Quantum mechanism helps agents combat “bad” social choice rules

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Quantum strategies have been successfully applied to game theory for years. However, as a reverse problem of game theory, the theory of mechanism design is ignored by physicists. In this paper, the theory of mechanism design is generalized to a quantum domain. The main result is that by virtue of a quantum mechanism, agents who satisfy a certain condition can combat “bad” social choice rules instead of being restricted by the traditional mechanism design theory.

Keywords: Quantum games; Mechanism design; Implementation theory.

1. Introduction

Game theory is a very useful tool for investigating rational decision making in conflict situations. It was first founded by von Neumann and Morgenstern. Since its beginning, game theory has been widely applied to many disciplines, such as economics, politics, biology and so on. Compared with game theory, the theory of mechanism design simply concerns the reverse question: given some desirable outcomes, can we design a game that produces them?

As Serrano has described, we suppose that the goals of a group of self-interested agents (or a society) can be summarized in a social choice rule (SCR). An SCR is a mapping that prescribes the social outcome (or outcomes) on the basis of agents’ preferences over the set of all social outcomes. The theory of mechanism design answers the important question of whether and how it is possible to implement different SCRs. According to Maskin and Sjöström, whether or not an SCR is implementable depends on which game theoretic solution concept is used (e.g., dominant strategies and Nash equilibrium). Reference 3 is a fundamental work in the field of mechanism design. It provides an almost complete characterization of social choice rules that are Nash implementable.

In 1999, some pioneering breakthroughs were made in the field of quantum games. The game proposed by Eisert et al showed fascinating “quantum advantages” as a result of a novel quantum Nash equilibrium. Benjamin and Hayden, Du et al, Flitney and Hollenberg investigated multiplayer quantum Prisoner’s Dilemma. Guo et al gave a detailed review on quantum games. As a comparison, so far the theory of mechanism design is still investigated only by economists. To the best of our knowledge, up to now, there is no research in the cross field between...
quantum mechanics and mechanism design. Motivated by quantum games, in this paper, we will investigate what will happen if agents can use quantum strategies in the theory of mechanism design.

Section 2 of this paper recalls some preliminaries of mechanism design published in Ref. 2, while Sec. 3 reformulates the Maskin’s mechanism as a physical mechanism and proves that they are equivalent to each other. Section 4 generalizes the physical mechanism to a quantum domain and proves that under a certain condition, an original Nash implementable social choice rule will no longer be implemented. Section 5 draws the conclusions.

2. Preliminaries

Let \( N = \{1, \ldots, n\} \) be a finite set of agents with \( n \geq 2 \) and \( A = \{a_1, \ldots, a_k\} \) be a finite set of social outcomes. Let \( T_i \) be the finite set of agent \( i \)'s types, and the private information possessed by agent \( i \) is denoted as \( t_i \in T_i \). We refer to a profile of types \( t = (t_1, \ldots, t_n) \) as a state. Let \( T = \prod_{i \in N} T_i \) be the set of states. At state \( t \in T \), each agent \( i \in N \) is assumed to have a complete and transitive preference relation \( \succeq^i_t \) over the set \( A \). We denote by \( \succeq^t = (\succeq^1_t, \ldots, \succeq^i_t, \ldots, \succeq^n_t) \) the profile of preferences in state \( t \). The utility of agent \( i \) for outcome \( a \) in state \( t \) is \( u_i(a, t) : A \times T \to R \), i.e., \( u_i(a, t) \geq u_i(b, t) \) if and only if \( a \succeq^i_t b \). We denote by \( \succ^i_t \) the strict preference part of \( \succeq^i_t \). Fixing a state \( t \), we refer to the collection \( E = < N, A, (\succeq^i_t)_{i \in N} > \) as an environment. Let \( \varepsilon \) be the class of possible environments. A social choice rule (SCR) \( F \) is a mapping \( F : \varepsilon \to 2^A \setminus \{\emptyset\} \). A mechanism \( \Gamma = ((M_i)_{i \in N}, g) \) describes a message or strategy set \( M_i \) for agent \( i \), and an outcome function \( g : \prod_{i \in N} M_i \to A \).

