Boundedness of the Riesz potential in central Morrey–Orlicz spaces

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Abstract

Boundedness of the maximal function and the Calderón–Zygmund singular integrals in central Morrey–Orlicz spaces were proved in papers by the second and third authors. The weak-type estimates have also been proven. Here we show boundedness of the Riesz potential in central Morrey–Orlicz spaces and the corresponding weak-type version.

1 Orlicz spaces and central Morrey–Orlicz spaces

First of all, we recall the definition of Orlicz spaces on $\mathbb{R}^n$ and some of their properties to be used later on (see [23] and [25] for details).

A function $\Phi : [0, \infty) \to [0, \infty]$ is called an Orlicz function, if it is an increasing continuous and convex function with $\Phi(0) = 0$. Each such a function $\Phi$ has an integral representation $\Phi(u) = \int_0^u \Phi'(t) \, dt$, where the right-derivative $\Phi'_+ (t)$ is a nondecreasing right-continuous function (see [23, Theorem 1.1]). We will write below estimates for everywhere differentiable Orlicz function $\Phi$, but then using the above integral representation, these estimates will be true for almost all $u > 0$ with its right-derivative $\Phi'_+$ instead of derivative $\Phi'$. Of course, we have estimates

$$\Phi(u) \leq u \Phi'(u) \leq \Phi(2u) \quad \text{for all} \quad u > 0. \quad (1)$$

If we want to include in the Orlicz spaces, for example, spaces $L^\infty(\mathbb{R}^n), L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ for $1 \leq p < \infty$, then we need to consider a broader class of functions than Orlicz’s functions, the so-called Young functions. A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function, if it is a nondecreasing convex function with $\lim_{u \to 0^+} \Phi(u) = \Phi(0) = 0$, and not identically 0 or $\infty$ in $(0, \infty)$. It may have jump up to $\infty$ at some point $u > 0$, but then it should be left continuous at $u$.

Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite complete nonatomic measure space and $L^0(\Omega)$ be the space of all $\mu$-equivalent classes of real-valued and $\Sigma$-measurable functions defined on $\Omega$.

For any Young function $\Phi$, the Orlicz space $L^\Phi(\Omega)$, which contains all $f \in L^0(\Omega)$ such that $\int_\Omega \Phi(\varepsilon |f(x)|) \, d\mu(x) < \infty$ for some $\varepsilon = \varepsilon(f) > 0$ with the Luxemburg–Nakano norm

$$\|f\|_{L^\Phi} = \inf \left\{ \varepsilon > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\varepsilon}\right) \, dx \leq 1 \right\}, \quad (2)$$

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is a Banach space (cf. [23, pp. 70–71], [25, pp. 15–16], [26, pp. 125–127] and [36, pp. 67–68]). The fundamental function of the Orlicz space $L^\Phi(\Omega)$ is

$$\varphi_{L^\Phi(\Omega)}(t) = \|\chi_A\|_{L^\Phi(\Omega)} = \|\chi_{[0,\mu(A)]}\|_{L^\Phi([0,\infty)}) = 1/\Phi^{-1}(1/t),$$

where $\chi_A$ is the characteristic function of the set $A \subset \Omega, t = \mu(A)$ and $\Phi^{-1}$ is the right-continuous inverse of $\Phi$ defined by $\Phi^{-1}(v) = \inf\{u \geq 0: \Phi(u) > v\}$ with $\inf \emptyset = \infty$.

To each Young function $\Phi$ one can associate another convex function $\Phi^*$, i.e., the complementary function to $\Phi$, which is defined by

$$\Phi^*(v) = \sup_{u \geq 0} [uv - \Phi(u)] \text{ for } v \geq 0.$$  

Then $\Phi^*$ is also a Young function and $\Phi^{**} = \Phi$. Note that $u \leq \Phi^{-1}(u)\Phi^{**}(u) \leq 2u$ for all $u > 0$.

We say that a Young function $\Phi$ satisfies the $\Delta_2$-condition and we write shortly $\Phi \in \Delta_2$, if $0 < \Phi(u) < \infty$ for $u > 0$ and there exists a constant $D_2 \geq 1$ such that

$$\Phi(2u) \leq D_2\Phi(u) \text{ for all } u > 0. \quad (3)$$

In this paper we consider Orlicz spaces $L^\Phi(\mathbb{R}^n)$ on $\mathbb{R}^n$ with the Lebesgue measure. Then we define the Morrey–Orlicz spaces $M^{\Phi,\lambda}(\mathbb{R}^n)$ and central Morrey–Orlicz spaces $M^{\Phi,\lambda}(0)$. In the 2000s, several authors (for example, F. Deringoz, V. S. Guliyev, J. J. Hasanov, T. Mizuhara, E. Nakai, S. Samko, Y. Sawano, H. Tanaka and others) defined Orlicz versions of the Morrey space, i.e., Morrey–Orlicz spaces, and investigated the boundedness for the Hardy–Littlewood maximal operator and other operators on them (see, for example, [11], [17], [18], [32], [37] and the references therein). The Orlicz version of central Morrey spaces, i.e., central Morrey–Orlicz spaces were defined in papers by the second and third authors. They investigated boundedness on central Morrey–Orlicz spaces of the Hardy–Littlewood maximal operator in paper [27] and also boundedness of the Calderón–Zygmund singular integrals on them in paper [28]. In this paper we present conditions under which the Riesz potential is bounded on central Morrey–Orlicz spaces.

For any Young function $\Phi$, number $\lambda \in \mathbb{R}$, a set $A \subset \mathbb{R}^n$ with $0 < |A| < \infty$ and for $f \in L^0(\mathbb{R}^n)$ let

$$\|f\|_{\Phi,\lambda,A} = \inf \left\{ \varepsilon > 0 : \frac{1}{|A|^\lambda} \int_A \Phi\left( \frac{|f(x)|}{\varepsilon} \right) dx \leq 1 \right\},$$

and the corresponding (smaller) expression

$$\|f\|_{\Phi,\lambda,A,\infty} = \inf \left\{ \varepsilon > 0 : \sup_{u > 0} \Phi\left( \frac{u}{\varepsilon} \right) \frac{1}{|A|^\lambda} d(f\chi_A, u) \leq 1 \right\},$$

where $d(f, u) = |\{x \in \mathbb{R}^n : |f(x)| > u\}|$. Note that $\|f\|_{\Phi,\lambda,A,\infty} \leq \|f\|_{\Phi,\lambda,A}$ provided that the expression on the right is finite. In fact, if $\|f\|_{\Phi,\lambda,A} < c$, then for arbitrary $u > 0$ we have

$$1 \geq \frac{1}{|A|^\lambda} \int_A \Phi\left( \frac{|f(x)|}{c} \right) dx \geq \frac{1}{|A|^\lambda} \int_{\{x \in A : |f(x)| > u\}} \Phi\left( \frac{|f(x)|}{c} \right) dx \geq \frac{1}{|A|^\lambda} \Phi\left( \frac{u}{c} \right) d(f\chi_A, u),$$

and $\|f\|_{\Phi,\lambda,A,\infty} \leq c$. Hence, $\|f\|_{\Phi,\lambda,A,\infty} \leq \|f\|_{\Phi,\lambda,A}$. 

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Using these notions and considering open balls \( B(x_0, r) \) with a center at \( x_0 \in \mathbb{R}^n \) and radius \( r > 0 \), i.e. \( B(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| < r \} \), and also open balls \( B(0, r) = B_r \) with a center at 0 we can define Morrey–Orlicz spaces \( M^{\Phi, \lambda}(\mathbb{R}^n) \) and weak Morrey–Orlicz spaces \( WM^{\Phi, \lambda}(\mathbb{R}^n) \):

\[
M^{\Phi, \lambda}(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{M^{\Phi, \lambda}} = \sup_{x_0 \in \mathbb{R}^n, r > 0} \|f\|_{\Phi, \lambda, B(x_0, r)} < \infty \right\}
\]

and

\[
WM^{\Phi, \lambda}(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{WM^{\Phi, \lambda}} = \sup_{x_0 \in \mathbb{R}^n, r > 0} \|f\|_{\Phi, \lambda, B(x_0, r), \infty} < \infty \right\}.
\]

Similarly, we can define central Morrey–Orlicz spaces \( M^{\Phi, \lambda}(0) \) and weak central Morrey–Orlicz spaces \( WM^{\Phi, \lambda}(0) \):

\[
M^{\Phi, \lambda}(0) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{M^{\Phi, \lambda}(0)} = \sup_{r > 0} \|f\|_{\Phi, \lambda, B_r} < \infty \right\}
\]

and

\[
WM^{\Phi, \lambda}(0) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{WM^{\Phi, \lambda}(0)} = \sup_{r > 0} \|f\|_{\Phi, \lambda, B_r, \infty} < \infty \right\}.
\]

All these spaces are Banach ideal spaces on \( \mathbb{R}^n \) (sometimes they are \( \{0\} \), that is, they contain only all functions equivalent to 0 on \( \mathbb{R}^n \)). Moreover, we have continuous embeddings

\[
M^{\Phi, \lambda}(\mathbb{R}^n) \hookrightarrow WM^{\Phi, \lambda}(\mathbb{R}^n), \quad M^{\Phi, \lambda}(0) \hookrightarrow WM^{\Phi, \lambda}(0) \quad \text{and also } \quad M^{\Phi, \lambda}(\mathbb{R}^n) \hookrightarrow M^{\Phi, \lambda}(0),
\]

\[
WM^{\Phi, \lambda}(\mathbb{R}^n) \hookrightarrow WM^{\Phi, \lambda}(0). \quad \text{Here and further, for two Banach ideal spaces } X \text{ and } Y, \text{ we use the symbol } X \hookrightarrow Y \text{ rather than } X \subset Y \text{ for continuous embedding. Moreover, the symbol } X \overset{C}{\hookrightarrow} Y \text{ indicates that } X \hookrightarrow Y \text{ with the norm of the embedding operator not bigger than } C, \text{ i.e., } \|f\|_Y \leq C \|f\|_X \text{ for all } f \in X.
\]

Note that Morrey–Orlicz spaces and central Morrey–Orlicz spaces are generalizations of Orlicz spaces and Morrey spaces (on \( \mathbb{R}^n \)). In particular, we can obtain the following spaces (see [27] for more details):

(i) (Orlicz and weak Orlicz spaces) If \( \lambda = 0 \), then

\[
M^{\Phi, 0}(\mathbb{R}^n) = M^{\Phi, 0}(0) = L^{\Phi}(\mathbb{R}^n) \quad \text{and} \quad WM^{\Phi, 0}(\mathbb{R}^n) = WM^{\Phi, 0}(0) = W L^{\Phi}(\mathbb{R}^n).
\]

(ii) (Beurling–Orlicz and weak Beurling–Orlicz spaces) If \( \lambda = 1 \), then

\[
M^{\Phi, 1}(\mathbb{R}^n) = B^{\Phi}(\mathbb{R}^n) \quad \text{and} \quad WM^{\Phi, 1}(\mathbb{R}^n) = WB^{\Phi}(\mathbb{R}^n).
\]

As for \( B^{\Phi}(\mathbb{R}^n) \) and \( WB^{\Phi}(\mathbb{R}^n) \), see [27].

