Queue Layouts of Two-Dimensional Posets

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Abstract. The queue number of a poset is the queue number of its cover graph when the vertex order is a linear extension of the poset. Heath and Pemmaraju conjectured that every poset of width $w$ has queue number at most $w$. The conjecture has been confirmed for posets of width $w = 2$ and for planar posets with 0 and 1. In contrast, the conjecture has been refused by a family of general (non-planar) posets of width $w > 2$.

In this paper, we study queue layouts of two-dimensional posets. First, we construct a two-dimensional poset of width $w > 2$ with queue number $2(w - 1)$, thereby disproving the conjecture for two-dimensional posets. Second, we show an upper bound of $w(w + 1)/2$ on the queue number of such posets, thus improving the previously best-known bound of $(w - 1)^2 + 1$ for every $w > 3$.

Keywords: poset · queue number · width · dimension · linear extension

1 Introduction

Let $G$ be a simple, undirected, finite graph with vertex set $V$ and edge set $E$, and let $\sigma$ be a total order of $V$. For a pair of distinct vertices $u$ and $v$, we write $u <_{\sigma} v$ (or simply $u < v$), if $u$ precedes $v$ in $\sigma$. We also write $[v_1, v_2, \ldots, v_k]$ to denote that $v_i$ precedes $v_{i+1}$ for all $1 \leq i < k$; such a subsequence of $\sigma$ is called a pattern.

Two edges $(u, v) \in E$ and $(a, b) \in E$ nest if $u <_{\sigma} a <_{\sigma} b <_{\sigma} v$. A $k$-queue layout of $G$ is a total order of $V$ and a partition of $E$ into subsets $E_1, E_2, \ldots, E_k$, called queues, such that no two edges in the same set $E_i$ nest. The queue number of $G$, $\text{qn}(G)$, is the minimum $k$ such that $G$ admits a $k$-queue layout. Equivalently, the queue number is the minimum $k$ such that there exists an order $\sigma$ containing no $(k+1)$-rainbow, that is, a set of edges $\{(u_i, v_i); i = 1, 2, \ldots, k + 1\}$ forming pattern $[v_1, \ldots, v_{k+1}, v_{k+1}, \ldots, v_1]$ in $\sigma$.

Queue layouts can be studied for partially ordered sets (or simply posets). A poset over a finite set of elements $X$ is a transitive and asymmetric binary relation $\prec$ on $X$. The main idea is that given a poset, one should lay it out respecting the relation. Two elements $a, b$ of a poset, $P = (X, \prec)$, are called comparable if $a \prec b$ or $b \prec a$, and incomparable, denoted by $a \parallel b$, otherwise. Posets are visualized by their diagrams: Elements are placed as points in the plane and whenever $a < b$ in the poset and there is no element $c$ with $a < c < b$, there is a curve from $a$ to $b$ going upwards (that is $y$-monotone); see Fig. 1a. Such relations, denoted by $a \prec b$, are known as cover relations; they are essential in the sense that they are not implied by transitivity. The directed graph implicitly
defined by such a diagram is the cover graph $G_P$ of the poset $P$. Given a poset $P$, a linear extension $L$ of $P$ is a total order on the elements of $P$ such that $a <_L b$, whenever $a <_P b$. Finally, the queue number of a poset $P$, denoted by $qn(P)$, is the smallest $k$ such that there exists a linear extension $L$ of $P$ for which the resulting layout of $G_P$ contains no $(k+1)$-rainbow; see Fig. 1c.

