Disproof of a conjecture by Erdős and Guy on the crossing number of hypercubes

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Abstract
Let \( Q_n \) be the \( n \)-dimensional hypercube, and let \( \text{cr}(Q_n) \) be the crossing number of \( Q_n \). Erdős and Guy in 1973 conjectured the following equality: \( \text{cr}(Q_n) = \frac{5}{32}4^n - \left\lfloor \frac{n^2 + 1}{2} \right\rfloor 2^{n-2} \). In this paper, we construct a drawing of \( Q_n \) with less crossings when \( n > 6 \), which implies that for \( n > 6 \) we have a strict inequality.

KEYWORDS
crossing number, drawing, hypercube

1 | INTRODUCTION

Let \( G \) be a simple connected graph with vertex set \( V(G) \) and edge set \( E(G) \). The crossing number of a graph \( G \), denoted \( \text{cr}(G) \), is the minimum possible number of edge crossings in a drawing of \( G \) in the plane. The notion of crossing number is a central one for Topological Graph Theory and has been studied extensively by mathematicians, including Erdős, Guy, Harary, Turán, Tutte, et al. (see [3,8,10–12]).

The investigation on the crossing number of graphs is a difficult problem. In 1973, Erdős and Guy [3] wrote, “Almost all questions that one can ask about crossing numbers remain unsolved.” Garey and Johnson [6] proved that computing the crossing number is NP-complete. In this field, Eggleton and Guy [2] in 1970 announced a drawing of the \( n \)-dim hypercube \( Q_n \) with exactly \( \frac{5}{32}4^n - \left\lfloor \frac{n^2 + 1}{2} \right\rfloor 2^{n-2} \) crossings, which implies

\[
\text{cr}(Q_n) \leq \frac{5}{32}4^n - \left\lfloor \frac{n^2 + 1}{2} \right\rfloor 2^{n-2},
\]
where \(Q_n\) denotes the \(n\)-dimensional hypercube.

Not long afterward, in 1971 a gap was found [7] in the drawing given by Eggleton and Guy. We still quote: “but a gap has been found in the description of the construction, so this must also remain a conjecture. We again conjecture equality in (1).” (Erdős and Guy [3]).

For a long time, the equality conjectured by Erdős and Guy was believed to be true, even though a proof appeared out of reach. The past results on \(cr(Q_n)\), including \(cr(Q_3) = 0\) (trivial), \(cr(Q_4) = 8\) (see [1]), and the obtained best drawings for \(Q_n\) with \(n = 5, 6, 7, 8\) (see [4,9]) seem to support the equality.

Very recently in 2008, Faria, Figueiredo, Sýkora, and Vrtko [5] announced a drawing for which the number of crossings coincides with \(\frac{5}{32}4^n - \frac{n^2+1}{2}2^{n-2}\), giving further support to the conjecture.

In this paper, we construct drawings of the hypercubes \(Q_n\) with, for \(n > 6\), fewer crossings, disproving the conjecture. We prove the following upper bound for the crossing number of hypercubes.

**Theorem 1.1.** \(\text{cr}(Q_n) \leq \frac{139}{896}4^n - \left[\frac{n^2+1}{2}\right]2^{n-2} + 4 \cdot 2^{\lambda_n} + n \left(\frac{4^n}{24 \cdot 576} - \frac{4^{\frac{n}{2}}}{6}\right)\), where

\[
\lambda_n = \begin{cases} 
0 & \text{if } 5 \leq n \leq 12, \\
1 & \text{if } n \geq 13.
\end{cases}
\]

Our proposed family of drawings for \(Q_n\), when \(n = 5\) and 6, has the exact number of crossings conjectured by Erdős and Guy, and when \(n > 6\) has less crossing number than that conjectured. In Section 4 of this paper, we will compare the asymptotic behavior of our proposed new function for the upper bound of \(\text{cr}(Q_n)\) to the previous function.

## 2 | NOTATION

A drawing of \(G\) is said to be a good drawing, provided that no edge crosses itself, no adjacent edges cross each other, no two edges cross more than once, and no three edges cross in a point. It is well known that the crossing number of a graph is attained only in good drawings of the graph. So, we always assume that all drawings throughout this paper are good drawings. Let \(D\) be a good drawing of the graph \(G\), and let \(A\) and \(B\) be two disjoint subsets of \(E(G)\). In the drawing \(D\), the number of the crossings formed by the edges of \(A\) with the edges of \(B\) is denoted by \(\nu_D(A, B)\). The number of the crossings between the edges of \(A\) is denoted by \(\nu_D(A)\). In what follows, \(\nu_D(E(G))\) is abbreviated to \(\nu(D)\) when it is unambiguous. Let \(u\) be a vertex of \(G\), and let \(U\) be a vertex subset of \(V(G)\). We define \(\mathcal{I}(u)\) to be the edge subset of \(E(G)\) consisting of all edges incident with \(u\). Let

\[
\mathcal{I}(U) = \bigcup_{u \in U} \mathcal{I}(u)
\]

and let

\[
\partial(U) = \mathcal{I}(U) \setminus E(U),
\]

where \(E(U) = \{uv \in E(G) : u, v \in U\}\).
The \( n \)-dimensional hypercube \( Q_n \) is a graph with the vertex set \( V(Q_n) = \{d_1d_2 \ldots d_n : d_i \in \{0, 1\}, i = 1, 2, \ldots, n\} \), for which any two vertices \( a = a_1a_2 \ldots a_n \) and \( b = b_1b_2 \ldots b_n \) are adjacent if and only if there exists a unique \( i \in \{1, 2, \ldots, n\} \) such that \( a_i \neq b_i \). In particular, if the unique \( i \in \{1, 2, \ldots, n\} \) with \( a_i \neq b_i \) is equal to \( n - 1 \), that is, \( a = a_1a_2 \ldots a_{n-2}a_{n-1}a_n \) and \( b = a_1a_2 \ldots a_{n-2}a_{n-1} \overline{a}_n \), we denote \( b = \overline{a} \), and conversely, \( a = \overline{b} \).

For any vertex \( a = a_1a_2 \ldots a_n \in V(Q_n) \) and any binary string \( x_1x_2 \ldots x_t \) of length \( t \), we define \( a^{(x_1x_2 \ldots x_t)} = a_1a_2 \ldots a_nx_1x_2 \ldots x_t \) to be the vertex of \( V(Q_{n+t}) \).

\[ \text{PROOF OF THEOREM 1.1} \]

In this section, we prove Theorem 1.1 by constructing a drawing, denoted \( \Gamma_n \), of \( Q_n \) with the desired crossings for every integer \( n \geq 5 \). The constructions of the drawing \( \Gamma_n \) are different according to the parity of \( n \). Hence, we shall introduce the constructions of \( \Gamma_n \) in Sections 3.1 and 3.2. In Section 3.3, we verify the constructed drawing \( \Gamma_n \) has the desired number of crossings.

3.1 Construction of the drawing \( \Gamma_n \) for all odd \( n \)

• Throughout this subsection, we use \( n \) as an odd integer no less than 5.

The desired drawing \( \Gamma_n \) of \( Q_n \) will be constructed recursively (see Figure 1 for the drawing \( \Gamma_5 \)). Note that the drawing \( \Gamma_5 \) for \( Q_5 \) we give here is different to the previous drawing for \( Q_5 \) in [4,9]. The drawing in Figure 1 has the same number of crossings as the drawing obtained in [4,9], however, it can help us to inductively construct our general drawings for \( Q_n \) with fewer crossings than the crossing number conjectured by Erdős and Guy.

The idea of constructing the drawing \( \Gamma_{n+2} \) from the obtained drawing \( \Gamma_n \) is as follows. By preprocessing the drawing \( \Gamma_n \), we get a new drawing of \( Q_n \), denoted \( \Gamma_n^* \), with some properties which are helpful for subsequent procedures. Replacing each vertex \( u \) in \( \Gamma_n^* \) by four new vertices \( u^{(00)}, u^{(10)}, u^{(11)}, u^{(01)} \in V(Q_{n+2}) \) in the “very small neighborhood” of the original location of \( u \), and replacing each edge \( uv \) in \( \Gamma_n^* \) by a “bunch” (in this paper, a bunch means edges drawn in parallel close to each other) of four new edges \( u^{(00)}v^{(00)}, u^{(10)}v^{(10)}, u^{(11)}v^{(11)}, u^{(01)}v^{(01)} \in E(Q_{n+2}) \) drawn along the original route of \( uv \), we get a transitional drawing of \( Q_{n+2} \), denoted \( \bar{\Gamma}_{n+2} \). Next, by modifying the “routes” of some edges in \( \bar{\Gamma}_{n+2} \), we decrease the crossings and obtain the desired drawing \( \Gamma_{n+2} \) of \( Q_{n+2} \). In what follows, we shall give a detailed description of the construction.
To accurately describe the process, all the drawings in this paper will be given in the two-dimensional Euclidean plane $\mathbb{R} \times \mathbb{R}$. For any vertex $u$ in some drawing, by $X_u$ and $Y_u$ we denote the $X$- and $Y$-coordinates of $u$ in $\mathbb{R} \times \mathbb{R}$. Three different drawings $\Gamma_n$, $\Gamma^*_n$, and $\bar{\Gamma}_n$ ($\Gamma_n$ exists only for $n \geq 7$ of $Q_n$ will appear later, for convenience, we use the notion $D_n$ to denote any one of the above three possible drawings $\Gamma_n$, $\Gamma^*_n$, and $\bar{\Gamma}_n$.

In this paper, any drawing $D_n$ shares the following inductive rule for the arrangements of vertices:

For $n = 5$, the locations of vertices in $D_5$ (including $\Gamma_5$ and $\Gamma^*_5$) are depicted in Figure 1, in particular,

$\{(X_u, Y_u) : u \text{ is a vertex in } D_5\} = \{-2, -1, 1, 2\} \times \{-4, -3, -2, -1, 1, 2, 3, 4\}$.

- For convenience, in the rest of this paper, any vertex $a = a_1a_2 \ldots a_n \in V(Q_n)$ drawn in figures will be represented by the corresponding decimal number $2^{n-1}a_1 + 2^{n-2}a_2 + \cdots + 2^0a_n$.
- Throughout this paper, we always let $\mathcal{N}$ denote some fixed large positive integer.

Suppose $n > 5$. Take an arbitrary vertex $u$ in $D_{n-2}$. The four vertices extended from $u$, say $u^{(00)}$, $u^{(10)}$, $u^{(11)}$, and $u^{(01)}$, in $D_n$ will be located in $\mathbb{R} \times \mathbb{R}$ such that

$$X_{u^{(00)}} = X_{u^{(10)}} = X_{u^{(11)}} = X_{u^{(01)}} = X_u$$

and

$$\begin{align*}
Y_{u^{(00)}} &= Y_u, \\
Y_{u^{(10)}} &= Y_u + \frac{Y_u - Y_u}{\lambda} \\
Y_{u^{(11)}} &= Y_u + 2 \cdot \frac{Y_u - Y_u}{\lambda} \\
Y_{u^{(01)}} &= Y_u + 3 \cdot \frac{Y_u - Y_u}{\lambda}.
\end{align*}$$

Figure 1 The drawing $\Gamma_5$ with 56 crossings
In other words, \( u^{(00)}, u^{(10)}, u^{(11)}, u^{(01)} \) in \( D_n \) are drawn at the line where \( u \) lies, and are located at the “very small neighborhood” of the original location of \( u \) in \( D_{n-2} \), noticing that \( N \) is some fixed large positive integer. To make it clear, we illustrate the above inductive rule in Figure 2.

