Extended MacMahon-Schwinger’s Master Theorem
and Conformal Wavelets in Complex Minkowski Space

M. Calixto$^{1,2}$ and E. Pérez-Romero$^2$

$^1$ Departamento de Matemática Aplicada, Facultad de Ciencias, Campus de Fuentenueva, 18071 Granada, Spain
$^2$ Instituto de Astrofísica de Andalucía (IAA-CSIC), Apartado Postal 3004, 18080 Granada, Spain

Abstract

We construct the Continuous Wavelet Transform (CWT) on the homogeneous space (Cartan domain) $\mathbb{D}_4 = SO(4,2)/(SO(4) \times SO(2))$ of the conformal group $SO(4,2)$ (locally isomorphic to $SU(2,2)$) in 1+3 dimensions. The manifold $\mathbb{D}_4$ can be mapped one-to-one onto the future tube domain $\mathbb{C}_4^+$ of the complex Minkowski space through a Cayley transformation, where other kind of (electromagnetic) wavelets have already been proposed in the literature. We study the unitary irreducible representations of the conformal group on the Hilbert spaces $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ and $L^2_h(\mathbb{C}_4^+, d\tilde{\nu}_\lambda)$ of square integrable holomorphic functions with scale dimension $\lambda$ and continuous mass spectrum, prove the isomorphism (equivariance) between both Hilbert spaces, admissibility and tight-frame conditions, provide reconstruction formulas and orthonormal basis of homogeneous polynomials and discuss symmetry properties and the Euclidean limit of the proposed conformal wavelets. For that purpose, we firstly state and prove a $\lambda$-extension of Schwinger’s Master Theorem (SMT), which turns out to be a useful mathematical tool for us, particularly as a generating function for the unitary-representation functions of the conformal group and for the derivation of the reproducing (Bergman) kernel of $L^2_h(\mathbb{D}_4, d\nu_\lambda)$. SMT is related to MacMahon’s Master Theorem (MMT) and an extension of both in terms of Louck’s $SU(N)$ solid harmonics is also provided for completeness. Convergence conditions are also studied.

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*Corresponding author: calixto@ugr.es, Manuel.Calixto@upct.es
1 Introduction

Since the pioneer work of Grossmann, Morlet and Paul [1], several extensions of the standard Continuous Wavelet Transform (CWT) on $\mathbb{R}$ (traditionally based on the affine group of time translations and dilations, see e.g. [2, 3]) to general manifolds $X$ have been constructed (see e.g. [4, 5] for general reviews and [6, 7] for recent papers on WT and Gabor systems on homogeneous manifolds). Particular interesting examples are the construction of CWT on spheres $S^{N-1}$, by means of an appropriate unitary representation of the Lorentz group in $N+1$ dimensions $SO(N, 1)$ [8, 9, 10], and on the upper sheet $\mathbb{H}^2_+$ of the two-sheeted hyperboloid $\mathbb{H}^2$ [11], or its stereographical projection onto the open unit disk $D_1 = SO(1, 2)/SO(2) = SU(1, 1)/U(1)$. The basic ingredient in all these constructions is a group of transformations $G$ which contains dilations and motions on $X$, together with a transitive action of $G$ on $X$.

In this article we shall consider the 15-parameter conformal group $G = SO(4, 2)$ in 1+3 dimensions and its natural action on the Minkowski spacetime. The fact that the conformal group contains space-time dilations and translations leads to a natural generalization of the standard CWT for signals on the real line to a higher dimensional manifold. Actually, the conformal group $SO(4, 2)$ consists of Poincaré transformations (space-time translations and Lorentz relativistic rotations and boosts), augmented by dilations and relativistic uniform accelerations, which can also be seen as a sort of local (point-dependent) scale transformations (see later on Section 5).

The conformal group $SO(4, 2)$ (or its four-covering $SU(2, 2)$) has been recognized as a symmetry of Maxwell theory of electromagnetism without sources since [12, 13]. Electromagnetic waves turn out to be written as superpositions of a particular set of conformal wavelets [14, 15, 16]. Thus, conformal wavelets provide a local spacetime-scale analysis of electromagnetic waves in much the same way as standard wavelets provide a time-scale analysis of time signals. In these works, electromagnetic waves are analytically continued or extended from real to complex spacetime and they are obtained from a single mother wavelet by applying conformal transformations of space and time.

Here we shall deal with a different type of conformal wavelets, although we shall work in complex spacetime too. Besides the above massless representations of $SO(4, 2)$ on the electromagnetic field, the conformal group has other representations with continuous mass spectrum labelled by the representations of the stability subgroup $SO(4) \times SO(2)$: the two spins $s_1, s_2 \in \mathbb{N}/2$ and the scale dimension $\lambda \in \mathbb{N}$ of the corresponding field [17].

We shall restrict ourselves to scalar fields ($s_1 = s_2 = 0$) for the sake of simplicity. After a reminder of these representations, we provide admissibility conditions, tight wavelet frames and reconstruction formulas for functions on the complex Cartan domain or Lie ball (see [18] for a general discussion on these classical complex domains)

$$D_4 = SO(4, 2)/(SO(4) \times SO(2)) = SU(2, 2)/SU(2) \times U(2),$$

which is the four-dimensional analogue of the open unit disk $D_1$ abovementioned. This domain can be mapped one-to-one onto the forward/future tube domain $\mathbb{C}_+^4$ (the four-
dimensional analogue of the Poincaré/Lobachevsky/hyperbolic upper half-plane \( \mathbb{C}_+ \) of the complex Minkowski space through a Cayley transformation. For completeness, we also provide an isometric (equivariant) map between the Hilbert spaces of holomorphic functions on \( \mathbb{D}_4 \) and \( \mathbb{C}_4^+ \), where we enjoy more physical intuition.

In order to prove admissibility conditions and reconstruction formulas, an extension of the traditional Schwinger’s Master Theorem (SMT) \([19]\) will show up as a useful mathematical tool for us. Schwinger’s inner product formula turns out to be essentially equivalent to MacMahon’s Master Theorem (MMT) \([20]\), which is one of the fundamental results in combinatorial analysis. A quantum analogue of the MMT has also been constructed \([21]\) and related to a quantum generalization of the boson-fermion correspondence of Physics. Moreover, an extension of the classical MMT \([20]\) was proved in \([22]\) by using the permutation group. The unification of SMT and MMT into a single form by using properties of the so-called \( SU(N) \) solid harmonics \([23, 24, 25]\) (a generalization of Wigner’s \( D \)-matrices for \( SU(2) \), see e.g. \([26]\)), was pointed out by Louck in \([23]\). The combined MacMahon-Schwinger’s Master Theorem provides a generating function for the diagonal elements, the trace, and the representation functions of the so-called totally symmetric unitary representations of the compact unitary group \( U(N) \) \([23, 24, 25]\).

In this article we shall state and prove a \( \lambda \)-extension of the SMT by using the above-mentioned \( SU(N) \) solid harmonics of \([24, 25]\). This \( \lambda \)-extension of the SMT will appear to be useful as a generating function for the unitary-representation functions of the non-compact special pseudo-unitary group \( SU(N,N) \) and for the computation of the reproducing (Bergman) kernel. We shall concentrate on the \( N = 2 \) case, i.e., on the conformal group \( SU(2,2) \) (the general case \( N \geq 2 \) is discussed in the Appendix \([A]\)), which will be essential in the development of conformal wavelets for fields with continuum mass spectrum.

The paper is organized as follows. In Section 2 we remind Schwinger and MacMahon’s Master Theorems and state and prove a \( \lambda \)-extension of Schwinger’s formula. The generalization to matrices \( X \) of size \( N \geq 2 \) is also discussed for completeness in the Appendix \([A]\). In order to be as self-contained as possible, in Section 3 we present the group-theoretical backdrop and leave for the Appendix \([A]\) a succinct exposition of the CWT on a general manifold \( X \), collecting the main definitions used in this paper. In Section 4 we briefly remind the CWT on \( \mathbb{R} \) and extend it to the Lobachevsky plane \( \mathbb{C}_+ \) and the open unit disk \( \mathbb{D}_1 \). The action of the affine group on \( \mathbb{C}_+ \) extends naturally to the conformal group \( SO(1,2) \) of the time axis \( \mathbb{R} \). This will serve us to introduce and establish a parallelism between standard and conformal wavelets in complex Minkowski space in Section 5. We shall eventually work in the Cartan domain \( \mathbb{D}_4 \), although we shall provide in Section 5.3 an (intertwiner) isometry between the Hilbert spaces of holomorphic functions on \( \mathbb{D}_4 \) and the future tube domain \( \mathbb{C}_+^4 \). The \( \lambda \)-extended SMT turns out to be a valuable mathematical tool inside \( \mathbb{D}_4 \) for proving admissibility and tight-frame conditions, reconstruction formulas and reproducing (Bergman) kernels in Section 5.2. We also discuss symmetry properties and comment on the Euclidean limit of the proposed wavelets in Sections 5.4 and 5.5 respectively. Section 6 is devoted to convergence considerations and Section 7 to conclusions and outlook. In Appendix C we prove orthonormality properties of a basis of
homogeneous polynomials introduced in Section 5.

2 Schwinger’s Master Theorem: an Extension

Schwinger’s inner product formula [19] can be stated as follows:

\begin{equation}
\mathcal{D}_{q_1,q_2}^j(X) = \sqrt{(j+q_1)!(j-q_1)!/(j+q_2)!(j-q_2)!} \sum_{k=\max(0,q_1+q_2)}^{\min(j+q_1,j+q_2)} \binom{j+q_2}{k} \binom{j-q_2}{k-q_1-q_2} \times x_{11}^{k-j+q_1-k} x_{12}^{j+q_2-k} x_{21}^{j-q_1-q_2} x_{22}^{k-q_1-q_2},
\end{equation}

the Wigner’s \( \mathcal{D} \)-matrices for \( SU(2) \) [20], where \( j \in \mathbb{N}/2 \) (the spin) runs on all non-negative half-integers and \( q_1, q_2 = -j, -j+1, \ldots, j-1, j \). Then the following identity holds:

\begin{equation}
e^{(\partial_u^i : X : \partial_v)} e^{(u : Y^T : v)} \bigg|_{u=v=0} = \sum_{j \in \mathbb{N}/2} t^{2j} \sum_{q=-j}^j \mathcal{D}_{qq}^j(X) = \det(I - tX)^{-1}
\end{equation}

where we denote by \((u : X : v) \equiv u X v^T = \sum_{i,j=1}^N u_i x_{ij} v_j \) and \( \partial_{u_i} \equiv \partial/\partial u_i \).

This formula turns out to be essentially equivalent to MacMahon’s Master Theorem:

\begin{equation}
\text{Theorem 2.2. (MMT) Let } X \text{ be an } N \times N \text{ matrix of indeterminates } x_{ij}, \text{ and } Y \text{ be the diagonal matrix } Y \equiv \text{diag}(y_1, y_2, \ldots, y_N). \text{ Then the coefficient of } y^\alpha \equiv y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_N^{\alpha_N} \text{ in the expansion of } \det(I - XY)^{-1} \text{ equals the coefficient of } y^\alpha \text{ in the product }
\end{equation}

\begin{equation}
\prod_{i=1}^N (x_{i1} y_1 + x_{i2} y_2 + \cdots + x_{iN} y_N)^{\alpha_i}.
\end{equation}

These abovementioned coefficients can be written in terms of the so-called \( SU(N) \) solid harmonics \( \mathcal{D}_{\alpha \beta}^\rho(X) \) (see [24, 25] and Appendix A for a general definition). \( SU(N) \) solid harmonics [103] are a natural generalization of the standard Wigner’s \( \mathcal{D} \)-matrices [3] to matrices \( X \) of size \( N \geq 2 \). In fact, replacing \( t X \) with \( X Y \) in (4) and using the multiplication property

\begin{equation}
\sum_{q'=-j}^j \mathcal{D}_{qq'}^{ij}(X) \mathcal{D}_{q'q''}^{ij}(Y) = \mathcal{D}_{qq''}^{ij}(XY)
\end{equation}

and the transpositional symmetry

\begin{equation}
\mathcal{D}_{qq'}^{ij}(Y) = \mathcal{D}_{q'q}^{ij}(Y^T),
\end{equation}

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we can restate MMT for $N = 2$ as:

$$\sum_{j \in \mathbb{N}/2} \sum_{q,q'} j D_{qq}(X) D_{j}^{j}(Y^T) = \det(I - XY)^{-1}. \tag{8}$$

Actually, MMT preceded Schwinger’s result by many years. Schwinger re-discovered the MMT in the context of his generating function approach to the angular momentum theory of many-particle systems. The unification into a single form by using properties of the $SU(N)$ solid harmonics was established by Louck in [23, 24, 25].

Wigner matrices $D_{qq}(X)$ are homogeneous polynomials of degree $2j$ in $x_{kl}$. Inspired by Euler’s theorem, we shall define the following differential operator:

$$D_\lambda f(t) \equiv (\lambda + t \frac{\partial}{\partial t}) f(t), \quad \lambda \in \mathbb{N} \tag{9}$$

which will be useful in the sequel. Now we are in condition to state and prove an extension of SMT 2.1. For the sake of completeness, a generalization for matrices $X$ of size $N \geq 2$ is also given in Appendix A.

**Theorem 2.3. (λ-Extended SMT)** For every $\lambda \in \mathbb{N}, \lambda \geq 2$ and every $2 \times 2$ matrix $X$, the following identity holds:

$$\sum_{j \in \mathbb{N}/2} (2j + 1) \lambda \sum_{n=0}^{\infty} t^{2j+2n} \left( n + \lambda - 2 \right) \left( n + 2j + \lambda - 1 \right) \det(X)^n \sum_{q=-j}^{j} D^{j}_{qq}(X) = \det(I - tX)^{-\lambda}. \tag{10}$$

**Proof:** We start from the basic SMT [2,1] and apply the operator $D_1$ on both sides of Eq. (4):

$$\sum_{j \in \mathbb{N}/2} (2j + 1) t^{2j} \sum_{q=-j}^{j} D^{j}_{qq}(X) = \frac{1 - t^2 \det(X)}{\det(I - tX)^2}. \tag{11}$$

Here we have used that

$$\det(I - tX) = 1 - \text{tr}(tX) + \det(tX)$$

and that $\text{tr}(X)$ and $\det(X)$ are homogeneous polynomials of degree 1 and 2, respectively. Making use of the expansion:

$$\frac{1}{1 - t^2 \det(X)} = \sum_{n=0}^{\infty} t^{2n} \det(X)^n, \tag{12}$$

the expression (11) can be recast as:

$$\sum_{j \in \mathbb{N}/2} (2j + 1) \sum_{n=0}^{\infty} t^{2j+2n} \det(X)^n \sum_{q=-j}^{j} D^{j}_{qq}(X) = \frac{1}{\det(I - tX)^2}. \tag{13}$$
This identity is a particular case of (10) for $\lambda = 2$. Now we shall proceed by induction on $\lambda$. Assuming that (10) is valid for every $\lambda \geq 2$ and applying the operator $D_\lambda$ on both sides of the equality (10), we arrive at:

$$\sum_{j \in \mathbb{N}/2} \frac{2j + 1}{\lambda - 1} \sum_{n=0}^{\infty} \frac{1}{\lambda - 2} \left( \frac{1}{\lambda - 2} \right) \left( \frac{1}{\lambda - 2} \right) (\lambda + 2j + 2n) t^{2j+2n}$$

$$\times \det(X)^n \sum_{q=-j}^{j} D_{qq}^{j}(X) = \frac{1 - t^{2} \det(X)}{\det(I - tX)^{\lambda+1}},$$

where we have made use again of (12). Considering (12) one more time, we can assemble the previous expression as:

$$\sum_{j \in \mathbb{N}/2} \frac{2j + 1}{\lambda - 1} \sum_{n,m=0}^{\infty} \left( \frac{1}{\lambda - 2} \right) \left( \frac{1}{\lambda - 2} \right) (\lambda + 2j + 2n) t^{2j+2(n+m)}$$

$$\times \det(X)^{n+m} \sum_{q=-j}^{j} D_{qq}^{j}(X) = \frac{\lambda}{\det(I - tX)^{\lambda+1}}.$$  \hspace{1cm} (14)

