Asymmetric regularization of the ground and excited state of the $^4$He nucleus

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Abstract

We find the threshold structure of the two- and three-nucleon systems, with the deuteron and $^3$H/$^3$He as the only bound nuclei, sufficient to predict a pair of four-nucleon states: a deeply bound state which is identified with the $^4$He ground state, and a shallow, unstable state at an energy $B^*_\alpha = [0.38 \pm 0.25]$ MeV above the triton-proton threshold which is consistent with data on the first excited state of the $^4$He. The analysis employs the framework of Pionless EFT at leading order with a generalized regulator prescription which probes renormalization-group invariance of the two states with respect to higher-order perturbations including asymmetrical disturbances of the short-distance structure of the interaction. In addition to this invariance of the bound-state spectrum and the diagonal $^3$H-$p$ $^1S_0$ phase shifts in the $^4$He channel with respect to the short-distance structure of the nuclear interaction, our multi-channel calculations with a resonating-group method demonstrate the increasing sensitivity of nuclei to the neutron-proton $P$-wave interaction. We show that two-nucleon phase shifts, the triton channel, and three-nucleon negative-parity channels are less sensitive with respect to enhanced two-nucleon $P$-wave attraction than the four-nucleon $^3$H-$p$ $^1S_0$ phase shifts.

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I. OVERTURE

The amount of complexity in a system grows with its number of constituents. It is a challenge for any theory to relate phenomena, which are absent in a system of $n$ particles but emerge in a $(n+1)$-particle system, to a set of parameters which characterizes the interaction between only $n$ particles. In nuclear physics, in particular, a useful theory should, at least, yield shallow two-nucleon states, a stable triton ($^3$H) and $^3$He, and a relatively deeply bound $^3$He particle. By now, one understands to construct such a theory without exchange particles, solely with neutron ($n$) and proton ($p$) degrees of freedom (pionless effective field theory [1–4] (EFT($\pi$)). This theory uses the $^3$H binding energy $\mathcal{B}$ as a renormalization condition and succeeds in the postdiction of a stable ground state of the four-nucleon system [5]. It finds the $^3$He particle deeply bound and thereby demonstrates the correlation of a system’s complex behavior with properties of its subsystems. Whether non-bound-state phenomena in complex nuclei are correlated similarly to two and three-nucleon properties or if these constitute genuine many-body properties is unknown.

It is the aim of this work to analyze one of these phenomena of the $\alpha$ nucleus in that light: the first excited 0$^+$ state in the $^4$He spectrum. We deem this observable of particular importance because pairs of a deep ground state and a shallow excited state with identical quantum numbers reoccur in larger nuclei ($^{12}$C, $^{16}$O) while in other systems (e.g., $^5$He, $^8$Be) no stable ground state is sustained below the threshold states. The missing bound ground state might be a hallmark of the fermion substructure of the nuclei which becomes relevant only for the latter, while $^{16}$O, for instance, is amenable to a description in terms of four interacting bosons (the four $\alpha$’s). For $^5$He, however, the analogy to five unitary bosons or even an $\alpha$ interacting with a neutron is erroneous – naively, because of the Pauli principle. Both treatments, as a five- or two-body system demand momentum-dependent interactions for the description of shallow states which are accompanied by an inclusion of radial and/or angular excitations. To relate these non-$S$-wave interactions model-independently to properties of the two- and/or $A$-body interaction is, to our knowledge, an open problem. It is in particular not understood if the insensitivity of low-energy amplitudes such as neutron-deuteron scattering, or the $\alpha$ ground state with respect to $P$-wave components of the two-nucleon (NN) interaction translates to the excited $\alpha$ state, which is a focus of this work.

A systematic approach is given by EFT($\pi$), which is an appropriate theory to analyze possible correlations of these shell-model characteristics to properties of an underlying interaction. At its leading order (LO), it comprises momentum-independent - and thus rotationally invariant in coordinate space - two and three-nucleon interactions. Most regularization schemes in numerical coordinate-space calculations smear the originally point-like nucleons over some volume and thereby induce non-zero matrix elements between nucleons in relative $L > 0$ waves. Although this is irrelevant for two-nucleon observables whose asymptotic states are $S$-wave projections, asymptotic $A > 2$ nucleon states have non-zero overlap with higher-partial-wave states on at least one of the relative coordinates. A renormalization-group (RG) analysis must therefore probe whether such a regulator-induced incorporation of higher-order operators is consistent with the power-counting of EFT($\pi$). All amplitudes which are non-zero because of $P$-wave matrix elements between nucleons must vanish with the removal of the regulator at LO.

As the admixture of higher-partial-wave interactions depends on the specific regularization and the tool which is employed to solve the few-body problem, we introduce a generalized regulator in order to assess the sensitivity of observables with respect to EFT-permitted higher-order contamination in a LO calculation. The regulator is specified by 2 parameters: the customary momentum cutoff $\Lambda$, and a measure $\epsilon$ of the strength by which nucleons are allowed to interact asymmetrically. This prescription allows for a more comprehensive RG analysis when it is impractical to vary $\Lambda$ over a broad range, e.g., if numerical tools are limited to certain cutoff values but can vary the strength of non-central operators with relative ease.

For the problem at hand, we find an excited 0$^+$ state of the $\alpha$ insensitive to an RG analysis in ($\Lambda, \epsilon$) space. This conclusion is based on the observed resemblance of the calculated triton-proton ($^3$H-$p$) phase shifts with data. With an iteration of higher-order operators whose coupling strength is smoothly increased from zero, we affect the attraction in two-nucleon $P$ waves. In this course, another excited state is introduced while the ground- and excited state remain invariant.

