Vector bundles on elliptic curve and Sklyanin algebras

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**Introduction**

In [4] we introduce the associative algebras $Q_{n,k}(E, \tau)$. Recall the definition. These algebras are labeled by discrete parameters $n, k$; $n, k$ are integers $n > k > 0$ and $n$ and $k$ have not common divisors. Then, $E$ is an elliptic curve and $\tau$ is a point in $E$. We identify $E$ with $\mathbb{C}/\Gamma$, where $\Gamma$ is a lattice.

Algebra $Q_{n,k}(E, \tau)$ is generated by $n$ generators $\{x_i\}$, $i \in \mathbb{Z}/n\mathbb{Z}$, which satisfy the relations:

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+(k-1)r}(0)}{\theta_{j-i-r}(\tau)\theta_{kr}(\tau)} x_{k(j-r)} x_{k(i+r)} = 0.$$  

Here all indices belong to $\mathbb{Z}/n\mathbb{Z}$, $\{\theta_j\}, j \in \mathbb{Z}/n\mathbb{Z}$ are $\theta$-functions of order $n$. Note that here $\tau \in \mathbb{C}$, we use the same symbol for $\tau \in E$ and for some preimage of this point in $\mathbb{C}$; $\theta_0, \ldots, \theta_{n-1}$ can be considered as sections of a line bundle of degree $n$ on $E$.

The main properties of algebras $Q_{n,k}(E, \tau)$.

(a) $Q_{n,k}(E, \tau)$ is a graded algebra $Q_{n,k}(E, \tau) = A_0 \oplus A_1 \oplus \ldots$. For generic $\tau$ is has the same size as the ring of polynomials. It means that $\dim A_l = n(n+1)\ldots(n+l-1)/l!$. If $\tau = 0$ then $Q_{n,k}(E, 0)$ is an algebra of polynomials in $n$ variables. So, $Q_{n,k}(E, \tau)$ for small $\tau$ is a flat deformation of the space of functions on $\mathbb{C}^n$.

(b) The finite Heisenberg group $\Gamma_n$ (with generators $\varepsilon_1, \varepsilon_2, \delta$, $\varepsilon_1^n = \varepsilon_2^n = \delta^n = 1$, $\delta$—central element, $\varepsilon_1\varepsilon_2 = \delta\varepsilon_2\varepsilon_1$) acts in $Q_{n,k}(E, \tau)$ by automorphisms. This action is compatible with the grading, and $A_1$ is an irreducible representation of $\Gamma_n$. 

1
(c) For generic $\tau$ the center $Z$ of $Q_{n,k}(\mathcal{E}, \tau)$ is a polynomial ring generated by $c$ elements of degree $n/c$, where $c$ is the maximal common divisor of $n$ and $k + 1$.

(d) **Characteristic manifold.** It is the set of modules $M$ over $Q_{n,k}(\mathcal{E}, \tau)$ of the simplest possible type. Such $M$ is graded, $M = M_0 \oplus M_1 \oplus \cdots$, $\dim M_i = 1$, and $M$ is generated by $M_0$. Characteristic manifold has a structure of algebraic variety, and we denote it by $\text{Ch}_{n,k}$. To describe $\text{Ch}_{n,k}$ we need the decomposition of $n/k$ into continuous fraction.

$$\frac{n}{k} = n_1 - \frac{1}{n_2 - \frac{1}{n_3 - \cdots - \frac{1}{n_p}}}$$

$n_i \geq 2, \ 1 \leq i \leq p$.

It is clear that such a decomposition is unique. Now consider the product of $p$ copies of $\mathcal{E}$, $\mathcal{E}^{(p)} = \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_p$. Let $\xi$ be a line bundle on $\mathcal{E}$ such that $\deg \xi = 1$ (in this case $\dim H^0(\xi) = 1$). Then construct the line bundle $\tilde{\xi}$ on $\mathcal{E}^{(p)}$, $\tilde{\xi} = \xi^{n_1+1} \boxtimes \xi^{n_2+2} \boxtimes \cdots \boxtimes \xi^{n_p+1}$, $\boxtimes$ means exterior tensor product. Denote by $\Delta_{i,i+1}$ the divisor on $\mathcal{E}^{(p)}$:

$$\Delta_{1,2}, \Delta_{2,3}, \cdots, \Delta_{p-1,p}$$

It is easy to see that $\dim H^0(\tilde{\xi}) = n$. The bundle $\tilde{\xi}$ determines the map $\mathcal{E}^{(p)} \to \mathbb{C}P^{n-1}$. The image is isomorphic to the characteristic manifold.

In this paper we examine these algebras and generalizations using the vector bundles on elliptic curves. To be more precise, the language of bundles on elliptic curves can be used for description of symplectic leaves of such algebras. Algebra $Q_{n,k}(\mathcal{E}, \tau)$ goes to abelian when $\tau \to 0$ so, the family of algebras $Q_{n,k}(\mathcal{E}, \tau)$ determines the hamiltonian structure on $\mathbb{C}^n$. We are interested in the symplectic leaves of this structure. Let us formulate the result. It is known that indecomposable bundles on an elliptic curve are labeled by two integers $(n, k)$, $k > 0$ and by a point of elliptic curve. (See §1). For such a bundle $\xi_{n,k}$, $n$ is its degree and $k$ is its rank. Let us fix an indecomposable bundle, and consider the moduli space $\text{Mod}(\xi_{0,1}; \xi_{n,k})$ of $k + 1$-dimensional bundles on elliptic curve with $1$-dimensional subbundle $(\nu, \rho)$, $\dim \rho = 1$, $\dim \nu = k + 1$, $\rho$ is trivial and $\nu/\rho \cong \xi_{n,k}$. Suppose that $n > 0$ (otherwise the space $\text{Mod}(\xi_{0,1}; \xi_{n,k})$ is empty). It is easy to see that $\text{Mod}(\xi_{0,1}; \xi_{n,k}) \cong P(\text{Ext}^1(\xi_{n,k}; \xi_{0,1}))$. Here $P$ means projective space
associated with vector space $\text{Ext}^1(\xi_{n,k}; \xi_{0,1})$. In other words, $\text{Mod}(\xi_{0,1}; \xi_{n,k})$ is the space of exact sequences:

$$0 \to \xi_{0,1} \to Y \to \xi_{n,k} \to 0$$

which are considered up to an isomorphism.

The hamiltonian structure on $\mathbb{C}^n$ arising in the classical limit $\tau \to 0$ of $Q_{n,k}(E, \tau)$ is homogeneous, so it determines the hamiltonian structure on the projective space $\mathbb{CP}^{n-1}$. Denote this structure by $h_{n,k}(E)$.

**Theorem 1.** The following decompositions of $\mathbb{CP}^{n-1}$ coincide: (a) the decomposition into the union of symplectic leaves of the structure $h_{n,k}(E)$. (b) $\mathbb{CP}^{n-1} \cong \text{Ext}^1(\xi_{n,k}; \xi_{0,1}) \cong \text{Mod}(\xi_{0,1}; \xi_{n,k})$. This moduli space is the union of strata. Each stratum corresponds to a type of $k+1$-dimensional bundle.

Hamiltonian structures on $\mathbb{CP}^{n-1}$ are the particular cases of the following general construction. Let $G$ be a semisimple Lie group and $P$-a parabolic subgroup in $G$; $\mathfrak{a}, \mathfrak{p}$-Lie algebras of $G$ and $P$. Algebra $\mathfrak{a}$ is a sum $n_- \oplus \mathfrak{a}_0 \oplus n_+$, where $\mathfrak{a}_0$ is a Levi subalgebra in $\mathfrak{p}$, $n_-$ is the radical in $\mathfrak{p}$, $\mathfrak{p} = n_- \oplus \mathfrak{a}_0$. The standard bialgebra structure of $\mathfrak{a}$, $\mathfrak{a} \to \Lambda^2 \mathfrak{a}$ can be restricted on $\mathfrak{p}$. It will be a non-trivial cocycle $p \to \Lambda^2 p$ which gives us an elements $\beta$ from $H^1(p, \Lambda^2 p)$ or from $\text{Ext}_p^1(p^*, p)$. Here we use the symbol $p$ for the adjoint representation. Consider now the moduli space $\text{Mod}(P,E)$ of $P$-bundles on $E$. Let $\mu$ be a point of $\text{Mod}(P,E)$. A tangent space $T_\mu$ at $\mu$ is isomorphic to the space $H^1(E, \mu(p))$. Here if $p$ is a representation of $p$ we denote by $\mu(p)$ the corresponding vector bundle on $E$. The element $\beta \in H^1(p, \Lambda^2 p) \cong \text{Ext}_p^1(p^*, p)$ determines the exact sequence of $p$-modules $0 \to p \to s \to p^* \to 0$ and the sequence of vector bundles on $E$: $0 \to \mu(p) \to \mu(s) \to \mu(p^*) \to 0$. The boundary homomorphism in the exact cohomology sequence is: $H^0(\mu(p^*)) \to H^1(\mu(p))$. Note that dual to $H^0(\mu(p^*))$ is $H^1(\mu(p) \otimes K) \cong H^1(\mu(p))$, because on elliptic curve the canonical bundle $K$ is trivial and we fix the trivialization. Therefore we got a map $(H^1(\mu(p)))^* \to H^1(\mu(p))$ which is skew-symmetric. This map is determined at each point $\mu \in \text{Mod}(P,E)$, so we construct a bivector field on $\text{Mod}(P,E)$ which is integrable. Note that our construction works on the non-singular part of the manifold $\text{Mod}(P,E)$. It is
easy to see that the symplectic leaves of this structure are exactly the fibers of the map 
\( \text{Mod}(P, \mathcal{E}) \to \text{Mod}(G, \mathcal{E}) \).

