REPRESENTATIONS OF THE AFFINE-VIRASORO ALGEBRA OF TYPE $A_1$

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Abstract. In this paper, we classify all irreducible weight modules with finite-dimensional weight spaces over the affine-Virasoro Lie algebra of type $A_1$.

1. Introduction

It is well known that the affine Lie algebras and the Virasoro algebra have been widely used in many physics areas and mathematical branches, and the Virasoro algebra served as an outer-derivative subalgebra plays a key role in representation theory of the affine Lie algebras. Their close relationship strongly suggests that they should be considered simultaneously, i.e., as one algebraic structure. Actually it has led to the definition of the so-called affine-Virasoro algebra [13, 16], which is the semidirect product of the Virasoro algebra and an affine Kac-Moody Lie algebra with a common center. Affine-Virasoro algebras sometimes are much more connected to the conformal field theory. For example, the even part of the $N = 3$ superconformal algebra is just the affine-Virasoro algebra of type $A_1$. Highest weight representations and integrable representations of the affine-Virasoro algebras have been studied in several papers (see [13, 8, 16, 12, 17, 19, 27, 28], etc.). All irreducible Harish-Chandra modules (weight modules with finite-dimensional weight spaces) with nonzero central actions the affine-Virasoro algebras were classified in [2]. However, up to now, all irreducible uniform bounded modules over these algebras are not yet classified.

In this paper, we classify all irreducible weight modules with finite-dimensional weight spaces over the affine-Virasoro Lie algebra of type $A_1$. Throughout this paper, $\mathbb{Z}$, $\mathbb{Z}^*$ and $\mathbb{C}$ denote the sets of integers, non-zero integers and complex numbers, respectively. $U(L)$ denote the universal enveloping algebra of a Lie

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algebra $L$. All modules considered in this paper are nontrivial. For any $\mathbb{Z}$-graded space $G$, we also use notations $G_+, G_-, G_0$ and $G_{(p,q)}$ to denote the subspaces spanned by elements in $G$ of degree $k$ with $k > 0$, $k < 0$, $k = 0$ and $p \leq k < q$, respectively.

2. Basics

In this section, we shall introduce some notations of the Virasoro algebra and affine-Virasoro algebras.

2.1. Virasoro algebra and twisted Heisenberg-Virasoro algebra. By definition, the Virasoro algebra $\text{Vir} := \mathbb{C}\{d_m, C \mid m \in \mathbb{Z}\}$ with bracket:

$$[d_m, d_n] = (n - m)d_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} C, \quad [d_m, C] = 0,$$

for all $m, n \in \mathbb{Z}$.

Let $\mathbb{C}[t, t^{-1}]$ be the Laurent polynomials ring over $\mathbb{C}$, then $\text{Der} \mathbb{C}[t, t^{-1}] = \mathbb{C}\{t^{m+1} \frac{d}{dt} \mid m \in \mathbb{Z}\}$ (also denote by $\text{Vect}(S^1)$, the Lie algebra of all vector fields on the circle).

$$\text{Vir} = \text{Der} \mathbb{C}[t, t^{-1}].$$

The twisted Heisenberg-Virasoro algebra $\mathcal{H}$ was first studied by Arbarello et al in [1], where a connection is established between the second cohomology of certain moduli spaces of curves and the second cohomology of the Lie algebra of differential operators of order at most one. By definition, $\mathcal{H}$ is the universal central extension of the following Lie algebra $\mathcal{D}$, which is the Lie algebra of differential operators order at most one.

**Definition 2.1.** As a vector space over $\mathbb{C}$, the Lie algebra $\mathcal{D}$ has a basis $\{d_n, Y_n \mid n \in \mathbb{Z}\}$ with the following relations

$$[d_m, d_n] = (n - m)d_{m+n}, \quad [d_m, Y_n] = nY_{m+n}, \quad [Y_m, Y_n] = 0,$$

for all $m, n \in \mathbb{Z}$.

Clearly, the subalgebra $H = \mathbb{C}\{Y_m \mid m \in \mathbb{Z}\}$ of $\mathcal{D}$ is centerless Heisenberg algebra and $W = \mathbb{C}\{d_m \mid m \in \mathbb{Z}\}$ is the Witt algebra (or centerless Virasoro algebra).
2.2. Affine-Virasoro algebra.

