Maximum Minimal Feedback Vertex Set: A Parameterized Perspective

Ajinkya Gaikwad¹, Hitendra Kumar¹, Soumen Maity¹,
Saket Saurabh²,³, and Shuvam Kant Tripathi¹

¹ Indian Institute of Science Education and Research, Pune, India
² The Institute of Mathematical Sciences, Chennai, India
³ University of Bergen, Bergen, Norway
ajinkya.gaikwad@students.iiserpune.ac.in; soumen@iiserpune.ac.in;
saket@imsc.res.in

Abstract. In this paper we study a maximization version of the classical Feedback Vertex Set (FVS) problem, namely, the Max Min FVS problem, in the realm of parameterized complexity. In this problem, given an undirected graph $G$, a positive integer $k$, the question is to check whether $G$ has a minimal feedback vertex set of size at least $k$. We obtain following results for Max Min FVS.

1. We first design a fixed parameter tractable (FPT) algorithm for Max Min FVS running in time $10^k n^{O(1)}$.

2. Next, we consider the problem parameterized by the vertex cover number of the input graph (denoted by $vc(G)$), and design an algorithm with running time $2^{O(vc(G) \log vc(G))} n^{O(1)}$. We complement this result by showing that the problem parameterized by $vc(G)$ does not admit a polynomial compression unless $\text{coNP} \subseteq \text{NP}/\text{poly}$.

3. Finally, we give an FPT-approximation scheme (fpt-AS) parameterized by $vc(G)$. That is, we design an algorithm that for every $\epsilon > 0$, runs in time $2^{O(vc(G))} n^{O(1)}$ and returns a minimal feedback vertex set of size at least $(1-\epsilon) opt$.

Keywords: Parameterized Complexity · FPT · vertex cover

1 Introduction

Feedback Vertex Set (FVS) together with Vertex Cover are arguably the two most well studied problems in parameterized complexity. In FVS, we are given an undirected graph $G$, a positive integer $k$, and the question is to check whether $G$ has a vertex set $S$ of size at most $k$ such that $G - S$ is a forest (an acyclic graph). The set $S$ is called feedback vertex set or fvs in short. Downey and Fellows [5], and Bodlaender [2] proposed the first two fixed parameter tractable (FPT) algorithms with the running time $O^*(2^{O(k \log k)})$. After a long series of improvements, the current champion algorithms are as follows. The fastest

\[^4\] The notation $O^*$ hides the polynomial factor in the running time.
known randomized algorithm is given by Li and Nederlof [11] and runs in time $O^*(2.7^k)$, and the fastest known deterministic algorithm, given by Iwata and Kobayashi [10], runs in time $O^*(3.460^k)$. Several minimization variants of FVS have been studied in the literature such as finding a set $S$ such that $G - S$ is acyclic and $G[S]$ is connected or $G[S]$ is an independent set. In this paper we consider a (not so well studied) maximization version of FVS, namely Max Min FVS. A set $S$ is called a minimal fvs, if $S$ is an fvs and for every $v \in S$, $S \setminus \{v\}$ is not an fvs. That is, no proper subset of $S$ is an fvs. It is not hard to see that if $S$ is a minimal FVS, then every $u \in S$ has a private cycle, that is, there exists a cycle in $G[(V(G) \setminus S) \cup \{u\}]$, which goes through $u$. Now we are ready to define the problem formally.

**Max Min FVS**

**Input:** An undirected graph $G = (V, E)$ and an integer $k \in \mathbb{N}$.

**Question:** Does $G$ admit a minimal FVS of size greater than or equal to $k$?

The graph parameter we discuss in this paper is vertex cover number.

**Definition 1.** A set $C \subseteq V$ is a vertex cover of $G = (V, E)$ if each edge $e \in E$ has at least one endpoint in $C$. The minimum size of a vertex cover in $G$ is the vertex cover number of $G$, denoted by $\text{vc}(G)$.

Lately, finding a large size minimal solution has attracted a lot of attention from the perspective of the Approximation Algorithms and Parameterized Complexity. Boria et al. [3] proved that for any constant $\epsilon > 0$, the optimization version of Max Min Vertex Cover is inapproximable within ratios $O(n^{0.55-\epsilon})$, unless P=NP. They complement this result by proving that Max Min Vertex Cover is approximable within ratio $O(n^{0.5})$ in polynomial time. This is in sharp contrast to the approximability of the classical Vertex Cover problem, for which an easy factor 2-approximation exists. This becomes even more interesting when we consider the optimization version of Max Min FVS. The Max Min FVS problem was first considered by Mishra and Sikdar [12], who showed that the problem does not admit an $n^{0.5-\epsilon}$ approximation (unless P=NP), and that it remains APX-hard even when the input graph is of degree at most 9. Dublois et al. [7] improved upon this by showing the first non-trivial polynomial time approximation for Max Min FVS with a ratio of $O(n^{3/2})$, as well as a matching hardness of approximation bound of $n^{3/2-\epsilon}$. Apart from these two problems, there are many other classical optimization problems that have recently been studied in the MAXMIN or MINMAX framework, such as Max Min Separator [9] and Max Min Cut [8].

