambda)| < \frac{1}{2}. \]

4. Consider the linear operator $T: l^\infty(\Lambda) \to l^\infty(\Lambda)$ defined by

\[ (Tc)_\lambda := \sum_{\lambda' \in \Lambda, \lambda' \neq \lambda} f_{\lambda'}(\lambda)c_{\lambda'}, \quad \lambda \in \Lambda, \quad c = \{c_{\lambda'} : \lambda' \in \Lambda\} \in l^\infty(\Lambda). \]

Clearly, $\|T\| < 1$. Hence the operator $T + I$ is surjective. Therefore, for any data $c = \{c_\lambda\} \in l^\infty(\Lambda)$ there exists a sequence $b = \{b_\lambda\} \in l^\infty(\Lambda)$ satisfying $(I + T)b = c$. Hence the function

\[ f(x) := \sum_{\lambda \in \Lambda} b_\lambda f_\lambda(x) \]

belongs to $B_S$ and solves the interpolation problem $f|_\Lambda = c$.

§6. Nonperiodic spectral gaps

Here we show that the periodicity of spectral gaps is crucial for the existence of discrete uniqueness sets. We consider spectra $S$ that are unions of disjoint intervals of given length. For simplicity, we assume that each interval has length one:

\[ S = \bigcup_{j=1}^{\infty} [\gamma_j, \gamma_j + 1]. \]  

We also assume that the distances $\xi_j$ between the intervals belong to a fixed interval, say $[2, 3]$:

\[ \xi_j := \gamma_{j+1} - \gamma_j - 1 \in [2, 3], \quad j \in \mathbb{N}. \]
Clearly, $S$ lies on the half-line $[\gamma_1, \infty)$ and admits the representation

$$S = \Gamma + [0, 1], \quad \Gamma := \bigcup_{j=1}^{\infty} \{\gamma_j\},$$

(16)

where $\Gamma$ satisfies $\delta(\Gamma) > 3$. Here $\delta(\Gamma)$ is the separation constant defined in (1).

We say that a u.d. set $\Gamma$ satisfies property (C) if it contains arbitrary long arithmetic progressions with rationally independent steps. More precisely, we assume the following.

**Property (C).** For every $m \in \mathbb{N}$ there exist rationally independent numbers $q_1, \ldots, q_m$, such that for every $N \in \mathbb{N}$ the set $\Gamma$ contains arithmetic progressions of length $N$ with differences $q_1, \ldots, q_m$. The latter means that there exist $a_1, \ldots, a_m$ such that

$$\bigcup_{j=1}^{m} \{a_j + q_j, a_j + 2q_j, \ldots, a_j + Nq_j\} \subset \Gamma.$$

**Theorem 7.** Assume $S$ is given in (14)–(16), where $\Gamma$ satisfies property (C). Then no u.d. set $\Lambda$ is a uniqueness set for the Sobolev space $W^\alpha_S$.

We can also check that, under the assumptions of Theorem 7, no u.d. set $\Lambda$ is a uniqueness set for the space $Y^S_S$.

**6.1. Interpolation sets.** The set $\Lambda$ is an interpolation set for the Paley-Wiener space $PW_S$ if for every sequence $\{c_\lambda, \lambda \in \Lambda\} \in l^2(\Lambda)$ there exists $f \in PW_S$ satisfying

$$f(\lambda) = c_\lambda, \quad \lambda \in \Lambda.$$

The following criterion is well known (see [8], Lecture 4, for instance).

**Lemma 4.** Let $S$ be a bounded set. Then a set $\Lambda$ is a set of interpolation for $PW_S$ if and only if there exists a constant $C > 0$ such that the inequality

$$\int_S \left| \sum_{\lambda \in \Lambda} c_\lambda e^{i2\pi \lambda t} \right|^2 dt \geq C \sum_{\lambda \in \Lambda} |c_\lambda|^2$$

holds for every finite sequence $c_\lambda$.

Theorem 7 is a direct corollary of the following lemma.

**Lemma 5** (Main Lemma). Assume that $S$ is a set as in Theorem 7. Then for every $\delta > 0$ there exists a bounded subset $S(\delta) \subset S$ such that every u.d. set $\Lambda$ satisfying $\delta(\Lambda) \geq \delta$ is a set of interpolation for $PW_{S(\delta)}$.

Consider the set $\Lambda \cup \{c\}$ for some point $c \notin \Lambda$. By Lemma 5, it is an interpolation set for $PW_{S(\delta)}$, for some bounded subset $S(\delta) \subset S$. Then there exists $f \in PW_{S(\delta)}$ satisfying $f(c) = 1$ and

$$f(\lambda) = 0, \quad \lambda \in \Lambda.$$

Now Theorem 7 follows from the observation that $PW_{S(\delta)} \subset W^\alpha_S$.
6.2. Proof of Lemma 5.

