Bounded operators on the weighted spaces of holomorphic functions on the unit Ball in $C^n$

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July 2, 2014

Abstract

Assuming that $S$ is the space of functions of regular variation, $\omega \in S$, $0 < p < \infty$, a function $f$ holomorphic in $B^n$ is said to be of Besov space $B_p(\omega)$ if

$$\|f\|_{B_p(\omega)}^p = \int_{B^n} (1 - |z|^2)^p |Df(z)|^p \frac{\omega(1 - |z|)}{(1 - |z|^2)^{n+1}} d\nu(z) < +\infty,$$

where $d\nu(z)$ is the volume measure on $B^n$ and $D$ stands for a fractional derivative of $f$.

We consider operators on $B_p(\omega)$ and show, that they are bounded.

AMS Subject Classification: 32C37, 47B38, 46T25, 46E15.

Key Words and Phrases: Weighted Besov spaces, Unit ball, Operator.

1 Introduction and basic constructions

Let $C^n$ denote the complex Euclidean space of a dimension $n$. For any points $z = (z_1, \ldots, z_n)$, $\zeta = (\zeta_1, \ldots, \zeta_n)$ in $C^n$, we define the inner product as $<z, \zeta> = z_1\overline{\zeta_1} + \ldots + z_n\overline{\zeta_n}$ and and note that $|z|^2 = |z_1|^2 + \ldots + |z_n|^2$. By $B^n = \{ z \in C^n, |z| < 1 \}$ and $C^n : S^n = \{ z \in C^n, |z| = 1 \}$ we denote the open unit ball and its boundary, i.e. the unit sphere, in $C^n$. Further, by $H(B^n)$ we denote the set of holomorphic functions on $B^n$ and by $H^\infty(B^n)$ the set of bounded holomorphic functions on $B^n$. 

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If \( f \in H(B^n) \), then \( f(z) = \sum_m a_m z^m \) (\( z \in B^n \)), where the sum is taken over all multiindices \( m = (m_1, \ldots, m_n) \) with nonnegative integer components \( m_k \) and \( z^m = z_1^{m_1} \cdots z_n^{m_n} \). Assuming that \( |m| = m_1 + \ldots + m_n \) and putting \( f_k(z) = \sum_{|m|=k} a_m z^m \) for any \( k \geq 0 \), one can rewrite the Taylor expansion of \( f \) as

\[
f(z) = \sum_{k=0}^{\infty} f_k(z),
\]

which is called homogeneous expansion of \( f \), since each \( f_k \) is a homogeneous polynomial of the degree \( k \). Further, for a holomorphic function \( f \) the fractional differential \( D^\alpha \) is defined as

\[
D^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha f_k(z),
\]

\[
D^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha f_k(z), \quad k = (k_1, \ldots, k_n), \quad z \in B^n.
\]

We consider the inverse operator \( D^{-\alpha} \) defined in the standard way:

\[
D^{-\alpha} D^\alpha f(z) = f(z).
\]

Particularly, \( D^1 f(z) = D f(z) \) if \( \alpha = 1 \).

The following properties of \( D \) are evident

1. \( DD^\alpha f(z) = D^{\alpha+1} f(z) \)
2. \( D^m (1-< z, \zeta >)^{-\alpha} \preceq (1-< z, \zeta >)^{-\alpha-m} \)

By \( d\nu \) we denote the volume measure on \( B^n \), normalized so that \( \nu(B^n) = 1 \), and by \( d\sigma \) the surface measure on \( S^n \), normalized so that \( \sigma(S^n) = 1 \).

Then following lemma, the proof of which can be found in [7] or [12], reveals the connection between these measures.