In some cases, we shall denote the four vertices \( u^{(00)}, u^{(10)}, u^{(11)}, u^{(01)} \) in Figure 2 to be \( (00), (10), (11), (01) \) respectively.

By the above inductive rule for the arrangements of vertices, we have the following.

Claim A. Let \( u_1 \) and \( u_2 \) be two adjacent vertices in \( D_n \). Then,

(i) either \( X_{u_1} = X_{u_2} \) or \( Y_{u_1} = Y_{u_2} \) holds;
(ii) if \( u_1 = \widehat{u}_2 \), then \( X_{u_1} = X_{u_2} \) and both vertices \( u_1, u_2 \) are drawn next to each other at the line \( x = X_{u_1} \);
(iii) \((Y_{u_1} - Y_{u_0}) \cdot (Y_{u_2} - Y_{u_0})\) is negative or positive according to \( X_{u_1} = X_{u_2} \) or \( Y_{u_1} = Y_{u_2} \).

Proof of Claim A. We first prove that Conclusion (ii) holds. Say \( u_1 = \widehat{u}_2 \). For \( n = 5 \), Conclusion (ii) holds by Figure 1. Hence, we only need to consider the case that \( n > 5 \), that is, \( n \geq 7 \) by recalling that \( n \) is odd in this subsection. Then there exists some vertex \( v \) in \( D_{n-2} \) such that

\[ \text{either } [u_1, u_2] = \{v^{(00)}, v^{(10)}\} \text{ or } [u_1, u_2] = \{v^{(11)}, v^{(01)}\}. \]

From Figure 2, we see immediately that \( u_1 \) and \( u_2 \) are drawn next to each other at the same line \( x = X_{u_1} = X_{u_2} \) in \( D_n \), which proves Conclusion (ii).

Next we shall prove Conclusions (i) and (iii) by induction on \( n \). Similarly as above, both conclusions for the case of \( n = 5 \) can be verified in Figure 1. We assume \( n \geq 7 \). Let

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**FIGURE 2** Inductive rule for the arrangement of vertices

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\[ u_1 = v_1^{(a_1a_2)} \]

and

\[ u_2 = v_2^{(b_1b_2)} \]

where \( v_1, v_2 \in V(Q_{n-2}) \) and \( a_1a_2, b_1b_2 \in \{00, 10, 11, 01\} \). Since \( u_1, u_2 \) are adjacent, we have that

\[ v_1 = v_2 \quad \text{or} \quad a_1a_2 = b_1b_2. \]

Then we distinguish two cases.

**Case 1.** \( v_1 = v_2 \).

By Conclusion (ii), we may assume without loss of generality that \( u_1 \neq \overline{u}_2 \), and thus,

either \( \{u_1, u_2\} = \{v_1^{(10)}, v_1^{(11)}\} \) or \( \{u_1, u_2\} = \{v_1^{(01)}, v_1^{(00)}\} \).

From Figure 2, we see that \( X_{u_1} = X_{u_2} = X_{v_1} \) and \( (Y_{\overline{u}_1} - Y_{u_1}) \cdot (Y_{\overline{u}_2} - Y_{u_2}) < 0 \), and that Conclusions (i) and (iii) follow.

**Case 2.** \( a_1a_2 = b_1b_2 \).

Since \( u_1 \) and \( u_2 \) are adjacent, it follows that \( v_1 \) and \( v_2 \) are adjacent vertices of \( V(Q_{n-2}) \). By applying the induction hypothesis for Conclusions (i) and (iii) on \( n - 2 \), we have that in \( D_{n-2} \), either

\[ X_{v_1} = X_{v_2} \quad \text{and} \quad (Y_{\overline{v}_1} - Y_{v_1}) \cdot (Y_{\overline{v}_2} - Y_{v_2}) < 0 \quad (4) \]

or

\[ Y_{v_1} = Y_{v_2} \quad \text{and} \quad (Y_{\overline{v}_1} - Y_{v_1}) \cdot (Y_{\overline{v}_2} - Y_{v_2}) > 0. \quad (5) \]

By Conclusion (ii), \( X_{\overline{v}_1} = X_{\overline{v}_2} \) and that \( \overline{v}_1 \) and \( \overline{v}_2 \) are drawn next to each other in the line \( x = X_{v_i} \) where \( i = 1, 2 \).

Suppose (4) holds. We conclude that \( X_{u_1} = X_{u_2} \) and \( (Y_{\overline{u}_1} - Y_{u_1}) \cdot (Y_{\overline{u}_2} - Y_{u_2}) < 0 \) (see Figure 3).

Suppose (5) holds. Since \( v_1 \) and \( v_2 \) are adjacent vertices of \( V(Q_{n-2}) \), we have that \( \overline{v}_1 \) and \( \overline{v}_2 \) are adjacent too. Since \( X_{v_1} \neq X_{v_2} \), it follows from Conclusion (ii) that \( X_{\overline{v}_1} \neq X_{\overline{v}_2} \). By applying the induction hypothesis for Conclusion (i) on \( n - 2 \), we have that \( Y_{\overline{v}_1} = Y_{\overline{v}_2} \).

Then we conclude that \( Y_{u_1} = Y_{u_2} \) and \( (Y_{\overline{u}_1} - Y_{u_1}) \cdot (Y_{\overline{u}_2} - Y_{u_2}) > 0 \) (see Figure 4). This proves Claim A.

Then, we shall characterize the drawing of the edges. As explained in the beginning of this subsection, for any odd number \( n \geq 5 \), the drawing of edges in \( \Gamma_{n+2} \) will be constructed from \( \Gamma_n \) inductively. We shall obtain the drawing of edges in \( \Gamma_{n+2} \) from \( \Gamma_n \) in three steps. We give a sketch of the three steps here, and give the accurate description later.
In the first step, we obtain another drawing of $Q_n$, denoted $\Gamma_n^*$, for which in the “very small neighborhood” of any vertex $u$ of $V(Q_n)$, the number of edges incident with $u$ which are drawn on the left of the line $x = X_u$ is almost the same as the number of edges incident with $u$ which are drawn on the right of the line $x = X_u$.

In the second step, each vertex $u$ in $\Gamma_n^*$ will be replaced by the $4$-cycle $uuuu_{(00)(00)(10)(11)(01)}$ drawn in the “very small neighborhood” of the original location of $u$ and “precisely” at the line $x = X_u$, and every edge $uv$ in $\Gamma_n^*$ will be replaced by a “bunch” of four new edges $uuuu_{(00)(00)(10)(10)(11)(11)(01)(01)}$ drawn along the original route of $uv$. This gives the transitional drawing $\Gamma_n^{*+2}$ of $Q_{n+2}$.

In the final step, we decrease the crossings in the drawing $\Gamma_n^{*+2}$ by adjusting the route of some edges “locally” or “globally” and get the final desired drawing $\Gamma_n^{*+2}$. Say $e$ is an edge of $\{uuuu_{(00)(00)(10)(11)(01)}, uuuu_{(01)(10)(11)(01)}\}$ to be adjusted in $\Gamma_n^{*+2}$. By saying to adjust the edge $e$ “locally,” we mean altering the drawing of the two parts of the edge $e$ which are located in the “very small neighborhoods” of both two 4-cycles $uuuu_{(00)(00)(10)(11)(01)}$ and $uuuu_{(01)(10)(11)(01)}$ outside both neighborhoods. By saying to adjust the edge $e$ “globally,” we mean altering the whole route of the edge $e$ and not necessarily drawing the edge $e$ still in its original bunch.

To proceed, we shall need some technical definitions.

**Definition 3.1.** Let $u$ be a vertex and $e$ an edge incident with $u$ in $D_n$. We call the edge $e$

- a left arc with respect to $u$, if the “starting part” of $e$ within the “very small neighborhood” of $u$ is drawn on the left of the line $x = X_u$;
- a right arc with respect to $u$, if the “starting part” of $e$ within the “very small neighborhood” of $u$ is drawn on the right of the line $x = X_u$;
- a below arc with respect to $u$, if the “starting part” of $e$ within the “very small neighborhood” of $u$ is drawn below the line $y = Y_u$;
- an above arc with respect to $u$, if the “starting part” of $e$ within the “very small neighborhood” of $u$ is drawn above the line $y = Y_u$. 

**FIGURE 3** Auxiliary diagrams illustrating Claim A

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In the first step, we obtain another drawing of $Q_n$, denoted $\Gamma_n^*$, for which in the “very small neighborhood” of any vertex $u$ of $V(Q_n)$, the number of edges incident with $u$ which are drawn on the left of the line $x = X_u$ is almost the same as the number of edges incident with $u$ which are drawn on the right of the line $x = X_u$.

In the second step, each vertex $u$ in $\Gamma_n^*$ will be replaced by the 4-cycle $uuuu_{(00)(00)(10)(11)(01)}$ drawn in the “very small neighborhood” of the original location of $u$ and “precisely” at the line $x = X_u$, and every edge $uv$ in $\Gamma_n^*$ will be replaced by a “bunch” of four new edges $uuuu_{(00)(00)(10)(10)(11)(11)(01)(01)}$ drawn along the original route of $uv$. This gives the transitional drawing $\Gamma_n^{*+2}$ of $Q_{n+2}$.

In the final step, we decrease the crossings in the drawing $\Gamma_n^{*+2}$ by adjusting the route of some edges “locally” or “globally” and get the final desired drawing $\Gamma_n^{*+2}$. Say $e$ is an edge of $\{uuuu_{(00)(00)(10)(11)(01)}, uuuu_{(01)(10)(11)(01)}\}$ to be adjusted in $\Gamma_n^{*+2}$. By saying to adjust the edge $e$ “locally,” we mean altering the drawing of the two parts of the edge $e$ which are located in the “very small neighborhoods” of both two 4-cycles $uuuu_{(00)(00)(10)(11)(01)}$ and $uuuu_{(01)(10)(11)(01)}$ outside both neighborhoods. By saying to adjust the edge $e$ “globally,” we mean altering the whole route of the edge $e$ and not necessarily drawing the edge $e$ still in its original bunch.

To proceed, we shall need some technical definitions.

**Definition 3.1.** Let $u$ be a vertex and $e$ an edge incident with $u$ in $D_n$. We call the edge $e$
Let \( L_{D_n}(u), R_{D_n}(u), B_{D_n}(u), A_{D_n}(u) \), be the set of edges incident with \( u \) which are left, right, below, above arcs with respect to \( u \) in the drawing \( D_n \), respectively.

Now let us give an example to illustrate the above points. It can be seen in Figure 5 that, \( e_1 \in L_{D_n}(u) \cap B_{D_n}(u) \), that is, the edge \( e_1 \) is both a left arc and a below arc with respect to \( u \) in the drawing \( D_n \); \( e_2 \in L_{D_n}(u) \cap A_{D_n}(u) \); \( e_3 \in R_{D_n}(u) \cap B_{D_n}(u) \); while the edge \( u\hat{u} \in B_{D_n}(u) \) and \( u\hat{u} \notin L_{D_n}(u) \cup R_{D_n}(u) \), that is, the edge \( u\hat{u} \) is just a below arc with respect to \( u \), and is neither a left arc nor a right arc with respect to \( u \) since the edge is drawn precisely along the line \( x = X_u \).

It is worthwhile to note that the above nature of any edge \( e \) with respect to any of its ends \( u \) is determined just by the route of the part of \( e \) which is located in the “very small neighborhood”

**Figure 4** Auxiliary diagrams illustrating Claim A
of u. That is also the reason why the edge $e_2$ is a left arc rather than a right arc with respect to u. It is easy to check that in Figure 5, $|\mathcal{L}_{D_n}(u)| = 2$, $|\mathcal{R}_{D_n}(u)| = 1$, $|B_{D_n}(u)| = 3$, $|A_{D_n}(u)| = 1$.

