Nonreciprocal radio-frequency-to-optical conversion with an optoelectromechanical system

Najmeh Eshaqi Sani,1 Stefano Zippilli,1 and David Vitali1,2,3

1Physics Division, School of Science and Technology, University of Camerino, I-62032 Camerino (MC), Italy
2INFN, Sezione di Perugia, via A. Pascoli, Perugia, Italy
3CNR-INO, L.go Enrico Fermi 6, I-50125 Firenze, Italy

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Nonreciprocal systems breaking time-reversal symmetry are essential tools in modern quantum technologies enabling the suppression of unwanted reflected signals or extraneous noise entering through detection ports. Here we propose a scheme enabling nonreciprocal conversion between optical and radio-frequency (rf) photons using exclusively optomechanical and electromagnetic interactions. The nonreciprocal transmission is obtained by interference of two dissipative pathways of transmission between the two electromagnetic modes established through two distinct intermediate mechanical modes. In our protocol, we apply a bichromatic drive to the cavity mode and a single-tone drive to the rf resonator, and use the relative phase between the drive tones to obtain nonreciprocity. We analytically show how to adjust the driving parameters for achieving nonreciprocal wavelength conversion and numerically show that this is possible within an experimentally feasible parameter regime.

I. INTRODUCTION

Reciprocity is the two-way symmetry of transmission of light/photon or sound/phonon between forward and backward paths and is a common useful property exploited in a plethora of devices. However, when the time-reversal symmetry or reciprocity is broken, one can have novel functionalities that have attracted a considerable attention in engineered photonic systems [1,6].

In fact, nonreciprocal transmission and amplification of signals are useful in communication, signal processing and measurement, because in nonreciprocal systems unwanted signals or spurious modes can be suppressed, thereby protecting the system from interference with extraneous noise [7]. Typically, nonreciprocal devices require an element breaking Lorentz reciprocal symmetry [8,9] such as a d.c. magnetic field, but this method typically require bulky elements which are hard to integrate and miniaturize. Therefore, there is a strong motivation to realize alternative and more flexible implementations of nonreciprocity [10]. Various nonreciprocal devices have been proposed and realized including magnetic materials [11-17], Josephson nonlinearities [18,19], using temporal modulation [20-25], physical rotation [26], chiral atomic states [27], and the quantum Hall effect [28].

Recently Ref. [29] has shown that a general recipe for obtaining nonreciprocal transmission is balancing any given coherer interaction with a properly tuned collective dissipative process. This insight led to propose and implement nonreciprocity using optomechanical devices where these ingredients are available and controllable. Multi-mode optomechanical and electromechanical schemes have been proposed to achieve nonreciprocity and directionality, with or without relying on the direct coherent coupling between the electromagnetic input and output modes [30-38]. Here, similarly to the approach used in Refs. [30,32] which does not require any direct interaction between electromagnetic modes, we consider a four-mode optoelectromechanical system composed of an optical cavity and an rf resonator, each coupled to two intermediate mechanical modes. Two distinct paths of transmission between the two electromagnetic modes through the two mechanical modes are established and their relative phase forms the basis of nonreciprocity and directionality. Differently from Ref. [32] which demonstrated the scheme in the microwave regime, here we exploit the possibility of mechanical modes to couple to fields of disparate wavelength, and we show the possibility of nonreciprocal optical-rf photon conversion. A similar, optical-microwave, four-modes nonreciprocal conversion scheme has been proposed in Ref. [30] which however considered a four-tone driving scheme, in which both the optical and the microwave cavity are bichromatically driven. Here we simplify such a scheme and we consider an rf resonator driven by a single tone. In an appropriate parameter regime where the rotating wave approximation (RWA) also is valid, the system effectively becomes nonreciprocal and the transmission between the cavity and rf resonator is directional. In this way one can add also the additional feature of nonreciprocity to the variety of optoelectromechanical devices which have been proposed and demonstrated for the transduction of rf and microwave signals to the optical domain [39-57].

The outline of the paper is as follows. In Sec. II, the system and its Hamiltonian are introduced. In Sec. III, the dynamics of our model described by Langevin-Heisenberg equations is studied, and the effective linearized model of the interacting four bosonic modes is obtained. In Sec. IV, we analyze analytically the possibility to achieve nonreciprocity with this system. Sec. V is devoted to the numerical analysis where we determine the conditions where nonreciprocal optical-rf conversion is achieved, while concluding remarks are given in Sec. VI.

II. THE SYSTEM

We consider a hybrid optoelectromechanical system composed of an optical cavity coupled by radiation pressure to a mechanical element able to sustain multiple vibrational modes, which is in turn capacitively coupled to an rf resonant LC circuit. Focusing on the case when only two nearby
vibrational modes are coupled to the optical and rf resonators, the total Hamiltonian of the system can be written as the sum of an optical, mechanical and electrical contribution respectively,

\[ \hat{H} = \hat{H}_{\text{opt}} + \hat{H}_{\text{mech}} + \hat{H}_{\text{LC}}. \]  

(1)

In more detail

\[ \hat{H}_{\text{opt}} = \hbar \omega_c(x_1, x_2) \hat{a}_1 \hat{a}_1^\dagger + \hbar \left[ E_1 e^{i(\omega_c t + \phi_{01})} + E_2 e^{i(\omega_c t + \phi_{02})} \right] \hat{a}_1^\dagger + h.c. \]  

(2)

\[ \hat{H}_{\text{mech}} = \sum_{j=1,2} \frac{\hat{p}_j^2}{2m_j} + \frac{m_j \omega_j^2 \hat{x}_j^2}{2} \]  

(3)

where we have considered a specific cavity mode, described by the photon annihilation (creation) operator \( \hat{a}_1 \) (\( \hat{a}_1^\dagger \)), with the usual bosonic commutation relations [\( \hat{a}_1, \hat{a}_1^\dagger \) = 1], and bichromatically driven at two frequencies \( \omega_{1,2} \), with corresponding driving rates given by \( E_j = \sqrt{2 \kappa_{\text{in}} p_j / \hbar \omega_{\text{LC}} j} \), with \( \kappa_{\text{in}} \) the \( j \)-th tone power and \( \omega_{\text{LC}} \) the cavity amplitude decay rate through the input port. The mechanical term is

\[ \hat{H}_{\text{LC}} = \frac{\hat{q}^2}{2L} + \frac{\hat{q}^2}{2C(x_1, x_2)} - \hat{q} V_{\text{AC}} \cos(\omega_{\text{xt}} t + \phi_x), \]  