Queue layouts of posets were first studied by Heath and Pemmaraju [5], who provided bounds on the queue number of posets in terms of their width, that is, the maximum number of pairwise incomparable elements. In particular, they observed that the size of a rainbow in a queue layout of a poset of width $w$ cannot exceed $w^2$, and therefore, $qn(P) \leq w^2$ for every poset $P$. Furthermore, Heath and Pemmaraju conjectured that $qn(P) \leq w$ for a width-$w$ poset $P$. The study of the conjecture received a notable attention in the recent years. Knauer, Micek, and Ueckerdt [6] confirmed the conjecture for posets of width $w=2$ and for planar posets with 0 and 1. Later Alam et al. [1] constructed a poset of width $w \geq 3$ whose queue number is $w+1$, thus refuting the conjecture for general non-planar posets. In the same paper Alam et al. improved the upper bound by showing that $qn(P) \leq (w-1)^2 + 1$ for all posets $P$ of width $w$. Finally, Felsner, Ueckerdt, and Wille [4] strengthened the lower bound by presenting a poset of width $w > 3$ with $qn(P) \geq w^2/8$.

In this short paper we refine our knowledge on queue layouts of posets by improving the known upper and lower bounds of the queue number of two-dimensional posets. Recall that the dimension of poset $P$ is the least positive integer $d$ for which there are $d$ linear extensions (realizers) $L_1, \ldots, L_d$ of $P$ so that $a < b$ in $P$ if and only if $a < b$ in $L_i$ for every $i \in \{1, \ldots, d\}$. Two-dimensional posets are described by realizers $L_1$ and $L_2$ and often represented by dominance drawings in which the coordinates of the elements are their positions in $L_1$ and $L_2$; see Fig. 1b. We emphasize that the existing lower bound constructions [1,4] are not two-dimensional. Thus, Felsner et al. [4] asked whether the conjecture of Heath and Pemmaraju holds for posets with dimension 2. Our first result answers the question negatively.

**Theorem 1.** There exists a two-dimensional poset $P$ of width $w > 1$ with $qn(P) \geq 2(w-1)$.

Observe that our construction and the proof of Theorem 1 for $w=3$ is arguably much simpler than the one of Alam et al. [1], which is based on a tedious case analysis. Thus, it can be interesting on its own right.

Next we study the upper bound on the queue number of two-dimensional posets. Our result is the following theorem, which is an improvement over the known $(w-1)^2 + 1$ bound of Alam et al. [1] for every $w > 3$.

**Theorem 2.** Let $P$ be a two-dimensional poset with realizers $L_1, L_2$. Then there is a layout of $P$ in at most $w(w+1)/2$ queues using either $L_1$ or $L_2$ as the vertex order.

The paper is structured as follows. In Section 3 we prove Theorem 1 and in Section 2 we prove Theorem 2. Section 4 concludes the paper with interesting open questions.
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2 An Upper Bound

Consider a two-dimensional poset $P = (X, \prec)$ of width $w \geq 1$ with realizers $L_1$ and $L_2$. In this section we study queue layouts of $P$ using vertex orders $L_1$ or $L_2$, which we call realizer-based. It is well-known that the elements of $P$ can be partitioned into $w$ chains, that is, subsets of pairwise comparable elements. We fix such a partition and treat it as a function $C : X \to \{1, \ldots, w\}$ such that if $C(u) = C(v)$ and $u \neq v$, then either $u \prec v$ or $v \prec u$.

We start with a property of a linear extension of a poset, whose proof follows directly from the absence of transitive edges in $G_P$. Recall that $\prec$ indicates cover relations of $P$, that is, edges of $G_P$.

**Proposition 1** A linear extension of a poset $P$ with chain partition $C$ does not contain pattern $[b_1, b_2, b_3]$, where $C(b_1) = C(b_2) = C(b_3)$ and $b_1 \prec b_3$.

The next observation, whose proof is immediate, provides a crucial property of realizer-based linear extensions of two-dimensional posets. In fact, a poset, $P$, admits a linear extension with such a property if and only if $P$ has dimension 2; see for example [3] where such linear extensions are called non-separating.

**Proposition 2** Consider a two-dimensional poset $P$ with realizers $L_1, L_2$ and chain partition $C$. Let $[a_1, b, a_2]$ be a pattern in $L_1$ (or $L_2$) with $C(a_1) = C(a_2)$. Then either $a_1 < b$ or $b < a_2$.

