Honeycomb lattice Kitaev model with Wen-Toric-code interactions, and anyon excitations.

Kazuhiko Minami

Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, Aichi, 464-8602, JAPAN.

Abstract

The honeycomb lattice Kitaev model $\mathcal{H}_K$ with two kinds of Wen-Toric-code four-body interactions $\mathcal{H}_{WT}$ is investigated exactly using a new fermionization method, and the ground state phase diagram is obtained. Six kinds of three-body interactions are also considered. A Hamiltonian equivalent to the honeycomb lattice Kitaev model is also introduced. The fermionization method is generalized to two-dimensional systems, and the two-dimensional Jordan-Wigner transformation is obtained as a special case of this formula. The model $\mathcal{H}_K + \mathcal{H}_{WT}$ is symmetric in four-dimensional space of coupling constants, and the anyon type excitations appear in each phase.

Keywords: Kitaev model, Wen model, toric-code model, new fermionization method, two-dimensional Jordan-Wigner transformation, ground state phase diagram, anyon.

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1 Introduction

Recently, a new fermionization formula was introduced [1], in which solvable Hamiltonians and the transformations to diagonalize them can be obtained simultaneously. The one-dimensional transverse Ising model, XY model, cluster model, the two-dimensional square lattice Ising model, and an infinite number of unsolved models were diagonalized by this formula. The Jordan-Wigner transformation is obtained as a special case of this treatment.[1][2]

The formula is summarized as follows: Let us consider a series of operators $\{\eta_j\}$ $(j = 1, 2, \ldots, M)$. The operators $\eta_j$ and $\eta_k$ are called 'adjacent' when $(j, k) = (j, j + 1)$ $(1 \leq j \leq M - 1)$, or $(j, k) = (M, 1)$. If the operators $\eta_j$ satisfy the relations

$$\eta_j \eta_k = \begin{cases} 1 & j = k \\ -\eta_j \eta_k & \eta_j \text{ and } \eta_k \text{ are adjacent} \\ \eta_j \eta_k & \text{otherwise}, \end{cases}$$

(1)

then we can introduce a solvable Hamiltonian

$$-\beta \mathcal{H} = \sum_{j=1}^{M} K_j \eta_j,$$

(2)
which can be mapped to the free fermion system by the transformation

\[ \varphi_j = \frac{1}{\sqrt{2}} e^{i \frac{\pi}{2} (j-1) \eta_0 \eta_1 \eta_2 \cdots \eta_j} \quad (0 \leq j \leq M), \]

where \( \eta_0 \) is an initial operator satisfying \( \eta_0^2 = -1, \eta_0 \eta_1 = -\eta_1 \eta_0, \) and \( \eta_0 \eta_k = \eta_k \eta_0 \) (2 \( \leq k \leq M \)). The operators \( \varphi_j \) satisfy \((-2i)\varphi_j \varphi_{j+1} = \eta_{j+1}, \) and

\[ \{ \varphi_j, \varphi_k \} = \varphi_j \varphi_k + \varphi_k \varphi_j = \delta_{jk}. \]

Hence the Hamiltonian (2) is expressed as a sum of two-body products of the fermion operators \( \varphi_j, \) and can be diagonalized.

The transformation (3) is automatically generated from the series of operators \( \{ \eta_j \}, \) and only the algebraic relations (1), together with the translational invariance, are needed to obtain the free energy. This procedure can be applied to any systems written by the operators that satisfy (1).

The one-dimensional XY model, and its generalizations[4] can be solved by this formula. In these cases, the transformation (3) results in the Jordan-Wigner transformation.

The one-dimensional cluster model with the next-nearest-neighbor interaction

\[ -\beta \mathcal{H} = \sum_{j=1}^{N} [K_1 \sigma^x_j \sigma^x_{j+1} \sigma^x_{j+2} + K_2 \sigma^x_{j+1} \sigma^x_{j+2} \sigma^x_{j+3}] \]

cannot be diagonalized by the Jordan-Wigner transformation. This model, however, can be decoupled into \( \mathcal{H} = \mathcal{H}_{\text{even}} + \mathcal{H}_{\text{odd}}, \) where \( j = \text{even} \) in \( \mathcal{H}_{\text{even}}, \) and \( j = \text{odd} \) in \( \mathcal{H}_{\text{odd}}, \) respectively. They satisfy \([\mathcal{H}_{\text{even}}, \mathcal{H}_{\text{odd}}] = 0, \) and \( \mathcal{H}_{\text{even}}, \) for example, is obtained from a series of operators

\[ \eta_{2j-1} = \sigma^x_{2j-1} \sigma^x_{2j} \sigma^x_{2j+1}, \quad \eta_{2j} = \sigma^x_{2j} \sigma^x_{2j+1} \sigma^x_{2j+2}, \]

which satisfy (1). In this case, the transformation (3) becomes

\[ \varphi_{2j} = \frac{1}{\sqrt{2}} \prod_{\nu=1}^{j} (1 - \sigma^z_{2\nu} - \sigma^x_{2\nu}) \sigma^x_{2j+1} \sigma^x_{2j+2}, \]

\[ \varphi_{2j+1} = \frac{1}{\sqrt{2}} \prod_{\nu=1}^{j} (1 - \sigma^z_{2\nu} \sigma^x_{2\nu}) \sigma^x_{2j+1} \sigma^x_{2j+2} \sigma^x_{2j+3} \]

\((j = 0, 1, 2, 3, \ldots), \)

which is apparently different from the Jordan-Wigner transformation, and the Hamiltonian (5) is diagonalized through this formula.[2][3]

In this paper, this formula is applied to two-dimensional systems. The transformation (3) is generally formulated for the square lattice. The Hamiltonian

\[ \mathcal{H} = \mathcal{H}_K + \mathcal{H}_3 + \mathcal{H}_{WT} \]

(8)
is transformed to the fermion system, and the ground state of $\mathcal{H}_K + \mathcal{H}_{WT}$ is exactly specified; here $\mathcal{H}_K$ denotes the Hamiltonian of the honeycomb lattice Kitaev model, $\mathcal{H}_3$ consists of the six kinds of three-body interactions, $\mathcal{H}_{WT}$ denotes the Hamiltonian of the Wen model which is equivalent to the Kitaev toric-code model. A Hamiltonian, which consists of the cluster-type chains coupled by the Ising interactions, is obtained as an system equivalent to $\mathcal{H}_K$. The ground-state phase diagram of $\mathcal{H}_K + \mathcal{H}_{WT}$ is obtained exactly, and it is depicted that the phase structure of gapped phases and gapless phases change with the rates of the interactions. The symmetry of the system is investigated, and it is derived that the system is symmetric in four-dimensional space of coupling constants. The anyon excitations exist in each phase.

In section 2, the honeycomb lattice Kitaev model and the three-body interactions are introduced. The Wen model is also introduced and the relation with the Kitaev toric-code model is considered. In section 3, the transformation (3) is generally formulated for the two-dimensional square lattice. A specific series of operators is then introduced to obtain and diagonalize the Hamiltonian (8). The transformation (3) in this case is found to be the two-dimensional Jordan-Wigner transformation. In section 4, the interactions are expressed by Majorana fermion operators. Operators that commute with the Hamiltonian are also introduced. In section 5, the series of operators is rearranged, and in section 6, a Hamiltonian equivalent to the honeycomb lattice Kitaev model is introduced. In section 7, $\mathcal{H}_K + \mathcal{H}_{WT}$ is diagonalized in a subspace containing one of the ground states. In section 8, the gapless condition is derived, and in section 9, the ground state phase diagram is obtained. Symmetries of the model is investigated and it is pointed out that the anyon excitations appear in each phase.