Lemma 6. Suppose \( \Gamma \) satisfies property (C). Then for every \( \varepsilon, 0 < \varepsilon < 1 \), there exist \( N \in \mathbb{N} \) and \( \eta_j \in \Gamma, j = 1, \ldots, N \), such that the exponential polynomial

\[
P(t) = \frac{1}{N} \sum_{j=1}^{N} e^{i \eta_j t}
\]

satisfies

\[
|P(t)| \leq \varepsilon, \quad \varepsilon < |t| < \frac{1}{\varepsilon}.
\]

(17)

Proof. 1. Fix any integer \( m > 1/\varepsilon \). Then fix numbers \( q_1, \ldots, q_m \) in the definition of property (C). Since the \( q_j \) are rationally independent, the set of points

\[
\left\{kq_j \in \left(\frac{-1}{\varepsilon}, \frac{1}{\varepsilon}\right)\right\}
\]

(18)

is separated, where \( k \in \mathbb{Z}, k \neq 0, j = 1, \ldots, m \). Hence the distance between any two points in this set exceeds some positive number \( \rho \). We can assume that \( \rho < \varepsilon \).

2. For \( n \in \mathbb{N} \) and \( q \geq 2 \) consider the \((1/q)\)-periodic exponential polynomial

\[
P_{n,q}(t) := \frac{1}{n} \sum_{j=0}^{n-1} e^{i 2 \pi jqt} = \frac{1}{n} \frac{e^{i 2 \pi nqt} - 1}{e^{i 2 \pi qt} - 1}.
\]

From the properties of the Dirichlet kernel it is well known that it satisfies

\[
|P_{n,q}(t)| \leq \rho, \quad \text{dist}\left(t, \left(\frac{1}{q}\right)\mathbb{Z}\right) \geq \rho,
\]

(19)

provided \( n \) is large enough.

3. Choose \( n \) so large that (19) holds with \( q = q_j, j = 1, \ldots, m \). Then, since the set (18) is \( \rho \)-separated, for every \( t \) satisfying \( \varepsilon < |t| < 1/\varepsilon \) the inequality

\[
|P_{n,q_j}(t)| < \varepsilon
\]

holds for all but at most one value of \( j \in \{1, \ldots, m\} \).

4. By property (C) there exist \( a_j \) such that \( a_j + kq_j \in \Gamma, k = 0, \ldots, n - 1 \). Set

\[
P(t) = \frac{1}{m} \sum_{j=1}^{m} e^{i 2 \pi a_j t} P_{n,q_j}(t).
\]

From Step 3 we see that

\[
|P(t)| < \frac{1 + (m-1)\varepsilon}{m} < \varepsilon, \quad \varepsilon < |t| < 1/\varepsilon,
\]

which completes the proof.
Proof of Lemma 5. Fix $\delta > 0$ and assume that a u.d. set $\Lambda$ satisfies $\delta(\Lambda) \geq \delta$. By Lemma 6, for every $0 < \varepsilon < 1$ there exists an exponential polynomial $P$ with frequencies in $\Gamma$ satisfying (17). We denote the set of its frequencies by $\Gamma_P \subset \Gamma$ and set

$$S(\delta) := \Gamma_P + [0, 1].$$

Clearly, $S(\delta)$ is a bounded subset of $S$.

Now we fix any positive smooth function $\Phi$ that vanishes outside $[0, 1]$ such that its Fourier transform $\varphi = \hat{\Phi}$ satisfies $\varphi(0) = 1$ and

$$\sup_{x \in \mathbb{R}} (1 + x^4)|\varphi(x)| < \infty. \quad (20)$$

Set

$$H(t) := \left( \Phi * \sum_{\gamma \in \Gamma_P} \delta_{\gamma} \right)(t) = \sum_{\gamma \in \Gamma_P} \Phi(t - \gamma).$$

Then the support of $H$ belongs to $S(\delta)$ and its Fourier transform is given by

$$h(x) := \widehat{H}(x) = P(x)\varphi(x).$$

Clearly, $h(0) = 1$.

