The double copy of the multipole expansion

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Abstract

We consider the classical double copy, that relates solutions of biadjoint scalar, gauge and gravity theories. Using a recently developed twistor expression of this idea, we use well-established techniques to show that the multipole moments of arbitrary vacuum type D gravity fields are straightforwardly mapped to their counterparts in gauge and biadjoint scalar theory by the single and zeroth copies. We cross-check our results using previously obtained results for the Kerr metric. Our results provide further physical intuition of how the double copy operates.

1 Introduction

There is mounting evidence that our various theories of fundamental physics are more closely connected than previously thought. In this paper, we will focus on a particular correspondence – the classical double copy – that relates solutions of the field equations in (non-)abelian gauge theories and gravity, as well as in a novel scalar theory with two different types of colour charge (biadjoint scalar field theory). Inspired by the original double copy for scattering amplitudes in the corresponding quantum field theories [1, 2] (which itself has a string theoretic origin [3]), the first classical double copy to appear was the Kerr-Schild double copy of ref. [4] (see refs. [5–18] for further developments). An alternative exact double copy procedure is the Weyl double copy of ref. [19] (see also refs. [14, 20, 23]). This uses the spinorial rather than tensorial formalism of General Relativity, and includes the Kerr-Schild double copy as a special case. To date, it constitutes the most general exact statement of the classical double copy, although other formalisms also offer useful alternative insights [24–35].

As well as practical applications of this correspondence (see e.g. ref. [36]), there are also important conceptual issues to address, including understanding the ultimate origin of the classical double

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copy itself. To this end, refs. \[37,38\] recently showed how one can derive both the form and scope of the Weyl double copy using well-established ideas from twistor theory \(39,41\) (see refs. \[42,44\] for pedagogical reviews of this subject), as well as showing that the Weyl copy is more general than previously thought. As well as applying new mathematical techniques, another useful method for extending our understanding of the double copy is to take known physical or mathematical properties in biadjoint scalar, gauge and gravity theories, and to see how they match up (or otherwise). Recent examples include properties of solutions at strong coupling \[45–48\], symmetries \[33,49–52\], and geometric / topological information \[8,53,54\]. Even more simply, the original Kerr-Schild double copy of ref. \[4\] told us that mass and energy map to charge in the gauge theory, which nicely mirrors the replacement of kinematic by colour information in scattering amplitudes \[1,2\].

In this paper, we extend these ideas by considering the well-known multipole expansion of classical solutions. For any given solution, one may define a series of higher-rank multipole tensors, which completely characterise the spatial and temporal distribution of charge (energy / momentum) in the gauge (gravity) theory respectively. The lowest-order results in this expansion constitute the total charge and mass mentioned above, and one may then ask whether higher-multipole moments are strict double copies of each other. It turns out that this can be easily addressed in the twistor picture of refs. \[37,38\]. As has been argued by Curtis \[55\], the multipole tensors can be replaced by multi-index spinor fields, each of which satisfies the twistor equation. As a consequence, one may instead describe the multipole moments of a given field purely in terms of higher-rank twistors, which are straightforwardly defined from the twistor-space functions describing the fields. We will combine this with the twistor-space double copy of refs. \[37,38\], and thus obtain an explicit statement that the multipole expansion double-copies, for arbitrary type \(D\) vacuum solutions. Our results provide a useful physical insight into how the double copy operates, and may well be relevant for thinking about further applications.

2 The twistor space double copy

In this section, we review salient details of the Weyl double copy, together with its twistor space incarnation, referring the reader to refs. \[37,38\] for full details. First, we recall that massless free spacetime fields can be represented by multi-index spinors \(\phi_{AB...C} (\tilde{\phi}_{A'B'...C'})\), representing the anti-self-dual (self-dual) parts of the field respectively. Index values run from 0 to 1, and may be raised, lowered and / or contracted using the Levi-Civita symbols \(\tilde{\epsilon}^{AB}\) etc. There are \(2n\) indices for a spin-\(n\) field, and the resulting quantities then satisfy a special case of the general massless free field equation

\[
\nabla^{AA'}\tilde{\phi}_{A'...C'} = 0, \quad \nabla^{AA'}\phi_{AB...C} = 0,
\]

where \(\nabla^{AA'}\) is the appropriate translation of the spacetime covariant derivative. These fields can be reinterpreted in twistor space \(\mathbb{T}\), corresponding to solutions of the twistor equation

\[
\nabla^{(A} \Omega^{B)} = 0 \quad \Rightarrow \quad \Omega^{A} = \omega^{A} - ix^{AA'}\pi_{A'}.
\]

