Graph functions maximized on a path

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Abstract

Given a connected graph $G$ of order $n$ and a nonnegative symmetric matrix $A = [a_{i,j}]$ of order $n$, define the function $F_A(G)$ as

$$F_A(G) = \sum_{1 \leq i < j \leq n} d_G(i,j) a_{i,j},$$

where $d_G(i,j)$ denotes the distance between the vertices $i$ and $j$ in $G$.

In this note it is shown that $F_A(G) \leq F_A(P)$ for some path of order $n$. Moreover, if each row of $A$ has at most one zero off-diagonal entry, then $F_A(G) < F_A(P)$ for some path of order $n$, unless $G$ itself is a path.

In particular, this result implies two conjectures of Aouchiche and Hansen:

- the spectral radius of the distance Laplacian of a connected graph $G$ of order $n$ is maximal if and only if $G$ is a path;
- the spectral radius of the distance signless Laplacian of a connected graph $G$ of order $n$ is maximal if and only if $G$ is a path.

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1 Introduction and main results

The aim of the present note is to give a general approach to problems like the following conjectures of Aouchiche and Hansen [1, 2]:

Conjecture 1 The largest eigenvalue of the distance Laplacian of a connected graph $G$ of order $n$ is maximal if and only if $G$ is a path.

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Conjecture 2 The largest eigenvalue of the distance signless Laplacian of a connected graph $G$ of order $n$ is maximal if and only if $G$ is a path.

First, let us introduce some notation and recall a few definitions. We write $\lambda(A)$ for the largest eigenvalue of a symmetric matrix $A$. Given a connected graph $G$, let $D(G)$ be the distance matrix of $G$, and let $T(G)$ be the diagonal matrix of the rowsums of $D(G)$. The matrix $D_L(G) = T(G) - D(G)$ is called the distance Laplacian of $G$, and the matrix $D_Q(G) = T(G) + D(G)$ is called the distance signless Laplacian of $G$. The matrices $D_L(G)$ and $D_Q(G)$ have been introduced by Aouchiche and Hansen and have been intensively studied recently, see, e.g., [1, 2, 3, 5, 7, 12].

Very recently, Lin and Lu [5] succeeded to prove Conjecture 2, but Conjecture 1 seems a bit more difficult and still holds. Furthermore, Conjectures 1 and 2 suggest a similar problem for the distance matrix itself. As it turns out such problem has been partially solved a while ago by Ruzieh and Powers [9], who showed that the largest eigenvalue of the distance matrix of a connected graph $G$ of order $n$ is maximal if $G$ is a path. The complete solution, however, was given more recently by Stevanović and Ilić [10].

Theorem 3 ([9],[10]) The largest eigenvalue of the distance matrix of a connected graph $G$ of order $n$ is maximal if and only if $G$ is a path.

These result are believed to belong to spectral graph theory, and their proofs involve non-negligible amount of calculations. Our goal is to show that all these results stem from a much more general assertion that has nothing to do with eigenvalues. To this end, we shall introduce a fairly general graph function and shall study its maxima.

1.1 The function $F_A(G)$ and its maxima

Let $G$ be a connected graph of order $n$. Write $d_G(i,j)$ for the distance between the vertices $i$ and $j$ in $G$, and let $A = [a_{i,j}]$ be a nonnegative symmetric matrix of order $n$. Define the function $F_A(G)$ as

$$F_A(G) = \sum_{1 \leq i < j \leq n} d_G(i,j) a_{i,j}.$$

Clearly $d_G(i,i) = 0$ for any $i \in V(G)$, so the diagonal of $A$ is irrelevant for $F_A(G)$.

In fact, the function $F_A(G)$ is quite mainstream, as it can be represented as

$$F_A(G) = \| A \circ D(G) \|_{l_1},$$

where $\circ$ denotes the entrywise Hadamard product of matrices, and $\| \cdot \|_{l_1}$ is the $l_1$ norm. This viewpoint suggests a number of extensions, which we shall investigate elsewhere.

