On quantum and parallel transport 
in a Hilbert bundle over spacetime

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Abstract

We study the Hilbert bundle description of stochastic quantum mechanics in curved spacetime developed by Prugovečki, which gives a powerful new framework for exploring the quantum mechanical propagation of states in curved spacetime. We concentrate on the quantum transport law in the bundle, specifically on the information which can be obtained from the flat space limit. We give a detailed proof that quantum transport coincides with parallel transport in the bundle in this limit, confirming statements of Prugovečki. We furthermore show that the quantum-geometric propagator in curved spacetime proposed by Prugovečki, yielding a Feynman path integral-like formula involving integrations over intermediate phase space variables, is Poincaré gauge covariant (i.e. is gauge invariant except for transformations at the endpoints of the path) provided the integration measure is interpreted as a “contact point measure” in the soldered stochastic phase space bundle raised over curved spacetime.

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1. Introduction

In this paper we investigate a new approach to quantum mechanics and quantum field theory in curved spacetime, developed by Prugovečki, which differs from the standard approach in two central ways. The first of these is that it uses the (special-) relativistic “stochastic” quantum mechanics developed by Prugovečki, which is based on a system of covariance for the Poincaré group based on positive operator valued measures of a phase space representation, rather than a system of imprimitivity constructed with projector valued measures [1] [2] (and references therein). The second major difference comes in the extension to curved spacetime, which uses a fibre bundle over spacetime whose standard fibre is the usual flat-space Hilbert or Fock space of the system considered [2] [3] [4] (and references therein). This allows the definition of a particle to be given in terms of the properties of the fibres above the points in spacetime, independently of the curvature of spacetime and of the acceleration of any individual observer, or of other global properties of the spacetime base. The curvature of spacetime plays its role in the propagation of states from the fibre above one point to that above another.

In the current paper we are specifically concerned with this “quantum propagation” in the bundle over spacetime, guided by the information which can be gained from the flat-space limit. Reference [4] postulates a Feynman path integral-like formula for the “quantum propagator” in curved spacetime. We show that this quantum-geometric propagator defined in [4] has the correct flat space limit and is Poincaré gauge covariant in curved spacetime (i.e. is gauge invariant except for transformations at the endpoints of the path). To obtain this result we use the invariance of the integration measure at the contact points of base space and fibre at the intermediate phase space integration points involved.

Standard problems in relativistic quantum mechanics include the choice of the Hilbert space, the definition of position and momentum operators, and the construction of a position probability density. In the case of a spinless particle in Minkowski space, the Hilbert space is conventionally chosen to be the positive frequency solutions of the Klein-Gordon equation. It is well known that the conserved current constructed from such a solution cannot be used to define a position probability density for the particle, because its “time” component is not positive definite. Newton and Wigner [5] constructed position and momentum operators, whose spectral resolution may be used to define probability densities on any constant-time hypersurface (in an inertial coordinate system). However, as is also well known, the resulting position probability densities do not obey relativistic causality. Recently Wald [6] has discussed the choice of the Hilbert space for a particle in a curved spacetime and background fields, emphasising its arbitrariness, and has given a generalisation of the Newton-Wigner operators to such cases. In quantum field theory in curved space time, there is an observer-dependent ambiguity in the vacuum state, and a corresponding ambiguity in the definition of a particle [7]. A celebrated consequence of this is the Unruh effect [8], which predicts that an observer accelerating in Minkowski space will see a thermal bath of particles. As this effect has yet to be observed experimentally, there remains the possibility that it is an artifact of inadequacies in the standard quantisation procedure.
In pioneering work, Prugovečki has sought to remove the incompatibilities between quantum mechanics and relativity, by incorporating the approximate nature of any physical measurement into quantum mechanics at a fundamental level \cite{1,2} (and references therein). The resulting stochastic quantum mechanics on phase space has significant advantages over the usual formulation, for example, it gives a covariant, conserved current whose time component is positive definite and so can be interpreted as a position probability density. In extending this approach to quantum mechanics and quantum field theory in curved spacetime, Prugovečki has introduced the Hilbert and Fock bundles over spacetime \cite{2,3,4}, where the fibre above each point of spacetime is a copy of the flat-space, stochastic Hilbert or Fock space, respectively. In the first-quantised theory, this avoids the problems of defining the Hilbert space and operators appropriate to curved spacetime. It also means that the probability density from the flat-space theory can be used to construct a probability density in curved spacetime (see section 3 below). There is however the new question of how a state in the fibre above any point is to propagate to a fibre above another point, to which the present paper is addressed. In geometro-stochastic quantum field theory, the use of the Fock bundle implies that there is a unique vacuum state in the fibre above each point, and a unique definition of a particle; the ambiguities of the usual approach are thereby eliminated.

The bundle approach can also be applied to the conventional formulations of quantum mechanics and quantum field theory simply by taking the standard fibre to be the conventional flat-space Hilbert or Fock space, respectively. In doing this one loses the advantages peculiar to Prugovečki’s stochastic quantum mechanics, such as the existence of a position probability density, but the other quoted advantages of using the bundle are preserved.

The recent work of Graudenz \cite{9} is also based on the use of a Fock bundle over spacetime. However it differs from the present work in its physical interpretation of the states and its treatment of their propagation.

In the present paper we consider specifically the first-quantized theory of a scalar particle of mass $m$ propagating on an arbitrary but fixed background spacetime. (The background gravitational field is neither quantized, nor is it altered by any back-reaction from the particle.) Our aim is to obtain information about the quantum propagation law on the Hilbert bundle over spacetime guided by the flat space limit of the theory. In this limit, we require the bundle description to coincide with Prugovečki’s (special-) relativistic stochastic quantum mechanics, which for convenience we will often refer to as the non-bundle description. Thus we take the non-bundle description as our reference point, and, although we summarise this theory, our aim is not to give a critical assessment of it.

The contents of the paper are as follows. In section 2 we give a self-contained summary of the relevant structures and properties of the relativistic stochastic quantum mechanical description of a free, spinless particle of mass $m$. We also introduce the Hilbert bundle structure used in the attempt to generalise this theory to a curved background spacetime. In section 3 we summarise Prugovečki’s prescription for the physical interpretation of the bundle formulation. In section 4 we discuss quantum transport on the Hilbert bundle, de-
scribing it in terms of maps between the fibres above pairs of points in spacetime. We define and discuss the quantum mechanical propagator obtained from these maps, thus making contact with the approach of [2] and [4]. We consider the flat space limit, and by comparing with the non-bundle description we deduce the required quantum transport law and propagator in this limit. In section 5 we discuss parallel transport on the Hilbert bundle, and show that in the flat space limit this coincides with quantum transport. This result is implicit in certain statements of reference [4] (e.g., the discussion following eq.(4.6.8)), but does not appear to have been explicitly stated or proven. In section 6 we obtain an integral representation for the quantum transport propagator in the flat space limit. We compare this with the flat-space limit of the quantum geometric propagator in curved spacetime defined in eq.(4.6.7a,b) of [4], verifying that they agree. In section 7 we make some remarks concerning the interpretation of the measure appearing in Prugovečki’s work, verifying in detail that the quantum propagator in curved spacetime is defined in a Poincaré gauge covariant manner. In section 8 we indicate the modifications to the foregoing needed to apply the bundle formalism to the conventional description of a Klein-Gordon particle. Section 9 contains the summary and conclusions.

Our notation is broadly compatible with that of references [2] and [4]. However we do not follow the convention of using boldface and normal fonts to distinguish certain objects represented by the same letter, since with some fonts this can be unclear. Rather we use an underline to make such distinctions.

2. Relativistic stochastic quantum mechanics on phase space and the Hilbert bundle over spacetime

We consider a spinless particle of mass $m$ propagating in Minkowski space. Let $k = (k^i) ∈ \mathbb{R}^4$, $i = 0, 1, 2, 3$, be the vector of components of the particle’s 4-momentum in an inertial frame. In terms of these components the forward mass hyperboloid $V^+_m$ is given by

$$V^+_m = \{ k ∈ \mathbb{R}^4 | k^2 = m^2, k^0 > 0 \},$$

where $k^2 = k.k = k_i k^i = (k^0)^2 - ((k^1)^2 + (k^2)^2 + (k^3)^2)$. We let $V^+$ denote $V^+_m$ in the case $m = 1$.

The quantum description of this particle is based on the Hilbert space $L^2(V^+_m)$, consisting of complex functions $\tilde{\psi}$ on $V^+_m$ which are square integrable with respect to the usual invariant measure

$$d\Omega_m(k) = d^3k/(2k^0).$$

We let $d\Omega(k)$ denote $d\Omega_m(k)$ in the case $m = 1$.

