Faster deterministic parameterized algorithm for
$k$-Path

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Abstract

In the $k$-Path problem, the input is a directed graph $G$ and an integer $k \geq 1$, and the goal is to decide whether there is a simple directed path in $G$ with exactly $k$ vertices. We give a deterministic algorithm for $k$-Path with time complexity $O^*(2.554^k)$. This improves the previously best deterministic algorithm for this problem of Zehavi [ESA 2015] whose time complexity is $O^*(2.597^k)$.

1 Introduction

In the $k$-Path problem, the input is a directed graph $G$ and an integer $k \geq 1$, and the goal is to decide whether there is a simple directed path in $G$ with exactly $k$ vertices. Several papers gave parameterized algorithm for this problem, both deterministic [1,3,4,8,10,12,13,15] and randomized [1,2,4,9–11,14]. See Table 1 for a summary of deterministic parameterized algorithms for $k$-Path. The fastest deterministic parameterized algorithm for $k$-Path was given by Zehavi [15] and its time complexity is $O^*(2.597^k)$. In this paper, we give a deterministic algorithm for $k$-Path with time complexity $O^*(2.554^k)$.

Our algorithm (as other algorithms for the $k$-Path problem) also solves a generalization of $k$-Path called Directed $k$-$(s,t)$-Path. In this problem, the input is a directed graph $G$, two vertices $s,t$, and an integer $k \geq 1$, and the goal is to decide whether there is a simple directed path from $s$ to $t$ in $G$ with exactly $k$ vertices. Our algorithm for Directed $k$-$(s,t)$-Path gives faster algorithms for the Long Directed $(s,t)$-Path and Long Directed Cycle problems. In the Long Directed $(s,t)$-Path problem, the input is a directed graph $G$, two vertices $s,t$, and an integer $k \geq 1$, and the goal is to decide whether there is a simple directed path from $s$ to $t$ in $G$ with at least $k$ vertices. In the Long Directed Cycle problem, the input is a directed graph $G$ and an integer $k \geq 1,
Table 1: Time complexities of deterministic algorithms for the $k$-Path problem.

| Reference                  | Running time               |
|----------------------------|----------------------------|
| Monien [12]                | $O^*(k!)$                  |
| Alon et al. [1]            | $O^*(c^k)$                 |
| Kneis et al. [10]          | $O^*(16^k)$                |
| Chen et al. [4]            | $O^*(4^k+o(k))$            |
| Fomin et al. [8]           | $O^*(2.851^k)$             |
| Fomin et al. [6], Shachnai and Zehavi [13] | $O^*(2.619^k)$ |
| Zehavi [15]                | $O^*(2.597^k)$             |
| This paper                 | $O^*(2.554^k)$             |

and the goal is to decide whether there is a directed cycle with at least $k$ vertices. Fomin et al. [7] gave algorithms to these two problems that use an algorithm for Directed $k$-(s,t)-Path as a black box. Using our algorithm instead the algorithm of Zehavi [15] improves the time complexity of the algorithms of Fomin et al. from $O^*(4.884^k)$ to $O^*(4.839^k)$.

Our algorithm can also be extended to solve a generalizations of $k$-Path called Weighted $k$-Path. In this problem, the input is a directed graph $G = (V, E)$, a weight function $w: V \to \mathbb{R}$, integer $k \geq 1$, and $W \in \mathbb{R}$. The goal is to decide whether there is a simple directed path in $G$ with $k$ vertices and weight at most $W$. To simplify the presentation, we will describe an algorithm for the unweighted problem.

Our algorithm is based on the algorithm of Zehavi [15], with a simple modification: replacing the universal family with a mixed universal family (see Definition 7). This causes several straightforward additional changes to the algorithm of [15] and its analysis. For completeness, we describe these changes in full.

2 Preliminaries

In this section we describe two tools, representative families and universal families, that will be used in our algorithm.

Representative families is a tool that is very useful in the design of parameterized algorithms (cf. [4]). In particular, it was used for giving efficient algorithms for $k$-Path in [6,8,13].

**Definition 1.** Let $U$ be a set and $S$ be a family of subsets of size $p$ of $U$. We say that $\hat{S} \subseteq S$ $q$-represents $S$ if for every set $B \subseteq U$ of size at most $q$, if there is a set $A \in S$ disjoint from $B$ then there is a set $\hat{A} \in \hat{S}$ disjoint from $B$.

**Theorem 2** (Fomin et al. [6], Shachnai and Zehavi [13]). There is an algorithm that given $c \geq 1$, integers $p$ and $k \geq p$, and a family $S$ of subsets of
size \( p \) of \( U \), constructs a family \( \hat{S} \subseteq S \) that \((k - p)\)-represents \( S \) with size \( \frac{(ck)^k}{p^p(ck - p)^k - p}2^{o(k)} \log |U| \). The construction time is \( O(|S|(ck/(ck - p))^k \cdot 2^{o(k)}) \log |U| + |S| \log |S|) \).

Suppose that \(|U| = n\) and the size of \( S \) satisfies the bound on the size of \( \hat{S} \) of the lemma, namely \(|S| = O^*(\frac{(ck)^k}{p^p(ck - p)^k - p}2^{o(k)})\). Then, the construction time of \( \hat{S} \)
is \( O^*(\frac{(ck)^k}{p^p(ck - p)^k - p}2^{o(k)}) = O^*(\phi_c(p/k)^k \cdot 2^{o(k)}) \), where \( \phi_c(\alpha) = \frac{c^{\alpha} - \alpha}{\alpha^{(c-\alpha)}2^\alpha} \). For simplicity, will assume that \( \phi_c(0) = 1 \) and \( \phi_1(1) = 1 \).

In order to obtain an improved algorithm for \( k \)-Path, Zehavi [15] used the following generalization of representative families.

