From parabolic to loxodromic gravity

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Abstract

Half of the Bondi-Metzner-Sachs (BMS) transformations consist of orientation-preserving conformal homeomorphisms of the extended complex plane known as fractional linear (or Möbius) transformations. These can be of 4 kinds, i.e. they are classified as being parabolic, or hyperbolic, or elliptic, or loxodromic, depending on the number of fixed points and on the value of the trace of the associated $2 \times 2$ matrix in the projective version of the $SL(2, \mathbb{C})$ group. The resulting particular forms of $SL(2, \mathbb{C})$ matrices affect also the other half of BMS transformations, and are used here to propose 4 realizations of the asymptotic symmetry group that we call, again, parabolic, or hyperbolic, or elliptic, or loxodromic. Within this framework it turns out that those fractional linear transformations of elliptic type which are periodic generate the BMS supertranslations. Moreover, an isomorphism exists between two copies of the subgroup of parabolic transformations and the 4-translations, whereas rotations and boosts correspond to elliptic and hyperbolic transformations, respectively.

In the last part of the paper, we prove that a subset of hyperbolic and loxodromic transformations, those having trace that approaches $\infty$, correspond to the fulfillment of limit-point condition for singular Sturm-Liouville problems. Thus, a profound link may exist between the language for describing asymptotically flat space-times and the world of complex analysis and self-adjoint problems in ordinary quantum mechanics.
I. INTRODUCTION

The asymptotic symmetry group of asymptotically flat space-times, originally discovered thanks to the work of Bondi, Metzner and Sachs (hereafter BMS), is still part of modern research in gravitational physics, thanks to the theoretical investigation of black hole physics, Hamiltonian methods and symmetry groups of various space-time models.

In particular, when a generic metric tensor is expressed in Bondi-Sachs coordinates \((u,r,x^B)\), it reads as

\[
g = e^{2\beta} \frac{V}{r} du \otimes du + e^{2\beta} (du \otimes dr + dr \otimes du) + \sum_{B,C=1}^{2} (dx^B - U^B du) \otimes (dx^C - U^C du). \tag{1.1}
\]

With this notation, \(x^B\) consists of the pair of coordinates \((\theta, \phi)\), and the BMS transformations are the diffeomorphisms which do not affect the asymptotic form of the metric of an asymptotically flat space-time. At a deeper level, the group of conformal transformations of future null infinity which preserve both angles and null angles is the BMS group.

Upon defining the complex stereographic coordinate

\[
\zeta \equiv e^{i\phi} \cot \frac{\theta}{2}, \tag{1.2}
\]

and considering the \(SL(2,\mathbb{C})\) matrix

\[
M_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \tag{1.3}
\]

the general form of a BMS transformation is well known to be (\(\overline{\zeta}\) being the complex conjugate of \(\zeta\))

\[
\zeta \rightarrow \frac{a\zeta + b}{c\zeta + d}, \quad (1.4)
\]

\[
u \rightarrow \frac{(1 + |\zeta|^2)[u + A(\zeta, \overline{\zeta})]}{|a\zeta + b|^2 + |c\zeta + d|^2}. \tag{1.5}
\]

Now the simple but non-trivial point of our paper is that the maps \((1.4)\) are the fractional linear transformations whose properties are well known in complex analysis and group theory, but have not been fully exploited by relativists and theoretical physicists. Section 2 outlines the well-established classification of maps \((1.4)\), while in Section 3 it is shown that an important correspondence exists between this classification and the usual translations,
rotations and boosts that allows to see the whole of the Poincaré group from a new perspective. Section 4 exploits this classification to obtain 4 basic forms of the maps (1.5), that we call parabolic, hyperbolic, elliptic, loxodromic. Section 5 obtains an intriguing link between fractional linear transformations and the geometry of singular Sturm–Liouville problems, while concluding remarks are presented in Section 6, and technical details are provided in the appendix.

