Synchronization of Linear Piecewise Chaotic Systems Using Sliding Mode Control

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Abstract. We present an algorithm to synchronize, under the master/slave configuration, a class of piecewise linear chaotic systems known as Sprott systems. The synchronization objective is to obtain identical synchronization between the master and slave systems in spite of the existence of external perturbations and parametric variations. The sliding control technique is used to design the coupling signal. This discontinuous controller renders the closed loop system robust with respect to matched bounded disturbances and to terms produced by parametric variations. The performance of the proposed controlled synchronization is illustrated numerically and experimentally.

1. Introduction
Synchronization can be defined as a phenomenon where two or more appropriately coupled systems undergo resembling evolution in time. Pecora and Carroll [5] showed that synchronization can be achieved in dynamical systems on chaotic regime. They constructed an unidirectional master/slave configuration where the slave is a suitably altered copy of the master system and the synchronization is achieved in terms of the time evolution of the difference between the states of the master and those of the slave system. Then, if the error dynamics are asymptotically stable, synchronization is guaranteed. Since the seminal paper by Pecora and Carroll [5] a large number of alternative approaches to achieved synchronization have been studied, see for example [4] and [7]. However, chaos synchronization is far from being straightforward because different aspects affect significantly the ability of systems to synchronize. A particularly important aspect to consider is the presence of disturbances and uncertainties, which are unavoidable parts of any practical implementation. Therefore, robustness is a very desirable characteristic of a synchronization approach.

Chaos synchronization has important applications, for example, many of the techniques to obtain controlled synchronization have been applied to synchronize chaotic systems to develop private communication systems. In this application the objective is to encode or encrypt information through a chaotic signal that will be sent to a receiver, where a chaotic system is synchronized to recover the information, see for example [1] and references therein.

Simplicity is always a desirable characteristic to consider in a practical implementation. As shown by the work of Sprott [6] a practical chaos generator can be constructed with a very simple circuit if the nonlinearity is a piecewise linear function. At the same time, the piecewise
linear nature of the system simplifies the analysis, since in this case the nonlinear system can be reformulated as a variable structure system consisting of linear parts with a given switching logic. One important problem is to synchronize this kind of systems. There are some proposals to solve it, see for example [3] where two Sprott circuits are synchronized using a feedback linearization control technique. In this work the synchronization is reached if the systems are identical and there are no parametric perturbations or external disturbances. These conditions on the systems are very hard to satisfy in practice.

One of the control techniques that displays good characteristics of robustness to parametric variations and external disturbances is the sliding mode control; this technique was used in [8] for the synchronization of chaotic systems with uncertainties. Another option to solve the problem of parametric uncertainties is the use of adaptive control; for example, the PID controller with an adaptable law has been proposed for the synchronization of mechanical systems [2].

In this paper we propose a technique to synchronize two Sprott systems. They must have the same structure, but not necessarily the same parametric values, and matched bounded perturbations could also be present. The interconnection scheme is the master/slave. We use a sliding mode control technique to design the coupling signal; therefore, the closed loop system is robust to external perturbations and parametric variations. In theory, we can attain asymptotic identical synchronization in spite of the existence of this kind of disturbances. However, in practice, as a result of a discontinuous coupling signal, there will be a small chattering in the synchronization errors. Nevertheless, in many applications this error may be negligible.

This paper will be outlined as follows. The statement of the problem is given in section 2, where the type of systems considered is defined, as well as the synchronization criterion. In section 3 the design of the coupling signal using the sliding mode technique is described. To illustrate the proposed synchronization technique in section 4 we present numerical and experimental results. Finally, the conclusions are given in section 5.

2. The synchronization problem
An important class of piecewise linear chaotic systems are the so called Sprott systems [6]. They are defined as

\[ \ddot{x} + a \dot{x} + x = G(x) \]  

where \( G(x) \) can take one of the following forms

\[
\begin{align*}
G(x) &= b |x| - c, \\
G(x) &= -b \max(x, 0) + c, \\
G(x) &= bx - c \text{sign}(x), \\
G(x) &= -bx + c \text{sign}(x),
\end{align*}
\]

where \( a, b \) and \( c \) are positive constants with adequate values for each case and \( \text{sign}(\cdot) \) is the signum function. This is the class of systems that we consider in this work.

Now define a master system

\[
\begin{align*}
\dot{x}_{1,m} &= x_{2,m}, \\
\dot{x}_{2,m} &= x_{3,m}, \\
\dot{x}_{3,m} &= -ax_{3,m} - x_{2,m} + G(x_{1,m}) + \gamma_m(t),
\end{align*}
\]

where \( \gamma_m(t) \) is a bounded perturbation term that is bounded, \( |\gamma_m(t)| \leq \delta_m \). Also define a slave
system
\begin{align*}
\dot{x}_{1,s} &= x_{2,s}, \\
\dot{x}_{2,s} &= x_{3,s}, \\
\dot{x}_{3,s} &= -ax_{3,s} - x_{2,s} + G(x_{1,s}) + \gamma_s(t) + v,
\end{align*}
where $\gamma_s(t)$ is also a bounded perturbation term, $|\gamma_s(t)| \leq \delta_s$ and $v$ is a coupling signal.

