RIPS-SEGEV TORSION-FREE GROUPS WITHOUT UNIQUE PRODUCT

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ABSTRACT. We explain and generalize Rips-Segev’s construction of torsion-free groups without unique product. We view these groups as given by graphical small cancellation presentations and prove that they are hyperbolic by extending Ollivier’s proof of a small cancellation theorem of Gromov to graphical small cancellation presentations over free products. We show that presentations of Rips-Segev’s groups are not generic among finite presentations of groups. We construct uncountably many non-isomorphic torsion-free groups without unique product.

The outstanding Kaplansky zero-divisor conjecture states that the group ring of a torsion-free group over an integral domain has no zero-divisors [Kap57, Kap70]. The conjecture is still open in general. Unique product groups (or groups with the unique product property, see Definition 2.1) are torsion-free groups that satisfy the conjecture [Coh74]. Examples of unique product groups are abelian and non-abelian free groups, nilpotent groups, and hyperbolic groups which act with large translation length on a hyperbolic space [Del97]. Answering a question of Passman [Pas77], Rips and Segev have provided first examples of torsion-free groups without the unique product property [RS87]. These groups and their Rips-Segev presentations have still not been systematically investigated.

Our motivation to study Rips-Segev groups comes from the following two open problems.

• Do Rips-Segev groups satisfy Kaplansky’s zero-divisor conjecture?
• Are unique product groups generic among finitely presented groups?

The second question is due to T. Delzant. We expect a positive answer to both questions. This would mean, in particular, that a generic finitely presented group satisfies Kaplansky’s zero-divisor conjecture.

Rips and Segev have defined their group by taking as relators words that are read on the cycles of a certain finite connected edge-labeled graph. The graph was designed to encode the non-unique product relations. On the other hand, to guarantee the final conclusion that their groups are torsion-free and without unique product, the authors have referred to the small cancellation theory [RS87, p. 123]. However, the statements they refer to either are not applicable (in fact, their presentations do not satisfy the classical $C'(p)$–condition for minimal sequences as described in [LS77, Ch. V.8] as relators have to be of length 4 in a free group) or their proofs are not present in the literature (the result [LS77, Th. 9.3, Ch. V] is available only in a specific case of the classical metric $C''(\lambda)$–condition over the free product and not under the $C''(\lambda)$–condition

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for minimal sequences as required by the Rips-Segev construction). In particular, contrary to the classical metric small cancellation conditions, the relators given by Rips-Segev can have long common parts, see our explanation in Section 1.2.

We adapt a new viewpoint on the Rips-Segev presentations. Namely, we show that they satisfy the graphical metric small cancellation condition with respect to the free product length (also known as the syllable-length) on the free product of certain torsion-free groups. We prove the following general result, of independent interest, which can be then applied to the Rips-Segev presentations.

**Theorem 1** (c.f. Theorems 1.13, 1.14). Let $G_1, \ldots, G_n$ be finitely generated groups. Let $\Omega$ be a family of finite connected graphs edge-labeled by $G_1 \cup \ldots \cup G_n$ so that the graphical metric small cancellation condition with respect to the free product length on the free product $G_1 \ast \ldots \ast G_n$ is satisfied. Let $G$ be the group given by the corresponding graphical presentation, that is, the quotient of $G_1 \ast \ldots \ast G_n$ subject to the relators being the words read on the cycles of $\Omega$.

Then $G$ satisfies a linear isoperimetric inequality with respect to the free product length. Moreover, $G$ is torsion-free whenever $G_1, \ldots, G_n$ are torsion-free; $G$ is Gromov hyperbolic whenever $G_1, \ldots, G_n$ are Gromov hyperbolic and $\Omega$ is finite.

Our theorem extends to the free product setting Gromov’s graphical small cancellation theorem which states that presentations with the graphical metric small cancellation condition define torsion-free hyperbolic groups [Gro03, Oll06]. This result also generalizes a theorem of Pankrat’ev who considered the free products of hyperbolic groups subject to relators satisfying the classical metric small cancellation condition [Pan99].

As a corollary, the preceding theorem provides all details to the Rips-Segev original construction, specifically, one can conclude that the Rips-Segev groups are indeed torsion-free. In addition, this yields the following result. It was probably known to experts but again the reference to the small cancellation conditions for minimal sequences (as stated in [RS87, p. 123]) is not sufficient (as it could lead to quadratic Dehn functions instead of linear ones which characterize hyperbolicity in the case of finitely presented groups).

**Theorem 2** (c.f. Theorem 2.13). The Rips-Segev torsion-free groups without the unique product property are Gromov hyperbolic.

Hyperbolic groups with large translation length have the unique product property [Del97]. Therefore, we conclude that the Rips-Segev groups are first examples of torsion-free hyperbolic groups which possess no action with large translation length on any hyperbolic space.

By [dlH88] and [Laf98], see also [Laf12], Gromov hyperbolic groups satisfy the (strong) Baum-Connes conjecture, and torsion-free groups satisfying the Baum-Connes conjecture satisfy the Kadison-Kaplansky conjecture stating that the reduced $C^*$-algebra of a torsion-free group has no non-trivial idempotents. The Rips-Segev groups therefore satisfy the Kadison-Kaplansky conjecture. However, this is, according to the present status of knowledge, not enough to imply the Kaplansky zero-divisor conjecture. On the other hand, the Kaplansky zero-divisor conjecture implies that there are no non-trivial idempotents in the group ring.

Extending our graphical viewpoint further, we define the generalized Rips-Segev presentations and construct many new torsion-free groups without unique product. Some of our new groups have infinitely many different pairs of subsets without the unique product property.

It is unknown whether or not there are countably many Rips-Segev groups up to isomorphism. We show that elementary Nielsen equivalences and conjugation do not induce isomorphisms of
Rips-Segev groups. This allows us to construct the following huge family of torsion-free groups without the unique product property.

**Theorem 3** (c.f. Theorem 3.5). There are uncountably many non-isomorphic torsion-free groups without the unique product property.

Finally, we prove that all currently known finite presentations of torsion-free groups without unique product are not generic among finite presentations of groups with respect to two different fundamental models of random finitely presented groups.

**Theorem 4** (c.f. Theorem 4.2). Generalized Rips-Segev presentations of torsion-free non-unique product groups are not generic in Gromov’s graphical model [Gro03, OW07] of finitely presented random groups.

**Theorem 5** (c.f. Theorem 4.3). Generalized Rips-Segev presentations of torsion-free non-unique product groups are not generic in Arzhantseva-Ol’shanskii’s few relators model [AO96] of finitely presented random groups.

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1. Small cancellation conditions

First we review essential aspects of the classical small cancellation theory. Then we present the graphical small cancellation conditions we use in our approach to the Rips-Segev presentations.

Let $X$ be an alphabet with $|X| \geq 2$. A word $w$ in $X \sqcup X^{-1}$ is reduced if a letter in $w$ is not followed by its inverse; $w$ is cyclically reduced if all of its cyclic permutations are reduced. Let $R$ be a set of cyclically reduced words in $X \sqcup X^{-1}$.

1.1. Reminder on the classical small cancellation theory. Let $F$ be a free group or a free product of finitely many groups. We suppose that $F$ is generated by $X$ and equipped with a length function $\ell$. We focus on basic length functions such as the word length metric and the free product length (also known as the syllable-length):

- If $F$ is a free group on $X$ and $w = x_{i_1}^{\epsilon_{i_1}} \cdots x_{i_j}^{\epsilon_{i_j}}$ is a reduced word in $X \sqcup X^{-1}$, then $|w| := j$ is the word length of $w$;
- If $F$ is a free product $G_1 \ast \cdots \ast G_n$ and $w$ is a reduced word in $X \sqcup X^{-1}$ representing a non-trivial element of $F$, then $w$ has a unique factorization $w = w_1 \cdots w_j$, where each $w_i \neq 1$ is in one of the factors $G_{k(i)}$ and no two subsequent $w_i$’s are in the same factor. Then $|w|_* := j$ is the free product length of $w$. For instance, if $g_i \in G_i$ then $|g_1 g_1|_* = 1$ and $|g_1 g_2 g_1 g_2^{-1}|_* = 3$. Observe that we always have $|w|_* \leq |w|$.

Let $G = \langle X | R_F \sqcup R \rangle$ be a group with generators $X$ subject to relators $R_F \sqcup R$, where $R_F$ is a set of relators of $F$ (it is empty whenever $F$ is free on $X$). That is, $G$ is the quotient of $F$ by the normal subgroup of $F$ generated by $R$.

A set of relators $R$ is symmetrized if with a word $r$ it contains all cyclic permutations of $r$ and of $r^{-1}$. Given a set $R$, one makes it symmetrized by adding inverses and cyclic permutations. We always assume that $R$ is symmetrized as both $R$ and its symmetrization define the same group $G$. 
A piece is a word $p$ such that $r_1 = pu_1$ and $r_2 = pu_2$ for distinct elements $r_1, r_2 \in R$.

A set of relators $R$ satisfies the metric $C'_\ell(\lambda)$–small cancellation condition for $\lambda > 0$ with respect to a length function $\ell$ if for every piece we have

$$\ell(p) < \lambda \min \{\ell(r) \mid r \in R\}.$$ 

In this case, we refer to the group $G$ as to a $C'_\ell(\lambda)$–small cancellation group. Its presentation $G = \langle X \mid R_F \sqcup R \rangle$ is called a $C'_\ell(\lambda)$–small cancellation presentation over a free group (resp. over a free product), whenever $F$ is a free group on $X$ (resp. whenever $F$ is a free product).

If we denote by $\Lambda$ the maximal piece length $\max \{\ell(p) \mid p \text{ is a piece}\}$ and by $\gamma$ the minimal relator length $\min \{\ell(r) \mid r \in R\}$, then the above condition can equivalently be stated as

$$\frac{\Lambda}{\gamma} < \lambda.$$

The $C'_\ell(\lambda)$–small cancellation condition has a geometric interpretation in the language of van-Kampen diagrams.

Let $D$ be a planar 2-complex. An edge is an 1-cell of $D$ (each edge is oriented). The set of all edges is denoted by $E(D)$. A face is a 2-cell of $D$ (it can be homeomorphic to a disc with holes). The number of all faces is denoted by $|D|$. The boundary $\partial D$ is the set of edges which intersect at most one face in $D$. An inner edge of $D$ is an edge which-meets $\partial D$ in at most one vertex. An inner segment of $D$ is a path of inner edges which all glue the same pair of faces. The exterior boundary $\partial_{ext}\Pi$ of a face $\Pi$ is the intersection of $\partial\Pi$ with $\partial D$. The inner boundary $\partial_{int}\Pi$ is the closure of $\partial\Pi - \partial_{ext}\Pi$. A face with non-empty exterior boundary is called an exterior face, otherwise it is called an inner face. A labeling of $D$ by $X$ is a map $\omega : E(D) \to X$ such that $\omega(e^{-1}) = \omega(e)^{-1}$. A labeling of $D$ (or a complex $D$ itself labeled by $X$) is called reduced if there is no cancellation in any word read on two subsequent edges.

The faces of a labeled complex $D$ are in $C'_\ell(\lambda)$–small cancellation, if for every inner segment $s$ in $D$ we have

$$\ell(s) < \lambda \min \{\ell(r) \mid r = \omega(\partial\Pi), \Pi \text{ is a face in } D\}.$$ 

In this case, we say that $D$ satisfies the $C'_\ell(\lambda)$–small cancellation condition.

