THE LEFT TAIL OF RENEWAL MEASURE

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Abstract. In the paper, we find exact asymptotics of the left tail of renewal measure for a broad class of two-sided random walks. We only require that an exponential moment of the left tail is finite. Through a simple change of measure approach, our result turns out to be almost equivalent to Blackwell’s Theorem.

1. Introduction

Let \((X_k)_{k \geq 1}\) be a sequence of independent copies of a random variable \(X\) with \(\mathbb{E}X > 0\) (we allow \(\mathbb{E}X = \infty\)). Further, define \(S_n = X_1 + \ldots + X_n,\ n \geq 1\) and \(S_0 = 0\). The measure defined by

\[
H(B) := \sum_{n=0}^{\infty} \mathbb{P}(S_n \in B), \quad B \in \mathcal{B}(\mathbb{R})
\]

is called the renewal measure of \((S_n)_{n \geq 1}\).

We say that the distribution of a random variable \(X\) is \(d\)-arithmetic (\(d > 0\)) if it is concentrated on \(d\mathbb{Z}\) and not concentrated on \(d'\mathbb{Z}\) for any \(d' > d\). A distribution is said to be non-arithmetic if it is not \(d\)-arithmetic for any \(d > 0\).

A fundamental result of renewal theory is the Blackwell Theorem (Blackwell [1953]): if the distribution of \(X\) is non-arithmetic, then for any \(h > 0\),

\[
\lim_{x \to \infty} \frac{H((x, x+h]) - h}{\mathbb{E}X} = 0
\]

If the distribution of \(X\) is \(d\)-arithmetic, then for any \(h > 0\),

\[
\lim_{n \to \infty} \frac{H((dn, dn + h]) - d|h/d|}{\mathbb{E}X} = 0
\]

The above results remain true if \(\mathbb{E}X = \infty\) with the usual convention that \(c/\infty = 0\) for any finite \(c\). In the infinite-mean case the exact asymptotics of \(H((x, x+h])\) are also known. Assume that \(X\) is a non-negative random variable with a non-arithmetic law such that \(\mathbb{P}(X > x) = L(x)x^{-\alpha}\) with \(\alpha \in (0, 1)\), where \(L\) is a slowly varying function. Then \(\mathbb{E}X = \infty\). If \(\alpha \in (1/2, 1)\), then without additional assumptions the so-called Strong Renewal Theorem holds, for \(h > 0\),

\[
\lim_{x \to \infty} m(x)H((x, x+h]) - \frac{h}{\Gamma(\alpha)\Gamma(2 - \alpha)} = 0
\]

where \(m(x) = \int_0^x \mathbb{P}(X > t)dt \sim L(x)x^{1-\alpha}/(1-\alpha) \to \infty\). Here and later on \(f(x) \sim g(x)\) means that \(f(x)/g(x) \to 1\) as \(x \to \infty\).

The case of \(\alpha \in (0, 1/2]\) is much harder and was completely solved just recently by Caravenna and Doney [2010]. It was shown that if \(\alpha \in (0, 1/2]\) and \(X\) is a non-negative random variable with regularly varying
tail, then (3) holds if and only if \cite[Proposition 1.11]{caravenna2016random}

\[
\lim_{\delta \to 0} \lim_{x \to \infty} \limsup_{n \to \infty} \frac{\delta_x}{x^2} \int_1^x \frac{F(x) - F(x-z)}{\mathcal{F}(z)x^2} \, dz = 0,
\]

where $F$ is the cumulative distribution function of $X$ and $\mathcal{F} = 1 - F$. It was already observed by \cite[Theorem 3.1]{kevei2016random} that this result generalizes to $X$ attaining negative values as well if additionally

\[
P(X \leq -x) = o(e^{-r x}) \quad \text{as } x \to \infty
\]

for some $r > 0$. This will be our setup. Full picture of SRT for random walks is also known \cite[Theorem 1.12]{caravenna2016random}.

