Wedge removability of metrically thin sets and application to the CR-meromorphic extension

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Abstract.
We give a wedge removability theorem for metrically thin sets of two codimensional Hausdorff null measure. Following [18], this removability theorem combined with the wedge removability theorem of [17] for closed subsets of two codimensional manifolds, gives a CR-meromorphic extension theorem in the greater codimensional case.

1 Introduction

1.1 CR submanifolds - For a smooth submanifold $M$ of an open subset of $\mathbb{C}^n$, let $T_p(M)$ be the real tangent space of $M$ at $p \in M$. In general, $T_p(M)$ is not invariant under the complex structure map $J$ for $T_p(\mathbb{C}^n)$. Therefore, we give special designation for the largest $J$-invariant subspace of $T_p(M)$. For a point $p \in M$, the complex tangent space of $M$ at $p$ is the vector space

$$H_p(M) = T_p(M) \cap J\{T_p(M)\}.$$

The totally real part of the tangent space of $M$ is the quotient space

$$X_p(M) = T_p(M)/H_p(M).$$

The complexifications of $T_p(M)$, $H_p(M)$ and $X_p(M)$ are denoted by $T_p(M) \otimes \mathbb{C}$, $H_p(M) \otimes \mathbb{C}$ and $X_p(M) \otimes \mathbb{C}$. The complex structure map $J$ on $T_p(\mathbb{R}^{2n})$ restricts to a complex structure map on $H_p(M) \otimes \mathbb{C}$ because $H_p(M)$ is $J$-invariant. So, $H_p(M) \otimes \mathbb{C}$ is the direct sum of the $+i$ and $-i$ eigenspace of $J$ which are denoted by $H^{1,0}_p(M)$ and $H^{0,1}_p(M)$. A submanifold $M$ in an open subset $U$ of $\mathbb{C}^n$ is called an embedded CR manifold or a CR submanifold of $U$ if $\dim_H H_p(M)$ is independent of $p \in M$. A CR submanifold $M$ is called generic if the CR dimension $\dim_H H_p(M)$ is minimal.

1.2 CR and CR-meromorphic functions - Let $M$ be a CR submanifold of an open subset of $\mathbb{C}^n$. A function $f : M \to \mathbb{C}$ is called a CR function if $\bar{\partial}f = 0$ (in
the current sense on $M$) for any $C^1$ vector field $L$ of $H^{1,0}(M)$. A closed subset $N$ of a manifold $X$ is called a \textit{scarred manifold} of dimension $p$ if there exists a closed subset $\tau \subset N$ ($\tau$ is called the \textit{scar set}) of $p$-dimensional Hausdorff $\mathcal{H}^p$ zero measure such that $N \setminus \tau$ is an oriented $p$ dimensional $C^1$ submanifold of $X \setminus \tau$ of locally finite $\mathcal{H}^p$ volume in $X$ and closed in the current sense ($d[N] = 0$). If the regular part of $N$ is a maximally complex or a CR submanifold, we will say that $N$ is a scarred maximally complex or CR submanifold.

**Definition 1** ([13]) Let $M$ be a CR manifold of dimension $p$. A $C^1$ CR map $f$ defined on an open and dense subset of $M$ and having values in $P_1(\mathbb{C})$ will be called a \textit{CR-meromorphic map} if the closure $\Gamma_f$ of its graph in $M \times P_1(\mathbb{C})$ is a $C^1$ scarred CR manifold of the same CR dimension than $M$.

This notion was first introduced by Harvey and Lawson [11] in the case $M$ was maximally complex. In that case, the meromorphic extension of CR-meromorphic maps is obtained by solving the boundary problem for their graph (see [11] [13]). In the case of non maximally complex CR manifold, the solving of the boundary problem seems not to apply. According to [18], in the case that a CR-meromorphic map take its values in a bounded domain of $\mathbb{C}$, it defines a CR current on $M$. Thus, from the representation theorem of Baouendi and Rothschild [3] we are reduced to a problem of extension of smooth CR functions defined on $M$. Let $f$ be a CR-meromorphic function defined on $M$ and let $x \in M$ be a point such that there exists a point $y \in P_1(\mathbb{C})$ such that $x \times y \not\in \Gamma_f$. Let $\phi_y : P_1(\mathbb{C}) \setminus y \rightarrow \mathbb{C}$ be the projective chart where $y$ is the infinite point and $\omega$ be a small enough neighborhood of $x$ in $M$ such that $\phi_y \circ f$ is bounded. Then $\phi_y \circ f$ defines a CR current on $\omega$. A point $p \in M$ is said an \textit{indeterminacy point} of $f$ if $\Gamma_f \supset \{p\} \times P_1(\mathbb{C})$. The \textit{indeterminacy set} $K$ of $f$ is the set of all indeterminacy points of $f$. $K$ is an obstruction to the extension. Indeed, if $x \in K$, the reduction to the case of CR currents cannot apply.

1.3 The Levi form - The local extension of CR functions or CR currents arise under convexity assumption of $M$. The more general notion of convexity where the extension arise is the notion of local and global minimality. However, as a first step lets recall to the notion of convexity given by the Levi form.

Suppose $M = \{\zeta \in \mathbb{C}^n; \rho_1(\zeta) = \ldots = \rho_d(\zeta) = 0\}$ is a smooth CR submanifold of $\mathbb{C}^n$, with $1 \leq d \leq n$. Let $p$ be a point in $M$ and suppose $\{\nabla \rho_1(p), \ldots, \nabla \rho_d(p)\}$ is an orthonormal basis for the normal space $N_p(M)$ of $M$. Then the \textit{extrinsic Levi form} is given by

$$\tilde{L}_p(W) = -\sum_{l=1}^{d} \left( \sum_{j,k=1}^{n} \frac{\partial^2 \rho_l(p)}{\partial \zeta_j \partial \zeta_k} w_j \bar{w}_k \right) \nabla \rho_l(p)$$

for $W = \sum_{k=1}^{n} w_k (\partial / \partial \zeta_k) \in H^{1,0}_p(M)$.

