Quantum D-branes and exotic smooth $\mathbb{R}^4$*

Torsten Asselmeyer-Maluga  
*German Aerospace center, Rutherfordstr. 2, 12489 Berlin  
torsten.asselmeyer-maluga@dlr.de

Jerzy Król  
*University of Silesia, Institute of Physics, ul. Uniwersytecka 4, 40-007 Katowice  
iriking@wp.pl

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In this paper, we present the idea that the formalism of string theory is connected with the dimension 4 in a new way, not covered by phenomenological or model-building approaches. The main connection is given by structures induced by small exotic smooth $\mathbb{R}^4$’s having intrinsic meaning for physics in dimension 4. We extend the notion of stable quantum D-branes in a separable noncommutative $C^*$ algebras over convolution algebras corresponding to the codimension-1 foliations of $S^3$ which are mainly connected to small exotic $\mathbb{R}^4$. The tools of topological K-homology and K-theory as well KK-theory describe stable quantum branes in the $C^*$ algebras when naturally extended to algebras. In case of convolution algebras, small exotic smooth $\mathbb{R}^4$’s embedded in exotic $\mathbb{R}^4$ correspond to a generalized quantum branes on the algebras. These results extend the correspondence between exotic $\mathbb{R}^4$ and classical D and NS branes from our previous work.

Keywords: exotic $\mathbb{R}^4$; quantum D-branes; qunatum D-branes in $C^*$-algebras.

1. Introduction

In this paper we further explore the relation between string theory and some 4-dimensional structures such that these structures do not appear as a result of usual compactification or model-buildings in string theory. These structures are rather involved in the mathematics of string theory but they are able to encode (in 4 dimensions) some dynamics of branes configuration and the geometry of certain string backgrounds. At the same time, these structures are of fundamental importance for 4-dimensional physics.

In our previous paper[13] we observed a correlation between D as well NS brane configurations in some backgrounds and the appearance of exotic smoothness on the topological $\mathbb{R}^4$. It is known that the $\mathbb{R}^4$ with standard smoothness structure

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is part of the string background. A variation of the brane configurations induce a change of the smoothness structure, i.e. one has to consider different smoothings of the $\mathbb{R}^4$. But this result is the unique feature of $\mathbb{R}^n$ holding only for $n = 4$ where a variety (actually a continuum) of smoothings of $\mathbb{R}^n$ must exist. In fact in any other dimension $n \neq 4$ there exists precisely a unique standard smooth $\mathbb{R}^n$. Physics corresponding to exotic smooth $\mathbb{R}^4$ has been gradually exhibited since the nineties.

In a recent series of papers, new aspects important for quantum gravity are being worked out directly.

The recognition of the role of exotic $\mathbb{R}^4$ in string theory relies so far on the following steps:

- Standard smooth $\mathbb{R}^4$ appears as a part of an exact string background;
- The process of changing the exotic smoothness on $\mathbb{R}^4$ is capable to encode the change in the configuration of specific D or NS branes;
- All exotic $\mathbb{R}^4$'s appearing in this setup are small exotic $\mathbb{R}^4$'s, i.e. a small exotic $\mathbb{R}^4$ embeds smoothly in the standard smooth $\mathbb{R}^4$ as open subsets.

Thus, string configurations can be expressed inherently in terms of 4-dimensional structures, i.e. exotic smooth $\mathbb{R}^4$'s are complex enough to encode some string configurations. Particularly all these phenomena disappear when one changes the smoothness to the standard one.

In this paper we consider the quantum regime of D-branes in string theory. Especially the correct setup for quantum branes is an open problem. However a natural proposal is the consideration of (non-commutative) $\mathbb{C}^\star$-algebras replacing (classical, submanifold-like) branes as well manifold spacetime. In the context of $\mathbb{C}^\star$-algebras there are many important counterparts of differential-geometric results including Poincaré duality, characteristic classes or the Riemann-Roch theorem. Especially one obtains a generalized formula for charges of quantum D-branes.

The basic technical ingredient of the analysis of small exotic $\mathbb{R}^4$'s is the relation between exotic (small) $\mathbb{R}^4$'s and non-cobordant codimension-1 foliations of $S^3$ as well groupes and wild embeddings as shown in [11]. The foliation of the 3-sphere is classified by the Godbillon-Vey class as element of the cohomology group $H^3(S^3, \mathbb{R})$. By using the $S^1$-gerbes it was possible to interpret the integral elements $H^3(S^3, \mathbb{Z})$ as characteristic classes of a $S^1$-gerbe over $S^3$. In the next section we will explain the whole complex of ideas more carefully. Then we present some facts and definitions of K-homology and KK-theory used to introduce stable D-branes as K-theory classes in terms of tachyons condensation. These K-theory classes can be naturally described by use of K-string theory (e.g. [5]). Furthermore there is a canonical interpretation for spectral triples including tachyon fields. This correspondence is further developed into the realm of noncommutative $\mathbb{C}^\star$-algebras, following e.g. [23,22], in section 3.2.

Now a natural interpretation of quantum stable D-branes is given by branes on the $\mathbb{C}^\star$-algebra. In fact a categorical description is necessary for an understanding of quantum D-branes: objects are quantum D-branes and the morphisms in the category are KK-theory classes. Then in section 4 we explore the notion of stable
D-branes in the convolution non-commutative algebra of the foliations representing exotic $\mathbb{R}^4$’s. In section 4.2 we establish the (partial) correspondence between stable D-branes as above and the net of exotic smooth $\mathbb{R}^4$’s embedded in some exotic $\mathbb{R}^4$. A discussion of the results closes the paper.

2. Exotic $\mathbb{R}^4$ and codimension-one foliations of the 3-sphere

The main line of the topological argumentation can be briefly described as follows:

(1) In Bizacas exotic $\mathbb{R}^4$ one starts with the neighborhood $N(A)$ of the Akbulut cork $A$ in the K3 surface $M$. The exotic $\mathbb{R}^4$ is the interior of $N(A)$.

(2) This neighborhood $N(A)$ decomposes into $A$ and a Casson handle representing the non-trivial involution of the cork.

(3) From the Casson handle we construct a grope containing Alexanders horned sphere.

(4) Akbuluts construction gives a non-trivial involution, i.e. the double of that construction is the identity map.

(5) From the grope we get a polygon in the hyperbolic space $\mathbb{H}^2$.

(6) This polygon defines a codimension-1 foliation of the 3-sphere inside of the exotic $\mathbb{R}^4$ with an wildly embedded 2-sphere, Alexanders horned sphere $A$.

(7) Finally we get a relation between codimension-1 foliations of the 3-sphere and exotic $\mathbb{R}^4$.

Now we will explain the details in this construction (see also 11).

