Precision metrology using weak measurements

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Framework : Our aim is to estimate a parameter associated with the interaction between two systems. To that end, we focus on the situation that one of them, called the QS, is a two-state system with eigenstates |−1⟩, |+1⟩ of

Weak measurements reveal partial information about a quantum state without “collapsing” it. This is done by coupling a measurement apparatus (MA) feebly to a test quantum system (QS), the dynamics of which is of interest. A procedure involves probing the QS at an intermediate stage between a pre-selected prepared state and a post-selected state which typically has little overlap with the prepared state [1]. A subsequent projective measurement on the MA yields an outcome known as the “weak value”. The fact that the weak value may lie outside the spectrum of the measurement operator by appropriately choosing the pre- and post-selected states leads itself to some interesting results. This phenomena has been used to study quantum effects such as quantum tunneling [2], Hardy’s paradox [3, 4], Leggett-Garg inequalities [5–8] and numerous others [9–16]. It has also been applied to the reconstruction of the wavefunctions of certain classes of quantum states [17–20].

Weak values may dramatically amplify the small perturbations of the meter state arising from the coupling between the QS and MA [21, 22]. This amplification makes weak measurements potentially useful in estimating the coupling strength with enhanced precision. In this guise, weak measurements have been used to estimate displacements on the order of angstroms [23], deflections on the order of femto-radians [24], small longitudinal phase shifts [25, 26], as well as to amplify single-photon nonlinearities [27], improve charge sensing in solid-state systems [28], and measure light chirality with a plasmonic slit [29].

Yet, the amplification effect of weak measurement comes at the cost of a reduced rate at which data can be acquired due to the requirement to select almost orthogonal pre- and post-selected states of the QS. This leads to a majority of trials being “lost”. Thus, the central question is whether the amplification effect of a weak measurement can overcome the corresponding reduction in the occurrence of such events to provide an estimation at a precision surpassing conventional techniques. This issue has garnered substantial interest recently. The signal-to-noise ratio of weak measurement in particular setups has been studied [30, 31]. It has been shown that post-selection does not amplify the information in the MA state [32, 33], even in the situation where practical experimental imperfections are presented [34, 35]. However, an unequivocal agreement as to the efficacy of weak measurements in precision metrology is still lacking. Our endeavor in this work is to provide such an answer.

In this paper, we show that post-selection does not enhance the precision of estimation, that weak measurements do not offer better precision relative to strong measurements, and that it is possible to beat the standard quantum limit and to achieve Heisenberg limit of quantum metrology with weak measurements using only classical resources. These seemingly opposed conclusions arise from a complete consideration of where the maximum information resides in the weak measurement protocol under various different conditions. Our results are valid both for single-particle MA states, in which the QS couples to a continuous degree of freedom of the MA, and for multi-particle states of a bosonic MA. In each case, non-classical correlations may be generated by the interaction between the QS and MA, though it is not always the case that this leads to quantum enhancement of precision. Our analysis properly counts the resources involved in the measurement process, enabling us to compare the precision of different measurement strategies and strengths using tools of classical and quantum Fisher information. Weak measurements have a rich structure, and offer some prospects for novel strategies for quantum-enhanced metrology. Nonetheless, we show that it requires a new approach to harness this potential.
an observable $\hat{S}$ with corresponding eigenvalues -1 and 1. The initial (pre-selected) state of the QS is prepared as $|\psi_j\rangle = \cos(\theta_j/2)|-1\rangle + \sin(\theta_j/2)e^{i\phi_j}|+1\rangle$. The initial state of the other system, called the MA, is $|\Phi_i\rangle$. The coupling strength $g$ to be estimated appears in the Hamiltonian $H = -g\delta(t-t_{0})\hat{S}M$ coupling MA to QS, where $M$ is an observable of the MA. After this interaction, the joint state of the MA and the QS is

$$|\Psi_j\rangle = \cos\frac{\theta_j}{2}|-1\rangle|\Phi_{-g}\rangle + \sin\frac{\theta_j}{2}e^{i\phi_j}|+1\rangle|\Phi_{+g}\rangle. \quad (1)$$

