Six dimensional Landau-Ginzburg-Wilson theory

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Abstract. We renormalize the six dimensional cubic theory with an $O(N) \times O(m)$ symmetry at three loops in the modified minimal subtraction (MS) scheme. The theory lies in the same universality class as the four dimensional Landau-Ginzburg-Wilson model. As a check we show that the critical exponents derived from the three loop renormalization group functions at the Wilson-Fisher fixed point are in agreement with the large $N$ $d$-dimensional critical exponents of the underlying universal theory. Having established this connection we analyse the fixed point structure of the perturbative renormalization group functions to estimate the location of the conformal window of the $O(N) \times O(2)$ model.
1 Introduction.

Scalar quantum field theories have been the subject of intense interest in recent years in the context of trying to develop our understanding of conformal field theories in dimensions greater than two using established concepts, \[1, 2, 3, 4, 5, 6, 7\], in a modern application, \[8, 9, 10, 11\]. One of the main aspects of this activity is in finding the conformal window of a theory where there are non-trivial fixed points of the $\beta$-function. In this window one in principle has a theory where ideas for extending Zamolodchikov's $c$-theorem, \[12\], to higher dimensions can be explored, for example, as well as other properties of strictly two dimensional conformal field theories. One of the first examples of a quantum field theory with a conformal window was that deriving from the Banks-Zaks fixed point in Quantum Chromodynamics (QCD), \[13, 14\]. For a range of the number of quark flavours, $N_f$, there is a non-trivial fixed point at a non-zero value of the strong coupling constant. This is established from two loop perturbation theory and the upper bound of the window follows from the one loop term. However, the lower bound has not been unambiguously resolved partly because near this lower end the value of the critical coupling constant is not small and the perturbative approximation appears to break down. Non-perturbative lattice studies have yet to provide definitive data which determine the location of the lower bound precisely. Until relatively recently these two methods of perturbative analysis of $\beta$-functions and lattice studies of the breakdown of the chiral limit were generally the main techniques to access conformal windows of a theory. A third approach was developed several years ago which falls under the banner of the conformal bootstrap technique, \[8, 9, 10, 11\]. It is a numerical analysis of the operator content of a scalar field theory and exploits the decoupling of operators from the spectrum in the limit to a fixed point. This allows one to obtain accurate values for scaling dimensions, \[10\].

To date one main application has been to study scalar field theories across different spacetime dimensions. Recent examples include trying to find the conformal window for $O(N)$ scalar $\phi^3$ theory in five dimensions, \[15, 16, 17, 18, 19\]. Originally Ma found that the conformal window was located at $N = 1038$, \[20\], in strictly six dimensions but the higher order terms in the $\epsilon$ in $d = 6 - 2\epsilon$ dimensions were computed to three, \[21\], and four, \[22\], loops. Using summation techniques the bound in five dimensions was reduced but not to the low values indicated by, for example, conformal bootstrap analyses, \[17\]. While such agreement between techniques is yet to be resolved for other quantities in the conformal window, such as estimates of critical exponents, there is very strong overlap in the values which suggest these complementary methods do provide a solid insight into the properties of these scalar quantum field theories.

One upshot of the reopening of studies in higher dimensional theories has been that higher order perturbative results have been computed beyond a few loops. For instance, the three loop result of \[23, 24\] from thirty years ago for six dimensional $\phi^3$ theory were only extended to four loops recently, \[22\]. Equally the five loop renormalization group function of four dimensional $\phi^4$ theory from the mid-1990's have now been extended to six loops in \[25, 26\] and to seven loops for the field anomalous dimension, \[27\]. Given the interest in the conformal bootstrap and its potential application to non-scalar theories or to scalar theories with symmetry other than $O(N)$ it is worthwhile providing higher loop perturbative results to complement recent, \[28\], and future bootstrap studies in various dimensions. Therefore the aim of this article is to renormalize the six dimensional extension of the Landau-Wilson-Ginzburg (LGW) model to three loops. In effect this is a $\phi^3$ type theory but endowed with an $O(N) \times O(m)$ symmetry. It has applications in condensed matter problems such as randomly dilute spin models, \[29, 30\]. A conformal bootstrap analysis has recently been provided from the conformal bootstrap technology, \[28\], and one aim is to provide data to complement similar bootstrap analyses in the future. In addition analysis for the $O(4) \times O(2)$ theory, which describes the chiral phase transition in two flavour QCD, has been discussed in \[31\]. A second motivation is to continue
exploring the tower of theories across the dimensions which are in the same universality class at the Wilson-Fisher fixed point, [32]. The LGW model with an $O(N) \times O(m)$ is a new example to continue this investigation which we will carry out in depth here. The extension of the four dimensional $O(N) \times O(m)$ symmetric theory of [30] to six dimensions is termed the ultraviolet completion of the theory. Once a theory has been constructed in a fixed dimension with the same symmetries as its lower dimensional counterpart the $\epsilon$ expansion of the critical exponents at the fixed point can be used to access non-perturbative fixed point properties in the lower dimensional partner. This ultraviolet-infrared connection was recognized earlier in [33, 34] but its power is being exploited in present analyses. To connect theories across dimensions requires a technique beyond perturbation theory since coupling constants become dimensionful outside their critical dimension. The interpolating expansion parameter has to be dimensionless and in the context of theories with an $O(N)$ symmetry the parameter is $1/N$ where $N$ is regarded as large. Earlier work by Vasil’ev’s group, [35, 36, 37], provided critical exponents to three orders in $1/N$ for the universal theory in $d$-dimensions with an $O(N)$ symmetry. In such exponents the origin of the $d$ dependence is not from dimensional regularization. Rather large $N$ Feynman integrals within the formalism of [35, 36, 37] are computed with analytic regularization. So the $d$ dependence in the exponents is a true reflection of the properties and structure of the universal theory in any dimension. Indeed any perturbative expansion of a critically evaluated renormalization group function in the large $N$ expansion agrees with the exponents of [35, 36, 37]. For the $O(N) \times O(m)$ extension this is also the case due to the computations of [38] in the large $N$ expansion and explicit four dimensional perturbation theory, [39, 40, 41, 42, 43, 44, 45]. Moreover, such a check will also be important in establishing the correctness of our three loop results for the six dimensional theory in the same universality class prior to analysing the renormalization group functions at criticality for a variety of values of $N$ and $m$. We will do this both in the fixed dimension of six as well as within the $\epsilon$ expansion. Moreover, we will draw similar conclusions to others, [28], where finding the specific location of the conformal window is not straightforward.

The paper is organized as follows. We construct the six dimensional version of the theory with an $O(N) \times O(m)$ symmetry which is in the same universality class as the four dimensional Landau-Ginzburg-Wilson model in section 2. The necessary large $N$ analysis which will allow us to confirm this is also reviewed in that section. The main three loop renormalization group functions are given in section 3 together with a summary of the technology required to determine them. From these results we derive the large $N$ critical exponents in section 4 in order to compare with known $O(1/N^2)$ exponents. Having verified agreement with known results we discuss the search for a conformal window in section 5. A detailed fixed point analysis at three loops for a variety of values of $N$ in provided in section 6. Concluding observations are given in section 7. Two appendices are provided. The first gives the remaining renormalization group functions which were not displayed in the main text for space reasons including the mass mixing matrix. While the other appendix provides the full spectrum of fixed points for $N = 1000$ as a complete example of the rich structure of this model.

2 Background.

As we will be considering the six dimensional model with $O(N) \times O(m)$ symmetry which is in the same universality class as the four dimensional theory with the same symmetry we begin by recalling the relevant aspects of the latter theory. In this case the Lagrangian involves a quartic interaction for a scalar field $\phi^i a$ where $1 \leq i \leq N$ and $1 \leq a \leq m$. Consequently the Lagrangian is, [38],

$$L^{(4)} = \frac{1}{2} \partial^\mu \phi^i a \partial_\mu \phi^j a + \frac{g_1}{4!} (\phi^i a \phi^j a)^2 + \frac{g_2}{4!} \left[ (\phi^i a \phi^j b)^2 - (\phi^i a \phi^j a)^2 \right]$$

(2.1)
where \( \tilde{g}_i \) are the couplings of the respective interactions. This version of the Landau-Ginzburg-Wilson theory is not the most useful for developing the large \( N \) expansion or indeed for seeing the connection with lower and higher dimensional theories. Instead it is better to reformulate \( \mathcal{L}^{(4)} \) in terms of cubic interactions by introducing a set of auxiliary fields \( \tilde{\sigma} \) and \( \tilde{T}^{ab} \) where the latter is antisymmetric and traceless in its \( O(m) \) indices. Then \( \mathcal{L}^{(4)} \) becomes,

\[
\mathcal{L}^{(4)} = \frac{1}{2} \partial^{\mu} \phi^{ia} \partial_{\mu} \phi^{ia} + \frac{1}{2} \tilde{\sigma} \phi^{ia} \phi^{ia} + \frac{1}{2} \tilde{T}^{ab} \phi^{ia} \phi^{ib} - \frac{3\tilde{\sigma}^2}{2\tilde{g}_1} - \frac{3\tilde{T}^{ab} \tilde{T}^{ab}}{2\tilde{g}_2} \tag{2.2}
\]

where we have introduced \( \tilde{g}_1 = g_1 + (m-1)\tilde{g}_2/m \) and \( \tilde{g}_2 = \tilde{g}_2, \) \( \tilde{g}_3, \tilde{g}_4, \) \( \tilde{g}_5 \). Here the coupling constants appear within the quadratic part of the Lagrangian which is the first step in constructing the critical exponents using the large \( N \) methods of \( \text{[35, 36]} \). However, for perturbative computations it is more appropriate for the couplings to appear with the actual interactions. So using a simple rescaling \( \mathcal{L}^{(4)} \) becomes

\[
\mathcal{L}^{(4)} = \frac{1}{2} \partial^{\mu} \phi^{ia} \partial_{\mu} \phi^{ia} + \frac{1}{2} \sigma^2 + \frac{1}{2} T^{ab} T^{ab} + \frac{1}{2} g_1 \sigma \phi^{ia} \phi^{ia} + \frac{1}{2} g_2 T^{ab} \phi^{ia} \phi^{ib}. \tag{2.3}
\]

In this formulation one can build the equivalent six dimensional theory based on the dimensionalities of the fields and ensuring that the Lagrangian is renormalizable. In \( d \)-dimensions the \( \phi^{ia} \) field has dimensions \( \frac{1}{2}d-1 \) while \( \sigma \) and \( T^{ab} \) are both dimension 2. Clearly (2.3) is renormalizable in four dimensions. The key to constructing the six dimensional extension is the retention of the two basic interactions of \( \phi^{ia} \) with the auxiliary fields. This means that the dimensionalities of all three fields are preserved at the connecting Wilson-Fisher fixed point in \( d \)-dimensions. However, the equivalent Lagrangian in six dimensions, \( \mathcal{L}^{(6)} \), has to have dimension 6 in order to retain a dimensionless action. Therefore the auxiliary field sector of (2.3) has to be replaced. This leads to

\[
\mathcal{L}^{(6)} = \frac{1}{2} \partial^{\mu} \phi^{ia} \partial_{\mu} \phi^{ia} + \frac{1}{2} \partial_{\mu} \sigma \partial^{\mu} \sigma + \frac{1}{2} \partial^{\mu} T^{ab} \partial_{\mu} T^{ab} + \frac{1}{2} g_1 \sigma \phi^{ia} \phi^{ia} + \frac{1}{6} g_2 \sigma^3 \\
+ \frac{1}{2} g_3 T^{ab} \phi^{ia} \phi^{ib} + \frac{1}{2} g_4 \sigma T^{ab} T^{ab} + \frac{1}{6} g_5 T^{ab} T^{ac} T^{bc} \tag{2.4}
\]

as the ultraviolet completion which is renormalizable in six dimensions and which should be equivalent to (2.3) in four dimensions at the Wilson-Fisher fixed point. As in previous extensions there are more interactions but also the \( \sigma \) and \( T^{ab} \) fields now cease being auxiliary fields and become propagating with fundamental propagators. The additional interactions which depend solely on \( \sigma \) and \( T^{ab} \) are referred to as spectators since they are only present in the critical dimension. The interactions with couplings \( g_1 \) and \( g_3 \) are the core ones which are present at the Wilson-Fisher fixed point through all the dimensions. They seed the universal theory in the sense that they determine the canonical dimensions of the fields. Thereby they induce the structure of the spectator interactions in each critical dimension by requiring renormalizability. Although our focus is primarily in relation to critical theories one could include masses for the three basic fields which would produce

\[
\mathcal{L}^{(6)}_m = \mathcal{L}^{(6)} - \frac{1}{2} m_1^2 \phi^{ia} \phi^{ia} - \frac{1}{2} m_2^2 \sigma^2 - \frac{1}{2} m_3^2 T^{ab} T^{ab} \tag{2.5}
\]

where \( m_i \) are the masses. Similar terms can be added to \( \mathcal{L}^{(4)} \).

To appreciate properties of these scalar models with an \( O(N) \times O(m) \) symmetry we recall relevant properties of the \( \beta \)-functions of (2.3) which have been computed to several loops orders, \( \text{[38, 35]} \). Although for the present purposes it is sufficient to quote the results to two loops which
are
\[
\beta_1(\bar{g}_1, \bar{g}_2) = \frac{1}{2}(d - 4)\bar{g}_1 + \frac{(mN + 8)}{6}\bar{g}_1^2 - \frac{1}{3}(m - 1)(N - 1)\bar{g}_2\left(\bar{g}_1 - \frac{\bar{g}_2}{2}\right)
\]
\[- \frac{1}{6}(3mN + 14)\bar{g}_1^3 + (m - 1)(N - 1)\left(\frac{11}{9}\bar{g}_1^2 - \frac{13}{12}\bar{g}_1\bar{g}_2 + \frac{5}{18}\bar{g}_2^2\right)\bar{g}_2
\]
\[+ O(\bar{g}_1^4)\] (2.6)
and
\[
\beta_2(\bar{g}_1, \bar{g}_2) = \frac{1}{2}(d - 4)\bar{g}_2 + 2\bar{g}_1\bar{g}_2 + \frac{1}{6}(m + N - 8)\bar{g}_2^2 - \frac{1}{18}(5mN + 82)\bar{g}_1^2\bar{g}_2
\]
\[+ \frac{1}{9}(5mN - 11(m + N) + 53)\bar{g}_1\bar{g}_2^2 - \frac{1}{36}[13mN - 35(m + N) + 99]\bar{g}_2^3
\]
\[+ O(\bar{g}_1^4)\] (2.7)
where the order symbol is understood to mean any combinations of the two coupling constants. For (2.3), (38, 35), there are several fixed points which are the free field Gaussian fixed point, that corresponding to the Heisenberg model and two where both critical couplings are non-zero. The fixed point corresponding to the Heisenberg case corresponds to \(\bar{g}_1 \neq 0\) and \(\bar{g}_2 = 0\) irrespective of whether \(m\) is set to unity or not. In the case when \(m \neq 1\) the parameter \(m\) always appears as a multiplier of \(N\). For the two fixed points where both critical couplings are non-zero one is known as the chiral stable (CS) fixed point and the other as the anti-chiral unstable (AU) one. For the Heisenberg fixed point in the context of the \((\bar{g}_1, \bar{g}_2)\)-plane it is actually a saddle-point and so is unstable to perturbations in the \(\bar{g}_2\) direction. In the reduction to the single coupling \(O(N)\) scalar theory the Heisenberg fixed point would be stable.

As we will be using the large \(N\) results, (38, 46), with which to compare our six dimensional perturbative results it is worthwhile recalling some of those results as well as giving a perspective on the fixed point structure. In the large \(N\) method of (35, 36) the critical exponents such as \(\eta\) and \(\omega\) are computed by analysing the skeleton Schwinger-Dyson equations at criticality. At that point the propagators and Green’s functions obey scaling law type forms where the powers are in effect the critical exponents. If one expands the exponent \(\eta\), for example, in powers of \(1/N\) where \(N\) is large,
\[
\eta = \sum_{i=1}^{\infty} \frac{\eta_i}{N^i}
\] (2.8)
then each term, \(\eta_i\), can be deduced from evaluating the relevant Feynman diagrams at each order of the \(1/N\) expansion. While such diagrams are divergent they are analytically regularized which means that the solution for each \(\eta_i\) and the other exponents are determined as functions of the spacetime dimension \(d\). Therefore these exponents, to as many orders in \(1/N\) as they can be computed, correspond to the exponents of the universal quantum field theory which underlies the Wilson-Fisher fixed point in \(d\)-dimensions. Thus when the exponents are expanded in powers of \(\epsilon\), where \(d = D - 2\epsilon\) and \(D\) is the critical dimension of a specific theory, then the \(\epsilon\) expansion will agree with the same expansion of the corresponding renormalization group function at the same fixed point. For theories such as (2.3) and (2.4) which are in the same universality class the large \(N\) critical exponents computed in (38, 46) reflect the three non-trivial fixed points noted above. The different solutions for the Heisenberg, AU and CS cases emerge from simple conditions which are best seen in the Lagrangian formulation involving the fields \(\sigma\) and \(T^{ab}\). These can be summarized by the vector \((\sigma, T^{ab})\) so that the Heisenberg fixed point is \((\sigma, 0)\), AU is \((0, T^{ab})\) and CS is \((\sigma, T^{ab})\) where a zero entry in the vector means the corresponding field is absent at that fixed point. In other words in the large \(N\) construction the critical exponents for a particular
fixed point are determined by including only those non-zero fields in the vector in the skeleton Schwinger-Dyson expansion.

