Radiative corrections to $\tau \rightarrow \pi(K)\nu_\tau[\gamma]$: a reliable new physics test

M. A. Arroyo-Ureña$^1$, G. Hernández-Tomí$^1$, G. López-Castro$^1$, P. Roig$^1$, and I. Rosell$^2$

1 Departamento de Física, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Apartado Postal 14-740, 07000 Ciudad de México, México and
2 Departamento de Matemáticas, Física y Ciencias Tecnológicas, Universidad Cardenal Herrera-CEU, CEU Universities, 46115 Alfar de la Patriarca, València, Spain

The ratios $R_{\tau/P} \equiv \Gamma(\tau \rightarrow P\nu_\tau[\gamma])/\Gamma(P \rightarrow \mu\nu_\mu[\gamma])$ provide sensitive tests of lepton universality $|g_\tau/g_\mu| = 1$ and are a useful tool for new physics searches. The radiative corrections to $R_{\tau/P}$ are computed following a large-$N_C$ expansion to deal with hadronic effects: Chiral Perturbation Theory is enlarged by including the lightest multiplets of spin-one heavy states such that the relevant Green functions are well-behaved at high energies. We find $\delta R_{\tau/P} = (0.68 \pm 0.57)\%$ and $\delta R_{\tau/K} = (0.97 \pm 0.58)\%$, which imply $|g_\tau/g_\mu| = 0.9964 \pm 0.0038$ and $|g_\tau/g_\mu|_K = 0.9857 \pm 0.0078$, compatible with and at 1.8σ of lepton universality, respectively. We test unitarity and bind non-standard effective interactions with the $\tau \rightarrow P\nu_\tau[\gamma]$ decays.

Introduction. – Lepton universality (LU) is a basic tenet of the Standard Model of particle interactions. A large diversity of weak interaction processes are compatible with the fact that lepton doublets have identical couplings $g_\ell$ to the $W$-boson. A few anomalies observed in semileptonic $B$ meson decays$^1$ seem to challenge this principle or require new non-universal weak interactions. Lower energy observables where very precise comparison are available unambiguously within the Standard Model of particles interactions. A theoretical point of view. Overcoming these problems is pressing from a theoretical point of view. Moreover, and as a by-product, an updated analysis of $\delta R_{\tau/P}$ would be useful to revisit the CKM unitarity test via $|V_{us}|$ in $\Gamma(\tau \rightarrow K\nu_\tau[\gamma]/\Gamma(\tau \rightarrow \pi\nu_\tau[\gamma])$ or $\Gamma(\tau \rightarrow K\nu_\tau[\gamma]/\Gamma(\tau \rightarrow \mu\nu_\mu[\gamma])$ and improve the constraints on possible non-standard interactions affecting this ratio$^{10,11}$.

2. Theory. First of all, the hadronic form factors modeled in Ref.$^4$ are different for real- and virtual-photon corrections. Furthermore, they do not satisfy the correct QCD short-distance behavior, violate unitarity, analyticity and the chiral limit at leading non-trivial orders, and use a cutoff to regulate the loop integrals, separating unphysically long- and short-distance corrections. Finally, the uncertainties quoted in Ref.$^4$ are unrealistic, since they are of the order of a purely Chiral Perturbation Theory result, that is, a computation which cannot include the $\tau$. Thus, a new analysis of $\delta R_{\tau/P}$ overcoming these problems is pressing from a theoretical point of view.

P \rightarrow \mu\nu_\mu[\gamma]. – The inclusive $P_{\tau\nu_\mu[\gamma]}$ decay rate can be analyzed unambiguously within the Standard Model (Chiral Perturbation Theory), being the estimation of the local counterterms the only model dependence. We follow the notation proposed in Ref.$^3$ and the numbers

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1 The strong helicity suppression of the $P_{\tau\nu_\mu[\gamma]}$ decays disfavors verifying $g_\tau = g_\mu$ similarly.
Table I. Numerical values for \( c_0^{(P)} \) of \[12\] [13] \( c_1^{(P)} \) from [15]. The uncertainties correspond to the input values \( L_0^r (\mu/m_\rho) = (6.9 \pm 0.7) \times 10^{-3} \), \( \gamma \equiv F_3^q (0.0)/F_1^q (0.0) = 0.465 \pm 0.005 \), and to the estimation of the counterterms \( (m, \text{from matching}) \), affecting only \( c_1^{(P)} \) and \( c_3^{(P)} \).

