Resilient Structural Stabilizability of Undirected Networks

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Abstract—In this paper, we consider the structural stabilizability problem of undirected networks. More specifically, we are tasked to infer the stabilizability of an undirected network from its underlying topology, where the undirected networks are modeled as continuous-time linear time-invariant (LTI) systems involving symmetric state matrices. Firstly, we derive a graph-theoretic necessary and sufficient condition for structural stabilizability of undirected networks. Then, we propose a method to determine the maximum dimension of stabilizable subspace solely based on the network structure. Based on these results, on one hand, we study the optimal actuator-disabling attack problem, i.e., removing a limited number of actuators to minimize the maximum dimension of stabilizable subspace. We show this problem is NP-hard. On the other hand, we study the optimal recovery problem with respect to the same kind of attacks, i.e., adding a limited number of new actuators such that the maximum dimension of stabilizable subspace is maximized. We prove the optimal recovery problem is also NP-hard, and we develop a \((1 - 1/e)\) approximation algorithm to this problem.

I. INTRODUCTION

In recent years, the control of networked dynamical systems has attracted a great amount of research interest [1–3]. It is of particular interest to study the asymptotic stabilizability of network control systems, i.e., the ability ensuring that all the system states can be steered to the origin by injecting proper controls, such as the undirected consensus network [1], voltage stabilization of grids [2], and formation control with undirected communication links [3].

The existing results on stabilizability analysis highly rely on the assumption that the system parameters can be exactly acquired, which is often violated in practice, (- see 4–6 and the references therein). It has been shown that the topological structure of a network, which can be obtained accurately, can be exploited to infer the required conditions to ensure the controllability of a network system efficiently [7, 9]. This motivates us to investigate the interplay between the network’s structure and the stabilizability of a network.

Assessing the stabilizability from the structural information on the system dynamics model has been an active topic of research [10–13]. However, in [10], the authors assumed no control input and proposed conditions on the sparsity pattern of symmetric state matrices such that a specific sparsity pattern sustains a Hurwitz stable state matrix. In addition, the problem considered in [11–13] is the arbitrary pole placement through output feedback, which is sufficient but not necessary for the stabilizability.

Stabilizability is a crucial concept in network security [14] and there has been a tremendous effort invested into the control of networks under malicious attacks [14–23]. The problems of adding extra actuators/sensors to ensure controllability/observability under attacks are addressed in [15, 16]. The problem of maintaining stabilization under the uncertain feedback-channel failure is considered in [17–18]. In [19, 20], the problem of optimal attack/recovery on structural controllability is investigated. Although the problems of stabilization under various attacks such as deception attack [14], replay attacks [21], denial-of-service [22] and destabilizing attacks [23], have been widely studied, the crucial problem of optimal attack against stabilizability by manipulating network topological structure, e.g., removing or adding actuators, has not been fully investigated. Moreover, to the best of the authors’ knowledge, our paper considers for the first time the problems of optimal attack and recovery on the stabilizable subspace of a network, i.e., the number of stabilizable states or nodes in a network.

Specifically, in this paper, we consider the structural stabilizability problem, and the contributions of this paper are four-fold. First, we derive a graph-theoretic necessary and sufficient condition for structural stabilizability of undirected networks. Second, we propose computationally efficient methods to determine the generic dimension of controllable subspace and the maximum stabilizable subspace of an undirected network system. Third, we formulate the optimal actuator-disabling attack problem, where the attacker disables a limited number of actuators such that the maximum stabilizable subspace is minimized. We prove this problem is NP-hard. Finally, we formulate the optimal recovery problem, where a defender activates a limited number of new actuators such that the dimension of the stabilizable subspace is maximized. We prove this problem is NP-hard, and we propose a \((1 - 1/e)\) approximation algorithm.

The rest of the paper is organized as follows. In Section II we formulate the problems considered in this paper. In Section III we recall several crucial preliminaries. We present the main results in Sections IV and V – the proofs are relegated to the Appendix. In Section VI we present examples to illustrate our results. Finally, Section VII concludes this paper.
II. Problem Formulations

We consider networks whose interconnection between states are captured by a symmetric linear time-invariant (LTI) system, described by

\[ \dot{x} = Ax + Bu, \quad (1) \]

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) are state vector and input vector, respectively. We refer to matrices \( A = A^T \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) as the state matrix and input matrix, respectively. Hereafter, we use the pair \((A, B)\) to represent the system \([\mathbf{I}]\).

In order to infer the properties of a system modeled by \([\mathbf{I}]\) from its structure, we introduce some necessary concepts on structured matrices.

**Definition 1** (Structured and Symmetrically Structured Matrices). A matrix \( \hat{M} \in \{0, \ast\}^{n \times m} \) is called a structured matrix, if \( [\hat{M}]_{ij} \), the \((i, j)\)-th entry of \( \hat{M} \), is either a fixed zero or an independent free parameter, denoted by \( \ast \). In particular, a matrix \( \hat{M} \in \{0, \ast\}^{n \times n} \) is symmetrically structured, if the value of the free parameter associated with \([\hat{M}]_{ij}\) is constrained to be the same as the value of the free parameter associated with \([\hat{M}]_{ji}\), for all \(i\) and \(j\).

We refer to \( \hat{M} \) as a numerical realization of a (symmetrically) structured matrix \( M \) if \( M \) is obtained by assigning real numbers to \( \ast \)-parameters in \( \hat{M} \).

Given a pair \((A, B)\), we let the pair \((\hat{A}, \hat{B})\) denote the structural pattern of the system \((A, B)\), where \( \hat{A} \in \{0, \ast\}^{n \times n} \) is a symmetrically structured matrix such that \([\hat{A}]_{ij} = \ast \) if \([A]_{ij} \neq 0 \) and \([A]_{ij} = 0 \) otherwise. The structured matrix \( \hat{B} \in \{0, \ast\}^{n \times m} \) is defined similarly.

Recall that a system is stabilizable if and only if the uncontrollable eigenvalues are asymptotically stable \([24]\). Hence, to study stabilizability, it is necessary to first investigate controllability. Next, we recall the notion of structural controllability.

**Definition 2** (Structural Controllability \([7]\)). A structural pair \((\hat{A}, \hat{B})\) is structurally controllable if there exists a numerical realization \((\hat{A}, \hat{B})\) such that the controllability matrix \(Q(\hat{A}, \hat{B}) := [B, AB, \cdots, \hat{A}^{n-1}B] \) has full rank.

