Color Superconducting Gap in Schwinger-Dyson Equation and Nonlocal Gauge Fixing

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Abstract

We solve Schwinger-Dyson (SD) equation for the color superconducting (CS) gap in nonlocal gauge which justifies the ladder approximation in subleading order calculation. The gap prefactor increases about 1.6 times to the leading order gap determined in Landau gauge.
§1. Introduction

High density quark matter is expected to be in CS phase. Many approach CS phase from the asymptotic densities since perturbative QCD can be used. Son showed that the gap increases due to the long-range magnetic interactions as density increases.\textsuperscript{1} And many works are done in order to determine the size of the gap in leading and subleading order.\textsuperscript{2)–5)}

In the leading order analysis of SD equation without vertex and wavefunction renormalization corrections, the gap prefactor is found to be gauge-dependent\textsuperscript{2), 3)} though gauge-dependence is negligible at extremely high densities. It’s not so surprise. The solutions of the SD equation with the bare vertex can be strongly gauge dependent even in QED without any matter\textsuperscript{6)} since the simple ladder approximation, where the full vertex function is approximated by the bare one, is not consistent with Ward-Takahashi (WT) identity.

We solve the SD equation for the CS gap, respecting gauge invariance as much as possible. We try to find the nonlocal gauge,\textsuperscript{7)} which makes the simple ladder approximation consistent with WT identity in the gap calculations to subleading order. We consider just color-flavor-locked CS phase here for simplicity but there is no difference in two flavor CS phase.

§2. Nonlocal gauge

The inverse propagator of the Nambu-Gorkov quark field $\Psi(x) \equiv (\psi(x), \psi_c(x))^T$ is

$$S^{-1}(p) = -i \begin{pmatrix} a(p) [(p_0 + \mu)\gamma^0 + b(p) \vec{p}] & -\Delta(p) \\ -\gamma^0 \Delta^\dagger(p)\gamma^0 & a(p) [(p_0 - \mu)\gamma^0 + b(p) \vec{p}] \end{pmatrix}, \quad (2.1)$$

with wavefunction renormalization constants $a(p), b(p)$ and the gap $\Delta$.

We want to find the nonlocal gauge where the wavefunction renormalization constants are unchanged by renormalization corrections, i.e., $a(p) = b(p) = 1$. In QED, such a nonlocal gauge enables us to use the bare vertex without breaking WT identity in all orders.\textsuperscript{7)} In contrast, WT identity of QCD (Slavonv-Taylor identity) is more complicated and nonlocal gauge does not guarantee that we can keep the bare vertex with gauge symmetry in all orders. However, we can keep the bare vertex $\gamma^\mu$ with WT identity in high density (so weak coupling) limit.

The QED-like diagram (Fig. 1a) gives the relations

$$(p - p')\mu \Lambda^{(a)\mu}(p, p') = a(p) \left[ (p_0 + \mu)\gamma^0 + b(p) \vec{p} \right] - a(p') \left[ (p'_0 + \mu)\gamma^0 + b(p') \vec{p} \right]$$,

where we suppress the color indices. We see that our nonlocal gauge where $a(p) = b(p) = 1$ makes $\Lambda^{(a)\mu} = \gamma^\mu$. The trigluonic diagram (Fig. 1b) contributes in high density effective
theory\(^2\)) as

\[ I_{\mu}^a \simeq g_s^3 \int \frac{l \cdot V}{l_\perp^2 + \Delta^2} \frac{\gamma_0 f^{abc} T^b T^c \left[ (c_1 l_\parallel + c_2 l_\perp + c_3 p^i + c_4 p^i') g_{\mu i} + c_5 (2l - p - p')_\mu \right]}{\left[ (\vec{l} - \vec{p})^2 + \pi M^2 |l_0 - p_0| / \left(2 |\vec{l} - \vec{p}|\right)\right] \left[ (\vec{l} - \vec{p'})^2 + \pi M^2 |l_0 - p'_0| / \left(2 |\vec{l} - \vec{p'}|\right)\right]} \]

\[ \sim i g_s^3 \gamma_0 V_\mu T^a \left(\frac{\Delta}{M}\right)^2 \ln (\Delta/\mu), \]

where \(c_1\) and \(c_5\) are 1 + \(O(\Delta^2/l_\perp^2)\) while all other \(c_i\)'s are \(O(\Delta^2/l_\perp^2)\). The screening mass \(M\) is given as \(g_s \mu \sqrt{N_f} / (2\pi)\) for \(N_f\) light quarks in the hard-dense-loop (HDL) approximation. The trigluonic contribution is suppressed, compared with QED-like diagram contribution.

We can use the simple ladder approximation for the SD gap equation to subleading order after determining the proper nonlocal gauge.

After projecting out the antiquarks by the on-shell projectors \(\Lambda_{\pm}^\pm \equiv \frac{1}{2} (1 \pm \frac{\gamma_0 \vec{p}}{|\vec{p}|})\), the equation\(^3\) for the local gauge where \(a(p) = b(p) = 1\) is given as

\[ -\frac{3}{2} \pi \alpha_s \int \frac{d^4 q}{(2\pi)^4} D_{\mu\nu}(q - p) \frac{q_0 + |\vec{q}| - \mu}{q_0^2 - \{|\vec{q}| - \mu\}^2 - \Delta^2} \text{tr} \left[ \gamma_\mu \gamma_0 A_\nu - \gamma_\nu \gamma_0 A_\mu^\pm \right] = 0 \quad (2.2) \]

where

\[ D_{\mu\nu}(k) = D^{(1)} O^{(1)}_{\mu\nu} + D^{(2)} O^{(2)}_{\mu\nu} + D^{(3)} O^{(3)}_{\mu\nu}, \quad (2.3) \]