An exotic $\mathbb{R}^4$ is a topological space with $\mathbb{R}^4$–topology but with a different (i.e. non-diffeomorphic) smoothness structure than the standard $\mathbb{R}^4_{std}$ getting its differential structure from the product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The exotic $\mathbb{R}^4$ is the only Euclidean space $\mathbb{R}^n$ with an exotic smoothness structure. The exotic $\mathbb{R}^4$ can be constructed in two ways: by the failure to arbitrarily split a smooth 4-manifold into pieces (large exotic $\mathbb{R}^4$) and by the failure of the so-called smooth h-cobordism theorem (small exotic $\mathbb{R}^4$). Here we will use the second method.

Consider the following situation: one has two topologically equivalent (i.e. homeomorphic), simple-connected, smooth 4-manifolds $M, M'$, which are not diffeomorphic. There are two ways to compare them. First one calculates differential-topological invariants like Donaldson polynomials 27 or Seiberg-Witten invariants 1. But there is another possibility: It is known that one can change a manifold $M$ to $M'$ by using a series of operations called surgeries. This procedure can be visualized by a 5-manifold $W$, the cobordism. The cobordism $W$ is a 5-manifold having the boundary $\partial W = M \sqcup M'$. If the embedding of both manifolds $M, M'$ in to $W$ induces homotopy-equivalences then $W$ is called an h-cobordism. Furthermore we assume that both manifolds $M, M'$ are compact, closed (no boundary) and simply-connected. As Freedman 29 showed a h cobordism implies a homeomorphism, i.e. h-cobordant and homeomorphic are equivalent relations in that case. Furthermore, for that case the mathematicians 11 are able to prove a structure theorem for such
h-cobordisms: Let $W$ be a h-cobordism between $M, M'$. Then there are contractable submanifolds $A \subset M, A' \subset M'$ together with a sub-cobordism $V \subset W$ with $\partial V = A \sqcup A'$, so that the h-cobordism $W \setminus V$ induces a diffeomorphism between $M \setminus A$ and $M' \setminus A'$. Thus, the smoothness of $M$ is completely determined (see also $2, 3$) by the contractible submanifold $A$ and its embedding $A \hookrightarrow M$ determined by a map $\tau : \partial A \to \partial A$ with $\tau \circ \tau = \text{id}_{\partial A}$ and $\tau \neq \pm \text{id}_{\partial A}$ ($\tau$ is an involution). One calls $A$, the Akbulut cork. According to Freedman [29], the boundary of every contractible 4-manifold is a homology 3-sphere. This theorem was used to construct an exotic $\mathbb{R}^4$. Then one considers a tubular neighborhood of the sub-cobordism $V$ between $A$ and $A'$. The interior $\text{int}(V)$ (as open manifold) of $V$ is homeomorphic to $\mathbb{R}^4$. If (and only if) $M$ and $M'$ are homeomorphic, but non-diffeomorphic 4-manifolds then $\text{int}(V) \cap M$ is an exotic $\mathbb{R}^4$. As shown by Bizaca and Gompf [17, 18] one can use $\text{int}(V)$ to construct an explicit handle decomposition of the exotic $\mathbb{R}^4$. We refer for the details of the construction to the papers or to the book [32]. The idea is simply to use the cork $A$ and add some Casson handle $CH$ to it. The interior of this construction is an exotic $\mathbb{R}^4$. Therefore we have to consider the Casson handle and its construction in more detail. Briefly, a Casson handle $CH$ is the result of attempts to embed a disk $D^2$ into a 4-manifold. In most cases this attempt fails and Casson [24] looked for a substitute, which is now called a Casson handle. Freedman [29] showed that every Casson handle $CH$ is homeomorphic to the open 2-handle $D^2 \times \mathbb{R}^2$ but in nearly all cases it is not diffeomorphic to the standard handle $[30, 31]$. The Casson handle is built by iteration, starting from an immersed disk in some 4-manifold $M$, i.e. a map $D^2 \to M$ with injective differential. Every immersion $D^2 \to M$ is an embedding except on a countable set of points, the double points. One can kill one double point by immersing another disk into that point. These disks form the first stage of the Casson handle. By iteration one can produce the other stages. Finally consider not the immersed disk but rather a tubular neighborhood $D^2 \times D^2$ of the immersed disk, called a kinky handle, including each stage. The union of all neighborhoods of all stages is the Casson handle $CH$. So, there are two input data involved with the construction of a $CH$: the number of double points in each stage and their orientation $\pm$. Thus we can visualize the Casson handle $CH$ by a tree: the root is the immersion $D^2 \to M$ with $k$ double points, the first stage forms the next level of the tree with $k$ vertices connected with the root by edges etc. The edges are evaluated using the orientation $\pm$. Every Casson handle can be represented by such an infinite tree.

The main idea is the construction of a grope, an infinite union of surfaces with non-vanishing genus, from the Casson handle. But the grope can be represented by a sequence of polygons in the two-dimensional hyperbolic space $\mathbb{H}^2$. This sequence of polygons is replaced by one polygon with the same area. From this polygon we can construct a codimension-one foliation on the 3-sphere as done by Thurston [40]. This 3-sphere is part of the boundary $\partial A$ of the Akbulut cork $A$. Furthermore one can show that the codimension-one foliation of the 3-sphere induces a codimension-one
foliation of $\partial A$ so that the area of the corresponding polygons agree.

Thus we are able to obtain a relation between an exotic $\mathbb{R}^4$ (of Bizaca as constructed from the failure of the smooth h-cobordism theorem) and codimension-one foliation of the $S^3$. Two non-diffeomorphic exotic $\mathbb{R}^4$ implying non-cobordant codimension-one foliations of the 3-sphere described by the Godbillon-Vey class in $H^3(S^3, \mathbb{R})$ (proportional to the area of the polygon). This relation is very strict, i.e. if we change the Casson handle then we must change the polygon. But that changes the foliation and vice versa. Finally we obtained the result:

The exotic $\mathbb{R}^4$ (of Bizaca) is determined by the codimension-1 foliations with non-vanishing Godbillon-Vey class in $H^3(S^3, \mathbb{R})$ of a 3-sphere seen as submanifold $S^3 \subset \mathbb{R}^4$. We say: the exoticness is localized at a 3-sphere inside the small exotic $\mathbb{R}^4$.

3. Towards quantum D-branes via K-theory

In this and subsequent sections we want to show that D-branes of string theory are related to exotic smooth $\mathbb{R}^4$'s also beyond the semi-classical limit, i.e. in the quantum regime of the theory where one should deal rather with quantum branes. What are quantum branes, is still in general an open and hard problem. One appealing proposition, relevant for this paper, is to consider branes in noncommutative spacetimes rather than on commutative manifolds or orbifolds. This leads to abstract D-branes in general noncommutative separable $C^*$ algebras as counterparts for quantum D-branes. The way from D-branes as submanifolds or K-homology classes and spaces to K-theory cycles, spectral triples and $C^*$ algebras is presented in the following subsections.