Next, we focus on the extremal points of $F_A(G)$, that is to say, we want to know which connected graphs $G$ of order $n$ satisfy the condition

$$F_A(G) = \max \{ F_A(H) : H \text{ is a connected graph of order } n \}.$$ 

In particular, we prove the somewhat surprising fact that for any admissible matrix $A$, the function $F_A(G)$ is always maximized by a path. More precisely the following theorem holds.
Conjecture 1 requires a slightly more careful approach.

Let \( G \) be a connected graph of order \( n \) and let \( A = [a_{i,j}] \) be a symmetric matrix of order \( n \). If \( A \) is nonnegative, then there is a path \( P \) with \( V(P) = V(G) \) such that

\[
\sum_{1 \leq i < j \leq n} d_G(i,j) a_{i,j} \leq \sum_{1 \leq i < j \leq n} d_P(i,j) a_{i,j}.
\]  

(1)

It is not hard to find nonnegative symmetric matrices \( A \) for which \( F_A(G) \) is maximized also by graphs other than paths. Thus, it is natural to attempt to characterize all symmetric, nonnegative matrices \( A \), for which \( F_A(G) \) is maximal only if \( G \) is a path. The complete solution of this problem seems difficult, so we shall give only a partial solution, sufficient for our goals.

Theorem 5 Let \( G \) be a connected graph of order \( n \) and let \( A = [a_{i,j}] \) be a symmetric nonnegative matrix of order \( n \). If each row of \( A \) has at most one zero off-diagonal entry, and \( G \) is not a path, then there is a path \( P \) with \( V(P) = V(G) \) such that

\[
\sum_{1 \leq i < j \leq n} d_G(i,j) a_{i,j} < \sum_{1 \leq i < j \leq n} d_P(i,j) a_{i,j}.
\]  

(2)

As yet we know of no application that exploits the full strength of Theorem 5. Indeed, to prove Conjectures 1 and 2, and Theorem 3, we shall use only the following simple corollary.

Corollary 6 Let \( G \) be a connected graph of order \( n \) and let \( A = [a_{i,j}] \) be a symmetric matrix of order \( n \). If each off-diagonal entry of \( A \) is positive, and \( G \) is not a path, then there is a path \( P \) with \( V(P) = V(G) \) such that \( F_A(P) > F_A(G) \).

1.2 Proofs of Conjectures 1 and 2, and Theorem 3

We proceed with the proof of Conjecture 2. Let \( G \) be a connected graph of order \( n \) for which \( \lambda(D^Q(G)) \) is maximal within all connected graphs of order \( n \). We shall prove that \( G \) is a path. Let \( x = (x_1, \ldots, x_n) \) be a unit eigenvector to \( \lambda(D^Q(G)) \). Since \( D^Q(G) \) is irreducible, the vector \( x \) is positive. Define an \( n \times n \) matrix \( A = [a_{i,j}] \) by letting \( a_{i,j} = (x_i + x_j)^2 \). Clearly \( A \) is symmetric and nonnegative. As is well-known,

\[
\lambda(D^Q(G)) = \langle D^Q(G) x, x \rangle = \sum_{1 \leq i < j \leq n} d_G(i,j) (x_i + x_j)^2 = F_A(G).
\]

Since each off-diagonal entry of \( A \) is positive, Corollary 6 implies that either \( G = P_n \) or there is a path \( P \) with \( V(P) = V(G) \) such that \( F_A(P) > F_A(G) \). The latter cannot hold as we would have

\[
\lambda(D^Q(G)) = F_A(G) < F_A(P) \leq \lambda(D^Q(P)),
\]

contrary to the choice of \( G \). Hence \( G = P_n \), completing the proof of Conjecture 2.

Theorem 3 can be proved in the same way, with \( A = [a_{i,j}] \) defined by \( a_{i,j} = x_i x_j \). However, Conjecture 1 requires a slightly more careful approach.