In the second equality we have written the general solution in Minkowski space, in terms of constant spinors which may be grouped together to make a 4-component twistor

\[
Z^{\alpha} = (\omega^{A}, \pi_{A'}).\n\]
A non-local map between spacetime and twistor space is established by requiring that the field in eq. (2) vanish, such that the twistor components satisfy the incidence relation
\[ \omega^A = i x^{AA'} \pi_{A'}. \] (4)

This is invariant under rescalings \( Z^\alpha \to \lambda Z^\alpha, \lambda \in \mathbb{C} \), such that we need only consider projective twistor space \( \mathbb{P} \mathbb{T} \). A point in spacetime corresponds to a Riemann sphere in \( \mathbb{P} \mathbb{T} \), also referred to as a (complex) line. An important result known as the Penrose transform expresses massless free spacetime fields satisfying eq. (1) via the contour integrals
\[ \bar{\phi}^{AB...C}(x) = \frac{1}{2\pi i} \oint_{\Gamma} d\pi^{E'} d\pi^{E} \pi^A \pi^{B'} ... \pi^{C'} [\rho_x f(Z^\alpha)], \] (5)

where \( \rho_x \) restricts all twistors to obey the incidence relation corresponding to spacetime point \( x \), and the contour \( \Gamma \) lies on the appropriate Riemann sphere. The combined integrand and measure must be invariant under rescalings \( Z^\alpha \to \lambda Z^\alpha \), which fixes the (holomorphic) function \( f(Z^\alpha) \) to have homogeneity \(-2\), \(-4\) and \(-6\) correspond to spacetime gravity fields in scalar, gauge and gravity theory respectively. Denoting the respective twistor functions by the subscripts \( \{ \text{scal.}, \text{EM}, \text{grav.} \} \) respectively, refs. [37,38] argued that one may define a gravity twistor function via \(^4\)
\[ f_{\text{grav.}}(Z^\alpha) = \frac{f^{(1)}_{\text{EM}}(Z^\alpha) f^{(2)}_{\text{EM}}(Z^\alpha)}{f_{\text{scal.}}(Z^\alpha)}, \] (6)

leading to the spacetime Weyl double copy formula
\[ \phi^{AB'C'D'}(x) = \frac{\phi^{(1)}_{AB'}(x) \phi^{(2)}_{C'D'}(x)}{\phi(x)} \] (7)

first presented in ref. [19]. Here \( \phi \) is a biadjoint scalar field, \( \phi^{(i)}_{AB'} \) an electromagnetic spinor, and \( \phi_{AB'C'D'} \) a Weyl spinor. The above discussion applies to the case of primed spinor fields in spacetime. For unprimed fields, one may consider the conjugate of eq. (2), whose solutions are associated with dual twistors \( W_\alpha \). The notion of the Penrose transform can be straightforwardly adapted from eq. (8):
\[ \phi_{AB...C}(x) = \frac{1}{2\pi i} \oint_{\Gamma} d\lambda_E d\lambda^E \lambda_A \lambda_B ... \lambda_C [\rho_x f(W_\alpha)], \] (8)

and the twistor double copy of eq. (6) similarly generalises. We will work with dual twistors by default in what follows, in order to match conventions with ref. [55].

