Some combinatorial matrices and their LU-decomposition

Abstract: Three combinatorial matrices were considered and their LU-decompositions were found. This is typically done by (creative) guessing, and the proofs are more or less routine calculations.

Keywords: Combinatorial matrix, LU-decomposition, Lehmer’s matrix, Fibonacci polynomials

MSC: 05A19; 15B36

1 Introduction

Combinatorial matrices often have beautiful LU-decompositions, which leads also to easy determinant evaluations. It has become a habit of this author to try this decomposition whenever he sees a new such matrix.

The present paper contains three independent ones collected over the last one or two years.

2 A matrix from polynomials with bounded roots

In [11] Kirschenhofer and Thuswaldner evaluated the determinant

\[ D_s = \det \left( \frac{1}{(2l)^2 - t^2(2l - 1)^2} \right)_{1 \leq i, j \leq s} \]

for \( t = 1 \). Consider the matrix \( M \) with entries \( \frac{1}{(2l)^2 - t^2(2l - 1)^2} \) where \( s \) might be a positive integer or infinity. In [11], the transposed matrix was considered, but that is immaterial when it comes to the determinant; we will treat the transposed matrix as well, but the results are slightly uglier.

The aim is to provide a completely elementary evaluation of this determinant which relies on the LU-decomposition \( LU = M \), which is obtained by guessing. The additional parameter \( t \) helps with guessing and makes the result even more general. We found these results:

\[
L_{i,j} = \frac{\prod_{k=1}^{j}((2j - 1)^2 t^2 - (2k)^2)}{\prod_{k=1}^{i}((2i - 1)^2 t^2 - (2k)^2)} \frac{(i + j - 2)!}{(i - j - 1)!},
\]

\[
U_{j,l} = \frac{t^{2j-2}(-1)^j 16^{j-1}(2j - 2)!}{\prod_{k=1}^{j}((2k - 1)^2 t^2 - (2l)^2)} \frac{(j + l - 1)!}{(j - l)!}.
\]

Note that

\[
\prod_{k=1}^{j}((2i - 1)^2 t^2 - (2k)^2) = (-1)^j 4^j \frac{\Gamma(j + 1 - t(i - \frac{1}{2})) \Gamma(j + 1 + t(i - \frac{1}{2}))}{\Gamma(1 - t(i - \frac{1}{2})) \Gamma(1 + t(i - \frac{1}{2}))}
\]

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and
\[
\prod_{k=1}^{j} ((2k-1)^2 t^2 - (2j)^2) = 4^j t^{2j} \frac{\Gamma(j + \frac{1}{2} + \frac{j}{2})}{\Gamma(j + \frac{1}{2})} \frac{\Gamma(j + \frac{1}{2} - \frac{j}{2})}{\Gamma(j + \frac{1}{2})};
\]

using these formulæ, \( L_{i,j} \) resp. \( U_{i,j} \) can be written in terms of Gamma functions.

The proof that indeed \( \sum_j L_{i,j} U_{j,i} = M_{i,j} \) is within the reach of computer algebra systems (Zeilberger’s algorithm). An old version of Maple (without extra packages) provides this summation.

As a bonus, we also state the inverses matrices:
\[
L_{i,j}^{-1} = \frac{\prod_{k=1}^{i-1} ((2j - 1)^2 t^2 - (2k)^2)}{\prod_{k=1}^{i-1} ((2i - 1)^2 t^2 - (2k)^2)} \frac{(-1)^{i+j}(2i-2)! (2j-1)!}{(i+j-1)!(i-j)!};
\]

and
\[
U_{i,j}^{-1} = \frac{\prod_{k=1}^{i-1} ((2k - 1)^2 t^2 - (2j)^2)}{\prod_{k=1}^{i-1} ((2i - 1)^2 t^2 - (2j)^2)} \frac{(-1)^{i+j}(2j)^2}{t^{2i-2}(2i-2)! (j+i)! (i-j)! 16^{i-1}};
\]

the necessary proofs are again automatic.

Consequently the determinant is
\[
D_s = \prod_{j=1}^{s} U_{j,j}.
\]

For \( t = 1 \), this may be simplified:
\[
D_s = \frac{1}{s!} \sum_{j=1}^{s} \frac{16^{j-1}(2j-2)! (2j-1)!}{(4j-4)! (4j-3)!} = \frac{4^s}{s!} \prod_{j=1}^{s} \frac{32^{j-1}(2j-1)!^2}{(4j-4)! (4j-3)!}
\]
\[
= \frac{4^s 16^s (s-1)!}{s!^2} \left( \prod_{j=1}^{s} \binom{4j}{2j} \binom{4j-2}{2j-1} \right) = \frac{4^s 16^s (s-1)!}{s!^2} \left( \prod_{j=1}^{s} \binom{2j}{j} \right)
\]
\[
= 16^s \frac{6(s-1)!}{s!^2} \left( \prod_{j=0}^{s-1} \binom{2j+1}{j} \right);
\]

the last expression was given in [11]. We used the notation \((2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)\).

