A model for $q^2\bar{q}^2$ systems, illustrated by an application to $K\bar{K}$ scattering

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Abstract

Here is presented a four-body potential model for $q^2\bar{q}^2$ systems which includes both the spin and flavour degrees of freedom, extending the formalism presented already in the spin independent situation. This allows an application to a realistic situation, which is chosen to be $K\bar{K}$ scattering. It is seen that because of the gluonic effects in this multi-quark system, the $K\bar{K}$ attraction resulting from the quark-exchange mechanism gets appreciably decreased compared to that emerging through the naive two-body potential approach. PCAS number(s):12.40.Qq, 13.75.Lb, 12.38.Lg, 14.40.Cs

1 Introduction

With strong evidence in favour of quark confinement both from experiment (failure to find free quarks) and theory [1, 2], we have some understanding of the quark-quark interaction for large distances. Using this, along with the understanding of the short distance quark-quark interaction obtained through

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perturbative QCD, different models of the quark-quark interaction have been used such as the MIT bag model \[\text{[3, 4]}\] and the constituent quark potential model. In the constituent quark potential model the quark-antiquark interaction is represented by a potential which is well motivated by QCD for small distances (given by the one-gluon exchange mechanism), but is modified so as to incorporate the confining potential as the limit for large distances. Moreover, the current masses of the QCD lagrangian are replaced by effective masses, termed \textit{constituent masses}, which are fitted, along with other parameters of the model, to experimentally known quantities related to some set of hadrons. For quarks of known (fitted) masses interacting through a space dependent potential, one can set up and solve a Schrödinger equation for their dynamics. In this way the constituent quark potential model, when improved to incorporate relativistic effects, explains in a consistent and unified way most of the observed mesonic states, from the pion to the upsilon, as quark-antiquark states with different values of orbital and radial quantum numbers \[\text{[5]}\].

But that is not enough. A successful model of strong interactions should be able to describe also possible systems having three or more quarks/antiquarks. It is not clear yet how, if at all, the quark potential model can be applied to multi-quark systems (with more than three quarks). Perhaps the simplest approach is to take the many-body hamiltonian as a sum of hamiltonians
corresponding to all pairs of particles involved, the basic method used in \(7, 8, 9\). But this has many theoretical as well as phenomenological (such as the van der Waals force problem \(10\)) flaws. Keeping this in mind, a \textit{four-body} potential model for a quark-exchange mechanism in \(q^2\overline{q}^2\) systems was proposed in \(11\) in a spin independent situation, taking into account the effects of the gluonic degrees of freedom as well in a non-trivial way. Only for small distances does this agree with the sum of two-body \((\mathbf{F}_i \cdot \mathbf{F}_j)\) potentials model. For, roughly speaking, interquark distances greater than 0.5 fm it qualitatively agrees with the flux tube model \(12, 13\) of the adiabatic surfaces of the gluonic field.

As shown in \(12\), there are three flux tube topologies for the \(N_q = N_{\overline{q}} = 2\) system, each of which will determine the ground state in a different region of configuration space. Since these are linearly independent of each other, we need a basis containing at least \textit{three} gluonic states to describe the gluonic field of this multiquark system. This is different to what the two-body potential model says for the \(q^2\overline{q}^2\) system; the two-body model would need only a \textit{two} dimensional colour basis \(7\). Thus in \(11\) the two-body potential model was written in a redundant colour basis

\[
|1\rangle_c = |1_{13}1_{24}\rangle_c, \quad |2\rangle_c = |1_{14}1_{23}\rangle_c \quad \text{and} \quad \quad |3\rangle_c = |\overline{3}_{12}3_{34}\rangle_c, \quad (1.1)
\]

corresponding to the three basic states in the flux tube model, and then the
suggested changes, forming the proposed model, were made. The (confinement) potential part of the hamiltonian of the system was written as

\[ V(q_1q_2\bar{q}_3\bar{q}_4) = \sum_{i<j} F_i \cdot F_j v_{ij}, \quad \text{with} \quad (1.2) \]

\[ v_{ij} = C r_{ij}^2 + \bar{C}. \quad (1.3) \]

This quadratic, rather than the theoretically better motivated coulomb-plus-linear, form of the two-body potential was used basically for computational convenience. Similarly for the kinetic energy part, the non-relativistic expression was used, also for simplicity and numerical convenience. These features have been retained in the present work as well.

The overlap matrix \( N \) in this redundant basis is as given by eq.(2.8) of [11], the potential matrix by eq.(3.10) there, and the matrix element of the non-relativistic kinetic energy operator between any two states is expressed through eq.(3.11) of the same (these are also given by eqs.(1.4),(1.6) and (1.5) below respectively, with \( f = 1 \)). The model proposed in [11] gives the following expressions for these matrices:

\[ N \rightarrow N(f) = \left( \begin{array}{ccc} 1 & \frac{1}{3} f & \frac{\sqrt{2}}{3} f \\ \frac{1}{3} f & 1 & -\frac{\sqrt{2}}{3} f \\ \frac{\sqrt{2}}{3} f & -\frac{\sqrt{2}}{3} f & 1 \end{array} \right), \quad (1.4) \]

\[ g\langle X'|H|X \rangle_g = g\langle X'|K|X \rangle_g + g\langle X'|V|X \rangle_g, \]

\[ g\langle X'|K|X \rangle_g = N(f)_{X',X}^{1/2} \left( \sum_i -\frac{\nabla_i^2}{2m_i} \right) N(f)_{X',X}^{1/2} \quad \text{and} \quad (1.5) \]
The basis now is actually $|1\rangle_g, |2\rangle_g$ and $|3\rangle_g$ instead of $|1\rangle_c, |2\rangle_c$ and $|3\rangle_c$; the new subscript $g$ refers to the gluonic degree of freedom, instead of the colour degree of freedom represented by the subscript $c$. $H$ is the Hamiltonian of the system, and $N(f)_{X',X}$ is the gluonic states overlap factor $g\langle X'|X\rangle_g$. The matrices in the above three equations are achieved by multiplying the two-body potential model expressions for the off-diagonal elements of overlap, kinetic energy and potential energy matrices respectively by a factor $f$ which tends to 1 for all $r_{ij} \ll b_s^{-1/2}$ and to 0 when any $r_{ij}$ becomes $\gg b_s^{-1/2}$. In the proposed model in [11], the diagonal elements were the same as obtained in the two-body model calculations, apart from the $\frac{5}{2}Df(1 - f)$ term in the 3,3 element of the proposed potential matrix (eq.(1.6)). The reasons for the introduction of this term are described in the discussion from eq.(4.18) till the end of section 4 of [11]. It should be clear from the discussion there that the model is actually defined in a basis which is not redundant even for all
This treatment of the diagonal and the off-diagonal elements of the two-body model based matrices was motivated by the work presented in [14, 15], along with the similarity for large interquark distances in space dependence of the diagonal elements of the $K, V$ and $N$ matrices in both the two-body potential and the flux tube model. The calculations reported in [14, 15] actually show that in the flux tube model the coupling of the three gluonic states $|1\rangle_g$, $|2\rangle_g$ and $|3\rangle_g$ decreases exponentially with inter-quark distances. For the space dependent factor $f$ multiplying the off-diagonal elements, the choice used in [11] was

$$f = \exp(-\bar{k} \sum_{i<j} r_{ij}^2), \quad \bar{k} = \frac{1}{6}kb_s \quad (1.7)$$

with $k$ a numerical coefficient. This is the simplest choice from a computational point of view. An alternative choice of $f$ is suggested in [16, 17]. Both of these forms have been studied in [18, 19, 20], which aim at extracting the gluon field overlap factor $f$ from a calculation using lattice Monte Carlo techniques. This work is in progress, but so far neither of these two candidates is preferred by the calculations.

Developing a formalism, based on the model proposed above, describing the dynamics of meson-meson interactions also requires taking into account the “slower motion”, the quark position dependence of the wave function.
This was done, for the spin-independent case, in sections 5 and 6 of the previous work [11]. There we used our model to specify a theoretically refined (but non-relativistic) hamiltonian of the $q^2ar{q}^2$ system in our gluonic basis, and solved \textit{approximately} for the wave function using the ‘resonating group method’ common in nuclear [21] and also recently in particle physics [22]. The approximation corresponded to specifying parts of the quark position dependent wave function before solving the Schrödinger equation for the rest.

In the present work we introduce the spin and flavour degrees of freedom, and as a first application apply the formalism to a physical meson-meson system, namely $K\bar{K}$. This is done in section 2. The solution for the total wave function of the system thus obtained gives us numerical results for the corresponding meson-meson phase shifts for the elastic as well as the non-elastic meson-meson scattering, along with a condition for the existence of a bound state of the whole system. These results are reported in section 3. This is followed by our conclusions.

2 \textbf{The model applied to $K\bar{K}$ systems}

In this section we present the formalism for a realistic situation. This means that along with the gluonic basis described in [11], we have to deal with the spin basis as well. Moreover, flavour dependence has to be considered.
The hamiltonian of the system, written now in this spin-gluonic basis, would also include the hyperfine term. To complete the formalism, we have to incorporate also the quark position dependence of the wave function. With quark contents of the $q^2\bar{q}^2$ system like that of $K\bar{K}$, the flavour wavefunction can be written generally as $l\bar{s}\bar{l}s$, with $l$ standing for a light (up or down) quark. We label these four particles $1,\bar{3},\bar{4}$ and $2$ respectively. Thus the pair $(2,\bar{3})$ would be composed of strange quark and antiquark, and $(1,\bar{4})$ of light quark and antiquark. In this way each of the particles 2 and 3 has a mass higher than of those belonging to the other pair (1 and $\bar{4}$) by a ratio which is the same as that of the strange quark mass $m_s$ to the up (or down) quark mass $m$. This mass ratio is denoted by $s$ in this paper. Anti-symmetrization of the total wave function is not necessary in the present case since we do not have any two identical fermions.

For the spin dependent part of the basis we use, as in the quark model, the states arising through the spins of the quarks only. So any pair of particles (quark or antiquark) would have a total spin of one or zero, thus forming a spin triplet or singlet respectively. This also means that the total spin of the whole $q^2\bar{q}^2$ system can have a value of zero, one or two. Being interested in the ground state of the $J^P = 0^+$ sector of the system, we focus on the spin states with the total spin of the system as zero. This is consistent because, as is mentioned later, tensor and spin-orbit forces are neglected in this work.
Hence, the hamiltonian separately conserves $L$ and $S$ i.e. the total orbital angular momentum and the total spin of the system. This allows us to restrict our considerations to the $S = 0$ sector, as mentioned above, meaning, in turn, that we will be dealing only with $L = 0$ spatial wave functions.

In every one of the three channels of the previous section (corresponding to the three gluonic states $|1\rangle_g, |2\rangle_g$ and $|3\rangle_g$), our four particles can be grouped into two mesonic sub-clusters. Each of these clusters may have a combined spin of zero or one and hence the $q\bar{q}q\bar{q}$ system may be composed of either two spin singlets or two triplets. This means that there may be two independent spin channels for each of the three gluonic channels above. Thus, there are six independent states of the system in hand. The corresponding six spin states are written in the notation of Appendix D of [7] (see Appendix A of the present paper for details) as:

In the first channel (with the gluonic part of the base state as $|1\rangle_g$):

$$|1S\rangle_s = |P_{13}P_{24}\rangle_s \quad \text{and} \quad |1T\rangle_s = |V_{13} \cdot V_{24}\rangle_s.$$  \hspace{1cm} (2.1)

In the second channel:

$$|2S\rangle_s = |P_{14}P_{23}\rangle_s \quad \text{and} \quad |2T\rangle_s = |V_{14} \cdot V_{23}\rangle_s.$$  \hspace{1cm} (2.2)

And in the third channel:

$$|3S\rangle_s = |S_{12}S_{34}\rangle_s \quad \text{and} \quad |3T\rangle_s = |A_{12} \cdot A_{34}\rangle_s.$$  \hspace{1cm} (2.3)
In this notation $S_{ij}$ and $A_{ij}$ stand for the scalar and axial vector spin wave functions respectively, and the pseudoscalar and vector spin wavefunctions $P_{ij}$ and $V_{ij}$ are defined in terms of their linear combinations.

Except in the diagonal terms corresponding to the $PP$ (i.e. pseudoscalar-pseudoscalar) sector of the second gluonic channel to be discussed below, the flavour wave functions are taken to be trivial everywhere, just giving rise to an isospin conserving factor as the overlap of any two of them. This happens in the absence of any mechanism for flavour change of a quark or antiquark. Actually, flavour changing is possible in any channel through annihilation of quarks and antiquarks of the same flavour. Our consideration of these processes in the $PP$ sector of the second channel only is, therefore, an approximation. This sector is singled out because it is here that the annihilation effects are apparently most significant and cannot be neglected in any realistic model of the processes involving mesons. This is because they are supposed to be responsible for the mass difference between the (pseudoscalar) isoscalar and isovector mesons. The annihilation effects are negligible in the (spin) vector-vector sector of the second channel, because of the small difference in mass of the spin one isoscalar and isovector mesons i.e. $\omega$ and $\rho$.

