Oscillatory Tunneling between Quantum Hall Systems

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Abstract

Electron tunneling between quantum Hall systems on the same two dimensional plane separated by a narrow barrier is studied. We show that in the limit where inelastic scattering time is much longer than the tunneling time, which can be achieved in practice, electrons can tunnel back and forth through the barrier continuously, leading to an oscillating current in the absence of external drives. The oscillatory behavior is dictated by a tunneling gap in the energy spectrum. We shall discuss ways to generate oscillating currents and the phenomenon of natural “dephasing” between the tunneling currents of edge states. The noise spectra of these junctions are also studied. They contain singularites reflecting the existence of tunneling gaps as well as the inherent oscillation in the system.

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In this paper, we study electron tunneling between quantum Hall (QH) systems separated by thin barriers. Examples of these systems are shown in figure 1 to figure 6. The thinness of the barrier allows an electron to tunnel through it many times before scattered away by inelastic effects. Oscillatory tunneling of this kind will occur if the inelastic scattering time $\tau_{in}$ is much longer than the tunneling time $\tau_T$,

$$\tau_{in} \gg \tau_T.$$  

(1)

The existence of oscillatory tunneling can be seen even in the semiclassical (SC) limit, where electron wave-packets moves in circular orbits with cyclotron frequency $\omega_c$. When the barrier is infinite, electrons will undergo a sequence of “reflected circular orbits” as shown in fig.1. In the absence of other scattering mechanisms, electrons having collided with the barrier once must collide with it again within the cyclotron period. As a result, they are forever captured by the barrier. (See fig. 5). When the barrier is reduced from infinity to a finite value, the captured electrons on one side of the barrier (say, $L$) can tunnel to the other side ($R$). Once tunneled, this electron will collide repeatedly with the barrier and eventually tunnel back to $L$. When eq.(1) is satisfied, this back and forth tunneling process can proceed without interruption, giving rise to an oscillating current in the absence of external drives.

While the SC picture captures the correct physics, it only tells half the story. In a quantum mechanical treatment, we shall see that different edge states tunnel with different frequencies. Thus, even in the absence of inelastic scattering, the tunneling current of different edge states will naturally dephase with each other. As a result, the total tunneling current will decrease in time. However, we show later that despite dephasing effects, there are ways to generate lasting current oscillations (thereby reflecting the oscillatory tunneling near the barrier) without the aid of an a.c. drive.

The crucial question is whether eq.(1) can be achieved. We shall argue below that this is possible at least for the case of integer filling. There are two sources of inelastic
scattering: Coulomb interaction between electrons on the same side of the barrier ("intra-region" interaction) and that on the different sides ("inter-region" interaction). Let us first consider noninteracting electrons and the limit of infinite barrier. The systems \( R \) and \( L \) on both sides of the barrier are now disconnected, reducing to two semi-infinite systems terminated by a hard wall. The Landau levels of such systems are well known, i.e. they bend upward as the barrier is approached. See figure 7. In the presence of intra-region interaction (but without inter-region interaction), and when the system has integer filling, the edge electrons will behave like a normal Fermi liquid. The lifetime \( \tau_{in} \) of the quasiparticles will then tend to infinity at the Fermi surface, and will dominate over any tunneling time \( \tau_T \) introduced by finite barriers. In other word, eq.(1) can always be satisfied near the Fermi surface when the system has integer filling, and that electron tunneling near the Fermi surface can be modelled by that of non-interacting systems. (Estimates of the tunneling time is given at the end of the paper). As we have seen in fig. 4, the Landau levels of \( R \) and \( L \) intersect because they all bend upward near the barrier. In the presence of tunneling, these intersections will turn into gaps, (see figure 8). As we shall see, the unusual features of these junctions are determined by these gaps.

What is more subtle is the effect of inter-region interactions. While it is obvious that the tunneling gap can withstand sufficiently weak inter-region interactions, the situation is less clear for large inter-region interactions. However, as we show later, it is possible to map our problem to a solvable model in one dimension (massive Thirring model). The exact solution of this model shows that the tunneling gap exists for arbitrary inter-region interaction. Although we have not yet been able to calculate the current responses for arbitrary inter-region interaction, the survival of the tunneling gap suggests that the tunneling characteristics of the non-interacting systems may also survive.

Before proceeding, we stress that the phenomena discussed here requires thin barriers. The junction used in many current experiments are produced by gate voltages and are much smoother than the barriers we consider here. Since magnetic lengths in a 10 Tesla field is about 80 Å, and that channels of 100 Å wide is feasible in current technology, the construction
of these junctions is possible. (See also Section VII for estimates of relevant parameters).

The rest of paper is organized as follows: In Section II, we discuss the energy spectra in the vicinity of the the barrier for a variety of external conditions. In Section III, we derive the effective Hamiltonian for the tunnel junction as well as the expression of tunneling current. In section IV, we suggest ways to generate oscillatory tunneling currents, and discuss the phenomenon of natural dephasing. In section V, we discuss the noise spectrum of the junction, which reflects directly the existence of tunneling gaps and the inherent natural oscillations of the system. In Section VI, we discuss the effect of inter-region Coulomb interaction. In Section VII, we give numerical estimates of various parameters.