Therefore it is a natural problem how to quantize such a hamiltonian structure on
the moduli space of \( P \)-bundles. Algebras \( Q_{n,k}(\mathcal{E}, \tau) \) give us a solution of this problem in a
very special case. In §2 we consider the case when \( P \) is a Borel subgroup in a Kac-Moody
group. (Kac-Moody group can be infinite-dimensional). (See point V of §2). The case
when \( P \) is a parabolic subgroup in \( \text{GL}_n \) will be the subject of our next paper.

Now about the structure of the text. In §1 we collect the facts about the bundles on
elliptic curve. The main problem which we deal with is: let \( A \) and \( B \) be two bundles on \( \mathcal{E} \),
then the space \( \text{Ext}^1(B, A) \) has a natural stratification. Each \( \beta \in \text{Ext}^1(B, A) \) determines
the exact sequence \( 0 \to A \to \xi(\beta) \to B \to 0 \). The bundle \( \xi(\beta) \) is a sum of indecomposable
bundles. We stratify the space \( \text{Ext}^1(B, A) \) according to the type of \( \xi(\beta) \). What is the
combinatorial structure of this stratification? In §2 we present some constructions of
algebras which in the simplest case give us \( Q_{n,k}(\mathcal{E}, \tau) \) when \( k = 1 \). In §3 we give the
description of symplectic leaves in twisted version of \( Q_{n,k}(\mathcal{E}, \tau) \) and discuss the case of a
general parabolic group.
§1. Indecomposable vector bundles on an elliptic curve

1. Description of the indecomposable bundles

Here we recall the results from [1].

Construction of the stable indecomposable bundles.

Fix two integers \( n, k, n \neq 0, k > 0 \) which are relatively prime and a line bundle \( \alpha \) on \( \mathcal{E} \) of degree \( n \cdot k \). If \( n > 0 \) then \( \dim H^0(\alpha) = n \cdot k, H^1(\alpha) = 0 \), if \( n < 0 \), then \( H^0(\alpha) = 0 \) and \( \dim H^1(\alpha) = n \cdot k \). In both cases Heisenberg group \( \Gamma_{n,k} \) (a central extension of \( \mathbb{Z}_{nk} \times \mathbb{Z}_{nk} \)) acts on cohomologies in irreducible way. The central element \( K \) of \( \Gamma_{n,k} \) acts by multiplication on the primitive root of unity \( \varepsilon, \varepsilon^{nk} = 1 \). Consider Heisenberg subgroup \( \Gamma_k \hookrightarrow \Gamma_{nk} \). The central element \( K' \) of \( \Gamma_k \) goes to \( K^n \). Consider also the irreducible representation of \( \Gamma_k, \Gamma_k \rightarrow \text{End} \, \pi \), where \( K' \) acts by \( \varepsilon^{-k} \). After tensoring \( \alpha \otimes \pi \) we get a \( k \)-dimensional bundle on \( \mathcal{E} \), where \( \Gamma_k \) acts by the natural way (on \( \alpha \) trough the embedding \( \Gamma_k \hookrightarrow \Gamma_{nk} \) on and on \( \pi \)). Central element acts by 1, so the group \( \mathbb{Z}_k \times \mathbb{Z}_k \) acts on \( \alpha \otimes \pi \). We can quote \( \mathcal{E}/\mathbb{Z}_k \times \mathbb{Z}_k \) and \( \alpha \otimes \pi/\mathbb{Z}_k \times \mathbb{Z}_k \). As a result we get an indecomposable \( k \)-dimensional vector bundle \( \xi_{n,k}(\alpha) \) on \( \mathcal{E} \cong \mathcal{E}/\mathbb{Z}_k \times \mathbb{Z}_k \). It is easy to see that \( \deg \xi_{n,k}(\alpha) = n, \dim H^0(\xi_{n,k}(\alpha)) = n \) and \( H^1(\xi_{n,k}(\alpha)) = 0 \) if \( n > 0 \), and \( H^0(\xi_{n,k}(\alpha)) = 0 \) and \( \dim H^1(\xi_{n,k}(\alpha)) = n \) if \( n < 0 \). The Heisenberg group \( \Gamma_n \) acts on cohomologies of \( \xi_{n,k}(\alpha) \) in irreducible way.

Construction of the semistable indecomposable bundles.

Fix again two integers \( n, k \) such that \( n = n_1 \cdot c, k = k_1 \cdot c, k > 0 \) and \( n_1 \) and \( k_1 \) have not common divisors . We denote by \( \mathcal{O} \) the trivial bundle on \( \mathcal{E} \) and by \( \mathcal{O}(s) \) the bundle of \( s \)-jets of \( \mathcal{O} \). It means that the fiber of \( \mathcal{O}(s) \) at the point \( a \in \mathcal{E} \) is a quotient \( \mathcal{O}(a)/\mathcal{O}(a)_s \). Here \( \mathcal{O}(a) \cong \mathbb{C}[[z]] \) is the space of sections of \( \mathcal{O} \) in the formal vicinity of \( a \) \( (z \) is a local coordinate in \( a \) and \( \mathcal{O}(a)_s = z^s\mathbb{C}[[z]] \subset \mathcal{O}(a) \). It is easy to see that \( \mathcal{O}(s) \) is an indecomposable \( s \)-dimensional bundle. We define \( \xi_{n,k}(\alpha) \) as \( \xi_{n_1,k_1}(\alpha) \otimes \mathcal{O}(c) \). If \( n = 0 \) then by \( \xi_{0,k}(\alpha) \) we denote the bundle of \( k \)-jets of a line bundle \( \alpha, \deg \alpha = 0 \). This is the full list of indecomposable bundles \( \{ \xi_{n,k}(\alpha) \}; n, k \in \mathbb{Z}, k > 0, rk \xi_{n,k}(\alpha) = k, \deg \xi_{n,k}(\alpha) = n, \alpha \) is a line bundle of degree \( n_1 \cdot k_1 \) where \( n_1 \cdot c = n, k_1 \cdot c = k; n_1, k_1 \) are relatively prime. We will call the pair of integers \( (n, k) \) the discrete parameters of the bundle and \( \alpha \)-the continuous parameter. In future we prefer to skip the continuous
parameters and write just $\xi_{n,k}$ if the dependence on $\alpha$ is evident. In the next proposition we collect some simple properties of $\xi_{n,k}$.

**Proposition.**

(a) $\xi^*_{n,k}(\alpha) = \xi_{-n,k}(\alpha^{-1})$.

(b) Suppose $n_2/k_2 > n_1/k_1$, then the dimension of $\text{Hom}(\xi_{n_1,k_1}; \xi_{n_2,k_2})$ is $n_2 \cdot k_1 - n_1 k_2$.

Ext$^1(\xi_{n_1,k_1}; \xi_{n_2,k_2}) = 0$

(c) Ext$^i(\xi_{n_1,k_1}; \xi_{n_2,k_2}) \cong$ Ext$^{1-i}(\xi_{n_2,k_2}; \xi_{n_1,k_1})$

(d) dim $\text{Hom}(\xi_{n,k}(\alpha_1); \xi_{n,k}(\alpha_2)) = c$, $c$ is the maximal common divisor of $n$ and $k$ if $\alpha_1 = \alpha_2$, otherwise it is zero.