If we define the scaling dimensions of the fields \( \phi^i \), \( \sigma \) and \( T^{ab} \) by \( \alpha \), \( \beta \) and \( \gamma \) respectively then

\[
\alpha = \mu - 1 + \frac{1}{2} \eta \ , \ \beta = 2 - \eta - \chi \ , \ \gamma = 2 - \eta - \chi T
\]

where \( d = 2\mu \), define the respective anomalous dimensions with \( \eta \) corresponding to that of \( \phi^i \). The exponents \( \chi \) and \( \chi_T \) correspond to the respective vertex anomalous dimensions of \( \sigma \) and \( T^{ab} \) with \( \phi^i \). These interactions are present in the universal theory. By contrast the spectator interactions, which involve only these two fields with themselves and not \( \phi^i \), are present in the different forms in the theory in each critical spacetime dimension \( D \). For completeness it is worth noting the leading large \( N \) critical exponent expressions. [35, 36, 38].

\[
\begin{align*}
\eta_{1}^{H} & = - \frac{4\Gamma(2\mu - 2)}{\Gamma(2 - \mu)\Gamma(\mu - 1)\Gamma(\mu - 2)\Gamma(\mu + 1)m} \\
\eta_{1}^{CS} & = - \frac{2(m + 1)\Gamma(2\mu - 2)}{\Gamma(\mu + 1)\Gamma(\mu - 1)\Gamma(\mu - 2)\Gamma(2 - \mu)} \\
\eta_{1}^{AU} & = - \frac{2(m - 1)(m + 2)\Gamma(2\mu - 2)}{m\Gamma(\mu + 1)\Gamma(\mu - 1)\Gamma(\mu - 2)\Gamma(2 - \mu)} \\
\chi_{1}^{H} & = - \frac{\mu(4\mu - 5)\eta_{1}^{H}}{\mu - 2} , \ \chi_{1}^{CS} = - \frac{\mu(4\mu - 5)\eta_{1}^{CS}}{\mu - 2} \\
\chi_{CS} & = - \frac{\mu(2\mu - 3)m + (4\mu - 5)\eta_{1}^{CS}}{(\mu - 2)(m + 1)} \\
\chi_{TU,1} & = - \frac{\mu(m - 2)[(m + 4)(2\mu - 3) + 1]\eta_{1}^{AU}}{(m - 1)(m + 2)(\mu - 2)}. \quad (2.10)
\end{align*}
\]

Higher order corrections are available in [35, 36, 37, 38, 46]. For the four dimensional theory the exponents corresponding to the critical slope of the \( \beta \)-functions have also been determined, [46], which also give an insight into the stability of each fixed point. With

\[
\omega = (\mu - 2) + \sum_{i=1}^{\infty} \frac{\omega_i}{N^i}
\]

then, [46],

\[
\begin{align*}
\omega_{+1}^{\text{Heis}} & = - \frac{4(2\mu - 1)^2\Gamma(2\mu - 2)}{\Gamma(2 - \mu)\Gamma(\mu - 1)\Gamma(\mu - 2)\Gamma(\mu + 1)mN} \\
\omega_{+1}^{AU} & = - \left[ 2\mu^2 - 3\mu - 1 + \frac{\mu(m - 2)[2\mu - 5 - 2(m + 4)(2\mu - 3)]}{(m - 1)(m + 2)} \right] \frac{\eta_{1}^{AU}}{N} \\
\omega_{+1}^{CS} & = \frac{(2\mu - 1)\eta_{1}^{CS}}{2(m + 1)(\mu - 2)N} \left[ m(\mu - 1)(\mu - 4) + (2\mu^2 - 7\mu + 4) \right. \\
& \quad \left. \pm \mu \left[ (m^2 - 1)(\mu - 1)^2 + 2(m - 1)(2\mu - 3)(\mu - 1) + (5\mu - 8)^2 \right] \right]. \quad (2.12)
\end{align*}
\]

where the \( \pm \) sign corresponds to two solutions in the CS case due to the presence of the two fields \( \sigma \) and \( T^{ab} \). For the other two fixed points there is only one solution since there is in effect only one coupling constant relevant at these respective points. These large \( N \) exponents in essence appear to provide a more fundamental insight into the critical point structure of the underlying universal theory in the large \( N \) expansion. Although it is worth emphasising that these results
are useful for checking explicit perturbative expressions it is the critical point structure of the \(O(N) \times O(m)\) theory for finite \(N\) which is our main focus.

In addition to the wave function and coupling constant renormalization we will also consider the renormalization of the three masses in (2.5) and determine the mixing matrix of anomalous dimensions to three loops. However, the comparison of the exponents derived from this matrix to the corresponding large \(N\) critical exponents is rather subtle which derives from the underlying operator. This is apparent in comparing the structure of the quadratic terms in \(\sigma\) and \(T^{ab}\) in (2.2), (2.3) and (2.5). In four dimensions the quadratic terms are present to implement the auxiliary field formulation of the quartic interaction. By contrast in six dimensions these fields have no auxiliary interpretation and the quadratic parts have to appear with a mass in order to have a consistent dimensionality. In terms of (2.2) the couplings \(g_1\) and \(g_2\) are not dimensionless away from four dimensions and can be interpreted as mass scales in higher dimensions. In other words at criticality the exponents \(\omega\) at each of the three fixed points will be related to the mass anomalous dimensions of \(\sigma\) and \(T^{ab}\) computed in perturbation theory using (2.4) and then evaluated at criticality. In reality it is not a direct relation since one is dealing with a mass mixing matrix. Instead one compares the appropriate exponent \(\omega\) with the eigen-anomalous dimension of the mixing matrix at criticality. The situation for the mass exponent of \(\phi^{ia}\) is slightly different. In perturbation theory the three mass operators have the same canonical dimension and hence the mixing matrix is \(3 \times 3\). In the large \(N\) expansion the canonical dimension of \(\frac{1}{2} \phi^{ia} \phi^{ia}\) differs from the other two mass operators and the critical exponent associated with the \(\phi^{ia}\) mass operator is not related to an \(\omega\) exponent. Instead like in the \(O(N)\) scalar theory the \(\phi^{ia}\) mass dimension is given by the anomalous dimension of the \(\sigma\) field. In other words it is proportional to the sum of \(\eta\) and \(\chi\).

3 Results.

We now turn to the derivation of the various renormalization group functions for (2.4) which builds essentially on the method developed in [22] for the \(O(N)\) case but with a minor caveat in the computation of several \(\beta\)-functions. The general procedure is to use an automated Feynman diagram approach where all the graphs are generated electronically using the QGRAF package, [47]. With indices appended to this output and Feynman rules substituted, the resulting scalar integrals are integrated by applying the integration by parts algorithm developed by Laporta, [48]. This reduces all the integrals to a basic set of what is termed master integrals whose \(\epsilon\) expansion is substituted at the final step. To implement the Laporta algorithm we have used the REDUCE version, [49, 50], and used the masters given in [51]. In [22] we checked that the three loop masters were consistent with the known four dimensional masters by applying the Tarasov method, [52, 53]. This is a way of relating \(d\)-dimensional Feynman integrals to \((d + 2)\)-dimensional ones. For all the renormalization group functions we determine we use the method of [54] to implement the renormalization in an automatic way. The Feynman diagrams are all evaluated as functions of the bare parameters, such as the coupling constants, and then these are replaced by their renormalized counterparts which involved the as yet undetermined counterterms. The counterterms are then chosen to render the appropriate Green’s function finite with reference to a particular scheme which throughout will be the \(\overline{\text{MS}}\) scheme. All the computations we carry out could not be possible without the use of the symbolic manipulation language FORM and its threaded version TFORM, [55, 56].

With the \(O(N) \times O(m)\) symmetry the Feynman rules for the propagators and vertices involving the field \(T^{ab}\) have an associated colour tensor. In other words the \(T^{ab}\) propagator will involve
the tensor, \[38\],

\[
P_{abcd} = \frac{1}{2} \left[ \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc} - \frac{2}{m} \delta_{ab} \delta_{cd} \right] \tag{3.1}
\]

which satisfies the trace properties

\[
P_{abcc} = P_{aacc} = 0, \quad P_{abcb} = \frac{(m-1)(m+2)}{2m} \delta_{ab}. \tag{3.2}
\]

It also satisfies the projector relations

\[
P_{abpq} P_{pqcd} = P_{abcd}, \quad P_{abpq} P_{cpdq} = \frac{(m-2)}{2m} P_{abcd}. \tag{3.3}
\]

Equipped with this the Feynman rule for the triple \(T^{ab}\) vertex involves the rank 6 colour tensor

\[
P_{3}^{abcdef} = P_{abpq} P_{cdpr} P_{efqr}. \tag{3.4}
\]

Consequently

\[
P_{3}^{abcdpq} P_{efpq} = P_{3}^{abcdef}, \quad P_{3}^{abcded} = \frac{(m-2)(m+4)}{4m} P_{abce}
\]

\[
P_{3}^{abpqrs} P_{cdpqrs} = \frac{(m-2)(m+4)}{4m} P_{abcd} \tag{3.5}
\]

for instance. Encoding these within a FORM module allows the group theory evaluation of the higher loop graphs to proceed more efficiently.

| Green’s function | One loop | Two loop | Three loop | Total |
|------------------|----------|----------|------------|-------|
| \(\phi^a \phi^b\) | 2        | 23       | 514        | 539   |
| \(\sigma \sigma\) | 3        | 19       | 343        | 365   |
| \(T^{ab}T^{cd}\) | 3        | 27       | 589        | 619   |
| \(T^{ab}T^{ia}T^{jb}\) | 5        | 137      | 4984       | 5126  |
| \(T^{ab}T^{cd}T^{ef}\) | 5        | 155      | 5857       | 6017  |
| Total            | 18       | 361      | 12287      | 12666 |

Table 1. Number of Feynman diagrams for each 2- and 3-point function.

The procedure we used to compute the large number of Feynman diagrams for the most part follows that described in [22] to which we refer the reader for the more technical aspects and focus on the amendments we made here to renormalize [25] at three loops. One useful technique which was exploited in [22] was that in addition to the wave function renormalization constants, the coupling constant and mass renormalization constants could be determined by purely evaluating the 2-point functions of each field. This was because each basic scalar propagator \(1/k^2\) could be replaced by

\[
\frac{\delta^{ij}}{k^2} \rightarrow \frac{\delta^{ij}}{k^2} + \frac{\lambda_1 \delta^{ij}}{(k^2)^2} + \frac{\lambda_2 g^{ij}k_e}{(k^2)^2} \tag{3.6}
\]

where the parameters \(\lambda_1\) and \(\lambda_2\) tag the mass operator insertion and 3-point vertex insertion both at zero momentum. The group structure of the general cubic theory is included on the final term and \(k_e\) is a fixed index corresponding to the external leg of that vertex. In performing this replacement one truncates the expansion at the linear term in \(\lambda_1\) as this reproduces all the relevant graphs for the respective mass operator and vertex renormalizations with one nullified external leg. This expansion does not lead to any problems in six dimensions as a \(1/(k^2)^2\) propagator is.
infrared safe unlike in four dimensions. We have recalled this procedure partly because we have exploited it to minimize the amount of computations we need to perform. Equally because it is not fully applicable to renormalizing (2.4) since it misses out certain graphs which involve the $\sigma T^{ab} T^{ab}$ vertex. While we used it for the mass mixing matrix for (2.5) the replacement does not generate all the vertex graphs for the renormalization of the couplings $g_3$ and $g_5$. Instead for the associated 3-point Green’s functions we had to generate all the Feynman diagrams separately using QGRAF and evaluate them with one nullified external vertex. While tedious there were no major difficulties. To gauge the size of the overall renormalization which was carried out, the number of graphs we computed for each Green’s function is given in Table 1.

The results of our computations are the renormalization group functions. As we will mainly focus our analysis on the $O(N) \times O(2)$ theory we record these, partly because of that but also due to space consideration, but note that the full $O(N) \times O(m)$ expressions are provided in the associated data file. First, the anomalous dimensions for the three fields are

$$\gamma_{\phi} (g_k) |_{m=2} = - \frac{1}{6} [g_1^2 + g_3^2] + \frac{1}{432} \left[ -22 N g_1^4 + 26 g_1^4 + 48 g_2 g_2 - 11 g_1^2 g_3^2 + 52 g_1^2 g_3^2 - 22 g_1^2 g_4^2 + 144 g_1 g_3 g_4 \
- 11 N g_3^4 - 22 g_4^2 - 22 g_4^2 g_4^2 \right] + \frac{1}{31104} \left[ 52 N g_1^4 g_1^6 - 464 N g_1^6 - 5184 g_3 - 9064 g_1^6 + 5292 N g_1^5 g_2 - 3264 g_1^5 g_2 \
- 772 N g_1^4 g_1^2 + 5184 g_3 g_1^4 g_1^2 - 11762 g_1^4 g_1^2 + 40 N g_1^4 g_3^2 + 15552 g_3 g_3 g_3 g_3 \
- 27192 g_1^4 g_3^2 + 104 N g_1^4 g_1^3 + 236 g_1^4 g_1^2 + 942 g_1^4 g_1^2 - 3264 g_1^3 g_3 g_3 \
+ 3388 g_1^3 g_3 g_1^2 + 5292 N g_1^3 g_3 g_3 - 9792 g_1^3 g_3 g_3 - 504 g_1^3 g_1^2 + 327 g_1^2 g_1^2 + g_1^2 g_1^2 g_1^2 \
+ 118 g_1^2 g_1^2 g_1^2 - 772 g_1^2 g_1^2 g_1^2 + 10368 g_1^2 g_1^2 g_1^2 + 23760 g_1^2 g_1^2 g_1^2 + 2904 g_1^2 g_1^2 g_1^2 \
+ 736 N g_1^3 g_1^3 g_4 - 2304 g_1^3 g_1^3 g_4 - 1648 g_1^3 g_1^3 g_4 + 20736 g_1^3 g_1^3 g_4 - 144 g_1^3 g_1^3 g_4 + 1194 g_1^3 g_1^3 g_4 \
- 756 g_1^3 g_1^3 g_4 + 1944 g_1^3 g_1^3 g_4 + 6408 g_1^3 g_1^3 g_4 - 412 g_1^3 g_1^3 g_4 + 1452 g_1^3 g_1^3 g_4 + 13 N g_4^2 g_4^2 - 1282 N g_4^2 g_4^2 + 5184 g_4^2 g_4^2 \
- 984 g_4^2 g_4^2 - 360 N g_4^2 g_4^2 - 3724 g_4^2 g_4^2 - 144 g_4^2 g_4^2 + O(g_5^5) \right]$$

$$\gamma_{\sigma} (g_k) |_{m=2} = \frac{1}{12} \left[ -2 N g_1^4 - g_2^2 - 2 g_2^2 \right] + \frac{1}{432} \left[ 4 N g_1^2 + 96 N g_1 g_2 - 22 N g_1 g_2^2 + 4 N g_1^2 g_3 + 96 N g_1 g_3 g_4 + 13 g_4^2 - 22 g_4^2 g_4 \right] \right] + \frac{1}{62208} \left[ -1048 N g_1^2 + 10368 g_3 - 17120 N g_1^2 + 4608 N g_1^2 N g_1^2 + 2112 N g_1^2 N g_1^2 \right. \
+ 12 N g_1^2 g_2^2 + 25920 g_3 N g_1^2 - 53292 N g_1^2 g_3 - 20736 g_3 N g_3 g_3 \right. \
- 34240 N g_1^2 N g_3^2 - 824 N g_1^2 g_4^2 - 3120 N g_1^2 g_4^2 - 2688 N g_3^2 g_3^3 \
+ 4608 N g_1^2 g_4^2 + 11712 N g_1^2 g_3^3 g_4 - 11712 N g_1^2 g_3^3 g_4 - 1904 N g_1^2 g_3^3 g_4 \
- 392 N g_1^2 g_3^3 g_4 + 24 N g_1^2 g_3^3 g_4 + 21104 N g_1^2 g_3^3 g_4 - 66672 N g_1^2 g_3^3 g_4 \
+ 4608 N g_1^2 g_3^3 g_4 - 5524 N g_1^2 g_3^3 g_4 + 3776 N g_1^2 g_3^3 g_4 + 41472 N g_1^2 g_3^3 g_4 \
- 77824 N g_1^2 g_3^3 g_4 - 824 N g_1^2 g_4^2 + 5808 N g_1^2 g_4^2 g_4 - 12240 N g_1^2 g_4^2 g_4 \
+ 2304 N g_1^2 g_4^2 g_4 + 672 N g_1^2 g_4^2 g_4 + 4992 N g_1^2 g_4^2 g_4 + 25920 N g_1^2 g_4^2 g_4 \
- 5195 g_4^2 + 1904 g_4^2 g_4^2 - 3120 g_4^2 g_4^2 - 1648 N g_2^2 g_4^2 + 25920 g_3 g_4^2 g_4 - 53280 g_2^2 g_4^2 \right. \
+ 4776 N g_2^2 g_4^2 g_4 + 6720 g_2^2 g_4^2 + 6 N g_2^2 g_4^2 - 8408 N g_2^2 g_4^2 + 680 N g_3^2 g_4^2 \right.$
\[
\gamma_T(g_i)|_{m=2} = \frac{1}{12} \left[ -N g_i^2 - 2g_i^4 \right] + O(g_i^6)
\]