**Definition 2.2.** Let $L$ be a finite-dimensional Lie algebra with a non-degenerated invariant normalized symmetric bilinear form $(\cdot, \cdot)$, then the affine-Virasoro Lie algebra is the vector space

$$L_{av} = L \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C \oplus \bigoplus_{i \in \mathbb{Z}} \mathbb{C}d_i,$$

with Lie bracket:

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m(x, y)\delta_{m+n,0}C,$$

$$[d_i, d_j] = (j-i)d_{i+j} + \frac{1}{12}(j^3 - j)\delta_{i+j,0}C,$$

$$[d_i, x \otimes t^m] = mx \otimes t^{m+i}, \quad [C, L_{av}] = 0,$$

where $x, y \in L$, $m, n, i, j \in \mathbb{Z}$ (if $L$ has no such form, we set $(x, y) = 0$ for all $x, y \in L$).

**Remark:** If $L = \mathbb{C}e$ is one dimensional, then $L_{av}$ is just the twisted Heisenberg-Virasoro algebra (one center element).

Now we only consider specially $L$ as the simple Lie algebra $\mathfrak{sl}_2 = \mathbb{C}\{e, f, h\}$. Then by Defintion 2.2, the corresponding affine-Virasoro algebra $\mathcal{L} := L_{av} = \mathbb{C}\{e_i, f_i, h_i, d_i, C \mid i \in \mathbb{Z}\}$, with Lie bracket:

$$[e_i, f_j] = h_{i+j} + i\delta_{i+j,0}C,$$

$$[h_i, e_j] = 2e_{i+j}, \quad [h_i, f_j] = -2f_{i+j},$$

$$[d_i, d_j] = (j-i)d_{i+j} + \frac{1}{12}(j^3 - j)\delta_{i+j,0}C,$$

$$[d_i, h_j] = jh_{i+j}, \quad [h_i, h_j] = 2i\delta_{i+j,0}C,$$

$$[d_i, e_j] = je_{i+j}, \quad [d_i, f_j] =jf_{i+j}, \quad [C, \mathcal{L}] = 0,$$

where $i, j \in \mathbb{Z}$.

**Remark.** In fact, $\mathcal{L}$ is the even part of the $N = 3$ superconformal algebra ([5]).

The Lie algebra $\mathcal{L} = \bigoplus_{i \in \mathbb{Z}} \mathcal{L}_i$ is a $\mathbb{Z}$-graded Lie algebra, where $\mathcal{L}_i = \mathbb{C}\{d_i, e_i, f_i, h_i, \delta_{0,i}C\}$.

The subalgebra $\mathcal{H}_X := \mathbb{C}\{X_i, d_i, C \mid i \in \mathbb{Z}\}$ for $X = e, f, h$ of $\mathcal{L}$ is isomorphic to the twisted Heisenberg-Virasoro algebra $\mathcal{H}$ (the only difference is the center element). Clearly $\mathfrak{h} := \mathbb{C}h_0 + \mathbb{C}C + \mathbb{C}d_0$ the Cartan subalgebra of $\mathcal{L}$. 
2.3. Harish-Chandra modules. For any \( L \)-module \( V \) and \( \lambda, \mu \in \mathbb{C} \), set \( V_{\lambda,\mu} := \{ v \in V \mid d_0v = \lambda v, h_0v = \mu v \} \), which is generally called the weight space of \( V \) corresponding to the weight \( \lambda, \mu \).

An \( L \)-module \( V \) is called a weight module if \( V \) is the sum of all its weight spaces.

A nontrivial irreducible weight \( L \)-module \( V \) is called of intermediate series if all its weight spaces are one-dimensional.

A weight \( L \)-module \( V \) is called a highest (resp. lowest) weight module with highest weight (resp. lowest weight) \( \lambda, \mu \in \mathbb{C} \), if there exists a nonzero weight vector \( v \in V_{\lambda,\mu} \) such that

1) \( V \) is generated by \( v \) as \( L \)-module;

2) \( L^+ v = 0 \) (resp. \( L^- v = 0 \)), where \( L^+ = L_+ + \mathbb{C} e_0 \) and \( L^- = L_- + \mathbb{C} f_0 \) (the notations \( L_+ = \sum_{i \geq 1} \mathcal{L}_i \), \( L_- = \sum_{i \leq -1} \mathcal{L}_i \) are introduced in Section 1).

