Lorentz violation dispersion relation and its application

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We derive a modified dispersion relation (MDR) in the Lorentz violation extension of quantum electrodynamics (QED) sector in the standard model extension (SME) framework. Based on the extended Dirac equation and corresponding MDR, we observe the resemblance of the Lorentz violation coupling with spin-gravity coupling. We also develop a neutrino oscillation mechanism induced by the presence of nondiagonal terms of Lorentz violation couplings in 2-flavor space in a 2-spinor formalism by explicitly assuming neutrinos to be Majorana fermions. We also obtain a much stringent bound (∼10^{-25}) on one of the Lorentz violation parameters by applying MDR to the ultrahigh energy cosmic ray (UHECR) problem.

Keywords: Lorentz violation; Majorana neutrino; modified dispersion relation; ultrahigh energy cosmic ray

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1. Introduction

In the development of physics, symmetry principle is a powerful tool in the construction and interpretation of physical laws of nature. Various efforts have been dedicated to the searches of new symmetry principle beyond standard model (SM) gauge symmetry and ordinary Lorentz symmetry, such as SU(5) and SO(10) in grand unified theory (GUT), or SO(32) and $E(8) \otimes E(8)$ in string theory. Aside from these gauge symmetries, ordinary Lorentz symmetry is also extended to SO(9, 1) in string theory or SO(10, 1) in M-theory.

On the other hand, symmetry principle is not implemented trivially in nature. The discovery of non-conservation of parity in 1957 makes people to realize that some sacred symmetry may be only a good approximation. And the later discovery of electroweak theory teaches us that symmetry could be hidden in vacuum condensation, in other words, it is realized through spontaneous symmetry breaking mechanism.

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Does similar situation happen in the case of Lorentz symmetry? This is a rather deep question since Lorentz symmetry is a fundamental spacetime symmetry and has been incorporated into the two cornerstones of current physics: general relativity and quantum field theory. The possibility to think of Lorentz symmetry breaking may be traced back to Dirac through reintroducing aether into the theory of electrodynamics in the early 1950s. There are also other perspectives related to Lorentz symmetry violation (LV). It is first demonstrated by Kostelecký and Samuel that spontaneous Lorentz symmetry breaking may happen in string field theory via unstable vacuum triggered by tachyon field. After then Kostelecký and Colladay incorporated Lorentz symmetry violation into the effective theory framework, which is the so called standard model extension (SME). In that work, spontaneous Lorentz symmetry breaking is triggered by nonzero vacuum expectation value (VEV) of a tensor field in underlying theory, and these VEV of tensor fields are incorporated with SM fields into all possible LV operators. In addition to string motivated LV, other approaches of quantum gravity also indicate some signatures of Lorentz violation. That include spin-network calculation in loop quantum gravity, foamy structure of spacetime, noncommutative quantum field theory and emergent gravity etc. However, without a complete theory of quantum gravity, all indications of LV above do not provide a firm and definite evidence that Lorentz symmetry is indeed breaking, or in other words, why it should not be an exact symmetry. However, we can take a positivism viewpoint that we can rely on experiments to verify or put bound on LV, as current experiments have already reached the sensitivity to Planck mass suppression (e.g., for dimensionless couplings, the sensitivity to Planck mass suppression means sensitivity to $\frac{m}{M_{\text{Planck}}} \sim 10^{-23}$).

The purpose of this paper is to derive a set of modified dispersion relations (MDR) in the framework of SME and explore their consequence in the propagation properties of free particles. So we first briefly review the basic principle of SME and its quantum electrodynamics (QED) subset in sections 2 and 3. Then, by focusing on the physical relevant LV couplings we derive the MDR with CPT even and CPT odd LV couplings respectively, together with referring their physical resemblance with other distinct physical subjects in section 4. In section 5 the MDR is applied to neutrino propagation and ultrahigh energy cosmic ray (UHECR) problems separately. In that section we formulate a neutrino oscillation mechanism in 2-spinor formalism and derive a much stringent bound on the LV coupling involved in UHECR problem. In section 6 we give a brief summary. The convention adopted in this paper is $\eta_{00} = +1$ for $\eta_{\mu\nu}$ and $\epsilon_{0123} = +1$ for $\epsilon_{\rho\sigma\mu\nu}$.

2. Principle of SME

The basic principle of standard model extension (SME) is that, SM is regarded as a leading order Lagrangian in the low energy effective field theory originating from a presumed existing fundamental theory. While the other terms are treated as perturbation denoting tiny departures from exact Lorentz symmetry. It is these
perturbation terms revealing the possible signature of physics beyond SM. And the whole Lagrangian could be written as

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \delta \mathcal{L},$$

(1)

where $\delta \mathcal{L}$ is the Lagrangian denoting tiny LV effects. Generally it has a form

$$\delta \mathcal{L} \supset \frac{\lambda}{M^k} \langle T \rangle \overline{\psi} \Gamma (i \partial) \chi,$$

(2)

where the Lorentz indices of VEV of tensor field $\langle T \rangle$ and partial differential operators $i \partial$ in (2) are suppressed. These indices are matched so that they are contracted exactly, which indicates that SME is apparent Lorentz covariant. This is an explicit assumption of SME (i.e. LV terms are required to be Lorentz covariant in their apparent Lorentz indices), and also a direct consequence of the assumption of spontaneous Lorentz symmetry breaking of an underlying Lorentz covariant theory, such as string field theory\textsuperscript{11}. However, this covariance property of LV operators should not be confused with particle Lorentz violation they indicated. According to the work of Kostelecky and Colladay\textsuperscript{11}, observer Lorentz symmetry is nothing but the equivalence relation of different coordinate choice, though appropriate choice of coordinate system would largely simplify our calculation and in some cases even would be helpful in the interpretation of corresponding physical properties. While particle Lorentz symmetry is a real symmetry concerning the properties of identical particles (or localized fields) with different spin orientation and momentum through particle rotation or boost performed in a specified inertial frame. In ordinary theory, this is just the symmetry classifying different species of identical particles. While in LV theory, particle Lorentz transformation leaves tensor VEV ($\langle T \rangle$) unchanged, thus changes the relation or interaction between SM fields with background tensor fields, so new phenomena may arise. In this sense particle defined as irreducible representation of Lorentz group is just a good approximation if particle Lorentz symmetry is indeed violated.

Aside from the requirement of observer Lorentz invariance, other restrictions may also help us to restrict or classify LV terms. We could require the theory to be gauge invariant under a particular gauge group transformations, e.g., gauge invariance under gauge group $SU(3)_{C} \otimes SU(2)_{L} \otimes U(1)_{Y}$, and that is why the theory is called SME. Dimensional counting may be used to classify various LV operators. If restricted to dimension 3 or 4 terms, this is the minimal version of SME originally appeared in\textsuperscript{11}. We note that dimension 5 operators are also classified recently\textsuperscript{12}. Hermiticity and energy positivity are also necessary to make the theory physically meaningful.

In addition to the above considerations, discrete symmetry transformation can be applied on the LV operators to classify them into CPT even and CPT odd classes. They form two special irreducible representations of homogeneous Lorentz group respectively. Using the convention of Coleman and Glashow\textsuperscript{10}, the general irreducible representation of homogeneous Lorentz group is marked by $(A, B)$, where $A, B$ are two angular momentum quantum numbers. So $(1, 1)$ is identified with CPT
even operators and represents traceless symmetric tensor of rank 2, while \(( 1 , 1 )\) is identified with CPT odd operators and represents four-vector. According to the argument of Coleman and Glashow\(^{10}\), the expectation value of \(( A , A )\) operator grows at large energy like \(E^2 A\). Thus CPT even operators dominate at high energies.

