Non-conformal brane plane waves and entanglement entropy

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Abstract

Following [arXiv:1202.5935 [hep-th]] and [arXiv:1212.4328 [hep-th]], we study non-conformal brane plane wave backgrounds dual to strongly coupled gauge theories with constant energy flux and holographic entanglement entropy for strip subsystems in them. We find that for the strip direction along the direction of the energy flux, the finite cutoff-independent part of entanglement entropy can be estimated in terms of a dimensionless combination of the energy density and the strip dimensions, alongwith an effective scale-dependent number of degrees of freedom. For the strip orthogonal to the flux direction, there are indications of phase transitions. We also briefly discuss NS5-brane backgrounds corresponding to plane wave states in little string theories.
1 Introduction and summary

Entanglement entropy has come to be a useful tool in discussions of nonrelativistic holography, in light of the simple prescription of Ryu-Takayanagi \[1, 2, 3\]. This states that the holographic entanglement entropy of a subsystem in the boundary \(d\)-dim conformal field theory is

\[ S_A = \frac{1}{4G_{d+1}} \text{Area}(\gamma_A), \]

where \(\gamma_A\) is a \((d-1)\)-dim minimal surface bounding the subsystem and extending into the bulk \(d+1\)-dim anti de Sitter spacetime. This prescription has been checked extensively with success. In the context of an explicit gauge/string realization of \(AdS_{d+1}/CFT_d\), we identify \(G_{d+1} = \frac{G_{10}}{V_{9-d}}\) after dimensionally reducing on the compact \((9-d)\)-dim space: correspondingly, the codim-2 minimal surface in question wraps this compact space too, and should now be thought of as an 8-dim surface.

It is natural to ask how this generalizes to non-conformal theories arising on non-conformal D-branes: \[2\] proposed that the entanglement entropy now is

\[ S_A = \frac{1}{4G_{10}} \int d^8x e^{-2\Phi} \sqrt{g}, \]

\(\Phi\) being the dilaton and \(g\) the string frame metric. This appears sensible and \[2\] studied examples vindicating this expression.

Some questions arise in this context. Firstly, the natural quantity in an M-theory context is the 11-dim Einstein metric, which under dimensional reduction (to a Type II string description) on the 11-th circle is again most naturally related to the 10-dim Einstein metric. Secondly, the natural generalization of the Ryu-Takayanagi prescription in any effective gravity model again involves the lower dimensional Einstein metric. Of relevance in this context are certain classes of lightlike \(AdS \times S\) deformations, \(AdS\) plane waves \[4, 5\] (see also \[6\]), which have simple dual microscopic interpretations as conformal field theory excited states with constant energy flux \(T_{++}\). Some of these exhibit deviations from the area law \[7\] of entanglement entropy holographically: see \[8\] for a systematic treatment of strip-subsystems using the covariant formulation \[9\]. Upon appropriate dimensional reduction, \(AdS\) plane waves become gravity backgrounds exhibiting hyperscaling violation: the \(AdS_5\) case exhibits logarithmic behaviour. Such spacetimes arise in effective Einstein-Maxwell-scalar theories \[10\] (see also \[11, 12\]) with some of them argued to have signatures of hidden Fermi surfaces \[13, 14\]. In this context again, the Einstein metric appears natural for holographic entanglement entropy e.g. \[13, 11\].

This suggests that the natural generalization of the Ryu-Takayanagi prescription in M-theory should be

\[ S_A = \frac{1}{4G_{11}} \int d^9x \sqrt{g} \]  

(1)

where \(g\) is the spatial Einstein metric induced on the 9-dim surface in the 11-dim bulk spacetime. This trivially agrees with the familiar expressions for M2- and M5-brane conformal theories after reducing on the spheres. Furthermore, a 9-dim minimal surface in M-theory
must wrap the 11-th circle to have a sensible interpretation as an 8-dim surface in the Type II description after dimensional reduction on the 11-th circle. This implies

\[ S_{11}^{A} \rightarrow \frac{1}{4G_{10}} \int d^{8}x \sqrt{g_{10}^{E}} = \frac{1}{4G_{10}} \int d^{8}x \ e^{-2\Phi} \sqrt{g_{10}^{st}}. \]  

