Regularization of Vacuum Fluctuations and Frame Dependence

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We discuss the regularization of vacuum fluctuations in a gravitational background with a sharp momentum cut-off. It is shown that general covariance is broken even by a 4-momentum cut-off and is related to the problem of frame choice. We find that in the case the regulator is related to a physical frame, a running of the cosmological constant is expected. This result is compared to the implications of other regularization procedures. Our main conclusion is that in any case the vacuum energy is a running quantity and we are able to compute the beta function for the scalar field in an FRW background.

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I. INTRODUCTION

In quantum field theories contact diagrams diverge. This problem was resolved by the technique of regularization and renormalization. It is important to understand what the reason for the divergences is and what lies at the core of the renormalization approach. Mathematically the divergences arise from multiplication of distributions which is a priori an undefined operation. To give an example consider scalar field theory with cubic self interaction. In position space for example a diagram with four external legs and a two off shell particles is proportional to \[ \int d^4x \int d^4y \phi^2(x)G^2(x-y)\phi^2(y). \] The problematic term is the square of \( G(x-y) \), the Greens function of the scalar field.

This problem can be addressed by regarding distributions as functions almost everywhere. In this particular case the expression is unproblematic everywhere except for the origin where \( x = y \) i.e. the UV limit in momentum space. The product thus can be defined on the quotient space \( D(\mathbb{R}^4)/D(\mathbb{R}^4 \setminus \{0\}) \) with \( D \) being the domain of functions the distribution can act on. This corresponds to subtracting the problematic point. Now the defined quantity has to be continued again to the whole space. This is done by adding a counter-term corresponding to the graph where the loop is only a point i.e \( \propto \lambda \phi^4 \). This procedure is equally valid on curved space-time backgrounds as shown in [1, 2]. The important fact is that \( \lambda \) is chosen “by hand” or fixed by experiment i.e. the field theory has no prediction at all for this value. We see that the very nature of renormalization is such that a prediction is impossible.

In the process of renormalization a technique is needed to quantify the degree of divergence of a diagram, called regularization. There are different schemes as dimensional or momentum regularization. To apply this procedure a regulator is introduced which is viewed as a mathematical tool and thus has to be removed at the end of the procedure.

The other way to view a regulator is to assume that the divergence itself is an artefact of the mathematical description, that physics is based on a fundamentally non singular theory and the regulator is a manifestation of a physical quantity. In the literature it has been argued that be embedding the quantum field theory QFT in some UV complete theory, where the regulator is a physical quantity, predictions for the vacuum energy can be made [3, 4]. Nevertheless, the underlying theory must respect the fundamental symmetries below the cut-off scale. Since we know that a momentum cut-off is only defined in a particular frame we expect violation of covariance as addressed in [5, 6]. We study this issue and compare the findings to other regularization schemes. We discuss physical implications at the end of this letter. The main result of the study is that in any case the vacuum energy has a QFT induced running and we will compute the beta function of this running in an FRW background for a scalar field.

II. VACUUM FLUCTUATIONS

Since the observation of Casimir forces [7] and recently even the dynamical Casimir effect [8] it is clear that the vacuum fluctuations of fields are not just mathematical peculiarities of the theory but physical reality. Thus we expect an impact of this vacuum energies on gravity. Since it is the energy momentum tensor which shapes our space-time we will make the attempt to calculate the contribution of vacuum fluctuations to this quantity. We will use the same approach as discussed in [3, 9].

A. The set up

Let us consider the minimal set up, where a massless scalar field \( \phi \) is minimally coupled to gravity, hence the Lagrangian we are working with is:
\[ \mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi \] (1)

We will use a semi classical approach to this problem, where a gravitational classical background is assumed to exist and the matter and metric perturbation fields are considered to be quantized on that background. We will demonstrate now the scalar field quantization. For the definition of quantum field operators we need a complete where a gravitational classical background is assumed to take us to define a scalar product by an orthogonal future directed flow \( u^\mu \). This enables us to define a scalar product

\[ \langle \phi | \psi \rangle = -i \int_\Sigma \sqrt{\mathcal{H}} (\phi \partial_\mu \psi - (\partial_\mu \phi) \psi^* ) u^\mu d\Sigma, \] (2)

here \( h \) is the metric induced on \( \Sigma \).

We assume the existence of the set of solutions orthonormal in the sense of this scalar product and define raising and lowering operators with respect to them. This means that we have selected a Fock space construction and have chosen a vacuum state. This construction is a direct consequence of the time-space split discussed above.