(4)

where each mechanical resonator has effective mass \( m_j \) (\( j = 1, 2 \)), displacement operator \( \hat{x}_j \) and conjugated momentum \( \hat{p}_j \), with commutation relations [\( \hat{x}_j, \hat{p}_j \) = \( i \hbar \delta_{ij} \)]. Finally the rf circuit term is

\[ \hat{H}_{\text{LC}} = \frac{\hat{q}^2}{2L} + \frac{\hat{q}^2}{2C(x_1, x_2)} - \hat{q} V_{\text{AC}} \cos(\omega_{\text{xt}} t + \phi_x), \]  

(5)

where \( L \) is the inductance of the rf resonator, the dynamical variables of the LC circuit are given by the total charge and flux operators \( \hat{q} \) and \( \hat{\phi} \) respectively, with commutation relation [\( \hat{q}, \hat{\phi} \) = \( i \hbar \)], and the rf resonator is driven by a single-tone drive at frequency \( \omega_{\text{xt}} \) with voltage amplitude \( V_{\text{AC}} \).

Such a configuration can be realized for example in the membrane-in-the-middle (MIM) optomechanical system case [59-62], i.e., a driven optical Fabry-Perot cavity with a thin semitransparent membrane inside. The membrane is metallized [49-55, 58-60] and capacitively coupled via an electrode to an LC resonant circuit formed by a coil and additional capacitors, see Fig. 1.

The optomechanical and electromechanical couplings arise due to the dependence of the cavity mode frequency \( \omega_c(x_1, x_2) \) and of the circuit capacitance \( C(x_1, x_2) \) respectively, upon the displacement \( x_j \) of the vibrational modes of the membrane. As in the scheme of Fig. 1, the effective capacitance of the circuit is the parallel of a tunable capacitor \( C_0 \) with the membrane capacitor formed by the metallized membrane and an electrode in front of it, \( C_m(x_1, x_2) \).

\[ C(x_1, x_2) = C_0 + C_m(x_1, x_2). \]  

(6)

The system Hamiltonian of Eq. 1 can be simplified by making two approximations: i) the two displacements \( x_j \) are typically small and one can develop both the cavity frequency and the capacitance at first order in \( x_j \); ii) one can neglect fast oscillating terms in the LC circuit driving. Moreover one can rewrite Eq. 1 in a more convenient form by introducing the phonon annihilation and creation operators \( \hat{b}_j \) and \( \hat{b}_j^\dagger \), \( j = 1, 2 \), such that

\[ \hat{x}_j \equiv x_{\text{pf}, j} \hat{b}_j + \hat{b}_j^\dagger, \]  

(7)

\[ \hat{p}_j \equiv p_{\text{pf}, j} \hat{b}_j - \hat{b}_j^\dagger, \]  

(8)

and the LC photon annihilation and creation operators \( \hat{a}_2 \) and \( \hat{a}_2^\dagger \), such that

\[ \hat{q} \equiv \hat{q}_{\text{pf} \rightarrow \text{LC}}, \]  

(9)

\[ \phi \equiv \phi_{\text{pf} \rightarrow \text{LC}}, \]  

(10)

where the rf resonant frequency of the LC circuit is defined as \( \omega_{\text{LC}}^{(0)} = 1 / \sqrt{L C(0, 0)} \), and moving, for the optical mode, to the frame rotating at the frequency halfway between the two driving tones, \( \omega_\ell = (\omega_{1,2} + \omega_{1,2}) / 2 \). One finally gets

\[ \hat{H} = \hbar \Delta_L \hat{a}_1^\dagger \hat{a}_1 + \hbar \sum_{j=1,2} \left( g_{0,j}(\hat{b}_j + \hat{b}_j^\dagger) \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \right) \]  

(11)

\[ - \hbar \sum_{j=1,2} \left( g_{2,j}(\hat{b}_j + \hat{b}_j^\dagger)(\hat{a}_2 + \hat{a}_2^\dagger)^2 - \hbar [\hat{V} e^{i\omega_{\text{xt} t}} \hat{a}_1^\dagger + h.c.]] \right) \]  

where we have introduced the bare cavity detuning \( \Delta_L \equiv \omega_c(0, 0) - \omega_\ell \), \( \omega_\ell \equiv \omega_{1,2} - \omega_{1,2} = -(\omega_{1,2} - \omega_{1,2}), \)
the single-photon optomechanical coupling rates $g_{0,j} \equiv \frac{\partial \bar{h}_{j}}{\partial \delta_{j^*}|_{\delta_{j}=0}}$, the single-photon electromechanical coupling rates $g_{0,2j} \equiv \frac{\partial \bar{h}_{j}}{\partial \tilde{X}_{j}}|_{\delta_j=0}$, the rf complex driving rate $V' \equiv (\tilde{q}_{j}/\sqrt{\nu_0/2}) e^{i\theta_{0}}$, and the complex optical driving rates $E_1 \equiv E_1 e^{i\theta_{1}}$ and $E_2 \equiv E_2 e^{i\theta_{2}}$.

### III. APPROXIMATED MODEL

We now derive the quantum Langevin equations for the system operators by supplementing the Heisenberg equations of motion stemming from Eq. (11) with fluctuation and dissipation terms describing the coupling of the two mechanical modes and of the two electromagnetic cavity modes with their own independent environment. We assume the ideal situation in which the optical cavity looses phonons only from the input port with amplitude decay rate $\kappa_{in} \equiv \kappa$, and it is characterized by the input noise operator $\hat{a}_{1,in}$. We introduce damping and Brownian noise in a similar way for the two mechanical resonators, with energy decay rates $\gamma_{m,j}$ and noise operators $\hat{b}_{j,in}$, $j = 1, 2$. For what concerns the LC circuit, we exploit the quantum electrical network theory of Ref. [65] and model dissipation with an RLC series circuit in which the input-output port is represented by an infinite transmission line with purely resistive characteristic impedance $Z = \sqrt{L_1/C_1}$, where $C_1$ and $L_1$ are the capacitance and the inductance per unit length along the transmission line, respectively. The input noise operator entering the circuit through the transmission line is $\hat{a}_{2,in}$. In an RLC series resonator the damping rate is $\gamma_{1c} \equiv Z/L_1$ and the rf-circuit quality factor is given by $Q_{1c} = \omega_{1c}^{(0)}/\gamma_{1c}$.

