Asymptotics for Bergman projections with smooth weights: a direct approach

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Abstract
We extend the direct approach to the semiclassical Bergman kernel asymptotics, developed recently in Deleporte et al. (Ann Fac Sci Toulouse Math, 2020) for real analytic exponential weights, to the smooth case. Similar to Deleporte et al. (2020), our approach avoids the use of the Kuranishi trick and it allows us to construct the amplitude of the asymptotic Bergman projection by means of an asymptotic inversion of an explicit Fourier integral operator.

Keywords Bergman kernel asymptotics · Strictly plurisubharmonic weight · Almost holomorphic extension

Mathematics Subject Classification 32A25 · 32W05 · 32W25 · 35S30

1 Introduction
Let $\Omega \subset \mathbb{C}^n$ be an open pseudoconvex domain and let $\Phi \in C^\infty(\Omega)$ be a strictly plurisubharmonic function. Exponentially weighted spaces of holomorphic functions of the form $H_\Phi(\Omega) = \text{Hol}(\Omega) \cap L^2(\Omega, e^{-2\Phi/h})$ occur naturally in analytic microlocal analysis [14, 16, 26, 28, 29], among other areas, when passing from the real to the complex domain by means of an FBI transform [28, 29]. Associated to the space $H_\Phi(\Omega)$ is the orthogonal (Bergman) projection

$$\Pi : L^2(\Omega, e^{-2\Phi/h}) \to H_\Phi(\Omega), \quad (1.1)$$
and complex microlocal techniques have long been known to be useful in the study of the asymptotic behavior of $\Pi$ in the semiclassical limit $h \to 0^+$. The existence of a complete asymptotic expansion for the Schwartz kernel of $\Pi$ close to the diagonal has been established in the pioneering contributions [4, 31], in the context of high powers of a holomorphic line bundle with positive curvature, over a complex compact manifold. See also [5, 7, 18, 23], as well as [12, 13] for the case of domains in $\mathbb{C}^n$. The original proofs in [4, 31] relied on the description of singularities of the Szegö kernel on the boundary of a strictly pseudoconvex smooth domain given by the Boutet de Monvel–Sjöstrand parametrix in the seminal work [3], which in turn depended, in particular, on the theory of Fourier integral operators with complex phase functions developed in [25]. More self-contained explicit approaches to the Bergman kernel asymptotics have subsequently been developed in [26, Section 3] and [1], with the former work starting with the approximate projection property $\tilde{\Pi}^2 = \tilde{\Pi} + O(h^\infty)$, while the approach of [1] focuses more directly on the reproducing property of the Bergman projection on the space $H_\Phi(\Omega)$.

To motivate a bit further, let us recall that the basic idea of the approach of [1] consists, roughly speaking, of expressing the identity operator on $H_\Phi(\Omega)$ in such a way that it automatically becomes the asymptotic Bergman projection, at least locally. This is accomplished, essentially, by first representing the identity as an $h$–pseudodifferential operator in the complex domain, and then passing to a non-standard phase via a suitable change of variables, usually referred to as the Kuranishi trick. See [28, Chapter 4], [1, Section 2.2], and also [15, Chapter 3] for the standard application of the Kuranishi trick to changes of variables for pseudodifferential operators in the real domain. Now the Kuranishi trick becomes somewhat complicated to execute in situations when the Levi form $i \partial \bar{\partial} \Phi \geq 0$ of the weight $\Phi$ becomes almost degenerate in some directions. Such nearly degenerate weights occur naturally, in particular, in the work in progress [17], devoted to a heat evolution approach to second microlocalization with respect to a real analytic hypersurface. A direct approach to the semiclassical asymptotics for Bergman projections, not relying upon any changes of variables, is therefore desirable, and it has recently been developed in [9] in the non-degenerate case, assuming that the weight $\Phi$ is real analytic. The approach of [9] has allowed, in particular, to give a quick proof of a result of [8, 27], stating that in the analytic case, the amplitude of the asymptotic Bergman projection is given by a classical analytic symbol. Our purpose here is to extend the approach of [9] to the case of weights that are merely $\mathcal{C}^\infty$ but not necessarily analytic. The following is the main result of this work.

**Theorem 1.1** Let $\Omega \subset \mathbb{C}^n$ be open and let $\Phi \in C^\infty(\Omega; \mathbb{R})$ be strictly plurisubharmonic in $\Omega$. Let $x_0 \in \Omega$. There exist a classical elliptic symbol $a(x, \tilde{y}; h) \in S^0_{\text{cl}}(\text{neigh}((x_0, \bar{x}_0), \mathbb{C}^{2n}))$ of the form

$$a(x, \tilde{y}; h) \sim \sum_{j=0}^{\infty} h^j a_j(x, \tilde{y}),$$

in $C^\infty$, with $a_j \in C^\infty(\text{neigh}((x_0, \bar{x}_0), \mathbb{C}^{2n}))$, holomorphic to $\infty$–order along the anti-diagonal $\tilde{y} = \bar{x}$, satisfying
\[(Aa)(x, \bar{x}; h) = 1 + \mathcal{O}(h^\infty), \quad x \in \text{neigh}(x_0, C^n), \quad (1.2)\]

where \(A\) is an elliptic Fourier integral operator, and small open neighborhoods \(U \Subset V \Subset \Omega\) of \(x_0\), with \(C^\infty\)-boundaries, such that the operator

\[\Pi_V u(x) = \frac{1}{h^n} \int_V e^{\frac{2}{h} \Psi(x, \bar{y})} a(x, \bar{y}; h) u(y) e^{-\frac{2}{h} \Phi(y)} dy d\bar{y} \quad (1.3)\]

satisfies

\[\Pi_V - 1 = \mathcal{O}(h^\infty): H_\Phi(V) \rightarrow L^2(U, e^{-2\Phi/h} L(dx)), \quad (1.4)\]

in the sense that the operator norm of \(\Pi_V - 1\) as a linear continuous map from \(H_\Phi(V)\) to \(L^2(U, e^{-2\Phi/h} L(dx))\) is \(\mathcal{O}(h^\infty)\). Here in (1.3), the \(C^\infty\) function \(\Psi\) is holomorphic to \(\infty\)-order along the anti-diagonal, \(\Psi(x, \bar{x}) = \Phi(x)\), and \(L(dx)\) is the Lebesgue measure on \(C^n\).

When proving Theorem 1.1, we proceed largely along the general lines of [9], and the essential new ingredient in the proofs is the use of the techniques of almost holomorphic extensions [20, 24, 25]. The lack of holomorphy causes some of the estimates and asymptotic constructions in the proofs to become a bit more explicit and refined, demanding a greater technical investment overall. Compared to [9], we also have to rely on the \(L^2\) estimates for the \(\tilde{\partial}\) operator [19] even more, to account for the fact that functions in the range of the operator \(\Pi_V\) in (1.3) are not quite holomorphic.

The plan of the paper is as follows: In Sect. 2, we introduce an explicit elliptic Fourier integral operator \(A\) in the complex domain, with the phase defined via an almost holomorphic extension of the weight \(\Phi\), and obtain a \(C^\infty\) symbol \(a\), holomorphic to \(\infty\)-order along the anti-diagonal, as a solution of (1.2). Let us observe that while the corresponding discussion in [9] in the analytic case makes use of the existence of a microlocal inverse of the corresponding analytic Fourier integral operator, here the asymptotic inversion of \(A\), with \(\mathcal{O}(h^\infty)\) errors, proceeds more directly by equipping the operator \(A\) with a suitable explicit contour of integration and by considering the stationary phase expansion for \(Aa\). Section 3 is devoted to showing the approximate reproducing property for the operator \(\Pi_V\) in (1.3), on the level of scalar products, \((\Pi_V u, v)_{L^2_\Phi(V)} = (u, v)_{H_\Phi(V)} + \mathcal{O}(h^\infty)\), for \(u, v \in H_\Phi(V)\), with \(v\) concentrated in a small neighborhood of \(x_0\). Similar to [9], the proof depends on a resolution of the identity and a contour deformation argument, with some additional care required due to the lack of holomorphy in (1.3). The proof of Theorem 1.1 is then concluded in Sect. 4, making use of the \(\tilde{\partial}\) techniques. Finally, in Sect. 5, we recall, following [1], the link between the operator \(\Pi_V\) in Theorem 1.1 and the orthogonal projection (1.1), showing that the kernels of (1.1) and (1.3) are close pointwise, locally.
2 Asymptotic inversion of a Fourier integral operator

The discussion in this section can be viewed as a natural analog in the $C^\infty$-setting of [9, Section 3], working systematically with almost holomorphic extensions of the weights, [20, 25]. Let $\Omega \subset \mathbb{C}^n$ be open, and let $\Phi \in C^\infty(\Omega; \mathbb{R})$ be strictly plurisubharmonic in $\Omega$,

$$
\sum_{j,k=1}^{n} \frac{\partial^2 \Phi}{\partial x_j \partial x_k}(x) \xi_j \overline{\xi}_k \geq c(x) |\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{C}^n,
$$

(2.1)

where $0 < c \in C(\Omega)$. Let $x_0 \in \Omega$. Identifying $\mathbb{C}^n$ with the anti-diagonal $\{(x, y) \in \mathbb{C}^{2n}; y = \overline{x}\}$, we see that exists $\Psi \in C^\infty(\text{neigh}(x_0), \mathbb{C}^{2n})$ such that

$$
\Psi(x, \overline{x}) = \Phi(x), \quad x \in \text{neigh}(x_0, \mathbb{C}^n),
$$

(2.2)

and for every $N$,

$$
\left(\partial_{\overline{x}_j} \Psi\right)(x, y) = O_N(|y - \overline{x}|^N), \quad \left(\partial_{\overline{y}_j} \Psi\right)(x, y) = O_N(|y - \overline{x}|^N), \quad 1 \leq j \leq n,
$$

(2.3)

locally uniformly, see [20, 24, 25], [11, Chapter 8]. Here we work with the usual operators,

$$
\partial_{\overline{x}_j} = \frac{1}{2} \left(\partial_{\text{Re} x_j} + i \partial_{\text{Im} x_j}\right), \quad \partial_{\overline{y}_j} = \frac{1}{2} \left(\partial_{\text{Re} y_j} + i \partial_{\text{Im} y_j}\right), \quad 1 \leq j \leq n.
$$

(2.4)

$$
\partial_{x_j} = \frac{1}{2} \left(\partial_{\text{Re} x_j} - i \partial_{\text{Im} x_j}\right), \quad \partial_{y_j} = \frac{1}{2} \left(\partial_{\text{Re} y_j} - i \partial_{\text{Im} y_j}\right), \quad 1 \leq j \leq n.
$$

(2.5)

We notice that (2.3) and [30, Lemma X.2.2] imply that for all $\alpha, \beta, \gamma, \delta \in \mathbb{N}^n$ we have

$$
\left(D^\alpha_{x} D^\beta_{\overline{y}} D^\gamma_{\overline{x}} D^\delta_{y} \partial_{\overline{x}_j} \Psi\right)(x, y) = O_{\alpha,\beta,\gamma,\delta,N}(|y - \overline{x}|^N), \quad 1 \leq j \leq n,
$$

(2.6)

$$
\left(D^\alpha_{x} D^\beta_{\overline{y}} D^\gamma_{\overline{x}} D^\delta_{y} \partial_{y_j} \Psi\right)(x, y) = O_{\alpha,\beta,\gamma,\delta,N}(|y - \overline{x}|^N), \quad 1 \leq j \leq n,
$$

(2.7)

locally uniformly.

Our starting point is the following classical estimate, see for instance [1]. Since related computations based on Taylor expansions will appear below, it will be natural to recall the proof.

