ON THE ESTIMATE OF DISTANCE TRAVELED
BY A PARTICLE IN A DISK-LIKE VORTEX PATCH

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Abstract. We consider the incompressible two-dimensional Euler equation in the plane in the case when
its initial vorticity is the characteristic function of a bounded open set. We show that the travel distance
grows linearly for most of fluid particles initially placed on the set when the area of the symmetric difference
between the set and a disk is small enough.

1. Introduction

We consider the incompressible 2D Euler equation in vorticity form in the whole plane:
\[ \partial_t \theta + u \cdot \nabla \theta = 0 \quad \text{for } x \in \mathbb{R}^2 \text{ and for } t > 0, \]
\[ \theta|_{t=0} = \theta_0 \quad \text{for } x \in \mathbb{R}^2 \]
where the Biot-Savart law is given by \( u = (K * \theta) \) with
\[ K(x) := \frac{1}{2\pi} \frac{x^\perp}{|x|^2} = \frac{1}{2\pi} \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right). \]

When \( \theta_0 \) lies on \( L^1 \cap L^\infty \), the existence and uniqueness of a global-in-time weak solution is due to
Yudovich [21].

In this paper, we are interested in estimating the distance traveled by a fluid particle. More precisely,
we consider the case when the initial data \( \theta_0 \) is the characteristic function \( \mathbb{1}_{\Omega_0} \) of a bounded open
set \( \Omega_0 \) in \( \mathbb{R}^2 \). Then the corresponding solution \( \theta \) is given by Yudovich theory, and it has the form
of \( \theta(t) = \mathbb{1}_{\Omega_t} \), where \( \Omega_t \) is defined by \( \Omega_t = \{ \phi_x(t) \in \mathbb{R}^2 | x \in \Omega_0 \} \) and \( \phi_x(\cdot) \) is the particle trajectory
of the particle whose initial position is at \( x \in \mathbb{R}^2 \), which can be obtained by solving the following
system of the ordinary differential equations:
\[ \frac{d}{dt} \phi_x(t) = u(t, \phi_x(t)) \quad \text{for } t > 0 \quad \text{and} \quad \phi_x(0) = x. \]

This is well-defined since the velocity field \( u \) allows a log-Lipschitz estimate (e.g. see the modern
texts [15], [17]). For \( t > 0 \) and \( x \in \mathbb{R}^2 \), we say that \( d_x(t) \) is the distance traveled by the particle,
whose initial position is at \( x \), up to time \( t \). More precisely, we define \( d_x(t) \) by
\[ d_x(t) := \int_0^t |u(s, \phi_x(s))| ds. \]

For instance, when \( \Omega_0 \) is the unit disk \( D \) centered at the origin, we easily compute (see Subsection 1.1)
\[ d_x(t) = \frac{|x|}{2} t \quad \text{for any } x \in D \text{ and for any } t > 0. \]

For a general bounded open set \( \Omega_0 \), we have, at least, a trivial linear upper bound
\[ d_x(t) \leq C t \quad \text{for any } x \text{ in the plane and for any } t > 0. \]

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where the above constant $C$ depends only on the Lebesgue measure $|\Omega_0|$ of the set $\Omega_0$. Indeed, the conservation of the total mass $|\Omega_t|$ in time due to the incompressibility of the fluid gives a uniform bound for $u$ (e.g. see estimate (9) with the substitution $f = \mathbb{1}_{\Omega_0}$).

The main result of this paper says that, for most of particles placed on the initial vortex patch in the beginning, the travel distance actually grows linearly in time when the initial patch is disk-like in the sense that the measure of the initial symmetric difference

$$\Omega_0 \triangle B_r := (\Omega_0 \setminus B_r) \cup (B_r \setminus \Omega_0)$$

is small enough. Here we denote $B_r := \{x \in \mathbb{R}^2 \mid |x| < r\}$ for $r > 0$. Without loss of generality, we will compare the initial patch $\Omega_0$ only with the unit disk

$$D := B_1$$

thanks to the scaling of the Euler equation.

