Dynamical phase transitions in quantum systems

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Abstract

We study generic features of open quantum systems embedded into a continuum of scattering wavefunctions and compare them with results discussed in optics. A dynamical phase transition may appear at high level density in a many-level system and also in a two-level system if the coupling $W$ to the environment is sufficiently large. $W$ contains both the selfenergy $W_{ii}$ of the state $i$ and the coupling $W_{ij}$ between the two states $i$ and $j \neq i$ of the system via the environment. It is complex. In the case of two neighboring states with equal widths, two singular (exceptional) points exist when $W_{ij}$ is imaginary. In the parameter range between these two points, the widths bifurcate as a function of a certain external parameter. Time reversal symmetry breaking and its violation in the neighborhood of exceptional points is discussed. Using a unitary representation of the $S$ matrix, the cross section is calculated for a two-level system and for different $W$, including at the exceptional point (double pole of the $S$ matrix). The results obtained for the transition of level repulsion at small (real) $W_{ij}$ to width bifurcation at large (imaginary) $W_{ij}$ show qualitatively the same features that are observed experimentally in the transition from Autler-Townes splitting to electromagnetically induced transparency in optics. Fermi’s golden rule holds only below the dynamical phase transition. It is violated beyond the dynamical phase transition.

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I. INTRODUCTION

Recently, dynamical phase transitions (DPTs) are considered in different open quantum systems. They appear mostly at high level density and are observed experimentally as well as theoretically by using different approaches. Common to all of them is a very robust spectroscopic redistribution that takes place in the system in a critical region of a certain control parameter. As a result of the spectroscopic redistribution, short-lived states appear together with long-lived ones which all have lost their spectroscopic relation to the original states in the subcritical parameter region.

A few examples are the following. A DPT is observed experimentally and explained theoretically in the spin swapping operation, where the Fermi golden rule is violated \cite{1,2}. The experimentally observed so-called phase lapses in quantum dots \cite{3} can be explained by means of a DPT described by the non-Hermiticity of the Hamilton operator \cite{4} and by width bifurcation caused by exceptional points (EPs) \cite{5,6}. A long-standing puzzle in nuclear physics are, on the one hand, the extremely long-lived compound nucleus states characteristic of nuclei at high level density and described well by means of statistical methods \cite{7,8} and, on the other hand, the shell model states with individual spectroscopic features at low level density \cite{9}. Obviously, a DPT occurs at high level density also in nuclei \cite{10}.

The DPTs observed in many-body open quantum systems show features which are similar to the Dicke superradiance known in optics \cite{11}. This has been shown some years ago \cite{12}: the DPT observed in a many-body open quantum system \cite{13} is analogue to the formation of the superradiant Dicke state in optics. The simple model used in this study is based on statistical assumptions and can, of course, not explain the sensitive dependence of the system properties on the variation of external parameters. It shows however the meaning of the imaginary part of the coupling term between system and environment for the formation of the so-called superradiant state in a many-body open quantum system. In further studies with a small number of states and complex coupling coefficient between system and environment, the properties of the long-lived (trapped or subradiant) states are considered \cite{14}.

The similarity between many-body open quantum systems and optics is observed also in other papers. Cooperative spontaneous emission is considered as a many-body eigenvalue problem in \cite{15}. As a result, some states decay much faster than the single-atom decay rate, while other states are trapped and undergo very slow decay. When the size of the
atomic cloud is small compared with the radiation wavelength, the eigenvalues have a large imaginary part which corresponds to a large frequency shift of the emitted radiation. The collective Lamb shift in single photon Dicke superradiance is formulated theoretically in [16]. Furthermore, Fermi’s golden rule does not adequately describe superradiance in optics [17], what is along the lines of observations in open quantum systems beyond the DPT [1, 2]. This fact suggests to consider the possibility that a DPT of the type discussed above for a many-body quantum system appears also in optics.

Traditionally, the feedback between a many-body quantum system and the environment of scattering wavefunctions is either neglected or related to the measuring process, or it is taken into account only in the numerical calculations. In a more fundamental description, system as well as environment are described quantum mechanically, i.e. the system is considered to be embedded into the environment of scattering wavefunctions, the Hamilton operator of the system is non-Hermitian and the equations are solved exactly without any statistical assumptions. The environment consisting of the continuum of scattering wavefunctions exists always and cannot be deleted. The influence of the continuum onto the states of the system has been shown explicitly in experiments with bound states in quantum point contacts [18–20]. Theoretically, the coupling between system and environment is contained in the (nonlinear) source term of the corresponding Schrödinger equation. This equation with source term can be exactly rewritten into a Schrödinger equation without source term but with a Hamilton operator $\mathcal{H}$ in which the interaction of the states via the environment is involved in the non-diagonal matrix elements [21]. That means the eigenvalues of the non-Hermitian operator $\mathcal{H}$ contain the information on the interaction of states via the environment, i.e. the feedback between system and environment.

The Feshbach projection operator technique [7] containing the non-Hermitian Hamilton operator $\mathcal{H}$, provides a convenient method to describe an open many-body quantum system. Using this method, first the energy-independent many-body problem of the system (with the Hermitian Hamiltonian $H_B$) is solved in the standard manner. These solutions provide the energies $E_i^B$ and wavefunctions $\Phi_i^B$ of the discrete states with inclusion of the so-called internal interaction. In a second step, the energy-dependent scattering wavefunctions $\xi_c^E$ of the environment are calculated and, further, the (energy-dependent) coupling matrix
The non-Hermitian Hamiltonian $\mathcal{H}$ contains the self-energy of the states in the diagonal matrix elements and the interaction of the different states via the environment (the so-called external interaction) in the nondiagonal matrix elements [21]. The Hamiltonian $\mathcal{H}$ reads

$$\mathcal{H} = H_B + V_{BC}G^{(+)}_{C}V_{CB}$$

where $V_{BC}$ and $V_{CB}$ stand for the coupling between system and environment and $G^{(+)}_{C}$ is the Green function in the subspace of scattering states. The non-Hermitian operator $\mathcal{H}$ can be diagonalized: the eigenvalues $E_i$ are complex and the eigenfunctions $\Phi_i$ are complex and biorthogonal.

The external interaction of the states via the continuum is complex, generally. The principal value integral is

$$\text{Re} \langle \Phi_B^i | \mathcal{H} | \Phi_B^j \rangle - E_i^B \delta_{ij} = \frac{1}{2\pi} \sum_{c=1}^{C} P \int_{\epsilon_c}^{\epsilon_c'} \text{d}E' \frac{\gamma^0_{ic} \gamma^0_{jc}}{E - E'}$$

and the residuum reads

$$\text{Im} \langle \Phi_B^i | \mathcal{H} | \Phi_B^j \rangle = -\frac{1}{2} \sum_{c=1}^{C} \gamma^0_{ic} \gamma^0_{jc}$$

where $C$ is the number of continua. It is $C = 1$ when the different states are coupled to one common continuum. The interaction of the states of the system with and via the environment is involved in the eigenvalues $E_i$ and eigenfunctions $\Phi_i$ of the Hamiltonian $\mathcal{H}$. It is therefore relatively easy, in this formalism, to study the influence of the environment onto the states of the system.

The non-Hermitian quantum physics is –in contrast to the widely spread meaning– not a further approximation introduced in the theory. It is an expression of the fact that the system considered is really open: it is embedded into the continuum of scattering wavefunctions which always exists, and its properties are influenced by the coupling to this environment. The interaction of the states via the environment is unimportant at low level density where it can be described by perturbation theory (if necessary). It becomes, however, decisive in the regime of overlapping resonances [22]. Here, the eigenvalues of $\mathcal{H}$ avoid crossing in
energy (level repulsion) and bifurcate in time (width bifurcation), respectively. This effect is caused mathematically by the existence of singular points (mostly called EPs) at which two eigenvalues of $\mathcal{H}$ coalesce, the two corresponding eigenfunctions are not orthogonal to one another but linearly dependent, and the phases of the eigenfunctions are not rigid. In solid state physics a similar effect with nonrigid phases is known and related to a magnetic impurity [23] (the difference between these two effects is discussed in section 3.3 of [22]). The existence of EPs has been proven experimentally by means of microwave billiards [24].