reported in Refs. [12] [13]:

\[
\Gamma_{\tau \rightarrow 3 \pi} = \tau \rightarrow P \nu_\tau [\gamma]. - \tau \text{ decays must be scrutinized by using an effective approach encoding the hadronization of the QCD currents and we consider here the same large-} N_C \text{ expansion of QCD used in Refs. [12] [13] to estimate the counterterms of } P_{\tau 2} [\gamma], \text{ quoted previously [16]. Similarly to [3], the decay rate can be organized as}
\]

\[
\Gamma_{\tau \rightarrow 3 \pi} = \Gamma_{\tau \rightarrow \gamma} \left( 1 + \frac{2 \alpha}{\pi} \log \frac{m_Z}{m_\rho} \right) \left( 1 + \frac{\alpha}{\pi} \frac{F(m_\mu^2/m_\rho^2)}{m_\rho^2} \right) \left( 1 - \frac{3 \alpha}{2 \pi} \log \frac{m_\rho}{M_\tau} + \delta_\tau |_{\text{rSD}} + \delta_\tau |_{\text{vSD}} \right),
\]

where again the point-like long-distance correction will be reported later, the structure dependent contributions have been split into the real-photon (rSD) and virtual-photon (vSD) corrections and \( \Gamma_{\tau \rightarrow \gamma} \) is the rate in absence of radiative corrections.

\[
\Gamma_{\tau \rightarrow \gamma} = \frac{G_F^2 |V_{dU}|^2 F_P^2}{8 \pi} m_\tau \left( 1 - \frac{m_\rho^2}{m_\tau^2} \right),
\]

being \( D = d, s \) for \( P = \pi, K \), respectively. The numerical values for \( c_0^{(P)} \) are reported in Table I of [12] [13]. Note that the most important uncertainties come from the estimations of the local counterterms, which were computed with a large-\( N_C \) expansion of QCD where Chiral Perturbation Theory is enlarged by including the lightest multiplets of spin-one heavy states such that the relevant Green functions are well-behaved at high energies [10].

The matrix element of the real-photon correction reads

\[
i \mathcal{M}[\tau(p_\tau) \rightarrow P(p)\nu_\tau(q)\gamma^*(k)] = G_F V_{dU} \epsilon_F P_M \gamma \bar{u}(q)(1 + \gamma_5) \frac{2 p^\mu}{2 p \cdot k + k^2} + \frac{2 p^\mu - k \gamma^\mu}{-2 p_\tau \cdot k + k^2} u(p_\tau) - G_F V_{dU} \epsilon_P \left\{ \frac{i F^\mu P}{2 F^\mu_P} (k^2) \epsilon_{\mu \nu \rho \sigma} k^\rho p^\sigma + F^\mu_P P_M (W^2, k^2) \frac{(W^2 + k^2 - m_\rho^2)}{k_\mu P_{\nu}} \gamma^\mu (1 - \gamma_5) u(p_\tau)\right\},
\]

where \( W^2 = (p_\tau - q)^2 = (p + k)^2 \) and \( \Gamma_p = -\epsilon_\mu^P (k) \) for an on-shell photon. In the first line the structure-independent contribution is shown [4], whereas in the second and third lines we give the structure-dependent contributions in terms of the relevant form factors, which encapsulate the hadronization of the related QCD currents. At leading order in the chiral expansion, the form factors \( A_2^P \) and \( A_4^P \) are not independent and can be written in terms of a single form factor \( B \) (only depending on \( k^2 \) and identical for \( P = \pi, K \) at this order): \( A_2^P (k^2) = -2B(k^2) \) and \( A_4^P (k^2) = -2B(k^2)/\left( W^2 - m_\rho^2 \right) \).

For virtual-photon corrections, we will focus on the
structure-dependent contributions (SD), as we agree with the
structure-independent ones calculated in Refs. [4].

The relevant amplitude, where one photon vertex is
tached to the structure-dependent contributions (SD), as we agree with
the structure-independent ones calculated in Refs. [4].

In our case at hand, the form factors $F_{V,A}(W^2, k^2)$ and
$B(k^2)$ can be taken from Refs. [17, 18].

\[
F_{V}(W^2, k^2) = \frac{-N_C M^4}{24 \pi^2 F_{\rho}(k^2 - M^2_{\rho}) (W^2 - M^2_{\rho})}, \\
F_{A}(W^2, k^2) = \frac{F_{\rho}}{2} \frac{M^2_{\rho} - 2M^2_{\rho} - k^2}{(M^2_{\rho} - k^2)(M^2_{\rho} - W^2)}, \\
B(k^2) = \frac{F_{\rho}}{M^2_{\rho} - k^2}.
\]

(10)

where well-behaved two- and three-point Green functions
are imposed and we consider the chiral and $U(3)$ flavor
limits. In [10] $M_V$ and $M_A$ stand for the vector and
axial-vector resonance masses, $M_V = M_\rho$, $M_A = M_{a_1}$
and $M_V = M_{K*}$, $M_A \sim M_{f_1}$ for the pion and kaon case,
respectively.