(iii) (classical Morrey, weak Morrey, central Morrey and weak central Morrey spaces) If \( \Phi(u) = u^p, 1 \leq p < \infty \) and \( \lambda \in \mathbb{R} \), then

\[
M^{\Phi, \lambda}(\mathbb{R}^n) = M^{p, \lambda}(\mathbb{R}^n), \quad WM^{\Phi, \lambda}(\mathbb{R}^n) = WM^{p, \lambda}(\mathbb{R}^n) \quad \text{and} \quad M^{\Phi, \lambda}(0) = M^{p, \lambda}(\mathbb{R}^n), \quad WM^{\Phi, \lambda}(0) = WM^{p, \lambda}(\mathbb{R}^n).
\]

Here \( M^{p, \lambda}(\mathbb{R}^n), WM^{p, \lambda}(\mathbb{R}^n), M^{p, \lambda}(0), WM^{p, \lambda}(0) \) are the classical Morrey, weak Morrey, central Morrey and weak central Morrey spaces, respectively.
We want to note that $M^{p, \lambda}(\mathbb{R}^n) \neq \{0\}$ if and only if $0 \leq \lambda \leq 1$ (see [6, Lemma 1]) and $M^{p, \lambda}(0) \neq \{0\}$ if and only if $\lambda \geq 0$ (see [4, 6, 7]). Moreover, $M^{p,0}(\mathbb{R}^n) = M^{p,0}(0) = L^p(\mathbb{R}^n)$ and $M^{p,1}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ (see [21, Theorem 4.3.6]). However, $L^\infty(\mathbb{R}^n) \hookrightarrow M^{p,1}(0)$ and the inclusion is strict. For example, in one-dimensional case $f(x) = \sum_{n=0}^{\infty} 2^{n/p} x_{[n,n+2^{-n}]}(|x|) \in M^{p,1}(0) \setminus L^\infty(\mathbb{R}^1)$. Of course, for $0 \leq \lambda \leq 1$ the inclusion

$$M^{p, \lambda}(\mathbb{R}^n) \hookrightarrow M^{p, \lambda}(0)$$

holds and is strict for $0 < \lambda < 1$ (a suitable example we can find in [20, p. 156]). It is also true that if $1 \leq p < q < \infty, 0 \leq \mu < \lambda < 1$ and $\frac{1-\lambda}{p} = \frac{1-\mu}{q}$, then

$$M^{p, \mu}(\mathbb{R}^n) \hookrightarrow M^{p, \lambda}(\mathbb{R}^n) \quad \text{and} \quad M^{q, \mu}(0) \hookrightarrow M^{p, \lambda}(0). \quad (8)$$

Both inclusions are proper (see, for example, [19]); the second embedding in (8) is also true for $1 < \lambda < \mu$.

The embeddings (8) follow by the Hölder–Rogers inequality with $\frac{1}{p} > 1$, since for any $x_0 \in \mathbb{R}^n$ we have

$$\int_{B(x_0, r)} |f(x)|^p \, dx \leq \left( \int_{B(x_0, r)} |f(x)|^q \, dx \right)^{p/q} |B(x_0, r)|^{1-p/q}$$

$$= \left( \frac{1}{|B(x_0, r)|^\mu} \int_{B(x_0, r)} |f(x)|^q \, dx \right)^{p/q} |B(x_0, r)|^{1-p/q+\mu p/q}$$

$$= \left( \frac{1}{|B(x_0, r)|^\mu} \int_{B(x_0, r)} |f(x)|^q \, dx \right)^{p/q} |B(x_0, r)|^{\lambda},$$

and from the fact that $1 - p/q + \mu p/q = (\mu - 1) p/q + 1 = -(1 - \lambda) + 1 = \lambda$.

If the supremum in definitions (4)–(7) is taken over all $r > 1$, then we will have corresponding definitions of non-homogeneous Morrey–Orlicz spaces, non-homogeneous weak Morrey–Orlicz spaces, non-homogeneous central Morrey–Orlicz spaces and non-homogeneous weak central Morrey–Orlicz spaces.

2 The Riesz potential in Lebesgue, Orlicz and Morrey spaces

The Riesz potential of order $\alpha \in (0, n)$ of a locally integrable function $f \in \mathbb{R}^n, n \geq 1$, is defined as

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy, \quad \text{for} \quad x \in \mathbb{R}^n. \quad (9)$$

The linear operator $I_\alpha$ plays a role in various branches of analysis, including potential theory, harmonic analysis, Sobolev spaces and partial differential equations. Therefore, investigations of the boundedness of the operator $I_\alpha$ between different spaces are important.

The classical Hardy–Littlewood–Sobolev theorem states that if $1 < p < q < \infty$, then a Riesz potential $I_\alpha$ is of strong-type $(p, q)$, that is, bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $1/q = 1/p - \alpha/n$. For $p = 1 < q < \infty$ Zygmund proved that $I_\alpha$ is of weak-type $(1, q)$, that is, bounded from $L^1(\mathbb{R}^n)$ to $WL^q(\mathbb{R}^n)$, where $1/q = 1 - \alpha/n$. The weak-$L^q$ space
that the quasi-norm 
\[ \| f \|_{q, \infty} = \sup_{t>0} t \{ x \in \mathbb{R}^n : |f(x)| > t \}^{1/q} \] is finite. The proofs of these results we can find in the books [13, pp. 125–127], [16, pp. 2–5], [38, pp. 117–121], [39, pp. 150–154] and [40, pp. 86–87].

The boundedness of \( I_\alpha \) from an Orlicz space \( L^\Phi(\mathbb{R}^n) \) to another Orlicz space \( L^\Psi(\mathbb{R}^n) \) was studied by Simonenko (1964), O’Neil (1965) and Torchinsky (1976) under some restrictions on the Orlicz functions \( \Phi \) and \( \Psi \). In 1999 Cianchi [10] gave a necessary and sufficient condition for the boundedness of \( I_\alpha \) from \( L^\Phi(\mathbb{R}^n) \) to \( L^\Psi(\mathbb{R}^n) \) and from \( L^\Phi(\mathbb{R}^n) \) to weak Orlicz space \( W L^\Psi(\mathbb{R}^n) \). Another sufficient conditions for boundedness of the Riesz operator \( I_\alpha \) (and even for a generalized fractional operator \( I_\rho \)) were given in 2001 by Nakai [30], [31]. Then in 2017, Guliyev–Deringoz–Hasanov in [18, Theorem 3.3] gave more readable necessary and sufficient conditions for the boundedness of \( I_\alpha \) from \( L^\Phi(\mathbb{R}^n) \) to \( W L^\Psi(\mathbb{R}^n) \) and from \( L^\Phi(\mathbb{R}^n) \) to \( L^\Psi(\mathbb{R}^n) \).

Results concerning boundedness of the Riesz potential between Morrey spaces were first obtained by Spanne with the Sobolev exponent \( 1/q = 1/p - \alpha/n \), and this result was published in 1969 by Peetre [35]: if \( 0 < \alpha < n, 1 < p < n(1-\lambda)/\alpha, 0 < \lambda < 1, 1/q = 1/p - \alpha/n \) and \( \lambda/p = \mu/q \), then the Riesz potential \( I_\alpha \) is bounded from \( M^{p,\lambda}(\mathbb{R}^n) \) to \( M^{q,\mu}(\mathbb{R}^n) \). Then in 1975 a stronger result was obtained by Adams [1], and reproved by Chiarenza–Frasca [2]. Adams proved boundedness of \( I_\alpha \) from \( M^{p,\lambda}(\mathbb{R}^n) \) to \( M^{q,\lambda}(\mathbb{R}^n) \) with a better exponent \( q_1 \), namely \( 1/q_1 = 1/p - \alpha/[n(1-\lambda)] \). Adams result is stronger than the Peetre–Spanne theorem because \( q < q_1 \) and \( (1-\mu)/q = (1-\lambda)/q_1 \), from which follows the embedding \( M^{q,\lambda}(\mathbb{R}^n) \hookrightarrow M^{q,\mu}(\mathbb{R}^n) \) and this means that the target space \( M^{q,\lambda}(\mathbb{R}^n) \) is smaller than target space \( M^{q,\mu}(\mathbb{R}^n) \) in the Peetre–Spanne result.

Central Morrey spaces \( M^{p,\lambda}(0) \) were first introduced in [14, p. 607] and in [2, p. 5] (see also [8, p. 257] and [13, p. 500] for \( \lambda = 1 \)). Result on the boundedness of the Riesz potential in these spaces was proved by Fu–Lin–Lu [12, Proposition 1.1]: if \( 1 < p < n(1-\lambda)/\alpha, 0 < \lambda < 1, 1/q = 1/p - \alpha/n \) and \( \lambda/p = \mu/q \), then the Riesz potential \( I_\alpha \) is bounded from \( M^{p,\lambda}(0) \) to \( M^{q,\mu}(0) \) (see also [3], where the result is proved even for more general local Morrey-type spaces). Komori-Furuya and Sato [22, Proposition 1] showed that Adams type result on boundedness in central Morrey spaces does not hold. They showed that if \( \frac{\lambda}{p} = \frac{1-\lambda}{q} \) and \( \alpha/n < 1/p - 1/q < \alpha/[n(1-\lambda)] \), then \( I_\alpha \) is not bounded from \( M^{p,\lambda}(0) \) to \( M^{q,\mu}(0) \) because \( \mu/q = \lambda/p - (1/p - \alpha/n - 1/q) < \lambda/p \).

We will generalize the last results to central Morrey–Orlicz spaces. In Theorem 2, the necessary conditions for boundedness of \( I_\alpha \) are given, and in Theorem 3 – sufficient conditions are presented.

In the proof of boundedness of the Riesz potential in the central Morrey–Orlicz spaces we will need some necessary estimates. We will present them in the next section.

