Additional Bending of Light in Sun’s Vicinity 
by its Interior Index of Refraction

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In the seventies, scientists observed discrepancies of the bending of light around the Sun based on Einstein’s prediction of the curvature of star light due to the mass of the Sun. We claim that the interior electromagnetic properties of the Sun influence the curvature of the light path outside the Sun as well. In this paper, we investigate the additional deflection of light in the vacuum region surrounding the Sun by its electromagnetic parameters. Starting with Maxwell’s equations, we show how the deflection of light passing the Sun depends on the electric permittivity and the magnetic permeability of the interior of the Sun. The electromagnetic field equations in Cartesian coordinates are transformed to the ones in an appropriately chosen Riemannian space. This coordinate transform is dictated by the introduction of a refractional potential. The geodetic lines with the shortest propagation time are constructed from this potential. As far as the deflection of light propagating along these geodetic lines is concerned, we show that the existence of a refractional potential influences the light path outside any object with a typical refractive index. Our results add new aspects to the bending of star light explained by general relativity. Some astrophysical observations, which cannot be explained by gravity in a satisfactory manner, are justified by the electromagnetic model. In particular, the frequency dependency of the light deflection is discussed. Our results show that the additional bending due to the refractive index is proportional to the third power of the inverse distance, while general relativity predicts that the bending due to the mass is proportional to the inverse distance.

I. INTRODUCTION

Albert Einstein [1] predicted the bending of light from a distant star by the mass of our Sun through the heaviness of light. The experimental verification in 1919 by Eddington, see [2], of the apparent position shift of the star on the firmament, corroborated the theory of general relativity of Einstein [3]. An overview of the 1919 measurements is given by Will [4]. Einstein explained the bending of the grazing starlight by the gravity of the Sun as it followed the curved geodetic path of light in four-dimensional space. A total deflection angle of 1.75 arcsec is arrived at. Woodward and Yourgrau [5, 6] discussed a paradox in the interaction of the gravitational and electromagnetic fields. To solve this paradox they introduced a frequency dependency of the speed of light in the gravitational field, while Treder [7] used a nonlinear generalizations of Maxwell’s dynamics in the general relativity. Merat et al [8] explained the effect on the light deflection close to the vicinity of the solar limb by introducing a dispersive layer.

The leading question is: are Maxwell’s equations able to explain the change of the light path passing an object? This investigation is the aim of the present paper. We start with Maxwell’s equations in Riemannian space. We consider a bounded object of general form and composition. Let us denote the fastest path of light waves as the geodetic line. We choose a non-trivial metric and we arrive at a simple representation of the electromagnetic equations. In that particular space, in a vacuum sub domain, the waves propagate with the velocity of light along straight lines, being the geodetic lines. Next we determine these geodetic lines in our Cartesian space and arrive at the conclusion that they directly follow from the Helmholtz decomposition theorem for the spatial coordinate changes as a function of refractive index with respect to the vacuum value. This leads to the introduction of a refractional potential.

In this paper, the deflection of star light passing the Sun is discussed. We consider a radially inhomogeneous sphere model with a certain refractive index depending on the radial position. We derive a simple relation, in which the total deflection angle is related to the mean value of the refractive index of the Sun. This refractive index is frequency dependent.

II. MAXWELL’S EQUATIONS IN TENSOR NOTATION

Light is an electromagnetic phenomenon. We consider waves with complex time factor exp(−iωt), where i is imaginary unit, ω is the radial frequency and t is the time. In a vacuum domain, with Cartesian coordinates \( x \in \mathbb{R}^3 \), we write Maxwell’s equation in the frequency domain as

\[
e_{ijk} \partial_j B_k + \frac{1}{c_0^2} i \omega E_i = 0, \\
e_{ijk} \partial_j E_k - i \omega B_i = 0,
\]

where \( E_j = E_j(x, \omega) \) is the electric field vector, \( B_j = B_j(x, \omega) \) is the magnetic field vector, \( c_0 \) is the velocity of...
light in vacuum, and $\epsilon_{ijk}$ is the Levi-Civita symbol. For repeated subscripts, Einstein’s summation convention is used.

In a subdomain $\mathcal{S}$ of $\mathbb{R}^3$, containing a material medium, we define the spatially dependent refractive index $n = n(x, \omega) = c_0 / c(x, \omega)$. Further $\mu = \mu(x, \omega)$ represents the spatially dependent magnetic permeability. Note that the wave velocity is given by $c = 1 / \sqrt{\mu \epsilon}$, where $\epsilon = \epsilon(x, \omega)$ is the spatially dependent electric permittivity. We neglect absorption, so that all material parameters are real valued.

Maxwell’s equations in $\mathcal{S}$ are given by

$$\mu \epsilon_{ijk} \partial_j \left( \frac{1}{\mu} B_k \right) + \frac{n^2}{c_0} \omega E_i = 0,$$

$$\epsilon_{ijk} \partial_j E_k - \iota \omega B_i = 0.$$  \hspace{1cm} (2)

For vacuum in the whole $\mathbb{R}^3$ we have $n = 1$ and the constant magnetic permeability $\mu = \mu_0$. Then, the geodetic lines are straight. In a vacuum domain outside $\mathcal{S}$ with a material medium, we are not allowed to conclude that the geodetic lines in that domain are straight. The geodetic lines are not equivalent to the ray paths in optics, which are defined using a high-frequency approximation of Maxwell’s equations. In the neighborhood of domain $\mathcal{S}$ with $n \neq 1$, these optical rays in vacuum remain straight when they pass $\mathcal{S}$, because within the ray approximation the interaction with matter in $\mathcal{S}$ is neglected. However, the presence of the object $\mathcal{S}$ leads to diffraction of the incident wave and this may influence the path of propagation. In fact, the geodetic line may become curved. Although, with the help of present-day computer codes a more or less complete solution of Maxwell’s equations is possible, the structure of the geodetic lines is hardly to observe from the numerical solution. We therefore investigate the nature of Maxwell’s equations in a different coordinate system.