II. FRACTIONAL LINEAR TRANSFORMATIONS

In complex analysis [28], a fractional linear transformation is an orientation-preserving conformal homeomorphism \( h \) of the extended complex plane \( \mathbb{C} \cup \{\infty\} \), such that
\[
h(z) = \frac{\tilde{\alpha} z + \tilde{\beta}}{\tilde{\gamma} z + \tilde{\delta}}, \quad \forall t \neq 0.
\] (2.1)
Since the ratio in (2.1) is independent of \( t \), one can exploit this to make sure that, eventually, the matrix of coefficients has unit determinant (we define \( \alpha \equiv t \tilde{\alpha}, \ldots, \delta \equiv t \tilde{\delta} \)), i.e.
\[
t^2 (\tilde{\alpha} \tilde{\delta} - \tilde{\beta} \tilde{\gamma}) = 1 \implies \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2, \mathbb{C}),
\] (2.2)
with associated
\[
h(z) = \frac{\alpha z + \beta}{\gamma z + \delta}.
\] (2.2)
The fixed points of \( h \) solve, by definition, the equation \( h(z) = z \), i.e., from (2.2)
\[
\gamma z^2 + (\delta - \alpha)z - \beta = 0,
\] (2.3)
which is solved by
\[
z = \frac{(\alpha - \delta) \pm \sqrt{(\alpha + \delta)^2 - 4(\alpha \delta - \beta \gamma)}}{2} = \frac{(\alpha - \delta) \pm \sqrt{(\alpha + \delta)^2 - 4}}{2},
\] (2.4)
where we have exploited the \( SL(2, \mathbb{C}) \) condition.

Thus, if
\[
(\alpha + \delta)^2 = 4 \implies |\alpha + \delta| = 2,
\] (2.5)
there exists only one fixed point \( z = \frac{(\alpha - \delta)}{2} \), and the map \( h \) is said to be parabolic. Now a theorem guarantees that every parabolic transformation can be mapped into a transformation whose only fixed point is at \( \infty \), i.e. \([28, 30]\)
\[
h_P(z) = z + \beta.
\] (2.6)
This is a translation in $\mathbb{C} \cup \{\infty\}$, and is not periodical. The representative matrix in $PSL(2, \mathbb{C})$ is

$$M_P = \begin{pmatrix} \pm 1 & \beta \\ 0 & \pm 1 \end{pmatrix}. \quad (2.7)$$

If the discriminant $(\alpha + \delta)^2 - 4$ in (2.4) does not vanish, two fixed points are instead found to occur, and the resulting homeomorphism $h$ can be mapped into a transformation with fixed points at 0 and $\infty$, i.e. $[29, 30]$

$$h(z) = \frac{\alpha}{\delta} z = k z, \quad \alpha \delta = 1. \quad (2.8)$$

Hence one finds

$$\frac{\alpha}{\delta} = \alpha^2 = k \implies \alpha = \sqrt{k} \implies j = \text{tr}(h) = \alpha + \delta = \alpha + \frac{1}{\alpha} = \sqrt{k} + \frac{1}{\sqrt{k}}. \quad (2.9)$$

Now one can distinguish three cases:

(i) If $k = |\kappa| > 0$, $h$ is said to be hyperbolic, and for it

$$h(z) = h_H(z) = |\kappa| z,$$

$$\text{tr}^2(h) - 4 = (\alpha + \delta)^2 - 4 > 0 \implies |\alpha + \delta| > 2, \quad (2.10)$$

and the corresponding matrix in $PSL(2, \mathbb{C})$ is

$$M_H = \begin{pmatrix} |\kappa| & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.11)$$

Equation (2.10) for $h_H(z)$ is a dilation of the plane, and under its action all lines passing through the origin remain fixed.

