BOUNDARY QUOTIENT GRAPHS AND THE GRAPH INDEX

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Abstract. In this paper, we will consider the boundary quotient graphs. Let $G$ be a finite directed graph with its vertex set $V(G)$ and its edge set $E(G)$. The boundary $\partial$ of $G$ is a subset of the vertex set $V(G)$. For the given boundary $\partial \subseteq V(G)$, we give an boundary quotient : if $v_1, v_2 \in \partial$, then $v_1 = v_2$, for all $v_1, v_2 \in \partial$. Then we can construct a new graph $G_\partial = G / \partial$ called the $\partial$-quotient graph of $G$. In Chapter 1, we restrict our interests to the finite simplicial directed graphs. We will observe some properties of $G / \partial$. In particular, we show that all total boundary quotient graphs has the same type, where total boundary $\partial$ is $V(G)$. Every total boundary quotient graph is graph-isomorphic to one-vertex-$|E(G)|$-loop-edge graph. In fact, every total boundary quotient graph of a finite directed graph is graph-isomorphic to the one-vertex-multi-loop-edge graph. This result shows that boundary quotient $\partial$ is not an invariants on finite simplicial directed graphs. However, we show that the “admissible” boundary quotient is an invariant on finite simplicial directed graphs with mixed maximal types. In Chapter 2, we consider arbitrary finite directed graphs and define the subgraph boundary quotient $\partial_H$ of the given graph $G$, where $H$ is a full subgraph of $G$. The subgraph boundary $\partial_H$ is defined by the set $V(H) \cup E(H)$. By identifying all element in $\partial_H$ in $G$, we can get the subgraph boundary quotient graph $G / \partial_H$. Define the subgraph index $Ind_G(\partial_H)$ of $G$ with respect to $H$, by the exponential of $V(G / \partial_H) \cup E(G / \partial_H)$. We will observe the properties of the subgraph boundary index of finite directed graphs. In particular, we can get that $Ind_G(\partial_H) = Ind_H(1)$, where 1 is the trivial graph.

In this paper, we will consider boundary quotient graphs. Let $G$ be a finite directed graph with its vertex set $V(G)$ and its edge set $E(G)$. A boundary $\partial$ of the graph $G$ is defined by a subset of $V(G)$. For the fixed boundary $\partial$ of $G$, the boundary quotient, also denoted by $\partial$, is defined by the following relations:

(i) All elements in $\partial$ are identified. i.e., if $v_1 \neq v_2 \in \partial$, then we assume that $v_1 = v_2$, for all $v_1, v_2 \in \partial$. In other words, the boundary $\partial$ makes a base point $v_\partial$, as the identified vertices by $\partial$.

(ii) If $v_1 \neq v_2$ in $\partial$ and if there exists an edge $e$ connecting $v_1$ and $v_2$, then the edge $e$ is replaced by a loop-edge concentrated on the base point $v_\partial$.

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The loop-edges in (ii) are called the $\partial$-loop-edges at the base point $v_\partial$. Also, the edges in $E(G)$, which are not affected by the boundary quotient $\partial$, are said to be non-$\partial$. We define a boundary quotient graph $G / \partial$ of $G$ by $\partial$, by the directed graph $Q$ with its vertex set

$$V(Q) = \{v_\partial\} \cup \{v \in V(G) : v \notin \partial\}$$

and its edge set

$$E(Q) = \{e \in E(G) : e \text{ is non-}\partial\}$$

$$\cup \{l : l \text{ is a } \partial\text{-loop-edges at } v_\partial\}.$$  

We say that the boundary $\partial$ is total, if $\partial = V(G)$.

In Chapter 1, we will consider some properties of boundary quotient graph $Q = G / \partial$. We will show that the total boundary quotient graph $G / \partial_t$ of a finite simplicial directed graph $G$ is graph-isomorphic to the one-vertex-$|E(G)|$-loop-edge graph $L_{|E(G)|}$, where $\partial_t$ is the total boundary of $G$. This shows us that the boundary quotient is generally not an invariant on finite simplicial directed graphs. For example, if $C_N$ is a one-flow circulant graph and if $T_{N+1}$ is a finite directed tree with $(N + 1)$-vertices, then clearly graphs $C_N$ and $T_{N+1}$ is not graph-isomorphic. However, the total boundary quotient graphs $C_N / \partial_{C_N}$ and $T_{N+1} / \partial_{T_{N+1}}$ are graph-isomorphic, because both of them are graph-isomorphic to the graph $L_N$, where $L_N$ is the one-vertex-$N$-loop-edge graph.

We also define so-called the admissible boundaries and the admissible boundary quotient graphs. Let $G$ be a finite simplicial directed graph. The admissible boundary $\partial_a$ of $G$ is defined by a boundary with the following condition:

$$\{v_1, v_2\} \subseteq \partial_a \iff \text{there is no finite path connecting } v_1 \text{ and } v_2 \text{ on } G.$$  

The admissible boundary quotient $\partial_a$ is in general not an invariant on finite simplicial directed graphs. However, we show that the admissible boundary quotient is an invariant on the finite simplicial connected directed graph with mixed maximal type.

In Chapter 2, we introduce the subgraph boundaries which is different from the boundaries defined in Chapter 1. Let $G$ be a finite directed graph (not necessarily simplicial) and let $H$ be a full subgraph of $G$. Then the boundary $\partial_H = V(H) \cup E(H)$ is called the subgraph boundary of $G$ with respect to $H$. Similar to Chapter 1, by identifying all elements in $\partial_H$ to the base point $v_{\partial_H}$, we can construct the subgraph boundary quotient graph $G / \partial_H$. Notice that,

$$|V(G / \partial_H)| = |V(G)| - |V(H)| + 1,$$

$$|E(G / \partial_H)| = |E(G)| - |E(H)|$$

and hence
$|V(G) \cup E(G)| = |V(G) \cup E(G)| - |V(H) \cup E(H)| + 1.$

By using the above quantity, we will define the subgraph boundary index $\text{Ind}_G(H)$ of a full subgraph $H$ of $G$, by

$$\text{Ind}_G(H) = \exp \left( |G / \partial_H| - 1 \right).$$

If we denote 1 as the trivial graph (the one-vertex-no-edge graph), then we can get that

$$\text{Ind}_G(H) = \frac{\text{Ind}_G(1)}{\text{Ind}_H(1)}.$$

We will consider this quantity in detail in Chapter 2.

1. **Boundary Quotient Graphs**

In this chapter, we will define a boundary quotient graph $G / \partial$ and observe some properties of it. Throughout this chapter, let $G$ be a finite simplicial directed graph with its vertex set $V(G)$ and its edge set $E(G)$. Since the graph $G$ is simplicial, this graph contains neither multiple edges between two vertices nor loop-edges. For example, all growing trees and all one-flow circulant graphs are simplicial graphs. The boundary $\partial$ of $G$ is defined by a subset of $V(G)$. The boundary quotient graph $G / \partial$ is defined by a graph $Q$ under the boundary quotient, also denoted by $\partial$.

1.1. **Boundary Quotient Graphs.**

Let $G$ be the given finite simplicial directed graph.

**Definition 1.1.** A boundary $\partial$ of $G$ is a subset of $V(G)$. If $\partial = V(G)$, we will say that $\partial$ is total in $G$. Otherwise, we say that $\partial$ is proper in $G$. The boundary quotient of $\partial$ is defined by the following relations:

(R1) All vertices in $\partial$ are identified to the point $v_\partial$. This new point $v_\partial$ is called the base point of $\partial$, i.e., if $v_1 \neq v_2$ in $\partial$, then identify $v_1$ and $v_2$ with the base point $v_\partial$.

(R2) If $v_1 \neq v_2$ in $\partial$ and if there exists a direct edge $e$ connecting $v_1$ and $v_2$, then identify $e$ to the loop-edge $l_e$ at the base point $v_\partial$. 
The boundary quotient with (R1) and (R2) is also denoted by \( \partial \), if there is no confusion.

We say that the edges of \( G \), which are not affected by the boundary quotient \( \partial \), are non-\( \partial \) edges. Also, the loop-edges constructed by (R2) of \( \partial \), are said to be \( \partial \)-loop-edges at the base point \( v_\partial \). With respect to the fixed boundary quotient \( \partial \) on the given graph \( G \), we will define the corresponding boundary quotient graph \( G / \partial \).

