ON CERTAIN HYPERPLANE ARRANGEMENTS AND COLORED GRAPHS

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Abstract. We exhibit a one-to-one correspondence between 3-colored graphs and subarrangements of certain hyperplane arrangements denoted \( J_n \), \( n \in \mathbb{N} \). We define the notion of centrality of 3-colored graphs, which corresponds to the centrality of hyperplane arrangements. Via the correspondence, the characteristic polynomial \( \chi_{J_n} \) of \( J_n \) can be expressed in terms of the number of central 3-colored graphs, and we compute \( \chi_{J_n} \) for \( n = 2, 3 \).

1. Introduction

A hyperplane arrangement is a finite set of affine hyperplanes in a real affine space. In this article, we shall consider a hyperplane arrangement problem of a specific type: Given a positive integer \( n \), let \([n]\) denote \( \{1, 2, 3, \ldots, n\} \). For each \( 1 \leq \alpha < \beta \leq n \), we define \( H_{\alpha\beta} := \{x \in \mathbb{R}^n \mid x_\alpha + x_\beta = 1\} = H_{\beta\alpha} \) which are said to be walls or hyperplanes of type I. For each \( i \in [n] \), define \( 0_i := \{x \in \mathbb{R}^n \mid x_i = 0\} \), and \( 1_i := \{x \in \mathbb{R}^n \mid x_i = 1\} \) which are said to be walls of type II. Let \( J_n \) denote the hyperplane arrangement consisting of all hyperplanes of type I or type II. We are interested in the number of the regions, i.e., the connected components of \( \mathbb{R}^n \setminus \bigcup_{H \in J_n} H \).

Our particular hyperplane arrangement has its origin in algebraic geometry, namely the number of certain moduli spaces [1, Problem 5.2]. But as we were researching for a solution, we discovered a neat relation with graph theory and soon this interplay has taken the center stage. The problem was then modified to better suit the graph theoretic approach. The arrangement \( J_n \) should remind experts of (deformations of) the braid arrangement, especially the well known Shi arrangement [3] which consists of walls of the forms \( x_i - x_j = 0 \) and \( x_i - x_j = 1 \). We hope to explore the relation to Shi arrangements in the future.

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We refer to Lecture 1 of Stanley’s chapter [5] on hyperplane arrangements for many fundamental results, but recall a few key notions here. Let \( B = \{ H_i \mid i \in I \} \) be a hyperplane arrangement where

\[
H_i = \left\{ x \in \mathbb{R}^n \left| \sum_{j=1}^{n} a_{ij} x_j = b_i \right. \right\}
\]

and \( I \) is an index set. The arrangement \( B \) is said to be central if the intersection of all hyperplanes in \( B \) is nonempty. The rank of a hyperplane arrangement is the dimension of the space spanned by the normal vectors to the hyperplanes in the arrangement.

**Definition 1.** Let \( A \) and \( b \) denote the matrices \((a_{ij})\) and \( b = (b_i), \ i = 1, \ldots, |B| \), respectively. The augmented matrix \( B = [A|b] \) will be called the matrix associated with the hyperplane arrangement \( B \) or simply the associated matrix of \( B \).

Since the row vectors of \( A \) are the normal vectors to the hyperplanes, rank of \( B \) equals rank (\( A \)). In central hyperplane arrangements, this also equals rank (\( B \)), since the associated matrix is consistent. Note that \( [A|b] \) completely determines \( B \), and giving a hyperplane arrangement is equivalent to giving its associated matrix.

Let \( A \) be a hyperplane arrangement. Then the characteristic polynomial of \( A \) is defined

\[
\chi_A(t) = \sum_{B} (-1)^{|B|} t^{n - \text{rank}(B)},
\]

where \( B \) runs through all central sub-arrangements of \( A \): In fact, the characteristic polynomial is defined using the Möbius function [5, Definition 1.3] and the equivalence is a theorem due to H. Whitney [2, Lemma 2.3], but this form suits our purpose just fine. The following is perhaps the most fundamental theorem when it comes to counting the number of regions.

