2-D soil slope-stability evaluation using vector-sum method

Mingwei Guo1,a*, Xuechao Dong1,2,b

1State Key Laboratory of Geomechanics and Geotechnical Engineering, Institute of Rock and Soil Mechanics, Chinese Academy of Sciences, Wuhan, Hubei, China
2University of Chinese Academy of Sciences, Beijing, China
aemail: mwguo@whrsm.ac.cn, bemail: dongxuechao18@mails.ucas.ac.cn
*Corresponding author: aemail: mwguo@whrsm.ac.cn

Abstract: Over the stress state of the sliding body artificially assumed in Limit equilibrium method, the real stress state from any numerical analysis can be imported into a limit equilibrium analysis where the normal stress and shear stress are computed with respect to any selected slip surface. Considering the vector characteristics of force and the strength reduction definition, vector sum method by strength reduction technique was proposed, in which the factor of safety (FOS) can be directly calculated by the force and moment equilibrium. Finally, numerical experiments with six possible conditions are used to verify this proposed method, and the results show that the FOS obtained by the proposed method is in good agreement with the average FOS by limit equilibrium methods.

1. Introduction
Slope stability analysis provides assessment of the stability of a slope in terms of the FOS and corresponding critical slip surface. It's therefore critical to define the definition of the FOS so that it can be effectively used in the slope design. It is well known that limit equilibrium method (LEM) and strength reduction method (SRM) are most widely used in practical slope engineering[Duncan, 1996].

LEM only considers sliding body as rigid body, divides the sliding body into many slices, and even doesn’t utilize the stress versus strain characteristics of soils. In order to obtain the FOS and make a multiple solution problem unique solution, many assumptions have to be made to establish the force or moment equations based on limit state of these slices, such as the magnitude, the direction, and the position of forces acting on the interface of slices [Lam and Fredlund 1993; Steward et al.2011;]. For 3-D problem, more assumptions are required to obtain the FOS, and only the FOS by rigorous method satisfying all six equilibrium conditions of sliding body can be recognized as the accurate solution[Zheng, 2009].

With SRM, the finite element technique is usually used, in which the cohesive force and internal friction coefficient of a material are always reduced by the same factor. When this factor is to bring a slope to failure, it is defined as the factor of safety (FOS) of the slope. This method was used as early as in 1975 by Zeinkiewicz et al, and has made great progress through extensive studies [Griffiths & Lane 1999; Cheng, et al., 2007;]. Although this method can automatically get the FOS and locate corresponding critical slip surface, the inherent theoretical defects in the SRM can’t guarantee the derived FOS with its corresponding slip surface is the global optimal solution rather than a suboptimal solution, and the SRM tends to reach premature nonconvergence under some complex conditions.
For overloading definition, the FOS is defined as the ratio of the total resisting force to the total driving force along a certain slip surface based on the stress state of the slope usually acquired from the finite element analysis, which is called enhanced limit method [Kim J.Y. and Lee S. R, 1997; Pham H.T. and Fredlund D.G, 2003]. Here, only the magnitude of forces in slope stability analysis, while the direction of force isn’t taken into account. It has been pointed out that the critical slip surface obtained by this method might be for the most part shallower than those obtained by the strength reserving definition with LEM [Zheng, et al,2005]. Considering the vector characteristics of the force, vector sum method was put forward by Ge (2008), and it has been developing in recent years (Guo, et.al,2019a,2019b). However, the global sliding direction was still assumed based on the knowledge of sliding failure characteristics in practical landslide.

In this paper, combining the vector characteristics of force and strength reduction definition, the vector sum method based on strength reduction technique is proposed. In this approach, not only the global sliding direction was deduced by rigorous theory, but also the moment equilibrium was also considered along the center of moment at the critical balance state of the slope.

2. Vector sum method

Fig.1 is a sketch of simple slope. Due to vector characteristics of force, all resisting forces along potential slip surface can be composed to be a resisting force vector if the vector composition law is used and correspondingly, the total driving force vector can also be obtained, then, the force equilibrium equation can be established in the global sliding direction when the soil shear strength is divided to bring the slope to the limit equilibrium state. Similarly, the moment equilibrium equation can also be established at the moment center. Therefore, the global FOS of the slope should be the smaller one by the force or moment equilibrium equation. In the following, the derivation process is shown in detail for the FOS based on strength reduction definition.

\[
\begin{align*}
F_f &= \frac{\int (\tau_{max}\vec{a}) \cdot (-\vec{d}) d_i}{\int (\sigma_{\tau}\vec{d}) d_i} \\
&= \frac{\int (\tau_{max}\vec{a}) \cdot (-\vec{d}) d_i}{\int (\sigma_{\tau}\vec{d}) d_i}
\end{align*}
\]

Where, \(\sigma_{\tau}\) is the shear stress acting on the bed rock by the sliding body at any point on the slip surface; \(\vec{a}\) is the unit direction of resisting shear stress along the slip surface; \(\vec{d}\) is the global sliding direction; \(\tau_{max}\) is the maximum resisting shear force along the tangential direction upward on a slip surface.