**Definition 1.2.** The boundary quotient graph \( G / \partial \) is a finite directed (non-simplicial) graph \( Q \) with its vertex set

\[
V(Q) = \{v_\partial\} \cup \{v \in V(G) : v \in V(G) \setminus \partial\}
\]

and its edge set

\[
E(Q) = \{e \in E(G) : e \text{ is non-\( \partial \)}\} \cup \{l : l \text{ is a } \partial \text{-loop-edge}\}.
\]

From now, for convenience, we will denote \( e = v_1ev_2 \), where \( e \) is an edge connecting \( v_1 \) and \( v_2 \) with the direction from \( v_1 \) to \( v_2 \). (i.e., \( e \) is an edge with its initial vertex \( v_1 \) and its terminal vertex \( v_2 \).) Also, if \( w = e_1 ... e_k \) is a finite path with its length \( k \), with the admissible edges \( e_1, ..., e_k \) and if \( e_1 = v_1e_1v' \) and \( e_k = v''e_kv_2 \), then we write \( w = v_1wv_2 \) to emphasize the initial and terminal vertices of \( w \). We give some fundamental examples of boundary quotient graphs.

**Example 1.1.** Let \( G \) be a graph with \( V(G) = \{v_1, v_2\} \) and \( E(G) = \{e = v_1ev_2\} \). Suppose we have the total boundary \( \partial = V(G) \). Then, by the boundary quotient \( \partial \), we can get the boundary quotient graph \( G / \partial \), as a graph \( Q \) with its vertex set \( V(Q) = \{v_\partial\} \) and \( E(Q) = \{l_e = v_\partial \ l_e \ v_\partial\} \), i.e., the graph \( Q \) is the one-vertex-one-loop-edge graph. The \( \partial \)-loop edge \( l_e \) is constructed from the edge \( e \).

**Example 1.2.** Let \( G \) be a graph with \( V(G) = \{v_1, v_2, v_3\} \) and \( E(G) = \{e_1 = v_1e_1v_2, e_2 = v_1e_2v_3\} \). Suppose we have a \( \partial_{12} = \{v_1, v_2\} \). Then the boundary quotient graph \( G / \partial_{12} \) is a graph \( Q_{12} \) with \( V(Q_{12}) = \{v_{\partial_{12}}, v_3\} \) and \( E(Q_{12}) = \{l_{e_1} = v_{\partial_{12}} \ l_{e_1} \ v_{\partial_{12}}, e_2\} \). Similarly, if we have the boundary \( \partial_{13} = \{v_1, v_3\} \), then we have the boundary quotient graph \( G / \partial_{13} \), as a graph \( Q_{13} \) with \( V(Q_{13}) = \{v_{\partial_{13}}, v_2\} \) and \( E(Q_{13}) = \{l_{e_2} = v_{\partial_{13}} \ l_{e_2} \ v_{\partial_{13}}, e_1\} \). Now, the boundary \( \partial \) is total in \( G \), i.e., \( \partial = V(G) \). Then the boundary quotient graph \( G / \partial \) is the graph \( Q \) with \( V(Q) = \{v_\partial\} \) and \( E(Q) = \{l_{e_1} = v_\partial \ l_{e_1} \ v_\partial, l_{e_2} = v_\partial \ l_{e_2} \ v_\partial\} \).

Define the one-flow circulant graph \( C_N \), as a graph \( K \) with its vertex set \( V(K) = \{v_1, ..., v_N\} \) and its edge set

\[
E(K) = \{e_j = v_je_jv_{j+1} : j = 1, ..., N, \text{ and } v_{N+1} \overset{\text{def}}{=} v_1\}.
\]
Example 1.3. Let $G$ be a one-flow circulant graph $C_3$. Suppose we have the total
boundary $\partial = V(G)$. Then the corresponding boundary quotient graph $G / \partial$ is the
graph $Q$ with $V(G) = \{v_0\}$ and $E(G) = \{l_{e_j} = v_0 l_{e_j} v_0 : j = 1, 2, 3\}$. So, the
graph $G / \partial$ is the one-vertex-three-loop-edge graph.

Let $G_1$ and $G_2$ be finite simplicial directed graphs and let $v_1$ and $v_2$ be arbitrary
fixed vertices of $G_1$ and $G_2$, respectively. Define the vertex-fixed glued graph
$G_1 \# v_1 \# v_2 G_2$ at the glued vertex $v_1 \# v_2$ by the graph $G$ with the following
conditions:

(C1) Identify $v_1$ and $v_2$. This identified vertex $v_1 \# v_2$ in $G$ is called the glued
vertex.

(C2) $V(G) = \{v_1 \# v_2\} \cup (V(G_1) \setminus \{v_1\}) \cup (V(G_2) \setminus \{v_2\})$.

(C3) $E(G) = E(G_1) \cup E(G_2)$.

Inductively, we can have the vertex-fixed glued graph $G_1 \# v_1 \# ... \# v_m G_m$, for $m \in \mathbb{N}$, at its glued vertex $v_1 \# ... \# v_m$. The vertex-fixed glued graphs are
depending on the choice of their glued vertices. i.e., the vertex-fixed glued graph
$G_1 \# v \# ... \# v G_m$ at the glued vertex $v = v_1 \# ... \# v_m$ and the vertex-fixed
glued graph $G_1 \# v' \# ... \# v' G_m$ at the glued vertex $v' = v'_1 \# ... \# v'_m$ are not
graph-isomorphic, in general, if $v_j \neq v'_j$, for some $j$ in $\{1, ..., m\}$.

Proposition 1.1. Let $G_j$ be one-flow circulant graph $C_{n_j}$, for all $j = 1, ..., m$.
Then the vertex-fixed glued graph $G_1 \# \ldots \# G_m$ is independent of the choice of
 glued vertex, up to graph-isomorphisms. □

The above proposition is easily proved, by the definition of the circulant graphs
and graph-isomorphisms.

Example 1.4. Let $G$ be a one-flow circulant graph $C_4$ and let the boundary $\partial$ is
$\{v_1, v_2\}$. Then the boundary quotient graph $G / \partial$ is the graph $Q$ with $V(Q) = \{v_0, v_3, v_4\}$
and $E(G) = \{l_{e_1} = v_0 l_{e_1} v_0, e_2, e_3, e_4\}$. Notice that this graph $Q$ is the
vertex-fixed glued graph $G_1 \# G_2$, where $G_1$ is the graph with $V(G) = \{v_0\}$ and
$E(G) = \{l_{e_1}\}$ and $G_2$ is the graph with $V(G_2) = \{v_0, v_2, v_3\}$ and $E(G) = \{e_2, e_3, e_4\}$. Moreover, the graph $G_1$ is graph-isomorphic to the one-vertex-one-loop-edge graph
and the graph $G_2$ is graph-isomorphic to the one-flow circulant graph $C_3$.

We will define the following special types on the finite directed simplicial graphs.

Definition 1.3. (1) Let $G$ be a graph which is graph-isomorphic to $C_n$, the one-flow
circulant graphs with $n$-vertices. Then the graph $G$ is said to be of type $C_n$.

(2) We say that a graph $G$ is of type $L_m$, if the graph $G$ is graph-isomorphic to
the one-vertex-$m$-loop-edge graph.

(3) A graph $G$ is of type $T$ if $G$ is isomorphic to a directed tree $T$. 
(4) A graph $G$ is of mixed type if there exists full subgraphs $G_1, \ldots, G_n$ of $G$ such that (i) $\{G_1, \ldots, G_n\}$ is the minimal covering of $G$ and (ii) $G_j$ is of type $C_{N_j}$ or of type $L_{N_j}$ or of type $T_j$, for $j = 1, \ldots, n$. (Since a graph $G$ is finite, we can always choose such finite covering consisted of full subgraphs.)

**Proposition 1.2.** Let $G$ be a one-flow circulant graph $C_N$ and let $\partial$ be the total boundary of $G$. Then the boundary quotient graph $G / \partial$ is the graph of type $L_N$.

**Proof.** Let $G$ be a one-flow circulant graph and let $\partial = V(G)$ be the total boundary of $G$. Then, by the boundary quotient, the graph $G / \partial$ is a graph $Q$ with $V(Q) = \{v_0\}$ and $E(Q) = \{e_j = v_0 \rightarrow v_j : j = 1, \ldots, N\}$, where $e_j$'s are edges $v = v_j e_j v_{j+1}$ in $E(G)$, for all $j = 1, \ldots, N$, with $v_{N+1} = 1$. Therefore, the graph $G / \partial$ is of type $L_N$.

