AREAS OF SPHERICAL POLYHEDRAL SURFACES WITH REGULAR FACES

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Abstract. For a finite planar graph, it associates with some metric spaces, called (regular) spherical polyhedral surfaces, by replacing faces with regular spherical polygons in the unit sphere and gluing them edge-to-edge. We consider the class of planar graphs which admit spherical polyhedral surfaces with the curvature bounded below by 1 in the sense of Alexandrov, i.e. the total angle at each vertex is at most $2\pi$. We classify all spherical tilings with regular spherical polygons, i.e. total angles at vertices are exactly $2\pi$. We prove that for any graph in this class which does not admit a spherical tiling, the area of the associated spherical polyhedral surface with the curvature bounded below by 1 is at most $4\pi - \epsilon_0$ for some $\epsilon_0 > 0$. That is, we obtain a definite gap between the area of such a surface and that of the unit sphere.

Keywords. combinatorial curvature, critical area, gap, spherical tiling with regular spherical polygons.

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1. Introduction

The combinatorial curvature for planar graphs, as the generalization of the Gaussian curvature for surfaces, was introduced by [30, 37, 14, 20]. Many interesting geometric and combinatorial results have been obtained since then, see e.g. [44, 40, 17, 4, 16, 25, 18, 38, 35, 5, 9, 8, 43, 7, 22, 24, 23, 32, 13].

Let $(V, E)$ be an undirected simple graph with the set of vertices $V$ and the set of edges $E$. The graph $(V, E)$ is called planar if it is topologically embedded into the sphere or the plane. We write $G = (V, E, F)$ for the combinatorial structure, or the cell complex, induced by the embedding where $F$ is the set of faces, i.e. connected components of the complement of the embedding image of the graph $(V, E)$ in the target. Two elements in $V, E, F$ are called incident if the closures of their images have non-empty intersection. We say that a planar graph $G$ is a planar tessellation if the following hold, see e.g. [23]:

(i) Every face is homeomorphic to a disk whose boundary consists of finitely many edges of the graph.
(ii) Every edge is contained in exactly two different faces.
(iii) For any two faces whose closures have non-empty intersection, the intersection is either a vertex or an edge.
Assumption 1. We only consider planar tessellations and call them planar graphs for the sake of simplicity. For a planar graph, we always assume that for any vertex $x$ and face $\sigma$,

$$\deg(x) \geq 3, \quad \deg(\sigma) \geq 3$$

where $\deg(\cdot)$ denotes the degree of a vertex or a face.

For a planar graph $G$, the combinatorial curvature at the vertex is defined as

$$\Phi(x) = 1 - \frac{\deg(x)}{2} + \sum_{\sigma \in F : x \in \sigma} \frac{1}{\deg(\sigma)}, \quad x \in V,$$

where the summation is taken over all faces $\sigma$ incident to $x$. To digest the definition, we endow the ambient space of $G$, $\mathbb{S}^2$ or $\mathbb{R}^2$, with a canonical piecewise flat metric and call it the (regular) Euclidean polyhedral surface, denoted by $S(G)$: Replace each face by a regular Euclidean polygon of side-length one and same facial degree, glue them together along their common edges, and define the metric on $\mathbb{S}^2$ or $\mathbb{R}^2$ via gluing metrics, see [6, Chapter 3]. It is well-known that the generalized Gaussian curvature on an Euclidean polyhedral surface, as a measure, concentrates on the vertices. And one is ready to see that the combinatorial curvature at a vertex is in fact the mass of the generalized Gaussian curvature at that vertex up to the normalization $2\pi$, see e.g. [3, 19].

We denote by

$$\mathcal{PC}_{>0} := \{ G = (V, E, F) : \Phi(x) > 0, \forall x \in V \}$$

the class of planar graphs with positive combinatorial curvature everywhere.

There are many examples in $\mathcal{PC}_{>0}$, e.g. Platonic solids, Archimedean solids, and Johnson solids [21, 41, 42], and see [35, 31, 13] for more. The complete classification of $\mathcal{PC}_{>0}$ is not yet known. Note that by Alexandrov’s embedding theorem [3], the Euclidean polyhedral surface $S(G)$ of a finite planar graph $G \in \mathcal{PC}_{>0}$ can be isometrically embedded into $\mathbb{R}^3$ as a boundary of a convex polyhedron.

We review some known results on the class $\mathcal{PC}_{>0}$. In [9], DeVos and Mohar proved that any graph $G \in \mathcal{PC}_{>0}$ is finite, which solves a conjecture of Higuchi [17], see [37, 38] for early results. In fact, for the set

$$P := \{ G : G \in \mathcal{PC}_{>0} \text{ is neither a prism nor an antiprism } \},$$

DeVos and Mohar proved that $\sharp P < \infty$ and proposed to determine the number

$$C_{S^2} := \max_{(V, E, F) \in P} \sharp V.$$

On the one hand, for the lower bound estimate of $C_{S^2}$ many authors [35, 31, 13, 33] attempted to construct large examples in this class, and finally found some examples possessing 208 vertices. On the other hand, in [9], DeVos and Mohar showed that $C_{S^2} \leq 3444$, which was improved to $C_{S^2} \leq 380$.
by Oh [32]. By a refined argument, in [13], Ghidelli completely solved the problem.

**Theorem 1.1.**

1. $C_{S^2} = 208$. [13, 31]
2. $\max \{\deg(\sigma) : \sigma \in F, (V, E, F) \in P\} \leq 41$. [13]

In this paper, we study spherical polyhedral surfaces with faces isometric to regular spherical polygons. Spherical polyhedral surfaces have been extensively studied in the literature, see e.g. [1, 34, 28, 6, 26, 27]. A regular spherical polygon is the domain in the unit sphere $S^2(1)$ defined by the intersection of finitely many hemispheres such that all side lengths and interior angles are equal respectively.

**Assumption 2.** We only consider regular spherical polygons in $S^2(1)$ which have at least three sides.

For any $n \geq 3$ and $0 < a \leq \frac{2\pi}{n}$, there is a regular spherical $n$-gon in $S^2(1)$ of side length $a$ contained in a hemisphere, denoted by $\Delta_n(a)$, which is unique up to the spherical isometry.