Post-selecting the QS in state $|\psi_f\rangle = \cos(\theta_f/2)|-1\rangle + \sin(\theta_f/2)e^{i\phi_f}|+1\rangle$ leads to the MA state

$$|\Phi_d\rangle = \frac{1}{\sqrt{p_d}}(\gamma_d^-|\Phi_{-g}\rangle + \gamma_d^+|\Phi_{+g}\rangle), \quad (2)$$

with $\gamma_d^- = \cos(\theta_j/2)\cos(\theta_f/2)$, $\gamma_d^+ = \sin(\theta_j/2)\sin(\theta_f/2)\exp(i\phi_j)$ and $\phi_d = \phi_i - \phi_f$. The probability of successful post-selection, i.e., the probability of obtaining $|\Phi_d\rangle$ is $p_d$. When the post-selection fails (with probability $p_f = 1 - p_d$), the MA state, which is not considered in the original protocol and is often ignored in experiments, is

$$|\Phi_r\rangle = \frac{1}{\sqrt{1-p_d}}(\gamma_r^-|\Phi_{-g}\rangle + \gamma_r^+|\Phi_{+g}\rangle), \quad (3)$$

where $\gamma_r^- = \cos(\theta_j/2)\sin(\theta_f/2)$, $\gamma_r^+ = -\sin(\theta_j/2)\cos(\theta_f/2)\exp(i\phi_j)$. Repeating the pre-select-coupling-post-select process $N$ times, yields $Np_d$ copies of $|\Phi_d\rangle$ and $N(1-p_d)$ copies of $|\Phi_r\rangle$. The best attainable precision in estimating $g$ is given by the Cramér-Rao bound $\Delta^2 g \geq 1/(NF_{tot})$ [36], where $F_{tot}$ is the sum total of the classical and quantum Fisher information (FI) contained at different stages of the pre-select–coupling–post-select process.

Depending on the estimation protocol, $F_{tot}$ may have different values. To date almost all applications of the weak measurement protocol to precision metrology focus on the amplification effect of weak values, which corresponds to considering the information about $g$ contained in the MA state $|\Phi_d\rangle$. In this situation $F_{tot} = p_dQ_d$, where $Q_d$ is the quantum FI of $|\Phi_d\rangle$, i.e. the maximum FI that can be achieved with the optimal measurement on $|\Phi_d\rangle$. $p_dQ_d$ can be viewed as the total information in the post-selected meter state. In addition, one may also monitor the failure mode $|\Phi_r\rangle$, as has been suggested, to achieve better precision in parameter estimation [37, 38] and state tomography [20]. The maximum information one can acquire from the failure mode is $(1-p_d)Q_r$, where $Q_r$ is the quantum FI of $|\Phi_r\rangle$. Finally, the distribution $\{p_d, 1-p_d\}$ of the post-selection process on QS also contains information about $g$. This distribution yields a classical FI $F_p$ which we refer to as the information in the post-selection process. If we account for all these contributions, we have (see supplementary material for a proof)

$$F_{tot} = p_dQ_d + (1-p_d)Q_r + F_p. \quad (4)$$

The whole process (post-selection plus measurements on the MA state) is a measurement on the joint state $|\Psi_j\rangle$ [39], therefore $F_{tot}$ is no larger than the quantum FI $Q_j$ of $|\Psi_j\rangle$, i.e. post-selection cannot increase the precision in estimating $g$. We note that $Q_d$ or $Q_r$ alone may be larger than $Q_j$ due to the amplification effect of weak values. Nevertheless this apparent gain of information, and thus attainable precision, is completely canceled by the small probability of successful post-selection. This analysis goes beyond previous studies [32, 33] by considering all the contributions to the total information, and thus provides a complete answer to Q1 posed in the abstract. In following sections we provide answers to Q2 and Q3 in both configuration and phase space interactions.

**Configuration space interactions**: We begin with the most widely used scenario in weak measurement [1, 9, 10, 21, 23–26, 28, 29] where both the QS and MA are single-particle states, possibly in different degrees of freedom of the same particle [40]. In this situation the MA is prepared in a Gaussian superposition state of two conjugate variables

$$|\Phi\rangle = \int dq \frac{1}{(2\pi \sigma^2)^{1/4}} \exp(-\frac{q^2}{4\sigma^2})|q\rangle \int dp \frac{(2\sigma^2)^{1/4}}{\pi^{1/4}} \sqrt{p_k} \exp(-\frac{p^2}{\sigma^2})|p\rangle$$