**Theorem** ([6]). Let \( A \) be a hyperplane arrangement in an \( n \)-dimensional real vector space. Let \( r(A) \) be the number of chambers and \( b(A) \) be the number of relatively bounded chambers. Then we have

1. \( b(A) = (-1)^n \chi(+1) \).
2. \( r(A) = (-1)^n \chi(-1) \).

In our case, we have \( \binom{n}{2} \) walls of type I and \( 2n \) walls of type II. All together there are \( N = \binom{n}{2} + 2n \) walls and a simple case by case analysis would require centrality examination and rank computation of \( 2^N \) subarrangements. The main theme of this paper is that, by a systematic use of symmetry and geometry, we can reduce that number significantly. Enumeration is further enhanced by using the notion of associated graphs (Definition 4) and associated matrices.
The graph theory approach is for the specific hyperplane arrangement problem studied in this paper. We associate a graph to each hyperplane subarrangement of $J_n$, and we translate the centrality of hyperplane arrangements in terms of graph properties (Definition 2 and Theorem 1). This makes enumeration of central subarrangements much more systematic and efficient: We work out the basic examples of two and three dimensional cases in Section 3 but without borrowing any significant results from graph theory. We employ more substantial graph theory results to attack the higher dimensional cases in the forthcoming manuscript [4]. Also, we believe that the method can be generalized to treat other hyperplane arrangements by considering the signed graphs [7], and this will be taken up in a future work.

2. Associated colored graphs

Let $(V, E)$ be a graph with vertices $V = [n]$ and the set of edges $E$. Let $v$ and $v'$ be vertices (which may be equal). A path is said to be even (resp., odd) if its length is even (resp., odd).

For the purpose of this paper, we shall consider $\{0, 1, \ast\}$-colored graphs on $V = [n]$ (ind. indicating no numeric value assigned). We shall let $\gamma: V \to \{0, 1, \ast\}$ denote the color function. A vertex $v$ with $\gamma(v) = \ast$ will be called not colored.

By a 3-colored graph, we shall always mean a $\{0, 1, \ast\}$-colored graph.

Definition 2. A 3-colored graph $([n], E)$ is said to be central if

1. if $v$ is colored, then it is not on a closed walk of odd length, and
2. $\gamma(v) = \gamma(v')$ (resp., $\gamma(v) \neq \gamma(v')$) for any pair of colored vertices $v, v'$ such that there is a $v - v'$ path of even (resp., odd) length.

Remark 1. (1) Note that the centrality condition determines the parity of the paths between any two given colored vertices: In a central graph, if a $v - v'$ path is even (resp., odd), so are all other $v - v'$ paths.

(2) Condition (1) of Definition 2 is equivalent to that every component $C$ with a colored vertex is bipartite. Suppose $C$ has a closed walk $\gamma$. Since $C$ is connected, there exists a path $\tau$ from a colored vertex $v$ to $\gamma$. Then traversing $\tau, \gamma,$ and $\tau^{-1}$ back to $v$ is an odd cycle which contains $v$, violating Condition (1). Converse is obvious.

Definition 3. For a subarrangement $A \subseteq J_n$, $I(A)$ denotes the set of indices $\tau \in [n]$ that appear in $A$. That is,

$$I(A) = \left( \bigcup_{H_{\alpha\beta} \in A} \{\alpha, \beta\} \right) \cup \left( \bigcup_{0, \alpha \in A} \{\alpha\} \right) \cup \left( \bigcup_{1, \beta \in A} \{\beta\} \right).$$

Definition 4. Let $A$ be a subarrangement of $J_n$ which does not contain both $0_i$ and $1_i$, $\forall i$. The associated graph $\Gamma_A$ of $A$ is a 3-colored graph with the vertex set $V(\Gamma_A) = I(A)$, and edge set $E(\Gamma_A) = \{\{\alpha, \beta\} : H_{\alpha\beta} \in A\}$ where the vertices are assigned exactly one of $\{0, 1, \ast\}$ in the obvious fashion: given
$i \in I(A)$, we assign 0 (resp., 1) to $i$ if $0_i \in A$ (resp., $1_i \in A$). If neither 0$_i$ nor 1$_i$ is in $A$, we assign $\ast$ to $i$, i.e., the vertex $i$ is not colored.