In the realm of parameterized complexity, Zehavi [13] studied Max Min Vertex Cover – find a minimal vertex cover of size at least $k$, if exists – and designed an algorithm with running time $O^*(2^{\text{vc}(G)})$, which is "almost optimal" unless Strong Exponential Time Hypothesis fails. For Max Min FVS, Dublois et al. [7] obtained a polynomial kernel of size $O(k^3)$. That is, they design a polynomial time algorithm that given an instance $(G, k)$ returns an equivalent
instance \((G', k')\) such that \(k' \leq k\) and \(|V(G') + E(G')| \leq O(k^3)\). This result is
the starting point of our work. There are results about kernelization of \textsc{Max Min Vertex Cover} and \textsc{Max Min FVS} in \cite{1}. In particular, they proved
that \textsc{Max Min VC} parameterized by vertex cover number does not admit a
polynomial kernel. This result is related to the kernelization of \textsc{Max Min FVS}
parameterized by vertex cover of our work.

1.1 Preliminaries

We only consider finite undirected graphs without loops or multiple edges, and
we denote an edge between two vertices \(u\) and \(v\) by \((u, v)\). A subgraph \(H\) of a
graph \(G\) is \textit{induced} if \(H\) can be obtained from \(G\) by deleting a set of vertices
\(D = V(G) \setminus S\), and we denote \(H = G[S]\). For a graph \(G\) and a set \(S \subseteq V(G)\),
we use the notation \(G - S = G[V(G) \setminus S]\), and for a vertex \(v \in V(G)\), we abbreviate
\(G \setminus \{v\}\) as \(G - v\). The \textit{(open) neighbourhood} \(N_G(v)\) of a vertex \(v \in V(G)\) is
the set \(\{u \mid (u, v) \in E(G)\}\). The \textit{closed neighbourhood} \(N_G[v]\) of a vertex \(v \in V(G)\) is the
set \(\{v\} \cup N_G(v)\). The \textit{degree} of \(v \in V(G)\) is \(|N_G(v)|\) and denoted by \(d_G(v)\). We
use \(d_S(v)\) to denote the degree of vertex \(v\) in \(G[S]\). For an integer \(n \geq 1\), we let
\([n]\) be the set containing all integers \(i\) with \(1 \leq i \leq n\). In a graph \(G\), \textit{contraction}
of an edge \(e = (u, v)\) is the replacement of \(u\) and \(v\) with a single vertex such that
edges incident to the new vertex are the edges other than \(e\) that were incident
with \(u\) or \(v\). The resulting graph, denoted \(G/e\), has one less edge than \(G\). We refer to Appendix A and \cite{4,6} for details on parameterized complexity.

1.2 Our results and methods

Using, the polynomial kernel, of size \(O(k^3)\), of Dubois et al. \cite{7}, we can design
an FPT algorithm for \textsc{Max Min FVS} running in time \(O^*(2^{O(k^3)})\) as follows.
For every vertex subset \(S\) of size at least \(k'\) of \(V(G')\) test whether \(S\) is a minimal
fvs or not. If we succeed for any \(S\), we have that \((G, k)\) is a yes instance, else, it
is a no instance. As our first result we improve upon this result and obtain the
following result.

\textbf{Theorem 1.} \textsc{Max Min FVS} \textit{can be solved in time} \(O^*(10^k)\).

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{fig1}
\caption{Graph \(G\) with \(\text{fvs}(G) = 1\) and \(\text{opt}(G) = 4\)}
\end{figure}
Proof of Theorem 1 follows the strategy used for designing an iterative compression based FPT algorithm for FVS. Let $\text{fvs}(G)$ denote the minimum size of a feedback vertex set of $G$, and let $\text{opt}(G)$ denote the maximum size of a minimal feedback vertex set of $G$. Clearly, $\text{fvs}(G) \leq \text{opt}(G)$. The gap between these quantities can be arbitrary large – as shown in Figure 1. Here, $\text{fvs}(G) = 1$, while $\text{opt}(G) = |V(G)| - 2$. Further, observe that the same example also shows that the gap between $\text{vc}(G)$ and $\text{opt}(G)$ can be arbitrary large. Here, $\text{vc}(G) = 2$, while $\text{opt}(G) = |V(G)| - 2$. The discussion above implies that for Max Min FVS both $\text{fvs}(G)$ and $\text{vc}(G)$ are interesting parameters to consider. For our second result, we consider Max Min FVS parameterized by $\text{vc}(G)$ and obtain the following result.

**Theorem 2.** Max Min FVS can be solved in time $O^*(2^{O(\log \text{vc}(G))})$.

The starting point of the algorithm is based on the natural partitioning ideas. However, to complete the algorithm we need to design an algorithm for Induced Forest Isomorphism parameterized by $\text{vc}(G)$, which could be of independent interest. This algorithm is the bottleneck in designing an algorithm for Max Min FVS running in time $O^*(2^{O(\log \text{vc}(G))})$. We complement this result by showing that Max Min FVS parameterized by $\text{vc}(G)$ does not admit a polynomial compression unless $\text{coNP} \subseteq \text{NP/poly}$. Note that $\text{vc}(G)$ can be much larger than $\text{opt}(G)$, for example in a cycle, and so Theorem 1 is not implied by Theorem 2. Finally, we show that if we allow a “small loss” then we can improve upon Theorem 2. That is, we design an FPT-approximation algorithm for Max Min FVS parameterized by $\text{vc}(G)$.

**Theorem 3.** Let $\epsilon > 0$ be a fixed constant. Then, there exists an algorithm for Max Min FVS, that runs in time $2^{O(\text{vc}(G))}n^{O(1)}$ and returns a minimal feedback vertex set of size at least $(1 - \epsilon)\text{opt}(G)$.