Calculation of $\delta R_{\tau/p} = \delta_\tau - \delta_{P\mu}$. — Adding up the
structure-independent terms obtained with both virtual
and real photons (SI, the point-like approximation), we
confirm the results given in Refs. [4]:

\[
\delta R_{\tau/p} |_{SI} = \frac{3}{2} \log \frac{M^2_{\rho}}{m^2_{\mu}} + \frac{3}{2} + g \left( \frac{m^2_{\mu}}{M^2_{\tau}} \right) - f \left( \frac{m^2_{\mu}}{m^2_{\rho}} \right),
\]

(11)

where $f(x)$ and $g(x)$ read:

\[
f(x) = 2 \left( \frac{1 + x}{1 - x} \right) \log(1 - x) - \frac{x(8 - 5x)}{2(1 - x)^2} \log x
+ 4 \frac{1 + x}{1 - x} \log x - \frac{x}{1 - x} \left( \frac{3}{2} + \frac{4}{3} \pi^2 \right),
\]

(12)

and

\[
g(x) = 2 \left( \frac{1 + x}{1 - x} \right) \log(1 - x) - \frac{x(2 - 5x)}{2(1 - x)^2} \log x
+ 4 \frac{1 + x}{1 - x} \log x + \frac{x}{1 - x} \left( \frac{3}{2} - \frac{4}{3} \pi^2 \right),
\]

(13)

being $\text{Li}_2(x) = - \int_0^x \frac{\log(1 - t)}{t} dt$. One gets $\delta R_{\tau/p} |_{SI} = 1.05\%$ and $\delta R_{\tau/K} |_{SI} = 1.67\%$.

The structure-dependent contributions with real photons (rSD) can be extracted from (7.14) and (7.16) of Ref. [17] for $P \rightarrow \mu \nu_{\mu} \gamma$ ($\delta_{P\mu}|_{rSD} = -1.3 \cdot 10^{-8}$ and
$\delta_{K\mu}|_{rSD} = -1.7 \cdot 10^{-5}$) and from Ref. [17] for $\tau \rightarrow P \nu_{\tau} \gamma$
($\delta_{\tau}\gamma|_{rSD} = 0.15\%$ and $\delta_{\tau K}|_{rSD} = (0.18 \pm 0.05)\%$). This
gives $\delta R_{\tau/p} |_{rSD} = 0.15\%$ and $\delta R_{\tau/K} |_{rSD} = (0.18 \pm 0.05)\%$, being the terms from the $P$ decay negligible.

In the case of $P_{\mu 2}$ the structure-dependent contribution
with virtual photons (vSD) can be extracted directly from [3] and the numerical values for $c_{\mu 2}$ of Table I
$\delta_{P\mu}|_{vSD} = (0.54 \pm 0.12)\%$ and $\delta_{K\mu}|_{vSD} = (0.43 \pm 0.12)\%$.

The new calculation we need to perform from scratch for the analysis of $R_{\tau/p}$ is the structure-dependent part

\[ F(x) \text{ and } G(x) \text{ of } [3] \text{ and } [4] \text{ can be related to } f(x) \text{ and } g(x) \text{ by } \]