\[
\gamma_T(g_i)|_{m=2} = \frac{1}{432} \left[ 2N g_i^2 g_2^2 - 22Ng_i^2 g_4^2 + 96Ng_i g_3 g_4 - 11g_2^2 g_4 + 48g_2 g_4^2 - 22Ng_3^2 - 11Ng_3 g_2^2 + 4g_4^2 \right]
\]

\[
+ \frac{1}{31104} \left[ -206N^2 g_1^2 g_3^2 + 2592\zeta_3 N g_1^2 g_3^2 - 4280Ng_1^2 g_3^2 + 52N^2 g_1^4 g_3^2 - 176Ng_1^2 g_3^4 + 1200N g_3^3 g_2 g_4 + 2904Ng_3^3 g_2 g_4^2 + 1152N^2 g_1^3 g_3 g_4^2 - 1344Ng_1^3 g_3^3 g_4 - 704Ng_3^2 g_2^2 g_4^2 - 772Ng_3^2 g_2^2 g_4^3 + 5184\zeta_3 N g_1^2 g_2 g_3^4 + 9576Ng_3^2 g_2^2 g_4^3 + 2388Ng_3^2 g_2^3 g_4 - 2556Ng_3^2 g_2^4 g_4 + 2168Ng_3^2 g_3^3 g_4 - 46Ng_3^2 g_3^4 + 15552\zeta_3 N g_1^3 g_3 g_4^3 - 33836Ng_1^3 g_3^3 g_4 + 340Ng_1^4 g_4^2 + 756Ng_1 g_2 g_3^4 + 5364Ng_1 g_2 g_3^4 g_4 + 576Ng_1 g_2^2 g_3^4 + 8376Ng_1 g_2^3 g_3^4 + 2496Ng_1 g_3^3 g_4^3 + 327Ng_1 g_3^4 g_4^2 + 942Ng_1 g_3^4 g_4^3 + 23Ng_1 g_3 g_4^2 + 5184g_3^2 g_4^2 + 12534Ng_1 g_2 g_3^4 g_4 + 576Ng_2 g_3^4 g_4^2 + 2028Ng_2 g_3^5 g_4 - 412Ng_2 g_3 g_4^2 + 2592Ng_3 g_4^4 - 5354Ng_3^2 g_3^2 g_4^2 - 2152Ng_3^2 g_3^4 g_4^2 - 36Ng_3^2 g_4^4 + 5184\zeta_3 g_3^6 - 9476g_3^6 + O(g_i^6) \right)
\]

(3.7)

where \(\zeta_3\) is the Riemann zeta function and the argument of the functions represents all five coupling constants. The five \(\beta\)-functions are of similar form and we note that

\[
\beta_1(g_i)|_{m=2} = \frac{1}{24} \left[ -2Ng_i^3 + 8g_i^3 + 12g_i g_2 - g_1 g_2^2 + 8g_1 g_3^2 - 2g_1 g_2^2 + 12g_1 g_4 \right]
\]

\[
+ \frac{1}{864} \left[ -172Ng_1^5 + 536Ng_1^4 g_2 - 360Ng_1^4 g_2^2 - 22Ng_1^3 g_2^2 - 628g_1^3 g_2^2 + 4Ng_1^3 g_3^2 - 1072Ng_1^3 g_2 g_3 + 40Ng_1^3 g_2^2 + 24Ng_1^3 g_2 g_3^2 + 16Ng_1^3 g_3^2 g_4 - 96Ng_1^3 g_2^2 g_4 + 600Ng_1^2 g_2^3 g_4 - 216Ng_1^2 g_2^4 g_4 + 13Ng_1^2 g_3^2 g_4 - 22Ng_1^2 g_3^3 g_4 - 64Ng_1 g_2 g_3^2 g_4 + 96Ng_1 g_2 g_3^3 g_4 - 22Ng_1 g_3^2 g_3 g_4 - 1256Ng_1 g_3^3 g_3 g_4 + 4g_1 g_4 - 108Ng_2 g_3^2 g_4 + 84Ng_1 g_4 g_3 + 24g_3 g_4^2 + 60g_3 g_4^3 \right]
\]

\[
+ \frac{1}{124416} \left[ 14648N^2 g_1^7 + 259200\zeta_3 N g_1^7 - 81376Ng_1^7 + 20736g_3 g_1^7 + 251360g_1^7 - 144Ng_1^6 g_2 - 31104\zeta_3 N g_1^6 g_2 + 249408N^6 g_1^6 g_2 + 186624\zeta_3 g_1^6 g_2 + 18000g_1^6 g_2 + 12N^2 g_1^2 g_2^2 + 25920\zeta_3 N g_1^2 g_2^2 - 107980N g_1^2 g_2^2 - 41472\zeta_3 g_1^2 g_2^2 + 358480g_1^2 g_2^2 + 20736\zeta_3 N g_1^3 g_2^2 - 106848Ng_1^3 g_2^2 + 62208\zeta_3 g_1^3 g_2^2 + 754080g_1^3 g_2^2 + 23496N g_1^3 g_2^2 - 15712g_1 g_2^2 g_4 - 9120g_1 g_2 g_3^2 g_4 + 124416\zeta_3 g_1 g_2 g_3^2 g_4 - 20736g_1 g_2^2 g_4^2 - 186624\zeta_3 g_1 g_2 g_3^3 g_4 + 160704g_1^4 g_3 g_4^2 + 435456\zeta_3 g_1 g_3 g_4^2 + 294249g_1 g_3^2 g_4 + 6624N g_1 g_3^2 g_4 - 50688g_1^3 g_2^2 g_4 + 1904N g_1^3 g_2^2 g_4 + 62208\zeta_3 g_1^3 g_2^2 g_4 - 392N g_1^3 g_2^2 g_4^2 + 158032g_1 g_2 g_3^2 g_4^2 + 24g_1 g_2 g_3^2 g_4^2 + 44032g_1 g_2 g_3^2 g_4^2 + 31104\zeta_3 g_1^3 g_2^2 g_4^2 + 98352g_1^2 g_2 g_3^2 g_4 - 82244\zeta_3 g_1^2 g_2 g_3^2 g_4 + 655776g_1 g_2^3 g_3^2 g_4 + 4608N g_1 g_2 g_3^2 g_4 - 124416\zeta_3 g_1 g_2 g_3^2 g_4 + 17664g_1 g_2 g_3^2 g_4 - 5524N g_1 g_2 g_3^2 g_4 + 373248\zeta_3 N g_1 g_3^2 - 27552g_1 g_3^2 - 124416\zeta_3 g_1 g_3^2 + 21801N g_1 g_3^2 + 41472\zeta_3 N g_1 g_3^2 + 62336g_1 g_3^2 - 165888\zeta_3 g_1 g_3^2 \right]
\]
and triple poles of all the underlying renormalization constants correctly emerge as predicted by

\[ \text{The remaining expressions are given in Appendix A where the mixing matrix of mass anomalous} \]

\[ \text{dimensions is also provided. One test of the expressions we have computed is that the double} \]

\[ \text{and triple poles of all the underlying renormalization constants correctly emerge as predicted by} \]

\[ \text{the renormalization group formalism. Equally we have checked the limit back to the pure } O(N) \]

\[ \text{theory where the } O(m) \text{ indices are completely passive and found agreement with} [21]. \text{The final} \]

\[ \text{checks which we have derive from the comparison with the large } N \text{ exponents which we devolve} \]

\[ \text{to the next section.} \]

\[ \text{4 Large } N \text{ analysis.} \]

\[ \text{Equipped with the explicit forms of the renormalization group functions we are in a position to} \]

\[ \text{check them against the large } N \text{ critical exponents for each of the three fixed points. In order to} \]

\[ \text{do this we follow the prescription introduced in} [21] \text{and define scaled coupling constants by} \]

\[ g_1 = i \sqrt{\frac{12 \epsilon}{mN}} x , \quad g_2 = i \sqrt{\frac{12 \epsilon}{mN}} y , \quad g_3 = i \sqrt{\frac{12 \epsilon}{N}} z , \quad g_4 = i \sqrt{\frac{12 \epsilon}{mN}} t , \quad g_5 = i \sqrt{\frac{12 \epsilon}{N}} w . \] (4.1)

\[ \text{With these we can deduce the location of each fixed point in a large } N \text{ expansion where each} \]

\[ \text{coefficient of the power of } 1/N \text{ is a function of } \epsilon \text{ having set } d = 6 - 2 \epsilon. \text{ Each of the three} \]

\[ \text{fixed points is defined by different field content and therefore for the Heisenberg and AU only several} \]

\[ \text{of the coupling constants are non-zero. From the respective } \beta \text{-functions we find} \]

\[ x = 1 + \left( 22 - \frac{155}{3} \epsilon + \frac{1777}{36} \epsilon^2 \right) \frac{1}{mN} . \]
and, finally, for the Heisenberg case which is consistent with [21] where the order symbol represents the truncation point for the two independent expansions. For AU we have

\[ x = y = t = 0 \]

\[ z = 1 + \left( 11 - 40 \frac{1}{m} + \frac{7}{2} m + \frac{299}{3} \frac{1}{m} - \frac{155}{6} \epsilon - \frac{25}{3} m \epsilon - \frac{3820}{36} \frac{1}{m} \epsilon^2 + \frac{1777}{72} \epsilon^2 + \frac{80}{9} m \epsilon^2 \right) \frac{1}{N} \]

\[ + \left( -\frac{477}{2} + 2400 \frac{1}{m^2} - 1320 \frac{1}{m} + \frac{231}{2} m + \frac{147}{8} m^2 - 14180 \frac{1}{m^2} \epsilon - 6959 \frac{1}{m} \epsilon + 1949 \epsilon \right) \frac{1}{N} \]

\[ - \frac{2555}{4} \frac{1}{m} \epsilon - 150 m \epsilon^2 + \frac{38755}{18} \frac{1}{m^2} \epsilon^2 - 20664 \zeta_3 \frac{1}{m} \epsilon^2 - \frac{136469}{36} \frac{1}{m} \epsilon^2 + 10296 \zeta_3 \frac{1}{m} \epsilon^2 \]

\[ - \frac{19919}{24} \epsilon^2 + 1242 \zeta_3 \epsilon^2 + \frac{123919}{144} m \epsilon^2 - 576 \zeta_3 m \epsilon^2 + \frac{23695}{72} m^2 \epsilon^2 \right) \frac{1}{N^2} + O \left( \epsilon^3; \frac{1}{N^3} \right) \]

\[ w = 6 + \left( 486 - 3240 \frac{1}{m} + 81 m + 5178 \frac{1}{m} \epsilon - 645 \epsilon - 150 m \epsilon - \frac{12105}{2} \frac{1}{m} \epsilon^2 + \frac{2781}{4} \epsilon^2 \right) \frac{1}{N} \]

\[ + \left( -118071 + 4874400 \frac{1}{m^2} - 1417320 \frac{1}{m} + 32283 m + \frac{10161}{4} m^2 - 1447068 \frac{1}{m^2} \epsilon \right) \frac{1}{N} \]

\[ + \left( 3945054 \frac{1}{m} \epsilon + 464454 \epsilon - \frac{186915}{2} m \epsilon - 10830 m^2 \epsilon + 16668989 \frac{1}{m^2} \epsilon^2 \right) \frac{1}{N} \]

\[ - \frac{755676 \zeta_3}{4} \frac{1}{m} \epsilon^2 - \frac{8635273}{2} \frac{1}{m^2} \epsilon^2 + 2238192 \zeta_3 \frac{1}{m} \epsilon^2 - \frac{2355195}{4} \epsilon^2 + 236196 \zeta_3 \epsilon^2 \]

\[ + \frac{915527}{8} m \epsilon^2 - 42120 \zeta_3 m \epsilon^2 + \frac{76709}{4} m^2 \epsilon^2 \right) \frac{1}{N^2} + O \left( \epsilon^3; \frac{1}{N^3} \right) \]

and, finally,

\[ x = 1 + \left( 11 + 11 m - \frac{155}{6} \epsilon - \frac{155}{6} m \epsilon + \frac{1777}{72} \epsilon^2 + \frac{1777}{72} m \epsilon^2 \right) \frac{1}{N} \]

\[ + \left( \frac{1563}{2} + 63 m - \frac{237}{2} m^2 - \frac{6855}{2} \epsilon - \frac{835}{2} m \epsilon + 435 m^2 \epsilon + \frac{35345}{9} \epsilon^2 - 2646 \zeta_3 \epsilon^2 \right) \frac{1}{N} \]

\[ + \frac{4085}{72} m^2 \epsilon^2 - 1602 \zeta_3 m \epsilon^2 - \frac{54101}{72} m^2 \epsilon^2 - 432 \zeta_3 m^2 \epsilon^2 \right) \frac{1}{N^2} + O \left( \epsilon^3; \frac{1}{N^3} \right) \]

\[ y = 6 + \left( 486 + 486 m - 645 \epsilon - 645 m \epsilon + \frac{2781}{4} \epsilon^2 + \frac{2781}{4} m \epsilon^2 \right) \frac{1}{N} \]

\[ + \left( 248949 + 133398 m + 30249 m^2 - 66067 \epsilon - 317175 m \epsilon - 58170 m^2 \epsilon + \frac{1419565}{2} \epsilon^2 \right) \frac{1}{N} \]

\[ + \frac{354780 \zeta_3 \epsilon^2 + \frac{1289545}{4} m \epsilon^2 - 215460 \zeta_3 m \epsilon^2 + \frac{205901}{4} m^2 \epsilon^2 - 58320 \zeta_3 m^2 \epsilon^2 \right) \frac{1}{N^2} + O \left( \epsilon^3; \frac{1}{N^3} \right) \]
\[ z = 1 + \left( 11 + \frac{7}{2} m - \frac{155}{6} \epsilon - \frac{25}{3} m \epsilon - \frac{1777}{72} \epsilon^2 + \frac{80}{9} m \epsilon^2 \right) \frac{1}{N} \\
+ \left( \frac{1563}{2} + \frac{231}{2} m + \frac{147}{8} m^2 - \frac{6855}{2} \epsilon - \frac{2555}{4} m \epsilon - 150 m^2 \epsilon + \frac{35345}{9} \epsilon^2 - 2646 \zeta_3 \epsilon^2 \right) \frac{1}{N^2} + O \left( \epsilon^3, \frac{1}{N^3} \right) \]

\[ t = 6 + \left( 486 + 216 m - 645 \epsilon - 315 m \epsilon + \frac{2781}{4} \epsilon^2 + \frac{1407}{4} m \epsilon^2 \right) \frac{1}{N} \]

\[ + \left( 248949 + 65988 m + 7389 m^2 - 660675 \epsilon - 168030 m \epsilon - 19545 m^2 \epsilon + \frac{1419565}{2} \epsilon^2 \\
- 354780 \zeta_3 \epsilon^2 + 183756 m \epsilon^2 - 99900 \zeta_3 m \epsilon^2 + 25357 m^2 \epsilon^2 - 11664 \zeta_3 m^2 \epsilon^2 \right) \frac{1}{N^2} \]

\[ + O \left( \epsilon^3, \frac{1}{N^3} \right) \]

\[ w = 6 + \left( 486 + 81 m - 645 \epsilon - 150 m \epsilon + \frac{2781}{4} \epsilon^2 + 180 m \epsilon^2 \right) \frac{1}{N} \]

\[ + \left( 248949 + 32283 m + \frac{10161}{4} m^2 - 660675 \epsilon - \frac{186915}{2} m \epsilon - 10830 m^2 \epsilon + \frac{1419565}{2} \epsilon^2 \\
- 354780 \zeta_3 \epsilon^2 + 915527 m \epsilon^2 - 42120 \zeta_3 m \epsilon^2 + 76709 m^2 \epsilon^2 \right) \frac{1}{N^2} + O \left( \epsilon^3, \frac{1}{N^3} \right) \]

for CS where all the couplings are active. With these particular values at each of the three fixed points we find agreement with the known large \( N \) exponents \([35, 36, 37, 38, 46]\) out to \( O(\epsilon^3) \). This includes the mass mixing matrix. However, the comparison with the mass dimension exponents is not straightforward since one has to compare with the anomalous dimensions of the eigenvalues of the mass mixing matrix \( \gamma_{ij}(g_i) \) evaluated at each critical point. For instance, at AU the exponent \( \omega_{\epsilon_1} \) is in precise agreement with the critical eigen-anomalous dimension. Equally at CS the exponents \( \eta + \chi \) and the linear combination \( \omega_{\epsilon_1} + \omega_{\epsilon_1} - 1 \) are also in exact correspondence with the \( O(\epsilon^3) \) terms of the eigen-anomalous dimensions. These nontrivial large \( N \) checks at each of the three fixed points on the three loop \( \overline{\text{MS}} \) renormalization group functions provide confidence that our perturbative computation is correct.

