Multisymmetric polynomials in dimension three

Mátéyás Domokos \(^a\) \(^*\) and Anna Puskás \(^b\)

\(^a\) Rényi Institute of Mathematics, Hungarian Academy of Sciences,
1053 Budapest, Reáltanoda utca 13-15., Hungary
E-mail: domokos@renyi.hu

\(^b\) Columbia University, Department of Mathematics,
MC 4406, 2990 Broadway, New York, NY 10027, USA.
E-mail: apuskas@math.columbia.edu

Abstract

The polarizations of one relation of degree five and two relations of degree six minimally generate the ideal of relations among a minimal generating system of the algebra of multisymmetric polynomials in an arbitrary number of three-dimensional vector variables. In the general case of \(n\)-dimensional vector variables, a relation of degree \(2n\) among the polarized power sums is presented such that it is not contained in the ideal generated by lower degree relations.

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1 Introduction

The symmetric group \(S_n\) acts on the \(n\)-dimensional complex vector space \(V := \mathbb{C}^n\) by permuting coordinates. Consider the diagonal action of \(S_n\) on the space \(V^m := V \oplus \cdots \oplus V\) of \(m\)-tuples of vectors from \(V\). The algebra of multisymmetric polynomials is the corresponding ring of invariants \(R_{n,m} := \mathbb{C}[V^m]^{S_n}\), consisting of the polynomial functions on \(V^m\) that are constant along the \(S_n\)-orbits.

In the special case \(m = 1\), \(R_{n,1}\) is a polynomial ring generated by the elementary symmetric polynomials (or by the first \(n\) power sums). It is classically known (see [14], [13], [16]) that the polarizations of the elementary symmetric polynomials constitute a minimal \(\mathbb{C}\)-algebra generating system of \(R_{n,m}\) for an arbitrary \(m\). The ideal of relations among these generators is not completely understood, although it was classically studied in [9], [10], [11], and in a couple of more recent papers (see the references in Section 2).

An explicit finite presentation of \(R_{n,m}\) by generators and relations is known, see Proposition 2.1 below. Note that the price for having a uniform description of the ideal of relations in Proposition 2.1 is the inclusion of redundant elements in the system of generators.

In the present paper for \(n = 3\) we determine a minimal system of generators of the ideal of relations among a minimal generating system of \(R_{3,m}\). We exploit the natural action of the general

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linear group \( GL_m \) on \( R_{n,m} \). Identify \( V^m \) with the space \( \mathbb{C}^{n \times m} \) of \( n \times m \) matrices. The complex general linear group \( GL_m \) acts on \( \mathbb{C}^{n \times m} \) by matrix multiplication from the right: \( x \mapsto xg^{-1} \) \((x \in \mathbb{C}^{n \times m}, g \in GL_m)\). As usual, this induces an action of \( GL_m \) on the coordinate ring \( \mathbb{C}[V^m] \) by linear substitution of variables. Since the action of \( GL_m \) on \( V^m \) commutes with the action of \( S_n \), the algebra \( R_{n,m} \) is a \( GL_m \)-submodule in the coordinate ring \( \mathbb{C}[V^m] \). In particular, we may choose a minimal system of homogeneous \( \mathbb{C} \)-algebra generators of \( R_{n,m} \), whose \( \mathbb{C} \)-linear span is a \( GL_m \)-submodule \( W_{n,m} \) in \( \mathbb{C}[V^m] \). Write \( S(W_{n,m}) \) for the symmetric tensor algebra of \( W_{n,m} \). (This is polynomial ring, with one variable associated to each element of a fixed basis of \( W_{n,m} \).) Endow the algebra \( S(W_{n,m}) \) with the \( GL_m \)-module structure induced by the representation of \( GL_m \) on \( W_{n,m} \). Consider the natural \( \mathbb{C} \)-algebra surjection \( \varphi : S(W_{n,m}) \rightarrow R_{n,m} \) extending the identity map on \( W_{n,m} \). The ideal \( \ker(\varphi) \) of relations among the chosen generators of \( R_{n,m} \) is a \( GL_m \)-submodule of \( S(W_{n,m}) \).

The coordinate ring \( \mathbb{C}[V^m] = \mathbb{C}[x_{ij} \mid i = 1, \ldots, n; j = 1, \ldots, m] \) is an \( nm \)-variable polynomial algebra, where \( x_{ij} \) stands for the \( ij \)th coordinate function on the \( j \)th vector component. Given a monomial \( w = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \) in the \( m \)-variable polynomial algebra \( \mathbb{C}[x_1, \ldots, x_m] \), set

\[
[w] := \sum_{i=1}^{n} x_1^{\alpha_1} \cdots x_m^{\alpha_m}.
\]

These elements of \( R_{n,m} \) are called the \textit{polarized power sums}, and

\[
\{[x_1^{\alpha_1} \cdots x_m^{\alpha_m}] \mid \sum_{j=1}^{m} \alpha_j \leq n\}
\]

is a minimal system of \( \mathbb{C} \)-algebra generators of \( R_{n,m} \).

Denote by \( W_{n,m} \) the subspace of \( R_{n,m} \) spanned by the set \( \{1\} \). This is a \( GL_m \)-submodule. In the case \( n = 3 \), we present three explicit elements in the kernel of the surjection \( \varphi : S(W_{3,m}) \rightarrow R_{3,m} \) such that each of them generates an irreducible \( GL_m \)-submodule in \( S(W_{3,m}) \), and the union of any \( \mathbb{C} \)-bases of these three irreducible \( GL_m \)-submodules constitutes a minimal generating system of the ideal \( \ker(\varphi) \) (see Theorem 3.1).

For arbitrary \( n \) we point out a connection between multisymmetric polynomials and vector invariants of the full orthogonal group, and use this to show that a homogeneous system of generators of the ideal of relations between the polarized power sums must contain a relation of degree \( 2n \) (see Theorem 3.2 for the precise statement).

## 2 Preliminaries

Denote by \( \mathcal{M}_m \) the set of monomials in the polynomial algebra \( \mathbb{C}[x_1, \ldots, x_m] \), and for a natural number \( d \) denote by \( \mathcal{M}_m^d \) the subset of monomials of degree at most \( d \). To each \( w \in \mathcal{M}_m \) associate an indeterminate \( t(w) \), and take the commutative polynomial algebra \( \mathcal{F}_{n,m}^d := \mathbb{C}[t(w) \mid w \in \mathcal{M}_m^d] \) in infinitely many variables. For each \( d \in \mathbb{N} \) it contains the subalgebra \( \mathcal{F}_{n,m} := \mathbb{C}[t(w) \mid w \in \mathcal{M}_m] \). In particular, we identify \( \mathcal{F}_{n,m}^n \) and \( S(W_{n,m}) \) in the obvious way: by definition, \( \{[w] \mid w \in \mathcal{M}_m^n\} \) is a \( \mathbb{C} \)-vector space basis of \( W_{n,m} \), and the map \( [w] \mapsto t(w) \) extends uniquely to a \( \mathbb{C} \)-algebra isomorphism \( S(W_{n,m}) \cong \mathcal{F}_{n,m}^n \). We denote by \( \varphi_{n,m} \) the surjection \( \varphi_{n,m} : \mathcal{F}_{n,m} \rightarrow R_{n,m} \) given by \( \varphi(t(w)) = [w] \) for all \( w \in \mathcal{M}_m \). The restriction of \( \varphi_{n,m} \) to \( \mathcal{F}_{n,m}^d \) will be denoted by \( \varphi_{n,m}^d \); it is a surjection onto \( R_{n,m} \) whenever \( d \geq n \). To simplify notation later in the text, we sometimes write \( \mathcal{F} \) instead of \( \mathcal{F}_{n,m}^n \) and \( \varphi \) instead of \( \varphi_{n,m}^n \).
We recall some elements in the kernel of \( \varphi_{n,m} \). By a distribution of the set \( \{1, \ldots, n+1\} \) we mean a set \( \pi := \{\pi_1, \ldots, \pi_h\} \) of pairwise disjoint non-empty subsets whose union is \( \bigcup_{i=1}^h \pi_i = \{1, \ldots, n+1\} \). Write \( D_{n+1} \) for the set of distributions of \( \{1, \ldots, n+1\} \). Take monomials \( w_1, \ldots, w_{n+1} \in M_m \), and set

\[
\Psi_{n+1}(w_1, \ldots, w_{n+1}) = \sum_{\pi \in D_{n+1}} \prod_{i \in \pi} (-1)(|\pi_i| - 1)! \cdot t(\prod_{s \in \pi} w_s). \tag{2}
\]

**Proposition 2.1** The kernel of the surjection \( \varphi_{n,n+2}^*: F_{n,m}^{n^2-n+2} \to R_{n,m} \) is generated as an ideal by the elements \( \Psi_{n+1}(w_1, \ldots, w_{n+1}) \), ranging over all choices of monomials \( w_i \in M_m \) with \( \deg(w_1 \cdots w_{n+1}) \leq n^2 - n + 2 \).

Proposition 2.1 is a special case of a result in \([6]\) dealing with vector invariants of a class of complex reflection groups. In loc. cit. we first gave a simple short proof of an infinite version about the kernel of \( F_{n,m} \to R_{n,m} \) (see also \([2], [3], [4], [15]\) for related work). Then we applied Derksen’s general degree bound for syzygies from \([5]\) and ideas of Garsia and Wallach from \([8]\) to derive in particular the above finite presentation of \( R_{n,m} \) in \([6]\).

To produce elements in \( \ker(\varphi) = \ker(\varphi_{n,m}^n) = \ker(\varphi_{n,m}) \cap F_{n,m}^n \) we shall start with the relations in Proposition 2.1 belonging to \( \ker(\varphi_{n,m}) \) and eliminate the variables \( t(w) \) with \( \deg(w) > n \). There is one exception, we construct an element \( J \) in \( \ker(\varphi_{n,n}^2) = \ker(\varphi) \cap F_{n,n}^2 \) by another method as follows:

\[
J := \det \begin{pmatrix}
t(x_1^2) & t(x_1x_2) & \cdots & t(x_1x_n) & t(x_1) \\
t(x_2x_1) & t(x_2^2) & \cdots & t(x_2x_n) & t(x_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
t(x_nx_1) & t(x_nx_2) & \cdots & t(x_n^2) & t(x_n) \\
t(x_1) & t(x_2) & \cdots & t(x_n) & n
\end{pmatrix}. \tag{3}
\]

**Proposition 2.2** The element \( J \) belongs to \( \ker(\varphi_{n,n}^2) \), and \( g \cdot J = \det^2(g)J \) for any \( g \in \GL_n \).

*Proof.* Applying \( \varphi \) to the entries of the \( (n+1) \times (n+1) \) matrix in (3) we get the matrix \( X^T \cdot X \), where

\[
X = \begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1n} & 1 \\
x_{21} & x_{22} & \cdots & x_{2n} & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_{n1} & x_{n2} & \cdots & x_{nm} & 1
\end{pmatrix},
\]

and \( X^T \) denotes the transpose of \( X \). Since \( X \) has size \( n \times (n+1) \), the rank of \( X^T X \) is at most \( n \), hence \( \det(X^T X) = 0 \), showing that \( J \in \ker(\varphi) \).