Rearranging series:

$$\sum_{n,m=0}^{\infty} a_n b^{n+m} = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} a_m \right) b^n,$$  \hspace{1cm} (15)

the identity (14) can be recast in the form:

$$\sum_{j \in \mathbb{N}/2} \frac{2j + 1}{\lambda - 1} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \frac{1}{\lambda - 2} \right) \left( \frac{1}{\lambda - 2} \right) (\lambda + 2j + 2m) t^{2j+2n}$$

$$\times \det(X)^{n} \sum_{q=-j}^{j} D_{qq}^{j}(X) = \frac{\lambda}{\det(I - tX)^{\lambda+1}}.$$ \hspace{1cm} (16)

It remains to prove the following combinatorial identity:

$$\frac{1}{\lambda - 1} \sum_{m=0}^{n} \left( \frac{1}{\lambda - 2} \right) \left( \frac{1}{\lambda - 2} \right) (\lambda + p + 2m)$$

$$= \left( \frac{n + \lambda - 1}{\lambda - 1} \right) \left( \frac{n + p + \lambda}{\lambda - 1} \right).$$ \hspace{1cm} (17)

We shall proceed by induction on $n$. Let us define both sides of the previous equality as the two sequences:

$$F(n) = \frac{1}{\lambda - 1} \sum_{m=0}^{n} \left( \frac{1}{\lambda - 2} \right) \left( \frac{1}{\lambda - 2} \right) (\lambda + p + 2m),$$

$$G(n) = \left( \frac{n + \lambda - 1}{\lambda - 1} \right) \left( \frac{n + p + \lambda}{\lambda - 1} \right).$$
It is easy to verify that $F(0) = G(0)$. Assuming that $F(n) = G(n)$, we ask whether $F(n+1) = G(n+1)$. Indeed, on the one hand

$$F(n+1) = \frac{1}{\lambda - 1} \sum_{m=0}^{n+1} \binom{m + \lambda - 2}{\lambda - 2} \binom{m + p + \lambda - 1}{\lambda - 2} (\lambda + p + 2m)$$

$$= F(n) + \frac{\lambda + p + 2(n+1)}{\lambda - 1} \left( \binom{n + \lambda - 1}{\lambda - 2} \binom{n + p + \lambda}{\lambda - 2} \right)$$

$$= F(n) + \frac{(\lambda + p + 2(n+1))(\lambda - 1)^2}{(\lambda - 1)(n+1)(n+p+2)} G(n)$$

and on the other hand

$$G(n+1) = \binom{n + \lambda}{\lambda - 1} \binom{n + p + \lambda + 1}{\lambda - 1} = \frac{(n + p + \lambda + 1)(n + \lambda)}{(n+1)(n+p+2)} G(n).$$

Realizing that

$$(n+1)(n+p+2) + (\lambda + p + 2n + 2)(\lambda - 1) = (n + p + \lambda + 1)(n + \lambda)$$

we arrive at $F(n+1) = G(n+1)$, which proves (17). Finally, inserting (17) in (16), we conclude that (10) is valid for $\lambda + 1$, thus completing the proof. 

3 The Group-Theoretical Backdrop

The usual CWT on the real line $\mathbb{R}$ is derived from the natural unitary representation of the affine or similitude group $G = SIM(1)$ in the space of finite energy signals $L^2(\mathbb{R}, dx)$ (see Section 4 for a reminder). The same scheme applies to the CWT on a general manifold $X$, subject to the transitive action, $x \rightarrow gx, g \in G, x \in X$, of some group of transformations $G$ which contains dilations and motions on $X$. We address the reader to the References [4, 5] for a nice and thorough exposition on this subject with multiple examples. For the sake of self-containedness, we also collect in the Appendix some basic definitions which are essential for our construction of conformal wavelets.

As already said in the Introduction, the CWT on spheres $X = \mathbb{S}^{N-1}$ has been constructed in [8, 9, 10] by means of an appropriate unitary representation of the Lorentz group in $N + 1$ (space-time) dimensions $G = SO(N, 1)$. The case of $G = SO(2, 1)$ is particularly interesting as it encompasses wavelets on the circle $\mathbb{S}^1$ and on the real line $\mathbb{R}$, associated to the continuous and discrete series representations, respectively (see [27] for a unified group-theoretical treatment of both type of wavelets inside $SL(2, \mathbb{R}) \simeq SO(1, 2)$). The group $SO(1, 2)$ (the conformal group in $0 + 1$-dimensions) has also been used to construct wavelets on the upper sheet $\mathbb{H}^+_{2}$ of the two-sheeted hyperboloid $\mathbb{H}^2$ [11], or its stereographical projection onto the open unit disk [1].
The (angle-preserving) conformal group in $N$ (space-time) dimensions is finite-dimensional except for $N = 2$. For $N \neq 2$, the conformal group $SO(N, 2)$ consists of Poincaré [space-time translations $b^\mu \in \mathbb{R}^N$ and restricted Lorentz $\Lambda^\mu_\nu \in SO^+(N - 1, 1)$] transformations augmented by dilations $(a \in \mathbb{R}_+)$ and relativistic uniform accelerations (special conformal transformations $c^\mu \in \mathbb{R}^N$) which, in $N$-dimensional Minkowski spacetime, have the following realization:

$$
x^\nu = x^\nu + b^\nu, \quad x'^\nu = \Lambda^\mu_\nu(x^\nu), \quad x^\mu = a x^\mu, \quad x'^\nu = \frac{x^\nu + c^\nu x^2}{1 + 2c c x^2},
$$

respectively. We are using the Minkowski metric $\eta^\mu_\nu = \text{diag}(1, -1, \ldots, -1)$ to rise and lower space-time indices and the Einstein summation convention $cx = c_\mu x^\mu$. The new ingredients with regard to the affine group $SIM(1)$ are the extension from time-translations by $b^0$ to $N$-translations by $b^\mu$, the addition of Lorentz transformations $\Lambda^\mu_\nu$ (rotations and boosts) and accelerations by $c^\mu$. Special conformal transformations can be seen as a sequence of inversions and translations by $c^\mu$ of the form:

$$
x^\mu \underset{\text{inv}}{\rightarrow} x^\mu \underset{\text{inv}}{\rightarrow} \frac{x^\mu + c^\mu x^2}{x^2} \rightarrow \frac{(x^\mu + c^\mu x^2)/x^2}{(x^\mu + x^2 c^\mu)^2/x^4} = \frac{x^\mu + c^\mu x^2}{1 + 2c c x^2}.
$$

They can also be interpreted as point-dependent (generalized gauge) dilations in the sense that, while standard dilations change the spacetime interval $ds^2 = dx^\mu dx_\mu$ globally as $ds^2 \rightarrow a^2 ds^2$, special conformal transformations scale the spacetime interval point-to-point as $ds^2 \rightarrow \sigma(x)^{-2} ds^2$, with $\sigma(x) = 1 + 2cx + c^2 x^2$. The same happens with the squared mass $m^2$, thus forcing a continuous mass spectrum unless $m = 0$, as for the electromagnetic field.

The infinitesimal generators of the transformations $[18]$ are easily deduced:

$$
P_\mu = \frac{\partial}{\partial x^\mu}, \quad M^\mu_\nu = x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu},
$$

$$
D = x^\mu \frac{\partial}{\partial x^\mu}, \quad K_\mu = -2x^\mu x^\nu \frac{\partial}{\partial x^\mu} + x^2 \frac{\partial}{\partial x^\mu},
$$

and they close into the conformal Lie algebra

$$
[M^\mu_\nu, M^\rho_\sigma] = \eta^\mu_\rho M^\nu_\sigma + \eta^\mu_\sigma M^\nu_\rho - \eta^\nu_\rho M^\mu_\sigma - \eta^\nu_\sigma M^\mu_\rho,
$$

$$
[P_\mu, M^\rho_\sigma] = \eta^\rho_\mu P_\sigma - \eta^\rho_\sigma P_\mu, \quad [P_\mu, P_\nu] = 0,
$$

$$
[K^\mu_\nu, M^\rho_\sigma] = \eta^\mu_\rho K_\sigma - \eta^\mu_\sigma K_\rho, \quad [K^\mu_\nu, K^\rho_\sigma] = 0,
$$

$$
[D, P_\mu] = -P_\mu, \quad [D, K^\mu_\nu] = K^\mu_\nu, \quad [D, M^\mu_\nu] = 0,
$$

$$
[K^\mu_\nu, P_\sigma] = 2(\eta^\mu_\sigma D + M^\mu_\nu).
$$

The quadratic Casimir operator:

$$
C_2 = D^2 - \frac{1}{2} M^\mu_\nu M^\mu_\nu + \frac{1}{2}(P_\mu K^\mu_\nu + K^\mu_\nu P^\mu_\nu),
$$

generalizes the Poincaré Casimir $P^2 = P_\mu P^\mu$ (the squared rest mass).

Any group element $g \in SO(4, 2)$ (near the identity element) could be written as the exponential map

$$
g = \exp(u), \quad u = \tau D + b^\mu P_\mu + c^\mu K_\mu + \omega^\mu_\nu M^\mu_\nu,
$$

8
of the Lie-algebra element $u$ (see [28, 29]). The compactified Minkowski space $\mathbb{M} = S^{N-1} \times_{\mathbb{Z}_2} S^1$, can be obtained as the coset $\mathbb{M} = SO(N, 2)/\mathbb{W}$, where $\mathbb{W}$ denotes the Weyl subgroup generated by $K_{\mu}, M_{\mu\nu}$ and $D$ (i.e., a Poincaré subgroup $\mathbb{P} = SO(N-1, 1)\otimes\mathbb{R}^N$ augmented by dilations $\mathbb{R}^+$). The Weyl group $\mathbb{W}$ is the stability subgroup (the little group in physical usage) of $x^\mu = 0$.

For $N = 2$, the group $SO(2, 2)$ is isomorphic to the direct product $SO(1, 2) \times SO(1, 2)$. It is well known that, in two dimensions, the conformal group is infinite dimensional. Actually, the splitting $SO(2, 2) = SO(1, 2) \times SO(1, 2)$ has to do with the separation into holomorphic and anti-holomorphic self-maps of the infinitesimal conformal isometries of a complex domain, the generators of which,

$$L_n = -z^{n+1} \frac{\partial}{\partial z}, \quad \bar{L}_n = -\bar{z}^{n+1} \frac{\partial}{\partial \bar{z}}, \quad z = x^1 + ix^0, \quad \bar{z} = x^1 - ix^0, \quad n \in \mathbb{Z},$$

(24)

close into the Witt algebra $[L_m, L_n] = (m - n)L_{m+n}$ (idem for $\bar{L}$). The conformal group in $N = 2 + 1$ dimensions, $SO(3, 2)$, is also the symmetry group of the anti de Sitter space in $3 + 1$-dimensions, $AdS_4 = SO(3, 2)/SO(3, 1)$, a maximally symmetric Lorentzian manifold with constant negative scalar curvature (i.e., the Lorentzian analogue of 4-dimensional hyperbolic space) which arises, for instance, as a vacuum solution of Einstein’s General Relativity field equations with a negative (attractive) cosmological constant (corresponding to a negative vacuum energy density and positive pressure).

We shall focus on the 15-parameter conformal group in $3 + 1$-dimensions, $SO(4, 2)$, which turns out to be locally isomorphic to the pseudo-unitary group

$$SU(2, 2) = \{ g \in \text{Mat}_{4\times4}(\mathbb{C}) : g^\dagger \Gamma g = \Gamma, \det(g) = 1 \}$$

(25)
of complex special $4 \times 4$ matrices $g$ leaving invariant the $4 \times 4$ hermitian form $\Gamma$ of signature $(++--)$. Here $g^\dagger$ stands for adjoint (or conjugate/hermitian transpose) of $g$ (it is also customary to denote it by $g^\ast$). Actually, the conformal Lie algebra (21) can be also realized in terms of the Lie algebra generators of the fundamental representation of $SU(2, 2)$, given by the following $4 \times 4$ matrices

$$D = \frac{\gamma^5}{2}, \quad M^{\mu\nu} = \frac{[\gamma^\mu, \gamma^\nu]}{4} = \frac{1}{4}\left( \begin{array}{cc} \sigma^\mu \tilde{\sigma}^\nu - \sigma^\nu \tilde{\sigma}^\mu & 0 \\ 0 & \tilde{\sigma}^\mu \sigma^\nu - \tilde{\sigma}^\nu \sigma^\mu \end{array} \right),$$

$$P^\mu = \gamma^\mu \frac{1 + \gamma^5}{2} = \left( \begin{array}{cc} 0 & \sigma^\mu \\ 0 & 0 \end{array} \right), \quad K^\mu = \gamma^\mu \frac{1 - \gamma^5}{2} = \left( \begin{array}{cc} 0 & 0 \\ 0 & \tilde{\sigma}^\mu \end{array} \right)$$

(26)

where

$$\gamma^\mu = \left( \begin{array}{cc} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{array} \right), \quad \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \left( \begin{array}{cc} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{array} \right),$$

denote the Dirac gamma matrices in the Weyl basis and

$$\sigma^0 \equiv I = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad \sigma^1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma^2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma^3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),$$

(27)
are the Pauli matrices (we are writing $\sigma^\mu \equiv \sigma_\mu$). Indeed, using standard properties of gamma and Pauli matrices, one can easily check that the choice (26) fulfils the commutation relations (21).

To be more precise, $SU(2, 2)$ is the four-cover of $SO(4, 2)$, much in the same way as $SU(2)$ is the two-cover of $SO(3)$. This local isomorphism between the conformal group $SO(N, 2)$ and the pseudo-unitary group $SU(M, M)$ only happens for $N = 1$ and $N = 4$ dimensions, where

$$SO(1, 2) = SU(1, 1)/\mathbb{Z}_2, \quad SO(4, 2) = SU(2, 2)/\mathbb{Z}_4.$$  \hspace{1cm} (28)

The $\lambda$-extension of the SMT given in Theorem 2.3 (and Theorem A.2) turns out to be closely related to the group $SU(2, 2)$ (and $SU(N, N)$ in general), providing a kind of generating function for the unitary-representation functions of this group (the discrete series, to be more precise). This formula will be a useful mathematical tool for us, specially in proving admissibility and tight-frame conditions and providing reconstruction formulas. From this point of view, the conformal group $SO(N, 2)$ in $N = 4$ dimensions is singled out from the general $N$-dimensional case, at least in this article.

Before tackling the construction of conformal wavelets in 8-dimensional complex Minkowski space in Section 5, we shall briefly remind the simpler case of the CWT on the time axis $\mathbb{R}$ and its extension to the Lobachevsky plane $\mathbb{C}_+$ and the open unit disk $\mathbb{D}_1$, which are homogeneous spaces of $SO(1, 2)$.

## 4 Wavelets for the Affine Group

Let us consider the affine or similitude group of translations and dilations in one dimension,

$$G = SIM(1) = \mathbb{R} \rtimes \mathbb{R}^+ = \{g = (b, a) / b \in \mathbb{R}, a \in \mathbb{R}^+\},$$

with group law ($g'' = g'g$):

$$a'' = a'a$$
$$b'' = b' + a'b$$

This group will serve us as an introduction for studying the most interesting case of the conformal group $G = SO(4, 2)$ as a “similitude” group of space-time, which will be considered in the next section.

