Snake states of neutral atom from synthetic gauge field in a ring-cavity

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We propose the creation of an atomic analogue of electronic snake states in which electrons move along one-dimensional snake-like trajectory in the presence of a suitable magnetic field gradient. To this purpose, we propose the creation of laser induced synthetic gauge field inside a three-mirror ring cavity and show that under appropriate conditions, the atomic trajectory in such configuration mimics snake-state like motion. We analyse this motion using semi-classical and full quantum mechanical techniques for a single atom. We provide a detailed comparison of the original electronic phenomena and its atomic analogue in terms of relevant energy and length scales and conclude by briefly pointing out the possibility of consequent study of ultra cold condensate in similar ring-cavity configuration.

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I. INTRODUCTION

The ability to control the dynamics of an ultra cold atomic system at nano-kelvin temperature using light-atom interaction provides a fertile ground for quantum simulation [1–3]. Particularly interesting are a wide range of exotic phenomena that arise in an electronic condensed matter system coupled to real electromagnetic field, can now be quantum simulated at a completely different energy and length scale with the creation of a wide variety of light-induced synthetic gauge field for such charge neutral atoms [4–12]. For example, synthetic gauge-field driven quantum simulations can validate the superfluidity in a strongly interacting neutral fermionic system driven through the BCS-BEC cross-over [13–15]; creation of synthetic spin-orbit coupling for charge neutral bosonic [16] or fermionic atoms [17] can enable the realisation of topological phases [18, 19] in such non-electronic charge neutral quantum materials; it can herald futuristic quantum technology through the realisation of atomtronic devices such as an atom-squid [20] - to name a few. The major focus in this endeavour was the creation of quantum Hall phases [21–23] that requires uniform synthetic magnetic field or other topological phases [24]. In comparison to these studies, much less investigation took place to explore such atomic dynamics in synthetic magnetic field that is nonuniform in space and the consequent quantum simulation of any electronic phenomena. In this work, we extend the ambit of such magnetic field driven quantum simulation by proposing the atomic analogue of snake states of electrons that describes one dimensional snake-like motion of an electron in a specific form of in-homogenous magnetic field that changes sign at a given point and has been recently realised in graphene p-n junction [25, 26].

Snake states are special one-dimensional current carrying states that occur in two-dimensional electron gas (2DEG) due to magnetic field gradient, and, exists at the boundary where the applied transverse magnetic field changes sign \((B = 0)\). They were first theoretically studied by Müller [27] who also pointed out their connection with the conventional quantum Hall edge states [28]. The first experimental confirmation of such states was carried in a 2DEG realised in GaAs-AlGaAs heterostructure by K. von Klitzing’s group [29]. These developments were subsequently followed by a substantial number of theoretical and experimental studies [30–37].

To realise such light induced synthetic gauge field that can create the atomic analogue of such snake states, we consider a single two-level atom interacting with two counter-propagating running wave modes in a three-mirror ring cavity (see Fig.1). In a cavity, the optical field is quantized [38] and there is appreciable back action of the atomic motion on the cavity modes [39] leading to novel quantum phases of many-body atomic system through self-organisation [40–43]. Recently, it has been suggested that ultra cold atomic system inside a single mode [11] or a multimode cavity [12] can be used to generate synthetic gauge field that are dynamical in nature. Both these studies carried out in a linear cavity.

Whereas a linear cavity supports only standing wave modes, in a three mirror ring cavity that we consider here, provides a more richer variety of options. The laser beam can be pumped-in from two different mirrors and they travel in opposite directions inside the cavity [44–57]. Additionally, a laser beam can also be pumped-in from the transverse direction by directly illuminating the atoms. Consequently a ring cavity supports pairs of counter-propagating travelling wave modes (clockwise and counter-clockwise) apart from the standing wave modes (sine and cosine) [44, 45]. Each of these two counter-propagating modes can be populated separately. The coherent scattering of photons by atoms results in redistribution of the photons in these two counter-propagating travelling modes [44, 46].

In this work, we consider such a ring-cavity set up to propose a non-uniform synthetic transverse gauge field for a single charge-neutral ultra cold atom that changes its sign about a point of symmetry where it vanishes, and the resulting snake trajectories of the charge-neutral atom in such gauge field. A brief outline of the paper is as follows. In section II, we describe the considered ring cavity system, its corresponding Hamiltonian and the dressed state energies of the atom in the presence of...
cavity fields. In section III, we derive the Schrödinger equation for the atomic wavefunction, and obtain the equation for the probability amplitude for the atom to be in the lowest energy dressed state. Due to the adiabatic following of the lowest energy dressed state, we get a vector potential which gives rise to a non-uniform magnetic field. In section IV, we obtain the trajectories of the atom moving in the presence of such a non-uniform magnetic field. In section V, we provide an analysis of the effective potential and the energy spectrum for the atom moving in such magnetic fields.

II. MODEL SYSTEM AND THE HAMILTONIAN

The total Hamiltonian describing the coupled atom-cavity system [58] depicted in Fig. 1 can be written as

\[ \hat{H}_{SP} = \hat{H}_A + \hat{H}_C + \hat{H}_{\text{int}}. \]  

(1)

Here the atomic and cavity part of the hamiltonian are respectively given as

\[ \hat{H}_A = \frac{\hat{\vec{P}}^2}{2m_a} + \hbar \omega_a \hat{\sigma}_z \]  

(2)

\[ \hat{H}_C = \hbar \omega_c (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) \]  

(3)

Here \(|g\rangle\) and \(|e\rangle\) are respectively the ground and excited state of the two-level atom with energy \(E_g\) and \(E_e\) respectively, \(\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|\) is the Pauli matrix, \(E_e - E_g = \hbar \omega_a\), \(\hat{\vec{p}} \) is the internal two-dimensional Hilbert space of the atom, and \(\vec{r}^*\) and \(\vec{P}\) are the atomic centre-of-mass co-ordinate and momentum. We also introduce \(|n_1, n_2\rangle\) to denote the number state for the photons of the two counter-propagating running wave modes as shown in Fig. 1 respectively labelled by the cavity field operators \(\hat{a}_1\) and \(\hat{a}_2\), respectively, which are degenerate with cavity resonance frequency \(\omega_c\). The single two-level atom is directly illuminated by a far detuned transverse pump beam, at frequency \(\omega_p\). The excited atom scatters the photons into these cavity modes.

The third term of \(\hat{H}_{SP}\) describes the dipole interaction between the atom, the cavity field and the pump field and is given as

\[ \hat{H}_{\text{int}} = \hat{H}_{A-P} + \hat{H}_{A-C} = -\vec{d} \cdot \vec{E}_P - \vec{d}_C \cdot \vec{E}_C \]  

(4)

where \(\vec{d} = d(\hat{\sigma}^+ + \hat{\sigma}^-)\) is the dipole operator with \(\hat{\sigma}^+ = |e\rangle\langle g|\) and, \(\hat{\sigma}^- = |g\rangle\langle e|\). \(\hat{H}_{A-P}\) and \(\hat{H}_{A-C}\) describe respectively the interaction between the atom with the pump field and the cavity field in one arm of the ring cavity, and their corresponding electric fields are given by

\[ \vec{E}_P(\vec{r}) = \hat{e}_x E_0 \cos(ky + \phi) (e^{-i\omega_p t} + e^{i\omega_p t}) \]  

\[ \vec{E}_C(\vec{r}) = \hat{e}_y \sqrt{\frac{\hbar \omega_c}{2 \epsilon_0 V}} e^{-y^2/w_0^2} (\hat{a}_1 e^{i k x} + \hat{a}_2 e^{-i k x} + \text{h.c.}) \]  

with the cavity field polarized along the \(y\)-direction and the pump field polarized along \(x\)-direction. Here \(\epsilon_0\) is the vacuum permittivity, \(V\) is the mode volume, \(w_0\) is the beam waist, \(k = \frac{2\pi}{\lambda_p}\) and \(\lambda_p\) is the pump wavelength and \(\omega_p\) is the pump frequency.

It may be pointed out that the cavity decay rate, \(\kappa\), and the atomic spontaneous emission rate, \(\Gamma\), is included in the master equation for the atom-cavity density matrix, \(\rho\), by using the Lindblad operators [40]-

\[ \dot{\rho} = -\frac{i}{\hbar}[\hat{H}_{SP}, \rho] + L \rho, \]  

where, \(L \rho = \kappa \sum_{n=1,2} \{ \hat{a}_n \rho \hat{a}^\dagger_n - \{ \hat{a}^\dagger_n \hat{a}_n, \rho \} \} + \Gamma \sum_{n=1,2} \{ \hat{a}^\dagger_n \rho \hat{a}_n - \{ \hat{a}_n \rho \hat{a}^\dagger_n, \rho \} \}. \) We will be working in the good-cavity regime, \(g_0 >> \kappa\) [51], where the photon is emitted and reabsorbed a number of times before it leaves the cavity through dissipation processes and gives rise to a coherent evolution of the system. Since we are primarily interested in studying the effect of the atom-cavity interaction on the atomic dynamics, we neglect the cavity decay rate and the atomic spontaneous decay rate in our current work.