The phase space of the particle is a subspace of the cotangent bundle over Minkowski space, which can be identified with a subspace of the tangent bundle, isomorphic to $\mathbb{R}^4 \times V^+_m$, i.e., the set of all pairs $(q, p)$, where $q ∈ \mathbb{R}^4$ are inertial coordinates on Minkowski space, and $p ∈ V^+_m$ as above [10]. We will work with the coordinates $(q, v)$, where $v = p/m ∈ V^+$.
is the vector of components of the particle’s 4-velocity. Following Prugovečki, it will be convenient to represent the point \((q, v)\) by the complex variable

\[ \zeta = q - i\ell v \in \mathbb{R}^4 \times iV^+ \subset \mathbb{C}^4, \tag{3} \]

where \(\ell\) is a real, positive constant.\(^1\)

Prugovečki gives a formulation of the quantum theory of the particle in which \(L^2(V_m^+)\) is mapped to a Hilbert space \(F\) of functions on phase space \([1]\) p.90, \([2]\) pp.82,138, \([3]\) eq.(2.2), \([4]\). That is, \(F\) consists of all functions \(\psi\) on phase space of the form

\[ \psi(\zeta) = \tilde{Z}_{\ell,m}^{-1/2} \int_{V_m^+} e^{-i\zeta \cdot k} \tilde{\psi}(k) d\Omega_m(k), \tag{4} \]

where \(\tilde{Z}_{\ell,m}\) is a real constant depending on \(\ell\) and \(m\), and \(\tilde{\psi} \in L^2(V_m^+)\). These wavefunctions satisfy the Klein-Gordon equation

\[ (\partial_\mu \partial^\mu + m^2)\psi(\zeta) = 0, \tag{5} \]

where \(\partial_\mu = \partial/\partial q^\mu\). The inner product on \(F\) is defined by

\[ \langle \psi_1 | \psi_2 \rangle = \int_{\Sigma^+} \psi_1^*(\zeta) \psi_2(\zeta) d\Sigma(\zeta). \tag{6} \]

This integral is carried out over any surface of the form

\[ \Sigma^+ = \sigma \times V^+, \tag{7} \]

where \(\sigma\) is an arbitrary Cauchy surface in Minkowski space \([1]\) p.113, \([2]\) pp.87,140. The measure may be written as

\[ d\Sigma(\zeta) = 2v^i d\sigma_i(q) d\Omega(v), \tag{8} \]

where \(d\sigma_i(q)\) are the components of the volume element on \(\sigma\).\(^2\)

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1 In Prugovečki’s formulation of geometro-stochastic quantum mechanics, \(\ell\) is a fundamental non-zero length scale, introducing a certain fuzziness in position and momentum resolution, which has a regularizing effect. Prugovečki suggests equating \(\ell\) with the Planck length, although for consistency with current particle physics data any length of the order of \(10^{-16}\) cm or smaller would be acceptable.

2 The different forms of \([4]\) given in the cited references are reconciled by the values of the overall normalisation coefficients used. Note that we follow \([4]\) here, in particular in the adoption of \(e^{-\ell v \cdot k}\) for the function playing the role of the momentum space resolution generator \(\eta\) in \([4]\) eq.(2.4.8) and \(f\) in \([2]\) eq.(5.1.2). This choice is made for convenience; our analysis is independent of the particular form adopted for the resolution generator.

3 The normalisation constant \(\tilde{Z}_{\ell,m}\) in \([4]\) is chosen so that the inner products on \(L^2(V_m^+)\) and \(F\) agree, under the map given by \([4]\).
The Poincaré group, ISO\(_0(3,1)\), is the set of all \(g = (b, \Lambda)\), where \(b \in \mathbb{R}^4\) and \(\Lambda \in \text{SO}_0(3,1)\), with the product
\[
(b, \Lambda)(b', \Lambda') = (b + \Lambda b', \Lambda\Lambda') .
\] (9)

Thus, for example,
\[
g^{-1} = (-\Lambda^{-1}b, \Lambda^{-1}) .
\] (10)
The action of ISO\(_0(3,1)\) on Minkowski space is given by
\[
(b, \Lambda)q = \Lambda q - b ,
\] (11)
where \(q \in \mathbb{R}^4\) and the minus sign in front of \(b\) arises because of the adopted frame transformation given in eq.(22) below. It will be convenient to use the same action of \(g\) on complex vectors such as \(\zeta = q - i\ell v\), i.e.,
\[
(b, \Lambda)\zeta = \Lambda \zeta - b = \Lambda q - b - i\ell \Lambda v .
\] (12)
The Hilbert space \(\mathcal{F}\) carries a unitary, irreducible, spin zero, phase-space representation \(U(g) = U(b, \Lambda)\) of the Poincaré group by \([1]\) p.92
\[
(U(g)\psi)(\zeta) = \psi(g^{-1}\zeta) .
\] (13)

There exists in \(\mathcal{F}\) a unique, rotationally invariant wavefunction \(\eta\), called the (phase space) resolution generator of the representation \([1]\) p.106, \([2]\) p.82. Let \(\Lambda_v\) be the Lorentz boost to the frame of an observer moving with 4-velocity \(v\). For all \(\zeta = q - i\ell v \in \mathbb{R}^4 \times iV^+\), define
\[
\phi_\zeta = U(-q, \Lambda_v)\eta .
\] (14)
Then
\[
\psi(\zeta) = \langle \phi_\zeta | \psi \rangle \quad \forall \psi \in \mathcal{F} .
\] (15)
The states \(\phi_\zeta\) constitute an overcomplete basis for \(\mathcal{F}\) (a coherent state basis). They give the following resolution of the identity operator \(1_\mathcal{F}\) on \(\mathcal{F}\):
\[
1_\mathcal{F} = \int_{\Sigma^+} |\phi_\zeta\rangle d\Sigma(\zeta) \langle \phi_\zeta | .
\] (16)

\(K^{(\ell)}\) defined by
\[
K^{(\ell)}(\zeta', \zeta) = \langle \phi_\zeta' | \phi_\zeta \rangle
\] (17)
is the propagator on \(\mathcal{F}\), i.e.,
\[
\psi(\zeta') = \int_{\Sigma^+} K^{(\ell)}(\zeta', \zeta)\psi(\zeta)d\Sigma(\zeta) .
\] (18)

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\(^4\) In the detailed analysis of this reference, \(\eta\) is the phase space resolution generator corresponding to the momentum space resolution generator \(\tilde{\eta} \in L^2(V^+_m)\) via \(\eta = \mathcal{W}_\eta\tilde{\eta}\), where \(\mathcal{W}_\eta\) is the map defined by \([3]\). Note that our \(\eta\) and \(\tilde{\eta}\) correspond respectively to \(\eta\) and \(\tilde{\eta}\) in this reference.
For later reference, we consider the action of $U(g)$ on $\phi_\zeta$, where $g = (b, \Lambda) \in \text{ISO}_0(3,1)$:

$$U(b, \Lambda)\phi_\zeta = U(b, \Lambda)U(-q, \Lambda_v)\eta = U(b - \Lambda q, \Lambda \Lambda v)\eta. \quad (19)$$

Using the facts that there exists a rotation $\Lambda_R$ (Wigner rotation) such that $\Lambda \Lambda v = \Lambda \Lambda v \Lambda_R$ (recall that $\Lambda \Lambda v$ means the boost of 4-velocity $\Lambda v$), and that $\eta$ is rotationally invariant, this gives

$$U(g)\phi_\zeta = U(-\Lambda q - b, \Lambda \Lambda v)\eta = \phi_{\Lambda q - b - i\ell \Lambda v} = \phi_{g \zeta}, \quad (20)$$

where (14) and (12) were used.

Prugovečki gives a physical interpretation of a state $\psi \in F$ as the probability amplitude for simultaneous measurements of the stochastic position and stochastic momentum of the particle [1] p.97, [2] pp.84,151,156. The amplitude squared $|\psi(\zeta)|^2$ of $\psi$ is to be interpreted as the probability density for the mean stochastic position $q$ and the mean stochastic momentum $mv$ of the particle. More precisely, the integral of this density over any Borel set in the space $\{q - i\ell v \mid q \in \sigma, v \in V^+\}$, where $\sigma$ is a Cauchy surface in Minkowski space, gives the probability that a measurement of the stochastic position and momentum at the “time” defined by $\sigma$ will give a result lying in this Borel set.

The integral over all momenta of $|\psi(\zeta)|^2$ gives a positive-definite density on configuration space, which is the zeroth component of a conserved current [1] p.111, [2] p.87. Thus this density can indeed be interpreted as a conserved, configuration space probability density. This of course is quite different from the usual treatment of the relativistic particle via the Klein-Gordon equation, where there is no conserved, positive-definite density on configuration space.

In the present work, we are primarily concerned with comparing the above description of the relativistic particle with the Hilbert bundle description summarised below. For this purpose we will compare $\psi(\zeta)$ with the corresponding quantity in the bundle description, specialising, however, to the flat space case. Equality of these objects is certainly sufficient to ensure the agreement of the physical predictions made by the two formulations.

