Exact Solution Versus Gaussian Approximation for a Non-Ideal Bose Gas in One-Dimension

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Abstract
We investigate ground-state and excitation spectrum of a system of non-relativistic bosons in one-dimension interacting through repulsive, two-body contact interactions in a self-consistent Gaussian mean-field approximation. The method consists in writing the variationally determined density operator as the most general Gaussian functional of the quantized field operators. There are mainly two advantages in working with one-dimension. First, the existence of an exact solution for the ground-state and excitation energies. Second, neither in the perturbative results nor in the Gaussian approximation itself we do not have to deal with the three-dimensional patologies of the contact interaction. So that this scheme provides a clear comparison between these three different results.

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1 Introduction

The non-relativistic interacting Bose gas is certainly among the most interesting problems in many-body physics. This system has been extensively investigated since fifties motivated strongly by the superfluid properties of helium [1]. The recent successful achievement of Bose-Einstein condensation of alkalis vapors [2] has renewed a great interest in theoretical studies of Bose gases [3].

For dilute gases the macroscopic condensate function is given by the Gross-Pitaevskii equation. The effects of noncondensate atoms can be calculated in the loop expansion in power of $\sqrt{\rho a^3}$, where $\rho$ is the density and $a$ is the $S$-wave scattering [4]. This perturbative method tends to be less reliable in the circumstance when higher trap densities are involved. On the other hand, non-perturbative approximation schemes have been developed in connection with the problem of self-interacting, relativistic Bose fields, which became relevant for inflationary models [5], and in the analysis of the transient phenomena in the collision of the complex nuclear system [6]. This problem has also been used as a testing ground for non-perturbative methods which could then be applied to more complicated systems of interacting
fields. In this context, the Gaussian variational approach \[7\] has received considerable attention, also in view of its relation to extended mean field methods traditionally employed in many-body physics \[8; 9\].

The main objective of this work is to compare the standard perturbation results with the unperturbative ones obtained with the Gaussian approximation in the case of a system of non-relativistic bosons in one dimension interacting through repulsive, two-body contact interactions. The use of an one-dimensional problem gives the advantage of dealing with a repulsive contact interaction without getting into problem due to divergences \[10; 11\]. Also the existence of an exact solution in this case show us clearly when the perturbation theory breaks down and how the upperbound non-perturbative results stands in this region.

The Gaussian mean-field approximation which we will employ here is based in terms of a time-dependent projection approach developed earlier for the nonrelativistic nuclear many-body dynamics by Nemes and de Toledo Piza \[12\]. This scheme allows one to formulate a mean-field expansion for the dynamics of the two-point correlation function from which one recovers the results of the Gaussian mean-field approximations in lowest order, i.e., this permits to include and to evaluate higher dynamical corrections effects to the simplest Gaussian mean-field approximation. Moreover, the expansion is energy-conserving (for closed system) to all orders \[13\]. The method was recently applied in the context of the field-theoretical models, namely, to treat an uniform relativistic (1+1)-dimensional self-interacting boson system described by the $\lambda\phi^4$ theory \[14\], to treat an uniform relativistic (1+1)-dimensional self-interacting fermion system described by the chiral Gross-Neveu model \[14\], and to treat an uniform (3+1)-dimensional relativistic interacting boson-fermion system described by the Plasma Scalar model \[15\].

Although we developed a temperature dependent formalism in this paper we will concentrate on the zero temperature case and calculate in Gaussian mean-field approximation the physical quantities related to the optimized free energy such as ground state energy and sound velocity to compare with pertubative and exact results given in literature \[16; 17\]. In Section 2, we define the problem and show its exact solution. Section 3 reviews the approximation used here and discusses the equilibrium solutions for the resulting variational equations. Through a truncated version of the Gaussian variational equation we obtain the perturbative results in Section 4. In Section 5
we show the numerical results obtained from the Gaussian and perturbative approximations and compare these with the exact calculations. Section 6 discusses the range of validity of these theoretical methods and main conclusions.

2 The problem and the exact solution

We consider an uniform and isotropic system of non-relativistic, interacting, spinless bosons in one-dimension described by the Hamiltonian,

\[ H = - \sum_i \left( \frac{\partial^2}{\partial x_i^2} \right) + 2c \sum_{i,j} \delta(x_i - x_j) \]  

with periodic boundary condition in region \( 0 \leq x_i \leq L \), where \( 2c > 0 \) is the amplitude of the repulsive delta function potential. The only dimensionless intensive variable in the theory is

\[ \gamma = \frac{c}{\rho} \]

where \( \rho = N/L \) is the density system.

