A simple out-of-equilibrium field theory formalism?

F. Gelis(1), D. Schiff(2), J. Serreau(2)

April 2001

Abstract

In this paper, we determine a condition of applicability of a very simple formalism for the calculation of a fast process taking place in a non-equilibrium medium, in which the particle distribution functions are frozen in time.

Indeed, a well known obstacle for the use of such a naive formalism is the appearance of the so-called “pinch singularities”. We argue that these potentially dangerous terms can be regularized, and that they are negligible if the characteristic time-scale of the process under study is small compared to the relaxation time of the system.

1 Introduction

The real-time formulation of thermal field theories encodes the interactions with the heat-bath through distribution functions that appear explicitly in the propagators of the fields. In thermal equilibrium, those distributions depend only on energy, and are the Bose-Einstein or Fermi-Dirac distributions.

This formalism has been thought to be also suitable for the study of processes taking place in a non-equilibrated medium, provided one replaces the equilibrium statistical weights by other functions of energy reflecting the new particle distributions. These functions are allowed to have a space-time dependence, but all the statistical weights of a diagram are taken at the same point in space-time, so that this coordinate is just a passive parameter in the calculation of a diagram. Therefore, the Feynman rules of this formalism remain formally similar to those of the initial real-time formalism. Of course, such a naive formalism is not expected to be universally valid to calculate out-of-equilibrium processes, but should be reasonably good for processes that are fast compared to the relaxation time of the medium. In other words, the approximation made by evaluating all the distribution functions at the same time in a diagram should be satisfactory if the distributions indeed change very little over the typical time it takes for this process to take place.

As a starting point of the discussion, let us come back to the work of Altherr and Seibert [6] who realized that this naive formalism is plagued by singular terms that seem to prevent its use for any practical purpose. More precisely, these terms are ill-defined products of propagators, with poles “pinching” the real energy axis, also known as “pinch singularities”. These pinch terms cancel exactly in equilibrium thanks to the Kubo-Martin-Schwinger (KMS) identities. However, the KMS relations, which are an expression of the detailed balance principle, are not satisfied out-of-equilibrium, and this is responsible for the non-cancellation of the pinch singularities.

Altherr then proposed to regularize the pinch terms by using effective propagators on which a width
has been resummed. Indeed, this width moves the poles of the propagator away from the real axis, and the pinch terms become finite. For a width $\Gamma$, they yield contributions proportional to $1/\Gamma$. However, the order of magnitude of these pinch contributions has not been investigated in a systematic way.

In fact, subsequent work by Bedaque[10] and Negawa[11, 12] shows that pinch terms are clearly related to neglecting space-time inhomogeneities. In a more sophisticated formalism based on Baym-Kadanoff equations, the would-be pinch terms are compensated by terms containing gradients of the distribution functions.

One could therefore assess the validity of the naive formalism by estimating the order of magnitude of the pinch terms, compared to the ordinary terms. Large pinch contributions would indicate that the simplification that led to this formalism is not applicable, and that we are in a situation where the relaxation of the medium has a non trivial interplay with the process one is calculating. On the other hand, negligible pinch terms are an indication that this naive formalism can be used.

Our strategy in this paper is as follows. We start from this naive formalism, and perform a consistency check by estimating the relative order of magnitude of the pinch terms on a very simple example. From there, we obtain a condition for these pinch terms to be negligible in front of the regular ones. This condition defines the domain of validity of the naive formalism, and coincides with intuition: pinch terms are negligible if the typical time scale of the process under study is much smaller than the relaxation time. In other words, if this condition is satisfied, the calculation of a fast process at a given time can be performed with out-of-equilibrium statistical weights that are “frozen” at that time and one is allowed to neglect pinch terms.

2 Out-of-equilibrium retarded-advanced formalism

We start by a generalization of the closed-time-path (CTP) formalism[1, 2, 3] in which one replaces the Bose-Einstein and Fermi-Dirac distributions by arbitrary functions[4, 5]. In addition to having an energy dependence different from the equilibrium one, these distribution functions may depend on a space-time coordinate. However, an extra simplification of the formalism we are using here is that all the statistical weights of a given diagram are evaluated at the same point in space-time. In other words, all gradients are neglected. For bosons[6], the four components of the free matrix propagator in this formalism are:

