Nonclassical steering with two-mode Gaussian states

Massimo Frigerio,1,11 Claudio Destri,1,11 Stefano Olivares,2,11 and Matteo G. A. Paris2,11

1 Dipartimento di Fisica dell’Università degli Studi di Milano-Bicocca, I-20126 Milano, Italy
2 Dipartimento di Fisica “Aldo Pontremoli” dell’Università degli Studi di Milano, I-20133 Milano, Italy

(Dated: May 8, 2020)

Singularity or negativity of Glauber P-function is a widespread notion of nonclassicality, with important implications in quantum optics and with the character of an irreducible resource. Here we explore how P-nonclassicality may be generated by conditional Gaussian measurements on bipartite Gaussian states. This nonclassical steering may occur in a weak form, which does not imply entanglement, and in a strong form that implies EPR-steerability and thus entanglement. We show that field quadratures are the best measurements to remotely generate nonclassicality, and exploit this result to derive necessary and sufficient conditions for weak and strong nonclassical steering. For two-mode squeezed thermal states (TMST), weak and strong nonclassical steering coincide, and merge with the notion of EPR steering. This also provides a new operational interpretation for P-function nonclassicality as the distinctive feature that allows one-party entanglement verification on TMSTs.

The classification of quantum correlations is a very active front of research since the early days of quantum mechanics. In this Letter, we investigate quantum steering, a class of asymmetric quantum correlations stronger than entanglement [1], but weaker than violation of Bell’s inequality [2, 3], that was introduced in relation to the EPR argument [4, 5], to indicate the possibility of one party into different quantum states by means of suitable measurements. Despite this early appearance, steering received firm mathematical bases only recently [6, 7], and we refer to this definition as EPR steering, particularly in the context of continuous-variable (CV) systems [8]. The central idea of EPR steering is to use the influence of the measurements performed by one party (say Alice) to convince the other party (say Bob) that the shared state was entangled: if the initial correlated state allows steering in the context of continuous-family of phase space quasiprobability distributions, known as s-ordered Wigner functions and defined according to [20]:

\[ W_s[\hat{\rho}](X) = \int_{\mathbb{R}^n} \frac{d^2\Lambda}{(2\pi)^n} e^{\frac{i}{2}A^T\Lambda + iA^T\chi_s} \chi[\hat{\rho}](\Lambda) \] (1)

for \( s \in [-1, 1] \). Here the characteristic function [29] is defined as \( \chi[\hat{\rho}](\Lambda) = \text{Tr}[\hat{\rho}e^{i\Lambda^T\hat{R}}] \), where \( \hat{R} = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n)^T \) is the vector of the canonical operators (or quadrature operators), related to the mode operators by \( \hat{x}_j = (\hat{a}_j + \hat{a}_j^\dagger)/\sqrt{2}, \hat{p}_j = -i(\hat{a}_j - \hat{a}_j^\dagger)/\sqrt{2} \). The case \( s = 1 \) corresponds precisely to the Glauber P-function, which is therefore the most singular of the family and can behave even more singularly than a tempered distribution. When the P-function of a CV quantum state \( \hat{\rho} \) is not positive semidefinite [30] and/or it is more singular than a delta distribution, the state is termed nonclassical [23, 31, 32]. The so-called nonclassical depth of a CV state \( \hat{\rho} \) is then the quantity \( t = \frac{1}{s_m} - s_m \), where \( s_m \) is the largest real number such that \( W_s[\hat{\rho}](X) \) is non-singular \( \forall s < s_m \). Thus \( \hat{\rho} \) is nonclassical if \( t > 0 \) and classical if \( t = 0 \).

Let us now consider a Gaussian state \( \hat{\rho}_{AB} \) of mode \( A \) controlled by Alice, and mode \( B \) controlled by Bob. We write its characteristic function as [29, 33]:

\[ \chi[\hat{\rho}_{AB}](\Lambda) = \exp\left\{ -\frac{1}{2} \Lambda^T \sigma \Lambda - i\Lambda^T \hat{R} \right\} \] (2)

where the covariance matrix (CM) reads \( \sigma_{jk} = \frac{1}{2}(\langle \hat{R}_j \hat{R}_k \rangle - \langle \hat{R}_j \rangle \langle \hat{R}_k \rangle) \), with \( \langle \hat{R} \rangle = \text{Tr}_{AB} \hat{\rho}_{AB} \hat{R} \). The
uncertainty relations (UR) may be recast into a constraint on the CM associated with physical states \([\sigma_i]_1\), i.e. \(\sigma + i \Omega / 2 \geq 0\), where \(\Omega = \oplus_{n=1}^{N} \omega\) (for \(n\) modes) and \(\omega = i \sigma_y\) is the standard symplectic form \([35]\).