**Theorem 1.1.** For any $R > 0$, there exist constants $\delta_0 > 0$, $C > 0$ and $c > 0$ such that if $\Omega_0 \subset B_R$ and if

$$|D \triangle \Omega_0| < \delta_0,$$

then the solution of (1) for the initial data $\theta_0 = \mathbb{1}_{\Omega_0}$ satisfies the following two properties:

(I) For any $\delta > 0$ satisfying $|D \triangle \Omega_0| \leq \delta \leq \delta_0$, there exists a set $H_\delta \subset \Omega_0$ such that

$$|H_\delta| \geq |\Omega_0| - C\delta^{1/4}$$

and

$$\limsup_{t \to \infty} \frac{d_x(t)}{t} \geq c\delta^{1/2}$$

for every $x \in H_\delta$.

(II) For any $\delta > 0$ satisfying $|D \triangle \Omega_0| \leq \delta \leq \delta_0$ and for each $T > 0$, there exists a set $H_{\delta,T} \subset \Omega_0$ such that

$$|H_{\delta,T}| \geq |\Omega_0| - C\delta^{1/4}$$

and

$$\frac{d_x(T)}{T} \geq c\delta^{1/4}$$

for every $x \in H_{\delta,T}$.

In the proof, we use the $L^1$-stability result in [19]. Indeed, when the initial patch is disk-like, we can show that for each fixed time moment, most of particles, which are initially placed on the initial set $\Omega_0$, should be detected in the annulus region $\{\epsilon \leq |x| < 1\}$ thanks to the incompressibility of the fluid. Here $\epsilon > 0$ will be taken small depending on $\delta > 0$. This gives Lemma 2.2. Then we prove that most of particles spend most of their life time in the annulus (e.g. see (15)). Since the speed induced by the exact disk patch is non-trivial within the region, we get the conclusion by using the stability result again.

For the vortex patch problem, there are many other interesting results including persistence of boundary regularity [4], [2] (or see the textbook [5] and references therein, also see [14] for a blow-up result in a modified SQG patch equation), existence of rotating patches [7], [8], [3], [11], which are so-called “V-states”, and stability of circular patches [20], [9], [19] (also see [11] for rectangular patches in a 2D infinite cylinder). Confinement of patch evolution, or support of positive vorticity evolution, is also interesting (e.g. see [16], [13], [18], [6], [12]).
Before proving our theorem, we present the simplest example for travel distance.

1.1. When $\Omega_0 = D$.
If we consider the initial data $\theta_0 = 1_D$, then the radial symmetry of the data implies that corresponding solution is stationary. The velocity induced by the stationary vorticity $\theta_t = 1_D$ is

\[ u(x) = \begin{cases} \frac{x_1}{2|x|^2} & \text{if } x \in D, \\ \frac{x_2}{2|x|^2} & \text{otherwise} \end{cases} \]

(e.g. see [15]). If we decompose into its radial part and tangential part: $u = u_{\text{rad}} \frac{x}{|x|} + u_{\text{tan}} \frac{x}{|x|}$, then we get $u_{\text{rad}} = 0$ and

\[ u_{\text{tan}}(x) = \begin{cases} \frac{|x|}{2} & \text{if } x \in D, \\ \frac{1}{2|x|} & \text{otherwise}. \end{cases} \]

It says that each particle in the disk just rotates in a constant angular velocity. Thus we simply have, for any $x \in D$ and for any $t > 0$,

\[ \frac{d_x(t)}{t} = \frac{|x|}{2} \]

and $d_x(\infty) = \infty$ unless $x = 0$.

2. Proof

2.1. $L^1$-stability of a disk patch.
The paper [19] showed the following $L^1$-stability of a circular vortex patch:

Lemma 2.1 (Theorem 3 in [19]). For any bounded open set $\Omega_0 \subset \mathbb{R}^2$ and for any $r > 0$, we have

\[ \|1_{\Omega_t} - 1_{B_r}\|_{L^1} \leq 4\pi \cdot \sup_{x \in \Omega_0 \Delta B_r} \|x^2 - r^2\| \cdot \|1_{\Omega_0} - 1_{B_r}\|_{L^1} \text{ for any } t > 0. \]

In [19], the authors used conservation of mass, momentum and moment of inertia to prove the above result.

