Robust Model Predictive Control for Linear Systems with State and Input Dependent Uncertainties

Danylo Malyuta¹, Behçet Açıkmeşe² and Martin Cacan³

Abstract—This paper presents a computationally efficient robust model predictive control law for discrete linear time invariant systems subject to additive disturbances that may depend on the state and/or input norms. Despite the dependency being non-convex, we are able to capture it exactly for input dependency and approximately for state dependency in at most a second order cone programming problem. The formulation has linear complexity in the planning horizon length. The approach is thus amenable to efficient real-time implementation with a guarantee on recursive feasibility and global optimality. Robust position control of a satellite is considered as an illustrative example.

I. INTRODUCTION

In this paper we develop a novel convex formulation for robust model predictive control (RMPC) of discrete linear time invariant systems with additive state and/or input dependent uncertainty. Several practical applications may benefit from this work. One example is satellite position control with a Gates model for thruster execution-error uncertainty that accounts for magnitude and pointing errors [1], [2]. Another example is control of a linearized system where linearization error is modeled as an additive perturbation growing with the distance from the linearization point. The same model can capture the state uncertainty associated with the delay induced by non-zero controller computation time. Last but not least, uncertainty of some robotic parallel manipulators can be handled by this formulation [3].

A major advantage of RMPC is its ability to guarantee by design that input and state constraints are satisfied for all uncertainty realizations. Several extensive survey papers cover available modeling assumptions and solution methods [4], [5], [6], [7]. We restrict our attention to discrete linear time invariant systems and focus on developing a real-time implementable algorithm on computationally constrained hardware. In our approach, the planning problem is open loop and we deal with a perfect model affected by additive bounded uncertainty. There is evidence that this description is a reasonable choice [8], [9], [10]. Several authors have considered closed loop formulations [11], [12], [13], [14], [15]. However, these suffer from combinatorial complexity in the control horizon length and problem dimension unless certain restrictive assumptions are made in terms of cost norm or feedback type.

Closest to our work is [16], [17], [18], [19] that use pre-computed constraint tightening factors to guarantee robustness to worst-case uncertainty through a set of linear constraints. The computational complexity is equivalent to nominal MPC and the online problem is at most a quadratic program (QP). A similar idea is exploited for nonlinear systems in [10].

Our main contribution is to extend this efficient RMPC formulation to an uncertainty model that handles state and input norm dependency. Dependent uncertainty has received some attention in nominal and robust nonlinear MPC [20], [21], [22]. These formulations, however, are conservative due to their use of a Lipschitz constant for constraint tightening, which considers uncertainty magnitude but not direction. Our formulation considers directionality via Hölder’s inequality and is therefore less conservative. Furthermore, we model the important case of input dependent uncertainty which we have not yet seen be addressed in literature. Certainly, no connection has yet been made between this formulation and convex handling of the Gates thruster execution-error model for satellite control. Our approach has linear complexity in the horizon length and is at most a second order cone programming (SOCP) problem. We can thus expect that the algorithm is amenable to real time implementation [23], [24], [25], [26]. Our approach is extensible to a semi-feedback RMPC formulation by pre-computing a static feedback component [12], [27] and can be embedded in tube MPC [28].

The paper is organized as follows. In Section II the uncertainty model and the resulting robust optimal control problem are introduced. In Section III a solution is presented as an RMPC law that is at most an SOCP problem. Section IV proves recursive feasibility and suggests a computationally efficient check of robust controlled invariance. An example is presented in Section V which illustrates the method’s effectiveness. Section VI discusses possible extensions and is followed by some concluding remarks in Section VII.

II. PROBLEM FORMULATION

This section describes the control problem and, in particular, defines the state and input dependent uncertainty model. We use the following notation. \( \mathbb{R} \) denotes the set of reals, \( \mathbb{R}_+ \) the set of non-negative reals and \( \mathbb{R}_{++} \) the set of positive reals. The same goes for \( \mathbb{Z} \), the set of integers. Unless otherwise specified, matrices are uppercase (e.g. \( A \)), scalars and vectors are lowercase (e.g. \( x \)) and sets are calligraphic uppercase (e.g. \( S \)). We reserve \( 0_{n\times m} \in \mathbb{R}^{n\times m} \) and \( I_n \in \mathbb{R}^{n\times n} \) for the zero and identity matrices respectively, where \( n = m = 3 \) by default when the subscripts are omitted. We

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input dependent component that is bounded by a non-convex component originating from a polytope and a state and φ non-decreasing such that φ where each function expressed as the Minkowski sum of a polytopic independent component.

