Renormalization of Heavy Quark Effective Field Theory: Quantum Action Principles and Equations of Motion

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Abstract

We discuss the quantum action principles and equations of motion for Heavy Quark Effective Field Theory. We prove the so-called equivalence theorem for HQEFT which states that the physical predictions of HQEFT are independent from the choice of interpolating fields. En passant we point out that HQEFT is in fact more subtle than the quantum mechanical Foldy-Wouthuysen transformation.

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1 Introduction

Heavy Quark Effective Field Theory (HQEFT) has arguably been the fastest growing sector of elementary particle theory in the four years since its introduction [1]. This can be ascribed to two features: on one hand the spin and flavor symmetries are phenomenologically successful [2] and, on the other hand, corrections to the symmetry limit \( m_Q \to \infty \) are well defined and therefore calculable. Given this state of affairs, it seems appropriate to investigate the theoretical formalism of HQEFT in all orders of the combined \( 1/m_Q, \alpha_S \) expansion.

It is only a slight exaggeration that Quantum Field Theory (QFT) has no physical content and is just the most powerful tool known for the calculation of \( S \)-matrix elements [3]. The fields and the off-shell values of their Green’s functions are not observable and therefore all QFTs which lead to identical \( S \)-matrices are physically equivalent. We are therefore free to choose the calculationally most convenient set of fields [4], similar to the choice of appropriate coordinates in classical physics. This observation has borne the extremely powerful concept of Effective Field Theory (EFT) [5] which underlies almost all phenomenologically successful applications of QFT.

It is however far from obvious how the general theorems [6] guaranteeing the independence of the \( S \)-matrix from the choice of interpolating fields translate to the intricate formalism of renormalized perturbation theory [7]. Nevertheless, the so-called equivalence theorem, which formalizes this independence, has been proved rigorously [8], using the renormalized quantum action principles [9, 10]. We shall extend this proof to HQEFT below.

In order to avoid any misconceptions, we should stress that the (colored) heavy quark fields \( h_v, \bar{h}_v \) of HQEFT are not interpolating fields for \( S \)-matrix elements in non-perturbative QCD. This rôle is played by the meson and baryon fields. However, it seems instructive to define the \( S \)-matrix for free quarks as an intermediate step in order to show how the equivalence theorem works in a perturbative calculation.

This note is organized as follows: to fix our notation and to introduce the technical subtleties of higher order calculations, we briefly describe HQEFT in section 2. The quantum action principles are introduced in section 3. We sketch the proof of the equivalence theorems for Green’s functions and \( S \)-matrix elements in sections 4 and 5, respectively. As an application, we discuss the reparametrization invariance of HQEFT in section 6 and clarify
the rôle of the classical equations of motion in section 7. Finally, we present our conclusions in section 8 and describe why HQEFT is more subtle than the quantum mechanical Foldy-Wouthuysen transformation.

2 Heavy Quark Effective Field Theory

Within the framework of perturbation theory, the heavy mass expansion is constructed by performing the following steps on the Green’s functions, i.e., Feynman diagrams of QCD: Heavy quark loops are integrated out and replaced by their expansion around vanishing momenta entering the loop. The momenta of the remaining heavy quark propagators (those connected to external heavy quark lines) are replaced by the residual momenta \( k^\mu = p^\mu - mv^\mu, v^2 = 1, v_0 > 0, \) with \( m \) the pole mass of the heavy quark. The antiparticle pole is eliminated by expanding the denominators around \( k = 0 \):

\[
\frac{1}{v \cdot k + k^2/2m + i\epsilon} = \frac{1}{v \cdot k + i\epsilon} \sum_{n=0}^{\infty} \left( \frac{-k^2/2m}{v \cdot k + i\epsilon} \right)^n. \tag{1}
\]

Although this procedure is legitimate for tree level amplitudes and for the regularized loop diagrams, the different high energy behaviour of the propagators

\[
G(k) = i \frac{m\not{v} + \not{k} + m}{(mv + k)^2 - m^2 + i\epsilon} \tag{2}
\]

vs.

\[
S_v(k) = \frac{iP_v^+}{v \cdot k + i\epsilon} \tag{3}
\]

in general destroys the equality of the full and effective theory amplitudes after renormalization. One has to correct for this by introducing matching contributions into the effective theory. This amounts to calculating the differences of the full and effective theory amplitudes for any amputated one-particle irreducible (1PI) diagram of the full theory and absorb them into a set of local operator insertions in the effective theory.