We begin with a more detailed motivation of the theoretical problem posed by the excited 0$^+$ $\alpha$ state. An ansatz for a solution is given after that, when we introduce a non-central regulator prescription for the nuclear theory without pions. A presentation of results obtained with this technique in the two, three, and four-nucleon systems follows with an emphasis on their respective sensitivity to unphysical short-distance distortions of the NN potential with a spin-orbit term.
II. THE FOUR-NUCLEON $J^P = 0^+$ CHANNEL

The Thomas collapse, Efimov’s limit cycle, and the Phillips correlation are representatives of the fundamental problem of whether microscopic theories are useful for the prediction of complex features of macroscopic systems. The correlation between the three and four-nucleon ground states is a rare example of an $A$-body phenomenon being constrained by observables which involve less than $A$ particles. Here, we want to study another complex observable in the four-nucleon system in order to further the understanding to what extent non-relativistic, particle-number-conserving theories can be used to predict many-body complexity from few-body properties. Specifically, the binding energy of the $α$ ground state $B_α$, which is large relative to the lowest breakup into a triton with $B_{3H} \approx 8.48$ MeV and a proton, has been related in Ref. [6] numerically to the np scattering lengths $a_s \approx -23.7$ fm, $a_t \approx 5.42$ fm and $B_{3H}$. We ask whether the second $0^+$ state ($B^*_α$) which is located about 0.4 MeV in the $^3\text{H-}p$ continuum is correlated to the same observables [7]. Ref. [8] finds such resonant states in the four-boson system interacting via a two-body interaction with infinite scattering length. The analysis of nuclei entails, in contrast, a large but finite two-body scattering length $a_2$, while the triton can still be thought of as the lowest three-body bound state in a finite neutron-neutron-proton spectrum which is the precursor of the infinite Efimov spectrum to develop for $a_2 \to \infty$.

A shallow bound state emerges naturally if the interaction between $^3\text{H}$ and a proton close to threshold is similar to that of the neutron with a proton close to zero energy: two particles which are treated as point-like on the energy and momentum scales involved and which are subject to very large $S$-wave scattering lengths. Compared with the neutron, the triton is larger and we expect the interaction with the proton to be less affected by the repulsive core. A small increase in the $^1S_0$ attraction would turn the virtual into a bound state and introduce a new virtual pole at $E = -\infty$ [9] which is far outside of the range of applicability of EFT(\#). The original virtual state would become more and more bound and finally settle to form the $α$ ground state while Simultaneously the newly created virtual state will approach threshold. There is no a priori reason why the virtual state should be close to threshold when the ground-state energy is in an EFT-consistent interval around data. Only if the effective $^3\text{H-}p$ interaction which emerges from the two and three-nucleon operators of EFT(\#) exhibits this feature, the theory has a chance to converge to the experimentally found shallow resonance.

III. ASYMMETRIC REGULARIZATION OF A NON-RELATIVISTIC THEORY

In the development of contact field theories for nuclei, cutoff schemes with a regularized delta-function: $\delta^{(3)}_\Lambda (r) \propto e^{-\Lambda^2/4r^2}$, provide an intuitive method to renormalize two-body amplitudes in coordinate space. This spherically-symmetric regulator admits more or less $S$-wave modes in the calculation of an observable. Useful theories are insensitive to additional modes which resolve structure below some radial separation. A general RG transformation would, in addition, assess the effect of small couplings between modes of different relative angular momentum. Such a generalized analysis is unnecessary for the low-energy two-nucleon system because a np $S$-wave state does not couple to higher partial waves at lowest order in EFT(\#). Amplitudes with more than three particles in the asymptotic states inevitably involve relative motions with non-zero angular momentum. A regulator which probes sensitivity with respect to short-distance structure more comprehensively is apt:

$$\delta^{(3)}_{\Lambda, r, \nabla, \sigma_{1,2}} (r) \propto e^{\sum_{i=1,2} \frac{\epsilon_{i}}{\Lambda(r)} \cdot \hat{O}_i (r, \nabla, \sigma_{1,2})} . \quad (1)$$

The general regulator may include an infinite number of higher-order operators $\hat{O}_i$ with increasing mass dimension $n(i)$ and dimensionless constants $\epsilon_i < \infty$. In this form, it satisfies the condition $\lim_{\Lambda \to \infty} \delta^{(3)}_{\Lambda, r, \nabla, \sigma_{1,2}} (r) = \delta (r)$ and respects the $SO(3) \otimes SU(2)$ symmetry of the Hamiltonian through constraints on $\hat{O}_i$. The customary form is $\hat{O} = r^2$ and $\epsilon = -1/4$. In addition, an explicit dependence on the relative coordinate between the interacting particles, $r$, the angular momentum associated with this coordinate, $L = -ir \times \nabla$, and the spin degrees of freedom, $\sigma_{1,2}$, is included here.

