Quasi-Projective Reduction of Toric Varieties

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Abstract

We define a quasi–projective reduction of a complex algebraic variety $X$ to be a regular map from $X$ to a quasi–projective variety that is universal with respect to regular maps from $X$ to quasi–projective varieties. A toric quasi–projective reduction is the analogous notion in the category of toric varieties. For a given toric variety $X$ we first construct a toric quasi–projective reduction. Then we show that $X$ has a quasi–projective reduction if and only if its toric quasi–projective reduction is surjective. We apply this result to characterize when the action of a subtorus on a quasi–projective toric variety admits a categorical quotient in the category of quasi–projective varieties.

Introduction

Let $X$ be a complex algebraic variety. We call a regular map $p:X \to X^{\text{qp}}$ from $X$ to a quasi–projective variety $X^{\text{qp}}$ a quasi–projective reduction of $X$ if every regular map $f:X \to Z$ to a quasi–projective variety $Z$ factors uniquely through $p$, i.e., there exists a unique regular map $\tilde{f}:X^{\text{qp}} \to Z$ such $f = \tilde{f} \circ p$.

As a first result of the present article we characterize when a given toric variety $X$ has a quasi–projective reduction. In Section 1 we construct a toric quasi–projective reduction of $X$, i.e., a toric morphism $q:X \to X^{\text{tpq}}$ to a quasi–projective toric variety $X^{\text{tpq}}$ such that every toric morphism from $X$ to a quasi–projective toric variety factors uniquely through $q$. Then we prove (see Section 2):

**Theorem 1.** A toric variety $X$ has a quasi–projective reduction if and only if its toric quasi–projective reduction $q:X \to X^{\text{tpq}}$ is surjective. If $q$ is surjective, then it is the quasi–projective reduction of $X$.

The above theorem implies in particular that every complete toric variety has a projective reduction. But as we show by an explicit example (see 3.1), the quasi–projective reduction
need not exist in general. We apply Theorem 1 to obtain a complete answer to the following problem, posed by A. Białynicki-Birula:

Let $X$ be a quasi–projective toric variety with acting torus $T$ and let $H \subseteq T$ be a subtorus. When does the action of $H$ admit a quotient in the category of quasi–projective varieties, i.e., an $H$-invariant regular map $s: X \to Y$ to a quasi–projective variety $Y$ such that every $H$-invariant regular map from $X$ to a quasi–projective variety factors uniquely through $s$?

In order to state our answer, let $s_1: X \to X\text{ }/\text{ }\text{tor }H$ denote the toric quotient (see [1]). Recall that $s_1$ is universal with respect to $H$-invariant toric morphisms. Moreover, let $q: X\text{ }/\text{ }\text{tor }H \to Y$ be the toric quasi–projective reduction. Then our result is the following (for the proof see Section 2):

**Theorem 2.** The action of $H$ on $X$ admits a quotient in the category of quasi–projective varieties if and only if $s := q \circ s_1$ is surjective. If $s$ is surjective, then it is the quotient for the action of $H$ on $X$.

Examples of quasi–projective toric varieties with a subtorus action admitting a quotient in the category of quasi–projective varieties are obtained from Mumford’s Geometric Invariant Theory. In 3.4 and 3.5 we discuss examples of subtorus actions that have no such quotient.

**Notation**

A **toric variety** is a normal algebraic variety $X$ endowed with an effective regular action of an algebraic torus $T$ that has an open orbit. We refer to $T$ as the **acting torus** of $X$. For every toric variety $X$ we fix a base point $x_0$ in the open orbit.

A regular map $f: X \to X'$ of toric varieties with base points $x_0$ and $x_0'$ respectively is called a **toric morphism** if $f(x_0) = f(x_0')$ and there is a homomorphism $\varphi: T \to T'$ of the acting tori such that $f(t \cdot x) = \varphi(t) \cdot f(x)$ holds for every $(t, x) \in T \times X$.