1.4 Minimality - One of the characteristic properties of CR manifold is that $H(M)$ is involutive. So, we will say that a curve $\gamma : [0, 1] \rightarrow M$ is a CR curve if
for all $t \in [0,1]$, the tangent to $\gamma$ at the point $\gamma(t)$ is in $H_{\gamma(t)}(M)$. We will say that two points are in the same CR orbit of M if they can be reached by a finite number of CR curves of $M$. Let $\{U_i\}_{i \in I}$ be a basis of neighborhoods of $p$ in $M$. For each $i \in I$, we can define the orbit of $p$ in $U_i$. The inductive limit of those orbit is well defined and does not depends of the family $\{U_i\}$ and is call the local CR orbit of $p$. A CR submanifold $M$ is said minimal at $p$ is the local CR orbit of $p$ is an open neighborhood of $p$ in $M$.

1.5 Wedge removable sets - Let $M$ be a generic CR submanifold of an open subset of $\mathbb{C}^n$ (i.e. $T_p(M) + JT_p(M) = T_p(\mathbb{C}^n)$). By a wedge of edge $M$ at $p$, we mean an open set in $\mathbb{C}^n$ of the form

$$W = \{z + \eta; z \in U, \eta \in C\}$$

for some open neighborhood $\omega$ of $p$ in $M$ and some truncated open cone $C$ in $N_p(M)$, i.e. the intersection of a open cone with a ball centered at 0. Let $K$ be a proper closed subset of $M$, $K$ is said W-removable at $p$ if there exists a wedge $W$ of edge $M$ at $p$ such that any CR function defined on $M \setminus K$ extends holomorphically to $W$.

The theory of CR removability theory has been first developed by the deep work of Jörcke [12, 13, 14]. In the hypersurface case, she proved the removability of proper closed subsets of 2 codimensional submanifolds of $M$. Then Chirka-Stout [6] proved the removability of closed subsets of two codimensional null Hausdorff measure. The CR removability results in the greater codimensional case was obtained by Merker [16] and Jörcke [14]. In [17], Merker and Porten proved the removability of proper closed subset of two codimensional submanifolds of $M$ and of closed subset of $M$ of finite three codimensional Hausdorff measure. All those removability theorems have been given under minimality assumption on $M$. Our main theorem gives a removability result for closed subsets of two codimensional Hausdorff null measure under convexity assumption given by the Levi form.

**Main theorem** Suppose $M$ is a generic, CR submanifold of an open set $U$ of $\mathbb{C}^n$ of class $C^4$ with $\text{dim}_\mathbb{R}M = 2n - d$ ($1 \leq d \leq n - 1$). Let $p_0$ a point of $M$ such that the convex hull of the image of the Levi form has nonempty interior. Then their exists a wedge $W$ of edge $M$ at $p_0$ such that if $K$ is a closed subset of $M$ of null Hausdorff $\mathcal{H}^{2n-d-2}$ measure, every CR function $f$ on $M \setminus K$ extends holomorphically to $W$.

In the case $K$ is empty, this is the wedge of the edge extension theorem of Airapetyan-Henkin [2] and Boggess-Polking [4]. Let $f$ be a CR-meromorphic map defined on a minimal generic manifold $M$ and $K$ be the indeterminacy set of $f$. If $x \notin K$, as explained in [18], $f$ extends meromorphically to a wedge of edge $M$ at $x$. Then, by the uniqueness of the extension, all the meromorphic extensions of $f$ coincides. Thus, deforming $M$ outside $K$ in the wedge where the extension arise, we can assume that $f$ is
the restriction of a meromorphic map defined in a neighborhood $U$ of $M \setminus K$. By Oka-Levi theorem, the envelope of meromorphy of an open subset of $\mathbb{C}^n$ is the same than its envelope of holomorphy. So it suffice to prove that the envelope of holomorphy of $U$ contains a wedge $W$ of edge $M$ to prove that $f$ extend meromorphically into $W$. As remarked in [18], $K$ is included (and of empty inside) in a scarred two codimensional submanifold of $M$. So applying our theorem and a removability theorem of [17] for proper closed subsets of two codimensional submanifolds of $M$ we obtain:

**Corollary 1** Suppose $M$ is a generic, $C^4$, CR submanifold of an open subset $U$ of $\mathbb{C}^n$ with $\dim \mathbb{R} M = 2n - d$ ($1 \leq d \leq n - 1$). Let $p_0$ a point of $M$ such that the convex hull of the image of the Levi form has nonempty interior. Then there exists a wedge $W$ of edge $M$ at $p_0$ such that every CR-meromorphic function defined on $M$ extends meromorphically to $W$.

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2 Bishop’s equation

2.1 Normal form for CR submanifolds -

**Lemma 1** Suppose $M$ is a generic, CR submanifold of $\mathbb{C}^n$ of class $C^k$ ($k \geq 2$) with $\dim \mathbb{R} M = 2n - d$ ($1 \leq d \leq n$). Suppose that $p_0$ is a point in $M$. There is a neighborhood $U$ of $p_0$ in $\mathbb{C}^n$ such that for any $p \in U \cap M$ there exists a biholomorphism $\Phi = \Phi_p : U \to \Phi\{U\} \subset \mathbb{C}^n$; and a function $h = h_p : \mathbb{R}^d \times \mathbb{C}^{n-d} \to \mathbb{R}^d$ of class $C^k$ with $h(0) = 0$ ($\Phi$ and $h$ depending in $C^k$ fashion of $p \in U \cap M$) such that

$$\Phi\{M \cap U\} = \{(x + iy, w) \in \Phi\{U\} \subset \mathbb{C}^d \times \mathbb{C}^{n-d}; y = h(x, w)\}$$

Furthermore

$$\frac{\partial^{|\alpha|+|\beta|} h(0)}{\partial x^\alpha \partial \bar{w}^\beta} = \frac{\partial^{|\alpha|+|\beta|} h(0)}{\partial x^\alpha \partial \bar{w}^\beta} = 0$$

for all $\alpha \in \mathbb{N}^d$, $\beta \in \mathbb{N}^{n-d}$ such that $0 \leq |\alpha| + |\beta| \leq 2$.