The scalar field operator can be written as a mode expansion with the raising and lowering operators

\[ \phi(x) = \int \frac{d^3k}{(2\pi)^3 2k} [a_k \phi_k(t) e^{ikx} + h.c.] . \] (3)

Varying the action w.r.t. \( \phi \) we obtain the equation of motion the field has to obey. The wave equation reads in the FRW background as:

\[ \Box \phi(x) = \nabla_\mu g^{\mu\nu} \partial_\nu \phi = -\partial_\mu^2 \phi - \Gamma_{0\lambda}^\mu \partial_\nu \phi + \frac{1}{a^2} \partial_i^2 \phi = 0. \] (4)

Which for the individual modes implies

\[ \partial_\eta^2 \phi_k + \frac{3}{a} \partial_\eta \phi_k + \frac{1}{a^2} k^2 \phi_k = 0 \quad \Leftrightarrow \quad \phi_k'' + 2H \phi_k' + k^2 \phi_k = 0 \] (5)

with \( \frac{d}{d\eta} := \frac{\partial}{\partial \eta} \) and \( \eta = \int \frac{dt}{a} \).

Defining \( \psi_k := \phi_k/a \) the equation reduces to

\[ \psi_k'' + \left( k^2 - \frac{a''}{a} \right) \psi_k = 0. \] (6)

Various solutions to this equation are known which form depends on the background type. Choosing a de Sitter background where \( a = -1/(H \eta) \) the solution is [3]

\[ \phi_k(\eta) = \frac{1}{a} \left( 1 - \frac{i}{k \eta} \right) e^{-ik \eta} = e^{-ik \eta} \left( \frac{i}{k} - n \right) H. \] (7)

**B. The energy momentum tensor**

In this set up, however, the particular dynamics of the field is not of interest but its average effect on the space time. Therefore, we seek an expression for the vacuum expectation value of the energy momentum of the field

\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\lambda \phi \partial^\lambda \phi. \] (8)

To define the energy density and pressure consistent with our time and space split we use the frame field \( u^\mu \) to define the time direction. Note that it fulfills the normalization condition \( u_\mu u^\mu = -N \) with \( N \) being the lapse function to stay general. Hence the orthogonal projector to this direction is defined by

\[ h^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}. \] (9)

And the induced metric \( h_{ij} \) on the spatial hypersurface \( \Sigma \) is the pullback of it, with \( X \) the coordinates on the 4 manifold and \( x \) coordinates on the 3 manifold

\[ h_{ij} = h_{\mu\nu} \frac{\partial X^\mu}{\partial x^i} \frac{\partial X^\nu}{\partial x^j} =: h_{\mu\nu} X_\mu X_\nu. \] (10)

We can now define the energy and pressure with respect to this direction by

\[ \rho = u^\mu u^\nu T_{\mu\nu} \quad \text{and} \quad 3p = h^{\mu\nu} h_{\mu\nu} T_{\mu\nu}. \] (11)

Using that \( X_\mu \partial_\mu = 0 \) which is just the orthogonality condition, \( \partial_\mu X_\mu = \partial_i \) and \( h_i^i = 3 \) we get

\[ \rho = N \int \frac{d^3k}{(2\pi)^3 2k} \{ \phi_k' + \frac{1}{2} \partial_\lambda \phi \partial^\lambda \phi \} \] (12)

\[ 3p = N \int \frac{d^3k}{(2\pi)^3 2k} \{ \partial_\lambda \phi \partial^\lambda \phi + \frac{3}{2} \partial_\lambda \phi \partial^\lambda \phi \} \] (13)

As shown in [3] [10] inserting the mode expansion in this expression and evaluating it between vacuum states with the condition \( \langle 0 | a_k a_k^\dagger | 0 \rangle = (2\pi)^3 2 \delta(k - k') \) gives the vacuum energy density and pressure:

\[ \rho_{\text{vac}} = \frac{N}{2} \int \frac{d^3k}{(2\pi)^3 2k} \left( |\phi_k|^2 + \frac{k^2}{a^2} |\phi_k|^2 \right), \] (14)

\[ p_{\text{vac}} = \frac{N}{2} \int \frac{d^3k}{(2\pi)^3 2k} \left( |\phi_k|^2 - \frac{k^2}{3a^2} |\phi_k|^2 \right). \] (15)