2. The Moduli space of $k$-dimensional bundles on an elliptic curve.

We know all indecomposable bundles, so each $k$-dimensional bundle of degree $n$ can be decomposed into the sum:

$$\xi_{n_1,k_1}(\alpha_1) \oplus \xi_{n_2,k_2}(\alpha_2) \oplus \ldots \oplus \xi_{n_p,k_p}(\alpha_p)$$

where $n_1 + n_2 + \ldots + n_p = n; k_1 + k_2 + \ldots + k_p = k$.

So as a set the moduli space $\text{Mod}_{n,k}$ of $k$-dimensional bundles of degree $n$ is the set of data \{ $n_1,k_1,\alpha_1; n_2,k_2,\alpha_2; \ldots; n_p,k_p,\alpha_p$ \}.

As usual we can organize the subset of semistable bundles into some kind of a manifold. (See for example [2])

**Proposition.** A decomposition (*) determines the semistable bundle if and only if $n_1/k_1 = n_2/k_2 = \ldots = n_p/k_p$. Such bundle is stable if and only if in each fraction $n_i/k_i, n_i$ and $k_i$ have not common divisors.

We will use the symbol Mod$^s_{n,k}$ for the moduli space of stable bundles. It is clear that Mod$^s_{n,k}$ is isomorphic to the symmetric power $S^cE, c = (n,k)$. Recall that if we have an arbitrary family of vector bundles parametrized by the manifold $N$ then the semistable bundles constitute an open subset $U \subset N$. Suppose that the dimension of bundles and the
degree are fixed. In this case we get the natural map \( U \to \text{Mod}^s_{n,k} \).

3. The Operator \( F \)

\( F \) is a well-known duality transform on the derived category of the coherent sheaves on an elliptic curve. Recall briefly the construction. Let \( \mathcal{E}' \) be the set of line bundles on \( \mathcal{E} \) of degree zero, \( \mathcal{E}' \cong \mathcal{E} \). There is a natural bundle \( P \) on the product \( \mathcal{E} \times \mathcal{E}' \). The fiber of \( P \) at the point \((a, b)\) is the fiber of the bundle \( b \) at the point \( a \). Let \( \pi_1, \pi_2 \) be two projections \( \mathcal{E} \times \mathcal{E} \to \mathcal{E} \): \( \pi_1(z_1, z_2) = z_1, \pi_2(z_1, z_2) = z_2 \). Let \( \nu \) be a coherent sheaf on \( \mathcal{E} \).

The transformation \( \nu \to F(\nu) \) is given by the formula: \( F(\nu) = \pi_2, * (P \otimes \pi_1^*(\nu)) \).

It is clear that \( F \) is a covariant functor on the derived category of coherent sheaves.

Properties of the functor \( F \).

(a) \( F(\xi_{n,k}(\alpha)) = \xi_{-k,n}(\alpha^{-1}) \) if \( n > 0 \)
(b) suppose that \( n_1, n_2 > 0 \), then \( \text{Hom}(\xi_{n_1,k_1}; \xi_{n_2,k_2}) = \text{Hom}(F(\xi_{n_1,k_1}); F(\xi_{n_2,k_2})) \)
\[ = \text{Hom}(\xi_{-k_1,n_1}; \xi_{-k_2,n_2}) = \text{Hom}(\xi_{k_2,n_2}; \xi_{k_1,n_1}) \]. The same is true if we replace here \( \text{Hom} \) by \( \text{Ext}^1 \).

4. Partial ordering on the moduli space of bundles.

Let \( \xi_{n_1,k_1}(\alpha_1) \oplus \xi_{n_2,k_2}(\alpha_2) \oplus \ldots \oplus \xi_{n_p,k_p}(\alpha_p) \) be a decomposition of some vector bundle \( B \). We will say that the bundle \( C \) is less than \( B \) (notation: \( C \prec B \)) if \( C \) belongs to the closure of \( B \). It means that there is a family \( X \) of bundles which satisfies the following properties.

The base \( M \) of the family \( X \) is connected. Let \( M_B \) and \( M_C \) be the subsets in \( M \) consisting of points \( p \in M \) such the the corresponding bundle \( X(p) \) is \( B \) or \( C \) respectively. Then \( M_C \) belongs to the closure of \( M_B \).

It is clear that the bundle \( B \) is a maximal element (it means that if \( B \prec C \) then \( C = B \)) if \( \alpha_1, \alpha_2, \ldots, \alpha_p \) are all different and \( n_1/k_1 = n_2/k_2 = \ldots = n_p/k_p \) and \( n_i, k_i \) have not common divisors. Note, that in this case the continuous parameters \( \alpha_1, \ldots, \alpha_p \) are the line bundles of the same degree.

Problem. Suppose we have two bundles \( B = \xi_{n_1,k_1}(\alpha_1) \oplus \ldots \oplus \xi_{n_p,k_p}(\alpha_p) \) and \( C = \xi_{\bar{n}_1,\bar{k}_1}(\beta_1) \oplus \ldots \oplus \xi_{\bar{n}_s,\bar{k}_k}(\beta_s) \). When \( C \prec B \)?
We restrict ourselves only by the case when all pairs \((n_i, k_i)\) have not common divisors and the same is true for the pairs \(\tilde{n}_i, \tilde{k}_i\). Let us reformulate the question a little. Let us fix a sequence of fractions \(\{\tau_1, \tau_2, \ldots, \tau_p\}\), \(\tau_i = n_i/k_i, (n_i, k_i) = 1\), suppose that \(\tau_1 \leq \tau_2 \leq \ldots \leq \tau_p\). We prefer to write \(\xi_{\tau_i}(\alpha_i)\) instead of \(\xi_{n_i,k_i}(\alpha_i)\). Suppose also that \(\alpha_i \neq \alpha_j\) if \(\tau_i = \tau_j\). Then we deform the bundle \(\xi_{\tau_1}(\alpha_1) + \ldots + \xi_{\tau_p}(\alpha_p)\). The result is a \(k_1 + k_2 + \ldots + k_p\) dimensional bundle \(\nu\) which admit a filtration \(\nu_0 \subset \nu_1 \subset \ldots \subset \nu_{p-1} \cong \nu\) such that

\[\nu_0 \cong \xi_{\tau_1}(\alpha_1);\ \nu_1/\nu_0 \cong \xi_{\tau_2}(\alpha_2);\ldots;\nu_{p-1}/\nu_{p-2} \cong \xi_{\tau_p}(\alpha_p).\]

Informally, \(p\) indecomposable bundles are glued together. The existence of such a filtration is evident. For example, after the deformation \(\xi_{\tau_1}(\alpha_1)\) remains to be a subbundle because \(\text{Ext}^0(\xi_{\tau_1}(\alpha_i), \xi_{\tau_1}(\alpha_1)) = 0\) if, \(i > 1\).

The bundle \(\nu\) can be decomposed into a sum of indecomposable ones. Suppose, that all indecomposable components of the decomposition are stable. So \(\nu = \xi_{\bar{\tau}_1}(\beta_1) \oplus \ldots \oplus \xi_{\bar{\tau}_s}(\beta_s), \bar{\tau}_1 \leq \bar{\tau}_2 \leq \ldots \leq \bar{\tau}_s\).

What sequences \(\{\bar{\tau}_1, \bar{\tau}_2, \ldots, \bar{\tau}_s\}\) can be obtained this way? To find an answer we need some preparations.

\(\Box\) Consider first the case of two bundles \(\xi_{\tau_1}(\alpha_1)\) and \(\xi_{\tau_2}(\alpha_2), 0 < \tau_1 < \tau_2, \tau_1 = n_1/k_1; \tau_2 = n_2/k_2\). The space \(W_{\tau_1, \tau_2} = \text{Ext}^1(\xi_{\tau_2}(\alpha_2), \xi_{\tau_1}(\alpha_1))\) has dimension \(n_2k_1 - k_2n_1\). Each vector \(\beta \in W_{\tau_1, \tau_2}\) determines the exact sequence:

\[0 \to \xi_{\tau_1}(\alpha_1) \to B[\beta] \to \xi_{\tau_2}(\alpha_2) \to 0.\]

Here \(B[\beta]\) is a \(k_1 + k_2\)-dimensional bundle which depends on \(\beta\). So, we have a map \(W_{\tau_1, \tau_2} \xrightarrow{1} \text{Mod}_{n_1+n_2,k_1+k_2}\).

