RIGHT NOETHERIAN SEMIGROUPS

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Abstract. A semigroup $S$ is right noetherian if every right congruence on $S$ is finitely generated. In this paper we present some fundamental properties of right noetherian semigroups, discuss how semigroups relate to their substructures with regard to the property of being right noetherian, and investigate whether this property is preserved under various semigroup constructions.

1. Introduction

A semigroup $S$ is said to be right noetherian if every right congruence on $S$ is finitely generated; equivalently, $S$ satisfies the ascending chain condition on its right congruences. Left noetherian semigroups are defined in an analogous way. In this paper we will only consider right noetherian semigroups.

For groups, being right noetherian means that every subgroup is finitely generated, and this is equivalent to being left noetherian. Groups with this property are commonly known as noetherian groups and have received a fair amount of attention. They were first studied systematically by Baer in [1], and it was shown in [6] that polycyclic groups are noetherian. Noetherian groups can have a rather complex structure, as explained in Chapter 9 of [14]. In particular, it is shown there that there exist infinite simple noetherian groups that are periodic or, on the other extreme, torsion-free [14, Theorem 28.1, Theorem 28.3]. However, every noetherian group is finitely generated.

The study of right noetherian semigroups was initiated by Hotzel in [7]. He obtained particularly comprehensive results regarding weakly periodic semigroups (which include all regular semigroups); in particular, he showed that if they are right noetherian, then they contain only finitely many right ideals and are finitely generated. The question of whether every right noetherian semigroup is finitely generated has been considered by several authors (see [2], [7], [10]) and remains an open problem. Right noetherian semigroups were further studied by Kozhukhov in [9], where he discusses the structure of right noetherian 0-simple semigroups and right noetherian inverse semigroups.

2010 Mathematics Subject Classification. 20M10, 20M18.

There are other uses of the term ‘right noetherian’ in semigroup theory: it may be used to refer to semigroups for which every right ideal is finitely generated, and to semigroups for which there is no infinite descending sequence with respect to the proper right-divisibility relation [5].
The purpose of this paper is to investigate how the property of being right noetherian is preserved under various semigroup constructions. It is structured as follows. In Section 2, we provide some fundamental facts about right noetherian semigroups; many of these results are folklore, but we gather them for completeness. In Section 3, we investigate whether the property of being right noetherian is inherited by substructures in certain situations. The remainder of the paper is concerned with various semigroup constructions. Specifically, we consider the following: direct products (Section 4), semidirect and wreath products (Section 5), 0-direct unions and (strong) semilattices of semigroups (Section 6), and semigroup free products and monoid free products (Section 7).

2. Fundamental Properties

We begin this section by presenting several standard equivalent characterisations of right noetherian semigroups in terms of the ascending chain condition and maximal condition on right congruences, paralleling the situation for rings.

Proposition 2.1. Let $S$ be a semigroup. Then the following are equivalent:

1. $S$ is right noetherian;
2. $S$ satisfies the ascending chain condition on its right congruences: every infinite ascending chain $\rho_1 \subseteq \rho_2 \subseteq \ldots$ of right congruences on $S$ eventually terminates;
3. every non-empty set of right congruences on $S$ has a maximal element.

Proof. (1) $\Rightarrow$ (2). Consider an infinite ascending chain

$$\rho_1 \subseteq \rho_2 \subseteq \ldots$$

of right congruences on $S$. Let $\rho$ be the right congruence $\bigcup_{i=1}^{\infty} \rho_i$. Since $S$ is right noetherian, we have that $\rho$ is generated by some finite set $X$. Let $k$ be minimal such that $X \subseteq \rho_k$. Then $\rho \subseteq \rho_k$. But $\rho_k \subseteq \rho$, so $\rho = \rho_k$. Hence, we have that $\rho_n = \rho_k$ for all $n \geq k$.

(2) $\Rightarrow$ (3). Suppose there exists a non-empty set $F$ of right congruences on $S$ with no maximal element. Choose $(a_1, b_1) \in \rho$, and let $\rho_1$ be the right congruence on $S$ generated by $(a_1, b_1)$. Since $\rho$ is not finitely generated, we have that $\rho_1 \neq \rho$. Choose $(a_2, b_2) \in \rho \setminus \rho_1$, and let $\rho_2$ be the right congruence on $S$ generated by $\{(a_1, b_1), (a_2, b_2)\}$. We have that $\rho_1 \subset \rho_2$. Continuing in this way, we have a non-empty set $\{\rho_1, \rho_2, \ldots\}$ of right congruences on $S$ with no maximal element.

(3) $\Rightarrow$ (1). Suppose there exists a non-finitely generated right congruence $\rho$ on $S$. Choose $(a_1, b_1) \in \rho$, and let $\rho_1$ be the right congruence on $S$ generated by $(a_1, b_1)$. Since $\rho$ is not finitely generated, we have that $\rho_1 \neq \rho$. Choose $(a_2, b_2) \in \rho \setminus \rho_1$, and let $\rho_2$ be the right congruence on $S$ generated by $\{(a_1, b_1), (a_2, b_2)\}$. We have that $\rho_1 \subset \rho_2$. Continuing in this way, we have a non-empty set $\{\rho_1, \rho_2, \ldots\}$ of right congruences on $S$ with no maximal element. $\square$
Now, let $S$ be a semigroup and let $X \subseteq S \times S$. We introduce the notation

$$\overline{X} = X \cup \{(x, y) : (y, x) \in X\},$$

which will be used throughout the paper. For $a, b \in S$, an $X$-sequence connecting $a$ and $b$ is any sequence

$$a = x_1 s_1, \ y_1 s_1 = x_2 s_2, \ y_2 s_2 = x_3 s_3, \ldots, \ y_k s_k = b,$$

where $(x_i, y_i) \in \overline{X}$ and $s_i \in S^1$ for $1 \leq i \leq k$.

We now introduce the following definition.

**Definition 2.2.** Let $S$ be a semigroup, let $X \subseteq S \times S$, and let $s, t \in S$. We say that $(s, t)$ is a consequence of $X$ if either $s = t$ or there exists an $X$-sequence connecting $s$ and $t$.

We have the following basic lemma.

**Lemma 2.3.** Let $S$ be a semigroup, let $X \subseteq S \times S$, let $\rho$ be a right congruence on $S$ generated by $X$, and let $s, t \in S$. Then $s \rho t$ if and only if $(s, t)$ is a consequence of $X$.

**Definition 2.4.** A semigroup $S$ is said to be weakly right noetherian if every right ideal of $S$ is finitely generated.

**Remark 2.5.** Being weakly right noetherian for a semigroup $S$ is equivalent to $S$ satisfying the ascending chain condition on right ideals.

**Lemma 2.6.** Let $S$ be a semigroup. If $S$ is right noetherian, then $S$ is also weakly right noetherian.

**Proof.** Let $I$ be a right ideal of $S$. Consider the Rees right congruence $\rho_I$ on $S$ given by

$$a \rho_I b \iff a = b \text{ or } a, b \in I.$$

Since $S$ is right noetherian, $\rho_I$ is generated (as a right congruence) by some finite set $X \subseteq I \times I$. We claim that $I$ is generated (as a right ideal) by the finite set

$$U = \{x \in I : (x, y) \in \overline{X} \text{ for some } y \in I\}.$$

Indeed, let $a \in I$, and choose $b \in I$ with $a \neq b$. Since $a \rho_I b$, there exists an $X$-sequence connecting $a$ and $b$. In particular, there exist some $x \in U$ and $s \in S^1$ such that $a = xs$. \[\square\]

**Corollary 2.7.** [7, Lemma 1.6] Let $S$ be a right noetherian semigroup. Then $S$ contains no infinite set of pairwise incomparable right ideals.

**Definition 2.8.** Let $S$ be a semigroup. An element $s \in S$ is said to be decomposable if $s \in S^2$. An element is indecomposable if it is not decomposable.

**Corollary 2.9.** Let $S$ be a right noetherian semigroup. Then $S$ has only finitely many indecomposable elements.
Proof. Since $S$ is weakly right noetherian, $S$ is finitely generated as a right ideal; that is, there exists a finite set $X \subseteq S$ such that $S = XS^1$. Therefore, we have that $S \setminus X \subseteq S^2$, so $S$ has at most $|X|$ indecomposable elements. □

Remark 2.10. The class of right noetherian semigroups is strictly contained in the class of weakly right noetherian semigroups. Indeed, every group is weakly right noetherian, but there exist groups that are not right noetherian.

It is clear that the property of being right noetherian is a finiteness condition for semigroups:

Lemma 2.11. Finite semigroups are right noetherian.

For commutative semigroups, one-sided congruences coincide with (two-sided) congruences. Right noetherian commutative semigroups have the following characterisation, which is due to Budach [3] (necessity) and Rédei [18] (sufficiency).

Theorem 2.12. Let $S$ be a commutative semigroup. Then $S$ is right noetherian if and only if $S$ is finitely generated.

For groups, being right noetherian is equivalent to every subgroup being finitely generated (or, equivalently, satisfying the ascending chain condition on subgroups).

Proposition 2.13. A group $G$ is right noetherian if and only if every subgroup of $G$ is finitely generated.

Proof. ($\Rightarrow$) Let $H$ be a subgroup of $G$. We have a right congruence $\rho$ on $G$ given by

$$a \rho b \iff ab^{-1} \in H.$$ 

Since $G$ is right noetherian, we have that $\rho$ is generated by some finite subset $U$. We shall show that $H$ is generated by the finite set

$$X = \{xy^{-1} : (x, y) \in U\}.$$

Let $h \in H$. We have that $h \rho 1$, so there exists an $X$-sequence

$$h = x_1g_1, y_1g_1 = x_2g_2, \ldots, y_kg_k = 1.$$ 

Hence, we have

$$h = (x_1y_1^{-1})x_2g_2 = \cdots = (x_1y_1^{-1})\cdots(x_{k-1}y_{k-1}^{-1})x_kg_k = (x_1y_1^{-1})\cdots(x_ky_k^{-1}) \in (X).$$

($\Leftarrow$) Let $\rho$ be a right congruence on $G$. The set

$$H = \{xy^{-1} : (x, y) \in \rho\}$$

is a subgroup of $G$, so it is generated by some finite set $X$. We claim that $\rho$ is generated by the set

$$U = \{(x, y) : xy^{-1} \in X\}.$$ 

Indeed, let $a \rho b$ with $a \neq b$. We have that $ab^{-1} \in H$, so hence

$$ab^{-1} = (x_1y_1^{-1})\cdots(x_ky_k^{-1})$$
for some $x_i y_i^{-1} \in X$. Therefore, we have a $U$-sequence
\[
a = x_1 (y_1^{-1} x_2 \ldots y_k^{-1} b), y_1 (y_1^{-1} x_2 \ldots y_k^{-1} b) = x_2 (y_2^{-1} x_3 \ldots y_k^{-1} b), \ldots, y_k (y_k^{-1} b) = b,
\]
as required. □

**Corollary 2.14.** Let $G$ be a group and let $H$ be a subgroup of $G$. If $G$ is right noetherian, then $H$ is right noetherian.

**Remark 2.15.** It is known that polycyclic groups (and hence supersolvable and finitely generated nilpotent groups) are right noetherian [3]. Also, there exist right noetherian groups that are not finitely presented; for example, Tarski groups (infinite groups $G$ all of whose non-trivial proper subgroups are cyclic of fixed prime order $p$), which Ol'Shanskii showed exist in [14, Chapter 9].

The following result shows that the property of being right noetherian is closed under homomorphic images (or equivalently quotients).

**Lemma 2.16.** Let $S$ be a semigroup and let $T$ be a homomorphic image of $S$. If $S$ is right noetherian, then $T$ is right noetherian.

**Proof.** Let $\rho$ be a right congruence on $T$. Let $\theta : S \to T$ be a surjective homomorphism, and define a right congruence $\rho'$ on $S$ by
\[
s \rho' t \iff s \theta \rho t \theta.
\]
Since $S$ is right noetherian, $\rho'$ is generated by a finite set $X$. We claim that $\rho$ is generated by the finite set
\[
Y = \{(x \theta, y \theta) : (x, y) \in X\}.
\]
Indeed, let $a \rho b$ with $a \neq b$. There exist $s, t \in S$ such that $a = s \theta$ and $b = t \theta$. Since $s \rho' t$ and $s \neq t$, there exists an $X$-sequence connecting $s$ and $t$. Applying $\theta$ to every term of this sequence yields a $Y$-sequence connecting $a$ and $b$. □

It is often the case in semigroup theory that for a semigroup $S$ and an ideal $I$ of $S$, if both $I$ and the Rees quotient $S/I$ have a property $P$, then $S$ also has property $P$. In particular, this holds for the properties of being finitely generated and finitely presented [4, Theorem 2.1]. The next result reveals that this is also the case for the property of being right noetherian.

