The uniqueness of tangent cones for Yang-Mills connections with isolated singularities

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Abstract

We proved a uniqueness theorem of tangent connections for a Yang-Mills connection with an isolated singularity with a quadratic growth of the curvature at the singularity. We also obtained control over the rate of the asymptotic convergence of the connection to the tangent connection. The rate of convergence can be strengthened when the tangent cone is integrable. There are parallel results for the cones at infinity of a Yang-Mills connection on an asymptotically flat manifold. We also gave an application of our methods to the Yang-Mills flow and proved that the Yang-Mills flow exists for all time and has asymptotic limit if the initial value is close to a smooth local minimizer of the Yang-Mills functional.

Introduction

The studies of singularities have been important in geometric analysis. Although in many cases our prime interests are in studying smooth solutions of certain geometric PDE’s, singular solutions arise naturally as the limits of sequences of smooth ones. The local behavior of a singularity is largely determined by the so called tangent cones. Tangent cones have been important in various kinds of regularity theories, and the use of them has been especially successful in the theory of minimal surfaces and harmonic maps (see for example the books [16], [18], [6] and the paper [14]).

The uniqueness of tangent cones at a singular point of a geometric object is equivalent to the asymptotic convergence of the object to a tangent cone. This case provides the simplest possible nontrivial singularity and the local singularity of the geometric object would be better understood if we also know the rate of its asymptotic convergence to the unique cone.

The important work by Leon Simon [15] gave a general asymptotic convergence theorem for the solutions to a class of nonlinear evolution equations which arise naturally in certain geometric variational problems, especially in problems associated with tangent cones. In the same paper, L. Simon applied the general theorem to show the uniqueness of tangent objects for minimal surfaces and energy minimizing maps with the assumption that there exists a tangent object with an isolated singularity. An earlier work of W. Allard and F. Almgren [1] proves the case of minimal surfaces with the integrability assumption on the tangent cone.

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This result and the methods of Leon Simon and its variants have been used and studied by many other people in different settings. The work Cheeger and Tian [3] proved a uniqueness theorem for cone structures at infinity of Ricci flat manifolds assuming that the cone at infinity being integrable. The work Morgan, Mrowka and Ruberman [2] studied the asymptotic limits of anti-self-dual connections with finite energy on a cylindrical 4-manifolds and gave applications to computations of Donaldson invariants.

In this paper, we proved the uniqueness of tangent cones of a Yang-Mills connection with an isolated singularity on manifolds of dimension greater than 4, under a quadratic growth assumption on the curvature of the connection (see Theorem 1). We estimated the rate of asymptotic convergence of the connection to its cone. On the other hand, we have a faster convergence rate would be fast if we assume the tangent cone is integrable. There are parallel results for the cones at infinity of a Yang-Mills connection on an asymptotically flat manifold. The author has constructed examples of Yang-Mills connections with given tangent cones and with different rates of convergence to the cones in [25].

The difficulties of our proof mainly lies in the degenerate elliptic nature of the Yang-Mills equation. Under a good transverse gauge, the Yang-Mills equation become elliptic. If we have long time existence of the gauge, then our result will follow from an application of L. Simon’s result. The long time existence of this gauge, however, depends on the asymptotic convergence which we want to show. Our strategy to solve this dilemma is to find a suitably defined gauge and prove the long time existence of the gauge and the solution in one shot. In the proof of long-time existence and convergence in Section 3, we modified L. Simon’s methods in [15] to our case. The method is to divide the existence interval of the solution into three parts on which the norms of the solution have different growth behaviors. Roughly speaking, the norm of the solution is exponentially decreasing on the first interval, it is changing slowly on the second interval, and it is exponentially increasing on the third interval. These behaviors are modelled on those of the solutions to the linearized equation. One can then control the norms of the solution on the three intervals using different techniques. The first interval is easy. For the second interval we used the variational inequality by L. Simon (see Section 3.3) and for the last interval we have to use the properties of the gauge we constructed in Section 2.3, especially the property that under this gauge the time derivative of the connection is uniformly small on the existence interval.

One might expect that there are similar results for tangent cones for more general (not necessarily isolated) singularities, say, singularities which are higher dimensional submanifolds or subvarieties. The methods of this paper, however, do not directly apply to the more general case.

An application of the gauge we used gives a result for the Yang-Mills flow. We showed that the Yang-Mills flow exists for all time and has an asymptotic limit if the initial value is close to a smooth local minimizer of the Yang-Mills functional (see Corollary 1).

This paper is organized into four parts. In Section 1 we review backgrounds, set up notations and state our main results. In Section 2, we describe our procedure of fixing a suitable transverse gauge and the related estimates for connections under this gauge. In Section 3 we prove an asymptotic convergence theorem of certain solutions of a class of evolution equations and this gives long time existence of the gauge constructed in Section 2 and convergence of the connection at once. We also showed how we control the rate of
the asymptotic convergence. Finally in Section 4, we proved the result stated above for the Yang-Mills flow.

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1 Background, Notations and Main Results

1.1 Background and main theorems

We shall give a very brief background here. One may find more detailed information about Yang-Mills connections in, for example, the excellent book \[4\] and some recent analytic results we used in \[20\]. We assume that $M$ is an $n$-dimensional manifold and $P$ is a principal bundle on $M$ with compact structure group $G$ which has Lie algebra $\mathfrak{g}$. Suppose $E$ is a vector bundle associated to $P$ with a linear representation $\rho : G \to \text{GL}(V)$, where $V$ is the fiber type of $E$. $\text{Aut}_P = P \times_{\text{Ad}} G$ is the principal bundle associated to the $\text{Ad}$ representation and we denote by $\mathfrak{g}_E$ the associated bundle to $\text{Aut}_P$ with the differential of $\rho$, $d\rho : \mathfrak{g} \to \text{End}(V)$. $\mathfrak{g}_E$ is a subbundle of $\text{End}(E)$. The connections considered in this paper will always be $G$-connections.

Let $A$ be a connection on $E$ which corresponds to a covariant derivative $\nabla_A : \Gamma(E) \to \Omega^1(E)$, which is $G$-equivariant. Under a trivialization $\phi : E|_U \to U \times V$ on an coordinate open subset $U \subset M$, $A$ has local expression $\nabla_A = d + A_\phi$, where $A_\phi \in \Omega^1(\mathfrak{g}_E|_U) \subset \Omega^1(\text{End}(E|_U))$. We often suppress the subscript $\phi$ and use $A$ for this local expression when the trivialization is clear in the context. The covariant differentiation $\nabla_A$ extends naturally to sections of various tensor bundles on $M$ with values in $E$ or $\text{End}(E)$. $d_A = \wedge \circ \nabla_A$ gives the coupled exterior differential on forms on $M$ with values in $E$ or $\text{End}(E)$. In particular, in local coordinates,

$$d_A(\alpha) = d\alpha + A \wedge \alpha,$$

$$d_A(\beta) = d\beta + [A, \beta],$$

for $\alpha \in \Omega^*(E)$ and $\beta \in \Omega^*(\text{End}(E))$.

For a connection $A$ on $E$, the operator $d_A \circ d_A : \Omega^*(E) \to \Omega^{*+2}(E)$ is given by the algebraic operator $F_A \wedge : \Omega^*(E) \to \Omega^{*+2}(E)$, where $F_A \in \Omega^2(\text{End}(E))$ is the curvature of $A$ and locally has the expression

$$F_A = dA + A \wedge A = dA + \frac{1}{2} [A, A].$$

The curvature $F_A$ satisfies the Bianchi identity

$$d_AF_A = 0.$$
Let \(\mathcal{A} = \{\text{all connections on } E\}\). The gauge group, \(\Gamma(\text{Aut } P)\), consists of bundle automorphisms of \(E\) that preserve the base. The gauge action of \(\Gamma(\text{Aut } P)\) on \(\mathcal{A}\) is given by

\[
\nabla_{g(A)}(v) = g \circ \nabla_A \circ g^{-1}(v),
\]

for \(g \in \Gamma(\text{Aut } P)\), \(A \in \mathcal{A}\) and \(v \in \Gamma(E)\). (Another convention in literature is to let \(\nabla_{g(A)} = g^{-1} \circ \nabla_A \circ g\).) Under local trivialization, if we write \(\nabla_A = d + A\), then \(\nabla_{g(A)} = d + g(A)\), then

\[
g(A) = gA g^{-1} - dg g^{-1}. \tag{1.5}\]

We have, as the curvature is a tensor,

\[
F_{g(A)} = gF_A g^{-1}. \tag{1.6}\]

Now we assume that \(M\) is oriented and is given a Riemannian structure \(g\) and assume that \(E\) is given a \(G\)-invariant metric \(h\). The Yang-Mills functional \(\text{YM} : \mathcal{A} \to \mathbb{R}\) is defined by

\[
\text{YM}(A) = \int_M |F_A|^2 dV_g.
\]

Here \(|\cdot|^2\) is given by the metrics \(g\) and \(h\).

We call \(A\) a Yang-Mills connection if and only if \(A\) is a critical point of \(\text{YM}\) on \(\mathcal{A}\). In other words, \(A\) is Yang-Mills if and only if for any continuously differentiable family of connections \(\{A_t\}_{-\varepsilon < t < \varepsilon}\),

\[
\left. \frac{d}{dt} \right|_{t=0} \text{YM}(A_t) = 0.
\]

A Yang-Mills connection \(A\) satisfies the following Yang-Mills equation, which is the Euler-Lagrange equation for \(\text{YM}\),

\[
d_A^* F_A = 0. \tag{1.7}\]

In (1.7), \(d_A^* : \Omega^*(\text{End}(E)) \to \Omega^{*+1}(\text{End}(E))\) is the formal adjoint of \(d_A\). The Bianchi identity (1.4) and Yang-Mills equation (1.7) together give a system of equations which is a nonlinear analogue of the equation for harmonic forms.

It follows by direct calculation that under local coordinates, if \(\alpha \in \Omega^p(\text{End}(E))\), then

\[
d_A^*(\alpha) = d^* \alpha + (-1)^{(p-1)n+1} \ast [A, \ast \alpha] = (-1)^{(p-1)n+1} \ast d_A \ast \alpha.
\]

By using variations generated by a vector field on \(M\), we have the following first variation formula for smooth Yang-Mills connections

\[
\int_M |F_A|^2 \text{div } X - 4 \sum_{1 \leq i < j \leq n} \langle F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) \rangle dV_g = 0. \tag{1.8}\]

This formula is true for any compactly supported \(C^1\) vector field \(X\) on \(M\).

Definition. A connection \(A\) (possibly with singularities) is stationary if the first variation
Proposition 1 If $E$ is a stationary connection. By using a cutoff of the radial vector field $X = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$, we have the following important monotonicity formula by Price [12] (see [20] for a variant version). Let $\text{injrad}(x)$ denote the injective radius of $x \in M$.

Proposition 1 If $A$ is a stationary connection on $E$, then for any $x \in M$, there exists positive constants $\Lambda = \Lambda(x)$ and $r_x < \text{injrad}(x)$ which only depend the supremum bound of curvature of $M$, such that if $0 < \sigma < \rho \leq r_x$, then

$$
\rho^{4-n} e^{\Lambda \rho} \int_{B_{\rho}(x)} |F_A|^2 \, dV_g - \sigma^{4-n} e^{\Lambda \sigma} \int_{B_{\sigma}(x)} |F_A|^2 \, dV_g
\geq 4 \int_{B_{\rho}(x) \setminus B_{\sigma}(x)} r^{4-n} \left| \frac{\partial}{\partial r} F_A \right|^2 \, dV_g
$$

(1.9)

Remark. We remark here that if $M \cong \mathbb{R}^n$ is the Euclidean space, then we can take $\Lambda = 0$ and $r_x = \infty$ in the above proposition, and furthermore, the equality in (1.9) holds. In this case, we have, by differentiating the equation (1.9)

$$
\rho^{5-n} \int_{\partial B_{\rho}(x)} |F_A|^2 \, d\sigma - (n - 4) \rho^{4-n} \int_{B_{\rho}(x)} |F_A|^2 = 4 \rho^{4-n} \int_{\partial B_{\rho}(x)} \left| \frac{\partial}{\partial r} F_A \right|^2 \geq 0.
$$

(1.10)

Using the compactness theorems for $L^p$ connections with $p > n$ in Uhlenbeck [24], it is easy to show the following proposition.

Proposition 2 Assume $\{A_i\}$ is a sequence of smooth Yang-Mills connections on a bundle $(E, h_i)$ over $(M, g_i)$, with $|F_{A_i}|$ locally uniformly bounded in $i$. Assume that $g_i$ and $h_i$ converges smoothly to metrics $g$ and $h$ on compact sets of $M$. Then there exist a subsequence $\{A_j\}$, gauge transformations $\sigma_j$ on $M$ and a smooth Yang-Mills connection $A$ on $E$ over $M$ such that on any compact set $K \subset M$, $\sigma_j(A_j)$ converges to $A$ in $C^\infty$ topology.

For more general compactness theorems with only uniform bounds on the $L^2$ norms of the curvatures, we need to use a priori estimates to bound the pointwise norm of curvature and the convergence will be possible only away from a blow-up set – where the curvature energy of the sequence concentrate. This blow-up and weak compactness is familiar in other settings, as in the theories of minimal surfaces and harmonic maps. We refer the reader to Nakajima [1] and Tian [24] for some general compactness theorems.

We now assume that $\dim M \geq 5$, $x_0 \in M$ and $A$ is a Yang-Mills connection on a bundle $E$ over $M \setminus \{x_0\}$, i.e., $A$ has an isolated singularity at $x_0$. By a standard cutoff argument, we may show that (1.8) is true for $A$ with any $C^1$ compactly supported vector field $X$ and $A$ is stationary on $M$. We also assume that the curvature of $A$ satisfies the following quadratic growth condition in a neighborhood $U$ of $x_0$:

$$
|F_A(x)| \leq Cr^{-2}, \quad \forall x \in U,
$$

(1.11)

where $r = \text{dist}(x, x_0)$ on $M$. We may identify $U$ and an open set, say, $B_1(0)$ in $T_{x_0}M \cong \mathbb{R}^n$, and denote the induced pullback metric on $B_1(0)$ also by $g$. For any $\lambda \in (0, 1)$, let
τ_λ : T_x M → T_x M be the scaling map given by τ_λ : v ↦ λv. We may also lift τ_λ to a map on the bundle E via a local trivialization. Define the scaling of the metrics g, h and the connection A by
\[ g_\lambda = \lambda^{-2} \tau_\lambda^* g, \quad h_\lambda = \tau_\lambda^* h, \quad A_\lambda = \tau_\lambda^* A \] (1.12)
Then for any 0 < λ < 1, A_λ is a stationary Yang-Mills connection with respect to g_λ with an isolated singularity at 0. Since A is defined on B_1(0) \ {0}, it follows that A_0 is defined on B_{1-\lambda}(0) \ {0}. The metrics g_i and h_i converges smoothly to the standard metrics g on \( \mathbb{R}^n \) and h on a trivial bundle coming from the metrics at 0. The compactness property Prop. 2 now implies that for any sequence \( \lambda_i \to 0 \), there is a subsequence that after gauge transformations, converges on compact sets of \( \mathbb{R}^n \setminus \{0\} \) to a smooth Yang-Mills connection A_0 on \( \mathbb{R}^n \setminus \{0\} \).

**Definition.** A_0 as above is called a tangent connection of the connection A at x_0.

As in the case of minimal surfaces and harmonic maps, we expect A_0 to be radially homogeneous, i.e., a cone. Indeed, we have the following property for a tangent connection at a point ([20, 5.3.1]).