Analogous to Euclidean polyhedral surfaces, we define spherical polyhedral surfaces associated to a planar graph. For any finite planar graph $G = (V, E, F)$ and $0 < a \leq \inf_{\sigma \in F} \frac{2\pi}{\deg(\sigma)}$, we replace each face by a regular spherical polygon in $S^2(1)$ of side length $a$, and glue them together along their common edges. This induces a metric structure on the ambient space of $G$, called spherical polyhedral surface of $G$ with side length $a$ and denoted by $S_a(G)$. We denote by $\text{Area}_a(G)$ the area of $S_a(G)$. For any $x \in V$, the total angle at $x$ measured in $S_a(G)$ is denoted by $\theta_a(x)$, and the angle defect at $x$ is defined as

$$K_a(x) = 2\pi - \theta_a(x).$$

By the spherical geometry, we yield the Gauss-Bonnet theorem on spherical polyhedral surfaces.

**Theorem 1.2.** For any finite planar graph $G = (V, E, F)$ and for any positive number $a \leq \inf_{\sigma \in F} \frac{2\pi}{\deg(\sigma)}$,

$$\text{Area}_a(G) + \sum_{x \in V} K_a(x) = 4\pi.$$

We say that a geodesic metric space $(X, d)$ has the (sectional) curvature bounded below by 1 in the sense of Alexandrov if it satisfies the Toponogov triangle comparison property with respect to the unit sphere, and denote by $\text{Alex}(1)$ the set of such spaces, see e.g. [6]. It is well-known that $S_a(G) \in \text{Alex}(1)$ if and only if

$$K_a(x) \geq 0, \quad \forall x \in V.$$

By Alexandrov and Pogorelov’s theorem, they can be embedded into $S^3$ as the boundary of convex polyhedron [11, 34]. In fact, for each vertex $x$ with $\theta_a(x) \leq 2\pi$, there is a neighborhood of $x$ in $S_a(G)$ which is isometric to a
neighborhood of a pole in the 1-suspension of a circle of length $\theta_a(x)$, and the result follows from \cite[Theorem 10.2.3]{6}. We denote by

$$\mathcal{PC}_{\geq 1} := \{G = (V, E, F) : \text{there exists } a > 0 \text{ such that } S_a(G) \in \text{Alex}(1)\}$$

the class of finite planar graphs whose spherical polyhedral surfaces have the curvature bounded below by 1 in the Alexandrov sense. We will prove that

$$\mathcal{PC}_{\geq 1} = \mathcal{PC}_{> 0},$$

see Proposition \cite[2.1]{7}. This suggests a possible way to study the class $\mathcal{PC}_{\geq 1}$ by known results on the class $\mathcal{PC}_{> 0}$, where the latter refers to the Euclidean setting.

In a planar graph, the pattern of a vertex $x$ is defined as a vector

$$\text{Ptn}(x) := (\deg(\sigma_1), \deg(\sigma_2), \cdots, \deg(\sigma_N)),$$

where $\{\sigma_i\}_{i=1}^{N}$ are the faces incident to $x$ ordered by $\deg(\sigma_1) \leq \deg(\sigma_2) \leq \cdots \leq \deg(\sigma_N)$, and $N = \deg(x)$.

**Definition 1.3.** For any vertex $x$ of pattern $(\deg(\sigma_1), \deg(\sigma_2), \cdots, \deg(\sigma_N))$ with positive combinatorial curvature, we define the critical side-length of the vertex $x$ (or for the pattern) by

$$a_c(x) = a_c(\deg(\sigma_1), \deg(\sigma_2), \cdots, \deg(\sigma_N))$$

$$:= \max \left\{ a \in \left[0, \frac{2\pi}{\deg(\sigma_N)}\right] : K_a(x) \geq 0 \right\}. \quad (3)$$

For a planar graph $G \in \mathcal{PC}_{\geq 1}$, the critical side-length of $G$ is defined as

$$a_c(G) := \min_{x \in V} a_c(x).$$

The critical area of $G$ is defined as

$$\text{Area}^{\text{cri}}(G) := \text{Area}_{a_c(G)}(G).$$

Clearly $\text{Area}_a(G)$ is monotonely increasing in $a$. So,

$$\text{Area}^{\text{cri}}(G) = \max\{\text{Area}_a(G) : S_a(G) \in \text{Alex}(1)\}.$$

We say that a planar graph $G$ admits a spherical tiling with regular spherical polygons if there is a spherical tiling with regular spherical polygons whose planar graph structure is isomorphic to $G$, and denote by $\mathcal{T}_{S2}$ the set of such planar graphs. One is ready to prove the following proposition.

**Proposition 1.4.** For a planar graph $G$, the following are equivalent:

1. $G \in \mathcal{T}_{S2}$.
2. There is some $a > 0$ such that $S_a(G)$ is isometric to the unit sphere, i.e. $K_a(x) = 0$ for all $x \in V$.
3. $a_c(x) = a_c(y)$ for all $x, y \in V$, and $K_{a_c(x)}(x) = 0$ for all $x \in V$.
4. $\text{Area}^{\text{cri}}(G) = 4\pi$. 
For any \( G \in \mathcal{T}_{S^2} \), consider the spherical tiling in the unit sphere. We observe that the convex hull of the vertices in \( \mathbb{R}^3 \) is a convex polyhedron with regular Euclidean polygons as faces. Combining the above proposition with the classification of such convex polyhedra \([21, 42, 41]\), we classify the class \( \mathcal{T}_{S^2} \) in the following theorem.

**Theorem 1.5.** The set \( \mathcal{T}_{S^2} \) consists of the following:

(a) the 5 Platonic solids,
(b) the 13 Archimedean solids,
(c) the infinite series of prisms and antiprisms,
(d) 22 Johnson solids \( J_1, J_3, J_6, J_{11}, J_{19}, J_{34}, J_{37}, J_{62}, J_{63}, J_{72}, J_{73}, J_{74}, J_{75}, J_{76}, J_{77}, J_{78}, J_{79}, J_{80}, J_{81}, J_{82}, J_{83} \).

**Remark 1.6.** The Johnson solids in the list (d) coincide with the Johnson solids with a circumsphere containing all vertices, except Johnson solids \( J_2, J_4, J_5 \), see Wikipedia [39]. \( J_2 \) (\( J_4, J_5 \), resp.) is excluded from the list (d) by Proposition 1.4, since it has 5 (8, 10, resp.) vertices x’s of pattern \((3, 3, 5) \) \((3, 4, 8), (3, 4, 10), \) resp.) such that \( K_{a_c}(x) > 0 \). Actually, such x’s are exactly the vertices of the unique 5 (8, 10, resp.)-gonal face \( F \) of \( J_2 \) (\( J_4, J_5 \), resp.), where \( F \) is not contained in a hemisphere. So \( F \) is not a regular spherical polygon.