$$
G_0^{++}(P, X) = \Delta_F(P) + 2\pi n(|p_0|, X)\delta(P^2 - m^2),
G_0^{--}(P, X) = \Delta_F^*(P) + 2\pi n(|p_0|, X)\delta(P^2 - m^2),
G_0^{+-}(P, X) = 2\pi(\theta(-p_0) + n(|p_0|, X))\delta(P^2 - m^2),
G_0^{-+} - G_0^{++} - G_0^{--} + G_0^{--} = 0,
$$

with $\Delta_F(P) \equiv i/(P^2 - m^2 + i\epsilon)$ and where $X$ denotes the point in space-time where the distribution functions are evaluated. In order to keep the notations compact, we have implicitly assumed that the distributions are isotropic in momentum space since we have not included a $p$ dependence. Since all the distribution functions will be evaluated at the same space-time point in the calculation of Feynman diagrams, we drop the variable $X$ in the following.

This formalism contains only these propagators and the vertices corresponding to the fundamental interactions. It does not contain any higher-order correlator reflecting the initial statistical distribution. In[13], it has been shown that this is achieved if one drops all the non Gaussian correlations coming from the initial density operator.

In this formalism, the vertices are the same as those of statistical equilibrium, i.e. $\lambda^{--} = -\lambda^{++} \neq 0$ while all the other components are vanishing. Pinch singularities appear as products of $\delta(P^2 - m^2)$, or as products of $\mathcal{P}/(P^2 - m^2)$ where $\mathcal{P}$ denotes the principal part. It is worth noting that this formalism requires only the distribution function $n(p_0)$ for positive arguments; it is customary to extend its definition to the complete real axis by requiring $n(-p_0) = -n(p_0)$ for bosons.

It is convenient to switch to the retarded-advanced basis[14, 15] where two of the components of the free

\footnote{We consider bosonic fields in this paper for the purpose of definiteness, but our arguments are valid for any type of field.}
advanced formalism is transforming. A choice leading to the retarded-advanced propagators with the same momentum. The transformation that leads to this formalism is a linear transformation of the $2 \times 2$ matrix made of the $G_0^{\pm}$:

$$G_0^{XY}(P) \equiv \sum_{a,b=\pm} U^{Xa}(P)U^{Yb}(-P)G_0^{ab}(P),$$  \hspace{1cm} (2)

where the capital indices $X, Y$ take the values $R$ or $A$, and where $U(P)$ is a $2 \times 2$ matrix specifying the transformation. A choice leading to the retarded-advanced formalism is

$$U(P) \equiv \begin{pmatrix} 1 & -1 \\ -n(-p_0) & -n(p_0) \end{pmatrix},$$  \hspace{1cm} (3)

where the first row is $R$, the second row is $A$, the first column is $+$ and the second column is $-$. After this transformation, the matrix propagator becomes:

$$G_0^{XY}(P) = \begin{pmatrix} 0 & G_0^R(P) \\ G_0^A(P) & 0 \end{pmatrix},$$  \hspace{1cm} (4)

with

$$G_0^A(P) \equiv G_0^{++}(P) - G_0^{-+}(P),$$

$$G_0^R(P) \equiv G_0^{+-}(P) - G_0^{-+}(P).$$  \hspace{1cm} (5)

In this formalism, the vertices $\lambda^{X...YZ}$ can be expressed in terms of $\lambda^{+-+-}$ and of the distribution functions $\lambda^R, \lambda^A$. In particular $\lambda^{AA} = 0$. However, contrary to what happens in equilibrium, we have $\lambda^{R^-R} \neq 0$. This is a consequence of the fact that the KMS relations do not hold out of equilibrium.

### 3 Resummation of a self-energy

As it stands now, the out-of-equilibrium retarded-advanced formalism is plagued by pinch singularities. These pinch terms can be made finite, as noted in [13], by resumming a width on the propagators. One can see this width as a purely mathematical device introduced to make the results finite, but it makes more sense to identify it with the usual collisional width of particles in a plasma. For instance, if the theory under consideration were QCD, this width would be of order $g^2 T \ln(1/g)$.