## 2 FPT algorithm parameterized by solution size

In this section we design an FPT algorithm for Max Min FVS. First we give an algorithm for Extension Max Min FVS. We start by defining Extension Max Min FVS. In this problem, as input instance $I = (G, W_1, W_2, k)$, we are given an undirected graph $G$, an integer $k$ and a minimal feedback vertex set $W = W_1 \cup W_2$ in $G$. We are also given a partition $(W_1, W_2)$ of the vertices of $W$. The objective is to decide if $G$ has a minimal feedback vertex set $S$ such that $W_1 \subseteq S$, $W_2 \cap S = \emptyset$ and $|S \setminus W_1| \geq k$, or correctly conclude that no such minimal feedback vertex set exists. We give an algorithm for Extension Max Min FVS running in time $3^k + \gamma(I)q^{O(1)}$, where $\gamma(I)$ is the number of connected components of $G[W_2]$. 
2.1 An algorithm for Extension Max Min FVS

Let $I = (G,W_1,W_2,k)$ be an instance of Extension Max Min FVS and let $H = G - W$ where $W = W_1 \cup W_2$ and $W_1 \cap W_2 = \emptyset$. We first give some reduction rules to simplify the input instance.

**Reduction EMMFVS 1** If $G - W_1$ has a vertex $v$ of degree at most 1, remove it from the graph.

Reduction EMMFVS 1 is safe, because for a given instance $(G,W_1,W_2,k)$ of Extension Max Min FVS, if the graph $G - W_1$ has a vertex of degree at most one, then this vertex is not part of any cycle in $G - W_1$. Thus, its removal does not change the solution.

**Reduction EMMFVS 2** If there is a vertex $v$ in $H$ such that $G[W_2 \cup \{v\}]$ contains a cycle, then include $v$ in $W_1$, and decrease the parameter by 1. That is, the new instance is $(G,W_1 \cup \{v\},W_2,k-1)$.

Reduction MMFVS 2 is safe. Suppose $G[W_2 \cup \{v\}]$ contains a cycle $C$. As the solution here has to be disjoint from $W_2$, the only way to destroy $C$ is to include $v$ in the solution.

**Reduction EMMFVS 3** If $(u,v) \in E(G)$ such that $N(u) \cap N(v) = \emptyset$, $d(v) = d(u) = 2$ and $u,v \notin W$, then contract $(u,v)$. That is, the new instance is $(G/(u,v),W_1,W_2,k)$.

Reduction Rule EMMFVS 3 is safe as any minimal feedback vertex set contains at most one of $u$ and $v$. This reduction rule is inspired from [7] and a formal proof of this rule is given in [7]. Furthermore, all reductions can be applied in polynomial time.

**Lemma 1.** Extension Max Min FVS can be solved in time $3^{k+\gamma(I)}n^{O(1)}$.

**Proof.** Let $(G,W_1,W_2,k)$ be the input instance. If $G[W_2]$ is not a forest then we return that $(G,W_1,W_2,k)$ is a no-instance. So from now onward we assume that $G[W_2]$ is a forest. We follow a branching technique with a measure function $\mu$. For instance $I = (G,W_1,W_2,k)$, we define its measure

$$\mu(I) = k + \gamma(I)$$

where $\gamma(I)$ is the number of connected components of $G[W_2]$. The algorithm first applies Reduction EMMFVS 1, EMMFVS 2, and EMMFVS 3 exhaustively. For clarity we denote the reduced instance by $(G,W_1,W_2,k)$. Since $W$ is a feedback vertex set, $H = G - W$ is a forest. Thus $H$ has a vertex of degree at most 1. In each tree of the forest $H$, arbitrarily pick one of its vertices as the root. Now we focus on a deepest leaf $v$ of any tree in $H$. Clearly $v$ has at least one neighbour in $W_2$, otherwise Reduction EMMFVS 1 would have been applied. We distinguish two cases based on the number of neighbours of $v$ in $W_2$.
Case 1. Assume that \( v \) has at least two neighbours in \( W_2 \). Since Reduction EMMFVS 2 cannot be applied, we have that no two neighbours of \( v \) belong to the same connected component of \( G[W_2] \). So, we can assume that all neighbours of \( v \) belong to different connected components of \( G[W_2] \). See Figure 2(a). Now we branch by including \( v \) in the solution in one branch and excluding it in the other branch. That is, we call the algorithm on instances \((G, W_1 \cup \{v\}, W_2, k-1)\) and \((G, W_1, W_2 \cup \{v\}, k)\). We check minimality of the partial solution in every branch. If one of these branches returns a solution, then we conclude that \((G, W_1, W_2, k)\) is a yes-instance, otherwise \((G, W_1, W_2, k)\) is a no-instance.

Case 2. Assume that \( v \) has exactly one neighbour in \( W_2 \). Let \( \pi(v) \) be the parent of \( v \) in \( H \). We now have a number of subcases and subsubcases to consider. Clearly, the degree of \( \pi(v) \) cannot be one in \( G - W_1 \), otherwise Reduction EMMFVS 1 would have been applied.

