The influence of Rashba spin-orbit interaction on the spin dynamics of a topologically disordered hopping system is studied in this paper. This is a significant generalization of a previous investigation, where an ordered (polaronic) hopping system has been considered instead. It is found, that in the limit, where the Rashba length is large compared to the typical hopping length, the spin dynamics of a disordered system can still be described by the expressions derived for an ordered system, under the provision that one takes into account the frequency dependence of the diffusion constant and the mobility (which are determined by charge transport and are independent of spin). With these results we are able to make explicit the influence of disorder on spin related quantities as, e.g., the spin life-time in hopping systems.

I. INTRODUCTION

Much attention has recently been devoted to the problem of utilizing electron spin in semiconductor electronics. An overview of this evolving subject of spintronics is given in Ref. 7. One central aim of these efforts is the (preferably) electrical generation, manipulation, and detection of spin currents.

While a large part of spintronics related literature is concerned with itinerant electron systems, the corresponding behavior of localized electrons is worth serious consideration, too. For instance, some proposals for solid state quantum computing are based on manipulating the electronic or nuclear spin state of impurities.2,3,4,5

One possibility of affecting the spin behavior through electrical means is by utilizing two-dimensional structures (semi-conductor interfaces or heterostructures) showing Rashba spin-orbit interaction (SOI).6 This paper considers a model, which consists of a two-dimensional system of localized electronic states. Charge transport is possible through thermal activation (hopping transport).7 The spin dynamics is assumed to be determined by Rashba SOI.

In a former publication8 we derived microscopic and macroscopic transport equations for charge and spin of an ordered (polaronic) hopping system with Rashba SOI. Here, we generalize this theory to the case of spatial (i.e. not energetic) disorder. A suitable reference system would be nearest-neighbor hopping between donors in a semi-conductor or between Anderson localized states. The central approximation concerning spin behavior in hopping systems, which applies to this investigation, consists in the assumption, that the spin degree of freedom is solely influenced by Rashba SOI, i.e., there must be no spin dependent scattering (only s-wave scattering, no magnetic scatterers). Furthermore, the Rashba length, i.e. the length scale over which the spin rotation occurs, must be large compared to a typical hopping length $R_t$.

In Sec. II the (disorder averaged) transport equations for spin are derived for a disordered hopping system and their relation to the charge transport equations is studied. The (averaged or macroscopic) evolution equations governing charge transport (drift-diffusion equations) keep their functional shape in the transition from an ordered to a disordered system, the only effect being, that the diffusion constant and the mobility become frequency dependent (or, expressed differently: the macroscopic evolution equations for the disordered system have memory, even if the microscopic equations did not).

We show, that the equations for spin dynamics due to Rashba SOI likewise keep their functional shape in the presence of disorder, provided the hopping length is sufficiently short. Furthermore, the only differences to the ordered case are again a frequency dependence of the diffusion constant and the mobility of the charge. Thus, provided the length scale for spin inversion due to Rashba SOI (which will be denoted as the “Rashba length” in the following) is large in comparison with the typical hopping length, the influence of disorder on the spin dynamics is entirely determined by the corresponding change of the charge transport coefficients.

Two different situations are considered in detail: The temporal evolution of an initial spin polarization is investigated (Sec. III) and the stationary state of a system with spin polarized boundary conditions is determined (Sec. IV). In both cases, the analytical treatment is compared with numerical simulations.

It is found, that the spin life-time strongly increases with increasing disorder. On the other hand, the spatial behavior of the diffusing spin in the stationary case is practically unaffected by the introduction of disorder. Additionally, the finding, previously made for an ordered Rashba hopping system, that there is a critical electrical field, above which the total spin polarization has an oscillatory component, is also stable against the introduction of topological disorder.

Section VII gives a summary of the results and discusses the experimental relevance. The two appendices present details of the derivation of the macroscopic spin evolution equation and its long wave-length limit.
II. THE TRANSPORT EQUATIONS

The equations which govern the evolution of charge and spin density of a hopping system with Rashba SOI have been derived in Refs. 8, 9. These microscopic transport equations are rate equations for the occupation numbers and the spin expectation values of the localized sites. It is found, that the transition matrix elements between different localized sites obtains the spin-dependent factor \( \exp(-i\sigma \cdot (K \times R_{m'm})) \) due to Rashba SOI. Here, \( \sigma \) denotes the vector of Pauli spin matrices, \( R_{m'm} = R_{m'} - R_m \) is the distance vector between sites \( m' \) and \( m \), and \( K \) is a vector normal to the two-dimensional plane, the length of which is proportional to the Rashba coupling strength. Its inverse \( 1/K = 1/|K| \) has the dimension of a length and will in the following be called Rashba length.

The derivation of the rate equations is subject to the following restrictions: (i) single particle approximation (i.e. no electron-electron interaction), (ii) Markovian rate equations (i.e. the relevant times are beyond the quantum kinetic regime), and (iii) the Rashba length is large compared with typical hopping length scales, \( KR_t \ll 1 \). In this paper, we furthermore restrict ourselves to: (iv) two site hopping probabilities. This gives the leading contribution to hopping transport, the small parameter being the ratio between the resonance integral and the energetic width of the set of electron states. Ohmic charge transport is adequately described by this approximation, but higher order effects, as e.g. the Hall effect, are neglected.

The corresponding rate equations read\[^8\]

\[
\frac{d}{dt} \rho_m = \sum_{m_1} \{ \rho_{m_1} W_{m_1 m} - \rho_m W_{mm_1} \}
\]

for the occupation numbers and

\[
\frac{d}{dt} \rho_m = \sum_{m_1} \{ \dot{D}_{m_1 m} \cdot \rho_{m_1} W_{m_1 m} - \rho_m W_{mm_1} \}
\]

for the spin orientation. Here, \( \rho_m \) is the occupation number of site \( m \), \( \rho_m \) is the expectation value of the spin operator on this site, \( W_{m_1 m} \) is the transition rate between sites \( m_1 \) and \( m \), and \( \dot{D}_{m_1 m} \) is a \( 3 \times 3 \)-matrix, which describes a rotation of the spin during the transition. An explicit expression of this matrix is given in Eq. \[15\].