We now summarise the Hilbert bundle structure used in the programme to generalise the above to the case of a particle moving on a curved spacetime [2] §5.1, [4] §§4.1-2. Let $(M, g^L)$ be a Lorentzian 4-manifold. A Poincaré frame $s$ for $T_x M$, $x \in M$, is of the form $s = (a^i, e_i)$, $i = 0, 1, 2, 3$, where $a \in T_x M$ and $(e_i)$ is an orthonormal frame (a Lorentz frame) for $T_x M$. The Poincaré frame bundle $PM$ over $M$ is the set of all $(x, s)$, where $x \in M$ and $s$ is a Poincaré frame for $T_x M$. $PM$ is a principal fibre bundle, having as its structure group the Poincaré group $G = \text{ISO}_0(3,1)$. Let $a =^e \in \mathbb{R}^4$ be the vector of
components \( a^i; \ i = 0, 1, 2, 3 \), and let \( \mathbf{a} \) be defined, following \[11\] by
\[
\mathbf{a} = -a^i e_i .
\] (21)

Then the right action \( R_{g^{-1}} \) of \( g^{-1} = (-\Lambda^{-1} b, \Lambda^{-1}) \in \text{ISO}_0(3,1) \) on \( PM \) is
\[
R_{g^{-1}}(x, \mathbf{a}, e_i) = (x, \Lambda a + b, e_j (\Lambda^{-1})^j_i) ,
\] (22)
where \( \Lambda a + b = -(\Lambda a + b)^i e'_i \), with \( e'_i = e_j (\Lambda^{-1})^j_i \).

Let the Hilbert bundle \( \mathcal{H} \) over \( M \) be a fibre bundle associated to \( PM \), having as its typical fibre the 1-particle Hilbert space \( \mathcal{F} \). \( \mathcal{H} \) may be viewed as the \( G \)-product \( PM \times_G \mathcal{F} \) of \( PM \) and \( \mathcal{F} \), i.e., the quotient of \( PM \times \mathcal{F} \) by an action of \( G \), defined as follows. Let \( x \in M, s \) be a Poincaré frame for \( T_x M \), and \( \psi \in \mathcal{F} \), so \( (x, s) \in PM \) and \( (x, s, \psi) \in PM \times \mathcal{F} \). An action of \( \text{ISO}_0(3,1) \) on \( PM \times \mathcal{F} \) is defined by
\[
(x, s, \psi) \mapsto (R_{g^{-1}}(x, s), U(g)\psi) \quad \forall g \in \text{ISO}_0(3,1) .
\] (23)

Let \([ (x, s, \psi) ] \) be the equivalence class of \((x, s, \psi)\) under this action. The equivalence classes \([ (x, s, \psi) ] \) are the elements of \( \mathcal{H} \). The projection \( \pi \) on \( \mathcal{H} \) is the obvious one,
\[
\pi : \mathcal{H} \to M \quad \text{by} \quad \pi : [(x, s, \psi)] \mapsto x .
\] (24)

For all \( x \in M \) and all Poincaré frames \( s \) for \( T_x M \), we can define the maps
\[
\sigma^s_x : \pi^{-1}(x) \to \mathcal{F} \quad \text{by} \quad \sigma^s_x : [(x, s, \psi)] \mapsto \psi \quad \forall \psi \in \mathcal{F} .
\] (25)

Note that each equivalence class in \( \pi^{-1}(x) \) has exactly one element \((x, s', \psi)\) such that \( s' = s \), so \( \sigma^s_x \) is indeed defined and single-valued on all of \( \pi^{-1}(x) \). Let \( s \) and \( s' \) be two Poincaré frames for \( T_x M \) and \( g \in \text{ISO}_0(3,1) \) such that
\[
(x, s') = R_{g^{-1}}(x, s) .
\] (26)

It follows from (23) and (25) that
\[
\sigma^{s'}_x = U(g) \cdot \sigma^s_x .
\] (27)

A Poincaré frame \( s \) for \( T_x M \) associates to each \( \Psi_x \in \pi^{-1}(x) \) an element \( \Psi^s_x \) of the standard fibre \( \mathcal{F} \) by
\[
\Psi^s_x = \sigma^s_x \Psi_x .
\] (28)

The minus sign in (21) is due to the fact that \( \mathbf{a} = (a^i) \) is a vector in \( T_x M \) pointing towards the origin of the frame \((e_i)\) since the components \( a^i \) parametrize an active translation of this frame as given by eq.(22) in the text. Eq.(21) then defines a vector \( \mathbf{a} \) pointing from the origin of the Poincaré frame \( s = (a, e_i) \) to the point of contact between \( T_x M \) and \( M \). Thus it is opposite to the convention adopted in \[2\] and \[4\] for the quantity denoted by boldface \( a \). For all other vectors \( \mathbf{v} \) referred to the frame \((e_i)\) we write \( \mathbf{v} = v^i e_i \).
Ψ_s^x is called the wavefunction of the state Ψ_x in the Poincaré gauge s. This induces an inner product on the fibres of the bundle ℋ by

\[ \langle \Psi'_x | \Psi_x \rangle = \langle \Psi'^s_x | \Psi^s_x \rangle \quad \forall \Psi'_x, \Psi_x \in \pi^{-1}(x) , \]  

where s is any Poincaré frame for T_xM, and the inner product on the right hand side is the one on the standard fibre defined by equation (1). It follows from (27) that \( \langle \Psi'_x | \Psi_x \rangle \) is independent of the choice of the frame s appearing in (29).

For all \( x \in M \), all Poincaré frames s for \( T_xM \), and all \( \zeta = q - i\ell v \in \mathbb{R}^4 \times iV_+ \) define

\[ \Phi^s_x(\zeta) = (\sigma^s_x)^{-1} \phi_\zeta \quad (\in \pi^{-1}(x)) , \]

where \( \phi_\zeta \) is a basis state in \( \mathcal{F} \) defined by (14). Combining (15), (28) and (30), it follows that the value of the wavefunction \( \Psi^s_x(\zeta) \) (corresponding to the state \( \Psi_x \) in the Poincaré gauge s) at \( \zeta \) is given by

\[ \Psi^s_x(\zeta) = \langle \Phi^s_x | \Psi_x \rangle \quad \forall \Psi_x \in \pi^{-1}(x) . \]

From the properties of the coherent states \( \phi_\zeta \), it follows that the \( \Phi^s_x \) for any s constitute an overcomplete basis for \( \pi^{-1}(x) \), and from (16) they give the following resolution of the identity map \( 1_x \) on \( \pi^{-1}(x) \):

\[ 1_x = \int_{\Sigma} |\Phi^s_\zeta \rangle d\Sigma(\zeta) \langle \Phi^s_\zeta | . \]

Let \( x, s, s' \) and \( g \) be as in (20). The gauge-dependence of the \( \Phi^s_\zeta \) is shown by

\[ \Phi^{s'}_\zeta = (\sigma^{s'}_x)^{-1} \phi_\zeta = (\sigma^s_x)^{-1} U(g^{-1}) \phi_\zeta = (\sigma^s_x)^{-1} \phi_{g^{-1}\zeta} = \Phi^s_{g^{-1}\zeta} , \]

where (24) and (20) were used.

3. Physical interpretation of sections of the Hilbert bundle

In the Hilbert bundle description, the physical state of the 1-particle system is described by a section \( \Psi \) of the bundle \( \mathcal{H} \), i.e.,

\[ \Psi : M \to \mathcal{H} , \quad \Psi : x \mapsto \Psi_x \in \pi^{-1}(x) . \]

Thus this description involves a vector \( \Psi_x \) in the fibre \( \pi^{-1}(x) \) above each point \( x \in M \), whereas in the non-bundle description the state is described by a single vector \( \psi \in \mathcal{F} \). This apparent oversupply of information contained in such a section is removed in its physical interpretation, because each state \( \Psi_x \) determines the stochastic phase space probability amplitude only at the corresponding base point \( x = \pi(\Psi_x) \).
The phase space of the particle is (up to a factor of $m$ in the momentum) the set of all pairs $(x,v)$, where $x \in M$ and $v \in V_x^+$, where $V_x^+$ is the forward 4-velocity hyperboloid at $x$, i.e.,
\[ V_x^+ = \{ v \in T_x M \mid g^L(v,v) = 1, \text{ v future directed} \} . \]

The phase space probability amplitude $\Psi(x,v)$ predicted by the section $\Psi$ is defined as follows [2] p.156. Let $s$ be a section of $PM$, i.e.,
\[ s : M \to PM \quad \text{by} \quad x \mapsto s(x) = (\underline{a}(x), e_i(x)) . \]

Thus $s$ provides a Poincaré gauge choice $s(x)$ at each $x \in M$, and we shall also refer to the section $s$ as a choice of gauge. Let $a(x) = (a^i(x)) \in \mathbb{R}^4$ be the vector of components $a^i(x)$ associated to $\underline{a}(x)$ as defined in (27), and let $v = (v^i) \in V^+$ be the vector of components of $\underline{v}$ defined by $\underline{v} = v^i e_i(x)$ (see footnote 5). Then the phase space probability amplitude predicted by $\Psi$ is
\[ \Psi(x,v) = \Psi_{x}^{s(x)}(\hat{\zeta}(x)) = (\sigma_{x}^{s(x)}(\Psi_{x}))(\hat{\zeta}(x)) , \]

where
\[ \hat{\zeta}(x) = -a(x) - i\ell v \in \mathbb{R}^4 \times iV^+ , \]
with $q^i = -a^i(x)$ denoting the point of contact in $T_x M$. It follows from (22), (27) and (28) that $\Psi(x,v)$ is independent of the choice of the gauge $s$ appearing in (27). Analogously to the non-bundle description, the amplitude squared $|\Psi(x,v)|^2$ of $\Psi(x,v)$ would be interpreted as the probability density on phase space.