We introduce a Riemannian space, where the distance element is defined as

$$ds = \sqrt{g_{ij} \, dx^i \, dx^j}. \hspace{1cm} (3)$$

In tensor notation, Maxwell’s equations are

$$\mu g_{il} \epsilon^{ijk} \partial_j \left( \frac{1}{\mu} B_k \right) + \frac{n^2}{c_0} \iota \omega E_i = 0,$$

$$g_{ii} \epsilon^{ijk} \partial_j E_k - \iota \omega B_i = 0,$$  \hspace{1cm} (4)

where $g_{ij}$ is the symmetric metric tensor, and where $E_i$, $B_i$, and $\overline{\partial}_j$ are the electric field vector, the magnetic field vector and the partial derivative in the Riemannian space, respectively. These vectors are defined as

$$\left\{ b_i, E_i, \overline{\partial}_j \right\} = \frac{\partial x^j}{\partial x^i} \left\{ B_j, E_j, \partial_j \right\}. \hspace{1cm} (5)$$

The permutation tensor $\epsilon^{ijk}$ is related to the Levi-Civita symbol as

$$\epsilon^{ijk} = \frac{1}{\sqrt{g}} \epsilon_{ijk}, \hspace{1cm} (6)$$

where $g$ is the determinant of the metric tensor $g_{ij}$. Note, that in our standard Cartesian space, $g_{ij} = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol. Using this definition in Eq. (4), we obtain

$$\mu \frac{g_{ii}}{\sqrt{g}} \epsilon^{ijk} \overline{\partial}_j \left( \frac{1}{\mu} B_k \right) + \frac{n^2}{c_0} \iota \omega E_i = 0,$$

$$\frac{g_{ii}}{\sqrt{g}} \epsilon^{ijk} \overline{\partial}_j E_k - \iota \omega B_i = 0.$$  \hspace{1cm} (7)

Our aim is to find that Riemannian space where the propagation paths are straight.

In the remainder of the paper, we omit the symbol $\omega$ to denote the frequency dependency of the field and material quantities. If necessary we only give their spatial dependency.

III. INVARIANCE OF DISTANCE ELEMENT

In view of the invariance of scalar distance element $ds$, we may conclude that

$$ds = \sqrt{dx^i \, dx^j} = \sqrt{g_{ij} \, dx^i \, dx^j}. \hspace{1cm} (8)$$

If we choose the simplest, non-trivial, case,

$$\frac{\partial x^i}{\partial x^j} = n \delta^i_j, \hspace{1cm} g_{ij} = \frac{1}{n^2} \delta_{ij},$$

where $\delta^i_j$ is the Kronecker tensor, we obtain

$$\overline{\partial}_j = \frac{1}{n} \partial_j, \hspace{1cm} (10)$$

and we arrive at the invariance,

$$\frac{1}{c(x)} \sqrt{dx^i \, dx^j} = \frac{1}{c_0} \sqrt{dx^i \, dx^j},$$

where we have used that $n(x) = c_0 / c(x)$. From this invariance we conclude that the travel time over every distance element in the chosen Riemannian space with vacuum velocity equals to the travel time over every distance element in the original Cartesian space with spatially dependent velocity. We emphasis that the transformation of coordinates is not a local transformation, but a global one. For example a change of coordinates in our object domain $\mathcal{S}$ influences coordinates outside $\mathcal{S}$ as well. This influence diminishes for larger distances from $\mathcal{S}$.

It is noted that for the present metric, we have $\sqrt{g} = 1/n^3$, and the Maxwell’s equations (7) simplify to

$$\mu \frac{g_{ii}}{\sqrt{g}} \epsilon_{ijk} \overline{\partial}_j \left( \frac{1}{\mu} B_k \right) + \frac{n^2}{c_0} \iota \omega E_i = 0,$$

$$n \epsilon_{ijk} \overline{\partial}_j \left( \frac{1}{n} E_k \right) - \iota \omega B_i = 0, \hspace{1cm} \overline{\mathbf{x}} \in \mathcal{S}, \hspace{1cm} (12)$$

and

$$\epsilon_{ijk} \overline{\partial}_j B_k + \frac{1}{c_0} \iota \omega E_i = 0,$$

$$\epsilon_{ijk} \overline{\partial}_j E_k - \iota \omega B_i = 0, \hspace{1cm} \overline{\mathbf{x}} \in \mathbb{R}^3 \setminus \mathcal{S}. \hspace{1cm} (13)$$
with \( B_k = \frac{1}{n} F_k \) and \( S \) is the transform domain of the object \( S \).

At this point we interpret these equations as follows. Before an electromagnetic wave in vacuum arrives at the domain \( S \), it satisfies Eq. (13) in which the geodetic lines are straight. After entering the domain \( S \), the wave satisfies Eq. (12) and meets a changing spatial curvature. The wave will be distorted not only inside \( S \), but also outside. In our world, we would say a secondary scattered field is excited as a consequence of the change in material parameters. Outside \( S \), this scattered field travels again along straight geodetic lines. Since in our Riemannian space the geodetic lines are straight, we may conclude that in the original Cartesian space these lines have to be curved. In this paper we only investigate the propagation of the primary wave, because the scattered waves arrive always later at a certain point of observation. To investigate this in more detail, we consider the coordinate transformation between the two spaces.