2 Hamiltonian

Let us consider the brick-wall lattice, shown in Fig.1 (see also Fig.5), with the interactions

$$-\beta \mathcal{H}_K = \sum_{l=1}^{M_2} \left[ K_x \sum_{j=\text{odd}}^{M_1-1} \sigma^x_j \sigma^x_{j+1} + K_y \sum_{j=\text{odd}}^{M_1-1} \sigma^y_j \sigma^y_{j+1} \right. $$

$$+ K_z \sum_{j=\text{odd}}^{M_1-1} \sigma^z_j \sigma^z_{j+1} \right) , \quad (9)$$

where $M_1$ is even and the summation is taken over all odd $j$. Hamiltonian (9) is the Kitaev model on the honeycomb lattice shown in Fig.2. The Kitaev model is introduced in [5], in which the ground state is specified, the phase diagram is obtained, and abelian anyon excitations in gapped phases, and non-abelian anyon excitations in gapless phases are found.
We will also introduce six kinds of three-body interactions shown in Fig. 3 as

\[-\beta H_3 = \sum_{l=1}^{M_2} \sum_{j=odd}^{M_2-1} \left[ K_1 \sigma^x_{j,l} \sigma^z_{j+1,l} \sigma^y_{j+2,l} + K_2 \sigma^y_{j-1,l} \sigma^z_{j,l} \sigma^x_{j+1,l} + K_3 \sigma^y_{j,l} \sigma^z_{j-1,l} \sigma^y_{j+1,l+1} + K_4 \sigma^z_{j-1,l} \sigma^y_{j,l+1} \sigma^y_{j,l+1} + K_5 \sigma^z_{j-2,l} \sigma^y_{j,l+1} \sigma^y_{j+l+1} + K_6 \sigma^z_{j-1,l} \sigma^y_{j,l+1} \sigma^x_{j+1,l+1} \right]. (10)\]

These three-body interactions are already investigated by several authors. Lee et al.[6] and Shi et al.[7] introduced the interactions $K_1$ and $K_2$, and Yu and Wang[8] and Yu[9] introduced from $K_3$ to $K_6$. Yu[9] also introduced various kinds of four-body and six-body interactions.

Let us here consider the Wen model[10]. The Hamiltonian is given by

\[-\beta H_{WT} = \sum_{l=1}^{M_2} \sum_{j=odd}^{M_2-1} \left[ L_1 \sigma^y_{j,l+1} \sigma^z_{j+1,l} \sigma^y_{j+1,l} \right] + \sum_{j=odd}^{M_2-1} \sum_{l=1}^{M_2} \left[ L_2 \sigma^y_{j+1,l+1} \sigma^z_{j+1,l+1} \sigma^y_{j+1,l+1} \right], (11)\]

and the interactions are shown in Fig.3. Wen originally introduced the case $L_1 = L_2$, and investigated the ground state quantum orders.

We will also consider the Kitaev toric-code model[11], which consists of two types of interactions, as shown in Fig 4. The spin variables are located on each edge. Let us consider the spins on the vertical edges and consider a canonical transformation

\[
\sigma^x_{jl} \rightarrow \sigma^z_{jl}, \quad \sigma^z_{jl} \rightarrow \sigma^x_{jl}, \quad \sigma^y_{jl} \rightarrow -\sigma^y_{jl},
\]

and next another transformation of all spins

\[
\sigma^x_{jl} \rightarrow \sigma^y_{jl}, \quad \sigma^y_{jl} \rightarrow \sigma^z_{jl}, \quad \sigma^z_{jl} \rightarrow \sigma^x_{jl},
\]

then we find the Wen model on the square lattice rotated by $\pi/4$ from the original square lattice. Thus these two models are in this sense equivalent (see also sec.7.2 of [5]).

In the case of the honeycomb lattice Kitaev model having only two-body interactions, the Hamiltonian commutes with the following operators associated to each hexagon

\[
W_{jl} = \sigma^x_{j-1,l} \sigma^y_{j+1,l+1} \sigma^z_{j+1,l+1} \sigma^x_{j+2,l+1} \sigma^y_{j+1,l+1} \sigma^z_{j+1,l+1}.
\]

Each $W_{jl}$ has the eigenvalues $w_{jl} = \pm 1$. It is easy to demonstrate that the Hamiltonian with the three-body interactions (10) and with the Wen-Toric-code four-body interactions (11) also commute with all $W_{jl}$. The eigenstates of the Hamiltonian may thus be labelled by the set of eigenvalues of $W_{jl}$, and the total Hilbert space is divided into subspaces labelled by $\{w_{jl}\}$. 
It should be noted that the interactions $L_1$ and $L_2$ are not independent. When we consider the product of the four-body terms, we find the following relation

\[
(L_1 \sigma^y \sigma^x \sigma^y \sigma^x _{j+1} \sigma^y \sigma^x _{j+2} \sigma^y \sigma^x _{j+1} \sigma^y)
= L_1 L_2 \sigma^y \sigma^x \sigma^y \sigma^x \sigma^y \sigma^x _{j+1} \sigma^y \sigma^x \sigma^y _{j+1} \sigma^y
= L_1 L_2 W_{jl}.
\]

3 Transformation

We will generalize (3), and formulate the fermionization transformation for the two-dimensional lattice. Let us introduce operators $\eta_{kl}$ on each row $l$. The operators $\eta_{kl}$ with fixed $l$ satisfy the condition (1), and $\eta_{kl}$ on different row $l$ commute with each other. The series of operators on the first row is $\eta_{11}, \eta_{21}, \ldots, \eta_{M1}$ with an initial operator $\eta_{01}$. The transformation is introduced as

\[
\varphi_{01} = \frac{1}{\sqrt{2}} e^{i \pi/2 (0-1) j} \eta_{01}, \quad \text{and} \quad \varphi_{j1} = \frac{1}{\sqrt{2}} e^{i \pi/2 (j-1) \eta_{01} \eta_{11} \cdots \eta_{j1}}, \quad (13)
\]

where $j = 1, 2, \ldots, M$. From (13) we obtain

\[
\varphi_{j1} \varphi_{j+11} = \frac{i}{2} \eta_{j+11}.
\]

At the end of the first row, we find

\[
\varphi_{M1} \varphi_{11} = \left( \frac{1}{\sqrt{2}} i^{M-1} \eta_{01} \eta_{11} \cdots \eta_{M1} \right) \left( \frac{1}{\sqrt{2}} i^{1-1} \eta_{01} \eta_{11} \right)
= i^{M-1} (\eta_{11} \cdots \eta_{M1}) \frac{1}{2} \eta_{11}
= (-1) i^{M} \eta_{11} \cdots \eta_{M1} \cdot \frac{i}{2} \eta_{11}.
\]

