Functional Equations Associated to Collatz-Type Maps on Integer Rings of Algebraic Number Fields

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Abstract

In 1995, Meinardus & Berg presented a reformulation of the Collatz Conjecture in terms of a functional equation in a single complex variable over the open unit disk. This paper generalizes that method to deal with not only a large class of Collatz-type maps defined on the integers, but further generalizations thereof to Collatz-type maps on the rings of integers on an arbitrary algebraic number field.

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1 Introduction

Fix an integer $\varrho \geq 2$, and consider a map $H : \mathbb{Z} \to \mathbb{Z}$ expressible in the form:

$$H(n) = \begin{cases} 
\frac{\alpha_0 n + b_0}{d_0} & \text{if } n = 0 \mod \varrho \\
\frac{\alpha_1 n + b_1}{d_1} & \text{if } n = 1 \mod \varrho \\
\vdots & \vdots \\
\frac{\alpha_{\varrho-1} n + b_{\varrho-1}}{d_{\varrho-1}} & \text{if } n = \varrho - 1 \mod \varrho
\end{cases}, \forall n \in \mathbb{Z} \quad (1)$$

where $\alpha_j$, $b_j$, and $d_j$ are integer constants (with $\alpha_j, d_j \geq 0$ for all $j$) so that the following two properties hold:

I. $\alpha_j, d_j > 0$ and $\gcd(\alpha_j, d_j) = 1$ for all $j \in \{0, \ldots, \varrho - 1\}$.

II. For each $j \in \{0, \ldots, \varrho - 1\}$, $\frac{\alpha_j n + b_j}{d_j}$ is a non-negative integer if and only if $n = j \mod \varrho$.

We write $\mu_j$ to denote the quantity:
\[ \mu_j \overset{\text{def}}{=} \frac{\eta a_j}{d_j}, \quad \forall j \in \{0, \ldots, \varrho - 1\} \] (2)

In particular, since \( a_j \) and \( d_j \) are always co-prime, note that \( d_j \) must be a divisor of \( \varrho \), which shows that \( \mu_j \) is an integer for all \( j \in \{0, \ldots, \varrho - 1\} \). In particular, the condition that, for \( n \in \mathbb{Z} \), \( \frac{a_j n + b_j}{d_j} \) is an integer if and only if \( n \) is congruent to \( j \mod \varrho \) then shows that \( \mu_j \) can never equal \( \varrho \) for any \( j \); if such a \( j \) existed, the co-primality of \( a_j \) and \( d_j \) would force \( a_j = d_j = 1 \), which then means that \( \frac{a_j n + b_j}{d_j} = n + b_j \), which is an integer even when \( n \) is not congruent to \( j \mod \varrho \).

Finally, we then have that:

\[ H(\varrho n + j) = \mu_j n + H(j), \quad \forall j \in \{0, \ldots, \varrho - 1\}, \quad \forall n \in \mathbb{Z} \] (3)

Maps of this type generalize the Collatz map:

\[ C(n) \overset{\text{def}}{=} \begin{cases} \frac{n}{2} & \text{if } n = 0 \mod 2 \\ 3n + 1 & \text{if } n = 1 \mod 2 \end{cases} \] (4)

as well as various generalizations thereof, such as the shortened \( ax + 1 \) maps (where \( a \) is an odd integer \( \geq 3 \)):

\[ T_a(n) \overset{\text{def}}{=} \begin{cases} \frac{n}{2} & \text{if } n = 0 \mod 2 \\ \frac{an + 1}{2} & \text{if } n = 1 \mod 2 \end{cases} \] (5)

(see, for instance, [?], which studies \( T_3 \) and \( T_5 \), \( n \)-furcations (considered by [2]), \((f, g, h)\)-maps (considered by [4]), and many of the myriad examples studied by K.R. Matthews and his colleagues (see [11, 8, 7, 9, 10]).

The subject of this paper, however, will be the generalizations of these maps to the ring of integers \( \mathcal{O}_F \) of an algebraic number field \( F \). To the author’s knowledge, the first such generalization was given by Leigh in 1983 (see Matthews’ slides [10] for more details), in which he formulated an analogue of the shortened Collatz map \( T_3 \) on the ring \( \mathbb{Z} \left[ \sqrt{2} \right] \):

\[ T(\alpha) \overset{\text{def}}{=} \begin{cases} \frac{\alpha}{\sqrt{2}} & \text{if } \alpha = 0 \mod \sqrt{2} \\ \frac{3\alpha + 1}{\sqrt{2}} & \text{if } \alpha = 1 \mod \sqrt{2} \end{cases}, \quad \forall \alpha \in \mathbb{Z} \left[ \sqrt{2} \right] \] (6)

A clearer picture of this map emerges when elements \( \alpha \in \mathbb{Z} \left[ \sqrt{2} \right] \) are represented as \( \alpha = x + y\sqrt{2} \), where \( x, y \in \mathbb{Z} \). Writing the \( \alpha \)s in terms of an integral basis of \( \mathbb{Z} \left[ \sqrt{2} \right] \) in this manner reveals that \( T \) can be viewed as a “two-dimensional” variant of a 2-Hydra map:

\[ T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{cases} \begin{bmatrix} y \\ \frac{x}{2} \end{bmatrix} & \text{if } x = 0 \mod 2 \\ \begin{bmatrix} 3y \\ \frac{3x + 1}{2} \end{bmatrix} & \text{if } x = 1 \mod 2 \end{cases} \] (7)
in the sense that it acts on coordinate pairs \((x, y) \in \mathbb{Z}^2\), rather than individual integers.

The author corresponded with Matthews several years ago, and learned that—to Matthews’ knowledge—outside of the work chronicled in [10], there are no other noteworthy examples of Hydra maps on algebraic number fields (among other settings), nor any studies thereof. The purpose of this paper is to establish a formalism for dealing with these maps, and to link them with a generalization of the technique of Meinardus and Berg, whereupon the study of the dynamics of the maps is reformulated in terms of functional equations satisfied by holomorphic functions \(f : \mathbb{D} \to \mathbb{C}\), where \(\mathbb{D}\) is the open unit disk in \(\mathbb{C}\).

2 Conventions and Multidimensional Notation

For the remainder of the paper, we work with a given number field \(F\) (a finite-degree field-extension of \(\mathbb{Q}\)) of degree \(d\). For brevity, we use the term number ring to refer to the ring of integers of a number field, and, as is traditional in algebraic number theory, we write \(\mathcal{O}_F\) to denote the ring of integers of \(F\) ("\(F\)-integers")—that is, those \(\alpha \in F\) whose minimal polynomial over \(\mathbb{Z}\) has a leading coefficient of 1. The guiding light of our approach is to view \(\mathcal{O}_F\) as an isomorphic copy of the \(d\)-dimensional lattice \(\mathbb{Z}^d\) embedded within \(F\), the field being viewed as a \(d\)-dimensional vector space over \(\mathbb{Q}\). To minimize any cumbersome aspects of notation, we introduce the following conventions:

- **Bold, lower case letters** (ex: \(j, a, n, x, z\), etc.) are reserved to denote tuples of finite length. In computations involving matrices, such tuples will always be treated as column vectors.

- **BOLD, UPPER CASE LETTERS** (ex \(A, D, P\), etc.) are reserved to denote \(d \times d\) matrices. In particular, \(D\) will be used for diagonal matrices, while \(P\) will be used for permutation matrices—matrices (whose entries are 0s and 1s) that give the so-called “defining representation” of the symmetric group \(S\) acting on \(\mathbb{R}^d\).

- Given integers \(d, \varrho \geq 2\), we write \(j \in (\mathbb{Z}/\varrho\mathbb{Z})^d\) to mean that \(j\) is a \(d\)-tuple of integers taken from the set \(\{0, \ldots, \varrho - 1\}\).

- We will utilize the Iverson bracket notation, where, given some statement \(S\), we write \([S]\) to denote the quantity which is 1 when \(S\) is true and 0 when \(S\) is false.

**Definition (Conventions for Multidimensional Algebra):** Let \(d\) be an integer, let \(s \in \mathbb{C}\), and let \(a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d),\) and \(c = (c_1, \ldots, c_d)\) be elements of \(\mathbb{C}^d\). Then, we use the following notational conventions:

I. \(a + b \overset{\text{def}}{=} (a_1 + b_1, \ldots, a_d + b_d)\)

II. \(ab \overset{\text{def}}{=} (a_1 b_1, \ldots, a_d b_d)\)

III. \(\frac{a}{b} \overset{\text{def}}{=} \left(\frac{a_1}{b_1}, \ldots, \frac{a_d}{b_d}\right)\)

IV. \(sa \overset{\text{def}}{=} (sa_1, \ldots, sa_d)\)
V. \( a + s \overset{\text{def}}{=} (a_1 + s, \ldots, a_d + s) \)

VI. \[
\sum_{\mathbf{k} = \mathbf{a}} \mathbf{b} \overset{\text{def}}{=} \sum_{k_1 = a_1}^{b_1} \cdots \sum_{k_d = a_d}^{b_d}
\]

VII. The expression “\( \mathbf{k} \in \{\mathbf{a}, \ldots, \mathbf{b}\} \)” means \( a_j \leq k_j \leq b_j \) for all \( j \in \{1, \ldots, d\} \).

VIII. (Roots of unity): In a minor abuse of notation, we will write \( e(\frac{\mathbf{a} \cdot \mathbf{k}}{\mathbf{b}}) \) to denote \( e(\frac{\mathbf{a} \cdot \mathbf{k}}{\mathbf{b}}) = e^{2\pi i (\frac{\mathbf{a} \cdot \mathbf{k}}{\mathbf{b}})} \).

IX. (Congruences): For any real number \( q \), and any \( \mathbf{a}, \mathbf{b} \in \mathbb{R} \):

\[
\mathbf{a} \equiv \mathbf{b} \iff a_\ell \equiv b_\ell \text{ mod } q, \forall \ell \in \{1, \ldots, d\}
\]

Additionally, for brevity, we will often write \( a = b \mod q \) as simply \( a \equiv b \).