To explain the second statement, let us describe the \( \GL_m \)-action on \( F = S(W_{n,m}) \) more explicitly. First of all, \( \GL_m \) acts by \( \mathbb{C} \)-algebra automorphisms on the \( m \)-variable polynomial ring \( \mathbb{C}[x_1, \ldots, x_m] \). Namely, \( g \in \GL_m \) sends the variable \( x_i \) to the \( i \)th entry of the row vector \( (x_1, x_2, \ldots, x_m) \cdot g \) (matrix multiplication). Note that the degree \( d \) homogeneous component of \( \mathbb{C}[x_1, \ldots, x_m] \) is \( \GL_m \)-stable, and as a \( \GL_m \)-module, it is isomorphic to the \( d \)th symmetric tensor power of the natural \( m \)-dimensional representation of \( \GL_m \) on the space \( \mathbb{C}^m \) of column vectors. Write \( U \) for the \( \mathbb{C} \)-linear span of \( M_m^m \) in \( \mathbb{C}[x_1, \ldots, x_m] \) (so \( U \) is the sum of the homogeneous components of degree \( 1, 2, \ldots, n \)). It is easy to see that the \( \mathbb{C} \)-linear map from \( U \to R_{n,m} \) induced by \( w \mapsto [w] \) (\( w \in M_m^m \)) is a \( \GL_m \)-module isomorphism, so we have \( U \cong W_{n,m} \) as a \( \GL_m \)-module. This shows that the effect of \( g \in \GL_m \) on a variable \( t(w) \) of \( F_{n,m}^m \) is given by the
formula \( g \cdot t(w) = t(g \cdot w) \), where for an arbitrary polynomial \( f = \sum_{w \in \mathcal{M}_m} a_w w \in \mathbb{C}[x_1, \ldots, x_m] \), we shall mean by \( t(f) \) the element \( \sum_{w \in \mathcal{M}_m} a_w t(w) \in \mathcal{F}_{n,m} \).

These considerations show that applying \( g \in \text{GL}_m \) to all entries of the matrix \( Y \) in (3) we get the matrix \((\tilde{g})^T Y \tilde{g}\), where \( \tilde{g} \) stands for the \((n + 1) \times (n + 1)\) block diagonal matrix

\[
\begin{pmatrix}
g & 0 \\
0 & 1
\end{pmatrix}
\]

Therefore our statement follows by multiplicativity of the determinant. \( \Box \)

The idea of applying the \( \text{GL}_m \)-module structure in the study of generators and relations of rings of invariants \( \mathbb{C}[V^m]^G \) (where \( G \) is a group of linear transformations on \( V \)) is well known, see for example section 5.2.7 in [7], or [1] for a recent application. We collect some necessary facts on representations of \( \text{GL}_m \) on polynomial rings.

The representation of \( \text{GL}_m \) on \( S(W_{n,m}) \) is a polynomial representation (cf. [12]). Recall that polynomial \( \text{GL}_m \)-modules are completely reducible. The isomorphism classes of irreducible polynomial representations are labeled by the set \( \text{Par}_m \) of partitions with at most \( m \) non-zero parts. By a partition \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \text{Par}_m \) we mean a decreasing sequence \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0 \) of non-negative integers, and write \( h(\lambda) \) for the number of non-zero parts of \( \lambda \). Given \( \lambda \in \text{Par}_m \) we denote by \( V_\lambda \) a copy of an irreducible polynomial \( \text{GL}_m \)-module labeled by \( \lambda \). For example, \( V_{(k)} \) is isomorphic to the degree \( k \) homogeneous component of \( \mathbb{C}[x_1, \ldots, x_m] \), the \( k \)th symmetric tensor power of the natural \( \text{GL}_m \)-module \( \mathbb{C}^m \), and \( V_{(1^k)} := V_{(1, \ldots, 1)} \cong \Lambda^k(\mathbb{C}^m) \), the \( k \)th exterior power of \( \mathbb{C}^m \). We have the \( \text{GL}_m \)-module isomorphism

\[
W_{n,m} \cong V_{(1)} \oplus V_{(2)} \oplus \cdots \oplus V_{(n)}.
\]

Write \( U_m \) for the subgroup of unipotent upper triangular matrices in \( \text{GL}_m \), and write \( T \cong (\mathbb{C}^\times)^m = \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times \) for the maximal torus consisting of diagonal matrices. Given a polynomial \( \text{GL}_m \)-module \( M \), we say that a non-zero \( v \in M \) is a highest weight vector of weight \( \lambda \) if \( v \) is fixed by \( U_m \). \( T \) stabilizes \( \mathbb{C} v \) and acts on it by the weight \( \lambda \), i.e. \( \text{diag}(z_1, \ldots, z_m) \cdot v = (z_1^{\lambda_1} \cdots z_m^{\lambda_m})v \). In this case the \( \text{GL}_m \)-submodule generated by \( v \) is isomorphic to \( V_\lambda \). The irreducible \( \text{GL}_m \)-module \( V_\lambda \) contains a unique (up to non-zero scalar multiples) vector fixed by \( U_m \), and (up to non-zero scalar multiples), it is the only vector in \( V_\lambda \) on which \( T \) acts by the weight \( \lambda \).

The action of \( T \) defines a \( \mathbb{Z}^m \)-grading on any \( \text{GL}_m \)-module \( M \): \( v \in M \) is multihomogeneous of multidegree \( \alpha = (\alpha_1, \ldots, \alpha_m) \) if \( \text{diag}(z_1, \ldots, z_m) \cdot v = (z_1^{\alpha_1} \cdots z_m^{\alpha_m})v \) for all \( \text{diag}(z_1, \ldots, z_m) \in T \). In particular, \( \mathbb{C}[V^m] \), \( R_{n,m} \), and \( \mathcal{F}_{n,m} \) become \( \mathbb{Z}^m \)-graded algebras this way, and the map \( \varphi : \mathcal{F}_{n,m} \to R_{n,m} \) is multihomogeneous. The polarized power sum \( [x_1^{\alpha_1} \cdots x_m^{\alpha_m}] \) is multihomogeneous of multidegree \( (\alpha_1, \ldots, \alpha_m) \).

The above \( \mathbb{Z}^m \)-grading is a refinement of the usual \( \mathbb{Z} \)-grading on the polynomial algebra \( \mathbb{C}[V^m] \). Similarly, \( R_{n,m} \) and \( \mathcal{F}_{n,m} \) are graded algebras, the degree of a multihomogeneous element of multidegree \( (\alpha_1, \ldots, \alpha_m) \) being \( \sum_{j=1}^m \alpha_j \). Denote by \( \mathcal{F}^{(+)} \) the sum of homogeneous components of positive degree in the graded algebra \( \mathcal{F} = \mathcal{F}_{n,m} \). Then \( \mathcal{F}^{(+)} \) is a maximal ideal and \( \mathcal{F} / \mathcal{F}^{(+)} \cong \mathbb{C} \). The ideal \( \ker(\varphi) = \ker(\varphi_{n,m}^\alpha) \) is homogeneous. By a minimal system of generators of \( \ker(\varphi) \) we mean a set of homogeneous elements that constitutes an irredundant generating system of the ideal \( \ker(\varphi) \). It is well known that a subset \( N \subset \ker(\varphi) \) of homogeneous elements is a minimal generating system of the ideal \( \ker(\varphi) \) if and only if \( N \) is a basis of a \( \mathbb{C} \)-vector space direct complement of \( \mathcal{F}^{(+)} \ker(\varphi) \) in \( \ker(\varphi) \). This shows that although the minimal generating system \( N \) is not unique, for each degree the number of elements in \( N \) of that degree is uniquely determined. Even more, we may assume that \( N \) spans a \( \text{GL}_m \)-submodule \( \text{Span}\{N\} \) of \( \mathcal{F} \), and the \( \text{GL}_m \)-module structure of \( \text{Span}\{N\} \) is uniquely determined by \( \ker(\varphi) \). Indeed, note that an irreducible \( \text{GL}_m \)-submodule of \( \mathcal{F} \) isomorphic to \( V_\lambda \) is contained in the homogeneous component of \( \mathcal{F} \) of degree \( |\lambda| = \sum_{j=1}^m \lambda_j \). Therefore we may take a \( \text{GL}_m \)-module direct complement of
\[ \mathcal{F}^{(+)} \ker(\varphi) \subseteq \ker(\varphi), \] and a homogeneous \( \mathbb{C} \)-basis of this complement is a minimal generating system of \( \ker(\varphi) \) with the desired properties.

**Definition 2.3** For \( \lambda \in \text{Par}_m \) denote by \( \text{mult}_{n,m}(\lambda) \) the multiplicity of the irreducible \( \text{GL}_m \)-module \( V_\lambda \) as a summand in the factor \( \text{GL}_m \)-module \( \ker(\varphi_{n,m}^n)/\ker(\varphi_{n,m}^n)(+) \cdot \ker(\varphi_{n,m}^n) \), and for a partition \( \lambda \) with more than \( m \) non-zero parts set \( \text{mult}_{n,m}(\lambda) = 0 \).

Next we recall that the multiplicities \( \text{mult}_{n,m}(\lambda) \) have only a mild dependence on the parameter \( m \). Note that a highest weight vector \( f \) of weight \( \lambda \) in \( \mathcal{F}_{n,m} \) is in particular multihomogeneous of multidegree \( \lambda \), hence is contained in the subalgebra \( \mathcal{F}_{n,h(\lambda)} \).

**Lemma 2.4** Let \( f \) be a multihomogeneous element of multidegree \( \lambda \) in \( \mathcal{F}_{n,h(\lambda)} \), and let \( m_1 \geq m_2 \geq h(\lambda) \) be positive integers. Then \( f \) is a highest weight vector for the action of \( \text{GL}_{n_1} \) on \( \mathcal{F}_{n_1,m_1} \) if and only if it is a highest weight vector for the action of \( \text{GL}_{n_2} \) on \( \mathcal{F}_{n_2,m_2} \).