Recently, the superradiant states in optical devices are studied in many papers. They appear coherently when the single parts (states) of the system come into contact with one another, e.g. [25, 26]. Further, long-lived subradiant states [27] and superradiant forward scattering [28] are observed. Correlated spontaneous emission from an ensemble of $N$ identical two-level atoms is considered in [29]. The transition between Autler-Townes splitting [30] and electromagnetically induced transparency shows features [31] which are similar to level repulsion and width bifurcation observed in a many-body system (figure 9 in [32]). A similar effect is observed in the electromagnetically induced transparency on a single artificial atom [33]. Also superradiance of quantum dots is experimentally found [34]. An enhancement of photon intensity in forced coupled quantum wells inside a semiconductor microcavity is discussed in [35]. Another interesting result is the observation of the collective Lamb shift [36, 37]. Also in this case, electromagnetically induced transparency is studied including its dependence on the sample geometry [38]. In a recent paper [39], the non-Hermitian formalism is used to design a plasmonic system for spatially manipulating light on the nanoscale, and the selective excitation of each individual element in the array is experimentally demonstrated.

The aim of the present paper is to simulate a DPT in an open quantum system by using a schematical model. The eigenvalues $\mathcal{E}_i$ of a non-Hermitian Hamilton operator $\mathcal{H}$ are traced as function of the distance to neighboring eigenstates. The non-diagonal matrix elements $\omega_{ij}$ of $\mathcal{H}$ simulate the coupling between the states $i$ and $j$ via the environment (continuum of scattering wavefunctions) and are complex. Hence, the eigenvalues $\mathcal{E}_{i,j}$ of $\mathcal{H}$ contain the influence of the environment onto the states of the system: level repulsion is caused by $\text{Re}(\omega_{ij})$ and width bifurcation by $\text{Im}(\omega_{ij})$. In realistic systems, $\text{Re}(\omega_{ij}) \gg \text{Im}(\omega_{ij})$ far from EPs while $\text{Im}(\omega_{ij}) \gg \text{Re}(\omega_{ij})$ near to an EP. Here, the eigenvalue trajectories avoid crossing and a transition from level repulsion in energy to width bifurcation takes place.
The paper is organized as follows. Sect. II sketches the formalism used in the present paper. The basis is the $2 \times 2$ Hamiltonian matrix $\mathcal{H}^{(2)}$ which describes two states of the system and their coupling to and via the environment (continuum of scattering wavefunctions). The physical meaning of all matrix elements is explained by using the non-Hermitian Hamilton operator (2) as well as the equations (3) and (4) which are characteristic of an open quantum system. Furthermore, the basic expressions such as EP, biorthogonality, phase rigidity, level repulsion, width bifurcation, mixing of the eigenfunctions are defined and an expression for the $S$ matrix is given. In Sect. III the corresponding Hamiltonian matrix $\mathcal{H}^{(N)}$ for an $N$ level system is written down and the physical meaning of the matrix elements is discussed.

In the following sections, some numerical results are given. The eigenvalues $\mathcal{E}_i$ and eigenfunctions $\Phi_i$ of the $2 \times 2$ Hamiltonian $\mathcal{H}^{(2)}$ with $N = 2$ levels are given in Sect. IV while those of $\mathcal{H}^{(N)}$ for $N = 3$ and 4 are shown in Sect. V. In Sect. VII numerical results on the influence of EPs onto the cross section are given.

The results are discussed in Sect. VII. Special attention is devoted first to symmetries around an EP and their role for an DPT. Secondly we point to a possible DPT in a two-level system at strong coupling to its environment. Furthermore, the experimental observation of the different DPTs is discussed. The last section VIII contains some conclusions drawn from the results given in the main part of the paper.

II. EIGENVALUES AND EIGENFUNCTIONS OF A NON-HERMITIAN OPERATOR IN THE NEIGHBORHOOD OF AN EXCEPTIONAL POINT

An open quantum system is embedded into an environment. The natural environment is the continuum of scattering wavefunctions that always exists. It can be modified, however not deleted. The Hamilton operator $\mathcal{H}$ of the system is non-Hermitian. Its eigenvalues contain the information on the interaction of states via the environment, i.e. the feedback between system and environment. The matrix elements of $\mathcal{H}$ consist formally of a first-order and a second-order interaction term. The first-order term stands for the direct (internal) interaction $V_{ij}$ between the two states $i$ and $j$. In atomic physics, the states with inclusion of the internal interaction are called usually dressed states. The second-order term describes the (external) interaction $\omega_{ij} \propto V_{ic} G_c V_{cj}$ via the continuum $c$ (where $G_c$ is the Green function in the continuum of scattering states). The second-order term consists of the principal value
integral \((3)\) and the residuum\((4)\). It is therefore complex, generally. Although it is of second order, it determines, under certain conditions, the dynamics of the system, as will be shown below. Furthermore, \(\omega_{ij}\) contains the interaction of the state \(i\) with the environment, i.e. the self-energy of the state which is analog to the Lamb shift known in atomic physics.

For illustration, let us simulate the Hamiltonian \(\mathcal{H}\) by considering explicitly only two states of the system, i.e. by the symmetric \(2 \times 2\) Hamiltonian

\[
\mathcal{H}^{(2)} = \begin{pmatrix} \epsilon_1 & \omega_{12} \\ \omega_{21} & \epsilon_2 \end{pmatrix}.
\]

(5)

It is assumed here that the direct internal interaction \(V_{ij}\) between the two states is involved in the energies \(\epsilon_i\) \((i = 1, 2)\) of the two states. The \(\omega_{12} = \omega_{21}\) describe the external interaction of the two states via the environment. The eigenvalues of \(\mathcal{H}^{(2)}\) are

\[
\mathcal{E}_{1,2} = \frac{\epsilon_1 + \epsilon_2}{2} \pm Z ; \quad Z = \frac{1}{2} \sqrt{(\epsilon_1 - \epsilon_2)^2 + 4 \omega_{12}^2}.
\]

(6)

As a function of a certain parameter, the levels repel each other in energy according to the value \(\text{Re}(Z)\) while the widths bifurcate corresponding to \(\text{Im}(Z)\). The two eigenvalue trajectories cross when \(Z = 0\), i.e. when \((\epsilon_1 - \epsilon_2)/2\omega_{12} = \pm i\). The crossing points are called mostly exceptional points (EPs) according to the definition given in \([40]\). At these singular points, the two eigenvalues coalesce, \(\mathcal{E}_1 = \mathcal{E}_2 \equiv \mathcal{E}_0\). In the vicinity of the crossing points, the dependence of the eigenvalue trajectories on a certain parameter is more complicated than far from them: the two levels approach each other in energy and the widths become equal so that \(\text{Re}(\mathcal{E}_1) \leftrightarrow \text{Re}(\mathcal{E}_2)\) and \(\text{Im}(\mathcal{E}_1) \leftrightarrow \text{Im}(\mathcal{E}_2)\) at the crossing point.

Generally, \(\mathcal{H}^{(2)}\) is a non-Hermitian operator, the unperturbed energies \(\epsilon_i\) and the interaction \(\omega_{ij}\) are complex, see \([2], [3]\) and \([4]\). The states can decay, in general, and the two eigenvalues \((6)\) can be written as

\[
\mathcal{E}_{1,2} = E_{1,2} - \frac{i}{2} \Gamma_{1,2} \quad (\text{with } \Gamma_{1,2} \geq 0).
\]

(7)

The widths \(\Gamma_i\) are proportional to the inverse lifetimes \(\tau_i^{-1}\) of the states, \(i = 1, 2\). The topological phase of the EP is twice the Berry phase (see section 2.5 of \([22]\)). This theoretical result is proven experimentally by means of a microwave cavity \([24]\).