**Lemma 1.** If \( f \) is a measurable function with summable modulus over \( B^n \), then

\[
\int_{B^n} f(z) d\nu(z) = 2n \int_0^1 r^{2n-1} dr \int_{S^n} f(r\zeta) d\sigma(\zeta).
\]

**Definition 1.** By \( S \) we denote the well-known class of all non-negative measurable functions \( \omega \) on \( (0,1) \) with

\[
\omega(x) = \exp \left\{ \int_x^1 \frac{\varepsilon(u)}{u} du \right\}, \quad x \in (0,1),
\]

where \( \varepsilon(u) \) is some measurable, bounded functions on \( (0,1) \) and \(-\alpha \omega \leq \varepsilon(u) \leq \beta \omega\).
Note that the functions of $S$ are called functions of regular variation (see [10]). Throughout the paper, we shall assume that $\omega \in S$. Throughout the paper the capital Letters $C(\ldots)$ and $C_k$ stand for different positive constants depending only on the parameters indicated.

We define the holomorphic Besov spaces on the unit ball as follows (see [3]).

**Definition 2.** Let $\omega \in S$, $0 < p < \infty$. A function $f \in H(B^n)$ is said to be of $B_p(\omega)$ if

$$M_p^p(\omega) = \int_{B^n} (1 - |z|^2)^p |Df(z)|^p \frac{\omega(1 - |z|)}{(1 - |z|^2)^{n+1}} d\nu(z) < +\infty.$$  

We introduce the norm in $H(B^n)$ as $\|f\|_{B_p(\omega)} = M_f(\omega)$ ($|f(0)|$ is not to be added since $Df = 0$ implies $f = 0$ for a holomorphic function $f$).

Besides, it is easy to check that if $p > 1$, $n = 1$ and $\omega(t) = 1$, then $B_p(\omega)$ becomes the classical Besov space (see [1], [5], [11]).

In particular, for $p = +\infty$ we shall write $B_{\infty}(\omega) = B_\omega$, where $B_\omega$ denotes the $\omega-$weighted Bloch space on the ball (see [3]).

In [6], [8], [9], one can see some other definitions and some characterizations of holomorphic Besov spaces on $B^n$.

Let $1 \leq p < \infty$ and let $f \in B_p(\omega)$. Further, let $m > -n/p - \beta_\omega/p$. Then the function $Df(z)$ has the representation

$$Df(z) = C(\pi, m) \int_{B^n} \frac{(1 - |\zeta|^2)^m Df(\zeta)}{(1 - \langle z, \zeta \rangle)^{n+1+m}} d\nu(\zeta), \quad z \in B^n,$$

where $C(n, m) = \frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)}$, follows as a simple consequence of that well known in the one-dimensional case (for details, see [2], [12]).

The following auxiliary lemma will be used.

**Lemma 2.** If $1 \geq p < \infty$ and $f \in B_p(\omega)$, then

$$|f(z)| \leq C(\pi, m) \int_{B^n} \frac{(1 - |\zeta|^2)^m}{|1 - \langle z, \zeta \rangle|^{n+m}} |Df(\zeta)| d\nu(\zeta)$$

for $m \in N$ and $m > -n/p - \beta_\omega/p$.  

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**Proof.** Obviously, \( f(z) = \int_0^1 Df(raz) \, dr \), and by (2) we get

\[
f(z) = C(\pi, m) \int_0^1 \int_{B^n} \frac{(1 - |z|^2)^m Df(\zeta)}{(1 - r < z, \zeta >)^{n+1+m}} \, d\nu(\zeta) \, dr
\]

\[
= C(\pi, m) \int_{B^n} (1 - |\zeta|^2)^m \frac{Df(\zeta) \int_0^1 \frac{d}{dr} \left( \frac{1}{(1 - r < z, \zeta >)^{n+1+m}} \right) \, dr}{(1 - r < z, \zeta >)^{n+1+m}} \, d\nu(\zeta)
\]

\[
= \tilde{C}(\pi, m) \int_{B^n} (1 - |\zeta|^2)^m ((1 - < z, \zeta >)^{n+\alpha} - 1) \, Df(\zeta) \, d\nu(\zeta).
\]

It is clear that \( ((1 - < z, \xi >)^{n+m+1} - 1)/ < z, \xi > \) is bounded in \( B^n \). Hence the desired statement follows. \( \square \)