Since \(\chi(\hat{P})(\Lambda)\) is a Gaussian function on phase space whenever \(\hat{P}\) is a Gaussian state, it is straightforward to conclude from Eq.\((2)\) and Eq.\((1)\) that \(\hat{P}\) is nonclassical if and only if the least eigenvalue of \(\sigma\) is smaller than \(\frac{1}{2}\). Examples of classical Gaussian states are coherent and thermal states, while squeezed vacuum states are always nonclassical. In the following, we will be interested in characterizing how quantum correlations in the joint Gaussian quantum state \(\hat{P}_{AB}\) may be exploited to influence the nonclassicality of one mode (say \(A\)) by Gaussian measurements on the other one (mode \(B\)). In doing so, Local Gaussian Unitary Transformations (LGUTs) do not affect these correlations, and therefore we may freely perform LGUTs on the two modes to bring \(\hat{P}_{AB}\) into a simpler form. In particular, by means of LGUTs a two-mode Gaussian state can always be brought into the so-called canonical form \([33,36,37]\), for which the CM \(\sigma\) can be decomposed in 2 \(\times\) 2 diagonal blocks \(\sigma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}\) with \(A = a \cdot I_2\), \(B = b \cdot I_2\) and \(C = \text{diag}(c_1,c_2)\), while \(a,b,c_1,c_2 \in \mathbb{R}\). We now note that the unconditional state of mode \(A\), defined either as the state that Alice uses to describe her mode without knowing anything about Bob’s mode or as the state she assigns to her mode by assuming that Bob has performed some measurement on his mode without letting her know the outcome, is given by \(\hat{P}_A = \text{Tr}_B[\hat{P}_{AB}]\) and has a CM \(\sigma_A = A\). Since the UR imply that \(a \geq \frac{1}{2}\) this means that \(\hat{P}_A\) must be classical. The same holds true for mode \(B\), thus we may say that given a two-mode Gaussian state \(\hat{P}_{AB}\) in canonical form, neither of the two modes has any intrinsic nonclassicality. Based on this observation, we advance the following definition:

**Definition 1.** A two-mode Gaussian state \(\hat{P}_{AB}\) in canonical form is called weakly nonclassically steerable (WNS) from mode \(B\) to mode \(A\) (\(B \rightarrow A\)) if there exists a Gaussian positive operator-valued measure (POVM) \(\{\hat{N}_\alpha\}_{\alpha \in \mathcal{C}}\) on mode \(B\) such that the conditional state of mode \(A\) after such measurement and communication of the outcome \(\alpha\):

\[
\hat{P}_{\alpha,A} = \frac{1}{p_\alpha} \text{Tr}_B \left[ \hat{P}_{AB} \left( I_A \otimes \hat{N}_\alpha \right) \right]
\]

is nonclassical, where \(p_\alpha = \text{Tr}_{AB}[\hat{P}_{AB}(I_A \otimes \hat{N}_\alpha)]\) is the probability of observing the outcome \(\alpha \in \mathcal{C}\).

Let us now deduce a simple criterion to discern weakly nonclassically steerable states, starting with the following proposition:

**Proposition 2.** The least classical (i.e. with highest possible nonclassical depth) conditional state \(\hat{P}_{\alpha,A}\) of mode \(A\) attainable with Gaussian measurements on mode \(B\) of a two-mode Gaussian state \(\hat{P}_{AB}\) in canonical form is reached by quadrature detection on mode \(B\), either of the \(\hat{x}_B\) quadrature if \(c_2 \geq |c_1|\), or of the \(\hat{p}_B\) quadrature otherwise.