3.1. D-branes on spaces: K-homology and KK-theory

The description of systems of stable Dp-branes of IIA,B string theories via K-theory of topological spaces can be extended toward the branes in noncommutative $C^*$ algebras. A direct string representation of the algebraic and K-theoretic ideas can be best explained in K-matrix string theory where tachyons are elements of the spectral triple representing the noncommutative geometry of the world-volumes for the configurations of branes.

First let us consider the case of a vanishing $H$-field. The charges of D-branes are classified by suitable K-theory groups, i.e. $K^0(X)$ in IIB and $K^1(X)$ in IIA string theories, where $X$ is the background manifold. On the other hand, world-volumes of Dp-branes correspond to the cycles of K-homology groups, $K_1(X), K_0(X)$, which are dual to the K theory groups. Let us see how K-cycles correspond to the configurations of D-branes.

A K-cycle on $X$ is a triple $(M, E, \phi)$ where $M$ is a compact $\text{Spin}^c$ manifold without boundary, $E$ is a complex vector bundle on $M$ and $\phi : M \to X$ is a continuous map. The topological K-homology $K_* (X)$ is the set of equivalence classes of the triples $(M, E, \phi)$ respecting the following conditions:
(i) \((M_1, E_1, \phi_1) \sim (M_2, E_2, \phi_2)\) when there exists a triple (bordism of the triples) \((M, E, \phi)\) such that \((\partial M, E|_{\partial M}, \phi|_{\partial M})\) is isomorphic to the disjoint union \((M_1, E_1, \phi_1) \cup (-M_2, E_2, \phi_2)\) where \(-M_2\) is the reversed Spin\(^c\) structure of \(M_2\) and \(M\) is a compact Spin\(^c\) manifold with boundary.

(ii) \((M, E_1 \oplus E_2, \phi) \sim (M, E_1, \phi) \cup (M, E_2, \phi)\),

(iii) Vector bundle modification \((M, E, \phi) \sim (\hat{M}, \hat{H} \otimes \rho^*(E), \phi \circ \rho)\).

\(\hat{M}\) is an even dimensional sphere bundle on \(M\), \(\rho: \hat{M} \to M\) projection, \(\hat{H}\) is a vector bundle on \(\hat{M}\) which gives the generator of \(K(S^{2p}_q) = \mathbb{Z}\) on every \(S^{2p}_q\) over each \(q \in M\).

The topological K-homology defined above has an abelian group structure where the sum is the disjoint union of cycles. The triples \((M, E, \phi)\) with \(M\) of even dimension determines \(K_0(X)\). Similarly, \(K_1(X)\) corresponds to odd dimensions of \(M\). Thus \(K_\ast(X)\) decomposes into a direct sum of abelian groups:

\[K_\ast(X) = K_0(X) \oplus K_1(X)\]

K-homology is dual to K-theory and the decomposition of \(K_\ast(X)\) is a direct consequence of Bott periodicity (see 15).

Now one can interpret the cycles \((M, E, \phi)\) as D-branes \(^{33}\). \(M\) is the worldvolume of the brane, \(E\) the Chan-Paton bundle on it and \(\phi\) gives the embedding of the brane into the (background) spacetime \(X\). Moreover, \(M\) has to wrap the Spin\(^c\) manifold \(^{28}\) and \(K_0(X)\) classifies stable D-branes configurations in IIB, and \(K_1(X)\) in IIA, string theories. The equivalences of K-cycles as formulated in the conditions (i)-(iii) correspond to natural relations for D-branes \(^{5}\). \(^{22}\).

The topological K-homology theory above can be obtained analytically (analytic K-homology theory). This theory is special, commutative, case of the following construction on general \(C^*\) algebras \(^{5}\). A Fredholm module over a \(C^*\) algebra \(A\) is a triple \((\mathcal{H}, \phi, F)\) such that

1. \(\mathcal{H}\) is a separable Hilbert space,
2. \(\phi\) is a * homomorphism between \(C^*\) algebras \(A\) and \(\mathcal{B}(\mathcal{H})\) of bounded linear operators on \(\mathcal{H}\),
3. \(F\) is self-adjoint operator in \(\mathcal{B}(\mathcal{H})\) satisfying

\[F^2 - id \in K(\mathcal{H}), \quad [F, \phi(a)] \in K(\mathcal{H}) \text{ for every } a \in \mathcal{A}\]

where \(K(\mathcal{H})\) are compact operators on \(\mathcal{H}\). Now let us see how a Fredholm module \((\mathcal{H}, \phi, F)\) describes certain configuration of IIA K-matrix string theory related to D-branes. To this end we consider the operators of the K-matrix theory \(\Phi^0, ..., \Phi^9\) (infinite matrices) acting on the Hilbert space \(\mathcal{H}\) as generating the \(C^*\) algebra \(A_M\) \(^{5}\). In the case of commuting \(\Phi^\mu\), hence commutative \(A_M\), we have the following correspondence (explaining the index \(M\) in \(A_M\)):


• Every commutative $C^*$ algebra is isomorphic to the algebra of continuous complex functions vanishing at infinity $C(M)$ on some locally compact Hausdorff space $M$ (Gelfand-Naimark theorem and Gelfand representation). A point $x \in M$ is determined by a character of $A_M$ which is a * homomorphism $\phi_x : A_M \to \mathbb{C}$.

• $M$ serves as a common spectrum for $\Phi^0, \ldots, \Phi^9$ and the choice of a point in $M$ represented as the eigenvalue of $\Phi^\mu$ fixes the position of the non BPS instanton along $x^\mu$.

• In this way $M$ is covered by the positions of infinite many non BPS instantons and serves as the world-volume of some higher dimensional D brane.

Now let us explain the role of the tachyon $T$. $T$ is a self-adjoint unbounded operator acting on the Chan-Paton Hilbert space $\mathcal{H}$. $A_M$ is a $C^*$ unital algebra generated by $\Phi^0, \ldots, \Phi^9$ which can be now a noncommutative algebra. The corresponding geometry of the world-volume $M$ would be a noncommutative geometry (in the sense of Connes) and given by some spectral triple. The spectral triple is in fact $(\mathcal{H}, A, T)$ which means that the following conditions are fulfilled:

$$T - \lambda \in \mathbf{K}(\mathcal{H})$$
for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $[a, T] \in \mathbf{B}(\mathcal{H})$ for every $a \in A_M$

These conditions are fulfilled in our case of K-matrix string theory for a tachyon field $T$, Chan-Paton Hilbert space $\mathcal{H}$ and $C^*$ algebra $A_M$ generated by $\Phi^\mu$. Thus the natural extension of the spacetime manifold as well D-brane world-volumes toward a noncommutative algebra and noncommutative world-volumes of branes (represented by spectral triples) can be described by (see e.g. 5):

1. Fix the (spacetime) $C^*$ algebra $\mathcal{A}$;
2. A * homomorphism $\phi : \mathcal{A} \to \mathbf{B}(\mathcal{H})$ generates the embedding of the D-brane world-volume $M$ and its noncommutative algebra $A_M$ as $A_M := \phi(\mathcal{A})$;
3. D-branes embedded in a spacetime $\mathcal{A}$ are represented by the spectral triple $(\mathcal{H}, A_M, T)$;
4. Equivalently, a D-brane in $\mathcal{A}$ is given by an unbounded Fredholm module $(\mathcal{H}, \phi, T)$.