**Lemma 3.** Let \( \omega \in S \) and let \( f \in B_p(\omega) \) for some \( 0 < p \leq 1 \). Then

\[
\left( \int_{B^n} |Df(z)|^\omega^{1/p} \frac{(1 - |z|)}{(1 - |z|)^n} \, d\nu(z) \right)^p \leq \int_{B^n} |Df(z)|^p \frac{(1 - |z|)^p \omega(1 - |z|)}{(1 - |z|)^{n+1}} \, d\nu(z)
\]

**Proof.** We have \( |Df(z)| = |Df(z)|^p |Df(z)|^{1-p} \). By Lemma 2 we get

\[
|Df(z)| \leq |Df(z)|^p \frac{\|f\|_{1-p, B_p(\omega)}^{1-p}}{\omega(1-p)^p (1 - |z|)^{1-p}}.
\]

Therefore

\[
|Df(z)|^{1-p} \frac{(1 - |z|)^{1-p} (1 - |z|)}{(1 - |z|)^n+1} \leq |Df(z)|^p \frac{\|f\|_{1-p, B_p(\omega)}^{1-p}}{\omega(1-p)^p (1 - |z|)^{n+1}},
\]

and by integration over \( B^n \) we get

\[
\int_{B^n} |Df(z)|^{1-p} \frac{(1 - |z|)^{1-p} (1 - |z|)}{(1 - |z|)^n+1} \, d\nu(z) \leq \|f\|_{1-p, B_p(\omega)}^{1-p} \int_{B^n} |Df(z)|^p \frac{\omega(1 - |z|)(1 - |z|)^p}{(1 - |z|)^{n+1}}.
\]

The proof is completed. \( \square \)

**Lemma 4.** Let \( \omega \in S, \ \alpha + 1 - \beta > 0, \) and \( \beta - \alpha > \alpha_\omega \). Then

\[
\int_{B^n} \frac{(1 - |\zeta|^2)^\omega(1 - |\zeta|)}{|1 - < z, \zeta >|^\beta+\alpha+1} \, d\nu(\zeta) \leq C(\alpha, \beta, \omega) \frac{\omega(1 - |z|^2)}{(1 - |z|^2)^{\beta+\alpha}}.
\]

**Proof.** By Lemma 1 for \( \beta > 0 \) we get

\[
\int_{B^n} \frac{(1 - |\zeta|^2)^\omega(1 - |\zeta|)}{|1 - < z, \zeta >|^\beta+\alpha+1} \, d\nu(\zeta) = 2n \int_0^1 r^{2n-1} (1 - r^2)^\omega(1 - r) \, dr \times
\]
\[
\int_{S^n} \frac{d\sigma(\zeta)}{1 - \zeta > |^{\beta+n+1}} \leq 2n \int_0^1 r^{2n-1} \frac{(1 - r^2)^\omega(1 - r)}{(1 - r|z|)^{\beta+1}} dr.
\]

In the last inequality we have used Theorem 1.12 from [12].

The problem is to estimate the last one dimensional integral. To this end we have

\[
\int_0^1 \frac{(1 - r^2)^\omega(1 - r) dr}{(1 - r|z|)^{\beta+1}} \leq \int_0^1 \frac{u^\omega(u) du}{(1 - |z| + u|z|)^{\beta+1}}
\]

\[
= \left\{ \int_0^{1-|z|} \frac{u^\omega(u) du}{(1 - |z| + u|z|)^{\beta+1}} + \int_{1-|z|}^{1} \frac{u^\omega(u) du}{(1 - |z| + u|z|)^{\beta+1}} \right\} = I_1 + I_2.
\]

First we estimate the integral \(I_1\).

\[
I_1 \leq \int_0^{1-|z|} \frac{u^\omega(u)}{(1 - |z|)^{\beta+1}} du = \frac{1}{(1 - |z|)^{\beta+1}} \int_0^{1-|z|} u^\omega(u) du =
\]

\[
\frac{(\alpha + 1)\omega}{(1 - |z|)^{\beta+1}} \left[ (1 - |z|)^{\alpha+1} \omega(1 - |z|) + \int_0^{1-|z|} u^\omega(u) \varepsilon(u) du \right]
\]