The self-energy that one has to resum in order to include the width has a peculiarity out-of-equilibrium: the component $\Sigma^{RR}$ do not vanish, contrary to what happens in equilibrium. Indeed, the matrix corresponding to the self-energy in the RA formalism is:

$$\Sigma^{XY}(P) = \begin{pmatrix} \Sigma^{RR}(P) & \Sigma^R(P) \\ \Sigma^A(P) & 0 \end{pmatrix}.$$  \hspace{1cm} (6)

This can be checked explicitly by using the previous Feynman rules for the retarded-advanced formalism, or can be understood by relating this component to the more familiar CTP formalism, which is done by means of the following relation:

$$\Sigma^{RR}(P) = n(p_0)\Sigma^{-+}(P) - (1 + n(p_0))\Sigma^{-+}(P).$$  \hspace{1cm} (7)

The right hand side of the previous equation is the usual collision term that appears in the Boltzmann equation, and it is known not to vanish out-of-equilibrium. Taking into account this extra component in the resummation, the resummed propagator is [4,14]:

$$G^{XY}(P) = \begin{pmatrix} 0 & \Sigma^A(P) \\ G^R(P) & -i\Sigma^{RR}(P)G^A(P)G^R(P) \end{pmatrix},$$  \hspace{1cm} (8)

where $G^A$ and $G^R$ are the resummed retarded and advanced propagators:

$$G^R(P) \equiv \frac{G_0^R(P)}{1 + iG_0^R(P)\Sigma^R(P)},$$

$$G^A(P) \equiv \frac{G_0^A(P)}{1 + iG_0^A(P)\Sigma^A(P)}.$$  \hspace{1cm} (9)

The important property of the resummed matrix propagator is that its $G^{AA}$ component is not zero out of equilibrium. Moreover, this component is coming from the pinch terms, regularized by the imaginary part of the self-energy $\Sigma$.

In the following, when we talk about the "contribution of pinch terms", we have in mind the contribution of the $G^{AA}$ component of the full propagator.
(which does not exist in equilibrium, and could explode if the width is going to zero).

4 Order of magnitude of pinch terms

It is now interesting to compare the relative order of magnitude of the regular components $G^R$ or $G^A$ and of the component $G^{AA}$ which contains the pinch terms. We have for instance:

$$G^{AA}(P) = -i\Sigma^{RR}(P)G^A(P) = \frac{\Sigma^{RR}(P)}{P^2 - M^2 - 2i\Gamma},$$

(10)

where $\Gamma$ is the width introduced on the propagator by the previous resummation, and $M^2 = m^2 + \text{Re}\Sigma_R$ is the resummed mass squared.

At this stage, it is very easy to express the previous ratio in terms of the various length scales of the problem. First, the width $\Gamma$ is a collision rate, and its inverse is the mean free path of the particle in the medium:

$$\Gamma \sim \lambda_{\text{mean}}^{-1}.$$  \hspace{1cm} (11)

The virtuality $P^2 - M^2$ can be related by the uncertainty principle to the typical lifetime of the off-shell state of momentum $P$. We can write:

$$P^2 - M^2 \sim p^0 \lambda_{\text{coh}}^{-1}.$$  \hspace{1cm} (12)

Physically, the space-time scale $\lambda_{\text{coh}}$ (usually called coherence length) one can define with the typical virtuality of the propagators inside a diagram is a measure of the typical time it takes for the process under study to take place.

We need also an estimate for $\Sigma^{RR}(P)$. This can be obtained if we recall that $\Sigma^{RR}(P)$ is the collision term of a Boltzmann equation for particles of energy $p_0$:

$$p_0 \frac{dn(p_0, t)}{dt} = \Sigma^{RR}(P),$$  \hspace{1cm} (13)

so that we can write:

$$\Sigma^{RR}(P) \sim p_0\lambda_{\text{non eq}}^{-1},$$  \hspace{1cm} (14)

where $\lambda_{\text{non eq}}$ is the scale characterizing the relaxation of the medium. Collecting everything, we find the relative order of magnitude of the pinch terms to be

$$\text{pinch} \sim \frac{\lambda_{\text{coh}}^{-1}}{\text{Max}(\lambda_{\text{mean}}^{-1}, \lambda_{\text{coh}}^{-1})}.$$  \hspace{1cm} (15)

We can now discuss different limits. If the microscopic scale $\lambda_{\text{coh}}$ is the smallest of the three scales: $\lambda_{\text{coh}} \ll \lambda_{\text{mean}}, \lambda_{\text{non eq}}$, we have:

$$\text{pinch} \bigg|_{\text{small } \lambda_{\text{coh}}} \sim \frac{\lambda_{\text{coh}}}{\lambda_{\text{non eq}}} \ll 1.$$  \hspace{1cm} (16)

In other words, the contribution of pinch terms is always negligible if the process under consideration is much faster than the relaxation of the medium. Practically, when this is the case, one can just discard the pinch terms.