A result of Prugovečki (discussion in [3] below eq.(3.5) and [4] p.178) states in the current context that if $\Psi_{x}^{s(x)}(-a(x) - i\ell v)$ for all $v \in V^+$ is fixed, then $\Psi_{x}^{s(x)}(q - i\ell v)$ for all $q \in \mathbb{R}^4$ is determined by analyticity. Combining this with (37), we see that there is actually a one-to-one correspondence between sections of $H$ and phase space probability amplitudes, i.e., not only does $\Psi_x$ determine $\Psi(x,v)$ via (37), but also $\Psi(x,v)$ (for all $v$) determines $\Psi_x$ by the result just quoted.

Prugovečki also gives a stronger probability interpretation of a section $\Psi_x$, [4] eq.(4.2.16), where $\Psi_x$ determines the relative probability density $\Psi(x',v)$ at points $x'$ near to $x$, up to some accuracy related to the curvature of $(M,g^L)$. This interpretation is stronger because it requires an approximate consistency between $\Psi_x$ and $\Psi_{x'}$ at nearby points $x$ and $x'$. We will not base what follows on this interpretation, although we will make a comment on it at the end of the next section.

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6 For convenience, we make the usual assumption that $PM$ admits a global section. This is physically desirable since it is associated with the existence of a spin structure on spacetime.
7 There is an interpretation, which we will not use, of $\Psi_x$ as a (gauge-independent) function on the space $T_x M \times V_x^+$, i.e., $\Psi(q,v) \in \mathbb{C}$, where $q \in T_x M$ and $v \in V_x^+$ [4] 4.2. In terms of this interpretation, (37) takes the form $\Psi(x,v) = \Psi_x(q(x),v)$, i.e., $\Psi(x,v)$ is obtained by restricting $\Psi_x$ to the point of contact $\underline{q} = \underline{a}(x)$ between $T_x M$ and $M$.
8 This is based on the interpretation of $\Psi_x$ as a function on $T_x M \times V_x^+$ mentioned in footnote 7. It uses the exponential map at $x$ to associate $x'$ with some $\underline{q} \in T_x M$, and then determines $\Psi(x',v)$ from the restriction of $\Psi_x$ to this value of $\underline{q}$.
4. Quantum transport in the Hilbert bundle

Sections 5.4 of [2] and 4.6 of [4] postulate explicit path integral-like formulae for “geometro-stochastic” and “quantum-geometric” forms of the quantum propagator in the bundle over curved spacetime. In this section we give a rather general definition of quantum transport and the associated propagator, without however postulating any explicit form in curved spacetime, and then go on to consider their specific forms in the flat space case.

In the non-bundle description, the propagation of the wavefunction in time is governed by the Klein-Gordon equation. This is supposed to be paralleled in the Hilbert bundle formalism by a law for the propagation of a state from the fibre above one point in the base to the fibre above another point, called quantum transport. That is, for pairs of points \( x', x \in M \), there is a map

\[
Q(x', x) : \pi^{-1}(x) \rightarrow \pi^{-1}(x') , \quad Q(x', x) : \Psi_x \mapsto Q(x', x)\Psi_x ,
\]

which implements quantum mechanical propagation in the bundle description. Thus, given an “initial” state \( \Psi_x \) in the fibre above some point \( x \), the maps \( Q(x', x) \) can be used to propagate this state to other points \( x' \), building up a section \( \Psi \) of \( \mathcal{H} \) (refer to equation (34)) by

\[
\Psi_{x'} = Q(x', x) \Psi_x .
\]

No path dependence in \( Q(x', x) \) for a path joining \( x' \) and \( x \) will be assumed, nor will any particular single path be considered. A priori we might expect that the \( Q \) maps will not be defined for all pairs \( (x', x) \), but that for causality reasons they might only be defined when \( x' \) is in the causal future of \( x \). In this case the resulting section \( \Psi \) would only be defined on part of \( M \). We will return to this point below.

The quantum transport propagator \( K^s(x', \zeta'; x, \zeta) \) from \( x \) to \( x' \) in the arbitrary gauge \( s \) is defined by

\[
K^s(x', \zeta'; x, \zeta) = \langle \Phi_{\zeta'}^{s(x')} | Q(x', x) | \Phi_{\zeta}^{s(x)} \rangle ,
\]

i.e., this is the matrix element of the map \( Q(x', x) \) between the fibre basis states \( \Phi_{\zeta'}^{s(x')} \in \pi^{-1}(x') \) and \( \Phi_{\zeta}^{s(x)} \in \pi^{-1}(x) \) defined in (33). Taking the inner product of both sides of (40) with \( \Phi_{\zeta'}^{s(x')} \) and using (31), (32) and (41) we find

\[
\Psi_{x'}^{s(x')} (\zeta') = \int_{\Sigma^+} K^s(x', \zeta'; x, \zeta) \Psi_x^{s(x)} (\zeta) d\Sigma(\zeta) ,
\]

i.e., \( K^s(x', \zeta'; x, \zeta) \) is the quantum propagator from \( x \) to \( x' \) for the wavefunction of the state in the gauge \( s \). (Note that since this expression is written in terms of wavefunctions,
i.e., elements of the standard fibre $F$, the measure $d\Sigma(\zeta)$ is that defined by \footnote{\text{[33]}}\footnote{\text{[11]}}\footnote{\text{[2] p.43, \text{[4]} eq.(4.1.13). (Compare also \text{[33]} and \text{[11]} were used. Thus the quantum propagator is dependent on the gauge $s$ only at the points $x'$ and $x$.}

For the rest of this section we will concentrate on the special case where the spacetime base $(M, g^L)$ is Minkowski space. In this case, the physical interpretation of a section of $\mathcal{H}$ given by \footnote{\text{[33]}}\footnote{\text{[11]}} implies a restriction on the sections of $\mathcal{H}$ which are allowed. This is because in flat space, the Hilbert bundle description must reproduce the set of possible phase space probability densities given by the non-bundle description. That is, the only allowed are those which predict an amplitude $\Psi(x, v)$ which corresponds to one of the states $\psi \in F$. This in turn gives information on the quantum transport law in the case when the base spacetime is Minkowski space, because the sections of $\mathcal{H}$ which it generates via \footnote{\text{[10]}} must satisfy this restriction. In the remainder of this section we give the corresponding quantum transport law and discuss some of its properties.

Since the base manifold $(M, g^L)$ is Minkowski space, we can introduce inertial coordinates. From now on we will use $x = (x^i) \in \mathbb{R}^4$ to denote the coordinates of a point in $M$ in some inertial coordinate system. At each point $x \in M$ we define an orthonormal frame $(e^L_i(x))$ by

$$e^L_i(x) = \frac{\partial}{\partial x^i} \bigg|_x$$

This of course is just the natural frame in our chosen coordinates $x$, and $e^L_i(x)$ is of course equal to the parallel transport of $e^L_i(x)$ from some fixed point $O \in M$ to $x$. Define the section $s_L$ of $PM$ by

$$s_L(x) = (a_L(x) = x^ie^L_i(x), e^L_i(x)) \quad \forall x \in M.$$  

Let us insert some general remarks at this place. One can obtain any frame $s(x) = (a(x), e_i(x))$ in $T_xM$ by a Poincaré transformation applied to the frame \footnote{\text{[2]}} $\mathcal{S} = (0, e^L_i(O))$ in $T_O M$ according to

$$(x, s(x)) = R_{g(x)^{-1}}(O, \mathcal{S})$$

\footnote{\text{One can consider $x \in \mathbb{R}^4$ to be the vector of components $x^i$ of a point $x \in M$ in a basis $(e_i(x))$ for $M$, considered as a vector space with origin $O$. The frame $(e_i(O))$ for $M$ is naturally identified with the Poincaré frame $\mathcal{S} = (0, e^L_i(O))$, and hence with the frames $s_L(x) = (x^ie^L_i(x), e^L_i(x))$ for all $x$, which all have their origins displaced to the point $O$. \text{[2] p.43, \text{[3]} eq.(4.1.13). (Compare also footnote 5 in this context.)}}$
where $\tilde{g}_{s(x)}^{-1} = (x + a(x), \Lambda(x))$. Eq. (47) is similar to (22) or (43), however, when the spacetime base is Minkowskian the Poincaré frames associated with different base points (here $O$ and $x$) can also be related by Poincaré transformations (denoted by a tilde) with translations given by $\tilde{a}(x) = x + a(x)$.