**Proposition 2.1** We have

$$
\Phi(x) + \Phi(y) - 2\text{Re} \, \Psi(x, \overline{y}) \asymp |x - y|^2, \quad x, y \in \text{neigh}(x_0, \mathbb{C}^n).
$$

(2.8)
Proof Using (2.2), (2.6), (2.7) we get

\[
\begin{align*}
(\partial_x \Psi)(x, \overline{x}) &= \partial_x \Phi(x), \\
(\partial_y \Psi)(x, \overline{x}) &= \partial_{\overline{x}} \Phi(x), \\
\Psi''_{xx}(x, \overline{x}) &= \Phi''_{xx}(x), \\
\Psi''_{xy}(x, \overline{x}) &= \Phi''_{x\overline{x}}(x),
\end{align*}
\]

(2.9)

By a Taylor expansion, we obtain then, using (2.2), (2.6), (2.7), (2.9), (2.10),

\[
\Psi(x + z, \overline{x} + w) = \Phi(x) + \partial_x \Phi(x) \cdot z + \partial_{\overline{x}} \Phi(x) \cdot w \\
+ \frac{1}{2} \left( \Phi''_{xx}(x)z \cdot z + 2 \Phi''_{x\overline{x}}(x)w \cdot z \\
+ \Phi''_{x\overline{x}}(x)w \cdot w \right) + O(|(z, w)|^3),
\]

and therefore,

\[
2 \text{Re} \Psi(x + z, \overline{x} + w) = 2 \Phi(x) + \partial_x \Phi(x) \cdot (z + \overline{w}) + \partial_{\overline{x}} \Phi(x) \cdot (\overline{z} + w) \\
+ \frac{1}{2} \left( 2 \text{Re} \left( \Phi''_{xx}(x)z \cdot z + \Phi''_{x\overline{x}}(x)\overline{w} \cdot \overline{w} \right) \\
+ 2 \Phi''_{x\overline{x}}(x)w \cdot \overline{z} \right) + O(|(z, w)|^3).
\]

(2.11)

Here we have used that \(\partial_{\overline{x}} \Phi(x) = \overline{\partial_x \Phi(x)}\). Using also the Taylor expansions

\[
\begin{align*}
\Phi(x + z) &= \Phi(x) + \partial_x \Phi(x) \cdot z + \overline{\partial_x \Phi(\overline{z})} \\
&+ \frac{1}{2} \left( \Phi''_{xx}(x)z \cdot z + 2 \Phi''_{x\overline{x}}(x)\overline{z} \cdot z \\
+ \Phi''_{x\overline{x}}(x)\overline{z} \cdot \overline{z} \right) + O(|z|^3), \quad (2.12) \\
\Phi(x + \overline{w}) &= \Phi(x) + \partial_x \Phi(x) \cdot \overline{w} + \overline{\partial_x \Phi(x)} \cdot w \\
&+ \frac{1}{2} \left( \Phi''_{xx}(x)\overline{w} \cdot \overline{w} + 2 \Phi''_{x\overline{x}}(x)w \cdot \overline{w} \\
+ \Phi''_{x\overline{x}}(x)w \cdot w \right) + O(|w|^3), \quad (2.13)
\end{align*}
\]

we get from (2.11), (2.12), and (2.13),

\[
\Phi(x + z) + \Phi(x + \overline{w}) - 2 \text{Re} \Psi(x + z, \overline{x} + w) = \Phi''_{x\overline{x}}(x)(w - \overline{z}) \cdot (\overline{w} - z) \\
+ O(|(z, w)|^3). 
\]

(2.14)

Hence we get for \(x, y \in \text{neigh}(x_0, \mathbb{C}^n)\),

\[
\Phi(x) + \Phi(y) - 2 \text{Re} \Psi(x, \overline{y}) = \Phi''_{x\overline{x}}(x_0)(\overline{y} - \overline{x}) \cdot (y - x) + O(|y - x|^3), \quad (2.15)
\]

and using the strict plurisubharmonicity of \(\Phi\) given in (2.1), we infer (2.8). \(\square\)
Let us set, following [9, Section 3],
\[ \varphi(y, \tilde{x}; x, \tilde{y}) = \Psi(x, \tilde{y}) - \Psi(x, \tilde{x}) - \Psi(y, \tilde{y}) + \Psi(y, \tilde{x}). \] (2.16)

We have \( \varphi \in C^\infty(\text{neigh}((x_0, \overline{x_0}); x_0, \overline{x_0}), C^{4n}) \). Of particular interest for us here are critical points of \((x, \tilde{y}) \mapsto \varphi(y, \tilde{x}; x, \tilde{y})\).

**Proposition 2.2** For each \((y, \tilde{x}) \in \text{neigh}((x_0, \overline{x_0}), C^{2n})\), the complex valued \( C^\infty \) function

\[ \text{neigh}((x_0, \overline{x_0}), C^{2n}) \ni (x, \tilde{y}) \mapsto \varphi(y, \tilde{x}; x, \tilde{y}) \]

has a unique critical point given by \((x, \tilde{y}) = (y, \tilde{x})\). The corresponding critical value is equal to 0, and when \( \tilde{x} = \tilde{y} \), the quadratic part of the Taylor expansion of the \( C^\infty \) function \( C^{2n} \ni (z, w) \mapsto \varphi(y, \tilde{x}; y + z, \tilde{x} + w) \) at \((z, w) = (0, 0)\) is the non-degenerate holomorphic quadratic form on \( C^{2n} \) given by \((z, w) \mapsto \Phi''_{xx}(y) w \cdot z\).

**Proof** We shall consider the Taylor expansion of the \( C^\infty \) function \( (z, w) \mapsto \varphi(y, \tilde{x}; y + z, \tilde{x} + w) \) at \((z, w) = (0, 0) \in C^{2n}\). Here \((y, \tilde{x}) \in \text{neigh}((x_0, \overline{x_0}), C^{2n})\). Let us write, using Taylor’s formula and the almost holomorphy of \( \Psi \) along the anti-diagonal, as in (2.3), (2.6), (2.7),

\[
\Psi(y + z, \tilde{x} + w) = \Psi(y, \tilde{x}) + \Psi'_x(y, \tilde{x}) \cdot z + \Psi'_y(y, \tilde{x}) \cdot \overline{w} \\
+ \Psi''_{yx}(y, \tilde{x}) w \cdot z + \Psi''_{yy}(y, \tilde{x}) w \cdot \overline{w} \\
+ \frac{1}{2} \left( \Psi''_{xx}(y, \tilde{x}) z \cdot z + 2 \Psi''_{xy}(y, \tilde{x}) w \cdot z + \Psi''_{yy}(y, \tilde{x}) w \cdot \overline{w} \right) \\
+ O_N(|y - \tilde{x}|^N)((z, w)|^2) + O(|z|^3). \] (2.17)

\[
\Psi(x, \tilde{x} + w) = \Psi(y, \tilde{x}) + \Psi'_x(y, \tilde{x}) \cdot w + \Psi'_y(y, \tilde{x}) \cdot \overline{w} \\
+ \frac{1}{2} \Psi''_{yy}(y, \tilde{x}) w \cdot w + O_N(|y - \tilde{x}|^N)|w|^2 + O(|w|^3). \] (2.18)

\[
\Psi(y + z, \tilde{x}) = \Psi(y, \tilde{x}) + \Psi'_x(y, \tilde{x}) \cdot z + \Psi'_y(y, \tilde{x}) \cdot \overline{w} \\
+ \frac{1}{2} \Psi''_{xx}(y, \tilde{x}) z \cdot z + O_N(|y - \tilde{x}|^N)|z|^2 + O(|z|^3). \] (2.19)

Here \( N \in \mathbb{N} \) is arbitrary. We get, using (2.16), (2.17), (2.18), (2.19),

\[
\varphi(y, \tilde{x}; y + z, \tilde{x} + w) = \Psi''_{xy}(y, \tilde{x}) w \cdot z + O_N(|y - \tilde{x}|^N)((z, w)|^2) \\
+ O(|z, w|^3). \] (2.20)

This shows in particular that \((z, w) = (0, 0)\) is a critical point of \((z, w) \mapsto \varphi(y, \tilde{x}; y + z, \tilde{x} + w)\), with the corresponding critical value being equal to 0. We notice also that the matrix \( \Psi''_{xy}(y, \tilde{x}) \) is invertible for \((y, \tilde{x}) \in \text{neigh}((x_0, \overline{x_0}), C^{2n})\). Restricting \((y, \tilde{x})\) in (2.20) to the anti-diagonal in \( C^{2n} \), i.e. letting \( \tilde{x} = \overline{y} \), and using (2.10), we get,

\[
\varphi(y, \overline{y}; y + z, \overline{y} + w) = \Phi''_{xx}(y) w \cdot z + O(|z, w|^3). \] (2.21)
Here, in view of (2.1), the holomorphic quadratic form \((z, w) \mapsto \Phi''_{x\overline{x}}(y) w \cdot z\) is non-degenerate on \(C^{2n}_{z,w}\), for \(y \in \text{neigh}(x_0, C^n)\), and this completes the proof. \(\square\)

It follows from the last observation in the proof of Proposition 2.2 that the pluriharmonic quadratic form \(q(z, w) := \text{Re} \left( \Phi''_{x\overline{x}}(x_0) w \cdot z \right)\) is non-degenerate on \(C^{2n}_{z,w}\), and hence necessarily of signature \((2n, 2n)\). Explicitly, we have

\[
q(z, \overline{z}) = \text{Re} \left( \Phi''_{x\overline{x}}(x_0) \overline{z} \cdot z \right) \asymp |z|^2, \quad z \in \mathbb{C}^n, \tag{2.22}
\]

and therefore

\[
q(iz, i\overline{z}) = -q(z, \overline{z}) \asymp -|z|^2, \quad z \in \mathbb{C}^n. \tag{2.23}
\]

Using the continuity of \(\Psi''_{xy}(y, \tilde{x})\) we conclude that for all \((y, \tilde{x}) \in \mathbb{C}^{2n}\) sufficiently close to \((x_0, \overline{x_0})\), we have

\[
\text{Re} \left( \Psi''_{xy}(y, \tilde{x}) \overline{z} \cdot z \right) \asymp |z|^2, \quad z \in \mathbb{C}^n, \tag{2.24}
\]

\[
\text{Re} \left( \Psi''_{xy}(y, \tilde{x}) i\overline{z} \cdot iz \right) \asymp -|z|^2, \quad z \in \mathbb{C}^n. \tag{2.25}
\]

Combining (2.20) with (2.24), (2.25), we get for all \((y, \tilde{x}) \in \text{neigh}((x_0, \overline{x_0}), \mathbb{C}^{2n})\),

\[
\text{Re} \varphi(y, \tilde{x}; y + z, \tilde{x} + \overline{z}) = \text{Re} \left( \Psi''_{xy}(y, \tilde{x}) \overline{z} \cdot z \right) + O_N(|\overline{y} - \tilde{x}|^N) |z|^2 + O(|z|^3) \\
\geq \frac{1}{C} |z|^2, \quad z \in \text{neigh}(0, \mathbb{C}^n), \tag{2.26}
\]

\[
\text{Re} \varphi(y, \tilde{x}; y + iz, \tilde{x} + i\overline{z}) = \text{Re} \left( \Psi''_{xy}(y, \tilde{x}) i\overline{z} \cdot iz \right) + O_N(|\overline{y} - \tilde{x}|^N) |z|^2 + O(|z|^3) \\
\leq -\frac{1}{C} |z|^2, \quad z \in \text{neigh}(0, \mathbb{C}^n). \tag{2.27}
\]

It follows from (2.20), (2.26), (2.27) that for each \((y, \tilde{x}) \in \text{neigh}((x_0, \overline{x_0}), \mathbb{C}^{2n})\), the critical point \((x, \tilde{y}) = (y, \tilde{x})\) of the real-valued \(C^\infty\) function \(\text{neigh}((x_0, \overline{x_0}), \mathbb{C}^{2n}) \ni (x, \tilde{y}) \mapsto \text{Re} \varphi(y, \tilde{x}; x, \tilde{y})\) is non-degenerate of signature \((2n, 2n)\).

Let \(\Gamma(y, \tilde{x}) \subset C^{2n}_{x,\overline{y}}\) be a smooth \(2n\)-dimensional contour of integration passing through the critical point \((x, \tilde{y}) = (y, \tilde{x})\) and depending smoothly on \((y, \tilde{x}) \in \text{neigh}((x_0, \overline{x_0}), \mathbb{C}^{2n})\), such that along \(\Gamma(y, \tilde{x})\), we have

\[
\text{Re} \varphi(y, \tilde{x}; x, \tilde{y}) \leq -\frac{1}{C} \text{dist} ((x, \tilde{y}), (y, \tilde{x}))^2. \tag{2.28}
\]

Following [28, Chapter 3], we shall say that \(\Gamma(y, \tilde{x})\) is a good contour for the \(C^\infty\) real-valued function \((x, \tilde{y}) \mapsto \text{Re} \varphi(y, \tilde{x}; x, \tilde{y})\). In particular, it follows from (2.27)
that the $2n$-dimensional affine contour

$$\Gamma(y, \tilde{x}) : \text{neigh}(0, C^n) \ni z \mapsto (y + z, \tilde{x} - z) \in C^{2n}_{x,y}$$

(2.29)

is good.

Proceeding similarly to [9, Section 3], we shall now introduce a suitable Fourier integral operator in the complex domain, with the function $\phi$ in (2.16) playing the role of the phase function, with no fiber variables present. To this end, let us first specify a suitable class of amplitudes. Let $a \in S^0_cl(\text{neigh}((x_0, \bar{x}_0), C^{2n}))$,

$$a(x, \tilde{y}; h) \sim \sum_{j=0}^{\infty} a_j(x, \tilde{y})h^j, \quad h \to 0^+,$$

(2.30)

be a classical $C^\infty$ symbol, with the asymptotic expansion (2.30) in $C^\infty(\text{neigh}((x_0, \bar{x}_0), C^{2n}))$, such that $a_j \in C^\infty(\text{neigh}((x_0, \bar{x}_0), C^{2n}))$ satisfy

$$\left( \partial_{x_0} a_j \right)(x, \tilde{y}) = \mathcal{O}(|\tilde{y} - x|^{\infty}), \quad \left( \partial_{\tilde{x}} a_j \right)(x, \tilde{y}) = \mathcal{O}(|\tilde{y} - x|^{\infty}),$$

$$j = 0, 1, 2, \ldots$$

(2.31)

Given a good contour $\Gamma(y, \tilde{x}) \subset C^{2n}_{x,y}$ for the $C^\infty$ function $(x, \tilde{y}) \mapsto \text{Re} \, \phi(y, \tilde{x}; x, \tilde{y})$ and an amplitude $a(x, \tilde{y}; h)$ satisfying (2.30), (2.31), we set

$$(A_\Gamma a)(y, \tilde{x}; h) = \frac{1}{h^n} \int_{\Gamma(y, \tilde{x})} e^{\frac{1}{2} \phi(y, \tilde{x}; x, \tilde{y})} a(x, \tilde{y}; h) \, dx \, d\tilde{y}.$$  

(2.32)

We have $A_\Gamma a \in C^\infty(\text{neigh}((x_0, \bar{x}_0), C^{2n}))$, and let us first check that the definition of $A_\Gamma a$ is essentially independent of the choice of a good contour, up to a rapidly vanishing error as $h \to 0^+$, provided that $(y, \tilde{x})$ is confined to the anti-diagonal.