### 3 Multipoles and the double copy

The idea of multipoles is familiar from Newtonian physics in three-dimensional Euclidean space. A stationary electrostatic or Newtonian potential \( \phi \) in a sourceless region satisfies Laplace’s equation

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4We have here skimmed over the fact that the twistor functions used throughout are not unique, and are instead representatives of cohomology classes. The product of eq. (5) is then interpreted to apply only to particular chosen representatives, as discussed in detail in ref. [38].
\( \nabla^2 \phi = 0 \), and may be expanded as \[ \phi = \frac{M}{r} + \frac{M_i x^i}{r^3} + \frac{M_{ij} x^i x^j}{r^5} + \ldots, \tag{9} \]

where \( r = (x^i x^i)^{1/2} \), and the multipoles \( \{ M_{ij...k} \} \) are constant tensors defined in terms of derivatives of the potential, evaluated at the origin \( \mathcal{O} \). Upon shifting to a different point, the multipole moments change in a way that involves only lower-order multipoles. The extension of these ideas to General Relativity has been discussed in refs. [56–58], for general asymptotically flat spacetimes. We will not need the full complication of the latter, given that we will be concerned with solutions of the massless free field equation of eq. (1) in Minkowski space. Given a constant unit timelike vector \( t^a \), one may then consider the 3-space orthogonal to this, with induced metric \[ h_{ab} = \eta_{ab} - t_a t_b. \tag{10} \]

Reference [56] then showed that an appropriate generalisation of the multipole tensors appearing in eq. (9) is provided by symmetric, trace-free tensor fields \( Q^{a_1...a_n}(x) \) satisfying

\[ \nabla_a Q^{a_1...a_n} = 0, \quad \nabla^m Q^{a_1...a_n} = \frac{n(2n-1)}{3} h^{m(a_1} Q^{a_2...a_n)} - \frac{n(n-1)}{3} Q^{m(a_3...a_n} h^{a_1 a_2)}, \tag{11} \]

where the \( n \)th such quantity is referred to as the \( 2^n \)-multipole tensor, and the second condition requires that the derivatives of multipole tensors depend only upon lower multipoles. This is the analogue of the “shifting the origin” property mentioned for Newtonian multipoles above, and ensures that the set of tensors \( \{ Q^{a_1...a_n} \} \) corresponds to the same solution of the field equation.

In the spinorial formalism, each spacetime index in eq. (11) will become a pair of spinor indices. Contracting with the timelike vector appearing there, one may define the symmetric spinor field

\[ \omega^{A_1...A_{2n}} = (6i)^n Q^{A_1...A_{n}B_1...B_{n}} t^{A_{n+1}}_{B_1} \ldots t^{A_{2n}}_{B_n}, \tag{12} \]

which turns out to solve a higher-rank generalisation of the twistor equation of eq. (2):

\[ \nabla^L (L' \omega^{A_1...A_{2n}}) = 0. \tag{13} \]

Thus, we can associate the spinors of eq. (12) with multi-index multipole twistors \( \{ Q^{a_1...a_{2n}} \} \). To see how this works in practice, consider a given physical spin-\( n \) field \( \Psi_{A_1 A_2...A_{2n}} \). Then one may define higher-spin fields iteratively by taking derivatives and contracting with the timelike vector appearing in eq. (12):

\[ \Psi^{(n)}_{A_1...A_{2n}} = t_{A V A_1} \nabla^{A'}_{A_2} [ \Psi^{(n-1)}_{A_3...A_{2n}} ]. \tag{14} \]

These constitute a spinorial analogue of the multiple derivatives appearing in the Newtonian formalism of eq. (14), whereby higher multipole moments contain more derivatives of the original potential. For a spin-1 field, one may write an explicit twistor space integral for the total conserved charge producing the field [59]:

\[ Q = -\frac{i}{4\pi^2} \int f(W_\alpha) d^4 W, \quad d^4 W = \frac{1}{4!} \epsilon^{\alpha \beta \gamma \delta} dW_\alpha dW_\beta dW_\gamma dW_\delta, \tag{15} \]