(ii) If $k \in \mathbb{C}$ and $|k| = 1$, one can write

$$k = e^{i\varphi} \implies j = e^{i\frac{\varphi}{2}} + e^{-i\frac{\varphi}{2}} = 2 \cos \frac{\varphi}{2}. \quad (2.12)$$

The transformation $h$ is then said to be elliptic, and for it

$$\text{tr}^2(h) = (\alpha + \delta)^2 < 4 \implies |\alpha + \delta| < 2, \quad (2.13)$$

$$h(z) = h_E(z) = e^{i\varphi} z, \quad (2.14)$$
with resulting matrix in $PSL(2, \mathbb{C})$

$$M_E = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.15)$$

The map $(2.14)$ can be periodic provided that there exists a natural number $l$ such that $\varphi = 2\pi l$. A normal elliptic [29] transformation $(2.14)$ is a rotation of the complex plane about the origin, with amplitude $\varphi$, and for it all circles centred at the origin remain fixed.

(iii) If $k = \rho e^{i\sigma} \in \mathbb{C}$, one finds from $(2.9)$

$$j = \left( \sqrt{\rho} + \frac{1}{\sqrt{\rho}} \right) \cos \frac{\sigma}{2} + i \left( \sqrt{\rho} - \frac{1}{\sqrt{\rho}} \right) \sin \frac{\sigma}{2} \in \mathbb{C}, \quad (2.16)$$

while if $k$ is $< 0$ one finds, again from $(2.9)$,

$$j = \frac{1 - |k|}{i\sqrt{|k|}} \in \mathbb{C}. \quad (2.17)$$

The resulting transformation $h_L$ is said to be loxodromic, and it reads as

$$h(z) = h_L(z) = \rho e^{i\sigma} z, \quad (2.18)$$

with corresponding matrix in $PSL(2, \mathbb{C})$

$$M_L = \begin{pmatrix} \rho e^{i\sigma} & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.19)$$

A loxodromic transformation is obtained by combining an elliptic and a hyperbolic transformation, with the same fixed points [29]. Some authors say that a non-elliptic transformation with exactly two fixed points is loxodromic, and that these include the hyperbolic transformations [30].

In figure 1 we describe the various families of fractional linear transformations in the complex-$j^2$ plane.

III. CONNECTION WITH THE POINCARÉ GROUP

The aim of this Section is to show that the classification of fractional linear transformations carried out in Sec. II has a deep relation with special relativity. In particular, we would like to show that the parabolic, hyperbolic and elliptic transformations are strictly
FIG. 1: If the squared trace $j^2$ is used for the classification, fractional linear transformation are elliptic if $j^2 < 4$, parabolic if $j^2 = 4$, hyperbolic if $j^2 > 4$, and loxodromic if $j^2$ is not real-valued.

related to the usual concepts of translations, boosts and rotations respectively, which are the building blocks of the Poincaré group. Hence, before proceeding to the classification of the BMS group that, as mentioned, is the asymptotic symmetry group of asymptotically flat space-times, we regard it interesting to see explicitly such a connection with flat space-time symmetry group. Let us start with translations.

Consider two parabolic transformations $h$ and $h'$, with parameters $\beta$ and $\delta$:

$$z' = h_P(z) = z + \beta, \quad (3.1a)$$

$$w' = h'_P(w) = w + \delta. \quad (3.1b)$$

Then, since $\mathbb{C} \simeq \mathbb{R}^2$, Eqs. (3.1a) and (3.1b) induce the four-translation

$$x'^\mu = x^\mu + a^\mu, \quad (3.2)$$

where $z = x^0 + ix^1$, $w = x^2 + ix^3$, $\beta = a^0 + ia^1$ and $\delta = a^2 + ia^3$. Hence, as we claimed, there exists an isomorphism between two copies of the subgroup of parabolic transformations and the 4-translations.

