On Global Boundedness, Stability and Almost Periodicity of Solutions for Heat Equations

By

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Abstract. This paper is concerned with the existence, stability and almost periodicity of global bounded solutions of linear and semilinear parabolic equations. We give a rather complete characterization to the linear equation \( u_t - \Delta u = \lambda u + f(x, t) \), in terms of the parameter \( \lambda \), of whether or not the global bounded solutions exist. Also, we obtain a complete characterization on the stability of global bounded solutions.

Key Words and Phrases. Global boundedness, Almost periodicity, Stability, Heat equations.

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1. Introduction

1.1. Statement of the main results

The purpose of this paper is to study the existence, stability and almost periodicity of global bounded solutions of the following semilinear parabolic equations subject to Dirichlet boundary condition on the whole real line

\[
\begin{aligned}
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u &= g(x, t, u) \quad \text{in } \Omega \times \mathbb{R}, \\
u(x, t) &= 0 \quad \text{on } \partial \Omega \times \mathbb{R},
\end{cases}
\end{aligned}
\]  

(1.1)

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with \( C^1 \) boundary \( \partial \Omega \), \( g \) is a measurable function in \( \Omega \times \mathbb{R} \times \mathbb{R} \) and satisfies some structural conditions. We also pay close attention to the following linear case:

\[
\begin{aligned}
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u &= \lambda u + f(x, t) \quad \text{in } \Omega \times \mathbb{R}, \\
u(x, t) &= 0 \quad \text{on } \partial \Omega \times \mathbb{R},
\end{cases}
\end{aligned}
\]  

(1.2)

where \( \lambda \in \mathbb{R} \) and \( f \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)) \) is a given function.

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For \(-\infty \leq s < \tau \leq +\infty\), denote \(Q^s_\tau = \Omega \times (s, \tau)\). We write \(u \in \dot{W}^{1,1}_2(Q^s_\tau)\) to mean \(u \in L^2((s, \tau); H^1_0(\Omega))\) and \(\partial u / \partial t \in L^2(Q^s_\tau)\). We say \(u\) is a solution of the problem (1.1) on \(Q^s_\tau\), provided that for any \(-\infty < s < \tau < +\infty\), \(u\) is a solution of (1.1) on \(Q^s_\tau\); namely, \(u \in C([s, \tau]; L^2(\Omega)) \cap L^2((s, \tau); H^1_0(\Omega))\) and for any function \(\varphi \in \dot{W}^{1,1}_2(Q^s_\tau)\) with \(\varphi(\cdot, s)|_\Omega = \varphi(\cdot, \tau)|_\Omega = 0\), the following integral equality holds

\[
\iint_{Q^s_\tau} \left(-u \frac{\partial \varphi}{\partial t} + \nabla u \nabla \varphi - g(x, t, u) \varphi \right) dx dt = 0.
\]

**Definition 1.1.** A function \(u\) is called a global \(L^2(\Omega)\)-bounded solution of the problem (1.1), if \(u\) is a solution of (1.1) on \(Q^s_\tau\) satisfying

\[
\sup_{t \in \mathbb{R}} \int_\Omega |u(x, t)|^2 dx < +\infty.
\]

Next we introduce definitions of almost periodic functions as in [12].

**Definition 1.2.** Let \(X\) be a Banach space. We say that a function \(u(\cdot, t) \in C(\mathbb{R}; X)\) is \(X\) almost periodic, denoted by \(u(\cdot, t) \in AP(X)\), if for any \(\varepsilon > 0\), the set

\[
T(\varepsilon, u) = \left\{ \tau \in \mathbb{R} : \sup_{t \in \mathbb{R}} ||u(\cdot, t + \tau) - u(\cdot, t)||_X < \varepsilon \right\}
\]

is relatively dense, i.e., there is a number \(l = l(\varepsilon) > 0\) such that any interval of length \(l\) contains at least one number from \(T(\varepsilon, u)\).

**Definition 1.3.** Let \(1 \leq q < +\infty\). We say that a function \(u(\cdot, t) \in L^q_{loc}(\mathbb{R}; X)\) is \(X\) almost periodic in the sense of Stepanov, denoted by \(u(\cdot, t) \in S^qAP(X)\), if for any \(\varepsilon > 0\), the set

\[
T(\varepsilon, u) = \left\{ \tau \in \mathbb{R} : \sup_{t \in \mathbb{R}} \left(\int_0^1 ||u(\cdot, t + \tau + s) - u(\cdot, t + s)||_X^q ds\right)^{1/q} < \varepsilon \right\}
\]

is relatively dense.

Obviously, if \(u(\cdot, t) \in AP(X)\), then \(u(\cdot, t) \in S^qAP(X)\), but not vice versa. Arrange the eigenvalues of the elliptic problem

\[
\begin{aligned}
-\Delta \psi(x) &= \lambda \psi(x) \quad \text{in } \Omega, \\
\psi(x) &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

in an ascending order \(0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots\). Let \(\{\psi_i\}_{i=1}^{+\infty}\) be the orthonormal basis of \(L^2(\Omega)\), and \(\psi_i(x) \in H^1_0(\Omega)\) be an eigenfunction of the problem (1.5)
corresponding to $\lambda_i$. Denote

$$f_i(t) = \int_\Omega f(x, t)\psi_i(x)\,dx, \quad F_2 = \sup_{i \in \mathbb{R}} \iint_{Q_{t_1}^{t_2}} |f(x, s)|^2\,dx\,ds.$$ 

Throughout this paper, we denote by $C$ a positive constant depending only on $\lambda$, $N$ and $\Omega$.

Our main results are the following theorems.