II. THE ENERGY SPECTRUM NEAR THE BARRIER

We have argued in Sec. I that when eq. (1) is satisfied, tunneling between QH systems with fully filled Landau levels can be modelled by that of non-interacting electrons. Although we have mentioned the general behavior of the spectrum in Section I, we shall give a detailed description here as we shall need it later. For simplicity, we shall focus on the setup in fig 3. The system is periodic in $y$, $\psi(x, y) = \psi(x, y + L)$. The Hamiltonian in the Landau gauge is

$$H = \frac{1}{2m} p_x^2 + \frac{1}{2m} \left( \frac{e}{c} B x \right)^2 + V(x),$$

where $B$ is the external magnetic field, and $V(x) = V_o > 0$ or 0 for $|x| > a$. (See fig. 4). The eigenstates are of the form $\psi_{n,k}(x, y) = L^{-1/2} e^{iky} u_{n,k}(x)$, $k = (2\pi m)/L$, where $m$ is an integer, and $u_{n,k}(x)$ is an eigenfunction of

$$H_k = \hbar \omega_c \left( -\frac{\ell^2}{2} \partial_x^2 + V_k(x) \right), \quad V_k(x) \equiv \frac{1}{2} \left( \frac{x}{\ell} - k\ell \right)^2 + V(x)/\hbar \omega_c,$$

with energy $E_{n,k}$. Here, $\ell = \sqrt{\hbar c/eB}$ is magnetic length and $\omega_c = (eB/mc)$ is the cyclotron frequency. Eq. (3) can be written as

$$H = \sum_k H_k \equiv \sum_{k,n=0}^{\infty} E_{n,k} a_{n,k}^+ a_{n,k},$$

(4)
where \( a_{n,k} \) is annihilation operator of \( \psi_{n,k} \). The existence of oscillatory tunneling near the barrier can be seen from the fact that \( V_k(x) \) reduces to a degenerate double well as \( k \to 0 \). It is well known that when an electron is placed in one side of the double well, it will tunnel back and forth between the wells with a frequency given by the excitation energy from the ground state to the first excited state.

Although both \( u_{n,k}(x,y) \) and \( E_{n,k} \) can be obtained by analytic methods [2], they can be easily understood in the limit of high barriers. When \( V_o = \infty \), \( L \) and \( R \) become two disconnected semi-infinite systems, \( H \to H_L + H_R \),

\[
H_L = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_{n,k}^L c_{n,k}^+ c_{n,k}, \quad H_R = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_{n,k}^R d_{n,k}^+ d_{n,k},
\]

where \( \epsilon_{n,k}^L \) and \( \epsilon_{n,k}^R \) are the Landau levels of \( L \) and \( R \) in the limit \( V_o = \infty \). \( c_{n,k} \) and \( d_{n,k} \) are the corresponding eigenstates. The behavior of the Landau levels \( \epsilon_{n,k}^L \) and \( \epsilon_{n,k}^R \) as a function of \( k \) have been studied by a number of authors [1] [2]. (See also fig. 7 and fig. 8). In the bulk of \( L, \ell k \ll -1, \epsilon_{n,k}^L = (n + 1/2)\hbar \omega_c \). \( \epsilon_{n,k}^L \) begins to deviate appreciably from its bulk value about a magnetic length away from the wall, \( \ell k \lesssim -1 \). The entire curve increases monotonically (to infinity) as \( k \) increases, passing through \( (2n + 3/2)\hbar \omega_c \) at the barrier, (i.e. when \( \ell k = -a/\ell \)). \( \epsilon_{n,k}^R \) has an identical behavior in the reverse \( k \) direction. When \( V_o \) is reduced from infinity to a finite value, the intersections of the spectra of \( L \) and \( R \) will turn into “tunneling” gaps. The two sets of energy curves \( \{ \epsilon_{n,k}^L \} \) and \( \{ \epsilon_{m',k}^R \} \) now turn into a single set \( \{ E_{n,k} \} \), which we shall refer to as the \( n \)-th Landau level of the entire system. Each curve \( E_{n,k} \) is a smooth function in \( k \). It reduces to \( \epsilon_{n,k}^L \) and \( \epsilon_{n,k}^R \) for \( \ell k << -1 \) and \( \ell k >> 1 \).

The qualitative features of the wavefunctions \( u_{n,k} \) can be determined from the effective potential \( V_k(x) \). (See figure [3].) If \( \phi_{n,k}^L(x) \) and \( \phi_{n,k}^R(x) \) are the eigenstates of \( L \) and \( R \) in the infinite barrier limit, (hence \( \phi_{n,k}^L(x) = 0 \) for \( x > 0 \), and \( \phi_{n,k}^R(x) = 0 \) for \( x < 0 \)), then for high (but finite) barriers, we have (see fig. [3])

\[
u_{0(1),k} \approx \left( \phi_{0,k}^L(x) + (-)\phi_{0,k}^R(x) \right) / \sqrt{2} \quad \text{for} \quad -1 << \ell k << 1, \tag{6}
\]

\[
u_{0(1),k} \approx \phi_{0(1),k}^L(x), \quad \text{for} \quad \ell k << -1; \tag{7}
\]
\[ u_{0(1),k}(x) \approx \pm \phi_{0(1),k}^R(x), \quad \text{for } \ell k \gg 1. \] (8)

In the absence of voltage bias between \( L \) and \( R \), it can be seen from fig. 8 that the lowest tunneling gap of \( L \) and \( R \) (which occurs at \( k = 0 \)) lie above the first “bulk” Landau level, i.e. \((3/2)\hbar \omega_c\). The location of the tunneling gap, however, can be easily changed by applying a voltage bias. (See fig. 10). Note that in the presence of a voltage bias \( V \), \( H \) is still diagonal in \( k \) and is still given by eq.(4) except that the spectrum \( E_{n,k} \) and the eigenstates \( a_{n,k} \) now functions of \( V \).

When the spins of the electrons are taken into account, the spectrum of \( L \) and \( R \) in the infinite barrier limit consists of two sets of Landau levels differing from each other by the Zeeman energy. Since \( V(x) \) does not flip spins, the intersections of the opposite spin Landau levels will not turn into gaps when \( V_o \) becomes finite.