**Definition 3.** Let \( U_1, \ldots, U_t \) be disjoint sets, \( p_1, \ldots, p_t, q_1, \ldots, q_t \) be non-negative integers, and \( S \) be a family of subsets of \( U = \bigcup_{i \leq t} U_i \) such that for every \( A \in S \), \(|A \cap U_i| = p_i \) for all \( i \leq t \). We say that \( \hat{S} \subseteq S \) \((q_1, \ldots, q_t)\)-represents \( S \), if for every set \( B \subseteq U \) for which \(|B \cap U_i| \leq q_i \) for all \( i \leq t \), if there is a set \( A \in S \) disjoint from \( B \) then there is a set \( \hat{A} \subseteq \hat{S} \) disjoint from \( B \).

**Theorem 4 (Zehavi [15]).** There is an algorithm that given \( c_1, \ldots, c_t \geq 1 \), integers \( p_1, \ldots, p_t, k_1, \ldots, k_t \), and a family \( S \) of subsets of \( U = \bigcup_{i \leq t} U_i \) such that for every \( A \in S \), \(|A \cap U_i| = p_i \) for all \( i \leq t \), constructs a family \( \hat{S} \subseteq S \) that \((k_1 - p_1, \ldots, k_t - p_t)\)-represents \( S \) with size \( \prod_{i \leq t}(\frac{(c_i k_i)^{k_i}}{p_i p_i (c_i k_i - p_i)^{k_i - p_i}2^{o(k_i)})} \cdot 2^{o(k_i)}) \cdot 2^{o(k_i)}) \). The construction time is \( O(|S|) \prod_{i \leq t}(\frac{(c_i k_i)^{k_i}}{p_i p_i (c_i k_i - p_i)^{k_i - p_i}2^{o(k_i)})} \cdot 2^{o(k_i)}) \cdot 2^{o(k_i)}) \). Then, the construction time of \( \hat{S} \) is \( O^*(\prod_{i \leq t}(\frac{(c_i k_i)^{k_i}}{p_i p_i (c_i k_i - p_i)^{k_i - p_i}2^{o(k_i)})} \cdot 2^{o(k_i)}) \cdot 2^{o(k_i)}) \) = \( O^*(\prod_{i \leq t}(\phi_c(p_i/k_i)^{k_i} \cdot 2^{o(k_i)}) \cdot 2^{o(k_i)}) \).

The algorithm of Zehavi [15] also uses universal families.

**Definition 5.** Let \( F \) be a family of subsets of a set \( U \), where \(|U| = n\). We say that \( F \) is an \((n, p, q)\)-universal family if for every disjoint sets \( A, B \subseteq U \) of sizes \( p \) and \( q \), respectively, there is a set \( F \in F \) such that \( A \subseteq F \) and \( B \cap F = \emptyset \).

**Lemma 6 (Fomin et al. [8]).** There is an algorithm that given integers \( n, p, q \), constructs an \((n, p, q)\)-universal family of size \( \prod_{i \leq t}(\frac{(c_i k_i)^{k_i}}{p_i p_i (c_i k_i - p_i)^{k_i - p_i}2^{o(k_i)})} \cdot 2^{o(k_i)}) \cdot 2^{o(k_i)}) \). The time is \( n \log n \).

In order to obtain our improved algorithm, we define a variant of universal families which we call mixed universal families.

**Definition 7.** Let \( F \) be a family of subsets of a set \( U \), where \(|U| = n\). We say that \( F \) is an \((n, p, q, \zeta)\)-mixed universal family if for every disjoint sets \( A, B \subseteq U \) of sizes \( p \) and \( q \), respectively, there is a set \( F \in F \) such that \(|A \setminus F| = \lfloor \zeta p \rfloor \) and \( B \cap F = \emptyset \).
We will prove the following lemma in Section 5.

**Lemma 8.** There is an algorithm that given $0 < \zeta < 1$, $0 < x < 1$, and integers $n, p, q$, constructs an $(n, p, q, \zeta)$-mixed universal family of size $O\left(\frac{1}{\eta^{p+q}} \cdot \log n\right)$ in $O\left(\frac{1}{\eta^{p+q}} \cdot 2^{2(p+q)} \cdot n \log n\right)$ time, where $\eta = \frac{1}{\zeta^{(1-\zeta)^{p+q}}}$.

Note that the size and the construction time are minimized when $x = \frac{(1-\zeta)p}{p+q}$.

### 3 Overview

Our algorithm is based on the algorithm of Zehavi [15]. In this section we give a high level description of the algorithm of [15] and of our algorithm.

The $k$-Path problem can be solved in $n^{k+O(1)}$ time by the following dynamic programming algorithm. Define $\mathcal{P}_{u,v}^i$ to be a family containing all sets $X \subseteq V$ such that $|X| = i$, $u, v \in X$, and there is a simple path from $u$ to $v$ whose set of vertices is precisely $X$. The families $\mathcal{P}_{u,v}^i$ are computed using the formula

$$\mathcal{P}_{u,v}^i = \bigcup_{w : (w,v) \in E} \bigcup_{X \in \mathcal{P}_{u,w}^{i-1}, v \notin X} (X \cup \{v\}).$$

To speed up this algorithm, instead of computing the families $\mathcal{P}_{u,v}^i$, compute families $\hat{\mathcal{P}}_{u,v}^i \subseteq \mathcal{P}_{u,v}^i$ that $(k-i)$-represents $\mathcal{P}_{u,v}^i$. The computation of $\hat{\mathcal{P}}_{u,v}^i$ is done as follows. First, compute

$$\mathcal{N}_{u,v}^i = \bigcup_{w : (w,v) \in E} \bigcup_{X \in \hat{\mathcal{P}}_{u,w}^{i-1}, v \notin X} (X \cup \{v\}).$$

