Developed RKM Method for Solving Ninth-Order Ordinary Differential Equations with Applications

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Abstract In this study, the development of direct explicit numerical integrators of RK-type for solving a class of ninth-order ordinary differential equations (ODEs) to develop the computational efficiency of the methods. The objective of this paper is to generalize the numerical integrators of RK type for solving classes of ODEs of orders up to eighth-order ODEs. Using the approach of Taylor-expansion, we have derived the order-conditions (OCs) for RKM integrators. Based on these OCs, developed method of ninth-order with five-stage has been derived. The zero stability of RKM integrator is proven. Stability polynomial of RKM integrator for test problem has been introduced. The advantage and the efficiency of the proposed method have been shown clearly using the numerical results which are agree well with analytical solutions due to the proposed method is more efficient and accurate method.

Keyword: RK; RKN; RKD; RKM; RKT; RKFD methods; Integrator; Class of ninth-order; Stage; ODE; Order conditions; Taylor expansion.

1 Introduction

Mathematics has several tools to describe the problems in real life, engineering and science. Ordinary and partial differential equations (ODEs, PDEs) are significant tools in applied mathematics. They played significant rule in describing the mathematical models in applications of engineering, science and economics. High-order DE arises in some fields of engineering and science such as nonlinear optics and quantum mechanics. The numerical or approximated solutions of PDEs or ODEs are very important in scientific computation, as they are widely used to model real life problems. For review of numerical methods for ODEs, [1, 2, 3, 4, 5, 6, 7, 8, 9] have derived numerical methods of RK-type for examining the solutions of ODEs of second-order up to eighth-order. However, to improve the computational efficiency of the indirect numerical methods for solving class of ninth-order ODEs, the indirect numerical methods improved to be direct,
more efficient and accurate methods. By Taylor expansion, we have derived the OCs for RKM integrator up to the 10th-order. According to these OCs, RKM method of fifth-stages and ninth-order has been derived. The novelty of this work is the generalization of numerical methods of RK-type for solving the ODEs of order less than eighth-order ODEs. Numerical implementations using Maple and MATLAB show that the numerical solutions of proposed integrator agree well with exact solutions due to the fact the proposed integrator is efficient and accurate integrator and it has less computational time comparing with classical methods. In this paper, explicit RKM method for solving ninth-order quasi-linear ODEs have studied.

2 Preliminary

In this section, we have introduced the definition of class of 9th-ODEs as the follows:

2.1 Class of Ninth-Order Ordinary Differential Equations

This subsection concerned with class of ninth-order ODEs which has no appearance up to eighth-order derivatives. This class can be written as follows:

\[ y^{(9)}(t) = g(t, y(t)); \quad t \geq t_0 \]  

(1)

Subject to initial condition: \( y(t_0) = \alpha_i; i = 0, 1, \ldots, 8 \) where \( f : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) and \( y(t) = [y_1(t), y_2(t), \ldots, y_N(t)] \). Knowing that, \( f \) is vector of independent variables of \( N \) components of the system of ODE (1). Some of engineers and scientists used to solve Equation (1) by multistep method (Lmm). Mostly, they used to solve ODEs of higher-order by converting them to equivalent system of first-order ODEs and they solved using a classical RK method [10]. However, it would be more efficient if eighth-order ODE can be solved using proposed direct RKM method. In this paper, we are concerned with direct explicit RKM integrators for solving class of eighth-order ODEs. Using Taylor series expansion approach, we have obtained the order-conditions of the proposed methods. Consequently, we have derived RKM integrator based on these algebraic OCs.