All the noise operators are uncorrelated from each other and characterized by thermal noise correlations at temperature $T$, where the only non-zero correlation functions are $\langle \hat{b}_{i \text{in}}(t) \hat{b}_{i \text{in}}^\dagger(t') \rangle = \langle \hat{b}_{i \text{in}}^\dagger(t) \hat{b}_{i \text{in}}(t') \rangle = \delta(t-t') = \frac{1 + \bar{n}_i}{2} \delta(t-t')$, and $\langle \hat{a}_{j \text{in}}(t) \hat{a}_{j \text{in}}^\dagger(t') \rangle = \langle \hat{a}_{j \text{in}}^\dagger(t) \hat{a}_{j \text{in}}(t') \rangle = \delta(t-t') = \frac{1 + \bar{n}_j}{2} \delta(t-t')$, with the number of thermal phonons given by $\bar{n}_i = \left\{ \exp[\hbar \omega_{i}/k_B T] - 1 \right\}^{-1}$, $j = 1, 2$, a similar expression for the mean thermal number of rf photons, $\bar{n}_2 = \left\{ \exp[\hbar \nu_0/2 k_B T] - 1 \right\}^{-1}$, while $\bar{n}_1 = 0$ because at optical frequencies $\hbar \nu_0 > k_B T$.

The quantum Langevin equations can then be written as

$$\dot{a}_1 = - (\kappa + i \Delta_1) a_1 - i [E_1 e^{i\theta_{1},t} + E_2 e^{i\theta_{2},t}] - i \sum_{j=1,2} g_{0,j} (\hat{b}_j + \hat{b}_j^\dagger) a_1 + \sqrt{2 \kappa} a_{1,in}$$

$$\dot{a}_2 = - \frac{\gamma_{1c}}{2} a_2 + 2i \sum_{j=1,2} g_{0,j} (\hat{a}_2 + \hat{a}_2^\dagger)(\hat{b}_j + \hat{b}_j^\dagger) + iV' e^{i\omega_0 t} + \sqrt{\gamma_{1c}} a_{2,in}$$

$$\dot{b}_j = - \frac{\gamma_{m,j}}{2} b_j - i g_{0,j} (\hat{a}_1 + i \hat{a}_1^\dagger)(\hat{a}_2 + \hat{a}_2^\dagger) + \sqrt{\gamma_{m,j}} b_{j,in}$$

where $\gamma_{1c}$ is the time-independent part of the mean mechanical amplitude $\beta_j(t)$; and where $\delta$ is a small detuning which is used to tune the non-reciprocity as discussed in the following sections. In this representation, the linearized quantum Langevin equations take the form

$$\dot{a}_1 = - (\kappa + i \Theta_1(t)) a_1 - i \sum_{j=1,2} \left[ G_{j1}^{\lambda}(t) \delta \hat{b}_j + G_{j1}^{\phi}(t) \delta \hat{b}_j^\dagger \right] + \sqrt{2 \kappa} a_{1,in}$$

$$\dot{a}_2 = - \frac{\gamma_{1c}}{2} a_2 + i \Theta_2(t) \dot{a}_2 + i \Theta_2(t) \dot{a}_2^\dagger - i \sum_{j=1,2} \left[ G_{j2}^{\lambda}(t) \delta \hat{b}_j + G_{j2}^{\phi}(t) \delta \hat{b}_j^\dagger \right] + \sqrt{\gamma_{1c}} a_{2,in}$$

$$\dot{b}_j = - \frac{\gamma_{m,j}}{2} b_j + i(1-1') \delta \dot{b}_j - i \sum_{\ell=1,2} \left[ G_{j\ell}^{\lambda}(t') \delta \hat{a}_\ell + G_{j\ell}^{\phi}(t') \delta \hat{a}_\ell^\dagger \right] + \sqrt{\gamma_{m,j}} b_{j,in}$$

where the time dependent coefficients can be expressed in terms of the mean field amplitudes $\alpha_j(t)$ and $\beta_j(t)$ as (see App. A)

$$\Theta_1(t) = - (2) \sum_{j=1,2} g_{0,j} \text{Re} \left[ \beta_j(t) - \beta_{j}^{(dc)} \right]$$

$$\Theta_2(t) = - (2) \sum_{j=1,2} g_{0,j} \text{Re} \left[ \alpha_j(t) - \alpha_{j}^{(dc)} \right]$$

$$G_{j1}^{\lambda}(t) = g_{0,j} |\alpha_j(t)| e^{i \lambda \left( \omega_{0} - (1-1') \delta \right) t}$$

$$G_{j1}^{\phi}(t) = - 4 g_{0,j} \text{Re} \left[ \alpha_j(t) \right] e^{i \phi \left( \omega_{0} - (1-1') \delta \right) t}$$

$$\Gamma(t) = - 4 \sum_{j=1,2} g_{0,j} \text{Re} \left[ \beta_j(t) \right] e^{i \phi \omega_{0} t}$$

### A. Linearization

First, we linearize the equations for the fluctuations by assuming sufficiently large mean amplitudes. In particular, we analyze the fluctuations in interaction picture with respect to the Hamiltonian $\hat{H}_{|0>}$ = $\hbar \omega_0 \hat{a}_1^\dagger \hat{a}_1 + \hbar \omega_{1c} \hat{a}_2^\dagger \hat{a}_2 + \hbar (\omega_1 + \delta) \hat{b}_1^\dagger \hat{b}_1 + \hbar (\omega_2 - \delta) \hat{b}_2^\dagger \hat{b}_2$, where $\Delta \equiv \Delta_1 + 2 \sum_{j=1,2} g_{0,j} \text{Re} \left[ \beta_{j}^{(dc)} \right]$ and $\omega_{1c} \equiv \omega_{1c}^{(0)} - 4 \sum_{j=1,2} g_{0,j} \text{Re} \left[ \beta_{j}^{(dc)} \right]$, with $\beta_{j}^{(dc)}$ the time-independent part of the mean mechanical amplitude $\beta_j(t)$; and where $\delta$ is a small detuning which is used to tune the non-reciprocity as discussed in the following sections.
In particular, we remark that these parameters can be expanded as a sum of many terms each oscillating at a different frequency as

\[ X(t) = \sum _{\xi } X_{\xi } e^{i\omega _{\xi } t} \quad \text{for } X \in \{ \Theta _{e}, G_{ij}^{(\alpha )}, \Gamma \} , \quad (15) \]

where the sum is over all the possible frequency components, \( \omega _{\xi }^{(X)} \), of each parameter, and \( X_{\xi } \) indicate the corresponding amplitudes (specific expressions for these quantities are reported in App. [A] see also Ref. [66]).