**Theorem 1.1.** Assume that $F_2 < +\infty$ and $\lambda_{i_0} < \lambda < \lambda_{i_0 + 1}$ for some positive integer $i_0$. Let

$$u_i(t) = \begin{cases} -\int_t^{+\infty} e^{(\lambda - \lambda_i)(t-s)}f_i(s)\,ds, & i = 1, 2, \ldots, i_0, \\ \int_t^{-\infty} e^{(\lambda - \lambda_i)(t-s)}f_i(s)\,ds, & i = i_0 + 1, i_0 + 2, \ldots. \end{cases}$$

Then

(i) $u(x, t) = \sum_{i=1}^{+\infty} u_i(t)\psi_i(x)$ is the unique global $L^2(\Omega)$-bounded solution of the problem (1.2), and we have the estimate

$$\sup_{i \in \mathbb{R}} \|\nabla u(\cdot, t)\|_{L^2(\Omega)}^2 + \sup_{i \in \mathbb{R}} \left\| \frac{\partial u(\cdot, s)}{\partial s} \right\|_{L^2(Q_{t_1}^{t_2})}^2 \leq CF_2. \tag{1.6}$$

(ii) Suppose $w$ is a solution of the problem (1.2) on $Q_{t_0}^{+\infty}$ with the initial value $w(x, t_0) = w_0(x) = \sum_{i=1}^{+\infty} a_i\psi_i(x) \in L^2(\Omega)$. If there exists a positive integer $j_0 \leq i_0$ such that $a_{j_0} \neq u_{j_0}(t_0)$, then

$$\|w(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \geq |a_{j_0} - u_{j_0}(t_0)|e^{(\lambda_{j_0} - \lambda)(t_0 - t)}, \quad t \geq t_0. \tag{1.7}$$

That is to say, $u$ is unstable for $\lambda_{j_0} < \lambda < \lambda_{j_0 + 1}$.

**Theorem 1.2.** Assume that $F_2 < +\infty$ and $\lambda < \lambda_1$. Let

$$u_i(t) = \int_t^{-\infty} e^{(\lambda - \lambda_i)(t-s)}f_i(s)\,ds, \quad i = 1, 2, \ldots.$$ 

Then

(i) $u(x, t) = \sum_{i=1}^{+\infty} u_i(t)\psi_i(x)$ is the unique global $L^2(\Omega)$-bounded solution of the problem (1.2), and the estimate (1.6) still holds.

(ii) Furthermore, $u$ is uniformly, globally and exponentially stable. More precisely, if $w$ is a solution of the problem (1.2) on $Q_{t_0}^{+\infty}$ with the initial value $w(x, t_0) = w_0(x) \in L^2(\Omega)$, then

$$\|w(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \leq \|w_0(\cdot) - u(\cdot, t_0)\|_{L^2(\Omega)}e^{(\lambda_{i_0} - \lambda)(t_0 - t)}, \quad t \geq t_0. \tag{1.8}$$

**Theorem 1.3.** Assume that $F_2 < +\infty$ and there exists a positive integer $k_0$ such that $\lambda = \lambda_{k_0}$.

(i) If $\int_0^t f_{k_0}(s)\,ds$ is unbounded on $\mathbb{R}$, then the problem (1.2) has no global $L^2(\Omega)$-bounded solution.
(ii) If \( \int_0^1 f_k(s) ds \) is bounded on \( R \), then the problem (1.2) admits infinitely many global \( L^2(\Omega) \)-bounded solutions, and any global bounded solution \( u \) necessarily has the form \( u(t) = \sum_{i=1}^{+\infty} u_i(t) \psi_i(x) \), where

\[
u_i(t) = \begin{cases} -\int_t^{+\infty} e^{(\lambda_i - \hat{\lambda}_1)(t-s)} f_i(s) ds, & i = 1, 2, \ldots, k_0 - 1, \\ \int_0^t f_k(s) ds + c, & i = k_0, \\ \int_{-\infty}^t e^{(\lambda_i - \hat{\lambda}_1)(t-s)} f_i(s) ds, & i = k_0 + 1, k_0 + 2, \ldots, 
\end{cases}
\]

and \( c \in R \) is any constant. Moreover, we have the estimates

\[
\sup_{t \in R} \|Vu(\cdot, t)\|^2_{L^2(\Omega)} \leq C \left( F_2 + \sup_{t \in R} \left( \int_0^t f_k(s) ds + e \right)^2 \right),
\]

and

\[
\sup_{t \in R} \left\| \frac{\partial u(\cdot, s)}{\partial s} \right\|^2_{L^2(\Omega(t, t))} \leq CF_2.
\]

(iii) If \( u \) is a global \( L^2(\Omega) \)-bounded solution of the problem (1.2) with \( \lambda = \hat{\lambda}_1 \), then \( u \) is stable but not asymptotically stable.

(iv) If \( u \) is a global \( L^2(\Omega) \)-bounded solution of the problem (1.2) with \( \lambda = \hat{\lambda}_k \) and \( k > 1 \), then \( u \) is unstable.

**Theorem 1.4.** Suppose \( \lambda \in R \), and \( u \) is the global \( L^2(\Omega) \)-bounded solution of the problem (1.2).

(i) If \( f \in S^2 AP(L^2(\Omega)) \), then \( u \in AP(H^1_0(\Omega)) \) and \( \partial u/\partial t \in S^2 AP(L^2(\Omega)) \).

(ii) If \( f \in AP(L^2(\Omega)) \), then \( u \in AP(H^1_0(\Omega)) \) and \( \partial u/\partial t \in AP(L^2(\Omega)) \).

The following assumptions will be accepted in regard to the problem (1.1):

(A1) \( g \) is a measurable function in \( \Omega \times R \times R \) satisfying one of the following inequalities, either

\[
-\infty < \hat{\sigma} \leq \lambda_1,
\]

or else

\[
\lambda_{i_0} < \hat{\sigma} \leq \lambda_{i_0+1}
\]

for some positive integer \( i_0 \). Here

\[
\hat{\sigma} = \text{essinf}_{(x, t) \in \Omega \times R, u \neq v \in R} \frac{g(x, t, u) - g(x, t, v)}{u - v},
\]

\[
\hat{\sigma} = \text{esssup}_{(x, t) \in \Omega \times R, u \neq v \in R} \frac{g(x, t, u) - g(x, t, v)}{u - v}.
\]
(A2) $g(x, t, \cdot)$ is differentiable at $u = 0$ uniformly in $\Omega \times R$, i.e.,

$$\lim_{u \to 0} \sup_{x \in \Omega, t \in R} \left| \frac{g(x, t, u) - g(x, t, 0)}{u} - \frac{\partial g}{\partial u}(x, t, 0) \right| = 0.$$ 

Define the function

$$\sigma(x, t, u) = \begin{cases} 
\frac{g(x, t, u) - g(x, t, 0)}{u}, & (x, t) \in Q_{-\infty}^{+\infty}, 0 \neq u \in R, \\
\frac{\partial g}{\partial u}(x, t, 0), & (x, t) \in Q_{-\infty}^{+\infty}, u = 0.
\end{cases}$$

Obviously, $\sigma(x, t, u) \in L^\infty(\Omega \times R \times R)$.

**Theorem 1.5.** Suppose that (A1) and (A2) are fulfilled and $g(x, t, 0) \in L^\infty(R; L^2(\Omega))$. Then

(i) There exists uniquely a global $L^2(\Omega)$-bounded solution $u$ to the problem (1.1).