X. (Dot Product):

\[
\langle \mathbf{a} | \mathbf{b} \rangle \overset{\text{def}}{=} \sum_{\ell = 1}^{d} a_\ell b_\ell
\]

XI. (Component-wise Exponentiation): \( \mathbf{a}^\mathbf{b} \overset{\text{def}}{=} \left( a_1^{b_1}, \ldots, a_d^{b_d} \right) \)

XII. We write \( \Pi(\mathbf{a}) \overset{\text{def}}{=} \prod_{j=1}^{d} a_j \).

Remark: In this section, our tuples will be either \( d \)-tuples of complex variables (\( \mathbf{z} = (z_1, \ldots, z_d) \)) or \( d \)-tuples of rational constants. Our matrices, on the other hand, will always have rational constants as entries. As such, we can view our matrices as linear operators on the Hilbert space \( \mathbb{R}^d \). Since the vector dot product \( \langle \cdot | \cdot \rangle \) is the inner product associated to this Hilbert space, the transposes of our matrices will coincide with their adjoints. That is, whenever \( \mathbf{A} \) has real entries, the following identity holds:

\[
\langle \mathbf{Aa} | \mathbf{b} \rangle = \langle \mathbf{a} | \mathbf{A}^T \mathbf{b} \rangle, \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^d
\]

### 3 Definitions of Hydra Maps on Number Rings

Fix an integral basis \( \mathcal{B} = \{\gamma_1, \ldots, \gamma_d\} \subseteq \mathcal{O}_\mathbb{F} \) for \( \mathbb{F} \). Then, we shall represent elements of \( \mathcal{O}_\mathbb{F} \) as elements of \( \mathbb{Z}^d \) by way of the identification:

\[
z \in \mathcal{O}_\mathbb{F} \iff (m_1, \ldots, m_d) \in \mathbb{Z}^d
\]

where \( m_1, \ldots, m_d \) are the unique integers so that \( z = \sum_{n=1}^{d} m_n \gamma_n \).

At this point, the reader should prepare themselves for an onslaught of definitions. This is done not for its own sake, or because the author pleasures in it, but because it is needed in order to give the underlying concepts rigorous footing. That being said, the definitions to come will hopefully be more easily digested if we first spend a moment discussing exactly what it is that these definitions have set out to generalize.
The idea is as follows: given an ideal \( \mathfrak{I} \subseteq \mathcal{O}_F \), we would like to choose a basis \( \mathcal{B} \) so that, given any \( d \)-tuple \( m \in \mathbb{Z}^d \) representing some element of \( \mathcal{O}_F \), we can determine the equivalence class in \( \mathcal{O}_F/\mathfrak{I} \) to which \( m \) belongs by computing the residues of \( m \)'s components modulo some integer \( \varrho \). For clarity, here are examples in one and two dimensions.

- Let \( d = 1 \), \( F = \mathbb{Q} \), and \( \mathcal{O}_F = \mathbb{Z} \), consider the ideal \( 3\mathbb{Z} \), and the quotient ring \( \mathbb{Z}/3\mathbb{Z} \). Here, we choose the basis \( \mathcal{B} = \{ 1 \} \) for \( \mathbb{Z} \). Here, each of the three equivalence classes of \( \mathbb{Z}/3\mathbb{Z} \) can be identified with one of the integers 0, 1, 2. Moreover, given any integer \( m \in \mathbb{Z} \), we can determine which of the three equivalence classes \( m \) belongs to simply by computing the residue of \( m \) mod 3; the value of that residue is then exactly the equivalence class of \( \mathbb{Z}/3\mathbb{Z} \) to which \( m \) belongs.

- Let \( d = 2 \), \( F = \mathbb{Q}(\sqrt{2}) \), \( \mathcal{O}_F = \mathbb{Z}[\sqrt{2}] \), consider the ideal \( \mathfrak{I} = \langle \sqrt{2} \rangle \) generated by \( \sqrt{2} \) along with the quotient ring \( \mathbb{Z}[\sqrt{2}]/\langle \sqrt{2} \rangle \). Here, we choose the basis \( \{ 1, \sqrt{2} \} \) for \( \mathbb{Z}[\sqrt{2}] \). Since \( \langle 2 \rangle \) is a proper ideal of \( \langle \sqrt{2} \rangle \), it follows that everything in \( \mathbb{Z}[\sqrt{2}] \) which is congruent to 0 mod 2 will also be congruent to 0 mod \( \sqrt{2} \). Additionally, since \( 2 \in \mathfrak{I} \), it follows that \( a + b\sqrt{2} \equiv a \) for all \( a, b \in \mathbb{Z} \). If \( a \) is even, then \( a \equiv 0 \). If \( a \) is odd, then \( a \equiv 1 \). Since \( \langle 2 \rangle \) is an additive subgroup of \( \langle \sqrt{2} \rangle \), the additive quotient group \( \mathbb{Z}/\sqrt{2}\mathbb{Z} \) will be a subgroup of \( \mathbb{Z}/2\mathbb{Z} \), and thus, \( a \equiv 1 \) will force \( a \equiv 1 \). Consequently, the equivalence class in \( \mathbb{Z}[\sqrt{2}]/\langle \sqrt{2} \rangle \) to which any given number \( a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}] \) belongs (the value of \( a + b\sqrt{2} \) “mod \( \sqrt{2} \)” is uniquely determined by the value of \( a \) mod 2. In particular, every \( a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}] \) is congruent to either 0 or 1 mod \( \sqrt{2} \), and the particular residue class to which \( a + b\sqrt{2} \) belongs is completely determined by the value of \( a \) mod 2, with:

\[
a + b\sqrt{2} \equiv 0 \Leftrightarrow a \equiv 0
\]

\[
a + b\sqrt{2} \equiv 1 \Leftrightarrow a \equiv 1
\]

As such, if we write \( \mathbb{Z}[\sqrt{2}] \) in \( \{ 1, \sqrt{2} \} \)-coordinates, the element \( m = (a, b) \in \mathbb{Z}^2 \) corresponding to the element \( a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}] \) represents a number congruent to 0 (resp., 1) mod \( \sqrt{2} \) if and only if the first component \( (a) \) of \( m \) is congruent to 0 (resp., 1) mod 2.

In general, since every ideal in a number ring is principal, given an ideal \( \mathfrak{I} = \langle \alpha \rangle \in \mathcal{O}_F \) and a basis \( \mathcal{B} \) of \( \mathcal{O}_F \), the idea is to convert the statement “\( z \in \mathcal{O}_F \) is congruent to \( \beta \mod \alpha \)” to an equivalent statement of the form “the \( \iota_1 \)th, \( \iota_2 \)nd, ..., \( \iota_d \)th components of \( m \in \mathbb{Z}^d \) satisfy the congruences \( m_{\iota_1} \equiv j_1, \ldots, m_{\iota_d} \equiv j_d \)” for some \( N \in \mathbb{N} \), some \( d' \in \{ 1, \ldots, d \} \), and some indices \( \iota_1, \ldots, \iota_d \in \{ 1, \ldots, d \} \).

**Definition:** Fix \( F \) and a basis \( \mathcal{B} \) of \( \mathcal{O}_F \) over \( \mathbb{Z} \).

1. For any \( m, n \in \mathbb{N} \), \( \mathfrak{J}_m(n) \) denotes the set of all \( m \)-tuples \( j = (j_1, \ldots, j_m) \) whose entries are integers in the set \( \{ 0, \ldots, n - 1 \} \).
II. We write \( \varphi_B : \mathbb{Q}^d \to \mathbb{F} \) be the linear-space isomorphism that sends each \( m \in \mathbb{Q}^d \) to the unique element of \( \mathbb{F} \) whose representation in \( B \)-coordinates is \( m \), and which satisfies the property that the restriction \( \varphi_B |_{\mathbb{Z}^d} \) of \( \varphi_B \) to \( \mathbb{Z}^d \) is an isomorphism of the groups \((\mathbb{Z}^d,+),(\mathbb{O}_B,+))\). We write \( \varphi_B^{-1} \) to denote the inverse of \( \varphi_B \); \( \varphi_B^{-1} \) outputs the \( d \)-tuple \( B \)-coordinate representation of the inputted \( z \in \mathbb{F} \).

III. Let \( N \in \mathbb{N}_1 \). A \( B \)-address \( \text{mod}(ulu) N \) of length \( d' \) is a pair \( A = (I,J) \), where:

- \( I \) (the indices of the address) is a non-empty subset of \( \{1,\ldots,d'\} \), enumerated in increasing order as \( I = \{i_1,\ldots,i_{d'}\} \).
- \( J \) (the residues of the address) is a sequence of \( d' \) elements of \( \{0,\ldots,N-1\} \), written \( J = \{j_1,\ldots,j_{d'}\} \).

IV. Given a \( B \)-address \( A = (I,J) \mod N \) of length \( d' \in \{1,\ldots,d\} \), a \( d' \)-tuple \( m = (m_1,\ldots,m_{d'}) \in \mathbb{Z}^{d'} \) is said to live at \( A \) (denoted \( m \prec A \)) whenever the components of \( m \) satisfy the following system of congruences:

\[
m_{i_1} \equiv j_1 \mod N
\]

\[
: 
\]

\[
m_{i_{d'}} \equiv j_{d'} \mod N
\]

V. Let \( \mathcal{I} \) be an ideal in \( \mathcal{O}_B \), let \( N \in \mathbb{N}_1 \), let \( L = |\mathcal{O}_B/\mathcal{I}| \), and list the equivalence classes of \( \mathcal{O}_B/\mathcal{I} \) as \( \mathcal{I}_0,\ldots,\mathcal{I}_{L-1} \), where \( \mathcal{I}_0 \) is the equivalence class of \( \mathcal{I} \) itself. We say that \( \mathcal{I} \) is \( B \)-representable \( \text{mod} N \) if there is a collection of \( L \) \( B \)-addresses \( A_0,\ldots,A_{L-1} \) (called the \( B \)-addresses of \( \mathcal{I} \text{ mod } N \)) so that for every \( \ell \in \{0,\ldots,L-1\} \) and every \( m \in \mathbb{Z}^{d'} \):

\[
\varphi_B(m) \in \mathcal{I}_\ell \iff m \prec A_\ell
\]

that is, \( m \) represents an element of \( \mathcal{I}_\ell \) if and only if \( m \) lives in \( A_\ell \).

More generally, we say that \( \mathcal{I} \) is \( B \)-representable if there is an \( N \) so that \( \mathcal{I} \) is \( B \)-representable \( \text{mod } N \).

VI. Let \( \mathcal{I} \) be an ideal in \( \mathcal{O}_B \). Then, the \( B \)-modulus of \( \mathcal{I} \) is the smallest positive integer \( N \) so that \( \mathcal{I} \) is \( B \)-representable \( \text{mod } N \).