(2)

The first expression involving the 10-dim Einstein metric is, happily, identical to the second one, proposed in [2], when we recall that \( g_{10}^{E} = e^{-\Phi/2} g_{10}^{st} \). Alternatively, the 11-dim metric is

\[ ds_{11}^{2} = e^{-2\Phi/3} ds_{10, st}^{2} + e^{4\Phi/3} (dx_{11} + A)^{2} \] : dimensionally reducing (1) on the 11-th circle gives

\[ S_{A} \rightarrow \int \frac{d^{8}xdx_{11}}{4G_{11}} \ \sqrt{g_{11,11}} \ e^{-42\Phi/3} \sqrt{g_{10}^{st}} = \frac{1}{4G_{10}} \int d^{8}x \ e^{-2\Phi} \sqrt{g_{10}^{st}}. \]  

(3)

The last expression pertains to Type II string/supergravity and is in fact the one proposed in [2], which we have recovered from our expression (1).

Proposing that the holographic entanglement entropy (1) (2) be expressed uniformly in terms of the Einstein metric in 10- or 11-dim also appears consistent with the expectation that the corresponding expression for entanglement entropy in a lower dimensional effective gravity theory obtained after dimensional reduction on some compact space involves the Einstein metric. The study of non-conformal brane ground states and associated hyperscaling violating spacetimes in [11] in a sense vindicates this.

In what follows, we use the above prescriptions to study entanglement entropy for strip-subsystems in plane wave excited states in non-conformal \( Dp \)-brane theories, following [4, 8]. These bulk spacetimes arise in certain zero-temperature double scaling limits of boosted black \( Dp \)-branes [5], and are dual to strongly coupled gauge theories with constant energy flux \( T_{++} \).

For the strip along the energy flux direction, the leading divergent part is the same as for ground states [2, 15]. The finite cutoff-independent piece [1, 2, 16] grows with subsystem size, in accord with the intuition that energy pumped into the system, and thereby entanglement, increases as the size increases. It can be written as a dimensionless combination of the energy density \( Q \) and the strip length/width, and further involves an effective scale-dependent number of degrees of freedom \( N_{eff}(l) = N^{2} \left( \frac{g_{YM}^{2}}{l_{p}^{3}} \right)^{\frac{p-3}{2}} \) which also appeared for ground states [15]. We also find consistency with the expectation that the finite cutoff-independent piece decrease under the renormalization group flow in the \( Dp \)-brane phase diagram. For the strip orthogonal to the flux direction, there are indications of phase transitions, constrained however by the multiple length scales here, unlike the conformal case. Finally we point out NS5-brane backgrounds describing plane wave excited states in the dual little string theories.
2 Plane wave excited states

2.1 Conformal theories

First we consider $AdS_{d+1}$ plane waves arising on the conformal $D3, M2, M5$-branes: the bulk spacetimes corresponding to normalizable deformations of $AdS_{d+1} \times S$ are

$$ds^2 = \frac{R^2}{r^2}(-2dx^+dx^- + dx_i^2 + dr^2) + R^2\tilde{Q}r^{d-2}(dx^+)^2 + R^2d\Omega^2. \quad (4)$$

These are likely $\alpha'$-exact backgrounds. The parameter $\tilde{Q} \gg 0$ gives rise to an energy-momentum density $T_{++}$ in the boundary CFT. For uniformizing with the non-conformal brane description to follow, it is useful to redefine the constant in the $g_{++}$ component as

$$\tilde{Q} = \frac{Q}{G_{d+1}} \quad (D3), \quad \tilde{Q} = \frac{Q}{N^{3/2}} \quad (M2), \quad \tilde{Q} = \frac{Q}{N^3} \quad (M5). \quad (5)$$