More generally, we can get that;

**Proposition 1.3.** (1) Let $G$ be a one-flow circulant graph $C_N$ and let $\partial_k = \{v_j, v_{j+1}, \ldots, v_{j+k-1}\}$ be a boundary consisting of $k$-vertices, where $1 \leq j < N$ and $2 \leq j + k < N + 1$. Then the boundary quotient graph $G / \partial_k$ is of mixed type $(L_k, C_{N-k})$.

(2) Let $G$ be a one-flow circulant graph $C_N$ and let $\partial_k = \{v_{j_1}, v_{j_2}, \ldots, v_{j_n}\}$, where $(j_1, \ldots, j_n)$ is a sequence in $\{1, \ldots, N\}$, for $n < N$ satisfying that $j_{k+1} = j_k + t_k$, for $t_k > 1$ and for $k = 1, \ldots, n$. Then the boundary quotient graph $G / \partial_k$ is of type $(C_{j_1-1}, C_{j_2-1}, C_{j_3-1}, \ldots, C_{N-j_n-1}, C_{N-j_n})$.

**Proof.** (1) Suppose $1 < j < N$ and $1 < j + k - 1 < N$. Consider the full subgraph $L$ of $G$ with its vertex set $V(L) = \partial_k$. Then, by the previous proposition, the boundary quotient graph $L / \partial$ is of type $L_k$. With respect to the base point $v_0$, there exists subgraph $K$ of type $(N - k)$ such that $G / \partial = \#_{v_0} K$. i.e., the boundary quotient graph $G / \partial$ is the glued graph $L \#_{v_0} K$ of maximal type, with the base point $v_0$, where $G_1$ is of type $L_k$ and $G_2$ is of type $C_{N-k}$.

(2) The boundary quotient graph $G / \partial_k$ satisfies that $G / \partial_k$ is graph-isomorphic to the glued graph $K_1 \#_{v_0} K_2 \#_{v_0} \ldots \#_{v_0} K_n$, with its glued vertex (which is the base point of $G / \partial_k$), where $K_1$ is of type $C_{j_1-1}$ and $K_i$ is of type $C_{j_i-1}$, for each $i = 1, \ldots, n - 1$, and $K_n$ is of type $C_{N-j_n}$.

Let $T_N$ be a finite directed tree with $N$-vertices. The given tree $T_N$ is said to be a growing tree if there exists the root vertex $v_0$, and there always exists a unique finite path $w_v = v_0 w v$ on $T_N$, for all other vertex $v \neq v_0$ in $V(G)$. Thus the growing tree $T_N$ has only one-flow from the root vertex $v_0$. 
Proposition 1.4. Let $T_N$ be a tree with $N$-vertices and let $\partial$ be the total boundary of $T_N$. Then the boundary quotient graph $T_N / \partial$ is of type $L_{N-1}$. □

The above proposition is proved by induction. The above proposition says that if $T$ is a tree and if $\partial = V(T)$ is the total boundary, then $T$ is of type $L_{|E(T)|}$. Also, it says that the total boundary quotient is not an invariants on finite simplicial graphs. Since the total boundary quotient of $C_{N-1}$ is also type $L_{N-1}$. Trivially the graphs $C_{N-1}$ and $T_N$ are not graph-isomorphic. But their total boundary quotient graphs are of type $L_{N-1}$. Thus the total boundary quotient is not an invariants on graphs.

Notice that every finite connected simplicial directed graph $G$ is graph-isomorphic the iterated glued graph

$$G_1 \#_{v_1} (G_2 \#_{v_2} (G_3 \#_{v_3} (G_4 \#_{v_4} ... (G_n \#_{v_n} G_{n+1}))))$$

of trees, circulant graphs and trivial graphs. Different from the vertex-fixed glued graph case, it is not necessary that $v_1 = v_2 = ... = v_n$. Remark that the iterated glued graphs of $G$ are not uniquely determined. But we can choose the maximal one-flow circulant full subgraphs in $G$ and the maximal sub-trees in $G$, in their neighborhoods. Also, if we fix the subset $V$ of the vertex set $V(G)$, then we can choose the maximal one-flow circulant full subgraphs and the maximal full sub-trees in $G$, with respect to the fixed vertices in $V$. For example, let $G$ be a graph with

$$V(G) = \{v_1, ..., v_8\}$$

and

$$E(G) = \begin{cases} e_1 = v_1e_1v_2, & e_2 = v_2e_2v_3, \\ e_3 = v_3e_3v_5, & e_4 = v_5e_4v_4, \\ e_5 = v_4e_5v_1, & e_6 = v_5e_6v_6, \\ e_7 = v_6e_7v_7, & e_8 = v_6e_8v_8 \end{cases}.$$  

Fix a boundary $\partial_1 = \{v_2, v_5, v_8\}$. Then, for the fixed vertices $v_2, v_5$ and $v_8$, we can choose the full subgraphs $G_1, G_2$ and $G_3$, as graphs with

$$V(G_1) = \{v_2, v_3, v_5\} \text{ and } E(G_1) = \{e_2, e_3\},$$
$$V(G_2) = \{v_1, v_2, v_4, v_5\} \text{ and } E(G_2) = \{e_1, e_4, e_5\},$$

and

$$V(G_3) = \{v_5, v_6, v_7, v_8\} \text{ and } E(G_3) = \{e_6, e_7, e_8\}.$$  

All those three full subgraphs are trees. Then the boundary quotient graph $G / \partial_1$ is the glued graph

$$G'_1 \#_{v_1} (G'_2 \#_{v_1} (G'_3 \#_{v_1} G_4)).$$
where \( G_1' \) is of type \( C_2 \), \( G_2' \) is of type \( C_3 \) and \( G_3' \) is of type \( C_2 \) and where \( G_4 \) is a tree with \( V(G_4) = \{v_6, v_7\} \) and \( E(G_4) = \{e_7\} \). Also, we have the following iterated glued graph of the above given graph \( G \);

\[
K_1 \#_{v_2} (K_2 \#_{v_3} (((K_3 \#_{v_4} (K_4 \#_{v_5} K_5 \#_{v_1} K_6)) \#_{v_4} K_7) \#_{v_6} K_8)),
\]

where

\[
\begin{align*}
V(K_1) &= \{v_1, v_2\} \quad \text{and} \quad E(K_1) = \{e_1\}, \\
V(K_2) &= \{v_2, v_3\} \quad \text{and} \quad E(K_2) = \{e_2\}, \\
V(K_3) &= \{v_3, v_4\} \quad \text{and} \quad E(K_3) = \{e_3\}, \\
V(K_4) &= \{v_4, v_5, v_6\} \quad \text{and} \quad E(K_4) = \{e_4, e_6\}, \\
V(K_5) &= \{v_1, v_5\} \quad \text{and} \quad E(K_5) = \{e_5\}, \\
V(K_6) &= \{v_1\} \quad \text{and} \quad E(K_6) = \emptyset, \\
V(K_7) &= \{v_4, v_6\} \quad \text{and} \quad E(K_7) = \{e_0\}
\end{align*}
\]

and

\[
V(K_8) = \{v_6, v_7, v_8\} \quad \text{and} \quad E(K_8) = \{e_7, e_8\}.
\]

The above example shows us that the iterated glued graph of the graph-isomorphic graph is not uniquely determined. However, we can choose the iterated glued graph of a given graph by the maximal \( CT \)-iterated glued graph.

**Definition 1.4.** Let \( G \) be a finite connected simplicial directed graph. A \( CT \)-iterated glued graph of \( G \) is the graph-isomorphic graph \( K = K_1 \#_{v_1} K_2 \ldots \#_{v_n} K_{n+1} \), where each \( K_j \) is a full subgraph of \( G \) which is of type \( C_N \) or of type \( T \) or of type 1, for \( j = 1, \ldots, n + 1 \). A graph is of type 1, if it is the trivial graph which is the one-vertex-no-edge graph. We say that the \( CT \)-iterated glued graph \( K \) is maximal if each gluing components \( K_j \)'s are the maximal full subgraphs satisfying the type, in its neighborhood of \( G \). The maximal type \((t_1, \ldots, t_{n+1})\) of \( G \) is defined by

\[
t_j = \begin{cases} 
C_N, & \text{if } K_j \text{ is of type } C_N, \\
T, & \text{if } K_j \text{ is of type } T, \\
1, & \text{if } K_j \text{ is the trivial graph,}
\end{cases}
\]

for \( j = 1, \ldots, n + 1 \). (Recall that, by definition, \( C_N \) is the one-flow circulant graph and \( T \) is a tree.) The type of the maximal \( CT \)-iterated glued graph is called the maximal type of \( G \).