The next useful property in the section holds for realizer-based linear extensions of two-dimensional posets.

**Proposition 3** Consider a two-dimensional poset $P$ with realizers $L_1, L_2$ and chain partition $C$. Then $L_1$ (or $L_2$) does not contain pattern $[a_1, b_2, a, a_2, b_1]$, where $C(a_1) = C(a_2) = C(a), C(b_1) = C(b_2)$, and $a_1 \prec b_1, b_2 \prec a_2$.

**Proof.** For the sake of contradiction, assume that $[a_1, b_2, a, a_2, b_1]$ is in $L_1$, with $C(a_1) = C(a_2) = C(a), C(b_1) = C(b_2)$, and $a_1 \prec b_1, b_2 \prec a_2$. Notice that $a_1 \parallel b_2$, 

Fig. 1: A two-dimensional poset of width 3, its dominance drawing, and a 2-queue layout
as otherwise we have $a_1 < b_2 < b_1$ and the edge $(a_1, b_1)$ is transitive. Hence by Proposition 2 applied to $[a_1, b_2, a]$, $b_2 < a$. Therefore, it holds that $b_2 < a < a_2$, which contradicts to non-transitivity of edge $(b_2, a_2)$.

Now we ready to prove the main result of the section.

Proof of Theorem 2. Assume that poset $P$ is partitioned into $w$ chains, and consider a maximal rainbow, denoted $T$, induced by the order $L_1$. We need to prove that $|T| \leq w(w+1)/2$.

First observe that the rainbow, $T$, does not contain two distinct edges $(a_1, b_1)$ and $(a_2, b_2)$ with $C(a_1) = C(a_2) = C(b_1) = C(b_2)$. Otherwise, the former edge nests the latter one and we have $a_1 < a_2 < b_2 < b_1$, which violates non-transitivity of $(a_1, b_1)$. Therefore, we already have $|T| \leq w^2$. (This is the argument of Heath and Pemmaraju for their original upper bound in [5])

Next we show two more configurations that are absent in $T$:

(i) For every pair of distinct chains, the rainbow does not contain edges $(a_1, b_1)$, $(b_2, a_2)$, and $(a_3, a_4)$ with $C(a_1) = C(a_2) = C(a_3) = C(a_4)$ and $C(b_1) = C(b_2)$. For a contradiction, assume the rainbow contains the three edges. By Proposition 1, edge $(a_3, a_4)$ cannot cover elements $a_1$ or $a_2$. Thus, $L_1$ contains pattern $[a_1, b_2, a_3, a_4, a_2, b_1]$ or $[b_2, a_1, a_3, a_4, b_1, a_2]$. Both patterns violate Proposition 3.

(ii) For every triple of distinct chains, the rainbow does not contain edges $(a_1, b_1)$, $(b_2, a_2)$, $(a_3, c_3)$, and $(c_4, a_4)$ with $C(a_1) = C(a_2) = C(a_3) = C(a_4)$, $C(b_1) = C(b_2)$, and $C(c_3) = C(c_4)$.

For a contradiction, assume $T$ contains the four edges. Consider the innermost edge in the rainbow; without loss of generality, assume the edge is $(a_1, b_1)$. Vertex $a_1$ is covered by two edges, $(a_3, c_3)$ and $(c_4, a_4)$, forming the pattern of Proposition 3; a contradiction.

Now observe that $T$ may contain at most $w$ unicolored edges (that is, $(u, v)$ such that $C(u) = C(v)$) and at most $w(w-1)$ bicolo red edges (that is, $(u, v)$ such that $C(u) \neq C(v)$).

On the one hand, if $T$ contains exactly $w$ uni-colored edges and $|T| > w(w + 1)/2$, then it must contain at least one pair of bi-colored edges $(a_1, b_1)$, $(b_2, a_2)$ with $C(a_1) = C(a_2)$, $C(b_1) = C(b_2)$. Together with the unicolored edge from chain $C(a_1)$, the triple forms the forbidden configuration (i).