## 5 Conformal window search.

One of our aims is to find the conformal window for (2.4). Given the nature of the renormalization group equations computed at three loops it transpires that pinning down the actual range of the conformal window is not straightforward. A similar observation was made in \([28]\) for the four dimensional \( O(N) \times O(3) \) case using the conformal bootstrap method. For the pure \( O(N) \) case, \([21]\), which has two coupling constants unlike our five here the conformal window was determined by solving the equations

\[ \beta_i(g_i) = 0 \]

where \( i = 1 \) and \( 2 \). As the generalization of these equations to five couplings is

\[ \beta_1(g_i) = \beta_2(g_i) = \ldots = \beta_5(g_i) = 0 \]

together with the Hessian it turned out our computer resources were not sufficient to solve the complete system numerically in general. Instead we have resorted to an alternative strategy
which could equally well have been applied to the pure $O(N)$ theory. One observation of \cite{21, 22} in respect of the conformal window in the $O(N)$ case was the nature of the fixed point spectrum above and below a conformal window boundary. At leading order the main window boundary is at $N_{cr} = 1038$, \cite{20}, for $O(N)$. Above this value of $N_{cr}$ there are fixed points with real couplings. By contrast below this point there are no real fixed points. Given this distinguishing property we have solved the equations \cite{5, 2} for fixed values of $N$ and then analysed the stability properties of the real solutions. The stability of a fixed point is determined by finding the eigenvalues of the stability matrix $\mathbf{S}$ at each real fixed point in turn for the chosen value of $N$ where $\mathbf{S}$ is defined by

$$
\mathbf{S} = \left( \frac{\partial^{2} \lambda_{j}}{\partial g_{ij}} \right). \tag{5.3}
$$

Specifically if all the eigenvalues are negative then this signifies ultraviolet (UV) stability, while if all eigenvalues are positive then that fixed point would be UV unstable and consequently infrared (IR) stable. Obtaining a mixed signature indicates that the fixed point is a saddle point. In the situation where the eigenvalues are zero, we can only conclude that the fixed point is marginal and beyond the linear approximation. We did not find any such cases for the values of $N$ analysed. While this may appear to be a tedious process for finding the conformal window boundary it turned out to be relatively quick since one can narrow the search area by a process of sectioning.

To illustrate the process we focus for the moment on the $O(N) \times O(2)$ theory. First, given the fact that there are more couplings in \cite{2, 3} the criteria defining the window boundary differs slightly from the properties of the $O(N)$ case. In order to define this we need to introduce a descriptive syntax which derives partly from the nature of the fixed points which emerge and the structure of the four dimensional $O(N) \times O(m)$ coupling constant plane. In \cite{2, 3} there were three non-trivial fixed points designated Heisenberg, anti-chiral unstable and chiral stable and they were associated with different combinations of the fields $\sigma$ and $T^{ab}$ that were active or not at a fixed point. Moreover with fewer couplings in four dimensions each type of fixed point had a definite stability which led to the notation AU or CS aside from the Heisenberg solution which was necessarily a saddle point. In our conformal window analysis of \cite{2, 3} we will retain our AU and CS syntax as well as Heisenberg but use it to represent the field content only. So, for instance, indicating an AU fixed point will mean that only interactions involving the $T^{ab}$ field are present while a CS type of fixed point will correspond to all interactions of \cite{2, 3} being active. This readjustment in syntax is necessary since, as will become clear, the fixed point structure is much richer than that of the six dimensional $O(N) \otimes \mathbf{3}$ theory and \cite{2, 3}. So we will refer to Heisenberg, AU and CS types of solutions. Illustrating this with the coupling vector $(g_{1}, g_{2}, g_{3}, g_{4}, g_{5})$ their characteristic critical coupling constant patterns respectively are $(x, y, 0, 0, 0)$, $(0, 0, z, 0, w)$ and $(x, y, z, t, w)$ where we mean that $x$, $y$, $z$, $t$ and $w$ are non-zero in these patterns. For simplicity we have omitted the constant of proportionality given in \cite{1, 1}. It is important to appreciate that for the Heisenberg, AU and CS patterns the actual fixed point which is present could actually be stable or unstable and not be related to the U or S of the label type. In one respect the emergence of these patterns within the perturbative context, where we are now working, should not be surprising as the fixed $N$ analysis has to at least contain the Heisenberg, AU and CS large $N$ solutions. With this syntax for \cite{2, 3} we can now give our criteria for the conformal window boundary. From the analysis we have carried out we regard a window boundary to be where there is a change in the number of a particular pattern of fixed point such as CS. We note that as in the $O(N)$ case various fixed point solutions are connected to each other via symmetries, \cite{21}, and so we focus on a representative fixed point of each such class in the discussion. We also find a large number of fixed points with complex and purely imaginary values which may indicate non-unitarity solutions or even that a limit cycle exists. In our discussions in this and the next section we will focus only on the real solutions for the critical couplings as they lead to clear stability properties.
As a first stage to our search strategy it is best to summarize the analysis for the upper boundary we found which was $N = 1105$ when $m = 2$. For the case of $N = 1106$ we have three CS type fixed points. One of these is UV stable which is at

$$
x = 1.024331 + 0.602917\epsilon - 618.493720\epsilon^2 + O(\epsilon^3) \\
y = 10.027831 - 224.568795\epsilon + 204744.131100\epsilon^2 + O(\epsilon^3) \\
z = 1.014679 + 0.242004\epsilon - 259.254500\epsilon^2 + O(\epsilon^3) \\
t = 8.413935 - 122.062932\epsilon + 110001.339800\epsilon^2 + O(\epsilon^3) \\
w = 7.750728 - 86.093662\epsilon + 77109.596670\epsilon^2 + O(\epsilon^3) \ . \quad (5.4)
$$

The corresponding critical exponents are

$$
\gamma^s_\phi = 0.002810\epsilon - 0.003531\epsilon^2 - 2.095198\epsilon^3 + O(\epsilon^4) \\
\gamma^s_\sigma = 1.158724\epsilon - 2.828644\epsilon^2 + 2307.673939\epsilon^3 + O(\epsilon^4) \\
\gamma^s_T = 1.093583\epsilon - 1.472805\epsilon^2 + 1165.028293\epsilon^3 + O(\epsilon^4) \ . \quad (5.5)
$$

The other two CS style fixed points are saddle points at

$$
x = 1.023546 - 0.790738\epsilon + 618.557767\epsilon^2 + O(\epsilon^3) \\
y = 10.288220 + 238.034889\epsilon - 204695.170900\epsilon^2 + O(\epsilon^3) \\
z = 1.014350 - 0.341297\epsilon + 259.727356\epsilon^2 + O(\epsilon^3) \\
t = 8.553710 + 126.145941\epsilon - 109987.441000\epsilon^2 + O(\epsilon^3) \\
w = 7.848666 + 87.779203\epsilon - 77103.604170\epsilon^2 + O(\epsilon^3) \quad (5.6)
$$

and

$$
x = -0.869900 - 0.200484\epsilon - 0.868576\epsilon^2 + O(\epsilon^3) \\
y = 20.723963 + 8.470150\epsilon - 14.322290\epsilon^2 + O(\epsilon^3) \\
z = 1.011451 - 0.019282\epsilon + 0.058843\epsilon^2 + O(\epsilon^3) \\
t = -4.381299 - 2.162646\epsilon - 6.897939\epsilon^2 + O(\epsilon^3) \\
w = 5.927808 + 0.692949\epsilon + 3.355853\epsilon^2 + O(\epsilon^3) \ . \quad (5.7)
$$

In addition there are three Heisenberg fixed points, one of which is UV stable at

$$
x = 1.010040 - 0.023705\epsilon + 0.020596\epsilon^2 + O(\epsilon^3) \\
y = 6.557735 - 0.940183\epsilon + 0.810426\epsilon^2 + O(\epsilon^3) \\
z = 0 \ , \ t = 0 \ , \ w = 0 \quad (5.8)
$$

with critical exponents

$$
\gamma^s_\phi = 0.000922\epsilon - 0.001777\epsilon^2 - 0.000152\epsilon^3 + O(\epsilon^4) \\
\gamma^s_\sigma = 1.039622\epsilon - 0.075355\epsilon^2 - 0.008779\epsilon^3 + O(\epsilon^4) \\
\gamma^s_T = 0 \ . \quad (5.9)
$$

The other two fixed points are saddle points and are located at

$$
x = 0.979414 - 0.003228\epsilon + 0.071572\epsilon^2 + O(\epsilon^3) \\
y = 17.380571 + 10.947386\epsilon + 21.645075\epsilon^2 + O(\epsilon^3) \\
z = 0 \ , \ t = 0 \ , \ w = 0 \quad (5.10)
$$
and
\[
\begin{align*}
x &= -0.857078 - 0.208350\epsilon - 0.632470\epsilon^2 + O(\epsilon^3) \\
y &= 19.745752 + 9.661778\epsilon - 2.588019\epsilon^2 + O(\epsilon^3) \\
z &= 0 , \ t = 0 , \ w = 0 .
\end{align*}
\] (5.11)

There was one AU fixed point which is UV stable at
\[
\begin{align*}
x &= 0 , \ y = 0 , \ t = 0 \\
z &= 0.998197 + 0.006635\epsilon - 0.008935\epsilon^2 + O(\epsilon^3) \\
w &= 5.367450 + 0.851212\epsilon - 1.446454\epsilon^2 + O(\epsilon^3)
\end{align*}
\] (5.12)

with critical exponents
\[
\begin{align*}
\gamma^*_\phi &= 0.001802\epsilon - 0.003273\epsilon^2 - 0.000708\epsilon^3 + O(\epsilon^4) \\
\gamma^*_\sigma &= 0 \\
\gamma^*_T &= 0.996396\epsilon + 0.006664\epsilon^2 + 0.002605\epsilon^3 + O(\epsilon^4)
\end{align*}
\] (5.13)

For values of \(N\) above 1106 the same pattern and number of Heisenberg, AU and CS fixed points emerge with the same stability structure. By contrast for \(N = 1105\) a different style of solution emerges. This is first seen in the CS type of fixed points in that we have only one such fixed point which is at
\[
\begin{align*}
x &= -0.869887 - 0.200513\epsilon - 0.868979\epsilon^2 + O(\epsilon^3) \\
y &= 6.558394 - 0.941587\epsilon + 0.811596\epsilon^2 + O(\epsilon^3) \\
z &= 0 , \ t = 0 , \ w = 0
\end{align*}
\] (5.14)

More crucially it is a saddle point. In other words there is no stable CS fixed point. So given this change in pattern we regard \(N = 1105\) as the bound for the conformal window in six dimensions. It is instructive to provide the picture for the other types of fixed points for \(N = 1105\). There are also three Heisenberg fixed points. The UV stable one is
\[
\begin{align*}
x &= 1.010049 - 0.023726\epsilon + 0.0200513\epsilon^2 + O(\epsilon^3) \\
y &= 6.558394 - 0.941587\epsilon + 0.811596\epsilon^2 + O(\epsilon^3) \\
z &= 0 , \ t = 0 , \ w = 0
\end{align*}
\] (5.15)

with critical exponents
\[
\begin{align*}
\gamma^*_\phi &= 0.000923\epsilon - 0.001779\epsilon^2 - 0.000152\epsilon^3 + O(\epsilon^4) \\
\gamma^*_\sigma &= 1.039662\epsilon - 0.075439\epsilon^2 - 0.008783\epsilon^3 + O(\epsilon^4) \\
\gamma^*_T &= 0
\end{align*}
\] (5.16)

while the other two fixed points are saddle points at
\[
\begin{align*}
x &= -0.857055 - 0.208383\epsilon - 0.632604\epsilon^2 + O(\epsilon^3) \\
y &= 19.736951 + 9.657499\epsilon - 2.589415\epsilon^2 + O(\epsilon^3) \\
z &= 0 , \ t = 0 , \ w = 0
\end{align*}
\] (5.17)
and
\[
x = 0.979447 - 0.003297 \epsilon + 0.071496 \epsilon^2 + O(\epsilon^3) \\
y = 17.371128 + 10.944494 \epsilon + 21.644028 \epsilon^2 + O(\epsilon^3) \\
z = 0, \ t = 0, \ w = 0.
\]
(5.18)

The one AU fixed point is UV stable and is located at
\[
x = 0, \ y = 0, \ t = 0 \\
z = 0.998195 + 0.006641 \epsilon - 0.008942 \epsilon^2 + O(\epsilon^3) \\
w = 5.367025 + 0.851662 \epsilon - 1.447623 \epsilon^2 + O(\epsilon^3)
\]
(5.19)

with critical exponents
\[
\gamma_\phi^* = 0.001803 \epsilon - 0.003276 \epsilon^2 - 0.000709 \epsilon^3 + O(\epsilon^4) \\
\gamma_\sigma^* = 0 \\
\gamma_T^* = 0.996393 \epsilon + 0.006670 \epsilon^2 + 0.002608 \epsilon^3 + O(\epsilon^4).
\]
(5.20)

For \( N < 1105 \) we applied our algorithm of section searching for changes in fixed point patterns but found no further boundaries. However the structure for certain fixed values of \( N \) will be recorded later for completeness. One observation on our window analysis is that the boundary at \( N = 1105 \) is not dissimilar to the leading order value of \( N_{cr}^{(1)} = 1038, [20] \), for the \( O(N) \) case. In \[21, 22\] the \( O(\epsilon^3) \) corrections to \( N_{cr}^{(2)} \) were computed and by using resummation methods a value of \( N_{cr}^{(2)} \) around 400 was found for the five dimensional theory. Clearly applying our section search method cannot be readily extended beyond the leading order which is for the strictly six dimensional theory. Instead solving \((5.2)\) simultaneously with \( \det(S) = 0 \) would be the way to extract such corrections but was beyond the range of our computational tools.

We close this section by briefly discussing a different tack for gaining more insight into the conformal window problem for \( O(N) \times O(m) \). As is apparent from the \( O(N) \times O(2) \) case the change in nature of the fixed points indicates a boundary. Moreover different types of (real) solutions emerge. Therefore, for the general \( O(N) \times O(m) \) case we searched for the conformal window for the AU pattern of couplings. In other words we set \( x = y = t = 0 \) at the outset for a selection of values of \( m \) and solved \((5.1)\). Included in this is the equation for the Hessian which allows us to determine the critical value of \( N \) defining the window boundary, which we will denote by \( N_{cr}^{(m)} \) for this AU pattern, without having to do a section search. To get a perspective on our results we have provided the leading order value for \( N_{cr}^{(m)} \) for various \( m \) in Table 2. As \( m \to \infty \) we found that \( N_{cr}^{(m)} \) asymptotes to a straight line. While this is only a partial picture for the situation for \( m > 2 \) one thing is evident which is that in six dimensions when \( m \geq 5 \) there should be a change in pattern for AU type fixed points for a fixed \( N \) search akin to that illustrated in our section based search for \( m = 2 \). This is in addition to the change in pattern for the other style of solutions. The solution given in Table 2 for \( m = 4 \) reflects that there was no solution rather than an exact value of zero. Although we have recorded 0 in the Table for that reason it does appear to be consistent with the monotonic increase in \( N_{cr}^{(m)} \) with \( m \). Since we are able to solve \((5.1)\) the three loop corrections to the leading order values in Table 2 have been determined. We found
\[
N_{cr}^{(1)} = -2946.134605 + 3951.961993 \epsilon + 2676.699839 \epsilon^2 + O(\epsilon^3) \\
z = 1.006955 - 0.008027 \epsilon + 0.012574 \epsilon^2 + O(\epsilon^3) \\
w = 8.952176 - 0.933006 \epsilon + 1.840946 \epsilon^2 + O(\epsilon^3) \\
N_{cr}^{(2)} = -1087.488959 + 1415.172128 \epsilon + 261.248651 \epsilon^2 + O(\epsilon^3)
\]
for a selection of \( m \) where the respective critical couplings have been displayed. This is an important point. While we have provided values for \( N_{cr}^{(m)} \) in (5.21) other solutions were found for each \( m \). In [21, 22] there were three solutions but the small \( N_{cr}^{(m)} \) solutions were discarded because they were negative or had complex critical couplings. We have followed the same reasoning here. The negative values for \( N_{cr}^{(m)} \) are in keeping with similar negative solutions for the eight dimensional ultraviolet completion of the \( O(N) \) sequence of theories, [23]. We have also excluded from this AU analysis values of \( N_{cr}^{(m)} \) which have large critical couplings as such values are clearly outside the perturbative approximation we are using.

| \( m \) | 1     | 2     | 3     | 4     | 5     | 6     | 10    | 20    | 30    | 40    | 50    |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( N_{cr}^{(m)} \) | **2946.1** | -1087.5 | -410.2 | 0     | 216.8 | 421.7 | 992.3 | 1999.6 | 2887.9 | 3746.3 | 4592.9 |

Table 2. Leading order value of \( N_{cr}^{(m)} \) for the conformal window for different values of \( m \).
6 Fixed point analysis.