B. Rotating wave approximation

Then, we neglect all the time dependent terms. To be specific, we note that when the system frequencies are selected such that (see Fig. 2)

\[ \Delta = \omega _{LC} - \omega _{+} \]

\[ \omega _{1} + \delta = \omega _{LC} \]

\[ \omega _{2} - \delta = \omega _{LC} + \omega _{X} \]

\[ \omega _{X} = 2 \omega _{+} \quad (16) \]

only the terms \( G_{ij}^{(+)}(t) \) in Eq. (14) have a time independent part. In the following we indicate the time independent part of \( G_{ij}^{(+)}(t) \) and of \( G_{ij}^{(-)}(t) \) with the symbols \( g_{ij} \) and \(-g_{ij}^{*} \) respectively (see App. [A] for details). Now, if all the non-zero frequencies are sufficiently large, i.e. [see Eq. (15)]

\[ |X_{\xi }|, \kappa , \gamma _{LC} \ll \omega _{\xi }^{(X)} \quad \text{for } X \in \{ \Theta _{e}, G_{ij}^{(\alpha )}, \Gamma \} , \quad (17) \]

then it is possible to neglect all the time dependent terms, such that the quantum Langevin equation for the fluctuations reduce to the form

\[ \dot{\delta \hat{a}}_1 = -\kappa \delta \hat{a}_1 - i \sum _{j=1,2} g_{1j} \delta \hat{b}_j + \sqrt{2\kappa} \delta \hat{a}_{1,in} \]

\[ \dot{\delta \hat{a}}_2 = -\frac{\gamma _{LC}}{2} \delta \hat{a}_2 + i \sum _{j=1,2} g_{2j} \delta \hat{b}_j + \sqrt{\gamma _{LC}} \delta \hat{a}_{2,in} \]

\[ \dot{\delta \hat{b}}_j = -\left[ \frac{\gamma _{m,j}}{2} \delta \hat{b}_j - i(1-Y)^j \right] \delta \hat{b}_j - i g_{1j} \delta \hat{a}_1 + i g_{2j} \delta \hat{a}_2 + \sqrt{\gamma _{m,j}} \delta \hat{b}_{j,in} . \quad (18) \]

We also note that this equation can be valid only if the detuning \( \delta \) is not too large, that is, it should be of the same order or smaller than the effective coupling coefficients

\[ |\delta| \leq |g_{ij}| . \quad (19) \]

C. Perturbative expansion in powers of the bare couplings

Finally, we compute explicit expressions for the interaction coefficients \( g_{ij} \) by expanding the mean amplitudes \( \alpha _{ij}(t) \) and \( \beta _{ij}(t) \) in powers of the bare interaction coefficients \( g_{0,ij} \). In particular, if the bare couplings are sufficiently small, then it is justifiable to consider only the corresponding leading terms, which, here, means to expand the mean amplitude up to second order. In this way we find the following approximated expressions

\[ g_{11} = -i g_{0,11} \chi _{1} \hat{E}_1 \]

\[ g_{12} = -2 g_{0,12} \chi _{1} ^{2} \left( |1|^2 \chi _{1} ^{2} E_1 ^{2} + \chi _{2} ^{2} E_2 ^{2} \right) \mu \]

\[ g_{21} = -32 g_{0,21} |1|^2 \chi _{1} ^{2} Im \{ \chi _{1} ^{2} E_1 E_2 ^{*} \} \mu \]

\[ g_{22} = -2 i g_{0,22} \chi _{1} ^{2} V V ^{*} \mu \]

where we have introduced the parameters

\[ \mu = g_{0,11} g_{0,21} Im \{ \chi _{1,m,1} \} + g_{0,12} g_{0,22} Im \{ \chi _{1,m,2} \} \]

and the susceptibilities

\[ \chi _{1} ^{1} \equiv \left( \kappa + i\Delta _{e} \right)^{-1} , \]

\[ \chi _{1} ^{LC} \equiv \left( \frac{\gamma _{LC}}{2} + i \omega _{LC} ^{(0)} \right)^{-1} , \]

\[ \chi _{m,j} ^{1} \equiv \left( \frac{\gamma _{m,j}}{2} + i \omega _{j} \right)^{-1} . \quad (20) \]

We note that the leading contribution in the couplings \( g_{11} \) and \( g_{22} \) are zeroth order terms which are the result of the direct optical driving with strength \( \hat{E}_1 \) and of the radio-frequency driving with strength \( V \) respectively. Instead the couplings \( g_{12} \) and \( g_{21} \) are both the results of the superposition of various second order processes which involve all the drivings (see also App. [A]). This entails that the amplitudes and phases of these four parameters cannot be controlled independently from one another. And, this makes the identification of regimes of non-reciprocity in this system more difficult as compared to similar results achieved for a system driven by four fields [30] [44].

IV. NONRECIROCITY

The equations (18) can be easily solved in Fourier space, and together with the standard input output relation \( \hat{a}_{in} = \sqrt{2\kappa} \hat{a}_{in} + \hat{a}_{in,1} \) and \( \hat{a}_{2,in} = \sqrt{\gamma _{LC}} \hat{a}_{2,in} + \hat{a}_{2,in,2} \), it is possible to express the output operators in terms of the input ones (see App. [B]). In general, each output operator can be expanded as

\[ \hat{a}_{out} (\omega ) = \sum _{p=1,2} \left[ S_{ij} \hat{a}_{in} (\omega ) + T_{ij} \hat{b}_{in} (\omega ) \right] . \quad (21) \]

Here we are interested in the coefficients \( S_{12} \) and \( S_{21} \), which describe, respectively, how a radio-frequency input signal is converted into an optical field, and conversely how an optical signal is converted into a radio-frequency field. Nonreciprocity correspond to the situation in which one of these two coefficients is zero while the other is finite. In general these quantities take the form

\[ S_{jk} = -\sqrt{2 \kappa \gamma _{LC}} \frac{F_{jk}}{D} \quad , \quad (24) \]

with

\[ F_{12} = g_{11} \chi _{1,m,1} g_{12} + g_{12} \chi _{1,m,2} g_{22} \]

\[ F_{21} = g_{11} \chi _{m,1} g_{21} + g_{12} \chi _{m,2} g_{22} \]

\[ F_{22} = g_{12} \chi _{m,1} g_{21} + g_{22} \chi _{m,2} g_{22} . \quad (25) \]
and phases of the driving fields, fulfill the following equations in interaction picture and where we have introduced the mechanical susceptibility $S$

Now, $\delta = -2.6 \text{kHz}, \gamma_{LC} = 78.8 \text{kHz}, \gamma_{m1} = \gamma_{m2} = 4 \text{kHz}, \kappa = 800 \text{kHz}, E_1 = 48.4 \text{GHz}$ and $E_2 = 97 \text{GHz}$. In (a) $\delta = -4 \text{kHz}, \gamma_{LC} = 90 \text{kHz}, \gamma_{m1} = 6 \text{kHz}, \kappa = 900 \text{kHz}, E_1 = 48.7 \text{GHz}$ and $E_2 = 97 \text{GHz}$. The mechanical frequencies are fixed by the resonance conditions (16).