We use parentheses to denote vertical stacking, e.g. $(1, 2, 3) \in \mathbb{R}^3$. Given a matrix $M$, row vector $M_i^T$ denotes its $i$-th row. We use $\| \cdot \|_p$ to denote a $p$-norm, e.g. $p = 1, 2, \infty$, and $| \cdot |$ to denote the absolute value. $\text{co} S$ denotes the convex hull of $S$. Direct set mapping is written as $MS = \{ Mx : x \in S \}$.

The support function of a set $S \subseteq \mathbb{R}^n$ along a direction $v \in \mathbb{R}^n$ is denoted by $\sigma_S(v) \triangleq \max_{z \in S} v^Tz$. The shorthand a : b denotes the integer sequence $a, \ldots, b$. $S^n$ denotes the Cartesian product performed $n$ times, e.g. $S^3 = S \times S \times S$.

Consider a discrete linear time invariant system with additive uncertainty:

$$x_{k+1} = Ax_k + Bu_k + Dp_k,$$

where $k \in \mathbb{Z}_+$, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $p_k \in \mathbb{R}^d$ are, respectively, the time step, state, input and uncertainty, while $A$, $B$ and $D$ are constant matrices of commensurate dimension. The following state and input constraints are to be respected for all time:

$$x_k \in \mathcal{X} \subseteq \mathbb{R}^n \quad \forall k \in \mathbb{Z}_+,$$

$$u_k \in \mathcal{U} \subseteq \mathbb{R}^m \quad \forall k \in \mathbb{Z}_+,$$

where $\mathcal{X}$ and $\mathcal{U}$ are compact convex polytopes that we express as follows:

$$\mathcal{X} \triangleq \{ x \in \mathbb{R}^n : Fx \leq f \},$$

$$\mathcal{U} \triangleq \{ u \in \mathbb{R}^m : Hu \leq h \},$$

where the rows of $F$, $H$ and the elements of $f$, $h$ define respectively the polytope facet normals and distances. Consider now the following uncertainty model:

$$p_k \in \mathcal{P}(x_k, u_k) \subseteq \mathbb{R}^d,$$

$$\mathcal{P}(x_k, u_k) \triangleq \{ Ww + \sum_{i=1}^{n_q} L_i q_i : \mathbb{R}^d : R w \leq r, \}

\|q_i\|_{p_{x,i}} \leq \phi_i(\|Fx_ix_k\|_{p_{x,i}}, \|Fu_i u_k\|_{p_{u,i}}),$$

$$i = 1, \ldots, n_q,$$

where each function $\phi_i : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is convex and non-decreasing such that $\phi_i(\|y\|_{p_{x,i}}, \|z\|_{p_{u,i}})$ is convex [29]. The uncertainty set is effectively the sum of an independent component originating from a polytope and a state and input dependent component that is bounded by a non-convex inequality. The following assumption is a consequence of the fact that unbounded uncertainties are not practical and cannot be handled by bounded control inputs.

**Assumption 1.** The set $\mathcal{P}(x_k, u_k)$ is compact for all $x_k \in \mathcal{X}$ and $u_k \in \mathcal{U}$.

We use $\| \cdot \|_2$ and a conic dependent component (blue).

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where $G \in \mathbb{R}^{n_x \times n}$ and $g \in \mathbb{R}^{n_g}$.

Since $\mathcal{I}$ is fRCI, for each $x_k \in \mathcal{I}$, $u_k \in U$ such that $x_{k+1} \in \mathcal{I}$ $\forall p_k \in P(x_k, u_k)$. We compute such a $u_k$ via a tightened state constraint in an optimization problem. Letting $x_k$ be the current state, the system (1) has the following impulse response over $t = 1 : N$ future time steps:

$$
x_{k+t} = A^t x_k + \sum_{i=0}^{t-1} B^i u_{k+i} + \sum_{i=0}^{t-1} D^i p_{k+i},
$$

(7)

where $B^i A = A^{i-1} B$ and $D^i A = A^{i-1} D$ (the index $t$ is omitted for notational simplicity). An invariance-preserving input sequence over the $N$-step planning horizon satisfies:

$$
G \left( A^t x_k + \sum_{i=0}^{t-1} B^i u_{k+i} + \sum_{i=0}^{t-1} D^i p_{k+i} \right) \leq g,
$$