Now, turn attention to rotations and boosts. It is well known that, in virtue of the isomorphism $L^\dagger_+ \simeq SL(2, \mathbb{C})/\mathbb{Z}_2$, where $L^\dagger_+$ denotes the connected component of $SO(1, 3)$, a rota-
tion of an angle $\phi$ and a boost of rapidity $\chi$ about an axis $\hat{n} = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)$ can be performed by using $SL(2, \mathbb{C})$ matrices

$$U_{\hat{n}}(\phi) = e^{\frac{i}{2} \hat{\sigma} \cdot \hat{n} \phi} = \mathbb{I} \cos(\phi/2) + i(\hat{\sigma} \cdot \hat{n}) \sin(\phi/2), \quad (3.3a)$$

$$H_{\hat{n}}(\chi) = e^{\frac{i}{2} \hat{\sigma} \cdot \hat{n} \chi} = \mathbb{I} \cosh(\chi/2) + (\hat{\sigma} \cdot \hat{n}) \sinh(\chi/2). \quad (3.3b)$$

Note that also $-U_{\hat{n}}$ and $-H_{\hat{n}}$ can be used as well.

We can take the traces to obtain

$$j_U \equiv \text{tr}(U_{\hat{n}}(\phi)) = 2 \cos(\phi/2) \implies j_U^2 \leq 4, \quad (3.4a)$$

$$j_H \equiv \text{tr}(H_{\hat{n}}(\chi)) = 2 \cosh(\chi/2) \implies j_H^2 \geq 4. \quad (3.4b)$$

Then it follows that rotations correspond to elliptic transformations and that boosts correspond to hyperbolic transformations. The limit cases in which in Eqs. (3.4) there holds the equality ($\phi = 2k\pi$, where $k \in \mathbb{Z}$, and $\chi = 0$) correspond again to parabolic transformations. We can eventually state that, from the above analysis, the whole Poincaré group can be expressed in terms of fractional linear transformations.

IV. FROM PARABOLIC TO LOXODROMIC BMS TRANSFORMATIONS

As is clear from (1.4), the choice of coefficients $a, b, c, d$ plays a role also in the other half of BMS transformations, given by Eq. (1.5). Thus, bearing in mind the matrices (2.7), (2.11), (2.15) and (2.19), our next logical step is to write Eq. (1.5) in the form

$$u \rightarrow F(\zeta, \bar{\zeta}) \cdot \left[ u + A(\zeta, \bar{\zeta}) \right], \quad (4.1)$$

where we distinguish 4 cases as follows.

(I) Parabolic BMS:

$$\zeta \rightarrow \pm \zeta + \beta, \quad (4.2)$$

$$u \rightarrow F_P \cdot \left[ u + A \right], \quad F_P = \frac{(1 + |\zeta|^2)}{[1 + |\pm \zeta + \beta|^2]}, \quad (4.3)$$

(II) Hyperbolic BMS:

$$\zeta \rightarrow |\kappa| \zeta, \quad (4.4)$$
\[ u \to F_H [u + A], \quad F_H = \frac{(1 + |\zeta|^2)}{(1 + |\kappa|^2 |\zeta|^2)}. \] (4.5)

(III) Elliptic BMS:

\[ \zeta \to e^{i\varphi} \zeta, \quad (4.6) \]

\[ u \to F_E [u + A], \quad F_E = \frac{1 + |\zeta|^2}{|\zeta e^{i\varphi}|^2 + 1} = 1. \] (4.7)

Interestingly, Eqs. (4.6) and (4.7) imply that an elliptic fractional linear transformation engenders a supertranslation of the \( u \) variable. A full supertranslation would require also

\[ \theta \to \theta, \phi \to \phi, \]

whereas (4.6) implies that \( \phi \to \phi + \varphi \). However, bearing in mind that \( \varphi \sim \varphi + 2\pi \), we obtain that a full supertranslation is generated by a periodic elliptic transformation. In fact, under such a transformation we also have \( \zeta' = e^{2\pi i} \zeta = \zeta \) and hence \( A(\zeta', \bar{\zeta}') = A(\zeta, \bar{\zeta}) \) (provided, of course, that \( A \) is single-valued).