**Proposition 2.17.** Let $S$ be a semigroup and let $I$ be an ideal of $S$. If both $I$ and $S/I$ are right noetherian, then $S$ is right noetherian.

**Proof.** Let $\rho$ be a right congruence on $S$. We have that $\rho_I$, the restriction of $\rho$ to $I$, is generated by a finite set $X$, since $I$ is right noetherian. Let $U = S \setminus I$. For each $s \in U$ such that $s \rho i$ for some $i \in I$, choose $\alpha(s) \in I$ such that $s \rho \alpha(s)$. Let $\rho'$ be the right congruence on $S/I$ generated by the set
\[
Y = (\rho \cap (U \times U)) \cup \{(s, 0) : s \in U, s \rho i \text{ for some } i \in I\}.
\]
Since $S/I$ is right noetherian, we have that $\rho'$ is generated by a finite set $Y' \subseteq Y$. Let $Y_1 = Y' \cap (U \times U)$ and let

$$Y_2 = \{(x, \alpha(x)) : (x, 0) \in Y'\}.$$ 

We now make the following claim.

**Claim.** Let $s \in S$ such that $s \rho i$ for some $i \in I$. Then there exists $s' \in I$ such that $(s, s')$ is a consequence of $Y_1 \cup Y_2$.

**Proof.** If $s \in I$, simply let $s' = s$. Suppose now that $s \in U$. We have that $s \rho 0$, so there exists a $Y'$-sequence

$$s = x_1s_1, y_1s_1 = x_2s_2, \ldots, y_ks_k = 0,$$

where $x_is_i \in U$ for $1 \leq i \leq k$. If $y_k \in U$, then we have a $Y_1$-sequence connecting $s$ and $y_ks_k \in I$. If $y_k = 0$, then $(s, x_ks_k)$ is a consequence of $Y_1$, and we obtain $\alpha(x_ks_k) \in I$ from $x_ks_k$ using $Y_2$. □

Returning to the proof of Proposition 2.17, we claim that $\rho$ is generated by the finite set $X \cup Y_1 \cup Y_2$.

Indeed, let $s \rho t$ with $s \neq t$. If there do not exist any $i \in I$ such that $(s, i) \in \rho$, then $s, t \in U$ and $(s, t)$ is a consequence of $Y_1$. Otherwise, by the above claim, there exist $s', t' \in I$ such that $(s, s')$ and $(t, t')$ are consequences of $Y_1 \cup Y_2$. Since $s' \rho t'$, we have that $(s', t')$ is a consequence of $X$. Hence, $(s, t)$ is a consequence of $X \cup Y_1 \cup Y_2$. □

It is well known that for modules over Noetherian rings (rings that satisfy the ascending chain condition on right ideals), finite generation and finite presentability are equivalent properties. Monoid acts play the analogous role in the theory of monoids as modules do in the theory of rings. In the following, we give some basic definitions about monoid acts. One may consult [8] for more information.

Let $S$ be a semigroup. A *(right)* $S$-act is a non-empty set $A$ together with a map

$$A \times S \to A, (a, s) \mapsto as$$

such that $a(st) = (as)t$ for all $a \in A$ and $s, t \in S$.

If $S$ is a monoid and $a1 = a$ for all $a \in A$, then $A$ is a *(monoid)* $S$-act. For instance, $S$ itself is an $S$-act via right multiplication.

An equivalence relation $\rho$ on an $S$-act $A$ is an *(S-act) congruence* on $A$ if $(a, b) \in \rho$ implies $(as, bs) \in \rho$ for all $a, b \in A$ and $s \in S$. Note that the congruences on the $S$-act $A$ are precisely the right congruences on $S$.

An $S$-act $A$ is *finitely generated* if there exists a finite subset $X \subseteq A$ such that $A = XS^1$, and $A$ is *finitely presented* if it is isomorphic to a quotient of a finitely generated free $S$-act by a finitely generated congruence.

Finite presentability of monoid acts has been systematically studied in [12], [13]. The following result provides a characterisation of right noetherian monoids in terms of finite presentability of their acts.
Proposition 2.18. The following are equivalent for a monoid $S$:

1. $S$ is right noetherian;
2. every cyclic $S$-act is finitely presented;
3. every finitely generated $S$-act is finitely presented.

Proof. (1) $\Rightarrow$ (2). Let $A$ be a cyclic $S$-act. We have that $A \cong S/\rho$ where $\rho$ is a right congruence on $S$ [8, Proposition 5.17]. Since $S$ is right noetherian, the right congruence $\rho$ is finitely generated, so $A$ is finitely presented.

(2) $\Rightarrow$ (3). We prove by induction on the number of generators. Let $A = \langle X \rangle$ where $|X| = n$. If $n = 1$, then $A$ is cyclic and hence finitely presented, so assume that $n > 1$. Assume that every finitely generated $S$-act with a generating set of size less than $n$ is finitely presented. If there exists some $x \in X$ such that $A$ is the disjoint union of its subacts $\langle x \rangle$ and $\langle X \setminus \{x\} \rangle$, then $A$ is finitely presented by [13, Corollary 5.9], since $\langle x \rangle$ and $\langle X \setminus \{x\} \rangle$ are finitely presented by assumption. Otherwise, choose $x \in X$ and let $B = \langle x \rangle$. Now $B$ is cyclic and hence finitely presented, and the Rees quotient $A/B$ is generated by the set $X \setminus \{x\}$, so it is finitely presented by assumption. It now follows from [13, Corollary 4.4] that $A$ is finitely presented.

(3) $\Rightarrow$ (1). For any right congruence $\rho$ on $S$, we have that the cyclic $S$-act $A = S/\rho$ is finitely presented, and hence $\rho$ is finitely generated by [13, Proposition 3.9]. □

3. Subsemigroups

In this section we study the relationship between a semigroup and its substructures with regard to being right noetherian. The following example demonstrates that subsemigroups (indeed, ideals) of right noetherian semigroups are not in general right noetherian.

Example 3.1. Let $S$ be the free commutative semigroup on $\{a, b\}$, and let $I$ be the ideal of $S$ generated by $\{a\}$. The set $\{ab^i : i \geq 0\}$ is a minimal (semigroup) generating set for $I$, so $I$ is not right noetherian by Theorem 2.12.

In the following we explore various situations in which the property of being right noetherian is inherited by subsemigroups.

The first situation we consider is where a subsemigroup has finite complement. It has been shown that various finiteness properties, including finite generation, finite presentability and periodicity, are inherited when passing to subsemigroups of finite complement and extensions; see [19] for more details. The following result shows that this also holds for the property of being right noetherian.

Theorem 3.2. Let $S$ be a semigroup with a subsemigroup $T$ such that $S \setminus T$ is finite. Then $S$ is right noetherian if and only if $T$ is right noetherian.
Proof. \((\Rightarrow)\) Let \(\rho\) be a right congruence on \(T\). Denote by \(\overline{\rho}\) the right congruence on \(S\) generated by \(\rho\). Since \(S\) is right noetherian, \(\overline{\rho}\) is generated by some finite subset \(X \subseteq \rho\). Define a set

\[ U = \{xs : (x, y) \in X \text{ for some } y \in T, s \in S \setminus T\}. \]

Note that \(U\) is finite, since \(X\) and \(S \setminus T\) are finite. We claim that \(\rho\) is generated by the finite set \(Y = X \cup (\rho \cap (U \times U))\).

Indeed, let \(a \rho b\) with \(a \neq b\). Since \(a \overline{\rho} b\), there exists an \(X\)-sequence

\[ a = x_1s_1, y_1s_1 = x_2s_2, \ldots, y ks_k = b, \]

where \((x_i, y_i) \in X\) and \(s_i \in S^1\) for \(1 \leq i \leq k\). If every \(s_i \in S^1\), then \((a, b)\) is a consequence of \(X\). Otherwise, let \(i\) be minimal such that \(s_i \in S \setminus T\), and let \(j\) be maximal such that \(s_j \in S \setminus T\). Then \((a, x_is_i)\) and \((y_js_j, b)\) are consequences of \(X\), and \((x_is_i, y_js_j) \in \rho \cap (U \times U)\), so \((a, b)\) is a consequence of \(Y\).

\((\Leftarrow)\) Let \(\rho\) be a right congruence on \(S\). Let \(\rho_T\) be the restriction of \(\rho\) to \(T\). Since \(T\) is right noetherian, we have that \(\rho_T\) is generated by a finite set \(X\).

Let \(V = S \setminus T\). For each \(s \in V\) such that \(s \rho_T t\) for some \(t \in T\), choose \(\alpha(s) \in T\) such that \(s \rho \alpha(s)\). We claim that \(\rho\) is generated by the finite set

\[ Y = X \cup \{(s, \alpha(s)) : s \in V, s \rho_T t \text{ for some } t \in T\} \cup (\rho \cap (V \times V))\].

Indeed, let \(s \rho_T t\) with \(s \neq t\). If \(s, t \in V\), then \((s, t) \in \rho \cap (V \times V)\). If \(s, t \in T\), then \(s \rho_T t\), so \((s, t)\) is a consequence of \(X\). Finally, suppose that \(s \in V\) and \(t \in T\). We have that \(\alpha(s) \rho s \rho t\), so \(\alpha(s) \rho_T t\), and hence \((\alpha(s), t)\) is a consequence of \(X\). Therefore, we have that \((s, t)\) is a consequence of \(Y\). \(\square\)

**Corollary 3.3.** A semigroup \(S\) is right noetherian if and only if \(S^1\) is right noetherian.

**Corollary 3.4.** A semigroup \(S\) is right noetherian if and only if \(S^0\) is right noetherian.

The next situation we consider is where the complement of a subsemigroup is a left ideal.

**Proposition 3.5.** Let \(S\) be a semigroup with a subsemigroup \(T\) such that \(S \setminus T\) is a left ideal of \(S\). If \(S\) is right noetherian, then \(T\) is right noetherian.

**Proof.** Let \(\rho\) be a right congruence on \(T\). Let \(\overline{\rho}\) denote the right congruence on \(S\) generated by \(\rho\). Since \(S\) is right noetherian, \(\overline{\rho}\) is finitely generated, so it is generated by some finite subset \(X \subseteq \rho\). We claim that \(\rho\) is generated by \(X\).

Indeed, let \(a \rho b\) with \(a \neq b\). Since \(a \overline{\rho} b\), there exists an \(X\)-sequence

\[ a = x_1s_1, y_1s_1 = x_2s_2, \ldots, y ks_k = b. \]
Since $a \in T$ and $S \setminus T$ is a left ideal of $S$, we must have that $s_1 \in T^1$. Therefore, we have that $y_1 s_1 \in T$, which in turn implies that $s_2 \in T^1$. Continuing in this way, we have that $s_i \in T^1$ for $1 \leq i \leq k$, so $(a, b)$ is a consequence of $X$. □

**Corollary 3.6.** Let $S$ be a semigroup with a subsemigroup $T$ such that $S \setminus T$ is a right noetherian ideal of $S$. Then $S$ is right noetherian if and only if $T$ is right noetherian.

**Proof.** The direct implication follows from Proposition 3.5. For the converse, let $I = S \setminus T$. Now $S/I \cong T \cup \{0\}$ is right noetherian by Corollary 3.4 since $T$ is right noetherian. It now follows from Proposition 2.17 that $S$ is right noetherian. □

**Remark 3.7.** If $S$ is a right noetherian semigroup with a right noetherian subsemigroup $T$ such that $I = S \setminus T$ is an ideal of $S$, it is not necessary that $I$ be right noetherian. Indeed, in Example 3.1 we have that $S$ and $T = S \setminus I \cong \mathbb{N}$ are both right noetherian, but $I$ is not right noetherian.

Let $S$ be a semigroup and let $X$ be a subset of $S$. We define the right stabiliser of $X$ in $S$ to be the set

$$\text{Stab}(X) = \{s \in S^1 : Xs = X\}.$$ 

It is clear that Stab$(X)$ is a submonoid of $S^1$.

Our next result states that the right stabiliser of any subset of a right noetherian semigroup is right noetherian.

**Proposition 3.8.** Let $S$ be a semigroup and let $X$ be a subset of $S$. If $S$ is right noetherian, then the right stabiliser $\text{Stab}(X)$ of $X$ is also right noetherian.

**Proof.** Let $\rho$ be a right congruence on Stab$(X)$. Let $\overline{\rho}$ denote the right congruence on $S^1$ generated by $\rho$. We have that $S^1$ is right noetherian by Corollary 3.3, so $\overline{\rho}$ is generated by some finite subset $Y \subseteq \rho$. We claim that $\rho$ is generated by $Y$.