An SCR \( F \) satisfies no-veto if, whenever \( a \succeq^i_b \) for all \( b \in A \) and for all agents \( i \) but perhaps one \( j \), then \( a \in F(E) \). An SCR \( F \) is monotonic if for every pair of environments \( E \) and \( E' \), and for every \( a \in F(E) \), whenever \( a \succeq^i t b \) implies that \( a \succeq^i t b \), there holds \( a \in F(E') \). We assume that there is complete information among the agents, i.e., the true state \( t \) is common knowledge among them. Given a mechanism \( \Gamma = ((M_i)_{i \in N}, g) \) played in state \( t \), a Nash equilibrium of \( \Gamma \) in state \( t \) is a strategy profile \( m^* \) such that: \( \forall i \in N, g(m^*(t)) \succeq^i_t g(m_i, m_{-i}^*(t)), \forall m_i \in M_i \). Let \( N(\Gamma, t) \) denote the set of Nash equilibria of the game induced by \( \Gamma \) in state \( t \), and \( g(N(\Gamma, t)) \) denote the corresponding set of Nash equilibrium outcomes. An SCR \( F \) is Nash implementable if there exists a mechanism \( \Gamma = ((M_i)_{i \in N}, g) \) such that for every \( t \in T \), \( g(N(\Gamma, t)) = F(t) \).

Maskin 3 provided an almost complete characterization of social choice rules that were Nash implementable. The main results of Ref. 3 are two theorems: (i) (Necessity) If an SCR \( F \) is Nash implementable, then it is monotonic. (ii) (Sufficiency) Let \( n \geq 3 \), if an SCR \( F \) is monotonic and satisfies no-veto, then it is Nash implementable. In order to facilitate the following investigation on quantum mechanism, we briefly recall the Maskin’s mechanism as follows 2:

Let \( \mathbb{Z}_+ \) be the set of non-negative integers. Considering the following mechanism \( \Gamma = ((M_i)_{i \in N}, g) \), where agent \( i \)'s message set is \( M_i = A \times T \times \mathbb{Z}_+ \), we denote a
3. Physical mechanism

It can be seen that in the Maskin’s mechanism, a message is an abstract mathematical notion. People usually neglect how it is realized physically. However, the world is a physical world. Any information must be related to a physical entity. Here we assume:

(i) Each agent has a coin and a card. The state of a coin can be head up or tail up (denoted as $H$ and $T$ respectively).

(ii) Each agent $i$ independently chooses a strategic action $\omega_i$ whether to flip his/her coin. The set of agent $i$’s action is $\Omega_i = \{\text{Not flip}, \text{Flip}\}$. An action $\omega_i \in \Omega_i$ chosen by agent $i$ is defined as $\omega_i : \{H, T\} \rightarrow \{H, T\}$. If $\omega_i = \text{Not flip}$, then $\omega_i(H) = H$, $\omega_i(T) = T$; If $\omega_i = \text{Flip}$, then $\omega_i(H) = T$, $\omega_i(T) = H$.

(iii) The two sides of a card are denoted as Side 0 and Side 1. The message written on the Side 0 (or Side 1) of card $i$ is denoted as $\text{card}(i, 0)$ (or $\text{card}(i, 1)$).

(iv) There is a device that can measure the state of $n$ coins and send messages to the designer.