(ii) If (1.11) holds, then the global $L^2(\Omega)$-bounded solution $u$ is uniformly, globally and exponentially stable, that is to say, for any $t_0 \in R$ and any $v_0(x) \in L^2(\Omega)$,

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^2(\Omega)} \leq \|u(\cdot, t_0) - v_0(\cdot)\|_{L^2(\Omega)} e^{(\lambda_1 - \sigma)(t_0 - t)}, \quad t \geq t_0,$$

where $v(x, t)$ is the weak solution of (1.1) on $Q_{t_0}^{+\infty}$ with the initial condition $v(x, t_0) = v_0(x)$.

(iii) If (1.12) holds, then the global $L^2(\Omega)$-bounded solution $u$ is unstable.

(iv) Furthermore, if $\sigma(\cdot, t, \cdot) \in AP(L^\infty(\Omega \times R))$ and $g(\cdot, t, 0) \in AP(L^2(\Omega))$, then $u(\cdot, t) \in AP(L^2(\Omega)) \cap S^2 AP(H^1_0(\Omega))$.

1.2. A brief survey of the related literature

The existence of global bounded solutions, and almost periodic solutions of the problem (1.2) has been of great interest to many authors over the past 30 years. In [1, 6, 11, 13–15, 18, 19], the authors consider the following abstract equation

$$(1.13) \quad u'(t) = Au(t) + h(t) \quad t \in R,$$

where $A : D(A) \subset X \to X$ is the generator of $C_0$-semigroup $U(t)$, and $h \in BC(R; X)$ ($X$ is a Banach space and $BC(R; X)$ stands for the space of all bounded and continuous functions on $R$). Prüss [18] proved that for each $h \in BC(R; X)$, there is precisely one global bounded mild solution $u \in BC(R; X)$.
of (1.13) if and only if

\[ \{ \mu \in \mathbb{C} : |\mu| = 1 \} \subset \rho(U(1)), \]

where \( \rho(U(1)) \) denotes the resolvent set of \( U(1) \). Moreover, \( u \) can be represented by

\begin{equation}
(1.14) \quad u(t) = \int_{-\infty}^{t} U(t - \tau)P h(\tau)d\tau - \int_{t}^{+\infty} U(t - \tau)(I - P)h(\tau)d\tau,
\end{equation}

where \( P \) is a dichotomic projection of \( U(t) \). Let \( X = L^2(\Omega), \quad A = A + \lambda : H^1_0(\Omega) \cap H^2(\Omega) \to L^2(\Omega), \) and \( h(t) = f(\cdot,t) \). The above result implies that (1.2) admits uniquely a global \( L^2(\Omega) \)-bounded solution \( u \) if \( \lambda \neq \lambda_i \) for any \( i \in \mathbb{N}^+ \). This is in accordance with the result on the existence of global bounded solutions described in Theorem 1.1 and Theorem 1.2. However, our expression of the global bounded solutions, which is deduced by using the Galerkin method, is more direct compared with the abstract integral form (1.14). In addition, we obtain the results on the stability and instability of the global bounded solutions.

Provided that \( A \) is a sectorial operator and \( \text{sp}(h) \cap \sigma(A) = \emptyset \) (where \( i = \sqrt{-1} \), and \( \sigma(A) \) denotes the spectrum of \( A \)), the authors showed in [13] that (1.13) has a unique mild solution \( u \in BC(\mathbb{R};X) \) such that \( \text{sp}(u) \subset \text{sp}(h) \), where \( \text{sp}(h) \) stands for the Carleman spectrum of \( h \). What is more, the results in [13] imply that if there exists a positive integer \( k_0 \) such that \( \lambda = \lambda_{k_0} \) and \( \int_0^t f_{k_0}(s)ds \) is bounded on \( \mathbb{R} \), then (1.2) admits a unique global \( L^2(\Omega) \)-bounded solution \( u \) satisfying \( \text{sp}(u) \subset \text{sp}(f) \). However we give an exact expression of infinitely many global \( L^2(\Omega) \)-bounded solutions to (1.2) in Theorem 1.3 provided that there exists a positive integer \( k_0 \) such that \( \lambda = \lambda_{k_0} \) and \( \int_0^t f_{k_0}(s)ds \) is bounded on \( \mathbb{R} \). Among all of the solutions, only one solution satisfies \( \text{sp}(u) \subset \text{sp}(f) \). Moreover, we show in Theorem 1.3 that the problem (1.2) has no global \( L^2(\Omega) \)-bounded solution in the case of \( \lambda = \lambda_{k_0} \) and \( \int_0^t f_{k_0}(s)ds \) is unbounded on \( \mathbb{R} \). This together with Theorem 1.4 improves the results in [6, 13–15, 18, 19].

It is worth to point out that we give a complete characterization on the stability of global bounded solutions based on the expression of solutions in Theorems 1.1–1.3. To our knowledge this is not discovered in the previous results on the subject. By the way, the hypothesis on \( h \) in [11, 13, 14, 18, 19] is \( h \in BC(\mathbb{R};X) \), while the assumption on \( f \) in (1.2) in the current paper is \( f \in L^2_{\text{loc}}(\mathbb{R};L^2(\Omega)) \) with \( \sup_{t \in \mathbb{R}} \int_{\Omega} |f(x,s)|^2dxds < +\infty \). Clearly, this condition on \( f \) is a little weaker.

The authors studied in [5, 16, 20] the existence of almost periodic solutions to semilinear parabolic equations, where the global \( L^2(\Omega) \)-boundedness was merely treated as a hypothesis and the authors offered no evidence for this.
We give in Theorem 1.5 a positive answer to the existence of global $L^2(\Omega)$-bounded solutions to the problem (1.1). It is worth mentioning that the existence of global bounded solutions and almost periodic solutions for abstract neutral differential equations have been investigated in many papers (see for instance [2–4, 8–10, 17], but our conditions (see (A1)) are more explicit, and we also give the stability of the global $L^2(\Omega)$-bounded solutions and almost periodic solutions.