The operator $(-1) i^{M} \eta_{11} \cdots \eta_{M1}$ commute with the Hamiltonian (2), is hermitian and has the eigenvalues $\pm 1$. The Hilbert space is divided into two subspaces corresponding to the eigenvalues $+1$ and $-1$. We assume the periodic boundary condition for $\eta_{j1}$, and thus introduce the boundary condition for $\varphi_{j1}$ as

\[
\varphi_{M+11} = \begin{cases} +\varphi_{11}, & (-1) i^{M} \eta_{11} \cdots \eta_{M1} = +1 \\ -\varphi_{11}, & (-1) i^{M} \eta_{11} \cdots \eta_{M1} = -1, \end{cases} \quad (14)
\]

in each eigenspace.
Next, let us consider the transformation for the second row. We will introduce the following factor that comes from the first row as
\[ H(1) = (-1)^M \eta_{11} \cdots \eta_{M1}. \]
Then the transformation for the second row is defined as
\[ \varphi_{2j} = H(1) \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \eta_{02} \eta_{12} \cdots \eta_{2j}. \] (15)
The transformation (15) is schematically written as
\[ \varphi_{2j} = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \eta_{02} \eta_{12} \cdots \eta_{2j} (-1)^M \eta_{11} \eta_{21} \cdots \eta_{M1}. \] (16)
Note that \( \eta_{02} \) is introduced in (16), though \( \eta_{01} \) is not introduced in \( H(1) \). The boundary condition for the second row is obtained from (14) replacing \( \varphi_{j1} \) by \( \varphi_{2j} \), and \( \eta_{j1} \) by \( \eta_{j2} \). There is no boundary in the first row.

Generally for the \( l \)-th row, the transformation is defined as
\[ \varphi_{jl} = \left( \prod_{r=1}^{l-1} H(r) \right) \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \eta_{0l} \eta_{1l} \cdots \eta_{jl}, \] (17)
where
\[ H(r) = (-1)^M \eta_{1r} \cdots \eta_{Mr}. \]
From (17) we obtain
\[ \varphi_{jl} \varphi_{j+1l} = \frac{i}{2} \eta_{j+1l}. \]
The boundary condition for the \( l \)-th row is obtained from (14) replacing \( \varphi_{j1} \) by \( \varphi_{jl} \), and \( \eta_{j1} \) by \( \eta_{jl} \). It is easy to convince from (17) that
\[ \{ \varphi_{jl}, \varphi_{km} \} = \delta_{jk} \delta_{lm}. \]
Let us here consider a specific series of operators
\[ \eta_{2j-1l} = \sigma^z_j \eta_l, \quad \eta_{2j} \eta_{j+1l} = \sigma^x_j \sigma^z_{j+1} \eta_l, \] (18)

[together with the initial operators \( \eta_{0l} = i \sigma^x_l \). The index \( j \) runs 1 ≤ \( j \) ≤ \( N \), where \( N \) is the number of sites in a row, and in this case we have \( M = 2N \). Then the transformations, for example with \( l = 2 \) and 1, are schematically written as
\[ \varphi_{2j-1} = \frac{1}{\sqrt{2}} \times \frac{\sigma^x_1 \sigma^z_2 \cdots \sigma^x_{j-1} \sigma^y_{j+1}}{\sigma^z_1 \sigma^x_2 \sigma^z_j \cdots \sigma^z_{N1}}, \]
\[ \varphi_{2j-2} = \frac{1}{\sqrt{2}} \times \frac{\sigma^x_1 \sigma^z_2 \cdots \sigma^x_{j-1} \sigma^y_{j+2}}{\sigma^z_1 \sigma^x_2 \sigma^z_{j+1} \cdots \sigma^z_{N1}}, \]
\[ \varphi_{2k-2} = \frac{1}{\sqrt{2}} \times \frac{\sigma^x_1 \sigma^z_2 \cdots \sigma^x_{j-1} \sigma^z_{k+2}}{\sigma^z_1 \sigma^x_2 \sigma^z_{j+1} \cdots \sigma^z_{k1}}, \] (19)
One may readily verify that the anti-commutation relation \( \{ \varphi_{jl}, \varphi_{km} \} = \delta_{jk} \delta_{lm} \)
comes from the anti-commutation relations \( \{ \sigma_{jl}^x, \sigma_{kl}^x \} = 0 \) and \( \{ \sigma_{kl}^z, \sigma_{kl}^z \} = 0 \).
This is the Jordan-Wigner transformation in two-dimension \([12]-[14]\), i.e. the
two-dimensional Jordan-Wigner transformation is obtained as a special case of
(17).

4 Operators and interactions

When we consider the series of operators (18), we find from (19), that \( \varphi_{jl} \) are
classified into two kinds of operators according to \( \rho = \text{odd} \) and \( \rho = \text{even} \). Thus
we will introduce a new notation

\[
\varphi_1(j, l) = \varphi_{2j-2}, \quad \varphi_2(j, l) = \varphi_{2j-1}.
\]

(20)

The operators and their relations are summarized in Table 1 and Fig.5. Inter-
actions in (9) are expressed, in terms of \( \varphi_{\alpha}(j, l) \), as

\[
\begin{align*}
K_x \sigma_j^x \sigma_{j+1}^x & = K_x (-2i) \varphi_2(j, l) \varphi_1(j + 1, l), \\
K_y \sigma_j^y \sigma_{j-1}^y & = K_y (-2i) \varphi_2(j, l) \varphi_1(j - 1, l), \\
K_{zj} \sigma_{j-1}^z \sigma_{j+1}^z & = K_z (+2i) \varphi_2(j - 1, l) \varphi_1(j - 1, l) \\
& \quad \times (+2i) \varphi_2(j, l + 1) \varphi_1(j, l + 1).
\end{align*}
\]

(21)

The first two interactions consist of two-body products of the operators \( \varphi_{\alpha}(j, l) \)
with uniform coupling constants, and thus can be diagonalized exactly.

The interaction \( K_z \sigma_{j-1}^z \sigma_{j+1}^z \) is expressed as a four-body product of \( \varphi_{\alpha}(j, l) \).
We can find, however, from Fig.5 that \( \varphi_2(j - 1, l) \varphi_1(j, l + 1) \) is disjoint from
other operators, and takes one of its eigenvalues, like a 'floating spin'. We find
\( (\varphi_2(j - 1, l) \varphi_1(j, l + 1))^2 = -\varphi_2(j - 1, l)^2 \varphi_1(j, l + 1)^2 = -(1/2)^2 \), and then the
eigenvalues are \( \pm i/2 \). Hence the product of operators \( \varphi_2(j - 1, l) \varphi_1(j, l + 1) \)
works as a constant in each eigenspace, and \( K_z \sigma_{j-1}^z \sigma_{j+1}^z \) is expressed as

\[
K_z \sigma_{j-1}^z \sigma_{j+1}^z = K_z (+4 \Psi_{jl}) \varphi_2(j, l + 1) \varphi_1(j - 1, l),
\]

where \( \Psi_{jl} = \varphi_2(j - 1, l) \varphi_1(j, l + 1) \).