Let us write $\tilde{g}_{s(x)}$ as a product $\tilde{g}_{s(x)} = g_{\Lambda(x)} \tilde{g}_{(x + a(x))}$ composed of a pure translation given by $(x + a(x), 1)$ and a pure Lorentz transformation given by $(0, \Lambda(x))$. We can then construct a frame in all the tangent spaces $T_xM$ which is parallel to the fixed Lorentz frame $e^L(O)$ at $O \in M$ and define a corresponding “parallel” Poincaré gauge by

$$ (x, s||(x)) = R_{g_{(x + a(x))}}^{-1} (O, \delta) .$$

(48)

For $a(x) = a_L(x) = -x$ this yields the global Lorentz gauge (46) on $PM$. However, the most general Poincaré frame $s(x) = (a(x), e_i(x))$ in $T_xM$ and the corresponding general gauge on $PM$ is obtained from the parallel gauge by an arbitrary $x$-dependent smooth $\Lambda$-transformation in $x \in M$

$$ (x, s(x)) = R_{\tilde{g}_{\Lambda(x)}}^{-1} (x, s||((x)))$$

(49)

generating the most general section of $PM$ in the Minkowski case.

A physically acceptable quantum propagation law is now defined by

$$ Q(x', x) = (\sigma^s_{x'}(x'))^{-1} \sigma^s_x(x) .$$

(50)

The proof that these $Q$ maps do generate an acceptable section $\Psi$ of $\mathcal{H}$ from any given “initial” state $\Psi_x \in \pi^{-1}(x)$ is as follows. The value $\Psi_{x'}$ of $\Psi$ at $x'$ is

$$ \Psi_{x'} = Q(x', x) \Psi_x = (\sigma^s_{x'}(x'))^{-1} \sigma^s_x(x) \Psi_x = (\sigma^s_{x'}(x'))^{-1} \sigma^s_{x'}(x') \Psi_{x'} .$$

(51)

where (28) was used. Inserting the choice $s = s_L$ in (37), we find that the phase space probability amplitude predicted by this section is

$$ \Psi(x', v) = (\sigma^s_{x'}(x'))(\tilde{\zeta}(x')) = \Psi^s_x(x)(\tilde{\zeta}(x')) ,$$

(52)

where (51) was used, and where in the gauge $s = s_L$ (38) gives

$$ \tilde{\zeta}(x') = -a_L(x') - i\ell v = x' - i\ell v .$$

(53)

Hence we find that

$$ \Psi(x', v) = \Psi^s_x(x)(x' - i\ell v) ,$$

(54)

i.e., the phase space probability amplitude predicted by $\Psi$ corresponds to the state $\Psi^s_x(x) \in \mathcal{F}$ in the non-bundle description. Thus $\Psi$ does indeed satisfy the requirement that it should correspond to a state in the non-bundle description, as discussed in the paragraph following
equation (14). Further, because of the observation at the end of the previous section that there is a one-to-one correspondence between phase space probability amplitudes and sections of $\mathcal{H}$, the quantum transport law defined by (50) is the unique law which satisfies this requirement of compatibility with the non-bundle description.

We note that this flat-space quantum transport law satisfies the composition law

$$Q(x', x) = Q(x', x_1).Q(x_1, x),$$

(55)

for any $x_1 \in M$. Substituting this and (52) in (51) gives

$$K^s(x', \zeta'; x, \zeta) = \int_{\Sigma^+} K^s(x', \zeta'; x_1, \zeta_1) K^s(x_1, \zeta_1; x, \zeta) d\Sigma(\zeta_1),$$

(56)

for any $x_1 \in M$.

We now derive an expression for the quantum transport propagator defined by (41), for the quantum transport law defined by (50). For the flat space case, it would be sufficient from $s_L$ by a general Poincaré transformation, i.e., both by a Lorentz transformation of the frame $(e^L_i(x))$ and by a shift of its origin in $T_xM$. Hence, let $s$ be an arbitrary gauge as in (17) or (19), and let $g_s(x) = (b(x), \Lambda(x)) \in \text{ISO}_0(3,1)$, then one obtains a general frame $s(x) = (a(x), e_i(x))$, with $a(x) = \Lambda(x)a_L(x) + b(x)$ and $e_i(x) = e^L_j(x)[\Lambda(x)^{-1}]_{ji}$, from $s_L(x)$ in the following manner

$$(x, s(x)) = R_{g_s(x)^{-1}}(x, s_L(x)).$$

(57)

Then

$$K^s(x', \zeta'; x, \zeta) = \langle \phi_\zeta | \sigma_{x'}^{s(x')} . (\sigma_{x'}^{s(x')} \sigma_{x}^{-1}) . (\sigma_{x}^{-1} \sigma_{x'}^{s(x')})^{-1} | \phi_\zeta \rangle$$

$$= \langle \phi_\zeta | U(g_s(x'))U(g_s(x)^{-1}) | \phi_\zeta \rangle,$$

(58)

where (30) and (27) were used. For example, in the gauge $s = s_L$ defined by (16), we have of course $g_s(x) = 1 \forall x$, so that

$$K^{s_L}(x', \zeta'; x, \zeta) = \langle \phi_\zeta | \phi_\zeta \rangle = K^{(f)}(\zeta', \zeta),$$

(59)

where $K^{(f)}(\zeta', \zeta)$ is the propagator in the non-bundle description defined in (17). Thus, in particular, we note that $K^{s_L}(x', \zeta'; x, \zeta)$ is actually independent of $x'$ and $x$.

We close the section with some comments. As mentioned in the previous section, Prugovečki also gives a stronger physical interpretation of a section of $\mathcal{H}$ (eq.(4.2.16)) than the one we have relied on here. In the flat space case, this interpretation allows the probability density $\Psi(x', \nu)$ for all $x' \in M$ to be calculated exactly from $\Psi_x$, and so it implies an exact consistency requirement between $\Psi_x$ and $\Psi_{x'}$ for all $x, x' \in M$. It can be shown that the sections $\Psi$ generated by the quantum transport law defined by (50) do satisfy this consistency condition.
At the beginning of this section we raised the point that the maps $Q(x', x)$ may not necessarily be defined for all pairs of points $x', x \in M$. However it is clear that the maps defined by (50) are well-defined for all such pairs of points, and that they satisfy (55) without specifying a path. This is physically reasonable in the flat-space case, since the fibre above each point $x$ is an exact copy of the non-bundle description of the particle. Thus the state $\Psi_x$ for any $x$ actually contains a complete description of the system, and it is therefore not surprising that it determines the states $\Psi_{x'}$ at all other points $x'$, not only at points in the causal future of $x$. This also raises the question of the physical status of the quantum transport law on the bundle. Should it be thought of as being analogous to the Klein-Gordon equation in that it determines some sort of causal, dynamical propagation of the state on the bundle, or is it more of the nature of a kinematical consistency condition, resulting from the geometrical structure of the Hilbert bundle formalism?

5. Parallel transport in the Hilbert bundle

We begin by summarising the relevant parts of the treatment of parallel transport given in [2] §§5.2–4 and [1] §§4.3–4.6, before going on to consider the flat space case.

The Levi-Civita connection $\omega'$ on the Lorentz frame bundle over $M$, consistent with the metric $g^L$, can be uniquely extended to an affine connection $\tilde{\omega}$ on the Poincaré frame bundle $PM$ over $M$ [12], [13] III.3. The result of this can be summarised as follows. Let $u$ be a section of the Lorentz frame bundle, i.e., $u : x \mapsto u(x) = (e_i(x)) \forall x \in M$, where $(e_i(x))$ is an orthonormal frame for $T_xM$. Then the usual connection 1-forms in the Lorentz gauge $u$ are given by the pull-back $u^*\omega'$ of $\omega'$ to $M$ under $u$. Let $\bar{s}$ be the section of $PM$ defined by

$$\bar{s}(x) = (0, u(x)) \ . \quad (60)$$

Then the pull-back of $\tilde{\omega}$ to $M$ under this section of $PM$ is

$$\bar{s}^*\tilde{\omega} = (\theta^i, u^*\omega') \ , \quad (61)$$

where $(\theta^i)$ is the coframe dual to the frame $(e_i)$. The pull-back $s^*\tilde{\omega}$ of $\tilde{\omega}$ under a general Poincaré gauge $s$ (equivalently, the value of $\tilde{\omega}$ on all of $PM$) follows by the usual gauge transformation properties, and yields $s^*\tilde{\omega} = (\tilde{\theta}^i, u^*\omega')$ with $\tilde{\theta}^i = \theta^i + \nabla a^i$ (compare [11]). Different extensions of $\omega'$ to a connection on $PM$ are thus obtained by replacing the $\theta^i$ in (61) by some other 1-forms $\tilde{\theta}^i$ (soldering forms).