Now we briefly mention the equivalent results for the transposed matrix:
\[
L_{i,j} = \frac{\prod_{k=1}^{i-1} ((2k-1)^2 t^2 - (2j)^2)}{\prod_{k=1}^{i-1} ((2k-1)^2 t^2 - (2j)^2)} \frac{(i+j-1)!}{(i-j+1)!}.
\]
\[
U_{i,j} = \frac{t^{2j-2} (16^{j-1} (2j-1)!)}{(j+i)!!} \frac{(2j-1)!}{(j+i+2)!}.
\]
\[
L_{i,j}^{-1} = \frac{\prod_{k=1}^{i-1} ((2k-1)^2 t^2 - (2j)^2)}{\prod_{k=1}^{i-1} ((2k-1)^2 t^2 - (2j)^2)} \frac{(-1)^{i+j} (2j)^2}{t^{2j-2} 16^{i-1} (2i-1)! (j+i+2)!};
\]
\[
U_{i,j}^{-1} = \frac{\prod_{k=1}^{i-1} ((2k-1)^2 t^2 - (2j)^2)}{\prod_{k=1}^{i-1} ((2k-1)^2 t^2 - (2j)^2)} \frac{(2j-1)!}{(2j-2)!} \frac{(-1)^{i+j} (2j)^2}{t^{2j-2} 16^{i-1} (2i-1)! (j+i+2)!};
\]
3 Lehmer’s tridiagonal matrix

Ekhad and Zeilberger [7] have unearthed Lehmer’s [12] tridiagonal $n \times n$ matrix $M = M(n)$ with entries (indexed by $1 \leq i, j \leq n$)

$$M_{i,j} = \begin{cases} 
1 & \text{if } i = j, \\
 z^{1/2} q^{(i-1)/2} & \text{if } i = j - 1, \\
 z^{1/2} q^{(i-2)/2} & \text{if } i = j + 1, \\
0 & \text{otherwise.}
\end{cases}$$

Note the similarity to Schur’s determinant

$$\text{Schur}(x) := \begin{vmatrix} 
1 & xq^{1+m} & \cdots \\
-1 & 1 & xq^{2+m} & \cdots \\
& -1 & 1 & xq^{3+m} & \cdots \\
& & -1 & 1 & xq^{4+m} & \cdots \\
& & & \ddots & \ddots & \ddots
\end{vmatrix}$$

that was used to great success in [9]. This success was based on the two recursions

$$\text{Schur}(x) = \text{Schur}(xq) + xq^{1+m} \text{Schur}(xq^2)$$

and, with

$$\text{Schur}(x) = \sum_{n=0} a_n x^n,$$

by

$$a_n = q^n a_n + q^{1+m} q^{2n-2} a_{n-1},$$

leading to

$$a_n = \frac{q^{n^2 + mn}}{(1-q)(1-q^2) \ldots (1-q^n)}.$$
It follows from the basic recursion of the Gaussian $q$-binomial coefficients [2] that

$$
\lambda(j) = \lambda(j - 1) - zq^{j-2}\lambda(j - 2).
$$

(1)

Then we have

$$
U_{j,j} = \frac{\lambda(j)}{\lambda(j - 1)}, \quad U_{j,j+1} = z^{1/2} q^{(j-1)/2},
$$

and all other entries in the $U$-matrix are zero. Further,

$$
L_{j,j} = 1, \quad L_{j+1,j} = z^{1/2} q^{(j-1)/2} \frac{\lambda(j) - 1}{\lambda(j)},
$$

and all other entries in the $L$-matrix are zero.

The typical element of the product $(LU)_{i,j}$, that is

$$
\sum_{1 \leq k \leq n} L_{i,k} U_{k,j}
$$

is almost always zero; the exceptions are as follows: If $i = j$, then we get

$$
L_{j,j} U_{j,j} + L_{j,j-1} U_{j-1,j} = \frac{\lambda(j) + zq^{j-2}\lambda(j - 2)}{\lambda(j - 1)} = 1,
$$

because of the above recursion (1). If $i = j - 1$, then we get

$$
L_{j-1,j-1} U_{j-1,j} + L_{j-1,j-2} U_{j-2,j} = z^{1/2} q^{(j-2)/2},
$$

and if $i = j + 1$, then we get

$$
L_{j+1,j+1} U_{j+1,j} + L_{j+1,j} U_{j,j} = z^{1/2} q^{(j-1)/2} \frac{\lambda(j - 1) - 1}{\lambda(j)} \frac{\lambda(j)}{\lambda(j - 1)} = z^{1/2} q^{(j-1)/2}.
$$