3 Proposed RKM Method

The proposed explicit RKM integrator with s-stage for solving ninth-order ODE (1) has the following formula where \( \mu = \frac{t - t_0}{\eta} \):

\[
y_{n+1} = y_n + \mu y'_n + \frac{\mu^2}{2} y''_n + \frac{\mu^3}{6} y'''_n + \frac{\mu^4}{24} y^{(4)}_n + \frac{\mu^5}{120} y^{(5)}_n + \frac{\mu^6}{720} y^{(6)}_n + \frac{\mu^7}{5040} y^{(7)}_n + \frac{\mu^8}{40320} y^{(8)}_n + \mu \sum_{i=1}^{s} b_i k_i, \\
\]

\[
y_{n+1}' = y_{n}' + \mu y_{n}'' + \frac{\mu^2}{2} y_{n}''' + \frac{\mu^3}{6} y_{n}^{(4)} + \frac{\mu^4}{24} y_{n}^{(5)} + \frac{\mu^5}{120} y_{n}^{(6)} + \frac{\mu^6}{720} y_{n}^{(7)} + \frac{\mu^7}{5040} y_{n}^{(8)} + \mu \sum_{i=1}^{s} b_{i}' k_{i}, \\
\]

\[
y_{n+1}'' = y_{n}'' + \mu y_{n}''' + \frac{\mu^2}{2} y_{n}^{(4)} + \frac{\mu^3}{6} y_{n}^{(5)} + \frac{\mu^4}{24} y_{n}^{(6)} + \frac{\mu^5}{120} y_{n}^{(7)} + \frac{\mu^6}{5040} y_{n}^{(8)} + \mu \sum_{i=1}^{s} b_{i}'' k_{i},
\]

(2)
\[ y_{n+1}^m = y_n^m + h_{y n}^m + \frac{h^2}{2} y_{n}^{m''} + \frac{h^3}{6} y_{n}^{m'''} + \frac{h^4}{24} y_{n}^{m'''} + \frac{h^5}{120} y_{n}^{m'''} + h^6 \sum_{i=1}^{s} y_i^{m''} k_i, \]  
\( m = 1 \)  
\[ y_{n+1}^{mm} = y_n^{mm} + h y_{n}^{mm} + \frac{h^2}{2} y_{n}^{mm'} + \frac{h^3}{6} y_{n}^{mm''} + \frac{h^4}{24} y_{n}^{mm''} + h^5 \sum_{i=1}^{s} y_i^{mm''} k_i, \]  
\[ y_{n+1}^{mmm} = y_n^{mmm} + h y_{n}^{mmm} + \frac{h^2}{2} y_{n}^{mmm'} + \frac{h^3}{6} y_{n}^{mmm''} + \frac{h^4}{24} y_{n}^{mmm''} + h^5 \sum_{i=1}^{s} y_i^{mmm''} k_i, \]  
\[ y_{n+1}^{mmmm} = y_n^{mmmm} + h y_{n}^{mmmm} + \frac{h^2}{2} y_{n}^{mmmm'} + \frac{h^3}{6} y_{n}^{mmmm''} + \frac{h^4}{24} y_{n}^{mmmm''} + h^5 \sum_{i=1}^{s} y_i^{mmmm''} k_i, \]  
\[ y_{n+1}^{mmmm'} = y_n^{mmmm'} + h \sum_{i=1}^{s} y_i^{mmmm'} k_i, \]

and

\[ k_1 = g(t_n, y_n), \]
\[ k_i = g(t_n + c_i h, y_n + h c_i y_n') + \frac{h^2}{2} c_i^2 y_n'' + \frac{h^3}{6} c_i^3 y_n''' + \frac{h^4}{24} c_i^4 y_n''' + \frac{h^5}{120} c_i^5 y_n'''
+ \frac{h^6}{720} c_i^6 y_n^{''''} + \frac{h^7}{5040} c_i^7 y_n^{'''''} + h^8 \sum_{j=1}^{i-1} a_{ij} k_j. \]

\( a_{ij}, b_i, b_i', b_i'', b_i''' \) are parameters of RKM integrator for weakly and \( h \) is the step-size. RKM is an implicit method if \( a_{ij} = 0 \) for \( i > j \) and otherwise RKM is explicit method. The parameters of RKM integrator have been expressed in Table 1: We have derived the OCs of RKM integrator for solving class of ninth-order ODEs by using the same approach of finding OCs of RKM methods for solving 3\( ^{rd} \)-order ODEs which derived by [1].