To conclude this section, we derive the expression for the current in the \( x \)-direction. If we define the number of particle to the left and to the right of the barrier as

\[ N_L(t) = \int_{-\infty}^{0} dx \hat{\psi}^+(x, y; t) \hat{\psi}(x, y; t), \quad N_R(t) = \int_{0}^{\infty} dx \hat{\psi}^+(x, y; t) \hat{\psi}(x, y; t) \] (9)

the current in \( x \) is then \( I(t) = e \dot{N}_L = -e \dot{N}_R \). Using the fact that \( \hat{\psi}(x, y, t) = \sum_{n,k} \left[ L^{-1/2} e^{iky} u_{n,k}(x) \right] a_{n,k} e^{-iE_{n,k} t/\hbar} \), we can write

\[ I(t) = \sum_k I_k(t) = \sum_k \frac{ie}{\hbar} \sum_{n,m} [E_{n,k} - E_{m,k}] g_{n,m}(k) a_{n,k}^+ e^{i[E_{n,k} - E_{m,k}] t/\hbar}; \] (10)

\[ g_{n,m}(k) = \int_{-\infty}^{0} u_{n,k}(x) u_{m,k}(x) dx. \] (11)

Note that only terms with \( n \neq m \) contribute to the current as \( g_{n,m}(k) \) reduces to the overlap of two orthogonal states in \( L \) or \( R \) in the \( V_o = \infty \) limit,

\[ g_{n,m}(k) \approx 0 \quad \text{for } \ell |k| > 1. \] (12)

For this reason, we can from now on focus on the range \( \ell |k| \leq 1 \) in eq.(10).

Limiting to the lowest two Landau levels, eq.(10) becomes
\[ I(t) = \sum_k \frac{ie}{\hbar} T_k a_{1,k}^+ a_{0,k} e^{i[E_{1,k} - E_{0,k}]t/\hbar} + \text{h.c.}, \quad T_k = [E_{1,k} - E_{0,k}] g_{1,0}(k). \quad (13) \]

For \( |k| \leq 1 \), eq. (13) implies

\[ g_{1,0}(k) \approx \frac{1}{2} \int_{-\infty}^{0} dx \left[ |\phi_{0,k}^{L}(x)|^2 - |\phi_{0,k}^{R}(x)|^2 \right] \quad (14) \]
\[ = \frac{1}{2} \int_{-\infty}^{0} dx |\phi_{0,k}^{L}(x)|^2 = \frac{1}{2}. \quad (15) \]

With eq. (13) and eq. (12), we have

\[ T_k \approx (E_{1,0} - E_{0,0})/2 \equiv \Delta_0/2 \quad \text{for } |k| << 1 \quad (16) \]
\[ \approx 0 \quad \text{for } |k| > 1. \quad (17) \]

III. EFFECTIVE TUNNELING HAMILTONIAN AND THE TUNNELING CURRENT

In this section and the next two, we shall focus on the tunneling between the lowest Landau level of \( L \) and \( R \). For simplicity, we shall also consider the case of zero bias. The results derived here can be generalized easily to other Landau levels and to non-zero bias.

The Hamiltonian of the entire system, eq. (11), now reduces to

\[ H = \sum_k \left( E_{0,k} a_{0,k}^+ a_{0,k} + E_{1,k} a_{1,k}^+ a_{1,k} \right). \quad (18) \]

As discussed in Sec.I, only those \( k \)'s in the range \( \ell|k| \leq 1 \) contribute to the current eq. (13). Within this range, \( E_{0,k} \) and \( E_{1,k} \) are close to \( \epsilon_{0,k}^L \) and \( \epsilon_{0,k}^R \) except at \( k = 0 \), (i.e. the intersection of \( \epsilon_{0,k}^L \) and \( \epsilon_{0,k}^R \)), where a gap \( \Delta_o \) is opened up. (See also fig. 8 and fig. 11). For later discussions, we define

\[ E_{1,k} - E_{0,k} \equiv E_k, \quad \Delta_o \equiv E_{k=0} = E_{1,0} - E_{0,0}. \quad (19) \]

The tunneling phenomenon contained in eq. (18) is more transparent if \( H \) is written in the form of a tunneling Hamiltonian. Defining energies \( \epsilon_{L,k}, \epsilon_{R,k} \), and tunneling matrix element \( T_k \) as
\[ \epsilon_{L,k} - \epsilon_{R,k} = \epsilon_{0,k}^L - \epsilon_{0,k}^R \equiv \epsilon_k, \]  
(20)

\[ \epsilon_{L(R),k} = \epsilon_{0,k}^L + \zeta_k, \quad \zeta_k = \frac{1}{2} \left[ E_{1,k} + E_{0,k} - \epsilon_{0,k}^L - \epsilon_{0,k}^R \right], \]  
(21)

\[ T_k = \frac{1}{2} \sqrt{E_k^2 - \epsilon_k^2}, \]  
(22)

eq.(18) can be written as

\[ H = H_o + H_T = \sum_k \left( \epsilon_{L,k} c_{L,k}^+ c_{L,k} + \epsilon_{R,k} c_{R,k}^+ c_{R,k} \right) - \sum_k \left( T_k c_{L,k}^+ c_{R,k} + h.c. \right), \]  
(23)

where

\[ \begin{pmatrix} c_L \\ c_R \end{pmatrix}_k = \hat{U}_k \begin{pmatrix} a_o \\ a_1 \end{pmatrix}_k, \quad \hat{U}_k = \begin{pmatrix} v & u \\ u & -v \end{pmatrix}_k, \]  
(24)

\[ u_k = \sqrt{\frac{1}{2} \left( 1 + \frac{\epsilon_k}{E_k} \right)}, \quad v_k = \sqrt{\frac{1}{2} \left( 1 - \frac{\epsilon_k}{E_k} \right)}. \]  
(25)