### 3. QED Subset of SME

In this section we present power-counting renormalizable QED subset satisfying all the requirements discussed in the above section. We can divide LV QED into pure photon part and fermion part. The interaction between them is included through covariant derivatives \( D^\nu = \partial^\nu + iq A^\nu \). For simplicity we confine ourselves to electrons though the equation derived below can be applicable to more general fermions which are not necessarily elementary particles. The LV QED Lagrangian is

\[
\delta L_{\text{QED}} = \delta L_{\text{photon}} + \delta L_{\text{electron}},
\]

where

\[
\delta L_{\text{photon}} \supset -\frac{1}{4} (k_F)_{\kappa\lambda\mu\nu} F^{\kappa\lambda} F^{\mu\nu} + \frac{1}{2} (k_{AF})_{\kappa\lambda\mu\nu} A^\kappa F^{\lambda\mu\nu},
\]

and

\[
\delta L_{\text{electron}} = \delta L_{\text{even}} + \delta L_{\text{odd}},
\]

\[
\delta L_{\text{even}} \supset -\frac{1}{2} H_{\mu\nu} \overline{\psi} \sigma^{\mu\nu} \psi + \frac{i}{2} \gamma^\mu \overline{\psi} \gamma^\nu D^\nu \psi + \frac{i}{2} d_{\mu\nu} \overline{\psi} \gamma_5 \gamma^\mu D^\nu \psi,
\]

\[
\delta L_{\text{odd}} \supset -a_\mu \overline{\psi} \gamma^\mu \psi - b_\mu \overline{\psi} \gamma_5 \gamma^\mu \psi,
\]

including those which are not directly deducible from terms compatible with electroweak structure

\[
\delta L_{\text{odd}} \supset \frac{i}{2} e^\nu \overline{\psi} D^\nu \psi - \frac{1}{2} f^\nu \overline{\psi} \gamma_5 D^\nu \psi + \frac{1}{4} i g^{\lambda\mu\nu} \overline{\psi} \sigma_{\lambda\mu} D^\nu \psi.
\]

All the coupling coefficients \(c, d, e, f, g, m_5, a, b,\) and \(H\) above are real and constant parameters required by the hermiticity of Lagrangian. They are related to VEV of tensor fields in the underlying theory. However, not all of them are physically observable, and some of them can be eliminated through field redefinition. This is the result of the fact that there is a spinor coordinate selection freedom, which implies that the mathematical expression of Lorentz invariant Dirac Lagrangian is not uniquely determined\(^{13}\). There exists an equivalent class of Dirac Lagrangian which are related by the fermion field redefinition of the form

\[
\Psi(x) = [1 + f(x, \partial)] \chi(x),
\]

where \(f(x, \partial)\) represents a general \(4 \times 4\) matrix function of the coordinates and derivatives. For example, we can choose \(f(x, \partial) = +ia \cdot x\), or its finite form \(\Psi(x) = \exp[ia \cdot x] \chi(x)\) to reproduce \(-a_\mu \overline{\psi} \gamma^\mu \psi\) from the conventional Dirac Lagrangian

\[
L_{\text{Dirac}} = \frac{1}{2} i \overline{\psi} \gamma^\mu \gamma_5 \partial_\mu \psi - m \overline{\psi} \psi.
\]
Thus some apparent (physical irrelevant) LV couplings can be accounted for by field redefinition of conventional Dirac spinors, and then can be absorbed into redefined fields through inverse transformation. However, this field redefinition or field transformation works effectively only in the absence of interaction with other fields or interaction between spinor components due to nonlocality problem, which were observed by Colladay and McDonald. Fortunately, since in the following we will focus on extended Dirac equation and its consequence, disregard photon parts and set the covariant derivatives $\nabla_\nu$ into partial derivatives $\partial_\nu$, that is to consider only free Dirac equations, so no such obstructions will meet when performing field redefinition to remove some couplings. After field redefinition, the simplified extended Dirac Lagrangian which contains only physically relevant parameters (some rearrangement is performed to include original Dirac Lagrangian to form a compact and elegant form) is written as

$$L_{\text{electron}} = \frac{1}{2} i \bar{\psi} \Gamma^\mu \partial_\mu - \bar{\psi} \tilde{M} \psi,$$

where

$$\Gamma^\mu = \gamma^\mu + c^{(\nu\mu)} \gamma_\nu + d^{\mu\nu} \gamma_5 \gamma_\nu + \frac{1}{2} g^{\nu\mu} \sigma_{\lambda\nu},$$

$$\tilde{M} = m + b_\mu \gamma_5 \gamma^\mu + \frac{1}{2} H_{\mu\nu} \sigma^{\mu\nu}. \quad (12)$$

All the coefficients in the above two equations (12) and (13) have been redefined, thus do not have the symmetry properties as their original ones (without tilde) in their corresponding Lorentz indices. For details, see Colladay et al. For simplicity we omit the “tilde” below and the reader should not confuse them with the original ones.

Using the Euler-Lagrangian equation

$$\frac{\partial L}{\partial \Psi^l} - \frac{\partial}{\partial \partial_\mu} \left( \frac{\partial L}{\partial (\partial_\mu \Psi^l)} \right) = 0, \quad (14)$$

we can obtain the extended Dirac equation below

$$[i(\gamma_\nu + c_{\mu\nu} \gamma^\mu + d_{\mu\nu} \gamma_5 \gamma^\mu + \frac{g_{\lambda\mu}}{2} \sigma^{\lambda\nu}) \partial_\nu - (m + b_\mu \gamma_5 \gamma^\mu + \frac{1}{2} H_{\mu\nu} \sigma^{\mu\nu})] \Psi(x) = 0. \quad (15)$$

In subsequent section, we will discuss this equation in detail.

4. Derivation of Dispersion Relation of Extended Dirac Equation

In order to get a modified dispersion relation, we proceed with the usual squaring procedure (which leads to Klein-Gordon equation when we apply it to the usual Dirac equation) to see the consequence when apply it to the extended Dirac equation (15). Multiplying (15) by $[-i \Gamma^\rho \partial_\rho - \tilde{M}]$, we get equation

$$[\Gamma^\rho \partial_\rho + i [\Gamma^\rho, M] \partial_\rho + M^2] = 0, \quad (16)$$
where
\[ \Gamma^{\nu \mu} \mu = \{ G^{(\rho)}(\nu) c^{(\nu \mu)} - d^{(\rho)} c^{(\nu \mu)} - 2i\gamma_5 \sigma_{\nu \sigma} d^{(\rho)} G^{(\sigma \rho)} - \epsilon_{\lambda \nu \sigma \alpha} g^{\lambda \rho} (G^{(\rho)} + \gamma_5 d^{(\rho)}) \gamma_5 \gamma^\alpha \]
\[ + \frac{1}{4} g^{\lambda \rho} g^{\sigma \rho} [i\epsilon_{\lambda \nu \sigma \alpha} \gamma_5 + (\eta_{\lambda \eta} \eta_{\mu} - \eta_{\sigma \nu} \eta_{\lambda})] \} \] \[ \partial_\rho \partial_\mu, \]
\[ M^2 = m^2 + m(2b_\mu \gamma_5 \gamma^\mu + H^{\phi \sigma} \sigma_\rho) - b_\mu \gamma^\mu - \epsilon^{\mu \nu \sigma \alpha} H_{\mu \nu} b_\sigma \gamma_\alpha \]
\[ + \frac{1}{4} (\gamma^\mu \gamma_\lambda \gamma_\sigma \gamma_\rho) \gamma_5 \] \[ \partial_\rho \partial_\mu [i\epsilon^{\mu \nu \sigma \alpha} \gamma_5 + (\eta^{\rho \mu} \eta^{\sigma \nu} - \eta^{\rho \nu} \eta^{\sigma \mu})], \]
\[ \{ \Gamma^{\rho \mu} G^{(\nu \mu)} = \eta^{\rho \mu} + d^{(\rho \mu) c^{(\nu \mu)} + 2e^{(\rho \mu)}. \] \[ (20) \]

Note that, with the symmetry property of \( \epsilon^{(\rho \mu) c^{(\nu \mu)} \), we have
\[ G^{(\nu \mu)} G^{(\nu \mu)} = \eta^{\rho \mu} + d^{(\rho \mu) c^{(\nu \mu)} + 2e^{(\rho \mu)}. \] \[ (21) \]

Obviously this squared extended Dirac equation with 2 classes of undetermined LV parameters \( (\epsilon^{(\rho \mu)} c^{(\nu \mu)} \), \( d^{(\rho \mu)}, H_{\rho \mu}; b_\mu, g_{\lambda \mu \nu} \) is too complicated to be diagonalized in spinor space by continuing the same procedure. However, since their CPT properties are distinct, we can discuss them separately. But we should keep in mind that this is just for convenience. There is no fundamental reason to forbid the appearance of the CPT odd operators unless one imposes CPT symmetry as a custodial symmetry survived after the Lorentz symmetry breaking. Note that Lorentz invariance is just one part of sufficient conditions to the proof of CPT theorem in the local field theory, not a necessary one. On the other hand, CPT violation conclusively leads to Lorentz violation in local field theory, a theorem proved by Greenberg. So we could have a Lorentz violating theory with only CPT even operators, in which CPT odd operators are all ruled out by CPT invariance. While in the theory with CPT odd operators (this theory would be automatically Lorentz violating, as guaranteed by Greenberg’s theorem), CPT even Lorentz violating operators would be induced via loops, so they must be much smaller in the naive analysis with the assumption of tree level disappearance of CPT even Lorentz violating operators, thus can be neglected at the tree level calculation, which will be the case of next subsection.