Based on aforementioned assumptions, we reformulate the Maskin’s mechanism $\Gamma = ((M_i)_{i \in N}, g)$ as a physical mechanism $\Gamma^P = ((S_i)_{i \in N}, G)$, where $S_i = \Omega_i \times C_i$, $C_i$ is agent $i$’s card set, $C_i = A \times T \times Z_+ \times A \times T \times Z_+$. A typical card written by agent $i$ is described as $c_i = (\text{card}(i, 0), \text{card}(i, 1))$, where $\text{card}(i, 0) = (a_i, t_i, z_i)$, $\text{card}(i, 1) = (a_i', t_i', z_i')$. A physical mechanism $\Gamma^P = ((S_i)_{i \in N}, G)$ describes a strategy set $S_i$ for agent $i$ and an outcome function $G : \prod_{i \in N} S_i \rightarrow A$. We shall use $S_{-i}$ to express $\prod_{j \neq i} S_j$, and thus, a strategy profile is $s = (s_i, s_{-i})$, where $s_i = (\omega_i, c_i) \in S_i$ and $s_{-i} = (\omega_{-i}, c_{-i}) \in S_{-i}$. A Nash equilibrium of $\Gamma^P$ played in state $t$ is a strategy profile $s^* = (s_1^*, \cdots, s_n^*)$ such that for any agent $i \in N$, $s_i \in S_i$, $G(s_1^*, \cdots, s_n^*) \geq_i G(s_i, s_{-i}^*)$. Figure 1 depicts the setup of a physical mechanism.

From the viewpoint of the designer, the physical mechanism works in the same manner as the Maskin’s mechanism. The working steps of the physical mechanism are shown as follows:

Step 1: Nature selects a state $t \in T$ and assigns $t$ to the agents. Each coin is set head up.

Step 2: In state $t$, if all agents agree that the social choice rule $F$ is Pareto-inefficient (or “bad”), i.e., there exist $i \in T$, $t \neq i$, $\tilde{a} \in F(\tilde{i})$ such that $\tilde{a} \succeq_{\tilde{i}} a \in F(t)$ for every $i \in N$, and $\tilde{a} \succ_j a \in F(t)$ for at least one $j \in N$, then go to Step 4.

Step 3: Each agent $i$ sets $c_i = ((a_i, t_i, z_i), (a_i, t_i, z_i))$ (where $a_i \in A$, $t_i \in T$, $z_i \in Z_+$),
$\omega_i = \text{Not flip}$. Go to Step 5.

Step 4: Each agent $i$ sets $c_i = ((\hat{a}, \hat{t}, 0), (a_i, t_i, z_i))$, then chooses a strategic action $\omega_i \in \Omega_i$ whether to flip coin $i$.

Step 5: The device measures the state of $n$ coins and sends card$(i, 0)$ (or card$(i, 1)$) as $m_i$ to the designer if coin $i$ is head up (or tail up). The designer receives the overall message $m = (m_1, \cdots, m_n)$ and let the final outcome be $G(s) = g(m)$ using rule (1), (2) and (3) defined in the Maskin’s mechanism. END.

![Fig. 1 The setup of a physical mechanism. Each agent has a coin and a card. The state of a coin can be head up or tail up. Each agent independently chooses a strategy whether to flip his/her coin.](image)

**Proposition 1:** Given an SCR $F$ and a state $t \in T$, $N(\Gamma^P, t)$ is equivalent to $N(\Gamma, t)$.

**Proof:** First, define a function $R : \{H, T\} \rightarrow \{0, 1\}$, $R(H) = 0$, $R(T) = 1$. For any $s^* = (s_1^*, \cdots, s_n^*) \in N(\Gamma^P, t)$ and $a = G(s^*)$, if $a$ is generated by Step 4 and 5, then for each agent $i$, let $m_i^* = \text{card}(i, R(\omega_i^* (H)))$; if $a$ is generated by Step 3 and 5, then for each agent $i$, let $m_i^* = \text{card}(i, 0)$. Obviously, $m^* = (m_1^*, \cdots, m_n^*) \in N(\Gamma, t)$.