The eigenfunctions of the non-Hermitian Hamilton operator \(\mathcal{H}^{(2)}\), Eq. \((5)\), are biorthogonal,

\[
\langle \Phi_k^* | \Phi_l \rangle = \delta_{k,l}.
\]

(8)
From these equations follows [22]

\[ \langle \Phi_k | \Phi_k \rangle \equiv A_k \geq 1 \]  
(9)

\[ \langle \Phi_k | \Phi_{l \neq k} \rangle = -\langle \Phi_{l \neq k} | \Phi_k \rangle \equiv B_k^l \ ; \ |B_k^l| \geq 0 . \]  
(10)

The eigenfunctions \( \Phi_i \) of \( \mathcal{H}^{(2)} \) can be represented in the set of basic wavefunctions \( \Phi_i^0 \) of the matrix \( [5] \) with vanishing non-diagonal matrix elements \( \omega_{ij} \),

\[ \Phi_i = \sum_{j=1}^{N=2} b_{ij} \Phi_{j}^0 . \]  
(11)

The \( b_{ij} \) are normalized according to the biorthogonality relations \( [8] \) of the wavefunctions \( \{\Phi_i\} \). They characterize the mixing of the eigenfunctions of \( \mathcal{H}^{(2)} \) due to the coupling of the states via the environment.

At the crossing point \( A_k^{(cr)} \rightarrow \infty \), \( |B_k^l^{(cr)}| \rightarrow \infty \) [22]. The relation between the eigenfunctions \( \Phi_1 \) and \( \Phi_2 \) of \( [5] \) at the crossing point is

\[ \Phi_1^{cr} \rightarrow \pm i \  \Phi_2^{cr} ; \quad \Phi_2^{cr} \rightarrow \mp i \  \Phi_1^{cr} \]  
(12)

according to analytical as well as numerical studies, see Appendix of [41] and section 2.5 of [22]. That means, the state \( \Phi_1 \) jumps at the EP via the chiral state \( \Phi_1 \pm i \  \Phi_2 \) to the state \( \pm i \  \Phi_2 \) [42]. From (12) follows:

(i) When two levels are distant from one another, their eigenfunctions are (almost) orthogonal, \( \langle \Phi_k^* | \Phi_k \rangle \approx \langle \Phi_k | \Phi_k \rangle = A_k \approx 1. \)

(ii) When two levels cross at the EP, their eigenfunctions are linearly dependent according to (12) and \( \langle \Phi_k^* | \Phi_k \rangle \equiv A_k \rightarrow \infty. \)

These two relations show that the phases of the two eigenfunctions relative to one another change when the crossing point is approached. This can be expressed quantitatively by defining the phase rigidity \( r_k \) of the eigenfunctions \( \Phi_k \),

\[ r_k \equiv \frac{\langle \Phi_k^* | \Phi_k \rangle}{\langle \Phi_k | \Phi_k \rangle} = A_k^{-1} . \]  
(13)

It holds \( 1 \geq r_k \geq 0. \) The non-rigidity \( r_k \) of the phases of the eigenfunctions of \( \mathcal{H}^{(2)} \) follows also from the fact that \( \langle \Phi_k^* | \Phi_k \rangle \) is a complex number (in difference to the norm \( \langle \Phi_k | \Phi_k \rangle \) which is a real number) such that the normalization condition \( [8] \) can be fulfilled only by the additional postulation \( \text{Im} \langle \Phi_k^* | \Phi_k \rangle = 0 \) (what corresponds to a rotation).
If \( r_k < 1 \), an analytical expression for the eigenfunctions as a function of a certain control parameter cannot generally be obtained. An exception is the special case \( \gamma_1 = \gamma_2 \) for which

\[
Z = \frac{1}{2} \sqrt{(e_1 - e_2)^2 + 4 \omega_{12}^2}.
\]

In this case, the condition \( Z = 0 \) can not be fulfilled if \( \omega_{12} = x \) is real due to

\[
(e_1 - e_2)^2 + 4 x^2 > 0.
\]  

(14)

The EP can be found only by analytical continuation into the continuum \([21, 22]\) and the two states avoid crossing. This is analogous to the avoided level crossings of discrete states.

The condition \( Z = 0 \) can be fulfilled however in the above case if \( \omega_{12} = i x \) is imaginary,

\[
(e_1 - e_2)^2 - 4 x^2 = 0 \rightarrow e_1 - e_2 = \pm 2 x,
\]

(15)

and two EPs appear. It holds further

\[
(e_1 - e_2)^2 > 4 x^2 \rightarrow Z \in \mathbb{R}
\]

(16)

\[
(e_1 - e_2)^2 < 4 x^2 \rightarrow Z \in \mathbb{I}
\]

(17)

independent of the parameter dependence \( e_i(a) \). In the first case, the eigenvalues \( E_i = E_i - i/2 \gamma_i \) differ from the original values \( \epsilon_i = e_i - i/2 \gamma_i \) by a contribution to the energies (level repulsion) and in the second case by a contribution to the widths (width bifurcation).

The width bifurcation starts at one of the EPs and becomes maximum in the middle between the two EPs. This happens at the crossing point \( e_1 = e_2 \) where \( \Delta \gamma/2 \equiv |\gamma_1/2 - \gamma_2/2| = 4 x \).

If \( r_k < 1 \), the Schrödinger equation contains nonlinear terms. According to (5), the Schrödinger equation with the unperturbed operator \( H_0 \equiv H(\omega_{12} = 0) \) and a source term arising from the interaction \( \omega_{12} = \omega_{21} \) of the states via the environment reads \([21]\)

\[
(H_0 - \epsilon_n) |\Phi_n\rangle = - \begin{pmatrix} \omega_{12} & 0 \\ 0 & \omega_{21} \end{pmatrix} |\Phi_n\rangle \equiv W |\Phi_n\rangle
\]

\[
= \sum_{k=1,2} \langle \Phi_k | W | \Phi_n \rangle \{ A_k | \Phi_k \rangle + \sum_{l \neq k} B_{kl}^l | \Phi_l \rangle \}.
\]

(18)

Here \( \langle \Phi_k | \Phi_l \rangle \equiv A_k \geq 1 \) according to (9) and \( \langle \Phi_k | \Phi_{l \neq k} \rangle = - \langle \Phi_{l \neq k} | \Phi_k \rangle \equiv B_{kl}^l \), \( |B_{kl}^l| \geq 0 \) according to (10). The \( A_k \) and \( B_{kl}^l \) characterize the degree of resonance overlapping. In the regime of overlapping resonances, \( 1 > A_k > 0 \), \( |B_{kl}^l| > 0 \), and equation (18) is nonlinear. The most important part of the nonlinear contributions is contained in

\[
(H_0 - \epsilon_n) |\Phi_n\rangle = \langle \Phi_n | W | \Phi_n \rangle |\Phi_n\rangle.
\]

(19)
which is a nonlinear Schrödinger equation. According to (18), the nonlinear Schrödinger equation (19) passes smoothly into the standard linear Schrödinger equation when $A_k \to 1$, $B_k \to 0$ and $r_k \to 1$.

The cross section can be calculated from the $S$ matrix, $\sigma(E) \propto |1 - S(E)|^2$. A unitary representation of the $S$ matrix in the case of two resonance states coupled to one common continuum of scattering wavefunctions reads [43]

$$S = \frac{(E - E_1 - \frac{i}{2} \Gamma_1) (E - E_2 - \frac{i}{2} \Gamma_2)}{(E - E_1 + \frac{i}{2} \Gamma_1) (E - E_2 + \frac{i}{2} \Gamma_2)}$$

(20)

where $E_i$ and $\Gamma_i$ are defined in (7) and $E$ is the energy of the system. This representation of the $S$ matrix contains the influence of EPs onto the cross section via the eigenvalues of $H^{(2)}$, Eq. (7). It provides reliable results therefore also when $r_k < 1$ [43].

At a double pole of the $S$ matrix (being an EP), the resonance line shape deviates from the Breit-Wigner one. In this case, the $S$ matrix reads [43]

$$S = 1 - 2i \frac{\Gamma_d}{E - E_d + \frac{i}{2} \Gamma_d} - \frac{\Gamma_d^2}{(E - E_d + \frac{i}{2} \Gamma_d)^2}$$

(21)

where $E_1 = E_2 = E_d$ and $\Gamma_1 = \Gamma_2 = \Gamma_d$. The second term corresponds to the usual linear term (however with the factor 2 in front) while the third term is quadratic. The interference between these two parts of the $S$ matrix has been illustrated by, e.g., Fig. 9 in [32] where the cross section is calculated for the case of two resonance states coupled to one decay channel. The asymmetry of the line shape of both peaks in the cross section at the double pole of the $S$ matrix is described by (21).