where $p$ and $q$ are, e.g., momentum and position or time and frequency. The interaction Hamiltonian between the QS and MA is chosen as $H = -g\delta(t-t_{0})\hat{S}\hat{q}$. Note that this interaction Hamiltonian entangles the QS with an external degree of freedom of the MA. It does not change the particle number distribution in the state of the MA. After the interaction and post-selection, the MA state becomes $|\Phi_k\rangle = \int dp \hat{\phi}(s(p), p)|p\rangle$ with

$$\hat{\phi}(s(p), p) = \frac{(2\sigma^2)^{1/4}}{\pi^{1/4}\sqrt{p_k}} \left[ \gamma_k^- e^{-\sigma^2(p-g)^2} + \gamma_k^+ e^{-\sigma^2(p-g)^2} \right]. \quad (5)$$

The probability of successful post-selection, i.e., the probability of obtaining $|\Phi_d\rangle$ is

$$p_d = \frac{1 + \cos\theta_j \cos\theta_f + \sin\theta_j \sin\theta_f \cos\phi_0 e^{-2s^2}}{2}, \quad (6)$$

with $s = g\sigma$ characterising the measurement strength. With Eqs. (6, 7) we can estimate $Q_d$, $Q_r$, and $F_p$ [39].

$$Q_d = \frac{4\sigma^2}{p_d} \left[ p_d + S (2s^2 - 1) - \frac{1}{p_d} S^2 s^2 \right],$$

$$Q_r = \frac{4\sigma^2}{1-p_d} \left[ 1 - p_d - S (2s^2 - 1) - \frac{1}{1-p_d} S^2 s^2 \right],$$

$$F_p = \frac{4\sigma^2 S^2 s^2}{p_d(1-p_d)}. \quad (8)$$
where \( S = e^{-2x^2} \sin \theta_i \sin \theta_f \cos \phi_0 \). Further the quantum FI of the joint meter-system state before post-selection is \( \Omega_j = 4\sigma^2 \). We can now calculate \( F_{\text{tot}} \) for different estimation strategies. In particular, if we take into account of all the contributions in Eq. (4), we have \( F_{\text{tot}} = \Omega_j \), i.e. we achieve the maximal precision. A commonly employed strategy in weak measurement precision metrology retains only the information in the successfully post-selected meter state. In this case, the complicated functional form of \( F_{\text{tot}} = p_dQ_d \) demands numerical maximization over \( \psi_i \) and \( \psi_f \). Nonetheless, some limits that may be obtained analytically allow us to answer Q2. In the weak measurement limit, defined as \( s \rightarrow 0 \)

\[
p_dQ_d = 2\sigma^2(1 + \cos \theta_i \cos \theta_f - \sin \theta_i \sin \theta_f \cos \phi_0),
\]

the maximum value of which is \( 4\sigma^2 \), attained when the pre- and post-selected states are asymptotically orthogonal, i.e., \( \langle \psi_i | \psi_f \rangle = 0 \). This coincides with the situation when the weak value is the largest. In the limit of strong measurement when \( s \gg 1 \)

\[
p_dQ_d = 2\sigma^2(1 + \cos \theta_i \cos \theta_f),
\]

which also attains the maximum of \( 4\sigma^2 \), but for the situation that both pre- and post-selected states are \( |+1 \rangle \) or \(-1 \rangle \). In both these limits, \( p_dQ_d = \Omega_j \), \( F_p = 0 \) and \( Q_r = 0 \). Thus, all the information is contained in the post-selected state, and not in either the post-selection process or the failure mode.

The conclusion is that when the uncertainty of the meter state \( \sigma \) is fixed, the precision in the weak measurement limit, that is, to estimate a small parameter \( g \) through pre-selection-coupling-post-selection, is no better than that in the strong case, that is, to estimate a large parameter. However, if the parameter to be estimated is fixed, the precision is always better if we use a meter state with smaller \( \sigma \). This answers Q2 for the configuration-space-interaction scenario.

The above analysis focuses on the effect of the uncertainty in the external degrees of freedom of the MA as in the previous works [21, 23–26, 28–31, 35, 37], showing that weak measurements may or may not offer an overhead advantage depending on the scenario. In quantum metrology, the relevant measure of the resource required to effect a measurement is the average number of photons \( (n) \) in the MA state. The scaling of the precision of estimation with respect to \( n \) is the signature of whether the system is capable of operating beyond the standard quantum limit (in which the FI scales linearly in \( n \)) and offering genuine quantum advantages. Since the interaction Hamiltonian does not change particle-number distributions, for QS and MA prepared in (multi-mode) coherent states with amplitude \( \alpha \), post-selected meter states are also multi-mode coherent states, and the FIs in Eqns. (9, 10) pick up an additional factor of \( n = |\alpha|^2 \). Thus the scalings are at the standard quantum limit, providing at most overhead advantages. This is the answer to Q3 for the configuration-space-interaction scenario.