This lemma can be found in [4], we just remark that the given construction of $\Phi$ and $h$ depends in $C^{k-1}$ fashion of $p \in U \cap M$. We will call the quadric associated to $h$ the quadric $q := \frac{\partial^2 h}{\partial w \partial \bar{w}}(0)$.

2.2 Bishop’s equation - Let $M$ be a generic CR submanifold of a neighborhood $U = U_1 \times U_2$ of the origin of $\mathbb{C}^n$ defined by the equation

$$M = \{(z = x + iy, w) : (x, w) \in U_1, y \in U_2; y = h(x, w)\}$$
where \( U_1 \) is an open neighborhood of the origin of \( \mathbb{R}^d \times \mathbb{C}^{n-d} \), \( U_2 \) is an open neighborhood of the origin of \( \mathbb{R}^d \), \( h : U_1 \rightarrow U_2 \) is of class \( C^k \) and \( h(0) = 0 \), \( Dh(0) = 0 \). Let note \( D \) the open unit disc of \( \mathbb{C} \), \( \bar{D} \) its closure and \( S^1 \) the unit circle of \( \mathbb{C} \). Given an analytic disc \( W : \bar{D} \rightarrow \mathbb{C}^{n-d} \), we wish to find an analytic disc \( G : \bar{D} \rightarrow \mathbb{C}^d \) so that the boundary of the disc \( A = (G, W) : \bar{D} \rightarrow \mathbb{C}^n \) is contained in \( M \). This mean that \( G \) must satisfy

\[
\text{Im} G(\zeta) = h(\text{Re} G(\zeta), W(\zeta)) \text{ for } \zeta \in S^1.
\]

This equation involves both \( u = \text{Re} G \) and \( v = \text{Im} G \). The above equation will be easier to solve by eliminating \( v \). To do this, we use the Hilbert transform which is defined as follows. If \( u : S^1 \rightarrow \mathbb{R}^d \) is a continuous function, then \( u \) extends to a unique harmonic function on the unit disc \( D \). This harmonic function has a unique harmonic conjugate in \( \bar{D} \) which vanishes at the origin. The Hilbert transform of \( u \) (denoted \( Tu : S^1 \rightarrow \mathbb{R}^d \)) is defined to be \( v|_{S^1} + c \) where \( c = -v(\zeta = 0) \). The function \( -iG = v - iu \) is also analytic so \( T(v|_{S^1}) = -u + x \) where \( x = u(\zeta = 0) \). Conversely, if \( u, v : S^1 \rightarrow \mathbb{R}^d \) are continuous functions with \( u = -Tv + x \), then \( u + iv : S^1 \rightarrow \mathbb{C}^d \) is the boundary values of a unique analytic disc \( G : \bar{D} \rightarrow \mathbb{C}^d \) with \( \text{Re} G(\zeta = 0) = x \). Suppose \( u + iv = G : \bar{D} \rightarrow \mathbb{C}^d \) is an analytic disc with \( v(e^{i\varphi}) = h(u(e^{i\varphi}), W(e^{i\varphi})) \) for \( 0 \leq \varphi \leq 2\pi \). We apply \(-T\) to both sides of this equation and obtain

\[
u(e^{i\varphi}) = -T(h(u, W))(e^{i\varphi}) + x \text{ for } 0 \leq \varphi \leq 2\pi
\]

where \( x \in \mathbb{R}^d \) is the value of \( u \) at \( \zeta = 0 \). The above equation will be referred to as Bishop’s equation. Conversely, suppose the analytic disc \( W : \bar{D} \rightarrow \mathbb{C}^{n-d} \) and the vector \( x \in \mathbb{R}^d \) are given, and suppose \( u : S^1 \rightarrow \mathbb{R}^d \) is a solution to Bishop’s equation. From the above discussion, the function

\[
\varphi \rightarrow u(e^{i\varphi}) + ih(u(e^{i\varphi}), W(e^{i\varphi}))
\]

is the boundary values of a unique analytic disc \( G : \bar{D} \rightarrow \mathbb{C}^d \). Since \( \text{Re} G(e^{i\varphi}) = u(e^{i\varphi}) \), the boundary of the analytic disc \( A = (G, W) : \bar{D} \rightarrow \mathbb{C}^n \) is contained in \( M \). Furthermore, \( \text{Re} G(\zeta = 0) = x \).

The Hilbert Transform is a smooth linear map from the space \( C^\alpha(S^1, \mathbb{R}^d) \) of \( \alpha \)-Hölderian maps from \( S^1 \) into \( \mathbb{R}^d \) to itself (with \( 0 < \alpha < 1 \)) (a proof can be found in [3]).

3 The convex quadric case

A quadric \( M \) is a submanifold given by the equation

\[
M = \{(z = x + iy, w) \in \mathbb{C}^d \times \mathbb{C}^{n-d}; y = q(w, \bar{w})\}
\]
where \( q : \mathbb{C}^{n-d} \times \mathbb{C}^{n-d} \to \mathbb{C}^d \) is a quadric form. Let us start with a given analytic disc \( W : \bar{D} \to \mathbb{C}^{n-d} \). The analytic disc \( W : \bar{D} \to \mathbb{C}^{n-d} \) is given by a convergent power series

\[
W_t(\zeta) = \sum_{j=0}^{\infty} t_j a_j \zeta^j, \quad t_j \in \mathbb{R}, \quad a_j \in \mathbb{C}^{n-d}, \quad \zeta \in \bar{D}.
\]

