Areas of existence of ruled surfaces

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Abstract. Ruled surfaces are used in mechanical engineering, in aircraft and motor-vehicle constructions, in agriculture, in construction and light industries. It is easier to denominate areas where ruled surfaces are not applied than to list them all. The value of ruled surfaces in human activities cannot be overestimated. In this paper the consideration of the formation of ruled surfaces with a uniform way of setting them, which was proposed earlier in two articles with the same title “General Principles of Defining Ruled Surfaces”, is continued.

The concept of a limit ruled surface is introduced in this article, and on the basis of a single method for defining ruled surfaces, the question of identifying the areas of their existence that lie between two limiting ruled surfaces is considered. These areas of existence are, in fact, a line congruence.

Keywords - surfaces; ruled surfaces; geometry; descriptive geometry; surface formation, line congruence.

1. Introduction

Ruled surfaces are used everywhere in the technologies [1-5], especially in manufacturing the parts with complex surfaces and in developing the technologies for the reproduction of complex surfaces using copiers on various machines, construction [6-8], design, agriculture [9]. Therefore, the papers dedicated to the design of ruled surfaces do not cease to be relevant in the scientific literature. Thus, in the paper [10], a method for obtaining double oblique cylindroids by immersing a curve in a hyperbolic linear congruence was considered. In the paper [11] the method for obtaining parametric equations of hyperbolic line congruence was proposed. Ruled surfaces were considered at different times in the papers [12-18], so it can be assumed that the relevance of theoretical research in the field of the theory of ruled surfaces does not decrease over time.

That is why the law of their formation was proposed in the paper [19], which covers all possible variants of ruled surfaces. In earlier published papers [19; 20] the options for setting ruled surfaces, which confirmed the proposed law of formation, which has the following interpretation, were given. Any ruled surface is defined by three guiding lines and three geometric conditions that characterize the ratio of the generator to these guides. Geometric conditions are the intersection with the guiding line, touching or intersecting at a certain acute angle with the guiding surface.

In papers [19; 20] the following points were considered:
1. All three guiding lines are the lines (15 examples).
2. Guiding lines are the surfaces and the lines (22 examples).
3. Guiding lines are only the surfaces (5 examples).

In this paper the confirmation of the validity of the law of formation with the development of the theory of ruled surfaces will be continued.
2. Formulation of the problem
There are a number of papers that are devoted to the straight line congruences [21-24], in which there is no identification of the areas of their existence. Therefore, there is a definite need to find the boundaries of these congruences and identify areas of existence of given congruences within which a ruled surface can be constructed.

3. Theory
3.1. Limiting ruled surface
We introduce the concept of a limiting ruled surface. The limiting ruled surface is defined by two guiding surfaces or lines $m, n$ and a set of director – planes $l$, each of which simultaneously touches both the guiding line $m$ and the guiding line $n$.

Each generator of the limiting ruled surface touches two guiding surfaces or lines $m, n$ and is tangent (in this limiting case — belongs) to the third moving director – plane $l$. The points of tangency $M$ and $N$ of the moving director – plane $l$ with guiding surfaces (Figure 1) or guiding lines (Figure 2) $m, n$ determine the position of the $MN$ line of a given ruled surface.

![Figure 1](image1.png)

![Figure 2](image2.png)
The guiding lines \(m, n\) define two sets of tangents to their director – planes \(l\) in the space \(\mathbb{R}^3\). The director – planes that have an external touch with the guiding lines \(m, n\) (Figure 1, a, 2, a) are referred to the first set. The director – planes that have an internal touch with the guiding lines \(m, n\) (Figure 1, b, 2, b) are referred to the second set. These two sets of director – planes \(l\) correspond to two sets of generators of the limiting ruled surface.

To construct the limiting ruled surface, it is enough to construct the plane \(l\) (Figure 1, 2), the tangent to both guiding lines \(m\) and \(n\), and to find the tangency points of \(M\) and \(N\).

The equation for the limiting ruled surface is found by solving the following system of equations:

\[
\begin{align*}
F_1(X_m, Y_m, Z_m) &= 0; \\
\frac{dF_1}{dX_m}(X-X_m) + \frac{dF_1}{dY_m}(Y-Y_m) + \frac{dF_1}{dZ_m}(Z-Z_m) &= 0; \\
F_2(X_n, Y_n, Z_m) &= 0; \\
\frac{dF_2}{dX_n}(X-X_n) + \frac{dF_2}{dY_n}(Y-Y_n) + \frac{dF_2}{dZ_n}(Z-Z_n) &= 0; \\
\frac{dF_1}{dX_m} - \frac{dF_1}{dY_m} &= \frac{dF_2}{dY_n} - \frac{dF_2}{dZ_n}, \\
\frac{dF_1}{dY_m} - \frac{dF_1}{dZ_m} &= \frac{dF_2}{dX_n} - \frac{dF_2}{dZ_n},
\end{align*}
\]

where (1) is the equation of the guiding surface \(m\);
(2) is the equation of the plane tangent at the point \(M\) to the guiding surface \(m\);
(3) is the equation of the guiding surface \(n\);
(4) is the equation of the plane tangent at the point \(N\) to the guiding surface \(n\);
(5, 6) are the equations determining parallelism, and for this case the combination of two planes (2) and (4), tangent, respectively, to the guiding surfaces \(m\) and \(n\);
(7, 8) are the equations of two projecting planes, the line of intersection of which is the generator \(MN\) of the ruled surface.

