CONJUGACY DEPTH FUNCTION FOR GENERALISED LAMPLIGHTER GROUPS

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ABSTRACT. In this article, we completely characterise the asymptotic behaviour of conjugacy separability for the lamplighter groups. More generally, we give exponential upper and lower bounds for all wreath products of finitely generated abelian groups where the acting group is infinite and base group is finite.

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1. Introduction

Studying infinite, finitely generated groups through their finite quotients is a common method in group theory. Groups in which one can distinguish elements using their finite quotients are called residually finite. Formally speaking, a group $G$ is said to be residually finite if for every pair of distinct elements $f, g \in G$ there exists a finite group $Q$ and a surjective homomorphism $\pi: G \to Q$ such that $\pi(f) \neq \pi(g)$ in $Q$. Group properties of this type are called separability properties and are usually defined by what types of subsets we want to distinguish. In this

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article, we study *conjugacy separability*, meaning that we will study groups in which one can distinguish conjugacy classes using finite quotients. To be more specific, a group $G$ is said to be conjugacy separable if for every pair of nonconjugate elements $f, g \in G$ there exists a finite group $Q$ and a surjective homomorphism $\pi : G \to Q$ such that $\pi(f)$ is not conjugate to $\pi(g)$ in $Q$.

1.1. Motivation. One of the original reasons for studying separability properties in groups is that they provide an algebraic analogue to decision problems in finitely presented groups. To be more specific, if $S \subseteq G$ is a separable subset such that $S$ is recursively enumerable and where one can always effectively construct the image of $S$ under the canonical projection onto a finite quotient of $G$, then one can then decide whether a word in the generators of $G$ represents an element belonging to $S$ simply by checking finite quotients. Indeed, it was proved by Mal’tsev [10], adapting the result of McKinsey [11] to the setting of finitely presented groups, that the word problem is solvable for finitely presented, residually finite groups in the following way. Given a finite presentation $\langle X \mid R \rangle$ and a word $w \in F(X)$ where $F(X)$ is the free group with the generating set $X$, one runs two algorithms in parallel. The first algorithm enumerates all the products of conjugates of the relators and their inverses and checks whether $w$ appears on the list whereas the second algorithm enumerates all finite quotients of $G$ and checks whether the image of the element of $G$ represented by $w$ is nontrivial. In other words, the first algorithm is looking for a witness of the triviality of $w$ whereas the second algorithm is looking for a witness of the nontriviality of $w$. Using an analogous approach, Mostowski [12] showed that the conjugacy problem is solvable for finitely presented, conjugacy separable groups. In a similar fashion, finitely presented, LERF groups have solvable generalised word problem meaning that the membership problem is uniformly solvable for every finitely generated subgroup. In general, algorithms that involve enumerating finite quotients of an algebraic structure are sometimes called algorithms of Mal’tsev-Mostowski type or McKinsey’s algorithms.

Given an algorithm, it is natural to ask how much computing power is necessary to produce an answer. In the case of algorithms of Mal’tsev-Mostowski type, one can measure their space complexity by the associated depth functions. Given a residually finite group $G$ with a finite generating set $S$, its residual finiteness depth function $RF_{G,S} : \mathbb{N} \to \mathbb{N}$ quantifies how deep within the lattice of normal subgroups of finite index of $G$ one needs to look to be able to decide whether or not a word of length at most $n$ represents a nontrivial element. In particular, if $w$ is a word in $S$ of length at most $n$, then either $G$ has a finite quotient of size at most $RF_{G,S}(n)$ in which the image of the element represented by $w$ is nontrivial, and if there is no finite quotient of size less than or equal to $RF_{G,S}(n)$ in which the image of $w$ is nontrivial, then $w$ must represent the trivial word. In particular, we see that the the residual finiteness depth function of $G$ with respect to the generating set $S$ fully determines the size of finite quotients McKinsey’s needs to generate in order to give produce an answer. Since every finite group can be fully described by its Cayley table, we see that the space complexity of the word problem of $G$ with respect to the generating set $S$ can be bounded from above by $(RF_{G,S}(n))^2$. Moreover, the notion of depth function can be generalised to different separability properties. In this note, we study conjugacy separability depth functions which denote as $Conj_{G,S}(n)$ which is a function that measures how deep within the lattice of normal subgroups of finite index one needs to go in order to be able to distinguish distinct conjugacy classes.
of elements of word length at most $n$ with respect to the finite generating subset $S$. Just like in computational complexity, we study these functions up to asymptotic equivalence. See subsection 2.1 for the precise definitions of depth functions and the corresponding asymptotic notions.

1.2. Statement of the results. Not much is known about the asymptotic behaviour of the function $\text{Conj}_{G,S}(n)$ for different classes of groups. The first result of this kind was by Lawton, Louder, and McReynolds [8] who showed that if $G$ is a nonabelian free group or the fundamental group of a closed oriented surface of genus $g \geq 2$, then $\text{Conj}_{G}(n) \preceq n^{2}$. For the class of finitely generated nilpotent groups, the second named author and Deré [4] showed that if $G$ is a finite extension of a finitely generated abelian group, then $\text{Conj}_{G}(n) \preceq (\log(n))^{d}$ for some natural number $d$, and when $G$ is a finite extension of a finitely generated nilpotent group that is not virtually abelian, then there exist natural numbers $d_{1}$ and $d_{2}$ such that $n^{d_{1}} \preceq \text{Conj}_{G}(n) \preceq n^{d_{2}}$. Finally, in [6], the authors of this note gave upper bounds for $\text{Conj}_{A \wr B}(n)$ of wreath products of conjugacy separable groups $A$ and $B$ which generalises Remeslennikov’s classification of conjugacy separable wreath products [13]. However, when applied directly to wreath products of abelian groups, the formulas given in [6] produce rather coarse upper bounds. Applying [6, Theorem C] to the lamplighter group $\mathbb{F}_{2} \wr \mathbb{Z}$, one can then demonstrate that that its conjugacy depth function can be bounded from above by the function $2n^{n^{2}}$. In this note, we only focus on conjugacy depth functions of wreath products of finitely generated abelian groups where the base group is finite. This restriction allows us to use more effective methods to obtain much better upper bounds than those presented in [6]. Additionally, we are able to use methods from commutative algebra to produce lower bounds and in the case of the lamplighter group, we fully determine the asymptotic equivalence class of its conjugacy depth function.

Before stating our main theorem, we introduce some notation. Letting $f,g: \mathbb{N} \to \mathbb{N}$ be non-decreasing functions, we write $f \preceq g$ if there is a constant $C \in \mathbb{N}$ such that $f(n) \leq Cg(Cn)$ for all $n \in \mathbb{N}$. If $f \preceq g$ and $g \preceq f$, we then write $f \approx g$. We denote $\mathbb{F}_{p}$ as the finite field of order $p$ where $p$ is prime.

The following theorem is the main result of this article.

**Theorem 1.1.** Let $A$ be a finite abelian group, and suppose that $B$ is a finitely generated abelian group of torsion free rank $k > 0$. If $k = 1$, i.e. if $B$ is virtually cyclic, then

$$\text{Conj}_{A \wr B}(n) \approx 2^{n}.$$

Otherwise,

$$2^{n} \preceq \text{Conj}_{A \wr B}(n) \preceq 2^{nk}.$$  

As a corollary, we are able to compute the precise asymptotic behaviour of conjugacy separability for the lamplighter groups.

**Corollary 1.2.** $\text{Conj}_{\mathbb{F}_{p} \wr \mathbb{Z}}(n) \approx 2^{n}$.

1.3. Outline of the paper. In Section 2, we recall standard mathematical notions and concepts that will be used throughout the paper. In particular, in subsection 2.1, we recall the notions of word length, depth functions and associated asymptotic notions; in subsection 2.2, we recall the basic terminology of wreath products of groups. Finally, in subsection 2.3, we recall the notion of Laurent polynomial rings.
and show that generalised lamplighter groups can be seen as subgroups of $\text{GL}(2, R)$ where $R$ is the Laurent polynomial ring over $\mathbb{F}_p$. We finish this section by giving a criterion for conjugacy for such groups purely in terms of commutative algebra.

In Section 3 we use methods from commutative algebra to produce lower bounds for the conjugacy depth functions for generalised lamplighter groups by constructing an infinite sequence of pairs of non-conjugate elements that require quotients whose size is exponential in word length in order to remain non-conjugate.

In Section 4 we use combinatorial methods together with the conjugacy criterion for wreath products of abelian groups to construct upper bounds for wreath products of abelian groups.

In Section 5 we combine the lower bounds obtained in Section 3 together with the upper bounds obtained in Section 4 to prove Theorem 1.1.

2. Preliminaries

We denote $\mathbb{F}_p$ as the finite field of $p$ elements where $p$ is prime. We denote $\text{Sym}(n)$ as the symmetric group on $n$ letters. For $x, y \in G$, we say that $x \sim_G y$ if there exists an element $z \in G$ such that $zxz^{-1} = y$ and suppress the subscript when $G$ is clear from context. Whenever the given group is abelian, we will use additive notation.

We say that a subgroup $H \leq G$ is conjugacy embedded in $G$ if for every $f, g \in H$ we have that $f \sim_H g$ if and only if $f \sim_G g$. Following the definition, one can easily check that the relation of being conjugacy embedded is transitive: if $A \leq B \leq C$ such that $A$ is conjugacy embedded in $B$ and $B$ is conjugacy embedded in $C$, then $A$ is conjugacy embedded in $C$.

Given a group $G$, we say that a subgroup $R \leq G$ is a retract of $G$ if there exists a surjective homomorphism $\rho: G \to R$ such that $\rho \mid_R = \text{id}_R$. We say that the map $\rho$ is the canonical retraction corresponding to $R$. The following remark follows straight from the definition of being a retract.

Remark 2.1. Let $G$ be a group, and let $R \leq G$ be a subgroup. If $R$ is a retract of $G$, then $R$ is conjugacy embedded in $G$.

When given a semi-direct product of abelian groups $A \rtimes B$, the next lemma allows us to reduce the study of conjugacy in $A \rtimes B$ to conjugacy in $A \rtimes (B/K)$ where $K$ is the kernel of the action of $B$ on $A$.