The matching works here order by order in the loop expansion because the subtraction of the expansion about zero residual momentum \( k_0^\mu = 0 \) improves the infrared (IR) behavior of the integrands, exactly in such a way as to render the corrections IR convergent \[11\]. Note that it is essential for \( v \) to be timelike for this to happen.
A tree level, the above procedure may be carried out in the functional integral of QCD by integrating over the lower components of the heavy quark field with respect to $v$ and thus constructing an effective field theory (HQEFT). The result

$$\mathcal{L}^{(0)}_v = \bar{h}_v i(v \cdot D) h_v + (\bar{h}_v iD^+_v + \bar{R}_v) \frac{1}{2m + i(v \cdot D) - i\epsilon} (iD^+_v h_v + R_v)$$

$$+ \bar{\rho}_v h_v + \bar{h}_v \rho_v,$$

containing the transverse derivative

$$D^+_v = \not{\!D} - \not{\!v}(v \cdot D).$$

is expressed in terms of the upper component field

$$h_v(x) = e^{inv \cdot x} P^+_v Q(x) \quad [P^\pm_v = (1 \pm \not{\!v})/2]$$

and the sources of upper and lower components

$$\rho_v(x) = P^+_v e^{inv \cdot x} \eta(x),$$

$$R_v(x) = P^-_v e^{inv \cdot x} \eta(x).$$

The latter are not needed for deriving $S$-matrix elements, but one might find it convenient to retain them in order to easily obtain the effective theory analogues of operator insertions.

Matching contributions enter in the same way as the tree-level quark-gluon vertex, so they can by included into (4) by introducing a generalized covariant derivative

$$i\not{\!D} = i\not{\!D} + O(\alpha_s)$$

with projections

$$iD^+_v = P^+_v i\not{\!D} P^+_v = (iv \cdot D) P^+_v + O(\alpha_s),$$

$$iD^\perp_v = P^+_v i\not{\!D} P^-_v + P^-_v i\not{\!D} P^+_v = iD^\perp_v + O(\alpha_s),$$

$$iD^-_v = P^+_v i\not{\!D} P^-_v = -(iv \cdot D) P^-_v + O(\alpha_s).$$

The effective Lagrangian which generalizes (4) is

$$\mathcal{L}_v = \bar{h}_v iD^+_v h_v + (\bar{h}_v iD^\perp_v + \bar{R}_v) \frac{1}{2m - iD^-_v} (iD^\perp_v h_v + R_v) + C_0$$

$$+ \bar{\rho}_v h_v + \bar{h}_v \rho_v,$$
where $C_0$ summarizes operators consisting only of light fields which are introduced by integrating out heavy quark loops in the full theory. This effective Lagrangian is valid to arbitrary (but finite) order in the $1/m$ and loop expansions.

One can easily reformulate HQEFT with an arbitrary residual mass $\delta m$ of the order $\Lambda_{\text{QCD}} \ll m$ by the change $D \rightarrow D - \delta m v$ and the induced changes in $D$. For generality, we include this residual mass term in the heavy quark propagator.

### 3 Quantum Action Principles

It is useful to establish some notation for a concise presentation of the Quantum Action Principles (QAP). For convenience, we shall denote all fields, heavy and light, generically by a scalar field $\phi$. The generating functional of the renormalized Green’s functions is given by

$$Z[j]_{\mathcal{L}} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \ldots dx_n j(x_1) \ldots j(x_n) \langle 0 | T [\phi(x_1) \ldots \phi(x_n)] | 0 \rangle_{\mathcal{L}(\phi)}.$$  \hspace{1cm} (12)

and we use the following shorthand for local operator insertions:

$$O(z) \downarrow Z[j]_{\mathcal{L}} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \ldots j(x_1) \ldots \langle 0 | T [O(z) \phi(x_1) \ldots ] | 0 \rangle_{\mathcal{L}(\phi)}. \hspace{1cm} (13)$$

By including appropriate source terms into the lagrangian, we can express the generating functionals formally as functional integrals

$$Z[j]_{\mathcal{L}} = \int \mathcal{D} \phi \ e^{i \int \mathcal{L}(\phi;j)}$$  \hspace{1cm} (14)

$$O(z) \downarrow Z[j]_{\mathcal{L}} = \int \mathcal{D} \phi \ O(z) \ e^{i \int \mathcal{L}(\phi;j)}, \hspace{1cm} (15)$$

but these expressions have to be renormalized properly.

