Matrix variate and tensor variate Laplace distributions

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Abstract

In this article, we define a matrix variate asymmetric Laplace distribution. We prove some properties of the matrix variate asymmetric Laplace distribution. We prove the relationship between the matrix variate asymmetric Laplace distribution and other distributions. We define a matrix variate generalized asymmetric Laplace distribution. We prove some properties of the matrix variate generalized asymmetric Laplace distribution. We prove the relationship between the matrix variate generalized asymmetric Laplace distribution and other distributions. We define a tensor variate asymmetric Laplace distribution. We prove some properties of the tensor variate asymmetric Laplace distribution. We prove the relationship between the tensor variate asymmetric Laplace distribution and other distributions. We define a tensor variate generalized asymmetric Laplace distribution. We prove some properties of the tensor variate generalized asymmetric Laplace distribution. We prove the relationship between the tensor variate generalized asymmetric Laplace distribution and other distributions.

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1 Matrix variate Laplace distributions

1.1 Introduction

The matrix variate asymmetric Laplace distribution is a continuous probability distribution that is a generalization of the multivariate asymmetric Laplace distribution to matrix-valued random variables. The matrix variate asymmetric Laplace distribution of a random matrix $X \in \mathbb{R}^{k \times n}$ can be written in the following notation:

$$\mathcal{MAL}_{k \times n} (M, \Sigma, \Psi)$$

or

$$\mathcal{MAL} (M, \Sigma, \Psi),$$

where $M \in \mathbb{R}^{k \times n}$ is a location matrix, $\Sigma \in \mathbb{R}^{k \times k}$ is a positive-definite scale matrix and $\Psi \in \mathbb{R}^{n \times n}$ is a positive-definite scale matrix.

The matrix variate generalized asymmetric Laplace distribution is a continuous probability distribution that is a generalization of the multivariate generalized asymmetric Laplace distribution to matrix-valued random variables. The matrix variate generalized asymmetric Laplace distribution of a random matrix $X \in \mathbb{R}^{k \times n}$ can be written in the following notation:

$$\mathcal{MGAL}_{k \times n} (M, \Sigma, \Psi, \lambda)$$

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or
\[ \mathcal{MGL}(M, \Sigma, \Psi, \lambda), \]
where \( M \in \mathbb{R}^{k \times n} \) is a location matrix, \( \Sigma \in \mathbb{R}^{k \times k} \) is a positive-definite scale matrix, \( \Psi \in \mathbb{R}^{n \times n} \) is a positive-definite scale matrix and \( \lambda > 0 \).

### 1.2 Matrix variate asymmetric Laplace distribution

**Definition 1.2.1.** \( X \sim \mathcal{MAL}(M, \Sigma, \Psi) \) if and only if the probability density function of the random matrix \( X \in \mathbb{R}^{k \times n} \) is given by

\[
f_X(x) = \frac{2 \exp \operatorname{tr}(\Psi^{-1}x^T\Sigma^{-1}M)}{(2\pi)^{kn/2} (\det \Psi)^{k/2} (\det \Sigma)^{n/2}} \left( \frac{\operatorname{tr}(\Psi^{-1}x^T\Sigma^{-1}x)}{2 + \operatorname{tr}(\Psi^{-1}M^T\Sigma^{-1}M)} \right)^{1/2 - kn/4} \times K_{1 - kn/2} \left( \sqrt{2 + \operatorname{tr}(\Psi^{-1}M^T\Sigma^{-1}M)} \operatorname{tr}(\Psi^{-1}x^T\Sigma^{-1}x) \right),
\]

where \( K_{1 - kn/2}(\cdot) \) is the modified Bessel function of the third kind.

**Theorem 1.2.1.** \( X \sim \mathcal{MAL}_{k \times n}(M, \Sigma, \Psi) \Leftrightarrow \operatorname{vec}(X) \sim \mathcal{AL}_{kn}(\operatorname{vec}M, \Psi \otimes \Sigma). \)

**Proof.** By Definition 2.3.1, \( X \sim \mathcal{MAL}_{k \times n}(M, \Sigma, \Psi) \) if and only if

\[
f_X(x) = \frac{2 \exp \operatorname{tr}(\Psi^{-1}x^T\Sigma^{-1}M)}{(2\pi)^{kn/2} (\det \Psi)^{k/2} (\det \Sigma)^{n/2}} \left( \frac{\operatorname{tr}(\Psi^{-1}x^T\Sigma^{-1}x)}{2 + \operatorname{tr}(\Psi^{-1}M^T\Sigma^{-1}M)} \right)^{1/2 - kn/4} \times K_{1 - kn/2} \left( \sqrt{2 + \operatorname{tr}(\Psi^{-1}M^T\Sigma^{-1}M)} \operatorname{tr}(\Psi^{-1}x^T\Sigma^{-1}x) \right),
\]

where \( K_{1 - kn/2}(\cdot) \) is the modified Bessel function of the third kind.

By the definition of the multivariate asymmetric Laplace distribution [3], \( \operatorname{vec}(X) \sim \mathcal{AL}_{kn}(\operatorname{vec}M, \Psi \otimes \Sigma) \) if and only if

\[
f_{\operatorname{vec}X}(\operatorname{vec}x) = \frac{2 \exp \left( (\operatorname{vec}x)^T(\Psi \otimes \Sigma)^{-1}(\operatorname{vec}M) \right)}{(2\pi)^{kn/2} (\det (\Psi \otimes \Sigma))^{1/2}} \left( \frac{(\operatorname{vec}x)^T(\Psi \otimes \Sigma)^{-1}(\operatorname{vec}x)}{2 + (\operatorname{vec}M)^T(\Psi \otimes \Sigma)^{-1}(\operatorname{vec}M)} \right)^{1/2 - kn/4} \times K_{1 - kn/2} \left( \sqrt{2 + (\operatorname{vec}M)^T(\Psi \otimes \Sigma)^{-1}(\operatorname{vec}M)} \left( (\operatorname{vec}x)^T(\Psi \otimes \Sigma)^{-1}(\operatorname{vec}x) \right) \right),
\]

where \( K_{1 - kn/2}(\cdot) \) is the modified Bessel function of the third kind.