Let us transform the exact sequence using the operator \(F\). We get:

\[0 \to \xi_{-\tau_1^{-1}}(\alpha_1^{-1}) \to FB[\beta] \to \xi_{-\tau_2^{-1}}(\alpha_2^{-1}) \to 0,\]

In other words, there is a map: \(W_{\tau_1^{-1}, \tau_2^{-1}} \xrightarrow{\Pi} \text{Mod}_{k_1-k_2,n_1+n_2}\).
Proposition. The following diagram is commutative:

\[
\begin{array}{ccc}
W_{\tau_1,\tau_2} & \xrightarrow{I} & \text{Mod}_{n_1+n_2,k_1+k_2} \\
\downarrow & & \downarrow F \\
W_{-\tau_1^{-1},-\tau_2^{-1}} & \xrightarrow{\Pi} & \text{Mod}_{-k_1-k_2,n_1+n_2}.
\end{array}
\]

The fibers of maps I, II form stratifications on the linear spaces \(W_{\tau_1,\tau_2}\) and \(W_{-\tau_1^{-1},-\tau_2^{-1}}\). Our proposition claims that these stratifications are identical. Now fix some line bundle \(\nu\) of degree \(l\). After tensoring \(\xi_{\tau_1}(\alpha_1)\) and \(\xi_{\tau_2}(\alpha_2)\) by \(\nu\) we get an isomorphism \(W_{\tau_1,\tau_2} \cong \text{Ext}^1(\xi_{\tau_2}(\alpha_2),\xi_{\tau_1}(\alpha_1))\) and \(W_{\tau_1+l,\tau_2+l} \cong \text{Ext}^1(\xi_{\tau_2+l}(\tilde{\alpha}_2),\xi_{\tau_1+l}(\tilde{\alpha}_1))\). Here \(\tilde{\alpha}_1\) and \(\tilde{\alpha}_2\) depend on \(\alpha_1, \alpha_2\) and \(\nu\). It is clear that the fibers of the maps of \(W_{\tau_1,\tau_2}\) and \(W_{\tau_1+l,\tau_2+l}\) onto the corresponding moduli spaces are the same. The result of these considerations is the following. Let \(\Gamma\) be the group \(SL_2(\mathbb{Z})\) which acts on the projective line in the usual way:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) \to \frac{a\tau+b}{c\tau+d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1.
\]

Then there is a commutative diagram:

\[
\begin{array}{ccc}
W_{\tau_1,\tau_2} & \xrightarrow{I} & \text{Mod} \\
\downarrow & & \downarrow \\
W_{g(\tau_1),g(\tau_2)} & \xrightarrow{\Pi} & \text{Mod}
\end{array}
\]

\(g \in SL_2(\mathbb{Z})\). Therefore, the stratifications on the spaces \(W_{\tau_1,\tau_2}\) and \(W_{g(\tau_1),g(\tau_2)}\) are the same.

We reformulate this is the next proposition.

Proposition. Suppose that \(\xi_{\tau_1}(\alpha_1) \oplus \xi_{\tau_2}(\alpha_2) \prec \xi_{u_1}(\beta_1) \oplus \xi_{u_2}(\beta_2) \oplus \ldots \oplus \xi_{u_s}(\beta_s)\). Here \(\tau_1, \tau_2, u_1, \ldots, u_s\) are rational numbers. Then for each \(g \in SL_2(\mathbb{Z})\)

\[
\xi_{g(\tau_1)}(\tilde{\alpha}_1) \oplus \xi_{g(\tau_2)}(\tilde{\alpha}_2) \prec \xi_{g(u_1)}(\tilde{\beta}_1) \oplus \xi_{g(u_2)}(\tilde{\beta}_2) \oplus \ldots \oplus \xi_{g(u_s)}(\tilde{\beta}_s).
\]

Note that continuous parameters change when we act by \(g\).

Rough stratification

Start with a bundle \(B = \xi_{\tau_1}(\alpha_1) \oplus \xi_{\tau_2}(\alpha_2) \oplus \ldots \oplus \xi_{\tau_p}(\alpha_p)\), \(\tau_1 \leq \tau_2 \leq \ldots \leq \tau_p, \tau_i = n_i/k_i, (n_i, k_i) = 1\). Let us form a moduli space \(\text{Mod}_{\tau_1,\tau_2,\ldots,\tau_p}\) which consists of the vector
bundles of dimension $k_1 + k_2 + k_3 + \ldots + k_p$ with a filtration $\nu_0 \subset \nu_1 \subset \nu_2 \subset \ldots \subset \nu_{p-1}$ such that $\varphi_i : \nu_i/\nu_{i-1} \cong \xi_{\tau_{i+1}}(\alpha_{i+1})$ and isomorphisms $\varphi_i$ are fixed. The moduli space $\text{Mod}_{\tau_1, \tau_2, \ldots, \tau_p}$ has a simple structure. Namely, it is a fibre bundle $\text{Mod}_{\tau_1, \tau_2, \ldots, \tau_p} \rightarrow \text{Ext}^1(\xi_{\tau_2}(\alpha_2), \xi_{\tau_1}(\alpha_1)) \oplus \text{Ext}^1(\xi_{\tau_3}(\alpha_3), \xi_{\tau_2}(\alpha_2)) \oplus \ldots \oplus \text{Ext}^1(\xi_{\tau_p}(\alpha_p), \xi_{\tau_{p-1}}(\alpha_{p-1})).$ Let $\tilde{M}$ be an arbitrary fiber of this bundle. Then it is itself a fibre bundle $\tilde{M} \rightarrow \text{Ext}^1(\xi_{\tau_3}(\alpha_3), \xi_{\tau_1}(\alpha_1)) \oplus \text{Ext}^1(\xi_{\tau_2}(\alpha_2)) \oplus \ldots \oplus \text{Ext}^1(\xi_{\tau_p}(\alpha_p), \xi_{\tau_{p-2}}(\alpha_{p-2})).$ Again we can consider the fiber of this bundle and it will be a map onto the sum of finite dimensional vector spaces.

For the sequence of rational numbers $\{s_1, s_2, \ldots, s_q\}, s_1 \leq s_2 \leq \ldots \leq s_q$ we define a submanifold $\text{Mod}^{s_1, s_2, \ldots, s_q}_{\tau_1, \tau_2, \ldots, \tau_p} \subset \text{Mod}_{\tau_1, \tau_2, \ldots, \tau_p}$. A point $y \in \text{Mod}_{\tau_1, \ldots, \tau_p}$ belongs to $\text{Mod}^{s_1, s_2, \ldots, s_q}_{\tau_1, \tau_2, \ldots, \tau_p}$ if the decomposition of the corresponding $k_1 + k_2 + \ldots + k_p$ bundle is $\xi_{s_1}(\beta_1) \oplus \xi_{s_2}(\beta_2) \oplus \ldots \oplus \xi_{s_q}(\beta_q)$, where $\beta_i \neq \beta_j$ if $s_i = s_j$. Manifold $\text{Mod}^{s_1, \ldots, s_q}_{\tau_1, \ldots, \tau_p}$ is defined as the closure of the set of such $y$. A group of symmetries of the bundle $B$ is $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* (p$ times) acts on $\text{Mod}_{\tau_1, \tau_2, \ldots, \tau_p}$. The submanifolds $M^{s_1, \ldots, s_q}_{\tau_1, \ldots, \tau_p}$ are stable with respect to this action. It is not hard to show that $M_{\tau_1, \ldots, \tau_p}$ is the union of all $M^{s_1, \ldots, s_q}_{\tau_1, \ldots, \tau_p}$. It is clear why we call this stratification “rough”. Suppose that $s_1 = s_2 = \frac{p}{q}$.

Let us consider the manifold $U$ which is $M^{s_1, \ldots, s_q}_{\tau_1, \ldots, \tau_p}$ minus all strata which are smaller. It is possible that $U$ contains the point such that the corresponding vector bundle is $\xi_{2p, 2q}(\beta_1) \oplus \xi_{s_3}(\beta_3) \oplus \ldots \oplus \xi_{s_q}(\beta_q)$. So we collect all such strata into one. The bundles of “generic” type where all components in the decomposition into a sum of indecomposable are stable constitute the open set in the strata.

© Characteristic manifold and the ordering on the set of sequences of rational numbers. The weak form of the question about the ordering on the set of types of bundles is following.

Definition. Suppose $\{\tau_1 \leq \tau_2 \leq \ldots \leq \tau_p\} = R$ and $\{s_1, \leq \ldots, \leq s_q\} = S$ are two sequences of rational numbers, $\tau_i = n_i/k_i, s_j = \bar{n}_j/\bar{k}_j$ such that $\Sigma n_i = \Sigma \bar{n}_i$ and $\Sigma k_i = \Sigma \bar{k}_i$. We will say that $R \prec S$ if the manifold $\text{Mod}^{s_1, \ldots, s_q}_{\tau_1, \ldots, \tau_p}$ is non-empty.