Indeed, let $a \rho b$ with $a \neq b$. Since $a\overline{\rho}b$, there exists a $Y$-sequence

$$a = x_1 s_1, y_1 s_1 = x_2 s_2, \ldots, y_k s_k = b,$$

where $(x_i, y_i) \in Y$ and $s_i \in S^1$ for $1 \leq i \leq k$. Since $a, x_1 \in \text{Stab}(X)$, we have that

$$Xs_1 = (Xx_1)s_1 = Xa = X,$$

so $s_1 \in \text{Stab}(X)$. Therefore, we have that $y_1 s_1 \in \text{Stab}(X)$, which in turn implies that $s_2 \in \text{Stab}(X)$. Continuing in this way, we have that $s_i \in T(X)$ for $1 \leq i \leq k$, so $(a, b)$ is a consequence of $Y$, as required. □

Of particular interest in semigroup theory are maximal subgroups, which coincide with the group $H$-classes. Schützenberger showed in [21] how one can assign a group to an arbitrary $H$-class, so as to reflect the group-like properties of that class. Here we give his construction; one may consult [11] for more details and basic properties of Schützenberger groups.
Let $S$ be a semigroup and let $H$ be an $H$-class of $S$. Define a relation $\sigma(H)$ on the right stabiliser $\text{Stab}(H)$ of $H$ by
\[(s, t) \in \sigma(H) \iff hs = ht \text{ for all } h \in H.\]
It is easy to see that $\sigma(H)$ is a congruence on $\text{Stab}(H)$, and it turns out that the quotient $\Gamma(H) = \text{Stab}(H)/\sigma(H)$ is a group. We call $\Gamma(H)$ the Schützenberger group of $H$.

Note that if $H$ is a group $H$-class (that is, a maximal subgroup), then it is isomorphic to $\Gamma(H)$ [11, Theorem 3.3].

By [9, Lemma 1.4], maximal subgroups of right noetherian semigroups are right noetherian. Since $\Gamma(H)$ is a quotient of $\text{Stab}(H)$, Proposition 3.8 and Lemma 2.16 together yield the following generalisation of this result.

**Corollary 3.9.** Let $S$ be a semigroup and let $H$ be an $H$-class of $S$. If $S$ is right noetherian, then the Schützenberger group $\Gamma(H)$ is also right noetherian.

We may now easily deduce that subgroups of right noetherian semigroups are right noetherian.

**Corollary 3.10.** [9, Lemma 1.4] Let $S$ be a semigroup and let $G$ be a subgroup of $S$. If $S$ is right noetherian, then $G$ is right noetherian.

**Proof.** Let $e$ be the identity of $G$. Now $G$ is a subgroup of the maximal subgroup $H_e \cong \Gamma(H_e)$. We have that $H_e$ is right noetherian by Corollary 3.9 and hence $G$ is right noetherian by Corollary 2.14. \qed

A semigroup $S$ is said to be regular if for every $s \in S$ there exists $x \in S$ such that $s = sx = xs$. An inverse semigroup is a regular semigroup whose idempotents commute.

We have the following characterisation, due to Kozhukhov, of right noetherian inverse semigroups.

**Theorem 3.11.** [9, Theorem 4.3] Let $S$ be an inverse semigroup. Then the following are equivalent:

1. $S$ is right noetherian;
2. $S$ has finitely many idempotents and all its maximal subgroups are right noetherian.

The above result leads to the following alternative characterisation of right noetherian inverse semigroups, which is a generalisation of Proposition 2.13.

**Corollary 3.12.** Let $S$ be an inverse semigroup. Then $S$ is right noetherian if and only if every inverse subsemigroup of $S$ is finitely generated.

**Proof.** ($\Rightarrow$) Let $T$ be an inverse subsemigroup of $S$. It follows from Theorem 3.11 that $T$ has finitely many idempotents and all its maximal subgroups are finitely generated, so $T$ is finitely generated by [20, Proposition 3.1].
We have that the semilattice $Y$ of idempotents of $S$ is finite (since it is finitely generated). Also, since all the subgroups of $S$ are finitely generated, we have that all the maximal subgroups of $S$ are right noetherian by Proposition 2.13. Hence, by Theorem 3.11, we have that $S$ is right noetherian. □

The next result shows that the property of being right noetherian is inherited by inverse subsemigroups.

**Proposition 3.13.** Let $S$ be a semigroup and let $T$ be an inverse subsemigroup of $S$. If $S$ is right noetherian, then $T$ is right noetherian.

**Proof.** We show that the semilattice $Y$ of idempotents of $T$ is right noetherian. It then follows from Theorem 2.12 that $Y$ is finitely generated and hence finite. Since, by Corollary 3.10, the maximal subgroups of $T$ are right noetherian, we then have that $T$ is right noetherian by Theorem 3.11.

So, let $\rho$ be a (right) congruence on $Y$. Let $\overline{\rho}$ denote the right congruence on $S$ generated by $\rho$. Since $S$ is right noetherian, $\rho$ is generated by some finite subset $X \subseteq \rho$. Let $U$ be the finite subsemigroup of $Y$ generated by the set $\{e \in Y : (e, f) \in \overline{X} \text{ for some } f \in Y\}$. We show that $\rho$ is generated by the set $Z = \rho \cap (U \times U)$. (Note that $X \subseteq Z$.)

Let $e \rho f$ with $e \neq f$. Since $e \overline{\rho} f$, there exists an $X$-sequence

$$e = e_1s_1, f_1s_1 = e_2s_2, \ldots, f_ks_k = f,$$

where $(e_i, f_i) \in \overline{X}$ and $s_i \in S^1$ for $1 \leq i \leq k$.

We claim that $e_1 \ldots e_is_i e \in Y$ for every $i \in \{1, \ldots, k\}$. Indeed, we have that $e_1s_1 e = e \in Y$ and, for $i \in \{2, \ldots, k\}$, we have

$$e_1 \ldots e_is_i e = e_1 \ldots e_{i-1}f_{i-1}s_{i-1}e = f_{i-1}(e_1 \ldots e_{i-1}s_{i-1})e,$$

so the claim follows by induction. We now have an $X$-sequence

$$e = e_1(e_1s_1e), f_1(e_1s_1e) = e_2(e_1e_2s_2e), \ldots, f_k(e_1 \ldots ekses_k e) = (e_1 \ldots e_k)(ef).$$

By a symmetrical argument, we also have that $(f, (f_1 \ldots f_k)(ef))$ is a consequence of $X$. Since $(e_1 \ldots e_k, f_1 \ldots f_k) \in Z$, it follows that $(e, f)$ is a consequence of $Z$. □

**Corollary 3.14.** Right noetherian semigroups contain no infinite semilattices.

It is well known that if a monoid $M$ has a non-unit $x$ such that $xy = 1$ for some $y \in M$, then the elements $y^lx^l$ are distinct and form a semilattice. Hence, we have:

**Corollary 3.15.** Let $M$ be a right noetherian monoid. Then every right (or left) unit of $M$ is in fact a (two-sided) unit. Equivalently, the complement of the group of units of $M$ is an ideal.
4. Direct products

In this section we investigate under what conditions the direct product $S \times T$ of two semigroups is right noetherian. Since $S$ and $T$ are homomorphic images of $S \times T$, by Lemma 2.16 we have:

**Lemma 4.1.** Let $S$ and $T$ be two semigroups. If $S \times T$ is right noetherian, then both $S$ and $T$ are right noetherian.

The direct product of two finitely generated semigroups is not necessarily finitely generated. Necessary and sufficient conditions for a direct product of two semigroups to be finitely generated were given in [17].

**Theorem 4.2.** [17, Theorem 2.1, Theorem 8.2] Let $S$ and $T$ be two semigroups with $S$ infinite.

1. If $T$ is infinite, then $S \times T$ is finitely generated if and only if both $S$ and $T$ are finitely generated and $S^2 = S$ and $T^2 = T$.

2. If $T$ is finite, then $S \times T$ is finitely generated if and only if $S$ is finitely generated and $T^2 = T$.

This raises the question as to whether there might exist a direct product $S \times T$ of two finitely generated right noetherian semigroups such that $S \times T$ is right noetherian but not finitely generated. However, this turns out to not be the case.

**Lemma 4.3.** Let $S$ and $T$ be two semigroups. If $S$ is infinite and $S \times T$ is right noetherian, then $T^2 = T$.

**Proof.** If $T^2 \neq T$, then $T$ has an indecomposable element $t$. Therefore, we have that $(s, t)$ is indecomposable for every $s \in S$, so $S \times T$ has infinitely many indecomposable elements. It follows from Corollary 2.9 that $S \times T$ is not right noetherian. 

**Corollary 4.4.** Let $S$ and $T$ be two finitely generated right noetherian semigroups. If the direct product $S \times T$ is right noetherian, then it is finitely generated.

It can now be easily seen that the converse of Lemma 4.1 does not hold. Indeed, for any infinite right noetherian semigroup $S$, the direct product $N \times S$, where $N$ is the free monogenic semigroup, is not right noetherian since $N$ contains an indecomposable element. We now present a more striking example:

**Example 4.5.** If $S$ is any infinite right noetherian semigroup and $T = \{a, b\}$ is the two-element right zero semigroup, then $S \times T$ is not right noetherian.

Let $\rho$ be the right congruence on $S \times T$ generated by the set

$$X = \{((s, a), (s, b)) : s \in S\}.$$ 

We claim that $X$ is a minimal generating set for $\rho$. Indeed, if $s \in S$ and we write $(s, a)$ as a product $(s, a) = uv$ for some $u, v \in S \times T$, we have that $u = (x, c)$ for some $x \in S$ and $c \in T$, and $v = (y, a)$ for some $y \in S$. Now, applying the generating pair
where $c \neq d \in T$, to $(s, a)$, we obtain $(x, d)(y, a) = (s, a)$. Therefore, we cannot have that $((s, a), (s, b))$ is a consequence of $X \setminus \{(s, a), (s, b)\}$.

**Remark 4.6.** If the semigroup $S$ in Example 4.5 is a group, then $S \times T$ is a completely simple semigroup with two maximal subgroups that are both right noetherian. Hence, Theorem 3.11 does not generalise to regular semigroups.

The direct product of a right noetherian semigroup with a right noetherian monoid may not be right noetherian; in fact, Example 4.5 shows that there exist finite semigroups $S$ such that the direct product of $S$ with any infinite monoid $M$ is not right noetherian. This leads us to make the following definition.

**Definition 4.7.** Let $S$ be a right noetherian semigroup. We say that $S$ preserves right noetherian in direct products if it satisfies the following: For every monoid $M$, the direct product $S \times M$ is right noetherian if and only if $M$ is right noetherian.

We will show in what follows that a large class of right noetherian semigroups have the above property. We have seen that it easy to find right noetherian semigroups that do not have the property, but it turns out that this is not the case for monoids, and so we raise the following question.

**Open Problem 4.8.** Do there exist right noetherian monoids $M$ and $N$ such that the direct product $M \times N$ is not right noetherian?

We now show how, given a right congruence on the direct product $S \times M$ of a semigroup $S$ and a monoid $M$, one can naturally construct a family of right congruences on $S$. So, let $\rho$ be a right congruence on $S \times M$. For each $m \in M$, we define a right congruence $\rho^S_m$ on $S$ by

$$s \rho^S_m t \iff (s, m) \rho (t, m).$$

To see that $\rho^S_m$ is a right congruence, let $s \rho^S_m t$ and $u \in S$. Since $\rho$ is a right congruence on $S$, we have that

$$(su, m) = (s, m)(u, 1) \rho (t, m)(u, 1) = (tu, m),$$

so $su \rho^S_m tu$.

When there is no danger of confusion, we will usually just write $\rho_m$ for $\rho^S_m$.

The following result shows that the property of preserving right noetherian in direct products is a finiteness condition for monoids.

**Theorem 4.9.** Let $M$ be a finite monoid. Then, for any semigroup $S$, the direct product $M \times S$ is right noetherian if and only if $S$ is right noetherian.

**Proof.** Let $S$ be a right noetherian semigroup, and let $\rho$ be a right congruence on $M \times S$. Since $S$ is right noetherian, each $\rho_m$ is generated by a finite set $X_m$. Let

$$Y_m = \{((m, x), (m, y)) : (x, y) \in X_m\}.$$
Now let $Q$ be the set
\[ \{(m,n) \in M \times M : m \neq n, (m,s) \rho (n,t) \text{ for some } s,t \in S\}. \]

For each pair $(m,n) \in Q$, we define a set
\[ I(m,n) = \{ s \in S : (m,s) \rho (n,t) \text{ for some } t \in S\}. \]

We claim that $I(m,n)$ is a right ideal of $S$. Indeed, let $s \in I(m,n)$ and $u \in S$. We then have that $(m,s) \rho (n,t)$ for some $t \in S$, and therefore
\[ (m,su) = (m,s)(1,u) \rho (n,t)(1,u) = (n,tu), \]
so $su \in I(m,n)$.

