Herzog–Schönheim conjecture, vanishing sums of roots of Unity and convex polygons

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ABSTRACT

Let $G$ be a group and $H_1, \ldots, H_s$ be subgroups of $G$ of indices $d_1, \ldots, d_s$, respectively. In 1974, M. Herzog and J. Schönheim conjectured that if $\{H_i x_i\}_{i=1}^{s}$, $x_i \in G$, is a coset partition of $G$, then $d_1, \ldots, d_s$ cannot be pairwise distinct. In this article, we present the conjecture as a problem on vanishing sum of roots of unity and convex polygons and prove some results using this approach.

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1. Introduction

Let $G$ be a group, $s$ a natural number, and $H_1, \ldots, H_s$ be subgroups of $G$. If there exist $x_i \in G$ such that $G = \bigcup_{i=1}^{s} H_i x_i$, and the sets $H_i x_i, 1 \leq i \leq s$, are pairwise disjoint, then $\{H_i x_i\}_{i=1}^{s}$ is a coset partition of $G$ (or a disjoint cover of $G$). In this case, all the subgroups $H_1, \ldots, H_s$ can be assumed to be of finite index in $G$ [21, 27]. We denote by $d_1, \ldots, d_s$ the indices of $H_1, \ldots, H_s$, respectively. The coset partition $\{H_i x_i\}_{i=1}^{s}$ has multiplicity if $d_i = d_j$ for some $i \neq j$. The Herzog–Schönheim conjecture is true for the group $G$, if any coset partition of $G$ has multiplicity.

If $G$ is the infinite cyclic group $\mathbb{Z}$, a coset partition of $\mathbb{Z}$ is $\{d_i \mathbb{Z} + r_i\}_{i=1}^{s}$, $r_i \in \mathbb{Z}$, with each $d_i \mathbb{Z} + r_i$ the residue class of $r_i$ modulo $d_i$. These coset partitions of $\mathbb{Z}$ were first introduced by Erdős [14] and he conjectured that if $\{d_i \mathbb{Z} + r_i\}_{i=1}^{s}$, $r_i \in \mathbb{Z}$, is a coset partition of $\mathbb{Z}$, then the largest index $d_i$ appears at least twice. Erdős’ conjecture was proved independently by Davenport and Rado, and independently by Mirsky and Newman using analysis of complex functions [13, 15, 27, 45]. Furthermore, it was proved that the largest index $d_i$ appears at least $p$ times, where $p$ is the smallest prime dividing $d_i$ [27, 40, 45], that each index $d_i$ divides another index $d_j$, $j \neq i$, and that each index $d_k$ that does not properly divide any other index appears at least twice [45]. We refer also to [31–34, 41] for more details on coset partitions of $\mathbb{Z}$ (also called covers of $\mathbb{Z}$ by arithmetic progressions) and to [17] for a proof of the Erdős’ conjecture using group representations.

In 1974, Herzog and Schönheim extended Erdős’ conjecture for arbitrary groups and conjectured that if $\{H_i x_i\}_{i=1}^{s}$, $x_i \in G$, is a coset partition of $G$, then $d_1, \ldots, d_s$ cannot be pairwise distinct [19]. In the 1980s, in a series of papers, Berger, Felzenbaum and Fraenkel studied the Herzog–Schönheim conjecture for arbitrary groups and conjectured that if $\{H_i x_i\}_{i=1}^{s}$, $x_i \in G$, is a coset partition of $G$, then $d_1, \ldots, d_s$ cannot be pairwise distinct [19].
conjecture [2–4] and in [5] they proved the conjecture is true for the pyramidal groups, a subclass of the finite solvable groups. Coset partitions of finite groups with additional assumptions on the subgroups of the partition have been extensively studied. We refer to [6, 18, 39, 42, 43, 44]. In [25], the authors very recently proved that the conjecture is true for all groups of order less than 1440.

The common approach to the Herzog–Schönheim conjecture is to study it in finite groups. Indeed, given any group $G$, every coset partition of $G$ induces a coset partition of a particular finite quotient group of $G$ with the same indices (the quotient of $G$ by the intersection of the normal cores of the subgroups from the partition) [21]. In [7, 8, 10], we adopt a completely different approach to the Herzog–Schönheim conjecture and in this article, we develop and deepen it further. Instead of finite groups, we consider free groups of finite rank and we develop new tools to the problem that permit us to give conditions on the coset partition of the free group that ensure it has multiplicity. This approach has the advantage that it permits to obtain results on the Herzog–Schönheim conjecture in both the free groups of finite rank and all the finitely generated groups. Indeed, any coset partition of a finitely generated group $G$ induces a coset partition of a free group with the same indices [7].

In order to study the Herzog–Schönheim conjecture in free groups of finite rank, we use the machinery of covering spaces. Let $X$ be the bouquet with $n$ leaves (or the wedge sum of $n$ circles). Its fundamental group is $F_n$, the free group of finite rank $n$. As $X$ is a “good” space (connected, locally path connected and semilocally 1-connected), $X$ has a universal covering space which can be identified with the Cayley graph of $F_n$, an infinite simplicial tree. Furthermore, there exists a one-to-one correspondence between the subgroups of $F_n$ and the covering spaces (together with a chosen point) of $X$, as we now explain.

For any subgroup $H$ of $F_n$ of finite index $d$, there exists a $d$-sheeted covering space $(\tilde{X}_H, p)$ with a fixed basepoint. We call it the Schreier graph of $H$ and denote it by $\tilde{X}_H$. It can be seen also as a finite complete bi-deterministic automaton; fixing the start and the end state at the basepoint, it recognizes the set of elements in $H$. It is called the Schreier coset diagram for $F_n$ relative to the subgroup $H$ [38, p. 107] or the Schreier automaton for $F_n$ relative to the subgroup $H$ [36, p. 102]. The $d$ vertices (or states) correspond to the $d$ right cosets of $H$, each edge (or transition) $Hg\rightarrow Hg^a, g \in F_n$; a generator of $F_n$, describes the right action of $a$ on $Hg$. If we fix the start state at $H$, the basepoint, and the end state at another vertex $Hg$, where $g$ denotes the label of some path from the start state to the end state, then this automaton recognizes the set of elements in $Hg$ and we call it the Schreier automaton of $Hg$ and denote it by $\tilde{X}_{Hg}$.

In general, for any automaton $M$, with alphabet $\Sigma$, and $d$ states (or a directed graph with $d$ vertices), there exists a square matrix $A$ of size $d \times d$, with entry $(A)_{ij} = a_{ij}$ equal to the number of directed edges from vertex $i$ to vertex $j$, $1 \leq i, j \leq d$. This matrix is non-negative and it is called the transition matrix [12]. If for every $1 \leq i, j \leq d$, there exists $m_{ij} \in \mathbb{Z}^+$ such that $(A^m)_{ij} > 0$, the matrix is irreducible. For an irreducible non-negative matrix $A$, the period of $A$ is the gcd of all $m \in \mathbb{Z}^+$ such that $(A^m)_{ij} > 0$ (for any $i$). If $i$ and $j$ denote, respectively, the start and end states of $M$, then $A_k$, the number of words of length $k$ (in the alphabet $\Sigma$) accepted by $M$, is equal to $(A^k)_{ij}$. The generating function of $M$ is defined by $p(z) = \sum_{k=0}^{\infty} A_k z^k$. It is a rational function: the fraction of two polynomials in $z$ with integer coefficients [12, 37, p. 575].