Similarly, we define structural stabilizability as follows:

**Definition 3** (Structural Stabilizability). A structural pair \((\hat{A}, \hat{B})\) is said to be structurally stabilizable if there exists a stabilizable numerical realization \((\hat{A}, \hat{B})\).

**Remark 1.** Stabilizability is not a generic property \([8]\), yet the structural stabilizability of \((\hat{A}, \hat{B})\) implies the existence of a numerical realization \((\hat{A}, \hat{B})\) such that \((\hat{A}, \hat{B})\) is stabilizable. It is a necessary condition for the stabilizability of any realization \((\hat{A}, \hat{B})\) of a structural pair \((A, B)\).

In the next two subsections, we will be focusing on two different main threats: (i) analysis, and (ii) design.

A. Analysis of Structural Stabilizability

In this subsection, we first formulate the problem of characterizing structural stabilizability using only the structural pattern of a pair, as stated below:

**Problem 1.** Given a continuous-time linear time-invariant pair \((A, B)\), we denote \((\hat{A}, \hat{B})\) the structural pattern of \((A, B)\), where \( \hat{A} \in \{0, \ast\}^{n \times n} \) is symmetrically structured. Find a necessary and sufficient condition such that \((\hat{A}, \hat{B})\) is structurally stabilizable.

In addition to the above problem, we also consider how “unstabilizable” a system is, when a system is not stabilizable. To characterize the “unstabilizability”, we propose using the dimension of the stabilizable subspace of a system, which can be stated as follows:

**Definition 4** (Stabilizable Subspace \([25]\)). Given a pair \((A, B)\), where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \), a set \( S \subseteq \mathbb{R}^n \) is said to be the stabilizable subspace of \((A, B)\) if for \( \forall x(0) \in S \), there exists a control input \( u(t) \in \mathbb{R}^m \), for \( t \geq 0 \), such that

\[
\lim_{t \to \infty} x(t) = 0.
\]

As a special case, if a pair \((A, B)\) is stabilizable, then \( S = \mathbb{R}^n \). Moreover, we aim to determine the maximum dimension of stabilizable subspace, denoted by \( \text{m-dim}(A, B) \), among all numerical realizations of \((A, B)\). Formally, we can state this problem as follows.

**Problem 2.** Given a structural pair \((\hat{A}, \hat{B})\), where \( \hat{A} \) is symmetrically structured, find \( \text{m-dim}(\hat{A}, \hat{B}) \).

Upon these problems that concern mainly with the analysis of structural stabilizability, we can now focus on the design aspect of these problems in the following subsection.

B. Optimal Actuator-Attack and Recovery Problems

Stabilizability plays a key role on network security \([14]\). In this paper, we also consider the network resilient problems. More specifically, we assume that an attacker aims to minimize the maximum dimension of the stabilizable subspace by removing a certain amount of actuation capabilities, i.e., inputs. We formalize this problem as follows.

**Problem 3** (Optimal Actuator-disabling Attack Problem). Consider a structural pair \((\hat{A}, \hat{B})\), where \( \hat{A} \in \{0, \ast\}^{n \times n} \) is symmetrically structured, and \( \hat{B} \in \{0, \ast\}^{n \times m} \) is a structured matrix. Let the set \( \Omega \) be \( \Omega = [m] \), where \( [m] := \{1, 2, \cdots, m\} \). Given a budget \( k \in \mathbb{N} \), find

\[
\mathcal{J}^* = \arg \min_{\mathcal{J} \subseteq \Omega} \text{m-dim}(\hat{A}, \hat{B}(\Omega \setminus \mathcal{J}))
\]

\[
s.t. \quad |\mathcal{J}| \leq k,
\]

where \( \hat{B}(\mathcal{I}) \in \{0, \ast\}^{n \times [\mathcal{I}]} \) is a matrix formed by the columns of \( \hat{B} \) indexed by \( \mathcal{I} \), for some \( \mathcal{I} \subseteq \Omega \).

In other words, the Problem 3 concerns about finding an optimal strategy to attack the stabilizability of a network using a fixed budget. Meanwhile, it is also of interest to
consider the perspective of a system’s designer (or, defender) that is concerned with the resilience of the network, i.e., how to maximize the dimension of stabilizable subspace by adding actuation capabilities (i.e., inputs) to the system:

**Problem 4 (Optimal Recovery Problem).** Consider a structural pair \((A, B)\), where \(A \in \{0, \ast\}^{n \times n}\) is symmetrically structured and \(B \in \{0, \ast\}^{n \times m}\) is structured. Let \(U_{\text{can}}\), where \(|U_{\text{can}}| = \ell\), be the set of candidate inputs that can be added to the system, and let \(B_{\text{can}} \in \{0, \ast\}^{n \times \ell}\) be the structured matrix characterizing the interconnection between new inputs and the states in the system. Given a budget \(k \in \mathbb{N}\), find

\[
J^* = \arg \max_{J \subseteq [\ell]} \text{m-dim}(A, [B, B_{\text{can}}(J)])
\]

\[
s.t. \ |J| \leq k,
\]

where \([B, B_{\text{can}}(J)] \in \{0, \ast\}^{n \times |J|}\) is a structured matrix formed by the columns in \(B_{\text{can}}(J)\) indexed by \(J\), and \([B, B_{\text{can}}(J)]\) is the concatenation of \(B\) and \(B_{\text{can}}(J)\).

By the duality between stabilizability and detectability [24], all the results obtained on stabilizability in this paper can be readily used to characterize detectability.

**III. Preliminaries**

To present solutions to Problems 1 – 4, we introduce some relevant notions in structural system theory and graph theory.

**A. Structural System Theory**

Consider a (symmetrically) structured matrix \(M\). Let \(n_M\) be the number of its independent \(\ast\)-parameters and associate with \(M\) a parameter space \(\mathbb{R}^{n_M}\). Let \(p_M = (p_1, \ldots, p_{n_M})^T \in \mathbb{R}^{n_M}\) to encode the values of the independent \(\ast\)-entries of \(M\) of a particular numerical realization \(\tilde{M}\). In what follows, a set \(V \subseteq \mathbb{R}^n\) is called a variety if there exist polynomials \(\varphi_1, \ldots, \varphi_k\), such that \(V = \{x \in \mathbb{R}^n : \varphi_i(x) = 0, \forall i \in [k]\}\), and \(V\) is proper when \(V \neq \mathbb{R}^n\). We denote by \(V^c = \mathbb{R}^n \setminus V\) its complement.

