Gravitational and Axionic Backgrounds for Four-dimensional Superstrings

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ABSTRACT

We construct new four-dimensional superstring vacua with extended superconformal symmetries. A non-trivial dilaton background implies the existence of Abelian killing symmetries. These are used to construct dual equivalent backgrounds in a way preserving the $N = 2$ superconformal invariance.

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1 Introduction

During recent years, a lot of efforts were focussed on the construction of $N = 2$ superconformal (heterotic) string backgrounds [1]. These backgrounds lead to target space supersymmetry and, consequently, the perturbative vacuum is guaranteed to be stable. In these works the discussion mainly concentrated on flat four-dimensional Minkowski space-time times an internal compact space without torsion and with constant dilaton field, i.e. tori, orbifolds and numerous Calabi-Yau spaces. However we like to construct supersymmetric string vacua with, in addition to the metric background, more general non-constant background fields. Moreover, to address certain important questions in quantum gravity one has to consider string backgrounds which describe four-dimensional curved space-times. In particular one is interested to construct exact superconformal field theories which correspond to four-dimensional black-hole backgrounds, cosmological scenarios or supersymmetric instanton type of solutions.

In this contribution we will report about a relatively systematic discussion [2] on supersymmetric string backgrounds with $N = 2$ or $N = 4$ superconformal symmetry, based on compact as well as non-compact spaces plus non-trivial antisymmetric tensor-field and non-constant dilaton. In contrast to the compact Calabi-Yau spaces, almost all backgrounds with non-trivial dilaton field will possess Killing symmetries. Many of such backgrounds exhibit singularities on some hypersurface in spacetime and can be regarded as generalizations of the two-dimensional black-hole considered in [3].

A key to the proper understanding of string propagation on curved spaces is provided by duality symmetries [4]. Duality symmetries relate different backgrounds which nevertheless correspond to the same superconformal field theory. Specifically, we will show that some interesting non-Kählerian $N = 4$ solutions, which describe four-dimensional axionic instantons, are dual-equivalent to four-dimensional, non-compact Ricci-flat Kähler spaces.

2 The $N = 2$ ($N = 4$) Background and $U(1)$ Duality Transformations

The most general $N = 2$ superspace action for $m$ chiral superfields $U_i$ ($i = 1, \ldots, m$) ($\bar{D}_+ U_i = 0$) and $n$ twisted chiral superfields $V_p$ ($p = 1, \ldots, n$) ($\bar{D}_+ V_p = D_- V_p = 0$) in two dimensions is determined by a single real function $K(U_i, \bar{U}_i, V_p, \bar{V}_p)$ [3]:

$$S = \frac{1}{2\pi\alpha'} \int d^2x D_+ D_- \bar{D}_+ \bar{D}_- K(U_i, \bar{U}_i, V_p, \bar{V}_p). \quad (2.1)$$
To see the background interpretation of the theory it is convenient to write down the purely bosonic part of the superspace action (2.1):

\[
S = -\frac{1}{2\pi\alpha'} \int d^2x [K_{u_i\bar{u}_j} \partial^\alpha u_i \partial_\alpha \bar{u}_j - K_{v_p\bar{v}_q} \partial^\alpha v_p \partial_\alpha \bar{v}_q + \epsilon_{ab}(K_{u_i\bar{v}_p} \partial_a u_i \partial_b \bar{v}_p + K_{v_p\bar{u}_i} \partial_a v_p \partial_b \bar{u}_i)],
\]

where \(K_{u_i\bar{u}_j} = \frac{\partial^2 K}{\partial u_i \partial \bar{u}_j}\), etc. Here \(u_i\) is the lowest component of the superfield \(U_i\) and so on. Thus, one recognizes that the first two terms in above equation describe the in general non-Kählerian metric background of the model (the metric is Kähler only when \(m = 0\) or \(n = 0\)). The \(\epsilon_{ab}\)-term in (2.2) provides the antisymmetric tensor field background. Of course, in order that these backgrounds provide consistent string solutions, they have to satisfy the string equation of motion, i.e. the vanishing of the \(\beta\)-function equations [7]. Including also the dilaton background \(\Phi(u_i, v_p)\), the \(\beta\)-function equations will lead to some differential equations for the two functions \(K\) and \(\Phi\) as we will discuss in the following. Moreover, the central charge defect \(\delta c = c - \frac{3D}{2}\) will be determined by \(K\) and \(\Phi\). In the presence of \(N = 4\) superconformal symmetry the solutions to lowest order in \(\alpha'\) are exact to all orders in a specific scheme, and \(\delta c\) remains zero to all orders.