In [8], we initiate the study of the transition matrices and generating functions of the Schreier automata in the context of coset partitions of the free group. Let $F_n = (\Sigma), \Sigma^*$ the free monoid generated by $\Sigma$. Let $\{H_i x_i\}_{i=1}^{\infty}$ be a coset partition of $F_n$ with $H_i < F_n$ of index $d_i > 1$, $x_i \in F_n, 1 \leq i \leq s$. Let $A_i$ denote the transition matrix of the Schreier graph of $H_i$, with period $h_i \geq 1$ and let $p_i(z)$ denote the generating function of the Schreier automaton of $H_i x_i, 1 \leq i \leq s$. For every $i$, $A_i$ is a non-negative irreducible matrix and there is an entry of $(A_i)^k$ equal to $A_i^{k, k}$, the number of words of length $k, k \geq 0$, in $H_i x_i \cap \Sigma^*$. Since $F_n$ is the disjoint union of the sets $\{H_i x_i\}_{i=1}^{\infty}$, each element in $\Sigma^*$ belongs to one and exactly one such set, so $n^k$, the number of
words of length \( k \) in \( \Sigma^* \), satisfies \( n^k = \sum_{i=1}^{i=s} A_{i,k} \), for every \( k \geq 0 \), and moreover \( \sum_{k=0}^{k=\infty} n^k z^k = \sum_{i=1}^{i=s} p_i(z) \).

Using this kind of counting argument, we prove that if \( h = \max\{h_i \mid 1 \leq i \leq s\} \) is greater than 1, then there is a repetition of the maximal period \( h > 1 \) and that, under certain conditions, the coset partition has multiplicity [8]. Furthermore, we recover the Davenport–Rado result (or Mirsky–Newman result) for the Erdős’ conjecture and some of its consequences. In this article, we deepen further study of the transition matrices and generating functions of the Schreier automata in the context of coset partitions of the free group. Indeed, using elements from the Perron–Frobenius theory of irreducible non-negative matrices, we study the behavior of the generating functions at some special poles.

For every \( 1 \leq i \leq s \), the transition matrix \( A_i \) of \( \tilde{X}_i \), the Schreier graph of \( H_i < F_n \), is a non-negative and irreducible matrix with Perron–Frobenius eigenvalue \( n \) (the number of free generators of \( F_n \)) (see Section 2.3). Indeed, as the sum of each row and each column in \( A_i \) is equal to \( n \), \( n \) is the positive simple eigenvalue of maximal absolute value of \( A_i \). If \( A \) is an irreducible non-negative matrix with period \( h > 1 \), then \( A \) has exactly \( h \) complex simple eigenvalues of maximal absolute value. Namely, \( n \omega^k, \ 0 \leq k \leq h - 1 \), where \( \omega = e^{2\pi i} \) is the root of unity of order \( h \). Moreover, the matrix \( A \) is similar to \( \omega A \), that is the spectrum of \( A \) is invariant under multiplication by \( \omega \). If \( A \) is an irreducible non-negative matrix with period \( h = 1 \), that is \( A \) is irreducible and aperiodic, then it satisfies many properties similar to those of the positive matrices.

Given a coset partition of \( F_n \), \( P = \{H_i x_i\}_{i=1}^{i=s} \), with \( H_i < F_n \) of index \( d_i > 1 \), \( x_i \in F_n \). We say that \( H_i \) has period \( h_i \), if its Schreier graph has a transition matrix with period \( h_i \), and we say that \( H_i \) has generating function \( p_i(z) \) if \( p_i(z) \) is the generating function of the Schreier automaton of \( H_i x_i \). We consider the case of coset partitions with at least one of the matrices \( A_i \) not aperiodic, that is \( h = \max\{h_i \mid 1 \leq i \leq s\} > 1 \). In this case, as said above, \( |J| > 1 \), where \( J = \{j \mid 1 \leq j \leq s, \ h_j = h\} \) [8]. For every \( j \in J \), the set \( \{1/n \omega^k \mid \omega = e^{2\pi i}/n, \ 0 \leq k \leq h - 1\} \), is a set of simple poles of \( p_j(z) \) and \( \sum_{j \in J} \text{Res}(p_j(z), 1/n) = 0 \), since \( \sum_{j=1}^{j=s} p_i(z) = \sum_{k=0}^{k=\infty} n^k z^k = 1 / (1 - nz) \) and \( \text{Res}(1/n, 1/n) = 0 \), where \( \text{Res}(f(z), z_0) \) denoted the residue of \( f(z) \) at \( z_0 \).

Using our computations of the residues at these simple poles, the equation \( \sum_{j \in J} \text{Res}(p_j(z), 1/n) = 0 \) is a vanishing sum of roots of unity of order \( h \) with positive integer coefficients. Furthermore, we show that the coset partition \( P \) induces a set of irreducible vanishing sums of roots of unity of order \( h \) with positive integer coefficients, that is vanishing sums of roots of unity with the property that no proper non-empty subsums vanish. A vanishing sum of roots of unity with positive integer coefficients can be interpreted as a convex \( k \)-sided polygon with integral sides whose angles are rational when measured in degrees, called a associated convex \( k \)-sided polygon [24, 35]. So, to each such sum there is an associated convex polygon.

**Theorem 1.** Let \( F_n \) be the free group on \( n \geq 2 \) generators. Let \( P = \{H_i x_i\}_{i=1}^{i=s} \) be a coset partition with \( H_i < F_n \) of index \( d_i \), \( x_i \in F_n \), and \( 1 < d_1 \leq \ldots \leq d_s \), with period \( h_i \geq 1, 1 \leq i \leq s \). Assume \( h > 1 \), where \( h = \max\{h_i \mid 1 \leq i \leq s\} \). Let \( \omega = e^{2\pi i} \). Let \( J = \{j \mid 1 \leq j \leq s, \ h_j = h\} \). Then, \( P \) induces a vanishing sum of roots of unity of order \( h \) of the following form, with \( m_j \) natural numbers, \( 0 \leq m_j \leq d_j \):

\[
\sum_{j \in J} \left( \prod_{i \in J \setminus \{j\}} d_i \right)^{m_j} = 0
\]

Furthermore, \( P \) induces a set of irreducible vanishing sums of roots of unity of order \( h \) of the following form, where \( J' \subseteq J \):
\[ \sum_{j \in J} \left( \prod_{i \in J', i \neq j} d_i \right) (\omega)^m_j = 0 \]

To each irreducible sum, there is associated a convex polygon \( \mathcal{G} \) with \( |J'| \) sides. Using the construction from Theorem 1, we can translate the HS conjecture as a problem in terms of planar geometry.

**Theorem 2.** Let \( F_n \) be the free group on \( n \geq 2 \) generators. Let \( P = \{ H_i \}_{i=1}^{s} \) be a coset partition with \( H_i < F_n \) of index \( d_i, x_i \in F_n, 1 < d_1 \leq \ldots \leq d_s \), with period \( h_i \geq 1, 1 \leq i \leq s \). Assume \( h > 1 \), where \( h = \max \{ h_i \mid 1 \leq i \leq s \} \). Then \( P \) has multiplicity if and only if there is an associated convex polygon \( \mathcal{G} \) with at least two edges of the same length.

Using the construction from Theorem 1, and general results on irreducible vanishing sum of roots of unity, we prove, under certain conditions on \( h \), that a coset partition has multiplicity.

**Theorem 3.** Let \( F_n \) be the free group on \( n \geq 2 \) generators. Let \( P = \{ H_i \}_{i=1}^{s} \) be a coset partition with \( H_i < F_n \) of index \( d_i, x_i \in F_n, 1 \leq i \leq s, 1 < d_1 \leq \ldots \leq d_s \). Let \( \tilde{X}_i \) denote the Schreier graph of \( H_i \) with transition matrix \( A_i \) of period \( h_i \geq 1, 1 \leq i \leq s \). Assume \( h > 1 \), where \( h = \max \{ h_i \mid 1 \leq i \leq s \} \). Then \( P \) has multiplicity in the following cases:

i. If \( h = p^n \), where \( p \) is a prime.
ii. If \( h = p^nq^m \), where \( p, q \) are primes.

Furthermore, in these cases, all the associated convex polygons are regular.

The article is organized as follows. In Section 2, we give some preliminaries on vanishing sums of roots of unity, automata and their generating functions, and non-negative irreducible matrices. In Section 3, we present some preliminary results on the Schreier automaton of a coset of a subgroup of \( F_n \), and on its generating function. We prove some properties of the generating function, and, in particular, we compute the residues of the generating function at some special poles. In Section 4, we present the irreducible vanishing sum of roots of unity and its associated polygon induced by a coset partition, and we prove the main results. In many places, we write the HS conjecture instead of the Herzog–Schönheim conjecture.