When these annihilation processes are incorporated, the flavour wave function gets mixed with the quark position dependent part. This follows
because the size of a mesonic cluster depends upon the masses of the quarks (antiquarks) it contains. Thus the $PP$ sector of the second channel is considered separately below. For all other channels, we can consider the quark position dependent part of the wave function—referred to as ‘quark wave function’ in the following—separately from the flavour part. This quark wave function is a function of four 3-vectors $r_1, r_2, r_3$ and $r_4$. These can be replaced by their combinations, with one of them as the overall centre-of-mass co-ordinate of the whole system $R_c$ and three others which are taken, here, to be different in different channels. Writing explicitly, these are:

In the first channel (with the gluonic part of the wave function as $|1\rangle_g$)

$$R_1 = \frac{r_1 + sr_3 - sr_2 - r_4}{1 + s}, \quad y_1 = r_1 - r_3 \quad \text{and} \quad z_1 = r_2 - r_4.$$  \hspace{1cm} (2.4)

In the second channel

$$R_2 = \frac{r_1 + r_4 - r_2 - r_3}{2}, \quad y_2 = r_1 - r_4 \quad \text{and} \quad z_2 = r_2 - r_3.$$  \hspace{1cm} (2.5)

And in the third channel

$$R_3 = \frac{r_1 + r_2 - r_3 - r_4}{1 + s}, \quad y_3 = r_1 - r_2 \quad \text{and} \quad z_3 = r_3 - r_4.$$  \hspace{1cm} (2.6)

The quark wave function in any channel is written, in the following, as a product of two factors, one being a function of $R_k$ and the other of $y_k$ and $z_k$ only, for $k = 1, 2$ or 3. The former is denoted by $\chi_{kl}(R_k)$, with $I$ designating the spin state (singlet-singlet or triplet-triplet), and the latter
by $\xi_k(y_k)\zeta_k(z_k)$, with $\xi_k(y_k)$ and $\zeta_k(z_k)$ corresponding to the two mesonic clusters of the channel $k$. The spatial dependence of these on $y_k$ and $z_k$ is taken to be gaussian in consistency with the choice of the quadratic form of the inter-quark potential in eq.(1.3). But the $\chi_{kI}$’s are treated as variational functions to be determined by solving the approximate coupled Schrödinger equations, using the ‘resonating group method’ [21].

With the above mentioned forms of the quark wave functions in different channels, the total state vector of the whole $q^2\bar{q}^2$ system is written as

$$|\Psi(q_1, q_2, \bar{q}_3, \bar{q}_4; g)\rangle = \sum_{kI} |k\rangle g|kI\rangle |k\rangle \psi_c(R_c)\chi_{kI}(R_k)|\xi_k(y_k)\zeta_k(z_k), \quad (2.7)$$

with

$$\xi_k(y_k) = \frac{1}{(2\pi d^2_{k1})^{3/4}}\exp[-y^2_k/4d^2_{k1}] \quad \text{and} \quad \zeta_k(z_k) = \frac{1}{(2\pi d^2_{k2})^{3/4}}\exp[-z^2_k/4d^2_{k2}]. \quad (2.8)$$

This is actually the case with annihilation neglected. In that situation the mesons represented by $\xi_1, \zeta_1, \xi_3$ and $\zeta_3$ have one light and one strange (antistrange) particle. On the other hand, that denoted by $\xi_2$ has both particles as the light ones (up and down) and $\zeta_2$ has both the quark and antiquark as the heavier ones. It follows thus from the properties of the solutions of a 3-d harmonic oscillator that $d_{11}, d_{12}, d_{31}$ and $d_{32}$ have a particular value, say, $d'$, $d_{21}$ has a different one, say, $d$, and $d_{22}$ differs from all these having a value
denoted by $d''$. Quantitatively

$$\frac{d''^2}{d^2} = \sqrt{\frac{m(m_s + m)}{2mm_s}} = \sqrt{s + \frac{1}{2s}}$$ and $$\frac{d''^2}{d^2} = \sqrt{\frac{2m}{2m_s}} = \sqrt{\frac{1}{s}}.$$  \hspace{1cm} (2.9)  

It must be pointed out that we are using the approximation of neglecting the spin dependence of the size of any cluster. For the absolute magnitudes of the sizes, the relation $d^2 = \sqrt{3}R_n^2/2$ is used relating the radius of a meson composed of the light mesons only to the r.m.s. charge radius $R_n = 0.6$ fm of a nucleon whose $qqq$ wave function is generated by the same quadratic confining potential.

The $PP$ (or $S$) sector of the second channel needs special consideration because of the annihilation and creation processes, making the pairs $(2, \bar{3})$ and $(1, \bar{4})$ mixtures of $s\bar{s}, u\bar{u}$ and $d\bar{d}$ in the flavour space. The particular combinations depend upon the physical mesons taking part in the scattering process which may be $\eta$ and/or $\eta'$. In the isovector sector only $(2, \bar{3})$ has an amplitude for going to, say, $u\bar{u}$ from its original flavour state as $s\bar{s}$, resulting in $\eta$ or $\eta'$. As the size, and hence the quark wavefunction, of a particular cluster is related to the masses of the quark it contains, these amplitudes for having different flavour contents imply that in this particular channel we cannot write the quark and flavour wavefunctions separately as it is possible for the other channels. Rather these two are mixed here, giving us the combined quark-flavour wavefunctions (except $\chi_{2S}(R_2)$) as
\[ |2S\rangle_{fq} = |M_{14}\rangle_{fq}|M_{23}\rangle_{fq} = \begin{cases} |\eta'_{14}\rangle_{fq}|\eta'_{23}\rangle_{fq} & \text{for } \eta'\eta' \text{ mesons} \\ |\eta_{14}\rangle_{fq}|\eta'_{23}\rangle_{fq} & \text{for } \eta\eta' \text{ mesons} \\ |\eta'_{14}\rangle_{fq}|\eta_{23}\rangle_{fq} & \text{for } \eta'\eta \text{ mesons} \\ |\pi_{14}\rangle_{fq}|\eta'_{23}\rangle_{fq} & \text{for } \pi\eta' \text{ mesons} \\ |\pi_{14}\rangle_{fq}|\eta_{23}\rangle_{fq} & \text{for } \pi\eta \text{ mesons} \end{cases} \] 

(2.10)

with

\[ |\eta_{ij}\rangle_{fq} = \cos \theta \frac{|d\bar{d} + u\bar{u}|_f}{\sqrt{2}} \xi_2(r_{ij}) - \sin \theta |s\bar{s}|_f \zeta_2(r_{ij}) \]

\[ |\eta'_{ij}\rangle_{fq} = \sin \theta \frac{|d\bar{d} + u\bar{u}|_f}{\sqrt{2}} \xi_2(r_{ij}) + \cos \theta |s\bar{s}|_f \zeta_2(r_{ij}) \]

(2.11)

\[ |\pi_{14}\rangle_{fq} = |\pi\rangle_f \xi_2(r_{14}) = |\pi\rangle_f \xi_2(y_{14}) \]

(2.12)

Here

\[ \xi_2(r_{ij}) = \frac{1}{(2\pi d_{21})^{3/4}} \exp[-r_{ij}^2/4d_{21}^2] \] and \[ \zeta_2(r_{ij}) = \frac{1}{(2\pi d_{22})^{3/4}} \exp[-r_{ij}^2/4d_{22}^2], \]

(2.13)

where \( d_{21} = d, d_{22} = d'' \), and \( \theta (= 34.7^\circ) \) is related to the mixing angle \( \theta_P = -20^\circ \) of flavour singlet and octet resulting in \( \eta \) and \( \eta' \) (see pages III.68-69 of [23]).

After this discussion of the total wavefunction, we write the Schrödinger equation for the system in hand:

\[ (H - E_c)|\Psi(q_1, q_2, \bar{q}_3, \bar{q}_4; g)\rangle = 0 \] 

(2.14)
where $H$ is the total hamiltonian and $E_c$ is the total centre-of-mass energy of the $q^2\bar{q}^2$ system. This also means that the overlap of $(H - E_c)|\Psi\rangle$ with an arbitrary variation $|\delta\Psi\rangle$ of the state vector $|\Psi\rangle$ vanishes. In $|\delta\Psi\rangle$ we consider, as in resonating group method calculations, only the variations in $\chi_{kI}$. Thus we write

$$
\langle \delta\Psi | H - E_c | \Psi \rangle = \sum_{kI} \int d^3 R_c d^3 R_k d^3 y_k d^3 z_k 
\psi_c(R_c) \delta \chi_{kI}(R_k) \xi_k(y_k) \zeta_k(z_k) f(k|s<kI|g(k|H - E_c|l)g|lJ)s|l) f(\psi_c(R_c) \chi_{IJ}(R_l) \xi_l(y_l) \zeta_l(z_l) = 0.
$$

(2.15)

To do the four space integrations implied in the overlap of the diquark di-antiquark position dependent $(H - E_c)|\Psi\rangle$ and $|\delta\Psi\rangle$, any of the three sets of three-vectors defined by eqs.\(2.4\),\(2.5\) and \(2.6\), along with $R_c$, can be used. The choice $R_k, y_k$ and $z_k$ has, however, a clear advantage. The arbitrary variations $\delta \chi_{kI}(R_k)$’s for different (but continuous!) values of $R_k$, and of $k$ and $I$ as well, are linearly independent and hence their coefficients in eq.\(2.15\) should be zero. With the trivial $R_c$ integration performed to give a finite result using, say, box normalization, this leads to

$$
\sum_{lI} \int d^3 y_k d^3 z_k \xi_k(y_k) \zeta_k(z_k) f(k|s<kI|g(k|H - E_c|l)g|lJ)s|l) f(\chi_{IJ}(R_l) \xi_l(y_l) \zeta_l(z_l) = 0,
$$

(2.16)

for $k,l=1,2$ or $3$ and $I, J = S$ or $T$ (except the special case of $k,l=2$ and
\[ I, J = S \text{ where a mixed quark-flavour wavefunction is used).} \]

In [11] (and also in the previous section) the suggested hamiltonian of the system was mentioned in the spin independent context. Despite the additional degrees of freedom incorporated here we proceed in the same fashion, considering first the hamiltonian in the two-body potential model limit and then modifying the off-diagonal elements. Now the spin dependent part of the hamiltonian, composed of the terms corresponding to hyperfine (contact as well as tensor) and spin-orbit interactions, has to be considered as well, along with that representing the annihilation effects. But our job here is simpler than this because of our constraint to the S-wave ground states, implying that the only additional spin-dependent term in the hamiltonian to be dealt with is that representing the hyperfine contact interaction.

For the hyperfine term in the two-body potential model limit we take the expression given by one gluon exchange (used, with some modifications, in [7]), and sum over all the pairs:

\[ V_{ij}^{\text{hyp}} = \sum_{i<j} V_{ij}^{\text{hyp}} = -\sum_{i<j} \mathbf{F}_i \cdot \mathbf{F}_j \frac{8\pi \alpha_s^{ij}}{3m_i m_j} \delta^3(\mathbf{r}_{ij}) \mathbf{S}_i \cdot \mathbf{S}_j. \]  

(2.17)

Numerical values of \( \alpha^{ij}_s \), fitted to the light meson spectroscopy below, will be taken as varying with the sum of the masses of the particles \( i \) and \( j \); thus each of them will eventually be replaced by \( \alpha^{ll}_s, \alpha^{ls}_s \text{ or } \alpha^{ss}_s \), \( l \) standing for a quark(antiquark) of light mass (i.e. up or down) and \( s \) for a strange (heavier)
The pair annihilation and creation effects (where considered) are represented by a Hamiltonian term \( V^a \) (denoted by \( H_A \) in the Hamiltonian appearing in [3]) operating in the flavour space only. This should be a sum of two terms belonging to the pairs \( 1\bar{4} \) and \( 2\bar{3} \). Thus

\[
V^a = V^a_{1\bar{4}} + V^a_{2\bar{3}},
\]

(2.18)

The matrix elements of \( V^a_{ij} \) are written as

\[
\begin{align*}
 f\langle u\bar{u}|V^a_{ij}|u\bar{u}\rangle_f &= f\langle d\bar{d}|V^a_{ij}|d\bar{d}\rangle_f = l, \\
 f\langle u\bar{u}|V^a_{ij}|s\bar{s}\rangle_f &= f\langle d\bar{d}|V^a_{ij}|s\bar{s}\rangle_f = \sqrt{l}\sqrt{n} \quad \text{and} \quad f\langle s\bar{s}|V^a_{ij}|s\bar{s}\rangle_f = n.
\end{align*}
\]

(2.19) (2.20)

in the corresponding flavour spaces. The mass (or flavour) dependence shown here is in qualitative agreement with the mass dependence of the annihilation term \( H_A \) appearing in [3]. In the above, \( l \) and \( n \) are phenomenological parameters to be fitted to the masses and flavour wave functions of \( \pi, \eta \) and \( \eta' \) mesons.

This specification of the Hamiltonian of the system means that \( H \) in eq. (2.16) is to be replaced by \( K + V^p + V^a + \sum_{i=1}^{4} m_i \), with \( V^p = V^{cf} + V^{hyp} \)

i.e. a sum of the confinement and hyperfine potentials. Using eqs. (1.2) and (2.17) for \( V^{cf} \) and \( V^{hyp} \) respectively, it can be seen easily that now the matrix elements of \( V^p \) in our spin basis, in the two-body potential model, would be
given by

\[ s\langle X|V^p|X'\rangle_s = \sum_{i<j} F_i \cdot F_j \langle V_{ij} \rangle_{X_s,X'_s}, \quad \text{with} \quad (2.21) \]

\[ (V_{ij})_{X_s,X'_s} = v_{ij} s\langle X|X'\rangle_s - \frac{8\pi\alpha_s^{ij}}{3m_im_j} \delta^3(r_{ij}) s\langle X|S_i \cdot S_j|X'\rangle_s. \quad (2.22) \]

In this form, \( s\langle X|V^p|X'\rangle_s \) is very similar to the expression for \( V \) appearing in eq.(1.2). So, as far as its matrix elements between the gluonic states are concerned, the whole formalism developed in [11] (also reported in the previous section) can be utilized, provided the spin state dependence of the matrix coefficients \( V_{ij} \), replacing \( v_{ij} \), of \( F_i \cdot F_j \) is taken care of. Thus, the matrix element of the \( V^p \) term in the Hamiltonian between any two gluonic-spin states appearing in eq.(2.16) is given by (see eq.(1.6) for comparison)

\[
 V^p = \begin{pmatrix}
 -\frac{4}{3}(V_{13} + V_{24})_{1,1} & \frac{4f}{9} \left( \begin{array}{c}
 V_{12} + V_{34} \\
 -V_{13} - V_{24} \\
 -V_{14} - V_{23}
 \end{array} \right)_{1,2} & \frac{2f}{3\sqrt{3}} \left( \begin{array}{c}
 -2(V_{13} + V_{24}) \\
 +V_{14} + V_{23} \\
 -V_{12} - V_{34}
 \end{array} \right)_{1,3} \\
 -\frac{4}{3}(V_{14} + V_{23})_{2,2} & \frac{2f}{3\sqrt{3}} \left( \begin{array}{c}
 2(V_{14} + V_{23}) \\
 +V_{12} + V_{34} \\
 -V_{24} - V_{13}
 \end{array} \right)_{2,3} \\
 \text{symmetric} & \frac{1}{3} \left( \begin{array}{c}
 2(V_{12} + V_{34}) \\
 +V_{13} + V_{24} \\
 +V_{14} + V_{23}
 \end{array} \right)_{3,3} & \frac{5}{2} I Df(1 - f)
 \end{pmatrix},
\]

in the gluonic-spin basis \( |1\rangle_g|1S\rangle_s, |1\rangle_g|1T\rangle_s, \ldots, |3\rangle_g|3T\rangle_s \).