When solving it is enough to use the system of only seven equations (1) – (7). Sequentially, excluding the unknowns \(X_m, Y_m, Z_m, X_n, Y_n, Z_n\), the equation (9) of the limiting ruled surface can be obtained:

\[
F(X, Y, Z) = 0
\]

With the help of limiting ruled surfaces, possible boundaries for the existence of ruled surfaces are defined, the tasks of which include guiding lines \(m\) and \(n\).

3.2. Areas of existence of ruled surfaces

Let two closed flat curved guiding lines \(m\) and \(n\) be given, the planes of which are perpendicular to one of the projection planes (in any case, such a plane can be introduced). Then the guiding lines are projected onto this plane of projections in the form of straight line segments (Figure 3).
At mutual intersection one part of the limiting ruled surface with generators, having external contact with guiding lines $m$ and $n$ (Figure 2, a) and the other part of the limiting ruled surface with generators, having internal contact with these guiding lines (Figure 2, b) divide $\mathbb{R}^3$ space into “ring domains”, parts of space, the shapes and sizes of which depend on the shapes of sizes and the relative position of the guiding lines $m$ and $n$. In general, these guiding lines can be any curved lines, located relative to each other arbitrarily.

Ring domains of space are divided into two groups. The first group includes ring domains in which straight lines can be drawn that have a tangency (intersection) with both guiding lines $m$ and $n$. In fig. 3, these areas are shaded and marked with a “+”. According to the height contingently they can be divided into the areas of the lower (left), middle and upper (right) belts. Straight lines tangent to both guiding lines $m$ and $n$ must necessarily go inside all three belts of the ring domains of the first group and have two points of contact with the self-intersection lines of the limit ruled surface.

The ring domains of the second group in Figure 3 are shown free from shading and marked with a “-”. The ring domains of the second group are separated from the ring domains of the first group by the limiting ruled surface. A straight line, although on a small area passing inside the ring domains of the second group, cannot have a simultaneous contact (intersection) with the curved guiding lines $m$ and $n$.

A straight line, throughout its length located within the limits bounded by the ring domains of the first group, will necessarily have tangency points (intersections) with both guiding lines $m,n$ and with the self-intersection lines of the limiting ruled surface. Consequently, an arbitrary third guiding line $l$, completely located within the volume of the ring domains of the first group, forms the specific ruled surface together with the guiding lines $m$ and $n$, the set of generators is distinguished from the set of straight lines that fill the ring domains of the first group. If at least part of the third guiding line $l$ is located within the volume of the ring domains of the second group, then this part of it will not participate in the formation of the ruled surface, and the ruled surface itself will be discontinued, limiting itself in two generators of the limiting ruled surface. If it turns out that the guiding line $l$ is completely located within the volume of the ring domains of the second group, then the ruled surface defined by the guiding lines $m$, $n$ and $l$ cannot exist. In the particular case, when the third guiding line, being inside the ring domains of the second group, touches the limiting ruled surface at one or two points, the ruled surface defined by the three guiding lines will turn into one or two straight lines.

In the case of specifying the guiding surfaces $m$ and $n$ (Figure 4), a similar pattern can be traced. The limiting ruled surfaces in contact with the guiding surfaces $m$ and $n$ divide $\mathbb{R}^3$ space into the ring
domains of the first group (are shaded and marked with “+”) and the ring domains of the second group (marked with “−”). However, now on the guiding lines \(m\) and \(n\) there appeared the strips of surfaces enclosed between the contact lines of the limiting ruled surface with the given guiding lines. In figure 4 these strips are covered with shading, different from the shading of the ring domains of the first group. Let’s call them strips of tangency.

Let’s take an arbitrary point on the strip of tangency of the guiding surface \(m\). If the guiding lines are the second-order surfaces, then in the general case two straight lines can be drawn through this point, each of which is tangent to both surfaces \(m\) and \(n\). Similar straight lines can be drawn through an arbitrary point taken on the strip of tangency of the guiding surface \(n\).

All the positions considered in relation to the formation of a ruled surface using the third guiding line \(l\) for figure 3, where the guiding lines \(m\) and \(n\) are curved lines, remain valid for figure 4, where the guiding lines \(m\) and \(n\) are the surfaces.