*Ref. [7] suggests such a universal, shallow excitation.
Explicitly, we choose
\[ \delta_{\Lambda,L,s_1,s_2}^{(3)}(r) \propto e^{-\frac{\Lambda^2}{2} r^2 + \frac{3}{2} \cdot L \cdot (s_1 + s_2)} \]  
and thereby to analyze sensitivity to spin-orbit distortions because firstly the two-nucleon LECs do not have to be re-calibrated. These LO LECs are fitted to S-wave observables which are unaffected by the \( L \cdot S \) term whose effect on higher partial waves also vanishes for \( \Lambda \to \infty \) because of the node of the radial wave function at zero distance. Second, it is the operator of lowest mass dimension which induces transitions between states of different orbital-angular momentum. In addition, one has to demand a small size of the operator matrix element which characterizes the amplitude of interest. For nuclear states with good total spin, angular momentum, and total momentum, for example, the condition
\[ \epsilon \times \langle 2S+1 L'_j | L \cdot S | 2S+1 L_j \rangle < 1 \]  
must be satisfied in order to use an iterated spin-orbit operator as a regulator. With \( \epsilon \) subject to this system-specific constraint, standard EFT(\( \alpha \)) which show that the proton-proton Coulomb interaction is a perturbation in light nuclear bound states, we set \( c \) to\( \epsilon \) and hence, there must be no contribution to amplitudes from the \( L \cdot S \) term whose effect on higher partial waves also vanishes for \( \Lambda \to \infty \) because of the node of the radial wave function at zero distance. Second, it is the operator of lowest mass dimension which induces transitions between states of different orbital-angular momentum. In addition, one has to demand a small size of the operator matrix element which characterizes the amplitude of interest. For nuclear states with good total spin, angular momentum, and total momentum, for example, the condition
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\[ \hat{H}_{\text{ncl}} = -\sum_i A_i \frac{\nabla_i^2}{2m} + \sum_{i<j} \left[ c_{ij}^S (1 - \sigma_i \cdot \sigma_j) + \epsilon c_{ij}^L (3 + \sigma_i \cdot \sigma_j + \epsilon L_{ij} \cdot S_{ij}) \right] e^{-\frac{\Lambda^2}{2} r_{ij}^2} + \sum_{i<j<k} d_{ijk}^A \left( \frac{1}{2} - \frac{1}{6} \tau_i \cdot \tau_j \right) e^{-\frac{\Lambda^2}{2} (r_{ij}^2 + r_{jk}^2)} . \]  
In this form, the spin-orbit interaction can be understood as an irrelevant operator, whose contribution vanishes because its LEC \( c_{ij}^L \epsilon \) is not renormalized. As for its higher mass dimension, the renormalization of \( c_{ij}^L \) is insufficient, and hence, there must be no contribution to amplitudes from the \( L \cdot S \) term in the limit \( \Lambda \to \infty \). For finite \( \Lambda \) and \( \epsilon \neq 0 \), the short-distance behavior of the EFT is distorted asymmetrically\(^4\). We expanded Eq. (2) for practical reasons and use the linear term of the exponential, only. This expansion introduces an upper bound for the second RG parameter \( \epsilon \) while in the exponent any \( \epsilon \) is admissible. With this approximation, Eq. (3) constrains \( \epsilon \), the iterated spin-orbit interaction represents an uncontrolled higher-order contribution. Following Refs. \(^11\) \(^11\), which show that the proton-proton Coulomb interaction is a perturbation in light nuclear bound states, we set \( \alpha_{\text{EM}} = 0 \). However, for scattering of charged nuclei with asymptotic center-of-mass momenta \( \lesssim 10 \) MeV, the Coulomb interaction has to be included non-perturbatively \(^4\). As the \(^3\)H(\( p,n \))\(^4\)He reaction is an integral part of any analysis of the \(^4\)He system, we elaborate on its effect when we discuss \(^3\)H-\( p \) scattering below in Fig. 7.

With the basic NN \( P \)-wave phase shifts, we illustrate the sensitivity of nuclear amplitudes with respect to either RG parameter in Fig. 4.

Varying \( \Lambda \) between \( 4 \) fm\(^{-1} \) and \( 15 \) fm\(^{-1} \) results in phase shifts within the gray area. In the zero-range limit \( \Lambda \to \infty \), the interaction does not affect this partial wave. The phase shifts converge to zero. On the other end, the relatively steep rise of the phase for \( \Lambda = 2 \) fm\(^{-1} \) (solid black line) does imply a minimal value for the cutoff in calculations with \( P \)-wave components in the asymptotic states which exceeds the naive two-body breakdown scale. As these higher-partial waves are elements of multi-nucleon states, results obtained with cutoffs only slightly larger than \( \Lambda \approx 0.5 \) fm\(^{-1} \), i.e., the pion-cut scale, are affected by unphysical poles.

Almost the same phase uncertainty mapped out by this \( \Lambda \) variation, can be parameterized by changing \( \epsilon \). This is shown by the red (blue) hatched area which results from an \( \epsilon \in [0,1] \) with a fixed \( \Lambda = 4(15) \) fm\(^{-1} \). The effect of the \( \epsilon \) variation diminishes with increasing \( \Lambda \). This is consistent with a unique zero-range, i.e., \( \Lambda \to \infty \), limit: Observables converge to the same value in this limit, regardless of the admixture of higher-order operators through the regulator via Eq. (1). A more comprehensive analysis of the sensitivity of NN \( P \)-wave observables follows in the discussion of Fig. 2.