Since $S$ is weakly right noetherian, we have that $I(m,n)$ is generated (as a right ideal) by some finite set $P(m,n)$. For each $p \in P(m,n)$, choose $\alpha_{m,n}(p) \in S$ such that
\[ (m,p) \rho (n,\alpha_{m,n}(p)). \]

Let $H$ be the finite set
\[ \{(m,p),(n,\alpha_{m,n}(p)) : (m,n) \in Q, p \in P(m,n)\}. \]

We claim that $\rho$ is generated by the finite set
\[ Y = \left( \bigcup_{m \in M} Y_m \right) \cup H. \]

Indeed, let $(m,s) \rho (n,t)$ with $(m,s) \neq (n,t)$. Suppose first that $m = n$. Since $s \rho t$ with $s \neq t$, there exists an $X_m$-sequence
\[ s = x_1s_1, y_1s_1 = x_2s_2, \ldots, x_ks_k = t. \]

Therefore, we have a $Y_m$-sequence
\[ (m,s) = (m,x_1)(1,s_1), (m,y_1)(1,s_1) = (m,x_2)(1,s_2), \ldots, (m,y_k)(1,s_k) = (m,t). \]

Now suppose that $m \neq n$. We claim that there exists $s' \in S$ such that the pair $((m,s),(n,s'))$ is a consequence of $H$. Indeed, since $s \in I(m,n)$, we have that $s = pu$ for some $p \in P(m,n)$ and $u \in S^I$. Let $s' = \alpha_{m,n}(p)u$. Either $u = 1$, in which case $s = p$ and $((m,s),(n,s')) \in H$, or we have an $H$-sequence
\[ (m,s) = (m,p)(1,u), (n,\alpha_{m,n}(p))(1,u) = (n,s'). \]

It now follows that $(n,s') \rho (n,t)$, so $s' \rho t$. Therefore, either $s' = t$ or there exists a $Y_n$-sequence connecting $(n,s')$ and $(n,t)$. Hence, $((m,s),(n,t))$ is a consequence of $Y$. $\square$

We now exhibit an example of a right noetherian semigroup with a finitely generated subsemigroup that is not right noetherian.
Example 4.10. Let $S$ be any infinite finitely generated right noetherian semigroup, and let $T$ be the right zero semigroup on $\{a, b\}$. By Theorem 4.9 we have that $S \times T^1$ is right noetherian. Now, we showed in Example 4.5 that the subsemigroup $S \times T$ of $S \times T^1$ is not right noetherian. However, since $T^2 = T$, the direct product $S \times T$ is finitely generated by Theorem 4.2.

We now prove some lemmas that will be useful in the remainder of the section.

Lemma 4.11. Let $S$ be a semigroup and let $T$ be a homomorphic image of $S$. If $S$ preserves right noetherian in direct products, then so does $T$.

Proof. Let $M$ be a right noetherian monoid. Since $S \times M$ is right noetherian and $T \times M$ is a homomorphic image of $S \times M$, we have that $T \times M$ is right noetherian by Lemma 2.16. \qed

Lemma 4.12. Let $S$ be a semigroup and let $M$ be a monoid. If both $S$ and $M$ preserve right noetherian in direct products, then so does the direct product $S \times M$.

Proof. Let $N$ be a right noetherian monoid. Since $M$ preserves right noetherian in direct products, we have that $M \times N$ is right noetherian. Now, by the assumption that $S$ preserves right noetherian in direct products, we have that $(S \times M) \times N \cong S \times (M \times N)$ is right noetherian. \qed

Lemma 4.13. Let $S$ be a semigroup and let $I$ be an ideal of $S$. If both $I$ and the Rees quotient $S/I$ preserve right noetherian in direct products, then so does $S$.

Proof. Let $M$ be a right noetherian monoid, and let $J = I \times M$. We have that $J$ is an ideal of $S \times M$, and $J$ is right noetherian since $I$ preserves right noetherian in direct products. We have a surjective homomorphism

$$(S/I) \times M \to (S \times M)/J, ([s]_I, m) \mapsto [(s, m)]_J.$$ 

Since $(S/I) \times M$ is right noetherian, we have that $(S \times M)/J$ is right noetherian by Lemma 2.16. It now follows from Proposition 2.17 that $S \times M$ is right noetherian. \qed

Lemma 4.14. Let $S$ be a semigroup. Then $S$ preserves right noetherian in direct products if and only if $S^0$ does.

Proof. Let $M$ be a right noetherian monoid, and let $J = I \times M$. We have that $J$ is an ideal of $S \times M$, and $J$ is right noetherian since $I$ preserves right noetherian in direct products. We have a surjective homomorphism

$$(S/I) \times M \to (S \times M)/J, ([s]_I, m) \mapsto [(s, m)]_J.$$ 

Since $(S/I) \times M$ is right noetherian, we have that $(S \times M)/J$ is right noetherian by Lemma 2.16. It now follows from Proposition 2.17 that $S \times M$ is right noetherian. \qed

Remark 4.15. The analogue of Lemma 4.14 for $S^1$ does not hold. Indeed, the two-element right zero semigroup $T$ does not preserve right noetherian in direct products but $T^1$ does.

Lemma 4.16. Let $M$ and $N$ be two right noetherian monoids, and let $\rho$ be a right congruence on $M \times N$. Then there exists a finite set $X \subseteq M$ with the following property: for each $m \in M$, there exist $x \in X$, $m' \in M$ such that $m = x m'$ and $\rho_m = \rho_{x^m}$.
Proof. First note that for any \( m, n \in M \), we have \( \rho_m \subseteq \rho_{mn} \). Suppose that there are infinitely many pairwise disjoint right congruences of the form \( \rho_m \). Let \( I_1 \) be the right ideal
\[
\{ m \in M : \rho_m \text{ is not the identity congruence on } N \}.
\]
Since \( M \) is weakly right noetherian, \( I_1 \) is generated by some finite set \( Y_1 \). Now, there exists \( y_1 \in Y_1 \) such that there are infinitely many pairwise disjoint right congruences of the form \( \rho_m \) where \( m \) is in the right ideal \( \langle y_1 \rangle \). Let \( I_2 \) be the right ideal
\[
\{ m \in \langle y_1 \rangle : \rho_m \neq \rho_{y_1} \}.
\]
We have that \( I_2 \) is generated by some finite set \( Y_2 \), and there exists \( y_2 \in Y_2 \) such that there are infinitely many pairwise disjoint right congruences of the form \( \rho_m, m \in \langle y_2 \rangle \). Continuing in this way, we have an infinite chain of right congruences
\[
\rho_{y_1} \subset \rho_{y_2} \subset \ldots.
\]
But \( N \) is right noetherian, so we have a contradiction. Hence, there exists a finite set \( U \) such that for each \( m \in M \), there exists \( u \in U \) with \( \rho_m = \rho_u \).

For each \( u \in U \), let \( I_u \) be the right ideal generated by the set \( \{ m \in M : \rho_m = \rho_u \} \). Since \( M \) is weakly right noetherian, we have that \( I_u \) is generated by some finite set \( Y_u \). Now set \( X = \bigcup_{u \in U} Y_u \). It is clear that \( X \) satisfies the condition in the statement of the lemma. \[\square\]

We now show that finitely generated commutative monoids preserve right noetherian in direct products.

**Theorem 4.17.** Let \( M \) be a finitely generated commutative monoid. Then, for any monoid \( N \), the direct product \( M \times N \) is right noetherian if and only if \( N \) is right noetherian.

**Proof.** Since \( M \) is a homomorphic image of the free commutative monoid \( \mathbb{N}_0^n \) on \( n \) generators for some \( n \), it follows from Lemma 4.11 and Lemma 4.12 that it is sufficient to prove that \( \mathbb{N}_0 \) preserves right noetherian in direct products.

So, let \( U \) be a right noetherian monoid, and let \( \rho \) be a right congruence on \( \mathbb{N}_0 \times U \). Let \( \rho' \) be the right congruence on \( U \) generated by the set
\[
R = \{(u, v) \in U \times U : (a, u) \rho (b, v) \text{ for some } a, b \in \mathbb{N}_0 \}.
\]
Since \( U \) is right noetherian, \( \rho' \) is generated by a finite set \( X \subseteq R \). For each pair \( (x, y) \in \overline{X} \), choose \( \alpha(x, y), \beta(x, y) \in \mathbb{N}_0 \) such that
\[
(\alpha(x, y), x) \rho (\beta(x, y), y),
\]
and let
\[
Y = \{(\alpha(x, y), x) \rho (\beta(x, y), y) : (x, y) \in \overline{X}\}.
\]
Let \( N \) be the maximum natural number in the set
\[
\{\alpha(x, y), \beta(x, y) : (x, y) \in \overline{X}\}.
\]
Now let \( Q \) be the set
\[
\{(a, b) \in \mathbb{N}_0 \times \mathbb{N}_0 : a < b < N, (a, u) \rho (b, v) \text{ for some } u, v \in U\}.
\]
For each pair \((a, b) \in Q\), we define a right ideal
\[
I(a, b) = \{u \in N : (a, u) \rho (b, v) \text{ for some } v \in U\}.
\]
Since \( U \) is weakly right noetherian, we have that \( I(a, b) \) is generated by some finite set \( P(a, b) \). For each \( p \in P(a, b) \), choose \( \alpha_{a,b}(p) \in U \) such that
\[
(a, p) \rho (b, \alpha_{a,b}(p)).
\]
Let \( H \) be the finite set
\[
\{(a, p), (b, \alpha_{a,b}(p)) : (a, b) \in Q, p \in P(a, b)\}.
\]
Now, for each \( a < N \), we have that \( \rho^U_a \) is generated by some finite set \( X^U_a \). Let
\[
Y^U_a = \{(a, x), (a, y) : (x, y) \in X^U_a\}.
\]
By Lemma 4.16 there exists a finite set \( S \) such that, for each \( u \in U \), there exist \( s \in S \) and \( u' \in U \) such that \( u = su' \) and \( \rho^{N_0}_u = \rho^{N_0}_s \). For each \( s \in S \), we have that \( \rho^{N_0}_s \) is generated by some finite set \( X^{N_0}_s \). Let
\[
Y^{N_0}_s = \{(s, x), (y, s) : (x, y) \in X^{N_0}_s\}.
\]
We claim that \( \rho \) is generated by the finite set
\[
Z = Y \cup H \cup \left( \bigcup_{a < N} Y^U_a \right) \cup \left( \bigcup_{s \in S} Y^{N_0}_s \right).
\]

Let \((a, u) \rho (b, v)\) with \((a, u) \neq (b, v)\). Suppose first that \( u = v \). Then \( a \rho^N_u b \). Now, there exist \( s \in S \) and \( u' \in U \) such that \( u = su' \) and \( \rho^{N_0}_u = \rho^{N_0}_s \). Therefore, we have that \((a, b)\) is a consequence of \( X^{N_0}_s \), and hence we have a \( Y^{N_0}_s \) connecting \((a, u) = (a, s)(0, u')\) and \((b, v) = (b, u) = (b, s)(0, u')\).

Now suppose that \( u \neq v \). We have that \( u \neq v \), so there exists an \( X \)-sequence
\[
u = x_1 u_1, y_1 u_1 = x_2 u_2, \ldots, y_k u_k = v.
\]
Let \( \alpha_i = \alpha(x_i, y_i) \) and \( \beta_i = \beta(x_i, y_i) \) for \( i \in \{1, \ldots, k\} \). Now let \( a_1 = a - \alpha_1 \) and \( a_i = \alpha_{i-1} + a_{i-1} - \alpha_i \) for \( j \in \{2, \ldots, k\} \). Suppose first that \( a_i \geq 0 \) for each \( i \in \{1, \ldots, k\} \). We then have a \( Y \)-sequence
\[
(a, u) = (\alpha_1, x_1)(a_1, u_1), (\beta_1, y_1)(a_1, u_1) = (\alpha_2, x_2)(a_2, u_2), \ldots, (\beta_k, y_k)(a_k, u_k) = (a', v),
\]
where \( a' = \beta_k + a_k \). Now, either \( a' = b \) or, by exactly the same argument as above, \((a', v), (b, v)\) is a consequence of \( Y^{N_0}_s \).

Now suppose that there exists \( i \in \{1, \ldots, k\} \) such that \( a_i < 0 \), and assume that \( i \) is minimal such that \( a_i < 0 \). Let \( b_k = b - \beta_k \) and \( b_i = a_{i+1} + b_{i+1} - \beta_i \) for \( i \in \{1, \ldots, k-1\} \). Using a similar argument as above, we have that either
is a consequence of $Z$, or there exists $j$ maximal such that $b_j < 0$. In the latter case, let

$$u' = x_i u_i, \quad a' = \beta_{i-1} + a_{i-1}, \quad v' = y_j u_j, \quad b' = \alpha_{j+1} + b_{j+1}.$$  

Note that $a', b' < N$. By a similar argument to the one in Theorem 4.9, we have that $((a', u'), (b', v'))$ is a consequence of $H \cup \bigcup_{a < N} Y_a^U$.