The above matrix should give \( 6 \times 6 = 36 \) matrix elements of the \( V^p \) operator in its 6-dimensional basis. But as written above, it has only \( nine \)
elements. Actually, every term in every matrix element of the above is meant to stand for a $2 \times 2$ matrix defined by

$$(V_{ij})_{k,l} = \begin{pmatrix} (V_{ij})_{kS, lS} & (V_{ij})_{kS, lT} \\ (V_{ij})_{kT, lS} & (V_{ij})_{kT, lT} \end{pmatrix},$$

(2.24)

for $k, l = 1, 2$ or 3. $(V_{ij})_{kI, lJ}$ is given through eq. (2.22) and $I$ multiplying the $D$ term in 3,3 element of eq. (2.23) is a $2 \times 2$ identity matrix. To determine the overlap of any two spin states and the corresponding matrix element of the operator $S_i \cdot S_j$, use has to be made of the definitions given through eqs. (2.1), (2.2) and (2.3) above. The results are reported in Appendix A.

As far as the spin dependence is concerned, the other terms in the Hamiltonian are unit operators. Using the results reported in the previous section for the matrix elements between gluonic states, we write eq. (2.10) as

$$\sum_{lJ} \int d^3R_l \left[ K_{kI, lJ}(R_k, R_l) + \gamma_{kI, lJ}^{\text{cf}}(R_k, R_l) + \gamma_{kI, lJ}^{\text{hyp}}(R_k, R_l) \right. \quad - \left. \left( E_c - \sum_{i=1}^{\frac{1}{4}} m_i \right) N_{kI, lJ}(R_k, R_l) \right] \chi_{lJ}(R_l) = 0, \quad (2.25)$$

for $l = 1, 2$ and 3 along with $J = S$ and $T$. This gives six equations, each for one of the six possible values of the pair of variables $k, I$ with $k = 1, 2$ or 3 and $I = S$ or $T$. Here $K_{kI, lJ}$, $\gamma_{kI, lJ}^{\text{cf}}$, $\gamma_{kI, lJ}^{\text{hyp}}$ and $N_{kI, lJ}$ are defined through the following equations:

$$\int d^3R_k' K_{kI, lJ}(R_k, R_k') \chi_{lJ}(R_l) = \int d^3y_k d^3z_k \xi_k(y_k)\zeta_k(z_k) K_{kI, lJ}(R_k) \xi_l(y_l)\zeta_l(z_l),$$

(2.26)
\[ \int d^3 \mathbf{R}_l^I V_{kI,lJ}^{cf} (\mathbf{R}_k, \mathbf{R}_l^I) \chi_{I,J}(\mathbf{R}_l^I) = \int d^3 \mathbf{y}_k d^3 \mathbf{z}_k \xi_k(\mathbf{y}_k) \zeta_k(\mathbf{z}_k) V_{kI,lJ}^{cf} (\mathbf{R}_k, \mathbf{R}_l^I) \chi_{I,J}(\mathbf{R}_l^I) \xi_l(\mathbf{z}_l) \]  

(2.27)

\[ \int d^3 \mathbf{R}_l^I V_{kI,lJ}^{hyp} (\mathbf{R}_k, \mathbf{R}_l^I) \chi_{I,J}(\mathbf{R}_l^I) = \int d^3 \mathbf{y}_k d^3 \mathbf{z}_k \xi_k(\mathbf{y}_k) \zeta_k(\mathbf{z}_k) V_{kI,lJ}^{hyp} (\mathbf{R}_k, \mathbf{R}_l^I) \chi_{I,J}(\mathbf{R}_l^I) \xi_l(\mathbf{z}_l) \]  

(2.28)

\[ \int d^3 \mathbf{R}_l^I N_{kI,lJ} (\mathbf{R}_k, \mathbf{R}_l^I) \chi_{I,J}(\mathbf{R}_l^I) = \int d^3 \mathbf{y}_k d^3 \mathbf{z}_k \xi_k(\mathbf{y}_k) \zeta_k(\mathbf{z}_k) N_{kI,lJ} (\mathbf{R}_k, \mathbf{R}_l^I) \chi_{I,J}(\mathbf{R}_l^I) \xi_l(\mathbf{z}_l) \]  

(2.29)

with \( K_{kI,lJ}, V_{kI,lJ}^{cf} \) and \( V_{kI,lJ}^{hyp} \) representing the matrix elements of the \( K, V^{cf} \) and \( V^{hyp} \) operators between the spin and gluonic states appearing in eq.(2.16). \( N_{kI,lJ} \) is the overlap of these states, calculated using the results mentioned in the previous section along with those in Appendix A for the spin overlap factor \( s(kI|lJ)_s \).

The spatial integrations appearing on the RHS of the eqs.(2.26) to (2.29) are done after substituting the expressions for \( K_{kI,lJ}, V_{kI,lJ}^{cf}, V_{kI,lJ}^{hyp} \) and \( N_{kI,lJ} \), obtained through the procedure outlined above. In these calculations the diagonal \((k = l)\) and off-diagonal \((k \neq l)\) cases were dealt with separately. In the former case, \( \chi_{I,J}(\mathbf{R}_l) \) is linearly independent of the integration variables \( \mathbf{y}_k \) and \( \mathbf{z}_k \) and thus was simply taken out of the integrations. In the case of off-diagonal terms \((with k \neq l)\), the integration variables \( \mathbf{y}_k \) and \( \mathbf{z}_k \) were replaced by their equivalent combinations with one identical to \( \mathbf{R}_l \), and the other one independent of it. Integrating out the vector independent of \( \mathbf{R}_l \), expressing the remaining one on the RHS (apart from \( \mathbf{R}_l \)) in terms of \( \mathbf{R}_k \),
and $R$, we got the results for $\mathcal{K}_{kI,lJ}$, $\mathcal{V}_{kI,lJ}^{cf}$, $\mathcal{V}_{kI,lJ}^{hyp}$ and $\mathcal{N}_{kI,lJ}$ after comparing with the LHS of the corresponding equation.

Where annihilation is considered, we have to use the combined quark and flavour wave function $|2S\rangle_{fq}$ defined through eq.(2.10). Thus in the diagonal term corresponding to the $2S$ channel we have, in place of the $k = l = 2$ and $I = J = S$ term in eq.(2.16),

$$\int d^3r_1 d^3r_2 d^3r_3 f_q \langle 2S | \langle 2S | g \langle 2 | H - E_c | 2 \rangle_g | 2S \rangle_s | 2S \rangle_{f_q} \chi_{2S}(R_2)$$

$$= \int d^3r_1 d^3r_2 d^3r_3 f_q \langle 2S | \langle 2S | g \langle 2 | K + V^p - (E_c - \sum_{i=1}^{4} m_i) | 2 \rangle_g | 2S \rangle_s | 2S \rangle_{f_q} \chi_{2S}(R_2)$$

$$+ \int d^3r_1 d^3r_2 d^3r_3 f_q \langle 2S | V^a | 2S \rangle_{f_q} \chi_{2S}(R_2). \quad (2.30)$$

As expressed through eq.(2.10), the form of $|2S\rangle_{f_q}$ depends upon the physical content of the $2S$ channel. This would result in different expressions for each of $\mathcal{K}_{2S,2S}$, $\mathcal{V}_{2S,2S}^{cf}$, $\mathcal{V}_{2S,2S}^{hyp}$ and $\mathcal{N}_{2S,2S}$ for different pairs of mesons in the channel. In the following calculations we restrict ourselves to just the lowest, in threshold energy, channels: $\eta\eta$ in the isoscalar, plus $\eta\pi$ and $\eta'\pi$ in the the isovector sector. This is done because of our special interest in the behaviour of the $K\bar{K}$ system near the threshold (see the next section).

The results thus obtained for these non-local kernels, for all the values of $k, l, I$ and $J$, appearing in eq.(2.25) are reported in Appendix B. Substitution of these in eq.(2.25) would gives six coupled equations. However we neglect
all the connections to the third gluonic channel, justified to some extent by
the absence of any significant effect of removing the third channel in the
spinless case (see fig. 4 of [11]). This leaves us with just four equations (two
of these are written below as eq. (2.33) and (2.34)). The off diagonal (i.e. with
$k \neq l$) terms in these equations tend to zero for large inter-cluster distances.
Thus for consistency with the observed meson spectroscopy we would require
the constant term in each of the diagonal parts to be identical to the sum
of masses of the mesons present in the corresponding channel. Fitting, in
this way, to the masses of $K, \eta, \eta', \pi, K^*, \omega$ (or $\rho$) and $\phi$ mesons, we get
the following values of the above mentioned free parameters of the formalism:

$$
m = 277 \text{ MeV}, \quad s = 1.955, \quad \bar{C} = 456 \text{ MeV},
$$

$$
\alpha_{ls}^{uls} = 1.583, \quad \alpha_{ss}^{uls} = 1.561, \quad \alpha_{ss}^{ss} = 1.501,
$$

$$
l = 272 \text{ MeV} \quad \text{and} \quad n = 67.4 \text{ MeV}. \quad (2.31)
$$

For $C$ (see eq.(1.3)), the equality of kinetic and potential energies of a har-
monic oscillator is used, giving us

$$
C = -\frac{1}{4d^2} \frac{3}{4} \omega_2 = -\frac{3}{16md^4} = -270 \text{ MeV/fm}^2. \quad (2.32)
$$

After this parameter fit (except for $\bar{k}$ to be discussed in the next sec-
tion), we write down two of the four coupled equations mentioned above.
The remaining two equations would involve vector mesons. These are not
incorporated beyond this stage because of our above mentioned neglect of
channels opening at energies significantly higher than the $K\bar{K}$ threshold.

$$
\left[ M_K + M_{\bar{K}} - \frac{1}{2\mu_{KK}} \sum_{R_1}^2 \chi_{1S}(R_1) \right] + e_0 \int d^3R_2 \left\{ \left[ -\frac{1}{2m} \left( q_{11}R_1^2 + q_{12}R_2^2 + q_{10} \right) + C(R_1) \right] \exp \left[ -e_2R_2^2 - e_1R_1^2 \right] 
- G(R_1, R_2) \right\} \chi_{2S}(R_2) = 0, \quad \text{and} \quad (2.33)
$$

$$
\left[ M_a + M_b - \frac{1}{2\mu_{ab}} \sum_{R_2}^2 \chi_{2S}(R_2) \right] + e_0 \int d^3R_1 \left\{ \left[ -\frac{1}{2m} \left( q_{21}R_1^2 + q_{22}R_2^2 + q_{20} \right) + C(R_1) \right] \exp \left[ -e_2R_2^2 - e_1R_1^2 \right] 
- G(R_1, R_2) \right\} \chi_{1S}(R_1) = 0, \quad (2.34)
$$

for $K\bar{K} \leftrightarrow ab$, where $a$ and $b$ are the two mesons in the second channel
($\eta\eta$, $\eta\pi$ or $\eta'\pi$). $C(R_1)$ and $G(R_1, R_2)$ appearing in the above are

$$
C(R_1) = \frac{1}{2} \left( b_1R_1^2 + b_0 \right) - \frac{1}{6} \left( E_c + \frac{8}{3}C - 2m(s + 1) \right) \quad (2.35)
$$

$$
G(R_1, R_2) = l_{10} \exp \left[ -(e_1 + e'_1 - l_{11})R_1^2 - (e_2 + l_{12})R_2^2 \right] \times
\left[ \alpha_s^{ls} \exp (l_{13}R_1 \cdot R_2) + \alpha_s^{ls} \exp (-l_{13}R_1 \cdot R_2) \right] + l_{20} \exp \left[ -(e_1 + e'_1 + l_{21})R_1^2 - e_2R_2^2 \right] \times
\left[ \alpha_s^{ls} \exp (l_{22}R_1^2) + s\alpha_s^{ls} \exp (-l_{22}R_1^2) \right] + l_{30} \exp \left[ -(e_1 + e'_1 + l_{31})R_1^2 - (e_2 + l_{32})R_2^2 \right] \times
\left[ \alpha_s^{ls} \exp (l_{33}R_1 \cdot R_2) + \alpha_s^{ls} \exp (-l_{33}R_1 \cdot R_2) \right]. \quad (2.36)
$$
Moreover,

\[ H = \frac{1}{6} \frac{8\pi}{3m^2s} \frac{(2\kappa)^{3/2}}{(2\pi d^2)^{3/2}}. \]  

(2.37)

It should be noted that in the coefficients of \( \nabla_R^2 R_1 \) and \( \nabla_R^2 R_2 \), the reduced masses of the two pseudoscalar mesons of the particular channel now appear. This is done so as to ensure that the terms involving \( \nabla_R^2 R_1 \) or \( \nabla_R^2 R_2 \) give the correct kinetic energy of the relative motion of the interacting physical mesons. Other symbols appearing in the above equations are defined in Appendix B at appropriate places.