It is clear that $\lim_{x \to \infty} H((-\infty, -x)) = 0$. There are considerably fewer papers dedicated to analysis of exact asymptotics of such object than of $H((x, x+h])$ as in Blackwell’s Theorem. Under some additional assumptions we know more about the asymptotic behaviour of the left tail. \cite{stone1965random} proved that if for some $r > 0$ (5) holds, then for some $r_1 > 0$,

\[
H((-\infty, -x)) = o(e^{-r_1 x}) \quad \text{as } x \to \infty.
\]

Stone’s result was strengthened by \cite{vanderGenugten1969random}, where exact asymptotics as well the speed of convergence of the remainder term are given for $d$-arithmetic and spread-out laws (i.e. laws, whose $n$th convolution has a nontrivial absolutely continuous part for some $n \in \mathbb{N}$). An important contribution regarding the asymptotics of the left tail of renewal measure was made by \cite{carlsson1983random}, who concerned with the case when $E|X|^m < \infty$ for some $m \geq 2$, but this does not fit well into our setup. We allow $E|X_+ = \infty$, but on the other hand we require that some exponential moments of $X_-$ exist. The results mentioned above were obtained using some analytical methods, whereas we will use a simple probabilistic argument, which boils down the asymptotics of $H((-\infty, -x))$ to the asymptotics of $\hat{H}((x, x+h])$, where $\hat{H}$ is some new (possibly defective, see below) renewal measure.

1.1. **Defective renewal measure.** For $\rho \in (0, 1)$ consider

\[
H_{\rho}(B) := \sum_{n=0}^{\infty} \rho^n \mathbb{P}(S_n \in B), \quad B \in \mathcal{B}(\mathbb{R}),
\]

where $(S_n)_{n \geq 1}$ is, as in the previous section, a random walk starting from 0. $H_{\rho}$ is called a defective renewal measure of $(S_n)_{n \geq 1}$. In contrast to the renewal measure, $H_{\rho}$ is a finite measure. Let $\tau$ be independent of $(S_n)_{n \geq 1}$ and $\mathbb{P}(\tau = n) = (1 - \rho)^{n-1} \rho^n$, $n = 0, 1, \ldots$. Then $H_{\rho}(B) = \mathbb{P}(S_\tau \in B)/(1 - \rho)$. It is well known that if the distribution of $S_1$ is subexponential, then $\mathbb{P}(S_\tau > x) \sim E\tau \mathbb{P}(S_1 > x)$. Here, we are interested in exact asymptotics of $H_{\rho}(B)$ when $B = (x, x+T]$ for any $T > 0$. In this context, local subexponentiality is the key concept \cite{asmussen2003phase}. Let $\mu$ be a probability measure on $\mathbb{R}$. For $T > 0$ we write $\Delta = (0, T]$ and $x + \Delta = (x, x+T]$. We say that $\mu$ belongs to the class $\mathcal{L}_\Delta$ if $\mu(x + \Delta) > 0$ for sufficiently large $x$ and

\[
\frac{\mu(x+s+\Delta)}{\mu(x+\Delta)} \to 1 \quad \text{as } x \to \infty,
\]

uniformly in $s \in [0, 1]$.

We say that $\mu$ is $\Delta$-subexponential if $F \in \mathcal{L}_\Delta$ and

\[
\mu^2(x+\Delta) \sim 2\mu(x+\Delta).
\]

Then we write $\mu \in \mathcal{S}_\Delta$. Finally, $\mu$ is called locally subexponential if $\mu \in \mathcal{S}_\Delta$ for any $T > 0$. We denote this class by $\mathcal{S}_{\text{loc}}$.

The following Theorem is an obvious conclusion from \cite[Theorem 1.1]{watanabe2009random}.
Theorem 1.1 Assume that $\mu$ is a probability measure on $\mathbb{R}$ such that
$$\int_{\mathbb{R}} e^{-\varepsilon x} \mu(dx) < \infty \quad \text{for some } \varepsilon > 0.$$ For $0 < \rho < 1$ define
$$\eta = \sum_{n=0}^{\infty} \rho^n \mu^* n.$$ Then $\mu \in S_{\Delta}$ if and only if $\eta/(1 - \rho) \in S_{\Delta}$ if and only if
$$\eta(x + \Delta) \sim \rho^{(1 - \rho)^2} \mu(x + \Delta).$$ Some examples of measures from $S_{\text{loc}}$ may be found in [Asmussen et al., 2003, Section 4].