When $\varepsilon$ is sufficiently small, from (17) and (20) we obtain

$$|h(x)| < \frac{\varepsilon}{1 + x^2} \quad \text{for all } |x| > \delta.$$

Using this estimate and assuming that $\varepsilon$ is sufficiently small, for every $\lambda \in \Lambda$ we get the estimate

$$\sum_{\mu \in \Lambda, \mu \neq \lambda} |h(\mu - \lambda)| < \sum_{\mu \in \Lambda, \mu \neq \lambda} \frac{\varepsilon}{1 + (\mu - \lambda)^2} < 2 \sum_{n \in \mathbb{N}} \frac{\varepsilon}{1 + (\delta n)^2} < \frac{1}{2}.$$

Set

$$M := \max_{t \in [0, 1]} |\Phi(t)|.$$

Then

$$\int_{S(\delta)} \left| \sum_{\lambda \in \Lambda} c_{\lambda} e^{i\lambda t} \right|^2 dt \geq \frac{1}{M} \int_{S(\delta)} \left| \sum_{\lambda \in \Lambda} c_{\lambda} e^{i\lambda t} \right|^2 H(t) dt$$

$$= \frac{1}{M} \left( \sum_{\lambda \in \Lambda} |c_{\lambda}|^2 + \sum_{\lambda, \mu \in \Lambda, \lambda \neq \mu} c_{\lambda} \overline{c}_{\mu} h(\lambda - \mu) \right)$$

$$\geq \frac{1}{M} \left( \sum_{\lambda \in \Lambda} |c_{\lambda}|^2 - \sum_{\lambda, \mu \in \Lambda, \lambda \neq \mu} \frac{|c_{\lambda}|^2 + |c_{\mu}|^2}{2} |h(\lambda - \mu)| \right)$$

$$= \frac{1}{M} \left( \sum_{\lambda \in \Lambda} |c_{\lambda}|^2 - \sum_{\lambda, \mu \in \Lambda, \mu \neq \lambda} |h(\lambda - \mu)| \right) > \frac{1}{2M} \sum_{\lambda \in \Lambda} |c_{\lambda}|^2.$$

By Lemma 4 this completes the proof of Lemma 5.
6.3. **Random spectra do not admit u.d. uniqueness sets.** Here we consider the situation when $S$ is a countable union of unit intervals, the distances between these intervals being randomly distributed. More precisely, below we assume that $S$ and $\Gamma$ are defined in (14) and (16), and that the $\xi_j := \gamma_{j+1} - \gamma_j - 1$ are independent random variables which are uniformly distributed over the interval $[2,3]$. With these assumptions we have the following theorem.

**Theorem 8.** With probability one no u.d. set $\Lambda$ is a uniqueness set for $W^{(a)}_S$.

**Proof.** Theorem 8 follows from the following claim, which is an analogue of the Main Lemma in §6.1: with probability one, for every fixed $\delta > 0$ there exists a bounded subset $S(\delta) \subset S$ such that every u.d. set $\Lambda$, $\delta(\Lambda) \geq \delta$, is a set of interpolation for $PW_{S(\delta)}$.

Recall that $S = \bigcup_{j=1}^{\infty} \{\gamma_j\} + [0,1]$, $\gamma_{j+1} - \gamma_j \in [3,4]$, $j \in \mathbb{N}$.

It is easy to see that given integers $k \geq 1$ and $N \geq 2$ and a number $q \in (3,4)$, the set

$$\{\gamma_k, \gamma_k + q, \ldots, \gamma_k + Nq\} + \left[\frac{1}{4}, \frac{3}{4}\right]$$

belongs to $S$ whenever

$$|\gamma_{k+j} - (\gamma_k + jq)| < \frac{1}{4}, \quad j = 1, 2, \ldots, N.$$

Recall also that $\gamma_{j+1} - \gamma_j$ is uniformly distributed over $[3,4]$. So, the probability that the latter inequalities hold true is positive and independent of $k$.

Now fix $m \in \mathbb{N}$ and $q_1, \ldots, q_m \in (3,4)$ in accordance with property (C). By the Borel-Cantelli lemma, with probability one there exist integers $k_1, \ldots, k_m$ such that the finite sequence

$$\Gamma^* := \bigcup_{j=1}^{m} \{\gamma_{k_j}, \gamma_{k_j} + q_j, \ldots, \gamma_{k_j} + Nq_j\}$$

satisfies

$$S(\delta) := \Gamma^* + \left[\frac{1}{4}, \frac{3}{4}\right] \subset S.$$ 

Now, choosing $m$ and $N$ sufficiently large, the proof proceeds in exactly the same way as the proof of the Main Lemma (see Lemma 5).

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**§ 7. Remarks**

**7.1. Multi-dimensional extensions.** All our one-dimensional results above admit multi-dimensional extensions. Here we give a very brief account of these extensions.