In our application, all but a finite number of the parameters \( \{a_0, a_1, \ldots\} \) and \( \{t_0, t_1, \ldots\} \) will vanish. In order for the set \( \{A(\zeta) = (G(\zeta), W(\zeta)); \zeta \in S^1\} \) to be contained in \( M \), the analytic disc \( G : \bar{D} \to \mathbb{C}^d \) must satisfy

\[
\text{Im} G(\zeta) = q(W(\zeta), \overline{W(\zeta)}) \quad \text{for} \quad \zeta \in S^1.
\]

Using the linearity and symmetry of \( q \), we find

\[
G(\zeta) = x + i \sum_{j=0}^{\infty} t_j^2 q(a_j, \bar{a}_j) + 2i \sum_{0 \leq k < j} t_j t_k q(a_j, \bar{a}_k) \zeta^{j-k}
\]

(with \( x \in \mathbb{R}^d \)) is, given the condition \( \text{Re} G(0) = x \), the unique solution of the equation \( \text{Im} G(\zeta) = q(W(\zeta), \overline{W(\zeta)}) \) for \( \zeta \in S^1 \). Then we have

\[
A(\zeta) \in M \quad \text{for} \quad \zeta \in S^1
\]

\[
A(\zeta = 0) = \left( x + i \sum_{j=0}^{\infty} t_j^2 q(a_j, \bar{a}_j), t_0 a_0 \right)
\]

We want to verify that when \( t \) moves, the boundaries of the obtained discs give us a submersion of the space of parameters. Take \( t_0 = 1 \). It suffice to verify that the matrix \( \mathcal{M} \) of the derivatives of \((W, W, \text{Re} G, v(0))\) (we recall that \( v(0) = \sum_{j=0}^{\infty} t_j^2 q(a_j, \bar{a}_j) \)) in \( t_i \) is of maximal rank. We have for \( \zeta \in S^1 \)

\[
\frac{\partial}{\partial t_j} v(0) = 2t_j q(a_j, \bar{a}_j)
\]

and

\[
\frac{\partial}{\partial t_j} \text{Re} G(\zeta) = \text{Re} \left( 2i \sum_{1 \leq k < j} t_k q(a_j, \bar{a}_k) \zeta^{j-k} + 2i \sum_{k>j} t_k q(a_k, \bar{a}_j) \zeta^{k-j} \right)
\]

\[
= i \left( \sum_{k<j} t_k q(a_j, \bar{a}_k) \zeta^{j-k} + \sum_{k>j} t_k q(a_k, \bar{a}_j) \zeta^{k-j} - \sum_{k<j} t_k q(a_j, \bar{a}_k) \zeta^{j-k} - \sum_{k>j} t_k q(a_k, \bar{a}_j) \zeta^{k-j} \right)
\]

\[
= i \left( \sum_{k<j} t_k q(a_j, \bar{a}_k) \zeta^{j-k} + \sum_{k>j} t_k q(a_k, \bar{a}_j) \zeta^{k-j} - \sum_{k<j} t_k q(a_k, \bar{a}_j) \zeta^{j-k} - \sum_{k>j} t_k q(a_j, \bar{a}_k) \zeta^{k-j} \right)
\]
We remark that the vectors \( \{ i(\sum_{k=1}^\infty t_k q(a_j, a_j)) \} \) are linear combinations of the vectors \( \{ \alpha_j \} \) and of the vectors \( \{ \alpha_j a_j \} \) (we recall that the \( \alpha_j \) are vectors and that \( q \) takes vectorial values). Then by adding this vectors to the derivative of \( \Re G(\zeta) \) we do not change the rank of the matrix of the derivatives in \( t_j \). So the rank of the matrix \( M \) of the derivatives of \( (W, \bar{W}, \Re G, v(0)) \) in \( t_j \), with \( \zeta \in S^1 \)

\[
M(a, \zeta, t) = \begin{pmatrix}
a_1 \zeta^1 & \ldots & a_j \zeta^j & \ldots \\
\bar{a}_1 \zeta^{-1} & \ldots & \bar{a}_j \zeta^{-j} & \ldots \\
\partial ReG(\zeta) & \ldots & \partial ReG(\zeta) & \ldots \\
2t_1 q(a_1, a_1) & \ldots & 2t_j q(a_j, a_j) & \ldots \\
\end{pmatrix}
\]

is of the same rank than the matrix

\[
M'(a, \zeta, t) = \begin{pmatrix}
a_1 \zeta^1 & \ldots & a_j \zeta^j & \ldots \\
\bar{a}_1 \zeta^{-1} & \ldots & \bar{a}_j \zeta^{-j} & \ldots \\
P_1(a, \zeta, t) & \ldots & P_j(a, \zeta, t) & \ldots \\
t_1 q(a_1, a_1) & \ldots & t_j q(a_j, a_j) & \ldots \\
\end{pmatrix}
\]

where

\[
P_j(a, \zeta, t) = 2 \sum_{k<j} (q(a_j, a_k) \zeta^{j-k} - q(a_k, a_j) \zeta^{k-j})
\]

is a polynomial in \( \zeta \) and \( \zeta^{-j} \) of degree \( j \) and depends only on \( (t_1, \ldots, t_{j-1}) \). Let note \( M'_N \) the matrix \( M' \) truncated at the \( N^{th} \) column.

**Lemma 2** For any fixed \( \zeta \in S^1 \) there exists \( N \in \mathbb{N} \), \( b = (b_j)_{1 \leq j \leq N} \) and \( s = (s_j)_{1 \leq j \leq N} \) such that the rank of \( M'_N(b, \zeta, s) \) is maximal.

