Semiclassical Quantization for the Motion of Guiding Center
Using the Coherent State Path Integral

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Abstract

A new form of the Bohr-Sommerfeld quantization is given of the motion of guiding center in strong magnetic field. This is obtained by the effective action for the degree of guiding center which is deduced from the coherent state path integral for the two types of motion under the mutual interaction; the cyclotron motion and the motion of guiding center.

I. INTRODUCTION

The physics of charged particles in strong magnetic field has been one of the central problems in quantum mechanics inspired by condensed matter physics [1]. The unique feature is that the system is described by two degrees of freedom, namely, the motion of cyclotron and guiding center, if one chooses the special type of gauge for writing the vector potential. In the case that we have the uniform magnetic field only, the guiding center degrees of freedom is not active; this is manifested by the infinite degeneracy of Landau levels. The degeneracy implies some symmetry which is governed by nothing but the guiding center coordinates. Now if a potential of non-magnetic origin is added, the guiding center degrees of freedom begins to be active. The energy exchange occurs between the motion of cyclotron and guiding center, namely, we expect the mixing between inter-Landau levels. Thus the problem is reduced to the study of the two systems which couple each other.

In this note we focus our attention on the motion of the guiding center. We are particularly interested in the effective action to get the semiclassical quantization rule. In order to
achieve this, we use the coherent state path integral. It is well known that both the motion of cyclotron and guiding center can be described by the raising and lowering operators for harmonic oscillator and hence the coherent state provides with a natural means for this specific problem. Concerning the semiclassical quantization, Entelis and Levit studied it in the framework of the Born-Oppenheimer approximation, in which the motion of guiding center and the cyclotron motion are treated as the slow and fast degrees of freedom respectively. The fast cyclotron motion is firstly handled to get the adiabatic effective action for the slow motion of guiding center that includes the adiabatic quantum phase factor. On the contrary, our method does not rely upon the use of the specific assumption of adiabaticity. This paper is a preliminary account of the basic idea and the more detailed argument will be given in the forthcoming paper.

II. COHERENT STATE

We shall start with a brief account for the coherent state representing the quantum state of the charged particle in uniform magnetic field. Let us introduce the guiding center coordinates

\[ X = x - \Pi_y/\mu\omega , \quad Y = y + \Pi_x/\mu\omega \]  

with \( \omega = eB/\mu c \) being the cyclotron frequency. \( \vec{\Pi} = \vec{p} + \frac{e}{c}\vec{A} \) represents the kinematical momentum where the vector potential is given by \( \vec{A} = (-\frac{1}{2}By, \frac{1}{2}Bx, 0) \). We have the commutation relation

\[ [X, Y] = \frac{i\hbar}{\mu\omega} \]  

Now we define the following boson creation and annihilation operators

\[
\begin{align*}
    a &= \frac{1}{\sqrt{2\mu\omega}}(\Pi_x - i\Pi_y) \\
    a^\dagger &= \frac{1}{\sqrt{2\mu\omega}}(\Pi_x + i\Pi_y) \\
    b &= \frac{1}{\sqrt{2\ell}}(X + iY) \\
    b^\dagger &= \frac{1}{\sqrt{2\ell}}(X - iY)
\end{align*}
\]
which satisfy the commutation relation; \([a, a^\dagger] = 1\) and \([b, b^\dagger] = 1\) and \(a\) and \(b\) commute each other, where \(\ell\) is the so-called magnetic length \(\ell \equiv \sqrt{\frac{\mu_0}{\omega^2}}\). In terms of these boson operators, the Hamiltonian in the uniform magnetic field is given by

\[
H_0 = \left( a^\dagger a + \frac{1}{2} \right) \hbar \omega
\]  

(4)

The angular momentum operator \(L_z = x p_y - y p_x\) is also written as

\[
L_z = (a^\dagger a - b^\dagger b) \hbar
\]  

(5)

Now let us define \(|N, m\rangle\) as the simultaneous eigenstate of \(H_0\) and \(L_z\),

\[
H_0 |N, m\rangle = \left( N + \frac{1}{2} \right) \hbar \omega |N, m\rangle
\]

\[
L_z |N, m\rangle = m \hbar |N, m\rangle
\]  