Definition. Characteristic manifold in $\text{Mod}_{\tau_1, \tau_2, \ldots, \tau_p}$ is the union of all strata $\text{Mod}^{s_1, s_2, \ldots, s_q}_{\tau_1, \tau_2, \ldots, \tau_p}$ such that if $\text{Mod}^{s_1, s_2, \ldots, s_q}_{\tau_1, \tau_2, \ldots, \tau_p} \supset \text{Mod}^{\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_l}_{\tau_1, \ldots, \tau_p} = Y$ then $Y$ is a one-point stratum.
Mod $\tau_1, \tau_2, \ldots, \tau_p$. Therefore the characteristic manifold consists of the “smallest” possible strata. If $R = \{\tau_1, \ldots, \tau_p\}$, and $S = \{s_1, \ldots, s_q\}$ is a stratum from the characteristic manifold we will write down $R \mapsto S$.

**Proposition.** Sequence $R < S$ if and only if there is a set of sequences $R \mapsto U_1 \mapsto U_2 \mapsto \ldots \mapsto U_l \mapsto S$.

**Description** of the relation $\mapsto$. Let $R = \{0, \tau\}$, $\tau > 1$. First we write down $\tau = \frac{n}{k}$ as a continuous fraction: $n/k = (n_1, n_2, \ldots, n_t)$. The notation $(n_1, n_2, \ldots, n_t)$ means:

$$(n_1, n_2, \ldots, n_t) = n_1 - \frac{1}{n_2 - \frac{1}{n_3 - \frac{1}{\ldots - \frac{1}{n_t}}}}$$

Here all $n_i \geq 2$. The numbers $n_1, \ldots, n_t$ are determined by $n, k$.

**Proposition.** The relation $\mapsto$ connects the pair $\{0, \tau\}$ with the only one sequence $\{s_1, \ldots, s_t\}$, where $s_1 = 1, s_2 = n_1 - 1, s_3 = (n_1, n_2 - 1), s_4 = (n_1, n_2, n_3 - 1), \ldots, s_t = (n_1, n_2, \ldots, n_t - 1)$.

Now suppose that the pair $R$ is $\{\tau_1, \tau_2\}$, $\tau_1 \leq \tau_2$. There is again only one sequence $S = \{s_1, \ldots, s_t\}$ such that $R \mapsto S$. To find it we can use the action of $SL(2, \mathbb{Z})$. So there is $g \in SL(2, \mathbb{Z})$ such that $g(\tau_1) = 0$, $g(\tau_2) > 1$ and we first transform the pair $\{\tau_1, \tau_2\}$ into $\{0, g(\tau_2)\}$, then go to the sequence $\{\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_t\}$ according to the previous proposition and after that we return back with help of $g^{-1}$: $\{s_1, \ldots, s_t\} = \{g^{-1}(\bar{S}_1), g^{-1}(\bar{S}_2), \ldots, g^{-1}(\bar{S}_t)\}$. Combining all this arguments we get the next proposition.

**Proposition.** The relation $\mapsto$ connects the pair $\{\tau_1, \tau_2\}$ with the only one sequence $\{s_1, s_2, \ldots, s_t\}$ which is calculated in the following way. Suppose first that the integreal part of $\tau_1$ is less then the integreal part of $\tau_2$. Write down $\tau_1$ as $m_1 + (m_2, m_3, \ldots, m_a)^{-1}$ and $\tau_2$ as $(n_1, n_2, \ldots, n_b)$. Then $\{s_1, s_2, \ldots, s_t\}$ is $\{m_1 + (m_2, m_3, \ldots, m_a - 1)^{-1}, m_1 + (m_2, m_3, \ldots, m_a - 1)^{-1}, \ldots, m_1 + (m_2 - 1)^{-1}, m_1 + 1, n_1 + 1, (n_1, n_2 - 1), (n_1, n_2, n_3 - 1), \ldots, (n_1, n_2, \ldots, n_b - 1)\}$. Here we suppose that $m_2, m_3, \ldots \geq 2$.

General case is clear because of the acting $SL_2(\mathbb{Z})$. 11
Remark. Suppose that \( \tau_1 = p_1/q_1, \tau_2 = p_2/q_2 \) such that \( p_2q_1 - q_2p_1 = 1 \) then the sequence \( \{s_1 \ldots s_t\} \) consists only of one term. So \( \{\tau_1, \tau_2\} \mapsto \{(p_1 + p_2)/(q_1 + q_2)\} \). Now we describe relation \( \mapsto \) in the general case.

**Proposition.** The sequences \( \{\tau_1, \ldots, \tau_p\} \mapsto \{s_1, s_2, \ldots, s_q\} \) if the sequence \( \{s_i\} \) can be obtained by the following construction. Let us fix some pair of \( \{\tau_i, \tau_{i+1}\} \). Then using the rule from the previous proposition write the sequence \( S = \{f_1, \ldots, f_l\}, \{\tau_i, \tau_{i+1}\} \mapsto S \).

Then the sequence \( \{s_1, s_2, \ldots, s_q\} \) will be

\[
\{\tau_1, \tau_2, \ldots, \tau_{i-1}, f_1, f_2, \ldots, f_l, \tau_{i+2}, \ldots, \tau_p\}.
\]

Using this rule it is possible to find all rough strata in the manifold \( \text{Mod}_{\tau_1, \tau_2, \ldots, \tau_s} \). They are labeled by the sequences \( \{s_1, \ldots, s_q\} \) which are smaller than \( \{\tau_1, \ldots, \tau_s\} \). All such sequences can be described with the help of the system of inequalities. We will write them in the special case of \((0, \tau)\)-pair.

**Proposition.** Let \( \tau \) be \( n/k, (n, k) = 1 \), then the sequences \( S = \{p_1/q_1, p_2/q_2, \ldots, p_l/q_l\}, (p_i, q_i) = 1, p_i/q_i \leq p_{i+1}/q_{i+1} \) for \( i = 1, 2, \ldots, l - 1 \) which satisfy the condition \((0, \tau) \prec S\) can be characterized as a solution of the following system:

\[
0 < p_1/q_1, p_l/q_l < n/k, p_1 > 0, p_2 + p_1q_2 - p_2q_1 > 0, p_3 + (p_1 + p_2)q_3 - p_3(q_1 + q_2) > 0 \ldots p_l + (p_1 + p_2 + \ldots + p_{l-1})q_l - p_l(q_1 + q_2 + \ldots + q_{l-1}) > 0;
\]

\[
p_1 + \cdots + p_l = n, q_1 + \cdots + q_l = k + 1.
\]

**Example (1).** The rough strata which are more than the fixed one constitute the set with partial ordering.

(a) \( N \) is even, then \( \{0, N\} \mapsto \{1, N - 1\} \mapsto \ldots \mapsto \{\frac{N}{2}, \frac{N}{2}\} \)

(b) \( N \) is odd then \( \{0, N\} \mapsto \{1, N - 1\} \mapsto \ldots \mapsto \{\frac{N-1}{2}, \frac{N+1}{2}\} \mapsto \{\frac{N}{2}\} \)

(c) \( \{0, \frac{5}{2}\} \mapsto \{1, 2, 2\} \mapsto \{\frac{3}{2}, 2\} \mapsto \{\frac{5}{2}\} \)

(d) \( \{0, \frac{17}{2}\} \mapsto \{1, 8, 8\} \mapsto \{2, 7, 8\} \mapsto \{3, 6, 8\} \mapsto \{4, 5, 8\} \mapsto \{\frac{9}{2}, 8\} \mapsto \{5, 5, 7\} \mapsto \{5, 6, 6\} \mapsto \{4, \frac{13}{3}\} \mapsto \{2, \frac{15}{2}\} \mapsto \{3, 7, 7\} \mapsto \{4, 6, 7\} \mapsto \{4, \frac{13}{3}\}\)
(e) In this point we start with a pair \(0, (n_1, n_2)\) where \(n_1, n_2\) are sufficiently big.

\[
\begin{align*}
&\{0, (n_1, n_2)\} \mapsto \{1, n_1 - 1, (n_1, n_2 - 1)\} \mapsto \{2, n_1 - 2, (n_1, n_2 - 1)\} \\
&\mapsto \{1, n_1 - \frac{1}{2}, (n_1, n_2 - 2)\} \mapsto \{2, n_1 - 1, n_1 - 1, (n_1, n_2 - 2)\} \\
&\mapsto \{1, n_1 - 1/3, (n_1, n_2 - 3)\} \mapsto \{5, n_1 - 5, (n_1, n_2 - 1)\} \\
&\mapsto \{4, n_1 - 4, (n_1, n_2 - 1)\} \mapsto \{4, n_1 - 3, n_1 - 1, (n_1, n_2 - 2)\} \\
&\mapsto \{3, n_1 - 2, n_1 - 1, (n_1, n_2 - 2)\} \mapsto \{3, n_1 - \frac{3}{2}, (n_1, n_2 - 2)\} \\
&\mapsto \{3, n_1 - 2, n_1 - \frac{1}{2}, n_1 - \frac{1}{n_2-3}\} \mapsto \{2, (n_1, 2, 2), (n_1, n_2 - 3)\} \\
&\mapsto \{2, n_1 - 1, n_1 - \frac{1}{2}, (n_1, n_2 - 3)\} \mapsto \{2, n_1 - 1, n_1 - \frac{1}{3}, (n_1, n_2 - 4)\} \\
&\mapsto \{1, n_1 - \frac{1}{3}, (n_1, n_2 - 5)\}
\end{align*}
\]

**Example (2).** Here we consider \((9/2)\) case. The space \(\text{Ext}^1(\xi_0, \xi_{9/2})\) is decomposed according to the type of corresponding bundle. So \(\mathbb{C}^0\) is a union of submanifolds.

