A note on \(W\) symmetry of \(N = 2\) gauge theory

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Abstract

The AGT correspondence indicates \(N = 2\) gauge theory possesses of \(W\) algebra symmetry. We study how the conformal block of Toda CFT gives the expectation value of Casimir operators of gauge theory. The \(A_2\) Toda CFT with \(W_3\) symmetry is taken as the main example.
1 Introduction

Some recent study reveals interesting relations between different subjects: the 4D gauge theory, 2D CFT, integrable theory, quantum algebra, etc. The Alday-Gaiotto-Tachikawa (AGT) correspondence, and its generalization by Wyllard, establishes a relation between \( \mathcal{N} = 2 \) supersymmetric gauge theory in the \( \Omega \) background and Liouville/Toda conformal field theory. The Nekrasov-Shatashvili (NS) correspondence relates the same \( \mathcal{N} = 2 \) gauge theory with certain quantum integrable models. All these subjects have some structures that were mostly studied in their own field before the recent discoveries, now it is an interesting problem to study how these properties are realized in each related subjects, and hope to gain some new insights.

At the moment, in the context of AGT we have explicit examples where the gauge theory partition function is identified with the CFT conformal block, the nonlocal operators of gauge theory are realized by CFT operators, and some other developments. In the context of NS we also have examples where the Bethe equation and Hamiltonians of integrable models are derived from the gauge theory partition function. In this paper, through some simple examples, we show how the Hamiltonians can be obtained from CFT conformal block. Although there are evidences that the whole class of \( \mathcal{N} = 2 \) theories are quantum integrable, we focus on particular theories, and it is enough to demonstrate the basic properties. The first is the SU(N) pure gauge theory, with the dual irregular conformal block of \( A_{N-1} \) Toda CFT, and in the semiclassical limit (or the minisuperspace limit of CFT) related to the quantized periodic Toda chain. The second one is the \( \mathcal{N} = 2^* \) gauge theory, the related CFT is the Toda CFT on torus, and they are related to the periodic elliptic Calogero-Moser model. For technical reasons, SU(3) gauge theory and the related \( W_3 \) algebra are our main examples.

In the next section we briefly review the Gaiotto’s irregular conformal block of Liouville CFT, its relation to the Gram/Shapovalov matrix. In section 3, 4 and 5 we show how the SU(3) gauge theory calculation and \( W_3 \) algebra of \( A_2 \) Toda CFT can be consistent with each other and hence with the requirements of quantum integrable chains.

2 Irregular conformal block and Gaiotto state

For SU(2) pure gauge theory, its Seiberg-Witten curve can be written in the form

\[
x^2 - \phi_2(z) = 0, \quad \phi_2(z) = \frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z}.
\]

where \( u \sim \langle \text{tr} \varphi \varphi \rangle \) is the Coulomb parameter, \( \varphi \) is the adjoint scalar in the vector supermultiplet. In the AGT correspondence, the Seiberg-Witten curve is quantized and pro-
moted to the operator equation $x^2 - \hat{\phi}_2(z) = 0$. The operators are understood as correlator of the corresponding Liouville CFT. The operator $\hat{\phi}_2(z)$ is identified with the energy-momentum operator of the CFT: $\hat{\phi}_2(z) = T(z) = \sum_n L_n z^{-n-2}$. Then (1) indicates there exists a state $|G\rangle$ that $L_1|G\rangle = \Lambda^2|G\rangle, L_n|G\rangle = 0, n \geq 2$, where $|G\rangle$ is the Gaiotto state of the Liouville CFT, a kind of coherent state, first constructed by Gaiotto in [4]. It can be expanded according to the level: $|G\rangle = |v_0\rangle + \sum_{k=1}^{\infty} \Lambda^k |v_k\rangle$, where the CFT expansion parameter $\Lambda$ is identified with gauge theory scale, and $|v_0\rangle$ is the highest weight state: $L_0|v_0\rangle = \Delta|v_0\rangle, L_n|v_0\rangle = 0, n \geq 1$. At each level $k$ the vector $v_k$ is linear combination of descendant states of the form $L_{-k_1} \cdots L_{-k_2} L_{-k_1} |v_0\rangle$ with $k_i \geq k_i - 1 \geq \cdots \geq k_1 > 0$ and $\sum k_i = k$. Gaiotto has given the explicit forms of $|G\rangle$ at the first few level, as later pointed out in [14], it can be systematically constructed from the Gram/Shapovalov matrix of the Liouville CFT.

If we associate the partition $(k_1, \cdots, k_2, k_1)$ with a Young diagram $Y$, then the vector $L_{-k_1} \cdots L_{-k_2} L_{-k_1} |v_0\rangle$ at level $k$ can be denoted as $L_{-Y} |v_0\rangle$. Then the level $k$ Gram/Shapovalov matrix is $K_k[\tilde{Y}; Y] = \langle v_0 | L_Y^i L_{-Y} | v_0 \rangle$, and the coefficients of the linear combination at level $k$ turn out to be elements of the inverse Gram/Shapovalov matrix at level $k$ [14]:

$$|v_k\rangle = \sum_{|Y| = -k} K_{-1}^{-1}[1^k; Y] L_{-Y} |v_0\rangle. \tag{2}$$

where $\tilde{Y} = [1^k]$ corresponds to vector $L_{-1}^{k} |v_0\rangle$. The norm of $|v_k\rangle$ is

$$\langle v_k | v_k \rangle = \sum_{|Y| = |\tilde{Y}| = -k} K_{-1}^{-1}[1^k; \tilde{Y}] K_{-1}^{-1}[1^k; Y] \langle v_0 | L_Y^i L_{-Y} | v_0 \rangle = K_{-1}^{-1}[1^k; [1^k]]. \tag{3}$$

In order to relate CFT data to gauge theory data, we identify the CFT parameters with the gauge theory parameters as $c = 1 + 6Q^2(\epsilon_1 \epsilon_2)^{-1}, \Delta = (Q^2/4 - a^2)(\epsilon_1 \epsilon_2)^{-1}$, where $Q = \epsilon_1 + \epsilon_2$ is the background charge of CFT, then $\langle v_k | v_k \rangle$ equals to the Nekrasov’s $k$-th instanton partition function $Z_k$ of $\mathcal{N} = 2$ gauge theory [11]. The norm of the Gaiotto state therefore equals to the full instanton partition function $Z_{\text{inst}}(a, q)$.