The kernels of the integrals appearing in the off-diagonal parts in the above coupled equations contain non-separable parts

\[ \exp(l_{13} \mathbf{R}_1 \cdot \mathbf{R}_2) \cdots \exp(l_{33} \mathbf{R}_1 \cdot \mathbf{R}_2). \]

The presence of these “non-separable potential” terms makes the solution of the coupled equations rather involved. To avoid that complication, we can solve our problem in the approximation of replacing these terms by their truncated expansions which would leave us with an inexact but easily manageable form of the equations. With that strategy in mind, the above equations were solved first for the case of no hyperfine interaction (thus avoiding non-separable terms) by setting \( H = 0 \) in the above two coupled equations. The method used for that is explained below for the full interaction case. The resulting phase shifts for this no hyperfine case (reported in the next section) are so small that it would be a good approximation to take the variational wave functions in the absence of
hyperfine interaction as the wave-functions corresponding to a freely propagating plane wave. Using this approximation for the variational wave functions \( \chi_{1S}(R_1) \) and \( \chi_{1S}(R_1) \) even in the presence of hyperfine interaction, we looked for a reasonable separable approximation to our non-separable potential terms. As far as the terms \( \exp(l_{13} R_1 \cdot R_2) \) and \( \exp(-l_{13} R_1 \cdot R_2) \) are concerned, it was seen to be a very good approximation to just replace them with the exponential expansion up to the second power in \( l_{13} R_1 \cdot R_2 \). But the terms multiplying \( l_{30} \) are not so easy to manage. Their separable approximation involved two parameters which had to be adjusted for each of value of the kinetic energy in the centre of mass frame. Written explicitly, our approximation has been to take

\[
l_{30} \exp \left[ -(e_1 + e'_1 + l_{31}) R_1^2 - (e_2 + l_{32}) R_2^2 \right] \times \frac{\alpha_{ls}^1 \exp(-l_{33} R_1 \cdot R_2) + \alpha_{ls}^2 \exp(l_{33} R_1 \cdot R_2)}{n_1 l_{30} \alpha_{ls}^1 \left\{ \exp \left[ -\tau_1 (e_1 + e'_1 + l_{31} + e_2 + l_{32} - l_{33}) (R_1^2 + R_2^2) \right] - \left[ -\tau_1 (e_1 + e'_1 + l_{31} + e_2 + l_{32} + l_{33}) (R_1^2 + R_2^2) \right] \right\} \approx n_1 l_{30} \alpha_{ls}^1 \left\{ \exp \left[ -\tau_1 (e_1 + e'_1 + l_{31} + e_2 + l_{32} + l_{33}) (R_1^2 + R_2^2) \right] \right\}, \tag{2.38}
\]

in \( G(R_1, R_2) \), and hence in the above two coupled equations. Here \( n_1 \) and \( \tau_1 \) are the above mentioned two energy dependent parameters. Actually these are to be used in eq.\( (2.33) \) only. Those appearing in eq.\( (2.34) \) are different and, therefore, are denoted by \( n_2 \) and \( \tau_2 \) instead.
Both sides of eq. (2.38), multiplied by $R_2^2 \chi_{2S}(R_2)$ and integrated over $R_2$, were plotted as functions of $R_1$. This comparison showed that in this way even the worst discrepancy could be reduced to less than 10 percent of the total hyperfine coupling for that particular value of $R_1$ and the on-shell momentum $p_c(2)$. After calculating the $T$ and $S$ scattering matrices, the results were checked for any possible deviation from unitarity of the $S$ matrix and symmetry of the $T$ matrix (required by “reciprocity” of inelastic scattering, see, for example, p.528 of [24]). The discrepancy was for some cases as bad as 30 percent, implying that the above approximation needs to be improved by, for example, iterating it many times. This improvement remains to be made, although it can be easily shown that this re-adjusting of the values of $n_1, n_2, \tau_1$ and $\tau_2$ (with improved functional dependences of $\chi_{1S}(R_1)$ and $\chi_{2S}(R_2)$) is not needed for the range of energy where our immediate interest lies (e.g. below the $K\bar{K}$ threshold in the first channel). This follows because the formal momentum space solutions (see eqs. (2.39) and (2.40) below) of the above coupled equations, for that range of energy, would be of the form

$$\chi(p) = -\frac{1}{\Delta(p)} \times \text{Constant}.$$  

Changing values of $n_1, \tau_1, n_2$ and $\tau_2$ in that situation would just affect the constant coefficients of $\frac{1}{\Delta_1(p_1)}$ and $\frac{1}{\Delta_2(p_2)}$, leaving the momentum, and hence the space, dependence of the solutions $\chi_{1S}$ and $\chi_{2S}$ unchanged.
With the above approximation the integrand appearing in the coupling terms in both of our coupled equations (2.33) and (2.34) are products of two factors, each of them is a function of $R_1$ or $R_2$. This means that in this form the two coupled equations can be solved exactly, using the method demonstrated in appendix B of \[1\]. The procedure used in the present case was to first write eqs.(2.33) and (2.34), in the approximate form (see eq.(2.38)), in momentum space. For incoming waves in the first channel, the formal momentum space solution of these equations would be

\[
\chi_{1s}(p_1) = \frac{\delta(p_1 - p_c(1))}{p_c^2(1)} - \frac{1}{\Delta_1(p_1)} \left[ Q_1^{(1)} A_2(e_2) + Q_2^{(1)} B_2(e_2) + Q_3^{(1)} A_2(e_2 + l_{12}) + Q_4^{(1)} B_2(e_2 + l_{12}) 
+ Q_5^{(1)} A_2(\tau_1 e_1 + e'_1 + e_2 + l_{31} - l_{33}) 
+ Q_6^{(1)} A_2(\tau_1 e_1 + e'_1 + e_2 + l_{32} + l_{33}) \right] 
\tag{2.39}
\]

\[
\chi_{2s}(p_2) = -\frac{1}{\Delta_2(p_2)} \left[ Q_1^{(2)} A_1(e_1) + Q_2^{(2)} B_1(e_1) 
+ Q_3^{(2)} A_1(e_1 + e'_1 + l_{21} - l_{22}) + Q_4^{(2)} A_1(e_1 + e'_1 + l_{21} + l_{22}) 
+ Q_5^{(2)} A_1(e_1 + e'_1 - l_{11}) + Q_6^{(2)} B_1(e_1 + e'_1 - l_{11}) 
+ Q_7^{(2)} A_1(\tau_2 e_1 + e'_1 + l_{31} + e_2 + l_{32} - l_{33}) 
+ Q_8^{(2)} A_1(\tau_2 e_1 + e'_1 + l_{31} + e_2 + l_{32} + l_{33}) \right]. 
\tag{2.40}
\]

The new symbols appearing in these equations are defined in Appendix C.
It is to be noted that in the above equations \( p_1 \) and \( p_2 \) have been replaced everywhere by \( p_1 \) and \( p_2 \) respectively, utilizing the spherical symmetry of our problem. Multiplying eq.\((2.39)\) by \( p_2^1 F_a (p_1, e_1), p_2^1 F_b (p_1, e_1), p_2^1 F_a (p_1, e_1 + e'_1 + l_{21} - l_{22}), p_2^1 F_a (p_1, e_1 + e'_1 + l_{21} + l_{22}), p_2^1 F_a (p_1, e_1 + e'_1 - l_{11}), p_2^1 F_b (p_1, e_1 + e'_1 - l_{11}), p_2^1 F_a \left( p_1, \tau_2 e_1 + e'_1 + l_{31} + e_2 + l_{32} - l_{33} \right) \) and \( p_2^1 F_a \left( p_1, \tau_2 e_1 + e'_1 + l_{31} + e_2 + l_{32} + l_{33} \right) \) in turn and integrating w.r.t. \( p_1 \) gives us 8 equations (\( F_a (p_1, x) \) and \( F_b (p_1, x) \) are the Fourier transforms of \( \exp[-xR_1^2] \) and \( R_1^2 \exp[-xR_1^2] \) respectively). Similarly multiplying eq.\((2.40)\) by the Fourier transforms \( p_2^2 F_a (p_2, e_2), p_2^2 F_b (p_2, e_2), p_2^2 F_a (p_2, e_2 + l_{12}), p_2^2 F_b (p_2, e_2 + l_{12}), p_2^2 F_a \left( p_2, \tau_1 e_1 + e'_1 + l_{31} + e_2 + l_{32} - l_{33} \right) \) and \( p_2^2 F_a \left( p_2, \tau_1 e_1 + e'_1 + l_{31} + e_2 + l_{32} + l_{33} \right) \) and integrating w.r.t. \( p_2 \) gives us 6 more equations. These 14 equations can be written as a matrix equation

\[
QU_1 = U_2 \quad (2.41)
\]
with

\[ U_2 = 4\pi \begin{pmatrix} 
F_a (p_c(1), e_1) \\
F_b (p_c(1), e_1) \\
F_a (p_c(1), e_1 + e'_1 + l_{21} - l_{22}) \\
F_a (p_c(1), e_1 + e'_1 + l_{21} + l_{22}) \\
F_a (p_c(1), e_1 + e'_1 - l_{11}) \\
F_b (p_c(1), e_1 + e'_1 - l_{11}) \\
F_a (p_c(1), \tau_2 e_1 + e'_1 + l_{31} + e_2 + l_{32} - l_{33}) \\
F_a (p_c(1), \tau_2 e_1 + e'_1 + l_{31} + e_2 + l_{32} + l_{33}) \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 
\end{pmatrix}, \quad (2.42) \]

\( Q \) a 14 \times 14 matrix containing many integrals, and \( U_1 \) a vector containing \( A_2(e_2), B_2(e_2), \cdots, A_2(\tau_1 e_1 + e'_1 + l_{31} + e_2 + l_{32} + l_{33}) \)

and \( A_1(e_1), B_1(e_1), \cdots, A_1(\tau_2 e_1 + e'_1 + l_{31} + e_2 + l_{32} + l_{33}) \) as its elements. Inverting the matrix \( Q \) gives these 14 elements of the \( U_1 \) vector. With these values in hand, all the quantities in the expressions for the variational functions \( \chi_{1S}(p_1) \) and \( \chi_{2S}(p_2) \) are known. So these can be now simply obtained by making the usual replacement of \( p_1 \) and \( p_2 \) by their on-shell values \( p_c(1) \) and \( p_c(2) \), defined by eqs.(C.10) and (C.11), respectively in eqs.(2.39) and (2.40).

From eqs.(2.39) and (2.40) the two \( T \) matrix elements \( T_{1,1} \) and \( T_{2,1} \), proportional to the coefficients of the non-relativistic Green operators \( -\frac{1}{\Delta_1(p_1)} \) and \( -\frac{1}{\Delta_2(p_2)} \) respectively, can be read off. These are reported in Appendix
D. Similarly, for incoming waves in channel 2, the use of $U_2$ as
\[
U_2 = 4\pi \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]
instead of that given by eq. (2.42), gives the two $T$ matrix elements $T_{2,2}$ and $T_{1,2}$ reported also in Appendix D.

For the total energy in the centre of mass frame above the higher threshold, both of the channels would be open. Thus for incoming waves in either of them, there would be a loss of flux in the incoming channel (with the total flux remaining conserved). Representing this inelasticity by a factor $\epsilon_k$, for $k = 1$ or 2, we can write
\[
S_{1,1} = 1 - 2i T_{1,1} = \epsilon_1 e^{2i\delta_1} \tag{2.44}
\]
\[
S_{2,2} = 1 - 2i T_{2,2} = \epsilon_2 e^{2i\delta_2}. \tag{2.45}
\]

For elastic scattering, $\epsilon_1$ or $\epsilon_2$ would be unity for incoming waves in channel 1 or 2 respectively.
Below the lower threshold the situation would be qualitatively different as, with both \( p_c(1) \) and \( p_c(2) \) being imaginary, \( \delta(p_1 - p_c(1)) \) and \( \delta(p_2 - p_c(2)) \) would not contribute to the integration over all the real values of \( p_1 \) and \( p_2 \) performed to arrive at eq. (2.41). Thus all the terms collected in the vector \( U_2 \) would be absent, leaving us instead with

\[
Q U_1 = 0. \tag{2.46}
\]

A non-trivial solution of this equation for the elements of the vector \( U_1 \) requires

\[
\text{det } Q = 0. \tag{2.47}
\]

giving us a condition for the existence of a bound state of the whole system.

### 3 Results

As should be clear from the expressions reported in Appendix B, our coupled equations can describe a number of meson-meson systems. Amongst these systems we have chosen \( K\bar{K} \), keeping in mind that it has been investigated by other groups using different models for the quark-quark interaction. An important issue is whether the whole \( K\bar{K} \) system has a bound state just below the \( K\bar{K} \) threshold or not. According to our method of answering this question, the condition for a non-trivial solution of the Schrödinger equation to exist for the total energy of the whole system below the lowest threshold
is the one given by eq. (2.47). The Q matrix mentioned there is actually a complicated function of the parameters of the formalism fitted above, except $\bar{k}$, and the total energy of the whole system. $\bar{k}$ (see eq. (1.7)) is the phenomenological parameter of our model of gluonic effects. Our numerical calculations, including calculation of the determinant of the Q matrix, were done for three values of $\bar{k}$ in turn. The value $\bar{k} = 0$ corresponds to a two-body potential model hamiltonian. On the other hand, $\bar{k} = 1/2 \text{ fm}^{-2}$ is emerging from the lattice gauge theory calculations [19] for rectangular configurations of quark positions. For other configurations, indications [20] are that the spatial decrease of gluonic topologies overlap may be slower, and thus we have also used an intermediate value ($\bar{k} = 1/6 \text{ fm}^{-2}$). Which of these, if any, would simulate the ”experimental” (lattice gauge theory calculations based) behaviour of the gluonic overlap is yet to be seen.

Our numerical calculations showed that for any value of $\bar{k}$, the above condition for the existence of a bound state (see eq. (2.47)) is not satisfied for any value of energy below the $K\bar{K}$ threshold. This is the situation in the isoscalar as well as in the isovector sector, whereas in the latter case all connections to the $\pi\eta$ channel are neglected as those would not affect the answer to the main question being discussed here. On the other hand, using a closely related model Weinstein and Isgur [7] get $K\bar{K}$ bound states in both the isoscalar and the isovector sectors, and conclude that the two scalar
meson resonances \( f_0(975) \) and \( a_0(980) \) can be explained as loosely bound \( K\bar{K} \) states. Their model corresponds, in some approximation (see the first paragraph of the next section), to ours in the limit \( \bar{k} = 0 \). Therefore it is interesting to see if, in the corresponding limit, we can get their results by varying our parameters. Our calculations showed that, in the \( \bar{k} = 0 \) limit, we need to multiply our total couplings of the two channels by a factor of 2.715 before we can get bound states in both the isoscalar and isovector sectors. Alternatively, we can get these bound states by multiplying only the hyperfine couplings by a factor of 1.926. We get bound states without hyperfine interaction as well, but for that an increase by a factor of 3.06 in the remaining couplings, with \( \bar{k} = 0 \), would be needed. This is one of the indications in our work that the hyperfine coupling is the main interaction arising through quark exchange, and that the hyperfine and other couplings arising through confinement etc. have opposite signs.