For the de Sitter solution using
\[ \dot{\phi}_k = \frac{1}{a} \phi_k' = -i \frac{k}{a^2} e^{ik/a} \Rightarrow |\dot{\phi}_k|^2 = \frac{k^2}{a^4} \]  
(16)

and \[ |\phi_k|^2 = \frac{1}{a^2} + \frac{H^2}{k^2}, \]  
(17)

one obtains:

\[ \rho_{\text{FRW}}^{\text{vac}} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \left( p^2 + \frac{H^2}{2} \right) \]  
(18)

\[ \rho_{\text{FRW}}^{\text{vac}} = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2} \left( \frac{k^2}{a^4} + \frac{H^2}{2a^2} \right), \]  
(19)

Note that \( k \) is just a mode label and that the physical momentum is \( p = k/a \), hence we made this change of variables to physical momenta. We note that in [11] the time dependence of the canonical momenta was eliminated by field redefinition which led to an additional term in the integrals proportional to \( H^2/k \) and thus consider Eq. (18) and (19) leading order approximations.

\[ \langle 0 | T_{\mu\nu} | 0 \rangle_{\text{vac}} = \langle 0 | T_{\mu\nu} | 0 \rangle_{\text{FRW}} - \langle 0 | T_{\mu\nu} | 0 \rangle_{\text{Minkowski}}. \]  
(20)

An important observation has been made in general relativity by Arnowit, Dewitt and Misner [12], when calculating in a canonical way the energy of space-times. They observed that one has to subtract from the energy of a curved space-time the energy of a space-time it approaches asymptotically. For example the energy of a Schwarzschild space-time is obtained after subtracting the Minkowski space-time energy. This indicates that gravity has a defined normalization of energy of space-times. This has been pointed out in connection with vacuum energy renormalization in [13]. The normalization of the vacuum energy momentum tensor was chosen to be zero in Minkowski space-time

\[ \langle 0 | T_{\mu\nu} | 0 \rangle_{\text{vac}} = \langle 0 | T_{\mu\nu} | 0 \rangle_{\text{FRW}} - \langle 0 | T_{\mu\nu} | 0 \rangle_{\text{Minkowski}}. \]  
(20)

One can also argue in the following way. Assume Minkowski space-time is a solution of a given Einstein equation. If this space-time had an energy, its energy distribution had the same symmetry as itself and hence would be space-time uniform. If this energy would couple to gravity by means of entering the energy momentum tensor in Einstein’s equation, the Minkowski space-time could not be a solution of this equation. Thus we arrive at a contradiction.

Also other approaches exist which would support the disappearance of the Minkowski vacuum energy, for instance the idea of degravitation due to a finite graviton width [14], which suggest that a space-time uniform source would not contribute to the energy momentum tensor on cosmological scales.

The subtraction of the Minkowski space has an intrinsic arbitrariness, since it is not specified what the relation between the frames is in which we compute the fluctuations. Therefore, a priori the FRW contribution can have a lapse \( N \), while by definition \( N = 1 \) in flat space-time thus we have:

\[ \rho_{\text{vac}} = \frac{N}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \left( p^2(1 - \frac{1}{N}) + \frac{H^2}{2} \right), \]  
(21)

\[ \rho_{\text{vac}} = -\frac{N}{3} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \left( -p^2(1 - \frac{1}{N}) + \frac{H^2}{2} \right). \]  
(22)

The choice of \( N \) determines the frame relation between the cut-off frame and the frame we evaluate the physical quantities in. It is clear that choosing the frame of the cut-off corresponds to \( N = 1 \) and the equations reduce to the expressions introduced in [13].
D. Breaking of general covariance

At this point we discuss the issue of general covariance the guiding principle of general relativity. This principle is used to construct the action of general relativity and it dictates that the Bianchi identity has to hold even off shell, it reads

\[ \nabla^\mu G_{\mu\nu} = \nabla^\mu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0. \] (23)

Thus, if Einstein’s equation holds general covariance symmetry this equation is equivalent to \( \dot{\rho} + 3H (\rho + p) = 0 \). Recall now that in our set up the background geometry is de Sitter (\( H = \text{const} \)) and the only source for the gravitational field is the vacuum energy momentum tensor. Thus if \( \Lambda = \text{const} \) and \( \dot{\rho} = 0 \) covariant conservation implies that \( \rho_{\text{vac}} = -p_{\text{vac}} \). To check this we turn now to the issue of regularization with the goal to extract finite physical values of the formally divergent expressions.