**Lemma 1** With notations as above and let \( r = \text{dist}(x,0) \), we have
\[ \frac{\partial}{\partial r} |F_{A_0}(x)| = 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \] (1.13)

**Proof.** The stationarity of A_0 follows from the stationarity of \( \tau_\lambda^* A \) and the strong convergence on compact sets of \( \mathbb{R}^n \setminus \{0\} \), or follows directly since singular set of A_0 is isolated, with codimension at least 5. Since \( \mathbb{R}^n \) has the flat metric, by the remark following Prop. 2, we may take \( \Lambda = 0 \) in the monotonicity formula (1.9) and
\[ \rho^{4-n} \int_{B_{\rho}(0)} |F_{A_0}|^2 \, dx - \sigma^{4-n} \int_{B_{\sigma}(0)} |F_{A_0}|^2 \, dx \]
\[ = 4 \int_{B_{\rho}(0) \setminus B_{\sigma}(0)} r^{4-n} \left| \frac{\partial}{\partial r} |F_{A_0}| \right|^2 \, dx \] (1.14)
However, for any \( \rho > 0 \), we have
\[ \rho^{4-n} \int_{B_{\rho}(0)} |F_{A_0}|^2 \, dx = \lim_{t \to \infty} (\lambda t \rho)^{4-n} \int_{B_{\lambda t \rho}(x)} |F_A|^2 \, dV_g \]
\[ = \lim_{r \to 0} r^{4-n} \int_{B_r(x)} |F_A|^2 \, dV_g \geq 0 \]
Therefore, both sides of (1.14) are zero and
\[ \int_{B_{\rho}(0) \setminus B_{\sigma}(0)} r^{4-n} \left| \frac{\partial}{\partial r} |F_{A_0}| \right|^2 \, dx = 0 \] (1.15)
This implies (1.13). q.e.d.
With the conclusion of the lemma, it is a standard fact that after a smooth gauge transformation on $\mathbb{R}^n \setminus \{0\}$, $A_0$ can be made into radially homogeneous, i.e., $A_0 = p^*(A'_0)$ for some smooth Yang-Mills connection $A'_0$ on the trivial bundle over $S^{n-1}$, where $p : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ is the natural projection. We shall assume later that all tangent connections are in this radially homogeneous form. With some abuse of terminology, we shall call this $A'_0$ a tangent connection or a tangent cone of $A$ at $x_0$.

**Remark.** Our definition of tangent connections are of course not the most general. In fact, using the general compactness theorems in [11] and [20] which only assumes a uniform bound on the $L^2$ norms of the curvature and the monotonicity formula, a tangent connection may be defined for a stationary Yang-Mills connection with a singularity set $S$ at a singular point $x_0 \in S$. We need some additional assumptions on $S$, though, for example, we may have to assume that $C(S) = \{x = x_0 + t(y - x_0) : y \in S, t \in (0, \infty)\}$ has zero $(n-4)$-dimensional Hausdorff measure, or assume that $S$ is stratified by manifolds with dimension not greater than $n-4$. These assumptions on $S$ are necessary because we don’t have the best possible removable singularity theorems for stationary Yang-Mills connections (for a good one, see [21]). In the case of Hermitian-Yang-Mills connections, we only need to assume the $(n-4)$-dimensional Hausdorff measure of $S$ is finite (see [22]).

Now we are ready to state our main theorems. We assume that $(M, g)$ is of dimension $n \geq 5$, $x_0 \in M$ and $(E, h)$ is a bundle over $M \setminus \{x_0\}$.

**Theorem 1** Let $A$ be a smooth Yang-Mills connection on the bundle $E$ over $M \setminus \{x_0\}$. Assume that there exists $C > 0$ and a neighborhood $U$ of $x$ such that

$$|F_A(x)| \leq Cr^{-2}, \forall x \in U,$$

where $r = \text{dist}(x, x_0)$. Then the tangent connection of $A$ is unique up to gauge transformations. In other words, there exists a smooth Yang-Mills connection on $E|_{S^{n-1}(x_0)}$ over the unit sphere around $x_0$ and a gauge transformation $\tau$ on $E|_M \setminus \{x_0\}$ such that

$$A(r) \xrightarrow{C^\infty} A_0 \quad \text{as } r \rightarrow 0,$$

where $A(r) = \tau^*_r(A)|_{S^{n-1}(x_0)}$ is the rescaled connection under the scaling map $\tau_r : B_{r^{-1}}(x_0) \rightarrow B_1(x_0)$ as before. Furthermore, there exists constants $C_k > 0$ and $\alpha > 0$ depending on $A$, such that

$$|A(r) - A_0|_{C^k(S^{n-1})} \leq C_k |\log r|^{-\alpha}.$$

We call a smooth Yang-Mills connection $A_0$ on $S^{n-1}$ integrable if for every solution $a \in \Omega^1(g_E)$ of

$$La = \Delta_{A_0} a + (-1)^n * [a, *F_{A_0}] = 0,$$

where $L$ is, as in Section 3.1, the linearization of $d^*_a d_a F_{A_0+a}$ at 0, there exists a path of Yang-Mills connections $A(t)$, $t \in (-\varepsilon, \varepsilon)$ with $A(0) = A_0$ such that

$$\frac{\partial}{\partial t} \bigg|_{t=0} A(t) = a.$$  \hspace{1cm} (1.17)

This has the geometric meaning that $A_0$ has an integrable neighborhood in the moduli space of smooth Yang-Mills connections on $S^{n-1}$ with tangent space at $A_0$ being given by the Jacobi fields at $A_0$, i.e., the solutions to (1.16).
Theorem 2 If in addition to the hypotheses of theorem 1, we assume that the tangent connection \( A_0 \) of \( A \) is integrable, then there exists \( \alpha > 0 \) and \( C_k > 0 \) such that we have the following better control of convergence,

\[
|A(r) - A_0|_{C^k(S^{n-1})} \leq C_k r^{-\alpha}.
\]

Remark. 1) We remark that in the proof of Theorem 1, the monotonicity formula is only used to prove the desired rate of convergence. In [25], the author was able to construct examples defined on compact manifolds. It would be interesting to construct examples defined on compact manifolds. Theorem 1 and Theorem 2 respectively. These examples are defined on a ball with certain boundary values and the construction is achieved by a deformation-perturbation method.

2) There are parallel results of Theorem 1 and 2 for the tangent cones at infinity of a Yang-Mills connection on bundles over \( \mathbb{R}^2 \). There are parallel results of Theorem 1 and 2 for the tangent cones at infinity of a Yang-Mills connection on bundles over \( \mathbb{R}^n \) (\( n \geq 5 \)), or more generally on an asymptotically flat manifold. We need to assume quadratic decay of the curvature there. The proofs will be the same. We leave the formulation to the reader.

1.2 Cylindrical coordinates and notations

Since our problem in Theorem 1 is of local nature, we may assume that \( E \) is a bundle over the disk \( B_2(0) \setminus \{0\} \). Consider the cylindrical coordinates, \( \phi : B_2(0) \to S^{n-1} \times [t_0, \infty), t_0 = -\log r_0 \) defined by

\[
\phi(x) = (\omega(x), t(x)) = \left( \frac{x}{|x|}, -\log |x| \right).
\]

We may assume that the disk \( B_2(0) \) has the standard metric and the bundle \( E \) has the product metric, because nonstandard metrics will only give a perturbation which is exponentially decaying in \( t \) as \( t \to \infty \) in the cylindrical coordinates and will not affect our proof later. We identify \( E \) with \( (\phi^*)^{-1}(E) \).

Assume that \( \tilde{A} \) is a Yang-Mills connection on the bundle over \( B_2(0) \setminus \{0\} \) and assume

\[
(\phi^*)^{-1}(\tilde{A}) = A(t) + \beta(t) dt,
\]

where \( A \in \Omega^1(\text{End } E), \beta \in \Omega^0(\text{End } E) \).

Lemma 2 The Yang-Mills equation \( d_A^* F_A = 0 \) is equivalent to the following system of equations,

\[
\tilde{A} - (n - 4) \tilde{A} - d_A^* F_A - d_A \beta + (n - 4) d_A \beta + (-1)^{n+1} \ast [\beta, \ast d_A \beta] = 0 \quad (1.18)
\]

\[
d_A^*(\tilde{A} - d_A \beta) = 0 \quad (1.19)
\]

Proof. Recall that \( d_A^* (\xi) = (-1)^{n(p-1)+1} \ast d \ast \xi + \ast [A, \ast \xi] \), for \( \xi \in \Omega^p(\text{End } E) \) over an \( n \)-dimensional manifold. In considering \( (\phi^*)^{-1} \circ d_A^* \), we know that \( d \) commutes with \( (\phi^*)^{-1} \), therefore we need to consider the behavior of \( \ast \) under \( (\phi^*)^{-1} \). We denote by \( g_0 \) the standard metric in \( \mathbb{R}^n \), by \( g \) the standard product metric on \( S^{n-1} \times \mathbb{R} \), and by \( \tilde{g} = (\phi^*)^{-1}(g_0) \) the pushforward metric of \( g_0 \). Denote by \( \ast_{n_0}, \ast_n, \tilde{\ast}_n \) and \( \ast \) the Hodge operator associated to respectively \( g_0, g, \tilde{g} \) and the standard metric \( S^{n-1} \). Let \( d_{n_0}, d_n \) and \( d \) be the exterior
differential on respectively $\mathbb{R}^n$, $S^{n-1} \times \mathbb{R}$ and $S^{n-1}$. If $\{d\omega, \ldots, d\omega, dt\}$ is a local orthonormal basis of $T^1(S^{n-1} \times \mathbb{R})$ with respect to $g$, by definition of $\phi$, $\{e^{-t}d\omega, e^{-t}dt\}$ constitutes an orthonormal basis for $\tilde{g}$. Hence

$$\tilde{g} = e^{2t}g, \quad \langle \alpha, \beta \rangle_{\tilde{g}} = e^{2t}d\beta \langle \alpha, \beta \rangle_g, \quad dV_{\tilde{g}} = e^{-nt}dV_g.$$  \hspace{1cm} (1.20)

Therefore

$$\alpha \wedge \tilde{\pi}_n \beta = \langle \alpha, \beta \rangle_{\tilde{g}}dV_{\tilde{g}} = e^{2t}d\beta \langle \alpha, \beta \rangle_g e^{-nt}dV_g = e^{-(n-2)td\beta} \alpha \wedge \ast_n \beta.$$

That is

$$\tilde{\pi}_n \beta = e^{-(n-2)td\beta} \ast_n \beta.$$

We can now carry out the calculation, first

$$d^*_AF_A = (-1)^{n+1}(\ast_n d_n \ast_n F_A + \ast_n [\tilde{A}, \ast_n F_A])$$

and we have

$$(\phi^*)^{-1}(F_A) = F_A - (\dot{A} - dA)dt.$$

For simplicity of notation, let $\eta = \dot{A} - dA$.

$$(\phi^*)^{-1}(\ast_n d_n \ast_n F_A) = \tilde{\pi}_n d_n \tilde{\pi}_n (F_A - \eta dt)$$

$$= e^{(n-2)t} \ast_n d_n e^{-(n-4)t} \ast_n (F_A - \eta dt)$$

$$= (-1)^n e^{2t}\{(\dot{\eta} - (n-4)\eta - d^*F_A + (\ast d \ast \eta) \wedge dt\}$$

$$(\phi^*)^{-1}(\ast_n [\dot{A}, \ast_n F_A]) = \tilde{\pi}_n [A + \beta dt, \tilde{\pi}_n (F_A - \eta dt)]$$

$$= e^{2t} \ast_n [A + \beta dt, \ast_n (F_A - \eta dt)]$$

$$= e^{2t}\{- \ast [A, \ast F_A] + \ast [\beta, \ast \eta] + (-1)^{n-1} \ast [A, \ast \eta]dt\}$$

Combining the above two equalities, we obtain

$$(\phi^*)^{-1}(d^*_AF_A) = -e^{2t}\{- (\dot{\eta} - (n-4)\eta - d^*_A F_A + (-1)^n \ast [\beta, \ast \eta]) + d_A^* \eta \wedge dt\}$$

$\dot{A}$ is Yang-Mills if and only if the $dt$ part and the part without $dt$ above are zero. This together with the observation that $[\dot{A}, \beta] = (-1)^n \ast [\beta, \ast \dot{A}]$ give the desired system (1.18) and (1.19). q.e.d.

We shall use the above cylindrical coordinates in our proof and consider solutions to the system (1.18) and (1.19). Our arguments in this and the next chapter will work for more general settings, so we assume in the following of this paper that $(E, h)$ is a Euclidean vector bundle on a compact $(n-1)$-dimensional Riemannian manifold $(M, g)$ $(n \geq 5)$ with a compact structure group $G$, and $P$ is the associated principal bundle. We let the bundle $E \times [t_0, \infty)$ and the manifold $M \times [t_0, \infty)$ have the product metrics.

We shall use letters with tildes, $\tilde{A}, \tilde{A}_t$, etc to represent connections on the bundle $E \times I$ on $M \times I$, where $I$ is an interval of possibly infinite length. And we use $A, A_0, B$, etc to represent connections on bundle $E$ on $M$.

For simplicity, we shall assume that $E$ is trivial. For nontrivial $E$, our arguments still work if we fix a smooth connection $A_0$ on $E$ and replace $A$ by $A - A_0$, $d$ and $d^*$ by $d_{A_0}$ and $d_{A_0}^*$ etc.
Definition. A connection $\tilde{A}$ on the bundle $E \times [t_0, \infty)$ a Yang-Mills connection if it satisfies the system (1.18) and (1.19).

Fix $\mu \in (0,1)$ and let $k$ be a nonnegative integer. We shall use in the following various Hölder norms and spaces. The norms may be defined by using partition of unity arguments, for example. The reason we use Hölder norms rather than Sobolev norms is that the restriction to submanifolds of $C^{k,\mu}$ functions are $C^{k,\mu}$ functions, which makes many statements simpler. However, it should be observed that the Sobolev norms also work for the proof with appropriate attention to the corresponding trace properties of Sobolev functions.

Let $A(t)$ be a section of a bundle over $M \times I$, with $I$ an interval. We shall use the following abbreviation,

$$|A(t)|_{C^{k,\mu}(I)} = |A(t)|_{C^{k,\mu}(M \times I)}.$$

Note that the $| \cdot |_{C^{k,\mu}}$ norm doesn’t count in the time derivatives. We define

$$|A(t)|_{C^{(k,1),\mu}(I)} = \sum_{0 \leq j \leq k, 0 \leq i + j \leq k+1} \| \nabla_M \frac{\partial^j}{\partial t^j} A(t) \|_{C^0} + \sum_{0 \leq j \leq k, 0 \leq i + j = k+1} \| \nabla_M \frac{\partial^j}{\partial t^j} A(t) \|_{C^{0,\mu}}$$

The spaces $C^{(k,1),\mu}(I)$ are defined by those bundle sections such that the norms in (1.21) are finite. The difference between $C^{(k,1),\mu}(I)$ and $C^{k+1,\mu}(I)$ norms lies in that in the former we only take up to $k$-th derivatives for the variable $t$, thus distinguishing the ‘time’ dimension $t$ from the ‘spatial’ dimensions on $M$. We define the following space of connections,

$$S^{k,\mu}(I) = \{ A + \beta dt : A(t) \in C^{k,\mu}(I), \beta(t) \in C^{(k-1,1),\mu}(I) \}.$$ (1.22)

and let the norm of $S^{k,\mu}(I)$ be

$$|A + \beta dt|_{S^{k,\mu}(I)} = |A|_{C^{k,\mu}(I)} + |\beta|_{C^{(k-1,1),\mu}(I)}$$ (1.23)

The reason for this definition will become clear in Section 2.2.

The natural space of gauges for connections in $C^{k,\mu}(M)$ is the space $C^{k+1,\mu}(M)$ of gauges. The natural space of gauges for connections in $S^{k,\mu}(I)$ is given by gauges in $C^{(k,1),\mu}(I)$, because we see from the formula

$$g(A + \beta dt) = gAg^{-1} - dg g^{-1} + (g\beta g^{-1} - \frac{\partial g}{\partial t} g^{-1})dt$$

(1.24)

that $C^{(k,1),\mu}(I)$ gauges is the minimal space of gauges that act on $S^{k,\mu}(I)$ continuously.

For notations of $L^2$ and Sobolev norms, by fixing smooth covariant derivatives on $E$ and
on $E \times I$, we define

$$\|A\| = \left( \int_M |A|^2 d\sigma \right)^{\frac{1}{2}}, \quad \|A\|_{H^1} = \left( \sum_{0 \leq i \leq l} \int_M |\nabla^i A|^2 d\sigma \right)^{\frac{1}{2}},$$

$$\|A(t)\|_I = \left( \int_{M \times I} |A(t)|^2 dt d\sigma \right)^{\frac{1}{2}}, \quad \|A(t)\|_{H^1(I)} = \left( \sum_{0 \leq i+j \leq l} \int_{M \times I} |\nabla^i \partial^j_t A|^2 dt d\sigma \right)^{\frac{1}{2}}.$$

Sometimes we abuse notation and use a connection $A$ on $M$ to represent the time independent connection $p^* (A)$ on $M \times I$; the meaning should be clear from the context.