Recalling the $N$-scaling of the entropy of the various conformal branes, we see $\tilde{Q}$ is the energy-momentum density per (nonabelian) degree of freedom. This gives

$$ds^2 = \frac{R^2}{r^2}(-2dx^+dx^- + dx_i^2 + dr^2) + \frac{G_{d+1}Q}{R^{d-3}}r^{d-2}(dx^+)^2 + R^2d\Omega^2, \quad (6)$$

where $G_{d+1}$ is the $d+1$-dim Newton constant, $Q$ is the energy-momentum density component $T_{++}$ in the boundary theory. These spacetimes are best thought of in the following way: start with $AdS_{d+1}$-Schwarzschild black branes boosted with parameter $\lambda$ (see e.g. [17])

$$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + \frac{\lambda}{2}\frac{rdn}{R^{d+n}} (dx^- + \lambda^{-1}dx^+)^2 + dx_i^2] + \frac{dr^2}{r^2(1 - \frac{r_0^d}{R^{d+n}})} + R^2d\Omega^2,$$

$$r_0^4 \sim G_{10}\varepsilon_4 \quad (D3), \quad r_0^6 \sim G_{11}\varepsilon_3 \quad (M2), \quad r_0^3 \sim G_{11}\varepsilon_6 \quad (M5), \quad (7)$$

where the terms arising from the finite temperature blackening factor $(1 - \frac{r_0^d}{R^{d+n}})$ (with $n = 4(D3), 6(M2), 3(M5)$) have been recast using the usual $(\frac{r}{R})^n \rightarrow \frac{r}{R}$ coordinate transformation (with $r = 0$ now being the $AdS$ boundary), and the temperature parameter $r_0$ expressed using the energy density $\varepsilon$ above extremality in the respective D3, M2 or M5-brane theories. Consider the (zero temperature, infinite boost) double scaling limit [5]

$$\lambda \rightarrow \infty, \quad \varepsilon_{p+1} \rightarrow 0, \quad \text{with} \quad \frac{\lambda^2\varepsilon_{p+1}}{2} = Q = \text{fixed}. \quad (8)$$

Of the finite temperature terms, the metric component $g_{++}$ alone survives in this limit, and the spacetimes (7) reduce to the $AdS$ plane wave spacetimes (6), after using

$$G_5 \sim G_{10}R_3^5 \quad \text{or} \quad G_{4,7} \sim G_{11}R_2^{7,4}, \quad \text{with} \quad R_{D3}^4 \sim g_sNl_s^4, \quad R_{M2}^6 \sim Nl_P^6, \quad R_{M5}^3 \sim Nl_P^3.$$
The entanglement entropy for ground states \((Q = 0)\) in the CFTs arising on the various conformal \((D3, M2, M5)\) branes with strip-shaped subsystems has the form

\[
S_A \sim \frac{R^{d-1}}{G_{d+1}} \left( \frac{V_{d-2}}{c_{d-2}^d} - c_d \frac{V_{d-2}}{r^{d-2}} \right),
\]

where \(c_d > 0\) is some constant, \(l\) the strip width and \(V_{d-2}\) the longitudinal size (we are interested in the scaling behaviours alone throughout this paper, so will not be careful with numerical coefficients). The first term represents the area law while the second term is a finite cutoff-independent part encoding a size-dependent measure of the entanglement. With \(Q \neq 0\), we have an energy flux in a certain direction: these are nonstatic spacetimes, and we use the covariant formulation of holographic entanglement entropy \([9]\). Consider the strip to be along the flux direction, \(i.e.\) with width along some \(x_i\) direction \([8]\). Then the leading divergent term is the same as for ground states. The width scales as \(l \sim r_\ast\) and the finite cutoff-independent piece in these excited states is

\[
\pm \sqrt{Q} v_{d-2} t^{2-\frac{d}{2}} \sqrt{\frac{R^{d-1}}{G_{d+1}}} [\pm : d < 4, \ - : d > 4] ;
\]

\[
\sqrt{Q} v_{2N} \text{ log}(l Q^{1/4}) \ (D3) ; \ \sqrt{Q} l N^{3/2} \ (M2) ; \ - \sqrt{Q} V_4 \sqrt{N^3} \ (M5).
\]

For the M5-theory, we have \(V_4 N^3 l \left( - \sqrt{Q} N + \frac{1}{p} \right) > 0 \) \(i.e.\) the finite part of EE for the excited state is larger than that for the ground state if \(l \ll \sqrt{N} Q^{-1/6}\). This may seem a bit surprising: perhaps the best interpretation is that these expressions are obtained in the large \(N\) approximation (where \(\sqrt{N} Q^{-1/6} \to \infty\)), with corrections at finite \(N\).