(8)

for all sequences $p_{k+i} \in P(x_{k+i}, u_{k+i})$, $i = 0 : t-1$ and $t = 1 : N$. Similarly to [16], [34] we reformulate this requirement by maximizing the left hand side of (8):

$$
G_j^T \bar{x}_{k+t} + \sum_{i=0}^{t-1} \max_{p_{k+i} \in P(x_{k+i}, u_{k+i})} G_j^T D^i p_{k+i} \leq g_j,
$$

(9)

for $j = 1 : n_g$ and $t = 1 : N$, where $\bar{x}_{k+t}$ denotes the nominal state after $t$ time steps:

$$
\bar{x}_{k+t} \triangleq A^t x_k + \sum_{i=0}^{t-1} B^i u_{k+i}.
$$

(10)

Note that the maximization term in (9) is the support function $\sigma_{D^i} P(x_{k+i}, u_{k+i})(G_j) \in \mathbb{R}_+$ where $G_j$ is $\mathcal{I}$’s $j$-th facet’s normal. This support function induces constraint tightening.

A complication arises in evaluating the support function due to an algebraic loop in the state dependent uncertainty. The set $\mathcal{P}(x_{k+i}, u_{k+i})$ depends on $x_{k+i}$ which itself depends on the uncertainty sequence over time steps $0 : i-1$ in the planning horizon. Thus, $p_{k+i}$ becomes dependent on $p_k, \ldots, p_{k+i-1}$ and the maximum value that it can take involves the maximization of a convex function over a non-convex domain. Simulation experience with the example in Section V shows that a convex upper bound to this maximization is too conservative, thus we prefer to simplify by considering instead the uncertainty set $\mathcal{P}(\bar{x}_{k+i}, u_{k+i})$ based on the nominal state. Note that the formulation remains exact over the entire planning horizon for input dependent uncertainty and, more importantly, is overall exact at $t = 1$ since $x_k$ is fixed. Therefore the application of $u_k$ guarantees that $x_{k+1} \in \mathcal{I}$. The same cannot be said however over the remaining $N-1$ steps because states with a potentially larger norm inducing a potentially larger uncertainty may occur.

Because RMPC is implemented in receding horizon fashion and $u_k$ is anyway the only input to be applied, we consider this to be a reasonable approach. With this in mind, the support function is simplified using the separable nature of (4b) and Hölder’s inequality:

$$
\sigma_{D^i} P(\bar{x}_{k+i}, u_{k+i})(G_j) = \sigma_{D^i} \mathcal{W}(G_j) + \sum_{l=1}^{n_g} \|G_j^T D^i L_l\|_{q_g} \phi_l(\|F_{x,l} \bar{x}_{k+i}\|_{p_x,l}, \|F_{u,l} u_{k+i}\|_{p_u,l}),
$$

(11)











Algorithm 1 Off-line and on-line RMPC steps.

Off-line:

if Corollary 1 is satisfied for $\mathcal{I} = \mathcal{X}$ then

$\mathcal{I} \leftarrow \mathcal{X}$

else

Compute $\mathcal{I}$ using the set-based algorithm in [31]

if Corollary 1 is not satisfied for $\mathcal{I}$ then

Report error “N is too large”

Store $\sigma_{D^i} \mathcal{W}(G_j)$ for $j = 1 : n_g$, $t = 1 : N$, $i = 0 : t-1$

On-line:

Obtain the current state $x$

Set $x_k \leftarrow x$, solve (13) and apply $u_k$

Sleep $T_s$ seconds $\triangleright$ Discretization time step

where the $q_l$-norm is dual to the $p_{q,l}$-norm in (4b), i.e. $1/q_l + 1/p_{q,l} = 1$, and we denote by $\mathcal{W} = \{Ww : \|W\|_{\infty} \leq r\}$ the polytopic independent uncertainty part of $\mathcal{P}(\bar{x}_{k+i}, u_{k+i})$. Note that (11) is not conservative since H"older’s inequality is tight [29]. Using (11), we can write (9) as a set of $Nn_g$ convex constraints:

$$
G_j^T \bar{x}_{k+t} + \sum_{i=0}^{t-1} \sum_{l=1}^{n_g} \|G_j^T D^i L_l\|_{q_g} \phi_l(\|F_{x,l} \bar{x}_{k+i}\|_{p_x,l}, \|F_{u,l} u_{k+i}\|_{p_u,l}) \leq g_j,
$$