Then, use Theorem 2 to compute a family $\hat{\mathcal{P}}_{u,v}^i$ that $(k-i)$-represents $\mathcal{N}_{u,v}^i$ (note that here $U = V$). The time complexity of the algorithm of Theorem 2 is roughly

$$O^*(|\mathcal{N}_{u,v}^i|(ck/(ck-i))^{k-i}) = O^*\left(\frac{(ck)^{2k-i}}{i^i(ck-i)^{2k-2i}}\right) = O^*(\phi_c(\alpha)^k),$$

where $\alpha = i/k$ (recall that $\phi_c(\alpha) = \frac{\alpha^{2-\alpha}(c-\alpha)^{2\alpha}}{\alpha^{2-\alpha}(c-\alpha)^{2\alpha}}$). The total time of the algorithm is thus $O^*(\max_{0 \leq \alpha \leq 1} \phi_c(\alpha)^k)$. The optimal choice for $c$ is $c = 1 + \frac{1}{\sqrt{5}} \approx 1.45$. For this choice of $c$, the function $f_c(\alpha)$ is maximized when $\alpha = 1 - \frac{1}{\sqrt{5}} \approx 0.55$, and $f_c(1 - \frac{1}{\sqrt{5}}) = 3/2 + \sqrt{5}/2 \approx 2.619$. Therefore, the running time of the algorithm is $O^*(2.619^k)$.

In order to reduce the time complexity, Zehavi [15] used the following approach. Suppose that $G$ contains a path of size $k$, and let $P = p_1, \ldots, p_k$ be such path. Suppose that we guessed a coloring of vertices of $G$ such that $\gamma/k$ vertices of
\( P \) are colored blue, and the rest \((1 - \gamma)k\) vertices of \( P \) are colored red, where \( \gamma \) is some constant. Assume that the blue vertices are distributed uniformly along \( P \), namely the number of blue vertices among \( p_1, \ldots, p_i \) is approximately \( \gamma i \) for all \( i \).

We will call this property the \textit{uniformity property}. We call a coloring that satisfies the two requirements above \textit{good}. Additionally, suppose we guessed a partition of the blue vertices into two sets, denoted \( L \) and \( R \), such that when traversing \( P \) from \( p_1 \) to \( p_k \), the first \( \frac{1}{2} \gamma k \) blue vertices of \( P \) are from \( L \), and the remaining \( \frac{1}{2} \gamma k \) blue vertices are from \( R \). Such a partition will be called good. The set \( U \) is partitioned into three disjoint subsets: \( U_1 = L, U_2 = R, \) and \( U_3 = V \setminus (L \cup R) \).

Now define families

\[
P_{u,v}^{j,j_2,s} = \{ X \in P_{u,v}^{j+j_2+s} : |X \cap L| = j, |X \cap R| = j_2, |X \setminus (L \cup R)| = s \}.
\]

Similarly to before, the algorithm builds families \( N_{u,v}^{j,j_2,s} \) and for each family \( N_{u,v}^{j,j_2,s} \) it uses Theorem 3 to generate a family \( \hat{P}_{u,v}^{j,j_2,s} \) that \((\frac{1}{2} \gamma k - j, \frac{1}{2} \gamma k - j_2, (1 - \gamma)k - s)\)-represents \( N_{u,v}^{j,j_2,s} \). The triplets \((j, j_2, s)\) for which we construct families \( \hat{P}_{u,v}^{j,j_2,s} \) are all triplets that satisfy the following properties.

1. \( 0 \leq j, j_2 \leq \frac{1}{2} \gamma k \) and \( 0 \leq s \leq (1 - \gamma)k - j - j_2 \).
2. If \( j < \frac{1}{2} \gamma k \) then \( j_2 = 0 \) (this property follows from the assumption that the partition \( L, R \) is good).
3. \( \frac{j + j_2}{s} \approx \frac{\gamma}{1 - \gamma} \) (this property follows from the uniformity property).

The time complexity of constructing a single family \( \hat{P}_{u,v}^{j,j_2,s} \) is roughly

\[
O^* \left( \frac{(c_1 \frac{1}{2} \gamma k)^{2 \frac{1}{2} \gamma k - j}}{j^2(c_1 \frac{1}{2} \gamma k - j)^{2 \frac{1}{2} \gamma k - 2j}} \cdot \frac{(c_2 \frac{1}{2} \gamma k)^{2 \frac{1}{2} \gamma k - j_2}}{j_2^2(c_2 \frac{1}{2} \gamma k - j_2)^{2 \frac{1}{2} \gamma k - 2j_2}} \cdot \frac{(c_3(1 - \gamma)k)^{2(1 - \gamma)k - s}}{s^2(c_3(1 - \gamma)k - s)^{2(1 - \gamma)k - 2s}} \right) = O^* \left( \frac{\phi_{c_1}(\alpha)^{1/2 \gamma k}}{\phi_{c_2}(\alpha_2)^{1/2 \gamma k}} \cdot \frac{\phi_{c_2}(\alpha_2)^{1/2 \gamma k}}{\phi_{c_3}(\beta)^{(1 - \gamma)k}} \right),
\]

where \( \alpha = j/(\frac{1}{2} \gamma k) \), \( \beta = j_2/(\frac{1}{2} \gamma k) \), and \( \beta = s/(1 - \gamma)k \). If \( c_1 = c_2 = c_3 = 1 + 1/\sqrt{\gamma} \) then each of \( \phi_{c_1}(\alpha), \phi_{c_2}(\alpha_2), \) and \( \phi_{c_3}(\beta) \) is maximized when the parameters \( \alpha, \alpha_2, \) and \( \beta \) are equal to \((1 - \frac{1}{\sqrt{\gamma}}) \approx 0.55\), respectively. However, this cannot occur simultaneously. Due to property 2 above, \( \alpha_2 = 0 \) when \( \alpha < 1 \), and \( \alpha = 1 \) when \( \alpha_2 > 0 \). Moreover, consider the case when \( \alpha = 0.55 \). Due to property 3 above, \( \beta = \frac{s}{(1 - \gamma)k} \approx \frac{j_2}{(1 - \gamma)k} \cdot \frac{1}{\gamma}(j + j_2) = \frac{1}{\gamma^2}(\frac{1}{2} \gamma k)\alpha = \alpha/2 = 0.275 \).