\(^*\)We use \( L_{ij} \cdot S_{ij} = \frac{1}{2}(r_i - r_j) \times (\nabla_i - \nabla_j) \cdot (\sigma_i + \sigma_j) \) and calibrate \( d_3^A \) to the deuteron binding energy \( B_D = 2.224 \) MeV, the singlet \( np \) scattering length \( a_s = -23.75 \) fm. In the fit of \( d_3 \) to \( B_{3\text{H}} = 8.482 \) MeV, the effect of the spin-orbit term is insignificant. Numerical values are given in Table 1.
IV. SPIN-ORBIT DEPENDENCE OF $A \leq 3$ NUCLEI

We first demonstrate the effect of a variation of the spin-orbit strength $\epsilon$ in the two and three-nucleon sector. To qualify this as analogous to the conventional probe of sensitivity to high-energy modes via a $\Lambda$ variation, the effect on two-nucleon $P$-wave systems and $^3H$ must be parametrically small. As there is no experimental evidence of shallow poles in the two-nucleon sector beside the ones corresponding to the deuteron and its virtual copy in the singlet channel, and because the EFT($\pi$) tenet demands higher-order contributions to be perturbative, $\epsilon$ must not create any of these states with typical momenta which are smaller than the EFT’s breakdown scale. This criterion is validated with the phase shifts as shown in Fig. 2 and complements the result presented in Fig. 1. In all three $S = 1$ NN $P$-wave channels, the considered $\epsilon$’s induce a spread of the phases ($^3P_0$: blue, $^3P_1$: gray, $^3P_2$: red) which precludes an emerging pole below 10 MeV. The attractive character in the $J = 0, 1$ channels is obscured by the contribution of the $c_{\Lambda T}^1$ contact term. With this effect removed, i.e., $c_{\Lambda T}^1 = 0$, we obtain the less opaque bands shown in Fig. 2. The interaction is relatively strong for $J = 0$ (faint blue band), and weaker but of similar significance, albeit of different sign, for $J = 1$ and $J = 2$. The band spreads are induced with $\epsilon \in [0, 1]$. At 10 MeV, the width of all bands is < 0.1 Deg and thus neither suggests a resonance below $E_{\text{c.m.}} = 10$ MeV. Hence, the effect of the spin-orbit distortion in the two-body sector is as small as required of a regulator. Emerging two-nucleon bound states with negative parity indicate that $\epsilon$ exceeds the regulator range. This occurs at $\epsilon \gtrsim 1.58$ in the $^3P_0$ $np$ channel as shown in Fig. 3 (blue solid line). From the appearance of the first bound state for an $\epsilon \gtrsim 1.58$, we can infer that $\langle \, ^3P_0 \mid \mathbf{L} \cdot \mathbf{S} \mid ^3P_0 \rangle = \mathcal{O}(10^{-1})$.

While the two-nucleon bound state (deuteron) contains only orbital $S$-wave components at LO in EFT($\pi$), nucleons in relative $L > 0$ states are part of the three-nucleon $S = \frac{1}{2}$ bound state (triton). In a properly renormalized EFT, these non-$S$-wave states which contribute, here, to observables via the $\mathbf{L} \cdot \mathbf{S}$ regulator and the finite cutoff, add to the overall theoretical EFT uncertainty. They must vanish in the $\Lambda \to \infty$ limit. The EFT($\pi$) expansion of the two-nucleon scattering amplitude, in particular, is ordered such that relative $P$-waves become relevant at $\mathcal{O}(a_s^{-3}/m^3)$, and thus we assume that ensuing non-zero effects of the spin-orbit distortion on a three-body bound state are also suppressed relative to the LO part of the interaction. To validate this claim, the dependence of eigenvalues of the Hamiltonian Eq. (4) on $\epsilon$ with $\Lambda = 6$ fm$^{-1}$ and LECs as given in Table I.

$^*S = 0$ matrix elements of the spin-orbit force vanish.
values are defined such that for $\epsilon < 0$, i.e., $\frac{\epsilon}{\pi}$ for $1_H$, the spin-orbit distortion has no significant effect on $^3\text{H}$ as calibrated at LO in EFT($\hat{p}$). An excited state emerges out of the $dn$ continuum and settles at $B_{3\text{H}}$ for $0.5 < \epsilon 1.5$. The gap between $B_{3\text{H}}$ and the ground state becomes very large relative to $B_{3\text{H}} - B_D$, and $^3\text{H}$ could be identified with an excited state since the effects of the deep ground state are small.

For $\epsilon \lesssim -2.5$, the only eigenvalue (solid red line) below the deuteron-neutron ($dn$) threshold (black solid line) is $B_{3\text{H}}$, i.e., the spin-orbit distortion has no significant effect on $^3\text{H}$ as calibrated at LO in EFT($\hat{p}$). An excited state emerges out of the $dn$ continuum and settles at $B_{3\text{H}}$ for $0.5 < \epsilon 1.5$. The gap between $B_{3\text{H}}$ and the ground state becomes very large relative to $B_{3\text{H}} - B_D$, and $^3\text{H}$ could be identified with an excited state since the effects of the deep ground state are small.

For $\epsilon \lesssim 1$, the only eigenvalue (solid red line) below the deuteron-neutron ($dn$) threshold (black solid line) is $B_{3\text{H}}$, i.e., the spin-orbit distortion has no significant effect on $^3\text{H}$ as calibrated at LO in EFT($\hat{p}$). An excited state emerges out of the $dn$ continuum and settles at $B_{3\text{H}}$ for $0.5 < \epsilon 1.5$. The gap between $B_{3\text{H}}$ and the ground state becomes very large relative to $B_{3\text{H}} - B_D$, and $^3\text{H}$ could be identified with an excited state since the effects of the deep ground state are small.