With the strip orthogonal to the flux direction, a phase transition was noted \([8]\): for large width \(l\), there is no connected surface corresponding to a spacelike subsystem.

### 2.2 Non-conformal Dp-brane theories

The string metric and dilaton for \(Dp\)-brane plane waves (with normalizable \(g_{++}\)-deformations) are

\[
\begin{align*}
&ds^2_{st} = \frac{r^{(7-p)/2}}{R_p^{(7-p)/2}} dx_i^2 + \frac{G_{10} Q_p}{R_p^{(7-p)/2}} \frac{(dx^+)^2}{r^{(7-p)/2}} + R_p^{(7-p)/2} r^{(7-p)/2} \frac{dr^2}{r^{(7-p)/2}} + R_p^{(7-p)/2} r^{(p-3)/2} d\Omega_8^{2-p}, \\
e^\Phi = g_s \left( \frac{R_p^{(7-p)}}{r^{(p-3)}} \right)^{\frac{3-p}{4}} ; \ g_Y^2 \sim g_s \alpha^{(p-3)/2} , \ R_p^{7-p} \sim g_Y^2 N \alpha^{5-p} \sim g_s N \alpha^{(7-p)/2}.
\end{align*}
\]

As for the conformal branes, the \(g_{++}\) term here has been obtained starting with the non-conformal finite temperature solutions \([18]\): using the coordinate \(r\) where \(U = \frac{r}{\alpha}\), the temperature parameter being \(r_0^{7-p} = (U_0 \alpha)^{7-p} \sim G_{10} \bar{e}_{p+1}\), we have the metric component
\[ g_{tt} = -\frac{r^{-(7-p)/2}}{R_p^{7-p}}(1 - \frac{r^{7-p}}{r_p^{7-p}}). \]

Rewriting in terms of lightcone variables \( t = \frac{x^+ + x^-}{\sqrt{2}}, \ x_p = \frac{x^+ - x^-}{\sqrt{2}} \), we obtain (11) by a zero temperature, infinite boost, double scaling limit \( \lambda \rightarrow \infty, r_0 \rightarrow 0 \) holding the boosted lightcone momentum density \( \frac{\lambda^2 x + 1}{r} \equiv Q_p \) fixed [5]. These describe strongly coupled Yang-Mills theories with constant energy flux \( T_{++} \).

The Einstein metric \( ds_E^2 = e^{-\Phi/2}ds_{st}^2 \), upon dimensional reduction on \( S^{8-p} \) and the \( x^+ \)-direction (compactified), gives rise to hyperscaling violating spacetimes \( ds^2 = r^{20/d}(-\frac{dt^2}{r^2} + \frac{dx^2 + dr^2}{r^2}) \), where \( t \equiv x^- \), with nontrivial Lifshitz (\( z \)) and hyperscaling violating (\( \theta \)) exponents \( \theta = \frac{p^2 - 6p + 7}{p - 5}, \ z = \frac{2(p - 6)}{p - 5} \) (see [3]).

We now calculate entanglement entropy for a strip subsystem: since this is a non-static space-time, we use the covariant formulation [9], following [8] for the conformal case. We consider a strip along the energy flux direction, with width along some time, we use the covariant formulation [9], following [8] for the conformal case. We consider a strip along the energy flux direction, with width along some \( x_i \)-direction, corresponding to a spacelike subsystem. The subsystem is specified by \( 0 \leq x_1 \leq l, \ (x^+, x^-) = (\alpha y, -\beta y) \), with the range of \( y \) and the other \( x_i \) coordinates being \(( -\infty, \infty) \) (with regulated lengths \( L_i \gg l \)). Then the entanglement entropy (11) (12) simplifies to

\[
S_A \sim \frac{V_{p-1}R_p^{7-p}}{G_{10}} \int_{r_*}^{r_0} dr \sqrt{2/\beta + \frac{G_{10}Q}{R_p^{7-p}}} \sqrt{1 + \frac{r^{7-p}}{R_p^{7-p}(x_1')}^2}, \tag{12}
\]