By the properties of the trace, Kronecker product and vectorization [2 p. 2-12],

\[
\begin{align*}
\operatorname{tr}(\Psi^{-1}A^T\Sigma^{-1}B) &= (\operatorname{vec}A)^T \operatorname{vec}(\Sigma^{-1}B\Psi) = (\operatorname{vec}A)^T (\Psi^{-1})^T \otimes (\Sigma^{-1}) (\operatorname{vec}B) = (\operatorname{vec}A)^T (\Psi^T \otimes \Sigma)^{-1} (\operatorname{vec}B) = (\operatorname{vec}A)^T (\Psi \otimes \Sigma)^{-1} (\operatorname{vec}B).
\end{align*}
\]

Therefore,

\[
\operatorname{tr}(\Psi^{-1}x^T\Sigma^{-1}x) = (\operatorname{vec}x)^T (\Psi \otimes \Sigma)^{-1} (\operatorname{vec}x),
\]
\[
\text{tr} \left( \Psi^{-1} M^T \Sigma^{-1} M \right) = (\text{vec} M)^T (\Psi \otimes \Sigma)^{-1} (\text{vec} M),
\]
and
\[
\text{tr} \left( \Psi^{-1} x^T \Sigma^{-1} M \right) = (\text{vec} x)^T (\Psi \otimes \Sigma)^{-1} (\text{vec} M).
\]

By the property of the determinant [2, p. 2-12],
\[
(\det \Psi)^{k/2} (\det \Sigma)^{n/2} = (\det (\Psi \otimes \Sigma))^{1/2}.
\]

Thus, \( X \sim MAL_{k \times n}(M, \Sigma, \Psi) \Leftrightarrow \text{vec}(X) \sim AL_{kn}(\text{vec} M, \Psi \otimes \Sigma). \)

**Theorem 1.2.2.** Let \( T \in \mathbb{R}^{k \times n} \). \( X \sim MAL_{k \times n}(M, \Sigma, \Psi) \) if and only if the characteristic function of the random matrix \( X \in \mathbb{R}^{k \times n} \) is given by
\[
\varphi_X(T) = \frac{1}{1 + \frac{1}{2} \text{tr} (\Psi^T \Sigma T) - i \text{tr} (M^T T)}.
\]

**Proof.** By Theorem 2.4.1
\[
X \sim MAL_{k \times n}(M, \Sigma, \Psi) \Leftrightarrow \text{vec}(X) \sim AL_{kn}(\text{vec} M, \Psi \otimes \Sigma).
\]

By the definition of the multivariate asymmetric Laplace distribution [3],
\[
\varphi_{\text{vec} X}(\text{vec} T) = \frac{1}{1 + \frac{1}{2} (\text{vec} T)^T (\Psi \otimes \Sigma) (\text{vec} T) - i (\text{vec} M)^T (\text{vec} T)}.
\]

Thus,
\[
\varphi_{\text{vec} X}(\text{vec} T) = \varphi_X(T) = \frac{1}{1 + \frac{1}{2} \text{tr} (\Psi^T \Sigma T) - i \text{tr} (M^T T)}.
\]

**Theorem 1.2.3.** Let \( Y \sim MAL_{k \times n}(M, \Sigma, \Psi) \), \( X \sim MN_{k \times n}(0, \Sigma, \Psi) \) and \( W \sim \text{Exp}(1) \), independent of \( X \), then
\[
Y = MW + W^{1/2}X.
\]

**Proof.** Let \( Y = MW + W^{1/2}X \).

Therefore,
\[
\text{vec} \ Y = \text{vec} \left( MW + W^{1/2}X \right) = (\text{vec} M) W + W^{1/2} \text{vec} \ X.
\]

By the definition of the matrix normal distribution [2],
\[
X \sim MN_{k \times n}(M, \Sigma, \Psi) \Leftrightarrow \text{vec}(X) \sim N_{kn}(\text{vec} M, \Psi \otimes \Sigma).
\]

By the theorem [3] Theorem 6.3.1],
\[
\text{vec} \ (Y) \sim AL_{kn}(\text{vec} M, \Psi \otimes \Sigma).
\]

By Theorem 2.4.1
\[
Y \sim MAL_{k \times n}(M, \Sigma, \Psi) \Leftrightarrow \text{vec} \ (Y) \sim AL_{kn}(\text{vec} M, \Psi \otimes \Sigma).
\]

**Theorem 1.2.4.** If \( X \sim MAL(M, \Sigma, \Psi) \), then the expected value of the random matrix \( X \in \mathbb{R}^{k \times n} \) is given by
\[
\mathbb{E}[X] = M.
\]
Proof. By Theorem 2.4.1

\[ X \sim MAL_{k \times n}(M, \Sigma, \Psi) \iff \text{vec}(X) \sim AL_{kn}(\text{vec } M, \Psi \otimes \Sigma). \]

By the property of the multivariate asymmetric Laplace distribution [3],

\[ E[\text{vec } X] = \text{vec } M. \]

Thus,

\[ E[X] = M. \]

Theorem 1.2.5. If \( X \sim MAL_{k \times n}(M, \Sigma, \Psi) \), then the variance-covariance matrix of the random matrix \( X \in \mathbb{R}^{k \times n} \) is given by

\[ K_{XX} = \Psi \otimes \Sigma + (\text{vec } M)(\text{vec } M)^T. \]

Proof. By Theorem 2.4.1

\[ X \sim MAL_{k \times n}(M, \Sigma, \Psi) \iff \text{vec}(X) \sim AL_{kn}(\text{vec } M, \Psi \otimes \Sigma). \]

By the property of the multivariate asymmetric Laplace distribution [3],

\[ K_{\text{vec } X, \text{vec } X} = \Psi \otimes \Sigma + (\text{vec } M)(\text{vec } M)^T. \]

Thus,

\[ K_{XX} = \Psi \otimes \Sigma + (\text{vec } M)(\text{vec } M)^T. \]

Theorem 1.2.6. Let \( D \) is a rectangular \( m \times k \) matrix, \( C \) is a rectangular \( n \times p \) matrix, \( m \leq k \) and \( p \leq n \), then

\[ X \sim MAL_{k \times n}(M, \Sigma, \Psi) \iff DXC \sim MAL_{m \times p}(DMC, D\Sigma D^T, C^T\Psi C). \]

Proof. The characteristic function of \( DXC \) is

\[ \varphi_{DXC}(T) = E \left[ e^{i \text{tr}(DXCT^T)} \right] = E \left[ e^{i \text{tr}(XCT^TD)} \right]. \]

By Theorem 2.4.2

\[ \varphi_{DXC}(T) = \frac{1}{1 + \frac{1}{2} \text{tr}(\Psi\Sigma^T\Sigma D\Sigma^T T^T) - i \text{tr}(CT^TDM)} \]

Thus,

\[ X \sim MAL_{k \times n}(M, \Sigma, \Psi) \iff DXC \sim MAL_{m \times p}(DMC, D\Sigma D^T, C^T\Psi C). \]

Corollary 1.2.1. Let \( D \) is a rectangular \( m \times k \) matrix and \( m \leq k \), then

\[ X \sim MAL_{k \times n}(M, \Sigma, \Psi) \iff DX \sim MAL_{m \times n}(DM, D\Sigma D^T, \Psi). \]