From the Eq.(1), it can be seen that the resisting and driving shear stresses have effect on the F_S when the force equilibrium of a slope is only considered. It can be easily gained for the shear stress along the slip surface in the Eq.(1) by finite element analysis. Moreover, the safety factor (Eq.1), can be directly obtained by integrals along the slip surface rather than iterative calculation.
The moment the potential energy of a slope changes into a relative minimum, it will have a stationary value, which offers the mathematical method to obtain the global sliding direction of the slope. For the global sliding direction, it can be rigorously determined by the principle of minimum potential energy [Guo, et.al, 2019a]. Then, for a simple slope (Fig. 1), the global sliding direction can be expressed using the slip angle $\theta$, which can be simplified by the Eq. (2).

$$\tan \theta = \frac{\int n_i n_j \sigma_{Mx} + (n_i^2 - 1)\sigma_{Mx} dl}{\int n_i n_j \sigma_{My} + (n_i^2 - 1)\sigma_{My} dl}$$

(2)

where, $n$ is the normal direction at the point M on the slip surface of the slope, $\sigma_{Mx}$ and $\sigma_{My}$ are the components of $\sigma_M$ on the axis X and Y.

According to the definition of moment, the magnitude of the moment of the acting force at certain point is directly proportional to the distance from the point to the force. It is defined as the cross product of the distance vector ($r$) and the acting force vector ($F$), and based on the right-hand rule, the moment can be calculated by:

$$M = r \times F$$

(3)

Then, the safety factor by the moment equilibrium can be deduced as the Eq.(4)[Guo, et.al, 2019b].

$$F_m = \frac{\int r \times (-\tau_{max} \hat{a}) dl}{\int (r \times \sigma) d l}$$

(4)

If the moment center is already determined, the safety factor by moment equilibrium can be easily calculated. Based on previous studies[Guo, et.al, 2019b], the moment center is suggested as the following: key points along the potential slip surface can be used to define a circle, which is straightforward, and the circle center can be considered as the center of the moment, which is proved to be an efficient technique to determine the moment center.

3. Examples

In order to verify the proposed approach, an example consisting of six different conditions and involving circular and composite slip surfaces is studied, which was also studied by Fredlund & Krahn(1977) using LEMs.

In this paper, the finite element technique is used to get the stress field of a slope composed of elasto-plastic materials. The ideal elasto-plastic constitutive model, Mohr-Coulomb yield criterion, and non-associated flow rule are used in the elasto-plastic finite element analysis, in addition, the dilation angle is assumed to be zero during elasto-plastic finite element computing. As we know, the Gaussian integration points are used to integrate the stiffness matrix in the finite element analysis, and the discontinuous stresses with low accuracy at the boundary of elements are computed by the direct calculation of stress integral. This study adopts the global stress smoothing technique to overcome the low accuracy deficiency. The stress at any position within an element can be calculated by:

$$\sigma = \sum_{i=1}^{n} N_i \sigma_i$$

(5)

$$N_i = 1/4 * (1 + \varepsilon_i \eta_i)(1 + \eta_i \varepsilon_i)$$

(6)

where, $n$ is the number of nodes of the element, $N_i$ is the shape function about a nodal point $i$, and $\sigma_i$ is the corresponding nodal stress, $(\varepsilon_i \eta_i)$ is the local coordinate at any position in isoparametric element, and $(\varepsilon_i \eta_i)$ is the local coordinate of the node $i$ in four-node isoparametric element.

3.1. Calculating model and conditions

Fig. 2 shows the geometry of the problem involving two cases. One is a circular failure surface, and the other is a composite failure surface. For both cases, there are three conditions that consider simple slope condition, a water condition with piezometric line (blue color in Fig.2) and a water condition with pore
water pressure coefficient \( r_u \), respectively. Table.1 shows the parameters of soil materials in this example. For the first case, only the soil material is used and it is a homogeneous slope with a circular failure surface. However, for the second case, the weak layer is added based on the first case, and the failure surface is composed by a circular surface and a weak layer.

According to the geometry of this problem (Fig.2), the finite element calculating model can be established, as shown in Fig.3. The number of element is 3111. For the boundary conditions, the bottom is fixed, and the lateral boundary is normally restricted. The ideal elasto-plastic constitutive model, the Mohr-Coulomb yield criterion, and the non-associated flow rule are used in the elasto-plastic finite element analysis.