On the contrary, when the coherence length is very large, this ratio is instead

$$\text{pinch} \bigg|_{\text{large } \lambda_{\text{coh}}} \sim \frac{\lambda_{\text{mean}}}{\lambda_{\text{non eq}}}.$$  \hspace{1cm} (17)

Usually, the remaining two scales are typically of the same order of magnitude since it takes a few times the mean free path to equilibrate the system.

We can now make a connection with the approaches of [10] and [11, 12], who investigated the interplay between pinch singularities and the relaxation towards equilibrium. We see that the ratio of Eq. (15) becomes infinite in the limit where $\lambda_{\text{non eq}}$ becomes small, which is precisely the limit where effects of the relaxation are very important. This is equivalent to saying that neglecting the relaxation in the bare formalism of section 2 is responsible of pinch singularities. However, we add the following precision: the importance of pinch terms depend on the typical time-scale associated with the process under consideration; and they are negligible whenever the process is much faster than the relaxation. In other words, the calculation of such a fast process at a given limit...
time can be done with formulas similar to the equilibrium ones, with “frozen” out-of-equilibrium distributions. In order to specify these distributions, one needs to solve an appropriate Boltzmann equation with given initial conditions.

In the case of a very slow process (Eq. (17)), one cannot disentangle the evolution of the medium from the process under study, and it becomes necessary to keep track of the gradients. In addition, it becomes very problematic to define quantities like production rates, since the usual formulas of thermal field theory give local rates.

5 Conclusions

In this paper, we have investigated the consistency of a naive out-of-equilibrium field theory formalism in which rates are calculated at a given time, neglecting all gradients. We have estimated the relative order of the pinch terms and found that that they are always negligible for processes that are characterized by a time-scale very short in front of the relaxation scale.

In other words, the naive simplification which consists in dropping all the gradients and in using distribution functions that are “frozen” in time leads to a consistent formalism provided the statistical weights have very small variations over the typical time it takes for the process under study to take place.

On the contrary, the situation is much more involved if this condition is not satisfied, i.e. for a process whose typical time-scale is comparable to the relaxation time itself. That this situation is difficult to handle should not come as a surprise since in this regime a local kinetic theory is not applicable: one has to go back to first principles and solve Baym-Kadanoff-type equations.

Acknowledgements

We would like to thank R. Baier for useful discussions. The work of F.G. is supported by DOE under grant DE-AC02-98CH10886. F.G. would also like to thank the European Center for Theoretical Studies in Nuclear Physics and related areas (ECT*, Trento) where part of this work has been performed, for its hospitality and support.

References

[1] J. Schwinger, J. Math. Phys. 2, 407 (1961).
[2] P.M. Bakshi, K.T. Mahanthappa, J. Math. Phys. 4, 1 (1963).
[3] L.V. Keldysh, Sov. Phys. JETP 20, 1018 (1964).
[4] N.P. Landsman, Ch.G. van Weert, Phys. Rep. 145, 141 (1987).
[5] K. Chou, Z. Su, B. Hao, L. Yu, Phys. Rep. 118, 1 (1985).
[6] T. Altherr, D. Seibert, Phys. Lett. B 333, 149 (1994).
[7] R. Kubo, J. Phys. Soc. Japan 12, 570 (1957).
[8] P.C. Martin, J. Schwinger, Phys. Rev. 115, 1342 (1959).
[9] T. Altherr, Phys. Lett. B 341, 325 (1995).
[10] P.F. Bedaque, Phys. Lett. B 344, 23 (1995).
[11] A. Niegawa, [hep-th/9810043].
[12] A. Niegawa, Prog. Theor. Phys. 102, 1 (1999).
[13] M. Le Bellac, H. Mabilat, Z. Phys. C 75, 137 (1997).
[14] P. Aurenche, T. Becherrawy, Nucl. Phys. B 379, 259 (1992).
[15] M.A. van Eijck, R. Kobes, Ch.G. van Weert, Phys. Rev. D 50, 4097 (1994).
[16] M.E Carrington, Hou Defu, M. Thoma, Eur. Phys. J. C 7, 347 (1999).
[17] P. Aurenche, F. Gelis, H. Zaraket, Phys. Rev. D 62, 096012 (2000).
[18] F. Gelis, Phys. Lett. B 493, 182 (2000).
[19] M. Le Bellac, Thermal Field Theory, Cambridge University Press, 1996.