4.1. CPT Odd

At first we write the field equation involving only CPT odd LV couplings
\[ [i\gamma \cdot \partial + \frac{i}{2} g_{\lambda \mu \nu} \sigma^{\lambda \mu} \partial_\nu - (m + \gamma_5 b \cdot \gamma)] \Psi(x) = 0, \] \[ (22) \]
then by multiplying (22) on the left with \[ -[(i\gamma \cdot \partial + \frac{i}{2} g_{\lambda \mu \nu} \sigma^{\lambda \mu} \partial_\nu - \gamma_5 b \cdot \gamma) - m], \] we obtain a quadratic equation
\[ \{ [i\eta_{\rho \mu} + \frac{1}{4} (i\epsilon^{\lambda \mu \nu \beta} \gamma_5 + (\eta^{\lambda \alpha} \eta^{\mu \beta} - \eta^{\lambda \beta} \eta^{\mu \alpha})) g_{\lambda \mu \nu} g_{\alpha \beta \rho}] \} \partial^\mu \partial^\rho + b^2 + m^2 \]
\[ + 2\gamma_5 \sigma_{\rho \nu} b^\rho \partial^\nu - i\epsilon^{\lambda \mu \nu \beta} b_\rho g_{\lambda \mu \nu} \gamma_\alpha \partial^\rho \} \Psi(x) = 0. \] \[ (23) \]
This equation can be rearranged by putting all diagonal terms (in spinor space) on one side, while leaving nondiagonal ones on the other side, that is
\[
\left\{ \partial^2 + \frac{1}{2} g_{\lambda\mu} g^{\mu\rho} \partial^\rho + b^2 + m^2 \right\} + 
\left\{ \frac{i}{4} \epsilon^{\lambda\mu\alpha\beta} g_{\lambda\mu\nu} (\gamma_5 g_{\alpha\beta\rho} \partial^\rho + 4 \gamma_\alpha b_\beta \partial^\nu + 2 \gamma_5 \sigma_{\rho\nu} b^\rho \partial^\nu \right\} \Psi(x) = 0.
\]
(24)

Proceeding with the same squaring procedure, in principle we can get an 8th order differential equation without the appearance of \( \Gamma \) structure matrices. However, since this routine is too tedious and makes physics obscure, we do not follow this way, rather we concentrate on the equation (22) itself. As noted in [11], \( g_{\lambda\mu\nu} \) may arise from interaction among fermion constituents for a composed fermion, thus is expected to be suppressed further more than other LV couplings, so we simply drop it in (22) and get
\[
\left[ i \gamma \cdot \partial - (m + \gamma_5 b \cdot \gamma) \right] \Psi(x) = 0.
\]
(25)

It would be easy to get a quartic order differential equation
\[
\left[ (\partial^2 + b^2 + m^2)^2 - 4((b \cdot (i\partial))^2 + b^2 \partial^2) \right] \Psi(x) = 0
\]
from equation (25). By using the Ansatz
\[
\Psi(x) \equiv \phi(p) \exp[-ip \cdot x],
\]
we finally get a modified dispersion relation
\[
((p^2 - b^2 - m^2)^2 + 4b^2 p^2 - 4(b \cdot p)^2) = 0.
\]
(28)

Note that, equation (28) is noninvariant under interchange \( p \rightarrow -p \), however it is invariant under simultaneous interchange \( p \rightarrow -p \) and \( b \rightarrow -b \). This is the common feature of CPT odd LV operators, which indicates a helicity dependence of energy levels. Taking into account of \( g_{\lambda\mu\nu} \) just implies a further splitting of energy degeneracy.

Before closing this subsection, we observe that even without LV, gravity can induce an equation of the same form as (25) except that, the constant vector \( b_\mu \) is replaced by a spacetime dependent vector \( B_a \), whose meaning will become clear later. This implies that gravitational field provides a practical global Lorentz symmetry breaking source. The above observation will be manifested by the derivation of the covariant Dirac equation
\[
L_{\text{spin-gravity}} = \sqrt{-g} \overline{\psi} \gamma^a \overline{D_a} \psi - m \overline{\psi} \psi,
\]
(29)

where the covariant derivative is \( \overline{D_a} = \overline{\partial_a} - \frac{1}{4} \epsilon^{bca} \sigma_{bc} \). Thus
\[
L_{\text{spin-gravity}} = \sqrt{-g} \overline{\psi} \gamma^a \cdot \overline{\partial_a} \psi - m \overline{\psi} \psi + \left[ \sqrt{-g} \overline{\psi} \gamma^a \sigma_{bc} \psi w_{bca} \right]
\]
\[= L_{\text{free}} + L_{\text{int}}, \]
(30)
where $L_{\text{int}} = \sqrt{-g} \gamma^a d_{bc} \psi w_{bca}$ can be shown equal to

$$L_{\text{int}} = L_{\text{VI}} + L_{\text{AI}} = \frac{i}{2} \sqrt{-g} \psi \eta^{a[b} \gamma^c \psi w_{bca|a} + \frac{\sqrt{-g}}{4} \psi \gamma_5 \gamma_d \psi \epsilon^{abcd} w_{[bc]} a, \quad (31)$$

where $w_{[bc]} = w_{bca}$ and $L_{\text{VI}}$ is an antihermitian term, thus vanishes automatically by the hermiticity requirement of the theory. While $L_{\text{AI}}$ can be shown equal to $\sqrt{-g} \psi \gamma_5 \gamma_d \psi B^d$, where

$$B^d = \frac{1}{4} \epsilon^{abcd} w_{[bc]} a = \frac{1}{4} \epsilon^{abcd} e_{bp} (e_{c} \epsilon_{c} e_{d} + \Gamma_{\mu \nu} e_{a} e_{c} \epsilon_{c} \epsilon_{\mu}). \quad (32)$$

Thus (29) could be rewritten in the form of

$$L_{\text{spin–gravity}} = L_{\text{free}} + L_{\text{AI}} = \det(e) \overline{\psi} \left[ (i \gamma \cdot \partial - m) + \gamma_5 \gamma_d B^d \right] \psi. \quad (33)$$

By identifying $B_a$ with $-b_a$, we see that the covariant Dirac equation

$$[(i \gamma \cdot \partial - m) + \gamma_5 \gamma_d B^d] \psi = 0 \quad (34)$$

is of the same form as (25).

As observed by Mohanty, Prasanna, and Lambiase\[13\], the spin-gravity coupling can induce leptogenesis in the presence of lepton number violation interactions, thus may help to resolve the net baryon asymmetry problem through the so called electroweak sphaleron process. Since equation (34) concerning spin-gravity coupling and equation (25) concerning LV vector coupling are of the same form, a non-vanishing CPT odd LV coupling, $b_\mu$, may also be a candidate in demonstrating these effects and thus provide a possible solution to asymmetry problem. So the formal similarity of the two equations suggests that $b_\mu$ and $-B_a$ may mimic the effects each other produced, thus experimental searches of the two may be complementary. In other words, experimental searches for curvature-spin coupling may also provide signals for LV bounds on $b_\mu$, and vice versa. However, there is a significant difference between the two. First, $b_\mu$ in LV case is treated as constant background field, while $-B_a$ generated by curvature couplings is spacetime dependent though in some cases can be treated as semi-classical background. Second, $b_\mu$ is a CPT odd LV coupling treated as an unaltered constant under CPT transformation, while $-B_a$ is generated from gravitational sources and should transform in the same way as the ordinary matter field under CPT, thus makes the corresponding operator CPT invariant, which could be easily seen from the fact that gravitational interaction respect CPT symmetry. Further more, though $-B_a$ breaks global Lorentz symmetry, it respects local Lorentz symmetry automatically in an appropriate free fall inertial frame, thus it is actually a local Lorentz invariant term. The last point is that, $-B_a$ is a universal coupling reflecting a curved spacetime effect provided we insist on the equivalence principle to be still hold in quantum-gravity interplay region, while there is no good reason to regard $b_\mu$ as universal. So experimentally we could distinguish
the two with different physical meanings by the effects produced by nonvanishing \( \delta b_i = b^i_j - b^j_i \) (where \( i \) and \( j \) refer to different flavors).

### 4.2. CPT Even

Next we turn to CPT even LV corrections to conventional Dirac equation and derive the corresponding dispersion relations.