In the above example, the first iterated glued graph of \( G \) is a \( CT \)-iterated glued graph which is not maximal. The second iterated graph of \( G \) is an iterated glued graph of \( G \) which is not a \( CT \)-iterated glued graph. The following glued graph of \( G \) is the maximal \( CT \)-iterated glued graph of \( G \);

\[
G_1 \#_{v_5} G_2
\]

with

\[
V(G_1) = \{v_1, v_2, v_3, v_4, v_5\} \quad \text{and} \quad E(G_1) = \{e_1, e_2, e_3, e_4, e_5\}
\]
and

\[ V(G_2) = \{v_5, v_6, v_7, v_8\} \quad \text{and} \quad E(G_2) = \{e_6, e_7, e_8\}, \]

where \( G_1 \) is of type \( C_5 \) and \( G_2 \) is of type \( T_4 \), where \( T_4 \) is the graph with \( V(T_4) = \{b_1, b_2, b_3, b_4\} \) and \( E(T_4) = \{f_1, f_2, f_3\} \), where \( f_1 = b_1 f_1 b_2, f_2 = b_2 f_2 b_3 \) and \( f_3 = b_2 f_3 b_4 \), i.e., the maximal \( CT \)-iterated glued graph \( G_1 \#_{v_3} G_2 \) of \( G \) is of its maximal type \((C_5, T_4)\).

In the above definition, we introduced the trivial graph. The trivial graphs are needed for finding the maximal \( CT \)-iterated glued graph. For example, let \( G \) be a graph with \( V(G) = \{v_1, v_2\} \) and \( E(G) = \{e_1 = v_1 e_1 v_2, e_2 = v_2 e_2 v_1\} \). This graph has the following maximal \( CT \)-iterated glued graph \( K \),

\[
K = K_1 \#_{v_2} K_2 \#_{v_1} K_3,
\]

with

\[
V(K_1) = \{v_1, v_2\} \quad \text{and} \quad E(K_1) = \{e_1\},
\]

\[
V(K_2) = \{v_1, v_2\} \quad \text{and} \quad E(K_2) = \{e_2\}
\]

and

\[
V(K_3) = \{v_1\} \quad \text{and} \quad E(K_3) = \emptyset.
\]

Remark that \( K_3 \) is the trivial graph of its type 1. So, this graph \( G \) is of mixed maximal type \((T, T', 1)\), where \( T = K_1 \) and \( T' = K_2 \). We can have the following lemma.

**Lemma 1.5.** Let \( G \) be a finite simplicial directed graph. Then there always exists the maximal \( CT \)-iterated glued graph. Furthermore, the maximal \( CT \)-iterated glued graph is uniquely determined, up to graph-isomorphisms.

**Proof.** Let \( G \) be the given graph. Then this graph is the disjoint union of \( G_1, ..., G_n \), where \( G_j \) is the connected component of \( G \), for \( j = 1, ..., n \). Since \( G \) is finite simplicial, all \( G_j \)'s are also finite simplicial. By the simplicity of \( G_j \), this graph \( G_j \) does not contain the loop-edges and multiple edges. So, graphically, we can choose the full subgraphs \( G_{j_1}, ..., G_{j_n} \), where \( G_{j_i} \) is of type \( C_{N_i} \) or of type \( T_{k_i} \) or of type 1, for \( i = 1, ..., n_j \) and for all \( j = 1, ..., n \). By the Axiom of Choice, we can choose such maximal family.

Now, assume that \( K_1 \) and \( K_2 \) are maximal \( CT \)-iterated glued graphs of \( G \). For convenience, suppose that the given graph \( G \) is connected. Let’s assume that \( K_1 \) and \( K_2 \) are not graph-isomorphic. Without loss of generality, we may suppose that \( K_1 \) (resp. \( K_2 \)) has \( K_{i_1}, ..., K_{i_s} \) (resp. \( K_{j_1}, ..., K_{j_t} \)) gluing components which are of type \( C_{N_1}, ..., C_{N_s} \) (resp. \( C_{N_1}, ..., C_{N_t} \)), respectively, and \( s \neq t \) in \( \mathbb{N} \). Also, assume that \( s < t \). By the maximality, \( K_2 \) cannot be the maximal \( CT \)-iterated glued graph of \( G \). \( \blacksquare \)
The above lemma says that every finite simplicial directed graph is graph-isomorphic to the maximal CT-iterated glued graph of maximal type \(((C_1, \ldots, C_n), (T_1, \ldots, T_k), (1, \ldots, 1))\). Therefore, we can conclude that the boundary quotient graph \(G / \partial\) is, in general, of maximal type \(((L_1, \ldots, L_m), (C_1, \ldots, C_n), (T_1, \ldots, T_k), (1, \ldots, 1))\).

**Theorem 1.6.** Let \(G\) be an arbitrary finite simplicial graph and let \(\partial\) be a boundary of \(G\). Then, in general, the quotient graph \(G / \partial\) is of maximal type \(((L_{m_1}, \ldots, L_{m_s}), (C_{N_1}, \ldots, C_{N_n}), (T_{k_1}, \ldots, T_{k_m}), (1, \ldots, 1))\), where \(|E(G)| = \sum_{p=1}^m m_p + \sum_{i=1}^n N_i + \sum_{r=1}^m k_r\).

**Proof.** Let \(G\) and \(\partial\) be given and assume that \(G\) is connected. By the previous lemma, the graph \(G\) is graph-isomorphic to the maximal CT-iterated glued graph of \(G_1, \ldots, G_n\), where \(G_j\) is either of type \(C_{n_j}\) or of type \(T_{k_j}\), for \(j = 1, \ldots, n\). By regarding each gluing component \(G_j\) as the full subgraph \(K_j\) of \(G\), we can get the subboundary \(\partial_{G_j}\), for each \(j\), i.e., \(\partial_{G_j} = V(G_j) \cap \partial\). Notice that we have the boundary quotient graph \(G / \partial\) is the vertex-fixed glued graph of \(G / \partial_j\), \(j = 1, \ldots, n\), with its glued vertex \(v_0 = v_{\partial_1} \# \cdots \# v_{\partial_n}\). By the previous propositions, if \(G_j\) is of type \(C_{N_j}\) (or of type \(T_{k_j}\)), then \(G_j / \partial_j\) is of maximal type of \((L_{m_j}, (C_{N_j,m_j}), (T_{j,k_j}, \ldots, T_{j,k_j}))\) for \(j = 1, \ldots, n\). Since \(G / \partial = (G_1 / \partial_1) \# v_{\partial_1} \cdots \# v_{\partial_n} (G_n / \partial_n)\), it is of maximal type with \(L_{m_j}\)'s, \(C_{N_j}\)'s and \(T_j\)'s. Remark that \(|E(G / \partial)| = |E(G)|\). Therefore,

\[|E(G)| = m + \sum_{i=1}^n N_i + \sum_{r=1}^m k_r.\]

Now, suppose that the graph \(G\) is the disjoint union of finite simplicial connected directed graphs \(G_1, \ldots, G_k\). Then, for each \(G_k\), we have the above result. And hence we can get the desired result. 

If we consider the total boundary quotient graph, we have the following simple results;

**Theorem 1.7.** Let \(G\) be a finite connected simplicial directed graph and let \(\partial\) be the total boundary of \(G\). Then the boundary quotient graph \(G / \partial\) is of type \(L_{|E(G)|}\).

**Proof.** By the previous lemma, \(G\) is graph-isomorphic to the maximal CT-iterated glued graph \(K\) with its gluing components \(K_1, \ldots, K_n\). Then each \(K_j\) is of type \(C_N\) or type \(T\) or of type 1. Notice that if \(\partial\) is total in \(G\), then

\[G / \partial = (K_1 / \partial_1) \# \cdots \# (K_n / \partial_n),\]

where the right-hand side is the vertex-fixed glued graph and where \(\partial_j = \partial \cap V(K_j) = V(K_j)\) for \(j = 1, \ldots, n\). If \(K_j\) is trivial, then \(K_j / \partial_j\) is trivial. If \(K_j\) is either of type \(C_N\) or of type \(T\), then \(K_j / \partial_j\) is of type \(L_{|E(K_j)|}\). Therefore, we can conclude the result.
1.2. General Total Boundary Quotient Graphs.

