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Uniqueness of solution of the unsteady filtration problem in heterogeneous porous media

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Abstract We establish uniqueness of the solution of the unsteady state dam problem in the heterogeneous and rectangular case assuming the dam wet at the bottom and dry near to the top.

Keywords Unsteady state dam problem · Fluid flow · Heterogeneous porous medium · Uniqueness of solution

Mathematics Subject Classification 35A02 · 35R35 · 76S05

1 Introduction

We consider a heterogeneous porous medium supplied by several reservoirs of a fluid, represented by a bounded domain $\Omega$ of $\mathbb{R}^n$ with locally Lipschitz boundary $\partial \Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ is the impervious part of the boundary, $\Gamma_2$ is the part in contact with either air or the fluid reservoirs.

The fluid infiltrates through $\Omega$ obeying to Darcy’s law

$$v = -a(x) \nabla (u + x_n),$$

where $a(x) = (a_{ij}(x))_{ij}$ is the $n \times n$ permeability matrix of the medium, $x = (x_1, ..., x_n)$, $v$ is the fluid velocity and $u$ its pressure.

We are concerned with the problem of finding the pressure $u$ and the saturation $\chi$ of the fluid inside $\Omega$. Using the mass conservation law, Darcy’s law, the boundary conditions and the initial data, we obtain the following strong formulation for our problem (see [3]):
where $\alpha, T$ are positive numbers, $Q = \Omega \times (0, T)$, $\Sigma_1 = \Gamma_1 \times (0, T)$ is the impervious part of $\partial \Omega$, $\Sigma_2 = \Gamma_2 \times (0, T)$ is the pervious part, $\Sigma_3 = \Sigma_2 \cap \{\phi > 0\}$ is the part covered by fluid, and $\Sigma_4 = \Sigma_2 \cap \{\phi = 0\}$ is the part where the fluid flows outside $\Omega$. $\phi$ is a nonnegative Lipschitz continuous function defined in $\overline{Q}$, $\nu$ is the outward unit normal vector to $\partial \Omega$, $a = (0, ..., 0, 1) \in \mathbb{R}^n$, $u_0, \chi_0 : \Omega \rightarrow \mathbb{R}$ are functions satisfying for a positive constant $M$

\[ 0 \leq u_0(x) \leq M, \quad 0 \leq \chi_0(x) \leq 1 \quad \text{for a.e. } x \in \Omega. \tag{1.1} \]

For $a(x)$, we assume that we have for two positive constants $\lambda$ and $\Lambda$

\[ \forall \xi \in \mathbb{R}^n, \quad \text{for a.e. } x \in \Omega \quad \lambda |\xi|^2 \leq a(x) \cdot \xi. \tag{1.2} \]

\[ \forall \xi \in \mathbb{R}^n, \quad \text{for a.e. } x \in \Omega \quad |a(x) \cdot \xi| \leq \Lambda |\xi|. \tag{1.3} \]

Moreover, we assume that

\[ \text{div}(a(x)e) \in L^2(\Omega). \tag{1.4} \]

Using the strong formulation, we are led to the following weak formulation

\[
\begin{aligned}
\text{(P)}
\end{aligned}
\]

\[
\begin{aligned}
\text{(i)} & \quad u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(1 - \chi) = 0 \quad \text{a.e. in } Q \\
\text{(ii)} & \quad u = \phi \quad \text{on } \Sigma_2 \\
\text{(iii)} & \quad \int_Q \left[ a(x)(\nabla u + \chi e) \cdot \nabla \xi - (\alpha u + \chi)t \xi \right] dx \, dt \\
& \quad \leq \int_{\Omega} (\chi_0 + \alpha u_0) \xi(x, 0) \, dx \\
& \quad \forall \xi \in H^1(Q), \quad \xi \leq 0 \quad \text{on } \Sigma_3, \quad \xi \geq 0 \quad \text{on } \Sigma_2, \quad \xi(x, T) = 0 \quad \text{for a.e. } x \in \Omega.
\end{aligned}
\]

For the existence of a solution of the problem $(P)$ in the homogeneous case ($a(x) = I_n$), we refer to [1, 8] respectively in the incompressible ($\alpha = 0$) and compressible ($\alpha > 0$) cases. For the heterogeneous case, we refer to [16] in a more general framework under Assumptions (1.1)–(1.4) for both incompressible and compressible cases. For the incompressible case with nonlinear Darcy’s law, we refer to [3, 11, 12] respectively for Dirichlet, Neuman and generalized boundary conditions. Regarding regularity of the solution, we refer to [1, 2], where it has been proved when $a(x) = I_n$ that $\chi \in C^0([0, T]; L^p(\Omega))$ for all $p \geq 1$ in both incompressible and compressible cases, and that $u \in C^0([0, T]; L^p(\Omega))$ for all $1 \leq p \leq 2$ in the compressible case. Extensions to the quasilinear and incompressible case were obtained in [11–13] in both homogeneous and nonhomogeneous frameworks. The authors of this paper were recently able to extend the above regularity result in [14] to a more general framework under weaker assumptions on the data.

In this paper, we are mainly concerned with the uniqueness of the solution of the problem $(P)$. This question was first addressed for a rectangular homogeneous dam in [7, 15] respectively for a formulation based on quasi-variational inequalities and for the formulation $(P)$ with a dam wet at the bottom and dry near to the top in the second case. Uniqueness of the solution for a homogeneous dam with general geometry was established by the method
of doubling variables in [1], but it is not obvious whether it works in the heterogeneous situation. Extensions to a quasilinear operator modeling incompressible fluid flow governed by a nonlinear Darcy’s law with Dirichlet, or Neuman boundary conditions were obtained in [11,12] respectively.

Our main result in this work is the uniqueness of problem \( (P) \) solution for a heterogeneous and rectangular porous medium assuming it remains wet near the bottom and dry close to the top. Our method is inspired by an idea from [7] in the homogeneous case and relies on solution regularity that has been recently obtained in [14]. Our uniqueness result is new in the heterogeneous and rectangular framework, but most likely the technique is limited to that particular geometry like shape.

2 Preliminary results

In this work we shall be interested in the following situation of a two dimensional rectangular dam \( \Omega = (0, L) \times (0, K) \), with \( L, K > 0 \), and \( \Gamma_1 = [0, L] \times \{0\}, \Gamma_2 = ([0] \times [0, K]) \cup ([0, L] \times \{K\}) \cup ([L] \times [0, K]) \) (see Fig. 1).