Lemma 2.2. Suppose that $A$ and $B$ are finitely generated abelian groups. Then $(a_1, b) \sim (a_2, b)$ in $A \rtimes B$ if and only if $(a_1, b) \sim (a_2, b) \mod (0, K)$ where $K$ is the kernel of the action of $B$ on $A$.

Proof. Since the backwards direction is clear, we may assume that $(a_1, b_1) \sim (a_2, b_2)$ in $A \rtimes B$. Suppose for a contradiction that there exists $(x, y) \in A \rtimes B$ and $k \in K$ such that $(x, y) \cdot (a_1, b) \cdot (x, y)^{-1} = (a_2, b) \cdot (0, k)$. We then have

$$(x, y) \cdot (a_1, b) \cdot (x, y)^{-1} = (a_2, b) \cdot (0, k)$$

$$(x + y \cdot a_1, yb) \cdot (-y^{-1} \cdot x, y^{-1}) = (a_2, b + k)$$

$$(x + y \cdot a_1 - b \cdot x, b) = (a_2, b + k).$$

Hence, we must have that $k = 0$. Therefore, we have $(x, y)(a_1, b) \cdot (x, y)^{-1} = (a_2, b)$ which is a contradiction. Hence, we have our claim. □
Given an abelian group $B$, we will use $\text{Tor}(B)$ to denote its torsion subgroup. If $B$ is finitely generated, then it uniquely splits as

$$B = \text{Tor}(B) \oplus \mathbb{Z}^k$$

for some $k \in \mathbb{N}$. We say that $k$ is the torsion-free rank of $B$. We define $\tau : B \to \text{Tor}(B)$ and $\phi : B \to \mathbb{Z}^k$ as the canonical retractions. Then every element $b \in B$ can be uniquely expressed as $b = \tau(b) + \phi(b)$. We say that $\tau(b)$ is the torsion part of $b$ and $\phi(b)$ is the torsion-free part of $b$.

For ease of notation, we will view direct sums of groups over an indexing set using finitely supported functions on the indexing set. More precisely, if $G = \bigoplus_{i \in I} A$ is a direct sum of copies of a group $A$ indexed by a set $I$, then for $f \in G$ we will write $f(i)$ to denote the $i$-th coordinate of $f$. In particular, elements in $G$ correspond to functions $f : I \to A$ where $f(i) = 1$ for all but finitely many elements in $I$.

2.1. Asymptotic notions and depth functions. Given a finitely generated group $G$ with finite generating subset $S$, we define the word length of $G$ with respect to $S$, denoted $\| : \|_S : G \to \mathbb{N} \cup \{0\}$, as

$$\|g\|_S = \min\{|w| : w \in F(S) \text{ and } w =_G g\}.$$

where $|w|$ denotes the word length of $w$ in $F(S)$. Word length is a standard tool in geometric group theory used to equip $G$ with a left invariant metric $d_S : G \times G \to \mathbb{N}$ given by $d_S(g_1, g_2) = \|g_1^{-1}g_2\|_S$. We use $B_{G,S}(n)$ to denote the ball of radius $n$ centered around the identity with respect to the finite generating subset $S$, i.e. $B_{G,S}(n) = \{g \in G \mid \|g\|_S \leq n\}$. When the finite generating subset is clear from context, we will instead write $B_G(n)$.

If $G$ is a finitely generated group, we define the function $D_G : G\{1\} \to \mathbb{N} \cup \{\infty\}$ as

$$D_G(g) = \min\{|G/N| : N \unlhd_{f.i.} G \text{ and } g \notin N\},$$

with the understanding that $D_G(g) = \infty$ if no such finite quotient exists. We say that $D_G(g)$ is the residual finiteness depth (depth for short) of $g \in G$. The depth of $g$ is the size of the smallest finite quotient of $G$ such that the image of $g$ under the canonical projection is not trivial. We say that $G$ is residually finite if $D_G(g) < \infty$ for all $g \in G\{1\}$. Given a finite generating subset $S$ for a residually finite group $G$, the residual finiteness depth function $\RF_{G,S} : \mathbb{N} \to \mathbb{N}$ is defined as

$$\RF_{G,S}(n) = \max\{D_G(g) : g \in B_{G,S}(n), g \neq 1\}.$$

The conjugacy separability depth function of $G$ is defined in a similar way. Let $f, g \in G$ be a pair of elements such that $f \not\sim_G g$. We then let

$$\text{CD}_G(f,g) = \min\{|G/N| : N \unlhd_{f.i.} G \text{ and } fN \not\sim_{G/N} gN\}$$

with the understanding that $\text{CD}_G(f,g) = \infty$ if no such finite quotient exists. We say that $\text{CD}_G(f,g)$ is the conjugacy separability depth (conjugacy depth for short) of the pair $(f,g)$ in $G$. Similar to the definition of residual finiteness, we say that $G$ is conjugacy separable if $\text{CD}_G(g,h) < \infty$ for all $h \in G$ such that $f \not\sim_G g$. Given a finite generating subset $S \subseteq G$ for a conjugacy separable group $G$, the conjugacy separability depth function $\text{Conj}_{G,S} : \mathbb{N} \to \mathbb{N}$ is defined as

$$\text{Conj}_{G,S}(n) = \max\{\text{CD}_G(f,g) : f, g \in B_{G,S}(n) \text{ and } f \not\sim_G g\}.$$

We note that both the functions $\RF_{G,S}(n)$ and $\text{Conj}_{G,S}(n)$ depend on the choice of the finite generating subset $S$. However, one can easily check that the asymptotic
behaviour does not. It is well known that a change of a finite generating subset
is a quasi-isometry. In particular, if \( S_1, S_2 \subset G \) are two finite generating subsets
of a group \( G \), then \(|g| \cdot |s_1| \approx |g| \cdot |s_2|\). The same holds for depth functions.

For non-decreasing functions \( f, g : \mathbb{N} \to \mathbb{N} \), we write \( f \preceq g \) if there is a constant \( C \in \mathbb{N} \) such that \( f(n) \leq Cg(Cn) \) for all \( n \in \mathbb{N} \), and if \( f \preceq g \) and \( g \preceq f \), we then write \( f \approx g \). It can then be shown that if \( G \) is a residually finite group with two finite
generating sets \( S_1, S_2 \), then \( RF_{G,S_1}(n) \approx RF_{G,S_2}(n) \); see [2] for more details. In the case when \( G \) is conjugacy separable, we also have \( \text{Conj}_{G,S_1}(n) \approx \text{Conj}_{G,S_2}(n) \); see [8] for more details. As we are only interested in the asymptotic behaviour of the
above defined functions, we will suppress the choice of generating subset whenever
we reference the depth functions or the word-length.

We conclude this subsection with the following easy lemma.

**Lemma 2.3.** Suppose that \( G \) is a finitely generated conjugacy separable group with
a finitely generated subgroup \( R \leq G \) such that \( R \) is a retract. Then \( R \) is conjugacy separable, and moreover, we have
\[
\text{Conj}_R(n) \preceq \text{Conj}_G(n).
\]

**Proof.** Let \( \rho : G \to R \) be the corresponding retraction. We start by showing there
is a finite generating set \( X \subseteq G \) such that \( X = X_R \cup X_K \), \( R = \langle X_R \rangle \), and \( \langle X_K \rangle \cap R = \{1\} \). Suppose that \( G = \langle X' \rangle \), where \( X = \{x_1, \ldots, x_m\} \). We set \( X_R = \{\rho(x_1), \ldots, \rho(x_m)\} \) and \( X_K = \{\rho(x_1)^{-1}x_1, \ldots, \rho(x_m)^{-1}x_m\} \). Clearly, \( R = \langle X_R \rangle \) and \( \langle X_K \rangle \leq \ker(\rho) \), so \( \langle X_K \rangle \cap R = \{1\} \). Clearly, \(|r||x_r| = |r||x| \) for every \( r \).

Now suppose that \( r_1, r_2 \in B_R(n) \) are given such that \( r_1 \neq_R r_2 \). Following the
previous paragraph together with remark 2.1, we see that \( r_1, r_2 \in B_G(n) \) and that \( r_1 \neq_G r_2 \). Thus, we will show that \( \text{CD}_R(r_1, r_2) \leq \text{CD}_R(r_1, r_2) \). Suppose that \( N_R \leq f,i R \) realises \( \text{CD}_R(r_1, r_2) \) and that there is \( N_G \leq f,i G \) such that \( r_1 N_G \neq r_2 N_G \) in \( G/N_G \). Since \( |R/(R \cap N_G)| \leq |G/N_G| \), we see that \( r_1(R \cap N_G) \neq r_2(R \cap N_G) \) in \( R/(R \cap N_G) \). It follows that
\[
|R/(R \cap N_G)| \preceq |G/N_G| < |R/N_R|,
\]
which contradicts the assumption that \( N_R \) realises \( \text{CD}_G(r_1, r_2) \). Therefore, we see
that \( \text{Conj}_R(n) \preceq \text{Conj}_G(n) \). \( \square \)

2.2. **Wreath products.** For groups \( A \) and \( B \), we denote the restricted wreath
product of \( A \) and \( B \), written as \( A \wr_{\times} B \), by
\[
A \wr_{\times} B = \left( \bigoplus_{b \in B} A \right) \times B,
\]
where \( B \) acts on \( \bigoplus_{b \in B} A \) via left multiplication on the coordinates. An element
\( f \in \bigoplus_{b \in B} A \) is understood as a function \( f : B \to A \) such that \( f(b) \neq 1 \) for only
finitely many \( b \in B \). With a slight abuse of notation, we will use \( A^B \) to denote \( \bigoplus_{b \in B} A \). The left action of \( B \) on \( A^B \) is then realised as \( b \cdot f(x) = f(bx) \).