As a result we get

\[
(\alpha + 1) \int_0^{1-|z|} u^\omega(u) du = (1 - |z|)^{\alpha+1} \omega(1 - |z|) + \int_0^{1-|z|} u^\omega(u) \varepsilon(u) du,
\]

and

\[
\int_0^{1-|z|} (\alpha + 1 - \varepsilon(u)) u^\omega(u) du = (1 - |z|)^{\alpha+1} \omega(1 - |z|).
\]

On the other hand

\[
\alpha + 1 - \beta \omega \leq \alpha + 1 - \varepsilon(u)
\]

which yields

\[
(\alpha + 1 - \beta \omega) \int_0^{1-|z|} u^\omega(u) du \leq (1 - |z|)^{\alpha+1} \omega(1 - |z|)
\]

or

\[
I_1 \leq C(\alpha, \beta, \omega) \frac{\omega(1 - |z|)}{(1 - |z|)^{\beta-\alpha}} \tag{3}
\]

Now we want to estimate \(I_2\). For \(|z| \geq 1/2\) we have

\[
I_2 \leq 2^{\beta-1} \int_{1-|z|}^{1} \frac{\omega(u)}{u^{\beta-\alpha+1}} = \frac{2^\beta}{\alpha - \beta - 1} \int_{1-|z|}^{1} \omega(u) du^{\alpha-\beta-1}
\]

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Then
\[
\int_{1-|z|}^{1} \frac{\omega(u) \epsilon(u)}{u^{\beta-\alpha}} du = \frac{\omega(1-|z|)}{(1-|z|)^{\beta-\alpha-1}} - \omega(1) - \int_{1-|z|}^{1} \frac{\omega(u) \epsilon(u)}{u^{\beta-\alpha}} du
\]
and it follows that
\[
\int_{1-|z|}^{1} \left(1 + \frac{\epsilon(u)}{\beta - \alpha - 1}\right) \frac{\omega(u)}{u^{\beta-\alpha}} du = \frac{\omega(1-|z|)}{(1-|z|)^{\beta-\alpha-1}} - \omega(1) \leq \frac{\omega(1-|z|)}{(1-|z|)^{\beta-\alpha-1}}.
\]

Then the inequality
\[
1 + \frac{\epsilon(u)}{\beta - \alpha - 1} \geq 1 - \frac{\alpha\omega}{\beta - \alpha - 1} > 0,
\]
gives us
\[
\int_{1-|z|}^{1} \frac{\omega(u)}{u^{\beta-\alpha}} du \leq C(\alpha, \beta, \omega) \frac{\omega(1-|z|)}{(1-|z|)^{\beta-\alpha-1}}.
\]

Summing up, from (3) and (4) we get the proof of Lemma 4.

**Lemma 5.** The following statement is true
\[
D(fg) = gDf + fDg - fg
\]

**Proof.** To this end we define the radial derivative \( R \) of \( f \) as follows
\[
(Rf)(z) = \sum_{k=1}^{\infty} k f_k(z), \quad z \in B^n
\]
or equivalently
\[
(Rf)(z) = \sum_{k=1}^{n} z_k \frac{\partial f(z)}{\partial z_k}, \quad z \in B^n.
\]

It is easy to note that \( R(fg) = gR(f) + fR(g) \) and for the operator \( D \) we have \( Df = f + Rf \). Combining the last equalities we get
\[
D(fg) = fg + gR(f) + fR(g).
\]

On the other hand we have \( R(f) = Df - f \) and \( R(g) = Dg - g \). Hence \( D(fg) = gDf + fDg - fg \).

\[\square\]
2 Bounded Operators on $B_p(\omega)$

In this Section first we consider the following operator

$$T_h^\alpha(f)(z) = \int_{B^n} \frac{(1 - |\xi|^2)^\alpha h(\xi)f(\xi)}{(1 - \langle z, \xi \rangle)^{n+\alpha+1}} d\theta(\xi), \quad \alpha > -1.$$ 