Our modification to the two-body potential model proposed in this paper (equivalent to using non-zero values of \( \bar{k} \)) implies a decrease in the \( K\bar{K} \) coupling. That means that we need to increase the couplings even more so as to get binding. The factors so needed for the different values of \( \bar{k} \) are reported in table (3.1), along with the corresponding energy values for the resulting binding.

In addition we report below (in tables (3.2) to (3.7)) the \( K\bar{K} \) phase shifts,
in the elastic as well as in the inelastic region, for the hyperfine interaction increased by the above mentioned factor of 1.926, for the different values of \( \tilde{k} \). With that increase in the coupling we get bound states of the whole \( K\bar{K} \) system for \( \tilde{k} = 0 \) in both the isoscalar and the isovector sectors, and what is explored here is just the effect of our proposed modification to the four-body potential. The numerical procedure to get these phase shifts was based on eqs. (2.44) and (2.45). Each of them is a complex equation and hence can be solved for the two quantities \( \epsilon_k \) (the inelasticity factor) and \( \delta_k \) (the phase shift), with \( k = 1 \) or 2, for each value of energy.

Also mentioned are the values of the inelastic phase shifts for the incoming waves in the other channel i.e. \( \eta \eta \) in the isoscalar channel and \( \pi \eta' \) in the isovector one. Moreover, the values of all of these phase shifts in the absence of hyperfine interaction (without any increase of coupling) are reported. However, it must be emphasized that because of the various approximations which we have used in this work, it would be improper to take these numbers

| \( \tilde{k} \) in fm\(^{-2} \) | increase to get binding | B.E. in the isovector channel | B.E. in the isoscalar channel | increase in the total coupling to get binding | increase without hyperfine to get binding |
|-----------------|----------------------|-----------------------------|-----------------------------|---------------------------------|---------------------------------|
| 0               | 1.926                | <1 MeV                      | 35 MeV                      | 2.715                           | 3.06                            |
| 1/6             | 3.00                 | <1 MeV                      | 53 MeV                      | 4.26                            | 6.57                            |
| 1/2             | 6.43                 | <1 MeV                      | 87 MeV                      | 8.89                            | 19.13                           |

Table 3.1: Increases in the couplings necessary to get binding.
as precise results of our model. One of the indications of this inaccuracy is the violation of unitarity resulting from our separable approximation to the actual non-separable terms in the integrands appearing in our coupled equations (2.33) and (2.34). As mentioned in the paragraph following eq.(2.38), this approximation affects badly our results for the phase shifts, although not our conclusions regarding $K\bar{K}$ binding. For the elastic region this unitarity violation manifests itself in the reported (see tables (3.2) to (3.7)) deviation from unity of the inelasticity factor $\epsilon_1$.

### 4 Conclusions

In this paper a formalism has been developed to deal with meson-meson systems having their dynamics resulting through quark exchange effects, and applied, as a first application to a realistic case, to $K\bar{K}$ systems. Here we

| $E_c$ (total cm energy) | full coupling | without hyperfine |
|------------------------|---------------|-------------------|
|                        | $\delta_1$ (degs) | $\epsilon_1$ | $\delta_2$ (degs) | $\delta_1$ (degs) | $\epsilon_1$ | $\delta_2$ (degs) |
| 997.0 MeV              | 102.09        | 0.729            | 1.11           | 1.00             |
| 1042.0 MeV             | 91.49         | 0.719            | 3.05           | 1.00             |
| 1092.0 MeV             | 96.56         | 0.726            | 4.05           | 1.00             |
| 1095.5 MeV             | 99.01         | 0.728            | 4.25           | 1.00             |
| 1142.0 MeV             | 91.74         | 155.94           | 3.94           | 1.35             |
| 1192.0 MeV             | 83.41         | 147.26           | 3.39           | 1.46             |

Table 3.2: The isovector sector phase shifts for $\vec{k}=0$. 


| $E_c$ (total cm energy) | full coupling | without hyperfine |
|-------------------------|---------------|-------------------|
|                         | $\delta_1$ (degs) | $\epsilon_1$ | $\delta_2$ (degs) | $\delta_1$ (degs) | $\epsilon_1$ | $\delta_2$ (degs) |
| 997.0MeV                | 157.18        | 0.953            | 2.29            | 1.00            |
| 1042.0MeV               | 125.92        | 0.804            | 6.56            | 1.00            |
| 1092.0MeV               | 118.85        | 0.779            | 9.61            | 1.00            |
| 1097.5MeV               | 119.76        | 0.781            | 10.73           | 1.00            |
| 1142.0MeV               | 112.10        | 142.78           | 9.26            | 4.38            |
| 1192.0MeV               | 105.26        | 127.53           | 7.33            | 4.45            |

Table 3.3: The isoscalar sector phase shifts for $\bar{k} = 0$.

| $E_c$ (total cm energy) | full coupling | without hyperfine |
|-------------------------|---------------|-------------------|
|                         | $\delta_1$ (degs) | $\epsilon_1$ | $\delta_2$ (degs) | $\delta_1$ (degs) | $\epsilon_1$ | $\delta_2$ (degs) |
| 997.0MeV                | 5.13          | 0.997            | 0.21            | 1.00            |
| 1042.0MeV               | 17.10         | 0.972            | 0.59            | 1.00            |
| 1092.0MeV               | 30.15         | 0.919            | 0.82            | 1.00            |
| 1097.5MeV               | 33.03         | 0.905            | 0.86            | 1.00            |
| 1142.0MeV               | 30.84         | 179.46           | 0.83            | 0.27            |
| 1192.0MeV               | 27.19         | 177.86           | 0.74            | 0.30            |

Table 3.4: The isovector sector phase shifts for $\bar{k} = 1/6$ fm$^{-2}$.

| $E_c$ (total cm energy) | full coupling | without hyperfine |
|-------------------------|---------------|-------------------|
|                         | $\delta_1$ (degs) | $\epsilon_1$ | $\delta_2$ (degs) | $\delta_1$ (degs) | $\epsilon_1$ | $\delta_2$ (degs) |
| 997.0MeV                | 17.21         | 0.972            | 0.40            | 1.00            |
| 1042.0MeV               | 50.39         | 0.821            | 1.18            | 1.00            |
| 1092.0MeV               | 79.08         | 0.725            | 1.79            | 1.00            |
| 1097.5MeV               | 87.37         | 0.715            | 1.98            | 1.00            |
| 1142.0MeV               | 86.34         | 151.52           | 1.86            | 0.90            |
| 1192.0MeV               | 78.69         | 140.20           | 1.56            | 0.96            |

Table 3.5: The isoscalar sector phase shifts for $\bar{k} = 1/6$ fm$^{-2}$.

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| $E_c$         | full coupling | without hyperfine | $\delta_1$ (degs) | $\epsilon_1$ (degs) | $\delta_2$ (degs) | $\delta_1$ (degs) | $\epsilon_1$ (degs) | $\delta_2$ (degs) |
|--------------|---------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| (total cm energy) |               |                   |                   |                   |                   |                   |                   |                   |
| 997.0MeV     | 0.58          | 1.000             | 0.02              | 1.00              |                   |                   |                   |                   |
| 1042.0MeV    | 1.96          | 1.000             | 0.06              | 1.00              |                   |                   |                   |                   |
| 1092.0MeV    | 3.38          | 0.999             | 0.09              | 1.00              |                   |                   |                   |                   |
| 1095.5MeV    | 3.62          | 0.999             | 0.10              | 1.00              |                   |                   |                   |                   |
| 1142.0MeV    | 4.15          |                   | 0.10              |                   |                   |                   |                   | 0.03              |
| 1192.0MeV    | 4.25          |                   | 0.09              |                   |                   |                   |                   | 0.04              |

Table 3.6: The isovector sector phase shifts for $\tilde{k} = 1/2$ fm$^{-2}$.

| $E_c$         | full coupling | without hyperfine | $\delta_1$ (degs) | $\epsilon_1$ (degs) | $\delta_2$ (degs) | $\delta_1$ (degs) | $\epsilon_1$ (degs) | $\delta_2$ (degs) |
|--------------|---------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| (total cm energy) |               |                   |                   |                   |                   |                   |                   |                   |
| 997.0MeV     | 1.17          | 1.000             | 0.04              | 1.00              |                   |                   |                   |                   |
| 1042.0MeV    | 4.18          | 0.998             | 0.12              | 1.00              |                   |                   |                   |                   |
| 1092.0MeV    | 8.01          | 0.994             | 0.19              | 1.00              |                   |                   |                   |                   |
| 1097.5MeV    | 9.22          | 0.991             | 0.22              | 1.00              |                   |                   |                   |                   |
| 1142.0MeV    | 9.28          |                   | 0.21              |                   |                   |                   |                   | 0.10              |
| 1192.0MeV    | 8.36          |                   | 0.19              |                   |                   |                   |                   | 0.12              |

Table 3.7: The isoscalar sector phase shifts for $\tilde{k} = 1/2$ fm$^{-2}$.  

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had to increase our resulting coupling by some numerical factor before we
could get a bound state of the whole system, even in the two-body potential
limit. The variational calculations based on the two-body potential reported
in [7] claim to get bound states of the whole $K\bar{K}$ system, concluding that
the two scalar meson resonances $f_0(975)$ and $a_0(980)$ can be explained as
loosely bound $K\bar{K}$ states in the isoscalar and the isovector sectors, respecti-
vely. Our detailed model of meson-meson dynamics, even in the two-body
potential model limit, is different to theirs, mainly because of our restricted
(i.e. only in the $2S$ diagonal term) incorporation of annihilation effects. This
neglect of the annihilation effects may have appreciably decreased the $K\bar{K}$
binding arising through our model. This is expected as the annihilation part
of the hamiltonian, incorporating the process $K\bar{K} \leftrightarrow \pi\pi$, was [25]
mainly responsible for the $K\bar{K}$ binding in the calculations reported in [7]. Moreover,
we might be underestimating the hyperfine interaction by treating this inter-
action partially as a perturbation, although in our work as well the hyperfine
coupling turned out to be the main interaction arising through the quark
exchange mechanism. It is difficult to say more about this problem unless
a more refined treatment of the hyperfine interaction, along with the anni-
hilation effects, is carried out. On the other hand, the fitting of the model
parameters in [4] includes adjusting the ranges and normalization of their
effective meson-meson potentials in an ad-hoc way, and it is not clear how
that affects their results.

Leaving these issues to some future work, we looked for any possible change in one of our parameters so as to get $K\bar{K}$ bound states in the limit where our model would roughly correspond to that used in \cite{7} (i.e. for $\bar{k} = 0$), and then determined, with the changed parameter, the effects of going beyond that limit i.e. of using our theoretically improved four-body potential rather than the naive two-body potential. This means the use of non-zero values of $\bar{k}$ in our terminology. This investigation showed the same trend as observed in the spin independent case reported in \cite{11}: increasing $\bar{k}$, i.e. decreasing the gluonic states overlap, results in a significantly weaker meson-meson interaction. This means that if we get a $K\bar{K}$ bound state in the two-body potential model limit, we do not necessarily get one with our QCD-inspired refinement of the $q^2\bar{q}^2$ potential.

Much improvement in the calculations can be made by going beyond the approximations we have used, giving quantitatively more precise results. But even without this being carried out, this work clearly indicates that the theoretical refinement of the four-body potential results in an appreciable decrease in a major part of the meson-meson interaction—enough to cast doubt on any result based on a naive two-body potential model.
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Appendix A: The spin basis

In the main part of this paper, we use the spin basis given through the eqs. (2.1), (2.2) and (2.3). The notation used in these equations is that of the Appendix D of [7]. In this notation, an orthonormal spin basis is $|S_{12}^{-}S_{34}^{-}\rangle_s$ and $|A_{12} \cdot A_{34}\rangle_s$, defined through eqs. (D3) and (D4) there. Our remaining spin base states are given in terms of these by eqs. (D5-D8) of the same. These equations give easily the overlaps of our spin base states, written as $\langle kI|lJ\rangle_s$ in the results mentioned in Appendix B, as elements of the overlap
matrix

\[
A = \begin{pmatrix}
1 & 0 & \sqrt{1/4} & -\sqrt{3/4} & \sqrt{1/4} & \sqrt{3/4} \\
1 & -\sqrt{3/4} & -\sqrt{1/4} & \sqrt{3/4} & -\sqrt{1/4} \\
1 & 0 & -\sqrt{1/4} & \sqrt{3/4} & -\sqrt{1/4} \\
1 & \sqrt{3/4} & -\sqrt{1/4} & 1 & 0 \\
1 & 0 & -\sqrt{1/4} & -\sqrt{1/4} & 1 \\
1 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

(A.1)

in the basis \(|1S\rangle_s, |1T\rangle_s, \cdots, |3T\rangle_s\).