Hence, $((a, u), (b, v))$ is a consequence of $Z$. \[\square\]

**Remark 4.18.** Theorem 4.17 does not generalise to finitely generated commutative semigroups. Indeed, we have already observed that $N \times M$ is not right noetherian for any infinite right noetherian monoid $M$.

Our next result states that right noetherian inverse semigroups preserve right noetherian in direct products.

**Theorem 4.19.** Let $S$ be a right noetherian inverse semigroup. Then, for any monoid $M$, the direct product $S \times M$ is right noetherian if and only if $M$ is right noetherian.

In order to prove Theorem 4.19, we first prove a couple of lemmas.

**Lemma 4.20.** Any right noetherian group $G$ preserves right noetherian in direct products.

**Proof.** Let $M$ be a right noetherian monoid, and let $\rho$ be a right congruence on $G \times M$. Define a right congruence $\rho'$ on $M$ by

$$m \rho' n \iff (g, m) \rho (h, n) \text{ for some } g, h \in G.$$  

Since $M$ is right noetherian, $\rho'$ is generated by some finite set $X$. For each element $(x, y) \in X$, choose $\alpha(x, y), \beta(x, y) \in G$ such that

$$(\alpha(x, y), x) \rho (\beta(x, y), y),$$  

and let ation

$$Y = \{((\alpha(x, y), x), (\beta(x, y), y)) : (x, y) \in X\}.$$  

By Lemma 4.16, there exists a finite set $S$ such that, for every $m \in M$, we have $m = sm'$ for some $m' \in M$ and $\rho_m = \rho_s$. Since $G$ is right noetherian, each $\rho_s$ is generated by some finite set $U_s$. For $s \in S$, let

$$V_s = \{(u, s), (v, s) : (u, v) \in U_s\}.$$  

We shall show that $\rho$ is generated by the finite set

$$Z = Y \cup \left( \bigcup_{s \in S} V_s \right).$$
Let \((g, m) \rho (h, n)\) with \((g, m) \neq (h, n)\). Assume that \(m \neq n\). Since \(m \rho' n\), there exists an \(X\)-sequence
\[m = x_1 m_1, y_1 m_1 = x_2 m_2, \ldots, y_k m_k = n.\]
Let \(\alpha_i = \alpha(x_i, y_i)\) and \(\beta_i = \beta(x_i, y_i)\) for \(i \in \{1, \ldots, k\}\), and let
\[g' = \beta_k \alpha_k^{-1} \ldots \beta_1 \alpha_1^{-1} g.\]
We then have a \(Y\)-sequence
\[(g, m) = (\alpha_1, x_1)(\alpha_1^{-1} g, m_1), (\beta_1, y_1)(\alpha_1^{-1} g, m_1) = (\alpha_2, x_2)(\alpha_2^{-1} \beta_1 \alpha_1^{-1} g, m_2), \ldots, (\beta_k, y_k)(\alpha_k^{-1} \ldots \beta_1 \alpha_1^{-1} g, m_2) = (g', n).\]
If \(g' = h\), then we are done, so assume that \(g' \neq h\). (If \(m = n\), simply let \(g' = g\).
Now, there exist \(s \in S\) and \(m' \in M\) such that \(n = sm'\) and \(\rho_n = \rho_s\). It follows
that \(g' \rho_s h\), and hence there exists a \(U_s\)-sequence
\[g' = u_1 g_1, v_1 g_1 = u_2 g_2, \ldots, v_l g_l = h.\]
Therefore, we have a \(V_s\)-sequence
\[(g', n) = (u_1, s)(g_1, m'), (v_1, s)(g_1, m') = (u_2, s)(g_2, m'), \ldots, (v_l, s)(g_l, m') = (h, n).\]
Hence, \(((g, m), (h, n))\) is a consequence of \(Z\). \(\square\)

The next lemma is concerned with Brandt semigroups, which are completely 0-simple inverse semigroups. They can also be characterised as follows.
Let \(G\) be a group and let \(I\) be a non-empty set. We define the following multiplication on \((I \times G \times I) \cup \{0\}\) :
\[(i, g, j)(k, h, l) = \begin{cases} (i, gh, l) & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}\]
The set \((I \times G \times I) \cup \{0\}\) is a semigroup under this multiplication, and we denote it by \(B(G, I)\). Brandt semigroups are precisely the semigroups isomorphic to some \(B(G, I)\) \([15, \text{Theorem II.3.5}]\).

**Lemma 4.21.** Let \(G\) be a right noetherian group and let \(I\) be a finite set. Then the Brandt semigroup \(S = B(G, I)\) preserves right noetherian in direct products.

**Proof.** Given Lemma 4.20, we may now assume that \(|I| \geq 2\). Let \(M\) be a right noetherian monoid, and let \(\rho\) be a right congruence on \(S \times M\). For each \(i \in I\), let \(G_i\) denote the subgroup \(\{i\} \times G \times \{i\} \cong G\), and let \(\rho_i\) be the restriction of \(\rho\) to \(G_i \times M\). Also, let \(\rho_0\) denote the restriction of \(\rho\) to \(\{0\} \times M\). By Lemma 4.20, we have that each \(\rho_i, i \in I \cup \{0\}\), is generated by a finite set \(X_i\).
Fix \(i_0 \in I\). Consider the set
\[P = \{(i, j) \in I \times I : i \neq j, ((i, 1, i_0), m) \rho ((i, g, i_0), n) \text{ for some } g \in G, m, n \in M\}.
For \((i, j) \in P\), let \(J(i, j)\) denote the set
\[\{m \in M : ((i, 1, i_0), m) \rho ((j, g, i_0), n) \text{ for some } g \in G, n \in M\}.\]
It is clear that \( J(i, j) \) is a right ideal of \( M \). Since \( M \) is weakly right noetherian, \( J(i, j) \) is generated (as a right ideal) by some finite set \( A(i, j) \). For each \( a \in A(i, j) \), choose \( \alpha_{i,j}(a) \in G \) and \( \beta_{i,j}(a) \in M \) such that
\[
\left( (i, 1, i_0), a \right) \rho \left( (j, \alpha_{i,j}(a), i_0), \beta_{i,j}(a) \right).
\]
Let \( H \) be the finite set
\[
\left\{ \left( (i, 1, i_0), (j, \alpha_{i,j}(a), i_0), \beta_{i,j}(a) \right) : (i, j) \in P, a \in A(i, j) \right\}.
\]
Now consider the set
\[
R = \{ i \in I : ((i, 1, i_0), m) \rho (0, n) \text{ for some } m, n \in M \}.
\]
For \( i \in R \), let \( J_i \) denote the set
\[
\{ m \in M : ((i, 1, i_0), m) \rho (0, n) \text{ for some } n \in M \}.
\]
Now \( J_i \) is a right ideal of \( M \), so it is generated by some finite set \( A_i \). For each \( a \in A_i \), choose \( \alpha_i \in M \) such that \( ((i, 1, i_0), a) \rho (0, \alpha_i) \), and let \( K \) be the finite set
\[
\left\{ \left( (i, 1, i_0), (0, \alpha_i) \right) : i \in R, a \in A_i \right\}.
\]
We claim that \( \rho \) is generated by the finite set
\[
X = \left( \bigcup_{i \in I \cup \{0\}} X_i \right) \cup H \cup K.
\]
Let \( u = (s, m) \) and \( v = (t, n) \), where \( s = (i, g, k) \) and \( t = (j, h, l) \), be such that \( u \rho v \) and \( u \neq v \).

Suppose first that \( k = l \). Multiplying on the right by \( ((k, g^{-1}, j), 1) \), we have that
\[
u' = ((i, 1, j), m) \rho ((j, hg^{-1}, j), n) = v'.
\]
We show that \( (u', v') \) is a consequence of \( X \). Then, multiplying on the right by \( ((j, g, k), 1) \), we have that \( (u, v) \) is a consequence of \( X \).

If \( i = j \), then \( (u', v') \) is a consequence of \( X_i \).

If \( i \neq j \), we have that \( m \in J(i, j) \), so \( m = am' \) for some \( a \in A(i, j) \) and \( m' \in M \). Now, applying an element of \( H \) to
\[
\left( ((i, 1, i_0), a) \left( (i_0, 1, i), m' \right) \right)
\]
we obtain
\[
\left( ((i, 1, i_0), a) \left( (i_0, 1, i), m' \right) \right)
\]
and we have that \( (u'', v') \) is a consequence of \( X_j \).

Now suppose that \( k \neq l \). Multiplying on the right by \( ((k, g^{-1}, i_0), 1) \), we have that
\[
\left( ((i, 1, i_0), m) \right) \rho (0, n).
\]
We have that \( m \in A_i \), so \( m = ap \) for some \( a \in A_i \) and \( p \in M \). Now, applying an element of \( K \) to

\[ u = ((i, 1, i_0), a)((i_0, g, k), p), \]

we obtain \( u' = (0, ap) \). Similarly, there exists \( n' \in M \) such that \( v' = (0, n') \) is obtained from \( v \) by an element of \( K \). Since \( u' \rho_0 v' \), we have that \( (u', v') \) is a consequence of \( X_0 \). Hence, \( (u, v) \) is a consequence of \( X \). \( \square \)

A principal series of a semigroup \( S \) is a finite chain of ideals

\[ K(S) = I_1 \subset I_2 \subset \ldots \subset I_n = S, \]

where \( K(S) \) is the unique minimal ideal of \( S \), called the kernel of \( S \), and, for each \( k \in \{1, \ldots, n - 1\} \), the ideal \( I_k \) is maximal in \( I_{k+1} \).

We are now in a position to prove Theorem 4.19.

**Proof of Theorem 4.19.** It follows from Theorem 3.11 that every right noetherian inverse semigroup has a principal series. We shall prove Theorem 4.19 by induction on the length of the principal series. We may assume that \( S \) has a zero 0 by Corollary 4.14 so let \( S \) have a principal series

\[ \{0\} \subset I_1 \subset I_2 \subset \cdots \subset I_n = S. \]

Now \( I_1 \) is completely 0-simple by \cite{9} Lemma 1.3], so it is a Brandt semigroup \( B(G, I) \). It follows from Theorem 3.11 that \( G \) is right noetherian and \( I \) is finite, so \( I_1 \) preserves right noetherian in direct products by Lemma 4.21. Suppose now that \( n > 1 \), and assume that any right noetherian inverse semigroup with a shorter principal series preserves right noetherian in direct products. The Rees quotient \( S/I_1 \) is right noetherian by Lemma 2.16, and it preserves right noetherian in direct products by the inductive hypothesis. It now follows from Lemma 4.13 that \( S \) preserves right noetherian in direct products. \( \square \)

**Remark 4.22.** Example 4.5 shows that Theorem 4.19 does not generalise to regular semigroups.

### 5. Semidirect and Wreath Products

In this section we consider semidirect products of semigroups and wreath products of monoids.

Let \( S \) and \( T \) be two semigroups. A *(left)* action of \( T \) on \( S \) is a map

\[ T \times S \to S, \quad (t, s) \mapsto t \cdot s \]

such that, for all \( s, s_1, s_2 \in S \) and \( t, t_1, t_2 \in T \),

\begin{align*}
(1) \quad t_1 \cdot (t_2 \cdot s) &= (t_1 t_2) \cdot s, \\
(2) \quad t \cdot (s_1 s_2) &= (t \cdot s_1)(t \cdot s_2). 
\end{align*}
If \( T \) is a monoid with identity 1, we also require that \( 1 \cdot s = s \) for all \( s \in S \).

The **semidirect product** of \( S \) and \( T \) (with respect to the given action) is the Cartesian product \( S \times T \) with multiplication given by

\[
(s_1, t_1)(s_2, t_2) = (s_1(t_1 \cdot s_2), t_1t_2)
\]

for all \( s_1, s_2 \in S \) and \( t_1, t_2 \in T \).

As observed in Section 5, a direct product being right noetherian implies that the factors are right noetherian. We have the following generalisation of this fact.

**Proposition 5.1.** Let \( S \) and \( T \) be two semigroups, and let \( U \) be a semidirect product of \( S \) and \( T \). If \( U \) is right noetherian, then both \( S \) and \( T \) are right noetherian.

**Proof.** Since \( U \) is a homomorphic image of \( U \), it follows from Lemma 2.16 that \( T \) is right noetherian.

Now let \( \rho \) be a right congruence on \( S \). We define a right congruence \( \rho' \) on \( U \) by

\[
(s_1, t_1) \rho' (s_2, t_2) \iff s_1 \rho s_2 \text{ and } t_1 = t_2.
\]

Since \( U \) is right noetherian, \( \rho' \) is generated by a finite set \( X \). We claim that \( \rho \) is generated by the finite set

\[
Y = \{(x, y) : ((x, t), (y, t)) \in X \text{ for some } t \in T\}.
\]

Let \( a \rho b \) with \( a \neq b \). Choose \( t \in T \). Then \( (a, t) \rho' (b, t) \), so there exists an \( X \)-sequence

\[
(a, t) = (x_1, t_1)(s_1, u_1), (y_1, t_1)(s_1, u_1) = (x_2, t_2)(s_2, u_2), \ldots, (y_k, t_k)(s_k, u_k) = (b, t).
\]

Hence, we have a \( Y \)-sequence

\[
a = x_1(t_1 \cdot s_1), y_1(t_1 \cdot s_1) = x_2(t_2 \cdot s_2), \ldots, y_k(t_k \cdot s_k) = b,
\]

so \( (a, b) \) is a consequence of \( Y \). \( \square \)

We now define the wreath product of two monoids; see [16] for more details.