(6)

which satisfy the following relations

\[
\begin{align*}
    a^\dagger |N, m\rangle &= (N + 1)^{\frac{1}{2}} |N + 1, m + 1\rangle \\
    a |N, m\rangle &= N^{\frac{1}{2}} |N - 1, m - 1\rangle \\
    b^\dagger |N, m\rangle &= (N - m + 1)^{\frac{1}{2}} |N, m - 1\rangle \\
    b |N, m\rangle &= (N - m)^{\frac{1}{2}} |N, m + 1\rangle
\end{align*}
\]  

(7)

Here \(N, m\) take the value such that \(N = 0, 1, 2, \cdots, \infty, m = -\infty, \cdots, N - 1, N\). This feature indicates that the Landau level labelled by the quantum number \(N\) has an infinite number of degeneracy. Thus \(|N, m\rangle\) can be constructed as follows;

\[
|N, m\rangle = \frac{(b^\dagger)^{N-m}(a^\dagger)^N}{N!(N-m)!} |0, 0\rangle
\]  

(8)

Now the coherent state \(|z, \xi\rangle\) is introduced such that

\[
a |z, \xi\rangle = z |z, \xi\rangle , b |z, \xi\rangle = \xi |z, \xi\rangle
\]  

(9)

with \(z, \xi\) represent the complex number. \(|z, \xi\rangle\) is constructed as

\[
|z, \xi\rangle = \exp \left( za^\dagger + \xi b^\dagger \right) |0, 0\rangle
\]  

(10)
Note that this forms the so-called "overcomplete set of states". The overlap between two CS’s is calculated as

$$\langle z', \xi' | z, \xi \rangle = \exp \left[ -\frac{1}{2} \left( |z|^2 + |z'|^2 + |\xi|^2 + |\xi'|^2 \right) + (z')^* z + (\xi')^* \xi \right]$$ \hspace{1cm} (11)

The most important property of the coherent state is the resolution of unity,

$$\int |z, \xi\rangle d\mu[z, \xi] \langle z, \xi| = 1 \hspace{1cm} (12)$$

and $d\mu(z, \xi)$ means the measure of the complex parameter space given as

$$d\mu[z, \xi] = \frac{dz^* dz^i d\xi^* d\xi^i}{\pi \pi} \hspace{1cm} (13)$$

The wave function representing the coherent state is written as

$$\psi_{z, \xi}(x, y) \equiv \langle x, y | z, \xi \rangle$$

$$= \frac{1}{\sqrt{2\pi \ell}} \exp \left\{ -\frac{1}{4\ell^2} \left[ x - \sqrt{2\ell} (iz + \xi) \right]^2 - \frac{1}{4\ell^2} \left[ y + \sqrt{2\ell} (z + i\xi) \right]^2 + iz\xi \right\} \hspace{1cm} (14)$$

which is used to evaluate the matrix elements of the Hamiltonian.

III. COHERENT STATE PATH INTEGRAL

We now consider the quantum transition in terms of the path integral using the coherent state constructed above. In the following argument we are concerned with the one-body problem for the charged particle in uniform magnetic field as well as the force of nonmagnetic origin; $H = H_0 + \phi(x, y)$. The nonmagnetic term $\phi(x, y)$ involves the coupling between the guiding center and cyclotron motion. Here we make the following remark. In order to deal with the path integral for two systems under the mutual interaction, the two steps procedure is adopted, namely, one first considers the transition amplitude as if one degree were fixed, and after having accomplished this, the transition amplitude for the remaining degree is considered. Now, we consider the transition amplitude

$$K = \langle \xi_f, z_f | \exp[-\frac{i}{\hbar} \hat{H} t] | \xi_i, z_i \rangle$$ \hspace{1cm} (15)
Following the usual procedure of the coherent state path integral, we first divide the time $t$ into $N$ small interval (and take the limit $N \to \infty$) and next insert the resolution of unity relation for the coherent state at each time division point, then $K$ is cast into the infinite dimensional integral

$$
K(\xi_f, z_f t | \xi_i, z_i, 0) = \lim_{N \to \infty} \int \cdots \int d\mu(\xi_k) \prod_{k=1}^{N-1} \left\langle \xi_k \right| \exp\left[-\frac{i}{\hbar} \hat{H} \epsilon \right] | \xi_{k-1} \rangle \tag{16}
$$