VII. Let \( \mathcal{I} \) be a \( B \)-representable ideal in \( \mathcal{O}_B \) of \( B \)-modulus \( \varrho \), and let \( A \) be one of the \( B \)-addresses of \( \mathcal{I} \). Observe that there is then a unique subset \( S \) of \((\mathbb{Z}/\varrho\mathbb{Z})^d \) (the set of all \( d \)-tuples of integers \( \{0,\ldots,\varrho-1\} \)) so that any given \( m \in \mathbb{Z}^d \) lives in \( A \) if and only if \( m \) is congruent mod \( \varrho \) to one of the elements of \( S \); that is, if and only if \( m \equiv j \mod \varrho \) for some \( j \in S \); that is, the congruences:

\[
m_1 \equiv j_1 \mod \varrho
\]

\[
: 
\]

\[
m_d \equiv j_d \mod \varrho
\]

hold true. The elements of \( S \) are called the tenants of \( \mathcal{I} \text{ in } A \); \( S \) is the set of tenants of \( \mathcal{I} \text{ in } A \).
denote the equivalence classes of \( O \) let the form:

\[ \mathbb{Z} \]

the set of non-positive integers is the half-lattice \( I \) of \( \{ \ldots, -1, 0 \} \), where the \( \alpha_\ell \) lives in \( I \) contained in \( \mathbb{Z} \). There are \( 49 \) distinct 4-tuples \( j \in J_4(7) \) so that \( m \equiv j \) implies \( m \prec A \).

**Definition (Oriented Half-Lattices):** Fix \( F \) (with \( [F : Q] = d \)) and \( B = \{ \gamma_1, \ldots, \gamma_d \} \).

I. We write \( O^+_{F,B} \) to denote the \( B \)-oriented half-lattice of \( F \)-integers: defined by:

\[
O^+_{F,B} := \left\{ \sum_{n=1}^d m_n \gamma_n : m_1, \ldots, m_d \in \mathbb{N}_0 \right\}
\]

(8)

In this notation, the set of non-negative integers is the half-lattice \( O^+_{Q,(1)} \), while the set of non-positive integers is the half-lattice \( O^+_{Q,(1)} \). where changing the basis (in this case, its orientation) cause a corresponding change in the location of the half-lattice within \( Z = O_Q \).

II. We write \( \mathbb{Z}^+_{F,B} \) to denote the isomorphic copy of \( O^+_{F,B} \) in \( \mathbb{Z}^d \); that is, \( \mathbb{Z}^+_{F,B} \) is the additive monoid of \( d \)-tuple \( B \)-coordinates of elements of \( O^+_{F,B} \).

**Definition (\( J \)-Hydra map):** Fix \( F, B, \) and a \( B \)-representable ideal \( J \subseteq O_F \), let \( \varphi \) denote the \( B \)-modulus of \( J \), and write \( J_0, \ldots, J_{L-1} \) (where \( L \) \( \varphi \)) to denote the equivalence classes of \( O_F / J \), with \( J_0 \) denoting the equivalence class of \( J \) itself.

Then, an \( J \)-Hydra map on \( O^+_{F,B} \) is a surjective map \( H : O^+_{F,B} \to O^+_{F,B} \) of the form:

\[
H(z) = \begin{cases} 
\frac{\alpha_0 z + \beta_0}{\delta_0} & \text{if } z \in J_0 \\
\vdots & \\
\frac{\alpha_{L-1} z + \beta_{L-1}}{\delta_{L-1}} & \text{if } z \in J_{L-1}
\end{cases}
\]

(9)

where the \( \alpha_\ell, \beta_\ell, \) and \( \delta_\ell \) are \( F \)-integers satisfying the following conditions:

i. \( \alpha_\ell, \delta_\ell \neq 0 \) for all \( \ell \in \{0, \ldots, L-1\} \).

ii. \( \gcd(\alpha_\ell, \delta_\ell) = 1 \) for all \( \ell \in \{0, \ldots, L-1\} \).

iii. For all \( \ell \in \{0, \ldots, L-1\} \), the quantity \( \frac{\alpha_\ell z + \beta_\ell}{\delta_\ell} \) is an element of \( O^+_{F,B} \) for all \( z \in O^+_{F,B} \) satisfying \( z \in J_\ell \).

iv. For all \( \ell \in \{0, \ldots, L-1\} \), the the matrix representation in \( B \)-coordinates on \( \mathbb{Q}^d \) of the “multiplication by \( \frac{\alpha_\ell}{\delta_\ell} \)” map on \( F \) is of the form \( DP \), where \( D, P \in \text{GL}_d(\mathbb{Q}) \) are, respectively, a diagonal matrix such that every entry on the diagonal of \( gD \) is a positive integer, and where \( P \) is a permutation matrix (that is, the matrix representation of a permutation of \( d \) objects).

v. For all \( \ell \in \{0, \ldots, L-1\} \), the ideal \( \langle \delta_\ell \rangle_{O_F} \) in \( O_F \) generated by \( \delta_\ell \) is contained in \( J \).

**Remark:** To streamline the discussion, we will speak of the “modulus” of the \( J \)-Hydra map to denote the \( B \)-modulus of \( J \). We shall use of the symbol \( \varphi \) to
denote this modulus; \( \varrho \), wherever it appears, will always denote the \( \mathcal{B} \)-modulus of the underlying ideal \( \mathfrak{I} \).

**Proposition (Lattice Analogues):** Let \( H : \mathcal{O}_{\mathfrak{I}, \mathcal{B}}^+ \to \mathcal{O}_{\mathfrak{I}, \mathcal{B}}^+ \) be an \( \mathcal{I} \)-Hydra map, where \( \mathcal{I} \) is of \( \mathcal{B} \)-modulus \( \varrho \geq 2 \). Then, there are unique \( d \times d \) matrices \( \{ A_j \}_j \in (\mathbb{Z}/\varrho \mathbb{Z})^d \subseteq \text{GL}_d(\mathbb{Q}) \) and \( d \times 1 \) column vectors \( \{ b_j \}_j \in (\mathbb{Z}/\varrho \mathbb{Z})^d \subseteq \mathbb{Q}^d \) so that the map \( H : \mathbb{Z}_{\mathfrak{I}, \mathcal{B}}^+ \to \mathbb{Z}_{\mathfrak{I}, \mathcal{B}}^+ \) defined by:

\[
H(m) = \sum_{j \in (\mathbb{Z}/\varrho \mathbb{Z})^d} \left[ m \equiv j \right] (A_j m + b_j), \quad \forall m \in \mathbb{Z}_{\mathfrak{I}, \mathcal{B}}^+
\]  

(10)

is \( \varphi_{\mathcal{B}} \)-conjugate of \( H \); that is \( H = \varphi_{\mathcal{B}} \circ H \circ \varphi_{\mathcal{B}}^{-1} \). In particular, for all \( j \in (\mathbb{Z}/\varrho \mathbb{Z})^d \), the \( d \)-tuple \( A_j m + b_j \) will be an element of \( \mathbb{Z}_{\mathfrak{I}, \mathcal{B}}^+ \) for all \( m \in \mathbb{Z}_{\mathfrak{I}, \mathcal{B}}^+ \) for which \( m \equiv j \).

For those who enjoy commutative diagrams, we have:

\[
\begin{array}{ccc}
\varphi_{\mathcal{B}}^{-1} & \downarrow & \varphi_{\mathcal{B}} \\
\mathcal{O}_{\mathfrak{I}, \mathcal{B}}^+ & \rightarrow & \mathcal{O}_{\mathfrak{I}, \mathcal{B}}^+ \\
\mathbb{Z}_{\mathfrak{I}, \mathcal{B}}^+ & \rightarrow & \mathbb{Z}_{\mathfrak{I}, \mathcal{B}}^+
\end{array}
\]

We call \( H \) the **lattice analogue** of \( H \). Likewise, we call \( H \) the **field analogue** of \( H \).

Proof: (Sketch) Let everything be as described above. Fix \( z = \sum_{n=1}^{d} c_n \gamma_n \in \mathcal{O}_{\mathfrak{I}, \mathcal{B}}^+ \), where the \( c_n \)'s are non-negative integers. Letting \( \mathfrak{I}_f \) denote the unique equivalence class of \( \mathfrak{I} \) in \( \mathcal{O}_{\mathfrak{F}} \) to which \( z \) belongs, it follows by definition of \( H \) that:

\[
H(z) = \frac{\alpha \ell z + \beta \ell}{\delta \ell}
\]

Now, consider \( \mathbb{F} \) as a \( d \)-dimensional linear space over \( \mathbb{Q} \), equipped with the coordinate system given by the basis \( \mathcal{B} = \{ \gamma_1, \ldots, \gamma_d \} \). In these coordinates, the “multiplication by \( \frac{\varphi_{\mathcal{B}}}{\delta \ell} \)” map on \( \mathbb{F} \) can be uniquely represented as left-multiplication by some \( A \in \text{GL}_d(\mathbb{Q}) \) with rational entries. As such, letting \( \mathbf{m} \) denote the coordinate \( d \)-tuple representing \( z \) (that is, \( \mathbf{m} = \varphi_{\mathcal{B}}^{-1}(z) \)), it follows that:

\[
\varphi_{\mathcal{B}}^{-1} \left( \frac{\alpha \ell z}{\delta \ell} \right) = A \mathbf{m}
\]

Letting \( \mathbf{b} = \varphi_{\mathcal{B}}^{-1} \left( \frac{\beta \ell}{\delta \ell} \right) \) be unique the coordinate \( d \)-tuple representing \( \frac{\beta \ell}{\delta \ell} \), we then have that:

\[
\varphi_{\mathcal{B}}^{-1}(H(z)) = \varphi_{\mathcal{B}}^{-1} \left( \frac{\alpha \ell z + \beta \ell}{\delta \ell} \right) = \varphi_{\mathcal{B}}^{-1} \left( \frac{\alpha \ell z}{\delta \ell} \right) + \varphi_{\mathcal{B}}^{-1} \left( \frac{\beta \ell}{\delta \ell} \right) = A \mathbf{m} + \mathbf{b}
\]

where we used the fact that, as defined, \( \mathcal{O}_{\mathcal{B}} \) is an isomorphism of the linear spaces \( \mathbb{F} \) and \( \mathbb{Q}^d \).