Corollary 1.2.2. Let \( C \) is a rectangular \( n \times p \) matrix and \( p \leq n \), then

\[ X \sim MAL_{k \times n}(M, \Sigma, \Psi) \iff XC \sim MAL_{k \times p}(MC, \Sigma, C^T\Psi C). \]
1.3 Matrix variate generalized asymmetric Laplace distribution

Definition 1.3.1. \( X \sim \mathcal{M} \mathcal{G} \mathcal{A} \mathcal{L}(M, \Sigma, \Psi, \lambda) \) if and only if the probability density function of the random matrix \( X \in \mathbb{R}^{k \times n} \) is given by

\[
f_X(x) = \frac{2 \exp \left( \text{tr} \left( \Psi^{-1} x^T \Sigma^{-1} M \right) \right)}{(2\pi)^{kn/2} \Gamma (\lambda) (\det \Psi)^{k/2} (\det \Sigma)^{n/2}} \left( \frac{\text{tr} \left( \Psi^{-1} x^T \Sigma^{-1} x \right)}{2 + \text{tr} \left( \Psi^{-1} M^T \Sigma^{-1} M \right)} \right)^{\lambda/2 - kn/4} \times K_{\lambda-kn/2} \left( \sqrt{2 + \text{tr} \left( \Psi^{-1} M^T \Sigma^{-1} M \right)} \frac{\text{tr} \left( \Psi^{-1} x^T \Sigma^{-1} x \right)}{\text{tr} \left( \Psi^{-1} x^T \Sigma^{-1} x \right)} \right),
\]

where \( K_{\lambda-ken/2}(\cdot) \) is the modified Bessel function of the third kind.

Theorem 1.3.1. \( X \sim \mathcal{M} \mathcal{G} \mathcal{A} \mathcal{L}_{k \times n}(M, \Sigma, \Psi, \lambda) \Leftrightarrow \text{vec} \,(X) \sim \mathcal{G} \mathcal{A} \mathcal{L}_{kn}(\text{vec} \,M, \Psi \otimes \Sigma, \lambda). \)

Proof. By Definition 1.3.1, \( X \sim \mathcal{M} \mathcal{G} \mathcal{A} \mathcal{L}_{k \times n}(M, \Sigma, \Psi, \lambda) \) if and only if

\[
f_X(x) = \frac{2 \exp \left( \text{tr} \left( \Psi^{-1} x^T \Sigma^{-1} M \right) \right)}{(2\pi)^{kn/2} \Gamma (\lambda) (\det \Psi)^{k/2} (\det \Sigma)^{n/2}} \left( \frac{\text{tr} \left( \Psi^{-1} x^T \Sigma^{-1} x \right)}{2 + \text{tr} \left( \Psi^{-1} M^T \Sigma^{-1} M \right)} \right)^{\lambda/2 - kn/4} \times K_{\lambda-ken/2} \left( \sqrt{2 + \text{tr} \left( \Psi^{-1} M^T \Sigma^{-1} M \right)} \frac{\text{tr} \left( \Psi^{-1} x^T \Sigma^{-1} x \right)}{\text{tr} \left( \Psi^{-1} x^T \Sigma^{-1} x \right)} \right),
\]

where \( K_{\lambda-ken/2}(\cdot) \) is the modified Bessel function of the third kind.

By the definition of the multivariate asymmetric Laplace distribution, \( \text{vec} \,(X) \sim \mathcal{G} \mathcal{A} \mathcal{L}_{kn}(\text{vec} \,M, \Psi \otimes \Sigma, \lambda) \) if and only if

\[
f_{\text{vec} \,x}(\text{vec} \,x) = \frac{2 \exp \left( (\text{vec} \,x)^T \left( \Psi \otimes \Sigma \right)^{-1} (\text{vec} \,x) \right)}{(2\pi)^{kn/2} \Gamma (\lambda) (\det \left( \Psi \otimes \Sigma \right))^{1/2}} \left( \frac{(\text{vec} \,x)^T \left( \Psi \otimes \Sigma \right)^{-1} (\text{vec} \,x)}{2 + (\text{vec} \,M)^T \left( \Psi \otimes \Sigma \right)^{-1} (\text{vec} \,M)} \right)^{\lambda/2 - kn/4} \times K_{\lambda-ken/2} \left( \sqrt{2 + (\text{vec} \,M)^T \left( \Psi \otimes \Sigma \right)^{-1} (\text{vec} \,M)} \left( (\text{vec} \,x)^T \left( \Psi \otimes \Sigma \right)^{-1} (\text{vec} \,x) \right) \right),
\]

where \( K_{\lambda-ken/2}(\cdot) \) is the modified Bessel function of the third kind.

By the properties of the trace, Kronecker product and vectorization, \( [2] \) p. 2-12,

\[
\text{tr} \left( \Psi^{-1} A^T \Sigma^{-1} B \right) = (\text{vec} \,A)^T \text{vec} \left( \Sigma^{-1} B \Psi^{-1} \right) = (\text{vec} \,A)^T \left( \left( \Psi^{-1} \right)^T \otimes \Sigma^{-1} \right) (\text{vec} \,B) = (\text{vec} \,A)^T \left( \Psi^T \otimes \Sigma \right)^{-1} (\text{vec} \,B) = (\text{vec} \,A)^T \left( \Psi \otimes \Sigma \right)^{-1} (\text{vec} \,B).
\]

Therefore,

\[
\text{tr} \left( \Psi^{-1} x^T \Sigma^{-1} x \right) = (\text{vec} \,x)^T \left( \Psi \otimes \Sigma \right)^{-1} (\text{vec} \,x),
\]

and

\[
\text{tr} \left( \Psi^{-1} M^T \Sigma^{-1} M \right) = (\text{vec} \,M)^T \left( \Psi \otimes \Sigma \right)^{-1} (\text{vec} \,M),
\]
By the property of the determinant [2, p. 2-12],

\[(\det \Psi)^{1/2} (\det \Sigma)^{n/2} = (\det (\Psi \otimes \Sigma))^{1/2}.\]

Thus, \(X \sim {\mathcal GAL}_{k \times n}(M, \Sigma, \Psi, \lambda) \iff \text{vec}(X) \sim {\mathcal GAL}_{kn}(\text{vec } M, \Psi \otimes \Sigma, \lambda). \)

**Theorem 1.3.2.** Let \(Y \sim {\mathcal GAL}_{k \times n}(M, \Sigma, \Psi, \lambda), X \sim {\mathcal N}_{k \times n}(0, \Sigma, \Psi)\) and \(W\) has the standard gamma distribution with shape parameter \(\lambda\), independent of \(X\), then

\[Y = MW + W^{1/2}X.\]

**Proof.** Let \(Y = MW + W^{1/2}X\).

Therefore, \(\text{vec } Y = \text{vec } \left( MW + W^{1/2}X \right) = (\text{vec } M) W + W^{1/2} \text{vec } X\).