**Proposition 2.3** There exists an open neighborhood $V_0 \Subset \Omega \subset C^n$ of $x_0$ such that for any two good contours $\Gamma(y, \tilde{y}), \Gamma_0(y, \tilde{y})$ for the function $(x, \tilde{y}) \mapsto \text{Re} \, \phi(y, \tilde{x}; x, \tilde{y})$, for $y \in V_0$, and any amplitude $a$ satisfying (2.30), (2.31), we have for $y \in V_0$,

$$\left(A_\Gamma a\right)(y, \tilde{y}; h) - \left(A_{\Gamma_0} a\right)(y, \tilde{y}; h) = \mathcal{O}(h^\infty),$$

(2.33)

in the $C^\infty(V_0)$ sense.

**Proof** Let us start by making some general remarks concerning good contours when parameters are present, closely related to the discussion in [28, Chapter 3], [10, Chapter 1]. Let $f(x, y), x \in \mathbb{R}^n, y \in \mathbb{R}^{2N}$, be a real-valued $C^\infty$ function in a neighborhood of $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^{2N}$. Assume that $f'_x(0, 0) = 0$ and that $f''_{yy}(0, 0)$ is non-degenerate of signature $(N, N)$. The implicit function theorem gives that the equation $f'_y(x, y) = 0$ uniquely defines a $C^\infty$ function $y = y(x)$ in a neighborhood of $0$, with $y(0) = 0$, and applying the Morse lemma with parameters [21, Appendix C], we obtain that there
exist new $C^\infty$ coordinates $z = z(y) = z(x, y)$ for $\mathbb{R}^{2N}_y$ near $y = 0$ when $x \in \mathbb{R}^n$ is small, such that writing $z = (t, s)$ with $t, s \in \mathbb{R}^N$, we have

$$f(x, y) = f(x, y(x)) + \frac{1}{2}(t^2 - s^2).$$

(2.34)

We may also recall from [21, Appendix C] that the new coordinates, which also depend on $x$, are centered at the critical point $y(x)$ and are of the form

$$z = Q(x, y)(y - y(x)),$$

(2.35)

where the matrix $Q(0, 0)$ is invertible. Assume next that $\Gamma(x) \subset \mathbb{R}^{2N}_y$ is a good contour for $y \mapsto f(x, y)$, depending smoothly on $x$, so that $\Gamma(x)$ passes through $y(x)$ and along $\Gamma(x)$, we have for all $x \in \mathbb{R}^n$ small enough,

$$f(x, y) \leq f(x, y(x)) - \frac{1}{C} |y - y(x)|^2, \quad y \in \Gamma(x),$$

(2.36)

for some $C > 0$. It follows from (2.34), (2.35), and (2.36) that along $\Gamma(x)$, we have

$$\frac{1}{2}(t^2 - s^2) \leq -\frac{1}{O(1)}(t^2 + s^2),$$

(2.37)

and by the implicit function theorem, we obtain therefore that in the Morse coordinates $z = (t, s)$, the contour $\Gamma(x)$ takes the form

$$t = g(s, x), \quad s \in \text{neigh}(0, \mathbb{R}^N),$$

(2.38)

where

$$|g(s, x)| \leq \alpha |s|, \quad \alpha < 1.$$

(2.39)

See also [28, Chapter 3]. Throughout the discussion above, $x \in \mathbb{R}^n$ varies in a sufficiently small neighborhood of the origin. Letting $z \mapsto y(z)$ be the inverse of the map $y \mapsto z(y)$, well defined for $x$ small, we obtain the following parametrization of the good contour $\Gamma(x)$,

$$\text{neigh}(0, \mathbb{R}^N) \ni s \mapsto y(g(s, x), s) \in \mathbb{R}^{2N}_y.$$  

(2.40)

A consequence of this discussion is that if $\Gamma_0(x) \subset \mathbb{R}^{2N}_y$ is another good contour for $y \mapsto f(x, y)$, depending smoothly on $x$, then its representation in the Morse coordinates $z = (t, s)$ takes the form

$$t = g_0(s, x), \quad s \in \text{neigh}(0, \mathbb{R}^N),$$

(2.41)

$$|g_0(s, x)| \leq \alpha |s|, \quad \alpha < 1,$$

(2.42)
and the contours $\Gamma(x)$, $\Gamma_0(x)$ are therefore homotopic via the deformation

$$\text{neigh}(0, \mathbb{R}^N) \times [0, 1] \ni (s, \theta) \mapsto y(\theta g(s, x) + (1 - \theta) g_0(s, x), s), \tag{2.43}$$

well defined for all $x$ small enough, with the $C^\infty$ dependence on $x$. It follows also from (2.34), (2.38), (2.39), (2.41), (2.42) that when $\theta \in [0, 1]$, the contour

$$\Gamma_\theta(x) : \text{neigh}(0, \mathbb{R}^N) \ni s \mapsto y(\theta g(s, x) + (1 - \theta) g_0(s, x), s), \tag{2.44}$$

is good for $y \mapsto f(x, y)$, uniformly in $\theta \in [0, 1]$ (and $x \in \mathbb{R}^n$ small enough).

We shall now apply the discussion above to the $C^\infty$ function

$$\text{neigh}((x_0, \bar{x}_0), \mathbb{C}^{2n}) \ni (x, \bar{y}) \mapsto \text{Re} \varphi(y, \bar{y}; x, \bar{y}), \tag{2.45}$$

with $y \in \text{neigh}(x_0, \mathbb{C}^n)$ playing the role of the parameters. Let $\Gamma = \Gamma(y)$, $\Gamma_0 = \Gamma_0(y)$ be two good contours for the function in (2.45), and let $\Gamma_\theta(y) \subset \mathbb{C}^{2n}_{x, \bar{y}}$ be the "intermediate" contour defined as in (2.44), for $\theta \in [0, 1]$, where $N = 2n$. Letting $G_{[0, 1]}(y) \subset \mathbb{C}^{2n}_{x, \bar{y}}$ be the $(2n + 1)$–dimensional contour formed by the union of the contours $\Gamma_\theta(y)$, for $\theta \in [0, 1]$, parametrized as in (2.43), we may write by an application of Stokes’ formula to the $(2n, 0)$–differential form $\omega = e^{\frac{2}{\pi} \varphi(y, \bar{y}; x, \bar{y})} \theta(x, \bar{y}; h) h \wedge d\bar{y}$,

$$\int_{\partial G_{[0, 1]}(y)} \omega = \int_{\partial G_{[0, 1]}(y)} \int_{G_{[0, 1]}(y)} d\omega = \int_{\partial G_{[0, 1]}(y)} \int_{G_{[0, 1]}(y)} d \left( e^{\frac{2}{\pi} \varphi(y, \bar{y}; x, \bar{y})} \theta(x, \bar{y}; h) \right) a(x, \bar{y}; h) h \wedge d\bar{y}$$

$$= \int_{\partial G_{[0, 1]}(y)} \int_{G_{[0, 1]}(y)} \partial_x \left( e^{\frac{2}{\pi} \varphi(y, \bar{y}; x, \bar{y})} \theta(x, \bar{y}; h) \right) a(x, \bar{y}; h) h \wedge d\bar{y}$$

$$+ \int_{\partial G_{[0, 1]}(y)} \int_{G_{[0, 1]}(y)} \partial_{\bar{y}} \left( e^{\frac{2}{\pi} \varphi(y, \bar{y}; x, \bar{y})} \theta(x, \bar{y}; h) \right) a(x, \bar{y}; h) h \wedge d\bar{y}. \tag{2.46}$$

Using (2.16) we compute, for $1 \leq j \leq n$,

$$\partial_{x_j} \left( e^{\frac{2}{\pi} \varphi(y, \bar{y}; x, \bar{y})} \theta(x, \bar{y}; h) \right) = e^{\frac{2}{\pi} \varphi(y, \bar{y}; x, \bar{y})} \frac{2}{\pi} \partial_x \varphi(y, \bar{y}; x, \bar{y}) \theta + \partial_{x_j} \theta(x, \bar{y}; h) \right)$$

$$= e^{\frac{2}{\pi} \varphi(y, \bar{y}; x, \bar{y})} \frac{2}{\pi} \partial_x \varphi(y, \bar{y})$$

$$- \psi(x, \bar{y}) \theta(a(x, \bar{y}; h) + \partial_{x_j} \theta(x, \bar{y}; h)), \tag{2.47}$$

and here we have in view of (2.3), for all $N$,

$$|\partial_{x_j} (\psi(x, \bar{y}) - \psi(x, \bar{y}))| \leq O(1) \left( |x - \bar{y}|^N + |x - y|^N \right)$$

$$\leq O(1) \left( |x - \bar{y}|^N + |\bar{y} - \bar{y}|^N + |x - y|^N \right)$$

$$\leq O(1) \left( |\bar{y} - \bar{y}|^N + |x - y|^N \right)$$

$$\leq O(1) \text{dist} ((x, \bar{y}), (y, \bar{y}))^N. \tag{2.48}$$
Furthermore, using (2.30) and (2.31) we get

\[ \left| \partial_x a(x, \tilde{y}; h) \right| \leq O_N(1) \left( |x - \tilde{y}|^N + h^N \right) \]
\[ \leq O_N(1) \left( \text{dist} ((x, \tilde{y}), (y, \overline{y}))^N + h^N \right). \quad (2.49) \]

We get therefore using (2.28), (2.47), (2.48), (2.49), uniformly along \( G_{[0,1]}(y) \),

\[ \left| \partial_x \left( e^{2 \theta \psi(y, x, \tilde{y})} a(x, \tilde{y}; h) \right) \right| \leq O_N(1) e^{2 \Re \psi(y, x, \tilde{y})} \]
\[ \left( \frac{1}{h} \text{dist} ((x, \tilde{y}), (y, \overline{y}))^N + h^N \right) \]
\[ O_N(1) e^{-\frac{1}{2} \dist ((x, \tilde{y}), (y, \overline{y}))^2} \left( \frac{1}{h} \text{dist} ((x, \tilde{y}), (y, \overline{y}))^N + h^N \right). \quad (2.50) \]

Using that

\[ e^{-t^2/Ch} N \leq O_N(1) h^{N/2}, \quad t \geq 0, \quad N = 1, 2, \ldots, \]

we conclude that we have uniformly along \( G_{[0,1]}(y) \), for all \( N \),

\[ \left| \partial_x \left( e^{2 \theta \psi(y, x, \tilde{y})} a(x, \tilde{y}; h) \right) \right| \leq O_N(1) h^N. \quad (2.51) \]

This bound is uniform in \( y \in \text{neigh}(x_0, \mathbb{C}^n) \). Next, when considering

\[ \partial_x \left( e^{2 \theta \psi(y, x, \tilde{y})} a(x, \tilde{y}; h) \right) = e^{2 \theta \psi(y, x, \tilde{y})} \left( \frac{2}{h} \partial_x \psi(y, x, \tilde{y}) a(x, \tilde{y}; h) + \partial_x a(x, \tilde{y}; h) \right) \]
\[ + \partial_x \psi(y, x, \tilde{y}) a(x, \tilde{y}; h) \]
\[ = e^{2 \theta \psi(y, x, \tilde{y})} \left( \frac{2}{h} \partial_x \psi(y, x, \tilde{y}) (\Psi(x, \tilde{y})) \right. \]
\[ \left. - \Psi(y, \tilde{y}) a(x, \tilde{y}; h) + a_x a(x, \tilde{y}; h) \right), \quad (2.52) \]

we write similarly to (2.48), for \( N = 1, 2, \ldots, \)

\[ \left| \partial_x (\Psi(x, \tilde{y}) - \Psi(y, \tilde{y})) \right| \leq O_N(1) \left( |x - \tilde{y}|^N + |y - \tilde{y}|^N \right) \]
\[ \leq O_N(1) \left( |x - y|^N + |y - \tilde{y}|^N + |y - \overline{y}|^N \right) \]
\[ \leq O_N(1) \left( |x - y|^N + |y - \overline{y}|^N \right) \]
\[ \leq O_N(1) \text{dist} ((x, \tilde{y}), (y, \overline{y}))^N. \quad (2.53) \]
Using also (2.30), (2.31), we conclude that

\[
\left| \partial_{\widetilde{y}} \left( e^{\frac{2}{h} \varphi(y, \widetilde{y}; x, y)} a(x, \widetilde{y}; h) \right) \right| \leq O_N(1) e^{-\frac{1}{e^{\varphi}} dist((x, \widetilde{y}), (y, \widetilde{y}))^2} \left( \frac{1}{h} \right)^{\text{dist}((x, \widetilde{y}), (y, \widetilde{y}))^N + hN} ,
\]

(2.54)

and therefore, similar to (2.51), we get uniformly along \( G_{[0,1]}(y) \), for all \( N \),

\[
\left| \partial_{\widetilde{y}} \left( e^{\frac{2}{h} \varphi(y, \widetilde{y}; x, y)} a(x, \widetilde{y}; h) \right) \right| \leq O_N(1) h^N.
\]

(2.55)

We get, combining (2.46), (2.51), and (2.55),

\[
\int_{\partial G_{[0,1]}(y)} e^{\frac{2}{h} \varphi(y, \widetilde{y}; x, y)} a(x, \widetilde{y}; h) dx \wedge d\widetilde{y} = O(h^\infty),
\]

(2.56)

uniformly for \( y \in \text{neigh}(x_0, C^n) \). Here we may write, with a suitable orientation,

\[
\partial G_{[0,1]}(y) = \Gamma(y) - \Gamma_0(y) + \Gamma_1(y),
\]

where

\[
\text{Re} \varphi(y, \widetilde{y}; x, \widetilde{y}) \leq -\frac{1}{C}, \quad (x, \widetilde{y}) \in \Gamma_1(y),
\]