**2. Preliminaries**

**2.1. About vanishing sum of roots of unity and their induced convex polygons**

We refer the reader to [22–24, 35]. A vanishing sum of roots of unity of length \( k \) is an equation of the form

\[ \sum_{j=1}^{j=k} a_j \zeta_j = 0 \quad (2.1) \]

where the \( a_j \) belong to \( \mathbb{C}^* \) and the \( \zeta_j \) are roots of unity. In [35] and [24], the authors consider such equations with \( a_j \) in \( \mathbb{Z} \) and we use their terminology. If the coefficients \( a_j \) are positive integers, then Equation 2.1 can be interpreted as a convex \( k \)-sided polygon with integral sides whose angles are rational when measured in degrees. In this case, Equation 2.1 is also called a \( k \)-sided polygon. The Equation 2.1 is called degenerate if two of the \( \zeta_j \) are equal. It is called irreducible if there is no relation \( \sum_{j=1}^{j=k} b_j \zeta_j = 0, b_j(a_j - b_j) = 0, 1 \leq j \leq k \), where at least one but not all \( b_j = 0 \), that is there is no proper non-empty subsum that vanishes. It is called primitive if \( \gcd(a_1, \ldots, a_k) = 1 \) and if there is no non-empty relation \( \sum_{j=1}^{j=k} b_j \zeta_j = 0 \), where \( b_j = 0 \) for at least
one $j$, that is $k$ is the minimal length of a vanishing sum with the $\zeta_p$. It is called minimal if there is no non-empty relation $\sum_{j=1}^{\overline{j=k}} b_j \zeta_j = 0$, where $0 \leq b_j \leq a_j$ for every $j$. Every polygon is a linear combination with positive coefficients of minimal polygons. A primitive equation $\sum_{j=1}^{\overline{j=k}} a_j \zeta_j = 0$ with positive coefficients is minimal [24, Th.3]. Every $k$-sided polygon may be obtained from a finite set of minimal polygons of $k$ or fewer sides and there is only a finite number of classes of congruent minimal polygons of given size [24].

**Theorem 2.1.** [24, Th.1] If $\sum_{j=1}^{\overline{j=k}} a_j \zeta_j = 0$, with $a_j \in \mathbb{Z}$, is irreducible, then there are distinct primes $p_1, p_2, \ldots, p_t$ where $p_1 < p_2 < \ldots < p_t \leq k$ and $p_1 p_2 \ldots p_t$-th roots of unity $\zeta_j$ such that $\zeta_j = \eta_j \zeta, 1 \leq j \leq k$, $\zeta$ any root of unity. Moreover, if $(*)$ is an irreducible polygon and if we cannot choose $p_i < k$, then we can choose $t=1$. In the latter case all $a_j$ are equal and $(*)$ represents a regular $k$-sided polygon.

From Theorem 2.1, any irreducible equation $\sum_{j=1}^{\overline{j=k}} a_j \zeta_j = 0$ can be transformed into an irreducible equation $\sum_{j=1}^{\overline{j=k}} a_j \zeta_j = 0$, where $\omega = e^{2\pi i / P}$ is a root of unity of order $P$ with $P$ a divisor of $p_1 p_2 \ldots p_t$ (up to rotation by some $\zeta$). In [24], there is a classification of the primitive and irreducible $k$-sided polygons with $k \leq 7$. Most of them are regular and in the other cases they have at least two edges of the same length.

In [22], the authors study the following question: given a natural number $h$, what are the possible values of the length $k$ of a vanishing sum of roots of unity of order $h$? They show that for any $h = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}$, where $p_1, \ldots, p_t$ are different primes, $k$ belongs to $\mathbb{N} p_1 + \ldots + \mathbb{N} p_t$ [22, p. 92]. This result implies that any non-empty vanishing sum of $h$-th roots of unity must have length at least $p_1$, where $p_1$ is the smallest prime dividing $h$. Furthermore, they show:

**Theorem 2.2.** [22, Th.2.2, Cor.3.4] Let $\sum_{j=1}^{\overline{j=k}} a_j \zeta_j = 0$ be a vanishing sum of roots of unity of order $h$, $\omega = e^{2\pi i / P}$, with $a_j \in \mathbb{Z}$. Then

i. If $h = p^n$, where $p$ is a prime. Then, up to a rotation, the only irreducible vanishing sums of roots of unity are $1 + \zeta_p + \ldots + \zeta_p^{p-1} = 0$, where $\zeta_p$ is the root of unity of order $p$.

ii. If $h = p^n q^m$, where $p, q$ are primes. Then, up to a rotation, the only irreducible vanishing sums of roots of unity are $1 + \zeta_p + \ldots + \zeta_p^{p-1} = 0$ and $1 + \zeta_q + \ldots + \zeta_q^{q-1} = 0$, where $\zeta_p$ and $\zeta_q$ are the roots of unity of order $p$ and $q$ respectively.

iii. If $h = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}$, where $p_1, \ldots, p_t$ are different primes, then any $\mathbb{Z}$-linear relation among the $h$-th roots of unity can be obtained from the basic relations $1 + \zeta_{p_i} + \ldots + \zeta_{p_i}^{p_i-1} = 0, 1 \leq i \leq t$, by addition, subtraction and rotation.

### 2.2. Automata and generating function of their language

We refer the reader to [36, p. 96], [11, p. 7], [12, 29, 30]. A finite state automaton is a quintuple $(S, \Sigma, \mu, Y, s_0)$, where $S$ is a finite set, called the state set, $\Sigma$ is a finite set, called the alphabet, $\mu : S \times \Sigma \to S$ is the transition function, $Y$ is a (possibly empty) subset of $S$ called the accept (or end) states, and $s_0 \in S$ is called the start state. It is a directed graph with vertices the states and each transition $s \to s'$ between states $s$ and $s'$ is an edge with label $a \in \Sigma$. The label of a path $p$ of length $n$ is the product $a_1 a_2 \ldots a_n$ of the labels of the edges of $p$. The finite state automaton $M = (S, \Sigma, \mu, Y, s_0)$ is deterministic if there is only one initial state and for every label from $S$, each state is the source of at most one edge with this label. In a deterministic automaton, a path is determined by its starting point and its label [36, p. 105]. It is co-deterministic if there is only one final state and for every label from $S$, each state is the target of at most one edge with this label. The automaton $M = (S, \Sigma, \mu, Y, s_0)$ is bi-deterministic if it is both deterministic and co-deterministic. An automaton $M$ is complete if for each state $s \in S$ and for each $a \in \Sigma$, there is exactly one edge from $s$ labeled $a$. 
Definition 2.3. Let $M = (S, \Sigma, \mu, Y, s_0)$ be a finite state automaton. Let $\Sigma^*$ be the free monoid generated by $\Sigma$. Let $\text{Map}(S, S)$ be the monoid consisting of all maps from $S$ to $S$. The map $\phi : \Sigma \rightarrow \text{Map}(S, S)$ given by $\mu$ can be extended in a unique way to a monoid homomorphism $\phi : \Sigma^* \rightarrow \text{Map}(S, S)$. The range of this map is a monoid called the transition monoid of $M$, which is generated by $\{\phi(a) \mid a \in \Sigma\}$. An element $w \in \Sigma^*$ is accepted by $M$ if the corresponding element of $\text{Map}(S, S)$, $\phi(w)$, takes $s_0$ to an element of the accept states set $Y$. The set $L \subseteq \Sigma^*$ accepted by $M$ is called the language accepted by $M$, denoted by $L(M)$.