In [29], Milka proved that in spherical three-dimensional space there exists only finite number of (combinatorial types of) convex polyhedra with equian-gular faces, except two infinite families – prisms and antiprisms. However, he did not use the critical side-length \( a_c \). For other classification results on spherical tilings, one refers to [11, 12, 15], for example. For a combinatorial approach to classify tilings of constant curvature spaces, see [10].

We are interested in critical areas of planar graphs in \( \mathcal{P}C_{\geq 1} \). In particular, we propose the following problem.

**Problem 1.7.** What are the following constants:

1. \( \text{Area}_{\text{min}} := \inf_{G \in \mathcal{P}C_{\geq 1}} \text{Area}^{\text{cri}}(G) \), and
2. \( \text{Area}_{\text{max}} := \sup_{G \in \mathcal{P}C_{\geq 1}\setminus \mathcal{T}_{S^2}} (\text{Area}^{\text{cri}}(G)) \).

Both constants in the above problem are attained by some specific graphs in \( \mathcal{P}C_{\geq 1} \), by Theorem 1.1 [1], since all prisms and antiprisms admit spherical tilings with regular spherical polygons and have critical area \( 4\pi \). In this paper, we will give quantitative bounds for the constants in the problem.

The first part of Problem 1.7 is devoted to the minimal critical area in the class \( \mathcal{P}C_{\geq 1} \). We prove the following theorem.

**Theorem 1.8.** \( 8.3755 \times 10^{-2} \leq \text{Area}_{\text{min}} \leq 2.0961 \times 10^{-1} \).

For the upper bound estimate in the theorem, we provide a graph in \( \mathcal{P}C_{\geq 1} \) with small critical area. For the lower bound estimate, we use the following local argument: For any graph \( G \in \mathcal{P}C_{\geq 1} \), we consider the local structures of vertices which attain the critical side-length of the graph. We bound the
critical area of the graph below by the summation of areas of faces incident
to such a vertex, and compute the results case by case. This is amenable
by the result of Ghidelli, Theorem 1.1 (2), which considerably reduces the
cases of possible vertex patterns, up to the facial degree 41.

Next, we consider the second part of Problem 1.7. By Proposition 1.4,
\[ \text{Area}^{\text{cri}}(G) = 4\pi, \quad \text{for any } G \in \mathcal{T}_{S^2}, \]
which attains the maximal critical area in the class \( \mathcal{P}C_{\geq 1} \), by the volume
comparison theorem for Alexandrov spaces with curvature at least 1, see [6].

The problem is to determine the maximum of critical areas in the class \( \mathcal{P}C_{\geq 1} \)
except spherical tilings \( \mathcal{T}_{S^2} \). We prove the following quantitative result.

**Theorem 1.9.** \( 4\pi - 2.5678 \times 10^{-1} \leq \widetilde{\text{Area}}_{\text{max}} \leq 4\pi - 1.6471 \times 10^{-5}. \)

The lower bound estimate follows from the calculation of critical area of
Johnson solid, \( J_{16} \). For the upper bound estimate, one observes that only
local arguments as in the proof of Theorem 1.8 are insufficient. Our strategy
is to reformulate the problem to a new one which can be estimated by local
arguments. One is ready to see that
\[ \widetilde{\text{Area}}_{\text{max}} = 4\pi - \epsilon_{\text{gap}}, \]
where
\[ \epsilon_{\text{gap}} := \inf_{G \in \mathcal{P}C_{\geq 1} \setminus \mathcal{T}_{S^2}} (4\pi - \text{Area}^{\text{cri}}(G)), \]
which is called the gap between the maximal critical area \( 4\pi \) and other
critical areas. Hence the above theorem is equivalent to the gap estimate,
\[ 1.6471 \times 10^{-5} \leq \epsilon_{\text{gap}} \leq 2.5678 \times 10^{-1}. \]

By the Gauss-Bonnet theorem, (2), we have
\[ \epsilon_{\text{gap}} = \inf_{G \in \mathcal{P}C_{\geq 1} \setminus \mathcal{T}_{S^2}} \sum_{x \text{ is a vertex of } G} K_{a_c(G)}(x). \]

Hence for the upper bound estimate, it suffices to obtain the lower bound
estimate of total angle defect for these planar graphs. This new problem fits
to local arguments, and we prove the results by enumerating all cases.

The following is a corollary of Theorem 1.9.

**Corollary 1.10.**
\[ G \in \mathcal{P}C_{\geq 1} \setminus \mathcal{T}_{S^2}, \ 0 < a \leq a_c(G) \implies \text{Area}_a(G) \leq 4\pi - 1.6471 \times 10^{-5}. \]

2. **Preliminaries**

Let \( G = (V, E, F) \) be a finite planar graph. Two vertices are called
neighbors if there is an edge connecting them. We denote by \( \text{deg}(x) \) the
degree of a vertex \( x \), i.e. the number of neighbors of a vertex \( x \), and by
\( \text{deg}(\sigma) \) the degree of a face \( \sigma \), i.e. the number of edges incident to a face \( \sigma \)
(equivalently, the number of vertices incident to \( \sigma \)).
For any spherical regular \( n \)-gon of side length \( a < \frac{2\pi}{n} \), \( \Delta_n(a) \), contained in a hemisphere of \( S^2(1) \), we denote by \( \beta = \beta_{n,a} \) the interior angle at corners of the \( n \)-gon. By the spherical cosine law for angles [2, p. 65],

\[
\cos \frac{a}{2} \sin \frac{\beta}{2} = \cos \frac{\pi}{n}.
\]

One is ready to see that \( \beta_{n,a} \) is monotonely increasing in \( a \) and

\[
\lim_{a \to 0} \beta_{n,a} = \frac{n - 2}{n} \pi,
\]

where the right hand side is the interior angle of a regular \( n \)-gon in the plane.

The area of \( \Delta_n(a) \) is given by

\[
\text{Area}(\Delta_n(a)) = n\beta - \left( n - 2 \right)\pi.
\]

We are ready to prove the Gauss-Bonnet theorem for spherical polyhedral surfaces.