Let \( M \) and \( N \) be two monoids. Let \( M^N \) denote the set of all functions from \( N \) to \( M \). For \( m \in M \), we let \( c_m \) denote the map in \( M^N \) that maps every element of \( N \) to \( m \). Under componentwise multiplication, \( M^N \) is a monoid with identity \( c_1 \).

For a function \( f \in M^N \), the **support** of \( f \) is the set

\[
\{n \in N : nf \neq 1_M\}.
\]

The set \( M^{(N)} \) of all \( f \in M^N \) with finite support is a submonoid of \( M^N \). Note that \( M^{(N)} \) and \( M^N \) are identical if \( N \) is finite.

We define an action of \( N \) on \( M^N \) as follows: Given a function \( f \in M^N \) and an element \( n \in N \), we define a function \( ^nf \in M^N \) by \( ^nf = (xn)f \) for all \( x \in N \). This action yields a semidirect product of \( M^N \) and \( N \), called the **unrestricted wreath product** of \( M \) and \( N \), which we denote by \( M \wr N \).

The **restricted wreath product** of \( M \) and \( N \), denoted by \( M \wr N \), is the subsemigroup of \( M \wr N \) generated by \( M^{(N)} \times N \). It is well known that if \( N \) is a group, then \( M \wr N = M^{(N)} \times N \). Also, if \( N \) is finite, then \( M \wr N = M \wr N \).
The next result provides some necessary conditions for the wreath product of two monoids to be right noetherian.

**Proposition 5.2.** Let $M$ and $N$ be two monoids. If the wreath product $M \wr N$ is right noetherian, then one of the following holds:

1. $M$ is trivial and $N$ is right noetherian;
2. $M$ is right noetherian and $N$ is finite.

**Proof.** If $M$ is trivial, then $N \cong M \wr N$ is right noetherian.

Suppose that $M$ is non-trivial. To prove that $N$ is finite, we assume for a contradiction that it is infinite. Let $1$ and $e$ be the identities of $M$ and $N$ respectively. Choose $m \in M \setminus \{1\}$ and a countably infinite set $\{n_1, n_2, \ldots\} \subseteq N$. For each $i \in \mathbb{N}$, we define a map

$$f_i : N \to M, n \mapsto \begin{cases} m & \text{if } n \in \{n_1, \ldots, n_i\} \\ 1 & \text{otherwise.} \end{cases}$$

Let $\rho$ be the right congruence on $M \wr N$ generated by the set

$$\left\{ ((f_i, e), (c_1, e)) : i \in \mathbb{N} \right\}.$$

Since $M \wr N$ is right noetherian, $\rho$ can be generated by some finite set

$$X = \left\{ ((f_i, e), (c_1, e)) : 1 \leq i \leq l \right\}.$$

Choose $i > l$, and let $f = f_i$ and $n = n_i$. We have an $X$-sequence

$$(f, e) = (x_1, e)(g_1, e), (y_1, e)(g_1, e) = (x_2, e)(g_2, e), \ldots, (y_k, e)(g_k, e) = (c_1, e).$$

Therefore, we have a sequence

$$f = x_1g_1, y_1g_1 = x_2g_2, \ldots, y_kg_k = c_1.$$

Since $nf = m$ and $nx_1 = 1$, we have $ng_1 = m$. Therefore, we have $n(y_1g_1) = m$, which in turn implies that $ng_2 = m$. Continuing in this way, we have that $ng_i = m$ for all $i \in \{1, \ldots, k\}$. But then we have

$$nc_1 = (ny_k)(ng_k) = m \neq 1,$$

which is a contradiction.

Since $N$ is finite, $M \wr N$ coincides with $M \wr N$. It follows from Proposition 5.1 that the monoid $M^N$ is right noetherian, and hence $M$, being a homomorphic image of $M^N$, is also right noetherian. $\square$

The following example shows that the converse of Proposition 5.2 does not hold, and the property of being right noetherian is not preserved under semidirect products of monoids.

**Example 5.3.** Let $M$ be an infinite right noetherian monoid such that $M \setminus \{1_M\}$ is a subsemigroup, and let $N$ be any finite monoid with a zero $0$. Note that if $M$
preserves right noetherian in direct products (for example, \(M = \mathbb{N}_0\)), then \(M^N\) is right noetherian. Consider the ideal 
\[ I = \{(f, 0) : f \in M^N\} \]
of \(M \setminus N\). Let \(X \times \{0\}\) be any set that generates \(I\) as a right ideal, and let \(S\) denote the infinite set 
\[ \{f \in M^N : 1_N f = 1_M\}. \]
We claim that \(S\) is contained in \(X\), so \(I\) is not finitely generated as a right ideal and \(M \setminus N\) is not weakly right noetherian. Indeed, if \(f \in S\), then \((f, 0) = (g, 0)(h, n)\) for some \(g \in X\), \(h \in M^N\) and \(n \in N\), so \(f = g^h = g_0^n h\). Therefore, we have
\[ 1_M = 1_N f = (1_N g)(1_N c_{0h}) = (1_N g)(0h), \]
which implies that \(0h = 1_M\) (since \(M \setminus \{1_M\}\) is an ideal of \(M\)) and hence \(f = g\).

For groups, the property of being right noetherian is preserved under semidirect products:

**Theorem 5.4.** Let \(G\) and \(H\) be two groups, and let \(U\) be a semidirect product of \(G\) and \(H\). Then \(U\) is right noetherian if and only if both \(G\) and \(H\) are right noetherian.

**Proof.** The direct implication follows from Proposition 5.1. For the converse, we need to show that every subgroup of \(U\) is finitely generated, so let \(S\) be a subgroup of \(U\). Define a set 
\[ G_S = \{g \in G : (g, 1) \in S\}. \]
Since \(G_S\) is a subgroup of \(G\) and \(G\) is right noetherian, we have that \(G_S\) is generated by some finite set \(X\).

Let \(H_S\) be the projection of \(S\) onto \(H\). Since \(H_S\) is a subgroup of \(H\) and \(H\) is right noetherian, we have that \(H_S\) is generated by some finite set \(Y\). For each \(y \in Y\), choose \(g_y \in G\) such that \((g_y, y) \in S\). We claim that \(S\) is generated by the finite set 
\[ Z = \{(x, 1) : x \in X\} \cup \{(g_y, y) : y \in Y\} \subseteq S. \]
Let \((g, h) \in S\). Now \(h = y_1 \ldots y_k\) for some \(y_i \in Y \cup Y^{-1}\). For \(i \in \{1, \ldots, k\}\), let \(u_i = g_{y_i}\) if \(y_i \in Y\) or \(u_i = g_{y_i}^{-1}\) if \(y_i \in Y^{-1}\). We have that
\[ (g, h)(u_k, y_k)^{-1} \ldots (u_1, y_1)^{-1} = (g', 1) \in S, \]
where 
\[ g' = g(y_1 \ldots y_{k-1} \cdot u_k^{-1}) \ldots (y_1 \cdot u_2^{-1})u_1^{-1}. \]
Since \(g' \in G_S\), we have \(g' = x_1 \ldots x_n\) for some \(x_i \in X \cup X^{-1}\). It now follows that 
\[ (g, h) = (x_1, 1) \ldots (x_n, 1)(u_1, y_1) \ldots (u_k, y_k) \in (Z), \]
as required. \qed

We now provide necessary and sufficient conditions for the wreath product of two groups to be right noetherian.
Theorem 5.5. Let $G$ and $H$ be two groups. Then the wreath product $G \wr H$ is right noetherian if and only if one of the following holds:

1. $G$ is trivial and $H$ is right noetherian;
2. $G$ is right noetherian and $H$ is finite.

Proof. The direct implication follows from Proposition 5.2, so we just need to prove the converse.

If $G$ is trivial and $H$ is right noetherian, then $G \wr H \cong H$ is right noetherian.

Suppose that $G$ is right noetherian and $H$ is finite. The wreath product of $G$ and $H$ is a semidirect product of $G^H$ and $H$, and $G^H$ is isomorphic to the direct product of $|H|$ copies of $G$, so it follows from Theorem 5.4 that $G \wr H$ is right noetherian. □

6. Unions of semigroups

In this section we study 0-direct unions and (strong) semilattices of semigroups.

We begin with 0-direct unions.

Definition 6.1. Let $U$ be a semigroup $S \cup T \cup 0$, where $S$ and $T$ are disjoint subsemigroups of $U$, 0 is a zero element disjoint from $S \cup T$, and $st = ts = 0$ for all $s \in S$ and $t \in T$. Then $U$ is the 0-direct union of $S$ and $T$.

The property of being right noetherian is preserved under 0-direct unions:

Proposition 6.2. Let $U$ be the 0-direct union of two semigroups $S$ and $T$. Then $U$ is right noetherian if and only if both $S$ and $T$ are right noetherian.

Proof. ($\Rightarrow$) Since $U$ is right noetherian and the sets $U \setminus S$ and $U \setminus T$ are ideals of $U$, we have that $S$ and $T$ are right noetherian by Proposition 3.5.

($\Leftarrow$) Since $T$ is right noetherian, the ideal $U \setminus S = T \cup \{0\}$ is right noetherian by Corollary 3.4. Therefore, since $S$ is right noetherian, we have that $U$ is right noetherian by Corollary 3.6. □

We now turn our attention to semilattices of semigroups.

Let $Y$ be a semilattice and let $(S_\alpha)_{\alpha \in Y}$ be a family of disjoint semigroups (resp. monoids), indexed by $Y$, such that $S = \bigcup_{\alpha \in Y} S_\alpha$ is a semigroup. If $S_\alpha S_\beta \subseteq S_{\alpha \beta}$ for all $\alpha, \beta \in Y$, then $S$ is called a semilattice of semigroups (resp. monoids), and we denote it by $S = S(Y, S_\alpha)$.

Now suppose that for each $\alpha, \beta \in Y$ with $\alpha \geq \beta$, there exists a homomorphism $\phi_{\alpha, \beta} : S_\alpha \to S_\beta$ satisfying the following:

- for each $\alpha \in Y$, the homomorphism $\phi_{\alpha, \alpha}$ is the identity map on $S_\alpha$;
- for each $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$, we have $\phi_{\alpha, \beta} \phi_{\beta, \gamma} = \phi_{\alpha, \gamma}$.

For $a \in S_\alpha$ and $b \in S_\beta$, we define

$$ab = (a \phi_{\alpha, \alpha})(b \phi_{\beta, \beta}).$$
With this multiplication, $S$ is a semilattice of semigroups (resp. monoids). In this case we call $S$ a strong semilattice of semigroups (resp. monoids) and denote it by $S = S(Y, S_\alpha, \phi_{\alpha,\beta})$.

In the remainder of this section, we investigate under what conditions a (strong) semilattice of semigroups or monoids is right noetherian.

Since the semilattice $Y$ is a homomorphic image of $S = S(Y, S_\alpha)$, by Lemma 2.16 and Corollary 3.14 we have:

**Lemma 6.3.** Let $S = S(Y, S_\alpha)$ be a semilattice of semigroups. If $S$ is right noetherian, then $Y$ is finite.

For a semilattice of semigroups $S(Y, S_\alpha)$ to be right noetherian, it is not required that all the $S_\alpha$ be right noetherian. In order to show this, we consider a construction involving semigroup acts.

**Construction 6.4.** Let $S$ be a semigroup, and let $A$ be an $S$-act disjoint from $S$ with action

$$A \times S \rightarrow A, (a, s) \mapsto a \cdot s.$$  

We define the following multiplication on the set $S \cup A$:

$$x \circ y = \begin{cases} 
xy & \text{if } x, y \in S \\
x \cdot y & \text{if } x \in A, y \in S \\
y & \text{if } x \in S \cup A, y \in A.
\end{cases}$$

With this operation, the set $S \cup A$ is a semilattice of two semigroups $S_\alpha = S$ and $S_\beta = A$ where $\alpha > \beta$, and we denote it by $U(S, A)$. Note that $A$ is an ideal of $U(S, A)$.