We also assume that

\[
\begin{align*}
    a(x)e & \in C^{0,1}(\Omega), \\
    \text{div}(a(x)e) & \geq 0 \text{ a.e. in } \Omega, \\
    \phi_0 \leq \phi & \leq \phi_1 \text{ on } \Sigma_2,
\end{align*}
\]  

(2.1)

(2.2)

(2.3)

where \( \phi_0 \) and \( \phi_1 \) are two nonnegative Lipschitz continuous functions defined on \( \Omega \) and satisfying for some \( \varepsilon_0 > 0 \) small enough

\[
\begin{align*}
    \phi_0(0, x_2) = \phi_0(L, x_2) &= (\varepsilon_0 - x_2)^+ = \varphi_0(x_2) \\
    \phi_1(0, x_2) = \phi_1(L, x_2) &= (K - \varepsilon_0 - x_2)^+ = \varphi_1(x_2) \\
    \phi_0(x_1, K) = \phi_1(x_1, K) &= 0.
\end{align*}
\]

(2.4)

Let us now denote by \((v_i, \gamma_i)\) the solution of the stationary problem corresponding to \( \phi_i, \ i = 0, 1 \) (see [5])

![Fig. 1 Two dimensional rectangular dam](image_url)
Find \((v_i, \gamma_i) \in H^1(\Omega) \times L^\infty(\Omega)\) such that:

(i) \(v_i \geq 0, \ 0 \leq \gamma_i \leq 1, \ v_i(1 - \gamma_i) = 0\) \ a.e. in \(\Omega\)

(ii) \(v_i = \phi_i\) on \(\Gamma_2\)

(iii) \(\int_{\Omega} a(x)(\nabla v_i + \gamma_i e) \cdot \nabla \xi dx \leq 0\)

\(\forall \xi \in H^1(\Omega), \ \xi = 0\) on \(\Gamma_2 \cap \{\phi_i \leq 0\}\), \(\xi \geq 0\) on \(\Gamma_2 \cap \{\phi_i = 0\}\).

Then we have the following uniqueness result.

**Theorem 2.1** Assume that (2.1)–(2.2) hold. Then the solution \((v_i, \gamma_i)\) of \((P^i_1)\) is unique and given by

\[(v_i, \gamma_i) = (\psi_i, \chi_{\{\phi_i > 0\}}).\]  

(2.5)

**Proof** When \(a(x)e \in C^1(\overline{\Omega})\), uniqueness of the solution can be obtained from [4] (see also [6]), since we cannot have pools in a rectangular dam. When \(a(x)e \in C^{0,1}(\overline{\Omega})\), one may combine Theorem 5.1 of [5] and the proof of Theorem 6.3 in [4] to establish the uniqueness.

Now we observe that we have for each \(k \in (0, K)\) and \(\xi \in H^1(\Omega)\)

\[
\int_{\Omega} a(x)(\nabla(k - x_2)^+ + \chi_{\{(k - x_2)^+ > 0\}} e) \cdot \nabla \xi dx = \int_{\Omega \cap \{x_2 < k\}} a(x)(-e + e) \cdot \nabla \xi dx = 0,
\]

which means that \(((k - x_2)^+, \chi_{\{(k - x_2)^+ > 0\}})\) is a solution of the stationary dam problem for the Dirichlet boundary data \((k - x_2)^+\) on \(\Gamma_2\). Due to the uniqueness of the solution of the problem \((P^i_1), (2.5)\) holds.

**Remark 2.1** Theorem 2.1 remains true without the regularity Assumption (2.1) provided the following assumptions on the permeability matrix hold (see [10])

\[
a_{12} = 0 \quad \text{a.e. in } \Omega,
\]

\[
\frac{\partial a_{22}}{\partial x_2} \geq 0 \quad \text{in } D'(\Omega).
\]

Next, we will construct a solution corresponding to a dam that is wet up to level \(x_2 = \epsilon_0\) and dry above \(x_2 = K - \epsilon_0\) over the whole interval \([0, T]\).

**Lemma 2.1** Assume that (2.1)–(2.4) hold and the initial data satisfies

\[
v_0 \leq u_0 \leq v_1 \quad \text{a.e. in } \Omega.
\]

(2.6)

\[
\gamma_0 \leq \chi_0 \leq \gamma_1 \quad \text{a.e. in } \Omega.
\]

(2.7)

Then there exists a solution \((u, \chi)\) of problem \((P)\) such that

\[
v_0 \leq u \leq v_1 \quad \text{a.e. in } Q
\]

(2.8)

\[
\gamma_0 \leq \chi \leq \gamma_1 \quad \text{a.e. in } Q.
\]

(2.9)

**Proof** Let \(v_{i\epsilon}\) be the solution of the approximating stationary problem \((P^i_{\epsilon})\), \(i = 0, 1\)
where \( H_\epsilon(s) = \min(1, s^+ / \epsilon) \) is an approximation of the Heaviside graph \( H(s) = [0, 1] \chi_{[0]} + \chi_{(0, \infty)} \).

Let \( u_\epsilon \) be the solution of the following approximating problem of the problem \((P)\) 

\[
\begin{align*}
(P_\epsilon) & \quad \text{Find } u_\epsilon \in H^1(Q) \text{ such that:} \\
& \quad (i) \quad u_\epsilon = \phi \text{ on } \Sigma_2 \\
& \quad (ii) \quad \int_Q \left[ a(x) \left( \nabla u_\epsilon + H_\epsilon(u_\epsilon)e \right) \cdot \nabla \xi + \epsilon u_\epsilon \xi_t - \frac{\partial G_\epsilon(u_\epsilon)}{\partial \xi} \right] dxdt \\
& \quad \quad + \int_\Omega G_\epsilon(u_\epsilon(x, T)) \xi(x, T) dx = \int_\Omega (au_0(x) + \chi_0(x)) \xi(x, 0) dxdy \\
& \quad \forall \xi \in H^1(Q), \xi = 0 \text{ on } \Sigma_2,
\end{align*}
\]

where \( u_0 = \min(u_0, v_{1\epsilon}) \) and \( \chi_0 = \min(\chi_0, H_\epsilon(v_{1\epsilon})) \), and \( G_\epsilon(s) = \alpha s + H_\epsilon(s) \).