where in the weak coupling limit, \(|k_0| \ll |\vec{k}|\),

\[ D^{(1)} = \frac{|\vec{k}|}{|\vec{k}|^3 + M_0^2 \Delta + \pi M^2 |k_4| / 2}, \quad D^{(2)} = \frac{1}{k_4^2 + \vec{k}^2 + 2M^2}, \quad D^{(3)} = \frac{\xi}{k_4^2 + \vec{k}^2}, \]

where \(M_0\) is Higgs-like gluon mass \(\sim g_s \mu / (2\pi)\).\(^3\) The polarization tensors are defined as

\[ O^{(1)} = P^\perp + \frac{(u \cdot k)^2}{(u \cdot k)^2 - k^2} P^\mu, O^{(2)} = P^\perp - O^{(1)}, \quad O^{(3)} = P^\parallel, \]
where \( u_\mu = (1, \vec{0}) \) and
\[
\begin{align*}
P^\perp_{\mu \nu} &= g_{\mu \nu} - \frac{k_\mu k_\nu}{k^2}, & P^\parallel_{\mu \nu} &= \frac{k_\mu k_\nu}{k^2}, \\
P^\mu_{\mu \nu} &= \frac{k_\mu k_\nu}{k^2} - \frac{k_\mu u_\nu + u_\mu k_\nu}{(u \cdot k)} + \frac{u_\mu u_\nu}{(u \cdot k)^2} k^2.
\end{align*}
\]

Now, we assume the (nonlocal) gauge fixing parameter \( \xi(k) \) has only temporal dependence, that is, \( \xi(k) \approx \xi(k_4) \). After angular integration with \( |\vec{p}|, |\vec{q}| \sim \mu, (2.2) \) becomes;
\[
I^{(1)} + I^{(2)} + I^{(3)} = 0 \tag{2.4}
\]
where
\[
I^{(1)} \approx \frac{2}{3} \ln \left[ \frac{(2\mu)^3}{M_0^2 + \pi M^2 |p_4 - q_4|/4} \right], I^{(2)} \approx 1, I^{(3)} \approx -\xi \ln \left[ \frac{(2\mu)^2}{|p_4 - q_4|^2} \right].
\]
We find the solution of Eq. (2.4),\(^8\)
\[
\xi \approx \frac{2}{3} \ln \frac{(2\mu)^3}{M_0^2 + \pi M^2 |p_4 - q_4|/4} \frac{\ln (2\mu)^2}{\ln |p_4 - q_4|^2}. \tag{2.5}
\]

\section{3. Gap equation}

With our nonlocal gauge (2.5) the SD equation for the gap is\(^3\)
\[
\Delta(p_4) = \frac{2}{3} \pi \alpha_s \int \frac{d^4 q}{(2\pi)^4} D^\xi_{\mu \nu}(q - p) \Delta(q) \frac{\text{tr} [\gamma^\mu A_q^+ \gamma^\nu A_q^-]}{q_0^2 - \{[\vec{q}] - \mu\}^2 - \Delta^2} \tag{3.1}
\]
\[
\sim \frac{2\alpha_s}{9\pi} \int_0^{p_4} dq_4 \frac{\Delta(q_4)}{\sqrt{q_4^2 + |\Delta(q_4)|^2}} \log \frac{\Lambda(p_4)}{p_4} + \frac{2\alpha_s}{9\pi} \int_{p_4}^3 dq_4 \frac{\Delta(q_4)}{\sqrt{q_4^2 + |\Delta(q_4)|^2}} \log \frac{\Lambda(q_4)}{q_4}
\]
where \( \Lambda \equiv e^{3\xi/2}(2\mu)^6/\sqrt{2\pi M^5} \). We can replace the integral equation by the differential equation;
\[
p d^2 \Delta dp^2 + \frac{d\Delta}{dp} + \frac{2\alpha_s}{9\pi} \frac{\Delta(p)}{\sqrt{p^2 + |\Delta|^2}} = 0.
\]
Here we neglect the contributions related with \( \frac{dA}{dp}, \frac{d^2A}{dp^2} \) since they are suppressed by the condition \( |p_4| \ll M_0, M \).

In IR region \((p \ll |\Delta|)\) with \( p \frac{d\Delta}{dp} |_{p=0} = 0, \Delta(p) = \Delta_0 J_0(\sqrt{\frac{8\alpha_s p}{9\pi M^2}}) \). And in UV region \((p \ll |\Delta|)\) with \( p \frac{d\Delta}{dp} |_{p=0} = 0, \Delta(p) = B \sin(\sqrt{\frac{2\alpha_s}{9\pi} \log \frac{4}{p}}) \). We can determine \( \Delta_0 \) and \( B \) by matching two solutions at \( p = \Delta_0 \). In the weak coupling limit,\(^3\)
\[
\Delta_0 \approx \frac{27\pi^4}{g_5^5} \mu e^{1 + \frac{3}{2g_5^2} e^{-\frac{2\alpha_s}{\sqrt{8\pi}}}}. \tag{3.2}
\]

Note that \( \xi \sim 1/3 \) in Eq. (2.5) with \( |p_4 - q_4| \sim \Delta \). So the vertex and wavefunction corrections increase the gap prefactor \( e^{1/2} \) times larger than the leading order result at Landau gauge \((\xi \neq 0)\).\(^4\)
§4. conclusion

We solve the SD equation for the CS gap after determining the nonlocal gauge where wavefunction renormalization corrections vanish. Our nonlocal gauge makes us be able to keep WT identity for the gap calculations to subleading order and to make the bare vertex (ladder) approximation available. The nonlocal gauge is \( \sim 1/3 \) near on-shell of the gap and increases the leading-order gap at Landau gauge by 2/3.

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