III. EIGENVALUES AND EIGENFUNCTIONS OF THE NON-HERMITIAN OPERATOR IN THE CASE WITH $N \geq 2$ STATES COUPLED TO ONE COMMON CONTINUUM

Knowing the properties of the eigenvalues and eigenfunctions of $H^{(2)}$, Eq. (5), in the neighborhood of an EP, it is interesting to study the more general case with $N > 2$ states which are coupled to one common continuum of scattering wavefunctions. For this purpose,
we consider an $N \times N$ matrix

$$
\mathcal{H}^{(N)} = \begin{pmatrix}
\epsilon_1 = \epsilon_1 + \omega_{11} & 0 & \cdots & \omega_{1N} \\
0 & \epsilon_2 = \epsilon_2 + \omega_{22} & \cdots & \omega_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_{N1} & \omega_{N2} & \cdots & \epsilon_N = \epsilon_N + \omega_{NN}
\end{pmatrix}
$$

(22)

the diagonal elements of which are the $N$ complex eigenvalues $\epsilon_i + \omega_{ii} \equiv \epsilon_i - i/2 \gamma_i$ of a non-Hermitian operator. The $\omega_{ii}$ are the so-called selfenergies of the states arising from their coupling to the environment of scattering wavefunctions into which the system is embedded. In atomic physics, these values are known as Lamb shift. Our calculations are performed with coupling matrix elements $\omega_{ii}$ the values of which do not depend on the parameter considered. In such a case, the $\omega_{ii}$ can considered to be included into the diagonal matrix elements, which read $\epsilon_i \equiv \epsilon_i + \omega_{ii} = \epsilon_i - i/2 \gamma_i$. The $\epsilon_i$ and $\gamma_i$ denote the energies and widths, respectively, of the $N$ states (including their selfenergies) without account of the interaction of the different states via the environment. The $\omega_{kN} = \omega_{Nk}$, $k = i, j$ simulate the interaction of the two states $i$ and $j \neq i$ via the common environment (consisting of one continuum of scattering wavefunctions), see Eqs. (2) to (4) [44]. This interaction is important only at high level density and near to an EP.

In the Feshbach projection operator technique, the internal interaction of two states $i$ and $j \neq i$ (which appears in the closed system described by the Hermitian Hamiltonian $H_B$) is taken into account by diagonalizing $H_B$, see Eq. (2). The eigenstates of $H_B$ are the so-called dressed states. The external interaction of these states (via the environment) is contained in the $\omega_{ij}$. The internal interaction can be caused only by some part of $\text{Re}(\omega_{ij})$ while the external interaction contains complex $\omega_{ij}$, see equations (3) and (4). Most interesting part of the external interaction is therefore $\text{Im}(\omega_{ij})$. It becomes important at high level density where the corresponding resonance states overlap.

In the present paper, we are interested in the case that all $N$ states are coupled to one another via the same common continuum of scattering wavefunctions. Such a case can be represented by (22), i.e. by assuming $\omega_{i \neq N} = \omega_{k \neq N} = 0$ while $\omega_{iN} = \omega_{Ni} \neq 0$ [44]. The results of some numerical studies will be given in Sect. V. They show width bifurcation under the influence of EPs also in a system with $N > 2$ states.

We add here some remarks on time reversal symmetry breaking in the neighborhood of an EP. An EP that is well separated from the influence of external sources (including the
influence caused by other resonance states), is highly symmetric when approached by varying a certain parameter. That means, the two states pass one into the other one according to (12) with an exchange of their wavefunctions, \( \Phi_{1}^{cr} \rightarrow \pm i \Phi_{2}^{cr} \) and \( \Phi_{2}^{cr} \rightarrow \mp i \Phi_{1}^{cr} \). At a certain finite distance from the EP, there are again two states with the wavefunctions \( |\Phi_{1}| \) and \( |\Phi_{2}| \), respectively.

This symmetry may be distorted under the influence of an external magnetic field as has been shown experimentally on a microwave cavity \[46\]. The magnetic field causes time reversal symmetry breaking. The symmetry may be disturbed also under the influence of another resonance state in the neighborhood due to the finite parameter range around the EP in which the wavefunctions of the two states are mixed with each other \[21\]. When the interaction of the third state is symmetric relative to the two crossing ones, the third state will appear as an observer and time reversal symmetry is not broken. Numerical examples of such a situation are shown in the transmission through a quantum dot \[47\] and also in the generic case studied in \[5\]. When the interaction of the third state with the two crossing ones is, however, not symmetrically, time reversal symmetry may be broken and may cause irreversible processes due to the nonlinear terms in the Schrödinger equation (19). Numerical examples will be shown in section IV.

IV. NUMERICAL RESULTS FOR THE EIGENVALUES AND EIGENFUNCTIONS OF \( \mathcal{H} \) WITH \( N = 2 \) STATES

The calculations are performed with the Hamiltonian \( \mathcal{H}^{(2)} \), Eq. (5). The \( \epsilon_{i} = \epsilon_{i}(a) = e_{i}(a) - \frac{i}{2} \gamma_{i} \) are the complex energies of two states, including their self-energies. The energies \( e_{i}(a) \) depend on a certain parameter \( a \) while the \( \gamma_{i} \) are fixed and constant in the parameter range considered. The \( \omega_{ij} = \omega_{ij}(a) \) stand for the external interaction of the two states via the environment. We simulate the fact that the wavefunctions of the two states are mixed in a finite parameter range around the critical value of their crossing \[21\] by assuming a Gaussian distribution for the coupling coefficients,

\[
\omega_{ik}(a) = \omega_{ki}(a) = \omega \cdot \exp \left[ - (\epsilon_{i}(a) - \epsilon_{k}(a))^{2} \right].
\]

The coupling coefficients \( \omega \) are complex, generally (see Sect. III).

We are interested, above all, in the situation at high level density where the resonance
states overlap and, according to Sect. 11, the influence of EPs onto the eigenvalues and eigenfunctions of $\mathcal{H}^{(2)}$ can be seen. In Figs. 1 and 2 we show results for the eigenvalues $\mathcal{E}_i = E_i - \frac{i}{2} \Gamma_i$ and the mixing coefficients $|b_{ij}|^2$, defined in (11), of two states $i = 1, 2$ for real, complex and imaginary $\omega$.

**FIG. 1:** Energies $E_i$ (full lines), widths $\Gamma_i/2$ and mixing coefficients $|b_{ij}|^2$ of $N = 2$ states coupled to $K = 1$ channel as a function of the parameter $a$. The parameters of the subfigures are $\omega = 0.05 i$ (left panel), $\omega = 0.025 (1 + i)$ (middle panel) and $\omega = 0.05$ (right panel). Further parameters: $\gamma_1/2 = \gamma_2/2 = 0.5$, $e_1 = 1 - a/2$; $e_2 = \sqrt{a}$. The dashed lines show $e_i(a)$.

In Fig. 1 the widths of both states are equal, $\gamma_1 = \gamma_2 = 0.5$. This case described by (14) to (17), is well reproduced in the numerical simulations. The case with real $\omega$ (right panel) shows the familiar picture of level repulsion and an exchange of the two states at the critical value $a_{cr}$ of the parameter $a$. The critical value $a_{cr}$ is determined by the crossing point of the two trajectories $e_i(a)$. The EP is beyond the parameter range shown in the figure as discussed below (14).

The picture with imaginary $\omega$ (left panel) is completely different from that with real $\omega$, in full agreement with Eqs. (15) to (17). The two EPs can be seen very clearly in the mixing
coefficients $|b_{ij}|^2$ which increase limitless in approaching the EPs (see Sect. II). According to (16) and (17), the energies $E_i$ of the two states are equal in the parameter range between the two EPs, and level repulsion appears only beyond the EPs. The widths $\Gamma_i$ show the opposite behavior: they are equal beyond the EPs but bifurcate in the parameter range between the two EPs. Here, small variations of the parameter $a$ cause large changes in the widths, especially near to the two EPs. It is remarkable that the width bifurcation increases in this region \textit{without any change of the coupling strength $\omega$}. The only value which is changed is the (external) parameter $a$.