As such, for each \( j \in (\mathbb{Z}/\varrho \mathbb{Z})^d \), let \( U_j \) denote \( \varphi_{\mathcal{B}} \left( \varrho \mathbb{Z}^d + j \right) \). Since the sets \( \{ \varrho \mathbb{Z}^d + j \}_j \) form a partition of \( \mathbb{Z}^d \)—and hence, of \( \mathbb{Z}_{\mathfrak{I}, \mathcal{B}}^+ \)—the \( U_j \)'s form
a partition of $O_{F,B}^+$. By construction, for each $j$, the set $U_j$ is contained in a unique equivalence class $I_j$ of $I$ in $O_{F,B}^+$. Consequently, by the argument given in the previous paragraph, for each $j \in (\mathbb{Z}/\mathbb{Z})^d$, there is a unique $A_j \in \text{GL}_d(\mathbb{Q})$ and a unique $d \times 1$ column vector $b_j$ with rational entries so that:

$$\varphi_B^{-1}(H(z)) = A_j \varphi_B^{-1}(z) + b_j, \forall z \in U_j$$

Multiplying by the Iverson bracket $[\varphi_B^{-1}(z) \equiv j]$ and summing over all $j \in (\mathbb{Z}/\mathbb{Z})^d$ gives:

$$\sum_{j \in (\mathbb{Z}/\mathbb{Z})^d} \left[\varphi_B^{-1}(z) \equiv j\right] (A_j \varphi_B^{-1}(z) + b_j) = \sum_{j \in (\mathbb{Z}/\mathbb{Z})^d} \left[\varphi_B^{-1}(z) \equiv j\right] \varphi_B^{-1}(H(z))$$

$$= \sum_{j \in (\mathbb{Z}/\mathbb{Z})^d} [z \in U_j] \varphi_B^{-1}(H(z))$$

$$\left(\{z \in U_j\} \in \{0, 1\}; \forall z \in F\right); \varphi_B^{-1} \left(\sum_{j \in (\mathbb{Z}/\mathbb{Z})^d} [z \in U_j] H(z)\right)$$

Hence:

$$\varphi_B^{-1}(H(z)) = \sum_{j \in (\mathbb{Z}/\mathbb{Z})^d} \left[\varphi_B^{-1}(z) \equiv j\right] (A_j \varphi_B^{-1}(z) + b_j), \forall z \in O_{F,B}^+$$

Replacing $z$ with $\varphi_B(m)$ (where $m \in Z_{F,B}^+$) gives:

$$\left(\varphi_B^{-1} \circ H \circ \varphi_B\right)(m) = \sum_{j \in (\mathbb{Z}/\mathbb{Z})^d} \left[ \bar{m} \equiv j \right] (A_j m + b_j), \forall m \in Z_{F,B}^+$$

as desired.

Q.E.D.

**Definition** (Multidimensional Hydra maps): An $3$-Hydra map on $Z_{F,B}^+$ (or, “on $Z^d$”, or, merely, “a $d$-dimensional Hydra map”) is a map $H : Z_{F,B}^+ \rightarrow Z_{F,B}^+$ of the form:

$$H(m) = \sum_{j \in (\mathbb{Z}/\mathbb{Z})^d} \left[ \bar{m} \equiv j \right] (A_j m + b_j), \forall m \in Z_{F,B}^+ \quad (11)$$

where $\{A_j\}_{j \in (\mathbb{Z}/\mathbb{Z})^d} \subseteq \text{GL}_d(\mathbb{Q})$ and $\{b_j\}_{j \in (\mathbb{Z}/\mathbb{Z})^d} \subseteq \mathbb{Q}^d$ are such that $H$ is the lattice analogue of an $3$-Hydra map on $O_{F,B}^+$. Since $\varphi_B$ is a ring isomorphism, there is a bijective correspondence between the orbit classes of an $3$-Hydra map on $O_{F,B}^+$ and those of its lattice analogue.
on $\mathbb{Z}_R^+$. Using an approach in the same spirit as the one given here, one can almost certainly construct lattice analogues for Hydra-like maps on an even more general type of space than number rings. All that is needed is a discrete, commutative, finitely-generated unital principal-ideal ring; it might also be interesting to see what happens when the ring is merely countably generated.

### 3.0.1 Some Examples

Both of these are taken from ([10]). Leigh (1983) considered the map $H: \mathbb{Z} [\sqrt{2}] \rightarrow \mathbb{Z} [\sqrt{2}]$ defined by:

$$H(z) = \begin{cases} \frac{z}{\sqrt{2}} & \text{if } z = 0 \mod \sqrt{2} \\ \frac{3z + 1}{\sqrt{2}} & \text{if } z = 1 \mod \sqrt{2} \end{cases}$$

Here, $\gamma_1 = 1$ and $\gamma_2 = \sqrt{2}$, $\mathcal{J} = \langle \sqrt{2} \rangle$. We covered this case in an earlier example; $\mathcal{I}$ is $\{1, \sqrt{2}\}$-representable, with modulus $\rho = 2$. That is, any $z = a + b\sqrt{2} \in \mathbb{Z} [\sqrt{2}]$ is congruent to either $0 \mod \sqrt{2}$ (which occurs if and only if $a \equiv 0$) or $1 \mod \sqrt{2}$ (which occurs if and only if $a \equiv 1$). Thus, we can write:

$$H(a + b\sqrt{2}) = \begin{cases} a + b\sqrt{2} & \text{if } a = 0 \mod 2 \\ \frac{a + b\sqrt{2} + 1}{\sqrt{2}} & \text{if } a = 1 \mod 2 \end{cases}$$

So, writing $\mathbf{m}$ to denote the element $(m_1, m_2) \in \mathbb{Z}^2$, we have that the lattice analogue of $H$ is the map:

$$H \left( \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \right) = \begin{cases} \begin{bmatrix} m_2 \\ \frac{m_2}{\sqrt{2}} \end{bmatrix} & \text{if } m_1 = 0 \mod 2 \\ \begin{bmatrix} 3m_2 \\ \frac{3m_2 + 1}{\sqrt{2}} \end{bmatrix} & \text{if } m_1 = 1 \mod 2 \end{cases}$$

and so:

$$H(\mathbf{m}) = \mathbf{m} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left[ \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \mathbf{m} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right] \left( \begin{bmatrix} 0 & 3 \\ \frac{1}{2} & 0 \end{bmatrix} \mathbf{m} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$
Note that both \( A_{(0,0)} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \) and \( A_{(1,0)} = \begin{bmatrix} 0 & 3 \\ \frac{3}{2} & 0 \end{bmatrix} \) are of the form \( DP \) (product of a diagonal matrix and a permutation):

\[
\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 0 & 3 \\ \frac{3}{2} & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}
\]

A second example is the map \( H : \mathbb{Z} \left[ \sqrt{3} \right] \to \mathbb{Z} \left[ \sqrt{3} \right] \) defined by:

\[
H(z) \overset{\text{def}}{=} \begin{cases} 
\frac{\sqrt{3}}{3} & \text{if } z = 0 \mod \sqrt{3} \\
\frac{\sqrt{3}}{3} & \text{if } z = 1 \mod \sqrt{3} \\
\frac{4z+1}{3\sqrt{3}} & \text{if } z = 2 \mod \sqrt{3}
\end{cases}
\]

Here \( \gamma_1 = 1 \) and \( \gamma_2 = \sqrt{3} \), \( \mathfrak{I} = \langle \sqrt{3} \rangle \); \( \mathfrak{I} \) is \( \{1, \sqrt{3}\}\)-representable, with modulus \( q = 3 \). Since:

\[
H \left( a + b\sqrt{3} \right) = \begin{cases} 
b + \frac{a}{3}\sqrt{3} & \text{if } a = 0 \mod 3 \\
b + \frac{a+1}{3}\sqrt{3} & \text{if } a = 1 \mod 3 \\
4b + \frac{4a+1}{3}\sqrt{3} & \text{if } a = 2 \mod 3
\end{cases}
\]

we have:

\[
H \left( \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \right) = \begin{cases} 
\begin{bmatrix} m_2 \\ \frac{m_1}{3} \end{bmatrix} & \text{if } m_1 = 0 \mod 3 \\
\begin{bmatrix} m_2 \\ \frac{m_1-1}{3} \end{bmatrix} & \text{if } m_1 = 1 \mod 3 \\
\begin{bmatrix} 4m_2 \\ \frac{4m_1+1}{3} \end{bmatrix} & \text{if } m_1 = 2 \mod 3
\end{cases}
\]

\[
= \begin{cases} 
\begin{bmatrix} 0 & 1 \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} & \text{if } m_1 = 0 \mod 3 \\
\begin{bmatrix} 0 & 1 \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix} & \text{if } m_1 = 1 \mod 3 \\
\begin{bmatrix} 0 & 4 \\ \frac{4}{3} & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \begin{bmatrix} 1/3 \end{bmatrix} & \text{if } m_1 = 2 \mod 3
\end{cases}
\]

and so:

\[
H \left( \mathbf{m} \right) = \left[ m \equiv \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right] \left[ \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & 0 \end{bmatrix} \right] \mathbf{m} + \left[ m \equiv \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right] \left( \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & 0 \end{bmatrix} \mathbf{m} + \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix} \right)
\]

\[
+ \left[ m \equiv \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right] \left( \begin{bmatrix} 0 & 4 \\ \frac{4}{3} & 0 \end{bmatrix} \mathbf{m} + \begin{bmatrix} 1/3 \end{bmatrix} \right)
\]
where, in all three branches, the permutation is \[
\begin{bmatrix}
0 & 1 \\
1 & 0 
\end{bmatrix}.
\]

4 Permutation Operators

4.1 The 1-Dimensional Case

Although the author independently discovered the idea of studying the Collatz map in the context of functional equations on the space of holomorphic functions on the open unit disk \(\mathbb{D}\), this approach was first noted by Meinardus & Berg (\[1\]), who noted that the Collatz Conjecture\(^1\) can be reformulated in terms of the solution set in \(\mathcal{A}(\mathbb{D})\) of a certain functional equation. Specifically:

4.1.1 - Functional Equation Collatz Conjecture: The following statements are equivalent:

I. Every positive integer is eventually iterated to 1 by \(C\) (or, equivalently, by \(T_3\))

II. The subspace of \(\mathcal{A}(\mathbb{D})\) consisting of all functions \(f(z)\) satisfying the functional equation:

\[
f(z) = f(z^2) + \frac{z^{-1/3}}{3} \sum_{k=0}^{2} \xi_3^{-2k} f\left(\xi_3^k z^{2/3}\right), \quad \forall |z| < 1 \quad (12)
\]

is spanned by the set \(\{1, \frac{z}{1-z}\}\); here, \(\xi_3 \overset{\text{def}}{=} e^{2\pi i/3}\).

Whereas Meinardus & Berg viewed their functional equations as stand-alone objects, this paper adopts the approach of interpreting functional equations such as (12) as equations for the fixed points of a linear operator on \(\mathcal{A}(\mathbb{D})\), the space of all holomorphic functions \(f: \mathbb{D} \to \mathbb{C}\). To that end, let \(H\) be a \(\varphi\)-Hydra map. We shall write \(N_0\) to denote the set of all integers \(\geq 0\).