The phases of \( c_{L,k} \) and \( c_{R,k} \) have been chosen so that \( u_k, v_k, \) and \( T_k \) are all real. [The relation between \( T_k \) defined in eq.(22) and that in eq.(13) will be clear shortly]. Eq.(25) also implies that

\[ T_k = u_k v_k E_k. \]  
(26)

Although strictly speaking \( \zeta_k \neq 0 \), it can be taken as zero as it is much smaller than \( \epsilon_{0,k}^{L(R)} \). As a result, \( \epsilon_{L,k}, \epsilon_{R,k}, c_{L,k}, \) and \( c_{R,k} \) are well approximated by \( \epsilon_{0,k}^L, \epsilon_{0,k}^R, c_{0,k}, \) and \( d_{0,k}, \) (see eq.(1)), even though they are not exactly the same. \( H_o \) in eq.(23) can therefore be interpreted as the Hamiltonian of \( L \) and \( R \) in the infinite barrier limit, and \( H_T \) describes the tunneling between them.

[There is another point worth noting. In the conventional tunneling Hamiltonian, the tunneling term \( H_T \) is usually written as \( \sum_{k,k'}(T_{k,k'} c_{L,k}^+ c_{R,k'} + h.c.), \) whereas in eq.(23) \( k \) is conserved during tunneling processes. This is entirely a consequence of the symmetry of the systems in fig. 2 and fig. 3.]
Next, we turn to the tunneling current. Defining the number of particles to the left and to the right as $N_L = \sum_k c_{L,k}^+ c_{L,k}$, and $N_R = \sum_k c_{R,k}^+ c_{R,k}$, the current in $x$ is then $I(t) = e\dot{N}_L = -e\dot{N}_R$, or explicitly,

$$I(t) = \frac{ie}{\hbar} \sum_k \left( T_k c_{L,k}^+(t) c_{R,k}(t) - h.c. \right). \quad (27)$$

Using eq. (25) and the fact that $a_{0(1),k}(t) = a_{0(1),k} e^{-iE_{0(1),k}t/\hbar}$, we can write eq. (27) as

$$I(t) = \frac{ie}{\hbar} \sum_k T_k \left( a_{1,k}^+ a_{0,k} e^{iE_{k}t/\hbar} + h.c. \right). \quad (28)$$

Using eq. (25) again, we can rewrite

$$I(t) = \frac{e}{\hbar} \sum_k \left[ \nu_k(t) \left( c_{R,k}^+ c_{R,k} - c_{L,k}^+ c_{L,k} \right) + \eta_k(t) \left( c_{L,k}^+ c_{R,k} + h.c. \right) \right], \quad (29)$$

where $\nu_k$ and $\eta_k$ are defined as

$$\nu_k(t) = 2(T_k u_k v_k) \sin(E_k t/\hbar), \quad \eta_k(t) = i T_k \left( e^{iE_k t/\hbar} u_k^2 + e^{-iE_k t/\hbar} u_k^2 \right). \quad (30)$$

Comparing eqs. (28) and (29) with eq. (13), one notes that these two definitions of $T_k$ are consistent if $g_{0,1}$ in eq. (13) is identified as $u_k v_k$ in eq. (26). From eq. (27), one can also see that the asymptotic form eq. (16) is also satisfied by both definitions.

The expressions eqs. (28) and (29) represent the major difference between oscillatory tunneling and the usual type of electron tunneling (such as those in normal and Josephson junctions), where first order perturbation theory in $H_T$ provides an adequate description of the tunneling current,

$$I(t) - I(0) = \frac{i}{\hbar} \int_{-\infty}^{t} [\tilde{I}(t), \tilde{H}_T(t')] dt', \quad \tilde{A}(t) \equiv e^{iH_0 t/\hbar} A e^{-iH_0 t/\hbar}. \quad (31)$$

In the conventional treatment, eq. (31), higher order terms corresponding to multitunneling processes are ignored. In contrast, the time dependences in eq. (27)-(29) are generated by the full Hamiltonian $H$, which is amount to extending the perturbation series eq. (31) to infinite order. (Note that the infinite series is necessary to generate a gap in the spectrum).

The reason that the perturbative result eq. (31) is applicable for most systems is because multitunneling processes are usually subpressed by quantum diffusion even in the absence
of inelastic scattering. In the usual case, tunneling takes place between electronic states that are extended over the bulk of the sample (L or R). Once tunneled across, the electron leaves the barrier on the time scale of quantum diffusion. The time to travel a distance comparable to the barrier width $a/v_F$, which is usually much shorter than the tunneling time. As a result, the tunneling current can be accounted for by first order perturbation theory. However, for the junctions we consider, tunneling takes place between edge states which are localized near the barrier. The electron has no where to go after tunneling but to tunnel back. The continuous back and forth tunneling renders the conventional scheme eq. (31) inadequate.