Next, for any $m^* = (m_1^*, \cdots, m_n^*) \in N(\Gamma, t)$, for each agent $i$, let $s_i^* = (\omega_i^*, c_i^*)$, where $\omega_i^* = \text{Not flip}$, $c_i^* = (m_i^*, m_i^*)$), then $s^* = (s_1^*, \cdots, s_n^*) \in N(\Gamma^P, t)$. □

| State $t_1$ | State $t_2$ |
|------------|------------|
| Apple      | Lily       | Cindy     | Apple      | Lily       | Cindy     |
| $a_3$      | $a_2$      | $a_1$     | $a_4$      | $a_3$      | $a_1$     |
| $a_1$      | $a_1$      | $a_3$     | $a_1$      | $a_2$      |
| $a_2$      | $a_4$      | $a_2$     | $a_2$      | $a_2$      | $a_3$     |
| $a_4$      | $a_3$      | $a_4$     | $a_3$      | $a_4$      |

| $F(t_1) = \{a_1\}$ | $F(t_2) = \{a_2\}$ |

**Example 1:** Let $N = \{\text{Apple, Lily, Cindy}\}$, $T = \{t_1, t_2\}$, $A = \{a_1, a_2, a_3, a_4\}$. In each state $t \in T$, the preference relations $(\succeq^t_i)_{i \in N}$ over the outcome set $A$ and the
corresponding SCR $F$ are given in Table 1. Obviously, $F$ is monotonic and satisfies no-veto. By Maskin’s theorem, $F$ is Nash implementable. The SCR $F$ is “bad” from the viewpoint of the agents because in state $t = t_2$, all agents unanimously prefer a Pareto-efficient outcome $a_1 \in F(t_1)$: for each agent $i$, $a_1 \succ^i_{t_2} a_2 \in F(t_2)$. Therefore when the true state is $t_2$, the physical mechanism enters Step 4.

Since every agent prefers $a_1$ to $a_2$ in state $t_2$, it seems that for each agent $i$, $(a, t, 0) = (a_1, t_1, 0)$ should be a unanimous $\text{card}(i, 0)$, and “Not flip” be the same strategic action. As a result, the outcome $a_1$ may be generated by rule (1). However, $\text{Apple}$ has an incentive to unilaterally deviate from $(a_1, t_1, 0)$ to $(a_4, *, *)$ by flipping her coin, since $a_1 \succ^\text{Apple}_{t_2} a_4, a_4 \succ^\text{Apple}_{t_1} a_1$; $\text{Lily}$ also has an incentive to unilaterally deviate from $(a_1, t_1, 0)$ to $(a_3, *, *)$ by flipping her coin, since $a_1 \succ^\text{Lily}_{t_1} a_3, a_3 \succ^\text{Lily}_{t_2} a_1$. $\text{Cindy}$ has no incentive to deviate from $(a_1, t_1, 0)$ because $a_1$ is her top-ranked outcome in two states. Therefore, $c^\text{Apple} = ((a_1, t_1, 0), (a_4, *, *))$, $c^\text{Lily} = ((a_1, t_1, 0), (a_3, *, *))$, $c^\text{Cindy} = ((a_1, t_1, 0), (a_1, t_1, 0))$.

Note that either $\text{Apple}$ or $\text{Lily}$ can certainly obtain her expected outcome only if just one of them flips her coin and deviates from $(a_1, t_1, 0)$ (If this case happens, rule (2) will be triggered). But this condition is unreasonable, because all agents are rational, nobody is willing to give up and let the others benefit. Therefore, both $\text{Apple}$ and $\text{Lily}$ will flip their coins and deviate from $(a_1, t_1, 0)$. As a result, rule (3) will be triggered. Since $\text{Apple}$ and $\text{Lily}$ both have a chance to win the integer game, the winner is uncertain. Consequently, the final outcome is uncertain between $a_3$ and $a_4$, denoted as $a_3/a_4$.

To sum up, although every agent prefers $a_1$ to $a_2$ in state $t = t_2$, $a_1$ cannot be generated in Nash equilibrium. Indeed, the Maskin’s mechanism makes the Pareto-inefficient outcome $a_2$ be Nash implementable in state $t = t_2$. The underlying reason is just the same as what we have seen in the well-known Prisoner’s Dilemma, i.e., the individual rationality is in conflict with the group rationality. In this sense, the agents cannot combat the “bad” SCR under the classical circumstance.