III. REGULARIZATION

The integrals we obtained for the vacuum energy and pressure are divergent. Therefore, a regularization procedure is needed to extract sensible values out of them. We will now distinguish two basic cases, one were we treat the regulator as a mathematical tool and the other where it is supposed to originate from a real physical length scale.

At first we consider the cut-off regulator and assume that it appears due to a finite length scale and truncates in this way the mode-sum. In flat space-time a covariant 4-momentum cut-off can be applied which is consistent with the symmetry of the theory. We will demonstrate now that on curved space-time this is not the case.

A. The 4-momentum cut-off

The vacuum integrals show quadratic divergences and as discussed above we introduce a momentum space cut-off \( \Lambda \). The interesting and distinguishing feature of this regularization is its physical meaning. Assuming a fundamental physics scale associated with the length \( 1/\Lambda \) one expects values which depend on the regulator to represent physical observable quantities which occur due to the micro physics of the system. That a Planck scale cut-off is related to the cosmological constant problem has been argued for in the literature [39]. As seen above we have the freedom to choose the frame relation between the cut-off and the observer frame. If we choose them to coincide the expressions read:

\[ \rho_{\text{vac}} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{H^2}{2p}. \] (24)

We note that the Hubble parameter in our vacuum integrals does not depend on the momentum, especially in the de Sitter case when it is constant, hence we write

\[ \rho_{\text{vac}} = \frac{1}{2} \left( \frac{H^2}{2} \right) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p}. \] (25)

Recall the relativistic invariant form of the integral as used in [15]

\[ \int \frac{d^3p}{(2\pi)^3} = i \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + i\epsilon}. \] (26)

Performing a Wick rotation and introducing \( \Lambda \) as the 4-momentum cut-off we obtain

\[ \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} = \frac{1}{4\pi^2} \Lambda^2. \] (27)

Thus, the vacuum energy has the value

\[ \rho_{\text{vac}} = \frac{1}{16\pi^2} H^2 \Lambda^2 \] (28)

and the pressure

\[ p_{\text{vac}} = -\frac{1}{16\pi^2} H^2 \Lambda^2. \] (29)

We observe that even with a 4-momentum cut-off, which respects the remaining Lorentz symmetry, general covariance is broken.

B. The Frame Choice

The issue of broken general covariance by the 3-momentum cut-off is discussed in [13], it is suggested to introduce non-covariant counter terms to remedy this problem. Those would compensate the breaking and the full theory is considered covariant again. The \( \Lambda^2 H^2 \) contribution renormalizes the Newton constant and is thus without physical consequences. At this point it is important to stress the difference between the gauge invariance of QFT and general covariance. As opposed to covariance under diffeomorphisms there is no physical gauge. However, it is possible that quantities depend on the frame as long as they transform correctly when the frame is changed. Furthermore, there are physically preferred frames whenever a material system is associated with it. As we are interested in the consequences of a cut-off stemming from a finite length scale, there is a preferred frame, that of the cut-off.

In [13] the counterterm is non-covariant, since by construction it has a different lapse dependence as the other terms and in the action and contains tensor components
as $R_{00}$. We will now show that there is a way to use the lapse dependence resulting from the subtraction of Minkowski contributions to make the theory covariant under diffeomorphisms even if it is not invariant. The main point is that the formal subtraction of the flat space contribution does not have to coincide with the frame of the FRW mode-sum construction. Therefore, the lapse in Eq. $(21)$ becomes an arbitrary function and can serve to restore the covariance with a constant $\Lambda$. It is obvious that if we choose in our deSitter set-up $N = 1/(1 + H^2/\Lambda^2) := N_0$ that $\rho_{\text{vac}} = -\rho_{\text{vac}}$ what is required for a constant $\Lambda$. Expanding in powers of $(H/\Lambda)^2$ the energy density is

$$\rho_{\text{deSitter}} = \frac{1}{32\pi^2} \left( H^2 \Lambda^2 - H^4 + O \left( \frac{1}{\Lambda^2} \right) \right). \quad (30)$$

We observe that in this set-up a constant term appears that is proportional to $H^4$, as in frameworks which respect general covariance. In the symmetric deSitter case with constant $H$ this has no further relevance. The renormalized vacuum energy is proportional to the Einstein tensor, covariantly conserved and thus again only renormalizes the Newton constant. Note that this result is only valid in the frame where $N = N_0$ and in all other frames due to covariant conservation the vacuum energy must be a function of time.