So far we assumed we guessed a good coloring of the vertices and a good partition of the blue vertices into sets \( L \) and \( R \). Since we want a deterministic algorithm, we need to deterministically generate several colorings such that at least one coloring is good, and for each coloring we need to generate several partitions such that at least one partition is good. Since we don’t know which are the
good coloring and partition, the algorithm performs the dynamic programming stage for every coloring and every partition. Therefore, the time complexity is multiplied by the number of colorings and partitions.

Given a good coloring, the generation of a good partition is done using an \((n, \frac{1}{2}\gamma k, \frac{1}{2}\gamma k)\)-universal family \(F\). For any \(F \in F\), define sets \(L, R\) by taking \(L\) to be the set of blue vertices that are contained in \(F\) and \(R\) be the set of blue vertices that are not contained in \(F\). By the definition of universal family, there is at least one good partition. By Lemma 6, the size of \(F\) is approximately \((\frac{\gamma k}{2}) \approx 2^{\gamma k}\), which means that the usage of a universal family increases the time complexity of the algorithm by a factor of \(2^{\gamma k}\). Note that in order to maximize the benefit of using generalized representative families, the value of \(\gamma\) should be large. However, the factor of \(2^{\gamma k}\) due to the universal set restricts the optimal choice of \(\gamma\) to be small (the algorithm of [15] uses \(\gamma = 0.084\)).

Consider now the generation of a good coloring. Unfortunately, the uniformity property prevents generation of a small set of colorings with at least one good coloring. Therefore, we will drop the requirement of uniformity and only require from a good coloring that the number of vertices of \(P\) that are colored blue is \(\gamma k\). It is easy to generate \(n\) colorings of \(V\) such that at least one coloring satisfies the property above. However, since we don’t have the uniformity property, the analysis above does not hold, and it is possible to have both \(\alpha_2\) and \(\beta\) equal to 0.55 at the same time. The solution to this problem is as follows: The path \(P\) is partitioned into \(1/\epsilon\) sub-paths of size \(\epsilon(k - 1) + 1\) each (we assume for simplicity that \(\epsilon(k - 1)\) is integer) such that the end vertex of the \(i\)-th path is the start vertex of the \((i + 1)\)-th path. We assume that the endpoints of these paths are colored red. Now partition the sub-paths into two sets \(P_L, P_R\) such that each set contains \(1/(2\epsilon)\) paths, and the number of blue vertices that are in the paths of \(P_L\) is \(\frac{1}{2}\gamma k\) (in general, such a partition may not be possible, but we assume for simplicity that such a partition exists). Suppose that we guessed a good partition of the blue vertices into sets \(L\) and \(R\), where a good partition is a partition in which the blue vertices of \(P_L\) are in \(L\) and the blue vertices of \(P_R\) are in \(R\). Now, let \(P_1, P_2, \ldots, P_{1/(2\epsilon)}\) be an ordering of the paths in \(P_L\) such that \(|P_{i-1} \cap L| \geq |P_i \cap L|\) for all \(i\). Similarly, let \(P_{1/(2\epsilon)+1}, \ldots, P_{1/\epsilon}\) be an ordering of the paths in \(P_R\) such that \(|P_{i-1} \cap R| \leq |P_i \cap R|\) for all \(i\). Let \(s_i, t_i\) be the first and last vertex of \(P_i\), respectively. Suppose that we guessed the vertices \(s_1, \ldots, s_{1/\epsilon}, t_1, \ldots, t_{1/\epsilon}\). The algorithm works in iterations, where in the \(i\)-th iteration the algorithm constructs the path \(P_i\) (or a path that represents \(P_i\)). The order property defined above serves as a replacement for the uniformity property. Namely, the worst case for the time complexity is when each path \(P_i\) contains \(\epsilon \gamma k\) blue vertices. Therefore, the analysis done under the uniformity property also applies here.

We say that the algorithm works in two stages where the first stage is finding the paths of \(P_L\) and the second stage is finding the paths of \(P_R\). Since the max-
imum of \( \phi_{c_3}(\beta) \) occurs when \( \beta \approx 0.55 \), the time complexity of the first stage is less than the time complexity of the second stage. In order to balance these time complexities, a non-balanced partition of the sub-paths into \( \mathcal{P}_l, \mathcal{P}_r \) is used. More precisely, let \( \delta \) be a (small) parameter. The sub-paths are partitioned into two sets \( \mathcal{P}_l, \mathcal{P}_r \) such that \( \mathcal{P}_l \) contains \( \left( \frac{1}{2} + \delta \right) \frac{1}{\epsilon} \) paths, and the number of blue vertices that are in the paths of \( \mathcal{P}_l \) is \( \left( \frac{1}{2} + \delta \right) \gamma k \).

Our algorithm In order to reduce the time complexity, instead of using a universal family to generate the partition \( L, R \), we use a mixed universal family. The size of the latter family is significantly smaller than the size of the former (see Lemma 6 and Lemma 8). Therefore, our algorithm uses \( \gamma = 1 \) (namely, all the vertices are blue) and thus it gains larger benefit from the usage of generalized representative families. Note that the usage of a mixed universal family means that the paths of \( \mathcal{P}_l \) can have vertices from \( L \) and \( R \). Therefore, \( \alpha_2 \) can be non-zero during the first stage of the algorithm.

4 Our algorithm

In this section we give a more detailed description of the algorithm and analyze its time complexity.