\footnote{Throughout the paper we use lower-case Latin, upper-case Latin and Greek indices for spacetime tensors, spacetime spinors and twistors respectively. Note, however, that the indices in eq. (9) run only over spatial components i.e. from 1 to 3.}
where an appropriate contour must be chosen, and where \( f(W_\alpha) \) is the twistor function corresponding to the spacetime field. Given a higher-spin field as in eq. (14), we can form multiple spin-1 fields by contracting with solutions of the twistor equation\(^6\) \( \{\alpha^{A_1\ldots A_{2n}}\} \):

\[
\Phi_{AB}^{(n)} = -i^{n+1} \alpha^{A_1\ldots A_{2n}} [\Psi_{A_{2n}A_{2n}}^{(n+1)}].
\]  

(16)

Each of these fields will have a conserved charge according to eq. (15), and we may collect together all such charges in the twistor-covariant form

\[
q(A^{\alpha_1\ldots\alpha_{2n}}) = \frac{i^{n+1}}{4\pi^2} \int W_{\alpha_1} \ldots W_{\alpha_{2n}} A^{\alpha_1\ldots\alpha_{2n}} f_{n+1}(W_\alpha) d^4W,
\]  

(17)

for symmetric twistors \( \{A^{\alpha_1\ldots\alpha_{2n}}\} \), where \( f_{n+1}(W_\alpha) \) is the twistor function corresponding to the spacetime higher-spin field \( (n+1)\Psi_{A_{2n}A_{2n}} \), and multipole index \( n \). Equation (17) defines a set of quantities dual to the \( \{A^{\alpha_1\ldots\alpha_{2n}}\} \):

\[
Q_{\alpha_1\ldots\alpha_{2n}} = \frac{i^{n+1}}{4\pi^2} \int W_{\alpha_1} \ldots W_{\alpha_{2n}} f_{n+1}(W_\alpha) d^4W,
\]  

(18)

which are the multipole twistors we have been seeking. Note that the iterative structure of the higher-spin fields in eq. (14) implies that the twistor functions \( \{f_{n+1}\} \) in eq. (18) can also be constructed iteratively, and there are various ways that this can be written. A fully invariant condition is \(^5\)

\[
f_n = i(R_\alpha W_\beta I^{\alpha\beta})^{-1} R_\gamma P^\gamma_\delta \frac{\partial f_{n-1}}{\partial W_\delta},
\]  

(19)

where we have introduced the so-called infinity twistors for Minkowski spacetime:

\[
I_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{A'B'} \end{pmatrix}, \quad I_{\alpha\beta} = \begin{pmatrix} \epsilon^{AB} & 0 \\ 0 & 0 \end{pmatrix},
\]  

(20)

\( R_\alpha \) is an arbitrary twistor, and we have introduced the projector \(^5\)

\[
\lambda P^\alpha_\beta = I^{\alpha\gamma} Q_{\gamma\beta},
\]  

(21)

where \( \lambda \) is the relevant mass or charge parameter for a given theory. One thus has \( \lambda = m \) in gravity, where \( m \) is the total mass of the system. In gauge or biadjoint theory, it will be the total charge of the system that is creating the field, which we will denote by \( q \) and \( y \) respectively.

### 3.1 The double copy of the multipole expansion

The multipole twistors introduced above allow us to address the double copy of the multipole expansion in a particularly compact and elegant way. Consider twistor functions corresponding to a biadjoint scalar, electromagnetic and gravity solution respectively, which we label by \( f_X(W_\alpha) \), \( X \in \{\text{scal.}, \text{EM}, \text{grav.}\} \). From each of these, one may define a set of higher-spin twistor functions according to the iterative procedure of eq. (19), denoted here by \( f_X^{(n)}(W_\alpha) \). By eq. (18), this immediately leads to a set of multipole twistors for each original spacetime field. This construction is shown in table \(^1\) where each column contains twistor functions of the same homogeneity, leading

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\(^6\)That the fields of eq. (16) indeed satisfy the massless free field equation of eq. (1) follows from eq. (13).
Table 1: Twistor functions occurring in the multipole expansion for three different theories. Here $f_X$ is a twistor function corresponding to a physical spacetime field in theory $X$, and $f^{(n)}_X$ is a higher-spin field generated from this by the iterative procedure of eq. (19).