We note $\|1_A - 1_B\|_{L^1} = \int_{A \Delta B} 1 \, dx = |A \Delta B|$ for any bounded open sets $A$ and $B$. In this paper, we will use the result (6) of the above lemma with the choice $r = 1$:

\[ |\Omega_t \triangle D| \leq 4\pi \cdot \sup_{x \in \Omega_0 \Delta D} \|x^2 - 1\| \cdot |\Omega_0 \triangle D| \text{ for any } t > 0 \]

in the following way:

Let $R > 0$. Assume $\Omega_0 \subset B_R$ with $\alpha_0 := |\Omega_0 \triangle D| > 0$. Consider the initial data $\theta_0 = 1_{\Omega_0}$ with its solution $\theta(t) = 1_{\Omega_t}$. Since $\sup_{\Omega_0 \Delta D} \|x^2 - 1\| \leq (R^2 + 1)$, the stability result (7) implies

\[ |\Omega_t \triangle D| \leq 2\sqrt{\pi(R^2 + 1)}\alpha_0. \]

On the other hand, we observe that there exists a constant $C_0 > 0$ such that

\[ \|K * f\|_{L^\infty} \leq C_0 \|f\|_{L^1}^{1/2} \|f\|_{L^\infty}^{1/2} \]

for any $f \in (L^1 \cap L^\infty)(\mathbb{R}^2)$. This can be proved by using the obvious estimate $|K(x)| \leq \frac{C}{|x|}$ of the Biot-Savart kernel (2) (or see Lemma 2.1. in [13]). In our setting, we have

\[ \|K * (1_{\Omega_t} - 1_D)\|_{L^\infty} \leq C_0 \|1_{\Omega_t} - 1_D\|_{L^1}^{1/2} \|1_{\Omega_t} - 1_D\|_{L^\infty}^{1/2} \leq C_1 \alpha_0^{1/4} \]

where

\[ C_1 = C_1(R) := C_0 \sqrt{2\pi(R^2 + 1)} > 0. \]
Since the vector field \((K * \mathbb{1}_D)\) has no radial component by (5) and the velocity \(u(t)\) from \(\theta(t) = \mathbb{1}_{\Omega_t}\) can be decomposed: \(u(t) = K * [\theta(t)] = K * \mathbb{1}_{\Omega_t} = K * \mathbb{1}_D + K * (\mathbb{1}_{\Omega_t} - \mathbb{1}_D)\), we obtain
\[
|u_{rad}(t, x)| \leq C_1 \alpha_0^{1/4}
\]
for any \(x \in \mathbb{R}^2\) and for any \(t > 0\). For the tangential component, we have, for \(x \in D\) and for \(t > 0\),
\[
|u_{tan}(t, x)| \geq \frac{|x|}{2} - C_1 \alpha_0^{1/4}.
\]
In this setting, we can prove the following lemma:

**Lemma 2.2.** Let \(0 < \epsilon < 1\). Suppose \(\Omega_0 \subset B_R\) and \(2\sqrt{\frac{\pi}{2}}(R^2 + 1) \cdot |\Omega_0 \triangle D| \leq \epsilon^2 \pi\). Then for each finite \(T > 0\), we have
\[
\int_{\Omega_0} \left(\frac{1}{T} \int_0^T \mathbb{1}_{B_{\epsilon \cup B_1^c}(\phi_x(t))} dt\right) dx \leq 2\epsilon^2 \pi.
\]

**Proof.** We define
\[
F(t, x) = \mathbb{1}_{B_{\epsilon \cup B_1^c}(\phi_x(t))} = \begin{cases} 
1 & \text{if either } |\phi_x(t)| < \epsilon \text{ or } |\phi_x(t)| \geq 1, \\
0 & \text{otherwise.}
\end{cases}
\]
For each \(t > 0\), since the velocity \(u\) is divergence-free, the flow map \(X(t, x) := \phi_x(t)\) is area preserving so that we have
\[
\int_{\Omega_0} F(t, x) dx = \int_{\Omega_0} \mathbb{1}_{B_{\epsilon \cup B_1^c}(\phi_x(t))} dx = \int_{\Omega_t \cap B_{\epsilon}} 1 dx + \int_{\Omega_t \cap B_1^c} 1 dx \\
\leq \int_{B_{\epsilon}} 1 dx + \int_{\Omega_t \cap D} 1 dx \leq |B_{\epsilon}| + |\Omega_t \triangle D| \\
\leq \epsilon^2 \pi + 2\sqrt{\frac{\pi}{2}}(R^2 + 1) \alpha_0 \leq 2\epsilon^2 \pi
\]
by (8) and by the assumption in this lemma. Let \(T > 0\). Then, by Fubini, we get \(\int_{\Omega_0} \int_0^T F(t, x) dt dx = \int_0^T \int_{\Omega_0} F(t, x) dx dt \leq 2\epsilon^2 \pi T\).