The **support** of \( f \) which is the set of elements on which \( f \) is not trivial will be
denoted as
\[
\text{supp}(f) = \{b \in B \mid f(b) \neq 1\}.
\]
The **range** of \( f \) will be denoted as
\[
\text{rng}(f) = \{f(x) \mid x \in B\}.
\]
Following the given notation, if $H \leq A$ and $K \leq B$, we will use $H^K$ to denote the subset

$$H^K = \{ f \in A^B | \text{supp}(f) \subseteq K \text{ and } \text{rng}(f) \subseteq H \}.$$  

Keeping this notation in mind, the wreath product $H \wr K$. The first part of the statement easily follows from Lemma 2.5 and Lemma in particular, and $A$ conjugacy embedded in the group $A$. Let $\rho: A \to B$ be such a function. We see that $\rho$ is a surjective homomorphism and $\rho|_1 = \text{id}_R$, and thus, it follows that $R$ is a retract of $A \wr B$. We finish by noting that Remark 2.1 implies $R$ is conjuguacy embedded in $A \wr B$. □

Suppose that $b \in B$ and $f \in A^B$ is a function with a finite support. We say that $f$ is minimal with respect to $b$ if all elements of $\text{supp}(f)$ lie in distinct cosets of $\langle b \rangle$ in $B$. We will say that an element $fb \in A \wr B$ is reduced if $f$ is minimal with respect to $b$.

The following lemma is a special case of [6] Lemma 5.13.

**Lemma 2.5.** Let $A, B$ be finitely generated groups, and suppose that $b \in B$ and $f: B \to A$ are given such that $fb \in B_{A|B}(n)$. Then there exists a constant $C$ independent of $b$ and $f$ and $f' \in A^B$ such that $f'b \sim fb$, $f'b$ is reduced, and $\|f'b\| \leq C\|fb\|$.

The following statement which provides a conjugation criterion for wreath products of abelian groups follows from [6] Lemma 5.14.

**Lemma 2.6.** Let $A, B$ be abelian groups, and let $G = A \wr B$ be their wreath product. Let $f_1, f_2 \in A^B$, $b_1, b_2 \in B$ be such that the elements $f_1b_1$ and $f_2b_2$ are reduced. Then $f_1b_1 \sim_G f_2b_2$ if and only if $b_1 = b_2$ and $f_1b \in f_2b^B$, i.e. there exists $c \in B$ such that $c\text{supp}(f_1) = \text{supp}(f_2)$ and $f_1(cx) = f_2(x)$ for all $x \in B$.

One interpretation of Lemma 2.6 is that by ensuring that we are only working with reduced elements of $A \wr B$, we only need to worry about them being conjugate by an element from $B$.

**Lemma 2.7.** Let $A$ be a finite abelian group where $p \mid |A|$. The group $F_p \wr \mathbb{Z}$ is conjugacy embedded in the group $A \wr \mathbb{Z}$ and $\text{Conj}_{F_p \wr \mathbb{Z}}(n) \leq \text{Conj}_{A \wr \mathbb{Z}}(n)$. In particular, $\text{Conj}_{F_p \wr \mathbb{Z}}(n) \leq \text{Conj}_{A \wr \mathbb{Z}}(n)$ where $B$ is an infinite, finitely generated abelian group.

**Proof.** The first part of the statement easily follows from Lemma 2.6 and Lemma 2.6.

To show that second part of the statement, first show that $F_p \wr \mathbb{Z}$ is not distorted in $A \wr \mathbb{Z}$. If $\mathbb{Z}/p^n\mathbb{Z} = \langle a \rangle$, we then see that $\mathbb{Z}/p\mathbb{Z} = \langle a^{p^n-1} \rangle$. Letting $X_p = \{a, a^{p^n-1}b \} \subseteq L_p$ and $X_p = \{a^{p^n-1}, b \} \subseteq L_p$, it then follows that $L_p = \langle X_p \rangle$.
and \( L_p = \langle X_p \rangle \). One can easily check that for any \( x \in L_p \) that \( \|x\|_{X_p} = \|x\|_{X_p'} \), and subsequently, \( B_{L_p,X_p}(n) \subseteq B_{L_p, X_p'}(n) \). Now suppose that \( x, y \in B_{\mathbb{F}_p, \mathbb{Z}}(n) \) are not conjugate. We then have that \( f, g \in B_{A|\mathbb{Z}}(n) \), and since \( \mathbb{F}_p \backslash \mathbb{Z} \) is conjugacy embedded into \( A \backslash \mathbb{Z} \), we have that \( f \cong_{A|\mathbb{Z}} g \). Suppose that \( N \trianglelefteq_{f.i.} A \backslash \mathbb{Z} \) realises \( \text{CD}_{A|\mathbb{Z}}(x, y) \). It then follows that \( x(N \cap (\mathbb{F}_p \backslash \mathbb{Z})) \) is not conjugate to \( y(N \cap (\mathbb{F}_p \backslash \mathbb{Z})) \) in \( \mathbb{F}_p \backslash \mathbb{Z}/((\mathbb{F}_p \backslash \mathbb{Z}) \cap N) \). We then have that
\[
\text{CD}_{\mathbb{F}_p}(x, y) \leq |\mathbb{F}_p \backslash \mathbb{Z}/((\mathbb{F}_p \backslash \mathbb{Z}) \cap N)| = \text{CD}_{A|\mathbb{Z}}(x, y),
\]
and as a consequence of the above inequality, we see that \( \text{Conj}_{\mathbb{F}_p,\mathbb{Z}}(n) \leq \text{Conj}_{A|\mathbb{Z}}(n) \). Since \( \mathbb{Z} \) is a retract of \( B \), Lemma \[2.4\] implies that \( \text{Conj}_{A|\mathbb{Z}}(n) \leq \text{Conj}_{A|\mathbb{B}}(n) \). Therefore, \( \text{Conj}_{\mathbb{F}_p,\mathbb{Z}}(n) \leq \text{Conj}_{A|\mathbb{B}}(n) \).

For an element in a wreath product, the next lemma relates the size of the support of its function part and the size of the elements in the range of the function part to the word length.

**Lemma 2.8.** Let \( A, B \) be finitely generated groups and let \( G = A \wr B \) be their wreath product. Then there exists a constant \( C > 0 \) such that if \( g = fb \) where \( f \in A^B \) and \( b \in B \), then
\[
\begin{align*}
\text{(i)} & \quad \text{supp}(f) \subseteq B_B(C\|g\|), \\
\text{(ii)} & \quad \text{rng}(f) \subseteq B_A(C\|g\|), \\
\text{(iii)} & \quad b \in B_B(C\|g\|).
\end{align*}
\]

**Proof.** Use \[3\] Theorem 3.4].

Given a wreath product \( A \wr B \) with a surjective homomorphism \( \pi : B \to \overline{B} \), we denote \( \overline{\pi} : A \wr B \to A \wr \overline{B} \) as the canonical extension of \( \pi \) to all of \( A \wr B \). Similarly, if \( \pi : A \to \overline{A} \) is a surjective homomorphism, we let \( \overline{\pi} : A \wr B \to \overline{A} \wr B \) as the canonical extension of \( \pi \) to all of \( A \wr B \).

### 2.3. Linear groups over Laurent polynomial rings.

Much of the following discussion, which includes undefined notation and terms, can be found in \[1\] 5 7. We will write \( \mathbb{F}_p[x] \) to denote the ring of polynomials in the variable \( x \) with coefficients in \( \mathbb{F}_p \), and we will use \( \mathbb{F}_p[x, x^{-1}] \) to denote the ring of Laurent polynomials over \( \mathbb{F}_p \).

We first note that \( \mathbb{F}_p[x, x^{-1}] \) is the localisation of the ring \( \mathbb{F}_p[x] \) on the set \( S = \{ x^m \mid m \in \mathbb{N} \} \). We then have that the ideals of \( \mathbb{F}_p[x, x^{-1}] \) are in one-to-one correspondence with ideals of \( \mathbb{F}_p[x] \) that don’t intersect the set \( S \). In particular, for any ideal \( \mathcal{I} \subset \mathbb{F}_p[x] \) where \( \mathcal{I} \cap S = \emptyset \), we have that \( \mathbb{F}_p[x, x^{-1}]/(\mathcal{I} S^{-1}) = \mathbb{F}_p[x, x^{-1}]/\mathcal{I} \). We finish by observing that the maximal ideals of \( \mathbb{F}_p[x, x^{-1}] \) can be written as \( \mathcal{I} = (f) \) where \( f \) is an irreducible polynomial not in \( S \). If \( k = \deg(f) \), then \( \mathbb{F}_p[x, x^{-1}]/\mathcal{I} = p^k \).

We now focus on the following representation of \( \mathbb{F}_p \backslash \mathbb{Z} \) as a group of matrices with coefficients in the ring \( \mathbb{F}_p[x, x^{-1}] \). First, let us define a function \( P : \mathbb{F}_p^\mathbb{Z} \to \mathbb{F}_p[x, x^{-1}] \) given by
\[
P(f) = \sum_{m \in \mathbb{Z}} f(m)x^m.
\]
One can easily check in the context of finitely supported functions that \( P \) is a bijection and for any \( r \in \mathbb{F}_p, \ f, g \in \mathbb{F}_p^\mathbb{Z}, \) and \( m \in \mathbb{Z} \) that the following holds:
\[
\begin{align*}
\text{(i)} & \quad P(rf) = rP(f), \\
\text{(ii)} & \quad P(f + g) = P(f) + P(g),
\end{align*}
\]
Lemma 2.9. The group $\mathbb{F}_p \wr \mathbb{Z}$ is isomorphic to the subgroup of $\text{GL}(2, \mathbb{F}_p[x, x^{-1}])$ given by
\[
L_p = \left\{ \begin{bmatrix} x^m & P \\ 0 & 1 \end{bmatrix} : m \in \mathbb{Z}, P \in \mathbb{F}_p[x, x^{-1}] \right\}.
\]

Proof. Let $\varphi : \mathbb{F}_p \wr \mathbb{Z} \rightarrow L_p$ be the map given by
\[
\varphi(fm) = \begin{bmatrix} x^m & P(f) \\ 0 & 1 \end{bmatrix}.
\]

It is easy to see that this map is bijective. Thus, we need to show that it is a homomorphism. This can be done directly by computation. Let $f_1, f_2 \in \mathbb{F}_p$ and $m_1, m_2 \in \mathbb{Z}$ be arbitrary. We can then write:
\[
\begin{align*}
\varphi((f_1 m_2) \cdot (f_2 m_2)) &= \begin{bmatrix} x^{m_1} & P(f_1) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^{m_2} & P(f_2) \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} x^{m_1} x^{m_2} & P(f_1) x^{m_1} P(f_2) \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} x^{m_1 + m_2} & P(f_1 + m_1 \cdot f_2) \\ 0 & 1 \end{bmatrix} \\
&= \varphi((m_1 + m_2, f_1 + m_1 \cdot f_2) = \varphi((f_1 m_1) \cdot (f_2 m_2)).
\end{align*}
\]

For the rest of this article, we will identify $\mathbb{F}_p \wr \mathbb{Z}$ with $L_p$ without mention and denote the element
\[
\begin{bmatrix} x^m & P \\ 0 & 1 \end{bmatrix}
\]
as $(P, m)$.

