On compact hypersurfaces in a Riemannian vector bundle with prescribed vertical Gaussian curvature

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Abstract. Let $M$ be a compact Riemannian manifold and $E$ a Riemannian vector bundle on $M$. We look for hypersurfaces of $E$ with a prescribed vertical Gaussian curvature. In trying to solve this problem fibre-wise, we loose the regularity of the resulting solution. To ensure the smoothness of the solution, we construct it as a radial graph over the unit sphere subbundle of $E$ and prove its existence by solving in this one a nonlinear partial differential equation of Monge-Ampère type.

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1 Introduction

Let $(M,g)$ be a compact Riemannian manifold of dimension $n \geq 1$, without boundary, and $(E,\tilde{g})$ a Riemannian vector bundle on $M$ of rank $m \geq 2$. Denote by $E_*$ the bundle $E$ with the zero section removed and by $\Sigma$ the corresponding unit sphere bundle. Let $VE$ be the vertical subbundle of $TE$ and $HE$ the horizontal subbundle of $TE$ associated to a metric-connexion on $(E,\tilde{g})$. For a hypersurface $\mathcal{Y}$ of $E$ for which each fibre $\mathcal{Y}_x$ is a hypersurface on the fibre $E_x$ of $E$, the value at a point $\xi \in \mathcal{Y} \cap E_x$ of the vertical Gaussian curvature of $\mathcal{Y}$ is the value at the point $\xi$ of the Gaussian curvature of $\mathcal{Y}_x$ when regarded as a hypersurface of $E_x$.

In this study, we are interested in finding an embedding $\mathcal{Y}$ of $\Sigma$ into $E_*$ admitting a prescribed vertical Gaussian curvature equal to $K$, a given strictly positive function on $E_*$. We look for $\mathcal{Y}$ as a radial graph constructed on $\Sigma$, that is a map of the form $\xi \in \Sigma \mapsto e^{u(\xi)}\xi$, where $u \in C^\infty(\Sigma)$ is an unknown function extended to $E_*$ by letting it be radially constant. When Greek indices are used, they designate vertical directions tangent to $\Sigma$, and will range from $n+1$ to $n+m-1$. The function $u$ must satisfy on $\Sigma$ the following degenerate equation of Monge-Ampère type:

\[
\det \left[ \left( \delta_\beta + D_\alpha u D_\beta u - D_\gamma D_\beta u \right) \right] = (1 + v_1)^{\frac{m+1}{2}} e^{(m-1)u} K(e^u \xi),
\]

where $D$ stands for the Sasaki connexion of the manifold $(E,G)$, $G$ is a Riemannian metric on $E$ for which the vertical and the horizontal distributions are orthogonal and

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\[ v_1 = \sum_{n+1 \leq a \leq n+m-1} D_a u D^a u. \] In studying (1.1), one needs a priori estimates on covariant derivatives of \( u \) till order three.

In case the ambient space is the Euclidean one, that is when \( M \) is reduced to a point, the vertical Gaussian curvature of an hypersurface of \( E \) is exactly its Gaussian curvature. The question was considered at first by Oliker [9] who gave sufficient conditions on the prescribed function ensuring the existence of a solution. In particular he assumes that there exist two real numbers \( r_1 \) and \( r_2 \) such that \( 0 < r_1 \leq 1 \leq r_2 \) and

\[ K(\xi) > \|\xi\|^{(1-m)} \text{ if } \|\xi\| < r_1; \quad K(\xi) < \|\xi\|^{(1-m)} \text{ if } \|\xi\| > r_2 \]

combined with the following monotonicity assumption:

\[ \frac{\partial [\rho^{n-1} K(\rho \xi) \rho]}{\partial \rho} \leq 0, \text{ for all } \xi \in \Sigma. \]

The latter gives uniqueness up to homothety. These conditions were subsequently simplified in [3] by Delanoë. In [1] Caffarelli, Nirenberg and Spruck were interested in finding embedded hypersurfaces of \( \mathbb{R}^m \) whose principal curvatures satisfy a prescribed relation. To a certain extent, this question is related to the Minkowski problem (see [2], [8], [10]) from which it differs by the way of parametrising, i.e. by a radial graph instead of the inverse of the Gauss map.

The problem we deal with here requires an approach different from that used in previous works because of the degeneracy of the equation. This involves only the vertical component of the Hessian of \( u \), so we have no control on the horizontal component of the second fundamental form of the hypersurface we look for, it has a strictly positive vertical Gaussian curvature but need not be convex. This makes the study of the question more interesting and shows to what extent it differs from the Euclidean case. Our first result, which is derived in an almost elementary setting, is to clarify this remark.

**Theorem 1.** Let \( K \in \mathcal{C}^\infty(E_\ast) \) be a strictly positive function which is constant on each fibre of \( E \). Then there exists a radial graph \( \mathcal{Y} \) on \( \Sigma \) whose vertical Gaussian curvature is given by \( K \). Moreover, if \( K \) is constant \( \mathcal{Y} \) may be convex and in case \( K \) is non constant, every such graph is non convex.

Our proof relies on direct computations. The hypothesis on the prescribed function \( K \) means that it is the vertical lift to \( E \) of a \( \mathcal{C}^\infty \) positive function on \( M \). Any such strictly positive function gives rise to a non convex hypersurface in \( E \) with strictly positive vertical Gaussian curvature. The next result is to show that assumption (1.3) does not assure uniqueness not even up to homothety.

**Theorem 2.** Let \( K \in \mathcal{C}^\infty(E_\ast) \) be a strictly positive function such that \( K(\xi) = K(\|\xi\|) \) for all \( \xi \). If there exists a real number \( r > 0 \) such that \( K(r \xi) = r^{(1-m)} \), for all \( \xi \in \Sigma \), then there exists a radial graph \( \mathcal{Y} \) on \( \Sigma \) whose vertical Gaussian curvature is given by \( K \). Such a graph may be chosen to be convex. Conversely if there exists a radial graph \( \mathcal{Y} \) on \( \Sigma \) with vertical Gaussian curvature given by \( K \), then there exists a real number \( r > 0 \) such that \( K(r \xi) = r^{(1-m)} \), for all \( \xi \in \Sigma \). Furthermore if \( K(r \xi) = r^{(1-m)} \), for all \( r > 0 \) and \( \xi \in \Sigma \), there exists an infinite number of non homothetic radial graphs \( \mathcal{Y} \) on \( \Sigma \) with vertical Gaussian curvature given by \( K \) but a unique, up to homothety, convex one.
To deal with the general case, for \( u \in C^\infty(\Sigma) \), we set
\[
N_1(u) = \det \left[ (\delta^j_i + D_i u D^j u - D^j_i u)_{1 \leq i, j \leq n} \right]
\]
and
\[
N_2(u) = \det \left[ (\delta^\beta_\alpha + D_\alpha u D^\beta u - D^\beta_\alpha u)_{n+1 \leq \alpha, \beta \leq n+m-1} \right].
\]
Applying a continuity method in the framework of \( C^\infty \) functions [4] via the Nash and Moser inverse function theorem, for the latter see R. Hamilton [6], we first prove the following existence result.

**Theorem 3.** Let \( f \in C^\infty(\Sigma) \) be a strictly positive function and \( \lambda \) be a strictly positive real number. Then there exists a unique solution \( u \in C^\infty(\Sigma) \) of the equations
\[
\begin{align*}
N_1(u) &= 1 \\
N_2(u) &= e^{\lambda u} f(\xi)(1 + |D^u u|^2)^{m+1/2}.
\end{align*}
\]
Moreover, for such a solution, the matrices \((G_{\alpha\beta} + D_\alpha u D^\beta u - D^\beta_\alpha u)_{n+1 \leq \alpha, \beta \leq n+m-1}\) and \((G_{ij} + D_i u D_j u - D^j_i u)_{1 \leq i, j \leq n}\) are positive definite.

Equation (1.1) does not give any information about the horizontal behaviour of the solution if there is any. To make up for this insufficiency, we use theorem 3 to assign particular values to the horizontal derivatives. On the other hand, equation (1.1) is not even locally invertible; to overcome this difficulty, we apply the fixed point theorem of Nagumo [7] to prove the following result.

**Theorem 4.** Let \( K \in C^\infty(E^*) \) be an everywhere strictly positive function. Assume that there exist two real numbers \( r_1 \) and \( r_2 \) satisfying \( 0 < r_1 \leq 1 \leq r_2 \) and such that inequalities (1.2) hold. Then there exists a radial graph \( \mathcal{Y} \) on \( \Sigma \) of class \( C^\infty \) whose vertical Gaussian curvature is given by \( K \) and such that \( r_1 \leq \|\xi\| \leq r_2 \) for all \( \xi \in \mathcal{Y} \).

The rest of this article is divided into four parts. First, we recall some preliminary results, which are needed to set the equation for prescribing the vertical Gaussian curvature on the unit sphere bundle \( \Sigma \). We derive this equation in the third section by pulling the expression of the vertical Gaussian curvature back from the hypersurface. In the forth part we give the a priori estimates required in proving theorems 3 and 4 and in the last one we put these a priori estimates together and prove the results.

## 2 Preliminaries and notations

1- Let \((M, g)\) be a Riemannian manifold of dimension \( n \geq 1 \) and denote by \( \nabla \) its Levi-Civita connexion. Let \((E, \tilde{g})\) be a Riemannian vector bundle on \( M \) of rank \( m \geq 2 \), \( \pi \) the projection of \( E \) on \( M \) and \( E_\pi \) the bundle \( E \) with the zero section removed. Denote by \( \tilde{\nabla} \) a metric-connexion on \((E, \tilde{g})\). Let \( U \) be an open set of \( M \) with coordinates \((x^i)_{1 \leq i \leq n}\) and over which \( E \) is trivial. The open set \( \pi^{-1}(U) \) may be equipped with a coordinate
system \((x^i, y^\alpha)\) with \(1 \leq i \leq n\) and \(n + 1 \leq \alpha \leq n + m\), where \((y^\alpha)_{n+1 \leq \alpha \leq n+m}\) are the fibre-coordinates with respect to a fixed frame \((s_\alpha)\) of \(E\) over \(U\).

We denote by \(\Gamma^k_{ij}, i, j, k \in \{1, ..., n\}\), the Christoffel symbols of the connexion \(\nabla : \nabla e_i = \partial / \partial x^i\), and by \(\Gamma^\beta_{\alpha i}, i \in \{1, ..., n\}\) and \(\alpha, \beta \in \{n + 1, ..., n + m\}\), the Christoffel symbols of the connexion \(\tilde{\nabla} : \tilde{\nabla} e_\alpha = \Gamma^\beta_{\alpha \iota} s_\beta\).

On the open set \(\pi^{-1}(U)\), we can then consider the following moving frame

\[
S = \{e_i, e_\alpha \mid i = 1, ..., n\ \text{and} \ \alpha = n + 1, ..., n + m\},
\]

where

\[
e_i = \frac{\partial}{\partial x^i} - y^\alpha \Gamma^\beta_{\alpha i} \frac{\partial}{\partial y^\beta}
\]

is the horizontal lift of \(e_i\) and \(e_\alpha = \partial / \partial y^\alpha\). Now, we equip the manifold \(E\) with a Riemannian structure given by the metric \(G\) defined by the following :

\[
G(e_i, e_j) = g(e_i, e_j), \quad G(e_\alpha, e_\beta) = \tilde{g}(s_\alpha, s_\beta), \quad G(e_i, e_\alpha) = 0
\]

and introduce the connexion \(D\) of Sasaki \([11]\) which is \(G\)-metric and defined as follow

\[
D_{e_i} e_j = \Gamma^k_{ij} e_k, \quad D_{e_\alpha} e_\beta = \Gamma^\beta_{\alpha i} e_\beta, \quad D_{e_i} e_\alpha = D_{e_\alpha} e_\beta = 0.
\]

The connexion \(D\) is not torsion free. In fact, if \(S^k_{hij}\) denotes the curvature components of \(\tilde{\nabla}\), when expressed in \(S\), the only non-zero components of the torsion \(T\) of \(D\) are those of the form

\[
T^e_{ij} = -y^\beta S^e_{\beta ij}.
\]

Let \(\tilde{g}_{\alpha \beta} = \tilde{g}(s_\alpha, s_\beta)\). Since \(\tilde{\nabla}\) is \(\tilde{g}\)-metric, we obtain

\[
\tilde{\nabla} e_\alpha \tilde{g}_{\alpha \beta} = \tilde{g} (\tilde{\nabla} e_\alpha s_\beta) + \tilde{g}(s_\alpha, \tilde{\nabla} e_\beta s_\beta) = \Gamma^\lambda_{\iota \alpha} \tilde{g}_{\lambda \beta} + \Gamma^\lambda_{\iota \beta} \tilde{g}_{\alpha \lambda}.
\]