Lieb and Liniger in Ref. [10] have shown that the exact ground state energy can be obtained by solving an inhomogeneous Fredholm equation of the second kind. So that in the thermodynamic limit (\( N, L \to \infty \) such that \( \rho = \) fixed constant) we can write the ground state energy per particle as

\[ \frac{E_0}{N} = \frac{1}{\rho} \int_{-K}^{K} f(k)k^2 dk \]  

where \( f(k) \) is the solution of

\[ 2c \int_{-K}^{K} \frac{f(p)}{c^2 + (p - k)^2} dp = 2\pi f(k) - 1 \]  

with the condition

\[ \int_{-K}^{K} f(k) dk = \rho \]  

For computational purposes we can change variables as follows

\[ k = Kx ; \ c = K\lambda ; \ f(Kx) = g(x) \]  

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in terms of which the Eqs.(2), (3) and (4) become, respectively,

$$\frac{E_0}{N} = \rho^2 \gamma^3 \int_{-1}^{1} g(x) x^2 dx$$  \hspace{1cm} (5) 

$$1 + 2\lambda \int_{-1}^{1} \frac{g(x)}{\lambda^2 + (x - y)^2} dx = 2\pi g(y)$$  \hspace{1cm} (6) 

$$\gamma \int_{-1}^{1} g(x) dx = \lambda.$$ \hspace{1cm} (7) 

For a fixed $\lambda$ we can calculate $g(x)$ having $\lambda(\gamma)$ and with it follows $\frac{E_0}{N}(\gamma)$ and $K(\gamma)$ [10]. Once we have the ground state energy it is possible to calculate the chemical potential

$$\mu = \frac{\partial E_0}{\partial N}.$$ \hspace{1cm} (8) 

From $\mu$ we can calculate the sound velocity derived from the macroscopic compressibility

$$v_s = 2 \left( \mu - \frac{1}{2} \gamma \frac{\partial \mu}{\partial \gamma} \right)^{1/2}.$$ \hspace{1cm} (9) 

Lieb, in another work [17], also has shown that this theory exhibit two types of excitations. The energy of the first type of excitation is given by

$$\epsilon_1(p) = -\mu + q^2 + 2 \int_{-k}^{K} k J(k) dk ,$$ \hspace{1cm} (10) 

where $\mu$ is the chemical potential and

$$p = q + \int_{-K}^{K} J(k) dk ,$$ \hspace{1cm} (11) 

while $J(k)$ is again a solution of a Fredholm equation

$$2\pi J(k) = 2\epsilon \int_{-K}^{K} \frac{J(r) dr}{\epsilon^2 + (k - r)^2} - \pi + 2 \tan^{-1}(q - k).$$ \hspace{1cm} (12)
The energy of the second type of excitation have the same structure and is such that

\[ \epsilon_2(p) = \mu - q^2 + 2\int_{-K}^{K} kG(k)dk \]  \tag{13}

\[ p = -q + \int_{-K}^{K} G(k)dk \]  \tag{14}

\[ 2\pi G(k) = 2c \int_{-K}^{K} \frac{G(r)dr}{c^2 + (k - r)^2} + \pi - 2\tan^{-1}(q - k) \]  \tag{15}

To solve these equations we use the same procedure used in the ground state energy. Here \( q \) plays the same role as \( \lambda \) in the other case.

3 Gaussian approximation

In second quantization, rewriting the Hamiltonian (1), in momentum representation with periodic boundary conditions in region \( L \), follows

\[ H = \sum_{\vec{k}} e(k)a_{\vec{k}}^\dagger a_{\vec{k}} + \frac{c}{L} \sum_{k_1k_2q} a_{k_1+q}^\dagger a_{k_2-q}^\dagger a_{k_2} a_{k_1} \]  \tag{16}

where \( e(k) = k^2 \) is the free particle kinetic energy, and the contact repulsive \( (c > 0) \) interaction potential between a pair of particles is \( 2c \delta(\vec{x} - \vec{x}') \). The \( a_{\vec{k}}^\dagger \) and \( a_{\vec{k}} \) are boson creation and annihilation operators, which satisfy the standard boson commutation relations at equal times.