Let \( G \) be a connected graph of order \( n \) such that \( \lambda(D^L(G)) \) is maximal among all connected \( n \) vertex graphs. We shall prove that \( G \) must be a path. Let \( x = (x_1, \ldots, x_n) \) be a unit
eigenvector to \( \lambda (D^L(G)) \) and define an \( n \times n \) matrix \( A = [a_{i,j}] \) by letting \( a_{i,j} = (x_i - x_j)^2 \). Clearly \( A \) is symmetric and nonnegative. Also, it is well-known that

\[
\lambda (D^L(G)) = \langle D^L(G) \mathbf{x}, \mathbf{x} \rangle = \sum_{1 \leq i < j \leq n} d_G(i,j)(x_i - x_j)^2 = F_A(G).
\]

However, at this stage we cannot rule out that \( A \) has numerous zero entries, and so Corollary 6 does not apply as before. Yet Theorem 4 implies that there is a path \( P \) with \( V(P) = V(G) \) such that \( F_A(P) \geq F_A(G) \); hence,

\[
\lambda (D^L(G)) = F_A(G) \leq F_A(P) \leq \lambda (D^L(P)).
\]

Due to the choice of \( G \), equalities should hold throughout the above line, implying that \( \mathbf{x} \) is an eigenvector to \( D^L(G) \). But in Theorems 4.4 and 4.6 of [7] Nath and Paul have established that all entries of an eigenvector to \( \lambda (D^L(P)) \) are different and so the off-diagonal entries of \( A \) are positive. Now we apply Corollary 6 and finish the proof as for Conjecture 2.

2 Proofs of the main theorems

For graph notation undefined here we refer the reader to [4]. For general properties of the distance Laplacian and the distance signless Laplacian the reader is referred to [1, 2, 3].

Here is some notation that will be used later in the proofs:
- \( P_n \) and \( C_n \) stand for the path and cycle of order \( n \);
- \( G - u \) denotes the graph obtained from \( G \) by removing the vertex \( u \);
- \( G - \{u, v\} \) denotes the graph obtained from \( G \) by removing the vertices \( u \) and \( v \).

We shall assume that any graph of order \( n \) is defined on the vertex set \([n] = \{1, \ldots, n\}\).

The proofs of Theorems 4 and 5 have the same general structure, but the latter requires a lot of extra details so it will be presented separately.

**Proof of Theorem 4** Note first that if \( H \) is a spanning tree of \( G \), then \( d_G(i,j) \leq d_H(i,j) \) for every \( i, j \in V(G) \); hence

\[
\sum_{1 \leq i < j \leq n} d_G(i,j) a_{i,j} \leq \sum_{1 \leq i < j \leq n} d_H(i,j) a_{i,j}.
\]

Therefore, we may and shall assume that \( G \) is a tree itself. We carry out the proof by induction on \( n \). If \( n \leq 3 \), every tree of order \( n \) is a path, so there is nothing to prove in this case. Assume now that \( n > 3 \) and the assertion holds for any \( n' \) such that \( n' < n \). Choose a vertex \( u \in V(G) \) of degree 1. By symmetry, we assume that \( u = n \), and let \( k \) be the single neighbor of \( u \); hence \( G - n \) is a tree of order \( n - 1 \).

Define a symmetric matrix \( A' = [a'_{i,j}] \) of order \( n - 1 \) as follows:

\[
a'_{i,j} = \begin{cases} 
  a_{i,j}, & \text{if } i \neq k \text{ and } j \neq k; \\
  a_{k,j} + a_{n,j}, & \text{if } i = k; \\
  a_{i,k} + a_{i,n}, & \text{if } j = k.
\end{cases}
\]
Clearly $A'$ is a symmetric nonnegative matrix. By the induction assumption there is a path $P'$ with $V(P') = V(G - n) = [n - 1]$ such that

$$\sum_{1 \leq i < j < n} d_{G-n}(i, j) a'_{i,j} \leq \sum_{1 \leq i < j < n} d_{P'}(i, j) a'_{i,j}. \quad (3)$$