**A Simple Model:** Near the intersection point, one can linearize the infinite barrier spectrum $\epsilon_k$ such that

$$\epsilon_k = v_F \hbar k$$

where $v_F$ is the Fermi velocity which is of the order of $\ell \omega_c$. When the tunneling is weak, $\Delta_o << \hbar \omega_c$, the region in $k$-space where $E_k$ differs significantly from $\epsilon_k$ is $\ell |k| \leq \frac{\Delta_o}{\hbar \omega_c}$. We can therefore model $E_k$ as

$$E_k^2 = \Delta_o^2 + \epsilon_k^2.$$  

(33)

In terms of this model, eq. (22) and eq. (26) become

$$T_k = \frac{1}{2} \Delta_o, \quad u_k v_k = \frac{\Delta_o}{2 E_k}.$$  

(34)

**IV. OSCILLATIONS OF THE TUNNELING CURRENT**

From eqs. (23) and (24), we see that the tunneling current is made up of different edge state components $I_k$, each of which oscillates at a different frequency $\omega_k = E_k/\hbar$. In this section, we discuss ways to generate natural current oscillations, and to discuss the dephasing between different current components. For simplicity, let both $L$ and $R$ have identical chemical potentials (i.e. $\mu_L = \mu_R = \mu$), and that $\mu$ is below the tunneling gap. (See
fig. [11]. The corresponding Fermi vectors in $L$ and $R$ are $-k_F$ and $k_F$ respectively. The quantum state of the system is then $|\Psi> = \prod_{k<-k_F} c_{L,k}^+ \prod_{p>k_F} c_{R,p}^+ |0>$. [That we take the initial state as $|\Psi>$ instead of the true ground state of the entire system $|\Psi_o> = \prod_{k} |k| a_{0,k}^+$, is because the relaxation from $|\Psi>$ to $|\Psi_o>$ requires inelastic processes, which are ineffective when eq.(1) is satisfied.] The tunneling current is

$$<I(t)> = \frac{e}{\hbar} \sum_{k} \nu_k(t) \left( <c_{R,k}^+ c_{R,k}> - <c_{L,k}^+ c_{L,k}> \right), \tag{35}$$

where the average is with respect to $|\Psi>$. It is clear that $|\Psi>$ will not generate any current as the current components in $L$ and $R$ cancel each other, ($\nu_k(t) = \nu_{-k}(t)$, hence)

$$<I(t)> = \frac{\hbar}{e} \left( \sum_{k<-k_F} - \sum_{k>k_F} \right) \nu_k(t) = 0.$$

The simplest way to generate a single (of a small number of) oscillating current component is to move all the edge states $k$ to the right by a small amount, i.e. shifting $k$ to $(k + \theta/L)$. This shift amounts to changing the periodic boundary condition of the wavefunction to $\psi(x, y) = e^{i\theta} \psi(x, y + L)$. Returning to the cylindrical geometry fig. [2], this change of boundary condition corresponds to passing a fraction $\left(\frac{\theta}{2\pi}\right)$ of a flux quantum through the center hole. When half of a flux quantum is passed, ($\theta = \pi$), we have in effect added an electron on top of the Fermi sea in $L$. (See fig. [11]). The tunneling current is therefore

$$<I(t)> = -\nu_{k_F+\pi/L} \approx -\nu_{k_F} = -\frac{e}{\hbar} 2(Tuv)_{k_F} \sin \left( E_{k_F} t / \hbar \right). \tag{36}$$

For chemical potentials slightly below the tunneling gap, $\ell k_F \ll 1$, eq.(26) implies $(Tuv)_{k_F} = 2E_{k_F} (uv)_{k_F}^2 = E_{k_F}$. We can then write eq.(36) in a very simple form

$$<I(t)> = -e \left( E_{k_F} / \hbar \right) \sin \left( E_{k_F} t / \hbar \right). \tag{37}$$

If, instead of pushing a flux quantum through $L$, we introduce a chemical potential difference between $L$ and $R$ at time $t = 0$, [ ($\mu_L = \mu_R$) $\rightarrow$ ($\mu_L = \mu + eV/2$, $\mu_R = \mu - eV/2$)]. The Fermi wavevectors in $L$ and $R$ are then changed to $-k_F + \delta k_F$ and $k_F + \delta k_F$, (see fig. [12]),

$$\delta k_F = eV / (\hbar v_F). \tag{38}$$
The quantum state in eq.(33) now becomes \( |\Psi > = (\Pi_{k < -k_F + \delta k_F} c_{L,k}^+)(\Pi_{p > k_F + \delta k_F} c_{R,p}^+)|0 > \).

The current at \( t > 0 \) is then

\[
< I(t) > = \frac{e}{\hbar} \sum_{[k]} \nu_k(t), \tag{39}
\]

where \([k]\) denotes the range of excited edge states \(|k + k_F| \leq \delta k_F\). Each of the \( \nu_k \) term oscillates with frequency \( E_k/\hbar \). If the entire range \([k]\) lies in the linear region of the spectrum, (hence \( E_k \approx \epsilon_k = \hbar v_F k \)), (see fig. 12), then the states at the opposite end of the interval will be the first ones to be out of dephase with each other, as they have maximum frequency difference. This takes place at time \( \tau_{dp}^{(i)} = \hbar/(E_{k_F}'\delta k_F) \approx \hbar/(eV/2) \), referred to as the “initial” dephasing time. As time increases, the coherence of the states in \([k]\) reduces as more and more states at different ends of the interval keep dephasing with each other. (See fig. 12). When \( t \sim \tau_{dp}^{(f)} = \hbar/[E_{k_F}'(2\pi/L)] = L/[2\pi v_F] \), referred to as the “final” dephasing time, only one or two states in the vicinity of \( k_F \) remain coherent.