For any directed graph with $d$ vertices or any finite state automaton $M$, with alphabet $\Sigma$, and $d$ states, there exists a square matrix $A$ of size $d \times d$, with $a_{ij}$ equal to the number of directed edges from vertex $i$ to vertex $j$, $1 \leq i, j \leq d$. This matrix is non-negative (i.e. $a_{ij} \geq 0$) and it is called the transition matrix (as in [12]) or the adjacency matrix (as in [37, p. 575]). For any $k \geq 1$, $(A^k)_{ij}$ is equal to the number of directed paths of length $k$ from vertex $i$ to vertex $j$. The function $p_{ij}(z) = \sum_{k=0}^{\infty} (A^k)_{ij} z^k$ is called the generating function of $M$ [37, p. 574]. As $\sum_{k=0}^{\infty} A^k z^k = (I - zA)^{-1}$, the generating function $p_{ij}(z)$ is equal to $((I - zA)^{-1})_{ij}$ and it satisfies:

Theorem 2.4. [37, p. 574] The generating function $p_{ij}(z)$ is given by

$$p_{ij}(z) = \frac{(-1)^{i+j} \det(I - zA : j, i)}{\det(I - zA)}$$

where $(B : j, i)$ denotes the matrix obtained by removing the $j$th row and $i$th column of $B$, $\det(I - zA)$ is the reciprocal polynomial of the characteristic polynomial of $A$. In particular, $p_{ij}(z)$ is a rational function.

Note that if $M$ is a bi-deterministic automaton with alphabet $\Sigma$, $d$ states, start state $i$, accept state $j$ and transition matrix $A$, then $(A^k)_{ij}$ is the number of words of length $k$ in the free monoid $\Sigma^*$ accepted by $M$.

### 2.3. Irreducible non-negative matrices

We refer to [1, Ch. 16], [26, Ch. 8], [20, pp. 536–551]. There is a vast literature on the topic. Let $A$ be a transition matrix of size $d \times d$ of a directed graph or an automaton with $d$ states, as defined in Section 2.2. The matrix $A$ is a non-negative matrix, that is $a_{ij} \geq 0$ for every $1 \leq i, j \leq d$. For any $k \geq 1$, $(A^k)_{ij}$ is equal to the number of directed paths of length $k$ from vertex $i$ to vertex $j$. If for every $1 \leq i, j \leq d$, there exists $m_{ij} \in \mathbb{Z}^+$ such that $(A^m)_{ij} > 0$, the matrix is irreducible. This condition is equivalent to the graph being strongly connected, that is any two vertices are connected by a directed path. For $A$ an irreducible non-negative matrix, the period of $A$ is the gcd of all $m \in \mathbb{Z}^+$ such that $(A^m)_{ij} > 0$ (for any $i$). If the period is 1, $A$ is called aperiodic. In [26], an irreducible and aperiodic matrix $A$ is called primitive and the period $h$ is called the index of imprimitivity.

Let $A$ be an irreducible non-negative matrix of size $d \times d$ with period $h \geq 1$ and spectral radius $r$. Then the Perron–Frobenius theorem states that $r$ is a positive real number and it is a simple eigenvalue of $A$, $\lambda_{PF}$, called the Perron–Frobenius eigenvalue. It satisfies $\sum_i a_{ij} \leq \lambda_{PF} \leq \sum_j a_{ij}$. The matrix $A$ has a right eigenvector $v_R$ with eigenvalue $\lambda_{PF}$ whose components are all positive and likewise, a left eigenvector $v_L$ with eigenvalue $\lambda_{PF}$ whose components are all positive. Both right and left eigenspaces associated with $\lambda_{PF}$ are one-dimensional.

Theorem 2.5. [16, Thm. V7] Let $A$ be a non-negative and irreducible matrix of size $d \times d$. Let $P(z) = (I - zA)^{-1}$. Then all the entries $p_{ij}(z)$ of $P(z)$ (as given in Theorem 2.4) have the same radius of convergence $\frac{1}{\lambda_{PF}}$, where $\lambda_{PF}$ is the Perron–Frobenius eigenvalue of $A$. 


Remark 2.6. The behavior of irreducible non-negative matrices depends strongly on whether the matrix is aperiodic or not. If \( h > 1 \), then \( A \) has exactly \( h \) complex simple eigenvalues with maximal absolute value: \( \lambda_{PF} \omega^j \), \( 0 \leq j \leq h - 1 \), where \( \omega = e^{2\pi i/h} \) is a primitive root of unity of order \( h \). Moreover, the matrix \( A \) is similar to \( \omega A \), that is the spectrum of \( A \) is invariant under multiplication by \( \omega \). Furthermore, the limit \( \lim_{k \to \infty} \frac{1}{k} \sum_{m=0}^{h-1} \frac{A^m}{\lambda_{PF}^m} \) exists and is equal to the \( d \times d \) matrix \( Q = v_R v_L \), with \( v_R \) of order \( d \times 1 \) and \( v_L \) of order \( 1 \times d \) such that \( v_L v_R = 1 \). If the matrix \( A \) is an irreducible and aperiodic non-negative matrix, then it satisfies many properties similar to those of the positive matrices. In particular, \( \lim_{k \to \infty} \frac{1}{k} \sum_{m=0}^{h-1} \frac{A^m}{\lambda_{PF}^m} \) exists and is equal to the \( d \times d \) matrix \( Q \) defined above. In this article, our main focus is on aperiodic non-negative matrices.

3. Preliminary results

3.1. The Schreier automaton of a coset of a subgroup of \( F_n \)

We now introduce the particular automata we are interested in, that is the Schreier coset diagram for \( F_n \) relative to the subgroup \( H \) [38, p. 107] or the Schreier automaton for \( F_n \) relative to the subgroup \( H \) [36, p. 102]. We refer to [7] for concrete examples.

Definition 3.1. Let \( F_n = \langle \Sigma \rangle \) and \( \Sigma^* \) the free monoid generated by \( \Sigma \). Let \( H < F_n \) of index \( d \). Let \((\tilde{X}_H, p)\) be the covering of the \( n \)-leaves bouquet with vertices \( x_1, x_2, \ldots, x_d \) and basepoint \( x_i \) for a chosen \( 1 \leq i \leq d \). Let \( t_j \in \Sigma^* \) denote the label of a path from \( x_i \) to \( x_j \). Let \( \tilde{X}_H \) be the Schreier coset diagram for \( F_n \) relative to the subgroup \( H \), with \( x_i \) representing the subgroup \( H \) and the other vertices representing the cosets \( Ht_j \) accordingly. We call \( \tilde{X}_H \) the Schreier graph of \( H \), with this correspondence between the vertices and the cosets \( Ht_j \) accordingly.

From its definition, \( \tilde{X}_H \) is a strongly connected graph with \( d \) vertices, so its transition matrix \( A \) is a non-negative and irreducible matrix of size \( d \times d \). As \( \tilde{X}_H \) is a directed \( n \)-regular graph, the sum of the elements at each row and at each column of \( A \) is equal to \( n \). So, from the Perron–Frobenius result for non-negative irreducible matrices, \( n \) is the Perron–Frobenius eigenvalue of \( A \), that is the positive real eigenvalue with maximal absolute value. If \( A \) has period \( h \geq 1 \), then \( \{n^{\frac{2\pi i m}{h}} | 0 \leq k \leq h - 1 \} \) is a set of simple eigenvalues of \( A \). Moreover, \( A \) is similar to the matrix \( A e^{2\pi i m} \), that is the set \( \{2\pi i m | 0 \leq k \leq h - 1 \} \) is a set of eigenvalues of \( A \), for each eigenvalue \( \lambda \) of \( A \).

Definition 3.2. Let \( F_n = \langle \Sigma \rangle \) and \( \Sigma^* \) the free monoid generated by \( \Sigma \). Let \( H < F_n \) of index \( d \). Let \( \tilde{X}_H \) be the Schreier graph of \( H \), with basepoint \( x_i \). Using the notation from Definition 3.1, for a fix \( j \), the Schreier automaton of \( Ht_j \), denoted by \( \tilde{X}_{Ht_j} \), is the automaton with start state \( x_i \) and end state \( x_j \). The language accepted by \( \tilde{X}_{Ht_j} \) is the set of elements in \( \Sigma^* \) that belong to \( Ht_j \). We call the elements in \( \Sigma^* \cap Ht_j \), the positive words in \( Ht_j \). The identity may belong to this set.