**Definition 1.1.** A van-Kampen diagram for a word $w$ (in letters from $X \sqcup X^{-1}$), over a set of relators $R$, is a finite planar connected and simply-connected 2-complex $D$ labeled by $X$ such that

- The boundary $\partial D$ is labeled by the word $w$;
- The faces of $D$ are simply-connected;
- The boundary $\partial\Pi$ of each face bears a word $r \in R$.

The fundamental van-Kampen lemma [LS77, Ch.V.1 & Ch.V.9] states that a word $w$ represents an element of the normal subgroup of $F$ generated by $R$ if and only if there exists a van-Kampen diagram for $w$ over $R$. If a set of relators $R$ satisfies the $C'_\ell(\lambda)$–small cancellation condition then the faces of any van-Kampen diagram over $R$ are in $C'_\ell(\lambda)$–small cancellation.

**Theorem 1.2** (Classical small cancellation lemma). Let $0 < \lambda \leq \frac{1}{6}$. Let $D$ be a finite planar labeled simply-connected 2-complex with simply-connected faces in $C'_\ell(\lambda)$–small cancellation.

- If $D$ has more than two faces then there are at least two exterior faces $\Pi$ such that

$$|\partial_{ext}\Pi|_{\ell} > (1 - 3\lambda)|\partial\Pi|_{\ell},$$

where $|\cdot|_{\ell}$ denotes the metric of a length function $\ell$.
\( \partial_{\text{int}} \Pi \) consists of at most three inner segments, and \( \partial_{\text{ext}} \Pi \) is connected.

- \( D \) satisfies the linear isoperimetric inequality
  \[
  |\partial D|_\ell \geq (1 - 6\lambda) \sum_{\Pi_i \text{ is a face in } D} |\partial \Pi_i|_\ell,
  \]
  the boundary \( \partial D \) is at least as long as (with respect to the length function \( \ell \)) the boundary \( \partial \Pi \) of a face \( \Pi \) in \( D \).

A well-known refinement of this theorem is the Greendlinger lemma [LS77, Th. 4.5, Ch. V].

**Corollary 1.3.** Let \( G \) be a \( C'_\ell(\frac{1}{6}) \)-small cancellation group. Then

- \( G \) satisfies a linear isoperimetric inequality with respect to \( \ell \);
- If \( F \) is torsion-free and no relator in \( R \) is a proper power, then \( G \) is torsion-free.

### 1.2. Rips-Segev’s small cancellation conditions.

An \( R \)-sequence for a word \( w \) is a sequence of relators \( r_1, \ldots, r_n \in R \) such that \( w = F \prod_{i=1}^{n} u_i r_i u_i^{-1} \), where \( u_i \in F \) and the equality is in \( F \). An \( R \)-sequence \( r_1, \ldots, r_n \) for \( w \) is called minimal if \( n \) is minimal among the \( R \)-sequences for \( w \).

A van-Kampen diagram for a word \( w = 1 \) is called minimal van-Kampen diagram if the number of its faces is minimal among the van-Kampen diagrams for \( w \). A minimal \( R \)-sequence \( r_1, \ldots, r_n \) for \( w \) corresponds to a minimal van-Kampen diagram for \( w \) the boundaries of whose faces are labeled \( r_1, \ldots, r_n \). Lyndon and Schupp studied minimal van-Kampen diagrams to solve the word and conjugacy problem for \( C'_\ell(\frac{1}{6}) \)-groups [Lyn66, Sch68]. In their terminology minimal van-Kampen diagrams are referred to as of diagrams of minimal \( R \)-sequences.

Appel-Schupp subsequently developed the following \( C(4) - T(4) \)-small cancellation for minimal sequences [AS72]. Their conditions imply small cancellation conditions only on minimal van-Kampen diagrams (in contrast to all van-Kampen diagrams in the classical \( C'_\ell(\lambda) \)-case).

Suppose that the set of relators \( R \) satisfies the following conditions.

1. The relators in \( R \) have length 4.
2. If \( r_1, r_2 \in R \) cancel two or more letters, then either \( r_2 = r_1^{-1} \) or \( r_1 r_2 \) is in \( R \).
3. If \( r_1, r_2, r_3 \in R \), and there is cancellation in all the products \( r_1 r_2, r_2 r_3 \), and \( r_3 r_1 \), then \( r_1 r_2 r_3 \) is a product of at most two elements of \( R \).

If the conditions (1)–(3) hold, the corresponding group presentation satisfies the \( C(4) - T(4) \)-small cancellation condition for minimal van-Kampen diagrams.

Conditions (1) and (2) unify to the \( C(4) \)-condition for minimal van-Kampen diagrams over \( R \), which states that every inner face of a minimal van-Kampen diagram has 4 inner segments. Condition (3) is called the \( T(4) \)-condition. Note that, as relators have length 4, the set \( R \) is finite whenever \( X \) is finite.

The given geometry of minimal van-Kampen diagrams under the \( C(4) - T(4) \)-small cancellation condition allows to apply a classical small cancellation lemma [LS77, Th. 6.3, 7.3, Ch. V], cf. Theorem 1.2(1). This solves the word and conjugacy problems for groups satisfying \( C(4) - T(4) \)-small cancellation for minimal van-Kampen diagrams. For instance, groups of tame alternating knots are of this type [AS72].

Gromov’s hyperbolicity of a finitely presented group is characterized by a linear word problem: given a finite presentation of the group the minimal van-Kampen diagrams over this presentation have to satisfy a linear isoperimetric inequality (w.r.t. the word length metric), c.f. Theorem 1.2(2). Note that a group with \( C(4) - T(4) \)-small cancellation for minimal van-Kampen
diagrams can have quadratic word problem, and thus the $C(4) - T(4)$-condition for minimal sequences is not sufficient to imply that the group is Gromov hyperbolic.

Rips-Segev [RS87, p. 123], to obtain the final conclusion that their groups are torsion-free and without unique product, refer to a $C([p/2])$-condition for minimal van-Kampen diagrams ($p$ is a large number depending on the graph which defines their group presentation) and wish to apply the results of the classical small cancellation theory [LS77]. However, such a condition is not described in [LS77]:

- In Lyndon-Schupp’s book only the $C(4)$-condition for minimal van-Kampen diagrams is treated. The above definition can however easily be generalized. Instead of the above condition (1), assume that $R$ has length $[p/2]$. Then condition (2) implies that minimal van-Kampen diagrams satisfy the $C([p/2])$-condition that every inner face has not less than $[p/2]$ inner segments. Then $R$ is finite, but Rips-Segev need to consider an infinite set $R$.

- However, in the Rips-Segev groups every minimal van-Kampen diagram $D$ over $R$ satisfies the $C([p/2])$-condition that every inner face has at least $[p/2]$ inner segments [RS87, L. 4]. Rips-Segev’s reference [LS77] would provide tools to study such diagrams.

In particular, Rips-Segev refer to the Greendlinger lemma [LS77, Th. 9.3, Ch. V] and the torsion theorem [LS77, Th. 10.1, Ch. V], which states that a $C'(1/8)$-group with the property that no relator is a proper power is torsion-free. Both results are only available in specific cases of the classical small cancellation conditions. The Greendlinger lemma is only available for the metric small cancellation conditions. Lyndon-Schupp’s torsion theorem [LS77, Th. 10.1, Ch. V] is not applicable if we replace the $C'(1/8)$-condition on all relators and not only on those including inner segments of a minimal diagram. So, it is not clear what are the additional assumptions needed. Tools and notations are missing.

To solve this problem, we consider a new small cancellation condition for minimal van-Kampen diagrams, which has not been investigated in the literature, but has implicitly been introduced in Rips-Segev’s paper.

A reduced word $p$ is a Rips-Segev piece if for two relators $r_1 = up$ and $r_2 = p^{-1}v$, the product $uv$ cancels neither to 1 nor to another relator, see [RS87, p. 117].

We say that $R$ satisfies Rips-Segev-$C'_\ell(\lambda)$-small cancellation, if for every Rips-Segev piece we have that
\[\ell(p) < \lambda \min\{\ell(r) \mid r \in R\}.\]

In this case, we refer to the group $G$ as to a Rips-Segev-$C'_\ell(\lambda)$-small cancellation group. Its presentation $G = \langle X, R_F \sqcup R \rangle$ is called a Rips-Segev-$C'_\ell(\lambda)$-small cancellation presentation.
over a free group (resp. over a free product), whenever $F$ is a free group (resp. whenever $F$ is a free product).

Rips-Segev had considered an infinite set $R$ of arbitrarily long relators. The set $R$ satisfies the Rips-Segev-$C'_s(1/[p/2])$–condition, which is the Rips-Segev-$C'_s((1/[p/2])$–condition over a free product and $\ell$ is the free product length. The relators have certain long common parts whose lengths are not controlled by the $C'_s(1/[p/2])$–condition.

However, minimal van-Kampen diagrams satisfy the $C'_s(1/[p/2])$–condition over a free product. In this sense, the Rips-Segev small cancellation condition is a generalization in the free product setting of the above $C'_s(1/[p/2])$– and $C'(p/2)$–small cancellation conditions for minimal van-Kampen diagrams.

Again, arguments that the classical results [LS77, Th. 9.3, Th. 10.1, Ch. V] are true for the Rips-Segev small cancellation are missing. We provide a full proof, and therefore all details necessary for the Rips-Segev construction.

- If a minimal van-Kampen diagram $D$ is reduced, this yields the Greendlinger lemma [LS77, L. 9.3, Ch. V]. By the proof of [LS77, L. 9.2(2), Ch. V], $D$ is reduced if inner segments of $D$ are pieces in the classical sense. Assume $u$ is not a Rips-Segev piece and glues $\Pi_1$ and $\Pi_2$ in $D$. Then we delete $u$ from $D$. By definition, the word $r_1r_2$ read on the new face is a relator. But then the minimal van-Kampen diagram for $w$ was not minimal, a contradiction. Hence, the Greendlinger lemma [LS77, L. 9.3, Ch. V] can be extended to the Rips-Segev small cancellation.

- The proof of the torsion-theorem cannot be obtained using Lyndon-Schupp’s arguments. In particular, Lyndon-Schupp’s proof uses that if $r = x^m a$ (in semi-reduced form), $m > 1$ and $r$ is not a proper power, then $x$ and $x^{m-1}$ are pieces [LS77, L. 10.2, Ch. V]. This is false in Rips-Segev’s situation.

We give a proof of the torsion-theorem for the Rips-Segev group presentations and our generalizations of these presentations. The proof is using the specific structure of Rips-Segev’s relators, see Theorem 1.14 and Remark 1.7 below. We employ methods recently developed in the context of the graphical small cancellation theory [Oll06]. Specifically, we use them in the context of the graphical small cancellation condition over a free product. This specific case of the graphical small cancellation conditions has not been considered before.