**Definition 3.2.** Let $e = u_1 u_2$ be an edge in $D_n$. We say that the edge $e$ is self-symmetric provided that the following conditions hold:

If $X_{u_1} = X_{u_2}$, then the edge $e$ is drawn symmetrically with respect to the line $y = \frac{Y_{u_1} + Y_{u_2}}{2}$, that is, the part drawn above is a reflection of the part drawn below with the “mirror” $y = \frac{Y_{u_1} + Y_{u_2}}{2}$, in particular, $e$ is a left (right) arc with respect to $u_1$ if and only if $e$ is a left (right) arc with respect to $u_2$, and $e$ is an above (below) arc with respect to $u_1$ if and only if $e$ is a below (above) arc with respect to $u_2$ (see Figure 6).

If $Y_{u_1} = Y_{u_2}$, then the edge $e$ is drawn symmetrically with respect to the line $x = \frac{X_{u_1} + X_{u_2}}{2}$, that is, the part drawn on the left is a reflection of the part drawn on the right with the “mirror” $x = \frac{X_{u_1} + X_{u_2}}{2}$, in particular, $e$ is a left (right) arc with respect to $u_1$ if and only if $e$ is a right (left) arc with respect to $u_2$, and $e$ is an above (below) arc with respect to $u_1$ if and only if $e$ is an above (below) arc with respect to $u_2$ (see Figure 7).

It can be seen that in the drawing $\Gamma_n$ given by Figure 1, every edge is self-symmetric. Furthermore, every edge is self-symmetric in any drawing $D_n$ of $Q_n$ for all $n \geq 5$ given in this paper. One basic principle in constructing $\Gamma_{n+2}$ from $\Gamma_n$ is to keep the adjusted edges self-symmetric after any adjustment in each one of the three steps, from $\Gamma_n$ to $\Gamma_n^*$, or from $\Gamma_n^*$ to

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**Figure 5** The natures of any edge with respect to its ends in $D_n$

**Figure 6** The edge $u_1 u_2$ is self-symmetric with $X_{u_1} = X_{u_2}$
In fact, any adjustment on some edge will be done with respect to its both ends synchronously to keep the adjusted edge self-symmetric. The main reason to keep edges self-symmetric is to make the four new edges $uv_{00}$, $uv_{01}$, $uv_{10}$, and $uv_{11}$ do not cross each other for any edge $uv$ in $\Gamma$ during the process of constructing $\Gamma_{n+2}$ from $\Gamma$, which shall be illustrated more specifically later.

Now we give a structure which is crucial in this paper to construct the drawing $\Gamma$ with the desired number of crossings.

**Definition 3.3.** Let $U = \{u_1, u_2, ... , u_8\}$ be a set of eight vertices in $D_n$. We say that the drawing of the edge set $I(U)$ forms a fundamental structure, provided that the drawing of $I(U)$ is given as either Diagram (1) or Diagram (2) in Figure 8, in particular, with

$$X_{u_1} = X_{u_5}, X_{u_2} = X_{u_6}, X_{u_3} = X_{u_7}, X_{u_4} = X_{u_8},$$

$$Y_{u_1} = Y_{u_2} = Y_{u_3},$$

$$Y_{u_4} = Y_{u_5} = Y_{u_7} = Y_{u_8},$$

and

$$|\partial(U) \cap A_{D_n}(u_i)| = |\partial(U) \cap B_{D_n}(u_i)| = \frac{n - 3}{2} \quad \text{(6)}$$

**FIGURE 7** The edge $u_1u_2$ is self-symmetric with $Y_{u_1} = Y_{u_2}$

**FIGURE 8** Fundamental structures

$\Gamma_{n+2}$, or from $\tilde{\Gamma}_{n+2}$ to $\Gamma_{n+2}$. In fact, any adjustment on some edge will be done with respect to its both ends synchronously to keep the adjusted edge self-symmetric. The main reason to keep edges self-symmetric is to make the four new edges $u^{(00)}v^{(00)}$, $u^{(10)}v^{(10)}$, $u^{(11)}v^{(11)}$, $u^{(01)}v^{(01)}$ do not cross each other for any edge $uv$ in $\Gamma$ during the process of constructing $\Gamma_{n+2}$ from $\Gamma$, which shall be illustrated more specifically later.

Now we give a structure which is crucial in this paper to construct the drawing $\Gamma$ with the desired number of crossings.
for $i \in [1, 8]$, that is, the number of edges of $\partial(U)$ which are above arcs with respect to $u_i$ and the number of edges of $\partial(U)$ which are below arcs with respect to $u_i$ are both $\frac{n - 3}{2}$.

Moreover, no vertex lies in the interior of the cycle $C$, where $C$ denotes any of the 4-cycles $u_1 u_2 u_3 u_4$, $u_5 u_4 u_3 u_2$, $u_3 u_5 u_6 u_7$, and $u_7 u_8 u_1 u_2$.

Furthermore, if the drawing of the edge set $\mathcal{I}(U)$ in $\mathcal{D}_n$ forms a fundamental structure, the edges of $E(U)$ are called fundamental edges.

- It can be verified in Figure 1 that there exist two fundamental structures in the drawing $\Gamma_5$, say $\mathcal{I}(U_1)$ and $\mathcal{I}(U_2)$, where $U_1$ and $U_2$ denote the set of eight vertices numbered as $2, 10, 26, 18, 22, 30, 14, 6$ and numbered as $3, 11, 27, 19, 23, 31, 15, 7$, respectively, now we are ready to characterize each step in detail.

**Step 1. Obtaining $\Gamma_n^*$ from $\Gamma_n$.**

In this step, we construct the drawing $\Gamma_n^*$ satisfying the following three properties:

**Property 1.** For any vertex $u$ in $\Gamma_n^*$, the number of left arcs and the number of right arcs with respect to $u$ are almost equal, precisely speaking, either

$$\left| L_{\Gamma_n^*}(u) \right|, \left| R_{\Gamma_n^*}(u) \right| = \left( \frac{n + 1}{2}, \frac{n - 1}{2} \right)$$

or

$$\left| L_{\Gamma_n^*}(u) \right|, \left| R_{\Gamma_n^*}(u) \right| = \left( \frac{n - 1}{2}, \frac{n + 1}{2} \right).$$

**Property 2.** In the drawing $\Gamma_n^*$, there exist $2^\frac{n+3}{2}$ fundamental structures.

**Property 3.** For any $n \in \{5, 7, 9\}$, there exists a decomposition of $V(Q_n)$ into several disjoint cycles which satisfy some conditions (called enclosed cycles which will be introduced later) say $C_1, C_2, ..., C_m$, in $\Gamma_n^*$ such that no fundamental edge is contained in any enclosed cycle, where $m \geq 1$.

The first thing worth noting is that the adjustments in this step change only the drawing of the adjusted edge within the “very small neighborhoods” of both its ends, in particular, the adjustments change only the nature of the adjusted edge with both its ends but keep the position relation between the adjusted edge and other edges unchanged from the topological point of view. To make it clear, we take $n = 9$ for example in Figure 9 to show the adjustments in this step. Suppose first in the drawing $\Gamma_9$, $\left| L_{\mathcal{D}_n}(u) \right| = 2$ and $\left| R_{\mathcal{D}_n}(u) \right| = 7$, that is, the number of left arcs and the number of right arcs with respect $u$ are not almost equal. By redrawing the edges $e_3$ and $e_4$ within the very small neighborhood of $u$ to make them to be the left arcs with respect to $u$, we make the number of left arcs and the number of right arcs with respect to $u$ to be almost equal. As seen in Figure 9, the adjustments change only the natures of $e_3$ and $e_4$ with respect to $u$ (actually, the natures with respect to another end of $e_3, e_4$ also need to be changed to keep $e_3, e_4$ self-symmetric, here we emphasize the adjustment only with respect to $u$) but not change the topological position of $e_3, e_4$ with other edges. In fact, the selection of the edges to be adjusted is decided by elaborate attempts instead of a general rule in the cases when $n \in \{5, 7, 9\}$.
For Property 2, as observed earlier, in the drawing $\Gamma_5$, there exist $2 = 2^{\frac{4}{2}}$ fundamental structures. To make $\Gamma^*_n$ satisfy Property 2, during the whole process of obtaining $\Gamma_{n+2}$ from $\Gamma_n$, we shall create two new fundamental structures out of any fundamental structures in $\Gamma_n$, and moreover, during the process of obtaining $\Gamma^*_n$ from $\Gamma_n$, we shall preserve the fundamental structures, that is, we should avoid adjusting any fundamental edge and changing the number of below arcs and the number of above arcs with respect to any vertex in some fundamental structure in $\Gamma^*_n$ (see Equation 6).

To illustrate Property 3, we need to give the definition of enclosed cycles.

**Definition 3.4.** Let $C$ be a cycle in $\Gamma^*_n$. We call $C$ an **enclosed cycle** provided that the following two conditions hold.

**Condition 1.** Let $e_1, e_2$ be two adjacent edges in the cycle $C$, say their common endpoint is $u$. Then one of the following types given in Figure 10 holds for the drawing within the very small neighborhood of $u$, where the bold lines represent the edges $e_1, e_2$.

**Condition 2.** Let $e = u_1u_2$ be an edge in the cycle $C$.

Suppose $Y_{u_1} = Y_{u_2}$, say $X_{u_1} < X_{u_2}$. Then the relation of $e$ with its both ends $u_1$ and $u_2$ is displayed as some diagram in Figure 11. In particular,

$$|L_{\Gamma_n}(u_1)| = |R_{\Gamma_n}(u_2)|$$

and

$$e \in B_{\Gamma_n}(u_1)(e \in A_{\Gamma_n}(u_1)) \text{ if and only if } e \in B_{\Gamma_n}(u_2)(e \in A_{\Gamma_n}(u_2))$$

and

$$e \in L_{\Gamma_n}(u_1)(e \in R_{\Gamma_n}(u_1)) \text{ if and only if } e \in R_{\Gamma_n}(u_2)(e \in L_{\Gamma_n}(u_2)).$$

Suppose $X_{u_1} = X_{u_2}$, say $Y_{u_1} > Y_{u_2}$. Then the relation of $e$ with its both ends $u_1$ and $u_2$ is displayed as some diagram in Figure 12. In particular,

$$|L_{\Gamma_n}(u_1)| = |L_{\Gamma_n}(u_2)|$$
Finding a decomposition of $V(Q_n)$ into several disjoint enclosed cycles is preparing, in fact, for decreasing the crossings within the neighborhoods of the small 4-cycle $u^{(01)}u^{(10)}u^{(11)}u^{(00)}$ during the process from $\Gamma_{n+2}^{*}$ to $\Gamma_{n+2}$, where $u$ denotes an arbitrary vertex in $\Gamma_n^*$ for $n \in \{5, 7, 9\}$, on which the specific operation will be displayed in Step 3. We cannot find a general rule on how to construct $\Gamma_n^*$ from $\Gamma_n$ when $n \in \{5, 7, 9\}$. Instead, the process of constructing $\Gamma_5^*, \Gamma_7^*, \Gamma_9^*$ from $\Gamma_5, \Gamma_7, \Gamma_9$ is designed by elaborate attempts. Therefore, we shall display the process for $n \in \{5, 7, 9\}$ by figures. As for the case of $n \geq 11$, to avoid wasteful duplication in Steps 2 and 3, we plan to leave the general rule on how to construct $\Gamma_n^*$ from $\Gamma_n$ to the later part of this subsection when we complete the description in Step 3.

\[ e \in B_{\Gamma_n^*}(u_1)(e \in A_{\Gamma_n^*}(u_1)) \text{ if and only if } e \in A_{\Gamma_n^*}(u_2)(e \in B_{\Gamma_n^*}(u_2)) \]

and

\[ e \in L_{\Gamma_n^*}(u_1)(e \in R_{\Gamma_n^*}(u_1)) \text{ if and only if } e \in L_{\Gamma_n^*}(u_2)(e \in R_{\Gamma_n^*}(u_2)). \]
From Figure 1 for the drawing $\Gamma_5$, Figure 13 for the drawing $\Gamma_5^*$, Figure 40 (note that Figures 40–43 are placed at the end of the paper) for the drawing $\Gamma_7$, Figure 41 for the drawing $\Gamma_7^*$, Figure 42 for the drawing $\Gamma_9$, Figure 43 for the drawing $\Gamma_9^*$, one can see the adjustments of constructing $\Gamma_n^*$ from $\Gamma_n$ for $n \in \{5, 7, 9\}$. In Figure 13, Figures 41 and 43 for the drawings $\Gamma_5^*, \Gamma_7^*, \Gamma_9^*$, the edges in any enclosed cycle are marked in bold.