The basic construction in the theory of toric varieties is to associate to a given fan $\Delta$ in an $n$-dimensional lattice an $n$-dimensional toric variety $X_\Delta$. The assignment $\Delta \mapsto X_\Delta$ is in fact an equivalence of categories (see e.g. [6] or [9]). For our construction of toric quasiprojective reductions we need the following generalization of the notion of a fan:
Let $N$ denote a $n$-dimensional lattice and set $N_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} N$. A \textit{quasi–fan} in $N$ is a finite set $\Delta$ of rational convex polyhedral cones in $N_{\mathbb{R}}$ such that for each $\sigma \in \Delta$ also every face of $\sigma$ is an element of $\Delta$ and any two cones of $\Delta$ intersect in a common face. So a quasi–fan is a fan if all its cones are strictly convex. For a quasi–fan $\Delta$, we denote by $\Delta^{\text{max}}$ the set of its maximal cones and by $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma$ its support.

For a homomorphism $F: N \to N'$ of lattices, let $F_{\mathbb{R}}$ denote the associated homomorphism of real vector spaces. A \textit{map of quasi–fans} $\Delta$ in $N$ and $\Delta'$ in $N'$ is a lattice homomorphism $F: N \to N'$ such that for every $\sigma \in \Delta$ there is a $\sigma' \in \Delta'$ with $F_{\mathbb{R}}(\sigma) \subset \sigma'$. As mentioned above, every map $F: \Delta \to \Delta'$ of fans gives rise to a toric morphism $f: X_{\Delta} \to X_{\Delta'}$.

Every quasi–fan $\Delta$ in $N$ defines in a canonical manner a fan: Let $V$ denote the intersection of all cones of $\Delta$. Then $V$ is a linear subspace of $N_{\mathbb{R}}$. Set $L := V \cap N$ and let $Q: N \to \tilde{N} := N/L$ denote the projection. Then the cones $Q_{\mathbb{R}}(\sigma), \sigma \in \Delta^{\text{max}}$, are the maximal cones of a fan $\tilde{\Delta}$ in $\tilde{N}$. We call $\tilde{\Delta}$ the \textit{quotient fan} of $\Delta$. By construction, $Q$ is a map of the quasi–fans $\Delta$ and $\tilde{\Delta}$.

\section{Construction of the Toric Quasi-Projective Reduction}

The construction of the toric quasi–projective reduction is done in the category of fans. Toric morphisms from a complete toric variety $X_{\Delta}$ to projective spaces are related to concave support functions of the fan $\Delta$. Since we also want to consider non-complete fans it is more natural to work with the following notion instead of support functions:

Let $\Delta$ be a quasi–fan in a lattice $N$. A finite family $\mathcal{U} := (u_i)_{i \in I}$ of linear forms $u_i \in M := \text{Hom}(N, \mathbb{Z})$ is called $\Delta$-\textit{concave}, if it satisfies the following condition: for every $\sigma \in \Delta^{\text{max}}$ there is an index $i(\sigma)$ such that

\[ u_{i(\sigma)}|_{\sigma} \leq u_i|_{\sigma} \text{ for all } i \in I. \]

Note that for two given $\Delta$-concave families $\mathcal{U} := (u_i)_{i \in I}$ and $\mathcal{U}' := (u'_j)_{j \in J}$ of linear forms the \textit{sum family}

\[ \mathcal{U} + \mathcal{U}' := (u_i + u'_j)_{(i,j) \in I \times J} \]

is again a $\Delta$-concave family. For a $\Delta$-concave family $\mathcal{U}$ let $P_{\mathcal{U}}$ denote the convex hull of $\mathcal{U}$. Then $P_{\mathcal{U}}$ is a lattice polytope in $M_{\mathbb{R}}$. Let $\Sigma_{\mathcal{U}}$ denote the normal quasi–fan of $P_{\mathcal{U}}$ in $N$. Recall that the faces $P'$ of $P$ correspond order-reversingly to the cones of $\Sigma_{\mathcal{U}}$ by

\[ P' \mapsto \tau_{P'} := \{ v \in N_{\mathbb{R}}; p'(v) \leq p(v) \text{ for all } (p', p) \in P' \times P \}. \]
If $u_1, \ldots, u_r$ denote the vertices of $P_U$, then the family $(u_i)_{i=1, \ldots, r}$ is strictly $\Sigma_U$-concave, i.e., on every relative interior $\tau^o_{\{u_i\}}$ the linear form $u_i$ is strictly smaller than the forms $u_j$ with $j \neq i$.