The left-invariant Haar measure is:

$$d\mu(g) = \frac{1}{a^2}da \wedge db$$

The representation

$$[U_\lambda(a, b)\phi](x) = a^{-\lambda}\phi\left(\frac{x - b}{a}\right) \equiv \phi(a, b)(x)$$ \hspace{1cm} (29)

of $G$ on $L^2(\mathbb{R}, dx)$ is unitary for $\lambda = \frac{1}{2} + is$. In fact, every $U_\lambda$ is unitarily equivalent to $U_{1/2}$ and one always works with $\lambda = \frac{1}{2}$. This representation is reducible and splits into
two irreducible components: the positive $\omega > 0$ and negative $\omega < 0$ frequency subspaces. Restricting oneself to the subspace $\omega > 0$, the admissibility condition (123) assumes the form
\[ \int_{0}^{\infty} \frac{\hat{\psi}(\omega)^2}{\omega} d\omega < \infty \]
where $\hat{\psi}$ stands for the Fourier transform of $\psi$. Given an admissible function $\psi \in L^2(\mathbb{R}, dx)$, the machinery of wavelet analysis proceeds in the usual way.

4.1 Wavelets on the Lobachevsky plane $\mathbb{C}_+$
An extension of the representation (29) of the affine group, this time on the space $L^2_h(\mathbb{C}_+, d\tilde{\nu}_\lambda)$ of square integrable holomorphic functions on the upper half complex plane (or forward tube domain)
\[ T_1 \equiv \mathbb{C}_+ \equiv \{ w = x + iy \in \mathbb{C} / \Im(w) = y > 0 \}, \quad (30) \]
is given by:
\[ \tilde{U}_\lambda(a, b)\phi(w) = a^{-\lambda} \phi(\frac{w-b}{a}). \quad (31) \]
This representation is unitary with respect to the scalar product:
\[ \langle \phi|\phi' \rangle = \int_{\mathbb{C}_+} \overline{\phi(w)\phi'(w)} d\tilde{\nu}_\lambda(w, \bar{w}), \quad d\tilde{\nu}_\lambda(w, \bar{w}) = \frac{2\lambda - 1}{4\pi} \Im(w)^{2(\lambda-1)}|dw|, \quad (32) \]
for any $\phi, \phi' \in L^2_h(\mathbb{C}_+, d\tilde{\nu}_\lambda)$, where we use $|dw|$ as a shorthand for the Lebesgue measure $d\Re(w) \wedge d\Im(w)$. Although all representations $\tilde{U}_\lambda, \lambda \geq 1$, are equivalent, they become inequivalent when the affine group is immersed inside the conformal group of the time axis $\mathbb{R}$, $SO(1, 2) \simeq SL(2, \mathbb{R}) \approx SU(1, 1)$. Actually, this will be the case with the conformal group $SO(4, 2)$ in the next section. This immersion of $SIM(1)$ inside $SL(2, \mathbb{R})$ is apparent for the Iwasawa decomposition KAN (see, for instance, [30]) when parameterizing and element $g \in SL(2, \mathbb{R})$ as:
\[ g = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1/\sqrt{a} & 0 \\ 0 & \sqrt{a} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos \theta & \frac{b \cos \theta - \sqrt{a} \sin \theta}{\sqrt{a}} \\ \frac{\sin \theta}{\sqrt{a}} & \sqrt{a} \cos \theta + \frac{b \sin \theta}{\sqrt{a}} \end{pmatrix} \]
where $\theta \in (-\pi, \pi]$ (see [27] for a unified group-theoretical treatment of wavelets on $\mathbb{R}$ and the circle $S^1$ inside $SL(2, \mathbb{R})$).

4.2 Wavelets on the open unit disk $\mathbb{D}_1$
There is a one-to-one mapping between the Lobachevsky plane $\mathbb{C}_+$ and the open unit disk (or Cartan domain)
\[ \mathbb{D}_1 = \{ z \in \mathbb{C}, |z| < 1 \}, \quad (33) \]
given through the Cayley transformation:
\[
z(w) = \frac{1 + iw}{1 - iw} \leftrightarrow w(z) = \frac{1 - z}{1 + z}.
\] (34)

Note that the (Shilov) boundary \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \) of \( D_1 \) is stereographically projected onto the boundary \( \mathbb{R} = \{ w \in \mathbb{C} : \Re(w) = 0 \} \) of \( \mathbb{C}_+ \) by \( w(e^{i\theta}) = \tan(\theta/2) \).

We can establish an isometry between \( L^2_h(\mathbb{C}_+, d\tilde{\nu}_\lambda) \) and the space \( L^2_h(D_1, d\nu_\lambda) \) of square integrable holomorphic functions on the unit disk \( D_1 \) with integration measure
\[
d\nu_\lambda(z, \bar{z}) = \frac{2\lambda - 1}{\pi} |1 - z\bar{z}|^{2(\lambda-1)}|dz|, \quad \lambda \geq 1,
\] (35)

where \( \bar{z} \) denotes complex conjugate. This isometry is given by the correspondence
\[
S_\lambda : L^2_h(D_1, d\nu_\lambda) \rightarrow L^2_h(\mathbb{C}_+, d\tilde{\nu}_\lambda)
\]
\[
\phi \mapsto S_\lambda \phi \equiv \tilde{\phi},
\]
with
\[
\tilde{\phi}(w) = 2^{2\lambda}(1 - iw)^{-2\lambda}\phi(z(w))
\] (36)
and \( z(w) \) given by (34). In fact, taking into account that \( |1 - z\bar{z}| = 2^2\Re(w)|1 - iw|^{-2} \) and the Jacobian determinant \( |dz|/|dw| = 2^2|1 - iw|^{-4} \), then
\[
\langle \phi | \phi' \rangle_{L^2_h(D_1)} = \frac{2\lambda - 1}{\pi} \int_{\mathbb{D}_1} \overline{\phi(z)} \phi'(z)(1 - z\bar{z})^{2(\lambda-1)}|dz|
\]
\[
= \frac{2\lambda - 1}{4\pi} \int_{\mathbb{C}_+} 2^{2\lambda}(1 - iw)^{-2\lambda}\phi(z(w))2^{2\lambda}(1 - iw)^{-2\lambda}\phi'(z(w))\Re(w)^{2(\lambda-1)}|dw|
\]
\[
= \int_{\mathbb{C}_+} \overline{\phi(w)}\tilde{\phi}(w)d\tilde{\nu}_\lambda(w, \bar{w}) = \langle \tilde{\phi} | \tilde{\phi}' \rangle_{L^2_h(\mathbb{C}_+)}.
\]

The constant factor \( (2\lambda - 1)/\pi \) of \( d\nu_\lambda(z, \bar{z}) \) is chosen so that the set of functions
\[
\varphi_n(z) \equiv \left( \frac{2\lambda + n - 1}{n} \right)^{1/2} z^n, \quad n = 0, 1, 2, \ldots,
\] (37)

constitutes an orthonormal basis of \( L^2_h(D_1, d\nu_\lambda) \), as can be easily checked by direct computation. These basis functions verify the following closure relation:
\[
\sum_{n=0}^{\infty} \overline{\varphi_n(z)}\varphi_n(z') = (1 - \bar{z}z')^{-2\lambda},
\] (38)
which is nothing other than the reproducing (Bergman) kernel of \( L^2_h(D_1, d\nu_\lambda) \) (see e.g. [4] for a general discussion on reproducing kernels). We shall provide a four-dimensional analogue of (37) and (38) in Eqs. (65) and (67), respectively, and prove the orthonormality
in Appendix C. The isometry $S_\lambda$ given by (36) maps the orthonormal basis (37) of $L^2_h(D_1, d\nu_\lambda)$ onto the orthonormal basis
\[ \tilde{\varphi}_n(w) = 2^{2\lambda}(1 - iw)^{-2\lambda}\varphi_n(z(w)), \quad n = 0, 1, 2, \ldots \] of $L^2_h(\mathbb{C}_+, d\tilde{\nu})$, which verify the reproducing kernel relation
\[ \sum_{n=0}^{\infty} \overline{\tilde{\varphi}_n(w)}\tilde{\varphi}_n(w') = \left(\frac{i}{2}(\tilde{w} - w')\right)^{-2\lambda}. \] (40)

Let us denote by $V_\lambda \equiv S_\lambda^{-1}\tilde{U}_\lambda S_\lambda$ the representation of the affine group on $L^2_h(D_1, d\nu_\lambda)$ induced from (31) through the isometry (36). More explicitly:
\[ [V_\lambda(a, b)\phi](z) = [S_\lambda^{-1}\tilde{U}_\lambda(a, b)\tilde{\phi}](z) = \alpha^{-\lambda}\left(1 - \frac{i w(z) - b}{a}\right)^{-2\lambda}\left(1 - i w(z)\right)\phi\left(\frac{w(z) - b}{a}\right). \]
This representation is, by construction, unitary on $L^2_h(D_1, d\nu_\lambda)$.

5 Wavelets for the Conformal Group $SO(4, 2)$

The four-dimensional analogue of the extension of the time axis $\mathbb{R}$ to the time-energy half-plane $\mathbb{C}_+$ is the extension of the Minkowski space $\mathbb{R}^4$ to the (eight-dimensional) future tube domain $\mathbb{C}_+^4$ of the complex Minkowski space $\mathbb{C}^4$ (see later on this Section).

The four-dimensional analogue of the one-to-one mapping between the half-plane $\mathbb{C}_+$ and the disk $D_1$ is now the Cayley transform (47) between $\mathbb{C}_+^4$ and the Cartan domain $D_4 = \text{SU}(2, 2)/\text{SU}(2)\times\text{SU}(2)$, the Shilov boundary of which is the compactified Minkowski space $U(2)$ (the four-dimensional analogue of the boundary $U(1) = \mathbb{S}^1$ of the disk $\mathbb{D}_1$). Let us see all this mappings and constructions in more detail.

5.1 Wavelets on the forward tube domain $\mathbb{C}_+^4$

The four-dimensional analogue of the upper-half complex plane (30) is the future/forward tube domain
\[ T_4 = \mathbb{C}_+^4 \equiv \{W = X + iY = w_\mu\sigma^\mu \in \text{Mat}_{2 \times 2}(\mathbb{C}) : Y > 0\} \] of the complex Minkowski space $\mathbb{C}^4$, with $X = x_\mu\sigma^\mu$ and $Y = y_\mu\sigma^\mu$ hermitian matrices fulfilling the positivity condition $Y > 0 \iff y^0 = 3(w^0) > ||\tilde{y}||$.

The domain $\mathbb{C}_+^4$ is naturally homeomorphic to the quotient $SU(2, 2)/SU(2)\times SU(2)$ in the realization of the conformal group in terms of $4 \times 4$ complex (block) matrices $f$ fulfilling
\[ G = SU(2, 2) = \left\{ f = \begin{pmatrix} R & iS \\ -iT & Q \end{pmatrix} \in \text{Mat}_{4 \times 4}(\mathbb{C}) : f^\dagger \Gamma f = \Gamma, \det(f) = 1 \right\}. \] (42)
with
\[ \Gamma = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \]
the time component of the Dirac \(4 \times 4\) matrices \(\gamma^\mu\) in the Weyl basis \((I = \sigma^0\) is the \(2 \times 2\) identity matrix and \(f^\dagger = f^*\) stands again for adjoint or conjugate/hermitian transpose of \(f\)). In general, \(\Gamma\) is a \(4 \times 4\) hermitian form of signature \((++--)\). The inverse element of \(f\) is then given by:
\[ f^{-1} = \gamma^0 f^\dagger \gamma^0 = \begin{pmatrix} Q^\dagger & -iS^\dagger \\ iT^\dagger & R^\dagger \end{pmatrix}. \]
The particular identification of \(\mathbb{C}_+^4\) with the coset \(SU(2, 2)/SU(2)^2\) is given through:
\[ W = W(f) = (S + iR)(Q + iT)^{-1} = (Q + iT)^{-1}(S + iR). \quad (43) \]
The left translation \(f' \to ff'\) of \(G\) on itself induces a natural left-action of \(G\) on \(\mathbb{C}_+^4\) given by:
\[ W = W(f') \to W' = W(ff') = (RW + S)(TW + Q)^{-1}. \quad (44) \]
Let us make use of the standard identification \(x^\mu \leftrightarrow X = x^\mu \sigma^\mu\) between the Minkowski space \(\mathbb{R}^4\) and the space of \(2 \times 2\) hermitian matrices \(X\), with \(\sigma^\mu\) the Pauli matrices \((27)\), and \(x^2 = x_\mu x^\mu = \text{det}(X)\) the Minkowski squared-norm. Setting \(W = x^\mu \sigma^\mu\), the transformations \((18)\) can be formally recovered from \((44)\) as follows:

i) Standard Lorentz transformations, \(x'^\mu = \Lambda_\mu^\nu (\omega)x^\nu\), correspond to \(T = S = 0\) and \(R = Q^{-1} = a^{1/2}I\), where we are making use of the homomorphism (spinor map) between \(SO^+(3, 1)\) and \(SL(2, \mathbb{C})\) and writing \(W' = RW'R^\dagger\), \(R \in SL(2, \mathbb{C})\) instead of \(x'^\mu = \Lambda_\mu^\nu x^\nu\).

ii) Dilations correspond to \(T = S = 0\) and \(R = Q^{-1} = a^{1/2}I\).

iii) Spacetime translations correspond to \(R = Q = I\) and \(S = b_\mu \sigma^\mu, T = 0\).

iv) Special conformal transformations correspond to \(R = Q = I\) and \(T = c_\mu \sigma^\mu, S = 0\) by noting that \(\text{det}(I + TW) = 1 + 2cx + c^2x^2\).

We shall give the next proposition without proof. Instead, we address the reader to its counterpart (Proposition 5.2) in the next Subsection for an equivalent proof.

**Proposition 5.1.** The representation of \(G\) on square-integrable holomorphic functions \(\varphi(W)\) given by
\[ [\tilde{U}_\lambda(f)\varphi](W) = \text{det}(R^\dagger - T^\dagger W)^{-\lambda}\varphi(((Q^\dagger W - S^\dagger)(R^\dagger - T^\dagger W)^{-1}\] is unitary with respect to the integration measure
\[ d\tilde{\nu}_\lambda(W, W^\dagger) \equiv \frac{c^*_\lambda}{2^4} \text{det}(\frac{i}{2}(W^\dagger - W))^{\lambda-4}|dW| = \frac{c^*_\lambda}{2^4} \Im(w)^{2(\lambda-4)}|dW|, \quad (45)\]
where \( \lambda \in \mathbb{N}, \lambda > 3 \) (the “scale or conformal dimension”) is a parameter labelling non-equivalent representations, \( c_\lambda \equiv (\lambda - 1)(\lambda - 2)^2(\lambda - 3)/\pi^4 \) and we are using \(|dW| = \bigwedge_{\mu=0}^{\lambda-3} d\Re(w_\mu)d\Im(w_\mu)\) as a shorthand for the Lebesgue measure on \( \mathbb{C}^4_+ \).

We identify the factor \( \mathcal{M}(f,W)^{1/2} = \det(R^\dagger - T^\dagger W)^{-\lambda} \) in (45) as a multiplier or Radon-Nikodym derivative, (remember the general definition in (12)). It generalizes the factor \( a^{-\lambda} \) in (31) by extending (global) standard dilations \( R = Q^{-1} = a^{1/2}I, T = S = 0 \) to (local/point-dependent) “generalized dilations” with \( T = c_\mu \sigma^\mu \). The representation (45) is a special (spin-less or scalar) case of the discrete series representations of \( SU(2, 2) \), which are characterized by \( \lambda \) and two spin labels \( s_1 \) and \( s_2 \). Decomposing the discrete series representations of \( SU(2, 2) \) into irreducible representations of the inhomogeneous Lorentz group leads to a continuous (Poincaré) mass spectrum [17].