### 3 Some technical results

To prove the main results of this paper, we need some technical calculations. In order not to hide the main ideas in proofs of the main results we collect such calculations in Lemma 1 below.
Lemma 1. Let $\Phi$ be a Young function, $\Phi^*$ its complementary function, $\lambda \geq 0$ and $r > 0$. Then

(i) $\int_{B_r} |f(x)|g(x)| \, dx \leq 2 |B_r|^{\lambda} \| f\|_{\Phi, \lambda, B_r} \| g\|_{\Phi^*, \lambda, B_r}.$

(ii) $\| \chi_{B(x_0, r)} \|_{\Phi^*, \lambda, B_r} \leq \frac{|B_r \cap B(x_0, r)|}{|B_r|^{\lambda}} \Phi^{-1} \left( \frac{|B_r|^{\lambda}}{|B_r \cap B(x_0, r)|} \right)$, where $B_r \cap B(x_0, r) \neq \emptyset$ for $x_0 \in \mathbb{R}^n$ and $r > 0$.

In particular, $\| \chi_{B_r} \|_{\Phi^*, \lambda, B_r} \leq \Phi^{-1} \left( \frac{|B_r|^{\lambda-1}}{|B_r|^{\lambda}} \right).$

(iii) $\| \chi_{B_r} \|_{\Phi, \lambda, B_r} = 1/\Phi^{-1} \left( \frac{|B_r|^{\lambda}}{|B_r \cap B(x_0, r)|} \right)$ and $\| \chi_{B_t} \|_{M^{\Phi, \lambda}(0)} = 1/\Phi^{-1} \left( \frac{|B_t|^{\lambda}}{|B_t|^{\lambda-1}} \right)$ for any $t > 0$.

(iv) $\| \chi_{B_t} \|_{\Phi, \lambda, B_t, \infty} = 1/\Phi^{-1} \left( \frac{|B_t|^{\lambda}}{|B_t \cap B(x_0, r)|} \right)$ and $\| \chi_{B_t} \|_{W^{2, \lambda}(0)} = 1/\Phi^{-1} \left( \frac{|B_t|^{\lambda}}{|B_t|^{\lambda-1}} \right)$ for any $t > 0$.

Proof. (i) This estimate was proved in [28] Lemma 2.6.

(ii) Since for $u > 0$ we have $\Phi^* \left( \frac{u}{\Phi^{-1}(u)} \right) \leq u$ (cf. Lemma 2.6 in [28]) it follows for $u = \frac{|B_r|^{\lambda}}{|B_r \cap B(x_0, r)|}$ that

$$
\int_{B_r} \Phi^* \left( \frac{\chi_{B(x_0, r)}(x)}{|B_r|^{\lambda}} \right) \, dx = \int_{B_r \cap B(x_0, r)} \Phi^* \left( \frac{|B_r|^{\lambda}}{\Phi^{-1} \left( \frac{|B_r|^{\lambda}}{|B_r \cap B(x_0, r)|} \right)} \right) \, dx \leq \frac{|B_r|^{\lambda}}{|B_r \cap B(x_0, r)|} \int_{B_r \cap B(x_0, r)} \, dx = |B_r|^{\lambda}.
$$

Hence, $\| \chi_{B(x_0, r)} \|_{\Phi^*, \lambda, B_r} \leq \Phi^{-1} \left( \frac{|B_r|^{\lambda}}{|B_r \cap B(x_0, r)|} \right)$, and (ii) follows.

(iii) Let $t > 0$. Since $\Phi(\Phi^{-1}(u)) \leq u$ for any $u > 0$ it follows that

$$
\int_{B_r} \Phi \left( \chi_{B_t}(x) \Phi^{-1} \left( \frac{|B_r|^{\lambda}}{|B_r \cap B_t|} \right) \right) \, dx = \int_{B_r \cap B_t} \Phi \left( \Phi^{-1} \left( \frac{|B_r|^{\lambda}}{|B_r \cap B_t|} \right) \right) \, dx \leq \int_{B_r \cap B_t} \frac{|B_r|^{\lambda}}{|B_r \cap B_t|} \, dx = |B_r|^{\lambda},
$$

and so $\| \chi_{B_r} \|_{\Phi, \lambda, B_t} \leq 1/\Phi^{-1} \left( \frac{|B_r|^{\lambda}}{|B_r \cap B_t|} \right)$. On the other hand,

$$
1 \geq \frac{1}{|B_r|^{\lambda}} \int_{B_r} \Phi \left( \frac{\chi_{B_t}(x)}{\| \chi_{B_r} \|_{\Phi, \lambda, B_r}} \right) \, dx = \Phi \left( \frac{1}{\| \chi_{B_t} \|_{\Phi, \lambda, B_r}} \right) \frac{|B_r \cap B_t|}{|B_r|^{\lambda}},
$$

or

$$
\frac{|B_r|^{\lambda}}{|B_r \cap B_t|} \geq \Phi \left( \frac{1}{\| \chi_{B_r} \|_{\Phi, \lambda, B_r}} \right).
$$
Since \( u \leq \Phi^{-1}(\Phi(u)) \) for any \( u > 0 \) such that \( \Phi(u) < \infty \) we obtain
\[
\Phi^{-1}\left( \frac{|B_r|^{\lambda}}{|B_r \cap B_t|} \right) \geq \Phi^{-1}\left( \frac{1}{\| \chi_{B_t} \|_{\Phi,\lambda,B_r}} \right) \geq \frac{1}{\| \chi_{B_t} \|_{\Phi,\lambda,B_r}}.
\]
which together with the previous estimate gives equality \( \| \chi_{B_t} \|_{\Phi,\lambda,B_r} = 1/\Phi^{-1}\left( \frac{|B_r|^{\lambda}}{|B_r \cap B_t|} \right) \).

Thus,
\[
\| \chi_{B_t} \|_{M^{\Phi,\lambda}(0)} = \sup_{r>0} \| \chi_{B_t} \|_{\Phi,\lambda,B_r} = \sup_{r>0} \frac{1}{\Phi^{-1}\left( \frac{|B_r|^{\lambda}}{|B_r \cap B_t|} \right)}
= \max\left[ \sup_{r \leq t} \frac{1}{\Phi^{-1}\left( \frac{|B_r|^{\lambda}}{|B_r \cap B_t|} \right)}, \sup_{r \geq t} \frac{1}{\Phi^{-1}\left( \frac{|B_r|^{\lambda}}{|B_r \cap B_t|} \right)} \right]
= \max\left[ \sup_{r \leq t} \frac{1}{\Phi^{-1}(|B_r|^{\lambda-1})}, \sup_{r \geq t} \frac{1}{\Phi^{-1}(|B_r|^{\lambda})} \right] = \frac{1}{\Phi^{-1}(|B_t|^{\lambda-1})},
\]
and point (iii) of the lemma has been proved.
(iv) For \( t > 0 \) we have
\[
\sup_{u>0} \Phi\left( \frac{u}{\varepsilon} \right) \frac{1}{|B_r|^{\lambda}} \left\{ x \in B_r : \chi_{B_t}(x) > u \right\} = \sup_{0<\varepsilon<1} \Phi\left( \frac{u}{\varepsilon} \right) \frac{|B_r \cap B_t|}{|B_r|^{\lambda}} = \Phi\left( \frac{1}{\varepsilon} \right) \frac{|B_r \cap B_t|}{|B_r|^{\lambda}}.
\]
Thus,
\[
\| \chi_{B_t} \|_{\Phi,\lambda,B_r,\infty} = \inf\{ \varepsilon > 0 : \Phi\left( \frac{1}{\varepsilon} \right) \frac{|B_r \cap B_t|}{|B_r|^{\lambda}} \leq 1 \}
\leq \inf\{ \varepsilon > 0 : \frac{1}{\varepsilon} \leq \Phi^{-1}\left( \frac{|B_r|^{\lambda}}{|B_r \cap B_t|} \right) \} \leq 1/\Phi^{-1}(\Phi(1/\varepsilon)),
\]
because \( 1/\varepsilon \leq \Phi^{-1}(\Phi(1/\varepsilon)) \). On the other hand, since \( 1 \geq \Phi\left( \frac{1}{\| \chi_{B_t} \|_{\Phi,\lambda,B_r,\infty} |B_r \cap B_t|} \right) \) it follows that
\[
\frac{1}{\| \chi_{B_t} \|_{\Phi,\lambda,B_r,\infty}} \leq \Phi^{-1}\left( \frac{1}{\| \chi_{B_t} \|_{\Phi,\lambda,B_r,\infty}} \right) \leq \Phi^{-1}\left( \frac{|B_r|^{\lambda}}{|B_r \cap B_t|} \right),
\]
which together gives the first equality in (iv). The second equality in (iv) has the same proof as the second equality in (iii).

\section{On the norm of the dilation operator in central Morrey–Orlicz spaces}

For any \( a > 0 \) and \( x \in \mathbb{R}^n \) we define the dilation operator \( D_a \) by
\[
D_a f(x) = f(ax), \quad f \in L^0(\mathbb{R}^n).
\]
The dilation operator is bounded in central Morrey–Orlicz spaces \( M^{\Phi,\lambda}(0) \) and we will calculate its norm. For this purpose quantity \( s_{\Phi^{-1}} \) is needed for the Orlicz function \( \Phi \):
\[
s_{\Phi^{-1}}(t) = \sup_{s>0} \frac{\Phi^{-1}(st)}{\Phi^{-1}(s)}, \quad t > 0.
\]
THEOREM 1. If $\Phi$ is an Orlicz function, $\lambda \geq 0$ and $a > 0$, then the operator norm of $D_a$ is

$$\|D_a\|_{M^{\Phi,\lambda}(0) \rightarrow M^{\Phi,\lambda}(0)} = s_{\Phi^{-1}}(a^{n(\lambda-1)}) .$$  \((11)\)

Proof. By definition of $s_{\Phi^{-1}}$, for any $s > 0, a > 0$, we have

$$\Phi^{-1}(a^{n(\lambda-1)}s) \leq s_{\Phi^{-1}}(a^{n(\lambda-1)}) \Phi^{-1}(s),$$

and so

$$\Phi\left(\frac{\Phi^{-1}(a^{n(\lambda-1)}s)}{s_{\Phi^{-1}}(a^{n(\lambda-1)})}\right) \leq \Phi\left(\Phi^{-1}(s)\right) = s.$$

For $a^{n(\lambda-1)}s = \Phi(u)$ we have $u = \Phi^{-1}(a^{n(\lambda-1)}s)$ and

$$\Phi\left(\frac{u}{s_{\Phi^{-1}}(a^{n(\lambda-1)})}\right) \leq a^{n(1-\lambda)}\Phi(u), \text{ for any } u > 0. \quad \text{(12)}$$

Therefore, from \((12)\) it follows that for any $f \in M^{\Phi,\lambda}(0)$ and $r > 0$,

$$\int_{B_r} \Phi\left(\frac{|D_a f(x)|}{s_{\Phi^{-1}}(a^{n(\lambda-1)}) \|f\|_{M^{\Phi,\lambda}(0)}}\right) dx = \int_{B_r} \Phi\left(\frac{|f(ax)|}{s_{\Phi^{-1}}(a^{n(\lambda-1)}) \|f\|_{M^{\Phi,\lambda}(0)}}\right) dx$$

$$= a^{-n} \int_{B_{ar}} \Phi\left(\frac{|f(y)|}{s_{\Phi^{-1}}(a^{n(\lambda-1)}) \|f\|_{M^{\Phi,\lambda}(0)}}\right) dy$$

$$\leq a^{-n} a^{n(1-\lambda)} \int_{B_{ar}} \Phi\left(\frac{|f(y)|}{\|f\|_{M^{\Phi,\lambda}(0)}}\right) dy \leq a^{-\lambda n} |B_{ar}|^{\lambda}$$

$$= a^{-\lambda n} v_n^{\lambda} (ar)^{\lambda n} = |B_r|^\lambda,$$

which means that $\|D_a f\|_{M^{\Phi,\lambda}(0)} \leq s_{\Phi^{-1}}(a^{n(\lambda-1)}) \|f\|_{M^{\Phi,\lambda}(0)}$. Here, $v_n = |B_1|$.