In Figure 42 for the drawing $\Gamma_9$, in Figure 43 for the drawing $\Gamma_9^*$, we depicted only a part of the whole drawing. It is mainly because the page size is limited. To make the adjustments to be seen, we display only the drawings of the edges incident with the vertices $u$ with $X_u \in \{-2, -1\}$ and $3 \leq Y_u \leq 4$ in both figures. The whole drawing of $\Gamma_9^*$ ($\Gamma_9$) can be divided into eight identical parts (see Figure 14 for a schematic illustration), denoted $L_1, L_2, L_3, L_4, R_1, R_2, R_3, R_4$, among which, $L_1$ and $L_2$ are reflections of each other (including which edges are marked in bold) with the “mirror” $y = \frac{5}{2}$; $L_3, L_4$ as a whole part is a reflection of $L_1, L_2$ as a whole part with the “mirror” $y = 0$; $R_1, R_2, R_3, R_4$ as a whole part is a reflection of $L_1, L_2, L_3, L_4$ as a whole part with the “mirror” $x = 0$. If fact, each one of the drawings $\Gamma_5, \Gamma_5^*, \Gamma_7, \Gamma_7^*$ we give in Figures 1, 13, 40, and 41 can be divided into eight identical parts in the way as stated above. In a later part of this subsection, we shall illustrate our constructions in Figures 23 (1) and (2), 24, 25, 26, 32, and 44 by giving the corresponding drawings of $\Gamma_5, \Gamma_5^*, \Gamma_7, \Gamma_7^*$ within the part $L_1$.
Step 2. Obtaining $\Gamma_{n+2}$ from $\Gamma_{n}^*$.  

Let $u_1u_2$ be an arbitrary edge in the drawing $\Gamma_{n}^*$. By the inductive rule for the arrangement of vertices (see Equations 2 and 3, and Figure 2), we can replace $u_1u_2$ by a bunch of four new edges $u_1^{(00)}u_2^{(00)}$, $u_1^{(10)}u_2^{(10)}$, $u_1^{(11)}u_2^{(11)}$, $u_1^{(01)}u_2^{(01)}$ such that the bunch is drawn along the original route of $u_1u_2$ in $\Gamma_{n}^*$. The local drawing (we call it “mesh-like structure”) around the eight vertices $u_1^{(00)}$, $u_1^{(10)}$, $u_1^{(11)}$, $u_1^{(01)}$ and $u_2^{(00)}$, $u_2^{(10)}$, $u_2^{(11)}$, $u_2^{(01)}$ in $\Gamma_{n+2}$ are as follows. We take an edge $u_1^{(ab)}u_2^{(ab)}$, for example, where $ab \in \{00, 10, 11, 01\}$. In the drawing $\Gamma_{n+2}$, the natures of the edge $u_1^{(ab)}u_2^{(ab)}$ with respect to its ends $u_1^{(ab)}$ and $u_2^{(ab)}$ are the same as the natures of the edge $u_1u_2$ with respect to vertices $u_1$ and $u_2$ in $\Gamma_{n}^*$. More precisely,
provided that

\[
(u_1, u_2) \in \mathcal{L}_{\Gamma_n}(u_i) \\
(u_1, u_2) \in \mathcal{R}_{\Gamma_n}(u_i), \ u_1, u_2 \in \mathcal{B}_{\mathcal{L}_{\Gamma_n}(u_i)}, \ u_1, u_2 \in \mathcal{A}_{\mathcal{R}_{\Gamma_n}(u_i)}.
\]

Moreover, the edges \(u_i^{(00)}u_j^{(10)}u_i^{(11)}u_j^{(01)}\) are drawn precisely along the line \(x = X_{u_i}\). The edge \(u_i^{(00)}u_j^{(01)}\) is drawn to be an arc on the right side or on the left side of the line \(x = X_{u_i}\) according to (7) or (8) holds for \(u_i\) in \(\Gamma_n^*\), respectively, where \(i = 1, 2\).

In Figure 15, Diagrams (1) and (2) depict the mesh-like structure formed by edges incident with \(u_i^{(00)}\), \(u_i^{(10)}\), \(u_i^{(11)}\), \(u_i^{(01)}\) within the neighborhood of the small 4-cycle \(u_i^{(00)}u_i^{(10)}u_i^{(11)}u_i^{(01)}\) in \(\Gamma_n\) for the case of \(|\mathcal{L}_{\Gamma_n}(u_i)|, |\mathcal{R}_{\Gamma_n}(u_i)| \in \left(\frac{n+1}{2}, \frac{n-1}{2}\right)\) and for the case of \(|\mathcal{L}_{\Gamma_n}(u_i)|, |\mathcal{R}_{\Gamma_n}(u_i)| = \left(\frac{n-1}{2}, \frac{n+1}{2}\right)\), respectively, where \(u_i\) denotes an arbitrary vertex in \(\Gamma_n^*\). On the other hand, during the process in Step 2 we see that the drawing \(\tilde{\Gamma}_{n+2}\) would inherit the characteristic of every edge being self-symmetric from \(\Gamma_n^*\). Moreover, the four edges \(u_i^{(00)}u_j^{(00)}u_i^{(10)}u_j^{(10)}u_i^{(11)}u_j^{(11)}u_i^{(01)}u_j^{(01)}\) extended from \(u_1u_2\) do not cross each other. This can be seen from Figure 16, in which we give the corresponding drawings of the four edges \(u_i^{(00)}u_j^{(00)}u_i^{(10)}u_j^{(10)}u_i^{(11)}u_j^{(11)}u_i^{(01)}u_j^{(01)}\) by taking the cases that the self-symmetric edge \(u_iu_2\) is depicted as Diagrams (1) and (4) in Figure 6 and Diagrams (1) and (3) in Figure 7 when \(Y_{u_i} < Y_{u_j}\), for example.

The process of obtaining \(\tilde{\Gamma}_7\) from \(\Gamma_5^*\) shown in Figures 23 and 24 will be helpful for us to understand accurately the adjustments described in this step.
Step 3. Obtaining $\tilde{\Gamma}_{n+2}$ from $\tilde{\Gamma}_{n+2}$.

To obtain the drawing $\Gamma_{n+2}$, we need to make two kinds of adjustments on the edges in $\tilde{\Gamma}_{n+2}$. The first kind is applied only on the edges in $\tilde{\Gamma}_{7}, \tilde{\Gamma}_{9}, \tilde{\Gamma}_{11}$, which are associated with the enclosed cycles in $\Gamma_{7}, \Gamma_{9}, \Gamma_{11}$, respectively. The second is applied on the edges in $\tilde{\Gamma}_{n+2}$ which are associated with the fundamental structures in $\Gamma_{n}$ for all $n \geq 5$.

Now we take the first kind of adjustments and suppose $n \in \{5, 7, 9\}$ at first. Let $u$ be an arbitrary vertex in the drawing $\Gamma_{n}$. Recall that the drawing $\Gamma_{n}$ has Property 3, that is, $u$ is in some enclosed cycle $C$ (see Figure 10 for the four types of the local drawing within the very small neighborhood of $u$ in $\Gamma_{n}$). Combined with the adjustment given in Step 2 (see Figure 15), we have that the mesh-like structures around the small 4-cycle $u(00)u(10)u(11)u(01)$ are depicted as in Figure 17 of which Diagrams (1)–(4) correspond to Types I–IV of Figure 10, where the two bunches marked in bold are the group of edges extended from the edges of some enclosed cycle in $\Gamma_{n}$.

Corresponding to each mesh-like structure depicted in Figure 17, we displayed in Figure 18 the adjusted mesh-like structure around $u(00)u(10)u(11)u(01)$ after the first kind of adjustments done. In particular, we reversed two edges from those two bunches within the neighborhoods of the small 4-cycle $u(00)u(10)u(11)u(01)$ and marked the two reversed edges in bold.

Combined with Condition 2 (see Figures 11 and 12) in the definition of enclosed cycles, Definition 3.4, we can derive that the adjusted edges still keep self-symmetric, and that the four edges in any bunch which is adjusted, do not cross each other and still are in a bunch along their original route before the adjustments done, what differ is the relative position of the four edges changed “a little.” We show this by Figures 19 and 20, in which each diagram corresponds to each diagram in Figures 11 and 12, respectively, where in Figures 19 and 20 the adjusted edges are marked in bold.

Next we take the second kind of adjustments in this step. This kind of adjustments is applied on edges in $\tilde{\Gamma}_{n+2}$ for all odd integers $n \geq 5$, which are associated with the fundamental structures in $\Gamma_{n}$ and will change the route of the adjusted edge “globally.” Take a vertex subset $U = \{u_1, u_2, \ldots, u_8\}$ of $V(Q_n)$ such that the drawing of $\mathcal{I}(U)$ forms a fundamental structure in $\Gamma_{n}$, which is drawn without loss of generality as Diagram (1) of Figure 8. We see that the edges of $E(\pi(U))$ in $\tilde{\Gamma}_{n+2}$ will be drawn as in Figure 21, where

$$\pi(U) = \bigcup_{i=1}^{8} \{u_i(00), u_i(10), u_i(11), u_i(01)\}$$
FIGURE 17  Mesh-like structures with depicting the edges extended from enclosed cycles

FIGURE 18  Adjusted mesh-like structures around $u^{(0)} u^{(10)} u^{(11)} u^{(0)}$
Recall that no fundamental edge belongs to any enclosed cycle in $\Gamma^*_n, \Gamma^*_n, \Gamma^*_n$ given in Property 3. On the other hand, the first kind of adjustments changed only the edges from the bunches extended from the edges of enclosed cycles in $\Gamma^*_n$ with $n \in \{5, 7, 9\}$. Hence, the drawing for the edges of $E(\pi(U))$ after the first kinds of adjustment on $\tilde{\Gamma}_{n+2}$ keeps the same as in Figure 21 for $n \in \{5, 7, 9\}$. Then we adjust the following edges:

$$P_2(u_1)P_2(u_4), P_3(u_1)P_3(u_4), P_3(u_4)P_2(u_4),$$
$$P_1(u_5)P_1(u_6), P_2(u_5)P_2(u_6), P_3(u_5)P_2(u_6),$$
$$P_1(u_8)P_1(u_8), P_2(u_8)P_2(u_8), P_3(u_8)P_3(u_8),$$
$$P_1(u_7)P_2(u_7), P_2(u_7)P_3(u_7), P_3(u_7)P_2(u_7),$$

which are shown as in Figure 22. It is worthwhile to note during the process of the second kind of adjustments, two new fundamental structures were created from the old one, which are emphasized in blue. Moreover, by (6), we can verify that...
and thus,

$$v_{\pi_{n+2}}(E(\pi(U)), \partial(\pi(U))) = v_{\pi_{n+2}}(E(\pi(U)), \partial(\pi(U)))$$
meanwhile, we can verify from Figures 21 and 22 that
\[ \nu_{\Gamma_{n+2}}(E(\pi(U))) = \nu_{\Gamma_{n+2}}^{\ast}(E(\pi(U))) - 8. \] (10)

This completes the description of the process in Step 3. To help the reader to understand the process described in the above three steps, we give Figures 23–26 to depict a complete process from \( \Gamma_5 \) to \( \Gamma_7^{\ast} \).