### IV. THE REFRACTIONAL POTENTIAL AND TENSION

Let the position vectors in the Cartesian space and the Riemannian space be given by \( x_j \) and \( x_{f_j} \), respectively. The values of these coordinates in the Cartesian coordinate system are related to each other as

\[
\mathbf{r}_j(x) = x_j + f_j(x) .
\]  
(14)

To derive an expression for the continuously differentiable function \( f_j \) in terms of the refractive index \( n \), we write

\[
f_j(x) = \mathbf{r}_j(x) - x_j ,
\]  
(15)

and apply the divergence and the curl operators to both sides of this relation. After using the inverse of the first relation of Eq. (13), we arrive at

\[
\partial_j f_j = 3(n - 1) \quad \text{and} \quad e_{ijk} \partial_i f_k = 0 .
\]  
(16)

Then, Helmholtz decomposition theorem for a curl-free field provides the non-trivial solution

\[
f_j = -\partial_j \Phi ,
\]  
(17)

where we define \( \Phi \) as the refractional potential, given by

\[
\Phi(x) = \int_{x' \in S} \frac{3 |n(x') - 1|}{4\pi|x - x'|} dV .
\]  
(18)

Obviously, \( f_j \) is the tension due to the difference in refractive index with respect to vacuum. We denote \( f_j \) as the refractional tension. This representation is valid under the condition that \( n - 1 \) vanishes at the boundary surface of the domain \( S \).

Before we continue with our analysis, we conclude that Eqs. (14), (17) and (18) define our spatial transformation from the \( x \)-space to the \( \mathbf{r} \)-space. This definition holds for any distribution of the refractive index inside domain \( S \). Note that the expression of the refractional potential yields a non-zero value outside \( S \) and this confirms that the refractive index distribution inside the object \( S \) not only determines the spatial coordinate transformation inside this object, but also outside. Hence, the geodetic lines in the vacuum domain around \( S \) are influenced by the inner refractive index of the object. It is obvious that \( \mathbf{r} \) is a nonlinear function of \( x \) and therefore it is difficult to find directly the geodetic lines. Therefore, we consider a piecewise-linear approximation of the geodetic path.

For a small perturbation of \( x_j \), from Eq. (14) it follows that

\[
\mathbf{r}_j(x + dx) = x_j + dx_j + f_j(x + dx) ,
\]  
(19)

and

\[
f_j(x + dx) = f_j(x) + (dx_k \partial_k) f_j(x) + O(|dx|^2) ,
\]  
(20)

where \( (dx_k \partial_k) \) denotes the spatial derivative in the direction of the perturbation. Substituting Eq. (20) into Eq. (19) and using Eq. (14), we arrive at

\[
\mathbf{r}_j(x + dx) = \mathbf{r}_j(x) + d\mathbf{r}_j(x) + O(|dx|^2) ,
\]  
(21)

where

\[
d\mathbf{r}_j(x) = dx_j + (dx_k \partial_k) f_j(x) .
\]  
(22)

For convenience, we introduce the curvature tensor \( C_{jk} \) as

\[
C_{jk} = \delta_{jk} + \partial_k f_j ,
\]  
(23)

so that Eq. (22) is written as

\[
d\mathbf{r}_j(x) = C_{jk} dx_k .
\]  
(24)

We remark that the trace of \( C_{jk} \) is equal to \( 3n \), where we used Eq. (16). Since the matrix is real and symmetric, an eigenvalue decomposition with positive eigenvalues exists and the sum of the eigenvalues is equal to the trace. Inspection of Eqs. (23) and (24) learns that Eq. (24) constitutes a local contravariant transformation. This implies that the eigenvectors are spanned by the unit vectors in the directions of the tension \( f \). One of the eigenvalues corresponds to the eigenvector \( f_j / |f| \), so that this eigenvalue \( \lambda \) satisfies

\[
C_{jk} \frac{f_k}{|f|} = \lambda \frac{f_j}{|f|} .
\]  
(25)

Use of Eq. (23) and contraction of the result with \( f_j / |f| \) leads to

\[
\lambda = 1 + \frac{f_k}{|f|} \partial_k |f| .
\]  
(26)

The procedure to determine the other two eigenvalues are discussed in our spherical example.
Next we consider the scalar arclength $\overline{ds}^2$ given by

$$\overline{ds}^2 = dx_i dx_i.$$ \hfill (27)

Using Eq. (24) we arrive at

$$\overline{ds}^2 = C_{jl} C_{jk} dx_l dx_k.$$ \hfill (28)

Introducing the unit vector $\hat{s}_k = dx_k/ds$, $\hat{s}_k\hat{s}_k = 1$, we write $\overline{ds} = \overline{d}s(x, \hat{s})$ as

$$\overline{ds} = \left[ C_{jl} C_{jk} \hat{s}_l \hat{s}_k \right]^{1/2} ds.$$ \hfill (29)

To investigate the dynamic behavior, see p. 114 of Born and Wolf [9], we consider the optical length of the geodetic path in a similar way as the refractive indices, which are invariant for the direction of the geodetic line, at this position. We note that this differential equation holds for refractive indices, which are invariant for the direction of the geodetic line.

We remark that the actual computation of this refractive index is simplified by employing an eigenvalue decomposition.