To conclude this section, let us note that groups defined by a Rips-Segev-$C'_s(1/\pi)$–presentation are not a priori Gromov hyperbolic. For Rips-Segev’s infinite presentation, the minimal van-Kampen diagrams have faces in $C'_s(1/\pi)$–small cancellation. Therefore they satisfy a linear isoperimetric inequality with respect to the free product metric, hence also for the word length metric, cf. Theorem 1.2(2). The problem is that minimal van-Kampen diagrams over a finite presentation do not satisfy a $C'_s(1/6)$–small cancellation condition as not all inner segments are controlled by the Rips-Segev small cancellation condition. Therefore, it can a priori not be concluded that the group is Gromov hyperbolic. Our approach also overcomes this issue.

1.3. Graphical group presentations. The degree $d(v)$ of a vertex $v$ in a graph is the number of edges at $v$. Let $\Omega$ be a finite connected reduced labeled (in the above introduced terminology) graph with no vertices of degree one. A cycle $c$ in $\Omega$ is a sequence of edges $e_1 = (v_1, w_1), \ldots, e_i = (v_i, w_i), \ldots, e_n = (v_n, w_n)$ of $\Omega$ such that $w_i = v_{i+1}$ and $v_1 = w_n$. It is simple if no edge $e$ or its inverse $e^{-1}$ occurs more than once in $c$. 

Definition 1.4. Let $C$ be a set of simple cycles generating the fundamental group of $\Omega$. Let $R$ be the set of words read on $C$. A group $G(\Omega) = \langle F \mid R F \sqcup R \rangle$ is the graphically presented group defined by $\Omega$ over $F$.

Note that $G(\Omega)$ does not depend on the choice of $C$. Since we can choose $C$ to be finite, the group $G(\Omega)$ is finitely presented whenever $F$ is a finitely presented group.

An immersion of labeled graphs is a locally injective graph morphism which preserves the labellings.

Definition 1.5. A piece in $\Omega$ is a labeled path which has at least two distinct immersions in $\Omega$.

Definition 1.6. A reduced labeling of $\Omega$ satisfies $Gr^r_t(\lambda)$–graphical small cancellation condition for $\lambda > 0$ with respect to a length function $\ell$ if

$$\Lambda < \lambda,$$

where

$$\Lambda := \max \{ \ell(p) \mid p \text{ is a piece in } \Omega \} \text{ and } \gamma := \min \{ \ell(r) \mid r \text{ is a label of a cycle in } \Omega \}.$$

In this case, we refer to the group $G(\Omega)$ as to a $Gr^r_t(\lambda)$–graphical small cancellation group. Its presentation $G(\Omega) = \langle X, R F \sqcup R \rangle$ is called a graphical $Gr^r_t(\lambda)$–small cancellation presentation over a free group (resp. over a free product), whenever $F$ is free group on $X$ (resp. whenever $F$ is a free product).

If $F$ is a free group on $X$ and the length function is the word length, then the $Gr^r_t(\lambda)$–graphical small cancellation condition reduces to the graphical small cancellation condition $Gr^r_t(\lambda)$ as defined by Gromov [Oll06].

Remark 1.7. In the context of graphically presented groups, the Rips-Segev-$C^r_t(\lambda)$–small cancellation should be thought of as of the $Gr^r_t(\lambda)$–small cancellation on $\Omega$.

More precisely, let $R$ be the set of words read on all simple cycles of $\Omega$. If all $r \in R$ are no proper powers and if $R$ satisfies the Rips-Segev $C^r_t(\lambda)$–small cancellation condition, then the corresponding reduced labeling of $\Omega$ satisfies $Gr^r_t(\lambda)$–small cancellation. On the other hand, if a reduced labeling of $\Omega$ satisfies $Gr^r_t(\lambda)$–small cancellation, then $R$ satisfies Rips-Segev-$C^r_t(\lambda)$–small cancellation. In particular, if $R$ satisfies only the Rips-Segev small cancellation condition, then we do not expect the corresponding group to be torsion-free. In general, we do not expect that groups with a finite Rips-Segev-$C^r_t(\lambda)$–small cancellation presentation have solvable word and conjugacy problem.

In this paper we present results on groups satisfying the graphical small cancellation conditions over a free product.

1.3.1. Graphical small cancellation lemma. Let $\Omega$ be endowed with a reduced labeling that satisfies graphical $Gr^r_t(\lambda)$–small cancellation. We fill cycles in $C$ with a disc. Let $\tilde{\Omega}$ denote the 2-complex so obtained.

Let $D$ be a van-Kampen diagram over $R$. Let $\Pi$ be a face of $D$ with boundary word $r$. The face $\Pi$ is a copy of the 2-cell in $\tilde{\Omega}$ with the same boundary word $r$. We say, $\Pi$ lifts to $\tilde{\Omega}$. The edges in the boundary of $\Pi$ lift to $\tilde{\Omega}$ with $\Pi$.

Remark 1.8. The lift of $\Pi$ is indeed unique. Given a second lift to $\tilde{\Omega}$, two distinct cycles in $\Omega$ had the label $r$. This contradicts the graphical small cancellation condition on $\Omega$. 

Definition 1.9. [Oll06] An edge which glues two faces $\Pi_1, \Pi_2$ in $D$ originates in $\Omega$ if its lifts with $\Pi_1$ and $\Pi_2$ coincide.

Lemma 1.10 ([Oll06, Corollary 14]). A simply-connected finite labeled 2-complex consisting of faces in $C'(\frac{1}{6})$–small cancellation can only contain simply-connected faces.

This lemma is an application of the classical small cancellation lemma, Theorem 1.2. Moreover, we observe that the word length metric can be replaced by an arbitrary length function $\ell$ in Olliver’s proof [Oll06, Corollary 14].

Let $D$ be a van-Kampen diagram over $R$. It is simply-connected by definition. Delete all edges in $D$ originating in $\Omega$. The remaining 2-complex $\tilde{D}$ is simply-connected with reduced boundary. The boundaries of its faces can be reduced [Oll06].

The words on the faces of $\tilde{D}$ correspond to reduced cycles in $\Omega$. The $Gr'_{\ell}(1/6)$–graphical small cancellation condition on $\Omega$ implies that the faces of $\tilde{D}$ are in $C'(\frac{1}{6})$–small cancellation. By the above lemma the faces of $\tilde{D}$ are simply-connected. We now apply the classical small cancellation lemma, Theorem 1.2.

Theorem 1.11 (Graphical small cancellation lemma). Let $0 < \lambda \leq \frac{1}{6}$. Let $D$ be a van-Kampen diagram over a $Gr'_{\ell}(\lambda)$–presentation.

- If $\tilde{D}$ has more than two faces then there are at least two exterior faces $\Pi$ in $\tilde{D}$ such that
  \[|\partial_{\text{ext}} \Pi|_\ell > (1 - 3\lambda) |\partial \Pi|_\ell,\]
  $\partial_{\text{int}} \Pi$ consists of at most three pieces, and $\partial_{\text{ext}} \Pi$ is connected.
- The following inequality is satisfied
  \[|\partial D|_\ell \geq (1 - 6\lambda) \sum_{\Pi_i \text{ is a face in } \tilde{D}} |\partial \Pi_i|_\ell,\]
  the boundary $\partial D$ is at least as long as (with respect to the length function $\ell$) the boundary $\partial \Pi$ of a face $\Pi$ in $\tilde{D}$.

Assume now that $\ell = |\cdot|_\ast$, and let $D$ be a minimal van-Kampen diagram over a $Gr'_{\ell}(\frac{1}{6})$–presentation. The faces of $D$ are van-Kampen subdiagrams of $D$ all of whose inner edges originate in $\Omega$.

Lemma 1.12. If all inner edges of $D$ originate in $\Omega$, then $D$ satisfies a linear isoperimetric inequality

\[|D| \leq C|\partial D|_\ast.\]

Proof. We extend the proof of [Oll06, Lemma 11] for the word length metric to free product length $|\cdot|_\ast$. It suffices to prove the claim for a special choice of $R$. Let $R$ be the words read on cycles $c$ such that $|c|_\ast \leq 3 \text{diam}(\Omega)$. We prove that a minimal van-Kampen diagram $D$ for $w$ over $R$ and with all its edges originating in $\Omega$ satisfies $|D| \leq \frac{3|w|_\ast}{\gamma}$, where $\gamma = \min\{|r|_\ast | r \text{ is the label of a cycle in } \Omega\}$.

If $|w|_\ast \leq 2 \text{diam}$, where $\text{diam} = \text{diam}(\Omega)$, there is a diagram with one face. If $2 \text{diam} \leq |w|_\ast \leq (n + 1) \text{diam}$,
$w$ is the concatenation of paths $w'$ and $w''$, so that $|w'|_* = 2 \text{diam}$. There is a path $x$ connecting the end and the starting vertex of $w'$ such that

$$|x|_* \leq |x| \leq \text{diam};$$

and w.l.o.g. (choose $w'$ such that $|w''|_* = |w|_* - |w'|_*$)

$$|xw''| \leq |w|_* - \text{diam} \leq n \text{diam}.$$

Observe that $|w'x|_* \leq 3 \text{diam}$. The word $w'x$ has a van-Kampen diagram consisting only of one face. Iterating yields that $D$ has at most $1 + \frac{|w|_*}{\text{diam}}$ faces. Since $\text{diam} \geq \frac{\text{girth}}{2} \geq \frac{\gamma}{2}$ and $|w|_* \geq \gamma$, we have that $|D| \leq \frac{3|w|_*}{\gamma}$.

1.3.2. **Main small cancellation theorems.** The above lemma and Theorem 1.11 implies the linear isoperimetric inequality for minimal van-Kampen diagrams for an arbitrary graphical small cancellation presentation: Given a minimal van-Kampen diagram $D$, some edges may originate in $\Omega$. We delete originating edges and obtain the diagram $\tilde{D}$. We apply Theorem 1.11 to $\tilde{D}$ and obtain the inequality $|\partial D|_\ell \geq (1 - 6\lambda) \sum_{\text{face in } \tilde{D}} |\partial \Pi_\ell|_\ell$. The faces of $\tilde{D}$ are van-Kampen subdiagrams of $D$ all of whose inner edges originate in $\Omega$. They all satisfy a linear isoperimetric inequality by the lemma. Combining the inequalities, we obtain a linear isoperimetric inequality for $D$ over the given presentation. This yields our first main result on the graphical small cancellation over a free product.

We therefore consider a possibly infinite family of finite connected graphs, which has globally bounded vertex degree. The graphical small cancellation condition as well as the definition of graphically presented groups extend to this situation in a straightforward way.

**Theorem 1.13.** Let $G_1, \ldots, G_n$ be finitely generated groups. Let $\Omega$ be a family of finite connected graphs with a reduced labeling by $G_1 \cup \ldots \cup G_n$. Suppose a labeling of $\Omega$ satisfies $Gr'_*\left(\frac{1}{6}\right)$ over the free product $G_1 \ast \cdots \ast G_n$. Let $G$ be the group presented by $G_1 \cup \ldots \cup G_n$ as generators and the words read on a family of simple cycles generating the fundamental group of $\Omega$ as relators. Let $D$ be a minimal van-Kampen diagram over the given presentation of $G$.

Then $D$ satisfies the linear isoperimetric inequality

$$|D| \leq C|\partial D|_*.$$

If $\Omega$ is finite and $G_1, \ldots, G_n$ are Gromov hyperbolic, then $G$ is Gromov hyperbolic.