To complete the whole process, as stated in Step 1, we still need to describe the general rule on how to obtain the desired drawing \( \Gamma_n^{\ast} \) from \( \Gamma_n \) when \( n \geq 11 \). Yet, there exist some differences between the process from \( \Gamma_{11} \) to \( \Gamma_{11}^{\ast} \) and the process from \( \Gamma_n \) to \( \Gamma_n^{\ast} \) for \( n \geq 13 \).

We first describe the rule for obtaining \( \Gamma_{11}^{\ast} \) from \( \Gamma_{11} \). To do this, we need to find a perfect matching, denoted \( \mathcal{M} \), in \( \Gamma_{11}^{\ast} \), which meets the following:

**Requirement for the matching \( \mathcal{M} \):** “Let \( u_1u_2 \) be an arbitrary edge of the matching \( \mathcal{M} \). Then \( u_1u_2 \) is the edge of some enclosed cycle with

\[ ((L_{\Gamma_1^{\ast}}(u_1), |R_{\Gamma_1^{\ast}}(u_1)|) = (4, 5) \text{ or } (|L_{\Gamma_1^{\ast}}(u_1), |R_{\Gamma_1^{\ast}}(u_1)|) = (5, 4) \]

according to

\[ u_1u_2 \in L_{\Gamma_1^{\ast}}(u_1) \text{ or } u_1u_2 \in R_{\Gamma_1^{\ast}}(u_1), \]

respectively, where \( i = 1, 2 \).”

Since every vertex belongs to some enclosed cycle in \( \Gamma_5^{\ast} \), by the definition of enclosed cycle (see Figures 11 and 12), we derive that there exists a unique such perfect matching \( \mathcal{M} \) in \( \Gamma_5^{\ast} \). Let \( u_1u_2 \) be an arbitrary edge of the matching \( \mathcal{M} \). We also see that the relative position of the edge \( u_1u_2 \) with its both ends \( u_1 \) and \( u_2 \) is depicted as Diagrams (3)–(6) in Figure 11 or Diagrams (1), (3), (6), and (8) in Figure 12. Hence, after the first kind of adjustments on the drawing \( \Gamma_{11} \), the obtained drawings for \( u_1^{(00)}u_2^{(00)}, u_1^{(10)}u_2^{(10)}, u_1^{(11)}u_2^{(11)}, u_1^{(01)}u_2^{(01)} \) are depicted as the corresponding Diagrams (3)–(6) in

**FIGURE 22** The drawing of \( E(\pi(U)) \) in \( \Gamma_{n+2} \) depicting the adjustments associated with fundamental-structures
FIGURE 23  Auxiliary drawings illustrating the process from $\Gamma_5$ to $\Gamma'_5$

FIGURE 24  A part of $\overline{\Gamma}_7$ illustrating the process from $\Gamma'_5$ to $\overline{\Gamma}_7$

FIGURE 25  A part of $\Gamma_7$ illustrating the process from $\overline{\Gamma}_7$ to $\Gamma_7$
Figure 19 or Diagrams (1), (3), (6), and (8) in Figure 20. By Property 3, no fundamental edge belongs to any enclosed cycle in $\Gamma_9^e$, we have that $u_1u_2$ is not a fundamental edge. Although there is a possibility that $u_i$ is an end of some fundamental edge in $\Gamma_9^e$, the second kind of adjustments on $\tilde{\Gamma}_{11}$ do not change the nature whether an edge $e \in E(u_i^{(ab)})$ is a left arc or a right arc with respect to $u_i^{(ab)}$, and naturally do not change the number of left arcs and the number of right arcs with respect to $u_i^{(ab)}$ (see Figure 22), where $i = 1, 2$ and $ab \in \{00, 10, 11, 01\}$. Moreover, observe that the second kind of adjustments does not involve any one of the following edges:

Now by taking the cases depicted as Diagram (3) in Figure 19 or as Diagram (1) in Figure 20 when the first kind of adjustments are taken on $\tilde{\Gamma}_{11}$, for example, we show the rule on how to construct $\Gamma_{11}^e$ from $\Gamma_{11}$. As stated above, although in both cases, the number of left arcs and the number of right arcs with respect to $u_i^{(ab)}$ in $\Gamma_{11}$ are the same as given by Diagram (3) in Figure 19 or as given by Diagram (1) in Figure 20 (the drawings before the second kind of adjustments are taken), the mesh-like structure around $u_i^{(00)}u_i^{(10)}u_i^{(11)}u_i^{(01)}$ has probably changed. We shall also see that the possible changes on the mesh-like structures do not affect our following effort to balance the number of left arcs and the number of right arcs. Hence, we shall illustrate only some edges of which the positions are definitely unaffected by the second kind of adjustments rather than giving the whole mesh-like structures.

Consider the case depicted as Diagram (3) in Figure 19. We check first that in $\Gamma_{11}$,
\[ (|L_{\Gamma_1}(P_1(u_1))|, |R_{\Gamma_1}(P_1(u_1))|) = \left( \frac{9 + 1}{2}, \frac{9 + 1}{2} \right) = (5, 5), \]
\[ (|L_{\Gamma_1}(P_2(u_1))|, |R_{\Gamma_1}(P_2(u_1))|) = \left( \frac{9 + 1}{2}, \frac{9 - 1}{2} \right) = (5, 4), \]
\[ (|L_{\Gamma_1}(P_3(u_1))|, |R_{\Gamma_1}(P_3(u_1))|) = \left( \frac{9 + 3}{2}, \frac{9 - 3}{2} \right) = (6, 3), \]
\[ (|L_{\Gamma_1}(P_4(u_1))|, |R_{\Gamma_1}(P_4(u_1))|) = \left( \frac{9 - 1}{2}, \frac{9 + 3}{2} \right) = (4, 6), \]

and that
\[ (|L_{\Gamma_1}(P_1(u_2))|, |R_{\Gamma_1}(P_1(u_2))|) = \left( \frac{9 + 1}{2}, \frac{9 + 1}{2} \right) = (5, 5), \]
\[ (|L_{\Gamma_1}(P_2(u_2))|, |R_{\Gamma_1}(P_2(u_2))|) = \left( \frac{9 - 1}{2}, \frac{9 + 1}{2} \right) = (4, 5), \]
\[ (|L_{\Gamma_1}(P_3(u_2))|, |R_{\Gamma_1}(P_3(u_2))|) = \left( \frac{9 - 3}{2}, \frac{9 + 3}{2} \right) = (3, 6), \]
\[ (|L_{\Gamma_1}(P_4(u_2))|, |R_{\Gamma_1}(P_4(u_2))|) = \left( \frac{9 + 3}{2}, \frac{9 - 1}{2} \right) = (6, 4). \]

Then we make the number of left arcs and the number of right arcs with respect to each one of \( P_1(u_1), P_2(u_1), P_3(u_1), P_4(u_1), P_1(u_2), P_2(u_2), P_3(u_2), P_4(u_2) \) almost equal by distorting seven edges (see Figure 27),

\[ P_1(u_1)P_2(u_1), P_2(u_1)P_3(u_1), P_3(u_1)P_4(u_1), \]
\[ P_1(u_2)P_2(u_2), P_2(u_2)P_3(u_2), P_3(u_2)P_4(u_2), \]
\[ P_4(u_1)P_4(u_2), \]

“locally,” and check that in \( \Gamma_{11}^* \),

\[ (|L_{\Gamma_{11}^*}(P_1(u_1))|, |R_{\Gamma_{11}^*}(P_1(u_1))|) = \left( \frac{9 + 1}{2}, \frac{9 + 3}{2} \right) = (5, 6), \]
\[ (|L_{\Gamma_{11}^*}(P_2(u_1))|, |R_{\Gamma_{11}^*}(P_2(u_1))|) = \left( \frac{9 + 1}{2}, \frac{9 + 3}{2} \right) = (5, 6), \]
\[ (|L_{\Gamma_{11}^*}(P_3(u_1))|, |R_{\Gamma_{11}^*}(P_3(u_1))|) = \left( \frac{9 + 3}{2}, \frac{9 + 1}{2} \right) = (6, 5), \]
\[ (|L_{\Gamma_{11}^*}(P_4(u_1))|, |R_{\Gamma_{11}^*}(P_4(u_1))|) = \left( \frac{9 + 1}{2}, \frac{9 + 3}{2} \right) = (5, 6), \]

and

\[ (|L_{\Gamma_{11}^*}(P_1(u_2))|, |R_{\Gamma_{11}^*}(P_1(u_2))|) = \left( \frac{9 + 3}{2}, \frac{9 + 1}{2} \right) = (6, 5), \]
\[ (|L_{\Gamma_{11}^*}(P_2(u_2))|, |R_{\Gamma_{11}^*}(P_2(u_2))|) = \left( \frac{9 + 3}{2}, \frac{9 + 1}{2} \right) = (6, 5), \]
\[ (|L_{\Gamma_{11}^*}(P_3(u_2))|, |R_{\Gamma_{11}^*}(P_3(u_2))|) = \left( \frac{9 + 1}{2}, \frac{9 + 3}{2} \right) = (5, 6), \]
\[ (|L_{\Gamma_{11}^*}(P_4(u_2))|, |R_{\Gamma_{11}^*}(P_4(u_2))|) = \left( \frac{9 + 3}{2}, \frac{9 + 1}{2} \right) = (6, 5). \]

Consider the case depicted as Diagram (1) in Figure 20. We check first that in \( \Gamma_{11} \),
and that

\[
\begin{align*}
(l \mathcal{L}_{\Gamma_{11}}(P_1(u_1)), l \mathcal{R}_{\Gamma_{11}}(P_1(u_1))) &= \left(\frac{9+1}{2}, \frac{9+1}{2}\right) = (5, 5), \\
(l \mathcal{L}_{\Gamma_{11}}(P_2(u_1)), l \mathcal{R}_{\Gamma_{11}}(P_2(u_1))) &= \left(\frac{9+1}{2}, \frac{9+1}{2}\right) = (5, 4), \\
(l \mathcal{L}_{\Gamma_{11}}(P_3(u_1)), l \mathcal{R}_{\Gamma_{11}}(P_3(u_1))) &= \left(\frac{9+1}{2}, \frac{9+1}{2}\right) = (6, 3), \\
(l \mathcal{L}_{\Gamma_{11}}(P_4(u_1)), l \mathcal{R}_{\Gamma_{11}}(P_4(u_1))) &= \left(\frac{9+1}{2}, \frac{9+1}{2}\right) = (4, 6),
\end{align*}
\]