The expression in the frame \(S\) of the components of the curvature tensor \(R\) of \(D\) are given by

\[
R_{dcab} = G \left( (D_{e_a} e_b - D_{e_b} e_a - D_{[e_a, e_b]} e_c) e_d, e_d \right), \quad R_{cab} = G^{de} R_{e cab}
\]

and standard computations yield that, for all \(1 \leq i, j \leq n\) and \(n + 1 \leq \alpha, \beta, \lambda, \mu \leq n + m\),

\[
R^i_{\alpha \beta \mu} = R^\lambda_{\alpha \beta \mu} = R^i_{\alpha \beta \mu} = R^\lambda_{\alpha \beta \mu} \equiv 0.
\]

Denoting by \(r\) the function \(r(\xi) = ||\xi||\) and by \(\nu\) the unit radial field which is given, on the open set \(\pi^{-1}(U)\), by

\[
\nu = r^{-1} y^\alpha \frac{\partial}{\partial y^\alpha} := r^{-1} y^\alpha e_\alpha.
\]
Let us compute $D_A\nu$ where $A = A^i e_i + A^a e_a$ is a vector field on $\pi^{-1}(U)$. At first, we have

$$D_Ar = D_A(r^2)^{1/2} = (1/2)(r^2)^{-1/2}D_Ar^2 = (1/2)r^{-1}D_A(\tilde{g}_{\alpha\beta} y^\alpha y^\beta)$$

$$= (1/2)r^{-1} \sum_{1 \leq i \leq n} A^i e_i, (\tilde{g}_{\alpha\beta} y^\alpha y^\beta) + (1/2)r^{-1} \sum_{n+1 \leq \lambda \leq n+m} A^\lambda \frac{\partial}{\partial y^\lambda}(\tilde{g}_{\alpha\beta} y^\alpha y^\beta)$$

$$= (1/2)r^{-1} \sum_{1 \leq i \leq n} A^i e_i, (\tilde{g}_{\alpha\beta} y^\alpha y^\beta) + r^{-1} \sum_{n+1 \leq \lambda \leq n+m} \tilde{g}_{\lambda\beta} A^\lambda y^\beta.$$

Relations (2.1) and the fact that $\tilde{\nabla}$ is $\tilde{g}$-metric imply that

$$e_i, (\tilde{g}_{\alpha\beta} y^\alpha y^\beta) = \frac{\partial}{\partial x^i} \tilde{g}_{\alpha\beta} y^\beta - y^\lambda \Gamma^\mu_{i\lambda} \tilde{g}_{\alpha\beta} \frac{\partial}{\partial y^\mu}(y^\alpha y^\beta) = 2\Gamma^a_{ia} \tilde{g}_{\beta\gamma} y^\gamma y^\beta - 2y^\lambda \Gamma^\mu_{i\lambda} \tilde{g}_{\alpha\beta} y^\alpha y^\beta = 0.$$

Therefore

$$D_Ar = r^{-1} \sum_{n+1 \leq \lambda \leq n+m} \tilde{g}_{\lambda\beta} A^\lambda y^\beta = G(A, \nu).$$

At present,

$$D_A\nu = -r^{-2}(D_Ar)y^\alpha e_\alpha + r^{-1}(D_A y^\alpha)e_\alpha + r^{-1}y^\alpha D_Ae_\alpha$$

$$= -r^{-1}G(A, \nu)\nu + r^{-1} \sum_{1 \leq i \leq n} A^i (D_{ei} y^\alpha e_\alpha + y^\alpha D_{ei} e_\alpha)$$

$$+ r^{-1} \sum_{n+1 \leq \lambda \leq n+m} A^\lambda (D_{e\lambda} y^\alpha e_\alpha + y^\alpha D_{e\lambda} e_\alpha).$$

Taking account of (2.2), we get

$$D_A\nu = r^{-1} \sum_{1 \leq i \leq n} A^i (-y^\lambda \Gamma^\mu_{i\lambda} \delta_\mu^\alpha e_\alpha + y^\alpha \Gamma^\mu_{i\alpha} e_\mu)$$

$$- r^{-1}G(A, \nu)\nu + r^{-1} \sum_{n+1 \leq \lambda \leq n+m} A^\lambda \delta_\lambda^\alpha e_\alpha$$

$$= r^{-1} \sum_{1 \leq i \leq n} A^i (-y^\lambda \Gamma^\mu_{i\lambda} e_\mu + y^\alpha \Gamma^\mu_{i\alpha} e_\mu)$$

$$- r^{-1}G(A, \nu)\nu + r^{-1} \sum_{n+1 \leq \lambda \leq n+m} A^\lambda e_\lambda.$$

Therefore

$$D_A\nu = -r^{-1}G(A, \nu)\nu + r^{-1} \sum_{n+1 \leq \lambda \leq n+m} A^\lambda e_\lambda. \quad (2.4)$$

2- Let $\Sigma = \{ \xi \in E \mid ||\xi|| = 1 \}$. The restriction of $\nu$ to $\Sigma$ is normal to $\Sigma$. So the tangent space to $\Sigma$ at $\xi \in \Sigma$ is a direct sum of the horizontal subspace $H_\xi E$ of $T_\xi E$ and the tangent
space of the fibre passing through $\xi$. This allows us to fix a local orthonormal frame field tangent to $E$ of the form

$$ \mathcal{R} = \{ e_i, e_\alpha, \nu \mid i = 1, \ldots, n \text{ and } \alpha = n + 1, \ldots, n + m - 1 \}, $$

where $e_i$ for $i \in \{1, \ldots, n\}$ is an horizontal vector field. Without losing generality, we can restrict ourself to the case when $e_i$ is the horizontal lift of the natural vector field $\partial/\partial x^i$ of $M$. For $\alpha = n + 1, \ldots, n + m - 1$, $e_\alpha$ is a vertical vector field. Let

$$ \mathcal{R}^* = \{ \omega^A \mid A \leq n + m \} $$

be the dual coframe. In the following, we will make use of the summation convention, when letters are used as indices they range from 1 to $n + m$ for an upper case Latin, from 1 to $n + m - 1$ for a lower case one and from $n + 1$ to $n + m - 1$ for a lower case Greek.

Applying $D$ to $e_A$, we get a 1-form on $E$ with values in $TE$. Expressing the result in $\mathcal{R}$ leads us to introduce the matrix $(\omega^A_B)$ of 1-forms uniquely defined by the equalities

$$ De_A = \omega_B^A \otimes e_B. $$

From (2.4), it follows that

$$ D\nu = 0 \text{ and } D e_\alpha = (1 - \mu_\alpha) r^{-1} e_\alpha, \text{ on } \Sigma_r, $$

where $\Sigma_r = \{ \xi \in E \mid \|\xi\| = r \}$ and $\mu_\alpha$ is a parameter that equals 1 if $a$ is a horizontal direction and zero if it is a vertical one. Inserting (2.5) into the previous relation, we see that, on $\Sigma_r$,

$$ \omega^{a}_{n+m} = (1 - \mu_a) r^{-1} \omega_a \text{ and } \omega^{a+m}_{n+m} = 0. $$

On the other hand, since $D$ is a metric connexion, it follows that

$$ G(D e_a e_b, \nu) = -G(e_a, D e_b \nu). $$

Therefore, by virtue of (2.5),

$$ \omega^{a+m}_{n+m}(e_b) = -(1 - \mu_b) r^{-1} G_{ab} \text{ for all } a, b \leq n + m - 1. $$

For later use, let us compute the components of the curvature tensor $\tilde{R}$ of $\Sigma$. Using Gauss equation and relation (2.7) above, which gives the components of the second fundamental form of $\Sigma$, we show that

$$ \tilde{R}_{dca}b = R_{dca}b + (1 - \mu_a)(1 - \mu_b)(G_{ad}G_{bc} - G_{ac}G_{bd}). $$

Therefore, we get the values of the curvature components that will be used in next computations:

$$ \tilde{R}_{a\beta\gamma} = \tilde{R}_{a\beta i} = \tilde{R}_{a\beta i} = 0, \text{ } n + 1 \leq \alpha, \beta, \gamma \leq n + m - 1 \text{ and } 1 \leq i, j \leq n, $$

and

$$ \tilde{R}_{a\beta\mu} = \delta^\lambda_\beta G_{a\mu} - \delta^\lambda_\mu G_{a\beta}, \text{ } n + 1 \leq \alpha, \beta, \lambda, \mu \leq n + m - 1. $$

3- Let \( u \in C^2(\Sigma) \) be a function extended to \( E_* \) in a radially constant way. The differential of \( u \) is given by
\[
du = \sum_{a=1}^{n+m-1} D_a u \omega^a.
\]
The component \( D_a u \) is homogeneous of degree \((\mu_a - 1)\). We also have
\[
D_{ab} u = D^2 u(e_a, e_b) = (D_{ea} Du)(e_b).
\]
Hence
\[
D_{ab} u = D_{ea} \left( Du(e_b) \right) - Du(D_{ea} e_b)
\]
and we easily check that \( D_{ab} u \) is homogeneous of degree \((\mu_a + \mu_b - 2)\). Since \( u \) is radially constant, we can write
\[
D_{av} u = D_{ea} \left( Du(\nu) \right) - Du(D_{ea} \nu) = -Du(D_{ea} \nu).
\]
Which in view of (2.5) implies that, for all \( a \leq n + m - 1 \),
\[
\tag{2.10}
D_{av} u = -(1 - \mu_a)r^{-1}D_a u \text{ on } \Sigma_r.
\]
Similar computations give
\[
\tag{2.11}
D_{\nu \nu} u = 0, \text{ on } \Sigma_r.
\]

3 Derivation of the equation

In this section, we use the method of moving frames to derive the expression of the vertical Gaussian curvature for the hypersurface under consideration. Using a homogeneity argument and pulling this expression back from the hypersurface will give us the desired equation on the unit sphere bundle \( \Sigma \). For this purpose, we keep all notations of the previous part. In particular, from the choice of the local orthonormal frame \( \mathcal{R} \) it follows that
\[
\mathcal{R}_1 = \{e_\alpha, \nu \mid \alpha = n + 1, \ldots, n + m - 1\}
\]
is a local orthonormal frame field tangent to fibres of \( E \). Let \( \overline{D} \) denotes the induced connexion on the fibre of \( E \). We look for a hypersurface \( \mathcal{Y} \) which is a radial graph over the unit sphere bundle that is an application of the form
\[
\mathcal{Y}(\xi) = e^{u(\xi)} \xi, \text{ for } \xi \in \Sigma,
\]
where \( u \in C^2(\Sigma) \) is a function extended to \( E_* \) by letting it be radially constant.

The definition of \( D \) implies that \( D_{ea} \nu \) is a vertical vector field. So, using (2.5), at a point of the fibre \( \mathcal{Y}_x = \mathcal{Y} \cap E_x \), we obtain
\[
\tag{3.1}
\overline{D}_{ea} \nu = D_{ea} \nu = e^{-u}e_\alpha.
\]
Therefore, we get
\[
\tag{3.2}
\overline{D}_{\alpha \nu} u = -e^{-u} \overline{D}_{\alpha} u \text{ for } n + 1 \leq \alpha \leq n + m - 1.
\]
Hence, taking (3.1) into account, the definition of the covariant derivative allows us to write

\[ D_{\nu}(\mathcal{D}_\alpha u) = \mathcal{D}_{\nu\alpha} u + \mathcal{D} u(\mathcal{D}_{\nu\alpha} e_\alpha) = 0. \]  

By a reasoning analogous to the one used to prove (3.1), we show that

\[ \mathcal{D}_{e_\alpha} e_\beta = \mathcal{D}_{e_\alpha} e_\beta = \sum_{n+1}^{n+m-1} \omega_{\beta}^n(e_\alpha)e_\gamma - e^{-u}G_{\alpha\beta}E, \]  