For completeness, we rewrite the CPT even Dirac Lagrangian below

\[
\mathcal{L}_{\text{electron}}^{\text{even}} = \frac{i}{2} \overline{\psi}(\eta_{\mu\nu} + c_{\mu\nu} - d_{\mu\nu} \gamma_5) \gamma^\mu \overrightarrow{\partial}^\nu \psi - \overline{\psi}(m + \frac{1}{2M} \tilde{\Delta}_{\mu\nu} \tilde{\sigma}^{\mu\nu}) \psi,
\]

where we replace \( H_{\mu\nu} \) by \( \frac{1}{M} \tilde{\Delta}_{\mu\nu} \), i.e.

\[
\tilde{\Delta}_{\mu\nu} = H_{\mu\nu} \times M. \tag{36}
\]

The meaning of this replacement will be clear later. Then we write down the corresponding field equation by setting \( g_{\lambda\mu\nu} \) and \( b_i \) equal to zero in (15), that is

\[
[i(\gamma_{\nu} + c_{\mu\nu} \gamma^\mu + d_{\mu\nu} \gamma_5 \gamma^\mu) \overrightarrow{\partial}^\nu - (m + \frac{1}{2} H_{\mu\nu} \tilde{\sigma}^{\mu\nu})] \Psi(x) = 0. \tag{37}
\]

Multiplying (37) on the left with \([i(\gamma_{\nu} + c_{\mu\nu} \gamma^\mu + d_{\mu\nu} \gamma_5 \gamma^\mu) \overrightarrow{\partial}^\nu - \frac{1}{2} H_{\mu\nu} \tilde{\sigma}^{\mu\nu} + m]\), we get the following equation

\[
\{ -G_{\mu\nu} G_{\rho\nu} - d_{\rho\nu} d_{\nu}^\rho - 2i \sigma_{\rho\sigma} \gamma_5 G_{\rho\nu} + d_{\rho\nu}^\sigma \} \partial^\mu \partial^\nu + i \epsilon_{\alpha\rho\sigma\beta} [\gamma_5 G_{\rho\nu} + d_{\rho\nu}^\sigma] \overrightarrow{\partial}^\mu \partial^\nu + \frac{1}{4} [ \epsilon_{\alpha\rho\sigma\beta} \gamma_5 H^{\alpha\rho} H^{\sigma\beta} + 2H^2] - m^2 \} \Psi(x) = 0. \tag{38}
\]

In deriving this equation, we use the anti-commutation relations

\[
\{ \gamma_{\nu}, \sigma_{\rho\sigma} \} = \{ \gamma_{\nu}, \frac{i}{2} [\gamma_{\rho} \gamma_{\sigma} - \gamma_{\sigma} \gamma_{\rho}] \} = -2 \epsilon_{\nu\rho\sigma\alpha} \gamma_5 \gamma^\alpha, \tag{39}
\]

\[
\{ \sigma_{\rho\sigma}, \sigma_{\mu\nu} \} = 2i \epsilon_{\rho\sigma\mu\nu} \gamma_5 + 2(\eta_{\mu\nu} \eta_{\sigma\rho} - \eta_{\nu\rho} \eta_{\sigma\mu}), \tag{40}
\]

which can be proven by direct calculation. Note that, for aethetical consideration, we have retained the term \(-2i \sigma_{\rho\sigma} d_{\rho\nu}^\mu \gamma_5 \gamma^\mu \partial^\mu \partial^\nu + 2i \epsilon_{\nu\rho\sigma\alpha} \gamma_5 \gamma^\alpha \partial^\mu \partial^\nu \) in \(-2i \sigma_{\rho\sigma} d_{\rho\nu}^\mu [\gamma_5 (c + \eta) \sigma_{\nu}^\alpha + \sigma_{\nu}^\sigma] \partial^\mu \partial^\nu \), which vanishes automatically for antisymmetric properties of \( \sigma_{\rho\sigma} \). In (38) we use \( G_{\mu\nu} \) defined in (20) instead of \( c_{\mu\nu} \), and this definition is triggered by the observation that the Minkowsky metric \( \eta_{\mu\nu} \) is always followed by \( c_{\mu\nu} \).

Eq. (38) shows that, it is hard to be diagonalized in 4-spinor space by the squaring procedure we used before. This is due to the entanglement between nondiagonal terms involving \( d_{\mu\nu} \) and \( H_{\mu\nu} \). So we can derive the modified dispersion relation by assuming \( d_{\mu\nu} = 0 \) or \( H_{\mu\nu} = 0 \) respectively.

When \( H_{\mu\nu} = 0 \) in (38), it leads to equation

\[
\{ [\eta_{\mu\nu} + c_{\mu\nu} \gamma_5 + 2c_{\mu\nu} - d_{\mu\nu} d_{\nu}^\rho \gamma_5 \gamma^\mu + d_{\rho\nu}^\sigma \} [\eta_{\alpha\beta} + c_{\alpha\beta} c_{\sigma\beta} + 2c_{\sigma\beta} d_{\sigma\beta} \gamma_5 \gamma^\mu \partial^\mu \partial^\nu + m^2] + 4 [d_{\rho\sigma} d_{\rho\nu} (c + \eta) \sigma_{\nu}^\alpha (c + \eta) \gamma_5 \gamma^\alpha \gamma_5 \gamma^\mu \partial^\mu \partial^\nu \partial^\alpha \partial^\beta \} \Psi(x) = 0. \tag{41}
\]
This remains to be a complicated equation. By using Ansatz (27) we can get a quartic-order dispersion relation

\[
\begin{align*}
&[(G_\rho^{\mu}G_\nu^{\rho} - d_{\rho\mu}d_{\sigma\nu})p^{\mu} p^{\nu} - m^2][(G_\sigma^{\rho}G_\beta^{\rho} - d_{\rho\sigma}d_{\beta\rho})p^{\alpha} p^{\beta} - m^2] \\
&+ 4(d_{\rho\mu}d_{\rho\nu}G_\sigma^{\rho}G_\sigma^{\nu}G_\sigma^{\beta} - d_{\rho\mu}d_{\rho\nu}G_\sigma^{\rho}G_\gamma^{\nu})p^{\mu} p^{\nu} p^{\alpha} p^{\beta} = 0,
\end{align*}
\]

(42)

and this is still a complicated equation. We can analyze the role of \(d_{\mu\nu}\) and \(c_{\mu\nu}\) separately in (42) by setting the opposite term equal to zero respectively.

Setting \(d_{\mu\nu} = 0\) corresponds to

\[
\begin{align*}
&[(\eta_{\mu\nu} + c_{\rho\mu}c_\nu^{\rho} + 2c_{\mu\nu})p^{\mu} p^{\nu} - m^2] = (G_{\rho\mu}G^{\rho}_{\nu} p^{\mu} p^{\nu} - m^2) \\
&\equiv (\tilde{G}_{\mu\nu} p^{\mu} p^{\nu} - m^2) = 0,
\end{align*}
\]

(43)

where at the last step we define \(\tilde{G}_{\mu\nu} \equiv G_{\rho\mu}G^{\rho}_{\nu}\). This definition makes (43) looking like a formally relativistic dispersion relation, except with Minkowsky metric \(\eta_{\mu\nu}\) replaced by \(\tilde{G}_{\mu\nu}\). From (43) and the definition (20) we see that \(c_{\mu\nu}\) behaves like a fluctuation of metric.

While setting \(c_{\mu\nu} = 0\) in (42) corresponds to

\[
[(p^2 + m^2 + d_{\rho\mu}d_{\nu\rho}p^{\mu} p^{\nu})^2 - 4(m^2 p^2 + (d_{\rho\mu}p^{\rho} p^{\mu})^2)] = 0,
\]

(44)

which is a quartic order equation, but could be solved formally by using triangle parametrization with

\[
\begin{align*}
X^2 &\equiv 4m^2 p^2, \\
Y^2 &\equiv 4(d_{\rho\mu}p^{\rho} p^{\mu})^2, \\
Z^2 &\equiv (p^2 + m^2 + d_{\rho\mu}d_{\nu\rho}p^{\mu} p^{\nu})^2.
\end{align*}
\]

(45)

Thus \(X^2 + Y^2 = Z^2\), which is just the identity of (44), and set

\[
Y = Z \sin[\theta], \quad X = Z \cos[\theta],
\]

(46)

with \(\theta \ll 1\) since \(d_{\mu\nu}\) constrained by experiment must be small.

When \(d_{\mu\nu} = 0\) in (38), it leads to equation

\[
\begin{align*}
&\{[(\eta_{\mu\nu} + c_{\rho\mu}c_\nu^{\rho} + 2c_{\mu\nu})\partial^{\mu}\partial^{\nu} - \frac{1}{2}H^2 + m^2]^2 + \\
&[\epsilon_{\alpha\beta\gamma\delta}c_{\gamma\varsigma\xi}H^{\alpha\beta}H^{\gamma\xi}(\frac{1}{16}H^{\gamma\delta}H^{\nu\delta} - \eta^{\gamma\delta}G_\gamma^{\rho}G_\varsigma^{\sigma}G_\xi^{\nu}\partial^{\rho}\partial^{\sigma})]\} \Psi(x) = 0,
\end{align*}
\]

(47)

where \(H^2 = H^{\varsigma\eta}H_{\varsigma\eta}\).