Note, that the two sets of equations are decoupled in the chosen approximation (due to the restriction to two site hopping probabilities). This means, that the charge (or particle) transport has no influence on the spin transport and vice versa. Thus, such an interesting phenomenon as the spin Hall effect, which already has been studied for ordered hopping systems\[^8\] and which has sparked controversial discussions recently\[^13\]--\[^15\],\[^16\],\[^17\], lies outside the scope of the theory presented here.

Reference \[^3\] utilized these rate equations in order to study the behavior of an ordered (polaronic) hopping system. Here, a topologically disordered hopping system is the subject of investigation. Thus, a disorder average has to be performed, in order to obtain macroscopic transport equations.

Appendix \[^A\] gives the technical details of the derivation of the macroscopic transport equations subject to the restrictions given above. It proceeds along the lines of a “generic” averaging procedure, following the approach used in Ref. \[^18\].

Since the disorder averaged system is homogeneous, it is convenient to work in wave-vector space. The disorder averaged rate equations in Fourier-Laplace space are obtained as

\[
[s + W(s|0) - W(s|q)] \rho(s|q) = \rho_0(q)
\]

and

\[
[s + W(s|0) - \int d^2q_1 W(s|q - q_1) \dot{D}(q_1)] \cdot \rho(s|q) = \rho_0(q).
\]

Here, \( s \) is the Laplace variable (related to time) and \( q \) is the (in-plane, i.e. two-dimensional) wave vector.

In the long wave-length limit the transition rates of an isotropic system can be expanded in the form

\[
W(s|q) = W(s|0) - D(s|q^2 - i\mu(s) q \cdot E),
\]

where the (generally frequency dependent) in-plane diffusion constant \( D \) and mobility \( \mu \) have been introduced. The quantity \( E \) is the in-plane electric field. In a system of reduced symmetry, \( D \) and \( \mu \) are tensors, but this case is not considered here. Note, that the asymmetry between in-plane and normal-to-plane diffusion and drift does not show here, since the wave vectors \( q \) are restricted to two dimensions.

As shown in App. \[^B\] the support of \( \dot{D}(q) \) lies within \( q < 2K \). Thus, provided \( K \) is sufficiently small, the long wave-length limit is also applicable to the integrand in Eq. \[^4\]. The condition \( KR_t \ll 1 \) already assures this, since wave-vectors corresponding to the (inverse of the) typical hopping length are within the long wave-length limit. This can safely be assumed, because already for charge transport, we are not concerned with effects due to higher than the second spatial derivative of \( \rho(r) \) in the corresponding evolution equation.

Using expression \[^3\], the integral in Eq. \[^4\] can be transformed into the \( 3 \times 3 \)-matrix

\[
[(W(0) - Dq^2 - i\mu q \cdot E) \dot{D}(r) - (2iDq - \mu E) \cdot \nabla \circ \dot{D}(r) + D \Delta \dot{D}(r)]_{r=0},
\]

where the \( s \)-dependence has been dropped from the notation. The symbol \( \circ \) denotes the dyadic product. Thus, only the first three moments of the rotation matrix \( \dot{D}(q) \) (expressed here as spatial derivatives at the origin of space) enter the calculations in the long wave-length limit.
Finally, inserting the rotation matrix $\hat{D}$ as given by Eq. (B1) one obtains the rate equations in the long-wavelength limit (see App. B for details of the calculation, $\hat{I}_3$ is the $3 \times 3$ identity matrix)

$$\begin{bmatrix} s + D(s)q^2 + i\mu(s)q \cdot E \end{bmatrix} \rho(s|q) = \rho_0(q)$$  \hspace{1cm} (7)

and

$$\begin{bmatrix} (s + D(s)q^2 + i\mu(s)q \cdot E + 4D(s)K^2) \hat{I}_3 + 4D(s)K \cdot \rho(s|q) \\ K \times (4iD(s)q - 2\mu(s)E) \end{bmatrix} \times \rho(s|q) = \rho_0(q).$$  \hspace{1cm} (8)

These evolution equations agree exactly with the corresponding equations of an ordered hopping system, derived in Ref. 8, except that here the diffusion constant $D$ and the mobility $\mu$ are frequency dependent. This frequency dependence is entirely determined by charge transport and does not depend on SOI within the approximations considered here. Thus, one finds that under the condition $KR_t \ll 1$, the spin dynamics is affected by disorder only through the change of $D$ and $\mu$. Note, that the above derivation is independent of the approximations used to derive $W(s|q)$ (the procedure used to obtain concrete expressions for $D(s)$ and $\mu(s)$). In particular, it is not restricted to continuous time random walk, the procedure further considered in Ref. 8 even though we followed the general approach used there.

For reference, the transport equations (7) and (8) transformed to real space- and time-coordinates read

$$\frac{d}{dt} \rho(t|r) = \int dt_1 \left[ D(t - t_1) \Delta \rho(t_1|r) - \mu(t - t_1) E \cdot \nabla \rho(t_1|r) \right],$$  \hspace{1cm} (9)

$$\frac{d}{dt} \rho(t|r) = \int dt_1 \left\{ D(t - t_1) \Delta \rho(t_1|r) - \mu(t - t_1) E \cdot \nabla \rho(t_1|r) - 4K^2 D(t - t_1) \rho(t_1|r) - 4K^2 D(t - t_1) e_z \rho_z(t_1|r) \\
- 4D(t - t_1) K [ \nabla \rho_z(t_1|r) - e_z \nabla \cdot \rho(t_1|r) ] - 2\mu(t - t_1) K [ e_z E \cdot \rho(t_1|r) - E \rho_z(t_1|r) ] \right\}. \hspace{1cm} (10)$$

### III. TOTAL SPIN EVOLUTION

In this section we consider the temporal evolution of the total spin magnetization. In this case we have to set $q = 0$ in Eqs. (7) and (8). Then, Eq. (7) immediately yields particle number conservation. Writing the corresponding equation for spin polarization in matrix form, while fixing the coordinate system such that $E \parallel e_z$ and $K \parallel e_z$, one obtains

$$\begin{bmatrix} s + 4DK^2 & 0 & -2\mu KE \\
0 & s + 4DK^2 & 0 \\
2\mu KE & 0 & s + 8DK^2 \end{bmatrix} \cdot \rho(s) = \rho_0. \hspace{1cm} (11)$$

One can see, that, in zero electric field $E = 0$, the spin components decay with two different time constants (taking $D$ for the moment as frequency independent): The $z$-component with $\tau_z = 1/8DK^2$, and the in-plane components with twice this value $\tau_2 = 1/4DK^2$, i.e. the spin component perpendicular to the plane decays two times faster than the in-plane components, a fact which has also been found for ordered hopping systems and itinerant electrons. Note, in particular, that the spin lifetime is inversely proportional to the diffusion constant. Since $D$ substantially decreases with increasing disorder in hopping systems, the spin lifetime will strongly increase with increasing disorder.