**Theorem 1.14.** Let $G_1, \ldots, G_n$ be finitely generated torsion-free groups. Let $\Omega$ be a family of finite connected graphs with a reduced labeling by $G_1 \cup \ldots \cup G_n$. Suppose $\Omega$ satisfies $Gr'_*\left(\frac{1}{6}\right)$ in the free product $G_1 \ast \cdots \ast G_n$. Let $G$ be the group presented by $G_1 \cup \ldots \cup G_n$ as generators and the words read on the cycles of $\Omega$ as relators.

Then $G$ is torsion-free.

**Proof.** We extend [LS77, Th 10.1, Ch. V].

Let $w$ be a word in the free group. Let $|z| > 1$ be an element of least length among all conjugates of $w$ in $G$ of order $n \geq 2$ in $G$. (All conjugates have the same order.) Let $D$ be a van-Kampen diagram for $z^n$. We can assume the boundaries of the faces of $\tilde{D}$ are not proper powers and not a concatenation of cycles in $\Omega$.

Indeed, let $\overline{\Pi}$ be a face in $\tilde{D}$, and $\overline{r}$ the label of $\partial \overline{\Pi}$. Assume $\overline{r} = a^m$, $m \geq 2$, and $\overline{r}$ is reduced. As all inner edges of $\tilde{D}$ are originating in $\Omega$, $\overline{r}$ is a word on a cycle in $\Omega$. If $a$ is not a simple
cycle in $\Omega$, $a$ and $a^{m-1}$ are pieces. This contradicts $Gr'_I(\frac{1}{6})$. Thus, $a$ is a simple cycle in $\Omega$. Then we replace the face $\overline{\Pi}$ by the $m$-rose consisting of $m$-faces labeled $a$ glued at one point. If $r$ is a concatenation of $m$ different cycles $c_i$, we replace $\overline{\Pi}$ by the $m$ rose consisting of $m$-faces with boundary $c_i$ glued at one point.

By Theorem 1.11, $z^n = uz'$, where $u$ is a subword of a word $r \in R$ which is not a concatenation of cycles, and such that $|u| > \frac{1}{2} |r|$. By the minimality condition on $z$, $u$ is not a subword of $z$. Then $u = z^m t$, where $m \geq 1$ and such that $t$ does not begin with a power of $z$. Write $z = ts$. Then $r = uv = (ts)^mtv$.

If $m > 1$, as $r$ is not a concatenation of smaller cycles, $(ts)^{m-1}, (ts)$, and $t$ are pieces. Then $u$ would be the product of three pieces. Thus $|u| < \frac{1}{2} |r|$, a contradiction.

Hence $r = tsv$. If $ts = tv$, $r$ and $z$ are powers of a common subword. This word would be a piece as $r$ is not a concatenation of smaller cycles. This contradicts the $Gr'_I(\frac{1}{6})$-condition. Thus, $t$ is a piece, and $|t| < \frac{1}{6} \min\{|r| \mid r \in R\}$. But Theorem 1.11, in addition to the existence of $u$, implies the existence of a word $u'$ such that $u'$ is a subword of $z' = t^{-1}$, $u'$ is a subword of $r' \in R$ and $|u'| > \frac{1}{2} |r'|$, contradicting that $t$ is a piece.

$\square$

2. Construction of Rips-Segev’s groups

Let $A$ and $B$ be nonempty finite sets of reduced words in $X \sqcup X^{-1}$. The product of $A$ and $B$ is the set $AB = \{ab \mid a \in A, b \in B\}$.

Let $G$ be a group generated by $X$. Let $A$ and $B$ be nonempty finite subsets of $G$. If an element $x$ in $AB$ has a unique expression in $G$ as a product $x = ab$ for $a \in A$ and $b \in B$, then $A$ and $B$ are said to have a unique product property or to be a unique product group.

**Definition 2.1.** If for all nonempty finite subsets $A$ of $G$ and for all nonempty finite subsets $B$ of $G$, the sets $A$ and $B$ have a unique product in $G$, then $G$ is said to have the unique product property or to be a unique product group.

We construct groups without the unique product property, that is, admitting at least two nonempty finite subsets which do not have a unique product in the group.

We first consider an instructive example. Let $A = \{a, ab\}$ and $B = \{1, b\}$. The graphical presentation of $AB$ is the subgraph of the Cayley graph of the free group on $a$ and $b$, which for all $x \in A$ and all $y \in B$ contains the simple paths which connect $x$ with the product $xy$. It has the following geometry.

There are two vertices marked by $o$ which represent unique products in the free group, whereas the vertex $\bullet$ represents a product which is not unique.

To obtain a group in which $A$ and $B$ do not have a unique product, we identify the vertices $o$. We obtain the following graph.
This graph has two vertices • each of which represent two different products in \( AB \). The group defined by this graph has the presentation \( \langle a, b \mid b^2 \rangle \). The subsets \( A \) and \( B \) do not have a unique product in this group.

We have therefore constructed a graphical presentation of a group without the unique product property.

Given nonempty finite sets \( A \) and \( B \) of reduced words in \( X \sqcup X^{-1} \), we formalize this construction of a graph of which no vertex represents a unique product in \( AB \). We then give conditions on such graphs which imply that the corresponding graphically presented groups do not have the unique product property.

### 2.1. Graphs encoding the non-unique product property.

Given two nonempty finite sets \( A \) and \( B \) of reduced words in \( X \sqcup X^{-1} \), we define \( \Omega \), the graphical presentation of \( AB \). The graph is the subgraph of the Cayley graph of the free group on \( X \) which for all \( a \in A \) and for all \( b \in B \) contains the simple paths which connect the identity with \( a \) and the simple paths which connect \( a \) with \( ab \). Every element of \( A \) and every product in \( AB \) is a distinguished vertex in this graph. A vertex \( o \) displays a unique product, whereas a • represents at least two products in \( AB \). In our figures, we omit the paths from the identity to vertices in \( A \).

**Example 2.2.** Let \( c \) be a reduced word in \( a \) and \( b \) and \( C \in \mathbb{N} \). Let \( v_i := ca^i \) and \( w_i := ca^ib \). Let \( A := \{v_0, v_1, v_2, \ldots, v_C-1\} \) and \( B := \{1, a, b, ab\} \). The product set

\[
AB = \{v_0, \ldots, v_C, w_0, \ldots, w_C\}
\]

has the following graphical presentation \( \Omega \).

![Graphical presentation of AB](image)

Starting with \( \Omega \), the graphical presentation of \( AB \), we now construct a graph in which no vertex represents exactly one product in \( AB \).

For each unique product \( x \) in the free group, choose \( y \in AB \) different from \( x \). Identify each pair of vertices \( x \) and \( y \) in \( \Omega \). Then we apply a folding: Identify each two subsequent edges, whenever the word read on these edges cancels to the empty word, and repeat this process as long as possible. The resulting graph is reduced. Finally, repeatedly delete all vertices of degree one. If a graph so obtained is non-trivial, we say it has a non-unique product geometry for \( A \) and \( B \).

Given a graph \( \Gamma \) which has a non-unique product geometry for \( A \) and \( B \), we denote by \( A' \) and \( B' \) the images of the sets \( A \) and \( B \) in the corresponding graphical group \( G(\Gamma) \).

**Proposition 2.3.** Let \( \Gamma \) be a graph which has a non-unique product geometry for \( A \) and \( B \). Suppose that the maps \( \iota_A : A \to A' \) and \( \iota_B : B \to B' \) are injective, then \( A \) and \( B \) do not have a unique product in \( G(\Gamma) \).

**Proof.** For every product \( x = ab \) which is unique in the free group (or in the free product of groups, respectively) we find a cycle \( ab(a'b')^{-1} \) in \( \Gamma \) such that \( a \neq a' \) in \( A \) and \( b \neq b' \) in \( B \). As \( \iota_A : A \to A' \) and \( \iota_B : B \to B' \) are injective, \( a \neq a' \) and \( b \neq b' \) in \( G(\Gamma) \). \( \square \)
2.2. **Generalized Rips-Segev graphs.** Let $G_1$ and $G_2$ be torsion-free groups. Let $a \neq 1 \in G_1$ and $b \neq 1 \in G_2$. Let $c_i$ denote elements of $G_1 * G_2$ specified below. Let $v_{il} := c_i a^l$ and $w_{il} := c_i a^l b$. Given a non-zero natural number $n$ and $n$ non-zero natural numbers $C_i$, we denote by $A$ the set

$$A = \bigcup_{i=1,...,n} \{v_{i0}, v_{i1}, v_{i2}, \ldots, v_{i(C_i-1)}\}$$

and let $B := \{1, a, b, ab\}$. Then

$$AB = \bigcup_{i=1,...,n} \{v_{i0}, v_{i1}, \ldots v_{iC_i}, w_{i0}, \ldots w_{iC_i}\}.$$

The graphical presentation of products $AB$ consists of the disjoint union of $n$ graphs as shown in Example 2.2 above. We call $\{v_{i0}, v_{i1}, \ldots, v_{iC_i}\}$ the $a$-line $i$. For each $i$ there are four vertices representing a unique product, $v_{i0}$, $v_{iC_i}$, $w_{i0}$ and $w_{iC_i}$.

For $n = 3$, and $C_1 = C_2 = C_3 = 4$, the Figure 1 below shows two examples of graphs which have a non-unique product geometry for $A$ and $B$.

![Figure 1](image-url)

**Figure 1.** Two graphs with a non-unique product geometry for $A$ and $B$, where $n = 3$, and $C_1 = C_2 = C_3 = 4$. The full $b$-edges are those glued in Step I below. The dashed $b$-edges are those glued in Step O below. In the graph on the right hand-side no $b$-edges are glued in Step O.

To investigate the graphs which have a non-unique product geometry for $A$ and $B$, we now further formalize the construction of such graphs.

**Step I**, see Figures 2, 3. For each of the vertices $w_{i0}$ and $w_{iC_i}$ we choose a vertex $v_{ij}$ such that $0 \leq I \leq C_j$ and the pairs $(j, I)$ are all different among themselves. Then we identify $w_{i0}$ (or $w_{iC_i}$) and $v_{ij}$.

The vertices which have been identified in Step I now represent at least two products of $AB$. We have possibly identified some vertices $w_{i0}$ or $w_{iC_i}$ with a vertex $v_{j0}$ or $v_{jC_i}$, see Figure 3. In a second step, we take care of the vertices $v_{j0}$ and $v_{jC_i}$, which have not been identified in Step I.

**Step O**, see Figure 4. If there is no vertex $w_{i0}$ or $w_{iC_i}$ which has been identified with $v_{j0}$ (or $v_{jC_i}$), we choose a vertex $w_{iO}$, so that $0 < O < C_i$ and the index pairs $(l, O)$ are all different among themselves. Then we identify $v_{j0}$ (or $v_{jC_i}$) and $w_{iO}$.
In the resulting graph every vertex represents at least two products in $AB$. Moreover, the graph is reduced. It has exactly $n$ of the $a$-lines.

We assume the choices have been made so that the resulting graph is connected. It is easy to see that this is always possible.

**Final step.** We delete all edges of degree one (all $b$-edges that have not been glued). Then we set $c_1 := 1$. Then $c_i$ is the word read on the shortest path in this graph with starting vertex the vertex representing $c_1$, and with terminal vertex the vertex representing $c_l$.

The graph so obtained has a non-unique product geometry for $A$ and $B$.