In this section we present a fixed point analysis for a variety of specific values of $N$. This includes the determination of those fixed points which are stable or otherwise, in order to give a flavour of the fixed point spectrum away from $N = 1105$. In addition we will indicate the potential for another conformal window boundary for non-CS type fixed points. While we focus on a selection of values of $N$ for a reader interested in exploring the solution space further the complete set of renormalization group functions for arbitrary $m$ can be analysed which are available in the attached data file. We will begin by looking at $N = 1000$ and then proceed to lower values of $N$. For $N = 1000$ we have one CS fixed point as expected which is a saddle point

\[
\begin{align*}
\gamma = & -0.868555 - 0.203744\epsilon - 0.915849\epsilon^2 + O(\epsilon^3) \\
y = & 19.811433 + 7.966436\epsilon - 15.205442\epsilon^2 + O(\epsilon^3) \\
z = & 1.012581 - 0.020942\epsilon + 0.066905\epsilon^2 + O(\epsilon^3) \\
t = & -4.342552 - 2.231269\epsilon - 7.303390\epsilon^2 + O(\epsilon^3) \\
w = & 5.911324 + 0.770705\epsilon + 3.795188\epsilon^2 + O(\epsilon^3). \\
\end{align*}
\]  

(6.1)

We also have three Heisenberg fixed points, one of which is UV stable at

\[
\begin{align*}
x = & 1.0111102 - 0.026162\epsilon + 0.022238\epsilon^2 + O(\epsilon^3) \\
y = & 6.637801 - 1.117476\epsilon + 0.962982\epsilon^2 + O(\epsilon^3) \\
z = & 0, \quad t = 0, \quad w = 0 \\
\end{align*}
\]  

(6.2)

with the critical exponents

\[
\begin{align*}
\gamma^\phi = & 0.001022\epsilon - 0.001981\epsilon^2 - 0.000145\epsilon^3 + O(\epsilon^4) \\
\gamma^\sigma = & 1.044358\epsilon - 0.085562\epsilon^2 - 0.009090\epsilon^3 + O(\epsilon^4) \\
\gamma^T = & 0. \\
\end{align*}
\]  

(6.3)

The other two fixed points

\[
\begin{align*}
x = & -0.854446 - 0.212078\epsilon - 0.647751\epsilon^2 + O(\epsilon^3) \\
y = & 18.789145 + 9.197094\epsilon - 2.733818\epsilon^2 + O(\epsilon^3) \\
z = & 0, \quad t = 0, \quad w = 0 \\
\end{align*}
\]  

(6.4)

and

\[
\begin{align*}
x = & 0.983210 - 0.011253\epsilon + 0.063259\epsilon^2 + O(\epsilon^3) \\
y = & 16.345805 + 10.658027\epsilon + 21.524495\epsilon^2 + O(\epsilon^3) \\
z = & 0, \quad t = 0, \quad w = 0 \\
\end{align*}
\]  

(6.5)

are saddle points. Again we also have one AU fixed point which is UV stable which is located at

\[
\begin{align*}
x = & 0, \quad y = 0, \quad t = 0 \\
z = & 0.998006 + 0.007339\epsilon - 0.009793\epsilon^2 + O(\epsilon^3) \\
w = & 5.318846 + 0.901757\epsilon - 1.582315\epsilon^2 + O(\epsilon^3) \\
\end{align*}
\]  

(6.6)

giving the critical exponents

\[
\begin{align*}
\gamma^\phi = & 0.001992\epsilon - 0.003615\epsilon^2 - 0.000789\epsilon^3 + O(\epsilon^4) \\
\gamma^\sigma = & 0. \\
\gamma^T = & 0.996016\epsilon + 0.007374\epsilon^2 + 0.003032\epsilon^3 + O(\epsilon^4). \\
\end{align*}
\]  

(6.7)
To illustrate the full spectrum of fixed points for a particular value of $N$ we have provided the remaining fixed points for $N = 1000$ in Appendix B. In addition to other real solutions which do not fit the Heisenberg, AU or CS pattern we record the complex solutions for completeness there.

Next examining the value of $N = 600$ in order to illustrate a change in the fixed point pattern, we have one CS saddle point solution at

$$
x = -0.862204 - 0.222460\epsilon - 1.255729\epsilon^2 + O(\epsilon^3) \\
y = 15.871131 + 5.764114\epsilon - 20.128689\epsilon^2 + O(\epsilon^3) \\
z = 1.020205 - 0.031243\epsilon + 0.133767\epsilon^2 + O(\epsilon^3) \\
t = -4.133094 - 2.641545\epsilon - 10.197488\epsilon^2 + O(\epsilon^3) \\
w = 5.794456 + 1.264133\epsilon + 7.071880\epsilon^2 + O(\epsilon^3) . \quad (6.8)
$$

In addition there are three Heisenberg fixed points with the one at

$$
x = 1.018022 - 0.037843\epsilon + 0.001985\epsilon^2 + O(\epsilon^3) \\
y = 7.507506 - 4.490389\epsilon + 9.490485\epsilon^2 + O(\epsilon^3) \\
z = 0 , \quad t = 0 , \quad w = 0 \quad (6.9)
$$

being UV stable giving

$$
\gamma_\phi^* = 0.001727\epsilon - 0.003465\epsilon^2 - 0.000010\epsilon^3 + O(\epsilon^4) \\
\gamma_\sigma^* = 1.083337\epsilon - 0.195953\epsilon^2 + 0.089727\epsilon^3 + O(\epsilon^4) \\
\gamma_T^* = 0 \quad (6.10)
$$

while the other two fixed points are saddle points at

$$
x = -0.839313 - 0.232926\epsilon - 0.736639\epsilon^2 + O(\epsilon^3) \\
y = 14.602366 + 7.174642\epsilon - 3.193826\epsilon^2 + O(\epsilon^3) \\
z = 0 , \quad t = 0 , \quad w = 0 \quad (6.11)
$$

and

$$
x = 1.007039 - 0.068389\epsilon + 0.063230\epsilon^2 + O(\epsilon^3) \\
y = 11.302398 + 11.995559\epsilon + 13.765855\epsilon^2 + O(\epsilon^3) \\
z = 0 , \quad t = 0 , \quad w = 0 . \quad (6.12)
$$

There is also have one AU fixed point which is UV stable at

$$
x = 0 , \quad y = 0 , \quad t = 0 \\
z = 0.996683 + 0.012238\epsilon - 0.015305\epsilon^2 + O(\epsilon^3) \\
w = 5.033838 + 1.165363\epsilon - 2.476601\epsilon^2 + O(\epsilon^3) \quad (6.13)
$$

with exponents

$$
\gamma_\phi^* = 0.003311\epsilon - 0.005969\epsilon^2 - 0.001383\epsilon^3 + O(\epsilon^4) \\
\gamma_\sigma^* = 0 \\
\gamma_T^* = 0.993377\epsilon + 0.012333\epsilon^2 + 0.006784\epsilon^3 + O(\epsilon^4) . \quad (6.14)
$$
We have recorded this spectrum to contrast it with that for \( N = 519 \). So for this value we then have one CS fixed point which is a saddle point at

\[
\begin{align*}
x &= -0.860686 - 0.228611\epsilon - 1.395472\epsilon^2 + O(\epsilon^3) \\
y &= 14.937476 + 5.238428\epsilon - 21.686769\epsilon^2 + O(\epsilon^3) \\
z &= 1.023093 - 0.0347571\epsilon + 0.165366\epsilon^2 + O(\epsilon^3) \\
t &= -4.069923 - 2.780163\epsilon - 11.373918\epsilon^2 + O(\epsilon^3) \\
w &= 5.750244 + 1.435664\epsilon + 8.435080\epsilon^2 + O(\epsilon^3) .
\end{align*}
\]

However, by contrast, we have one Heisenberg fixed point which is a saddle point at

\[
\begin{align*}
x &= -0.834431 - 0.239444\epsilon - 0.765623\epsilon^2 + O(\epsilon^3) \\
y &= 13.591847 + 6.689873\epsilon - 3.246649\epsilon^2 + O(\epsilon^3) \\
z &= 0 , \ t = 0 , \ w = 0 .
\end{align*}
\]

In addition there is one UV stable AU type fixed point at

\[
\begin{align*}
x &= 0 \ , \ y = 0 \ , \ t = 0 \\
z &= 0.996169 + 0.014150\epsilon - 0.017238\epsilon^2 + O(\epsilon^3) \\
w &= 4.941667 + 1.239792\epsilon - 2.803418\epsilon^2 + O(\epsilon^3) .
\end{align*}
\]

giving critical exponents

\[
\begin{align*}
\gamma_{\phi}^* &= 0.003824\epsilon - 0.006875\epsilon^2 - 0.0016281\epsilon^3 + O(\epsilon^4) \\
\gamma_{\sigma}^* &= 0 \\
\gamma_{T}^* &= 0.992352\epsilon + 0.014277\epsilon^2 + 0.008615\epsilon^3 + O(\epsilon^4) .
\end{align*}
\]

So between \( N = 519 \) and \( N = 600 \) the behaviour of a Heisenberg type fixed point changes. This seems to indicate that a conformal window type region exists with respect to the Heisenberg structure and thus there is a new window between 519 and 600. However, its actual location is not of major significance in the context of (2.4) as this in effect corresponds to the original Heisenberg model with no \( T^{ab} \) field.

The final case we consider in detail in our excursion through fixed values of \( N \) is \( N = 2 \). It is of potential interest since for this value in a variety of models a supersymmetric solution emerged, [21 58 59]. We have three CS fixed points all of which are saddle points at

\[
\begin{align*}
x &= -0.454392 - 1.128422\epsilon - 10.883437\epsilon^2 + O(\epsilon^3) \\
y &= 0.673205 + 1.783387\epsilon + 15.854883\epsilon^2 + O(\epsilon^3) \\
z &= 0.318954 + 0.395758\epsilon + 3.102196\epsilon^2 + O(\epsilon^3) \\
t &= 0.379850 + 0.510247\epsilon + 4.361634\epsilon^2 + O(\epsilon^3) .
\end{align*}
\]

The value for the coupling \( w \) has not been provided with the others as a novel feature emerged for this set. It transpired that there were three fixed points with the same \( x, y, z \) and \( t \) values but differing only in the \( w \) value. Therefore, we note these values separately as

\[
w \in \{0.717916 + 0.824313\epsilon + 5.193907\epsilon^2 + O(\epsilon^3), \\
-0.267715 - 0.020190\epsilon + 0.685452\epsilon^2 + O(\epsilon^3), \\
-0.450201 - 0.479646\epsilon - 3.632751\epsilon^2 + O(\epsilon^3)\} .
\]
There was one Heisenberg fixed point, which is a saddle point

\[
x = -0.470736 - 0.73744 \epsilon - 5.70852 \epsilon^2 + O(\epsilon^3)
\]
\[
y = 0.762184 + 0.999917 \epsilon + 6.174478 \epsilon^2 + O(\epsilon^3)
\]
\[
z = 0, \ t = 0, \ w = 0.
\]  

(6.21)

In addition we found one AU fixed point which is UV stable

\[
x = 0, \ y = 0, \ t = 0
\]
\[
z = 0.577350 + 1.507526 \epsilon + 19.533564 \epsilon^2 + O(\epsilon^3)
\]
\[
w = 0.800625 + 1.806817 \epsilon + 27.377665 \epsilon^2 + O(\epsilon^3)
\]  

(6.22)

with critical exponents

\[
\gamma_\phi^* = \gamma_T^* = 0.333333 \epsilon + 1.333333 \epsilon^2 + 22.148148 \epsilon^3 + O(\epsilon^4)
\]
\[
\gamma_\sigma^* = 0.
\]  

(6.23)

One property of the emergent supersymmetric solutions found in earlier work, \[21, 58, 59\], was that critical couplings were equivalent. For this AU solution a different feature is apparent which is that the exponents of $\phi^ia$ and $T^{ab}$ are equal.

7 Discussion.

We have provided a comprehensive three loop analysis of the extension of the four dimensional Landau-Ginzburg-Wilson $O(N) \times O(m)$ symmetric theory to six dimensions. One aspect of our study was to investigate the ultraviolet completion beyond four dimensions. The main interest previously had been in $O(N)$ symmetric theories and our extension \[2.4\] fits in with the vision of how to proceed. Briefly this requires a common interaction which seeds the theories in various even dimensions along the thread of Wilson-Fisher fixed points in $d$-dimensions. Each fixed dimension Lagrangian is required to be renormalizable in its critical dimension which requires the addition of matter field independent extra interactions. With the increase in dimension the number of these so-called spectator interactions increases. For \[2.4\] overall there are five interactions each with its own coupling. One reassuring aspect of our computations is the verification that the three loop renormalization group functions are consistent with the large $N$ critical exponents of \[83\]. These exponents are determined in the underlying universal theory and as $1/N$ is a dimensionless coupling constant it transcends a specific dimension. In other words the exponents contain information on the respective renormalization group functions in all the theories connected via the Wilson-Fisher fixed point thread. Indeed verifying that our three loop perturbative results were consistent with the results of \[83\] was an important check.

One consequence of the larger number of coupling constants is a richer spectrum of fixed points for specific values of $N$ and $m$. While our analysis of this concentrated on $m = 2$ we do not expect that the general picture of fixed points differs conceptually for higher values of $m$. Instead the boundary values will be at different values of $N$ as our AU study for various values of $m$ illustrated. Our $m = 2$ analysis was similar to the $O(N)$ case of \[21\] with real and complex critical couplings with the latter corresponding to non-unitary theories. However, for real solutions we were able to isolate fixed points which had a structure in keeping with the phase plane in the four dimensional model. In other words there are Heisenberg, AU and CS type solutions which depend on which combination of $\sigma$ and $T^{ab}$ fields are active and their stability was studied for certain values of $N$. One of the main areas of interest in the $O(N)$ and $O(N) \times O(m)$ symmetric theories is whether
there is a fixed point in the five dimensional theory and if so what is the conformal window. In [15] a bootstrap study indicated that this was not an easy exercise from the lower dimensional point of view unless one was examining AU type coupling patterns. Our investigation left us with a similar point of view. Although we were able to narrow down the leading order value of the window for CS solutions for $m = 2$. By contrast we were able to solve the AU set of equations and found that a window exists above $m = 5$. However, we have provided the full data from our renormalization group functions which can be mined for future studies for other values of $m$.

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## A Remaining renormalization group functions.

For completeness we record the remaining $\beta$-functions for the $O(N) \times O(2)$ theory for comparison with [38] as well as various other renormalization group functions. The $\beta$-functions are