**Proof.** Let us denote by \(\sigma_c\) the CM of the conditional state: one can show that it does not depend on the outcome \(\alpha\), but just on the CM of the POVM performed on \(B\). Therefore, \(\hat{P}_{AB}\) in canonical form is WNS if and only if there exists a Gaussian POVM such that the least eigenvalue of \(\sigma_c\) is smaller than \(\frac{1}{2}\). The effects of the most general Gaussian POVM on a single mode may be written as \(\hat{N}_\alpha = D(\alpha)\hat{P}_G D(\alpha)^\dagger / \pi\) where \(D(\alpha) = \exp \{a \hat{a}^\dagger - \alpha^* \hat{a}^\dagger \hat{a}\}\) is the displacement operator and \(\hat{P}_G\) is a single-mode Gaussian state with \(\langle \hat{R} \rangle = 0\). Furthermore, we may choose the following convenient parametrization for the CM \(\sigma_M\) of \(\hat{P}_G\):

\[
\sigma_M = \frac{1}{2\mu_\mu} \begin{pmatrix} 1 + \kappa_\sigma \cos \phi & -\kappa_\sigma \sin \phi \\ -\kappa_\sigma \sin \phi & 1 - \kappa_\sigma \cos \phi \end{pmatrix}
\]

where \(\mu = \text{Tr}[\hat{P}_G^2] \in [0,1]\) is the purity of \(\hat{P}_G\), \(\mu_\sigma = [1 + 2 \sin^2 r_m]^{-1}\), \(\kappa_\sigma = \sqrt{1 - \mu_\sigma^2}\), \(r_m\) being the squeezing parameter of the state, and \(\phi \in [0,2\pi]\) is a phase. According to a well-known result \([33,38,39]\), the conditional CM is then given by the Schur complement \([10]\) of \(\sigma\) with respect to \((\mathbb{B} + \sigma_M)\), explicitly \(\sigma_c = A - C^T (B + \sigma_M)^{-1} C\). Since \(A\) is diagonal, the minimum \(\lambda_m\) (over all possible CMs \(\sigma_M\)) of the smallest eigenvalue of \(\sigma_c\) is attained for the supremum of the greatest eigenvalue of \(C^T (B + \sigma_M)^{-1} C\), which is positive semidefinite. By explicit calculation, this supremum requires \(\phi = 0\) if \(|c_2| \geq |c_1|\), and \(\phi = \pi\) otherwise. The resulting expression is a monotonic decreasing function of \(\mu_\sigma\), since one can see by inspection that its first derivative with respect to \(\mu_\sigma\) is always nonpositive. Therefore, one needs to set \(\mu_\sigma = 0\) in order to attain the supremum and in this limit the value of \(\mu_\sigma\) becomes irrelevant. The limit \(\mu_\sigma \to 0\) makes the Gaussian POVM \(\hat{N}_\alpha\) to collapse into the spectral measure of the \(\hat{x}(\hat{P})\) quadrature for \(\phi = 0(\pi)\).

This result immediately leads us to the aforementioned criterion:

**Proposition 3.** A two-mode Gaussian state \(\hat{P}_{AB}\) in canonical form is WNS (\(B \rightarrow A\)) if and only if the parameters of its CM satisfy:

\[
a - c^2 / b < 1 / 2, \quad c = \max \{|c_1|, |c_2|\}
\]

**Proof.** Let us suppose that \(c = |c_2| \geq |c_1|\), so that we can fix \(\phi = 0\) in Eq.\((4)\). Then, for \(\mu_\sigma \to 0\), one can explicitly compute \(\lambda_m = a - c^2 / b\). But the initial state \(\hat{P}_{AB}\) is WNS if and only if the least classical conditional state is nonclassical, which amounts to \(\lambda_m < 1 / 2\), as stated by Eq.\((5)\). Otherwise, if \(c = |c_1| > |c_2|\), one should choose \(\phi = \pi\) to arrive at the same conclusion.
We call this property weak nonclassical steering because it does not imply entanglement. Indeed, there are (non isolated) choices for the values of \(a, b, c_1, c_2\) that correspond to physical states \((\sigma > 0\) and fulfilling UR) that are separable and WNS, e.g. \(a = b = 13.9, c_1 = 4.6, c_2 = -13.7\). Besides, there exist WNS states with \(c_1c_2 > 0\), which is a sufficient condition for separability. Motivated by these results, we introduce the following more stringent notion of nonclassical steering:

**Definition 4.** A two-mode Gaussian state \(\hat{\rho}_{AB}\) in canonical form is called strongly nonclassically steerable (SNS) \((B \to A)\) if the measurement of any quadrature on mode \(B\) generates a nonclassical conditional state of mode \(A\).