4.1.2 - Definition (Permutation operators): We write \(Q_H: \mathcal{A}(\mathbb{D}) \to \mathcal{A}(\mathbb{D})\) to denote the linear operator acting by way of the formula:

\[
Q_H \left\{ \sum_{n=0}^{\infty} c_n z^n \right\} \overset{\text{def}}{=} \sum_{n=0}^{\infty} c_{H(n)} z^n \quad (13)
\]

We call \(Q_H\) the permutation operator induced by \(H\), on account of their re-arrangement of the positions of the coefficients in a power series.

4.1.3 - Definition (Set-series): For any set \(V \subseteq N_0\), we write:

\[
\varsigma_V(z) \overset{\text{def}}{=} \sum_{v \in V} z^v = \sum_{n=0}^{\infty} 1_V(n) z^n \quad (14)
\]

where \(1_V(n)\) is 1 when \(n \in V\) and is 0 otherwise. We call \(\varsigma_V(z)\) \((\varsigma)\) the set-series of \(V\).

\(^1\)Every positive integer is eventually iterated to 1 by the map \(C\).
4.1.4 - Proposition (Action of $Q_H$ on set-series): For any $V \subseteq \mathbb{N}_0$, we have:

$$Q_H \{\varsigma_V\} (z) = \varsigma_{H^{-1}(V)} (z)$$

where $H^{-1}(V)$ is the pre-image of $V$ under $H$.

Proof:

$$Q_H \{\varsigma_V\} (z) = \sum_{n=0}^{\infty} 1_V (H(n)) z^n = \sum_{n=0}^{\infty} 1_{H^{-1}(V)} (n) z^n = \varsigma_{H^{-1}(V)} (z)$$

Q.E.D.

4.1.5 - Theorem (Characterization of Ker $(Q_H - 1)$): Let $H : \mathbb{N}_0 \to \mathbb{N}_0$ be any surjective map, and enumerate the at most countably many distinct irreducible orbit classes of $H$ in $\mathbb{N}_0$ as $V_0, V_1, \ldots$. Then, every element of Ker $(Q_H - 1)$ is of the form:

$$\sum_{n=0}^{\infty} c_n \varsigma_{V_n} (z)$$

for some complex constants $c_0, c_1, \ldots$.

Proof: A set $V \subseteq \mathbb{N}_0$ is an orbit class of $H$ precisely when $H^{-1}(V) = V$. Since $Q_H \{\varsigma_V\} = \varsigma_{H^{-1}(V)}$, it then follows that $\varsigma_{V_n}$ is fixed by $Q_H$ for all $n$. For the other direction, let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an element of $A(\mathbb{D})$ which is fixed by $Q_H$. Then, let $C$ denote the set of values attained by the $c_n$s, and, for each $c \in C$, let $N_c$ denote the set of all $n \in \mathbb{N}_0$ for which $c_n = c$. Then, we can write:

$$f(z) = \sum_{c \in C} c \varsigma_{N_c} (z)$$

and hence:

$$\sum_{c \in C} c \varsigma_{H^{-1}(N_c)} (z) = Q_H \{f\} (z) = f(z) = \sum_{c \in C} c \varsigma_{N_c} (z)$$

which forces $H^{-1}(N_c) = N_c$, which shows that, for each $c$, $N_c$ is an orbit class of $H$, and is hence either an irreducible orbit class (one of the $V_n$s), or is the union of irreducible orbit classes $V_{n_1}, V_{n_2}, \ldots$, in which case:

$$\varsigma_{N_c} (z) = \bigcup_k V_{n_k} (z) = \sum_k \varsigma_{V_{n_k}} (z)$$

Regardless, this shows that $f$ is then of the form (16).

Q.E.D.

In light of this relationship between Ker $(Q_H - 1)$ and the orbit classes of $H$ in $\mathbb{N}_0$, it is clear that the task of characterizing the orbit classes of $H$ is then equivalent to solving the functional equation:

$$f(z) = Q_H \{f\} (z)$$

for $f \in A(\mathbb{D})$. Since our $H$ is a $\varrho$-Hydra map, one can compute $Q_H \{f\} (z)$ in closed form:
4.1.6 - Proposition (Formula for $Q_H$): Let $H$ be a $\varrho$-Hydra map. Then:

$$Q_H \{ f \} (z) = \sum_{j=0}^{g-1} z^{\varrho j} \sum_{k=0}^{\mu_j - 1} e\left( -\frac{kH(j)}{\mu_j} \right) f\left( e\left( \frac{k}{\mu_j} \right) z^{\varrho/\mu_j} \right), \forall |z| < 1, \forall f \in \mathcal{A}(\mathbb{D})$$  \hspace{1cm} (18)

Proof: Letting $f(z) = \sum_{n=0}^{\infty} c_n z^n$, we have that:

$$Q_H \{ f \} (z) = \sum_{n=0}^{\infty} c_{H(n)} z^n$$

(split $n$ mod $\varrho$): $$\sum_{n=0}^{g-1} \sum_{j=0}^{\mu_j - 1} c_{H(n+j)} z^{\varrho n + j}$$

(use (3)): $$\sum_{j=0}^{g-1} \sum_{n=0}^{\infty} c_{\mu_j n + H(j)} z^{\varrho n + j}$$

$$= \sum_{j=0}^{g-1} \sum_{n=0}^{\infty} c_{\mu_j n + H(j)} \left( z^{\varrho/\mu_j} \right)^{\mu_j n}$$

Observing the Series multisection identity:

$$\sum_{n=0}^{\infty} c_n z^{\varrho n + b} = \frac{1}{a} \sum_{k=0}^{a-1} \left( -\frac{kb}{a} \right) f\left( e\left( \frac{k}{a} \right) z \right), \forall |z| < 1$$  \hspace{1cm} (19)

for any integers $a \geq 1$ and $b \in \{ 0, \ldots, a - 1 \}$, and writing:

$$\varpi_{a,b} \{ f \} (z) = \frac{1}{a} \sum_{k=0}^{a-1} e\left( -\frac{kb}{a} \right) f\left( e\left( \frac{k}{a} \right) z \right)$$  \hspace{1cm} (20)

to denote a series multisection operator, we have that:

$$Q_H \{ f \} (z) = \sum_{j=0}^{g-1} z^{-\varrho/\mu_j} \sum_{n=0}^{\infty} c_{\mu_j n + H(j)} \left( z^{\varrho/\mu_j} \right)^{\mu_j n + H(j)}$$

$$= \sum_{j=0}^{g-1} z^{-\varrho/\mu_j} \varpi_{\mu_j,H(j)} \{ f \} \left( z^{\varrho/\mu_j} \right)$$

Here:

$$\varpi_{\mu_j,H(j)} \{ f \} \left( z^{\varrho/\mu_j} \right) = \frac{1}{\mu_j} \sum_{k=0}^{\mu_j - 1} e\left( -\frac{kH(j)}{\mu_j} \right) f\left( e\left( \frac{k}{\mu_j} \right) z^{\varrho/\mu_j} \right)$$
Finally:
\[ j - \frac{a}{\mu_j} H(j) = j - \frac{d_j}{a_j} \left( \frac{j a_j + b_j}{d_j} \right) = j - \left( j + \frac{b_j}{a_j} \right) = -\frac{b_j}{a_j} \]

and so:
\[
Q_H \{ f \}(z) = \sum_{j=0}^{g-1} z^{j - \frac{a}{\mu_j} H(j)} \mathcal{A}_{\mu_j, H(j)} \{ f \} \left( z^{\theta/\mu_j} \right) \\
= \sum_{j=0}^{g-1} \frac{z^{-b_j}}{\mu_j} \sum_{k=0}^{\mu_j-1} e^{-\frac{k}{\mu_j}} H(j) f \left( e^{\frac{k}{\mu_j}} z^{\theta/\mu_j} \right)
\]

which is the desired identity.

Q.E.D.

The study of \( Q_H \) is an entire discipline unto itself, and shall be pursued by the author in later publications. For the remainder of the present paper, we content ourselves with rehashing the above work in the context of multidimensional Hydra maps.

4.2 The \( d \)-Dimensional Case

In formulating Permutation operators induced by Hydra maps on Number Rings, in doing so, the greater richness of algebraic possibilities buried within Hydra maps on number rings come into full flower. Group actions on our number rings will be at the heart of the matter. In the course of determining the permutation operators, we will examine two particular types of actions on \( \mathcal{O}_F \): \( S_d \), the symmetric group on \( d \) elements; and actions of additive quotient groups of the form \( \mathcal{O}_F / I \), where—as usual—\( I \) is an ideal in \( \mathcal{O}_F \). As with the multisection of power series in the previous subsection, all of these group actions will have explicit realizations as transformations of set-series.

4.2.1. - Definition (Fourier set-series): For purposes of notation, rather than work with multivariate power series, we will instead work with holomorphic Fourier series defined on the \( d \)-dimensional upper half-plane:

\[
\mathbb{H}_{+i}^d = \{ (z_1, \ldots, z_d) \in \mathbb{C}^d : \text{Im}(z_j) > 0, \ \forall j \in \{1, \ldots, d\} \} 
\]  

(21)

The analogue of \( \varsigma_V(z) \) will then be the Fourier set-series:

\[
\psi_V(z) \overset{\text{def}}{=} \sum_{v \in V} e^{2\pi i v z} 
\]  

(22)

holomorphic on \( \mathbb{H}_{+i} \). The multivariate Fourier set-series is then:

\[
\psi_V(z) \overset{\text{def}}{=} \sum_{v \in V} e^{2\pi i (v \cdot z)} 
\]  

(23)

holomorphic on \( \mathbb{H}_{+i}^d \).
4.2.1 - Definition (Permutation Operators on Number Rings): Let $H$ be an $I$-Hydra map on $\mathbb{Z}_F^+$. Then, the Permutation Operator induced by $H$ is the linear operator $\mathcal{Q}_H : A(\mathbb{H}_+^d) \to A(\mathbb{H}_+^d)$ defined on the space of all holomorphic functions on $\mathbb{H}_+^d$ by way of the formula:

$$
\mathcal{Q}_H \left\{ \sum_{n \in \mathbb{N}_0^d} c_n e^{2\pi i \langle n | z \rangle} \right\} \overset{\text{def}}{=} \sum_{n \in \mathbb{N}_0^d} c_{H(n)} e^{2\pi i \langle n | z \rangle}
$$

where, recall, $\mathbb{N}_0^d$ is the set of all $d$-tuples of non-negative integers.