**Proof.** First, let suppose \( \zeta = 1 \). Let suppose, that for all \( N \in \mathbb{N} \), \( b = (b_j)_{1 \leq j \leq N} \) and \( s = (s_j)_{1 \leq j \leq N} \) the rank of \( M'_N(b, 1, s) \) is not maximal. Let suppose \( N, b, s \) are chosen such that the rank \( r \) of \( M'_N(b, 1, s) \) is the greater rank reached by those matrixs. As the rank is never maximal we have \( r < 2n \). Then there exists a non null \((1, 2n)\) matrix \((\alpha, \beta, \gamma, \delta) \in \mathbb{C}^{n-d} \times \mathbb{C}^{n-d} \times \mathbb{C}^d \times \mathbb{C}^d \) such that \((\alpha, \beta, \gamma, \delta) \times M'_N(b, 1, s) = 0 \). So for any families \( b = (b_j)_{N \leq j \leq N'} \) and \( s = (s_j)_{N \leq j \leq N'} \), we have \((\alpha, \beta, \gamma, \delta) \times M'_N(b, 1, s) = 0 \) because \( M'_N(b, 1, s) \) is a sub-matrix of \( M'_N(b, 1, s) \).

As the \( 2n - d \) first lines of the matrix are independent of \( s_{N'} \), we have \( \delta \times (s_{N'}, q(b_{N'}, \bar{b}_{N'})) = 0 \) for all \( N' > N, s_{N'} \in \mathbb{R} \) and \( b_{N'} \in \mathbb{C}^{n-d} \). From the hypothesis that the convex hull of the image of \( q \) is an open cone of \( \mathbb{R}^d \), the previous equality implies that \( \delta = 0 \). As the \( 2n - 2d \) first lines of the matrix are independent of \( s_{N'-1} \), we have \( \gamma \times s_{N'-1}(q(b_{N'}, \bar{b}_{N'-1}) - q(b_{N'-1}, \bar{b}_{N'})) = 0 \) for all \( N' > N + 1, s_{N'-1} \in \mathbb{R} \) and \( b_{N'-1}, b_{N'} \in \mathbb{C}^{n-d} \). Choosing for example \( b_{N'} = ib_{N'-1} \) we find \( q(b_{N'}, \bar{b}_{N'-1}) - q(b_{N'-1}, \bar{b}_{N'}) = 2i q(b_{N'-1}, \bar{b}_{N'-1}) \). So, from the hypothesis that the convex hull of the image of \( q \) is an open cone of \( \mathbb{R}^d \) the equality implies that \( \gamma = 0 \). Then, it is obvious that \( \alpha = \beta = 0 \) and we have the contradiction.
Let choose \( k \in \mathbb{N} \), \( b^* = (b_j^*)_{1 \leq j \leq k} \) and \( s^* = (s_j^*)_{1 \leq j \leq k} \) such that the rank of \( \mathcal{M}'_{k}(b^*, 1, s^*) = 2n \), i.e. is maximal. Let \( \zeta \) fixed, there exists \( r \in \mathbb{N}^+ \) such that \( |\zeta^{m} - 1| < \epsilon \) for all \( 1 \leq m \leq k \) for \( \epsilon > 0 \) small enough.

For all \( 1 \leq m \leq k \), let choose \( b_{mr} = b_{m}^* \) and \( s_{mr} = s_{m}^* \) and \( b_{j} = s_{j} = 0 \) for all \( j \notin \{r, 2r, ..., kr\} \). Then, by continuity, for \( \epsilon \) small enough we have

\[
\text{rank}\mathcal{M}'_{kr}(b, \zeta, s) = \text{rank}\mathcal{M}'_{k}(b^*, 1, s^*) = 2n.
\]

\[\square\]

**Proposition 1** There exists \( N \in \mathbb{N} \), \( a = (a_j)_{j \leq N} \) and an open cone \( \Omega \subset \mathbb{R}^N \) such the rank of \( \mathcal{M}(a, t, \zeta) \) is maximal for any \( t \in \Omega \) and \( \zeta \in S^1 \).

**Proof.** We just remark that \( \text{rank} \mathcal{M}_{N}(a, \zeta, t) = \text{rank} \mathcal{M}_{N}(a, \zeta, \lambda t) \) for all \( \lambda \in \mathbb{R} \backslash \{0\} \). So it is enough to prove our proposition for \( \Omega \) any non-empty open set in \( \mathbb{R}^n \) and by continuity it suffice to find \( N, b \) and \( t \) fixed such that for any \( \zeta \in S^1 \), \( \text{rank} \mathcal{M}_{N}(a, \zeta, t) = 2n \).

For all \( \zeta \in S^1 \), let choose \( N_{\zeta}, \{b_{j}(\zeta)\}_{1 \leq j \leq N_{\zeta}} \) and \( \{s_{j}(\zeta)\}_{1 \leq j \leq N_{\zeta}} \) verifying the previous lemma. By continuity, there exists a neighborhood \( V_{\zeta} \) of \( \zeta \) in \( S^1 \) such that \( \text{rank} \mathcal{M}_{N_{\zeta}}(b_{j}(\zeta), \zeta, s_{j}(\zeta)) = 2n \) for all \( \zeta' \in V_{\zeta} \). Let \( \{\zeta_1, ..., \zeta_k\} \subset S^1 \) a finite subset such that \( S^1 \subset \bigcup_{m=1}^{k} V_{\zeta_m} \) where \( V_{\zeta_m} \) is a relatively compact open subset of \( V_{\zeta_m} \). Let \( N = N_{\zeta_1} + ... + N_{\zeta_k}, a_j = b_{j-(N_{\zeta_1} + ... + N_{\zeta_m})}(\zeta_m), t_j = \theta_m s_{j-(N_{\zeta_1} + ... + N_{\zeta_m})}(\zeta_m) \) for all \( N_{\zeta_1} + ... + N_{\zeta_m} < j \leq N_{\zeta_1} + ... + N_{\zeta_m+1}, 0 < \theta_m << \theta_{m+1} \) and \( 0 \leq m \leq k - 1 \). There is still to show that \( \text{rank}\mathcal{M}'_{N_{\zeta}}(a, \zeta, t) = 2n \) for all \( \zeta \in S^1 \). Let \( \zeta \in V_{\zeta_m} \), we have \( \text{rank} \mathcal{M}'_{N_{\zeta}}(a, \zeta, t) \geq \text{rank} \mathcal{M}'_{N_{1} + ... + N_{m}}(a, \zeta, t) \). The rank of the second matrix is greater or equal to the one of the sub-matrix formed by the \( N_{m} \) last columns. As the \( \theta_m \) are chosen such that \( \theta_{m-1} << \theta_m \), the rank of this sub-matrix is equal to the rank of \( \mathcal{M}'_{N_{m}}(b(\zeta_m), \zeta, s(\zeta_m)) \), i.e. is maximal. \[\square\]