Using formal manipulations of the functional integrals (14) and (15) we can derive the three quantum action principles for the generating functionals immediately:

1. invariance under variations of the quantized fields $\phi$, i.e. field equations:

$$\delta \mathcal{L}(z) \downarrow Z[j]_{\mathcal{L}} = 0, \quad \delta \mathcal{L}(\phi) = \frac{\delta \mathcal{L}(\phi)}{\delta \phi} \delta \phi \hspace{1cm} (16)$$
\[
\delta \phi(x) = (P(\phi))(x)\delta \epsilon
\]  
(17)

with \(P(\phi)\) an arbitrary local polynomial in \(\phi\) and its derivatives.

2. change under variation of external fields:
\[
\frac{\delta L}{\delta \chi}(x) \downarrow Z[j; \chi]_L = -i\frac{\delta Z[j; \chi]_L}{\delta \chi}(x)
\]  
(18)

3. change under variation of parameters in the lagrangian
\[
\frac{\partial L}{\partial \eta} \downarrow Z[j]_L(\eta) = -i\frac{\partial Z[j]_L(\eta)}{\partial \eta}.
\]  
(19)

After renormalization the three QAPs (16), (19), and (19) are valid without changes if the dimensional renormalization scheme is used \([12, 10]\). In other renormalization schemes (e.g. BPHZ \([13]\)) there are (calculable) normal product corrections \([9]\).

Since dimensional regularization is defined with the help of the Schwinger parameter representation, which is also convenient for practical calculations, we define the parameter representation for diagrams containing heavy particles:

Any internal light line \(\ell\) is assigned a parameter \(\alpha_\ell\) and an auxiliary vector \(u_\ell\). The propagator is represented by
\[
\frac{iD(k_\ell)}{k_\ell^2 - m_\ell^2 + i\epsilon} = \int_0^\infty d\alpha_\ell D(-i\partial/\partial u_\ell) \exp i(\alpha_\ell k_\ell^2 + u_\ell \cdot k_\ell - \alpha_\ell[m_\ell^2 - i\epsilon])\bigg|_{u_\ell=0}.
\]  
(20)

where \(D(k_\ell)\) may contain algebraic objects like \(\gamma\) matrices, Lorentz tensors etc. Any internal heavy line \(h\) is assigned a parameter \(\beta_h\) and an auxiliary vector \(u_h\). The propagator is represented by
\[
\frac{iP_v^+}{v_h \cdot k_h - \delta m_h + i\epsilon} = P_v^+ \int_0^\infty d\beta_h \exp i([\beta_h v_h + u_h] \cdot k_h - \beta_h[\delta m_h - i\epsilon])\bigg|_{u_h=0}.
\]  
(21)

Any vertex factor \(V(p_i, k_i)\) with \(p_i, k_i\) being the momenta of the attached external and internal lines, respectively, is translated into \(V(p_i, -i\partial/\partial u_i)\). Finally, the Gaussian momentum integrations are formally carried out.
The requirement that no heavy particle loops are included ensures that all momentum integrations are indeed Gaussian. In the following we consider only the case of one heavy quark with velocity $v$. The generalization to the multiparticle sector is straightforward if the velocities of the heavy particles do not coincide.

Here we shall not present a proof of theorem I of [10], i.e. the consistency of dimensional renormalization for HQEFT in the parametric representation. We do not expect a failure of a translation of the proof in [10], and we hope to come back to the purely technical details of such a proof in a later publication.

In order to establish the first QAP (16) in HQEFT, we have to specify the separation of the Lagrangian $\mathcal{L}$ into the free ($\mathcal{L}_0$) and interaction ($\mathcal{L}_{\text{int}}$) parts. We take $\mathcal{L}_0$ to be equal to

$$\mathcal{L}_0 = \bar{h}_v (iv \cdot \partial - \delta m) h_v$$

(22)

corresponding to the propagator (3). The QAP (16) then says

$$\left( \bar{h}_v P(\phi) (iv \cdot \partial - \delta m) h_v \right) (z) \Downarrow Z[j]_\mathcal{L} = - \left( \bar{h}_v P(\phi) \frac{\delta \mathcal{L}_{\text{int}}}{\delta h_v} \right) (z) \Downarrow Z[j]_\mathcal{L}$$

(23)

where now $\phi$ denotes the light fields of the theory only. In order to remain in the one heavy particle sector, we insist that the field variations be linear in the heavy quark field.