The following lemma allows us to understand finite quotients of $\mathbb{F}_p \wr \mathbb{Z}$ in terms of the cofinite ideals of $\mathbb{F}_p[x, x^{-1}]$. For the following lemma, we identify $\mathbb{F}_p[x, x^{-1}]$ with the normal subgroup of $\mathbb{F}_p \wr \mathbb{Z}$ given by elements of the form $(P, 0)$ where $P \in \mathbb{F}_p[x, x^{-1}]$.

Lemma 2.10. Let $N \trianglelefteq \mathbb{F}_p \wr \mathbb{Z}$. Then $N \cap \mathbb{F}_p[x, x^{-1}]$ is an ideal in $\mathbb{F}_p[x, x^{-1}]$. In particular, if $N \trianglelefteq_{f,i} \mathbb{F}_p \wr \mathbb{Z}$, then $N \cap \mathbb{F}_p[x, x^{-1}]$ is a cofinite ideal of $\mathbb{F}_p[x, x^{-1}]$.

Proof. Let $M = N \cap \mathbb{F}_p[x, x^{-1}]$. We note for $(P, 0) \in M$ that
\[
(0, m)(P, 0)(0, -m) = (x^m P, 0) \in M
\]
since $M$ is normal. In particular, we have that $M$ is closed under multiplication by $x^m$ in $\mathbb{F}_p[x, x^{-1}]$ for all $m \in \mathbb{Z}$. Additionally, for $(P_1, 0), (P_2, 0) \in M$ we have that
\[
(P_1, 0)(P_2, 0) = (P_1 + P_2, 0).
\]
That implies $M$ is closed under addition. It is clear for $r \in R$ and $P \in \mathbb{F}_p[x, x^{-1}]$ that $(rP, 0) \in M$ since multiplying $P$ by $r$ is the same as adding $r$ copies of $P$. Thus, for $(P, 0) \in M$ and a general element $\sum_{m \in \mathbb{Z}} a_m x^m$ of $\mathbb{F}_p[x, x^{-1}]$, we may write
\[
P \cdot \sum_{m \in \mathbb{Z}} a_m x^m = \sum_{m \in \mathbb{Z}} a_m x^m P \in M.
\]
Thus, $M$ is an ideal in $\mathbb{F}_p[x, x^{-1}]$. Moreover, the second part of the statement immediately follows.
The following lemma gives the explicit expression for the conjugacy class of an arbitrary element of $\mathbb{F}_p \wr \mathbb{Z}$.

**Lemma 2.11.** For $(P, m) \in \mathbb{F}_p \wr \mathbb{Z}$, its conjugacy class is given by
\[
\{(x^fP + (x^m - 1)Q, m) \mid \ell \in \mathbb{Z}, Q \in \mathbb{F}_p[x, x^{-1}]\}.
\]

**Proof.** Let $Q \in \mathbb{F}_p[x, x^{-1}]$ and $\ell \in \mathbb{Z}$ be arbitrary. Then we have
\[
(Q, \ell)(P, m)(Q, \ell)^{-1} = (Q + x^fP, \ell + m)\left((-x^{-\ell}Q, -\ell)\right)
\]
\[
= (Q + x^fP - x^{\ell+m}x^{-\ell}Q, \ell + m - \ell)
\]
\[
= (Q + x^fP - x^mQ, m)
\]
\[
= (x^fP + (1 - x^m)Q, m).
\]

Since $Q$ was arbitrary, we may replace it by $-Q$ allowing us to write
\[
(Q, \ell)(P, m)(Q, \ell)^{-1} = (x^fP + (x^m - 1)Q, m).
\]

From here, our statement is clear. □

Let $S$ be a finite generating subset for $\mathbb{F}_p \wr \mathbb{Z}$. For a Laurent polynomial $P \in \mathbb{F}_p[x, x^{-1}]$, we finish this section with the following lemma which bounds the coefficients of $P$ in terms of the word length of $P$ with respect to $S$ when viewed as an element of $\mathbb{F}_p \wr \mathbb{Z}$. Its proof is a straightforward consequence of Lemma 2.8.

**Lemma 2.12.** Let $S$ be a finite generating subset for $\mathbb{F}_p \wr \mathbb{Z}$. There exists a constant $C > 0$ satisfying the following. Let $x = (P, v)$ where $P \in \mathbb{F}_p[x, x^{-1}]$ and $v \in \mathbb{Z}$. If $ax^\ell$ is a monomial in $P$, then $|\ell| \leq C\|x\|_S$ and $|a| \leq p$. 

### 3. Lower bounds

In this section, we provide asymptotic lower bounds for $\text{Conj}_{\mathbb{F}_p \wr \mathbb{Z}}(n)$.

**Proposition 3.1.** Let $A$ be a finite abelian group and $B$ be an infinite, finitely generated abelian group. Then $2^n \preceq \text{Conj}_{A \wr B}(n)$.

**Proof.** By Lemma 2.7, we may assume that $A \cong \mathbb{F}_p$ for some prime and that $B \cong \mathbb{Z}$. We need to find an infinite sequence of pairs of elements $\{f_i, g_i\}_{i=1}^\infty$ such that
\[
(i) \lim_{i \to \infty} \max \{\|f_i\|, \|g_i\|\} = \infty,
(ii) f_i \cong g_i,
(iii) p^{C}\max\{\|f_i\|, \|g_i\|\} \leq \text{CD}_{\mathbb{F}_p \wr \mathbb{Z}}(f_i, g_i),
\]
where $C > 0$ is some constant.

Let $\{q_i\}_{i=1}^\infty$ be an enumeration of the set of primes greater than $p$ such that $p$ is a primitive root mod $q_i$. In this case, it is well known that $\psi_{q_i}(x) = \sum_{i=1}^{q_i-1} x^i$ is an irreducible polynomial over $\mathbb{F}_p$. Let
\[
f_i = (x^{q_i} - 1, q_i) \quad \text{and} \quad g_i = (x - 1 + x^{q_i} - 1, q_i).
\]

Let us consider the quotient $\mathbb{F}_p[x, x^{-1}]/(\psi_{q_i}(x)) \times (\mathbb{Z}/q_i\mathbb{Z})$ with the associated projection map $\pi_i$. We then see that
\[
|\mathbb{F}_p[x, x^{-1}]/(\psi_{q_i}(x)) \times (\mathbb{Z}/q_i\mathbb{Z})| = q_ip^{q_i-1}
\]
and that
\[
\pi_i(f_i) = (0, 0) \quad \text{and} \quad \pi(g_i) = (x - 1, 0) \neq (0, 0).
\]
It follows that \( \pi(f_i) \sim \pi(g_i) \). Subsequently, we see that \( f_i \) and \( g_i \) are not conjugate in \( F \) and that \( \text{CD}_{F,\Z}(f_i, g_i) \leq p^{|g_i|} \).

To finish, we need to demonstrate that \( p^{|g_i|} \leq \text{CD}_{F,\Z}(f_i, g_i) \) for all \( i \). In other words, we need to show that if \( N \trianglelefteq f, \) \( F \) is a normal subgroup in \( N \) is a proper ideal in \( F \). In particular, \( \mathcal{J}_N = N \cap F_p[x, x^{-1}] \) is an ideal in \( F_p[x, x^{-1}] \) where \( \mathcal{J}_N \) is a finite dimensional module over \( F_p \). In particular, \( \mathcal{J}_N \times N \cap \Z \leq N \) is a normal subgroup in \( F_p \) such that if \( f_i \sim g_i \mod N \), then \( f_i \sim g_i \mod N \). Thus, we may assume that \( N \equiv \mathcal{J} \times \mathcal{J} \mod \Z \) for some \( d \in \N \). If \( \mathcal{J}_N = F_p[x, x^{-1}] \), then \( (F_p \times \Z) \) is a finite abelian group. In particular, we have that \( \mathcal{J}_N \) is a proper ideal in \( F_p[x, x^{-1}] \). Moreover, we have that \( |F_p[x, x^{-1}] / \mathcal{J}_N| < p^{|g_i|} \).

Since \( F_p[x, x^{-1}] \) is a localisation of a PID, it is a PID. Therefore, there exists a polynomial \( R \in F[x] \) such that \( \mathcal{J}_N = (R) \). Thus, we note that one of the following cases must hold:

\[
\text{gcd}(x^{q_i} - 1, R) = \begin{cases} 
  x^{q_i} - 1, & \\
  \psi_{q_i}, & \\
  x - 1, & \\
  1, & 
\end{cases}
\]

Let us first note that \( F_p[x, x^{-1}] / (R) \leq (F_p[x, x^{-1}] \times \Z) / N \). We see that we may ignore the first two cases, as in both we have that \( p^{|g_i|} \leq |F_p \times \Z / N| \).