In fact, (1) also indicates the following relation,

$$\frac{\langle G | L_0 | G \rangle}{\langle G | G \rangle} = 2u. \tag{4}$$

This is very easy the verify. Using the Virasoro relation (see (24) in section four), we have $L_0 L_{-k_1} \cdots L_{-k_2} L_{-k_1} |\Delta\rangle = (\Delta + k) |\Delta\rangle$. Therefore

$$\frac{\langle G | L_0 | G \rangle}{\langle G | G \rangle} = \sum_{k=0}^{\infty} \langle v_k | \Delta + k | v_k \rangle \Lambda^{4k} = \Delta + \Lambda^4 \frac{\partial}{\partial \Lambda^4} \ln Z_{\text{inst}} = 2u \tag{5}$$

$$\frac{\langle G | L_0 | G \rangle}{\langle G | G \rangle} = \sum_{k=0}^{\infty} \langle v_k | \Delta + k | v_k \rangle \Lambda^{4k} = \Delta + \Lambda^4 \frac{\partial}{\partial \Lambda^4} \ln Z_{\text{inst}} = 2u \tag{5}$$
If we make the relation between $u$ and $<\text{tr}\varphi^2>$ precise, it is $2u = (−\epsilon_1\epsilon_2)^{-1}(<\text{tr}\varphi^2> − \frac{1}{4}Q^2)$, then the relation above is the Matone’s relation of $\mathcal{N} = 2$ gauge theory\cite{15, 16}.

The above is the story for the SU(2) pure gauge theory and its irregular Liouville conformal block. Generalization of this construction to other cases has been studied, including theories with mass deformations, with other gauge groups, see for example\cite{4, 18, 19, 20, 21}. For example, the pure SU(N) gauge theory should be related to irregular conformal block of the $A_{N−1}$ Toda CFT, the corresponding conformal algebra is the $W_N$ algebra\cite{24}. The Seiberg-Witten curve can be written as\cite{25, 26}

$$x^N − \phi_2(z)x^{N−2} − \phi_3(z)x^{N−3} − \cdots − \phi_N(z) = 0,$$

with

$$\phi_s(z) = \frac{2u_s}{z^s}, \quad s = 2, 3, \cdots N − 1, \quad \phi_N(z) = \frac{\Lambda^N}{z^{N+1}} + \frac{2u_N}{z^{N}^2} + \frac{\Lambda^N}{z^{N−1}}. \quad (6)$$

here $u_s \sim \text{tr}\varphi^s$ are the Coulomb parameters. After promoting this relation to operator relation of Toda CFT, the operators $\hat{\phi}_s(z)$ are identified with spin-$s$ currents $W^{(s)}(z)$ of the CFT\cite{27, 28}, among of them $W^{(2)}(z) = T(z)$ is the energy-momentum current. As the mode expansion of currents are $W^{(s)}(z) = \sum_{n\in\mathbb{Z}} W^{(s)}_n z^{−(n+s)}$, then the Gaiotto states can be consistently constructed from the constraints,

$$W_n^{(s)}|G\rangle = 0, \quad n \geq 1, \quad s = 2, 3, \cdots, N−1,$n

$$W_1^{(N)}|G\rangle = \Lambda^N|G\rangle, \quad W_n^{(N)}|G\rangle = 0, \quad n \geq 2. \quad (7)$$

It is shown in\cite{21} that at each level the coefficients of linear combination are elements of the inverse Gram/Shapovalov matrix, and $\langle v_k|v_k\rangle$ equals to the element $K_{−1}^{−1}((W^{(N)}_1)^k, W^{(N)}_1)^k)$, and should equal to the instanton partition function $Z_k$, if the CFT parameters are properly identified with gauge theory parameters,

$$\langle G|G\rangle = Z^{\text{inst}}, \quad \text{i.e.} \quad \langle v_k|v_k\rangle = Z_k, \quad k \geq 1, \quad (8)$$

We should also expect the following relation holds,

$$\frac{\langle G|W_n^{(s)}\rangle G\rangle}{\langle G|G\rangle} = 2u_s, \quad s = 2, 3, \cdots N. \quad (9)$$

For various case, it has been shown in\cite{18, 19, 20, 21, 22, 23} by some explicit calculations that the Gaiotto state of Toda CFT indeed gives gauge theory instanton partition. In the case of pure gauge theory, it is simply the norm of $|G\rangle$ as in(8), and in fact it is indeed equal to a particular element of the inverse Gram/Shapovalov matrix. In the next two section, we will show the $W_3$ CFT also gives Casimirs of SU(3) pure gauge theory as in(9), and in fact also directly related to elements of the inverse Gram/Shapovalov matrix. The $N = 2^*$ gauge theory and the CFT on the torus are also discussed.
3 Instanton calculation for SU(3) gauge theory

The Nekrasov instanton partition function for \( k \)-instanton sector of SU(N) pure gauge theory can be evaluated by the formula\[^{[12]}\]:

\[
Z_k = \sum_{\{\sum |Y_\alpha| = k\}} \prod_{\alpha,\beta=1}^N \prod_{s \in Y_\alpha} \prod_{s' \in Y_\beta} \frac{1}{E_{\alpha\beta}(s)(\epsilon_+ - E_{\beta\alpha}(s'))},
\]

(10)

with \( \epsilon_+ = \epsilon_1 + \epsilon_2 \), and

\[
E_{\alpha\beta}(s) = a_{\alpha\beta} - h_{\beta}(s)\epsilon_1 + (v_\alpha(s) + 1)\epsilon_2,
\]

(11)

the v.e.v of the adjoint scalar satisfy \( \sum_{i=1}^N a_i = 0 \). The sum is running over all possible partitions symbolled by the Young diagrams \( \{Y_\alpha\} \) with \( \sum |Y_\alpha| = k \). (These Young diagrams have no direct relation with Young diagrams associated to descendant states on the CFT side.) Moreover, the expectation value of the Casimirs \( \text{tr} \varphi^m \) can be calculated by\[^{[34, 35]}\]

\[
< \text{tr} \varphi^m > = \frac{1}{Z_{\text{inst}}} \sum_{k=0}^{\infty} \sum_{\{\sum |Y_\alpha| = k\}} \frac{\text{ch}_m(\{Y_\alpha\})}{\prod_{\alpha,\beta=1}^N \prod_{s \in Y_\alpha} \prod_{s' \in Y_\beta} E_{\alpha\beta}(s)(\epsilon_+ - E_{\beta\alpha}(s'))} \Lambda^{2kN}.
\]