For the matrix elements of the \(S_i \cdot S_j\) operators, for different values of the indices \(i\) and \(j\), in our spin basis, we also used the results expressed through eqs.(D9-D11) of the Appendix D of [7]. Some of the results obtained in this way are reported here:

\[
|1S\rangle_s = s\langle P_{13} P_{24}\mid |1S\rangle_s = s\langle P_{13} P_{24}\mid \]

\[
|1T\rangle_s = s\langle V_{13} \cdot V_{24}\mid |1T\rangle_s = s\langle V_{13} \cdot V_{24}\mid \]

(A.2)

(A.3)

(A.4)
\[ s\langle 1S \rangle_s = \left[ \begin{array}{c} S_1 \cdot S_2 \\ S_1 \cdot S_3 \\ S_1 \cdot S_4 \\ S_2 \cdot S_3 \\ S_2 \cdot S_4 \\ S_3 \cdot S_4 \end{array} \right] \]

\[ s\langle 2S \rangle_s = \left[ \begin{array}{c} S_1 \cdot S_2 \\ S_1 \cdot S_3 \\ S_1 \cdot S_4 \\ S_2 \cdot S_3 \\ S_2 \cdot S_4 \\ S_3 \cdot S_4 \end{array} \right] \]

\[ s\langle P_{14} P_{23} \rangle_s = \left[ \begin{array}{c} P_{14} P_{23} \end{array} \right] \]

\[ \langle 2S \rangle_s = \left[ \begin{array}{c} P_{14} P_{23} \end{array} \right] \]

\[ (A.5) \]

\[ (A.6) \]

Appendix B: The kernels of the integro-differential equations

The results for (in general non-local) kernels appearing in eq. (2.25), calculated using the procedure outlined in the two paragraphs following this equation, are:

\[ \mathcal{N}_{kI,kJ} = \delta_{IJ} \delta(R_k - R'_k), \quad \text{for all values of } k \text{ and } J. \quad (B.1) \]

\[ \mathcal{K}_{kI,kJ} = \delta_{IJ} \delta(R_k - R'_k) \left[ \frac{3}{4} |\omega_{k1} + \omega_{k2}| - \frac{f_k}{2m} \sum_{a} r_a^2 \right], \quad (B.2) \]

for all values of \( k, I \) and \( J \), except those corresponding to the \( 2S \) diagonal term. Here

\[ \omega_{k1} = \frac{g_k}{2md_{k1}^2} \quad \text{and} \quad \omega_{k2} = \frac{h_k}{2md_{k2}^2}, \quad \text{along with} \quad (B.3) \]
\[
\begin{align*}
f_1 &= f_3 = \frac{2}{s+1}, \\
g_1 &= g_3 = h_1 = h_3 = \frac{s+1}{s}, \\
f_2 &= \frac{s+1}{2s}, \\
g_2 &= 2, \\
h_2 &= \frac{2}{s},
\end{align*}
\]  \hspace{1cm} \text{(B.4)}

\(d_{k1}\) and \(d_{k2}\), for \(k = 1, 2\) or \(3\), are the same ones which appear in eq.(2.8).

\[
\begin{align*}
\mathcal{V}^{cf}_{1I,1J} &= \delta_{IJ} \delta(R_1 - R'_1) \left[ -\frac{8}{3} \bar{C} - 4C[d_{11}^2 + d_{12}^2] \right], \quad \text{for } I,J = S \text{ or } T. \, \text{Similarly,} \quad \text{(B.5)} \\
\mathcal{V}^{cf}_{2I,2J} &= \delta_{IJ} \delta(R_2 - R'_2) \left[ -\frac{8}{3} \bar{C} - 4C[d_{21}^2 + d_{22}^2] \right] \quad \text{(B.6)} \\
\mathcal{V}^{cf}_{3I,3J} &= \delta_{IJ} \delta(R_3 - R'_3) \times \\
& \quad \left\{ -\frac{8}{3} \bar{C} - \frac{4}{3} CR_3^2 - 6Cd' - 2Cd' \left( \frac{s-1}{s+1} \right)^2 \\
& \quad + \frac{5}{2} \frac{D}{(s+1)^3} \left[ 8kd'^2(s^2 + 1) + (s+1)^2 \right]^{3/2} \times \\
& \quad \exp \left[ \frac{4k(s-1)^2R_3^2}{8kd'^2(s^2 + 1) + (s+1)^2} \left( \frac{4kd'^2}{8kd'^2(s^2 + 1) + (s+1)^2} - \frac{1}{(s-1)^2} \right) \right] \quad \text{and} \\
& \quad \frac{5}{2} \frac{D}{(s+1)^3} \left[ 8kd'^2(s^2 + 1) + (s+1)^2 \right]^{3/2} \times \\
& \quad \exp \left[ \frac{8k(s-1)^2R_3^2}{16kd'^2(s^2 + 1) + (s+1)^2} \left( \frac{8kd'^2}{16kd'^2(s^2 + 1) + (s+1)^2} - \frac{1}{(s-1)^2} \right) \right] \right\}. \quad \text{(B.7)}
\end{align*}
\]

(See the discussion before eq.(2.9) for \(d'\)).
\[\nu_{1S,1S}^{hyp} = -\frac{8\pi}{3m^2s} \delta(R_1 - R'_1) \left[ \frac{\alpha_{s}^{13}}{(2\pi d_{11}^2)^{3/2}} + \frac{\alpha_{s}^{24}}{(2\pi d_{12}^2)^{3/2}} \right] \] (B.8)

\[\nu_{1T,1T}^{hyp} = -\frac{1}{3} \nu_{1S,1S}^{hyp} \] (B.9)

\[\nu_{2T,2T}^{hyp} = \frac{1}{3} \frac{8\pi}{3m^2s} \delta(R_2 - R'_2) \left[ \frac{s \alpha_{s}^{11}}{(2\pi d_{11}^2)^{3/2}} + \frac{\alpha_{s}^{23}}{s(2\pi d_{12}^2)^{3/2}} \right] \] (B.10)

\[\nu_{3S,3S}^{hyp} = -\frac{1}{3} \frac{8\pi}{3m^2s} \delta(R_3 - R'_3) \frac{1}{(2\pi d^2)^{3/2}} \left[ \alpha_{s}^{12} + \alpha_{s}^{34} \right] \] (B.11)

\[\nu_{3S,3T}^{hyp} = \nu_{3T,3S}^{hyp} = -\frac{1}{4\sqrt{3}} \frac{8\pi}{3m^2s} \delta(R_3 - R'_3) \frac{1}{(2\pi d^2)^{3/2}} \times \]
\[
\left\{ \frac{\alpha_{s}^{13} + \alpha_{s}^{24}}{(s^2 + 1)} \right\} \frac{(s + 1)^2}{2s^2} \exp \left[ -\frac{R_3^2}{2d^2} \frac{(s + 1)^2}{2s^2} \right] - s \alpha_{s}^{14} \frac{(s + 1)^2}{2s^2} \exp \left[ -\frac{R_3^2}{2d^2} \frac{(s + 1)^2}{2s^2} \right] - \frac{\alpha_{s}^{23}}{s} \frac{(s + 1)^2}{2} \exp \left[ -\frac{R_3^2}{2d^2} \frac{(s + 1)^2}{2} \right] \right\} \] (B.12)

\[\nu_{3T,3T}^{hyp} = -\frac{1}{6} \frac{8\pi}{3m^2s} \delta(R_3 - R'_3) \frac{1}{(2\pi d^2)^{3/2}} \times \]
\[
\left\{ -\left[ \alpha_{s}^{12} + \alpha_{s}^{34} \right] + \frac{\alpha_{s}^{13} + \alpha_{s}^{24}}{(s^2 + 1)} \right\} \frac{(s + 1)^2}{2s^2} \exp \left[ -\frac{R_3^2}{2d^2} \frac{(s + 1)^2}{2s^2} \right] + s \alpha_{s}^{14} \frac{(s + 1)^2}{2s^2} \exp \left[ -\frac{R_3^2}{2d^2} \frac{(s + 1)^2}{2s^2} \right] + \frac{\alpha_{s}^{23}}{s} \frac{(s + 1)^2}{2} \exp \left[ -\frac{R_3^2}{2d^2} \frac{(s + 1)^2}{2} \right] \right\} \] (B.13)

\[N_{k1,lJ} = e_{0s} \langle k|I|2,J \rangle_{N_{k1,lJ}} \exp \left[ -e_{1} R_{k}^2 - e_{2} R_{3}^2 \right] \] for \( k = 1 \) or 3 (B.14)

\[N_{2l,lJ} = e_{0s} \langle 2I|l,J \rangle_{N_{2l,lJ}} \exp \left[ -e_{1} R_{l}^2 - e_{2} R_{2}^2 \right] \] for \( l = 1 \) or 3 (B.15)

\[N_{k1,lJ} = s \langle k|I|J \rangle_{N_{k1,lJ}} \left( \frac{s + 1}{64s^3} \right) \frac{1}{\pi d^2(1 + 4k d^2)} \left( \frac{1}{2d^2} \right)^{3/2} \times \]
\[
\exp \left\{ -\left( \frac{s+1}{2} \right)^2 \left( \frac{1+8\bar{k}d'^2}{4d'^2} \right) \left[ \frac{s^2+1}{s^2} \left( R_k^2 + R_l^2 \right) + \frac{2s^2-1}{s^2} R_k \cdot R_l \right] \right\}
\]

for \( k, l = 1 \) or \( 3 \), but with \( k \neq l \). Here (B.16)

\[
e_0 = (s+1)^{9/4} s^{-15/8} 2^{3/4} \left( \pi \kappa d'^2 \right)^{-3/2}
\]

(B.17)

\[
e_1 = \frac{1}{4d'^2} \left( \frac{s+1}{2} \right)^2 \left[ \gamma - \frac{\lambda^2}{\kappa} \right]
\]

(B.18)

\[
e_2 = 4\bar{k} + \frac{1}{2d'^2} \sqrt{\frac{2s}{s+1}}, \quad \text{with} \quad (B.19)
\]

\[
\kappa = 8\bar{k}d'^2 \left[ \frac{s^2+1}{s^2} \right] + 1 + s^{-3/2} \left[ \frac{(s+1)^2}{\sqrt{2(s+1)}} + 1 \right]
\]

(B.20)

\[
\lambda = 8\bar{k}d'^2 \left[ \frac{s^2-1}{s^2} \right] + 1 + s^{-3/2} \left[ \frac{s^2-1}{\sqrt{2(s+1)}} - 1 \right]
\]

(B.21)

\[
\gamma = 8\bar{k}d'^2 \left[ \frac{s^2+1}{s^2} \right] + 1 + s^{-3/2} \left[ \frac{(s-1)^2}{\sqrt{2(s+1)}} + 1 \right].
\]

(B.22)

\[
\mathcal{K}_{kI,2J} = -\frac{e_0}{2m_s} \langle kI|2J \rangle_{s} N^0_{k,2} \left[ q_{11} R_k^2 + q_{12} R_2^2 + q_{10} \right] \exp \left[ -e_1 R_k^2 - e_2 R_2^2 \right]
\]

for \( k = 1 \) or \( 3 \) (B.23)

\[
\mathcal{K}_{2I,lJ} = -\frac{e_0}{2m_s} \langle 2I|lJ \rangle_{s} N^0_{l,2} \left[ q_{21} R_l^2 + q_{22} R_2^2 + q_{20} \right] \exp \left[ -e_1 R_l^2 - e_2 R_2^2 \right]
\]

for \( l = 1 \) or \( 3 \) (B.24)

\[
\mathcal{K}_{kI,lJ} = -\frac{1}{2m_s} \langle kI|lJ \rangle_{s} N^0_{k,l} \left( s+1 \right)^6 \left[ \frac{1}{\pi d'^2(1+4kd'^2)} \right]^{3/2} \times
\]

\[
\begin{align*}
&\left( \frac{s+1}{2} \right)^4 \left[ \frac{s-1}{s+1} \right]^2 R_k^2 \left( \frac{8(s-1)^2}{(s+1)^3} \left[ \frac{4}{(s-1)^2} + \frac{1}{s} \right]^2 \left[ \frac{1+4kd'^2}{2d'^2} \right]^2 \right. \\
&+ 32s \left( s+1 \right)^4 \frac{1+8\bar{k}d'^2}{8d'^2} \left. \right]
\end{align*}
\]

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\[
-2 \left( \frac{s-1}{s+1} \right) R_k \cdot R_l \left( \frac{8(s-1)^2}{(s+1)^2} \frac{1}{s} \left[ 1 + 4\bar{k}d^2 \right] \right)^2 \\
+ \frac{32s}{(s+1)^2} \frac{(s+1)^3}{s^2} \left[ 1 + 8\bar{k}d^2 \right] \left[ \frac{1}{8d^2} \right] \left( \frac{(s+1)^2}{s^2} \right) \left[ (s+1)^2 + (1+s^2)8\bar{k}d^2 \right] \\
+ R_l^2 \left( \frac{8(s-1)^2}{(s+1)^3} \frac{1}{s^2} \left[ 1 + 4\bar{k}d^2 \right] \right)^2 + \frac{32s}{(s+1)^3} \left[ 1 \right] \left[ \frac{(s+1)^2}{8d^2} \right] \left( \frac{(s+1)^2}{s^4} \right) \left[ (s+1)^2 + (1+s^2)8\bar{k}d^2 \right] \\
-6 \left( \frac{s+1}{4} \right) \left[ 8(s-1)^2 \right] \left[ \frac{1}{(s+1)^3} \right] \left[ \frac{4}{(s-1)^2} \right] \left[ 1 + 4\bar{k}d^2 \right] \\
+ \frac{32s}{(s+1)^3} \left[ \frac{1}{s^2} \right] \left[ (s+1)^2 + (1+s^2)8\bar{k}d^2 \right] - 3 \left( \frac{s+1}{2s} \right) \left[ \frac{1}{2d^2} \right] \left[ \frac{1}{2d^2} \right] \right) \\
\exp \left\{- \left( \frac{s+1}{2} \right)^2 \left[ \frac{1}{4d^2} \right] \left[ \frac{s^2 + 1}{s^2} \left( R_k^2 + R_l^2 \right) + 2 \frac{s^2}{s^2} - 1 \frac{R_k \cdot R_l}{\bar{d}^2} \right] \right\}
\]
for \( k, l = 1 \) or \( 3 \), but with \( k \neq l \). Here (B.25)

\[
q_{11} = \left( \frac{s+1}{2} \right)^4 \left\{ 8(s-1)^2 \left[ \frac{1}{(s+1)^2} \left( \frac{8\bar{k}}{s} + \frac{1}{(s-1)^2} \right) \right] \right.
- \left( \frac{\lambda}{\kappa} - \frac{s-1}{s+1} \right) \left( \frac{2\bar{k}}{s} + \frac{1}{2d^2\sqrt{s}} \right) \right)^2 \\
+ \frac{32s}{(s+1)^3} \left[ \left( \frac{s-1}{s+1} \right) \left( \frac{2\bar{k}}{s} + \frac{1}{2d^2\sqrt{s}} \right) \right] \\
- \left( \frac{\lambda}{\kappa} - \frac{s-1}{s+1} \right) \left( \frac{\bar{k}s^2 + 1}{s^2} + \frac{s^{-3/2} + 1}{4d^2} \right) \right\} \quad (B.26)
\]