Subcase 2.1. Assume that the degree of \( \pi(v) \) in \( G - W_1 \) is two. Then both \( v \) and \( \pi(v) \) are of degree two, and hence must have a common neighbour in \( W_2 \), otherwise Reduction EMMFVS 3 would have been applied. See Figure 2(b). Clearly, every solution of the EXTENSION MAX MIN FVS instance \((G, W_1, W_2, k)\) contains either \( v \) or \( \pi(v) \). Note that, without loss of generality we can add \( v \) inside the solution and keep \( \pi(v) \) outside the solution. Therefore, we get reduced EXTENSION MAX MIN FVS instance \((G, W_1 \cup \{v\}, W_2 \cup \{\pi(v)\}, k - 1)\). We check minimality of this partial solution.

Subcase 2.2. Assume that the degree of \( \pi(v) \) in \( G - W_1 \) is at least 3. We split this subcase into two subsubcases.

Subsubcase 2.2.1. Assume that \( \pi(v) \) has no neighbours in \( W_2 \). As the degree of \( \pi(v) \) in \( G - W_1 \) is at least three and it has no neighbours in \( W_2 \), it has at least two children. Without loss of generality, suppose \( \pi(v) \) has two children \( v \) and \( v' \). Observe that \( v' \) is a leaf node, otherwise \( v \) is not a deepest leaf in \( H \). The degree of \( v' \) cannot be one in \( G - W_1 \), otherwise Reduction EMMFVS 1 would have been applied. Also the degree of \( v' \) cannot be more than one in \( W_2 \) as otherwise Case 1 will be applicable. Therefore \( v' \) has exactly one neighbour in \( W_2 \). Similarly, we can argue that \( v \) has exactly one neighbour in \( W_2 \). See Figure 2(c). In this case we make three branches by including one of \( v \) and \( v' \) in the solution and excluding \( \{v, v', \pi(v)\} \) in the other branch. That is, we get instances \((G, W_1 \cup \{v\}, W_2, k - 1)\), \((G, W_1 \cup \{v'\}, W_2, k - 1)\) and \((G, W_1, W_2 \cup \{v, v', \pi(v)\}, k)\). We check minimality of the partial solution in every branch. Notice that we have not considered a branch where \( \pi(v) \) is inside the solution. For the sake of contradiction, assume that \( \pi(v) \) is inside the solution \( S \). We will prove that starting from \( S \), we can construct another solution \( S' \) such that \( |S'| \geq |S| \) but \( \pi(v) \notin S' \). First we observe that if \( \pi(v) \) is in \( S \), none of its children are in \( S \). This is because each child of \( \pi(v) \) has degree two in \( G - W_1 \) and one of its neighbours is \( \pi(v) \). That means every cycle that contains a child of \( \pi(v) \) also contains \( \pi(v) \). Next
assuming that $\pi(v)$ is inside the solution $S$, there must exist a private cycle $C$ of $\pi(v)$. Note that $C$ must contain at least one child of $\pi(v)$ as it does not have a neighbour in $W_2$. Without loss of generality, let that child be $v$. In this case we will replace $\pi(v)$ by $v$, that is, $S' = (S \setminus \{\pi(v)\}) \cup \{v\}$. We observe that the solution will satisfy minimality condition because every cycle that is hit by $v$ is also hit by $\pi(v)$.

**Subsubcase 2.2.2.** Assume that $\pi(v)$ has at least one neighbour in $W_2$. Note that $\pi(v)$ has at least one child. See Figure 2(d). In this case we get three branches by including one of $v$ and $\pi(v)$ in the solution and excluding $\{v, \pi(v)\}$ in the other branch. That is, we get instances $(G, W_1 \cup \{v\}, W_2, k - 1)$, $(G, W_1 \cup \{\pi(v)\}, W_2, k - 1)$ and $(G, W_1, W_2 \cup \{v, \pi(v)\}, k)$. We check minimality of the partial solution in every branch.

The algorithm stops when $k = 0$ or $H = \emptyset$. If $k = 0$ at some leaf node in the search tree, then we conclude that the given instance of EXTENSION MAX MIN FVS is a yes-instance. Otherwise, it is a no-instance. *Note that at each branch of Case 1, Subcase 2.1, Subsubcase 2.2.1 and 2.2.2, we check the minimality of the partial solution, that is, we check if for every vertex $w \in W_1$ whether there exists a cycle containing $w$ in $G - \{W_1 \setminus w\}$. If the partial solution of a branch is not minimal, we discard that branch. To estimate the running time of the algorithm for instance $I = (G, W_1, W_2, k)$, we use the measure $\mu(I)$ as defined at the beginning of the proof. Observe that Reductions EMMFVS 1, EMMFVS 2, and EMMFVS 3 do not increase the measure. Now we see how $\mu(I)$ changes.*
when we branch. In Case 1, when \( v \) goes to the solution, \( k \) decreases by 1 and \( \gamma(I) \) remains the same. Thus \( \mu(I) \) decreases by 1. In the other branch, \( v \) goes into \( W_2 \), then \( k \) remains the same and \( \gamma(I) \) decreases at least by 1. Thus \( \mu(I) \) decreases at least by 1. Thus we have a branching vector \((1, \geq 1)\) for branching in Case 1. In Subcase 2.1, when \( v \) or \( \pi(v) \) goes to the solution, \( k \) decreases by 1 and \( \gamma(I) \) remains the same. Thus \( \mu(I) \) decreases by 1. In Subsubcase 2.2.1, clearly \( \mu(I) \) decreases by 1 in the first and second branch as \( k \) value decreases by 1. In the third branch, when we include \( \{v, \pi(v), v'\} \) in \( W_2 \), \( \gamma(I) \) drops at least by 1 and \( k \) remains the same, therefore \( \mu(I) \) decreases at least by 1. Thus, we have a branching vector \((1, 1, \geq 1)\). Similarly, we have a branching vector \((1, 1, \geq 1)\) for Subsubcase 2.2.2. As the maximum number of branches is 3, the running time of our algorithm is \( 3^{\mu(I)}n^{O(1)} \). Since we have \( \mu(I) = k + \gamma(I) \), the running time of our algorithm is \( 3^{k+\gamma(I)}n^{O(1)} \). \(\square\)