The situation is more interesting in the general FRW background. The analogous computation for Matter Dominated (MD, $c_i = 2/3$) and Radiation Dominated (RD, $c_i = 1$) epochs were performed in $[3]$ and including the lapse dependence read as follows

$$\rho_{\text{vac}} = \frac{1}{16\pi^2} \left( \frac{1}{2} \Lambda^4 (N - 1) + H^2 \Lambda^2 N \right) \quad (31)$$

$$\rho_{\text{vac}} = \frac{1}{16\pi^2} \left( \frac{1}{6} \Lambda^4 (N - 1) + c_i H^2 \Lambda^2 N \right). \quad (32)$$

The requirement of covariant conservation and a constant cut-off in form of $\dot{\rho} + 3H(\rho + p) = 0$ sets conditions on the lapse function. Inserting the time scaling of $H \propto 1/t$, leading to $\dot{H} = -H^2$ we have

$$N(t) = \frac{-(2H^2 + \Lambda^2) \dot{N} + 4H \Lambda^2}{6(c_i + 1)H^3 + 4H \left( \Lambda^2 + \dot{H} \right)} \quad (33)$$

$$= \frac{-(2H^2 + \Lambda^2) \dot{N} + 4H \Lambda^2}{6(c_i + 1)H^3 + 4H (\Lambda^2 - H^2)}. \quad (32)$$

With the scaling behaviour of $H$ the differential equation can be solved and expanded in powers of $\Lambda$ leading to

$$N(t) \approx 1 - \frac{3H^2}{\Lambda^2} + \frac{3 \log \left( \frac{2\Lambda^2}{H^2} \right)}{\Lambda^4} H^4 \quad (\text{MD}), \quad (33)$$

$$N(t) \approx 1 - \frac{4H^2}{\Lambda^2} + \frac{8 \log \left( \frac{2\Lambda^2}{H^2} \right)}{\Lambda^4} H^4 \quad (\text{RD}). \quad (34)$$

Which implies for the energy density

$$\rho_{\text{vac}}^{\text{MD}} = \frac{1}{16\pi^2} \left\{ -\frac{H^2 \Lambda^2}{2} + 3H^4 \left( \log \left( \frac{2\Lambda^2}{H^2} \right) - 1 \right) + O \left( \frac{1}{\Lambda^2} \right) \right\}, \quad (35)$$

$$\rho_{\text{vac}}^{\text{RD}} = \frac{1}{16\pi^2} \left\{ -H^2 \Lambda^2 + 4H^4 \left( \log \left( \frac{\Lambda^2}{2H^2} \right) - 1 \right) + O \left( \frac{1}{\Lambda^2} \right) \right\}. \quad (36)$$

Again the contribution proportional to $H^2$ renormalizes the Newton constant, while the $H^4$ contribution acquires a logarithmic part analogously to covariant methods. The Logarithmic terms have the Hubble parameter in the argument, which is proportional to $1/t$ in the matter and radiation dominated epoch. Thus we are left with a logarithmic running of the Lagrangian quantities, which is a known situation in other QFTs. Note that the beta function in the case of a mass less scalar is found to be proportional to $H^4$.

C. Interpretation

There are two possible approaches to regularization, as mentioned in the introduction. The one is to use it as a mathematical tool without physical meaning. Then, requiring that the regularization procedure respects the underlying symmetries of the theory, it is legitimate to introduce appropriate counterterms which counterbalance the initial symmetry violation. In that case the $H^2\Lambda^2$ terms are absorbed in the Newton constant since the only covariantly conserved tensor with $T_{00} \propto H^2$ is the Einstein tensor. In that case general covariance acts in the same way as a protective symmetry for the vacuum energy as the conformal symmetry in the case of a scalar mass. After the absorption of the quadratic divergences in the Newton constant the vacuum energy is running logarithmically with a beta function proportional to $H^4$ in the mass less case.