As would be expected, the multidimensionality makes computing the explicit formula (and thus, the functional equations) for these operators is not as straightforward as the one-dimensional case addressed in the previous subsection. As mentioned before, everything we do will be done while working over $\mathbb{N}_0^d$ (that is, $\mathbb{Z}_F^+$, where we identify $\mathbb{N}_0^d$ with $\varphi^{-1}_F \left( \mathcal{O}_F^+ \right)$), rather $\mathcal{O}_F^+$. As such, we will adopt the following conventions for the remainder of this section:

- Let $a$ denote a $d \times 1$ column vector of the form:

$$
a \overset{\text{def}}{=} \begin{bmatrix} a_1 \\
\vdots \\
a_d \end{bmatrix}
$$

whose entries are positive integers.

- Let $D$ denote the $d \times d$ diagonal matrix:

$$
D \overset{\text{def}}{=} \begin{bmatrix} a_1 & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & a_d \end{bmatrix}
$$

where $a_1, \ldots, a_d \in \mathbb{N}_1$ are the entries of $a$.

- Let $P$ be a $d \times d$ permutation matrix.

- Let $b$ be an arbitrary $d \times 1$ column vector:

$$
b \overset{\text{def}}{=} \begin{bmatrix} b_1 \\
\vdots \\
b_d \end{bmatrix}
$$

where $b_1, \ldots, b_d \in \mathbb{N}_0$.

- For the chosen $D$ and $P$, let $A$ denote the matrix $A = DP$.

The first order of business is to generalize the action decomposition operators. Here, the analogue of the action of $\mathbb{Z}/a\mathbb{Z}$ on $\mathbb{Z}$ is that of $\mathbb{Z}^d/A\mathbb{Z}^d$ on $\mathbb{Z}^d$.

4.2.2 - Definition (Action Decomposition Operators):

We write $\varpi_{A,b} : A(\mathbb{H}_+^d) \to A(\mathbb{H}_+^d)$ to denote the linear operator:
\[ \varpi_{A, b} \{ \psi \} (z) \overset{\text{def}}{=} \sum_{n \in \mathbb{N}_0^d} c_{A_n + b} e^{2\pi i (A_n + b) z} \quad \forall \psi (z) = \sum_{n \in \mathbb{N}_0^d} c_n e^{2\pi i n z} \in \mathcal{A} \left( \mathbb{H}_+^d \right) \]  

\textbf{4.2.3 - Proposition:}  
\[ \varpi_{D, b} = \varpi_{D, 0} \]  

Proof: By definition:
\[ \varpi_{D, b} \{ \psi \} (z) = \sum_{n \in \mathbb{N}_0^d} c_{Dn + b} e^{2\pi i (Dn + b) z} \]

Letting \( m = Pn \), this becomes:
\[ \varpi_{A, b} \{ \psi \} (z) = \sum_{P^{-1}m \in \mathbb{N}_0^d} c_{Dm + b} e^{2\pi i (Dm + b) z} \]

Since \( P \) is a bijection of \( \mathbb{N}_0^d \):
\[ \sum_{P^{-1}m \in \mathbb{N}_0^d} = \sum_{m \in P(\mathbb{N}_0^d)} = \sum_{m \in \mathbb{N}_0^d} \]

and so:
\[ \varpi_{D, b} \{ \psi \} (z) = \sum_{m \in \mathbb{N}_0^d} c_{Dm + b} e^{2\pi i (Dm + b) z} = \varpi_{D, 0} \{ \psi \} (z) \]

where the right-most equality follows by definition of \( \varpi_{D, 0} \).

Q.E.D

\textbf{4.2.4 - Proposition (Action Decomposition Operator Formulae over Number Rings):}  
\[ \varpi_{A, b} \{ \psi \} (z) = \frac{1}{|\det A|} \sum_{k=0}^{a-1} e \left( \frac{-bk}{a} \right) \psi \left( z + \frac{k}{a} \right) , \forall \psi \in \mathcal{A} \left( \mathbb{H}_+^d \right) \]  

(27)

where, recall, \( e \left( \frac{-bk}{a} \right) = e^{2\pi i \left( \frac{-b}{a} \right) k} \).

Proof: From the 1-dimensional case, we know that :
\[ \sum_{n=0}^{\infty} c_{an+b} e^{2\pi i (an+b) z} = \frac{1}{a} \sum_{k=0}^{a-1} e \left( \frac{-bk}{a} \right) \sum_{n=0}^{\infty} c_n e^{2\pi i n (z + \frac{k}{a})} \]
As such:

\[ \varpi_{A,b} \{ \psi \} (z) = \varpi_{DP,b} \{ \psi \} (z) \]

(Proposition ?) \[ = \varpi_{D,b} \{ \psi \} (z) \]

\[ = \sum_{n \in \mathbb{N}_0^d} c_{Dn+b} e^{2\pi i(Dn+b)z} \]

(Dn = an) \[ = \sum_{n \in \mathbb{N}_0^d} c_{an+b} e^{2\pi i(an+b)z} \]

\[ = \sum_{n \in \mathbb{N}_0^d} c_{a_1n_1+b_1, \ldots, a_dn_d+b_d} \prod_{\ell=1}^{d} e^{2\pi i(a_\ell n_\ell + b_\ell)z_\ell} \]

(Use 1-dimensional case) \[ = \sum_{n \in \mathbb{N}_0^d} c_1 \prod_{\ell=1}^{d} \frac{1}{a_\ell} \sum_{k_\ell=0}^{a_\ell-1} e \left( - \frac{b_\ell k_\ell}{a_\ell} \right) e^{2\pi i n_\ell \left( z_\ell + \frac{b_\ell}{a_\ell} \right)} \]

\[ = \sum_{n \in \mathbb{N}_0^d} \frac{c_n}{|\det A|} \sum_{k=0}^{a-1} e \left( - \frac{b k}{a} \right) e^{2\pi i \left( n z + \frac{b}{a} \right)} \]

\[ = \frac{1}{|\det A|} \sum_{k=0}^{a-1} e \left( - \frac{b k}{a} \right) \sum_{n \in \mathbb{N}_0^d} c_n e^{2\pi i (n z + \frac{b}{a})} \]

Q.E.D.

4.2.5 - Theorem (Fourier Permutation Operator Formula, Ver. 1): Let \( H \) be an \( \mathcal{I} \)-Hydra map on \( O_{\mathcal{F}, \mathcal{G}} \), where \( \mathcal{I} \) is of \( \mathcal{B} \)-modulus \( \rho \). Then, \( \mathcal{L}_H \) is given by the formula:

\[ \mathcal{L}_H \{ \psi \} (z) = \sum_{j \in (\mathbb{Z} / \rho \mathbb{Z})^d} e^{-2\pi i (b_j^j A_j^{-T} z)} \varpi_{e A_j H(j)} \{ \psi \} \left( A_j^{-T} z \right) \] (28)

for all \( \psi (z) \in \mathcal{A} (\mathbb{H}_{+1}^d) \) and all \( z \in \mathbb{H}^d_{+1} \).

Proof: Let \( \psi (z) = \sum_{n \in \mathbb{N}_0^d} c_n e^{2\pi i (n z)} \) be a function in \( \mathcal{A} (\mathbb{H}_{+1}^d) \). Then:

\[ \mathcal{L}_H \{ \psi \} (z) = \sum_{n \in \mathbb{N}_0^d} c_{H(n)} e^{2\pi i (n z)} \]
Just as \( \mathbb{N}_0 \) can be partitioned into its \( q \) distinct equivalence classes modulo \( q \):

\[
\mathbb{N}_0 = \bigcup_{j=0}^{q-1} (q\mathbb{N}_0 + j)
\]

we can partition \( \mathbb{N}^d_0 \) into its \( q^d \) distinct equivalence classes modulo \( q \):

\[
\mathbb{N}^d_0 = \bigcup_{j \in (\mathbb{Z}/q\mathbb{Z})^d} (q\mathbb{N}^d_0 + j)
\]

Consequently, for any function \( f : \mathbb{N}^d_0 \to \mathbb{N}^d_0 \):

\[
\sum_{n \in \mathbb{N}^d_0} f(n) = \sum_{n \in \mathbb{N}^d_0} \sum_{j \in (\mathbb{Z}/q\mathbb{Z})^d} f(qn + j)
\]

As such:

\[
\mathcal{Q}_H \{ \psi \} (z) = \sum_{n \in \mathbb{N}^d_0} c_{H(n)} e^{2\pi i (n|z)} = \sum_{n \in \mathbb{N}^d_0} \sum_{j \in (\mathbb{Z}/q\mathbb{Z})^d} c_{H(qn + j)} e^{2\pi i ((qn + j)|z)}
\]

Since \( qn + j \equiv j \) for all \( n \in \mathbb{N}^d_0 \), it follows that:

\[
H(qn + j) = A_j (qn + j) + b_j
\]

\[
= qA_jn + A_jj + b_j
\]

\[
= qA_jn + H(j)
\]

and so:

\[
\sum_{n \in \mathbb{N}^d_0} \sum_{j \in (\mathbb{Z}/q\mathbb{Z})^d} c_{H(qn + j)} e^{2\pi i ((qn + j)|z)} = \sum_{n \in \mathbb{N}^d_0} \sum_{j \in (\mathbb{Z}/q\mathbb{Z})^d} c_{qA_jn + H(j)} e^{2\pi i ((qA_jn + H(j))|z)}
\]

As with the case over \( \mathbb{Z} \), to finish, we need only make the \( qn + j \) in the exponent into \( qA_jn + H(j) \); then, we can condense the resultant expression into the action decomposition operator \( \mathbb{C}_{qA_j, H(j)} \):

\[
\sum_{n \in \mathbb{N}^d_0} \sum_{j \in (\mathbb{Z}/q\mathbb{Z})^d} c_{qA_jn + H(j)} e^{2\pi i ((qA_jn + H(j))|z)}
\]

To do this, we use the invertibility of the \( A_j \)'s and their adjoints, along with the bilinearity of the dot product:

\[
\langle qn + j | z \rangle = \left\langle A_j^{-1} A_j (qn + j) | z \right\rangle
\]

\[
= \left\langle A_j (qn + j) | A_j^{-T} z \right\rangle
\]

\[
= \left\langle A_j (qn + j) + b_j - b_j | A_j^{-T} z \right\rangle
\]

\[
= \left\langle A_j (qn + j) + b_j | A_j^{-T} z \right\rangle - \left\langle b_j | A_j^{-T} z \right\rangle
\]

\[
= \left\langle qA_jn + H(j) | A_j^{-T} z \right\rangle - \left\langle b_j | A_j^{-T} z \right\rangle
\]
and so:

\[
\sum_{n \in \mathbb{N}} \sum_{j \in (\mathbb{Z}/\mathbb{Z})^d} c_{\partial A_j n + H(j)} e^{2\pi i (an + jz)} =
\sum_{j \in (\mathbb{Z}/\mathbb{Z})^d} e^{-2\pi i (b_j | A_j^{-T} z)} \sum_{n \in \mathbb{N}} c_{\partial A_j n + H(j)} e^{2\pi i (\partial A_j n + H(j) | A_j^{-T} z)}
\]

\[
\sum_{j \in (\mathbb{Z}/\mathbb{Z})^d} e^{-2\pi i (b_j | A_j^{-T} z)} \varphi_{\partial A_j, H(j)} (\psi) \left( A_j^{-T} z \right)
\]

So:

\[
\mathcal{D}_H (\psi) (z) = \sum_{j \in (\mathbb{Z}/\mathbb{Z})^d} e^{-2\pi i (b_j | A_j^{-T} z)} \varphi_{\partial A_j, H(j)} (\psi) \left( A_j^{-T} z \right)
\]

as desired.