By the definition of the matrix normal distribution [2],

\[X \sim {\mathcal N}_{k \times n}(M, \Sigma, \Psi) \iff \text{vec } (X) \sim {\mathcal N}_{kn}(\text{vec } M, \Psi \otimes \Sigma).\]

By the theorem [4],

\[\text{vec } (Y) \sim {\mathcal GAL}_{kn}(\text{vec } M, \Psi \otimes \Sigma, \lambda).\]

By Theorem 1.3.1

\[Y \sim {\mathcal GAL}_{k \times n}(M, \Sigma, \Psi, \lambda) \iff \text{vec } (Y) \sim {\mathcal GAL}_{kn}(\text{vec } M, \Psi \otimes \Sigma, \lambda).\]

**Theorem 1.3.3.** Let \(T \in \mathbb{R}^{k \times n}\). \(X \sim {\mathcal GAL}_{k \times n}(M, \Sigma, \Psi, \lambda)\) if and only if the characteristic function of the random matrix \(X \in \mathbb{R}^{k \times n}\) is given by

\[
\varphi_X (T) = \left( \frac{1}{1 + \frac{1}{2} \text{tr} (\Psi^T T) - i \text{tr} (M^T T)} \right)^\lambda.
\]

**Proof.** By Theorem 1.3.1

\[X \sim {\mathcal GAL}_{k \times n}(M, \Sigma, \Psi, \lambda) \iff \text{vec } (X) \sim {\mathcal GAL}_{kn}(\text{vec } M, \Psi \otimes \Sigma, \lambda).\]

By the definition of the multivariate generalized asymmetric Laplace distribution [4],

\[
\varphi_{\text{vec } X} (\text{vec } T) = \left( \frac{1}{1 + \frac{1}{2} (\text{vec } T)^T (\Psi \otimes \Sigma) (\text{vec } T) - i (\text{vec } M)^T (\text{vec } T)} \right)^\lambda.
\]

Thus,

\[
\varphi_{\text{vec } X} (\text{vec } T) = \varphi_X (T) = \left( \frac{1}{1 + \frac{1}{2} \text{tr} (\Psi^T T) - i \text{tr} (M^T T)} \right)^\lambda.
\]

**Theorem 1.3.4.** \(X \sim {\mathcal GAL}_{k \times n}(M, \Sigma, \Psi, 1) \iff X \sim {\mathcal M}_{k \times n}(M, \Sigma, \Psi).\)
Proof. By Theorem 1.3.3, $X \sim \mathcal{MGAL}_{k \times n}(M, \Sigma, \Psi, 1)$ if and only if

$$\varphi_X(T) = \left( \frac{1}{1 + \frac{1}{2} \text{tr}(\Psi^T \Sigma T) - i \text{tr}(M^T T)} \right)^1 = \left( \frac{1}{1 + \frac{1}{2} \text{tr}(\Psi^T \Sigma T) - i \text{tr}(M^T T)} \right).$$

By Theorem 2.4.2, $X \sim \mathcal{MAL}_{k \times n}(M, \Sigma, \Psi)$ if and only if

$$\varphi_X(T) = \left( \frac{1}{1 + \frac{1}{2} \text{tr}(\Psi^T \Sigma T) - i \text{tr}(M^T T)} \right).$$

Thus, $X \sim \mathcal{MGAL}_{k \times n}(M, \Sigma, \Psi, 1) \iff X \sim \mathcal{MAL}_{k \times n}(M, \Sigma, \Psi).$ \hfill \qed

**Theorem 1.3.5.** If $X \sim \mathcal{MGAL}(M, \Sigma, \Psi, \lambda)$, then the expected value of the random matrix $X \in \mathbb{R}^{k \times n}$ is given by

$$\mathbb{E}[X] = \lambda M.$$

**Proof.** By Theorem 1.3.4,

$$X \sim \mathcal{MGAL}_{k \times n}(M, \Sigma, \Psi, \lambda) \leftrightarrow \text{vec}(X) \sim \mathcal{GAL}_{kn}(\text{vec}M, \Psi \otimes \Sigma, \lambda).$$

By the property of the multivariate generalized asymmetric Laplace distribution \cite{4},

$$\mathbb{E}[\text{vec}X] = \lambda \text{vec}M = \text{vec}(\lambda M).$$

Thus,

$$\mathbb{E}[X] = \lambda M.$$ \hfill \qed

**Theorem 1.3.6.** If $X \sim \mathcal{MGAL}_{k \times n}(M, \Sigma, \Psi, \lambda)$, then the variance-covariance matrix of the random matrix $X \in \mathbb{R}^{k \times n}$ is given by

$$K_{XX} = \lambda \left( \Psi \otimes \Sigma + (\text{vec}M)(\text{vec}M)^T \right).$$

**Proof.** By Theorem 1.3.4,

$$X \sim \mathcal{MGAL}_{k \times n}(M, \Sigma, \Psi, \lambda) \leftrightarrow \text{vec}(X) \sim \mathcal{GAL}_{kn}(\text{vec}M, \Psi \otimes \Sigma, \lambda).$$

By the property of the multivariate asymmetric generalized Laplace distribution \cite{4},

$$K_{\text{vec}X, \text{vec}X} = \lambda \left( \Psi \otimes \Sigma + (\text{vec}M)(\text{vec}M)^T \right).$$

Thus,

$$K_{XX} = \lambda \left( \Psi \otimes \Sigma + (\text{vec}M)(\text{vec}M)^T \right).$$ \hfill \qed

**Theorem 1.3.7.** Let $D$ is a rectangular $m \times k$ matrix, $C$ is a rectangular $n \times p$ matrix, $m \leq k$ and $p \leq n$, then

$$X \sim \mathcal{MGAL}_{k \times n}(M, \Sigma, \Psi, \lambda) \iff DXC \sim \mathcal{MGAL}_{m \times p}(DMC, D\Sigma D^T, C^T \Psi C, \lambda).$$
Proof. The characteristic function of $\mathbf{DXC}$ is
\[
\varphi_{\mathbf{DXC}}(\mathbf{T}) = \mathbb{E} \left[ e^{i \text{tr}(\mathbf{DXC}^T \mathbf{T})} \right] = \mathbb{E} \left[ e^{i \text{tr}(\mathbf{XCT}^T \mathbf{T})} \right].
\]
By Theorem 1.3.3,
\[
\varphi_{\mathbf{DXC}}(\mathbf{T}) = \left( \frac{1}{1 + \frac{1}{\lambda} \text{tr} (\Psi \mathbf{CT}^T \mathbf{DS} \mathbf{T}^T \mathbf{T}) - i \text{tr} (\mathbf{CT}^T \mathbf{DM})} \right)^\lambda = \left( \frac{1}{1 + \frac{1}{\lambda} \text{tr} (\mathbf{C}^T \Psi \mathbf{CT}^T \mathbf{DS} \mathbf{T}^T \mathbf{T}) - i \text{tr} (\mathbf{T}^T \mathbf{DM})} \right)^\lambda.
\]
Thus,
\[
\mathbf{X} \sim \mathcal{MGAL}_{k \times n} (\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D) \iff \mathbf{DXC} \sim \mathcal{MGAL}_{m \times p} (\mathbf{DM}, \mathbf{DSD}^T, \mathbf{C}^T \Psi \mathbf{C}, \lambda).
\]