Lemma 3.3. Let \( H < F_n \) of index \( d \), with Schreier graph \( \tilde{X}_H \) and transition matrix \( A \) with period \( h \geq 1 \). Then the following properties hold:

i. \( \lambda_{PF} = n \)

ii. The vector \( v_R = (1, 1, \ldots, 1)^T \) of order \( 1 \times d \) is a right eigenvector of \( \lambda_{PF} \) whose components are all positive.

iii. The vector \( v_L = \frac{1}{d} (1, 1, \ldots, 1) \) of order \( d \times 1 \) is a left eigenvector of \( \lambda_{PF} \) whose components are all positive and such that \( v_L v_R = 1 \).

iv. The matrix \( Q = v_R v_L \), with \( v_R \) of size \( d \times 1 \) and \( v_L \) of order \( 1 \times d \) such that \( v_L v_R = 1 \), is of order \( d \times d \) with all entries equal \( \frac{1}{d} \).

v. If \( h = 1 \), then \( \lim_{k \to \infty} \frac{A^k}{\lambda_{PF}^k} = Q \).

vi. If \( h > 1 \), then \( \lim_{k \to \infty} \frac{1}{k+1} \sum_{m=0}^{h-1} \frac{A^m}{\lambda_{PF}^m} = Q \).
Proposition 3.5. [8] Let $H < F_n$ of index $d$, with Schreier graph $\tilde{X}_H$ and transition matrix $A$ with period $h \geq 1$. Let $p_{ij}(z)$ denote the generating function of the Schreier automaton, with $i$ and $j$ the start and end states respectively. Then

i. for every $1 \leq i, j \leq d$, $\frac{1}{h}$ is the radius of convergence of $p_{ij}(z)$.
ii. for every $1 \leq i, j \leq d, \{\frac{1}{h} \epsilon^{zh} | 0 \leq k \leq h-1\}$ is a set of simple poles of $p_{ij}(z)$ of minimal absolute value.
iii. for $|z| < \frac{1}{h}$ and every $1 \leq i \leq d$,

$$\sum_{j=1}^{d} p_{ij}(z) = \frac{1}{1 - nz}$$

Definition 3.6. For $1 \leq i, j \leq d$, we define $m_{ij}$, $0 \leq m_{ij} \leq d$, to be the minimal natural number such that $(A^{m_{ij}})_{ij} \neq 0$.

By definition, if $i \neq j$, then $m_{ij}$ is the minimal length of a directed path from $i$ to $j$ in $\tilde{X}_H$ and if $i = j$, then $m_{ij} = 0$. Note that if $H$ is a subgroup of $\mathbb{Z} = \langle 1 \rangle$ of index $d$, its Schreier graph is a directed loop of length $d$ with each edge labeled by 1 and its transition matrix $A$ is a permutation...
matrix with period \(d\) and \(m_{ij} = \tau\), where \(d\mathbb{Z} + \tau\) is the coset with \(i\) and \(j\) the start and end states respectively.

Whenever \(h > 1\), only for the exponents \(m_{ij} + rh, r \geq 0, (A^{m_{ij} + rh})_{ij} \neq 0\), that is only positive words of length \(m_{ij} + rh\) are accepted by the Schreier automaton, with \(i\) and \(j\) the start and end states respectively. The proportion of words of length \(m_{ij} + rh\), for very large \(r\), accepted by the automaton is computed in [9]:

**Lemma 3.7.** [9] Let \(H < F_n\) be of index \(d\), with Schreier graph \(\tilde{X}_H\) and transition matrix \(A\) with period \(h > 1\). Then, the following properties hold:

i. \((A^r)_{ij} = 0\), whenever \(r \not\equiv m_{ij} (\text{mod} \ h)\), \(1 \leq i, j \leq d\).

ii. For every \(1 \leq i, j \leq d\),
\[
\lim_{r \to \infty} \frac{(A^{m_{ij} + rh})_{ij}}{n^{m_{ij} + rh}} = \frac{h}{d}
\]

iii. \(h\) divides \(d\).

**Example 3.8.** Consider \(\tilde{X}_N\) as described in Figure 2. If we consider the Schreier automaton \(\tilde{X}_Na\) that accepts the positive words in the coset \(Na\), then \(m_{12} = 1\). For \(\tilde{X}_Na, m_{14} = 1\). Clearly, \(m_{11} = 0\). As, \(2, -2, 0, 0\) are the eigenvalues of \(A_{Na} (1 - 2z), (1 + 2z), 1, 1\) are the eigenvalues of \(I - zA\) and \(\det(I - zA) = (1 - 4z^2)\). The generating functions are \(p_{11}(z) = \frac{1 - 2z^2}{1 - 4z^2}\), \(p_{12}(z) = p_{14}(z) = \frac{z}{1 - 4z^2}\), and \(p_{13}(z) = \frac{2z^2}{1 - 4z^2}\).

In the following lemma, we compute the residue of the generating function at the simple pole \(\frac{1}{n}\) and show it is a function of \(n\) and \(d\) only.

**Lemma 3.9.** Let \(H < F_n\) of index \(d\), with Schreier graph \(\tilde{X}_H\) and transition matrix \(A\) with period \(h \geq 1\). Let \(p_{ij}(z)\) denote the generating function of the Schreier automaton, with \(i\) and \(j\) the start and end states, respectively. Then, for every \(1 \leq i, j \leq d\),
\[
\text{Res} \left( p_{ij}(z), \frac{1}{n} \right) = - \left( \frac{1}{nd} \right)
\]

**Proof.** Let \(1 \leq i \leq d\). Clearly, \(\lim_{z \to \frac{1}{n}} \frac{p_{ij}(z)}{p_{ik}(z)}\) is a finite number, for any \(1 \leq j, k \leq d\), since \(\frac{1}{n}\) is a simple pole of both. Furthermore, \(p_{ij}(z)\) and \(p_{ik}(z)\) are asymptotically equivalent, that is \(\lim_{z \to \frac{1}{n}} \frac{p_{ij}(z)}{p_{ik}(z)} = 1\), for any \(1 \leq j, k \leq d\). Indeed, as \(z \to \frac{1}{n}\), the generating functions \(p_{ij}(z) \to \infty\) and \(p_{ik}(z) \to \infty\) at the same rate, since \(p_{ij}(z)\) and \(p_{ik}(z)\) are the generating functions of different cosets of the same subgroup. We prove that formally. If \(h > 1\), then from Lemma 3.7(i), for \(z \in D, \) where \(D = \{ z \in \mathbb{C} \mid |z| < \frac{1}{n}\}:
\[
p_{ij}(z) = \sum_{r=0}^{\infty} (A^{m_{ij} + rh})_{ij} z^{m_{ij} + rh}
\]
\[
p_{ik}(z) = \sum_{r=0}^{\infty} (A^{m_{ik} + rh})_{ik} z^{m_{ik} + rh}
\]

From Lemma 3.7(ii), \(\lim_{r \to \infty} (A^{m_{ij} + rh})_{ij} = \frac{b}{d}\), that is \(\lim_{z \to \frac{1}{n}} \lim_{r \to \infty} (A^{m_{ij} + rh})_{ij} z^{m_{ij} + rh} = \lim_{r \to \infty} \lim_{z \to \frac{1}{n}} (A^{m_{ij} + rh})_{ij} z^{m_{ij} + rh} = \frac{b}{d}\). Also, \(\lim_{z \to \frac{1}{n}} \lim_{r \to \infty} (A^{m_{ik} + rh})_{ik} z^{m_{ik} + rh} = \frac{b}{d}\). So, \(\lim_{z \to \frac{1}{n}} \frac{p_{ij}(z)}{p_{ik}(z)} = 1\). If \(h = 1\), then \(p_{ij}(z) = \sum_{r=0}^{\infty} (A^r)_{ij} z^r\) and \(p_{ik}(z) = \sum_{r=0}^{\infty} (A^r)_{ik} z^r\). From Lemma 3.3(v), \(\lim_{z \to \frac{1}{n}} \lim_{r \to \infty} (A^r)_{ij} z^r = \lim_{z \to \frac{1}{n}} \lim_{r \to \infty} (A^r)_{ik} z^r = \frac{1}{d}\), for every \(1 \leq i, j, k \leq d\).
d. So, \( \lim_{z \to 1} \frac{p_u(z)}{p_k(z)} = 1 \). As \( \lim_{z \to 1} \frac{p_u(z)}{p_k(z)} = \frac{\text{Res}(p_u(z), \frac{1}{n})}{\text{Res}(p_k(z), \frac{1}{n})} \), \( \text{Res}(p_u(z), \frac{1}{n}) = \text{Res}(p_k(z), \frac{1}{n}) \). From Prop. 3.5(iii), \( \sum_{j=1}^{d} \text{Res}(p_j(z), \frac{1}{n}) = \text{Res}(\frac{1}{1-z^k}, \frac{1}{n}) \), so for all \( 1 \leq j \leq d \),

\[
\sum_{j=1}^{d} \text{Res}(p_j(z), \frac{1}{n}) = -\left(\frac{1}{n}\right)
\]

that is, \( \text{Res}(p_j(z), \frac{1}{n}) = -\frac{1}{n} \left(\frac{1}{h}\right) \).