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V. 4 NUCLEONS

In the two- and three-nucleon channels considered above, there is no data which indicates a shallow resonant state. The α particle is thus the smallest nucleus which sustains such a state close to its lowest break-up threshold. The RG invariant existence of such a state at LO in EFT(\(\pi\)) is subject of the following analyses of the \(\epsilon\) sensitivity of the four-nucleon bound (Fig. 5) and scattering (Fig. 6) systems.

- Bound-state spectrum - The effect of the spin-orbit interaction on the \(P\)-wave components in the α channel is shown in Fig. 5. The dependence of the eigenvalues of the \(A = 4\) Hamiltonian on \(\epsilon\) is similar to the triton
FIG. 5. Energy eigenvalues (Ground State: solid; EXcited state: dashed) in the four-nucleon \( J^p = 0^+ \) channel as a function of the spin-orbit component of the regulator for \( \Lambda = 4 \text{ fm}^{-1} \) (blue), \( \Lambda = 6 \text{ fm}^{-1} \) (red), and \( \Lambda = 8 \text{ fm}^{-1} \) (green). A dashed horizontal line indicates the \(^3\text{H}\)–nucleon threshold.

channel (see Fig. 3). Specifically, we find that below a \( \Lambda \)-dependent critical value, the calculated spectrum contains the ground state (solid line) in Fig. 3 and an accumulation of eigenstates whose energy cannot be discriminated numerically from \( B_{^3\text{H}} \). At the critical spin-orbit strength, an excited \( 0^+ \) state (dashed lines) below the \(^3\text{H}-\text{p}\) threshold emerges. The observed values, \( \epsilon_c(\Lambda = 4 \text{ fm}^{-1}) \sim 0.58 \), \( \epsilon_c(\Lambda = 6 \text{ fm}^{-1}) \sim 0.67 \), and \( \epsilon_c(\Lambda = 8 \text{ fm}^{-1}) \sim 0.72 \) suggest almost the value predicted above with pair counting: \( \lim_{\Lambda \to \infty} \epsilon_c \sim 0.9 \) (see the intersect of the respective blue, red, and green dashed lines in Fig. 5 with the black dashed line). The limit is near the point where a three-nucleon \( \frac{1}{2}^- \) state becomes bound. The excited state’s structure is accordingly a proton orbiting in a \( P \)-wave around a negative parity triton. In comparison, the dominant structure of the ground state is a proton in a \( S \)-wave relative to a \(^3\text{H}\) core. The \( \epsilon_c \) in the four-body system is smaller compared with the critical values in the 2, and three-body systems for \( \Lambda < \infty \).

These results are consistent with the interpretation of the iterated spin-orbit term as part of EFT(\#) for \( \epsilon < \epsilon_c \). The increasing \( L \cdot S \) sensitivity of nuclear systems with their particle number stems from the matrix element of the spin-orbit operator (see Eq. 3). The contribution of a two-nucleon pair in an eigenstate of the spin-orbit operator to the energy of an \( A \)-body state is expected to scale with \( A \). Therefore, any attractive spin-orbit interaction will eventually bind an \( A \)-body nucleus, regardless of how small it is in the two-nucleon system. In practice, the different \( \epsilon_c \) pertinent to the two, three, and four-body observables allow for a parameterization of an interaction which describes consistently the two, three, and four-nucleon system, like EFT(\#), with a handle on the character of a shallow \( 0^+ \) state in the \( \alpha \) channel.

- Elastic scattering - The four-nucleon bound states emerge for \( \epsilon \) strengths which are compatible with an interpretation of the spin-orbit distortion as a regulator effect in the two- and three-nucleon systems. Therefore, we cannot rule out that these states evolve from former physical resonances, and that the \( \epsilon \) dependence indicates the need for a modified EFT power counting. In this section, we refute this possibility. We demonstrate that besides the excited states which emerge with increasing \( \epsilon \) another, \( \epsilon \) invariant \( 0^+ \) resonance exists which we identify with the physical state. The character of the latter cannot be inferred from the spectrum as obtained in Fig. 3. The LO EFT(\#) analysis of the \(^4\text{He}\) scattering system, below, provides comprehensive evidence for both, the existence of a resonant state, and its RG invariance in contrast to the excitations which exhibit such an \( \epsilon \)/RG dependence.

The results are shown in Fig. 6. We focus on energies around the \(^3\text{H}-\text{p}\) threshold where experiment locates the excited \( \alpha \) state. In this interval, we consider two coupled channels, \(^3\text{H}-\text{p}\) (solid phase shifts) and \(^3\text{He}-\text{n}\)

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*From \( \Lambda \in [4,12] \text{ fm}^{-1} \), a polynomial extrapolation yields \( \lim_{\Lambda \to \infty} B_{\alpha} \sim 28.9 \text{ MeV} \) which is approached from below.

†See Refs. [7, 13] for other investigations which report on an excited shallow state.
Now, we choose to extract the EFT prediction for $B/\pi$ electroweak iso-spin dependent components at LO EFT (dashed phase shifts). We remove the degeneracy of these channels which follows from the absence of strong- and electroweak iso-spin dependent components at LO EFT (Eq. (4)) by shrinking the variational basis for $^3$He. This enforces different boundary conditions, namely for a neutron sufficiently far from the $^3$He, the latter is stable with an energy approximately 0.6 MeV less than $B_{3H}$. We do thereby not change the variational basis (see also Appendix 1M) which is used to approximate the wave function in a region where the particles interact strongly. Resonant behavior is thus modeled the same way as it is for degenerate asymptotic states, but effects on the phase shifts from the channel coupling are disentangled from that of the resonant states.