The frequency dependence of $D$ complicates the matter, but its overall effect is to further increase the decay time constant for later times (see the discussion in Sec. V). Thus, the spin decay slows down further in the progress of time.

Even in a finite electric field, the in-plane component of $\rho$ perpendicular to the field follows a decay law $[\rho_y(t) \rho_z(t) \rho_z(t)] \parallel e_z$ in Eq. (11), whereas the other two components are coupled and have the solution

$$\begin{bmatrix} \rho_x(s) \\
\rho_y(s) \\
\rho_z(s) \end{bmatrix} = \frac{1}{\det(\Pi)} \Pi \cdot \begin{bmatrix} \rho_{x0} \\
\rho_{y0} \\
\rho_{z0} \end{bmatrix}, \hspace{1cm} (12)$$

with the matrix

$$\Pi = \begin{bmatrix} s + 8DK^2 & 2\mu KE \\
-2\mu KE & s + 4DK^2 \end{bmatrix}. \hspace{1cm} (13)$$

When $D$ and $\mu$ do not depend on frequency, this corresponds to a sum of exponential functions in time, so that the spin components are either hyperbolic or trigonometric functions of time, times an exponential decay factor (see Ref. 8). Here, in the disordered case, $D$ and $\mu$ are frequency dependent, and the transformation of the solution of Eq. (11) into $t$-space is impossible without choosing a specific model for the frequency dependence $D(s)$ and $\mu(s)$ beforehand.
For large times, $D$ and $\mu$ approach their respective DC-values, i.e., the asymptotic behavior of $\rho$ for large times is easily obtained. On the other hand, due to the exponential decay, the large time behavior is only relevant, if the time constant for DC-behavior is smaller than the time constant for spin decay $1/4DK^2$.

The solution for initial condition $\rho_0 = e_z$ with time-independent $D$ and $\mu$ is given for future reference

$$\rho_x(t) = e^{-6DK^2t} \frac{\epsilon}{\sqrt{e^2 - 1}} \sin(2DK^2 \sqrt{e^2 - 1} t)$$

$$\rho_y(t) = e^{-6DK^2t} \left[ \cos(2DK^2 \sqrt{e^2 - 1} t) - \frac{1}{\sqrt{e^2 - 1}} \sin(2DK^2 \sqrt{e^2 - 1} t) \right].$$

The dimensionless electric field is $\epsilon = \mu E/(DK)$. This quantity is time-independent if the Einstein relation between $D$ and $\mu$ is valid, even when $D$ and $\mu$ themselves depend on time. In this case, $\epsilon$ furthermore does not depend on the disorder, which also strongly affects the quantities $D$ and $\mu$ themselves.

The solution here is written with trigonometric functions, which is the appropriate form for $\epsilon > 1$. In the case $\epsilon < 1$, hyperbolic functions have to be used instead. Thus, the dimensionless electric field $\epsilon$ discriminates between two different behaviors of the total spin polarization: exponential decay in small electric fields and an additional oscillation in large electric fields. The occurrence of one or the other regime can be tuned by varying (one or several of) the electric field, the temperature, and the Rashba length.

Note, that the conclusions of this section remain the same, whether one considers a single polarized spin initially placed at the origin, or a homogeneously polarized system.

### IV. STATIONARY STATE WITH BOUNDARY CONDITIONS

As a second case, we determine the stationary state of a system with an in-plane electric field and spin injection. Taking, as before, $E \propto e_x$, and boundaries parallel to the $y$-axis, the charge and spin densities can only depend on the $x$-coordinate. Denoting the derivative with respect to $x$ by a prime, one obtains

$$0 = D_0 \rho''(x) - \mu_0 E \rho'(x)$$

$$0 = D_0 \rho''(x) - \mu_0 E \rho'(x) - 4D_0 K^2 \rho(x) - 4D_0 K^2 e_x \rho_z(x) - 4D_0 K \left[ e_x \rho_z'(x) - e_y \rho_z'(x) \right] - 2\mu_0 KE \left[ e_z \rho_x(x) - e_x \rho_z(x) \right].$$

Here, $D_0 = \int_0^\infty dt D(t)$ and $\mu_0 = \int_0^\infty dt \mu(t)$ are the DC-values of the corresponding quantities.

Specifically, if one considers a half-plane, and takes the boundary conditions at $x = 0$ and $x = \infty$ to be $\rho(0) = \rho(\infty) = \rho_0$, $\rho(0) = e_z$, and $\rho(\infty) = 0$, the solution then reads $\rho(x) = \rho_0$,

$$\rho_x(x) = e^{-x/\lambda} A x \sin(x/\Lambda),$$

$$\rho_y(x) = 0,$$

$$\rho_z(x) = e^{-x/\lambda} [\cos(x/\Lambda) - A x \sin(x/\Lambda)].$$

where the dimensionless electric field $\epsilon = \mu_0 E/(D_0 K)$ assumes the DC-value and is the parameter determining the quantities $\omega_{\pm} = \sqrt{\pm e^2 - 8 \over 2} + 1 \sqrt{e^2 + 16 \epsilon^2 + 512}$, the amplitudes $A_x = \frac{e^2 + 32}{\omega_{++}}$, and $A_z = \frac{2 \omega_{--}}{3 \omega_{++}}$, the decay length $\lambda = 2(\omega_{+} - \epsilon)^{-1} / K$ and the oscillation length $\Lambda = 2(\omega_{-} K)$. Note, that the disorder enters only through the ratio $\mu_0/D_0$. Thus, provided the Einstein relation between $D_0$ and $\mu_0$ is valid, the spatial behavior of $\rho(x)$ does not depend on the disorder, the relevant length scale being determined mainly by $K$, since the (dimensionless) quantity $\omega_{-}$ only very weakly depends on its sole parameter $\epsilon$ (specifically, $3.95 < \omega_{-} \leq 4$).