The operators \( \Psi_{jl} \) commute with all the interactions in \( (9), (10), \) and \( (11) \),
and have simple relation with the operators \( W_{jl} \) given in (12). Let us express
\( \Psi_{jl} \) in terms of the spin operators as

\[
\Psi_{jl} = \varphi_2(j - 1, l) \varphi_1(j, l + 1) \\
= \varphi_2j^{-3 l} \varphi_2 j^{-2 l + 1} \\
= \frac{i}{2} \times \frac{\sigma_j^x}{\sigma_j^x} \sigma_{j-1}^z \sigma_{j+1}^z \sigma_{j+1}^z \sigma_{j+1}^z.
\]

(22)

From (22), we find

\[
\Psi_{jl} \Psi_{jl+2} = \varphi_2j^{-3 l} \varphi_2 j^{-2 l + 1} \varphi_2 j^{-1 l} \varphi_2 j^{-2 l + 1}.
\]

7
and another interaction

One can similarly introduce other three-body interactions as follows:

Let us next consider the three-body interactions shown in Fig.6. The interactions are expressed in terms of $\varphi_{a}(j, l)$ as

$$K_{1}\sigma_{j}^{x}i\sigma_{j+1}^{y}\sigma_{j+2}^{y} = (-2i)\varphi_{2}(j, l)\varphi_{2}(j + 2, l),$$

$$K_{2}\sigma_{j}^{y}i\sigma_{j+1}^{x}\sigma_{j+2}^{x} = (+2i)\varphi_{1}(j - 1, l)\varphi_{1}(j + 1, l),$$

and, for example,

$$K_{6}\sigma_{j}^{x}i\sigma_{j+2}^{y}i\sigma_{j+1}^{x}i\sigma_{j+1}^{y}i\sigma_{j+1}^{x}i\sigma_{j+2}^{y} = K_{6}(-i)\sigma_{j-1}^{y}\cdot\sigma_{j}^{x}\cdot\sigma_{j+1}^{y}\cdot\sigma_{j+1}^{y}i\sigma_{j+1}i\sigma_{j+1}i\sigma_{j+2}^{y} = K_{6}(-i)(+2i)\varphi_{2}(j - 1, l)\varphi_{1}(j, l)\cdot\varphi_{2}(j, l + 1)\varphi_{1}(j, l + 1)\cdot(-2i)\varphi_{2}(j, l + 1)\varphi_{1}(j, l + 1).$$

(23)

We again find $\varphi_{2}(j - 1, l)\varphi_{1}(j, l + 1) = \Psi_{j, l}$, which works as a constant, and also $\varphi_{2}(j, l + 1)^{2} = 1/2$. Then (23) can be expressed as

$$K_{6}\sigma_{j}^{x}i\sigma_{j+1}^{y}i\sigma_{j+2}^{x}i\sigma_{j+2}^{y} = K_{6}(+4\Psi_{j, l})\varphi_{1}(j - 1, l)\varphi_{1}(j + 1, l + 1).$$

One can similarly introduce other three-body interactions as follows:

$$K_{3}\sigma_{j}^{y}i\sigma_{j-1}^{x}i\sigma_{j+1}^{y} = K_{3}(-i)\sigma_{j}^{y}\cdot\sigma_{j-1}^{x}\cdot\sigma_{j}^{y}\cdot\sigma_{j+1}^{y} = K_{3}(+4\Psi_{j, l})\varphi_{2}(j, l)\varphi_{2}(j, l + 1),$$

$$K_{4}\sigma_{j}^{x}i\sigma_{j+1}^{y}i\sigma_{j+1}^{x}i\sigma_{j+1}^{y} = K_{4}(+i)\sigma_{j-1}^{x}\cdot\sigma_{j}^{y}\cdot\sigma_{j}^{y}\cdot\sigma_{j+1}^{y} = K_{4}(-4\Psi_{j, l})\varphi_{1}(j - 1, l)\varphi_{1}(j - 1, l + 1),$$

and

$$K_{5}\sigma_{j}^{x}i\sigma_{j-1}^{y}i\sigma_{j+1}^{x} = K_{5}(+i)\sigma_{j-2}^{x}\sigma_{j-1}^{y}\cdot\sigma_{j-1}^{y} = K_{5}(-4\Psi_{j, l})\varphi_{2}(j - 2, l)\varphi_{2}(j, l + 1).$$

The four-body interaction $L_{1}$ is expressed, as shown in Fig.7, as

$$L_{1}\sigma_{j}^{y}i\sigma_{j+1}^{x}i\sigma_{j+2}^{y}i\sigma_{j+2}^{x}i\sigma_{j+1}^{y}i\sigma_{j+1}^{x} = L_{1}\sigma_{j}^{y}i\sigma_{j+1}^{x}i\sigma_{j+2}^{y}i\sigma_{j+1}^{x}i\sigma_{j+1}^{y}i\sigma_{j+1}^{x} = L_{1}(+2i)\varphi_{2}(j, l + 1)\varphi_{1}(j, l + 1)\varphi_{2}(j, l + 1)\varphi_{1}(j, l + 1)\varphi_{2}(j, l + 1)\varphi_{1}(j, l + 1) = L_{1}(+4\Psi_{j, l})\varphi_{2}(j, l)\varphi_{1}(j + 1, l + 1),$$

and another interaction $L_{2}$ is

$$L_{2}\sigma_{j+1}^{y}i\sigma_{j+2}^{x}i\sigma_{j+2}^{y}i\sigma_{j+1}^{x} = L_{2}\sigma_{j+1}^{y}i\sigma_{j+2}^{x}i\sigma_{j+2}^{y}i\sigma_{j+1}^{x} = L_{2}(-4\Psi_{j, l})\varphi_{2}(j, l)\varphi_{1}(j + 1, l + 1).$$
5 Generalizations

We introduced a series of operators \( \eta_{2j-1} = (+2i)\varphi_2(j)\varphi_1(j) \) and \( \eta_{2j} = (-2i)\varphi_2(j)\varphi_1(j+1) \). Let us then consider another series of operators

\[
\eta_1, \quad \eta_2\eta_3\eta_4, \quad \eta_5, \quad \eta_6\eta_7\eta_8, \quad \eta_9, \quad \ldots,
\]

and generally

\[
\bar{\eta}_{2j-1} = \eta_{4j-3} = (+2i)\varphi_2(2j-1)\varphi_1(2j-1),
\]
\[
\bar{\eta}_{2j} = \eta_{4j-2}\eta_{4j-1}\eta_{4j} = (-2i)\varphi_2(2j-1)\varphi_1(2j+1).
\]

The series \( \{\bar{\eta}_j\} \) satisfies the condition (1) and generates solvable Hamiltonians. Similarly let us consider series of operators with periodic structures as

\[
\varphi_2(\rho)\varphi_1(\rho + k), \quad \varphi_2(\rho)\varphi_1(\rho + k + l),
\]
\[
\varphi_2(\rho + l)\varphi_1(\rho + k + l), \quad \varphi_2(\rho + l)\varphi_1(\rho + k + 2l),
\]
\[
\varphi_2(\rho + 2l)\varphi_1(\rho + k + 2l), \quad \varphi_2(\rho + 2l)\varphi_1(\rho + k + 3l),
\]
\[
\ldots
\]

Generally, we can introduce series of operators that satisfy (1) as

\[
\bar{\eta}_{2j-1} = (\pm 2i)\varphi_2(\rho + (j - 1)l)\varphi_1(\rho + (j - 1)l + k),
\]
\[
\bar{\eta}_{2j} = (\pm 2i)\varphi_2(\rho + (j - 1)l)\varphi_1(\rho + jl + k),
\]

where the sign of the factors \((\pm 2i)\) are arbitrary, \(l\) and \(k\) are integers, \(l \geq 1\), and \(\rho = 1, 2, \ldots, l\) is fixed in each series. The series with different \(\rho\) commute with each other.