Now we consider the simple case of a single \(U(1)\) isometry assuming that the potential \(K\) has one Killing symmetry, \(R = Z + \bar{Z}\):

\[
K = K(Z + Z, Y_i, \bar{Y}_i, V_p, \bar{V}_p),
\]

where \(Z\) and \(Y_i\) are chiral fields, whereas \(V_p\) are twisted chiral fields. (Of course the discussion holds in the same way if \(Z\) is a twisted chiral field.) Then the ‘dual’ potential has the following form [3, 4]

\[
\tilde{K}(R, Y_i, \bar{Y}_i, V_p, \bar{V}_p, \Psi + \bar{\Psi}) = K - R(\Psi + \bar{\Psi}),
\]

where \(\Psi\) a twisted chiral field. Varying the action with respect to \(\Psi\) gives back the original theory. On the other hand one can equally well consider the constraint coming from the variation with respect to \(Z\), [3] \(\frac{\partial S}{\partial Z} = 0\), and the dual theory is obtained as a Legendre transform of \(K\) where the independent variable are \(\psi, y_i\) and \(v_p\).

### 3 Kähler Spaces without Torsion and their Duals

If the antisymmetric tensor field vanishes the space is Kählerian and the metric is given in terms of the Kähler potential by the standard formula \(G_{ij} = G_{\bar{i}\bar{j}} = 0, G_{ij} = K_{u_i\bar{u}_j}\). Then the Ricci-tensor takes its well-known form \(R_{u_i\bar{u}_j} = -\partial_{u_i} \partial_{\bar{u}_j} U\), \(R_{a\bar{u}_j} = 0\) with \(U = \log \det K_{u_i\bar{u}_j} = \frac{1}{2} \log \det G\). The only condition for conformal invariance is \(\beta_{\mu\nu}^G = 0\) which here implies

\[
\Phi = \frac{1}{2} U + f(u_i) + \bar{f}(\bar{u}_i),
\]

(3.1)
and
\[ \nabla_{u_i} \partial_{u_j} \Phi = \nabla_{u_i} \partial_{\bar{u}_j} \Phi = 0. \tag{3.2} \]
where \( f \) is an arbitrary holomorphic function. As described in [2] it is not difficult to show that the vanishing of the holomorphic double derivative on the dilaton implies, for non-trivial dilaton, that there is a generic Killing symmetry in the Kähler metric as well as in the dilaton. Then, in a special coordinate system the compatibility of the equations (3.1) and (3.2) along with our freedom to perform Kähler transformations implies that

\[ K = K(z + \bar{z}, y_i, \bar{y}_i), \quad \Phi = \partial_z K = \partial_{\bar{z}} K \tag{3.3} \]

and
\[ \Phi = \frac{1}{2} U + C(z + \bar{z}) \tag{3.4} \]
where \( C \) is any real number. We can take (3.4) as the equation specifying the dilaton in terms of the metric and then (3.4) becomes a non-linear differential equation for the Kähler potential

\[ \det[K_{u_i \bar{u}_j}] = \exp[-2C(z + \bar{z}) + K_z + K_{\bar{z}}] \tag{3.5} \]

generalizing the CY condition.

Let us consider a special class assuming that the model has a \( U(N) \) isometry, i.e. \( K = K(x), \Phi = \Phi(x) \) with \( x = \sum_{i=1}^{N} |u_i|^2 \). For \( N > 1 \), the linear term in the dilaton, eq.(3.4), is not allowed by the \( U(N) \) isometry and the dilaton field becomes \( \Phi = \frac{1}{2} U = \frac{1}{2} \log[(K')^{N-1}(K' + xK'')] \) (\( K' = \frac{\partial K}{\partial x} \)). Let us define the following function: \( Y(x) = xK'(x) \). Now we have to insert this ansatz into the field equation, and the solution takes the following form:

\[ e^Y \sum_{m=0}^{N-1} \frac{(-1)^m Y^m}{m!} = A + Bx^N. \tag{3.6} \]

Here \( A \) and \( B \) are arbitrary parameters. The dilaton can be also expressed entirely of \( Y \) as \( \Phi = \frac{1}{2} U = -\frac{1}{2} Y + \text{const.} \)

Let us now consider four-dimensional backgrounds, namely we consider solutions of the form eq. (3.6) with \( N = 2 \). Using \( Y \) together with the overall phase \( \theta \) as (real) coordinates, the metric then reads:

\[ ds^2 = \frac{(dY)^2}{4f(Y)} + \frac{f(Y)}{4} \left( d\theta - i \bar{y} dy - y d\bar{y} \right)^2 
+ \frac{Y}{(1 + y \bar{y})^2} dy d\bar{y}, \quad f(Y) = \frac{2(Ae^{-Y} + Y - 1)}{Y}. \tag{3.7} \]

This metric in the \((\theta, \psi, \phi)\) subspaces is a deformation of the fibration of \( S^3 \) over \( S^2 \), whose line element is manifest in (3.7). The space possesses a generic singularity for \( Y \to \infty \), and another singularity at \( Y = 0 \), if \( A \neq 1 \).