**Theorem 1.** Let $0 < p < \infty$, $h \in H^1(B^n)$. Then

1. if $T_h^\alpha$ is bounded on $B_p(\omega)$ then $h \in H^\infty(B^n)$.
2. conversely,

a) if $1 \leq p < \infty$ and $h \in H^\infty(B^n)$ then $T_h^\alpha: B_p(\omega) \to B_p(\omega)$

b) if $0 < p < 1$ and $h \in H^\infty(B^n)$ then $T_h^\alpha: B_p(\omega) \to B_p(\omega^*)$, where $\omega^*(t) = t^{(\alpha+n+1)(1-p)}\omega(t)$ and $m > -n/p - \beta_\omega/p$.

**Proof.** 1. Let $T_h^\alpha$ be bounded on $B_p(\omega)$. We take

$$f_\tau(\xi) = \frac{1}{(1 - \langle \xi, \tau \rangle)^{\alpha+n+1}},$$

where $\tau \in [0,1]^n$ is a parameter. We calculate

$$T_h^\alpha(f)(z) = \int_{B^n} \frac{(1 - |\xi|^2)^\alpha h(\xi)d\theta(\xi)}{(1 - \langle z, \xi \rangle)^{n+\alpha+1}(1 - \langle \xi, \tau \rangle)^{n+\alpha+1}}$$

Then we get

$$||T_h f_\tau||_{B_p(\omega)} = ||h(z)|| \cdot ||f_\tau||_{B_p(\omega)} \leq ||T_h|| \cdot ||f||_{B_p(\omega)}$$

and hence $|h(\tau)| \leq ||T_h||$. Replacing $f_\tau(z)$ by $f_\tau(e^{i\theta}z)$ we get $|h(\tau e^{i\theta})| \leq ||T_h||$ which implies $h \in H^\infty(B^n)$.

2. Conversely, a) let $p \geq 1$ and $h \in H^\infty(B^n)$. We show that $T_h^\alpha(f) \in B_p(\omega)$ for any $f \in B_p(\omega)$. To this end by Lemma 2 we use the inequality

$$|f(\xi)| \leq C(\pi, m) \int_{B^n} \frac{(1 - |t|^2)^m|Df(t)|}{|1 - \langle \xi, t \rangle|^{m+n}} d\theta(t)$$

which implies that

$$|f(\xi)|^p \leq \frac{C(\pi, m)}{(1 - |\xi|^2)^{(m-1)p/\alpha}} \int_{B^n} \frac{(1 - |t|^2)^{mp}|Df(t)|^p}{|1 - \langle \xi, t \rangle|^{m+n}} d\theta(t)$$

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Then for $p > 1$, by Holder's inequality, we get,

$$|T^\alpha f(z)|^p \leq C(\pi, m) \left( \int_{B^n} \frac{(1 - |\xi|^2)^\alpha |f(\xi)| \cdot |h(\xi)| d\theta(\xi)}{|1 - \langle z, \xi \rangle|^{n+2+\alpha}} \right)^p \leq C(\pi, m) \frac{||h||_\infty}{(1 - |z|^2)^{p/q}} \int_{B^n} \frac{(1 - |\xi|^2)^\alpha |f(\xi)|^p d\theta(\xi)}{|1 - \langle z, \xi \rangle|^{n+2+\alpha}}.$$

Then we have

$$I \equiv C(\pi, m) \int_{B^n} |D^\alpha \overline{h} f(z)|^p \frac{\omega(1 - |z|) d\theta(z)}{(1 - |z|^2)^{n+1-p}} \leq C(\pi, m) \int_{B^n} (1 - |t|^2)^{mp} |Df(t)|^p \int_{B^n} \frac{(1 - |\xi|^2)^\alpha (m-1)p/q}{|1 - \langle \xi, t \rangle|^{n+m}} \int_{B^n} \frac{\omega(1 - |z|) d\theta(z) d\theta(\xi) d\theta(t)}{(1 - |z|^2)^{n+1-p+p/q} |1 - \langle z, \xi \rangle|^{n+2+\alpha}}$$