The term rank [20] of (a symmetrically) structured matrix \(M\), denoted as \(\tau(M)\), is the largest integer \(k\) such that, for some suitably chosen distinct rows \(\{i_\ell\}_{\ell=1}^k\) and distinct columns \(\{j_\ell\}_{\ell=1}^k\), all of the entries \(\{[M]_{i_\ell j_\ell}\}_{\ell=1}^k\) are \(\ast\)-entries. Additionally, a symmetrically (structured) matrix \(M \in \{0, \ast\}^{n \times m}\) is said to have generic rank \(k\), denoted as \(\text{g-rank}(M) = k\), if there exists a numerical realization \(M\) of \(M\), such that \(\text{rank}(M) = k\). Note that, if \(\text{g-rank}(M) > 0\), then the set of parameters describing all possible realizations when \(\text{rank}(M) < \text{g-rank}(M)\) form a proper variety, [27].

Given a structural pair \((\tilde{A}, \tilde{B})\), where \(\tilde{A} \in \{0, \ast\}^{n \times n}\) is symmetrically structured, \((\tilde{A}, \tilde{B})\) is said to be irreducible, if there does not exist a permutation matrix \(P\) such that

\[
P\tilde{A}P^T = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix}, \quad P\tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix},
\]

where \(\tilde{A}_{11} \in \{0, \ast\}^{p \times p}\), and \(\tilde{B}_1 \in \{0, \ast\}^{p \times m}\).

**B. Graph Theory**

Given a digraph \(\mathcal{D} = (V, \mathcal{E})\), a path \(P\) in \(\mathcal{D}\) is an ordered sequence of distinct vertices \(\mathcal{P} = (v_1, \ldots, v_k)\) with \(v_1, \ldots, v_k \in V\) and \((v_i, v_{i+1}) \in \mathcal{E}\) for all \(i = 1, \ldots, k-1\). Given a set \(S \subseteq V\), we denote the in-neighbour set of \(S\) by \(\mathcal{N}(S) = \{v_i \in V : (v_i, v_j) \in \mathcal{E}, v_j \in S\}\).

Given a directed graph \(\mathcal{D} = (V, \mathcal{E})\) and two sets \(S_1, S_2 \subseteq V\), we define the associated bipartite graph of \(\mathcal{D}\) by \(B(S_1, S_2, E_{S_1, S_2})\), whose vertex set is \(S_1 \cup S_2\) and edge set is \(E_{S_1, S_2} = \{(s_1, s_2) \in \mathcal{E} : s_1 \in S_1, s_2 \in S_2\}\). Given \(B(S_1, S_2, E_{S_1, S_2})\), and a set \(S \subseteq S_1\) or \(S \subseteq S_2\), we define the bipartite neighbor set of \(S\) as \(\mathcal{N}_B(S) = \{(j, i) : (j, i) \in E_{S_1, S_2}, i \in S\}\). A matching \(M\) is a set of edges in \(E_{S_1, S_2}\) that do not share vertices, i.e., given edges \(e = (s_1, s_2)\) and \(e' = (s_1', s_2')\), \(e, e' \in M\) only if \(s_1 \neq s_1'\) and \(s_2 \neq s_2'\). A matching is said to be maximum if it is a matching with the maximum number of edges among all possible matchings. Given a matching \(M\), two vertices \(s_1\) and \(s_2\) are matched if \(e = (s_1, s_2) \in M\). The vertex \(v\) is said to be right-unmatched (respectively, left-unmatched) with respect to a matching \(M\) associated with \(B(S_1, S_2, E_{S_1, S_2})\) if \(v \in S_2\) (respectively, \(v \in S_1\)) and \(v\) does not belong to an edge in the matching \(M\).

Given a structural pair \((\tilde{A}, \tilde{B})\), where \(\tilde{A} \in \{0, \ast\}^{n \times n}\) is symmetrically structured and \(\tilde{B} \in \{0, \ast\}^{n \times m}\) is structured, we associate \((\tilde{A}, \tilde{B})\) with a directed graph \(\mathcal{D}(\tilde{A}, \tilde{B}) = (X \cup U, \mathcal{E}_X, X, \mathcal{U}, X)\), where the vertex sets \(X = \{x_i\}_{i=1}^n\) and \(U = \{u_j\}_{j=1}^m\) are the set of state vertices and input vertices, respectively; and the edge set \(\mathcal{E}_X, \mathcal{E}_U\) are the set of edges between state vertices and the set of edges between input vertices and state vertices, respectively. In particular, a state vertex \(x_i \in X\) is said to be (input-reachable) if there exists a path from the input vertex \(u_j \in U\) to it. We also associate \((\tilde{A}, \tilde{B})\) with a bipartite graph \(B(\tilde{A}, \tilde{B}) = (X \cup U, \mathcal{E}_X, X, \mathcal{U}, X)\), which we refer to as the system bipartite graph.

**IV. Analysis of Structural Stabilizability**

In what follows, we have two subsections where we address Problems 1 and 2. Specifically, in Section IV-A we obtain Theorem 1 that characterizes the solutions to Problem 1 whereas in Section IV-B Theorem 2 gives a characterization of the maximum dimension of stabilizable subspace, which addresses Problem 2.

**A. Graph-Theoretic Conditions on Structural Stabilizability**

Since the stabilizability concerns the stability of the uncontrollable part of \((\tilde{A}, \tilde{B})\), it is necessary to first characterize the controllable and uncontrollable parts from the structural information contained in the pair \((\tilde{A}, \tilde{B})\). We recall a lemma from [28] that characterizes controllable modes for the numerical realizations of a structural pair.

**Lemma 1** [28]. Given a structural pair \((\tilde{A}, \tilde{B})\), where \(\tilde{A} \in \{0, \ast\}^{n \times n}\) is symmetrically structured, and \(\tau(\tilde{A}) = k\),
if \((\bar{A}, \bar{B})\) is irreducible, then there exists a proper variety \(V \subset \mathbb{R}^{n_\bar{A}+n_\bar{B}}\), such that for any numerical realization \((\bar{A}, \bar{B})\) with \([\bar{p}_\bar{A}, \bar{p}_\bar{B}]\) \(\in V^c\), \(\bar{A}\) has \(k\) nonzero, simple and controllable modes.