Prugovecki chooses to use $\tilde{\omega}$ as the connection on $PM$ corresponding to the metric $g^L$ on $M$. Thus the additional components of $\tilde{\omega}$ compared to $\omega'$ are regarded as a mathematical device, rather than as being physical fields which might be influenced by matter via some source equation. They are chosen to be certain fixed 1-forms on $PM$, which merely provide a means of extending the Levi-Civita connection to $PM$ and so defining parallel transport on $H$. So, essentially, $\bar{s}^*\tilde{\omega}$ defined in (61) is used. The association of $H$ to $PM$ then determines a parallel transport law on $H$ in the usual way (e.g., [14] p.369). Thus for any
pair of points \( x', x \in M \) and any curve \( \gamma \) joining them, there is a corresponding parallel transport map

\[
\tau_\gamma(x', x) : \pi^{-1}(x) \to \pi^{-1}(x') , \quad \tau_\gamma(x', x) : \Psi_x \mapsto \tau_\gamma(x', x) \Psi_x .
\]  

(62)

Let \( s \) be some Poincaré frame for \( T_x M \), and \( \Phi_\xi \in \pi^{-1}(x) \) be one of the corresponding basis states defined by (60). Then from (25) and the definition of parallel transport in an associated bundle, it follows that

\[
\tau_\gamma(x', x) \Phi_\xi^s = \Phi_\xi^{\tau_\gamma(x', x)s} ,
\]

where on the right hand side \( \tau_\gamma(x', x) \) is now used to denote parallel transport in the principal bundle \( PM \) under the connection \( \tilde{\omega} \), i.e., \( \tau_\gamma(x', x)s \) denotes the result of the parallel transport of \( s \) from \( x \) to \( x' \) along \( \gamma \). By the use of (32), an arbitrary state \( \Psi_x \in \pi^{-1}(x) \) can be expanded in terms of the \( \Phi_\xi^s \) for any \( s \). Therefore the parallel transport of an arbitrary state can be written as

\[
\tau_\gamma(x', x) \Psi_x = \int_{\Sigma^+} \Phi_\xi^{\tau_\gamma(x', x)s} d\Sigma(\zeta) ,
\]

(64)

where \( s \) is any Poincaré frame for \( T_x M \) and \( \Psi_x^s(\zeta) \) is given by (31).

The parallel transport propagator \( K_\gamma^s(x', \zeta' ; x, \zeta) \) from \( x \) to \( x' \) along \( \gamma \) in the arbitrary gauge \( s \) is defined by

\[
K_\gamma^s(x', \zeta' ; x, \zeta) = \langle \Phi_\xi^{s(x')} | \tau_\gamma(x', x) | \Phi_\xi^{s(x)} \rangle ,
\]

i.e., this is the matrix element of the map \( \tau_\gamma(x', x) \), given by (63), between the fibre basis states \( \Phi_\xi^{s(x')} \in \pi^{-1}(x') \) and \( \Phi_\xi^{s(x)} \in \pi^{-1}(x) \), defined in (30). Its definition is of course exactly analogous to that of the quantum transport propagator \( K^s(x', \zeta' ; x, \zeta) \) in (41), and it plays the same role for parallel transport as that object does for quantum transport. In particular, it too satisfies equations corresponding to (12) (with \( \Psi_{x'} \) defined via (62) rather than (10)) and (14).

As in the previous section, we now concentrate on the special case where \( (M, g^L) \) is Minkowski space. As parallel transport is path-independent we will drop the subscript \( \gamma \) on \( \tau_\gamma(x', x) \) and \( K_\gamma^s(x', \zeta' ; x, \zeta) \). In this case there is a remarkable connection between quantum and parallel transport. Under the affine connection described by (60) and (61), it can be shown that the section \( s_L \) of \( PM \) defined by (10) has vanishing covariant derivative, so that

\[
\tau(x', x) s_L(x) = s_L(x') .
\]

(66)

Using (63) it follows that

\[
\tau(x', x) \Phi_\xi^{s_L(x)} = \Phi_\xi^{s_L(x')} \quad \forall \zeta \in \mathbb{R}^4 \times iV^+ .
\]

(67)

Now using the formula (50) for the quantum transport map \( Q(x', x) \) in the flat space case, and the definition (30) of \( \Phi_\xi \), we see that

\[
\tau(x', x) \Phi_\xi^{s_L(x)} = Q(x', x) \Phi_\xi^{s_L(x)} .
\]

(68)
Since the $\Phi^{sL}_{\zeta}(x)$ form a basis for $\pi^{-1}(x)$, it follows that quantum and parallel transport coincide in the flat space case,

$$\tau(x', x) = Q(x', x) \ .$$

(69)

The corresponding propagators defined by (41) and (65) are of course also equal, in any gauge $s$,

$$K^s(x', \zeta'; x, \zeta) = K^s(x', \zeta'; x, \zeta) \ .$$

(70)

If the Levi-Civita connection were extended to a connection on $PM$ having a non-flat translational part, in distinction to the affine connection described by (61), then the relation (70) would not hold.

As an aside, we remark that combining (70) and (59) gives

$$K^{sL}(x', \zeta'; x, \zeta) = K^{L}(\zeta', \zeta) \ .$$

(71)

This is in conflict with equation (4.6.5) of reference [4], which in the present notation would be $K^{sL}(x', \zeta'; x, \zeta) = K^{L}(x' + \zeta', x + \zeta)$. For example, (71) implies that $K^{sL}(x', \zeta'; x, \zeta)$ is independent of $x'$ and $x$, but the cited equation from reference [4] implies an $x'$ and $x$ dependence.

We return briefly to the question of the physical status of the quantum transport law raised at the end of the previous section. We have shown here that in the flat space case, quantum transport coincides with parallel transport. This of course is governed by a first-order differential equation, the geodesic equation, which does not have the character of a wave equation enforcing the causality properties of the underlying spacetime metric. Rather, the causality properties of this description of the particle are already encoded in any individual state in the fibre above any point $x \in M$, each such state giving a complete description of the system. This seems to favour an interpretation of the role of quantum transport as a *kinematical consistency condition* rather than as a dynamical evolution law.

### 6. Path integral formula for quantum transport in the flat space case

We will now show that the quantum propagator $K^s(x', \zeta'; x, \zeta)$ in flat space satisfies an equation which is similar to (56), but where the integral involves a spacelike surface in the base spacetime, rather than being an integral over a $\zeta$ variable for a fixed point in the base spacetime. Combining (56) and (44) we find

$$K^s(x', \zeta'; x, \zeta) = \int_{\Sigma^+} K^{sL}(x', g_s(x')^{-1}\zeta'; x_1, g_s(x_1)^{-1}\zeta_1) \ K^{sL}(x_1, g_s(x_1)^{-1}\zeta_1; x, g_s(x)^{-1}\zeta) \ d\Sigma(\zeta_1) \ .$$

(72)

11 This conflict has recently been removed in Ref. [15].
where \( s \) is an arbitrary Poincaré gauge, \( s_L \) is defined by (10), and \( g_s \) is defined by (24). Using the Poincaré invariance of the measure \( d\Sigma \) (equivalently the unitarity of the representation (13)) we can eliminate \( g_s(x_1) \), giving

\[
K^s(x', \zeta'; x, \zeta) = \int_{\Sigma^+} K^{s_L}(x', g_s(x')^{-1} \zeta'; x_1, \zeta_1) K^{s_L}(x_1, \zeta_1; x, g_s(x)^{-1} \zeta) \, d\Sigma(\zeta_1). \tag{73}
\]

Now we use the fact, noted after equation (29), that \( K^{s_L}(x', \zeta'; x, \zeta) \) is independent of \( x' \) and \( x \), writing (73) in terms of \( K^{(i)} \), to allow us to relabel \( q_1 \) in \( \zeta_1 = q_1 - i\ell v_1 \) as \( x_1 \), which in the gauge \( s_L \) is identical to \( -a_L(x_1) \). (Recall that \( x_1 = (x_1^i) \in \mathbb{R}^4 \) is the vector of coordinates of a point in \( M \), rather than being the abstract vector \( x \) appearing in footnote 10, so the equation \( x_1 = q_1 \) is well-defined.) This gives

\[
K^s(x', \zeta'; x, \zeta) = \int_{\Sigma^+} K^{s_L}(x', g_s(x')^{-1} \zeta'; x_1, x_1 - i\ell v_1) \, K^{s_L}(x_1, x_1 - i\ell v_1; x, g_s(x)^{-1} \zeta) \, d\Sigma(x_1 - i\ell v_1). \tag{74}
\]

Thus the integral on the right hand side has been recast in a form such that the \( x_1 \) integral can be thought of as going over a spacelike surface \( \sigma \) in the base Minkowski space-time (where \( \Sigma^+ = \sigma \times V^+ \), see (1)), albeit that this arose rather trivially from the fact that the integrand in (73) was independent of \( x_1 \). Using (11) again, we write the integral back in terms of propagators in the original gauge \( s \),

\[
K^s(x', \zeta'; x, \zeta) = \int_{\Sigma^+} K^s(x', \zeta'; x_1, g_s(x_1))(x_1 - i\ell v_1)) \, K^s(x_1, g_s(x_1)(x_1 - i\ell v_1); x, \zeta) \, d\Sigma(x_1 - i\ell v_1). \tag{75}
\]