3.1 Projection technique and approximation scheme

In this section we introduce the time-dependent projection technique which will permit to obtain closed approximations to the dynamics of a system. It has been developed earlier in the context of nonrelativistic nuclear many-body dynamics and was recently applied in the quantum-field theoretical context. It allows for the formulation of a mean-field expansion for the dynamics of the two-point correlation function from which one recovers the results of the Gaussian mean-field approximations in lowest order. If
carried to higher orders it allows for the inclusion and evaluation of higher
dynamical correlation corrections to the simplest mean-field approximation.

We begin by describing the quantum state of the system in Heisenberg
picture in terms of a many-body density operator \( \mathcal{F} \), a time independent,
non-negative, Hermitian operator with unit trace. \( \mathcal{F} \) may in general involve
statistical mixtures, in particular in the boson number operator, in the spirit
of a grand canonical description. Our implementation of the Gaussian ap-
proximation will consist in decomposing the full density \( \mathcal{F} \) as

\[
\mathcal{F} = \mathcal{F}_0(t) + \mathcal{F}'(t) ,
\]

where \( \mathcal{F}_0(t) \) is a Gaussian ansatz which achieves a Hartree-Fock factorization
of traces involving more than two field operators. The most general hermitian
Gaussian density \( \mathcal{F}_0(t) \) have the form of a exponential of linear and bilinear
expressions in the fields normalized to unit trace \[18, 19\]. In the mome
ntum basis, it reads

\[
\mathcal{F}_0 = \frac{\exp \left( \sum_{(k_1, k_2)} A_{k_1, k_2} a_{k_1}^\dagger a_{k_2} + \left( B_{k_1, k_2} a_{k_1}^\dagger a_{k_2}^\dagger + \text{h.c.} \right) + \left( C_{k_1} a_{k_1}^\dagger + \text{h.c.} \right) \right)}{\text{Tr} \left\{ \exp \left( \sum_{(k_1, k_2)} A_{k_1, k_2} a_{k_1}^\dagger a_{k_2} + \left( B_{k_1, k_2} a_{k_1}^\dagger a_{k_2}^\dagger + \text{h.c.} \right) + \left( C_{k_1} a_{k_1}^\dagger + \text{h.c.} \right) \right) \right\}} ,
\]

where the matrix \( A_{k_1, k_2} \) is hermitian. The parameters in Eq.(18) are fixed by
requiring that mean values in \( \mathcal{F}_0 \) of expressions that are bilinear in the fields
reproduce the corresponding \( \mathcal{F} \) averages [see Eqs.(21) and (22) below].

The uniformity and isotropy assumptions we make allow us to restrict
this general ansatz so that \( A_{k_1, k_2} \) is diagonal, \( B_{k_1, k_2} \) vanishes unless \( k_1 = -k_2 \)
and both of these matrices and the \( C_{k_1} \) depend only on the magnitudes of
the momentum vectors. Finally, by a general canonical transformation of
the Bogolyubov type, the time-dependent matrix \( \mathcal{F}_0 \) can be diagonalized \[4\].
In this case, the diagonalization is achieved by defining transformed boson
operators as

\[
\eta_k = x_k b_k + y_k b^\dagger_{-k}
\]

\[
\eta_{-k}^\dagger = x_k b_k + y_k b_{-k}
\]
where

\[ b^e_k = a^e_k - \Gamma^e_k \]

\[ \Gamma_k = \langle a^e_k \rangle \]

We also have used the isotropy of the uniform system to make the c-number transformation parameters \( x_k, y_k \) and \( \Gamma_k \) dependent only on the magnitude of \( \vec{k} \). In order for this transformation to be canonical we have still to impose on the \( x_k \) and \( y_k \) the usual normalization condition

\[ |x_k|^2 - |y_k|^2 = 1 \quad (19) \]

In terms of the Bogolyubov quasi-boson operators, the trace-normalized Gaussian density operator is now written explicitly as

\[ \mathcal{F}_0 = \prod_{\vec{k}} \frac{1}{1 + \nu_k} \left( \frac{\nu_k}{1 + \nu_k} \right)^{\eta^\dagger_{\vec{k}} \eta_{\vec{k}}} \cdot (20) \]

Straightforward calculation shows that

\[ Tr(\eta^\dagger_{\vec{k}_1} \eta_{\vec{k}_2} \mathcal{F}_0) = \nu_{k_1} \delta(\vec{k}_1 - \vec{k}_2) \quad (21) \]

so that the \( \nu_k \) are positive quantities corresponding to mean occupation numbers of the \( \eta \)-bosons. One also finds that

\[ Tr[(x_{\vec{k}}^* a_{\vec{k}} + y_{\vec{k}}^* a_{-\vec{k}}^\dagger)^n \mathcal{F}_0] = Tr[(\eta_{\vec{k}}^* + A_k)^n \mathcal{F}_0] = (A_k)^n \quad (22) \]

with \( A_k = x_{\vec{k}}^* \Gamma_{\vec{k}} + y_{\vec{k}}^* \Gamma_{-\vec{k}}^* \) so that non vanishing values of the \( \Gamma_k \) correspond to coherent condensates of unshifted Bogolyubov transformed bosons. We again invoke the uniformity of the system to impose

\[ \Gamma_k = \delta_{k,0} \Gamma_0 \quad (23) \]

in the calculations to follow.