### 2.2 An algorithm for Max Min FVS

Given an input instance \((G, k)\), greedily find a minimal feedback vertex set \( W \) of \( G \). If \( |W| \geq k \), then \((G, k)\) is a yes-instance. Otherwise, we have a minimal feedback vertex set \( W \) of size at most \( k - 1 \), that is, \( \gamma(I) \leq k - 1 \) and the goal is to decide whether \( G \) has a minimal feedback vertex set \( S \) of size at least \( k \).

We do the following. We guess the intersection of \( S \) with \( W \), that is, we guess the set \( W_1 = S \cap W \), and reduce parameter \( k \) by \( |W_1| \). For each guess of \( W_1 \), we set \( W_2 = W \setminus W_1 \) and solve EXTENSION Max Min FVS on the instance \((G, W_1, W_2, k - |W_1|)\). If for some guess, \( G \) has a minimal feedback vertex set \( S \) such that \( W_1 \subseteq S, W_2 \cap S = \emptyset \) and \( |S \setminus W_1| \geq k - |W_1| \), then we conclude that the given instance of Max Min FVS is a yes-instance. Otherwise, we conclude that the given instance of Max Min FVS is a no-instance. The number of all guesses is bounded by \( \sum_{i=0}^{k-1} \binom{k-1}{i} \). We have an algorithm solving EXTENSION Max Min FVS in time \( 3^{k+\gamma(I)}n^{O(1)} = g^n n^{O(1)} \) as \( k + \gamma(I) \leq 2k - 1 \). Therefore we have an algorithm solving Max Min FVS in time

\[
\sum_{i=0}^{k-1} \binom{k-1}{i} g^{k-i} n^{O(1)} = 10^kn^{O(1)}.
\]

Thus we obtain Theorem 1.

### 3 FPT algorithm parameterized by vertex cover number

In this section we prove that Max Min FVS is FPT when parameterized by vertex cover number \( \text{vc}(G) \).

**Proof (The Proof of Theorem 2).** If \( G = (V, E) \) has a vertex \( v \) of degree at most 1, remove it from the graph. We find a vertex cover \( C \) of size at most \( \text{vc}(G) \)
of the reduced graph $G$. For our purpose, a standard branching algorithm with $O(2^{\text{vc}(G)n})$ running time is sufficient (see e.g. [4]). We denote by $I$ the independent set $V \setminus C$. We next guess $C_{in} = S \cap C$, where $S$ is a largest minimal FVS. There are at most $2^{\text{vc}(G)}$ candidates for $C_{in}$ as each member of $C$ has two options: either in $C_{in} = S \cap C$ or $C_{out} = S \cap C$. Clearly, $C_{out} = S \cap C$ contains the vertices of $C$ which are outside the solution. If $G[C_{out}]$ is not a forest then return that $C_{in}$ is a wrong guess and reject it. So from now onwards we assume that $G[C_{out}]$ is indeed a forest. Next we check minimality of $C_{in}$, that is, for each $v$ in $C_{in}$, whether $G[V \setminus C_{in} \cup \{v\}]$ has a cycle containing $v$. If the minimality of $C_{in}$ is not satisfied then return that $C_{in}$ is a wrong guess.

**Outline of the algorithm:** Given a guess $C_{in}$, our goal is to find a largest minimal FVS containing $C_{in}$. We look for a set $Z \subseteq I$ of vertices which can be added to $C_{out}$ so that $G[C_{out} \cup Z]$ remains a forest and every vertex $v \in V \setminus Z$ has at least two neighbours in some component of $G[C_{out} \cup Z]$. This is why every vertex in $I \setminus Z$ must be included in the solution. Finally the algorithm outputs $C_{in} \cup (I \setminus Z)$.

**Algorithm to find $Z$:** If $G - C_{in}$ has a vertex $v$ of degree at most 1, remove it from the graph. If there is a vertex $v$ in $I$ such that $G[C_{out} \cup \{v\}]$ contains a cycle, then include $v$ in the solution. Suppose $S'$ is a minimal FVS such that $C_{in} \subseteq S'$. We know $G - S'$ is a forest $F$. Suppose $F$ has exactly $q$ trees $T_1, T_2, \ldots, T_q$. Note that the number of trees in $F$ is at most $\text{vc}(G)$, that is, $q \leq \text{vc}(G)$. We guess a partition $P = \{P_1, P_2, \ldots, P_q\}$ of $C_{out}$. We say the partition $P = \{P_1, P_2, \ldots, P_q\}$ corresponds to trees of $F$, if $P_i = C_{out} \cap V(T_i)$. For each part $P_i$ there must exist a set $Z_i \subseteq I$ of vertices such that $G[P_i \cup Z_i] = T_i$. Otherwise $P$ is a wrong guess. Note that $Z = \bigcup_{i=1}^{q} Z_i$. There are at most $\text{vc}(G)^{2\text{vc}(G)}$ candidates for $P$ and we try out all guesses.