We first discuss diagonal phase shifts (the off-diagonal phases/mixing angles are analyzed in the Appendix) as shown in Fig. 6 for $\Lambda \in [4,12]$ fm$^{-1}$ (color-coded) with $\epsilon$-regulator parameters outside an epsilon surrounding of the critical values, $|\epsilon - \epsilon_c| \gtrsim 10^{-2}$. The phases rise from zero energy (relative to $B_{3H}$) up to a discontinuity before a cusp marks the opening of the $^3$He-n channel (barely visible kink at, e.g., 0.6 MeV for $\Lambda = 8$ fm$^{-1}$ at the onset of the dashed line). The shape of the discontinuity is characteristic of a threshold behavior in a multi-channel problem. We conclude from the absence of other particle-stable nuclei with $\Lambda \leq 4$ besides the ones which are included in our calculation, that the spikes correspond to resonant states.

The position of the resonance/discontinuity moves away from threshold with increasing $\Lambda$. This motion correlates with the deliberately chosen gap between the two open channels which widens also with increasing $\Lambda$. The relative position of the spike between zero energy and the threshold of the $^3$He-n channel is visibly the same for the three $\Lambda$ values. Hence, we postdict the physical resonance with $J^\pi = 0^+$ between the $^3$He-p and $^3$He-n threshold independently of $\Lambda$. We quantify the prediction by first extrapolating from the three calculated cutoff values the limit

$$\lim_{\Lambda \to \infty} \frac{B_\alpha^*}{B_{3H} - B_{3He}} = 0.5 \ .$$

(5)

Now, we choose to extract the EFT prediction for $B_\alpha^*$ where the experimental value for the energy difference between the two thresholds is reproduced exactly, namely: $B_{3H} - B_{3He} \approx 0.76$ MeV, and we obtain

$$B_\alpha^*(\text{EFT}(\pi)) = [0.38 \pm 0.15_{\text{EFT}} \pm 0.1_{\text{RGM}}] \text{ MeV} \ .$$

(6)

The EFT uncertainty is inferred from the difference of the ratio in Eq. (5) between $\Lambda = 4$ fm$^{-1}$ and $\Lambda = 12$ fm$^{-1}$. It is consistent with an a-priori estimate $1/3$ of the LO result, using the canonical estimate for the expansion parameter of EFT(\pi). The RGM uncertainty represents a conservative estimate of the numerical method. With this uncertainty, the EFT(\pi) prediction is consistent with the experimental value: $B_\alpha^*(\text{exp.}) = 0.395(20)$ MeV [15].

In addition to the invariance with respect to spatially symmetric transformations as parameterized by $\Lambda$, we investigate the sensitivity of the resonance with respect to the spin-orbit distortion. The phase shifts as shown in Fig. 6 do not change significantly for any $|\epsilon - \epsilon_c| \gtrsim 10^{-2}$. Only if the spin-orbit distortion is tuned to a critical value, spikes, similar to the displayed ones, indicate the presence of the negative-parity sub-threshold state. We refrain to display these additional discontinuities in the figure because the shown spikes and the associated states which cause the steep rise of the phases are unaffected, i.e., phase shifts for $|\epsilon - \epsilon_c| \gtrsim 10^{-2}$ are indistinguishable from $\epsilon = 0$. It is the physical $0^+$ excited state which causes the rise. From this stability of the phase shifts and the $A$ invariance, we conclude that this state is RG invariant. In other words, an excited four-nucleon state in the $\alpha$ channel with $J^\pi = 0^+$ is well described by EFT(\pi), and as such correlated with three low-energy data points, e.g., the deuteron binding energy, the $np$ singlet scattering length, and the $B_{3H}$.

Below the $^3$He-n threshold, we identify a second discontinuity for each cutoff (spikes of dashed/solid lines). These discontinuities are found at the same energy in the $^3$H-p phase shifts and indicate another threshold. Although, a $0^-$ resonance, approximately 0.42 MeV below the $^3$He-n threshold is well established [15], this is not the state responsible for the calculated spikes here. We rule out this possibility by choosing $\epsilon \approx 0$, thereby turning off the coupling between positive and negative parity states at $\Lambda = 12$ fm$^{-1}$. Hence, any effect of a negative-parity state on the considered scattering problem which defines asymptotic states of positive parity is absent by construction. As the spikes do not disappear at $\epsilon = 0$, we interpret the second set of discontinuities as the iso-spin mirror of the first $0^+$. A possible explanation for its presence is the absence of the Coulomb interaction in the proton-proton system in our analysis. If this state has largest overlap with a $^3$H-p configuration, the repulsion would further destabilize it and thereby diminish its effect on the phases in the energy range considered here. We test this conjecture by including the Coulomb interaction non-perturbatively. All other numerical and physical parameters are retained. With its full strength, the effects of the Coulomb interaction dominate the phase shift behavior. To study the essence of the effect without making the identification of resonant behavior impractical,
we increase the Coulomb interaction gradually from zero strength by modifying the fine structure constant to $\kappa \times \alpha_{\text{EM}}$ and take $\kappa \in \{0.02, 0.4, 1\}$.