The problem is to design a coupling signal $v$ to obtain synchronization between the master and slave systems. The synchronization objective is given by
\[ \lim_{t \to \infty} \|x_m(t) - x_s(t)\| = 0. \tag{2} \]

To solve this problem define the error variables $e_1 = x_{1,m} - x_{1,s}$, $e_2 = x_{2,m} - x_{2,s}$ and $e_3 = x_{3,m} - x_{3,s}$ with dynamics given by
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= e_3, \\
\dot{e}_3 &= -ae_3 - e_2 + G(x_{1,m}) - G(x_{1,m} - e_1) + \gamma_m(t) - \gamma_s(t) - v,
\end{align*}
Now the problem is to design a coupling signal $v$ such that the origin of the error space will be an asymptotically stable equilibrium point.

3. Design of the coupling signal

We design the coupling signal $v$ based on the sliding mode control technique. Consider a discontinuity surface defined by
\[ S = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0, \tag{4} \]
where $\alpha_i \in \mathbb{R}$, $i = 1, 2, 3$ are constants. On the other hand, we define a coupling signal $v$ given by
\[ v(x_m, e) = F(x_m, e) \text{ sign}(S), \tag{5} \]
where $F(\cdot)$ is, in general, a piecewise smooth function.

In the following subsections we present a way to design the surface $S$ (4) and the coupling signal $v(\cdot)$ (5) so that the problem (2) will be solved.

3.1. Sliding surface design

The sliding surface must be designed such that, when the trajectories arrived to the discontinuity surface defined by $S = 0$, they must be directed to the origin of the state space of the error variables; i.e., a sliding mode is presented.

One way to define the behavior of system (3) when its trajectories slide in the surface $S$ is using the equivalent control approach. This equivalent control $v_{eq}$ can be seen as the average control when the trajectories of the system are in the surface $S$ [9].

The value of $v_{eq}$ is found from the equation $\dot{S} = 0$ which, from equations (3)-(4), it has the form
\[ \dot{S} = \sum_{i=2}^{3} \alpha_{i-1} e_i + \alpha_3 (-ae_3 - e_2 + G(x_{1,m}) - G(x_{1,m} - e_1) + \gamma_m(t) - \gamma_s(t) - v_{eq}) = 0. \tag{6} \]
If $\alpha_3 = 1$, the equivalent control is given by
\[ v_{eq} = (\alpha_2 - a) e_3 + (\alpha_1 - 1) e_2 + G(x_{1,m}) - G(x_{1,m} - e_1) + \gamma_m(t) - \gamma_s(t). \tag{7} \]
Substituting $v_{eq}$ into (3) gives
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= e_3 \\
\dot{e}_3 &= -\alpha_1 e_2 - \alpha_2 e_3.
\end{align*}
(8)
(9)
(10)

The last 2 equations form a linear system decoupled from $e_1$, with the form
\begin{equation}
\dot{\tilde{e}} = A\tilde{e},
\end{equation}
(10)
where
\begin{equation}
\tilde{e} = [e_2, e_3]^T
\end{equation}
(11)
and
\begin{equation}
A = \begin{bmatrix} 0 & 1 \\ -\alpha_1 & -\alpha_2 \end{bmatrix}.
\end{equation}
(12)

System (10) has a unique equilibrium point at the origin $\tilde{e} = 0$ and is exponentially stable if the constants $\alpha_i > 0$ for $i = 1, 2$ are chosen such that matrix $A$ is strictly Hurwitz. Furthermore, by (4) we see that also $e_1 \to 0$. Thus, the trajectories in the discontinuity surface will go to the origin of the error space.

Now, we will find conditions so that the discontinuity surface will be attractive; i.e., to find the conditions on $v(\cdot)$ such that the trajectories out of the surface $S$ go to this surface in a finite time.