Now, at a point \( x \in M \), the tangent space of the fibre \( Y_x \) is spanned by the vectors \( \{E_\alpha := \mathcal{D} \mathcal{Y}(e_\alpha) = e_\alpha + e^u \mathcal{D}_\alpha u e_\alpha\} \). Therefore, on \( Y_x \), the induced metric is given by

\[ h_{\alpha\beta} = G(E_\alpha, E_\beta) = G_{\alpha\beta} + e^{2u} \mathcal{D}_\alpha u \mathcal{D}_\beta u \]  

and the unit vector field

\[ \nu = f(\nu - e^u \mathcal{D}_\alpha u e_\alpha), \quad f = (1 + e^{2u \mathcal{D}_\alpha u \mathcal{D}_\alpha u})^{-\frac{1}{2}} \]  

is normal to \( Y_x \). In view of the equalities \( \mathcal{D}_{\nu} u = 0, \mathcal{D}_{\nu} \nu = 0 \), from (3.3) and (3.4) we obtain

\[ \mathcal{D}_{E_\alpha} E_\beta = \sum_{n+1}^{n+m-1} \omega_{\beta}^n(e_\alpha)e_\gamma - e^{-u}G_{\alpha\beta}E + e^u(e_\alpha \mathcal{D}_\beta u) \nu \]

\[ + \mathcal{D}_\beta u e_\alpha + \mathcal{D}_\alpha u e_\beta + e^u \mathcal{D}_\alpha u \mathcal{D}_\beta u \nu. \]  

But the definition of the covariant derivative gives

\[ e_\alpha \mathcal{D}_\beta u = \mathcal{D}_\alpha u + \sum_{n+1}^{n+m-1} \omega_{\beta}^n(e_\alpha) \mathcal{D}_\gamma u. \]

Reporting into (3.5), we obtain

\[ \mathcal{D}_{E_\alpha} E_\beta = \omega_{\beta}^n(e_\alpha)E_{\gamma} + \mathcal{D}_\beta u E_\alpha + \mathcal{D}_\alpha u E_\beta \]

\[ + e^{-u}[-h_{\alpha\beta} + e^{2u} \mathcal{D}_\alpha u \mathcal{D}_\beta u] \nu. \]

Hence

\[ G(\mathcal{D}_{E_\alpha} E_\beta, \nu) = f e^{-u}[-h_{\alpha\beta} + e^{2u} \mathcal{D}_\alpha u \mathcal{D}_\beta u]. \]

This gives the components of the second fundamental form of \( Y_x \) when \( Y_x \) is regarded as a hypersurface of \( E_x \). Namely

\[ I_{\alpha\beta} = f e^{-u}(h_{\alpha\beta} - e^{2u} \mathcal{D}_\alpha u \mathcal{D}_\beta u). \]

Therefore, the Gaussian curvature of \( Y_x \) at the point \( e^u x \in Y_x \) is

\[ G_x(e^u x) = e^{-(m-1)u} \left( 1 + e^{2u} |\mathcal{D} u|^2 \right)^{-\frac{n+1}{2}} \det \left( \delta_{\alpha\beta} + e^{2u} \mathcal{D}_\alpha u \mathcal{D}_\beta u - e^{2u} \mathcal{D}_\alpha u \mathcal{D}_\beta u \right). \]
The definition of the covariant derivative gives, for \( \alpha, \beta \in \{n + 1, \ldots, n + m - 1\} \),
\[
D_{\alpha \beta} u = e_\alpha < Du, e_\beta > - < Du, D_{e_\alpha} e_\beta > .
\]

On the other hand, since the radial derivative of \( u \) is identically equal to zero, it is clear that
\[
< Du, e_\beta >= < Du, e_\beta >
\]
and taking into account the definition of the connexion \( D \) which implies that \( D_{e_\alpha} e_\beta \) is a vertical vector field, we can write
\[
< Du, D_{e_\alpha} e_\beta >= < Du, D_{e_\alpha} e_\beta >.
\]

Hence, using Gauss equation, we get \( D_{\alpha \beta} u = D_{\alpha \beta} u \). Inserting into (3.6), we finally arrive at the following expression of the Gaussian curvature of \( Y_x \) at the point \( e^u \xi \in Y_x \):
\[
(3.7) \quad G_x(e^u \xi) = e^{-(m-1)u}(1 + e^{2u}|Du|^2)^{-\frac{m+1}{2}} \cdot \det \left( \delta_{\alpha}^\beta + D_{\alpha} u D_{\beta} u - D_{\alpha}^2 u \right).
\]
But the value of the vertical Gaussian curvature \( G^v \) of the graph \( Y \) at the point \( e^u \xi \) is defined to be
\[
G^v(e^u \xi) = G_x(e^u \xi) \text{ if } e^u \xi \in Y_x.
\]

On the other hand, taking into account the homogeneity of the covariant derivatives of \( u \), we can equate their values on \( Y_x \) and \( \Sigma_x \). Therefore, by pulling back (3.7), we obtain the desired expression of the vertical Gaussian curvature of the hypersurface \( Y \) by mean of its values on \( \Sigma \):
\[
(3.8) \quad G^v(e^u \xi) = (1 + |D^v u|^2)^{-\frac{m+1}{2}} e^{-(m-1)u} \cdot \det \left( \delta_{\alpha}^\beta + D_{\alpha} u D_{\beta} u - D_{\alpha}^2 u \right),
\]
where \( |D^v u|^2 = D_{\alpha}^2 u \).

In the sequel, we denote
\[
N_1(u) = \det \left( \left( \delta_{ij}^\alpha + D_i u D_j u - D_{ij} u \right)_{1 \leq i,j \leq n} \right)
\]
and
\[
N_2(u) = \det \left( \left( \delta_{\alpha \beta}^\alpha + D_{\alpha} u D_{\beta} u - D_{\alpha \beta} u \right)_{n+1 \leq \alpha,\beta \leq n+m-1} \right).
\]

We also denote by \( G'_u \) the covariant 2-tensor whose components, \( G'_{ab} = G'(e_a, e_b) \), are given by
\[
G'_{ij} = G_{ij} + D_i u D_j u - D_{ij} u \text{ for } 1 \leq i, j \leq n,
\]
\[
G'_{\alpha \beta} = G_{\alpha \beta} + D_{\alpha} u D_{\beta} u - D_{\alpha \beta} u \text{ for } n + 1 \leq \alpha, \beta \leq n + m - 1
\]
and
\[
G'_{\alpha i} = G'_{i \alpha} = 0 \text{ for } 1 \leq i \leq n \text{ and } n + 1 \leq \alpha \leq n + m - 1.
\]

The function \( u \) is said to be admissible if the tensor \( G'_u \) is positive definite. This allows us to view \( G'_u \) as a new Riemannian metric on \( \Sigma \).

Finally, by an adapted frame to \( u \) we mean a \( G \)-orthonormal one that diagonalises the Riemannian tensor \( G'_u \).
4 A priori estimates

Lemma 1. Any admissible function $u \in \mathcal{C}^2(\Sigma)$ satisfies the following estimate

$$\max \left( \sum_{i=1}^{n} D_i u D_i^i u, \sum_{\alpha=n+1}^{n+m-1} D_\alpha u D_\alpha^\alpha u \right) \leq e^{2 \text{osc}(u)} - 1.$$ 

Proof. Let $u \in \mathcal{C}^2(\Sigma)$ be an admissible function and set $w = e^{-u}$. Easy computations show that the matrices

$$(wG_{ij} + D_{ij} w)_{1 \leq i,j \leq n}$$

and

$$(wG_{\alpha\beta} + D_{\alpha\beta} w)_{n+1 \leq \alpha,\beta \leq n+m-1}$$

are positive definite. At a point $X_0 \in \Sigma$ where the function $\Omega_1 = w^2 + \sum_{i=1}^{n} D_i w D_i^i w$ attains its maximum, since $D\Omega_1(X_0) = 0$ in a $G$-orthonormal frame that diagonalises the symmetric matrix $(D_{ij} w)$ we get for all horizontal direction, $i \in \{1, \ldots, n\}$,

$$D_i w (w + D_i w) = 0.$$ 

Thus $D_i w(X_0) = 0$. Since, for all $X \in \Sigma$, we have $\Omega_1(X) \leq \Omega_1(X_0)$, we see that

$$w^2 + \sum_{i=1}^{n} D_i w D_i^i w \leq w^2(X_0)$$

from which, we conclude, in view of the definition of $w$, that

$$\sum_{i=1}^{n} D_i u D_i^i u \leq e^{2 \text{osc}(u)} - 1.$$ 

Arguing analogously at a point where the function $\Omega_2 = w^2 + D_\alpha w D_\alpha w$ attains its maximum, we show that

$$\sum_{\alpha=n+1}^{n+m-1} D_\alpha u D_\alpha^\alpha u \leq e^{2 \text{osc}(u)} - 1$$

which ends the proof.

Lemma 2. Let $F \in \mathcal{C}^3(\Sigma \times \mathbb{R})$ be a strictly positive function and $u \in \mathcal{C}^5(\Sigma)$ be an admissible solution of the equation

$$(4.1) \quad \mathcal{N}_1(u)\mathcal{N}_2(u) = \left(1 + |D^u u|^2 \right)^\frac{n+1}{2} F(\xi, u).$$

Assume that there exists a positive real number $C_0$ such that

$$(4.2) \quad e^{-C_0} \leq e^u \leq e^{C_0}.$$
Let $L = \{ \xi \in E \mid e^{-C_0} \leq \| \xi \| \leq e^{C_0} \}$ and denote by $\triangle u = G^{ab}D_{ab}u$ the Laplacian of $u$ with respect to the metric $G$. Then there exist positive constants $C_1, C_1', b$ such that

$$(4.3) \quad 0 < C_1 \leq n + m - 1 + \| Du \|^2 - \triangle u \leq C_1'$$

and

$$b^{-1}G \leq C_u' \leq bG,$$

where $C_1 = (n + m - 1)(\min L)^{(n+m-1)^{-1}}$. The constants $C_1'$ and $b$ depend on $\| u \|_{\infty}$, $\max L$, $\| F \|_{\mathcal{C}^2(L)}$ and the geometry of $(\Sigma, G, D)$.

**Proof.** The equivalence between the two metrics becomes an obvious fact once assertion (4.3) of the lemma is established. So the a priori bound on the $\mathcal{C}^2$-norm of $u$ comes down to establishing (4.3).

Making use of equation (4.1), the admissibility of $u$ and the arithmetic and geometric means inequality, we may write, in an adapted frame to $u$,

$$n + m - 1 + \| Du \|^2 - \triangle u = G^{ab}G_{ab} = \sum_{a=1}^{n+m-1}(1 + \| D_a u \|^2 - D_a u)$$

$$\geq (n + m - 1) \prod_{a=1}^{n+m-1}(1 + \| D_a u \|^2 - D_a u)$$

$$= (n + m - 1) \left[ F(\xi, u)(1 + \| D^2 u \|^2) (\frac{n+m-1}{2}) \right]$$

which in view of the compactness of $L$ implies that

$$(4.4) \quad 0 < C_1 = (n + m - 1) \left( \min L \right) (\frac{1}{n+m-1}) \leq n + m - 1 + \| Du \|^2 - \triangle u.$$

On the other hand, denoting $\triangle' u = G^{ab}D_{ab}u$, we also have

$$(4.5) \quad n + m - 1 + \triangle' u = G^{ab}G_{ab} + G^{ab}D_a u D_b u > 0.$$

In view of (4.4), the proof reduces to establishing an a priori bound from below on $\triangle u$. For this purpose, let $b > 0$ be a fixed real number such that

$$n + m - 1 + \| Du \|^2 \leq b$$

and consider the function

$$(4.6) \quad \Gamma = (b - \triangle u) \exp [k \| Du \|^2 + e^{l(u+C_0)}],$$

where $k, l > 0$ are real numbers to be fixed below. Let $\xi \in \Sigma$ be a point where $\Gamma$ attains its maximum and suppose that

$$(4.7) \quad - \triangle u(\xi) \geq 1.$$

Hence, writing $\triangle' \log(\Gamma) \leq 0$ at $\xi$, we get

$$- \triangle' \frac{\triangle u}{b - \triangle u} - \frac{G^{ab}D_a \triangle u D_b \triangle u}{(b - \triangle u)^2} + l^2 e^{l(u+C_0)} G^{ab} D_a u D_b u$$

$$+ le^{l(u+C_0)} \triangle' u + 2k G^{ab} D_{abc} u D^c u + 2k G^{ab} G^{cd} D_{ac} u D_{bd} u \leq 0.$$
Next, covariantly differentiating twice the equation (4.1), we obtain

\begin{equation}
G^{ab}(D_{da}uD_bu + D_auD_{db}u - D_{dab}u) = D_d \log(F) + (m + 1) \frac{D_auD_{d}^2u}{1 + |D^v u|^2}
\end{equation}

and

\begin{equation}
G^{ab}(2D_{ca}uD_bu + 2D_{ca}uD_{db}u - D_{cdab}u) = G^{ae}G^{fb}K_{eaf}K_{dab} + D_{cd} \log(F)
\end{equation}

\begin{equation}
+ (n + 1) \frac{D_{ca}uD^a u + D_{ca}uD_{d}^2u}{1 + |D^v u|^2} - 2(m + 1) \frac{D_{ca}uD^a uD_{db}uD^\beta u}{(1 + |D^v u|^2)^2},
\end{equation}

where \(K_{cab} = D_{ca}uD_bu + D_auD_{cb}u - D_{cab}u\). Hence contracting by \(G^{cd}\) we obtain:

\begin{equation}
-G^{ab}G^{cd}D_{cdab}u = -2G^{ab}G^{cd}D_{ca}uD_{db}u - 2G^{ab}G^{cd}D_{cdab}u
+ \triangle \log(F) + (m + 1) \frac{G^{cd}D_{ca}uD^a u + D_{ca}uD^a u}{1 + |D^v u|^2}
\end{equation}

\begin{equation}
+ (n + 1) \frac{G^{cd}D_{ca}uD^a uD_{d}^2uD^\beta u}{(1 + |D^v u|^2)^2},
\end{equation}

where \(K = G^{cd}G^{ae}G^{fb}K_{eaf}K_{dab}\). The previous expression will imply the desired expression of \(\triangle' \alpha' u\) by permutations of the covariant derivatives. The last gives rise to terms involving torsion and curvature components. Recall that standard computations show that

\begin{equation}
D_{ab}u = D_{ba}u + T^e_{ba}D_e u
\end{equation}

and

\begin{equation}
D_{abc}u = D_{bac}u + R^e_{cba}D_e u + T^e_{ba}D_{ce} u.
\end{equation}