Using Eq. (20), we find that they can be expressed as

$$E_{12} = s\chi_{m,2} \left[ \frac{\nu}{\mu} + r \right] \frac{\chi_{LC}}{\chi_{1}} e^{i\phi} + r - \mathcal{G}$$

$$E_{21} = s^\ast \chi_{m,2} \left[ \frac{\nu}{\mu} - \frac{1}{\chi_1} r \right] \frac{\chi_{LC}}{\chi_{1}} e^{-i\phi} - \frac{1}{\chi_1} r - \mathcal{G}$$

where

$$\varphi = 2(\phi_{11} - \phi_{12} - \phi_{X}) - \pi$$

$$\mathcal{G} = \left| \frac{\chi_{LC}}{\chi_1} V' \right|^2$$

$$r = \frac{4 i\mu \text{Im} (\text{ex}) \text{Re} (\text{ex}) \chi_{m1}}{i\chi_1 \text{Re} (\text{ex}) \text{Re} (\text{ex})}$$

$$s = -4 i\mu \text{Re} (\text{ex}) \chi_1^2 | \text{ex} |^2 \chi_{LC} | \text{ex} |_1 \frac{1}{\chi_1}$$

Now, $S_{12} = 0$ when the phase $\varphi$ and the parameter $\mathcal{G}$ defined in Eq. (30), which can be tuned by controlling the amplitudes and phases of the driving fields, fulfill the following equations

$$e^{i\phi} = \frac{\chi_{LC}}{\chi_1} \frac{\nu}{\mu} + r$$

$$\mathcal{G} = \frac{\nu}{\mu} + 2 \text{Re} (r)$$

At the same time we find that using these relations in the equation for $F_{21}$ [see Eq. (29)],

$$F_{21} = -s^\ast \chi_{m,2} 2 \left( \frac{\nu}{\mu} + \text{Re} (r) \right) \left( \frac{\nu}{\mu} + r \right)$$

which is, in general, different from zero and indicates that the system can be non-reciprocal.

Finally we note that an important parameter used to achieve non-reciprocity is the detuning $\delta$. In fact, if $\delta = 0$ (and $\gamma_{m1} = \gamma_{m2}$), then $\chi_{m1} = \chi_{m2}$. In this case, it is easy to check (see Eq. (25) and (26)) that whenever $S_{12} = 0$ also $S_{21} = 0$, and the transmission in this system becomes reciprocal.

V. NUMERICAL RESULTS

We have verified the non-reciprocity in this system numerically. Specifically, we have set the driving fields powers and phases such that the conditions (31) for the suppression of $S_{12}$ are fulfilled and then we have numerically maximized the corresponding value of $S_{21}$ over various system parameters. In doing this we have checked throughout that the selected system parameters were consistent with our approximations. In particular, the fulfillment of the conditions for the validity of the rotating wave approximation expressed by Eq. (17) (see also App. A) poses a strong limitation on the range of parameters which we can safely explore with our approximated model. As a result, although we have verified the non-reciprocity, we have found that, in the valid regime of parameters when the conversion (transmission) in the forward direction is suppressed $S_{12} = 0$, the corresponding conversion in the backward direction is always relatively low.

In details, for the results of Fig. 3 we have set the values of the optical driving amplitudes $E_j$, of the rf cavity frequency $\omega_{LC}$, of the optical detuning $\Delta$, and of the frequency of the rf driving, and we have chosen the driving phases and the amplitude $|V'|$ which suppress $S_{12}$, according to Eq. (31), at zero.

FIG. 3. Transmission coefficient in the forward ($|S_{12}|^2$) and in the backward ($|S_{21}|^2$) directions, when the values of $|V'|$ and $\varphi$ are set to fulfill Eq. (31) for $\omega = 0$. In both plots $\omega_{LC} = 6 \text{MHz}$, $\Delta = -4 \text{MHz}$, $\omega_{rf} = 20 \text{MHz}$, $\omega_{011} = 8 \text{Hz}$, $\omega_{012} = 20 \text{Hz}$, $\omega_{021} = 20 \text{Hz}$ and $\omega_{022} = 4 \text{Hz}$. In (a) $\delta = -2.6 \text{kHz}, \gamma_{LC} = 78.8 \text{kHz}, \gamma_{m1} = \gamma_{m2} = 4 \text{kHz}, \kappa = 800 \text{kHz}, E_1 = 48.4 \text{GHz}$ and $E_2 = 97 \text{GHz}$. In (b) $\delta = -4 \text{kHz}, \gamma_{LC} = 90 \text{kHz}, \gamma_{m1} = 6 \text{kHz}, \kappa = 900 \text{kHz}, E_1 = 48.7 \text{GHz}$ and $E_2 = 97 \text{GHz}$. The mechanical frequencies are fixed by the resonance conditions (16).
frequency $\omega = 0$. Then, we have maximized the value of $S_{21}$ over the detuning $\delta$, and the damping rate of the rf-circuit $\gamma_{LC}$. In this way the values of the mechanical frequencies are fixed by the resonance conditions expressed by Eq. (16).

These plots demonstrate the non-reciprocal conversion of an electromagnetic signal from the radio-frequency to the optical regime in an optoelectromechanical system which use only three driving fields.

VI. CONCLUSIONS

In conclusion, we have analyzed the possibility of achieving non-reciprocal transmission and rf-to-optical conversion in an optoelectromechanical system composed of an optical cavity, a rf LC-circuit and two mechanical resonators.