The definitions in §1 can be extended to the multi-dimensional situation. In particular, given a set $S \subset \mathbb{R}^p$, the Paley-Wiener space $PW_S$ consists of the
(p-dimensional) inverse Fourier transforms of the $L^2(\mathbb{R}^p)$-functions which vanish almost everywhere outside $S$. A set $\Lambda \subset \mathbb{R}^p$ is uniformly discrete (u.d.) if the infimal distance between distinct points in it is positive. A u.d. set $\Lambda$ has uniform density $D(\Lambda)$ if

$$\text{Card}(\Lambda \cap ([0,r]^p + s)) = r^p D(\Lambda) + o(r^p) \quad \text{uniformly on } s \text{ as } r \to \infty.$$ 

Here $s = (s_1, \ldots, s_p) \in \mathbb{R}^p$ and

$$[0,r]^p + s = \{x = (x_1, \ldots, x_p) \in \mathbb{R}^p : s_j \leq x_j \leq s_j + r, j = 1, \ldots, p\}.$$ 

We denote the $p$-dimensional measure of a set $S \subset \mathbb{R}^p$ by $|S|$. We denote the ‘projection’ of the set $S \subset \mathbb{R}^p$ onto the cube $[0,a]^p$, where $a$ is a positive number, by $S_a$:

$$S_a := (S + a\mathbb{Z}^p) \cap [0,a]^p.$$ 

We now formulate a multi-dimensional variant of Theorem 1. Suppose that $s_1, \ldots, s_p$ are real numbers linearly independent over the set of integers. Then

$$\Lambda := \{m_1 + s_1 2^{-|m_1|} \cdots m_p, \ldots, m_p + s_p 2^{-|m_1|} \cdots |m_p|, (m_1, \ldots, m_p) \in \mathbb{Z}^p\}$$

is a uniqueness set for $\text{PW}_S$, for every bounded set $S \subset \mathbb{R}^n$ satisfying $|S_1| < 1$.

The proof of this result goes along the same lines as the proof of Theorem 2 in [9]. Choosing the numbers $s_j$ small, one can make the set $\Lambda$ in the above result an arbitrarily small perturbation of the lattice $\mathbb{Z}^p$.

Also Theorem 2, as noted in [7], admits an extension to several dimensions.

For every set $S \subset \mathbb{R}^p$ of finite measure there exists a u.d. set $\Lambda \subset \mathbb{R}^p$, $D(\Lambda) = |S|$, that is a uniqueness set for $\text{PW}_S$.

We can introduce spaces $W^{(\alpha)}_S$ and $Y_S$ as follows: the space $W^{(\alpha)}(\mathbb{R}^p)$, $\alpha > p/2$, consists of the functions $f$ on $\mathbb{R}^p$ whose Fourier transform $F$ vanishes outside $S$ and satisfies

$$\|F\|^2 := \int_{\mathbb{R}^p} (1 + |t|^{2\alpha})|F(t)|^2 \, dt < \infty,$$

where $|t|^2 = t_1^2 + \cdots + t_p^2$ and $dt = dt_1 \cdots dt_p$.

The space $Y_S(\mathbb{R}^p)$ consists of the continuous functions $f$ satisfying

$$\sup_{x \in \mathbb{R}^p} (1 + |x|^{2p})|f(x)| < \infty, \quad |x|^2 := x_1^2 + \cdots + x_p^2,$$

and such that the Fourier transform $F$ vanishes outside $S$.

We can check that both Lemmas 1 and 2 admit multi-dimensional extensions. This allows us to obtain the following version of Theorems 3 and 4.

Assume that $S \subset \mathbb{R}^p$ is such that $|S_a| < a^p$ for some $a$ $> 0$. Then the spaces $W^{(\alpha)}_S(\mathbb{R}^p)$, $\alpha > p/2$, and $Y_S(\mathbb{R}^p)$ admit a u.d. uniqueness set.

The $p$-dimensional versions of Theorems 5–8 also hold true.
7.2. Questions. We have left several open problems.

1. In connection with Theorems 1 and 2, we can ask: does there exist a u.d. set \( \Lambda \), \( D(\Lambda) = 1 \), that is a uniqueness set for \( PW_S \), for every set \( S \subset \mathbb{R} \), \( |S| < 1 \)?

2. The following question arises in connection with Theorems 3 and 4: let \( S \subset \mathbb{R} \) be a set with periodic weak gaps; does the space \( PW_S \cap C(\mathbb{R}) \) admit a u.d. uniqueness set?

3. It also would be interesting to know whether Theorem 2 remains true for the Fourier transforms of integrable functions. Let \( S \subset \mathbb{R} \) be a set of finite measure. Is it true that the space

\[
\hat{L}_S := \{ f = \hat{F} : F \in L^1(\mathbb{R}), F = 0 \text{ almost everywhere outside } S \}
\]

admits a u.d. uniqueness set?

Theorem 5 implies that the answer is ‘yes’ whenever \( S_a \) is not dense on \([0, a]\) for some \( a \).

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