Thus the classification of stable D-branes in $\mathcal{A}$ is given by the classification of Fredholm modules $(\mathcal{H}, \phi, T)$ using analytical K-homology. In the particular case of commutative $C^*$ algebras based on the isomorphism of the topological and analytical K-homology groups, we have the classification of stable D-branes in terms of K-cycles, as was already discussed. In terms of K-matrix string theory we can say that stable configurations of D-instantons determine the stable higher dimensional D-branes which are K-homologically classified as above.5

Now let us turn to a more general situation than K-string theory of D-instantons, i.e. backgrounds given by non-BPS Dp-branes or non-BPS Dp-Dp-branes in type II string theory. Then the stable configurations of Dq-branes are classified by gen-
eralized K-theory namely Kasparov KK-theory \[5\]. As in the case of D-branes in a \( C^* \) algebra \( \mathcal{A} \) corresponding to Fredholm modules, one defines an odd Kasparov module \( (\mathcal{H}_B, \phi, T) \), where \( \mathcal{H}_B \) is an countable Hilbert module over the \( C^* \)-algebra \( B \), by

- a \( \ast \)-homomorphism from \( \mathcal{A} \) to the \( C^* \) algebra of bounded linear operators on \( \mathcal{H}_B \), \( \phi : \mathcal{A} \rightarrow B(\mathcal{H}_B) \);
- a self-adjoint operator \( T \) from \( B(\mathcal{H}_B) \) satisfying:

\[
T^2 - 1 \in K(\mathcal{H}_B) \text{ and } [T, \phi(a)] \in K(\mathcal{H}_B) \text{ for every } a \in \mathcal{A},
\]

where \( K(\mathcal{H}_B) \) is \( B \otimes K \). \( (\mathcal{H}_B, \phi, T) \) is in fact a family of Fredholm modules on the algebra \( B \). When \( B = \mathbb{C} \) we have an ordinary Fredholm module as before. The homotopy equivalence classes of odd Kasparov modules \( (\mathcal{H}_B, \phi, T) \) determine elements of \( KK^1(\mathcal{A}, B) \). Also one defines even Kasparov classes \( KK^0(\mathcal{A}, B) = KK(\mathcal{A}, B) \) as homotopy equivalence classes of the triples \( (\mathcal{H}_B^{(0)} \oplus \mathcal{H}_B^{(1)}, \phi^{(0)} \oplus \phi^{(1)}, \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}) \). A natural \( \mathbb{Z}_2 \) grading appears due to the involution \( \mathcal{H}_B^{(0)} \oplus \mathcal{H}_B^{(1)} \rightarrow \mathcal{H}_B^{(0)} \oplus -\mathcal{H}_B^{(1)} \).

Now one obtains the classification pattern for branes in spaces. Let us introduce non-BPS unstable \( D_p \)-branes wrapping the \( p+1 \)-dimensional world-volume \( B \). Then stable \( D_q \)-branes configurations embedded in a space \( A \) transverse to \( B \) corresponding to (are classified by) the classes of \( KK^1(A, B) \) (we identify the commutative algebras \( C(A), C(B) \) with \( A, B \) correspondingly). Similarly, given non-BPS unstable \( D_p \)-\( \overline{D}_p \)-branes system, then stable \( D_q \)-branes embedded in \( A \) transverse to \( B \) (\( p+1 \)-dimensional world-volumes) are classified by elements of \( KK^0(A, B) \). The case of even \( KK^0(A, B) \) contains the \( \mathbb{Z}_2 \) grading as corresponding to the Chan-Paton indices of \( D_p \) and \( \overline{D}_p \)-branes.

### 3.2. D-branes on separable \( C^* \) algebras and KK-theory

Thus the classification of D-branes in a spacetime manifold is given by KK-theory as sketched in the previous subsection. This can be extended over noncommutative spacetimes and noncommutative D-branes both represented by separable \( C^* \) algebras as already can be seen from the appearance of tools of KK-theory. First let us reformulate the “classic” case of spaces in a way allowing this extension \[9\].

In case of type II superstring theory, let \( X \) be a compact part of a spacetime manifold, i.e. \( X \) is a compact spin\(^c \) manifold again with no background \( H \)-flux. As we saw, a flat D-brane in \( X \) is a Baum-Douglas K-cycle (\( W, E, f \)). Here \( f : W \hookrightarrow X \) is the embedding of the closed spin\(^c \) submanifold \( W \) of \( X \) and \( E \rightarrow W \) is a complex vector bundle with connection (Chan-Paton gauge bundle). It follows from the Baum-Douglas construction that \( E \) determines the stable class in the K-theory group \( K^0(W) \) and all K-cycles form an additive category under disjoint union. Now, the set of all K-cycles classes up to a kind of gauge equivalence as in Baum-Douglas construction, gives the K-homology of \( X \). This K-homology is also the
set of stable homotopy classes of Fredholm modules taken over the commutative $C^*$ algebra $C(X)$ of continuous functions on $X$. This defines the correspondence (isomorphism) between a K-cycle $(W, E, f)$ and the unbounded Fredholm module $(\mathcal{H}, \rho, D^W_E)$. Here $\mathcal{H}$ is the separable Hilbert space of square integrable spinors on $W$ taking values in the bundle $E$, i.e. $L^2(W, S \otimes E)$, $\rho : C(X) \to \mathcal{B}(\mathcal{H})$ is the representation of the $C^*$ algebra $C(X)$ in $\mathcal{H}$ such that $C(X) \ni g \to a_{g\circ f} \in \mathcal{B}(\mathcal{H})$ where $a_{g\circ f}$ is the operator of point-wise multiplication of functions in $L^2(W, S \otimes E)$ by the function on $W$, $g \circ f$, and $f : W \to X$. $D^W_E$ is the Dirac operator twisted by $E$ corresponding to the spin$^c$ structure on $W$. Given the K-theory class of the Chan-Paton bundle $E$, i.e. $[E] \in K^0(W)$, then the dual K-homology class of a D-brane, $[W, E, f]$ uniquely determines $[E]$. In that way D-branes determine K-homology classes on $X$ which are dual to K-theory classes from $K^r(W)$ where $r$ is the transversal dimension for the brane world-volume $W$. This K-theory class is derived from the image of $[E] \in K^0(W)$ by the Gysin K-theoretic map $f$. As we discussed already, the odd and even classes of K-homology $K_*(X)$ correspond to the parity of the dimension of $W$. The K-cycle $(W, E, f)$ corresponds to a Dp-brane and its gauge equivalence is given by Baum-Douglas construction using the conditions (i)-(iii) in section 3.1. Thus we have $22$.