Lemma 1 shows that the irreducibility of \((\bar{A}, \bar{B})\) guarantees that all the non-zero modes of \((\bar{A}, \bar{B})\) are controllable generically. Subsequently, we can claim that given an irreducible pair \((\bar{A}, \bar{B})\), if for any numerical realization \((\bar{A}, \bar{B})\) there exists an uncontrollable eigenvalue, then that uncontrollable eigenvalue is 0. This implies that \((\bar{A}, \bar{B})\) is not stabilizable. Therefore, if a pair \((\bar{A}, \bar{B})\) is irreducible but not structurally controllable, then \((\bar{A}, \bar{B})\) is not structurally stabilizable. Hence, we have the following lemma.

**Lemma 2.** Given an irreducible structural pair \((\bar{A}, \bar{B})\), where \(\bar{A} \in \{0, \ast\}^{n \times n}\) is symmetrically structured, then \((\bar{A}, \bar{B})\) is structurally stabilizable if and only if \((\bar{A}, \bar{B})\) is structurally controllable.

While Lemma 2 is a condition for structural stabilizability when \((\bar{A}, \bar{B})\) is irreducible, we should also consider the case when \((\bar{A}, \bar{B})\) is reducible. By the definition of reducibility, \((\bar{A}, \bar{B})\) can be permuted to the form of (4). In order for \((\bar{A}, \bar{B})\) to be structurally stabilizable, it is required that there exists a numerical realization \(\bar{A}_{22}\) whose eigenvalues of are all negative. Summarizing these two arguments, it is equivalent to say that whether there exists a negative definite numerical realization \(\bar{A}_{22}\) determines whether the structural pair is stabilizable. Consequently, it is important to determine when the above claim is true, as follows.

**Lemma 3.** Given a reducible structural pair \((\bar{A}, \bar{B})\), where \(\bar{A} \in \{0, \ast\}^{n \times n}\) is in the form of (4). Then there exists a numerical realization \(\bar{A}_{22}\) which is negative definite if and only if the diagonal entries of \(\bar{A}_{22}\) are all \(\ast\)-entries.

Combining Lemmas 2 and 3 we have an algebraic condition for structural stabilizability. In what follows, we present graph-theoretic interpretation of these conditions.

**Theorem 1.** Consider a structural pair \((\bar{A}, \bar{B})\), where \(\bar{A}\) is symmetrically structured. Let \(D(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X}, \mathcal{X}} \cup \mathcal{E}_{\mathcal{U}, \mathcal{X}})\) be the digraph associated with \((\bar{A}, \bar{B})\), and \(X_r \subseteq \mathcal{X}\) and \(X_u \subseteq X\) be the subset of state vertices which are input-reachable and input-unreachable, respectively. The \((\bar{A}, \bar{B})\) is structurally stabilizable if and only if the following two conditions hold simultaneously in \(D(\bar{A}, \bar{B})\):

1) the vertex \(x_i\) has a self-loop, \(\forall x_i \in X_u:\)
2) \(|\mathcal{N}(S)| \geq |S|, \forall S \subseteq X_r.

Essentially, to ensure structural stabilizability, two conditions should hold simultaneously: (i) every unreachable state vertex should have a self-loop, and (ii) the reachable part of the system should be structurally controllable [23].

Next, we utilize Theorem 1 to characterize the maximum dimension of the stabilizable subspace.

**B. Maximum Dimension of Stabilizable Subspace**

Similar to the previous subsection, we will first consider the case when \((\bar{A}, \bar{B})\) is irreducible, then extend the solution approach to the general case.

By Lemma 2 when \((\bar{A}, \bar{B})\) is irreducible, the \((\bar{A}, \bar{B})\) is structurally controllable if and only if it is structurally stabilizable. This motivates us to consider the relationship between controllable subspace and stabilizable subspace. Moreover, it is shown in [24] that the maximum dimension of controllable subspace is equal to the generic dimension of stabilizable subspace of a structural pair without symmetric parameter constraints. We may suspect that equality also holds when symmetric parameter dependency is considered. Motivated by this intuition, we first study the generic dimension of the controllable subspace, and then extend the derived results to obtain a solution of Problem 2.

Given a structured pair \((\bar{A}, \bar{B})\), where \(\bar{A}\) is symmetrically structured, if there exists a proper variety \(V \subset \mathbb{R}^{n_\bar{A}+n_\bar{B}}\), such that \(\text{rank}(Q(\bar{A}, \bar{B})) = k\) when \([\bar{p}_\bar{A}, \bar{p}_\bar{B}]\) \(\in V^c\), then we say the generic dimension [27] of controllable subspace of \((\bar{A}, \bar{B})\), denoted as \(d_c\), is \(k\). For almost all numerical realizations \((\bar{A}, \bar{B})\) with \([\bar{p}_\bar{A}, \bar{p}_\bar{B}]\) \(\in \mathbb{R}^{n_\bar{A}+n_\bar{B}}\) (except for a proper variety, e.g., \([\bar{p}_\bar{A}, \bar{p}_\bar{B}]\) \(\in V\)), the dimension of controllable subspace is \(d_c\).

We characterize the generic dimension of controllable subspace of a structural pair involving a symmetrically structured matrix by the following lemma.

**Lemma 4.** Given an irreducible structural pair \((\bar{A}, \bar{B})\), where \(\bar{A} \in \{0, \ast\}^{n \times n}\) is symmetrically structured and \(\bar{B} \in \{0, \ast\}^{n \times m}\) is structured, the generic dimension of controllable subspace equals to the term rank of \([\bar{A}, \bar{B}]\), i.e., the concatenation of matrices \(\bar{A}\) and \(\bar{B}\).

When \((\bar{A}, \bar{B})\) is reducible, we can permute \((\bar{A}, \bar{B})\) to obtain the form in (4). By Definition 2 and Theorem 1, the maximum dimension of the stabilizable subspace should be the sum of the generic dimension of controllable subspace and the maximum number of negative eigenvalues over all the numerical realizations of the uncontrollable part. This can be formalized in the following result.

**Theorem 2.** Consider a structural pair \((\bar{A}, \bar{B})\), where \(\bar{A} \in \{0, \ast\}^{n \times n}\) is symmetrically structured. Then,

1) if \((\bar{A}, \bar{B})\) is irreducible, then the maximum dimension of stabilizable subspace of \((\bar{A}, \bar{B})\) equals to the generic dimension of controllable subspace of \((\bar{A}, \bar{B})\);

2) if \((\bar{A}, \bar{B})\) is reducible, then we permute the matrix \(\bar{A}\) into the form (4). The m-dim(\(\bar{A}, \bar{B}\)) equals to \(\text{t-rank}([\bar{A}_{11}, \bar{B}_{11}]) + k\), where \(k\) is the total number of \(\ast\)-entries in the diagonal of \(\bar{A}_{22}\).