Since \(\Omega_0, g_0 << \omega_a, \omega_c\) \((g_0 \sim \text{GHz}, \omega_a, \omega_c \sim \text{THz})\) [46, 59], we can apply the rotating-wave approximation (RWA) to \(\hat{H}_{A-P}\) (see Appendix A) and to \(\hat{H}_{A-C}\) (see Appendix B). The resulting Hamiltonian for the atom-field system after RWA becomes

\[ \hat{H}_{SP} = \hat{H}_A + \hat{H}_C + \hat{H}_{\text{int}}. \]  

\[ = \frac{\hat{\vec{P}}^2}{2m_a} + \frac{\hbar \omega_a \hat{\sigma}_z}{2} + \hbar \omega_c (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) \]  

\[ + \hbar \Omega(y) \left( \hat{\sigma}^+ e^{-i\omega_p t} + \hat{\sigma}^- e^{i\omega_p t} \right) \]  

\[ + \frac{\hbar g(y)}{2} \left[ \hat{\sigma}^+ \hat{a}^\dagger_1 e^{ikx} + \hat{\sigma}^- \hat{a}_2 e^{-ikx} \right] + \frac{\hbar g(y)}{2} \left[ \hat{\sigma}^+ \hat{a}^\dagger_2 e^{-ikx} + \hat{\sigma}^- \hat{a}_1 e^{ikx} \right] \]  

(5)

where \(\Omega(y) = \Omega_0 \cos(ky + \phi)\), \(\Omega_0 = \frac{-\hat{e}_y E_0}{\hbar} \) is the Rabi frequency, \(\phi\) is the pump phase, \(g(y) = \frac{-\hat{e}_y}{\hbar} \sqrt{\frac{\hbar \omega_c}{2 \epsilon_0 V}} e^{-y^2/w_0^2} = g_0 e^{-y^2/w_0^2} \) and \(g_0 = \frac{-\hat{e}_y}{\hbar} \sqrt{\frac{\hbar \omega_c}{2 \epsilon_0 V}} \).

The time-dependence of the hamiltonian (5) can be removed through a unitary transformation by the operator

\[ \hat{U}(t) = e^{-i\omega_p t(\hat{a}^\dagger_1 + \hat{a}^\dagger_2 \hat{a}_1 + \hat{a}^\dagger_2 \hat{a}_2)} \]  

(6)

that takes it to the rotating frame of the pump field. The Hamiltonian \((\hat{H}_{SP})\) for the atom-field system after this transformation can then be written as [60] (see Appendix C for details) -

\[ \hat{H}_{RF} = \hat{H}_0 + \hat{H}_I \]  

(7)
FIG. 1: (Color online) (a) Schematic for a single two-level atom trapped inside a ring cavity. The atom is directly illuminated by a far red-detuned transverse pump laser beam with Rabi frequency, $\Omega_0$, which drives the off-resonant atomic transition between the internal states $|g\rangle$ and $|e\rangle$. The atom scatters the pump photons into either of the two cavity modes, labelled by the operators, $\hat{a}_1$ and $\hat{a}_2$. $\kappa$ is the cavity field decay rate. (b) Schematic for snake trajectories of a single atom inside the ring cavity in presence of a non-uniform synthetic magnetic field which arises due to the adiabatic following of the lowest energy dressed state. The magnetic field flips direction as $y$ changes sign and the blue and red trajectories are for two different strengths of the magnetic field.

where

$$\hat{H}_0 = \frac{\hat{p}^2}{2m_a} - \hbar \Delta_c \left( \hat{a}_1^{\dagger} \hat{a}_1 + \hat{a}_2^{\dagger} \hat{a}_2 \right)$$

$$\hat{H}_I = -\frac{\hbar \Delta_a}{2} \hat{\sigma}_z + \hbar \Omega(y) \left( \hat{\sigma}^+ + \hat{\sigma}^- \right) + \hbar g(y) \left[ \hat{\sigma}^+ \hat{a}_1 e^{ikx} + \hat{\sigma}^+ \hat{a}_2 e^{-ikx} + \hat{\sigma}^- \hat{a}_1^{\dagger} e^{-ikx} + \hat{\sigma}^- \hat{a}_2^{\dagger} e^{ikx} \right]$$

(8)

Diagonalizing $\hat{H}_I$ in these bare states basis, we get the following eigenstates that are called dressed states $|D_1\rangle$, $|D_2\rangle$ and $|D_3\rangle$, for the coupled atom-photon system, with $E_1$, $E_2$ and $E_3$ as their eigenvalues, respectively.

where $\Delta_a = \omega_p - \omega_a$ is the atom-pump detuning and $\Delta_c = \omega_p - \omega_c$ is the cavity-pump detuning. In our case where only a single pump laser is used, we set $\Delta_c = 0$ by considering the cavity is in resonance with the pump.

The second and third term of the $\hat{H}_I$ contains all the terms that represent atom-photon interaction. In the space spanned by the atom-photon bare-states, namely $|e, n_1, n_2\rangle$, $|g, n_1 + 1, n_2\rangle$ and $|g, n_1, n_2 + 1\rangle$, $\hat{H}_I$ contains off-diagonal terms and can be written as

$$\hat{H}_I = \begin{bmatrix}
-\frac{\hbar \Delta_a}{2} & \hbar g(y) e^{-ikx} \sqrt{n_1 + 1} & \hbar g(y) e^{-ikx} \sqrt{n_2 + 1} \\
\hbar g(y) e^{ikx} \sqrt{n_1 + 1} & \frac{\hbar \Delta_a}{2} & \hbar \Omega(y) \left( \hat{\sigma}^+ + \hat{\sigma}^- \right) \\
\hbar g(y) e^{ikx} \sqrt{n_2 + 1} & 0 & \frac{\hbar \Delta_a}{2}
\end{bmatrix}$$

(9)

They are

$$E_1 = -\frac{\hbar}{2} G; \quad |D_1\rangle = \begin{bmatrix}
- (G + \Delta_a) e^{-ikx} \\
\frac{g(y)}{2} e^{ikx} \sqrt{n_1 + 1} \\
\frac{g(y)}{2} e^{-ikx} \sqrt{n_2 + 1}
\end{bmatrix} A_1$$

(10a)
FIG. 2: Color online: Dressed-state energies ($E_1$, $E_2$ and $E_3$), as a function of the atom-pump detuning, $\Delta_a/2\pi$ for $g_0/2\pi = 0$ (solid lines) and for $g_0/2\pi = 50\,\text{GHz}$ (dotted lines). $E_2 < 0$ for $\Delta_a < 0$ and $E_2 > 0$ for $\Delta_a > 0$ for $g_0/2\pi = 0$ and $g_0/2\pi = 50\,\text{GHz}$. Refer text for the parameters used.

\[
E_2 = \frac{h\Delta_a}{2}; \quad |D_2\rangle = \begin{bmatrix} 0 \\ -e^{-2ikx}\sqrt{\frac{n_2+1}{n_1+n_2+2}} \\ \sqrt{\frac{n_1+1}{n_1+n_2+2}} \\ 0 \\ -e^{-2ikx}\cos\theta_{n_1,n_2} \\ \sin\theta_{n_1,n_2} \end{bmatrix} \quad (10b)
\]

\[
E_3 = \frac{h}{2}G; \quad |D_3\rangle = \begin{bmatrix} \frac{(G-\Delta_a)^{-1}e^{-ikx}}{2g(y)\sqrt{n_1+n_2+e^{-2ikx}}} \\ \frac{2g(y)e^{-2ikx}}{B_1B_2} \\ \frac{2g(y)e^{-2ikx}}{B_1B_2} \\ e^{-ikx}\cos\phi_{n_1,n_2} \\ e^{-2ikx}\sin\phi_{n_1,n_2}\sin\theta_{n_1,n_2} \\ \sin\phi_{n_1,n_2}\cos\theta_{n_1,n_2} \end{bmatrix} \quad (10c)
\]

where
\[
G = \sqrt{\Delta_a^2 + 4g^2(y)(n_1+n_2+2)} \\
A_1 = \sqrt{2G(G+\Delta_a)}; \quad B_1 = \sqrt{2G(G-\Delta_a)}
\]

\[
\sin\theta_{n_1,n_2} = \sqrt{\frac{n_1+1}{n_1+n_2+2}}; \quad \cos\theta_{n_1,n_2} = \sqrt{\frac{n_2+1}{n_1+n_2+2}}
\]

\[
\sin\phi_{n_1,n_2} = \frac{G+\Delta_a}{2G}; \quad \cos\phi_{n_1,n_2} = \frac{G-\Delta_a}{2G}
\]

\[
\theta_{n_1,n_2} = \tan^{-1}\left(\frac{n_1+1}{n_2+1}\right) \\
\phi_{n_1,n_2} = \tan^{-1}\left(\frac{G+\Delta_a}{G-\Delta_a}\right)
\]