If \( \xi \in H^1(Q), \xi = 0 \text{ on } \Sigma_2 \), we have from \((P_\epsilon^{ii})\) by taking into account the fact that \( v_{1\epsilon} \) is independent of \( t \):

\[
\int_Q \left[ a(x) \left( \nabla v_{1\epsilon} + H_\epsilon(v_{1\epsilon})e \right) \cdot \nabla \xi + \xi_t - \left( \alpha v_{1\epsilon} + H_\epsilon(v_{1\epsilon}) \right) \xi_t \right] dxdtdt \\
+ \int_\Omega \left( \alpha v_{1\epsilon} + H_\epsilon(v_{1\epsilon}) \right) \xi(x, T) dx = \int_\Omega \left( \alpha v_{1\epsilon} + H_\epsilon(v_{1\epsilon}) \right) \xi(x, 0) dx. \tag{2.10}
\]

For \( \delta > 0 \), the function \( \xi_\delta = \frac{(u_\epsilon - v_{1\epsilon} - \delta)^+}{u_\epsilon - v_{1\epsilon}} \) belongs to \( H^1(Q) \) and satisfies \( \xi_\delta = 0 \) on \( \Sigma_2 \) since \( \phi \leq \phi_1 \) on \( \Sigma_2 \). Writing \((2.10)\) and \((P_\epsilon^{ii})\) for \( \xi = \xi_\delta \) and subtracting the two identities from each other, we get by taking into account \((2.6)-(2.7)\)

\[
\begin{align*}
& \int_Q \left[ a(x) \left( \nabla (u_\epsilon - v_{1\epsilon}) + (H_\epsilon(u_\epsilon) - H_\epsilon(v_{1\epsilon}))e \right) \cdot \nabla \xi_\delta + \epsilon (u_\epsilon - v_{1\epsilon}) \xi_\delta_t \right] dxdtdt \\
& \quad - \left( \alpha (u_\epsilon - v_{1\epsilon}) + H_\epsilon(u_\epsilon) - H_\epsilon(v_{1\epsilon}) \right) \xi_\delta_t dxdydt \\
& \quad + \int_\Omega \left( \alpha (u_\epsilon(x, T) - v_{1\epsilon}) + H_\epsilon(u_\epsilon(x, T)) - H_\epsilon(v_{1\epsilon}) \right) \xi_\delta(x, T) dx \\
& \quad = \int_\Omega \left( \alpha (u_0(x) - v_{1\epsilon}) + \chi_0 - H_\epsilon(v_{1\epsilon}) \right) \xi_\delta(x, 0) dx \leq 0. \tag{2.11}
\end{align*}
\]

By Lemma 2.1 of [16], we obtain from \((2.11)\)

\[
u_{1\epsilon} \leq v_{1\epsilon} \quad \text{a.e. in } Q \tag{2.12}
\]

and by the monotonicity of \( H_\epsilon \), we get

\[
H_\epsilon(u_\epsilon) \leq H_\epsilon(v_{1\epsilon}) \quad \text{a.e. in } Q. \tag{2.13}
\]

We recall that from the proof of existence (see [8] or [16] for example), we know that we have up to a subsequence

\[
u_{1\epsilon} \rightharpoonup v_1 \quad \text{weakly in } L^2(0, T; H^1(\Omega)). \tag{2.14}
\]

\[
H_\epsilon(u_\epsilon) \rightharpoonup \chi \quad \text{weakly in } L^2(Q). \tag{2.15}
\]

where \((u, \chi)\) is a solution of problem \((P)\).

Similarly, we have since the solution of problem \((P_1^\epsilon)\) is unique

\[
v_{1\epsilon} \rightharpoonup v_1 \quad \text{weakly in } H^1(\Omega). \tag{2.16}
\]

\[
H_\epsilon(v_{1\epsilon}) \rightharpoonup \gamma_1 \quad \text{weakly in } L^2(\Omega). \tag{2.17}
\]

\(\chi\) denotes the Heaviside function.
Now, let $\xi \in \mathcal{D}(Q)$ with $\xi \geq 0$. Passing to the limit, we obtain by using (2.12)–(2.17)
\[
\int_Q (v_1 - u)\xi \, dx \, dt = \lim_{\epsilon \to 0} \int_Q (v_{1\epsilon} - u_\epsilon)\xi \, dx \, dt \geq 0,
\]
\[
\int_Q (\gamma_1 - \chi)\xi \, dx \, dt = \lim_{\epsilon \to 0} \int_Q (H_\epsilon(v_{1\epsilon}) - H_\epsilon(u_\epsilon))\xi \, dx \, dt \geq 0,
\]
which leads to
\[
u \leq v_1 \quad \text{a.e. in } Q \tag{2.18}
\]
\[
\chi \leq \gamma_1 \quad \text{a.e. in } Q. \tag{2.19}
\]
Similarly, for $\delta > 0$ the function $\xi_\delta = (v_{0\epsilon} - u_\epsilon - \delta)^+$ belongs to $H^1(Q)$ and satisfies $\xi_\delta = 0$ on $\Sigma_2$ since $\phi_0 \leq \phi$ on $\Sigma_2$. Then by taking into account (2.6)–(2.7), we get
\[
\int_Q \left[ a(x) \left( \nabla (v_{0\epsilon} - u_\epsilon) + (H_\epsilon(v_{0\epsilon}) - H_\epsilon(u_\epsilon))e \right) \cdot \nabla \xi_\delta + \epsilon (v_{0\epsilon} - u_\epsilon) \xi_\delta t \right] \, dx \, dt 
\]
\[
\quad - \left( \alpha(v_{0\epsilon} - u_\epsilon) + H_\epsilon(v_{0\epsilon}) - H_\epsilon(u_\epsilon) \right) \xi_\delta t \right] \, dx \, dt 
\]
\[
\quad + \int_\Omega \left( \alpha(v_{0\epsilon} - u_\epsilon(x, T)) + H_\epsilon(v_{0\epsilon}) - H_\epsilon(u_\epsilon(x, T)) \right) \xi_\delta(x, T) \, dx 
\]
\[
= \int_\Omega \left( \alpha(v_{0\epsilon} - u_\epsilon) + H_\epsilon(v_{0\epsilon}) - \chi_0 \right) \xi_\delta(x, 0) \, dx \leq 0. \tag{2.20}
\]
By Lemma 2.1 of [16], we obtain from (2.20)
\[
v_{0\epsilon} \leq u_\epsilon \quad \text{a.e. in } Q \tag{2.21}
\]
and by the monotonicity of $H_\epsilon$, we get
\[
H_\epsilon(v_{0\epsilon}) \leq H_\epsilon(u_\epsilon) \quad \text{a.e. in } Q. \tag{2.22}
\]
Arguing as above and using (2.21)–(2.22), we obtain by passing to the limit up to a subsequence, that we have for any $\xi \in \mathcal{D}(Q)$ with $\xi \geq 0$
\[
\int_Q (u - v_0)\xi \, dx \, dt = \lim_{\epsilon \to 0} \int_Q (u_\epsilon - v_{0\epsilon})\xi \, dx \, dt \geq 0,
\]
\[
\int_Q (\chi - \gamma_0)\xi \, dx \, dt = \lim_{\epsilon \to 0} \int_Q (H_\epsilon(u_\epsilon) - H_\epsilon(v_{0\epsilon}))\xi \, dx \, dt \geq 0,
\]
which leads to
\[
v_0 \leq u \quad \text{a.e. in } Q \tag{2.23}
\]
\[
\gamma_0 \leq \chi \quad \text{a.e. in } Q. \tag{2.24}
\]
Combining (2.18)–(2.19) and (2.23)–(2.24), we obtain (2.8)–(2.9). \qed