The other way to view the regulator is an effect of a physical finite length scale, which can be manifest in the low energy observables. In that case still the underlying symmetries have to be respected. The momentum cut-off can be made compatible with Lorentz symmetry by taking the 4-momentum regulator. We showed in this article, however, that it still violates general covariance. A way to solve this issue is to connect this cut-off to a physical frame. In that sense the theory regulator is not diffeomorphism invariant but transforms correctly as a physical system and is thus covariant. We show that in that case the energy-momentum contribution is covariantly conserved by assigning a particular frame to the cut-off. It is the only frame, with $N = N_0$, where the cut-off is a constant. In all other frames the requirement of general covariance implies that $\Lambda$ must not be constant any-more. Therefore, for an observer for which
the frame does not coincide with $N_0$, the vacuum energy contribution has a time dependence stemming from a time varying cut-off $\Lambda(t)$, which can be computed from general covariance, knowing the observers’ frame. This treatment would in general mean that the physical vacuum is a preferred reference frame and could in principle itself lead to observable effects as discussed in [16]. The running of the vacuum energy with $\Lambda$ is now a physical observable since the cut-off is not an auxiliary parameter but a physical scale. Interesting observations are, that even if in one epoch the observer sees a constant energy density in the frame $N_0$, at the transition to an other epoch the frame of constant energy density is different and thus for the observer the energy density begins to vary. Therefore, in the case of regularization by a physical finite length scale there are two sources of a running vacuum energy. One is the logarithmic running and the other is a frame effect, which is stronger if the observers’ frame differs significantly form the frame of constant vacuum energy.

**D. Comparison to other methods**

There are methods to regularize vacuum fluctuations in a gravitational background which respect general covariance, as discussed in [9] [17] [19]. For example the outcome of the point splitting method in [9] for a scalar field in deSitter space is that its energy-momentum tensor adds a finite contribution to the cosmological constant proportional to the Ricci curvature squared and thus to $H^4$. This shows that the finite energy density contribution is inversely proportional to the radius of the space-time to the fourth power. This type of behaviour can be expected from a Casimir energy. The divergent contributions renormalize the Cosmological constant, the Newton constant and a logarithmically divergent term adds to the coefficient of the $O(R^2)$ terms in the gravitational action.

In [18] the calculation is performed to linear order in $R$ and the finding is that the perturbation method in linear curvature approximation always results in renormalization of the cosmological constant and Newton constant, which is consistent with [9]. The authors of [19] perform a non covariant cut-off regularization, but apply non-covariant counterterms to restore the initial symmetry and come to the same result. One could speculate that there is a mechanism similar to the one proposed in [20] of a UV complete theory where these terms originate from and just appear non-covariant.

The procedure presented in this letter can be viewed in two ways. There is an intriguing possibility that the cut-off frame is physical and thus a preferred frame in the well defined sense. In this situation there are frames in which the requirement of covariant conservation induces a running of the constants in the Lagrangian. For instance if the observer is in the frame $N_0^{MD}$ during matter domination and $\Lambda$ appears constant, at the transition to radiation domination it will become time-dependent.

This happens since the frame of constant $\Lambda$, $N_0^{RD}$, differs from the one in the matter dominated epoch. This running would appear in addition to the logarithmic running of the parameter of the $H^4$ term.

If the cut-off scale is not related to a physical finite length and thus to a frame, then it just restores covariant conservation of the regularized quantities by hand, as a non-covariant counterterm, and leads to renormalization of the bare quantities in the gravitational action. It is interesting that finite and divergent terms of order $H^4$, even not included initially, reappear after regularization. Note that for a mass-less scalar terms appear, which are of the form $H^4 \log(H^2/\Lambda^2)$. These terms imply a logarithmic running of the vacuum energy with the beta function proportional to $H^4$ for a mass-less scalar field.

**E. Dimensional Regularization**

After discussing the possible implications of a finite length scale on the vacuum energy we turn in a different direction and assume that the regulator has no physical meaning at all and only helps to extract the relevant physical information of a formally divergent quantity. We choose dimensional regularization, since its major advantage is, that it does not violate general covariance. This approach leads to logarithmic scale dependence of the energy density only, as discussed in [13] [21]. It has been argued by [22] [25] that this regularization procedure might be singled out in case conformal symmetry is a fundamental symmetry of nature. The reason is that dimensional regularization minimally violates conformal symmetry by logarithmic terms, which are a manifestation of the conformal anomaly. If other regularization schemes are used, the counterterms have to be adjusted such that only the logarithmic terms violate conformal symmetry. Therefore, dimensional regularization is not special but automatically meets the requirement without further adjustments. This procedure protects the Higgs mass in the case, that there are no new quantum fields at a much higher scale [22].