If, in addition, all weight spaces \( V_{\lambda,\mu} \) of a weight \( L \)-module \( V \) are finite-dimensional, the module \( V \) is called a Harish-Chandra module. Clearly, a highest (lowest) weight module is a Harish-Chandra module.

For a weight module \( V \), we define

\[
\text{Supp}(V) := \{ \lambda \in \mathbb{C} \mid V_{\lambda} = \bigoplus_{\mu \in \mathbb{C}} V_{\lambda,\mu} \neq 0 \}. \tag{2.5}
\]

Obviously, if \( V \) is an irreducible weight \( L \)-module, then there exists \( \lambda \in \mathbb{C} \) such that \( \text{Supp}(V) \subset \lambda + \mathbb{Z} \). So \( V = \bigoplus_{i \in \mathbb{Z}} V_i \) is a \( \mathbb{Z} \)-graded module, where \( V_i = V_{\lambda+i} \).

Kaplansky-Santharoubane [15] in 1983 gave a classification of \( \text{Vir} \)-modules of the intermediate series. There are three families of indecomposable modules with each weight space is one-dimensional:

1) \( \mathcal{A}_{a,b} = \sum_{i \in \mathbb{Z}} \mathbb{C} v_i; \quad d_m v_i = (a + i + bm)v_{m+i}; \)

2) \( \mathcal{A}(a) = \sum_{i \in \mathbb{Z}} \mathbb{C} v_i; \quad d_m v_i = (i + m)v_{m+i} \) if \( i \neq 0 \), \( d_m v_0 = m(m+a)v_m; \)

3) \( \mathcal{B}(a) = \sum_{i \in \mathbb{Z}} \mathbb{C} v_i; \quad d_m v_i = iv_{m+i} \) if \( i \neq -m \), \( d_m v_{-m} = -m(m+a)v_0 \), for some \( a, b \in \mathbb{C} \), where \( C \) acts trivially on the above modules.

It is well-known that \( \mathcal{A}_{a,b} \cong \mathcal{A}_{a+1,b}, \forall a, b \in \mathbb{C} \), then we can always suppose that \( a \notin \mathbb{Z} \) or \( a = 0 \) in \( \mathcal{A}_{a,b} \). Moreover, the module \( \mathcal{A}_{a,b} \) is simple if \( a \notin \mathbb{Z} \) or \( b \neq 0, 1 \). In the opposite case the module contains two simple subquotients namely the trivial module and \( \mathbb{C}[t, t^{-1}]/\mathbb{C} \). It is also clear that \( \mathcal{A}_{0,0} \) and \( \mathcal{B}(a) \) both have \( \mathbb{C} v_0 \) as a submodule, and their corresponding quotients are isomorphic, which we denote by \( \mathcal{A}'_{0,0} \). Dually, \( \mathcal{A}_{0,1} \) and \( \mathcal{A}(a) \) both have \( \mathbb{C} v_0 \) as a quotient module, and their corresponding submodules are isomorphic to \( \mathcal{A}'_{0,0} \). For convenience, we simply write \( \mathcal{A}'_{a,b} = \mathcal{A}_{a,b} \) when \( \mathcal{A}_{a,b} \) is irreducible.
All Harish-Chandra modules over the Virasoro algebra were classified in [22] in 1992. Since then such works were done on the high rank Virasoro algebra in [20] and [26], the Weyl algebra in [25].

**Theorem 2.3.** [22] Let $V$ be an irreducible weight Vir-module with finite-dimensional weight spaces. Then $V$ is a highest weight module, lowest weight module, or Harish-Chandra module of intermediate series.

**Theorem 2.4.** [21] If $V$ is an irreducible weight module with finite-dimensional weight spaces over $D$, then $V$ is a highest or lowest weight module or the Harish-Chandra module of uniformly bounded. Moreover, any uniformly bounded module is a Harish-Chandra module of intermediate series.

**Remarks.** (1) The Harish-Chandra module of the intermediate series over $D$ is induced by $A_{a, b, c} = \sum_{i \in \mathbb{Z}} C v_i$ with $Y_n v_i = cv_{n+i}$ for some $c \in \mathbb{C}$. Denote this module by $A_{a, b, c}$. It is well-known that $A_{a, b, c} \cong A_{a+1, b, c}, \forall a, b, c \in \mathbb{C}$. Moreover, the module $A_{a, b, c}$ is simple if $a \notin \mathbb{Z}$ or $b \neq 0, 1$ or $c \neq 0$. We also use $A'_{a, b, c}$ to denote by the simple subquotient of $A_{a, b, c}$ as in the Virasoro algebra case.