where \( r_0 \) is the UV cutoff and \( r_* \) is the turning point for the extremal surface \( x_1(r) \). The lightlike limit \( \beta = 0 \) corresponds to constant-\( x^- \) (null time) slicing: this is natural from the point of view of constant time slices in the lower dimensional theory obtained by \( x^+ \)-dimensional reduction, with time defined as \( t \equiv x^- \). However such a null entanglement entropy is tricky to define ab initio. We therefore study spacelike subsystems on a constant time slice corresponding to \( \alpha = \beta = 1 \): the scaling estimates below for \( l(r_*) \) and \( S_A^{\text{finite}} \) are as in the null case. Analysing (12) gives the extremal surface and associated entanglement

\[
\frac{dx_1}{dr} = \frac{AR_p^{7-p}}{r^{8-p}\sqrt{2 + \frac{G_{10}Q}{R_p^{7-p}} - \frac{A^2R_p^{7-p}}{r^{9-p}}}}, \quad S_A \sim \frac{V_{p-1}R_p^{7-p}}{G_{10}} \int_{r_*}^{r_0} dr \frac{2 + \frac{G_{10}Q}{R_p^{7-p}} - \frac{A^2R_p^{7-p}}{r^{9-p}}}{\sqrt{2 + \frac{G_{10}Q}{R_p^{7-p}} - \frac{A^2R_p^{7-p}}{r^{9-p}}}}. \tag{13}
\]

For \( Q = 0 \), these expressions reduce to those for the ground state: recall that for the non-conformal \( Dp \)-branes, the gauge coupling \( g_{YM}^2 = g_s\alpha'_{(p-3)/2} \) is dimensionful and the entanglement entropy for strip-subsystems has the form [2] [15]

\[
S_A = N_{\text{eff}}(\epsilon) V_{d-2} \frac{c_d - c_{d-2}}{c_{d-2}} V_{d-2} \frac{l_{d-2}}{l_{d-2}}, \quad N_{\text{eff}}(\epsilon) = N^2\left( \frac{g_{YM}^2N_\epsilon^{(p-3)}}{\epsilon^{p-3}} \right)^\frac{p-8}{p-5}. \tag{14}
\]

The two terms here again reflect the local area law and the finite cutoff-independent piece, but with a scale-dependent number of degrees of freedom \( N_{\text{eff}} \) [15]. The cutoff \( \epsilon \equiv \frac{1}{r_0} \) here is written in terms of the non-conformal \( Dp \)-brane supergravity radius/energy variable
$u = \frac{r_{(5-p)/2}}{R_p^{(5-p)/2}}$ introduced in $[19]$. Thus $\frac{1}{r} = \frac{r_{(5-p)/2}}{R_p^{(5-p)/2}}$.

With $Q \neq 0$, from $[13]$, the divergent ultraviolet behaviour of $S_A$ (large $r$) is seen to be the same as for the ground state

$$S_{A}^{\text{div}} \sim \frac{V_{p-1} R_{p}^{(5-p)/2}}{r_{*}^{p-1}} \frac{V_{d-2}}{\epsilon^{d-2}} ,$$

in accord with the expectation that the excited state leaves unaffected the short distance behaviour, while introducing long range correlations. The finite part can be estimated by realising as in $[8]$ that the turning point can be approximated for large $Q$ as $\frac{G_{10} Q}{r_{*}^{p}} \sim \frac{\Lambda^{p} R_{p}^{(5-p)/2}}{r_{*}^{p}}$, which then, using $[13]$ along with $l \equiv \Delta x_{1}$, gives the scaling estimates

$$l \sim \frac{R_{p}^{(5-p)/2}}{r_{*}^{p-1}}, \quad S_{A}^{\text{finite}} \sim \frac{V_{p-1} \sqrt{Q}}{(3-p)\sqrt{G_{10}} r_{*}^{(3-p)/2}} \sim \frac{V_{p-1} \sqrt{Q}}{(3-p)\sqrt{G_{10}}} \frac{R_{p}^{(7-p)/2}/(5-p)}{l^{(p-3)/(5-p)}} .$$