On the other hand, if $T$ contains at most $w - 1$ uni-colored edges and $|T| > w(w+1)/2$, then $T$ contains two pairs of bi-colored edges, as in configuration (ii); a contradiction.

This completes the proof of the theorem. \qed

Notice that the bound of Theorem 2 is worst-case optimal, as we show next.

Lemma 1. There exists a two-dimensional poset of width $w \geq 1$, denoted $R_w$, with realizers $L_1, L_2$ such that its layout with vertex order $L_1$ contains a $(w(w + 1)/2)$-rainbow.
Fig. 2: A 2-dimensional poset of width $w \geq 1$, $R_w$, with a realizer-based order containing a $(w(w + 1)/2)$-rainbow, which is comprised of $w$ thick edges that nest all edges of $R_{w-1}$.

Proof. The poset $R_w$ is built recursively. For $w = 1$, the poset consists of two comparable elements. For $w > 1$, we assume that $R_{w-1}$ is constructed and described by realizers $L_{w-1}^1$ and $L_{w-1}^2$. The poset $R_w$ is constructed from $R_{w-1}$ by adding $2w$ elements.

Assume $|L_{w-1}^1| = n$ and the elements of $R_{w-1}$ are indexed by $w + 1, \ldots, w + n$. We set $L_1^w$ to the identity permutation and use $L_2^w = L_2^{w-1} \cup (1, n + 2w, 2, n + 2w - 1, \ldots, w, n + w + 1)$, where $\cup$ denotes the concatenation of the two orders. Fig. 2 illustrates the construction. It is easy to verify that the width of the new poset is exactly $w$. Observe that in the layout of $R_w$ with order $L_1^w$, edges $(1, n + 2w), \ldots, (w, n + w + 1)$ form a $w$-rainbow and nest all edges of $R_{w-1}$. Therefore, the layout contains a $(w(w + 1)/2)$-rainbow, as claimed.

We remark that Lemma 1 provides a poset whose queue layout with one of its realizers contains a $(w(w + 1)/2)$-rainbow. It is straightforward to extend the construction (by concatenating $R_w$ with its dual) so that both realizer-based vertex orders yield a rainbow of that size. However, the queue number of the poset (and the proposed extension) is at most $w$, which is achieved with a different, non-realizer-based, vertex order. Thus, a more delicate construction is needed to force a larger rainbow in every linear extension of a poset.

3 A Lower Bound

In this section we provide a new counter-example to the conjecture of Heath and Pemmaraju [5] by describing a two-dimensional poset of width $w \geq 3$ whose queue number exceeds $w$. The poset, denoted $P_w$, is constructed recursively. The base case, $P_2$, is a four-element poset with $L_1 = (1, 2, 3, 4)$ and $L_2 = (2, 1, 4, 3)$; see Fig. 3b. The step of the construction is illustrated in Fig. 3c. Poset $P_w$
Fig. 3: A counter-example to the conjecture of Heath and Pemmaraju [5]: A two-dimensional poset, $P_w$, of width $w \geq 3$ with queue number exceeding $w$

consists of a copy of $P_{w-1}$, a copy of the poset $R_{w-1}$ utilized in Lemma 1, the duals of the two posets, and a chain of additional elements. Recall that the dual of a poset, $P$, is the poset, $\overline{P}$, on the same set of elements such that $x < y$ in $P$ if and only if $y < x$ in $\overline{P}$ for every pair of the elements $x$ and $y$.