Gauge transformations are usually assumed to be in the natural space of gauges for the connections they act on; hence by $\Gamma(\text{Aut } P)$ we often mean $C^{k+1,\mu}(M, \text{Aut } P)$, etc. If $A$ is a connection on $M$, we let $\text{Stab}(A) = \{ \sigma \in \Gamma(\text{Aut } P) : \sigma(A) = A \}$ be the stabilizer of $A$ in the gauge group.

Because the cylindrical coordinates $\phi$ is a conformal map, the quadratic growth hypothesis on the curvature in Theorem 1 translates to the uniform bound condition on curvature

$$|F_{\tilde{A}}| = (|F_A|^2 + |\dot{A} - d_A \beta|^2)^{1/2} \leq C,$$

for a connection $\tilde{A}$ on $E \times [0, \infty)$. For a Yang-Mills connection $\tilde{A}$ with (1.24), Prop 2 implies the following compactness property,

**Lemma 3** Given $L > 0$. For any sequence of positive numbers $\{R_i\}$ going to infinity, there exists a subsequence $i'$, gauge transformations $\sigma_{i'} \in \Gamma(\text{Aut } P \times [R_{i'}, R_{i'} + L])$ and a Yang-Mills connection $A$ on $M$ of $\tilde{A}$, such that

$$\lim_{i' \to \infty} |\sigma_{i'}(\tilde{A}) - A|_{C^{k,\mu}([R_{i'}, R_{i'} + L])} = 0$$

## 2 Constructions of Gauges

In this section we constructed the ‘standard form’ gauges for Yang-Mills connections with an isolated singularity in cylindrical coordinates with certain estimates on norms of the connection. These estimates will be important for us to prove the long-time existence of this gauge and convergence of the connection in the next section.

### 2.1 Some bounds of gauges

We give two well-known bounds about norms of gauges and connections in the following lemma.

**Lemma 4** Let $\tilde{A}_0$ and $\tilde{A}_1$ be connections in $C^{k,\mu}(I)$. $k \geq 0$, $0 \leq \mu \leq 1$. Assume that $\sigma$ is a gauge on the bundle $E \times I$, we have

a) If $\max\{|\tilde{A}_o|_{C^{k,\mu}(I)}, |\sigma(\tilde{A}_0)|_{C^{k,\mu}(I)}\} \leq C$, then there exists $C_1 = C_1(k, C)$ such that

$$|\sigma|_{C^{k+1,\mu}(I)} \leq C_1$$

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b) There exists \( C = C(k) > 0 \) independent of \( A_i \) and \( \sigma \) such that

\[
|\sigma(\tilde{A}_1) - \sigma(\tilde{A}_0)|_{C^{k,\mu}(I)} \leq C(|\sigma|_{C^{k,\mu}(I)})^2 |\tilde{A}_0 - \tilde{A}_1|_{C^{k,\mu}(I)}
\]

Proof. a) We have

\[
d_{\tilde{A}_0}\sigma = (\sigma(\tilde{A}_0) - \tilde{A}_0) \cdot \sigma
\]

We also observe the fact that \(|\sigma|_{C^0(I)} \leq C\) because the structure group \( G \) of \( \text{Aut} P \) is compact. Therefore we can apply a bootstrapping procedure to (2.1) and prove the required estimates. This estimate is standard, see for example \[4, 2.3.7\].

b) We have

\[
\sigma(\tilde{A}_1) - \sigma(\tilde{A}_0) = \sigma \cdot (\tilde{A}_1 - \tilde{A}_0) \cdot \sigma^{-1}
\]

Notice that \( d(\sigma^{-1}) = -\sigma^{-1} d\sigma \cdot \sigma^{-1} \) and therefore,

\[
|\sigma^{-1}|_{C^{k,\mu}(I)} \leq C(k)|\sigma|_{C^{k,\mu}(I)}
\]

Combining (2.2) and (2.3), we have the required bounds. q.e.d.

Remark. The above lemma also holds if we look at other suitable spaces of connections and natural spaces of gauges acting on them. For example, the conclusion of the lemma hold if we replace the \( C^{k,\mu}(I) \) norm for connections and the \( C^{k+1,\mu}(I) \) norms for gauges by the \( S_k,\mu(I) \) norm for connections and the \( C^{(k,1),\mu}(I) \) and \( C^{(k,0),\mu}(I) \) norms for gauges respectively. Similarly, the lemma holds for the \( C^{k,\mu}(M) \) norm for connections and the \( C^{k+1,\mu}(M) \) and \( C^{k,\mu}(M) \) norm of gauges. We shall frequently use these bounds in the following, often implicitly.

Now we assume that \( \tilde{A} \) is a smooth Yang-Mills connection on \( E \times [t_0, \infty) \) with \( \sup |F_{\tilde{A}}| \leq C \). In the following lemma we give a bound on the set of tangent connections of \( \tilde{A} \). Here we regard tangent connections of \( \tilde{A} \) as Yang-Mills connections on \( E \) over \( M \) by time independence.

**Lemma 5** Let \( \mathcal{C} = \{ \{A, \text{set of tangent connections of } \tilde{A} \} \} \), then there exists \( c_1 = c_1(k, \tilde{A}) > 0 \) such that for any \( A \in C \), there exists \( g \in C^{k+1,\mu}(\text{Aut} P) \) with \( |g(A)|_{C^{k,\mu}} \leq c_1 \).

Proof. If \( A_0 \in \Gamma(M, \Lambda^1(\mathfrak{g}_E)) \) is a tangent Yang-Mills connection of \( \tilde{A} \), then the bound on the curvature of \( \tilde{A} \) and the smooth convergence in Prop 4 imply that \( \sup_{x \in M} |F_{A_0}(x)| \leq C \). Let \( A_i \) be a sequence of tangent cones for \( \tilde{A} \), then we may apply Prop 2 to obtain a subsequence \( A_{i'} \) and gauge transformations \( g_{i'} \), such that \( g_{i'}(A_{i'}) \) converges smoothly to a tangent cone \( A_0 \) of \( \tilde{A} \). This compactness property implies the bound in the lemma, for any \( k \geq 0 \). q.e.d.

The following lemma will be useful in Section 2.3.

**Lemma 6** Given \( L > 0, \varepsilon > 0 \), there exists \( R_0 = R_0(\varepsilon, L) > 0 \) such that if \( R \geq R_0 \), then there exists a gauge transformation \( g \in \Gamma(\text{Aut} P \times [R, R + L]) \) and a tangent connection \( A \) of \( \tilde{A} \), such that

\[
|g(\tilde{A}) - A|_{S^{k,\mu}([R, R+L])} \leq \varepsilon,
\]

\[
|A|_{C^{k,\mu}} \leq c_1,
\]

where \( c_1 = c_1(k, \tilde{A}) \) is the constant given in Lemma 4.
Proof. Assume that the lemma is not true, then there exist $\varepsilon_j \to 0$ and $R_j \to \infty$ such that there does not exist tangent Yang-Mills connection $A$ such that (2.4) and (2.5) are true for $R$ replaced by $R_j$. By Lemma 3, there exist a subsequence $\{j^\prime\}$ and $g_{j^\prime} \in \Gamma(\text{Aut } P \times [R_{j^\prime}, R_{j^\prime} + L])$ such that $g_{j^\prime}(\tilde{A})|_{[R_{j^\prime}, R_{j^\prime} + L]} \overset{\text{S}k,\mu}{\to} A$, a tangent connection of $\tilde{A}$. By virtue of Lemma 3, up to a further $C^{k+1,\mu}$ gauge transformation on $M$, we may assume $|A|_{C^k,\mu} \leq c_1$. Now taking $j^\prime$ large will give us a contradiction. q.e.d.

2.2 Connections in standard form

For Yang-Mills connections, there is a standard way of fixing gauges, i.e. the Coulomb gauge (also called the Hodge gauge). We say that $B$ is in the Coulomb gauge relative to $A$ if

$$d_A^\ast (B - A) = 0$$

(see [4, 2.3.1] or [5] for example). The above Coulomb gauge equation and the Yang-Mills equation (2.7) form an elliptic system. In the following, we find a suitable gauge which gives rise to an elliptic system for a connection $\tilde{A}$ on $E \times I$, $I$ being an interval of possibly infinite length. Our choices of gauges are based on those in Morgan, Mrowka and Ruberman [9, 2.4.3]. The norms in the following proposition are natural to our setting.

Proposition 3 Assume that $A_0$ is a smooth connection on $E$ and $\tilde{A} = A(t) + \beta(t) dt$ is a smooth connection on $E \times I$. There exist $\varepsilon_1 = \varepsilon_1(A_0) > 0$ and $C = C(A_0) > 0$ such that if

$$|\tilde{A} - A_0|_{\text{S}^k,\mu(I)} = |A(t) - A_0|_{C^k,\mu(I)} + |\beta(t)|_{C^k,\mu(I)} \leq \varepsilon_1,$$

then there exists a gauge transformation $g \in C^{(k,1),\mu}(I)$, such that $g(\tilde{A}) = A_1(t) + \beta_1(t) dt$ satisfies

$$d_{A_0}^\ast (A_1(t) - A_0) = 0,$$

$$\beta_1(t) \in \text{Ker}(d_{A_0}),$$

$$|g(\tilde{A}) - A_0|_{\text{S}^k,\mu(I)} = |A_1(t) - A_0|_{C^k,\mu(I)} + |\beta_1(t)|_{C^k,\mu(I)} \leq C(1 + |I|)^2 |\tilde{A} - A_0|_{\text{S}^k,\mu(I)},$$

$$|g - Id|_{C^{(k,1),\mu}(I)} \leq C(1 + |I|) |\tilde{A} - A_0|_{\text{S}^k,\mu(I)},$$

where $\text{Ker}(d_{A_0}) = \text{Ker}(d_{A_0} : \Gamma(g_E) \to \Omega^1(g_E)) \subset \Gamma(g_E)$. Furthermore, $g$ is unique up to the pullback of an element of $\text{Stab}(A_0)$, i.e., up to the composite with $p^\ast(\tau) \in \Gamma(\text{Aut } P \times I)$, where $p : \text{Aut } P \times I \to \text{Aut } P$ is the projection and $\tau \in \text{Stab}(A_0) \subset \Gamma(\text{Aut } P)$.

Following the terminology in [4], we make the following definition

Definition. We call the choice of gauge $g(\tilde{A}) = A_1(t) + \beta_1(t) dt$ satisfying (2.7) and (2.8) in the above lemma a standard form of $\tilde{A}$ around $A_0$.

Notice that the condition $d_{A_0}^\ast (A(t) - A_0) = 0$ means $A(t)$ is in Coulomb gauge relative to $A_0$. In the lemma, if $|I| = \infty$, then (2.9) is vacuous.
Proof.  We shall find the desired gauge in two steps.

Step 1. We find a gauge under which the Coulomb gauge condition (2.7) and the bounds (2.9) and (2.10) are satisfied. Consider the gauge action map at \( A_0 \),

\[
\Phi : C^{k+1,\mu}(A) \times C^{k,\mu}(A,\lambda(E)) \to C^{k,\mu}(A,\lambda(E))
\]

\[
(g, a) \mapsto -d_{A_0}g^{-1} + g a^{-1} = g(A_0 + a) - A_0
\]

The differential of this map at \( (0,0) \) is

\[
D\Phi : C^{k+1,\mu}(\lambda(E)) \times C^{k,\mu}(\lambda(E)) \to C^{k,\mu}(\lambda(E))
\]

\[
(h, b) \mapsto -d_{A_0}h + b
\]

We can write down explicitly a right inverse of \( D\Phi \)

\[
(D\Phi)^{-1} : C^{k,\mu}(\lambda(E)) \to (C^{k+1,\mu}(\lambda(E)) \cap \text{Ker}(d_{A_0})) \times (C^{k,\mu}(\lambda(E)) \cap \text{Ker}(d_{A_0}^*)
\]

by \((D\phi)^{-1}(a) = (h, b)\), where

\[
h = (d_{A_0}^*d_{A_0})^{-1}(d_{A_0}^*a), \quad b = a + d_{A_0}h.
\]

Here we used the fact that

\[
d_{A_0}^*d_{A_0} : C^{k+1,\mu}(\lambda(E)) \cap \text{Ker}(d_{A_0})^\perp \to C^{k-1,\mu}(\lambda(E)) \cap \text{Im}(d_{A_0}^*)
\]

(2.11)

is invertible and has a continuous inverse. Using the implicit function theorem, we see that a right inverse \( \Psi \) of \( \Phi \) can be defined in a neighborhood of \( (0,0) \). It is easy to see that map

\[
\Psi : C^{k,\mu}(\lambda(E)) \supset U \to C^{k+1,\mu}(A) \times C^{k,\mu}(\lambda(E)) \cap \text{Ker}(d_{A_0}^*)
\]

is of class \( C^k \) (note that the space of connections is affine here). If \( \Psi(a) = (g, b) \), then \( A_0 + a = g(A_0 + b) \) and \( b \in \text{Ker}(d_{A_0}^*) \). This \( \Psi \) gives the well-known local slice structure for the configuration space of connections under gauge actions.

Now we assume that \( \tilde{A} = A(t) + \beta(t)dt = A_0 + a(t) + \beta(t)dt \) satisfies (2.10). In particular, \( |a(t)|_{C^{k,\mu}} \leq \varepsilon_1 \), \( \forall t \in I \). If \( \varepsilon_1 \) is sufficiently small, for each fix \( t \), we may define gauge transformation \( g(t) \) by requiring that \( g(t)(A_0 + a(t)) = A_1(t) = A_0 + a_1(t) \) with \( a_1(t) \in \text{Ker}(d_{A_0}^*) \), i.e., by letting

\[
\Psi(a(t)) = (g^{-1}(t), a_1(t)).
\]

Although \( g(t) \) is obtained separately for each \( t \), we may view \( g \) as a gauge transformation on \( E \times I \), and we have

\[
g(\tilde{A}) = A_0 + a_1(t) + (g(t)\beta(t)g(t)^{-1} - \frac{\partial}{\partial t}g(t)g^{-1})dt
\]

Now that (2.7) is satisfied, we need to show that \( g \in C^{(k,1),\mu}(I) \) and the bounds (2.9) and (2.10) hold. We observe first that by the continuity of \( \Psi \) on Hölder spaces,

\[
\sup_{t \in I} |g(t) - \text{Id}|_{C^{k+1,\mu}} \leq C \sup_{t \in I}|a(t)|_{C^{k,\mu}} \leq C |\tilde{A} - A_0|_{C^{k,\mu}(I)}.
\]

(2.12)
By differentiating the identity \( \Psi(a(t)) = (g^{-1}(t), a_1(t)) \), we have
\[
-g^{-1} \frac{\partial}{\partial t} g(t) g^{-1} = D_1 \Psi(a(t)) \left( \frac{\partial}{\partial t} a(t) \right)
\] (2.13)
Since \( \frac{\partial}{\partial t} a(t) \in C^{k-1,\mu}(\Lambda^1(\mathfrak{g}_E)) \) and
\[
D_1 \Psi : C^{k-1,\mu}(\Lambda^1(\mathfrak{g}_E)) \to C^{k,\mu}(\text{Aut } P)
\]
is a continuous map, we have
\[
\sup_{t \in I} |\frac{\partial}{\partial t} g|_{C^{k,\mu}} \leq C \sup_{t \in I} |\frac{\partial}{\partial t} a(t)|_{C^{k-1,\mu}} \leq C|\tilde{A} - A_0|_{\mathcal{S}^{k,\mu}(I)}
\] (2.14)
By taking further derivatives of (2.13) and using (2.6), we have
\[
\sup_{t \in I} \frac{\partial^j}{\partial t^j} g|_{C^{k+1-\mu}} \leq C|\tilde{A} - A_0|_{\mathcal{S}^{k,\mu}(I)}, \quad \text{for } 1 \leq j \leq k.
\] (2.15)
(2.14) and (2.13) now imply (2.9) and (2.10). This completes the first step.