### 3.3. The residue of the generating function at special poles

In this subsection, we consider \( H < F_n \) of index \( d \), with Schreier graph \( \tilde{X}_H \) and transition matrix \( A \) with period \( h > 1 \). Let \( p_j(z) \) denote the generating function of the Schreier automaton, with \( i \) and \( j \) the start and end states, respectively. In the following lemmas, we compute the residue of \( p_j(z) \) at the \( h \) poles of the form \( \frac{1}{n} \omega^k \), where \( \omega = e^{\frac{2\pi i}{h}} \), a primitive root of unity of order \( h \), \( 0 \leq k \leq h - 1 \). It is done in several steps in the following technical lemmas.

**Lemma 3.10.** Let \( p_j(z) = \frac{M(z)}{D(z)} \) denote the generating function of the Schreier automaton, with \( i \) and \( j \) the start and end states, respectively, where \( M(z) \) is the numerator and \( D(z) = \det(I-zA) \) the denominator. Let \( q \) be any rational number, \( \mid q \mid < \frac{1}{h} \). Then

1. \( p_j(q\omega) = \omega^{m_j} p_j(q) \)
2. \( p_j(q\omega^j) = (q\omega^j)^{m_j} p_j(q) \)
3. \( D(q\omega^j) = D(q), \ 1 \leq \ell \leq h-1 \)
4. \( M(q\omega) = \omega^{m_j} M(q) \)
5. \( m_j \) is the multiplicity of 0 as a root of \( M(z) \) and as a zero of \( p_j(z) \).

**Proof.** To shorten, we write \( m \) instead of \( m_j \).

1. (i) For \( \mid z \mid < \frac{1}{h}, p_j(z) = \sum_{r=0}^{\infty} (A^{m+hr})_{ij} z^{m+hr} \). So, \( p_j(q\omega) = \sum_{r=0}^{\infty} (A^{m+hr})_{ij} q^{m+hr} \omega^{m+hr} = \omega^m \sum_{r=0}^{\infty} (A^{m+hr})_{ij} q^{m+hr} = \omega^m p_j(q) \), since \( \omega^h = 1 \). That is, \( p_j(q\omega) = \omega^{m_j} p_j(q) \) and with the same proof: \( p_j(q\omega^j) = (q\omega^j)^{m_j} p_j(q) \).
2. (ii) Since \( h > 1 \), for any eigenvalue \( \lambda \neq 0 \) of \( A \) with algebraic multiplicity \( n_\lambda \), \( \lambda \omega^j \) is an eigenvalue of \( A \) and \( 1 - z\lambda \omega^j \) is an eigenvalue of \( I-zA \) (with same multiplicity \( n_\lambda \), for every \( 0 \leq j \leq h-1 \). When 0 is an eigenvalue of \( A \), the corresponding eigenvalue of \( I-zA \) is 1. So,

\[
D(z) = \det(I-zA) = \prod_{\lambda} \prod_{r=0}^{r=h-1} (1-z\lambda \omega^j)
\]

\[
D(q) = \prod_{\lambda} \prod_{r=0}^{r=h-1} (1-q\lambda \omega^j)
\]

\[
D(q\omega^j) = \prod_{\lambda} \prod_{r=0}^{r=h-1} (1-q\lambda \omega^{j+r})
\]

As \( \omega^h = 1, D(q\omega^j) = D(q) \) for every \( 1 \leq \ell \leq h-1 \).

3. (iv) By definition, \( p_j(z) = \frac{M(z)}{D(z)} \), so \( M(q\omega) = p_j(q\omega)D(q\omega) = \omega^{m_j} p_j(q)D(q) \) from (i) and (ii), that is \( M(q\omega) = \omega^{m_j} M(q) \).
4. (v) From (i), \( p_j(z) = \sum_{r=0}^{\infty} (A^{m+hr})_{ij} z^{m+hr} = z^m \sum_{r=0}^{\infty} (A^{m+hr})_{ij} z^{hr} \), that is \( m \) the multiplicity of 0 as a zero of \( p_j(z) \) and as a root of \( M(z) \).\]
From Lemma 3.10(ii), we recover the fact that if \( \frac{1}{n} \) is a simple pole of \( p_{ij}(z) \) then \( \frac{1}{n} e^{2\pi i} \) is also a simple pole of \( p_{ij}(z) \), for every \( 0 \leq \ell \leq h - 1 \). Indeed, Lemma 3.10(ii) implies

\[
\lim_{z \to e^{2\pi i n}} p_{ij}(z) = \omega_{ij}^{m_{ij}} \lim_{z \to e^{2\pi i n}} p_{ij}(z)
\]

**Lemma 3.11.** Let \( p_{ij}(z) = \frac{M(z)}{D(z)} \). Then, for every non-zero eigenvalue \( \lambda \) of \( A \) and every \( 1 \leq \ell \leq h - 1 \),

\[
\text{Res} \left( \frac{1}{D(z)}, \frac{1}{\lambda} \omega^\ell \right) = \omega^\ell \text{ Res} \left( \frac{1}{D(z)}, \frac{1}{\lambda} \right)
\]

In particular,

\[
\text{Res} \left( \frac{1}{D(z)}, \frac{1}{n} \omega^\ell \right) = \omega^\ell \text{ Res} \left( \frac{1}{D(z)}, \frac{1}{n} \right)
\]

**Proof.** As in the proof of Lemma 3.10, \( D(z) = \prod_{\rho} \prod_{r=0}^{r=h-1} (1 - z \rho \omega^r) \). So, \( \frac{1}{\lambda} \omega^\ell \) is a pole of \( \frac{1}{D(z)} \), for every eigenvalue \( \rho \neq 0 \) of \( A \) and every \( 0 \leq r \leq h - 1 \). Consider \( \lambda \neq 0 \) a particular eigenvalue of \( A \) and let \( d(z) = \prod_{\rho \neq \lambda} \prod_{r=0}^{r=h-1} (1 - z \rho \omega^r) \), so \( D(z) = \prod_{r=0}^{r=h-1} (1 - z \omega^r) d(z) \). It holds that

\[
d \left( \frac{1}{\lambda}, \omega^\ell \right) = \prod_{\rho \neq \lambda} \prod_{r=0}^{r=h-1} \left( 1 - \frac{\rho}{\lambda} \omega^r \right)
\]