We now formally describe the construction. Denote by $L_1(P), L_2(P)$ the two realizers of a two-dimensional poset $P$. Let $\cup$ denote the concatenation of two sequences, and let $(x_1, x_2, \ldots) \cup (y_1, y_2, \ldots)$ denote the interleaving of two equal-length sequences, that is, $(x_1, y_1, x_2, y_2, \ldots)$. Assume that $R_{w-1}$ contains $r$ elements. Then we set

$L_1(P_w) = (x_1, \ldots, x_r+1) \cup b \cup s \cup y_1 \cup (L_1(R_{w-1}) \cup (y_2, \ldots, y_r+1)) \cup \left( L_1(P_{w-1}) \cup a \cup L_1(P_{w-1}) \cup L_1(R_{w-1}) \cup t \right)$, and

$L_2(P_w) = s \cup L_2(R_{w-1}) \cup L_2(P_{w-1}) \cup a \cup L_2(P_{w-1}) \cup \left( (x_1, \ldots, x_r) \cup L_2(R_{w-1}) \cup x_{r+1} \cup t \cup b \cup (y_1, \ldots, y_r+1) \right)$. 

We refer to Fig. 3 for the illustration of the construction and to Fig. 5 for the instance of $P_3$. Now we prove that the constructed poset has queue number at least $2w - 2$.

Proof of Theorem 1. It is easy to verify that the constructed poset, $P_w$, is two-dimensional and has width exactly $w$. Furthermore, the poset is dual to itself, that is, $P_w = \overline{P_w}$ with $a$ and $b$ being the fixed points. Thus, we may assume that in the linear extension corresponding to the optimal queue layout of the poset, element $a$ precedes $b$ and we have $s < \cdots < a < b < y_1 < \cdots < y_{r+1}$. Next
we consider the queue layout induced by the elements $s, R_{w-1}, P_{w-1}, a,$ and $y_1, \ldots, y_{r+1}$; see Fig. 4.

We prove the theorem by induction. For $w = 2$, the claim holds trivially. For $w > 2$, we assume that $qn(P_{w-1}) \geq 2(w - 2)$ and distinguish two cases depending on the size of the maximum rainbow, $T$, formed by edges $(s, y_1), (v_1, y_2), \ldots, (v_r, y_{r+1})$, where $v_i, 1 \leq i \leq r$ are elements of $R_{w-1}$:

- if $|T| \geq 2$, then $qn(P_w) \geq qn(P_{w-1}) + |T| \geq 2w - 1$, as all edges of $P_{w-1}$ are nested by edges of $T$;
- if $|T| = 1$, then the elements of $R_{w-1}$ must appear in the order induced by $L_1(R_{w-1})$, since otherwise at least two of the edges of $T$ nest. By Lemma 1, the edges of $R_{w-1}$ form a $(w(w - 1)/2)$-rainbow. The rainbow is covered by edge $(s, y_1)$, which yields $qn(P_w) \geq (w(w - 1)/2) + 1 \geq 2(w - 1)$ for $w \geq 3$.

This completes the proof of Theorem 1.

4 Conclusions

We disproved the conjecture of Heath and Pemmaraju for two-dimensional posets and answered a question posed by Felsner et al. \[4\]. A number of intriguing problems in the area remain unsolved.

- Is it possible to get a subquadratic upper bound on the queue number of two-dimensional posets of width $w$? A poset of Felsner et al. \[4\] that requires $w^2/8$ queues in every linear extension is not two-dimensional, which leaves a hope for an asymptotically stronger result than the one given by Theorem 2.
- What is the queue number of two-dimensional posets of width $3$? By Theorem 1 and the result of Alam et al. \[1\], the value is either 4 or 5.
- Queue layouts of graphs are closely related to so-called track layouts, which are connected with the existence of low-volume three-dimensional graph drawings \[2,7\]. In particular, every $t$-track (undirected) graph has a $(t - 1)$-queue layout, and every $q$-queue (undirected) graph has track number at most $4q \cdot 4q^{(2q-1)(4q-1)}$. We think it is interesting to study the relationship between the two concepts for directed graphs and posets.

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Additional Illustrations

Fig. 5: A two-dimensional poset with 38 elements and width 3. The queue number of the poset is exactly 4; the lower bound is shown in Theorem 2, and the upper bound is verified computationally via an open source SAT-based solver available at http://be.cs.arizona.edu