2. Main result

Assume that $X$ is a random variable with $\mathbb{E}X \in (0, \infty]$. We define the Laplace transform of the distribution of $X$ by
$$g(\theta) := \mathbb{E} e^{-\theta X}.$$ Function $g$ is convex and lower-semicontinuous. We are interested in a situation of an exponentially decaying left tail, that is,
$$\mathbb{P}(X < 0) > 0 \quad \text{and} \quad g(\theta) < \infty \quad \text{for some } \theta > 0.\quad (8)$$ Under (8) we define
$$\kappa := \sup\{\theta > 0 : g(\theta) < 1\} \quad \text{and} \quad \rho := g(\kappa).$$ Since $g'(0) = -\mathbb{E}X < 0$, $\kappa$ is strictly positive. Moreover, we have $g(\theta) \to \infty$ as $\theta \to \infty$ and thus $\kappa$ is finite. In general we have $0 < \rho \leq 1$ and a sufficient condition for $\rho = 1$ is that $g(\theta) < \infty$ for all $\theta > 0$.

Theorem 2.1 Assume $X$ is a random variable with a positive (possibly infinite) expectation such that $8$ holds. Let $H$ be the renewal measure of $(S_n)_{n \geq 0}$, where $S_n = \sum_{k=1}^{n} X_k$ for $n \in \mathbb{N}$, $S_0 = 0$ and $X_k$ are independent copies of $X$. Define $\kappa$ and $\rho$ as in (9).

(a) Assume that $\rho = 1$.

(a-i) Assume that $X$ has a non-arithmetic distribution. Then
$$\lim_{x \to \infty} e^{\kappa x} H((\infty, -x)) = \frac{1}{\kappa g'(\kappa)} \in [0, \infty).$$ Moreover, if
$$\mathbb{E}e^{-\kappa X} 1_{(-X > t)} \sim \frac{L(t)}{t^\alpha}$$ for some $\alpha \in (0, 1)$ and a slowly varying function $L$, then $g'(\kappa) = \infty$. For $\alpha \in (0, 1/2]$, assume additionally that $F(t) = \mathbb{E}e^{-\kappa X} 1_{(-X \leq t)}$ satisfies (4). In such case,
$$e^{\kappa x} H((\infty, -x)) \sim \frac{1}{\Gamma(\alpha)\Gamma(2 - \alpha)} \frac{1}{km(x)},$$ where $m(x) \sim L(x)x^{1-\alpha}/(1 - \alpha)$.

(a-ii) Assume that $X$ has a $d$-arithmetic distribution. Then
$$\lim_{n \to \infty} e^{\kappa d n} H((\infty, -nd)) = \frac{d}{(e^{\kappa d} - 1)g'(\kappa)}.$$
(b) If \( \rho < 1 \), then
\[
\lim_{x \to \infty} e^{\kappa x} H((-\infty, -x)) = 0.
\]
Moreover, if \( \rho^{-1}E e^{-\kappa X} 1_{(-\infty , \cdot )} \in S_{loc} \), then
\[
e^{\kappa x} H((-\infty, -x)) \sim \frac{E e^{-\kappa X} 1_{x < -x + 1}}{\kappa(1 - \rho)^2}.
\]

**Remark 2.2** Condition (10) is implied by
\[
\mathbb{P}(-X > t) = \frac{\alpha L(t)}{\kappa} t^{\beta + \gamma}, \quad t > 0.
\]
Indeed, for any slowly varying function \( L \) and \( \beta < -1 \), [Bingham et al., 1989, Proposition 1.5.10] asserts that
\[
\int_{\rho}^\infty t^\beta L(t)dt \sim x^\beta + 1 L(x)/( - \beta - 1).
\]

**Remark 2.3** Under the same assumptions, a stronger result concerning (a-ii) is proved in [van der Genugten, 1969, Theorem 2] (the remainder term is also exponential).