**Remark 2.** In the form (4), the index of columns of \(\bar{A}_{11}\) are corresponding to input-reachable state vertices in \(D(\bar{A}, \bar{B})\), and the index of columns of \(\bar{A}_{22}\) are corresponding to the input-unreachable state vertices in \(D(\bar{A}, \bar{B})\). The input-reachable/unreachable vertices can be identified by...
A natural approximation solution to optimal design problems is through greedy algorithms \cite{32}. Although greedy algorithms may not provide an optimal solution, under specific objective functions of the problem, a suboptimal solution with suboptimally guarantees can be provided. Specifically, a particular class of problem with such properties is called submodularity function problems, defined as follows.

**Definition 6** (Max-k-Union Problem \cite{31}). Given a universe \( \mathcal{U}_S = \{S_i\}_{i=1}^p \) and an integer \( k \in \mathbb{Z}^+ \), find

\[
\mathcal{L}^* = \arg \max_{L \subseteq \{\ell_i\}_{i=1}^k} \left| \bigcup_{i=1}^k S_{\ell_i} \right|
\]

s.t. \( L \subseteq [p] \).

Thus, we obtain the following theorem.

**Theorem 5.** The Optimal Recovery Problem (Problem 4) is NP-hard.

Although the problem is NP-hard, that does not imply that all instances of the problem are equally difficult. As a consequence, we now propose to characterize the approximability of Problem 3. We first consider a subclass of instances of Problem 3, which satisfy the following assumption.

**Assumption 1.** The symmetrically structured matrix \( \bar{A} \in \{0, \ast\}^{n \times n} \) is such that for any \( S \subseteq X \), where \( X \) is the set of state vertices in the state digraph \( \mathcal{D}(\bar{A}), |X(S)| \geq |S| \). In addition, there exists no vertex with self-loop in \( \mathcal{D}(\bar{A}) \).

Assumption 1 ensures that in the bipartite graph associated with \( \mathcal{D}(\bar{A}) \), there is no right-unmatched vertex with respect to any maximum matching, i.e., the Condition-2) in Theorem 1 is always satisfied. In addition, by Assumption 1, it follows that the diagonal entries of \( \bar{A} \) satisfy \( \bar{A}_{ii} = 0 \), for \( \forall i \in [n] \). We then have the following theorem.

**Theorem 4.** Under Assumption 1 denote by \( m_1 \) the total number of sets (i.e., \( \{S_i\}_{i=1}^{m_1} \)) in an instance of Min-k-Union problem, and \( m_2 \) the total number of candidate inputs in an instance of Problem 3. Additionally, let \( \rho : \mathbb{Z} \rightarrow \mathbb{R} \). Then, there exists a \( \rho(m_1) \)-approximation algorithm for Min-k-Union problem if and only if there exists a \( \rho(m_2) \)-approximation algorithm for Problem 3.

As a result of Theorem 4, Problem 3 is at least as hard as the Min-k-Union problem.

**B. Solution to Problem 2**

To investigate the computation complexity of obtaining a solution to Problem 2, we take a similar strategy to that used in the previous section, i.e., we first consider the following special instance: the pair \( (\bar{A}, \bar{B}) \) satisfies the Assumption 1. In this case, we will show that Problem 4 is equivalent to adding a limited number of actuators to maximize the total number of input-reachable state vertices, which is similar to the Max-k-Union problem, stated as follows.

**Definition 6** (Max-k-Union Problem \cite{31}). Given a universe \( \mathcal{U}_S = \{S_i\}_{i=1}^p \) and an integer \( k \in \mathbb{Z}^+ \), find

\[
\mathcal{L}^* = \arg \max_{L \subseteq \{\ell_i\}_{i=1}^k} \left| \bigcup_{i=1}^k S_{\ell_i} \right|
\]

s.t. \( L \subseteq [p] \).

Thus, we obtain the following theorem.

**Theorem 5.** The Optimal Recovery Problem (Problem 4) is NP-hard.

A natural approximation solution to optimal design problems is through greedy algorithms \cite{32}. Although greedy algorithms may not provide an optimal solution, under specific objective functions of the problem, a suboptimal solution with suboptimally guarantees can be provided. Specifically, a particular class of problem with such properties is called submodularity function problems, defined as follows.

**Definition 7** (Submodular function \cite{32}). Let \( \Omega \) be a nonempty finite set. A set function \( f : 2^\Omega \rightarrow \mathbb{R} \), where \( 2^\Omega \) denotes the power set of \( \Omega \), is a submodular function if for every \( \mathcal{J}_1, \mathcal{J}_2 \subseteq \Omega \) with \( \mathcal{J}_1 \subseteq \mathcal{J}_2 \) and every \( i \in \Omega \setminus \mathcal{J}_2 \), we have \( f(\mathcal{J}_2 \cup \{i\}) - f(\mathcal{J}_2) \leq f(\mathcal{J}_1 \cup \{i\}) - f(\mathcal{J}_1) \).

The greedy algorithm \cite{32} achieves a \( (1 - 1/e) \)-factor approximation to the optimal solution provided that the objective function is submodular. In this paper, we show that the objective function in Problem 3 is submodular; hence, the greedy algorithm provides a constant factor guarantee to the optimal solutions.

**Theorem 6.** Algorithm 1 returns a \( (1 - 1/e) \)-approximation of the optimal solution to Problem 2.

**Remark 3.** In \cite{33}, the authors argue that insofar there is no constant factor approximation to the Min-k-Union problem. Thus, together with Theorem 4, we cannot use the greedy algorithm to approximate Problem 3 with guarantee.

**VI. ILLUSTRATIVE EXAMPLES**

In this section, we present examples to illustrate our results on structural stabilizability and approximation solution to Problem 4.
Algorithm 1 \((1-1/e)\) approximation solution to Problem 4

Input: The pair \((\bar{A}, \bar{B})\), \(\mathcal{U}_{can} \in \{0, \ast\}^{m \times m'}\), and the budget \(k\);

Output: Suboptimal solution \(\mathcal{J}\);

1: Initialize \(\mathcal{J} \leftarrow \emptyset\), \(\mathcal{L} \leftarrow \{m'\}^\star:\mathcal{L}\) is the set of indexes of new actuators in \(\mathcal{U}_{can}\), the set of new actuators that can be added to the system.
2: for iteration \(i \in [k]\) do
3:  for each \(j \in \mathcal{L}\) do
4:    \(d_j \leftarrow \text{m-dim}(\bar{A}, \bar{B}, \mathcal{B}_{\text{can}}(\mathcal{J} \cup \{j\}))\);
5:  end for
6:  \(I \leftarrow \{i: d_i = \max\{d_j\}_{j \in \mathcal{J}}\}\);
7:  Pick \(a \in I\);
8:  \(\mathcal{J} \leftarrow \mathcal{J} \cup \{j\}\);
9:  \(\mathcal{L} \leftarrow \mathcal{L} \setminus \{j\}\);
10: end for
11: return \(\mathcal{J}\)

Figure 1: In this figure, we depict the structure of \(\mathcal{D}(\bar{A}, \bar{B})\). The red vertex labeled by \(u_1\) and black vertices labeled by \(x_2, \ldots, x_{11}\) are the input vertex and state vertices, respectively. The black arrows represent the edges from input vertex to state vertices, as well as edges between state vertices.