**Figure 2.** Step I: The vertices $w_j i C_i$ and $v_j l$ have been identified.

**Figure 3.** Step I: The vertices $w_i C_i$ and $v_j C_j$ have been identified.

**Figure 4.** Step O: The vertices $w_l O$ and $v_j 0$ have been identified.
Each a-line \( \ell \) in this graph has \( s_\ell \) vertices \( v_{i_1}, \ldots, v_{i_{s_\ell}} \) arising from Step I, and \( t_\ell \) vertices \( v_{i_1}, \ldots, v_{i_{t_\ell}} \) arising from Step O. Both \( s_\ell \) and \( t_\ell \) can be zero.

Let us denote the tuple \((I_1, \ldots, I_{i_{s_\ell}}, O_{i_{t_\ell}}, C_i)\) by \((I_1, O_1, C_i)\). Let \((I, O, C)\) denote the disjoint union \( \bigsqcup_{i=1}^n (I_1, O_1, C_i)\).

We call the graph a \textit{generalized Rips-Segev graph with coefficients \((I, O, C)\)}. We order the vertices of a generalized Rips-Segev graph with coefficients \((I, O, C)\), so that \(0 \leq I_1 < I_2 < \ldots < I_{i_{s_\ell}} \leq C_i\) and \(0 < O_1 < O_2 < \ldots < O_{i_{t_\ell}} < C_i\).

Below we show a local picture of a typical a-line in a generalized Rips-Segev graph which has \( s = 3 \) and \( t = 2 \).

\[ \text{Figure 5. Typical a-line.} \]

There can be extremal cases. It can happen that \( I_{j_1} = O_{j_2} \) for some \( k, l \):

\[ \text{Figure 6. } I_{j_1} = O_{j_2} \text{ for some } k, l. \]

If \( s = 2, t = 0, I_{j_1} = c_i \) and \( I_{j_2} = C_i \), the a-line looks as follows:

\[ \text{Figure 7. } s = 2, t = 0, I_{j_1} = c_i \text{ and } I_{j_2} = C_i. \]

A generalized Rips-Segev graph has at least \( 2n \) and at most \( 4n \) b-edges. If the number of b-edges is \( 2n \), then for each \( i \) there are two coefficients \( I_{i_{d_1}} \) and \( I_{i_{d_2}} \) which equal to \( 0 \) and \( C_i \) respectively. If the number of b-edges is \( 4n \), no coefficient \( I_i \) is \( 0 \) or \( C_i \) for all \( i \).
Let us now describe conditions on \( \Gamma \) which imply that \( A \) and \( B \) do not have a unique product in \( G(\Gamma) \).

**Proposition 2.4.** Suppose \( \Gamma \) is a generalized Rips-Segev graph for \( A \) and \( B \) whose the labeling satisfies the \( Gr'_{1/6} \)-graphical small cancellation condition. Then \( A \) and \( B \) do not have a unique product in the corresponding torsion-free group \( G(\Gamma) \).

**Proof.** Let \( \Gamma \) be as above. Observe that it is connected. Let \( R \) be the set of the words read on all reduced cycles of \( \Gamma \). Denote by \( A' \) and \( B' \) the image of \( A \) and \( B \) in the corresponding group \( G(\Gamma) \). We show using the graphical small cancellation condition that the maps \( \iota_A : A \to A' \) and \( \iota_B : B \to B' \) are injective. Therefore, \( A \) and \( B \) do not have a unique product in \( G(\Gamma) \) by Proposition 2.3.

Indeed, suppose that \( a_1 \neq a_2 \in A \) and \( a_1 =_{G(\Gamma)} a_2 \) where the equality is in \( G(\Gamma) \). Let \( p \) be a path on \( \Gamma \) connecting \( a_1 \) and \( a_2 \). Let \( x \) be the label of \( p \). As the label of \( \Gamma \) is reduced, the word \( x \) is non-trivial, and \( x =_{G(\Gamma)} 1 \), where the equality is in \( G(\Gamma) \). Let \( D \) be a minimal van-Kampen diagram for \( x \) over \( R \). Then \( D \) has no inner edges originating in \( \Gamma \).

The graphical small cancellation lemma, Theorem 1.11, implies that \( |x|_* \geq 6 \).

We choose \( p \) so that the number of faces of \( D \) is minimal among the number of faces of minimal van-Kampen diagrams for paths connecting \( a_1 \) with \( a_2 \) in \( \Gamma \).

If \( \Pi \) is an exterior face in \( D \) let \( y \) denote the word on \( \partial_{ext} \Pi \). If the lift of \( \Pi \) to \( \Gamma \) is such that the lift of \( y \) with \( \Pi \) coincides with the lift of \( x \) to \( p \), then \( \Pi \) is called originating with \( x \).

In \( D \) there is no face originating with \( x \). If there was such a face, we could remove it from \( D \). The boundary word of the diagram so obtained then has a lift to \( \Gamma \). This diagram has a lesser number of faces, a contradiction to the choice of \( p \).

The path \( p \) lies on a cycle \( c = pp' \) in \( \Gamma \). Let \( c \) be labeled \( w \) and \( p' \) be labeled \( x' \) so that \( w = xx' \).

We can assume that \( |x|_* \leq |x'|_* \). We have that \( |w|_* \geq |x|_* + |x'|_* - 2 \). Let \( D' \) be a van-Kampen diagram for \( w \) which globally lifts to \( \Gamma \), i.e. all inner edges of \( D' \) originate in \( \Gamma \). Let us glue \( D \) and \( D' \) along \( x \). The diagram so obtained is denoted \( D'' \). The boundary word of \( D'' \) coincides with \( x' \).

As in \( D \) there are no faces originating with \( x \), the inner segments of \( D'' \) which lie on \( x \) are all pieces. Therefore, the small cancellation lemma implies that the boundary length \( |x'|_* \) of \( D'' \) is at least the boundary length \( |w|_* \) of \( D' \). This means that \( |x'|_* \geq |w|_* \geq |x|_* + |x'|_* - 2 \). This inequality contradicts the implication of the small cancellation lemma that \( |x|_* \geq 6 \).

If \( b_1 \neq b_2 \) in \( B \) and \( b_1 =_{G(\Gamma)} b_2 \), then \( |b_1 b_2^{-1}|_* < 3 \). This is a contradiction to the small cancellation lemma. \( \square \)

We have just proved a special case for graphical small cancellation over free products of a theorem of Gromov [Gro03, Oll06, Gru13] which states that, if \( F \) is a free group and \( \Gamma \) satisfies the \( Gr'_{1/6} \)-condition with respect to the word length metric, the map of \( \Gamma \) into the Cayley graph of \( G(\Gamma) \) is an isometric embedding.

In view of Proposition 2.4, we determine conditions so that the labeling of a generalized Rips-Segev graph satisfies \( Gr'_{1/6} \)-graphical small cancellation.

Let \( \Gamma \) be a generalized Rips-Segev graph with coefficients \((I, O, C)\). For each \( a \)-line \( i \), we consider the edge distances between the vertices

\[
\ell_i(v_{iI_1}, v_{iI_2}, \ldots, v_{iI_{i-1}}, v_{iO_1}, v_{iO_2}, \ldots, v_{iO_{O_i}}, v_{iC_i}).
\]
For example, if $s_i = t_i = 2$, then these are the numbers

\[ I_1, I_2, O_1, O_2, C_i, |C_i - O_2|, |C_i - O_1|, |C_i - I_2|, |C_i - I_1|, \]
\[ |O_2 - O_1|, |O_2 - I_2|, |O_2 - I_1|, |O_1 - I_2|, |O_1 - I_1|, |I_2 - I_1|. \]

To describe the global structure of $\Gamma$, let $\mathcal{D}$ be the disjoint union of all these numbers for all $a$-lines of $\Gamma$. We call $\mathcal{D}$ the distances of $b$-edges of $\Gamma$, or, given a set of coefficients $(I, O, C)$ the distances of the numbers $(I, O, C)$.

Given a number $P$, its multiplicity $m_P$ is the number of the occurrences of $P$ in $\mathcal{D}$. Denote by $M$ the sum $\sum_{P : p > 0, m_p \geq 2} m_P$. For example, given $(I_1, I_1, O_1, O_1, C_1) = (1, 1, 2, 5, 9)$ and $(I_1, I_2, O_2, O_2, C_2) = (21, 21, 41, 41, 100)$, then $m_1 = 4$, $m_4 = 4$, $m_8 = 2$ and $M = 10$.

Let us detect which paths in $\Gamma$ are pieces in the sense of Definition 1.5.

Let $p$ be a reduced path in $\Gamma$. Then there are $\varepsilon_i = \pm 1$ and integers $P_i$ such that

\[ P = a^{P_0} b^{\varepsilon_1} a^{P_1} b^{\varepsilon_2} a^{P_2} \ldots a^{P_{j-1}} b^{\varepsilon_j} a^{P_j}. \]

Note that the exponents $P_i$ can be zero. If $P_i$ is zero, then $\varepsilon_i = \varepsilon_{i+1}$. If $0 < i < j$ and $P_i \neq 0$, then $|P_i|$ is a number in $\mathcal{D}$.

If $|P| < \max\{C_1\}$, then $a^P$ is a piece. Clearly, if $\varepsilon = \pm 1$, then $b^\varepsilon$ and $b^{2\varepsilon}$ are pieces. Further pieces are of type $a^{P_0} b^\varepsilon a^{P_1}$ and $a^{P_0} b^{2\varepsilon} a^{P_1}$. Assume that a non-zero number $P$ has $m_P \geq 2$ in $\mathcal{D}$. Then $a^{P_0} b^\varepsilon a^{P_1}$ can be a piece.

Recall that $\Lambda$ denotes the maximal piece length given by $\max\{|p|_* | p \text{ is a piece in } \Gamma\}$.

**Lemma 2.5.** Let $\Gamma$ be a generalized Rips-Segev graph $\Gamma$. Then

\[ \Lambda \leq 2M + 3. \]

**Proof.** Let $p = a^{P_0} b^{\varepsilon_1} a^{P_1} b^{\varepsilon_2} a^{P_2} \ldots a^{P_{j-1}} b^{\varepsilon_j} a^{P_j}$ be a reduced path in $\Gamma$. We can assume that $P_0$ and $P_j$ are non-zero. For $0 < i < j$, if $P_i \neq 0$, the exponent $|P_i|$ is a number in $\mathcal{D}$. Assume $N$ of the exponents $P_i$ are non-zero. Then $|p|_* = 2N + 3$.

If $p$ is a piece, it had two immersions in $\Gamma$. Hence, the non-zero exponents of $p$ had multiplicity at least 2. A piece can therefore have at most $M$ non-zero exponents. This yields the claim. \( \square \)

The minimal cycle length $\gamma$ of $\Gamma$ was defined to be $\min\{|r|_* | r \text{ is a simple cycle in } \Gamma\}$. We estimate $\gamma$ in terms of the minimal number of $b$-edges on simple cycles in $\Gamma$. Therefore, we encode the $b$-edges in $\Gamma$ in a second graph, which we call the graph underlying $\Gamma$.

**Definition 2.6.** The graph underlying $\Gamma$ has $n$ vertices $c_1, \ldots, c_n$, and its edges $(c_i, c_j)$ are in bijection with the $b$-edges that connect the $a$-lines $i$ and $j$ in $\Gamma$.