Figure 2 Example problem

**Table 1** Parameters of materials

| Material     | \( c \)/kPa | \( \varphi \)/(°) | \( \varphi^0 \)/° | \( \gamma^0 \)/kN·m\(^{-3}\) | \( E \)/kPa | \( \mu \) |
|--------------|-------------|------------------|------------------|-----------------|------------|------|
| Soil         | 29.3        | 20               | 0                | 19.2            | 8.0E4      | 0.3  |
| Weak layer   | 0.0         | 10               | 0                | 20.0            | 8.0E4      | 0.3  |

3.2 Calculating results and comparison analysis

Table.2 shows the results calculated using this proposed method. From the Table.2, it can be seen that for any condition, the FOS by force equilibrium control the global stability, i.e. the failure in this example is mainly due to the sliding failure along the potential slip surface rather than the rotating damage controlled by moment equilibrium. For the circular slip surface, the FOS changes from 2.137 with a simple slope condition to 1.805 under a water condition with \( r_u = 0.25 \) and correspondingly, the global sliding direction changes from -17.04° to -16.07°. Similarly, for the composite surface, the FOS changes from 1.428 with only weak layer to 1.160 under a water condition with \( r_u = 0.25 \) and correspondingly, the global sliding direction changes from -31.41° to -31.59°.

**Table 2** Calculating results by VSM

| Example problem     | Simple slope | Same as column 2 with a thin weak layer | Same as Column 2 with \( r_u = 0.25 \) | Same as column 3 with \( r_u = 0.25 \) for both materials | Same as column 2 with a piezometric line | Same as column 3 with a piezometric line |
|---------------------|--------------|-----------------------------------------|----------------------------------------|--------------------------------------------------|----------------------------------------|----------------------------------------|
| Global sliding direction(°) | -17.04       | -31.41                                  | -16.07                                 | -31.59                                           | -18.98                                | -33.37                                |
| FOS by force equilibrium | 2.137        | 1.428                                    | 1.805                                  | 1.160                                            | 1.828                                 | 1.323                                 |
| FOS by moment equilibrium | 2.297        | 1.482                                    | 1.944                                  | 1.211                                            | 1.981                                 | 1.363                                 |

The six possible conditions of geometry, soil properties, water conditions, and the comparison of the FOS are presented in Table.3. From Table.3, it can be seen that different FOS were obtained using
different LEMs. Moreover, the difference among the LEMs was also analyzed in detail [Fredlund & Krahn(1977)]. For these FOS using LEMs, the average FOS under each condition can be considered as the reference answer.

Table 3  comparison of factors of safety for example problem

| Example problem                      | Simple slope | Same as column 2 with a thin weak layer | Same as column 2 with $r_u = 0.25$ | Same as column 3 with $r_u = 0.25$ for both materials | Same as column 2 with a piezometric line | Same as column 3 with a piezometric line |
|-------------------------------------|--------------|-----------------------------------------|-----------------------------------|------------------------------------------------------|----------------------------------------|----------------------------------------|
| Ordinary method                     | 1.928        | 1.288                                   | 1.607                             | 1.029                                                | 1.693                                  | 1.171                                  |
| Simplified Bishop method            | 2.080        | 1.377                                   | 1.766                             | 1.124                                                | 1.834                                  | 1.248                                  |
| Spencer's method                    | 2.073        | 1.373                                   | 1.761                             | 1.118                                                | 1.830                                  | 1.245                                  |
| Janbu's simplified method           | 2.041        | 1.448                                   | 1.735                             | 1.191                                                | 1.827                                  | 1.333                                  |
| Janbu's rigorous method             | 2.008        | 1.432                                   | 1.708                             | 1.162                                                | 1.776                                  | 1.298                                  |
| Morgenstern-Price method            | 2.076        | 1.378                                   | 1.765                             | 1.124                                                | 1.833                                  | 1.250                                  |
| Average of 2D methods               | 2.034        | 1.383                                   | 1.724                             | 1.124                                                | 1.799                                  | 1.258                                  |
| Proposed method (VSM)               | 2.137        | 1.428                                   | 1.805                             | 1.160                                                | 1.828                                  | 1.323                                  |
| Differences between 2D LEMs and VSM | 5.06%        | 3.25%                                   | 4.70%                             | 3.20%                                                | 1.61%                                  | 5.17%                                  |

In this example, the FOS obtained by proposed method for each condition is slightly larger than those obtained by LEMs, either with or without a weak layer inside the slope. The most difference is only 5.17% between the average FOS of LEMs and VSM (Table.3). Fig.4 and 7 show the comparison of FOS using LEMs and VSM, and the horizontal axis means calculating condition for any case, i.e. the "simple" is for gravity condition only, the "piezometric line" means a water condition with the piezometric line(blue color in Fig.2) and $r_u=0.25$ means a water condition with $r_u=0.25$.