### V. NUMERICAL SIMULATION

Two kinds of numerical simulations are performed. First, the temporal evolution of a single spin is determined (calculation A, corresponding to Sec. III) by solving the differential equations. This yields, after ensemble averaging, the temporal behavior of an initially localized $z$-spin. Secondly, the system is subjected to boundary conditions, which correspond to spin injection, and the spatial behavior of spin polarization in the stationary state is determined (calculation B, corresponding to Sec. IV). This is done by solving a linear system of equations. Again, a disorder average must then be performed. The disorder averages are performed over about 50 or 100 realizations for calculation A or B, respectively. In each case, the statistical error is of the order of the symbol $\epsilon$ in the figures. The number of sites in the calculations varies between about 1600 and 5000. It has been taken care, that the results are not affected by boundary effects.

The transition rates for the simulation are taken to be

$W_{m_1 m_2} = \nu_0 e^{-\alpha R_{m_1 m_2}},$

where $\nu_0$ is the attempt frequency (subsuming, e.g., the temperature dependence of the rates) and determines the numerical time scale. The parameter $\alpha$ is the inverse localization length of a single impurity state. Denoting by $N$ the (areal) density of localized states, the quantity $\alpha N^{-1/2}$ provides a measure of the disorder.

Calculations A and B complement each other. Calculation A allows to study the temporal behavior, but the numerical complexity forbids to investigate systems with large disorder (large $\alpha N^{-1/2}$). Large disorder implies a large dispersion (typically many orders of magnitude) of
relevant hopping times. Since the numerical time step for a differential equation solver has to be related to the smallest relevant time scale, but the long time physics is related to the largest relevant time scale, the study of systems with large disorder leads to an enormous increase in the number of time steps to be calculated, if one is interested in the long time behavior. Furthermore, for larger disorder, the typical behavior can only be found in systems, which are spatially more extended. This increases the numerical expense even further. Thus, only relatively small values of \( \alpha N^{-1/2} \) and/or \( tv_0 \) can be investigated by a numerical integration of the rate equations.

On the other hand, calculation B immediately gives the \( t \to \infty \) behavior with a reasonable effort even for systems with large disorder. However, the temporal behavior is not accessible.

To be specific, in calculation A we take the initial condition \( \rho_0 = e_z \). That means, we consider a system with an initial spin polarization in the \( z \)-direction. The three components of the calculated total spin polarization are shown in Fig. 1 for the disorder parameter \( \alpha N^{-1/2} = 3 \). Figures 2 and 3 give the \( z \)-component for several disorder parameters.

For a comparison with the analytical expressions, the time-dependent diffusivity and mobility have to be determined. In method A, where the time evolution of each system is known, effective quantities are determined by \( D_{\text{eff}}(t) = \langle r^2 \rangle / (4t) = \langle \sum_m \rho_m(t) R_m^2 \rangle / (4t) \) and \( \mu_{\text{eff}}(t) = \langle \dot{\rho} \rangle / (Et) = \langle \sum_m \rho_m(t) \dot{R}_m \cdot e_z \rangle / (Et) \) from charge diffusion and drift. The brackets \( \langle \cdot \cdot \cdot \rangle \) denote the disorder average. Method B poses a problem here, since the diffusion co-efficient cannot be directly measured in this case. Fortunately only the quotient \( \mu / D \), which could be obtained using the Einstein relation, enters the analytic expressions.

The relation between the effective quantity \( D_{\text{eff}}(t) \) and those occurring in the analytical expressions \( D(t) \), the inverse Laplace transform of \( D(s) \) is unfortunately not simple, but rather involves differentiation operations: \( D(t) = \text{Re} \left[ \frac{1}{\pi} L \left[ D_{\text{eff}}(t) \right] \right] \). A corresponding expression connects \( \mu_{\text{eff}} \) and \( \mu \). Numerical differentiation of noisy data (the disorder averaged \( D_{\text{eff}} \)) is highly unstable, so that the direct comparison of the numerical results and the analytic expressions (but using the numerically obtained \( D \) and \( \mu \)) is not reasonable. The situation were much improved, if a general model for the behavior of \( D(t) \) for the system under consideration were available, but no such model is available for the range of parameters considered here.

Thus, we take a different approach. The asymptotic values of \( D_{\text{eff}}(t) \) and \( \int_0^t dt_1 D(t_1) \) are the same for very small \( (tv_0 < 1) \) or for very large times (DC-behavior). In between, for moderate times, the detailed behavior differs, but, since they change monotonically over several orders of magnitude, see Fig. 4, we expect the Markovian limit of the transport equations to be valid in the following sense. Because \( D(t) \) and \( \mu(t) \) decrease over many orders of magnitude with increasing \( t \), the main contribution to the time integrals in Eqs. (9) and (10) comes from small times \( t < 1 / \nu_0 \). The charge and spin density is taken out of the integral (as in the derivation of a Markovian equation), the remaining time integrals replaced by \( \int_0^t dt D(t) \approx D_{\text{eff}}(t) \), and for \( \mu \) accordingly.

The resulting equations are solved analytically, but using the numerically obtained (from charge transport) quantities \( D_{\text{eff}}(t) \) and \( \mu_{\text{eff}}(t) \). The analytic solutions calculated in this way are displayed in Figs. 4 to 8 for comparison. One can see, that the agreement with the numerical simulation is reasonable.