Fact 1: There is a one-to-one correspondence between flat D-branes in $X$, modulo Baum-Douglas equivalence, and stable homotopy classes of Fredholm modules over the algebra $C(X)$.

In the presence of a non-zero B-field on $X$, which is a $U(1)$-gerbe with connection represented by the characteristic class in $H^3(X, \mathbb{Z})$ $^{22}$, one can define twisted D-brane on $X$ as $^{22}$.

**Definition 3.1.**

A twisted D-brane in a B-field $(X, H)$ is a triple $(W, E, \phi)$, where $\phi : W \to X$ is a closed, embedded oriented submanifold with $\phi^* H = W_3(W)$, and $E$ is the Chan-Paton bundle on $W$, i.e. $E \in K^0(W)$, and $W_3(W)$ is the 3-rd integer Stiefel-Whitney class of the normal bundle of $W$, $W_3(W) \in H^3(W, \mathbb{Z})$.

By the cancellation of the Freed-Witten anomaly, the condition in the definition is really necessary. Let $H \in H^3(X, \mathbb{Z})$ represents the NS-NS $H$-flux. Since $W_3(W)$ is the obstruction to the existence of a spin$^c$ structure on $W$ $^{34}$, in the case of $W_3(W) = 0$ one has flat D-branes in $X$. Thus equivalence classes of twisted D-branes on $X$ are represented by twisted topological K-homology $K_*(X, H)$ which is dual to the twisted K-theory $K^*(X, H)$.

Now, in the case of $S^3$ and integral classes $H \in H^3(S^3, \mathbb{Z})$, one has some exotic $\mathbb{R}^4$-s which is determined by the class $H$, when $S^3$ is a part of the boundary of the Akbulut cork $^{12}$. This is the same class $H$ which twists the K-theory leading to $K^*(S^3, H)$. We can also represent the $U(1)$ gerbes with connection on $S^3$, by the bundles $\mathcal{E}_H$ of algebras over $S^3$, such that the sections of the bundle $\mathcal{E}_H$ define the noncommutative, twisted algebra $C_0(X, \mathcal{E}_H).$ The Dixmier-Douady class of $\mathcal{E}_H$, $\delta_H(\mathcal{E}_H)$, is again $H \in H^3(S^3, \mathbb{Z})$ $^{10}$. $^{16}$. $^{38}$. 

The important relation is the following (22, Proposition 1.15):

Fact 2: There is a one-to-one correspondence between twisted D-branes in \((X, H)\) and stable homotopy classes of Fredholm modules over the algebra \(C_0(X, \mathcal{E}_H)\).

Since the algebra \(C_0(X, \mathcal{E}_H)\) determines its stable homotopy classes of the Fredholm modules on it, then in the case \(X = S^3\) one has the correspondence:

A. Let the exotic smooth \(\mathbb{R}^4\)'s are determined by the integral third classes \(H \in H^3(S^3, \mathbb{Z})\). Then, these exotic smooth \(\mathbb{R}^4\)'s correspond one-to-one to the sets of twisted D-branes in \((S^3, H)\), provided \(S^3\) is a part of the boundary of the Akbulut cork.

Thus, given the complete collection of twisted D-branes in \((S^3, H)\), which take values in \(K_*(S^3, H)\), one can determine, in principle, the corresponding exotic \(\mathbb{R}^4\). This exotic \(\mathbb{R}^4_H\) corresponds to the class \([H] \in H^3(S^3)\) and the class \(H\) twists the K-homology as dual to the twisted K-theory \(K^*(S^3, H)\). In the following we try to convince the reader that the correspondence of D-branes to 4-exotics can be extended to more general cases with a closer relation.

Remembering that \(S^3 \subset \mathbb{R}^4\) is a part of the Akbulut cork of the exotic structure, our previous observation can be restated as:

B. The change of the exotic smoothness of \(\mathbb{R}^4, \mathbb{R}^4_{H_1} \rightarrow \mathbb{R}^4_{H_2}, H_1, H_2 \in H^3(S^3, \mathbb{Z}), H_1 \neq H_2\), corresponds to the change of the curved backgrounds \((S^3, H_1) \rightarrow (S^3, H_2)\) hence the sets of stable D-branes.

This motivates the formulation:

C. Some small exotic smoothness appearing on \(\mathbb{R}^4, \mathbb{R}^4_{H_1}\), can destabilize (or stabilize) D-branes in \((S^3, H_2)\), where \(S^3 \subset \mathbb{R}^4\) lies at the boundary of the Akbulut cork of \(\mathbb{R}^4_{H_1}\). We say that D-branes configurations in \((S^3, H_2)\) are 4-exotic-sensitive.

Next we extend the formalism of D-branes in spaces to quantum D-branes in general \(C^*\) algebras including the correspondence described above. There were developed recently impressive counterparts of a variety of topological, geometrical and analytical results, like Poincaré duality, characteristic classes and the Riemann-Roch theorem, in \(C^*\) algebras. Besides the generalized formula for charges of quantum D-branes in a noncommutative separable \(C^*\) algebrabras was worked out \(23, 22\). Thus one obtains a suitable framework for considering the quantum regime of D-branes. Therefore we will try to find a relation to 4-exotics also in this quantum regime of D-branes.

Following \(5, 23, 22\), one can choose an obvious substitute for the category of quantum D-branes: the category of separable \(C^*\) algebras where the morphisms are elements of some KK-theory group. This means that for a pair \((\mathcal{A}, \mathcal{B})\) of separable \(C^*\) algebras the morphisms \(h : \mathcal{A} \rightarrow \mathcal{B}\) is lifted to the element of the group \(KK(\mathcal{A}, \mathcal{B})\). Thus we can consider a generalized D-brane in a separable \(C^*\) algebra \(\mathcal{A}\) as corresponding to the lift \(h! : \mathcal{A} \rightarrow \mathcal{B}\) where \(\mathcal{B}\) represents a quantum D-brane.

More precisely following \(22\), let us consider a subcategory \(\mathcal{C}\) of the category of \(C^*\) separable algebras and their morphisms, which consists of strongly K-oriented morphisms. Therefore there exists a contravariant functor \(! : \mathcal{C} \rightarrow KK\) such that
$C \ni f : A \to B$ is mapped to $f! \in KK_d(B, A)$, here $KK$ is the category of separable $C^*$ algebras with KK classes as morphisms. Strongly K-oriented morphisms and the functor $!$ are subjects to the following conditions:

1. Identity morphism $id_A : A \to A$ is strongly K-oriented (SKKO) for every separable $C^*$ algebra $A$ and $(id_A)! = 1_A$. Also, the 0-morphism $0_A : A \to A$ is SKKO and $(0_A)! = 0 \in KK(A, A)$.