Following the proof of Proposition 3, we immediately conclude:

**Proposition 5.** A two-mode Gaussian state \(\hat{\rho}_{AB}\) in canonical form is SNS \((B \to A)\) if and only if the parameters of its CM satisfy:

\[
a - c^2/b < 1/2, \quad c' = \min(|c_1|, |c_2|)
\]  

(6)

**Proof.** The least nonclassical conditional state is reached, among all quadrature measurements, by the “wrong” choice of phase \((\phi = \pi\) for \(|c_2| \geq |c_1|\) and \(\phi = 0\) otherwise). Therefore, it is sufficient to demand that the minimum eigenvalue of \(\sigma_c\) is less than \(1/2\) also in this case, thereby arriving at Ineq. (6).

In order to generalize these definitions from two-mode Gaussian states in canonical form to all Gaussian states of two modes, we should take into account (local) single-mode squeezing transformations, which may alter the nonclassicality of each mode independently of their quantum correlations. However, since any two-mode Gaussian state can be brought to its unique canonical form through LGUTs without altering the correlations, we can extend the definitions in the following way:

**Definition 6.** A generic two-mode Gaussian state \(\hat{\rho}_{AB}\) is called weakly (strongly) nonclassically steerable if the unique Gaussian state \(\hat{\rho}'_{AB}\) in canonical form related to \(\hat{\rho}_{AB}\) by LGUTs is weakly (strongly) nonclassically steerable.

In order to extend also the results regarding the necessary and sufficient conditions for WNS/SNS, we need to specify the effect of LGUTs on \(\sigma_c\). Any Gaussian unitary transformation is implemented by a symplectic linear transformation in the phase space formalism, and vice versa. Therefore a LGUT on a two-mode system is described by an element \(S_A \oplus S_B\) acting on quantum phase space, where \(S_{A(B)} \in SL_{A(B)}(2)\). The \(2 \times 2\) blocks of a generic \(\sigma\) are transformed according to:

\[
A' = S_A A S_A^T, \quad B' = S_B B S_B^T, \quad C' = S_A C S_B^T \tag{7}
\]

Let us now suppose that \(S_A \oplus S_B\) brings the initial \(\sigma\) in canonical form, so that \(A' = a'T_2, B' = b'T_2\) and \(C' = \text{diag}(c'_1, c'_2)\). The conditional CM \(\sigma_c\) resulting from a Gaussian measurement with CM \(\sigma_M\) on the initial state with CM \(\sigma\) can be rearranged as:

\[
\sigma_c = S_A^T \left[ A' - C' (B' + \sigma'M')^{-1} C'^T \right] S_A \tag{8}
\]

where \(S_A\) is the CM of \(S_{A(B)}\), which is a sufficient condition for separability. Therefore, it is sufficient to demand that the CM of \(\sigma_c\) is less than \(1/2\) also in this case, thereby arriving at Ineq. (6).

In order to generalize these definitions from two-mode Gaussian states in canonical form to all Gaussian states of two modes, we should take into account (local) single-mode squeezing transformations, which may alter the nonclassicality of each mode independently of their quantum correlations. However, since any two-mode Gaussian state can be brought to its unique canonical form through LGUTs without altering the correlations, we can extend the definitions in the following way:

**Definition 6.** A generic two-mode Gaussian state \(\hat{\rho}_{AB}\) is called weakly (strongly) nonclassically steerable if the unique Gaussian state \(\hat{\rho}'_{AB}\) in canonical form related to \(\hat{\rho}_{AB}\) by LGUTs is weakly (strongly) nonclassically steerable.