The width bifurcation appears also when the coupling coefficients $\omega$ are complex (middle panel). However, the EPs can not be seen in the parameter range shown (as in the case with real $\omega$, right panel). In all cases, the wavefunctions are mixed completely at the critical value $a_{cr}$ that is determined by the crossing point of the two trajectories $e_i(a)$. Only when $\omega$ is imaginary (left panel), the complete mixing occurs in a finite range of $a$.

Fig. 2 shows that the main results obtained for $\gamma_1 = \gamma_2 = 0.5$ (Fig. 1) remain also when the two $\gamma_i$ differ from one another ($\gamma_1 = 0.53$, $\gamma_2 = 0.55$). For real $\omega$ (right panel), the energy trajectories avoid crossing and the states are exchanged at $a = a_{cr}$. However, the
FIG. 3: The same as Fig. 1 but $\omega = 0.5 \, i$ (left panel), $\omega = 0.25 \, (1 + i)$ (middle panel), $\omega = 0.5$ (right panel), and $e_2 = a$.

widths cross freely at the critical point. This fact is very well known from many studies performed during last years.

When $\omega$ is imaginary (left panel), the energies of the two states are equal to one another in a certain finite parameter range. Here, the widths bifurcate although the coupling strength $\omega$ is fixed. Further, the wavefunctions of the two states are mixed in the regime with width bifurcation. Although the EPs can not be seen in the figure, the mixing coefficients $|b_{ij}|^2$ point to their existence not far from the considered parameter range (they can be found by means of a second parameter). However, the two states are not exchanged.

The energy trajectories avoid crossing and the widths bifurcate in a finite parameter range around $a = a_{cr}$ also in the case when $\omega$ is complex (middle panel). In this case, the states are exchanged in a similar manner as in the case with real $\omega$ (right panel).

The width bifurcation increases with increasing coupling strength $\omega$ between system and environment. In Fig. 3, results are shown that are obtained for a value of $\omega$ which is larger by a factor of ten than that used for the calculations in Fig. 1. The distance between the
two EPs is larger in this case (Fig. 3 left panel) than in the case with smaller \( \omega \) (Fig. 1 left panel), what is in agreement with (17). Furthermore, the coupling to the continuum causes a much stronger width bifurcation (Fig. 3b) than in the foregoing case (Fig. 1b). The width of one state increases at the cost of the width of the other state. Finally, the width of one state approaches zero and that of the other state 2. The difference between \( \Gamma_1/2 \) and \( \Gamma_2/2 \) is about \( \sum_{i=1}^{2} \gamma_i/2 = 2 \cdot 0.5 = 1 \). By further increasing of \( \omega \), the system consisting of two decaying states will crash.

The results for complex and real coupling strengths (Fig. 3 middle and right panels, respectively) show a corresponding behavior as those with imaginary coupling strength (Fig. 3 left panel). In any case, the eigenvalues and eigenfunctions of \( \mathcal{H}^{(2)} \) are influenced by the two EPs in a larger parameter range than in Fig. 1.

It should be underlined here that far from the critical parameter range shown in Figs. 1–3, the \( \mathcal{E}_i \) trajectories approach the \( \epsilon_i \) trajectories. Here, the states are exchanged, at most, and the wavefunctions are normalized in the standard manner.

V. NUMERICAL RESULTS FOR THE EIGENVALUES AND EIGENFUNCTIONS OF \( \mathcal{H} \) WITH \( N > 2 \) STATES

In this section, we show results obtained for the eigenvalues \( \mathcal{E}_i = E_i - \frac{i}{2} \Gamma_i \) of (22) and for the mixing coefficients \( |b_{12}|^2 \), defined in (11) with \( N > 2 \), for real, complex and imaginary coupling coefficients. As in Sect. IV for \( N = 2 \) states, we consider \( \epsilon_i = \epsilon_i(a) = e_i(a) - \frac{i}{2} \gamma_i \) where the energies \( e_i \) depend on a certain parameter \( a \) and the \( \gamma_i \) are constant in the considered parameter range. The fact that the wavefunctions of the states are mixed in a finite parameter range around their crossing [21] is simulated, in analogy to Sect. IV, by assuming the Gaussian distribution (23) for the nonvanishing coupling coefficients \( \omega_{ik} \) with \( i = N \) or \( k = N \).

In Figs. 4 and 5 we show some results for \( N = 3 \). The results are more complicated than in the case with only two crossing states. They can be understood, however, fully on the basis of the results discussed in Sect. IV.

Fig. 4 shows results with the same widths of all states, \( \gamma_1 = \gamma_2 = \gamma_3 = 0.5 \). When \( \omega \) is real (right panel) the eigenvalue trajectories avoid crossing at different critical parameter values. However, far from the critical parameter range with crossing eigenvalue trajectories,
FIG. 4: Energies $E_i$ (full lines), widths $\Gamma_i/2$ and mixing coefficients $|b_{ij}|^2$ of $N = 3$ states coupled to $K = 1$ channel as a function of the parameter $a$. The parameters of the subfigures are $\omega = 0.05 \ i$ (left panel); $\omega = 0.025 \ (1 + i)$ (middle panel) and $\omega = 0.05$ (right). Further parameters: $\gamma_1/2 = \gamma_2/2 = \gamma_3/2 = 0.5$, $e_1 = 1 - a/4.5$; $e_2 = 1.1 - a/2$; $e_3 = \sqrt{a}$. The dashed lines show $e_i(a)$.

two states are exchanged while the middle one remains in the middle. Its trajectory is somewhat influenced in the critical region by those of the two neighboring states. This result differs from those shown in [5] with $N > 2$ where the parameter dependence of all energy trajectories is the same. In [5], the trajectory of the third (middle) state is not at all influenced by those of the neighboring states due to the high symmetry around all the EPs in this case. The third state is, therefore, not involved in the redistribution processes in the critical region and appears as an observer state. When however the parameter dependence of the different energy trajectories is not the same, the wavefunctions of the three states are mixed in the critical parameter range, see Fig. 4i of the present paper with real $\omega$. That means, the third state loses its role as an observer state in relation to the energy trajectories when the symmetry around the EPs is disturbed under the influence of neighboring states.

The figures with imaginary $\omega$ (left panel of Fig. 4) can also be understood by using the results shown in Sect. IV Also in the case with $N = 3$ states, two EPs can be seen. The
parameter range between them is crossed by the energy trajectory of the third state, and altogether two states are exchanged. The widths bifurcate in the range between the two EPs, however the width of one of the states remains constant in this range. This behavior is similar to that studied in [5] with $N > 2$ in which the parameter dependence of all energy trajectories is the same. That means, also in the present calculation the third state does not lose its role as an observer state in relation to the width trajectories. The symmetry around the EPs is not disturbed under the influence of neighboring states due to the fact that, in the present calculation, the widths of all states are equal to one another. Further, the mixing coefficients $|b_{ij}|^2$ (Fig. 4c) point to another EP between the two well expressed ones at the corners where width bifurcation starts and ends. This result can be understood in the following manner. When the distance between the two EPs at the corners becomes larger, the parameter range with width bifurcation splits into two regions. Finally, there are two separated regions with width bifurcation with altogether four EPs at the four corners.

Most important result with imaginary $\omega$ (left panel of Fig. 4) is that the amount of width bifurcation depends strongly on the parameter $a$, especially in the very neighborhood of the two EPs. Width bifurcation increases between the two EPs without any enhancement of the coupling strength $\omega$. This result discussed in Sect. [15] for $N = 2$ states, holds also when the number of states is larger than 2.

Additionally, we show the results with complex $\omega$ in Fig. 4, middle panel. The energy trajectories are similar to those obtained with real $\omega$. The width trajectories show width bifurcation, however with a shift of the parameter values for the maximal and minimal value of $\Gamma$ relative to one another. Interesting is the appearance of an additional region with width bifurcation at the parameter value where two levels avoid crossing. The corresponding two EPs can be seen in the $|b_{ij}|^2$ (Fig. 4f).