This discussion can be somewhat improved, without the need to introduce a specific model for \( D(t) \). Figure 4 shows, that approximately

\[
D_{\text{eff}}(t) = \begin{cases} 
D_\infty, & (tv_0 < 1); \\
D_\infty (tv_0)^{-\kappa}, & (tv_0 > 1); \\
D_0, & (tv_0 \gg 1),
\end{cases}
\]

with constants \( D_\infty \) (infinite frequency) and \( D_0 \) (DC value). (The third case cannot really be deduced from Fig. 4, but it is justified for any system which does not have an infinite memory.) The exponent is about \( \kappa \approx 1/4 \) for \( \alpha N^{-1/2} = 3 \) and \( \kappa \approx 1/2 \) for \( \alpha N^{-1/2} = 7 \). Small and large times [the first and third case of Eq. (20)] thus permit the approximation \( \int_0^t dt_1 D(t_1) \approx D_{\text{eff}}(t) \). On the other hand, for medium times [the second case of Eq. (20), which corresponds to the relevant time scale of the numerical calculations], one obtains

\[
\int_0^t dt_1 D(t_1) \approx (1 - \kappa)D_{\text{eff}}(t).
\]

Thus, the effective \( D \) for the analytic expression has to be slightly reduced (by the factor \( 1 - \kappa \)). This has the effect [see Eqs. (14) and (15)] of decreasing the decay rate and also the frequency, in effect moving the zero crossings to larger times. Calculations show, that this model of \( D_{\text{eff}}(t) \) is too crude to give a better agreement with the numerics compared with the simple model advertised in the previous paragraph. The general trend of diminishing the decay rate and the frequency, however, is in accord with the numerical evidence (see Fig. 1).

Note, that the oscillatory behavior of the total spin polarization for \( \epsilon > 1 \), which has been predicted for ordered hopping systems, survives the introduction of disorder. Furthermore, Figs. 2 and 3 show, that increasing disorder (increasing \( \alpha N^{-1/2} \)) leads to an increase of the spin lifetime. This can be readily explained by noting the strong decrease of the diffusion constant in the transport equations, a conclusion, which is confirmed by the agreement between numerical simulation and analytical calculation.

The results of calculation B (spin injection with boundary conditions) are shown in Fig. 8. The only parameter of the analytical expressions is the ratio \( D_0 / \nu_0 \) which is assumed to take the value given by the Einstein relation \( (kT / e) \). The agreement to the analytical expressions (18) shown in Fig. 4 is convincing. Note, that in this case,
Other possible explanations.

A deviation is only found in very large electrical fields (\(\epsilon \gg 1\), see Fig. 6). The spatial spin coherence in calculation B diminishes with increasing \(\alpha N^{-1/2}\) in large electrical fields. The numerical data can still be fitted to the analytical model, but with a larger value of \(\epsilon\) as given by the Einstein relation (not shown). This indicates, that the ratio \(D_0/\mu_0\) increases beyond the equilibrium value \(kT/e\) in large fields, but otherwise, the spin behavior is still described by Eq. (15). This is indicative of a transition from isotropic to directed percolation in a growing electric field. This transition leaves the diffusion constant nearly unaffected, while the mobility decreases. This scenario can explain the observed behavior qualitatively, but further research is necessary to illuminate the true physical basis of the observed behavior and exclude other possible explanations.

VI. DISCUSSION

We have derived macroscopic spin transport equations for spatially disordered hopping systems with Rashba SOI. It is found, that the introduction of disorder leaves the vectorial structure of these equations intact, the only effect being that diffusion constant and mobility become frequency dependent (or, expressed differently, that the transport equations obtain memory). This frequency dependence is already determined by the charge transport behavior and does not depend on the specifics of spin transport. The derivation of this relation between ordered and disordered hopping spin transport is subject to the approximation, that the Rashba length (the length scale of spin precession) is large against a typical hopping length. Furthermore, only two-site hopping processes can be dealt with in the present treatment, thus excluding, e.g., the discussion of a possible spin Hall effect.

Other effects, previously predicted for ordered spin hopping, also occur for disordered spin hopping. For instance, the spin decay is exponential in small electric fields, whereas it obtains an oscillatory component in large electric fields. Thus, there is a finite critical field, dividing both regimes, also in the disordered case. It is also conceivable that the behavior “oscillates” between exponential and vibrational behavior. But this can be excluded, so far as \(D(t)\) and \(\mu(t)\) change monotonically with time.

In comparison to the ordered case, the spin life-time is shown to be strongly increased by the introduction of disorder. This is explained by the significantly reduced diffusion constant, which enters the life-time reciprocally and which sharply decreases with increasing disorder.

For the stationary state of spins injected through a boundary into a topologically disorder hopping system, it is found, that the spatial behavior of the spin polarization is largely unaffected by varying the disorder. An exception to this insensitivity is the systematic reduction of spin polarization with increasing disorder in large in-plane electric fields \(\epsilon \gg 1\). But even here, the numerical data indicate, that the phenomenological description of spin transport, which is derived in this paper, is still valid. The deviation appears to be due to the invalidity of the Einstein relation between diffusion co-efficient and mobility in strong electric fields for the considered system, and is not a consequence of the peculiarities of spin transport.

A comment concerning the required smallness of the Rashba interaction is in order here. For the value \(K = \sqrt{N}/10\) (the largest value used in the simulations) the spin reverses over a distance of the order of about ten times the mean distance between impurities. The typical hopping length also is several mean distances long. Thus, the central condition \(KR_t \ll 1\) is hardly valid. This (large) value of \(K\) has been taken for numerical reasons, but even so, the agreement with theory is very good. This shows, that the phenomenological description of spin transport derived above is quite robust even for (relatively) large values of the Rashba interaction strength. Smaller (more realistic) values of the Rashba coupling \(K\) render the approximation all the more valid.

Finally, a few words concerning the experimental relevance of this work. Usually given values for the Rashba SOI strength (10^{-9} eV cm) lead to a Rashba length of the order of 1/K \(\approx 100\) Å, if one assumes that the effective electron mass is equal to its elementary mass. This length scale is in the range of typical hopping lengths. Thus, depending on the system under consideration, the theory presented here is a candidate for a description of the physical behavior. For the value of the Rashba coupling given above and at a temperature of 10 K, the critical electrical field \(\epsilon = 1\), which differentiates between exponential and oscillatory behavior of the total spin polarization, corresponds to a field of about 10^3 V/cm in natural units.