Let \( \delta, \zeta, \epsilon \) be some constants to be determined later. To simplify the presentation, we define the following variables:

\[
m_l = \left( \frac{1}{2} + \delta \right) \left( \frac{1}{\epsilon} - 1 \right)
\]

\[
m_r = \left( \frac{1}{2} - \delta \right) \left( \frac{1}{\epsilon} - 1 \right)
\]

\[
\text{Psize} = \lfloor \epsilon (k - 1) \rfloor - 1
\]

\[
\text{Psize}_{\text{mid}} = k - \frac{1}{\epsilon} - 1 - (m_l + m_r) \cdot \text{Psize}
\]

\[
\text{Lnum}_{\text{base}} = \left\lfloor \left( \frac{1}{2} + \delta \right) \left( k - \frac{1}{\epsilon} - 1 \right) \right\rfloor
\]

\[
\text{Rnum}_{\text{base}} = \left\lceil \left( \frac{1}{2} - \delta \right) \left( k - \frac{1}{\epsilon} - 1 \right) \right\rceil
\]

\[
\text{Rnum'} = \lfloor \zeta \cdot \text{Lnum}_{\text{base}} \rfloor
\]

\[
\text{Lnum} = \text{Lnum}_{\text{base}} - \text{Rnum'}
\]

\[
\text{Rnum} = \text{Rnum}_{\text{base}} + \text{Rnum'}
\]

\[
\text{Lnum}_i = \frac{i}{m_l} (\text{Lnum} - \text{Psize}_{\text{mid}})
\]

Similarly to Zehavi [15], we define a problem called \textsc{Cut} \( k \)-\textsc{Path} (we note that the definition here is different than the one in [15]). The input to this problem is a di-
directed graph $G = (V, E)$, a partition of $V$ into disjoint sets $L, R$, a sequence of distinct vertices $s_1, \ldots, s_{m_l+1+m_r}$, and a sequence of distinct vertices $t_1, \ldots, t_{m_l+1+m_r}$. The two sequences above satisfy $|\{s_1, \ldots, s_{m_l+1+m_r}\} \setminus \{t_1, \ldots, t_{m_l+1+m_r}\}| = 1$. We denote $V_e = \{s_1, \ldots, s_{m_l+1+m_r}, t_1, \ldots, t_{m_l+1+m_r}\}$.

The goal of the problem is to decide whether there are paths $P_1, \ldots, P_{m_l+1+m_r}$ with the following properties.

1. For all $i$, the first vertex of $P_i$ is $s_i$ and the last vertex is $t_i$.
2. For all $i$, the internal vertices of $P_i$ are disjoint from $V_e$.
3. For all $i \neq j$, the internal vertices of $P_i$ are disjoint from the internal vertices of $P_j$.
4. For all $i \neq m_l + 1$, the number of internal vertices of $P_i$ is $P_{\text{size}}$.
5. The number of internal vertices of $P_{m_l+1}$ is $P_{\text{size}_{\text{mid}}}$.
6. For every $i \leq m_l$, the number of internal vertices of $P_1, \ldots, P_i$ that are in $L$ is at least $L_{\text{num}}$.
7. The number of internal vertices of $P_1, \ldots, P_{m_l+1}$ that are in $L$ is $L_{\text{num}}$.
8. For every $i \leq m_r$, the internal vertices of $P_{m_l+1+i}$ are from $R$.

We note that property 6 follows from the assumption that the paths $P_1, \ldots, P_{m_l}$ are ordered such that $|P_{i-1} \cap L| \geq |P_i \cap L|$ for every $i \leq m_l$. We also note that properties 7 and 8 imply that the number of internal vertices in all the paths that are in $L$ is $L_{\text{num}}$. Therefore, the number of internal vertices in all the paths that are in $R$ is $R_{\text{num}}$.

The Cut $k$-Path problem can be solved using the algorithm of [15] with minor modifications. The algorithm consists of three stages.

The first stage constructs the paths $P_1, \ldots, P_{m_l}$. This stage builds a table $M$ in which $M[i, j, j_2, v]$ is a family that $(L_{\text{num}} - j, R_{\text{num}} - j_2)$-represents the family of all sets of the form $(P_1 \cup \cdots \cup P_i) \setminus V_e$, where $P_1, \ldots, P_i$ are paths such that

- $P_1, \ldots, P_{i-1}$ satisfy properties 1, 4, and 6.
- $P_1, \ldots, P_i$ satisfy properties 2 and 3.
- The first vertex of $P_i$ is $s_i$ and the last vertex of $P_i$ is $v$.
- The number of internal vertices of $P_1, \ldots, P_i$ that are in $L$ and $R$ is $j$ and $j_2$, respectively.
The indices \( i, j, j_2 \) have the following ranges: \( 1 \leq i \leq m_l \), \( Lnum_{i-1} \leq j \leq Lnum \), and \( 1 + (i - 1) \cdot Psiz - j \leq j_2 \leq i \cdot Psiz - j \). Moreover, the range of \( v \) is \( v \in V \setminus V_e \). The computation of an entry \( M[i, j, j_2, v] \) is done as follows. Suppose \( j + j_2 > 1 + (i - 1) \cdot Psiz \). In this case perform

\[
M[i, j, j_2, v] = \begin{cases} 
\{A \cup \{v\} : A \in \bigcup_{u \in E} M[i, j - 1, j_2, u]\} & \text{if } v \in L \\
\{A \cup \{v\} : A \in \bigcup_{u \in E} M[i, j, j_2 - 1, u]\} & \text{otherwise}
\end{cases}
\]

Then, use Theorem 4 to find a family that \((Lnum - j, Rnum - j_2)\)-represents \( M[i, j, j_2, v] \), and replace \( M[i, j, j_2, v] \) with this family. Theorem 4 is applied with \( U_1 = L \) and \( U_2 = R \), and with constants \( c_1, c_2 \).