**Example 6.5.** Let $A$ be a free cyclic $\mathbb{Z}$-act disjoint from $\mathbb{Z}$. Then $U(\mathbb{Z}, A)$ is the monoid defined by the presentation

$$\langle a, b, c | ab = ba = 1, ac = c^2 = c \rangle.$$  

Notice that $A$ is a right zero subsemigroup of $U(S, A)$. It is easy to see that infinite right zero semigroups are not right noetherian, so the following result yields the desired counterexample.

**Proposition 6.6.** Let $S$ be a semigroup, let $A$ be an $S$-act disjoint from $S$, and let $U = U(S, A)$. Then $U$ is right noetherian if and only if $S$ is right noetherian and $A$ is a finitely generated $S$-act.

**Proof.** ($\Rightarrow$) Since $U \setminus S = A$ is an ideal of $U$, we have that $S$ is right noetherian by Proposition 4.5.

Let $\rho$ be the right congruence on $U$ generated by $A \times A$. Since $U$ is right noetherian, $\rho$ is generated by some finite set $Y \subseteq A \times A$. We claim that $A$ is generated by the finite set

$$X = \{x \in A : (x, y) \in \overline{Y} \text{ for some } y \in A\}.$$  


Let \( a \in A \), and choose \( b \in A \setminus \{a\} \). Since \( a \rho b \), there exists a \( Y \)-sequence
\[
a = x_1u_1, y_1u_1, \ldots, y_ku_k = b
\]
of shortest possible length connecting \( a \) and \( b \). If \( u_1 \in A \), then \( a = u_1 = y_1u_1 \), which is a contradiction. Hence, we have \( u_1 \in S^1 \) and \( a = x_1u_1 \in XS^1 \).

\((\Leftarrow)\) Let \( \rho \) be a right congruence on \( U \), and let \( X \) be a finite generating set for \( A \). Let \( Q(X) \) denote the set of \( x \in X \) such that \( x \cdot s \rho t \) for some \( s, t \in S \). For each \( x \in Q(X) \), we define a right ideal
\[
I_x = \{s \in S : x \cdot s \rho t \text{ for some } t \in S\}.
\]
We have that \( S \) is weakly right noetherian by Lemma 2.6, so \( I_x \) is generated as a right ideal by some finite set \( P_x \). For each \( p \in P_x \), choose \( \alpha_x(p) \in S \) such that \( x \cdot s \rho \alpha_x(p) \), and let
\[
H = \{(x \cdot p, \alpha_x(p)) \in A \times S : x \in Q(X), p \in P_x\}.
\]
Now, let \( R(X) \) denote the set of pairs \( (x, y) \in X \times X \) with \( x \neq y \) such that \( x \cdot s \rho y \cdot t \) for some \( s, t \in S \). For each pair \( (x, y) \in R(X) \), we define a right ideal
\[
I(x, y) = \{s \in S : x \cdot s \rho y \cdot t \text{ for some } t \in S\}.
\]
We have that \( I(x, y) \) is generated by some finite set \( P(x, y) \). For each \( p \in P(x, y) \), choose \( \alpha_{x,y}(p) \in S \) such that \( x \cdot s \rho y \cdot \alpha_{x,y}(p) \), and let
\[
K = \{(x \cdot p, y \cdot \alpha_{x,y}(p)) \in A \times A : (x, y) \in R(X), p \in P(x, y)\}.
\]
Now, let \( \rho_S \) be the restriction of \( \rho \) to \( S \). Since \( S \) is right noetherian, we have that \( \rho_S \) is generated by some finite set \( Y \).

For each \( x \in X \), we define a right congruence \( \rho_x \) on \( S \) by
\[
s \rho_x t \iff x \cdot s \rho x \cdot t.
\]
We have that \( \rho_x \) is generated by some finite set \( Y_x \). Let
\[
Z_x = \{(x \cdot y, x \cdot y') \in A \times A : (y, y') \in Y_x\}.
\]
We claim that \( \rho \) is generated by the finite set
\[
Z = H \cup K \cup Y \cup \left( \bigcup_{x \in X} Z_x \right).
\]
Let \( u \rho v \) with \( u \neq v \). There are three cases.

**Case 1:** \( u, v \in S \). Since \( u \rho_S v \), we have that \( (u, v) \) is a consequence of \( Y \).

**Case 2:** \( u \in A \), \( v \in S \). We have that \( u = x \cdot s \) for some \( x \in X \) and \( s \in S \). Since \( s \in I_x \), we have that \( s = ps' \) for some \( p \in P_x \) and \( s' \in S \). We obtain \( t = \alpha_x(p)s' \in S \) from \( u \) using \((x \cdot p, \alpha_x(p)) \in H \), and we have that \((t, v)\) is a consequence of \( Y \).

**Case 3:** \( u, v \in A \). We have that \( u = x \cdot s \) and \( v = y \cdot t \) for some \( x, y \in X \) and \( s, t \in S \). If \( x = y \), then \( s \rho_x t \), so there exists a \( Y_x \)-sequence connecting \( s \) and \( t \). Therefore, there clearly exists a \( Z_x \)-sequence connecting \( u \) and \( v \).
Now suppose that $x \neq y$. Since $s \in I(x, y)$, we have that $s = pr$ for some $p \in P(x, y)$ and $r \in S$. Letting $s' = \alpha_x y(p)r \in S$, we obtain $y_s'v$ from $u$ using $(x-p, y\alpha_x y(p)) \in K$, and we have that $(y_s', v)$ is a consequence of $Z_y$. 

**Remark 6.7.** The semigroup $U(S, A)$ can also be viewed as the disjoint union of $S$ and infinitely many copies of the trivial semigroup. Therefore, the finiteness of $I$ is not necessary for a disjoint union of semigroups $S = \bigcup_{i \in I} S_i$ to be right noetherian.

**Remark 6.8.** If $S$ is regular, then $U(S, A)$ is regular, so right noetherian regular semigroups can have infinitely many idempotents, and Proposition 3.13 does not generalise to regular subsemigroups.

In the case that a semigroup $S$ is a monoid, a necessary condition for $S = S(Y, S_a)$ to be right noetherian is that $S_\beta$ is right noetherian.

**Proposition 6.9.** Let $S = S(Y, S_a)$ be a semilattice of semigroups, let $\beta \in Y$, and suppose that $S_\beta$ is a monoid. If $S$ is right noetherian, then $S_\beta$ is right noetherian.

**Proof.** Let $1_\beta$ be the identity of $S_\beta$. Now let $\rho$ be a right congruence on $S_\beta$. Denote by $\overline{\rho}$ the right congruence on $S$ generated by $\rho$. Since $S$ is right noetherian, $\overline{\rho}$ is generated by some finite set $X \subseteq \rho$. We claim that $\rho$ is generated by $X$. Indeed, let $a \rho b$ with $a \neq b$. Then $a \overline{\rho} b$, so there exists an $X$-sequence

\[ a = x_1 s_1, y_1 s_1 = x_2 s_2, \ldots, y_k s_k = b, \]

where $(x_i, y_i) \in \overline{X}$ and $s_i \in S^1$ for $1 \leq i \leq k$. Therefore, we have a sequence

\[ a = x_1 (1_\beta s_1), y_1 (1_\beta s_1) = x_2 (1_\beta s_2), \ldots, y_k (1_\beta s_k) = b. \]

We show that $1_\beta s_i \in S_\beta$ for all $i \in \{1, \ldots, k\}$, so that $(a, b)$ is a consequence of $X$, as required. So, for each $i \in \{1, \ldots, k\}$, let $s_i \in S^1_{\alpha_i}$ where $\alpha_i = \beta$ if $s_i = 1$. Since $a \in S_\beta$ and $a = x_1 s_1 \in S_{\beta \alpha_1}$, we have that $\beta \alpha_1 = \beta$. Since $x_2 s_2 \in S_{\beta \alpha_2}$ and

\[ x_2 s_2 = y_1 s_1 \in S_{\beta \alpha_1} = S_\beta, \]

it follows that $\beta \alpha_2 = \beta$. Continuing in this way, we have that $\beta \alpha_i = \beta$, and hence $1_\beta s_i \in S_\beta$, for all $i \in \{1, \ldots, k\}$.

We have shown that a semilattice of semigroups $S(Y, S_a)$ being right noetherian does not imply that every $S_\alpha$ is right noetherian. In the opposite direction, however, we have the following result.

**Proposition 6.10.** Let $S = S(Y, S_a)$ be a semilattice of semigroups. If $Y$ is finite and each $S_\alpha$ is right noetherian, then $S$ is also right noetherian.

**Proof.** Let $\rho$ be a right congruence on $S$. For each $\alpha \in Y$, let $\rho_\alpha$ denote the restriction of $\rho$ to $S_\alpha$. Since $S_\alpha$ is right noetherian, we have that $\rho_\alpha$ is generated by a finite set $X_\alpha$. Let $Q$ denote the set

\[ \{(\alpha, \beta) \in Y \times Y : \alpha \neq \beta, \ a \rho b \text{ for some } a \in S_\alpha, b \in S_\beta\}. \]
For each pair \((\alpha, \beta) \in Q\), let \(I_{\alpha, \beta}\) denote the right ideal of \(S_{\alpha}\) generated by the set
\[
U_{\alpha, \beta} = \{ a \in S_{\alpha} : a \rho b \text{ for some } b \in S_{\beta} \}.
\]
Since \(S_{\alpha}\) is weakly right noetherian, we have that \(I_{\alpha, \beta}\) is generated by some finite subset \(U'_{\alpha, \beta} \subseteq U_{\alpha, \beta}\). For each \(u \in U'_{\alpha, \beta}\), choose \(\lambda_{\alpha, \beta}(u) \in S_{\beta}\) such that \(u \rho \lambda_{\alpha, \beta}(u)\).

We now define a set
\[
H = \{ (u, \lambda_{\alpha, \beta}(u)) : (\alpha, \beta) \in Q, u \in U'_{\alpha, \beta} \}.
\]
We claim that \(\rho\) is generated by the finite set
\[
Y = \left( \bigcup_{\alpha \in Y} X_{\alpha} \right) \cup H.
\]
Indeed, let \(a \rho b\) with \(a \in S_{\alpha}, b \in S_{\beta}\) and \(a \neq b\). Set \(\gamma = \alpha \beta = \beta \alpha\).

We claim that there exist \(c \in S_{\beta} \cup S_{\gamma}\) and \(d \in S_{\alpha} \cup S_{\gamma}\) such that \((a, c)\) and \((b, d)\) are consequences of \(H\). Indeed, assuming that \(\alpha \neq \gamma\) (if \(\alpha = \gamma\), simply let \(a = c\)), we have that \(a \in I_{\alpha, \beta}\), so there exist \(u \in U'_{\alpha, \beta}\) and \(s \in S_{\gamma}^1\) such that \(a = us\). Now, applying the generating pair \((u, \lambda_{\alpha, \beta}(u)) \in H\), we obtain \(c = \lambda_{\alpha, \beta}(u)s \in S_{\beta} \cup S_{\gamma}\).

A similar argument proves the existence of \(d\).

There are now two cases to consider.

Case 1: \(c \in S_{\beta}\) or \(d \in S_{\alpha}\). If \(s \in S_{\beta}\), then \(b \rho c\), so \((b, c)\) is a consequence of \(X_{\beta}\), and hence \((a, b)\) is a consequence of \(H \cup X_{\beta}\). Likewise, if \(d \in S_{\alpha}\), then \((a, b)\) is a consequence of \(H \cup X_{\alpha}\).

Case 2: \(c, d \in S_{\gamma}\). We have that \(c \rho d\), so that \((c, d)\) is a consequence of \(X_{\gamma}\), and hence \((a, b)\) is a consequence of \(H \cup X_{\gamma}\).

From Propositions 6.9 and 6.10 we obtain:

**Corollary 6.11.** A semilattice of monoids \(S = S(Y, S_{\alpha})\) is right noetherian if and only if \(Y\) is finite and each \(S_{\alpha}\) is right noetherian.

The final result of this section provides necessary and sufficient conditions for a strong semilattice of semigroups to be right noetherian.

**Theorem 6.12.** A strong semilattice of semigroups \(S = S(Y, S_{\alpha}, \phi_{\alpha, \beta})\) is right noetherian if and only if \(Y\) is finite and each \(S_{\alpha}\) is right noetherian.

**Proof.** The reverse implication follows from Proposition 6.10.

For the direct implication, suppose that \(S\) is right noetherian. We have that \(Y\) is finite by Lemma 6.3. To each \(S_{\alpha}\) we adjoin an identity \(1_{\alpha}\). For each \(\alpha, \beta \in Y\) with \(\alpha \geq \beta\), we extend \(\phi_{\alpha, \beta}\) to a homomorphism \(\phi'_{\alpha, \beta} : S_{\alpha}^1 \to S_{\beta}^1\) by setting \(1_{\alpha} \phi'_{\alpha, \beta} = 1_{\beta}\). We now have a strong semilattice of monoids \(T = S(Y, S_{\alpha}^1, \phi'_{\alpha, \beta})\). Since \(T \setminus S\) is finite, we have that \(T\) is right noetherian by Theorem 3.2. It now follows from Corollary 6.11 that each \(S_{\alpha}^1\) is right noetherian, and hence each \(S_{\alpha}\) is right noetherian by Corollary 3.3.