That is, \( d \left( \frac{1}{\lambda} \omega^\ell \right) = d \left( \frac{1}{\lambda} \right) \), since \( w^h = 1 \). Now, with \( I = \{ 0 \leq r \leq h - 1 \mid r \neq h - \ell \} \), we have

\[
\text{Res} \left( \frac{1}{D(z)}, \frac{1}{\lambda} \right) = \lim_{z \to \frac{1}{\lambda}} \left( z - \frac{1}{\lambda} \right) \frac{1}{D(z)} = \lim_{z \to \frac{1}{\lambda}} \left( -\frac{1}{\lambda} \right) (1 - \lambda z) \frac{1}{D(z)}
\]

\[
= \left( -\frac{1}{\lambda} \right) \frac{1}{d \left( \frac{1}{\lambda} \right)} \prod_{r=1}^{r=h-1} (1 - \omega^r)
\]

\[
\text{Res} \left( \frac{1}{D(z)}, \frac{1}{\lambda} \omega^\ell \right) = \lim_{z \to \frac{1}{\lambda} \omega^\ell} \left( z - \frac{1}{\lambda} \omega^\ell \right) \frac{1}{D(z)} = \lim_{z \to \frac{1}{\lambda} \omega^\ell} \left( -\frac{1}{\lambda} \omega^\ell \right) (1 - \lambda \omega^{h-\ell} z) \frac{1}{D(z)}
\]

\[
= \omega^\ell \left( -\frac{1}{\lambda} \right) \frac{1}{d \left( \frac{1}{\lambda} \omega^\ell \right)} \prod_{r \in I} (1 - \omega^{\ell+r})
\]

As \( d \left( \frac{1}{\lambda} \omega^\ell \right) = d \left( \frac{1}{\lambda} \right) \), it remains to show \( \prod_{r \in I} (1 - \omega^{\ell+r}) = \prod_{r=1}^{r=h-1} (1 - \omega^r) \). Since \( w^h = 1 \), \( \prod_{r \in I} (1 - \omega^{\ell+r}) = \prod_{r=1}^{r=h-1} (1 - \omega^r) \). So, \( \text{Res} \left( \frac{1}{D(z)}, \frac{1}{\lambda} \omega^\ell \right) = \omega^\ell \text{ Res} \left( \frac{1}{D(z)}, \frac{1}{\lambda} \right) \), for every eigenvalue \( \lambda \neq 0 \) of \( A \) and every \( 1 \leq \ell \leq h - 1 \) and in particular, for the Perron–Frobenius eigenvalue \( n \), that is \( \text{Res} \left( \frac{1}{D(z)}, \frac{1}{n} \omega^\ell \right) = \omega^\ell \text{ Res} \left( \frac{1}{D(z)}, \frac{1}{n} \right). \)

**Lemma 3.12.** For every \( 1 \leq i, j \leq d, p_{ij}(z) = \frac{M(z)}{D(z)} \) satisfies:

i. \( \text{Res} (p_{ij}(z), \frac{1}{n} \omega) = -\left( \frac{1}{n} \right) \omega^{m_{ij}+1} \)

ii. \( \text{Res} (p_{ij}(z), \frac{1}{n} \omega^j) = -\left( \frac{1}{n} \right) \omega^{m_{ij}+1} \)
**Proof.**

(i) \( \text{Res} \left( p_{ij}(z), \frac{1}{n} \right) = \lim_{z \to \frac{1}{n}} \left( z - \frac{1}{n} \right) \frac{M(z)}{D(z)} = \text{Res} \left( \frac{1}{D(z)}, \frac{1}{n} \right) M \left( \frac{1}{n} \right) \)

(ii) \( \text{Res} \left( p_{ij}(z), \frac{1}{n} \omega^f \right) = \omega^f \text{Res} \left( \frac{1}{D(z)}, \frac{1}{n} \right) (\omega^f)^{m_{ij}} M \left( \frac{1}{n} \right) = (\omega^f)^{m_{ij}+1} \text{Res} \left( p_{ij}(z), \frac{1}{n} \right) = -\left( \frac{1}{nd} \right) (\omega^f)^{m_{ij}+1} \)

From Lemma 3.10(iv), \( M(\frac{1}{n} \omega) = \omega^{m_{ij}} M(\frac{1}{n}) \) and \( \text{Res}(\frac{1}{D(z)}, \frac{1}{n} \omega) = \omega \text{ Res}(\frac{1}{D(z)}, \frac{1}{n}) \), from Lemma 3.11, so \( \text{Res}(p_{ij}(z), \frac{1}{n} \omega) = \omega^{m_{ij}+1} \text{ Res}(p_{ij}(z), \frac{1}{n}) = -\left( \frac{1}{nd} \right) \omega^{m_{ij}+1} \), with the last equality from Lemma 3.9.

**Example 3.13.** Consider \( \tilde{X}_N \) as described in Figure 2. The generating functions are \( p_{11}(z) = \frac{22z}{1-2z}, \ p_{12}(z) = p_{14}(z) = \frac{2z}{1-2z} \), and \( p_{13}(z) = \frac{2z}{1-2z} \). For all \( 1 \leq j \leq 4 \), \( \text{Res}(p_{ij}(z), \frac{1}{n}) \) is calculated as follows: \( \text{Res}(p_{11}(z), \frac{1}{2}) = \text{Res}(p_{13}(z), \frac{1}{2}) = \frac{1}{2} \) and \( \text{Res}(p_{12}(z), \frac{1}{2}) = \text{Res}(p_{14}(z), \frac{1}{2}) = -\frac{1}{8} \).

**4. The HS conjecture as a problem in vanishing sum of roots of unity and convex polygons**

Let \( F_n \) be the free group on \( n \geq 2 \) generators. Let \( \{H,x_i\}_{i=1}^s \) be a coset partition with \( H_i < F_n \) of index \( d_i \), \( x_i \in F_n, 1 \leq i \leq s \), and \( 1 < d_1 \leq \ldots \leq d_s \). Let \( \tilde{X}_i \) denote the Schreier graph of \( H_i \), with transition matrix \( A_i \) of period \( h_i \geq 1, 1 \leq i \leq s \). Let \( \tilde{X}_{H,x_i} \) denote the Schreier automaton of \( H_i x_i \), with generating function \( p_i(z), 1 \leq i \leq s \). Let \( m_i \) be the minimal natural number such that \( (A_i^{m_i})_{k,l} \neq 0 \), where \( k_i \) and \( l_i \) denote the initial and final states corresponding to \( \tilde{X}_{H,x_i} \). In this section, we assume \( h = \max \{ h_i \ | \ 1 \leq i \leq s \} > 1 \). We denote \( \omega = e^{\frac{2\pi}{h}} \) and \( J = \{ j \ | \ h_j = h, 1 \leq j \leq s \} \). Under these conditions, we prove in [8]:

**Proposition 4.1.** [8] Assume \( h > 1 \), where \( h = \max \{ h_i \ | \ 1 \leq i \leq s \} \). Let \( J = \{ j \ | \ 1 \leq j \leq s, h_j = h \} \). Then there is a repetition of the maximal period, that is \( | J | \geq 2 \).
4.1. Proof of Theorem 1

We show that Proposition 4.1 induces a set of irreducible vanishing sums of roots of unity of order \( h \) that describe convex polygons, which is the content of Theorem 1. The following proposition is a slightly more extended version of Theorem 1.

**Proposition 4.2.** Assume \( h > 1 \), where \( h = \max\{h_i \mid 1 \leq i \leq s\} \). Let \( J = \{j \mid 1 \leq j \leq s, \ h_j = h\} \). The coset partition \( P = \{H_i x_i \}_{i=1}^{s}, x_i \in F_n \) induces a vanishing sum of roots of unity of order \( h \)

\[
\sum_{j \in J} \left( \prod_{i \in J, i \neq j} d_i \right) (\omega)^{m_j} = 0
\]

Furthermore, \( P \) induces

i. a set of irreducible vanishing sums of roots of unity of order \( h \) of the following form, where \( J' \subseteq J \):

\[
\sum_{j \in J'} \left( \prod_{i \in J', i \neq j} d_i \right) (\omega)^{m_i} = 0
\]

ii. a set of convex polygons, where each polygon \( \mathcal{P} \) has \( |J'| \) sides.

iii. each \( J' \) satisfies \( |J'| \geq p \), where \( p \) is the smallest prime dividing \( h \).