Step 2. By the first step, we may assume \( \tilde{A} = A(t) + \beta(t) dt = A_0 + a(t) + \beta(t) dt \), \( a(t) \in \text{Ker}(d_{A_0}^* \beta) \) and (2.6) holds. We decompose \( \beta(t) \) as follows,
\[
\beta(t) = \beta_0(t) + \beta_1(t), \quad \beta_0(t) \in \text{Ker}(d_{A_0}), \quad \beta_1(t) \in \text{Ker}(d_{A_0})^\perp.
\]
Since \( \beta(t) \in C^{(k-1,1),\mu} \), by (2.11), we have \( \beta_1 = (d_{A_0}^* d_{A_0})^{-1}(d_{A_0} \beta) \in C^{(k-1,1),\mu}(I) \) and \( \beta_0 = \beta - \beta_1 \in C^{(k-1,1),\mu}(I) \) and
\[
|\beta_1(t)|_{C^{(k-1,1),\mu}(I)} + |\beta_0(t)|_{C^{(k-1,1),\mu}(I)} \leq C|\tilde{A} - A_0|_{\mathcal{S}^{k,\mu}(I)}.
\] (2.16)
By standard ODE theory, the following linear ordinary differential equation
\[
\begin{cases}
\frac{\partial}{\partial t} g = g \beta_0(t), & \forall t \in I \\
g(0) = \text{Id}.
\end{cases}
\] (2.17)
has a solution \( g \in C^{k,\mu}(\text{Aut } P \times I) \) (note \( g(t) \) have one more derivative in \( t \) than \( \beta_0(t) \)). Since \( \beta_0 \in \text{Ker}(d_{A_0}^*) = \text{the Lie algebra of Stab}(A_0) \), it follows that \( g(t) \in \text{Stab}(A_0), \forall t \in I \).
\[
\text{i.e. } g(t)(A_0) = A_0, \text{ or equivalently},
\]
\[
d_{A_0} g(t) = 0.
\] (2.18)
Applying \( g = g(t) \) to \( \tilde{A} \), we have by (2.17),
\[
g(\tilde{A}) = g(t)(A(t)) + (g\beta(t) g^{-1} - \frac{\partial}{\partial t} g^{-1}) dt = A_0 + a_1(t) + g(t)(\beta_1(t)) dt,
\]
where \( a_1(t) \in \text{Ker}(d_{A_0}^*) \) and \( g(t)(\beta_1(t)) = g(t)\beta_1(t) g(t)^{-1} \). The following lemma implies that \( g(\beta_1(t)) \in \text{Ker}(d_{A_0})^\perp \), hence \( g(\tilde{A}) \) is in the standard form around \( A_0 \).
Lemma 7 Assume that \(A_0\) is a connection on \(E\), \(g \in \text{Stab}(A_0)\) and \(A \in A_0 + \text{Ker}(d_{A_0}^*)\), then
\[
g(A) \in A_0 + \text{Ker}(d_{A_0}^*).\]

If \(\beta \in \text{Ker}(d_{A_0}^*) \subset \Gamma(g_E)\), then
\[
g(\beta) = g\beta g^{-1} \in \text{Ker}(d_{A_0}^*).\]

Proof. Since \(g(A_0) = A_0\), we have
\[
d_{A_0}^*(g(A) - A_0) = d_{g(A_0)}^*(g(A) - g(A_0)) = g \circ d_{A_0}^* \circ g^{-1} (g(A) - A_0)g^{-1} = 0
\]
Hence \(g(A) \in A_0 + \text{Ker}(d_{A_0}^*)\). Similar to above, we can prove \(g \in \text{Stab}(A_0)\) preserves \(\text{Ker}(d_{A_0}^*) \subset \Gamma(g_E)\). Since gauge action preserves the metric on \(E\), \(g\) also preserves \(\text{Ker}(d_{A_0}^*)\).

Next we show that the gauge \(g = g(t)\) from \((2.17)\) satisfies \((2.9)\) and \((2.10)\). Differentiating \((2.17)\) on \(M\) and integrating in \(t\), we have
\[
|g - \text{Id}|_{C^{k,\mu}(I)} \leq C(\beta_0|_{C^{(k-1,1),\mu}(I)} + \int |\beta_0|_{C^{(k-1,1),\mu}(I)} dt) 
\]
\[
\leq C(1 + |I|)|\tilde{A} - A_0|_{S^{k,\mu}(I)}. \tag{2.19}
\]

By differentiation of \((2.18)\) with respect to \(t\), we have
\[
d_{A_0} \frac{\partial g(t)}{\partial t} = 0. \tag{2.20}
\]
The equations \((2.19)\), \((2.18)\) and \((2.20)\) and the smoothness of \(A_0\) enable us to bootstrap on the derivatives of \(g\) on \(M\). Hence we have \(g \in C^{(k,1),\mu}(I)\) and the following estimates.
\[
|g - \text{Id}|_{C^{(k,1),\mu}(I)} \leq C|g - \text{Id}|_{C^{k,\mu}(I)} \leq C(1 + |I|)|\tilde{A} - A_0|_{S^{k,\mu}(I)} \tag{2.21}
\]

Because \(g(t) \in \text{Stab}(A_0)\) by \((2.18)\),
\[
|g(\tilde{A}) - A_0|_{S^{k,\mu}(I)} = |g(t)(A(t) - A_0)|_{C^{k,\mu}(I)} + |g\beta_1 g^{-1}|_{C^{(k-1,1),\mu}(I)}
\]
by \((2.16)\) \leq C(1 + |I|)g_{C^{k,\mu}(I)}^2|\tilde{A} - A_0|_{S^{k,\mu}(I)}
by \((2.19)\) \leq C(1 + |I|)^2|\tilde{A} - A_0|_{S^{k,\mu}(I)}

The uniqueness of the standard form gauge (up to \(\text{Stab}(A_0)\)) is not hard and is left to the reader. q.e.d.

Remark. Note that in Step 1 of the proof above, we can not improve the number of time derivatives of the gauge action \(g\) to be more than that of the connection, this is the reason why we work with \(C^{(k,1),\mu}(I)\) gauges and \(S^{k,\mu}\) connections, which at the beginning may seem strange.
Lemma 8 Assume that $I$ is an interval of length $L$, $0 < \varepsilon' \leq \tau' \leq \tau < 1$, $\tau$ is sufficiently small depending on $A_0$ and $L$, $\tilde{A}$ is a connection on $E \times I$. $A_0, A_1$ are Yang-Mills connections on $M$ with

$$|A_1 - A_0|_{C^{k,\mu}} \leq \tau'$$  \hspace{1cm} (2.22)
$$|\tilde{A} - A_1|_{S^{k,\mu}(I)} \leq \varepsilon'$$  \hspace{1cm} (2.23)

Then there exist gauges $\tilde{\eta} \in C^{(k,1),\mu}(I \times \text{Aut} P)$ and $\eta \in C^{k+1,\mu}(\text{Aut} P)$ such that $\tilde{\eta}(\tilde{A})$, $\eta A_1$ are respectively in standard form and Coulomb gauge around $A_0$, and

$$|\eta(A_1) - A_0|_{C^{k,\mu}} \leq C\tau'$$
$$|\tilde{\eta}(\tilde{A}) - \eta(A_1)|_{S^{k,\mu}(I)} \leq C\varepsilon'$$

Proof. We follow the two-step construction of standard form gauges around $A_0$ in the first step of the proof of Prop. 3. By keeping track of the norms, it is easy to prove for some constant $C$.

We state in the following lemma some elliptic estimates for Yang-Mills connections in standard form gauges and Coulomb gauges.

Lemma 9 a) Assume that $A_0$ is a smooth connection on $E$ over $M$ and $\tilde{A} = A(t) + \beta(t)\mathrm{d}t = A_0 + a(t) + \beta(t)\mathrm{d}t \in S^{k,\mu}(I)$ is a Yang-Mills connection in standard form around $A_0$. Then the following hold:
1) There exists $\varepsilon_1 = \varepsilon_1(A_0) > 0$, such that if $|a(t)|_{C^{k,\mu}(I)} < \varepsilon_1 < 1$, we have

$$|eta(t)|_{C^{(k-1,1),\mu}(I)} \leq C|a(t)|_{C^{k-1,\mu}(I)}|\dot{a}|_{C^{k-2,\mu}(I)}$$  \hspace{1cm} (2.24)

for some constant $C = C(A_0, k) > 0$. 2) There exists $\varepsilon_2 = \varepsilon_2(A_0) > 0$ such that if $|\tilde{A} - A_0|_{S^{k,\mu}(I)} < \varepsilon_2$, then for any $s \in (0, (b-a)/2)$,

$$|\tilde{A} - A_0|_{S^{k,\mu}(I_s)} \leq C(s)||a(t)||_{L^2(M \times I)}$$  \hspace{1cm} (2.25)

for some constant $C(s) = C(A_0, k, s) > 0$, where $I_s = [a + s, b - s]$ if $I = [a, b]$.

b) Assume that $A_1$ is a Yang-Mills connection on $M$ in Coulomb gauge around a Yang-Mills connection $A_0$ then there exists $\varepsilon = \varepsilon(A_0) > 0$, such that if $|A_1 - A_0|_{C^{1,\mu}} < \varepsilon$, then

$$|A_1 - A_0|_{C^{k,\mu}} \leq C(k)||A_1 - A_0||, \forall k \geq 0$$  \hspace{1cm} (2.26)

Proof. Proof of a). Let $\dot{a}(t) = \frac{\partial}{\partial t} a(t)$. It follows from (2.7) that $d^*_{A_0} \dot{a} = 0$. From one of the Yang-Mills equations (1.19), we have $d^*_{A_0}(\dot{a} - d_A\beta) = 0$, hence

$$\Delta_A \beta = d^*_{A_0} d_A \beta = d^*_{A_0} \dot{a} = d^*_{A_0} \dot{a} - *([a, \ast \dot{a}]) = -*[a, \ast \dot{a}]$$  \hspace{1cm} (2.27)

For fixed $t$, if $|A(t) - A_0|_{C^{k,\mu}}$ is sufficiently small, then $\Delta_A(t) : C^{k,\mu} \cap \text{Ker}(d_{A_0})^\perp \rightarrow C^{k-2,\mu} \text{Im}(d^*_{A_0})$ is invertible. We denote the inverse by $G_A$. Hence

$$\beta = -G_A([a, \ast \dot{a}])$$  \hspace{1cm} (2.28)
Hence if \(|a|_{C^{k,\mu}(I)} < e_1 < 1\), we have from (2.23) the estimates (2.24). We note that \(\beta(t)\) has one less derivative in \(t\) than \(a(t)\). We also note that \(\beta\) is of quadratic nature in terms of \(a\).

Substitute (2.23) into the Yang-Mills equation (1.18) and use the Coulomb gauge condition (2.7), we have

\[
\dot{a} - (n-4)\dot{a} - d^*_A F_{A0+a} - d_{A0} d^*_A a + d_A ((G_A(*a, *\dot{a}))' - (n-4)d_A G_A(*a, *\dot{a})) + (-1)^{k+1} *[G_A(*a, *\dot{a}), *d_A G_A(*a, *\dot{a})] = 0. \tag{2.29}
\]

We observe that the left hand side of (2.29) is an (pseudo-differential) elliptic operator of \(a\) on \(M \times I\) if \(|a|_{C^{k,\mu}(I)}\) is sufficiently small. If \(\dot{\hat{A}} - A_0|_{S^{k,\mu}(I)}\) is sufficiently small, we have the following a priori interior estimates for connections in standard form,

\[
|a(t)|_{S^{k,\mu}(I_s)} \leq C(s)||a(t)||_I, \tag{2.30}
\]

where \(I_s = |a + s, b - s|\) if \(I = |a, b|\) and \(s < \frac{b-a}{2}\). Combining the estimate (2.24), we obtain the estimate (2.25).

**Proof of b).** The Yang-Mills equation and the Coulomb gauge condition give

\[
d^*_A F_{A0+a} + d_{A0} d^*_A a = 0 \tag{2.31}
\]

From which and bootstrapping, we obtain easily the estimate (2.26). q.e.d.

### 2.3 Standard form gauges with bounds for Yang-Mills connections

We shall prove the following key proposition, which provides us with a good gauge (actually, the standard form around some tangent connection) on an interval and a lower bound of the connection at the end of the maximal existence interval of such a gauge. This design of gauge is motivated by those in Cheeger and Tian [3].

**Proposition 4** Assume \(\hat{A}\) is a Yang-Mills connection on \(E \times [t_0, \infty)\) with \(\sup_{M \times [t_0, \infty]} |F_{\hat{A}}| \leq C\). We fix a tangent Yang-Mills connection \(A_0\) of \(\hat{A}\). Given \(L > 0\), \(R_1 > 0\), there exists \(\tau_1 = \tau_1(A_0, L) > 0\) and a function \(\epsilon_1 = \epsilon_1(\tau)\) satisfying \(0 < \epsilon_1(\tau) < \tau\) such that for \(0 < \tau \leq \tau_1\) and \(0 < \epsilon \leq \epsilon_1(\tau)\), there exist \(R \geq R_1\), integer \(2 < N \leq \infty\), and gauge transformation \(g \in \Gamma(\text{Aut}P \times [R, R'])\), where \(R' = R + \frac{2}{3}NL\), such that the following (a) and (b) hold:

(a) \(g(\hat{A}) - A_0|_{C^{k,\mu}([R, R+2L])} = A(t) - A_0|_{C^{k,\mu}([R, R+2L])} + |\beta(t)|_{C^{(k-1,1),\mu}([R, R+2L])} \leq \epsilon\),

(b) The following estimates hold,

\[
|g(\hat{A}) - A_0|_{S^{k,\mu}([R, R+2L])} = |A(t) - A_0|_{C^{k,\mu}([R, R+2L])} + |\beta(t)|_{C^{(k-1,1),\mu}([R, R+2L])} \leq \epsilon, \tag{2.32}
\]

\[
|\partial^t g(\hat{A})|_{S^{k-1,\mu}([R, R'])} = |\partial^t A(t)|_{C^{k-1,\mu}([R, R'])} + |\partial^t \beta(t)|_{C^{(k-2,1),\mu}([R, R'])} \leq \epsilon, \tag{2.33}
\]

In particular, \(|A(t) - A_0|_{C^{k,\mu}([R, R'])} \leq \tau\) and \(|\partial^t A(t)|_{C^{k-1,\mu}([R, R'])} \leq \epsilon\).

Furthermore, if \(N\) is the maximal integer such that the above gauge \(g\) can be extended
to \([R, R + \frac{2}{3}NL]\) and (a) and (b) are satisfied, and \(N < \infty\), then we have (c)

\[
\sup_{[R', L, R']} |A(t) - A_0|_{C^{k,\mu}} \geq c_2 \tau,
\]  

for some constant \(c_2 = c_2(A_0, L) \in (0, 1)\).

**Remarks.** (1) The condition on \(\varepsilon\) and \(\tau\) means that \(\varepsilon\) is sufficiently small relative to \(\tau\) and \(\tau\) is sufficiently small relative to 1. As a simplification, it can just be said as “If \(0 < \varepsilon << \tau << 1\) depending on \(A_0, L\), then . . . ”. We shall use this expression later to simplify our statements.

(2) (2.32) means that the connection is initially very close to the tangent connection \(A_0\). (2.33) means that the connection stays bounded to \(A_0\) on the existence interval \([R, R']\).

(2.34) means that the times derivative of the connection \(A_k\) keeps being very small on \([R, R']\).

(2.35) means that the connection is at a distance away from \(A_0\) at the end of the existence interval of the constructed gauge. In other words, if the connection stays within a certain distance to \(A_0\), then the gauge would exist up to infinity. As we shall see later, (2.34) will play an important role in our proof of Theorem 1 and is well worth our efforts here.

**Proof.** Assume given \(L > 0\) and \(R_1 > 0\). We consider numbers \(\varepsilon_2, \varepsilon, \tau\) with \(0 < \varepsilon_2 < \varepsilon < \tau\), which will be determined later. Fix

\[R > \max\{R_1, R_0(\varepsilon_2, 3L) + L\},\]

with \(R_0\) as in Lemma 6. Choose intervals

\[I_i = [R + \frac{2iL}{3} - L, R + \frac{2iL}{3}], \quad 0 \leq i < \infty.\]

These intervals are chosen so that \(|I_i| = L, |I_i \cap I_{i+1}| = \frac{1}{3}L\) and \(I_i \cap I_{i+2} = \emptyset\). We have the following claim.