$\theta_{n_1,n_2}$ and $\phi_{n_1,n_2}$ are called the mixing angles, while the laser phase is given by $\chi = kx$. The subscripts $n_1$ and $n_2$ denote the photon number dependence of the mixing angles. The atom-pump detuning gets modified in the presence of atom-field coupling and this modification is directly proportional to the cavity mode-function, $g(y)$ and to the number of photons in the two cavity modes. This dependence of the atom-pump detuning on the cavity photon number will later play an important role in the simulation of dynamical gauge fields for a single atom moving in the presence of two counter-propagating running wave modes inside the ring cavity. When the atom-field coupling, $g(y)$, goes to zero, then the dressed states come back to $|g,n_1+1,n_2\rangle$, $|e,n_1,n_2\rangle$, and $|g,n_1,n_2+1\rangle$.

We plot the dressed state energies ($E_1$, $E_2$ and $E_3$) as a function of atom-pump detuning $\Delta_a$, for different values of the Rabi-frequency, $g_0$, in Fig.(2). $E_2$ is independent of the Rabi-frequency, $g_0$, so we get, $E_2(\frac{g_0}{2\pi} = 0) = E_2(\frac{g_0}{2\pi} = 50\,\text{Hz})$. For $g_0 = 0$, we get a linear plot shown by the solid lines. As soon as we include $g_0 \neq 0$, the linear plot becomes non-linear due to the presence of the atom-field coupling (shown by dotted lines). The energy gap between the energy levels, $E_1$ and $E_3$, increases as we increase the value of the Rabi-frequency.

In the next section, we will solve the time-independent Schrödinger equation to find out the probability amplitude for finding the particle in the lowest energy dressed state, $|D_1\rangle$. The adiabatic evolution of the system in the lowest energy subspace will result in the appearance of a geometric vector potential and a scalar potential which will now depend on the number of photons in the two cavity modes and therefore, the resulting magnetic field will be dynamical.

III. EQUATION OF MOTION FOR THE ATOMIC WAVE-FUNCTION

Using the dressed state basis ($|D_j\rangle$ basis) for the internal Hilbert space of the atom at any point $\vec{r}$, we can write the full state vector of the atom as

\[
|\Psi(\vec{r},t)\rangle = \sum_{j=1,2,3} \psi_j(\vec{r},t)|D_j\rangle \quad (11)
\]
Suppose at an initial time, $t$, the atom is prepared in the lowest energy dressed state, $|D_1\rangle$. We define the recoil energy $\left(E_R = \frac{\hbar^2 k^2}{2m_a}\right)$, as the kinetic energy of an atom initially at rest when it absorbs or emits a single photon. The frequency scale corresponding to $E_R$ is $\omega_R$ and the frequency scale corresponding to the dressed state energies $(E_1, E_2, E_3)$ is $\omega_D$.

Typically the frequency scale of the atomic motion $\omega_R \sim 4kHZ$ and the frequency scale of the dressed state energies $\omega_D \sim 300GHz$. Thus the velocity of the atom can be considered so small, that it will continue to remain proportional to $|D_1\rangle$ at all times. This allows us to do the adiabatic approximation [9, 61–70]. Here, the position of the atom, $\vec{r}$, corresponds to the slow variables of the system, while, the dressed states $|D_1\rangle$, $|D_2\rangle$ and $|D_3\rangle$ correspond to the fast variables.

Since these dressed states are position-dependent, therefore, $P$ will act on both parts of the wave-function, differentiating them with respect to $\vec{r}$. Thus the action of the momentum operator on the total wave-function $|\Psi(\vec{r}, t)\rangle$ (see Appendix D)-

$$\hat{P}|\Psi(\vec{r}, t)\rangle = -i\hbar\vec{\nabla}\sum_{j=1,2,3} \psi_j(\vec{r}, t)|D_j\rangle = \sum_{j,l} \left[\vec{p}\delta_{l,j} - \vec{A}_{l,j}\right] \psi_j |D_l\rangle$$

(12)

where $\vec{A}_{l,j} = i\hbar \langle D_l|\vec{\nabla}|D_j\rangle$ and $\vec{p} = -i\hbar\vec{\nabla} = \text{does not act on the spinorial part}$. The kinetic energy has the following form (see Appendix D for details)-

$$\frac{\hat{P}^2}{2m_a}|\Psi(\vec{r}, t)\rangle = \frac{1}{2m_a} \hat{P} \left( \sum_{j,l=1,2,3} \left[\vec{p}\delta_{l,j} - \vec{A}_{l,j}\right] \psi_j |D_l\rangle \right)$$

$$= \frac{1}{2m_a} \sum_{j,l,m=1,2,3} \left\{ \left[\vec{p}\delta_{l,j} - \vec{A}_{l,j}\right] \left[\left(\vec{p}\delta_{m,l} - \vec{A}_{m,l}\right) \psi_j \right] \right\} |D_m\rangle$$

The Schrödinger equation for the total wave-function in the dressed state basis

$$i\hbar \frac{\partial}{\partial t}|\Psi(\vec{r}, t)\rangle = \hat{H}_{RF}|\Psi(\vec{r}, t)\rangle$$

(13)

Using the adiabatic approximation, we assume that $\psi_2, \psi_3 = 0$ and project Eq.(13) to the eigen space of the lowest energy dressed state, $|D_1\rangle$, to obtain -

$$\langle D_1| i\hbar \frac{\partial}{\partial t}|\Psi(\vec{r}, t)\rangle = \langle D_1| \hat{H}_{RF}|\Psi(\vec{r}, t)\rangle$$

(14)

Hence, $\vec{A}_{1,1}$ appears with the momentum and so, it acts as a synthetic vector potential while $\vec{A}_{2,1}$ and $\vec{A}_{3,1}$ will contribute to the scalar potential term. The synthetic scalar potential is given by $W = \frac{\hbar}{2m_a} \left( |\vec{A}_{2,1}|^2 + |\vec{A}_{3,1}|^2\right)$, which arises due to the adiabatic following of the dressed state, $|D_1\rangle$. In addition, the last term of Eq.(15), $-\hbar G/2$, acts as a deep trapping potential for the atomic centre-of-mass motion. The important dynamics of the system are governed by the vector potential, $\vec{A}_{1,1}$. The scalar potential, the vector potential and the corresponding magnetic field obtained here depends on the photon number in the cavity which is a dynamic quantity.

The full expression for the vector potential in Eq.

$$\Rightarrow i\hbar \frac{\partial}{\partial t} \psi_1(\vec{r}, t) = H_S \psi_1(\vec{r}, t) = \left[ \frac{1}{2m_a} \left\{ \left(\vec{p} - \vec{A}_{1,1}\right)^2 + |\vec{A}_{2,1}|^2 + |\vec{A}_{3,1}|^2\right\} - \frac{\hbar G}{2}\right] \psi_1(\vec{r}, t).$$

(15)
recoil momentum, $\hbar$ coupling through potential is directed along the $B$-axis and has the form
\[
\mathbf{A}_{1,1} = i\hbar \langle D_1 | \hat{\nabla} | D_1 \rangle
\Rightarrow A_y(y)\ddot{x} = \hbar k \left( \frac{G + \Delta_a}{2G} + \frac{8g^2(y)(n_1 + 1)}{2G(G + \Delta_a)} \right) \dot{x} \tag{16}
\]
The vector potential is similar like one in Landau gauge having only one component, but symmetric in $y$ and hence gives rise to an anti-symmetric magnetic field that is shown in Fig. (4a). It may be pointed out that such inhomogeneous synthetic gauge field can be created using different methods [71]. However in the presence of a ring cavity the strength of such gauge field depends on the photon numbers, $n_1$ and $n_2$ and the atom-photon coupling through $G$ and thus can be made more tunable. It is calculated in units of $B_0/l_B$ which has dimensions of recoil momentum, $\hbar k$.