Corollary 1.3.1. Let $\mathbf{D}$ is a rectangular $m \times k$ matrix and $m \leq k$, then
\[
\mathbf{X} \sim \mathcal{MGAL}_{k \times n} (\mathbf{M}, \Sigma, \Psi, \lambda) \iff \mathbf{DXC} \sim \mathcal{MGAL}_{m \times n} (\mathbf{DM}, \mathbf{DSD}^T, \mathbf{C}^T \Psi \mathbf{C}, \lambda).
\]

Corollary 1.3.2. Let $\mathbf{C}$ is a rectangular $n \times p$ matrix and $p \leq n$, then
\[
\mathbf{X} \sim \mathcal{MGAL}_{k \times n} (\mathbf{M}, \Sigma, \Psi, \lambda) \iff \mathbf{XC} \sim \mathcal{MGAL}_{k \times p} (\mathbf{MC}, \Sigma, \mathbf{C}^T \Psi \mathbf{C}, \lambda).
\]

2 Tensor variate Laplace distributions

2.1 Introduction

The tensor variate asymmetric Laplace distribution is a continuous probability distribution that is a generalization of the matrix variate asymmetric Laplace distribution to tensor-valued random variables. The tensor variate asymmetric Laplace distribution of a random order-D tensor $\mathbf{X}$, with dimensional lengths $n_1 \times n_2 \times \cdots \times n_D = \mathbf{n}$ can be written in the following notation:
\[
\mathcal{TAL}_\mathbf{n} (\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D)
\]
or
\[
\mathcal{TAL} (\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D),
\]
where $\mathbf{M}$ is a location tensor, $\Sigma_i$ is a positive-definite scale matrix.

The tensor variate generalized asymmetric Laplace distribution is a continuous probability distribution that is a generalization of the matrix variate generalized asymmetric Laplace distribution to tensor-valued random variables. The tensor variate generalized asymmetric Laplace distribution of a random order-D tensor $\mathbf{X}$, with dimensional lengths $n_1 \times n_2 \times \cdots \times n_D = \mathbf{n}$ can be written in the following notation:
\[
\mathcal{GAL}_\mathbf{n} (\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, \lambda)
\]
or
\[
\mathcal{GAL} (\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, \lambda),
\]
where $\mathbf{M}$ is a location tensor, $\Sigma_i$ is a positive-definite scale matrix and $\lambda > 0$. 
2.2 Tensor variate asymmetric Laplace distribution

Definition 2.2.1. Let $\mathbf{X}$ be a random order-$D$ tensor, with dimensional lengths $n_1 \times n_2 \times \cdots \times n_D$, with realization $\mathbf{x}$. $\mathbf{X} \sim \mathcal{TAL}(\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D)$ if and only if the characteristic function of the random tensor is given by

$$\varphi_{\mathbf{X}}(\mathbf{t}) = 1 + \frac{1}{2} (\mathbf{t} \mathbf{x})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \mathbf{M} \mathbf{t} - i (\mathbf{t} \mathbf{x})^T \mathbf{M} \mathbf{x}.$$

Proof.

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{2 \exp \left( (\mathbf{x})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \mathbf{M} \right)}{(2\pi)^{n^*/2} \prod_{i=1}^D (\det \Sigma_i)^{n^*/(2n_i)}} \left( \frac{(\mathbf{x})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \mathbf{x}}{2 + (\mathbf{x})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \mathbf{x}} \right)^{1/2-n^*/4} \times K_{1-n^*/2} \left( \left( \frac{2 + (\mathbf{M})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \mathbf{M} \mathbf{t}}{2 + (\mathbf{x})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \mathbf{x}} \right)^{1/2-n^*/4} \right),$$

where $n^* = \prod_{i=1}^D n_i$ and $K_{1-n^*/2}()$ is the modified Bessel function of the third kind.

Theorem 2.2.1. Let $\mathbf{X}$ be a random order-$D$ tensor, with dimensional lengths $n_1 \times n_2 \times \cdots \times n_D$, with realization $\mathbf{x}$. $\mathbf{X} \sim \mathcal{TAL}_{\mathbf{n}}(\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D) \iff \mathbf{x} \sim \mathcal{AL}_{\mathbf{n}} \left( \mathbf{M}, \bigotimes_{i=1}^D \Sigma_i \right).$

Proof.

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{2 \exp \left( (\mathbf{x})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \mathbf{M} \right)}{(2\pi)^{n^*/2} \prod_{i=1}^D (\det \Sigma_i)^{n^*/(2n_i)}} \left( \frac{(\mathbf{x})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \mathbf{x}}{2 + (\mathbf{x})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \mathbf{x}} \right)^{1/2-n^*/4} \times K_{1-n^*/2} \left( \left( \frac{2 + (\mathbf{M})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \mathbf{M} \mathbf{t}}{2 + (\mathbf{x})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \mathbf{x}} \right)^{1/2-n^*/4} \right),$$

where $n^* = \prod_{i=1}^D n_i$ and $K_{1-n^*/2}()$ is the modified Bessel function of the third kind.

Thus, $\mathbf{X} \sim \mathcal{TAL}_{\mathbf{n}}(\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D) \iff \mathbf{x} \sim \mathcal{AL}_{\mathbf{n}} \left( \mathbf{M}, \bigotimes_{i=1}^D \Sigma_i \right)$. 

Theorem 2.2.2. Let $\mathbf{T}$ be a random order-$D$ tensor. $\mathbf{X} \sim \mathcal{TAL}_{\mathbf{n}}(\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D)$ if and only if the characteristic function of the random tensor is given by

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \frac{1}{1 + \frac{1}{2} (\mathbf{t} \mathbf{x})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \mathbf{M} \mathbf{t} - i (\mathbf{t} \mathbf{x})^T \mathbf{M} \mathbf{x}}.$$
Proof. By Theorem 2.4.1

\[ \mathcal{X} \sim \mathcal{T} \mathcal{A} \mathcal{L}_n (\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D) \iff \text{vec } \mathcal{X} \sim \mathcal{A} \mathcal{L}_{n^*} \left( \text{vec } \mathcal{M}, \bigotimes_{i=1}^D \Sigma_i \right). \]