To show that \((11)\) holds we consider the characteristic function $\chi_{B_t}(x)$ of the ball $B_t$, $t > 0$. Note that $D_a \chi_{B_t}(x) = \chi_{B_t/a}(x)$. Moreover, by Lemma 1(iii) we get

$$\sup_{t>0} \frac{\|D_a \chi_{B_t}\|_{M^{\Phi,\lambda}(0)}}{\|\chi_{B_t}\|_{M^{\Phi,\lambda}(0)}} = \sup_{t>0} \frac{\Phi^{-1}(|B_t|^{\lambda-1})}{\Phi^{-1}(|B_t/a|^{\lambda-1})} = \sup_{t>0} \frac{\Phi^{-1}(v_n^{\lambda-1} t^{\lambda-1})}{\Phi^{-1}(v_n^{\lambda-1})}$$

$$= \sup_{s>0} \frac{\Phi^{-1}(s)}{\Phi^{-1}(s/a^{n(1-\lambda)})} = \frac{\Phi^{-1}(s/a^{n(1-\lambda)})}{\Phi^{-1}(s/a^{n(1-\lambda)})} = s_{\Phi^{-1}}(a^{n(\lambda-1)}).$$

This brings us to \((11)\).

\(\square\)

5 The Riesz potential in central Morrey–Orlicz spaces—necessary conditions

We begin to study the boundedness of the Riesz potential, first finding the necessary conditions for its boundedness.
THEOREM 2. Let \( 0 < \alpha < n, \Phi, \Psi \) be Orlicz functions and \( 0 \leq \lambda, \mu < 1 \).

(i) If the Riesz potential \( I_\alpha \) is bounded from \( M^{\Phi,\lambda}(0) \) to \( M^{\Psi,\mu}(0) \), then there are positive constants \( C_1, C_2 \) such that

\(\begin{align*}
&\text{(a) } u^\Phi \Phi^{-1}(u^\lambda - 1) \leq C_1 \Psi^{-1}(u^\mu - 1) \text{ for any } u > 0, \\
&\text{(b) } s_{\Psi^{-1}}(u^\mu - 1) \leq C_2 u^\Phi s_{\Phi^{-1}}(u^\lambda - 1) \text{ for any } u > 0.
\end{align*}\)

(ii) If there exists a small constant \( c > 0 \) such that \( c \leq \frac{\omega_n^{\lambda/\mu}}{v_n-1} \) with \( v_0 = 1 \) and

\[ \liminf_{t \to \infty} \frac{\Phi^{-1}(ct^\lambda)}{\Psi^{-1}(t^\mu)} = \infty, \]

then \( I_\alpha \) is not bounded from \( M^{\Phi,\lambda}(0) \) to \( M^{\Psi,\mu}(0) \).

Proof. (i) (a) Let \( t > 0 \) and \( x \in B_t \). In this case we have

\[ I_\alpha \chi_{B_t}(x) = \int_{B_t} |x - y|^{\alpha - n} \, dy \geq (2t)^{\alpha - n} |B_t| = v_n 2^{\alpha - n} t^\alpha \]

or

\[ t^\alpha \chi_{B_t}(x) \leq \frac{2^{n - \alpha}}{v_n} I_\alpha \chi_{B_t}(x) \chi_{B_t}(x). \]

Then

\[ \|t^\alpha \chi_{B_t}\|_{M^{\Psi,\mu}(0)} \leq \frac{2^{n - \alpha}}{v_n} \|I_\alpha \chi_{B_t}\|_{M^{\Phi,\lambda}(0)} \leq \frac{2^{n - \alpha}}{v_n} C \|\chi_{B_t}\|_{M^{\Phi,\lambda}(0)}, \]

and by the Lemma \([\Pi]\) (iii) we obtain

\[ \frac{t^\alpha}{\Psi^{-1}(|B_t|^{\mu - 1})} \leq \frac{2^{n - \alpha}}{v_n} C \frac{1}{\Phi^{-1}(|B_t|^{\lambda - 1})}, \]

which means

\[ \frac{t^\alpha}{\Psi^{-1}(v_n^{\mu - 1} t^{(\mu - 1)n})} \leq \frac{2^{n - \alpha}}{v_n} C \frac{1}{\Phi^{-1}(v_n^{\lambda - 1} t^{(\lambda - 1)n})}. \]

Thus,

\[ t^{\alpha/n} \Phi^{-1}(v_n^{\lambda - 1} t^{\lambda - 1}) \leq \frac{2^{n - \alpha}}{v_n} C \frac{1}{\Psi^{-1}(v_n^{\mu - 1} t^{\mu - 1})}, \]

which by a simple change of variables can be rewritten as

\[ u^{\alpha/n} \Phi^{-1}(u^{\lambda - 1}) \leq C_1 \Psi^{-1}(u^{\mu - 1}) \text{ for any } u > 0, \]

where \( C_1 = 2^{n - \alpha} v_n^{\alpha/n - 1} C \).

(i) (b) First, note that we have identity

\[ I_\alpha(D_t f)(x) = t^{-\alpha} D_t(I_\alpha f)(x) \text{ for any } t > 0. \]

In fact,

\[ I_\alpha(D_t f)(x) = \int_{\mathbb{R}^n} \frac{f(ty)}{|x - y|^{n - \alpha}} \, dy = t^{-\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|y - tx|^{n - \alpha}} \, dy = t^{-\alpha} D_t(I_\alpha f)(x). \]
Now, let $f \in M^{\Phi, \lambda}(0)$. Using the above identity and applying Theorem 1 we obtain
\[
\|I_{\alpha}(D_t f)\|_{M^{\Phi, \mu}(0)} = t^{-\alpha} \|D_t(I_{\alpha} f)\|_{M^{\Phi, \mu}(0)} = t^{-\alpha} s_{\psi^{-1}(t^{n(\mu-1)})} \|I_{\alpha} f\|_{M^{\Phi, \mu}(0)}.
\]
Assumption of boundedness of $I_{\alpha}$ and reuse of Theorem 1 gives
\[
\|I_{\alpha} f\|_{M^{\Phi, \mu}(0)} = \frac{t^\alpha}{s_{\psi^{-1}(t^{n(\mu-1)})}} \|I_{\alpha}(D_t f)\|_{M^{\Phi, \mu}(0)} \leq \frac{t^\alpha}{s_{\psi^{-1}(t^{n(\mu-1)})}} C \|D_t f\|_{M^{\Phi, \lambda}(0)} = C \frac{t^\alpha}{s_{\psi^{-1}(t^{n(\mu-1)})}} s_{\psi^{-1}(t^{n(\lambda-1)})} \|f\|_{M^{\Phi, \lambda}(0)},
\]
or
\[
\|I_{\alpha} f\|_{M^{\Phi, \mu}(0)} \leq C \frac{u^{\alpha/n} s_{\psi^{-1}(u^{\lambda-1})}}{s_{\psi^{-1}(u^{\mu-1})}} \|f\|_{M^{\Phi, \lambda}(0)} \text{ for any } u > 0.
\]
Thus,
\[
\|I_{\alpha} f\|_{M^{\Phi, \mu}(0)} \leq C \inf_{u > 0} \frac{u^{\alpha/n} s_{\psi^{-1}(u^{\lambda-1})}}{s_{\psi^{-1}(u^{\mu-1})}} \|f\|_{M^{\Phi, \lambda}(0)}.
\]
We must have that $\inf_{u > 0} \frac{u^{\alpha/n} s_{\psi^{-1}(u^{\lambda-1})}}{s_{\psi^{-1}(u^{\mu-1})}} = C > 0$ since otherwise $I_{\alpha} f = 0$ and we get a contradiction. Therefore,
\[
s_{\psi^{-1}(u^{\mu-1})} \leq C c u^{\alpha/n} s_{\psi^{-1}(u^{\lambda-1})} \text{ for any } u > 0.
\]
(ii) We follow the same argument as in [22, Proposition 1]. Let $R \geq 1$, $x_R = (R, 0, ..., 0) \in \mathbb{R}^n$ and $f_R(x) = \chi_{B(x_R, 1)}(x)$. Then
\[
\|f_R\|_{M^{\Phi, \lambda}(0)} = \sup_{r > 0} \inf_{\varepsilon > 0} \left\{ \frac{1}{|B_r|^\lambda} \int_{B_r} \Phi \left( \frac{\chi_{B(x_R, 1)}(x)}{\varepsilon} \right) dx \leq 1 \right\}
\]
\[
= \sup_{r > 0} \inf_{\varepsilon > 0} \left\{ \frac{1}{|B_r|^\lambda} \int_{B_r \cap B(x_R, 1)} \Phi \left( \frac{1}{\varepsilon} \right) dx \leq 1 \right\}
\]
\[
= \sup_{r > 0} \inf_{\varepsilon > 0} \left\{ \frac{|B_r \cap B(x_R, 1)|}{|B_r|^\lambda} \Phi \left( \frac{1}{\varepsilon} \right) \leq 1 \right\}
\]
\[
= \sup_{r > R-1} \inf_{\varepsilon > 0} \left\{ \frac{|B_r \cap B(x_R, 1)|}{|B_r|^\lambda} \Phi \left( \frac{1}{\varepsilon} \right) \leq 1 \right\},
\]
because if $0 < r \leq R - 1$ then $|B_r \cap B(x_R, 1)| = 0$. Thus,
\[
\|f_R\|_{M^{\Phi, \lambda}(0)} = \sup_{r > R-1} \frac{1}{\Phi^{-1}(\frac{|B_r|^\lambda}{|B_r \cap B(x_R, 1)|})}.
\]
We will consider two cases: $R - 1 < r < R$ and $r \geq R$. In the first case, using calculations from [6, p. 161], we can prove that for $n \geq 2$
\[
|B_r \cap B(x_R, 1)| \leq 2^\frac{n}{\alpha} v_{n-1} \left( \frac{r}{R} \right)^n,
\]
and so
\[ \frac{|B_r|^\lambda}{|B_r \cap B(x_R, 1)|} \geq \frac{v_n^{\lambda} r^{\lambda n}}{2^n v_{n-1} r^n} \geq \frac{v_n^{\lambda}}{2^n v_{n-1}} R^{\lambda n}. \]