The magnetic field corresponding to this vector potential is directed along the $z$-axis and has the form -
\[
\hat{\nabla} \times \mathbf{A}_{1,1} = B(y)\hat{z}
\]
with,
\[
B(y) = 4(n_1 - n_2)B_0 \frac{\Delta_a g_0^2}{G^3} \frac{y}{w_0} \exp(-\frac{2y^2}{w_0^2}) \tag{17}
\]

In the subsequent sections, we shall discuss the dynamics of a single atom in the presence of magnetic fields arising from such vector potentials first using a semiclassical method followed by a full quantum mechanical treatment.

**IV. TRAJECTORIES OF A SINGLE ATOMIC MOVING IN PRESENCE OF A NON-UNIFORM MAGNETIC FIELD**

In this section, we treat the equations of motion for a single atom moving in the presence of the non-uniform magnetic field, $B_z(y)\hat{z}$, which arises due the vector potential in Eq. (15) using semiclassical method. This enables us to find the trajectories of the atom moving in such a magnetic field and to identify them as snake orbits. In the next section we provide a full quantum mechanical treatment of the same problem. The Hamiltonian for the system can be written as -
\[
\hat{H} = \frac{\hat{p}^2}{2m_a} + \hat{V}(\hat{r}) \tag{19}
\]
where $B_0 = \frac{\hbar k}{w_0}$ and has the dimension of $[MT^{-1}]$. It defines the unit and the dimension of the synthetic magnetic field and the synthetic magnetic length is given by $l_B = \sqrt{\frac{\hbar}{eB_0}}$. The spatially varying magnetic field in the unit of $B_0$ is plotted in Fig. (4b). The magnetic field flips direction as $y$ changes sign. The strength of the magnetic field depends on the difference between the number of photons in the two cavity modes, $n_1$ and $n_2$. As the difference between $n_1$ and $n_2$ changes sign, the resulting magnetic field also flips the direction from $-\hat{z}$ to $\hat{z}$. The dependence of the magnetic field on the atom-field coupling, $g(y)$, gives it a Gaussian variation and it attains its peak value $B \sim \pm 0.03 B_0$ at $y \sim \pm 20 l_B$ for $g_0 = 2\pi \times 120 \text{ GHz}$.

The accompanying synthetic scalar potential, $W$, which appears in the equation for the probability amplitude, $\psi_1$, has a dependence on $y$. It is measured in units of energy, $E_0 = \frac{\hbar^2 k^2}{2ma^2}$ and is shown in Fig. (4c). The profile is given by
\[
W(y) = \frac{\hbar^2 k^2}{2ma} \frac{g^2(y)}{G^2(n_1 + n_2 + 2)} \left( n_1 + n_2 + 2 \right)^2 \left( \frac{\Delta_a}{k} \right)^2 \left( \frac{2y}{Gw_0} \right)^2 + \left( n_2 - n_1 \right)^2 + \frac{8G(n_1 + 1)(n_2 + 1)}{(G + \Delta_a)} \tag{18}
\]

where the effective potential $\tilde{V}(\hat{r})$ is given as
\[
\tilde{V}(\hat{r}) = \sum_{j=1,2,3} E_j(\hat{r}) \hat{Q}_j(\hat{r}) \tag{20}
\]
with $\hat{Q}_j(\hat{r}) = |D_j(\hat{r})\rangle \langle D_j(\hat{r})|$ is the projector onto the $j^{th}$-eigenstate of the atom-field coupling. We can define an $\hat{r}$ dependent force operator in this Hilbert space as
\[
\hat{F}(\hat{r}) = -\hat{\nabla} \tilde{V}(\hat{r})
\]
\[
= -\sum_{j} \left[ \left( \nabla E_j(\hat{r}) \right) \hat{Q}_j(\hat{r}) + E_j(\hat{r}) \left( \nabla \hat{Q}_j(\hat{r}) \right) \right] \tag{21}
\]

The expectation value $\langle \hat{F} \rangle$ is then given by $\langle \hat{F} \rangle = \langle \Psi | \hat{F}(\hat{r}) | \Psi \rangle$ where $|\Psi(\hat{r},t)\rangle = \sum_{j=1,2,3} \psi_j(t) |D_j(\hat{r}(t))\rangle$ comes out to be [72] -
\[
\langle \hat{F}(\hat{r},t) \rangle = -\sum_{j} |\psi_j|^2 \nabla E_j + \sum_{j,k} \psi_j^* \psi_k (E_k - E_j) |D_k| \nabla |D_j\rangle
\]

The semi-classical equation of motion for the atom in the lowest energy dressed state can now be given as (see Appendix F and [72] for details)
\[
m_a \frac{d\vec{v}}{dt} = -\nabla E_1 - \nabla W(\hat{r}) + \vec{v} \times \vec{B}(\hat{r}) \tag{22}
\]
For electrons $[27, 32]$  

| Quantity                        | For electrons $[27, 32]$ | For atoms $[MT^{-1}]$ |
|--------------------------------|--------------------------|-----------------------|
| Magnetic field                 | $B_0 = 2 \, T$           | $B_0 \sim 3.4 \times 10^{-23} \, kgs^{-1}$; $B_{peak} \sim 0.03 \, B_0$ |
| Dimensions of magnetic field   | $[MA^{-1}T^{-2}]$        |                       |
| Magnetic length                | $l_B = \frac{\hbar}{eB_0} \sim 18 \, nm$ | $l_B = \frac{\hbar}{eB_0} = \frac{\hbar}{2eB_0} \sim 2 \, \mu m$ |
| Cyclotron frequency            | $\omega_c = \frac{eB_0}{m_e} \sim 352 \, GHz$ | $\omega_{a0} = \frac{B_0 m_a}{eB_0} \sim 232 \, Hz$; $\omega_{cp} = \frac{B_{peak} m_a}{eB_0} \sim 7 \, Hz$ |
| Temperature                    | $T_0 = \frac{\hbar \omega_c}{k_B} \sim 3 \, K$ | $T_0 = \frac{\hbar \omega_c}{k_B} \sim 2 \, nK$; $T_p = \frac{\hbar \omega_c}{k_B} \sim 0.05 \, nK$ |
| Radius of the particle trajectory | $r_c \sim 0.5 \, nm$   | $r_a \sim 30 \, \mu m$ |
| Reference energy               | $E_0 \sim 0.12 \, meV$  | $E_0 \sim 7.67 \times 10^{-11} \, meV$ |
| Energy gap between Landau levels | $\delta_e \sim 0.13 \, E_0$ | $\delta_a \sim 0.03 \, E_0$ |

TABLE I: Comparison of the numerical values of quantities measured in typical condensed matter experiments and the numerical values obtained in our system.

FIG. 3: *Color online*: Snake states for a single atom in the presence of a non-uniform synthetic magnetic field, in the vicinity of $\hat{z} = 0$. The magnetic field $B(y)$ changes direction from $-\hat{z}$ as $\hat{y}$ changes sign. *Blue* (solid line) and *red* (dashed lines) trajectories correspond to two magnetic field strengths which depend on the atom-photon coupling strengths, $g_0/2\pi$. Refer text for the parameters used in the figures.

We solve the Eqs. $(24a-24b)$ to obtain the values of $x(t)$ and $y(t)$ and plot the trajectory of the atom in the $x - y$ plane as shown in Fig. (3), for 2 different values of the atom-photon coupling, $g_0$. At $y = 0$, $B(y) = 0$ and the atom experiences magnetic field with alternating directions on both sides of this line and consequently it moves in a snake trajectory which drifts along the $-\hat{x}$ direction, i.e., in a direction perpendicular to the synthetic field gradient $[27, 74]$. The radius of curvature for the particle trajectory is inversely proportional to the synthetic magnetic field $(r \propto \frac{1}{B(y)})$. Therefore, for large/small values of the magnetic field, the particle will trace a trajectory with a small/large radius of curvature and this results in the particle scanning a finite region in the $x - y$ plane. The appearance of snake-states gives rise to a non-vanishing flow of atoms along the $-\hat{x}$ direction.

The snake trajectories of electrons had been widely studied in various condensed matter systems consisting of two-dimensional electron gas $[27, 30, 32, 37]$, graphene $p - n$ junctions $[25, 26]$, etc. It may be noted that the dimensions of the magnetic field for an electron, $[MA^{-1}T^{-2}]$, includes the dimensions of charge, $([q] = [AT])$. Since an atom is charge neutral, the dimension of synthetic magnetic field for an atom, $[MT^{-1}]$. For the typical system parameters it will be therefore useful to compare the values of the various relevant quan-
tities in these two completely different system. As we demonstrate in the Table I in the relevant experimental range the synthetic magnetic field on the atom is relatively weaker, making the size of the trajectory much larger than the electron trajectory in a condensed matter system. In the next section, we shall do a full quantum mechanical treatment to understand the single particle energy spectrum for such snake orbits.