(IV) Loxodromic BMS:

\[ \zeta \to \rho e^{i\sigma} \zeta, \quad (4.8) \]

\[ u \to F_L [u + A], \quad F_L = \frac{(1 + |\zeta|^2)}{(1 + \rho^2 |\zeta|^2)}. \] (4.9)

We remark the complete formal analogy between \( F_H \) and \( F_L \), in agreement with the previously stated (but apparently unrelated) property, according to which loxodromic transformations include the hyperbolic family [30].

V. FRACTIONAL LINEAR TRANSFORMATIONS AND SINGULAR STURM-LIOUVILLE PROBLEMS

Since our paper is aimed at finding links between well-established but apparently unrelated branches of mathematics and physics, we here consider the fractional linear transformations that pertain to singular Sturm-Liouville problems, and then try to exploit the classification studied in Section 2.
A linear homogeneous second-order differential equation in one real variable $x$ can be always reduced to the (canonical) form \[ L Q u(x) = 0, \quad L Q \equiv -\frac{d^2}{dx^2} + Q(x), \] while a regular Sturm-Liouville problems consists of an eigenvalue equation for the linear operator $L_q \equiv -\frac{d^2}{dx^2} + q(x)$, i.e.

\[ L_q u(x) = \lambda u(x) \quad x \in [a, b], \tag{5.2} \]

where $q$ is a suitably smooth potential term, supplemented by the following boundary conditions at the ends of the interval (hereafter $\omega \in [0, \pi$ and $\eta \in [0, \pi]$):

\[ u(a) \cos(\omega) + u'(a) \sin(\omega) = 0, \tag{5.3} \]
\[ u(b) \cos(\eta) + u'(b) \sin(\eta) = 0. \tag{5.4} \]

For the problem (5.2)-(5.4), there exists a discrete spectral resolution of the one-dimensional Laplace-type operator $L_q$, with a countable infinity of real eigenvalues, and the eigenfunctions forming a complete basis for the space $L^2(a, b)$.

The theory of singular Sturm-Liouville problems deals with the case where the closed interval $[a, b]$ is replaced by the interval $[0, \infty]$, by studying a sequence of intervals $[a, b]$ as $b \to \infty$. The pioneering work of Weyl made it possible to prove the following results \[33, 34\]:

**Theorem 1.**

If $\text{Im}(\lambda) \neq 0$, and $\chi, \psi$ are the linearly independent solutions of the eigenvalue equation $L_q u = \lambda u$ which satisfy

\[ \chi(0, \lambda) = \sin(\omega), \quad \chi'(0, \lambda) = -\cos(\omega), \tag{5.5} \]
\[ \psi(0, \lambda) = \cos(\omega), \quad \psi'(0, \lambda) = \sin(\omega), \tag{5.6} \]

then there exists a function $m$ of $\lambda$ such that, for all $\lambda \in \mathbb{C} - \mathbb{R}$, the linear combination

\[ y(x, \lambda) = \chi(x, \lambda) + m(\lambda)\psi(x, \lambda) \tag{5.7} \]

satisfies the real boundary condition

\[ y(b) \cos(\eta) + y'(b) \sin(\eta) = 0. \tag{5.8} \]
Upon defining
\[ A \equiv \chi(b, \lambda), \quad B \equiv \chi'(b, \lambda), \quad C \equiv \psi(b, \lambda), \quad D \equiv \psi'(b, \lambda), \quad (5.9) \]
\[ z \equiv \cot \eta, \quad (5.10) \]
the boundary condition (5.8) leads to the fractional linear transformation
\[ m = -\frac{Az + B}{Cz + D}, \quad (5.11) \]
and hence
\[ z = -\frac{Dm + B}{Cm + A}, \quad (5.12) \]
where \( AD - BC = 1 \) by virtue of Theorem 1, as is proved in Ref. [34]. The reality condition for the variable \( z \) defined in (5.10) is therefore the equation of a circle \( C_b \):
\[ |m|^2 - \frac{(BC - \bar{A}D)}{(CD - CD)}m - \frac{(A\bar{D} - B\bar{C})}{(CD - CD)}m + \frac{(\bar{A}B - A\bar{B})}{(CD - CD)} = 0, \quad (5.13) \]
having centre
\[ M_b = \frac{A\bar{D} - B\bar{C}}{(CD - CD)}, \quad (5.14) \]
and radius
\[ r_b = \frac{1}{|CD - \bar{CD}|}, \quad (5.15) \]
The following theorem holds:

**Theorem 2.**
As \( b \) approaches \( \infty \), either \( C_b \rightarrow C_\infty \), a limit circle, or \( C_b \rightarrow m_\infty \), a limit point. In the limit circle case, all solutions of the eigenvalue equation \( L_q u = \lambda u \) are of class \( L^2(0, \infty) \). If \( \text{Im}(\lambda) \neq 0 \), only one solution of \( L_q u = \lambda u \) is of class \( L^2(0, \infty) \) in the limit-point case.

At this stage, on the right-hand side of Eq. (5.11) we multiply numerator and denominator by a non-vanishing complex number \( \tau \), and obtain
\[ m = \frac{\alpha z + \beta}{(\gamma z + \delta)}, \quad (5.16) \]
where
\[ \alpha \equiv -\tau A, \quad \beta \equiv -\tau B, \quad \gamma \equiv \tau C, \quad \delta \equiv \tau D. \quad (5.17) \]
Thus, in light of the condition $AD - BC = 1$, the matrix on the right-hand side of (5.16) is in $SL(2, \mathbb{C})$ if and only if $\tau = \pm i$. With the resulting form of the matrix, i.e.

$$
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} = \begin{pmatrix}
\mp iA & \mp iB \\
\pm iC & \pm iD
\end{pmatrix},
$$

we can exploit the classification of section 2 from which, for the trace $j \equiv \alpha + \delta$, we obtain the correspondence \cite{28, 29}

- $j \in \mathbb{R}$, $|j| < 2 \implies$ elliptic,
- $j \in \mathbb{R}$, $|j| = 2 \implies$ parabolic,
- $j \in \mathbb{R}$, $|j| > 2 \implies$ hyperbolic,
- $j \in \mathbb{C} - \mathbb{R} \implies$ loxodromic.

From Eq. (5.15), the limit-point case, for which the radius $r_b$ of the circle $C_b$ approaches 0, reduces to

$$
|\bar{C}D - C\bar{D}| = |\bar{\gamma} \delta - \gamma \bar{\delta}| \to \infty,
$$

i.e.

$$
|\text{Im}(\bar{\gamma} \delta)| \to \infty \implies |\text{Im}(\bar{\alpha} - \bar{j})| \to \infty.
$$

This shows that the loxodromic or hyperbolic sectors, which contain also fractional linear transformations having $|j| \to \infty$, may lead to shrinkage to zero of $r_b$ and hence to the limit-point condition. Alternatively, one might assume that $|\gamma| \to \infty$, but this is incompatible with the desire of having square-integrable eigenfunctions of the operator $L_q$, as is clear from (5.9) and (5.17).

On the other hand, as far as the limit-circle condition is concerned, all elliptic and parabolic fractional linear transformations are acceptable.

Although the established correspondence between singular Sturm-Liouville problems and fractional linear transformations is not in the 1 − 1 form, it shows (in our opinion) an intriguing link between the modern theory of ordinary differential equations on the one hand, and fractional linear and BMS transformations on the other hand.