For the third case, we have that \( x - 1 \in \mathcal{J}_N \) which implies that

\[
\pi_N(f_i) = (x^{q_i} - 1 \mod \mathcal{J}_N, q_i \mod t) = ((x - 1)\psi_{q_i}(x) \mod \mathcal{J}_N, q_i \mod t) = (0, q_i \mod t).
\]

Similarly, we have

\[
\pi_N(g_i) = (x - 1 + (x - 1)\psi_{q_i}(x) \mod \mathcal{J}, q_i \mod t) = (0, q_i \mod t).
\]

Hence, \( \pi_N(f_i) = \pi_N(g_i) \). Thus, we may assume that \( \text{gcd}(x^{q_i} - 1, R) = 1 \). Following Lemma 2.11, we can write the conjugacy class of \( f_i \) as

\[
\{ (x^n(x^{q_i} - 1) + (x^{q_i} - 1)\lambda, q_i) \mid n \in \Z, \lambda \in F_p[x, x^{-1}] \}.
\]

In order for \( f_i \sim g_i \mod N \), we need to have

\[
x - 1 + x^{q_i} - 1 \in \{ x^n(x^{q_i} - 1) + (x^{q_i} - 1)\lambda \mid n \in \Z, \lambda \in F_p[x, x^{-1}] \} \mod (R).
\]

The above is equivalent to

\[
x - 1 \in \{ (x^n + \lambda - 1)(x^{q_i} - 1) \mid n \in \Z, \lambda \in F_p[x, x^{-1}] \} \mod (R).
\]

Using basic algebra, we see that the above is equivalent to

\[
x - 1 \in \{ \lambda(x^{q_i} - 1) \mid \lambda \in F_p[x, x^{-1}] \} \mod (R).
\]

Thus, we have that \( f_i \sim g_i \mod N \) if and only if \( x - 1 \in (x^{q_i} - 1) \mod (R) \).

Since \( \text{gcd}(x^{q_i} - 1, R) = 1 \), there exist polynomials \( \alpha, \beta \in F_q[X] \) such that

\[
(x^{q_i} - 1)\alpha + R\beta = 1.
\]
By multiplying through by $x - 1$, we may write

\[ x - 1 = (x - 1)(x^q - 1)\alpha + (x - 1)R\beta. \]

Reducing mod $(R)$, we may write

\[ x - 1 = (x - 1)(x^q - 1) \mod (R). \]

We see that $f_i \sim g_i \mod N$, and therefore,

\[ p^n < C\text{D}_{\mathbb{F}_p\mathbb{Z}}(f_i, g_i) \leq q_i p^{q_i - 1}. \]

\[ 2^n \leq C\text{D}_{\mathbb{F}_p\mathbb{Z}}(n) \]

since $2^n \approx p^n$. \qed

4. Upper bounds

The aim of this section is to construct upper bounds for the conjugacy depth function of a wreath product $A \wr B$ of finitely generated abelian groups where the base group $A$ is finite. The idea is to show that we can always find a reasonably small quotient of the acting group $B$ such that Lemma 2.6 can be used to demonstrate that the images of the elements are not conjugate. Recall that one of the assumptions of Lemma 2.6 is that we are working with reduced elements, i.e. the elements of the supports lie in distinct cosets of the acting element. Thus, in order to ensure we have reduced elements, we show in subsection 4.1 how to construct a finite quotient of finitely generated abelian group that separates finite subsets and infinite cyclic subgroups. Subsection 4.2 then deals with the conditions that Lemma 2.6 uses to establish non-conjugacy. In particular, we show that if a quotient of a finitely generated abelian group is of sufficient size, then certain finite subsets do not become translates of each other. Finally, subsection 4.3 combines these methods to construct a finite quotient preserving non-conjugacy of our given non-conjugate elements and gives an upper bound on its size in terms of their word lengths.

Before we proceed, we recall some notation. If $B$ is a finitely generated abelian group, we may write $B = \text{Tor}(B) \times \mathbb{Z}^k$ where $\text{Tor}(B)$ is the subgroup of finite order elements of $B$ and $k$ is the torsion-free rank of $B$. Letting $\phi: B \to \mathbb{Z}^k$ and $\tau: B \to \text{Tor}(B)$ denote the natural projections, we may then write every $x \in B$ uniquely as $x = \phi(x) + \tau(x)$ where we refer to $\phi(x)$ as the torsion-free part of $x$ and $\tau(x)$ as the torsion part of $x$. When given a vector $b = (b_1, \ldots, b_k) \in \mathbb{Z}^k$, we denote $\gcd(b) = \gcd(b_1, \ldots, b_k)$.

4.1. Simultaneous cosets. In this subsection, we study effective separability of cosets of cyclic subgroups in finitely generated abelian groups. Given an infinite, finitely generated abelian group $G$, an element $b \in G$, and a finite subset $S \subseteq B_G(\ell)$, we give an upper bound in terms of $\|b\|$ and $\ell$ on the size of a finite quotient of the group $G$ such that each pair of cosets of the cyclic subgroup generated by $b$ corresponding to two distinct elements in $S$ remain distinct. In the following arguments, we use the observation that $s_1(b) = s_2(b)$ if and only if $s_1^{-1}s_2 \in (b)$.

The following lemma is important for the proof of Lemma 4.3.
Lemma 4.1. Let $b \in \mathbb{Z}$ be an integer where $b \in [-n, n]$ and $S \subseteq \mathbb{Z}$ be a subset such that $S \subseteq [-Cn, Cn]$ for some constant $C > 0$. Suppose that $m = 2|b|c$ for some number $c > 0$ satisfying $|bc| > Cn$, and let $\pi : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be the canonical projection. Then for every $s \in S$, we have that $\pi(s) \in \langle \pi(b) \rangle$ in $\mathbb{Z}/m\mathbb{Z}$ if and only if $s \in \langle b \rangle$ in $\mathbb{Z}$. Furthermore, if $\pi(s) \in \langle \pi(b) \rangle$, then $\pi(s) = t\pi(b)$ for the smallest integer $t$ with respect to the absolute value such that $s = tb$. In particular, $|t| \leq m$.

Proof. Observe that the map $\pi : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is injective on the interval $[-|b| - 1, c|b|]$. We then note that

$$\pi((b)) = \pi(b\mathbb{Z}) = \pi(\{-(c - 1)|b|, -(c - 2)|b|, \ldots, -|b|, 0, |b|, \ldots, (c - 1)|b|, c|b|\}).$$

Thus, if $\pi(s) \in \langle \pi(b) \rangle$ for some $s \in S$, then $s \in \langle b \rangle$ since the map $\pi$ is injective on the interval $[-|b| - 1, c|b|]$.

Finally, suppose that $\pi(s) \in \langle \pi(b) \rangle$ and that $\pi(s) = a\pi(b)$ in $\mathbb{Z}/m\mathbb{Z}$ where $a \in \mathbb{Z}$ is the smallest such value with respect to the absolute value. Following the previous argument, it follows that $ab \in [-|b| - 1, c|b|]$, and therefore, we have $s = ab$ in $\mathbb{Z}$.

To deal with the higher-dimensional cases, we first prove two technical lemmas.

Lemma 4.2. Let $b = (b_1, \ldots, b_k) \in \mathbb{Z}^k$ be non-trivial, and let $\varphi : \mathbb{Z}^k \to \mathbb{Z}$ be the homomorphism given by $\varphi_u(b) = u \cdot b$ where $\cdot$ is the dot product of vectors. Then there are vectors $\lambda^{(1)}, \ldots, \lambda^{(k-1)} \in \mathbb{Z}^k$ such that $\ker(\varphi_u) = \langle \lambda_1, \ldots, \lambda_{k-1} \rangle$ and $\|\lambda^{(i)}\| \leq 2^{k-1}\|b\|$ for all $i$.

Proof. We define vectors $b^{(1)}, \ldots, b^{(k-1)}$ in the following way:

$$b^{(1)} = (-b_2, b_1, 0, \ldots, 0),$$

$$\vdots$$

$$b^{(k-1)} = (0, \ldots, 0, -b_{k-1}).$$

We set $\lambda^{(1)} = \frac{1}{\gcd(-b_2, b_1)} b^{(1)}$, and note that if $k = 2$, then $\ker(\varphi_u) = \langle \lambda^{(1)} \rangle$. Since $\|\lambda^{(1)}\| \leq \|b\|$, we are done. For $k > 2$, we will inductively build a generating set for $\ker(\varphi_u)$ satisfying the statement of the lemma. We start with some basic observations.

By construction, we have that $b^{(1)}, \ldots, b^{(k-1)} \in \ker(\varphi_u)$. Let $\Lambda_i$ be the maximal subgroup of $\mathbb{Z}^k$ of rank $i$ that contains $b^{(1)}, \ldots, b^{(i)}$. Since the vectors $b^{(1)}, \ldots, b^{(i)}$ are linearly independent over $\mathbb{R}$, we immediately see that $\Lambda_1 \leq \cdots \leq \Lambda_{k-1} = \ker(\varphi_u)$ and that $\Lambda_i/\Lambda_{i-1} \cong \mathbb{Z}$ for every $i = 2, \ldots, k-1$.

Now assume that we already have a set of generators for $\Lambda_{i-1}$ which we denote as $\lambda^{(1)}, \ldots, \lambda^{(i-1)}$. By construction, the elements $\{\lambda^{(1)}, \ldots, \lambda^{(i-1)}\}$ satisfy $\Lambda_j = \langle \lambda_1, \ldots, \lambda_j \rangle$ for all $j < i$ where $\|\lambda_j\| \leq 2^{i-1-j}\|b_j\|$ for $1 \leq j \leq i - 1$. Denote $L_i = \langle \Lambda_{i-1}, b^{(i)} \rangle$. Since $\Lambda_{i-1} \leq \langle \Lambda_{i-1}, b^{(i)} \rangle \leq \Lambda_i$, we see that $\Lambda_i/L_i$ is a finite cyclic group. Furthermore, a preimage of some of it’s generator must be contained within the $i$-dimensional parallelogram given by the vectors $\lambda^{(1)}, \ldots, \lambda^{(i-1)}, b^{(i)}$. In particular, we see that $\|\lambda^{(i)}\| \leq \|b^{(i)}\| + \sum_{j=1}^{i-1} \|\lambda^j\|$. One can then easily check that

$$\|\lambda^{(i)}\| \leq \|b^{(i)}\| + \sum_{j=1}^{i-1} 2^{i-1-j}\|b^{(j)}\|.$$ 

Noting that $\|b^{(i)}\| = |b_i| + |b_{i+1}|$, we see that $\|\lambda^{(i)}\| \leq 2^{i-1}\|b\|$ as desired. □
Lemma 4.3. Let $k > 1$, and suppose that $b = (b_1, \ldots, b_k) \in \mathbb{Z}^k$ is a vector where $\gcd(b_1, \ldots, b_k) = 1$. Then $b$ belongs to some free base of $\mathbb{Z}^k$, and moreover, there exists a matrix $T \in \text{GL}_k(\mathbb{Z})$ such that $T(b) = (1, 0, \ldots, 0)$ and $T([-1, 1]^k) \subseteq [2^{k-1} \|b\|, 2^{k-1} k \|b\|]^k$.