For \( n = 1 \) and \( R - 1 < r < R \) with \( v_0 = 1 \) we have
\[ \frac{|B_r|^\lambda}{|B_r \cap B(x_R, 1)|} = \frac{(2r)^\lambda}{r^{\lambda + 1}} = \frac{2^\lambda r^{\lambda - 1}}{r^{\lambda + 1}} > \frac{2^\lambda R^{\lambda - 1}}{r^{\lambda + 1}} \]
\[ \geq \frac{2^\lambda R^{\lambda - 1} v_n}{R v_n - 1} > 2^\lambda R^\lambda \geq \frac{v_n^{\lambda}}{2 v_n^\lambda R^\lambda}. \]

In the second case, \( |B_r \cap B(x_R, 1)| \leq |B(x_R, 1)| = v_n \) and
\[ \frac{|B_r|^\lambda}{|B_r \cap B(x_R, 1)|} \geq \frac{v_n^{\lambda} r^{\lambda n}}{v_n^{\lambda - 1} R^{\lambda n}}. \]

Thus,
\[ \| f_R \|_{M^{-\lambda}(0)} \leq \max \left[ \frac{1}{\Phi^{-1}(\frac{v_n^{\lambda} R^{\lambda n}}{2^n v_{n-1} R^{\lambda n}})}, \frac{1}{\Phi^{-1}(v_n^{\lambda - 1} R^{\lambda n})} \right]. \]

Since \( \frac{v_n^{\lambda - 1}}{v_n^\lambda} \geq \sqrt{\frac{v_n}{2\pi}} \) with \( v_0 = 1 \) (see [3, Theorem 2]), it follows that \( \frac{2^\lambda v_n^{\lambda - 1}}{v_n^\lambda} \geq 1 \) and then \( \frac{v_n^{\lambda}}{2^n v_{n-1}} \leq v_n^{\lambda - 1} \), which gives
\[ \| f_R \|_{M^{-\lambda}(0)} \leq \frac{1}{\Phi^{-1}(\frac{v_n^{\lambda} R^{\lambda n}}{2^n v_{n-1} R^{\lambda n}})}. \]

Next, we will estimate \( I_\alpha f_R \). If \( x, y \in B(x_R, 1) \) then \( |x - y| \leq 2 \) and we obtain
\[ I_\alpha f_R(x) = \int_{\mathbb{R}^n} \frac{\chi_{B(x_R, 1)}(y)}{|x - y|^{n-\alpha}} dy = \int_{B(x_R, 1)} |x - y|^{\alpha - n} dy \]
\[ \geq 2^{\alpha - n} |B(x_R, 1)| \chi_{B(x_R, 1)}(x) = 2^{\alpha - n} v_n \chi_{B(x_R, 1)}(x). \]

Thus,
\[ \| I_\alpha f_R \|_{M^{-\mu}(0)} = \sup_{r > 0} \| I_\alpha f_R \|_{\Psi_\mu, B_r} \geq \| I_\alpha f_R \|_{\Psi_\mu, B_{R+1}} \]
\[ = \inf \left\{ \varepsilon > 0 : \int_{B_{R+1}} \Psi \left( \frac{|I_\alpha f_R(x)|}{\varepsilon} \right) dx \leq |B_{R+1}|^\mu \right\}. \]

Since \( x \in B(x_R, 1) \) and \( B_{R+1} \cap B(x_R, 1) = B(x_R, 1) \) it follows that
\[ \| I_\alpha f_R \|_{M^{-\mu}(0)} \geq \inf \left\{ \varepsilon > 0 : \int_{B(x_R, 1) \cap B_{R+1}} \Psi \left( 2^{\alpha - n} v_n / \varepsilon \right) dx \leq |B_{R+1}|^\mu \right\} \]
\[ = \frac{2^{\alpha - n} v_n}{\Psi^{-1} \left( |B_{R+1}|^\mu \right)} \geq \frac{2^{\alpha - n} v_n}{\Psi^{-1}(v_n^{\mu - 1} (R + 1)^\mu)} \geq \frac{2^{\alpha - n} v_n}{\Psi^{-1}(v_n^{\mu - 1} 2^\mu R^\mu)}. \]
Making the substitution \( t^\mu = v_n^{\mu - 1/2} R^{2n} \) we obtain

\[
\frac{\|I_\alpha f_R\|_{M^{\Phi, \mu}(0)}}{\|f_R\|_{M^{\Phi, \lambda}(0)}} \geq 2^{\alpha - n} v_n \frac{\Phi^{-1} \left( \frac{v_n^{\mu - 1/2} R^{2n}}{R^{2n}} \right)}{\Psi^{-1}(v_n^{\mu - 1/2} R^{2n})} = 2^{\alpha - n} v_n \frac{\Phi^{-1} \left( \frac{1}{v_n^{1/2 - \mu}} t^{\lambda} \right)}{\Psi^{-1}(t^{\mu})} \\
\geq 2^{\alpha - n} v_n \frac{\Phi^{-1} \left( \frac{1}{v_n^{1/2 - \mu}} t^{\lambda} \right)}{2^{\alpha + \lambda n} v_n} \geq 2^{\alpha - 2n - \lambda n} v_n \frac{\Phi^{-1} \left( ct^{\lambda} \right)}{\Psi^{-1}(t^{\mu})},
\]

and

\[
\liminf_{R \to \infty} \frac{\|I_\alpha f_R\|_{M^{\Phi, \mu}(0)}}{\|f_R\|_{M^{\Phi, \lambda}(0)}} \geq 2^{\alpha - 2n - \lambda n} v_n \liminf_{t \to \infty} \frac{\Phi^{-1}(ct^{\lambda})}{\Psi^{-1}(t^{\mu})} = \infty.
\]

Thus, the operator \( I_\alpha \) is not bounded from \( M^{\Phi, \lambda}(0) \) to \( M^{\Phi, \mu}(0) \) for \( n \geq 1 \).

\[ \square \]

6 The Riesz potential in central Morrey–Orlicz spaces – sufficient conditions

We want to prove boundedness of the Riesz potential \( I_\alpha \), \( 0 < \alpha < n \), between two different central Morrey–Orlicz spaces. First of all, we show the following well-definedness of \( I_\alpha f \) when \( f \in M^{\Phi, \lambda}(0) \).

**Lemma 2.** Let \( 0 < \alpha < n \), \( \Phi \) be an Orlicz function and \( 0 \leq \lambda < 1 \). Then for \( f \in M^{\Phi, \lambda}(0) \) the Riesz potential \( I_\alpha f \) is well-defined.

**Proof.** We will prove this lemma using the same arguments that were presented in the proof in [29, Theorem 2.1]. Let \( f \in M^{\Phi, \lambda}(0) \), \( r > 0 \) and \( x \in B_r \), and let

\[
I_\alpha f(x) = I_\alpha(f \chi_{B_{2r}})(x) + I_\alpha(f(1 - \chi_{B_{2r}}))(x). \tag{13}
\]

Since \( f \chi_{B_{2r}} \in L^1(\mathbb{R}^n) \), the first term is well-defined. For the second term, its well-definedness is shown in the proof of Lemma 3 below. Further, since for \( 0 < s < r \),

\[
f \chi_{B_{2s}} + f(1 - \chi_{B_{2s}}) = f \chi_{B_{2r}} + f(1 - \chi_{B_{2r}}),
\]

it follows that for \( x \in B_s \subset B_r \),

\[
I_\alpha(f \chi_{B_{2s}})(x) + I_\alpha(f(1 - \chi_{B_{2s}}))(x) = I_\alpha(f \chi_{B_{2r}})(x) + I_\alpha(f(1 - \chi_{B_{2r}}))(x).
\]

This shows that \( I_\alpha f \) is independent of \( B_r \) containing \( x \). Thus, \( I_\alpha f \) is well-defined on \( \mathbb{R}^n \). \[ \square \]

Now we will present sufficient conditions on spaces so that the operator \( I_\alpha \) is bounded between distinct central Morrey–Orlicz spaces. In the proofs of these estimates we will use estimates from [27] for the Hardy–Littlewood maximal operator. The **Hardy–Littlewood maximal operator** \( M \) is defined for \( f \in L^1_{loc}(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \) by

\[
Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy.
\]
Then, for an Orlicz function $\Phi$ and $0 \leq \lambda \leq 1$, this operator $M$ is bounded on $M^{\Phi,\lambda}(0)$, provided $\Phi^* \in \Delta_2$, that is, there exists a constant $C_0 > 1$ such that
\[
\|Mf\|_{M^{\Phi,\lambda}(0)} \leq C_0 \|f\|_{M^{\Phi,\lambda}(0)} \quad \text{for all } f \in M^{\Phi,\lambda}(0)
\] (see [27, Theorem 6(i)]). Moreover, $M$ is bounded from $M^{\Phi,\lambda}(0)$ to $W M^{\Phi,\lambda}(0)$, that is, there exists a constant $c_0 > 1$ such that $\|Mf\|_{W M^{\Phi,\lambda}(0)} \leq c_0 \|f\|_{M^{\Phi,\lambda}(0)}$ for all $f \in M^{\Phi,\lambda}(0)$ (see [27, Theorem 6(ii)]).

**THEOREM 3.** Let $0 < \alpha < n$, $\Phi, \Psi$ be Orlicz functions and either $0 < \lambda, \mu < 1$, $\lambda \neq \mu$ or $\lambda = \mu = 0$. Assume that there exist constants $C_3, C_4 \geq 1$ such that
\[
\int_0^\infty t^\frac{\alpha}{n} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} \leq C_3 \Psi^{-1}(u^{\mu-1}) \quad \text{for all } u > 0
\]
and
\[
\int_0^\infty t^\frac{\alpha}{n} \Phi^{-1}\left(\frac{r^\lambda}{t}\right) \frac{dt}{t} \leq C_4 \Psi^{-1}\left(\frac{r^\mu}{u}\right) \quad \text{for all } u > 0 \text{ and for all } r > 0.
\]

(i) If $\Phi^* \in \Delta_2$, then $I_\alpha$ is bounded from $M^{\Phi,\lambda}(0)$ to $M^{\Psi,\mu}(0)$, that is, there exists a constant $C_5 \geq 1$ such that $\|I_\alpha f\|_{M^{\Psi,\mu}(0)} \leq C_5 \|f\|_{M^{\Phi,\lambda}(0)}$ for all $f \in M^{\Phi,\lambda}(0)$.