A. Maximum Dimension of Stabilizable Subspace

We consider a structural pair \((\bar{A}, \bar{B})\), where \(\bar{A} \in \{0, \ast\}^{11 \times 11}\) is symmetrically structured and \(\bar{B} \in \{0, \ast\}^{1 \times 1}\) is structured.

\[
\bar{A} = \begin{bmatrix}
0 & a_{12} & 0 & a_{14} & a_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{12} & 0 & 0 & 0 & a_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{14} & 0 & 0 & 0 & 0 & a_{17} & 0 & 0 & 0 & 0 & 0 \\
a_{15} & a_{24} & 0 & 0 & 0 & 0 & a_{26} & 0 & 0 & 0 & 0 \\
a_{16} & 0 & 0 & 0 & 0 & 0 & 0 & a_{27} & 0 & 0 & 0 \\
a_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{28} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{30} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{31} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{101} \\
a_{101} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{101} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{1011} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad \bar{B} = \begin{bmatrix}
b_{11} \\
b_{41} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \end{bmatrix}
\]

(7)

We depict the digraph representation of the structural pair \((\bar{A}, \bar{B})\), denoted by \(\mathcal{D}(\bar{A}, \bar{B})\), in Figure 1. Since \(x_3\) and \(x_7\) are unreachable vertices and they do not have self-loops, the pair \((\bar{A}, \bar{B})\) is not structurally stabilizable due to Theorem 1. Furthermore, the total number of right-matched (with respect to any maximum matching in the associated bipartite graph \(\mathcal{B}(\bar{A}, \bar{B})\)) reachable vertices is 3, and the total number of unreachable vertices with self-loop is 2. Therefore, by invoking Theorem 2, we conclude that the maximum stabilizable subspace is \(3 + 2 = 5\).

B. Optimal Recovery Problem

Now, we present an example to illustrate the use of Algorithm 1. Consider again the structural pair \((\bar{A}, \bar{B})\) specified in (7). As noted in the last subsection, \((\bar{A}, \bar{B})\) is not structurally stabilizable. We let \(\mathcal{U}_{can} = \{u_i\}_{i=2}^{7}\) be the set of candidate actuators that can be added into the system and associate it with the structured matrix \(\bar{B}_{\text{can}} \in \{0, \ast\}^{6 \times 6}\), of which nonzero entries are captured by the red edges of the digraph \(\mathcal{D}(\bar{A}, [\bar{B}, \bar{B}_{\text{can}}])\) depicted in Figure 2.

We have obtained in the last subsection that \(\text{m-dim}(\bar{A}, \bar{B})\) is 5. Suppose we have a budget \(k = 3\), then Problem 4 consists in adding 3 actuators from \(\mathcal{U}_{can}\) into the system such that the maximum stabilizable subspace is maximized. In the first iteration of Algorithm 1, \(u_4\) is selected because \(\text{m-dim}(\bar{A}, [\bar{B}, \bar{B}_{\text{can}}((\{4\})]) = 4 > \text{m-dim}(\bar{A}, \bar{B}) \geq \text{m-dim}(\bar{A}, \bar{B}, \mathcal{U}_{can}, \{\{4\})]\) \(\forall u_i \in \mathcal{U}_{can}\). Similarly, in the second iteration, \(u_3\) is selected by Algorithm 1. This results in \(\text{m-dim}(\bar{A}, [\bar{B}, \bar{B}_{\text{can}}((\{3, 4\})]) = 10\). Finally, \(u_7\) is selected and \(\text{m-dim}(\bar{A}, [\bar{B}, \bar{B}_{\text{can}}((\{3, 4, 7\})]) = 11\). Since the maximum possible stabilizable subspace is always less than or equal to the total number of states, in this example, Algorithm 1 returns an optimal solution to Problem 4.

VII. CONCLUSION

In this paper, we studied the structural stabilizability problem of undirected networked dynamical systems. We proposed a computationally-efficient graph-theoretic method to derive the maximum dimension of stabilizable subspace of an undirected network. In addition, we formulated the optimal actuator-disabling attack problem and optimal recovery problem. We proved that these two problems are NP-hard. Finally, we developed a \((1 - 1/e)\) approximation algorithm for the optimal recovery problem.

In the future, we will focus on developing approximation algorithms for the optimal actuator-disabling attack problem. Furthermore, it would also be of interest to relax the assumption on symmetry, and extend the results in this paper to directed networked systems.

APPENDIX

Proof of Lemma 2 First, we notice that sufficiency follows from the fact that structural controllability ensures that almost surely there exists numerical realization ensuring controllability, which implies that any desired state can be attained by a finite sequence of inputs. Therefore, if there was not one such sequence, then the uncontrollable subspace is nonempty, and the only way to ensure that we can take the state to the origin is when the subspace is stable. a control input driving the states to the origin in finite time. Necessity follows by contrapositive argument. Suppose \((\bar{A}, \bar{B})\) is irreducible but not structurally controllable, then by Theorem 1 in [28], there exists a set \(S \subseteq X\) such that \(|\mathcal{N}(S)| < |S|\), which implies that \(\text{g-rank}([\bar{A}, \bar{B}]) < n\). For \(\forall [\bar{P}_A, \bar{P}_B] \in \mathbb{R}^{n \times n}\), \(\exists v \in \mathbb{C}^n\), such that \(v^T [\bar{A}, \bar{B}] = 0\),
i.e., $v^T \tilde{A} v = v^T 0$. Consequently, there exists a zero eigenvalue which is not controllable, hence not stabilizable. □

**Proof of Lemma 2 (If)** Let us construct a numerical realization $\tilde{A}_{22}$ by assigning zero value to off-diagonal *-entries of $A_{22}$, and negative values to *-entries on the diagonal. In this case, matrix $A_{22}$ is negative definite diagonal matrix.