Then we balance the number of left arcs and the number of right arcs with respect to each vertex of \(P_1(u_1), P_2(u_1), P_3(u_1), P_4(u_1), P_3(u_2), P_2(u_2), P_3(u_2), P_4(u_2)\) by distorting seven edges (see Figure 28),

\[
\begin{align*}
P_1(u_1)P_2(u_1), P_2(u_1)P_3(u_1), P_3(u_1)P_4(u_1), \\
P_1(u_2)P_2(u_2), P_2(u_2)P_3(u_2), P_3(u_2)P_4(u_2), \\
P_3(u_1)P_4(u_1),
\end{align*}
\]

“locally,” and check that in \(\Gamma_1^{*}\),

\[
\begin{align*}
(l \mathcal{L}_{\Gamma_{11}^*}(P_1(u_1)), l \mathcal{R}_{\Gamma_{11}^*}(P_1(u_1))) &= \left(\frac{9+1}{2}, \frac{9+1}{2}\right) = (5, 6), \\
(l \mathcal{L}_{\Gamma_{11}^*}(P_2(u_1)), l \mathcal{R}_{\Gamma_{11}^*}(P_2(u_1))) &= \left(\frac{9+1}{2}, \frac{9+1}{2}\right) = (5, 6), \\
(l \mathcal{L}_{\Gamma_{11}^*}(P_3(u_1)), l \mathcal{R}_{\Gamma_{11}^*}(P_3(u_1))) &= \left(\frac{9+1}{2}, \frac{9+1}{2}\right) = (6, 5), \\
(l \mathcal{L}_{\Gamma_{11}^*}(P_4(u_1)), l \mathcal{R}_{\Gamma_{11}^*}(P_4(u_1))) &= \left(\frac{9+1}{2}, \frac{9+1}{2}\right) = (5, 6),
\end{align*}
\]
We see that the adjusted edges above remain self-symmetric. Since $\mathcal{M}$ is a perfect matching and $u_1u_2$ is an arbitrary edge of the matching $\mathcal{M}$, we have that the number of left arcs and the number of right arcs with respect to each vertex in $\Gamma_{11}$ have been balanced, and moreover, the above adjustments do not affect the number of fundamental structures in $\Gamma_{11}$, which is still the same as in $\Gamma_{11}$, that is, the obtained drawing $\Gamma_{11}^*$ from $\Gamma_{11}$ satisfies the required Properties 1, 2.

Let $n \geq 13$. It remains to introduce the general rule on how to construct $\Gamma_{n}^*$ from $\Gamma_n$. Let $u$ be an arbitrary vertex in the drawing $\Gamma_{n-2}^*$. Note that the mesh-like structure around $u^{(00)}u^{(10)}u^{(11)}u^{(01)}$ in $\Gamma_n$ is depicted as Figure 29. Since only the second kind of adjustments may be taken on edges in $\Gamma_n$ (without the first kind of adjustments), due to the same reason as above for the rule of obtaining $\Gamma_{11}^*$ from $\Gamma_{11}$, the number of left arcs and the number of right arcs with respect to each vertex of $u^{(00)}$, $u^{(10)}$, $u^{(11)}$, $u^{(01)}$ in $\Gamma_n$ is the same as ones in $\Gamma_n^*$, and moreover, the small 4-cycles $u^{(00)}u^{(10)}u^{(11)}u^{(01)}$ are not affected by the second kind of adjustment. For the cases when the mesh-like structure around $u^{(00)}u^{(10)}u^{(11)}u^{(01)}$ in $\Gamma_n$ is given as Diagram (1) or Diagram (2) in Figure 29, we check that in $\Gamma_n$.
respectively. Then we balance the number of left arcs and the number of right arcs with respect to each vertex of \( u^{(00)}, u^{(10)}, u^{(11)}, u^{(01)} \) by distorting three edges (see Figure 30),

\[
\begin{align*}
(|L_{n}^{*}(P_1(u))|, |R_{n}^{*}(P_1(u))|) &= \left( \frac{n-1}{2}, \frac{n-1}{2} \right), \\
(|L_{n}^{*}(P_2(u))|, |R_{n}^{*}(P_2(u))|) &= \left( \frac{n-1}{2}, \frac{n-3}{2} \right), \\
(|L_{n}^{*}(P_3(u))|, |R_{n}^{*}(P_3(u))|) &= \left( \frac{n-3}{2}, \frac{n-1}{2} \right), \\
(|L_{n}^{*}(P_4(u))|, |R_{n}^{*}(P_4(u))|) &= \left( \frac{n-1}{2}, \frac{n-1}{2} \right),
\end{align*}
\]

or

\[
\begin{align*}
(|L_{n}^{*}(P_1(u))|, |R_{n}^{*}(P_1(u))|) &= \left( \frac{n-1}{2}, \frac{n-1}{2} \right), \\
(|L_{n}^{*}(P_2(u))|, |R_{n}^{*}(P_2(u))|) &= \left( \frac{n-3}{2}, \frac{n-1}{2} \right), \\
(|L_{n}^{*}(P_3(u))|, |R_{n}^{*}(P_3(u))|) &= \left( \frac{n-1}{2}, \frac{n-1}{2} \right), \\
(|L_{n}^{*}(P_4(u))|, |R_{n}^{*}(P_4(u))|) &= \left( \frac{n-1}{2}, \frac{n-1}{2} \right),
\end{align*}
\]

FIGURE 29 The drawing of mesh-like structure around \( u^{(00)}u^{(10)}u^{(11)}u^{(01)} \) in \( \Gamma_n \) constructed from \( \Gamma_{n-2}^* \).
or

\[
\begin{align*}
|L_{\Gamma_n^*}(P_1(u))|, |R_{\Gamma_n^*}(P_1(u))| &= \left(\frac{n+1}{2}, \frac{n-1}{2}\right), \\
|L_{\Gamma_n^*}(P_2(u))|, |R_{\Gamma_n^*}(P_2(u))| &= \left(\frac{n-1}{2}, \frac{n+1}{2}\right), \\
|L_{\Gamma_n^*}(P_3(u))|, |R_{\Gamma_n^*}(P_3(u))| &= \left(\frac{n-1}{2}, \frac{n+1}{2}\right), \\
|L_{\Gamma_n^*}(P_4(u))|, |R_{\Gamma_n^*}(P_4(u))| &= \left(\frac{n+1}{2}, \frac{n-1}{2}\right),
\end{align*}
\]

for each case, respectively. Similarly as above, we have that each edge in $\Gamma_n^*$ keeps self-symmetric and the drawing $\Gamma_n^*$ satisfies Properties 1 and 2 as desired.

This completes the whole inductive process of constructing $\Gamma_{n+2}$ from $\Gamma_n$ for all odd $n \geq 5$.

### 3.2 Construction of the drawing $\Gamma_{n+1}$ out of $\Gamma_n^*$ for all odd $n \geq 5$

Let $n \geq 5$ be an odd integer. We shall construct the desired drawing $\Gamma_{n+1}$ directly from the drawing $\Gamma_n^*$ given in Section 3.1. To make the process clear, we give a part of the drawing $\Gamma_6$ in Figure 32 and the drawing $\Gamma_8$ in Figure 44 to illustrate the process of constructing $\Gamma_6$ from $\Gamma_5^*$ (see the corresponding Figure 23 (2)), and the process of constructing $\Gamma_8$ from $\Gamma_7^*$ (see the corresponding Figure 26), respectively.

In general, the process of constructing $\Gamma_{n+1}$ from the drawing $\Gamma_n^*$ is as follows.

Let $u$ be an arbitrary vertex in the drawing $\Gamma_n^*$. We locate the two new vertices $u^{(0)}$ and $u^{(1)}$ in $\Gamma_{n+1}$ with

\[
\begin{align*}
X_{u^{(0)}} &= X_{u^{(1)}} = X_u, \\
Y_{u^{(0)}} &= Y_u, \\
Y_{u^{(1)}} &= Y_u + \frac{Y_r - Y_s}{N},
\end{align*}
\]

and the new edge $u^{(0)}u^{(1)}$ drawn precisely at the line $x = X_u$.

Let $u_1u_2$ be an arbitrary edge in the drawing $\Gamma_n^*$. We draw the two edges $u_1^{(0)}u_2^{(0)}$ and $u_1^{(1)}u_2^{(1)}$ in $\Gamma_{n+1}$ to be a “bunch” such that the bunch is along the original route of $u_1u_2$ in $\Gamma_n^*$. In particular, the natures of each one, say $u_1(a)u_2(a)$ where $a \in \{0, 1\}$, of the two edges $u_1^{(0)}u_2^{(0)}$, $u_1^{(1)}u_2^{(1)}$, with respect to both its ends $u_1^{(a)}$, $u_2^{(a)}$ in $\Gamma_{n+1}$ are the same as the natures of $u_1u_2$ with respect to both its ends.
Similarly as in Step 2 of Section 3.1, since the edge $u_1 u_2$ is self-symmetric in $\Gamma_n^*$, we have that the two edges $u_1^{(0)} u_2^{(0)}$, $u_1^{(1)} u_2^{(1)}$ do not cross each other, that is,

$$\nu_{\Gamma_{n+1}^*}(\{(u_1^{(0)}, u_2^{(0)}), (u_1^{(1)}, u_2^{(1)})\}) = 0.$$  \hfill (11)

Moreover, we see that the mesh-like structure formed around $u^{(0)} u^{(1)}$ is depicted as Diagram (1) or Diagram (2) in Figure 31 according to $(L_{\Gamma_n^*}(u), R_{\Gamma_n^*}(u)) = \left(\frac{n+1}{2}, \frac{n-1}{2}\right)$ or $(L_{\Gamma_n^*}(u), R_{\Gamma_n^*}(u)) = \left(\frac{n-1}{2}, \frac{n+1}{2}\right)$, respectively. This completes the description of the process in this subsection.

### 3.3 Calculations of the number of crossings in the desired drawing

In this subsection, we shall calculate the number of crossings in the drawing $\Gamma_n$ constructed in Sections 3.1 and 3.2.

The following Lemma 3.1 can be found in [5]. For the reader's convenience, we give a proof below.

**Lemma 3.1.** Let $n \geq 5$ be an odd integer. Let $m_n$ be the number of crossings given in each diagram of Figure 31. Let $M_n$ be the number of crossings given in each diagram of Figure 15. Let $\tilde{M}_n$ be the number of crossings given in each diagram of Figure 18. Then

(i) $m_n = \left(\frac{n+1}{2}\right) + \left(\frac{n-1}{2}\right),$

(ii) $M_n = \frac{4}{2} \cdot \left(\frac{n+1}{2}\right) + \frac{4}{2} \cdot \left(\frac{n-1}{2}\right) + (n - 1)$,
and

\[(iii) \widehat{M}_n = M_n - 1.\]

Proof.

(i) We take Diagram (1) of Figure 31 for example to show the calculations, because Diagram (2) is just a reflection of Diagram (1) and definitely has the same number of crossings. Notice that any two distinct bunches which lie on the left side of the line \(x = X_u\) (i.e., numbered from 1 to \(\frac{n+1}{2}\)) have exactly one crossing. This implies that the number of crossings formed by the bunches on the left side of the line \(x = X_u\) is

\[
\binom{n+1}{2}/2.
\]

Similarly, the number of crossings formed by the bunches on the right side of the line \(x = X_u\) (i.e., numbered from \(\frac{n+3}{2}\) to \(n\)) is

\[
\binom{n-1}{2}/2.
\]

Hence, this gives \(m_n = \binom{n+1}{2}/2 + \binom{n-1}{2}/2\).
(ii) We take Diagram (2) of Figure 15 for example to show the calculations. Notice that any two distinct bunches which are numbered from 1 to $\frac{n-1}{2}$ have exactly $6 = \binom{4}{2}$ crossings, and thus, the number of crossings formed by the bunches (numbered from 1 to $\frac{n-1}{2}$) is

$$\binom{4}{2} \cdot \binom{\frac{n-1}{2}}{2}.$$ 

Moreover, the arc $P_1(u)P_3(u)$ has exactly two crossings with each bunch numbered from 1 to $\frac{n-1}{2}$, that is, the corresponding number of crossings is

$$2 \cdot \frac{n-1}{2} = n - 1.$$ 

In a similar observation, we have that the number of crossings formed by the bunches numbered from $\frac{n+1}{2}$ to $n$ is

$$\binom{4}{2} \cdot \binom{\frac{n+1}{2}}{2}.$$ 

Therefore, this gives $M_n = \binom{4}{2} \cdot \binom{\frac{n+1}{2}}{2} + \binom{4}{2} \cdot \binom{\frac{n-1}{2}}{2} + (n - 1)$.