This proves that indeed $LU = M$. Therefore for the determinant of the Lehmer matrix $M$ we obtain the expression

$$
\prod_{j=1}^{n} \frac{\lambda(j)}{\lambda(j - 1)} = \frac{\lambda(n)}{\lambda(0)} = \sum_{0 \leq k \leq n/2} \left[ \begin{array}{c} n-k \\ k \\ \end{array} \right] (-1)^k q^{k(k-1)} z^k.
$$

Taking the limit $n \to \infty$, leads to the old result by Lehmer for the determinant of the infinite matrix:

$$
\lim_{n \to \infty} \det(M(n)) = \sum_{k \geq 0} \frac{(-1)^k q^{k(k-1)} z^k}{(q; q)_k}.
$$

Remarks.

1. For $q = 1$, Lehmer’s determinant plays a role when enumerating lattice paths (Dyck paths) of bounded height, or planar trees of bounded height, see [6, 8, 10].

2. Recursions as in (1) have been studied in [3, 4, 13] and are linked to so-called Schur polynomials [15].

4 Matrices for Fibonacci polynomials

Cigler [5] introduced several matrices that have Fibonacci polynomials as determinants; we will only treat two of them as showcases.

The Fibonacci polynomials are

$$
F_n(x) = \sum_h \binom{n-h}{h} x^{n-2h};
$$

our answers will come out in terms of the related polynomials

$$
f_n = \sum_h \binom{n+h}{2h} x^h.$$
where we write \( X = x^2 \) for simplicity. It is easy to check that

\[
f_n = (X + 2)f_{n-1} - f_{n-2},
\]

for instance by comparing coefficients.

The first matrix is

\[
M = \left( \begin{array}{c} i - 1 \frac{X}{j} + \frac{i + 1}{j + 1} \end{array} \right)_{0 \leq i, j \leq n}
\]

and we will determine its LU-decomposition \( M = LU \).

We obtained

\[
L_{i,j} = \frac{\left( \frac{i + 1}{2h} \right)X^h + \left( \frac{j}{2h-1} \right)X^h}{\sum_h \left( \frac{i + 1}{2h} \right)X^h} = \left( \frac{i}{j} \right) + \frac{i}{j + 1} \frac{f_{j+1}}{f_j}
\]

and

\[
U_{j,i} = \frac{\left( \frac{i + 1}{2h} \right)X^h}{\sum_h \left( \frac{i + 1}{2h} \right)X^h} = \frac{f_{j+1}}{f_j},
\]

\[
U_{j,i} = (-1)^{i+j} \sum_h \left( \frac{j}{2h-1} \right)X^h = (-1)^{i+j} \left( \frac{f_{j+1}}{f_j} - 1 \right), \quad j < i.
\]

For a proof, we do this computation:

\[
\sum_j L_{i,j}U_{j,i} = L_{i,1}U_{1,1} + \sum_{0 \leq j < i} L_{i,j}U_{j,i}
\]

\[
= \left[ \left( \frac{i}{l} \right) + \frac{i}{l + 1} \right] \frac{f_{j+1}}{f_j} + \sum_{0 \leq j < l} \left[ \left( \frac{i}{j} \right) + \frac{i}{j + 1} \right] \frac{f_{j+1}}{f_j} \left(-1\right)^{i+j} \left( \frac{f_{j+1}}{f_j} - 1 \right)
\]

\[
= \left( \frac{i}{l} \right) \frac{f_{j+1}}{f_j} + \frac{i}{l + 1} \sum_{0 \leq j < l} \left( \frac{i}{j} \right) \frac{f_{j+1}}{f_j} \left(-1\right)^{i+j} + \sum_{0 \leq j < l} \left( \frac{i}{j + 1} \right) \frac{f_{j+1}}{f_j} \left(-1\right)^{i+j}
\]

\[
- \sum_{0 \leq j < l} \left( \frac{i}{j} \right) \left(-1\right)^{i+j} - \sum_{0 \leq j < l} \left( \frac{i}{j + 1} \right) \frac{f_{j+1}}{f_j} \left(-1\right)^{i+j}
\]

\[
= \left( \frac{i}{l + 1} \right) + \left( X + 2 \right) \sum_{0 \leq j < l} \frac{i}{j} \left(-1\right)^{i+j} + \sum_{0 \leq j < l} \left( \frac{i}{j + 1} \right) \left(-1\right)^{i+j} - \sum_{0 \leq j < l} \left( \frac{i}{j + 1} \right) \left(-1\right)^{i+j}
\]

\[
- \sum_{0 \leq j < l} \left( \frac{i}{j} \right) \left(-1\right)^{i+j} - \sum_{0 \leq j < l} \left( \frac{i}{j + 1} \right) \frac{f_{j+1}}{f_j} \left(-1\right)^{i+j}
\]