(12)

where \( q_{\text{in}} = \Lambda^{2N} \) is the instanton expansion parameter, and

\[
\text{ch}_m(\{Y_\alpha\}) = \frac{1}{m!} \sum_{\alpha=1}^N \left( a_{\alpha}^m - \sum_{s(i_\alpha,j_\alpha) \in Y_\alpha} [(a_\alpha + j_\alpha \epsilon_1 + i_\alpha \epsilon_2)^m - (a_\alpha + j_\alpha \epsilon_1 + (i_\alpha - 1)\epsilon_2)^m - (a_\alpha + (j_\alpha - 1)\epsilon_1 + i_\alpha \epsilon_2)^m + (a_\alpha + (j_\alpha - 1)\epsilon_1 + (i_\alpha - 1)\epsilon_2)^m] \right) \] \quad (13)

It can be read from the degree \( m \) part of the Chern character of the equivalent bundle \( \text{Ch}_\varphi(\mathcal{E}) \),

\[
\text{ch}_m(\{Y_\alpha\}) = \sum_{\alpha}^N [e^{a_\alpha} - (1 - e^{\epsilon_1})(1 - e^{\epsilon_2}) \sum_{s(i_\alpha,j_\alpha) \in Y_\alpha} e^{a_\alpha + (j_\alpha - 1)\epsilon_1 + (i_\alpha - 1)\epsilon_2}]_m
\]

(14)

In the NS correspondence, \( < \text{tr} \varphi^m > \) as functions of the quasimomenta \( a_i \), are the quantized Hamiltonians of Toda chain. The quasimomenta \( a_i \) are constrained by the Bethe equation that determine the critical points of the gauge theory prepotential\[^{[3]}\].

For SU(3) theory, the nontrivial independent Casimires are \( \text{tr} \varphi^2 \) and \( \text{tr} \varphi^3 \). It is simple to see for \( m = 2 \) we have \( \text{ch}_2(\{Y_\alpha\}) = \frac{1}{2} \sum_{\alpha} a_\alpha^2 - k\epsilon_1 \epsilon_2 \), then \[^{[12]}\] leads to the Matone’s relation, the information of \( < \text{tr} \varphi^2 > \) can be derived from the partition function. This conclusion is valid for SU(N) theory\[^{[16]}\]. More information can be found in \( < \text{tr} \varphi^3 > \),

\[
\text{ch}_3(\{Y_\alpha\}) = \frac{1}{6} \sum_{\alpha=1}^N (a_\alpha^3 - 3\epsilon_1 \epsilon_2 \sum_{s \in Y_\alpha} [2a_\alpha + (2j_\alpha - 1)\epsilon_1 + (2i_\alpha - 1)\epsilon_2])
\]

(15)
Let us denote
\[
\text{tr} \varphi^m_k = \sum_{\{\sum |Y_\alpha| = k\}} \frac{\text{ch}_m(\{Y_\alpha\})}{\prod_{\alpha, \beta=1}^N \prod_{s \in Y_\alpha} \prod_{s' \in Y_\beta} E_{\alpha \beta}(s)(\epsilon_+ - E_{\beta \alpha}(s'))}.
\] (16)

Then with no instanton correction, \( k = 0 \), we have
\[
\text{tr} \varphi^3_{k=0} = \frac{1}{2}(a_1^3 + a_2^3 + a_3^3) = -\frac{1}{2}a_1a_2(a_1 + a_2).
\] (17)

For one instanton correction, \( k = 1 \), we have contributions from the partitions \([1],[\emptyset, \emptyset]\), and from its permutations obtained by \(a_1 \leftrightarrow a_2, a_1 \leftrightarrow a_3\). They contribute
\[
\text{tr} \varphi^3_{k=1} = \frac{1}{6} \sum_{\alpha=1}^3 a_1^3 + a_2^3 + a_3^3 - 3\epsilon_1\epsilon_2(2\alpha + \epsilon_+)

\quad \times \frac{9a_1a_2(a_1 + a_2)}{(a_1 - a_2)^2 - \epsilon_+^2)((2a_1 + a_2)^2 - \epsilon_+^2)((a_1 + 2a_2)^2 - \epsilon_+^2)}.
\] (18)

For two instanton correction, \( k = 2 \), we have partitions \([1,1],[\emptyset, \emptyset],[11],[\emptyset, \emptyset],[2],[\emptyset, \emptyset]\), and their permutations. They contribute
\[
\text{tr} \varphi^3_{k=2} = \frac{1}{2}a_1a_2(a_1 + a_2)Z_3 - \frac{27a_1a_2(a_1 + a_2)(a_1^2 + a_1a_2 + a_2^2 - \epsilon_+^2 + \epsilon_1\epsilon_2)(8a_1^6 + 24a_1^4a_2 + \cdots)}{\epsilon_1\epsilon_2 \prod_{\alpha < \beta}(a_{\alpha \beta}^2 - \epsilon_+^2)(a_{\alpha \beta}^2 - (\epsilon_+ + \epsilon_1)(\epsilon_+ + \epsilon_2)^2)}.
\] (19)

### 3.1 \( \mathcal{N} = 2^* \) gauge theory

The \( \mathcal{N} = 2^* \) gauge theory couples an adjoint matter to the vector multiplet. If the physical mass is \( m^* \), then it is the parameter \( m = m^* + (\epsilon_1 + \epsilon_2)/2 \) appears in the Nekrasov instanton partition function. It can be evaluated by the formula\(^{13}\),
\[
Z_k = \sum_{\{\sum |Y_\alpha| = k\}} \prod_{\alpha, \beta=1}^N \prod_{s \in Y_\alpha} \prod_{s' \in Y_\beta} \frac{(E_{\alpha \beta}(s) - m)(\epsilon_+ - E_{\beta \alpha}(s') - m)}{E_{\alpha \beta}(s)(\epsilon_+ - E_{\beta \alpha}(s'))},
\] (20)