(iii) We take Diagram (3) of Figure 18 for example show the calculations. Notice that we obtain this diagram from Diagram (3) of Figure 17 by distorting two edges. For the notational convenience, we put both diagrams into a new figure (see Figure 33) and marked the adjusted edges in colors. Note that we distort only two edges, say $e_1$ and $e_2$ (emphasized in green and blue, respectively) during the process from Diagram (a) to Diagram (b) in Figure 33. To show Conclusion (iii), we need to calculate the crossings formed by $e_1$ and by $e_2$ in both Diagrams (a) and (b).

**Figure 33** Auxiliary drawings for the calculation to show $\tilde{M}_n = M_n - 1$
Observe Diagram (a). The edge $e_1$ has exactly three crossings with each bunch numbered from 2 to $\frac{n+1}{2}$. The edge $e_2$ has exactly one crossing with each bunch numbered from $\frac{n+5}{2}$ to $n$, and moreover, has one crossing with the arc $P_1(u)P_4(u)$. Hence, the total number of crossings formed by $e_1$ and by $e_2$ within the mesh is

$$3 \cdot \frac{n-1}{2} + \left( \frac{n-3}{2} + 1 \right) = 2n - 2. \quad (12)$$

Now observe Diagram (b). The edge $e_1$ has exactly three crossings with each bunch numbered from $\frac{n+5}{2}$ to $n$, and has two crossings with the bunch numbered $\frac{n+3}{2}$ (crossed the two edges which lie in the bunch numbered $\frac{n+3}{2}$ and are incident to $P_3(u)$ and $P_4(u)$, respectively). The edge $e_2$ has exactly one crossing with each bunch numbered from 2 to $\frac{n+1}{2}$. Hence, the total number of crossings formed by $e_1$ and $e_2$ within the mesh is

$$\left( 3 \cdot \frac{n-3}{2} + 2 \right) + \frac{n-1}{2} = 2n - 3. \quad (13)$$

Then Conclusion (iii) follows from (12) and (13) readily. \hfill \square

To make the calculations clear, we also give the drawings (see Figures 34–39) and the corresponding number of crossings (see Table 1) of the mesh-like structures as depicted in Figures 15 and 31 with $n \in \{7, 9, 11\}$.

**Lemma 3.2.** For odd integers $n \geq 5$, we define a sequence of numbers $A_n$ satisfying the recurrence relation and the initial condition as follows:

**FIGURE 34** The drawing of mesh-like structure around $u^{(0)}u^{(1)}u^{(2)}u^{(3)}$ in $\bar{\Gamma}_6$ constructed from $\Gamma^*_7$
\[ A_{n+2} = 16 \cdot A_n + 2^n \cdot \left( \binom{4}{2} \cdot \left( \frac{n+1}{2} \right) + \binom{4}{2} \cdot \left( \frac{n-1}{2} \right) + (n-1) \right) \]

\[ - \varepsilon_n \cdot 2^n - 8 \cdot 2^{\frac{n-1}{2}}, \]

where

\[ \varepsilon_n = \begin{cases} 1 & \text{if } n \in \{5, 7, 9\}, \\ 0 & \text{if } n \geq 11. \end{cases} \]
Then we have

\[
A_n = \begin{cases} 
\frac{139}{896} \cdot 4^n - \left(\frac{n^2 + 1}{2}\right)2^{n-2} + \frac{1}{7} \cdot 2^{n+1} & \text{if } n \in \{5, 7, 9, 11\}, \\
\frac{139}{896} \cdot 4^n - \left(\frac{n^2 + 1}{2}\right)2^{n-2} + \frac{1}{7} \cdot 2^{n+1} + \frac{2^n - 1}{3} \cdot 2^{n-2} & \text{otherwise.}
\end{cases}
\]

Proof. For odd integers \( n \geq 5 \), we define a sequence of numbers \( B_n \) satisfying the recurrence relation and the initial condition given by
By (14)–(17), we see that

\[ A_n = B_n \text{ for } n \in \{5, 7, 9, 11\}. \]  

Now we show that

\[ B_n = \frac{139}{896} 4^n - \left( \frac{n^2 + 1}{2} \right) 2^{n-2} + \frac{1}{7} \cdot 2^{\frac{n+1}{2}} \]  

for any odd integer \( n \geq 5 \).  

We shall prove (19) by induction on \( n \). If \( n = 5 \), we verify that

\[ B_5 = \frac{139}{896} 4^5 - \left( \frac{5^2 + 1}{2} \right) 2^{5-2} + \frac{1}{7} \cdot 2^{\frac{5+1}{2}} = 56, \]  
done. Suppose \( n \geq 5 \) is an odd integer and (19)
holds for \(n\). It suffices to prove (19) holds for \(n + 2\). By (16) and the hypothesis, we have that

\[
B_{n+2} = 16 \cdot B_n + 2^n \cdot \left( \frac{4}{2} \cdot \left( \frac{n+1}{2} \right) + \frac{4}{2} \cdot \left( \frac{n-1}{2} \right) + (n - 1) \right) - 2^n - 8 \cdot 2^{\frac{n+1}{2}}
\]

\[
= 16 \cdot \left( \frac{139}{896} 4^n - \frac{n^2 + 1}{2} \cdot 2^{n-2} + \frac{1}{7} \cdot 2^{\frac{n+1}{2}} \right) + 2^n \cdot \left( \frac{4}{2} \cdot \left( \frac{n+1}{2} \right) \right) + (n - 1) - 2^n - 8 \cdot 2^{\frac{n+1}{2}}
\]

\[
= \left( \frac{139}{896} 4^{n+2} - (2n^2 + 2) \cdot 2^n + \frac{8}{7} \cdot 2^{\frac{n+1}{2}} \right) + 2^n \cdot \left( \frac{3n^2 - 4n + 1}{2} \right)
\]

\[
= \frac{139}{896} 4^{n+2} - 2^n - 2^{\frac{n+3}{2}}
\]

which proves (19).

Next we show that

\[
A_n = B_n + \frac{2^{n-11} - 1}{3} \cdot 2^{n-2} \text{ for any odd integer } n \geq 13. \tag{20}
\]

By induction on \(n\). If \(n = 13\), it follows from (14)–(16) and (18) that

\[
A_n = A_{13}
\]

\[
= 16 \cdot A_{11} + 2^{11} \cdot \left( \frac{4}{2} \cdot \left( \frac{11+1}{2} \right) + \frac{4}{2} \cdot \left( \frac{11-1}{2} \right) + (11 - 1) \right) - 8 \cdot 2^{\frac{11-3}{2}}
\]

\[
= 16 \cdot B_{11} + 2^{11} \cdot \left( \frac{4}{2} \cdot \left( \frac{11+1}{2} \right) + \frac{4}{2} \cdot \left( \frac{11-1}{2} \right) + (11 - 1) \right) - 8 \cdot 2^{\frac{11-3}{2}}
\]

\[
= \left[ 16 \cdot B_{11} + 2^{11} \cdot \left( \frac{4}{2} \cdot \left( \frac{11+1}{2} \right) + \frac{4}{2} \cdot \left( \frac{11-1}{2} \right) + (11 - 1) \right) - 2^{11} - 8 \cdot 2^{\frac{11-3}{2}} \right] + 2^{11}
\]

\[
= B_{13} + 2^{11}
\]

\[
= B_n + \frac{2^{n-11} - 1}{3} \cdot 2^{n-2},
\]
done. Suppose \( n \geq 13 \) is an odd integer and (20) holds for \( n \). It suffices to prove (20) holds for \( n + 2 \). By (14)–(16) and the hypothesis, we conclude that

\[
A_{n+2} = 16 \cdot A_n + 2^n \cdot \left( \binom{4}{2} \cdot \left( \frac{n+1}{2} \right) + \binom{4}{2} \cdot \left( \frac{n-1}{2} \right) + (n-1) \right) - 8 \cdot 2^{n-3}
\]

\[
= 16 \cdot \left( B_n + \frac{2^{n+1} - 1}{3} \cdot 2^{n-2} \right) + 2^n \cdot \left( \binom{4}{2} \cdot \left( \frac{n+1}{2} \right) + \binom{4}{2} \right) \cdot \left( \frac{n-1}{2} \right) + (n-1) \right) - 8 \cdot 2^{n-3}
\]

\[
= 16 \cdot B_n + \frac{2^{n+2} - 1}{3} \cdot 2^n + 2^n \cdot \left( \binom{4}{2} \cdot \left( \frac{n+1}{2} \right) + \binom{4}{2} \cdot \left( \frac{n-1}{2} \right) + (n-1) \right) - 8 \cdot 2^{n-3}
\]

\[
= 16 \cdot B_n + 2^n \cdot \left( \binom{4}{2} \cdot \left( \frac{n+1}{2} \right) + \binom{4}{2} \cdot \left( \frac{n-1}{2} \right) + (n-1) \right) - 2^n - 8 \cdot 2^{n-3}
\]

\[
= B_{n+2} + \frac{2^{n+2} - 1}{3} \cdot 2^n
\]

\[
= B_{n+2} + \frac{2^{n+2} - 1}{3} \cdot 2^{n+2-2},
\]

which proves (20).

Then the lemma follows from (18) to (20) readily. □

Now we are in a position to give the detailed calculation of \( \nu(\Gamma_n) \). By the process described in Section 3.1, we conclude that for all odd integers \( n \geq 5 \),

\[
\nu(\Gamma_n^*) = \nu(\Gamma_n) \quad (21)
\]

and

\[
\nu(\Gamma_{n+2}^*) = 16 \cdot \nu(\Gamma_n^*) + 2^n \cdot M_n, \quad (22)
\]
where \(16 = 4^2\) in the term \(16 \cdot \nu\left(\Gamma^*_{n}\right)\) is the crossings produced by any two bunches, say 
\[ u_1^{(00)}u_2^{(00)}, u_1^{(10)}u_2^{(10)}, u_1^{(11)}u_2^{(11)}, u_1^{(01)}u_2^{(01)} \]

and 
\[ u_3^{(00)}u_4^{(00)}, u_3^{(00)}u_4^{(00)}, u_3^{(00)}u_4^{(00)}, u_3^{(00)}u_4^{(00)} \]

with \(u_1u_2\) and \(u_3u_4\) cross in \(\Gamma^*_{n}\), and where the term \(2^n \cdot M_n\) is the total crossings in the mesh-like structure around \(u^{(00)}u^{(10)}u^{(11)}u^{(01)}\) for all vertices \(u\) in \(\Gamma^*_{n}\).