By Theorem 4 the time complexity of the first stage is \( O^*(X_1^{2o(k)}) \), where

\[
X_1 = \max_{i=1}^{m_l} \max_{j=1}^{Lnum_{i-1}} \max_{j_2=1}^{1+(i-1)\cdot Psiz-j} \frac{(c_1 \cdot Lnum)^{2 \cdot Lnum - j}}{j^i (c_1 \cdot Lnum - j)^{2 \cdot Lnum - 2j}} \cdot \frac{(c_2 \cdot Rnum)^{2 \cdot Rnum - j_2}}{j_2^{j_2} (c_2 \cdot Rnum - j_2)^{2 \cdot Rnum - 2j_2}}.
\]

Our goal is to estimate \( Y_1 = X_1^{1/k} \). For this purpose, we redefine the values of the following variables:

\[
m_l = \left(\frac{1}{2} + \delta\right) \frac{1}{\epsilon} \\
m_r = \left(\frac{1}{2} - \delta\right) \frac{1}{\epsilon} \\
Psiz = \epsilon k \\
Lnum_{base} = (\frac{1}{2} + \delta) k \\
Rnum_{base} = (\frac{1}{2} - \delta) k \\
Rnum' = \zeta (\frac{1}{2} + \delta) k \\
Lnum = (1 - \zeta) (\frac{1}{2} + \delta) k \\
Rnum = \left(\frac{1}{2} - \delta + \zeta (\frac{1}{2} + \delta)\right) k \\
Lnum_i = \frac{i + 1}{m_l} \cdot Lnum
\]

Note that since we can assume that \( k \) is large enough and that \( \epsilon \) is small enough, the value of \( Y_1 \) for the new definitions of the variables is arbitrarily close to the value of \( Y_1 \) for the old definitions. Define \( \alpha = \frac{j}{Lnum} \) and \( \alpha_2 = \frac{j_2}{Rnum} \). The range of \( j \) in the definition of \( X_1 \) (over all \( i \)) is \( 0 \leq j \leq Lnum \). Therefore, \( 0 \leq \alpha \leq 1 \). The range of \( j \) in the second maximum in the definition of \( X_1 \) implies
that $j \geq \text{Lnum}_{i-1} = \frac{i}{m_j} \cdot \text{Lnum}$. Therefore, $i \leq m_i \cdot \frac{\text{Lnum}}{2} = (\frac{1}{2} + \delta)\frac{1}{\epsilon} \cdot \alpha$. The range of $j_2$ in the third maximum implies that

$$j_2 \leq i \cdot \text{Psize} - j \leq (\frac{1}{2} + \delta)\frac{1}{\epsilon} \cdot \epsilon k - \text{Lnum} \cdot \alpha$$

$$= (\frac{1}{2} + \delta) \alpha k - (1 - \zeta)(\frac{1}{2} + \delta)k \cdot \alpha = \zeta(\frac{1}{2} + \delta)\alpha k.$$ 

Therefore,

$$\alpha_2 = \frac{j_2}{\text{Rnum}} \leq \frac{\zeta(\frac{1}{2} + \delta)}{\frac{1}{2} - \delta + \zeta(\frac{1}{2} + \delta)} \cdot \alpha.$$ 

We obtain that

$$Y_1 = \max_{0 \leq \alpha \leq 1} \max_{0 \leq \alpha_2 \leq 1} \frac{\phi_{c_1}(\alpha)^{\text{Lnum}/k} \cdot \phi_{c_2}(\alpha_2)^{\text{Rnum}/k}}{\frac{\zeta(\frac{1}{2} + \delta)}{\frac{1}{2} - \delta + \zeta(\frac{1}{2} + \delta)} \cdot \alpha}$$

$$= \max_{0 \leq \alpha \leq 1} \max_{0 \leq \alpha_2 \leq 1} \frac{\phi_{c_1}(\alpha)^{(1-\zeta)(\frac{1}{2} + \delta)} \cdot \phi_{c_2}(\alpha_2)^{\frac{1}{2} - \delta + \zeta(\frac{1}{2} + \delta)}}{\frac{\zeta(\frac{1}{2} + \delta)}{\frac{1}{2} - \delta + \zeta(\frac{1}{2} + \delta)}}.$$ 

The second stage of the algorithm constructs the path $P_{m_{i+1}}$ (see [15] for more details). This stage does not affect the time complexity of the algorithm. Finally, the third stage constructs the paths $P_{m_{i+2}}, \ldots, P_{m_{i+1+m_r}}$. This stage constructs a table $K[i, j_2, v]$ where $1 \leq i \leq m_t$, $1 + (m_t + i - 1) \cdot \text{Psize} + \text{Psize}_{mid} - \text{Lnum} \leq j_2 \leq (m_t + i) \cdot \text{Psize} + \text{Psize}_{mid} - \text{Lnum}$, and $v \in V \setminus V_e$. The time complexity of the third stage is $O^*(X_22^{\alpha(k)})$, where

$$X_2 = \max_{i=1}^{m_r} \max_{j_2=1+(m_t+i-1) \cdot \text{Psize} + \text{Psize}_{mid} - \text{Lnum}} \frac{(c'_2 \cdot \text{Rnum})^{2 \cdot \text{Rnum} - j_2}}{j_2^2 \cdot (c'_2 \cdot \text{Rnum} - j_2)^2 \cdot \text{Rnum} - 2j_2^2}.$$ 

Under the simplified definitions of the variables, we have that the range of $j_2$ over all $i$ satisfies $j_2 \geq m_t \cdot \text{Psize} - \text{Lnum} = \text{Rnum}'$. Therefore $\alpha_2 \geq \frac{\text{Rnum}'}{\text{Rnum}} = \frac{\zeta(\frac{1}{2} + \delta)}{\frac{1}{2} - \delta + \zeta(\frac{1}{2} + \delta)}$. Let $Y_2 = X_2^{1/k}$. We obtain that

$$Y_2 = \max_{\frac{\zeta(\frac{1}{2} + \delta)}{\frac{1}{2} - \delta + \zeta(\frac{1}{2} + \delta)} \leq \alpha_2 \leq 1} \phi_{c'_2}(\alpha_2)^{\frac{1}{2} - \delta + \zeta(\frac{1}{2} + \delta)}.$$ 