(12)

for $j = 1 : n_g$ and $t = 1 : N$. Importantly, $\sigma_{D^i} \mathcal{W}(G_j)$ can be pre-computed offline leading to a more efficient implementation. The RMPC on-line optimization problem to be implemented in receding horizon fashion can thus be written as:

$$
\min_{x_{k+1}, \ldots, x_{k+N-1}, u_k, \ldots, u_{k+N-1}} J(x_{k+1}, \ldots, x_{k+N}, u_k, \ldots, u_{k+N-1})
$$

(13)

subject to (2b) and (12).

where the cost function $J : (\mathbb{R}^n)^N \times (\mathbb{R}^m)^N \rightarrow \mathbb{R}$ is a design choice which should be convex. If $J$ is linear and the 1- or $\infty$-norms are used for all $\phi_l$ in (12), this is a linear program (LP). If $J$ is quadratic, it is a QP. In all cases when the 2-norm is used in the $\phi_l$ functions, (13) is an SOCP. Note that all of these problem types are convex and have efficient solvers available that guarantee convergence to the global optimum when the feasible set is non-empty (which it is certified to always be the case in Section V). Furthermore, the constraint count of (13) is $O(N)$ owing to the welcome property that the effect of worst-case uncertainty on a discrete time linear system is explicitly given by (11), avoiding a combinatorial search. Algorithm 1 summarizes the off-line and on-line steps that make up the full RMPC controller.

IV. MAIN RESULTS

In this section, Theorem I proves that the control law (13) is recursively feasible and Corollary I provides an alternative
sufficient condition for certifying $X$ to be fRCI. This helps to avoid laborious set-based approaches for computing $I$.

**Theorem 1.** The optimization problem (13) is recursively feasible if and only if it is feasible at the vertices of $I$.

**Proof.** Let $V \triangleq \{v_1, ..., v_M\}$ be the set of vertices of $I$. The forward implication is trivial. If (13) is recursively feasible then it is feasible in particular when $x_k \in V$. For the reverse implication, suppose that we solve (13) with $x_k$ set to each vertex $v_m$, $m = 1 : M$, and obtain associated optimal input sequences $u_{k+1}^m$, $t = 0 : N - 1$, and nominal state sequences $x_{k+1}^m$, $t = 0 : N$, as given by (10):

$$x_{k+1}^m = A^t v_m + \sum_{i=0}^{t-1} B_i^A \sigma_{k+i}^m.$$ 

Consider now a state $x_k \in I$. Since $I$ is a convex polytope, we can express $x_k$ as a convex combination of $I$'s vertices:

$$x_k = \sum_{m=1}^M \theta_m v_m, \quad \sum_{m=1}^M \theta_m = 1, \quad \theta_m \geq 0 \quad \forall m = 1 : M.$$ 

It is now possible to sum the constraint (12) applied at each vertex, weighted by $\theta_m$, to obtain:

$$G_j^I \sum_{m=1}^M \theta_m x_{k+1}^m + \sum_{i=0}^{t-1} \sum_{i=1}^{t-1} \sigma_{D_i^j}(G_j) +$$

$$\sum_{i=0}^{t-1} \sum_{i=1}^{t-1} \sum_{m=1}^M \theta_m \sigma_{D_i^j} \phi_l(\|F_{x,i} x_{k+1}^m\|_{p_x,i}) \phi_l(\|F_{u,i} u_{k+1}^m\|_{p_u,i}).$$

Because each $\phi_l$ is convex, it follows from Jensen’s inequality that:

$$\phi_l(\|F_{x,i} x_{k+1}^m\|_{p_x,i}) \phi_l(\|F_{u,i} u_{k+1}^m\|_{p_u,i}) \leq$$

$$\sum_{m=1}^M \theta_m \phi_l(\|F_{x,i} x_{k+1}^m\|_{p_x,i}) \phi_l(\|F_{u,i} u_{k+1}^m\|_{p_u,i}),$$

and as a result we have:

$$G_j^I \sum_{m=1}^M \theta_m x_{k+1}^m + \sum_{i=0}^{t-1} \sum_{i=1}^{t-1} \sigma_{D_i^j}(G_j) +$$

$$\sum_{i=0}^{t-1} \sum_{i=1}^{t-1} \sum_{m=1}^M \theta_m \sigma_{D_i^j} \phi_l(\|F_{x,i} x_{k+1}^m\|_{p_x,i}) \phi_l(\|F_{u,i} u_{k+1}^m\|_{p_u,i}),$$

**Corollary 1.** Let $I = X$. A sufficient condition for $I$ to be fRCI is for (13) to be feasible at its vertices. The condition becomes also necessary when $N = 1$.