Now assume that $\Delta$ is a fan. Call a subset $R$ of the set $\Delta^{(1)}$ of extremal rays of $\Delta$ indecomposable, if for every $\Delta$-concave family $\mathcal{U}$ the set $R$ is contained in some maximal cone of $\Sigma_{\mathcal{U}}$. Let $R_1, \ldots, R_k$ be the maximal indecomposable subsets of $\Delta^{(1)}$.

**1.1 Lemma.** There exists a $\Delta$-concave family $\mathcal{U}$ such that every $R_i$ is the intersection of $\Delta^{(1)}$ with some maximal cone of $\Sigma_{\mathcal{U}}$.

**Proof.** For every decomposable subset $S$ of $\Delta^{(1)}$ choose a $\Delta$-concave family $\mathcal{U}_S$ such that $S$ is not contained in any maximal cone of $\Sigma_{\mathcal{U}_S}$. Let $\mathcal{U}$ be the sum of these families $\mathcal{U}_S$. Then $P_{\mathcal{U}}$ is the Minkowski-Sum of the $P_{\mathcal{U}_S}$. Consequently $\Sigma_{\mathcal{U}}$ is the common refinement of the $\Sigma_{\mathcal{U}_S}$. This readily yields the claim. 

A $\Delta$-concave family $\mathcal{U}$ with the property of Lemma 1.1 will be called generic. As a consequence of the above lemma we obtain the following statement for the sets $\varrho_i := \text{conv}\left(\bigcup_{\varrho \in R_i} \varrho\right)$.

**1.2 Remark.** The $\varrho_i$ are the maximal cones of a quasi–fan $\Sigma$ in $N$. The lattice homomorphism $\text{id}_N$ is a map of the quasi–fans $\Delta$ and $\Sigma$. Moreover, if $\mathcal{U}$ is a generic $\Delta$-concave family, then $\text{id}_N$ is an affine map of the quasi–fans $\Sigma$ and $\Sigma_{\mathcal{U}}$.

Here a map $F$ of quasi–fans $\Delta$ in $N$ and $\Delta'$ in $N'$ is called affine if for every maximal cone $\sigma'$ of $\Delta'$ the set $F^{-1}(\sigma') \cap |\Delta|$ is a (maximal) cone of $\Delta$. Note that a map of fans is affine if and only if the associated toric morphism is affine.

Now we construct the quasi–projective toric reduction of a toric variety $X_{\Delta}$ defined by the fan $\Delta$. Let $V$ denote the minimal cone of the quasi–fan $\Sigma$ determined by $\Delta$ as in Remark 1.2. Set $L := N \cap V$, let $Q: N \to \tilde{N} := N/L$ be the projection and denote by $\tilde{\Delta}$ the quotient-fan of $\Sigma$.

**1.3 Proposition.** The toric morphism $q: X_{\Delta} \to X_{\tilde{\Delta}}$ associated to $Q$ is the toric quasi–projective reduction of $X_{\Delta}$.

**Proof.** First we show that $X_{\tilde{\Delta}}$ is in fact quasi–projective. Choose a generic $\Delta$-concave family $\mathcal{U} = (u_\sigma)_{\sigma \in \Delta^{\max}}$. Let $V_1$ denote the minimal cone of the quasi–fan $\Sigma_{\mathcal{U}}$. Set $L_1 :=$
N ∩ V₁, let P: N → \overline{N} := N/L₁ be the projection and denote the quotient-fan of Σᵤ by \overline{Δ}.