More generally we can consider, from (3), series of operators

\[
\varphi_{\tau_1}\varphi_{\tau_2}, \quad \varphi_{\tau_2}\varphi_{\tau_3}, \quad \varphi_{\tau_3}\varphi_{\tau_4}, \quad \varphi_{\tau_4}\varphi_{\tau_5}, \quad \ldots
\]

and generally introduce \( \bar{\eta}_j = (\pm 2i)\varphi_{\tau_j}\varphi_{\tau_{j+1}} \), where all \(\tau_j\) are different with each other. Then the series (27) satisfies (1).

6 Equivalent Hamiltonian

As an example of (26), let us consider (24), which is a special case of (26) with \(\rho = 1, \ l = 2, \) and \(k = 0\). Let us consider the series of operators (18) in Table 1, and in this case we have

\[
\bar{\eta}_{1l} = \sigma_{1l}^\dagger, \quad \bar{\eta}_{2l} = -\sigma_{1l}^\dagger\sigma_{2l}^\dagger\sigma_{3l}^\dagger, \quad \bar{\eta}_{3l} = \sigma_{3l}^\dagger, \quad \bar{\eta}_{4l} = -\sigma_{3l}^\dagger\sigma_{4l}^\dagger\sigma_{5l}^\dagger,
\]

and generally

\[
\bar{\eta}_{2j-1 l} = \sigma_{2j-1 l}^\dagger, \quad \bar{\eta}_{2j l} = -\sigma_{2j-1 l}^\dagger\sigma_{2j l}^\dagger\sigma_{2j+1 l}^\dagger.
\]
The initial operators can be chosen as \( \bar{\eta}_{jl} = i\sigma_{jl}^x \). Following (9) and (21), we will introduce the Hamiltonians

\[
\mathcal{H}_x = K_x \sum_{l=1}^{M_2} \sum_{j=odd}^{M_1-1} (-2i) \bar{\phi}_2(j, l) \bar{\phi}_1(j + 1, l),
\]

\[
= K_x \sum_{l=1}^{M_2} \sum_{j=odd}^{M_1-1} (-\sigma_{2j-1}^x \sigma_{2j}^z \sigma_{2j+1}^z),
\]

\[
\mathcal{H}_y = K_y \sum_{l=1}^{M_2} \sum_{j=odd}^{M_1-1} (-2i) \bar{\phi}_2(j, l) \bar{\phi}_1(j - 1, l),
\]

\[
= K_y \sum_{l=1}^{M_2} \sum_{j=odd}^{M_1-1} (-\sigma_{2j-3}^y \sigma_{2j-2}^z \sigma_{2j-1}^z),
\]

\[
\mathcal{H}_z = K_z \sum_{l=1}^{M_2} \sum_{j=odd}^{M_1-1} (+2i) \bar{\phi}_2(j - 1, l) \bar{\phi}_1(j - 1, l)
\]

\[
\times (+2i) \bar{\phi}_2(j, l + 1) \bar{\phi}_1(j, l + 1),
\]

\[
= K_z \sum_{l=1}^{M_2} \sum_{j=odd}^{M_1-1} \sigma_{2j-3}^z \sigma_{2j-1}^z
\]

where \( \bar{\phi}_\alpha(j, l) \) are obtained from (3) and (20) replacing \( \eta_{jl} \) by \( \bar{\eta}_{jl} \).

The interactions are shown in Fig. 8. The sum \( \mathcal{H}_x + \mathcal{H}_y \) is the Hamiltonian of parallel spin chains with the cluster-type interactions, and \( \mathcal{H}_z \) is the Hamiltonian of the Ising interactions between these chains. The total Hamiltonian \( \mathcal{H}_{K2} = \mathcal{H}_x + \mathcal{H}_y + \mathcal{H}_z \) is equivalent to the honeycomb lattice Kitaev model (9) (there is no difference when two Hamiltonians are written in terms of \( \phi_\alpha(j, l) \) and \( \bar{\phi}_\alpha(j, l) \)). We can also find other equivalent Hamiltonians from the series of operators (6) in Table 1.

### 7 Diagonalization

We will derive the phase structure of the case with the interactions \( K_x, K_y, K_z \), and \( L_1 \) and \( L_2 \). In this case, Lieb’s theorem[15] applies and it is proved that one of the ground states is found in the subspace where \( \Psi_{jl} = -i/2 \) for all \( j \) and \( l \). In this subspace, the translational invariance of \( \Psi_{jl} \) enables us to derive the ground state energy explicitly.

We need \( \phi_2(j, l) \) with odd \( j \), and \( \phi_1(j, l) \) with even \( j \). Let us consider Fourier transformations

\[
\phi_2(j, l) = \frac{1}{\sqrt{M_1}} \frac{1}{\sqrt{M_2}} \sum_{-\pi \leq q_1 < \pi \atop -\pi \leq q_2 < \pi} e^{iq_1 k} e^{iq_2 l} c_2(q_1, q_2),
\]

where \( \bar{\phi}_\alpha(j, l) \) and \( \phi_\alpha(j, l) \) are obtained from (3) and (20) replacing \( \eta_{jl} \) by \( \bar{\eta}_{jl} \).
where \( j = 2k - 1 \) is odd, and
\[
\varphi_1(j, l) = \frac{1}{\sqrt{M_2}} \frac{1}{\sqrt{M_2}} \sum_{-\pi \leq q_1 < \pi} \sum_{-\pi \leq q_2 < \pi} e^{iq_1 k} e^{iq_2 l} e^{iq_1 / 2} c_1(q_1, q_2),
\]  
(29)

where \( j = 2k \) is even. The factor \( e^{iq_1 / 2} \) in (29) is introduced so as to have a symmetric form of the Hamiltonian later in (33). The operators \( c_\alpha(q_1, q_2) \) are the fermi operators satisfying \( \{ c_\alpha^\dagger(p_1, p_2), c_\beta(q_1, q_2) \} = \delta_{\alpha\beta} \delta_{p_1 q_1} \delta_{p_2 q_2} \) and \( \{ c_\alpha(p_1, p_2), c_\beta(q_1, q_2) \} = 0 \). The inverse transformation for \( \varphi_2(j, l) \) is
\[
c_2(q_1, q_2) = \frac{1}{\sqrt{M_2}} \frac{1}{\sqrt{M_2}} \sum_{k=1}^{M_1/2} \sum_{l=1}^{M_2} e^{-iq_1 k} e^{-iq_2 l} \varphi_2(j, l).
\]

A similar relation appears for \( \varphi_1(j, l) \) with an additional phase factor \( e^{-iq_1 / 2} \). The operators \( \varphi_\alpha(j, l) \) (\( \alpha = 1, 2 \)) defined in (17), (18) and (20) satisfy \( \varphi_\alpha^\dagger(j, l) = \varphi_\alpha(j, l) \), and thus we find that \( c_\alpha(-q_1, -q_2) = c_\alpha^\dagger(q_1, q_2) \). Then the term proportional to \( K_x \) in (9) is expressed, from (21), (28) and (29), as
\[
K_x(-2i) \frac{1}{\pi} \sum_{-\pi \leq q_1 < \pi} \sum_{-\pi \leq q_2 < \pi} c_2(q_1, q_2) e^{-iq_1 / 2} c_1^\dagger(q_1, q_2).
\]

(30)