The “remainder” density \( \mathcal{F}'(t) \), defined by Eq.(17), is a traceless, pure correlation density. As already remarked, a crucial point to observe is that \( \mathcal{F}'_0(t) \) can be written as a time-dependent projection of \( \mathcal{F} \), i.e.,
\( \mathcal{F}_0(t) = \mathcal{P}(t)\mathcal{F} \) with \( \mathcal{P}(t)\mathcal{P}(t) = \mathcal{P}(t) \). \( \text{(24)} \)

In order to completely define this projector we require further that it satisfies

\[ i\dot{\mathcal{F}}_0(t) = [\mathcal{P}(t), \mathcal{L}] \mathcal{F} = [\mathcal{F}_0(t), \mathcal{H}] + \mathcal{P}(t) [H, \mathcal{F}] \, , \text{(25)} \]

where \( \mathcal{L} \) is the Liouvillian defined as

\[ \mathcal{L} \cdot = [H, \cdot] \, , \text{(26)} \]

\( H \) being the Hamiltonian of the field. Eq.(25) is just the Heisenberg picture counterpart of the condition \( \dot{\mathcal{P}}(t)\mathcal{F} = 0 \) which has been used to define \( \mathcal{P}(t) \) in the Schrödinger picture \[13\]. It is possible to show that conditions (24) and (25) make \( \mathcal{P}(t) \) unique and to obtain an explicit form for this object in terms of the quasi-boson operators and of the natural orbital occupations see Refs. \[9, 12, 15\].

The existence of the projector \( \mathcal{P}(t) \) allows one to obtain an equation relating the correlation part \( \mathcal{F}'(t) \) to the Gaussian part \( \mathcal{F}_0(t) \) of the full density. This can be immediately obtained from Eqs.(17), (24) and (25) and reads

\[ (i\partial_t + \mathcal{P}(t)\mathcal{L}) \mathcal{F}'(t) = (\mathcal{I} - \mathcal{P}(t)) \mathcal{L}\mathcal{F}_0(t) \, . \text{(27)} \]

This equation has the formal solution

\[ \mathcal{F}'(t) = \mathcal{G}(t,0)\mathcal{F}'(0) - i \int_0^t dt' \mathcal{G}(t,t') (\mathcal{I} - \mathcal{P}(t')) \mathcal{L}\mathcal{F}_0(t') \, , \text{(28)} \]

where the first term accounts for initial correlations possibly contained in \( \mathcal{F} \).

The object \( \mathcal{G}(t,t') \) is the time-ordered Green’s function

\[ \mathcal{G}(t,t') = T \left( \exp \left[ i \int_{t'}^t d\tau \mathcal{P}(\tau)\mathcal{L} \right] \right) \, . \text{(29)} \]
We see thus that \( F'(t) \), and therefore also \( F \) [see Eq.(17)], can be formally expressed in terms of \( F_0(t') \) (for \( t' \leq t \)) and of initial correlations \( F'(0) \).

This allows us to express also the dynamics of the system as functionals of \( F_0(t') \) and of the initial correlations. Since, on the other hand, the reduced density \( F_0(t') \) is expressed in terms of the one-boson densities alone, we see that the resulting equations are essentially closed equations. Note, however, that the complicated time dependence of the field operators is explicitly probed through the memory effects present in the expression (28) for \( F'(t) \). Approximations are therefore needed for the actual evaluation of this object.

A systematic expansion scheme for the memory effects has been discussed in Refs. [9, 12, 13, 15]. The lowest order correlation corrections to the pure mean field approximation, in which \( F' \) is simply ignored, correspond to replacing the full Heisenberg time-evolution of operators occurring in the collision integrals by a mean-field evolution governed by

\[
H_0 = \mathcal{P}^\dagger(t)H .
\]

Consistently with this approximation, \( \mathcal{L} \) is replaced in (25) and (26) by \( \mathcal{L}_0 = [H_0, \cdot] \). In this way correlation effects are treated to second order in \( H \) in the resulting collision integrals.