(ii) The operator $I_\alpha$ is bounded from $M^{\Phi,\lambda}(0)$ to $W M^{\Psi,\mu}(0)$, that is, there exists a constant $c_5 \geq 1$ such that $\|I_\alpha f\|_{W M^{\Psi,\mu}(0)} \leq c_5 \|f\|_{M^{\Phi,\lambda}(0)}$ for all $f \in M^{\Phi,\lambda}(0)$.

**Remark 1.** The same conclusions hold for non-homogeneous versions of $M^{\Phi,\lambda}(0)$ and $M^{\Psi,\mu}(0)$.

**Remark 2.** From the estimate (15) we get the inequality (i) from Theorem 2. Namely, using the concavity of the function $\Phi^{-1}$ we get
\[
\int_0^\infty t^\frac{\alpha}{n} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} \geq \int_0^{2u} t^\frac{\alpha}{n} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} \geq u^\frac{\alpha}{n} \Phi^{-1}((2u)^{\lambda-1}) \ln 2 \geq \frac{\ln 2}{2^{1-\lambda}} u^\frac{\alpha}{n} \Phi^{-1}(u^{\lambda-1}).
\]

**Remark 3.** Note that if either $\lambda = \mu > 0$ or $\lambda = 0$ and $\mu > 0$, then estimate (16) doesn’t hold.

**Remark 4.** If $\lambda = \mu = 0$, then inequalities (15) and (16) are the same. Moreover, condition (15) in this case is a sufficient condition for boundedness of $I_\alpha$ from Orlicz space $L^{\Phi}(\mathbb{R}^n)$ to weak Orlicz space $WL^{\Phi}(\mathbb{R}^n)$, and if additionally $\Phi^* \in \Delta_2$ then $I_\alpha$ is bounded from Orlicz space $L^{\Phi}(\mathbb{R}^n)$ to Orlicz space $L^{\Psi}(\mathbb{R}^n)$ (proof we can find, for example, in [18, Theorem 3.3]).

In the proof of Theorem 3 the following lemmas are important.

**Lemma 3.** Let $0 < \alpha < n$, $\Phi, \Psi$ be Orlicz functions and $0 \leq \lambda, \mu < 1$. If the estimate (15) holds, then there exists a constant $C_6 \geq 1$ such that
\[
\Psi\left(\frac{\int_{\mathbb{R}^n \setminus B_r} |f(y)| \frac{dy}{|y|^{n-\alpha}}}{C_6 \|f\|_{M^{\Phi,\lambda}(0)}}\right) \leq |B_r|^{\mu-1} \quad \text{for all } f \in M^{\Phi,\lambda}(0) \text{ and } r > 0.
\]
Proof. We prove this lemma using the same arguments as in the proof of Theorem 7.1 in [33] and Lemma 2 in [28]. From the Lemma 1 and the condition (15) it follows that

\[
\int_{\mathbb{R}^n \setminus B_r} \frac{|f(y)|}{|y|^{\alpha}} dy \leq 2^{n-\alpha} v_n^{1-\alpha/n} \sum_{j=1}^{\infty} \frac{1}{|B_{2^jr}|^{1-\alpha/n}} \int_{B_{2^jr}} |f(y)| dy
\]

\[
\leq 2^{n-\alpha} v_n^{1-\alpha/n} \sum_{j=1}^{\infty} 2 |B_{2^jr}|^{\lambda-1+\alpha/n} \|f\|_{\Phi,\lambda,B_{2^jr}} \|1\|_{\Phi^*,\lambda,B_{2^jr}}
\]

\[
\leq C_7 \sum_{j=1}^{\infty} |B_{2^jr}|^{\lambda-1+\alpha/n} \|f\|_{\Phi,\lambda,B_{2^jr}} \cdot \frac{\Phi^{-1}(|B_{2^jr}|^{\lambda-1})}{|B_{2^jr}|^{\lambda-1}}
\]

\[
\leq \frac{C_7}{n \ln 2} \|f\|_{M^{\Phi,\lambda}(0)} \sum_{j=1}^{\infty} |B_{2^jr}|^{\alpha/n} \Phi^{-1}(|B_{2^jr}|^{\lambda-1}) \int_{|B_{2^jr}|} ^{\infty} \frac{dt}{t |B_{2^jr}|}
\]

\[
\leq C_8 \|f\|_{M^{\Phi,\lambda}(0)} \int_{|B_r|} ^{\infty} t^{\alpha/n} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t}
\]

\[
\leq C_3 C_8 \Psi^{-1}(|B_r|^{\mu-1}) \|f\|_{M^{\Phi,\lambda}(0)},
\]

where \(C_7 = 2^{n-\alpha+1} v_n^{1-\alpha/n}\) and \(C_8 = \frac{2^n}{n \ln 2} C_7\). Thus, with \(C_6 = C_3 C_8\), we obtain

\[
\Psi \left( \int_{\mathbb{R}^n \setminus B_r} \frac{|f(y)|}{|y|^{\alpha}} dy \right) \leq \Psi \left( \Psi^{-1}(|B_r|^{\mu-1}) \right) = |B_r|^{\mu-1},
\]

and this is precisely the assertion of Lemma 3. \(\square\)

Now we pass to the next lemma which plays a crucial role in the proof of the main result.

**Lemma 4.** Let \(0 < \alpha < n, \Phi, \Psi\) be Orlicz functions, \(\Phi^* \in \Delta_2\) and either \(0 < \lambda, \mu < 1, \lambda \neq \mu\) or \(\lambda = \mu = 0\). If the estimate (10) holds, then there exists a constant \(C_9 \geq 1\) such that

\[
\int_{B_r} \Psi \left( \int_{B_{2r}} \frac{|f(y)|}{x - y|^{n-\alpha}} dy \right) \frac{dx}{C_9 \|f\|_{M^{\Phi,\lambda}(0)}} \leq |B_r|^\mu, \quad \text{for all } f \in M^{\Phi,\lambda}(0) \text{ and } r > 0.
\]

**Proof.** Let \(f \in M^{\Phi,\lambda}(0)\). We write \(I_\alpha(f \chi_{B_{2r}})\) as follows

\[
I_\alpha(f \chi_{B_{2r}})(x) = \int_{B_{2r}} \frac{|f(y)|}{x - y|^{n-\alpha}} dy = \int_{|x-y| \leq \delta} \frac{|f(y)| \chi_{B_{2r}}(y)}{|x-y|^{n-\alpha}} dy + \int_{|x-y| > \delta} \frac{|f(y)| \chi_{B_{2r}}(y)}{|x-y|^{n-\alpha}} dy =: J_1 f(x) + J_2 f(x),
\]

where \(\delta > 0\) will be defined later on. It is known that

\[
J_1 f(x) \leq C_{10} |B_d|^\Phi M(f \chi_{B_{2r}})(x),
\]

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where $C_{10} = \frac{2^n}{2^a - 1} C_7$. Note that for any parameters $u > 0$ and $r > 0$ we have

$$\int_0^\infty t^{n/2} \Phi^{-1}(\frac{r}{t}) \frac{dt}{t} \geq \int_0^{2u} t^{n/2} \Phi^{-1}(\frac{r}{t}) \frac{dt}{t} \geq \ln 2 \cdot u^{n/2} \Phi^{-1}(\frac{r}{2u}) \geq \frac{1}{2} u^{n/2} \Phi^{-1}(\frac{r}{u}).$$

Thus, applying (16) we obtain

$$J_1 f(x) \leq \frac{2}{\ln 2} C_4 C_{10} \frac{\Psi^{-1}(\frac{|B_{2r}|^\alpha}{|B_2|})}{\Phi^{-1}(\frac{|B_{2r}|^\lambda}{|B_2|})} M(f \chi_{B_{2r}})(x).$$

Following Hedberg’s method we get for $J_2 f(x)$

$$J_2 f(x) = \sum_{k=1}^\infty \int_{2^{k-1} \delta < |x - y| \leq 2^k \delta} \frac{|f(y) \chi_{B_{2r}}(y)|}{|x - y|^{n-\alpha}} dy \leq \sum_{k=1}^\infty (2^{k-1} \delta)^{\alpha - n} \int_{|x - y| \leq 2^k \delta} |f(y) \chi_{B_{2r}}(y)| dy = \sum_{k=1}^\infty (2^{k-1} \delta)^{\alpha - n} \int_{B_{2r}} |f(y) \chi_{B(x, 2^k \delta)}(y)| dy.$$

From Lemma[1] (i) and (ii) it follows that

$$J_2 f(x) \leq 2 |B_{2r}|^\lambda \|f\|_{\Phi,\lambda,B_{2r}} \sum_{k=1}^\infty (2^{k-1} \delta)^{\alpha - n} \|B_{x, 2^k \delta}\|_{\Phi^*, \lambda,B_{2r}} \leq 2^{n-\alpha + 1} \|f\|_{\Phi,\lambda,B_{2r}} \sum_{k=1}^\infty (2^k \delta)^{\alpha - n} |B_{2r} \cap B(x, 2^k \delta)| \Phi^{-1} \left(\frac{|B_{2r}|^\lambda}{|B_{2r} \cap B(x, 2^k \delta)|}\right).$$

Taking into account that $u \Phi^{-1}(1/u)$ is increasing and $|B_{2r} \cap B(x, 2^k \delta)| \leq |B(x, 2^k \delta)|$ we obtain

$$J_2 f(x) \leq 2^{n-\alpha + 1} \|f\|_{\Phi,\lambda,B_{2r}} \sum_{k=1}^\infty (2^k \delta)^{\alpha - n} \Phi^{-1} \left(\frac{|B_{2r}|^\lambda}{|B(x, 2^k \delta)|}\right) |B(x, 2^k \delta)| = 2^{n-\alpha + 1} v_n \|f\|_{\Phi,\lambda,B_{2r}} \sum_{k=1}^\infty (2^k \delta)^{\alpha - n} \Phi^{-1} \left(\frac{|B_{2r}|^\lambda}{|B_{2^k \delta}|}\right) = \frac{C_7}{n \ln 2} \|f\|_{\Phi,\lambda,B_{2r}} \sum_{k=1}^\infty |B_{2^k \delta}|^{\alpha - n} \Phi^{-1} \left(\frac{|B_{2r}|^\lambda}{|B_{2^k \delta}|}\right) \int \frac{dt}{t}$$

$$\leq \frac{C_7}{n \ln 2} \|f\|_{\Phi,\lambda,B_{2r}} \sum_{k=1}^\infty |B_{2^k \delta}|^{\alpha - n} \Phi^{-1} \left(\frac{|B_{2r}|^\lambda}{|B_{2^k \delta}|}\right) \int \frac{dt}{t}.$$
\[
\leq C_8 \|f\|_{\Phi,\lambda, B_{2r}} \int_{|B_\delta|}^{\infty} t^{-\sigma} \Phi^{-1} \left( \frac{|B_{2r}|^\lambda}{t} \right) dt \leq C_4 \frac{C_8 \|f\|_{M^\Phi,\lambda(0)}}{\Psi^{-1} \left( \frac{|B_{2r}|^\mu}{|B_\delta|} \right)}.
\]