**Claim 1.** There exist constant \(c_2 = c_2(A_0, L)\), integer \(2 < K \leq \infty\), Yang-Mills connections \(A_i \in C^{k,\mu}\) in Coulomb gauge around \(A_0\) and gauge \(g\) on \(M \times \bigcup_{r=1}^{K} I_i\), such that \(g(\tilde{A}) = A(t) + \beta(t)dt\) is in standard form around \(A_0\),

\[
|g(\tilde{A}) - A_i|_{g^{k,\mu}(I_i)} \leq \varepsilon, \quad 0 \leq i \leq K,
\]

\[
|A_i - A_0|_{C^{k,\mu}} \leq \frac{\tau}{2}, \quad 0 \leq i \leq K,
\]

and additionally, if \(K\) is maximal as such and \(K < \infty\), then

\[
\sup_{t \in I_K} |A(t) - A_0|_{C^{k,\mu}} \geq c_2 \tau,
\]

We first show that the claim implies our proposition. Assume the claim is true, we take

\(N = K\). (2.36) at \(i = 0\) gives (2.32). Differentiation of (2.36) with respect to \(t\) gives (2.34). Combining (2.36) and (2.37) gives (2.33). (2.38) gives (2.35).
In the following, we prove the above claim by induction on \( i \). For \( 0 \leq i \leq 3 \), we note that by Lemma 3, there exists gauge \( g \) on \([R - L, R + 2L] = \bigcup_{0 \leq i \leq 3} I_i \) such that \( g(\hat{A}) \) is in standard form around \( A_0 \) and
\[
|g_0(\hat{A}) - A_0|_{S^k,\mu(R - L, R + 2L)} \leq \varepsilon_2 \leq \varepsilon
\]
Hence we may let \( A_i = A_0 \), \( 0 \leq i \leq 3 \). It is obvious that if \( \varepsilon \leq \frac{c_2}{2} \), then (2.36), (2.37) are satisfied for \( 0 \leq i \leq 3 \).

Assume that \( i \geq 4 \) and the claim is true for all integers \( j \) such that \( j \leq i \). In the proof following, the constants \( C \) we used only depend on \( L \) and \( A_0 \). Assume that the induction hypothesis gives
\[
g(\hat{A}) = A(t) + \beta(t)dt
\]
on \([R - L, R + 2L] \) and Yang-Mills connections \( A_j \) for \( 0 \leq j \leq i \) on \( M \) such that (2.36) and (2.37) hold. If \( \sup_{t \in I_i} |A(t) - A_0|_{C^k,\mu} \geq c_2 \), where \( c_2 = c_2(A_0, L) \) is to be fixed later, then we are done by setting \( K = i \). So we assume
\[
|A_i - A_0|_{C^k,\mu} \leq c_2 \tau.
\]  
We have by (2.36) and (2.37),
\[
|A_i - A_0|_{C^k,\mu} \leq 2c_2 \tau, 
\]
then (2.38), (2.39) are satisfied for \( 3 \leq i \leq 4 \).

Lemma 6 gives a tangent Yang-Mills connection \( A_i+1 \), and a gauge transformation \( g' \) on \( I_{i+1} \) such that
\[
|g'(\hat{A}) - A_i+1|_{S^k,\mu(I_{i+1})} \leq \varepsilon_2, 
\]
\[
A_i+1 |_{C^k,\mu} \leq c_1, 
\]
Let \( t_i = R + \frac{2i}{i+1} \) be the middle point of \( I = I_i \cap I_{i+1} \) and define \( h = g(t_i) \cdot g'(t_i)^{-1} \). By (2.34), (2.37), (2.41) and (2.42), we have
\[
|g'(t_i)(A(t_i))|_{C^k,\mu} \leq |A_i+1|_{C^k,\mu} + C\varepsilon_2 \leq C, 
\]
\[
|h(g'(t_i)(A(t_i)))|_{C^k,\mu} = |g(t_i)(A(t_i))|_{C^k,\mu} \leq |A_i|_{C^k,\mu} + C\varepsilon \leq C.
\]
Hence by Lemma 4, we have
\[
|h|_{C^k+1,\mu} \leq C
\]  
Therefore,
\[
|h(A_i+1) - A_i|_{C^k,\mu} \leq |h(A_i+1) - h(g'(t_i)(A(t_i)))|_{C^k,\mu}
\]
\[
+ |g(t_i)(A(t_i)) - A_i|_{C^k,\mu} \leq C\varepsilon_2 + C'\varepsilon \leq C\varepsilon
\]  
Let \( g'' = hg' \) on \( I_{i+1} \), then by (2.40) and (2.41),
\[
|g''(\hat{A}) - A_i+1|_{S^k,\mu(I_{i+1})} = |h \cdot (g'(\hat{A}) - A_i+1) \cdot h^{-1}|_{S^k,\mu(I_{i+1})} \leq C\varepsilon_2
\]
\[
|h(A_i+1) - A_0|_{C^k,\mu} \leq |h(A_i+1) - A_i|_{C^k,\mu} + |A_i - A_0|_{C^k,\mu}
\]
\[
\leq C\varepsilon_2 + 2c_2 \tau \leq Cc_2 \tau
\]
if \( \varepsilon \leq c_2 \tau \).

We observe from (2.45) and (2.46) that \( g''(\tilde{A}) \) and \( h(A'_{i+1}) \) satisfies (2.36) and (2.37) in the claim. But \( g''(\tilde{A}) \) is not in the standard form around \( A_0 \) yet.

In view of (2.45) and (2.46), we can apply Lemma \( \mathcal{L} \) with \( \tilde{A}, A_1, I \) and \( \varepsilon', \tau' \) there replaced by \( g''(A), hA'_{i+1}, I_{i+1}, \varepsilon_2 \) and \( c_2 \tau \). Thus there exists gauges \( \tilde{\eta} \) on \( I_{i+1} \) and \( \eta \) on \( M \) respectively, such that

\[
|\tilde{\eta}g''(\tilde{A}) - \eta(h(A'_{i+1}))|_{S^{k,\mu}(I_{i+1})} \leq c_3 \varepsilon_2 \\
|\eta h((A'_{i+1})) - A_0|_{C^{k,\mu}} \leq c_4 c_2 \tau
\]

Compare \( g \) and \( \tilde{\eta}g'' \) on \( I = I_{i+1} \cap I_i \), we see that they both make \( \tilde{A} \) into standard form around \( A_0 \), hence by the uniqueness of standard form gauges in Prop. \( \mathcal{L} \), they must differ by a pull back of \( \sigma \in \text{Stab}(A_0) \cap C^{k+1,\mu}(M) \), i.e. \( g = \sigma \tilde{\eta}g'' \) on \( I \). Therefore we can extend \( g \) to be over \( I_{i+1} \) by letting

\[ g|_{I_{i+1}} = \sigma \tilde{\eta}g'' \]

Now we let \( A_{i+1} = \sigma \eta hA'_{i+1} \), then \( g(\tilde{A}) \) and \( A_{i+1} \) are respectively in standard form and Coulomb gauge around \( A_0 \), and

\[
|g(\tilde{A}) - A_{i+1}|_{S^{k,\mu}(I_{i+1})} \leq C |\tilde{\eta}g''(\tilde{A}) - \eta hA'_{i+1}|_{S^{k,\mu}(I_{i+1})} \leq c_5 \varepsilon_2 \\
|A_{i+1} - A_0|_{C^{k,\mu}} \leq C |\eta hA'_{i+1} - A_0|_{C^{k,\mu}} \leq c_6 c_2 \tau
\]

Finally, if we choose \( c_2, \varepsilon_2 \) sufficiently small such that \( c_6 c_2 \leq \frac{1}{2} \), \( c_5 \varepsilon_2 \leq \varepsilon \) and \( 0 \leq \varepsilon \leq \tau \) \( \ll 1 \) satisfying all the requirements in the proof above, then (2.36) and (2.37) are true for \( i+1 \) and the induction step of the claim follows. q.e.d.

### 2.4 \( L^2 \) estimates of solutions at the end of maximal existence interval

In the following proposition, we show that in the gauge constructed in Prop. \( \mathcal{L} \) near the end of the maximal existence interval \([R, R']\) (if this is finite), the \( L^2 \) norms of the connection is bounded away from zero and in average do not change much with respect to time \( t \).

**Proposition 5** Given \( L > 0 \) and \( 1 > \eta > 0 \), if \( 0 \leq \varepsilon \ll \tau \ll 1 \) depending on \( A_0, L \) and \( \eta \), and \( \tilde{A} = A(t) + \beta(t)dt \) is a Yang-Mills connection and in the gauge on \([R, R'] = [R, R + \frac{2}{3} NL]\) with respect to \( \varepsilon, \tau \) as in Prop. \( \mathcal{L} \) and assume that \( N < \infty \) is maximal there so that (c) in Prop. \( \mathcal{L} \) holds, then

\[
\sup_{t \in [R' - L, R']} \|A(t) - A_0\| \leq (1 + \eta) \sup_{t \in [R' - 2L, R' - L]} \|A(t) - A_0\|, \quad (2.47)
\]

\[
\sup_{t \in [R' - L, R']} \|A(t) - A_0\| \geq c \tau, \quad (2.48)
\]

for some constant \( c = c(L, \eta, A_0) \).

**Proof.** Since \( N < \infty \), (c) in Prop. \( \mathcal{L} \) gives

\[
\sup_{t \in [R' - L, R']} |A(t) - A_0|_{C^{k,\mu}} \geq c_2 \tau \quad (2.49)
\]

for \( c_2 = c_2(A_0, L) > 0 \). We first show the following claim.
Claim 2 Assume that \( A(t) + \beta(t)dt \) is a connection in the gauge of Prop. \( \Box \), then there exists a Yang-Mills connection \( A_1 \) with

\[
\sup_{t \in [R' - 2L, R']} |A(t) - A_1|_{C^k, \mu} \leq \delta(\varepsilon) \tag{2.50}
\]

\[
d^*_{A_0} (A_1 - A_0) = 0 \tag{2.51}
\]

where \( \delta \) is an increasing function with \( \lim_{\epsilon \to 0} \delta(\varepsilon) = 0 \).

**Proof of Claim.** Assume that the claim is not true, then there exists a sequence \( \varepsilon_j \to 0 \) and a sequence of Yang-Mills connections \( \tilde{A}_j = A_j(t) + \beta_j(t)dt \) on \( I = [R' - 2L, R'] \) in standard form gauge around \( A_0 \) and satisfy conditions in Prop. \( \Box \) (b), i.e.

\[
|\tilde{A}_i - A_0|_{S^k, \mu(I)} \leq \tau, \tag{2.52}
\]

\[
|\frac{\partial}{\partial t} \tilde{A}_j|_{S^{k-1, \mu}(I)} \leq \varepsilon_j, \tag{2.53}
\]

and there exists \( \varepsilon_0 > 0 \) such that for any \( j \) there does not exist a Yang-Mills connection \( A_1 \) such that (2.50) and (2.51) are true with \( \delta(\varepsilon) \) and \( A(t) \) replaced by \( \varepsilon_0 \) and \( A_j(t) \). By compactness of \( S^k \), taking a subsequence, we may assume \( \tilde{A}_j \to \tilde{A}' = A'(t) + \beta'(t)dt \) in \( S^k(I) \) and it follows that \( \tilde{A}' \) is also a Yang-Mills connection in standard form around \( A_0 \) on \( I \) and

\[
|\tilde{A}'|_{S^k, \mu(I)} \leq \lim_{i \to \infty} \inf \tilde{A}_j|_{S^k, \mu(I)} \leq \tau
\]

Taking limit as \( j \to \infty \) in (2.53), we have \( \tilde{A}'(t) = 0, \beta'(t) = 0 \). Hence, from the Yang-Mills equation (1.19), we have

\[
d^*_{A'} d_{A'} \beta' = 0
\]

Since \( A' \) is close to \( A_0 \) in \( C^k \) norm, \( d^*_{A'} d_{A'} \) would be invertible on the space \( \text{Ker}(d_{A_0}) \) while \( \beta' \in \text{Ker}(d_{A_0}) \), hence \( \beta' \equiv 0 \). From the other Yang-Mills equation (1.18), we have \( A' \) is Yang-Mills, hence \( \tilde{A}' \) is in fact the pullback of a Yang-Mills on \( E \). Now (1.18) implies that

\[
d^*_{A_j(t)} F_{A_j(t)} = d_1(t) + d_2(t) + d_3(t), \tag{2.54}
\]

where

\[
\sup_{t \in I} |d_1(t)|_{C^{k-2, \mu}} = \sup_{t \in I} |\tilde{A}_j(t) + \tilde{A}_j(t)|_{C^{k-2, \mu}} \leq 2\varepsilon_j \text{ by (2.53)}
\]

\[
\sup_{t \in I} |d_2(t)|_{C^{k-2, \mu}} = \sup_{t \in I} |d_{A_j} \beta_j(t)|_{C^{k-2, \mu}} \leq C\varepsilon_j \text{ by (2.53)}
\]

\[
\sup_{t \in I} |d_3(t)|_{C^{k-2, \mu}} = \sup_{t \in I} (\mu - 4)d_{A_j} \beta_j(t) + (-1)^n + 1 * [\beta_j(t), *d_{A_j} \beta_j(t)]|_{C^{k-2, \mu}}
\]

\[
\leq C \sup_{t \in I} |\beta_j(t)|_{C^{k-1, \mu}} := C\delta_j
\]

where \( \delta_j = \sup_{t \in I} |\beta_j(t)|_{C^{k-1, \mu}} \to 0 \) since \( \tilde{A}_j \to A' \) in \( S^k(I) \). The left hand side of (2.54) is a uniformly elliptic second order linear operator acting on \( A_j(t) - A' \) with coefficients bounded in \( C^{k, \mu}(M) \) uniformly in \( t \). Hence we can apply elliptic estimates and bootstrapping to get

\[
\sup_{t \in I} |A_j(t) - A'|_{C^k, \mu} \leq C(\varepsilon_j + \delta_j + \sup_{t \in I} |A_j(t) - A'|_{C^k, \mu}) \to 0
\]

But this is a contradiction to our assumption at beginning. q.e.d.
Let $A_1$ be as in the above claim. Let $t_0 \in [R' - L, R']$ such that $|A_0 - A(t_0)|_{C^{k, \mu}} = \sup_{t \in [R' - L, R']} |A_0 - A(t)|_{C^{k, \mu}}$, then by (2.49) and (2.50), we have

$$|A_1 - A_0|_{C^{k, \mu}} \geq |A_0 - A(t_0)|_{C^{k, \mu}} - |A(t_0) - A_1|_{C^{k, \mu}} \geq c_2 \tau - \delta(\varepsilon) \geq \frac{1}{2} c_2 \tau$$

(2.55)

if $\delta(\varepsilon) \leq \frac{1}{2} c_2 \tau$. Now we have for any $t \in [R' - 2L, R']$,

$$|A(t) - A_0|_{C^{k, \mu}} \geq |A_1 - A_0|_{C^{k, \mu}} - |A_1 - A(t)|_{C^{k, \mu}} \geq \frac{1}{2} c_2 \tau - \delta(\varepsilon) \geq \frac{1}{4} c_2 \tau$$

if $\delta(\varepsilon) \leq \frac{1}{4} c_2 \tau$. By a priori elliptic estimates for Yang-Mills connections in standard form [2.24], we have

$$\sup_{t \in [R' - 2L, R' - L]} \|A(t) - A_0\| \geq C^{-1} |A(R' - \frac{3L}{2}) - A_0|_{C^{k, \mu}} \geq \frac{1}{4} C^{-1} c_2 \tau = c_3 \tau$$

$$\sup_{t \in [R' - L, R']} \|A(t) - A_0\| \geq C^{-1} |A(R' - \frac{L}{2}) - A_0|_{C^{k, \mu}} \geq c_3 \tau$$

$$\sup_{t \in [R' - L, R']} \|A(t) - A_0\| \leq \sup_{t \in [R' - 2L, R' - L]} \|A(t) - A_0\| + 2CL \left| \frac{\partial}{\partial t} A(t) \right|_{C^0([R' - 2L, R'])} \leq (1 + \eta) \sup_{t \in [R' - 2L, R' - L]} \|A(t) - A_0\|$$

if $4CL \varepsilon \leq \eta c_3 \tau$. This implies that if $0 < \varepsilon << \tau << 1$ depending on $A_0$, $L$ and $\eta$, then (2.47) and (2.48) hold. q.e.d.

### 3 An asymptotic convergence result for a class of evolution equations

By fixing the gauge as in the last section, we can reduce the main theorem into an asymptotics problem for a certain nonlinear elliptic evolution equation quite similar to the case treated in [13, Theorem 1]. The difference from the case there is that we only have estimates and growth control (as given by (2.34), (2.47) and (2.48)) near the end of our existence interval $[R, R']$, not on the whole interval. However, to the up side, we have the bound

$$\left| \frac{\partial}{\partial t} A(t) \right|_{C^{k-1, \mu}}(t) \leq \varepsilon$$

which allows us to compare norms of connections at different points on the time interval. We prove a general asymptotic convergence theorem relevant to our case and complete the proof of Theorem 1 and Theorem 2 in this section.