By the definition of the multivariate asymmetric Laplace distribution \[3\], \( \text{vec } \mathcal{X} \sim \mathcal{A} \mathcal{L}_{n^*} \left( \text{vec } \mathcal{M}, \bigotimes_{i=1}^D \Sigma_i \right) \) if and only if

\[ \varphi_{\text{vec } \mathcal{X}} (\text{vec } \mathcal{X}) = \frac{1}{1 + \frac{1}{2} (\text{vec } \mathcal{X})^T \left( \bigotimes_{i=1}^D \Sigma_i \right) \text{vec } \mathcal{X} - i (\text{vec } \mathcal{M})^T \text{vec } \mathcal{X}}. \]

Thus,

\[ \varphi_{\text{vec } \mathcal{X}} (\text{vec } \mathcal{X}) = \varphi (\mathcal{X}) = \frac{1}{1 + \frac{1}{2} (\text{vec } \mathcal{X})^T \left( \bigotimes_{i=1}^D \Sigma_i \right) \text{vec } \mathcal{X} - i (\text{vec } \mathcal{M})^T \text{vec } \mathcal{X}}. \]

Theorem 2.2.3. Let \( \mathcal{Y} \sim \mathcal{T} \mathcal{A} \mathcal{L}_n (\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D) \), \( \mathcal{X} \sim \mathcal{T} \mathcal{N}_n (\mathcal{O}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D) \) and \( W \sim \text{Exp}(1) \), independent of \( \mathcal{X} \), then

\[ \mathcal{Y} = \mathcal{M} \mathcal{W} + W^{1/2} \mathcal{X}. \]

Proof. Let \( \mathcal{Y} = \mathcal{M} \mathcal{W} + W^{1/2} \mathcal{X}. \) Therefore,

\[ \text{vec } \mathcal{Y} = \text{vec} \left( \mathcal{M} \mathcal{W} + W^{1/2} \mathcal{X} \right) = (\text{vec } \mathcal{M}) W + W^{1/2} \text{vec } \mathcal{X}. \]

By the definition of the tensor normal distribution \[4\] Theorem 2.1,[

\[ \mathcal{X} \sim \mathcal{T} \mathcal{N}_n (\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D) \iff \text{vec } \mathcal{X} \sim \mathcal{N}_{n^*} \left( \text{vec } \mathcal{M}, \bigotimes_{i=1}^D \Sigma_i \right). \]

By the theorem \[3\] Theorem 6.3.1,

\[ \text{vec } \mathcal{Y} \sim \mathcal{A} \mathcal{L}_{n^*} \left( \text{vec } \mathcal{M}, \bigotimes_{i=1}^D \Sigma_i \right). \]

By Theorem 2.4.1

\[ \text{vec } \mathcal{Y} \sim \mathcal{A} \mathcal{L}_{n^*} \left( \text{vec } \mathcal{M}, \bigotimes_{i=1}^D \Sigma_i \right) \iff \mathcal{Y} \sim \mathcal{T} \mathcal{A} \mathcal{L}_n (\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D). \]

Theorem 2.2.4. Let \( \mathcal{X} \) is a random order-D tensor, with dimensional lengths \( n_1 \times n_2 \times \cdots \times n_D \), with realization \( \mathcal{X} \). \( \mathcal{X} \sim \mathcal{T} \mathcal{A} \mathcal{L}_n (\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D) \) if and only if the probability density function is given by

\[ f_{\mathcal{X}} (\mathcal{X}) = \frac{2 \exp \left( (\text{vec } \mathcal{X})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \text{vec } \mathcal{M} \right)}{(2\pi)^{n^*/2} \prod_{i=1}^D (\det \Sigma_i)^{n^*/(2n_i)}} \left( \frac{(\text{vec } \mathcal{X})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \text{vec } \mathcal{X}}{2 + (\text{vec } \mathcal{M})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \text{vec } \mathcal{M}} \right)^{1/2-n^*/4} \times K_{1-n^*/2} \left( \left( 2 + (\text{vec } \mathcal{M})^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \text{vec } \mathcal{M} \right) \left( \text{vec } \mathcal{X} \right)^T \left( \bigotimes_{i=1}^D \Sigma_i \right)^{-1} \text{vec } \mathcal{X} \right), \]

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where $n^* = \prod_{i=1}^{D} n_i$ and $K_{1-n^*/2}(\cdot)$ is the modified Bessel function of the third kind.

By the definition of the tensor variance gamma distribution $\mathcal{TVVG}$, $X \sim \mathcal{TVVG}(\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, 1)$ if and only if

$$f_X(x) = \frac{2 \exp \left( (\text{vec } x)^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right)^{-1} \text{vec } \mathcal{M} \right)}{(2\pi)^{n^*/2} \prod_{i=1}^{D} (\det \Sigma_i)^{n^*/(2n_i)}} \times K_{1-n^*/2} \left( \sqrt{2 + (\text{vec } \mathcal{M})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right)^{-1} \text{vec } \mathcal{M}} \right) \frac{1}{2-n^*/4} \left( (\text{vec } x)^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right)^{-1} \text{vec } x \right)^{1/2-n^*/4},$$

where $n^* = \prod_{i=1}^{D} n_i$ and $K_{1-n^*/2}(\cdot)$ is the modified Bessel function of the third kind.

Thus, $X \sim \mathcal{TVVG}(\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D) \iff X \sim \mathcal{TVVG}(\mathcal{O}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, 1).$

**Theorem 2.2.5.** If $X \sim \mathcal{TAL}(\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D)$, then the expected value of the random tensor $X$ is given by

$$E[X] = \mathcal{M}.$$  

**Proof.** By Theorem 2.4.1

$$X \sim \mathcal{TAL}(\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D) \iff \text{vec } X \sim \mathcal{AL}(\text{vec } \mathcal{M}, \bigotimes_{i=1}^{D} \Sigma_i).$$

By the property of the multivariate asymmetric Laplace distribution [3],

$$E[\text{vec } X] = \text{vec } \mathcal{M}.$$

Thus,

$$E[X] = \mathcal{M}.$$  

**Theorem 2.2.6.** If $X \sim \mathcal{TAL}(\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D)$, then the variance-covariance matrix of the random tensor $X$ is given by

$$K_{XX} = \bigotimes_{i=1}^{D} \Sigma_i + \text{vec } \mathcal{M} (\text{vec } \mathcal{M})^T.$$  

**Proof.** By Theorem 2.4.1

$$X \sim \mathcal{TAL}(\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D) \iff \text{vec } X \sim \mathcal{AL}(\text{vec } \mathcal{M}, \bigotimes_{i=1}^{D} \Sigma_i).$$

By the property of the multivariate asymmetric Laplace distribution [3], if $\text{vec } X \sim \mathcal{AL}(\text{vec } \mathcal{M}, \bigotimes_{i=1}^{D} \Sigma_i)$, then

$$K_{\text{vec } X, \text{vec } X} = \bigotimes_{i=1}^{D} \Sigma_i + \text{vec } \mathcal{M} (\text{vec } \mathcal{M})^T.$$