V. ENERGY SPECTRUM FOR AN ATOM IN A NON-UNIFORM MAGNETIC FIELD

We will now analyse the behaviour of an atom moving in the presence of a synthetic non-uniform magnetic field given by Eq.(17). The related energy spectrum of an electron in the presence of a uniform magnetic field $(\vec{B} = B_0 \hat{z})$, was studied by Landau almost a century back, using a particular choice of vector potential called Landau gauge. The eigenfunctions of the resulting Schrödinger equation are labelled by magnetic quantum number, $n$, and one of the momentum components, say $p_x = \hbar k_x$. The magnetic length, $l_B = \frac{\hbar}{\sqrt{\mu_B}}$, gives the spatial variation of the electronic wave-function. The energy levels known as Landau levels have energy $E_n = (n + \frac{1}{2}) \hbar \omega_c$ which are independent of $k_x$ and hence degenerate with the degree of degeneracy given by $\frac{1}{2n+1}$. The presence of a non-uniform magnetic field lifts the degeneracy of the Landau levels and results in a finite dispersion, $E_n(k_x)$ and this dispersion is plotted in Fig. 5. Here we provide an analysis of this dispersion.

In Eq.(16) $\tilde{A}_x(y)$ is independent of $x$, $\Rightarrow |\vec{p}_x, H_S\rangle = 0$. Hence the wave-function for the atom in the lowest energy ($|D_1\rangle$) subspace can be written as -

$$\psi_1(x, y) = e^{ik_x x} \phi(y)$$

(25)

We substitute the expression for $\psi_1(x, y)$ in Eq.(15), to obtain (Appendix G for details)-

$$\frac{\partial^2 \phi(y)}{\partial y^2} + \left[ \frac{E}{E_0} - V_{eff}(y) \right] \phi(y) = 0$$

(26)

where

$$V_{eff}(y) = V_{k_x}(y) + V_{tr}(y)$$

(27)

with

$$V_{k_x}(y) = (\tilde{k}_x - A_x(y))^2$$

(28)

$$V_{tr}(y) = \frac{W - \frac{hG}{2}}{E_0}$$

(29)

with $l_B = \sqrt{\frac{\pi}{\mu_B}} = \sqrt{\frac{\mu_0 \lambda_B}{2\pi}}$; $E_0 = \frac{\hbar^2}{2m_0 \lambda_B^2}$; $\tilde{k}_x = k_x l_B$; $\tilde{y} = \frac{y}{l_B}$. This is the Schrödinger equation for a single atom moving in the presence of an effective potential, $V_{eff}(y)$. Numerical values of the parameters used are $\Delta_2 = -2\pi \times 128$, $GHz$, $g_0 = 2\pi \times 120 GHZ$, $n_1 - n_2 = 1$, $\lambda_B = 780.5$ nm, $w_0 = 25.3$ $\mu$m and $B_0 \sim 3.4 \times 10^{-23}$ $kg s^{-3}$. The magnetic length $l_B \sim 2\mu$m, gives the spatial extent of the atomic wave-function in the considered parameter regime. Length and momentum values are, therefore, expressed in terms of $l_B$ and $l_B^{-1}$, respectively, and the energy values in the unit of $E_0/h = \frac{\hbar}{2m_0 l_B^2} \sim 116$ Hz. The total trap potential which is responsible for trapping the atom in the lowest energy dressed state can be approximated by a box potential with a constant depth, $V_0 \sim 4$ $GHz$, and a strong confinement along the $y$-direction at $y = \pm 35 l_B$ and can be eliminated for the purpose of studying the dynamics in the synthetic magnetic field ($\sim 100$Hz) represented by $V_{k_x}(y)$.

Therefore, the dynamics is almost entirely governed by $V_{k_x}(y)$ as shown in Fig. 5 where we plot the energy eigenvalues for an atom and relate them to the shape change of such effective $k_x$-dependent potential [27]. For a given $k_x$ the potential is symmetric function $y$. But the potential profile changes in a nontrivial and asymmetric way as one changes $k_x$. This may be contrasted with the behavior of the corresponding potential for the Landau problem in the presence of a uniform magnetic field (in Landau gauge) which is a symmetric function of both $\tilde{k}_x$ and $\tilde{y}$.

At a given value $\tilde{k}_x = \tilde{k}_x^{(c)}$, $V_{k_x}(\tilde{y})$ changes its shape. For the values used in our representative plot $\tilde{k}_x^{(c)} = 14.76$. At further higher value of $\tilde{k}_x$ the double well potential structure gets converted in a barrier potential. We plot the potentials $V_{k_x}(\tilde{y})$ as a function of $\tilde{y}$ for several representative values ($(\tilde{k}_x^{(1)}(14.7) < \tilde{k}_x^{(2)}(14.76) < \tilde{k}_x^{(3)}(15.0) < \tilde{k}_x^{(4)}(15.1))$ to demonstrate this change in the profile of the potential with changing $\tilde{k}_x$. This potential is symmetrical about $\tilde{y} = 0$. For $\tilde{k}_x < \tilde{k}_x^{(c)}$, it forms a single well with the minima of the potential lying at $\tilde{y} = 0$. For $\tilde{k}_x \geq \tilde{k}_x^{(c)}$, there are two symmetrically located minima at $\tilde{y} = \pm \tilde{y}_1$ around a maxima at $\tilde{y} = 0$. At the two minima, $V_{k_x}(\tilde{k}_x, \tilde{y}_1) = 0$. And as mentioned for higher values of $\tilde{k}_x$, the central maxima at $\tilde{y} = 0$ grows and effectively turns the potential in a potential barrier. The locations of the double minima points $\tilde{y}_1$ as a function of $\tilde{k}_x$ are plotted in Fig.6.

The energy spectrum for the atom is shown in Fig.5. We only plot few lowest energy levels that brings forth the quantum mechanical behaviour of a neutral atom in such synthetic non-uniform magnetic field. Energy bands of the magnetic problem in Eq.26 are labelled by the band index "n". We continue to call each such bands as Landau levels even though they have now free particle like dispersion. The minima of the lowest energy band lies at $\tilde{k}_x = 14.76$ and as the band index increases for higher excited states, the minima of the energy band shifts to a higher value of $\tilde{k}_x$.

To understand these energy bands better, we note that these energy bands show a strong asymmetry in $\tilde{k}_x$ which corresponds to the asymmetry in $V_{\tilde{k}_x}(\tilde{y})$ as a function of
FIG. 4: Color online: Geometric potentials and the resulting magnetic field, for two values of the atom-photon coupling strengths, arising due to the adiabatic following of the lowest energy dressed state, $|D_1\rangle$. Fig.(4a) Vector potential $A_x(\bar{y})$ (in units of $B_0 l_B$) as a function of $\bar{y}$. Fig.(4b) Magnetic field $B_z(\bar{y})$ (in units of $B_0 = \frac{\hbar k_w}{E_0}$) as a function of $\bar{y}$. Fig.(4c) Scalar potential $W(\bar{y})$ (in units of $E_0$) as a function of $\bar{y}$. Refer text for the parameters used.

FIG. 5: Color online: Energy, $E$ (in units of $E_0 = \frac{k^2}{2ma^2 l_B^2}$), as a function of $\bar{k}_x$. The effective potential, $V_{\bar{k}_x}(\bar{y})$, as a function of $\bar{y}$ is shown for 4 values of $\bar{k}_x$ referred in text and shows the transition of the effective potential from a single well to a double-well to a barrier. Refer text for the parameters used.

$\bar{k}_x$. For $\bar{k}_x > \bar{k}_x^{(c)}$, the effective potential has a double well structure that is symmetrically located about a central potential barrier at $\bar{y} = 0$. The states localised at the minima of each well form their own Landau level $E_n(\bar{k}_x)$ with parabolic dispersion like a free particle and are degenerate. With decreasing $\bar{k}_x$, the central barrier is getting lowered and there will be tunnelling across this barrier. It is because of this tunnelling the degeneracy

FIG. 6: Color online: $\bar{y}_1$ as a function of $\bar{k}_x$ which shows the double well minima at $\pm \bar{y}_1$ for $\bar{k}_x^{(2)}$ and $\bar{k}_x^{(3)}$. At $\bar{k}_x^{(4)}$, a barrier is formed which separates minima points. Since the barrier width does not increase with increase in $\bar{k}_x$, therefore, the minima points remain at a constant separation with increasing $k_x$. Refer text for the parameters used.
of the two branches of \( E_n(\vec{k}_x) \) will be lifted splitting it into symmetric and antisymmetric states separated by a gap. This explains the splitting of the energy levels for \( \vec{k}_x < \vec{k}_x^{(c)} \).