In the previous section, we only considered the finite simplicial directed graphs. We say that the directed graph $G$ is finite if $|V(G)| < \infty$ and $|E(G)| < \infty$. However, they may have the loop-edges and multiple edges between two vertices. However, we can extend the above results in Section 1.1 to the general finite graph cases. Moreover, we have that

**Theorem 1.8.** Let $G$ be a finite directed graph and $\partial$, the total boundary of $G$. Then the total boundary quotient graph $G / \partial$ of $G$ is of type $L_{|E(G)|}$. □

We cannot use the maximal CT-iterated glued graph technique to prove the above general case. But we can use the edge-iterated glued graph of $G$. If there are multiple edges $e_1, ..., e_k$ connecting $v_1$ and $v_2$, with same direction. Take the full subgraph $K_j$ with $V(K_j) = \{v_1, v_2\}$ and $E(K_j) = \{e_j\}$, for $j = 1, ..., k$. If we identify the vertex $v_1$ and $v_2$, then we have $(K_1 / \{v_1, v_2\}) \# v_1 \# v_2 ... \# v_1 \# v_2 (K_k / \{v_1, v_2\})$ of type $L_k$. So, if we construct the edge-iterated glued graph of a finite directed graph $G$, with its gluing components which are generated by edges like above $K_j$'s, then we can prove the above theorem. Again, notice that if $G$ is graph-isomorphic to the edge-iterated glued graph $K$ with its gluing components $K_1, ..., K_N$, then the total boundary quotient graph $G / \partial$ satisfies that

$$G / \partial = (K_1 / \partial_1) \# v_\partial ... \# v_\partial (K_N / \partial_N),$$

where $\partial_1, ..., \partial_N$ are total boundaries of $K_1, ..., K_N$, respectively, and where $v_\partial$ is the base point $v_{\partial_1} \# ... \# v_{\partial_N}$.

1.3. Admissible Boundary Quotient Graphs.

In this section, we will define and observe the admissible boundaries of finite simplicial directed graphs and the corresponding boundary quotient graphs.

**Definition 1.5.** Let $G$ be a finite simplicial directed graph and let $\partial_a$ be a boundary of $G$. The boundary $\partial_a$ is said to be an admissible boundary if $\partial_a$ is the boundary of $G$ satisfying the following conditions;

(1) there is no admissible finite paths connecting $v_1$ and $v_2$, for all pair $(v_1, v_2) \in \partial_a \times \partial_a$ such that $v_1 \neq v_2$.

(2) the set $\partial_a$ is the maximal subset of $V(G)$ satisfying the condition (1).
We can get that the boundary quotient $\partial_a$ is an invariant on finite simplicial directed graphs with its mixed maximal types. First let’s show the following lemma.

**Lemma 1.9.** Let $G$ be a finite simplicial connected directed graph which is graph-isomorphic to its maximal CT-iterated glued graph $K$ of its mixed maximal type $((C_{N_1}, \ldots, C_{N_n}), (T_1, \ldots, T_r), (1, \ldots, 1))$. If $G_1, \ldots, G_r$ are full subgraphs of $G$ with their type $T_1, \ldots, T_r$, respectively, as the gluing components of $K$, then the admissible boundary $\partial_a$ of $G$ is contained in $\bigcup_{j=1}^{r} V(G_j)$. In particular, $\partial_a = \bigcup_{j=1}^{r} (V(G_j) \cap \partial_a)$.

**Proof.** Remark that the given graph $G$ is connected. Since $G$ is finite simplicial, there is a unique maximal CT-iterated glued graph $K$ graph-isomorphic to $G$. Let $K_1, \ldots, K_n$ be the gluing components of $K$. Then each $K_j$ is graph-isomorphic to a full subgraph of $G$ and it is of type $C_{N_j}$ or of type $T_j$ or of type 1. Assume that $K_j$ is of type $C_{N_j}$. Then, for any pair $\{v_1, v_2\}$ in $V(K_j) \times V(K_j)$, there always exists a finite path connecting $v_1$ and $v_2$ in $K_j$, because $K_j$ is a circulant graph. So, $\partial_a \cap V(K_j) = \emptyset$, for such $j$. Now, suppose that $K_j$ is of type $T$. Then $\partial_a \cap K_j$ is either empty or non-empty (See the following example). Therefore,

$$\partial_a = \bigcup_{i=1}^{r} (\partial_a \cap V(K_j)),$$

where $K_j$ is of type $T_j$, for all $i = 1, \ldots, r$. ■

The above lemma shows how we can determine the admissible boundaries for the simplicial connected directed graph $G$ is the subset of the disjoint union of the vertex sets of gluing components of type $T$.

**Example 1.5.** Consider the following three non-isomorphic trees $T_1$, $T_2$ and $T_3$, where

$$V(T_1) = \{v_1, v_2, v_3\} \text{ and } E(T_1) = \{e_1 = v_1^1v_1^2, e_2 = v_1^1v_2^3\}$$
$$V(T_2) = \{v_1^2, v_2^3, v_3^3\} \text{ and } E(T_2) = \{e_3 = v_2^1v_3^2, e_4 = v_2^3v_3^3\}$$
$$V(T_3) = \{v_1^3, v_2^3, v_3^3\} \text{ and } E(T_3) = \{e_1 = v_2^1v_1^3, e_2 = v_3^3v_3^3\}.$$

Then the admissible boundaries $\partial_{a_1}$, $\partial_{a_2}$ and $\partial_{a_3}$ of $G_1$, $G_2$ and $G_3$ are

$$\partial_{a_1} = \emptyset, \quad \partial_{a_2} = \{v_2, v_3\} \quad \text{and} \quad \partial_{a_3} = \{v_2, v_3\}.$$

So, the admissible boundary quotient graph $T_1 / \partial_{a_1} = T_1$ and the admissible boundary quotient graphs $T_2 / \partial_{a_2}$ and $T_3 / \partial_{a_3}$ are graphs with

$$V(T_2 / \partial_{a_2}) = \{v_2^1, v_2\} \text{ and } E(T_2 / \partial_{a_2}) = \{e_1, e_2\},$$

where $e_1 = v_2^1v_2v_{a_2}$ and $e_2 = v_2^1e_2v_{a_2}$, and
In the above example, we can observe that the admissible boundary quotient graphs $T_2 / \partial a_2$ and $T_3 / \partial a_3$ are graph-isomorphic, via the graph isomorphism $g : T_2 / \partial a_2 \to T_3 / \partial a_3$ mapping $v_1^2 \mapsto v_{\partial a_3}$ and $v_2 \mapsto v_3^1$. So, unfortunately, the admissible boundary quotient is not an invariant on the finite simplicial directed graphs.

**Remark 1.1.** The admissible boundary quotient is not an invariant on finite directed trees and hence it is not an invariant on finite simplicial connected directed graphs.

Recall that we say a finite simplicial directed graph $G$ is of maximal type if it is graph-isomorphic to the maximal CT-iterated glued graph $K$ and each gluing component is of type $C_N$ or of type $T$ or of type 1. The given graph $G$ is said to be of **mixed maximal type**, if there exists at least one distinct pair $(K_1, K_2)$ of the gluing components such that $K_1$ and $K_2$ have different types. i.e., if $K_1$ is of type $C_N$ (or $T$), then $K_2$ is of type $T'$ (resp. $C_M$). As we have seen before, the admissible boundary quotient $\partial a$ on finite simplicial directed graph is not an invariant. However, we can get the following result:

**Theorem 1.10.** The admissible boundary quotient $\partial a$ is an invariant on finite simplicial connected directed graphs of **mixed maximal type**. i.e., if $G_1$ and $G_2$ are graph-isomorphic finite simplicial directed graphs and if $\partial a_1$ and $\partial a_2$ are corresponding admissible boundaries of $G_1$ and $G_2$, respectively, then the boundary quotient graphs $G_1 / \partial a_1$ and $G_2 / \partial a_2$ are also graph-isomorphic. And the converse is also true.