\[\square\]

2.2. **Proof of (I) in Theorem 1.1.** Travel distance in infinite time.

**Proof of (I) in Theorem 1.1.** Let \(R > 0\). First, we take \(\delta_0 := \left(\min\left\{\frac{1}{8C_1}, \frac{1}{2} \sqrt{2\sqrt{\frac{\pi}{2}}(R^2 + 1) \alpha_0}\right\}\right)^4 > 0\) where \(C_1 = C_1(R) > 0\) is defined in (10). Assume \(\Omega_0 \subset B_R\) and \(\alpha_0 := |\Omega_0 \triangle D| < \delta_0\) and take any \(\delta > 0\) satisfying \(\alpha_0 \leq \delta \leq \delta_0\). Define \(\epsilon = \epsilon(\delta) := C_2 \cdot \delta^{1/4}\) where \(C_2 := \max\{4C_1, \sqrt{2\sqrt{\frac{\pi}{2}}(R^2 + 1) \alpha_0}\} > 0\). Clearly, we have \(0 < \epsilon \leq 1/2\),
\[
C_1 \alpha_0^{1/4} \leq C_1 \delta^{1/4} \leq \frac{\epsilon}{4}
\]
and
\[
2\sqrt{\frac{\pi}{2}}(R^2 + 1) \alpha_0 \leq 2\sqrt{\frac{\pi}{2}}(R^2 + 1) \delta \leq \epsilon^2 \pi.
\]
For each \(n \in \mathbb{N}\) and for \(x \in \Omega_0\), we define \(\psi_n(x) := \frac{1}{n} \int_0^n \mathbb{1}_{D \setminus B_{\epsilon}(\phi_x(t))} dt\). We observe \(0 \leq \psi_n \leq 1\).
By Lemma 2.2 due to (13), we have
\[
\int_{\Omega_0} \psi_n dx = \int_{\Omega_0} \left(\frac{1}{n} \int_0^n \mathbb{1}_{D \setminus B_{\epsilon}(\phi_x(t))} dt\right) dx = \int_{\Omega_0} \frac{1}{4} \int_0^n \left(1 - \mathbb{1}_{B_{\epsilon \cup B_1^c}(\phi_x(t))}\right) dt dx \geq |\Omega_0| - 2\epsilon^2 \pi.
\]
We put \( f_n(x) = \sup_{m \geq n} \psi_m(x) \) for each \( n \). Then we notice that \( 0 \leq f_n \leq 1 \) and \( \{f_n\}_{n=1}^{\infty} \) decays pointwise to some function \( f \geq 0 \). By Dominated Convergence Theorem (e.g. see [10]), we have \( \int_{\Omega_0} f \, dx = \lim_{n \to \infty} \int_{\Omega_0} f_n \, dx \). From \( \int_{\Omega_0} f_n \, dx \geq \int_{\Omega_0} \psi_n \, dx \geq |\Omega_0| - 2\varepsilon^2 \pi \), we get
\[
\int_{\Omega_0} f \, dx \geq |\Omega_0| - 2\varepsilon^2 \pi.
\]
By putting \( H := \{x \in \Omega_0 \mid f(x) > \varepsilon\} \), we have
\[
|\Omega_0| - 2\varepsilon^2 \pi \leq \int_{\Omega_0} f \, dx = \int_{\Omega_0 \cap \{f \leq \varepsilon\}} f \, dx + \int_{\Omega_0 \cap \{f > \varepsilon\}} f \, dx \leq \epsilon |\Omega_0| + |H|.
\]
Thus we get
\[
|H| \geq |\Omega_0| - \epsilon(|\Omega_0| + 2\varepsilon \pi) \geq |\Omega_0| - \epsilon(|D| + |D \triangle \Omega_0| + 2\pi) \geq |\Omega_0| - \epsilon(\pi + \delta_0 + 2\pi) \geq |\Omega_0| - C\delta^{1/4}
\]
for some \( C > 0 \) by \( \epsilon \sim \delta^{1/4} \). We observe
\[
f(x) = \limsup_{n \to \infty} \frac{1}{n} \int_0^n 1_{D \setminus B_1}(\phi_x(t)) \, dt.
\]
Thus for each \( x \in H \), there is an increasing sequence \( \{n_k(x)\}_{k=1}^{\infty} \) in \( \mathbb{N} \) such that \( \lim_{k \to \infty} n_k(x) = \infty \) and
\[
\frac{1}{n_k(x)} |\{t \in [0, n_k(x)] \mid |\phi_x(t)| < 1\}| = \frac{1}{n_k(x)} \int_0^{n_k(x)} 1_{D \setminus B_1}(\phi_x(t)) \, dt \geq \epsilon.
\]
On the other hand, whenever \( (t, x) \) satisfies \( \epsilon \leq |\phi_x(t)| < 1 \), we have
\[
|u(t, \phi_x(t))| \geq |u_{tan}(t, \phi_x(t))| \geq \frac{|\phi_x(t)|}{2} - C_1\alpha_0^{1/4} \geq \frac{\epsilon}{2} - C_1\delta^{1/4} \geq \frac{\epsilon}{4}
\]
by (11) and (12). Thus, for any \( x \in H \) and for any \( k \in \mathbb{N} \), we have
\[
d_x(n_k(x)) = \int_0^{n_k(x)} |u(s, \phi_x(s))| \, ds \geq (n_k(x) \cdot \epsilon) \cdot \frac{\epsilon}{4}
\]
which implies \( \limsup_{t \to \infty} \frac{d_x(t)}{t} \geq \frac{\epsilon^2}{4} \) for each \( x \in H \). Since \( \epsilon \sim \delta^{1/4} \), we get (3).