5.2 Wavelets on the Cartan domain \( \mathbb{D}_4 \)

Instead of working in the forward tube domain \( \mathbb{C}^4_+ \), we shall choose for convenience a different eight-dimensional space \( \mathbb{D}_4 \) generalizing the (two-dimensional) open unit disk \( \mathbb{D}_1 \) in (33), where we shall take advantage of the full power of the \( \lambda \)-extension of the SMT given by the formula (10). Both spaces, \( \mathbb{C}^4_+ \) and \( \mathbb{D}_4 \), are related by a Cayley-type transformation, which induces an isomorphism between the corresponding Hilbert spaces of square-integrable holomorphic functions on both manifolds (see later on Section 5.3).

5.2.1 Cayley transform and \( \mathbb{D}_4 \) as a coset of \( SU(2, 2) \)

The four-dimensional analogue of the map (34) form the Lobachevsky plane \( \mathbb{C}_+ \) onto the unit disk \( \mathbb{D}_1 \) is now the Cayley transformation (and its inverse):

\[
\begin{align*}
W & \rightarrow Z(W) = (I - iW)^{-1}(I + iW) = (I + iW)(I - iW)^{-1}, \\
Z & \rightarrow W(Z) = i(I - Z)(I + Z)^{-1} = i(I + Z)^{-1}(I - Z),
\end{align*}
\]

that maps (one-to-one) the forward tube domain \( \mathbb{C}^4_+ \) onto the Cartan complex domain defined by the positive-definite matrix condition:

\[
\mathbb{D}_4 = \{ Z \in \text{Mat}_{2 \times 2}(\mathbb{C}) : I - ZZ^\dagger > 0 \}.
\]

Note that the (Shilov) boundary

\[
\mathbb{D}_4 = U(2) = \{ Z \in \text{Mat}_{2 \times 2}(\mathbb{C}) : ZZ^\dagger = Z^\dagger Z = I \} = S^3 \times \mathbb{Z}_2 \times S^1
\]

of \( \mathbb{D}_4 \) is a compactification of the real Minkowski space

\[
\mathbb{M}_4 = \{ W \in \text{Mat}_{2 \times 2}(\mathbb{C}) : W^\dagger = W \},
\]

i.e., the boundary of \( \mathbb{C}^4_+ \) (see e.g. [18]). The restriction of the Cayley map \( Z \rightarrow W(Z) \) to \( Z \in U(2) \) is precisely the stereographic projection of \( U(2) \) onto \( \mathbb{M}_4 \).
The Cartan domain $D_4$ is naturally homeomorphic to the quotient $SU(2, 2)/S(U(2)^2)$ in the new realization of:

$$G = SU(2, 2) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Mat}_{4 \times 4}(\mathbb{C}) : g^\dagger \gamma^5 g = \gamma^5, \det(g) = 1 \right\},$$  \hspace{1cm} (49)

with

$$\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

the fifth Dirac $4 \times 4$ gamma matrix in the Weyl basis ($I = \sigma^0$ denotes again the $2 \times 2$ identity matrix). The inverse element of $g$ is now:

$$g^{-1} = \gamma^5 g^\dagger \gamma^5 = \begin{pmatrix} A^\dagger & -C^\dagger \\ -B^\dagger & D^\dagger \end{pmatrix}. $$

Both realizations (42) and (49) are related by the map

$$f \rightarrow g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \Upsilon^{-1} f \Upsilon = \frac{1}{2} \begin{pmatrix} R + iS - iT + Q & -R + iS + iT + Q \\ -R - iS - iT + Q & R - iS + iT + Q \end{pmatrix},$$  \hspace{1cm} (50)

with

$$\Upsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}.$$  

The particular identification of $D_4$ with the coset $SU(2, 2)/S(U(2)^2)$ is given by [see later on Eq. (59) for more details]:

$$Z(g) = BD^{-1}, \quad Z^\dagger(g) = CA^{-1}.$$  \hspace{1cm} (51)

Actually, making explicit the matrix restrictions $g^\dagger \gamma^5 g = \gamma^5$ in (49):

$$g^{-1}g = I_{4 \times 4} \Leftrightarrow \begin{cases} D^\dagger D - B^\dagger B = I \\ A^\dagger A - C^\dagger C = I \\ A^\dagger B - C^\dagger D = 0 \end{cases}$$  \hspace{1cm} (52)

and

$$gg^{-1} = I_{4 \times 4} \Leftrightarrow \begin{cases} DD^\dagger - CC^\dagger = I \\ AA^\dagger - BB^\dagger = I \\ AC^\dagger - BD^\dagger = 0 \end{cases}$$  \hspace{1cm} (53)

the positive matrix condition in (48) now reads

$$I - ZZ^\dagger = I - A^{-1} A C^\dagger C A^{-1} = (AA^\dagger)^{-1} > 0,$$

(54)

where we have used the second condition in (52). Moreover, using the identification (51) and the first condition in (52), we can see that

$$\det(ZZ^\dagger) = \det(B^\dagger B) \det(I + B^\dagger B)^{-1} < 1,$$

\hspace{1cm} (55)
This determinant restriction can also be proved as a direct consequence of the positive-definite matrix condition $I - ZZ^\dagger > 0$. In fact, the characteristic polynomial

$$
\det((1 - \rho)I - ZZ^\dagger) = 1 - \text{tr}(\rho I + ZZ^\dagger) + \det(\rho I + ZZ^\dagger) = \rho^2 - (2 - \text{tr}(ZZ^\dagger))\rho + \det(I - ZZ^\dagger)
$$

yields the eigenvalues

$$
\rho_\pm = \frac{2 - \text{tr}(ZZ^\dagger) \pm \sqrt{\Delta}}{2}, \quad \Delta \equiv (2 - \text{tr}(ZZ^\dagger))^2 - 4\det(I - ZZ^\dagger).
$$

Since $I - ZZ^\dagger$ is hermitian and positive definite, its eigenvalues $\rho_\pm$ are real and positive. The condition $\rho_- > 0$ implies that:

$$
2 - \text{tr}(ZZ^\dagger) > 0 \Rightarrow \text{tr}(ZZ^\dagger) < 2,
$$

and the fact that $\Delta \geq 0$ gives:

$$
0 \leq \Delta = \text{tr}(ZZ^\dagger)^2 - 4\det(ZZ^\dagger) \Rightarrow \det(ZZ^\dagger) \leq \frac{1}{4}\text{tr}(ZZ^\dagger)^2 < 1,
$$

where we have used (57) in the last inequality. From (57), we can regard $\mathbb{D}_4$ as an open subset of the eight-dimensional ball with radius $\sqrt{2}$. All those bounds for $Z \in \mathbb{D}_4$ will be useful for proving convergence conditions later on Section 6. See also the Appendix C for a suitable parametrization of $Z$ when computing scalar products.

### 5.2.2 Haar measure, unitary representation and reproducing kernel

Any element $g \in G$ admits a Iwasawa decomposition of the form

$$
g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \Delta_1 & Z\Delta_2 \\ Z^\dagger\Delta_1 & \Delta_2 \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix},
$$

with

$$
\begin{align*}
\Delta_1 &= (AA^\dagger)^{1/2} = (I - ZZ^\dagger)^{-1/2}, \quad U_1 = \Delta_1^{-1}A, \\
\Delta_2 &= (DD^\dagger)^{1/2} = (I - Z^\dagger Z)^{-1/2}, \quad U_2 = \Delta_2^{-1}D.
\end{align*}
$$

This decomposition is adapted to the quotient $\mathbb{D}_4 = G/H$ of $G = SU(2,2)$ by the maximal compact subgroup $H = S(U(2)^2)$; that is, $U_1, U_2 \in U(2)$ with $\det(U_1U_2) = 1$. In order to release $U_{1,2}$ from the last determinant condition, we shall work from now on with $G = U(2,2)$ and $H = U(2)^2$ instead. Likewise, a parametrization of any $U \in U(2)$, adapted to the quotient $S^2 = U(2)/U(1)^2$, is (the Hopf fibration)

$$
U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \delta & -z\delta \\ \bar{z}\delta & \delta \end{pmatrix} \begin{pmatrix} e^{i\beta_1} & 0 \\ 0 & e^{i\beta_2} \end{pmatrix},
$$

17
where \( z = b/d \in \overline{C} \simeq S^2 \) (the one-point compactification of \( C \) by inverse stereographic projection), \( \delta = (1 + zz)^{-1/2} \) and \( e^{i\beta_1} = a/|a|, e^{i\beta_2} = d/|d| \).

The left-invariant Haar measure [the exterior product of left-invariant one-forms \( g^{-1}dg \)] of \( G \) proves to be:

\[
d\mu(g) = d\mu(g)|_{G/H} d\mu(g)|_H, \\
d\mu(g)|_{G/H} = \det(I - ZZ^\dagger)^{-4}|dZ|, \\
d\mu(g)|_H = dv(U_1)dv(U_2),
\]

where we are denoting by \( dv(U) \) the Haar measure on \( U(2) \), which [using (60)] can be in turn decomposed as:

\[
dv(U) \equiv dv(U)|_{U(2)/U(1)^2} dv(U)|_{U(1)^2}, \\
dv(U)|_{U(2)/U(1)^2} = dv(U)|_{S^2} \equiv ds(U) = (1 + z\bar{z})^{-2}|dz|,
\]

where \(|dz|\) and \(|dZ|\) denote the Lebesgue measures in \( C \) and \( C^4 \), respectively.

Let us consider the space of holomorphic functions \( \phi(Z) \) on \( \mathbb{D}_4 \).

**Proposition 5.2.** For any group element \( g \in G \), the following (left-)action

\[
\phi_g(Z) \equiv [U_g(g)](Z) = \det(D^\dagger - B^\dagger Z)^{-\lambda}\phi(Z'), \\
Z' = g^{-1}Z = (A^\dagger Z - C^\dagger)(D^\dagger - B^\dagger Z)^{-1}
\]

defines a unitary irreducible square integrable representation of \( G \) on \( L^2_h(\mathbb{D}_4, dv_\lambda) \) under the invariant scalar product

\[
\langle \phi|\phi' \rangle = \int_{\mathbb{D}_4} \overline{\phi(Z)}\phi'(Z)dv_\lambda(Z, Z^\dagger), \\
dv_\lambda(Z, Z^\dagger) \equiv c_\lambda \det(I - ZZ^\dagger)^{-4}|dZ|,
\]

for any \( \lambda \in \mathbb{N}, \lambda \geq 4 \) (the “scale or conformal dimension”), where \( c_\lambda = \pi^{-4}(\lambda - 1)(\lambda - 2)^2(\lambda - 3) \) is chosen so that the unit function, \( \phi(Z) = 1, \forall Z \in \mathbb{D}_4 \), is normalized, i.e. \( \langle \phi|\phi \rangle = 1 \).

**Proof:** One can easily check by elementary algebra that \( U_\lambda(g)U_\lambda(g') = U_\lambda(gg') \). In order to prove unitarity, i.e. \( \langle \phi_g|\phi_g \rangle = \langle \phi|\phi \rangle \) for every \( g \in G \), we shall make use of (52) and (53). In fact:

\[
\det(I - Z'Z^\dagger) = |\det(D^\dagger - B^\dagger Z)|^{-2}\det(I - ZZ^\dagger),
\]

and the Jacobian determinant

\[
|dZ| = |dZ'||\det(D^\dagger - B^\dagger Z)|^8,
\]

give the isometry relation \( \|\phi_g\|^2 = \|\phi\|^2 \). Now taking \( g' = g^{-1} \) implies the unitarity of \( U_\lambda \).

For the computation of \( c_\lambda \) and other orthonormality properties see Appendix C.
In the next Section, we shall provide an isomorphism between $L_h^2(D_4, d\nu)$ and $L_h^2(C_4^+, d\bar{\nu})$, where we enjoy more physical intuition.

In order to prove admissibility conditions in Section 5.2.3 it will be convenient to give an orthonormal basis of $L_h^2(D_4, d\nu_\lambda)$.

**Proposition 5.3.** The set of homogeneous polynomials of degree $2j + 2m$:

$$
\varphi_{q_1,q_2}^{j,m}(Z) \equiv \sqrt{\frac{2j+1}{\lambda-1}} \left(\frac{m+\lambda-2}{\lambda-2}\right) \left(\frac{m+2j+\lambda-1}{\lambda-2}\right) \det(Z)^m D_{q_1,q_2}^j(Z),
$$

$m \in \mathbb{N}, j \in \mathbb{N}/2, q_1, q_2 = -j, -j + 1, \ldots, j - 1, j,$

constitutes an orthonormal basis of $L_h^2(D_4, d\nu_\lambda)$, that is:

$$
\langle \varphi_{q_1,q_2}^{j,m} | \varphi_{q_1',q_2'}^{j',m'} \rangle = \delta_{j,j'} \delta_{m,m'} \delta_{q_1,q_1'} \delta_{q_2,q_2'}. \tag{66}
$$

Note that the number of linearly independent polynomials $\prod_{i,j=1}^2 z_{ij}^{n_{ij}}$ of fixed degree of homogeneity $n = \sum_{i,j=1}^2 n_{ij}$ is $(n+1)(n+2)(n+3)/6$, which coincides with the number of linearly independent polynomials (65) with degree of homogeneity $n = 2m + 2j$. This proves that the set of polynomials (65) is a basis for analytic functions $\phi \in L_h^2(D_4, d\nu_\lambda)$. Moreover, this basis turns out to be orthonormal. We address the interested reader to the Appendix for a proof. We prefer to omit it here in order to make the presentation more dynamic.

Note also the close resemblance between the definition (65) and the left-hand side of the equality (10) in the $\lambda$-extended SMT 2.3. In fact, taking $tX = Z^\dagger Z'$ in (10) and using the properties (6) and (7) of Wigner’s $D$-matrices, we can prove the following closure relation for the basis functions (65):

$$
\sum_{j \in \mathbb{N}/2} \sum_{m=0}^\infty \sum_{q,q'=-j}^j \varphi_{q,q}^{j,m}(Z) \varphi_{q',q}^{j,m}(Z') = \det(I - Z^\dagger Z')^{-\lambda}, \tag{67}
$$

which is nothing other than the reproducing (Bergman) kernel in $L_h^2(D_4, d\nu_\lambda)$ (see e.g. [4] for a general discussion on reproducing kernels). Note that, although the scalar product (64) is only valid for $\lambda \geq 4$, the expression (67) is formally valid for $\lambda \geq 2$, since we are just using in it the requirements of the $\lambda$-extended SMT 2.3. The case $\lambda = 2$ is related to the Szegö kernel (see e.g. [13]).