In this system the mechanical resonators mediate an indirect interaction between optical cavity and LC circuit, and the non-reciprocity relies on the interference between different transmission processes mediated by the two mechanical resonators and which result in different relative phases in the forward and backward directions.

We have demonstrated that non-reciprocity is achievable also when only three fields (two optical and one rf) are used to drive the system.

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Appendix A: Approximations

The average amplitude of the electromagnetic and mechanical fields, $a_j = \hat{a}_j - \delta \hat{a}_j$ and $b_j = \hat{b}_j - \delta \hat{b}_j$, fulfill the equations [see Eq. (12)]

$$\begin{align*}
\dot{a}_1 &= -(\kappa + i \Delta_L) a_1 - i [E_1 e^{i\omega_{1,t}} + E_2 e^{-i\omega_{1,t}}] \\
\dot{a}_2 &= -(\gamma_{LC} + i \omega_0(0)) a_2 + 2 i \sum_{j=1,2} g_{0,j} (\alpha_2 + \alpha_2^\dagger) (b_j + b_j^\dagger) + iV'e^{i\omega_{1,t}} \\
\dot{b}_j &= -(\gamma_{m,j}\omega_0(0) + i \omega_0(0)) b_j - ig_{0,j} |a_1|^2 \\
&\quad + i g_{0,j} (\alpha_2 + \alpha_2^\dagger)^2.
\end{align*}$$

(A1)

(A2)

(A3)

The corresponding solutions enter into the equations for the fluctuations, $\delta \hat{a}_1$ and $\delta \hat{b}_j$, as modulations of the interaction coefficients between different operators. When these rescaled interaction coefficients are sufficiently large, it is legitimate to linearize these equations, by neglecting non-linear terms in the fluctuations. In this way we find the linearized quantum Langevin equations for the fluctuations

$$\begin{align*}
\delta \dot{a}_1 &= - \left\{ \kappa + i \left[ \Delta_L + 2 \sum_{j=1,2} g_{0,j} (\delta \hat{b}_j + \delta \hat{b}_j^\dagger) \right] \right\} \delta \hat{a}_1 \\
&\quad - i \alpha_1(t) \sum_{j=1,2} g_{0,j} (\delta \hat{b}_j + \delta \hat{b}_j^\dagger) + \sqrt{2\kappa} \delta \hat{a}_{1,n} \\
\delta \dot{a}_2 &= - \left\{ \frac{\gamma_{LC}}{2} + i \left[ \omega_0(0) - 4 \sum_{j=1,2} g_{0,j} (\delta \hat{b}_j + \delta \hat{b}_j^\dagger) \right] \right\} \delta \hat{a}_2 \\
&\quad + 4 i \sum_{j=1,2} g_{0,j} (\delta \hat{b}_j + \delta \hat{b}_j^\dagger) + \sqrt{\gamma_{LC}} \delta \hat{a}_{2,n} \\
\delta \dot{b}_j &= - \left\{ \frac{\gamma_{m,j}}{2} + i \omega_0(0) \right\} \delta \hat{b}_j - ig_{0,j} \left[ \alpha_1(t) \delta \hat{a}_1 + \alpha_2^\dagger(t) \delta \hat{a}_1^\dagger \right] \\
&\quad + 2 i g_{0,j} (\delta \hat{a}_2(t)) (\delta \hat{a}_2 + \delta \hat{a}_2^\dagger) + \sqrt{\gamma_{m,j}} \delta \hat{b}_{j,n} \tag{A4}
\end{align*}$$

which are equivalent to the equations in interaction picture reported in the main text [see Eq. (9) and (10)]. As discussed in the main text, we have evaluated explicit expressions for the coefficients $[4]$ by solving the equations for the mean amplitudes $[A1]$. This can be done recursively by expanding the amplitudes in powers of the bare interaction coefficients $g_{0,j}$. When the interactions coefficients $g_{0,j}$ are sufficiently small it is possible to consider only the first few terms of this expansion and neglect the rest. Here we consider coefficients up to second order [66]. We find that in the long time limit the mean amplitudes, $\alpha_1(t)$ and $\beta_j(t)$, are composed of a sum of terms which oscillate at multiples of the driving frequencies and at their sums end differences.

Specifically, we obtain the following expressions for the mean fields in the long time limit, up to the second order in $g_{0,j}$

$$\begin{align*}
\alpha_1(t) &\approx \left[ a_1^{(0)} + a_1^{(2)} e^{i\omega_{1,t}} + a_1^{(0)} e^{-i\omega_{1,t}} \right. \\
&\quad + a_1^{(2)} e^{-3i\omega_{1,t}} + a_1^{(2)} e^{-3i\omega_{1,t}} \\
&\quad + a_1^{(2)} e^{i(2\omega_{0,X}+\omega_{1,t})} + a_1^{(2)} e^{i(2\omega_{0,X}+\omega_{1,t})} \\
&\quad + a_1^{(2)} e^{i(2\omega_{0,X}+\omega_{1,t})} + a_1^{(2)} e^{i(2\omega_{0,X}+\omega_{1,t})} \\
&\quad + a_1^{(2)} e^{i(2\omega_{0,X}+\omega_{1,t})} + a_1^{(2)} e^{i(2\omega_{0,X}+\omega_{1,t})} \\
&\left. + a_1^{(2)} e^{i(2\omega_{0,X}+\omega_{1,t})} + a_1^{(2)} e^{i(2\omega_{0,X}+\omega_{1,t})} \right] \\
\beta_j(t) &\approx \beta_j^{(1)} e^{2i\omega_{1,t}} + \beta_j^{(1)} e^{-2i\omega_{1,t}} \\
&\quad + \beta_j^{(2)} e^{2i\omega_{1,t}} + \beta_j^{(2)} e^{-2i\omega_{1,t}} \tag{A5}
\end{align*}$$

where the zero-th order terms are

$$\begin{align*}
a_1^{(0)} &= -i \chi_1 E_1 \\
a_1^{(1)} &= -i \chi_1 E_1 \\
a_1^{(2)} &= i \chi_{LC} V',
\end{align*}$$