(2) All indecomposable Harish-Chandra modules of the intermediate series over $D$ were classified in [18].

**Lemma 2.5.** Let $V$ be a uniformly bounded weight $D$-module. Then $V$ has an irreducible submodules $V' \cong A'_{a,b,c}$ for some $a, b, c \in \mathbb{C}$.

**Proof.** Consider $V$ as a Vir-module. From representation theory of Vir ([15]), we have $\dim V_{\lambda+n} = p$ for all $\lambda + n \neq 0$. We have a Vir-submodule filtration

$$0 = W^{(0)} \subset W^{(1)} \subset W^{(2)} \subset \cdots \subset W^{(p)} = V,$$

where $W^{(1)}, \cdots, W^{(p)}$ are Vir-submodules of $V$, and the quotient modules $W^{(i)}/W^{(i-1)} \cong A'_{a_i, b_i}$ for some $a_i, b_i \in \mathbb{C}$.

Now any $D$-submodule filtration is also a Vir-submodule filtration and then its length is finite. So $V$ has an irreducible $D$-submodule $V'$, which is also a uniformly bounded. So by Theorem 2.3, $V' \cong A'_{a, b, c}$ for some $a, b, c \in \mathbb{C}$. □

3. **The case of $\dim L = 2$**

Let $T$ be a 2-dimensional nontrivial Lie algebra. Then we can suppose that $T = \mathbb{C}\{h, e\}$ with $[h, e] = 2e$. Clearly, there exists an invariant symmetric bilinear form $(\cdot, \cdot)$ on $T$ given by

$$(h, h) = 2, \ (h, e) = (e, e) = 0.$$
In this case, the Lie algebra $\mathcal{T}_2 := T_{av}$ is generated by \{\(d_n, e_n, h_n, C \mid n \in \mathbb{Z}\}\)

\[
[d_m, d_n] = (n - m)d_{m+n} + \frac{1}{12}(n^3 - n)\delta_{m+n,0}C, \quad [d_m, e_n] = ne_{m+n},
\]

\[
[d_m, h_n] = nh_{m+n}, \quad [h_m, e_n] = 2e_{m+n},
\]

\[
[e_m, e_n] = 0, \quad [h_m, h_n] = 2m\delta_{m+n,0}C,
\]

for all \(m, n \in \mathbb{Z}\).

Clearly, $\mathcal{T}_2$ is a subalgebra of $\mathcal{L}$.

**Proposition 3.1.** Let $V$ be a uniformly bounded irreducible $\mathcal{T}_2$-module. Then $V = \sum \mathbb{C}v_i \cong \mathcal{A}_{a,b,c}$ is the Harish-Chandra module of intermediate series with $h_n v_i = cv_{n+i}$ and $e_n v_i = 0$.

**Proof.** From representation theory of Vir ([15]), we have $C = 0$.

Clearly, $\mathbb{C}\{h_0, e_0\}$ is a 2-dimensional solvable Lie subalgebra of $\mathcal{T}_2$. So we can choose an irreducible $\mathbb{C}\{h_0, e_0\}$-submodule $\mathbb{C}\{v\}$ of $V$ such that $h_0 v = c_1 v$ and $e_0 v = 0$ for some $c_1 \in \mathbb{C}$.

Now the Lie subalgebra $\mathcal{H}_e := \mathbb{C}\{d_n, e_n \mid n \in \mathbb{Z}\}$ is isomorphic to the Lie algebra $\mathcal{D}$ defined in Section 2 (see Definition 2.1). Set \(U = U(\mathcal{H}_e)v\), which is $\mathcal{H}_e$-module generated by $v$. By Lemma 2.5, we can choose an irreducible $\mathcal{H}_e$-submodule $V' = \sum u_i$ of $U$ with $e_n u_i = du_{i+n}$ for some $d \in \mathbb{C}$ and for all $n, i \in \mathbb{Z}$.