#### 3.1 A type of nonlinear evolution equations

Let $E$ be a vector bundle on Riemannian manifold $M$ and let $\mathcal{E}$ be a functional of ‘energy type’ defined for sections $a \in C^1(M, E)$ by

$$\mathcal{E}(a) = \int_M F(x, a, \nabla a)$$

(3.1)
where $F = F(x, z, p)$ for $x \in M$, $z \in E$, $p \in T_x M \otimes E_x$ depend smoothly on $(x, z, p)$ and $F$ is uniformly convex in the $p$ variable for $p \in T_x M \otimes E_x$ and $|z|, |p|$ small. We also require that $F$ has analytic dependence on $(z, p) \in E \times T_x M \otimes E_x$ with uniform bounds on $F$ and its derivatives in $z, p$ for sufficiently small $|z|, |p|$ . By this we mean that there exists $c_0 > 0$ such that

$$F(x, z + \sum_i \lambda_i w_i, p + \sum_j \lambda_j q_j)$$

$$= \sum_{|\alpha| \geq 0} F_\alpha(x, z, w_1, \ldots, w_m, p, q_1, \ldots, q_{m'}) \lambda^\alpha,$$

where $0 \leq i \leq m$, $0 \leq j \leq m'$, $m$ and $m'$ being the dimension of fibers of bundle $E$ and $TM \otimes E$. $|z|, |w|, |p_i|, |q_j| \leq c_0$, $z, w \in E_x, p_i, q_j \in T_x M \otimes E_x$, $\lambda = (\lambda_1, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_{2m'}) \in \mathbb{R}^{m+m'}$, $|\lambda| \leq 1$. The above expansion and its derivatives (in terms of $z$ and $p$ only) should converge absolutely in the above domain of variables and there are uniform bounds

$$\sup_{|\alpha| + |\beta| = j, |z|, |p| \leq c_0} |D^\alpha_z D^\beta_p F(x, z, p)| \leq c(j).$$

(3.2)

The Euler-Lagrange operator for $\mathcal{E}(a)$, denoted by $\mathcal{M}(a)$, is uniquely characterized by

$$-(\mathcal{M}(a), b)_{L^2(M)} = \frac{d}{ds} \mathcal{E}(a + sb)|_{s=0}.$$  

(3.3)

In other words, $\mathcal{M}(a) = - \grad \mathcal{E}(a)$). For simplicity, we also require that

$$\mathcal{M}(0) = 0,$$

i.e. 0 is a critical point of $\mathcal{E}$. By uniform convexity, we have that $\mathcal{M}(a)$ is a second order quasi-linear operator which is uniformly elliptic for $|a|_{C^1(M)}$ sufficiently small. Thus the linearization (the Jacobi operator at 0)

$$L a := \frac{d}{ds} \mathcal{M}(su)|_{s=0}.$$  

(3.4)

is a second order elliptic self-adjoint linear operator.

We shall fix $k \geq 2$ and $\mu > 0$ in this section for the consideration of $C^{k, \mu}$ norms. Assume that $a(t) \in C^{k, \mu}(M \times I, E \times I)$ is a section of $E \times I$. Fix $\gamma > 0$ a constant and denote $\frac{2a(t)}{\gamma}$ by $\dot{a}(t)$. We consider the following equation

$$\ddot{a}(t) - \gamma \dot{a}(t) + N(a(t)) + G_1(\dot{a}(t)) + G_2(\ddot{a}(t)) = 0.$$  

(3.5)

In the equation we require that $N(a)$ is a second-order quasi-linear differential operator with smooth coefficients and $N(a)$ is uniformly elliptic if $|a|_{C^1(M)}$ is sufficiently small and $N$ approximates $\mathcal{M}$ well in the following sense,

$$\|\mathcal{M}(a) - N(a)\| \leq \frac{1}{4} \min\{\|N(a)\|, \|\mathcal{M}(a)\|\},$$  

(3.6)
for any $a \in C^{k,\mu}(M)$ with $|a|_{C^{k,\mu}} \leq \sigma$, where $\sigma = \sigma(\mathcal{E})$ is a constant. Let $G_0(a) := N(a) - L(a)$ ($L$ as in (3.4)). We require in addition that
\[
G_i : C^{l,\mu}(M \times I) \to C^{l,\mu}(M \times I) \quad \text{for } 0 \leq i \leq 2, 0 \leq l \leq k - 2
\] (3.7)
are continuously differentiable linear maps and the derivatives
\[
D^\beta G_i : C^{l,\mu}(M \times I) \to C^{l,\mu}(M \times I), \quad \forall 0 \leq |\beta| \leq k - 2, 0 \leq l \leq k - |\beta| - 2
\] (3.8)
are continuous linear maps for $0 \leq |\beta| \leq k - 2$, $0 \leq l \leq k - |\beta| - 2$ with the following uniform bounds on the operator norms:
\[
\|D^\beta G_i(a)\|_{\text{op}} \leq C|a|_{C^{k,\mu}}, \quad \forall a \in C^{k,\mu}(M).
\] (3.9)
In fact, what we have in mind is that $G_i$’s are some pseudo-differential operators which may have nonsmooth symbols.

If $a = a(t) \in C^{k,\mu}(M \times [t_1, t_2])$ is a solution of (3.3) with $|a|_{C^{k,\mu}(I)} \leq C$, then it follows from the standard Schauder theory and Sobolev theory (for example, in [3] and [10]), that for $\sigma \in (0, \frac{1}{2}(t_2 - t_1))$ and $0 \leq l \leq k - 2$,
\[
|a|_{C^{l,\mu}([t_1, t_2] + \sigma)} \leq c_l \sigma^{-(l+\mu)}|a|_{[t_1, t_2]},
\] (3.10)
Example. If $\tilde{A} = A(t) + \beta(t)dt = A_0 + a(t) + \beta(t)dt$ is a Yang-Mills connection in the standard form around a Yang-Mills connection $A_0$. We claim that $a(t)$ satisfies an equation of the form (3.3).

Let $\mathcal{E}(a) = YM(A_0 + a)$ in this case. We note that $\mathcal{E}(a)$ is uniformly convex if we restrict $a(t) \in \text{Ker}(d_{A_0}^*)$ throughout. $\mathcal{E}(a)$ is an analytic functional because the space of connections is affine. We note that in contrast, the energy for harmonic maps, which is analytic only when the metric on the target is analytic. In the Yang-Mills case here, the linear fibers of the bundle may be thought as the ‘target’ space, which have standard, hence analytic metrics. By (1.18) and (1.19) and the fact $\beta = -G_A([a, *\dot{a}])$ ($G_A = (\Delta_A)^{-1}$), see the proof of Lemma 4; $a(x, t)$ satisfies an equation of the form of (3.3), where
\[
\gamma = (n - 4)
\]
\[
N(a) = - (d_{A_0}^* + a)F_{A_0 + a} + d_{A_0}d_{A_0}^*a
\]
\[
L(a) = -(d_{A_0}d_{A_0}^*a + d_{A_0}^*d_{A_0}a + (-1)^n [a, *F_{A_0}])
\]
\[
G_0(a) = N(a) - L(a) = (-1)^n [a, *d_{A_0}a + a \wedge a] + d_{A_0}d_{A_0}^*a
\]
\[
G_1(\dot{a}) = d_A G_A([\dot{a}, *\dot{a}]) - (n - 4)d_A G_A([a, *\dot{a}])
\]
\[
+ (-1)^{n+1} [G_A([a, *\dot{a}]), *d_A G_A([a, *\dot{a}])]
\]
\[
G_2(\ddot{a}) = d_A G_A([a, *\ddot{a}])
\]

We regard only one $\dot{a}$ in each term of $G_1$ as the variable, others (dependent on $a$ and $\dot{a}$) will be seen as parts of coefficients of $G_1$. Similarly, we only regard the $\dot{a}$ in $G_2$ as the variable. By the smoothing properties of the Green’s operator $G_A$, and (3.3), it is easy to see that the operators $N, G_i$ satisfy the stated properties if $|a|_{C^{k,\mu}(I)} \leq \tau$ and hence the additional requirement following (3.5) are satisfied.
Assume that $\varepsilon, \tau, \eta, L$ are positive constants, $\varepsilon \leq \tau$, $a = a(x, t) \in C^{k,\mu}(M \times [R, R'])$, where $R' - R \geq 3L$ and possibly $R' = \infty$.

**Definition.** We call $a = a(x, t)$ $(\varepsilon, \tau, \eta, L)$-bounded on $[R, R']$ if there exists $0 < c = c(\eta, L) < 1$ such that

$$|a|_{C^{k,\mu}([R,R'])} \leq \tau$$  \hspace{1cm} (3.11)

$$|\frac{\partial a}{\partial t}|_{C^{k-1,\mu}([R,R'])} \leq \varepsilon$$  \hspace{1cm} (3.12)

$$|a|_{C^{k,\mu}([R,R+2L])} \leq \varepsilon$$  \hspace{1cm} (3.13)

and additionally, if $R' < \infty$ then $a$ satisfies,

$$\sup_{[R'-L,R']} \|a(t)\| \geq c\tau.$$  \hspace{1cm} (3.14)

This definition, of course, is motivated by our estimates on connections in Section 2.3 and Section 2.4. We state our result about solutions to (3.5) as follows.

**Theorem 3** Fix $k \geq 5$ and $\mu > 0$. Given $L > 0$, there exists $\eta \in (0,1)$ such that if $0 < \varepsilon << \tau << 1$ depending on $E$, $L$ and $\eta$, $a(t)$ is an $(\varepsilon, \tau, \eta, L)$-bounded solution to (3.5) and

$$E(a(t)) - E(0) \geq -c\varepsilon,$$  \hspace{1cm} (3.15)

then $R' = \infty$, $|a|_{C^{k,\mu}([R,\infty))} \leq \tau$ and there exists a $C^{2,\mu}$ critical point $\omega$ of $E$ such that

$$\lim_{t \to \infty} |a(t) - w|_{C^{2,\mu}} = 0$$  \hspace{1cm} (3.16)

### 3.2 Growth estimates

Let $E$ be a functional as in the last subsection. Recall that $L$ is the linearization of the Euler-Lagrange operator $M = -\text{grad} \ E$. Assume that $\mu_1 \leq \mu_2 \leq \ldots$ and $\phi_1, \phi_2, \ldots$ are the complete set of eigenvalues and the corresponding orthonormal (in $L^2$ norm) eigenfunctions of the operator $L$ on $\Gamma(E)$. Let

$$\lambda_{\gamma}^\pm = \frac{1}{2}(\gamma \pm \sqrt{\gamma^2 - 4\mu_i}).$$  \hspace{1cm} (3.17)

Every solution of the linear evolution equation

$$L(a) = \ddot{a} - \gamma \dot{a} + L(a) = 0,$$  \hspace{1cm} (3.18)

can be written as

$$a(x, t) = \sum_{i \in I_1} (a_i \cos \alpha_i t - b_i \sin \alpha_i t) \frac{\gamma^i}{2} \phi_i(x)$$

$$+ \sum_{i \in I_2} (a_i + b_i t) e^{\frac{\gamma^i t}{2}} \phi_i(x) + \sum_{i \in I_3} (a_i e^{\lambda_{\gamma}^+ t} + b_i e^{\lambda_{\gamma}^- t}) \phi_i(x)$$  \hspace{1cm} (3.19)
for suitable constants \(a_i, b_i\), where
\[
I_1 = \{i : \mu_i < -\frac{\gamma}{4}\}, \quad \alpha_i = \text{Im} \lambda_i^+,
\]
\[
I_2 = \{i : \mu_i = -\frac{\gamma}{4}\},
\]
\[
I_3 = \{i : \mu_i > -\frac{\gamma}{4}\}.
\]
And the \(L^2\) norm square of \(a(\cdot, t)\) can be written as
\[
\|a(t)\|^2 = \sum_{i \in I_1} (a_i \cos \alpha_i t - b_i \sin \alpha_i t)^2 e^{\gamma t} + \sum_{i \in I_2} (a_i + b_i t)^2 e^{\gamma t} + \sum_{i \in I_3} (a_i e^{\lambda_i^+ t} + b_i e^{\lambda_i^- t})^2.
\] (3.20)

We let
\[
\delta_1 = \min\{s : s \in \{\text{Re} \lambda_i^\pm\}, s > 0\}, \quad \delta_2 = \min\{|s| : s \in \{\text{Re} \lambda_i^\pm\}, s < 0\}
\] (3.21)

We have

**Lemma 10** [15, Lemma 2] Assume \(a \in C^{k,\mu}(M \times [0,3L])\) satisfy (3.3), where \(G_i\) satisfies properties above. For any given \(L > 0, 1 > \eta > 0, \delta < \frac{1}{4} \min\{\delta_1, \delta_2\}\), where \(\delta_1, \delta_2\) as in (3.21), there exists \(\tau_0 = \tau_0(L, \eta) > 0\), such that if \(|a|_{C^{k,\mu}([0,3L])} \leq \tau_0\), then the following are true. By denoting \(S(j) = \sup_{t \in [j-1,L]J} \|a(t)\|\), we have

(i) \(S(2) \geq e^{\delta L/2} S(1) \Rightarrow S(3) \geq e^{(\delta_1 - \delta)L} S(2)\)
(ii) \(S(2) \geq e^{-(\delta_2 - \delta)L} S(1) \Rightarrow S(3) \geq (1 - \eta) S(2)\)
(iii) \(S(2) \geq e^{\delta L/2} S(3) \Rightarrow S(1) \geq e^{(\delta_2 - \delta)L} S(2)\)
(iv) \(S(2) \geq e^{-(\delta_1 - \delta)L} S(3) \Rightarrow S(1) \geq (1 - \eta) S(2)\)
(v) If \(S(2) \geq \max \{e^{-(\delta_1 - \delta)L} S(1), e^{-(\delta_2 - \delta)L} S(1)\}\), then \(S(2) \leq (1 + \eta) \inf_{t \in [L,2L]} \|a(t)\|\)
and \(\|\dot{a}(t)\|_{C^1} \leq \eta \|a(t)\|\), \(\forall t \in [L,2L]\).

**Sketch of proof:** First we can prove that (i)-(v) hold for solutions to the linear equation (3.18) by using the expression of the \(L^2\) norm of (3.20). Then by a blow-up argument we can prove that for \(\tau_0\) sufficiently small, the lemma holds for solutions to (3.3). In the proof, we need the regularity properties of solutions to (3.3). q.e.d.

The next proposition is a simpler form of Theorem 4 in [15].

**Proposition 6** Assume \(a\) on \([R, R+NL]\) satisfies (3.3) as above. For \(0 < \eta < 1\), there exists \(\tau_0 > 0\), such that if \(0 < |a|_{C^{k,\mu}([R,R+NL])} < \tau_0\), then there exists integers \(1 \leq k_1 \leq k_2 \leq N - 1\) such that the following hold, where \(S(j) = \sup_{t \in [j-1,L]J} \|a(t)\|\).

(a) \(S(j) \leq e^{-(\delta_2 - \delta)L} S(j-1), \text{ for } 1 \leq j \leq k_1 - 1\).
(b) \(\|a(t_1)\| \leq (1 + \eta) \|a(t_2)\|, \text{ for } t_1, t_2 \in [k_1 L, (k_2 - 1)L], \text{ and } |t_1 - t_2| \leq L\)
\(\|\dot{a}(t)\| \leq \eta \|a(t)\|, \text{ for } t \in [k_1 L, (k_2 - 1)L]\).
(c) \(S(j) \geq e^{(\delta_1 - \delta)L} S(j-1), \text{ for } k_2 + 1 \leq j \leq N - 1\).
Proof. We set
\[ k_1 = \min_{1 \leq j \leq k-1} \{ S(j) \geq e^{-(\delta_2-\delta)L}S(j-1) \} \]
\[ k_2 = \min_{k_1 \leq j \leq N-1} \{ S(j+1) \geq e^{(\delta_1-\delta)L}S(j) \} \]
And it is easy to check the theorem holds by repetitive use of the previous lemma. q.e.d.

This proposition allows us to conceptually divide the existence interval into three parts according to the ‘growth rate’ of the \( L^2 \) norms of \( a \). In the first part which corresponds to case (a), the \( L^2 \) norm of \( a \) is decreasing exponentially (in average); in case (b), the \( L^2 \) norm changes slowly in proportion; in case (c), it is growing exponentially.