In diagrammatic language, the content of (23) is that attaching the factor $(iv \cdot \partial - \delta m)$ to a vertex is equivalent to contracting the neighbouring heavy quark line to a point. In the parametric representation, we have to show that

$$\left( v \cdot \frac{\partial}{\partial u_h} - i \delta m \right) I(p, u, \alpha, \beta) = \frac{\partial}{\partial \beta_h} I(p, u, \alpha, \beta)$$

(24)

which is true since apart from a global factor $\exp -i \sum_h \beta_h \delta m$, the dimensionally regularized integrand $I$ depends on $u_h$ and $\beta_h$ only in the combination $\beta_h v + u_h$. The desired equality follows from the fact that

$$\int_0^\infty d\beta_h \frac{\partial}{\partial \beta_h} I = -I(\beta_h = 0),$$

(25)

since $I$ falls off exponentially at the upper limit.
The proof that the QAP remains valid after renormalization proceeds exactly as in [10]. The only potential obstruction are diagrams such as depicted in Fig. 1a, where the subdiagram enclosed by dashed lines has no equivalent if the center line of the diagram is contracted to a point (cf. also Fig. 1 of [10]). The corresponding counterterm diagram would contribute to the left-hand side of (23), but not to the right-hand side. However, this counterterm diagram (Fig. 1b) necessarily contains a heavy quark loop and thus vanishes identically.

The proof of the second QAP (18) is simple combinatorics since $L_0$ is independent of the external fields. The same applies to the third QAP (19, 27) if $L_0$ is independent of the parameter $\eta$. Therefore we shall restrict ourselves to the case where the varied parameter is the velocity of the heavy quark

$$v \rightarrow v + \eta \delta v \quad \text{with} \quad v \cdot \delta v = 0; \quad (26)$$

the case $\eta = \delta m$ can be treated similarly.

We have to prove:

$$\frac{\partial}{\partial \eta} Z[j]_\mathcal{L} = - \int dz (\bar{h} v \delta v \cdot \partial h_v)(z) \downarrow Z[j]_\mathcal{L} + i \frac{\partial \mathcal{L}_{\text{int}}}{\partial \eta} \downarrow Z[j]_\mathcal{L}. \quad (27)$$

The second term on the right-hand side is cancelled exactly by the terms on the left-hand-side where the derivative acts on the vertices. Furthermore, acting with $\partial/\partial \eta$ on the propagator part of the amplitude results in terms of the form

$$\frac{\partial}{\partial \eta} \int_0^\infty d\beta I(u + \beta [v + \eta \delta v], \beta \delta m, \ldots) \bigg|_{u=0} \bigg|_{u'=0} = \int_0^\infty d\beta \beta \delta v \cdot \frac{\partial}{\partial u} I(u + \beta [v + \eta \delta v], \beta \delta m, \ldots) \bigg|_{u=0} \bigg|_{u'=0} \quad (28)$$

The last expression corresponds to a Feynman diagram where two heavy lines are joined by the quadratic vertex $-\bar{h} v \delta v \cdot \partial h_v$, which is equivalently obtained from the first term of the right-hand side of (27). Simple combinatorics completes the proof for the regularized amplitudes, and the validity of (27) after renormalization is proved as described in [10].
4 The Equivalence Theorem for Green’s Functions

The equivalence theorem asserts the following relations among Green’s functions

$$\langle 0 | T [φ(x_1) \ldots φ(x_n)] | 0 \rangle_{L(φ)} = \langle 0 | T [φ'(x_1) \ldots φ'(x_n)] | 0 \rangle_{L'(φ)=L(φ')}, \quad (29)$$

of fields which are related by a reparametrization

$$φ(x) → φ'(x) = φ(x) + ηF(φ(x)) \quad (30)$$
in all orders of a simultaneous expansion in the numbers of loops and powers of η.