\[
q_{12} = 4 \left( \frac{s+1}{2s} \right) \left[ 2\bar{k} + \frac{1}{2d^2} \sqrt{\frac{2s}{s+1}} \right] \quad (B.27)
\]

\[
q_{10} = -\frac{3}{2} \left( \frac{s+1}{2} \right)^2 \left[ 8(s-1)^2 \left( \frac{8\bar{k}}{s} + \frac{1}{(s-1)^2} \right) \right] \\
+ \frac{32s}{(s+1)^3} \left( \frac{\bar{k}s^2 + 1}{s^2} + \frac{s^{-3/2} + 1}{4d^2} \right) \\
+ \frac{3d^2}{2\kappa} \left( \frac{s+1}{s+1} \right)^2 \left( \frac{2\bar{k}}{s} + \frac{1}{2d^2\sqrt{s}} \right)^2 
\]
\begin{align}
\mathcal{V}_{k,l,J}^{cf} &= -\frac{8}{3} C N_{k,l,J} + s \langle kI|lJ \rangle \lambda \mathcal{V}_{k,l}^{cf}, \quad \text{with} \quad \text{(B.32)}
\end{align}

\begin{align}
\mathcal{V}_{1,2}^{cf} &= \mathcal{V}_{2,1}^{cf} = e_0 [b_1 R_1^2 + b_0] \exp \left[ -e_1 R_1^2 - e_2 R_2^2 \right] \quad \text{(B.33)}
\end{align}

\begin{align}
\mathcal{V}_{2,3}^{cf} &= \mathcal{V}_{3,2}^{cf} = \frac{2}{3 \sqrt{3}} C e_0 \frac{(s+1)^2}{2 s^2} \left\{ \left( s^2 + s + 1 \right) \frac{\lambda^2}{\kappa^2} - 2(s^2 - 1) \frac{\lambda}{\kappa} + s^2 - s + 1 \right\} R_3^2 \\
&\quad + \frac{(s^2 + s + 1) 24 d^2}{(s + 1)^2 \kappa} \right\} \exp \left[ -e_1 R_3^2 - e_2 R_2^2 \right] \quad \text{(B.34)}
\end{align}

\begin{align}
\mathcal{V}_{1,3}^{cf} &= \mathcal{V}_{3,1}^{cf} = -\frac{2}{3 \sqrt{3}} C \frac{(s+1)^6}{64 s^3} \left[ \frac{1}{\pi d^2 (1 + 4 k d^2)} \right] \left[ \frac{1}{\pi d^2 (1 + 4 k d^2)} \right]^{3/2} \times \\
&\quad \left\{ \frac{1}{2 s^2} \left( \frac{s+1}{2} \right)^2 \right\} \left[ (s+1) R_3 + (s-1) R_1 \right] \left[ \frac{9 d^2}{1 + 4 k d^2} \right] \times \\
&\quad \left\{ \frac{1}{2 s^2} \left( \frac{s+1}{2} \right)^2 \right\} \left[ \frac{1 + 8 k d^2}{4 d^2} \right] \left[ \frac{s^2 + 1}{s^2} \left( R_1^2 + R_3^2 \right) + 2 \frac{s^2 - 1}{s^2} \cdot R_1 \cdot R_3 \right] \right\}.
\end{align}

(B.35)
Here the new definitions used are

\[
b_1 = -\frac{4}{9} C \frac{(s + 1)^4}{4s^2} \left[ \frac{\lambda}{\kappa} - \frac{s - 1}{s + 1} \right]^2 \tag{B.36}
\]

\[
b_0 = -\frac{8}{3} C \left( \frac{s + 1}{s} \right)^2 \frac{d^2}{\kappa}. \tag{B.37}
\]

\[
\mathcal{V}^{hyp}_{1S,2S} = \frac{\mathcal{V}^{hyp}_{2S,1S}}{6 \sqrt{3} m^2 s (2\pi d^2)^{3/2}} \left[ L_{11} + L_{12} + L_{13} \right] \times \exp \left[ -(e_1 + e_1') R_1^2 - e_2 R_2^2 \right] \tag{B.38}
\]

\[
\mathcal{V}^{hyp}_{1S,2T} = \frac{\mathcal{V}^{hyp}_{2T,1S}}{6 \sqrt{3} m^2 s (2\pi d^2)^{3/2}} \left[ -3 L_{11} + L_{12} + L_{13} \right] \times \exp \left[ -(e_1 + e_1') R_1^2 - e_2 R_2^2 \right] \tag{B.39}
\]

\[
\mathcal{V}^{hyp}_{1T,2S} = \frac{\mathcal{V}^{hyp}_{2S,1T}}{6 \sqrt{3} m^2 s (2\pi d^2)^{3/2}} \left[ L_{11} + L_{12} + L_{13} \right] \times \exp \left[ -(e_1 + e_1') R_1^2 - e_2 R_2^2 \right] \tag{B.40}
\]

\[
\mathcal{V}^{hyp}_{1T,2T} = \frac{\mathcal{V}^{hyp}_{2T,1T}}{6 \sqrt{3} m^2 s (2\pi d^2)^{3/2}} \left[ L_{11} + L_{12} + 5 L_{13} \right] \times \exp \left[ -(e_1 + e_1') R_1^2 - e_2 R_2^2 \right] \tag{B.41}
\]

\[
\mathcal{V}^{hyp}_{3S,2S} = \frac{\mathcal{V}^{hyp}_{2S,3S}}{4 \sqrt{3} m^2 s (2\pi d^2)^{3/2}} \left[ L_{21} + 2 L_{22} + L_{23} \right] \times \exp \left[ -(e_1 + e_1') R_3^2 - e_2 R_2^2 \right] \tag{B.42}
\]

\[
\mathcal{V}^{hyp}_{3S,2T} = \frac{\mathcal{V}^{hyp}_{2T,3S}}{12 \sqrt{3} m^2 s (2\pi d^2)^{3/2}} \left[ -L_{21} - 2 L_{22} + 3 L_{23} \right] \times \exp \left[ -(e_1 + e_1') R_3^2 - e_2 R_2^2 \right] \tag{B.43}
\]

\[
\mathcal{V}^{hyp}_{3T,2S} = \frac{\mathcal{V}^{hyp}_{2S,3T}}{12 \sqrt{3} m^2 s (2\pi d^2)^{3/2}} \left[ L_{21} - 6 L_{22} + L_{23} \right] \times \exp \left[ -(e_1 + e_1') R_3^2 - e_2 R_2^2 \right] \tag{B.44}
\]
\[ \mathcal{V}_{3T,2T}^{hyp} = \frac{e_0}{\sqrt{3}} \frac{8\pi}{3m^2s} \frac{(2\kappa)^{3/2}}{(2\pi d^2)^{3/2}} [5L_{21} + 2L_{22} + L_{23}] \times \exp\left[-(e_1 + e'_1)R_3^2 - e_2R_2^2\right] \] (B.45)

\[ \mathcal{V}_{1S,3S}^{hyp} = \mathcal{V}_{3S,1S}^{hyp} = -\frac{1}{\sqrt{3}} \frac{8\pi}{3m^2s} \frac{1}{(2\pi d^2)^{3/2}} \frac{(s + 1)^6}{64s^3} \left[2L_{31} + L_{32} + L_{33}\right] \times \exp\left\{ -\left(\frac{s + 1}{2}\right)^2 \left(\frac{1 + 8k d^2}{4d^2}\right) \frac{s^2 + 1}{s^2} (R_1^2 + R_3^2) + 2\frac{s^2 - 1}{s^2} R_1 \cdot R_3 \right\} \] (B.46)

\[ \mathcal{V}_{1S,3T}^{hyp} = \mathcal{V}_{3S,1T}^{hyp} = -\frac{1}{\sqrt{3}} \frac{8\pi}{3m^2s} \frac{1}{(2\pi d^2)^{3/2}} \frac{(s + 1)^6}{64s^3} \left[2L_{31} - L_{32} - L_{33}\right] \times \exp\left\{ -\left(\frac{s + 1}{2}\right)^2 \left(\frac{1 + 8k d^2}{4d^2}\right) \frac{s^2 + 1}{s^2} (R_1^2 + R_3^2) + 2\frac{s^2 - 1}{s^2} R_1 \cdot R_3 \right\} \] (B.47)

\[ \mathcal{V}_{1S,3S}^{hyp} = \mathcal{V}_{3S,1T}^{hyp} = -\frac{1}{\sqrt{3}} \frac{8\pi}{3m^2s} \frac{1}{(2\pi d^2)^{3/2}} \frac{(s + 1)^6}{64s^3} \left[-2L_{31} - L_{32} + 3L_{33}\right] \times \exp\left\{ -\left(\frac{s + 1}{2}\right)^2 \left(\frac{1 + 8k d^2}{4d^2}\right) \frac{s^2 + 1}{s^2} (R_1^2 + R_3^2) + 2\frac{s^2 - 1}{s^2} R_1 \cdot R_3 \right\} \] (B.48)

\[ \mathcal{V}_{1S,3T}^{hyp} = \mathcal{V}_{3S,1T}^{hyp} = -\frac{1}{\sqrt{3}} \frac{8\pi}{3m^2s} \frac{1}{(2\pi d^2)^{3/2}} \frac{(s + 1)^6}{64s^3} \left[2L_{31} + 5L_{32} + L_{33}\right] \times \exp\left\{ -\left(\frac{s + 1}{2}\right)^2 \left(\frac{1 + 8k d^2}{4d^2}\right) \frac{s^2 + 1}{s^2} (R_1^2 + R_3^2) + 2\frac{s^2 - 1}{s^2} R_1 \cdot R_3 \right\} \] (B.49)

Here

\[ L_{11} = l_{10} \exp\left[l_{11}R_1^2 - l_{12}R_2^2\right] \left(\alpha_{s}^{13} \exp[l_{13}R_1 \cdot R_2] + \alpha_{s}^{24} \exp[-l_{13}R_1 \cdot R_2]\right) \] (B.50)

\[ L_{12} = l_{20} \exp\left[-l_{21}R_1^2\right] \left(\alpha_{s}^{14} \exp[l_{22}R_1^2] + s\alpha_{s}^{23} \exp[-l_{22}R_1^2]\right) \] (B.51)

\[ L_{13} = l_{30} \exp\left[l_{31}R_1^2 - l_{32}R_2^2\right] \left(\alpha_{s}^{12} \exp[-l_{33}R_1 \cdot R_2] + \alpha_{s}^{34} \exp[l_{33}R_1 \cdot R_2]\right) \] (B.52)

\[ L_{21} = l_{30} \exp\left[l_{31}R_3^2 - l_{32}R_2^2\right] \left(\alpha_{s}^{13} \exp[-l_{33}R_3 \cdot R_2] + \alpha_{s}^{24} \exp[l_{33}R_3 \cdot R_2]\right) \] (B.53)

\[ L_{22} = l_{20} \exp\left[l_{21}R_3^2\right] \left(\alpha_{s}^{14} \exp[l_{22}R_3^2] + s\alpha_{s}^{23} \exp[-l_{22}R_3^2]\right) \] (B.54)

\[ L_{23} = l_{10} \exp\left[l_{11}R_3^2 - l_{12}R_2^2\right] \left(\alpha_{s}^{12} \exp[l_{13}R_3 \cdot R_2] + \alpha_{s}^{34} \exp[-l_{13}R_3 \cdot R_2]\right) \] (B.55)
\[ L_{31} = \left[ \alpha_s^{13} + \alpha_s^{23} \right] \exp \left\{ -\frac{(s + 1)^4 1 + 4 \bar{k} d^2}{16 s^2} \frac{d^2}{d^2} \left[ \left( \frac{s - 1}{s + 1} \right) R_1 + R_3 \right]^2 \right\} \quad (B.56) \]

\[ L_{32} = \left[ \frac{\pi d^2}{1 + 4 \bar{k} d^2} \right]^{3/2} \left[ \frac{2}{s + 1} \right] \left\{ s \alpha_s^{14} \delta(R_1 + R_3) + \alpha_s^{23} \delta(R_1 - R_3) \right\} \quad (B.57) \]

\[ L_{33} = \left[ \alpha_s^{12} + \alpha_s^{33} \right] \exp \left\{ -\frac{(s + 1)^4 1 + 4 \bar{k} d^2}{16 s^2} \frac{d^2}{d^2} \left[ R_1 + \left( \frac{s - 1}{s + 1} \right) R_3 \right]^2 \right\}, \quad \text{with} \quad (B.58) \]

\[ e_1' = \frac{1}{4d^2} \left( \frac{s + 1}{2} \right)^2 \frac{\lambda^2}{\kappa} \quad (B.59) \]

\[ l_{10} = \left( \frac{s}{s + 1} \right)^3 \quad (B.60) \]

\[ l_{11} = \frac{s^2 - 1}{16d^2} \left[ 2\lambda - \kappa \left( \frac{s - 1}{s + 1} \right) \right] \quad (B.61) \]

\[ l_{12} = \frac{\kappa}{d^2} \left( \frac{s}{s + 1} \right)^2 \quad (B.62) \]

\[ l_{13} = \frac{s}{2d^2} \left[ \lambda - \kappa \left( \frac{s - 1}{s + 1} \right) \right] \quad (B.63) \]

\[ l_{20} = \frac{s}{8} \quad (B.64) \]

\[ l_{21} = \frac{\kappa}{4d^2} \left( \frac{s + 1}{2} \right)^2 \quad (B.65) \]

\[ l_{22} = \frac{\lambda}{2d^2} \left( \frac{s + 1}{2} \right)^2 \quad (B.66) \]

\[ l_{30} = \left( \frac{s}{s - 1} \right)^3 \quad (B.67) \]

\[ l_{31} = \frac{1}{4d^2} \left( \frac{s + 1}{2} \right)^2 \left( \frac{s + 1}{s - 1} \right) \left[ -2\lambda + \kappa \left( \frac{s + 1}{s - 1} \right) \right] \quad (B.68) \]

\[ l_{32} = \frac{\kappa}{d^2} \left( \frac{s}{s - 1} \right)^2 \quad (B.69) \]

\[ l_{33} = \frac{s}{2d^2} \left( \frac{s + 1}{s - 1} \right) \left[ -\lambda + \kappa \left( \frac{s + 1}{s - 1} \right) \right]. \quad (B.70) \]