Using the expressions above for $R_{p}, G_{10}$ etc in terms of gauge theory parameters $g_{YM}, N$, this finite part can be simplified and recast as $(p \neq 3)$

$$S_{A}^{\text{finite}} \sim \frac{1}{3-p} \frac{V_{p-1} \sqrt{Q}}{l^{(p-3)/2}} N \left(\frac{g_{YM}^{2} N}{l^{p-3}}\right)^{\frac{p-3}{5-p}} \sim \frac{\sqrt{N_{\text{eff}}(l)}}{3-p} \frac{V_{p-1} \sqrt{Q}}{l^{(p-3)/2}} ,$$

$$N_{\text{eff}}(l) = N^{2} \left(\frac{g_{YM}^{2} N}{l^{p-3}}\right)^{\frac{p-3}{5-p}} .$$

$N_{\text{eff}}(l)$ is the scale-dependent number of degrees of freedom $[14]$ involving the dimensionless coupling at scale $l$. Recalling that $Q$ is the energy density in $p+1$-dim, the second factor is recognized as the natural dimensionless combination of $V_{p-1}, Q, l$, given the expectation that the entanglement is proportional to $V_{p-1}, \sqrt{Q}$. Then the finite part suggests that these plane wave states are an effective chiral subsector in the gauge theory. Along the lines of $[5]$, a scale-dependent redefinition $Q = \tilde{Q} N_{\text{eff}}(l)$ can be devised to recast the finite part as

$$\sim N_{\text{eff}}(l) \frac{V_{p-1} \sqrt{Q}}{l^{(p-3)/2}} .$$

The energy-momentum density $\tilde{Q}$ is then the energy-momentum density per nonabelian degree of freedom, but involving the dimensionless coupling at scale $l$.

It is interesting to study some specific Dp-brane theories in particular comparing with their UV/IR conformal phases as in their phase diagram $[18]$. This vindicates the intuition that the finite part of entanglement decreases under renormalization group flow.

**D2-M2:** We have

$$S_{A}^{\text{fin}} \sim V_{1} \sqrt{I} \sqrt{Q} \left\{ \frac{N^{2}}{(g_{YM}^{2} N)^{1/2}} \right\}^{1/3} \quad (D2); \quad S_{A}^{\text{fin}} \sim V_{1} \sqrt{I} \sqrt{Q} \sqrt{N^{3/2}} \quad (M2) .$$

We see as expected that $S_{A}^{\text{fin}}$ for the plane wave states is greater than that for the ground states for the same strip geometry. Noting the IIA regime of validity $g_{YM}^{2} N^{1/5} \ll \frac{r}{\alpha'} \ll g_{YM}^{2} N$
for the turning point $r_*$, we find $1 \ll g_{YM}^2 N l_{D2} \ll N^{6/5}$. Thus $N^{3/2} \ll N^2 / (g_{YM}^2 N l_{D2})^{1/3} \ll N^2$, i.e. the finite entanglement in the D2-supergravity phase is in between the free 2+1-dim SYM phase (UV) and the M2-phase (IR), consistent with the expectations of the thinning of degrees of freedom under renormalization group flow. This also suggests that the free 2+1-dim SYM entanglement in these plane wave excited states is possibly $\sim V_1 \sqrt{I/\sqrt{Q}} \sqrt{N^2}$.

**D4-M5**: We have

$$S_{D4-M5}^{\text{fin}} \sim -\frac{V_3 \sqrt{Q}}{\sqrt{l}} \sqrt{N^2 g_{YM}^2 N l} \quad (D4) \quad \text{and} \quad S_{D4-M5}^{\text{fin}} \sim -\sqrt{Q} \frac{V_3}{l} \sqrt{N^3} \quad (M5). \quad (19)$$

The finite parts are actually the same expression for both the D4-supergravity and M5-phases, if we recognize that the D4-branes are M5-branes wrapping the 11th circle (size $r_{11} = g_s l_{s} = g_{YM}^2$) and $V_4 = V_3 l_{11}$, $Q_{D4} = Q_{M5} l_{11}$. The IIA regime of validity $\frac{1}{g_{YM}^2 N} \ll l_{\alpha} \ll N^{1/3}$ implies $1 \ll g_{YM}^2 N l \ll N^{2/3}$. The free 5d SYM entanglement is possibly $-\frac{V_3}{\sqrt{l}} \sqrt{N^2}$.