When replacing \(i\partial^{\mu} \rightarrow p^{\mu}\) in (47), we can solve the corresponding quartic order equation in the quadratic form

\[
\begin{align*}
p^2 + (c_{\rho\mu}c_\nu^{\rho} + 2c_{\mu\nu})p^{\mu} p^{\nu} + \frac{1}{2}H^2 - m^2 = 0,
\end{align*}
\]

(48)

\[
\epsilon_{\alpha\beta\gamma\delta}c_{\gamma\varsigma\xi}H^{\alpha\beta}H^{\gamma\xi}(\frac{1}{16}H^{\gamma\delta}H^{\nu\delta} + \eta^{\gamma\delta}G_\gamma^{\rho}G_\varsigma^{\sigma}G_\xi^{\nu}\partial^{\rho}\partial^{\sigma}) = 0,
\]

(49)
where (49) should be interpreted as a constraint equation on $H$. It is possible to separate (48) from (47) only in the case of the assumption that (49) is semi-positive defined. This assumption is satisfied definitely when the vector defined below

$$J_\beta \equiv \frac{1}{2} \epsilon_{\alpha \rho \sigma \beta} H^{\alpha \rho} G^{\sigma \nu} p^\nu$$

(50)
is a timelike or lightlike vector, which can be seen from the expression (49). In equation (49) we see $H$ appears with a totally antisymmetric tensor $\epsilon_{\gamma \zeta \eta \delta}$, its appearance reminisces us the anomaly expression in the presence of gauge field, that is

$$\mathcal{A}(x) = -\frac{1}{16\pi} \epsilon_{\mu \nu \rho \sigma} F_{\mu \nu}^a(x) F_{\rho \sigma}^a(x) \text{tr}[t_\alpha t_\beta]$$

So we guess the physical effects produced by $H^{\mu \nu}$ defined in (36) (rather than $H^{\mu \nu}$) resemble that of electromagnetic field. This is indeed confirmed by the operator $-\frac{1}{2M} \bar{\psi} \gamma^\mu \gamma^\nu \psi$ in the Lagrangian (35), which is nothing but the “Pauli term” appeared in the Lagrangian form. This term could give an additional contribution to fermion magnetic moment (anomalous magnetic moment), thus could be constrained by muon "g-2" experiment.

5. Applications

Since we have already derived a set of modified dispersion relations induced by various LV couplings, we can see what novel physical consequences these relations can lead to. It is well known that $E^2 = \vec{p}^2 + m^2$ is a fundamental equation in conventional physics, so modification of this equation is expected to have a wide impact on high energy physics at which LV effects are expected to be less suppressed than at low energies. Indeed, introducing even minuscule LV would lead to processes conventionally forbidden at high energies, or accumulating unexpected observable effects when particles propagate through cosmological distance, or even lead to some processes allowed at intermediate energy range while forbidden at higher and lower energies. For example, radiative muon decay $\mu \to e + \gamma$, neutron stability, at ultrahigh energies, and vacuum dispersion and birefringence, vacuum photon splitting, and photon decay, etc.

However, for the dispersion relations we derived, there are two points to be clarified. Firstly, there is a preferred frame in which each relation has a most simplified form. So when we use this form of relation, we implicitly presume that a preferred inertial frame has been chosen. Secondly, as we previously commented, the LV couplings involved are redefined couplings, while this redefinition only works properly in the absence of interactions. So, strictly speaking, these relations are only applicable to propagating problems where particles involved could be considered as free fermions, though no obstruction would meet in deriving a dispersion relation from the interaction equation, where partial derivatives are replaced by covariant derivatives and radiative corrections are include to calculate a complete propagator. Then, since CPT-odd equation resembles the covariant Dirac equation with spin-gravity coupling involved, which had been extensively discussed elsewhere, and CPT-odd operator grows with energy increase much slower than CPT-even one,
which has been discussed in the end of section II, we focus our attention on CPT-even couplings and discuss the implication of the corresponding MDR in neutrino and ultrahigh energy cosmic ray problems.

5.1. Neutrino Oscillation

Neutrino oscillation might be the only definite signal indicating physics beyond SM, and has been extensively discussed in the literature. While most of them focus on neutrino with mass nondegenerate scenario and use Dirac equation as a starting point, which makes the assumption of neutrino classification unclear. In our derivation, we assume neutrinos to be Majorana spinors from the beginning, thus it is suited to be described in a 2-spinor formalism. As this assumption indicates, we should first reduce the Dirac equation into a 2-spinor form. Beginning with (37) by ignoring $H_{\mu \nu}$ for simplicity, we rewrite the equation as

$$i(\gamma_{\nu} + c_{\mu \nu} \gamma^\mu + d_{\mu \nu} \gamma^5 \gamma^\mu) \overleftrightarrow{\partial^\nu} - m] \Psi(x) = 0.$$  \hfill (51)

For convenience, we redefine these LV couplings in a manifest V-A form resembling the V-A theory, which is a low energy effective field theory of electroweak theory. The definition is

$$g_{L}^{\mu \nu} \equiv (c - d)_{\mu \nu}, \quad g_{R}^{\mu \nu} \equiv (c + d)_{\mu \nu}.$$  \hfill (52)

With this definition, equation (51) could be written as

$$i(\gamma_{\nu} + g_{L}^{\mu \nu} \frac{1 - \gamma^5}{2} \sigma_{\mu} + g_{R}^{\mu \nu} \frac{1 + \gamma^5}{2} \gamma^\mu) \overleftrightarrow{\partial^\nu} - m] \Psi(x) = 0.$$  \hfill (53)

Using the projection operator $\frac{1 + \gamma^5}{2}$ and the definition of Majorana spinor $\Psi \equiv \Psi^c = \mathcal{C} \Psi^\dagger$, we can derive from (53) the corresponding equation

$$i(\sigma_{\nu} + c_{\mu \nu} \sigma^\mu) \overleftrightarrow{\partial^\nu} \phi - im\sigma^2 \phi^* = 0$$  \hfill (54)

satisfied by Majorana 2-spinors, where $\sigma^\mu \equiv (-1, \sigma)$ and $\phi$ is the reduced wave function. For details, see Appendix. Rewrite (54) in the form of Schrödinger equation

$$i \frac{\partial}{\partial t} \phi = \frac{1}{i} \sigma \cdot \overleftrightarrow{\nabla} \phi - ic_{\mu \nu} \sigma^\mu \overleftrightarrow{\partial^\nu} \phi,$$  \hfill (55)

where we already assumed neutrino to be massless fermion, as our derivation of neutrino oscillation will not be based on mass nondegenerate scenario. Note that $\phi$ is a simple notation of a column of 2-spinors, and in our case, we consider only two flavors as an illustration, so $\phi \equiv \left( \begin{array}{c} |\nu_{\mu}\rangle \\ |\nu_{\tau}\rangle \end{array} \right)$, and the corresponding Hamiltonian

$$\hat{H} = \frac{1}{2} \overleftrightarrow{\sigma} \cdot \overleftrightarrow{p} - ic_{\mu \nu} \sigma^\mu \partial^\nu$$

should be regarded as a $2 \times 2$ matrix operator in flavor space. Using the Ansatz $\phi(x) \equiv \phi(p) \exp[-ip \cdot x]$, we can write the Hamiltonian in momentum space as

$$\hat{H} = \begin{pmatrix} \overleftrightarrow{\sigma} \cdot \overleftrightarrow{p} - (c_{\mu \nu})_{11} \sigma^\mu \ p^\nu & -(c_{\mu \nu})_{12} \sigma^\mu \ p^\nu \\ -(c_{\mu \nu})_{21} \sigma^\mu \ p^\nu & \overleftrightarrow{\sigma} \cdot \overleftrightarrow{p} - (c_{\mu \nu})_{22} \sigma^\mu \ p^\nu \end{pmatrix}.$$  \hfill (56)
We can diagonalize this Hamiltonian by a rotation matrix $R$. Choose a specific reference frame in which rotation invariance still holds, then we can assume $(c_{ij})_{\alpha\beta} = k_{\alpha\beta}\delta_{ij}$ (where $i$ and $j$ run over 1 to 3, $\alpha$ and $\beta$ run over 1 to 2), and that all the other terms are zero. Since Lorentz violation is stringently restricted to be tiny, $k_{\alpha\beta} \ll 1$, we can simply drop the diagonal terms of $(c_{\mu\nu})_{\alpha\beta}$ in (56), and assume $k_{12} = k_{21} = k$. Then we can get the corresponding eigenvalues of Hamiltonian (56) as

$$\lambda_1 = 1 + k, \quad \lambda_2 = 1 - k,$$

and the corresponding rotation matrix

$$R \equiv \begin{pmatrix} \cos[\theta] - \sin[\theta] \\ \sin[\theta] \cos[\theta] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

which just corresponds to set the rotation angle in (58) as $\theta = \frac{\pi}{4}$. Then we can get the relation between energy eigenvector $R\phi$ with $\phi$

$$(|\nu_\mu\rangle, |\nu_\tau\rangle) = \left( \begin{pmatrix} \cos[\theta] \\ \sin[\theta] \end{pmatrix}, \begin{pmatrix} \sin[\theta] \\ -\sin[\theta] \cos[\theta] \end{pmatrix} \right).$$