(The Poincaré invariance of the measure \( d\Sigma \) cannot be used to eliminate \( g_s(x_1) \) from this expression. The obvious substitution \( x'_1 - i\ell v_1 = g_s(x_1)(x_1 - i\ell v_1) \) of course eliminates it from the arguments in which it currently appears, but it reappears upon substituting the two arguments which involve \( x_1 \) alone, since \( x_1 = g_s(x_1)^{-1} x'_1 \) (see (11), (12)).) Now we simplify the expression \( g_s(x)(x - i\ell v) \). Let the arbitrary gauge \( s \) be \( s(x) = (\mathbf{a}(x), e_i(x)) \), as in (36), and let \( g_s(x) = (b(x), \Lambda(x)) \). From (12) we have, since \( x = -a_L(x) \) in the gauge \( s_L \) according to (10),

\[
g_s(x)(x - i\ell v) = -\Lambda(x)a_L(x) - b(x) - i\ell \Lambda(x)v. \tag{76}
\]

Then from (17) and the definition (22) of the right action of \( \text{ISO}_0(3,1) \) on \( PM \), we find that

\[
a(x) = \Lambda(x)a_L(x) + b(x). \tag{77}
\]

We emphasise that this formula for \( a(x) \) holds for each \( x \), and merely displays the relationship between the components of \( a(x) \) corresponding to the arbitrary gauge \( s \), and the components of \( a_L(x) \) corresponding to the gauge \( s_L \), which relationship is given at each
point $x$ by the Poincaré transformation $g_s(x) = (b(x), \Lambda(x))$. Combining these results and writing it in terms of vectors of components $a^i(x)$ and $v^i$ respectively gives
\[ g_s(x)(x - i\ell v) = -a(x) - i\ell\bar{v}(x) , \] (78)
where $\bar{v}(x) = \Lambda(x)v$. Here the vector $-a(x)$ denotes the point of contact of $T_x M$ and $M$ in the gauge $s$ in $T_x M$ with $M$ being Minkowski space in the present case. Furthermore, $\Lambda(x)$ relating $v$ and $\bar{v}$ in (78) is the Lorentz transformation by which the special “parallel” gauge $s_L$ deviates from the general Poincaré gauge $s$.

Substitution in (78) now gives
\[ K^s(x', \zeta'; x, \zeta) = \int_{\Sigma^+} K^s(x', \zeta'; x_1, -a(x_1) - i\ell\bar{v}(x_1)) \]
\[ \times \int_{\Sigma^+} K^s(x_1, -a(x_1) - i\ell\bar{v}(x_1); x, \zeta) d\Sigma(-a(x_1) - i\ell\bar{v}(x_1)) , \] (79)
where we have used (78) and the Poincaré invariance of the measure to put it into the form $d\Sigma(-a(x_1) - i\ell\bar{v}(x_1)) = d\Sigma(x_1 - i\ell v_1)$. Finally, this expression can be substituted into itself an arbitrary number of times to obtain
\[ K^s(x', \zeta'; x, \zeta) = \int_{\Sigma^+} \cdots \int_{\Sigma^+} K^s(x', \zeta'; x_{N-1}, \tilde{\zeta}_{N-1}) \]
\[ \times \prod_{n=N-1}^1 K^s(x_n, \tilde{\zeta}_n; x_{n-1}, \tilde{\zeta}_{n-1}) d\Sigma(-a(x_n) - i\ell\bar{v}_n) , \] (80)
where
\[ \tilde{\zeta}_n = -a(x_n) - i\ell\bar{v}_n , \quad n = 1, \ldots, N - 1 , \] (81)
and
\[ x_0 = x , \quad \tilde{\zeta}_0 = \zeta , \] (82)
for any integer $N \geq 1$.

In reference [4] equations (4.6.7a,b), Prugovečki postulates a Feynman path integral-like equation for the quantum transport propagator in the Hilbert bundle over a curved base spacetime. The flat space limit of this equation should of course be consistent with our equation (80) just derived. We now examine this consistency. The use of the parallel transport propagator $K^s$ rather than $K^s$ in the integrand of [4] eq.(4.6.7a) is of course justified, since these coincide in the flat space case (equation (70)). Further, each of the surfaces $\Sigma^+_n$, $n = 1, \ldots, N - 1$, in (80) is of the form $\sigma_n \times V^+$, where $\sigma_n$ is an arbitrary Cauchy surface in Minkowski space (see equation (7)). One possible choice would be to take these surfaces to be time-ordered, discrete leaves in a foliation of a region of spacetime bounded by two (non-intersecting) spacelike hypersurfaces $\sigma_0$ and $\sigma_N$, with $\sigma_0$ containing $x$ and $\sigma_N$ containing $x'$. Then considering the limit $N \to \infty$ would indeed produce a path integral-like formula of the form of [4] eq.(4.6.7a) provided the measure $d\Sigma(x_n, \bar{v}_n)$ appearing there is identified in flat space with the measure $d\Sigma(-a(x_n) - i\ell\bar{v}_n)$ on the right hand side of (80) where $-a(x_n)$ denotes, as mentioned, the coordinates of the point $x \in M \equiv \mathbb{R}^4$ with regard to the affine base in $T_x M$ in the gauge $s$. 

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7. Possible extensions of quantum transport to curved spacetime

The approach of reference [4] to postulating a definition for the quantum transport propagator in the Hilbert bundle over a curved spacetime base is motivated by the form of this propagator in the flat space case. The flat space quantum propagator can be written in a way involving the parallel transport propagator which, it is argued, has a natural extension to curved space, obtained simply by replacing the flat space parallel transport propagator by the one for curved space. However, this generalisation to curved spacetime is not immediate. The difficulty is the non-existence in the case of curved spacetime of a unique gauge choice analogous to $s_L$ in the flat space case. This is because the path-dependence of parallel transport makes it impossible to construct a Lorentz frame at each point, such that the parallel transport of the frame from one point to any other point gives the frame at that point, independent of the path taken.

There is one further point to be considered. In generalizing the expression for the quantum propagator from a flat spacetime base to a curved spacetime base the hypersurfaces $\Sigma^+ = \sigma \times V^+$ defined in (7), with $\sigma$ being a space-like surface in Minkowski space, has to be turned into a spacelike hypersurface in curved spacetime representing a leaf of a foliation of the underlying curved manifold. It is not clear from the beginning whether the invariant measure (8) defining our Hilbert space $\mathcal{F}$, which carries a unitary irreducible representation of the Poincaré group of spin $s=0$ and mass $m$, is immediately interpretable as a measure on the different hypersurfaces $\sigma_n \times V^+$ where the $\sigma_n$ are given by a particular foliation of the curved spacetime base, and whether the whole procedure of defining a quantum propagator on curved spacetime is strongly dependent on the foliation chosen.

Let us now generalize eq.(80) in a way which is in accord with Prugovečki’s proposal presented in [4] eq.(4.6.7a,b) yielding, indeed, a Poincaré gauge covariant definition (see eq.(12)) of a quantum-geometric propagator defined on a one-particle (first quantized) Hilbert bundle over curved spacetime which is based on parallel transport along geodesic paths denoted by $\gamma(x_n, x_{n-1})$ between points $x_{n-1} \in \sigma_{n-1}$ and $x_n \in \sigma_n$ of two adjacent leaves of a foliation. We show that the resulting Feynman-like path integral expression involving integrations over intermediate points on $\sigma_n \times V^+$, with the $\sigma_n$ being, as mentioned, hypersurfaces of a foliation in the spacetime base rather than hypersurfaces in the fibre, is a Poincaré gauge covariant expression; or, more precisely, obeys the relation

$$K^{s'}(x', \zeta'; x, \zeta) = K^s(x', g(x)^{-1}\zeta'; x, g(x)^{-1}\zeta)$$

with $g(x) = (b(x), \Lambda(x))$ and $s$ and $s'$ related as in (20). In accord with [4] eq.(4.6.7a,b) we thus write in the gauge $s$ for the quantum propagator on a curved base manifold $M$

$$K^s(x', \zeta'; x, \zeta) = \lim_{\epsilon \to +0} \int K^s_{\gamma(x',x_{N-1})}(x', \zeta'; x_{N-1}, \hat{\zeta}_{N-1}) \prod_{n=N-1}^{1} K^s_{\gamma(x_n,x_{n-1})}(x_n, \hat{\zeta}_n; x_{n-1}, \hat{\zeta}_{n-1}) d\Sigma(x_n, \bar{v}_n),$$

In [4] the geodesic arc between $x_{n-1}$ and $x_n$ was denoted by $\gamma(x_{n-1}, x_n)$.
with \( \epsilon = (t' - t)/N \) and \( \hat{\zeta}_n \) given by
\[
\hat{\zeta}_n = -a(x_n) - i\ell \bar{v}_n, \quad n = 1, \ldots, N - 1, \quad x_0 = x, \quad \hat{\zeta}_0 = \zeta, \tag{85}
\]
and where we have introduced a measure \( d\Sigma(x_n, \bar{v}_n) \) to be defined below. In [84] the parallel transport propagator between two points on adjacent leaves of a foliation connected by a geodesic arc \( \gamma(x_n, x_{n-1}) \) from \( x_{n-1} \) to \( x_n \) is denoted by \( K_{\gamma(x_n, x_{n-1})}^s(x_n, \hat{\zeta}_n; x_{n-1}, \hat{\zeta}_{n-1}) \) where the suffix \( \gamma \) is a reminder of the path dependence. Eq. (22) implies that two gauges \( s \) and \( s' \) are related by
\[
s(x) = (a(x) = -a^i(x)e_i(x), e_i(x)), \quad s'(x) = (a'(x) = -a'^i(x)e'_i(x), e'_i(x))
\]
with \( a'^i = [\Lambda(x)]^i_j a^j(x) + b^i(x) \) and \( e'_j = e_j(x)[\Lambda(x)^{-1}]^i_j \)
\[
\tag{86}
\]
and the corresponding \( \zeta \) variables defined with respect to the frames \( s(x) \) and \( s'(x) \) transforming as
\[
\zeta'(x) = (g(x)\zeta(x) = \Lambda(x)q(x) - b(x) - i\ell \Lambda(x)\bar{v}(x).
\]
\[
\tag{87}
\]
We mention again that the point \( q(x) = -a(x) \) is the coordinate description of the point of contact \( x \in M \) as determined in the gauge \( s \) in \( T_x M \) and analogously in the gauge \( s' \).