As we go to higher values of \( \vec{k}_x \), the potential forms a strong barrier and this is visible in the energy dispersion where the energy bands become degenerate in pairs. An approximate expression for the energy can be given as

\[
E_n(\vec{k}_x) \simeq \left(n + \frac{1}{2}\right) \frac{\hbar}{m_a} B(\vec{y}_1)
\]

where \( \vec{y}_1 \) is the solution of \( \left( \vec{k}_x - \frac{\Delta x(x)}{\Delta y(y)} \right) = 0 \). In the expression for \( E_n(\vec{k}_x) \), we have replaced the magnetic field in the cyclotron frequency, \( \omega_{cp} = \frac{\hbar_1(\vec{y}_1)}{m_a} \), by the expression obtained in Eq.(17). The spacing between the energy levels in this case is \( \delta_n \) and is of the order of 0.03\( E_0 \) (corresponds to a temperature of 0.03 \( nK \)) which is much smaller than the corresponding energy difference in an electron system, denoted by \( \delta_x \) and has a value of 0.13\( E_0 \) (corresponds to a temperature of 0.2 \( K \)).

VI. CONCLUSIONS

In this work, we have used atom-photon coupling inside a ring-cavity and adiabatic approximation to eliminate higher energy dressed states to propose a non-uniform geometric gauge field that supports the atomic analogue of electronic snake states. The current study is done in a non-interacting picture and we primarily investigate the single atom-dynamics for such atomic snake states and we obtain - 46, 59, and \( \vec{a} \) is the Rabi frequency and \( \phi \) is the pump phase. Since the atom-pump coupling strength \( \Omega_0 \) is of the order of \( \sim GHz \), \( \omega_a, \omega_p \sim THz \) \( [46, 59] \), and \( |\omega_a - \omega_p| \ll |\omega_a + \omega_p| \), we can use the rotating wave approximation. So we move to the interaction picture and the time evolution of the atomic field operators is found by solving the Heisenberg equation \( i \hbar \frac{d}{dt} \hat{a}^{\pm} = \left[ \hat{a}^{\pm}, \hat{H}_A \right] = \left[ \hat{a}^{\pm}, \frac{\omega_a^2}{2m} + \hbar \omega_a \right] \) and we obtain - 47, 59, \( \omega_a \) is the atomic resonance frequency. So, the atom-pump field Hamiltonian in the interaction picture is

\[
\hat{H}_{A-P} = \hbar \Omega(y) \left[ \hat{\sigma}^+(0) e^{(\omega_a - \omega_p)t} + \hat{\sigma}^+(0) e^{(\omega_a + \omega_p)t} \right]
\]

In the interaction picture, we can see that there are two types of terms with frequencies \( \omega_a - \omega_p \) and \( \omega_a + \omega_p \). For \( \omega_a \sim \omega_p \), the fast oscillating terms with frequency \( \omega_a + \omega_p \)

Appendix A: Atom-pump field interaction

In this section, we derive the expression for the atom-pump field interaction Hamiltonian, \( \hat{H}_{A-P} \) in the rotating-wave approximation as given in Eq.(5). The pump field is polarized along the \( x \)-direction and the expression for the electric field is given by -

\[
\vec{E}_p = \hat{e}_x E_0 \cos(ky + \phi)(e^{-i\omega_p t} + e^{i\omega_p t})
\]

Then, \( \hat{H}_{A-P} \) takes the following form -

\[
\hat{H}_{A-P} = -\vec{a} \cdot \vec{E}_p
= -\vec{a} \left( \hat{\sigma}^+ \cdot \hat{\sigma}^+ \hat{\sigma}^- \cdot \hat{\sigma}^- \right) \cdot \hat{\sigma}_x E_0 \cos(ky + \phi)(e^{-i\omega_p t} + e^{i\omega_p t})
= \hbar \Omega(y) \left[ \hat{\sigma}^+ e^{-i\omega_p t} + \hat{\sigma}^- e^{i\omega_p t} \right]
+ \hat{\sigma}^- e^{-i\omega_p t} + \hat{\sigma}^- e^{i\omega_p t}
\]

where \( \Omega(y) = \Omega_0 \cos(ky + \phi), \Omega_0 = \frac{\Delta x(x)}{\Delta y(y)} \) is the Rabi frequency and \( \phi \) is the pump phase. Since the atom-pump coupling strength \( \Omega_0 \) is of the order of \( \sim GHz \), \( \omega_a, \omega_p \sim THz \) \( [46, 59] \), and \( |\omega_a - \omega_p| \ll |\omega_a + \omega_p| \), we can use the rotating wave approximation. So we move to the interaction picture and the time evolution of the atomic field operators is found by solving the Heisenberg equation \( i \hbar \frac{d}{dt} \hat{a}^{\pm} = \left[ \hat{a}^{\pm}, \hat{H}_A \right] = \left[ \hat{a}^{\pm}, \frac{\omega_a^2}{2m} + \hbar \omega_a \right] \) and we obtain - 47, 59, \( \omega_a \) is the atomic resonance frequency. So, the atom-pump field Hamiltonian in the interaction picture is

\[
\hat{H}_{A-P} = \hbar \Omega(y) \left[ \hat{\sigma}^+(0) e^{(\omega_a - \omega_p)t} + \hat{\sigma}^+(0) e^{(\omega_a + \omega_p)t} \right]
+ \hat{\sigma}^+(0) e^{-i(\omega_a + \omega_p)t} + \hat{\sigma}^+(0) e^{i(\omega_a - \omega_p)t}
\]
can be neglected as compared to the slow oscillating terms with frequency \( \omega_a - \omega_p \). This is called the rotating wave approximation (RWA). We transform back from the interaction picture to the Schrödinger picture to get back the time-independent atomic field operators, \( \hat{\sigma}^\pm \), which appear in \( \hat{H}_{A-P} \). Therefore, the final form of the atom-pump field interaction in the Schrödinger picture is -

\[
\hat{H}_{A-P} = \hbar \Omega(y) \left( \hat{\sigma}^+ e^{-i\omega_p t} + \hat{\sigma}^- e^{i\omega_p t} \right)
\]  

(A4)

Appendix B: Atom-cavity field interaction

In this section, we find out the form of the atom-cavity field interaction, \( \hat{H}_{A-C} \) in the rotating wave approximation as given in Eqn.(5). \( \hat{H}_{A-C} \) describes the interaction between the atoms and the cavity field. The cavity field is polarized along the \( y \)-direction and the expression for the electric field in one arm of the ring cavity is given by -

\[
\tilde{E}_C(r,t) = e_y \sqrt{2\hbar \omega_c} e^{-y^2/\omega_0^2} \left( \hat{a}_1 e^{ikx} + \hat{a}_2 e^{-ikx} \right) + e_y \sqrt{2\hbar \omega_c} e^{-y^2/\omega_0^2} \left( \hat{a}_1^\dagger e^{-ikx} + \hat{a}_2^\dagger e^{ikx} \right)
\]  

(B1)

So again to get a clearer picture, we move to the interaction picture. The atomic field operators are given as \( \hat{\sigma}^\pm(t) = \hat{\sigma}(t) e^{\pm i\omega_0 t} \). The time evolution of the cavity field operators is found by solving the Heisenberg equation \( \hbar^2 \frac{\partial \hat{a}_1, \hat{a}_2}{\partial t} = [\hat{a}_1, \hat{H}_C] = [\hat{a}_1, \hbar \omega_c \hat{a}_1^\dagger \hat{a}_1 + \hbar \omega_c \hat{a}_2^\dagger \hat{a}_2] \) and we obtain - \( \hat{a}_1, \hat{a}_2(t) = \hat{a}_1, \hat{a}_2(0) e^{-i\omega_0 t} \). Similarly, for \( \hat{a}_1, \hat{a}_2 \), we get - \( \hat{a}_1^\dagger(t) = \hat{a}_1^\dagger(0) e^{i\omega_0 t} \), where \( \omega_0 \) is the cavity resonance frequency. Therefore, the atom-cavity field interaction in the interaction picture is -

\[
\hat{H}_{A-C}^I = -\vec{d} \cdot \vec{E}_C
\]

= \( \hbar g(y) \left[ \hat{\sigma}^+(t) \hat{a}_1(t) e^{ikx} + \hat{\sigma}^+(t) \hat{a}_2(t) e^{-ikx} \right. \\
+ \left. \hat{\sigma}^+(t) \hat{a}_1^\dagger(t) e^{-ikx} + \hat{\sigma}^+(t) \hat{a}_2^\dagger(t) e^{ikx} \right. \\
+ \left. \hat{\sigma}^-(t) \hat{a}_1(t) e^{ikx} + \hat{\sigma}^-(t) \hat{a}_2(t) e^{-ikx} \right. \\
+ \left. \hat{\sigma}^-(t) \hat{a}_1^\dagger(t) e^{-ikx} + \hat{\sigma}^-(t) \hat{a}_2^\dagger(t) e^{ikx} \right]
\]  