(2.57)

for some \( C > 0 \), when \( y \in \text{neigh}(x_0, C^n) \). It follows that

\[
(A_\Gamma a)(y, \widetilde{y}; h) - (A_{\Gamma_0} a)(y, \widetilde{y}; h) = O(h^\infty),
\]

(2.58)

uniformly for \( y \in \text{neigh}(x_0, C^n) \). Let us see, finally, that the relation (2.58) holds in the \( C^\infty \) sense, i.e. also for the derivatives of \( A_\Gamma a - A_{\Gamma_0} a \). To this end, we observe first that for all \( \alpha, \beta \in \mathbb{N}^n \) there exists \( M_{\alpha\beta} \geq 0 \) such that

\[
\partial^\alpha_y \partial^\beta_{\widetilde{y}} (A_\Gamma a - (A_{\Gamma_0} a)) (y, \widetilde{y}; h) = O_a(1) h^{-M_{\alpha\beta}} .
\]

(2.59)

Combining (2.58), (2.59) with the convexity estimates for the derivatives of a smooth function [15, Chapter 1], we conclude that

\[
\partial^\alpha_y \partial^\beta_{\widetilde{y}} (A_\Gamma a - (A_{\Gamma_0} a)) (y, \widetilde{y}; h) = O(h^\infty),
\]

(2.60)

uniformly, after an arbitrarily small decrease of the neighborhood of \( x_0 \in C^n \) where (2.58) holds. The proof is complete. \( \Box \)

Remark The proof of Proposition 2.3 shows that in order for the function \((A_\Gamma a)(y, \widetilde{x}; h)\) to be independent of the choice of a good contour \( \Gamma \), up to \( O(h^\infty) \), it is essential that \((y, \widetilde{x})\) should be confined to the anti-diagonal, so that \( \widetilde{x} = \widetilde{y} \).
We shall next proceed to establish the existence of a complete asymptotic expansion for \( (A_\Gamma a)(y, \vec{y}; h) \), as \( h \to 0^+ \). When doing so, thanks to Proposition 2.3, it will be convenient to work with the particular choice of the good contour \( \Gamma(y, \vec{y}) \) given in (2.29). Using the parametrization of \( \Gamma(y, \vec{y}) \) given in (2.29) and (2.32), we get

\[
A_\Gamma a(y, \vec{y}; h) = \frac{C_n}{h^n} \int_U e^{i \frac{f(y, z)}{h}} b(y, z; h) \, L(dz)
\]  
(2.61)

Here \( f(y, z) = -2i \varphi(y, \vec{y}; y + z, \vec{y} - \vec{z}), b(y, z; h) = a(y + z, \vec{y} - \vec{z}; h), \) and \( L(dz) \) is the Lebesgue measure on \( \mathbb{C}^n \). Furthermore, the constant \( C_n \neq 0 \) in (2.61) depends on the dimension \( n \) only and the region of integration \( U \subset \mathbb{C}^n \) is a small neighborhood of the origin. Using (2.21) we see that

\[
f(y, z) = 2i \Phi'''_{x\bar{\tau}}(y) \bar{z} \cdot z + \mathcal{O}(|z|^3),
\]  
(2.62)

so that in particular

\[
\text{Im } f(y, z) \geq \frac{C}{h^2} |z|^2, \quad z \in U,
\]  
(2.63)

for some \( C > 0 \) and all \( y \in \mathbb{C}^n \) close enough to \( x_0 \). For future reference, we shall now proceed to compute \( \det (\nabla^2 f(y, 0)/i) \), where the Hessian \( \nabla^2 z \) is taken in the real sense of \( \mathbb{R}^{2n} \cong \mathbb{C}^n \), so that \( \nabla^2 f(y, 0)/i \) is a real symmetric \( 2n \times 2n \) matrix. Writing \( \mathbb{C}^n \ni z = t + is, t, s \in \mathbb{R}^n \), we see that the quadratic part of the Taylor expansion of \( z \mapsto f(y, z)/i \) at the origin, given in (2.62), is of the form

\[
2 \Phi''_{x\bar{\tau}}(y) \bar{z} \cdot z = 2 \left( \Phi''_{x\bar{\tau}}(y) t \cdot t + \Phi''_{x\bar{\tau}}(y) s \cdot s + i \Phi''_{x\bar{\tau}}(y) t \cdot s - i \Phi''_{x\bar{\tau}}(y) s \cdot t \right)
= 2 \left( A_1 t \cdot t + A_1 s \cdot s - 2 A_2 t \cdot s \right).
\]  
(2.64)

Here we have written \( \Phi''_{x\bar{\tau}}(y) = A_1 + i A_2 \), with \( A_1, A_2 \) being \( n \times n \) real matrices and observed that since \( \Phi''_{x\bar{\tau}}(y) \) is Hermitian, we have \( A_1^t = A_1, A_2^t = -A_2 \). We get

\[
2 \Phi''_{x\bar{\tau}}(y) \bar{z} \cdot z = 2 \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix} \cdot \begin{pmatrix} t \\ s \end{pmatrix},
\]  
(2.65)

and writing

\[
\Phi''_{x\bar{\tau}}(y) \bar{z} \cdot z = \frac{1}{2} \begin{pmatrix} \Phi''_{x\bar{\tau}}(y) & 0 \\ 0 & \Phi''_{x\bar{\tau}}(y) \end{pmatrix} \begin{pmatrix} \bar{z} \\ z \end{pmatrix} \cdot \begin{pmatrix} \bar{z} \\ z \end{pmatrix},
\]  
(2.66)

\[
\begin{pmatrix} \bar{z} \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix}, \quad \begin{pmatrix} \bar{z} \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix},
\]  
(2.67)

we obtain the factorization

\[
\frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \Phi''_{x\bar{\tau}}(y) & 0 \\ 0 & \Phi''_{x\bar{\tau}}(y) \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}.
\]  
(2.68)
It follows from (2.62), (2.65), and (2.68) that
\[
\det \left( \frac{\nabla^2 z f(y, 0)}{i} \right) = 2^{4n} \det \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix} = 2^{4n} \left( \det (\Phi''_{xx}(y)) \right)^2, \tag{2.69}
\]
which is a smooth strictly positive function near \( y = x_0 \). For future reference, let us also compute the quadratic form
\[
\left( \nabla^2 z f(y, 0) \right)^{-1} \begin{pmatrix} t \\ s \end{pmatrix} \cdot \begin{pmatrix} t \\ s \end{pmatrix} = \frac{1}{2^2 i} \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}^{-1} \begin{pmatrix} t \\ s \end{pmatrix} \cdot \begin{pmatrix} t \\ s \end{pmatrix}, \tag{2.70}
\]
dual to \( \nabla^2 z f(y, 0) \). To this end, a simple computation using (2.68) shows that
\[
\left( A_1 A_2 - A_2 A_1 \right)^{-1} = \left( i \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \Phi''_{xx}(y) \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} i \\ 0 \end{pmatrix} \cdot \begin{pmatrix} i \\ 0 \end{pmatrix}, \tag{2.71}
\]
and therefore we get
\[
\left( \nabla^2 z f(y, 0) \right)^{-1} \begin{pmatrix} t \\ s \end{pmatrix} \cdot \begin{pmatrix} t \\ s \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} \Phi''_{xx}(y) \end{pmatrix}^{-1} \left( \begin{pmatrix} i \\ 0 \end{pmatrix} \cdot \begin{pmatrix} i \\ 0 \end{pmatrix} \right), \tag{2.72}
\]
Here the quadratic form in (2.72) can be regarded as the symbol of the second order constant coefficient differential operator on \( C^n_\zeta \) given by
\[
\frac{2}{i} \begin{pmatrix} \Phi''_{xx}(y) \end{pmatrix}^{-1} D_\zeta \cdot D_\bar{\zeta} = 2i \begin{pmatrix} \Phi''_{xx}(y) \end{pmatrix}^{-1} \partial_\zeta \cdot \partial_{\bar{\zeta}}, \tag{2.73}
\]
where \( D_\zeta = \frac{1}{i} \partial_\zeta = \frac{1}{2}(D_t - i D_s) \), \( D_{\bar{\zeta}} = \frac{1}{i} \partial_{\bar{\zeta}} = \frac{1}{2}(D_t + i D_s) \).
It follows from (2.62), (2.63), and (2.69) that we are in the position to apply complex stationary phase in the form given in [21, Theorem 7.7.5] to derive a complete asymptotic expansion for \( A_{\Gamma}a(y, \bar{y}; h) \) given in (2.61), as \( h \to 0^+ \). We obtain therefore that there exist differential operators \( L_{j, x}(D) \) in \((t, s)\) of order \( 2j \), which are \( C^\infty \) functions of \( y \in \text{neigh}(x_0, C^n) \), such that for each \( N \) we have uniformly for \( y \in \text{neigh}(x_0, C^n) \),
\[
(A_{\Gamma}a)(y, \bar{y}; h) = \sum_{j=0}^{N-1} h^j \left( L_{j, x}(D)b \right)(y, 0) + \mathcal{O}_N(h^N). \tag{2.74}
\]
Let us also recall from [21, Theorem 7.7.5], using also (2.73), the following explicit expressions for the operators \( L_j \),

\[
(L_j, y(D)b)(y, 0) = \frac{C_n \pi^n}{2^n \det(\Phi''_{x\bar{x}}(y))} \sum_{v=\mu=j} \sum_{\mu \geq \frac{3\mu}{2}} \frac{i^{v-j}}{\mu! v!} \left( (\Phi''_{x\bar{x}}(y))^{-1} \partial_z \cdot \partial_{\bar{\tau}} \right)^v (g^\mu b)(y, 0)
\]

where

\[
g(y, z) = f(y, z) - 2i\Phi''_{x\bar{x}}(y)\bar{z} \cdot z = O(|z|^3).
\]

In particular,

\[
L_{0, y} = \frac{C_n \pi^n}{2^n \det(\Phi''_{x\bar{x}}(y))}
\]

satisfies

\[
\frac{1}{C} \leq |L_{0, y}| \leq C, \quad y \in \text{neigh}(x_0, \mathbb{C}^n).
\]

The expansion (2.74) can be differentiated any number of times with respect to \( y, \bar{y} \). Following [9, Section 3], our purpose is now to show that there exists an amplitude \( a(x, \bar{y}; h) \in S_0^0(\text{neigh}((x_0, \bar{x}_0), \mathbb{C}^{2n})) \) satisfying (2.30), (2.31) such that

\[
(A_{\Gamma} a)(y, \bar{y}; h) = 1 + O(h^\infty),
\]

for \( y \in \text{neigh}(x_0, \mathbb{C}^n) \). Looking for \( a \) in the form (2.30), we may write in view of (2.74),

\[
(A_{\Gamma} a)(y, \bar{y}; h) \sim \sum_{\ell=0}^\infty h^\ell c_\ell(y), \quad c_\ell(y) = \sum_{j+k=\ell} (L_{k,y}(D)b_j)(y, 0),
\]

where \( b_j(y, z) = a_j(y + z, \bar{y} - \bar{z}) \). Using the expansion (2.79), we shall determine successively \( a_0, a_1, \ldots \) satisfying (2.31), so that

\[
c_0(y) = L_{0, y} a_0(y, \bar{y}) = 1, \quad c_\ell(y) = \sum_{j+k=\ell} (L_{k,y}(D)b_j)(y, 0) = 0, \quad \ell \geq 1.
\]

First, (2.80) determines the \( C^\infty \) function \( a_0(y, \bar{y}) \) uniquely, in view of (2.76), (2.77), and taking an almost holomorphic extension from the anti-diagonal, we obtain

\[
a_0(x, \bar{y}) \in C^\infty(\text{neigh}((x_0, \bar{x}_0), \mathbb{C}^{2n})),
\]
satisfying (2.31) for \( j = 0 \). Assume next that \( a_0, a_1, \ldots, a_{M-1} \), satisfying (2.31), have been determined so that (2.81) holds for \( \ell \leq M - 1 \). To determine \( a_M \), we consider the equation (2.81) with \( \ell = M \), writing

\[
c_M(y) = L_{0,y}a_M(y, \overline{y}) + \sum_{j+k=M-j=M} (L_{k,y}(D)b_j)(y, 0) = 0. \tag{2.82}
\]

Here we may notice that the expression in the sum in (2.82) only depends on the values of the \( a_j \)'s along the anti-diagonal, for \( j \leq M - 1 \). Indeed, for each \( \alpha \in \mathbb{N}^n \), we have in view of the almost holomorphy of \( a_j = a_j(x, \overline{y}) \) along the anti-diagonal given in (2.31), for \( j \leq M - 1 \),

\[
(D^\alpha z b_j)(y, 0) = D^\alpha z (a_j(y + z, \overline{y} - \overline{z}))|_{z=0} \\
= (D^\alpha z a_j)(y, \overline{y}) = D^\alpha y (a_j(y, \overline{y})), \tag{2.83}
\]

\[
(D^\alpha x b_j)(y, 0) = D^\alpha x (a_j(y + z, \overline{y} - \overline{z}))|_{z=0} \\
= (-1)^\alpha (D^\alpha y a_j)(y, \overline{y}) = (-1)^\alpha D^\alpha y (a_j(y, \overline{y})). \tag{2.84}
\]

It follows that (2.82) has a unique \( C^\infty \) solution \( a_M(y, \overline{y}) \), for \( y \in \text{neigh}(x_0, C^n) \), and we may then take an almost holomorphic extension. Modifying the choice of an almost holomorphic extension of \( a_M(y, \overline{y}) \) will only affect \( (A_\Gamma a)(y, \overline{y}; h) \) by a term which is \( O(h^\infty) \).