**Proof of Theorem 1.2.** For any face \( \sigma \in F \), the area of \( \sigma \) in \( S_a(G) \) is given by

\[
\deg(\sigma)\beta_{\deg(\sigma),a} - (\deg(\sigma) - 2)\pi.
\]

Hence by the counting argument,

\[
\text{Area}_a(G) = \sum_{\sigma \in F} \deg(\sigma)\beta_{\deg(\sigma),a} - \sum_{\sigma \in F} (\deg(\sigma) - 2)\pi
\]

\[
= \sum_{x \in V} \theta_a(x) - \pi \sum_{\sigma \in F} \deg(\sigma) + 2\pi\#E
\]

\[
= -\sum_{x \in V} K_a(x) + 2\pi(\#V - \#E + \#F) = -\sum_{x \in V} K_a(x) + 4\pi.
\]

This proves Theorem 1.2. \( \square \)

**Proposition 2.1.** \( \mathcal{PC}_{\geq 1} = \mathcal{PC}_{> 0} \).

**Proof.** For any \( G \in \mathcal{PC}_{\geq 1} \), there is a \( a > 0 \) such that \( S_a(G) \in \text{Alex}(1) \). For any face \( \sigma \in F \) with \( \deg(\sigma) = n \), we know that the interior angle of \( \Delta_n(a) \) is greater than that of an Euclidean \( n \)-gon. Consider the Euclidean polyhedral surface \( S(G) \). Note that the total angle at each vertex in \( S(G) \) is less than that in \( S_a(G) \). This yields that \( G \in \mathcal{PC}_{> 0} \). This proves that \( \mathcal{PC}_{\geq 1} \subset \mathcal{PC}_{> 0} \).

For the other direction, let \( G \in \mathcal{PC}_{> 0} \). Note that by \( \[5\] \), for each vertex \( x \in V \), there is a small constant \( a(x) \) such that the total angle \( \theta_{a(x)}(x) < 2\pi \). Since the graph is finite, we can choose a small constant \( a \) such that

\[
S_a(G) \in \text{Alex}(1),
\]

which proves that \( G \in \mathcal{PC}_{\geq 1} \). \( \square \)
For a vertex $x$ of the pattern $(f_1, f_2, \cdots, f_N)$, with $N = \deg(x)$ and $\{f_i\}_{i=1}^N$ are the degrees of faces incident to $x$. Then for any $0 < a \leq \frac{2\pi}{f_N}$, the total angle at the vertex $x$ in $S_a(G)$ for some graph $G$ is given by

$$\theta_a(x) = \sum_{i=1}^{N} 2 \arcsin \frac{\cos \frac{\pi}{f_i} \cos \frac{a}{2}}{\cos \frac{\pi}{2}}.$$  

To determine the critical side-length of the vertex, we have two cases. If $\theta_{\frac{2\pi}{f_N}}(x) < 2\pi$, then $a_c(x) = \frac{2\pi}{f_N}$. Otherwise, $a_c(x)$ is the unique solution to the following equation

$$\theta_a(x) = 2\pi.$$  

Now we prove Proposition 1.4.

**Proof of Proposition 1.4.** (1) $\Rightarrow$ (2) : This is trivial.

(2) $\Rightarrow$ (3) : This follows from the monotonicity of $\theta_a(x)$ in $a$ for any $x \in V$.

(3) $\Rightarrow$ (4) : Since $S_{a_c(G)}(G)$ has the curvature at least 1 and is smooth at each vertex, hence it is locally isometric to a domain in $S^2(1)$. This implies that $S_{a_c(G)}(G)$ is isometric to $S^2(1)$. Hence $\text{Area}^{\text{cri}}(G) = 4\pi$.

(4) $\Rightarrow$ (1) : We know that $S_{a_c(G)}(G)$ has the curvature at least 1 and $\text{Area}^{\text{cri}}(G) = 4\pi$. Hence the rigidity of Bishop-Gromov volume comparison for Alexandrov surfaces with the curvature at least 1 yields that $S_{a_c(G)}(G)$ is isometric to $S^2(1)$, see e.g. [6, Exercise 10.6.12].

Next, we give the proof of Theorem 1.5.

**Proof of Theorem 1.5.** For any $G$ admitting a spherical tiling with regular spherical polygons, $S_{a_c(G)}(G)$ is the spherical tiling. Consider the convex hull $A$ of the vertex set $V$ in $S_{a_c(G)}(G)$ in $\mathbb{R}^3$. We obtain a convex polyhedron $A$ in $\mathbb{R}^3$ such that all faces are regular Euclidean polygons. The classification of such polyhedra in $\mathbb{R}^3$ was obtained by [21, 42, 41]. Any convex polyhedron in $\mathbb{R}^3$ such that all faces are regular Euclidean polygons is one of the following:

(a) the 5 Platonic solids,
(b) the 13 Archimedean solids,
(c) the infinite series of prisms and antiprisms, and
(d) the 92 Johnson solids.

It is easy to see that the examples in (a), (b), (c) admit spherical tilings with regular spherical polygons. For example, the antiprism of $2n$ ($n > 3$) vertices admits a spherical tilings with $2n$ regular triangles and $2$ regular $n$-gons having the vertices $P_i$ ($i = 0, 1, \ldots, 2n-1$) such that (1) the distance between $P_i$ and the north pole is $\pi/2 + (-1)^i \arctan \left( \frac{1}{2} \sqrt{2 - 4 \cos^2 \frac{\pi}{n} + 2 \cos \frac{3\pi}{n}} \right)$, (2) the longitude of $P_i$ is $\pi i/n$, and (3) $P_i$ is adjacent to $P_{i+1 \mod 2n}$ and to $P_{i+2 \mod 2n}$. The side-length for $n = 5$ is indeed the side-length $\arctan 2$ of the regular spherical icosahedron on the unit sphere. These are all computed by a mathematics software Maple. To complete the classification of spherical
tilings with regular spherical polygons \( T_{32} \), it suffices to check Johnson solids case by case using the property (3) in Proposition 1.4, see Table 1 for some part of results. This proves Theorem 1.5. □

| Johnson Solid | the vertex patterns | \( a_c \) |
|---------------|--------------------|--------|
| \( J_1 \)     | \((3,3,3,3),(3,3,4)\) | \( \pi/2 \) |
| \( J_3 \)     | \((3,3,4,4),(3,4,6)\) | \( \pi/3 \) |
| \( J_6 \)     | \((3,3,5,5),(3,5,10)\) | \( \pi/5 \) |
| \( J_{11} \)  | \((3,3,3,3),(3,3,3,5)\) | \( \arctan(2) \) |
| \( J_{19} \)  | \((3,4,4,4),(4,4,8)\) | \( 2\arccos \frac{\sqrt{12-2\sqrt{2(1+3\sqrt{2})}}}{17} \) |
| \( J_{62} \)  | \((3,3,3,3),(3,3,3,5),(3,5,5)\) | \( \arctan(2) \) |
| \( J_n \) (76 ≤ \( n \) ≤ 83) | \((3,4,4,5),(4,5,10)\) | \( 2\arccos \frac{\sqrt{75-10\sqrt{5}15}}{205} \) |