In order to extend also the results regarding the necessary and sufficient conditions for WNS/SNS, we need to specify the effect of LGUTs on \(\sigma_c\). Any Gaussian unitary transformation is implemented by a symplectic linear transformation in the phase space formalism, and vice versa. Therefore a LGUT on a two-mode system is described by an element \(S_A \oplus S_B\) acting on quantum phase space, where \(S_{A(B)} \in SL_{A(B)}(2)\). The \(2 \times 2\) blocks of a generic \(\sigma\) are transformed according to:

\[
A' = S_A A S_A^T, \quad B' = S_B B S_B^T, \quad C' = S_A C S_B^T \tag{7}
\]

Let us now suppose that \(S_A \oplus S_B\) brings the initial \(\sigma\) in canonical form, so that \(A' = a'T_2, B' = b'T_2\) and \(C' = \text{diag}(c'_1, c'_2)\). The conditional CM \(\sigma_c\) resulting from a Gaussian measurement with CM \(\sigma_M\) on the initial state with CM \(\sigma\) can be rearranged as:

\[
\sigma_c = S_A^T \left[ A' - C' (B' + \sigma'M')^{-1} C'^T \right] S_A \tag{8}
\]

where \(S_A\) is the CM of \(S_{A(B)}\), which is a sufficient condition for separability. Therefore, it is sufficient to demand that the CM of \(\sigma_c\) is less than \(1/2\) also in this case, thereby arriving at Ineq. (6).

In order to generalize these definitions from two-mode Gaussian states in canonical form to all Gaussian states of two modes, we should take into account (local) single-mode squeezing transformations, which may alter the nonclassicality of each mode independently of their quantum correlations. However, since any two-mode Gaussian state can be brought to its unique canonical form through LGUTs without altering the correlations, we can extend the definitions in the following way:

**Definition 6.** A generic two-mode Gaussian state \(\hat{\rho}_{AB}\) is called weakly (strongly) nonclassically steerable if the unique Gaussian state \(\hat{\rho}'_{AB}\) in canonical form related to \(\hat{\rho}_{AB}\) by LGUTs is weakly (strongly) nonclassically steerable.

In order to extend also the results regarding the necessary and sufficient conditions for WNS/SNS, we need to specify the effect of LGUTs on \(\sigma_c\). Any Gaussian unitary transformation is implemented by a symplectic linear transformation in the phase space formalism, and vice versa. Therefore a LGUT on a two-mode system is described by an element \(S_A \oplus S_B\) acting on quantum phase space, where \(S_{A(B)} \in SL_{A(B)}(2)\). The \(2 \times 2\) blocks of a generic \(\sigma\) are transformed according to:

\[
A' = S_A A S_A^T, \quad B' = S_B B S_B^T, \quad C' = S_A C S_B^T \tag{7}
\]

Let us now suppose that \(S_A \oplus S_B\) brings the initial \(\sigma\) in canonical form, so that \(A' = a'T_2, B' = b'T_2\) and \(C' = \text{diag}(c'_1, c'_2)\). The conditional CM \(\sigma_c\) resulting from a Gaussian measurement with CM \(\sigma_M\) on the initial state with CM \(\sigma\) can be rearranged as:

\[
\sigma_c = S_A^T \left[ A' - C' (B' + \sigma'M')^{-1} C'^T \right] S_A \tag{8}
\]

where \(S_A\) is the CM of \(S_{A(B)}\), which is a sufficient condition for separability. Therefore, it is sufficient to demand that the CM of \(\sigma_c\) is less than \(1/2\) also in this case, thereby arriving at Ineq. (6).
of explicit examples, we now show that this is not the case. It suffices to consider Gaussian states in canonical form with $a = (n + 2)/(2n + 1)$, $b = n$, $c_1 = (2n)^{-1/2}$ and $c_2 = -[2n/(2n + 1)]^{1/2}$, for any integer $n > 2$. By direct computation one shows that the CMs are $\geq 0$ and obeying the UR. They are also WNS because they fulfill Ineq. (5). However, their GQDs $D_{A/B}$ and $D_{B/A}$ may attain arbitrarily small values in the limit $n \to +\infty$.