The discussion above may be summarized in the following theorem, which is the main result of this section.

**Theorem 2.4** There exists an elliptic symbol \( a(x, \overline{y}; h) \in S^0_{\text{cl}}(\text{neigh}((x_0, \overline{x}_0), C^{2n})) \) of the form

\[
a(x, \overline{y}; h) \sim \sum_{j=0}^{\infty} a_j(x, \overline{y})h^j, \tag{2.85}
\]

in \( C^\infty \), with \( a_j \in C^\infty(\text{neigh}((x_0, \overline{x}_0), C^{2n})) \) satisfying

\[
(\partial_\overline{\alpha} a_j)(x, \overline{y}) = O(|\overline{y} - \overline{x}|^\infty), \quad (\partial_y a_j)(x, \overline{y}) = O(|\overline{y} - \overline{x}|^\infty), \quad j = 0, 1, 2, \ldots, \tag{2.86}
\]

such that we have

\[
(A_\Gamma a)(y, \overline{y}; h) = \frac{1}{h^n} \iiint_{\Gamma(y, \overline{y})} e^{2\pi i \varphi(y, \overline{y}; x, \overline{y})} a(x, \overline{y}; h) \, dx \, d\overline{y} \\
= 1 + O(h^\infty), \quad y \in \text{neigh}(x_0, C^n), \tag{2.87}
\]

in the \( C^\infty \) sense. Here \( \Gamma(y, \overline{y}) \) is a good contour for the function \( (x, \overline{y}) \mapsto \text{Re} \varphi(y, \overline{y}; x, \overline{y}) \). The restrictions of the \( a_j \)'s to the anti-diagonal \( \overline{y} = \overline{x} \) are uniquely
determined, for \( j \geq 0 \), and we have
\[
a_0(x, \bar{x}) = A_n \det \left( \Phi''_{x\bar{x}}(x) \right), \quad x \in \text{neigh}(x_0, \mathbb{C}^n),
\]
with \( A_n \neq 0 \) depending on \( n \) only.

**Remark** The elliptic symbol \( a(x, \tilde{y}; h) \) constructed in Theorem 2.4 is unique in the following sense: assume that \( b(x, \tilde{y}; h) \in S^0_{cl}(\text{neigh}((x_0, \bar{x})), \mathbb{C}^{2n}) \) of the form
\[
b(x, \tilde{y}; h) \sim \sum_{j=0}^{\infty} b_j(x, \tilde{y})h^j,
\]
such that
\[
(\partial_x b_j)(x, \tilde{y}) = O(|\tilde{y} - \bar{x}|^\infty), \quad (\partial_{\tilde{y}} b_j)(x, \tilde{y}) = O(|\tilde{y} - \bar{x}|^\infty), \quad j = 0, 1, 2, \ldots,
\]
satisfies \((A_{\Gamma} b)(y, \bar{y}; h) = 1 + O(h^\infty)\), for a good contour \( \Gamma \). We have then
\[
a(x, \tilde{y}; h) - b(x, \tilde{y}; h) = O\left(|\tilde{y} - \bar{x}|^\infty + h^\infty\right).
\]

### 3 Approximate reproducing property in the weak sense

Let \( V \Subset \Omega \) be a small open neighborhood of \( x_0 \in \Omega \), with \( C^\infty \)-boundary. Let \( \Psi \in C^\infty(\text{neigh}((x_0, \bar{x})), \mathbb{C}^{2n}) \) be an almost holomorphic extension of the \( C^\infty \) strictly plurisubharmonic weight function \( \Phi \), so that (2.2), (2.3) hold. We may assume that \( \Psi \), as well as the classical \( C^\infty \) symbol \( a \), introduced in Theorem 2.4 and satisfying (2.87), are defined in a neighborhood of the closure of \( V \times \rho(V) \). Here \( \rho(x) = \bar{x} \) is the map of complex conjugation.

Let us set
\[
\bar{\Pi}_V u(x) = \frac{1}{h^n} \int_V e^{\frac{2}{h} \Psi(x, \bar{y})} a(x, \tilde{y}; h)u(y)e^{-\frac{2}{h} \Phi(y)} dy d\bar{y},
\]

\[u \in L^2_\Phi(V) := L^2(V, e^{-2\Phi/h} L(dx)).\]

It follows from Proposition 2.1 and the Schur test that
\[
\bar{\Pi}_V = O(1) : L^2_\Phi(V) \to L^2_\Phi(V).
\]

Furthermore, combining (2.3), (2.85), (2.86) with the Schur test, we obtain
\[
\bar{\partial} \circ \bar{\Pi}_V = O(h^\infty) : L^2_\Phi(V) \to L^2_{\Phi,(0,1)}(V).
\]
Here the target space is a space of \((0, 1)\)-forms on \(V\). Letting \(u \in L^2_\Phi(V)\) be holomorphic, we can express \(\tilde{\Pi}_V u\) in the polarized form, as a contour integral in \(C_{\sqrt{h}}^{2n}\) of a \((2n, 0)\)-form,

\[
\tilde{\Pi}_V u(x) = \frac{1}{h^n} \int_{\Gamma} e^{\frac{2}{h} (\Psi(x, \bar{y}) - \Psi(y, \bar{y}))} a(x, \bar{y}; h) u(y) \, dy \, d\bar{y},
\]

\[u \in H_\Phi(V) := \text{Hol}(V) \cap L^2_\Phi(V).\]  

(3.4)

Here the contour of integration \(\Gamma \subset V \times \rho(V)\) is given by

\[
\Gamma = \{ \bar{y} = y, \ y \in V \},
\]

(3.5)

and we observe that the restriction of the closed \((2n, 0)\)-form \(dy \wedge d\bar{y}\) on \(C_{\sqrt{h}}^{2n}\) to the anti-diagonal \(\bar{y} = y\) agrees with \(dy \wedge dy\), a non-vanishing multiple of the Lebesgue volume form on \(\mathbb{C}^n\).

The purpose of this section is to show that the operator \(\tilde{\Pi}_V\) in (3.4) satisfies an approximate reproducing property, in the weak formulation. Specifically, we shall prove that for a suitable class of \(u, v \in H_\Phi(V)\), the sesquilinear form

\[
H_\Phi(V) \times H_\Phi(V) \ni (u, v) \mapsto (\tilde{\Pi}_V u, v)_{L^2_\Phi(V)}
\]

agrees, up to an \(O(h^{-\infty})\)-error, with the scalar product \((u, v)_{H_\Phi(V)}\). In [9], we have observed that this result cannot be expected to hold if \(u, v\) are general elements of \(H_\Phi(V)\), and similar to [9], we shall demand that \(v\) should belong to an exponentially weighted space of holomorphic functions of the form \(H_{\Phi_1}(V)\), where \(\Phi_1 \leq \Phi\), with strict inequality away from a small neighborhood of \(x_0\). The following theorem is the main result of this section.

**Theorem 3.1** There exists a small open neighborhood \(W \Subset V\) of \(x_0\) with \(C^\infty\)-boundary such that for every \(\Phi_1 \in C(\Omega; \mathbb{R})\), \(\Phi_1 \leq \Phi\), with \(\Phi_1 < \Phi\) on \(\Omega \setminus \overline{W}\), and every \(N \in \mathbb{N}\) there exists \(C_N\) such that for all \(u \in H_\Phi(V)\), \(v \in H_{\Phi_1}(V)\), we have

\[
| (\tilde{\Pi}_V u, v)_{L^2_\Phi(V)} - (u, v)_{H_\Phi(V)} | \leq C_N h^N \| u \|_{H_\Phi(V)} \| v \|_{H_{\Phi_1}(V)}.
\]

(3.7)

Following [9, Section 4], the proof of Theorem 3.1 will proceed by a contour deformation argument. Compared with the analytic case treated in [9], here, when justifying the contour deformation, we shall have to take into account the lack of holomorphy in the integrand in (3.4), giving rise to an additional correction term, to be estimated. Let \(W \Subset V_1 \Subset V_2 \Subset V\) be open neighborhoods of \(x_0\) with \(C^\infty\)-boundaries, with \(W\) to be chosen small enough, and let \(\Phi_1 \in C(\Omega; \mathbb{R})\) be such that

\[
\Phi_1 \leq \Phi \text{ in } \Omega, \quad \Phi_1 < \Phi \text{ on } \Omega \setminus \overline{W}.
\]

(3.8)
Arguing as in [9, Section 4], we find that the scalar product
\[(\tilde{\Pi}_V u, v)_{L^2_{\Phi}(V)} = \int_V \tilde{\Pi}_V u(x) \overline{v(x)} e^{-2\Phi(x)/h} L(dx), \quad u \in H_{\Phi}(V), \quad v \in H_{\Phi_1}(V),\] 
\tag{3.9}

takes the form
\[(\tilde{\Pi}_V u, v)_{L^2_{\Phi}(V)} = \int_{V_1} \tilde{\Pi}_{V_2} u(x) \overline{v(x)} e^{-2\Phi(x)/h} L(dx) + O(1)e^{-\frac{1}{\tau_h}} \| u \|_{H_{\Phi}(V)} \| v \|_{H_{\Phi_1}(V)}. \tag{3.10}\]

Here and below $C > 0$ is independent of $u$, $v$, and similar to (3.4), we have written
\[\tilde{\Pi}_{V_2} u(x) = \frac{1}{\hbar^n} \int_{\Gamma_{V_2}} e^{\frac{i}{2h}(\Psi(x, \bar{y}) - \Psi(y, \bar{x}))} a(x, \bar{y}; h) u(y) dy \, d\bar{y}. \tag{3.11}\]

Next, an application of [9, Proposition 2.2] gives that there exists $\eta > 0$ such that
\[v(x) = \int_V v_z(x) d\bar{z} + O(1)\| v \|_{H_{\Phi}(V)} e^{\frac{1}{h}(\Phi(x) - \eta)}, \quad x \in V_1. \tag{3.12}\]

Here
\[v_z(x) = \frac{1}{(2\pi \hbar)^n} e^{\frac{i}{h}(x - z) \cdot \theta(x, z)} v(z) \chi(z) \det (\partial_x \theta(x, z)) \in \text{Hol}(V), \tag{3.13}\]

and $\theta(x, z)$ depends holomorphically on $x \in V$ with
\[- \text{Im} \left( (x - z) \cdot \theta(x, z) \right) + \Phi(z) \leq \Phi(x) - \delta |x - z|^2, \quad x, z \in V, \tag{3.14}\]

for some $\delta > 0$. The function $\chi \in C_0^\infty(V; [0, 1])$ in (3.13) satisfies $\chi = 1$ in $V_2$.

The resolution of the identity (3.12), (3.13), (3.14) is valid for an arbitrary element of $H_{\Phi}(V)$, and restricting the attention to $v \in H_{\Phi_1}(V)$, we get, letting $W \in W_1 \subseteq V_1$,
\[v(x) = \int_{W_1} v_z(x) d\bar{z} + O(1)\| v \|_{H_{\Phi_1}(V)} e^{\frac{1}{h}(\Phi(x) - \frac{1}{\tau_h})}, \quad x \in V_1. \tag{3.15}\]

We get, combining (3.10), (3.15), and (3.2),
\[\quad (\tilde{\Pi}_V u, v)_{L^2_{\Phi}(V)} = \int_{W_1} \int_{V_1} \tilde{\Pi}_{V_2} u(x) \overline{v_z(x)} e^{-2\Phi(x)/h} L(dx) d\bar{z} d\bar{z} + O(1)e^{-\frac{1}{\tau_h}} \| u \|_{H_{\Phi}(V)} \| v \|_{H_{\Phi_1}(V)} \]
\[= \int_{W_1} (\tilde{\Pi}_{V_2} u, v_z)_{L^2_{\Phi}(V_1)} d\bar{z} d\bar{z} + O(1)e^{-\frac{1}{\tau_h}} \| u \|_{H_{\Phi}(V)} \| v \|_{H_{\Phi_1}(V)}. \tag{3.16}\]
Similar to [9], the advantage of the representation (3.16) lies in the good localization properties of the holomorphic functions \( v_z \), for \( z \in W_1 \), in view of (3.14). The following key observation is analogous to [9, Proposition 4.2].

**Proposition 3.2** Let \( \delta > 0 \) be small and let us set for \( z \in V, (x, \tilde{x}, y, \tilde{y}) \in V \times \rho(V) \times V \times \rho(V) \subset C^{4n} \),

\[
G_z(x, \tilde{x}, y, \tilde{y}) = 2\text{Re} \, \Psi(x, \tilde{y}) - 2\text{Re} \, \Psi(y, \tilde{y}) + \Phi(y) + F_z(\tilde{x})
\]

where

\[
F_z(\tilde{x}) = \Phi(\tilde{x}) - \delta |\tilde{x} - \overline{z}|^2.
\]

The \( C^\infty \) function \( G_z \) has a non-degenerate critical point at \((z, \overline{z}, z, \overline{z})\) of signature \((4n, 4n)\), with the critical value 0. The following submanifolds of \( C^{4n} \) are good contours for \( G_z \) in a neighbourhood of \((z, \overline{z}, z, \overline{z})\), i.e. they are both of dimension \(4n\), pass through the critical point, and are such that the Hessian of \( G_z \) along the contours is negative definite:

1. The product contour

\[
\Gamma_V \times \Gamma_V = \{(x, \tilde{x}, y, \tilde{y}); \tilde{x} = x, \tilde{y} = y, x \in V, y \in V\}.
\]

2. The composed contour

\[
\{(x, \tilde{x}, y, \tilde{y}); (y, \tilde{x}) \in \Gamma_V, (x, \tilde{y}) \in \Gamma(y, \tilde{x})\}.
\]

Here \( \Gamma(y, \tilde{x}) \subset C^{2n}_{x, y} \) is a good contour for the \( C^\infty \) function \( x, \tilde{y} \mapsto \text{Re} \, \varphi(y, \tilde{x}; x, \tilde{y}) \) described in (2.28), see also Proposition 2.2.