To the authors knowledge, the value of the Rashba coupling strength in hopping systems has not been investigated up to now. It may well be, that this value is smaller than for itinerant electrons, in which case the estimate given above for the Rashba length (100 Å) must be increased. Then, it is even easier to comply with the condition \(KR_t \ll 1\). A further advantage of this case would be, that a modification of the Rashba coupling strength by an external gate electrode would require smaller applied voltages.
APPENDIX A: THE MACROSCOPIC TRANSPORT EQUATIONS FOR SPIN DENSITY

The aim is to derive macroscopic transport equations starting from the microscopic equations \(1\), the master equation for particle (respectively charge) transport, and \(2\), the equation governing the spin evolution. In the following, we show that under the approximation of a large Rashba length \(KR_l \ll 1\), the disorder average of Eq. \(1\) also gives an averaged equation for spin, where the averaged equations have the same relation to each other, as is the case for Eqs. \(1\) and \(2\). In order to elucidate the correspondence between the disorder averages of both equations, in the following, we first consider a “generic” averaging procedure for charge transport, and then explore the outcome of the same procedure as applied to spin transport.

The “generic” averaging procedure follows the procedure used by Klafter and Silbey\(^{15}\) which they employed to study the connection between the continuous time random walk approach and the exact (generalized) master equation. We assume, that the spatially disordered system can be adequately represented using an ordered host lattice with very small lattice constant. Then only some sites of the host lattice correspond to the original sites in the disordered system, whereas all other host lattice sites are unavailable in a specific disorder realization. The transition rates \(W_{m_1m}\) (where \(m_1\) and \(m\) correspond to sites of the original [disordered] model) can be generalized to transition rates \(V_{n_1n}\) on the host lattice, such that \(V_{n_1n}\) vanishes, except when both sites are available as defined above. Using this ordered representation of the originally disordered problem, one can construct a formal solution and thereafter perform the disorder average (see Ref. \(18\)). In this way, one obtains a generalized master equation (GME) for the (averaged or macroscopic) disordered system, which is formally exact. Approximations (as, e.g., the continuous time random walk considered in Ref. \(18\)) are only needed thereafter, in order to obtain explicit expressions for the transition rates of the GME and thus be able to calculate solutions of the GME. In the following paragraph we will show that, provided \(KR_l \ll 1\), the derivation of the formally exact GME for charge transport [starting from Eq. \(1\)] can nearly unaltered be applied to spin transport [starting from Eq. \(2\)], too. In this way, any approximations, which are applied to the charge transport GME, yield immediately a corresponding spin transport GME without the need to introduce further approximations (except \(KR_l \ll 1\) of course).

We denote the (disordered) transition rates on the ordered host lattice (with site indices \(n\), etc.) by

\[
V_{n_1n} = W_{m_1m}\delta_{n_1m}\delta_{nm}
\tag{A1}
\]

and introduce diagonal elements by

\[
V_{nn} = -\sum_{m_1} W_{mm_1}\delta_{nm},
\tag{A2}
\]

where \(\delta_{nm}\) is equal to unity or to zero, when the host lattice site \(n\) coincides or not with an original site \(m\) in the current disorder realization, respectively. Then, the charge transport rate equation can be written as

\[
\frac{d}{dt}\rho_n(t) = \sum_{n_1} V_{n_1n}\rho_{n_1}.
\tag{A3}
\]

These equations can equivalently be expressed as a matrix equation, such that, if \(N\) denotes the number of sites of the host lattice, the \(\rho_n\) become an \(N\)-vector \(\rho\) and correspondingly \(V_{n_1n}\) becomes an \(N\times N\)-matrix \(V\). The rate equation then takes the simple form \(d\rho/dt = V\rho\) or, in Laplace space \(\mathcal{L}\{s\} = \mathcal{L}\{V\}\rho\), where the index 0 marks the initial conditions. The averaged solution is formally obtained as

\[
\langle\rho(s)\rangle = \mathcal{L}\left\{[s - V^{-1}]\rho_0 = [s - M(s)^{-1}\rho_0, \tag{A4}
\right.
\]

where \(\langle\ldots\rangle\) denotes disorder average. The second equation constitutes the definition of the self-energy \(M\), the elements of which are the (as yet exact, but only formally defined) transition rates of the GME

\[
s\rho_n(s) - \sum_{n_1} M_{n_1n}(s)\rho_{n_1}(s) = \rho_{n_0}, \tag{A5}
\]

Conservation of probability allows to write this equation in the usual form

\[
s\rho_n(s) - \rho_{n_0} = \sum_{n_1} [M_{n_1n}(s)\rho_{n_1}(s) - M_{nn_1}(s)\rho_n(s)], \tag{A6}
\]

where the sum now excludes the term \(n_1 = n\). Note, that this equation lives on an ordered (quasi-continuous) lattice in contrast to Eq. \(1\). One can see, that the formal structure of Eq. \(4\) has survived the averaging, except that the transition rates have become frequency dependent. Thus, even though the microscopic equations \(1\) are Markovian, the macroscopic (averaged) equations can be non-Markovian.

Let us now execute the same procedure for the spin density. The quantity \(\rho_n\) is a 3-vector, thus \(\rho\) lives in the Cartesian product space of 3-vectors and \(N\)-vectors. The transition rates are products \(D_{n_1n}V_{n_1n} = V_{n_1n}\), where \(D_{n_1n}\) determines the action in 3-space and \(V_{n_1n}\) is identical to the corresponding quantity defined above for charge transport. Note, that we do not need a special rule for the “diagonal elements”, because \(D_{nn}\) is the unity matrix. Thus, the formal solution is

\[
\langle\mathcal{P}(s)\rangle = \mathcal{L}\left\{[s - V^{-1}]\mathcal{P}_0 = [s - M(s)^{-1}\mathcal{P}_0, \tag{A7}
\right.
\]

defining the quantity \(M\).

We now introduce the assumption, that a product of several rotation matrices \(D\) obeys the relation

\[
D_{m_1m_2} \cdot \ldots \cdot D_{m_kn} \approx D_{mn}. \tag{A8}
\]
This equation is valid to first order in $K$. Therefore, the condition $KR_l \ll 1$, where $R_l$ is a relevant length scale of the hopping transport, assures the applicability of the approximation. The fact, that Eq. (A8) is valid only to first order in $K$, gives the reason for the restriction to two-site hopping probabilities in this paper. Three-site probabilities yield interesting physics as, e.g., the spin Hall effect, but those effects are of higher order in $K$.