Table 1. The critical side-length \( a_c \) of the Johnson solids having a circumsphere containing all vertices and more than two vertex patterns. See Theorem 1.5 and Remark 1.6.

By Theorem 1.1 [2], for any graph in \( PC_{\geq 0} \) the maximal degree of faces is at most 41. Motivated by this result, we say a vertex pattern \((f_1,f_2,\ldots,f_N)\) is *admissible* if it has positive combinatorial curvature and \( f_N \leq 41 \). We denote by \( AdP \) the set of admissible vertex patterns. The set \( AdP \) consists of the following 342 tuples, derived from the list of vertex patterns with positive combinatorial curvature [9, 8]. Table 2 is obtained from [29, Table 1]

| \((3,3,k), 3 \leq k \leq 41\) | \((3,10,k), 10 \leq k \leq 14\) | \((5,6,k), 6 \leq k \leq 7\) |
| \((3,4,k), 4 \leq k \leq 11\) | \((3,11,k), 11 \leq k \leq 13\) | \((3,3,3,k), 3 \leq k \leq 41\) |
| \((3,5,k), 5 \leq k \leq 41\) | \((4,4,k), 4 \leq k \leq 41\) | \((3,3,4,k), 4 \leq k \leq 11\) |
| \((3,6,k), 6 \leq k \leq 41\) | \((4,5,k), 5 \leq k \leq 19\) | \((3,3,5,k), 5 \leq k \leq 7\) |
| \((3,7,k), 7 \leq k \leq 41\) | \((4,6,k), 6 \leq k \leq 11\) | \((3,4,4,k), 4 \leq k \leq 5\) |
| \((3,8,k), 8 \leq k \leq 23\) | \((4,7,k), 7 \leq k \leq 9\) | \((3,3,3,3,k), 3 \leq k \leq 5\) |
| \((3,9,k), 9 \leq k \leq 17\) | \((5,5,k), 5 \leq k \leq 9\) |

Table 2. Admissible patterns \( AdP \).

under the restriction [29, Table 3], by ignoring the order of numbers in each tuple.

We can calculate critical side-length for all admissible vertex patterns by using the formulas (6) and (7). One is ready to prove the following:
Proposition 2.2.

\[ \max_{p \in \mathcal{A}} a_c(p) = \arccos \frac{-1}{3}. \]

The maximum is attained only at the pattern \( p = (3, 3, 3) \).

In fact, \( \arccos(-1/3) \) is the side-length of the spherical tiling by 4 congruent regular triangles.

3. Proof of Theorem 1.8

In this section, we consider the estimates for the minimal critical area in the class \( \mathcal{PC}_{\geq 1} \) and prove Theorem 1.8.

For the upper bound estimate of the minimal critical area in the class \( \mathcal{PC}_{\geq 1} \), we consider an example in the class which has small critical area, see Figure 1. This example was initially constructed by Ghidelli [13] to show that there exists a graph \( \Gamma \in \mathcal{PC}_{>0} \) having a vertex of pattern \( (3, 7, 41) \).

Actually, the set of vertices in \( \Gamma \) consists of 2 vertices \( u \) of pattern \( (3, 5, 7) \), 6 vertices \( v \) of pattern \( (3, 3, 5, 5) \), 53 vertices \( w \) of pattern \( (3, 3, 5, 7) \), 8 vertices \( x \) of pattern \( (3, 5, 41) \), 11 vertices \( y \) of pattern \( (3, 3, 3, 41) \), and 22 vertices \( z \) of pattern \( (3, 7, 41) \). By the numerical computation,

\[
\begin{align*}
    a_c(\Gamma) &= \min(a_c(u), a_c(v), a_c(w), a_c(x), a_c(y), a_c(z)) \\
               &= \min(0.86961 \cdots, \pi/5, 0.22634 \cdots, 2\pi/41, 0.15291 \cdots, 0.030382 \cdots) \\
               &= a_c(z) = 0.030382 \cdots =: a_0.
\end{align*}
\]

The set of faces in \( \Gamma \) consists of 61 triangles, 15 pentagons, 11 heptagons and a 41-gon. Hence

\[
\text{Area}^{\text{cri}}(\Gamma) = 61 \text{Area}(\Delta_3(a_0)) + 15 \text{Area}(\Delta_5(a_0)) \\
+ 11 \text{Area}(\Delta_7(a_0)) + \text{Area}(\Delta_{41}(a_0)) \\
= 2.0961 \times 10^{-1}.
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{A planar graph \( \Gamma \in \mathcal{PC}_{\geq 1} \) has small critical area. \( \text{Pttn}(u) = (3, 5, 7), \text{Pttn}(v) = (3, 5, 5, 5), \text{Pttn}(w) = (3, 3, 5, 7), \text{Pttn}(x) = (3, 5, 41), \text{Pttn}(y) = (3, 3, 3, 41), \) and \( \text{Pttn}(z) = (3, 7, 41) \).}
\end{figure}
For the lower bound estimate of the minimal critical area, we shall estimate the area from below for any graph $G$ in the class $PC \geq 1$. For the graph $G$, let $x_0$ be a vertex such that $a_c(x_0) = a_c(G)$. It is obvious that

$$\text{Area}^{\text{cri}}(G) \geq \sum_{x_0 \in \sigma_i} \text{Area}_{a_c}(\sigma_i).$$

That is, we give the lower bound estimate via the summation of areas of faces incident to $x_0$. So that

$$(8) \quad \text{Area}_{\text{min}} \geq \min_{\text{Pttn}(x) \in \text{AdP}} \sum_{x \in \sigma_i} \text{Area}_{a_c}(\sigma_i).$$

By the numerical computation, we enumerate all critical side-lengths for admissible vertex patterns. For each admissible vertex pattern, we calculate the summation of areas of faces in the pattern using the critical side-length of this pattern, as in (8). Then the minimum of calculated results gives the lower bound of the minimal critical area, $8.3755 \times 10^{-2}$, see the Appendix for the algorithms (Section 5.2).