\[ F(x) = \frac{3}{2} \log x + f(x)/2 + 13/8 - \pi^2/3 \text{ and } G(x) = g(x)/2 + 19/8 - \pi^2/3. \]
with virtual photons for $\tau \to P\nu_\tau$, corresponding to the Feynman diagram of figure 3. Inserting the form factors of (10) into (8)—a tedious calculation whose technical details will be explained deeper in a forthcoming longer article [19]—yields our results
\[
\delta_{\tau P}|_{\text{VSD}} = -(0.48 \pm 0.56)\%, \quad \delta_{\tau K}|_{\text{VSD}} = -(0.45 \pm 0.57)\%. \tag{14}
\]
Accordingly, we find
\[
\delta R_{\tau/\pi}|_{\text{VSD}} = -(1.02 \pm 0.57)\%, \quad \delta R_{\tau/K}|_{\text{VSD}} = -(0.88 \pm 0.58)\%. \tag{15}
\]
A reliable estimation of the uncertainties in (14) is fundamental, since it is the most important source of error in $\delta R_{\tau/P}$. Keep in mind the great difference with the $P_{\alpha\beta}$ decays: (a) in $P$ decays the calculation is performed within Chiral Perturbation Theory (ChPT), so the unknown local counterterms can be determined by matching ChPT with the effective approach at higher energies, the large-$N_C$ extension including the first resonances we have quoted previously; (b) in $\tau$ decays, and due to the energy scale at hand, the calculation is done directly with the large-$N_C$ extension of ChPT, so the matching procedure to estimate the unknown counterterms is not possible anymore. Bearing in mind this handicap, we have estimated the uncertainties of $\delta_{\tau P}|_{\text{VSD}}$ by considering two ingredients. First of all, and in order to assess the model-dependence of the effective approach, we have also calculated $\delta_{\tau P}|_{\text{VSD}}$ with a less general scenario where only well-behaved two-point Green functions and a reduced resonance Lagrangian is used; consequently, the form factors of (10) are different [17, 18] and we take as a first source of error in (14) one half of the deviation in $\delta_{\tau P}|_{\text{VSD}}$ between the two scenarios, resulting in $\pm 0.22\%$ for the pion and $\pm 0.24\%$ for the kaon. Secondly, and in order to estimate the unknown local counterterms in $\delta_{\tau P}|_{\text{VSD}}$, whose dependences on the renormalization scale are known from our calculation, we have considered as the second source of uncertainty in (14) one half of the running of the counterterms between 0.5 and 1.0 GeV, giving $\pm 0.52\%$ [19]. Adding quadratically these two uncertainties yields the errors of (14): $\pm 0.56\%$ and $\pm 0.57\%$ for the pion and the kaon case, respectively.

Results—In Table I the different contributions to $\delta R_{\tau/P}$ are summarized, leading to our final result:
\[
\delta R_{\tau/\pi} = (0.18 \pm 0.57)\%, \quad \delta R_{\tau/K} = (0.97 \pm 0.58)\%. \tag{16}
\]
with dominant uncertainties coming from $\delta_{\tau P}|_{\text{VSD}}$. These results should be compared with the previous ones of Refs. [4], $\delta R_{\tau/\pi} = (0.16 \pm 0.14)\%$ and $\delta R_{\tau/K} = (0.90 \pm 0.22)\%$. Although their central values agree remarkably, this is merely a coincidence, as the one-sigma confidence intervals agree only at the $25(38)\%$ level for the $\pi(K)$ case. In our understanding uncertainties were underestimated in Refs. [4], since they have approximately the size which would be expected in a purely Chiral Perturbation Theory computation. Besides, it is important to stress again that the hadronization of the QCD currents used in that work differs for real- and virtual-photon corrections, does not satisfy the high-energy behavior dictated by QCD, violates unitarity, analyticity and the chiral limit, and a cutoff is used to regulate the loop integrals, splitting unphysically long- and short-distance regimes.

Our results can also be used to compute the radiative corrections of the individual $\tau \to P\nu_\tau[\gamma]$ decays, $\Gamma_{\tau P|\gamma}^{(0)} = \Gamma_{\tau P|\gamma}^{(0)}(1 + \delta_{\tau P})$, where $\Gamma_{\tau P|\gamma}^{(0)}$ was defined in [10], $S_{\text{ew}} = 1.0232$ denotes the resummed universal short-distance electroweak corrections [3] and $\delta_{\tau P}$ includes all remaining radiative SI and SD corrections. Considering [5], $\delta_{\tau P}$ is given by
\[
\delta_{\tau P} = \alpha \frac{2\pi}{2} \left( g \left( \frac{m_\nu^2}{M_\tau^2} \right) + 1 - \frac{2\pi^2}{3} - 3 \log \frac{m_\nu}{M_\tau} \right)
+ \delta_{\tau P}|_{\text{VSD}} + \delta_{\tau P}|_{\text{VSD}}. \tag{17}
\]
From our results one finds $\delta_{\tau P} = (-0.24 \pm 0.56)\%$ and $\delta_{\tau K} = (-0.15 \pm 0.57)\%$.

Considering our results and the current experimental information [7] in [1], it is found
\[
\frac{g_\tau}{g_\mu} \left|_\pi \right. = 0.9964 \pm 0.0028_{\text{th}} \pm 0.0025_{\text{exp}} = 0.9964 \pm 0.0038,
\frac{g_\tau}{g_\mu} \left|_K \right. = 0.9857 \pm 0.0028_{\text{th}} \pm 0.0072_{\text{exp}} = 0.9857 \pm 0.0078, \tag{18}
\]
compatible with and at 1.8σ of lepton universality, respectively.