(B2)

where \( g(y) = -\frac{\vec{d} \cdot \vec{E}_y}{\hbar} \sqrt{\frac{2\hbar \omega_c}{e}} e^{-y^2/\omega_0^2} = g_0 e^{-y^2/\omega_0^2} \). If \( \omega_a \sim \omega_c \), then the terms with \( e^{\pm i(\omega_a - \omega_c)t} \) will have small transition amplitudes that are proportional to \( \frac{1}{\omega_a + \omega_c} \). Therefore, the fast oscillating terms with frequency \( \omega_a + \omega_c \) can be neglected as compared to the slow oscillating terms with frequency \( \omega_a - \omega_c \). This is called the rotating wave approximation. We transform back to the Schrödinger picture to obtain the final form of the atom-field interaction [78-80] -

\[
\hat{H}_{A-C} = \hbar g(y) \left[ \hat{\sigma}^+ \hat{a}_1 e^{ikx} + \hat{\sigma}^+ \hat{a}_2 e^{-ikx} \right. \\
+ \left. \hat{\sigma}^- \hat{a}_1^\dagger e^{-ikx} + \hat{\sigma}^- \hat{a}_2^\dagger e^{ikx} \right]
\]

Appendix C: Going to the rotating frame of the pump field

In this section, we provide the detailed steps to derive Eqn. (7). We show how the observables, the states and the Schrödinger equation transform under the unitary transformation given by Eqn. (6) of the main text and we also find out the expression for the single particle Hamiltonian, \( \hat{H}_{SP} \), in the rotating frame of the pump field. The unitary operator mentioned in Eqn. (6) is -

\[
\hat{U}(t) = e^{-i\omega_p t (\sigma_+ + \sigma_+ \sigma_2)}
\]

Under this unitary transformation, the observables transform as -

\[
\hat{O}_{RF} = \hat{U}^\dagger \hat{O} \hat{U}
\]

(C1)

The states transform as -

\[
|\Psi_{RF}\rangle = \hat{U}^\dagger |\Psi\rangle
\]

(C2)

In the rotating frame, the Schrödinger equation transforms as -

\[
\hbar i \frac{\partial}{\partial t} |\Psi_{RF}\rangle = \hbar \left[ \frac{\partial}{\partial t} \left( \hat{U}^\dagger |\Psi\rangle \right) \right] = \hat{H}_{RF} |\Psi_{RF}\rangle
\]

where

\[
\hat{H}_{RF} = \frac{-\hbar \omega_p \hat{\sigma}_z}{2} - \hbar \omega_c \hat{a}_1^\dagger \hat{a}_1 - \hbar \omega_p \hat{a}_2^\dagger \hat{a}_2 + \hat{U}^\dagger \hat{H}_{SP} \hat{U}
\]

(C3)

is the single-particle Hamiltonian in the rotating frame of the pump field. We now find \( \hat{U}^\dagger \hat{H}_{SP} \hat{U} \). We use the Baker-Hausdorff formula to find out the transformation of the operators appearing in \( \hat{H}_{SP} \).

Baker-Hausdorff formula

\[
\hat{e}^{i\hat{G}\lambda} A e^{-i\hat{G}\lambda} = A + i \lambda \left[ \hat{G}, \hat{A} \right] + \frac{i^2 \lambda^2}{2!} \left[ \hat{G}, \left[ \hat{G}, \hat{A} \right] \right] + \ldots + \frac{i^n \lambda^n}{n!} \left[ \hat{G}, \left[ \hat{G}, \ldots \left[ \hat{G}, \hat{A} \right] \right] \right] + \ldots
\]

where \( \hat{G} \) is Hermitian and \( \lambda \) is a real parameter. The operators in \( \hat{H}_{SP} \) transform as -

\[
\hat{U}^\dagger \hat{a}_1 \hat{U} = \hat{a}_1 e^{i\omega_p t}
\]

\[
\hat{U}^\dagger \hat{a}_2 \hat{U} = \hat{a}_2 e^{i\omega_p t}
\]

\[
\hat{U}^\dagger \hat{\sigma}_+ \hat{U} = \hat{\sigma}_+ e^{i\omega_p t}
\]

\[
\hat{U}^\dagger \hat{\sigma}_- \hat{U} = \hat{\sigma}_- e^{i\omega_p t}
\]

\[
\hat{U}^\dagger \hat{\sigma}_z \hat{U} = \hat{\sigma}_z
\]
Therefore, we get -

\[ \hat{U}^{\dagger} \hat{H}_{SP} \hat{U} = \hat{H}_{SP} \]

Therefore, the single particle Hamiltonian, \( \hat{H}_{RF} \), in the rotating frame of the pump field is -

\[
\begin{align*}
\hat{H}_{RF} &= \frac{\hbar^2}{2m_a} \hat{\jmath} - \frac{\hbar \Delta_\alpha}{2} - \hbar \Delta_c \left( \hat{a}_1^{\dagger} \hat{a}_1 + \hat{a}_2^{\dagger} \hat{a}_2 \right) \\
&\quad + \hbar \Omega(y) \left( \hat{\sigma}^+ + \hat{\sigma}^- \right) \\
&\quad + \hbar g(y) \left( \hat{\sigma}^+_\downarrow \hat{a}_{1e}^{\dagger} e^{ikx} + \hat{\sigma}^+_\uparrow \hat{a}_{2e}^{\dagger} e^{-ikx} \right) \\
&\quad + \hat{\sigma}^- \hat{a}_{1e}^{\dagger} e^{-ikx} + \hat{\sigma}^- \hat{a}_{2e}^{\dagger} e^{ikx} \quad (C4)
\end{align*}
\]

where \( \Delta_\alpha = \omega_p - \omega_a \) is the atom-pump detuning and \( \Delta_c = \omega_p - \omega_c \) is the cavity-pump detuning.

Appendix D: Action of momentum operator on the atomic wave-function

In this section, we provide the detailed derivation of Eq.(12) of the main text to find out the action of the momentum operator, \( \hat{P} \), on the atomic wave-function, \( \Psi(\vec{r}, t) \).

\[
\hat{P} |\Psi(\vec{r}, t)\rangle = -i\hbar \vec{\nabla} \sum_{x} \frac{\psi_j(\vec{r}, t)}{D_j} \]

\[
= -i\hbar \sum_{j} \left[ \left( \nabla \psi_j(\vec{r}, t) \right) D_j + \psi_j(\vec{r}, t) \nabla \left| D_j \right| \right] \\
= \sum_{j} \left[ -i\hbar \nabla \psi_j(\vec{r}, t) \sum_{l} \left| D_l \right| (D_l |D_j) \right] \\
= \sum_{j,l} \left[ \vec{p} \delta_{l,j} - \vec{A}_{l,j} \right] \psi_j |D_l\rangle \\
\]

where \( \vec{A}_{l,j} = i\hbar (D_l |\nabla| D_j) \) is the vector potential and \( \vec{p} = -i\hbar \vec{\nabla} \) does not act on the spinorial part. The kinetic energy term can be written as -

\[ \frac{\hbar^2}{2m_a} |\Psi(\vec{r}, t)\rangle = \frac{1}{2m_a} \sum_{l=1,2,3} \left[ \vec{p} \delta_{l,j} - \vec{A}_{l,j} \right] \psi_j |D_l\rangle \]

We can write down a matrix \( \vec{A} \) whose components are given as -

\[ \vec{A}_{l,j} = i\hbar (D_l |\nabla| D_j) \]

\[ \vec{A} = \begin{bmatrix}
(D_1 |\nabla| D_1) & (D_1 |\nabla| D_2) & (D_1 |\nabla| D_3) \\
(D_2 |\nabla| D_1) & (D_2 |\nabla| D_2) & (D_2 |\nabla| D_3) \\
(D_3 |\nabla| D_1) & (D_3 |\nabla| D_2) & (D_3 |\nabla| D_3)
\end{bmatrix} = \begin{bmatrix}
hk \left( G + \Delta_\alpha \right) + \frac{8g^2(y)(n_1+1)}{2G(G+\Delta_\alpha)} \hat{x} - \frac{4\hbar g(y) \sqrt{(n_1+1)(n_2+1)}}{A_1 \sqrt{n_1+n_2+2}} \hat{x} \\
- \frac{4\hbar g(y) \sqrt{(n_1+1)(n_2+1)}}{A_1 \sqrt{n_1+n_2+2}} \hat{x} \\
- \frac{4\hbar g(y) \sqrt{(n_1+1)(n_2+1)}}{A_1 \sqrt{n_1+n_2+2}} \hat{x} + \frac{4\hbar g(y) \sqrt{(n_1+1)(n_2+1)}}{A_1 \sqrt{n_1+n_2+2}} \hat{x}
\end{bmatrix}
\]