Proof. [9] Theorem 9] implies there are integers $a_1, \ldots, a_k \in \mathbb{Z}$ such that

$$\sum_{i=1}^k a_i b_i = \gcd(b_1, \ldots, b_k) = 1$$

and where $\max\{|a_i|\} \leq \frac{1}{k} \max\{|b_i|\}$. Thus, we denote $a = (a_1, \ldots, a_k) \in \mathbb{Z}^k$.

Let $\varphi_k : \mathbb{Z}^k \to \mathbb{Z}$ be the linear map given by $\varphi_k(x) = x \cdot b$. Lemma 4.2] implies there are vectors $\lambda^{(1)}, \ldots, \lambda^{(k-1)} \in B_{2^k}(2^{k-1} \mathbb{Z}) \subseteq [2^{k-1} \|b\|, 2^{k-1} k \|b\|]^k$ that freely generate $\ker(\varphi_k)$. We can then form the matrix $T$ by setting the first row to be equal to the vector $a$ and the remaining $k-1$ vectors to be equal to the vectors $\lambda^{(1)}, \ldots, \lambda^{(k-1)}$, respectively. By construction, we see that $T(b) = (1, 0, \ldots, 0)$. Since $\text{Im}(\varphi_k) = \langle \varphi_k(a) \rangle$, we see that $\mathbb{Z}^k \subseteq \langle b \rangle \oplus \ker(\varphi_k)$ which implies that the row vectors of $T$ generate $\mathbb{Z}^k$. Therefore, $T \in \text{GL}_k(\mathbb{Z})$.

To finish the proof, we recall that $a \in [-\frac{1}{2} \|b\|, \frac{1}{2} \|b\|]^k$ and $\|\lambda^{(i)}\| \leq 2^{k-1} \|b\|$ for all $i$. In particular, this means that for every $i, j \in \{1, \ldots, k\}$ we have the $(i, j)$-th entry of $T$ which we denote as $T_{i,j}$ satisfies $|T_{i,j}| \leq 2^{k-1} \|b\|$. Therefore, we have

$$T([-1, 1]^k) \subseteq [-2^{k-1} k \|b\|, 2^{k-1} k \|b\|]^k. \quad \Box$$

Lemma 4.4. Let $k > 1$ and $n \in \mathbb{N}$ be fixed. Let $S \subseteq \mathbb{Z}^k$ and $b = (b_1, \ldots, b_k) \in \mathbb{Z}^k$ be given such that $b \in B_{2^k}(n)$ and where $S \subseteq B_{2^k}(Cn)$ for some $C > 0$. Then for every integer $m$ divisible by $c = \gcd(b_1, \ldots, b_k)$ and where $m$ satisfies the inequality $m > 2^k k C n^2$, the homomorphism $\pi : \mathbb{Z}^k \to (\mathbb{Z}/m\mathbb{Z})^k$ given by reduction of each coordinate mod $m$ is injective on the set $S$ and for every $s \in S$ we have that $\pi(s) \in \langle \pi(b) \rangle$ if and only if $s \in \langle b \rangle$. Furthermore, if $\pi(s) \in \langle \pi(b) \rangle$, then there is an integer $t \in \mathbb{Z}$ such that $s = tb$ and $|t| \leq m/c$.

Proof. Set $b' = \frac{1}{k} b$ and denote $b' = (b'_1, \ldots, b'_k)$. Since $\gcd(b'_1, \ldots, b'_k) = 1$, Lemma 4.3 implies there exists a matrix $T \in \text{GL}_k(\mathbb{Z})$ such that $T(b') = e_1$ and $T([-1, 1]^k) \subseteq [-2^{k-1}kn, 2^{k-1}kn]$ where $\{e_1, \ldots, e_k\}$ is the canonical free basis of $\mathbb{Z}^k$. Since $T$ is an automorphism of $\mathbb{Z}^d$ we see that $T(s) \in \langle T(b) \rangle = \langle e_1 \rangle$ if and only if $s \in \langle b \rangle$. We note that $T(S) \subseteq T(B_{2^d}(Cn)) \subseteq T([-Cn, Cn]^k) \subseteq [-2^{k-1}kCn^2, 2^{k-1}kCn^2]^k$.

Set $m = 2cl$ where $l \in \mathbb{N}$ is the smallest natural number such that $cl > 2^{k-1} k C n^2$, and denote $K = m \mathbb{Z}^k \subseteq \mathbb{Z}^k$. By construction, we have that $m \leq 2^{k+1} k C n^2$. Therefore, we see that the projection $\pi_K : \mathbb{Z}^k \to \mathbb{Z}^k/m\mathbb{Z}^k$ is injective on the hypercube $H = [-m+1,m]^k$. In particular, since $S \subseteq H$, for any $s \in S$ we have that $T(s)(m\mathbb{Z}^k) \subseteq \langle e_1 \rangle K$ if and only if $T(s) \in \langle e_1 \rangle$. It then follows that $\pi(s) \notin \langle \pi(b) \rangle$ whenever $s \notin \langle b' \rangle$.

Now suppose that $s \in \langle b' \rangle$, i.e. $T(s) \in \langle e_1 \rangle$. In this case, we may retract onto the first coordinate and assume that we are working in $\mathbb{Z}$. The rest of the statement then follows by Lemma 4.3. \Box

Lemma 4.5. Let $B$ be a finitely generated infinite abelian group of torsion free rank $k$. Let $b \in B_B(n)$ and $S \subseteq B_B(Cn)$ be given for some $C > 0$.\]
If $k = 1$, assume that $m \in \mathbb{N}$ satisfies $m \geq 2Cn$ and where both $\|\phi(b)\|$ and $\exp(T)$ divide $m$. If $k \geq 2$, assume that $m \in \mathbb{N}$ is such that $m \geq k^2Cn^2$ and both $c = \gcd(\phi(b))$ and $\exp(\text{Tor}(B))$ divide $m$. Then the homomorphism

$$
\pi: \text{Tor}(B) \oplus \mathbb{Z}^k \to \text{Tor}(B) \oplus (\mathbb{Z}/m\mathbb{Z})^k
$$

defined as the identity on $\text{Tor}(B)$ and as the coordinate-wise projection on $\mathbb{Z}^k$ is injective on the set $S$ and for every $s \in S$ we have that $\pi(s) \in \langle \pi(b) \rangle$ if and only if $s \in \langle b \rangle$.

**Proof.** Since the result follows from Lemma 4.4 when $B$ is torsion free, we may assume that $\text{Tor}(B) \neq 0$. Set $e = \exp(\text{Tor}(B))$.

The main argument of the proof when $k = 1$ is analogous to the case when $k \geq 2$, but instead of Lemma 4.4 one would use Lemma 4.1. For this reason, we leave proof in the case when $k = 1$ as an exercise.

Denote $e = \exp(\text{Tor}(B))$, and suppose that $m > 0$ and $\pi: \text{Tor}(B) \oplus \mathbb{Z}^k \to \text{Tor}(B) \oplus (\mathbb{Z}/m\mathbb{Z})^k$ are as in the statement of the lemma. Now let us assume that $\pi(s) \in \langle \pi(b) \rangle$ for some $s \in S$. Thus, there is some $t \in \mathbb{N}$ such that $\pi(s) = t\pi(b)$ which we pick to be as small possible. In particular, we see that $t \leq \gcd(m,e) = m$.

We write:

$$
\pi(s) = t\pi(b)
$$

$$
\tau(s) + \pi(\phi(s)) = t\tau(s) + t\pi(\phi(b)),
$$

from which immediately see that $\tau(s) = t\tau(b)$ and $\pi(\phi(s)) = t\pi(\phi(b))$. Following Lemma 4.4 we see that $\phi(s) = t\phi(b)$. Therefore, we may write

$$
s = t\tau(b) + t\phi(b) = tb.
$$

4.2. **Translations.** We say that an ordered list $X = (x_1, \ldots, x_m) \subseteq G$ is a translate of an ordered list $Y = (y_1, \ldots, y_n) \subseteq G$ if $n = m$ and there exists a permutation $\sigma \in \text{Sym}(n)$ such that

$$
x_{\sigma(1)}y_1^{-1} = \cdots = x_{\sigma(n)}y_n^{-1}.
$$

In this case, we say that $\sigma$ realises a translation of $X$ onto $Y$. We have the following lemma.

**Lemma 4.6.** Let $G_1, G_2$ be groups and let $G = G_1 \times G_2$ be their direct product with canonical projections $\pi_1: G \to G_1$ and $\pi_2: G \to G_2$. If $X,Y \subseteq G$ are two finite subsets, then the set $X$ is a translate of the set $Y$ if and only if $|X|,|Y| = n$ for some $n$ and there exists $\sigma \in \text{Sym}(n)$ such that $\sigma$ realises a translation of the list $(\pi_1(x_1), \ldots, \pi_1(x_n))$ onto the list $(\pi_1(y_1), \ldots, \pi_1(y_n))$ and also $\sigma$ realises a translation of the list $(\pi_2(x_1), \ldots, \pi_2(x_n))$ onto the list $(\pi_2(y_1), \ldots, \pi_2(y_n))$ where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ are some enumerations of the the sets $X$ and $Y$, respectively.