**Remark 2.4** If \( X \) has a non-arithmetic distribution, for any \( \delta > 0 \), we obtain “more local” behaviour:
\[
\lim_{x \to \infty} e^{\kappa x} H((-\infty - \delta, -x)) = \frac{1 - e^{-\delta x}}{\kappa g'(\kappa)}.
\]

**Proof of Theorem 2.1** Note that \( g'(-\kappa) = -E X e^{\theta X} \) is positive (1 = \( g(0) = g(\kappa) \)) and \( \theta \) is convex, but may infinite.

Define \( F_n = \sigma(X_1, \ldots, X_n) \) and let \( F_\infty \) be the smallest \( \sigma \)-field containing all \( F_n \). On \( (\Omega, F_\infty) \) we define a new measure \( \mathbb{Q} \) via projections
\[
\mathbb{Q}((X_1, \ldots, X_n) \in B) = \rho^{-n}E e^{-\kappa S_n} 1_{(-\infty, \infty)} \in B, \quad B \in \mathcal{B}(\mathbb{R}^n),
\]
where \( S_n = X_1 + \ldots + X_n \), \( n \in \mathbb{N} \) and \( S_0 = 0 \). By the definition of \( \rho \), \( \mathbb{Q} \) is a probability measure. Moreover, \( (X_n)_{n \geq 1} \) is an id sequence under \( \mathbb{Q} \) as well. Let \( \mathbb{E}_\mathbb{Q} \) denote the corresponding expectation. For any Borel function \( f: \mathbb{R} \to \mathbb{R}_+ \) one has
\[
\mathbb{E}_\mathbb{Q} f(S_n) = \rho^n E_{\mathbb{Q}} e^{-\kappa S_n} f(-S_n).
\]
Thus,
\[
\mathbb{P}(S_n < -x) = \rho^n \mathbb{E}_\mathbb{Q} e^{-\kappa S_n} 1_{S_n > x} = \rho^n \int_{(x, \infty)} e^{-\kappa t} Q^n_x(dt).
\]
Moreover, observe that \( \mathbb{E}_\mathbb{Q} X = -E X e^{\kappa X} = g'(\kappa) \in (0, \infty) \), thus \( (S_n)_n \) has a positive drift under \( \mathbb{Q} \) as well. Hence, for \( x > 0 \),
\[
H((-\infty, -x)) = H_p((-\infty, -x)) = \sum_{n=1}^\infty \mathbb{P}(S_n < -x) = \int_{(x, \infty)} e^{-\kappa t} H_Q(dt),
\]
where \( H_Q = \sum_{n=0}^\infty \rho^n Q^n_x \) is the (defective if \( \rho < 1 \)) renewal measure of \( (S_n)_{n \geq 0} \) under \( \mathbb{Q} \).

Writing \( e^{-\kappa t} = \kappa \int_t^\infty e^{-\kappa s} ds \), through Tonelli’s Theorem, we arrive at key identity:
\[
H_p((-\infty, -x)) = \kappa \int_x^\infty e^{-\kappa s} H_Q((x, s]) ds = \kappa e^{-\kappa x} \int_0^\infty e^{-\kappa h} H_Q((x, x + h]) dh.
\]

Consider first the case of \( \rho = 1 \). For any renewal measure \( H \) we have \( H((x, x + h])) \leq \alpha h + \beta \) for some \( \alpha, \beta > 0 \) and all \( x \), thus by Lebesgue’s Dominated Convergence Theorem and (11),
\[
\lim_{x \to \infty} e^{\kappa x} H_p((-\infty, -x)) = \kappa \int_0^\infty e^{-\kappa h} \lim_{x \to \infty} H_Q((x, x + h]) dh = \frac{1}{\kappa E_{\mathbb{Q}} X}.
\]
which gives the first part of (a-i). For (a-ii) use (2), instead of (1).