An important feature of this scheme (which holds also for higher orders of the expansion [13]) is that the mean energy is conserved, namely

\[
\frac{\partial}{\partial t} \langle H \rangle = 0
\]

where

\[
\langle H \rangle = Tr \ H F_0(t) + Tr \ H F'(t) .
\]

In the following sections we apply the general expressions obtained in above to treat a uniform boson system described by the Hamiltonian given in Eq.(16). We will consider only the lowest (mean-field) approximation, corresponding to \( \mathcal{F}'(t) = 0 \). Collisional correlations will be treated elsewhere.

### 3.2 Temperature time-dependent Gaussian treatment

First, it is important to note that the truncated density \( F_0 \) in general breaks the global gauge symmetry of \( H \), which is responsible for the conservation of
the number of $a$-bosons. It is possible to verify that assumption calculating the dispersion

$$\langle N^2 \rangle - \langle N \rangle^2 = Tr[F_0 \sum_{\vec{k}, \vec{k}'} a_{\vec{k}}^\dagger a_{\vec{k}} a_{\vec{k}'}^\dagger a_{\vec{k}'}] - \{Tr[F_0 \sum_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}}]\}^2.$$  

Calculating the traces above we have

$$\langle N^2 \rangle - \langle N \rangle^2 = 2|\Gamma_0|^2|\nu_0| \nu_0 + |y_0|^2(1 + \nu_0)| - 2\Gamma_0^2 x_0 y_0^* (1 + 2\nu_0) -$$

$$-2\Gamma_0^2 x_0 y_0^* (1 + 2\nu_0) + |\Gamma_0|^2 + \sum_{\vec{k}} |x_{\vec{k}}|^2 \nu_{\vec{k}} + (1 + \nu_{\vec{k}})|y_{\vec{k}}|^2 +$$

$$+ \sum_{\vec{k}} \{||x_{\vec{k}}|^2 \nu_{\vec{k}} + (1 + \nu_{\vec{k}})|y_{\vec{k}}|^2|^2 + |x_{\vec{k}}|^2 |y_{\vec{k}}|^2 (1 + 2\nu_{\vec{k}})\}.$$

Furthermore, mean values of many-boson operators taken with respect to $F_0$ will contain no irreducible many-body parts, so that the replacement of $F$ by $F_0$ amounts to a mean field approximation. The states described by $F_0$ have therefore to be interpreted as “intrinsic” mean field states.

In order to develop a finite temperature treatment within this framework, we look for variational extrema of an approximation $\Omega$ to the grand potential written in terms of the Gaussian density operator $F_0$ as

$$\Omega = Tr[F_0 (H - \mu N + KT \ln F_0)]$$ (30)

with

$$N = \sum_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}}^,$$

where $K$ is the Boltzmann constant and the Lagrange multipliers $\mu$ and $T$ will play the role of chemical potential and temperature respectively. The replacement of the full density operator $F$ by $F_0$ implies not only truncation of correlation energies but also replacement of the full entropy by the mean field entropy $S_0 = -KT \ln \Omega[F_0]$. Variations will be taken with respect to the parameters which determine the Gaussian density $F_0$, and the value of $\mu$ is fixed by choosing the particle density of the system. A detailed justification
of this general procedure has been given [10, 11], who have shown that it corresponds to the optimal determination (in the variational sense) of the grand partition function (hence of the grand potential) for the problem when one restricts oneself to trial density operators of the form given in Eq. (20). We have to emphasize that the use of this functional Ω does not assure that we will have the best variational results for other observables. See discussion in Secs. 5 and 6.

An important simplification occurs in the case of stationary problems. In this case $x_k$, $y_k$ and $\Gamma_0$ can be taken to be real and we can use a simple parametric representation that automatically satisfies the canonicity condition. It reads

$$x_k = \cosh \sigma_k, \quad y_k = \sinh \sigma_k.$$  

(31)

It is then straightforward to evaluate the traces involved in Ω to obtain

$$\Omega = \sum_k \left( e(k) - \mu + \frac{4c\Gamma_0^2}{L} \right) \left[ \frac{(1 + 2\nu_k) \cosh 2\sigma_k - 1}{2} - \mu \Gamma_0^2 + \frac{c\Gamma_0^4}{L} \right] - \frac{c\Gamma_0^2}{L} \sum_k (1 + 2\nu_k) \sinh 2\sigma_k \left\{ \frac{(1 + 2\nu_k) \cosh 2\sigma_k - 1}{2} \right\}^2 + \frac{c}{4L} \left\{ \sum_k (1 + 2\nu_k) \sinh 2\sigma_k \right\}^2 - KT \sum_k [(1 + \nu_k) \ln(1 + \nu_k) - \nu_k \ln \nu_k].$$