**Proof.** The sufficient condition follows from Theorem 1 and the fact that (12) yields the fRCI property by design whenever (13) is recursively feasible. The necessity of this condition when $N = 1$ is a consequence of that no conservatism in the planning problem is then present.

**V. ILLUSTRATIVE EXAMPLE**

In this section the control law presented in Section III is applied and evaluated on the problem of satellite robust position control in a two vehicle formation in low Earth orbit. We are able to capture the Gates thruster execution-error input dependent uncertainty via (15b), which is of significant practical interest because the Gates model is popular for spacecraft reaction control system (RCS) analysis [2], [35].

Consider a two satellite formation as shown in Figure 2 with a leader $c_l$, in a circular orbit with radius $a$ and a follower $c_f$. The follower’s translation in a local vertical local horizontal (LVLH) frame is given by the Clohessy-Wiltshire equations (time is omitted for notational simplicity):

$$\ddot{x} = 3\omega_0^2 x + 2\omega_0 \dot{y} + u_x + w_x,$$

$$\ddot{y} = -2\omega_0 \dot{x} + u_y + w_y,$$

$$\ddot{z} = -\omega_0^2 z + u_z + w_z,$$

where $\omega_0 = \sqrt{\mu/a^3}$ and $\mu$ is the standard gravitational parameter. Let $r = (x, y, z) \in \mathbb{R}^3$ denote the follower’s position relative to the leader, where the coordinate $x$ is not to be confused with the state vector. Using the state $x \triangleq (\rho, \dot{\rho}) \in \mathbb{R}^6$, input $u = (u_x, u_y, u_z) \in \mathbb{R}^3$ and exogenous
disturbance \( w = (w_x, w_y, w_z) \in \mathbb{R}^3 \), the dynamics (15a)-(15c) take on the familiar linear form:

\[
\dot{x} = Ax + Bu + Ew,
\]

where \( x \) is the state vector, \( A, B, E \) are matrices, and \( u \) and \( w \) are inputs and disturbances, respectively.

We use an impulsive control input in which the RCS system is assumed to induce an instantaneous velocity increment \( \Delta v(\tau) \) at time \( \tau \) via the control input \( u(t) = \Delta v(\tau) \delta(t - \tau) \), where \( \delta \) is the Dirac delta, every \( T_s \) seconds. Meanwhile, \( w \) is assumed to be a constant acceleration over the \( T_s \) silent time period. Using these assumptions, (16) is discretized via the standard linear system solution [36]:

\[
x_{k+1} = Ax_k + Bu_k + EW_k,
\]

where \( A = e^{AT_s}, B = ABc, E = \int_0^{T_s} e^{A(T_s - t)} \text{B} \text{d}t \).

We model three uncertainty sources:

1. Atmospheric drag, modeled as an independent component \( w_k \in \{ v : ||v||_\infty \leq w_{\text{max}} \} \).
2. Additive input error using the Gates model. Because (45) does not capture directional dependency, we confine ourselves to an isotropic description. In fact this is anyway the best modeling choice if the satellite’s design is unknown [1]. The error term is \( u_k = v_k^F + v_k^\text{prop} \) where \( v_k^F \in \{ v : ||v||_2 \leq \sigma_{\text{fix}} \} \) and \( v_k^\text{prop} \in \{ v : ||v||_2 \leq \sigma_{\text{prop}} ||u_k||_2 \} \);
3. Additive state estimation error \( e_k = e_k^F + (I, 0) e_k^\text{pos} + (0, I) e_k^\text{vel} \) where:

\[
e_k^F \in \{ e \in \mathbb{R}^6 : \| [I \ 0] e \|_\infty \leq p_{\text{max}}, \| [0 \ I] e \|_\infty \leq v_{\text{max}} \},
\]

\[
e_k^\text{pos} \in \{ e \in \mathbb{R}^3 : \| e \|_\infty \leq \sigma_{\text{pos}} ||I \ 0|| x_k\|_2 \},
\]

\[
e_k^\text{vel} \in \{ e \in \mathbb{R}^3 : \| e \|_\infty \leq \sigma_{\text{vel}} ||[0 \ I] x_k\|_2 \}.
\]