**Remark 2.2** Assume that $a(x)e \in C^{0,1}(\overline{\Omega})$. Then we get from (2.8)–(2.9) taking into account (2.5)
\[
u(x, t) > 0 \quad \text{if } 0 < x_2 < \epsilon_0 \tag{2.25}
\]
\[
u(x, t) = \chi(x, t) = 0 \quad \text{if } K - \epsilon_0 < x_2 < K. \tag{2.26}
\]
3 Uniqueness of the solution in rectangular dams

In this section we assume that

\[ a(x) \in C^{0,1}(\Omega), \quad \text{with} \quad N = \sup_{i,j,k} |(a_{ij})_{x_k}|_{\infty}. \tag{3.1} \]

\[ a(x) \text{ is a symmetric matrix.} \tag{3.2} \]

Here is our main result.

**Theorem 3.1** Assume that (2.2) and (3.1)–(3.2) hold. Then the solution of the problem (P) associated with the initial data \((u_0, \chi_0)\) and satisfying (2.25)–(2.26) is unique.

Let \((u_1, \chi_1)\) and \((u_2, \chi_2)\) be two solutions of the problem (P) satisfying (2.25)–(2.26). Set

\[ w = u_1 - u_2 \quad \text{and} \quad \eta = a w + \chi_1 - \chi_2. \]

We consider the following problem

Find \(v \in L^2(0, T; H^1(\Omega))\) such that:

\[ \text{div}(a(x)\nabla v) = -\eta \quad \text{in} \quad \Omega \quad \text{for each} \quad t \in [0, T] \tag{3.3} \]

\[ v = 0 \quad \text{on} \quad \Gamma_2 \tag{3.4} \]

\[ a(x)\nabla v \cdot \nu = 0 \quad \text{on} \quad \Gamma_1. \tag{3.5} \]

Then we have

**Lemma 3.1** There exists a unique weak solution of the problem (3.3)–(3.5).

**Proof** First, we observe (see [14]) that \(a u_i + \chi_i \in C^0([0, T]; L^2(\Omega)), i = 0, 1.\) As a consequence, we have \(\eta \in C^0([0, T]; L^2(\Omega)).\) Let \(V = \{ v \in H^1(\Omega) / v = 0 \text{ on } \Gamma_2 \}.\) Then \(V\) is a Hilbert space under the \(H^1(\Omega)\) norm, and by applying Lax-Milgram’s Theorem and taking into account (1.2)–(1.3), there exists for each \(t \in [0, T]\) a unique solution \(v(x, t)\) of the following problem

\[ \begin{cases} v(., t) \in V \quad & \int_{\Omega} a(x)\nabla v(x, t) \cdot \nabla \xi \, dx = \int_{\Omega} \eta(x, t) \xi \, dx \quad \forall \xi \in V. \tag{3.6} \end{cases} \]

Choosing \(\xi \in \mathcal{D}(\Omega)\) in (3.6), we obtain (3.3) in \(\mathcal{D}'(\Omega)\) and therefore in \(C^0([0, T]; L^2(\Omega)).\) (3.4) is satisfied in the trace sense. Writing (3.6) for \(\xi \in C^\infty(\bar{\Omega})\) with \(\xi = 0\) on \(\Gamma_2,\) and taking into account (3.3), we obtain (3.5) in \(H^{-1/2}(\Gamma_1)\).

Choosing \(v\) as a test function in (3.6) and using (1.2), Hölder and Poincaré’s inequalities, we obtain

\[ \int_{\Omega} |\nabla v(x, t)|^2 \, dx \leq \frac{1}{\lambda^2} \int_{\Omega} |\eta(x, t)|^2 \, dx. \tag{3.7} \]

Integrating (3.7) over the interval \([0, T]\) and using the fact that \(\chi_i \in L^\infty(Q), u_i \in L^\infty(0, T; L^\infty(\Omega))\) (see [14]), we obtain

\[ \int_Q |\nabla v(x, t)|^2 \, dx \, dt \leq \frac{1}{\lambda^2} \int_0^T \int_{\Omega} |\eta(x, t)|^2 \, dx \, dt \leq \frac{T|\eta|_{L^\infty(0, T; L^2(\Omega))}}{\lambda^2}. \]

Using Poincaré’s inequality, we obtain \(v \in L^2(0, T; H^1(\Omega)).\) Hence \(v\) is the unique solution of (3.3)–(3.5). □
Remark 3.1 By the regularity theory (see [9] for example), the solution \( v \) of the problem (3.3)–(3.5) satisfies \( v \in L^2(0, T; C^1(\Omega \cup \Gamma_1 \cup \Gamma_2)) \cap L^2(0, T; H^2(\Omega \cup \Gamma_1 \cup \Gamma_2)). \)