The results obtained for $N = 3$ states with different widths are shown in Fig. 5. Here $\gamma_1 = 0.53$, $\gamma_2 = 0.54$, $\gamma_3 = 0.55$. In difference to the results shown in Fig. 4, the symmetry around the EPs in relation to the widths is eliminated in these calculations.

For real $\omega$ (right panel of Fig. 5), we see level repulsion similar to that in the corresponding figure 4 with equal widths $\gamma_i$. The $\Gamma_i$ trajectories show free crossings at parameter values where the energy trajectories $E_i$ avoid crossing. The results point to a new EP at a small $a$ value where both $E_i$ and $\Gamma_i$ of both states are nearly the same.

Also the results for imaginary $\omega$ (left panel of Fig. 5) point to the new EP at a small
FIG. 5: The same as Fig. 4 but \( \gamma_1/2 = 0.53; \gamma_2/2 = 0.54; \gamma_3/2 = 0.55 \)

a value. Most important is however that width bifurcation appears in almost the same parameter range as in the corresponding Fig. 4b with equal \( \gamma_i \). Also in Fig. 5b, width bifurcation changes by varying the parameter \( a \) (especially in the very neighborhood of the EPs) by keeping fixed the coupling strength \( \omega \).

The results for complex \( \omega \) show level repulsion and width bifurcation. However also in this case the maximal and minimal values of the widths are at slightly different parameter values \( a \).

Neither for imaginary nor for complex or real \( \omega \), an observer state is observed in Fig. 5. This is due to the fact that the symmetry around the EPs is disturbed in these calculations not only in relation to the energy trajectories but also in relation to the width trajectories. Such a situation is, of course, more realistic than that with highly symmetrical energy and width trajectories.

The results of calculations with \( N > 3 \) states and different widths \( \gamma_i \) show all the characteristic features discussed above for \( N = 3 \) states, see e.g. Fig. 6. Most important result is that width bifurcation occurs in the very neighborhood of EPs and remains in a comparably large parameter range when \( \omega \) is imaginary (Fig. 6 left panel). Here, the wavefunctions are
FIG. 6: Energies $E_i$ (full lines), widths $\Gamma_i/2$ and mixing coefficients $|b_{ij}|^2$ of $N = 4$ states coupled to $K = 1$ channel as a function of the parameter $a$. The parameters of the subfigures are $\omega = 0.05 \ i \ (a,b,c); \ \omega = 0.025 \ (1 + i) \ (d,e,f)$ and $\omega = 0.05 \ (g,h,i)$. Further parameters: $\gamma_1/2 = 0.53; \ \gamma_2/2 = 0.54; \ \gamma_3/2 = 0.55; \ \gamma_4/2 = 0.56; \ e_1 = 1.2 - 0.7a; \ e_2 = 1.2 - 0.6a; \ e_3 = 1 - 0.5a; \ e_4 = \sqrt{a}$. The dashed lines show $e_i(a)$.

strongly mixed. The eigenvalue trajectories obtained with real $\omega$ (Fig. 6 right panel) are completely different from those with imaginary $\omega$ (Fig. 6 left panel).

VI. INFLUENCE OF EXCEPTIONAL STATES ONTO THE CROSS SECTION WITH $N = 2$ STATES

We consider now the influence of EPs onto the cross section of a reaction with $N = 2$ resonance states. We are interested, above all, in the transition from the case in which the two resonance states are well isolated from one another to the case where they are strongly overlapping. In the first case, the coupling of the two states via the continuum of scattering states is (almost) negligible. According to the results obtained and discussed in the foregoing
sections, it plays, however, an important role in the second case because the eigenvalues and
eigenfunctions of $\mathcal{H}^{(2)}$ are strongly influenced by EPs at high level density.

The cross section is calculated from $\sigma(E) \propto |1 - S(E)|^2$ with the $S$ matrix (20) which
contains the eigenvalues of $\mathcal{H}^{(2)}$. The eigenvalues and eigenfunctions of $\mathcal{H}^{(2)}$ are calculated
as a function of a certain parameter $a$ in a similar manner as in Sect. IV. The coupling
matrix elements are

$$\omega_{12}(y) = \omega_{21}(y) = \tilde{\omega} \cdot \left(\sqrt{1 - y^2} + iy\right); \quad \tilde{\omega} = \omega_0 \cdot \exp\left[-(\epsilon_2(a) - \epsilon_1(a))^2\right]. \quad (24)$$

Here the ratio between the real and imaginary parts is explicitly expressed by the factor
$\sqrt{1 - y^2} + iy$. The limiting cases $y = 0$ and $y = 1$ correspond to real and imaginary coupling
coefficients, respectively. The $\tilde{\omega}$ depend on the parameter $a$ and the $\omega_0$ are real numbers
that characterize merely the coupling strength. We mention here, that the assumption (24)
for the $\tilde{\omega}$ is not decisive for the results obtained. Qualitatively the same results are obtained
with, e.g., a linear relation between the real and imaginary parts of $\tilde{\omega}$.

We calculate the cross section for the case of equal widths of the two states, $\gamma_1 = \gamma_2$.
According to (15), (16) and (17), we have two EPs. Between the two EPs, it is $y = 1$ and
the widths bifurcate. This analytical result agrees with those of numerical studies, see Sect. IV.

In the following, the two EPs are denoted by EP1 and EP2 and the corresponding values
of the parameter $a$ by $a_1$ and $a_2$, respectively. Near to EP1 and EP2, i.e. at the corners of
the parameter range $a_1 \leq a \leq a_2$, width bifurcation increases quickly. The parameter value
with maximal width bifurcation (in the middle between EP1 and EP2) will be denoted by
$a_0$. As follows immediately from (3) and (11) as well as from (17), it is $y = 1$ for all parameter
values $a_1 \leq a \leq a_2$ (under the condition $\gamma_1 = \gamma_2$). For parameter values beyond this region,
$\omega_{ik}$ is complex. For resonance states that are well separated in energy, $y \to 0$.

In Figs. 7.a-f and 8.a-f, we show the energies $E_i$, widths $\Gamma_i$ and mixing coefficients $|b_{ij}|^2$
as a function of $y$ calculated with $\omega_0 = 0.4$ and $\omega_0 = 0.5$, respectively. Similar pictures
are obtained for $\omega_0 = 0.1$, 0.3, 0.6 and 0.7. When $a = a_1$ or $a = a_2$, the characteristic features of an EP can be seen at $y = 1$: $E_1 \to E_2; \ \Gamma_1 \to \Gamma_2$ and $b_{ij} \to \infty$ (Figs. 7a,c,e and 8a,c,e). When $a_1 < a < a_2$, the EPs do not show up in the figures (Figs. 7b,d,f and 8b,d,f). However, the wavefunctions $\Phi_i$ of both states are, in this case, strongly mixed in
the basic wavefunctions $\Phi_j^0$ (Figs. 7f and 8f). This result corresponds to those shown in
FIG. 7: Energies $E_i$, widths $\Gamma_i/2$ and mixing coefficients $|b_{ij}|^2$ of $N = 2$ states coupled to $K = 1$ channel as a function of $\gamma$ and cross section with 2 resonances as a function of $E$ for the small value $\omega_0 = 0.4$. The value $a = 0.26666$ corresponds to an EP ($a = a_1$, left panel) and the value $a = 0.8$ to the maximum width bifurcation ($a = a_0$, right panel). The cross section is calculated at $\gamma = 0$ (dashed), $\gamma = 0.5$ (dash-dotted) and $\gamma = 1$ (solid). The further parameters are $e_1 = 1.2 - 0.5 a$; $e_2 = a$; $\gamma_1/2 = \gamma_2/2 = 0.5$.