$$
\beta_2(g_i)|_{m=2} = \frac{1}{8} \left[8N g_1^3 - 2Ng_1^2g_2 + 3g_2^3 - 2g_2g_1^2 + 8g_1^2\right]
+ \frac{1}{288} \left[-48N g_1^2 - 644N g_1^4g_2 - 120Ng_1^3g_2^2 - 48Ng_1^4g_2^3 + 62Ng_1^2g_2g_3^2 - 64Ng_1^2g_2g_4 - 96Ng_1g_2g_3g_4 - 216Ng_1g_3g_4^2 - 125g_5^2 + 62g_3g_4^2 - 120g_2g_4^3 - 22Ng_2g_3g_4^2 - 644g_2g_4^3 + 84Ng_2g_3^2g_4 - 48g_1^4\right]
+ \frac{1}{41472} \left[101784N^2g_1^7 + 68448Ng_1^4g_2 - 153896N^2g_1^6g_2 + 10368\zeta_3Ng_1^6g_2 + 118816Ng_1^6g_2 + 45216Ng_1^5g_2^2 + 124416\zeta_3Ng_1^5g_2^2 + 50592Ng_1^5g_2^2 + 136896Ng_1^5g_2^2 + 1920Ng_1^5g_2^2 - 3156Ng_1^5g_2^2 + 88128\zeta_3Ng_1^5g_2^2 + 255780Ng_1^4g_2^3 + 20736\zeta_3Ng_1^4g_2^3 + 111200Ng_1^4g_2^3 + 16928Ng_1^5g_2^3 + 504Ng_1^5g_2^3 + 108864Ng_1^4g_2^3 + 108864Ng_1^4g_2^3 - 14172\zeta_3Ng_1^4g_2^3 - 17376Ng_1^3g_2^4 - 45888Ng_1^3g_2^4 + 45216Ng_1^3g_2^4 + 248832\zeta_3Ng_1^3g_2^4 + 37632Ng_1^3g_2^4 - 175986N^2g_1^2g_3g_4^2 + 55392N^2g_1^3g_3g_4^2 + 10560Ng_1^3g_3^2g_4 + 269760Ng_1^3g_3^2g_4 + 108864Ng_1^3g_3^2g_4 - 12544Ng_1^2g_2^5 + 4264Ng_1^2g_2^5 + 6312Ng_1^2g_2^5 + 155520\zeta_3Ng_1^2g_2^5 + 174344Ng_1^2g_2^5 + 54216Ng_1^2g_2^5 - 5524Ng_1^2g_2^5 + 3776Ng_1^2g_2^5 + 476928\zeta_3Ng_1^2g_2^5 + 181376Ng_1^2g_2^5 + 65912Ng_1^2g_2^5 - 17424Ng_1^2g_2^5 - 25632Ng_1^2g_2^5 + 273936Ng_1^2g_2^5 + 1920Ng_1^2g_2^5 - 14064Ng_1^2g_2^5 + 3312Ng_1^2g_2^5 + 2304Ng_1^2g_2^5 - 672Ng_1g_2g_3^4 - 124416\zeta_3Ng_1g_2g_3^4 - 30912Ng_1g_2g_3^4 + 20304Ng_1g_2g_3^4 - 35424Ng_1g_2g_3^4 - 61344Ng_1g_2g_3^4 + 12960\zeta_3g_2^3 + 33085g_2^3 - 12544g_2^3 + 4172\zeta_3g_2^3 - 17376g_2^3 + 5648Ng_2g_3^2g_4 + 88128\zeta_3g_3^2g_4 + 252624g_3^2g_4 - 24600Ng_2g_3^2g_4 + 124416\zeta_3g_3^2g_4 + 95808g_3^2g_4 + 6N^2g_2g_3g_4 - 8408Ng_2g_3^2g_4 - 4360Ng_2g_3^2g_4 + 10368\zeta_3g_3^2g_4 - 35080g_2g_3g_4\right].
$$
\[
\beta_3(g_i)_{m=2} = \frac{g_3}{24} \left[ 8g_4^2 + 24g_1g_4 - N g_3^2 - 4g_3^2 - 2g_4^2 \right] \\
+ \frac{g_3}{864} \left[ 40N g_1^4 - 536g_1^2 - 120g_1^3g_2 + 168N g_1^3g_4 - 480g_1^3g_4 + 20g_1^2g_2^2 - 648g_1^2g_24 + 214N g_1^2g_2^2 + 56g_1^2g_2^2 - 22N g_1^2g_2^4 - 1256g_1^2g_2^4 + 84g_1g_2^4 + 216g_1^4g_2^4 + 180N g_1g_2^4g_4 + 24g_1g_2^4g_4 + 120g_1^3g_4 - 11g_2^4g_4 + 48g_2g_4^4 - 44N g_4^4 - 476g_4^4 - 11Ng_4^3g_4 - 260g_4^2g_4^2 + 4g_4^2 \right] \\
+ \frac{g_3}{62208} \left[ -688N g_1^6 - 22480N g_1^6 + 10368g_1^6 + 125680g_1^6 - 10440N g_1^5g_2 + 62208c_3g_1^2g_2^3 + 11856g_1^5g_2^3 - 1584N g_1^7 + 62208g_3g_1^5g_4 + 40176N g_1^5g_4 + 62208c_3g_1^4g_4 + 41712g_1^4g_4 + 1312N g_1^4g_2^2 - 20736c_3g_1^4g_2^2 + 74048g_1^4g_2^2 - 45216N g_1^4g_2^2 + 127440g_1^4g_2^2 + 6562N g_1^3g_2^4 + 189216c_3N g_1^3g_2^4 + 4032N g_1^3g_2^4 - 93312c_3g_1^3g_2^4 + 131298g_1^3g_2^4 + 52N g_1^2g_2^4 + 40452N g_1^2g_2^4 + 158032g_1^2g_2^4 - 11436g_1^2g_2^4 + 5712N g_1^2g_2^4 + 62208c_3g_1^2g_2^4 + 52752g_1^2g_2^4 - 31104c_3g_1^2g_2g_4 + 15456N g_1^2g_2g_4 - 16176g_1^2g_2g_4 + 30104N g_1^2g_2g_4 + 186624c_3N g_1^2g_2g_4 - 212856N g_1^2g_2g_4 + 26208c_3g_1^2g_2g_4 + 25680g_1^2g_2g_4 + 2712g_1^2g_4 + 53216c_3g_1^2g_4 + 124416g_1^2g_4 + 80592g_1^2g_4 - 204g_2^4 + 32114c_3g_1^2g_4 + 9360g_1^2g_4 + 3281N g_1^2g_4 + 2828g_1^2g_4 - 772g_1^2g_4 + 9360g_1^2g_4 + 5168g_1^2g_4 + 5184c_3N g_1^2g_4 + 9360g_1^2g_4 + 5184c_3N g_1^2g_4 - 6984N g_1^2g_4 + 41472c_3N g_1^2g_4 + 25056g_1^2g_4 + 2388N g_1^2g_4 + 62208c_3N g_1^2g_4 + 503904g_1^2g_4 + 828N g_1^2g_4 + 828N g_1^2g_4 + 56812g_1^2g_4 + 280544g_1^2g_4 + 340g_1^2g_4 + 32352g_1^2g_4 + 1716g_1^2g_4 + 1716g_1^2g_4 + 13320g_1^2g_4 + 13320g_1^2g_4 - 180N g_1^2g_4 + 8364g_1^2g_4 - 62208c_3g_1^2g_4 + 96912g_1^2g_4 + 7140g_1^2g_4 - 62208c_3g_1^2g_4 + 7740N g_1^2g_4 + 121056g_1^2g_4 + 5520N g_1^2g_4 + 62208c_3g_1^2g_4 + 129984g_1^2g_4 + 62208c_3g_1^2g_4 + 56812g_1^2g_4 + 129984g_1^2g_4 + 76536g_1^2g_4 - 22320g_1^2g_4 + 180N g_1^2g_4 - 17184N g_1^2g_4 + 121056g_1^2g_4 + 5520N g_1^2g_4 + 327g_1^2g_4 + 942g_1^2g_4 + 23N g_1^2g_4 + 2560g_1^2g_4 + 5184c_3N g_1^2g_4 + 12534g_1^2g_4 + 576N g_1^2g_4 + 24312g_1^2g_4 + 2028g_1^2g_4 + 386N g_1^2g_4 - 2592c_3N g_1^2g_4 + 18434N g_1^2g_4 + 30368g_1^2g_4 + 23512g_1^2g_4 + 13N^2g_1^2g_4 - 8416g_1^2g_4 + 93312c_3g_1^2g_4 + 70304g_1^2g_4 + 36N g_1^2g_4 + 62208c_3g_1^2g_4 + 9648g_1^2g_4 + 5184c_3g_1^2g_4 - 9476g_1^2g_4 + O(g_1^6) \right]

\beta_4(g_i)_{m=2} = \frac{1}{24} \left[ -2N g_1^4g_4 + 12N g_1^3g_2^2 + g_2^4g_4 + 12g_2^2g_4^2 - 2N g_3^4g_4 + 6g_4^3 \right] \\
+ \frac{1}{864} \left[ 4N g_1^4g_4 + 96N g_1^3g_2^2g_4 - 72N g_1^3g_2^2g_4 + 216N g_1^3g_2^2g_4 - 22N g_1^2g_2^4 + 324N g_1^2g_2^4 + 168N g_1^2g_2^4 - 1288N g_1^2g_2^4 + 40N g_1^2g_2^4 - 216N g_1g_2^4g_4 + 144N g_1g_2^4g_4 - 36N g_1g_2^4g_4 + 13g_4^2 - 24g_4^2 - 650g_2^3g_4 + 42N g_2g_3g_2^2 - 96g_2g_4^4 - 152N g_3g_4 + 40N g_3g_4^3 - 708g_4^3 \right]
\[
\beta_5(g_i)_{m=2} = \frac{1}{8} \left[ 4N g_3^3 - N g_3^2 g_5 + 10 g_4 g_5^2 - 3 g_5^3 \right] + \frac{1}{1152} \left[ -96 N g_1^2 g_3^3 + 8 N g_1^2 g_2 g_5 + 248 N g_2^2 g_4 g_5 - 2592 N g_1 g_3^2 g_4 - 480 N g_1^2 g_3 g_4 g_5 + 124 g_2 g_3 g_5 - 1248 g_2 g_3^2 g_4 + 336 g_3 g_5^2 + 344 g_4^3 g_5 - 432 g_3 g_4^2 + 324 N g_3^3 g_5 - 292 N g_3^2 g_4 g_5 - 126 N g_3 g_5^2 - 286 g_4 g_5 g_6 + 2340 g_3^5 \right] - 513 g_5^5 \]

\[
+ \frac{1}{165888} \left[ 3840 N g_1^2 g_3^3 + 136896 N g_1^4 g_3 - 1648 N g_1^2 g_3^5 + 20736 \zeta_3 N g_1^4 g_3^5 + 34240 N g_1^2 g_3^5 - 5920 N g_1^2 g_3^4 g_5 + 17056 N g_1^4 g_3^4 g_5 - 19008 N g_1^2 g_3^4 g_5 + 9600 N g_1^4 g_3^5 g_4 + 5256 N g_1^2 g_3^4 g_5 - 130176 N g_1^4 g_3^5 g_4 + 581760 N g_1^4 g_3^4 g_5 - 183552 N g_1^2 g_3^4 g_5 - 165888 \zeta_3 N g_1^4 g_3^4 g_5 + 30528 N g_1^4 g_3^4 g_5 + 1920 N g_1^2 g_3^4 g_5 \right]
\]
\[ -824N g_1^2 g_2^2 g_3^3 g_5 + 16672N g_1^2 g_2^2 g_4^3 g_5 + 311040N g_1^2 g_2 g_3^4 g_4 \]
\[ -207360 \zeta_3 N g_1^2 g_2^3 g_4^3 g_5 + 161856N g_1^2 g_2^3 g_4^3 g_5 - 107616N g_1^2 g_2^4 g_3^3 g_5 \]
\[ + 217728N^2 g_1^2 g_3^3 g_4^2 - 119616N g_1^2 g_3^3 g_5^2 - 98208N^2 g_1 g_3^3 g_4 g_5 - 37952N g_1 g_3^3 g_4 g_5 \]
\[ + 27072N^2 g_1^2 g_3^3 g_4^2 + 995328 \zeta_3 N g_1^2 g_3^3 g_4^2 + 21120N g_1^2 g_3^3 g_4^2 \]
\[ + 64800N g_1^2 g_3^3 g_4^2 - 6704N^2 g_1^2 g_3^3 g_4^2 + 622080\zeta_3 N g_1^2 g_3^3 g_4^2 \]
\[ + 1466912N g_1^2 g_3^3 g_4^2 - 6984N g_1^2 g_3^3 g_4^2 - 114592N g_1^2 g_3^3 g_4^2 \]
\[ + 74448N g_1^2 g_3^3 g_4^2 - 65088N g_1^2 g_3^3 g_4^2 + 31680N g_1^2 g_3^3 g_4^2 \]
\[ + 497664\zeta_3 N g_1^2 g_3^3 g_4^2 + 300672N g_1 g_3^3 g_5^2 - 42912N g_1^2 g_3^3 g_4^2 \]
\[ - 220608N^2 g_1 g_3^3 g_4 - 248832\zeta_3 N g_1 g_3^3 g_4 - 45504N g_1 g_3^3 g_4 \]
\[ + 58752N^2 g_1 g_3^3 g_4 - 497664\zeta_3 N g_1 g_3^3 g_4 - 259944N g_1 g_3^3 g_4 \]
\[ + 248832\zeta_3 N g_1 g_3^3 g_4 + 1087488N g_1 g_3^3 g_4 - 559872\zeta_3 N g_1 g_3^3 g_4 \]
\[ - 909792N g_1 g_3^3 g_4 - 497664\zeta_3 N g_1 g_3^3 g_4 - 14592N g_1 g_3^3 g_4 \]
\[ + 186624\zeta_3 N g_1 g_3^3 g_4 - 72576N g_1 g_3^3 g_4 - 424849^2 g_3 g_5 \]
\[ - 82944\zeta_3 g_1^3 g_4 g_5 - 38544g_1^3 g_4 g_5 + 13536N g_1^3 g_4 g_5 - 3352N^2 g_1^3 g_4 g_5 \]
\[ + 290304\zeta_3 g_1^3 g_4 g_5 + 549300g_1^3 g_4 g_5 + 37224g_1^3 g_4 g_5 - 82944g_1^3 g_4 g_5 \]
\[ - 76032N g_2 g_3 g_4^3 - 6336N g_2 g_3 g_4^3 + 497664\zeta_3 g_2 g_3 g_4^3 \]
\[ + 444768g_2 g_3 g_4^3 - 311040\zeta_3 g_2 g_3 g_4^3 - 387504g_2 g_3 g_4^3 \]
\[ + 6816N^2 g_3^7 - 124416\zeta_3 N g_3^7 + 275712N g_3^7 + 64528N^2 g_3^7 \]
\[ + 207360\zeta_3 N g_3^7 g_5 + 15920N g_3^7 g_5 + 27072N^2 g_3^7 g_5 \]
\[ - 248832\zeta_3 N g_3^7 g_5 + 87936N g_3^7 g_5 - 30456N^2 g_3^7 g_5 + 279936\zeta_3 N g_3^7 g_5 \]
\[ - 5184N^2 g_3^7 g_5 - 4648N^2 g_1^4 g_5 g_5 - 62208\zeta_3 N g_3^7 g_5 - 167936N g_3^7 g_5 \]
\[ + 2376N^2 g_3^7 g_5 + 139968\zeta_3 N g_3^7 g_5 + 199728N g_3^7 g_5 - 12096N g_3^7 g_5 \]
\[ + 186624\zeta_3 N g_3^7 g_5 + 16848N g_3^7 g_5 - 77760\zeta_3 N g_3^7 g_5 \]
\[ + 19116N g_3^7 g_5 + 96864N g_3^7 g_5 + 90360N g_3^7 g_5 - 28026N g_3^7 g_5 \]
\[ + 373248\zeta_3 g_3^7 g_5 + 1293344g_3^7 g_5 - 933120\zeta_3 g_3^7 g_5 - 1361376g_3^7 g_5 \]
\[ + 629856\zeta_3 g_3^7 g_5 + 760428g_3^7 g_5 - 104976\zeta_3 g_3^7 g_5 - 137295g_5^7 \]
\[ + O(g_5^9) . \] (A.1)

The elements of the mass mixing matrix are

\[ \gamma_{11}(g_1)_{m=2} = \frac{1}{3} [g_1^2 + g_3^2] \]
\[ \left[ \frac{1}{216} \begin{array}{c}
-44N g_1^4 - 134g_1^4 - 30g_1^3 g_2 + 5g_1^3 g_2 + 268g_1 g_2^2 + 10g_1 g_2^2 - 90g_1 g_3 g_4 \\
- 22N g_1^4 + 4g_1^4 + 10g_1^3 g_3 \end{array} \right] \]
\[ + \frac{1}{15552} \begin{array}{c}
3212N^2 g_1^6 + 31104\zeta_3 N g_1^6 - 8032N g_1^6 + 2592\zeta_3 g_1^6 + 31420g_1^6 \\
- 15552\zeta_3 N g_1^5 g_2 + 4518N g_1^5 g_2 + 15552\zeta_3 g_1^5 g_2 - 2964g_1^5 g_2 \\
+ 7852N g_1^5 g_2 - 5184\zeta_3 g_1^5 g_2 + 18512g_1^5 g_2 - 9076N g_1^5 g_2^2 + 7776\zeta_3 g_1^5 g_2^2 \\
+ 94260N g_1^5 g_2^2 + 3040N g_1^5 g_2^2 - 1964g_1^5 g_2^2 - 2859g_1^5 g_2^2 + 15552\zeta_3 g_1^5 g_2^2 \\
- 2964g_1^5 g_2^2 - 1578g_1^4 g_3 g_2 g_3 - 15552\zeta_3 N g_1^4 g_3 g_2 g_3 + 4518N g_1^4 g_3 g_2 g_3 \\
+ 46656\zeta_3 g_1^4 g_3 g_2 g_3 - 8892g_1^4 g_3 g_2 g_3 - 4140g_1^4 g_3 g_2 g_3 - 51g_1^4 g_2^2 - 982g_1^4 g_2^2 \\
+ 328g_1^4 g_2^2 - 10368\zeta_3 g_1^4 g_2^2 + 38988g_1^4 g_2^2 g_4 - 1032g_1^4 g_2^2 g_4 \\
+ 46656\zeta_3 N g_1^2 g_4^3 - 2972N g_1^2 g_4^3 - 15552\zeta_3 g_1^2 g_4^3 + 27252g_1^2 g_4^3 \end{array} \]
\[ \gamma_{12}(g_i)_{|m=2} = \frac{N g_i^2}{12} + \frac{N}{864} \left[ -2g_1^4 - 18g_2^3g_3 - 3g_1^2g_2^2 - 2g_1^2g_3^2 - 18g_1^2g_3^2 - 3g_3^2g_4^2 \right] \]
\[ + \frac{N}{864} \left[ 2308N g_1^6 + 1426g_6^6 - 1984N g_1^6g_2 + 1822g_1^6g_2^2 + 282N g_1^4g_2^2 + 864\zeta_3g_4^6 \right. \]
\[ + 1430g_1^4g_2^2 + 2852g_1^2g_2^2 + 40g_1^4g_2^2 + 864\zeta_3g_1^3g_3^2 + 1420g_1^2g_2^3 - 2020g_1^2g_2^3 + 204g_1g_2^4 \]
\[ - 904g_1^3g_2^2 + 1426g_1^3g_3^2 + 1512g_1^3g_4^2 - 21g_1^2g_2^4 - 300g_1^2g_2^4 + 282g_1^2g_2^4 + 1728\zeta_3g_1^2g_2^3 + 1260g_1^2g_2^3g_3 + 1080g_1^2g_2^3g_3 \]
\[ + 756g_1g_2^4g_3 + 1728\zeta_3g_1^2g_2^3g_4 + 1260g_1^2g_2^3g_3 + 1080g_1^2g_2^3g_3 - 1098g_1^2g_2^3g_3 \]
\[ + 1154N g_1^4g_3^2 + 220g_1g_4^4 + 4060g_1g_2^3g_3^2 + 756g_1^2g_2^3g_4 + 864\zeta_3g_1^3g_2^3g_4 \]
\[ + 828g_1^2g_2^3g_4 + 3456\zeta_3g_1^2g_2^3g_4 + 1332g_1^2g_2^3g_4 - 992N g_1^2g_3^3g_4 \]
\[ - 356g_1^4g_3^3 + 1796g_1^2g_3^3g_4 + 334g_2^2g_3^3g_4 - 342g_2^2g_3^3g_4 + 141N g_4^4 \]
\[ - 246g_2^4 + 66g_2^4g_1^3 + O(g_5^8) \]