Since Σᵤ induces a strictly \overline{Δ}-concave family, the associated toric variety \textit{X}_{\overline{\Sigma}} is projective. The minimal cone \textit{V} of Σ is contained in \textit{V}_₁, so we obtain a lattice homomorphism G: \overline{N} → \overline{N} with G ∘ Q = P. By construction, G is an affine map of the fans \overline{Δ} and \overline{\Sigma}. So the associated toric morphism \textit{g}: \textit{X}_{\overline{\Delta}} → \textit{X}_{\overline{\Sigma}} is affine. Since \textit{X}_{\overline{\Sigma}} is projective we can use [4], Chap. II, Th. 4.5.2, to conclude that \textit{X}_{\overline{\Delta}} is quasi–projective.

Now we verify the universal property of \textit{q}. Let \textit{f}: \textit{X}_{\overline{\Delta}} → \textit{X}' be a toric morphism to a quasi–projective toric variety \textit{X}'. We may assume that \textit{f} arises from a map \textit{F}: N → N' of fans \overline{\Sigma} and \overline{\Delta}'. Choose a polytopal completion \Delta'' of \Delta'. By suitable stellar subdivisions (see [5], p. 72) we achieve that every maximal cone of \Delta'' contains at most one maximal cone of \Delta'.

Let (u_{σ''})_{σ'' ∈ \Sigma''_{\text{max}}} be a strictly \Delta''-concave family. Then the linear forms u_{σ''} ∘ \textit{F} form a \Delta-concave family. Let σ ∈ \Sigma_{\text{max}}. By construction, σ is mapped by \textit{F}_R into some cone of \Delta''. Moreover, σ is the convex hull of certain extremal rays of \Delta, so \textit{F}_R(σ) is in fact contained in a maximal cone of \Delta'. Hence \textit{F} is a map of the quasi–fans Σ and \Delta'. In particular we have \textit{F}(L) = 0. Thus there is a map \textit{F}: \overline{N} → N' of the fans \overline{\Delta} and \overline{\Delta}' with \textit{F} = \textit{\overline{F}} ∘ Q. The associated toric morphism \textit{\overline{f}}: \textit{X}_{\overline{\Delta}} → \textit{X}' yields the desired factorization of \textit{f}. □

2 Proof of the Theorems

Let \textit{X} be a toric variety with acting torus \textit{T} and assume that \textit{H} ⊂ \textit{T} is an algebraic subgroup. Let \textit{Z} be an arbitrary quasi–projective variety. We need the following decomposition result for regular maps:

2.1 Proposition. Let \textit{f}: \textit{X} → \textit{Z} be an \textit{H}-invariant regular map. Then there exist a locally closed subvariety \textit{W} of some \mathbb{P}_r, an \textit{H}-invariant toric morphism \textit{g}: \textit{X} → \mathbb{P}_r with \textit{g}(\textit{X}) ⊂ \textit{W} and a regular map \textit{h}: \textit{W} → \textit{Z} such that \textit{f} = \textit{h} ∘ \textit{g}.

Proof. In a first step we consider the special case that \textit{Z} = \mathbb{P}_m and \textit{X} is an open toric subvariety of some \mathbb{C}^n. Then there are polynomials \textit{f}_0, \ldots, \textit{f}_m ∈ \mathbb{C}[z_1, \ldots, z_n] having no common zero in \textit{X} such that \textit{f}(z) = [\textit{f}_0(z), \ldots, \textit{f}_m(z)] holds for every \textit{z} ∈ \textit{X}. Clearly we may assume that the \textit{f}_i have no non-trivial common divisor.
Since $f$ is $H$-invariant, every $f_i / f_j$ is an $H$-invariant rational function on $\mathbb{C}^n$. Thus, using $1 \in \gcd(f_0(z), \ldots, f_m(z))$ we can conclude that there is a character $\chi: H \to \mathbb{C}^*$ satisfying $f_i(h \cdot x) = \chi(h) f_i(x)$ for every $i$ and every $(h, x)$ in $H \times X$.