**Proof.** It is straightforward to see that $X$ is a translate of $Y$ only if $|X| = |Y|$ which implies we may assume that $|X|,|Y| = n$ for some $n \in \mathbb{N}$. As mentioned above, we have that $X$ is a translate of $Y$ if and only if there is $\sigma \in \text{Sym}(n)$ such that

$$
x_{\sigma(1)}y_1^{-1} = \cdots = x_{\sigma(n)}y_n^{-1}.
$$

If terms of Cartesian coordinates, this means that

$$
\pi_1(x_{\sigma(1)})\pi_1(y_1^{-1}) = \cdots = \pi_1(x_{\sigma(n)})\pi_1(y_n^{-1}),
$$

$$
\pi_2(x_{\sigma(1)})\pi_2(y_1^{-1}) = \cdots = \pi_2(x_{\sigma(n)})\pi_2(y_n^{-1}).
$$

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That is equivalent to saying that $\sigma$ realises a translation of $(\pi_1(x_1), \ldots, \pi_1(x_n))$ onto $(\pi_1(y_1), \ldots, \pi_1(y_n))$ and that $\sigma$ realises a translation of the list $(\pi_2(x_1), \ldots, \pi_2(x_n))$ onto the list $(\pi_2(y_1), \ldots, \pi_2(y_n))$. 

This next lemma tells us there exists a constant $\ell$ such that when given two finite subsets in $\mathbb{Z}^k$ they are translations of each other if and only if their images are translations in the group $(\mathbb{Z}/4\mathbb{Z})^k$ where we reduce each coordinate mod $4\ell$.

Lemma 4.7. Suppose that $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ are two finite ordered lists in $\mathbb{Z}^k$ such that $x_i, y_i \in [-\ell + 1, \ell - 1]^d$ for every $i$ for some $\ell \in \mathbb{N}$.

Let $\pi: \mathbb{Z}^k \to (\mathbb{Z}/c\mathbb{Z})^k$ be the homomorphism given by reducing each coordinate mod $c$ where $c \geq 4\ell$. Then the list $\pi(X) = (\pi(x_1), \ldots, \pi(x_n))$ is a translate of $\pi(Y) = (\pi(y_1), \ldots, \pi(y_n))$ in $(\mathbb{Z}/c\mathbb{Z})^k$ if and only if $X$ is a translate of $Y$ in $\mathbb{Z}^k$.

Furthermore, for all $\sigma \in \text{Sym}(n)$, we have that $\sigma$ realises a translation of $\pi(X)$ onto $\pi(Y)$ if and only if $\sigma$ realises a translation of $X$ onto $Y$.

Proof. We will only prove the ‘furthermore part of the statement since the first part follows from it.

Since the image of a translate is a translate of an image, the implication from left to right holds trivially. Therefore, we need only consider the other direction. Suppose that $c \geq 4\ell$. For every $\sigma \in \text{Sym}(n)$, define

$$D_\sigma(X, Y) = \{x_{\sigma(i)} - y_i \mid i = 1, \ldots, n\},$$

and define $D_\sigma(\pi(X), \pi(Y))$ analogously. We observe that $\sigma$ realises a translation of $X$ onto $Y$ if and only if $|D_\sigma(X, Y)| = 1$. Similarly, $\sigma$ realises a translation of $\pi(X)$ onto $\pi(Y)$ if and only if $|D_\sigma(\pi(X), \pi(Y))| = 1$. We also have that $D_\sigma(X, Y) \subseteq [-2\ell + 1, 2\ell - 1]^k$. We see that $\pi$ is injective on $[-2\ell + 1, 2\ell - 1]^k$, so $\pi$ is injective on $D_\sigma(X, Y)$ for every $\sigma \in \text{Sym}(n)$. Since $\pi$ is a homomorphism, we see that $D_\sigma(\pi(X), \pi(Y)) = \pi(D_\sigma(X, Y))$ for all $\sigma \in \text{Sym}(n)$, and hence, we have that $|D_\sigma(\pi(X), \pi(Y))| = |D_\sigma(X, Y)|$. In particular, we see that $|D_\sigma(\pi(X), \pi(Y))| = 1$ if and only if $|D_\sigma(X, Y)| = 1$. Thus, $\sigma$ realises a translation of $\pi(X)$ onto $\pi(Y)$ if and only if $\sigma$ realises a translation of $X$ onto $Y$. 

Lemma 4.8. Let $B$ be a finitely generated abelian group of torsion-free rank $k$, and suppose that $X, Y \subseteq B_+(\ell)$ are given. If $c \geq 4\ell$, then the homomorphism $\pi: B \to \text{Tor}(B) \times (\mathbb{Z}/c\mathbb{Z})^k$ given by the identity on $\text{Tor}(B)$ and by the coordinate-wise projection mod $c$ on the torsion-free part of $B$ is injective on $X \cup Y$. Moreover, $\pi(X)$ is a translate of $\pi(Y)$ if and only if $X$ is a translate of $Y$ in $\mathbb{Z}^k$. Furthermore, for all permutations $\sigma$, we have that $\sigma$ realises a translation of $\pi(X)$ onto $\pi(Y)$ if and only if $\sigma$ realises a translation of $X$ onto $Y$.

Proof. Suppose $c$ and $\pi: B \to \text{Tor}(B) \times (\mathbb{Z}/c\mathbb{Z})^k$ are given as in the statement of the lemma. We then have $\pi(X), \pi(Y) \subseteq [-\ell, \ell]^k$ which implies that $\pi$ is injective on $X \cup Y$. In particular, if $|X| \neq |Y|$, then $|\pi(X)| \neq |\pi(Y)|$, and subsequently, $\pi(X)$ is not a translate of $\pi(Y)$. Therefore, we may assume that $|X| = |Y|$.

Let $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ be enumerations of $X$ and $Y$, respectively. Following Lemma 4.6, we see that $\pi(X)$ is a translate of $\pi(Y)$ if and only if the list given by $\{\pi(\phi(x_1)), \ldots, \pi(\phi(x_m))\}$ is a translation of the list given by $\{\pi(\phi(y_1)), \ldots, \pi(\phi(y_n))\}$ and the list $\{\pi(\tau(x_1)), \ldots, \pi(\tau(x_m))\}$ is a translation of the list $\{\pi(\tau(y_1)), \ldots, \pi(\tau(y_n))\}$ where the translation is realised by the same permutation. However, by Lemma 4.7, we see that a permutation $\sigma \in \text{Sym}(m)$
realises a translation of the list given by \{\pi(\phi(x_1)), \ldots, \pi(\phi(x_m))\} onto the list \{\pi(\phi(y_1)), \ldots, \pi(\phi(y_m))\} if and only if it realises a translation of the list given by \{\phi(x_1), \ldots, \phi(x_m)\} onto the list given by \{\phi(y_1), \ldots, \phi(y_m)\}. Since \pi is defined as the identity on \text{Tor}(B), we see that \pi(X) is a translate of \pi(Y) if and only if X is a translate of Y, which concludes the proof.

\[
\pi
\]

4.3. Constructing the upper bounds. Using Lemma 4.5 and Lemma 4.2, the next proposition demonstrates when given non-conjugate elements \(x, y\) in a wreath product of abelian groups \(A \wr B\), there exists a finite quotient \(\overline{B}\) of \(B\) such that the images of \(x\) and \(y\) in \(A \wr \overline{B}\) remain non-conjugate.

**Proposition 4.9.** Let \(A\) be a finite abelian group and \(B\) be an infinite, finitely generated abelian group. Let \(f, g: B \to A\) finitely supported functions and \(b \in B\) an element such that \(fb, gb \in B_{A\wr B}(n)\) and \(fb \not\sim_{A\wr B} gb\). Then there exists a surjective homomorphism \(\pi: B \to \overline{B}\) to a finite group such that \(\pi(fb) \not\sim \pi(gb)\) in \(A \wr \overline{B}\). Moreover, there exists a constant \(C > 0\) independent of \(n\) such that if \(B\) has torsion free rank 1, then we have \(|\overline{B}| \leq Cn\), and if \(B\) is of torsion-free rank \(k > 1\), we then have \(|\overline{B}| \leq Cn^{2k}\).

**Proof.** Following Lemma 2.5, we may assume that both the functions \(f\) and \(g\) are given such that the elements \(fb\) and \(gb\) are reduced, i.e. the individual elements of their respective supports lie in distinct cosets of \((b)\) in \(B\).

Following Lemma 2.6, there are two cases to distinguish:

(i) \(\text{supp}(f)\) is not a translate of \(\text{supp}(g)\) in \(B\),

(ii) for every \(a \in B\) such that \(a + \text{supp}(f) = \text{supp}(g)\), there exists some \(x \in \text{supp}(g)\) such that \(f(x + a) \neq g(x)\).

We will construct a finite quotient \(\overline{B}\) such that the images of \(fb\) and \(gb\) are still reduced in \(A \wr \overline{B}\) whether (i) or (ii) is the case.

Lemma 2.8 implies that there is constant \(C_1 > 0\) such that

\[
\{b\} \cup \text{supp}(f) \cup \text{supp}(g) \subseteq B_B(C_1n).
\]

In particular, we see that

\[
\phi(\text{supp}(f)), \phi(\text{supp}(g)) \subseteq B_{2k}(C_1n) \subseteq [-C_1n, C_1n]^k
\]

where \(\phi: B \to \mathbb{Z}^k\) is the torsion free projection. Set \(\ell = C_1n\). It then follows that \(\phi(\text{supp}(f)), \phi(\text{supp}(g)) \subseteq [-\ell, \ell]^k\). We set \(S = \{s_2 - s_1 \mid s_1, s_2 \in \text{supp}(f) \cup \text{supp}(g)\}\)

and see that \(S \subseteq B_B(2\ell)\) and \(\|b\| \in B_B(\ell)\). Finally, we set \(e = \exp(\text{Tor}(B))\).

If \(k = 1\), let \(m \in \mathbb{N}\) be smallest possible such that \(m > 4\ell\) and where both \(e\) and \(\|\phi(b)\|\) divide \(m\). It is straightforward to see that \(m \leq 8\ell\). If \(k \geq 2\), let \(m \in \mathbb{N}\) be smallest possible such that \(m > k2^{k-1}2\ell^2\) and where both \(e\) and \(\gcd(\phi(b))\) divide \(m\). Without loss of generality, we may assume that \(\gcd(\phi(b))\) divides \(\|\phi(b)\|\). In particular, we see that \(m < k2^{k-1}2\ell^2\).