\[ \gamma_{13}(g_i)_{|m=2} = \frac{1}{2}N g_2^3 + \frac{N}{24} \left[ -2g_2^3 - 6g_2^3g_3 - 36g_2^3g_3 + 4g_4^3 - 3g_2^3g_4^2 \right] \]
\[ + \frac{N}{1728} \left[ 796N g_1^4g_2^2 + 1426g_1^4g_2^2 + 376N g_1g_4^4 - 600g_1^4g_2^2 - 198g_1^3g_2^3 \right. \]
\[ - 1728\zeta_3g_4^3g_2^3 + 1656g_4^3g_2^3 + 1984N g_1^3g_2^3 + 4040g_1^3g_2^3 + 4040g_1^3g_2^3 \]
\[ + 3456\zeta_3g_1^3g_2^3g_3 + 3456\zeta_3g_1^3g_2^3g_4 + 20g_1^2g_2^3g_3 + 20g_1^2g_2^3g_3 + 20g_1^2g_2^3g_3 + 3456\zeta_3g_1^2g_2^3g_4 \]
\[ - 864g_1^2g_2^3g_3 + 1512N g_1^2g_2^3g_4 + 1209g_1g_2^4g_4 + 188N g_1^2g_2^3g_4 + 836g_1^2g_2^3g_4 \]
\[ + 1600g_1^2g_2^3g_4 + 132g_2^4 + 1452g_1^2g_2^3g_4 + 1728\zeta_3g_1^2g_2^3g_4 \]
\[ + 1092g_1^2g_2^3g_4 - 992N g_1^2g_3^4g_4 + 226g_1^2g_3^4g_4 + 359g_1^2g_3^4g_4 + 47g_2^2g_3^4g_4 \]
\[ - 540g_2g_3^4g_4^2 + 432N g_3^6 - 864\zeta_3g_3^6 + 2258g_3^6 + 94N g_3^4g_4 \]
\[ + 780g_3^4g_1^4 + 822g_3^4g_1^4 + O(g_5^8) \]

\[ \gamma_{21}(g_i)_{|m=2} = \frac{g_i^2}{2} \]
\[ + \frac{1}{72} \left[ 14N g_1^4 - 20g_4^4 - 54g_1^2g_2^3 - 2g_1^2g_2^3 - 20g_1^2g_2^3 + 14g_2^4 - 54g_1^2g_3^4 - 9g_2^2g_3^4 \right] \]
\[ + \frac{1}{10368} \left[ -396N g_1^6 - 15552\zeta_3N g_1^6 + 17596N g_1^6 + 5184\zeta_3g_1^6 + 3476g_6^6 \right. \]
\[ - 9792N g_1^5g_2 + 17532g_1^5g_2 - 500N g_1^4g_2 + 10368\zeta_3g_1^4g_2^2 + 10054g_1^4g_2^2 + 848N g_1^4g_3 + 10368\zeta_3g_1^4g_3 + 6952g_1^4g_3 + 792N g_1^4g_3^2 - 676g_1^4g_3^2 \]
\[ + 5184\zeta_3g_1^4g_3 + 864\zeta_3g_1^4g_3^2 + 13824g_1^3g_2^3g_3 - 3888g_1^3g_2^3g_3 - 2640N g_1^3g_2^3g_4 \]
\[ + 24948\zeta_3g_1^3g_2^3g_4 - 10368\zeta_3g_1^3g_2^3g_4 + 7344g_1^3g_2^3g_4 - 2592\zeta_3g_1^3g_2^3g_4 + 2801g_2^4 \]
\[ - 696g_2^4g_3^2 - 300g_2^4g_3^2 - 500g_2^4g_3^2 + 10368\zeta_3g_1^2g_2^3g_4 + 12744g_1^2g_2^3g_4 \]
\[ - 5904g_2^3g_3^3 - 7776\zeta_3g_1^2g_3^4 + 8374N g_1^2g_3^4 + 3040g_1^2g_3^4 - 704N g_1g_2^4g_3^4 + 20736\zeta_3g_1^2g_2^3g_4^2 + 17600g_1^2g_2^3g_4^2 - 5184\zeta_3g_1^2g_2^3g_4^2 + 10532g_1^2g_2^3g_4^2 \]
\[ - 2250g_1^2g_2^3g_4^2 + 15552\zeta_3g_1^2g_2^3g_4^2 + 7128g_1^2g_2^3g_4^2 - 3576N g_1g_2^3g_4^2 \]
\[ - 10368\zeta_3g_1^2g_2^3g_4^2 + 3024g_2^4g_3^2 + 5184\zeta_3g_1^2g_2^3g_4^2 - 3672g_1^2g_2^3g_4^2 \]
\[ + 984\zeta_3g_1^2g_2^3g_4^2 - 5184\zeta_3g_2^3g_4^2 + 4968g_2^3g_4^2 + 300N g_3^4g_1^4 \]
\[ - 5184\zeta_3g_3^4g_1^2 + 5856g_3^4g_1^2 - 672g_3^2 + O(g_5^8) \]
\[ \gamma_{22}(g_i)_{|m=2} = \frac{1}{12} \left[ -2N g_1^2 + 5g_2^2 - 2g_1 \right] + \frac{1}{216} \left[ -160N_1 g_1^2 - 60N g_1^2 g_2 + 52N g_1^2 g_2^2 + 2N g_1^2 g_3^2 + 48N g_1 g_2^2 g_4 - 97g_2^4 \\
+ 52g_2^2 g_3^2 - 60g_2 g_3^3 - 11g_3^2 g_4^2 - 160g_4^4 \right] + \frac{1}{62208} \left[ -82472N_1 g_1^6 + 10368\zeta_3 N g_1^6 + 55600N_1 g_1^6 + 45216N^2 g_1^6 g_2 \\
+ 124416\zeta_3 N g_1^5 g_2 - 28128N g_1^5 g_2^2 - 4740N^2 g_1^5 g_2^2 + 57024\zeta_3 N g_1^5 g_2^3 \\
+ 308076N g_1^4 g_2^2 + 20736\zeta_3 N g_1^4 g_2^3 + 38480N g_1^4 g_2^4 - 33368N g_1^4 g_2^4 \\
- 62208\zeta_3 N g_1^3 g_2^4 - 22992N g_1^3 g_2^5 + 45888N g_1^3 g_2^6 + 45216N g_1^3 g_2^6 g_4 \\
+ 124416\zeta_3 N g_1^3 g_2^6 g_4 - 24672N g_1^3 g_2^7 g_4 - 98208N g_1^3 g_2^7 g_4 - 19768N g_1^5 g_2^8 \\
+ 6592N g_1^5 g_2^9 g_4 - 9408N g_1^5 g_2^9 g_4 - 155520\zeta_3 N g_1^5 g_2^9 g_4 \\
+ 174384N g_1^2 g_2^9 g_4 + 45216N g_1^3 g_2^9 g_4 - 5524N g_2^2 g_4^3 - 3776N g_2^3 g_4^3 \\
+ 259200\zeta_3 N g_1^3 g_2^9 g_4 - 51776N g_1^3 g_2^10 g_4 - 33368N g_1^3 g_2^10 g_4 \\
- 24000N g_1^5 g_2^10 g_4 + 3312N g_1 g_2^11 g_4 + 2304N^2 g_1 g_2^11 g_4 - 672N g_1 g_2^11 g_4 \\
- 62208\zeta_3 N g_1^2 g_2^11 g_4 + 17952N g_1^2 g_2^12 g_4 + 18144\zeta_3 g_2^2 g_4 + 52225g_2^6 \\
- 19768g_2^2 g_4^2 - 62208\zeta_3 g_3^2 g_4^3 + 22992g_2^3 g_4^3 + 9296N g_2^3 g_4^3 \\
+ 57024\zeta_3 g_3^2 g_4^4 + 303336g_2^4 g_4^4 + 24600N g_2^4 g_4^4 + 124416\zeta_3 g_2^5 g_4^5 \\
+ 17088g_2^5 g_4^5 + 6N^2 g_2^1 g_4^2 - 8408N g_2^2 g_4^2 - 1840N g_2^3 g_4^2 \\
+ 10368\zeta_3 g_2^4 g_4 - 26872g_2^6 g_4^5 + O(g_8) \right] \]

\[ \gamma_{23}(g_i)_{|m=2} = \frac{g_4^2}{2} + \frac{1}{144} \left[ -54N g_1^4 g_3^2 + 28N g_1^4 g_3^2 g_4 - 36N g_1 g_2^2 g_4 - 4g_2^2 g_4^2 - 108g_2 g_4^3 + 7N g_3^2 g_4^2 \\
- 12g_4^4 \right] + \frac{1}{10368} \left[ -2712N^2 g_1^2 g_3^2 + 6000N g_1^2 g_3^2 - 396N^2 g_1 g_3^2 g_4 - 5184\zeta_3 N g_1^2 g_3^2 g_4 \\
+ 10928N g_1^2 g_3^2 g_4 - 6480N g_1^2 g_2^2 g_4 - 5904N g_1^2 g_2^2 g_4 + 2256N^2 g_1^2 g_2^2 g_4 \\
+ 20736\zeta_3 g_3^3 g_4^3 - 14400N g_1^3 g_3^2 g_4 - 10368\zeta_3 N g_1^3 g_3^2 g_4 + 7344N g_1^3 g_3^2 g_4 \\
+ 912N g_1^2 g_3^2 g_4 - 500N g_1 g_3^2 g_4^2 + 15552\zeta_3 N g_1 g_3^2 g_4^2 + 8424N g_1^2 g_4^2 g_4 \\
- 3888N g_1^2 g_4^2 g_4 - 1620N^2 g_1^2 g_4^2 g_4 - 5184\zeta_3 N g_1^2 g_4^2 g_4 + 5196N g_1^2 g_4^2 g_4 \\
- 132N^2 g_1^3 g_3^2 g_4 + 2592\zeta_3 N g_1^3 g_3^2 g_4 + 37030N g_1^3 g_3^2 g_4^2 - 1468N g_1^3 g_3^2 g_4^2 \\
- 2112N g_1 g_3 g_2^2 g_4 + 5184\zeta_3 N g_1 g_3 g_2^2 g_4 + 5616N g_1 g_3 g_2^2 g_4 \\
+ 564N^2 g_1 g_3 g_2^2 g_4 - 10368\zeta_3 N g_1 g_3 g_2^2 g_4 + 3744N g_1 g_3 g_2^2 g_4 - 8064N g_1 g_3 g_2^2 g_4 \\
- 2592\zeta_3 g_1 g_3 g_2^2 g_4 + 2801N g_2^2 g_4^2 + 5184\zeta_3 g_2^2 g_4^2 + 864g_2^2 g_4^2 + 216N g_2^2 g_3^2 g_4 \\
+ 10368\zeta_3 g_2^2 g_3^2 g_4 + 9554g_2^2 g_4^3 - 1188N g_2 g_3^2 g_4 + 7740g_2 g_4^5 - 33N g_4 g_3 g_4^2 \\
+ 1256N g_4^2 g_4^2 + 632N g_4^2 g_4^2 - 10368\zeta_3 g_4^2 g_4^2 + 20676g_4^6 \right] + O(g_8) \]

\[ \gamma_{31}(g_i)_{|m=2} = \frac{g_3^2}{2} + \frac{1}{72} \left[ -20g_1^2 g_4^2 + 18g_1^2 g_4^2 - 108g_1 g_2^2 g_4 + 7N g_3^4 - 2g_3^4 + 5g_3^2 g_4^2 \right] + \frac{1}{10368} \left[ -5184\zeta_3 N g_1^4 g_3^2 + 9404N g_1^4 g_3^2 + 5184\zeta_3 g_1^4 g_3^2 + 3476g_1^4 g_3^2 + 2256N g_1^4 g_3^2 \\
- 1464g_1^4 g_2^2 + 3708g_1^4 g_2^2 - 5184g_1^4 g_2^2 - 7152N g_1^4 g_2^2 \\
+ 27648g_1^4 g_2^2 + 20736\zeta_3 g_1^4 g_2^2 + 3456g_1^4 g_2^2 - 374g_1^4 g_2^2 + 5016g_1^4 g_2^2 \\
+ 10368\zeta_3 g_1^4 g_2^2 g_4 + 9504g_1^4 g_2^2 g_4 - 10368\zeta_3 g_1^4 g_2^2 g_4 + 9936g_1^4 g_2^2 g_4 \right] \]
\[ \gamma_{32}(g_i)_{m=2} = g_i^2 + \frac{1}{24} \left[ -18N_1g_i^2 - 12Ng_1g_3 - 6g^2_2g_4^2 - 36g_2g_4^3 + 7N_3g_3^2 - 4g^4_1 \right] \]
\[ + \frac{1}{864} \left[ 1010N_1g_1^2g_4^2 + 756N_1g_1^4g_4^2 + 1728\zeta_3N_1g_1^2g_2g_3^2 - 900N_1g_1^2g_2g_3^3 \right. \]
\[ - 1080N_1g_1^3g_2g_3^2 + 1560Ng_1^3g_2g_3g_4 + 1512N_1g_1^3g_3^2 + 332\zeta_3N_1g_1^2g_2g_3^2 \]
\[ + 846N_1g_1^2g_2g_3^2 + 282N_1g_1^2g_2g_3g_4 + 2592\zeta_3N_1g_1^2g_2g_3g_4 + 2268N_1g_1^2g_2g_3g_4 \]
\[ - 904N_1g_1^2g_2g_3^3 - 496N_1g_1^2g_3^4 - 178N_1g_1^2g_3^4 + 3911N_1g_1^2g_3^4 + 40N_1g_1^2g_3^4 \]
\[ - 216N_1g_1^2g_2g_3^3 - 1782N_1g_1^2g_2g_3g_4 + 792N_1g_1^2g_2g_3g_4 + 282N_1g_1^2g_3^4 \]
\[ - 492N_1g_1^2g_2g_3^3 - 1344N_1g_1^2g_2g_3g_4 + 864\zeta_3g_3^2g_4^3 + 1420g_2g_3^4 \]
\[ - 204N_1g_1^2g_2g_3^3 + 864g_3^3g_4^3 + 1712g_3^2g_4^3 - 70N_1g_1^2g_2g_3^3 - 162g_2g_3^4 \]
\[ - 33N_1^2g_3^2g_4^2 + 1276Ng_3^2g_3^2g_4^2 - 37Ng_3^2g_3^2g_4^2 + 3734g_4^6 \right] + O(g_1^8) \\
\[ \gamma_{33}(g_i)_{m=2} = \frac{1}{12} \left[ -Ng_3^2 + 4g_1^2 \right] \]
\[ + \frac{1}{432} \left[ 2N_1g_1^2g_3^2 + 20Ng_1^2g_3^2 - 12Ng_1g_3g_4 + 10g_2g_4^2 - 60g_2g_4^3 - 76Ng_3^4 \right. \]
\[ + 31Ng_3^2g_4^2 - 356g_4^2 \right] \]
\[ + \frac{1}{31104} \left[ -206N_1^2g_1^2g_3^2 + 2592\zeta_3N_1g_1^2g_3^2 - 4280N_1g_1^2g_3g_4 + 344N_1^2g_1g_4^2 + 968Ng_1^4g_4^2 \right. \]
\[ + 1200N_1g_1^3g_2g_3^3 - 2064Ng_1^3g_2g_3g_4 + 4536N_1^2g_1^3g_3g_4 - 12144N_1g_1^3g_3g_4 \]
\[ - 8280N_1g_1^2g_2g_3^3 - 103N_1g_1^2g_2g_3g_4 + 656N_1g_1^2g_2g_3g_4 - 10368\zeta_3N_1g_1^2g_2g_3g_4 \]
\[ + 532N_1g_1^2g_2g_3g_4 - 3156N_1g_1^2g_2g_3g_4 - 7416N_1^2g_1^2g_3g_4 + 13436N_1g_1^2g_3g_4 \]
\[ - 442N_1g_1^2g_2g_3g_4 + 113644N_1g_1^2g_3g_4 + 2152N_1g_1^2g_3g_4 + 2268N_1g_1^2g_3g_4 \]
\[ - 7308N_1g_1^2g_2g_3g_4 + 3960N_1^2g_1^2g_3g_4 + 31104\zeta_3N_1g_1^2g_3g_4 \]
\[ - 16680N_1g_1^2g_3^2g_4^2 - 31104\zeta_3N_1g_1^2g_3^2g_4^2 - 15216N_1g_1^2g_3g_4^2 + 162g_2g_3^4 \]
\[ - 5718g_2g_3^4 - 221Ng_2g_3^2g_4^2 - 10368\zeta_3g_2g_3^2g_4^2 + 5278g_2g_3^2g_4^2 \]
\[ + 1080Ng_2g_3^2g_4^2 + 3108g_2g_3^2g_4^2 - 3292N_2^2g_3^2g_4^2 + 18144\zeta_3N_1g_3^2g_4^2 - 1214N_1g_3^2g_4^2 \]
\[ - 284N_2g_3^2g_4^2 + 31104\zeta_3N_1g_3^2g_4^2 + 24248N_2g_3^2g_4^2 - 10644N_2g_3^2g_4^2 \]
\[ + 67392\zeta_3g_3^6 + 53200g_3^6 \right] + O(g_1^8) . \tag{A.2} \]