Thus,

$$K_{XX} = K_{\text{vec } X, \text{vec } X} = \bigotimes_{i=1}^{D} \Sigma_i + \text{vec } \mathcal{M} (\text{vec } \mathcal{M})^T.$$
2.3 Tensor variate generalized asymmetric Laplace distribution

Definition 2.3.1. Let $\mathbf{X}$ is a random order-$D$ tensor, with dimensional lengths $n_1 \times n_2 \times \cdots \times n_D$, with realization $\mathfrak{x}$. $\mathbf{X} \sim \mathcal{TGAL} \left( \mathfrak{m}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, \lambda \right)$ if and only if the probability density function is given by

$$
\begin{align*}
    f_{\mathbf{X}} ( \mathfrak{x} ) &= \frac{2 \exp \left( (\text{vec} \mathfrak{x})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right)^{-1} \text{vec} \mathfrak{m} \right)}{(2\pi)^{n^*/2} \Gamma (\lambda) \prod_{i=1}^{D} (\det \Sigma_i)^{n^*/(2n_i)}} \cdot \frac{2 + (\text{vec} \mathfrak{m})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right)^{-1} \text{vec} \mathfrak{x}}{2 (\text{vec} \mathfrak{m})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right)^{-1} \text{vec} \mathfrak{m}} \right) \lambda^{n^*/4} \times K_{\lambda-n^*/2} \left( \sqrt{ \left( 2 + (\text{vec} \mathfrak{m})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right)^{-1} \text{vec} \mathfrak{m} \right) \left( (\text{vec} \mathfrak{x})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right)^{-1} \text{vec} \mathfrak{x} \right) - \lambda^2 n^*/4 } \right),
\end{align*}
$$

where $n^* = \prod_{i=1}^{D} n_i$ and $K_{\lambda-n^*/2} (\cdot)$ is the modified Bessel function of the third kind.

2.4 Relationship to multivariate generalized asymmetric Laplace distribution

Theorem 2.4.1. Let $\mathbf{X}$ is a random order-$D$ tensor, with dimensional lengths $n_1 \times n_2 \times \cdots \times n_D$, with realization $\mathfrak{x}$. $\mathbf{X} \sim \mathcal{TGAL_n} \left( \mathfrak{m}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, \lambda \right)$ $\iff$ $\text{vec} \mathbf{X} \sim \mathcal{GAL}_{n^*} \left( \text{vec} \mathfrak{m}, \bigotimes_{i=1}^{D} \Sigma_i, \lambda \right)$.

Proof.

$$
\begin{align*}
    f_{\text{vec} \mathbf{X}} ( \text{vec} \mathfrak{x} ) &= \frac{2 \exp \left( (\text{vec} \mathfrak{x})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right)^{-1} \text{vec} \mathfrak{m} \right)}{(2\pi)^{n^*/2} \Gamma (\lambda) \prod_{i=1}^{D} (\det \Sigma_i)^{n^*/(2n_i)}} \cdot \frac{2 + (\text{vec} \mathfrak{m})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right)^{-1} \text{vec} \mathfrak{x}}{2 (\text{vec} \mathfrak{m})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right)^{-1} \text{vec} \mathfrak{m}} \right) \lambda^{n^*/4} \times K_{\lambda-n^*/2} \left( \sqrt{ \left( 2 + (\text{vec} \mathfrak{m})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right)^{-1} \text{vec} \mathfrak{m} \right) \left( (\text{vec} \mathfrak{x})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right)^{-1} \text{vec} \mathfrak{x} \right) - \lambda^2 n^*/4 } \right),
\end{align*}
$$

where $n^* = \prod_{i=1}^{D} n_i$ and $K_{\lambda-n^*/2} (\cdot)$ is the modified Bessel function of the third kind.

By the definition of the multivariate generalized asymmetric Laplace distribution \[4\], $\text{vec} \mathbf{X} \sim \mathcal{GAL}_{n^*} \left( \text{vec} \mathfrak{m}, \bigotimes_{i=1}^{D} \Sigma_i, \lambda \right)$ if and only if

Thus, $\mathbf{X} \sim \mathcal{TGAL_n} \left( \mathfrak{m}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, \lambda \right)$ $\iff$ $\text{vec} \mathbf{X} \sim \mathcal{GAL}_{n^*} \left( \text{vec} \mathfrak{m}, \bigotimes_{i=1}^{D} \Sigma_i, \lambda \right)$.

\[\square\]
Theorem 2.4.2. Let $\mathbf{T}$ be an $n$-dimensional order-D tensor. $\mathbf{X} \sim T_{\mathcal{GAL}}(\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, \lambda)$ if and only if the characteristic function of the random tensor is given by

$$
\varphi_{\mathbf{X}}(\mathbf{t}) = \left( \frac{1}{1 + \frac{1}{2} (\text{vec } \mathbf{T})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right) \text{vec } \mathbf{I} - i (\text{vec } \mathbf{M})^T \text{vec } \mathbf{I}} \right)^{\lambda}.
$$

Proof. By Theorem 2.4.1

$$
\mathbf{X} \sim T_{\mathcal{GAL}}(\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, \lambda) \iff \text{vec } \mathbf{X} \sim \mathcal{GAL}_{n^*}(\text{vec } \mathbf{M}, \bigotimes_{i=1}^{D} \Sigma_i, \lambda).
$$

By the definition of the multivariate generalized asymmetric Laplace distribution [4], vec $\mathbf{X} \sim \mathcal{GAL}_{n^*}(\text{vec } \mathbf{M}, \bigotimes_{i=1}^{D} \Sigma_i, \lambda)$ if and only if

$$
\varphi_{\text{vec } \mathbf{X}}(\text{vec } \mathbf{t}) = \left( \frac{1}{1 + \frac{1}{2} (\text{vec } \mathbf{t})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right) \text{vec } \mathbf{I} - i (\text{vec } \mathbf{M})^T \text{vec } \mathbf{I}} \right)^{\lambda}.
$$

Thus,

$$
\varphi_{\text{vec } \mathbf{X}}(\text{vec } \mathbf{t}) = \varphi_{\mathbf{X}}(\mathbf{t}) = \left( \frac{1}{1 + \frac{1}{2} (\text{vec } \mathbf{t})^T \left( \bigotimes_{i=1}^{D} \Sigma_i \right) \text{vec } \mathbf{I} - i (\text{vec } \mathbf{M})^T \text{vec } \mathbf{I}} \right)^{\lambda}.
$$

Theorem 2.4.3. Let $\mathbf{Y} \sim T_{\mathcal{GAL}}(\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, \lambda)$, $\mathbf{X} \sim T_{\mathcal{N}}(\mathbf{0}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D)$ and $W$ has the standart gamma distribution with shape parameter $\lambda$, independent of $\mathbf{X}$, then

$$
\mathbf{Y} = \mathbf{M}W + W^{1/2} \mathbf{X}.
$$

Proof. Let $\mathbf{Y} = \mathbf{M}W + W^{1/2} \mathbf{X}$.