**Phase-space interactions:** We now consider a scenario that can change the particle-number distribution to see if the scaling of the measurement precision can surpass the classical limit. The initial state of the QS \( \langle \psi_i \rangle \) is the same as before, while the MA is prepared in a coherent state \( |\alpha \rangle \). Under the interaction unitary

\[
U = \exp \left[ ig\hat{S}\hat{n} \right]
\]

which is a conditional phase rotation that can be realized, for example, through cross-phase modulation between two light beams [27]. The joint state after the interaction is

\[
|\Psi\rangle = \cos \frac{\theta_i}{2} |\alpha e^{-ig}\rangle + \sin \frac{\theta_i}{2} e^{i\phi_0} |\alpha e^{ig}\rangle.
\]

The meter states after post-selection are \( (k = d, r) \)

\[
|\Phi_k\rangle = \frac{1}{\sqrt{p_k}} \left( \gamma_k^- |\alpha e^{-ig}\rangle + \gamma_k^+ |\alpha e^{ig}\rangle \right).
\]

The probability of obtaining this state, the expressions for the quantum FI of the these states, and the classical FI of the post-selection process are all given in [39].

The quantum FI of the system-meter state in Eq. (12) is [39]

\[
\Omega_j = 4n^2 \sin^2 \theta_i + 4n,
\]

where \( n = |\alpha|^2 \) is again the mean photon number (or energy) of the meter state. \( \Omega_j \) is the maximum amount of information attainable from this experiment, and can exhibit quantum scaling \( (\sim n^2) \) depending on the initial system state. The expression for \( \Omega_j \) immediately suggests that \( \theta_i = 0, \pi \) will never provide a better-than-classical scaling. These are the two cases when the initial state is an eigenstate of \( \hat{S} \), so that no entanglement is generated between the QS and MA. Indeed, for \( |\psi_i\rangle = (|+1\rangle + |\pm1\rangle) \), \( p_dQ_d = 2n(1 \pm \cos \theta_f) \), \( (1 - p_d)Q_r = 2n(1 \mp \cos \theta_f) \), both of which scale as the standard quantum limit and \( F_p = 0 \). Thus, \( F_{\text{tot}} = 4n \), but the information may equally be present in the successful or the failed post-selection mode. This is important since the failed post-selected mode is generally discarded completely [21, 23–29].

In contrast, the maximal \( \Omega_j \) is found for \( \theta_i = \pi/2 \). We immediately find that \( \theta_f = 0, \pi \) provides no better than classical scalings either. Thus, we set \( \theta_f = \pi/2 \) as well, and find that as \( g \rightarrow 0 \), it leads to

\[
F_p = 4n^2.
\]

This result shows that quantum enhanced scaling can be attained in the sensing of the Hamiltonian coupling parameter \( g \) in a weak measurement setup. On the other hand, in this same situation the quantum FI for both the
successful and failed post-selection mode scale classically; 
\[ p_d Q_d = 4n \sin^2(\phi_0/2) \text{ and } (1 - p_d) Q_f = 4n \cos^2(\phi_0/2). \]
This shows that \( p_d Q_d \) achieves its maximum when \( \phi_0 \to \pi \), i.e. \( \psi_i \) and \( \psi_f \) are orthogonal. Note also that if we take into account of all the contributions we have \( F_{tot} = Q_j \).
This is a particularly interesting situation since most, if not all, earlier experiment using weak measurement considered only the information \( Q_d \) contained in the successfully post-selected MA state. Yet, as we show here, the post-selection distribution \( F_j \) has much more information, indeed scales at the Heisenberg limit.