Now we choose \( \delta > 0 \) such that
\[
Mf(x) = \Phi^{-1} \left( \frac{|B_{2r}|^\lambda}{|B_\delta|} \right),
\]
where the constant \( C_0 \) is from (14). Then
\[
J_1 f(x) \leq \frac{2}{\ln 2} C_4 \frac{C_{10} C_0 \|f\|_{M^\Phi,\lambda(0)}}{\Psi^{-1} \left( \frac{|B_{2r}|^\mu}{|B_\delta|} \right)},
\]
and
\[
\int_{B_{2r}} \frac{|f(y)|}{|x - y|^{n - \alpha}} dy = J_1 f(x) + J_2 f(x)
\]
\[
\leq \left( \frac{2}{\ln 2} C_4 \frac{C_{10} C_0 + C_4 C_8}{M^\Phi,\lambda(0)} \right) \|f\|_{M^\Phi,\lambda(0)} \Psi^{-1} \left( \frac{|B_{2r}|^\mu}{|B_\delta|} \right).
\]
Thus, with \( C_{11} = 2 C_4 \max \left( \frac{2}{\ln 2} C_0 C_{10}, C_8 \right) \) we obtain
\[
\int_{B_{2r}} \frac{|f(y)|}{|x - y|^{n - \alpha}} dy \leq C_{11} \|f\|_{M^\Phi,\lambda(0)} \Psi^{-1} \left( \frac{|B_{2r}|^\mu \Phi}{C_0 \|f\|_{M^\Phi,\lambda(0)}} \right).
\]
Then
\[
\Psi \left( \frac{\int_{B_{2r}} \frac{|f(y)|}{|x - y|^{n - \alpha}} dy}{C_{11} \|f\|_{M^\Phi,\lambda(0)}} \right) \leq |B_{2r}|^{-\mu} \Phi \left( \frac{Mf(x)}{{\|Mf\|_{M^\Phi,\lambda(0)}}} \right)
\]
\[
= 2^{n(\mu - \lambda)} |B_r|^{-\mu} \Phi \left( \frac{Mf(x)}{{\|Mf\|_{M^\Phi,\lambda(0)}}} \right).
\]
Finally, with \( C_9 = 2^{n(\mu - \lambda)} C_{11} \) we get
\[
\frac{1}{|B_r|^\mu} \int_{B_r} \Psi \left( \frac{\int_{B_{2r}} \frac{|f(y)|}{|x - y|^{n - \alpha}} dy}{C_9 \|f\|_{M^\Phi,\lambda(0)}} \right) dx \leq \frac{1}{|B_r|^\lambda} \int_{B_r} \Phi \left( \frac{Mf(x)}{{\|Mf\|_{M^\Phi,\lambda(0)}}} \right) dx \leq 1
\]
and we arrive to the statement of this lemma. \( \square \)

**Proof of Theorem** (i) Let \( 0 < \alpha < n \) and \( 0 \leq \lambda < 1, 0 < \mu < 1 \). Let also \( f \in M^\Phi,\lambda(0) \) and \( r > 0 \). Since \( I_\alpha f \) is well-defined by Lemma 2, we prove only that
\[
\|I_\alpha f\|_{M^\Phi,\mu(0)} \leq C_5 \|f\|_{M^\Phi,\lambda(0)}.
\]
Now, by (13), for $C_5 = 2 \max(C_9, 2^{n-\alpha}C_6)$, it follows that

$$
\int_{B_r} \Psi \left( \frac{|I_\alpha f(x)|}{C_5 \|f\|_{M^{\Phi, \lambda}(0)}} \right) \, dx
\leq \frac{1}{2} \int_{B_r} \Psi \left( \frac{|I_\alpha (f(1-\chi_{B_{2r}}))(x)|}{C_5 \|f\|_{M^{\Phi, \lambda}(0)}} \right) \, dx + \frac{1}{2} \int_{B_r} \Psi \left( \frac{|I_\alpha (f(1-\chi_{B_{2r}}))(x)|}{2^{n-\alpha}C_6 \|f\|_{M^{\Phi, \lambda}(0)}} \right) \, dx
=: \frac{1}{2} (I_1 + I_2).
$$

By Lemma 4 for all $r > 0$

$$
I_1 \leq |B_r|^\mu.
$$

Next, we estimate $I_2$. Since for $x \in B_r$,

$$
|I_\alpha (f(1-\chi_{B_{2r}}))(x)| \leq 2^{n-\alpha} \int_{\mathbb{R}^n \setminus B_{2r}} \frac{|f(y)|}{|y|^{n-\alpha}} \, dy,
$$

it follows from Lemma 3 that

$$
I_2 \leq \int_{B_r} \Psi \left( \frac{\int_{\mathbb{R}^n \setminus B_{2r}} \frac{|f(y)|}{|y|^{n-\alpha}} \, dy}{C_6 \|f\|_{M^{\Phi, \lambda}(0)}} \right) \, dx \leq |B_{2r}|^{\mu-1} \cdot |B_r| \leq |B_r|^\mu.
$$

Hence, we obtain

$$
\frac{1}{|B_r|^\mu} \int_{B_r} \Psi \left( \frac{|I_\alpha f(x)|}{C_5 \|f\|_{M^{\Phi, \lambda}(0)}} \right) \, dx < 1.
$$

Thus, we get

$$
\|I_\alpha f\|_{M^{\Phi, \mu}(0)} \leq C_5 \|f\|_{M^{\Phi, \lambda}(0)}.
$$

(ii) Similarly to the previous case, by (13), we obtain for $u > 0$

$$
\Psi \left( \frac{|I_\alpha f(x)|}{c_5 \|f\|_{M^{\Phi, \lambda}(0)}} \right)
\leq \frac{1}{2} \Psi \left( \frac{|I_\alpha (f\chi_{B_{2r}})(x)|}{c_9 \|f\|_{M^{\Phi, \lambda}(0)}} \right) + \frac{1}{2} \Psi \left( \frac{|I_\alpha (f(1-\chi_{B_{2r}}))(x)|}{2^{n-\alpha+1}C_6 \|f\|_{M^{\Phi, \lambda}(0)}} \right)
=: \frac{1}{2} (I_3 + I_4),
$$

with $c_5 = 2 \max(c_9, 2^{n-\alpha+1}C_6)$, $c_9 = 2^{n(\mu-\lambda)+1}c_{11}$ and $c_{11} = 2C_{11} \max \left( \frac{2}{m_2}c_0 C_{10}, C_8 \right)$.

Since $\Psi(u) \, d(g, u) = v \, d(g, \Psi^{-1}(v)) = v \, d(\Psi(g), v)$ for any $u > 0$ with $v = \Psi(u)$ and

$$
d \left( \Psi \left( \frac{|I_\alpha f(x)|}{c_5 \|f\|_{M^{\Phi, \lambda}(0)}} \right), u \right) \leq d(I_3, u) + d(I_4, u),
$$

it follows that

$$
\sup_{u > 0} \frac{\Psi(u)}{|B_r|^\mu} \, d \left( \frac{|I_\alpha f(x)|}{c_5 \|f\|_{M^{\Phi, \lambda}(0)}}, u \right) \leq \sup_{u > 0} \frac{u}{|B_r|^\mu} \, d(I_3, u) + \sup_{u > 0} \frac{u}{|B_r|^\mu} \, d(I_4, u).
$$
From the proof of Lemma 4 for all \( r > 0 \)
\[
I_3 = \Psi \left( \frac{|I_0(f \chi_{B_r})(x)|}{c_9 \| f \|_{M^p, \lambda}(0)} \right) \leq \frac{1}{2} |B_r|^{\mu - \lambda} \Phi \left( \frac{Mf(x)}{\| Mf \|_{W M^p, \lambda}(0)} \right)
\]
and
\[
\sup_{u > 0} \frac{u}{|B_r|^\mu} d(I_1, u) \leq \sup_{u > 0} \frac{u}{|B_r|^\mu} d \left( \frac{1}{2} |B_r|^{\mu - \lambda} \Phi \left( \frac{Mf(x)}{\| Mf \|_{W M^p, \lambda}(0)} \right), u \right)
\]
\[
= \frac{1}{2} \sup_{u > 0} \frac{u}{|B_r|^\lambda} d \left( \Phi \left( \frac{Mf(x)}{\| Mf \|_{W M^p, \lambda}(0)} \right), u \right)
\]
\[
= \frac{1}{2} \sup_{u > 0} \frac{\Phi(u)}{|B_r|^\lambda} d \left( \frac{Mf(x)}{\| Mf \|_{W M^p, \lambda}(0)}, u \right) \leq \frac{1}{2}.
\]

For \( I_4 \), using Lemma 3 we obtain
\[
I_4 = \Psi \left( \frac{|I_0(f(1 - \chi_{B_r}))(x)|}{2^n - \alpha + C_6 \| f \|_{M^p, \lambda}(0)} \right) \leq \frac{1}{2} |B_r|^{\mu - 1}
\]
and
\[
\sup_{u > 0} \frac{u}{|B_r|^\mu} d(I_4, u) \leq \sup_{u > 0} \frac{u}{|B_r|^\mu} d \left( \frac{1}{2} |B_r|^{\mu - 1}, u \right)
\]
\[
= \frac{1}{2} \sup_{u > 0} \frac{u}{|B_r|^\mu} d \left( 1, u \right) \leq \frac{1}{2}.
\]
Thus,
\[
\sup_{u > 0} \frac{\Psi(u)}{|B_r|^\mu} d \left( \frac{|I_0 f(x)|}{c_5 \| f \|_{M^p, \lambda}(0)}, u \right) \leq 1
\]
and \( \| I_0 f \|_{W M^p, \mu}(0) \leq c_5 \| f \|_{M^p, \lambda}(0) \). \( \square \)

**Example 1.** Let \( 0 < \alpha < n, 1 < p < \frac{n(1-\lambda)}{\alpha}, 0 \leq \lambda < 1, \) and
\[
\Phi(u) = u^p, \quad \Psi(u) = u^q \quad \text{with} \quad 1 < p < q < \infty.
\]

Then \( \Phi^*(u) = (p - 1) p^{-\nu} u^p \), where \( 1/p + 1/p' = 1 \) and \( \Phi^*(2u) = 2^q \Phi^*(u) \), that is, \( \Phi^* \in \Delta_2 \).