By Properties 2 and 3 holding for \(\Gamma^*_{n}\), since we applied the first kind of adjustments for \(\bar{\Gamma}_{n+2}\) when \(n \in \{5, 7, 9\}\) and applied the second kind of adjustments for \(\bar{\Gamma}_{n+2}\) with all odd \(n \geq 5\), it follows from Conclusion (iii) of Lemma 3.1 and (9) and (10) that 
\[ \nu(\Gamma_{n+2}) = \nu(\bar{\Gamma}_{n+2}) - \epsilon_n \cdot 2^n - 8 \cdot 2^{n-\frac{3}{2}}, \]

where 
\[ \epsilon_n = \begin{cases} 
1 & \text{if } n \in \{5, 7, 9\}, \\
0 & \text{if } n \geq 11. 
\end{cases} \]

Combined with (21) and (22) and Conclusion (ii) of Lemma 3.1, we have

\[
\nu(\Gamma_{n+2}) = 16 \cdot \nu(\Gamma_n) + 2^n \cdot \left( \left(\begin{array}{c} 4 \\ 2 \end{array}\right) \cdot \left(\frac{n+1}{2}\right) + \left(\begin{array}{c} 4 \\ 2 \end{array}\right) \cdot \left(\frac{n-1}{2}\right) + (n-1) \right) - \epsilon_n \cdot 2^n - 8 \cdot 2^{n-\frac{3}{2}}. \tag{23}
\]

Similarly as above, by (11), Conclusion (i) of Lemma 3.1 and the process described in Section 3.2, we can derive that for all odd number \(n \geq 5\),

\[
\nu(\Gamma_{n+1}) = 4 \cdot \nu(\Gamma^*_{n}) + 2^n \cdot \left( \frac{(n+1)/2}{2} + \frac{(n-1)/2}{2} \right) = 4 \cdot \nu(\Gamma_n) + 2^{n-2} \cdot (n-1)^2. \tag{24}
\]

Set 
\[ \lambda_n = \begin{cases} 
0 & \text{if } 5 \leq n \leq 12, \\
1 & \text{if } n \geq 13. 
\end{cases} \]

By Lemma 3.2, we have that for any odd integer \(n \geq 5\),
\[
\nu(\Gamma_n) = \frac{139}{896} 4^n - \left( \frac{n^2 + 1}{2} \right) 2^{n-2} + \frac{1}{7} \cdot 2^{n+1} + \lambda_n \cdot \frac{2^{n-1} - 1}{3} \cdot 2^{n-2}
\]

\[
= \frac{139}{896} 4^n - \left( \frac{n^2 + 1}{2} \right) 2^{n-2} + \frac{4}{7} \cdot 2^{n-3} + \lambda_n \cdot \left( \frac{2^{n-1} - 2^{n-2}}{3} - \frac{2^{n-2}}{3} \right)
\]

\[
= \frac{139}{896} 4^n - \left( \frac{n^2 + 1}{2} \right) 2^{n-2} + \frac{4}{7} \cdot 2^{3 \left( \frac{n+1}{2} \right) - n} + \lambda_n \cdot \left( \frac{2^n - 2^{n-1}}{2^{13} \cdot 3} - \frac{2^{n-1}}{6} \right)
\]

\[
= \frac{139}{896} 4^n - \left( \frac{n^2 + 1}{2} \right) 2^{n-2} + \frac{4}{7} \cdot 2^{3 \left( \frac{n+1}{2} \right) - n} + \lambda_n \cdot \left( \frac{4^n}{24576} - \frac{4^{\frac{1}{2}}}{6} \right)
\]

(25)

Notice that

\[
\lambda_{n-1} = \lambda_n \quad \text{for any even integer } n \geq 6.
\]

Combined with (24) and (25), we have that for any even integer \( n \geq 6 \),

\[
\nu(\Gamma_n) = 4 \cdot \nu(\Gamma_{n-1}) + 2^{n-3} \cdot (n - 2)^2
\]

\[
= 4 \cdot \left( \frac{139}{896} 4^{n-1} - \left( \frac{(n-1)^2 + 1}{2} \right) 2^{n-3} + \frac{4}{7} \cdot 2^{3 \left( \frac{n+1}{2} \right) - (n-1)} + \lambda_{n-1} \left( \frac{4^{n-1}}{24576} - \frac{4^{\frac{1}{2}}}{6} \right) \right)
\]

\[
+ 2^{n-3} \cdot (n - 2)^2
\]

\[
= 4 \cdot \left( \frac{139}{896} 4^{n-1} - \left( \frac{n^2 - 2n + 2}{2} \right) 2^{n-3} + \frac{4}{7} \cdot 2^{3 \left( \frac{n+1}{2} \right) - (n-1)} + \lambda_n \left( \frac{4^{n-1}}{24576} - \frac{n-2}{6} \right) \right)
\]

\[
+ (n^2 - 4n + 4) \cdot 2^{n-3}
\]

\[
= 4 \cdot \left( \frac{139}{896} 4^{n-1} - \left( \frac{n^2 - 2n + 2}{2} \right) 2^{n-3} + \frac{4}{7} \cdot 2^{3 \left( \frac{n+1}{2} \right) - (n-1)} + \lambda_n \left( \frac{4^{n-1}}{24576} - \frac{n-2}{6} \right) \right)
\]

\[
+ \frac{n^2 - 4n + 4}{2} \cdot 2^{n-2}
\]

\[
= \frac{139}{896} 4^n - \left( n^2 - 2n + 2 \right) \cdot 2^{n-2} + \frac{4}{7} \cdot 2^n + \lambda_n \left( \frac{4^n}{24576} - \frac{4^2}{6} \right) + \frac{n^2 - 4n + 4}{2} \cdot 2^{n-2}
\]

\[
= \frac{139}{896} 4^n + \left( \frac{n^2 - 4n + 4}{2} - (n^2 - 2n + 2) \right) \cdot 2^{n-2} + \frac{4}{7} \cdot 2^n + \lambda_n \left( \frac{4^n}{24576} - \frac{4^2}{6} \right)
\]

\[
= \frac{139}{896} 4^n - \frac{n^2}{2} \cdot 2^{n-2} + \frac{4}{7} \cdot 2^3 \cdot 2^{-n} + \lambda_n \left( \frac{4^n}{24576} - \frac{4^2}{6} \right)
\]

\[
= \frac{139}{896} 4^n - \left( \frac{n^2 + 1}{2} \right) 2^{n-2} + \frac{4}{7} \cdot 2^{3 \left( \frac{n+1}{2} \right) - n} + \lambda_n \left( \frac{4^n}{24576} - \frac{4^{\frac{1}{2}}}{6} \right)
\]

(26)

Combined (25) and (26), we conclude that for all integers \( n \geq 5 \),

\[
\text{cr}(Q_n) \leq \nu(\Gamma_n) = \frac{139}{896} 4^n - \left( \frac{n^2 + 1}{2} \right) 2^{n-2} + \frac{4}{7} \cdot 2^{3 \left( \frac{n+1}{2} \right) - n} + \lambda_n \left( \frac{4^n}{24576} - \frac{4^{\frac{1}{2}}}{6} \right)
\]

completing the calculations.
4 | CONCLUDING REMARKS

We first remark that the drawing for $Q_n$ constructed in this manuscript is not optimal for $n \geq 11$, that is, the upper bound given in Theorem 1.1 can be improved slightly by still applying that kind of adjustments in Step 3 for the progress of obtaining $\Gamma_{n+2}$ from $\overline{\Gamma}_{n+2}$ for odd integers $n \geq 11$. In fact, for $n \geq 11$ we still can find several disjoint enclosed cycles, say $C_1, \ldots, C_m$ in $\overline{\Gamma}_n$, unfortunately, with $V(C_i) \cup \cdots \cup V(C_m) \subseteq V(Q_n)$. Theoretically, that would be insignificant since we have no general rule for finding those enclosed cycles for all $n \geq 11$. Indeed, the idea “enclosed cycles” essentially is just an improved variant of the method employed in [5]. The idea employed by Faria, de Figueiredo, Sýkora, and Vrto in [5] enlightened us to obtain the present drawing of $Q_n$ in this manuscript. So, the most important contribution on the Erdős and Guy’s problem was made by the four authors in [5]. We remark that our drawing managed to use the idea of “a fundamental structure” to decrease the crossings furthermore.

Note that the drawing in this manuscript is better than the conjectured crossings by Erdős and Guy. Observed that the two coefficients $\frac{139}{896}$ (for the case $5 \leq n \leq 12$) and $\frac{1}{24,576}$ (for the case $n \geq 13$) of the leading term $4^n$ in Theorem 1.1 are all less than that coefficient $\frac{5}{32} = \frac{140}{896} = \frac{26,880}{172,032}$ conjectured by Erdős and Guy. In particular, denote $f(n) = \frac{5}{32} 4^n - \frac{n^2 + 1}{2} 2^{n-2}$ to be the function of the conjectured values, and denote

$$\Delta(n) = f(n) - \nu(\Gamma_n)$$

to be the difference function of the conjectured values and the number of crossings in our constructed drawing, that is,

$$\Delta(n) = f(n) - \nu(\Gamma_n) = \left( \frac{5}{32} 4^n - \frac{n^2 + 1}{2} 2^{n-2} \right) - \left( \frac{139}{896} 4^n - \frac{n^2 + 1}{2} 2^{n-2} + \frac{4}{7} \cdot 2^{\frac{n}{2}} - n \right) + \lambda_n \left( \frac{4^n}{24576} - \frac{1}{6} \right) + \lambda_n \left( \frac{4^n}{896} - \frac{1}{6} \right)$$

$$= \left( \frac{5}{32} - \frac{139}{896} \right) 4^n + \lambda_n \left( \frac{4^n}{24576} - \frac{1}{6} \right)$$

| $n$  | Conjectured values | Our results | $\Delta$ |
|-----|--------------------|-------------|---------|
| 5   | 56                 | 56          | 0       |
| 6   | 352                | 352         | 0       |
| 7   | 1760               | 1744        | 16      |
| 8   | 8192               | 8128        | 64      |
| 9   | 35,712             | 35,424      | 288     |
| 10  | 151,040            | 149,888     | 1152    |
| 11  | 624,128            | 619,456     | 4672    |
| 12  | 2,547,712          | 2,529,024   | 18,688  |
| 13  | 10,311,680         | 10,238,848  | 72,832  |
FIGURE 40  The drawing $\Gamma_7$
FIGURE 41  The drawing $\Gamma^*_\gamma$
FIGURE 42  The partial drawing of $\Gamma_9$
FIGURE 43  The partial drawing of $\Gamma_9^*$
FIGURE 44  The partial drawing of $\Gamma_5$
For $n \geq 13$, since $\lambda_n = 1$, it follows that

$$\Delta(n) = \left(\frac{5}{32} - \frac{139}{896} - \frac{1}{24,576}\right)4^n + \left(\frac{4|\frac{n}{2}|}{6} - \frac{4}{7} \cdot 2^3\left|\frac{n}{2}\right| - n\right)$$

$$= \left(\frac{26,680}{172,032} - \frac{26,680}{172,032} - \frac{7}{172,032}\right)4^n + \left(\frac{4|\frac{n}{2}|}{6} - \frac{4}{7} \cdot 2^3\left|\frac{n}{2}\right| \cdot 2^3\left|\frac{n}{2}\right| - n\right)$$

$$= \frac{185}{172,032}4^n + 4\left|\frac{n}{2}\right|\left(\frac{1}{6} - \frac{4}{7} \cdot 2^3\left|\frac{n}{2}\right| - n\right)$$

$$> \frac{185}{172,032}4^n.$$

Moreover, we conclude that

$$\Delta(n) = \frac{185}{172,032}4^n + O(2^n)(n \to \infty).$$

Finally, we close this paper by Table 2 to show the difference between the number of crossings in our drawing and the values conjectured by Erdős and Guy for $5 \leq n \leq 13$ (Figures 40–44).

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