The infinitesimal factor can be approximated in the case of the limit $\epsilon \to 0$;

$$
\left\langle \xi_k \right| \exp\left[-\frac{i}{\hbar} \hat{H} \epsilon \right] | \xi_{k-1} \rangle \simeq \left\langle \xi_k \right| (1 - \frac{i}{\hbar} \hat{H} \epsilon) | \xi_{k-1} \rangle = \left\langle \xi_k \right| \xi_{k-1} \right\rangle \left(1 - \frac{i}{\hbar} \epsilon \left\langle \xi_k \right| \xi_{k-1} \right\rangle \right) \tag{17}
$$

The first factor is just the overlap between the infinitesimal interval, which is explicitly written as

$$
\left\langle \xi_k \right| \xi_{k-1} \rangle = \exp[\xi_k^* \xi_{k-1} - \frac{|\xi_k|^2}{2} - \frac{|\xi_{k-1}|^2}{2}] \tag{18}
$$

By using this form, the propagator turns out to be

$$
K(\xi_f, z_f t : \xi_i, z_i, 0) = \int \exp\left[\frac{i}{\hbar} \int T(z_f, z_i) \mathcal{D}\mu[\xi(t), \xi^*(t)] \right] \tag{19}
$$

$T(z_f, z_i)$ means the amplitude for the transition from the initial state $|z_i\rangle$ to the final state $|z_f\rangle$ under the motion of the guiding center $\xi(t)$, namely,

$$
T(z_f, z_i) = \left\langle z_f \left| T \exp\left[-\frac{i}{\hbar} \int_0^T h(a^t, a; \xi(t), \xi^*(t)) dt \right] \right| z_i \right\rangle \tag{20}
$$

Here $h(a^t, a; \xi, \xi^*) \equiv \left\langle \xi \left| H(a^t, a, b^t, b) \right| \xi^* \right\rangle$, which implies that in calculating the transition amplitude for the cyclotron motion the degree of guiding center is replaced as if it were the $c$-number coordinate. Thus, if using the relation of the resolution of unity holding for $|z\rangle$,

$T(z_f, z_i)$ is written in terms of the path integral over the $z(t)$ space

$$
T(z_f, z_i) = \int \exp\left[\frac{i}{\hbar} \int_0^T \frac{ih}{2} \{ (z^* \dot{z} - c.c) - (z^* z + \frac{1}{2}\hbar \omega - \left\langle z \right| h \left\langle z \right\rangle \} \right] \mathcal{D}\mu(z(t)) \tag{21}
$$

where the expectation value $\left\langle z \right| h \left| z \right\rangle$ is given by

$$
\Phi(z, z^*, \xi, \xi^*) = \left\langle z \right| h \left| z \right\rangle = \frac{\left\langle z, \xi \left| \phi(x, y) \right| z^*, \xi \right\rangle}{\left\langle z, \xi \right| z, \xi \right\rangle} \tag{22}
$$
IV. SEMICLASSICAL QUANTIZATION FORMULA

We shall consider the semiclassical limit of the propagator given by the form (19). The semiclassical limit means that $\hbar \to 0$. Noting that the propagator (18) consists of integral over two different degrees of freedom, we take the semiclassical limit in two-step procedure, namely, in the first step, the semiclassical limit is taken for the integration for the cyclotron degree, thus

$$T(z_f, z_i) = \exp[\frac{i}{\hbar} \Gamma(\tilde{C})] \exp[-\frac{i}{\hbar} \int_0^T \langle z | \hat{\hbar}(\xi(t), \xi^*(t)) | z \rangle \, dt]$$  \hspace{1cm} (23)

where $\Gamma(\tilde{C})$

$$\Gamma(\tilde{C}) = \int \langle z | i\hbar \frac{\partial}{\partial t} | z \rangle \, dt$$  \hspace{1cm} (24)

and $\tilde{C}$ represents the orbit satisfying the action principle

$$\delta \int_0^T \langle z | i\hbar \frac{\partial}{\partial t} - \hat{\hbar}(\xi(t), \xi^*(t)) | z \rangle \, dt = 0.$$  \hspace{1cm} (25)