**Proof** We have the Taylor expansions at \((x, \tilde{x}, y, \tilde{y}) = (z, \overline{z}, z, \overline{z}) \in C^{4n}\),

\[
2\text{Re} \, \Psi(x, \tilde{y}) - 2\text{Re} \, \Psi(y, \tilde{y}) = 2\Phi(z) + 2\text{Re} \left( \Phi'_z(z) \cdot (x - z) \right.
\]

\[
+ \Phi'_z(z) \cdot (\tilde{y} - \overline{z}) \bigg)
\]

\[
- 2\Phi(z) - 2\text{Re} \left( \Phi'_z(z) \cdot (y - z) + \Phi'_z(z) \cdot (\tilde{y} - \overline{z}) \right)
\]

\[
+ \mathcal{O}((x - z, \tilde{y} - \overline{z}, y - z)^2)
\]

\[
= 2\text{Re} \left( \Phi'_z(z) \cdot (x - z) - \Phi'_z(z) \cdot (y - z) \right)
\]

\[
+ \mathcal{O}((x - z, \tilde{y} - \overline{z}, y - z)^2),
\]

\[
\Phi(y) + F_z(\tilde{x}) = 2\Phi(z) + 2\text{Re} \left( \Phi'_z(z) \cdot (x - z) \right.
\]

\[
+ \Phi'_z(z) \cdot (\tilde{x} - z) \bigg)
\]

\[
+ \mathcal{O}((y - z, \tilde{x} - \overline{z})^2), \quad (3.22)
\]

\[
2\text{Re} \, \Psi(x, \tilde{x}) = 2\Phi(z) + 2\text{Re} \left( \Phi'_z(z) \cdot (x - z) + \Phi'_z(z) \cdot (\tilde{x} - \overline{z}) \right)
\]

\[
+ \mathcal{O}((x - z, \tilde{x} - \overline{z})^2), \quad (3.23)
\]
Here we have also used the almost holomorphy of $\Psi$. We get, combining (3.21), (3.22), (3.23) and using (3.17),

\[
G_z(x, \tilde{x}, y, \tilde{y}) = O\left(\text{dist}((x, \tilde{x}, y, \tilde{y}), (z, \bar{z}, z, \bar{z}))^2\right).
\] (3.24)

The point $(z, \bar{z}, z, \bar{z})$ is therefore a critical point of $G_z$ with the critical value 0. When showing that it is non-degenerate of signature $(4n, 4n)$, we observe that in view of Proposition 2.1, we have, using the expression for $G_z$ on the first line of (3.17),

\[
G_z(x, \tilde{x}, y, \tilde{y}) \leq -\frac{1}{C} |y - x|^2 - \delta |x - z|^2 \leq -\frac{1}{C} |x - z|^2 - \frac{1}{C} |y - z|^2.
\] (3.25)

The contour (3.19) is therefore good for $G_z$, and using also the fact that the quadratic part of the Taylor expansion of $G_z$ at the point $(z, \bar{z}, z, \bar{z})$ is a plurisubharmonic quadratic form on $\mathbb{C}^{4n}$, we conclude that $(z, \bar{z}, z, \bar{z})$ is a non-degenerate critical point of $G_z$, of signature $(4n, 4n)$. The verification of the fact that the contour (3.20) is also good for $G_z$ is performed exactly as in the proof of Proposition 4.2 of [9], using the expression for $G_z$ given on the second line of (3.17). The proof is complete. \[\square\]

We can now carry out the contour deformation argument for the scalar product $(\tilde{\Pi}_V u, v_z)$, alluded to above.

**Proposition 3.3** Let $v_z$ be of the form (3.13), (3.14). There exists an open neighborhood $W_1 \subset V_1$ of $x_0$ such that for all $u \in H_{\Phi}(V)$, we have,

\[
(\tilde{\Pi}_V u, v_z)_{L^2_{\Phi}(V_1)} = (u, v_z)_{H_{\Phi}(V_1)} + O(h^{-\infty})|u||v(z)| e^{-\Phi(z)/h},
\] (3.26)

uniformly in $z \in W_1$.

**Proof** Writing the Lebesgue measure on $\mathbb{C}^n$ in the form $L(dx) = C_n dx d\bar{x}$, let us express the scalar product in the space $H_{\Phi}(V_1)$ in the polarized form,

\[
(f, g)_{H_{\Phi}(V_1)} = \int_{V_1} f(x) \overline{g(x)} e^{-\frac{2\Phi(x)}{h}} L(dx)
= C_n \int_{\Gamma_{V_1}} f(x) \overline{g^*(\tilde{x})} e^{-\frac{2}{h} \Psi(x, \tilde{x})} dx d\bar{x}.
\] (3.27)

Here the contour $\Gamma_{V_1}$ is defined as in (3.5) and we have also set

\[
g^*(\tilde{x}) = \overline{g(\tilde{x})} \in H_{\Phi}(\rho(V_1)), \quad \Phi(\tilde{x}) = \Phi(\bar{x}).
\] (3.28)
In view of (3.11) and (3.27), we may write

\[
\begin{align*}
(\tilde{\Pi} V, V_z)_{L^2_\Phi(V_1)} &= \frac{C_n}{h^n} \int_{\Gamma_{V_1}} \left( \int_{\Gamma_{V_2}} e^{2\pi \Psi(x, \tilde{y}) - \Psi(y, \tilde{y})} a(x, \tilde{y}; h) u(y) \, dy \, d\tilde{y} \right) \\
&= \int_{\Gamma_{V_1} \times \Gamma_{V_2}} \omega.
\end{align*}
\]  

(3.29)

Here \( \omega \) is the \((4n, 0)\)-differential form on \( C^{4n} \) of the form

\[
\omega = f(x, \tilde{x}, y, \tilde{y}) \, dx \wedge d\tilde{x} \wedge dy \wedge d\tilde{y},
\]  

(3.30)

where \( f \in C^\infty(V \times \rho(V) \times V \times \rho(V)) \) is given by

\[
f(x, \tilde{x}, y, \tilde{y}) = \frac{C_n}{h^n} e^{2\pi \Psi(x, \tilde{y}) - \Psi(y, \tilde{y})} a(x, \tilde{y}; h) u(y) v_z(\tilde{x}) e^{-\frac{2}{\pi} \Psi(x, \tilde{x})}.
\]  

(3.31)

When estimating \( f \), we notice first, in view of (3.13), (3.14),

\[
\left| v_z^*(\tilde{x}) \right| \leq \frac{O(1)}{h^n} |v(z)| e^{-\Phi(z)/h} e^{F_z(\tilde{x})/h}, \quad \tilde{x} \in \rho(V_1),
\]  

(3.32)

where \( F_z \) is the strictly plurisubharmonic function in \( \rho(V_1) \) given in (3.18). Combining (3.32) with [9, Proposition 2.3], we get

\[
|f(x, \tilde{x}, y, \tilde{y})| \leq \frac{O(1)}{h^{3n}} |u|_{H^\Phi(V)} |v(z)| e^{-\Phi(z)/h} e^{G_z(x, \tilde{x}, y, \tilde{y})/h},
\]  

(3.33)

for \((x, \tilde{x}, y, \tilde{y}) \in V_1 \times \rho(V_1) \times V_2 \times \rho(V_2)\), with \( G_z(x, \tilde{x}, y, \tilde{y}) \) given in (3.17). Proposition 3.2 tells us that the contour \( \Gamma_1 := \Gamma_{V_1} \times \Gamma_{V_2} \) and the composed contour \( \Gamma_2 \) defined in (3.20) are both good for \( G_z \), and as reviewed in the proof of Proposition 2.3, there exists therefore a \( C^\infty \) homotopy between the contours \( \Gamma_1, \Gamma_2 \), well defined for all \( z \) in a small neighborhood of \( x_0 \), passing through good contours only, uniformly for \( z \) close enough to \( x_0 \). Let \( \Sigma \subset C^{4n} \) be the \((4n + 1)\)-dimensional contour of integration naturally associated to the homotopy above. We have, with a suitable orientation,

\[
\partial \Sigma - (\Gamma_1 - \Gamma_2) \subset \left\{ (x, \tilde{x}, y, \tilde{y}); \ G_z(x, \tilde{x}, y, \tilde{y}) \leq -\frac{1}{O(1)} \right\},
\]  

(3.34)
uniformly for all $z$ in a small neighborhood of $x_0$. We may write therefore, using Stokes’ formula, (3.33), and (3.34), for all $z$ in a small neighborhood of $x_0$,

$$\int \int \int \omega - \int \int \int \omega = \int \int \int \Sigma d\omega + O(1)\| u \|_{H_\Phi(V)} |v(z)| e^{-\Phi(z)/h} e^{-1/Ch}$$

$$= \int \int \int \Sigma \partial f \wedge dx \wedge d\tilde{x} \wedge dy \wedge d\tilde{y} + O(1)\| u \|_{H_\Phi(V)} |v(z)| e^{-\Phi(z)/h} e^{-1/Ch}. \quad (3.35)$$

Using (3.31), we compute

$$\partial_{\tilde{x}} f (x, \tilde{x}, y, \tilde{y}) = \frac{C_n}{h^n} e^{\frac{1}{h^2} (\Psi(x, \tilde{y}) - \Psi(y, \tilde{y}) - \Psi(x, \tilde{x}))} u(y) v_z^a (\tilde{x}) \left( \frac{2}{h} (\partial_{\tilde{x}} \Psi(x, \tilde{y}) - \partial_{\tilde{x}} \Psi(x, \tilde{x})) a + \partial_{\tilde{x}} a \right), \quad (3.36)$$

and recalling (2.3), (2.86), we infer for all $N \in \mathbb{N}$,

$$|\partial_{\tilde{x}} f| \leq \frac{O_N(1)}{h^{3n+1}} \| u \|_{H_\Phi(V)} |v(z)| e^{-\Phi(z)/h} e^{G_z(x, \tilde{x}, y, \tilde{y})/h}$$

$$\left( |\tilde{x} - \tilde{y}|^N + |\tilde{x} - x|^N + |y - \tilde{y}|^N + h^N \right) \leq \frac{O_N(1)}{h^{3n+1}} \| u \|_{H_\Phi(V)} |v(z)| e^{-\Phi(z)/h} e^{-\text{dist}((x, \tilde{x}, y, \tilde{y}), (z, \tilde{z}, z, \tilde{z}))^2/Ch}$$

$$\left( |\tilde{x} - \tilde{y}|^N + |\tilde{x} - x|^N + h^N \right). \quad (3.37)$$

Here we write

$$|\tilde{x} - \tilde{y}|^N \leq (|x - z| + |y - \tilde{z}|)^N \leq O_N(1) \left( \text{dist} ((x, \tilde{x}, y, \tilde{y}), (z, \tilde{z}, z, \tilde{z})) \right)^N, \quad (3.38)$$

and similarly,

$$|\tilde{x} - \tilde{z}|^N \leq O_N(1) \left( \text{dist} ((x, \tilde{x}, y, \tilde{y}), (z, \tilde{z}, z, \tilde{z})) \right)^N. \quad (3.39)$$

We get therefore, combining (3.37), (3.38), (3.39),

$$|\partial_{\tilde{x}} f (x, \tilde{x}, y, \tilde{y})| \leq O(h^\infty) \| u \|_{H_\Phi(V)} |v(z)| e^{-\Phi(z)/h}. \quad (3.40)$$

Similar computations and estimates show that the bound (3.40) is also valid for $\partial_{\tilde{y}} f$, $\partial_{\tilde{x}} f$, and $\partial_y f$. We conclude that there exists a small open neighborhood $W_1 \subseteq V_1$ of $x_0$ such that for all $z \in W_1$, the right hand side of (3.35) is of the form

$$O(h^\infty) \| u \|_{H_\Phi(V)} |v(z)| e^{-\Phi(z)/h}. \quad (3.41)$$
It follows therefore from (3.29), (3.35), and (3.41) that for all \( z \in W_1 \), the scalar product \((\tilde{\Pi}_V u, v_z)_{L^2_\Phi(V_1)}\) is equal to

\[
C_n \int \int_{\Gamma V_1} \left( \frac{1}{h^n} \int \int_{\Gamma (y, \tilde{x}) \cap (V_1 \times \rho(V_1))} e^{\frac{2}{h} \Psi(y, \tilde{x}, x, \tilde{y})} a(x, \tilde{y}; h) d x d \tilde{y} \right) \cdot u(y) v_z^n(\tilde{x}) e^{-\frac{2}{h} \Psi(y, \tilde{x})} d y d \tilde{x} + O(h^\infty) ||u||_{H_\Phi(V)} ||v(z)|| e^{-\frac{\Phi(z)}{h}}. 
\]  

(3.42)

Here in the contour of integration in the inner integral we have \((y, \tilde{x}) \in \Gamma V_1 \iff \tilde{x} = y, y \in V_1\), and by Theorem 2.4 we obtain therefore that the inner integral is equal to \(1 + O(h^\infty)\), provided that the neighborhood \(V_1\) is small enough. The integral (3.42) is therefore equal to

\[
(u, v_z)_{H_\Phi(V_1)} + O(h^\infty) ||u||_{H_\Phi(V)} ||v(z)|| e^{-\frac{\Phi(z)}{h}}, 
\]  

(3.43)

uniformly for \( z \in W_1 \). The proof is complete.