(a) origin where decomposition is \(\xi_0 \oplus \xi_{9/2}\)

(b) characteristic manifold. It has dimension three, and the bundle is a sum \(\xi_1(\alpha_1) \oplus \xi_4(\alpha_2) \oplus \xi_4(\alpha_3)\), if \(\alpha_3 \neq \alpha_4\) and \(\xi_1(\alpha_1) \oplus \xi_{9/2}(\alpha_2)\) if \(\alpha_2 = \alpha_3\), where the sum \(\alpha_1 + \alpha_2 + \alpha_3 = \mu\) is fixed and equal to the determinant of the bundle \(\xi_{9/2}\).

(c) next piece is a 5-dimensional manifold, where decomposition is \(\xi_2(\alpha_1) \oplus \xi_3(\alpha_2) \oplus \xi_4(\alpha_3), \alpha_1 + \alpha_2 + \alpha_3 = \mu\). The set which corresponds to fixed \(\alpha_1, \alpha_2, \alpha_3\) has dimension 3.

(d) two 6-dimensional manifolds where decomposition is \(\xi_2(\alpha_1) \oplus \xi_{7/2}(\alpha_2), \alpha_1 + 2\alpha_2 = \mu\) and \(\xi_{5/2}(\alpha_1) \oplus \xi_4(\alpha_2), 2\alpha_1 + \alpha_2 = \mu\).
(e) 9-dimensional part, where decomposition is $\xi_3(\alpha_1) \oplus \xi_3(\alpha_2) \oplus \xi_3(\alpha_3)$, $\alpha_1 + \alpha_2 + \alpha_3 = \mu$.

If $\alpha_1, \alpha_2, \alpha_3$ are all different then corresponding submanifold in $\mathbb{C}^9$ has dimension 9.

Suppose $\alpha_1 = \alpha_2$. Then for fixed $\alpha_1 = \alpha_2, \alpha_3 \neq \alpha_1$ the submanifold has dimension 5.

The stratum where we have $\xi_{6/2}(\beta) \oplus \xi_3(\alpha)$ has dimension 6. Here $\beta$ and $\alpha$ are chosen such that $\text{Hom}(\xi_{6/2}(\beta), \xi_3(\alpha)) = 0$. If $\text{Hom}(\xi_{6/2}(\beta), \xi_3(\alpha)) = \mathbb{C}$ then the dimension is 4.

The last possibility $\xi_3(\alpha) \oplus \xi_3(\alpha) \oplus \xi_3(\alpha)$. Here $3\alpha = \mu$ and for each such $\alpha$ (we have 9 possibilities) it is a line in $\mathbb{C}^9$.

§2. Construction of the algebras

I. The “symmetric” algebra associated with $R$-matrix (due to Lusztig · · ·)

Fix the vector space $V$ and $R$-matrix $V \otimes V \to V \otimes V$. We define first the so-called “exchange” algebra associated with $R$. Consider the algebra $A$ with the space of generators $V[i], i \in \mathbb{Z}$. Each $V[i]$ is isomorphic to $V$. Note, that the $R$-matrix defines a subspace $W \in V \otimes V \oplus V \otimes V$ which consists of pairs $v_1 \otimes v_2 - R(v_1 \otimes v_2)$; let $W_{ij}$ be the same space in $V[i] \otimes V[j] \oplus V[j] \otimes V[i]$. The defining relations in $A$ are: (a) $V[i] \otimes V[i]$, it means that $v_1 v_2 = 0$ if $v_1, v_2 \in V[i]$. (b) $W_{ij} \subset V[i] \otimes V[j] \oplus V[j] \otimes V[i], i < j$. It is easy to see that as a vector space $A = \oplus V[i_1] \otimes V[i_2] \otimes \cdots \otimes V[i_n]$ where the sum goes over all sequences $i_1 < i_2 < \cdots < i_n$. If we replace the direct sum by the product we get an algebra $\bar{A} = \prod V[i_1] \otimes \cdots \otimes V[i_n]$. The element from $\bar{A}$ is a function from the set of sequences $\{i_1 < i_2 < \cdots < i_n\}$ into $V \otimes V \otimes \cdots \otimes V$ (n times). The product of such infinite expressions is well-defined. Now let us construct the maps $\theta_n : V \otimes \cdots \otimes V \to \bar{A}$. For $v \in V$ we denote by $v[i]$ the corresponding element from $V[i]$. The maps are: $\theta_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \sum v_1[i_1] \cdot v_2[i_2] \cdots \cdot v_n[i_n], i_1 < i_2 < \cdots < i_n$. Denote by $T(V)$ the direct sum $\mathbb{C} \oplus V \oplus V \otimes V \oplus \cdots, \{\theta_n\}$ constitute a map $T(V) \to \bar{A}$ and the image is a subalgebra. Our notation for $T(V)$ with this multiplicative structure will be $T_R(V)$. A “Symmetric” algebra associated with $R$-matrix is a sub-algebra $S_R(V) \subset T_R(V)$ which is generated by $V \subset T(V)$. If $R$ is just permutation the $T_R(V)$ is a commutative algebra and $S_R(V)$ is a polynomial ring. If $R$ is unitary in the sense that $R^2 = 1$ then $S_R(V)$ has the same size as a polynomial ring. In the case when $R^2 \neq 1$, $S_R(V)$ is bigger than a polynomial algebra, and for “generic” $R$, $S_R(V)$ is isomorphic to a free algebra.
Algebras $T_R(V)$ and $S_R(V)$ bear the natural structure of the braided Hopf algebras. Namely fix in $\bar{A}$ two subalgebras $T_1$ and $T_2$. The first one is spanned by the elements $\sum v_1[i_1] \cdot v_2[i_2] \cdots v_n[i_n], i_1 < i_2 < \cdots < i_n \leq 0$. The second has a basis $\sum v_1[i_1] \cdot v_2[i_2] \cdots v_m[i_m], 0 < i_1 < i_2 < \cdots < i_m$. It is clear that $T_1$ and $T_2$ are isomorphic to $T_R(V)$, $T_1$ and $T_2$ generate in $\bar{A}$ the subalgebra $T_R(V) \otimes T_R(V)$. This algebra has the same size as a usual tensor product $T_R(V) \otimes T_R(V)$, but the multiplication is twisted in the evident way. (For example, the spaces $V \otimes 1$ and $1 \otimes V$ from the $T_R(V) \otimes T_R(V)$ are not commute but satisfy $R$-matrix permutation relation). $T_R(V) \otimes T_R(V)$ is called the braided tensor product of two copies of $T_R(V)$. Algebra $T_R(V)$ is a subalgebra in $T_1 \otimes T_2$, therefore we get a comultiplication map $T_R(V) \rightarrow T_R(V) \otimes T_R(V)$. The same arguments can be applied to $S_R(V)$. As a result $S_R(V)$ has a structure of braided Hopf algebra. It means that there is a comultiplication map $\Delta : S_R(V) \rightarrow S_R(V) \otimes S_R(V)$ where $\otimes$ is the twisted tensor product and $\Delta$ on the generators is given by the formula $\Delta(v) = v \otimes 1 + 1 \otimes v$.