For a muon-type neutrino emitted, the neutrino state after evolving through a time interval $t$ is determined by

$$|\nu(t)\rangle = (\sin[\theta] \exp[-i\delta E \cdot t]|\nu_2\rangle + \cos[\theta]|\nu_1\rangle) \exp[-iE_1t]$$

$$= \frac{1}{2} \sin[2\theta](\exp[-i\delta E \cdot t] - 1)|\nu_\tau\rangle$$

$$+ (1 + \sin[\theta]^2(\exp[-i\delta E \cdot t] - 1))|\nu_\mu\rangle) \exp[-iE_1t],$$

so the flavor transition probability is given by

$$P_{\mu \rightarrow \tau}(t) = |\langle \nu_\tau|\nu(t)\rangle|^2 = \sin^2[2\theta] \sin^2[\frac{\delta E \cdot t}{2}]$$

$$\simeq \sin^2[2\theta] \sin^2[\frac{|\vec{p}|k \cdot t}{2}],$$

where at the last step we used (57) and $\delta E = E_2 - E_1 = -2k|\vec{p}|$. Including the mass terms in (55) and (56) just complicates our formula without any principled difficulty. Note that the presence of nondiagonal tensor couplings ($(c_{\mu\nu})_{\alpha\beta} \neq 0$ for $\alpha \neq \beta$) in flavor space is essential for this Lorentz violation induced neutrino oscillation scenario. Though this is a rather simple model to illustrate neutrino oscillation caused by tiny Lorentz violation, we can still gain some insight by comparing it with experiments. The MINOS experiments reported their oscillation fit results with $\sin^2[2\theta_{23}] > 0.84$ with 90% confidence level and $\Delta m^2_{23} = 2.38^{+0.20}_{-0.16} \times 10^{-3}$eV$^2$ with 68% confidence level and they analyzed the data with the same two flavor $\nu_\mu \rightarrow \nu_\tau$ oscillation assumption. We find that the oscillation angle $\theta = \frac{\pi}{4}$ is consistent with MINOS results, even very close to it. From the squared mass difference ($\sim 10^{-3}$eV$^2$) and the robust bound from cosmology on the sum of neutrino mass, we can estimate the neutrino mass to be around 0.1 eV order. Then Lorentz violation
coupling would contribute to an effective mass term as can be seen from (54) when neutrino is significantly energetic for $k \cdot |\vec{p}| \sim m_{\text{neutrino}}$. In the MINOS neutrino experiment, the peak in neutrino energy spectrum is around 3 GeV (low-energy beam) to 7.8 GeV (high-energy beam) \textsuperscript{20}, so we can give a rather rough bound on the size of Lorentz violation coupling $k \sim \frac{m_{\text{neutrino}}}{|\vec{p}|} \sim \frac{0.1 \text{eV}}{1 \text{GeV}} \sim 10^{-10}$, if Lorentz violation contribution to neutrino oscillation in MINOS experiment is comparable to the non-degenerate mass contribution. In principle, this bound could be restricted to more stringent accuracy of order $k \sim 10^{-22} \sim \frac{\Delta m_{23}^2}{E^2}$ by dimensional analysis. Since this estimate of the size of Lorentz violation coupling is just the inverse of $\gamma (= \frac{E}{m})$ factor of the high energy neutrino and depends on the assumption of comparably contribution of Lorentz violation, to obtain more accurate estimate of the order of this LV coefficient we need to take into account the mass effect (i.e., (54) is used) and give a more reasonable weight on Lorentz violation contribution by using experiment data (e.g., $\Delta m_{23}^2$) directly or even by matching the whole energy spectrum. We noticed that some more comprehensive works \textsuperscript{30} have already been done in the three flavor case which involves the whole renormalizable LV operators ($c, d, a, b, e, f, g$, and $H$), though this was done under some reasonable perturbative expansion (since Lorentz violation correction would be tiny) in order to get the required effective Hamiltonian which controls neutrino propagation effects. In their first paper \textsuperscript{30}, a general framework in the LV induced neutrino oscillation was given and some definite signals in experimental searches for Lorentz violation in neutrino sector were classified and examined. Their subsequent works focused on particular models where the number of nonzero LV parameters were reduced significantly \textsuperscript{30}. In this sense, our work is just a illustration or toy-model, however, its simplicity makes the oscillation mechanism induced by tiny Lorentz violation more obviously and the assumption of the neutrino property (Majorana neutrino) more apparently. Further it could be generalized to more practical model (3-flavors) directly by taking into account mass terms since the pure Lorentz violation (i.e., massless neutrino case) model may not be a practical solution to globally fit all neutrino oscillation data from solar, reactor and atmosphere neutrino experiments \textsuperscript{21}. However, whether the generalized form could accommodate with the experiment data or not still lacks checking.

Some remarks should be said about the transition probability (61) which is proportional to neutrino energy, instead of inverse proportional to it as in the case of mass nondegenerate scenario. This property is the common feature of all non-standard neutrino oscillation scenarios and reminisces us the neutrino oscillation induced by equivalent principle violation \textsuperscript{22}. The formula of which is exactly the same as (61), except replacing $2k$ with $\frac{h_{00}}{\rho}$, where $h_{00} = -2\phi = \frac{2GMa}{R}$ is the 00-component of metric fluctuation, and $\alpha$ is the post Newtonian parameter (in general relativity, $\alpha = 1$ and is universal). This similarity is not an accident, since Lorentz violation is assumed as a remnant of quantum gravity, and in quantum region there is some indication that equivalence principle is violated. We guess that
the equivalence principle violation may indicate Lorentz violation at least locally, as the equivalence principle ensures the existence of local inertial frames, which is the premise of local Lorentz transformation. Actually, Lorentz violation must be followed by equivalence principle violation, an issue recently clarified in [28]. Furthermore, we can see from (52) that nonvanishing $d_{\mu\nu}$ gives rise to different couplings to left and right handed Dirac fermions, so it may induce observable effects in energy splitting between different helicities.

5.2. Ultrahigh Energy Cosmic Ray

Ultrahigh energy cosmic ray (UHECR) provides a natural source of high energy particles, with energies up to $10^{19}$ eV, much higher than that of energetic particles generated by man-made accelerator. But the energy of UHECR reached earth can not be much higher than that, as it is predicted to be terminated at around $5 \times 10^{19}$ eV for the energy lose in the collision of UHECR particle with CMB photons by Greisen, Zatsepin and Kuz’m [24], which is known as the GZK cutoff. Similar situation happens in the collision of TeV γ ray with infrared photons. However, this prediction has not yet been confirmed by experiments. AGASA, Fly’s Eye both claimed that they observed events with energies nearer or above this cutoff [25], while HiRes [26] and Pierre Auger [27] experiments recently claimed the observation of the cutoff. This unsettled problem has stimulated many attempts to resolve it, including active galactic nuclei (which is favored by Pierre Auger experiment), primary flux of magnetic monopoles, “Z-boson bursts” produced by collision of UHE neutrino with relic neutrino nearby, pseudo-complex extension of standard field theory [29] etc. Of course, LV also provides a possible candidate to extend or entirely rule out this cutoff [10]. In this paper, we follow the general analysis of Coleman and Glashow [10] and show that the LV coupling in modified dispersion relation (43) could be constrained either from the absence or the presence of GZK cutoff.

We take the common assumption that the identity of UHECR are protons, and analyze the pion-nucleon resonance formation reaction $P + \gamma (\text{CMB}) \rightarrow \Delta (1232)$, which is the dominant contribution to GZK cutoff. This reaction is possible if and only if $E_0 \geq E_{\text{min}}(\vec{P}_0)$, where $E_0$ is the total energy of initial particles and $E_{\text{min}}(\vec{P}_0)$ denotes the minimum total energy of the final products, whose total momentum is equal to initial total momentum $\vec{P}_0$, which is implicitly assumed from energy momentum conservation. So the reaction is kinematically allowed by the condition

$$\omega + E_p \geq E_\Delta,$$

where $\omega$ is the energy of CMB photon, $E_p$ and $E_\Delta$ are the energies of proton and $\Delta$ resonance, with the subscripts denoting proton and $\Delta$ resonance respectively. We rewrite dispersion relation (43) in a relativistic form

$$E_a^2 = \vec{P}_a^2 + m_a^2,$$
where $c_a$ is the maximal attainable velocity for the $a$th particle defined in \cite{10}. In the case we considered, it just requires the definition below

$$c_a \equiv \frac{1}{1 + c_{\mu_0}^0}, \quad (64)$$

where $c_{\mu_0}^0$ denotes the 00 component of $c_{\mu\nu}$ for the $a$th fermion, and other components of which are assumed to be zero. We rewrite (62) in terms of $E_p$ and $\omega$,

$$\omega + E_p \geq \sqrt{(|\vec{P}_p| - \omega)^2 c_\Delta^2 + m_\Delta^2 c_\Delta^4}. \quad (65)$$