**Proof.** This is well known, and follows directly from the rule giving the action of the unipotent subgroup \( U_{m_1} \) (resp. \( U_{m_2} \)) on \( \mathcal{F}_{n_1,m_1} \) (resp. \( \mathcal{F}_{n_2,m_2} \)). \( \square \)

**Corollary 2.5** Let \( \lambda \) be a partition. Then we have

\[
\text{mult}_{n,m}(\lambda) = \begin{cases} 
\text{mult}_{n,h(\lambda)}(\lambda) & \text{if } m \geq h(\lambda); \\
0 & \text{if } m < h(\lambda).
\end{cases}
\]

**Proof.** The case \( m < h(\lambda) \) is trivial by definition of \( \text{mult}_{n,m}(\lambda) \). Suppose \( m > h(\lambda) \). Denote by \( K_{n,m} \) the kernel of \( \varphi : \mathcal{F}_{n,m}^n \to \text{R}_m \), and write \( K_{n,h(\lambda)} := K_{n,m} \cap \mathcal{F}_{n,h(\lambda)}^n \) for the kernel of the restriction of \( \varphi \) to \( \mathcal{F}_{n,h(\lambda)}^n \). Let \( N \) be a \( \text{GL}_{h(\lambda)} \)-module complement of \( (\mathcal{F}_{n,h(\lambda)}^n)^+ K_{n,h(\lambda)} \) in \( K_{n,h(\lambda)} \). Decompose \( N \) as a direct sum \( \bigoplus N_i \) of irreducible \( \text{GL}_{h(\lambda)} \)-modules, and take a highest weight vector \( f_i \) in \( N_i \) for all \( i \). Then by Lemma 2.4 the \( f_i \) are highest weight vectors for the action of \( \text{GL}_m \) on \( \mathcal{F}_{n,m}^n \). Moreover, taking into account the \( \mathbb{Z}^m \)-grading, one can easily show that \( N \cap (\mathcal{F}_{n,m}^n)^+ K_{n,m} = 0 \), hence the \( f_i \) are linearly independent modulo \( (\mathcal{F}_{n,m}^n)^+ K_{n,m} \). This shows the inequality \( \text{mult}_{n,h(\lambda)}(\lambda) \leq \text{mult}_{n,m}(\lambda) \). The proof of the reverse inequality is similar. \( \square \)

To simplify notation, write

\[
\text{mult}_{n}(\lambda) := \text{mult}_{n,h(\lambda)}(\lambda).
\]

The numbers \( \text{mult}_{n}(\lambda) \) (all but finitely many of them are zero) carry all the sensible numerical information on a minimal generating system of the kernel of \( \varphi : \mathcal{F}_{n,m}^n \to \text{R}_m \) by Corollary 2.5. Moreover, setting

\[
h(\ker(\varphi)) := \max\{ h(\lambda) \mid \text{mult}_{n}(\lambda) \neq 0 \}
\]

we have the following:

**Corollary 2.6** The kernel of \( \varphi : \mathcal{F}_{n,m}^n \to \text{R}_m \) is generated as a \( \text{GL}_m \)-stable ideal by its intersection with \( \mathcal{F}_{n,h(\ker(\varphi))}^n \).

**Proof.** Let \( \oplus_i M_i \) be a \( \text{GL}_m \)-module direct complement of \( \mathcal{F}^{(+)} \ker(\varphi) \) in \( \ker(\varphi) \), where the \( M_i \) are irreducible \( \text{GL}_m \)-modules. Let \( f_i \) be a highest weight vector in \( M_i \). Then all the \( f_i \) belong to \( \mathcal{F}_{n,h(\ker(\varphi))}^n \), and the \( \text{GL}_m \)-module \( \oplus_i M_i \) generated by the set \( \{ f_i \} \) generates \( \ker(\varphi) \) as an ideal. \( \square \)
3 Main results

First specialize to \( n = 3 \). The fundamental elements \( \Psi(w_1, w_2, w_3, w_4) := \Psi_3(w_1, w_2, w_3, w_4) \) of \( \ker(\varphi_{3,m}) \) defined in (2) (here \( w_1, w_2, w_3, w_4 \in \mathcal{M}_m \)) take the form

\[
\Psi(w_1, w_2, w_3, w_4) = -6t(w_1w_2w_3w_4) + 2t(w_1w_2w_3)t(w_4) + 2t(w_1w_3w_4)t(w_2) + 2t(w_2w_3w_4)t(w_1) + t(w_1w_2)t(w_3w_4) + t(w_1w_3)t(w_2w_4) + t(w_1w_4)t(w_2w_3)
\]

Next we define an element \( J_{3,2} \) and an element \( J_{4,2} \) in \( \mathcal{F}_{3,2} \); they have multidegree \((3, 2)\), and \((4, 2)\), respectively. To simplify notation, we write \( x, y \) instead of \( x_1, x_2 \), so \( \mathcal{M}_2 \) consists of monomials in the commuting indeterminates \( x, y \).

\[
J_{3,2} := \frac{1}{2} \left( 3\Psi(xy, x, x, y) - 3\Psi(x, x, x, y^2) + \Psi(x, x, x, y)t(y) - \Psi(x, x, y, y)t(x) \right)
\]

\[
J_{4,2} := 3\Psi(xy, xy, x, x) - 3\Psi(x, x, x, xy^2) + 2\Psi(x, x, y)t(xy) - \Psi(x, x, y)t(x^2) - \Psi(x, x, y)t(x^2) - t(x^2)t(y^2) - t(x^3)t(y^2) - t(x^3)t(x^2)t(y^2)
\]

The elements \( J_{3,2} \) and \( J_{4,2} \) are both contained in \( \mathcal{F}_{3,2}^3 \) (i.e. they do not involve variables \( t(w) \) with \( \deg(w) > 3 \)). Indeed, direct calculation shows that

\[
J_{3,2} = 6t(x^2)t(xy) - 3t(xy^2)t(x^2) - 2t(x^2)t(y^2) + t(x^2)(x^2)^2 - 4t(xy^2)t(x^2) + 2t(xy)t(x^2)t(y^2) - 3t(x^3)t(y^2) + 4t(x^2)t(x^2)t(y^2) - t(x^3)t(y^2) - t(x^3)t(x^2)t(y^2)
\]

and

\[
J_{4,2} = 6t(x^2)y^2 + t(xy)^2t(x^2) - 3t(xy)^2t(x^2) - 6t(x^3)t(xy)^2 + 2t(x^2)t(xy)^2t(x^2) + 4t(x^3)t(xy)^2t(y)
\]

Finally, denote by \( J_{2,2,2} \) the element for \( n = 3 \) defined by (3) in general. Clearly, \( J_{2,2,2} \) belongs to \( \mathcal{F}_{2,3}^3 \) and has multidegree \((2, 2, 2)\).

**Theorem 3.1** We have

\[
\mathrm{mult}_3(\lambda) = \begin{cases} 1 & \text{for } \lambda = (3, 2), \lambda = (4, 2), \text{ and } \lambda = (2, 2, 2); \\ 0 & \text{for all other } \lambda. \end{cases}
\]

For \( m \geq 2 \) the elements \( J_{3,2}, J_{4,2} \in \mathcal{F}_{3,m}^3 \) generate irreducible \( \mathrm{GL}_m \)-submodules \( N_{3,m}^m \cong V_{(3,2)} \), \( N_{4,m}^m \cong V_{(4,2)} \) in \( \ker(\varphi) \), and for \( m \geq 3 \) the element \( J_{2,2,2} \in \mathcal{F}_{3,m}^3 \) generates an irreducible \( \mathrm{GL}_m \)-submodule \( N_{2,2,2,m}^m \cong V_{(2,2,2)} \) in \( \ker(\varphi) \). Furthermore, choose arbitrary \( \mathcal{C} \)-bases \( G_{(3,2)}^m, G_{(4,2)}^m, G_{(2,2,2)}^m \) in \( N_{3,m}^m, N_{4,m}^m, N_{2,2,2,m}^m \).

Set \( G^m := G_{(3,2)}^m \cup G_{(4,2)}^m \cup G_{(2,2,2)}^m \) when \( m \geq 3 \) and \( G^2 := G_{(3,2)}^2 \cup G_{(4,2)}^2 \). Then \( G^m \) is a minimal generating system of the kernel of the surjection \( \varphi : \mathcal{F}_{3,m}^3 \rightarrow R_{3,m} \) for any \( m \geq 2 \).
In classical language (see for example [16]), the elements in the GL\(_m\)-module generated by \(f \in \ker(\varphi)\) are called the polarizations of \(f\). So Theorem 3.1 can be paraphrased as follows: the polarizations of \(J_{3,2}, J_{4,2}, J_{2,2,2}\) minimally generate the ideal of relations among the polarized power sums of degree at most three in dimension three.

For an arbitrary \(n\) we show that the ideal of relations between the polarized power sums cannot be generated in degree strictly less than \(2n\):

**Theorem 3.2** Suppose \(m \geq n\). The element \(J \in \ker(\varphi^{n}_{n,m})\) given in (3) does not belong to the ideal \((F^{n}_{n,m})^{(+)} \cdot \ker(\varphi^{n}_{n,m})\). In particular, denoting by \(2^n := (2, \ldots , 2) \in \ Par_n\) the partition with \(n\) non-zero parts, all equal to \(2\), we have \(\text{mult}_n(2^n) = 1\).

Denote by \(\beta(n, m)\) the largest degree of an element in a minimal generating system of the ideal \(\ker(\varphi^{n}_{n,m})\) of relations between the polarized power sums. We summarize our present knowledge of \(\beta(n, m)\). The general upper bound

\[
\beta(n, m) \leq n^2 - n + 2
\]

is pointed out in [6]. By Theorem 3.2 we have the general lower bound

\[
\beta(n, m) \geq 2n \text{ for } m \geq n.
\]

We have \(\beta(2, m) = 4\) for all \(m \geq 2\) (see for example [5]), and by Theorem 3.1 we have

\[
\beta(3, m) = 6 \text{ for all } m \geq 2.
\]

Note that both for \(n = 2\) and \(n = 3\) the exact value of \(\beta(n, m)\) agrees with the general lower bound \(2n\) established here. Moreover, both for \(n = 2\) and \(n = 3\) we have the equality \(h(\ker(\varphi^{n}_{n,m})) = n\) when \(m \geq n\).

4 Reduction to \(m = 4\)

**Proposition 4.1** If \(V_\lambda\) occurs as an irreducible GL\(_m\)-module summand in the degree \(d\) homogeneous component of \(F^{n}_{n,m}\), then \(h(\lambda) \leq \frac{d+1}{2}\).