(32)

In a similar way the number constraint $Tr[N \mathcal{F}_0] = \langle N \rangle$ evaluates to

$$\langle N \rangle = \Gamma_0^2 + \sum_k \left[ \frac{(1 + 2\nu_k) \cosh 2\sigma_k - 1}{2} \right].$$

(33)

### 3.3 Equilibrium solutions

Equations determining the form of the Gaussian density $\mathcal{F}_0$ appropriate for thermal equilibrium are in general derived by requiring Ω, given by (32), to
be stationary under arbitrary variations of $\Gamma_0$, $\sigma_k$ and $\nu_k$. Variation with respect to $\Gamma_0$ gives the gap equation

$$\Gamma_0 \left\{ \frac{4c}{L} \Gamma_0^2 - 2\mu - 4cA + 8cB \right\} = 0 ,$$

where we define in the thermodynamic limit, $\sum_k \rightarrow \frac{L}{2\pi} \int_{-\infty}^{+\infty} dk$, the quantities $A$, $B$ and $C$ given by

$$A = \frac{1}{4\pi} \int_{-\infty}^{\infty} (1 + 2\nu_k) \sinh 2\sigma_k\, dk$$

$$B = \frac{1}{4\pi} \int_{-\infty}^{\infty} [(1 + 2\nu_k) \cosh 2\sigma_k - 1]\, dk$$

$$C = \frac{1}{4\pi} \int_{-\infty}^{\infty} e(k)[(1 + 2\nu_k) \cosh 2\sigma_k - 1]\, dk .$$

The gap equation, Eq.(34), admits a solution with a non-vanishing value of $\Gamma_0$ (condensed phase) obtained by requiring that the expression in curly brackets vanishes. This condensed phase solution involves the number constraint, Eq.(33), in addition to the values of $\nu_k$ and $\sigma_k$, which are determined by the remaining variational conditions on $\Omega$. In order to simplify the algebraic work involved in the study of this class of solutions, it is sometimes convenient to use the Eq.(33) to eliminate $\Gamma_0$ from $\Omega$, which then becomes

$$\Omega = F - \mu \langle N \rangle .$$

This identifies the free energy $F$ as

$$F = LC - 2c\rho(A - B)L + c(\rho^2 + A^2 - B^2)L + 2cABL$$

$$- \frac{LKT}{2\pi} \int_{-\infty}^{\infty} [(1 + \nu_k) \ln(1 + \nu_k) - \nu_k \ln \nu_k]\, dk .$$

For condensed phase, the chemical potential follows from Eqs.(33) and (34)
\[ \mu = 2c(\rho - A + B) \] (36)

Extremizing \( F \) by setting derivatives with respect to \( \sigma_k \) and \( \nu_k \) equal to zero one gets

\[ \tanh 2\sigma_k = \frac{2c[\rho - B - A]}{e(k) + 2c[\rho - B + A]} \] (37)

and

\[ \nu_k = \frac{1}{\{\exp[\sqrt{\Delta}/KT] - 1\}} \] , (38)

where

\[ e_g(k) = \sqrt{\Delta} = \sqrt{e(k)^2 + 4ce(k)[\rho - B + A] + 16c^2[\rho - B]A} \] (39)

corresponds to the excitation spectrum in our approximation. This result can be confirmed through an RPA calculation Ref. [20]. Finally, \( A \) and \( B \) can be found from gap equations (37-38), for given values of \( \rho \) and \( T \), by solving numerically the system of equations

\[
\begin{align*}
A &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ \frac{2c[\rho - B - A]}{\sqrt{\Delta}} \left[ 1 + \frac{2}{\{\exp[\sqrt{\Delta}/KT] - 1\}} \right] \right\} dk \\
B &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ \frac{e(k) + 2c[\rho - B + A]}{\sqrt{\Delta}} \left[ 1 + \frac{2}{\{\exp[\sqrt{\Delta}/KT] - 1\}} \right] - 1 \right\} dk
\end{align*}
\] (40)

Therefore, in terms of the values of \( A \) and \( B \), we can calculate \( C \) and the thermodynamic functions also numerically.