**Lemma 2.7.** Let $\Gamma$ be a generalized Rips-Segev graph with underlying graph of girth $2g$, then

\[ \gamma \geq 2g - 2m_0 - 1. \]

**Proof.** Suppose $2g > 2m_0 - 1$, otherwise the claim is trivial. Let $p = a^{P_0} b^{2\varepsilon} a^{P_1} b^{2\varepsilon} a^{P_2} \ldots a^{P_{g-1}} b^{2\varepsilon}$ be a reduced simple path. If $m_0 = 0$ all exponents $P_i \neq 0$ and $p$ is a shortest path passing $2g$ $b$-edges by construction. Then $|p|_* = 2g - 1$. Thus $2g - 1 \leq \gamma$ in this case. If $m_0 > 0$, at most $m_0$ exponents among the $P_i$ can be zero. Every additional zero exponent $P_i$ decreases $|p|_*$ by two or one. Thus $2g - 2m_0 - 1 \leq \gamma$. \( \square \)
In a generalized Rips-Segev graph, \( m_0 \leq M \). Indeed, if \( 0 \in \mathcal{D} \), then, for some \( i, 1 \leq i \leq n, 0 \) is either among
\[
I_{i1}, I_{i2}, \ldots I_{is_i}
\]
or among the distances
\[
|C_i - I_{is_i}|, \ldots, |C_i - I_{i1}|, |O_{it_i} - I_{is_i}|, \ldots, |O_{it_i} - I_{i1}|, \ldots, |O_{it_i} - I_{i1}|.
\]
In the first case \( m_{C_i} \geq 2 \), in the second case two coefficients among \((I_i, O_i, C_i)\) equal say \( P \), so \( m_p \geq 2 \).
Combining the above inequalities, we obtain the following condition on \( \Gamma \) to satisfy the \( \text{Gr}^*_1(\frac{1}{6})\)-graphical small cancellation condition.

**Proposition 2.8.** If a generalized Rips-Segev graph has an underlying graph with a girth larger than \( 14M + 19 \), then this generalized Rips-Segev graph satisfies \( \text{Gr}^*_1(\frac{1}{6})\)-small cancellation.

Let \( \Omega = (\Gamma_j)_{j \geq 1} \) be a family of graphs, such that all graphs \( \Gamma_j \) are generalized Rips-Segev graphs which satisfy the assumptions of the above Proposition 2.8. In addition suppose that for all \( j_1 \neq j_2 \) no non-zero number in the distances of \( b \)-edges of \( \Gamma_{j_1} \) coincides with a number in the distances of \( b \)-edges of \( \Gamma_{j_2} \). Then a graphical group presentation which corresponds to \( \Omega \) is called a generalized Rips-Segev presentation.

The given conditions ensure that two graphs \( \Gamma_{j_1} \) and \( \Gamma_{j_2} \) of a generalized Rips-Segev presentation \( \Omega \) do not share a piece of length 4. Taking Proposition 2.8 into account, this implies that \( \Omega \) satisfies the \( \text{Gr}^*_1(\frac{1}{6})\)-graphical small cancellation condition. Thus, the group defined by \( \Omega \) is torsion-free and without the unique product property. In particular, if \( \Gamma_i \) is a generalized Rips-Segev graph for \( A_i \) and \( B_i \), then Lemma 2.4 implies that \( A_i \) and \( B_i \) do not have a unique product in \( G(\Omega) \). If the factors of \( F \) are Gromov hyperbolic and \( \Omega \) is finite, then \( G(\Omega) \) is Gromov hyperbolic. In addition, such a group can have infinitely many different pairs of finite subsets without a unique product.

### 2.3. The underlying graph and examples of generalized Rips-Segev presentations.

We show the existence of generalized Rips-Segev graphs with coefficients \((I, O, C)\) the girth of whose underlying graph is at least \( 14M + 19 \). We do so by constructing labeled graphs which underlay generalized Rips-Segev graphs.

Let \( \Gamma \) be a generalized Rips-Segev graph with coefficients \((I, O, C)\). We denote the graph underlying \( \Gamma \) by \( \Phi(\Gamma) \), see Definition 2.6. Note that \( \Phi(\Gamma) \) has \( n \) vertices. The number of its edges is at least \( 2n \) and at most \( 4n \). The degree of the vertex \( c_i \) in \( \Phi(\Gamma) \) corresponds to the total number of \( b \)-edges on the \( a \)-line \( i \). In particular, \( d(c_i) \geq 4 \). For each \( i \) one of the following conditions holds.

1. \( 0 < I_{i1}, \ldots, I_{is_i} < C_i \) and \( d(c_i) = s_i + t_i + 4 \). The \( a \)-line \( i \) contributes 4 \( b \)-edges to the total number of \( b \)-edges.
2. Exactly one coefficient \( I_{is} \) equals either 0 or \( C_i \) and \( d(c_i) = s_i + t_i + 3 \). The \( a \)-line \( i \) contributes 3 \( b \)-edges to the total number of \( b \)-edges.
3. Two coefficients \( I_{is_1} \) and \( I_{is_2} \) equal 0 and \( C_i \) and \( d(c_i) = s_i + t_i + 2 \). The \( a \)-line \( i \) contributes 2 \( b \)-edges to the total number of \( b \)-edges.

Condition (1) corresponds to Figure 5. Figure 7 corresponds to condition (3).

The graph \( \Phi(\Gamma) \) is equipped with a labeling by words in the free group on \( a \) and \( b \), which is inherited by the one-to-one correspondence of the edges of \( \Phi(\Gamma) \) to the \( b \)-edges of \( \Gamma \). Let \((c_i, c_j)\)
be an edge in $\Phi(\Gamma)$. It corresponds say to the $b$-edge $(v_{i1}, v_{j2})$. Then the label of $(c_i, c_j)$ is the word $a^i b a^{-1} b^{-1}$.

Replace each edge $(c_i, c_j)$ labeled $a^i b a^{-1} b^{-1}_j$ with the path $a^i b a^{-1} b^{-1}_j$, see Figure 8. Then fold the graph so obtained. We refer to this reduction procedure as of $\{a, b\}$–reduction. The following proposition is immediate.

**Proposition 2.9.** The $\{a, b\}$–reduction of $\Phi(\Gamma)$ coincides with the generalized Rips-Segev graph $\Gamma$.

Let $\Phi$ be a finite connected graph on $n$ vertices and vertex degree 8. We say $(I, O, C) = \bigsqcup_{i=1}^n (I_{i1}, I_{i2}, O_{i1}, O_{i2}, C_i)$ is suitable for $\Phi$ if for all $i$ the condition (1) holds.

The idea is the following. If $\Phi$ has girth $g \geq 18$ find coefficients $(I, O, C)$ suitable for $\Phi$ such that $g \geq 14M + 19$. Then label $\Phi$ by words in the free group on $a$ and $b$ so that the $\{a, b\}$–reduction of the graph so obtained is a generalized Rips-Segev graph with coefficients $(I, O, C)$. This generalized Rips-Segev graph then satisfies the assumptions of Proposition 2.8 and therefore $Gr^{*}_n \left( \frac{1}{6} \right)$–small cancellation.

**Lemma 2.10.** For all natural numbers $d > 1$, there are finite connected graphs with vertex degree $2d$ which have arbitrarily large girth.

**Proof.** Let $B$ be a ball of radius $r$ in the free group on $n$ generators. The free group is residually finite. There is a normal finite index subgroup $N$ in the free group, so that $B \cap N$ is the identity. The free group on $n$ generators is the fundamental group of a bouquet of $n$ circles. The covering space of a bouquet of $n$ circles corresponding to $N$ is $2n$-regular and has girth at least $2r$. \[\square\]

In particular, there are finite connected graphs with vertex degree 8 and arbitrarily large girth. These graphs have $4n$ edges. Moreover, such a graph is the covering space of a free group on $x_1, x_2, x_3, x_4$. This means that for all $1 \leq j \leq 4$ every vertex $c_i$ has an edge $x_{ij} := (c_{ij}, c_i)$ and an edge $y_{ij} := (c_i, c_{kij})$ such that $y_{ij} = x_{kij}$.

**Example 2.11.** The coefficients

| $I_1$ | $I_2$ | $O_1$ | $O_2$ | $C$ |
|-------|-------|-------|-------|-----|
| 10    | 100   | 1000  | 10000 | $10^5$ |
| $10^6$ | $10^7$ | $10^8$ | $10^9$ | $10^{10}$ |
| $10^{11}$ | $10^{12}$ | $10^{13}$ | $10^{14}$ | $10^{15}$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $10^{5i-4}$ | $10^{5i-3}$ | $10^{5i-2}$ | $10^{5i-1}$ | $10^{5i}$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

satisfy $M = 0$. For arbitrarily large $n$, the coefficient consisting of $n$ different above lines are suitable for a graph of $n$ vertices and vertex degree 8.
For all $1 \leq i \leq n$, $1 \leq j \leq 4$ choose numbers $l_{ij}$ and $k_{ij}$ such that every vertex $c_i$ of $\Phi$ has an edge $x_{ij} := (c_{l_{ij}}, c_i)$ and an edge $y_{ij} := (c_i, c_{k_{ij}})$ and such that $y_{ij} = x_{k_{ij}}$. Then we also have that $x_{ij} = y_{l_{ij}}$.

Assume $(I, O, C)$ is a set of coefficients suitable for $\Phi$. The rules of how to obtain the labeling $L$ of $\Phi$ are implicit. For all $i$ we set $L(x_{i1}) = ba^{-I_{i1}}, L(x_{i2}) = a^{C_{i1}ba^{-I_{i2}}}, L(y_{i3}) = a^{O_{i1}ba^{-C_{i3}}} and L(y_{i4}) = a^{O_{i2}b}$. As $y_{ij} = x_{k_{ij}}$ and $x_{ij} = y_{l_{ij}}$, this yields a labeling of $\Phi$.

Figure 9 shows the local picture of a graph $\Phi$ of vertex degree 8, which is labeled with respect to suitable coefficients $(I, O, C)$. By construction we now have the following observation.

**Proposition 2.12.** The $\{a, b\}$–reduction of $\Phi$ labeled $L$ is a generalized Rips-Segev graph with coefficients $(I, O, C)$ and underlying graph $\Phi$.

By the above lemma, there are graphs $\Phi$ with vertex degree 8 and girth at least 18. Example 2.11 yields infinitely many choices of coefficients $(I, O, C)$ suitable for $\Phi$ such that $M = 0$. By the procedure explained above we then obtain a graph underlying a generalized Rips-Segev graph which satisfies $Gr'_{\frac{1}{6}}$–graphical small cancellation. Rips-Segev original graphs [RS87] correspond to such underlying graphs. (Our explicit choice for the coefficients $(I, O, C)$ is different.) In particular, the $\{a, b\}$–reduction of Rips-Segev original graphs yields generalized Rips-Segev graphs which satisfy $Gr'_{\frac{1}{6}}$–graphical small cancellation. We conclude the following.