**Proof.** From Proposition 3.5(iii), it results that \( \sum_{j \in J} \text{Res}(p_j(z), \frac{1}{n} w^j) = 0 \), for every \( \ell \) with \( \gcd(\ell, h) = 1 \). In particular, \( \sum_{j \in J} \text{Res}(p_j(z), \frac{1}{n} w) = 0 \). From Lemma 3.12, \( \sum_{j \in J} \text{Res}(p_j(z), \frac{1}{n} w) = -\sum_{j \in J} \left( \prod_{i \in J \setminus j} d_i \right) (\omega)^{m_j} \), that is, after reduction, we have \( \sum_{j \in J} \left( \prod_{i \in J \setminus j} d_i \right) (\omega)^{m_j} = 0 \) and after multiplication by \( \prod_{j \in J} d_j \), we have \( \sum_{j \in J} \left( \prod_{i \in J, i \neq j} d_i \right) (\omega)^{m_j} = 0 \), a vanishing sum of roots of unity of order \( h \) with positive integer coefficients.

(i), (ii) If this vanishing sum of roots of unity of order \( h \) is not irreducible, then it admits several proper irreducible non-empty subsums that vanish. For each irreducible vanishing subsum, there exists \( J' \subset J \), such that this irreducible subsum has the form \( \sum_{j \in J'} \left( \prod_{i \in J', i \neq j} d_i \right) (\omega)^{m_j} = 0 \) (obtained after simplification by \( \prod_{i \in J \setminus J'} d_i \)).

Since all the coefficients in the vanishing sum are positive integers, each such irreducible vanishing sum describes a convex polygon \( \mathcal{P} \) with \( |J'| \) sides (see Section 2.1). This polygon may be degenerate, as it may occur that \( m_{j_1} = m_{j_2} \), for some \( j_1, j_2 \in J' \).

(iii) From [22, p. 92], any non-empty vanishing sum of \( h \)-th roots of unity must have length at least \( p \), where \( p \) is the smallest prime dividing \( h \), so \( |J'| \geq p \). \( \square \)

Note that from Proposition 4.2, the number of repetitions of the maximal period \( h \) is at least \( p \), where \( p \) is the smallest prime dividing \( h \). In particular, in the case of \( Z \), the maximal period is \( d_s \), so we recover the following result for \( Z \): the largest index \( d_s \) appears at least \( p \) times, where \( p \) is the smallest prime dividing \( d_s \) [27, 40, 45].

**Example 4.3.** Let \( L = \langle a^2, b, aba \rangle \) be a subgroup of index 2 in \( F_2 \). Let \( N \) be the subgroup described in Figure 2. Consider the following coset partition: \( \{H_{i} x_i \}_{i=1}^{3}, \) with \( H_{1} x_1 = L, H_2 x_2 = Na, H_3 x_3 = Nab \), that is \( F_2 = L \cup Na \cup Nab \). From Proposition 4.1, there is a repetition of the maximal period 2, that is \( h_2 = h_3 = 2 \). It holds that \( h_1 = 1, m_2 = 1, m_3 = 2 \) and the induced vanishing sum of roots of order 2 is \( d_1 \omega + d_2 \omega^2 = 0, \omega = \varepsilon^2 \) (since \( \frac{1}{d_1} \omega^{m_1} + \frac{1}{d_2} \omega^{m_2} = \frac{1}{d_1} \omega + \frac{1}{d_2} \omega^2 = 0 \)), which implies \( d_2 = d_3 \). Clearly, the induced polygon here is degenerate.
4.2. Proof of Theorem 2

We show now that a coset partition \( P \) has multiplicity if and only if one of the polygons obtained from the induced vanishing sum have at least two edges of the same length.

**Proof of Theorem 2.** From Proposition 4.2, each irreducible non-empty subsum \( \sum_{j \in J} (\prod_{i \in J} d_i) (\omega)^{m_j} = 0 \), with \( J' \subseteq J \), describes a convex polygon \( \mathcal{P} \) with \( |J'| \) sides. Moreover, the length of each edge has the form \( \prod_{i \in J} d_i \), for some \( j \in J' \). So, \( \mathcal{P} \) has at least two edges of the same length if and only if \( \prod_{i \in J} d_i = \prod_{i \in J} d_i \), for some \( j, k \in J', j \neq k \), that is if and only if \( d_j = d_k \).

From the proof of Proposition 4.2, \( \omega \) is a root of the polynomial \( g(z) = \sum_{j \in J} (\prod_{i \in J} d_i) (z)^{m_j} \). As the cyclotomic polynomial of order \( h \), \( \Phi_h(z) \), is the minimal polynomial of \( \omega \) in \( \mathbb{Q}(z) \) and \( g(z) \) in \( \mathbb{Q}(z) \), \( \Phi_h(z) \) divides \( g(z) \) and every primitive root of unity of order \( h \) is a root of \( g(z) \). Clearly, this implies \( \max\{m_j \mid j \in J'\} \) is at least \( \varphi(h) \), the Euler function of \( h \). So, we proved the following corollary:

**Corollary 4.4.** Let \( g(z) = \sum_{j \in J} (\prod_{i \in J} d_i) (z)^{m_j} \) in \( \mathbb{Q}(z) \) as defined above. Then, \( \Phi_h(z) \), the cyclotomic polynomial of order \( h \), divides \( g(z) \). Moreover, \( \max\{m_j \mid j \in J'\} \) is at least \( \varphi(h) \), the Euler function of \( h \).

4.3. Proof of Theorem 3

Using the above construction, and general results on irreducible vanishing sum of roots of unity of order \( h \), we prove, under certain conditions on \( h \), that all the associated polygons are regular and this implies the coset partition has multiplicity.

**Proof of Theorem 3.** Let \( J = \{j \mid 1 \leq j \leq s, h_j = h\} \). Let \( \sum_{j \in J} (\prod_{i \in J} d_i) (\omega)^{m_j} = 0 \) be the induced vanishing sum of roots of order \( h \) induced by \( P \), and \( \sum_{j \in J} (\prod_{i \in J} d_i) (\omega)^{m_j} = 0 \), where \( J' \subseteq J \), an irreducible sub-sum. We consider its induced polygon \( \mathcal{F} \) and apply Theorem 2.2. Indeed, if \( h = p^n \), where \( p \) is a prime, then, up to a rotation, the only irreducible vanishing sums of roots of unity are \( 1 + \zeta_p + \ldots + \zeta_p^{p-1} = 0 \), where \( \zeta_p \) is the root of unity of order \( p \). That is, all the edges of \( \mathcal{F} \) have equal length and from Theorem 2, \( P \) has multiplicity. If \( h = p^n q^m \), where \( p, q \) are primes, then, up to a rotation, the only irreducible vanishing sums of roots of unity are \( 1 + \zeta_p + \ldots + \zeta_p^{p-1} = 0 \) and \( 1 + \zeta_q + \ldots + \zeta_q^{q-1} = 0 \), where \( \zeta_p \) and \( \zeta_q \) are the roots of unity of order \( p \) and \( q \) respectively. That is, all the edges of \( \mathcal{F} \) have equal length. In both cases, all the induced polygons are regular. \( \square \)

In Example 4.3, \( h = 2 \) and the induced vanishing sum of roots of order 2, \( d_2 \omega + d_2 \omega^2 = 0 \), is simply \( 1 + \zeta_2^2 \) times an integer number \( (d_2 = d_3) \).

Note that if \( h = p_1^{m_1} p_2^{m_2} \ldots p_t^{m_t} \), where \( p_1, p_2, \ldots, p_t \), \( t > 2 \), are different primes, then, from Theorem 2.2, an irreducible vanishing sum of roots of unity of order \( h \) can be obtained from \( \sum_{j=1}^{t} \left( \sum_{k=1}^{m_j} \epsilon_{i,k} \zeta_{i,k} \right) (1 + \zeta_{p_i} + \zeta_{p_i}^2 + \ldots + \zeta_{p_i}^{p_i-1}) = 0 \), where \( \epsilon_{i,k} \in \{-1,0,1\}, \zeta_{i,k} \) is any root of unity, \( \zeta_{p_i} \) is the root of unity of order \( p_i \), \( 1 \leq i \leq t, 1 \leq k \leq \epsilon_i \). In this case, we do not know how to prove that at least one of the polygons associated to the irreducible vanishing sums of roots of unity has at least two sides of the same length.


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