For SU(3) theory, the one instanton contribution gives the partition function
\[
Z_1 = \frac{3(m - \epsilon_1)(m - \epsilon_2)(4a_1^6 + 12a_1^5a_2 - 3a_1^4a_2^2 + \cdots + 2m^5 + 6\epsilon_1^2 - 6\epsilon_2^2)}{\epsilon_1\epsilon_2((a_1 - a_2)^2 - \epsilon_+^2)((2a_1 + a_2)^2 - \epsilon_+^2)((a_1 + 2a_2)^2 - \epsilon_+^2)}.
\] (21)

The the expectation value of Casimirs of \( \mathcal{N} = 2^* \) theory is give by a formula similar to\(^{12}\), now with
\[
\text{tr} \varphi^m_k = \sum_{\{\sum |Y_\alpha| = k\}} \left( \prod_{\alpha, \beta=1}^N \prod_{s \in Y_\alpha} \prod_{s' \in Y_\beta} \frac{(E_{\alpha \beta}(s) - m)(\epsilon_+ - E_{\beta \alpha}(s') - m)}{E_{\alpha \beta}(s)(\epsilon_+ - E_{\beta \alpha}(s'))} \right) \text{ch}_m(\{Y_\alpha\}).
\] (22)
\[ \langle \text{tr} \varphi^2 \rangle \] leads to the Matone’s relation as pure gauge theory case. The one instanton result for \[ \langle \text{tr} \varphi^3 \rangle \] is
\[
\text{tr} \varphi^3_{k=1} = -\frac{1}{2} a_1 a_2 (a_1 + a_2) Z_1 - \frac{3(m - \epsilon_1)(m - \epsilon_2)(6m^2 a_1^2 a_2 + 6m^2 a_1 a_2^2 + \cdots - 7\epsilon_1 \epsilon_2^6 - \epsilon_2^7)}{2((a_1 - a_2)^2 - \epsilon_2^2)((2a_1 + a_2)^2 - \epsilon_2^2)((a_1 + 2a_2)^2 - \epsilon_2^2)}.
\]

Let us denote the coupling of \( \mathcal{N} = 2^* \) theory by \( q \), then in the decoupling limit \( q \to 0, m \to \infty \), while keep \( qm^6 = \Lambda^6 \), the results for \( N = 2^* \) theory reduce to that for pure gauge theory.

### 4 Irregular conformal block for \( A_2 \) Toda CFT

According to Wyllard’s generalization[2] of the AGT[1], SU(3) gauge theories are related to the \( A_2 \) Toda CFT on surfaces with punctures, the underlying conformal algebra is the \( W_3 \) algebra, generated by the spin two current \( W^{(2)} = T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \) and the spin three current \( W^{(3)} = W(z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-3} \). The \( W_3 \) algebra is
\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0},
\]
\[
[L_m, W_n] = (2m - n)W_{m+n},
\]
\[
[W_m, W_n] = \frac{9}{2} \left( \frac{c}{3 \cdot 5!} m(m^2 - 1)(m^2 - 4) \delta_{m+n,0} + (m - n) \left[ \frac{16}{22} + 5c \Lambda_{m+n} \right] \right)
+ \left( \frac{(m + n + 2)(m + n + 3)}{15} - \frac{(m + 2)(n + 2)}{6} \right) L_{m+n} \right].
\]

with
\[
\Lambda_n = \sum_{m \in \mathbb{Z}} : L_m L_{n-m} : + \frac{x_2}{5} L_n,
\]
where for even \( n \): \( x_{2l} = (1-l)(1+l) \), and for odd \( n \): \( x_{2l+1} = (1-l)(2+l) \). The central charge is \( c = 2 + 24Q^2/(\epsilon_1 \epsilon_2) \) with \( Q = \epsilon_1 + \epsilon_2 \) the background charge of Toda CFT.

For the pure gauge theory, there exists the \( A_2 \) irregular conformal block and the related Gaiotto state. Again, the Gaiotto state can be expanded as \( |G\rangle = |v_0\rangle + \Lambda^3 |v_1\rangle + \Lambda^6 |v_2\rangle + \cdots \), at each level \( |v_k\rangle \) is linear combination of vectors of the form \( L_{-Y_1} W_{-Y_2} |v_0\rangle \) with \( |Y_1| + |Y_2| = k \). Here \( L_{-Y_1} = L_{-k_1} \cdots L_{-k_2} L_{-k_1} \) and similarly \( W_{-Y_2} = W_{-k_1'} \cdots W_{-k_2'} W_{-k_1'} \). We always write the product of generators \( L_{-n} \) on the left of the product of \( W_{-n} \). \( |v_0\rangle \) is the highest weight state satisfying
\[
L_0 |v_0\rangle = \Delta |v_0\rangle, \quad L_n |v_0\rangle = 0, \quad n \geq 1,
\]
and
\[
W_0 |v_0\rangle = w |v_0\rangle, \quad W_n |v_0\rangle = 0, \quad n \geq 1,
\]
From the singular structure of the Seiberg-Witten curve of the SU(3) pure gauge theory, the Gaiotto state should satisfy:

\[ L_n |G\rangle = 0, \quad n \geq 1 \]
\[ W_1 |G\rangle = \Lambda^3 |G\rangle, \quad W_n |G\rangle = 0, \quad n \geq 2. \]

Then we can determine |G⟩ level by level, using its Gram/Shapovalov matrix. The Gram/Shapovalov matrix is

\[ K(\tilde{Y}_1, \tilde{Y}_2; Y_1, Y_2) = \langle v_0 | W_{\tilde{Y}_1} \cdot L_{-Y_1} W_{-Y_2} | v_0 \rangle. \]

The first two level CFT data are given in [18, 19], we also give them in the Appendix. At the level \( k = 1 \), we have

\[ |v_1\rangle = \frac{1}{9(D\Delta^2 - w^2)}(-3wL_{-1} + 2\Delta W_{-1})|v_0\rangle, \]