3.2. Conditions for the existence of a sliding mode
Consider the following criterion given in [9]:
If
\begin{equation}
S \dot{S} < 0 \quad \forall S \neq 0 \text{ and } \forall t \geq 0,
\end{equation}
then the surface $S$ is a sliding surface. In our case, we have the following
\begin{equation}
S \dot{S} = S ((\alpha_2 - a) e_3 + (\alpha_1 - 1) e_2) + S (G(x_{1,m}) - G(x_{1,m} - e_1) + \gamma_m (t) - \gamma_s (t)) - F(x_m, e) |S|.
\end{equation}

Let us suppose that $F(\cdot) > 0$ for all $x_m, e$ and $t \geq 0$, then
\begin{equation}
S \dot{S} \leq |S| ((\alpha_2 - a) e_3 + (\alpha_1 - 1) e_2 + G(x_{1,m}) - G(x_{1,m} - e_1) + \gamma_m (t) - \gamma_s (t)) - F(x_m, e)
\end{equation}
if we find a function $F(x_m, e)$ such that
\begin{equation}
F(x_m, e) > |\alpha_2 - a| |e_3| + |\alpha_1 - 1| |e_2| + |G(x_{1,m}) - G(x_{1,m} - e_1)| + \delta_m + \delta_s
\end{equation}
\forall e, x_m \text{ and } \forall t \geq 0; \text{ then, the synchronization problem is solved.}

On the other hand, the condition about the convergence to the surface $S$ in a finite time is also satisfied with the control input given by (5). We can prove the last statement by using the following criterion given in [9].
If
\begin{equation}
\lim_{S \to 0^-} \dot{S} > 0 \text{ & } \lim_{S \to 0^+} \dot{S} < 0
\end{equation}
then the convergence to the surface $S$ is in a finite time. Both conditions in (15) are satisfied in global or local form, this depends on the function $F(\cdot)$. 

4. Synchronization of two Sprott circuits

In this section we illustrate the performance of the proposed synchronization technique with numerical and experimental results. Consider a master and slave systems given by

\[ \begin{aligned}
\dot{x}_{1,m} &= x_{2,m}, \\
\dot{x}_{2,m} &= x_{3,m}, \\
\dot{x}_{3,m} &= -1.2x_{1,m} - x_{2,m} - 0.6x_{3,m} + 2\text{sign}(x_{1,m}), \\
\dot{x}_{1,s} &= x_{2,s}, \\
\dot{x}_{2,s} &= x_{3,s}, \\
\dot{x}_{3,s} &= -1.2x_{1,s} - x_{2,s} - 0.6x_{3,s} + 2\text{sign}(x_{1,s}) + v.
\end{aligned} \]

where \( v \) is a coupling signal. Define the error variables \( e_1 = x_{1,m} - x_{1,s}, e_2 = x_{2,m} - x_{2,s} \) and \( e_3 = x_{3,m} - x_{3,s} \) with dynamics given by

\[ \begin{aligned}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= e_3, \\
\dot{e}_3 &= -1.2e_1 - e_2 - 0.6e_3 + 2\text{sign}(x_{1,m}) - 2\text{sign}(x_{1,m} - e_1) - v.
\end{aligned} \]

The coupling signal \( v \) has the form

\[ v = F(t, e) \text{sign}(S), \]

where \( S \) is

\[ S = \alpha_1 e_1 + \alpha_2 e_2 + e_3, \]

and the function \( F(t, e) \) has the form

\[ F(t, e) = \beta_1 |e_1| + \beta_2 |e_2| + \beta_3 |e_3| + \beta_4. \]

We took the following parameters: \( \alpha_1 = 6, \alpha_2 = 10, \beta_1 = 7, \beta_2 = 3, \beta_3 = 13 \) and \( \beta_4 = 5.5. \)

The numerical results are the following. Figure 1 shows the phase portrait of the master system, where a chaotic behavior can be observed. Figure 2 shows the synchronization errors before and after the coupling signal is applied at \( t = 50 \text{ sec}. \) As we can see, the synchronization errors are large when the coupling signal is not present, but converge to zero when it is applied.

The experimental results are now described. The electronic circuit that solves equations (16) and (17) \( (V = 0 \text{ for (16)}) \) is shown in Figure 3. Figure 4 shows the experimental results, which show also a convergence to a zero synchronization error.

5. Conclusions

In this paper we have proposed a new algorithm to synchronize two piecewise linear chaotic systems called Sprot systems. The conditions to apply this algorithm is that the master and slave systems must have the same nonlinear function \( G(x) \), and the perturbations satisfy the matching conditions.

In theory, this algorithm guarantee zero synchronization error; however, due to high frequency components in the coupling signal, in practice the synchronization errors display small chattering, giving an approximate synchronization. The magnitude of the chattering in the error state is, in general, directly related to the disturbance terms in the closed loop system. In our examples, the disturbance terms are delays in the interface card and parametric variations in the circuits, as well as modeling errors. However, in many applications these errors can be small enough such that the synchronization algorithm can be applied.

A restriction of this algorithm is that it needs full knowledge of the state vector of both systems. When we do not have full access of these states, we need to design observers, which is a hard task for nonlinear systems.
Figure 1. Phase portrait of the master system.

Figure 2. Numerical results. Synchronization errors before and after the coupling signal is applied (50 sec.)
Figure 3. Sprot circuit for $G(x) = -bx + c\text{sign}(x)$.

Figure 4. Experimental results. Synchronization errors and coupling signal.
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