We concentrate on field reparametrizations which are linear in the heavy field $\bar{h}_v$, but may be nonlinear in the other fields of the theory:

$$\bar{h}_v(x) → \bar{h}'_v(x) = \bar{h}_v(x) (1 + ηF(φ(x), \partial)) \quad (31)$$

The structure of the heavy quark Lagrangian allows to partially integrate so that the derivatives in $F$ act on $h_v$ and the light fields only. This simplifies the notation in the following arguments. Of course, the proof can be applied to $h_v$ also. Simultaneous changes in $h_v$ and $\bar{h}_v$ may be splitted into subsequent reparametrizations of $h_v$ and $\bar{h}_v$.

We want to show

$$\langle 0 | T [\bar{h}_v(x) \ldots] | 0 \rangle_{L(\bar{h}_v,h_v)} = \langle 0 | T [\bar{h}'_v(x) \ldots] | 0 \rangle_{L'(\bar{h}_v,h_v)=L(\bar{h}'_v,h_v)} \quad (32)$$

where it is understood that the Green’s functions are evaluated to some finite order $η^n$. Following Lam [8], we define

$$Z(η) = Z[j]_{L(\bar{h}_v + η\bar{h}_vF(φ,\partial),h_v)}. \quad (33)$$

The first QAP (16) tells us that

$$\bar{h}_vP(φ) \frac{δL'}{δh_v} \downarrow Z(η) = 0. \quad (34)$$

for any local polynomial $P(φ)$ in the light fields. From (31) this is equivalent to

$$\bar{h}_vP(φ) \frac{δL'}{δh_v} \downarrow Z(η) = -η \bar{h}_vP(φ) F(φ) \frac{δL'}{δh'_v} \downarrow Z(η). \quad (35)$$
After iterating this equation \( n \) times by replacing \( P(\phi) \) by \( P(\phi) F(\phi)^k \) for \( k = 1, 2, \ldots n \) on the left-hand side, we arrive at

\[
\bar{h}_v P(\phi) \frac{\delta L'}{\delta h'_v} \downarrow Z(\eta) = (-\eta)^{n+1} \bar{h}_v P(\phi) F(\phi)^n F(\phi) \frac{\delta L'}{\delta h'_v} \downarrow Z(\eta).
\]  

Since we are interested in the variation of \( Z(\eta) \) with \( \eta \), we use the third QAP (19) to get

\[
\frac{d}{d\eta} Z(\eta) = i \frac{\delta L'}{\delta \eta} \downarrow Z(\eta) = i \bar{h}_v F(\phi) \frac{\delta L'}{\delta h'_v} \downarrow Z(\eta).
\]  

(37)

Applying (36) with \( P = F \), we see that the generating functional \( Z(\eta) \) is independent of \( \eta \) in any finite order \( n \):

\[
\frac{d}{d\eta} Z(\eta) = O(\eta^{n+1}).
\]  

(38)

Taking derivatives with respect to the sources completes the proof of the equivalence theorem (32) for renormalized Green’s functions in the effective theory. Operator insertions as those responsible for weak decays may also be considered by coupling them to some source.

We emphasize that this proof holds to finite order in \( \eta \) and in the dimensional renormalization scheme (MS or \( \overline{\text{MS}} \)) only. In other renormalization schemes (e.g. BPHZ, lattice) there are corrections to (32). The same is true for resummed variations which modify the propagator.

5 The Equivalence Theorem for the \( S \)-Matrix

After discussing the proper definition of \( S \)-matrix elements in HQEFT, we shall derive the equivalence theorem for these \( S \)-matrix elements. In perturbation theory (or for that matter in any non-confining theory like QED), there are asymptotic states corresponding to single heavy particles. Translating the familiar LSZ reduction formula for a massive fermion in momentum space

\[
S(p, \ldots) = \frac{-i}{\sqrt{Z_Q}} \bar{u}(p/m) (\not{p} - m) \ldots \int dx e^{ip \cdot x} \langle 0 | T [Q(x) \ldots] | 0 \rangle \bigg|_{p^2 = m^2}
\]  

(39)
to the effective theory, and setting $\delta m = 0$, we use
\[ \bar{u}(v + k/m) = \bar{u}(v) \frac{\hat{p} + 1 + \frac{k}{m}}{2\sqrt{1 + (v \cdot k)/2m}} \] (40)
to arrive at
\[ S(k, \ldots) = \frac{-i}{\sqrt{Z_Q \tilde{Z}(k)}} \bar{u}(v) \left( v \cdot k + \frac{k^2}{2m} \right) \int dx e^{ik \cdot x} \langle 0 \| T [h_v(x) \cdots] \| 0 \rangle \bigg|_{v = -k^2/2m} \] (41)