Consequently, we deduce that

\begin{equation}
G^{cd}D_{cd}u = D_a \triangle u + G^{cd}T^e_{ac}D_{ed} u + G^{cd}T^e_{ad}D_{ce} u + G^{cd}(R^e_{dac} + D_eT^e_{ad})D_e u.
\end{equation}

Inserting into (4.10) yields

\begin{equation}
-G^{ab}G^{cd}D_{cdab}u = \triangle \log(F) - 2G^{ab}D_{ca}uD^a_b u - 2G^{ab}D_a \triangle uD_b u
+ (m + 1) \frac{D_{ca}uD^a u + D_{ca}uD^a u}{1 + |D^v u|^2}
- 2(m + 1) \frac{D_{ca}uD^a uD_{d}^2uD^\beta u}{(1 + |D^v u|^2)^2}
\end{equation}

\begin{equation}
+ (m + 1) \frac{R^e_{ac}D_e uD^a u}{1 + |D^v u|^2} - 2G^{ab}T^e_{ac}(D_e u + D^e_u)D_b u
- 2G^{ab}(R^e_{ac} + D^eT^e_{ac})D_e uD_b u + K.
\end{equation}

Furthermore, combining (4.11), (4.12) and the following relation

\begin{equation}
D_{abcd}u = D_{bacd}u + R^e_{cba}D_e u + R^e_{dca}D_e u + T^e_{ba}D_{ecd} u,
\end{equation}
we check that
\begin{equation}
D_{cdab}u = D_{abcd}u + T^e_{bc}D_{aebd}u + T^e_{ac}D_{ebd}u + T^e_{bad}D_{cae}u + T^e_{ad}D_{cebd}u \\
+ (R^e_{db} + D_cT^e_{ba})D_{ae}u + (R^e_{dac} + D_cT^e_{ad})D_{be}u \\
+ (R^e_{bad} + D_aT^e_{bd})D_{ce}u + (R^e_{bac} + D_aT^e_{ba})D_{de}u \\
+ (D_aR^e_{dbc} + D_cR^e_{bad} + D_cT^e_{bd} + D_cT^f_{ad}T^e_{bf})D_{ce}u.
\end{equation}

(4.15)

Therefore, taking into account (4.11) and (4.12), equality (4.13) leads to
\begin{equation}
-G^{ab}G^{cd}D_{cdab}u = \Delta \log(F) - 2G^{ab}D_{ca}uD^c_bu - 2G^{ab}D_a\Delta uD_{b}u \\
+ (m + 1) \frac{D_a\Delta uD^a_u + D_{ca}uD^a_{ca}u}{1 + |D^v_u|^2} - 2(m + 1) \frac{D_{ca}uD^a_uD^a_{g}uD^g_u}{(1 + |D^v_u|^2)^2} \\
+ (m + 1) \frac{R_{acc}^eD_euD^a_u}{1 + |D^v_u|^2} + K + 4G^{ab}G^{cd}T^e_{ac}D_{dc}u + E_1 + E_2,
\end{equation}

(4.16)

where
\begin{align*}
E_1 &= G^{ab}G^{cd}[(R^e_{bad} + D_aT^e_{bd})D_{ce}u + (R^e_{bac} + D_aT^e_{ba})D_{ed}u] \\
&+ 2G^{ab}G^{cd}T^e_{ac}[T^f_{be}D_{df}u + T^f_{db}D_{ef}u - 2D_{de}uD_{b}u]
\end{align*}

and
\begin{align*}
E_2 &= 2G^{ab}G^{cd}[(R^e_{dac} + D_cT^e_{ad} + T^f_{ac}T^e_{df})D_{be}u - D_{b}uD_{e}u] \\
&+ G^{ab}G^{cd}(D_aR^e_{db} + D_cR^e_{bad} + D_cT^e_{bd} + D_cT^f_{ad}T^e_{bf})D_{ce}u \\
&+ G^{ab}G^{cd}T^e_{ac}(R^f_{cd} + R^f_{bd} + D_bT^f_{de} + D_cT^f_{db} + 2D_yT^f_{de}D_{f}u \\
&+ G^{ab}G^{cd}T^e_{ac}(D_{de}T^f_{bd} + T^f_{de}T^g_{bf})D_{g}u.
\end{align*}

In view of (4.7), the relation $G_{ab}^c = G_{ab} + D_duD_b - D_{ab}u$ and the choice of the real $b$, there exist positive constants $C_2$ and $C_3$, independent of $u$, such that
\begin{equation}
|E_1| \leq C_2(b - \Delta u)(1 + G^{ab}G_{ab})
\end{equation}

(4.17)

and
\begin{equation}
|E_2| \leq C_3(1 + G^{ab}G_{ab}).
\end{equation}

(4.18)

On the other hand, it is clear that
\begin{equation}
G^{cd}D_{ca}uD^a_uD_{d\beta}uD^\beta_u \leq |D^v_u|^2G^{ab}G^{cd}D_{ac}uD_{bd}u
\end{equation}

and
\begin{equation}
G^{ab}G^{cd}D_{ac}uD_{bd}u \leq (b - \Delta u)G^{ab}G^{cd}D_{ac}uD_{bd}u.
\end{equation}

(4.19)
Inserting these inequalities into (4.16), by virtue of (4.5), (4.7) and (4.11), we easily deduce the existence of a positive constant, $C_4$, such that

$$-\triangle'\triangle u \geq K + 4G^{ab}G^{cd}T_{ac}^{e}D_{bde}u - 2(m + 2)(b - \triangle u)G^{ab}G^{cd}D_{ac}uD_{bd}u$$

(4.20)

$$-2G^{ab}D_{a}b\triangle uD_{b}u + (m + 1)(1 + |D^{v}u|^{2})^{-1}D_{a}\triangle uD^{a}u + \triangle \log(F) - C_4(b - \triangle u)(1 + G^{ab}G_{ab}).$$

Now, contracting (4.9) by $D^{d}u$, we get

$$G^{ab}D_{dab}uD^{d}u = 2G^{ab}D_{da}uD^{d}uD_{b}u - D_{d}\log(F)D^{d}u - \frac{(m + 1)D_{d}|D^{v}u|^{2}D^{c}u}{2(1 + |D^{v}u|^{2})}.$$ 

Using (4.11) and (4.12), we show that

$$G^{ab}D_{ab}uD^{d}u = G^{ab}D_{a}|Du|^{2}D_{b}u - D_{d}\log(F)D^{d}u + E_{3}$$

(4.21)

$$-\frac{m + 1}{2}(1 + |D^{v}u|^{2})^{-1}D_{a}|Du|^{2}D^{a}u,$$

where $E_{3}$ is given by

$$E_{3} = -G^{ab}\left[(R_{bad}^{c} + D_{a}T_{bd}^{c} + T_{ad}^{c})D_{ac}u + 2G_{be}T_{ad}^{e}\right]D^{d}u + 2T^{a}D^{d}u.$$

Thus, there exists a positive constant, say $C_{5}$, such that

(4.22) $$|E_{3}| \leq C_{5}(1 + G^{ab}G_{ab}).$$

Hence, combining (4.20) and (4.21), we obtain, by (4.22), the following relation :

$$-\frac{\triangle'\triangle u}{b - \triangle u} + 2kG^{ab}D_{abc}uD^{u} \geq \frac{K}{b - \triangle u} - 2(m + 2)G^{ab}D_{ac}uD_{b}u$$

(4.23)

$$-\frac{m + 1}{1 + |D^{v}u|^{2}}\left(-\frac{D_{\lambda}\triangle u}{b - \triangle u} + kD_{\lambda}|Du|^{2}\right)D^{\lambda}u + \frac{4G^{ab}G^{cd}T_{ac}^{e}}{b - \triangle u}D_{bde}u$$

$$+ 2G^{ab}\left(-\frac{D_{a}\triangle u}{b - \triangle u} + kD_{a}|Du|^{2}\right)D_{b}u + \frac{\triangle \log(F)}{b - \triangle u} - C_{6}(1 + G^{ab}G_{ab}),$$

where $C_{6}$ is a positive constant independent of $u$. But, at the point $\xi$, where $\Gamma$ attains its maximum, the gradient of the function $\Gamma$ must vanish. Taking the logarithmic derivative of $\Gamma$, we get

(4.24) $$\frac{-D_{a}\triangle u}{b - \triangle u} + kD_{a}|Du|^{2} = -le^{l(u+C_{0})}D_{a}u.$$ 

So that (4.23) leads to the following inequality

$$-\frac{\triangle'\triangle u}{b - \triangle u} + 2kG^{ab}D_{abc}uD^{u} \geq \frac{K}{b - \triangle u} - 2(m + 2)G^{ab}D_{ac}uD_{b}u$$

(4.25)

$$+ \frac{4G^{ab}T_{ac}^{e}}{b - \triangle u}D_{bde}u + \frac{\triangle \log(F)}{b - \triangle u} - 2le^{l(u+C_{0})}G^{ab}D_{a}uD_{b}u - C_{6}(1 + G^{ab}G_{ab}).$$
Now, we expand the following term

$$K' = G^{cd} G^{ef} G^{ab} \left[ K_{dab} + (b - \triangle u)^{-1} D_a \triangle u G_{db} - 2T_{ab}^{hc} G_{bh} \right]$$

$$\times \left[ K_{ce} + (b - \triangle u)^{-1} D_c \triangle u G_{cf} - 2T_{ce}^{hc} G_{hf} \right].$$

Taking into account the choice of the real $b$, (4.11), (4.12) and the following two relations

$$C_{ab} = G_{ab} + D_a u D_b - D_{ab} u \quad \text{and} \quad G^{ab} G_{cd} = \delta^a_d,$$

a direct computation leads to the following

$$K' \leq K - (b - \triangle u)^{-1} G^{tab} D_a \triangle u D_b \triangle u + 4G^{tab} G^{cd} T_{ac}^{e} D_{be} u$$

$$+ 2(b - \triangle u)^{-1} \left\{ (1 + |D u|^2 + \triangle u) G^{tab} D_a \triangle u D_b u - D^a \triangle u D_a u \right\}$$

$$- G^{tab} G^{cd} D_a \triangle u \left[ (R_{cde} + D_b T_{cd}^e) D_e u + (T_{be} D_e D_c u + T_{cb} D_e u) \right]$$

$$- 2G^{tab} D_a \triangle u \left[ T_{be}^c + (T_{be} D^e - G^{cd} T_{be}^f T_{cd}^e) D_e u \right]$$

$$- 4G^{tab} D_a u (T_{be}^c D^e + T_{bc} D^e D_e u) - 4G^{tab} T_{ba} D_e u$$

$$+ 4G^{cd} G^{ef} G_{ab} T_{ac}^{e} T_{bd}^{f} + 4G^{tab} T_{ba} D_e u.$$

Thus, taking into account (4.2) and the $C^1$ a priori estimate of Lemma 1, we conclude to the existence of a positive constant $C_7$ independent of $u$ such that

$$K' \leq K - (b - \triangle u)^{-1} G^{tab} D_a \triangle u D_b \triangle u + 4G^{tab} G^{cd} T_{ac}^{e} D_{be} u$$

$$+ 2(b - \triangle u)^{-1} G^{tab} D_a \triangle u \left[ (1 + |D u|^2 + \triangle u - G^{cd} T_{ac}^{e} D_e u) D_b u \right]$$

$$- 2T_{be} - (R_{cde} + G^{cd} D_b T_{cd}^e + 2T_{be}^c D^e - 2G^{cd} T_{be}^f T_{cd}^e) D_e u$$

$$- G^{cd} (T_{cd}^e G_{be} + T_{be}^e D_e u) + C_7(b - \triangle u)(1 + G^{tab} G_{ab})$$

$$- 2(b - \triangle u)^{-1} D_a \triangle u (D^a u - G^{cd} T_{ac}^{e} D_b u).$$

Using (4.24), (4.26) and taking (4.7) into account, we easily arrive at the existence of positive constants $C_8$ and $C_9$, independent of $u$, such that

$$K' \leq K - (b - \triangle u)^{-1} G^{tab} D_a \triangle u D_b \triangle u + 4G^{tab} G^{cd} T_{ac}^{e} D_{be} u$$

$$+ C_8 \left[ k + l e^{l(u+C_0)} \right] (b - \triangle u)(1 + G^{tab} D_a u D_b u) + C_9(b - \triangle u) G^{tab} G_{ab}.$$