Within the approximation (A8), it can be shown, that the spin dependence of the transition matrix $\hat{M}$ is given by

$$\hat{M}_{n_1n} = \hat{M}_{n_1n} \hat{D}_{n_1n}$$

(A9)

and the rate equation of the averaged system reads

$$s\rho_n(s) - \rho_{n0} = \sum_{n_1} [\hat{M}_{n_1n}(s)\hat{D}_{n_1n}\rho_{n_1}(s) - \hat{M}_{n_n}(s)\rho_n(s)].$$

(A10)

The approximate validity of Eq. (A9) can be seen by representing the inverse matrix $[s - (\hat{V})^{-1}$ occurring in Eq. (A7) as a geometric series. In each term of this series, the product $\hat{M} s\rho_n(s)\hat{D}_{n_1n}$ occurs, and is accordingly simplified. The disorder average then yields an expression with the structure

$$\frac{\hat{D}_{n_1n}}{s} \left(1 - \frac{1}{s}\langle\hat{V}\rangle_{n_1n} + \frac{1}{s^2}\langle\hat{V}^2\rangle_{n_1n} + \ldots\right)$$

(A11)

The series of averaged powers of $\hat{V}$ are equal to the expression $[s - \hat{M}]^{-1}$ from charge transport, which can similarly be expanded into a geometric series, and the overall $D$-factor split [as if Eq. (A8) is read from right to left], so that each term $\hat{D}_{n_1n}(\hat{M}^k)_{n_1n}$ of the new geometric series becomes $(\hat{M}^k)_{n_1n}$, and we thereby obtain Eq. (A9).

Another way to derive the approximation, which uses the Zwanzig projection operator method, is the following. Introducing a projection operator $P$ as effecting the disorder average

$$PA = \langle A \rangle,$$

(A12)

the self-energy can be expressed as (see, e.g., Ref. [18])

$$\hat{M} = \langle\hat{V}\rangle + \left\langle \delta\hat{V}\left[s - (1 - P)\hat{V}\right]^{-1}\delta\hat{V}\right\rangle,$$

(A13)

where $\delta\hat{V} = \hat{V} - \langle\hat{V}\rangle$. Here, only one term, namely $[s - (1 - P)\hat{V}]^{-1}$, has to be expanded in a geometric series. The application of approximation (A8) again leads to the conclusion (A8).

Homogeneity of the averaged system implies, that $\hat{M}_{n_1n}$ only depends on the difference vector between the sites $M_{n_1n} = M(R_{n_1n})$. Thus, it is convenient to work in wave-vector space. The disorder averaged rate equations in Fourier-Laplace space then become the Eqs. (9) and (11), where we have replaced the symbol $M$ by $W$, with the understanding that the quantities $W_{m_1m}$ are the transition rates for the disordered system, whereas $W(s|q)$ or $W(s|\mathbf{r})$ are the transition rates for the averaged — and thus homogeneous — system.

**APPENDIX B: THE RATE EQUATIONS IN THE LONG WAVE-LENGTH LIMIT**

The following expressions are written in the right-handed orthonormal basis $e_r = r/r, e_{K \times r} = K \times r/Kr$, and $e_z = K/K$.

The spin rotation matrix for a transition between sites $m$ and $m_1$ depends only on the distance vector between both sites $\hat{D}_{m_1m} = \hat{D}(R_{m_1m})$. The explicit expression for this quantity is

$$\hat{D}(r) = \hat{I}_\alpha + \sin(2Kr)(e_z \circ e_z - e_z \circ e_z) + (\cos(2Kr) - 1)(e_z \circ e_z + e_r \circ e_r).$$

(B1)

First, we show that the Fourier transform $\hat{D}(q)$ has finite support, specifically, it is restricted to the disc $|q| \leq 2K$. The (lengthy) calculation is not given here in detail, but proceeds in the following way: The (divergent) Fourier integral of $\hat{D}(r)$ is made convergent by introducing the factor $e^{-\alpha r}$ with a positive parameter $\alpha$. The corresponding transform $\hat{D}_\alpha(q)$ can be calculated analytically and consists of rational functions of $q$ and $\alpha$, except for non-analytic factors of $[q^2 + \alpha^2]^{-1/2}$ and $[q^2 + (\alpha \pm 2K i)^2]^{-1/2}$, which only have cuts and singularities in the disc $q \leq 2K$. The subsequent limit $\alpha \to 0^+$ shows that $\hat{D}(q)$ is a sum of $\delta$-distributions and their derivatives with support within this disc. Specifically, for $q > 2K$ one obtains $\lim_{\alpha \to 0^+} \hat{D}_\alpha(q) = 0$, the zero matrix. Thus,

$$\hat{D}(q) = 0 \text{ for } q > 2K.$$  

(B2)

Next, the first and second spatial derivative of $\hat{D}(r)$ are calculated. Recall, that the vector $r$ is two-dimensional. Thus, e.g., $\nabla \circ e_r = e_r \circ e_r + e_{K \times r} \circ e_{K \times r}$, where one should note, that the $z$-components are missing. Keeping this in mind, we have $\nabla \circ e_r = e_{K \times r} \circ e_{K \times r}/r$. Of course $\nabla \circ e_z = 0$, and thus $\nabla \circ e_{K \times r} = \nabla \circ (e_z \times e_r) = -e_{K \times r} \circ e_r/r$. Replacing the dyadic product by a scalar product (corresponding to a tensor contraction) immediately gives the additional relations $\nabla \cdot e_r = 1/r$ and $\nabla \cdot e_{K \times r} = 0$.