We note \( A(p, a, t) = (G(p, a, t), W(p, a, t)) \) the attached discs given by the previous proposition where \( p = (x, a_0) \in \mathbb{R}^d \times \mathbb{C}^{n-d}, a = (a_1, ..., a_N) \) and \( t = (t_1, ..., t_N) \).

From \[\|\] the image, for all \( t \in \Omega \), of \( A(p, a, t)(0) \) contains a wedge \( \mathcal{W} = \omega + \mathcal{C} \) of edge \( M \) at 0. Let \( \Omega_{z} \) the set of \( t \in \Omega \) such that \( A(p, a, t)(0) = z \) where \( z = p + \eta \) with \( p \notin K, \eta \in \mathcal{C} \) and \( K \) a closed subset of \( M \) verifying the hypothesis of the theorem 1. From our proposition, the map \( (u, W)(p, a, .): \Omega_{z} \times S^1 \rightarrow \mathbb{R}^d \times \mathbb{C}^{n-d} \) is a submersion. So, their exists discs, attached to \( M \), passing through \( z \), isotopic to the point \( p \) such that their boundary is in \( M \backslash K \). Here, we say that a disc \( A \) is isotopic to a point \( p \in M \backslash K \), if there exists a continuous family of discs attached to \( M \backslash K \) and of class \( C^2 \) that contains \( A \) and \( p \). This implies the theorem 1 in the quadric case. The details can be find at the end of the next section.

### 4 The convex case

Let \( p_0 \in M, U, \Phi = \Phi_p \) and \( h = h_p \) verifying lemma 1. Let \( U^p := \Phi_p(U) \), \( M^p := \Phi_p(M \cap U) \), \( K^p = \Phi_p(K) \), \( \Pi_1 : \mathbb{C}^n \rightarrow \mathbb{R}^d \times \mathbb{C}^{n-d} \) be the projection
\[ \Pi_1(u + iv, w) = (u, w) \text{ and } \Pi_2 : \mathbb{C}^n \to \mathbb{R}^d \text{ be the projection } \Pi_2(u + iv, w) = v. \]

Let \( \Pi : \mathbb{R}^d \setminus 0 \to S^{d-1} \) be the usual projection into the unit sphere \( S^{d-1} \) for \( \mathbb{R}^d \) and \( \Pi' := \Pi|_M \). Let \( N \in \mathbb{N} \), \( a_j \in \mathbb{C}^{n-d} \) and \( t_j \in \mathbb{R}^N \) fixed. We consider the analytic disc \( W(t, \zeta) : \mathbb{D} \to \mathbb{C}^{a_n} \) with \( W(t, \zeta) = \sum t_j \zeta^j \). Then if \(|t_ja_j| \) is small enough, Bishop’s equation (for \( M^p \) with the condition \( u(0) = W(0) = 0 \)) admits a unique solution depending in a \( C^4 \) fashion of \( t_j \). Let

\[ \mathcal{M} := \left( \frac{\partial W(t, \zeta)}{\partial t_j}, \frac{\partial W(t, \zeta)}{\partial t_j}, \frac{\partial u(t, \zeta)}{\partial t_j}, \frac{\partial v(t, 0)}{\partial t_j} \right)_{1 \leq j \leq N} \]

**Proposition 2** Let suppose that \( \Gamma_{p_0} \) has non empty interior. Then there exists \( \omega \) a neighborhood of \( p_0 \) in \( M \), \( N \in \mathbb{N} \), \( a_j \in \mathbb{C}^{n-d} \), \( \Omega \subset \mathbb{R}^N \) a truncated open cone such that for every \( p \in \omega, t \in \Omega, \zeta \in S^1 \) the rank of the matrix \( \mathcal{M} \) is maximal.

**Proof.** We call \( q \) (resp. \( q_0 \)) the quadric form associated to \( h = h_p \) (resp. \( h_{p_0} \)) defined in the section 2.1. Let define \( \mathcal{O}_\alpha(\mathbb{D}, \mathbb{C}^{n-d}) = C_\alpha(\mathbb{D}, \mathbb{C}^{n-d}) \cap \mathcal{O}(\mathbb{D}, \mathbb{C}^{n-d}) \) the space of holomorphic maps from \( \mathbb{D} \) into \( \mathbb{C}^{n-d} \) which are \( \alpha \)-holderian up to the boundary and

\[ \mathcal{B}(\bar{B}^{2n-d}, \mathbb{R}^d) = C^k(\bar{B}^{2n-d}, \mathbb{R}^d) \cap \{ \frac{\partial^{[a] + [\beta]}}{\partial x^a \partial y^\beta} g = 0, 0 \leq |a| + |\beta| \leq 2 \} \]

where \( \mathbb{B}^{2n-d} \) is the closed unit ball of \( \mathbb{R}^{2n-d} \). Let consider the map

\[ \mathcal{F} : C^\alpha(\bar{D}, \mathbb{R}^d) \times \mathcal{O}_\alpha(\bar{D}, \mathbb{C}^{n-d}) \times \mathcal{B}(\bar{B}^{2n-d}, \mathbb{R}^d) \to C^\alpha(\bar{D}, \mathbb{R}^d) \]

\[ \mathcal{F}(u, W, g) := u + T(g(u, W)) \]