**Proof.** ($\Rightarrow$) Suppose $G_1$ and $G_2$ are graph-isomorphic and assume that $g : G_1 \to G_2$ is the graph-isomorphism. Then, by definition, the map $g$ is a bijection between $V(G_1)$ and $V(G_2)$, preserving the admissibility on $G_1$. Take the admissible boundaries $\partial a_1$ and $\partial a_2$ of $G_1$ and $G_2$, respectively. Since $g$ preserves the admissibility, we can get that $g(\partial a_1) = \partial a_2$, by the maximality of $\partial a_1$ and $\partial a_2$. Therefore, $G_2 / \partial a_2 = g(G_1) / g(\partial a_1)$. They are graph-isomorphic via $g^* : G_1 / \partial a_1 \to G_2 / \partial a_2$, where

$$g^*(v) = \begin{cases} g(v) & \text{if } v \in V(G_1) \setminus \partial a_1 \\ v_{\partial a_2} & \text{if } v = v_{\partial a_1}. \end{cases}$$

Remark that $V(G_1 / \partial a_1) = (V(G_1) \setminus \partial a_1) \cup \{v_{\partial a_1}\}$. Since $g(\partial a_1) = \partial a_2$, the isomorphism $g^*$ is well-determined by $g$. (Notice that we use neither the assumption that $G_1$ and $G_2$ are of mixed maximal type nor $G_1$ and $G_2$ are connected. See the next proposition.)
(⇐) Assume that $G_1$ and $G_2$ are not graph-isomorphic. Recall that every finite simplicial directed graph is graph-isomorphic to its unique maximal $CT$-iterated glued graph. Take the finite directed graphs $K_1$ and $K_2$ which are the maximal $CT$-iterated glued graphs of $G_1$ and $G_2$, respectively. Suppose that $K_j$ has its gluing components $K_{j1}, ..., K_{jn_j}$, for $j = 1, 2$. Since $G_1$ and $G_2$ are not graph-isomorphic, $K_1$ and $K_2$ are also not graph-isomorphic. In other words, $K_1$ and $K_2$ have different types. Assume that the gluing components $K_{1r_1}$ are of type $T_{r_1}$, in $\{K_{11}, ..., K_{1n_1}\}$ and $K_{2r_2}$ are of type $T_{s_2}$, in $\{K_{21}, ..., K_{2n_2}\}$. By the previous lemma, we have that

$$\partial_{a_j} = \bigcup_i (\partial_a \cap V(K_{jr_i})), \text{ for } j = 1, 2.$$ 

In other words, the admissible boundary quotient $\partial_{a_j}$ does not act on the gluing components of type $C_N$. So, the admissible boundary quotient graphs of $G_1$ and $G_2$ are not graph-isomorphic, since $K_1$ and $K_2$ have the different type with same number of vertices. $\blacksquare$

In the previous theorem, we show that if two finite simplicial connected directed graphs with the mixed maximal types are graph-isomorphic, then their admissible boundary quotient graphs are also graph-isomorphic. In the next proposition, we will consider the general case when we just have two graph-isomorphic finite simplicial directed graphs (which are not necessarily of mixed maximal type). We can easily verify that their admissible boundary quotient graphs are also graph-isomorphic, by (⇒) in the proof of the previous theorem. In (⇒) of the previous proof, we did not use the assumption that $G_1$ and $G_2$ are of mixed maximal type. Therefore, by (⇒) of the previous proof, we have;

**Proposition 1.11.** Let $G_j$ be a finite simplicial directed graph (not necessarily connected or mixed maximal type) and let $\partial_{a_j}$ be the admissible boundary of $G_j$, for $j = 1, 2$. If $G_1$ and $G_2$ are graph-isomorphic, then $G_1 / \partial_{a_1}$ and $G_2 / \partial_{a_2}$ are graph-isomorphic. $\square$

Again, remark that the converse of the previous proposition does not hold true, by the previous example. The previous theorem provides the condition which makes the converse of the previous proposition hold true. The condition we found is when $G_1$ and $G_2$ are connected and they are of mixed maximal type.

**Example 1.6.** (1) Let $G = G_1 \#_{v_3} G_2$ be a glued graph of $G_1$ and $G_2$, with its glued vertex $v_3$, where $G_1$ is the graph of type $C_3$ with $V(G_1) = \{v_1, v_2, v_3\}$ and $E(G_1) = \{e_1, e_2, e_3\}$, and $G_2$ is the graph of type $T$, with $V(G_2) = \{v_3, v_4, v_5\}$ and $E(G_2) = \{e_4 = v_3 v_4, e_5 = v_3 v_5, e_3\}$, i.e., the graph $G$ has the maximal $CT$-iterated glued graph of mixed maximal type $(C_3, T)$. Then the admissible boundary $\partial_a = \{v_3, v_5\}$. So, we have the admissible quotient graph $G / \partial_a$ having its glued graph $K_1 \#_{v_3} K_2$, where $K_1$ is the full subgraph $G_1$ in $G$ and $K_2$ is the graph with $V(K_2) = \{v_{\partial_a}, v_3\}$ and $E(K_2) = \{f_1, f_2\}$, where $f_1 = v_3 f_1 v_{\partial_a}$ and $f_2 = v_3 f_2 v_{\partial_a}$. 

(2) Now, let $G' = G_1' \#_v G_2'$ be a glued graph of $G_1'$ and $G_2'$, with its glued vertex $v_3$, where $G_1'$ is the graph of type $C_3$ with $V(G_1') = \{v_1, v_2, v_3\}$ and $E(G_1') = \{e_1, e_2, e_3\}$, and $G_2'$ is the graph of type $T'$, with $V(G_2') = \{v_4, v_5, v_6\}$ and $E(G_2') = \{e_4 = v_4 e_5 v_6, e_5 = v_5 e_5 e_3\}$, i.e., the graph $G'$ has the maximal CT-iterated glued graph of mixed maximal type $(C_3, T')$. Then the admissible boundary $\partial_{a_1} = \{v_4, v_5\}$. So, we have the admissible quotient graph $G / \partial_{a_1}$ having its glued graph $K_1' \#_v K_2'$, where $K_1'$ is the full subgraph $G_1'$ in $G'$ and $K_2$ is the graph with $V(K_2) = \{v_{\partial_{a_1}}, v_3\}$ and $E(K_2) = \{f_1, f_2\}$, where $f_1 = v_{\partial_{a_1}} f_1 v_3$ and $f_2 = v_{\partial_{a_1}} f_2 v_3$.

(3) The gluing components $G_2$ of $G$ in (1) and $G'_2$ of $G'$ in (2) are graph-isomorphic, by the previous example. However, the admissible boundary quotient graphs $G / \partial_{a_1}$ in (1) and $G' / \partial_{a_1}$ in (2) are not graph-isomorphic.

(4) It is easy to check that the one-flow circulant graphs $C_{N_1}$ and $C_{N_2}$ are graph-isomorphic if and only if $N_1 = N_2$ in $\mathbb{N} \setminus \{1\}$. Moreover, the admissible boundaries $\partial_1$ and $\partial_2$ of $C_{N_1}$ and $C_{N_2}$ are empty. Therefore the admissible boundary quotient graphs of them are $C_{N_1}$ and $C_{N_2}$, respectively. Thus, anyway, the boundary quotient is an invariant on finite one-flow circulant graphs.

2. Subgraph Boundary Quotient Graphs and Subgraph Boundary Index

Let $G$ be a finite directed graph such that $|V(G)| < \infty$ and $|E(G)| < \infty$ (not necessarily simplicial and connected). In this chapter, we will define the graph index $\text{Ind}_G(H)$ of the graph $G$ with respect to its full subgraph $H$. To do that we will define the subgraph boundary quotient graph of $G / \partial_H$, where $\partial_H$ is the subgraph boundary of $H$.

**Definition 2.1.** Let $G$ be an arbitrary finite directed graph and let $H$ be a full subgraph of $G$. Define the subgraph boundary $\partial_H$ by the set $V(H) \cup E(H)$.

Now, we will define the subgraph boundary quotient graph $G / \partial_H$.

**Definition 2.2.** Let $G$ be a finite directed graph and $H$, a full subgraph and let $\partial_H$ be the subgraph boundary. Define the subgraph boundary quotient graph $G / \partial_H$, by the directed graph with

$$V(G / \partial_H) = \{v_{\partial_H}\} \cup (V(G) \setminus V(H)),$$

and

$$E(G / \partial_H) = E(G) \setminus E(H),$$
with the subgraph boundary quotient; if \( x_1 \neq x_2 \) in \( \partial H \), then identify \( v_{\partial_H} = x_1 = x_2 \), for all such pair \((x_1, x_2)\) in \( \partial H \times \partial H \).

**Definition 2.3.** By \( |K| \), we will denote the size \( |V(K) \cup E(K)| \) of the set \( V(K) \cup E(K) \) for all finite directed graphs \( K \). Let \( G \) be a finite directed graph and \( H \), a full subgraph and let \( \partial H \) be the subgraph boundary and \( G / \partial H \), the corresponding subgraph boundary quotient graph of \( G \) with respect to \( H \). The number \( \exp (|G / \partial H| - 1) \equiv \exp \frac{|G|}{|H|} - 1 \) is called the subgraph boundary index of \( G \) with respect to \( H \), and it is denoted by \( \text{ind}_G(H) \).