Note that \( c_\alpha(-q_1, -q_2) = c_\alpha^\dagger(q_1, q_2) \) and \( c_\alpha(q_1, q_2) \) creates and annihilates the same particle indexed by \( \alpha \) and \( (q_1, q_2) \). Thus the summation should be restricted to the region, for example, \( 0 \leq q_1 < \pi \) and \( -\pi \leq q_2 < \pi \), instead of \( -\pi \leq q_1 < \pi \) and \( -\pi \leq q_2 < \pi \). The term (30) is then written as
\[
K_x(-2i) \sum_{0 \leq q_1 < \pi} \sum_{-\pi \leq q_2 < \pi} (c_2(q_1, q_2) e^{-iq_1 / 2} c_1^\dagger(q_1, q_2) + c_2^\dagger(q_1, q_2) e^{iq_1 / 2} c_1(q_1, q_2)).
\]

(31)

The total Hamiltonian is expressed as
\[
-\beta \mathcal{H} = \sum_{0 \leq q_1 < \pi} \sum_{-\pi \leq q_2 < \pi} \left[ h_{11} c_1^\dagger(q_1, q_2) c_1(q_1, q_2) + h_{22} c_2^\dagger(q_1, q_2) c_2(q_1, q_2) + h_{12} c_1^\dagger(q_1, q_2) c_2(q_1, q_2) + h_{21} c_2^\dagger(q_1, q_2) c_1(q_1, q_2) \right],
\]

where
\[
\begin{align*}
h_{11} &= K_x(-2i) (e^{iq_1} - e^{-iq_1}) + K_y(-4\Psi)(e^{iq_2} - e^{-iq_2}) + K_z(+4\Psi)(e^{iq_1} + e^{iq_2} - e^{-iq_1} - e^{-iq_2}), \\
h_{22} &= K_x(-2i) (e^{iq_1} - e^{-iq_1}) + K_y(+4\Psi)(e^{iq_2} - e^{-iq_2}) + K_z(-4\Psi)(e^{iq_1} + e^{iq_2} - e^{-iq_1} - e^{-iq_2}), \\
h_{21} &= +K_x(-2i) e^{iq_1 / 2} + K_y(-2i) e^{-iq_1 / 2} + K_z(+4\Psi) e^{-iq_1 / 2} e^{-iq_2} + (L_1 + L_2)(-4\Psi) e^{iq_1 / 2} e^{iq_2}, \\
h_{12} &= h_{21}^\dagger,
\end{align*}
\]

(32)
and $\Psi_{jl} = \Psi = -i/2$. The Hamiltonian is also expressed, with the basis states $c^\dagger_2(q_1, q_2)c^\dagger_1(q_1, q_2)|0\rangle$, $c^\dagger_2(q_1, q_2)|0\rangle$, $c^\dagger_1(q_1, q_2)|0\rangle$, and $|0\rangle$, where $|0\rangle$ is the vacuum, as

$$-\beta \mathcal{H} = \sum_{0 \leq q_1 < \pi} \begin{pmatrix}
    h_{11} + h_{22} & 0 & 0 & 0 \\
    0 & h_{22} & h_{21} & 0 \\
    0 & h_{12} & h_{11} & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}.$$

In our case, where the three-body terms are absent, the energy eigenvalues are

$$\lambda = 0, \ 0, \text{ and } \pm \sqrt{4kk^*},$$

where

$$k = K_x e^{iq_1/2} + K_y e^{-iq_1/2} + K_z e^{-i(q_1/2 + q_2)} + L e^{i(q_1/2 + q_2)},$$

$$L = -(L_1 + L_2).$$

### 8 Gapless condition

We will consider the gapless condition that $\sqrt{4kk^*}$ in (34) becomes zero with some $q_1$ and $q_2$. Let $K_x = \beta J_x$, $K_y = \beta J_y$, $K_z = \beta J_z$, and $L = \beta J_4$. Four terms from (35),

$$J_x e^{iq_1/2} + J_y e^{-iq_1/2} \quad \text{and} \quad J_z e^{-i(q_1/2 + q_2)} + J_4 e^{i(q_1/2 + q_2)},$$

form two ellipses on the complex plane of the variable $z = x + iy$. The first two terms including $J_x$ and $J_y$ form

$$\frac{x^2}{(J_x + J_y)^2} + \frac{y^2}{(J_x - J_y)^2} = 1,$$

where $0 \leq q_1/2 < \pi/2$ corresponds to a part of the ellipse. The latter two terms including $J_z$ and $J_4$ form

$$\frac{x^2}{(J_4 + J_z)^2} + \frac{y^2}{(J_4 - J_z)^2} = 1,$$

where $0 \leq q_1/2 < \pi/2$ and $-\pi \leq q_2 < \pi$ corresponds to the full ellipse. The condition is satisfied if (36) and (37) are simultaneously satisfied with some real $(x, y)$. From (36) and (37) we obtain

$$\Phi(\pm) x^2 = \phi(\pm), \quad \Phi(\pm) y^2 = \phi(\pm),$$

where

$$\Phi(\pm) = \left(\frac{J_x - J_y}{J_x + J_y}\right)^2 - \left(\frac{J_4 - J_z}{J_4 + J_z}\right)^2,$$

$$\phi(\pm) = (J_x \pm J_y)^2 - (J_4 \pm J_z)^2.$$
Now let us consider the conditions
\[(X1) \quad \Phi(-+) > 0 \quad \text{and} \quad \phi(-) > 0,\]
\[(X2) \quad \Phi(-+) < 0 \quad \text{and} \quad \phi(-) < 0,\]
and
\[(Y1) \quad \Phi(+-) > 0 \quad \text{and} \quad \phi(+) > 0,\]
\[(Y2) \quad \Phi(+-) < 0 \quad \text{and} \quad \phi(+) < 0.\]

The equations (38) are satisfied with real \(x\) and \(y\) if
\[
((X1) \text{ or } (X2)) \quad \text{and} \quad ((Y1) \text{ or } (Y2)).
\] (39)

Because of the fact that \(\Phi(+-) > 0\) and \(\Phi(-+) < 0\) are equivalent, and that \(\Phi(+-) < 0\) and \(\Phi(-+) > 0\) are equivalent, (39) is equivalent to
\[
((X1) \text{ and } (Y2)) \quad \text{or} \quad ((X2) \text{ and } (Y1)).
\] (40)

Because of the fact that \(\phi(-) > 0\) and \(\phi(+) < 0\) yield \(\Phi(-+) > 0\), and that \(\phi(-) < 0\) and \(\phi(+) > 0\) yield \(\Phi(-+) < 0\), (40) is equivalent to
\[
(\phi(-) > 0 \text{ and } \phi(+) < 0) \quad \text{or} \quad (\phi(-) < 0 \text{ and } \phi(+) > 0).
\] (41)

The condition (41) determines the gapless region. From (36) and (37), we find that all the boundaries of the gapless region determined by (41) are gapless. We will here consider the following two cases:

**Case I.** \(J_x \geq 0, \ J_y \geq 0, \ J_z \geq 0, \ J_x + J_y + J_z = 1, \text{ and } J_4 \geq 0.\) In this case (41) is written as
\[
(J_x - J_y - J_4 + J_z)(J_x - J_y + J_4 - J_z) > 0,
\]
\[
\text{and} \quad J_x + J_y - J_4 - J_z < 0,
\]
or
\[
(J_x - J_y - J_4 + J_z)(J_x - J_y + J_4 - J_z) < 0,
\]
\[
\text{and} \quad J_x + J_y - J_4 - J_z > 0.
\]

When \(J_4 = 0\), we find the phase diagram obtained by Kitaev[5].