Therefore,

$$
\text{vec } \mathbf{Y} = \text{vec } \left( \mathbf{M}W + W^{1/2} \mathbf{X} \right) = (\text{vec } \mathbf{M}) W + W^{1/2} \text{vec } \mathbf{X}.
$$

By the definition of the tensor normal distribution [1, Theorem 2.1],

$$
\mathbf{X} \sim T_{\mathcal{N}}(\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D) \iff \text{vec } \mathbf{X} \sim \mathcal{N}_{n^*}(\text{vec } \mathbf{M}, \bigotimes_{i=1}^{D} \Sigma_i).
$$

By the theorem [1],

$$
\text{vec } \mathbf{Y} \sim \mathcal{GAL}_{n^*}(\text{vec } \mathbf{M}, \bigotimes_{i=1}^{D} \Sigma_i, \lambda).
$$

By Theorem 2.4.1

$$
\text{vec } \mathbf{Y} \sim \mathcal{GAL}_{n^*}(\text{vec } \mathbf{M}, \bigotimes_{i=1}^{D} \Sigma_i, \lambda) \iff \mathbf{Y} \sim T_{\mathcal{GAL}}(\mathbf{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, \lambda).
$$
Theorem 2.4.4. If $\mathcal{X} \sim T_{GAL}(\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, \lambda)$, then the expected value of the random tensor $\mathcal{X}$ is given by

$$\mathbb{E}[\mathcal{X}] = \lambda \mathcal{M}.$$ 

Proof. By Theorem 2.4.1

$$\mathcal{X} \sim T_{GAL_n}(\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, \lambda) \Leftrightarrow \text{vec} \mathcal{X} \sim \mathcal{GAL}_n^*(\text{vec } \mathcal{M}, \bigotimes_{i=1}^{D} \Sigma_i, \lambda).$$

By the property of the multivariate generalized asymmetric Laplace distribution [4], if $\text{vec} \mathcal{X} \sim \mathcal{GAL}_n^*(\text{vec } \mathcal{M}, \bigotimes_{i=1}^{D} \Sigma_i, \lambda)$, then

$$\mathbb{E}[\text{vec } \mathcal{X}] = \lambda \text{vec } \mathcal{M} = \text{vec } (\lambda \mathcal{M}).$$

Thus,

$$\mathbb{E}[\mathcal{X}] = \lambda \mathcal{M}. \quad \square$$

Theorem 2.4.5. If $\mathcal{X} \sim T_{GAL_n}(\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, \lambda)$, then the variance-covariance matrix of the random tensor $\mathcal{X}$ is given by

$$K_{\mathcal{X} \mathcal{X}} = \lambda \left( \bigotimes_{i=1}^{D} \Sigma_i + (\text{vec } \mathcal{M}) (\text{vec } \mathcal{M})^T \right).$$

Proof. By Theorem 2.4.1

$$\mathcal{X} \sim T_{GAL_n}(\mathcal{M}, \Sigma_1, \Sigma_2, \ldots, \Sigma_D, \lambda) \Leftrightarrow \text{vec } \mathcal{X} \sim \mathcal{GAL}_n^*(\text{vec } \mathcal{M}, \bigotimes_{i=1}^{D} \Sigma_i, \lambda).$$

By the property of the multivariate generalized asymmetric Laplace distribution [4], if $\text{vec } \mathcal{X} \sim \mathcal{GAL}_n^*(\text{vec } \mathcal{M}, \bigotimes_{i=1}^{D} \Sigma_i, \lambda)$, then

$$K_{\text{vec } \mathcal{X}, \text{vec } \mathcal{X}} = \lambda \left( \bigotimes_{i=1}^{D} \Sigma_i + (\text{vec } \mathcal{M}) (\text{vec } \mathcal{M})^T \right).$$

Thus,

$$K_{\mathcal{X} \mathcal{X}} = K_{\text{vec } \mathcal{X}, \text{vec } \mathcal{X}} = \lambda \left( \bigotimes_{i=1}^{D} \Sigma_i + (\text{vec } \mathcal{M}) (\text{vec } \mathcal{M})^T \right). \quad \square$$

3 Conclusions

In this article, we have defined the matrix variate asymmetric Laplace distribution. We have proved some properties of the matrix variate asymmetric Laplace distribution. We have proved the relationship between the matrix variate asymmetric Laplace distribution and the multivariate asymmetric Laplace distribution.

Based on all of the above, we can argue that the matrix variate asymmetric Laplace distribution is a generalization of the multivariate asymmetric Laplace distribution.
In this article, we have defined the matrix variate generalized asymmetric Laplace distribution. We have proved some properties of the matrix variate generalized asymmetric Laplace distribution. We have proved the relationship between the matrix variate generalized asymmetric Laplace distribution and the multivariate generalized asymmetric Laplace distribution. We have proved the relationship between the matrix variate generalized asymmetric Laplace distribution and the matrix variate asymmetric Laplace distribution.

Based on all of the above, we can argue that the matrix variate generalized asymmetric Laplace distribution is a generalization of the multivariate generalized asymmetric Laplace distribution. Also we can argue that the matrix variate generalized asymmetric Laplace distribution is a generalization of the matrix variate asymmetric Laplace distribution.

We have obtained results that do not contradict each other and agree with the theory of matrix distributions.

In this article, we have defined the tensor variate asymmetric Laplace distribution. We have proved some properties of the tensor variate asymmetric Laplace distribution. We have proved the relationship between the tensor variate asymmetric Laplace distribution and the multivariate asymmetric Laplace distribution. Also we can argue that the matrix variate asymmetric Laplace distribution is a special case of the tensor variate asymmetric Laplace distribution if $D = 2$.

Based on all of the above, we can argue that the tensor variate asymmetric Laplace distribution is a generalization of the multivariate asymmetric Laplace distribution.

In this article, we have defined the tensor variate generalized asymmetric Laplace distribution. We have proved some properties of the tensor variate generalized asymmetric Laplace distribution. We have proved the relationship between the tensor variate generalized asymmetric Laplace distribution and the multivariate generalized asymmetric Laplace distribution. Also we can argue that the matrix variate generalized asymmetric Laplace distribution is a special case of the tensor variate generalized asymmetric Laplace distribution if $D = 2$.

Based on all of the above, we can argue that the tensor variate generalized asymmetric Laplace distribution is a generalization of the multivariate generalized asymmetric Laplace distribution. Also we can argue that the tensor variate generalized asymmetric Laplace distribution is a generalization of the tensor variate asymmetric Laplace distribution.

We have obtained results that do not contradict each other and agree with the theory of tensor distributions.

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