For interaction strengths \( g > 0 \), the relative contributions of the different terms in \( F_{tot} \) change. In Fig. (1), we plot these variations for \( \phi_0 = \pi \). Exploiting the symmetry, we only plot the results in \( g = \{0, \pi/2\} \). As shown earlier, for \( g \to 0 \) the main contribution comes from \( F_p \), the classical FI in the post-selection distribution. As \( g \) increases, \( F_p \) falls, and the information in the post-selected states for both successful and failed QS measurement outcomes rises. For \( g = \pi/2 \), we plot the contributions in greater detail in Fig. (2) for \( \phi_0 = 0 \). For this case, \( F_p = 0 \) while \( (1 - p_d)Q_f, p_d Q_d \) are almost equal. Indeed, \( p_d Q_d - (1 - p_d)Q_f = 4n(n - 1) \exp(-2n) \). For \( n \gg 1 \), up to a small exponential correction, there is thus much information in the successful post-selection mode as in the failed mode, and both of them scale better than the classical scaling given by the standard quantum limit. For a weak coherent meter state with \( n \ll 1 \), the successful mode has no information. In all cases, the total \( F_{tot} \) still matches the maximum quantum FI attainable, that is \( Q_j \). These results provide answers to Q2 and Q3 for the conditional-phase-shift scenario.

**Discussion and Conclusions:** It is perhaps unsurprising that the Heisenberg limit for estimating the coupling parameter \( g \) in the conditional phase shift interaction can be attained when the system-meter coupling is strong, since in that case, the post-selected MA states are Schrödinger-cat states. That is, the measurement protocol produces highly non-classical states in the joint system. In the case of weak coupling (\( g \to 0 \)), however, the the post-selected MA states are classical, and the Heisenberg scaling arises only in the post-selection process itself. How this conditioning step using a classical MA state achieves a precision beyond the standard quantum limit is therefore an interesting open question.

Our calculations show that not only the failed post-selection mode but the post-selection process itself in a weak measurement contains important information. The analysis provide answers to the three long standing questions in the study of weak measurement posed in the abstract: (A1) post-selection can not enhance the measurement precision even when all the contributions from both the post-selection process and post-selected probe states are taken into account; (A2) when using the same resources, weak measurement does not give improved precision over strong measurement when both measurements are optimized. In particular, this result applies to all previous experiments that have explored weak-measurement enhancements to precision metrology. (A3) weak measurement that modifies the particle number distribution of the meter state can yield quantum-enhanced precision though no non-classical states need to be involved. These results highlight the rich structure of the weak measurement and shed new light on both the understanding of quantum measurement protocols and the development of new technologies for practical quantum metrology.
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SUPPLEMENTARY MATERIAL

Derivation of $F_{tot}$

The quantum FI $Q_k$ of $|\Phi_k\rangle$ ($k = d, r$) is given by

$$Q_k = 4 \left[ \left( \frac{d|\Phi_k\rangle}{dg} \right) \left( \frac{d\langle \Phi_k |}{dg} \right) - \langle \Phi_k | \left( \frac{d|\Phi_k\rangle}{dg} \right) \right]^2. \quad (16)$$

$Q_k$ can be achieved with the optimal POVMs. Assume the optimal measurement for $|\Phi_d\rangle$ is $\{\Pi_d^1, \ldots, \Pi_d^n\}$ with the probabilities of each outcome $P(1|\text{detect}), \ldots, P(V|\text{detect})$, where $P(v|\text{detect}) = \langle \Phi_d | \Pi_d^v | \Phi_d \rangle$, for $v = 1 \cdots U$. \quad (17)

Then we have

$$Q_d = \sum_{v=1}^{V} \frac{1}{P(v|\text{detect})} \left( \frac{d(P(v|\text{detect}))}{dg} \right)^2. \quad (18)$$

Similarly, the optimal measurement for $|\Phi_r\rangle$ is $\{\Pi_r^1, \ldots, \Pi_r^n\}$ with the probabilities of each outcome $P(1|\text{reject}), \ldots, P(W|\text{reject})$. Then post-selection on the QS state followed by the optimal measurement on the MA states can be considered as a POVM performed on the joint state $|\Psi\rangle$ with $\{|\psi_f\rangle\langle \psi_f | \otimes \Pi_r^1, \ldots, \psi_f\rangle\langle \psi_f | \otimes \Pi_r^n\}$, where $|\psi_f\rangle$ is the state of QS when post-selection fails. The probabilities associated with each outcome are $\{p_d \times P(1|\text{detect}), \ldots, p_d \times P(V|\text{detect}), (1 - p_d) \times P(1|\text{reject}), \ldots, (1 - p_d) \times P(W|\text{reject})\}$. The Fisher information is given by

$$F_{tot} = \sum_{v=1}^{V} \frac{1}{p_d} \left( \frac{d(p_d P(v|\text{detect}))}{dg} \right)^2 + \sum_{v=1}^{W} \frac{1}{(1 - p_d)} \left( \frac{d((1 - p_d) P(w|\text{reject}))}{dg} \right)$$

$$= p_d Q_d + (1 - p_d) Q_r + F_p,$$

where

$$F_p = \frac{1}{p_d} \left( \frac{dp_d}{dg} \right)^2 + \frac{1}{1 - p_d} \left( \frac{d(1 - p_d)}{dg} \right)^2. \quad (20)$$