**Case II.** \(J_x \geq 0, \ J_y \geq 0, \ J_4 \geq 0, \ J_z = -J, \ J \geq 0, \text{ and } J_x + J_y + J = 1.\) In this case, it can be derived that \(((X1) \text{ and } (Y2))\) cannot be satisfied, and (41) is written as
\[
(J_x - J_y - J_4 - J)(J_x - J_y + J_4 + J) < 0,
\]
\[
\text{and} \quad (1 - J_4)(J_x + J_y + J_4 - J) > 0.
\] (42)

The condition (42) is satisfied only when \(J_4 < 1.\)
9 Symmetries and Anyon excitations

The phase diagram is shown in Fig. 9 and Fig. 10. The triangle on the upper half plane corresponds to the case $J_x \geq 0$, $J_y \geq 0$, $J_z = J \geq 0$, and $J_4 \geq 0$, and the triangle on the lower half plane corresponds to the case $J_x \geq 0$, $J_y \geq 0$, $J_z = -J \leq 0$, and $J_4 \geq 0$. The interactions are normalized as $J_x + J_y + J = 1$.

Four corners are the points with the interactions $X$: $(J_x, J_y, J_z) = (1, 0, 0)$, $Y$: $(J_x, J_y, J_z) = (0, 1, 0)$, $Z_+ : (J_x, J_y, J_z) = (0, 0, 1)$, $Z_- : (J_x, J_y, J_z) = (0, 0, -1)$.

Two additional horizontal lines indicate $J_z = (1 - J_4)/2$ and $J = (1 + J_4)/2$.

Other two lines indicate $J_x = (1 - J_4)/2$ and $J_y = (1 + J_4)/2$.

As a result, the system is invariant changing the signs of arbitrary possible cases.

In case of $J_4 = 0$, the system is invariant changing the signs of arbitrary two interactions. The system with an even number of positive interactions are, therefore, equivalent to each other, and the system with an odd number of positive interactions are equivalent to each other. Thus the phase diagram given in Fig. 10, with $J_x, J_y, J_4 \geq 0$ and with $J_z \geq 0$ or $J_z \leq 0$, classifies all the possible cases.

In the first diagram in Fig. 10.

Next let us consider the degeneracy of the ground state. In case of $(J_x, J_y, J_z) = (J_4, (1 - J_4)/2, (1 - J_4)/2)$, two ellipses (36) and (37) become identical. In this case, for all $q_1$ there exists $q_2$ with which (34) becomes zero, and hence the ground state is highly degenerate.
It is easy to check that the summation over the region \((1 - J_4)/2, (1 - J_4)/2, J_4)\), both (36) and (37) become finite intervals on the real axis, and in these two cases, the ground state is also highly degenerate.

When \(J_4 = 1/3\), we have a symmetric point \(J_x = J_y = J_z = J_4 = 1/3\), where above three points become identical, as shown in the third diagram in Fig 10. In this case, (36) and (37) become finite intervals on the real axis.

When we consider the case with uniform \(\Psi_{ij}\), we can find the ground state in this subspace, and because of the translational invariance, the Hamiltonian can be diagonalized in the momentum representation. In this subspace, the Hamiltonian is expressed symmetrically as the sum in (30) and sums coming from other interactions. Let us consider the replacement of the variables

\[(q_1, q_2) \mapsto (-q_1, -q_2)\]  \hspace{1cm} (43)

and accordingly

\[(J_x, J_y) \mapsto (J_y, J_x), \quad (J_z, J_4) \mapsto (J_4, J_z),\]

\[c^+_1(-q_1, -q_2) = c_1(q_1, q_2) \mapsto \hat{c}^+_1(q_1, q_2),\]

\[c_2(-q_1, -q_2) = c^+_2(q_1, q_2) \mapsto \hat{c}_2(q_1, q_2).\]

Then the range of the summation \(-\pi \leq q_1 < \pi\) and \(-\pi \leq q_2 < \pi\) in (30) is invariant, and \(\hat{c}_1\) and \(\hat{c}_2\) satisfy the fermion anticommutation relations. We find from \(h_{21}\) and \(h_{12}\) in (32) that the Hamiltonian \(H(J_x, J_y, J_z, J_4)\) and \(H(J_y, J_x, J_4, J_z)\) are equivalent. This is the particle-hole transformation.

In this sense, the model with the interactions \(J_x = J_y, J_z \neq 0, J_4 = 0\), and the model with \(J_x = J_y, J_z = 0, J_4 \neq 0\) are equivalent.

Kitaev\cite{5} considered the large \(J_z\) limit of the honeycomb lattice Kitaev model, and derived an effective Hamiltonian that consists of the Wen-type four-body interaction \(J_4\) (see (37) in [5]). In this effective Hamiltonian, vortices are generated by two kinds of string operators, and an additional sign appears from each cross point of the strings when one interchange the positions of two excitations. In this sense the excitations are regarded as anyons.

This fact is consistent with our argument that the large \(J_z\) region is equivalent to the large \(J_4\) region, and we thus also find that the abelian anyons appear in the large \(J_4\) region as well as in the large \(J_z\) region.

Let us again consider the replacement of the variables that

\[(q_1, q) \mapsto (q_1, q) \text{ where } q = q_1/2 + q_2,\]  \hspace{1cm} (44)

and accordingly

\[(J_x, J_y) \mapsto (J_x, J_4), \quad (J_z, J_4) \mapsto (J_z, J_y),\]

\[c^+_1(q_1, q_2) = c^+_1(q_1, q - q_1/2) \mapsto \hat{c}^+_1(q_1, q),\]

\[c_2(q_1, q_2) = c_2(q_1, q - q_1/2) \mapsto \hat{c}_2(q_1, q).\]

It is easy to check that the summation over the region \(-\pi \leq q_1 < \pi\) and \(-\pi \leq q_2 < \pi\) is equivalent to the summation over \(-\pi \leq q_1 < \pi\) and \(-\pi \leq q < \pi\),
because of the periodic structure of the system with period $2\pi$. The operators $\tilde{c}_1$ and $\tilde{c}_2$ satisfy the fermion anticommutation relations, and we find from (32) that the Hamiltonian $H(J_x, J_y, J_z, J_4)$ and $H(J_z, J_x, J_4, J_y)$ are equivalent.

In this sense, from (43) and (44), the model with the interactions $J_y = J_z = J_4 = 0$, and the model with $J_x = J_z = J_4 = 0$, are equivalent to the case $J_x = J_y = J_4 = J_z = 0$.