### 5.2.3 Admissibility condition, tight frame and reconstruction formula

**Theorem 5.4.** The representation (63) is square integrable, the constant unit function $\psi(Z) = \varphi_{0,0}^0(Z) = 1, \forall Z \in D_4$ being an admissible vector (fiducial state or mother wavelet), i.e.:

$$
c_{\psi} = \int_G |\langle U_\lambda(g)\psi | \psi \rangle|^2 d\mu(g) < \infty \tag{68}
$$
and the set of coherent states (or wavelets) \( F = \{ \psi_g = U_\lambda(g) \psi, g \in G \} \) constituting a continuous tight frame in \( L^2_h(D_4, d\nu_\lambda) \) satisfying the resolution of the identity:

\[
\mathcal{A} = \int_G |\psi_g\rangle\langle\psi_g| d\mu(g) = c_\psi I.
\]  

(69)

**Proof:** Using the extended SMT 2.3 for \( tX = D^{-1} CZ \), we can expand

\[
\psi_g(Z) = \det(D^\dagger - B^\dagger Z)^{-\lambda} = \det(D^\dagger)^{-\lambda} \det(I - (BD^{-1})^\dagger Z)^{-\lambda} = \det(D^\dagger)^{-\lambda} \sum_{j=0}^\infty \frac{2j + 1}{\lambda - 1} \sum_{n=0}^\infty \left( \frac{n + \lambda - 2}{\lambda - 2} \right) \left( \frac{n + 2j + \lambda - 1}{\lambda - 2} \right) \times \det((BD^{-1})^\dagger Z)^n \sum_{q=-j}^j D_{qq}^j((BD^{-1})^\dagger Z).
\]  

(70)

Now, taking into account that \( \det((BD^{-1})^\dagger Z)^n = \det((BD^{-1})^\dagger)^n \det(Z)^n \) and the property (6) for \( D_{jqq}^n((BD^{-1})) = \sum_{q'=j}^j D_{jqq'}^n((BD^{-1})^\dagger) \)

we recognize the orthonormal basis functions (65) in the expansion (70), so that we can write the coherent states (wavelets) as:

\[
\psi_g(Z) = \sum_{j \in \mathbb{N}/2} \sum_{n=0}^\infty \sum_{q=-j}^j \hat{\psi}_{q',q}^{j,n}(g) \varphi_{q',q}^{j,n}(Z)
\]  

(72)

with “Fourier” coefficients

\[
\hat{\psi}_{q',q}^{j,n}(g) = \det(D^\dagger)^{-\lambda} \sqrt{\frac{2j + 1}{\lambda - 1} \sum_{n=0}^\infty \left( \frac{n + \lambda - 2}{\lambda - 2} \right) \left( \frac{n + 2j + \lambda - 1}{\lambda - 2} \right) \det((BD^{-1})^\dagger)^n} \times \sum_{q=-j}^j D_{qq}^j((BD^{-1})^\dagger) = \det(D)^{-\lambda} \varphi_{q',q}^{j,n}(BD^{-1}).
\]  

(73)

Using the orthogonality properties (66) of the basis functions (65), we can easily compute

\[
|\langle U_\lambda(g) \psi | \psi \rangle|^2 = |\langle \psi | \varphi_{00}^{00} \rangle|^2 = |\hat{\psi}_{00}^{00}(g)|^2 = \det(D^\dagger)^{-\lambda} = \det(I - \tilde{Z} \tilde{Z}^\dagger)^\lambda,
\]  

(74)

where we have defined \( \tilde{Z} \equiv BD^{-1} \) and used the first condition in (52). Using the Haar measure (61), the admissibility condition (68) gives:

\[
c_\psi = \int_{G/H} d\mu(g)|_{G/H} \det(I - \tilde{Z} \tilde{Z}^\dagger)^\lambda \int_H dv(U_1) dv(U_2) = c_\lambda^{-1} \left( \frac{(2\pi)^3}{2} \right)^2 < \infty,
\]  

(75)
where we have identified \( d\mu(g)|_{G/H} \), \( \det(I - \tilde{Z}\tilde{Z}^\dagger) = c_\lambda^{-1}d\nu_\lambda(\tilde{Z}, \tilde{Z}^\dagger) \) and taken into account that \( \int_{D_4} d\nu_\lambda(Z, Z^\dagger) = 1 \) and

\[
v(U(2)) = \int_{U(2)} d\nu(U) = \int \frac{|dz|}{(1 + z^2)^2} d\beta_1 d\beta_2 = \frac{(2\pi)^3}{2} \int_0^\infty \frac{dx}{(1 + x)^2} = \frac{(2\pi)^3}{2},
\]

(2\pi times the area of the 3-sphere \( S^3 = SU(2) \) of unit radius).

Now it remains to prove that the resolution operator (69) is a multiple of the identity \( \mathcal{I} \) in \( L^2_h(D_4, d\nu_\lambda) \). For this purpose, we shall compute its matrix elements:

\[
\langle \varphi_{j_1, q_2} | \varphi_{j_2, q_2}' \rangle = \int_G \langle \varphi_{j_1, q_2} | \varphi_{j_2, q_2}' \rangle d\mu(g) = \int_G \hat{\varphi}_{j_1, q_2}(g) \hat{\varphi}_{j_2, q_2}'(g) d\mu(g) = v(U(2))^2 \int_{G/H} d\mu(g)|_{G/H} \det(I - \tilde{Z}\tilde{Z}^\dagger) \hat{\varphi}_{j_1, q_2}(Z) \hat{\varphi}_{j_2, q_2}'(\tilde{Z}) = c_\psi \delta_{j,j'} \delta_{\lambda,\lambda'} \delta_{q_1,q_2}' \delta_{q_2,q_2}',
\]

where we have used (73), the orthogonality properties (66) of the basis functions (65) and the fact that \( G/H = D_4 \). \( \blacksquare \)

The reconstruction formula (128) here adopts the following form:

\[
\phi(Z) = \int_G \Phi_\psi(g) \psi_g(Z) d\mu(g),
\]

with wavelet coefficients

\[
\Phi_\psi(g) = \frac{1}{c_\psi} \langle \psi_g | \phi \rangle = \frac{1}{c_\psi} \int_{D_4} \det(D - Z^\dagger B)^{-\lambda} \phi(Z) d\nu_\lambda(Z, Z^\dagger).
\]

Expanding \( \phi \) in the basis (65)

\[
\phi(Z) = \sum_{j \in \mathbb{N}/2} \sum_{n=0}^\infty \sum_{q,q'=-j}^j \hat{\phi}_{j,n}^{j,m} \varphi_{j,n}^{j,m}(Z),
\]

and using (72) together with the orthogonality properties (66), we can write the wavelet coefficients (79) in terms of the Fourier coefficients \( \hat{\phi}_{j,n}^{j,m} \) as:

\[
\Phi_\psi(g) = \frac{1}{c_\psi} \sum_{j \in \mathbb{N}/2} \sum_{m=0}^\infty \sum_{q,q'=-j}^j \hat{\phi}_{j,m}^{j,n} \varphi_{j,m}^{j,n}(g) = \frac{1}{c_\psi} \sum_{j \in \mathbb{N}/2} \sum_{m=0}^\infty \sum_{q,q'=-j}^j \det(D)^{-\lambda} \varphi_{j,m}^{j,n}(BD^{-1}) \hat{\phi}_{j,m}^{j,n}.
\]

### 5.3 Isomorphism between \( L^2_h(D_4, d\nu) \) and \( L^2_h(\mathbb{C}_+^4, d\bar{\nu}) \)

For completeness, we shall give an isometry between \( L^2_h(D_4, d\nu) \) and \( L^2_h(\mathbb{C}_+^4, d\bar{\nu}) \) which allows us to translate mathematical properties and constructions from one space into the other.
Proposition 5.5. The correspondence

\[ S_\lambda : L^2_\lambda (\mathbb{D}_4, d\nu_\lambda) \rightarrow L^2_\lambda (\mathbb{C}_4^+, d\tilde{\nu}_\lambda) \]

\[ \phi \mapsto S_\lambda \phi \equiv \tilde{\phi}, \]

with

\[ \tilde{\phi}(W) = 2^{2\lambda} \det(I - iW)^{-\lambda} \phi(Z(W)) \] (80)

and \( Z(W) \) given by the Cayley transformation(47), is an isometry. Actually

\[ \langle \phi | \phi' \rangle_{L^2_\lambda (\mathbb{D}_4, d\nu_\lambda)} = \langle S_\lambda \phi | S_\lambda \phi' \rangle_{L^2_\lambda (\mathbb{C}_4^+, d\tilde{\nu}_\lambda)}, \] (81)

Moreover, \( S_\lambda \) is an intertwiner (equivariant map) of the representations (63) and (45), that is:

\[ U_\lambda = S_\lambda^{-1} \tilde{U}_\lambda S_\lambda. \] (82)

Proof: The left-hand side of the equality (81) is explicitly written as:

\[ \langle \phi | \phi' \rangle_{L^2_\lambda (\mathbb{D}_4, d\nu_\lambda)} = \int_{\mathbb{D}_4} \overline{\phi(Z)} \phi'(Z) c_\lambda \det(I - ZZ^\dagger)^{\lambda-4} |dZ|. \]

Taking into account that

\[ \det(I - ZZ^\dagger) = \det(2i(W^\dagger - W)) | \det(I - iW)|^{-2} \]

and the Jacobian determinant

\[ |dZ| = 2^{12} | \det(I - iW)|^{-8} |dW|, \]

then

\[ d\nu_\lambda(Z, Z^\dagger) = c_\lambda \det(I - ZZ^\dagger)^{\lambda-4} |dZ| = \]

\[ = 2^{4\lambda-4} | \det(I - iW)|^{-2\lambda} c_\lambda \det(\frac{i}{2}(W^\dagger - W))^{\lambda-4} |dW| = \]

\[ = 2^{4\lambda} | \det(I - iW)|^{-2\lambda} d\tilde{\nu}_\lambda(W, W^\dagger), \]

which results in:

\[ \int_{\mathbb{D}_4} \overline{\phi(Z)} \phi(Z) d\nu_\lambda(Z, Z^\dagger) = \int_{\mathbb{C}_4^+} \overline{\tilde{\phi}(W)} \tilde{\phi}(W) d\tilde{\nu}_\lambda(W, W^\dagger), \]

thus proving (81).

The intertwining relation (82) can be explicitly written as:

\[ [U_\lambda \phi](Z) = \det(D^\dagger - B^\dagger Z)^{-\lambda} \phi((A^\dagger Z - C^\dagger)(D^\dagger - B^\dagger Z)^{-1}) = \]

\[ [S_\lambda^{-1} \tilde{U}_\lambda \tilde{\phi}](Z) = \det(I - iW)^{-\lambda} \det(R^\dagger - T^\dagger W)^{-\lambda} \det(I - iW')^{-\lambda} \phi(Z(W')), \] (83)
where \( W' = (Q^tW - S^t)(R^t - T^tW)^{-1} \). On the one hand, we have that the argument of \( \phi \) is:

\[
Z(W') = (I + iW')(I - iW')^{-1}
\]

\[
= ((R^t - T^tW) + i(Q^tW - S^t)) ((R^t - T^tW) - i(Q^tW - S^t))^{-1}
\]

\[
= ((R^t - iS^t) + i(Q^t + iT^tW)) ((R^t + iS^t) - i(Q^t - iT^t)W)^{-1} .
\]

Taking now into account the map (50) we have:

\[
Z(W') = ((A^t - C^t) + i(A^t + C^t)W) ((D^t - B^t) - i(D^t + B^t)W)^{-1}
\]

\[
= (A^t(I + iW) - C^t(I - iW)) (D^t(I - iW) - B^t(I + iW))^{-1}
\]

\[
= (A^tZ - C^t) (D^t - B^tZ)^{-1} ,
\]

as desired. On the other hand, we have that

\[
(I - iW')(R^t - T^tW) = (R^t - T^tW) - i(Q^tW - S^t) = (R^t + iS^t) - i(Q^t - iT^t)W
\]

\[
= (D^t - B^t) - i(D^t + B^t)W = D^t(I - iW) - B^t(I + iW) = (D^t - B^tZ)(I - iW)
\]

which implies

\[
\det(I - iW)^\lambda \det(R^t - T^tW)^{-\lambda} \det(I - iW')^{-\lambda} = \det(D^t - B^tZ)^{-\lambda} \]

(84)

That is, the equality of multipliers in (83).

As a direct consequence of Proposition 5.5, the set of functions defined by

\[
\tilde{\varphi}_{q_1,q_2}^{j,m}(W) \equiv 2^{2\lambda} \det(I - iW)^{-\lambda} \varphi_{j,m}^{q_1,q_2}(Z(W)),
\]

(85)

with \( \varphi_{q_1,q_2}^{j,m} \) defined in (65), constitutes an orthonormal basis of \( L^2_h(\mathbb{C}_+^4, d\tilde{\nu}_\lambda) \) and the closure relation

\[
\sum_{j \in \mathbb{N}/2} \sum_{m=0}^\infty \sum_{q,q'=-j}^j \tilde{\varphi}_{q,q'}^{j,m}(W) \varphi_{q',q'}^{j,m}(W') = \det(i/2(W^t - W'))^{-\lambda},
\]

(86)

gives the reproducing (Bergman) kernel in \( L^2_h(\mathbb{C}_+^4, d\tilde{\nu}_\lambda) \).

The isometry (80) also allows us to translate the results of Theorem 5.4 form \( L_h^2(\mathbb{D}_4, d\nu_\lambda) \) into \( L^2_h(\mathbb{C}_+^4, d\tilde{\nu}_\lambda) \). Indeed, from (80) we conclude that the function \( \tilde{\psi} \in L^2_h(\mathbb{C}_+^4, d\tilde{\nu}_\lambda) \) given by:

\[
\tilde{\psi}(W) = 2^{2\lambda} \det(I - iW)^{-\lambda}
\]

(87)

is admissible. The construction of a tight frame and a reconstruction formula form this mother wavelet parallels (69) and (78), respectively.
5.4 Symmetry properties of the proposed conformal wavelets

When working with wavelets on the sphere [8, 9, 10] it is customary to take axisymmetric (or zonal) wavelets, that is, admissible vectors \( \psi \) which are invariant under rotations around the (namely) z-axis, although more general implementations including directional spherical wavelets are also possible (see e.g. [31]). Let us discuss the symmetry properties of our proposed admissible wavelets \( (87) \). Applying a general \( SU(2, 2) \)-transformation \( (45) \) to \( (87) \) gives:

\[
\tilde{U}_\lambda(f)\tilde{\psi}(W) = 2^{2\lambda} \det(R^\dagger - T^\dagger W)^{-\lambda} \det(I - iW')^{-\lambda},
\]

\[
W' = (Q^\dagger W - S^\dagger)(R^\dagger - T^\dagger W)^{-1}
\]

Using the identity \( (84) \) we have:

\[
\tilde{U}_\lambda(f)\tilde{\psi}(W) = \det(D^\dagger - B^\dagger Z)^{-\lambda}\tilde{\psi}(W),
\]

which leaves invariant \( \tilde{\psi} \) (up to a global phase) if:

\[
B = 0 \Rightarrow C = 0 \Rightarrow S = -T, \quad Q = R,
\]

where we have used \( (52), (53) \) and \( (50) \). Thus, the elements \( f \in SU(2, 2) \) of the form

\[
f = \begin{pmatrix} R & iS \\ iS & R \end{pmatrix}
\]

(91)

leave invariant \( (87) \). The constraints \( f^\dagger \gamma^0 f = \gamma^0 \) imply:

\[
S^\dagger S + R^\dagger R = I, \quad S^\dagger R = R^\dagger S.
\]

(92)

For \( S = 0 \), \( R \) is unitary. For \( R = I \), \( S \) is hermitian with \( S^2 = 0 \). The last condition is satisfied for translations \( S = b_\mu \sigma^\mu \) along null (light-like) vectors \( b^2 = b_\mu b^\mu = \det(S) = 0 \). This leaves us a 7-dimensional subgroup of \( SU(2, 2) \), isomorphic to \( S(U(2) \times U(2)) \), as the isotropy subgroup of the admissible vector \( (87) \). Any other basis state \( (85) \) could be used as a fiducial state to construct oriented wavelets.

In the next Figure we provide a visualization of this wavelet (modulus and argument) for the particular case of \( W = w\sigma^0 \) (temporal part), \( w \equiv x + iy \), for which \( \psi(W) = 2^{2\lambda}(1 - iw)^{-2\lambda} \) reduces to \( \tilde{\varphi}_0(w) \) in \( (39) \). We take \( \lambda = 1 \) for simplicity.
5.5 The Euclidean limit

We have seen that the Shilov boundary of $D_4$ is the compactified Minkowski space $U(2) = S^3 \times \mathbb{Z}_2$, $S^1$ (the four-dimensional analogue of the boundary $U(1) = S^1$ of the disk $D_1$). One expects the wavelet transform on $S^N$ to behave locally (at short scales or large values of the radius $\rho$) like the usual (flat) wavelet transform on $\mathbb{R}^N$. Indeed, in [27], one of the authors and collaborators discussed the Euclidean limit (infinite radius) for wavelets on $S^1$. The procedure parallels that of Ref. [9] for wavelets on $S^2$. In these references, the Euclidean limit is formulated as a contraction at the level of group representations. Let us restrict ourselves, for the sake of simplicity, to the conformal group $SO(1,2)$ in 1+0 (temporal) dimensions. The realistic 1+3 dimensional case $SO(4,2)$, although technically more complicated, follows similar guidelines and will be left for future work.