If we ignore the meter state when the post-selection fails, the whole process can still be considered as a POVM performed on the joint state with $\{|\psi_f\rangle\langle \psi_f | \otimes \Pi_r^1, \ldots, \psi_f\rangle\langle \psi_f | \otimes \Pi_r^n\}$, where $|\psi_f\rangle$ is the state of QS when post-selection fails. The probabilities associated with each outcome are $\{p_d P(1|\text{detect}), \ldots, p_d P(V|\text{detect}), 1 - p_d\}$. The total FI is given by

$$F_{tot} = p_d Q_d + F_p. \quad (21)$$

Quantum FI of the joint system-meter state in Eq. (12)

As shown in Sec., post-selection on the QS state followed by the measurement on the MA state can be considered as a measurement on the joint state $|\Psi_j\rangle$. Therefore, we can estimate the quantum FI of $|\Psi_j\rangle$, which will give us an upper bound on the precision, though it may not be achievable, in that $|\psi_f\rangle$ may not be the optimal.

We have

$$\frac{d|\Psi_j\rangle}{dg} = \cos \frac{\theta_i}{2} (|\alpha e^{-ig} \rangle - |\alpha e^{ig} \rangle) + \sin \frac{\theta_i}{2} e^{i\phi} (|\alpha e^{ig} \rangle + |\alpha e^{-ig} \rangle) - e^{ig} \sin \frac{\theta_i}{2} e^{i\phi} |\alpha \rangle + 1 |\alpha \rangle$$

$$\frac{d\langle \Psi_j \rangle}{dg} = -i|\alpha|^2 \left( \cos^2 \frac{\theta_i}{2} - \sin^2 \frac{\theta_i}{2} \right) = -i \cos \theta_i. \quad (24)$$

So we have the quantum FI of the joint state

$$Q_j = 4 \left[ \left( \frac{d|\Psi_j\rangle}{dg} \right) \left( \frac{d\langle \Psi_j \rangle}{dg} \right) - \langle \Psi_j | \left( \frac{d|\Psi_j\rangle}{dg} \right) \right]^2$$

$$= 4 n^2 \sin^2 \theta_i + 4n. \quad (25)$$

Quantum and classical FI for weak measurement in phase space

The probability of successful post-selection $p_d$, that is, of obtaining the state in Eq. (13) is being given by

$$p_d = \frac{1 + A \cos(B - 2g) + C}{2}, \quad (26)$$

where $A = \sin \theta_1 \sin \theta_f \exp(-2n \sin^2 g)$, $B = n \sin 2g + 2g + \phi_0$, $C = \cos \theta_1 \cos \theta_f$, and $n = |\alpha|^2$. The classical FI in this post-selection process is

$$F_p = \frac{4n^2 A^2 \sin^2 B}{1 - (A \cos(B - 2g) - C)^2}. \quad (27)$$

The quantum FI for the state after successful post-selection is
\[ Q_d = \frac{4}{p_d} \left\{ \frac{n^2}{2} (1 + C - A \cos B) + \frac{n^2}{2} (1 + C - A \cos (B + 2g)) - \frac{1}{p_d} \frac{n^2}{4} (\cos^2 \theta_i + \cos^2 \theta_f + 2C + A^2 \sin^2 B) \right\}. \quad (28) \]

The quantum FI for the state after failed post-selection is

\[ Q_r = \frac{4}{1 - p_d} \left\{ \frac{n^2}{2} (1 - C - A \cos B) + \frac{n^2}{2} (1 - C + A \cos (B + 2g)) - \frac{1}{1 - p_d} \frac{n^2}{4} (\cos^2 \theta_i + \cos^2 \theta_f - 2C + A^2 \sin^2 B) \right\}. \quad (29) \]