We have used the notations from [18], \( D = \frac{32}{22+5c}(\Delta + \frac{1}{5}) - \frac{1}{5} \). Note that the two coefficients of \( L_{-1} \) and \( W_{-1} \) are \( c_1 = K_{[1]}^{-1}(\emptyset, [1]; [1], \emptyset) \) and \( c_2 = K_{[1]}^{-1}(\emptyset, [1]; \emptyset, [1]) \) respectively. Therefore \( |v_1\rangle \) coincide with the level one null state. It can be shown that if we make the identification

\[ \Delta = \frac{a_1^2 + a_2^2 + a_1a_2 - \epsilon_1^2}{-\epsilon_1 \epsilon_2}, \quad w = 6(\frac{6}{22 + 5c})^{1/2} \frac{a_1a_2(a_1 + a_2)}{(-\epsilon_1 \epsilon_2)^{3/2}}, \quad c = 2 - 24 \frac{\epsilon_1^2}{-\epsilon_1 \epsilon_2}, \]

then \( \langle v_1 | v_1 \rangle \) gives the one instanton partition function \( Z_1 \) of gauge theory,

\[ \langle v_1 | v_1 \rangle = \frac{2\Delta}{9(D\Delta^2 - w^2)} = K_{[1]}^{-1}(\emptyset, [1]; \emptyset, [1]) = (\epsilon_1 \epsilon_2)^3(\frac{22 + 5c}{6^3})Z_1 = \tilde{Z}_1. \]

At the level \( k = 2 \), we have

\[ |v_2\rangle = (c_1^{[2]} L_{-2} + c_2^{[2]} L_{-1} + c_3^{[2]} L_{-1} W_{-1} + c_4^{[2]} W_{-2} + c_5^{[2]} W_{-1}|v_0\rangle, \]

where \( c_i^{[2]} \) are the last column of the inverse Gram/Shapovalov matrix \( c_i^{[2]} = c_{Y_1, Y_2}^{[2]} = K_{[2]}^{-1}(\emptyset, [1^2]; Y_1, Y_2) \). Its norm satisfies

\[ \langle v_2 | v_2 \rangle = K_{[2]}^{-1}(\emptyset, [1^2]; \emptyset, [1^2]) = (\epsilon_1 \epsilon_2)^6(\frac{22 + 5c}{6^3})^2Z_2 = \tilde{Z}_2, \]

The level \( k = 3 \) Gram/Shapovalov matrix is given in [20], if we write

\[ |v_3\rangle = (c_1^{[3]} L_{-3} + c_2^{[3]} L_{-2} L_{-1} + c_3^{[3]} L_{-1}^2 + c_4^{[3]} L_{-2} W_{-1} + c_5^{[3]} L_{-1} W_{-1} + c_6^{[3]} L_{-1} W_{-2} \]
\[ + c_7^{[3]} L_{-2} W_{-1} + c_8^{[3]} W_{-3} + c_9^{[3]} W_{-2} W_{-1} + c_{10}^{[3]} W_{-1}|v_0\rangle, \]
with the coefficients $c_{i}^{[3]} = c_{Y_{1}, Y_{2}}^{[3]} = K_{[3]}^{-1}(\emptyset, [1^{3}]; Y_{1}, Y_{2})$, then

$$
\langle v_{3}|v_{3}\rangle = K_{[3]}^{-1}(\emptyset, [1^{3}]; \emptyset, [1^{3}]) = (\epsilon_{1}\epsilon_{2})^{3} \left( \frac{22 + 5c}{63} \right)^{3} Z_{3} = \widetilde{Z}_{3}, \quad (37)
$$

Now let us check (9). First look at $\langle G|L_{0}|G \rangle$. From the $W_{3}$ algebra (24), $[L_{0}, L_{-n}] = nL_{-n}, [L_{0}, W_{-n}] = nW_{-n}$ it is easy to see $L_{0}L_{-Y_{1}}W_{-Y_{2}}|v_{0}\rangle = (\Delta + k)L_{-Y_{1}}W_{-Y_{2}}|v_{0}\rangle$ if $|Y_{1}| + |Y_{2}| = k$. Therefore, $L_{0}|v_{k}\rangle = (\Delta + k)|v_{k}\rangle$. The precise relation between $u$ and $<\text{tr}\varphi^{2}>$ for SU(3) theory is $2u = (-\epsilon_{1}\epsilon_{2})^{-1}(<\text{tr}\varphi^{2}> - Q^{2})$, then the Matone's relation follows as in the Liouville case.

Then, let us look at $\langle G|W_{0}|G \rangle$. Now apply the operator $W_{0}$ on $|v_{k}\rangle$ always results in $W_{0}|v_{k}\rangle = w|v_{k}\rangle + |\tilde{v}_{k}\rangle$ with $|\tilde{v}_{k}\rangle$ a vector also at the level $k$. For example,

$$
|\tilde{v}_{1}\rangle = \frac{9}{2}Dc_{1}^{[1]}L_{-1}L_{-1} + 2c_{1}^{[1]}W_{-1}|v_{0}\rangle,
$$

$$
|\tilde{v}_{2}\rangle = \frac{288\Delta}{22 + 5c}c_{2}^{[2]}L_{-2}L_{-1} + \left( \frac{9}{2}Dc_{2}^{[2]} + \frac{144}{22 + 5c}c_{4}^{[2]} \right) L_{-1}L_{-1} + (4c_{2}^{[2]} + 9Dc_{5}^{[2]}) L_{-1}W_{-1} + (4c_{2}^{[2]} + 2c_{3}^{[2]} + \frac{9}{2}Dc_{5}^{[2]}W_{-2} + 2c_{3}^{[2]}W_{-2}^{2})|v_{0}\rangle. \quad (38)
$$

It turns out that the product $\langle v_{k}|\tilde{v}_{k}\rangle$ is two times an element of the inverse Gram/Shapovalov matrix. And $\langle v_{k}|W_{0}|v_{k}\rangle$ is proportional to the $k$-instanton contribution to $<\text{tr}\varphi^{3}>$ of gauge theory, up to a factor. The direct calculation gives

$$
\langle v_{1}|W_{0}|v_{1}\rangle = \frac{2w(\Delta - 3)}{9(D\Delta^{2} - w^{2})} = w\widetilde{Z}_{1} + 2K_{[1]}^{-1}([1], \emptyset, [1]),
$$

$$
\langle v_{2}|W_{0}|v_{2}\rangle = w\widetilde{Z}_{2} + 2K_{[2]}^{-1}([1], [1]; \emptyset, [1^{2}]),
$$

$$
\langle v_{3}|W_{0}|v_{3}\rangle = wK_{[3]}^{-1}([1], [1^{3}]; \emptyset, [1^{3}]) + 2K_{[3]}^{-1}([1], [1^{2}]; \emptyset, [1^{3}]), \quad (41)
$$