2. Proofs of Theorems

Proof of Theorem 1.1. (i) We first claim $u \in L^\infty(\mathbb{R}; H_0^1(\Omega))$. To prove this, we compute by Hölder’s inequality that

$$
\lambda_i(u_i(t))^2 \leq \lambda_i \int_{-\infty}^{t} e^{(\lambda - \lambda_i)(t-s)} ds \int_{-\infty}^{t} e^{(\lambda - \lambda_i)(t-s)} f_i^2(s) ds
$$

$$
= \frac{\lambda_i}{\lambda_i - \lambda} \sum_{k=0}^{+\infty} e^{(\lambda - \lambda_i)(t-s)} f_i^2(s) ds \quad i \geq i_0 + 1.
$$

Recalling that $\lambda_i \leq \lambda_{i+1}$ ($i = 1, 2, \ldots$) and $\lambda_0 < \lambda < \lambda_{i_0+1}$, we derive

$$
\frac{\lambda_i}{\lambda_i - \lambda} = 1 + \frac{\lambda}{\lambda_i - \lambda} \leq 1 + \frac{\lambda}{\lambda_{i_0+1} - \lambda} = \frac{\lambda_{i_0+1}}{\lambda_{i_0+1} - \lambda} \quad \text{for } i \geq i_0 + 1,
$$

and

$$
e^{(\lambda - \lambda_i)(t-s)} \leq e^{\frac{\lambda}{\lambda_{i_0+1} - \lambda} (t-s)} \quad \text{for } s \in [t-k-1, t-k], i \geq i_0 + 1.
$$

Consequently it follows from the above inequalities that

$$
\lambda_i(u_i(t))^2 \leq \frac{\lambda_{i_0+1}}{\lambda_{i_0+1} - \lambda} \sum_{k=0}^{+\infty} e^{\frac{\lambda}{\lambda_{i_0+1} - \lambda} (t-k-s)} f_i^2(s) ds, \quad i \geq i_0 + 1.
$$

Hence we obtain for any $n > i_0 + 1$,

\begin{equation}
\sum_{i=i_0+1}^{n} \lambda_i(u_i(t))^2 \leq \frac{\lambda_{i_0+1}}{\lambda_{i_0+1} - \lambda} \sum_{k=0}^{+\infty} e^{\frac{\lambda}{\lambda_{i_0+1} - \lambda} (t-k-s)} \left( \sum_{i=i_0+1}^{n} f_i^2(s) ds \right)
\end{equation}

\begin{align*}
&\leq \frac{\lambda_{i_0+1}}{\lambda_{i_0+1} - \lambda} \sum_{k=0}^{+\infty} e^{\frac{\lambda}{\lambda_{i_0+1} - \lambda} (t-k-s)} F_2 \\
&= \frac{\lambda_{i_0+1}}{(\lambda_{i_0+1} - \lambda)(1 - e^{\frac{\lambda}{\lambda_{i_0+1} - \lambda}})} F_2.
\end{align*}
Similarly, we have

\[
\sum_{i=1}^{l_0} \lambda_i (u_i(t))^2 \leq \sum_{i=1}^{l_0} \lambda_{i_0} \int_t^{+\infty} e^{(\lambda - \lambda_{i_0})(t-s)} ds \int_t^{+\infty} e^{(\lambda - \lambda_{i_0})(t-s)} f_i^2(s) ds \\
\leq \frac{\lambda_{i_0}}{\lambda - \lambda_{i_0}} \sum_{i=1}^{l_0} \sum_{k=0}^{l_0} e^{-k(\lambda - \lambda_{i_0})} \int_t^{t+k+1} f_i^2(s) ds \\
\leq \frac{\lambda_{i_0} F_2}{(\lambda - \lambda_{i_0})(1 - e^{\lambda_{i_0} - \lambda})}.
\]

From (2.1), (2.2) and the arbitrariness of \( n \) and \( t \), we find that our claim holds and

\[
\sup_{t \in \mathbb{R}} \| \nabla u(t, \cdot) \|_{L^2(\Omega)}^2 \leq CF_2. \tag{2.3}
\]

(ii) Secondly, we check that \( \partial u / \partial t \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)) \). Applying Hölder’s inequality we can get that

\[
(\lambda - \lambda_i)^2 \int_t^{t+1} \left( \int_{-\infty}^t e^{(\lambda - \lambda_i)(t-s)} f_i(s) ds \right)^2 dt \\
\leq (\lambda_i - \lambda) \int_t^{t+1} \int_{-\infty}^t e^{(\lambda - \lambda_i)(t-s)} f_i^2(s) ds dt, \quad i \geq i_0 + 1.
\]

Then it follows from Fubini’s theorem that

\[
\sum_{i=1}^{l_0} \lambda_i (u_i(t))^2 dt \leq (\lambda_i - \lambda) \left( \int_{-\infty}^t \int_t^{t+1} e^{(\lambda - \lambda_i)(t-s)} f_i^2(s) ds dt ds \\
+ \int_t^{t+1} \int_s^{t+1} e^{(\lambda - \lambda_i)(t-s)} f_i^2(s) ds ds \right) \\
= \int_{-\infty}^t \left( e^{(\lambda - \lambda_i)(t-s)} - e^{(\lambda - \lambda_i)(t+1-s)} \right) f_i^2(s) ds \\
+ \int_t^{t+1} \left( 1 - e^{(\lambda - \lambda_i)(t+1-s)} \right) f_i^2(s) ds \\
\leq \sum_{k=0}^{l_0} \left( e^{k(\lambda - \lambda_i)} - e^{(k+2)(\lambda - \lambda_i)} \right) \int_t^{t+1} f_i^2(s) ds \\
+ \int_t^{t+1} f_i^2(s) ds, \quad i \geq i_0 + 1.
\]
Here we use the fact in the above inequality that for $i \geq i_0 + 1$,
\[
\begin{align*}
\{ & e^{(\lambda - \lambda_i)(s-t)} \leq e^{k(\lambda - \lambda_i)} \quad \text{for } s \in [\tau - k, \tau - k], \\
& e^{(\lambda - \lambda_i)(s+1-t)} \geq e^{(k+2)(\lambda - \lambda_i)} \quad \text{for } s \in [\tau - k - 1, \tau - k].
\}
\end{align*}
\]

Note that
\[
\frac{d}{dt}(e^{k(\lambda - \lambda_i)} - e^{(k+2)(\lambda - \lambda_i)}) = e^{k(\lambda - \lambda_i)}((k+2)e^{2(\lambda - \lambda_i)} - k) < 0
\]

for sufficiently large $t$ and any $k \geq 1$. Thus there exists an integer $k_0 > i_0 + 1$ such that
\[
e^{k(\lambda - \lambda_i)} - e^{(k+2)(\lambda - \lambda_i)} \leq e^{k(\lambda - \lambda_{i_0})} - e^{(k+2)(\lambda - \lambda_{i_0})}, \quad \forall k \geq 1, i \geq k_0.
\]