2. If $f : A \to B$ is SKKO then $f^\circ : A^\circ \to B^\circ$ is also SKKO, and $(f^\circ)! = (f!)^\circ$ where $A^\circ$ is the opposite $C^*$ algebra to $A$, i.e. the algebra having the same underlying vector space but the reversed product.

3. Any morphism $f : A \to B$ is SKKO, provided $A$ and $B$ are strong Poincaré dual (PD) algebras. Then $f!$ is determined as:

$$f! = (-1)^d_A \Delta_A \otimes_{A^0} [f^0] \otimes_{B^0} \Delta_B$$

here $[f]$ is the class of $f : A \to B$ in $KK(A, B)$. $\Delta_A$ is the fundamental class in $KK_d(A \otimes A^0, C) = K^{d_A}(A \otimes A^0)$, $\Delta_A^\vee$ its dual class in $KK_{-d_A}(C, A \otimes A^0) = K_{-d_A}(A \otimes A^0)$ which exists by strong PD [23].

K-orientability was introduced in its original form, by A. Connes to define the analogue of the spin$^c$ structure for noncommutative $C^*$ algebras (see also [23] and the next section). The formulation of K-orientability and strong PD $C^*$ algebras are crucial ingredients of noncommutative versions of Riemann-Roch theorem, Poincaré-like dualities, Gysin K-theory map and allows to formulate a very general formula for noncommutative D-brane charges [22,23,39]. Let us notice that if both $A$ and $B$ are PD algebras then any morphism $f : A \to B$ is K-oriented and the K-orientation for $f$ is given in (1).

In the special case of the proper smooth embedding $f : W \to X$ of codimension $d$ between the smooth compact manifolds $W, X$, we choose the normal bundle $\tau$ over $W$ to be spin$^c$, where $\tau$ is given by $TX = \tau \oplus f_* (TW)$. When $X$ is also spin$^c$ then the spin$^c$ condition on $\tau$ for vanishing $H$-flux in type II string theory formulated on $X$ is the Freed-Witten anomaly cancellation condition [23]. In this case any D-brane in $X$, given by the triple $(W, E, f)$, determines the KK-theory element $f! \in KK(C(W), C(X))$. The construction of K-orientation $f : M \to X$, between smooth compact manifolds, can be extended to smooth proper maps which are not necessary embeddings. Thus the general condition for K-orientability gives the correct analogue for stable D-branes in $C^*$ algebras.

**Definition 3.2.**

A generalized stable quantum D-brane on a separable $C^*$ algebra $A$, represented by a separable $C^*$ algebra $B$, is given by the strongly K-oriented homomorphism of $C^*$ algebras, $h_B : A \to B$. The K-orientation means that there is the lift $(h_B)! \in KK(B, A)$ where $!$ fulfills the functoriality condition as in (1).

This approach to quantum D-branes is a natural extension of the string formalism over $C^*$ algebras replacing spaces and branes, which is currently a conjectural
framework. This framework exceeds both the dynamical Seiberg-Witten limit of superstring theory (inducing noncommutative brane world-volumes) and the geometrical understanding of branes, and places itself rather in a deep quantum regime of the theory \(^3\). On the other hand in such a formal quantum limit of string theory one can observe the relation with 4-dimensional exotic open smooth structures, which relies on the natural relation of exotic \(R^4\) with \(C^*\) algebras of the foliations.

4. Exotic \(R^4\) and branes in \(C^*\) algebras

4.1. Exotic \(R^4\) and stable D-branes configurations on foliated manifolds

Now we want to tackle the problem to describe stable states of D-branes in a more general geometry than used for spaces, namely the geometry of foliated manifolds. The interesting case for us is a codimension-1 foliation of the 3-sphere \(S^3\) admitting a noncommutative geometry as we will show now. In general, to every foliation \((V, F)\) one can associate its noncommutative \(C^*\) algebra \(C^*(V, F)\), on the other hand a foliation determines its holonomy groupoid \(G\) and the topological classifying space \(BG\). Both cases, topological K-homology of \(G\) and \(C^*\) algebraic K-theory, are in fact dual. Analogously to our previous discussion of branes as K-cycles on \(X\), let us start with K-homology of \(G\) and define D-branes as K-cycles in \(G\):

A K-cycle on a foliated geometry \(X = (V, F)\) is a triple \((M, E, \phi)\) where \(M\) is a compact manifold without boundary, \(E\) is a complex vector bundle on \(M\) and \(\phi: M \to BG\) is a smooth K-oriented map. Due to the K-orientability in the presence of canonical G-bundle \(\tau\) on \(BG\), the condition of Spin\(^c\) structure on \(M\) is lifted to the Spin\(^c\) structure on \(TM \oplus \phi^*\tau\)\(^\[25\]\).

The topological K-homology \(K_{*, \tau}(X) = K_{*, \tau}(BG)\) of the foliation \((V, F)\) is the set of equivalence classes of the above triples, where the equivalence respects the following conditions:

(i) \((M_1, E_1, \phi_1) \sim (M_2, E_2, \phi_2)\) when there is a triple (bordism of the triples) \((M, E, \phi)\) such that \((\partial M, E_{|\partial M}, \phi_{|\partial M})\) is isomorphic to the disjoint union \((M_1, E_1, \phi_1) \cup (-M_2, E_2, \phi_2)\) where \(-M_2\) is the reversed Spin\(^c\) structure of \(TM_2 \oplus \phi_2^*\tau\) and \(M\) is a compact manifold with boundary.

(ii) \((M, E_1 \oplus E_2, \phi) \sim (M, E_1, \phi) \cup (M, E_2, \phi)\).

(iii) Vector bundle modification \((M, E, \phi) \sim (\tilde{M}, \tilde{H} \otimes \rho^*(E), \phi \circ \rho)\) similarly as in the case of manifolds.

As in the case of spaces (manifolds) and the corresponding K-homology groups representing stable D-branes of type II superstring theory (see section 3.1), we generalize stable D-branes as being represented by the above triples in case of the geometry of foliated manifolds.

**Theorem 4.1.**

The class of generalized stable D-branes on the \(C^*\) algebra \(C^*(S^3, F_1)\) (of the
codimension 1 foliation of $S^3$) corresponding to the $K$-homology classes $K_{*, \tau}(S^3/F)$ determines an invariant of exotic smooth $\mathbb{R}^4$.