What about higher level? The Gaiotto state $|v_{k}\rangle$ can be constructed by:

$$
|v_{k}\rangle = \sum c_{Y_{1}, Y_{2}}^{[k]}L_{-Y_{1}}W_{-Y_{2}}|v_{0}\rangle = \sum K_{[k]}^{-1}(\emptyset, [1^{k}]; Y_{1}, Y_{2})L_{-Y_{1}}W_{-Y_{2}}|v_{0}\rangle, \quad (42)
$$

and based on the observation of the first three level results, we conjecture they satisfy

$$
\langle v_{k}|v_{k}\rangle = K_{[k]}^{-1}(\emptyset, [1^{k}]; \emptyset, [1^{k}]) = (\epsilon_{1}\epsilon_{2})^{3k} \left( \frac{22 + 5c}{63} \right)^{k} Z_{k}, \quad (43)
$$

$$
\langle v_{k}|W_{0}|v_{k}\rangle = wK_{[k]}^{-1}(\emptyset, [1^{k}]; \emptyset, [1^{k}]) + 2K_{[k]}^{-1}([1], [1^{k-1}]; \emptyset, [1^{k}]) = -2\left(\frac{63}{22 + 5c}\right)^{1/2}(-\epsilon_{1}\epsilon_{2})^{-3/2} \left( \frac{22 + 5c}{63} \right)^{k} \text{tr}\varphi_{k}^{3}. \quad (44)
$$
Here $K^{-1}_{[k]}(0,[k];0,[k])$ is related to states $W^k_{-1}|v_0\rangle$ and its conjugate; $K^{-1}_{[k]}([1],[1-k];0,[k])$ is related to $W^k_{-1}|v_0\rangle$ and conjugate of $L_{-1}W^{-1}_{-1}|v_0\rangle$. Then we can express the expectation value $<\text{tr}\varphi^3>$ in terms of CFT data.

In order to rewrite the above relations as (8) and (9), we need to scale $|v_k\rangle$ to absorb the factor $(\epsilon_1\epsilon_2)^{3k(2\epsilon_2+5c)}$, and scale $W_0$ to absorb the factor $-2(\frac{6^3}{22+5c})^{1/2}(-\epsilon_1\epsilon_2)^{-3/2}$, then we have $\langle v_k|v_k\rangle = Z_k$ and $\langle v_k|W_0|v_k\rangle = \text{tr}\varphi^3_k$.

## 5 Torus one point correlator

For the $A_2$ Toda CFT, its one point correlator on the torus $\langle V_m(1)\rangle_{g=1}$ is given by

$$\langle V_m(1)\rangle_{g=1} = \int d\alpha C(\alpha, \alpha_m, 2Q - \alpha)|q^\Delta_{\alpha} \mathcal{F}^m_{\alpha}(q)|^2,$$

where $C(\alpha, \alpha_m, 2Q - \alpha)$ is the structure constant of three point function, and $\alpha$ is the momentum of intermediate channel state, $\alpha_m$ is the momentum of external state. We use $q$ as the expansion parameter for CFT because, as will be clear later, it is the same as the instanton expansion parameter of $\mathcal{N} = 2^*$ gauge theory. The one point conformal block is

$$\mathcal{F}^m_{\alpha}(q) = \sum_{\tilde{Y}_{1,2}} K^{-1}(\tilde{Y}_{1,2}; Y_{1,2}) \frac{\langle L_{-1}W_{-1}|v_0(0)\rangle|V_m(1)\rangle|L_{-1}W_{-1}|v_0(\infty)\rangle}{\langle v_0(0)|V_m(1)\rangle|v_0(\infty)\rangle} q^{Y_1+Y_2}.$$

where $V_m(1)$ is the external state related to mass deformation of gauge theory. According to the proposal of Wyllard$^2$, in order to establish the AGT relation for the $W_3$ CFT, the momentum $\alpha$ takes generic value and $\Delta, w$ associated to it are identified with gauge theory parameters as in$^{[32]}$, the external state $V_m(1)$ should be constrained by the semi-null condition

$$(L_{-1} - \frac{2\Delta_m}{3w_m})|V_m(1)\rangle = 0. \quad (47)$$

With this condition, $C(\alpha, \alpha_m, 2Q - \alpha)$ is known$^{[29, 30]}$, and higher point correlators of primary fields can be evaluated through the three point correlators, therefore solve the $W_3$ CFT under this condition. Apply $W_1$ on this condition we have $D_m\Delta_m^2 - w_m^2 = 0$. We find the following appropriate identification,

$$\Delta_m = \frac{3m(Q - m)}{\epsilon_1\epsilon_2}, \quad w_m = \frac{32}{22+5c}(\Delta_m + \frac{1}{5}) - \frac{1}{5}\frac{1}{2}\Delta_m. \quad (48)$$

We need the vertex $\Gamma(\tilde{Y}_{1,2}; 0; Y_{1,2}) = \langle L_{-1}W_{-1}|v_0(0)\rangle|V_m(1)\rangle|L_{-1}W_{-1}|v_0(\infty)\rangle$ to compute $\mathcal{F}^m_{\alpha}(q)$, see$^{[17]}$ for the CFT technique about it. The first few level vertexes are given in$^{[20]}$. As the elements of the (inverse) Gram/Shapovalov matrix is only nonzero for $|\tilde{Y}_{1}| + |\tilde{Y}_{2}| = |Y_{1}| + |Y_{2}|$, only the vertex at the level $[1, 1]$ in$^{[20]}$ is useful to check the torus one point block.
When we compare the conformal block with the instanton partition function, as the AGT[1] demonstrated, there is an U(1) factor presented. For the case of \( W_3 \) CFT, the relation should be

\[
Z_{SU(3)}^{\text{inst}}(a, m, q) = \left[ \prod_{i=1}^{\infty} (1 - q^i) \right]^{-\frac{3m(Q-m)}{\epsilon_1 \epsilon_2}} F^m_\alpha(q).
\]

(49)

It is easy to verify this for \( k = 1 \) with parameter identification(32).