Basically, the virtual refractive index $\overline{n}^{\varepsilon} (x, \hat{s})$ controls the path of the geodetic line in a similar way as the refractive index $n(x)$ controls the path of optical rays. Note that the virtual refractive index is not only determined by the local position of the geodetic line, but it also depends on the direction of the geodetic line at this position. We construct this geodetic line by considering the explicit Euler integration of the classic differential equation for the evolution of an optical ray path, see p. 121 of Born and Wolf [9], but we replace the physical refractive index $n$ by the virtual counterpart $\overline{n}^{\varepsilon}$, viz.

$$\frac{d[n^{\varepsilon}(x, \hat{s}) \hat{s}_j]}{ds} = \partial_j n^{\varepsilon}, \quad \text{with} \quad \hat{s}_j = \frac{dx_j}{ds},$$ \hfill (31)

where $x_j = x_j(s)$ is the trajectory of the geodetic line and $s$ is the parametric distance along this trajectory, while $\hat{s}_j$ is the tangential unit vector along the geodetic line. We note that this differential equation holds for refractive indices, which are invariant for the direction of the geodetic path. However, the explicit Euler integration updates the ray position and ray direction in such a way that only the previous information of position and direction is used over the pertaining path segment. During each integration step, the path directions do not change. This is consistent with the assumption of a piecewise linear approximation of the geodetic path of Eq. (19).

For a rotationally symmetric configuration the present analysis simplifies. For this specific case, we shall discuss the construction of the geodetic lines in full detail.

V. RADially inHomoGENEOus MEedium

We consider a radially inhomogeneous spherical medium. Introducing spherical coordinates (see Appendix A), the refractive potential and tension are determined in closed form. In this case, the tension depends on $R$ only and is directed in the radial direction. Hence, $f_\theta = f_\alpha = 0$ and the radial component is given by, see Eq. (A8),

$$f_R(R) = \frac{3}{R^2} \int_0^R [n(r) - 1] r^2 dr.$$ \hfill (32)

The Cartesian components of the tension are obtained as

$$f_k = \frac{x_k}{R} f_R(R).$$ \hfill (33)

From Eq. (26) it follows that the eigenvalue in the radial direction is given by

$$\lambda_R = 1 + \partial_R f_R,$$ \hfill (34)

while the eigenvalues in the tangential directions follow from the trace

$$\lambda_R + \lambda_\theta + \lambda_\phi = 3n.$$ \hfill (35)

In view of the axial symmetry of our configuration, the tangential eigenvalues are the same. We therefore confine our analysis to the plane in which the geodetic path is defined. Hence, we suffice with the computation of $\lambda_\theta$, viz.,

$$\lambda_\theta = (3n - \lambda_R)/2.$$ \hfill (36)

The eigenvalues depend only on $\partial_R f_R$. From Eq. (16) we observe that

$$\frac{1}{R^2} \partial_R (R^2 f_R) = 3(n-1),$$ \hfill (37)

or

$$\partial_R f_R = 3(n-1) - \frac{2 f_R}{R}.$$ \hfill (38)

From Eq. (34) it follows that

$$\lambda_R = 1 - 2 \frac{f_R}{R} + 3(n-1)$$ \hfill (39)

and

$$\lambda_\theta = 1 + \frac{f_R}{R}.$$ \hfill (40)

The virtual refractive index is then obtained as, cf. Eq. (30),

$$\overline{n}^{\varepsilon} = \left[ \left( 1 - 2 \frac{f_R}{R} + 3(n-1) \right) \frac{\hat{s}_R^2}{\hat{s}_R^2} + \left( 1 + \frac{f_R}{R} \right)^2 \frac{\hat{s}_\theta^2}{\hat{s}_\theta^2} \right]^{1/2},$$ \hfill (41)

where $\hat{s}_R = \cos(\theta - \alpha)$ and $\hat{s}_\theta = \sin(\theta - \alpha)$. Here, $\theta$ is the angle between $\hat{r}$ and the $x_1$-direction, while $\alpha$ is the angle between $\hat{s}$ and the $x_1$-direction.
Coordinate transformation for a homogeneous sphere in vacuum

To illustrate the coordinate transformation, we consider the simplest case of a homogeneous sphere in vacuum. Its radius is given by \( R_S \), the inner refractive index \( n = n_S \) is constant and the outer refractive index is equal to one. Equations (15) and (33) enable us to compute the position vector \( x_j \) in the Cartesian domain from the position vector \( \vec{x}_j \) as,

\[
x_k = \vec{x}_k - f_k.
\]

Then, Eq. (32) transfers into

\[
f_R(R) = \begin{cases} 
(n_S - 1) R, & R \leq R_S, \\
(n_S - 1) \frac{R_S^3}{R^2}, & R \geq R_S. 
\end{cases}
\]

Note that inside the homogeneous sphere the coordinate transformation is linear, while outside this sphere it is non-linear. We compute the spherical wave fronts in the Riemannian \( \vec{x} \)-space and the corresponding wave fronts in the Cartesian \( x \)-space. The wave fronts move sinusoidally in time, with wavenumber \( k_0 = \omega/c_0 \). For a source located at \( x' \), the wave fronts \( u \) as function of the spatial positions at zero time instant are given by

\[
u(\vec{x}) = \frac{\cos(k_0|\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|}, \quad \text{in Riemannian space},
\]

\[
u(x) = \frac{\cos(k_0|x - x'|)}{|x - x'|}, \quad \text{in Cartesian space},
\]

with \( x_j = \vec{x}_j - f_j(\vec{x}) \) and \( x'_j = \vec{x}'_j - f_j(\vec{x}') \). In the Riemannian space we compute the wave field on a regular grid for \( \vec{x} \), see top picture of FIG. 1. With Eq. (42), we obtain \( x \) on an irregular grid of the Cartesian space. Interpolation to a regular grid in the Cartesian space provides the bottom picture of FIG. 1. The influence of the coordinate transform to the wave fronts is obvious: it seems that the wave field bends around the sphere.