During the dephasing period, \( \tau_{dp}^{(i)} < t < \tau_{dp}^{(f)} \), the summand in eq.(39) is sufficiently smooth that the sum can be approximated by the integral

\[
< I(t) > = -\frac{e}{\hbar} \frac{L}{2\pi} \int_{-k_F-\delta k_F}^{-k_F+\delta k_F} (2T_k u_k v_k) \sin \left( \frac{E_k t}{\hbar} \right) dk. \tag{40}
\]

Expanding the integrand about \( k_F \), eq.(40) becomes

\[
< I(t) > = -\frac{e}{\hbar} \frac{L}{2\pi} (2T u)_{k_F} \sin \left( E_{k_F} t/\hbar \right) \frac{2\sin \left( [E_{k_F}' \delta k_F t]/\hbar \right)}{E_{k_F}' t/\hbar} + O(t^{-2}) + ..., \tag{41}
\]

Initially, (for \( t \approx 0 \)), eq.(40) gives

\[
< I(t) > \approx -\frac{e}{\hbar} (L\delta k_F/2\pi)(2T u)_{k_F} \sin \left( E_{k_F} t/\hbar \right), \tag{42}
\]

which is the single electron current eq.(37) multiplied by the number of electrons participating in tunneling, \( (L\delta k_F/2\pi) \). Dephasing effect causes this current to decrease as \( 1/t \), (see eq.(41) ). At time \( t \approx \tau_{dp}^{(f)} \), most of the terms in eq.(39) have undergone many oscillations except for a few terms near \( k = 0 \). The magnitude of the current is then reduced to that comparable to a single electron, eq.(37).
Let us consider a different situation where initially \( k_F = 0 \). The range \([k]\) is then symmetric about \( k = 0 \). (See fig. 12). The tunneling current eq. (39) becomes

\[
<I(t)> = \frac{2e}{\hbar} \sum_{0 \leq k < \delta k_F} \nu_k(t),
\]

(43)

The largest frequency difference among different \( k \) terms is still \( eV/\hbar \), whereas the minimum frequency difference becomes \( \delta \omega_{k=0} = (E_{2\pi/L} - E_0)/\hbar = \frac{1}{2} \left( \hbar^2 v_F^2 / \Delta_o \right) (2\pi/L)^2 / \hbar \). Therefore, we still have \( \tau^{(i)}_{dp} = \hbar/(eV) \), while the final dephasing time becomes \( \tau^{(f)}_{dp} = 2\pi / \delta \omega_{k=0} \). For \( \tau^{(i)}_{dp} > t > \tau^{(f)}_{dp} \), eq. (43) can be written as

\[
<I(t)> = -\frac{eL}{\hbar} \int_{-\delta k_F}^{\delta k_F} dk \sin \left( \left| \Delta_o + \frac{1}{2} E''_{k=0} k^2 \right| t/\hbar \right) + O(...)
\]

\[
= -\frac{e}{\hbar} \left( \frac{L\delta k_F}{2\pi} \right) \int [2\Delta_o \sin \left( \Delta_o t/\hbar \right)] \sqrt{\frac{\pi \hbar \Delta_o}{(eV)^2 t} C \left[ \sqrt{\frac{(eV)^2 t}{\pi \hbar \Delta_o}} \right]},
\]

(44)

(45)

where \( C(x) \equiv \int_0^x \cos(u^2)du \) is the Fresnel integral which approaches 1/2 as \( x \to \infty \). In deriving eq. (45), we have made use of eq. (38).

From eq. (43), we can see that as the chemical potential \( \mu \) sweeps through the gap, the dephasing processes slows down, changing from \( t^{-1} \) to \( t^{-1/2} \) for large \( t \). The final dephasing time \( \tau^{(f)}_{dp} = \hbar / \delta \omega_{k=0} = (L/2\pi v_F)(\Delta_o/[\hbar v_F(2\pi/L)]) \) is much longer than that in the previous case, \( (L/2\pi v_F) \), as the factor \( \Delta_o/[\hbar v_F(2\pi/L)] \) is typically much larger than 1. (See also Section VII).

V. NOISE SPECTRUM

The oscillatory tunneling of the edge states can also be detected through the noise spectrum, \( S(\omega) = \int_{-\infty}^{\infty} S(t)e^{i\omega t}dt \), \( S(t) = \frac{1}{2} [I(t),I(0)]_+ \). When both \( L \) and \( R \) have identical chemical potentials \( \mu \), eq. (29) and eq. (30) imply that

\[
S(t) = \left( \frac{e}{\hbar} \right)^2 \sum_k \text{Re}[\eta_k^*(t)\eta_k(0)] = \left( \frac{e}{\hbar} \right)^2 \sum_k |T_k|^2 \cos \left( \frac{E_k t}{\hbar} \right)
\]

\[
\sum_k \cdot \cdot \cdot \equiv \sum_k \cdot \cdot \cdot \left( f(\epsilon_L,k)\bar{f}(\epsilon_R,k) + f(\epsilon_R,k)\bar{f}(\epsilon_L,k) \right)
\]

(46)

(47)
where \( f(x) = \left( e^{(x-\mu)/k_BT} + 1 \right)^{-1} \) is the Fermi function, \( T \) is the temperature, and \( \mu \) is the chemical potential. The noise spectrum is

\[
S(\omega) = \left( \frac{e}{\hbar} \right)^2 \sum_k |T_k|^2 \pi \left[ \delta(\omega - E_k/\hbar) + \delta(\omega + E_k/\hbar) \right]
\]

At \( T = 0 \), we have \( \sum_k \to (L/2\pi) \left( \int_{-\infty}^{k_F} + \int_{k_F}^{\infty} \right) \). For \( \omega > 0 \), we have

\[
S(\omega) = \left( \frac{e}{\hbar} \right)^2 L \int_{k_F}^{\infty} |T_k|^2 \delta(\omega - E_k/\hbar) dk
\]

\[
= \left( \frac{e}{\hbar} \right)^2 L\hbar \left( \frac{|T_k|^2}{dE_k/dk} \right)_{h\omega=E_k}
\]

\[
= 0 \quad \text{for } h\omega < E_{k_F}
\]