Let $S_n$ be the set of all subarrangements of the hyperplane arrangements $J_n$ satisfying the conditions of the Definition 4, in particular, those arrangements which do not contain both 0$_i$ and 1$_i$, $\forall i$. Then we have the following lemma:

**Lemma 1.** There is a one-to-one correspondence between $S_n$ and the set of all 3-colored graphs on $[n]$.

**Proof.** Given any 3-colored graph $\Gamma = ([n], E)$ with the color function $\gamma : [n] \rightarrow \{0, 1, \ast\}$, we associate the hyperplane arrangement $A_\Gamma := \{\{x_i = \gamma(i)\}_{\gamma(i) \neq \ast}\} \cup \{H_{ij}\}_{(i,j) \in E}$.

This clearly is inverse to the association $A \mapsto \Gamma_A$ defined in Definition 4. □

Note that for the graph $\Gamma$ associated with a central arrangement $A$, there are no isolated non-colored vertices and two adjacent colored vertices are of different colors.

**Theorem 1.** Let $A \in S_n$. Then $A$ is central if and only if $\Gamma_A$ is central.

**Proof.** The corresponding colored graph $\Gamma_A$ can be decomposed into three subgraphs $\Gamma', \Gamma''$, $\Gamma'''$:

1. (graph of the first kind) $\Gamma'$ is the union of colorless connected components;
2. (graph of the second kind) $\Gamma''$ is the union of isolated colored vertices;
3. (graph of the third kind) $\Gamma''' = \Gamma \setminus (\Gamma' \cup \Gamma'')$ is the union of the connected components with at least one colored vertex and at least one edge.

Accordingly, we can decompose $A$ into $A' \cup A'' \cup A'''$ such that the subarrangements $A', A'', A'''$ correspond to $\Gamma'$, $\Gamma''$, $\Gamma'''$, respectively. For example, the hyperplane arrangement (Figure 1)

\[ A = \{H_{12}, H_{23}, 1, 3, 0_5, H_{46}, 1_7, H_{8,9}\} \]

decomposes into

\[ \{H_{46}, H_{89}\} \cup \{0_5, 1_7\} \cup \{H_{12}, H_{23}, 1_3\} \]

whose associated colored graph decomposition is

1. $\Gamma' = (\{4, 6, 8, 9\}, \{\{4, 6\}, \{8, 9\}\})$;
2. $\Gamma'' = (\{5, 7\}, \emptyset), \gamma(5) = 0, \gamma(7) = +1$;
3. $\Gamma''' = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}\}), \gamma(3) = +1$.

Since the index sets $I(A')$, $I(A'')$ and $I(A''')$ are all disjoint from each other, $A$ is central if and only if each of them is separately central. It is also easy to see that $A'$ and its associated graph $\Gamma_A'$ are central: $\Gamma_A'$ is trivially central since the centrality condition is vacuous for a non-colored graph. As for the hyperplane arrangement $A'$, setting all variables $x_i, i \in I(A')$, equal to 1/2 satisfies all hyperplane equations in $A'$. Note that $A''$ is also always central.
except the obvious non-central ones containing both 0, and 1, for some \( i \), which we have discarded in the beginning (definition of \( S_n \)). Hence, without loss of generality, we may assume that \( A = A'' \) so that its associated graph is connected and each vertex is connected to a distinguished colored vertex \( i_0 \) by a path, and that \( \gamma(i_0) = 1 \).