| C | A |
|---|---|
| y\(^p\) | y\(^p\) |
| y\(^p\) | y\(^p\) |
| y\(^p\) | y\(^p\) |
| y\(^p\) | y\(^p\) |
| y\(^p\) | y\(^p\) |
| y\(^p\) | y\(^p\) |
| y\(^p\) | y\(^p\) |

Table 1: Butcher Table of RKM method
3.1 The Order-Conditions Derivation of RKM Methods

To evaluate the parameters of RKM integrators in formulae (2)-(12). Firstly, we have to determine the OCs of the numerical integrators indicated by (2)-(12) using the approach of Taylor’s series expansion using Maple software. Secondly, solve these OCs to obtain the parameters of RKM integrators in formulae (2)-(12). Hence, the algorithm of finding the OCs as follows:

1. Using Taylor expansion, we expand $y^{(m)}_{i+1} = y^{(m)}(x_i + h)$ as follows:

$$
y^{(m)}_{i+1} = y^{(m)}_i + hy^{(m+1)}_i + \frac{h^2}{2}y^{(m+2)}_i + \frac{h^3}{6!}y^{(m+3)}_i + \cdots + \frac{h^n}{n!}y^{(m+n)}_i + \cdots
$$

2. Expand Equation (12) for $i = 2, 3, \ldots, s$.

3. Compare the two expansions of $y^{(m)}_{i+1}$ in step 1 and step 2 to obtain the following order conditions with respect to $y^{(m)}_i$ for $m = 0, 1, \ldots, 8; n = 0, 1, \ldots, m$

The OCs have derived as following:

\[ y : \sum b_i = \frac{1}{5040}, \sum b_i c_i = \frac{1}{40320}, \sum b_i c^2_i = \frac{1}{181440}, \sum b_i c^3_i = \frac{1}{604800}, \]  
\[ y' : \sum b'_i = \frac{1}{720}, \sum b'_i c_i = \frac{1}{5040}, \sum b'_i c^2_i = \frac{1}{20160}, \sum b'_i c^3_i = \frac{1}{60480}, \sum b'_i c^4_i = \frac{1}{151200}. \]  
\[ y'' : \sum b''_i = \frac{1}{720}, \sum b''_i c_i = \frac{1}{5040}, \sum b''_i c^2_i = \frac{1}{20160}, \sum b''_i c^3_i = \frac{1}{60480}, \sum b''_i c^4_i = \frac{1}{151200}. \]  
\[ y''' : \sum b'''_i = \frac{1}{120}, \sum b'''_i c_i = \frac{1}{720}, \sum b'''_i c^2_i = \frac{1}{2520}, \sum b'''_i c^3_i = \frac{1}{6720}, \sum b'''_i c^4_i = \frac{1}{15120}, \sum b'''_i c^5_i = \frac{1}{30240}. \]  
\[ y^{'''} : \sum b^{'''}_i c^6_i = \frac{1}{55440}. \]  
\[ y^{''''} : \sum b^{''''}_i = \frac{1}{24}, \sum b^{''''}_i c_i = \frac{1}{120}, \sum b^{''''}_i c^2_i = \frac{1}{360}, \sum b^{''''}_i c^3_i = \frac{1}{840}, \sum b^{''''}_i c^4_i = \frac{1}{1680}, \sum b^{''''}_i c^5_i = \frac{1}{3024}. \]  
\[ y^{'''''} : \sum b^{'''''}_i c^6_i = \frac{1}{7920}. \]  
\[ y^{''''''} : \sum b^{''''''}_i = \frac{1}{6}, \sum b^{''''''}_i c_i = \frac{1}{24}, \sum b^{''''''}_i c^2_i = \frac{1}{60}, \sum b^{''''''}_i c^3_i = \frac{1}{120}, \sum b^{''''''}_i c^4_i = \frac{1}{210}, \sum b^{''''''}_i c^5_i = \frac{1}{336}. \]  
\[ y^{'''''''} : \sum b^{'''''''}_i c^6_i = \frac{1}{990}. \]  
\[ y^{''''''''} : \sum b^{'''''''}_i c^7_i = \frac{1}{7920}, \sum b^{'''''''}_i c^8_i = \frac{1}{990}. \]  
\[ y^{''''''''' :} \sum b^{'''''''''_i c^9_i = \frac{1}{42}. \]  
\[ \sum b^{'''''''''_i = \frac{1}{56} \sum b^{'''''''''_i c^7_i = \frac{1}{72} \sum b^{'''''''''_i c^8_i = \frac{1}{90} \sum b^{'''''''''_i c^9_i = \frac{1}{110} \sum_{i=2,j=1}^{5,4} b^{'''''''''_{ij} a_{ij} = \frac{1}{40320}. \]
3.2 Derivation of RKM Methods