□
7. Free products

Let $S$ and $T$ be two semigroups with presentations $〈X \mid R〉$ and $〈Y \mid S〉$ respectively. The semigroup free product of $S$ and $T$ is the semigroup defined by the presentation $〈X,Y \mid R, S〉$. If $S$ and $T$ are monoids, then the monoid free product of $S$ and $T$ is the monoid defined by the presentation $〈X,Y \mid R, S, 1_S = 1_T〉$.

In the following we provide necessary and sufficient conditions for the semigroup (resp. monoid) free product of two semigroups (resp. monoids) to be right noetherian.

**Theorem 7.1.** Let $S$ and $T$ be two semigroups. Then the semigroup free product of $S$ and $T$ is right noetherian if and only if both $S$ and $T$ are trivial.

**Proof.** Let $U$ be the semigroup free product of $S$ and $T$.

$(\Rightarrow)$ Suppose that $T$ is non-trivial. Choose $a \in S$ and $b, c \in B$. For $i \in \mathbb{N}$, let $u_i = (ab)^iaca$. Let $I$ be the right ideal of $U$ generated by the set $X = \{u_i : i \in \mathbb{N}\}$. For any $i \in \mathbb{N}$, the element $u_i$ cannot be written as $u_ju$ for any $j \neq i$ and $u \in U$, so $X$ is a minimal generating set for $I$ and $I$ is not finitely generated. Hence, we have that $U$ is not (weakly) right noetherian.

$(\Leftarrow)$ The semigroup $U$ has the presentation $\langle e, f \mid e^2 = e, f^2 = f \rangle$.

Let $a_1 = ef$, $a_2 = fe$, $a_3 = efe$ and $a_4 = fef$, and let $U_i = \{a_i^n : n \in \mathbb{N}\}$. Note that each $U_i$ is isomorphic to $\mathbb{N}$. Let $U' = \bigcup_{i=1}^{4} U_i$. We shall show that $U'$ is right noetherian. Since $U \setminus U' = \{e, f\}$ is finite, it then follows from Theorem 3.2 that $U$ is right noetherian.

Let $\rho$ be a right congruence on $U'$. For each $i \in \{1, \ldots, 4\}$, let $\rho_i$ be the restriction of $\rho$ to $U_i$. Since $U_i$ is right noetherian, $\rho_i$ is generated by some finite set $X_i$. We now make the following claim.

**Claim.** Let $i, j \in \{1, \ldots, 4\}$ with $i \neq j$ such that $a_i^M \rho a_j^N$ for some $M, N \in \mathbb{N}$.

1. If $i \in \{1, 4\}$ and $j \in \{2, 3\}$, then $a_i^m \rho a_j^M$ for all $m \geq M$ and $a_i^n \rho a_j^N$ for all $n \geq N$.

2. If $i = 1, j = 4$, or $i = 2, j = 3$, then $a_i^{M+r} \rho a_j^{N+r}$ for any $r \geq 0$.

**Proof.** To prove (1), we just consider the case where $i = 1$ and $j = 2$. By induction we have that $a_1^n \rho a_2^{M}$ for all $m \geq M$:

$$a_1^{M+k} = a_1^M a_1^k \rho a_2^N a_1^k = a_2^N f a_1^{k-1} \rho a_1^M f a_1^{k-1} = a_1^{M+k-1}.$$ 

Similarly, we have that $a_2^n \rho a_2^N$ for all $n \geq N$.

To prove (2), we just consider the case where $i = 1, j = 4$. For $r \geq 0$, we have

$$a_1^{M+r} = a_1^M a_1^r \rho a_4^N a_1^r = a_4^{N+r},$$

as required. \qed
Let \( i, j \in \{1, \ldots, 4\} \) with \( i < j \). If \( a_i^m \rho a_j^n \) for some \( m, n \in \mathbb{N} \), let

\[
M_{ij} = \min\{m \in \mathbb{N} : a_i^m \rho a_j^n \text{ for some } n \in \mathbb{N}\}, \quad N_{ij} = \min\{n \in \mathbb{N} : a_i^{M_{ij}} \rho a_j^n\},
\]

and let \( X_{ij} = \{a_i^{M_{ij}}, a_j^{N_{ij}}\} \). If there do not exist any \( m, n \in \mathbb{N} \) such that \( a_i^m \rho a_j^n \), let \( X_{ij} = \emptyset \). We claim that \( \rho \) is generated by the finite set

\[
X = \left( \bigcup_{i=1}^{4} X_i \right) \cup \left( \bigcup_{1 \leq i < j \leq 4} X_{ij} \right).
\]

Indeed, let \( i, j \in \{1, \ldots, 4\} \) with \( a_i^m \rho a_j^n \) for some \( m, n \in \mathbb{N} \) and \( a_i^m \neq a_j^n \).

If \( i = j \), then \( (a_i^m, a_j^n) \) is a consequence of \( X_i \).

Assume that \( i < j \). Suppose first that \( i = 1, j \in \{2, 3\} \) or \( i = 2, j = 4 \). We have that \( a_i^m \rho a_i^{M_{ij}} \) by the above claim, so \( (a_i^m, a_i^{M_{ij}}) \) is a consequence of \( X_i \), and likewise \( (a_j^n, a_j^{N_{ij}}) \) is a consequence of \( X_j \). Therefore, \( (a_i^m, a_j^n) \) is a consequence of \( X_i, X_j \), and \( X_{ij} \).

Now suppose that \( i = 1, j = 4 \) or \( i = 2, j = 3 \). We have that \( m = M_{ij} + r \) for some \( r \geq 0 \). From the proof of (2) in the above claim, we obtain \( a_j^{N_{ij}+r} \rho a_j^n \) from \( a_i^m \) using \( X_{ij} \). Since \( a_j^{N_{ij}+r} \rho a_j^n \), we have that \( (a_j^{N_{ij}+r}, a_j^n) \) is a consequence of \( X_j \). Therefore, \( (a_i^m, a_j^n) \) is a consequence of \( X_{ij} \) and \( X_j \).

\[\square\]

**Theorem 7.2.** Let \( M \) and \( N \) be two monoids. Then the monoid free product of \( M \) and \( N \) is right noetherian if and only if one of the following holds:

1. \( M \) is right noetherian and \( N \) is trivial, or vice versa;
2. both \( M \) and \( N \) contain precisely two elements.

**Proof.** Let \( U \) be the monoid free product of \( M \) and \( N \). If \( N \) is trivial, then \( U \) is isomorphic to \( M \), so we assume that both \( M \) and \( N \) are non-trivial.

\((\Rightarrow)\) Suppose that \( U \) is right noetherian, and yet \( |N| \geq 3 \). Choose non-identity elements \( a \in M \) and \( b, c \in N \). We consider two cases.

**Case 1:** \( a, b, c \) are units. We have a free subgroup \( G = \langle ab, ac \rangle \) of \( U \). Since \( G \) has subgroups of infinite rank, it is not right noetherian, contradicting Corollary 3.10

**Case 2:** at least one of \( a, b, c \) is not a unit, and hence not a right unit by Corollary 3.15

For \( i \in \mathbb{N} \), let \( u_i = (ba)^i cabac \). Let \( I \) be the right ideal of \( U \) generated by the set \( \{u_i : i \in \mathbb{N}\} \). We claim that \( I \) is not finitely generated, contradicting that \( U \) is (weakly) right noetherian. Indeed, if \( I \) were finitely generated, then it could be generated by a finite set

\[
X = \{u_i : 1 \leq i \leq k\}.
\]

Let \( i \in \{1, \ldots, k\} \). Since at least one of \( a, b, c \) is not right invertible, it is easy to see that, for every \( u \in U \), the first \( 2i + 2 \) free factors of \( u_i u \) are \( b, a, \ldots, b, a, c, a \); in particular, the factor in position \( 2i + 1 \) is \( c \). On the other hand, the factor in
position $2i + 1$ of $u_{k+1}$ is $b$. Therefore, there do not exist any $i \in \{1, \ldots, k\}$ and $u \in U$ such that $u_{k+1} = u_i u$, so $I$ is not generated by $X$.

$(\Rightarrow)$ Now suppose that $|M| = |N| = 2$. There are two monoids with two elements, namely $\mathbb{Z}_2$ and $\{0, 1\}$.

If $M$ and $N$ are both isomorphic to $\mathbb{Z}_2$, then it is well known that $U$ is isomorphic to a semidirect product of $\mathbb{Z}$ and $\mathbb{Z}_2$, so $U$ is right noetherian by Theorem 5.3.

If $M$ and $N$ are both isomorphic to $\{0, 1\}$, then $U$ is isomorphic to $V^1$ where $V$ is the semigroup free product of two trivial semigroups, so $U$ is right noetherian by Theorem 7.1 and Corollary 3.3.

Finally, suppose that $M = \mathbb{Z}_2$ and $N = \{0, 1\}$. Then $U$ has the presentation

$$\langle a, b \mid a^2 = 1, b^2 = b \rangle.$$  

Let $u_1 = ab$, $u_2 = ba$ and $u_3 = bab$, and let $U_i = \{u_n^i : n \geq 0\}$. Also, let $U_4 = \{u_i^1 : i \geq 1\}$. We have that $U = \bigcup_{i=1}^4 U_i$, and each $U_i$ is isomorphic to $\mathbb{N}_0$.

Let $\rho$ be a right congruence on $U$. For each $i \in \{1, 2, 3\}$, let $\rho_i$ be the restriction of $\rho$ to $U_i$. Since $U_i$ is right noetherian, $\rho_i$ is generated by some finite set $X_i$. Let $i, j \in \{1, 2, 3\}$ with $i < j$. If $u_i^m \rho u_j^n$ for some $m, n \in \mathbb{N}$, let

$$M_{ij} = \min\{m \in \mathbb{N} : u_i^m \rho u_j^n \text{ for some } n \in \mathbb{N}\}, \quad N_{ij} = \min\{n \in \mathbb{N} : u_i^m \rho u_j^n \},$$

and let $X_{ij} = \{(u_i^{M_{ij}}, u_j^{N_{ij}})\}$. If there do not exist $m, n \in \mathbb{N}$ such that $u_i^m \rho u_j^n$, let $X_{ij} = \emptyset$.

Suppose that $u_i^m \rho u_j^n a$ for some $m, n \geq 1$. Let $M$ be minimal such that $u_i^M \rho u_j^n a$ for some $n \geq 1$, and let $N$ be minimal such that $u_i^M \rho u_j^N a$. By multiplying on the right by $b$ and by $u_1$, we obtain $u_1^M \rho u_1^{N+1}$ and $u_1^{M+1} \rho u_1^N$ respectively. In particular, we have that $u_1^M \rho u_1^{M+2k}$ for all $k \in \mathbb{N}_0$. Now let

$$H = \{(u_1^M, u_1^{N+1}), (u_1^{M+1}, u_1^N)\}.$$  

If there do not exist $m, n \in \mathbb{N}$ such that $u_i^m \rho u_j^n a$, let $H = \emptyset$.

We claim that $\rho$ is generated by $X \cup H$ where

$$X = \left( \bigcup_{i=1}^3 X_i \right) \cup \bigcup_{1 \leq i < j \leq 3} X_{ij}.$$  

Let $u \rho v$ with $u \neq v$. If $u, v \in \bigcup_{i=1}^3 U_i$, then a similar argument to the one in Theorem 7.1 proves that $(u, v)$ is a consequence of $X$. Assume therefore that $v \in U_4$, so $v = u_i^m a$ for some $n \in \mathbb{N}$. There are two cases to consider.

Case 1: $u \in U_i, i \in \{2, 3, 4\}$. We have that $ua \in \bigcup_{i=1}^3 U_i$, $va = u_1^n \in U_1$ and $ua \rho va$. Therefore, $(ua, va)$ is a consequence of $X$. Hence, multiplying on the right by $a$, we have that $(u, v)$ is a consequence of $X$.

Case 2: $u \in U_1$. Since $u_1^M \rho u_1^{M+2k}$ for all $k \in \mathbb{N}_0$, we have that $u_1 \rho u_1^r$ where $r \in \{M, M + 1\}$, so $(u, u_1^r)$ is a consequence of $X_1$. Using $H$, we obtain from $u_1^r$ the element $u_1^s a$ where $s \in \{N, N + 1\}$. Finally, by the same argument as in Case 1, we have that $(u_1^s a, v)$ is a consequence of $X$.  

$\square$
Remark 7.3. Since the group free product coincides with the monoid free product for groups, it follows from Theorem [7.2] that the free product of two non-trivial groups is right noetherian if and only if both groups are isomorphic to $\mathbb{Z}_2$.

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