**Example 1.** Consider the case when $R$ is diagonal. So $V$ has a basis $\{v_1, v_2, \cdots, v_n\}$ such that $R(v_i \otimes v_j) = C_{ij}(v_j \otimes v_i), C_{ij} \neq 0$. If matrix $\{C_{ij}\}$ satisfies the symmetry condition $C_{ij}C_{ji} = 1$ then the algebra $S_R(V)$ is an algebra of skew polynomials with generators $V_i$ and relations $v_i v_j = C_{ij} v_j v_i$. If $C_{ij} = B_{ij} \cdot D_{ij}$ where $D_{ij}D_{ji} = 1$ then the “symmetric” algebras which correspond to $\{C_{ij}\}$ and $\{B_{ij}\}$ are very close. More precisely: let $S(C)$ will be the first and $S(B)$-the second. All algebras $S(C), S(B)$ and the algebra of skew polynomials $S(D)$ have a $\Gamma$-grading, where $\Gamma$ is a lattice with basis $\{e_i\}; \deg v_i = e_i$. Then $S(C)$ is a subalgebra in $S(B) \otimes S(D)$ which is generated by the products $v_i \otimes v'_i$ ($v_i$-generators of $S(B), v'_i$-of $S(D)$). It is evident the $S(C)$ has the same size as $S(B)$ and coincides with the subalgebra in $S(B) \otimes S(D)$ of elements of degree zero ($\deg v_i = e_i, \deg v'_i = -e_i$). This all means that the most interesting case is when the matrix $\{C_{ij}\}$ is symmetric. In this case if we write down $C_{ij} = q^{c_{ij}},$ where $\{c_{ij}\}$ is a matrix of scalar products of the system of simple roots of some Kac-Moody algebra, then algebra $S_R(V)$ is isomorphic to the universal enveloping algebra of the maximal nilpotent subalgebra in the $q$-deformed Kac-Moody algebra.

II. Algebra which is defined by diagonal $R$-matrix in “continuous” case.
We apply now the scheme of the point I in the case when $R$ is diagonal but $V$ is the space of functions in one variable. If $V$ is the space of polynomials then $V \otimes V$ is the space of polynomials in two variables. We do not want to specify what functions are in $V$. And we denote by $V \otimes V$ the space of functions in two variables. Now fix a function $\lambda(x, y)$ in two variables and let $R$-matrix $R_\lambda$ acts $V \otimes V \to V \otimes V$ by the formula $R_\lambda(g(x, y)) = g(y, x) \cdot \lambda(x, y) \cdot \lambda^{-1}(y, x), g(x, y) \in V \otimes V$. Formally $R_\lambda \cdot R_\lambda^{-1} = 1$, but the algebra $S_{R_\lambda}(V)$ can be more complicated than the ring of skew polynomials. The reason is that we have to replace the usual tensor product by the “extended” one $\tilde{\otimes}$. First we construct the bigger algebra $\tilde{S}_{R_\lambda}(V) = \mathbb{C} \oplus S_1 \oplus S_2 \oplus \cdots, S_j$ consists of the symmetric functions in $j$ variables. The product is given by the formula:

$$f \ast g(X_1, \cdots, X_{\alpha+\beta}) = \frac{1}{\alpha! \cdot \beta!} \text{Symm} \left[ \prod_{1 \leq i \leq \alpha \atop \alpha < j \leq \alpha+\beta} \lambda(X_i, X_j) \right] \cdot f(X_1, \cdots, X_\alpha) \cdot g(X_{\alpha+1} \cdots, X_{\alpha+\beta}).$$

Here symbol Symm means the symmetrization with respect to the symmetric group acting on the set of $\alpha + \beta$ variables.

**Definition.** Algebra $S_{R_\lambda}(V)$ (to simplify the notations $S_\lambda$) is the subalgebra in $\tilde{S}_{R_\lambda}(V)$ generated by $S_1$.

**Examples.** If the function $\lambda = \frac{x - qy}{x-y}$ then the algebra $S_\lambda$ coincides with a subalgebra in the universal enveloping of $q$-deformed $\hat{sl}_2$. This subalgebra is a $q$-version of the universal enveloping algebra of the algebra of currents on the line into the maximal nilpotent subalgebra of $sl_2$.

We need also the shifted version of the “symmetric” algebra. Suppose the $R$-matrix $R : V \otimes V \to V \otimes V$ admit symmetries. In the simplest case it means that some operator $A : V \otimes V$ has a property $R \circ (A \otimes A) = (A \otimes A) \circ R$. Operator $A$ is extended to the automorphism $A : S_R(V) \to S_R(V)$. In such a situation we can add an element $a$ to the algebra $S_R(V)$ with the permutation relations: $a \cdot u = A(u) \cdot a$, where $u \in S_R(V)$. As a result we get semi-direct product $\mathbb{C}[a] \ltimes S_R(V)$. The shifted algebra $S_{R,A}(V)$ is by definition a subalgebra in $\mathbb{C}[a] \ltimes S_R(V)$ which is generated by the subspace $a \cdot V$. It is clear
that $S_{R,A}(V)$ has the same size as $S_R(V)$. Let us apply this construction in the case, when $V$ is the space of functions on the line and $A$ is a shift $f(z) \to f(z - p)$. This operator commutes with diagonal $R$-matrix if $\lambda(x, y) = \lambda(x + p, y + p)$. Suppose $\lambda$ satisfies this condition, then the product in $S_\lambda$ is deformed in the following way:

$$f \star g(x_1, x_2, \cdots, x_{\alpha + \beta}) = \frac{1}{\alpha!\beta!} \operatorname{Symm}\left( \prod_{1 \leq i < j \leq \alpha + \beta} \lambda(x_i, x_j) f(x_1, \cdots, x_\alpha) g(x_{\alpha + 1} - \alpha p, x_{\alpha + 2} - \alpha p, \cdots, x_{\alpha + \beta} - \alpha p) \right).$$

We denote the “shifted” algebra by $S_{\lambda, p}$.

III.

**Proposition.** Suppose that $\tau$ is not a point of finite order in $\mathcal{E}$. The algebra $Q_{n,1}(\mathcal{E}, \tau)$ is a subalgebra in $S_{\lambda, p}$, where $p = 2\tau$ and $\lambda(x, y) = \theta(x - y - n\tau)/\theta(x - y)$. The space of generators of $Q_{n,1}(\mathcal{E}, \tau)$ is the space of $\theta$-functions of order $n$.

Here $\theta$- is a $\theta$-function of order one which has zero at the origin. The meaning of this proposition is that the algebra $Q_{n,1}(\mathcal{E}, \tau) = \mathbb{C} \oplus B_1 \oplus B_2 \oplus \cdots$ has the following realization. The space of generators $B_1$ is identified with the space $H^0(\xi)$, where $\xi$ is a line bundle of degree $n$. The space $B_j$ is isomorphic to the space of sections of the line bundle $S^p \xi^{(1-p)\tau}$. Here $\xi^{(1-p)\tau}$ is the line bundle $\xi$ shifted by the map $\mathcal{E} \to \mathcal{E}, x \to x + (p - 1)\tau$; $S^p \xi^{(1-p)\tau}$ is a symmetric power of $\xi^{(1-p)\tau}$, therefore the sections of $S^p \xi^{(1-p)\tau}$ are symmetric $\theta$-functions of order $n$ in $p$ variables.

Now we present the construction of the algebra $Q_{n,1}(\mathcal{E}, \tau)$ which is the version of the constructions from point I (with some modifications).

Let $A$ be an algebra with two generators $e, u$ and relations $e \cdot u = (u + (n - 2)\tau) \cdot e, \tau \in \mathbb{C}$. Define the braided tensor product of two copies of $A : A_1$ and $A_2$, namely, $A_1 \otimes A_2$ is an algebra with generators $e_1, u_1; e_2, u_2$ and relations: $e_1 u_1 = (u_1 + (n - 2)\tau)e_1, e_2 u_2 = (u_2 + (n - 2)\tau)e_2, e_1 u_2 = (u_2 - 2\tau)e_1, e_2 u_1 = (u_1 - 2\tau)e_2, [u_1, u_2] = 0$, and $e_1 e_2 = -\exp(2\pi i(u_2 - u_1)) e_2 e_1$. We have two maps $\mu_1 : A \otimes A \cong (A_1 \otimes 1) \cdot (1 \otimes A_2) \to A_1 \otimes A_2$ and $\mu_2 : (1 \otimes A_2) \cdot (A_1 \otimes 1) \cong A \otimes A \to A_1 \otimes A_2$. The map $\mu_1 \circ \mu_2^{-1} : A \otimes A \to A \otimes A$ is a sort of $R$-matrix and it is possible to apply the technique from I. The result is that $Q_{n,1}(\mathcal{E}, \tau)$ is a subalgebra in $A \otimes A \otimes \cdots \otimes A$. 