Now, let us denote by \( \tilde{g} \) the mean value with respect to \( t \) of a function \( g(x, t) \) defined by
\[
\tilde{g}(x, t) = \frac{1}{h} \int_t^{t+h} g(x, s) ds.
\]
Then we have
\[
\tilde{g} \to g \quad \text{as} \quad h \to 0
\]
\[
\frac{\partial \tilde{g}}{\partial t} = \frac{1}{h}(g(x, t + h) - g(x, t)). \tag{3.8}
\]
Moreover, it is easy to check that
\[
div(a(x)\nabla \tilde{v}) = -\tilde{\eta} \quad \text{in} \quad \Omega \quad \text{for all} \quad t \in [0, T] \tag{3.9}
\]
\[
\tilde{v} = 0 \quad \text{on} \quad \Gamma_2 \tag{3.10}
\]
\[
a(x)\nabla \tilde{v}.v = 0 \quad \text{on} \quad \Gamma_1. \tag{3.11}
\]
Since \( \chi_i = 1 \) \((i = 1, 2)\) in a neighborhood of \( \Gamma_1 \), we obtain from \((P)iii\)
\[
(au_i + \chi_i)t = div(a(x)(\nabla u_i + \chi_i e)) \quad \text{in} \quad \mathcal{D}'(Q),
\]
\[
a(x)(\nabla u_i + e).v = 0 \quad \text{on} \quad \Gamma_1, \quad i = 1, 2. \tag{3.12}
\]
Using the fact that \( u_1 = u_2 \) on \( \Sigma_2 \), and writing the previous two equations for \((u_1, \chi_1)\) and \((u_2, \chi_2)\) and subtracting them from each other, we get
\[
(aw + \chi_1 - \chi_2)t = div(a(x)(\nabla w + (\chi_1 - \chi_2)e)) \quad \text{in} \quad \mathcal{D}'(Q). \tag{3.13}
\]
\[
w = 0 \quad \text{on} \quad \Sigma_2 \tag{3.14}
\]
\[
a(x)(\nabla w).v = 0 \quad \text{on} \quad \Sigma_1. \tag{3.15}
\]
Then we have

Lemma 3.2 For \( h \) small enough we have
\[
\tilde{\eta}_t = div(a(x)(\nabla \tilde{w} + (\tilde{\chi}_1 - \tilde{\chi}_2)e)) \quad \text{in} \quad \mathcal{D}'(Q) \tag{3.16}
\]
\[
\tilde{w} = 0 \quad \text{on} \quad \Sigma_2 \tag{3.17}
\]
\[
a(x)\nabla \tilde{w}.v = 0 \quad \text{on} \quad \Sigma_1. \tag{3.18}
\]

Proof (3.16) and (3.17) are a direct consequence of (3.13) and (3.14). To establish (3.15), let \( \zeta \in \mathcal{D}(Q) \) such that for some \( \tau_0 > 0 \), \( supp(\zeta) \subset \Omega \times (\tau_0, T - \tau_0) \). We denote by \( \tilde{\zeta} \) the function defined by \( \tilde{\zeta}(x, t) = \frac{1}{h} \int_{t-h}^{t} \zeta(x, s) ds \). Since for \(|h| < \tau_0/2\), the functions \( \pm \tilde{\zeta} \) are test functions for problem \((P)\), we obtain for \( i = 1, 2 \)
\[
\int_Q a(x)(\nabla u_i + \chi_i e) \cdot \nabla \tilde{\zeta} dx dt = \int_Q (au_i + \chi_i) \tilde{\zeta} dx dt. \tag{3.18}
\]
For the right hand side of (3.18), we have by using change of variables
\[
\int_Q (\alpha u_i + \chi_i) \tilde{\zeta}_i dx dt = \int_Q (\alpha u_i + \chi_i) \frac{1}{h} \zeta(x, t) dx dt - \int_Q (\alpha u_i + \chi_i) \frac{1}{h} \zeta(x, t-h) dx dt
\]
\[
= \int_Q (\alpha u_i + \chi_i)(x, t) \frac{1}{h} \zeta(x, t) dx dt
\]
\[
- \int_Q \int_{-h}^{T-h} \frac{1}{h} (\alpha u_i(x, t+h) + \chi_i(x, t+h)) \zeta(x, t) dx dt
\]
\[
= \int_Q (\alpha u_i + \chi_i)(x, t) \frac{1}{h} \zeta(x, t) dx dt
\]
\[
- \int_Q \int_{-h}^{T} \frac{1}{h} (\alpha u_i(x, t+h) + \chi_i(x, t+h)) \zeta(x, t) dx dt
\]
\[
= - \int_Q (\alpha u_i + \chi_i)(x, t - h) \zeta(x, t) dx dt.
\]

For the left hand side of (3.18), we have by integrating by parts
\[
\int_Q a(x) (\nabla u_i + \chi_i e) \cdot \nabla \tilde{\zeta}_i dx dt = - \int_Q \left[ \int_{0}^{t} a(x)(\nabla u_i + \chi_i e) dx \right] \cdot \nabla \tilde{\zeta}_i dx dt
\]
\[
= - \int_Q \left[ \int_{0}^{T} a(x)(\nabla u_i + \chi_i e) dx \right] \cdot \nabla \left( \frac{1}{h} (\zeta(x, t) - \zeta(x, t-h)) \right) dx dt
\]
\[
= - \int_Q \frac{1}{h} \left[ \int_{0}^{T} a(x)(\nabla u_i + \chi_i e) dx \right] \cdot \nabla \zeta(x, t) dx dt
\]
\[
+ \int_Q \frac{1}{h} \left[ \int_{0}^{T} a(x)(\nabla u_i + \chi_i e) dx \right] \cdot \nabla \zeta(x, t-h) dx dt
\]
\[
= - \int_Q \frac{1}{h} \left[ \int_{0}^{T} a(x)(\nabla u_i + \chi_i e) dx \right] \cdot \nabla \zeta(x, t) dx dt
\]
\[
+ \int_Q \frac{1}{h} \left[ \int_{0}^{T} a(x)(\nabla u_i + \chi_i e) dx \right] \cdot \nabla \zeta(x, t) dx dt
\]
\[
= \int_Q \frac{1}{h} \left[ \int_{0}^{T} a(x)(\nabla u_i + \chi_i e) dx \right] \cdot \nabla \zeta(x, t) dx dt
\]
\[
= \int_Q a(x)(\nabla \tilde{u}_i + \tilde{\chi}_i e) \cdot \nabla \zeta(x, t) dx dt.
\]

Then we deduce from (3.18)–(3.20) that
\[
\int_Q a(x)(\nabla \tilde{u}_i + \tilde{\chi}_i e) \cdot \nabla \zeta(x, t) dx dt = - \int_Q (\alpha u_i + \chi_i) \zeta(x, t) dx dt.
\]