This together with (2.4) leads to for any $n > k_0 + 1$,
\[
\sum_{i=k_0}^{n} (\lambda - \lambda_i)^2 \int_{\tau}^{\tau+1} (u_i(t))^2 dt \leq \sum_{k=1}^{\infty} (e^{k(\lambda - \lambda_{i_0})} - e^{(k+2)(\lambda - \lambda_{i_0})}) \int_{\tau-k}^{\tau-k+1} \left( \sum_{i=k_0}^{n} f_i^2(s) \right) ds
\]
\[
+ \int_{\tau-1}^{\tau+1} \left( \sum_{i=k_0}^{n} f_i^2(s) \right) ds
\]
\[
\leq CF_2.
\]

We conclude from the estimates (2.1) and (2.2) that
\[
\sum_{i=1}^{\infty} (\lambda - \lambda_i)^2 (u_i(t))^2 = \sum_{i=1}^{i_0} (\lambda - \lambda_i)^2 \frac{\lambda_i(u_i(t))^2}{\lambda_i} + \sum_{i=1}^{i_0-1} (\lambda - \lambda_i)^2 \lambda_i(u_i(t))^2 \leq CF_2.
\]

Combining the above two inequalities, we deduce by the arbitrariness of $n$ and $\tau$ that
\[
(2.5) \quad \sup_{t \in R} \int_{\tau}^{\tau+1} \sum_{i=1}^{\infty} (\lambda - \lambda_i)^2 (u_i(t))^2 dt \leq CF_2.
\]

Noting that
\[
(2.6) \quad \frac{du_i}{dt} = (\lambda - \lambda_i)u_i(t) + f_i(t), \quad i = 1, 2, \ldots,
\]
we derive
\[
\left( \frac{du_i}{dt} \right)^2 \leq 2(\lambda - \lambda_i)^2 (u_i(t))^2 + 2(f_i(t))^2.
\]
This and (2.5) together imply \( \partial u / \partial t \in L^2_{\text{loc}}(R; L^2(\Omega)) \), and
\[
\sup_{t \in R} \left\| \frac{\partial u}{\partial t}(\cdot, s) \right\|_{L^2(Q_t^{i+1})}^2 \leq CF_2.
\]

(iii) Next we intend to verify that \( u \) is just the solution of the problem (1.2) on \( Q^{+\infty} \). For any \(-\infty < t_1 < t_2 < +\infty \), let \( \Phi^{(i)}(x, t) = \sum_{i=1}^r \theta_i(t) \psi_i(x) \), where \( \theta_i(t) \) is smooth function in \([t_1, t_2]\) and \( \Phi^{(i)}(x, t_1) = \Phi^{(i)}(x, t_2) = 0 \) a.e. in \( \Omega \). Recall the definition of \( u \) and (2.6), then we have
\[
- \iint_{Q_t^i} u \frac{\partial \Phi^{(i)}}{\partial t} \, dx dt + \iint_{Q_t^i} \nabla u \nabla \Phi^{(i)} \, dx dt = \iint_{Q_t^i} (\dot{\lambda} u + \ddot{f}) \Phi^{(i)} \, dx dt.
\]
The above integral equality still holds for each \( \Phi \in \dot{W}^{1,1}_2(Q_t^i) \) with \( \Phi(x, t_1) = \Phi(x, t_2) = 0 \), hence \( u \) is a global \( L^2(\Omega) \)-bounded solution of the problem (1.2).

(iv) Now let us prove the uniqueness of global \( L^2(\Omega) \)-bounded solutions. Let \( u, v \) be two global \( L^2(\Omega) \)-bounded solutions of the problem (1.2). For any \( \Phi \in \dot{W}^{1,1}_2(Q_t^i) \) \((t_1 < t_2)\), the following equality holds
\[
\int_{\Omega} (u(x, s) - v(x, s)) \Phi(x, s) \, dx - \int_{Q_t^i} \left( (u - v) \frac{\partial \Phi}{\partial t} - \nabla (u - v) \nabla \Phi \right) \, dx dt
\]
\[
= \iint_{Q_t^i} \dot{\lambda}(u - v) \Phi \, dx dt + \int_{\Omega} (u(x, t_1) - v(x, t_1)) \Phi(x, t_1) \, dx, \quad t_1 \leq s \leq t_2.
\]
Write \( v_i(t) = \int_{\Omega} v(x, t) \psi_i(x) \, dx \) and set \( \Phi = \psi_i(x) \) in the above equality, then we derive from the arbitrariness of \( t_1 \) and \( t_2 \) that
\[
(2.7) \quad \frac{d}{dt} (u_i(t) - v_i(t)) + (\dot{\lambda}_i - \dot{\lambda}) (u_i(t) - v_i(t)) = 0, \quad i = 1, 2, \ldots.
\]
Hence \( u_i(t) - v_i(t) = (u_i(t_0) - v_i(t_0)) e^{(\dot{\lambda}_i - \dot{\lambda})(t_0 - t)} \) for any \( t_0 \in R \). Then it follows from the unboundedness of \( e^{(\dot{\lambda}_i - \dot{\lambda})(t_0 - t)} \) on \( R \) and the boundedness of \( u_i(t) - v_i(t) \) on \( R \) that \( u_i(t_0) - v_i(t_0) = 0 \). Therefore from the arbitrariness of \( t_0 \) we have \( u_i(t) \equiv v_i(t) \) for any \( i = 1, 2, \ldots \). Thus \( u(x, t) = v(x, t) \) a.e. in \( Q^{+\infty} \).

(v) Finally, let us show that the solution of (1.2) is unstable. Let \( w \) be a solution of the problem (1.2) on \( Q^{+\infty}_0 \) with the initial value \( w(x, t_0) = \sum_{i=1}^{+\infty} a_i \psi_i(x) \in L^2(\Omega) \). Write \( w_i(t) = \int_{\Omega} w(x, t) \psi_i(x) \, dx \). Similar to the proof of (2.7), we can get that
\[
(2.8) \quad \frac{d}{dt} (w_i(t) - u_i(t)) + (\dot{\lambda}_i - \dot{\lambda}) (w_i(t) - u_i(t)) = 0, \quad t \geq t_0, \; i = 1, 2, \ldots.
\]
If there exists a positive integer \( j_0 \leq i_0 \) such that \( a_{j_0} \neq u_{j_0}(t_0) \), then it follows from (2.8) that
\[
w_{j_0}(t) - u_{j_0}(t) = (a_{j_0} - u_{j_0}(t_0))e^{(\lambda_{j_0} - \lambda)(t_0 - t)}, \quad t \geq t_0.
\]
Thus
\[
\|w(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \geq |a_{j_0} - u_{j_0}(t_0)|e^{(\lambda_{j_0} - \lambda)(t_0 - t)}, \quad t \geq t_0.
\]
The proof is complete.