VI. GEODETIC LINES OUTSIDE AN INHOMOGENEOUS SPHERE IN VACUUM

We now consider a radially inhomogeneous sphere with radius \( R_S \). Let us define the mean value of the refractive index of the sphere as \( n_S \). Outside the sphere, the second expression for the tension of Eq. (43) do not change, i.e.,

\[
f_R(R) = (n_S - 1) \frac{R_S^3}{R^2}, \quad R > R_S.
\]

We have computed some geodetic lines by solving the differential equation of Eq. (31), using the method described below this equation. Using our deflection angle \( \alpha \), this set of differential equations in the plane \( x_3 = 0 \) is written as

\[
\frac{d(n^S \sin \alpha)}{ds} = \partial_2 n^S, \\
\frac{d(n^S \cos \alpha)}{ds} = \partial_1 n^S.
\]

These equations holds for refractive indices, which do not depend on the direction of the geodetic path. To include the directional dependence on the direction of the path, we solve this set of first order ordinary differential equations using Euler’s method with step size \( \Delta s \). At each step of this explicit scheme, the path direction is determined by the one of the previous step and it does not change over the path segment \( \Delta s \). In our linear approximation, the rotational factors \( \hat{s}_R^2 \) and \( \hat{s}_S^2 \) do not change over each path segment and depend only on \( R \). This means that the spatial derivatives of the virtual refractive index in the plane \( x_3 = 0 \) are given by

\[
\partial_1 n^S = \frac{x_1}{R} \partial_R n^S = A \cos \theta / n^S, \\
\partial_2 n^S = \frac{x_2}{R} \partial_R n^S = A \sin \theta / n^S,
\]
with

\[
A = 6 \left( 1 - 2 \frac{f_R}{R} \right) \frac{f_R}{R^2} \int_{0}^{R} R^2 - 3 \left( 1 + \frac{f_R}{R} \right) \frac{f_R}{R^2} \int_{0}^{R} R^2 \int_{0}^{R} \]

where \( f_R \) is given by the second relation of Eq. (43).

The Euler method is a first-order method, which means that the local error per step is proportional to the square of the step size, and the global error over the total path is proportional to the step size \( \Delta s \). Within this first-order approximation, we may take the virtual refractive index outside the differentiation with respect to \( s \) and divide both sides of Eq. (46) by \( n^g \). To reduce the errors, one may apply a so-called predictor-corrector method. Since the spatial variation of the geodetic line is very small and exhibits only some variation during the passage along the sphere, the extra corrector step is not necessary. We use a step size of \( \Delta s = 0.01 R_S \). After carrying out step (2a), we update the values for the virtual refractive index and its spatial derivatives and we return to step (1). The recursion is terminated, when the geodetic line has reached the boundary of our window of observation. Step (2a) seems superfluous, but we need the expression in our asymptotic analysis for small \( \alpha \).

The geodetic line is constructed via the following recursive scheme:

**Step 1**:

\[
x_2 := x_2 + \sin \alpha \Delta s,
\]

\[
x_1 := x_1 + \cos \alpha \Delta s,
\]

**Step 2**:

\[
\sin \alpha := \sin \alpha + \left( \frac{\partial_2 n^g}{n^g} \right) \Delta s,
\]

\[
\cos \alpha := \cos \alpha + \left( \frac{\partial_1 n^g}{n^g} \right) \Delta s,
\]

**Step 2a**:

\[
\alpha := \arctan \left( \frac{\sin \alpha + \left( \frac{\partial_2 n^g}{n^g} \right) \Delta s}{\cos \alpha + \left( \frac{\partial_1 n^g}{n^g} \right) \Delta s} \right).
\]

In FIG. 2 we show some numerical results of geodetic lines constructed for \( n_g = 1.5 \). In its top figure, the phenomenon of bending of the geodetic lines located outside the sphere is clearly visible. We also applied the construction of these geodetic lines with a predictor-corrector method and we did not observe visible differences. To gain some insight on the influence of the virtual refractive index on the course of the geodetic path, we picture in the middle figure its value along the path. Approaching the sphere, the virtual velocity \( v = c_0 / n^g \) increases and the geodetic line bends away from the sphere. Subsequently, the virtual velocity decreases in the radial direction and the geodetic line bends towards the sphere. The curvature of the wave fronts in the bottom figure of FIG. 2 agrees with this phenomenon. We observe that the presence of the sphere is noticeable in the horizontal range of \((-5R_S, 5R_S)\). Outside this range the virtual refractive index tends to 1 and the geodetic lines become straight. In the bottom figure, we present the cumulative deflection angles for the different geodetic paths.