Writing the last equation for \( i = 1, 2 \) and subtracting the two equations, we get
\[
\int_Q a(x)(\nabla \tilde{w} + (\tilde{\chi}_1 - \tilde{\chi}_2) e) \cdot \nabla \zeta(x, t) dx dt = - \int_Q \tilde{\eta}_t \zeta(x, t) dx dt
\]
which is (3.15).

\[\square\]

To prove Theorem 3.1, we need two more lemmas.
Lemma 3.3
\[
\frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} a(x) \nabla \tilde{v}. \nabla \tilde{v} dx + \int_{\Omega} \tilde{w} \eta dx = \int_{\Omega} \alpha \tilde{w} (a_{11} \tilde{v}_{x_1} + a_{22} \tilde{v}_{x_2}) dx \\
+ \int_{\Omega} (a_{11} \tilde{v}_{x_1} + a_{12} \tilde{v}_{x_2}) (a_{12} \tilde{v}_{x_1} + a_{22} \tilde{v}_{x_2}) dx \\
+ \int_{\Omega} (a_{12} \tilde{v}_{x_1} + a_{22} \tilde{v}_{x_2}) (a_{12} \tilde{v}_{x_1} + a_{22} \tilde{v}_{x_2}) dx.
\]
(3.21)

Proof From (3.9) and (3.15) we derive
\[
-d \text{div}(a(x) \nabla \tilde{w}) = d \text{div}(a(x) (\nabla \tilde{w} + (\tilde{x}_1 - \tilde{x}_2)e)) \text{ in } \mathcal{D}'(\Omega).
\]
(3.22)
Using (3.10)–(3.11), and taking into account (3.2), we obtain
\[
\left< -d \text{div}(a(x) (\nabla \tilde{v})_r), \tilde{v} \right> = \int_{\Omega} a(x) \nabla \tilde{v}_r. \nabla \tilde{v} dx - \int_{\partial \Omega} a(x) \nabla \tilde{v}_r. \nu \tilde{v} d\sigma(x) \\
= \int_{\Omega} a(x) \nabla \tilde{v}_r. \nabla \tilde{v} dx = \frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} a(x) \nabla \tilde{v}. \nabla \tilde{v} dx.
\]
(3.23)
Similarly, we get by using (3.10), (3.17), and taking into account (3.2)
\[
\langle \text{div}(a(x) \nabla \tilde{w}), \tilde{v} \rangle = -\int_{\Omega} a(x) \tilde{w}. \nabla \tilde{v} dx + \int_{\partial \Omega} a(x) \tilde{w}. \nu \tilde{v} d\sigma(x) \\
= -\int_{\Omega} a(x) \tilde{w}. \nabla \tilde{v} dx = -\int_{\Omega} a(x) \nabla \tilde{v}. \nabla \tilde{w} dx
\]
which can be written using (3.9) as
\[
\left< \text{div}(a(x) \nabla \tilde{w}), \tilde{v} \right> = -\int_{\Omega} \tilde{w} \eta dx.
\]
(3.24)
Similarly, using (3.10) and the fact that \( u_1, u_2 \) satisfy (2.25)–(2.26), we get for \( \Omega_0 = (0, L) \times (\epsilon_0, K - \epsilon_0) \)
\[
\langle \text{div}((\tilde{x}_1 - \tilde{x}_2) a(x) e), \tilde{v} \rangle = -\int_{\Omega_0} (\tilde{x}_1 - \tilde{x}_2) a(x) e. \nabla \tilde{v} dx \\
= \int_{\Omega_0} (-\tilde{\eta} + \alpha \tilde{w}) a(x) e. \nabla \tilde{v} dx \\
= \int_{\Omega_0} \text{div}(a(x) \nabla \tilde{v}) a(x) e. \nabla \tilde{v} dx + \int_{\Omega_0} \alpha \tilde{w} a(x) e. \nabla \tilde{v} dx \\
= \int_{\Omega_0} \text{div}(a(x) \nabla \tilde{v}) a(x) e. \nabla \tilde{v} dx + \int_{\Omega_0} \alpha \tilde{w} a(x) e. \nabla \tilde{v} dx \\
= \int_{\Omega_0} (a_{11} \tilde{v}_{x_1} + a_{12} \tilde{v}_{x_2}) (a_{12} \tilde{v}_{x_1} + a_{22} \tilde{v}_{x_2}) dx \\
+ \int_{\Omega_0} (a_{12} \tilde{v}_{x_1} + a_{22} \tilde{v}_{x_2}) (a_{12} \tilde{v}_{x_1} + a_{22} \tilde{v}_{x_2}) dx \\
+ \int_{\Omega_0} \alpha \tilde{w} (a_{12} \tilde{v}_{x_1} + a_{22} \tilde{v}_{x_2}) dx.
\]
(3.25)
Hence by combining (3.22)–(3.25), we get (3.21). □
Lemma 3.4 There exists a positive constant $C$ such that

$$\frac{\partial}{\partial t} \int_{\Omega} a(x) \nabla \tilde{v} \cdot \nabla \tilde{v} dx + 2 \int_{\Omega} \tilde{w}(\tilde{x}_1 - \tilde{x}_2) dx \leq C \int_{\Omega} a(x) \nabla \tilde{v} \cdot \nabla \tilde{v} dx. \quad (3.26)$$

The proof of Lemma 3.4 requires a lemma.

Lemma 3.5 There exists a positive constant $C$ such that

$$\int_{\epsilon_0}^{K-\epsilon_0} |\nabla \tilde{v}(0, x_2)|^2 dx_2 \leq C \int_{\Omega} |\nabla \tilde{v}|^2 dx \quad (3.27)$$

$$\int_{\epsilon_0}^{K-\epsilon_0} |\nabla \tilde{v}(L, x_2)|^2 dx_2 \leq C \int_{\Omega} |\nabla \tilde{v}|^2 dx \quad (3.28)$$

$$\int_{0}^{L} |\nabla \tilde{v}(x_1, \epsilon_0)|^2 dx_1 \leq C \int_{\Omega} |\nabla \tilde{v}|^2 dx \quad (3.29)$$

$$\int_{0}^{L} |\nabla \tilde{v}(x_1, K - \epsilon_0)|^2 dx_1 \leq C \int_{\Omega} |\nabla \tilde{v}|^2 dx. \quad (3.30)$$

Proof (i) Since $\tilde{v} \in C^1(\tilde{\Omega}_0)$, we have

$$\int_{\epsilon_0}^{K-\epsilon_0} |\nabla \tilde{v}(0, x_2)|^2 dx_2 = \lim_{\delta \to 0} \frac{1}{\delta} \int_{0}^{\delta} \int_{\epsilon_0}^{K-\epsilon_0} |\nabla \tilde{v}|^2 dx \quad \text{for } \delta_1 > 0 \text{ enough small} \quad (3.27)$$

which gives (3.27). In the same way we establish (3.28), (3.29), and (3.30).