**(Only if)** We approach the proof by contrapositive. Let $m$ be the dimension of $A_{22}$, and $\{v_i\}_{i=1}^m$ be the standard basis in $\mathbb{R}^m$. Suppose there exists a fixed zero $[A_{22}]_{ii} = 0$, then $v_i^T \tilde{A}_{22} v_i = [A_{22}]_{ii} = 0$, for all numerical realizations of $A_{22}$; hence, $\tilde{A}_{22}$ is not negative definite. □

**Proof of Theorem 7** (If) Without loss of generality, suppose $(\tilde{A}, \tilde{B})$ can be transformed to the form of (4). Suppose for $\forall S \subseteq \mathcal{X}$, $|\mathcal{N}(S)| \geq |S|$, then the input reachable subsystem $(\tilde{A}_{11}, \tilde{B}_1)$ is structurally controllable. If for $\forall x_i \in \mathcal{X}$, $x_i$ has self-loop in $\mathcal{D}(\tilde{A}, \tilde{B})$, then $[\tilde{A}]_{ii}$ is a *-entry. Let us assign negative numerical weights to all the *-entries of $\tilde{A}$ that correspond to the self-loop of all $x_i \in \mathcal{X}_c$. Then, the input-unreachable part of the system, $\tilde{A}_{22}$, is a negative definite diagonal matrix. Thus, we have shown that there exists a numerical realization $(\tilde{A}, \tilde{B})$, such that the uncontrollable part is asymptotically stable. Hence, the system is structurally stabilizable.

(Only if) The necessity can be proved by contrapositive. Suppose there exists a state vertex $x_i \in \mathcal{X}_c$ that $[\tilde{A}]_{ii} = 0$, then, by Lemma 3 any numerical realization $(\tilde{A}, \tilde{B})$ has an uncontrollable non-negative eigenvalue. Furthermore, assume there exists $S \subseteq \mathcal{X}$ such that $|\mathcal{N}(S)| < |S|$, then by Lemma 2 $(\tilde{A}, \tilde{B})$ is not structurally stabilizable. □

**Proof of Lemma 2** Suppose $\text{t-rank}([\tilde{A}, \tilde{B}]) = k$, then there exists a set $T \subseteq [n]$, such that for $\forall S \subseteq \mathcal{X}_T = \{x_i \in \mathcal{X}: i \in T\}$, $|\mathcal{N}(S)| \geq |S|$. By Theorem 2 in [29], $(\tilde{A}, \tilde{B})$ is structurally target controllable with respect to $T$, which implies that there exists a numerical realization $(\tilde{A}, \tilde{B})$ with $[\tilde{p}_A, \tilde{p}_B] \in V^r(w)W^c$, where $V$ and $W$ are proper varieties in $\mathbb{R}^{r \times n + 0}$ defined in the proof of Theorem 2 in [29], such that the dimension of the controllable subspace is $k$, i.e., almost surely the dimension of controllable subspace of a numerical realization $(\tilde{A}, \tilde{B})$ is $k$. We have the generic dimension of controllable subspace of $(\tilde{A}, \tilde{B})$, $\text{d}_c = \text{t-rank}([\tilde{A}, \tilde{B}])$. □

**Proof of Theorem 2** Without loss of generality, there exists only two cases: either $(\tilde{A}, \tilde{B})$ is irreducible or not. In the first case, by Lemma 1 Lemma 2 and Lemma 4, the generic dimension of controllable subspace of $(\tilde{A}, \tilde{B})$ is $\text{t-rank}([\tilde{A}, \tilde{B}])$, and if $\text{t-rank}([\tilde{A}, \tilde{B}]) < n$, then for any numerical realization $(\tilde{A}, \tilde{B})$, there are $(n-k)$ zero uncontrollable eigenvalues. Therefore, the maximum dimension of stabilizable subspace of $(\tilde{A}, \tilde{B})$ equals to $\text{t-rank}([\tilde{A}, \tilde{B}])$: In the other case, permute $(\tilde{A}, \tilde{B})$ to the form of (4) and let $k$ be the number of *-entries in the diagonal of $\tilde{A}_{22}$. By Theorem 1 we have $\text{m-dim}(\tilde{A}, \tilde{B}) = \text{t-rank}([\tilde{A}_{11}, \tilde{B}_1]) + k$. □

1To prove Lemma 4 here we use the results on structural target controllability. Due to page limitations, please refer to [29] for more details.

**Proof of Theorem 3** First, by Theorem 2 given a structural pair $(\tilde{A}, \tilde{B})$, it takes polynomial time to compute the maximum dimension of the stabilizable subspace, which implies that Problem 3 is in NP. We prove the NP-hardness of Problem 3 by (polynomially) reducing Min-k-Union problem to instances of Problem 3.

Suppose that we have a universe set $U_S = \{S_i\}_{i=1}^p$, and an integer $k \in \mathbb{Z}^+$, for which we need to select $k$ subsets in $\{S_i\}_{i=1}^p$ such that $|\bigcup_{i=1}^{k} S_i|$. Let $\ell = |U_S|$ and define the state vertex set as $\mathcal{X} = \{x_i\}_{i=1}^{2\ell}$, and input vertex set as $U = \{u_i\}_{i=1}^{\ell}$. Next, we can construct a set of directed edges between state vertices, $\mathcal{E}_{X,X} = \{(x_i, x_{i+n}), (x_{i+n}, x_i)\}_{i=1}^{2\ell}$, and a set of directed edges between input and state vertices, $\mathcal{E}_{U,X} = \{(u_i, x_i) : i \in [\ell], j \in S_i\}$. Next, let the attack budget be $c = p - k$. In our constructed instance of Problem 3, we aim to remove $c$ actuators from $\{u_i\}_{i=1}^p$ such that the maximum dimension of the stabilizable subspace is minimized. Subsequently, we claim that an optimal solution of the constructed instance of Problem 3 enables us to retrieve an optimal solution to the Min-k-Union problem.