4. Quantum mechanism

In 2007, Flitney and Hollenberg investigated Nash equilibria in $n$-player quantum Prisoner’s Dilemma. Following their procedures, we define:

$$\hat{\omega}(\theta, \phi) \equiv \begin{bmatrix} e^{i\phi} \cos(\theta/2) & i \sin(\theta/2) \\ i \sin(\theta/2) & e^{-i\phi} \cos(\theta/2) \end{bmatrix},$$

$$\Omega \equiv \{ \hat{\omega}(\theta, \phi) : \theta \in [0, \pi], \phi \in [0, \pi/2] \}, \hat{J} \equiv \cos(\gamma/2)\hat{I} \otimes n + i \sin(\gamma/2)\hat{\sigma}_x \otimes n, \hat{\rho} \equiv \hat{\omega}(0, 0), \hat{D}_n \equiv \hat{\omega}(\pi, \pi/n), \hat{C}_n \equiv \hat{\omega}(0, \pi/n).$$

In order to generalize the physical mechanism to a quantum domain, we revise the assumptions (i) and (ii) of the physical mechanism as follows:

1) Each agent $i$ has a quantum coin $i$ (qubit) and a classical card $i$. The basis vectors $|C\rangle \equiv (1, 0)^T, |D\rangle \equiv (0, 1)^T$ of a quantum coin denote head up and tail up respectively.
2) Each agent independently performs a local unitary operation on his/her own quantum coin. The set of agent i’s operation is $\hat{\Omega}_i = \hat{\Omega}$. A strategic operation chosen by agent $i$ is denoted as $\hat{\omega}_i \in \hat{\Omega}_i$. If $\hat{\omega}_i = \hat{I}$, then $\hat{\omega}_i(\ket{C}) = \ket{C}$, $\hat{\omega}_i(\ket{D}) = \ket{D}$; if $\hat{\omega}_i = \hat{D}_n$, then $\hat{\omega}_i(\ket{C}) = \ket{D}$, $\hat{\omega}_i(\ket{D}) = \ket{C}$. $\hat{I}$ denotes “Not flip”, $\hat{D}_n$ denotes “Flip”.

Based on aforementioned amendments, we generalize the physical mechanism $\Gamma^P = ((S_i)_{i \in N}, G)$ to a quantum mechanism $\Gamma^Q = ((\hat{S}_i)_{i \in N}, \hat{G})$, which describes a strategy set $\hat{S}_i = \hat{\Omega}_i \times C_i$ for each agent $i$ and an outcome function $\hat{G} : \otimes_{i \in N} \hat{\Omega}_i \times \prod_{i \in N} C_i \rightarrow A$. We shall use $\hat{S}_{-i}$ to express $\otimes_{j \neq i} \hat{\Omega}_j \times \prod_{j \neq i} C_j$, and thus, a strategy profile is $\hat{s} = (\hat{s}_i, \hat{s}_{-i})$, where $\hat{s}_i \in \hat{S}_i$ and $\hat{s}_{-i} \in \hat{S}_{-i}$. A Nash equilibrium of a quantum mechanism $\Gamma^Q$ played in state $t$ is a strategy profile $\hat{s}^* = (\hat{s}_1^*, \ldots, \hat{s}_n^*)$ such that for any agent $i \in N$, $\hat{s}_i \in \hat{S}_i$, $\hat{G}(\hat{s}_1^*, \ldots, \hat{s}_n^*) \succeq_i \hat{G}(\hat{s}_1, \ldots, \hat{s}_n)$. Figure 2 depicts the set-up of a quantum mechanism. Its working steps are as follows:

Step 1: Nature selects a state $t \in T$ and assigns $t$ to the agents. The state of every quantum coin is set as $\ket{C}$. The initial state of the $n$ quantum coins is $\ket{\psi_0} = \ket{C \cdots CC}$.