(A6)
the first order terms are
\[ \beta^{(1)}_{j} = -i \chi_{-j} \left[ g_{0,j} \left( \alpha^{(0)}_{-j} + \alpha^{(0)}_{j} \right) - 2g_{0,j} \alpha_{-j}^{(0)} \right] \]
\[ \beta^{(1)}_{j-2X} = -i \chi_{-j} g_{0,j} \alpha_{j}^{(0)} \]
\[ \beta^{(1)}_{j+2X} = i \chi_{j} g_{0,j} \alpha_{-j}^{(0)} \]
\[ \beta^{(1)}_{j-2X} = i \chi_{j} g_{0,j} \alpha_{j}^{(0)} \]  \( \text{(A7)} \)
and the second order terms are
\[ a_{1,2}^{(2)} = -i \chi \sum_{j=1}^{2} g_{0,1,j} \left[ 2a_{1,0}^{(0)} \text{Re} \left( \beta^{(0)}_{j} \right) + a_{1,2}^{(0)} \right] \]
\[ a_{1,3}^{(2)} = -i \chi \sum_{j=1}^{2} g_{0,1,j} \left[ \alpha_{j}^{(0)} + a_{1,2}^{(0)} \right] \]
\[ a_{1,2x}^{(2)} = -i \chi \sum_{j=1}^{2} g_{0,1,j} \left[ \alpha_{j}^{(0)} + a_{1,2x}^{(0)} \right] \]
\[ a_{2,2x}^{(2)} = -i \chi \sum_{j=1}^{2} g_{0,2,j} \left[ \alpha_{j}^{(0)} + \alpha_{j+1}^{(0)} \right] \]
\[ a_{2,3x}^{(2)} = -i \chi \sum_{j=1}^{2} g_{0,2,j} \left[ \alpha_{j}^{(0)} + \alpha_{j+1}^{(0)} \right] \]
\[ a_{2,2x}^{(2)} = 2i \chi \sum_{j=0}^{2} g_{0,2,j} \left[ a_{2,2x}^{(0)} + \alpha_{j+1}^{(0)} \right] \]
\[ a_{2,3x}^{(2)} = 2i \chi \sum_{j=0}^{2} g_{0,2,j} \left[ a_{2,3x}^{(0)} + \alpha_{j+1}^{(0)} \right] \]  \( \text{(A8)} \)
Correspondingly we find that the time dependent coefficients \( \Theta^{\ell}(t) \) of the linearized quantum Langevin equations \( \text{(13)} \) can be written as sums of many terms oscillating at different frequencies as in Eq. \( \text{[15]} \). To be specific, we find that the shifts of the electromagnetic frequencies can be written as
\[ \Theta^{\ell}(t) = \sum_{\xi \in \{+,X\}} \sum_{\xi \in \{\pm\}} \Theta^{\ell,\xi} e^{i \omega^{\ell,\xi} t} \quad \text{for} \quad \ell \in \{1, 2\} \]  \( \text{(A9)} \)
with frequencies
\[ \omega^{\ell,\pm} = \pm 2 \omega_{\ell} \quad \text{for} \quad \xi \in \{+,X\} \]  \( \text{(A10)} \)
and corresponding coefficients
\[ \Theta^{\ell+,\pm} = \frac{\omega^{\ell,\pm}}{2} \sum_{j=1}^{n} g_{0,j} \left[ \beta_{j+2}^{(1)} + \beta_{j+2}^{(1)} \right] \]
\[ \Theta^{\ell,X,\pm} = \frac{\omega^{\ell,\pm}}{2} \sum_{j=1}^{n} g_{0,j} \left[ \beta_{j+2}^{(1)} + \beta_{j+2}^{(1)} \right] \]  \( \text{(A11)} \)
The field enhanced interaction strengths are
\[ G_{1,0}^{(1)}(t) = \sum_{c=0,2x} \sum_{\xi \in \{\pm\}} G_{1,0,c}^{(1)} e^{i \omega^{(1)} c \xi t} \quad \text{for} \quad j \in \{1, 2\} \]
\[ G_{2,0}^{(1)}(t) = \sum_{c=0,2x} \sum_{\xi \in \{\pm\}} G_{2,0,c}^{(1)} e^{i \omega^{(1)} c \xi t} + \sum_{\xi \in \{\pm\}} G_{2,0,2x}^{(1)} e^{i \omega^{(1)} c \xi t} \]
\[ + G_{2,0,-2x}^{(1)} e^{i \omega^{(1)} c \xi t} \quad \text{for} \quad j \in \{1, 2\} \]  \( \text{(A12)} \)
with frequencies
\[ \omega_{c,\pm}^{(1)} = \Delta - \omega_{c} + \chi_{\pm} \quad \text{for} \quad \xi \in \{0, \pm 2\} \]  \( \text{and} \quad \xi \in \{\pm\} \)
\[ \omega_{c,\pm}^{(2)} = \omega_{c} \pm \omega_{c} \quad \text{for} \quad \xi \in \{\pm, X\} \]
\[ \omega_{c,\pm}^{(2)} = \omega_{c} \pm \omega_{c} + \xi \omega_{c} \quad \text{for} \quad \xi \in \{\pm\} \]
\[ \omega_{c,\pm}^{(2)} = \omega_{c} \pm \omega_{c} + \xi \omega_{c} + 2 \omega_{c} \quad \text{for} \quad \xi \in \{\pm\} \]
\[ \omega_{c,\pm}^{(2)} = \omega_{c} \pm \omega_{c} + \xi \omega_{c} + 2 \omega_{c} \quad \text{for} \quad \xi \in \{\pm\} \]  \( \text{(A13)} \)
where
\[ \bar{\omega}_{c} = \omega_{c} - (1)^{\prime} \delta_{c} \]  \( \text{(A14)} \)
and corresponding coefficients
\[ G_{1,0,\pm}^{(1)} = g_{0,0} \left[ a_{1,0}^{(0)} + a_{1,2}^{(2)} \right] \quad \text{for} \quad \xi \in \{\pm\} \]
\[ G_{1,0,\xi}^{(2)} = g_{0,0} a_{1,0,\xi}^{(2)} \quad \text{for} \quad \xi \in \{\pm\} \]
\[ G_{1,2,\xi,\pm}^{(2)} = g_{0,0} a_{1,2,\xi,\pm}^{(2)} \quad \text{for} \quad \xi \in \{\pm\} \]  \( \text{(A15)} \)
and
\[ G_{2,2,\xi,-\xi}^{(2)} = -2g_{0,2,j} \left[ a_{2,2,\xi,-\xi}^{(2)} - a_{2,2,\xi,-\xi}^{(2)} \right] \quad \text{for} \quad \xi \in \{\pm\} \]
\[ G_{2,2,\xi,\xi}^{(2)} = -2g_{0,2,j} \left[ a_{2,2,\xi,\xi}^{(2)} + a_{2,2,\xi,\xi}^{(2)} \right] \quad \text{for} \quad \xi \in \{\pm\} \]
\[ G_{2,2,\xi,-\xi}^{(2)} = -4g_{0,2,j} \left[ a_{2,2,-\xi,-\xi}^{(2)} + a_{2,2,-\xi,-\xi}^{(2)} \right] \quad \text{for} \quad \xi \in \{\pm\} \]  \( \text{(A16)} \)
And finally the self interaction strength of the rf mode can be expressed as
\[ \Gamma(t) = \Gamma_{0} e^{i \omega_{0} t} + \sum_{c(\pm,X)} \sum_{\xi \in \{\pm\}} \Gamma_{c,\xi} e^{i \omega^{(1)} c \xi t} \]  \( \text{(A17)} \)
with frequencies
\[ \omega^{(1)}_{0} = 2 \omega_{0} \]
\[ \omega^{(1)}_{c} = 2 \left( \omega_{0} \pm \omega_{c} \right) \quad \text{for} \quad \xi \in \{+,X\} \]  \( \text{(A18)} \)
and corresponding coefficients
\[ \Gamma_{0} = -2 \sum_{j=1}^{2} g_{0,j} \text{Re} \left[ \beta_{j}^{(1,c)} \right] \]
\[ \Gamma_{c,\pm} = \frac{1}{2} \Theta_{2,c,\pm} \quad \text{for} \quad \xi \in \{+, X\} \]  \( \text{(A19)} \)
We note that when the resonance conditions \( \text{[16]} \) are fulfilled, the frequencies
\[ \omega_{0,1}^{(1)} = \Delta - \omega_{0} + \omega_{c} \]
\[ \omega_{0,2}^{(1)} = \Delta - \omega_{0} + 2 \omega_{c} - \omega_{c} \]
\[ \omega_{0,2}^{(2)} = \omega_{0} \pm \omega_{c} \pm \omega_{c} + 2 \omega_{c} \]
\[ \omega_{0,1}^{(2)} = \omega_{0} \pm \omega_{c} \pm \omega_{c} + 2 \omega_{c} \]
\[ \omega_{0,2}^{(2)} = \omega_{0} \pm \omega_{c} \pm \omega_{c} + 2 \omega_{c} \]  \( \text{(A20)} \)
are zero. All the other frequencies, instead, are different from zero. The frequencies in Eq. (A20) correspond, respectively, to the coefficients
\begin{align*}
G_{1,1,0,+}^{(-)} &= g_{111} [a_{1,+}^{(0)} + a_{1,+}^{(2)}] \\
G_{1,2,2,-}^{(-)} &= g_{122} [a_{1,2,-}^{(0)} + a_{1,2,-}^{(2)}] \\
G_{2,1,-,+}^{(-)} &= -4 g_{21} R \left[a_{2,1,-}^{(2)} + a_{2,1,-}^{(2)} \right] \\
G_{2,2,0,+}^{(-)} &= -2 g_{222} [a_{2,0,+}^{(0)} + a_{2,0,+}^{(2)} + a_{2,0,+}^{(2)}].
\end{align*}