Using Lemma 4 we obtain furthermore

$$I \leq C(\pi, m) \int_{B^n} (1 - |t|^2)^{mp} |Df(t)|^p \int_{B^n} \frac{(1 - |\xi|^2)^\alpha \omega(1 - |\xi|) d\theta(z) d\theta(\xi)}{|1 - \langle \xi, t \rangle|^{n+m}} \leq C(\pi, m) ||h||_\infty \int_{B^n} \frac{\omega(1 - |\xi|) d\theta(\xi)}{(1 - |\xi|^2)^{m-1+p/q+m+n}} \leq C(\pi, m) ||f||_{B_p(\omega)} ||h||_\infty.$$
Let now $p = 1$. We have

$$
\int_{B^n} |D\Gamma_h^\alpha f(z)| \frac{\omega(1 - |z|) d\theta(z)}{(1 - |z|^2)^n} \leq C(\pi, m) \|h\|_\infty \int_{B^n} \frac{(1 - |w|^2)^m |Df(w)|}{|1 - \langle \xi, w \rangle|^{m+n}} \int_{B^n} \frac{(1 - |\xi|^2)^\alpha}{|1 - \langle \xi, w \rangle|^{\alpha+\alpha+2}} \int_{B^n} \omega(1 - |\xi|) d\theta(z) d\theta(\xi) d\theta(w)
$$

$$
= C(\pi, m) \|h\|_\infty \int_{B^n} (1 - |w|^2)^m |Df(w)| \int_{B^n} \frac{(1 - |\xi|^2)^\alpha}{|1 - \langle \xi, w \rangle|^{\alpha+\alpha+2}} \int_{B^n} \omega(1 - |z|) d\theta(z) d\theta(\xi) d\theta(w)
$$

$$
C(\pi, m) \leq \|h\|_\infty \int_{B^n} (1 - |w|^2)^m |Df(w)| \int_{B^n} \frac{(1 - |\xi|^2)^\alpha \omega(1 - |\xi|) d\theta(\xi) d\theta(w)}{|1 - \langle \xi, w \rangle|^{\alpha+\alpha+2}}
$$

$$
C(\pi, m) \leq \|h\|_\infty \int_{B^n} (1 - |w|^2)^m |Df(w)| \omega(1 - |w|) (1 - |w|^2)^\alpha d\theta(w)
$$

$$
= \int_{B^n} \frac{|Df(w)| \omega(1 - |w|)}{(1 - |w|^2)^n} d\theta(w) = C(\pi, m) \|f\|_{B_p(\omega)} \|h\|_\infty.
$$

b) Let $0 < p < 1$. Using Lemma 3 we get

$$
|f(\xi)|^p \leq C(\pi, m) \int_{B^n} \frac{(1 - |t|^2)^{mp+p} |Df(t)|^p}{|1 - \langle \xi, t \rangle|^{m+n+1}} d\theta(t)
$$

and

$$
|D\Gamma_h^\alpha f(z)|^p \leq C(\pi, m) \|h\|_\infty \int_{B^n} \frac{(1 - |\xi|^2)^{p(\alpha+1)} |f(\xi)|^p d\theta(\xi)}{|1 - \langle z, \xi \rangle|^{n+2+\alpha}}.
$$
As in the case of \( p > 1 \) by Lemma \[ \text{Lemma 5} \] we get

\[
\int_{B^n} |DT_h^n f(z)|^p \omega^*(1 - |z|) d\theta(z) \frac{((1 - |z|^2)^{\alpha+1})}{(1 - |z|^2)^{n+1-p}}
\]

\[
C(\pi, m) \leq \int_{B^n} (1 - |t|^2)^{mp+p} |Df(t)|^p \int_{B^n} \frac{(1 - |\xi|^2)^{p(\alpha+1)}}{|1 - \langle \xi, t \rangle|^{m+n+1}}
\]

\[
\int_{B^n} \omega^*(1 - |t|) d\theta(t) d\theta(t)
\]