For \( 0 < \alpha < 1 \), \( \mathcal{F} \) is of class \( C^1 \) and

\[ \frac{\partial \mathcal{F}}{\partial u}(0, 0, q_0) = \text{Id} \]

From the implicit functions theorem, in a neighborhood of \((0, 0, q_0)\), there exists a unique solution \( u(W, g) \) of the equation \( \mathcal{F}(u, W, g) = 0 \) depending in a \( C^3 \) fashion of \( W \) and \( g \). Let \( \mathcal{M}_g \) be the matrix defined as above for the manifold \( \{y = g(x, w)\} \) and \( \Theta \) be a compact of non empty inside \( \Theta \) of \( \Omega_0 \cap S^{N-1} \) where \( S^{N-1} \) is the unit sphere of \( \mathbb{R}^N \) and \( \Omega_0 \) and \( N \) are defined in the proposition 1 for \( q = q_0 \). Then for \(|g - q_0| < \epsilon \) with \( \epsilon \) small enough and \( t \in \Theta \), the rank of \( \mathcal{M}_g \) is still maximal. Let \( \omega \) be a neighborhood of \( p_0 \) in \( M \) small enough such that for \( p \in \omega \) we have \(|g - q_0| < \epsilon / 2 \). Let \( 0 < \lambda < 1 \) and set \( h_\lambda(x, w) := \frac{1}{\lambda} h(\lambda x, \lambda w) \).

From lemma 1, their exists \( \delta > 0 \) such that \(|h_\lambda - q| < \epsilon / 2 \) for \( 0 < \lambda < \delta \). So, for all \( t \in \Theta \), the rank of \( \mathcal{M}_{h_\lambda} \) is maximal for the submanifold \( \{y = h_\lambda(x, w)\} \). So, for all \( t \in \lambda \Theta \), the rank of \( \mathcal{M} \) is maximal for \( M^p \). Then it suffice to take \( \Omega = \{t \in \lambda \Theta; 0 < \lambda < \delta\} \).
From lemma 1, \( \{ \Phi_p^{-1} \}_{p \in \omega} \) is a family of holomorphic maps depending of \( C^{k-1} \) fashion of \( p \in \omega \). So, the coefficients of the Taylor expansion of \( \Phi_p^{-1} \) at 0 also depends of \( C^{k-1} \) fashion of \( p \in \omega \). Without loss of generality, we can assume that \( p_0 = 0 \) and that \( T_M(p_0) = \{ \text{Im} u = 0 \} \). Let \( \Omega \subset S^{d-1} \) be a small neighborhood of a point \( \theta \in S^{n-d} \). The map

\[
F = (F_1, F_2, F_3) \colon \varepsilon, \varepsilon[\times \Omega \times \omega \rightarrow \mathbb{R} \times S^{d-1} \times M
\]

defined by

\[
F_3(\lambda, \theta, p) = \Pi^{-1} \circ \Pi_1 \circ \Phi_p^{-1}(i\lambda \theta, 0)
\]

\[
F_1(\lambda, \theta, p) = \begin{cases} |\Pi_2(\Phi_p^{-1}(i\lambda \theta, 0) - F_3(\lambda, \theta, p))| & \text{if } \lambda > 0 \\ -|\Pi_2(\Phi_p^{-1}(i\lambda \theta, 0) - F_3(\lambda, \theta, p))| & \text{if } \lambda < 0 
\end{cases}
\]

\[
F_2(\lambda, \theta, p) = \begin{cases} \tilde{\Pi}(\Phi_p^{-1}(i\lambda \theta, 0) - F_3(\lambda, \theta, p)) & \text{if } \lambda > 0 \\ -\tilde{\Pi}(\Phi_p^{-1}(i\lambda \theta, 0) - F_3(\lambda, \theta, p)) & \text{if } \lambda < 0 
\end{cases}
\]

for \( \varepsilon > 0 \) small enough and \( \lambda \neq 0 \), extends at \( \lambda = 0 \) as a \( C^1 \) map. Indeed, we have \( \Phi_p^{-1} = L + r \) with \( L \) a linear map and \( r \in \mathcal{O}(\lambda^2) \) when \( \lambda \) tends to 0. As for \( \Phi_p^{-1} \) replaced by \( L, F \) extends at \( \lambda = 0 \) as a \( C^1 \) map, this is also true for \( \Phi_p^{-1} \).

**Lemma 3** For every truncated open cone \( C \subset \mathbb{R}^d \)

\[
\bigcup_{p \in \omega} \Phi_p^{-1}(\tilde{C})
\]

contains a wedge \( \mathcal{W} \) of edge \( M \) at \( p_0 \) where \( \tilde{C} := \{ z = (i\eta, v) \in \mathbb{C}^d \times \mathbb{C}^{n-d} \} \). Moreover, this union form a sheeting of class \( C^{k-1} \) of \( \mathcal{W} \).

**Proof.** Let \( \theta \in S^{d-1} \) fixed and \( \Omega \subset S^{d-1} \) be a small enough neighborhood of \( \theta \) such that \( \delta \Omega \subset C \) for \( \delta > 0 \) small enough. So, in a neighborhood of the point \( (0, \theta, p_0) \), \( F \) is homeomorphic to its image for all \( \theta \in S^{d-1} \). This is obvious because \( \Pi^{-1} \circ \Pi_1 \circ \Phi_p^{-1}(0, \cdot) \) is the identity map of \( \omega \) and \( \tilde{F}(\cdot, p_0) := (F_1(\cdot, p_0), F_2(\cdot, p_0)) \) is a homeomorphism because \( \Phi_{p_0} \) is a conformal map (as \( \Phi_{p_0} \) is a biholomorphism). As the image of \( F(0, \cdot) \) is in \( \{ 0 \} \times S^{d-1} \times M \), \( F([0, \varepsilon[\times \Omega \times \omega) \) contains \( V^+ := V \cap \{ \mathbb{R}^+ \times S^{d-1} \times M \} \) where \( V \) is a neighborhood of \( F(0, \theta, p_0) \), we have that \( \bigcup_{p \in \omega} \Phi_p^{-1}(\tilde{C}) \) contains the set

\[
\mathcal{W} := \{ p + \eta \text{ with } \eta = \lambda \theta \text{ and } (\lambda, \theta, p) \in V^+ \}
\]

which contains a wedge \( \mathcal{W} \) of edge \( M \). \( \square \)