**Proof of Theorem 1.2.** The proof of Theorem 1.2 is the same as that of Theorem 1.1, and we omit the details.

**Proof of Theorem 1.3.** (i) Suppose first \( \int_0^t f_{k_0}(s)ds \) is unbounded on \( R \) and \( u \) is a solution of the problem (1.2) on \( Q_{+\infty}^+ \). Then similar to the proof of (2.8), we have
\[
\begin{align*}
du_i(t) + (\lambda_i - \lambda)u_i(t) &= f_i(t), \quad t \in R, i \neq k_0; \\
du_{k_0}(t) &= f_{k_0}(t), \quad t \in R.
\end{align*}
\]
(2.9)
Consequently \( u_{k_0}(t) = \int_0^t f_{k_0}(s)ds + c \) is unbounded on \( R \), and \( u \) is not a global \( L^2(\Omega) \)-bounded solution of the problem (1.2). Hence the assertion (i) of Theorem 1.3 holds.

(ii) Suppose now \( \int_0^t f_{k_0}(s)ds \) is bounded on \( R \), and \( u \) is a global \( L^2(\Omega) \)-bounded solution of the problem (1.2). Let \( u = \sum_{i=1}^{+\infty} u_i(t)\psi_i(x) \), then \( u_i(t) \) is the bounded solution of the problem (2.9). By the theory of ordinary differential equations we conclude that \( u_i(t) \) necessarily has the form
\[
u_i(t) = \begin{cases} 
\int_t^{+\infty} e^{(\lambda_i - \lambda)(t-s)}f_i(s)ds, & i = 1, 2, \ldots, k_0 - 1, \\
\int_0^t f_{k_0}(s)ds + c, & i = k_0, \\
\int_{-\infty}^t e^{(\lambda_i - \lambda)(t-s)}f_i(s)ds, & i = k_0 + 1, k_0 + 2, \ldots.
\end{cases}
\]
On the other hand, similar to the proof of Theorem 1.1, we can verify that any \( u \) having the above form is necessarily a global \( L^2(\Omega) \)-bounded solution of the problem (1.2) satisfying (1.9) and (1.10). This proves assertion (ii).

(iii) Next we suppose \( u \) is a global \( L^2(\Omega) \)-bounded solution of the problem (1.2) and \( k_0 = 1 \), we will prove the stability of \( u \). Now let \( w(x, t), w_i(t), a_i \) be the same as that in Theorem 1.1. Then we have
\[
\begin{align*}
(2.10) \quad 
w_i(t) - u_i(t) &= e^{(\lambda_i - \lambda)(t_0 - t)}(w_i(t_0) - u_i(t_0)), \quad t \geq t_0, i \geq 2; \\
w_1(t) - u_1(t) &= c, \quad t \geq t_0,
\end{align*}
\]
which leads to
\[ \|w(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)}^2 = \sum_{i=1}^{+\infty} |w_i(t) - u_i(t)|^2 \leq \|w(\cdot,t_0) - u(\cdot,t_0)\|_{L^2(\Omega)}^2, \quad t \geq t_0. \]

Hence for any \( \varepsilon > 0 \), there exists \( \delta = \varepsilon \) such that
\[ \sup_{t \geq t_0} \|w(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} < \varepsilon \]
for any \( w(x,t_0) \) satisfying \( \|w(\cdot,t_0) - u(\cdot,t_0)\|_{L^2(\Omega)} < \delta \). That is to say \( u \) is stable. We also derive from (2.10) that
\[ \|w(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)}^2 \geq |w_1(t) - u_1(t)|^2 = c^2, \quad t \geq t_0. \]
Thus \( u \) is not asymptotically stable.

(iv) At last, if \( u \) is a global \( L^2(\Omega) \)-bounded solution of the problem (1.2) and \( k_0 > 1 \), we can prove it is unstable by using the arguments given in the proof of (v) in Theorem 1.1, and we omit the details here. The proof is complete. \( \square \)

**Proof of Theorem 1.4.** (i) First assume \( f \in S^2AP(L^2(\Omega)) \), \( \lambda \neq \lambda_i \) \( (i = 1, 2, \ldots) \), and \( u \) is the global \( L^2(\Omega) \)-bounded solution of the problem (1.2). Letting \( \tau \in T(\varepsilon, f) \), we derive from (1.6) that
\[ \sup_{t \in \mathbb{R}} \|\nabla(u(\cdot,t + \tau) - u(\cdot,t))\|_{L^2(\Omega)}^2 + \sup_{t \in \mathbb{R}} \iint_{Q_{t+\tau}^{i+1}} \left| \frac{\partial u}{\partial s} (x,s + \tau) - \frac{\partial u}{\partial s} (x,s) \right|^2 \, dx \, ds \leq C \sup_{t \in \mathbb{R}} \iint_{Q_{t+\tau}^{i+1}} \left| f(x, s + \tau) - f(x, s + t) \right|^2 \, dx \, ds \leq C \varepsilon^2. \]
This implies \( u \in AP(H_0^1(\Omega)) \) and \( \partial u / \partial t \in S^2AP(L^2(\Omega)) \).

(ii) Let us next consider the situation that \( f \in S^2AP(L^2(\Omega)) \) and \( \lambda = \lambda_{k_0} \). Since \( f_{k_0}(t) \in S^2AP(\mathbb{R}) \) and \( \int_0^t f_{k_0}(s) \, ds \) is bounded, it is easy to verify that \( u_{k_0}(t) = \int_0^t f_{k_0}(s) \, ds + c \in AP(\mathbb{R}) \) (see [7, p. 7] for details). Hence we conclude from (1.9) and (1.10) that \( u \in AP(H_0^1(\Omega)) \) and \( \partial u / \partial t \in S^2AP(L^2(\Omega)) \).