\[
C(\pi, m) \leq \int_{B^n} (1 - |t|^2)^{mp+p} |Df(t)|^p \int_{B^n} \frac{(1 - |\xi|^2)^{p(\alpha+1)}}{|1 - \langle \xi, t \rangle|^{m+n+1}}
\]

\[
C(\pi, m) \leq \int_{B^n} (1 - |t|^2)^{mp+p} |Df(t)|^p \omega^*(1 - |t|) d\theta(t)
\]

\[
C(\pi, m) \leq ||h||_\infty \int_{B^n} \frac{(1 - |t|^2)^{mp+p} |Df(t)|^p \omega^*(1 - |t|) d\theta(t)}{|1 - |\xi|^2|^{n+1-p}} \leq C(\pi, m) ||f||_{B_p(\omega)}^p ||h||_\infty.
\]

\[ \square \]

**Theorem 2.** Let \( H^\infty(B^n) \). Then \( M_h \) is a bounded operator \( B_p(\omega) \rightarrow B_p(\omega) \).

**Proof.** Using Lemma \[ \text{Lemma 5} \] we show that

\[
\int_{B^n} |Df(z)|^p |g(z)|^p \frac{\omega(1 - |z|)}{(1 - |z|^2)^{n+1-p}} d\theta(z) < \infty
\]

(3)

\[
\int_{B^n} |f(z)|^p |Dg(z)|^p \frac{\omega(1 - |z|)}{(1 - |z|^2)^{n+1-p}} d\theta(z) < \infty
\]

(4)

\[
\int_{B^n} |f(z)|^p |g(z)|^p \frac{\omega(1 - |z|)}{(1 - |z|^2)^{n+1-p}} d\theta(z) < \infty
\]

(5)

The proof of (3) is evident.

Proof of (4). First we show that

\[
|Dg(z)| \leq \frac{C||g||_\infty}{1 - |z|^2}, \quad z \in B^n,
\]

if \( g \in H^\infty(B^n) \). To this end we take the ball \( B^n(z) = \{ w \in B^n, |w - z| < n - |z|/2 \} \) and use the Cauchy inequality. In the case \( p > 1 \) we have

\[
|f(z)|^p \leq \left( C(\pi, m) \int_{B^n} \frac{(1 - |\xi|^2)^m |Df(\xi)| d\theta(\xi)}{|1 - \langle m, \xi \rangle|^{m+n}} \right)^{\frac{p}{\gamma_p}}
\]

\[
\leq \frac{C(\pi, m)}{(1 - |z|^2)^{\gamma_p/q}} \int_{B^n} \frac{(1 - |\xi|^2)^{m-\gamma} (1 - |\xi|^2)^{p+\gamma_p} |Df(\xi)|^{\gamma_p} d\theta(\xi)}{1 - \langle z, \xi \rangle^{m+n}}
\]

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Then for (4) we get

\[
\int_{B^n} |f(z)|^p |Dg(z)|^p \frac{\omega(1 - |z|)}{(1 - |z|^2)^{n+1-p}} d\theta(z)
\]

\[
\leq C(\pi, m)\|g\|^p_{\infty} \int_{B^n} (1 - |\xi|^2)^{m+(p-1)(\gamma+1)} |Df(\xi)|^p \int_{B^n} \frac{\omega(1 - |z|^2)d\theta(z)d\theta(\xi)}{|1 - \langle z, \xi \rangle|^{n+m(1 - |z|^2)^{n+2-p+\gamma p/q}}}
\]

\[
\leq C(\pi, m)\|g\|_{\infty} \int_{B^n} (1 - |\xi|^2)^{m+(p-1)(\gamma+1)} |Df(\xi)|^p \omega(1 - |\xi|) \frac{|Df(\xi)|^p \omega(1 - |\xi|)}{(1 - |\xi|)^{m+n-p+\gamma p/q}} d\theta(\xi)
\]

\[
= C(\pi, m)\|g\|_{\infty} \int_{B^n} |Df(\xi)|^p \omega(1 - |\xi|)(1 - |\xi|)^p d\theta(\xi)
\]

\[
= \int_{B^n} |Df(\xi)|^p \omega(1 - |\xi|)(1 - |\xi|)^p d\theta(\xi) \leq \|f\|^p_{B_p(\omega)} C(\pi, m)\|g\|^p_{\infty}
\]