Here the kinematical factor
\[ \tilde{Z}(k) = 1 + \frac{v \cdot k}{2m} \] (42)
follows from the matching condition (40) for the heavy particle wave functions. Alternatively, we can derive it from the residue of the heavy particle two-point function $\langle 0 \| T [h_v(x) h_v(y)] \| 0 \rangle$ at the one heavy particle pole. The additional term $k^2/2m$ in (41) cancels the multiple poles in the heavy-quark Green’s functions that arise from insertions of the $1/m$ terms in the Lagrangian into external lines.

Having defined the $S$-matrix for heavy quark states, the proof that the $S$-matrix is unchanged by reparametrizations, i.e., that the $S$-matrix elements calculated from the original HQEFT are the same as those calculated with the new Lagrangian obtained by the change of variables (31), proceeds exactly as in ordinary quantum field theory [8]. The additional terms in the Green’s functions on the right-hand side of (42) contribute in such a way that the residue of the pole at $v \cdot k + k^2/2m = 0$ (which appears as a series of multiple poles in the effective theory) is multiplied by some function of $k$ identical for all Green’s functions of the heavy quark. This function only modifies the factor $\tilde{Z}(k)$ in (41) and therefore does not contribute to $S$-matrix elements. Any contribution which has no pole at $v \cdot k + k^2/2m = 0$ vanishes after applying (41).

In the real world heavy quarks come in bound states, and the $S$-matrix is defined in terms of those. Denoting a heavy (scalar) bound state with mass $M$ generically by $B$, a typical matrix element is given by
\[ \langle B | \mathcal{O} | X \rangle = \frac{1}{\sqrt{Z_B}} \langle 0 \| J_B \| B \rangle \langle B | \mathcal{O} | X \rangle \] (43)
\[
\begin{align*}
&= \frac{-i}{\sqrt{Z_B}} \left( P^2 - M^2 \right) \int dx \, e^{iP \cdot x} \langle 0 \left| T \left[ J_B(x) \mathcal{O}(0) \right] \right| X \rangle |_{P^2 = M^2} \\
\end{align*}
\]

where
\[
Z_B = \left( P^2 - M^2 \right) \int dx \, e^{iP \cdot x} \langle 0 \left| T \left[ J_B(x) J_B(0) \right] \right| 0 \rangle |_{P^2 = M^2} \quad (44)
\]

and \( J_B \) is any operator such that \( \langle 0 \left| J_B(x) \right| B \rangle \) does not vanish. In the effective theory we have
\[
\begin{align*}
\langle B \left| \mathcal{O} \right| X \rangle &= \frac{-i}{\sqrt{Z_B^{\text{eff}}}} \left( v \cdot k + \frac{k^2}{2m} - \frac{M^2 - m^2}{2m} \right) \\
&\times \int dx \, e^{i k \cdot x} \langle 0 \left| T \left[ J_B^{\text{eff}}(x) \mathcal{O}(0) \right] \right| X \rangle |_{v \cdot k + \frac{k^2}{2m} - \frac{M^2 - m^2}{2m}} ,
\end{align*}
\]

where \( J_B^{\text{eff}} \) is some interpolating field of the effective theory, and the definition of \( Z_B \) is modified analogously
\[
Z_B^{\text{eff}} = \left( v \cdot k + \frac{k^2}{2m} - \frac{M^2 - m^2}{2m} \right) \\
\times \int dx \, e^{i k \cdot x} \langle 0 \left| T \left[ J_B^{\text{eff}}(x) J_B^{\text{eff}}(0) \right] \right| 0 \rangle |_{v \cdot k + \frac{k^2}{2m} - \frac{M^2 - m^2}{2m}} . 
\quad (46)
\]

The difference of hadron and quark masses has to be treated in the same perturbative expansion
\[
M = m + \bar{\Lambda} + \frac{\lambda}{2m} + \ldots 
\quad (47)
\]
so that multiple poles in the matrix element due to the mass shift are cancelled separately in any order of the \( 1/m \) expansion. Of course, one could rearrange the series to be an expansion in \( 1/M \) or any other physically sensible mass by suitable choice of \( m \).