The variation equation yields the equation of motion for the complex variables $z, z^*$, which leads to the classical equation of motion in the coupled form

$$\begin{cases}
  i\hbar \dot{z}^* + \hbar \omega z^* + \frac{\partial \Phi}{\partial \dot{z}} = 0 \\
  -i\hbar \dot{z} + \hbar \omega z + \frac{\partial \Phi}{\partial \dot{z}^*} = 0
\end{cases}$$  \hspace{1cm} (26)

Note that the solution of these equations are given in terms of the functions of the orbit $\xi(t)$. Thus the evolution operator $K$ becomes

$$K_{eff}(T) = \int \exp[\frac{i}{\hbar} S_{eff}] \prod d\mu[C]$$  \hspace{1cm} (27)

where the effective action is given by

$$S_{eff} = S_0 + \hbar \Gamma - \int_0^T \langle z | \hat{\hbar} | z \rangle \, dt$$  \hspace{1cm} (28)

with $S_0 = \int \frac{i\hbar}{2}(\xi^\dagger \dot{\xi} - c.c) dt$. If we note that $z$ and $z^*$ are given by the functions of the orbit $\xi$, the canonical term is written as
\[ \langle z | i\hbar \frac{\partial}{\partial t} | z \rangle = \left\{ \langle z | i\hbar \frac{\partial}{\partial z} | z \rangle \frac{\partial z}{\partial \xi} + \langle z | i\hbar \frac{\partial}{\partial z^*} | z \rangle \frac{\partial z^*}{\partial \xi} \right\} \frac{d\xi}{dt} \]  

(29)

The canonical term in the effective action for the motion of guiding center is thus changed by an amount of \( \omega \), where

\[ \omega = (A^i_z + A^i_{z^*})d\xi_i. \]  

(30)

Here \( A \) represents the so-called "connection field" that is defined as

\[ \begin{align*}
A^i_z &= \left\langle z | i\hbar \frac{\partial}{\partial z} | z \rangle \frac{\partial z}{\partial x_i} \\
A^i_{z^*} &= \left\langle z | i\hbar \frac{\partial}{\partial z^*} | z \rangle \frac{\partial z^*}{\partial x_i}
\end{align*} \]  

(31)

where \( x_i \) denotes the abbreviation of the guiding center coordinate (X,Y). The appearance of the phase \( \Gamma \) arises from taking the semiclassical limit of the expression of the time-ordered product in (20). This process is a counterpart of the adiabatic process which picks up a specific path from an infinite sum given by the time-ordered product. [4]

Now if we use the effective action of the form (28), we can obtain the semiclassical quantization formula for the motion of guiding center. We first consider the semiclassical approximation for the effective propagator; \( K_{sc}^{eff}(T) \). Having carried out it, we next take the Fourier transform of \( K_{sc}^{eff}(T) \) to apply the stationary phase approximation over the \( T \)-integral, and finally taking into account of the contribution of multiple traversals of the basic periodic orbit, we have

\[ K_{sc}(E) = \sum \frac{\exp[iW(E)]}{1 - \exp[iW(E)]} \]  

(32)

with \( W(E) = \oint \frac{i\hbar}{2} (\xi^* \dot{\xi} - c.c) dt + \Gamma(C) \). Then, the quantization rule should follow from poles on the E-plane

\[ \oint \frac{i\hbar}{2} (\xi^* \dot{\xi} - c.c) dt = (n - \frac{\Gamma(C)}{2\pi})2\pi\hbar \]  

(33)

which just gives the corrected Bohr-Sommerfeld formula including the geometric phase \( \Gamma \). \( C \) represents the periodic orbits determined by the action principle \( \delta S_{eff} = 0 \), which turns out to be
\[ \begin{cases} \dot{i\hbar}\xi^* + \frac{\partial \Phi}{\partial \xi} = 0 \\ -i\hbar\dot{\xi} + \frac{\partial \Phi}{\partial \xi^*} = 0 \end{cases} \] (34)

The formula (33) is just a generalization of the previously result that is the one including the effect of the adiabatic phase.