4. Proof of Theorem 1.9

In this section, we estimate the first gap $\varepsilon_{\text{gap}}$ and prove Theorem 1.9.

For the lower bound estimate of Theorem 1.9 we calculate the critical area of Johnson solid, $J_{16}$. The set of vertices in $J_{16}$ consists of 10 vertices $x$ of pattern $(3, 3, 4, 4)$ and 2 vertices $y$ of pattern $(3, 3, 3, 3)$. By the numerical computation,

$$a_c(J_{16}) = \min(a_c(x), a_c(y)) = \min(1.0472, 1.1071) = a_c(x) = 1.0472 =: a_0.$$  

Since the set of faces in $J_{16}$ contains 10 triangles and 5 squares,

$$\text{Area}^{\text{cri}}(J_{16}) = 10\text{Area}(\Delta_3(a_0)) + 5\text{Area}(\Delta_4(a_0)) = 4\pi - 2.5678 \times 10^{-1}.$$  

Actually, $\text{Area}^{\text{cri}}(J_{16}) \geq \text{Area}^{\text{cri}}(J)$ for any Johnson solid $J$, according to a numerical computation.

For the upper bound estimate of Theorem 1.9 we are based on (4) and the finiteness of $PC_{>0} \setminus \mathcal{T}_{\geq 2}$. We estimate total angle defect $\sum_{x \in G} K_{a_c(G)}(x)$ from below, for any $G = (V, E, F) \in PC_{>0} \setminus \mathcal{T}_{\geq 2}$. For the graph $G$, let $x_0$ be a vertex such that $a_c(x_0) = a_c(G)$, i.e. for any $y \in V$, $a_c(y) \geq a_c(x_0)$. For simplicity, we denote by $a_0 := a_c(G)$. By Proposition 2.2, $\text{Pttn}(x_0) \neq (3, 3, 3)$ since there are at least two different critical side-lengths for vertices in $G$ by (3) in Proposition 1.4. Now we have two cases: $K_{a_0}(x_0) = 0$ and $K_{a_0}(x_0) > 0$.

Consider the case $K_{a_0}(x_0) = 0$. Since $\text{Area}^{\text{cri}}(G) < 4\pi$, by the Gauss-Bonnet theorem 2, $\sum_{x \in V} K_{a_0}(x) > 0$. Hence, for some vertex $y \in V$,

$$K_{a_0}(y) > 0, \quad \text{and} \quad a_c(y) \geq a_0.$$
Therefore, we obtain a positive lower bound of \( \sum_{x \in V} K_{a_0}(x) \):

\[
\begin{align*}
K_{a_0}(x_0), & \quad (K_{a_0}(x_0) > 0); \\
\min \{K_{a_0}(y) > 0 : y \in V, a_c(y) \geq a_c(x_0)\}, & \quad (K_{a_0}(x_0) = 0).
\end{align*}
\]

By replacing vertices \( x, y \) by vertex patterns \( p, q \) in the lower bound \((9)\) and then minimizing it over \( p \in \mathcal{AdP} \setminus \{(3,3,3)\} \), we obtain the lower bound \( 1.6471 \times 10^{-5} \) given in Theorem 1.9. See the Appendix for the algorithm (Section 5.3).

5. Appendix

We prove Proposition 2.2 in Subsection 5.1 and present symbolic computations and numerical computations in the proof of Theorem 1.8 and Theorem 1.9 The computations are carried out by a mathematics software Maple.

5.1. Proof of Proposition 2.2. For all \( p = (f_1, f_2, \ldots, f_n) \in \mathcal{AdP} \) and \( q = (g_1, g_2, \ldots, g_m) \in \mathcal{AdP} \), we write \( p \leq_{emb} q \) if there are positive integers \( 1 \leq j_1 < j_2 < \cdots < j_n \leq m \) such that \( f_i \leq g_{j_i} \) (\( 1 \leq i \leq n \)). In this case, \( f_n \leq g_m \), because \( f_i \leq f_j \) and \( g_i \leq g_j \) for \( i < j \). The binary relation \( \leq_{emb} \) is a partial ordering on \( \mathcal{AdP} \). Let \( <_{emb} \) be the strict part \( \leq_{emb} \setminus = \).

**Lemma 5.1.** Assume \( p = (f_1, \ldots, f_n) \in \mathcal{AdP} \) and \( q = (g_1, \ldots, g_m) \in \mathcal{AdP} \).

1. \( K_a(p) \) is a strictly decreasing, continuous function of \( a \):

\[
a \in [0, 2\pi/f_n] \mapsto K_a(p) = 2\pi - \sum_{i=1}^{n} 2 \arcsin \frac{\cos \frac{\pi}{f_i}}{\cos \frac{\pi}{2}}.
\]

2. If \( p <_{emb} q \) and \( a \in [0, 2\pi/g_m] \), then \( K_a(p) > K_a(q) \).

**Proof.** As \( a \in [0, 2\pi/f_n] \) and \( f_i \geq 3 \), we have \( \pi/f_i \geq \pi/f_n \geq a/2 \). So, \( \arcsin(\cos(\pi/f_i)/\cos(a/2)) \) is defined. The other assertions are clear. \( \square \)

**Lemma 5.2.** If \( p = (f_1, \ldots, f_n) \in \mathcal{AdP} \), \( a_c(p) \) of Definition 1.3 is well-defined. Moreover, if \( K_{2\pi/f}(p) > 0 \), then \( a_c(p) = 2\pi/f_n \). If \( K_{2\pi/f}(p) \leq 0 \), then \( a_c(p) \) is the unique solution \( a \) such that \( K_a(p) = 0 \) and \( a \in (0, 2\pi/f_n] \).