(iii) Suppose now \( f \in AP(L^2(\Omega)) \) and \( \lambda_{k_0} < \lambda < \lambda_{k_0+1} \). Let \( \tau \in T(\varepsilon, f) \). Like the proof of (ii) in Theorem 1.1, we can derive
\[ \sup_{t \in \mathbb{R}} \left\| \frac{\partial u}{\partial t} (\cdot, t + \tau) - \frac{\partial u}{\partial t} (\cdot, t) \right\|_{L^2(\Omega)}^2 \leq C \sup_{t \in \mathbb{R}} \left\| f(\cdot, t + \tau) - f(\cdot, t) \right\|_{L^2(\Omega)}^2 \leq C \varepsilon^2, \]
and so \( \partial u / \partial t \in AP(L^2(\Omega)) \).
As for the case that \( f \in AP(L^2(\Omega)) \) and \( \lambda = \lambda_{k_0} \), the proof is similar to (iii), hence we omit it. The proof is complete.

Now we give the following lemma which plays an important role in the proof of Theorem 1.5. Define the equivalent norm of \( L^\infty(R; L^2(\Omega)) \) by

\[
(2.11) \quad \|u\| = \sup_{t \in R} \left( \sum_{i=1}^{k_0} \left( \int_{\Omega} u \psi_i \, dx \right)^2 \right)^{1/2} + \sup_{t \in R} \left( \int_{\Omega} u \psi_i \, dx \right)^{1/2},
\]

where \( u \in L^\infty(R; L^2(\Omega)) \).

While \( \|u\|_{L^\infty(R; L^2(\Omega))} \) represents the formal case.

**Lemma 2.1.** Suppose that \( f \in L^\infty(R; L^2(\Omega)) \), and \( u \) is the global \( L^2(\Omega) \)-bounded solution of the problem (1.2).

(i) If \( \lambda_{k_0} < \lambda < \lambda_{k_0+1} \), then

\[
(2.12) \quad \|u\| \leq \max \left\{ \frac{1}{\lambda_{k_0+1} - \lambda}, \frac{1}{\lambda - \lambda_{k_0}} \right\} \|f\|.
\]

(ii) If \( \lambda < \lambda_1 \), then

\[
(2.13) \quad \|u\|_{L^\infty(R; L^2(\Omega))} \leq \frac{1}{\lambda_1 - \lambda} \|f\|_{L^\infty(R; L^2(\Omega))}.
\]

**Proof.** Let us first suppose \( \lambda_{k_0} < \lambda < \lambda_{k_0+1} \). We compute by Hölder’s inequality that

\[
(u_i(t))^2 \leq \int_{-\infty}^{t} e^{(\lambda - \lambda_i)(t-s)} ds \int_{-\infty}^{t} e^{(\lambda - \lambda_i)(t-s)} f_i^2(s) ds
\]

\[
= \frac{1}{\lambda_i - \lambda} \int_{-\infty}^{t} e^{(\lambda - \lambda_i)(t-s)} f_i^2(s) ds
\]

\[
\leq \frac{1}{\lambda_{k_0+1} - \lambda} \int_{-\infty}^{t} e^{(\lambda - \lambda_{k_0+1})(t-s)} f_i^2(s) ds
\]

for \( i \geq i_0 + 1 \).

Hence we obtain for any \( n > i_0 + 1 \),

\[
\sum_{i=i_0+1}^{n} (u_i(t))^2 \leq \frac{1}{\lambda_{k_0+1} - \lambda} \int_{-\infty}^{t} e^{(\lambda - \lambda_{k_0+1})(t-s)} \left( \sum_{i=i_0+1}^{n} f_i^2(s) \right) ds
\]

\[
\leq \frac{1}{\lambda_{k_0+1} - \lambda} \int_{-\infty}^{t} e^{(\lambda - \lambda_{k_0+1})(t-s)} \left( \sum_{i=i_0+1}^{+\infty} f_i^2(s) \right) ds.
\]
From the arbitrariness of \( n \) we have
\[
\sum_{i=0}^{+\infty} (u_i(t))^2 \leq \frac{1}{\lambda_{ib_0+1} - \lambda} \int_{-\infty}^{t} e^{(\lambda - \lambda_{ib_0+1})(t-s)} \left( \sum_{i=0}^{+\infty} f_i^2(s) \right) ds
\]
\[
\leq \frac{1}{(\lambda_{ib_0+1} - \lambda)^2} \sup_{t \in \mathbb{R}} \left( \sum_{i=0}^{+\infty} f_i^2(t) \right).
\]

Now we compute
\[
\sum_{i=1}^{ib_0} (u_i(t))^2 \leq \sum_{i=1}^{ib_0} \frac{1}{\lambda_i - \lambda_{ib_0}} \int_{t}^{+\infty} e^{(\lambda_i - \lambda_{ib_0})(t-s)} f_i^2(s) ds
\]
\[
\leq \frac{1}{\lambda - \lambda_{ib_0}} \int_{t}^{+\infty} e^{(\lambda - \lambda_{ib_0})(t-s)} \left( \sum_{i=1}^{ib_0} f_i^2(s) \right) ds
\]
\[
\leq \frac{1}{(\lambda - \lambda_{ib_0})^2} \sup_{t \in \mathbb{R}} \left( \sum_{i=1}^{ib_0} f_i^2(t) \right).
\]

It follows from the above two inequalities that the assertion (i) holds.

As before, the assertion (ii) follows as well. The proof is complete.

\[\square\]

**Proof of Theorem 1.5.** Suppose that the assumptions (A1) and (A2) are fulfilled, and \( g(x, t, 0) \in L^{\infty} (\mathbb{R}; L^2(\Omega)) \).

(i) We will prove the existence of the global \( L^2(\Omega) \)-bounded solution by the contraction mapping principle. We first consider the case that (1.12) holds. Assume \( w(x, t) \in L^{\infty} (\mathbb{R}; L^2(\Omega)) \), then so is \( g(x, t, w(x, t)) \). It follows from Theorem 1.1 that the problem
\[
(2.14) \quad \begin{cases}
\frac{\partial u}{\partial t} - Au = \lambda u + g(x, t, w) - \lambda w, & \text{in } \Omega \times \mathbb{R}, \\
u = 0, & \text{on } \partial \Omega \times \mathbb{R}
\end{cases}
\]
admits uniquely a global \( L^2(\Omega) \)-bounded solution \( u \in L^{\infty} (\mathbb{R}; H^1_0(\Omega)) \), where \( \lambda = (\lambda_{ib_0} + \lambda_{ib_0+1})/2 \). Define the mapping \( \mathcal{G} : L^{\infty} (\mathbb{R}; L^2(\Omega)) \rightarrow L^{\infty} (\mathbb{R}; L^2(\Omega)) \) as follows:
\[
\mathcal{G}(w) = u, \quad w \in L^{\infty} (\mathbb{R}; L^2(\Omega)),
\]
where \( u \) is the global bounded solution of the problem (2.14). Given \( w_1, w_2 \in L^{\infty} (\mathbb{R}; L^2(\Omega)) \), we get from (2.12) that
\[ \| \mathcal{G}(w_1) - \mathcal{G}(w_2) \| \leq \frac{2}{\lambda_{i0+1} - \lambda_{i0}} \| g(\cdot, \cdot, w_1) - g(\cdot, \cdot, w_2) - \dot{\lambda}(w_1 - w_2) \| \]
\[ \leq \frac{2 \max\{ \ddot{\lambda} - \dot{\sigma}, \dot{\sigma} - \dot{\lambda} \}}{\lambda_{i0+1} - \lambda_{i0}} \| w_1 - w_2 \|. \]