Table 2: Multipole twistors arising from the twistor functions of table 1.

to the same spin fields in position space. We show the multipole twistors that arise from these in table 2. Note that a gravitational monopole contribution is not directly obtained from the corresponding twistor functions. As the total mass, however, it is obtainable from the angular momentum twistor $Q_{\alpha_1\alpha_2}$. For the $n = 1$ case, one finds integrals expressing the total charge generated by the biadjoint or EM field. From $n = 2$ upwards, there are multipole twistors in all three theories. For a given set of functions $\{f_X\}$ related by the twistor-space double copy, we can then associate each column of table 2 with a classical double copy triple, as shown in figure 1.

The physical interpretation of the identifications in figure 1 is straightforward. Consider, for example, the 2-multipole tensors $Q_X^{\alpha\beta}$. This represents the angular momentum in gravity [40], whereas in electromagnetism it is the charge dipole tensor, as expected given that the single copy turns mass into charge. Likewise, for the higher multipoles, the single copy replaces the relevant spatiotemporal distribution of mass / momentum with that of charge, with a further replacement to “biadjoint charge” in the zeroth copy.
It is one thing to formally identify the multipole twistors in different theories, as we have done in figure [1]. It is quite another thing to say that the multipole twistors in the different theories are the same, up to simple mass and charge replacements. Remarkably, this strong statement indeed turns out to be true for fields related by the original type D Weyl double copy of ref. [19], as we discuss in the following section.

3.2 Multipole moments of type D solutions

As discussed in ref. [60] and reviewed in refs. [37,38], all vacuum type D solutions arise from twistor functions of the form

\[ f_{\text{grav}} = (A^{\alpha\beta}W_\alpha W_\beta)^{-3}, \]  

where \( A^{\alpha\beta} \) is a constant twistor that can be taken to be symmetric. We see that eq. (22) has homogeneity \(-6\) under rescalings of \( W_\alpha \), as required for a gravity solution. Furthermore, it has two poles in twistor space, which give rise, after performing the Penrose transform of eq. (8) to position space, to the two two-fold degenerate principal spinors of the Weyl spinor that characterise it as being of type D. It turns out that the twistor \( A^{\alpha\beta} \) can be straightforwardly related to the 2-multipole twistor for this field. Substituting eq. (22) into eq. (18) for \( n = 1 \), one may carry out the integral using a special case of

\[ \oint W_\alpha W_\beta ... W_\alpha W_\beta \frac{(-n+2)}{n!} B_{(\alpha_1 \alpha_2} \ldots B_{\alpha_{2n-1} \alpha_{2n})}, \]

where \( B_{\alpha\beta} \) is the inverse of \( A^{\alpha\beta} \), and \( \Delta \) the determinant of the latter. One finds

\[ Q_{\alpha\beta} = \frac{\pi}{8i\Delta} B_{\alpha\beta}, \quad Q^{\alpha\beta} = \frac{8i\Delta}{\pi} A^{\alpha\beta}. \]

Given the general type D gravity twistor function of eq. (22), one may also identify the single and zeroth copies, giving rise to a gauge and biadjoint scalar field in spacetime respectively. As explained in refs. [37,38], these are

\[ f_{\text{scal}} = N_0 (A^{\alpha\beta}W_\alpha W_\beta)^{-1}, \quad f_{\text{EM}} = N_1 (A^{\alpha\beta}W_\alpha W_\beta)^{-2}. \]