The result follows from the fact that $K_{*, \tau}(S^3/F)$ is isomorphic to $K_{*, \tau}(BG)$ and this determines a class of stable D-branes in $(S^3, F)$. The foliations $(S^3, F)$ correspond to different smoothings on $\mathbb{R}^4$.

4.2. The net of exotic $\mathbb{R}^4$'s and quantum D-branes in $C^*(S^3, F)$

The extension of string theory and D-branes to general noncommutative separable $C^*$ algebras can be considered as an approach to quantum D-branes where D-branes are also represented by noncommutative separable $C^*$ algebras. A category of D-branes in a quantum regime, is the category of separable $C^*$ algebras where the morphisms are elements of KK-theory groups. For a pair $(A, B)$ of separable $C^*$ algebras the morphisms $h : A \to B$ belong to $KK(A, B)$. Abstract quantum D-branes in a separable $C^*$ algebra $A$ correspond to $\phi : A \to B$ where $B$ is the algebra representing a quantum D-brane and $\phi$ is a strongly K-oriented map. A general formula for RR charges in the noncommutative setting was obtained for these branes in [25, 22].

D-branes, as considered in the previous subsection, correspond to the lifted KK-theory classes. That is, if the D-brane corresponds to the triple $(M, E, f)$ and $f : M \hookrightarrow G = V/F$ is a K-oriented map then $f! \in KK(M, V/F)$ represents the D-brane (see [25]). More generally (still following [25]), given a K-oriented map $f : X \to Y$, one can define (under certain conditions) a push forward map $f!$ in K-theory. The very important property of the analytical group $K(V/F)$ of the foliation $(V, F)$ is its ,,wrong way“ (Gysin) functoriality, i.e. one associates to each K-oriented map $f : V_1/F_1 \to V_2/F_2$ of leaf spaces an element $f!$ of the Kasparov group $KK(C^*(V_1; F_1); C^*(V_2; F_2))$.

Now given a small exotic $\mathbb{R}^4$, say $e_1$, embedded in some small exotic $\mathbb{R}^4$, $e$, both are represented by the $C^*$ algebras of the codimension-1 foliations of $S^3$, $C^*(V_1; F_1)$ and $C^*(V; F)$ respectively. The embedding $i : e_1 \to e$ determines the corresponding K-oriented map of the leaf spaces $f_i : S^3/F_1 \to S^3/F$ and the KK-theory lift $f_i! \in KK(C^*(V_1; F_1); C^*(V; F))$. According to definition [25, 22] from section 3.2 we obtain

**Theorem 4.2.**

Let $e$ be an exotic $\mathbb{R}^4$ corresponding to the codimension-1 foliation of $S^3$ which gives rise to the $C^*$ algebra $A_e$. The exotic smooth $\mathbb{R}^4$ embedded in $e$ determines a generalized quantum D-brane in $A_e$.

Given exotic $\mathbb{R}^4$'s, $\{e_a, a \in I\}$, all embedded in $e$, one has the family of $C^*$ algebras, $\{A_a, a \in I\}$, of the codimension-1 foliations of $S^3$, $a \in I$. Now the embeddings $e_a \to e$ determine the corresponding K-oriented maps of the leaf spaces as before, and the $\phi_a$-homomorphisms of algebras $\phi_a : A_e \to A_a$. The corresponding classes in KK-theory $KK(A_e, A_a)$ represent the quantum D-branes in $A_e$. □
However, the correspondence in the theorem is many-to-one and an exotic smooth $\mathbb{R}^4$ embedded in $e$ can be represented (non-uniquely) by stable D-brane in $\mathcal{A}_e$, and not all abstract D-branes in the algebra $\mathcal{A}_e$ are represented by some exotic $e' \subset e$. Still one can consider D-branes represented by exotic $e_a$ in $e$ as carrying 4-dimensional, hence potentially physical, information. This is a kind of special „superselection” rule in superstring theory and will be discussed separately.

5. Discussion and conclusions

In this paper we give further evidences that string theory is indeed related to 4-dimensional nonstandard smoothness of open manifolds like $\mathbb{R}^4$. Our concern here was the quantum limit of D-branes. We show that, on the formal level, there are strong correlations between formalism of quantum D-branes in a quantum spacetime, both represented by separable $C^*$-algebras, and exotic smooth $\mathbb{R}^4$'s. These $\mathbb{R}^4$'s are also represented by $C^*$-algebras and embedded in some exotic $\mathbb{R}^4$. These $C^*$-algebras are the convolution algebras of the codimension 1 foliations of the 3-sphere when $S^3$ is taken as a part of the boundary of the Akbulut cork for the small exotic $\mathbb{R}^4$. Thus we model quantum D-branes in a quantum spacetime by exotic $\mathbb{R}^4$'s embedded in an exotic $\mathbb{R}^4$. When the target „spacetime” $\mathbb{R}^4$ is taken to be the standard one, which is always possible since exotic $\mathbb{R}^4$'s are small, one recovers the correlation with „classic” configurations of D and NS branes in certain string backgrounds, as was described in our previous paper. Thus the way to abstract algebraic setting of $C^*$-algebras and quantum D-branes generalizes the correspondence of branes (represented by submanifolds or K-homology classes) with exotic $\mathbb{R}^4$ seen as smooth submanifolds of the standard $\mathbb{R}^4$. These two-facets of exotic $\mathbb{R}^4$, namely the $C^*$ algebraic and smooth (sub)manifold, are crucial for exhibiting the full range of a correspondence to string theory. When smoothness on $\mathbb{R}^4$ is standard we lose the string information as encoded in 4-structures. Thus, one could in some important cases translate stringy situations into 4-smooth setting and conversely and this is not a duplicate of existing approaches in string theory. Thus we gain the additional and independent channel leading to 4-dimensions from string theory. The crucial is whether this 4-dimensional data carry information on physics in 4-dimensions. This important point was considered already in a series of research papers as well as in a textbook. In we showed that exotic smoothness of an open 4-region in spacetime have the same effect as the existence of magnetic monopoles, i.e. exotic smoothness induces the quantization condition for the electric charge. Moreover, one can consider exotic $\mathbb{R}^4$'s as quantum object, i.e. the spacetime induces the quantization processes.

However, the full-fledged presentation of the relation of exotic $\mathbb{R}^4$, and other open smooth 4-manifolds, with string theory is out of reach for the authors at present. We think that new analytical and topological tools are needed. In the forthcoming paper we will present an effort into this direction and try to understand quantum branes as a kind of wild embeddings based on the smoothness of 4-manifolds. Thus
the point of the question „Is it possible that string theory deals with 4-dimensional structures directly neither by implementing compactifications nor by phenomenological models-building, and these structures would have a physical meaning?” should be further explored and studied. As we emphasized in the previous paper [14] this effort should help with understanding both 4-dimensional physics as appearing from string theory and exotic open 4-manifolds in mathematics [13].