In view of the positivity of $K'$, this inequality implies that

$$K + 4G^{tab} G^{cd} T_{ac}^{e} D_{be} u \geq (b - \triangle u)^{-1} G^{tab} D_a \triangle u D_b \triangle u$$

$$- (b - \triangle u) \left[ C_8 \left( k + l e^{l(u+C_0)} \right) (1 + G^{tab} D_a u D_b u) + C_9 G^{tab} G_{ab} \right].$$
Combining with (4.25), we obtain

\[
\frac{-\triangle' \triangle u}{b - \triangle u} - \frac{G^{ab} D_a \triangle u D_b \triangle u}{(b - \triangle u)^2} + 2k G^{ab} D_{abc} u D^c u \geq \frac{\triangle \log(F)}{b - \triangle u}
\]

\[-2(m + 2)G^{ab} D_{ac} u D_b \triangle u - (C_0 + C_9)(1 + G^{ab} G_{ab})
\]

\[-(2 + C_8) \left[ k + l e^{l(u+C_0)} \right] (1 + G^{ab} D_a u D_b u).
\]

Substituting into (4.8) and taking (4.5) into account, we get the following

\[
2(k - m - 2)G^{ab} G^{cd} D_{ac} u D_{bd} u + \left[ l e^{l(u+C_0)} - C_6 - C_9 \right] G^{ab} G_{ab}
\]

\[
+ \left[ l e^{l(u+C_0)} - (2 + C_8) \left( k + l e^{l(u+C_0)} \right) \right] G^{ab} D_a u D_b u + \frac{\triangle \log(F)}{b - \triangle u} \leq C_{10},
\]

where \(C_{10} = C_6 + C_9 + (n + m - 1)l e^{2lC_0} + (2 + C_8)(k + l e^{2lC_0})\). Select \(k = m + 2\) so that the component of \(G^{ab} G^{cd} D_{ac} u D_{bd} u\) vanishes. Choosing \(l\) real sufficiently large, inequality (4.27) yields

\[
(4.28)
G^{ab} G_{ab} \leq C_{10} - \frac{\triangle \log(F)}{b - \triangle u}.
\]

On the other hand, by virtue of estimate (4.2), the development of \(\triangle \log(F)\) shows that there exists a positive constant \(C_{11}\) such that

\[
\left| \frac{\triangle \log(F)}{b - \triangle u} \right| \leq C_{11}.
\]

Therefore inequality (4.28) leads to the following

\[
(4.29)
G^{ab} G_{ab} \leq C_{12} := C_{10} + C_{11}.
\]

Finally, using equation (4.1), satisfied by \(u\), and the arithmetic and geometric means inequality, we check that

\[
-\triangle u \leq (1 + |D^v u|^2) \frac{m+1}{n} F \left( \xi, u(\xi) \right) \left( \frac{G^{ab} G_{ab}}{n + m - 2} \right)^{n+m-2}.
\]

Taking into account (4.2) and the \(C^1\) a priori estimate stated in Lemma 1, inequality (4.29) implies that \(b - \triangle u(\xi) \leq C_{13}\), where \(C_{13}\) depends on \(n, m, C_0, C_{12}\) and \(\sup_L \hat{F}\). The definition (4.6) of \(\Gamma\) allows us to conclude the proof of (4.3). The equivalence between the metrics \(G\) and \(G'_a\) is clear.

**Lemma 3.** Keeping all the notations of the previous Lemma, let \(u \in C^5(\Sigma)\) be an admissible solution of (4.1) satisfying (4.2). Denote

\[
\Omega^2 = G^{ab} G^{cd} G^{ef} D_{ace} u D_{bdf} u \text{ and } \hat{F} = \log \left( 1 + |D^v u|^2 \right)^{\frac{m+1}{n+1} F(\xi, u)}.
\]
(i) There exist two positive constants $k_1$ and $k_2$ such that:

\[ \Delta' \Omega^2 + 2G^{ab}G^{cd}G^{ef}D_{ace}F_{bdf}u - 2G^{ab}D_a \Omega^2 D_b u \]

\[ + H^{abdeefij}D_{ace}uD_{bdf}uD_{ij} \geq -k_1(1 + \Omega^3) \]

and

\[ \Delta' \Omega^2 \geq -k_2(1 + \Omega^3) - (m + 1) \frac{D_a \Omega^2 D^a u}{1 + |D^v u|^2} + 2G^{ab}D_a \Omega^2 D_b u. \]

The components of the tensor $H$ are given by

\[ H^{ab...ij} = G^{aj}G^{ib}G^{cd}G^{ef} + G^{ab}G^{cj}G^{id}G^{ef} + G^{ab}G^{cd}G^{ej}G^{if}. \]

(ii) We have:

\[ \|\Omega\|_{C^0(\Sigma)} < \infty \text{ and, for any } \alpha \in [0, 1], u \text{ is uniformly bounded in } C^{3,\alpha}(\Sigma). \]

**Proof.** 1- Inequality (4.31) is an immediate consequence of (4.30), it follows by simply expanding the terms involving $\tilde{F}$. So let us show how (ii) follows from (4.31). In view of Lemma 2, there exist positive constants $C_1$, $C_1'$ and $b$ such that

\[ 0 < C_1 \leq n + m - 1 + |Du|^2 - \Delta u \leq C_1' \text{ and } b^{-1}G \leq G'(u) \leq bG. \]

Expanding the following positive term

\[ G^{ab}G^{cd}G^{ef} \left( D_{ace}u - \frac{1}{n + m - 1} G_{ce}D_a \Delta u \right) \times \left( D_{bdf}u - \frac{1}{n + m - 1} G_{df}D_b \Delta u \right) \]

and using (4.32), we show that there exists a positive constant $C_2$ such that

\[ (4.34) \quad G^{ab}D_a \Delta u D_b \Delta u \leq C_2 \Omega^2. \]

On the other hand, by Cauchy’s inequality and (4.33), formula (4.16) of the proof of Lemma 2 says that there exist two positive constants $C_3$ and $C_4$ such that

\[ (4.35) \quad - \Delta' \Delta u \geq -C_3 + C_4 \Omega^2 + (m + 1) \frac{D_a \Delta u D^a u}{1 + |D^v u|^2} - 2G^{ab}D_a \Delta u D_b u. \]

Set $\Gamma = \Omega - l \Delta u$, where $l > 0$ is a real number. The relation

\[ \Delta' \Omega^2 = 2\Omega \Delta' \Omega + 2G^{ab}D_a \Omega D_b \Omega, \]

joined to (4.31) and (4.35) implies the existence of positive constants $C_5$ and $C_6$ such that

\[ 2\Omega \Delta' \Gamma \geq -C_5 - C_6 \Omega^3 + lC_4 \Omega^3 - 2G^{ab}D_a \Omega D_b \Omega \]

\[ - \frac{2(m + 1)\Omega}{1 + |D^v u|^2} D_a \Gamma D^a u + 4\Omega G^{ab}D_a \Gamma D_b u. \]

At a point $\xi \in \Sigma$, where $\Gamma$ attains its maximum, we have

\[ 0 \geq 2\Omega \Delta' \Gamma \text{ and } D_a \Gamma = 0. \]
So that inequality (4.36) allows us to write

\[ 0 \geq -C_5 - C_6 \Omega^3 + l C_4 \Omega^3 - 2l^2 G^{ab} D_a \Delta u D_b \Delta u. \]

Selecting \( l = (C_4)^{-1}(1 + C_5 + C_6) \) and inserting inequality (4.34) into this last one, we easily obtain \( \Omega \leq \max(1, 2l^2 C_2) \). From this we can easily conclude \( \Omega \leq C_7 \).

Now, covariantly differentiating once equation (4.1), we get

\[ \Delta' (D_A u) = H_A, \]

where the right side \( H_A \) involves only covariant derivatives of \( u \) of order less or equal to two. Therefore \( \| H_A \|_{\| \cdot \|_{\alpha, (\Sigma)}} \leq C_9 \), and one can use Schauder’s inequalities to deduce that \( \| D u \|_{\| \cdot \|_{2,\alpha, (\Sigma)}} \leq C_{10} \).

2- In this paragraph we describe the steps needed to establish inequality (4.30). At first, we can write

\[ (4.37) \quad \Delta' \Omega^2 = \sum_{i=1}^{5} K_i. \]

The tensor \( H \) is given by (4.32) and the terms \( (K_i)_{1 \leq i \leq 5} \) are defined as follow:

\[ K_1 = 2 G^{kl} G^{ab} G^{cd} G^{ef} D_{kl} D_{ace} D_{bdf} u, \]
\[ K_2 = 2 G^{kl} G^{ab} G^{cd} G^{ef} D_{kace} D_{bdf} u, \]
\[ K_3 = -2 G^{kl} H^{ab...ij} (D_{li} D_{j} u + D_{l} D_{ij} u - D_{lij} u) \times (D_{kace} D_{bdf} u + D_{ace} D_{kbf} u), \]
\[ K_4 = -G^{kl} H^{ab...ij} (2 D_{kli} D_{j} u + 2 D_{ki} D_{lj} u - D_{klij} u) D_{ace} D_{bdf} u, \]
\[ K_5 = -G^{kl} D_{k} H^{ab...ij} (D_{li} D_{j} u + D_{l} D_{ij} u - D_{lij} u) D_{ace} D_{bdf} u. \]

Let us write \( U \approx V \) to say that \( U \) and \( V \) are equivalent; i.e. if there exists a universal positive constant \( c \) such that: \( |U(u) - V(u)| \leq c(1 + \Omega^3) \) and use the convention of summing repeated indices from 1 to \( n + m - 1 \).

1. Study of \( K_5 \). Expanding \( D_k H^{ab...ij} \) and denoting in a \( G' \)-orthonormal frame

\[ A = D_{kab} u D_{kca} u D_{lca} u D_{lbd} u, \]
\[ B = D_{kab} u D_{kca} u D_{lab} u D_{lcd} u. \]

Using formulas (4.11) and (4.12) of the proof of Lemma 2, we see that

\[ (4.38) \quad K_5 \approx 6A + 6B. \]

2. Study of \( K_4 \). At first, we see that

\[ K_4 \approx G^{kl} H^{ab...ij} D_{klij} u D_{ace} u D_{bdf} u. \]

Thus, covariantly differentiating twice the equation satisfied by \( u \), we can show that:

\[ K_4 + H^{ab...ij} D_{ij} F D_{ace} u D_{bdf} u + H^{ab...ij} G^{kp} G^{ql} D_{jkl} u D_{iqp} u D_{ace} u D_{bdf} u \approx \]
\[ \approx G^{kl} H^{ab...ij} D_{ace} u D_{bdf} u (D_{klij} u - D_{lij} u). \]
Arguing as in the proof of Lemma 2, we see that the right hand side term is equivalent to zero. On the other hand, using (4.11) and (4.12) of the proof of Lemma 2, the sum of the terms in \((D^3 u)^4\) in the left hand side is equivalent to \(3B\). Therefore

\[(4.39)\]

\[K_4 + H^{ab \ldots ij} D_{ij} \tilde{F} D_{ace} u D_{bdf} u + 3B \simeq 0.\]

(iii) Study of \(K_3\). By virtue of formulas (4.11) and (4.12), and denoting in a \(G'\)-orthonormal frame

\[C = D_{lab} u D_{lcad} u D_{abcd} u,\]

we see that

\[(4.40)\]

\[K_3 \simeq 12C + 2P_{abcd} D_{abcd} u,\]

where \(P_{abcd}\) are linear combinations of the derivatives of \(u\) of order three, its components depend upon the curvature and torsion components as well as the covariant derivatives of \(u\) of order less than two.