**Algorithm to find $Z_i$:** Consider $i$th part $P_i$ of $P$. Note that $G[P_i]$ is a collection of trees. Given $P_i$, we want to have a set $Z_i \subseteq I$ of vertices such that $G[P_i \cup Z_i]$ is a tree. This can happen in different ways. For example, it may be the case that only one vertex $z$ of $I$ connects all trees of $G[P_i]$ to form a single tree. It may be the case that we need $s_i > 1$ vertices of $I$ to connect all trees of $G[P_i]$ to form a single tree. We further guess a partition $P_i = \{P_{i1}, P_{i2}, \ldots, P_{in}\}$ of $P_i$ into $s_i$ parts. For each $P_i$ there are at most $\text{vc}(G)^{|P_i|}$ possible partitions. Given a partition $P_i = \{P_{i1}, P_{i2}, \ldots, P_{in}\}$ of $P_i$ we want to have a set $Z_i = \{z_{i1}, z_{i2}, \ldots, z_{in}\}$ of vertices such that $z_{ij} \in I$ is adjacent to exactly one vertex of every tree in $P_{ij}$. Thus $G[z_{ij} \cup P_{ij}]$ forms a tree. Next we guess how these $s_i$ trees are joined to form a single tree. We need $s_i - 1$ cross edges of the form $(z_{ij}, v_{ik})$ where $v_{ik} \in P_{ik}$, $j \neq k$ to join $s_i$ trees. See Figure 3. There are $s_i(s_i - 1)$ cross edges of the form $(z_{ij}, v_{ik})$ where $v_{ik} \in P_{ik}$, $j \neq k$. Thus $s_i - 1$ edges can be selected in at most $(s_i^2)^{s_i}$ many ways. So the total number of selections of cross edges for all $P_i$’s together is at most $\prod_{i=1}^{q} (s_i^2)^{s_i} \leq \text{vc}(G)^{2\text{vc}(G)}$. Thus there are total $\text{vc}(G)^{4\text{vc}(G)}$ guesses for the structure of trees involving vertices $V(P_i) \cup Z_i$ and cross edges for all $i$ combined. A selection of $s_i - 1$ cross edges is a valid selec-
Fig. 3. An illustration for partition of $P_i$ into four parts $P_{i1}, P_{i2}, P_{i3}, P_{i4}$; cross edges $(z_{i1}, v_{i2}), (z_{i1}, v_{i4}), (z_{i4}, v_{i3})$ are shown in orange. For simplicity trees in each part are singleton.

Fig. 4. Illustration of Case 1 and Case 2.

*Case 1:* Assume that $P_{ij}$ contains at least two trees. Then there is a cycle $C$ containing $x_2$ where the remaining vertices of $C$ are outside the solution; see Figure 4(a). Therefore $x_2$ must go to the solution in order to destroy $C$.

*Case 2:* Assume that $P_{ij}$ contains exactly one tree. Recall that if $G - C_{in}$ has a vertex $v$ of degree at most 1, then we remove it from the graph. Therefore $x_1$ and $x_2$ have degree at least two. Since $d(x_1), d(x_2) \geq 2$, both of them must
have a neighbour in some \( P_{ik} \) where \( j \neq k \), and \((u_{ij}, v_{ik})\) is a cross edge in the
given guess. Therefore, the graph \( G[V(P_{ij}) \cup V(P_{ik}) \cup \{x_1, x_2, x_3\}] \) has a cycle
\( C \) containing \( x_2 \) where \( x_3 \) is a candidate vertex for \( z_{ik} \) and \( x_3 \) is outside the
solution. See Figure 4(b). Note that \( C \) contains \( x_2 \) where the remaining vertices
are outside the solution. So \( x_2 \) must be inside the solution in order to destroy
\( C \); this proves the claim.

So while choosing a candidate for \( z_{ij} \), we only make sure that the remaining
candidates which are going to the solution do not disturb the minimality of \( C_{in} \).
If there is no such candidate then we return that it is a wrong guess. Clearly,
it takes polynomial time to check if \( Z_i \) exists for all \( i \). For a given guess \( C_{in} \), if
\( Z = \bigcup_{i=1}^{n} Z_i \) exists in \( I \) then we see that \( C_{in} \cup (I \setminus Z) \) forms a minimal feedback
vertex set containing \( C_{in} \).

Given \( C_{in} \), in order to compute \( Z \) we consider at most \( \text{vc}(G)^{4\text{vc}(G)} \) guesses.
Thus given \( C_{in} \) we can compute \( Z \) in time \( \text{vc}(G)^{4\text{vc}(G)} n^{O(1)} \). Given \( C_{in} \), the
above algorithm returns either a minimal FVS containing \( C_{in} \) or returns \( C_{in} \)
is a wrong guess. Finally we consider the maximum size solution obtained over
all guesses. As there are \( 2^{\text{vc}(G)} \) candidates for \( C_{in} \), we can solve the problem in
\( \text{vc}(G)^{5\text{vc}(G)} n^{O(1)} \) time.

4 No polynomial kernel parameterized by \( \text{vc}(G) \)

We proved that MAX MIN FVS is FPT when parameterized by vertex cover
number \( \text{vc}(G) \), and in this section we show kernelization hardness of the problem.