We have here included arbitrary constant normalisation factors in the scalar and electromagnetic functions, which are in any case not fixed in the Weyl double copy of ref. [19]. Physically, one may absorb such constants by redefining the total amount of charge in a particular solution, but we will fix them shortly. Let us now construct and compare the multipole twistors from these solutions. For each field, we may construct higher-spin twistor functions using the procedure of eq. (19). Starting with the gravity function from eq. (22), one finds

\[ f^{(3)}_{\text{grav}} = -\frac{3i}{\alpha} (R_\alpha W_\beta I^{\alpha\beta})^{-1} (A^{\rho\lambda}W_\rho W_\lambda)^{-4} R_\gamma I^{\gamma\tau} Q_{\tau\delta} A^{\delta\sigma} W_\sigma, \]

where we have used eq. (21). We may now use eq. (24), which yields

\[ f^{(3)}_{\text{grav}} = 3 \left( -\frac{\pi}{4\Delta m} \right) (A^{\rho\lambda}W_\rho W_\lambda)^{-4}, \]

such that iterating this procedure leads to the formula

\[ f^{(n)}_{\text{grav}} = \left( -\frac{\pi}{4\Delta m} \right)^{n-2} \frac{n!}{2} (W_\alpha W_\beta A^{\alpha\beta})^{-(n+1)}, \]
as quoted in ref. [55]. Note that placing \( n = 2 \) in this formula reproduces the original gravity twistor function \( f_{\text{grav}}(W_\alpha) \) itself. We may find the multipole twistors of eq. (18) using eq. (23, 24), yielding

\[
Q_{\alpha_1 \ldots \alpha_{2n}}^{\text{grav}} = \frac{1}{2} \frac{1}{(2m)^{n-1}} \frac{(2n)!}{n!} Q_{\alpha_1 \alpha_2 \ldots \alpha_{2n}}^{\text{grav}},
\]

(29)

In principle, one may convert these multipole twistors back into multipole tensors. For the Kerr solution, a set of scalar multipole moments has been defined in the literature [58]. Let \( \tilde{z}^a \) be a vector aligned with the axis of rotation of the black hole, and \( \Lambda \) be the point at infinity after conformal compactification of the spacetime. Then the multipole moments are given by

\[
Q_n = \frac{1}{n!} Q_{\alpha_1 \ldots \alpha_n} \tilde{z}^{\alpha_1} \ldots \tilde{z}^{\alpha_n} \bigg|_\Lambda,
\]

(30)

where the notation on the right-hand side denotes that this be evaluated at \( \Lambda \) itself. As stated in ref. [55], the multipole twistors of eq. (29) do indeed reproduce the known multipole moments of the Kerr solution, first found in ref. [58].

We may carry out the above procedure for the biadjoint scalar and gauge theory twistor functions of eq. (25), and the resulting higher spin twistor functions are given by

\[
f_{\text{scal}}^{(n)} = \mathcal{N}_0 \left( -\frac{\pi}{4 \Delta y} \right)^{n} \frac{n!}{2} (W_\alpha W_\beta A^{\alpha\beta})^{-(n+1)}
\]

\[
f_{\text{EM}}^{(n)} = \mathcal{N}_1 \left( -\frac{\pi}{4 \Delta q} \right)^{n-1} \frac{n!}{2} (W_\alpha W_\beta A^{\alpha\beta})^{-(n+1)},
\]

(31)

where we have replaced the mass \( m \) in the gravity solution with the charge \( q \) in gauge theory, and biadjoint charge \( y \) in the scalar theory. These functions reproduce the original fields for \( n = 0 \) and \( n = 1 \) respectively. We may choose to fix the arbitrary normalisation constants \( \mathcal{N}_i \) by requiring that the 2-multipole (dipole) tensor in each theory is simply related by replacing

\[
m \rightarrow q \rightarrow y.
\]

(32)

in going from gravity to gauge theory, to biadjoint theory. This determines

\[
\mathcal{N}_0 = \left( -\frac{\pi}{4 \Delta y} \right)^{-2}, \quad \mathcal{N}_1 = \left( -\frac{\pi}{4 \Delta q} \right)^{-1},
\]