Now every $f_i$ is a sum of monomials $q_{i1}, \ldots, q_{ir}$. Note that also each of the monomials $q_{ij}$ is homogeneous with respect to the character $\chi$. Moreover, since the $f_i$ have no common zero in $X$, neither have the $q_{ij}$. Set $r := \sum r_i$ and define a toric morphism

$$g: X \to \mathbb{P}_r, \quad x \mapsto [q_{01}(x), \ldots, q_{0r_0}(x), \ldots, q_{m1}(x), \ldots, q_{mr_m}(x)].$$

Then $g$ is $H$-invariant. In order to define an open subset $W$ of $\mathbb{P}_r$ and a regular map $h: W \to \mathbb{P}_m$ with the desired properties, consider the linear forms

$$L_i: [z_{01}, \ldots, z_{0r_0}, \ldots, z_{m1}, \ldots, z_{mr_m}] \mapsto z_{i1} + \ldots + z_{ir_i}$$

on $\mathbb{P}_r$. Set $W := \mathbb{P}_r \setminus V(\mathbb{P}_r; L_1, \ldots, L_m)$ and

$$h: W \to \mathbb{P}_m, \quad [z] \mapsto [L_0(z), \ldots, L_m(z)].$$

Since the $f_i$ have no common zero in $X$ we obtain $g(X) \subset W$. Moreover, by construction we have $f = h \circ g$. So the assertion is proved for the case that $Z = \mathbb{P}_m$ and $X$ is an open toric subvariety of some $\mathbb{C}^n$.

In a second step assume that $Z$ is arbitrary but $X$ again is an open toric subvariety of some $\mathbb{C}^n$. Choose a locally closed embedding $\iota: Z \to \mathbb{P}_m$. By Step one we obtain a decomposition of $f' := \iota \circ f$ as $f' = h' \circ g$, where $g: X \to \mathbb{P}_r$ is an $H$-invariant toric morphism such that $g(X)$ is contained in an open subset $W'$ of $\mathbb{P}_r$ and $h': W' \to \mathbb{P}_m$ is regular.

Then $W := h'^{-1}(\iota(Z))$ is a locally closed subvariety of $W'$. Moreover we have $g(X) \subset W$ and there is a unique regular map $h: W \to Z$ with $h' = \iota \circ h$. It follows that $f = h \circ g$ is the desired decomposition.

Finally, let also $X$ be arbitrary. As described in [3], there is an open toric subvariety $U$ of some $\mathbb{C}^n$ and a surjective toric morphism $p: U \to X$ such that $p$ is the good quotient of $U$ by some algebraic subgroup $H_0$ of $(\mathbb{C}^*)^n$. Consider $f' := f \circ p$. Then $f'$ is invariant by the action of $H' := \pi^{-1}(H)$, where $\pi$ denotes the homomorphism of the acting tori associated to $p$.

By the first two steps we can decompose $f'$ as $f' = h' \circ g'$ with an $H'$-invariant toric morphism $g': U \to \mathbb{P}_r$ and a regular map $h: W \to Z$, where $W \subset \mathbb{P}_r$ is locally closed with $g'(U) \subset W$. Since $g'$ is $H'$-invariant, it is also invariant by the action of $H_0$. Thus there
is a unique toric morphism \( g: X \to \mathbb{P}_r \) such that \( g' = g \circ p \). Since \( p \) is surjective, \( g \) is \( H \)-invariant and we have \( f = h \circ g \) which is the desired decomposition of \( f \).

**Proof of Theorem 1.** Let us first assume that the toric quasi–projective reduction \( q: X \to X^{\text{qp}} \) is surjective. Let \( f: X \to Z \) be a regular map to a quasi–projective variety \( Z \). We have to show that \( f \) factors uniquely through \( q \).

By Proposition 2.1 there is a toric morphism \( g: X \to X' \) to a projective toric variety \( X' \), and a rational map \( h: X' \to Z \) which is regular on \( g(X) \) such that \( f = h \circ g \). Now there is a toric morphism \( \tilde{g}: X^{\text{qp}} \to X' \) such that \( g = \tilde{g} \circ q \). Since \( q \) was assumed to be surjective, we have \( \tilde{g}(X^{\text{qp}}) = g(X) \), and hence \( f \) factors through \( q \).