Step 2: In state $t$, if all agents agree that the social choice rule $F$ is “bad”, i.e., there exist $\hat{t} \in T, \hat{t} \neq t, \hat{a} \in F(\hat{t})$ such that $\hat{a} \succeq_j^t a \in F(t)$ for at least one $j \in N$, then go to Step 4.

Step 3: Each agent $i$ sets $c_i = ((a_i, t_i, z_i), (a_i, t_i, z_i))$ (where $a_i \in A, t_i \in T, z_i \in \mathbb{Z}_+$), $\hat{\omega}_i = \hat{I}$. Go to Step 7.

Step 4: Each agent $i$ sets $c_i = ((\hat{a}, \hat{t}, 0), (a_i, t_i, z_i))$. Let $n$ quantum coins be entangled by $\hat{J}. \ket{\psi_1} = \hat{J}\ket{C \cdots CC}$.

Step 5: Each agent independently performs a local unitary operation $\hat{\omega}_i$ on his/her own quantum coin. $\ket{\psi_2} = [\hat{\omega}_1 \otimes \cdots \otimes \hat{\omega}_n] \hat{J}\ket{C \cdots CC}$.

Step 6: Let $n$ quantum coins be disentangled by $\hat{J}^+. \ket{\psi_3} = \hat{J}^+ [\hat{\omega}_1 \otimes \cdots \otimes \hat{\omega}_n] \hat{J}\ket{C \cdots CC}$.
Step 7: The device measures the state of \( n \) quantum coins and sends \( \text{card}(i,0) \) (or \( \text{card}(i,1) \)) as \( m_i \) to the designer if the state of quantum coin \( i \) is \( |C \) (or \( |D \)).

Step 8: The designer receives the overall message \( m = (m_1, \cdots, m_n) \) and let the final outcome \( \hat{G}(\hat{s}) = g(m) \) using rules (1), (2) and (3) defined in the Maskin’s mechanism. END.

Note that if \( \Omega_i \) is restricted to be \( \{I, \hat{D}_n\} \), then \( \hat{\Omega}_i \) is equivalent to \{Not flip, Flip\}. In this way, a quantum mechanism is degenerated to a physical mechanism.

Given \( n \) \( (n \geq 3) \) agents, consider the pay-off to the \( n \)th agent, we denote by \( \$_{C\cdots CC} \) the expected pay-off when all agents choose \( I \) (the corresponding collapsed state is \( |C \cdots CC \)) and denote by \( \$_{C\cdots CD} \) the expected pay-off when the \( n \)th agent chooses \( \hat{D}_n \) and the first \( n - 1 \) agents choose \( I \) (the corresponding collapsed state is \( |C \cdots CD \)). \$_{D\cdots DD} \) and \$_{D\cdots DC} \) are defined similarly. Unlike Flitney and Hollenberg’s requirements on the pay-offs, for the case of quantum mechanism, the requirements on the pay-offs are described as condition \( \lambda \):

(i) \( \lambda_1 \): Given a state \( t \) and an SCR \( F \), there exist \( \hat{t} \in T, \hat{t} \neq t, \hat{a} \in F(\hat{t}) \) such that \( \hat{a} \geq^\dagger a \in F(t) \) for every \( i \in N, \hat{a} \geq^\dagger a \in F(t) \) for at least one \( j \in N, \) and the number of agents that encounter a preference change around \( \hat{a} \) in going from state \( \hat{t} \) to \( t \) is larger than unity. Denote by \( \ell \) the number of these agents. Without loss of generality, let these \( l \) agents be the last \( l \) agents among \( n \) agents.

(ii) \( \lambda_2 \): Consider the pay-off to the \( n \)th agent, \( \$_{C\cdots CC} > \$_{D\cdots DD} \), i.e., he/she prefers the expected payoff of a certain outcome (generated by rule 1) to the expected pay-off of an uncertain outcome (generated by rule 3).