The result is shown in Fig. 7 for $\Lambda = 4$ fm$^{-1}$. Compared with the Coulomb-less results (blue lines in Fig. 6), the results are qualitatively similar below $\approx 1$ MeV. Namely, for both attenuation factors (0.02 (black) and 0.4 (gray)), we observe a steep rise of the $^3\text{H}-\text{p}$ phase shifts at threshold, which we defined as zero energy. A kink signals the $^3\text{He}-\text{n}$ threshold. For the weaker Coulomb repulsion, this kink is hardly visible but apparent for the stronger repulsion where we identify it as a feature independent of the initial rise, i.e., the $0^+$ resonance. In contrast to the Coulomb-less phases, the resonant rise in the $^3\text{He}-\text{n}$ phases cannot be found at a comparable energy to the one in the $^3\text{H}-\text{p}$ channel. The small hump which can be observed for both Coulomb strengths in the $^3\text{He}-\text{n}$ phases can be interpreted as the remnant of a threshold to a stable or resonant state. Adopting the latter, it would correspond to the explanation given above, of a resonant state with a dominant triton-proton component, which is close to threshold if its constituents do not repel each other, but which becomes increasingly unstable with this repulsion.

VI. EPILOGUE

We find that the first excited, unstable state of the $\alpha$ can be predicted solely from characteristics of the two- and three-nucleon subsystems: the deuteron, the virtual singlet neutron-proton state, and the triton. The energy of the state is approximately 0.38 MeV above the triton-proton threshold, and its behavior as a shallow resonance is found independent of a set of regulator types for two- and three-nucleon contact interactions which comprise the pionless formulation of a microscopic nuclear theory. We thus show that this theory does not only correlate the $\alpha$ ground state to properties of its subsystems, but that the pair of a relatively deeply bound $\alpha$ and an unbound excited state is a consequence of an almost unitary two-body system and a fine-tuned three-body bound state.

The implied renormalization-group invariance was probed with a generalization of the regulator function. This enables the assessment of the sensitivity of observables with respect to asymmetric distortions of the short-distance structure of the interaction. This generalized regulator explains the independence of low-energy $A \leq 4$ observables from spin-orbit interactions of a certain strength. The calculated dependence of $A \leq 4$ observables on relatively strong, two-body spin-orbit interactions does suggest a correlation between a negative-parity three-nucleon state and four-nucleon systems.

Scattering phase shifts in the $\alpha$ channel around the $^3\text{H}-\text{p}$ and $^3\text{He}-\text{n}$ thresholds were calculated to identify near-threshold states in the $\alpha$. These calculations represent the first application of EFT($\pi$) to this scattering system.
and thereby extend the usefulness of the theory to more complex phenomena besides ground-state properties. Amongst the two near-threshold states, we find the physical, positive parity state invariant with respect to the generalized regulator.

**APPENDIX I: THE RGM CALCULATION AND INTERACTION PARAMETERS**

All observables in this work represent solutions of the stationary Schrödinger equation using the Hamiltonian in Eq. (4). We obtain them with a method based on the concept of resonating groups of particles (original idea: [16, 17]; specific implementation (RGM): [18, 19]). We specify only the most complicated calculation of this work, namely, four-nucleon scattering.

In the resonating-group ansatz for the wave function

\[ \Psi = \mathcal{A} \left\{ \sum_i \phi^{(i)}_I \phi^{(i)}_{II} F^{(i)}(R_i) \right\} \]  

we consider components corresponding to a fragmentation of the four-body system into $^3\text{H}-p$, $^3\text{He}-n$, and deuteron-deuteron ($d$-$d$). Each fragment wave function $\phi$ has appropriate $j^\pi_I/II$ quantum numbers coupled from total spin $S$ and angular momentum $L$. The two fragment spins are coupled to a channel spin with values which allow for its coupling with the angular momentum of the relative motion between the fragments, $F(R)$, to $J^\pi = 0^+$. The $\phi$'s follow the (LS)J coupling scheme. The deuteron resides in a pure relative $S$-wave ($L = 0$), while we consider components in the $^3\text{H}/^3\text{He}$ wave function which carry an angular momentum of $L = 1$ on both Jacobi coordinates. With these constraints, we consider all intermediate couplings and expand the radial dependencies in a Gaussian basis. The width sets for the intermediate $(s_1s_2)^{s_{12}=1}$ coupling within $^3\text{H}$, e.g., is obtained from a 20-dimensional $S$-wave deuteron.

The scattering problem is solved by “freezing” the wave functions of the fragments for those channels $i$ in Eq. (4) which are needed to define the asymptotic behavior. For a nucleon impinging with less than 10 MeV on a $^3\text{H}$ or $^3\text{He}$, we thus include as open S-wave channels: $^3\text{H}-p$, $^3\text{He}-n$, and $d$-$d$. In these cases, the free relative motion $F$ is expanded in a set of 20 Gaussians in the region where the effective fragment-fragment interaction is non-zero. To account for any deviation from this free motion within the interaction region, we add products of single components of the fragments with square-integrable relative function $F$. We build a complete basis in this way until the lowest eigenvalues in the resulting spectrum are stable at the 100 keV level.