One can now easily calculate
\[ \nabla \odot \tilde{D}(\mathbf{r}) = 2K \cos(2Kr) \mathbf{e}_r \odot (\mathbf{e}_r \odot \mathbf{e}_z - \mathbf{e}_z \odot \mathbf{e}_r) + \frac{\sin(2Kr)}{r} \mathbf{e}_{K \times r} \odot (\mathbf{e}_{K \times r} \odot \mathbf{e}_z - \mathbf{e}_z \odot \mathbf{e}_{K \times r}) \]

\[ - 2K \sin(2Kr) \mathbf{e}_r \odot (\mathbf{e}_z \odot \mathbf{e}_z + \mathbf{e}_r \odot \mathbf{e}_r) + \frac{\cos(2Kr) - 1}{r} \mathbf{e}_{K \times r} \odot (\mathbf{e}_{K \times r} \odot \mathbf{e}_r + \mathbf{e}_r \odot \mathbf{e}_{K \times r}) \]  

(B3)

and

\[ \Delta \tilde{D}(\mathbf{r}) = \left( -4K^2 \sin(2Kr) + 2K \frac{\cos(2Kr)}{r} - \frac{\sin(2Kr)}{r^2} \right) \mathbf{e}_r \odot \mathbf{e}_z - \mathbf{e}_z \odot \mathbf{e}_r \\
- \left( 4K^2 \cos(2Kr) + 2K \frac{\sin(2Kr)}{r} \right) \mathbf{e}_z \odot \mathbf{e}_z \odot \mathbf{e}_r - \mathbf{e}_r \odot \mathbf{e}_{K \times r} \odot \mathbf{e}_{K \times r} \]  

(B4)

The limit \( \mathbf{r} \to 0 \) needs a special comment, because one formally obtains expressions as, e.g.,

\[ \nabla \odot \tilde{D}(\mathbf{r}) \big|_{\mathbf{r}=0} = 2K \mathbf{e}_r \odot (\mathbf{e}_r \odot \mathbf{e}_z - \mathbf{e}_z \odot \mathbf{e}_r) \]

\[ + 2K \mathbf{e}_{K \times r} \odot (\mathbf{e}_{K \times r} \odot \mathbf{e}_z - \mathbf{e}_z \odot \mathbf{e}_{K \times r}) \]  

(B5)

which contain the basis vectors \( \mathbf{e}_r \) and \( \mathbf{e}_{K \times r} \) which are indeterminate in the limit \( \mathbf{r} = 0 \). This seeming puzzle is solved by the observation, that these basis vectors only occur in combinations, which yield the unit matrix \( \hat{I}_2 \) in two dimensions.

Thus, one obtains

\[ \mathbf{a} \cdot \nabla \odot \tilde{D}(\mathbf{r}) \big|_{\mathbf{r}=0} = 2K (\mathbf{a} \odot \mathbf{e}_z - \mathbf{e}_z \odot \mathbf{a}) \]  

(B6)

where \( \mathbf{a} \) is an arbitrary vector, and

\[ \Delta \tilde{D}(\mathbf{r}) \big|_{\mathbf{r}=0} = -4K^2 \hat{I}_3 - 4K^2 \mathbf{e}_z \odot \mathbf{e}_z \]  

(B7)

Inserting these expressions into Eq. (B5) gives Eq. (B).
FIG. 1: Temporal evolution of the components of the total spin polarization for a topologically disordered hopping system from numerical simulation: $\rho_x(t)$ (filled squares), $\rho_y(t)$ (filled triangles), $\rho_z(t)$ (filled circles). For comparison the values computed from the analytical expressions (14) and (15), where $D$ and $\mu$ are replaced by the corresponding time-dependent quantities $D_{\text{eff}}(t)$ and $\mu_{\text{eff}}(t)$ [obtained numerically from the charge transport in the same ensemble of disordered systems], are also displayed: $\rho_x$ (empty squares), $\rho_y \equiv 0$, $\rho_z$ (empty circles). The statistical error is of the order of the symbol size. The parameters are $\alpha N^{-1/2} = 3$, $\epsilon = 20$, and $K = \sqrt{N}/10$. 
FIG. 2: Temporal evolution of the $z$-component of the total spin polarization for three values of $\alpha N^{-1/2} = 3$ (full circles), 4 (full triangles), 5 (full diamonds). For comparison, the analytic solution for $\alpha N^{-1/2} = 3$ is also given (empty circles). A quite high value of the electric field ($\epsilon = 200$) has been chosen in order to numerically reach the oscillatory regime (Note the negative value of $\rho_z$ at large times, which is not possible for plain decay.). In contrast to Fig. 1, the time axis is logarithmic. Thus, with increasing disorder the spin life-time is strongly enhanced. $K = \sqrt{N}/10$.

FIG. 3: Temporal evolution of the $z$-component of total spin polarization for $\epsilon = 0$. Numerical simulation (full symbols) for $\alpha N^{-1/2} = 3$ (circles), 4 (triangles), 5 (diamonds) and analytical calculations (empty symbols, respectively) are shown. (On this scale, the empty diamonds are indistinguishable from the full ones.) $K = \sqrt{N}/10$. 
FIG. 4: Time dependence of the effective diffusion constant $D_{\text{eff}}(t)$ for disorder parameters $\alpha N^{-1/2} = 3$ (circle), 5 (diamond), and 7 (square), numerically determined from charge transport in a topologically disordered hopping system. The scale of the ordinate is arbitrary.

FIG. 5: Spin components in the stationary state for a disordered hopping system with spin injection simulated by the boundary condition $\rho(x = 0) = e_z$. The results of numerical simulation $\rho_x$ (square), $\rho_y$ (triangle), $\rho_z$ (circle), compare well with the analytical results Eq. (18): $\rho_x$ (dotted), $\rho_z$ (dash-dotted). The $y$-components of the spin are zero within the statistical error. The parameters $\alpha N^{-1/2} = 7$, $\epsilon = 0.232$, and $K = \sqrt{N}/20$. The dimensionless length $x = |r \cdot e_x|/K$ measures the distance from the boundary and the in-plane electrical field is aligned perpendicular to the boundary at $x = 0$. 
FIG. 6: The x-component of the spin in the stationary state for three values of $\alpha \mathcal{N}^{-1/2} = 3$ (circle), 5 (diamond), 7 (square), and the analytical result Eq. (18) (full line). The situation is analogous to Fig. 5 except for a stronger electric field $\epsilon = 232$. One can see, that for large $\epsilon$, the magnitude of the spin decreases with increasing disorder, and differs increasingly from the analytical results. (The z-component behaves analogously.) A discussion of this behavior is given in the text.