Let \( K \) be a finite directed graph. Then, since \( K \) is finite the cardinality

\[
|K| = |V(K) \cup E(K)| = |V(K)| + |E(K)| < \infty.
\]

If \( G \) is a finite directed graph and \( H \) is a full subgraph of \( G \) and if \( G / \partial_H \) is the corresponding subgraph boundary quotient graph, then

\[
(2.1) \quad \text{Ind}_G(H) \overset{\text{def}}{=} \exp (|G / \partial_H| - 1) \leq \exp |G| < \infty.
\]

More generally, we can get the following proposition;

**Proposition 2.1.** Let \( G \) be a finite directed graph and \( H \), a full subgraph of \( G \). Then the subgraph boundary index \( \text{Ind}_G(H) = \frac{\text{Ind}_G(1)}{\text{Ind}_H(1)} \), where 1 means the trivial graph (i.e., 1 is the one-vertex-no-edge graph).

**Proof.** By definition, \( \text{Ind}_G(H) = \exp (|G / \partial_H| - 1) \), where \( \partial_H \) is the subgraph boundary. It suffices to show that \( |G / \partial_H| = |G| - |H| + 1 \). It is trivial by definition of the subgraph admissible quotient graph \( G / \partial_H \). Observe that

\[
|G / \partial_H| = |V(G / \partial_H)| + |E(G / \partial_H)|
= |V(G)| - |V(H)| + |\{v_{\partial_H}\}| + |E(G)| - |E(H)|
= (|V(G)| + |E(G)|) - (|V(H)| + |E(H)|) + 1
= |G| - |H| + 1.
\]

Thus we have that \( |G / \partial_H| - 1 = |G| - |H| \). By taking exponential on both sides, we have

\[
\text{Ind}_G(H) = \exp (|G| - |H|) = \frac{\exp |G|}{\exp |H|}.
\]

For the trivial full subgraph 1, By the very definition, \( G / \partial_1 = G \) and hence

\[
\text{Ind}_G(1) = \exp (|G / \partial_1| - 1) = e^{-1} \exp |G|,
\]
where $\partial_1$ is the subgraph boundary of 1. Similarly, $Ind_H(1) = e^{-1} \exp |H|$. Therefore, we can get that

$$Ind_G(H) = \frac{Ind_G(1)}{Ind_H(1)}.$$  

By the previous proposition, we can get the following simple results:

**Corollary 2.2.** Let $G$ be a finite directed graph and $H$, a full subgraph in $G$.

(1) $Ind_G(H) = 1$ if and only if $H = G$.
(2) $Ind_G(H) > 1$ if and only if $H$ is properly contained in $G$.

**Proof.** The subgraph boundary index $Ind_G(H)$ is the number $\exp (|G / \partial_H| - 1)$, where $\partial_H$ is the subgraph boundary of $G$ with respect to $H$. By the previous proposition, $Ind_G(H) = \frac{Ind_G(1)}{Ind_H(1)}$, where 1 means the trivial graph.

(1) Suppose that $Ind_G(H) = 1$. Equivalently,

$$Ind_G(1) = Ind_H(1) \iff \exp |G| = \exp |H|.$$  

So, we have that $|G| = |H|$. Since $H$ is a full subgraph of $G$, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Thus the condition says that

$$|V(G)| + |E(G)| = |V(H)| + |E(H)|.$$  

By the disjointness of the vertex set and the edge set, $|V(H)| = |V(G)|$ and $|E(H)| = |E(G)|$, and hence $V(H) = V(G)$ and $E(H) = E(G)$. Therefore, $H = G$. Conversely, if $H = G$, then

$$Ind_G(H) = Ind_G(G) = \frac{Ind_G(1)}{Ind_G(1)} = 1.$$  

(2) Suppose that $Ind_G(H) > 1$. Then, by the previous proposition, we have that

$$Ind_G(H) = \frac{Ind_G(1)}{Ind_H(1)} > 1 \iff Ind_G(1) > Ind_H(1) \iff |G| > |H|.$$  

Thus $H$ is proper full subgraph of $G$. The converse clearly holds true.  

**Proposition 2.3.** Let $G$ be a finite directed graph and $H_1$ and $H_2$, full subgraphs. Then $|H_1| = |H_2|$ if and only if $Ind_G(H_1) = Ind_G(H_2)$.

**Proof.** ($\Rightarrow$) Let $G$, $H_1$ and $H_2$ be given as above. By the previous proposition,
\[(2.2) \quad \text{Ind}_G(H_1) = \frac{\text{Ind}_{G_1}(1)}{\text{Ind}_{G_2}(1)} = \frac{\text{Ind}_{G_1}(1)}{\text{Ind}_{G_2}(1)} = \text{Ind}_G(H_2).\]

We can get that \(\text{Ind}_{H_j}(1) = \exp (|H_j| - 1)\), for \(j = 1, 2\). Since \(|H_1| = |H_2|\), the subgraph boundary indices \(\text{Ind}_{H_1}(1)\) and \(\text{Ind}_{H_2}(1)\) coincide. Thus the second equality of (2.2) holds true.

\((\Leftarrow)\) Trivial, by the definition of the subgraph boundary quotient index.

Let \(G_1\) and \(G_2\) be finite directed graphs and assume that they are graph-isomorphic via the graph-isomorphism \(g : G_1 \to G_2\). Let \(H_1\) be a full subgraph of \(G_1\). Then the image \(g(H_1)\) of \(H_1\) is also a full subgraph of \(G_2\). The next theorem shows that the subgraph boundary index \(\text{Ind}\) is preserved up to graph-isomorphisms.

**Theorem 2.4.** Let \(G\) be a finite directed graph and \(H\), a full subgraph. Suppose that the graph \(G'\) is graph-isomorphic to \(G\). Then \(\text{Ind}_G(H) = \text{Ind}_{G'}(H')\), where \(H'\) is the image of \(H\), of the corresponding graph-isomorphism, in \(G'\).

**Proof.** Let \(g : G \to G'\) be a graph-isomorphism. Then \(g\) preserves the vertex set and the admissibility of \(G\). So, if \(H\) is a full subgraph of \(G\), then the image \(H' = g(H)\) of \(G'\) is also a full subgraph and moreover \(H'\) is graph-isomorphic to \(H\), via \(g^{-1} |_{H'}\). Therefore, \(|H| = |H'|\). By the little modification of the previous proposition, \(\text{Ind}_G(H) = \text{Ind}_{G'}(H')\).

By the previous theorem, if \(G_1\) and \(G_2\) are finite graphs and if there exists a graph-homomorphism \(g : G_1 \to G_2\) such that (i) \(g(V(G_1)) \subseteq V(G_2)\) and (ii) \(g\) preserves the admissibility of \(G_1\) in \(G_2\). Then we can regard the image \(g(G_1)\) of \(G_2\) as the full subgraph of \(G_2\). So, we can define the index of \(\text{Ind}_{G_2}(G_1)\) by the subgraph boundary index \(\text{Ind}_{G_2}(g(G_1))\). i.e.,

\[\text{Ind}_{G_2}(G_1) \overset{def}{=} \exp (|G_2 / \partial g(G_1)| - 1).\]

**Definition 2.4.** Let \(G_1\) and \(G_2\) be finite directed graphs. Then the boundary index \(\text{Ind}_{G_2}(G_1)\) is defined by

\[\text{Ind}_{G_2}(G_1) \overset{def}{=} \begin{cases} \text{Ind}_{G_2}(g(G_1)) & \text{if } \exists \text{ homomorphism } g : G_1 \to G_2 \\ 0 & \text{otherwise.} \end{cases}\]

By definition, we have the following theorem.