The proposed principal volumetric diagrams of the domains of existence of ruled surfaces allow having preliminary information about the shape and extent of the projected ruled surface.

3.3. Equidistant ruled surface

The functional features of some machines require providing radial spaces between the surfaces of their working bodies. In this case, there is a challenge for designers to develop and create mutually dependent surfaces with ensuring regulated constant distances between them.

Consider the formation of a ruled surface that meets the requirements.

Let the ruled surface be given by three guiding lines, of which \(m\) is a straight line, \(n\) is a curved line, and \(l\) is a surface (Figure 5).

![Figure 5](image)

The conditions characterizing the position of a given ruled surface relative to its guiding lines are that a minimum distance between any of its generators and each of the guiding lines \(m, n, l\) must be equal to the values of \(a, b, c\) (Figure 5).

Let’s construct auxiliary surfaces \(m_0, n_0, l_0\), equidistant from the given surfaces \(m, n, l\). The distance from each point of the surface \(m_0\) to the guiding line \(m\) is equal to the value of \(a\), the distance from each point of the surface \(n_0\) to the guiding line \(n\) is equal to the value of \(b\), the distance from each point of the surface \(l_0\) to the guiding line \(l\) is equal to the value of \(c\).

The surfaces \(m_0, n_0, l_0\) will be taken as the main guiding lines. Regarding these main guiding lines a ruled surface is constructed on the condition that each of its generators touches all three main guiding surfaces. In this case, the smallest ring domains between the constructed ruled surface and the guiding lines \(m, n, l\), equal to the values of \(a, b, c\) respectively, are ensured. When \(a = b = c\), the same radial spaces will be provided between the ruled surface and each of the guiding lines \(m, n, l\).
To derive the equidistant ruled surface equation, firstly, it is necessary to derive the equations for all the main guiding surfaces \( m, n, l \), equidistant to the given guiding lines \( m, n, l \). Let’s consider in a general form the derivation of the equation of the surface \( m \), using the following system of equations:

\[
Z_m = f(X_m, Y_m); \quad (10)
\]

\[
\frac{X-X_m}{\frac{dx_m}{dm}} = \frac{Y-Y_m}{\frac{dy_m}{dm}} = \frac{Z-Z_m}{\frac{dz_m}{dm}}; \quad (11, 12)
\]

\[
(X-X_m)^2 + (Y-Y_m)^2 + (Z-Z_m)^2 = a^2, \quad (13)
\]

where (10) is the equation of the given guiding surface \( m \);

(11, 12) are the equations of two projecting planes whose intersection line is the normal to the guiding surface \( m \);

(13) is the equation characterizing the distance between two guiding surfaces, given by \( m \) and the main \( m \), in the direction of the normal one.

When solving this system of four equations, successively eliminating the unknowns \( X_m, Y_m, Z_m \), we obtain equation (14) of the main guiding surface \( m \):

\[
Z = f(X, Y) \quad (14)
\]

The received equations of the main guiding surfaces are substituted into the system of equations (15) - (24), which was presented in part 1 of the study [18]:

\[
F_1(X_m, Y_m, Z_m) = 0; \quad (15)
\]

\[
\frac{dF_1}{dX_m}(X-X_m) + \frac{dF_1}{dY_m}(Y-Y_m) + \frac{dF_1}{dZ_m}(Z-Z_m) = 0; \quad (16)
\]

\[
F_2(X_n, Y_n, Z_n) = 0; \quad (17)
\]

\[
\frac{dF_2}{dX_n}(X-X_n) + \frac{dF_2}{dY_n}(Y-Y_n) + \frac{dF_2}{dZ_n}(Z-Z_n) = 0; \quad (18)
\]

\[
F_3(X_l, Y_l, Z_l) = 0; \quad (19)
\]

\[
\frac{dF_3}{dX_l}(X-X_l) + \frac{dF_3}{dY_l}(Y-Y_l) + \frac{dF_3}{dZ_l}(Z-Z_l) = 0; \quad (20)
\]

\[
\frac{X-X_m}{X_n-X_m} = \frac{Y-Y_m}{Y_n-Y_m} = \frac{Z-Z_m}{Z_n-Z_m}, \quad (21, 22)
\]

\[
\frac{X-X_m}{X_l-X_m} = \frac{Y-Y_m}{Y_l-Y_m} = \frac{Z-Z_m}{Z_l-Z_m}. \quad (23, 24)
\]

Solving this system, the equation equidistant to the guiding lines \( m, n, l \) of the desired ruled surface of the form (14) is obtained.

**Conclusion**

In the considered examples, once again you can verify the correctness of the proposed law of the formation of ruled surfaces. The proposed principle does not contradict the well-known theory of parametric geometry [25].

As it has already been stated in [19], the proposed principle for specifying ruled surfaces can be used in the educational process in explaining acquisition and assignment, as well as in making complex surfaces in various fields.
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