It is an interesting observation that general covariance has the same protective effect when it comes to the cosmological constant, as that of conformal symmetry for the Higgs mass. However, we know that general covariance is a symmetry of nature and does not need to be postulated in addition. While for the Higgs mass to be stable one needs to assume that there is no very high scale with QFT degrees of freedom the requirement for the vacuum energy is different. The condition is that there is no hard cut-off scale connected to a finite physical length. Given that, the renormalization conditions needed to ensure general covariance allow only vacuum contributions which are logarithmic and induce a dependence of the vacuum energy on the renormalization scale $\mu$ [19] [21].
Minkowski

First we would like to recall the calculation in Minkowski space-time for a massive scalar field. After the decomposition in plane waves and projecting out the energy density term from the energy-momentum tensor one obtains [19]

\[
\rho_{\text{vac}} = \frac{\mu^{d-4}}{2 (2\pi)^{(d-1)/2}} \int d^{d-1}p \omega(p) \quad (37)
\]

\[
= \frac{\mu^4}{2 (2\pi)^{(d-1)/2}} \frac{\Gamma(-d/2)}{\Gamma(-1/2)} \left( \frac{m}{\mu} \right)^4 \quad (38)
\]

Which by use of the identity \( \Gamma(-1/2) = -2\Gamma(1/2) \) results in \( \rho_{\text{vac}} = -\rho_{\text{vac}} \) which makes the vacuum contribution proportional to the metric tensor in Minkowski space-time. Here the scale \( \mu \) has been introduced for technical reasons and has no physical meaning.

Friedrich-Robertson-Walker

We are now interested in the case of curved space-time, particularly in the FRW. Again a massive scalar field \( \phi \) is considered in an FRW background. For the dimensional regularization procedure a quantization as applied in [19] is extremely useful, since the time dependence in the canonical momenta is removed. Here instead of the primary field \( \phi \) a re-scaled field \( \chi = a^{3/2} \phi \) is considered. This leads to an action, which reads

\[
S_\chi = \int d^4x \frac{1}{2} \left( \partial_\mu \chi \partial^\mu \chi - m^2 \chi^2 + A(t) \chi^2 \right) \quad (39)
\]

with \( A(t) = \frac{3}{4} \left( 3H^2 + 2\dot{H} \right) \).

Now when the field is expanded in modes

\[
\chi = \int \frac{d^3k}{(2\pi)^{3/2}} \left( a_k u_k e^{ik \cdot x} + a_k^* u_k^* e^{-ik \cdot x} \right), \quad (40)
\]

and the functions \( u_k \) satisfy

\[
(\partial_t^2 + w_k^2)u_k = 0 \quad \text{with} \quad w_k^2 = p^2/a^2 + m^2 - A(t) \quad (41)
\]

The solution has the form

\[
u_k = \frac{1}{\sqrt{2W}} \exp \left( i \int_0^t W(s) ds \right). \quad (42)
\]

In contrast to the previous cases there is no exact analytical solution available and approximations have to be made. On physical grounds it is justified to perform an adiabatic expansion with \( \frac{a}{a} << k \). There are two cases, as discussed in [11].

If \( m = 0 \) the geometric factors transform \( A(t) \rightarrow R/6 \) where \( R \) is the Ricci curvature. And in the massive case one has the ansatz \( u_k = i\omega u_k \) with the approximation \( W = \omega_k \).

In both cases we are left with a time dependent mass correction in the dispersion relation

\[
\Omega(p, t) = \sqrt{(p/a)^2 + m^2 - A(t)}. \quad (43)
\]

This can be immediately substituted in Eq. (37). As discussed in [11] we perform the subtraction of the flat space contribution. Since dimensional regularization respects general covariance there is no ambiguity in this case regarding the subtraction of the Minkowski contribution and the lapse does not play a role. Due to the absence of any effect of the frame dependence, the regulator is now considered purely technical and we obtain

\[
\rho_{\text{vac}} = \frac{\mu^{4-d}}{2 (2\pi)^{(d-1)/2}} \int d^{d-1}p \Omega(p, t) = \frac{\mu^4}{2 (2\pi)^{(d-1)/2}} \frac{\Gamma(-d/2)}{\Gamma(-1/2)} \left( \frac{m}{\mu} \right)^4 \Omega(p,t) \quad (44)
\]

\[
= \frac{\mu^4}{2 (2\pi)^{(d-1)/2}} \frac{\Gamma(-d/2)}{\Gamma(-1/2)} \left( \frac{m}{\mu} \right)^4 \left( A(t)^2 \right) \quad (45)
\]