(iii) \( \lambda_3 \): Consider the pay-off to the \( n \)th agent, \( \$_{C\cdots CC} > \$_{C\cdots CD}[1 - \sin^2 \gamma \sin^2(\pi/l)] + \$_{D\cdots DC} \sin^2 \gamma \sin^2(\pi/l) \).

**Proposition 2:** For \( n \geq 3 \), given a state \( t \in T \) and a “bad” SCR \( F \) (from the viewpoint of agents) that is monotonic and satisfies no-veto, by virtue of a quantum mechanism \( \Gamma^Q = ((\hat{S}_i)_{i \in N}, \hat{G}) \), agents satisfying condition \( \lambda \) can combat the “bad” SCR \( F \), i.e., there exists \( \hat{s} \in \mathcal{N}(\Gamma^Q, t) \) such that \( \hat{G}(\hat{s}) \notin F(t) \).

**Proof:** Given a state \( t \) and a “bad” SCR \( F \), since condition \( \lambda_1 \) is satisfied, then there exist \( \hat{t} \in T, \hat{t} \neq t, \hat{a} \in F(\hat{t}) \) such that \( \hat{a} \geq^\dagger a \in F(t) \) for every \( i \in N, \hat{a} \geq^\dagger a \in F(t) \) for at least one \( j \in N, \) and the number of agents that encounter a preference change around \( \hat{a} \) in going from state \( \hat{t} \) to \( t \) is larger than unity, i.e., \( l \geq 2 \). Let these \( l \) agents be the last \( l \) agents among \( n \) agents. Hence, the quantum mechanism enters Step 4. Each agent \( i \) sets \( c_i = ((\hat{a}, \hat{t}, 0), (a_i, t_i, z_i)) \). Let \( c = (c_1, \cdots, c_n) \).

Consider the pay-off to the \( n \)th agent (denoted as Laura), when she plays \( \hat{\omega}(\theta, \phi) \) while the first \( n - l \) agents play \( \hat{I} \) and the middle \( l - 1 \) agents play \( \hat{C}_l = \hat{\omega}(0, \pi/l) \), according to Ref. 9,

\[
\begin{align*}
\left(\$_{\text{Laura}}\right) &= \$_{C\cdots CC} \cos^2(\theta/2)[1 - \sin^2 \gamma \sin^2(\phi - \pi/l)] \\
&+ \$_{C\cdots CD} \sin^2(\theta/2)[1 - \sin^2 \gamma \sin^2(\pi/l)] \\
&+ \$_{D\cdots DC} \sin^2(\theta/2) \sin^2 \gamma \sin^2(\pi/l) \\
&+ \$_{D\cdots DD} \cos^2(\theta/2) \sin^2 \gamma \sin^2(\phi - \pi/l)
\end{align*}
\]
Since condition $\lambda_2$ is satisfied, then $\$_{C \cdots CC} > \$_{D \cdots DD}$. Laura chooses $\phi = \pi/l$ to minimize $\sin^2(\phi - \pi/l)$. As a result,
\[
\langle \$_{Laura} \rangle = \$_{C \cdots CC} \cos^2(\theta/2) + \$_{C \cdots CD} \sin^2(\theta/2)[1 - \sin^2 \gamma \sin^2(\pi/l)] + \$_{D \cdots DC} \sin^2(\theta/2) \sin^2 \gamma \sin^2(\pi/l)
\]
Since condition $\lambda_3$ is satisfied, then Laura prefers $\theta = 0$, which leads to $\langle \$_{Laura} \rangle = \$_{C \cdots CC}$. In this case, $\hat{\omega}_{Laura}(\theta, \phi) = \hat{\omega}(0, \pi/l) = \hat{C}_1$.