Sect. IV  
In Figs. 7 and 8 also the cross section $\sigma(E)$ for three different values of the parameter $a$ is shown. The value $\omega_0 = 0.4$ of the coupling strength in Fig. 7 is relatively small: at $\omega_0 = 0.1$, the calculated cross section does not at all depend on $a$, while a small variation of the cross section with different $a$ values can be seen at $\omega_0 = 0.3$ (as additional calculations...
FIG. 8: Energies $E_i$, widths $\Gamma_i/2$ and mixing coefficients $|b_{ij}|^2$ of $N = 2$ states coupled to $K = 1$ channel as a function of $y$ and cross section with 2 resonances as a function of $E$ for the value $\omega_0 = 0.5$. The value $a = 0.133333$ (left panel) corresponds to an EP ($a = a_1$) and the value $a = 0.9$ (right panel) corresponds to width bifurcation ($a \neq a_0$ and $a_1 < a < a_2$). The cross section is calculated at $y = 0$ (dashed), $y = 0.5$ (dash-dotted) and $y = 1$ (solid). The further parameters are $e_1 = 1.2 - 0.5 a$; $e_2 = a$; $\gamma_1/2 = \gamma_2/2 = 0.5$.

have shown). This variation is however smaller than that in a calculation with $\omega_0 = 0.4$ (Fig. 7). It increases further when $\omega_0 = 0.5$ (Fig. 8).

The line shape obtained for $a = a_1$ (or $a = a_2$) at $\omega_0 = 0.4$ and $\omega_0 = 0.5$ is shown in Fig. 7g and Fig. 8g, respectively. It is described well by Eq. (21): The cross section calculated at an EP ($a = a_1$ and $a = a_2$, respectively) is an interference picture of two resonances with
a relatively broad dip between the two bumps. This picture is obtained analytically as well as numerically at a double pole of the $S$ matrix \cite{47, 48} (which is nothing but an EP, see Sect. II).

When $a_1 < a < a_2$, a narrow dip appears in the cross section, see Fig. \ref{fig:8}h for $\omega_0 = 0.5$. The dip is somewhat broader when $\omega_0 < 0.5$, see Fig. \ref{fig:7}h for $\omega_0 = 0.4$.

Most interesting result is therefore the following. In the case with equal widths $\gamma_i$ (and $\omega_0 \approx \gamma_i/2$), the dip between the two maxima in the cross section is narrower when $a_1 < a < a_0$ (or $a_2 > a > a_0$) than at $a_1$ (or $a_2$) although the coupling strength $\omega_0$ is the same in both cases (fixed at a sufficiently high value). A narrow dip appears due to the interference of
two states with very different widths. It corresponds therefore to a large width bifurcation.

Thus, the results obtained for the cross section are in agreement with all the results shown and discussed in Sect. IV (and also with those for $N > 2$ states discussed in Sect. V). The width bifurcation starts beyond the EP without any enhancement of the coupling strength between system and environment.

In Fig. 9, the cross section is shown as a function of $y$ and $E$ calculated at $\omega_0 = 0.5$. The other parameters are similar to those in Figs. 7 and 8. At small values $y$, two well separated resonances appear in the cross section. Here, the resonance states do not overlap and $y \approx 0$. With increasing degree of overlapping, $y$ increases and reaches the value $y = 1$ at the EP. The value $y = 1$ remains constant for all parameter values $a \rightarrow a_0$. Here width bifurcation appears and the dip between the two maxima in the cross section becomes narrower.

It should be underlined that the maximum height of the resonance peaks in the cross section is normalized to 4 for all values of the parameter $a$ according to (20). An enhancement of the cross section with $y \rightarrow 1$ does therefore not appear in Fig. 9. This is in contrast to calculations for a realistic system, see [49] where the cross section is calculated without the normalization in (20).

VII. DISCUSSION OF THE RESULTS

A. Time reversal symmetry breaking and dynamical phase transitions

In the present paper, the properties of EPs and their influence on the dynamics of open quantum systems are considered. Since EPs are singular points, analytical studies are possible only in a few special cases, see Sect. II. The results of these analytical studies agree well with those obtained numerically. This confirms our method to simulate the main features of the dynamics of an open quantum system by considering the Hamiltonian matrix (5) and (22), respectively. The physical meaning of all matrix elements is derived from the basic equations (11) to (1) that characterize open quantum systems, i.e. quantum systems that are embedded into a continuum of scattering wavefunctions. This environment exists always. It can be changed by means of some external parameters, however never be deleted. Of special interest is the fact that the coupling matrix elements between system and environment are complex, generally, with the real part arising from the principal value integral (3) and...
the imaginary part originating from the residuum (4). We are interested in the symmetry properties around an EP which play an important role for time reversal symmetry and its violation in a dynamical phase transition (DPT). Furthermore, we consider a possible DPT in a two-level system if the coupling between system and environment is sufficiently large.

Let us first discuss the influence of symmetries around EPs onto width bifurcation in a many-level system. In the neighborhood of an EP, we disturb first the symmetry in energy (Fig. 4) and then in time (inverse proportional to the decay width, Fig. 5) by the existence of a nearby state. Such a situation corresponds to that characteristic of a realistic system. The results (Figs. 4 to 6, \( N = 3 \) and 4, respectively) show that all states take part in the process of width bifurcation, if the symmetries around the EPs are disturbed. We conclude therefore that in a realistic system at high level density a short-lived state appears as the result of a stepwise enhancement of its width at the cost of the widths of other states which decouple more or less from the environment. This result confirms the assumption made earlier that a DPT occurs at high level density by successive resonance trapping [50]. Our present results show that resonance trapping takes place, all in all, in a certain finite parameter range where the symmetries around the EPs are violated by nearby states due to the high level density, and the phase rigidity \( r_i \) of the participating states \( i \) is reduced (\( r_i < 1 \)).

As to a two-level system at large coupling strength \( \omega \), the number of EPs is two, and the surrounding area of each of the two EPs is not symmetrical. The EPs are approached by decreasing level repulsion on the one side and by increasing width bifurcation on the other side, see Eqs. (16) and (17) and the corresponding Fig. 3 (left panel). Thus also in this case with only two EPs, the symmetries in relation to energy and time around the EPs are violated.

Our numerical results (Sect. VI, Figs. 7 to 9) show further the new finding that a DPT may arise also at very low level density (even at \( N = 2 \)), if the interaction between system and environment is sufficiently strong (\( \omega_0 \approx \gamma_i/2 \)). In such a case, the parameter range between the two EPs is large and width bifurcation is strong (Fig. 3). It starts beyond the EP without any further amplifying of the coupling strength \( \omega \). Finally, width bifurcation creates one short-lived state together with a very narrow long-lived state. Due to interferences, a bump with a narrow dip can be seen in the cross section.

In both cases, width bifurcation occurs in a certain finite parameter range. The difference between the two cases is that the short-lived state results from successive crossings with many
other states in the first case while it results from only one crossing in the second case (under the condition that the interaction \( \omega \) between system and environment is sufficiently strong).

Further, width bifurcation causes an enhanced transmission through the system in the neighborhood of an EP (but not at the EP itself) in both cases. It appears when \( r_i < 1 \) for the phase rigidity of the participating states \( i \) and is limited in a natural manner in a realistic system. In a many-level system, the relation between phase rigidity and transmission is studied numerically in, e.g., [49]. The physical limit in the two-level-system follows from the fact that the \( S \) matrix describes a system with \textit{decaying} states, i.e. \( \Gamma_i \geq 0 \) for all states \( i \).

B. Dynamical phase transitions and Dicke superradiance

Interesting is a comparison of the results obtained in the present paper with experimental data. DPTs occurring in many-particle open quantum systems at high level density are studied in different papers, see e.g. [3, 4, 10]. In presence of an environment, the oscillatory dynamics of a quantum two-level system can undergo a DPT to a non-oscillatory phase as shown in [1, 2]. Surely, the Dicke superradiance known since many years [11], can also be considered to be a DPT as suggested by the results shown in the present paper and discussed in the following.

Many years ago, Fano studied two nearly resonant modes that decay via a common channel [53]. He showed that the shared decay channel yields additional cross coupling between the modes by means of which the asymmetric line shape for electron scattering from helium can be explained. According to present-day studies, the line shape of two nearly resonant states with almost the same lifetimes is described by the \( S \) matrix (21) at (or near to) a double pole. As shown in Sects. IV and VI the interference picture may change to a bump with an extremely sharp dip by varying a certain parameter but keeping constant the coupling strength \( \omega \) between system and environment (if it is sufficiently strong, \( \omega \approx \gamma_i/2 \)). The cross coupling between the two modes vanishes at smaller values \( \omega \) where two separated modes can be seen in the cross section. The first case is the scenario of width bifurcation beyond the EP while the second one corresponds to level repulsion below the EP.