Using the simple model at the end of Sec.III, eq.(50) and eq.(51) becomes

\[
S(\omega) = \left( \frac{e}{\hbar} \right)^2 \frac{L\Delta o^2}{v_F} \frac{\omega}{\sqrt{\omega^2 - (\Delta_o/\hbar)^2}} \quad \text{for } \hbar\omega \leq E_{k_F}
\]

\[
= 0 \quad \text{otherwise.}
\]

Note that \( k_F \) (hence \( E_{k_F} \)) depends on \( \mu \). When \( \mu \) lies outside the gap, \( k_F \leq 0 \), and \( S(\omega) \) shows a cusp at \( \omega = E_{k_F}/\hbar \). When \( \mu \) lies inside the gap, \( S(\omega) \) shows a square root divergence at \( \omega = \Delta_o/\hbar \). (See fig. 13)

The noise in the tunneling current will generate a similar noise spectrum \( S_H(\omega) \) in the Hall current \( I_H \). The Hall current in \( L \) is \( I_H = L^{-1}(e/\hbar) \sum_k (\partial \epsilon_k^L / \partial k ) c^+_{L,k} c_{L,k} \), and \( S_H \) is defined as \( S_H(t) = \frac{1}{2} < [\delta I_H(t), \delta I_H]_+ > \), where \( \delta I_H = I_H^— - < I_H > \). Using eq.(24), it is straightforward to work out this noise spectrum,

\[
S_H(\omega) = \left( \frac{e}{2\hbar L} \right)^2 \left( \frac{L}{2\pi} \right)^2 \int_{k_F}^{\infty} \left( \frac{\partial \epsilon_k}{\partial k} \right)^2 (u_k v_k)^2 \pi \delta(\omega - E_k/\hbar).
\]

where we have used the fact that \( 2\epsilon_{L,k} = \epsilon_k \) for unbiased junctions. Using eq.(24) and eq.(32), we have

\[
S_H(\omega) = \frac{1}{8} \left( \frac{v_F}{L\omega} \right)^2 S(\omega).
\]

The noise spectrum of the Hall current is proportional to that of the tunneling current. While \( S_H(\omega) \) may be difficult to measure in geometries like fig. 3, it is easy to measure in
the junctions shown in fig. 3 by measuring the noise spectrum of the Hall voltage, which is simply $(\hbar/e)^2 S_H(\omega)$. Even though the junctions in fig. 3 and fig. 5 are not the same, the physics of oscillatory tunneling are identical in both cases. The noise spectrum of the tunneling current in fig. 5 should have a divergence as any self oscillating system does, which should show up in the noise spectrum of the Hall voltage.

VI. INTERACTIONS BETWEEN EDGE STATES ON DIFFERENT SIDES OF THE BARRIER

So far, we have ignored interactions between edge states on different sides of the barrier. When these interactions are included, the effective Hamiltonian eq.(23) becomes

$$H = H_o + H_T + H_{int}, \quad H_{int} = \epsilon^{-1} \sum_q U(q) \rho_L(q) \rho_R(-q), \quad (56)$$

where $\rho_L(q) = \sum_K c_{k+q,L(R)}^+ c_{k,L(R)}$, $U(q) = \int_{-L/2}^{L/2} e^{-iqy}(4a^2 + y^2)^{-1/2} dy$, and $\epsilon$ is a dielectric constant. Eq.(56) is precisely the massive Thirring model and its spectrum can be solved exactly by the Bethe Ansatz. It is known from the exact solution that there is always a gap in the spectrum for all $U$. Since the singularities in the noise spectrum and the minimum frequency of the oscillatory tunneling current are due to the existence of the tunneling gap, we expect these features will persist in the presence of interaction effects.

VII. ESTIMATES OF THE KEY PARAMETERS

Numerical estimates for the parameters in Sec.IV are given in Table 1. In these estimates, we take $m = 0.067m_e$, where $m_e$ is the mass of the electron. The barrier height has been taken as 1 ev. The tunneling gap $\Delta_o$ is calculated by the quasiclassical method, and is given by

$$\Delta_o = \frac{\hbar \Omega}{\pi} e^{-\frac{\hbar}{\Omega}} \int_{-a}^{a} |\rho|^2 dx \quad (57)$$
where $2\pi/\Omega$ is period of the classical trajectory, $\Omega \approx (3/2)\omega_c$. The momentum $|p|$ is $p = \sqrt{2m(V_o - E)} \approx \sqrt{2mV_o}$ since $E \sim \hbar \omega_c$ and $V_o >> E$. The dephasing time labelled “linear” and “quadratic” refer to the cases in Sec.IV where the range of $|k|$ states covers the linear and quadratic part of the spectrum, (fig.12). We see from these estimates that eq.(1) can be satisfied when $\tau_{in} >> 10^{-11}\text{sec}$.