In the main text we have used the symbols \(g_\ell\) to indicate these coefficients, specifically, we have considered only the leading terms and we have used these definitions
\begin{align*}
g_{11} &\equiv g_{111} a_{1,+}^{(0)} \\
g_{12} &\equiv g_{122} [a_{1,2,-}^{(0)} + a_{1,2,-}^{(2)}] \\
g_{21} &\equiv 4 g_{21} R \left[a_{2,1,-}^{(2)} + a_{2,1,-}^{(2)} \right] \\
g_{22} &\equiv 2 g_{222} a_{2,0,+}^{(0)},
\end{align*}
which are equal to the definitions in Eq. (20). In our numerical simulations we have verified that all the other coefficients in Eqs. (A9), (A12) and (A17) are much smaller than the corresponding frequencies, i.e. \(|X_{\ell,j}| \ll \omega_{\ell,j}\), for \(X \in \{\Theta, G^{(\pm)}, \Gamma\}\) and for all corresponding \(\ell, j, \zeta\) and \(\xi\).

**Appendix B: The model in Fourier space**

Eq. (18) can be easily solved in Fourier space. To be specific one can express the mechanical operators in terms of the susceptibilities \(\chi\) and of the optical and rf mode operators as
\begin{align*}
\delta \hat{b}_1 &\equiv \chi_{m,1} \left[ -ig_1 \hat{a}_1 + ig_2 \hat{a}_2 + \sqrt{\gamma_m} \hat{b}_{1,\text{in}} \right] \\
\delta \hat{b}_2 &\equiv \chi_{m,2} \left[ -ig_1 \hat{a}_1 + ig_2 \hat{a}_2 + \sqrt{\gamma_m} \hat{b}_{2,\text{in}} \right].
\end{align*}

These expressions can be replaced into the equation for the electromagnetic fields and one obtain the following set of closed equations
\begin{align*}
\frac{-i \omega}{\delta \hat{b}_2(\omega)} &= \left( \frac{-i \kappa + ig_1^2 \hat{a}_{1,\text{in}} + ig_2^2 \hat{a}_{2,\text{in}}}{g_1^2 \hat{a}_{1,\text{in}} + g_2^2 \hat{a}_{2,\text{in}}} \right) \delta \hat{a}_1(\omega) \\
&\quad + \left( \frac{\sqrt{2} \kappa \hat{a}_{1,\text{in}} - ig_1 \hat{a}_{1,\text{in}} \sqrt{\gamma_m} \hat{b}_{1,\text{in}}} {\sqrt{\gamma_m} \hat{b}_{1,\text{in}} + ig_2 \hat{a}_{2,\text{in}}} \right) \delta \hat{b}_1(\omega)
\end{align*}
from which one finds the expressions of the modes operators in terms of the input noise operators. Finally using the input-output relations
\begin{align*}
\hat{a}_{1,\text{out}}(\omega) &= \sqrt{\kappa} \delta \hat{a}_1(\omega) + \hat{a}_{1,\text{in}}(\omega) \\
\hat{a}_{2,\text{out}}(\omega) &= \sqrt{\gamma_m} \delta \hat{a}_2(\omega) + \hat{a}_{2,\text{in}}(\omega)
\end{align*}
one finds the expressions for the output fields \(\hat{a}\), and in particular the expressions for the transmission coefficients \(\Gamma\).