Q.E.D.

4.2.6 - Corollary (Fourier Permutation Operator Formula, Ver. 2):
Let \( H \) be an \( \mathcal{I} \)-Hydra map on \( \mathcal{O}^{d, d}_+ \), where \( \mathcal{I} \) is of \( B \)-modulus \( \mathcal{L} \). For each \( j \), writing:

\[
A_j = D_j P_j
\]

where:

\[
D_j \overset{\text{def}}{=} \begin{bmatrix} a_{j:1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_{j:d} \end{bmatrix}
\]

(where the non-zero entries are positive rational numbers so that \( \varrho a_{j: \ell} \in \mathbb{N}_1 \) for all \( \ell \in \{1, \ldots, d\} \)), define:

\[
a_j \overset{\text{def}}{=} \begin{bmatrix} a_{j:1} \\ \vdots \\ a_{j:d} \end{bmatrix}
\]

and:

\[
m_j \overset{\text{def}}{=} \varrho a_j = (\mu_{j:1}, \ldots, \mu_{j:d}) \overset{\text{def}}{=} (\varrho a_{j:1}, \ldots, \varrho a_{j:d})
\]

Then, \( \mathcal{D}_H \) is given by the formula:

\[
\mathcal{D}_H \{ \psi \} (z) = \sum_{j \in (\mathbb{Z}/\mathbb{Z})^d} e^{-2\pi i (b_j | A_j^{-T} z)} \frac{m_j^{d-1}}{\Pi (m_j)} \sum_{k=0}^{m_j-1} e \left( \frac{-H (j) k}{m_j} \right) \psi \left( A_j^{-T} z + \frac{k}{m_j} \right)
\]

for all \( \psi (z) \in \mathcal{A} (\mathbb{H}^d, \mathbb{H}) \) and all \( z \in \mathbb{H}^d \), where, recall, \( \Pi (m_j) \) denotes the product of the entries of \( m_j \). Note, moreover, that \( \Pi (m_j) = \varrho^d \det D_j \).

Proof: We begin with the formula from 4.2.5:.

\[
\mathcal{D}_H \{ \psi \} (z) = \sum_{j \in (\mathbb{Z}/\mathbb{Z})^d} e^{-2\pi i (b_j | A_j^{-T} z)} \varphi_{\partial A_j, H(j)} (\psi) \left( A_j^{-T} z \right)
\]
where \( \psi(z) \in \mathcal{A}(\mathbb{H}^d_{+}) \) is arbitrary. Since \( gA_j = gD_jP_j \), where \( gD_j \) is a diagonal matrix whose diagonal entries are positive integers, we have that:

\[
\mathcal{D}(gA_j, H(j) \{ \psi \} (A_j^{-T}z) = \mathcal{D}(gD_j, H(j) \{ \psi \} (A_j^{-T}z)
\]

As such, noting that \( m_j \) is then the column vector whose entries are the diagonal entries of \( gD_j \), we can write:

\[
\mathcal{D}(H \{ \psi \} (z) = \sum_{j \in (\mathbb{Z}/g\mathbb{Z})^d} e^{-2\pi i (b_j|A_j^{-T}z)} \mathcal{D}(gD_j, H(j) \{ \psi \} (A_j^{-T}z)
\]

\[
= \sum_{j \in (\mathbb{Z}/g\mathbb{Z})^d} \frac{e^{-2\pi i (b_j|A_j^{-T}z)}}{|\det(gD_j)|} \sum_{k=0}^{m_j-1} e\left(-\frac{H(j)k}{m_j}\right) \psi\left(A_j^{-T}z + \frac{k}{m_j}\right)
\]

and so:

\[
\mathcal{D}(H \{ \psi \} (z) = \sum_{j \in (\mathbb{Z}/g\mathbb{Z})^d} \frac{e^{-2\pi i (b_j|A_j^{-T}z)}}{\Pi(m_j)} \sum_{k=0}^{m_j-1} e\left(-\frac{H(j)k}{m_j}\right) \psi\left(A_j^{-T}z + \frac{k}{m_j}\right)
\]

which is the desired formula.

Q.E.D.

One of the most significant distinctions between Hydra maps on \( \mathbb{Z}^d_{+} \), with those on \( \mathbb{Z} \) is that the former have multiple dimensions to work with. Not only do we need to worry about how the values of points' components change, we also need to account for motions of those orbits with respect to the different coordinate axes—that is, following the values as they hopping from one coordinate direction to another.

Recall that \( \Theta_d \) denotes the symmetric group on \( d \) objects (a group of order \( d! \)).

4.2.7 - Definitions:

I. We write \( \hat{\Theta}_d \) to denote the dual of \( \Theta_d \)—that is, the group (under pointwise multiplication) of all group homomorphisms \( \chi : \Theta_d \to \mathbb{C} \).

II. For every \( \chi \in \hat{\Theta}_d \), define the \( \chi \)-action decomposition operator \( \mathcal{D}_\chi : \mathcal{A}(\mathbb{H}^d_{+}) \to \mathcal{A}(\mathbb{H}^d_{+}) \) by:

\[
\mathcal{D}_\chi \{ \psi \} (z) = \frac{1}{d!} \sum_{P \in \Theta_d} \chi(P) \psi(Pz)
\]

where the sum is taken over all elements \( P \) of \( \Theta_d \), viewed here as \( d \times d \) permutation matrices. We call \( \mathcal{D}_\chi \{ \psi \} \) the \( \chi \)-component of \( \psi \) over \( \Theta_d \).

Remark: As a shorthand, we will sometimes write \( \psi_\chi \) to denote \( \mathcal{D}_\chi \{ \psi \} .

4.2.8 - Proposition: Let \( \psi \in \mathcal{A}(\mathbb{H}^d_{+}) \) be arbitrary.
I. \( (\mathcal{G}_d\text{-automorphy of } \varpi_\chi \{ \psi \} (z)) \) For all \( \chi \in \hat{\mathcal{G}}_d \), the function \( \varpi_\chi \{ \psi \} (z) \) is an \( \mathcal{G}_d \)-automorphic form, meaning that there is a function \( f_\chi : \mathcal{G}_d \to \partial \mathbb{D} \) (called a factor of automorphy) so that the functional equation:

\[
\varpi_\chi \{ \psi \} (Pz) = f_\chi (P) \varpi_\chi \{ \psi \} (z), \quad \forall z \in \mathbb{H}_d^+, \forall \chi \in \hat{\mathcal{G}}_d
\]

holds for all \( P \in \mathcal{G}_d \).

II. (Recovery Formula)

\[
\psi (z) = \sum_{\chi \in \hat{\mathcal{G}}_d} \varpi_\chi \{ \psi \} (z), \quad \forall z \in \mathbb{H}_d^+
\]

Proof: Computation.

Q.E.D

Example: Letting 1 denote the trivial character:

\[
\varpi_1 \{ \psi \} (z) = \frac{1}{d!} \sum_{P \in \mathcal{G}_d} \psi (Pz)
\]

and hence:

\[
\varpi_1 \{ \psi \} (Pz) = \varpi_1 \{ \psi \} (z), \quad \forall P \in \mathcal{G}_d
\]

is then completely symmetric. On the other hand, letting sgn denote the character which outputs the signature of a given permutation \( P \), the function:

\[
\varpi_{\text{sgn}} \{ \psi \} (z) = \frac{1}{d!} \sum_{P \in \mathcal{G}_d} \text{sgn}(P) \psi (Pz)
\]

is then an alternating function—swapping the positions of any two entries of \( z \) then changes the value of \( \varpi_{\text{sgn}} \{ \psi \} (z) \) by a factor of \(-1\).

4.2.9 - Lemma: Let \( H : \mathbb{N}_0^d \to \mathbb{N}_0^d \) be a map, and let \( \psi \in \mathcal{A} (\mathbb{H}_d^+) \).

I. If \( \psi \in \text{Ker} (1 - \mathcal{Q}_H) \), then \( \psi \circ P \in \text{Ker} (1 - \mathcal{Q}_H) \) for all \( P \in \mathcal{G}_d \).

II. If \( \psi \in \text{Ker} (1 - \mathcal{Q}_H) \), then \( \varpi_\chi \{ \psi \} \in \text{Ker} (1 - \mathcal{Q}_H) \) for all \( \chi \in \hat{\mathcal{G}}_d \).