**Theorem 2.13.** Rips-Segev torsion-free groups without unique product are Gromov hyperbolic.
The family underlying the generalized Rips-Segev graphs is much more general than the family of Rips-Segev’s original graphs. To obtain explicit examples of \( Gr'_{s}(\frac{1}{8}) \)-generalized Rips-Segev graphs, let \( \Phi \) be a graph with vertex degree 8 and girth \( g \geq 14M' + 19 \), \( M' > 1 \). Example 2.11 of coefficients can easily be modified to obtain coefficients suitable for \( \Phi \) such that \( M = M' \). One only needs to rearrange or replace some numbers in Example 2.11. If \( M' = 2 \), we can for instance replace \( 10^{10} \) by \( 10^{15} \).

To obtain a generalized Rips-Segev graph whose underlying graph some vertices \( c_{i} \) satisfy conditions (2) or (3), we take a set of coefficients which satisfies \( \delta(c_{i}) = s_{i} + t_{i} + 4 \) but in contrast to condition (1), for all \( i \) we allow 
\[ I_{11} = 0 \] (or \( C_{i} \)) or \( I_{22} = C_{i} \) (or 0). We choose \((I,O,C)\) such that \( M' \geq M \). Then label \( \Phi \) as before. Finally, if one of the coefficients \( I_{11}, I_{22} \) is \( C_{i} \), delete the edge \( x_{i3} \). If one of the coefficients \( I_{11}, I_{22} \) is 0, delete the edge \( x_{i4} \). This ensures that \( \Phi \) has exactly one \( x_{il} \) with a label ending \( a^{-C_{i}} \) and exactly one edge \( x_{ij} \) with a label ending \( b \). The deletion of edges increases the girth of \( \Phi \). The \( \{a,b\} \)-reduction of \( \Phi \) is a family of generalized Rips-Segev graphs with the desired properties.

Further generalizations can be obtained as follows. Let \( \Phi \) be a graph of vertex degree \( 2m \), \( m > 4 \), and girth at least 18. For all \( 1 \leq j \leq m \) every vertex \( c_{i} \) has an edge \( x_{ij} := (c_{ij}, c_{i}) \) and an edge \( y_{ij} := (c_{ij}, c_{ki}) \) such that \( y_{ij} = x_{kij} \). Suppose \( \Phi \) has \( n \) vertices. We say \((I,O,C) = \bigwedge_{i=1}^{n} (I_{1i}, \ldots, I_{mi}, O_{1i}, \ldots, O_{mi}, C_{i})\) is suitable for \( \Phi \) if for all \( i \) the condition (1) holds. Let \((I,O,C)\) be suitable for \( \Phi \) and let \( M = 0 \). We label the graph \( \Phi \) extending the above rules: For \( 5 \leq h \leq m \), the label \( L(x_{ih}) = a^{C_{ihu}} ba^{-f_{i}(h-2)} \) or \( ba^{-f_{i}(h-2)} \). For \( m + 5 \leq l \leq 2m \), the label \( L(y_{lh}) = a^{O_{i}(h-m-2)} ba^{-c_{ihu}} \) or \( a^{O_{i}(h-m-2)} b \). Finally delete edges so that each vertex has exactly one edge \( y_{il} \) with a label starting \( b \), exactly one edge \( y_{il} \) with a label starting \( a^{-C_{i}} \), exactly one edge \( x_{il} \) with a label ending \( a^{-C_{i}} \) and exactly one edge \( x_{il} \) with a label ending \( b \). The girth increases when we delete edges. The \( \{a,b\} \)-reduction of a graph so obtained is a generalized Rips-Segev graph whose underlying graphs are not necessarily regular.

So far we have constructed finite Rips-Segev presentations. Now we turn to infinite ones.

**Theorem 2.14.** There are infinitely many infinite generalized Rips-Segev presentations of torsion-free groups without unique product.

**Proof.** Let \((\Phi_{i})_{i \geq 1}\) be a family of finite connected graphs of vertex degree 8 and of girth larger than 18. Example 2.11 provides infinitely many families of coefficients \((I,O,C)_{i}\), so that \((I,O,C)_{i}\) is suitable for \( \Phi_{i} \) and so that no non-zero coefficient occurs more than once among the distances of \( \prod_{i}(I,O,C)_{i}(I,O,C)_{i} \). Each graph \( \Phi_{i} \) can be labeled as above. The \( \{a,b\} \)-reduction of \( \Phi_{i} \) is a generalized Rips-Segev graph. We showed the existence of infinitely many such families of graphs which satisfy the \( Gr'_{s}(\frac{1}{8}) \)-condition. \( \square \)

This yields many torsion-free groups without the unique product property which are defined by a generalized Rips-Segev presentation. Thus, we have many presentations defined by generalized Rips-Segev graphs giving torsion-free groups without the unique product property. The remaining part of this paper treats the following two questions: How many groups defined by generalized Rips-Segev presentations are there? How large is the size of the set of the finite generalized Rips-Segev presentations among finite presentations of groups?
3. The Number of Generalized Rips-Segev Groups

We apply the generalized graphical small cancellation lemma, Theorem 1.11, to groups given by generalized Rips-Segev presentations defined over a suitable family of generalized Rips-Segev graphs such that the labeling of this family satisfies the Gr′(1/6)–condition. Let D be a van-Kampen diagram over a group presentation given by a generalized Rips-Segev graph Γ. Recall that D denotes the diagram obtained by deleting the edges originating in Γ.

**Lemma 3.1.** Let D be a van-Kampen diagram over a generalized Rips-Segev presentation. Assume D has more than one face. Then there is at least one exterior face Π with the following property. There is a non-zero coefficient P in D such that \( b^1a^Pb^{i_2} \) is a subword of \( \partial_{\text{ext}}\Pi \).

**Proof.** Let Γ be a generalized Rips-Segev graph with M non-zero coefficient of multiplicity \( \geq 2 \), and an underlying graph of girth at least \( 14M + 19 \). Then Γ satisfies the Gr′(1/6)–graphical small cancellation condition by Proposition 2.8.

Given that D has more than one face, Theorem 1.11 implies that there is an exterior face Π such that \( |\partial_{\text{ext}}\Pi|_s > \frac{|\Pi|_s}{2} \).

If \( |\partial_{\text{int}}\Pi|_s \geq 6 \), then by the above inequality we have \( |\partial_{\text{ext}}\Pi|_s \geq 4 \). This implies our claim in this case. If \( |\partial_{\text{int}}\Pi|_s < 6 \), as the underlying graph has girth at least \( 14M + 19 \), the minimal cycle length \( \gamma \geq 18 \) and \( |\partial_{\text{ext}}\Pi|_s \geq 12 \). Hence, the claim holds.

**Proposition 3.2.** Let \( \Gamma \) and \( \Gamma' \) be two generalized Rips-Segev graphs such that all non-zero coefficients in the disjoint union of the distances of b-edges of \( \Gamma \) and \( \Gamma' \) are pairwise distinct. Then the identity map \( a \mapsto a, \ b \mapsto b \), does not induce an isomorphism between \( G(\Gamma) \) and \( G(\Gamma') \).

In other words, the corresponding Rips-Segev torsion-free groups \( G(\Gamma) \) and \( G(\Gamma') \) without the unique product property are not isomorphic as marked groups.

**Proof.** Assume that the identity induces an isomorphism \( G(\Gamma) \to G(\Gamma') \). Analyze van-Kampen diagrams D of relators r for \( G(\Gamma) \) over the relators of \( G(\Gamma') \). More precisely, r is a reduced word

\[
r = a^{P_0}b^{i_1}a^{P_1}b^{i_2} \ldots a^{P_{l-1}}b^{i_l}
\]

read on a reduced cycle of \( \Omega \). Now consider a van-Kampen diagram \( \tilde{D} \) for r over the relators of \( G(\Gamma') \). There has to be more than one face in \( \tilde{D} \), otherwise r can be read on a reduced cycle of \( \Gamma' \), a contradiction. By the above lemma, there is an exterior face Π and a non-zero coefficient Q among the distances of b-edges of \( \Gamma' \) such that \( b^{i_1}a^Qb^{i_2} \) is a subword of \( \partial_{\text{ext}}\Pi \). Therefore, in r there is an exponent \( P_i = Q \). This is a contradiction to the choice of the Rips-Segev presentations.

We can easily extend the above argument to prove the following observation.

**Remark 3.3.** Let \( \Gamma \) and \( \Gamma' \) be two generalized Rips-Segev graphs such that all non-zero coefficients in the disjoint union of the distances of b-edges of \( \Gamma \) and \( \Gamma' \) are pairwise distinct, do not differ by \( \pm 1 \), and do not equal to 1 or 2.

- Single elementary Nielsen equivalences do not induce isomorphisms of \( G(\Gamma) \) and \( G(\Gamma') \).
- If in addition no number in the distances of b-edges of \( \Gamma \) is a multiple of a number in the distances of b-edges of \( \Gamma' \), then there is no integer P so that the map \( a \mapsto ba^Pb^{-1} \) \( b \mapsto b \) is an isomorphism of \( G(\Gamma) \) and \( G(\Gamma') \).
Let \((A, B, A', B')\) be two pairs of finite subsets without unique product in a group \(G\). We say that \((A, B, A', B')\) are different pairs of subsets without unique product, whenever there is no pair of finite subsets \((A_0, B_0)\) without unique product in \(G\) such that the relations which describe the non-unique product property of \((A, B, A', B')\) in \(G\) can be expressed by finite products of conjugates of the relations describing the non-unique product property of \(A_0, B_0\) in \(G\).

We emphasize that in generalized Rips-Segev groups all relations making a pair of sets \((A', B')\) a non-unique product pair are consequences of the labels on the cycles of the defining generalized Rips-Segev graphs.

**Corollary 3.4.** The group defined by an infinite generalized Rips-Segev presentation contains infinitely many different pairs of subsets without unique product in the group.

In contrast, a generalized Rips-Segev group defined by one generalized Rips-Segev graph has no pairs of subsets without unique product different from \((A, B, A', B')\).

**Theorem 3.5.** There are uncountably many non-isomorphic torsion-free groups without the unique product property.

**Proof.** We adapt a standard argument. Let \(\Omega := (\Gamma_i)_{i \in \mathbb{Z}}\) be an infinite family of generalized Rips-Segev graphs such that the labeling of this family satisfies the \(Gr'_{i}^\ast (\frac{1}{g})\)–condition. The corresponding group \(G(\Omega)\) is torsion-free and without the unique product property. For every subset \(I \subseteq \mathbb{Z}\), we let \(\Omega_I := (\Gamma_i)_{i \in I}\). The groups \(G_I := G(\Omega_I)\) are torsion-free and without unique product. Taking into account Proposition 3.2 (and Example 2.11 as well as the proof of Theorem 2.14), we can assume that the groups \(G_I\) are pairwise non-isomorphic as marked groups. As a finitely generated group has only countably many pairs of generators, there are uncountably many non-isomorphic \(G_I\). □

In particular, there are uncountably many non-isomorphic torsion-free non-unique product groups which have a generalized Rips-Segev presentation.

An interesting open question is whether or not there are infinitely many non-isomorphic finitely presented Rips-Segev groups. In the next section, we show that the class of finitely presented Rips-Segev groups is small when considered within certain models of random finitely presented groups.

4. **Genericity and Generalized Rips-Segev Presentations**

**Lemma 4.1.** A generalized Rips-Segev graph with underlying graph of girth \(g\) has a reduced path

\[ a^{P_0}b^\varepsilon a^{P_1}b^\varepsilon a^{P_2} \ldots a^{P_{g-1}}b^\varepsilon a^{P_g}, \]

where \(\varepsilon = \pm 1\) and \(P_i \neq 0\) for all \(0 < i < g\).