**Corollary 2** There exists a wedge \( \mathcal{W} \) of edge \( M \) at \( p_0 \) such that for all \( K \subset M \) (not necessary closed) of null \( \mathcal{H}^{2n-d-1} \) measure and for every \( z \in \mathcal{W} \) there exists an analytic disc attached to \( M \) of class \( C^k \) passing through \( z \) and which border does not meet \( K \).
Proof. From lemma 3, it suffice to prove that for a fixed point \( p \in \omega \) and for all \( z \in \{A(t,0) : t \in \Omega\} \), there exists an analytic disc attached to \( M^p \), of class \( C^d \), passing through \( z \) and whose border does not meet \( K^p \). Where \( A(t,\zeta) := (G(t,\zeta),W(t,\zeta)) \) is the analytic disc attached to \( M^p \) obtained by the Bishop’s equation with \( (\text{ReG},W) = (0,0) \). Indeed, from \([9]\) \{\( A(t,0) : t \in \Omega \)\} contains an open cone of \( \mathbb{R}^d \).

Let \( \Theta \) and \( \delta \) be as defined here above. Let define the map

\[
\Psi : [0,\delta] \times \Theta \times S^1 \to \mathbb{R}^d \times \mathbb{C}^{n-d}
\]

\[
\Psi(\lambda,\theta,\zeta) = (v(\lambda\theta,0),u(\lambda\theta,\zeta),W(\lambda\theta,\zeta)).
\]

where \( G = u + iv \). From the proposition 2, for all \( \eta \in \{v(t,0); t \in \Omega\} \)

\[
E_\eta := \{(\lambda,\theta) \in [0,\delta] \times \Theta \text{ such that } v(\lambda\theta,0) = \eta\}
\]

is a submanifold of codimension \( d \) of \( [0,\delta] \times \Theta \), i.e. of dimension \( N - d \) and

\[
\Psi_\eta : E_\eta \times S^1 \to \mathbb{R}^d \times \mathbb{C}^{n-d}
\]

\[
\Psi_\eta(\lambda,\theta,\zeta) = (u(\lambda\theta,\zeta),W(\lambda\theta,\zeta))
\]

is a submersion. We remark that the map \( \Psi_\eta(\lambda,\theta,\cdot) \) is the projection of the border of the attached disc on \( \mathbb{R}^d \times \mathbb{C}^{n-d} \). It follows that \( \mathcal{H}^{N-d}(\Psi_\eta^{-1}(\Pi_1(K^p))) = 0 \) and (from \([9]\) 2.10.25) that for \( \mathcal{H}^{N-d} \)-almost all \( (\lambda,\theta) \in E_\eta, \Psi_\eta(\lambda,\theta,S^1) \cap \Pi_1(K^p) = \emptyset \).

So, for such a \((\lambda,\theta)\), the attached disc does not meet \( K \).

\[
\Box
\]

Corollary 3 Let \( K \) be a compact subset of \( M \) of null \( 2n - d - 2 \) dimensional Hausdorff measure. Then there exists an open truncated cone \( C \subset \mathbb{R}^d \) such that for all \( z \in \Phi^p_\eta^{-1}(\{0\} \times C \times \{0\}) \) with \( p \notin K \), there exists an analytic disc attached to \( M \setminus K \) passing through \( z \) and isotopic to \( p \).

Proof. With the same notations of the proof of the previous corollary, from lemma 1 we have that \( \mathcal{H}^{N-1}(\bigcup_{\eta \in \mathbb{R}^d} \Psi_\eta^{-1}(\Pi_1(K^p))) = 0 \). So, \( \bigcup_{\eta \in \mathbb{R}^d} \Psi_\eta^{-1}(\Pi_1(K^p)) \) does not disconnect \( [0,\delta] \times \Theta \). So, for all \( p \in \omega \setminus K \) and \( z \in \{A(t,0), t \in \Omega\} \) there exists an analytic disc attached to \( M^p \setminus K^p \) passing through \( z \) and isotopic to 0 and this implies the corollary because \( \Phi_p \) is a local biholomorphism.

\[
\Box
\]

Proof of theorem 1. From corollary 3 and the continuity principle \([16]\), \( f \) extends holomorphically in a neighborhood of each point of \( \bigcup_{p \in \omega \setminus K} \Phi^p_\eta^{-1}(C) \). Moreover, this extension is univalued. Indeed, the value of the extension is given by the Cauchy formula and is locally constant in the space of parameters \( E_\eta \) of the attached discs. As \( K \) is of null Hausdorff codimension \( 2 \), the set of parameters of which attached discs intersect \( K \) does not disconnect \( E_\eta \). For \( \lambda \) small enough, the situation is close to the quadratic case, so there exists \( \epsilon \) such that \( E^\lambda_\eta = E_\eta \cap \{||\lambda|| < \epsilon\} \) is connected. So the extension given by the Cauchy formula is
univalent in all the connected component of $E^\lambda_\eta$ in $E_\eta$. Doing so for all $\eta$ we obtain the unicity of the extension.

Let $V = \bigcup_{p \in \omega} \Phi_p^{-1}(C)$ and $S = \bigcup_{p \in K} \Phi_p^{-1}(\{0\} \times C \times \{0\})$. From lemma 3, $\mathcal{H}^{2n-2}(S) = 0$ and $f$ extends holomorphically in $V \setminus S$. By Hartog’s theorem, $f$ extends holomorphically in $V$. Lemma 3 proves that $V$ contains a wedge $\mathcal{W}$ of edge $M$ at $p_0$ and independent of $f$.

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