Recently, the transition from Autler-Townes-splitting (ATS) to electromagnetically induced transparency (EIT) is considered theoretically [51] as well as experimentally [31].
Two very different processes are examined for the case that the transparency of an initially absorbing medium for a probe field is increased by means of a control field. The explanation is based on the absorption in a Λ-type configuration. At very low control intensity, EIT occurs while ATS corresponds to the appearance of two dressed states at large control intensity. The aim of [51] and [31] is to discriminate between these two phenomena.

Let us translate our formalism to the language of a Λ-type system with the following three states: the ground state $|g\rangle$, the excited state $|e\rangle$ and the state $|s\rangle$ driven by the control field. The detuning $\delta$ from resonance provides the energy dependence of the cross section (see Eq. (20) for the $S$ matrix) where the two states $|g\rangle$ and $|e\rangle$ appear as resonances at the energies $E_g$ and $E_e$. The widths of the two resonance states are $\Gamma_g$ and $\Gamma_e$, respectively. These values are obtained theoretically from the complex eigenvalues $E_i \equiv E_i - i/2 \Gamma_i$ of the non-Hermitian operator $\mathcal{H}^{(2)}$, see Eq. (7). Not only the width $\Gamma_e$ of the state $|e\rangle$ is different from zero but also the width $\Gamma_g$ of the state $|g\rangle$ is non-vanishing, due to the coupling to the common environment which causes the self-energy to be complex (if the coupling is sufficiently strong). This fact corresponds to the result obtained in [15] that the emitted radiation has a large frequency shift when the size of the atomic cloud is small compared with the radiation wavelength. We arrive therefore at the following picture.

The strong control field corresponds to a weak coupling $\omega$ between system and environment, since the state $|e\rangle$ may decay to both the ground state $|g\rangle$ and the state $|s\rangle$. The interaction $\omega$ is real, predominantly, and the two states repel each other in energy.

A weak control field, however, is in accordance with a strong coupling strength $\omega$, since the state $|e\rangle$ can decay to the ground state $|g\rangle$ without any essential competition with another decay channel. In this case, the interaction $\omega$ is imaginary and width bifurcation occurs beyond the EP (without any further enhancement of $\omega$ as shown in Sect. VI).

Thus, we see level repulsion at a strong control field while width bifurcation (broad bump with a narrow dip) dominates at a weak control field. The first case corresponds to ATS and the second case to EIT. The transition between these two scenarios occurs smoothly (Figs. 7 to 9). Below the EP $\omega_{ik} \rightarrow 0$, and $\omega_{ii} \neq 0$ is the complex Lamb shift.

This result agrees with the conclusion drawn in [17] that Fermi’s golden rule does not adequately describe Dicke’s superradiance. It is in qualitative agreement also with the results obtained and discussed in [51] and [31] where the two phases (EIT model and ATS model, respectively) are considered separately. The EIT model first dominates in the low
Rabi frequency region. With increasing control Rabi frequency $\Omega$, the likelihood of the EIT model decreases, and the ATS model dominates for larger $\Omega$. The EIT/ATS model transition takes place in a certain critical area of the control frequency $\Omega$.

Instead of the control Rabi frequency $\Omega$, the ratio of $\text{Im}(\omega_{ij})$ to $\text{Re}(\omega_{ij})$ controls the cross section in Figs. 7 to 9 where $\omega_{ij}$ is the coupling coefficient between system and environment. According to the definition in Eq. (24), $y = 1$ corresponds to an imaginary coupling coefficient $\omega_{ij}$, $i \neq j$, and $y = 0$ to a real one. In the first case, width bifurcation dominates and in the second one level repulsion. The transition between these two scenarios is caused by a singularity (EP), as shown and discussed in Sect. VI.

The exact definition of $y$ is insignificant as additional calculations have shown. Important is only that level repulsion dominates at small $y$ and width bifurcation at large $y$. In the first case $r_i \approx 1$, the two resonance states (including the complex Lamb shift) do (almost) not overlap and the eigenfunctions $\Phi_i$ are (almost) orthogonal to one another. In the second case, $r_i \to 0$, the two states overlap and interfere strongly, and the $\Phi_i$ are related to one another according to Eq. (12).

In contrast to the Rabi frequency $\Omega$, the parameter $y$ cannot be measured directly in an experiment. It is a theoretical value that follows from the mathematical description of quantum systems embedded into the continuum of scattering wavefunctions. One of the relevant measurable values is width bifurcation. Starting from two states with approximately equal widths $\gamma_1 \approx \gamma_2$, the difference between the two widths increases up to $|\Gamma_1 - \Gamma_2| \to \gamma_1 + \gamma_2$ with $\Gamma_1 \to \gamma_1 + \gamma_2$ and $\Gamma_2 \to 0$. At the same time, the system becomes almost transparent. This transition happens at a finite value of the coupling strength $\omega_{ij}$ (causing cooperative emission).

The variation of the coupling strength $\omega$ between the two-level system and the environment can be achieved not only by means of the control field driving the state $|s\rangle$ as discussed above. EIT is observed experimentally also in the cooperative emission where a broad bump with a narrow dip appears. Also in this case, EIT occurs at strong coupling $\omega$.

VIII. CONCLUSIONS

In the present paper, DPTs in open quantum systems and their relation to singular points (EPs) are considered. We found that a DPT may appear at different coupling strengths of
the system to its environment.

- A DPT may occur at high level density by successive enhancement of the width of one of the states at the crossings with nearby states. Due to width bifurcation, the neighboring states decouple (more or less) from the environment.

- A DPT appears also in a two-level system if the coupling strength to the environment is sufficiently high. Due to width bifurcation, the width of one of the states may approach zero.

In both cases, the DPT occurs in a finite parameter range where \( r_i < 1 \) for the phase rigidity of the participating states \( i \). The basic process is width bifurcation (caused by the existence of EPs) by means of which states with very different lifetime are created. In both cases, the neighborhood of the EPs is non-symmetric in relation to energy and time (Sect. VII A). Due to this symmetry violation, the DPT is an irreversible process.

The DPT may explain many results observed experimentally which are counterintuitive at first glance, e.g. [3, 4] in a many-level system and [1, 2] in a two-level system. According to the results of the present paper, the violation of Fermi’s golden rule in Dicke’s superradiance [17] may be explained also by a DPT.

Due to the relation of a DPT to irreversible processes, it is interesting to prove experimentally the influence of a nearby state onto the symmetry properties of an EP. The experiment can be performed on a microwave cavity in a similar manner as the study of time reversal symmetry breaking at an EP under the influence of a magnetic field [46]. Our results suggest that time reversal symmetry will be broken also without any magnetic field. It can be violated solely by the existence of another state in the neighborhood of the EP where \( r_i < 1 \) holds for the phase rigidity. Under this condition, the phases of the eigenfunctions \( \Phi_i \) relative to one another are not fixed and time reversal symmetry may be broken.

While DPTs are known to appear in open quantum systems and related (qualitatively) to singular points (EPs) since several years (references can be found in Section 6 of [41] and in Section 5 of [52], see also [1–4, 10]), the Dicke’s superradiance is referred to singular points first in the present paper. The results should be proven by further experimental studies. On the one hand, the DPT in systems with cooperative emission (such as EIT) needs to be considered in more detail, see Sect. VII B. On the other hand, the symmetries around EPs...
and their violation by nearby states in an $N$-level system should be investigated as discussed above.

The results are important not only for fundamental questions of quantum mechanics but also for applications where long-lived states play an important role, e.g. for the storage of information. As shown in the present paper, states with a long lifetime may occur due to width bifurcation in a two-level system at a coupling strength $\omega$ that is large and causes coherent emission, and too small in order to destroy the whole system. In such a case, the wavefunction of the long-lived (subradiant) state is well defined.

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