Now suppose that \( p: X \to X^{\text{qp}} \) is a quasi–projective reduction. Then clearly \( p \) is surjective and \( X^{\text{qp}} \) is normal. Moreover, there is an induced action of the torus \( T \) on \( X^{\text{qp}} \) making \( p \) equivariant. We claim that this action is regular:

According to Proposition 2.1 choose a toric morphism \( g: X \to X' \) to a projective toric variety \( X' \), and a rational map \( h: X' \to X^{\text{qp}} \) such that \( g(X) \) is contained in the domain \( W' \) of definition of \( h \) and \( p = h \circ g \). By the universal property of the toric quasi–projective reduction \( q: X \to X^{\text{qp}} \) there is a toric morphism \( \tilde{g}: X^{\text{qp}} \to X' \) such that \( g = \tilde{g} \circ q \).

Moreover, by the universal property of \( p \), there is a regular map \( \alpha: X^{\text{qp}} \to X^{\text{qp}} \) such that \( q = \alpha \circ p \). Note that \( \alpha(X^{\text{qp}}) \subset q(X) \) and \( \tilde{g}(q(X)) \subset W' \). So, using surjectivity of \( p \) and equivariance of \( p \) and \( q \), we obtain for a given pair \((t, y) \in T \times X^{\text{qp}}\) the equality

\[
t \cdot y = h(\tilde{g}(t \cdot \alpha(y))).
\]

This implies regularity of the induced \( T \)-action on \( X^{\text{qp}} \). It follows that \( X^{\text{qp}} \) is in fact a toric variety and \( p \) is a toric morphism. Thus we obtain a toric morphism \( \beta: X^{\text{qp}} \to X^{\text{qp}} \) with \( p = \beta \circ q \). By uniqueness of the factorizations we obtain that \( \alpha \) and \( \beta \) are inverse to each other, i.e., \( q \) is also a quasi–projective reduction of \( X \).

**Proof of Theorem 2.** Suppose first that \( s: X \to Y \) is a quotient for the action of \( H \) on \( X \) in the category of quasi–projective varieties. As above we see that \( Y \) is a toric variety and \( s \) is a surjective toric morphism. The universal property of the toric quotient \( s_1: X \to X^{\text{tor}}_H \) yields a toric morphism \( q: X^{\text{tor}}_H \to Y \) such that \( s = q \circ s_1 \). Clearly \( q \) satisfies the universal property of the toric quasi–projective reduction of \( X^{\text{tor}}_H \).

Now let \( s_1: X \to X^{\text{tor}}_H \) denote the toric quotient, \( q: X^{\text{tor}}_H \to Y \) the toric quasi–projective reduction and assume that \( q \circ s_1 \) is surjective. Let \( f: X \to Z \) be an \( H \)-invariant
regular map to a quasi–projective variety. Choose a decomposition \( f = h \circ g \) as in Proposition 2.1. Then, by the universal properties of toric quotient and quasi–projective reduction there is a regular map \( \overline{g} \) with \( g = \overline{g} \circ q \circ s_1 \). Since \( q \circ s_1 \) is surjective, \( h \) is defined on \( \overline{g}(Y) \). Thus \( f = (h \circ \overline{g}) \circ (q \circ s_1) \) is the desired factorization of \( f \).

The above proofs yield in fact the following generalization of Theorems 1 and 2: Let \( X \) be any toric variety and let \( H \) be an algebraic subgroup of the acting torus \( T \) of \( X \). Call an \( H \)-invariant regular map \( p: X \to X_{qp}^{\text{op}} \) to a quasi–projective variety \( X_{qp}^{\text{op}} \) an \( H \)-invariant quasi–projective reduction if it is universal with respect to \( H \)-invariant regular maps from \( X \) to quasi–projective varieties.