In FIG. 3 we mimic the bending of light by the Sun. The mean refractive index of the Sun is close to one. For grazing incidence, its value is chosen such that a total deflection angle of 1.75 arcsec is obtained. This is equivalent by taking \( n_g = 1 + 0.53 \times 10^{-6} \). A comparison with FIG. 2 shows that in the top figure of FIG. 3 the deflection of the geodetic lines is hardly visible. The same applies for the shift of the maximum values of the virtual refractive index in the middle figure. The curves are
start with the expression for $A$ of Eq. (45). For small values of $n_S - 1$, we only retain the terms linear in $f_R$, i.e.,

$$A = (n_S - 1) \left[ -3 + 9 \cos^2(\theta - \alpha) \right] \frac{R_S^3}{R^4}.$$  \hspace{1cm} (50)

Further, in the region around the sphere, where $\theta$ has values around $\theta = \frac{1}{2} \pi$, we neglect the influence of $\alpha$. Outside this region, the values of $A$ become negligible. Next we consider the relation for $\alpha$ of Eq. (49). Within our approximations already made, we take $\sin \alpha \approx \alpha$, $\cos(\theta - \alpha) \approx \cos \theta$ and $n^8 \approx 1$. Subsequently, we expand the quotient of step (2a) of Eq. (49) in terms of small $\Delta s$ to obtain the cumulative deflection angle,

$$\alpha := \alpha + (\partial_2 n^k - \partial_1 n^k) \Delta s \approx \alpha + [A \sin \theta - A \cos \theta] \Delta s,$$  \hspace{1cm} (51)

where we have used Eq. (47). With $\Delta s \approx \Delta x_1$ and similar type of approximations made before, the updates for the spatial coordinates become

$$x_2 \approx x_2(0) \quad \text{and} \quad x_1 := x_1 + \Delta x_1.$$  \hspace{1cm} (52)

Since $x_2$ is considered to be constant, we write the radial coordinate as $R = x_2(0)/\sin \theta$. The asymptotic expression for $A$ becomes

$$A \approx (n_S - 1) \left[ -3 + 9 \cos^2 \theta \right] \frac{R_S^3}{x_2(0)} \sin^4 \theta.$$  \hspace{1cm} (53)

Combining all these approximations, we observe that Eqs. (51) and (52) represent the numerical counterpart to calculate the following integral for the total deflection in the negative $x_2$-direction as:

$$d^{EM} = -\int_{-\infty}^{\infty} [A \sin \theta - A \cos \theta] \sin \theta \sin \theta \sin \theta \sin \theta = -2 \int_{0}^{\infty} \frac{1}{R} \sin \theta \sin \theta \sin \theta \sin \theta,$$  \hspace{1cm} (54)

because the first term of the integrand is a symmetric function of $x_1$, while the second one is asymmetric. Further, from $x_1 = x_2(0)/\tan \theta$ follows that $\sin^2 \theta \sin \theta \sin \theta \sin \theta = -x_2(0) \sin \theta \sin \theta \sin \theta \sin \theta$, and the integral is rewritten as

$$d^{EM} = 2(n_S - 1) \left[ R_S \right] \frac{R_S^3}{x_2(0)} \int_{0}^{\pi} \sin^2 \theta \sin \theta \sin \theta \sin \theta.$$  \hspace{1cm} (55)

The integral can be calculated analytically and is equal to 4/5. The asymptotic formula for the total deflection is finally obtained as

$$d^{EM} = \frac{8}{5} (n_S - 1) \frac{1}{(R_0/R_S)^3}, \quad \text{for small } n_S - 1,$$  \hspace{1cm} (56)

where $R_0 \approx x_2(0)$ is the smallest value of $R$ on the geodetic line. The value of $R_0/R_S$ is often denoted as the impact parameter.

In TABLE I we present the values for the ratio of the deflection angle obtained numerically ($d^{EM}_{num}$) and the one obtained analytically ($d^{EM}_{asym}$) using the asymptotic approximations. We note that the closer this ratio is to
one, the better the asymptotic approach. We observe that the discrepancies in the approximations are of the order of $n_S - 1$. For increasing values of $R$ the approximations improve as well; the worst case appears for grazing incidence ($R_0/R_S = 1$); but we may conclude that, for values of $n_S - 1 < 10^{-3}$, the approximation is fully justified. Note that for the Sun case, deflection angles of the order of $1.75$ arcsec are arrived at, which correspond to $n_S - 1 \approx 10^{-6}$. Hence, there is no doubt about the validity of our asymptotic analysis.

From Eq. (56) we conclude that deflection angle is proportional to $(R_0/R_S)^{-3}$, while general relativity predicts a dependency of $(R_0/R_S)^{-1}$, see Fig. 1 of Biswas et al [10]. For convenience, we denote the electromagnetic contribution, the near-field term of the bending, while the gravitational contribution dominates the far-field term.

### VII. VALIDATION ON HISTORICAL DATA

At this point, we return to the work of Merat et al [8]. They conclude on basis of radio deflection observations [11], that for $R_0 < 5R_S$ deviations from the Einstein prediction become statistically significant. They have collected the whole set of star deflection data into 4 samples. The weighted mean of the distance $R_0/R_S$ of each sample has been given, together with the mean deviation of light deflections from the GR prediction. In FIG. 4, the red squares denote the total deflection data values, including the GR predictions. In the left picture of FIG. 4, the GR curve itself is shown as the black curve. The differences of the data with the GR curve amounts to $0.139$, $0.081$, $0.023$ and $0.013$, respectively. The mean square
of these residuals with respect to the total deflection error amounts to 31%.

A. Influence of the EM tension

To improve the GR reflection model, we assume that the total deviation data is a linear superposition of the GR curve \((R_0/R_S)^{-3}\) and our EM curve \((R_0/R_S)^{-3}\). We define

\[
d_{EM} = d^{tot} - \frac{1.75 \text{ arcsec}}{(R_0/R_S)} = \frac{B}{(R_0/R_S)^3}.
\] (57)

To find the unknown factor \(B\), we carry out a least-square fit, which minimizes the deviations and the four data points of \([3]\). Substituting the resulting value of \(B\) in Eq. (57), the total deflection function \(d^{GR} + d^{EM}\) is presented as the red line in the left picture of FIG. 4. The discrepancies between the four data points and this red curve amounts to -0.0107, 0.0549, 0.0137 and 0.0091, respectively. The mean square of these residuals with respect to the total deflection error amounts to 14%. These discrepancies with respect to the four data points may be explained as a Corona effect outside the domain \(S\) of the Sun.