V. SIMPLE EXAMPLE

We consider the case of the central force; \( \phi(x, y) = \phi(r), r = \sqrt{x^2 + y^2} \) for which we get \( \Phi(z, z^*, \xi, \xi^*) = \Phi(|\chi|^2) \) where \( \chi \) and hence the equation of motion yields

\[ \begin{cases} i\hbar \dot{\chi}^* + \hbar \omega z^* + \frac{i}{2}(-iz^* + \xi) \frac{d\Phi}{d|\chi|^2} = 0 \\ -i\hbar \dot{\chi} + \hbar \omega z - \frac{i}{2}(iz + \xi^*) \frac{d\Phi}{d|\chi|^2} = 0 \\ i\hbar \dot{\xi}^* + 1 \frac{2}{2}(iz^* + \xi) \frac{d\Phi}{d|\chi|^2} = 0 \\ -i\hbar \dot{\xi} + 1 \frac{2}{2}(-iz^* + \xi) \frac{d\Phi}{d|\chi|^2} = 0 \end{cases} \] (35)

As a special case of the lowest Landau level; \( N = 0 \) namely, \( z = z^* = 0 \), the Lagrangian turns out to be

\[ L = \frac{i\hbar}{2}(\xi^* \dot{\xi} - \dot{\xi}^* \xi) - \frac{1}{2} \hbar \omega - \Phi(|\xi|^2) \] (36)

which yields the equation of motion

\[ \begin{cases} i\hbar \dot{\xi}^* + 1 \frac{2}{2}\xi^* \frac{d\Phi}{d|\xi|^2} = 0 \\ -i\hbar \dot{\xi} + 1 \frac{2}{2}\xi \frac{d\Phi}{d|\xi|^2} = 0 \end{cases} \] (37)

from which we get \( \frac{d(|\xi|^2)}{dt} = 0 \), thus \( |\xi|^2 = \text{const} \). Physically speaking, the restriction to the lowest Landau level means the strong magnetic field limit \( B \rightarrow \infty \) or in terms of the cyclotron frequency, \( \omega \rightarrow \infty \) (\( \omega \propto B \)). Now we shall consider the Bohr-Sommerfeld quantization, which is obtained as special case of (33), namely, by putting \( z = z^* = 0, \)

\[ \frac{i\hbar}{2} \int_C (\xi^* d\xi - \xi d\xi^*) = 2\pi m \hbar \] (38)
If we put \( \xi = \frac{1}{\sqrt{2}}(X + iY) \), \( \xi^* = \frac{1}{\sqrt{2}}(X - iY) \), this can be rewritten as the form

\[
\frac{i}{\ell} \oint_C \frac{i}{2} (X dY - Y dX) = -\frac{h}{2\ell} \int_S 2dX \land dY
\]

\[
= -2\pi \hbar |\xi|^2
\]

(39)

Thus, we have

\[ |\xi|^2 = -m \]

and hence the energy spectrum is obtained as

\[ E_m = \frac{1}{2} \hbar \omega - \Phi(-m) \] (40)

Note that the above quantization rule gives nothing but the angular momentum quantization, since \( |\xi|^2 \) is given as the expectation value of \( L_z \) with respect to the coherent state.

As a special case, we consider the Coulomb potential for the interaction \( \phi(r) = -\frac{e^2}{r} \), for which \( \Phi(|\xi|) \) becomes

\[ \Phi(|\chi|) = \frac{e^2 \pi}{\sqrt{2\pi \ell}} e^{-|\chi|^2/2} I_0(|\chi|^2/2) \] (41)

where \( I_0(|\xi|) \) stands for the modified Bessel function. The energy spectrum is obtained by substituting the relation \( |\xi|^2 = -m \).

VI. FINAL REMARKS

In this note we have developed a basic idea for the effective action for the motion of guiding center to yield the semiclassical quantization rule. The formula presented here is mainly concerned with the single particle in strong magnetic field together with the potential of non-magnetic origin. Here an outline is given of the possible problems that are expected to be developed on basis of the present general formalism. (i): To take into account of the cyclotron motion in the quantization rule, which is to consider the effect of the geometric phase \( \Gamma \) after all. (ii): To extend the case of the lowest Landau level to the many particle case, which may be regarded as a similarity with the vortex gas for which the force is given by the logarithmic potential [7]. (iii): In relation to the many-particle extension, it may be
interesting to consider the charged particle with two kind of charges, namely, with opposite charges each other, which may be analogous to the two kind of vortices with opposite vortex strength. These will be left as subjects for the future study.
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