(iv) Study of \(K_1\). Covariantly differentiating three times the equation satisfied by \(u\), we eliminate the order five covariant derivatives of \(u\). In fact, we find

\[G^{bcd}_{ab} G^{def}_{cd} \tilde{G} D_{acek} u D_{bdf} u \simeq 2G^{cd}_{ab} G^{def}_{cd} \tilde{G} D_{acek} u D_{bdf} u\]

\[+ G^{cd}_{ab} G^{def}_{cd} \tilde{G} \tilde{G} (D_{ai} u D_{k} u + D_{i} u D_{aj} u - D_{aij} u) D_{acek} u D_{bdf} u\]

\[+ G^{cd}_{ab} G^{def}_{cd} \tilde{G} \tilde{G} (D_{k} u D_{i} u + D_{k} u D_{cj} u - D_{ci} u) D_{acek} u D_{bdf} u\]

\[+ G^{cd}_{ab} G^{def}_{cd} \tilde{G} \tilde{G} (D_{ac} u D_{i} u + D_{k} u D_{ci} u - D_{ek} u) D_{acek} u D_{bdf} u\]

\[= G^{cd}_{ab} G^{def}_{cd} \tilde{G} \tilde{G} D_{acek} u D_{ci} u D_{bdf} u\]

\[= G^{cd}_{ab} G^{def}_{cd} \tilde{G} \tilde{F} D_{bdf} u.\]

Thus, a permutation of the order of covariant derivatives of \(u\) allows us to write:

\[G^{cd}_{ab} G^{def}_{cd} \tilde{G} D_{klacc} u D_{bdf} u \simeq G^{cd}_{ab} D_{k} \Omega^2 D_{i} u\]

\[-G^{cd}_{ab} G^{def}_{cd} \tilde{G} \tilde{G} (D_{k} u D_{i} u + D_{k} u D_{cj} u - D_{ci} u) D_{acek} u D_{bdf} u\]

\[-3C + G^{cd}_{ab} G^{def}_{cd} \tilde{G} \tilde{G} Q_{kace} D_{bdf} u\]

\[-G^{cd}_{ab} G^{def}_{cd} \tilde{G} \tilde{G} D_{ace} \tilde{F} D_{bdf} u,\]

where \(Q_{kace}\) are linear combinations of the derivatives of \(u\) of order three, its components depend upon the curvature and torsion components as well as the covariant derivatives of \(u\) of order less or equal to two.

Expanding \(D_{a}(G^{cd}_{k} G^{cd}_{i})\) and using (4.11) and (4.12), we finally arrive at

\[(4.41)\]

\[K_1 \simeq 2G^{cd}_{ab} \Omega^2 D_{b} u + 2G^{cd}_{ab} G^{def}_{cd} \tilde{G} \tilde{G} Q_{kace} D_{bdf} u\]

\[-4A - 6C - 2G^{cd}_{ab} G^{def}_{cd} \tilde{G} \tilde{G} D_{ace} \tilde{F} D_{bdf} u.\]
(v) Inserting (4.38), (4.39), (4.30) and (4.31) into (4.37), we obtain
\[
\Delta' \Omega^2 \simeq 2 G_{ab} \Omega^2 D_b u + 2 (P + Q)_{abcd} D_{abcd} u + 2 A
\]
\[+ 3B + 6C + K_2 - H^{ab...ij} D_{ij} \tilde{F} D_{ace} u D_{bdf} u \]
\[-2G_{ab} G^{cd} G^{ef} D_{ace} \tilde{F} D_{bdf} u.\]
Let us introduce the tensors \( U \) and \( V \) which are defined in a \( G' \)-orthonormal frame by
\[
U_{abcd} = D_{abcd} u + D_{lab} u D_{bcd} u + (P + Q)_{abcd}
\]
and
\[
V_{abcd} = D_{abcd} u + D_{lab} u D_{bcd} u + D_{lac} u D_{lbd} u.
\]
We easily check that
\[
\|U\|^2 \simeq \frac{1}{2} K_2 + 2C + B + 2(P + Q)_{abcd} D_{abcd} u
\]
and, using once more relation (4.12) of the proof of Lemma 2, we obtain
\[
\|V\|^2 \simeq \frac{1}{2} K_2 + 2A + 2B + 4C.
\]
Thus, (4.42) becomes
\[
\Delta' \Omega^2 \simeq 2 G_{ab} \Omega^2 D_b u - H^{ab...ij} D_{ij} \tilde{F} D_{ace} u D_{bdf} u
\]
\[+ \|U\|^2 + \|V\|^2 - 2G_{ab} G^{cd} G^{ef} D_{ace} \tilde{F} D_{bdf} u
\]
and (4.30) follows from this relation in view of the positivity of \( \|U\|^2 \) and \( \|V\|^2 \).

5 Proof of the results

5.1 Proof of theorem 1

In order to end the proof of theorem 1, let us consider the function
\[
u = -\log(K).
\]
From the assumption on \( K \) we see that \( u = u \circ \pi \) in \( E_\ast \). Let \( \pi_\ast \) stand for the tangent map of \( \pi \). From the usual rules of differentiation it follows that
\[
Du(Y) = Du(\pi_\ast(Y)) \text{ for } Y \in TE.
\]
Recall that for any \( \xi \in E_\ast \) the vertical subspace \( V_\xi E \) of \( T_\xi E \) is exactly the kernel of \( \pi_\ast|_{T_\xi E} \). Thus, for all vertical direction \( \alpha \), we get \( D_\alpha u = 0 \) and, since \( D_{e_\alpha e_\beta} \) is a vertical vector field for \( \alpha, \beta \in \{n + 1, \ldots, n + m - 1\} \), the definition of the covariant derivative implies that \( D_{\alpha \beta} u = 0 \). Inserting this into equation (3.8), we deduce that
\[
G^u(e^u \xi) = e^{-(m-1)u(\xi)} = K(\xi)
\]
as claimed.

Now, if $K$ is constant, then the radial graph $\mathcal{Y} : \xi \mapsto e^{u(\xi)}\xi$, with $u$ as above, is a convex hypersurface whose vertical Gaussian curvature is given by $K$. Conversely, assume that there exists a function $u \in C^\infty(\Sigma)$ such that

$$\det \left( \delta^\beta_\alpha + D_\alpha u D^\beta u - D^\beta_\alpha u \right) = \left(1 + |D^\nu u|^2\right)^{\frac{m+1}{2}} e^{(m-1)u} K(\xi)$$

and that the hypersurface $\mathcal{Y}$ given by

$$\mathcal{Y}(\xi) = e^{u(\xi)}\xi, \text{ for } \xi \in \Sigma,$$

is convex so that its second fundamental tensor is a 2-covariant positive one. We need to compute the components of this tensor. Keeping all previous notations and following the formalism of moving frame we check that these are given by the $(n+m-1) \times (n+m-1)$-matrix

$$g' = \left( (1 - \mu_a)G_{ab} + (1 - 2\mu_a)\tilde{D}_a u \tilde{D}_b u - \tilde{D}_{ab} u \right)_{1 \leq a, b \leq n+m-1},$$

where $\tilde{D}_a u = e^{\mu_a u} D_a u$ and $\tilde{D}_{ab} u = e^{(\mu_a + \mu_b)u} D_{ab} u$. By the assumption on $\mathcal{Y}$, the matrix $g'$ is positive. Particularly, for any real $\varepsilon > 0$, the $n \times n$-matrix

$$\left( \varepsilon G_{ab} - D_a u D_b u - D_{ab} u \right)_{1 \leq a, b \leq n}$$

is positive definite. Now, consider the function

$$\Gamma(u) = -\varepsilon u + G^{ab}(\mu_a D_a u)(\mu_b D_b u)$$

at a point $\xi_0 \in \Sigma$ where it attains its maximum. For all horizontal direction $i \in \{1, ..., n\}$, we get $D_i \Gamma = 0$ so that, in an adapted frame to $u$, we can write

$$D_i u (-\varepsilon + D_i u) = 0.$$

Taking (5.2) into account, for all horizontal direction $i$, we obtain $D_i u(\xi_0) = 0$. Consequently $\Gamma(u)(\xi_0) = -\varepsilon u(\xi_0)$ and since $\Gamma(u)$ attains its maximum at $\xi_0$, we conclude that there exists a positive constant $C$ such that, for any $\varepsilon > 0$,

$$G^{ab}(\mu_a D_a u)(\mu_b D_b u) \leq C\varepsilon.$$

From this we conclude that the horizontal component of the gradient of $u$ is identically equal to zero. Using the definition of the covariant derivative, we show that, for any horizontal direction $i$, $1 \leq i \leq n$, and any vertical direction $\alpha$, $n+1 \leq \alpha \leq n+m-1$,

$$D_{i\alpha} u = D_{\alpha i} u = 0.$$

The first equality in (5.3) follows from the fact that torsion components of the form $T^a_{ia}$ are zero. Taking into account relation (2.8), we see that $D_{i\alpha \beta} u = D_{\alpha \beta i} u$. Now, for any vertical directions $\alpha, \beta$ and any horizontal one $i$, the definition of the covariant derivative yields

$$D_{i\alpha \beta} u = 0.$$

Therefore, covariantly differentiating the equation (5.1) in any horizontal direction, we see that the horizontal gradient of $K$ is identically equal to zero and as it is constant on each fibre of $E$, the vertical component of its gradient is also identically equal to zero. We then conclude that $K$ is constant.
5.2 Proof of theorem 2

Relying crucially on computations from section 2, we deduce that the vertical Gaussian curvature of the bundle $\Sigma_r$ is $r^{-(m-1)}$. The bundle $\Sigma_r$ is closed and convex. Now suppose that there exist a radial graph over $\Sigma$ with vertical Gaussian curvature given by $K$. Then there exists $u \in \mathcal{C}^\infty(\Sigma)$ satisfying (5.1). Set

$$\psi(r) = r^{m-1}K(r), \quad R_1 = \min_\Sigma e^u \quad \text{and} \quad R_2 = \max_\Sigma e^u.$$  

From (5.1), we see that

$$\psi(R_1) \leq 1 \quad \text{and} \quad \psi(R_2) \geq 1$$

thus, by continuity, there exists a real $r > 0$ such that $\psi(r) = 1$.

Now suppose that, for any $r > 0$ and any $\xi \in \Sigma$,

(5.4) \[ K(r\xi) = K(r) = \frac{1}{r^{m-1}}. \]

If $u \in \mathcal{C}^\infty(\Sigma)$ satisfies (1), then for any non constant $v \in \mathcal{C}^\infty(M)$, the function $\tilde{u} = u + v \circ \pi$ is also a solution of (5.1) and clearly their graphs are not homothetic. Thus, there is an infinite number of non homothetic closed hypersurfaces with vertical Gaussian curvature given by $K$. But two convex closed hypersurfaces with vertical Gaussian curvature given by $K$ are necessarily homothetic. In fact, if $\mathcal{Y}$, the radial graph of a function $u \in \mathcal{C}^\infty(\Sigma)$, is such an hypersurface, then

(5.5) \[ \det \left( \delta^\beta_\alpha + D_\alpha u D^\beta u - D^\beta_\alpha u \right) = \left( 1 + |D^v u|^2 \right)^{m+12}. \]

Recall from the proof of theorem 2 that the convexity of the hypersurface $\mathcal{Y}$ implies the nullity of the horizontal gradient of $u$. On the other hand, at a point $\xi \in \Sigma$ where $u$ attains its maximum, and in a $G$-orthonormal frame in which the symmetric matrix $[D_\alpha u D_\beta u - D_\alpha \beta u]$ is diagonal, it is clear that the eigenvalues of the matrix

$$G'_{\alpha} = G_{\alpha\beta} + D_\alpha u D_\beta u - D_\alpha \beta u$$

are strictly positive at $\xi$. By continuity, $[(G'_u)_{\alpha\beta}]$ must be positive definite everywhere. Let us introduce the operator $\triangle'$ by setting $\triangle' u = G'^{\alpha\beta} D_{\alpha\beta} u$. Covariantly differentiating (5.5), we get

$$G'^{\alpha\beta} (D_{\lambda\alpha} u D_\beta u + D_\alpha u D_{\lambda\beta} u - D_{\lambda\alpha \beta} u) = \frac{m + 1}{2} \frac{D_\lambda |D^v u|^2}{1 + |D^v u|^2}$$

and contracting by $D^\lambda u$, we obtain

(5.6) \[ G'^{\alpha\beta} D_{\lambda\beta} u D^\lambda u = G'^{\alpha\beta} D_\alpha |D^v u|^2 D_\beta u - \frac{(m + 1)}{2} \frac{D_\lambda |D^v u|^2}{1 + |D^v u|^2}. \]