B. Influence of the Corona

In the Corona, we only take into account the local effect of the refractive index of the Corona. In order to include the plasma effects of the Corona, we start with the refractive index described as a superposition of powers of \(R_S/R\), with constant factors \(\eta_p\), viz.

\[
n_C(R) - 1 = \sum_p \eta_p \left(\frac{R_S}{R}\right)^p, \quad p > 1, \quad R > R_S.
\] (58)

The data under consideration are obtained for \(R > 3R_S\) and we employ the refractive index described in \([3]\), viz.,

\[
n_C(R) - 1 = \eta_{p_1} \left(\frac{R_S}{R}\right)^{p_1} + \eta_{p_2} \left(\frac{R_S}{R}\right)^{p_2}, \quad \frac{R_S}{R} > 3,
\] (59)

where \(p_1 = 6\) and \(p_2 = 2.33\). From this Appendix B, we conclude that the electromagnetic deflection may be written as

\[
d_{EM} = \frac{C_{p_1}}{(R_0/R_S)^{p_1}} + \frac{C_{p_2}}{(R_0/R_S)^{p_2}}.
\] (60)

For the range of \(R_0 > 3R_S\) we determine the coefficients \(C_{p_1}\) and \(C_{p_2}\) by a least-square fitting of Eq. (60) to the four data points given by Merat et al \([3]\).

In the middle picture of FIG. 4 the deflection by the local coronal medium is presented as the blue dashed line. The discrepancies between the four data points and this blue dashed curve are -0.0002, 0.0071, -0.0119 and -0.0048, respectively. The mean square of these residuals with respect to the total deflection error amounts to 5%.

C. Influence of the EM tension and the Corona

In the integral expression of \(f_R\), see Eq. (62), for \(R > R_S\), we subtract in the integrand \(n(r)-1\) the coronal contribution, \(n_C(r)-1\), so that the integral is restricted to the range of \(0 < r < R^S\). Then, the EM deflection is given by Eq. (57). Following the pure gravity light bending theory of Maccone \([12]\), we also denote this as the naked-Sun situation. For small deflections, we take a linear superposition of the naked-Sun part and the mantle part (the Corona). We conclude that the total electromagnetic deflection may be written as

\[
d_{EM} = \frac{B}{(R_0/R_S)^3} + \frac{C_{p_1}}{(R_0/R_S)^{p_1}} + \frac{C_{p_2}}{(R_0/R_S)^{p_2}}.
\] (61)

In a least-square fitting procedure to the data, we observed that the system matrix is heavily ill-posed and impossible to invert numerically. A stable result is obtained by preconditioning. We rewrite Eq. (61) as

\[
d_{EM} = \frac{B}{(R_0/R_S)^3} \left[ 1 + \frac{C_1}{(R_0/R_S)^{p_1-3}} + \frac{C_2}{(R_0/R_S)^{p_2-3}} \right],
\] (62)

where \(C_1 = C_{p_1}/B\) and \(C_2 = C_{p_2}/B\). This nonlinear equation is solved with an iterative Gauss-Newton method. As starting values we take zero values for \(C_1\) and \(C_2\) and determine \(B\) by a direct least-square minimization. After carrying out a few Gauss-Newton iterations a stable result is obtained. The resulting deflection is plotted as the green line in the right picture of FIG. 4. The discrepancies between the four data points and this red curve amounts to -0.0000, 0.0008, -0.0036 and 0.0040, respectively. The mean square of these residuals with respect to the total deflection error amounts to 2%.

To judge the value of the different results, in Table II we present the values of the three parameters \(B, C_{p_1}\) and \(C_{p_2}\) (in arcsec) substituted into Eq. (62) to plot the three curves of Fig. 5. For convenience we also present the ratio of \(C_{p_1}/C_{p_2}\).

| | naked Sun (red curve) | mantle (blue dashed) | naked Sun + mantle (green curve) |
|---|---|---|---|
| \(B\) | 6.04 | 0 | 36.8 |
| \(C_{p_1}\) | 0 | -274 | -3.17 \times 10^4 |
| \(C_{p_2}\) | 0 | 5. | -1.59 \times 10^2 |
| \(C_{p_1}/C_{p_2}\) | - | -50 | 199 |
D. Frequency dependent bending

Consistent with the frequency dependent refractive index \( n_C = n_C(\omega) \) of the coronal medium also the interior refraction index \( n_S = n_S(\omega) \) of the Sun is frequency dependent. The additional EM deflection \( d_{\text{EM}} \) is linearly related to these frequency-dependent refractive indices. As far as the frequency-dependent refractive index of Sun’s interior is concerned, the outer layer can be represented by a refractive index that differs from the other inner layers. This will change its mean value \( n_S \).

VIII. CONCLUSIONS

In this paper we demonstrated that apart from a gravitational type and a coronal type of bending along the Sun, Maxwell’s equations predict an additional type of bending by the existence of a refractional tension. The latter is caused by the presence of a non-zero refractive index of the Sun’s interior medium. This electromagnetic addition to the GR tension has been verified on data from historical astrophysical measurements without changing the GR tension. It has been shown that the additional EM tension is an essential ingredient of the prediction of the interstellar wave propagation paths. The influence of the refractional tension becomes more significant for observations closer to the Sun.