Due to the definition of \( \dot{\sigma} \) and \( \dot{\lambda} \), it is easily seen that \( 2 \max\{ \ddot{\lambda} - \dot{\sigma}, \dot{\sigma} - \dot{\lambda} \} / (\lambda_{i0+1} - \lambda_{i0}) < 1 \). Therefore \( \mathcal{G} \) is a contraction mapping and has a unique fixed point.

In the case that (1.11) holds, the proof of existence of global \( L^2(\Omega) \)-bounded solution is similar to the above case, except that we use (2.13) and let \( \dot{\lambda} = (\ddot{\sigma} + \dot{\sigma})/2 \) in (2.14).

(ii) Next we intend to prove the stability of the global \( L^2(\Omega) \)-bounded solution \( u \) of (1.1). Fix \( t_0 \in R \) and choose \( v_0(x) \in L^2(\Omega) \) such that
\[ u(x, t_0) > v_0(x) \quad \text{a.e. } x \in \Omega. \]
Let \( v(x, t) \) be the weak solution of (1.1) on \( Q_{t_0}^{+\infty} \) with the initial condition \( v(x, t_0) = v_0(x) \). Then by the comparison principle we have
\[ u(x, t) \geq v(x, t) \quad \text{a.e. } (x, t) \in Q_{t_0}^{+\infty}. \]
According to the definition of the solution we easily get that
\[ \frac{d}{dt} \int_{\Omega} (u - v) \psi_1 \, dx + \int_{\Omega} \nabla(u - v) \nabla \psi_1 \, dx \]
\[ = \int_{\Omega} (g(x, t, u) - g(x, t, v)) \psi_1 \, dx, \quad t \geq t_0. \]
Obviously
\[ \int_{\Omega} \nabla(u - v) \nabla \psi_1 \, dx = \lambda_1 \int_{\Omega} (u - v) \psi_1 \, dx. \]
Since \( \psi_1(x) > 0 \) in \( \Omega \), we discover from (1.12) and (2.16) that
\[ (g(x, t, u) - g(x, t, v)) \psi_1(x) \geq \dot{\sigma}(u - v) \psi_1(x). \]
Our substituting the above inequality and (2.18) into (2.17) yields
\[ \frac{d}{dt} \int_{\Omega} (u - v) \psi_1 \, dx \geq (\ddot{\sigma} - \dot{\lambda}_1) \int_{\Omega} (u - v) \psi_1 \, dx, \quad t \geq t_0. \]
Consequently
\[ \int_{\Omega} (u(x, t) - v(x, t)) \psi_1(x) \, dx \geq e^{(\ddot{\sigma} - \dot{\lambda}_1)(t - t_0)} \int_{\Omega} (u(x, t_0) - v_0(x)) \psi_1(x) \, dx, \quad t \geq t_0. \]
Whence
\[ \|u(\cdot, t) - v(\cdot, t)\|_{L^2(\Omega)} \geq e^{(\sigma - \lambda_1)(t - t_0)} \int_{\Omega} (u(x, t_0) - v_0(x)) \psi_1(x) dx, \quad t \geq t_0. \]

Thus for any \( \varepsilon > 0 \), we can choose \( v_0 \) satisfying (2.15) and \( \|v_0(\cdot) - u(\cdot, t_0)\|_{L^2(\Omega)} < \varepsilon \) such that
\[ \|u(\cdot, t) - v(\cdot, t)\|_{L^2(\Omega)} \to +\infty \quad \text{as} \quad t \to +\infty. \]

Hence the global \( L^2(\Omega) \)-bounded solution \( u \) of (1.1) is unstable.

In the case that (1.11) holds, the proof of the stability is simple, hence we omit it.

(iii) Finally, let us show that the global \( L^2(\Omega) \)-bounded solution is almost periodic. Suppose now \( \sigma(\cdot, t, \cdot) \in AP(L^\infty(\Omega \times R)), \ g(\cdot, t, 0) \in AP(L^2(\Omega)) \) and (1.12) holds. Since
\[
\begin{align*}
|g(x, t + \tau, w(x, t + \tau)) - g(x, t, w(x, t))| &\leq |g(x, t + \tau, w(x, t + \tau)) - g(x, t + \tau, w(x, t))| \\
&\quad + |g(x, t + \tau, w(x, t)) - g(x, t, w(x, t))| \\
&\leq \sigma|w(x, t + \tau) - w(x, t)| + |\sigma(x, t + \tau, w(x, t)) - \sigma(x, t, w(x, t))| |w(x, t)| \\
&\quad + |g(x, t + \tau, 0) - g(x, t, 0)|,
\end{align*}
\]
we deduce that \( g(\cdot, t, w(\cdot, t)) \in AP(L^2(\Omega)) \), provided \( w(\cdot, t) \) belongs to \( AP(L^2(\Omega)) \). Define the mapping
\[ G : AP(L^2(\Omega)) \to AP(L^2(\Omega)) \]
as follows:
\[ G(w) = u, \quad w \in AP(L^2(\Omega)), \]
where \( u \) is the \( L^2(\Omega) \)-almost periodic solution of the problem (2.14). Similarly, we can prove by using the arguments as in the proof of (i) that \( G \) is a contraction mapping and has a unique fixed point. Hence the global \( L^2(\Omega) \)-bounded solution is \( L^2(\Omega) \)-almost periodic.

Likewise we can prove the almost periodicity of the global \( L^2(\Omega) \)-bounded solution for the case that \( \sigma(\cdot, t, \cdot) \in AP(L^\infty(\Omega \times R)), \ g(\cdot, t, 0) \in AP(L^2(\Omega)) \) and (1.11) holds. The proof is complete. \( \square \)

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