**Proof.** By Lemma 5.1 (1), the argument \( U \) of the maximum in (3) is a compact set. By the definition of admissible patterns and the limiting behavior in (5), one is ready to see that \( a_c(p) = \max U > 0 \). The latter part is due to Lemma 5.1 (1). \( \square \)

**Lemma 5.3.** \( p <_{emb} q \implies a_c(p) > a_c(q) \).

**Proof.** Let \( p = (f_1, \ldots, f_n) \) and \( q = (g_1, \ldots, g_m) \). Then \( a_c(p) = \max \{a \in [0, 2\pi/f_n] : K_a(p) \geq 0\} \geq \max \{a \in [0, 2\pi/g_m] : K_a(p) \geq 0\} \) by \( f_n \leq g_m \). This is greater than or equal to \( \max \{a \in [0, 2\pi/g_m] : K_a(q) \geq 0\} = a_c(q) \) by Lemma 5.1 (2). Moreover, \( a_c(p) > a_c(q) \) by Lemma 5.1. \( \square \)
Proof of Proposition 2.2. Among \(\mathcal{A}d\mathcal{P}\), the pattern \((3, 3, 3)\) is the least with respect to the partial ordering \(\leq_{emb}\). By Lemma 5.3, the conclusion follows. \(\square\)

5.2. An algorithm to compute the lower bound of \(\text{Area}_{\min}\) of Theorem 1.8. We will compute

\[
\min_{(f_1, \ldots, f_n) \in \mathcal{A}d\mathcal{P}} \sum_{i=1}^{n} \text{Area}(\Delta f_i(a_c(p))).
\]

To improve the accuracy of numerical computation of the critical side-length \(a_c(p)\) of vertex patterns \(p \in \mathcal{A}d\mathcal{P}\), we enumerate

\[
M := \{ (f_1, \ldots, f_n) \in \mathcal{A}d\mathcal{P} \mid K_{2\pi/ f_n}(f_1, \ldots, f_n) > 0 \}.
\]

Lemma 5.4. \(M\) consists of the following 103 patterns: \((3, 3, k)\) \((5 \leq k \leq 41)\), \((3, 4, k)\) \((7 \leq k \leq 41)\), and \((3, 5, k)\) \((11 \leq k \leq 41)\).

Proof. For any increasing sequence \((f_1, \ldots, f_n)\) of positive integers,

\[
K_{2\pi/ f_n}(f_1, \ldots, f_n) = \pi - 2 \sum_{i=1}^{N-1} \arcsin \frac{\cos(\pi/ f_i)}{\cos(\pi/ f_n)}.
\]

Note that for any \((f_1, f_2, f_3) \in \mathcal{A}d\mathcal{P}\)

\[
(10) \quad K_{2\pi/ f_5}(f_1, f_2, f_3) \leq 0 \iff \cos^2 \frac{\pi}{f_1} + \cos^2 \frac{\pi}{f_2} \geq \cos^2 \frac{\pi}{f_3}.
\]

\(M\) contains the 103 patterns, by (10) and \(K_{2\pi/ 4}(3, 3, 4) = K_{2\pi/ 6}(3, 4, 6) = K_{2\pi/ 10}(3, 5, 10) = 0\). We will prove that \(M\) is contained in the 103 patterns. For two increasing sequences \((f_1, \ldots, f_n)\) and \((g_1, \ldots, g_m)\) of positive integers, define a partial ordering

\[
(f_1, \ldots, f_n) \leq (g_1, \ldots, g_m) \iff (f_1, \ldots, f_{n-1}) \leq_{emb} (g_1, \ldots, g_{m-1}), f_n \geq g_m.
\]

Then,

\[
(f_1, \ldots, f_n) \leq (g_1, \ldots, g_m) \implies K_{2\pi/ f_n}(f_1, \ldots, f_n) \geq K_{2\pi/ g_m}(g_1, \ldots, g_m).
\]

It is sufficient to show \(K_{2\pi/ f_n}(f_1, \ldots, f_N) \leq 0\) for all \(\leq\)-minimal \((f_1, \ldots, f_N) \in \mathcal{A}d\mathcal{P} \setminus M\). All such \((f_1, \ldots, f_N)\) are \((3, 6, 41)\), \((4, 4, 41)\), and \((3, 3, 3, 41)\).

By (10), \(K_{2\pi/ 41}(3, 6, 41), K_{2\pi/ 41}(4, 4, 41) < 0\). The vertex pattern of the 41-gonal antiprism is \(p = (3, 3, 3, 41) \in \mathcal{A}d\mathcal{P}\). By Theorem 1.5, the 41-gonal antiprism is in \(T_{2\pi}^2\). By Proposition 1.4, \(K_{a_c(p)}(p) = 0\). By Lemma 5.1 (1) and \(a_c(p) \leq 2\pi/ 41, K_{2\pi/ 41}(3, 3, 3, 41) \leq 0\). \(\square\)

For all 239 patterns \(p = (f_1, \ldots, f_n) \in \mathcal{A}d\mathcal{P} \setminus M\), in order to compute \(a_c(p)\), we numerically solve equations \(K_a(p) = 0\) and \(a \in (0, 2\pi/ f_n)\), by using a floating-point arithmetic solver \texttt{fsolve} already built in Maple. In Maple, the number of digits carried in float is 10, unless otherwise we change. However, because of numerical error, \texttt{fsolve} is not able to solve the equation for \(p = (3, 5, 9)\). In this case, we use a symbolic solver \texttt{solve} built in Maple, to solve the equation \(K_a(3, 5, 9) = 0\). Then we verify the output
of $\text{solve}$, by computing the numerical value of the output of $\text{solve}$, and plotting a monotone function $2\pi - \sum_{i=1}^{n} 2 \arcsin \left( \frac{\cos(\pi/f_i)}{\cos(a/2)} \right)$ over $a \in (0, 2\pi/f_n)$.

We should compute the minimum of a finite list of numbers, under numerical error. To control the precision of computation of $\text{Maple}$, we will compute the second minimum, and the third minimum, and so on. These will be computed by a sorting algorithms, which puts elements of the list in a certain order. For the complexity and mathematical properties of various sorting algorithm, see [36].

By the computation,

\[
\min_{(f_1, \ldots, f_n) \in \mathcal{A}_P} \sum_{i=1}^{n} \text{Area}(\Delta_{f_i}(a_c(f_1, \ldots, f_n))) = 8.3755 \cdots \times 10^{-2}
\]

which is achieved only by $p = (3, 11, 13)$. It is sufficiently distant from the second minimum $1.2823 \cdots \times 10^{-1}$, which is achieved only by $p = (3, 7, 41)$. This gives the lower bound of $\text{Area}_{\min}$ given in Theorem 1.8.