Let us now use (83) as well as (86) and (87) to convert (84) into an expression for \( K_{s'} \). We obtain (with \( x_0 = x \) and \( g(x_0)\zeta_0 = \zeta \)
\[
K_{s'}(x', \zeta'; x, \zeta) = \lim_{\epsilon \to +0} \int \prod_{n=N-1}^1 K_{\gamma(x_n, x_{n-1})}^s(x_n, \hat{\zeta}_n; x_{n-1}, \hat{\zeta}_{n-1}) d\Sigma(x_n, \bar{v}_n).
\]
\[
\tag{88}
\]
Now we need to evaluate \( g(x_n)\hat{\zeta}_n \) for the intermediate points which is, for \( g(x_n) = (b(x_n), \Lambda(x_n)) \) using (83) and (12), given by
\[
g(x_n)\hat{\zeta}_n = -\Lambda(x_n)a(x_n) - b(x_n) - i\ell \Lambda(x_n)\bar{v}_n = -a'(x_n) - i\ell \bar{v}'_n = \hat{\zeta}'_n.
\]
\[
\tag{89}
\]
According to (86) this is the gauge transformed variable \( \hat{\zeta}'_n = \zeta'(x_n) \) with \( \bar{v}'_n = \Lambda(x_n)\bar{v}_n \). It is now seen from (87) together with (89) that the quantum propagator for a curved base manifold \( M \) is definable in a Poincaré gauge invariant manner provided the measure \( d\Sigma(x_n, \bar{v}_n) \) is Poincaré invariant.

The original measure \( d\Sigma(\zeta) = d\Sigma(q, v) \) defined in (8) is Poincaré invariant. Clearly, the measure denoted by \( d\Sigma(x_n, \bar{v}_n) \) in (84) and (88) cannot be invariant against arbitrary reparametrizations of the base manifold given in terms of changes of the atlas on \( M \). The only way the notation \( d\Sigma(x_n, \bar{v}_n) \) for the measure makes sense is by interpreting \( x_n \) as
the point of contact of $T_xM$ with the base $M$ as given in terms of the coordinates of this point in the gauge $s$ in the local tangent space $T_xM$, which is given by $-a(x_n)$, i.e., by interpreting $d\Sigma(x_n, \bar{v}_n)$ as $d\Sigma(-a(x_n), \bar{v}_n) = d\Sigma(\xi_n)$. Of course, there does not exist now a gauge $s_L$ as in the Minkowski case, described in sections 4 and 6 above, where $x$ is equal to $-a_L(x)$ for all $x \in \mathbb{R}^4$. Interpreting the measure appearing in (88) in this manner as a “contact point measure” one can show that this measure is Poincaré gauge invariant since then one has using $d\Sigma(\zeta) = d\Sigma(g(x)\zeta)$:

$$d\Sigma(x_n, \bar{v}_n) \equiv d\Sigma(-a(x_n), \bar{v}_n) = d\Sigma(g(x_n)(-a(x_n)), g(x_n)\bar{v}_n) = d\Sigma(-a'(x_n), \bar{v}'_n) \equiv d\Sigma(x_n, \bar{v}'_n)$$

(90)

with the primed quantities given as in (89).

Hence we come to the conclusion that (84) can, with the help of (89) and (90), be written in terms of the primed variables belonging to the gauge $s'$ showing that (84) is a Poincaré gauge covariant quantum-geometric propagator defined on a first quantized Hilbert bundle over curved spacetime with the measure $d\Sigma(x_n, \bar{v}_n)$ interpreted as a “contact point measure” in the above described sense. Clearly, eq. (84) is a particular proposal for a Poincaré gauge invariant quantum propagator on a curved base which is motivated by the flat space result investigated in section 6 depending, moreover, on the particular foliation chosen for the spacetime base.

In concluding this section we would like to add one further remark. In order to interpret the “contact point measure” $d\Sigma(x_n, \bar{v}_n)$ as a bona fide measure definable on a general curved spacetime base one would have to present an analytic proof that it is possible to go over in a smooth manner on $M$, from a truncated measure defined on the local fibres of the phase space bundle

$$\tilde{E} = \tilde{E}(M, F = \mathbb{R}^4 \times V^+, \text{ISO}_0(3,1)),$$

(91)

soldered to the base $M$ through the subspace $\mathbb{R}^4$ of $F$ – i.e., from a measure restricted to the point of contact of fibre and base in $\tilde{E}$ – to a smooth Poincaré invariant measure defined on the base with values in $V^+_x$. (Compare also footnotes 7 and 8 above in this context.) The soldering maps could be used in building up a hypersurface $\sigma$ in the spacetime base from the surface $\Sigma^+_x$ defined in the local fibres of $\tilde{E}$ using analyticity in the affine variable $q \in T_xM$ in order to construct a Cauchy surface in $M$ generating a foliation. To our knowledge, investigations in this direction have not been made.

8. Modifications to incorporate the usual description of the Klein-Gordon particle

The Hilbert bundle description can equally well be applied in the context of the conventional description of relativistic quantum mechanics, without introducing the full stochas-

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13 Despite the complex notation adopted for $\zeta = q - i\ell v$ following [4] we may denote the measure $d\Sigma(\zeta)$ equivalently by $d\Sigma(q, v)$. 

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tic phase-space quantum mechanics (and thereby of course foregoing its advantages). We briefly sketch this here.

The conventional, configuration-space description of the Klein-Gordon particle is obtained by applying the Fourier transform to the momentum-space wavefunctions, instead of equation (4). That is, instead of $F$, the Hilbert space is $F'$, consisting of the states $\psi$ on $\mathbb{R}^4$ of the form

$$\psi(q) = (2\pi)^{-3/2} \int_{V^+_m} e^{-i q \cdot \tilde{\psi}(k)} d\Omega_m(k),$$

(92)

where $\tilde{\psi} \in L^2(V^+_m)$. $F'$ carries a spin zero, irreducible representation of the Poincaré group, and a Hilbert bundle with typical fibre $F'$, associated to the Poincaré frame bundle, can be constructed as before, including the analogues of the canonical maps $\sigma^*_{\xi}$ (equation (25)). In this case, there is no (covariant) configuration space probability interpretation for the states $\psi$ in $F'$. Nevertheless, one can make a connection between the bundle and non-bundle formulations in the same fashion as before, by defining the configuration space amplitude corresponding to a section of the (new) Hilbert bundle via equation (37), with the $\psi$ variable suppressed. Despite the absence of a physical (probability) interpretation for such amplitudes, it is natural to require each such amplitude to agree with some wavefunction in $F'$. Perhaps the strongest argument for this is that we are effectively considering the one particle sector of the second quantized theory, and agreement of the analogous amplitudes in the non-bundle and bundle descriptions of the second quantized theory is required for physical reasons. The succeeding analysis proceeds as before, with the change that the momentum variables are everywhere suppressed.

9. Conclusion

We have examined in detail the Hilbert bundle formulation of the relativistic phase space quantum theory of a spinless particle. In the flat space limit we deduced the required quantum transport law and propagator on the bundle by comparison with Prugovečki’s stochastic phase space quantum mechanics. We showed that this coincides with parallel transport on the bundle (in the flat space limit). We found furthermore, that a formula for the quantum propagator in curved spacetime, which was postulated in [4], is Poincaré gauge covariant (as stated in [3]), provided the measure is interpreted as a “contact point measure” on the soldered bundle $\tilde{E}$. The problems associated with this interpretation were briefly mentioned. Finally, we sketched briefly how the Hilbert bundle construction can also be applied to the conventional description of the Klein-Gordon particle.

It is apparent that the Hilbert bundle approach gives an elegant and powerful new framework for quantum field theory in curved spacetime which has fundamental advantages over the conventional approach. We believe that further useful work can be done on the question of specifying the integration measure for the quantum propagation law in curved spacetime, for example along the lines mentioned in section 7 above.

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