**Proof.** We indicate how to find such a path. Start at \(v_{i0}\) or \(v_{iC_i}\). Follow the \(b\)-edge pointing to the \(a\)-line \(i\) to the vertex \(v_{jO}\) (or possibly \(v_{j0}, v_{jC_j}\)). Then go along the \(a\)-line \(j\) towards an (the other) end of \(j\). We reach the \(a\)-line \(l\) at \(v_{lO}\) or (or possibly \(v_{l0}, v_{lC_l}\)). Then go to an (the other) end \(v_{l0}, v_{lC_l}\). As the girth of the underlying graph is \(g\), we can go along at least \(g\) \(b\)-edges before coming back to the \(a\)-line \(i\). □

If \(\Delta\) is a graph, then let \(\Delta^j\) denote the \(j\)-subdivision of \(\Delta\). This is the graph obtained by replacing each edge of \(\Delta\) with \(j\) edges.
Theorem 4.2. Generalized Rips-Segev presentations are not generic in Gromov’s graphical model [Gro03, OW07] of finitely presented random groups:

For any \( v \in \mathbb{N} \) and for any \( C \geq 1 \), there exists an integer \( j_0 \) such that for any \( j \geq j_0 \), for any family of graphs \( \Delta = (\Delta_i) \) of girth \( \delta = (\delta_i) \) satisfying the conditions

1. Each vertex of \( \Delta \) is of valency at most \( v \);
2. \( \text{diam}(\Delta) \leq C\delta \) for all \( i \),

with probability tending to 1 with exponential asymptotics as \( \delta \to \infty \), the folded graph \( \overline{\Delta}^j \) obtained by a random labeling of \( \Delta^j \) contains no generalized Rips-Segev graph as a subgraph.

Proof. For any small \( \beta > 0 \), there is a number \( j_0 \) such that, for all \( j \geq j_0 \), the girth of the folded graph \( \overline{\Delta}^j \) is at least \( \delta = (\eta - \beta)\delta(\Delta) \). Here \( 0 < \eta < 1 \) is the gross cogrowth of a finitely generated free group [OW07, Proposition 7.8].

We show that the folded graph \( \overline{\Delta}^j \) obtained by a random labeling of \( \Delta^j \) contains no generalized Rips-Segev graph with exponential asymptotics as \( \delta \to \infty \). As usual, we denote by \( \Delta \) also a member of the family \( (\Delta_i) \). If \( \overline{\Delta}^j \) contains a generalized Rips-Segev graph, by Lemma 4.1 there is a path \( p \) of length \( \delta \) in \( \overline{\Delta}^j \), bearing a word of type

\[ w = a^{P_1}b^\varepsilon a^{P_2}b^\varepsilon a^{P_3} \ldots a^{P_{k-1}}b^\varepsilon a^{P_k}, \]

where \( \varepsilon = \pm 1 \) and \( P_i \neq 0 \) for all \( 0 < i < k \), see Figure 10. We have to estimate the number of such paths in \( \overline{\Delta}^j \).

First, we show that the number of the words \( w \) read on a path of length \( \bar{\delta} \) is at most \( 2^{\bar{\delta}+2} \). In fact, there are 2 possibilities for \( \varepsilon \). There are at most \( \sum_{k=0}^{\bar{\delta}/2} \binom{\bar{\delta}}{k} \) possibilities for the vertices \( o \) in between \( b \) and \( a \). Such a choice determines the lengths of the dotted \( a \)-paths. For every vertex \( o \) we have to choose an orientation for \( a \). This gives at most

\[ 2^{\sum_{k=0}^{\bar{\delta}/2} \binom{\bar{\delta}}{k}} = 2 \cdot 2^{\bar{\delta}/2} \cdot \sum_{k=0}^{\bar{\delta}/2} \binom{\bar{\delta}}{k} \leq 2 \cdot 2^{\bar{\delta}+1} \]

possibilities for \( w \) read on a path of length \( \bar{\delta} \) in \( \overline{\Delta}^j \).

Our path is the folding of a path of length at least \( \bar{\delta} = (\eta - \beta)\delta j \) in \( \Delta^j \). The number of paths of length \( \geq (\eta - \beta)\delta j \) in \( \Delta^j \) is at most \( C\delta j^2 v^{C\delta + C^2} \). Thus there are at most \( C\delta j^2 v^{C\delta + C^2} \) possibilities for the occurrence of an above path \( p \) in \( \overline{\Delta}^j \).

The number of reduced words of length \( \bar{\delta} = (\eta - \beta)\delta j \) is at least \( 3^{(\eta - \beta)\delta j} \). Thus the probability that a randomly labeled graph \( \overline{\Delta}^j \) is a generalized Rips-Segev graph is bounded by

\[ C\delta j^2 v^{2C^2} \frac{3^{(\eta - \beta)\delta j}}{3^{(\eta - \beta)\delta j}}. \]

\(^1\)Note however that we consider a “two generator model” for quotients of the free product \( F = G_1 \ast G_2 \), in the sense that the words in \( R \) are generated by concatenating the words \( a^\pm 1 \in G_1 \) and \( b^\pm 1 \in G_2 \). In particular, we consider random labellings of \( \Delta^j \) by the words \( a \in G_1 \) and \( b \in G_2 \), and therefore random quotients of \( F \).
We choose \( j \) so large that \( v^2C \left( \frac{2}{3} \right)^j < 1 \). As \( \delta \to \infty \), the probability that \( \Delta \) is a generalized Rips-Segev graph tends to zero with exponential asymptotics.

\[ \Box \]

**Theorem 4.3.** Generalized Rips-Segev presentations are not generic in Arzhantseva-Ol’shanskii’s few-relator model [AO96] of finitely presented random groups:

For all \( n \in \mathbb{N} \), the probability that a randomly chosen group presentation among all group presentations

\[ \langle a, b | r_1, \ldots, r_n; r_1 \text{ a cyclically reduced word in } a, b; | r_i | \leq t \] 

is a generalized Rips-Segev presentation tends to zero with exponential asymptotics as \( t \to \infty \).

**Proof.** We show that the number of words of type

\[ w = a^{P_0}b^\varepsilon a^{P_1}b^\varepsilon a^{P_2} \ldots a^{P_{k-1}}b^\varepsilon a^{P_k}, \]

where \( \varepsilon = \pm 1 \), \( P_i \neq 0 \) for all \( 0 < i < k \) and length \( \leq t \), is at most \( 4t \cdot 2^t \). In fact, there are 2 possibilities for the orientation of \( b \). There are at most

\[ \sum_{l=0}^{t} \sum_{k=0}^{l/2} \binom{l}{k} \leq \sum_{k=0}^{t/2} \binom{t}{k} \]

possibilities for the vertices \( o \) in between \( b \) and \( a \). These choices include all such words with length between 0 and \( t \). Such a choice determines the lengths of the dotted \( a \)-paths. For every vertex \( o \) we have to choose an orientation for \( a \). This gives at most

\[ 2t \sum_{k=0}^{t/2} \binom{t}{k} 2^k \leq 2t \cdot 2^{t/2} \cdot \sum_{k=0}^{t/2} \binom{t}{k} \leq 2t \cdot 2^{t+1} \]

possibilities for such a word of length \( \leq t \).

The probability that a random presentation in the above sense contains a relator \( w \) is at most

\[ \frac{4t2^t \cdot 2^{n-1}3^{(n-1)t}}{3n(t-1)} \]

and tends to zero with exponential asymptotics as \( t \to \infty \).

\[ \Box \]

These results suggest that unique product groups are generic among finitely presented groups. This would in particular imply that the Kaplansky zero divisor conjecture is generic among finitely presented groups.

**REFERENCES**

[AS72] K. I. Appel and P. E. Schupp, *The conjugacy problem for the group of any tame alternating knot is solvable*, Proc. Amer. Math. Soc. 33 (1972), 329–336.

[AO96] G. N. Arzhantseva and A. Yu. Ol’shanski˘ı, *Generality of the class of groups in which subgroups with a lesser number of generators are free*, Mat. Zametki 59 (1996), no. 4, 489–496, 638; English transl., Math. Notes 59 (1996), no. 3-4, 350–355.

[Coh74] J. M. Cohen, *Zero divisors in group rings*, Comm. Algebra 2 (1974), 1–14.

[Del97] T. Delzant, *Sur l’anneau d’un groupe hyperbolique*, C. R. Acad. Sci. Paris Sér. I Math. 324 (1997), no. 4, 381–384.

[Gro03] M. Gromov, *Random walk in random groups*, Geom. Funct. Anal. 13 (2003), no. 1, 73–146.

[Gru13] D. Gruber, *Groups with graphical C(6) and C(7) small cancellation presentations*, Trans. Amer. Math. Soc. (2013), to appear.
[dlH88] P. de la Harpe, Groupes hyperboliques, algèbres d’opérateurs et un théorème de Jolissaint, C. R. Acad. Sci. Paris Sér. I Math. 307 (1988), no. 14, 771–774.

[Kap57] I. Kaplansky, Problems in the theory of rings. Report of a conference on linear algebras, June, 1956, pp. 1-3, 1957, pp. v+60.

[Kap70] ______, “Problems in the theory of rings” revisited, Amer. Math. Monthly 77 (1970), 445–454.

[Laf98] V. Lafforgue, Une démonstration de la conjecture de Baum-Connes pour les groupes réductifs sur un corps p-adique et pour certains groupes discrets possédant la propriété (T), C. R. Acad. Sci. Paris Sér. I Math. 327 (1998), no. 5, 439–444.

[Laf12] ______, La conjecture de Baum-Connes à coefficients pour les groupes hyperboliques, J. Noncommut. Geom. 6 (2012), no. 1, 1–197.

[LS77] R. C. Lyndon and P. E. Schupp, Combinatorial group theory, Springer-Verlag, Berlin, 1977. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[Lyn66] R. C. Lyndon, On Dehn’s algorithm, Math. Ann. 166 (1966), 208–228.

[OW07] Y. Ollivier and D. T. Wise, Kazhdan groups with infinite outer automorphism group, Trans. Amer. Math. Soc. 359 (2007), no. 5, 1959–1976.

[Oll06] Y. Ollivier, On a small cancellation theorem of Gromov, Bull. Belg. Math. Soc. Simon Stevin 13 (2006), no. 1, 75–89.

[Pan99] A. E. Pankrat’ev, Hyperbolic products of groups, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 2 (1999), 9–13, 72; English transl., Moscow Univ. Math. Bull. 54 (1999), no. 2, 9–12.

[Pas77] D. S. Passman, The algebraic structure of group rings, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1977.

[Pro88] S. D. Promislow, A simple example of a torsion-free, nonunique product group, Bull. London Math. Soc. 20 (1988), no. 4, 302–304.

[RS87] E. Rips and Y. Segev, Torsion-free group without unique product property, J. Algebra 108 (1987), no. 1, 116–126.

[Sch68] P. E. Schupp, On Dehn’s algorithm and the conjugacy problem, Math. Ann. 178 (1968), 119–130.

[Val02] A. Valette, Introduction to the Baum-Connes conjecture, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2002.

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