On the other hand, for each $j \in V(G - n)$, the shortest path between $n$ and $j$ contains $k$, so

$$d_G(j, n) = d_{G-n}(j, k) + 1.$$

Hence we see that

$$\sum_{1 \leq i < j \leq n} d_G(i, j) a_{i,j} = \sum_{j=1}^{n-1} d_G(j, n) a_{j,n} + \sum_{1 \leq i < j < n} d_{G-n}(i, j) a_{i,j}$$

$$= \sum_{j=1}^{n-1} (d_{G-n}(k, j) + 1) a_{j,n} + \sum_{1 \leq i < j < n} d_{G-n}(i, j) a_{i,j}$$

$$= \sum_{j=1}^{n-1} a_{n,j} + \sum_{1 \leq i < j < n} d_{G-n}(i, j) a'_{i,j}.$$

Now, (3) implies that

$$\sum_{1 \leq i < j \leq n} d_G(i, j) a_{i,j} \leq \sum_{j=1}^{n-1} a_{j,n} + \sum_{1 \leq i < j < n} d_{P'}(i, j) a'_{i,j}. \quad (4)$$

Further, write $T$ for the tree obtained from the path $P'$ by joining $n$ to the vertex $k \in V(P')$. As before, we see that

$$\sum_{1 \leq i < j \leq n} d_T(i, j) a_{i,j} = \sum_{j=1}^{n-1} d_T(j, n) a_{j,n} + \sum_{1 \leq i < j < n} d_{T-n}(i, j) a_{i,j}$$

$$= \sum_{j=1}^{n-1} (d_{P'}(j, k) + 1) a_{j,n} + \sum_{1 \leq i < j < n} d_{P'}(i, j) a_{i,j}$$

$$= \sum_{j=1}^{n-1} a_{j,n} + \sum_{1 \leq i < j < n} d_{P'}(i, j) a'_{i,j}.$$

Hence, (4) implies that

$$\sum_{1 \leq i < j \leq n} d_G(i, j) a_{i,j} \leq \sum_{1 \leq i < j \leq n} d_T(i, j) a_{i,j}.$$

If $T = P_n$, there is nothing to prove, so suppose that $T \neq P_n$. To complete the proof we shall show that we can join $n$ to one of the ends of $P'$ so that $F_A(T)$ will not decrease.
By symmetry, assume that the vertex sequence of the path \( P' \) is precisely \( 1, 2, \ldots, n - 1 \); thus the neighbor \( k \) of \( n \) satisfies \( 1 < k < n - 1 \). Write \( A_0 \) for the principal submatrix of \( A \) in the first \( n - 1 \) rows and note that

\[
F_A(T) = \sum_{i=1}^{k} (k - i + 1) a_{i,n} + \sum_{i=k+1}^{n-1} (i - k + 1) a_{i,n} + F_{A_0}(P').
\]

Next, delete the edge \( \{n, k\} \) in \( T \), add the edge \( \{n, 1\} \), and write \( T_1 \) for the resulting path. If \( F_A(T_1) > F_A(T) \), the proof is completed, so let us assume that \( F_A(T_1) \leq F_A(T) \). Since

\[
F_A(T_1) = \sum_{i=1}^{n-1} i a_{i,n} + F_{A_0}(P'),
\]

we see that,

\[
\sum_{i=1}^{k-1} (k - i + 1) a_{i,n} + \sum_{i=k}^{n-1} (i - k + 1) a_{i,n} \geq \sum_{i=1}^{n-1} i a_{i,n}
\]

and so

\[
\sum_{i=1}^{k-1} (k - 2i + 1) a_{i,n} \geq (k - 1) (a_{k,n} + \cdots + a_{n-1,n}).
\]

Hence,

\[
(k - 1) (a_{1,n} + \cdots + a_{k-1,n}) \geq (k - 1) (a_{k,n} + \cdots + a_{n-1,n}). \quad (5)
\]