For the second part of (a-i), observe that

\[ Q(X > t) = E e^{-\kappa X} 1_{\{X > t\}} = \frac{L(t)}{t^\alpha}, \]

thus the result follows by the Strong Renewal Theorem.

If \( \rho < 1 \), then \( H_Q \) is a finite measure and (11) follows again by Lebesgue’s Dominated Convergence Theorem.

Consider now the case, when \( Q X \in S_{loc} \). Since \( E Q e^{-\kappa X} = 1 < \infty \), by Theorem 1.1, we have

\[ (1 - \rho)H_Q \in S_{loc} \quad \text{and} \quad H_Q((x, x + 1]) \sim \rho \frac{1}{1 - \rho} H_Q((x, x + 1]). \]  

Define \( L(y) := H_Q((\log y, \log y + 1]) \). By (7), \( L \) is a slowly varying function. Moreover, for any \( h > 0 \)

\[ \frac{H_Q((x, x + h])}{H_Q((x, x + 1])} \leq \sum_{n=1}^{[h]} \frac{H_Q((x + n - 1, x + n])}{H_Q((x, x + 1])} = \sum_{n=1}^{[h]} \frac{L(e^{\epsilon + n - 1})}{L(e^{\epsilon})}. \]

By Potter bounds ([Bingham et al., 1989, Theorem 1.5.6]) for any \( \varepsilon > 0 \) and \( C > 1 \) there exists \( x_0 \) such that for \( x > x_0 \),

\[ H_Q((x, x + h]) \sim h H_Q((x, x + 1)). \]

Indeed, for \( h = k/n \in \mathbb{Q}_+ \) one gets

\[ H_Q((x, x + \frac{k}{n})] = \sum_{i=1}^{k} H_Q((x + \frac{i-1}{n}, x + \frac{i}{n}]]) \sim k H_Q((x, x + \frac{1}{n}]). \]

and

\[ H_Q((x, x + \frac{1}{n}]) \sim \frac{1}{n} \sum_{i=1}^{n} H_Q((x + \frac{i-1}{n}, x + \frac{i}{n}]) = \frac{1}{n} H_Q((x, x + 1]). \]

By monotonicity, \( f(h) := \lim_{x \to \infty} \frac{H_Q((x, x + h])}{H_Q((x, x + 1])} \) exists for all \( h > 0 \) and \( f(h) = h \). Thus, by (12) and Lebesgue’s Dominated Convergence Theorem we conclude that

\[ \lim_{x \to \infty} e^{\kappa x} \frac{H_Q((-\infty, -x])}{H_Q((x, x + 1])} = \kappa \int_0^\infty e^{-\kappa h} \lim_{x \to \infty} \frac{H_Q((x, x + h])}{H_Q((x, x + 1])} dh = 1/\kappa. \]

The use of (13) completes the proof. \( \square \)

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**References**

S. Asmussen, S. Foss, and D. Korshunov. Asymptotics for sums of random variables with local subexponential behaviour. *J. Theoret. Probab.*, 16(2):489–518, 2003.

N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1989.

D. Blackwell. Extension of a renewal theorem. *Pacific J. Math.*, 3:315–320, 1953.

F. Caravenna and R. A. Doney. Local large deviations and the strong renewal theorem. *arXiv:1612.07635*, pages 1–44, 2016.
H. Carlsson. Remainder term estimates of the renewal function. *Ann. Probab.*, 11(1):143–157, 1983.
P. Kevei. A note on the Kesten-Grincevičius-Goldie theorem. *Electron. Commun. Probab.*, 21:1–12, 2016.
C. Stone. On moment generating functions and renewal theory. *Ann. Math. Statist.*, 36:1298–1301, 1965.
B. B. van der Genugten. Asymptotic expansions in renewal theory. *Compositio Math.*, 21:331–342, 1969.
T. Watanabe and K. Yamamuro. Local subexponentiality of infinitely divisible distributions. *J. Math- for-Ind.*, 1:81–90, 2009.

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