Proof of Lemma 3.4. We shall estimate the three integrals in the right hand side of (3.21). First, we obtain by applying Young’s inequality and using (1.2)–(1.3)

$$\int_{\Omega_0} \alpha \tilde{w}(a_{12} \tilde{v}_{x_1} + a_{22} \tilde{v}_{x_2}) dx \leq \frac{\alpha}{2} \int_{\Omega_0} \tilde{w}^2 dx + \frac{\alpha}{2} \int_{\Omega_0} (a_{12} \tilde{v}_{x_1} + a_{22} \tilde{v}_{x_2})^2 dx \quad (3.31)$$

Next, we have

$$\int_{\Omega_0} (a_{11} \tilde{v}_{x_1} + a_{12} \tilde{v}_{x_2}) dx = \int_{\Omega_0} (a_{11} \tilde{v}_{x_1}) (a_{12} \tilde{v}_{x_1}) dx$$

$$+ \int_{\Omega_0} (a_{11} \tilde{v}_{x_1}) (a_{22} \tilde{v}_{x_2}) dx + \int_{\Omega_0} (a_{12} \tilde{v}_{x_2}) (a_{12} \tilde{v}_{x_1}) dx + \int_{\Omega_0} (a_{12} \tilde{v}_{x_2}) (a_{22} \tilde{v}_{x_2}) dx \quad (3.32)$$

$$= I_1 + I_2 + I_3 + I_4.$$
Let us estimate the integrals $I_i$. Expanding and integrating by parts, and using (1.2)–(1.3), (3.1), and (3.28), we obtain for a positive constant $C_1$

$$
I_1 = \int_{\Omega_0} (a_{11})_{x_1} a_{12} \tilde{v}_{x_1}^2 dx + \frac{1}{2} \int_{\Omega_0} a_{11} a_{12} (\tilde{v}_{x_1})_{x_1} dx
= \int_{\Omega_0} (a_{11})_{x_1} a_{12} \tilde{v}_{x_1}^2 dx - \frac{1}{2} \int_{\Omega_0} (a_{11} a_{12})_{x_1} \tilde{v}_{x_1}^2 dx
+ \frac{1}{2} \int_{\epsilon_0}^{K-\epsilon_0} (a_{11} a_{12})_{x_1} (L, x_2) dx_2 - \frac{1}{2} \int_{\epsilon_0}^{K-\epsilon_0} (a_{11} a_{12} \tilde{v}_{x_1}^2) (0, x_2) dx_2
\leq \frac{1}{2} \int_{\Omega_0} (a_{11})_{x_1} a_{12} \tilde{v}_{x_1}^2 dx - \frac{1}{2} \int_{\Omega_0} a_{11} (a_{12})_{x_1} \tilde{v}_{x_1}^2 dx + \frac{\Lambda^2}{2} \int_{\epsilon_0}^{K-\epsilon_0} \tilde{v}_{x_1}^2 (L, x_2) dx_2
\leq \Lambda N \int_{\Omega_0} \tilde{v}_{x_1}^2 dx + \frac{\Lambda^2}{2} \int_{\epsilon_0}^{K-\epsilon_0} \tilde{v}_{x_1}^2 (L, x_2) dx_2 \leq C_1 \int_{\Omega} a(x) \nabla \tilde{v} \cdot \nabla \tilde{u} dx. \quad (3.33)
$$

Note that since $\tilde{v}(0, x_2) = \tilde{v}(L, x_2) = 0$ for $0 < x_2 < K$, we have

$$
\tilde{v}_{x_2} (0, x_2) = \tilde{v}_{x_2} (L, x_2) = 0 \quad \text{for} \quad 0 < x_2 < K. \quad (3.34)
$$

Integrating by parts and using (3.34), we obtain

$$
I_2 = -\int_{\Omega_0} a_{11} \tilde{v}_{x_1} (a_{22} \tilde{v}_{x_2})_{x_1} dx = -\int_{\Omega_0} a_{11} (a_{22})_{x_1} \tilde{v}_{x_1} \tilde{v}_{x_2} dx - \frac{1}{2} \int_{\Omega_0} a_{11} a_{22} (\tilde{v}_{x_1})_{x_2} dx
= -\int_{\Omega_0} a_{11} (a_{22})_{x_1} \tilde{v}_{x_1} \tilde{v}_{x_2} dx + \frac{1}{2} \int_{\Omega_0} a_{11} (a_{22})_{x_2} \tilde{v}_{x_1}^2 dx + \frac{1}{2} \int_{\Omega_0} (a_{11})_{x_2} a_{22} \tilde{v}_{x_1}^2 dx
- \frac{1}{2} \int_{0}^{L} (a_{11} a_{22} \tilde{v}_{x_1}^2) (x_1, K - \epsilon_0) dx_1 + \frac{1}{2} \int_{0}^{L} (a_{11} a_{22} \tilde{v}_{x_1}^2) (x_1, \epsilon_0) dx_1. \quad (3.35)
$$

Using Young’s inequality, (1.2)–(1.3), (3.1), and (3.29), we obtain from (3.35), for a positive constant $C_2$

$$
I_2 \leq \Lambda N \int_{\Omega_0} |\tilde{v}_{x_1}| \cdot |\tilde{v}_{x_2}| dx + \Lambda N \int_{\Omega_0} \tilde{v}_{x_1}^2 dx + \frac{\Lambda^2}{2} \int_{0}^{L} \tilde{v}_{x_1}^2 (x_1, \epsilon_0) dx_1
\leq \frac{3\Lambda N}{2} \int_{\Omega_0} |\nabla \tilde{v}|^2 dx + \frac{\Lambda^2}{2} \int_{0}^{L} \tilde{v}_{x_1}^2 (x_1, \epsilon_0) dx_1 \leq C_2 \int_{\Omega} a(x) \nabla \tilde{v} \cdot \nabla \tilde{u} dx. \quad (3.36)
$$