Now, delete the edge \( \{n, k\} \) in \( T \), add the edge \( \{n, n-1\} \), and write \( T_2 \) for the resulting path. If \( F_A(T_2) > F_A(T) \), the proof is completed, so let us assume that \( F_A(T_2) \leq F_A(T) \). Since

\[
F_A(T_2) = \sum_{i=1}^{n-1} (n - i) a_{i,n} + F_{A_0}(P'),
\]

we see that

\[
\sum_{i=1}^{k-1} (k - i + 1) a_{i,n} + \sum_{i=k}^{n-1} (i - k + 1) a_{i,n} \geq \sum_{i=1}^{n-1} (n - i) a_{i,n},
\]

and so

\[
\sum_{i=k}^{n-1} (2i - k - n + 1) a_{i,n} \geq (n - k - 1) (a_{1,n} + \cdots + a_{k-1,n}).
\]

Hence,

\[
(n - k - 1) (a_{k,n} + \cdots + a_{n-1,n}) \geq (n - k - 1) (a_{1,n} + \cdots + a_{k-1,n}).
\]

This inequality, together with \((5)\), implies that

\[
a_{k,n} + \cdots + a_{n-1,n} = a_{1,n} + \cdots + a_{k-1,n},
\]

and that \( F_A(T_1) = F_A(T) \) and \( F_A(T_2) = F_A(T) \). This completes the induction step and the proof of Theorem \([4]\). \( \square \)
2.1 Proof of Theorem 5

Most of the proof of Theorem 5 deals with the case of $G$ being a tree, so we extract this part in Theorem 7 below. The general case will be deduced later by different means.

For convenience write $N(n)$ for the class of all symmetric nonnegative matrix of order $n$ such that each row of $A$ has at most one zero off-diagonal entry.

**Theorem 7** Let $G$ be a tree of order $n$. If $A \in N(n)$ and $G \neq P_n$, then there exists a path $P$ with $V(P) = V(G)$ such that $F_A(G) < F_A(P)$.

**Proof** Our proof is by induction on $n$ and is structured as the proof of Theorem 4. If $n \leq 3$, every tree of order $n$ is a path, so there is nothing to prove in this case. For technical reason we would like to give a direct proof for $n = 4$ as well. There are two trees of order 4 - a path and a star. Assume that $G$ is a star, and by symmetry suppose that 2 is its center. We have

$$F_A(G) = 2a_{1,1} + a_{1,2} + 2a_{1,3} + a_{1,2} + a_{2,3} + 2a_{1,3}.$$ 

Remove the edge $\{4, 2\}$ and add the edge $\{4, 1\}$, thus obtaining a path $G_1$. Assume for a contradiction that $F_A(G) \geq F_A(G_1)$, which implies that $a_{4,1} \geq a_{4,2} + a_{4,3}$. Now, remove from $G$ the edge $\{4, 2\}$ and add the edge $\{4, 3\}$, thus obtaining a path $G_2$. Assume for a contradiction that $F_A(G) \geq F_A(G_2)$, which implies that $a_{4,3} \geq a_{4,2} + a_{4,1}$. We conclude that $a_{4,2} = 0$. By symmetry, we also get $a_{1,2} = 0$ and $a_{3,2} = 0$; hence $A$ has a zero row, contradicting the hypothesis. Thus, $G$ is a path.

Assume now that $n \geq 5$ and the assertion of Theorem holds for any $n'$ such that $n' < n$. Let $G$ be tree for which $F_A(G)$ attains a maximum. We shall prove that $G = P_n$. Choose a vertex $u \in V(G)$ of degree 1. By symmetry, we assume that $u = n$, and let $k$ be the single neighbor of $u$; hence $G - n$ is a tree of order $n - 1$.