3.3 Variational inequalities for analytical functionals

The inequalities in the following proposition are infinite dimensional generalizations given by Leon Simon \[15\] of Lojasiewicz inequalities with regard to critical points of analytic functions. Let \( \mathcal{E}(a) \) be an analytic elliptic functional for \( a \in C^1(M) \) as in Section 3.1.

Proposition 7 \[15, Theorem 3\] There exist constants \( 0 < \theta < \frac{1}{2}, 2 \leq \gamma, 0 < \sigma \) depending only on \( \mathcal{E} \), such that if \( |a|_{C^{k,\mu}} < \sigma \), then
\[ \|M(a)\| \geq \left( \inf_{\zeta \in S} \|a - \zeta\| \right)^\gamma \] (3.22)
where \( S = \{ \zeta \in C^{k,\mu}(E) : |\zeta|_{C^{k,\mu}(\Sigma)} < c_0, M(\zeta) = 0 \} \), and
\[ \|M(a)\| \geq |\mathcal{E}(a) - \mathcal{E}(0)|^{1-\theta} \] (3.23)

Let \( \sigma, \theta \) in this section be the same as in Prop. 7. Now assume that \( a \in C^{k,\mu}(M \times [t_1, t_2]) \) satisfies the following equation
\[ \dot{a} = N(a) + R(a) \] (3.24)
where \( N(a) \) approximates well the gradient \( M \) of \( \mathcal{E} \) in the sense of (3.6) and
\[ \|R(a)(t)\| \leq \frac{1}{2} \|\dot{a}(t)\|, \quad \forall t \in [t_1, t_2] \] (3.25)
We have

Lemma 11 \[15, Lemma 1\] Suppose \( a \) satisfies \( |a(t)|_{C^{k,\mu}([t_1, t_2])} \leq \sigma \), and suppose that for some constant \( \varepsilon > 0 \),
\[ \mathcal{E}(a(t)) > \mathcal{E}(0) - \varepsilon \] for all \( t \in [t_1, t_2] \)
(3.26)
Then
\[ \int_{t_1}^{t_2} \|\dot{a}(t)\| dt \leq C\theta^{-1}(|\mathcal{E}(a(t_1)) - \mathcal{E}(0)|^{\theta} + \varepsilon^{\theta}), \] (3.27)
In particular
\[ \sup_{t \in [t_1, t_2]} \|a(t) - a(t_1)\| \leq C\theta^{-1}(|\mathcal{E}(a(t_1)) - \mathcal{E}(0)|^{\theta} + \varepsilon^{\theta}). \] (3.28)
The next lemma is a gauge invariant form of the inequality (3.23) for connections and was essentially proven in [9].

**Lemma 12** Let $E$ be a vector bundle on $M$. $A$ is a $C^{k,\mu}$ connection on $E$ and $B$ is a smooth Yang-Mills connection on $E$. There exists $\varepsilon_3 > 0$ and $\theta \in (0, \frac{1}{2})$, such that if $|A - B|_{C^{k,\mu}} < \varepsilon_3$, then the following inequality holds,

$$
\left( \int_M |F_A|^2 - |F_B|^2 d\sigma \right)^{1-\theta} \leq 2d_A^*F_A. \quad (3.29)
$$

**Proof.** If $\varepsilon_3$ is sufficiently small, by a gauge transformation to the Coulomb gauge around $B$, we may assume that $A = B + a$ with $a \in \text{Ker}(d_B^*)$, and $|a|_{C^{k,\mu}} < C\varepsilon_3$. Let $U$ be a small $C^{k,\mu}$ neighborhood of 0 in $\text{Ker}(d_B^*) \subset \Omega^1(\text{End} E|_M)$ and consider $\mathcal{E} : U \to \mathbb{R}$ by $\mathcal{E}(b) = \text{YM}(B + b)$, $\forall b \in U$. Let $\mathcal{M}$ be the Euler-Lagrange operator of $\mathcal{E}$ on $U$. We claim that if $U$ is sufficiently small, $A = B + a$, $a \in U$, then

$$
\|d_A^*F_A - \mathcal{M}(a)\| \leq \frac{1}{4} \min\{\|\mathcal{M}(a)\|, \|d_A^*F_A\|\}. \quad (3.30)
$$

Indeed, since $(\mathcal{M}(a), b)_{L^2} = (d_A^*F_A, b)_{L^2}$, $\forall b \in \text{Ker}(d_B^*)$ and $\mathcal{M}(a) \in \text{Ker}(d_B^*)$, it follows that $\mathcal{M}(a) = p_{\text{Ker}(d_B^*)}(d_A^*F_A)$, where $p$ is the $L^2$ projection. Let $G_B = d_B^*d_B^{-1}$ as in the proof of Lemma 8, we can easily see that

$$
\mathcal{M}(a) = p_{\text{Ker}(d_B^*)}(d_A^*F_A) = d_A^*F_A - d_BG_Bd_A^*F_A \quad (3.31)
$$

where we used the identity

$$
d_A^*d_A^*F_A = \{F_A, F_A\} = 0. \quad (3.32)
$$

where $\{ , \}$ is defined by Lie bracket on the bundle parts and Riemannian product on the form part and hence is skew-symmetric. From (3.31) and the smoothing properties of $G_B$ it is easy to derive (3.30) if $|a|_{C^{k,\mu}}$ is small. Apply (3.23) to $\mathcal{E}$ and use (3.30) (note that $\mathcal{E}$ is only defined on $\text{Ker}(d_B^*)$, not on the space of all sections; however Prop. 9 is still true in this case because $\text{Ker}(d_B^*)$ is an analytic submanifold of the Hölder spaces of sections), we obtain that there exists $\theta \in (0, \frac{1}{2})$, such that

$$
\|\mathcal{E}(a) - \mathcal{E}(0)\|^{1-\theta} \leq 2\|d_A^*F_A\|. \quad (3.29)
$$

This last inequality is exactly (3.29) which we want to prove. q.e.d.

### 3.4 Proof of Theorem 3

Our method of proving Theorem 3 follows mostly the methods of proving Theorem 1 in [15] with modifications to our case. We first apply the growth estimates in Section 3.2 to the time derivative $\dot{a}(t)$ of the solution to (3.3) and divide the interval into three parts. We estimate the integration of $\|\dot{a}(t)\|$ on these three parts respectively and thus obtain a bound on $\|a(t)\|$, which then gives long-term existence of $a(t)$ and convergence. The estimates of
||\hat{a}(t)|| on the three parts use the estimates from Section 3.3 and the assumptions on the solution in Theorem 3, especially the condition (3.12).  

Assume $R' < \infty$. We may assume that $R' = R + NL$ for $N \geq 2$ by changing $R$ within an amount of $L$ if necessary. It is important to note that by differentiation of (1.18), we have $\hat{a}$ satisfies an equation of (3.5) form on $[R, R']$, with the same $\tau$ (we need to change $C^{k, \mu}$ norms to $C^{k-1, \mu}$ norms for $\hat{a}$; since $k \geq 5$, this is still in the regular range, an our previous results hold without change). Note that $\hat{a}$ in general doesn't satisfy an equation in the form of (3.5).  

Assume $\tau < \tau_0$, where $\tau_0$ is as in Prop. 3. Then by applying Prop. 3 to $\hat{a}(t)$, we have $1 \leq k_1 \leq k_2 \leq N-1$ for $\hat{a}$ such that the conclusions of Prop. 3 hold. We adopt the notation

$$S(j, \hat{a}) = \sup_{t \in [R+(j-1)L, R+jL]} ||\hat{a}(t)||$$ (3.33)

We shall assume in the rest of our proof that $\tau \geq \varepsilon^{\alpha}$ for the constant $\alpha = \frac{\theta}{8}$, where $\theta$ is the constant in Lemma 11. We have the following claim,

Claim 3 There exists a constant $C$ depending on the functional $E$ such that

$$||a(t)|| \leq C\varepsilon^{2\alpha}, \quad \forall t \in [R, R + NL].$$ (3.34)

Proof of Claim. First we consider the claim for $t \in [R, R + (k_1-1)L]$, by using Prop. 3 (a), we have

$$||\hat{a}(t)|| \leq S\left(\left(\left[t-R\right]/L\right)+1, \hat{a}\right) \leq e^{-(\delta_2-\delta)(t-L)}S(1, \hat{a}) \leq e^{-\delta(t-L)}\varepsilon$$ (3.35)

Hence for $t \in [R, R + (k_1-1)L]$,

$$||a(t)|| \leq ||a(R)|| + \int_0^t ||\hat{a}||dt \leq \varepsilon + \delta^{-1}e^{\delta L}\varepsilon \leq \varepsilon^{\frac{1}{2}},$$ (3.36)

if $\varepsilon^{\frac{1}{2}} \leq (1 + \delta^{-1}e^{\delta L})^{-1}$.

Next we consider the case $t \in [R + (k_1-1)L, R + (k_2 + 1)L]$. By using the fact that $|a(t)|_{C^{k, \mu}} \leq \tau$ and $||\hat{a}(t)|| \leq 1/|\hat{a}(t)||$ which follows from (b) in Prop. 3 if we take $\eta = 1/8$ there, we see that Lemma 11 applies to $\hat{a}$ on $[R + k_1L, R + (k_2 - 1)L]$ as long as $\varepsilon$ and $\tau$ are sufficiently small. Therefore

$$\int_{R+k_1L}^{R+(k_2-1)L} ||\hat{a}||dt \leq C(||E(a(R + k_1 L)) - E(0)||^{\theta} + \varepsilon^{\theta})$$ (3.37)

$$\leq C(|a(R + k_1 L)|^{\theta} + \varepsilon^{\theta})$$ by (3.11)

$$\leq C\left(\sup_{t \in [R+k_1L-1,R+k_1L+1]} ||a(t)||^{\theta} + \varepsilon^{\theta}\right)$$ by (3.10)

$$\leq C\varepsilon^{\theta} \leq C\varepsilon^{2\alpha}$$ by (3.38)

and hence for $\tau \in [R + (k_1-1)L, R + (k_2 + 1)L]$, by (3.12) $||\hat{a}|| \leq C\varepsilon$ and we have

$$||a(t)|| \leq ||a(R + k_1 L)|| + C\varepsilon^{2\alpha} + C\varepsilon$$ (3.38)

$$\leq ||a(R + (k_1 - 1)L)|| + C\varepsilon + C\varepsilon^{2\alpha}$$

$$\leq C\varepsilon^{\frac{1}{2}} + C\varepsilon + C\varepsilon^{2\alpha} \leq C\varepsilon^{2\alpha}.$$
Finally, we consider the case \( t \in [R + (k_2 + 1)L, R + NL] \). It follows from Prop. 3 (c) that
\[
S(j, \dot{a}) \geq e^{(\delta_1 - \delta)L}S(j - 1, \dot{a}), \forall k_2 + 1 \leq j \leq N - 1.
\]
We also know from (3.12) that \( S(N - 1, \dot{A}) \leq C\varepsilon \). Hence for any \( t \in (R + (j - 1)L, R + jL] \), where \( k_2 + 1 \leq j \leq N - 1 \), we have
\[
\|\dot{a}(t)\| \leq S(j, \dot{a}) \leq e^{-(\delta_1 - \delta)(N - 1 - j)L}S(N - 1, \dot{a}) \leq Ce^{-(\delta_1 - \delta)(R + (N - 2)L - t)\varepsilon}.
\]
It follows that
\[
\int_{R + (k_2 + 1)L}^{R + (N - 1)L} \|\dot{a}(t)\| dt \leq \int_{R + (k_2 + 1)L}^{R + (N - 1)L} Ce^{-(\delta_1 - \delta)(R + (N - 2)L - t)\varepsilon} dt \leq C\varepsilon. \tag{3.39}
\]
Hence for any \( t \in [R + (k_2 + 1)L, R + NL] \), we have
\[
\|a(t)\| \leq \|a(R + (k_2 + 1)L)\| + C\varepsilon \leq C\varepsilon^{2q} \tag{3.40}
\]
q.e.d.

Now (3.34) implies in particular that
\[
\sup_{t \in [R + (N - 1)L, R + NL]} \|a(t)\| \leq C\varepsilon^{2q} \tag{3.41}
\]
which gives a contradiction to our assumption (??) if \( \varepsilon \) is sufficiently small relative to \( \tau \). Therefore we must have \( R' = \infty \). Hence by (3.11), \( |a|_{C^2,\mu([R,\infty])} \leq \tau \) and by (3.12), \( \|\dot{a}(t)\| \leq \varepsilon \) for \( t \in [R,\infty) \). It follows that \( k_2 = \infty \), otherwise Prop. 3 (c) implies \( \|\dot{a}(t)\| \) is going to infinity for a sequence of \( t \). Now (3.36) and (3.37) imply that
\[
\int_{R}^{\infty} \|\dot{a}(t)\| dt < \infty.
\]
Hence there exists \( w \in L^2 \) such that \( a(t) \to w \) in \( L^2 \) as \( t \to \infty \). Uniform bounds on \( |a(t)|_{C^3,\mu} \) implies that for a sequence \( t_i \to \infty \), \( a(t_i) \to w \) in \( C^2,\mu \). In fact, compactness implies that \( a(t) \to w \) in \( C^2,\mu \) as \( t \to \infty \). Hence \( w \in C^2,\mu \) and taking the limit of \( \frac{\|\dot{a}(t)\|}{\|a(t)\|^{\alpha}} \) as \( t_i \to \infty \), we see that \( w \) is a critical point of \( E \). This finishes the proof of Theorem 3.

### 3.5 Proofs of Theorem 1 and Theorem 2

In this section we apply Theorem 3 to prove the convergence of the connection to its tangent connection, hence the uniqueness of tangent connections in Theorem 1. We shall use the monotonicity formula and Lemma 12 of Yang-Mills connections to show the desired rate of convergence in Theorem 1. The idea of using monotonicity formula and Lemma 12 comes from Leon Simon’s work [16, 3.10 - 3.15], where energy minimizing harmonic maps with a tangent map which has an isolated singularity is treated.

As before, assume that in the cylindrical coordinates, \( \dot{A} = A(t) + \beta(t)dt = A_0 + a(t) + \beta(t)dt, t \in [R, R'] \) is a connection on \( S^{n-1} \times [R, R'] \), is in the standard form gauge given in Prop. 3, then the conclusions of Prop. 1 and Prop. 2 imply that \( a(t) \) is a \( (\varepsilon, \tau, \eta, L) \)-bounded solution of an equation in the form of (3.5) on the interval \([R, R']\).
condition in Theorem \[3\] left is the energy lower bound \((3.13)\). This can be achieved by choosing the tangent connection \(A_0\) suitably at the beginning. We note that the set of energies of tangent connections of \(\tilde{A}\) is bounded from below. We choose \(A_0\) with energy very close to the infimum, then it is easy to see \((3.15)\) must be satisfied. Hence Theorem \[3\] applies to give the long-time existence of the standard form gauge and the convergence of the connections \(A(t)\) to a tangent connection \(A'_0\) as \(t \to \infty\). It is easy to see that any other tangent connection must be gauge equivalent to this \(A'_0\).