Now write \( H = \Gamma H^0 \) with a finite subgroup \( \Gamma \) and a subtorus \( H^0 \) of \( T \). Let \( g: X \to X' \) denote the geometric quotient for the action of \( \Gamma \) on \( X' \). Then \( g \) is a toric morphism. Hence there is an induced action of \( H^0 \) on \( X' \). Let \( s_1: X' \to X' / H^0 \) be the toric quotient for this action and let \( q: X' / H^0 \to Y \) be the quasi–projective toric reduction. Then we obtain:

**Theorem 3.** \( X \) has an \( H \)-invariant quasi–projective reduction if and only if \( q \circ s_1 \) is surjective. If so, then \( q \circ s_1 \circ g \) is the \( H \)-invariant quasi–projective reduction of \( X \).

### 3 Examples

We first give an example of a 3-dimensional toric variety \( X_\Delta \) that admits no quasi–projective reduction. This variety is an open toric subvariety of the minimal example for a smooth complete but non-projective toric variety presented in [9], Section 2.3.

**3.1 Example.** Let \( e_1, e_2 \) and \( e_3 \) denote the canonical basis vectors of the lattice \( \mathbb{Z}^3 \). Consider the vectors

\[
\begin{align*}
v_1 &:= -e_1, & v'_1 &:= e_2 + e_3, \\
v_2 &:= -e_2, & v'_2 &:= e_1 + e_3, \\
v_3 &:= -e_3, & v'_3 &:= e_1 + e_2.
\end{align*}
\]

Let \( \Delta \) be the fan in \( \mathbb{Z}^3 \) with the maximal cones

\[
\tau_1 := \text{cone}(v_1, v'_3), \quad \tau_2 := \text{cone}(v_2, v'_1) \quad \text{and} \quad \tau_3 := \text{cone}(v_3, v'_2).
\]
We claim that the toric quasi–projective reduction \( q \) of \( X_\Delta \) is the toric morphism associated to \( \text{id}_X \) interpreted as a map from \( \Delta \) to the fan \( \tilde{\Delta} \) having as its maximal cones

\[
\sigma_1 := \text{cone}(v_1, v_3, v_1', v_3'), \quad \sigma_2 := \text{cone}(v_1, v_2, v_1', v_2') \quad \text{and} \quad \sigma_3 := \text{cone}(v_2, v_3, v_2', v_3').
\]

Note that \( q \) is not surjective. In order to prove that \( q \) is the toric quasi–projective reduction of \( X_\Delta \), we have to show that every \( \Delta \)-concave family \( (u_i)_{i=1,2,3} \) can be extended to a \( \tilde{\Delta} \)-concave family. Note that \( v_1 + v_3' \) equals \( v_3 + v_1' \) and hence we have

\[
u_1(v_1) + u_1(v_3') = u_1(v_3) + u_1(v_1') \geq u_3(v_3) + u_2(v_1').
\]

Similarly we obtain

\[
\begin{align*}
u_2(v_2) + u_2(v_1') &= u_2(v_1) + u_2(v_2') \geq u_1(v_1) + u_3(v_2'), \\
u_3(v_3) + u_3(v_2') &= u_3(v_3') + u_3(v_2) \geq u_1(v_3') + u_2(v_2).
\end{align*}
\]

Summing over these three inequalities, we arrive at an identity, and therefore the inequalities are in fact equalities. This implies

\[
\begin{align*}
u_1(v_1) = u_2(v_1), \quad &u_1(v_1') = u_2(v_1'), \\
u_1(v_3) = u_3(v_3), \quad &u_1(v_3') = u_3(v_3'), \\
u_2(v_2) = u_3(v_2), \quad &u_2(v_2') = u_3(v_2'). \quad \checkmark
\end{align*}
\]

In the above example the quasi–projective toric reduction has a trivial kernel, and the variety \( X_{\tilde{\Delta}} \) has the same dimension as \( X_\Delta \). For the complete case we have more generally:
3.2 Remark. Let $\Delta$ be a complete fan in a lattice $N$. Then $\dim X_\Delta = \dim X^{\text{qp}}_\Delta$ holds if and only if $\Delta$ can be defined via a subdivision of a lattice polytope in $N_\mathbb{R}$.

The next example is taken from the book of Fulton. It shows that in general a complete toric variety is very far from its projective reduction.