**B Complex Solutions.**

In this appendix we provide the remaining spectrum of fixed points for \( N = 1000 \) and \( m = 2 \) as an example of the mix of purely real, purely imaginary or fully complex fixed points. As in the main text we exclude solutions related by symmetries. First, there are five sets of complex fixed points of the CS type. The first is at

\[
x = (0.030988i - 0.117059) + (0.225504 - 0.026447i)e + (0.141512 + 0.0220556i)e^2 + O(e^3)
\]

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\[ y = (10.616979i + 7.410016) + (-0.306607 + 5.10124i)\epsilon + (-3.30970 + 5.02141i)\epsilon^2 + O(\epsilon^3) \]
\[ z = (0.041381i + 0.950398) + (0.073839 - 0.000086i)\epsilon + (-0.076593 + 0.240769i)\epsilon^2 + O(\epsilon^3) \]
\[ t = (-3.598246i + 9.770473) + (2.741560 - 4.671500i)\epsilon + (17.262953 - 21.70028i)\epsilon^2 + O(\epsilon^3) \]
\[ w \in \{(12.52843i - 8.437892) + (-0.081315 + 6.488068i)\epsilon + (79.099646 + 19.016683i)\epsilon^2 + O(\epsilon^3), \\
(−8.157796i + 0.215249) + (10.458651 + 2.563357i)\epsilon + (−13.448748 + 143.623375i)\epsilon^2 + O(\epsilon^3), \\
(−4.370634i + 8.222643) + (−1.841641 - 7.930799i)\epsilon + (5.191836 - 47.804171i)\epsilon^2 + O(\epsilon^3)\} \]

where we have grouped three solutions together given that the only difference is in the location of the \( w \) coupling. The remainder are

\[ x = (-0.003823i + 1.005167) + (-0.054618 + 0.012486i)\epsilon + (0.018976 + 0.143820i)\epsilon^2 + O(\epsilon^3) \]
\[ y = (2.003185i + 18.658117) + (5.392312 + 1.086468i)\epsilon + (14.851433 - 19.827819i)\epsilon^2 + O(\epsilon^3) \]
\[ z = (0.008575i + 1.025642) + (-0.085717 - 0.009151i)\epsilon + (-0.006651 - 0.017253i)\epsilon^2 + O(\epsilon^3) \]
\[ t = (-4.466842i + 8.052554) + (4.210604 + 3.034436i)\epsilon + (8.671652 + 12.119131i)\epsilon^2 + O(\epsilon^3) \]
\[ w \in \{(15.494072i - 6.618774) + (6.586489 - 2.039527i)\epsilon + (160.338535 - 22.392846i)\epsilon^2 + O(\epsilon^3), \\
(−2.718385i + 6.100016) + (3.357728 + 1.279947i)\epsilon + (12.015204 + 7.083669i)\epsilon^2 + O(\epsilon^3), \\
(−12.775687i + 0.518757) + (0.842674 + 1.030180i)\epsilon + (56.581857 + 57.905307i)\epsilon^2 + O(\epsilon^3)\} \]
\[
(-3.266530i - 2.690567) + (1.337926 + 0.955634i) \epsilon + (33.890419 + 26.868026i) \epsilon^2 + O(\epsilon^3)
\]

\[\begin{align*}
x &= (-0.555294i - 1.513381) + (-23.114452 + 24.950148i) \epsilon \\
&\quad + (3858.041480 + 2762.998578i) \epsilon^2 + O(\epsilon^3) \\
y &= (34.90363i - 21.842603) + (890.252852 + 680.087930i) \epsilon \\
&\quad + (74550.859506 - 131881.754919i) \epsilon^2 + O(\epsilon^3) \\
z &= (0.549418i + 1.214497) + (24.255203 - 14.435870i) \epsilon \\
&\quad + (-2470.779282 - 3096.675176i) \epsilon^2 + O(\epsilon^3) \\
t &= (-29.55207i + 15.18796) + (-850.256009 - 539.584442i) \epsilon \\
&\quad + (-56438.610729 + 124039.167626i) \epsilon^2 + O(\epsilon^3) \\
w &\in \{(42.91263i - 23.15191) + (1127.488695 + 813.413116i) \epsilon \\
&\quad + (82884.915451 - 172830.465074i) \epsilon^2 + O(\epsilon^3), \\
(0.409667i + 1.270019) + (35.031670 + 22.655513i) \epsilon \\
&\quad + (1486.251233 - 4603.586046i) \epsilon^2 + O(\epsilon^3), \\
(43.3229i + 21.88189) + (-1155.605341 - 813.415117i) \epsilon \\
&\quad + (-84174.229119 + 179594.341857i) \epsilon^2 + O(\epsilon^3)\}\} \tag{B.5}
\]

and
\[
\begin{align*}
x &= (-0.006320i - 1.030086) + (0.107095 + 0.043240i) \epsilon + (-0.036554 + 0.122859i) \epsilon^2 \\
&\quad + O(\epsilon^3) \\
y &= (2.138785i - 9.592803) + (-5.961631 - 11.215238i) \epsilon + (-22.616924 - 62.894519i) \epsilon^2 \\
&\quad + O(\epsilon^3) \\
z &= (0.002767i + 1.0176108) + (-0.059037 - 0.021032i) \epsilon + (0.043635 - 0.043068i) \epsilon^2 \\
&\quad + O(\epsilon^3) \\
t &= (1.198860i - 8.255330) + (-2.187227 - 7.108184i) \epsilon + (-8.358400 - 32.699994i) \epsilon^2 \\
&\quad + O(\epsilon^3) \\
w &\in \{(13.36369i - 4.799020) + (10.845688 - 1.270978i) \epsilon + (139.104437 - 28.027224i) \epsilon^2 \\
&\quad + O(\epsilon^3), \\
(-12.50645i - 2.865408) + (-1.424629 - 4.030314i) \epsilon + (67.742495 + 17.073470i) \epsilon^2 \\
&\quad + O(\epsilon^3), \\
(-0.857238i + 7.664428) + (1.116387 + 5.387238i) \epsilon + (4.573636 + 22.217349i) \epsilon^2 \\
&\quad + O(\epsilon^3)\}\} \tag{B.6}
\]

There were several sets where some of the fixed points were either real or imaginary in addition to one being fully complex since we found the solutions
\[
\begin{align*}
x &= 0.114419i - 0.553587i \epsilon + 6.740110i \epsilon^2 + O(\epsilon^3) \\
y &= 22.486625i - 31.203475i \epsilon + 603.274688i \epsilon^2 + O(\epsilon^3) \\
z &= 1.070456 - 0.504440 \epsilon + 9.518801 \epsilon^2 + O(\epsilon^3) \\
t &= -11.601013i + 36.808098 \epsilon - 722.589642i \epsilon^2 + O(\epsilon^3) \\
w &\in \{(22.427674i - 1.617314) + (1.687428 - 38.335633i) \epsilon \\
&\quad + (255.539793 + 718.517860i) \epsilon^2 + O(\epsilon^3), \\
3.234627 + 8.891244 \epsilon - 144.067750 \epsilon^2 + O(\epsilon^3)\}\} \tag{B.8}
\]
and

\begin{align*}
  x &= -0.868555 - 0.203744\epsilon - 0.915849\epsilon^2 + O(\epsilon^3) \\
  y &= 19.811433 + 7.966436\epsilon - 15.205442\epsilon^2 + O(\epsilon^3) \\
  z &= 1.012581 - 0.020942\epsilon + 0.066905\epsilon^2 + O(\epsilon^3) \\
  t &= -4.342552 - 2.231269\epsilon - 7.303390\epsilon^2 + O(\epsilon^3) \\
  w &= (15.014668i - 2.955662) + (4.805741 + 1.917129i)\epsilon \\
  & \quad + (111.835693 - 52.975405i)\epsilon^2 + O(\epsilon^3) .
\end{align*}

(B.9)

This completes the set of all CS type solutions in addition to those in section 6.

For the remaining solutions we found at least one of the critical couplings was zero. First we group those solutions where the couplings are either real or imaginary. We found

\begin{align*}
  x &= 0 \ , \ t = 0 \\
  y &= 14.907120i + 11.502407i\epsilon - 10.399304i\epsilon^2 + O(\epsilon^3) \\
  z &= 0.998006 + 0.007339\epsilon - 0.009793\epsilon^2 + O(\epsilon^3) \\
  w &\in \{(15.559942i - 2.659423) + (4.519271 + 2.244641i)\epsilon \\
  & \quad + (110.184708 - 62.342815i)\epsilon^2 + O(\epsilon^3), \\
  & \quad 5.318846 + 0.901757\epsilon - 1.582315\epsilon^2 + O(\epsilon^3)\} \\
\end{align*}

(B.10)

and

\begin{align*}
  x &= 0 \ , \ z = 0 \ , \ t = 0 \\
  y &= 14.907120i + 11.502407i\epsilon - 10.399304i\epsilon^2 + O(\epsilon^3) \\
  w &\in \{10.540926 + 8.344899\epsilon - 16.563109\epsilon^2 + O(\epsilon^3), \ 0\} .
\end{align*}

(B.11)

Included in the first set is a complex \(w\) coupling. However, the second solution of each set are examples which are similar to pure \(\phi^3\) theory when its coupling is purely imaginary. That particular \(O(N)\) model described the Lee-Yang edge singularity problem. \cite{60}. Also the solutions of (B.11) correspond to the Lagrangian without a \(\phi^a\) field. In the case of the only non-zero coupling \(y\) this is the pure cubic theory involving only the \(\sigma\) field. The remaining solutions with any complex roots all have a vanishing critical \(z\) coupling and are

\begin{align*}
  x &= (0.102517i - 1.069257) + (-0.187674 - 0.485282i)\epsilon \\
  & \quad + (6.388101 + 3.227418i)\epsilon^2 + O(\epsilon^3) \\
  y &= (8.388316i + 4.182323) + (-31.690792 - 0.719078i)\epsilon \\
  & \quad + (394.496262 - 256.273660i)\epsilon^2 + O(\epsilon^3) \\
  z &= 0 \\
  t &= (-12.684990i - 5.204227) + (23.207111 + 4.566306i)\epsilon \\
  & \quad + (-270.175902 + 154.130691i)\epsilon^2 + O(\epsilon^3) \\
  w &\in \{(13.394284i + 8.214396) + (-30.663870 - 17.742257i)\epsilon \\
  & \quad + (479.083975 - 270.124833i)\epsilon^2 + O(\epsilon^3), \ 0\}
\end{align*}

(B.12)

and

\begin{align*}
  x &= 0 \ , \ z = 0 \\
  y &= (12.918644i + 4.311494) + (2.006636 + 9.961474i)\epsilon
\end{align*}

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and solutions we have which together with the other real solutions correspond to saddle point structures. For the paired
an internal consistency in our analysis of solution. The remaining unpaired solutions are
Lagrangians emerge similar to (B.11) ought not to come as a surprise and should be regarded as
pair of fields \( \{ \phi^{ia}, \sigma \} \) ought not to come as a surprise and should be regarded as
theory which has a zero \( \beta \)-function. That such solutions representing the sum of independent
of all the real solutions we record we found only the first two correspond to stable fixed points
which are
\[
x = 1.011102 - 0.026162\epsilon + 0.022238\epsilon^2 + O(\epsilon^3)
y = 6.637801 - 1.117476\epsilon + 0.962982\epsilon^2 + O(\epsilon^3)
z = 0 , \quad t = 0
w = 10.540926 + 8.344899\epsilon - 16.563109\epsilon^2 + O(\epsilon^3)
\] (B.14)
and
\[
x = 0 , \quad y = 0 , \quad z = 0 , \quad t = 0
w = 10.540926 + 8.344899\epsilon - 16.563109\epsilon^2 + O(\epsilon^3)
\] (B.15)
The final solution corresponds to the pure \( T^{ab} \) theory when \( m = 2 \) but the \( (0,0,0,0,w) \) structure
could be analysed in isolation for arbitrary \( m \). However, the stability of these two solutions, in contrast to the remaining real solutions which are not stable, appears to be driven by the
vanishing of the couplings \( g_2 \) and \( g_4 \). In this case there is no interaction whatsoever between the
pair of fields \( \{ \phi^{ia}, \sigma \} \) and \( T^{ab} \) which is apparent from (2.4). In other words one is dealing with
with an other set. There were four such cases.

The remainder of the solutions are real but interesting patterns emerge in several cases. First
we record the fixed points where there is no pairing with another set. There were four such cases.
Of all the real solutions we record we found only the first two correspond to stable fixed points
these two solutions, in contrast to the remaining real solutions which are not stable, appears to be driven by the
vanishing of the couplings \( g_2 \) and \( g_4 \). In this case there is no interaction whatsoever between the
pair of fields \( \{ \phi^{ia}, \sigma \} \) and \( T^{ab} \) which is apparent from (2.4). In other words one is dealing with a
partitioned Lagrangian and the coupling constant space is also partitioned. So the stability here
is a reflection of the stability of the two separate Lagrangians. In the second of these two solutions
the situation is effectively trivial since it reflects that one of the two Lagrangians is a free field
theory which has a zero \( \beta \)-function. That such solutions representing the sum of independent
Lagrangians emerge similar to (B.11) ought not to come as a surprise and should be regarded as
an internal consistency in our analysis of solution. The remaining unpaired solutions are
\[
x = -0.854446 - 0.212078\epsilon - 0.647751\epsilon^2 + O(\epsilon^3)
y = 18.789145 + 9.197094\epsilon - 2.733818\epsilon^2 + O(\epsilon^3)
z = 0 , \quad t = 0
w = 10.540926 + 8.344899\epsilon - 16.563109\epsilon^2 + O(\epsilon^3)
\] (B.16)
and
\[
x = 0.983210 - 0.011253\epsilon + 0.063259\epsilon^2 + O(\epsilon^3)
y = 16.345805 + 10.658027\epsilon + 21.524495\epsilon^2 + O(\epsilon^3)
z = 0 , \quad t = 0
w = 10.540926 + 8.344899\epsilon - 16.563109\epsilon^2 + O(\epsilon^3)
\] (B.17)
which together with the other real solutions correspond to saddle point structures. For the paired
solutions we have
\[
x = 0.986386 + 0.006824\epsilon + 0.023890\epsilon^2 + O(\epsilon^3)
\]
\[ y = 3.882413 + 3.856888\epsilon + 1.139387\epsilon^2 + O(\epsilon^3) \]
\[ z = 0 \]
\[ t = -6.810006 + 3.229867\epsilon + 5.755896\epsilon^2 + O(\epsilon^3) \]
\[ w \in \{13.726061 + 3.729174\epsilon - 29.707955\epsilon^2 + O(\epsilon^3), 0\} \quad (B.18) \]

and

\[ x = 0, \quad z = 0 \]
\[ y = 17.222217 + 37.874140\epsilon + 404.846200\epsilon^2 + O(\epsilon^3) \]
\[ t = -13.657731 - 33.103835\epsilon - 376.517213\epsilon^2 + O(\epsilon^3) \]
\[ w \in \{20.542650 + 42.608899\epsilon + 533.442168\epsilon^2 + O(\epsilon^3), 0\} . \quad (B.19) \]

The final three pairings exhibit a novel feature in that in each set the critical \( x \) and \( t \) couplings are equal. This is clear since we found

\[ x = -0.854046 - 0.211934\epsilon - 0.647273\epsilon^2 + O(\epsilon^3) \]
\[ y = 18.789012 + 9.196991\epsilon - 2.731101\epsilon^2 + O(\epsilon^3) \]
\[ z = 0 \]
\[ t = -0.854046 - 0.211934\epsilon - 0.647273\epsilon^2 + O(\epsilon^3) \]
\[ w \in \{-10.598432 + 8.327096\epsilon - 16.655991\epsilon^2 + O(\epsilon^3), 0\} \quad (B.20) \]

and

\[ x = 1.010586 - 0.0261234\epsilon + 0.022211\epsilon^2 + O(\epsilon^3) \]
\[ y = 6.633618 - 1.114926\epsilon + 0.960725\epsilon^2 + O(\epsilon^3) \]
\[ z = 0 \]
\[ t = 1.010586 - 0.026124\epsilon + 0.022211\epsilon^2 + O(\epsilon^3) \]
\[ w \in \{10.616993 + 8.279650\epsilon - 16.867303\epsilon^2 + O(\epsilon^3), 0\} . \quad (B.22) \]

While our focus here was on the \( O(1000) \times O(2) \) theory it represents a snapshot of the spectrum of potential solutions for the general symmetry group. What has emerged are real fixed points in addition to the Heisenberg, AU and CS type which were motivated by the four dimensional theory.

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