We can now finish the proof of Theorem 3.1 as in [9, Section 4], letting \(W \subset W_1\), where \(W_1\) is as in Proposition 3.3. We get, in view of (3.16) and (3.26),

\[
(\tilde{\Pi}_V u, v)_{L^2_\Phi(V)} = \int_{W_1} (u, v_z)_{H_\Phi(V_1)} d z d \tilde{z} + O(h^\infty) ||u||_{H_\Phi(V)} ||v||_{H_\Phi(V_1)}. 
\]  

(3.44)

Using (3.15), we can also write

\[
(u, v)_{H_\Phi(V)} = \int_{W_1} (u, v_z)_{H_\Phi(V_1)} d z d \tilde{z} + O(1) e^{-\frac{1}{c^2}} ||u||_{H_\Phi(V)} ||v||_{H_\Phi(V_1)}. 
\]  

(3.45)

The proof of Theorem 3.1 is complete.

4 Completing the proof of Theorem 1.1

Let us first pass from the scalar products in Theorem 3.1 to weighted \(L^2\) norm estimates. This will be done similarly to [9, Section 5], with appropriate modifications to accommodate the fact that the operator \(\tilde{\Pi}_V\) in (3.1) does not quite produce holomorphic functions. To this end, let \(0 \leq \chi_1 \in C^\infty(\Omega; \mathbb{R})\) be such that \(\chi_1 > 0\) on \(\Omega \setminus \overline{W}\), where \(W \subset V\) is as in Theorem 3.1, and let us set

\[
\Phi_1(x) = \Phi(x) - \delta \chi_1(x),
\]  

(4.1)

for \(\delta > 0\) small enough. In particular, \(\Phi_1\) is strictly plurisubharmonic in \(V\),

\[
\sum_{j,k=1}^n \frac{\partial^2 \Phi_1}{\partial x_j \partial \bar{x}_k}(x) \xi_j \bar{\xi}_k \geq \frac{|\xi|^2}{O(1)}, \quad x \in V, \quad \xi \in \mathbb{C}^n.
\]  

(4.2)
In what follows, without loss of generality, we shall assume that the bounded open set $V$ is convex, and we may even take it to be an open ball centered at $x_0$. Using Proposition 2.1 together with the Schur test, we obtain

$$
\Pi_V = \mathcal{O}(1) : L^2_{\Phi_1}(V) \to L^2_{\Phi_2}(V).
$$

(4.3)

Here $\Phi_2 = \Phi - \chi_2$, where $0 \leq \chi_2 \in C(\Omega; \mathbb{R})$ is given by

$$
\chi_2(x) = \inf_{y \in V_0} \left( \frac{|x - y|^2}{2C} + \delta \chi_1(y) \right),
$$

(4.4)

with $C > 0$ and $V_0$ being an open ball centered at $x_0$ such that $V \Subset V_0 \Subset \Omega$. In particular, we see that

$$
\Phi_2 \leq \Phi \text{ in } \Omega, \quad \Phi_2 < \Phi \text{ on } \Omega \setminus \overline{W},
$$

(4.5)

and

$$
\Phi_1 \leq \Phi_2 \text{ in } V.
$$

(4.6)

Let us now take a closer look at the function $\chi_2$ in (4.4). When doing so, let us observe first that the infimum in (4.4) is attained at a unique point in $V_0$, in view of the strict convexity of the function $V_0 \ni y \mapsto F_x(y) := \frac{|x - y|^2}{2C} + \delta \chi_1(y)$, for $\delta > 0$ small enough. On the other hand, when $x \in V$, the function $F_x(y)$ has a unique critical point $y_c(x)$ in $V_0$, for $\delta > 0$ small enough, which is a non-degenerate local minimum. Indeed, we have

$$
F'_x(y) = 0 \iff CF'_0(y) = x,
$$

and the map $V_0 \ni y \mapsto CF'_0(y) = y + C \delta \chi_1'(y)$ is a $C^\infty$ diffeomorphism from $V_0$ onto its image, which contains the open ball $V$ as a relatively compact subset, for $\delta > 0$ small enough. We have

$$
y_c(x) = x - C \delta \chi_1'(x) + \mathcal{O}(\delta^2), \quad x \in V,
$$

(4.7)

and we conclude therefore that

$$
\chi_2(x) = F_x(y_c(x)) = \delta \chi_1(x) - \frac{C}{2} \delta^2 |\chi_1'(x)|^2 + \mathcal{O}(\delta^3), \quad x \in V.
$$

(4.8)

In particular, we have $|| \Phi - \Phi_2 ||_{C^2(V)} = \mathcal{O}(\delta)$ is small enough, so that the function $\Phi_2$ is strictly plurisubharmonic in $V$,.

$$
\sum_{j,k=1}^n \frac{\partial^2 \Phi_2}{\partial x_j \partial \overline{x}_k}(x) \xi_j \overline{\xi}_k \geq \frac{|\xi|^2}{\mathcal{O}(1)}, \quad x \in V, \quad \xi \in \mathbb{C}^n.
$$

(4.9)
Next, we let
\[
\Pi_{\Phi_2} : L^2_{\Phi_2}(V) \rightarrow H_{\Phi_2}(V)
\]  
(4.10)
be the orthogonal projection. We shall make use of the following observation.

**Proposition 4.1** We have
\[
\Pi_{\Phi_2}\tilde{\Pi}_V - \tilde{\Pi}_V = \mathcal{O}(h^\infty) : L^2_{\Phi_1}(V) \rightarrow L^2_{\Phi_2}(V).
\]  
(4.11)

**Proof** Using (2.3), (2.30), (2.31), and Proposition 2.1 together with the Schur test, we observe that
\[
\partial \circ \tilde{\Pi}_V = \mathcal{O}(h^\infty) : L^2_{\Phi_1}(V) \rightarrow L^2_{\Phi_2}(0,1)(V),
\]  
(4.12)
where the target space in (4.12) is a space of $(0, 1)$–forms. Given \( f \in L^2_{\Phi_1}(V) \), the solution of the equation
\[
\partial u = \partial \tilde{\Pi}_V f
\]  
(4.13)
of the minimal \( L^2_{\Phi_2}(V) \)-norm is given by \( (1 - \Pi_{\Phi_2})\tilde{\Pi}_V f \), and an application of Hörmander’s \( L^2 \)-estimates for the \( \partial \)-operator in the open convex set \( V \) and the strictly plurisubharmonic weight \( \Phi_2 \), [22, Proposition 4.2.5], gives that
\[
|| (1 - \Pi_{\Phi_2})\tilde{\Pi}_V f ||_{L^2_{\Phi_2}(V)} \leq \mathcal{O}(h^{1/2}) ||\partial \tilde{\Pi}_V f ||_{L^2_{\Phi_2}(V)} \leq \mathcal{O}(h^\infty) || f ||_{L^2_{\Phi_1}(V)}.
\]  
(4.14)
Here we have also used (4.12). The proof is complete. \( \Box \)

It is now easy to derive a suitable estimate for the operator \( \tilde{\Pi}_V - 1 \), proceeding as in [9]. Let \( u \in H_{\Phi_1}(V) \), where \( \Phi_1 \in C^\infty(\Omega; \mathbb{R}) \) is given by (4.1), and let us apply Theorem 3.1, with
\[
v = \Pi_{\Phi_2}((\tilde{\Pi}_V - 1)u) = \Pi_{\Phi_2}\tilde{\Pi}_V u - u \in H_{\Phi_2}(V)
\]  
(4.15)
and with \( \Phi_2 \) in place of \( \Phi_1 \). Here in the second equality in (4.15) we have also used (4.6). We obtain, using also (4.3),
\[
\left|((\tilde{\Pi}_V - 1)u, \Pi_{\Phi_2}((\tilde{\Pi}_V - 1)u))_{L^2_{\Phi_2}(V)}\right| \leq \mathcal{O}(h^\infty)||u||_{H_{\Phi_1}(V)}||\Pi_{\Phi_2}\tilde{\Pi}_V u - u||_{H_{\Phi_2}(V)} \\
\leq \mathcal{O}(h^\infty)||u||_{H_{\Phi_1}(V)}^2.
\]  
(4.16)
Next, we write
\[
\Pi_{\Phi_2}((\tilde{\Pi}_V - 1)u) = (\tilde{\Pi}_V - 1)u + Ru,
\]  
(4.17)
where \( R = \Pi_{\Phi_2} \widetilde{\Pi}_V - \widetilde{\Pi}_V \). We get, combining (4.16), (4.17), and using that \( \Phi_j \leq \Phi \), for \( j = 1, 2 \), together with (3.2),

\[
\| (\widetilde{\Pi}_V - 1) u \|_{L^2_{\Phi}(V)}^2 \leq \mathcal{O}(h^\infty) \| u \|_{H_{\Phi_1}(V)}^2 + \mathcal{O}(1) \| u \|_{H_{\Phi}(V)} \| R u \|_{L^2_{\Phi}(V)}^2
\]

\[
\leq \mathcal{O}(h^\infty) \| u \|_{H_{\Phi_1}(V)}^2 + \mathcal{O}(1) \| u \|_{H_{\Phi}(V)} \| R u \|_{L^2_{\Phi}(V)}^2
\]

\[
\leq \mathcal{O}(h^\infty) \| u \|_{H_{\Phi_1}(V)}^2 + \mathcal{O}(h^\infty) \| u \|_{H_{\Phi}(V)} \| u \|_{H_{\Phi_1}(V)}
\]

\[
\leq \mathcal{O}(h^\infty) \| u \|_{L^2_{\Phi}(V)}^2.
\] (4.18)

Here in the penultimate inequality we have also used Proposition 4.1. We obtain therefore that

\[
\| (\widetilde{\Pi}_V - 1) u \|_{L^2_{\Phi}(V)} \leq \mathcal{O}(h^\infty) \| u \|_{H_{\Phi_1}(V)}, \quad u \in H_{\Phi_1}(V),
\] (4.19)

where \( \Phi_1 \in C^\infty(\Omega; \mathbb{R}) \) is of the form (4.1). The bound (4.19) is completely analogous to the estimate (5.5) in [9, Section 5], and we may therefore conclude the proof of Theorem 1.1 by repeating the arguments of [9, Section 5], which are based on the \( \overline{\partial} \)-techniques, exactly as they stand. Letting \( U \subset W \subset V \) be an open neighborhood of \( x_0 \) with \( C^\infty \)-boundary, we get therefore,

\[
\| (\widetilde{\Pi}_V - 1) u \|_{L^2_{\Phi}(U)} \leq \mathcal{O}(h^\infty) \| u \|_{H_{\Phi}(V)}, \quad u \in H_{\Phi}(V).
\] (4.20)

The proof of Theorem 1.1 is complete.

### 5 From asymptotic local to global Bergman kernels

The purpose of this section is to establish a link between the operator \( \widetilde{\Pi}_V \) in (1.3), enjoying the local approximate reproducing property (1.4), and the orthogonal projection

\[
\Pi : L^2(\Omega, e^{-2\Phi/h} L(d\mu)) \rightarrow H_{\Phi}(\Omega).
\] (5.1)

When doing so, we shall follow [1] closely, where the Bergman projection was considered in the context of high powers of a holomorphic line bundle over a complex compact manifold. The discussion below is therefore essentially well known, and is given here mainly for the completeness and convenience of the reader. See also [12, 13].

We shall assume in what follows that the open set \( \Omega \subset \mathbb{C}^n \) is pseudoconvex. It will also be convenient for us to choose a local polarization \( \Psi \) of \( \Phi \in C^\infty(\Omega) \) satisfying (2.2), (2.3), such that the Hermitian property

\[
\Psi(x, y) = \overline{\Psi(y, x)}, \quad (x, y) \in \text{neigh}((x_0, \overline{x_0}), \mathbb{C}^{2n}_{x,y})
\] (5.2)
holds. Indeed, if $\Psi(x, y)$ satisfies (2.2), (2.3), then so does $\overline{\Psi(y, x)}$, and replacing $\Psi(x, y)$ by $(\Psi(x, y) + \overline{\Psi(y, x)})/2$, we obtain (5.2).

Our starting point is the following well known result, allowing us to pass to pointwise estimates from the weighted $L^2$ estimates in Theorem 1.1.