The time complexity of the algorithm for Cut $k$-Path is $O^*(\max(Y_1, Y_2)2^{O(k)})$. The algorithm for $k$-PATH generates $O(|F| \cdot n^{O(k)})$ instances of Cut $k$-Path, where $F$ is an $(n, \text{Lnum}, \text{Rnum}, \zeta)$-mixed universal family. By Lemma 8 we have that $|F| = O^*(Y_3^{2^{O(k)}})$ where

$$Y_3 = \frac{\zeta^\zeta (1 - \zeta)^{(1-\zeta)}^{\frac{1}{2} + \delta} \cdot \cdot \cdot (1 - \zeta)(1 - \zeta)^{\frac{1}{2} + \delta)}{(1 - \zeta)(\frac{1}{2} + \delta)(1 - (1 - \zeta)(\frac{1}{2} + \delta))^{rac{1}{2} - \delta + \zeta(\frac{1}{2} + \delta)}}.$$
Therefore, the time complexity of the algorithm for \( k \)-\textsc{Path} is \( O^*(\max(Y_1, Y_2)^k \cdot Y_3^{2^o(k)}) \). We now choose the following parameters in order to minimize the time complexity: \( \delta = -0.00467, \zeta = 0.712, c_1 = 1.136, c_2 = 1.645, \) and \( c_2' = 1 + \frac{1}{\sqrt{5}} \approx 1.447 \). Under this choice of parameters, \( \max(Y_1, Y_2) \cdot Y_3 < 2.5537 \). The value of \( Y_1 \) is maximized when \( \alpha \approx 0.864 \) and \( \alpha_2 \approx 0.356 \). The value of \( Y_2 \) is maximized when \( \alpha_2 = 1 - \frac{1}{\sqrt{5}} \approx 0.553 \).

\section{Proof of Lemma \ref{lem:5}}

Our proof is based on the construction of separating collection of Fomin et al. \cite{6}. We give below three lemmas that correspond to Lemmas 4.5, 4.8, and 4.11 of \cite{6}. Since our lemmas are very similar to the lemmas of \cite{6}, we only sketch the proofs of our lemmas. We first give a construction of a mixed universal set with size below the size specified in Lemma \ref{lem:5} but with very large construction time.

\textbf{Lemma 9.} There is an algorithm that given \( 0 < \zeta < 1, 0 < x < 1, \) and integers \( n, p, q, \zeta \), constructs an \((n, p, q, \zeta)\)-mixed universal family of size 

\[ O\left( \frac{1}{\eta^p x (1-\zeta)p (1-x)^{q+\zeta p}} \cdot 2^{o(p)} (p+q) \log n \right). \]

\textbf{Proof.} We first show using the probabilistic method that there is an \((n, p, q, \zeta)\)-mixed universal family of the required size. Let

\[ t = \left\lceil \frac{2^{o(p)}}{\eta^p x (1-\zeta)p (1-x)^{q+\zeta p}} (p+q+1) \log n \right\rceil. \]

Create a family \( \mathcal{F} = \{ F_1, \ldots, F_t \} \), where each \( F_i \) is constructed by adding every element of \( U \) to \( F_i \) with probability \( x \) independently of the other elements. For fixed \( i \), a fixed set \( A \subseteq U \) of size \( p \), and a fixed set \( B \subseteq U \setminus A \) of size \( q \), the probability that \( |A \setminus F_i| = |\zeta p| \) and \( B \cap F_i = \emptyset \) is

\[ \binom{n}{|\zeta p|} x^{p-|\zeta p|} (1-x)^{q+|\zeta p|} \geq \eta^p/2^{o(p)} \cdot x^{(1-\zeta)p} (1-x)^{q+\zeta p}. \]

Thus, the probability that there is no \( F_i \) for which \( |A \setminus F_i| = \zeta p \) and \( B \cap F_i = \emptyset \) is at most

\[ (1 - \eta^p/2^{o(p)} \cdot x^p (1-x)^q)^t \leq e^{-(p+q+1) \log n} = \frac{1}{n^{p+q+1}}. \]

The number of ways to choose the sets \( A \) and \( B \) is at most \( n^{p+q} \). It follows that the probability that \( \mathcal{F} \) is not an \((n, p, q, \zeta)\)-mixed universal family is at most \( 1/n \).

In order to construct an \((n, p, q, \zeta)\)-mixed universal family of size \( t \), enumerate all \( \binom{2^n}{t} \leq 2^{2^n} \) possible families \( \mathcal{F} \) of size \( t \) and check for each \( \mathcal{F} \) whether it is an \((n, p, q, \zeta)\)-mixed universal family.

\textbf{Lemma 10.} Suppose that there is a construction of \((n, p, q, \zeta)\)-mixed universal families with size \( s(n, p, q, \zeta) \) and construction time \( t(n, p, q, \zeta) \). Then there is a construction of \((n, p, q, \zeta)\)-mixed universal families with size

\[ s'(n, p, q, \zeta) \leq s((p+q)^2, p, q, \zeta) \cdot (p+q)^{O(1)} \cdot \log n. \]
and construction time
\[ t'(n, p, q, \zeta) = O((p + q)^2 \cdot p, q, \zeta)) + s((p + q)^2, p, q, \zeta) \cdot (p + q)^{O(1)} \cdot n \log n. \]