**Theorem 2.5.** Let \(G_1\) and \(G_2\) be finite directed graphs. Then \(\text{Ind}_{G_2}(G_1) = 1 = \text{Ind}_{G_1}(G_2)\) if and only if \(G_1\) and \(G_2\) are graph-isomorphic.
Proof. (⇒) Assume that $\text{Ind}_{G_2}(G_1) = 1$. This means that there exists a graph-homomorphism $g : G_1 \to G_2$ such that the boundary index $\text{Ind}_{G_2}(G_1)$ is the subgraph boundary index $\text{Ind}_{G_2}(g(G_1))$, defined by $\exp (\langle G_2 / \partial_g(G_1) \rangle - 1) = 1$. Since $\text{Ind}_{G_2}(g(G_1)) = \frac{\text{Ind}_{G_2}(1)}{\text{Ind}_{g(G_1)}(1)}$, if this quantity is 1, then $\text{Ind}_{G_2}(1) = \text{Ind}_{g(G_1)}(1)$, and hence $G_2 = g(G_1)$. Equivalently, the graph-homomorphism $g$ is a graph-isomorphism from $G_1$ to $G_2$. Also, by defining the graph-isomorphism $g^{-1} : G_2 \to G_1$, we can get that $\text{Ind}_{G_1}(g^{-1}(G_2)) = 1$. So, $\text{Ind}_{G_1}(G_2) = 1$, too.

(⇐) Assume that $G_1$ and $G_2$ are graph-isomorphic with its graph-isomorphism $g : G_1 \to G_2$. Then the image $g(G_1)$ is the full subgraph of $G_2$, moreover $g(G_1) = G_2$. Therefore,

$$\frac{\text{Ind}_{G_2}(1)}{\text{Ind}_{g(G_1)}(1)} = \frac{\exp(|G_2| - 1)}{\exp(|g(G_1)| - 1)} = 1,$$

and hence $\text{Ind}_{G_2}(g(G_1)) = 1$. So, $\text{Ind}_{G_2}(G_1) = 1$. Also, $\text{Ind}_{G_1}(G_2) = 1$, via the graph-isomorphism $g^{-1}$.

We can verify the range of the subgraph boundary index $\text{Ind}_G(\cdot)$ of the graph $G$, as a function defined on the set of all full subgraphs of $G$.

**Proposition 2.6.** The image of the subgraph boundary index $\text{Ind}_G(\cdot)$, as a full-subgraph-valued function, is contained in the closed interval $[1, e^{|G|-1}]$. □

More explicitly, we have the following theorem.

**Theorem 2.7.** The image of the subgraph boundary index $\text{Ind}_G(\cdot)$ is contained in $\{e^{|G|-k} : k = 1, ..., |G|\}$.

Proof. By definition, if $H$ is a full subgraph of $G$, then $\text{Ind}_G(H) = \exp (|G / \partial_H| - 1)$. Furthermore, $\text{Ind}_G(H) = \frac{\text{Ind}_G(1)}{\text{Ind}_H(1)} = \frac{e^{|G|-1}}{e^{|H|-1}}$. Consider a full subgraph $L$ such that $V(L) = \{v\}$ and $E(L) = \{l\}$, where $l$ is the loop-edge concentrated on $v$. Then $\text{Ind}_L(1) = e^{|L|-1} = e$. Now, let $K$ be a full subgraph with $V(K) = \{v_1, v_2\}$ and $E(K) = \{e\}$, where $e$ is the edge connection $v_1$ and $v_2$, with direction. Then $\text{Ind}_K(1) = e^{|K|-1} = e^2$. If $H$ is $G$, itself, then we have $\text{Ind}_G(H) = 1$. Also, if $H$ is trivial, then $\text{Ind}_G(H) = e^{|G|-1}$. □

Let’s consider the chain of full subgraphs in the given graph $G$. We denote the relation $[H$ is a full subgraph of $G]$ by $[H < G]$. The finite inclusions

$$K_1 < K_2 < ... < K_n < G$$

is called the chain of full subgraphs.
**Proposition 2.8.** Let $G$ be a finite directed graph and let $K < H < G$ be a chain of full subgraphs of $G$. Then $\text{Ind}_G(K) = \text{Ind}_G(H) \cdot \text{Ind}_H(K)$.

**Proof.** We have that

$$\text{Ind}_G(K) = \frac{\text{Ind}_G(1)}{\text{Ind}_K(1)} = \left(\frac{\text{Ind}_G(1)}{\text{Ind}_H(1)}\right) \div \left(\frac{\text{Ind}_H(1)}{\text{Ind}_K(1)}\right)$$

$$= \text{Ind}_G(H) \cdot \text{Ind}_H(K).$$

More generally, we have the following corollary.

**Corollary 2.9.** Let $G$ be a finite directed graph and let $K_1 < \ldots < K_n < G$ be a chain of full subgraphs. Then

$$\text{Ind}_G(K_1) = \prod_{j=2}^{n+1} \text{Ind}_{K_j}(K_{j-1}),$$

with $K_{n+1} \overset{\text{def}}{=} G$. □

Let $K_1 < K_2 < \ldots < K_n < G$ be a chain of full subgraphs. Then we have the following dual structure of it;

$$(G / ∂_{K_n}) < (G / ∂_{K_{n-1}}) < \ldots < (G / ∂_{K_2}) < (G / ∂_{K_1}).$$

This is called the dual chain of the given chain.

**Lemma 2.10.** Let $K < H < G$ be a chain of full subgraphs. Then $(G / ∂_H) < (G / ∂_K)$.

**Proof.** Clearly, $\text{Ind}_G(H) \leq \text{Ind}_G(K)$. Equivalently, $\text{Ind}_H(1) \geq \text{Ind}_K(1)$. Also since $∂_K \subset ∂_H$, the subgraph boundary quotient graphs $G / ∂_H$ and $G / ∂_K$ satisfy the full-subgraph-inclusion $(G / ∂_H) < (G / ∂_K)$. □

The above lemma shows that the dual chain of the chain of full subgraphs is well-defined, as a chain of full subgraphs.

**Proposition 2.11.** Let $K < H < G$ be a chain of full subgraphs. Then

$$\text{Ind}_{G / ∂_K}(G / ∂_H) = \text{Ind}_H(K).$$
Proof. By the previous lemma, we have the full-subgraph-inclusion \((G / \partial H) < (G / \partial K)\), as the dual chain of \(K < H < G\). The subgraph boundary index \(\text{Ind}_{G / \partial K}(G / \partial H)\) is determined by

\[
\text{Ind}_{G / \partial K}(G / \partial H) = \frac{\text{Ind}_{G / \partial K}(1)}{\text{Ind}_{G / \partial H}(1)} = \frac{\exp(|G / \partial K| - 1)}{\exp(|G / \partial H| - 1)}
\]

\[
= \frac{\text{Ind}_{G}(K)}{\text{Ind}_{G}(H)} = \left( \frac{\text{Ind}_{G}(1)}{\text{Ind}_{G}(1)} \right) / \left( \frac{\text{Ind}_{H}(1)}{\text{Ind}_{H}(1)} \right) = \left( \frac{\text{Ind}_{H}(1)}{\text{Ind}_{H}(1)} \right)
\]

\[
= \frac{\text{Ind}_{H}(1)}{\text{Ind}_{K}(1)} = \text{Ind}_{H}(K).
\]

Therefore, \(\text{Ind}_{G / \partial K}(G / \partial H) = \text{Ind}_{H}(K)\). □

By the previous proposition, generally, we can get that:

**Corollary 2.12.** Let \(K_1 < ... < K_n < G\) be a chain of full subgraphs and let \((G / \partial K_n) < ... < (G / \partial K_1)\) be the corresponding dual chain. Then

\[
\text{Ind}_{G / \partial K_i}(G / \partial K_j) = \text{Ind}_{K_j}(K_i),
\]

for all \(i \leq j\) in \(\{1, ..., n\}\). □

**Remark 2.1.** As we observed before, we can apply the above results to the general chain of graphs. Assume that we have a chain of graphs \(G_1 < ... < G_{n+1}\). Here \(G_i < G_{i+1}\) means that there exists a graph-homomorphism \(g_i : G_i \rightarrow G_{i+1}\) such that \(g_i(G_i)\) is a full subgraph of \(G_{i+1}\), for all \(i = 1, ..., n\). Then we have its dual chain \((G_{n+1} / \partial g_i(G_i)) < ... < (G_{n+1} / \partial g_i(G_1)) < (G_{n+1} / \partial g_i(G_1))\), where \(i : G_1 \rightarrow G_1\) is the identity graph-isomorphism. Then we have the following boundary index relations:

1. \(\text{Ind}_{G_{n+1}}(G_1) = \prod_{j=2}^{n+1} \text{Ind}_{G_j}(G_{j-1})\).

2. \(\text{Ind}_{G_{n+1} / \partial G_i}(G_{n+1} / \partial G_i) = \text{Ind}_{G_j}(G_i),\) for all \(i \leq j\) in \(\{1, ..., n + 1\}\).

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