For threshold reaction, the initial proton momentum $\vec{P}_p$ is collinear with that of $\Delta$ resonance while anticollinear with that of CMB photon, which has already been used in (65), i.e., the substitution of $\vec{P}_\Delta = \vec{P}_p - \omega$. Squaring (65), we have

$$E_p^2 (1 + \frac{c_\Delta}{c_p})(1 - \frac{c_\Delta}{c_p}) + 2\omega (E_p + |\vec{P}_p| c_\Delta^2) + \omega^2 (1 - c_\Delta^2)$$

$$+ \left( m_p^2 - m_\Delta^2 \frac{c_\Delta^2}{c_p^2} c_\Delta^2 \right)^2 \geq 0. \quad (66)$$

For ultrahigh energy proton: $E_p \sim |\vec{P}_p|$. As LV coupling should be much smaller than 1, we take the approximation that $c_\Delta \sim 1$ and $1 + \frac{c_\Delta}{c_p} \sim 2$. Substituting this approximation into (66) leads to a quadratic order inequality of $E_p$

$$2E_p^2 (1 - \frac{c_\Delta}{c_p}) + 4\omega E_p + K \geq 0, \quad (67)$$

where

$$K \equiv \left( m_p^2 - m_\Delta^2 \frac{c_\Delta^2}{c_p^2} c_\Delta^2 \right)^2 \geq 0. \quad (68)$$

Thus the threshold energy that the reaction kinematically allows is the small positive root of (67) when equality is hold. With the assumption

$$1 - \frac{c_\Delta}{c_p} > 0, \quad (69)$$

and quadratic order equation (67) (with equality hold), we obtain two roots with opposite signs. One is negative but with larger absolute value, the other is positive which gives the threshold energy, i.e., pion-nucleon resonance formation reaction is kinematically allowed only for energy above this positive value. On the other hand, with the assumption

$$1 - \frac{c_\Delta}{c_p} < 0, \quad (70)$$

we obtain two positive values. The small one is the threshold energy, while the larger one is the terminating energy of this reaction, which means that the formation reaction is kinematically allowed in an intermediate energy band. This is a striking feature of Lorentz violation corrections to the familiar particle reaction process previously referred and was systematically discussed in \cite{10}, so GZK problem in this
case provides one concrete illustration of the analysis in \[^{10}\]. However, both cases give the same threshold formula, that is

\[
E_p = \frac{\omega \sqrt{1 - 1/2 \omega^2 K^2} (1 - \frac{c_{\Delta}}{c_p}) - \omega}{1 - \frac{c_{\Delta}}{c_p}}
\]

\[
\simeq - \frac{K}{4\omega} - \frac{K^2}{32\omega^3} (1 - \frac{c_{\Delta}}{c_p}) + \ldots,
\]

(71)

where the first term at the last step is the conventional threshold energy in the absence of LV, and the other terms are tiny LV corrections. Substituting \(E_{\text{thre}} = -\frac{K}{2\omega} = 5 \times 10^{19} \text{ eV}^{24}\) and the experimentally observed threshold energy \(E_p = 5.6 \times 10^{19} \text{ eV}^{25}\) into (71), we can deduce the bound

\[
1 - \frac{c_{\Delta}}{c_p} = - \frac{2w(E_p - E_{\text{thre}})}{E_{\text{thre}}^2}.
\]

(72)

Since \(\Delta\) resonance is not a spin-1/2 fermion, we simply assume \(c_{\Delta} = 1\), and substituting (64) into (72) yields the bound on LV coupling \(c_{00} \sim 10^{-25}\), which is more stringent than that derived in \[^{10}\] by two orders of magnitude. This is consistent with our expectation since we adopt the data confirming GZK cutoff. Note that this bound is of importance only in the sense of order of magnitude, since it cannot be fixed firmly from the location of GZK cutoff alone, many other effects could compensate for some amount of Lorentz violation, for example, the uncertainty of source distribution. However, the bound we obtained is strict enough and has already reached the Planck mass suppression sensitivity \(10^{-23}\), which indicates that Lorentz violation in dimension 4 operators (for proton-LV tensor coupling) is indeed too minuscule to be detected.

6. Summary

Searching for Lorentz violation (LV) experimentally or theoretically has received much attention in recent years. As QED has been tested to a marvelous accuracy, it is expected to be an ideal research area to probe the minuscule trace of LV both in theory and in experiment. In this paper we studied various modified dispersion relations (MDR) derived from extended QED by the assumption that a particular set of LV couplings is nonzero. This is a reasonable assumption since if a fundamental theory really violates Lorentz symmetry, the corresponding tensor coupling in low energy effective theory should contain less parameters than what we presented here. On focusing these LV couplings in MDR or extended Dirac equation, we observe the similarity of some LV couplings with the spin-gravity couplings or metric couplings in covariant Dirac equation. This resemblance may indicate a deep physical relevance of these LV couplings with that in the quantum-gravity interplay region, since Lorentz violation is assumed as a remnant signal of quantum gravity.

In addition to that, this similarity has also been observed in the application to the neutrino sector. We also derived an oscillation formalism by explicitly assuming
neutrino as Majorana spinor. We found that the nondiagonal terms of LV couplings in flavor space involved in neutrino sector could explain neutrino oscillation even in the massless case. Though this possibility has been comprehensively discussed by several authors, and even have already been partly tested in some experiment such as LSND, it is the first time, as far as we known, to derive the oscillation by explicitly assuming neutrino as Majorana spinor in 2-spinor formalism, thus our model could be viewed as a toy-model simply demonstrating neutrino oscillation induced by Lorentz violation. We also made a rough estimate on the size of LV couplings involved as \( k \sim 10^{-10} \). We note that the LV couplings for different species of fermions involved in specific problems are defined independently since we cannot calculate them from an underlying concrete model displaying this LV effective Lagrangian as the low energy limit after spontaneous Lorentz symmetry breaking. In principle this oscillation formalism can be generalized to the 3 flavor case by taking into account the mass effect. Then by comparing it with the neutrino energy spectrum obtained experimentally rather than only with the mixing angle and mass square difference, we expect that more accurate bounds on LV couplings could be obtained in neutrino sector.

By application of MDR to GZK problem, we derived a much stringent bound on the order of magnitude of LV parameters of protons (\( c_p^0 \sim 10^{-25} \)) from the recent observation in HiRes and Pierre Auger experiments and we note that more bounds could be obtained on neutrons by taking account of some exotic process (such as proton vacuum Cerenkov radiation) in the analysis of the propagation of ultrahigh energy cosmic rays when Lorentz violation is present. There are also many stringent bounds on the magnitude of various LV parameters up to date. We observe that most of these bounds are actually the bounds on the difference of LV parameters to different species of particles, as in the case of nonuniversal gravity coupling induced neutrino oscillation or the maximal attainable velocity analysis developed in. They just indicate (from the opposite side) that the difference of Lorentz violation tensor couplings to different species of particles is rather small, in other words, they strongly suggest that the tensor field triggering Lorentz violation in the underlying theory couples to the standard model field universally, at least for dimension 3/4 operators. So whether Lorentz symmetry is just a perfectly good approximate symmetry is still an open question.

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Appendix A.

To get an equation describing Majorana 2-spinor in (54), we begin with (53) describing Dirac 4-spinor. Actually, we could use 2-spinor or 4-spinor formalism to describe either Dirac or Majorana fermions. However, for a Majorana fermion the independent degrees of freedom are 2, thus not all of its 4 components in 4-spinor formalism are independent, while for a Dirac fermion, it needs at least 2 different 2-spinors. So for a Majorana fermion, it is adequate to be described by 2-spinor formalism, while 4-spinor formalism is suitable to describe a Dirac fermion. In order to get the suited equation describing Majorana 2-spinor from (53), we need to use the projection operator \[ \frac{1}{2} \left( \gamma^5 \right) \] and the Majorana spinor definition \( \Psi^c \equiv \Psi^T \) to project equation (53) from \( 4 \times 4 \) matrix space to the irreducible \( 2 \times 2 \) subspace.