Suppose we have a feasible solution $U^*_c = \{u_i\}_{i=1}^p$, then if we consider $\mathcal{L} = [p] \setminus \{\ell_i\}_{i=1}^{p-k}$, we have that $\mathcal{L}$ is a feasible solution of Min-k-Union problem. Moreover, suppose $U^*_c = \{u_i\}_{i=1}^p$ is a minimum solution to Problem 5 but $\mathcal{L} = [p] \setminus \{\ell_i\}_{i=1}^{p-k}$ is not an optimal solution of Min-k-Union problem, then $\mathcal{L}' = \{\eta_i\}_{i=1}^{k+1}$ would be a solution to Min-k-Union problem such that $|\bigcup_{i=1}^{k+1} S_i| < |\bigcup_{i \in \mathcal{L}} S_i|$. Notice that the maximum stabilizable subspace by removing $U^*_c$ is smaller than the maximum stabilizable subspace when removing $\mathcal{L}'$, which contradicts $U^*_c$ is an optimal solution. □

**Proof of Theorem 4** (i) Consider an instance of Problem 3 under Assumption 1. We associate the structural pair $(\tilde{A}, \tilde{B})$, where $\tilde{A} \in \{0, \star\}^{n \times n}$ and $\tilde{B} \in \{0, \star\}^{n \times m}$, with a digraph $\mathcal{D}(\tilde{A}, \tilde{B})$. For each $i \in [m]$, we let $S_i$ be the set of state vertices which are reachable from the input $u_i$. By Assumption 4 we have

$$\text{m-dim}(\tilde{A}, \tilde{B}(\mathcal{J})) = |\bigcup_{j \in \mathcal{J}} S_j|.$$ (8)
We let $U_S = \bigcup_{i=1}^{m} S_i$. Suppose the budget in Problem 3 is $k$, then Problem 3 is to find $(m-k)$ sets $S_{i1}, \ldots, S_{im-k}$ among $\{S_i\}_{i=1}^{m}$ such that $|\bigcup_{i=1}^{m-k} S_{i}|$, i.e., the number of reachable state vertices, is minimized. By Definition 5, we see that in this case Problem 3 is equivalent to the Min-k-Union problem, in which we are given sets $\{S_i\}_{i=1}^{m}$ and we aim to find $(m-k)$ sets $\{S_{i1}, \ldots, S_{im-k}\}$ of $\{1,2, \ldots, m\}$, such that $|\bigcup_{i=1}^{m-k} S_{i}|$ is minimized. If there exists a $\rho(m)$-approximation algorithm for the Min-k-Union problem, i.e., $\sum_{J \in \mathcal{J}} S_j \leq \rho(m) \sum_{J \in \mathcal{J}} S_j$, then, $\sum_{J \in \mathcal{J}} S_j$ is the set of all input-reachable state vertices, and

$$m \text{-dim}(\bar{A}, \bar{B}(\mathcal{J})) = \sum_{J \in \mathcal{J}} S_j$$

where $\mathcal{J}^{*}$ is an optimal solution to the Min-k-Union problem. From the above reasoning, we have that $B(\mathcal{J}^{*})$ is also an optimal solution to Problem 3 and $m \text{-dim}(\bar{A}, \bar{B}(\mathcal{J})) \leq \rho(m) \cdot (\sum_{J \in \mathcal{J}^{*}} S_j) = \rho(m) \cdot m \text{-dim}(\bar{A}, \bar{B}(\mathcal{J}^{*}))$.

(iii) Conversely, if there exists a $\rho(m)$-approximation algorithm for Problem 3 under Assumption 1, then from (9) and the above reasoning, there also exists a $\rho(m)$-approximation algorithm for Min-k-Union problem.

**Sketch of Proof of Theorem 5** Since a solution of Problem 4 can be verified in polynomial time, the Problem 4 is in NP. We can prove the NP-hardness by reducing a general instance of the Max-k-Union problem to an instance of Problem 4. Suppose we have a ground set $U_S = \{S_i\}_{i=1}^{m}$, and an integer $k \in \mathbb{N}$. The constrained maximum set coverage problem is to select $k$ subsets in $U_S$ such that $|\bigcup_{i=1}^{k} S_i|$ is maximized. Following a similar construction and reasoning taken in the proof of Theorem 5, we can prove that the Max-k-Union problem can be reduced to Problem 4 in polynomial time.

**Proof of Theorem 6** Consider a structural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0, *\}^{n \times n}$ is symmetrically structured and $\bar{B} \in \{0, *\}^{n \times m}$ is structured. We let $U$ denote the input vertices corresponding columns of $\bar{B}$, and let $U_\text{can}$, where $|U_\text{can}| = m'$, be the set of new actuators that can be added to the system. We associate with the set $U_\text{can}$ the structured matrix $\bar{B}_\text{can} \in \{0, *\}^{n \times m'}$. Define a function $f : \mathcal{J} \subseteq [m'] \rightarrow m \text{-dim}(\bar{A}, \bar{B}, B_\text{can}(\mathcal{J}))$. We first prove that if $f(\mathcal{J})$ is a submodular function, and then we show that Algorithm 1 returns a $(1 - 1/e)$-approximation solution.

Let $R_0$ be the set of state vertices with self-loop in $D(\bar{A}, \bar{B})$. Consider two sets $\mathcal{J}_1, \mathcal{J}_2$, where $\mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq [m']$. We let $R(\mathcal{J})$ be the set of reachable right-matched state vertices which have no self-loop, with respect to a maximum matching in $B(\bar{A}, \bar{B}, B_\text{can}(\mathcal{J}))$. By definition of $f(\mathcal{J})$, we have $f(\mathcal{J}) = |R_0| + |R(\mathcal{J})|$. Let $i \in [m'] \setminus \mathcal{J}_2$. Then,

$$f(\mathcal{J}_2 \cup \{i\}) - f(\mathcal{J}_1 \cup \{i\}) = (|R(\mathcal{J}_2 \cup \{i\})| + |R_0|) - (|R(\mathcal{J}_1 \cup \{i\})| + |R_0|)$$

where the inequality at the third row of (10) holds due to $\mathcal{J}_1 \subseteq \mathcal{J}_2$ and the fact that introducing one new input can at most make one right-unmatched vertex of a maximum matching in a bipartite graph $B(\bar{A}, \bar{B})$ be right-matched [13]. Swapping the terms in (10), it yields

$$f(\mathcal{J}_2 \cup \{i\}) - f(\mathcal{J}_2) \leq f(\mathcal{J}_1 \cup \{i\}) - f(\mathcal{J}_1),$$

which shows that $f(\mathcal{J})$ is a submodular function. Moreover, by Theorem 1, we have that $f(\mathcal{J})$ is a monotonically increasing submodular function.

Because $f(\mathcal{J})$ is a monotonically increasing submodular function, by a similar technique taken in the proof of Proposition 5.1, we can show that Algorithm 1 returns a $(1 - 1/e)$-approximation solution to Problem 4. □

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