On the other hand, the gap equation (34) may also admit, besides the condensed phase, the trivial solution \( \Gamma_0 = 0 \), which corresponds to a non-condensed phase. For this phase, we extremize \( \Omega \), given in Eq.(32), by setting derivatives with respect to \( \sigma_k \) and \( \nu_k \) equal to zero one gets

\[ \tanh 2\sigma_k = \frac{-2cA}{e(k) - \mu + 2cB} \] (41)
and

\[ \nu_k = \frac{1}{\{\exp[\sqrt{\Delta}/KT] - 1\}^n}. \] (42)

where

\[ e_g(k) = \sqrt{\Delta} = e(k) - \mu + 4c\rho. \] (43)

For non-condensed phase \((\Gamma_0 = 0)\) the only possible solution, from gap equations (41-42), is \(A = 0\) following that \(\sigma_k = 0\). Therefore, from Eqs.(33) and (42-43), we have the solution

\[ \rho = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{\exp[(k^2 - \mu + 4c\rho)/KT] - 1}, \] (44)

which serves to determine \(\mu\). That is the standard Hartree-Fock result which corresponds to a shift in the chemical potential when compared with the ideal Bose gas [21]. In this phase, we show in Ref. [20] that when \(T \to 0\) we have \(\mu \to 4c\rho\).

Particularly in this paper we are interested on properties for \(T = 0\), since in this case we can compare our results with perturbative and exact solutions existent in the literature [16, 17]. When \(T = 0\), the ground state energy is then given by

\[ F(T = 0) = \frac{E}{N} = \frac{C}{\rho} - 2c(A - B) + \frac{c}{\rho}(\rho^2 + A^2 - B^2) + 2\frac{c}{\rho}AB. \] (45)

We still have to decide what phase shall we use when \(T = 0\). To do so, let us examine the chemical potential \(\mu\) since from Eq.(36). When \(A > 0\) follows from Eq.(36) that \(\mu_{T\neq0} < 4c\rho = \mu_{T=0}\) and it is easy to check that for a given \(\rho\) at \(T = 0\) there are always \(A\) and \(B\) that are solutions of the system (40). This does not happen in the case of non-condensed phase at \(T = 0\) [14]. Therefore, we conclude that the stable phase at \(T = 0\) for any \(\rho\) is the one with \(\Gamma_0 \neq 0\).

Finally, in this calculation \(A, B\) and \(C\) are finite meaning that no renormalization is needed. This is a quite different result from the three-dimensional case Ref. [10, 20].
4 Independent $\eta$-bosons - Perturbative results

In Ref. [10] it is shown that from a convenient truncation of the Gaussian approximation in the three-dimension problem, it is possible to obtain the usual perturbative results in the parameter $2c$. Here we will follow the discussion of Ref. [16, 17].

This truncation of the Gaussian approximation consists in neglecting all terms representing interactions between $\eta$-bosons, what amounts to dropping all double integrals in Eq.(35). The variational conditions on $\sigma_k$ and $\nu_k$ appear then as

$$\tanh 2\sigma_k = \frac{2c\rho}{k^2 + 2c\rho}$$

and

$$\nu_k = \frac{1}{\{\exp[\sqrt{k^4 + 4c\rho k^2/KT}] - 1\}}$$

which gives us the usual phonon excitation spectrum

$$\epsilon(p) = \sqrt{p^4 + 4c\rho p^2}.$$  

(48)

When we using these results for calculating $F$, again stressing that there is no need for regularization here, we obtain

$$\frac{E}{N} = \rho^2 \gamma \left(1 - \frac{4\sqrt{\gamma}}{3\pi}\right).$$

(49)

Using Eq.(8) and Eq.(9) for the chemical potential and the sound velocity we get

$$\mu = 2\gamma \left(1 - \frac{\sqrt{\gamma}}{\pi}\right) \rho^2$$

(50)

$$v_s = 2\rho \left[\left(\gamma - \frac{1}{2\pi\gamma^{\frac{3}{2}}}\right)^{\frac{1}{2}}\right].$$

(51)
We can check that the sound velocity obtained from the excitation spectrum Eq.(48) defined by

\[ v_s = \lim_{p \to 0} \frac{\partial e(p)}{\partial p} \]  
(52)

gives us \( v_s = 2\sqrt{\gamma} \), that is exactly one order lower of the one obtained through the compressibility Eq.(51).