First, we give the projected wave function and \( \Gamma \) matrices below

\[
\Psi^L \equiv \frac{1}{2} \left( 1 - \gamma^5 \right) \Psi, \quad \Psi^R \equiv \frac{1}{2} \left( 1 + \gamma^5 \right) \Psi; \\
\gamma^L \equiv \frac{1}{2} \left( 1 - \gamma^5 \right) \gamma, \quad \gamma^R \equiv \frac{1}{2} \left( 1 + \gamma^5 \right) \gamma.
\]

Using these definitions to rewrite (53) in the form

\[
\left[ i \left( \gamma^R_{\nu} + g^{R}_{\mu \nu} \gamma^R_{\mu} \right) \overrightarrow{\partial^\nu} \Psi^L + i \left( \gamma^L_{\nu} + g^{L}_{\mu \nu} \gamma^L_{\mu} \right) \overrightarrow{\partial^\nu} \Psi^R - m \left( \Psi^R + \Psi^L \right) \right] = 0 \quad (A.3)
\]

and multiplying (A.3) from the left with \( \gamma^5 \), we have

\[
\left[ i \left( \gamma^R_{\nu} + g^{R}_{\mu \nu} \gamma^R_{\mu} \right) \overrightarrow{\partial^\nu} \Psi^L - i \left( \gamma^L_{\nu} + g^{L}_{\mu \nu} \gamma^L_{\mu} \right) \overrightarrow{\partial^\nu} \Psi^R - m \left( \Psi^R - \Psi^L \right) \right] = 0 \quad (A.4)
\]

From (A.3) and (A.4) we can get two independent equations

\[
\left[ i \left( \gamma^R_{\nu} + g^{R}_{\mu \nu} \gamma^R_{\mu} \right) \overrightarrow{\partial^\nu} \Psi^L - m \Psi^R \right] = 0, \\
\left[ i \left( \gamma^L_{\nu} + g^{L}_{\mu \nu} \gamma^L_{\mu} \right) \overrightarrow{\partial^\nu} \Psi^R - m \Psi^L \right] = 0
\]

(A.5)

for left handed and right handed fermions respectively, with the mass term mixing each other. The two above equations are not independent from each other for a Majorana fermion. We can take complex conjugate operation on (53) and charge conjugate operation \( \Psi^c = \overline{\Psi^T} \) on wavefunction to get a charge conjugate equation of the same fermion field

\[
\left[ i \left( \gamma^R_{\nu} + g^{R}_{\mu \nu} \gamma^R_{\mu} \right) \overrightarrow{\partial^\nu} \Psi^L - m \Psi^R \right] = 0, \\
\left[ i \left( \gamma^L_{\nu} + g^{L}_{\mu \nu} \gamma^L_{\mu} \right) \overrightarrow{\partial^\nu} \Psi^R - m \Psi^L \right] = 0
\]

(A.6)

which is just (53) with \( g^{L}_{\mu \nu} \) and \( g^{R}_{\mu \nu} \) interchanged. Since the charge conjugate field for a Majorana field is just the original field multiplied with a phase factor (see (A.7)), the field equation satisfied for the charge conjugate field should be the same, thus impose the condition \( g^{L}_{\mu \nu} = g^{R}_{\mu \nu} \), i.e., \( g^{L}_{\mu \nu} = g^{R}_{\mu \nu} = c_{\mu \nu} \). Then we can use the same procedure above in getting (A.5) to get a set of corresponding equations for (A.6), which is nothing but the same equations of (A.5). Note that for Majorana field, equation (A.5) contains actually just one independent equation, the lower one is...
the equivalent form of the upper one. Using the condition satisfied by Majorana fermion below

\[ \Psi(x) = \eta \Psi^c(x), \quad (A.7) \]

where \( \eta \) is a phase factor (for simplicity, we take it equal to 1), we can get the relation

\[ \Psi^R(x) = \frac{1 + \gamma_5}{2} \Psi^c(x) = -\gamma_0 C \Psi^{\ast L}(x). \quad (A.8) \]

Then substituting (A.8) into (A.5), we get the equation below

\[ i(\eta \nu + c_{\mu \nu} \eta^\mu) \overset{\rightarrow}{\partial^\nu} \phi + m \phi^\ast = 0, \quad (A.9) \]

where

\[ \eta^\mu = C^{-1} \gamma^0 \gamma^R \mu \quad (A.10) \]

\[ \phi(x) = \Psi^L(x). \quad (A.11) \]

Since (A.9) is expressed in \( 4 \times 4 \) matrix space, thus could be reducible. We can find an irreducible representation of \( \eta^\mu \) in \( 2 \times 2 \) matrix space, that is setting \( \eta^\mu = i \sigma^2 \sigma^\mu \), where \( \sigma^\mu \equiv (-1, \overrightarrow{\sigma}) \). So in 2-spinor irreducible space, (A.9) becomes

\[ i(\sigma_\nu + c_{\mu \nu} \sigma^\mu) \overset{\rightarrow}{\partial^\nu} \phi - im \sigma^2 \phi^\ast = 0, \quad (A.12) \]

which is just (54).

We can derive this equation satisfied by Majorana fermion from a more manifest way in displaying its Majorana feature, i.e., rewriting the modified Dirac Lagrangian in 2-component formalism and using the neutrality condition to drop the coupling terms between the two fermions. We first give the 2-component formalism of modified Dirac Lagrangian, which is obtained by expressing

\[ \psi_D = \begin{pmatrix} \chi \\ \eta^\mu \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overrightarrow{\sigma^\mu} & 0 \end{pmatrix} \quad (A.13) \]

in the Lagrangian (35), where \( \sigma^\mu \equiv (-1, \overrightarrow{\sigma}), \overrightarrow{\sigma}^\mu \equiv (-1, \overrightarrow{\sigma}) \). The resulting Lagrangian (only contains Lorentz violating couplings) is

\[ \mathcal{L}_{\text{Dirac}} = -i \eta(c + d)_{\mu \nu} \sigma^\mu \eta^\nu \partial^\nu \chi - i \overrightarrow{\chi}(c - d)_{\mu \nu} \overrightarrow{\sigma^\mu} \partial^\nu \chi \\
+ i H_{\mu \nu}(\sigma^\mu \chi + \chi \sigma^\mu \eta^\nu). \quad (A.14) \]

Note \( \sigma^{\mu \nu} \) appears here is defined to be \( \sigma^{\mu \nu} = \frac{1}{4}(\sigma^\mu \overrightarrow{\sigma^\nu} - \sigma^\nu \overrightarrow{\sigma^\mu}), \) where \( \overrightarrow{\sigma} = \frac{1}{2}(\overrightarrow{\sigma}^\mu \overrightarrow{\sigma}^\nu - \overrightarrow{\sigma}^\nu \overrightarrow{\sigma}^\mu) \), rather then that appeared in the text, which could be relabeled by \( \Sigma^{\mu \nu} = \frac{1}{2}[\sigma^\mu, \eta^\nu] = 2i \begin{pmatrix} \sigma^{\mu \nu} & 0 \\ 0 & \overrightarrow{\sigma^{\mu \nu}} \end{pmatrix} \). For any Dirac spinor \( \psi_D \), we can construct two corresponding Majorana spinors as

\[ \psi_{M1} = \frac{1}{\sqrt{2}}(\psi_D + \psi_D^\ast), \quad \psi_{M2} = \frac{-i}{\sqrt{2}}(\psi_D - \psi_D^\ast), \quad (A.15) \]
where $\psi_D^c = C \overline{\psi_D}^T$ is just the charge conjugate fermion field. Using this equation we can decompose the two Weyl-spinors as two decoupled Majorana spinors in the absence of additional inter-couplings (e.g., LV-couplings) by the deduced equation below

$$\chi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \quad \eta = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2),$$

(A.16)

where $\Psi_{Mi} = \begin{pmatrix} \psi_i \\ \bar{\psi}_j \end{pmatrix}$, $i = 1, 2$.

Then the corresponding Lagrangian (not the full one (35)) is

$$L_{\text{Dirac}} = -ic_{\mu\nu} \sum_{i=1}^{2} (\psi_i \sigma^\mu \partial^\nu \overline{\psi}_i) + d_{\mu\nu}(\psi_1 \sigma^\mu \partial^\nu \overline{\psi}_2 - \psi_2 \sigma^\mu \partial^\nu \overline{\psi}_1) + H_{\mu\nu}(\psi_2 \sigma^{\mu\nu} \psi_1 + \overline{\psi}_2 \sigma^{\mu\nu} \overline{\psi}_1),$$

(A.17)

where we have used the fact $\psi \sigma^{\mu\nu} \psi = 0$. So we can explicitly see the couplings between two Weyl-spinors arising from $d_{\mu\nu}$ and $H_{\mu\nu}$. In the Majorana theory, we can simply drop them. Thus the corresponding full Majorana-Lagrangian in 2-component theory is

$$L_{\text{Majorana}}^{2} = i(\eta_{\mu\nu} + c_{\mu\nu}) \partial^\nu \psi \sigma^\mu \overline{\psi} + \frac{m}{2}(\psi \psi + \overline{\psi} \overline{\psi}),$$

(A.18)

and the corresponding 4-component form is

$$L_{\text{Majorana}}^{4} = \frac{i}{2} \overline{\Psi}(\gamma^\mu + c^{\mu\nu} \gamma^\nu) \partial_{\mu} \Psi - \frac{m}{2} \overline{\Psi} \Psi.$$

(A.19)

From the Majorana Lagrangian (A.18), we can deduce an equation

$$i(\eta_{\mu\nu} + c_{\mu\nu}) \sigma^\mu \partial^\nu \overline{\psi} - m\psi = 0.$$

(A.20)

From the definition $\Psi^{c}_M = \lambda \Psi_M$, where $\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$, and the convention $\lambda = 1$ being chosen (which could be confirmed from equation (A.15)), we can deduce

$$\psi = i\sigma^\nu \overline{\psi}.$$

(A.21)

Substituting this equation to (A.20), we again obtain equation (A.12).

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