**D1**: We have $S_{D1}^{\text{fin}} \sim l \sqrt{Q} \sqrt{\frac{N^2}{g_{YM}^2 N l_{D2}^2/2}}$, which grows as $\sqrt{l}$ (including the $l$-dependence in $N_{\text{eff}}$). The regime of validity $g_{YM} N^{1/6} \ll l_{\alpha} \ll g_{YM} \sqrt{N}$ implies $1 \ll g_{YM}^2 N l^2 \ll N^{4/3}$.

### 2.3 Strip orthogonal to wave

The subsystem $A$ lying on a constant time-$t$ slice is $\Delta x^+ = -\Delta x^- = \frac{l}{\sqrt{2}}$ with $-\infty < x_t < \infty$. The (covariant) entanglement entropy functional for the extremal surface $x^+(r), x^-(r)$ is

$$S_A \sim \frac{V_{p-1} R_{p-1}^{l-p}}{G_{10}} \int_{r_*}^{r_0} dr \int dr \sqrt{1 - 2r^{l-p} (\partial_r x^+) (\partial_r x^-) + \frac{G_{10} Q}{R_{p-1}^{l-p}} (\partial_r x^+)^2} \quad (20),$$

giving

$$S_A \sim \frac{V_{p-1} R_{p-1}^{l-p}}{G_{10}} \int_{r_*}^{r_0} dr \int dr \frac{AB}{\sqrt{A^2 B^2 - 2B_{r-p}^{l-p} + \frac{G_{10} Q}{R_{p-1}^{l-p}}}}, \quad (21)$$

\[
    \Delta x^+ \quad \frac{2}{2} = \int_{r_*}^{r_0} dr \frac{dr}{\sqrt{A^2 B^2 r^{16-2p} + \frac{G_{10} Q}{R_{p-1}^{l-p}} - 2B_{r-p}^{l-p}}}, \quad \Delta x^- \quad \frac{2}{2} = \int_{r_*}^{r_0} dr \frac{dr}{\sqrt{A^2 B^2 r^{16-2p} + \frac{G_{10} Q}{R_{p-1}^{l-p}} - 2B_{r-p}^{l-p}}}.
\]

This structure is similar to the conformal case [8]. Large width $l$ is obtained only when the denominator function $A^2 B^2 r^{16-2p} + \frac{G_{10} Q}{R_{p-1}^{l-p}} - 2B_{r-p}^{l-p}$ acquires a double zero at $r = r_*$. This gives $B = \frac{8-p \frac{G_{10} Q}{R_{p-1}^{l-p}}}{B_{r-p}^{l-p}}, \quad A^2 = \frac{(7-p)(9-2p)}{(8-p)^2} \frac{1}{G_{10} R_{p-1}^{l-p}}$, for which $\Delta x^+, \Delta x^-$ both become positively divergent, incommensurate with a spacelike subsystem. As for the conformal case, this suggests a phase transition due to an upper bound on the width: no connected surface corresponding to a spacelike subsystem exists beyond this critical width, which has a scaling estimate $l_c \sim \frac{R_{p-1}^{l-p}}{r_* \sqrt{N}} \sim (g_{YM}^2 N \sqrt{N^{1/3}} (\frac{Q}{N^2})^{\frac{1}{(7-p)}})$.

There are multiple length scales here unlike the conformal case, constraining the phase transition structure: e.g. for D2-branes, the IIA supergravity regime of validity for $r_{*,c}$ (and so $l_c$) gives $\frac{(g_{YM}^2 N^3)}{N^2} \lesssim Q \lesssim (g_{YM}^2 N)^3 N^2.$
2.4 NS5-brane plane waves

NS5-branes in certain decoupling limits have been argued to be dual to nonlocal 6-dim “little string” theories \cite{20} (see e.g. \cite{21, 22, 23, 24, 25} for further studies). Here we identify plane wave states in these theories. Starting with the finite temperature NS5-brane system before decoupling

\[ ds^2 = -\left(1 - \frac{r_0^2}{r^2}\right)dt^2 + \left(1 + \frac{N\alpha'}{r^2}\right)\left(\frac{dr^2}{1 - \frac{r_0^2}{r^2}} + r^2d\Omega_3^2\right) + \sum_{i=1}^{5} dy_i^2, \quad e^{2\Phi} = g_s^2\left(1 + \frac{N\alpha'}{r^2}\right), \]  