Let us now focus on the relevant class of two-mode squeezed thermal states (TMST). The parameters of their CMs are given by ($r \in \mathbb{R}^+$):

$$
a/b = \frac{1}{2}(1 + N_A + N_B) \cosh 2r \pm \frac{1}{2}(N_A - N_B)
$$

$$
c = c_1 = -c_2 = \frac{1}{2}(1 + N_A + N_B) \sinh 2r
$$

(9)

where $N_i$ ($i = A, B$) denotes the average number of thermal photons in each mode. Since TMST are all and only those states whose CM is in canonical form with the additional constraint that $c_1 = -c_2 = c$, evidently the conditions for WNS and SNS coincide for them: the most nonclassical conditional state on mode $A$ is obtained by any quadrature measurement on mode $B$. From the proof of Theorem 4 it is also clear that TMST states are EPR-steerable from one mode to the other if and only if they are nonclassically steerable (strongly and therefore also weakly) in the same direction. This observation provides a new, somehow surprising, role for the notion of P-nonclassicality: it is the property that Alice should check, after Bob’s measurement on his mode, to certify that the shared TMST state is indeed entangled; we note that this fact could find applications in one-sided device-independent quantum key distribution [9]. Note that the universal steerability condition for TMST states becomes $\cosh 2r > 1 + 2N_A(1 + 2N_B)/(1 + N_A + N_B)$, which is readily interpreted as a lower bound on the two-mode squeezing needed to make the TMST steerable $B \rightarrow A$.

In order to illustrate nonclassical steering for TMST states, we employ plots of triangoloids. Consider the conditional CM of mode $A$ parameterized by $(\mu_c, \mu_{sc}, \phi_c)$ as in Eq. (4). For TMST it is possible to compute the functional dependence of these parameters on the initial TMST parameters $N_A, N_B, r$ and the POVM’s parameters $\mu_A, \mu_s, \phi$. In particular we found that $\phi_c = \phi$, thus the phase may be discarded. For a fixed TMST state, we can thus plot the region of achievable conditional states in the $(\mu_c, \mu_{sc})$-space, as obtained by considering all the POVM’s parameters $\mu$ and $\mu_s$. These are the curvilinear triangles (triangoloids) in Fig. 1, where we also displayed the nonclassical region (light-brown region), i.e., those parameters corresponding to nonclassical states [39]. The TMST state associated with a given triangoloid is nonclassically steerable $B \rightarrow A$ when the triangoloid intersects the nonclassical region, as in the right panel of Fig. 1. We shaded the intersection area according to the nonclassical depths, with lighter regions for higher $\ell$. As it may be also appreciated graphically, the decisive point for nonclassical steering of a TMST is the blue, lower vertex of the triangoloid, attained by quadrature detection on mode $B$: if this point is outside the nonclassical region, all other points of the triangoloid are outside too. Notice that the equivalence of EPR steering and nonclassical steering for TMSTs has a neat graphical interpretation: the light-brown nonclassical region is the largest region such that a TMST whose triangoloid intersects it is necessarily entangled.

![Figure 1](image)

**FIG. 1.** (Left): Triangoloid for TMST state with $N_A = N_B = 4.5$ and $r = 1.2$, $\mu_c$ is the purity of the conditional state, while $\mu_{sc} = (1 + 2 \sinh^2 r)\cosh(2r)$ quantifies squeezing of the conditional state. The light-brown region contains all nonclassical conditional states. (Right): triangoloid for $N_A = N_B = 0.75$ and $r = 1.2$.

The rightmost, red side of the triangoloids is attained by projective measurements on squeezed displaced vacuum states (squeezing increases, i.e. $\mu_s \to 0$ from the upper red point to the lower blue one). Notice also that the uppermost, green side, obtained by non-squeezed measurements ($\mu_s = 1$) is always at $\mu_{sc} = 1$, i.e. the associated conditional states are always classical.

As a final comment, we should mention that the quantities on the left sides of (5) and (6) are the conditional variances appearing in the Reid EPR-criterion [49, 50], whose test is already experimentally accessible [51, 52]. This is in agreement with the well-known result stating that quadrature measurements are the best choice for Gaussian EPR steering [53]. In turn, WNS amounts to ask that at least one of such variances is smaller than the vacuum value, whereas SNS requires the same to be true for both these variances separately. EPR-steerability instead asks that the product of them is smaller than the value attained by the same quantity on the vacuum [54].

This suggests a new hierarchy of steering concepts in the Gaussian landscape, with WNS being the weakest type, weaker than entanglement, and SNS the strongest, while EPR steering is in between, stronger than entanglement but weaker than SNS.