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TABLE I. Parameters in Section IV

\begin{table}[h]
\centering
\begin{tabular}{|l|c|c|}
\hline
 & \multicolumn{2}{c|}{\textit{2a} = 100\text{"A}} & \textit{2a} = 60\text{"A} \\
\hline
\hline
\omega_c (sec^{-1}) & $2.6 \times 10^{13}$ & \text{same} \\
\hline
(ev) & $1.7 \times 10^{-2}$ & \text{same} \\
\hline
(^oK) & 200 & \text{same} \\
\hline
\ell (\text{"A}) & 81 & \text{same} \\
\hline
\Delta_o (ev) & $1.2 \times 10^{-4}$ & $6.6 \times 10^{-4}$ \\
\hline
\Delta_o (^oK) & 1.4 & 7.6 \\
\hline
\Delta_o / \hbar \omega_c & $7.1 \times 10^{-3}$ & $3.8 \times 10^{-2}$ \\
\hline
\tau_T (sec) & $3.3 \times 10^{-11}$ & $6.3 \times 10^{-12}$ \\
\hline
$\frac{h \nu \ell}{\Delta_o 2 \pi L}$ & $7.1 \times 10^{-4}$ & $1.3 \times 10^{-4}$ \\
\hline
linear $\tau_{dp}^{(f)}$ (sec) & $4.7 \times 10^{-8}$ & $4.7 \times 10^{-8}$ \\
\hline
quadratic $\tau_{dp}^{(f)}$ & $1.3 \times 10^{-4}$ & $7.1 \times 10^{-4}$ \\
\hline
\end{tabular}
\caption{Parameters in Section IV}
\end{table}

B = 10 Tesla

L = 1cm
FIGURES

FIG. 1. A quantum Hall junction with a circular barrier. The trajectory of a semiclassical electron is indicated by arrows. Once tunnelled across the barrier, the electron will repeat a similar reflected circular motion on the other side and eventually tunnel back.

FIG. 2. Schematic representation of the quantum mechanical edge states near the barrier. The dotted lines labelled $a$ and $b$ denote two neighboring edge states, which in general have different tunnelling rates across the barrier. As the flux $\phi$ through the hole at the center increases, the edge states will move outward, thereby increasing their tunneling rates. State $b$ will evolve to $a$ when $\phi = 2\pi$.

FIG. 3. A rectangular version of the QH junction in Figure 1. The dashed lines means that the system is periodic in $y$ with period $L$.

FIG. 4. A quantum Hall weak link. The thick black lines represent hard walls, i.e. infinite potentials. As we shall see, oscillatory tunneling of the electrons across the weak link produces a singularity in the noise of the tunneling current.

FIG. 5. A junction similar to Figure 4 except that both $R$ and $L$ are multiply connected geometries. The current oscillation across the junction will generate an oscillation of Hall voltage across each ring.

FIG. 6. In the absence of external electric fields, a semiclassical electron will be forever captured by the barrier once its trajectory is intercepted by it.

FIG. 7. The spectrum of the rectangular QH system in figure 3 in the infinite barrier limit [1]: Because of translational invariance in $y$, the spectrum can be labelled by the $y$-momentum $k$ and the Landau level index $n$. The width of the barrier is $2a$. Its boundaries are denoted in dimensionless units $(-a/\ell, a/\ell)$, where $\ell$ is the magnetic length. The spectra of $L$ and $R$, $\epsilon_{n,k}^L$ and $\epsilon_{n,k}^R$, are represented by solid and dashed curves. They rise near the barrier (i.e. $|k| \to 0$) and intersect each other. See also Section II for a detailed discussion.
FIG. 8. The intersections of the spectra in figure 7 turn into gaps as the infinite barrier becomes finite. The spectrum of the entire system will be denoted as $E_{n,k}$. They are continuous curves that reduce to $\epsilon^L_{n,k}$ and $\epsilon^R_{n,k}$ as one approaches the bulk, i.e. as $k << -\ell^{-1}$ and $k >> \ell^{-1}$, where $\ell$ is the magnetic length.

FIG. 9. A schematic representation of the wavefunctions of the ground state $u_{0,k}(x)$ and first excited state $u_{1,k}(x)$ of the entire system $L + R$ in figure 8.

FIG. 10. The spectrum of the entire system in the presence of a voltage bias. The lowest tunneling gap is now moved below the first bulk Landau level.

FIG. 11. Both (a) and (b) show the spectrum near the lowest tunneling gap in the region $\ell k \leq 1$. Part (a) is a schematic representation of the initial state $|\Psi\rangle$ discussed in Section IV when both $L$ and $R$ have identical chemical potentials $\mu$. Occupied (unoccupied) states are indicated by solid (empty) circles. The spacing in $k$ is $2\pi/L$. The states $c_{L,k}$ and $c_{R,k}$ are linear combinations of $a_{0,k}$ and $a_{1,k}$. As the boundary condition in $y$ changes, (corresponding threading a flux $\theta/2\pi$ through the hole in fig. 2), all $k$ states move to the right, i.e. 4 and 3 towards 3 and 2, 1’ and 2’ towards 2’ and 3’, etc. Part (b) shows the location of the $k$ states in (a) after half a flux is passes through the hole, i.e. $k \to k + \pi/L$. This results in an excess edge electron on top of the Fermi surface in $L$.

FIG. 12. When a chemical potential difference $eV$ is imposed between $R$ and $L$ at time $t = 0$, the Fermi wavevectors in $L$ and $R$ is changed to $-k_F + \delta k_F$ and $k_F + \delta k_F$. The range of $k$ contributing to the tunneling current is given by $|k + k_F| < \delta k_F$, which is denoted as $[k]$. In (a), the range $[k]$ lies in the linear portion of the spectrum. The tunneling current decays as $t^{-1}$. In (b), $[k]$ includes the intersection point, the tunneling current decays as $t^{-1/2}$, inside the gap.

FIG. 13. Noise spectra at different chemical potentials: When $\mu$ is below or above the gap, (a) and (c), only states in the range $|k| > k_F$ contribute to the noise. The noise spectrum has a cusp. When $\mu$ is inside the gap, all states contribute to the current. The noise spectrum has a square root singularity.
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