**Theorem 4.** MAX MIN FVS parameterized by \( \text{vc}(G) \) does not admit a poly-
nomial compression unless \( \text{coNP} \subseteq \text{NP/poly} \).

**Proof.** We give a polynomial parameter transformation (PPT) from the MAX
MIN VC problem. Given an instance \((G, k)\) of MAX MIN VC, we construct in
polynomial time an instance \((G', k')\) of MAX MIN FVS as follows. We start with
the graph \( G \). We add a new vertex \( x \) and make it adjacent to every vertex of \( G \).
We add a set \( V_x = \{x_1, x_2, \ldots, x_{n+3}\} \) of new vertices and make \( x \) adjacent to
every vertex of \( V_x \). Furthermore, we add a new vertex \( y \) and make \( y \) adjacent to
every vertex of \( V_x \). This completes the construction of \( G' \). It is easy to see that
\( \text{vc}(G') \leq \text{vc}(G) + 2 \). Finally we set \( k' = k + n + 2 \). We claim that \( G \)
contains a minimal vertex cover of size at least \( k \) if and only if \( G' \) contains a minimal
feedback vertex set of size at least \( k' \).

Suppose \( C \) is a minimal vertex cover in \( G \) such that \( |C| \geq k \). We observe
that the set \( C \cup \{V_x \setminus \{x_1\}\} \) forms a minimal feedback vertex set of size at least
\( k' \) in \( G' \). Conversely, suppose that \( G' \) has a minimal feedback vertex set \( S \)
of size at least \( k' \). First we see that \( x \not\in S \). This is true because if \( x \in S \) then
\( (V_x \cup \{y\}) \cap S = \emptyset \) as the vertices in \( V_x \cup \{y\} \) are not part of any cycle in \( G' \setminus x \).
This implies that \( |S| \leq n + 1 \) which is a contradiction. Similarly, we can argue
that \( y \not\in S \). As we know that \( \{x, y\} \cap S = \emptyset \), we must have \( |S \cap V_x| = n + 2 \). This
implies that $|S \cap V(G)| \geq k$. Now, we show that $C = S \cap V(G)$ is a minimal vertex cover of $G$. It is easy to see that $C = S \cap V(G)$ is a vertex cover of $G$ otherwise $S$ will not be a feedback vertex set of $G'$. For the sake of contradiction assume that $C \setminus \{x\}$ is also a vertex cover of $G$. In this case, we observe that $x$ is not part of any cycle in $G' - (S \setminus \{x\})$ which is a contradiction. Therefore $C$ is a minimal vertex cover of size at least $k$ in $G$. \hfill \Box

5 An fpt-AS for Max Min FVS parameterized by $\text{vc}(G)$

An fpt-approximation scheme (fpt-AS) with parameterization $\kappa$ is an algorithm whose input is an instance $x \in I$ and an $\epsilon > 0$, and it produces a $(1 - \epsilon)$-approximate solution in time $f(\epsilon, \kappa(x)) \cdot |x|^{O(1)}$ for some computable function $f$. In this section we prove Theorem 3.\hfill \Box

Proof (The Proof of Theorem 3). We present an $f(\epsilon, \text{vc}(G)) \cdot n^{O(1)}$ time algorithm that produces a $(1 - \epsilon)$-approximate solution for the problem, where $n$ is the number of vertices in the input graph $G$. We assume that we have a minimum vertex cover $C$ of size $\text{vc}(G)$ of the input graph $G = (V, E)$. We denote by $I$ the independent set $V \setminus C$. Our goal here is to find a largest minimal FVS $S$ with $C_{in} = S \cap C$, where $C_{in} \subseteq C$ is given. That is, we guess the intersection of $S$ with vertex cover $C$. There are $2^{\text{vc}(G)}$ possible guesses. Clearly, $C_{out} = S \cap C$ contains the vertices of $C$ which are outside the solution. If $G[C_{out}]$ is not a forest then return that $C_{in}$ is a wrong guess and reject it. So from now onwards we assume that $G[C_{out}]$ is indeed a forest. We give some reduction rules to simplify the input instance.

Reduction EMMFVS 4 If there is a vertex $u \in I$ with at most one neighbour in $C_{out}$, delete $u$.

Reduction EMMFVS 5 If there is a vertex $u \in I$ such that $G[C_{out} \cup \{u\}]$ contains a cycle, then include $u$ in the solution and delete $u$.

Fig. 5. Here $Q(a) = \{C_1, C_2, C_3\}$, $Q(b) = \{C_1, C_2, C_4\}$, $Q(c) = \{C_1, C_3\}$, $Q(d) = \{C_2, C_3, C_4\}$. Note that $S_a = \{b, c, d\}$.
The algorithm first applies Reductions EMMFVS 1, EMMFVS 4, and EMMFVS 5 exhaustively. Every vertex \( u \in I \) has at least two neighbours in \( C_{\text{out}} \), otherwise Reduction EMMFVS 4 would have been applied. Since Reduction EMMFVS 4 cannot be applied, we have that no two neighbours of \( u \in I \) belong to the same connected component of \( G[C_{\text{out}}] \). On the reduced instance, we run the following greedy algorithm. Suppose that \( C_{\text{out}} \) has connected components \( C_1, C_2, \ldots \). We say connected component \( C_i \) is a neighbour of \( u \in I \), that is \( C_i \in Q(u) \), if \( G \) contains an edge \((u, v)\) for some \( v \in C_i \). We pick an arbitrary vertex \( u \in I \) and define \( S_u = \{ x_i \in I : |Q(u) \cap Q(x_i)| \geq 2 \} \). See Figure 5.