Now, we shall prove the assertion of the theorem for any such hyperplane arrangement. Suppose that the associated graph is central. We define a new coloring \( \gamma^* \) as follows. If \( j \) is colored, set \( \gamma^*(j) = \gamma(j) \). For a non-colored vertex \( j \), choose a path \( \sigma \) from \( j \) to \( i_0 \) and assign \( \gamma^*(j) = \gamma(i_0) = 1 \) if the path is even and \( \gamma^*(j) = 0 \) otherwise. Note that all paths from \( j \) to \( i_0 \) have the same parity since two paths of different parity would give a closed walk of odd length from \( i_0 \) to \( i_0 \), and this violates the centrality of \( \Gamma_A \).

Let \( x^* \) be a point whose coordinates satisfy \( x_j^* = \gamma^*(j) \) for any \( j \in I(A) \). We claim that \( x^* \) satisfies all hyperplane equations. Let \( H_\alpha \beta \in A \). Suppose there is an even path \( \{v_0, v_1, \ldots, v_{2k}\} \) corresponding to \( x^{v_0} + x^{v_1} = 1 \). Then joining the edge \( \{\beta, \alpha\} \) to \( \sigma \) creates an odd path from \( \beta \) to \( i_0 \) which implies that \( x^\beta = 0 \). Thus \( x^\beta + x^* = 1 \). The other case where \( \gamma(\alpha) = 0 \) is proved similarly.

Conversely, suppose that \( A \) is central. If there is an even path \( \{v_0, v_1, \ldots, v_{2k}\} \) and \( \gamma(v_0) = 0 \), then the hyperplanes \( x_{v_k} + x_{v_{k+1}} = 1 \) corresponding to edges successively determine \( x_{v_1} = 1, x_{v_2} = 0, \) etc., so that \( x_{v_{2k}} = 0 = \gamma(v_0) \). Likewise, if there is an odd path \( \{v_0, v_1, \ldots, v_{2k+1}\} \), we have \( \gamma(v_0) \neq \gamma(v_{2k+1}) \). It follows that any two paths between two colored vertices have the same parity.

\[ \square \]

3. Arrangements in the plane and space

In this section, we use our main theorem and analyze the dimension 2 and 3 cases. We present them here while promising deeper results in a forthcoming paper where we use graph theoretic approach to give a formula for computing the characteristic polynomial in terms of the number of bipartite graphs of given rank and size.
Definition 5.  
(1) We let $r_{\epsilon,\nu}^n$ denote the number of central graphs on $[n]$ with precisely $\epsilon$ edges and $\nu$ colored vertices. When there is no danger of confusion, we drop the superscript $n$.

(2) A graph with $\epsilon$ edges and $\nu$ colored vertices will be called an $(\epsilon, \nu)$ graph.

Note that $\epsilon, \nu$ are the numbers of type I and of type II hyperplanes respectively, hence the order $(\epsilon, \nu)$.

3.1. $n = 2$

$\mathcal{J}_2 = \{H_{12}, 0_1, 1_1, 0_2, 1_2\}$. The central subarrangements $\mathcal{B}$ are enumerated as follows, according to the cardinality.

(1) $|\mathcal{B}| = 1$: $\mathcal{B}$ contains either a single colored vertex, or an edge. There are five such cases.

(2) $|\mathcal{B}| = 2$: \{H_{12}, 1_1\}, \{H_{12}, 0_1\}, \{H_{12}, 1_2\}, \{0_1, 0_2\}, \{1_1, 1_2\}. These are of rank 2.

(3) $|\mathcal{B}| = 3$: \{H_{12}, 1_1, 0_2\}, \{H_{12}, 1_2, 0_1\}. These are of rank 2.

(4) $|\mathcal{B}| \geq 4$: There are no central subarrangements with $\geq 4$ hyperplanes.

With the trivial central arrangement with $|\mathcal{B}| = 0$, the characteristic polynomial is

$$\chi_{\mathcal{J}_2}(t) = ((-1)^28 + (-1)^32) \cdot t^{2-2} + (-1) \cdot 5t^{2-1} + t^{2-0} = 6 - 5t + t^2.$$ 

The number of chambers is $\chi(-1) = 12$ and the number of bounded chambers is $\chi(1) = 2$.