Using similar arguments as for the free quark case, it becomes obvious that any reparametrization of the heavy quark field that changes the form of the interpolating field \( J_B \) has no effect on the ratio (44). Since the choice of \( J_B \) has been arbitrary from the beginning, one might as well keep the definition of \( J_B \) fixed while calculating a matrix element using the reparametrized Lagrangian. The equivalence theorem ensures that masses and other properties of bound states remain unchanged. However, since nonperturbative QCD
cannot be formulated using the MS or \( \overline{\text{MS}} \) scheme, in order to calculate a particular matrix element one first has to match renormalization schemes (e.g. lattice versus \( \overline{\text{MS}} \)). In a general scheme the equivalence theorem does not hold without corrections, so one will in general obtain different matching contributions in different versions of the effective theory.

6 Reparametrization Invariance

As an application of the QAPs and the equivalence theorem, we discuss the implication of the Luke-Manohar reparametrization invariance \([16]\) on the effective Lagrangian. For simplicity, we keep \( \delta m \) equal to zero in this section.

An infinitesimal change in the velocity \( v \) used in the derivation of (11)

\[
v \rightarrow v + \delta v \quad \text{with} \quad v \cdot \delta v = 0
\]

causes a corresponding change in the field \( h_v \)

\[
h_{v+\delta v} = e^{im(\delta v \cdot x)} P^+_{v+\delta v}(h_v + H_v).
\]

Since the \( H_v \) field is integrated out, this implies

\[
\delta h_v = \left( im(\delta v \cdot x) + \frac{\delta \phi}{2} + \frac{\delta \tilde{\phi}}{2} \frac{1}{2m - iD^+_v - iD^+_{\perp}} \right) h_v.
\]

Inserting this variation into the effective Lagrangian \([11]\), a straightforward calculation shows that \( \mathcal{L}_v \) is reparametrization invariant

\[
\delta \mathcal{L}_v = 0
\]

if and only if

\[
\delta(iD_v + m\phi) = -[iD_v + m\phi, im(\delta v \cdot x)].
\]

Assuming gauge invariance, this condition is satisfied if and only if the generalized covariant derivative \( D_v \) depends on the ordinary covariant derivative \( D \) and the velocity \( v \) in the combination

\[
iD_v + m\phi = f(v + iD/m)
\]
This is exactly what one would expect since \(mv + k\) is the full momentum of the heavy quark, the quantity that enters into the matching calculation. In particular, since
\[
i\mathcal{D} = m(\dot{\phi} + i\mathcal{D}/m) - m\dot{\phi}, \quad (54)
\]
reparametrization invariance is valid on tree level, as it has been shown already in [17].

The QAPs imply that the relations following from this type of reparametrization invariance remain valid after renormalization, if the division of the effective Lagrangian into renormalized part and counterterms is carried out using the \(\overline{\text{MS}}\) or \(\text{MS}\) renormalization scheme. In other schemes reparametrization invariance may be broken.

7 Equations of motion

The physical content of the first QAP ([16 23]) is that naive application of the classical equations of motion
\[
(i\mathcal{V} \cdot \text{D} - \delta m)h_v = -\frac{\delta}{\delta h_v} \mathcal{L}^{(1)} \quad (55)
\]
is justified in the \(\overline{\text{MS}}\) renormalization scheme, where \(\mathcal{L}^{(1)}\) denotes the terms of order \(1/m\) and higher in the effective Lagrangian. However, as it stands the QAP applies only to single operator insertions such as weak currents. If multiple operator insertions \(\mathcal{O}_i\) are considered, the easiest way to obtain the correct terms is to couple sources \(j_i\) to them and include them in the effective Lagrangian. In higher orders of the loop expansion, 1PI diagrams containing multiple insertions generate counterterms nonlinear in the sources, so that the effective Lagrangian is some polynomial in \(j_i\). Employing the equivalence theorem, one can look for a reparametrization of the heavy quark field that results in a particular simplification. If the Lagrangian contains an operator of the form
\[
\mathcal{O} = \bar{h}_v F(\phi, iD) (i\mathcal{V} \cdot \text{D} - \delta m)h_v, \quad (56)
\]
one may set
\[
h_v \to (1 - F(\phi, iD)) h_v, \quad (57)
\]
so that the variation of the lowest-order term \(\mathcal{L}^{(0)}\) eliminates \(\mathcal{O}\) in favor of a tower of new higher-dimensional operators. This method may be used
to eliminate all terms containing \((iv \cdot D - \delta m)\) from the interaction Lagrangian. However, the familiar Luke-Manohar reparametrization invariance is obscured in this process. Furthermore, the kinematical normalization factor \(\tilde{Z}(k)\) in (42) has to be recalculated, but this is only relevant for free heavy quarks as external states.