The limit-point condition considered in Section 5 is of particular interest because, for eigenvalue problems on $(0, \infty)$, the limit-point condition at 0 and $\infty$ is the necessary and sufficient condition (see appendix) for proving essential self-adjointness on $C_0^\infty(0, \infty)$ of the
radial part of the quantum mechanical Hamiltonian in a central potential \([35, 36]\). In other words, relying upon separately well-established properties of real, complex and functional analysis on the one hand and asymptotic structure of general relativity on the other hand, we are suggesting that a profound link may exist between self-adjoint problems in ordinary quantum mechanics and the fractional linear and hence BMS transformations of general relativity in the large-trace loxodromic and hyperbolic sectors.

Further applications of projective \(SL(2, \mathbb{C})\) transformations have been discovered in Ref. \([37]\), where the authors study three-dimensional anti-de Sitter space and find that the conformal boundary is acted upon precisely by elements of \(PSL(2, \mathbb{C})\).

VI. CONCLUDING REMARKS AND OPEN PROBLEMS

As far as we can see, a synthesis of our findings is as follows.

The fractional linear transformations (hereafter FLT) of complex analysis are the appropriate tool for expressing several concepts, i.e.

(i) The 4-translations, rotations and boosts of special relativity correspond to 2 copies of parabolic FLT, or elliptic FLT, or hyperbolic FLT, respectively.

(ii) The parabolic through loxodromic FLT engender the bigger group of parabolic through loxodromic BMS transformations. In particular, the FLT which are both elliptic and periodic generate BMS supertranslations.

(iii) The limit-circle condition of singular Sturm-Liouville problems corresponds to elliptic and parabolic FLT, plus loxodromic and hyperbolic FLT with finite trace \(j\). The limit-point condition of singular Sturm-Liouville problems corresponds instead to loxodromic and hyperbolic FLT having trace that approaches \(\infty\).

This means that, for example, a Hamiltonian with equal but non-vanishing deficiency indices \([35]\) in ordinary quantum mechanics is “dual” to elliptic and parabolic BMS transformations, including BMS supertranslations. On the other side, an essentially self-adjoint Hamiltonian is “dual” to the large-trace subset of loxodromic and hyperbolic BMS transformations.

If the mathematical language of asymptotically flat space-times is naturally “dual” to the singular Sturm-Liouville problems of ordinary quantum mechanics, the important question
arises of whether the language of full general relativity is the unexpected gateway to the world of full quantum field theory, and which tools replace FLT in the affirmative case.

In a different framework, an example of gateway between general relativity and quantum fields is provided by the massless Rarita-Schwinger equations. When these are studied in curved space-time, the integrability conditions for finding gauge-invariant solutions of such equations lead to Ricci flatness, which is equivalent to solving the vacuum Einstein equations with the exception of two-dimensional space-time. Thus, maybe a new perspective in theoretical physics might be the task of establishing correspondences between different areas of classical and quantum physics, rather than the attempt of quantizing or dequantizing. We hope that the resulting landscape awaiting discovery, if it exists, will become accessible in the years to come. Another related question is whether a rigorous theory of discrete gravity can be developed with the help of discrete subgroups of the group of all FLT, with the related parabolic, elliptic, hyperbolic and loxodromic sectors.

Appendix A: The Weyl limit-point limit-circle criterion for self-adjointness

The function $V$ is in the limit-circle at 0 if for some $\lambda$, and therefore all $\lambda$, all solutions of the equation

$$\left[-\frac{d^2}{dx^2} + V(x)\right] \varphi(x) = \lambda \varphi(x)$$  \hspace{1cm} (A1)

are square-integrable at 0.

If $V$ is not in the limit-circle case at 0, it is said to be in the limit-point at 0.

According to the Weyl limit-point limit-circle criterion, if $V$ is a continuous real-valued function on the open interval $(0, \infty)$, the operator

$$P \equiv -\frac{d^2}{dx^2} + V(x)$$  \hspace{1cm} (A2)

is essentially self-adjoint on $C_0^\infty(0, \infty)$ (i.e. it is closable and its closure is self-adjoint therein) if and only if $V$ is in the limit-point case both at 0 and at $\infty$.

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