Moreover, $V = \sum U(H)u_i$, where $H = \mathbb{C}\{h_i \mid i \in \mathbb{Z}\}$. Clearly $e_0$ is nilpotent on some element $u_i \in V'$, so $d = 0$. It is $e_n V' = 0$, and then $e_n V = 0$ since $e_nh_{i}u_{j} = h_{i}e_{n}u_{j} - 2e_{i+j}u_{i} = 0$ for all $n \in \mathbb{Z}$. Now the irreducibility of $V$ as $\mathcal{T}_2$-module is equivalent to that of $V$ as $\mathcal{H}_h$-module, where $\mathcal{H}_h = \mathbb{C}\{d_n, h_n \mid n \in \mathbb{Z}\}$ is also isomorphic to the Lie algebra $\mathcal{D}$. By Theorem 2.4, $V = \sum v_i$ is the Harish-Chandra module of intermediate series with $h_n v_i = cv_{n+i}$ for some $c \in \mathbb{C}$ and for all $n, i \in \mathbb{Z}$. \qed

4. Harish-Chandra modules over the affine-Virasoro algebra $\mathcal{L}$

**Theorem 4.1.** Let $V$ be an irreducible weight $\mathcal{L}$-module with finite-dimensional weight spaces. If $V$ is not a highest and lowest module, then $V$ is uniformly bounded.

**Proof.** From Section 2, we can suppose that $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is an irreducible Harish-Chandra $\mathcal{L}$-module without highest and lowest weights. We shall prove that for any $i \in \mathbb{Z}^*$, $k \in \mathbb{Z}$,

\[
d_i|_{V_k} \oplus d_{i+1}|_{V_k} \oplus e_i|_{V_k} \oplus f_i|_{V_k} \oplus h_i|_{V_k}: V_k \to V_{k+1} \oplus V_{k+i+1} \quad (4.1)
\]
is injective. In particular, by taking $i = -k$, we obtain that $\dim V_k$ is uniformly bounded.

In fact, suppose there exists some $v_0 \in V_k$ such that
\[
d_i v_0 = d_{i+1} v_0 = e_i v_0 = f_i v_0 = h_i v_0 = 0. \tag{4.2}
\]
Without loss of generality, we can suppose $i > 0$. Note that when $\ell \gg 0$, we have
\[
\ell = n_1 i + n_2 (i + 1)
\]
for some $n_1, n_2 \in \mathbb{N}$, from this and the relations in the definition, one can easily deduce that $d_\ell, e_\ell, f_\ell, h_\ell$ can be generated by $d_i, d_{i+1}, e_i, f_i, h_i$. Therefore there exists some $N > 0$ such that
\[
d_\ell v_0 = e_\ell v_0 = f_\ell v_0 = h_\ell v_0 = 0 \quad \text{for all} \quad \ell \geq N. \tag{4.3}
\]
This means
\[
V = U(L_0 + L_{[N, +\infty)}) v_0 = 0, \tag{4.4}
\]
where $L_{[N, +\infty)} = \bigoplus_{i \geq N} L_i$.

Since $L = L_{[1, N]} + L_0 + L_- + L_{[N, +\infty)}$, using the PBW theorem and the irreducibility of $V$, we have
\[
V = U(L) v_0 = U(L_{[1, N]}) U(L_0 + L_-) U(L_{[N, +\infty)}) v_0
\]
\[
= U(L_{[1, N]}) U(L_0 + L_-) v_0. \tag{4.5}
\]

Note that $V_+$ is a $L_+$-module. Let $V_+'$ be the $L_+$-submodule of $V_+$ generated by $V_{[0, N]}$.

Now prove that
\[
V_+ = V_+'. \tag{4.6}
\]
In fact, let $x \in V_+$ be of degree $k$. If $0 \leq k < N$, then by definition $x \in V_+$. Suppose that $k \geq N$. By (4.5), $x$ is a linear combination of the form $u_i x_i$ with $u_i \in L_{[1, N]}$ and $x_i \in V$, where $i$ is in a finite subset of $\mathbb{Z}_+$. For any $i \in I$, the degree $\deg u_i$ of $u_i$ satisfies $1 \leq \deg u_i < N$, so $0 < \deg x_i = k - \deg u_i < k$. By inductive hypothesis, $x_i \in V_+'$, and thus $x \in V_+'$. So (4.6) holds.