This is a set of independent and dependent polytopic and ellipsoidal uncertainties that is readily described by (45). In particular, \( d = 21, q_1 = 4, w = (w_k, e_k^F), q_2 = v_k^F, q_3 = e_k^\text{pos}, q_4 = e_k^\text{vel}, p_{q_1} = p_{q_2} = 2, p_{q_3} = p_{q_4} = 4, \phi_1 = \sigma_{\text{fix}}, \phi_2 = u_k \rightarrow \sigma_{\text{prop}} ||u_k||_2, \phi_3 : x_k \rightarrow \sigma_{\text{pos}} ||[I \ 0] x_k\|_2, \phi_4 : x_k \rightarrow \sigma_{\text{vel}} ||[0 \ I] x_k\|_2 \) and the following matrices:

\[
D = [E -A B B -A -A] \quad W = (I_9, 0_{18 \times 9}) \quad R = (I_9, -I_9)
\]

\[
r = (w_{\text{max}}, p_{\text{max}}, v_{\text{max}}, -w_{\text{max}}, -p_{\text{max}}, -v_{\text{max}}, 1)
\]

\[
L_1 = (0_{9 \times 3}, I, 0_{15 \times 3}), L_2 = (0_{12 \times 3}, I, 0_{12 \times 3}), L_3 = (0_{15 \times 3}, I, 0_{9 \times 3}), L_4 = (0_{24 \times 3}, I),
\]

where \( I \in \mathbb{R}^3 \) is the one vector. We use the following cost function:

\[
J = \sum_{t=0}^{N-1} \tilde{u}_k^T \tilde{u}_k + \lambda \tilde{\dot{x}}_{k+t+1}^T \tilde{\dot{x}}_{k+t+1},
\]

where \( \tilde{u}_{k+t} \) and \( \tilde{\dot{x}}_{k+t+1} \) are the scaled input and nominal state such that they attain plus or minus unity at the boundary of their respective constraint polytopes \( \mathcal{U} \) and \( \mathcal{X} \) while \( \lambda = 0.003 \) is a manually chosen trade-off weight. The input penalty reflects a minimum-fuel type problem and the state penalty endows the finite horizon control law with an otherwise lacking long-term knowledge that the origin corresponds to minimal fuel usage, since nominally it takes zero control action to remain there. In this problem, \( \mathcal{X} \) in (2a) takes on the direct interpretation of a maximum control error specification. We use the following numerical values:

\[
\mathcal{X} = \{ x \in \mathbb{R}^6 : \| [I \ 0] x \|_\infty \leq 10 \text{ cm}, \| [0 \ I] x \|_\infty \leq 1 \text{ mm/s} \}
\]

\[
\mathcal{U} = \{ u \in \mathbb{R}^3 : \| u \|_\infty \leq 2 \text{ mm/s} \}
\]

\[
\mu = 3.986 \cdot 10^{14} \text{ m}^3/\text{s}^2 \quad a = 6793.137 \text{ km} \quad T_s = 100 \text{ s}
\]

\[
w_{\text{max}} = 50 \text{ mm/s}^2 \quad \sigma_{\text{fix}} = 1 \text{ mm/s} \quad \rho_{\text{max}} = 0.4 \text{ cm}
\]

\[
v_{\text{max}} = 4 \mu \text{ m/s} \quad \sigma_{\text{pos}} = \tan \frac{\pi}{180} \quad \sigma_{\text{vel}} = 0.02 \sigma_{\text{vel}} = 0.001.
\]

We compare the following three controllers:

1. Nominal MPC: ignores the uncertainty, effectively removing the summations in (12);
2. Conservative RMPC: replaces \( \phi_3 \) in (4b) by its maximum over \( x_k \in \mathcal{X} \) and \( u_k \in \mathcal{U} \), which results in a conservative independent uncertainty approximation;
3. Our RMPC law (13).