**Proof.** Denote by \(S^k(M)\) the \(k\)th symmetric tensor power of the GL\(_m\)-module \(M\). The degree \(d\) homogeneous component of \(F^{n}_{n,m}\) is isomorphic as a GL\(_m\)-module to

\[
\bigoplus_{d_1+2d_2+3d_3+\cdots+nd_n=d} S^{d_1}(V_{(1)}) \otimes S^{d_2}(V_{(2)}) \otimes \cdots \otimes S^{d_n}(V_{(n)}).
\]

Note that \(S^{d_i}(V_{(1)}) \cong V_{(d_i)}\), and for \(i \geq 2\), \(S^{d_i}(V_{(i)})\) is a GL\(_m\)-submodule of \(V_{(i)} \otimes \cdots \otimes V_{(i)}\) (\(d_i\) tensor factors), which involves only summands \(V_\lambda\) with \(h(\lambda) \leq d_i\) by Pieri’s rule (I.5.16 in [12]). One concludes by the Littlewood-Richardson rule (I.9.2 in [12] that for the irreducible constituents \(V_\lambda\) of (4) we have \(h(\lambda) \leq 1 + d_2 + d_3 + \cdots + d_n \leq \frac{d+1}{2}\) (the latter inequality follows from \(d = \sum_{i=1}^{n} id_i\)). \(\square\)

**Proposition 4.2** We have the inequality \(h(\ker(\varphi)) \leq (n^2 - n + 2)/2\). Consequently, the kernel of \(\varphi : F_{n,m} \rightarrow R_{n,m}\) is generated as a GL\(_m\)-stable ideal by its intersection with \(F^{n}_{n,(n^2-n+2)/2}\).
Proof. We know from Proposition 2.1 that \( \ker(\varphi) \) is generated as an ideal by the sum \( M \) of its homogeneous components of degree \( \leq n^2 - n + 2 \). Decompose \( M \) as the direct sum \( \bigoplus_i M_i \) of irreducible \( \GL_m \)-modules. By Proposition 4.1 \( M_i \cong V_{\lambda_i} \) for some \( \lambda_i \in \Par_m \) with \( h(\lambda_i) \leq (n^2 - n + 2)/2 \). Since \( M \) contains a \( \GL_m \)-module direct complement of \( (F_{n,m})^{(+)} \ker(\varphi) \) in the \( \GL_m \)-module \( \ker(\varphi) \), it follows that \( \text{mult}_{n,m}(\lambda) = 0 \) when \( h(\lambda) > \frac{n^2 - n + 2}{2} \). This shows the inequality \( h(\ker(\varphi)) \leq n^2 - n + 2 \), implying by Corollary 2.6 the second statement. \( \square \)

For \( n = 3 \) we have \( \frac{n^2 - n + 2}{2} = 4 \), hence by Proposition 4.2 it is sufficient to prove Theorem 5.1 in the special case \( m = 4 \).

5 Minimality

Throughout this section we assume \( n = 3 \). First we determine the \( \GL_m \)-module structure of the kernel \( K_{3,m} \) of \( \varphi : \mathcal{F}^3_{3,m} \to R_{3,m} \) up to degree 6. Denote by \( K_{3,m}^{(d)} \) the degree \( d \) homogeneous component of \( K_{3,m} \). Note that similarly to Corollary 2.8 one has that the multiplicity of \( V_\lambda \) as a summand in the \( \GL_{h(\lambda)} \)-module \( K_{3,h(\lambda)} \) is the same as the multiplicity of \( V_\lambda \) in the \( \GL_m \)-module \( K_{3,m} \) for an arbitrary \( m \geq h(\lambda) \).

Proposition 5.1 We have \( K_{3,m}^{(d)} = 0 \) for \( d \leq 4 \), and for \( d = 5,6 \) the following \( \GL_m \)-module isomorphisms hold:

\[
K_{3,m}^{(5)} \cong V_{(3,2)} \quad \text{for } m \geq 2;
K_{3,m}^{(6)} \cong 2 \cdot V_{(4,2)} + V_{(3,3)} + V_{(3,2,1)} + V_{(2,2,2)} \quad \text{for } m \geq 3.
\]

Proof. The fact that \( K_{3,m}^{(d)} = 0 \) for \( d \leq 4 \) follows for example from Proposition 2.1. Denote by \( (\mathcal{F}^3_{3,m})^{(d)} \) and \( R_{3,m}^{(d)} \) the degree \( d \) homogeneous component of \( \mathcal{F}^3_{3,m} \) and \( R_{3,m} \). By formula (4) we have

\[
(\mathcal{F}^3_{3,m})^{(5)} \cong S^8(V_{(1)}) + S^3(V_{(1)}) \otimes V_{(2)} + S^2(V_{(1)}) \otimes V_{(3)} + V_{(2)} \otimes V_{(3)} + V_{(1)} \otimes S^2(V_{(2)}),
\]

whereas

\[
(\mathcal{F}^3_{3,m})^{(6)} \cong V_{(6)} + V_{(4)} \otimes V_{(2)} + V_{(3)} \otimes V_{(3)} + V_{(2)} \otimes S^2(V_{(2)})
+ V_{(1)} \otimes V_{(2)} \otimes V_{(3)} + S^3(V_{(2)}) + S^2(V_{(3)}).
\]

By Pieri’s rule and some known plethysm formulae (see Section I.8 in [12]) one derives

\[
(\mathcal{F}^3_{3,m})^{(5)} \cong 5 \cdot V_{(5)} + 4 \cdot V_{(4,1)} + 4 \cdot V_{(3,2)} + V_{(2,2,1)}
\]

and

\[
(\mathcal{F}^3_{3,m})^{(6)} \cong 7 \cdot V_{(6)} + 5 \cdot V_{(5,1)} + 8 \cdot V_{(4,2)} + V_{(4,1,1)} + 2 \cdot V_{(3,3)} + 2 \cdot V_{(3,2,1)} + 2 \cdot V_{(2,2,2)}.
\]

To determine the \( \GL_m \)-module structure of \( R_{3,m} \) we start from the action of \( \GL_3 \times \GL_m \) on \( \mathbb{C}[V^m] = \mathbb{C}[x_{ij} \mid i = 1, 2, 3, \ j = 1, \ldots, m] \) by \( \mathbb{C} \)-algebra automorphisms given on the generators as follows: \( (g, h) \in \GL_3 \times \GL_m \) sends \( x_{ij} \) to the \((i, j)\)-entry of the matrix \( g^T(x_{ij})_{i=1,2,3}^j=m \). Cauchy’s formula (I.4.3 in [12]) tells us the \( \GL_3 \times \GL_m \)-module structure of \( \mathbb{C}[V^m] \):

\[
\mathbb{C}[V^m] \cong \bigoplus_{\lambda \in \Par_{\text{min}(3,m)}} V_\lambda \otimes V_\lambda.
\]
Consequently, the multiplicity of $V_{\lambda}$ in $R_{3,m}$ equals $\dim_{C}(V_{\lambda}^{S_3})$, where we identify $S_3$ with the subgroup of permutation matrices in $GL_3$, and we write $V_{\lambda}^{S_3}$ for the subspace of $S_3$-fixed points in the $GL_3$-module $V_{\lambda}$. By the Jacobi-Trudi Formula (I.3.4 in [12]), the character of an element $g \in S_3$ on $V_{\lambda}$ (where $h(\lambda) \leq 3$) equals the determinant of the $3 \times 3$ matrix whose $(i,j)$-entry is $h_{\lambda,-i-j}(z_1, z_2, z_3)$, where $h_{k}(z_1, z_2, z_3)$ is the $k$th complete symmetric polynomial in the eigenvalues $z_1, z_2, z_3$ of $g \in S_3 < GL_3$. On the other hand, if $g \in S_3 < GL_3$ has eigenvalues $z_1, z_2, z_3$, then $h_{k}(z_1, z_2, z_3)$ equals the number of monomials of degree $k$ in the variables $x_1, x_2, x_3$ fixed by $g$ (where $S_3$ acts by permuting the variables). Based on this one can quickly compute the $GL_m$-module structure of $R_{3,m}$, and gets

$$R_{3,m}^{(5)} \cong 5 \cdot V_{(3)} + 4 \cdot V_{(1,1)} + 3 \cdot V_{(3,2)} + V_{(2,2,1)}$$

and

$$R_{3,m}^{(6)} \cong 7 \cdot V_{(6)} + 5 \cdot V_{(5,1)} + 6 \cdot V_{(4,2)} + V_{(4,1,1)} + V_{(3,3)} + V_{(3,2,1)} + V_{(2,2,2)}.$$ 

Since the multiplicity of $V_{\lambda}$ in $K_{3,m}$ equals the difference of the multiplicities of $V_{\lambda}$ in $F_{3,m}^3$ and in $R_{3,m}$, the statement follows. □

**Proposition 5.2** $J_{3,2}, J_{4,2}, J_{2,2,2}$ are highest weight vectors for the action of $GL_m$ on $K_{3,m}$, and none of them is contained in the ideal $(F_{3,m}^3)^{(+)} K_{3,m}$.

Proof. Recall that the multidegree of any element of $V_{\lambda}$ is lexicographically smaller than $\lambda$. Therefore Proposition [5.1] shows that the multihomogeneous component of multidegree $(3,2)$ in $K_{3,m}$ is one-dimensional, and its non-zero elements are the highest weight vectors of the summand $V_{(3,2)}$. On the other hand, $J_{3,2}$ belongs to $K_{3,m}$ and has multidegree $(3,2)$, so it is a highest weight vector. Moreover, since $K_{3,m}$ does not contain elements of degree less than five, we conclude that $J_{3,2}$ is not contained in $(F_{3,m}^3)^{(+)} K_{3,m}$.

Similarly, an inspection of the decomposition of $K_{3,m}$ given in Proposition [5.1] shows that the multihomogeneous component of multidegree $(4,2)$ is two-dimensional, and all its non-zero elements are highest weight vectors generating a submodule isomorphic to $V_{(4,2)}$. Consequently, $J_{4,2}$ is a highest weight vector.

By Proposition [2.2] we know that $g \cdot J_{2,2,2} = \det^2(g)J_{2,2,2}$ for any $g \in GL_3$, hence in the special case $m = 3$, $J_{2,2,2}$ spans a one-dimensional $GL_3$-submodule isomorphic to $V_{(2,2,2)}$. Consequently, $J_{2,2,2}$ is a highest weight vector for any $m \geq 3$ by Lemma [2.4].

Since the minimal degree of an element of $K_{3,m}$ is 5, we have

$$K_{3,m}^{(6)} \cap (F_{3,m}^3)^{(+)} K_{3,m} = \sum_{j=1}^{m} t(x_j)K_{3,m}^{(5)}.$$ 

Note that $J_{4,2}$ contains the term $6t(x_1^2x_2)^2$ and $J_{2,2,2}$ contains the term $3t(x_1x_2)t(x_1x_3)t(x_2x_3)$. We conclude that none of them is contained in $(F_{3,m}^3)^{(+)} K_{3,m}$. □

Denote by $N_{3,m}(3,2), N_{4,m}(3,2), N_{m}(2,2,2)$ the $GL_m$-submodules in $F_{3,m}^3$ generated by $J_{3,2}, J_{4,2}, J_{2,2,2}$.

**Corollary 5.3** $N_{m}(3,2), N_{4,m}(4,2),$ and $N_{m}(2,2,2)$ are irreducible $GL_m$-submodules of $K_{3,m}$ isomorphic to $V_{(3,2)}, V_{(4,2)},$ and $V_{(2,2,2)}$. Moreover, the intersection of $N_{m}(3,2) + N_{m}(4,2) + N_{m}(2,2,2)$ and the ideal $(F_{3,m}^3)^{(+)} K_{3,m}$ is zero.
Proof. Taking into account the multidegrees of $J_{3,2}, J_{4,2}, J_{2,2,2}$, the first statement immediately follows from Proposition [7.2]. Moreover, $N_{(3,2)}^{m}, N_{(4,2)}^{m}, N_{(2,2,2)}^{m}$ are pairwise non-isomorphic irreducible $GL_{m}$-modules and none of them is contained in the $GL_{m}$-module $(F_{3,3}^{3})^{(+)}K_{3,m}$ (again by Proposition [7.2]), hence their sum $N_{(3,2)}^{m} + N_{(4,2)}^{m} + N_{(2,2,2)}^{m}$ is disjoint from $(F_{3,3}^{3})^{(+)}K_{3,m}$ by basic principles of representation theory.