The electromagnetic deflection (including the coronal one) is frequency-dependent and dominant in the near- and mid-field, while the GR contribution is frequency independent and it dominates the far-field. Future research is extremely important for gravity lensing and interstellar communication experiments, where accurate electromagnetic predictions of possible interstellar pathways are sought.

We conclude our paper by mentioning that a scaled experiment is possible by using a voluminous object with a noticeable refractive index. This will potentially verify the electromagnetic deflection outside the object, since the gravitational and coronal components can be neglected in this case.

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Appendix A: The refractive potential and the tension for a radially inhomogeneous sphere and its derivatives

For a radially inhomogeneous sphere, the refractive potential of Eq. (18) can be calculated analytically. We introduce spherical coordinates for the observation points $x$ as

$$x_1 = R \sin \theta \cos \phi, \quad x_2 = R \sin \theta \sin \phi, \quad x_3 = R \cos \theta,$$

(A1)

and spherical coordinates for the integration points $x'$ as

$$x'_1 = r \sin \theta' \cos \phi', \quad x'_2 = r \sin \theta' \sin \phi', \quad x'_3 = r \cos \theta'.$$

(A2)

For convenience we take the polar axis in the direction of $x$. Then, the Cartesian distance and the volume element become

$$|x - x'| = \left[ R^2 + r^2 - 2 R r \cos \theta' \right]^{\frac{1}{2}},$$

$$dV = r^2 \sin \theta' \, dr \, d\theta \, d\phi'.$$

(A3)

In the resulting integral we first carry out the integration with respect to $\phi'$; this merely amounts to a multiplication by a factor of $2\pi$, so that Eq. (18) transfers into

$$\Phi(R, \theta, \phi) = \frac{3}{2} \int_0^{\infty} \left[ n(r) - 1 \right] r^2 \, dr \left[ \sin \theta' \right] \left[ R^2 + r^2 - 2 R r \cos \theta' \right]^{\frac{1}{2}} d\theta'.$$

Next we carry out the integration with respect to $\theta'$, which is elementary. After this, we have

$$\Phi(R, \theta, \phi) = \frac{3}{2} \int_0^{\infty} \left[ n(r) - 1 \right] r^2 \left[ \frac{R + r}{R r} - \frac{|R - r|}{R r} \right] \, dr,$$

(A4)

which shows that $\Phi$ is independent of $\theta$ and $\phi$. Taking into account the meaning of $|R - r|$, we obtain

$$\Phi(R) = \frac{3}{R} \int_0^R \left[ n(r) - 1 \right] r^2 \, dr + 3 \int_R^{\infty} \left[ n(r) - 1 \right] r \, dr.$$

(A5)

Note that this expression holds for all $R$, if $n(r) = O(r^{-2})$ when $r$ tends to infinity.

The gradient of the potential is directed in the radial direction. Hence $\nabla \Phi = (d\Phi/dR) \mathbf{i}_R$. Applying Leibniz’ rule for differentiation of an integral to Eq. (A5) yields

$$\frac{d\Phi}{dR} = -\frac{3}{R^2} \int_0^R \left[ n(r) - 1 \right] r^2 \, dr$$

$$+ \frac{3}{R} \left[ n(R) - 1 \right] R^2 - 3\left[ n(R) - 1 \right] R,$$

(A6)

which simplifies to

$$\frac{d\Phi}{dR} = -\frac{3}{R^2} \int_0^R \left[ n(r) - 1 \right] r^2 \, dr.$$  

(A7)

With this result, the tension $F = f_R \mathbf{i}_R = -(d\Phi/dR) \mathbf{i}_R$ is obtained as

$$f_R(R) = \frac{3}{R^2} \int_0^R \left[ n(r) - 1 \right] r^2 \, dr.$$  

(A8)

Appendix B: Deflection due to presence of the Corona

Let us consider the coronal refractive index for a particular term in which the radial dependence is given a certain inverse power of $p$, viz.,

$$n_C(R) = 1 + \eta_p \left( \frac{R_S}{R} \right)^p,$$

(B1)

Then, the partial derivatives with respect to $x_1$ and $x_2$ are obtained as

$$\partial_1 n_C = \frac{x_1}{R} \partial_R n_C, \quad \partial_2 n_C = \frac{x_2}{R} \partial_R n_C,$$

(B2)

where

$$\partial_R n_C(R) = -\eta_p \frac{R_S^p}{R^{p+1}}.$$  

(B3)

Similar as before, for small values of $\eta$, the cumulative deflection angle is given by

$$\alpha = \alpha + (\partial_2 n^p - \partial_1 n^p) \Delta s$$

$$= \alpha - \eta_p \frac{R_S^p}{R^{p+1}} (\sin \theta - \cos \theta) \Delta s,$$

(B4)

and the total deflection caused by the presence of the Corona becomes

$$d_{EM} = -\eta_p \left[ \frac{R_S}{x_2(0)} \right]^p F(p),$$

(B5)

in which

$$F(p) = \int_0^\pi \sin^p \theta - \sin^p \theta \, d\theta = 2 \int_0^{\frac{\pi}{2}} \sin^p \theta \, d\theta$$

$$= \sqrt{\pi} \Gamma\left( \frac{1}{2} + \frac{1}{2} p \right).$$

(B6)

Evaluating the Gamma functions for $p = 6$ and $p = 2.33$, we obtain $F(6) = 5\pi/16$ and $F(2.33) = 1.4792$. 

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