From the Table.3 and Fig.4 and 7, the results demonstrate that for the circular failure surface, the FOS by VSM is slightly larger than that by any of LEMs, but the most significant difference is 5.06% for any condition in case 1. Similarly, for the composite failure surface, except the FOS by Janbu’s simplified method, the FOS obtained by proposed method are also slightly larger than those by any LEM, and the most significant difference is only 5.17% for any condition in case 2. Therefore, it can be concluded that the proposed method in this paper can effectively assess the global stability of this slope and gives consistent FOS by LEMs for any condition.
4. Conclusion

(1) Considering vector characteristics of force and strength reduction definition of FOS, vector sum method was proposed in this paper, and the global FOS can be directly determined by both force equilibrium of sliding body on the sliding direction and moment equilibrium at the center of the moment. In addition, for noncircular slip surface, this approach still has clearly physical meaning and can compute the FOS explicitly based on the stress state of slope by finite element method.

(2) More importantly, whether the critical slip surface is a circle or other complex slip lines, the global sliding direction was theoretically determined for any selected slip surface, which reveals how to place anti-slide pile in order to measure the landslide.

(3) The proposed method is applied in an example involving six possible conditions. And the calculating results show that the FOSs of the proposed approach are in good agreement with the average FOS obtained by popular LEMs, which demonstrates the accuracy of the proposed method.

Acknowledgements

The authors gratefully acknowledge the financial support of the China Scholarship Council and National Science Foundation of China under Grants No.51674239.

Reference

[1] Cheng, Y.M., Lansivaara, T., Wei, W. (2007) Two-dimensional slope stability analysis by limit equilibrium and strength reduction methods. Computers and geotechnics, 34:137-150.

[2] Duncan, J.M. (1996) State of the art: limit equilibrium and finite-element analysis of slopes. Journal of Geotechnical engineering, 122:577-596.

[3] Griffiths, D., Lane, P. (1999) Slope stability analysis by finite elements. Geotechnique, 49:387-403.

[4] Fredlund, D.G., Krahn, J. (1977) Comparison of slope stability methods of analysis. Canadian geotechnical journal, 14:429-439.

[5] Ge X.R. (2008) Deformation control law of rock fatigue failure, real-time X-ray CT scan of geotechnical testing, and new method of stability analysis of slopes and dam foundations. Chinese Journal of Geotechnical Engineering, 30:1–20.

[6] Guo, M., Li, C., Wang, S., Yin, S., Liu, S., Ge, X. (2019) Vector-sum method for 2D slope stability analysis considering vector characteristics of force. International Journal of Geomechanics, 19:04019058.1-04019058.11.

[7] Guo, M., Liu, S., Yin, S., Wang, S. (2019) Stability analysis of the Zhangmu multi-layer landslide using the vector sum method in Tibet, China. Bulletin of Engineering Geology and the Environment, 78:4187-4200.

[8] Kim, J.Y., Lee, S.R. (1997) An improved search strategy for the critical slip surface using finite element stress fields. Computers and Geotechnics, 21:295-313.

[9] Lam, L., Fredlund, D. (1993) A general limit equilibrium model for three-dimensional slope stability analysis. Canadian geotechnical journal, 30:905-919.
[10] Guo, M., Wang, S., Ge, X., Li, C., Zheng, H. (2013) A new practical method for two-dimensional slope stability analysis. Disaster Advances, 6:258-269.

[11] Pham, H.T., Fredlund, D.G. (2003) The application of dynamic programming to slope stability analysis. Canadian Geotechnical Journal, 40:830-847.

[12] Steward, T., Sivakugan, N., Shukla, S., Das, B. (2011) Taylor’s slope stability charts revisited. International Journal of Geomechanics, 11:348-352.

[13] Zeinkiewicz, O.C., Humpheson, C., Lewis, R.W. (1975) Associated and non-associated visco-plasticity in soils mechanics. Journal of Geotechnique, 25:671-689.

[14] Zheng, H., Liu, D., Li, C. (2005) Slope stability analysis based on elasto-plastic finite element method. International Journal for Numerical Methods in Engineering, 64:1871-1888.

[15] Zheng, H., Tham, L.G. (2009) Improved Bell's method for the stability analysis of slopes. International Journal for Numerical and Analytical Methods in Geomechanics, 33:1673-1689.