Choose arbitrary $C$-bases $G^{m}_{(3,2)}$, $G^{m}_{(4,2)}$, and $G^{m}_{(2,2,2)}$ in $N_{(3,2)}^{m}$, $N_{(4,2)}^{m}$, and $N_{(2,2,2)}^{m}$, and set $G^{m} := G^{m}_{(3,2)} \cup G^{m}_{(4,2)} \cup G^{m}_{(2,2,2)}$. Then by Corollary [7.3] $G^{m}$ can be extended to a minimal system of generators of the ideal $K_{3,m}$. To prove that $G^{m}$ is actually a minimal system of generators of $K_{3,m}$, it is sufficient to show that the ideal $K_{3,m}$ can be generated by $|G^{m}|$ elements.

Recall that the dimension of the $GL_{m}$-module $V_{\lambda}$ (where $\lambda \in \text{Par}_{m}$) equals

$$d_{\lambda}(m) := \prod_{1 \leq i < j \leq m} \frac{\lambda_{i} - \lambda_{j} + j - i}{j - i}$$

by the Weyl dimension formula (see for example (7.1.17) in [7]), and so

$$|G^{m}| = d_{(3,2)}(m) + d_{(4,2)}(m) + d_{(2,2,2)}(m).$$

6 Hironaka decomposition

It is well known that

$$P := \{[x_{j}], [x_{j}^{2}], [x_{j}^{3}] \mid j = 1, \ldots, m\}$$

is a homogeneous system of parameters in $R_{3,m}$. Write $C[P]$ for the $C$-subalgebra of $R$ generated by $P$. It is a polynomial ring in the $3m$ generators, and $R$ is a finitely generated $C[P]$-module. Moreover, since $R$ is Cohen-Macaulay, it is a free $C[P]$-module. A set $S \subset R_{3,m}$ of homogeneous elements constitutes a free $C[P]$-module generating system of $R$ if and only if if the image of $S$ is a $C$-vector space basis of the factor algebra $R/(P)$ of $R$ modulo the ideal $(P)$ generated by $P$. The elements of $P$ (respectively $S$) are referred to as the primary (respectively secondary) generators of $R_{3,m}$, and

$$R_{3,m} = \bigoplus_{S \subset S} C[P] \cdot S \quad \text{(5)}$$

the Hironaka decomposition of $R_{3,m}$. Write $Q := \{[x_{1}^{\alpha_{1}} \cdots x_{m}^{\alpha_{m}}] \mid \alpha_{1} + \cdots + \alpha_{m} \leq 3\}$ for the chosen minimal $C$-algebra generating system of $R_{3,m}$. We have $Q \supset P$, and we may assume that $S$ consists of products of powers of the elements of $Q$ (in particular, then $S$ consists of multihomogeneous elements, and the empty product $1 \in S$). Recall that the Hilbert series of an $N_{0}^{m}$-graded vector space $A := \bigoplus_{\alpha} A^{\alpha}$ with $\text{dim}_{C}(A^{\alpha}) < \infty$ is the formal power series in $Z[[t_{1}, \ldots, t_{m}]]$ defined by

$$H(A; t_{1}, \ldots, t_{m}) := \sum_{\alpha = (\alpha_{1}, \ldots, \alpha_{m})} \text{dim}_{C}(A^{\alpha}) t_{1}^{\alpha_{1}} \cdots t_{m}^{\alpha_{m}}.$$ 

It follows from [5] that

$$H(R_{3,m}; t_{1}, \ldots, t_{m}) = \frac{H(\text{Span}_{C}(S); t_{1}, \ldots, t_{m})}{\prod_{j=1}^{m}(1 - t_{j})(1 - t_{j}^{2})(1 - t_{j}^{3})} \quad \text{(6)}$$

where $\text{Span}_{C}(S)$ is the $C$-subspace in $R$ spanned by $S$ (since $S$ consists of multihomogeneous elements, it is $N_{0}^{m}$-graded). On the other hand, the Hilbert series of $R$ can be explicitly calculated
(see Section 7), and from this we know the number of elements of \( S \) having multidegree \( \alpha \) for each \( \alpha \).

The following two statements provide our basis to find a complete system of relations.

**Lemma 6.1** Fix a positive integer \( d \), and let \( S \) be a finite set of monomials in the elements of \( Q \), each element of \( S \) having degree at most \( d \), and suppose that \( S \) satisfies the following:

(i) \( 1 \in S \), and \( Q \setminus P \subseteq S \);

(ii) For each \( e = 1, \ldots, d \), the number of degree \( e \) elements in \( S \) equals the number of degree \( e \) elements in a system of secondary generators of \( R_{3,m} \).

(iii) For any \( s \in S \) and \( q \in Q \setminus P \) with \( \deg(s \cdot q) \leq d \) there exist scalars \( \gamma_\alpha \in \mathbb{C} (\alpha \in S, \deg(a) = \deg(sq)) \) with

\[
s \cdot q - \sum_{\deg(a) = \deg(sq)} \gamma_\alpha a \in (P).
\]

Then \( S \) can be extended to a system \( S \) of secondary generators of \( R_{3,m} \) such that \( S \) coincides with the subset of degree \( \leq d \) elements of \( S \).

**Proof.** A straightforward induction on the degree. \( \Box \)

We shall use the following notation: for \( f_1, f_2 \in R \), we write \( f_1 \equiv f_2 \) if \( f_1 - f_2 \in (P) \). For example, (7) reads as

\[
s \cdot q \equiv \sum_{\deg(a) = \deg(sq)} \gamma_\alpha a,
\]
and it means that there exists a multihomogeneous element \( r_s^q \) in the kernel \( K_{3,m} \) of the surjection \( \varphi: F_{3,m}^3 \to R_{3,m} \) such that

\[
r_s^q - s^* \cdot q^* + \sum_{\deg(a) = \deg(sq)} \gamma_\alpha a^* \in \sum_{j=1}^m \sum_{k=1}^3 F_{3,m} \cdot t(x_j^k)
\]

where given a product \( c = [w_1] \cdots [w_l] \) of the generators \( [w_i] \in Q \) we write \( c^* := t(w_1) \cdots t(w_l) \in F \). (Of course, \( r_s^q \) is not unique, it is determined modulo the intersection of \( K_{3,m} \) and the ideal on the right hand side of (8).)

**Lemma 6.2** Suppose that the assumptions of Lemma 6.1 hold. For all \( s \in S \), \( q \in Q \setminus P \) with \( \deg(qs) \leq d \) choose an element \( r_s^q \) satisfying (8). Then the ideal generated by the \( r_s^q \) contains all homogeneous components of \( K_{3,m} \) up to degree \( d \).

**Proof.** Straightforward. \( \Box \)

We shall use also the special case \( n = 3 \) of Lemma 6.1 from [6]:

**Lemma 6.3** If the monomial \( x_1^{\alpha_1} \cdots x_m^{\alpha_m} \in M_m \) has degree at least 3 in one of the variables \( x_1, \ldots, x_m \), and \( \alpha_1 + \cdots + \alpha_m \geq 4 \), then \( [x_1^{\alpha_1} \cdots x_m^{\alpha_m}] \equiv 0. \)
7 Hilbert series

In this section we express the Hilbert series of $R_{3,m}$ in a form that is practical to evaluate for small $m$. The symmetric group $S_3$ has three irreducible complex characters: $\chi_0$, the trivial character, $\chi_1$, the character of the 2-dimensional irreducible representation, and $\chi_2$, the sign character. Denote by $C$ the character ring of $S_3$; i.e. $C$ is the subring of the algebra of central functions on $S_3$ generated by $\chi_0, \chi_1, \chi_2$. It is a free $Z$-module spanned by $\chi_0, \chi_1, \chi_2$. The multiplication in $C$ is given as follows: $\chi_0$ is the identity element, $\chi_1^2 = \chi_0 + \chi_2$, $\chi_2^2 = \chi_0$, $\chi_1 \cdot \chi_2 = \chi_1$. For a graded $S_3$-module $A := \bigoplus_{k=0}^{\infty} A^{(k)}$ we set $H_{\chi_i}(A; t) := \sum_{k=0}^{\infty} \text{mult}_{\chi_i}(A^{(k)}) t^k$, where $\text{mult}_{\chi_i}(A^{(k)})$ denotes the multiplicity of the irreducible representation with character $\chi_i$ as a summand of $A^{(k)}$. Moreover, set

$$H_{S_3}(A; t) := \sum_{i=0}^{2} \chi_i H_{\chi_i}(A; t) \in C[[t]].$$

One defines the Hilbert series of a multigraded $S_3$-module in a similar way. Clearly, the Hilbert series of the multigraded vector space $R^m$ coincides with the coefficient of $\chi_0$ in

$$H_{S_3}(C[V^m]; t_1, \ldots, t_m) \in \sum_{i=0}^{2} \mathbb{Z}[t_1, \ldots, t_m] \chi_i.$$

We have the isomorphism $C[V^m] \cong C[V] \otimes \cdots \otimes C[V]$, hence

$$H_{S_3}(C[V^m]; t_1, \ldots, t_m) = \prod_{j=1}^{m} H_{S_3}(C[V]; t_j)$$

where multiplication is understood in the ring of formal power series $C[[t_1, \ldots, t_m]]$ with coefficients in the character ring $C$ of $S_3$. It is well known that

$$H_{S_3}(V; t) = \frac{\chi_0 + (t + t^2) \chi_1 + t^3 \chi_2}{(1-t)(1-t^2)(1-t^3)}.$$

Taking into account (6) we conclude that the Hilbert series of a system of multihomogeneous secondary generators $S$ (defined in Section 6) equals the coefficient of $\chi_0$ in

$$\prod_{j=1}^{m} (\chi_0 + (t_j + t_j^2) \chi_1 + t_j^3 \chi_2) \in \sum_{i=0}^{2} \mathbb{Z}[t_1, \ldots, t_m] \chi_i. \quad (9)$$

8 The cases $m = 2, 3, 4$

In this section we prove that the kernel $K_{3,4}$ of the surjection $\varphi : \mathcal{F}_{3,4}^{3} \to R_{3,4}$ can be generated by

$$d_{(3,2)}(4) + d_{(4,2)}(4) + d_{(2,2,2)}(4) = 60 + 126 + 10 = 196$$

elements. By the concluding remarks in Section 5 it follows that Theorem 3.1 holds in the special case $m = 4$. This finishes the proof of Theorem 3.1 for arbitrary $m$ by the concluding remark of Section 4.

To simplify notation, we shall write $x, y, z, w$ instead of $x_1, x_2, x_3, x_4$. 