Appendix E: Projection on to the lowest energy dressed state

In this section, we provide a detailed derivation of Eq.(15) of the main text starting from Eq.(14) -

\[ \langle D_1 |i\hbar \frac{\partial}{\partial t} |\Psi(\vec{r}, t)\rangle = \langle D_1 |i\hbar \frac{\partial}{\partial t} \sum_{j} \psi_j(\vec{r}, t) |D_j\rangle \]

LHS of Eq.(E1) gives -

\[ \langle D_1 |i\hbar \frac{\partial}{\partial t} |\Psi(\vec{r}, t)\rangle = \langle D_1 |i\hbar \frac{\partial}{\partial t} \sum_{j} \psi_j(\vec{r}, t) |D_j\rangle \]

\[ = i\hbar \sum_{j} \langle D_1 | \frac{\partial}{\partial t} \psi_j(\vec{r}, t) |D_j\rangle \]

\[ = \langle D_1 | i\hbar \frac{\partial}{\partial t} \sum_{j} \psi_j(\vec{r}, t) |D_j\rangle \]
RHS of Schrödinger Eq.(E1) is -

\[ \langle D_1 | \hat{H}_{RF} | \Psi(\vec{r},t) \rangle = \langle D_1 \left[ \left( \frac{\hat{P}^2}{2m_\alpha} \right) + \hat{H}_I \right] | \Psi(\vec{r},t) \rangle \]  

(E3)

The second term of Eq.(E3), gives -

\[ \langle D_1 | \hat{H}_I | \Psi(\vec{r},t) \rangle = \frac{1}{2m_\alpha} \sum_{j=1,i,m=1,2} \left( \left( \rho \delta_{i,j} - \bar{A}_{i,j} \right) \left( \rho \delta_{m,l} - \bar{A}_{m,l} \right) \psi_j \right) \langle D_1 | D_m \rangle \]  

(E4)

\[ \langle D_1 | \hat{P}^2 | \Psi(\vec{r},t) \rangle = \frac{1}{2m_\alpha} \left( \rho - \bar{A}_{1,1} \right)^2 + |\bar{A}_{2,1}|^2 + |\bar{A}_{3,1}|^2 \psi_1(\vec{r},t) \]  

(E5)

The first term of Eq.(E3) gives -

\[ \langle D_1 | \hat{H}_1 | \Psi(\vec{r},t) \rangle = -\frac{\hbar G}{2} \psi_1(\vec{r},t) \]  

From Eq.(E2) and Eq.(E3), we obtain the equation for the probability amplitude, \( \psi_1 \), to find the atom in the lowest energy dressed state, \( E_1 \) -

\[ i\hbar \frac{\partial}{\partial t} \psi_1(\vec{r},t) = H_S \psi_1(\vec{r},t) = \left[ \frac{1}{2m_\alpha} \left( \left( \rho - \bar{A}_{1,1} \right)^2 + |\bar{A}_{2,1}|^2 + |\bar{A}_{3,1}|^2 \right) - \frac{\hbar G}{2} \right] \psi_1(\vec{r},t) \]  

(E6)

\[ i\hbar \frac{\partial}{\partial t} \Psi(\vec{r},t) = \hat{V}(\vec{r},t) | \Psi(\vec{r},t) \rangle \]  

(F2)

and

\[ |\dot{\Psi}(\vec{r},t)\rangle = \frac{d}{dt} \left( \sum_j \psi_j(t) |D_j(\vec{r}(t))\rangle \right) = \sum_j \dot{\psi}_j |D_j\rangle + \sum_j \psi_j \vec{v} \cdot \hat{\nabla} D_j \]  

(F3)

We get the corresponding equation of motion for \( \psi_j \) as -

\[ \dot{\psi}_j = -iE_j \frac{\psi_j}{\hbar} + \sum_k \psi_j \vec{v} \cdot \langle D_j | \hat{\nabla} D_k \rangle \]  

(F4)

\[ \dot{\vec{v}} = -i \left( E_1 - \vec{v} \cdot \bar{A} \right) \psi_1 / \hbar \]  

(F5)

At order zero, we get \( \psi_1 = e^{-iE_1 t / \hbar} \) and \( \psi_2, \psi_3 = 0 \).

At first order in \( \vec{v} \), the equation of motion for \( \psi_1 \) is -

\[ \psi_j(T) \simeq i\hbar \frac{\vec{v} \cdot \langle D_j | \hat{\nabla} D_1 \rangle}{E_j - E_1} e^{-iE_1 T / \hbar} \]  

(F6)

Appendix F: Equations of motion for the snake trajectories

In this section, we provide a detailed derivation of Eq.(22). The expectation value of the force operator is -

\[ \langle \hat{F}(\vec{r},t) \rangle = -\sum_j |\psi_j|^2 \nabla E_j + \sum_{j,k} \psi_j^\dagger \psi_k (E_k - E_j) \langle D_k | \hat{\nabla} D_j \rangle \]  

(F1)

To obtain the force at first order in \( \vec{v} \), we have to find out all the coefficients, \( \psi_j \), at first order. The Schrödinger equation for \( |\Psi(\vec{r},t)\rangle \) is \[72\] -

\[ i\hbar |\dot{\Psi}(\vec{r},t)\rangle = \hat{V}(\vec{r},t) |\Psi(\vec{r},t)\rangle \]  

At order zero, we get \( \psi_1 = e^{-iE_1 t / \hbar} \) and \( \psi_2, \psi_3 = 0 \).

At first order in \( \vec{v} \), the equation of motion for \( \psi_1 \) is -

\[ \dot{\psi}_1 = -i \left( E_1 - \vec{v} \cdot \bar{A} \right) \psi_1 / \hbar \]  

(F5)

whose solution is a number of modulus 1. For \( j \neq 1 \) (assuming adiabatic motion of the atom, i.e., \( T (E_j - E_1) / \hbar >> 1 \)), we get -

\[ \psi_j(T) \simeq i\hbar \frac{\vec{v} \cdot \langle D_j | \hat{\nabla} D_1 \rangle}{E_j - E_1} e^{-iE_1 T / \hbar} \]  

(F6)

We see that first term of \( \langle \hat{F} \rangle \) in Eq.(F1), has no first-order component in \( \vec{v} \), since the contributions of \( \psi_j \)s for \( j \neq 1 \) are at least of order 2 and the contribution of \( \psi_1 \) is independent of \( \vec{v} \). In the second term of Eq.(F1), the relevant terms are the ones where one of two indices \( j \) or \( k \) is 1. Applying the closure property, and keeping terms upto first order in \( \vec{v} \), we get \( \langle \hat{F} \rangle = i\hbar \hat{\nabla} D_1 \left( \vec{v} \cdot \hat{\nabla} D_1 \right) + c.c. \). This expression gives the Lorentz force \( \vec{v} \times \vec{B} \) in the equation of motion.
Appendix G: Schrödinger equation for a particle in a non-uniform magnetic field

In this section, we provide a detailed derivation of Eq. (26) of the main text, which is the Schrödinger equation for a particle moving in the presence of a non-uniform magnetic field. We substitute \( \psi_1(x, y) = e^{i k_x x} \phi(y) \), in Eq. (15), to obtain:

\[
\left[ \frac{1}{2m_a} \left( -i \hbar \frac{\partial}{\partial y} \right)^2 + \frac{1}{2m_a} (\hbar k_x - A_x(y))^2 \right] \phi(y) = E \phi(y)
\]

\[
\Rightarrow \frac{\partial^2 \phi(y)}{\partial y^2} - \left( k_x - \frac{A_x(y)}{\hbar} \right)^2 \phi(y) + \frac{2m_a}{\hbar^2} \left[ E - W + \frac{hG}{2} \right] \phi(y) = 0
\]

\[
\Rightarrow \frac{\partial^2 \phi(y)}{\partial y^2} - \left( l_B k_x - \frac{A_x(y) l_B}{\hbar} \right)^2 \phi(y) + \frac{2m_a l_B^2}{\hbar^2} \left[ E - W + \frac{hG}{2} \right] \phi(y) = 0
\]

\[
\Rightarrow \frac{\partial^2 \phi(\bar{y})}{\partial \bar{y}^2} + \left[ E_{\text{eff}} - V_{\text{eff}}(\bar{y}) \right] \phi(\bar{y}) = 0 \quad (G1)
\]

This is the Schrödinger equation of a single atom moving in the presence of a non-uniform magnetic field.

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