Note that if \( u \) is not included in \( S \) then all the vertices of \( S_u \) must be included in \( S \). The intention is that if \( |S_u| \geq 2 \), then we prefer not to include \( u \) in \( S \), and hence include all the vertices of \( S_u \) in \( S \) as it is a maximization problem. But while including the vertices of \( S_u \) in the solution, we need to be careful about whether the inclusion of \( S_u \) in the solution, disturbs the minimality property of \( C_{\text{in}} \). This can be verified by checking if each vertex in \( C_{\text{in}} \) still has a private cycle after the inclusion of \( S_u \) in the solution. Based on the above observations, we propose the following algorithm. We pick an arbitrary vertex \( u \in I \), compute \( S_u \) and check the minimality of \( C_{\text{in}} \) assuming \( S_u \) is included in the solution. If the minimality of \( C_{\text{in}} \) is preserved then we set \( S = S \cup S_u \), \( I = I \setminus (S_u \cup \{u\}) \), and \( C_{\text{out}} = C_{\text{out}} \cup \{u\} \), that is, we include \( S_u \) in the solution and move \( u \) to \( C_{\text{out}} \). As \( u \) has neighbours in at least two connected components of \( G[C_{\text{out}}] \), when we move \( u \) to \( C_{\text{out}} \), the number of components in \( G[C_{\text{out}}] \) drops by at least 1. The algorithm again applies Reductions EMMFVS 4 and EMMFVS 5 exhaustively as \( C_{\text{out}} \) has been modified. On the other hand, if the minimality of \( C_{\text{in}} \) is not preserved then we set \( S = S \cup \{u\} \) and \( I = I \setminus \{u\} \). We repeat the above until \( I \) becomes empty.

There are \( 2^{\text{vc}(G)} \) candidates for \( C_{\text{in}} \); for each guess the above algorithm returns a minimal FVS and finally we consider the maximum size solution obtained over all guesses. Suppose the algorithm outputs \( S \). Let \( S_{\text{opt}} \) be an optimum solution. We claim that \( |S| \geq |S_{\text{opt}}| - \text{vc}(G) \). Let \( C_{\text{in}} = C \cap S_{\text{opt}} \). Clearly, we have \( |S_{\text{opt}}| \leq |C_{\text{in}}| + |I| \). Recall that the greedy algorithm adds all vertices from \( I \) to the solution except the ones that are moved to \( C_{\text{out}} \). We claim that there are at most \( \text{vc}(G) \) vertices from \( I \) that are moved to \( C_{\text{out}} \) and therefore not added to the solution. Every time a vertex is moved to \( C_{\text{out}} \), the number of connected components in \( G[C_{\text{out}}] \) drops by at least one. After moving \( \text{vc}(G) - 1 \) vertices, the number of connected components in \( G[C_{\text{out}}] \) becomes one. Therefore, we have moved at most \( \text{vc}(G) \) vertices to \( C_{\text{out}} \). This proves that \( |S| \geq |C_{\text{in}}| + I - \text{vc}(G) \geq |S_{\text{opt}}| - \text{vc}(G) \).

Next, we propose an FPT approximation scheme. Given an input graph \( G \), we first ask whether \( G \) has a minimal FVS of size at least \( \frac{\text{vc}(G)}{e} \). Note that this can be answered in time \( 10^{\text{vc}(G)} n^{O(1)} \) using the FPT algorithm proposed in Section 2. If this is a no-instance then we can find \( \text{opt}(G) \) in the same time by repetitively using the FPT algorithm. If this is a yes-instance, then obviously
opt(G) ≥ \frac{\text{vc}(G)}{\epsilon}. Hence the value of the constructed solution \( S \) is at least
\[
\frac{\text{opt}(G) - \text{vc}(G)}{\text{opt}(G)} \geq 1 - \frac{\text{vc}(G)}{\text{opt}(G)} \geq 1 - \epsilon
\]
times the optimum. That is, the constructed solution \( S \) is a \((1 - \epsilon)\)-approximate solution. \( \square \)

6 Conclusion and Open Problems

In this paper, we have studied \textsf{Max Min FVS} parameterized by the solution size and the vertex cover number of the input graph. We gave a single exponential time algorithm for \textsf{Max Min FVS} parameterized by solution size and a \( \mathcal{O}^*(2^{O(\text{vc}(G) \log \text{vc}(G))}) \) time algorithm parameterized by \( \text{vc}(G) \). Finally we proposed an FPT-AS parameterized by \( \text{vc}(G) \) with better running time \( 2^{O\left(\frac{\text{vc}(G)}{\epsilon}\right)} n^{O(1)} \). We list some nice problems that emerge from the results here: can our algorithm parameterized by \( \text{vc}(G) \) be made \( \mathcal{O}(\text{vc}(G)) \) time algorithm, for a fixed constant \( c \)? A \( 2^{O(\omega(1))} n^{O(1)} \) time algorithm seems possible where \( \omega \) is the treewidth of the input graph. Therefore, it would be interesting to see if the idea in Theorem 2 can be extended to study \textsf{Max Min FVS} parameterized by \( \text{fvs}(G) \).

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