Define a symmetric matrix $A' = [a'_{ij}]$ of order $n - 1$ as follows

$$a'_{ij} = \begin{cases} a_{ij}, & \text{if } i \neq k \text{ and } j \neq k; \\ a_{k,j} + a_{n,j}, & \text{if } i = k; \\ a_{i,k} + a_{i,n}, & \text{if } j = k. \end{cases}$$

Clearly $A' \in N(n - 1)$. Suppose that $G - n \neq P_{n-1}$. By the induction assumption, there is a path $P'$ with $V(P') = V(G - n) = [n - 1]$ such that

$$\sum_{1 \leq i < j < n} d_{G - n}(i, j) a'_{ij} < \sum_{1 \leq i < j < n} d_{P'}(i, j) a'_{ij}.$$ 

Hence, as in the proof of Theorem 4 we find that

$$F_A(G) = \sum_{j=1}^{n-1} a_{j,n} + \sum_{1 \leq i < j < n} d_{G - n}(i, j) a'_{ij} < \sum_{j=1}^{n-1} a_{j,n} + \sum_{1 \leq i < j < n} d_{P'}(i, j) a'_{ij}.$$ 

Now, join $n$ to $k$, and write $T$ for the obtained tree. As before, we see that

$$F_A(T) = \sum_{j=1}^{n-1} a_{j,n} + \sum_{1 \leq i < j < n} d_{P'}(i, j) a'_{ij} > F_A(G).$$
This contradicts the assumption that $F_A(G)$ is maximal. Therefore $G - n = P_{n-1}$.

By symmetry, assume that the vertex sequence of the path $G - n$ is precisely $1, 2, \ldots, n - 1$. If $k = 1$ or $k = n - 1$, we see that $G = P_n$, so let us assume that $1 < k < n - 1$. To complete the proof we shall show that we can join $n$ to $1$ or to $n - 1$ so that $F_A(G)$ will increase.

Write $A_0$ for the principal submatrix of $A$ in the first $n - 1$ rows and note that

$$F_A(G) = \sum_{i=1}^{n-1} (k - i + 1) a_{i,n} + \sum_{i=k+1}^{n-1} (i - k + 1) a_{i,n} + F_{A_0}(G - n).$$

Next, delete the edge $\{n, k\}$ in $G$, add the edge $\{n, 1\}$, and write $G_1$ for the resulting path. Since $F_A(G)$ is maximal, we see that $F_A(G_1) \leq F_A(G)$. From

$$F_A(G_1) = \sum_{i=1}^{n-1} i a_{i,n} + F_{G - n}(A_0)$$

it follows that,

$$\sum_{i=1}^{k-1} (k - i + 1) a_{i,n} + \sum_{i=k}^{n-1} (i - k + 1) a_{i,n} \geq \sum_{i=1}^{n-1} i a_{i,n},$$

and so

$$\sum_{i=1}^{k-1} (k - 2i + 1) a_{i,n} \geq (k - 1) (a_{k,n} + \cdots + a_{n-1,n}).$$

Hence, letting

$$S_1 = -2 \sum_{i=1}^{k-1} (i - 1) a_{i,n}$$

we see that

$$(k - 1) (a_{1,n} + \cdots + a_{k-1,n}) + S_1 \geq (k - 1) (a_{k,n} + \cdots + a_{n-1,n}) \quad (6)$$

Finally, delete the edge $\{n, k\}$ in $G$, add the edge $\{n, n - 1\}$, and write $G_2$ for the resulting path. Since $F_A(G)$ is maximal, we see that $F_A(G_2) \leq F_A(G)$. From

$$F_A(G_2) = \sum_{i=1}^{n-1} (n - i) a_{i,n} + F_{G - n}(A_0)$$

it follows that

$$\sum_{i=1}^{k} (k - i + 1) a_{i,n} + \sum_{i=k+1}^{n-1} (i - k + 1) a_{i,n} \geq \sum_{i=1}^{n-1} (n - i) a_{i,n},$$