3.3 Example. Consider the complete fan in $\mathbb{Z}^3$ obtained by taking the cones over the faces of the standard cube with vertices $(\pm 1, \pm 1, \pm 1)$. Deform this fan into a new complete fan $\Delta$ in $\mathbb{Z}^3$ by moving the vertex $(1,1,1)$ to $(1,2,3)$. The only support functions of $\Delta$ are the linear functions in $M$ (see [6], p. 26). So $X^{\text{qp}}_\Delta$ is a point. ♦

Now we turn to quotients of a quasi–projective toric variety $X$ with acting torus $T$ by subtori $H \subset T$. Examples of such quotients are obtained by Mumford’s Geometric Invariant Theory:

For the sake of simplicity assume $X = \mathbb{P}^n$. Then the choice of a lifting of the $T$-action to $\mathbb{C}^{n+1}$ yields a notion of $H$-semistability. The set $X^{\text{ss}} \subset X$ of $H$-semistable points is $T$-invariant and there is a quotient $X^{\text{ss}} \to Y$ in the category of quasi–projective varieties for the action of $H$ on $X^{\text{ss}}$ (see [8], also [7] and [2]).

3.4 Example. Let $\Delta$ be the fan in $\mathbb{R}^4$ that has $\sigma_1 := \text{cone}(e_1, e_2)$ and $\sigma_2 := \text{cone}(e_3, e_4)$ as maximal cones. Then $X_\Delta$ is an open toric subvariety of $\mathbb{C}^4$ with acting torus $T = \mathbb{C}^*^4$. Define a projection $S_1: \mathbb{Z}^4 \to \mathbb{Z}^3$ by setting

$$S_1(e_1) := e_1, \quad S_1(e_2) := e_2, \quad S_1(e_3) := e_3, \quad S_1(e_4) := e_1 + e_2.$$ 

Then $S_1(e_1), \ldots, S_1(e_4)$ generate $\tau := \text{cone}(e_1, e_2, e_3) \subset \mathbb{R}^3$. The faces $\text{cone}(e_1, e_3)$ and $\text{cone}(e_2, e_3)$ of $\tau$ are not containd in $S_1(|\Delta|)$.

By [1], the toric morphism $s_1: X_\Delta \to X_\tau$ defined by $S_1$ is the toric quotient for the action of the subtorus $H \subset T$ corresponding to the sublattice $\text{ker}(S_1)$ of $\mathbb{Z}^4$. In particular, $s_1$ is not surjective. So the action of $H$ on $X_\Delta$ has no quotient in the category of quasi–projective varieties. ♦
Note that surjectivity of the toric quotient $s_1: X \to X_{\text{tor}} / H$ does not imply the existence of a quotient in the category of quasi–projective varieties:

3.5 Example. Let $\Delta'$ be the fan in $\mathbb{R}^3$ with the maximal cones

$$
\tau_1 := \text{cone}(e_1, e_2), \quad \tau_2 := \text{cone}(e_3, e_4), \quad \tau_3 := \text{cone}(e_5, e_6).
$$

Then the associated toric variety $X_{\Delta'}$ is an open toric subvariety of $\mathbb{C}^6$. In the notation of Example 3.1, define a projection $S_1: \mathbb{Z}^6 \to \mathbb{Z}^3$ by

$$
S_1(e_1) := -e_1, \quad S_1(e_2) := v'_1, \\
S_1(e_3) := -e_2, \quad S_1(e_4) := v'_2, \\
S_1(e_5) := -e_3, \quad S_1(e_6) := v'_3.
$$

Then $S_1$ is a map of the fan $\Delta'$ and the fan $\Delta$ of 3.1, in fact the (surjective) toric morphism $s_1: X_{\Delta'} \to X_{\Delta}$ associated to $S_1$ is the toric quotient of the action of the subtorus $H$ of $\mathbb{C}^*^6$ corresponding to the sublattice $\ker(S_1) \subset \mathbb{Z}^6$. Since the quasi–projective reduction of $X_{\Delta}$ is not surjective, there is no quotient in the category of quasi–projective varieties for the action of $H$ on $X_{\Delta'}$.

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