(22)

defining lightcone coordinates \( t = \frac{x^+ + x^-}{\sqrt{2}} \), \( x_5 = \frac{x^+ - x^-}{\sqrt{2}} \), after a lightlike boost \( x^\pm \to \lambda^\pm x^\pm \) with parameter \( \lambda \), we identify a new double scaled zero-temperature decoupling limit that results in plane wave like excited states in the little string theory on the NS5-branes, distinct from the Hagedorn temperature limit \cite{21}. The temperature parameter above is \( r_0^2 = G_{10}\mu \).

Under the double scaling limit \( \lambda \to \infty \), \( g_s \to 0 \), with \( \frac{\lambda^2 g_s^2}{2} \equiv Q \) fixed (with dimensions of boundary energy density), the near horizon spacetime becomes

\[ ds_{st}^2 = -2dx^+dx^- + \frac{Q\alpha'^4}{r^2}(dx^+)^2 + \sum_{i=1}^{4} dy_i^2 + N\alpha'\frac{dr^2}{r^2} + N\alpha'd\Omega_3^2, \quad e^{2\Phi} = g_s^2\frac{N\alpha'}{r^2}, \]  

(23)
an asymptotically linear dilaton background with “normalizable” null deformation (vanishing as \( r \to \infty \)). It can be checked independently that this is a solution to the NS-NS sector spacetime equations: the only new contribution here to the string frame equations describing the NS5-brane is \( R_{++} = -2\nabla_+\nabla_+\Phi = +2\Gamma_+\partial_+\Phi = -\frac{2\alpha'^4}{N\alpha'} \), which is consistent. This is another way to find this solution, and is consistent with S-duality of the D5-brane plane waves earlier (for Type IIB). These lightlike deformations are likely \( \alpha' \)-exact and supersymmetric.

Dimensionally reducing the Einstein metric on \( S^3 \) and the \( x^+ \)-direction (compactified) gives \( ds_{E,6d}^2 \sim r^{1/2}\left(-\frac{r^2}{Q\alpha'}dt^2 + dy_i^2 + N\alpha'\frac{dr^2}{r^2}\right) \), which is not in hyperscaling violating form.

The double scaling limit here and the resulting NS5-brane plane wave excited states appear distinct from previous double scaled limits of little string theory \cite{24}: it will be interesting to explore these NS5-brane or little string excitations further.

Here we briefly discuss holographic entanglement entropy for these theories: for a space-like strip-subsystem of width \( l \), as before we have \( S_A \sim \frac{V_A\frac{\alpha'}{G_{10}}}{2}\int dr r\sqrt{2 + \frac{Q\alpha'^4}{r^2}}\sqrt{1 + \frac{r^2}{N\alpha'}(x')^2} \), giving \( \frac{dr}{dx} = \frac{\sqrt{N\alpha'}}{r}\frac{\Delta\sqrt{N\alpha'}}{\sqrt{2r^4 + Q\alpha'^4r^2 - A^2N\alpha'}} \), \( S_A \sim \frac{V_A\frac{\alpha'}{G_{10}}}{2}\int \frac{dr r(2 + \frac{Q\alpha'^4}{r^2})}{\sqrt{2r^4 + Q\alpha'^4r^2 - A^2N\alpha'}} \), and so (with \( U = \frac{r}{\alpha'} \)) the scaling estimates \( l \sim \sqrt{N\alpha'} \), \( S_{UV}^{A} \sim \frac{V_A\frac{\alpha'}{g_s^2}}{g_s^2}U_0^2 \), \( S_{A}^{finite} \sim \frac{V_A\frac{\sqrt{N\alpha'}}{g_s^2}}{g_s^2}U_* \). We see that (as for the ground state \cite{21, 15}) \( l \) degenerates, being fixed to the scale of nonlocality (independent of \( r_* \)) inherent in these theories.
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