Eq. (4.6) means that $V_+$ is finite generated as $L$-module. Choose a basis $B$ of $V_{[0, N]}$, then for any $x \in B$, we have $x = u_x v_0$ for some $u_x \in U(L)$. Regarding $u_x$ as a polynomial with respect to a basis of $L$, by induction on the polynomial degree and using $[u, w_1 w_2] = [u, w_1] w_2 + w_1 [u, w_2]$ for $u \in L$, $w_1, w_2 \in U(L)$, we see that there exists a positive integer $k_x$ large enough such that $k_x > N$ and $[L_{[k_x, +\infty)}, u_x] \subset U(L)L_{[N, +\infty)}$. 


Then by (4.3), $\mathcal{L}_{[k_x, +\infty)}x = [\mathcal{L}_{[k_x, +\infty)}, u_x]v_0 + u_x\mathcal{L}_{[k_x, +\infty)}v_0 = 0$. Take $k = \max\{k_x, x \in B\}$, then

$$\mathcal{L}_{[k, +\infty)}V_+ = \mathcal{L}_{[k, +\infty)}U(\mathcal{L}_+)V_{[0, N)} = U(\mathcal{L}_+)\mathcal{L}_{[k, +\infty)}V_{[0, N)} = 0.$$  

Since $\mathcal{L}_+ \subset \mathcal{L}_{[k, +\infty)} + [\mathcal{L}_{-k', 0}, \mathcal{L}_{[k, +\infty)}]$ for some $k' > k$, we get $\mathcal{L}_+ V_{[k', +\infty)} = 0$. Now if $x \in V_{[k'+N, +\infty)}$, by (4.5), it is a sum of elements of the form $u_jx_j$ such that $u_j \in \mathcal{L}_{[1, +\infty)}$ and then $x_j \in V_{[k', +\infty)}$, and thus $u_jx_j = 0$. This prove that $V$ has no degree $\geq k' + N$.

Now let $p$ be the maximal integer such that $V_p \neq 0$, then since the four-dimensional subalgebra $\mathbb{C}\{d_0, h_0, e_0, C\}$ has a two-dimensional solvable subalgebra $\mathbb{C}\{h_0, e_0\}$ and two central elements $\{d_0, C\}$, so there exists a common eigenvector $w$ of $\mathfrak{h} = \mathbb{C}\{d_0, h_0, C\}$ in $V_p$ with $e_0w = 0$. It is $\mathcal{L}^+w = 0$. Then $w$ is a highest weight vector of $\mathcal{L}$, this contradicts the assumption of the Theorem. □

5. REPRESENTATIONS OF THE LIE ALGEBRA $\mathcal{L}$

Now we shall consider uniformly bounded irreducible weight modules over $\mathcal{L}$.

Let $M(\lambda)$ be the finite-dimensional irreducible highest weight $\mathfrak{sl}_2$-module with highest weight $\lambda$, then $L(M(\lambda)) := M(\lambda) \otimes \mathbb{C}[t, t^{-1}]$ becomes an irreducible $\mathcal{L}$-module by the actions as follows:

$$d_m(u \otimes t^i) = (a + bm + i)u \otimes t^{m+i},$$
$$x_m(u \otimes t^i) = (x \cdot u) \otimes t^{m+i},$$

for any $u \in M(\lambda)$ and for some $a, b \in \mathbb{C}$.

**Remark.** $L(M(\lambda))$ irreducible iff $M(\lambda)$ is a nontrivial $\mathfrak{sl}_2$-module or $a \not\in \mathbb{Z}$ or $b \neq 0, 1$. We also use $L(M(\lambda))'$ to denote the irreducible submodule or subquotient of $L(M(\lambda))$.

**Proposition 5.1.** Let $V$ be a uniformly bounded irreducible weight $\mathcal{L}$-module. Then $V$ is isomorphic to $L(M(\lambda))'$ for some finite-dimensional irreducible $\mathcal{L}$-module $M(\lambda)$.

**Proof.** Similarly to Proposition 3.1, one has $C = 0$. Clearly, $\mathcal{T}_2 = \mathbb{C}\{d_n, h_n, e_n, C \mid n \in \mathbb{Z}\}$ is a subalgebra of $\mathcal{L}$.