---

* Electronic address: m.frigerio49@campus.unimib.it
* Electronic address: claudio.destri@unimib.it
1. DERIVATION OF ANALYTICAL EXPRESSIONS FOR TRIANGOLOID PLOTS

The covariance matrix (CM) $\sigma_c$ of the conditional state $\hat{\rho}_{c,\alpha}$, being single-mode, can be written in the following form:

$$
\sigma_c = \frac{1}{2\mu_c\mu_{sc}} \begin{pmatrix}
1 + \kappa_{sc}\cos\phi & -\kappa_{sc}\sin\phi \\
-\kappa_{sc}\sin\phi & 1 - \kappa_{sc}\cos\phi
\end{pmatrix}
$$

(S1)

where $\mu_c$ is the purity of the conditional state, $\mu_{sc} = (1+2 \sinh^2 r_c)^{-1}$ quantifies the amount of single-mode squeezing $r_c$, while $\phi_c$ is the squeezing phase and finally $\kappa_{sc} = \sqrt{1 - \mu_{sc}^2}$ for brevity. The eigenvalues of $\sigma_c$ are $\lambda_{\pm} = \frac{1 \pm \kappa_{sc}}{2\mu_c\mu_{sc}}$, so that, in particular, the conditional state is nonclassical if and only if:

$$
\lambda_- = \frac{1 - \kappa_{sc}}{2\mu_c\mu_{sc}} < \frac{1}{2} \Rightarrow \mu_{sc} < \frac{2\mu_c}{1 + \mu_c}
$$

(S2)

which defines implicitly the nonclassical region. Note that the nonclassicality of $\hat{\rho}_{c,\alpha}$ does not depend on $\phi_c$, so we can focus just on $\mu_c$ and $\mu_{sc}$, which can be retrieved from Eq.(S1) using the following relations:

$$
\det[\sigma_c] = (2\mu_c)^{-2}, \quad \text{Tr}[\sigma_c] = (\mu_c\mu_{sc})^{-1}
$$

(S3)

According to the Schur complement formula, the conditional CM $\sigma_c$ for a Gaussian measurement described by $\mu,\mu_s$ on mode B of a TMST state with parameters $N_A, N_B, r$, is given by:

$$
\sigma_c = a \cdot I_2 - c^2 \left[ \sigma_1 \cdot (b \cdot I_2 + \sigma_M)^{-1} \cdot \sigma_1 \right]
$$

(S4)

where $\sigma_1 = \text{diag}(1,-1)$, $\sigma_M$ is the measurement’s CM according to Eq.(4) and $a, b$ and $c$ are the parameters of the TMST state’s CM, defined in Eq.(9) of the main text. We now define two new parameters to simplify the calculations:

$$
\alpha := b + 1, \quad \beta := \frac{\kappa_s}{2\mu_s}
$$

(S5)

with $\kappa_s = \sqrt{1 - \mu_s^2}$ as in Eq.(4). Noting that $\alpha > \beta \geq 0$, we may write:

$$
(b \cdot I_2 + \sigma_M)^{-1} = \frac{1}{\alpha^2 - \beta^2} \begin{pmatrix}
\alpha + \beta\cos\phi & -\beta\sin\phi \\
-\beta\sin\phi & \alpha - \beta\cos\phi
\end{pmatrix}
$$

which can be inserted in Eq.(S4) to arrive at:

$$
\sigma_c = a \cdot I_2 - \frac{c^2}{\alpha^2 - \beta^2} \begin{pmatrix}
\alpha - \beta\cos\phi & -\beta\sin\phi \\
-\beta\sin\phi & \alpha + \beta\cos\phi
\end{pmatrix}
$$

(S6)

At this point, $\phi$ is still the phase of the measurement. However, we can now apply Eq.(S3) and solve for $\mu_c$ and $\mu_{sc}$ to get the final result:

$$
\mu_c = \frac{1}{2} \sqrt{\frac{\alpha^2 - \beta^2}{(c^2 - aa)^2 - a^2\beta^2}}
$$

$$
\mu_{sc} = \frac{\sqrt{(\alpha^2 - \beta^2)(c^2 - aa)^2 - a^2\beta^2}}{a(\alpha^2 - \beta^2) - \alpha c^2}
$$

(S7)

and we see that $\mu_c$ and $\mu_{sc}$ are independent of $\phi$, so it must be that $\phi_c = \phi$ and the phase becomes irrelevant for the conditional nonclassicality, hence for the whole (nonclassical) steering process with TMST states.