![Figure 2](image)

**Figure 2.** $\mathcal{J}_2$ has 12 regions and 2 bounded regions.

3.2. $n = 3$

In this case, $\mathcal{J}_3$ has three hyperplanes of type I and six hyperplanes of type II. Obviously, having both $0_i$ and $1_i$ for a given $i$ makes the arrangement not central. Hence any central subarrangement has no more than three hyperplanes of type II, so a central subarrangement $\mathcal{B}$ satisfies $|\mathcal{B}| \leq 6$. We enumerate all central graphs, besides the trivial one with $|\mathcal{B}| = 0$. 

![Figure 3](image)
(1) \(|B| = 1\): \(B\) either contains a single colored vertex, or an edge. There are nine such cases.

(2) \(|B| = 2\): \(B\) may have two edges and no colored vertices, or an edge and a colored vertex, or two colored vertices. These are all rank 2 arrangements.

(a) \((\epsilon = 0)\): \(\binom{3}{0} \cdot 2 \cdot 2 = 12\) subarrangements;

(b) \((\epsilon = 1)\): There are \(\binom{7}{1} \cdot (2 + 2) = 12\) central \((1,1)\) graphs with the colored vertex incident on the edge. There are six \((1,1)\) graphs with the colored vertex not incident on the edge.

(c) \((\epsilon = 2)\): These are three labeled trees on \([3]\).

(3) \(|B| = 3\): These are all of rank 3 except the six \((1,2)\) graphs with an edge with both vertices colored (with opposing colors) which are of rank 2.

(a) \((\epsilon = 0)\): \(2^3\) cases, obviously;

(b) \((\epsilon = 1)\): Choose an edge and either color the two vertices incident upon the edge \((\binom{3}{1} \cdot 2 = 6\) cases, this is essentially the case of two vertices) or color one vertex incident on the edge and the other vertex not on the edge \((\binom{3}{2} \cdot 2^3 = 24\) cases);

(c) \((\epsilon = 2)\): Two edges and one colored vertex (of either color). There are \(\binom{3}{3} \cdot 3 \cdot 2 = 18\) cases;

(d) \((\epsilon = 3)\): There is only 1 case.

(4) \(|B| = 4\): \(\epsilon \geq 1\) since otherwise a vertex must have two colors. These are all of rank 3, and there are 30 central graphs corresponding to \(B\) with \(|B| = 4\).

(a) \((\epsilon = 1)\): There are two ways to color the two vertices incident on the one existing edge, and two ways to color the vertex not incident on the edge. Hence \(\binom{4}{1} \cdot 2 \cdot 2 = 12\) central \((1,3)\) graphs.

(b) \((\epsilon = 2)\): There are two edges incident on two colored vertices and a non-colored vertex. The one non-colored vertex can be at an endpoint of the path or not. There are \(\binom{4}{2} \cdot 2 \cdot 2 = 12\) and \(\binom{4}{2} \cdot 2 \cdot 2^2 = 24\) central \((2,2)\) graphs of each kind.

(c) \((\epsilon = 3)\): There are no central graphs of this type. A closed path from the colored vertex to itself is a cycle of length 3.

(5) \(|B| = 5\): There are no \((3,2)\) central graphs. There are six central \((2,3)\) graphs.

(6) There are no central subarrangements with \(\geq 6\) hyperplanes since it would have to have either a cycle of length three with colored vertices or a vertex with two colors.

We summarize our findings in the table below.

| rank \(|B|\) | 1 | 2 | 3 | 4 | 5 |
|-----------|---|---|---|---|---|
| 1         | 9 | 0 | 0 | 0 | 0 |
| 2         | 0 | 33| 6 | 0 | 0 |
| 3         | 0 | 0 | 51| 30| 6 |
The characteristic polynomial is
\[\chi_{\mathcal{J}_3}(t) = t^3 - 9t^2 + 27t - 27.\]

Interestingly, the polynomials for \(n = 2, 3\) factor into linear forms. In general, this turns out not to be the case \(n \geq 4\) which will be illustrated in our forthcoming manuscript [4]. It would be interesting to understand when the characteristic polynomial factors.

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