\[
= \left( \frac{i}{l + 1} \right) + \left( X + 2 \right) \sum_{0 \leq j < l} \frac{i}{j} \left(-1\right)^{i+j} + \sum_{0 \leq j < l} \left( \frac{i}{j + 1} \right) \left(-1\right)^{i+j} - \sum_{0 \leq j < l} \left( \frac{i}{j + 1} \right) \left(-1\right)^{i+j} - \sum_{0 \leq j < l} \left( \frac{i}{j} \right) \left(-1\right)^{i+j}
\]

\[
- \sum_{0 \leq j < l} \left( \frac{i}{j} \right) \left(-1\right)^{i+j} + \left( \frac{i}{l} \right) + \sum_{1 \leq j < l} \frac{i}{j} \frac{f_{j+1}}{f_j} \left(-1\right)^{i+j}
\]

\[
= X \left( \frac{i - 1}{l} \right) + \left( \frac{i}{l + 1} \right) + \left( \frac{i}{l} \right) + \sum_{0 \leq j < l} \frac{i}{j} \left(-1\right)^{i+j} - \sum_{1 \leq j < l} \frac{i}{j} \left(-1\right)^{i+j} - \left(-1\right)^j
\]

\[
= X \left( \frac{i - 1}{l} \right) + \left( \frac{i + 1}{l + 1} \right).
\]

The determinant is then \( U_{0,0}U_{1,1} \ldots U_{n-1,n-1} \), and by telescoping

\[
\sum_h \left( \frac{n + h}{2h} \right)X^h = \sum_h \left( \frac{2n - h}{h} \right)X^{2n - 2h} = F_{2n}(x).
\]
For completeness, we also factor the transposed matrix as $LU = M'$:

$$L_{i,j} = (-1)^{i+j} \frac{\sum_h (j + h) X^h}{\sum_h (2h+1) X^h}, \quad \text{for } j < i,$$

$$L_{i,i} = 1,$$

and

$$U_{j,i} = \frac{\binom{i}{j} \sum_h (2h+1) X^h + \binom{i+1}{j+1} \sum_h (2h+1) X^h}{\sum_h (2h+1) X^h}.$$

Now we move to the second matrix:

$$M = \begin{pmatrix} \binom{i}{j} X & \binom{i+2}{j+1} \end{pmatrix} \times_{0 \leq i, j \leq n}.$$

We find

$$L_{i,j} = \frac{\binom{i+1}{j+1} \sum_h (2h+1) X^h + \binom{i}{j} \sum_h (2h+1) X^h}{\sum_h (2h+1) X^h}$$

and

$$U_{j,i} = \frac{\sum_h (2h+1) X^h}{\sum_h (2h+1) X^h}, \quad \text{for } j > i + 1, \quad U_{j,j} = 0 \text{ for } j > i + 2.$$

For a proof, we compute

$$\sum_j L_{i,j} U_{j,l} = \frac{\binom{i+1}{l+1} \sum_h (2h+1) X^h + \binom{i}{l} \sum_h (2h+1) X^h}{\sum_h (2h+1) X^h}$$

and

$$\sum_h \left( \frac{l+1+h}{2h+1} \right) X^h \sum_j L_{i,j} U_{j,l} = \frac{\binom{i+2}{l+1} \sum_h (2h+1) X^h + \binom{i+1}{l} \sum_h (2h+1) X^h}{\sum_h (2h+1) X^h}$$

$$\sum_h \left( \frac{l+1+h}{2h+1} \right) X^h \sum_j L_{i,j} U_{j,l} = \frac{\binom{i+2}{l+1} \sum_h (2h+1) X^h + \binom{i+1}{l} \sum_h (2h+1) X^h}{\sum_h (2h+1) X^h}$$

and therefore

$$\sum_j L_{i,j} U_{j,l} = \binom{i+2}{l+1} + \binom{i}{l} X.$$
as required. The determinant is then

\[ \sum_h \binom{n + 1 + h}{2h + 1} X^h = \sum_h \binom{n + 1 + h}{n-h} X^h = \sum_j \binom{2n + 1 - j}{j} x^{2n-2j} = x^{-1} F_{2n+1}(x^2). \]

For the transposed matrix \( LU = M^t \), we find

\[ L_{i,i-1} = \sum_h \binom{i+h}{2h+1} X^h, \]
\[ L_{i,i} = 1, \quad L_{i,j} = 0 \quad \text{for } j < i - 1, \]

and

\[ U_{i,j} = \frac{\binom{i+j}{j-1} \sum_h \binom{i+1+h}{2h+1} X^h + \binom{i}{j} \sum_h \binom{i+1+h}{2h} X^h}{\sum_h \binom{i+1+h}{2h+1} X^h}. \]

For completeness, we mention another recent paper about matrices and Fibonacci polynomials: [1].

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