Consider $V$ as a $\mathcal{T}_2$-module, similarly to Lemma 2.5, $V$ has an irreducible uniformly bounded submodule $V'$. By Proposition 3.1, $V' = \sum \mathbb{C}v_i$ of $V$ with $h_nv_i = cv_{n+i}$ and $e_nv_i = 0$ for some $c \in \mathbb{C}$. Moreover, $V = U(F)V'$, where $F = \mathbb{C}\{f_i \mid i \in \mathbb{Z}\}$ is the Lie subalgebra of $\mathcal{L}$ generated by $f_i$ for all $i \in \mathbb{Z}$.
If $c = 0$, then $e_i f_j v_k = [e_i, f_j] v_k + f_j e_i v_k = 0$ for all $i, j, k \in \mathbb{Z}$. So $EV = 0$, where $E = \mathbb{C} \{e_i \mid i \in \mathbb{Z}\}$ is the Lie subalgebra of $L$ generated by $e_i$ for all $i \in \mathbb{Z}$.

Then the irreducibility of $V$ as $L$-module is equivalent to that of $V$ as $T'_2$-module, where the subalgebra $T'_2 := \mathbb{C} \{d_n, h_n, f_n, C \mid n \in \mathbb{Z}\}$ is also isomorphic to the Lie algebra $T_2$. By Proposition 3.1, $V$ is the Harish-Chandra module of intermediate series with $f_n v_i = 0$ for all $n, i \in \mathbb{Z}$. The theorem is proved. So we can suppose that $c \neq 0$.

Fix $k \in \mathbb{Z}$, for any $i \in \mathbb{Z}$, $v_i = \frac{1}{e_i} h_{i-k} v_k$, so $U(F) v_i \subseteq U(H, F) v_k$, where $U(H, F)$ is the universal enveloping algebra of the Lie subalgebra generated by $h_i, f_i$ for all $i \in \mathbb{Z}$ of $L$. So

$$V = U(H, F) v_k,$$

(5.1)

for any $k \in \mathbb{Z}$. Then for any $v \in V$ there exists $n \in \mathbb{Z}_+$ such that $e_i^nv = 0$ since $e_i v_k = 0$. It is that each $e_i$ is locally nilpotent on $V$.

Replacing $T_2 = \mathbb{C} \{d_n, h_n, e_n, C \mid n \in \mathbb{Z}\}$ by the subalgebra $T'_2 := \mathbb{C} \{d_n, h_n, f_n, C \mid n \in \mathbb{Z}\}$, we see that each $f_i$ is locally nilpotent on $V$. So $V$ is an integrable weight $L$-module.

By (5.1) we know that $V$ becomes an irreducible module over the loop algebra $L(L) = \mathbb{C} \{e_i, f_i, h_i, d_0 \mid i \in \mathbb{Z} \}$. Moreover, $V$ is an integrable weight $L(L)$-module. So by [3] (or see [4], [6]), $V \cong M(\lambda, \bar{a}) = L(\otimes_{i=1}^k M(\lambda_i))$ as $L(L)$-modules, where $\lambda = (\lambda_1, \cdots, \lambda_k)$ and $\bar{a} = (a_1, \cdots, a_k)$. However, by $[d_i, h_j] = jh_{i+j}$, we have $k = 1$ and $\bar{a} = 1$. So as an irreducible $L(L)$-module, $V \cong L(M(\lambda))'$ for some highest weight $\lambda$ of $sl_2$.

Now we only need consider the actions of $d_n$ on $V$. Suppose that $\sum_{i \in \mathbb{Z}} C v \otimes t^i$ is a Vir-module of intermediate series, then

$$d_n f_0 v \otimes t^i = f_0 d_n v \otimes t^i = (a + bn + i) f_0 v \otimes t^{n+i}.$$ 

So $\sum_{i \in \mathbb{Z}} C f_0 v \otimes t^i$ is a Vir-module of intermediate series. \hfill \Box

Combining with Theorem 4.1, we get the following result.

**Theorem 5.2.** Let $V$ be an irreducible weight $L$-module with finite-dimensional weight spaces. Then $V$ is a highest weight module or a lowest weight module or isomorphic to $L(M(\lambda))'$ for some finite-dimensional irreducible $L$-module $M(\lambda)$.