5.3. An algorithm to compute the upper bound of $\text{Area}_{\max}$ of Theorem 1.9. Based on the inequality (9) and the discussion just after it, we will compute the minimum element in the following finite set

\[
\bigcup_{p \in \mathcal{A}_P, p \neq (3,3,3)} \left( \left\{ K_{a_c(p)}(p) : K_{a_c(p)}(p) > 0 \right\} \cup \left\{ K_{a_c(q)}(q) > 0 : q \in \mathcal{A}_P, a_c(q) \geq a_c(p), K_{a_c(p)}(p) = 0 \right\} \right)
\]

and the set of $(p, q)$ with $K_{a_c(p)}(q)$ is the minimum.

To decide the condition $K_{a_c(p)}(p) = 0$, we have only to check $p \notin \mathcal{M}$, because of Lemma 5.4. This causes no numerical error.

Even if $K_{a_c(p)}(q) = 0$, a numerical computation of $K_{a_c(p)}(q)$ could result in a very small positive, erroneous value. To circumvent this, we use the following:

**Definition 5.5.** Let the set $Z$ of 9 2-sets $\{p, q\}$ consist of $\{(3, 4, 5), (4, 4, 4)\}$ and any two vertex patterns of Johnson solid $J_n$ for some $n$ with $n = 1, 3, 6, 11, 19, 62$ or $76 \leq n \leq 83$. See Table 7.

**Lemma 5.6.** For any $p \in \mathcal{A}_P \setminus \mathcal{M}$ and for any $q \in \mathcal{A}_P$, we have $K_{a_c(p)}(q) = 0$, if $p = q$ or $\{p, q\} \in Z$.

**Proof.** By Lemma 5.4, if $p = q$, then $K_{a_c(p)}(p) = 0$. Consider the case that $\{p, q\} = \{(3, 4, 5), (4, 4, 4)\} \in Z$. Note that $a_c(4, 4, 4) = 2 \arccos \left( \sqrt{6}/3 \right)$. We claim $K_{a_c(4,4,4)}(3, 4, 5) = 0$. It is because

\[
K_{a_c(4,4,4)}(3, 4, 5) = \frac{4\pi}{3} - 2 \arcsin \frac{\sqrt{6}}{4} - 2 \arcsin \frac{\sqrt{5} + 1}{8} \sqrt{6} = 0,
\]
if and only if
\[ \frac{2}{3} \pi - \arcsin \left( \sqrt{6}/4 \right) = \arcsin \left( \left( \sqrt{5} + 1 \right) \sqrt{6}/8 \right), \]
which is equivalent to
\[ \sin \left( \frac{\pi}{3} + \arcsin \left( \sqrt{6}/4 \right) \right) = \left( \sqrt{5} + 1 \right) \sqrt{6}/8. \]

So \( K_{d,\{4,4,4\}}(3, 4, 5) = 0 \). Because \( K_d(3, 4, 5) \) is monotone, we have \( a_c(4, 4, 4) = a_c(3, 4, 5) \). So, \( K_{a_c,\{3,4,5\}}(4, 4, 4) = K_{a_c,\{4,4,4\}}(4, 4, 4) = 0 \). In the other case, \( K_{a_c,p}(q) = 0 \) is by Theorem 1.5.

In a numerical computation of the minimum of the finite set \( \{12\} \), numerical errors of \( K_{a_c,p}(q) \) and \( a_c(p) \) matter for deciding the two inequalities \( K_{a_c,p}(q) > 0 \) and \( a_c(q) \geq a_c(p) \) in \( \{12\} \). So, we will numerically compute the triples \((p, q, k)\) in the following set \( S(\epsilon) \) such that the third entry is the first minimum or the second minimum among all \( k' \) with \((p', q', k') \in S(\epsilon)\).

\[ S(\epsilon) := \{(p, p, 2\pi / 41) : p = (f_1, f_2, 41) \in M \} \]
\[ \cup \left\{ (p, q, K_{a_c,p}(q)) : K_{a_c,p}(q) > \epsilon, q \in \text{AdP} \setminus \{p\}, \{p, q\} \notin Z, a_c(q) \geq \epsilon + a_c(p), p \in \text{AdP} \setminus M \setminus \{(3,3,3)\} \right\}, \]

where \( \epsilon \) of the two inequalities \( K_{a_c,p}(q) > \epsilon \) and \( a_c(q) \geq \epsilon + a_c(p) \) is any real number.

Note that
\[ |S(\epsilon)| \geq (\sharp M) + (\sharp \text{AdP} - 1) \cdot (\sharp \text{AdP} - \sharp M - 1) - 2\sharp Z = 81243. \]

As the “margin” \( \epsilon \in \mathbb{R} \) of the inequalities decreases, the set \( S(\epsilon) \) becomes large. A numerical computation shows that the smallest third entries of \( S(\pm 10^{-5}) \) are both \( 1.6471 \cdots \times 10^{-5} \) and are achieved only by \( K_{a_c,\{3,7,29\}}(3, 9, 16) \), while the second smallest third entries of \( S(\pm 10^{-5}) \) are both \( 1.7919 \times 10^{-5} \) and are achieved only by \( K_{a_c,\{4,4,28\}}(5, 5, 9) \). The numerical computations are done by Maple with the number of digits carried in float being 10. The smallest third entries \( 1.6471 \cdots \times 10^{-5} \) are sufficiently distant from the second smallest third entries \( 1.7919 \times 10^{-5} \). So, the minimum of \( S(0) = \{12\} \) is \( 1.6471 \cdots \times 10^{-5} \) and is achieved only by \( K_{a_c,\{3,7,29\}}(3, 9, 16) \). Hence

\[ \tilde{\text{Area}}_{\text{max}} \leq 4\pi - K_{a_c,\{3,7,29\}}(3, 9, 16) = 4\pi - 1.6471 \cdots \times 10^{-5}. \]

This completes the proof of Theorem 1.9.

The critical side-lengths of the argmin vertex patterns \( p = (3, 7, 29), q = (3, 9, 16) \) are very close: \( 0.14267 \cdots = a_c(3, 7, 29) < a_c(3, 9, 16) = 0.14269 \cdots \) and there is no \( r \in \text{AdP} \) such that \( a_c(3, 7, 29) < a_c(r) < a_c(3, 9, 16) \), according to a numerical computation.

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