Proof of (5).

\[
\int_{B^n} |f(z)|^p |g(z)|^p \frac{\omega(1 - |z|)}{(1 - |z|^2)^{n+1-p}} d\theta(z)
\]

\[
C(\pi, m)\|g\|^p_{\infty} \leq \int_{B^n} (1 - |\xi|^2)^{m+(p-1)(\gamma+1)} |Df(\xi)|^p \int_{B^n} \frac{\omega(1 - |z|)d\theta(z)d\theta(\xi)}{|1 - \langle z, \xi \rangle|^{m+n(1 - |z|^2)^{n+1-p+\gamma p/q}}}
\]

\[
\leq \int_{B^n} (1 - |\xi|^2)^{m+(p-1)(\gamma+1)} |Df(\xi)|^p \omega(1 - |\xi|) \frac{|Df(\xi)|^p \omega(1 - |\xi|)}{(1 - |\xi|)^{m+n-p+\gamma p/q}} d\theta(\xi)
\]

\[
= C(\pi, m)\|g\|_{\infty} \int_{B^n} |Df(\xi)|^p \omega(1 - |\xi|)(1 - |\xi|)^p d\theta(\xi)
\]

\[
\leq C(\pi, m)\|g\|_{\infty} \int_{B^n} (1 - |\xi|^2)^{m+(n+1)p} |Df(\xi)|^p d\theta(\xi)
\]

\[
\leq C(\pi, m)\|g\|_{\infty} \int_{B^n} (1 - |\xi|^2)^{m+(n+1)p} |Df(\xi)|^p \omega(1 - |\xi|) \frac{|Df(\xi)|^p \omega(1 - |\xi|)}{(1 - |\xi|^2)^{n+1-p}} d\theta(\xi)
\]

Let \(0 < p \leq 1\). Then by Lemma 4

\[
|f(z)|^p \leq C(\pi, m) \int_{B^n} (1 - |\xi|^2)^{m+(n+1)p} |Df(\xi)|^p d\theta(\xi)
\]

Then for (4) we get

\[
\int_{B^n} |f(z)|^p |Dg(z)|^p \frac{\omega(1 - |z|)}{(1 - |z|^2)^{n+1-p}} d\theta(z)
\]

\[
\leq C(\pi, m)\|g\|^p_{\infty} \int_{B^n} (1 - |\xi|^2)^{m+(n+1)p} |Df(\xi)|^p \frac{|Df(\xi)|^p \omega(1 - |\xi|)}{(1 - |\xi|^2)^{n+1-p}} d\theta(\xi)
\]

\[
= C(\pi, m)\|g\|_{\infty} \int_{B^n} |Df(\xi)|^p \omega(1 - |\xi|)(1 - |\xi|^2)^{n+(n+1)p} d\theta(\xi) = C(\pi, m)\|g\|_{\infty} \|f\|^p_{B_p(\omega)}
\]
Finally we obtain (5)

\[
\int_{B^n} |f(z)|^p |g(z)|^p \frac{\omega(1 - |z|)}{(1 - |z|^2)^{n+1-p}} d\theta(z)
\]

\[
\leq C(\pi, m) \|g\|_\infty \int_{B^n} \frac{(1 - |\xi|^2)^{mp+(n+1)p} |Df(\xi)|^p \omega(1 - |\xi|)}{(1 - |\xi|^2)^{n+1}} \frac{\omega(1 - |\xi|)}{(1 - |\xi|^2)^{(m+1)p-n-1-p}}
\]

\[
\leq C(\pi, m) \|f\|_{B^p(\omega)}^p \|g\|_\infty^p
\]

Summing ab, we get the proof of theorem.

\[\square\]

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