Using the following assumption: \( y^{(m)} = \frac{(1-c_i)^{(8-m)}}{(8-m)}y_i^{(8-m)} \), for \( m = 0, 1, \ldots, 8 \) and \( i = 1, 2, \ldots, s \), we have derived RKM method using the OCs (13)-(19) using Maple software. The coefficients of proposed RKM method denoted by \( a_{ij}, c_i \) and \( y_i^{(m)} \) for \( m = 0, 1, \ldots, 8 \) and \( i = 1, 2, \ldots, s \) for five-stage, nine-order RJKM integrator have been evaluated and tabled in Table 2.

| \( i \) | \( j \) | \( a_{ij} \) | \( c_i \) | \( y_i^{(m)} \) |
|-------|-------|-------------|--------|-------------|
| 0     | 0     | 0           | 0      | 0           |
| 1     | \( \frac{1}{2} \) | \( \frac{-160+372\sqrt{21}}{21} \) | \( \frac{1}{2} \) | \( \frac{-88905600}{210}+\frac{23\sqrt{21}}{3} \) |
| \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{-1920+1008\sqrt{21}}{7}+\frac{23\sqrt{21}}{3} \) | \( \frac{-88905600}{210}+\frac{23\sqrt{21}}{3} \) | \( \frac{1}{2} \) |
| \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{-1920+1008\sqrt{21}}{7}+\frac{23\sqrt{21}}{3} \) | \( \frac{-88905600}{210}+\frac{23\sqrt{21}}{3} \) | \( \frac{1}{2} \) |

Table 2: The Butcher Table for the RKM Method of Five-Stages and Eleventh-Order.
4 Stability of the RKM Method

In this section, we have studied the stability of the RKM method

4.1 Zero-Stability of RKM Method

Definition 4.1 RKM method is said to be zero-stable if it satisfied \(-1 < \xi \leq 1\). [11] 

The zero-stability is significant tool for examining the stability and convergence properties of the method. The characteristic polynomial for this method is \(\rho(\xi) = (\vartheta - 1)^9\). However, RKM integrator is zero-stable since the roots are \(\vartheta = 1; 9 - \text{times}\) are less or equal to one.

4.2 Absolute-Stability of RKM Method

Definition 4.2 RKM integrator is said to be absolutely-stable if all the roots lies within the unit circle [11]. The absolute-stability for RKD integrator has studied by [12] and the stability-regions for RKT and RKD methods have been compared. The following test problem \(w^{(9)}(\iota) = -\chi w(\iota)\) has been used to introduce the absolute stability for RKM method. The stability-function associated with RKM method is given by, 

\[ \Theta(\lambda, h) = \lambda_0(h)\lambda^9 + \lambda_1(h)\lambda^8 + \lambda_2(h)\lambda^7 + \lambda_3(h)\lambda^6 + \lambda_4(h)\lambda^5 + \lambda_5(h)\lambda^4 + \lambda_6(h)\lambda^3 + \lambda_7(h)\lambda^2 + \lambda_8(h)\lambda + \lambda_9(h). \]

5 Applications

In this section, we prove that the efficiency of the proposed method for solving some problems. The numerical results of these problem comparing with their analytical solutions have been showing in Figure 2.