The effect on the $\alpha$ of channels which couple negative parity $^3\text{H}/^3\text{He}$ via a relative $P$-wave to $0^+$ was probed and found sufficiently accounted for by the $(l_1l_2)^{L=0,1,2}$ components of $^3\text{H}$. As these negative-parity three-nucleon states where also not part of the study by themselves, we abstained from including states which resemble this coupling scheme in the four-nucleon calculations.

| $\Lambda$ [fm$^{-1}$] | $c_1^\alpha$ | $c_2^\alpha$ | $d_1^\alpha$ |
|----------------------|-------------|-------------|-------------|
| 2.00                 | -142.364    | -106.279    | 68.4883     |
| 4.00                 | -505.164    | -434.958    | 677.799     |
| 6.00                 | -1090.58    | -986.252    | 2652.65     |
| 8.00                 | -1898.62    | -1760.16    | 7816.23     |
| 10.0                 | -2929.28    | -2756.69    | 20483.2     |
| 12.0                 | -4182.37    | -3975.62    | 50939.9     |
| 15.0                 | -6479.58    | -6221.50    | 195570.     |

**Table I.** Numerical values of low-energy constants in MeV for EFT(#) calibrated to $\alpha_s \approx -23.8$ fm, $B_0 = 2.22$ MeV, and $B_{3\text{H}} = 8.48$ MeV.
APPENDIX II: MULTI-CHANNEL EFFECTIVE-RANGE EXPANSION

The discussion of Figs. 6 and 7 above concentrates on the behavior of the diagonal part of the S-matrix. Below, we analyze the off-diagonal elements, i.e., how strongly do the two channels couple as a function of energy? It is instructive to begin the discussion with an analogy. The deuteron is a bound state in which the two constituents – neutron and proton – move “most of the time” with zero angular momentum relative to each other. This is characteristic for the nuclear interaction which favors spatial rotational symmetry by a relatively weak coupling between the two-nucleon $^3S_1$ and $^3D_1$ states. For the numerical values of the standard parameters for the corresponding two-channel scattering matrix, this weak coupling of angular-momentum channels translates to a relatively small mixing angle $\epsilon$ and relatively small eigenphase shifts, describing asymptotic two-nucleon states in a relative $D$-wave, compared with those eigenphases which parametrize zero-angular-momentum scattering. For the generic two-channel S-matrix, we adopt the following standard:

$$S = \begin{pmatrix} \eta_{11}e^{2i\delta_{11}} & \eta_{12}e^{2i\delta_{12}} \\ \eta_{21}e^{2i\delta_{21}} & \eta_{22}e^{2i\delta_{22}} \end{pmatrix} = \begin{pmatrix} \cos \epsilon - \sin \epsilon & e^{2i\alpha} \\ \sin \epsilon & \cos \epsilon \end{pmatrix} \begin{pmatrix} 0 & e^{2i\beta} \\ e^{-2i\beta} & 0 \end{pmatrix} \begin{pmatrix} \cos \epsilon & \sin \epsilon \\ -\sin \epsilon & \cos \epsilon \end{pmatrix}. \quad (8)$$

In our analysis, we do not consider channels which differ by the angular momentum of the relative motion but in the nuclear composition of the fragments. As we consider the four-nucleon scattering process in the vicinity of the triton-proton threshold, only two two-fragment channels – $^3\text{H}-p$ and $^3\text{He}-n$ – are energetically accessible. It is helpful to think of the $^3\text{H}-p$ arrangement as the analog of the $S$-wave”, and $^3\text{He}-n$ as the $D$-wave analog of the deuteron. The existence of the excited, resonant state was deduced from the diagonal phases $\delta_{11/22}$, as shown in Fig. 6 and 7 only, without need to investigate the strength parameters $\eta$. Row and column indices specify the asymptotic states, $\delta_{12}$ parametrizes, e.g., the probability to detect a free $^3\text{He}-n$ state with relative energy $E_{ch}$ if a proton hits a triton with a corresponding $E'_{ch}$.

When we do include the coupling strengths $\eta$ – in Fig. 8 we use the more common parameterization with eigenphases $\delta_{\alpha/\beta}$ and mixing angle $\epsilon$ – we notice two peculiarities: first, in contrast to the mixing of angular momentum channels in the deuteron ground state, the eigenstates of the four-nucleon process are superpositions of the $^3\text{H}-p$ and $^3\text{He}-n$ configurations with similar weight – if a neutron impinges on a proton in a relative $S$-wave, it is very likely to emerge in an $S$-wave while the probability of it being deflected into a $D$-wave is small; the probability to detect $^3\text{He}$ and a free neutron after the collision of a neutron with $^3\text{H}$, in contrast, is almost as high as an elastic reaction. In terms of S-matrix parameters: the mixing angle (red) rises quickly to a value close to $\pi/4$ in Fig. 8. Second, at an energy between 0.15 MeV and 0.2 MeV, the two channels decouple and scatter elastically. The scattered waves are not phase shifted ($\delta_{\alpha/\beta} \approx 0$) and the nucleons are not rearranged ($\epsilon \approx 0$), i.e., at this energy, the nuclei scatter elastically like classical particles. In the vicinity of this peculiar point, the phase shifts behave with energy reminiscent of the dependence of energy eigenvalues of a two-level system around an avoided level crossing. Here, we find the effect of the scattering process on one eigenstate insignificant (gray phase for $E \lesssim 0.17$ MeV and black phase for $E \gtrsim 0.17$ MeV) compared to that on the other eigenstate. It is beyond the scope of this study to analyze the sensitivity of that phenomenon with respect to cutoff variations and the proton-proton Coulomb interaction. Finally, note the cusp in the mixing angle around the energy where we identified the resonant state from the diagonal phases (red, dashed line at $\approx 9$ MeV).

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FIG. 8. Eigenphases (black/gray) and mixing angle (red, dashed) for the coupled $^3\text{H}-p$, $^3\text{He}-n$ two-channel scattering system ($J^\pi = 0^+$). The energy is defined relative to the $^3\text{H}-p$ threshold (blue abscissa label at 0.067 MeV), $\Lambda = 8$ fm$^{-1}$, and $\alpha_{\text{EM}} = 0$.

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