Let us denote simply by $P = P_0$ and $K = K_0$ the temporal components of $P_\mu$ and $K_\mu$ (the generators of spacetime translations and accelerations). The Lie algebra commutators of $SO(1,2)$ are [remember the general $N$-dimensional case (21)]:

$$[D,P] = -P, \quad [D,K] = K, \quad [K,P] = 2D. \quad (93)$$

A contraction $G'$ of the Lie algebra $G = so(1,2)$ along sim(1) (generated by $P$ and $D$) can be constructed through the one-parameter family of invertible linear mappings $\pi_\rho : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \rho \in [1, \infty)$ defined by:

$$\pi_\rho(D) = D, \quad \pi_\rho(P) = P, \quad \pi_\rho(K) = \rho^{-1}K, \quad (94)$$

such that the Lie bracket of $G'$ is:

$$[X,Y]' = \lim_{\rho \rightarrow \infty} \pi_\rho^{-1} [\pi_\rho X, \pi_\rho Y], \quad (95)$$

with $[\cdot,\cdot]$ the Lie bracket (93) of $G$. The resulting $G'$ commutators are:

$$[D,P]' = -P, \quad [D,K]' = K, \quad [K,P]' = 0. \quad (96)$$

The contraction process is lifted to the corresponding Lie groups $G' = \mathbb{R}^2 \ltimes \mathbb{R}^+$ and $G = SO(1,2)$ by considering the exponential mapping $e^{\pi_\rho}$. The idea is that the representation of $G$ contract to the usual wavelet representation of the affine group $SIM(1)$ in the following sense:

**Definition 5.6.** Let $G'$ be a contraction of $G$, defined by the contraction map $\Pi_\rho : G' \rightarrow G$, and let $\mathcal{U}'$ be a representation of $G'$ in a Hilbert space $\mathcal{H}'$. Let $\{\mathcal{U}_\rho\}, \rho \in [1, \infty)$ be a one-parameter family of representations of $G$ on a Hilbert space $\mathcal{H}_\rho$, and $\iota_\rho : \mathcal{H}_\rho \rightarrow \mathcal{D}_\rho$ a linear injective map from $\mathcal{H}_\rho$ onto a dense subspace $\mathcal{D}_\rho \subset \mathcal{H}'$. Then we shall say that $\mathcal{U}'$ is a contraction of the family $\{\mathcal{U}_\rho\}$ if there exists a dense subspace $\mathcal{D}' \subset \mathcal{H}'$ such that, for all $\phi \in \mathcal{D}'$ and $g' \in G'$, one has:

- For every $\rho$ large enough, $\phi \in \mathcal{D}_\rho$ and $\mathcal{U}_\rho(\Pi_\rho(g'))\iota_\rho^{-1}\phi \in \iota_\rho^{-1}\mathcal{D}_\rho$.
\[ \lim_{\rho \to \infty} ||\iota_\rho \mathcal{U}_\rho(\Pi_\rho(g'))\iota_\rho^{-1}\phi - \mathcal{U}'(g')\phi||_{\mathcal{H}'} = 0, \forall g' \in G'. \]

More precisely, one can prove that:

**Theorem 5.7.** The representation \[ \mathcal{U}'(b,a)\phi(x) = \frac{1}{\sqrt{a}} \phi\left(\frac{x+b}{a}\right) \] of the affine group \( \text{SIM}(1) \) is a contraction of the one-parameter family \( \mathcal{U}_\rho \) of representations of \( \text{SO}(1,2) \) on \( \mathcal{H}_\rho \) as \( \rho \to \infty \). That is:

\[ \lim_{\rho \to \infty} ||\iota_\rho \mathcal{U}_\rho(\Pi_\rho'(b,a))\iota_\rho^{-1}\phi - \mathcal{U}'(b,a)\phi||_{\mathcal{H}'} = 0, \forall (b,a) \in \mathbb{R} \times \mathbb{R}^+, \]  

where \( \Pi_\rho' : \text{SIM}(1) \to \text{SO}(1,2)/\mathbb{R} \) is the restricted contraction map, with \( \sigma' : G'/\mathbb{R} \to G' \) a given section.

This construction can be straightforwardly extended to \( G = \text{SO}(4,2) \), the contraction \( G' \) being the so-called \( G_{15} \) group of Ref. [32]. A thorough discussion of the Euclidean limit of the conformal wavelets constructed in this paper falls beyond the scope of this article and will be left for future work [33]. Here we just wanted to give a flavor of it.

### 6 Convergence Remarks

Schwinger’s Theorem 2.1 and its extension 2.3 have been stated in the sense of generating functions in terms of formal power series in some indeterminates. From this point of view, we have disregarded convergence issues. However, infinite series expansions like for instance (12) would require in particular that \( |t|^2 |\det(X)| < 1 \). We shall prove that such convergence requirements, together with additional restrictions coming from the basic Theorem 2.1, are automatically fulfilled inside the complex domain \( \mathbb{D}_4 \) for \( tX = \tilde{Z}^\dagger Z \), with \( \tilde{Z} = BD^{-1} \) in the expansion (70). Let us state these convergence requisites.

**Proposition 6.1.** A sufficient condition for the convergence of the expansions (7) and (10) for \( t = 1 \) is that:

\[ |x_{11}| < 1, \ |x_{22}| < 1, \ |x_{12}x_{21}| < 1, \ |\det(X)| < 1. \]  

**Proof:** Looking at the explicit expression of Wigner’s \( \mathcal{D} \)-matrices (3)

\[ \mathcal{D}^j_{q,q}(X) = \sum_{k = \max(0,2q)}^{j+q} \binom{j+q}{k} \binom{j-q}{k-2q} x_{11}^j (x_{12}x_{21})^{j+q-k} x_{22}^{k-2q} \]  

we conclude that it is enough to have: \( |x_{11}| < 1, \ |x_{22}| < 1 \) and \( |x_{12}x_{21}| < 1 \), for the convergence of (4) for \( t = 1 \), because their exponents run up to infinity independent of each other. Moreover, if we require convergence in the expansions (10) and (12) for \( t = 1 \), then \( |\det(X)| < 1 \) is needed too. ■

We shall see that \( Z \) and \( \tilde{Z} \) fulfil (98), but before we shall prove that
Proposition 6.2. For any matrix $Z \in \mathbb{D}_4$ we have that the squared norm of their rows is lesser than 1, that is:

$$|z_{11}|^2 + |z_{12}|^2 < 1, \quad |z_{21}|^2 + |z_{22}|^2 < 1.$$  (100)

Proof: The positivity condition (54) says that

$$\det(I - ZZ^\dagger) > 0 \Leftrightarrow |z_{11}\bar{z}_{21} + z_{12}\bar{z}_{22}|^2 < (1 - |z_{11}|^2)(1 - |z_{12}|^2).$$  (101)

Hence, the last two factors must be either positive or negative. Supposing that both factors were negative would contradict $\text{tr}(ZZ^\dagger) < 2$ in (57). Therefore, we conclude that both factors are positive.

Let us remind that, since $Z$ and $\tilde{Z}$ belong to $\mathbb{D}_4$, they must satisfy $|\det(Z)| < 1$ and $|\det(\tilde{Z})| < 1$, as we saw in (55) and (58). Now we are in condition to prove that:

Proposition 6.3. The matrix $X = \tilde{Z}^\dagger Z$ verifies the convergence conditions (98) for every $Z, \tilde{Z} \in \mathbb{D}_4$ and, therefore, the expansion (70) is well defined for $\tilde{Z} = BD^{-1}$.

Proof: The conditions (100) imply in particular that $|z_{11}| < 1, |z_{12}| < 1, |z_{21}| < 1, |z_{22}| < 1$. Using this fact, the triangle inequality and taking into account that $Z, \tilde{Z} \in \mathbb{D}$ verify (100) and the determinant restriction (58), we arrive to:

$$|x_{11}| = |\tilde{z}_{11}\tilde{z}_{11} + \tilde{z}_{12}\tilde{z}_{21}| \leq |\tilde{z}_{11}\tilde{z}_{11}| + |\tilde{z}_{12}\tilde{z}_{21}| < |\tilde{z}_{11}| + |\tilde{z}_{12}| < |\tilde{z}_{11}|^2 + |\tilde{z}_{12}|^2 < 1,$$

$$|x_{22}| = |\tilde{z}_{21}\tilde{z}_{12} + \tilde{z}_{22}\tilde{z}_{22}| \leq |\tilde{z}_{21}\tilde{z}_{12}| + |\tilde{z}_{22}\tilde{z}_{22}| < |\tilde{z}_{21}| + |\tilde{z}_{22}| < |\tilde{z}_{21}|^2 + |\tilde{z}_{22}|^2 < 1,$$

$$|x_{12}| = |\tilde{z}_{11}\tilde{z}_{12} + \tilde{z}_{12}\tilde{z}_{22}| \leq |\tilde{z}_{11}\tilde{z}_{12}| + |\tilde{z}_{12}\tilde{z}_{22}| < |\tilde{z}_{11}| + |\tilde{z}_{12}| < |\tilde{z}_{11}|^2 + |\tilde{z}_{12}|^2 < 1,$$

$$|x_{21}| = |\tilde{z}_{21}\tilde{z}_{11} + \tilde{z}_{22}\tilde{z}_{21}| \leq |\tilde{z}_{21}\tilde{z}_{11}| + |\tilde{z}_{22}\tilde{z}_{21}| < |\tilde{z}_{21}| + |\tilde{z}_{22}| < |\tilde{z}_{21}|^2 + |\tilde{z}_{22}|^2 < 1,$$

$$|\det(X)| = |\det(\tilde{Z}^\dagger Z)| = |\det(\tilde{Z}^\dagger)||\det(Z)| = \det(\tilde{Z}^\dagger \tilde{Z})^{1/2} \det(ZZ^\dagger)^{1/2} < 1,$$  (102)

which proves the convergence conditions (98).■

7 Conclusions and Outlook

We have constructed the CWT on the Cartan domain $\mathbb{D}_4 = U(2, 2)/U(2)^2$ of the conformal group $SO(4, 2) = SU(2, 2)/\mathbb{Z}_4$ in 3+1 dimensions. The manifold $\mathbb{D}_4$ can be mapped one-to-one onto the future tube domain $\mathbb{C}_4^+$ of the complex Minkowski space through a Cayley transformation, where we enjoy more physical intuition. This construction paves the way towards a new analysis tool of fields in complex Minkowski space-time with continuum mass spectrum in terms of conformal wavelets. It is traditional in Relativistic Particle Physics to analyze fields or signals (for instance, elementary particles) in Fourier (energy-momentum) space. However, like in music where there are no infinitely lasting sounds, particles are created and destroyed in nuclear reactions. A wavelet transform based on the conformal group provides a way to analyze wave packets localized in both: space and time. Important developments in this direction have also been done in [14 15 16] for electromagnetic (massless) signals.
In the way, we have stated and proved a $\lambda$-extension of the Schwinger’s formula. This extension turns out to be a useful mathematical tool for us, specially as a generating function for the unitary-representation functions of $SU(2, 2)$, the derivation of the reproducing (Bergman) kernel of $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ and the proof of admissibility and tight frame conditions. The generalization of this theorem to matrices $X$ of size $N \geq 2$ follows similar guidelines and the particular details are discussed in the Appendix A using the general $SU(N)$ solid harmonics $D_{\alpha\alpha}^p(X)$ of Louck [25]. This result could be of help in studying the discrete series (infinite-dimensional) representations of the non-compact pseudo-unitary groups $SU(N, N)$.

The next step should be the discretization problem. References [34, 35, 36] give us the general guidelines to construct discrete (wavelet) frames on the sphere and the hyperboloid and [37] on the Poincaré group. The conformal group is much more involved, though in principle the same scheme applies.

Looking for further potential applications of the conformal wavelets constructed in this article, we think that they could be of use in analyzing renormalizability problems in relativistic quantum field theory. When describing space and time as a continuum, certain statistical and quantum mechanical constructions are ill defined. In order to define them properly, the continuum limit has to be taken carefully starting from a discrete approach. There is a collection of techniques used to take a continuum limit, usually referred as “renormalization rules”, which determine the relationship between parameters in the theory at large and small scales. Renormalization rules fail to define a finite quantum theory of Einstein’s General Relativity, one of the main breakthroughs in Theoretical Physics. The replacement of classical (commutative) space-time by a quantum (non-commutative) space-time promises to restore finiteness to quantum gravity at high energies and small (Planck) scales, where geometry becomes also quantum (non-commutative) [38]. Conformal wavelets could also be here of fundamental importance as an analysis tool.

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A Extended MacMahon-Schwinger’s Master Theorem for Matrices of Size \( N \geq 2 \)

We have shown the utility of the Theorem 2.3 in dealing with unitary representations of \( SU(2,2) \), in particular, in proving the admissibility condition 5.4. We would like to have a generalization of Theorem 2.3 for matrices of arbitrary size \( N \), since it would be a valuable tool as a generating function for the unitary-representation functions of \( SU(N,N) \).

The first step is to generalize Wigner \( \mathcal{D} \)-matrices. This generalization has been done in the literature (see [25] and references therein) by the so-called \( SU(N) \) solid harmonics \( \mathcal{D}_{\alpha\beta}(X) \) defined as:

\[
\mathcal{D}_{\alpha\beta}(X) \equiv \sqrt{\alpha!\beta!} \sum_{A \in M^p_N(\alpha,\beta)} \frac{X^A}{A!}
\]

where the following space saving notations are employed: \( A \) is a \( N \times N \) matrix in the non-negative integers \( a_{ij} \); \( A! \equiv \prod_{i,j=1}^N a_{ij}! \); \( X^A \equiv \prod_{i,j=1}^N x_{ij}^a \); \( \alpha \equiv (\alpha_1, \alpha_2, ..., \alpha_N) \) is a sequence of \( N \) nonnegative integers that sum to \( p \) (i.e., a composition of \( N \) into \( p \) nonnegative parts), shortly \( \alpha \vdash p \); \( \alpha! \equiv \prod_{i=1}^N \alpha_i! \); \( M^p_N(\alpha,\beta) \) denotes the set of all matrices \( A \) such that the entries in row \( i \) sum to \( \alpha_i \) and those in column \( j \) sum to \( \beta_j \), with \( \alpha \vdash p \) and \( \beta \vdash p \). Hence, \( \mathcal{D}_{\alpha\beta}(X) \) are homogeneous polynomials of degree \( p \) in the indeterminates \( x_{ij} \).

The particular identification with Wigner’s \( \tilde{\mathcal{D}} \)-matrices for \( X \) of size \( N = 2 \) is given by \( \tilde{\mathcal{D}}_{q_1,q_2}(X) = \mathcal{D}_{\alpha\beta}(X) \) with \( p = 2j, \alpha, \beta \vdash 2j, \alpha = (j+q_1,j-q_1), \beta = (j+q_2,j-q_2) \).

Matrices \( A \in M^p_N(\alpha,\beta) \) can be then indexed by an integer \( k \)

\[
A^{(k)} \equiv \begin{pmatrix} k & j+q_1-k \\ j+q_2-k & k-q_1-q_2 \end{pmatrix}
\]

with \( \max(0,q_1+q_2) \leq k \leq \min(j+q_1,j+q_2) \).