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8.1 The case \( m = 2 \)

(This case is sketched in [6].) By (9) we have

\[
H(S; t, u) = 1 + tu + t^2u + tu^2 + t^2u^2 + t^3u^3.
\]

Set

\[
S := \{1, [xy], [x^2y], [xy^2], [xy]_2, [x^2y][xy^2] \}.
\]

The equality \( \varphi(\Psi(x, x, y, y)) = 0 \) yields the congruence

\[
[x^2y^2][xy] \equiv \frac{1}{3}[xy]^2
\]

(10) and the equality \( \varphi(\Psi(xy, x, x, y)) = 0 \) yields

\[
6[xy \cdot x \cdot y] \equiv 4[xy^2][xy].
\]

It follows by Lemma 6.3 that

\[
[x^2y][xy] \equiv 0.
\]

(11)

As explained before Lemma 6.2 to the congruence (11) there belongs an element \( r_{[xy]} \in K_{3,2} \).

By symmetry in \( x \) and \( y \), we have also the congruence and the corresponding relation:

\[
[x^2y][xy] \equiv 0 \quad \text{implied by} \quad r_{[xy]} \in K_{3,2}.
\]

We have the congruence

\[
6[xy \cdot xy \cdot x \cdot x] \equiv 4[xy^2][xy] + 2[x^2y]^2
\]

(obtained by substituting the factors of \( xy \cdot xy \cdot x \cdot x \) on the left hand side into \( \Psi \)), yielding by Lemma 6.3

\[
[x^2y]^2 \equiv 0 \quad \text{and} \quad r_{[xy]} \in K_{3,2}.
\]

By symmetry, \( [xy^2]^2 \equiv 0 \) from \( r_{[xy^2]} \in K_{3,2} \). Finally, we have

\[
6[xy \cdot xy \cdot x \cdot y] \equiv 5[x^2y^2][xy] + 2[x^2y][xy^2] - [xy]^3
\]

and taking into account Lemma 6.3 and (10) we get

\[
[xy]^3 = -3[x^2y][xy^2] \quad \text{and} \quad r_{[xy]} \in K_{3,2}.
\]

Clearly we can choose \( r_{[xy]} = 0 \). Multiplying the congruence (11) by \( [xy^2] \) we get \( [xy][x^2y][xy^2] \equiv 0 \), hence we can choose

\[
r_{[xy]}[x^2y][xy^2] := t(xy^2) \cdot r_{[xy]}[xy^2].
\]

Similarly, it is easy to see that for the remaining \( s \in S \) and \( q \in Q \setminus P \) the element \( r_{[q]} \) can be chosen from the ideal generated by the 5 elements of \( K_{3,2} \) introduced already. It follows by Lemma 6.1 that \( S \) is a system of secondary generators of \( R_{3,2} \), and by Lemma 6.2 the ideal \( K_{3,2} \) is generated by

\[
r_{[xy]}, \quad r_{[xy^2]}, \quad r_{[x^2y]}, \quad r_{[xy][xy]}, \quad r_{[xy^2]}[xy^2], \quad r_{[xy^2]}[x^2y], \quad r_{[xy^2]}[x^2y].
\]

Moreover, since \( d_{(3,2)}(2) + d_{(4,2)}(2) = 2 + 3 = 5 \), the above is a minimal system of generators of the ideal \( K_{3,2} \).
8.2 The case \(m = 3\)

In Table 1 we collect the monomials in the elements of \(Q \setminus P\) of descending multidegree \(\alpha\) with all \(\alpha_i > 0\), up to total degree 8. Monomials congruent to 0 are indicated by \(*\) (and we indicate by \(\cong\) the multidegrees where all monomials are congruent to 0), and the symbol \(~\) indicates that some non-zero scalar multiples of the given monomials are congruent modulo \((P)\). Table 2 should be interpreted as follows: its second line for example says that in multidegree \((3,1,1)\) we have the congruence \([x^2y][xz] + [x^2z][xy] \equiv 0\). Hence we may choose a multihomogeneous element \(r_{3,1,1} \in K_{3,3}\) of multidegree \((3,1,1)\) differing from \(t(x^2y)t(xz) + t(x^2z)t(xy)\) by an element of the ideal of \(F_3^3\) generated by \(t(x^i), t(y^i), t(z^i), i = 1,2,3\). (From now on we change the notation for the relations, the lower indices indicate their multidegree.)

Note that \(\text{Span}_C(P)\) and the ideal \((P)\) are not \(\text{GL}_m\)-submodules in \(R_{n,m}\). However, they are \(S_m\)-submodules, where we think of the symmetric group \(S_m\) as the subgroup of \(\text{GL}_m\) consisting of permutation matrices. Observe that some of the congruence in Table 2 are symmetric or skew symmetric in two variables, and some is symmetric in \(x, y, z\). We may assume that the corresponding elements \(r_{3,1,1}, r_{2,2,1}\), etc. are chosen so that they also have the corresponding symmetry or skew-symmetry. The last two columns of Table 2 contain the number of \(S_3\)-translates (resp. \(S_3\)-translates) of \(G\) the relation listed in the third column, where we count the \(S_m\)-translates up to non-zero scalar multiples, so by the above observation the number of \(S_m\)-translates of \(r\) equals the index in \(S_m\) of the stabilizer of the congruence corresponding to \(r\).

Denote by \(G\) the relations listed in the third column of Table 2 and all their \(S_3\)-translates. Note that the sum of the numbers in the last but one column of Table 2 equals the cardinality of \(G\), so \(|G| = 43\).

Using the \(S_3\)-translates of the relations in Table 2 one can easily justify the \(*\) symbols and the equivalences \(~\) in Table 1. This means that up to total degree 8, all monomials (having descending multidegree) in \(Q \setminus P\) can be reduced to linear combinations of the monomials given in Table 3 (For multidegrees with \(\alpha_3 = 0\), this was shown already in section 8.1) One can easily see from (9) that for each descending multidegree \(\alpha\), the number of elements in Table 3 with multidegree \(\alpha\) coincides with the coefficient of \(t_1^{\alpha_1}t_2^{\alpha_2}t_3^{\alpha_3}\) in \(H(S; t_1, t_2, t_3)\). Define \(S\) as follows: in descending multidegrees its elements are listed in Table 3 and if \(\beta\) is a multidegree in the \(S_3\)-orbit of some descending multidegree \(\alpha\), then choose a permutation \(\pi \in S_3\) with \(\beta_i = \alpha_{\pi(i)}\) and include in \(S\) the images under \(\pi\) of the elements of multidegree \(\alpha\) in Table 3 (Of course, the set \(S\) is not uniquely defined: for certain multidegrees, say for multidegree \((1,3,1)\) we may choose for \(\pi\) the transposition \((12)\) or the three-cycle \((123)\). However, this does not influence the arguments below.) Since the set \(G\) is (essentially) \(S_3\)-stable, it follows that up to degree \(\leq 8\), all monomials in \(Q \setminus P\) can be reduced to linear combinations of \(S\) using the relations in \(G\). Moreover, \(H(S; t_1, t_2, t_3) = H(S; t_1, t_2, t_3)\). Consequently, by Lemmas 6.1 and 6.2 \(S\) is a system of secondary generators, and \(G\) generates the ideal \(K_{3,3}\) up to degree 8. We know from Proposition 2.1 that \(K_{3,3}\) is generated in degree \(\leq 8\), hence \(G\) generates \(K_{3,3}\). Since \(d_{(3,2)}(3) + d_{(4,2)}(3) + d_{(2,2,2)}(3) = 15 + 27 + 1 = 43 = |G|\), we conclude that \(G\) is a minimal system of generators of the ideal \(K_{3,3}\).

We finish this Section with the proof of the congruences in Table 2. The relations \(r_{3,2}, r_{4,2}, r_{3,3}\) were explained in section 8.1. The relation \(\varphi(\Psi(w_1, w_2, w_3, w_4)) = 0\) implies a congruence of multidegree \(\text{multideg}(w_1w_2w_3w_4)\) of the form

\[
[w_1 \cdot w_2 \cdot w_3 \cdot w_4] \equiv \cdots
\]

where \(\cdots\) is a linear combination of monomials in elements \([u]\) with \(\text{deg}(u) < \text{deg}(w_1w_2w_3w_4)\), which can be written as a polynomial in the elements of \(Q \setminus P\) using (12) below.
$r_{3,1,1}$: $\varphi(\Psi(x^2, x, y, z)) = 0$ and $[x^3yz] \equiv 0$ (by Lemma 6.3) imply

$$0 \equiv 6[x^3yz] = 6[x^2 \cdot x \cdot y \cdot z] \equiv [x^2y][xz] + [x^2z][xy].$$

$r_{2,2,1}^{(1)}, r_{2,2,1}^{(2)}$: Eliminate $[x^2y^2z]$ from the following consequences of the fundamental relation:

$$6[x \cdot y \cdot z \cdot xy] \equiv 3[xy][xyz] + [xz][xy^2] + [x^2y][yz]$$

$$6[x^2 \cdot y \cdot y \cdot z] \equiv 2[x^2y][yz]$$

$$6[x \cdot x \cdot y^2 \cdot z] \equiv 2[xy^2][xz]$$

$r_{4,1,1}$: We have $0 \equiv [x^4yz] = [x^2 \cdot x^2 \cdot y \cdot z] \equiv \frac{1}{3}[x^2y][x^2z]$. 

$r_{3,2,1}^{(2)}$: Follows by $0 \equiv 6[x^3yz^2] = 6[x^2 \cdot x \cdot y^2 \cdot z] \equiv [x^2y^2][xz] + [x^2z][yx^2]$ and (10).

$r_{3,2,1}^{(1)}$: The congruences $0 \equiv 6[x^3yz^2] = 6[xyz \cdot x \cdot x \cdot y] \equiv 2[x^2y][xyz] + 2[x^2yz][xy]$, and

$$[x^2yz] \equiv \frac{1}{3}[xy][xz]$$

(12)

yield $[xy]^2[xz] + 3[xyz][x^2y] \equiv 0$, and this and $r_{3,2,1}^{(2)}$ imply $r_{3,2,1}^{(1)}$. 

$r_{2,2,2}^{(1)}, r_{2,2,2}^{(2)}$: Eliminate $[x^2y^2z^2]$ from the congruences

$$6[x \cdot x \cdot y^2 \cdot z^2] \equiv 2[xy^2][xz^2]$$

$$6[x^2 \cdot y \cdot y \cdot z^2] \equiv 2[x^2y][yz^2]$$

$$6[xyz \cdot x \cdot y \cdot z] \equiv 2[xyz]^2 + [x^2y^2][yz] + [xy^2z][xz] + [xyz^2][xy]$$

$$6[xy \cdot x \cdot y \cdot z^2] \equiv 3[xy][xyz^2] + [x^2y][yz^2] + [xy^2][xz^2]$$

and use (12).

8.3 The case $m = 4$

The arguments are the same as in section 8.2. We just give the corresponding tables and prove the new congruences (involving all the four variables). Tables 4, 5, and 6 deal only with multidegrees $\alpha$ with all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$, since the remaining multidegrees have been taken care in section 8.2.