2.3. Proof of (II) in Theorem [1.1] Travel distance in finite time.

Proof of (II) in Theorem [1.1] We take the same \( \delta_0, \alpha_0, \delta, \epsilon \) as in the proof of Theorem [1.1] so that we have \( 0 < \epsilon \leq 1/2 \), (12) and (13).

Let \( T > 0 \). By Lemma [2.2] due to (13), we have
\[
\int_{\Omega_0} \left( \frac{1}{T} \int_0^T 1_{B_1 \cup B_1^C}(\phi_x(t)) \, dt \right) \, dx \leq 2\varepsilon^2 \pi.
\]
By Chebyshev (e.g. see [10]), we get \( \{x \in \Omega_0 \mid G_T(x) \geq \gamma \} \leq 2\pi \).

We put \( H_T := \{x \in \Omega_0 \mid G_T(x) < \epsilon \} \). Then we observe \( |H_T| \geq |\Omega_0| - 2\pi \geq |\Omega_0| - C\delta^{1/4} \) for some \( C > 0 \) by \( \epsilon \sim \delta^{1/4} \). We also observe that for any \( x \in H_T \),
\[
|\{t \in [0, T] \mid |\phi_x(t)| < \epsilon \text{ or } |\phi_x(t)| \geq 1\}| = \int_0^T 1_{B_1 \cap B_1^C}(\phi_x(t)) \, dt = T \cdot G_T(x) < \epsilon T.
\]
Thus for each \( x \in H_T \), we get
\[
|\{t \in [0, T] \mid |\phi_x(t)| < \epsilon \}| \geq (1 - \epsilon)T.
\]
By (11) and (12), if \( \varepsilon \leq |\phi_x(t)| < 1 \), then we get
\[
|u(t, \phi_x(t))| \geq \varepsilon \frac{t}{2}
\]
as in (14). Thus, for any \( x \in H_T \),
we get
\[
d_x(T) \geq (1 - \varepsilon)T \cdot \frac{\varepsilon}{2} \geq T \cdot \frac{\varepsilon}{8}
\]
because of \( \varepsilon \leq 1/2 \). Since \( \varepsilon \sim \delta^{1/4} \), we obtain (14).

\[\Box\]

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