In fact, any allowed reparametrization is of the form (57) and thus equivalent to the application of the equations of motion: The additional operator introduced into the Lagrangian is

\[
- \frac{\delta \mathcal{L}}{\delta h_v} F(\phi, iD) h_v. \tag{58}
\]

The equivalence theorem tells us that the vanishing of this operator works for an arbitrary number of insertions, whereas the QAP applies only to single insertions.

On the other hand, reparametrizations are as well possible in the full theory, if the additional terms introduced in the QCD Lagrangian are treated perturbatively. Since the effective theory may be constructed also from the reparametrized Lagrangian by calculating the appropriate matching corrections, we have the following situation: The equivalence theorem in the full and effective theories, respectively, guarantees that the matching contributions do not depend on in which theory reparametrizations are carried out, if we use dimensional renormalization in both cases. In ordinary field theory one usually considers only field redefinitions that are relativistically invariant

\[
Q(x) \rightarrow Q(x) + F(\phi, \partial) Q(x). \tag{59}
\]

Such redefinitions may be mirrored in the effective theory; we observe that they preserve the Luke-Manohar reparametrization invariance of the effective Lagrangian. Other transformations correspond to Lorentz non-invariant reparametrizations of the full theory.

8 Conclusions

In this letter we have established the quantum action principles (QAPs) in HQEFT which justify formal manipulations of the functional integral. Using the QAPs, we have proved the equivalence theorem for the effective theory
and thus have shown under what conditions field redefinitions in HQEFT are viable. A particular class consists of transformations corresponding to the Lorentz invariance of QCD — the Luke-Manohar reparametrization invariance of HQEFT. In practice, the equivalence theorem serves as a guideline how the classical equations of motion manifest themselves once radiative corrections and renormalization are taken into account.

From our discussion it should have become clear that — unless one is willing to handle normal product corrections — it is necessary to use a MS-like renormalization scheme, and to make sure that the free Lagrangian (i.e., the propagator) is not affected. In particular, this applies to attempts to derive the heavy quark Lagrangian employing the Foldy-Wouthuysen transformation familiar from single-particle quantum mechanics [18]. By its very nature this transformation modifies the quark propagator in the full theory and thus violates the hypothesis of the equivalence theorem. Since the authors of [18] do not discuss how to calculate the necessary normal product corrections, the argument given in [18] is not a complete derivation of HQEFT. On the other hand, the Lagrangian provided in [18] can be reduced to the tree-level Lagrangian (4) by a series of field redefinitions which are compatible with the equivalence theorem of the effective theory\footnote{These redefinitions change the normalization of the heavy quark field. The authors of [18] have argued that the admissible reparametrizations have to preserve the normalization of the fields. While this is applicable to wave functions in single-particle quantum mechanics, the particular normalization of quantized fields cancels in the calculation of $S$-matrix elements in the full theory [13, 13] as well as in the effective theory [11, 11].}. Thus beyond tree-level the Foldy-Wouthuysen approach can only be justified if it is amended by some prescription to calculate matching corrections, which will then yield the same results as the conventional approach [14, 11].

To summarize, the form of the HQEFT Lagrangian is by no means unique, but as long as a MS-like renormalization scheme is used, different versions can be mapped onto each other by perturbative field redefinitions. However, in any renormalization scheme where the QAPs hold in the weak sense only (i.e., normal product corrections arise), one has to perform a new matching calculation if one wants to carry out reparametrizations. In particular, in a general scheme the classical equations of motion cannot be naively applied, and reparametrization invariance is lost beyond leading order in the $\alpha_s$ and $1/m$ expansions.
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List of Figures

1. Potential obstruction in the proof of the QAP. The open square denotes the operator insertion on the left-hand side of (23).
This figure "fig1-1.png" is available in "png" format from:

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