Then we may consider how to realize \( \langle G|W_0^{(s)}|G \rangle \) in the CFT conformal block. When we derive the 1-point conformal block \[46\] we actually sew two legs of the same pants. Now let us twist a leg by the operator \( W_0^{(s)} \) first, then sew the leg with an untwisted leg.

Denote the \( L_0 \) twisted sewing,

\[
L^m_\alpha(q) = \sum_{Y_1, Y_2} K^{-1}(\tilde{Y}_1, \tilde{Y}_2; Y_1, Y_2) \frac{\langle L_{-Y_1} W_{-\tilde{Y}_2} v_0(0) | V_m(1) | L_0 L_{-Y_1} W_{-Y_2} v_0(\infty) \rangle}{\langle v_0(0) | V_m(1) | v_0(\infty) \rangle} q^{Y_1+Y_2},
\]

(50)

it is easy to see

\[
\frac{L^m_\alpha(q)}{F^m_\alpha(q)} = \Delta + q \frac{\partial}{\partial q} \ln F^m_\alpha(q).
\]

(51)

Then we can relate \( < \text{tr} \varphi^2 > \) of \( \mathcal{N} = 2^* \) theory and CFT data as

\[
(-\epsilon_1 \epsilon_2)^{-1} < \text{tr} \varphi^2 > = (\Delta - \frac{Q^2}{\epsilon_1 \epsilon_2}) + q \frac{\partial}{\partial q} \ln Z^{\text{inst}} = (1 - \frac{3m(Q-m)}{\epsilon_1 \epsilon_2}) \frac{1-E_2(q)}{24} - \frac{Q^2}{\epsilon_1 \epsilon_2} + \frac{L^m_\alpha(q)}{F^m_\alpha(q)}.
\]

(52)

We have used the fact about the Einstein series \( E_2(q) = 1 - 24 \sum_{i=1}^{\infty} iq^i/(1 - q^i) \).

Then consider the \( W_0 \) twisted sewing,

\[
W^m_\alpha(q) = \sum_{Y_1, Y_2} K^{-1}(\tilde{Y}_1, \tilde{Y}_2; Y_1, Y_2) \frac{\langle L_{-Y_1} W_{-\tilde{Y}_2} v_0(0) | V_m(1) | W_0 L_{-Y_1} W_{-Y_2} v_0(\infty) \rangle}{\langle v_0(0) | V_m(1) | v_0(\infty) \rangle} q^{Y_1+Y_2},
\]

(53)

\( W^m_\alpha(q) \) would be more complex than \( L^m_\alpha(q) \), we do not obtain a general result for arbitrary level \( k \), but we observe the following relation for the first level,

\[
-2(\frac{6^3}{22 + 5c})^{1/2}(-\epsilon_1 \epsilon_2)^{-3/2} \left( \text{tr} \varphi^3_{k=1} - \frac{3}{2} Q(m - \epsilon_1)(m - \epsilon_2) \right) = \prod_{i=1}^{\infty} (1 - q^i) \frac{3m(Q-m)}{\epsilon_1 \epsilon_2}^{-1} W^m_\alpha(q) \big|_{k=1},
\]

(54)

and obviously the following relation also holds,

\[
-2(\frac{6^3}{22 + 5c})^{1/2}(-\epsilon_1 \epsilon_2)^{-3/2} \text{tr} \varphi^3_{k=0} = \prod_{i=1}^{\infty} (1 - q^i) \frac{3m(Q-m)}{\epsilon_1 \epsilon_2}^{-1} W^m_\alpha(q) \big|_{k=0}.
\]

(55)
Without several higher level data we are unable to determine the general relation, but we expect on the left hand side there are new terms come from an U(1) factor similar to the terms on the right hand side in (52). The important point is that this factor involves $Q(m - \epsilon_1)(m - \epsilon_2)$ and is independent of $a_i$.

6 Conclusion

Since the AGT proposal\[1\], there has appeared some strong evidences that the $\mathcal{N} = 2$ gauge theory in the $\Omega$ background has the $W$ symmetry. In this paper we provide an observation, through the SU(3) gauge theory and $W_3$ algebra, that the Casimirs of gauge theory can be obtained from the CFT data, consistent with the form of Seiberg-Witten curve. Hopefully, this would be true for general cases. The $W_N$ algebra contains the commutation $[L_m, W^{(s)}_m] = ((s - 1)m - n)W^{(s)}_{m+n}$ for $s \geq 3$, so $[L_0, W^{(s)}_0] = 0$, this fact supports the expectation that $W^{(s)}_0$ would give all other Hamiltonian as in (9), up to factors independent of the quasimomenta. However, it is very hard to demonstrate the details for general case along this way because the full commutation relations for $W_N$ algebra would be very complicated.

Although there exists the free field realization of the $W_N$ algebra, the higher spin currents can be constructed from free fields through the quantum Miura transform, and in fact some properties relevant for AGT can be obtained from this construction\[21, 31\], however, a general treatment that incorporate the $W_N$ symmetry into the gauge theory context has not been presented. Therefore, an understanding of the $W$ symmetry in $\mathcal{N} = 2$ theory from a more fundamental level is surely desired. On the physics side, it is the mysterious six dimensional (0, 2) superconformal theory that inspired some recent progress on $\mathcal{N} = 2$ gauge theory\[25, 1\], hence it might be worth to pursue an explanation from six dimensional perspective, see some attempts in\[32, 33\].