As mentioned in the text, in the case of the diagonal term corresponding to the 2S channel, the expressions depend upon the physical content of this
channel. Thus we have:

\[ \mathcal{N}_{2S,2S} = \delta(R_2 - R'_2), \]  
(B.71)

( irrespective of the physical pseudoscalar mesons present).

\[ \mathcal{K}_{2S,2S}(\eta\eta) = \delta(R_2 - R'_2) \times \]

\[ \left[ \frac{3}{4} (\omega_{21}^l + \omega_{22}^l) \cos^2 \theta + (\omega_{21}^s + \omega_{22}^s) \sin^2 \theta \right] 
- \frac{1}{2m} (f_2^{ll} \cos^2 \theta + 2f_2^{ls} \cos^2 \theta \sin^2 \theta + f_2^{ss} \sin^4 \theta) \nabla^2 R_2 \right], \]
(B.72)

with

\[ \omega_{21}^l = \frac{g_2^l}{2md_{21}^2} = \frac{g_2^l}{2md^2}, \quad \omega_{21}^s = \frac{g_2^s}{2md_{22}^2} = \frac{g_2^s}{2md_{22}^2}, \]  
(B.73)

\[ \omega_{22}^l = \frac{h_2^l}{2md_{21}^2} = \frac{h_2^l}{2md^2} \quad \text{and} \quad \omega_{22}^s = \frac{h_2^s}{2md_{22}^2} = \frac{h_2^s}{2md_{22}^2}. \]  
(B.74)

(Here \( f_2^{ll} = 1, \quad f_2^{ls} = \frac{1}{2}(1 + 1/s), \quad f_2^{ss} = 1/s \))

\[ \mathcal{K}_{2S,2S}(\pi\eta) = \delta(R_2 - R'_2) \times \]

\[ \left[ \frac{3}{4} (\omega_{21}^l + \omega_{22}^l \cos^2 \theta + \omega_{22}^s \sin^2 \theta) 
- \frac{1}{2m} (f_2^{ll} \cos^2 \theta + f_2^{ls} \sin^2 \theta) \nabla^2 R_2 \right]. \]  
(B.76)

\[ \mathcal{K}_{2S,2S}(\pi\eta') = \delta(R_2 - R'_2) \times \]
\[
\left[ \frac{3}{4} \omega_{21} \cos^{2} \theta + \omega_{22}^{s} \cos^{2} \theta \right] - \frac{1}{2m} \left( f_{ \frac{1}{2} }^{l} \sin^{2} \theta + f_{ \frac{1}{2} }^{l} \cos^{2} \theta \right) \nabla_{R_{2}}^{2} \right]. 
\] (B.77)

\[
V_{2S,2S}^{ef}(\eta\eta) = \delta(R_{2} - R_{2}') \left[ \frac{8}{3} \hat{C} - 2 \times 4C[d_{21}^{2} \cos^{2} \theta + d_{22}^{2} \sin^{2} \theta] \right] \] (B.78)

\[
V_{2S,2S}^{ef}(\pi\eta) = \delta(R_{2} - R_{2}') \left[ \frac{8}{3} \hat{C} - 4C[d_{21}^{2} + d_{22}^{2} \cos^{2} \theta + d_{22}^{2} \sin^{2} \theta] \right] \] (B.79)

\[
V_{2S,2S}^{ef}(\pi\eta') = \delta(R_{2} - R_{2}') \left[ \frac{8}{3} \hat{C} - 4C[d_{21}^{2} + d_{22}^{2} \sin^{2} \theta + d_{22}^{2} \cos^{2} \theta] \right]. \] (B.80)

\[
V_{2S,2S}^{hyp}(\eta\eta) = -\frac{8\pi}{3m^{2}S} \delta(R_{2} - R_{2}') \left[ \frac{s(\alpha_{s}^{14(l)} + \alpha_{s}^{23(l)} \cos^{2} \theta)}{2(\pi d_{21}^{2})^{3/2}} + \frac{(\alpha_{s}^{14(s)} + \alpha_{s}^{23(s)} \sin^{2} \theta)}{2(\pi d_{22}^{2})^{3/2}} \right] \] (B.81)

\[
V_{2S,2S}^{hyp}(\pi\eta) = -\frac{8\pi}{3m^{2}S} \delta(R_{2} - R_{2}') \left[ \frac{sa_{s}^{14(l)} \cos^{2} \theta}{(2\pi d_{21}^{2})^{3/2}} + \frac{sa_{s}^{23(s)} \sin^{2} \theta}{(2\pi d_{22}^{2})^{3/2}} + \frac{\alpha_{s}^{23(s)} \cos^{2} \theta}{s(2\pi d_{22}^{2})^{3/2}} \right]. \] (B.82)

\[
V_{2S,2S}^{hyp}(\pi\eta') = -\frac{8\pi}{3m^{2}S} \delta(R_{2} - R_{2}') \left[ \frac{sa_{s}^{14(l)} \cos^{2} \theta}{(2\pi d_{21}^{2})^{3/2}} + \frac{sa_{s}^{23(s)} \sin^{2} \theta}{(2\pi d_{22}^{2})^{3/2}} + \frac{\alpha_{s}^{23(s)} \cos^{2} \theta}{s(2\pi d_{22}^{2})^{3/2}} \right]. \] (B.83)

\[
V_{2S,2S}^{a}(\eta\eta) = 2 \times \delta(R_{2} - R_{2}') \left[ 2l \cos^{2} \theta - 2\sqrt{2ln} \cos \theta \sin \theta + n \sin^{2} \theta \right] \] (B.84)

\[
V_{2S,2S}^{a}(\pi\eta) = \delta(R_{2} - R_{2}') \left[ 2l \cos^{2} \theta - 2\sqrt{2ln} \cos \theta \sin \theta + n \sin^{2} \theta \right] \] (B.85)

\[
V_{2S,2S}^{a}(\pi\eta') = \delta(R_{2} - R_{2}') \left[ 2l \sin^{2} \theta + 2\sqrt{2ln} \cos \theta \sin \theta + n \cos^{2} \theta \right]. \] (B.86)

**Appendix C: The definitions used in the momentum space solutions**

The following definitions are used in writing the momentum space solutions (2.39) and (2.40) of the coupled equations.

\[
Q_{1}^{(1)} = \left[ -\frac{1}{2m} \frac{q_{11}}{6} + \frac{b_{1}}{2} \right] F_{b}(p_{1}, e_{1}) + \left[ -\frac{1}{2m} \frac{q_{10}}{6} + \frac{b_{0}}{2} - \frac{E_{c}'}{6} \right] F_{a}(p_{1}, e_{1}) \]

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\[ -\alpha_s^H l_{20} F_a(p_1, e_1 + e'_1 + l_{21} - l_{22}) - \alpha_s^{ss} l_{20} F_a(p_1, e_1 + e'_1 + l_{21} + l_{22}) \]

\[ Q_2^{(1)} = -\frac{1}{2m} \frac{q_{12}}{6} F_a(p_1, e_1) \text{ nonumber} \]  
\[ Q_3^{(1)} = -2 \alpha_s^{ls} l_{10} F_a(p_1, e_1 + e'_1 - l_{11}) \]

\[ Q_4^{(1)} = -\frac{1}{3} \alpha_s^{ls} l_{10} F_b(p_1, e_1 + e'_1 - l_{11}) \]

\[ Q_5^{(1)} = -\alpha_s^{ls} l_{10} F_b(p_1, e_1 + e'_1 + l_{31} + e_2 + l_{32} - l_{33}) \]

\[ Q_6^{(1)} = -\alpha_s^{ls} l_{30} F_a(p_1, \tau_1 e_1 + e'_1 + l_{31} + e_2 + l_{32} + l_{33}) \]

\[ Q_1^{(2)} = -\frac{1}{2m} \frac{q_{22}}{6} F_b(p_2, e_2) + \left[ -\frac{1}{2m} \frac{q_{20}}{6} + \frac{b_0}{2} - \frac{E'_c}{6} \right] F_a(p_2, e_2) \]

\[ Q_2^{(2)} = \left[ -\frac{1}{2m} \frac{q_{21}}{6} + \frac{b_1}{2} \right] F_a(p_2, e_2) \]

\[ Q_3^{(2)} = -\alpha_s^H l_{20} F_a(p_2, e_2) \]

\[ Q_4^{(2)} = -\alpha_s^{ss} l_{20} F_a(p_2, e_2) \]

\[ Q_5^{(2)} = -2 \alpha_s^{ls} l_{10} F_a(p_2, e_2 + l_{12}) \]

\[ Q_6^{(2)} = -\frac{1}{3} \alpha_s^{ls} l_{10} F_b(p_2, e_2 + l_{12}) \]

\[ Q_7^{(2)} = -\alpha_s^{ls} l_{30} F_a(p_2, \tau_2 e_1 + e'_1 + l_{31} + e_2 + l_{32} - l_{33}) \]

\[ Q_8^{(2)} = -\alpha_s^{ls} l_{30} F_a(p_2, \tau_2 e_1 + e'_1 + l_{31} + e_2 + l_{32} + l_{33}) \]

\[ E'_c = E_c + \frac{8}{3} C - 2m(s + 1) \]

\[ A_k(x) = e_0 \int d^3 R_k \exp \left[ -x R_k^2 \right] \chi_{ks}(R_k) \]

\[ B_k(x) = e_0 \int d^3 R_k \exp \left[ -x R_k^2 \right] R_k^2 \chi_{ks}(R_k) \]
\[
\chi_{kS}(p_k) = \int \frac{d^3R_k}{(2\pi)^{3/2}} \exp [ip_k \cdot R_k] \chi_{kS}(R_k), \quad (C.7)
\]

for \( k = 1, 2 \). \( F_a(p_k, x) \) and \( F_b(p_k, x) \) are similar FTs of \( \exp [-xR_k^2] \) and \( R_k^2 \exp [-xR_k^2] \) respectively. Moreover

\[
\Delta_1(p_1) = \frac{p_1^2}{2\mu_{KK}} + M_K + M_{\bar{K}} - E_c - i\varepsilon \quad (C.8)
\]

\[
\Delta_2(p_2) = \frac{p_2^2}{2\mu_{ab}} + M_a + M_b - E_c - i\varepsilon \quad (C.9)
\]

\[
p_c(1) = \sqrt{2\mu_{KK}(E_c - M_K - M_{\bar{K}})} \quad (C.10)
\]

\[
p_c(2) = \sqrt{2\mu_{ab}(E_c - M_a - M_b)}. \quad (C.11)
\]

**Appendix D: The elements of the \( T \) matrix**

Consistent with our definition of the \( T \) matrix (see eqs. (2.44) and (2.45)), the four elements of the \( 2 \times 2 \) \( T \) matrix are (these can be read off from eqs. (2.39) and (2.40)):

\[
T_{1,1} = 2\mu_{KK} \frac{\pi}{2} p_c(1) \left[ Q_1^{(1)} A_2(e_2) + Q_2^{(1)} B_2(e_2) + Q_3^{(1)} A_2(e_2 + l_{12}) + Q_4^{(1)} B_2(e_2 + l_{12}) \\
+ Q_5^{(1)} A_2(\tau_1 e_1 + e_1' + l_{31} + e_2 + l_{32} - l_{33}) \\
+ Q_6^{(1)} A_2(\tau_1 e_1 + e_1' + l_{31} + e_2 + l_{32} + l_{33}) \right] \quad (D.1)
\]

\[
T_{2,1} = 2\mu_{ab} \frac{\pi}{2} p_c(1) \sqrt{\frac{v_2}{v_1}} \left[ Q_1^{(2)} A_1(e_1) + Q_2^{(2)} B_1(e_1) \\
+ Q_3^{(2)} A_1(e_1 + e_1' + l_{21} - l_{22}) + Q_4^{(2)} A_1(e_1 + e_1' + l_{21} + l_{22}) \\
+ Q_5^{(2)} A_1(e_1 + e_1' - l_{11}) + Q_6^{(2)} B_1(e_1 + e_1' - l_{11}) \\
+ Q_7^{(2)} A_1(\tau_2 e_1 + e_1' + l_{31} + e_2 + l_{32} - l_{33}) \right]
\]
\[ T_{2,2} = 2\mu_{ab} \frac{\pi}{2} p_e(2) \left[ Q_1(2) A_1(e_1) + Q_2(2) B_1(e_1) \right. \]
\[ + Q_3(2) A_1(e_1 + e'_1 + l_{21} - l_{22}) + Q_4(2) A_1(e_1 + e'_1 + l_{21} + l_{22}) \]
\[ + Q_5(2) A_1(e_1 + e'_1 + l_{11}) + Q_6(2) B_1(e_1 + e'_1 - l_{11}) \]
\[ + Q_7(2) A_1(e_1 + e'_1 + l_{31} + e_2 + l_{32} - l_{33}) \]
\[ + Q_8(2) A_1(e_1 + e'_1 + l_{31} + e_2 + l_{32} + l_{33}) \] \hspace{1cm} (D.2)

\[ T_{1,2} = 2\mu_{KK} \frac{\pi}{2} p_e(1) \sqrt{\frac{v_1}{v_2}} \left[ Q_1(1) A_2(e_2) + Q_2(1) B_2(e_2) + Q_3(1) A_2(e_2 + l_{12}) + Q_4(1) B_2(e_2 + l_{12}) \right. \]
\[ + Q_5(1) A_2(e_1 + e'_1 + l_{31} + e_2 + l_{32} - l_{33}) \]
\[ + Q_6(1) A_2(e_1 + e'_1 + l_{31} + e_2 + l_{32} + l_{33}) \] \hspace{1cm} (D.3)

with \( p_1 \) and \( p_2 \) in \( Q_1^{(1)} \cdots Q_8^{(2)} \) (see eqs. (C.2) and (C.3)) replaced by \( p_e(1) \) and \( p_e(2) \) respectively. It was checked numerically that \( T_{1,2} = T_{2,1} \), fulfilling the requirement of “reciprocity” in an inelastic scattering (see p.528 of [24]).

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