Proof: Let \( \psi (z) = \sum_{n \in \mathbb{N}_0^d} e^{2\pi i (n,z)} \in \mathcal{A} (\mathbb{H}_d^+) \) be fixed by \( \mathcal{Q}_H \).
I. Let \( P \in \mathcal{S}_d \). Then:

\[
\mathcal{D}_H \{ \psi \circ P \} (z) = \mathcal{D}_H \left\{ \sum_{n \in \mathbb{N}_0^d} c_n e^{2 \pi i \langle n | Pz \rangle} \right\} \\
= \mathcal{D}_H \left\{ \sum_{n \in \mathbb{N}_0^d} c_n e^{2 \pi i \langle P^T n | z \rangle} \right\} \\
(\mathbf{m} = \mathbf{P}^T \mathbf{n}) : \mathcal{D}_H \left\{ \sum_{\mathbf{m} \in \mathcal{P}^T (\mathbb{N}_0^d)} c_{\mathbf{P}^T \mathbf{m}} e^{2 \pi i \langle \mathbf{m} | z \rangle} \right\} \\
= \mathcal{D}_H \left\{ \sum_{\mathbf{m} \in \mathbb{N}_0^d} c_{\mathbf{P}^T \mathbf{m}} e^{2 \pi i \langle \mathbf{m} | z \rangle} \right\} \\
= \sum_{\mathbf{m} \in \mathbb{N}_0^d} c_{\mathbf{H}(\mathbf{P}^T \mathbf{m})} e^{2 \pi i \langle \mathbf{m} | z \rangle} \\
= \sum_{\mathbf{m} \in \mathbb{N}_0^d} c_{(\mathcal{H} \circ \mathbf{P}^T)(\mathbf{m})} e^{2 \pi i \langle \mathbf{m} | z \rangle} \\
= \mathcal{D}_{\mathcal{H} \circ \mathbf{P}^T} \left\{ \sum_{\mathbf{m} \in \mathbb{N}_0^d} c_{\mathbf{m}} e^{2 \pi i \langle \mathbf{m} | z \rangle} \right\} \\
= \mathcal{D}_{\mathcal{H} \circ \mathbf{P}^T} \{ \psi \} (z)
\]

Next, for any \( A \subseteq \mathbb{N}_0^d \):

\[
\mathcal{D}_H \{ \psi_A \} (z) = \mathcal{D}_H \left\{ \sum_{\mathbf{n} \in \mathbb{N}_0^d} [\mathbf{n} \in A] e^{2 \pi i \langle \mathbf{n} | z \rangle} \right\} \\
= \sum_{\mathbf{n} \in \mathbb{N}_0^d} [\mathcal{H} (\mathbf{n}) \in A] e^{2 \pi i \langle \mathbf{n} | z \rangle} \\
= \sum_{\mathbf{n} \in \mathbb{N}_0^d} \left[ \mathbf{n} \in \mathcal{H}^{-1} (A) \right] e^{2 \pi i \langle \mathbf{n} | z \rangle} \\
= \psi_{\mathcal{H}^{-1} (A)} (z)
\]
So, letting $V$ be $H$-invariant (that is, $H^{-1}(V) = V$):

$$
\mathcal{D}_H \{\psi_V \circ P\}(z) = \mathcal{D}_{H \circ P^{-T}} \{\psi_V\}(z) = \psi_{(H \circ P^{-T})^{-1}(V)}(z) = \psi_{P^{-T}(H^{-1}(V))}(z)
$$

$$(H^{-1}(V) = V); \quad \psi_{P^{-T}(V)}(z) = \sum_{n \in N^d} [n \in P^T(V)] e^{2\pi i \langle n|z \rangle}
$$

$$
= \sum_{n \in N^d} [P^{-T}n \in V] e^{2\pi i \langle n|z \rangle}
$$

$$(m = P^{-T}n); \quad = \sum_{m \in P^{-T}(N^d)} [m \in V] e^{2\pi i \langle m|Pz \rangle}
$$

$$
= \sum_{m \in N^d} [m \in V] e^{2\pi i \langle m|Pz \rangle}
$$

$$
= (\psi_V \circ P)(z)
$$

and so, $\psi_V \in \text{Ker}(1 - \mathcal{D}_H)$ implies $\psi_V \circ P \in \text{Ker}(1 - \mathcal{D}_H)$. This shows that a basis of $\text{Ker}(1 - \mathcal{D}_H)$ (namely, the set Fourier set-series for the orbit classes of $H$ in $\mathbb{Z}_{d,H}^d = N^d$) gets mapped into $\text{Ker}(1 - \mathcal{D}_H)$ by the “pre-composition by $P$” operator. As such, $\psi \circ P \in \text{Ker}(1 - \mathcal{D}_H)$ for all $\psi \in \text{Ker}(1 - \mathcal{D}_H)$ and all $P \in \hat{G}_d$. ✓

II. Let $\chi \in \hat{G}_d$ be arbitrary. Then:

$$
\varpi_\chi \{\psi\}(z) = \frac{1}{d!} \sum_{P \in \hat{G}_d} \chi(P) (\psi \circ P)(z)
$$

By the linearity of $\mathcal{D}_H$:

$$
\mathcal{D}_H \{\varpi_\chi \{\psi\}\}(z) = \mathcal{D}_H \left\{ \frac{1}{d!} \sum_{P \in \hat{G}_d} \chi(P) (\psi \circ P) \right\}(z)
$$

$$
= \frac{1}{d!} \sum_{P \in \hat{G}_d} \chi(P) \mathcal{D}_H \{\psi \circ P\}(z)
$$

(by (I), $\mathcal{D}_H \{\psi \circ P\} = \psi \circ P$);

$$
= \frac{1}{d!} \sum_{P \in \hat{G}_d} \chi(P) (\psi \circ P)(z)
$$

$$
= \varpi_\chi \{\psi\}(z)
$$

and so, $\varpi_\chi \{\psi\}(z)$ is fixed by $\mathcal{D}_H$ for all $\chi \in \hat{G}_d$, as desired. ✓

Q.E.D.

The simplicity of the above proof belies the utility of the result thus proved. The presence of $A_j^{-T} = D_j P_j^{-T}$ on the left-hand side of (30) shows that our
functional equations will, in general, have variables swapped around. For example, when \( H \) is Leigh’s map, the functional equation for the fixed points of \( \mathcal{D}_H \) is:

\[
\psi(z, w) = \psi(w, 2z) + \frac{e^{-\frac{2\pi i}{3}}}{9} \sum_{j=0}^{2} \sum_{k=0}^{2} \xi_3^k \psi \left( \frac{w + j}{3}, \frac{2z + k}{3} \right) \tag{31}
\]

where, again, \( \xi_3^k = e^{2k\pi i/3} \). The permuted variables needlessly complicate things. Fortunately, Lemma 4.2.9 lets us dispense with that complication by considering \( \psi_{\chi} \) in lieu of \( \psi \). In this particular case, \( \hat{\mathcal{G}}_2 \cong \mathbb{Z}/2\mathbb{Z} \), a finite abelian group. Consequently, \( \hat{\mathcal{G}}_2 \cong \hat{\mathcal{G}}_2 \). Thus, there are exactly two components of any \( \psi \); these turn out to be the symmetric component:

\[
\psi_{\text{Id}} (z, w) = \psi(z, w) - \psi(w, z)
\]

and the anti-symmetric component:

\[
\psi_{\text{sgn}} (z, w) = \psi(z, w) + \psi(w, z)
\]

where \( \text{Id} \) and \( \text{sgn} \) are the group homomorphisms which send the non-identity element of \( \mathcal{G}_2 \) to \( 1 \in \partial \mathbb{D} \) and \( -1 \in \partial \mathbb{D} \), respectively.

As such, replacing \( \psi \) in (31) by \( \psi_{\text{Id}} \) lets us write:

\[
\psi_{\text{Id}} (z, w) = \psi_{\text{Id}} (2z, w) + \frac{e^{-\frac{2\pi i}{3}}}{9} \sum_{j=0}^{2} \sum_{k=0}^{2} \xi_3^k \psi_{\text{Id}} \left( \frac{2z + k}{3}, \frac{w + j}{3} \right)
\]

whereas replacing \( \psi \) with \( \psi_{\text{sgn}} \) lets us write:

\[
-\psi_{\text{sgn}} (z, w) = \psi_{\text{sgn}} (2z, w) + \frac{e^{-\frac{2\pi i}{3}}}{9} \sum_{j=0}^{2} \sum_{k=0}^{2} \xi_3^k \psi_{\text{sgn}} \left( \frac{2z + k}{3}, \frac{w + j}{3} \right)
\]

Since the factor of automorphy corresponding to \( \text{sgn} \) is \(-1\), note that taking absolute values of both equations yields the same formal expression:

\[
|f(z, w)| = \left| f(2z, w) + \frac{e^{-\frac{2\pi i}{3}}}{9} \sum_{j=0}^{2} \sum_{k=0}^{2} \xi_3^k f \left( \frac{2z + k}{3}, \frac{w + j}{3} \right) \right|
\]

As such, if we take the absolute value of (31), we can then freely permute the variables of \( \psi \) in the resultant equation.

To finish, we now establish the generalization of the results from Section 4.1: 4.2.10 - **Proposition** (Action of \( \mathcal{D}_H \) on Fourier set-series): For any \( V \subseteq \mathbb{N}_0 \), we have:

\[
\mathcal{D}_H \{ \psi_V \} (z) = \psi_{H^{-1}(V)} (z) \tag{32}
\]
Proof:

\[
Q_H \{ \psi_V \} (z) = \sum_{n \in \mathbb{N}_0^d} 1_V (H (n)) e^{2\pi i (n | z)} = \sum_{n \in \mathbb{N}_0^d} 1_{H^{-1} (V)} (n) e^{2\pi i (n | z)} = \psi_{H^{-1} (V)} (z)
\]

Q.E.D.

4.2.11 - Theorem (Characterization of Ker \((Q_H^{-1})\)): Let \(H : \mathbb{N}_0^d \to \mathbb{N}_0^d\) be any surjective map, and enumerate the at most countably many distinct irreducible orbit classes of \(H\) in \(\mathbb{N}_0^d\) as \(V_0, V_1, \ldots\). Then, every element of Ker \((Q_H^{-1})\) is of the form:

\[
\sum_{n=0}^{\infty} c_n \psi_{V_n} (z)
\]

for some complex constants \(c_0, c_1, \ldots\).

Proof: A set \(V \subseteq \mathbb{N}_0^d\) is an orbit class of \(H\) precisely when \(H^{-1} (V) = V\). Since \(Q_H \{ \psi_V \} = \psi_{H^{-1} (V)}\), it then follows that \(\psi_{V_n}\) is fixed by \(Q_H\) for all \(n\).

For the other direction, let \(\Psi (z) = \sum_{n \in \mathbb{N}_0^d} c_n e^{2\pi i (n | z)}\) be an element of \(A (\mathbb{N}_0^d)\) which is fixed by \(Q_H\). Then, let \(C\) denote the set of values attained by the \(c_n\)s, and, for each \(c \in C\), let \(N_c\) denote the set of all \(n \in \mathbb{N}_0^d\) for which \(c_n = c\). Then, we can write:

\[
\Psi (z) = \sum_{c \in C} c \psi_{N_c} (z)
\]

and hence:

\[
\sum_{c \in C} c \psi_{H^{-1} (N_c)} (z) = Q_H \{ \Psi \} (z) = \Psi (z) = \sum_{c \in C} c \psi_{N_c} (z)
\]

which forces \(H^{-1} (N_c) = N_c\), which shows that, for each \(c\), \(N_c\) is an orbit class of \(H\), and is hence either an irreducible orbit class (one of the \(V_n\)s), or is the union of irreducible orbit classes \(V_{n_1}, V_{n_2}, \ldots\), in which case:

\[
\psi_{N_c} (z) = \psi_{\bigcup_k V_{n_k}} (z) = \sum_k \psi_{V_{n_k}} (z)
\]

Regardless, this shows that \(\Psi\) is then of the form \((33)\).

Q.E.D.

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