In each case, however, the satellite is acted upon by all three uncertainty sources described above. It turns out that for this problem, Corollary 1 succeeds and so \( \mathcal{I} = \mathcal{X} \). To demonstrate the conservatism exhibited by the conservative RMPC law, consider Figure 3 which shows the three controllers’ transient response when starting from a vertex of \( \mathcal{I} \). While the nominal MPC law briefly exits \( \mathcal{I} \), the conservative RMPC law tends to drive the satellite more quickly into the interior.
of $\mathcal{I}$ because it assumes more uncertainty than necessary. Our RMPC law is somewhere in between, both staying within $\mathcal{I}$ and not avoiding the boundaries of $\mathcal{I}$ unnecessarily.

The RMPC law (13) enables the control engineer to work with a richer set of feasible parameters. For example, the required position accuracy can be increased to 5 cm without changing any other parameter, while doing so with the conservative RMPC law requires using $N \leq 2$. Indeed, the conservative RMPC law otherwise does not pass Corollary 1 while [31] computes $\mathcal{X} = \mathcal{I}$ but carries the aforementioned $N = 1$ assumption, which we can increase to at most $N = 2$. This is just one example of how (13) is feasible for a wider range of parameter combinations.

Next, we compare the fuel consumption and computational efficiency of the three controllers. Fuel consumption is quantified by the sum of velocity increment magnitudes commanded per year as obtained from a linear regression while computational efficiency is measured by the time taken to solve (13). Since both quantities are affected by the uncertainty realization, we run a set of 500 Monte Carlo simulations where the satellite is initialized at the origin and is controlled for a duration of 4 orbits. Figure 4 shows a Gaussian kernel density plot for fuel consumption, with the initial transient removed. There is no statistically significant difference between the conservative and our RMPC while nominal MPC uses on average 9 mm/s/year less fuel (95% confidence level). Figure 5 shows the solver time distribution when run on an Intel Core i5-7200 CPU with a 2.5 GHz clock and 8 GB RAM, showing that all three algorithms have comparable solver times. The distributions reflect the problem difficulty hierarchy, wherein our RMPC law is the most difficult due to second order cone constraints and thus requires more time to solve. On average, nominal MPC takes 0.6 ms, conservative RMPC takes 2 ms, our RMPC takes 2.4 ms and these differences are all statistically significant at the 99.9% confidence level. Generally speaking, SOCP problems are amenable to real-time implementation and we expect this to be the case here [23], [24], [25], [26].

Our RMPC guarantees state and input constraint satisfaction for worst-case uncertainty, is on-par with conservative RMPC for fuel consumption, is practically equally suitable for real-time application, is less conservative and thus enables more valid parameter combinations. It therefore has a net advantage over nominal MPC and conservative RMPC.

VI. FUTURE WORK

This section outlines some future questions and possible extensions to the RMPC law (13). One possible line of work is to extend the noise model (4b) to handle input and state vector direction dependence. Indeed, to reduce conservatism some applications involve direction-dependent noise such as the full version of the Gates model [1]. Some applications also define a non-convex $\mathcal{U}$, for example $\mathcal{U} = \{0\} \cup (\mathcal{U}_i \setminus \mathcal{U}_o)$ where $\mathcal{U}_i \subset \mathcal{U}_o$ are two convex sets such that $\mathcal{U}_i$ defines the set of non-zero input values that the actuators cannot reproduce. This occurs due to e.g. digitization effects or, for satellites, the RCS thruster impulse bit. While the state constraint (12) can handle this situation, Theorem 1 does not apply for a non-convex $\mathcal{U}$ since its proof requires convexity. Furthermore, (25) becomes a non-convex constraint. This problem was previously handled in continuous time via lossless convexification [37], [38], [39] and mixed integer programming [40], however no proof of recursive feasibility has yet been proposed for uncertain discrete time systems. Last but not least, the computation of $\mathcal{I}$ via [31] is expensive and sometimes ineffective for the high-dimensional systems dealt with here. Moreover, it requires over-approximating noise as independent, over-approximating ellipsoids as polytopes and carries the assumption that $N = 1$. Recent work on controllable set computation for convex optimal control problems has the potential to speed up and remove this conservatism [41].

VII. CONCLUSION

This paper introduced a state and input dependent uncertainty model and a corresponding computationally efficient robust receding horizon control law based on convex programming. We have shown that the control law is recursively feasible. Simulations using a high dimensional satellite
system demonstrate that the approach is more versatile than a conservative one by reducing conservatism and thus increasing the set of feasible controller parameters while maintaining performance measures like fuel consumption and computational efficiency.

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