In the original Kitaev model, in the gapless phases, an external field opens an energy gap, and the string operators generate vortices that behave as anyons. For the purpose to investigate this phenomena, let us consider the Fourier transformation in whole the Hilbert space. (Note that the Fourier transformation itself is always possible even if $\Psi_{jl}$ are not uniform, though the Hamiltonian $H$ cannot be simplified in the subspace where $H$ does not have translational invariance.) It can be verified that the operators $\varphi_\alpha(j, l)$ are transformed as

\[
\varphi_\alpha(j, l) \mapsto \varphi_\alpha(-j, -l)
\]

and

\[
\varphi_\alpha(j, l) \mapsto \varphi_\alpha(j - l/2, l)
\]

with the transformations (43) and (44), respectively. So (43) and (44) correspond to change of locations in real space. The spin operators can be expressed by $\varphi_\alpha(j, l)$ as

\[
\sigma^x_{jl} = \eta_{2j-1}l = (2i)\varphi_2(j, l)\varphi_1(j, l),
\]

\[
\sigma^y_{jl} = \sqrt{2}\left(\prod_{r=1}^{l-1} \prod_{k=1}^{N} \eta_{2k-1}r\right)\left(\prod_{k=1}^{j-1} \eta_{2k-1}l\right)\varphi_1(j, l),
\]

\[
\sigma^y_{jl} = \sqrt{2}\left(\prod_{r=1}^{l-1} \prod_{k=1}^{N} \eta_{2k-1}r\right)\left(\prod_{k=1}^{j-1} \eta_{2k-1}l\right)\varphi_2(j, l).
\]

The Zeeman term and the string operators are, therefore, transformed together with $\varphi_\alpha(j, l)$. In the subspace where $\Psi_{jl}$ are uniform, the Hamiltonian $H$ is decomposed as (33) according to the wave numbers. The Zeeman term and the string operators do not commute with $\Psi_{jl}$, thus they are not simple in this momentum bases, they change their locations, and generate anyons.

10 Conclusion

At last we would like to note an interesting methodology presented in [16] and [17], in which isomorphisms of algebras that are generated from interactions are considered, and equivalences and mappings are investigated. In [16], results of [5] and [14] on the honeycomb lattice Kitaev model was rederived by the algebraic isomorphism, and in [17], the Jordan-Wigner transformation is generated in an iterative way in the case of the XY chain. The basic idea in these papers that the algebraic structure of interactions determine the spectrum of the model is common to our formula. In the present paper, however, the transformation (3) is explicitly given for the series of operators that satisfy (1), and the two-dimensional systems are investigated.

In summary, we obtain the exact ground state phase diagram of the honeycomb lattice Kitaev model with the Wen-Toric-code four-body interactions, and find that the structure of the system is symmetric in four-dimensional space.
The fermionization transformation (3) is generally formulated for two-dimensional systems. The construction of the series of operators that satisfy (1) is also generalized, and a model equivalent to the Kitaev model is introduced. We also find that the anyon excitations appear in all of the regions shown in the phase diagram, they can be transformed each other.

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Table 1: Relations between \( \varphi \), \( \eta \), and \( \sigma_k \), obtained by (3) and (20). Operators \( \eta \) are defined in (18) and in (6). In the case of the series (18), for example, \((+2i) \varphi \varphi = \eta \eta = \sigma_j \), and in the case of the series (6), \((+2i) \varphi \varphi = \eta \eta = \sigma_j \).

Table: Relations between \( \varphi \), \( \eta \), and \( \sigma_k \), obtained by (3) and (20). Operators \( \eta \) are defined in (18) and in (6). In the case of the series (18), for example, \((+2i) \varphi \varphi = \eta \eta = \sigma_j \), and in the case of the series (6), \((+2i) \varphi \varphi = \eta \eta = \sigma_j \).

| Fermi operators | original operators | series (18) | series (6) |
|-----------------|--------------------|-------------|-------------|
| \((+2i) \varphi \varphi\) | \(\eta j \eta j + 1 \eta j + 2 \) | \(-\sigma_j^x \sigma_j^z \) | \(-\sigma_{j+1}^z \sigma_{j+2}^x \) |
| \((-2i) \varphi \varphi\) | \(\eta j \) | \(\sigma_j^y \) | \(\sigma_{j+1}^z \) |
| \((+2i) \varphi \varphi\) | \(\eta j - 1 \) | \(\sigma_j^y \) | \(\sigma_{j+2}^z \) |
| \((-2i) \varphi \varphi\) | \(\eta j - 3 \eta j - 2 \eta j - 1 \) | \(\sigma_{j-1}^y \) | \(\sigma_{j+3}^z \) |
| \((+2i) \varphi \varphi\) | \(\eta j - 5 \eta j - 4 \eta j - 3 \eta j - 2 \eta j - 1 \) | \(-\sigma_{j-2}^y \) | \(-\sigma_{j+4}^z \) |
| \((+2i) \varphi \varphi\) | \((+i) \eta j \) | \(-\sigma_j^y \) | \(\sigma_{j+2}^z \) |
| \((-2i) \varphi \varphi\) | \((+i) \eta j \) | \(\sigma_j^z \) | \(\sigma_{j+2}^z \) |
| \((+2i) \varphi \varphi\) | \((+i) \eta j + 1 \) | \(\sigma_j^y \) | \(\sigma_{j+3}^z \) |
| \((-2i) \varphi \varphi\) | \((+i) \eta j + 1 \) | \(\sigma_j^z \) | \(\sigma_{j+3}^z \) |

Figure 1: Honeycomb lattice Kitaev model on the brick-wall lattice.
Figure 2: Two-body interactions of the honeycomb lattice Kitaev model.

Figure 3: Three-body and Wen-Toric-code four-body interactions.
Figure 4: Kitaev toric-code model.

\[
\begin{align*}
(j - 2, l + 1) &= (-2i)\varphi_2(j, l + 1)\varphi_1(j - 1, l + 1) \\
&\quad \times \varphi_1(j - 1, l + 1) \times \varphi_1(j + 1, l + 1) \times \varphi_1(j + 1, l + 1) \\
&= \sigma^y_{j-1} \sigma^y_{l+1} \sigma^y_{j+1} \sigma^y_{l+1} = \sigma^y_{l+1} \sigma^y_{l+1} \sigma^y_{j+1} \sigma^y_{l+1} \\
&= (+2i)\varphi_2(j - 1, l)\varphi_1(j - 1, l) \\
&\quad \times \varphi_1(j - 1, l) \times \varphi_1(j - 1, l) \\
&= (-2i)\varphi_2(j - 2, l) = (-2i)\varphi_2(j - 1, l) = (-2i)\varphi_2(j - 1, l) \\
&\quad \times \varphi_1(j - 1, l) \times \varphi_1(j - 1, l) \times \varphi_1(j + 1, l)
\end{align*}
\]

Figure 5: Spin-spin interactions and corresponding Majorana operators \(\varphi_\alpha(j, l)\) on the lattice.
Figure 6: Three-body interactions $K_1$, $K_2$, and $K_6$.

Figure 7: Four-body interaction $L_1$. 

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Figure 8: A model equivalent to the honeycomb lattice Kitaev model. The model is composed of the cluster and the Ising interactions.

\[
\begin{align*}
J_x &= \frac{1}{2}(1 - J_4) \\
J_y &= \frac{1}{2}(1 - J_4) \\
J_z &= \frac{1}{2}(1 - J_4) \\
J &= \frac{1}{2}(1 + J_4) \\
J_x &= \frac{1}{2}(1 + J_4) \\
J_y &= \frac{1}{2}(1 + J_4) \\
Z^+ &= \\
Z^- &=
\end{align*}
\]

Figure 9: Four lines in Fig.10.
Figure 10: The ground state phase diagram of the honeycomb lattice Kitaev model with Wen-Toric-code four-body interactions, where gapped regions are colored by gray.