From the expressions (2.8) and (2.9) of the curvature components, we can write

$$G'^{\alpha\beta} D_{\lambda\beta} u D^\lambda u = G'^{\alpha\beta} D_{\alpha\beta\lambda} u D^\lambda u + G'^{\alpha\beta} D_\alpha u D_\beta u - |D^v u|^2 G'^{\alpha\beta} G_{\alpha\beta}. $$
Inserting into (5.6), and evaluating at a point $\xi \in \Sigma$ where the function
\[ \Gamma = \frac{1}{2} G_{\alpha\beta} D_\alpha u D_\beta u \]
attains its maximum, we get
\[ G_{\alpha\beta} D_\alpha u D_\beta u = |D^v u|^2 G_{\alpha\beta} - G_{\alpha\beta} D_\alpha u D_\beta u. \]
But, at the point $\xi$, we must have $\Delta \Gamma \leq 0$. Thus
\[ |D^v u|^2 G_{\alpha\beta} - G_{\alpha\beta} D_\alpha u D_\beta u \leq 0. \]
In a $G$-orthonormal frame in which the symmetric matrix $(D_\alpha u D_\beta u - D_{\alpha\beta} u)$ is diagonal, we can write
\[ G_{\alpha\beta} D_\alpha u D_\beta u = \sum_{\alpha = n+1}^{n+m-1} \frac{|D_\alpha u|^2}{1 + |D_\alpha u|^2 - D_{\alpha\alpha} u} \leq |D^v u|^2 G_{\alpha\beta} G_{\alpha\beta}. \]
Therefore, the relation (5.7) implies
\[ 0 \leq G_{\alpha\beta} G_{\lambda\mu} D_{\alpha\lambda} u D_{\beta\mu} u \leq 0 \]
and in particular, at the point $\xi$, we must have for any directions $\alpha$ and $\beta$,
\[ D_{\alpha\beta} u = 0. \]
Inserting into (5.5), we obtain
\[ 1 + |D^v u|^2 = \det(\delta^\beta_\alpha + D_\alpha u D_\beta u) = (1 + |D^v u|^2)^{\frac{m+1}{2}}. \]
Hence $|D^v u|^2(\xi) = 0$ and then $|D^v u|^2 = 0$ everywhere. Thus, taking into account the fact that the horizontal gradient of $u$ is identically equal to zero, the function $u$ must be a constant. This ends the proof of the theorem.

### 5.3 Proof of theorem 3

Let $f \in C^\infty(\Sigma)$ be a strictly positive function and $\lambda > 0$ a real number. We want to solve the equations
\[ \begin{cases} \mathcal{N}_1(u) = 1 \\ \mathcal{N}_2(u) = e^{-\lambda u} f(\xi)(1 + |D^v u|^2)^{\frac{m+1}{2}}. \end{cases} \tag{5.8} \]
Remark first that a solution of (5.8), if there is any, is necessarily admissible. In fact, at a point $\xi \in \Sigma$ where $u$ attains its maximum, and in a frame adapted to $u$, it is clear that the eigenvalues of the tensor $G''_{\alpha\beta}$ are strictly positive at $\xi$. By continuity and since $\mathcal{N}_1(u) > 0$ we see that the matrix $[(G''_{\alpha\beta})_{\alpha\beta}]_{1 \leq i, j \leq n}$ must be positive definite everywhere. We also have $\mathcal{N}_2(u) > 0$ so that the matrix $[(G''_{\alpha\beta})_{\alpha\beta}]_{n+1 \leq \alpha, \beta \leq n+m-1}$ is positive definite everywhere.

On the other hand, if a solution exists, it is unique for if there exist two solutions $u_1$ and $u_2$, we put $u = u_1 - u_2$ and let $\xi \in \Sigma$ be a point where $u$ attains its minimum.
In an orthonormal frame for \([G'_{u_1}]_{\alpha\beta}\) that diagonalises \([G'_{u_2}]_{\alpha\beta}\) we have \((G'_{u_2})_{\alpha\beta} = (G'_{u_1})_{\alpha\beta} + D_{\alpha\beta} u\). Thus, we may write:

\[
\frac{N_2(u_2)}{N_2(u_1)} = \prod_{\alpha=\alpha+1}^{n+m-1} (1 + D_{\alpha\alpha} u) = e^{\lambda u}.
\]

Taking into account the fact that \(D_{\alpha\alpha} u(\xi) \geq 0\), the last equality implies : \(u(\xi) \geq 0\). Hence \(u_1 - u_2 \geq 0\) everywhere. By an analogous argument, but at a point where \(u\) attains its maximum, we get the inequality in the other sense. Thus, the functions \(u_1\) and \(u_2\) must be equal.

To treat the existence part, we will use the continuity method in the framework of \(C^\infty\) functions [4]. This method consists of the following steps.

For \(t \in [0,1]\), consider the family of equations

\[
\begin{align*}
N_1(u) &= 1 \\
N_2(u) &= e^{-\lambda u} f(\xi)(1 + |D^v u|^2)^{m+1} t.
\end{align*}
\]

(5.9)

By the previous discussion, any solution \(u_t\) of (5.9) is admissible. So let \(T\) be the set of \(t \in [0,1]\) for which such a solution exists in \(C^\infty(\Sigma)\).

Observe that the function \(u_0 = 0\) is a solution of (5.9) for \(t = 0\). Hence \(T\) is not empty. If we prove that \(T\) is open and closed in \([0,1]\) then \(T = [0,1]\) and the equation (5.8) is solved in \(C^\infty(\Sigma)\). Recall that the \(C^\infty\) regularity follows from the well known regularity theory of elliptic equations.

For the closeness, we have to establish, for any \(k \geq 0\), a uniformly bound of the \(C^k\)-norm of any admissible solution of (5.9). The \(C^0\) estimate \(|u_t| \leq \lambda^{-1}\|\log(f)\|_\infty\) of any solution of (5.9) is immediate, for if \(\xi_1 \in \Sigma\) is a point where a solution \(u_t\) of (5.9) attains its maximum, then in a frame adapted to \(u_t\) we get, for any \(\alpha\), \(D_{\alpha\alpha} u_t(\xi_1) = 0\) and \(D_{\alpha\alpha} u_t(\xi_1) \leq 0\). So (5.9) implies :

\[
0 \leq \log[N_2(u_t)](\xi_1) = -\lambda u_t + t \log[f(\xi_1)].
\]

Hence \(u_t \leq \lambda^{-1}\|\log(f)\|_\infty\). We complete our \(C^0\) estimate by an analogous reasoning, considering a point where \(u_t\) attains its minimum.

The first, second and third order a priori estimates are given in Lemmas 1, 2 and 3 and the higher order estimates are established by induction; we apply the maximum principle as in lemma 3 to functional depending on a norm of derivatives of order \(k\) we want to bound and on already bounded quantities. These estimates may be recovered by Schauder’s inequalities applied inductively to equations obtained by covariantly differentiating the initial one. In fact, it follows from the \(C^3\)-estimates that the components of the linearised operator are bounded in \(C^{0,\alpha}(\Sigma)\). Thus keeping the same notations as in lemma 3 and assuming that

\[
\|u\|_{C^{k,\alpha}(\Sigma)} \leq C_k, \ k \geq 3,
\]

where \(u \in C^\infty(\Sigma)\) is an admissible solution of the equation

\[
N_1(u)N_2(u) = F(\xi, Du, u).
\]
Covariantly differentiating \((k-1)\) times this equation, on any open set \(U\) with coordinates, we get
\[
\triangle'(D_{a_1\ldots a_{k-1}}u) = H_{a_1\ldots a_{k-1}},
\]
where the right side \(H_{a_1\ldots a_{k-1}}\) involves only covariant derivatives of \(u\) of order less or equal to \(k\). Therefore \(\|H_{a_1\ldots a_{k-1}}\|_{\mathcal{C}^{\alpha}(\Sigma)} \leq C_{\text{ste}}\). We can apply Schauder’s inequalities to deduce that for any compact \(K \subset U\), we have \(\|D^{(k-1)}u\|_{\mathcal{C}^{\alpha}(K)} \leq C'_{k+1}\). Consequently, \(\|u\|_{\mathcal{C}^{k+1,\alpha}(\Sigma)} \leq C_{k+1}\).

To show that \(T\) is open, let \(A^\infty(\Sigma)\) be the set of admissible functions \(u \in \mathcal{C}^\infty(\Sigma)\) and denote by \(\Theta\) the following subset:
\[
\Theta = \{u \in A^\infty(\Sigma) | N_1(u) = 1\}.
\]
The set \(\Theta\) is a hypersurface of \(A^\infty(\Sigma)\), this will be shown below. Now let \(\Gamma\) be the functional defined on \(\Theta \times [0, 1]\) by
\[
\Gamma(u, t) = \log |N_2(u)| + \lambda u - t \log \left[f(\xi)(1 + |D^\alpha u|^2)^{\frac{m+1}{2}}\right].
\]
The function \(\Gamma\) is continuously differentiable and its differential at \(u \in \Theta\) is a linear operator from \(B^\infty(\Sigma)\) into \(\mathcal{C}^\infty(\Sigma)\) given by
\[
d_u \Gamma(w) = G^{\alpha\beta}(2D_\alpha u D_\beta w - D_\alpha w) + \lambda w - t(m+1) (D^\alpha u D_\alpha w) \frac{1}{1 + |D^\alpha u|^2},
\]
where
\[
B^\infty(\Sigma) = \left\{w \in \mathcal{C}^\infty(\Sigma) | \sum_{1 \leq i,j \leq n} G^{ij}(2D_i u D_j w - D_{ij} w) = 0\right\}.
\]
For \(u \in \Theta\), \(d_u \Gamma\) is invertible. In fact, its null space is trivial since \(\lambda > 0\). So we are done if we prove that, for any \(v \in \mathcal{C}^\infty(\Sigma)\), there exists a solution \(w \in B^\infty(\Sigma)\) of the equation
\[
d_u \Gamma(w) = v.
\]
The proof is the same for arbitrary values of the parameter \(t\). So assume \(t = 0\), set \(r\) for the function \(r(\xi) = \|\xi\|\) for all \(\xi \in E_s\), extend \(u\) and \(v\) to \(E_s\) as radially constant functions and set \(\bar{\Sigma} = \{\xi \in E | 1 \leq \|\xi\| \leq 2\}\). Let us also denote by \(L[u]\) the linear operator defined by
\[
L[u](w) := r \sum_{1 \leq i,j \leq n} G^{ij}_{r,u} D_{ij} w + r^2 \sum_{m+1 \leq \alpha, \beta \leq n+m-1} G^{\alpha\beta}_{r,u} D_{\alpha\beta} w - \lambda w,
\]
where
\[
G^{\alpha\beta}_{r,u} = G^{\alpha\beta} + r^2 D^\alpha u D^\beta u - r^2 D_{\alpha\beta} u.
\]
For \(s \in [0, 1]\) and \(y \in \mathcal{C}^\infty(\bar{\Sigma})\), let \(H_s y = w_s\) be the unique solution of the problem
\[
\begin{cases}
D_{\nu} w_s + L[u](w_s) - a w_s = sV(y) + D_{\nu} y + rBD_{\nu} y & \text{in } \bar{\Sigma} \\
D_{\nu} w_s = 0 & \text{on } \partial \bar{\Sigma},
\end{cases}
\]
where \(\nu\) is the unit radial field, \(a \geq 0\) is a real number,
\[
V(y) = 2r \sum_{1 \leq i,j \leq n} G^{ij}_{u} D_i u D_j y + 2r^2 \sum_{n+1 \leq \alpha, \beta \leq n+m-1} G^{\alpha\beta}_{r,u} D_{\alpha} u D_{\beta} y - v(1 + ar)
\]
and
\[ B = m - 1 + r^2 \sum_{n+1 \leq a \leq n+m-1} (D_\alpha u D_\alpha u - D_\alpha^2 u). \]

The function \( \tilde{y} \) stands for the extension as a radially constant function to \( \tilde{\Sigma} \) of the restriction to \( \Sigma \) of \( y \).