We can also consider how the integrable hierarchy of $\mathcal{N} = 2$ gauge theory\[34\] can be realized in CFT. Turning on higher Casimirs in the gauge theory Lagrangian results in the deformed partition function $Z(\vec{t}, a, q)$ which can be evaluated by the localization method. The final partition function is just multiplying the N-tuple Young diagram contribution of the undeformed theory by a factor $\exp(\sum t_m \text{ch}_m(\{Y_{a_f}\}))$\[34, 35\]. Naively, we may think on the CFT side this is achieved by considering the correlator $\langle G|\exp(\sum t_s W^{(s)}(0))|G\rangle$. But this does not work. The operator $L_0$ is diagonal in each subspace of level $k$ in the Verma module, therefore can be “exponentialized”, but from (38) we know that $W_0$ and zero modes of higher spin currents are not diagonal in the subspace. The first few level calculation for the $W_3$ CFT shows $\langle G|\exp(t_3 W_0)|G\rangle$ and $Z(t_3, a, q)$ are not equal(turning on $t_2$ only trivially shift the gauge coupling $\tau$). It would be interesting to make this clear.
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7 Appendix: Gram/Shapovalov matrix

The Gram matrix of Shapovalov form is a symmetric sesquilinear form defined on the Verma module, it is block diagonal with each block corresponds to the level $k$ subspace of the Verma module. So we discuss the matrix at each level separately. At the level $k$ it is denoted by $K^{[k]}$, its elements are denoted by $K^{[k]}_{ij}$, we denote the cofactor matrix of $K^{[k]}$ by $\tilde{K}^{[k]}$. Then the inverse Gram/Shapovalov matrix is denoted by $K_{[k]}^{-1}$, and its elements are $K_{[k]}^{-1}_{ij} = \frac{\tilde{K}^{[k]}_{ij}}{\det(K^{[k]})}$.

For SU(2) gauge theory, the corresponding CFT is Liouville CFT with Virasoro symmetry. At the level $k = 1$, the Gram/Shapovalov matrix is one dimensional, $K^{A_1}_{[1]} = 2\Delta$. At the level $k = 2$, it is

$$ K^{A_1}_{[2]} = \begin{pmatrix} 4\Delta + \frac{c}{2} & 6\Delta \\ 6\Delta & 4\Delta(2\Delta + 1) \end{pmatrix} \quad (56) $$

At the level $k = 3$, it is

$$ K^{A_1}_{[3]} = \begin{pmatrix} 6\Delta + 2 & 10\Delta & 24\Delta \\ 10\Delta & 8\Delta(\Delta + 1) + c\Delta & 12\Delta(3\Delta + 1) \\ 24\Delta & 12\Delta(3\Delta + 1) & 24\Delta(\Delta + 1)(2\Delta + 1) \end{pmatrix} \quad (57) $$

For SU(3) gauge theory, the corresponding CFT is $A_2$ Toda CFT with $W_3$ symmetry. At the level $k = 1$, denote the base vector $L_{-1}|v_0\rangle, W_{-1}|v_0\rangle$ by $|i\rangle, i = 1, 2$, then we have

$$ K^{A_2}_{[1]} = \begin{pmatrix} 2\Delta & 3w \\ 3w & \frac{3}{2}D\Delta \end{pmatrix} \quad (58) $$

At the level $k = 2$ denote the base vector $L_{-2}|v_0\rangle, L_{-1}|v_0\rangle, L_{-1}W_{-1}|v_0\rangle, W_{-2}|v_0\rangle, W_{-2}^2|v_0\rangle$ by $|i\rangle, i = 1, 2, 3, 4, 5$, then we have

$$ K^{A_2}_{[2]} = \begin{pmatrix} 4\Delta + \frac{c}{2} & 6\Delta & 9w & 6w & \frac{45}{2}D\Delta \\ 6\Delta & 4\Delta(2\Delta + 1) & 6w(2\Delta + 1) & 12w & 27D\Delta + 18w^2 \\ 9w & 6w(2\Delta + 1) & 9D\Delta^2 + 9D\Delta + 9w^2 & 18D\Delta & \frac{45}{2}w(3D + 1) \\ 6w & 12w & 18D\Delta & 9\Delta(D + 1) & \frac{27}{2}D(3D + 1) \\ \frac{45}{2}D\Delta & 27D\Delta + 18w^2 & \frac{27}{2}D(2\Delta + 3) & \frac{27}{2}w(3D + 1) & \frac{81}{4}D^2\Delta(2\Delta + 1) + \frac{648D\Delta(\Delta + 1) + 4w^2}{22+5c} \end{pmatrix} \quad (59) $$

In this notation, $K^{[2]}_{[2]}(\emptyset, [1]^2; \emptyset, [1]^2) = K^{[2]}_{[2]55}$, and $K^{[2]}([1], [1]; \emptyset, [1]^2) = K^{[2]}_{[2]35}$. As has been shown in [3] that $\det(K^{[2]}_{A_2})$ is factorizable, related to the gauge theory expression through
the identification,\[\det(K^A_2) = 2^{16}3^8(22+5c)^{-4}((\epsilon_1\epsilon_2)^{-12}\prod_{\alpha<\beta}(a^2_{\alpha\beta}-\epsilon_+^2)(a^2_{\alpha\beta}-(\epsilon_+^{\beta}+\epsilon_1^{\beta})^2)(a^2_{\alpha\beta}-(\epsilon_+^{\beta}+\epsilon_2^{\beta})^2). (60)\]

A useful fact for study \[\langle v_2|W_0|v_2 \rangle\] is the element \[\tilde{K}^{-1}_{[2]35}, \text{ as we have} \]
\[\tilde{K}^{-1}_{[2]35} = 2^{4}3^5(22+5c)^{-2}w(\Delta-1)(22w^2+5cw^2-2\Delta^2+c\Delta^2-32\Delta^3)\]
\[(2c+c^2-44w^2-10cw^2-28\Delta-12c\Delta+c^2\Delta+40\Delta^2+16c\Delta^2+64\Delta^3),(61)\]
therefore write in the gauge theory parameters we have
\[K^{-1}_{[2]35} = -((\epsilon_1\epsilon_2)^2(\frac{-4\epsilon_1\epsilon_2-15Q^2}{3})^{3/2}a_1a_2(a_1+a_2)(a_1^2+a_1a_2+a_2^2-\epsilon_+^2+\epsilon_1\epsilon_2)(8a_1^6+24a_1^5a_2+\cdots)\]
\[\prod_{\alpha<\beta}(a^2_{\alpha\beta}-\epsilon_+^2)(a^2_{\alpha\beta}-(\epsilon_+^{\beta}+\epsilon_1^{\beta})^2)(a^2_{\alpha\beta}-(\epsilon_+^{\beta}+\epsilon_2^{\beta})^2)\]
\[8a_1^6+24a_1^5a_2-6a_1^4a_2^2+\cdots-2160\epsilon_1^2\epsilon_2^4-825\epsilon_1^2\epsilon_2^5-128\epsilon_2^6. (62)\]

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