The relation \( \rho_{\text{vac}} \approx -\rho_{\text{vac}} \) has to be understood approximately, since contrary to the previous treatments the solution to the equation of motion is found in an adiabatic expansion. However, for a slowly varying Hubble scale the variation of \( \rho_{\text{vac}} \) is only \( \propto m^2 H(t)^2 \) and \( \propto H(t)^4 \) for \( m = 0 \) and thus small, consistent with the requirements of the expansion. The terms without renormalization scale dependence are absorbed in the bare cosmological and Einstein constant and have no physical implication. As mentioned above the scale \( \mu \) itself has no physical meaning, but the computation allows us to extract the running of the cosmological constant with the energy scale when we fix its value at one specific scale. This would lead us to the situation known in QFT, where there is no particular scale prediction but an evolution with the energy scale. The energy density is given by

\[
\rho_{\text{vac}} = \beta_\Lambda \log \left( \frac{|m^2 - A(t)|}{\mu} \right) + A(\mu_0), \quad (46)
\]

\[
\text{with} \quad \mu_0 = m^2 - A(t_0).
\]

Note that in the case of a non static geometry the beta function is time dependent

\[
\beta_\Lambda = \frac{1}{64\pi^2} \left( A(t)^2 - 2A(t)m^2 \right), \quad (47)
\]
and in the case of a mass-less field as discussed above $A_0(t) = R/6$ and

$$\beta_\Lambda = \frac{A_0(t)^2}{64\pi^2}. \quad (48)$$

Note that this result is consistent with the cut-off regularization procedure, where for a mass-less scalar the beta function was proportional to $H^4$. This beta function together with the insight that the vacuum energy in any case experiences a time evolution is the main finding of this paper.

The usual difficulty to interpret the renormalization scale dependence of the vacuum energy is that $\mu$ has no physical meaning a priori. In some situations it is convenient to identify it with the momentum transfer in a given process. However, the situation in the case of the cosmological constant is more subtle. It might be appropriate to identify $\mu$ in that case with the temperature, but this choice is by far not unique. The interesting finding here is that even without any physical interpretation, treating $\mu$ as a purely auxiliary scale, there is a time dependence in the vacuum energy. This stems from the time dependence of the geometric quantity $A(t)$.

The above result is not dependent on the renormalization procedure and is reproduced, if general covariance is restored by hand and the terms with restored covariance absorbed in the constants of the Lagrangian. The sole physical observable is a time dependence of the cosmological constant, if fixed at an initial time $t_0$ with $\mu$ being an arbitrary scale.

It is tentative to employ classical scale invariance at a high scale, for example Planck scale, which would lead to a $\rho_{\text{vac}}(t_0) = 0$ boundary condition, as proposed in [26]. It should be studied in detail, whether this might lead to predictions consistent with cosmological observations.

**IV. CONCLUSION**

In this letter we discussed the cut-off regularization of vacuum fluctuations in a gravitational background. As well known a 3-momentum cut-off violates the Lorentz symmetry. This can be remedied by applying a 4-momentum cut-off in flat space-time. We show that in a general background covariance is violated even by the 4-momentum regulator.

We tried to come closer to the understanding of the issue of breaking general covariance. We observe that if the cut-off is connected to a physical frame and regarded as a covariant, rather than invariant, quantity, there exist a frame where the cut-off is constant. This is not the case for all other frames and knowing the relation of the frame of the observer to the frame of the cut-off, the time dependence of $\Lambda$ is determined by general covariance. This implies that in general the vacuum effects induce a running of the quantities in the Lagrangian. The issues of running CC in QFT have been discussed in [27] and references therein. An interesting physical question is, what happens at the transition from one cosmological epoch to another since the frames associated with a constant cut-off are different for Matter-, Radiation-domination and deSitter.

If there is no physical frame associated with the cut-off method described here leads to the same findings as described in the literature by renormalizing the bare Lagrangian quantities. With the important observation which holds independent of the regulator used, that after performing the subtraction of the flat space contributions the dominant term in the beta function computed for the cosmological constant is time dependent and running effects can lead to observable phenomenology.

Concluding we find that in any situation the vacuum energy is not a constant. Neither when the cut-off is a physical regulator, nor when there is no physical finite length scale. In any case a logarithmic running is found which can be accompanied by additional running effect at the transition of one cosmological epoch to another in case there is a fundamental length scale. As the main result we are able to compute the beta function for this running in an FRW geometry given by Eq. [47] and Eq. (48). A running of the vacuum energy, as discussed in [28], opens the possibility to define a time associated with this effect and to construct a cosmic clock. This ultimately leads to the conclusion that by studying quantum field theory effects on the vacuum energy we find that it has to evolve in time and thus itself defines a time measure.

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