Example 5.1 (Linear ODE) 

\[ y^{(9)}(\zeta) = -y(\zeta); \quad 0 < \zeta \leq 1. \]

subject to ICs: \(y^{(i)}(0) = (-1)^i; i = 0, 1, \ldots, 8.\)

The exact solution is \(y(\zeta) = e^{-\zeta}\)

Example 5.2 (Linear ODE) 

\[ y^{(9)}(\zeta) = -512y(\zeta); \quad 0 < t \leq 1. \]

subject to ICs: \(y^{(i)}(0) = (-2)^i; i = 0, 1, \ldots, 8.\)

The exact solution is \(y(\zeta) = e^{-2\zeta}\)
Example 5.3 (Non Constant Coefficients ODE)

\[ y^{(9)}(\zeta) = (-30240\zeta + 80640\zeta^3 - 48384\zeta^5 + 9216\zeta^7 - 512\zeta^9)y(\zeta); \quad 0 < \zeta \leq 1. \]

**subject to ICs:** \( y^{(8)}(0) = -14y^{(6)}(0) = -140y^{(4)}(0) = -840y^{(2)}(0) = -1680y(0) = 1680. \)

\( y^{(2i+1)}(0) = 0 \) for \( i = 0, 1, 2, 3 \)

*The exact solution is* \( y(\zeta) = e^{-\zeta^2} \)

Example 5.4 (Linear ODE)

\[ y^{(9)}(\zeta) = -\sin(\zeta), \quad 0 < \zeta \leq 1. \]

**subject to ICs:** \( y^{(2i)}(0) = (-1)^i \) and \( y^{(2i+1)}(0) = 0 \) for \( i = 0, 1, 2, 3 \).

*The exact solution is* \( y(\zeta) = \cos(\zeta) \).

Example 5.5 (Non Linear ODE)

\[ y^{(9)}(\zeta) = \frac{40320}{(1 + \zeta)^9}, \quad 0 < \zeta \leq 1. \]

**subject to ICs:** \( y(0) = 0, y^{(n)}(0) = (-1)^{n-1}(n-1)! \) for \( n = 1, 2, \ldots, 8 \)

*The exact solution is* \( y(\zeta) = \log(1 + \zeta) \)

Example 5.6 (Linear ODE with Relatively Long Interval)

\[ y^{(9)}(\zeta) = -0.0000000001e^{-\zeta^6}, \quad 0 < \zeta \leq 1. \]

**subject to ICs:** \( y^{(i)}(0) = (-0.1)^i; i = 0, 1, \ldots, 6 \).

*The exact solution is* \( y(\zeta) = e^{-\zeta^6} \)

6 Conclusion and Discussion

In this paper, the algebraic equations of OCs of RKM integrator for solving ninth-order quasi-linear ODEs have been derived. According to Taylor expansion, the proposed RKM method has been derived. The objective of this work is to construct explicit integrator for solving ninth-order, quasi-linear ODEs of RK-type by generalizing the integrators which are used for solving classes of order less than eighth-order ODEs. Ninth-order with five-stage RKM method for solving this class of ODEs has derived. Numerical solutions using the RKM method have been compared with analytical solutions in figures 1-6 which show that the approximated solutions of the implementations are more accurate and efficient as well-known existing integrators due to RKM integrate require less function evaluations.
Figure 1: Comparisons on approximated solutions versus exact solutions for Problems (a) 1, (b) 2, (c) 3, (d) 4, (e) 5 and (f) 6
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