The multiplication property (6) and the transpositional symmetry (7) for Wigner matrices are still valid for \( SU(N) \)-solid harmonics as:

\[
\sum_{\sigma \vdash p} \mathcal{D}^p_{\alpha\sigma}(X) \mathcal{D}^p_{\sigma\beta}(Y) = \mathcal{D}^p_{\alpha\beta}(XY)
\]

and

\[
\mathcal{D}^p_{\alpha\beta}(Y) = \mathcal{D}^p_{\beta\alpha}(Y^T)
\]

(see [25] for a combinatorial proof).

Moreover, for general \( N \times N \) matrices \( X \), the determinant \( \det(I-X) \) can be expanded in terms of sums of all principal \( q \)-th minors of \( X \) as

\[
\det(I-X) = \sum_{q=0}^N (-1)^{N+q} \sum_{\alpha \vdash q} \partial^q_x \det(X),
\]
where the \( N \)-dimensional multi-index \( \alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_N) \) is a partition of \( q \) with \( \alpha_i \in \{0, 1\} \), a fact that we now symbolize as \( \alpha \vdash q \); \( x \equiv (x_{11}, x_{22}, \ldots, x_{NN}) \) and \( \partial_x^\alpha = \prod_{i=1}^N \partial_{x_{ii}}^{\alpha_i} \).

Let us define the sum of all principal \((N - q)\)-th minors of \( X \) by

\[
T_q(X) \equiv \sum_{\alpha \vdash N - q} \partial_x^\alpha \det(X).
\]

They are homogeneous polynomials of degree \( q = 0, 1, \ldots, N \) in the indeterminates \( x_{ij} \).
For example: \( T_0(X) = 1 \), \( T_1(X) = \text{tr}(X) \), \ldots, \( T_N(X) = \det(X) \). Thus, \( \det(I - X) \) can be written in terms of these homogeneous polynomials as:

\[
\det(I - X) = \sum_{q=0}^N (-1)^q T_q(X). \tag{107}
\]

Other possibility could be to use Waring’s formulas [25].

To arrive at the \( \lambda \)-extended MacMahon-Schwinger’s Master Theorem (MSMT) for \( N \times N \) matrices, we shall now proceed step by step from \( \lambda = 2 \) to general \( \lambda \). Before, let us explicitly write down the generalization of the Theorem 2.1 to matrices of general size \( N \).

**Theorem A.1.** (MSMT) The identity

\[
\sum_{p=0}^\infty t^p \sum_{\alpha \vdash p} D_{\alpha \alpha}^p (tX) = \det(I - tX)^{-1} \tag{108}
\]

holds for any \( N \times N \) matrix \( X \).

The action of the operator \( D_1 \) on both sides of the Basic MacMahon-Schwinger’s formula (108) now gives:

\[
\sum_{p=0}^\infty (p + 1) \sum_{\alpha \vdash p} D_{\alpha \alpha}^p (tX) = \frac{1 - \sum_{q=2}^N (-1)^q (q - 1) T_q(tX)}{\det(I - tX)^2} = \frac{1 - \sum_{q=2}^N \hat{T}_q(tX)}{\det(I - tX)^2}, \tag{109}
\]

where we have defined \( \hat{T}_q(X) \equiv (-1)^q (q - 1) T_q(X) \). We can bring the numerator of the right-hand side of (109) back to the left-hand side by using the expansion:

\[
\frac{1}{1 - \sum_{q=2}^N \hat{T}_q(tX)} = \sum_{p=0}^\infty (\sum_{q=2}^N \hat{T}_q(tX))^p = \sum_{\gamma=0}^\infty \left( \sum_{j=2}^N \gamma_j \right) \hat{T}(tX)^\gamma, \tag{110}
\]

where we have used the following shorthand for

\[
\hat{T}(X)^\gamma \equiv \hat{T}_2(X)^{\gamma_2} \hat{T}_3(X)^{\gamma_3} \ldots \hat{T}_N(X)^{\gamma_N}, \quad \left( \sum_{j=2}^N \gamma_j \right) \equiv \frac{(\sum_{j=2}^N \gamma_j)!}{\gamma_2! \ldots \gamma_N!}. \tag{111}
\]
Note that $\hat{T}(X)^\gamma$ are homogeneous polynomials of degree $\sum_{j=2}^N j \gamma_j$ in $x_{ij}$. Inserting the expansion (110) in (109) we conclude that

$$\sum_{p=0}^\infty (p+1) \sum_{\gamma=0}^\infty \left( \sum_{j=2}^N \frac{\gamma_j}{\gamma} \right) \hat{T}(tX)^\gamma \sum_{\alpha+p} \mathcal{D}_{\alpha\alpha}^p (tX) = \frac{1}{\det(I-tX)^2}. \quad (112)$$

This is the generalization of (13) for general $N$. Let us proceed by applying $D_2$ on both sides of the identity (112):

$$\sum_{p=0}^\infty (p+1) \sum_{\gamma=0}^\infty \left( \sum_{j=2}^N \frac{\gamma_j}{\gamma} \right) (p+2 + \sum_{j=2}^N j \gamma_j) \hat{T}(tX)^\gamma \sum_{\alpha+p} \mathcal{D}_{\alpha\alpha}^p (tX) = 2 \frac{1 - \sum_{q=2}^N \hat{T}_q(tX)}{\det(I-tX)^3}. \quad (113)$$

Using again (110), we have

$$\sum_{p=0}^\infty \frac{p+1}{2} \sum_{\gamma=0}^\infty \sum_{\gamma'=0}^\infty \left( \sum_{j=2}^N \frac{\gamma_j}{\gamma} \right) \left( \sum_{j=2}^N \frac{\gamma_j'}{\gamma'} \right) (p+2 + \sum_{j=2}^N j \gamma_j) \hat{T}(tX)^{\gamma+\gamma'} \sum_{\alpha+p} \mathcal{D}_{\alpha\alpha}^p (tX) = \frac{1}{\det(I-tX)^3}. \quad (114)$$

Rearranging series as in (15) and making the change of $(N-1)$-dimensional multi-index: $\sigma \equiv \gamma + \gamma'$, we obtain

$$\sum_{p=0}^\infty \frac{p+1}{2} \sum_{\sigma=0}^\infty \left\{ \sum_{\gamma=0}^\sigma \left( \sum_{j=2}^N \frac{\gamma_j}{\gamma} \right) \left( \sum_{j=2}^N \frac{\gamma_j'}{\gamma'} \right) (p+2 + \sum_{j=2}^N j \gamma_j) \right\} \hat{T}(tX)^\sigma \sum_{\alpha+p} \mathcal{D}_{\alpha\alpha}^p (tX) = \frac{1}{\det(I-tX)^3}. \quad (114)$$

Applying now $D_3$ on both sides of (113) results:

$$\sum_{p=0}^\infty \frac{p+1}{2} \sum_{\gamma=0}^\infty \sum_{\gamma'=0}^\infty \left( \sum_{j=2}^N \frac{\gamma_j}{\gamma} \right) \left( \sum_{j=2}^N \frac{\gamma_j'}{\gamma'} \right) (p+2 + \sum_{j=2}^N j \gamma_j) (p+3 + \sum_{j=2}^N j (\gamma_j + \gamma'_j)) \hat{T}(tX)^{\gamma+\gamma'} \sum_{\alpha+p} \mathcal{D}_{\alpha\alpha}^p (tX) = 3 \frac{1 - \sum_{q=2}^N \hat{T}_q(tX)}{\det(I-tX)^4}. \quad (114)$$

and using again (110) we get:

$$\sum_{p=0}^\infty \frac{p+1}{3!} \sum_{\gamma=0}^\infty \sum_{\gamma'=0}^\infty \sum_{\gamma''=0}^\infty \left( \sum_{j=2}^N \frac{\gamma_j}{\gamma} \right) \left( \sum_{j=2}^N \frac{\gamma_j'}{\gamma'} \right) \left( \sum_{j=2}^N \frac{\gamma_j''}{\gamma''} \right) (p+2 + \sum_{j=2}^N j \gamma_j) \hat{T}(tX)^{\gamma+\gamma'+\gamma''} \sum_{\alpha+p} \mathcal{D}_{\alpha\alpha}^p (tX) = \frac{1}{\det(I-tX)^4}. \quad (115)$$
Rearranging series and making the change $\sigma \equiv \gamma + \gamma' + \gamma''$, we can recast the last expression as:

$$
\sum_{p=0}^{\infty} \frac{p+1}{3} \sum_{\lambda=0}^{\infty} \frac{1}{\sum_{\sigma=0}^{\lambda-2} \sum_{\gamma=0}^{\sigma-\gamma} \sum_{\gamma'=0}^{\sigma-\gamma'} \gamma''} \left( \sum_{j=2}^{N} j \gamma_j \right) \left( \sum_{j=2}^{N} j \gamma_j' \right) \left( \sum_{j=2}^{N} (\sigma_j - \gamma_j - \gamma_j') \right) 
\times (p + 2 + \sum_{j=2}^{N} j \gamma_j) (p + 3 + \sum_{j=2}^{N} j (\gamma_j + \gamma_j')) \right) \widehat{T}(tX)^{\sigma} \sum_{\alpha=0}^{p} D_{\alpha\alpha}^p (tX) = \frac{1}{\text{det}(I - tX)^{\lambda}}.
$$

If we repeat the process $(\lambda - 4)$ more times, then we arrive at the following identity:

$$
\sum_{p=0}^{\infty} \frac{p+1}{\lambda - 1} \sum_{\sigma=0}^{\lambda-2} \sum_{\gamma=0}^{\sigma-\gamma} \sum_{\gamma'=0}^{\sigma-\gamma'} \gamma'' \left( \sum_{j=2}^{N} j \gamma_j \right) \left( \sum_{j=2}^{N} j \gamma_j' \right) \left( \sum_{j=2}^{N} (\sigma_j - \gamma_j - \gamma_j') \right) 
\times (p + k + 2 + \sum_{j=2}^{N} \sum_{i=0}^{k} j \gamma_j^{(i)}) \widehat{T}(tX)^{\gamma + \gamma' + \cdots + \gamma^{(\lambda-2)}} \sum_{\alpha=0}^{p} D_{\alpha\alpha}^p (tX) = \frac{1}{\text{det}(I - tX)^{\lambda}}.
$$

Making once more the change $\sigma = \gamma + \gamma' + \cdots + \gamma^{(\lambda-2)}$, we can write:

$$
\sum_{p=0}^{\infty} \frac{p+1}{\lambda - 1} \sum_{\sigma=0}^{\lambda-2} \sum_{\gamma=0}^{\sigma-\gamma} \sum_{\gamma'=0}^{\sigma-\gamma'} \gamma'' \left( \sum_{j=2}^{N} j \gamma_j \right) \left( \sum_{j=2}^{N} j \gamma_j' \right) \left( \sum_{j=2}^{N} (\sigma_j - \gamma_j - \gamma_j') \right) 
\times (p + k + 2 + \sum_{j=2}^{N} \sum_{i=0}^{k} j \gamma_j^{(i)}) \widehat{T}(tX)^{\sigma} \sum_{\alpha=0}^{p} D_{\alpha\alpha}^p (tX) = \frac{1}{\text{det}(I - tX)^{\lambda}}.
$$

where we have defined the following coefficients:

$$
C_{p,s}^{\lambda,\sigma} = \frac{1}{(\lambda - 2)!} \sum_{\gamma=0}^{\sigma-\gamma} \sum_{\gamma'=0}^{\sigma-\gamma'} \gamma'' \left( \sum_{j=2}^{N} j \gamma_j \right) \left( \sum_{j=2}^{N} j \gamma_j' \right) \left( \sum_{j=2}^{N} (\sigma_j - \gamma_j - \gamma_j') \right) 
\times \prod_{k=0}^{\lambda-3} \left( \sum_{j=2}^{N} j \gamma_j^{(k)} \right) (p + k + 2 + \sum_{j=2}^{N} \sum_{i=0}^{k} j \gamma_j^{(i)}).
$$

In order to account for the particular coefficients $C_{p,s}^{2,\sigma}$ and $C_{p,s}^{3,\sigma}$, given inside curly brackets in (112) and (114), we must understand in (116) that: 1) summations on $\gamma^{(k)}$ with $k < 0$ are absent, 2) empty or nullary sums are zero and 3) empty or nullary products are 1, as customary.

Summarizing, we can enunciate the following:

**Theorem A.2.** (\(\lambda\)-extended MSMT) For every \(\lambda \in \mathbb{N}, \lambda \geq 2\) and every \(N \times N\) matrix \(X\), the following identity holds:

$$
\sum_{p=0}^{\infty} \frac{p+1}{\lambda - 1} \sum_{\sigma=0}^{\lambda-2} \sum_{\gamma=0}^{\sigma-\gamma} \sum_{\gamma'=0}^{\sigma-\gamma'} \gamma'' \left( \sum_{j=2}^{N} j \gamma_j \right) \left( \sum_{j=2}^{N} j \gamma_j' \right) \left( \sum_{j=2}^{N} (\sigma_j - \gamma_j - \gamma_j') \right) 
\times (p + \sum_{j=2}^{N} \sum_{i=0}^{k} j \gamma_j^{(i)}) \widehat{T}(tX)^{\sigma} \sum_{\alpha=0}^{p} D_{\alpha\alpha}^p (X) = \text{det}(I - tX)^{-\lambda},
$$

with $C_{p,s}^{\lambda,\sigma}$ given by (116) and \(\widehat{T}(X)^{\sigma}\) by (117).
The expression (117) generalizes (10) for matrices $X$ of arbitrary size $N$. In fact, for $N = 2$, the coefficient (116) reduces to:

$$C_{p,\sigma}^\lambda = \left(\frac{\lambda - 2 + \sigma_2}{\lambda - 2}\right)\left(\frac{\lambda - 1 + p + \sigma_2}{\lambda - 2}\right),$$

which agrees with (10). We have also been able to find simplifications of $C_{p,\sigma}^\lambda$ in the following cases (we take the binomials in the generalized sense $\binom{n}{m} = \frac{n(n-1)\ldots(n-m+1)}{m!}$ to account for fractional $n$):

i) For $2 \leq \lambda \leq 5$, the coefficients (116) are given by:

$$C_{p,\sigma}^\lambda = \left(\sum_{j=2}^{N} \frac{\sigma_j}{\lambda - 2}\right) \left(\frac{\lambda - 2 + \sum_{k=2}^{N} \sigma_k}{\lambda - 2}\right) \left(\frac{\lambda - 1 + p + \frac{1}{2} \sum_{k=2}^{N} k\sigma_k}{\lambda - 2}\right)$$

$$+ \left(\frac{\lambda - 2 + p + \frac{1}{2} \sum_{k=2}^{N} k\sigma_k}{\lambda - 4}\right) \frac{1}{4!} \sum_{k=3}^{N} (k - 2)k\sigma_k.$$

ii) For $N = 3$ and $\lambda \geq 2$, the coefficients (116) can be obtained through the expression:

$$C_{p,\sigma}^\lambda = \left(\frac{\sigma_2 + \sigma_3}{\lambda - 2}\right) \left(\frac{\lambda - 2 + \sigma_2 + \sigma_3}{\lambda - 2}\right)$$

$$\times \left(\frac{\lambda - i + p + \sigma_2 + \frac{3}{2} \sigma_3}{\lambda - 2i}\right) \prod_{j=1}^{i-1} \frac{\sigma_3 - 2(j - 1)}{8j}$$

where we have defined $\xi \equiv \text{Odd}(\lambda)$, that is, $\xi = 0$ when $\lambda$ is even and $\xi = 1$ when odd. See that (119) reduces to (118) for $\sigma_3 = 0$.