The estimate (15) holds since for all \( u > 0 \) we have
\[
\int_u^\infty t^{\alpha/n} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} = \int_u^\infty t^{\frac{\alpha}{n} + \frac{1}{p} - 1} \frac{dt}{t} = \frac{1}{1 - \frac{1}{p} - \frac{\alpha}{n}} u^{\frac{\alpha}{n} + \frac{1}{p} - 1}
\]
where the last integral is convergent since \( p < \frac{n(1-\lambda)}{\alpha} \). If \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \) and \( \frac{1}{p} = \frac{\alpha}{q} \), then \( \frac{\alpha}{n} + \frac{1}{p} - 1 = \frac{1}{p} - \frac{\alpha}{n} = \frac{q}{p} - \frac{\alpha}{q} = \frac{1}{q} \) and
\[
\int_u^\infty t^{\alpha/n} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} = \frac{q}{1 - \frac{1}{\mu}} u^{\frac{1}{\mu} - 1} = \frac{q}{1 - \frac{1}{\mu}} \Psi^{-1}(u^{\mu - 1}),
\]
that is, the estimate (15) holds. Also estimate (16) holds since for all \( u, r > 0 \)
\[
\int_u^r t^{\alpha/n} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} = r^{\frac{1}{p} - \frac{1}{\mu}} \int_u^{r^{\frac{1}{p} - \frac{1}{\mu}}} t^{\frac{\alpha}{n} - \frac{1}{p}} \frac{dt}{t} = \frac{r^{\frac{1}{p} - \frac{1}{\mu}}}{p - \frac{1}{n} - \frac{1}{p}} u^{\frac{1}{p} - \frac{1}{\mu}} = q r^{\frac{1}{q} - 1/u} = q \Psi^{-1}(r^{\mu - 1}).
\]

From the Theorem 3, we get the Spanne–Peetre type result proved in [12, Proposition 1.1], that is, the Riesz potential \( I_\alpha \) is bounded from \( M^p, \lambda(0) \) to \( M^p, \mu(0) \) under the conditions \( 1 < p < \frac{n(1-\lambda)}{\alpha}, 0 \leq \lambda < 1, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \) and \( \frac{\alpha}{p} = \frac{\alpha}{q} \).
Remark 5. It is easy to see that for $0 \leq \lambda < 1$ if $\Phi_1, \Phi_2$ are two Orlicz functions and there exists a constant $k > 0$ such that $\Phi_2(u) \leq \Phi_1(ku)$ for all $u > 0$, then $\|f\|_{\Phi_2,\lambda,A} \leq k \|f\|_{\Phi_1,\lambda,A}$ provided the right side is finite. Furthermore, $M^{\Phi_1,\lambda}(\mathbb{R}^n) \hookrightarrow M^{\Phi_2,\lambda}(\mathbb{R}^n)$ and $M^{\Phi_1,\lambda}(0) \hookrightarrow M^{\Phi_2,\lambda}(0)$. Hence it follows that if two Orlicz functions $\Phi_1, \Phi_2$ are equivalent, i.e. there exist positive constants $k_1, k_2$ such that $\Phi_1(k_1 u) \leq \Phi_2(u) \leq \Phi_1(k_2 u)$ for all $u > 0$, then $M^{\Phi_1,\lambda}(\mathbb{R}^n) = M^{\Phi_2,\lambda}(\mathbb{R}^n)$ and $M^{\Phi_1,\lambda}(0) = M^{\Phi_2,\lambda}(0)$ with equivalent norms.

Example 2. Let $0 < \alpha < n, 0 \leq \lambda < 1, 1 < p < \frac{n(1-\lambda)}{\alpha}, a > 0$ and

$$\Phi^{-1}(u) = \begin{cases} u^{\frac{1}{\lambda}} & \text{for } 0 \leq u \leq 1, \\ u^{\frac{1}{\lambda}} (1 + \ln u)^{-a} & \text{for } u \geq 1, \end{cases} \quad \Psi^{-1}(u) = u^{\frac{1}{\mu}} \text{ with } 1 < p < q < \infty.$$

If $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, \frac{\lambda}{p} = \frac{\mu}{q}$, then condition (15) is satisfied. Really, for $u \geq 1$ we have equality as in the Example 1. If $0 < u < 1$, then using the fact that function $(1 + \ln t^{\lambda-1})^{-\alpha}$ is strictly increasing of variable $t$ on $(0, 1]$ we get $(1 + \ln t^{\lambda-1})^{-\alpha} \leq 1$ for $0 < t \leq 1$ and so

$$\int_u^\infty t^{\alpha/n} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} = \int_u^1 t^{\frac{\alpha}{\lambda} + \frac{1}{p} - \frac{\alpha}{n}} (1 + \ln t^{\lambda-1})^{-\alpha} \frac{dt}{t} + \int_1^\infty t^{\frac{\alpha}{\lambda} + \frac{1}{p} - \frac{\alpha}{n}} \frac{dt}{t} \leq \int_u^\infty t^{\frac{\alpha}{\lambda} + \frac{1}{p} - \frac{\alpha}{n}} \frac{dt}{t} = \frac{u^{\frac{\alpha}{\lambda} + \frac{1}{p} - \frac{\alpha}{n}}}{1 - \frac{\alpha}{n}} = \frac{q}{1 - \mu} u^{\frac{\mu - 1}{\mu}} = q \Psi^{-1}(u^{\mu-1}),$$

that is, the estimate (15) holds. Next, we consider condition (16). If $u \geq r^\lambda$, then

$$\int_u^\infty t^{\frac{\alpha}{\lambda}} \Phi^{-1}(r^\lambda) \frac{dt}{t} = r^\lambda \int_u^\infty t^{\frac{\alpha}{\lambda} - \frac{1}{p}} \frac{dt}{t} = \frac{r^\lambda}{p - \frac{\alpha}{n}} u^{\frac{\alpha}{\lambda} - \frac{1}{p}} = q r^\lambda u^{-\frac{\alpha}{n}} = q \Psi^{-1}(r^\lambda u).$$

Let now $0 < u < r^\lambda$. Then, $(1 + \ln t^{\lambda-1})^{-\alpha} \leq 1$ as an increasing function of $t$ on $(0, r^\lambda]$ and since $u < t \leq r^\lambda$, we have

$$\int_u^\infty t^{\frac{\alpha}{\lambda}} \Phi^{-1}(r^\lambda) \frac{dt}{t} = r^\lambda \int_u^\infty t^{\frac{\alpha}{\lambda} - \frac{1}{p}} (1 + \ln r^\lambda) \frac{dt}{t} = r^\lambda \int_r^\infty t^{\frac{\alpha}{\lambda} - \frac{1}{p}} \frac{dt}{t} \leq r^\lambda \int_u^\infty t^{\frac{\alpha}{\lambda} - \frac{1}{p}} \frac{dt}{t} = \frac{r^\lambda}{p - \frac{\alpha}{n}} u^{\frac{\alpha}{\lambda} - \frac{1}{p}} = q r^\lambda u^{-\frac{\alpha}{n}} = q \Psi^{-1}(r^\lambda u),$$

that is, the estimate (16) holds. The function $\Phi^{-1}$ is increasing, unbounded, obviously concave on $(0, 1)$ and concave for large $u$. Therefore, there exists a concave function on $(0, \infty)$ which is equivalent to $\Phi^{-1}$ and so $\Phi$ is equivalent to an Orlicz function. Also we have equivalence

$$\Phi(u) \approx \begin{cases} u^p & \text{for } 0 \leq u \leq 1, \\ u^p (1 + \ln u)^{ap} & \text{for } u \geq 1. \end{cases}$$
Moreover, since

\[ s_{\Phi^{-1}}(t) = \begin{cases} \frac{t^{1/p}(1 - \ln t)^a}{t^{1/p}} & \text{for } 0 < t \leq 1, \\ t^{1/p} & \text{for } t \geq 1, \end{cases} \]

it follows that the Matuszewska–Orlicz index \( \beta_{\Phi^{-1}} = \frac{1}{p} \) and so \( 1 = \frac{1}{\beta_{\Phi}} + \frac{1}{\alpha_{\Phi}} = \frac{1}{\beta_{\Phi^*}} + \beta_{\Phi^{-1}} = 1 - \frac{1}{\beta_{\Phi^*}} + \frac{1}{p} \) or \( \beta_{\Phi^*} = \frac{p}{p-1} < \infty \), which means that \( \Phi^* \in \Delta_2 \) (for definitions and properties of indices – see [25, pp. 87–89]). Thus, by Remark 5 the space \( M^{\Phi,\lambda}(0) \) is a Banach space and by Theorem 3 the Riesz potential \( I_\alpha \) is bounded from \( M^{\Phi,\lambda}(0) \) to \( M^{\Psi,\mu}(0) = M^{q,\mu}(0) \).

**Example 3.** Let \( 0 < \alpha < n, 0 \leq \lambda < 1, 1 < p < \frac{n(1-\lambda)}{\alpha}, 0 \leq b \leq a \) and

\[ \Phi^{-1}(u) = u^{\frac{1}{p}} (1 + |\ln u|)^{-a} \quad \text{and} \quad \Psi^{-1}(u) = u^{\frac{1}{q}} (1 + |\ln u|)^{b} \quad \text{for} \quad u > 0. \]

If \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, \frac{\lambda}{p} = \frac{\mu}{q} \), then conditions (a), (b) of Theorem 2(i) and (15), (16) are satisfied. The calculations are similar to those in Example 2 so we will omit them here. Observe only that

\[ s_{\Phi^{-1}}(t) = t^{1/p}(1 + |\ln t|)^{a}, \quad s_{\Psi^{-1}}(t) = t^{1/q}(1 + |\ln t|)^{b}. \]

Then, the functions \( \Phi^{-1}, \Psi^{-1} \) are increasing, unbounded and concave near 0 and for large \( u \), and so the inverses \( \Phi, \Psi \) are equivalent to Orlicz functions. Thus, by Remark 5 the spaces \( M^{\Phi,\lambda}(0), M^{\Psi,\mu}(0) \) are Banach spaces and by Theorem 3 the Riesz potential \( I_\alpha \) is bounded from \( M^{\Phi,\lambda}(0) \) to \( M^{\Psi,\mu}(0) \).

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