The congruence in Table 4 corresponding to $r_{2,1,1,1}^{(0)}$ is symmetric in the variables $z, w$, hence the number of its $S_4$-translates is $\frac{24}{2} = 12$. The same holds for $r_{3,1,1,1}$ and $r_{2,1,1,1}^{(3)}$. The congruence corresponding to $r_{2,2,1,1}^{(1)}$ is symmetric in the variables $x, y$ and also in the variables $z, w$, hence the number of its $S_4$-translates is $\frac{24}{2} = 6$. The congruence corresponding to $r_{2,2,1,1}^{(2)}$ is unchanged if we simultaneously interchange $x, y$ and $z, w$, hence the number of its $S_4$-translates is $\frac{24}{2} = 12$.

The set $G$ of all $S_4$-translates of the relations listed in Tables 2 and 4 has cardinality $|G| = 196$. (This agrees with $d(3,2)(4) + d(4,2)(4) + d(2,2,2) = 60 + 126 + 10 = 196$.) Table 5 together with Table 6 give a system of secondary generators up to degree 8 in descending multidegrees.

An inspection of Tables 4 and 6 shows that up to degree 8, all monomials in the elements of $Q \setminus P$ with descending multidegree can be reduced to a linear combination of elements listed...
in Table 5 using the $S_4$-translates of the relations in Tables 2 and 4. We just give some sample examples:

$$[x^2y][xz][xw] \equiv -[x^2z][xy][xw] \equiv [x^2w][xy][xz] \equiv -[x^2y][xz][xw]$$

by the relations $r_{3,1,1}$, $r_{3,0,1,1}$, $r_{3,1,0,1}$, hence all the above products are congruent to 0. The relations $\sim$ in multidegree $(3,3,1,1)$ can be derived as follows (at each congruence we indicate the relation whose $S_m$-translate is used):

$$- \frac{1}{3}[xy]^2[xz][yw] \overset{(1)}{=} \frac{1}{3}[xy]^2[xyz][yw] \overset{(2)}{=} \frac{1}{3}[x^2z][xy^2][yw] \overset{(1)}{=} -[x^2z][y^2w][xy]$$

$$\overset{r_{3,1,1}}{=} [x^2y][y^2w][xz] \overset{r_{3,1,1}^{(1)}}{=} [xy^2][xyw][xz] \overset{r_{3,1,1}^{(2)}}{=} [x^2y][xyw][yz] \overset{r_{3,2,1}^{(1)}}{=} [x^2w][xy^2][yz]$$

$$\overset{r_{3,1,1}}{=} -[x^2w][y^2z][xy] \overset{r_{3,1,1}}{=} [x^2y][y^2z][xw] \overset{r_{3,1,1}^{(1)}}{=} [x^2y][xzw][xw] \overset{r_{3,2,1}^{(2)}}{=} \frac{1}{3}[xy]^2[xw][yz]$$

Furthermore,

$$-[xy]^3[zw] \overset{\text{r}_{3,3}^{(1)}}{=} 3[x^2y][xy^2][zw] \overset{r_{2,1,1,1}^{(1)}}{=} 3[xyz][xy^2][xw] + 3[xyw][xy^2][xz]$$

hence by the above long chain of congruences we conclude

$$[xy]^3[zw] \equiv -6[x^2y][xyz][yw].$$

Finally we verify the four-variable relations.

$r_{2,1,1,1}$: Various substitutions into the fundamental relation yield

$$6[x^2 \cdot y \cdot z \cdot w] \equiv [x^2y][zw] + [x^2z][yw] + [x^2w][yz] \quad (13)$$

$$6[x \cdot xy \cdot z \cdot w] \equiv [x^2y][zw] + 2[xy][xzw] + [xz][xyw] + [xw][xyz] \quad (14)$$

$$6[x \cdot y \cdot xz \cdot w] \equiv [x^2z][yw] + [xy][xzw] + 2[xz][xyw] + [xw][xyz] \quad (15)$$

$$6[x \cdot y \cdot z \cdot xw] \equiv [x^2w][yz] + [xy][xzw] + [xz][xyw] + 2[xw][xyz] \quad (16)$$

Now $\frac{1}{6}((13) + (14) - (15) - (16))$ gives $r_{2,1,1,1}$.

$r_{2,2,1,1}$, $r_{2,2,1,1}$, $r_{2,2,1,1}$: To make calculations more transparent, introduce the following temporary notation for the monomials of $Q \setminus P$ of multidegree $(2,2,1,1)$: $a_1 = [xy]^2[yz]$, $a_2 = [xy][xz][yw]$, $a_3 = [xy][yz][xw]$, $b_1 = [x^2y][yzw]$, $b_2 = [x^2y][xzw]$, $c_1 = [x^2z][y^2w]$, $c_2 = [x^2w][y^2z]$, $d = [xyz][xyw]$. Using $(10)$, $(12)$, their $S_4$-translates, and

$$6[xyzw] \equiv [xy][zw] + [xz][yw] + [xw][yz] \quad (17)$$

various substitutions into the fundamental relation yield
\[
6[xy \cdot xz \cdot y \cdot w] \equiv \frac{1}{3}a_1 + \frac{1}{3}a_2 + \frac{1}{3}a_3 + b_2 + d \quad (18)
\]
\[
6[xy \cdot x \cdot yz \cdot w] \equiv \frac{1}{3}a_1 + \frac{1}{3}a_2 + \frac{1}{3}a_3 + b_1 + d \quad (19)
\]
\[
6[x^2y \cdot y \cdot z \cdot w] \equiv \frac{1}{3}a_1 + \frac{1}{3}a_2 + \frac{1}{3}a_3 + 2b_1 \quad (20)
\]
\[
6[xy \cdot xy \cdot z \cdot w] \equiv \frac{2}{3}a_2 + \frac{2}{3}a_3 + 2d \quad (21)
\]
\[
6[xyz \cdot x \cdot yw \cdot y] \equiv \frac{1}{6}a_1 + \frac{1}{2}a_2 + \frac{1}{6}a_3 + c_1 + d \quad (22)
\]
\[
6[xw \cdot x \cdot yz \cdot y] \equiv \frac{1}{6}a_1 + \frac{1}{6}a_2 + \frac{1}{2}a_3 + c_2 + d \quad (23)
\]

Equations (18), (19), (20) imply
\[
b_1 \equiv b_2 \equiv d \quad (24)
\]
(in particular, relation \(r_{2,2,1,1}^{(3)}\)). From equations (24), (18), and (21) we conclude
\[
a_1 \equiv a_2 + a_3. \quad (25)
\]
Taking the difference of (21) and (22), and eliminating \(a_1\) by (25) we get
\[
\frac{1}{3}a_3 \equiv c_1 - d. \quad (26)
\]
Taking the difference of (21) and (23), and eliminating \(a_1\) by (25) we get
\[
\frac{1}{3}a_2 \equiv c_2 - d, \text{ hence } r_{2,2,1,1}^{(2)}. \quad (27)
\]
Finally, (25), (26), (27) yield
\[
a_1 \equiv 3c_1 + 3c_2 - 6d, \text{ hence } r_{2,2,1,1}^{(1)}. \quad (28)
\]

\(r_{3,1,1,1}:\)
\[
0 \equiv 6[x^3yzw] = 6[x \cdot xy \cdot xz \cdot w] = \frac{2}{3}[xy][xz][xw] + [x^2y][xzw] + [x^2z][xyw] \quad (29)
\]
Permuting \(y, z, w\) we get
\[
0 \equiv \frac{2}{3}[xy][xz][xw] + [x^2y][xzw] + [x^2z][xyw] \quad (29)
\]
\[
0 \equiv \frac{2}{3}[xy][xz][xw] + [x^2y][xzw] + [x^2z][xyw] \quad (30)
\]
Now (28) + (29) - (30) gives \(r_{3,1,1,1}\).
9 The proof of Theorem 3.2

First we point out that the kernel ker(ϕ^2_{n,m}) of the restriction of ϕ to \( F^2_{n,m} \) is described by the second fundamental theorem for vector invariants of the full orthogonal group (cf. Theorem 2.17.A in [16]).

Proposition 9.1 The ideal ker(ϕ^2_{n,m}) is (minimally) generated by the GL_m-submodule of \( F^2_{n,m} \) spanned by J.

Proof. Denote by O(V) the orthogonal group, i.e. O(V) consists of the linear transformations of V = \( \mathbb{C}^n \) preserving the standard quadratic form \( (v_1, \ldots, v_n) \mapsto \sum_{i=1}^{n} v_i^2 \). The orthogonal complement \( V_0 \) of \((1, \ldots, 1) \in V \) consists of the vectors in V with zero coordinate sum. We identify the stabilizer of \((1, \ldots, 1) \) in O(V) with O(V_0) in the obvious way. Note that the elements of S_n as transformations on V do belong to O(V_0). As an immediate corollary of the first fundamental theorem on vector invariants of the orthogonal group (cf. Theorem 2.11.A in [16]) we conclude that \( \varphi(F^2_{n,m}) = \mathbb{C}[V^m]^{O(V_0)} \subset R_{n,m} \). Set \( b_{ij} := [x_i x_j] - \frac{1}{n} [x_i] [x_j] \) for \( 1 \leq i, j \leq m \). The projection from \( V \rightarrow V_0 \) with kernel spanned by \((1, \ldots, 1) \) induces an identification of \( \mathbb{C}[V^m]^{O(V_0)} \) with the subalgebra of \( \mathbb{C}[V^m]^{O(V_0)} \) generated by the \( b_{ij} \). Moreover, \( \mathbb{C}[V^m]^{O(V_0)} \) is an m-variable polynomial ring over \( \mathbb{C}[V^m]^{O(V_0)} \) generated by \([x_1], \ldots, [x_m]\). Denote by L the subalgebra of \( F^2_{n,m} \) generated by \( u_{ij} := t(x_i x_j) - \frac{1}{n}(x_i) t(x_j), 1 \leq i, j \leq n \). By the above considerations, the ideal ker(ϕ^2_{n,m}) is generated by the kernel of the restriction \( \varphi|_L \) of \( \varphi \) to L. Now the kernel of \( \varphi|_L : L \rightarrow \mathbb{C}[V^m]^{O(V_0)} \), \( u_{ij} \mapsto b_{ij} \) is given by Theorem 2.17.A in [16], stating that it is generated by the polarizations of J.

Let I be a highest weight vector in ker(ϕ) of weight \( 2^n = (2, \ldots, 2) \). We claim that I necessarily belongs to \( F^2_{n,m} \), and if I is contained in \( F^2_+ \), ker(ϕ), then I is necessarily contained in \( (F^2_{n,m})^+ \cdot \ker(\varphi^2_{n,m}) \). It is clear that Theorem 3.2 follows from this claim and Proposition 9.1.