Next we proceed to show the rate of convergence for \(A(t) + \beta(t) \, dt \to A_0\) as \(t \to \infty\). Without loss of generality, we may assume \(R = 0\) and \(\tilde{A}_1 = \phi^* (\tilde{A})\) is the original connection on \(B_1(0) \setminus \{0\}\). Since \(\tilde{A}_1\) is stationary, by monotonicity formula,

\[
4 \int_{B_{\rho}(0)} r^{4-n} |\frac{\partial}{\partial r}| F_{\tilde{A}_1}|^2 \, dx
\leq \lim_{\sigma \to 0} \left\{ \int_{B_{\rho}(0)} \rho^{4-n} |F_{\tilde{A}_1}|^2 \, dx - \int_{B_{\frac{\rho}{2}}(0)} \sigma^{4-n} |F_{\tilde{A}_1}|^2 \, dx \right\}
= \lim_{\lambda_i \to 0} \left\{ \int_{B_{\rho}(0)} \rho^{4-n} |F_{\tilde{A}_1}|^2 \, dx - \int_{B_{\frac{\rho}{4}}(0)} \lambda_i^{4-n} |F_{\tilde{A}_1}|^2 \, dx \right\}
= \lim_{\lambda_i \to 0} \left\{ \int_{B_{\rho}(0)} \rho^{4-n} |F_{\tilde{A}_1}|^2 \, dx - \int_{B_{\frac{\rho}{2}}(0)} |F_{\tilde{A}_1}|^2 \, dx \right\}
= \int_{B_{\rho}(0)} \rho^{4-n} (|F_{\tilde{A}_1}|^2 - |F_{A_0}|^2) \, dx
\leq \frac{1}{n-4} \int_{\partial B_{\rho}(0)} \rho^{5-n} (|F_{\tilde{A}_1}|^2 - |F_{A_0}|^2) \, d\sigma \tag{3.42}
\]

where the last inequality follows from monotonicity formula \((1.10)\) and the fact that \(\frac{\partial}{\partial r}|F_{A_0}| = 0\). We note that there is no loss of curvature energy (on bounded sets) by Prop. \[2\]. Under our cylindrical coordinates, let \(\eta(t) = a(t) - d_A \beta(t) \, dt\) and \(T = -\log(\rho) \geq R\), by change of variables, \((3.42)\) becomes, for any \(T \in [R, R']\),

\[
\int_T^\infty ||\eta(t)||^2 \, dt \leq \frac{1}{n-4} \int_{S^{n-1}} |F_{A}(T) - \eta(T) dt|^2 - |F_{A_0}|^2 \, d\sigma \tag{3.43}
= C \int_{S^{n-1}} |F_{A}(T)|^2 - |F_{A_0}|^2 \, d\sigma + C ||\eta(T)||^2
\]

Since \(|A(t) - A_0|_{C^{k,\mu}} = |a(t)|_{C^{k,\mu}} \leq \tau\), we can apply Lemma \([12]\) to the right hand side of \((3.43)\) and obtain

\[
\int_T^\infty ||\eta(t)||^2 \, dt \leq C \|d^\prime_0 F_{A(T)}\|^\frac{1}{1-\theta} + C \|\eta(T)||^2 \tag{3.44}
\]

From \((1.18)\), we have

\[
||d^\prime_0 F_{A(T)}|| \leq ||\dot{a}(t)|| + ||\ddot{a}(t)|| + ||d_A \dot{\beta}(t)|| + ||d_A \beta(t)|| + C\tau ||\beta(t)|| \tag{3.45}
\]

\((2.23)\) implies that

\[
|\beta(t)|_{C^{k,\mu}} + |\dot{\beta}(t)|_{C^{k,\mu}} \leq C\tau |\dot{a}|_{C^{k,\mu}}. \tag{3.46}
\]
The elliptic estimates (3.10) implies that
\[ |\dot{a}(t)|_{C^{k,\mu}} + |\ddot{a}(t)|_{C^{k,\mu}} \leq C \|\dot{a}\|_{[t-1,t+1]} \]  
(3.47)

Applying (3.46) and (3.47) to the right hand side of (3.45) gives us
\[ \|d^*\dot{A}(t)F(t)\| \leq \|\ddot{a}(t)\| + \|\dot{a}(t)\| + C\tau |\dot{a}(t)|_{C^{k,\mu}} \leq C\|\dot{a}\|_{[t-1,t+1]} \]  
(3.48)

We have also from elliptic estimates for Sobolev norms applied to the equation \( d^*Ad_A\beta = -*[a,*\dot{a}] \), that
\[ \|d_A\beta(t)\| \leq C\tau \|\dot{a}(t)\| \]  
(3.49)

If \( \tau \) is small, (3.49) implies
\[ \frac{1}{2}\|\dot{a}(t)\| \leq \|\eta(t)\| \leq 2\|\dot{a}(t)\| \]  
(3.50)

Putting together (3.47), (3.48) and (3.50) and plugging in both sides (3.44), we have
\[ \int_t^\infty \|\dot{a}\|^2 ds \leq C(\int_{T-1}^{T+1} \|\dot{a}\|^2)^{\frac{1}{2(1-\theta)}} + C \int_{T-1}^{T+1} \|\dot{a}\|^2 \leq C(\int_{T-1}^{T+1} \|\dot{a}\|^2)^{\frac{1}{2(1-\theta)}} \]  
(3.51)

where \( \theta \in (0,\frac{1}{2}) \) depend only on \( A_0 \). (3.51) gives an integral decay estimate for \( \|\dot{a}\| \). Recall that \( |a(t)|_{C^{k,\mu}} \leq \tau \), for \( R < t < \infty \). Now it is an easy analytical exercise to show (for example, as in [16, 3.15]) that there exists \( T_1 > 0, \alpha > 0 \), such that
\[ \int_t^\infty \|\dot{a}(s)\|ds < C t^{-\alpha}, \quad \text{for } t \geq T_1 \]  
(3.52)

Therefore
\[ \|A(t) - A_0\| \leq C t^{-\alpha}, \quad \text{for } t \geq T_1 \]  
(3.53)

and by elliptic estimates,
\[ |\tilde{A}(t) - A_0|_{C^{k,\mu}} \leq C(k) t^{-\alpha}, \quad \text{for } t \geq T_1. \]  
(3.54)

The desired rate of convergence is obtained and the proof of Theorem 1 is finished.

With the convergence from Theorem 1 and the integrability assumption, the fast convergence of Theorem 2 is a well-known result (see for example the proof of Theorem 1 (i) in [2]). We remark here that a proof of Theorem 2 without using the variational inequalities in Section 3.3 is possible. In fact, the variational approach may be totally avoided in this case as in Cheeger and Tian [3], where integrability of the cone is assumed.
4 A result of existence and convergence for Yang-Mills flows

In this section we give an application of the previous methods to Yang-Mills flows. We shall show that a flow which starts from a connection sufficiently close (in smooth norms) to a smooth local minimizer of the Yang-Mills functional will converge asymptotically to a smooth Yang-Mills connection near the minimizer. Our method, like before, still consists of two steps, first we choose a suitable gauge, and then we use the result for parabolic evolution equations (Theorem 2) in [15].

Consider the following Yang-Mills flow equation for connections on bundle $E$ on Riemannian manifolds $M$

$$\frac{\partial}{\partial t} A(t) = -d_A^* F_A(t)$$

(4.1)

Idealistically, if (4.1) has a solution $A(t)$ on $[0, \infty)$, the limit of $A(t)$ at $\infty$ should be a Yang-Mills connection. Then this will give us a way to homotopically deform an arbitrary connection into a Yang-Mills connection and hopefully we can have a Morse theory suitably defined. However, the long-range existence of solutions of (4.1) as well as the existence and regularity of the limit in general are not at all obvious. Nonetheless, near a local minimizer of the Yang-Mills connection, we are able to show the flow does exist for all time and converges.

We first note that (4.1) is not parabolic due to the fact that $d_A^* F_A$ is not elliptic in $A$. As before, we hope to use the Coulomb gauge to make the equation parabolic. We note that (4.1) actually implies

$$d_A^* (\dot{A}) = 0,$$ \hspace{1cm} (4.2)

where $\dot{A} = \frac{\partial}{\partial t} A(t)$. This follows from $d_A^* d_A F_A = 0$ by (3.32).

Assume $A_0$ is a fixed smooth Yang-Mills connection, we have the following theorem now. Fix $l$ integer such that $H^l(S^n-1) \subset C^{3,\mu}(S^{n-1})$.

**Theorem 4** There exists $\varepsilon = \varepsilon(A_0) > 0$, $\alpha = \alpha(A_0) > 0$ such that for any given smooth $a_0 \in \Omega^1(g_E)$ with $\|a_0\|_{H^{l+2}} < \varepsilon$, there is a $T_s > 0$ and $A(t)$, a $C^\infty(M \times [0, T_s])$ solution of (4.1) satisfying $A(0) = A_0 + a_0$, $\sup_{[0,T_s]} \|A(t) - A_0\|_{H^l} < \varepsilon^\alpha$ and either

$$T_s < \infty \text{ and } \lim_{t \uparrow T_s} YM(A(t)) \leq YM(A_0) - \varepsilon$$

(4.3)

or

$$T_s = \infty \text{ and } \lim_{t \to \infty} (|\dot{A}(t)|_{C^1} + |A(t) - A_1|_{C^2}) = 0$$

(4.4)

where $A_1$ is a smooth Yang-Mills connection on $M$.

Assume $I$ is an interval of possibly infinite length and $A(t) + \beta(t)dt$ is a smooth connection on $E \times I$, where $A(t)$ are connections on $E$ and $\beta(t) \in \Gamma(g_E)$. We also assume that under a gauge transformation $g \in \Gamma(\text{Aut } P \times I),$

$$g(A(t) + \beta(t)dt) = A_1(t)$$

(4.5)
where \( A_1(t) \) are connections on \( E \) and satisfies \((\text{4.1})\) and hence \((\text{4.2})\) for \( t \in I \). From \((\text{4.5})\), we obtain,

\[
\begin{align*}
A_1 &= gAg^{-1} - dg\ g^{-1} & (\text{4.6}) \\
\frac{\partial}{\partial t}g &= g\beta & (\text{4.7})
\end{align*}
\]

We observe that \((\text{4.6})\) implies that \( A_1 \) and \( A \) are gauge equivalent connections on \( M \). Substitute \((\text{4.6})\) and \((\text{4.7})\) into \((\text{4.1})\) and \((\text{4.2})\), by straightforward computation, we obtain the following equations for \( A \) and \( \beta \)

\[
\dot{A} = -d_A^*F_A + d_A\beta
\]

\[
d_A^*(\dot{A} - d_A\beta) = 0
\]

We shall view \((\text{4.8})\) and \((\text{4.9})\) (which is implied by \((\text{4.8})\)) as equations for the connection \( A + \beta dt \) on \( M \times I \). It is easy to see that they are gauge invariant equations for gauge transformations on bundle \( E \times I \), i.e., if \( g \in \Gamma(\text{Aut} \ P \times I) \), \( g(A + \beta dt) = A_1 + \beta_1 dt \) then \( A_1 + \beta_1 dt \) also satisfies \((\text{4.8})\) and \((\text{4.9})\). This point of view enables us to consider as before a standard form of \( A + \beta dt \) around a connection \( A_0 \) on \( E \) and make the system \((\text{4.8})\) parabolic in \( A \) and \((\text{4.9})\) elliptic in \( \beta \). If \( A + \beta dt = A_0 + a(t) + \beta(t)dt \) is under such a standard form, i.e. \( d_{A_0}^*a = 0 \) and \( \beta \in \text{Ker}(d_{A_0})^\perp \), then we may solve \( \beta \) from \((\text{4.9})\) and by substituting in \((\text{4.8})\), rewrite \((\text{4.8})\) as

\[
\dot{a} = -d_{A_0}^*F_{A_0} + d_{A_0}(a) + d_AG_A([a, *\dot{a}])
\]

where \( G_A = (\Delta_A)^{-1} : \text{Im}(d_A^*) \to \text{Ker}(d_{A_0})^\perp \).

From above, we observe that instead of proving Theorem \( \text{4} \) in terms of \((\text{4.1})\), it suffices to prove the same conclusions hold for \( A(t) = A_0 + a(t), a(t) \in \text{Ker}(d_{A_0}^*) \) with \( a(t) \) being solution to \((\text{4.10})\) with initial value \( a_0 \) (up to a gauge transformation, we may assume \( d_{A_0}^*a_0 = 0 \)). For if we prove the latter, a solution to \((\text{4.1})\) may be obtained by

\[
\tilde{A}(t) = g(A(t) + \beta(t)), \quad \frac{\partial}{\partial t}g = g\beta
\]

where \( \beta = G_A([a, *\dot{a}]) \). We can show that \( g(t) \to g_0 \) for some \( g_0 \) and \( \dot{g} \to 0 \) in \( C^k \) as \( t \to \infty \), therefore \( \tilde{A}(t) \) will have limit at infinity \( g_0(\tilde{A}_1) \) if \( A_1 \) is the limit of \( A(t) \) and the same conclusions hold for \( \tilde{A} \).

After this observation, notice that \((\text{4.10})\) is essentially in the form of equation \((0.1)\) in the details here. We have the following obvious corollary from Theorem \( \text{4} \).

**Corollary 1** If \( A_0 \) is a smooth local minimizer of Yang-Mills functional on \( E \). Then there exists \( \varepsilon = \varepsilon(A_0) > 0 \) and \( \alpha = \alpha(A_0) > 0 \) such that for any given smooth \( a_0 \in \Omega^1(\mathcal{E}) \) with \( \|a_0\|_{H^{1+2}} < \varepsilon \), there is a \( A(t) \), a \( C^\infty(M \times [0, \infty)) \) solution of \((\text{4.4})\) satisfying \( A(0) = A_0 + a_0 \), and

\[
\lim_{t \to \infty} (|\dot{A}(t)|_{C^1} + |A(t) - A_1|_{C^2}) = 0
\]

where \( A_1 \) is a smooth Yang-Mills connection on \( M \) with \( |A_1|_{H^1} \leq \varepsilon^\alpha \).

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References

[1] W. K. Allard and F. Almgren, On the radial behavior of minimal surfaces and the uniqueness of their tangent cones, *Ann. of Math.* (2) **113** (1981), 215–265.

[2] D. Adams and L. Simon, Rates of asymptotic convergence near isolated singularities of geometric extrema, *Indiana Univ. Math. J.* **37** (1988), 225–254.

[3] J. Cheeger and G. Tian, On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay, *Invent. Math.* **118** (1994), 493–571.

[4] S. K. Donaldson and P. B. Kronheimer, “The geometry of four-manifolds”, Oxford University Press, New York, 1990.

[5] D. S. Freed and K. K. Uhlenbeck, “Instantons and four-manifolds”, Math. Sci. Res. Inst. Publ., 1, Springer-Verlag, New York, 1984.

[6] E. Giusti, “Minimal surfaces and functions of bounded variation”, Monographs in Mathematics, 80, Birkhäuser Verlag, 1984.

[7] D. Gilbarg and N. S. Trudinger, “Elliptic partial differential equations of second order”, Second edition, Springer-Verlag, New York, 1983.

[8] R. Harvey and H. B. Lawson, Calibrated geometries, *Acta Math.* **148** (1982), 47–157.

[9] J. W. Morgan, T. Mrowka and D. Ruberman, “The $L^2$-moduli space and a vanishing theorem for Donaldson polynomial invariants”, International Press, Cambridge, MA, 1994.

[10] C. B. Morrey, “Multiple integrals in the calculus of variations”, Springer-Verlag, New York, 1966.

[11] H. Nakajima, Compactness of the moduli space of Yang-Mills connections in higher dimensions, *J. Math. Soc. Japan* **40** (1988), 383-392.

[12] P. Price, A monotonicity formula for Yang-Mills fields, *Manuscripta Math.* **43** (1983), 131-166.

[13] R. M. Schoen, Analytic aspects of the harmonic map problem, in “Seminar on nonlinear partial differential equations” (S. S. Chern, Ed.), pp. 321–358, Springer, New York, 1984.

[14] R. Schoen and K. Uhlenbeck, A regularity theory for harmonic maps, *J. Differential Geom.* **17**, No. 2 (1982), 307–335.

[15] L. Simon, Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems, *Ann. of Math.* (2) **118**, No. 3 (1983), 525–571.

[16] L. Simon, “Theorems on regularity and singularity of energy minimizing maps”, Lecture in Mathematics, ETH Zürich, 1996.

[17] L. Simon, Isolated singularities of extrema of geometric variational problems, in “Lecture Notes in Math. 1161”, pp. 206-277, Springer, 1985.

[18] L. Simon, “Lectures on geometric measure theory”, Australian National University, 1983.

[19] R. Schoen and S. T. Yau, “Lectures on harmonic maps”, International Press, Cambridge, MA, 1997.

[20] G. Tian, Gauge Theory and Calibrated Geometry, I, *Ann. Math.* (2) **151** (2000), 193-268.

[21] T. Tao and G. Tian, A singularity removal theorem for Yang-Mills fields in higher dimensions, preprint, 2001, available at [http://math.stanford.edu/~byang](http://math.stanford.edu/~byang).
[22] G. Tian and B. Yang, Compactification of the moduli spaces of vortices and coupled vortices, preprint, 2001, accepted by *J. Reine Angew. Math.*

[23] K. K. Uhlenbeck, Removable singularities in Yang-Mills fields, *Comm. Math. Phys.* **83**, No.1 (1982), 11–29.

[24] K. K. Uhlenbeck, Connections with $L^p$ bounds on curvature., *Comm. Math. Phys.* **83**, No. 1 (1982), 31–42.

[25] B. Yang, Construction of Yang-Mills connections with given asymptotic tangent cones, preprint, 2001, available at [http://math.stanford.edu/~byang](http://math.stanford.edu/~byang)

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