**Remark.** The unitary highest weight modules over the affine-Virasoro algebra $L$ were considered in [13, 12].
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REFERENCES

[1] E. Arbarello; C. De Concini; V.G. Kac; C. Procesi, Moduli spaces of curves and representation theory, Comm. Math. Phys., 117 (1988), 1–36.
[2] Y. Billig, A category of modules for the full toroidal Lie algebra, International Mathematics Research Notices, 2006, 46 pp
[3] V. Chari, Integrable representations of affine Lie-algebras, Invent. Math. 85 (2) (1986), 317–335.
[4] V. Chari; A. Pressley, New unitary representations of loop groups, Math. Ann. 275 (1986), 87–104.
[5] S. Cheng; N. Lam, Finite conformal modules over the $N = 2, 3, 4$ superconformal algebras. J. Math. Phys. 42 (2) (2001), 906–933.
[6] S. Eswara Rao, On the representations of loop algebras, Comm. Algebra 21 (1993), 2131–2153.
[7] S. Eswara Rao, Classification of irreducible integrable modules for multi-loop algebras with finite-dimensional weight spaces, J. Algebra 246 (2001), 215–225.
[8] S. Eswara Rao; C. Jiang, Classification of irreducible integrable representations for the full toroidal Lie algebras, J. Pure Appl. Algebra 200 (1-2) (2005), 71–85.
[9] I.B. Frenkel; V.G. Kac, Basic representations of affine Lie algebras and dual resonance models, Invent. Math. 62 (1980), 23–66.
[10] X. Guo; R. Lv; K. Zhao, Simple Harish-Chandra modules, intermediate series modules, and Verma modules over the loop-Virasoro algebra, Forum Math. 23 (2011), 1029–1052.
[11] C. Jiang; D. Meng, Integrable representations for generalized Virasoro-toroidal Lie algebras, J. Algebra 270 (1) (2003), 307–334.
[12] C. Jiang; H. You, Irreducible representations for the affine-Virasoro Lie algebra of type $B_t$, Chinese Ann. Math. Ser. B 25 (3) (2004), 359–368.
[13] V. G. Kac, Highest weight representations of conformal current algebras, Symposium on Topological and Geometric Methods in Field Theory, Espoo, Finland, World Scientific, p. 3–16, 1986.
[14] V. G. Kac, Infinite-dimensional Lie Algebras, 3rd ed., Cambridge Univ. Press, Cambridge, U.K., 1990.
[15] I. Kaplansky; L. J. Santharoubane, Harish-Chandra modules over the Virasoro algebra, Infinite-dimensional groups with applications (Berkeley, Calif. 1984), 217–231, Math. Sci. Res. Inst. Publ., 4, Springer, New York, 1985.
[16] G. Kuroki, Fock space representations of affine Lie algebras and integral representations in the Wess-Zumino-Witten models, Comm. Math. Phys. 142 (3) (1991), 511–542.
[17] D. Liu; N. Hu, Vertex representations for toroidal Lie algebra of type G2, J. Pure Appl. Algebra 198 (1-3) (2005), 257–279.
[18] D. Liu; C. Jiang, Harish-Chandra modules over the twisted Heisenberg-Virasoro algebra, J. Math. Phys., 49 (1) (2008), 012901, 13 pp.
[19] X. Liu; M. Qian, Bosonic Fock representations of the affine-Virasoro algebra, J. Phys. A, 27 (5) (1994), 131–136.
[20] R. Lv; K. Zhao, Classification of irreducible weight modules over higher rank Virasoro algebras, Adv. Math. 201 (2) (2006), 630–656.
[21] R. Lv; K. Zhao, Classification of irreducible weight modules over the twisted Heisenberg-Virasoro algebra, Comm. Contemp. Math. 12 (2) (2010), 183–205.
[22] O. Mathieu, Classification of Harish-Chandra modules over the Virasoro Lie algebra, Invent. Math. 107 (1992), 225–234.
[23] R. V. Moody; S. Eswara Rao; T. Yomonuma, Toroidal Lie algebras and vertex representations, Geom. Ded. 35 (1990), 287–307.
[24] S. Eswara Rao; R. V. Moody, Vertex representations for n-affine-Virasoro Lie algebras and a generalization of the Virasoro algebra, Comm. Math. Phys. 159 (1994), 239–264.
[25] Y. Su, Classification of quasifinite modules over the Lie algebras of Weyl type, Adv. Math. 174 (2003), 57–68.
[26] Y. Su, Classification of Harish-Chandra modules over the higher rank Virasoro algebras, Comm. Math. Phys. 240 (2003), 539–551.
[27] M. Wakimoto, Lectures on infinite-dimensional Lie Algebra, World Scientific Publishing Co., Inc. 2001.
[28] L. Xia; N. Hu, Irreducible representations for Virasoro-toroidal Lie algebras. J. Pure Appl. Algebra 194 (1-2) (2004), 213–237.

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