For a proof, we use similar arguments to those of Gilbarg and Trüdinger [5], theorem 6.31. Moreover, since problem (5.11) is uniformly elliptic, if \( \mathcal{B} \) is a bounded subset of \( \mathcal{C}^\infty(\Sigma) \), there exist a sequence of positive real numbers \( C_k \) such that
\[ \|w_s\|_{\mathcal{C}^k(\tilde{\Sigma})} \leq C_k \text{ for any } (s, y) \in [0, 1] \times \mathcal{B}. \]

It follows that the operator \( H \) defined on \([0, 1] \times \mathcal{C}^\infty(\tilde{\Sigma}) \) by \( H(s, y) = y - H_s y \) is compact. Our aim is to solve equation
\[ H(s, w) = 0, \text{ for } s \in [0, 1]. \]

Any solution of (5.13) is a radial constant. In fact, such a solution satisfies:
\[ L[u](w) - aw = sV(w) + rBD_\nu w \text{ in } \tilde{\Sigma} \]
\[ D_\nu w = 0 \text{ on } \partial\tilde{\Sigma}. \]

Recall from section 2 that
\[ D_{e_a}(r_\nu) = (1 - \mu_a)e_a, \]
where \( \mu_a \) is equal to 1 or 0 accordingly to whether \( e_a \) is horizontal or vertical. Hence, using the definition of covariant derivative, we can write:
\[ D_{e_a}(rD_\nu w) = D^2w(e_a, r_\nu) + Dw(D_{e_a}(r_\nu)) = rD_\nu (D_a w) + (1 - \mu_a)D_a w. \]

From this, one deduces that
\[ rD_\nu (D_a w) = D_{e_a}(rD_\nu w) - (1 - \mu_a)D_a w. \]

Using again the definition of the covariant derivative, we can at first write:
\[ D_{ab}(rD_\nu w) = D^2(rD_\nu w)(e_a, e_b) = e_a[D(rD_\nu w)(e_b)] - D(rD_\nu w)(D_{e_a}e_b) \]
and then
\[ D_{ab}(rD_\nu w) = e_a[D_{e_b}(Dw(\nu))] - (D_{e_a}e_b) Dw(\nu) \]
\[ = D_{e_a}(D^2w(e_b, r\nu) + D_{e_a}Dw(D_{e_b}(\nu))) \]
\[ - D^2w(D_{e_a}e_b, r\nu) - Dw(D_{D_{e_a}e_b}(\nu)) \]
Therefore
\[ D_{ab}(rD_\nu w) = D^3w(e_a, e_b, r\nu) + D^2w(e_b, D_{e_a}(r\nu)) + D^2w(e_a, D_{e_b}(r\nu)) \]
\[ + Dw(D_{e_a}(D_{e_b}(\nu))) - Dw(D_{D_{e_a}e_b}(\nu)). \]
But relation (5.15) implies that
\[ Dw \left( D_{e_a}(D_{e_b}(r\nu)) \right) = (1 - \mu_b)Dw \left( D_{e_a}e_b \right), \]
and
\[ Dw \left( D_{D_{e_a}e_b}(r\nu) \right) = (1 - \mu_b)Dw \left( D_{e_a}e_b \right) \]
which in view of the definition of the connexion \( D \) and the definition of the covariant derivative implies that
\[ (5.17) \quad D_{ab}(rD_{\nu}w) = rD_{\nu}(D_{ab}w) + (2 - \mu_a - \mu_b)D_{ab}w. \]
In deriving this equality, we use the expression (2.8) related to the curvature of \( D \), in view of which the equality \( D_{ab\nu}w = D_{\nu ab}w \) holds.

Finally, we radially differentiate (5.14) and multiply the result by \( r \). Since \( D_{\nu}r = 1 \), \( D_{\nu}w = D_{\nu}w = 0 \) and taking (5.16) and (5.17) into account, we get
\[ (5.18) \quad L[u](rD_{\nu}w_s) + r \sum_{1 \leq i,j \leq n} G_{ij}^{\alpha \beta}uD_{ij}w_s - arD_{\nu}w_s = BrD_{\nu}w_s \]
\[ + Br^2D_{\nu\nu}w_s + s \left[ 2r \sum_{1 \leq i,j \leq n} G_{ij}^{\alpha \beta}uD_{ij}\tilde{w}_s - arv \right]. \]

The equation of Gauss together with the computations of the second section show that for functions \( w_1, w_2 \in \mathcal{C}^2(\tilde{\Sigma}) \) having the same values on \( \Sigma \), we have
\[ (5.19) \quad D_{ab}w_1 = D_{ab}w_2 + (1 - \mu_a)G_{ab}D_{\nu}(w_1 - w_2), \quad a, b \leq n + m - 1, \text{ on } \Sigma. \]
Since \( w_s = \tilde{w}_s \) and \( D_{\nu}w_s = 0 \) everywhere on \( \Sigma \), equation (5.18) when restricted to \( \Sigma \) implies the following :
\[ (5.20) \quad \sum_{1 \leq i,j \leq n} G_{ij}^{\alpha \beta}u(2sD_{ij}D_{\alpha\beta}\tilde{w}_s - D_{ij}\tilde{w}_s) = sv. \]
On the other hand, using (5.19) and taking (5.20) into account, we deduce from (5.14) that everywhere on \( \Sigma \), we have
\[ (5.21) \quad \sum_{n+1 \leq \alpha,\beta \leq n+m-1} G_{\alpha\beta}^{\alpha\beta}(2sD_{\alpha\beta}D_{\alpha\beta}\tilde{w}_s - D_{\alpha\beta}\tilde{w}_s) + (\alpha + \lambda)\tilde{w}_s = sv. \]
Since \( \tilde{w}_s \) is a radially constant function, a combination of (5.20) and (5.21) shows that \( \tilde{w}_s \) is another solution of (5.14). Hence, the maximum principle implies that \( \tilde{w}_s = w_s \) everywhere on \( \tilde{\Sigma} \) and then \( w_s \) is a radial constant.

Now, restricting (5.14) to \( \Sigma \), using the maximum principle and the classical theory of uniformly elliptic equations [5], we establish the existence of a sequence of real numbers \( R_k \) such that
\[ (5.22) \quad \|w_s\|_{\mathcal{C}^k(\Sigma)} < R_k, \text{ for any } k \geq 0. \]
Letting 
\[ B = \{ w \in \mathcal{C}^\infty(\tilde{\Sigma}) \mid \|w\|_{\mathcal{C}^k(\tilde{\Sigma})} < R_k, \text{ for any } k \geq 0 \}, \]
by virtue of (5.22), equation (5.13) has no solutions on the boundary of \( B \) for any \( s \). From Nagumo’s theorem [7], the degree of \( H \) at the origin with respect to \( B \) is invariant
\[ d(H(s,.),0,B) = d(H(0,.),0,B) = \gamma. \]
For \( s = 0 \), \( w_0 = 0 \) is the unique solution of (5.13) and standard computations show that for \( y \in \mathcal{C}^\infty(\tilde{\Sigma}) \), we have
\[ d_0^0(H_0 y) = w, \]
where \( w \) is the unique solution of
\[ \begin{cases} 
D_{\nu\nu}w + \sum_{a,b=1}^{n+m-1} G^{ab} D_{ab}w - (a + \lambda)w = D_{\nu\nu}y + r(m-1)D_{\nu}y \text{ in } \tilde{\Sigma} \\
D_{\nu}w = 0 \text{ on } \partial \tilde{\Sigma}.
\end{cases} \]

Arguing as above, we see that \( \ker(Id - d_0 H_0) = \{0\} \). On the other hand, it is an easy fact to show that \( d_0^0 H_0 \) is compact. Therefore, it follows from Fredholm’s theorem and from the regularity of elliptic equations that \( Id - d_0 H_0 \) is surjective. Thus, \( Id - d_0 H_0 \) is invertible and 0 is regular for \( Id - H_0 \). Particularly, \( \gamma = \pm 1 \) in (5.23) and \( H_1 \) has a fixed point \( w \) solution in \( \Sigma \) of the following equations:
\[ \begin{cases} 
\sum_{1 \leq i,j \leq n} G^{ij}_{u}(2D_i u D_j w - D_{ij}w) = a v \\
\sum_{n+1 \leq \alpha,\beta \leq n+m-1} G^{\alpha\beta}_{u} (2D_{\alpha} u D_{\beta} w - D_{\alpha\beta}w) + (a + \lambda)w = v.
\end{cases} \]
Now, suppose \( a \neq 0 \) and choose a function \( v \) which does not vanish identically on \( \Sigma \). Then, for any \( u \in \Theta \), the first equation in (5.24) gives us the existence of a function \( w \in \mathcal{C}^\infty(\Sigma) \) such that
\[ \sum_{1 \leq i,j \leq n} G^{ij}_{u}(2D_i u D_j w - D_{ij}w) \neq 0. \]
Hence \( \Theta \) is a hypersurface.

To conclude, if \( a = 0 \), we see from (5.24) that \( w \in B^\infty(\Sigma) \) solves (5.10). Hence, for any \( u \in \Theta \), \( d_u \Gamma \) is invertible as a linear operator from \( B^\infty(\Sigma) \) to \( \mathcal{C}^\infty(\Sigma) \). We can then use in an obvious way the implicit function theorem of Nash-Moser [6] to show that \( T \) is open in \( [0,1] \). This ends the proof of the theorem.

### 5.4 Proof of theorem 4

To prove this theorem, we apply the same topological argument as that used in the openness part of the proof of the previous theorem. Recall that we are given a strictly positive function \( K \in \mathcal{C}^\infty(E_a) \) and we suppose there exists two real numbers \( r_1 \) and \( r_2 \) such that \( 0 < r_1 \leq 1 \leq r_2 \) and
\[ \begin{cases} 
K(\xi) > (\|\xi\|)^{-m-1} \text{ if } \|\xi\| < r_1 \\
K(\xi) < (\|\xi\|)^{-m-1} \text{ if } \|\xi\| > r_2.
\end{cases} \]
Section 3 says that the problem we want to solve is equivalent to solving in $\mathcal{C}^\infty(\Sigma)$ the following equation
\begin{equation}
\mathcal{N}_2(u) = e^{(m-1)u} K(e^u \xi)(1 + |D^v u|^2)^{\frac{m+1}{2}}.
\end{equation}

For this purpose, to any $t \in [0,1]$ and any $w \in \mathcal{C}^\infty(\Sigma)$, we associate $H_t w = u_t$ where $u_t$ is the unique admissible solution of
\begin{equation}
\begin{cases}
\mathcal{N}_1(u) = 1 \\
\mathcal{N}_2(u) = e^{-u} [e^{mw} K(e^w \xi)]^t (1 + |D^v u|^2)^{\frac{m+1}{2}}.
\end{cases}
\end{equation}

Let $\mathcal{B}$ be a bounded subset of $\mathcal{C}^\infty(\Sigma)$. Theorem 3 ensures the existence of $u_t$ as well as that of a sequence of positive real numbers $C_k$ such that
\begin{equation}
\|u_t\|_{\mathcal{C}^k(\Sigma)} \leq C_k \text{ for any } (t,w) \in [0,1] \times \mathcal{B}.
\end{equation}

Thus, the operator $H$ defined by setting
\[ H(t,w) = H_t w \text{ for } (t,w) \in [0,1] \times \mathcal{C}^\infty(\Sigma) \]
is compact. Since $H_0(w) = 0$ for all $w \in \mathcal{C}^\infty(\Sigma)$, the existence of a fixed point of $H(1,\cdot)$, solution of equation (5.26), reduces in view of the theorem of Nagumo [7], to establishing that the set
\[ C = \{ u \in \mathcal{C}^\infty(\Sigma) : H(t,u) = u, \ t \in [0,1] \} \]
is bounded in $\mathcal{C}^\infty(\Sigma)$.

To deal with the $\mathcal{C}^0$-estimate, notice that any function $u \in C$ satisfies
\begin{equation}
\mathcal{N}_2(u) = e^{(t-1)u} [e^{(m-1)u} K(e^u \xi)]^t (1 + |D^v u|^2)^{\frac{m+1}{2}}.
\end{equation}

So, at a point $\xi_1 \in \Sigma$ where $u$ attains its maximum and since in a frame adapted to $u$, we have $D_\alpha u(\xi_1) = 0$ and $D_{\alpha\alpha} u(\xi_1) \leq 0$, (5.29) yields
\[ 1 \leq e^{(t-1)u} [e^{(m-1)u} K(e^u \xi)]^t. \]

Thus, if $u \geq \log(r_2)$, by assumption (5.25) we have
\[ 1 < e^{(t-1)u} \leq (r_2)^{t-1} \leq 1, \]
which is a contradiction. Thus $u \leq u(\xi_1) \leq \log(r_2)$. The lower bound $u \geq \log(r_1)$ is obtained in a similar way.

The a priori estimates till order three are given in Lemmas 1, 2 and 3 and the higher order estimates may be established by induction as in the proof of the previous theorem. Consequently $H_1$ has a fixed point $u \in \mathcal{C}^\infty(\Sigma)$ which is an admissible solution of (5.26). This completes the proof of the theorem.
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