**Proof.** Let \( f_1, \ldots, f_t : U \to [1, \ldots, (p + q)^2] \) be a \((p + q)\)-perfect family of hash functions. Namely, for every \( S \subseteq U \) of size \( p + q \), there is a function \( f_i \) such that the restriction of \( f_i \) to \( S \) is injective. There is a construction of \( f_1, \ldots, f_t \) with \( t = (p + q)^{O(1)} \cdot \log n \) and \( O((p + q)^{O(1)} \cdot n \log n) \) construction time \([1]\). Construct a \((p + q)^2, p, q, \zeta)\)-mixed universal family \( \hat{F} \) with size \( s((p + q)^2, p, q, \zeta) \). Then define
\[
\mathcal{F} = \bigcup_{i \leq t} \{ f_i^{-1}(\hat{F}) : \hat{F} \in \hat{F} \}
\]
where \( f_i^{-1}(\hat{F}) = \{ s \in U : f_i(s) \in \hat{F} \} \). We now show that \( \mathcal{F} \) is an \((n, p, q, \zeta)\)-mixed universal family. Let \( A, B \subseteq U \) be disjoint sets of sizes \( p \) and \( q \), respectively. There is a function \( f_i \) such that \( f_i \) restricted to \( A \cup B \) is injective. Let \( \hat{A} = \{ f_i(s) : s \in A \} \) and \( \hat{B} = \{ f_i(s) : s \in B \} \). The sets \( \hat{A}, \hat{B} \) are disjoint and their sizes are \( p \) and \( q \), respectively. Therefore, there is a set \( \hat{F} \in \hat{F} \) such that \( |\hat{A} \setminus \hat{F}| = [\zeta p] \) and \( \hat{B} \cap \hat{F} = \emptyset \). Let \( F = f_i^{-1}(\hat{F}) \). We have that \( |A \setminus F| = |\hat{A} \setminus \hat{F}| = [\zeta p] \) (since for every \( s \in A, s \in F \) if and only if \( f_i(s) \in \hat{F} \) and \( B \cap \hat{F} = \emptyset \)). \( \blacksquare \)

Let \( Z_{s,t}^p \) be a set containing tuples \( (p_1, \ldots, p_t) \) for every integers \( p_1, \ldots, p_t \) satisfying \( 0 \leq p_i \leq s \) for all \( i \), and \( \sum_{i=1}^t p_i = p \).

**Lemma 11.** Suppose that there is a construction of \((n, p, q, \zeta)\)-mixed universal families with size \( s(n, p, q, \zeta) \) and construction time \( t(n, p, q, \zeta) \). Then there is a construction of \((n, p, q, \zeta)\)-mixed universal families with size
\[
s'(n, p, q, \zeta) \leq n^{O(t)} \sum_{(p_1, \ldots, p_t) \in Z_{s,t}^p} \prod_{i=1}^t s(n, p_i, s - p_i, \zeta)
\]
and construction time
\[
t'(n, p, q, \zeta) = O \left( \int \sum_{\substack{\hat{p} \leq s, \hat{q} \leq p \\text{ s.t.} \ s - \hat{p} \leq q}} t(n, \hat{p}, s - \hat{p}, \zeta) \right) + s'(n, p, q, \zeta) \cdot n^{O(1)}
\]
where \( s = \lfloor \log^2(p + q) \rfloor \) and \( t = \lceil \frac{p + q}{s} \rceil \).

**Proof.** Without loss of generality assume that \( U = \{1, \ldots, n\} \). Let \( \mathcal{P}_{t}^n \) be the set of all partitions of \( U \) into \( t \) disjoint intervals. For a family of sets \( \mathcal{A} \) and a set \( B \), let \( \mathcal{A} \cap B = \{ A \cap B : A \in \mathcal{A} \} \). For two family of sets \( \mathcal{A} \) and \( \mathcal{B} \), let \( \mathcal{A} \circ \mathcal{B} = \{ A \cup B : A \in \mathcal{A}, B \in \mathcal{B} \} \).

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For every integer \( \hat{p} \) that satisfies \( 0 \leq \hat{p} \leq s \), \( \hat{p} \leq p \), and \( s - \hat{p} \leq q \), construct an \((n, \hat{p}, s - \hat{p}, \zeta)\)-mixed universal family \( F_{\hat{p}} \). Now define

\[
F = \bigcup_{j \leq n} \bigcup_{(U_1, \ldots, U_t) \in \mathcal{P}_t^n} \bigcup_{(p_1, \ldots, p_t) \in \mathbb{Z}^t_{s,t} \text{ s.t. } \forall i, s - p_i \leq q} \left( (F_{p_1} \cap U_1) \circ \cdots \circ (F_{p_t} \cap U_t) \right) \cap \{1, \ldots, j\}.
\]

We now show that \( F \) is an \((n, p, q, \zeta)\)-mixed universal family. Let \( A, B \subseteq U \) be disjoint sets of sizes \( p \) and \( q \), respectively. There is \((U_1, \ldots, U_t) \in \mathcal{P}_t^n\) such that \( |(A \cup B) \cap U_i| = \left\lceil \frac{p + q}{t} \right\rceil = s \) for all \( i < t \) and \( |(A \cup B) \cap U_t| \leq s \). Define \( A_i = A \cap U_i \) for all \( i \leq t \), and \( p_i = |A_i| \). We also define \( B_i = B \cap U_i \) for all \( i < t \). Moreover, \( B_t \) is a set obtained by taking \( B \cap U_t \) and adding \( s - p_t - |B \cap U_t| \) arbitrary elements of \( U \setminus U_t \) to the set. For every \( i \), there is a set \( F_i \in F_{p_i} \) such that \( |A_i \setminus F_i| = |\zeta_{p_i}| \) and \( B_i \cap F_i = \emptyset \). Let \( F_0 = \bigcup_{i \leq t} (F_i \cap U_i) \). We have that \( B \cap F_0 = \emptyset \). Additionally, \( |A \setminus F_0| = \sum_{i \leq t} |A_i \setminus F_i| = \sum_{i \leq t} |\zeta_{p_i}| \leq |\zeta p| \). Therefore, there is \( j \) such that \( F = F_0 \cap \{1, \ldots, j\} \) satisfies \( |A \setminus F| = |\zeta p| \) and \( B \cap F = \emptyset \). By definition, \( F \in F \).

Lemma 8 now follows from Lemmas 9, 10, and 11 (see [6]).

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