By symmetry, in Steps 4 and 5, if the $n$ agents choose $\hat{s}^* = (\hat{\omega}^*, c)$, where $\hat{\omega}^* = (\hat{I}, \cdots, \hat{I}, \hat{C}_1, \cdots, \hat{C}_1)$ (the first $n - l$ agents choose $\hat{I}$, the other $l$ agents choose $\hat{C}_1$), then $\hat{s}^* \in N(\Gamma^Q, t)$. In Step 7, the corresponding collapsed state of $n$ quantum coins is $[C \cdots CC]$ and $m_i = (\hat{a}, \hat{t}, 0)$ for each agent $i \in N$. Consequently, in Step 8, $\hat{G}(\hat{s}^*) = g(m) = \hat{a} \notin F(t)$.

Let us reconsider Example 1. The quantum mechanism enters Step 4 when the true state is $t_2$. Since both Apple and Lily encounter a preference change around $a_1$ in going from state $t_1$ to $t_2$, condition $\lambda_1$ is satisfied. $c_{\text{Apple}} = ((a_1, t_1, 0), (a_4, *, *))$, $c_{\text{Lily}} = ((a_1, t_1, 0), (a_3, *, *))$, $c_{\text{Cindy}} = ((a_1, t_1, 0), (a_1, t_1, 0))$. Let Cindy be the first agent. For any agent $i \in \{\text{Apple}, \text{Lily}\}$, let her be the last agent. Consider the pay-off to the third agent, suppose $\$_{CCC} = 3$ (the corresponding outcome is $a_1$), $\$_{CCD} = 5$ (the corresponding outcome is $a_4$ if $i = \text{Apple}$, and $a_3$ if $i = \text{Lily}$), $\$_{DCC} = 0$ (the corresponding outcome is $a_3$ if $i = \text{Apple}$, and $a_4$ if $i = \text{Lily}$), $\$_{DDD} = 1$ (the corresponding outcome is $a_3/a_4$). Hence, condition $\lambda_2$ is satisfied, and condition $\lambda_3$ becomes: $3 \geq 5[1 - \sin^2 \gamma \sin^2(\pi/2)]$. If $\sin^2 \gamma \geq 0.4$, condition $\lambda_3$ is satisfied. According to Proposition 2, the message corresponding to $\hat{s}^* \in N(\Gamma^Q, t)$ is $m = (m_1, m_2, m_3)$, where $m_1 = m_2 = m_3 = (a_1, t_1, 0)$. Consequently, $\hat{G}(\hat{s}^*) = g(m) = a_1 \notin F(t) = \{a_2\}$.

To help the reader understand the aforementioned result, let the SCR in Table 1 be “No smoking”. Let $a_1$ and $a_2$ denote “Smoke” and “Drink” respectively, then everybody prefers smoking to drinking in state $t_2$. According to the traditional theory of mechanism design, the “No smoking” SCR can always be Nash implemented because it is monotonic and satisfies no-veto. However, by virtue of quantum strategies, the agents can combat the “No smoking” SCR!

**Remark:** In Maskin and Sjöström, the authors used a modulo game instead of the integer game. The rule 3 is replaced by “(3) In all other cases, $g(m) = a_j$, for $j \in N$ such that $j = (\sum_{i \in N} z_i)(\text{mod } n)$”. Similar to aforementioned analysis, it can be derived that the results of this paper still hold.

5. Conclusion

In conclusion, this paper considers what will happen if agents can use quantum strategies in the theory of mechanism design. Two results are obtained: (i) We find that the success of the Maskin’s mechanism is built on an underlying Prisoner’s Dilemma. (ii) Under the classical circumstance, if an SCR is monotonic and satisfies
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no-veto, then no matter whether it is “bad” or not (from the viewpoint of the agents), it can be Nash implemented. However, we find that when the additional condition $\lambda$ is satisfied, an original Nash implementable “bad” SCR will no longer be Nash implementable in the context of a quantum domain.

van Enk and Pike \textsuperscript{11} pointed out that in quantum games, quantum strategies simply constructed a new game and solved it, not the original game. However, from the viewpoint of the designer, the interface between agents and the designer in the quantum mechanism is the same as that in the Maskin’s mechanism. Therefore, from the viewpoint of agents, quantum mechanism helps them combat “bad” social choice rules specified by the designer.

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