**Proposition 5.1** Let $V_1 \subseteq V_2 \subseteq \Omega$ be open. There exists $C > 0$ such that for all $f \in L^2_{\Phi}(V_2)$ satisfying $\bar{h} \partial f \in L^\infty(V_2)$ and all $h > 0$ small enough, we have

$$|f(x)| \leq C \left( \sup_{y \in V_2} |h \partial f(y)| e^{-\Phi(y)/h} + h^{-n} ||f||_{L^2_{\Phi}(V_2)} \right) e^{\Phi(x)/h}, \ x \in V_1. \quad (5.3)$$

**Proof** Let $h_0 > 0$ be such that $B(x, h_0) = \{y \in \mathbb{C}^n, \ |y - x| < h_0 \} \subset V_2$, for all $x \in V_1$. An application of [21, Lemma 15.1.8] gives for all $h \in (0, h_0]$,

$$|f(x)| \leq C \left( \sup_{B(x, h)} |h \partial f(y)| + h^{-n} ||f||_{L^2(B(x, h))} \right), \ x \in V_1. \quad (5.4)$$

Using that

$$e^{\Phi(y)/h} \leq \mathcal{O}(1)e^{\Phi(x)/h}, \ \ y \in B(x, h),$$

for $x \in V_1$, we obtain (5.3). \qed

We shall apply Proposition 5.1 to a function of the form

$$f = \tilde{\Pi}_V u - u, \ \ u \in H_{\Phi}(V), \quad (5.5)$$

satisfying

$$||f||_{L^2_{\Phi}(U)} \leq \mathcal{O}(h^{\infty}) ||u||_{H_{\Phi}(V)}, \quad (5.6)$$

in view of (1.4). As for the control of $\bar{h} \partial f = h \partial \tilde{\Pi}_V u$ in $L^\infty(U)$, we have in view of (2.3), (2.85), (2.86), for $N = 1, 2, \ldots$,

$$|h \partial f(x)| \leq \mathcal{O}(1) h^{-n} \int_{\mathbb{C}^n} e^{2 \Re \Psi(x, y)} |x - y|^N |u(y)| e^{-\frac{x}{h} \Phi(y)} L(dy)
\leq \mathcal{O}(1) e^{\Phi(x)/h} h^{-n} \int_{\mathbb{C}^n} e^{-|x-y|^2/Ch} |x - y|^N |u(y)| e^{-\frac{x}{h} \Phi(y)} L(dy)
\leq \mathcal{O}(h^{N-n}) e^{\Phi(x)/h} ||u||_{H_{\Phi}(V)}, \ x \in U. \quad (5.7)$$

Here we have also used Proposition 2.1 and the Cauchy–Schwarz inequality. Letting $\tilde{U} \Subset U$ be an open neighborhood of $x_0$, and combining Proposition 5.1 with (5.6), (5.7), we get

$$|\tilde{\Pi}_V u(x) - u(x)| \leq \mathcal{O}(h^{\infty}) e^{\Phi(x)/h} ||u||_{H_{\Phi}(V)}, \ x \in \tilde{U}. \quad (5.8)$$
We obtain therefore the following approximate local reproducing property,

\[
    u(x) = \frac{1}{h^n} \int_V e^{\frac{2}{h} \Psi(x, \overline{y})} a(x, \overline{y}; h) u(y) e^{-\frac{2}{h} \Phi(y)} L(dy)
    + \mathcal{O}(h^\infty) e^{-\frac{\Phi(x)}{h}} \| u \|_{H_\Phi(V)}, \quad x \in \widetilde{U},
\]

valid for all \( u \in H_\Phi(V) \). In particular, we can apply (5.9) to \( u \in H_\Phi(\Omega) \), and when doing so, let us recall that

\[
    |u(x)| \leq \mathcal{O}(1) h^{-n} e^{-\frac{\Phi(x)}{h}} \| u \|_{H_\Phi(\Omega)}, \quad x \in V,
\]

in view of [9, Proposition 2.3]. Let \( \chi \in C_0^\infty(V; [0, 1]) \) be such that \( \chi = 1 \) near \( \overline{U} \).

Using (5.10) and Proposition 2.1, we see that

\[
    \left| \frac{1}{h^n} \int_V e^{\frac{2}{h} \Psi(x, \overline{y})} (1 - \chi(y)) a(x, \overline{y}; h) u(y) e^{-\frac{2}{h} \Phi(y)} L(dy) \right|
    \leq \mathcal{O}(1) e^{-\frac{1}{c_n} h^{-\frac{\Phi(x)}{h}}} \| u \|_{H_\Phi(\Omega)}, \quad x \in \widetilde{U}.
\]

We get, combining (5.9) and (5.11), when \( u \in H_\Phi(\Omega) \),

\[
    u(y) = \int_{\Omega} \tilde{K}(y, z) \chi(z) u(z) e^{-\frac{2}{h} \Phi(z)} L(dz)
    + \mathcal{O}(h^\infty) e^{-\frac{\Phi(x)}{h}} \| u \|_{H_\Phi(\Omega)}, \quad y \in \widetilde{U},
\]

where

\[
    \tilde{K}(y, z) = \frac{1}{h^n} e^{\frac{2}{h} \Psi(y, \overline{z})} a(y, \overline{z}; h), \quad (y, z) \in V \times V.
\]

Next, arguing as in [27, Section 5], [6, Appendix A], we see that the Schwartz kernel of the orthogonal projection in (5.1) is of the form \( K(x, \overline{y}) e^{-2\Phi(y)/h} \), where \( K(x, z) \in \text{Hol}(\Omega \times \overline{\Omega}) \) satisfies

\[
    y \mapsto K(x, \overline{y}) \in H_\Phi(\Omega), \quad x \mapsto K(x, \overline{y}) \in H_\Phi(\Omega).
\]

Here we have written \( \overline{\Omega} = \{ x \in \mathbb{C}^n; \overline{x} \in \Omega \} \). Following [1] and applying (5.12) to the function \( y \mapsto K(y, \overline{x}) \in H_\Phi(\Omega) \), we get

\[
    K(y, \overline{x}) = \int_{\Omega} \tilde{K}(y, z) \chi(z) K(z, \overline{x}) e^{-\frac{2}{h} \Phi(z)} L(dz)
    + \mathcal{O}(h^\infty) e^{-\frac{\Phi(x)}{h}} \| K(\cdot, \overline{x}) \|_{H_\Phi(\Omega)}, \quad y \in \widetilde{U}.
\]

Here we have

\[
    \| K(\cdot, \overline{x}) \|_{H_\Phi(\Omega)} \leq \mathcal{O}(1) h^{-n/2} e^{-\frac{\Phi(x)}{h}}, \quad x \in \widetilde{U},
\]
see [2, Chapter 4], and combining (5.15) and (5.16), we infer that

\[
K(y, x) = \int_{\Omega} \tilde{K}(y, z) \chi(z) \tilde{K}(z, x) e^{-\frac{2}{\hbar} \Phi(z)} L(dz) + O(h^\infty) e^{\frac{\Phi(x) + \Phi(y)}{\hbar}}, \quad x, y \in \tilde{U}.
\]

(5.17)

Taking the complex conjugates in (5.17) and using the Hermitian property \( K(x, y) = K(y, x) \), we get

\[
K(x, y) = \int_{\Omega} K(x, z) \chi(z) \hat{K}(z, y) e^{-\frac{2}{\hbar} \Phi(z)} L(dz)
\]

\[
+ O(h^\infty) e^{\frac{\Phi(x) + \Phi(y)}{\hbar}}, \quad x, y \in \tilde{U},
\]

(5.18)

where, in view of (5.13) and (5.2),

\[
\hat{K}(z, y) = \frac{1}{\hbar^n} e^{\frac{2}{\hbar} \Psi(z, y)} b(z, y; h), \quad b(z, y; h) = a(y, z; h).
\]

(5.19)

Writing

\[
\Pi u(x) = \int_{\Omega} K(x, y) u(y) e^{-\frac{2}{\hbar} \Phi(y)} L(dy), \quad u \in L^2(\Omega, e^{-\frac{2}{\hbar} \Phi} L(dx)),
\]

(5.20)

we may express (5.18) as follows,

\[
K(x, y) = \Pi(\hat{K}(-, y) \chi)(x) + O(h^\infty) e^{\frac{\Phi(x) + \Phi(y)}{\hbar}}, \quad x, y \in \tilde{U}.
\]

(5.21)

Here we would like to show that \( \Pi(\hat{K}(-, y) \chi)(x) \) is close to \( \hat{K}(x, y) \chi(x) = \tilde{K}(x, y) \) for \( x \in \tilde{U} \), and to this end we follow an argument in [1], see also Proposition 4.1. The function

\[
\Omega \ni x \mapsto u_y(x) = \hat{K}(x, y) \chi(x) - \Pi(\hat{K}(-, y) \chi)(x)
\]

(5.22)

is the solution of the \( \overline{\partial} \)–equation

\[
\overline{\partial} u_y = \overline{\partial}(\hat{K}(\cdot, y) \chi) = \hat{K}(\cdot, y) \overline{\partial} \chi + \chi \overline{\partial} \hat{K}(\cdot, y),
\]

(5.23)

in \( \Omega \) of the minimal \( L^2(\Omega, e^{-\frac{2}{\hbar} \Phi} L(dx)) \) norm, and therefore, by Hörmander’s \( L^2 \)–estimates for the \( \overline{\partial} \) operator, see [22, Proposition 4.2.5]), we get for any \( y \in \tilde{U} \),
\[
\int_{\Omega} |u_y(x)|^2 e^{-2\Phi(x)/h} \, L(dx) \leq O(h) \int_{\Omega} \frac{1}{c(x)} |\overline{\partial}_x (\hat{K}(x, y) \chi(x))|^2 e^{-2\Phi(x)/h} \, L(dx) \\
\leq O(h) \left( \int_{\Omega} |\nabla \chi(x)|^2 |\hat{K}(x, y)|^2 e^{-2\Phi(x)/h} \, L(dx) + \int_{\Omega} \chi(x) |\overline{\partial}_x \hat{K}(x, y)|^2 e^{-2\Phi(x)/h} \, L(dx) \right) (5.24)
\]

Here we get, in view of (5.19) and Proposition 2.1,
\[
\int_{\Omega} |\nabla \chi(x)|^2 |\hat{K}(x, y)|^2 e^{-2\Phi(x)/h} \, L(dx) \leq O(1) \int_{V \setminus \hat{U}} |\hat{K}(x, y)|^2 e^{-2\Phi(x)/h} \, L(dx) = O(1) e^{2\Phi(y)/h} e^{-1/Ch}, \quad y \in \hat{U}. (5.25)
\]

Due to the almost holomorphy of \(\Psi\) and the symbol \(b(x, \hat{y}; h)\) in (5.19) near the anti-diagonal,
\[
|\overline{\partial}_x \hat{K}(x, y)| \leq \frac{O_N(1)}{h^{n+1}} e^{\frac{2}{\lambda} \text{Re} \Psi(x, \overline{y})} |x - y|^N, \quad N = 1, 2, \ldots, (5.26)
\]
and therefore, by another application of Proposition 2.1,
\[
\int_{\Omega} \chi(x) |\overline{\partial}_x \hat{K}(x, y)|^2 e^{-2\Phi(x)/h} \, L(dx) \leq O(h^\infty) e^{2\Phi(y)/h}, \quad y \in \hat{U}. (5.27)
\]
Combining (5.24), (5.25), and (5.27), we get
\[
\| u_y \|_{L^2(\Omega)}^2 \leq O(h^\infty) e^{\Phi(y)/h}, \quad y \in \hat{U}. (5.28)
\]
It finally remains for us to pass from the weighted \(L^2\)–bound (5.28) on \(u_y\) to a pointwise estimate. To this end, we shall apply Proposition 5.1 to \(u_y\), with \(V_2 = U, \ V_1 = \hat{U}\). Recalling (5.22) and using that \(\chi = 1\) on \(U\), we see that we only have to estimate \(h \overline{\partial}_z u_y(z) = h \overline{\partial}_z \hat{K}(z, \overline{y})\) for \(z \in U, \ y \in \hat{U}\). Using (5.26) and Proposition 2.1, we obtain that
\[
|h \overline{\partial}_z u_y(z)| \leq O(h^\infty) e^{(\Phi(z) + \Phi(y))/h}, \quad y \in \hat{U}, \quad z \in U. (5.29)
\]
Combining (5.3), (5.28), and (5.29), we get
\[
|u_y(x)| \leq O(h^\infty) e^{(\Phi(x) + \Phi(y))/h}, \quad x, y \in \hat{U}, (5.30)
\]
and therefore, using (5.21), (5.22), and (5.30), we infer that
\[
K(x, \overline{y}) = \hat{K}(x, \overline{y}) + O(h^\infty) e^{(\Phi(x) + \Phi(y))/h}, \quad x, y \in \hat{U}. (5.31)
\]
Recalling also (5.19), we obtain
\[
K(x, \overline{y}) = \overline{K}(y, x) + \mathcal{O}(h^\infty)e^{(\Phi(x) + \Phi(y))/h}, \quad x, y \in \tilde{U},
\]
(5.32)
and taking the complex conjugates and using the Hermitian symmetry of $K$, we get
\[
K(y, x) = \tilde{K}(y, x) + \mathcal{O}(h^\infty)e^{(\Phi(x) + \Phi(y))/h},
\]
(5.33)
uniformly for $x, y \in \tilde{U}$. Switching the variables $x$ and $y$ in (5.33), we may therefore summarize the discussion in this section in the following well known result, see [1, 4, 5, 7, 13, 18, 31].

**Proposition 5.2** Let $\Omega \subset \mathbb{C}^n$ be open pseudoconvex, let $\Phi \in C^\infty(\Omega)$ be strictly plurisubharmonic so that (2.1) holds, and let $K(x, \overline{y})e^{-2\Phi(y)/h}$ be the Schwartz kernel of the orthogonal projection (5.1). Let $x_0 \in \Omega$ and let $\tilde{U} \Subset U \Subset V \Subset \Omega$ be small open neighborhoods of $x_0$, where $U, V$ are as in Theorem 1.1. We have
\[
e^{-\Phi(x)/h} \left( K(x, \overline{y}) - \frac{1}{h^n} e^{2\Psi(x, \overline{y})} a(x, \overline{y}; h) \right) e^{-\Phi(y)/h} = \mathcal{O}(h^\infty),
\]
(5.34)
uniformly for $x, y \in \tilde{U}$. Here $\Psi \in C^\infty(\text{neigh}((x_0, \overline{x_0}), \mathbb{C}^{2n}))$ is a polarization of $\Phi$ and the classical symbol $a \in S^0_{cl}(\text{neigh}((x_0, \overline{x_0}), \mathbb{C}^{2n}))$ has been introduced in (1.2).

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