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References

1. S. Akbulut. Lectures on Seiberg-Witten invariants. *Turkish J. Math.*, 20:95–119, 1996.
2. S. Akbulut and K. Yasui. Corks, plugs and exotic structures. *Journal of Gokova Geometry Topology*, 2:40–82, 2008. [arXiv:0806.3010].
3. S. Akbulut and K. Yasui. Knotted corks. *J Topology*, 2:823–839, 2009. [arXiv:0812.5098].
4. J.W. Alexander. An example of a simple-connected surface bounding a region which is not simply connected. *Proceedings of the National Academy of Sciences of the United States*, 10:8 – 10, 1924.
5. T. Asakawa, S. Sugimoto, and S. Terashima. D-branes, matrix theory and K-homology. *JHEP*, 03:034, 2002. [arXiv:hep-th/0108085].
6. T. Asselmeyer. Generation of source terms in general relativity by differential structures. *Class. Quant. Grav.*, 14:749 – 758, 1996.
7. T. Asselmeyer-Maluga. Exotic smoothness and quantum gravity. *Class. Quantum Grav.*, 27:165002, 2010. [arXiv:1003.5506v1 [gr-qc]].
8. T. Asselmeyer-Maluga and C.H. Brans. Cosmological anomalies and exotic smoothness structures. *Gen. Rel. Grav.*, 34:1767–1771, 2002.
9. T. Asselmeyer-Maluga and C.H. Brans. *Exotic Smoothness and Physics*. WorldScientific Publ., Singapore, 2007.
10. T. Asselmeyer-Maluga and J. Król. Gerbes on orbifolds and exotic smooth $R^4$. subm. to Comm. Math. Phys., arXiv: 0911.0271, 2009.
11. T. Asselmeyer-Maluga and J. Król. Gerbes, SU(2) WZW models and exotic smooth $R^4$. submitted to Comm. Math. Phys., arXiv: 0904.1276, 2009.
12. T. Asselmeyer-Maluga and J. Król. Exotic smooth $R^4$, noncommutative algebras and quantization. arXiv: 1001.0882, 2010.
13. T. Asselmeyer-Maluga and J. Król. Small exotic smooth $R^4$ and string theory. In *International Congress of Mathematicians ICM 2010 Short Communications Abstracts Book*, Ed. R. Bathia, page 400, Hindustan Book Agency, 2010.
14. T. Asselmeyer-Maluga and J. Król. Exotic smooth $R^4$ and certain configurations of NS and D branes in string theory. arXiv: 1101.3169, 2011.
15. M. Atiyah. *K-Theory*. W.A. Benjamin, New York–Amsterdam, 1967.
16. M.F. Atiyah and G.B. Segal. Twisted K-theory. *Ukrainian Math. Bull.*, 1, 2004. arXiv: math/0407054.
17. Z. Bizaca. A handle decomposition of an exotic $\mathbb{R}^4$. *J. Diff. Geom.*, 39:491 – 508, 1994.
18. Ž. Bižaca and R Gompf. Elliptic surfaces and some simple exotic $\mathbb{R}^4$'s. *J. Diff. Geom.*, 43:458–504, 1996.
19. C.H. Brans. Exotic smoothness and physics. *J. Math. Phys.*, 35:5494–5506, 1994.
20. C.H. Brans. Localized exotic smoothness. *Class. Quant. Grav.*, 11:1785–1792, 1994.
21. C.H. Brans and D. Randall. Exotic differentiable structures and general relativity. *Gen. Rel. Grav.*, 25:205, 1993.
22. J. Brodzki, V Mathai, J. Rosenberg, and R. J. Szabo. D-branes, KK-theory and duality on noncommutative spaces. *J. Phys. Conf. Ser.*, 103:012004, 2008. arXiv:hep-th/0709.2128.
23. J. Brodzki, V Mathai, J. Rosenberg, and R. J. Szabo. D-branes, RR-fields and duality on noncommutative manifolds. *Commun. Math. Phys.*, 277:643, 2008. arXiv:hep-th/0607020.
24. A. Casson. *Three lectures on new infinite constructions in 4-dimensional manifolds*, volume 62. Birkhäuser, progress in mathematics edition, 1986. Notes by Lucian Guilou, first published 1973.
25. A. Connes. A survey of foliations and operator algebras. *Proc. Symp. Pure Math.*, 38:521–628, 1984. see www.alainconnes.org.
26. C. Curtis, M. Freedman, W.-C. Hsiang, and R. Stong. A decomposition theorem for h-cobordant smooth simply connected compact 4-manifolds. *Inv. Math.*, 123:343–348, 1997.
27. S. Donaldson and P. Kronheimer. *The Geometry of Four-Manifolds*. Oxford Univ. Press, Oxford, 1990.
28. D.S. Freed and E. Witten. Anomalies in string theory with D-branes. *Asian J. Math*, 3:819–851, 1999. arXiv:hep-th/9907189.
29. M.H. Freedman. The topology of four-dimensional manifolds. *J. Diff. Geom.*, 17:357–454, 1982.
30. R. Gompf. Infinite families of casson handles and topological disks. *Topology*, 23:395–400, 1984.
31. R. Gompf. Periodic ends and knot concordance. *Top. Appl.*, 32:141–148, 1989.
32. R.E. Gompf and A.I. Stipsicz. *4-manifolds and Kirby Calculus*. American Mathematical Society, 1999.
33. J.A. Harvey and G. Moore. Noncommutative tachyons and K-theory. *J. Math. Phys.*, 42:2765, 2001. arXiv:hep-th/0009030.
34. F. Hirzebruch and H. Hopf. Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten. *Math. Annalen*, 136:156, 1958.
35. J. Król. (Quantum) gravity effects via exotic $\mathbb{R}^4$. *Ann. Phys. (Berlin)*, 19:No. 3–5, 355–358, 2010.
36. J. Sladkowski. Exotic smoothness, noncommutative geometry and particle physics. *Int. J. Theor. Phys.*, 35:2075–2083, 1996.
37. J. Sladkowski. Gravity on exotic $\mathbb{R}^4$ with few symmetries. *Int. J. Mod. Phys. D*, 10:311–313, 2001.
38. R. J. Szabo. D-branes, tachyons and K-homology. *Mod.Phys.Lett.*, pages 2297–2316, 2002. arXiv:hep-th/0209210.
39. R. J. Szabo. D-branes and bivariant K-theory. Based on invited lectures given at the workshop "Noncommutative Geometry and Physics 2008 - K-Theory and D-Brane ", February 18-22 2008, Shonan Village Center, Kanagawa, Japan. To be published in the volume Noncommutative Geometry and Physics III by World Scientific, arXiv: 0809.3029, 2008.
40. W. Thurston. Noncobordant foliations of $S^3$. *BAMS*, 78:511 – 514, 1972.