**B Continuous Wavelet Transform on a Manifold: a Brief**

The usual CWT on the real line $\mathbb{R}$ is derived from the natural unitary representation of the affine group $G = SIM(1)$ in the space of finite energy signals $L^2(\mathbb{R}, dx)$ (see Section 4 for a reminder). The same scheme applies to the CWT on a general manifold $\mathbb{X}$, subject to the transitive action, $x \rightarrow gx, g \in G, x \in \mathbb{X}$, of some group of transformations $G$ which contains dilations and motions on $\mathbb{X}$. If the measure $d\nu(x)$ in $\mathbb{X}$ is $G$-invariant (i.e. $d\nu(gx) = d\nu(x)$), then the natural left action of $G$ on $L^2(\mathbb{X}, d\nu)$ given by:

$$[U(g)\phi](x) = \phi(g^{-1}x), \quad g \in G, \phi \in L^2(\mathbb{X}, d\nu),$$

defines a unitary representation, that is:

$$\langle U(g)\varphi | U(g)\phi \rangle = \langle \varphi | \phi \rangle \equiv \int_{\mathbb{X}} \overline{\varphi(x)}\phi(x)d\nu(x).$$
When $d\nu$ is not strictly invariant (i.e. $d\nu(gx) = M(g,x)d\nu(x)$), we have to introduce a multiplier (Radon-Nikodym derivative)

$$[U(g)\phi](x) = M(g,x)^{1/2}\phi(g^{-1}x), \quad g \in G, \phi \in L^2(\mathbb{X}, d\nu),$$ (121)

in order to keep unitarity. The fact that $U(g)U(g') = U(gg')$ (i.e. $U$ is a representation of $G$) implies cohomology conditions for multipliers, that is:

$$M(gg', x) = M(g, x)M(g', g^{-1}x).$$ (122)

Consider now the space $L^2(G,d\mu)$ of square-integrable complex functions $\Psi$ on $G$, where $d\mu(g) = d\mu(g')$, $\forall g' \in G$, stands for the left-invariant Haar measure, which defines the scalar product

$$(\Psi|\Phi) = \int_G \overline{\Psi(g)}\Phi(g)d\mu(g).$$

A non-zero function $\psi \in L^2(\mathbb{X}, d\nu)$ is called admissible (or a fiducial vector) if $\Psi(g) = \langle U(g)\psi|\psi \rangle \in L^2(G,d\mu)$, that is, if

$$c_\psi = \int_G \overline{\Psi(g)}\Psi(g)d\mu(g) = \int_G |\langle U(g)\psi|\psi \rangle|^2d\mu(g) < \infty.$$ (123)

A unitary representation for which admissible vector exists is called square integrable. For a square integrable representation, besides Eq. (123) the following property holds (see [1]):

$$\int_G |\langle U(g)\psi|\phi \rangle|^2d\tilde{\mu}(g) < \infty, \forall \phi \in L^2(\mathbb{X}, d\nu).$$ (124)

Let us assume that the representation $U$ is irreducible, and that there exists a function $\psi$ admissible, then a system of coherent states (CS) of $L^2(\mathbb{X}, d\nu)$ associated to (or indexed by) $G$ is defined as the set of functions in the orbit of $\psi$ under $G$

$$\psi_g \equiv U(g)\psi, \quad g \in G.$$

There are representations without admissible vectors, since the integration with respect to some subgroup diverges. In this case, or even for convenience when admissible vectors exist, we can restrict ourselves to a suitable homogeneous space $Q = G/H$, for some closed subgroup $H$. Then, the non-zero function $\psi$ is said to be admissible $\text{mod}(H,\sigma)$ (with $\sigma : Q \to G$ a given section) and the representation $U$ square integrable $\text{mod}(H,\sigma)$, if the condition

$$\int_Q |\langle U(\sigma(g))\psi|\phi \rangle|^2d\hat{\mu}(g) < \infty, \forall \phi \in L^2(\mathbb{X}, d\nu)$$ (125)

holds, where $d\hat{\mu}$ is a measure on $Q$ “projected” from the left-invariant measure $d\mu$ on the whole $G$ (see [39] for more details on this projection procedure). Note that this more general definition of square integrability includes the previous one for the trivial subgroup $H = \{e\}$ and $\sigma$ the identity function. The notions of square integrability and admissibility $\text{mod}(H,\sigma)$ were introduced in [40] (see also [4]).
The scalar product \( \langle \Phi | L | \psi \Phi \rangle \) expands \( \varphi \) form an overcomplete set in \( L^2(X, d\nu) \).

The condition (125) could also be written as an “expectation value”

\[
0 < \int_Q |\langle U(\sigma(q))\psi|\phi_\sigma\rangle|^2 d\bar{\mu}(q) = \langle \phi | A_\sigma | \phi \rangle < \infty, \ \forall \phi \in L^2(X, d\nu),
\]

where \( A_\sigma = \int_Q \langle \psi_{\sigma(q)}|\psi_{\sigma(q)}\rangle d\bar{\mu}(q) \) is a positive, bounded, invertible operator. If the operator \( A_\sigma^{-1} \) is also bounded, then the set \( F_\sigma = \{ |\psi_{\sigma(q)}\rangle, q \in Q \} \) is called a frame, and a tight frame if \( A_\sigma \) is a positive multiple of the identity, \( A_\sigma = cI, c > 0 \).

To avoid domain problems in the following, let us assume that \( \psi \) generates a frame (i.e., that \( A_\sigma^{-1} \) is bounded). The Coherent State map is defined as the linear map

\[
T_\psi : L^2(X, d\nu) \longrightarrow L^2(Q, d\bar{\mu}),
\]

\[\phi \longmapsto T_\psi \phi \equiv \Phi_\psi, \quad (127)\]

with \( \Phi_\psi(q) = \frac{\langle \psi_{\sigma(q)}|\phi \rangle}{\sqrt{\bar{\mu}(q)}} \). Its range \( L^2_\psi(Q, d\bar{\mu}) \equiv T_\psi(L^2(X, d\nu)) \) is complete with respect to the scalar product \( (\Phi|\Phi')_\psi \equiv (\Phi | T_\psi A_\sigma^{-1} T_\psi^{-1} \Phi')_Q \) and \( T_\psi \) is unitary from \( L^2(X, d\nu) \) onto \( L^2_\psi(Q, d\bar{\mu}) \). Thus, the inverse map \( T_\psi^{-1} \) yields the reconstruction formula

\[
\phi = T_\psi^{-1} \Phi_\psi = \int_Q \Phi_\psi(q) A_\sigma^{-1} \psi_{\sigma(q)} d\bar{\mu}(q), \ \Phi_\psi \in L^2_\psi(Q, d\bar{\mu}),
\]

which expands \( \phi \) in terms of coherent states (wavelets) \( A_\sigma^{-1} \psi_{\sigma(q)} \) with wavelet coefficients \( \Phi_\psi(q) = \langle T_\psi | \phi \rangle(q) \). These formulas acquire a simpler form when \( A_\sigma \) is a multiple of the identity, as it is precisely the case considered in this article.

## C Orthonormality of Homogeneous Polynomials

In order to prove the orthonormality relations (60), we shall adopt the following decomposition for a matrix \( Z \in \mathbb{D}_4 \)

\[
Z = U_1 \Xi U_2^\dagger,
\]

where \( U_{1,2} \in U(2)/U(1)^2 \) [as in (60) with \( \beta_1 = \beta_2 = 0 \)] and \( \Xi = \text{diag}(\xi_1, \xi_2) / \xi_1, \xi_2 \in \mathbb{D}_1 \). This parametrization ensures that \( Z \in \mathbb{D}_4 \); in fact

\[
I - ZZ^\dagger = U_1 (I - \Xi \Xi^\dagger) U_1^\dagger > 0
\]

(129) since the eigenvalues are \( 1 - |\xi_{1,2}|^2 > 0 \).

Let us perform this change of variables in the invariant measure (61) of \( L^2_\hbar(\mathbb{D}_4, d\nu_\lambda) \). On the one hand, the Lebesgue measure on \( \mathbb{C}_4 \) can be written as:

\[
|dZ| = J|d\xi_1| |d\xi_2| ds(U_1) ds(U_2),
\]

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with $ds(U_{1,2})$ defined in (62) and $J = \frac{1}{2}((|\xi_1|^2 - |\xi_2|^2)^2$ is the Jacobian determinant. The Lebesgue measures $|d\xi_{1,2}|$ and $|dz_{1,2}|$ will be written in polar coordinates $\xi_k = \rho_k e^{i\theta_k}$ and $z_k = r_k e^{i\alpha_k}$, $k = 1, 2$. On the other hand, the weight factor in (64) adopts the form

$$
\det(I - ZZ^\dagger)^{\lambda-4} = ((1 - \rho_1^2)(1 - \rho_2^2))^{\lambda-4} \equiv \Omega(\rho),
$$

so that the invariant measure of $\mathcal{L}^2_h(\mathbb{D}_4, d\nu)$ reads:

$$
d\nu(Z, Z^\dagger) = c_\lambda J(\rho)\Omega(\rho)|d\xi_1||d\xi_2|ds(U_1)ds(U_2) = c_\lambda J(\rho)\Omega(\rho)\rho_1 d\theta_1 \rho_2 d\rho_2 d\theta_2 (1 + r_1^2)^{-2} r_1 d\rho_1 (1 + r_2^2)^{-2} r_2 d\rho_2 d\alpha_2,
$$

with $0 \leq \rho_{1,2} < 1$, $0 \leq r_{1,2} < \infty$, $0 \leq \theta_{1,2} < 2\pi$, $0 \leq \alpha_{1,2} < 2\pi$. Let us call

$$
\mathcal{N}_{j,m} \equiv \sqrt{\frac{2j + 1}{\lambda - 1} \left( \frac{m + \lambda - 2}{\lambda - 2} \right) \left( \frac{m + 2j + \lambda - 1}{\lambda - 2} \right)}
$$

the normalization constants of the basis functions (55). We want to evaluate:

$$
\langle \varphi_{q_1,q_2}^{j,m} | \varphi_{q_1',q_2'}^{j',m'} \rangle = \mathcal{N}_{j,m} \mathcal{N}_{j',m'} \int_{\mathbb{D}_4} d\nu(Z, Z^\dagger) \det(Z)\mathcal{D}_{q_1,q_2}^j(\Xi) \mathcal{D}_{q_1',q_2'}^{j'}(\Xi) \det(\Xi)^m \det(\Xi)^{m'}
$$

$$
\times \int_{\mathbb{S}^2} ds(U_1) \mathcal{D}_{q_1,q_2}^j(\Xi) \mathcal{D}_{q_1',q_2'}^{j'}(\Xi) \int_{\mathbb{S}^2} ds(U_2) \mathcal{D}_{q_2,q_2}^j(\Xi) \mathcal{D}_{q_2',q_2'}^{j'}(\Xi)
$$

Using determinant properties, Wigner’s $\mathcal{D}$-matrix properties (6) and (7), and the fact that $\det(U_{1,2}) = 1$ and $\Xi$ is diagonal, the previous expression can be restated as:

$$
\langle \varphi_{q_1,q_2}^{j,m} | \varphi_{q_1',q_2'}^{j',m'} \rangle = \sum_{q=-j}^j \sum_{q'=-j'}^j c_\lambda \int_{\mathbb{D}_4^2} J \Omega(\rho)|d\xi_1||d\xi_2| \mathcal{D}_{q,q}^j(\Xi) \mathcal{D}_{q',q'}^{j'}(\Xi) \det(\Xi)^m \det(\Xi)^{m'}
$$

$$
\times \int_{\mathbb{S}^2} ds(U_1) \mathcal{D}_{q_1,q_2}^j(\Xi) \mathcal{D}_{q_1',q_2'}^{j'}(\Xi) \int_{\mathbb{S}^2} ds(U_2) \mathcal{D}_{q_2,q_2}^j(\Xi) \mathcal{D}_{q_2',q_2'}^{j'}(\Xi)
$$

Let us start evaluating the first integral. For the diagonal matrix $\Xi$ we have that $\mathcal{D}_{q_1,q_2}^j(\Xi) = \delta_{q_1,q_2} \xi_1^{q_1} \xi_2^{q_2}$, and

$$
\mathcal{D}_{q,q}^j(\Xi) \mathcal{D}_{q',q'}^{j'}(\Xi) \det(\Xi)^m \det(\Xi)^{m'} = \xi_1^{j+q} \xi_2^{j'} \xi_1^{q} \xi_2^{q'} = \rho_1^{j+q+q'+m+m'} \rho_2^{j'+q'-q+m'-m} e^{i(j-j'+q-q'-m'-m)\theta_1} e^{i(j-j'+q-q'-m'-m)\theta_2}.
$$

Integrating out angular variables gives the restrictions

$$
\int_0^{2\pi} \int_0^{2\pi} \mathcal{D}_{q,q}^j(\Xi) \mathcal{D}_{q',q'}^{j'}(\Xi) \det(\Xi)^m \det(\Xi)^{m'} d\theta_1 d\theta_2 = 4\pi^2 \delta_{q,q'} \delta_{j+m,j'+m'} \rho_1^{2(j+q+m)} \rho_2^{2(j-q+m)}.
$$

Integrating the radial part:

$$
4\pi c_\lambda \int_0^1 \int_0^1 J(\rho) \Omega(\rho) \rho_1^{2(j+q+m)} \rho_2^{2(j-q+m)} \rho_1 d\rho_1 \rho_2 d\rho_2 = \frac{(j+m)^2 + (j+m+2q^2 + 1)\lambda - 5q^2 - 1}{\pi^2(\lambda - 1)(j+m+q+\lambda-1)(j+m-q+\lambda-1)} \equiv \mathcal{R}_{j,m}^q
$$

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Putting all together in (132) we have:

\[
\langle \varphi_{j,m}^{q_1,q_2} | \varphi_{j',m'}^{q_1',q_2'} \rangle = \delta_{j,j'} \delta_{m,m'} \sum_{q=-\min(j,j')}^{\min(j,j')} \mathcal{R}_{j+m, q}^q \\
\times \int_{S^2} ds(U_1) \mathcal{D}_{q_1,q}^j(U_1) \mathcal{D}_{q_1,q}^{j'}(U_1) \int_{S^2} ds(U_2) \mathcal{D}_{q_2,q}^j(U_2) \mathcal{D}_{q_2,q}^{j'}(U_2)
\]

The last two integrals are easily computable. Actually they are a particular case of the orthogonality properties of Wigner’s \( D \)-matrices. More explicitly:

\[
\int_{S^2} ds(U) \mathcal{D}_{q_1,q_2}^j(U) \mathcal{D}_{q_1,q_2}^{j'}(U) = \int_0^\infty \int_0^{2\pi} \frac{r dr d\alpha}{(1 + r^2)^2} \mathcal{D}_{q_1,q_2}^j(U) \mathcal{D}_{q_1,q_2}^{j'}(U) = \delta_{j,j'} \delta_{q_1,q_1'} \frac{\pi}{2j + 1}.
\]

Going back to (134) it results:

\[
\langle \varphi_{j,m}^{q_1,q_2} | \varphi_{j',m'}^{q_1',q_2'} \rangle = \delta_{j,j'} \delta_{m,m'} \delta_{q_1,q_1'} \delta_{q_2,q_2'} \left( \frac{N_{j,m}}{2j + 1} \right)^2 \sum_{q=-j}^{j} \pi^2 \mathcal{R}_{j+m, q}^q.
\]

Finally, taking into account the combinatorial identity:

\[
\sum_{q=-j}^{j} (\lambda - 1) \pi^2 \mathcal{R}_{j+m, q}^q = \frac{2j + 1}{(m+\lambda-2)(m+2j+\lambda-1)}
\]

and the explicit expression of the normalization constants \( N_{j,m} \), we arrive at the orthonormality relations (66).

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