An interesting application is the unitarity test (see e. g. [20] and references therein) from the ratio
\[
\frac{\Gamma(\tau \to K\nu_\tau[\gamma])}{\Gamma(\tau \to \pi\nu_\tau[\gamma])} = \frac{|V_{us}|^2 F_K^2 (1-m_K^2/M_\tau^2)^2}{|V_{ud}|^2 F_\pi^2 (1-m_\pi^2/M_\tau^2)^2} (1+\delta), \tag{19}
\]
where, as a result of our calculation,
\[ \delta = \frac{\alpha}{2\pi} \left\{ g \left( \frac{m_K^2}{M^2} \right) - g \left( \frac{m_\pi^2}{M^2} \right) \right\} + \delta_{\tau K |_{\text{SD}}} - \delta_{\tau \pi |_{\text{SD}}} + \delta_{\tau K |_{\text{vSD}}} - \delta_{\tau \pi |_{\text{vSD}}} \]
+ (0.10 \pm 0.80)\% . \quad (20)

Using the FLAG 2+1+1 result for the meson decay constants ratio \( F_K / F_\pi = 1.1932 \pm 0.0019 \), \[7\] and masses and branching ratios from the PDG \[7\], one gets
\[ \frac{|V_{us}|}{|V_{ud}|} = 0.2288 \pm 0.0010 \text{th} \pm 0.0017 \text{exp} = 0.2288 \pm 0.0020 , \]
which is 2.1σ away from unitarity according to \(|V_{ud}| = 0.97373 \pm 0.00031 \). \[22\]

Alternatively, we can extract \(|V_{us}|\) directly from the \( \tau \to K \nu_\tau [\gamma] \) decays, \( \Gamma_{\tau K_{2[\gamma]}} = \Gamma_{\tau K_{2[\gamma]}}^{(0)} S_{\text{ew}} (1 + \delta_{\tau K}) \). Using our value of \( \delta_{\tau K} \) given after \[17\], \( \sqrt{2} F_K = (155.7 \pm 0.3) \text{ MeV} \), \[21\], \( S_{\text{ew}} = 1.0232 \) \[3\] and masses and branching ratios from the PDG \[7\], this yields
\[ |V_{us}| = 0.2220 \pm 0.0008 \text{th} \pm 0.0016 \text{exp} = 0.2220 \pm 0.0018 , \]
at 2.6σ from unitarity, according to \(|V_{ud}| = 0.97373 \pm 0.00031 \). \[22\]

We can also bind the effective couplings characterizing non-standard interactions from the \( \tau \to P \nu_\tau [\gamma] \) decays,
\[ \Gamma(\tau \to P \nu_\tau [\gamma]) = \Gamma_{\tau P}^{(0)} \left| \frac{V_{ud}}{V_{ud}} \right|^2 S_{\text{ew}} \left( 1 + \delta_{\tau P} + 2 \Delta^P \right) , \]
being \( D = d, s \) for \( P = \pi, K \), respectively. \( \Delta^P \) captures the new physics corrections and is given by \[11\]:
\[ \Delta^P = \varepsilon_L - \varepsilon_R - \varepsilon_{\pi} - \frac{m^2_{\pi}}{M (m_u + m_d)} \varepsilon_P . \]

Using \(|V_{ud}| = 0.97373 \pm 0.00031 \), \[22\], masses and branching ratios from the PDG \[7\], \( \sqrt{2} F_\pi = (130.2 \pm 0.8) \text{ MeV} \) and \( \sqrt{2} F_K = (155.7 \pm 0.3) \text{ MeV} \) from Ref. \[21\], \( S_{\text{ew}} = 1.0232 \) \[3\] and our values of \(|V_{us}|/|V_{ud}|\) in \[21\] and \( \delta_{\tau P} \) after \[17\], we find
\[ \Delta^{\tau \pi} = -(0.15 \pm 0.72) \cdot 10^{-2} , \quad \Delta^{\tau K} = -(0.36 \pm 1.18) \cdot 10^{-2} , \]
which update the results in Refs. \[11\] for \( u \leftrightarrow d \) and \( u \leftrightarrow s \) transitions, respectively. These values are reported in the \( \overline{\text{MS}} \)-scheme and at a scale of \( \mu = 2 \text{ GeV} \).

In conclusion, our final result for \( \delta_{\tau K/P} \) is consistent with the previous literature \[3\], but with much more robust assumptions, yielding a reliable uncertainty. Extracted ratios of lepton couplings are compatible with lepton universality (pion case) and at 1.8σ (kaon case) and can also be used for testing CKM unitarity and binding effective non-standard interactions, as we have illustrated.

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