Applying eq.(2.4), the dual metric is given as

\[ ds^2 = \frac{d\psi d\bar{\psi}}{f(Y)} + \frac{Y}{(1 + y \bar{y})^2} dy d\bar{y}, \tag{3.8} \]
whereas the dual dilaton and antisymmetric tensor field look like \( \tilde{\Phi} = -\frac{1}{2} \log [e^Y f(Y)] \), \( \tilde{B}_{\psi \bar{\psi}} = 2\bar{y}/(1 + \bar{y}) \). The dual space has curvature singularities at \( Y = -\infty, 0 \) and, for generic values of \( A \) at the zeros of \( f(Y) \).

4 Four-dimensional Non-Kählerian Spaces with Torsion and their Duals

To start with non-vanishing antisymmetric tensor fields we restrict ourselves to the simplest case, namely four-dimensional target spaces, i.e. \( m = n = 1 \). A particular simple class of solutions of the \( \beta \)-function equations [2] is then given by functions \( K \) which satisfy the four-dimensional Laplace equation

\[
(\partial_u \partial_{\bar{u}} + \partial_v \partial_{\bar{v}})K = 0. \tag{4.1}
\]

In addition, the dilaton field is simply given as

\[
2\Phi = \log K_{\bar{u}u} + \text{constant}. \tag{4.2}
\]

Eqs. (4.1) and (4.2) imply that \( \delta c = 0 \) and these backgrounds are expected to have \( N = 4 \) superconformal symmetry. This observation is consistent with the fact that eq. (4.1) is the generalization of the hyper-Kähler condition for spaces with antisymmetric tensor field. The form of the dilaton field has the important consequence that the four-dimensional metric in the Einstein frame is flat: \( G_{\mu\nu}^{\text{Einstein}} = e^{-2\Phi} G^{g}_{\mu\nu} = \delta_{\mu\nu} \). In fact, the solutions of the dilaton equation (4.2) have a very close relation to the axionic instantons of [8]. Specifically, if we assume that the theory possesses two \( U(1) \) isometries, i.e. \( \Phi = \Phi(u + \bar{u}, v + \bar{v}) \), equation (4.2) implies the following relation:

\[
d\Phi = \pm \frac{1}{2} e^{-2\Phi} H^*. \tag{4.3}
\]

This relation is nothing else than the self-duality condition on the dilaton-axion field. Its solutions are called axionic instantons.

Let us now construct the dual spaces for the solutions of the Laplace equation with isometries. We will perform a duality transformation on the chiral \( U \)-field replacing it by a twisted chiral field \( \Psi \). The Legendre transformed potential \( \tilde{K} \) will only contain twisted fields and will be therefore a true Kähler function leading to a non-compact Kähler space without torsion. Doing the Legendre transform we obtain the following line element

\[
ds^2 = \frac{1}{K_{\bar{u}u}} (dz - K_{u\bar{v}}dv)(d\bar{z} - K_{\bar{u}v}d\bar{v}) - K_{v\bar{v}}dvd\bar{v} \tag{4.4}
\]

where \( K(u + \bar{u}, v, \bar{v}) \) is the original quasi-Kähler potential and \( z, \bar{z} \) are the dual coordinates defined via the Legendre transform \( z + \bar{z} = K_u \). The Laplace equation implies that the determinant of the Kähler metric (4.4) is constant so we obtain a Ricci flat Kähler
manifold. The dual dilaton is consequently constant. The metric (4.4) describes a large class of 4-d non-compact Calabi-Yau manifolds, which are also hyper-Kähler. The associated \( \sigma \)-models have \( N=4 \) superconformal symmetry and \( \delta c = 0 \). The manifolds have generically asymptotically flat regions as well as curvature singularities.

Let us study now the special case of two isometries, i.e. \( K(u + \bar{u}, v + \bar{v}) \). If we parameterize, \( u = r_1 + i\theta, v = r_2 + i\phi \) then \( K \) is of the form \( K(r_1, r_2) = iT(r_1 + ir_2) - i\bar{T}(r_1 - ir_2) \). Introducing a new complex coordinate \( z = r_1 + ir_2 \), we can write the metric (4.4) in the following suggestive form

\[
d s^2 = \frac{ImT}{2} d\bar{z} d\bar{z} + \frac{2}{ImT} (d\theta + Td\phi)(d\bar{\theta} + \bar{T}d\bar{\phi}) \tag{4.5}
\]

where \( T(z) \) is an arbitrary meromorphic function. Now the interpretation of the metric (4.5) is straightforward: If we take \( \theta, \phi \) to be angular variables, then they parameterize a 2-d torus, with modulus \( T(z) \) which depends holomorphically on the rest of the coordinates and conformal factor proportional to \( 1/ImT \).

5 Conclusions

We have examined some four-dimensional superstring backgrounds with \( N=2 \) and \( N=4 \) superconformal symmetry (classical solutions to superstring theory). We show that there exists a plethora of such theories with non-trivial metric, dilaton and antisymmetric tensor field. It is a very interesting problem to find the exact \( N = 2 \) and \( N = 4 \) superconformal field theories which correspond to our general solutions as it was already done for particular backgrounds in [4]. Upon analytic continuation of the Euclidean solutions we expect to obtain many cosmological solutions to superstring theory whose spacetime properties deserve further study.

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