To prove this claim, given a polynomial GL_m-module U and a partition λ in Par_m, denote by \( \lambda(U) \) the λ-isotypic component of U (i.e. the sum of the GL_m-submodules of U isomorphic to the irreducible GL_m-module \( V_\lambda \)). Write \( \lambda \subset \mu \) (where \( \lambda, \mu \in \text{Par}_m \)) if \( \lambda_i \leq \mu_i \) for \( i = 1, \ldots, m \). It follows from Pieri's rule that denoting by A the ideal in \( F \) generated by the t(w) with \( \deg(w) \geq 3 \), we have \( A \subset \sum_{\lambda \subset \mu} \lambda(F) \). Since \( F^2_{n,m} \) is a GL_m-module direct complement of A, the 2^n-isotypic component of \( F \) is contained in \( F^2_{n,m} \). In particular, I belongs to \( F^2_{n,m} \).

Again by Pieri’s rule, the ideal generated by \( \lambda(F) \) is contained in \( \sum_{\lambda \subset \mu} \mu(F) \). So if I \( \in F^2_+ \), ker(ϕ), then I \( \in \sum_{\lambda \subset \mu} F^2_+ \cdot \lambda(\ker(\varphi)) \cdot \lambda \). Again since \( F^2_{n,m} \) is a GL_m-module complement of A, we conclude that \( \lambda(\ker(\varphi)) \leq F^2_{n,m} \) whenever \( \lambda \leq 2^n \), hence I \( \in \sum_{\lambda \subset 2^n} F^2_+ \cdot \lambda(\ker(\varphi^2_{n,m})) \). Using the retraction \( F \rightarrow F^2_{n,m} \) with kernel A, we conclude that I is contained in \( (F^2_{n,m})^+ \cdot \ker(\varphi^2_{n,m}) \).
| Multidegree | 3-variable monomials in the elements of $Q \setminus P$ |
|-------------|--------------------------------------------------|
| (1, 1, 1)   | $xyz$                                           |
| (2, 1, 1)   | $xy \parallel xz$                              |
| (3, 1, 1)   | $[x^2y] \parallel [x^2z] \sim [x^2z] \parallel [xy]$ |
| (2, 2, 1)   | $x^2y \parallel [y^2] \parallel [xz]$, $[xyz] \parallel [xy]^*$ |
| (4, 1, 1)*  | $x^2y \parallel [x^2z]^*$                       |
| (3, 2, 1)   | $[xyz]^2 \sim [x^2y] \parallel [y^2z] \sim [x^2z] \parallel [xy]^*$, $[xy] \parallel [yz] \parallel [xz]^*$ |
| (2, 2, 2)   | $[xyz]^2 \sim [x^2y] \parallel [y^2z] \sim [x^2z] \parallel [xy]^*$, $[xy] \parallel [yz] \parallel [xz]^*$ |
| (4, 2, 1)*  | $[xy]^2 \parallel [x^2z]^*$, $[x^2y] \parallel [xy] \parallel [xz]^*$ |
| (3, 3, 1)*  | $[xyz] \parallel [xy]^2$, $[x^2y] \parallel [xy] \parallel [yz]^*$, $[xy] \parallel [x^2z]^*$ |
| (3, 2, 2)   | $[xyz]^2 \sim [x^2y] \parallel [y^2z] \sim [x^2z] \parallel [xy]^*$, $[xy] \parallel [xz] \parallel [yz]^*$ |
| (5, 1, 1)*  | $[x^2y]^2 \parallel [x^2z]^*$, $[x^2y] \parallel [x^2z] \parallel [xy]^*$ |
| (4, 3, 1)*  | $[xyz]^2 \parallel [xy]^2 \parallel [xz]^*$, $[x^2y] \parallel [xy] \parallel [xz]^*$, $[x^2z] \parallel [xy]^2$, $[xy] \parallel [x^2z]^*$ |
| (4, 2, 2)*  | $[x^2y]^2 \parallel [x^2z]^*$, $[x^2y] \parallel [x^2z] \parallel [xy]^*$, $[x^2z] \parallel [y^2z]^*$, $[x^2z] \parallel [y^2z]^*$ |
| (3, 3, 2)*  | $[xyz]^2 \sim [x^2y] \parallel [y^2z] \sim [x^2z] \parallel [xy]^*$, $[xy] \parallel [xz] \parallel [yz]^*$ |

Table 1: 3-variable monomials in $Q \setminus P$

| Multidegree | Congruence | Relation | $\# \{S_3 \text{ translates}\}$ | $\# \{S_4 \text{ translates}\}$ |
|-------------|------------|----------|----------------------------------|----------------------------------|
| (3, 2, 0)   | $[xy] \parallel [x^2y] \equiv 0$               | $r_{3,2}$ | 6                                | 12                               |
| (3, 1, 1)   | $[x^2y] \parallel [xz] + [x^2z] \parallel [xy] \equiv 0$ | $r_{3,1,1}$ | 3                                | 12                               |
| (2, 2, 1)   | $[x^2y] \parallel [yz] - [xy^2] \parallel [xz] \equiv 0$ | $r_{2,2,1}^{(3)}$ | 3                                | 12                               |
| (2, 2, 2)   | $[xy] \parallel [yz] \parallel [xz] \equiv 0$   | $r_{2,2,2}^{(2)}$ | 3                                | 12                               |
| (4, 2, 0)   | $[x^2y]^2 \parallel [x^2z] \equiv 0$           | $r_{4,2}$ | 6                                | 12                               |
| (4, 1, 1)   | $[x^2y] \parallel [x^2z] \equiv 0$             | $r_{4,1,1}$ | 3                                | 12                               |
| (3, 3, 0)   | $[xy]^3 + 3[x^2y] \parallel [xy^2] \equiv 0$   | $r_{3,3}$ | 3                                | 6                                |
| (3, 2, 1)   | $[x^2y] \parallel [yz] - [x^2z] \parallel [xy^2] \equiv 0$ | $r_{3,2,1}^{(3)}$ | 6                                | 24                               |
| (3, 2, 1)   | $[x^2y]^2 \parallel [xz] + 3[x^2z] \parallel [xy^2] \equiv 0$ | $r_{3,2,1}^{(2)}$ | 6                                | 24                               |
| (2, 2, 2)   | $[xy] \parallel [yz] \parallel [xz] \equiv 0$  | $r_{2,2,2}^{(2)}$ | 3                                | 12                               |

Table 2: Relations in the case $m = 3$
### Multidegree

| Degree | Multidegree | Secondary generators |
|--------|-------------|----------------------|
| 0      | (0,0,0)     | 1                    |
| 2      | (1,1,0)     | xy                   |
| 3      | (2,1,0)     | $x^2y$               |
| 3      | (1,1,1)     | xyz                  |
| 4      | (2,2,0)     | $xy^2$               |
| 4      | (2,1,1)     | $xy|xz$              |
| 5      | (3,1,1)     | $x^2yz$              |
| 5      | (2,2,1)     | $x^2y|yz$            |
| 6      | (3,3,0)     | $x^2y|xy^2$          |
| 6      | (3,2,1)     | $x^2y|xyz$           |
| 6      | (2,2,2)     | $xyz^2$              |
| 7      | (3,2,2)     | $x^2y|xz|yz$         |

Table 3: Secondary generators in the case $m = 3$

### Congruence

| Multidegree | Congruence | Relation | % of $S_4$-translates |
|-------------|------------|----------|-----------------------|
| (2,1,1)     | $x^2y|zw \equiv xy|zx + xy|yw|zx$ | $r_{2,1,1}$ | 12 |
| (3,1,1)     | $xy|zx|yw \equiv -3x^2y|zw$ | $r_{3,1,1}$ | 12 |
| (2,2,1)     | $xy|zw \equiv 3x^2y|yw + 3x^2w|y^2z - 6xyz|xyw$ | $r_{2,2,1}$ | 6 |
| (2,2,1)     | $xy|yw \equiv 3x^2w|y^2z - 3xyz|xyw$ | $r_{2,2,1}$ | 12 |
| (2,2,1)     | $x^2y|yw \equiv [xyz][xyw]$ | $r_{2,2,1}$ | 12 |

Table 4: 4-variable relations in the case $m = 4$

### Multidegree

| Secondary generators |
|----------------------|
| (1,1,1)              | $xy|zw$, $xz|yw$, $xw|yz$ |
| (2,1,1)              | $xy|xw$, $xyw|xz$, $xyz|xw$ |
| (3,1,1)              | $x^2y|zw$ |
| (2,2,1)              | $x^2y|zw$, $x^2w|y^2z$, $xyz|xyw$ |
| (3,2,1)              | $x^2y|yw$ |
| (2,2,2)              | $x|zw$, $x^2z|yz$, $z^2|yw$, $x^2w|y^2z|zw$ |

Table 5: 4-variable secondary generators
| Multidegree | 4-variable monomials in the elements of $Q \setminus P$ |
|-------------|-----------------------------------------------------|
| (1, 1, 1)   | $xy|zw|, |xz|yw|, |xz|yw| |
| (2, 1, 1, 1)| $[xy|xzw|, |xz|xyw|, |xw|xyyz|, |
|             | $x^2|yw|, |x^2|yw|, |z^2|y^2w| |
| (3, 1, 1, 1)| $x^2y|xz|yw| \sim |x^2z|y^2x|yw| \sim |x^2y|xyz|yw| \sim |xy|xzw| |
| (2, 2, 1, 1)| $[xyz|xyw| \sim |x^2y|yzw| \sim |x^2z|y^2w| |x^2w|y^2z| |
|             | $[xy^2|zw|, |x^2y|zw|, |xly|xzw| |
| (4, 1, 1, 1)*| $[x^2y|xz|zw|^*, |x^2z|xy|yw|^*, |x^2w|xy|zw|^* |
| (3, 2, 1, 1)| $[x^2y|xz|yw|^*, |x^2y|xz|yw|^*, |x^2y|xz|yw|^*, |x^2y|xz|yw|^* |
|             | $[x^2y|yw|z^2|w|^*, |x^2y|yw|z^2|w|^*, |x^2y|yw|z^2|w|^*, |x^2y|yw|z^2|w|^* |
| (2, 2, 1)| $[x^2y|yw|^*, |x^2y|yw|^*, |x^2y|yw|^*, |x^2y|yw|^* |
| (5, 1, 1, 1)*| $[$ |x^2y|^2|zw|^*, |x^2y|^2|zw|^*, |x^2y|^2|zw|^*, |x^2y|^2|zw|^* |
| (4, 2, 1, 1)*| $[x^2y|^2|zw|^*, |x^2y|^2|zw|^*, |x^2y|^2|zw|^*, |x^2y|^2|zw|^* |
| (3, 3, 1, 1)| $[x^2y|^2|zw|^*, |x^2y|^2|zw|^*, |x^2y|^2|zw|^*, |x^2y|^2|zw|^* |
| (3, 2, 2, 1)| $[x^2y|^2|zw|^*, |x^2y|^2|zw|^*, |x^2y|^2|zw|^*, |x^2y|^2|zw|^* |
| (2, 2, 2, 2)| $[$ |x^2y|^2|zw|^*, |x^2y|^2|zw|^*, |x^2y|^2|zw|^*, |x^2y|^2|zw|^* |

Table 6: 4-variable monomials in $Q \setminus P$
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