Expanding and integrating by parts, and using Young’s inequality, (1.2)–(1.3), (3.1), and (3.30), we obtain for a positive constant $C_3$

$$
I_3 = \int_{\Omega_0} (a_{12})_{x_1} a_{12} \tilde{v}_{x_1} \tilde{v}_{x_2} dx + \frac{1}{2} \int_{\Omega_0} a_{12}^2 (\tilde{v}_{x_1})_{x_2} dx
= \int_{\Omega_0} (a_{12})_{x_1} a_{12} \tilde{v}_{x_1} \tilde{v}_{x_2} dx - \frac{1}{2} \int_{\Omega_0} (a_{12}^2)_{x_2} \tilde{v}_{x_1}^2 dx
+ \frac{1}{2} \int_{0}^{L} (a_{12}^2 \tilde{v}_{x_1}^2) (x_1, K - \epsilon_0) dx_1 - \frac{1}{2} \int_{0}^{L} (a_{12}^2 \tilde{v}_{x_1}^2) (x_1, \epsilon_0) dx_1
\leq \frac{\Lambda N}{2} \int_{\Omega_0} \tilde{v}_{x_1}^2 dx + \frac{\Lambda^2}{2} \int_{0}^{L} \tilde{v}_{x_1}^2 (x_1, K - \epsilon_0) dx_1
\leq C_3 \int_{\Omega} a(x) \nabla \tilde{v} \cdot \nabla \tilde{u} dx. \quad (3.37)
$$
Integrating by parts and using (3.34), and taking into account (1.2)–(1.3) and (3.1), we obtain for a positive constant $C_4$

$$I_4 = \int_{\Omega_0} (a_{12})_{x_1} a_{22} \bar{v}_{x_2}^2 dx + \frac{1}{2} \int_{\Omega_0} a_{12} a_{22} (\bar{v}_{x_2}^2)_{x_1} dx$$
$$= \int_{\Omega_0} (a_{12})_{x_1} a_{22} \bar{v}_{x_2}^2 dx - \frac{1}{2} \int_{\Omega_0} (a_{12} a_{22})_{x_1} \bar{v}_{x_2}^2 dx$$
$$= \frac{1}{2} \int_{\Omega_0} (a_{12})_{x_1} a_{22} - a_{12} (a_{22})_{x_1} \bar{v}_{x_2}^2 dx \leq C_4 \int_{\Omega} a(x) \nabla \tilde{\nu}, \nabla \nu dx. \quad (3.38)$$

To estimate the last integral in the right hand side of (3.21), we integrate by parts and use (3.29–3.30) and (1.2). We obtain for a positive constant $C_5$

$$\int_{\Omega_0} (a_{12} \tilde{\nu}_{x_1} + a_{22} \tilde{\nu}_{x_2}) (a_{12} \tilde{\nu}_{x_1} + a_{22} \tilde{\nu}_{x_2}) dx = \frac{1}{2} \int_{\Omega_0} ((a_{12} \tilde{\nu}_{x_1} + a_{22} \tilde{\nu}_{x_2})^2)_{x_2} dx$$
$$= \frac{1}{2} \int_{0}^{L} (a_{12} \tilde{\nu}_{x_1} + a_{22} \tilde{\nu}_{x_2})^2 (x_1, K - \epsilon_0) dx_1 - \frac{1}{2} \int_{0}^{L} (a_{12} \tilde{\nu}_{x_1} + a_{22} \tilde{\nu}_{x_2})^2 (x_1, \epsilon_0) dx_1$$
$$\leq \frac{1}{2} \int_{0}^{L} ((a_{12} \tilde{\nu}_{x_1})^2 + (a_{22} \tilde{\nu}_{x_2})^2) (x_1, K - \epsilon_0) dx_1$$
$$\leq \alpha^2 \int_{0}^{L} ((\tilde{\nu}_{x_1})^2 + (\tilde{\nu}_{x_2})^2) (x_1, K - \epsilon_0) dx_1 \leq C_5 \int_{\Omega} a(x) \nabla \tilde{\nu}, \nabla \nu dx. \quad (3.39)$$

Finally, combining (3.21), (3.31)–(3.33), and (3.35)–(3.39), we get for a positive constant $C$

$$\frac{\partial}{\partial t} \int_{\Omega} a(x) \nabla \tilde{\nu}, \nabla \nu dx + 2 \int_{\Omega} \tilde{\nu} (\chi_1 - \chi_2) dx \leq C \int_{\Omega} a(x) \nabla \tilde{\nu}, \nabla \nu dx,$$ which is (3.26). □

Proof of Theorem 3.1. First, integrating (3.26) from 0 to $t$ and letting $h \to 0$, we get

$$\int_{\Omega} a(x) \nabla v, \nabla v dx + 2 \int_{0}^{t} \int_{\Omega} w (\chi_1 - \chi_2) dx ds \leq C \int_{0}^{t} \int_{\Omega} a(x) \nabla v, \nabla v dx ds. \quad (3.40)$$

Next, we observe that since $u_i \in H(\chi_i)$ a.e. in $Q$, we have

$$w (\chi_1 - \chi_2) \geq 0 \quad \text{a.e. in } Q. \quad (3.41)$$

Setting $F(t) = \int_{0}^{t} \int_{\Omega} a(x) \nabla v, \nabla v dx ds$, we deduce from (3.40)–(3.41) that

$$F'(t) \leq CF(t) \quad \forall t \in [0, T]. \quad (3.42)$$

Integrating (3.42), we get since $F(0) = 0$, $0 \leq F(t) \leq F(0)e^{Ct} = 0 \quad \forall t \in [0, T]$, or

$$\int_{0}^{t} \int_{\Omega} a(x) \nabla v, \nabla v dx ds = 0 \quad \forall t \in [0, T].$$

Using (1.2), we obtain $\nabla v = 0$ a.e. in $Q$. Taking into account that $v = 0$ on $\Gamma_2 \subset \partial Q$ and the connectedness of $\Omega$, we obtain $v = 0$ in $Q$. Going back to (3.3), we obtain $\eta = 0$ a.e. in $Q$, which reads

$$w + \chi_1 - \chi_2 = 0 \quad \text{in } Q. \quad (3.43)$$

Multiplying (3.43) by $w$, we get $aw^2 + w (\chi_1 - \chi_2) = 0$ a.e. in $Q$. Taking into account (3.41), we obtain $w^2 = 0$ a.e. in $Q$, or $u_1 = u_2$ a.e. in $Q$. Finally, we obtain from (3.43) that $\chi_1 = \chi_2$ a.e. in $Q$. This achieves the proof. □
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