and so

$$\sum_{i=k}^{n-1} (2i - k - n + 1) a_{i,n} \geq (n - k - 1) (a_{1,n} + \cdots + a_{k-1,n}).$$
Hence, letting
\[
S_2 = -2 \sum_{i=k}^{n-1} (n-i-1) a_{i,n}
\]
\[
(n-k-1) (a_{k,n} + \cdots + a_{n-1,n}) + S_2 \geq (n-k-1) (a_{1,n} + \cdots + a_{k-1,n}).
\]
Comparing this inequality with (6), in view of \(S_1 \leq 0\) and \(S_2 \leq 0\), we find that
\[
a_{k,n} + \cdots + a_{n-1,n} = a_{1,n} + \cdots + a_{k-1,n} \quad \text{and} \quad S_1 = S_2 = 0.
\]
Hence,
\[
a_{2,n} = \cdots = a_{k,n} = 0 \quad \text{and} \quad a_{k,n} = \cdots = a_{n-2,n} = 0.
\]
Since \(n-3 \geq 2\), among the off-diagonal entries of the \(n\)'th row of \(A\), there are two that are zero, contrary to the hypothesis. Therefore, \(G = P_n\), completing the induction step and the proof of Theorem 7. ✷

Armed with Theorem 7, we are able the complete the proof of Theorem 5.

**Proof of Theorem 5** First we shall prove Theorem 5 if \(G\) is a unicyclic graph, i.e., if \(G\) has exactly \(n\) edges. Thus, let \(G\) be a connected unicyclic graph of order \(n \geq 3\). It is known that \(G\) contains a single cycle. If \(G\) is not the cycle \(C_n\) itself, then \(G\) contains a spanning tree \(H\) with maximum degree \(\Delta (H) \geq 3\); thus \(H \neq P_n\). Hence, Theorem 7 implies that there is a path \(P\) with \(V (P) = V (G)\) such that
\[
F_A (G) = \sum_{1 \leq i < j \leq n} d_G (i,j) a_{i,j} \leq \sum_{1 \leq i < j \leq n} d_H (i,j) a_{i,j} < \sum_{1 \leq i < j \leq n} d_P (i,j) a_{i,j}.
\]
If \(G\) is the cycle \(C_n\) itself, let \(i, j, k\) be three consecutive vertices along the cycle. The removal of the edge \(\{i, j\}\) increases the distance between \(i\) and \(j\), i.e.,
\[
d_G (i,j) < d_G - \{i,j\} (i,j)
\]
and on the other hand
\[
F_A (G) \leq F_A (G - \{i,j\}).
\]
If \(F_A (G) < F_A (G - \{i,j\})\), the theorem is proved, otherwise \(F_A (G) = F_A (G - \{i,j\})\) and so \(a_{i,j} = 0\). By the same token we obtain \(a_{j,k} = 0\); hence among the off-diagonal entries of the \(k\)'th row of \(A\) there are two that are zero, contrary to the hypothesis. So the theorem holds for unicyclic graphs.

Finally, note that any connected graph \(G\) that is not a tree contains a connected unicyclic spanning subgraph \(H\) or is unicyclic itself. Hence, if \(G\) is not a tree, then \(F_A (G) \leq F_A (H)\) for some connected unicyclic \(H\), and thus there is a path \(P\) with \(V (P) = V (G)\) such that
\[
F_A (G) \leq F_A (H) < F (P).
\]
The proof of Theorem 5 is completed. ✷
3 Concluding remarks

Results similar to Theorem 3 have been known for the adjacency matrix, the Laplacian, and the signless Laplacian of a connected graph \( G \):

**Theorem 8 ([6])** The largest eigenvalue of the adjacency matrix of a connected graph \( G \) of order \( n \) is minimal if and only if \( G \) is a path.

**Theorem 9 ([8])** The largest eigenvalue of the Laplacian of a connected graph \( G \) of order \( n \) is minimal if and only if \( G \) is a path.

**Theorem 10 ([11])** The largest eigenvalue of the signless Laplacian of a connected graph \( G \) of order \( n \) is minimal if and only if \( G \) is a path.

In the light of the present note we would like to raise the following question:

**Question.** Is there a result similar to Theorem 5 that implies Theorems 8, 9, and 10?

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