5  Numerical Results

We did all our calculations for a fixed \( \rho = 1 \). For the exact solution we follow the steps of Ref.[16] solving numerically the Eqs.(5),(6) and (7). For the Gaussian approximation we solve the system of Eqs.(39) and (40) taking \( T = 0 \), and we calculate the ground state energy given by the Eq.(45). Finally for the perturbative results we use closed expression (49). All this results are shown in Fig.1, where we plot the ground state energy as a function of \( \gamma \). We clearly see the upperbound Gaussian results and, as expected, satisfactory results for low \( \gamma \) of both perturbative and Gaussian calculations.

For the exact and Gaussian sound velocity we differentiate numerically the respective ground state curve. For the perturbative solution we have two possibilities one using the compressibility through Eq.(51) or in lower order using the excitation spectrum Eq.(48). We plot the sound velocity as a function of \( \gamma \) in Fig.2. We note that the Gaussian result is also an upperbound and for this range of \( \gamma \) the results obtained from perturbative theory using the compressibility and the exact solution are graphically indistinguishable.

Finally for the excitation spectrum we solve numerically (10), (11) and (12) for the first type of excitations and (13), (14) and (15) for the second type. To calculate the excitation spectrum in the Gaussian approximation using Eq.(39), once we have solved the system (40). For the perturbative theory we use Eq.(48). In Fig.3 and Fig.4 we plot the excitation energy as a function of the momentum for \( \gamma = 0.787094 \) and \( \gamma = 3.07725 \) respectively.

Neither perturbative nor Gaussian approximation describe the exact second type of excitations and the Gaussian approximation introduces a gap in the excitation energy that gets bigger if we go to the non-perturbative limit.
6 Conclusion

In summary, we have used a one-dimensional problem to understand, in a clear way, the range of validity of the perturbative and the Gaussian variational approximation in the context of interacting Bose gas. The methods were applied to calculate the grand-potential functional and compared to the exact results. In this way we avoid problems related to renormalization \cite{10} and have the exact solution for the ground state energy and excitation spectrum \cite{10, 17}.

We were able to see in Fig. 1 that the ground state energy for low values of $\gamma$ both perturbative and Gaussian (non-perturbative) results are quite good. As $\gamma$ increases we have a region where the exact solution is in between the two approximations. For higher values of $\gamma$ the perturbative results break down completely and the Gaussian approximation still gives us an upper bound result. This shows us that for high values of $\gamma$ a self-consistent non-perturbative calculation should be used. For the sound velocity, Fig. 2, we see that both perturbative and Gaussian results are upper bound approximation to the exact result. From Figs. 3 and 4, we see a very interesting feature of the Gaussian approximation, namely, it produces a coherent macroscopic occupation for $k = 0$. This occupation is artificial because as we can see from the exact solution there is no macroscopic occupation for $k = 0$.

Therefore, the results that are related directly from the thermodynamic potential, such as the ground state energy and the sound velocity, obtained through the compressibility, are good upperbounds. Yet the gap in the excitation spectrum, that came from the occupations, are very different from the exact solution. So, the fact that using the optimal determination of the grand potential does not assure that we will have the best variational result for the occupations \cite{19}. Thus, for better results in other observables we should involve ourselves in a more difficult task of finding the best functional related to this specific observable with the same trial density operator of the form given in Eq.(20). Finally we want to point out that the formalism developed here allows us to go in straight forward way to $T \neq 0$ still keeping the variational characteristic of having an upperbound result \cite{10}.

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7 Figure captions

Fig.1. The ground-state energy as a function of $\gamma = c/\rho$. The full curve 1 gives the exact solution obtained numerically. Curve 2 shows the Gaussian approximation also obtained numerically. Curve 3 is the result of perturbation theory.

Fig.2. The sound velocity $v_s$, as a function of $\gamma = c/\rho$. Curve 1 is the result obtained from perturbative theory using the excitation spectrum. Curve 2 derived from the macroscopic compressibility using the Gaussian approximation. Curves 3 and 4, that is graphically indistinguishable in this region, are the exact result and the perturbation solution obtained through the macroscopic compressibility.

Fig.3 The excitation energy as a function of the momentum for $\gamma = 0.787094$. The full curves 1 and 2 correspond to the two types of excitations (exact result). Curve 3 corresponds to the Gaussian spectrum and curve 4 is the perturbative spectrum.

Fig.4 The excitation energy as a function of the momentum for $\gamma = 3.07725$. The full curves 1 and 2 correspond to the two types of excitations (exact result). Curve 3 corresponds to the Gaussian spectrum and curve 4 is the perturbative spectrum.