Following Lemma 4.3, we see that the homomorphism \(\pi: \text{Tor}(B) \oplus \mathbb{Z}^k \to \text{Tor}(B) \oplus (\mathbb{Z}/m\mathbb{Z})^k\) defined as the identity on \(\text{Tor}(B)\) and as the coordinate-wise reduction \(m\) on \(\mathbb{Z}^k\) is injective on the set \(S\) and for every \(s \in S\) we have that \(\pi(s) \in (\pi(b))\) if and only if \(s \in (b)\). In particular, this means that for every \(s, s' \in \text{supp}(f) \cup \text{supp}(g)\) we have that \(\pi(s)(\pi(b)) = \pi(s')(\pi(b))\) if and only if \(s(b) = s'(b)\). Set \(\overline{B} = \text{Tor}(B) \oplus (\mathbb{Z}/m\mathbb{Z})^k\), and let \(\overline{\pi}: A \wr B \to A \wr \overline{B}\) be the canonical extension of \(\pi\) to the whole of \(A \wr B\). From the construction of the map \(\pi\), we see that \(\overline{\pi}\) is injective on \(\text{supp}(f) \cup \text{supp}(g)\). Therefore, it follows that
supp(\(\tilde{\pi}(f)\)) = \pi(supp(f)) and supp(\(\tilde{\pi}(g)\)) = \pi(supp(g)). Furthermore, we see that for every two \(s, s' \in supp(f) \cup supp(g)\) we have that \(\tilde{\pi}(s)(\tilde{\pi}(b)) = \tilde{\pi}(s')(\tilde{\pi}(b))\) in \(B\) if and only if \(s(b) = s'(b)\) in \(B\). In particular, we see that the elements \(\tilde{\pi}(fb)\) and \(\tilde{\pi}(gb)\) are in reduced form.

Since the elements \(\tilde{\pi}(fb)\) and \(\tilde{\pi}(gb)\) are in reduced form, we may use Lemma 2.6 to check whether or not they are conjugate in \(A/\overline{B}\). We note that regardless of whether \(k = 1\) or \(k \geq 2\), we have that \(m \geq 4\ell\). Additionally, if \(|supp(f)| \neq |supp(g)|\), then \(|supp(\tilde{\pi}(f))| \neq |supp(\tilde{\pi}(g))|\). Subsequently, since supp(\(\tilde{\pi}(f)\)) is not a translate of supp(\(\tilde{\pi}(g)\)), we have that \(\tilde{\pi}(fb)\) is not conjugate to \(\tilde{\pi}(gb)\) by Lemma 2.6. Therefore, we may assume that \(|supp(f)| = |supp(g)|\).

Let supp(\(f\)) = \(\{x_1, \ldots, x_m\}\) and supp(\(g\)) = \(\{y_1, \ldots, y_m\}\). Via Lemma 4.7, we see that supp(\(\tilde{\pi}(f)\)) is a translate of supp(\(\tilde{\pi}(g)\)) if and only if \(\{\pi(\phi(x_1)), \ldots, \pi(\phi(x_m))\}\) is a translation of \(\{\pi(\phi(y_1)), \ldots, \pi(\phi(y_m))\}\) and \(\{\pi(\tau(x_1)), \ldots, \pi(\tau(x_m))\}\) is a translation of \(\{\pi(\tau(y_1)), \ldots, \pi(\tau(y_m))\}\) where the translation is realised by the same permutation. Lemma 4.7 implies that a permutation \(\sigma \in Sym(m)\) realises a translation of \(\{\pi(\phi(x_1)), \ldots, \pi(\phi(x_m))\}\) onto \(\{\pi(\phi(y_1)), \ldots, \pi(\phi(y_m))\}\) if and only if it realises a translation of \(\{\phi(x_1), \ldots, \phi(x_m)\}\) onto the list \(\{\phi(y_1), \ldots, \phi(y_m)\}\). Since \(\pi\) is defined as the identity on Tor(\(B\)), we see that supp(\(\tilde{\pi}(f)\)) is a translate of supp(\(\tilde{\pi}(g)\)) if and only if supp(\(f\)) is a translate of supp(\(g\)). Therefore, if supp(\(f\)) is not a translate of supp(\(g\)) in \(B\), we see by Lemma 2.6 that \(\tilde{\pi}(fb)\) is not conjugate to \(\tilde{\pi}(gb)\). Thus, we may suppose that supp(\(\tilde{\pi}(f)\)) is a translate of supp(\(\tilde{\pi}(g)\)).

Now suppose that \(a + supp(f) = supp(g)\) for some \(a \in B\). By assumption, there exists some \(x \in supp(g)\) such that \(f(x + a) \neq g(x)\). As mentioned before, Lemma 4.7 implies that every translation of supp(\(\tilde{\pi}(f)\)) onto supp(\(\tilde{\pi}(g)\)) must have already occurred in \(B\). By the construction of \(\pi\), we see that for every \(x \in Tor(B) \times [-m, m]^k\) we have \(\tilde{\pi}(f)(x + K) = f(x)\) and \(\tilde{\pi}(g)(x + K) = g(x)\). We see that for every \(a \in Tor(B) \times \mathbb{Z}/m\mathbb{Z}\) such that \(\pi(a) + supp(\tilde{\pi}(f)) = supp(\tilde{\pi}(g))\), there must exist \(\widetilde{a} \in supp(\tilde{\pi}(f))\) such that \(\tilde{\pi}(f)(\pi(a) + \widetilde{a}) = \tilde{\pi}(g)(x)\). That means that \(\tilde{\pi}(fb)\) is not conjugate to \(\tilde{\pi}(gb)\).

If \(k = 1\), set \(C = 8C_1 e \cdot Tor(B)\). We then have

\[
|B| = m \cdot |Tor(B)| \leq (8C_1 e n) \cdot |Tor(B)| = Cn.
\]

If \(k \geq 2\), set \(C = k^{2k+2(k+1)}C_1^{2k} e^{2k} \cdot |Tor(B)|\). We then have

\[
|B| = m^k \cdot |Tor(B)| \leq (k^{2k+1}2(C_1 n)^2)^k \cdot |Tor(B)| = Cn^{2k}
\]
which concludes our proof.

As an immediate consequence of Proposition 4.9 we get the following upper bound for wreath products of abelian groups with finite base group.

**Proposition 4.10.** Let \(A\) be a finite abelian group and \(B\) be a finitely generated abelian group of torsion free rank \(k\). If \(k = 1\),

\[
Conj_{A^B}(n) \preceq 2^n.
\]

Otherwise, for \(k > 1\), we have

\[
Conj_{A^B}(n) \preceq 2^{n^{2k}}.
\]

**Proof.** Suppose that \(f, b \in A^B\) are finitely supported functions and \(b, c \in A\) are elements such that \(fb, bc \in B_G(n)\) and where \(fb \not\sim_G gc\).
Suppose first that $b \neq c$. Since $b - c \in B_{B}(2n)$, [2] Corollary 2.3 implies there exists a constant $C_{1} > 0$ and a surjective homomorphism $\varphi : B \to Q$ such that $\varphi(b) \neq \varphi(c)$ and where $|Q| \leq C_{1} \log(C_{1}n)$. Since $Q$ is abelian, we have that $\varphi(b)$ and $\varphi(c)$ are non-conjugate. By composing $\varphi$ with the projection of $A \wr B$ onto $B$, which we also denote $\varphi$, we have a surjective homomorphism $\varphi : A \wr B \to Q$ such that $\varphi(fb) \sim \varphi(gc)$ and where $|Q| \leq C_{1} \log(C_{1}n)$. Therefore, we may assume that $b = c$.

Following Proposition 4.9 we have two cases. When the torsion free rank is $1$, we see that there exists a finite abelian group $\mathcal{G}$ together with a surjective homomorphism $\phi : A \wr B \to \mathcal{G}$ such that $|\mathcal{G}| \leq C_{2}n$ and where $\pi(fb)$ is not conjugate to $\pi(gc)$ in $A \wr \mathcal{G}$ for some constant $C_{2} > 0$. We see that

$$|A \wr \mathcal{G}| = |A|^{2} |\mathcal{G}| \leq |A|^{C_{2}n} C_{2}n.$$ Interpreting the size of $|A \wr \mathcal{G}|$ as a function of $n$, we get that

$$|A \wr \mathcal{G}| \leq |\mathcal{G}| \leq |A|^{C_{2}n} C_{2}n \leq |A|^{|n} \leq 2^{n}.$$ Subsequently, we see that $\text{Conj}_{C}(n) \leq 2^{n}$.

When the torsion free rank is greater than $1$, we see that there exists a finite abelian group $\mathcal{G}$ together with a surjective homomorphism $\phi : A \wr B \to \mathcal{G}$ such that $|\mathcal{G}| \leq (2C_{2})^{2k}$ and where $\pi(fb)$ is not conjugate to $\pi(gc)$ in $A \wr \mathcal{G}$ for some constant $C_{2} > 0$. We see that $|A \wr \mathcal{G}| \leq 2^{n}$. Consequently, we see that $\text{Conj}_{C}(n) \leq 2^{n^{2k}}$. □

5. Proof of the main theorems

This section is devoted to the proof of Theorem 1.1 which we restate for the reader’s convenience.

**Theorem 1.1** Let $A$ be a finite abelian group, and suppose that $B$ is a finitely generated abelian group of torsion free rank $k > 0$. If $k = 1$, i.e. if $B$ is virtually cyclic, then

$$\text{Conj}_{A \wr B}(n) \approx 2^{n}.$$ Otherwise,

$$2^{n} \leq \text{Conj}_{A \wr B}(n) \leq 2^{n^{k}}.$$ 

**Proof.** We have by Proposition 4.10 gives our upper bound, and by Proposition 3.1 we have that $2^{n} \leq \text{Conj}_{A \wr B}(n)$. Therefore, we are done. □

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