(33)

after which comparison of eq. (31) with eq. (28) shows that all higher-spin twistor functions agree across all three theories, so that one may simply replace the multipole twistors of eq. (29) with the gauge and biadjoint scalar counterparts

\[
Q_{\alpha_1 \ldots \alpha_{2n}}^{\text{scal}} = \frac{1}{2} \frac{1}{(2y)^{n-1}} \frac{(2n)!}{n!} Q_{(\alpha_1 \alpha_2 \ldots \alpha_{2n})}^{\text{scal}},
\]

\[
Q_{\alpha_1 \ldots \alpha_{2n}}^{\text{EM}} = \frac{1}{2} \frac{1}{(2q)^{n-1}} \frac{(2n)!}{n!} Q_{(\alpha_1 \alpha_2 \ldots \alpha_{2n})}^{\text{EM}},
\]

(34)

As a direct consequence, the multipole moments of the gauge and biadjoint scalar fields corresponding to a given gravity field from eq. (22) precisely match, after making the necessary mass-to-charge
replacements. Our twistor analysis has applied for an arbitrary quadratic form in eq. (22) which, as explained in refs. [37, 38], is a general statement for any (vacuum type D) spacetime entering the original Weyl double copy of ref. [19]. In particular, this must apply to the Kerr solution, and there is a novel cross-check one may perform. The gauge theory counterpart of this solution is the $\sqrt{\text{Kerr}}$ solution discussed above, and its electromagnetic monopole moments have not previously been calculated directly. However, one may instead consider a charged Kerr black hole, otherwise known as a Kerr-Newman black hole [61, 62]. This is a solution of the Einstein-Maxwell equations, and as such consists of a metric plus a gauge field. The $\sqrt{\text{Kerr}}$ solution can be obtained by setting the mass of the solution to zero, leaving a gauge field living in Minkowski space, and which is known to correspond to the single copy of the gravity solution. The combined gravitational and electromagnetic multipole moments of the Kerr-Newman solution have been calculated in ref. [63]. The gravity moments agree with the pure Kerr solution, and the electromagnetic ones are simply obtained by the mass-to-charge replacement of eq. (32). This indeed verifies our results. Further support comes from recent studies showing that the multipole expansion for $\sqrt{\text{Kerr}}$ and Kerr can be obtained from scattering amplitudes that manifestly double-copy [64].

4 Discussion

In this paper, we have considered whether the multipole expansions of fields in biadjoint scalar, gauge and gravity theory can be related by the classical double copy. By combining a twistor formulation of the multipole expansion [55] with a recently developed twistor language for the classical double copy [37, 38], we have shown that the multipole moments for arbitrary type D vacuum solutions indeed match up in different theories, subject to appropriate mass / charge replacements. Our results provide a nice illustration of the efficiency of the twistor double copy, but are of interest in their own right. It is often the case that a single copy of a given gravity solution can be found, but not easily interpreted. A canonical case of this is the single copy of the Kerr black hole, first formally identified in ref. [4], and denoted as $\sqrt{\text{Kerr}}$ in subsequent literature (see e.g. refs. [64, 66]). It is known that this solution occurs by replacing the source for the Kerr black hole (a rotating disk of mass) with a similar gauge theory source (a rotating disk of charge). However, the nature of the sources is subtly different in the two theories [4], such that it is not clear what impact this has on the fields themselves. Multipole moments, however, allow us to fully characterise the structure of fields in a gauge-invariant way. Thus, the fact that the multipole moments for the Kerr and $\sqrt{\text{Kerr}}$ solutions are essentially identical tells us a great deal of information about how to physically interpret the single copy, by recycling our intuition gathered from the Kerr black hole. Furthermore, the fact that our results apply for any type D vacuum solution makes this a rather powerful statement, that may well help in interpreting and extending the double copy in future.

Another nice aspect of our results is that the multipole expansion in biadjoint theory also matches that in the gauge and gravity theories, for the wide class of solutions we have considered. This adds a powerful weight to the observations made in refs. [37, 38], namely that the twistor double copy allows us to understand the inverse zeroth copy from biadjoint scalar theory to gauge theory. That is, we have seen directly that the multipoles of vacuum type D gravity solutions and their single copies are essentially inherited directly from a much simpler scalar theory! It is interesting to ponder what other physical quantities can be phrased in such an appealing manner.
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