17
Proposition. Let $B$ be an algebra with generators $\{e_\alpha, u_\alpha\}, \alpha \in I$, (the set of indices $I$ can be infinite). The relations are: $e_\alpha u_\beta = (u_\beta - 2\tau)e_\alpha, (\alpha \neq \beta), e_\alpha u_\alpha = (u_\alpha + (n - 2)\tau)e_\alpha, e_\alpha e_\beta = -e^{-2\pi i(u_\beta-u_\alpha)}\theta(u_\alpha-u_\beta-nt\tau)\epsilon_\beta e_\alpha, u_\alpha u_\beta = u_\beta u_\alpha$. There is a homomorphism from the algebra $Q_{n,1}(\mathcal{E}, \tau)$ into $B$. The generator $f$ from $Q_{n,1}(\mathcal{E}, \tau)$ (which is $\theta$-function of order $n$) goes to $\sum_{\alpha} f(u_\alpha) \cdot e_\alpha \in B$.

The realization of $Q_{n,1}(\mathcal{E}, \tau)$ as a space of symmetric $\theta$-functions we call the functional realization. The center of $Q_{n,1}(\mathcal{E}, \tau)$ has a nice description in these terms.

Proposition. (a) Suppose $n$ is even. Then the center $Z$ of algebra $Q_{n,1}(\mathcal{E}, \tau)$ is a polynomial algebra with two generators of degree $n/2$. This space of generators coincides with the subspace $W_1 \subset S^{n/2}\xi^{(1-n/2)\tau}$ which consists of $\theta$-functions $g(z_1, \cdots, z_{n/2})$ such that $g(z_1, z_1 - n\tau, z_3, \cdots, z_{n/2}) = 0$. The quadratic expressions of generators constitute a 3-dimensional subspace $W_2 \subset S^{n}\xi^{(1-n)\tau}; W_2$ is a space of functions $g(z_1, \cdots, z_n)$ with condition $g(z_1, z_1 - n\tau, z_1 - 2n\tau, z_4, \cdots, z_n) = 0; W_s \subset S^{ns/2}\xi^{(1-ns/2)\tau}$ is a space of $g(z_1, \cdots, z_{ns/2})$, the condition is: $g(z_1, z_1 - n\tau, \cdots, z_1 - sn\tau, z_{s+2}, \cdots, z_{ns/2}) = 0$.

(b) Suppose $n$ is odd. Then the center $Z$ of the algebra $Q_{n,1}(\mathcal{E}, \tau)$ is generated by one element $\Delta$ of degree $n$, $\Delta$ is represented by the function $g(z_1, \cdots, z_n)$ which is zero on the diagonal $(z_1, z_1 - n\tau, z_1 - 2n\tau, z_4, \cdots, z_n)$. There is only one function (up to a constant) with this property. Note that element $\Delta^l$ is a function in $n \cdot l$ variables $z_1, \cdots, z_{nl}$. It is zero if $z_1 = z_2 + n\tau, z_2 = z_3 + n\tau, \cdots z_{2l} = z_{2l+1} + n\tau$.

IV. Generalization.

We apply now our construction “with function $\lambda$” to find new algebras. Let $a$ be the following Lie superalgebra: $a = a_1 \oplus a_2, a_1$ is odd part, $a_2$ is even, $[\alpha, \beta] = 0$ if $\alpha, \beta \in a_2$ or $\alpha \in a_1, \beta \in a_2$. We identify the space $a_1$ with the space of sections of the line bundle $\xi, \deg \xi = n$, and $a_2 = H^0(\xi^2)$. So, $\dim a_1 = n, \dim a_2 = 2n$. The bracket $a_1 \otimes a_1 \to a_2$ is the product of the sections. The universal enveloping algebra of $a$ is a graded algebra (deg $a_1 = 1, \deg a_2 = 2$) and its Hilbert function is equal to $(1+t)^n(1-t^2)^{-2n} = (1-t)^{-n}(1-t^2)^{-n}$. We want to construct now the elliptic deformation $T_n(\mathcal{E}, \tau)$ of the $U(a)$. The algebra $T_n(\mathcal{E}, \tau) = C \oplus A_1 \oplus A_2 \oplus \cdots$, where $A_1 = H^0(\xi), A_\rho$
is the space of meromorphic sections of the bundle $\Lambda^p \xi^{- (p-1) \tau}$ on $S^p E = E^p / S_p$, which satisfy the properties: 1) $f(z_1, \ldots, z_p)$ is skew-symmetric and has a pole of order $\leq 1$ on the diagonal $z_1 = z_2$ and is holomorphic outside the diagonal; 2) $f(z_1, \ldots, z_p) = 0$ if $z_1 = z_2 + n\tau = z_3 + 2n\tau$. The multiplication of two elements $f \in A_\alpha, g \in A_\beta$ is given by the formula:

$$
\begin{align*}
   f * g(x_1, x_2, \ldots, x_{\alpha+\beta}) & = \frac{1}{\alpha!\beta!} \text{Symm} \left( \prod_{\substack{1 \leq i \leq \alpha \leq \alpha+\beta \leq \beta}} \frac{\theta(x_i - x_j + 2n\tau)\theta(x_i - x_j - n\tau)}{\theta(x_i - x_j)^2} \right) f(x_1, \ldots, x_\alpha) \\
   & \cdot g(x_{\alpha+1} + 2\alpha\tau, x_{\alpha+2} + 2\alpha\tau, \ldots, x_{\alpha+\beta} + 2\alpha\tau)
\end{align*}
$$

Here $\text{Symm}$ is the operator of skew-symmetrization. It is a homomorphism from $T_n(E, \tau)$ into the algebra with generators $\{u_\alpha, e_\alpha\}, \alpha \in I$ (I-an arbitrary set of indices) and relations:

$$
\begin{align*}
   [u_\alpha, u_\beta] = 0; e_\alpha u_\beta = (u_\beta + 2\tau)e_\alpha, \text{ in this relation } \alpha \neq \beta; e_\alpha u_\alpha = (u_\alpha - (2n - 2)\tau)e_\alpha; \\
   e_\alpha e_\beta = -e^{4\pi i (u_\beta - u_\alpha)} \cdot \frac{\theta(u_\alpha - u_\beta + 2n\tau)\theta(u_\alpha - u_\beta - n\tau)}{\theta(u_\beta - u_\alpha + 2n\tau)\theta(u_\beta - u_\alpha - n\tau)} e_\beta e_\alpha
\end{align*}
$$

The generator $f \in A_1 = H^0(\xi)$ goes to the sum $\sum_\alpha f(u_\alpha) \cdot e_\alpha$.

**Remark 1.** Our definition of the algebra $T_n(E, \tau)$ works in the case when $\tau$ is not a point of finite order in $E$. If $\tau$ is a point of finite order we can define $T_n(E, \tau)$ as a limit. But the procedure of finding out the limit can be performed in different ways. For example, if $\tau \to 0$ we can get in the limit the $U(\mathfrak{a})$ or the skew-commutative algebra with $n$ odd generators of degree 1 and $2n$ even generators of degree 2. If $\tau \neq 0$ is a point of second order and $n$ is odd, then $T_n(E, \tau)$ degenerates into polynomials in $n$ generators of degree 1 and $n$ generators of degree 2.

**Remark 2.** If $n = 3$ algebra $T_3(E, \tau)$ is generated by three elements of degree one $\{x_\alpha, \alpha \in \mathbb{Z}_3\}$ and relations can be written as:

$$
\begin{align*}
   \theta_0(4\tau)x_\alpha^2 + \theta_1(4\tau)x_\alpha x_{\alpha+1} + \theta_2(4\tau)x_{\alpha+1}x_{\alpha+2} & = C_\alpha, \alpha = 0, 1, 2 \\
   e^{6\pi i \theta_0(3\tau)x_\beta} C_{\alpha+\beta} & = \theta_\alpha(\tau)\theta_1(2\tau)\theta_2(2\tau) C_{\alpha+\beta} x_\beta + \theta_{\alpha+1}(\tau)\theta_0(2\tau) \theta_1(2\tau) \times C_{\alpha+\beta+1}x_{\beta+1} + \theta_{\alpha+2}(\tau) \cdot \theta_2(2\tau) \theta_0(2\tau) C_{\alpha+\beta+2} x_{\beta+2}; \alpha, \beta \in \mathbb{Z}_3.
\end{align*}
$$
It is clear that the quotient $T_3(\mathcal{E}, \tau)/J \cong Q_3(\mathcal{E}, 4\tau)$, where $J$ is an ideal generated by $C_\alpha$. Note also that for generic $\tau$ algebra $T_3(\mathcal{E}, \tau)$ has two central elements of degree 3 and 6 and center is generated by these elements.

V. Further Generalizations.

Let $\Gamma$ be a root system, $\{\delta_1, \cdots, \delta_h\} \subset \Gamma^+$-simple positive roots, $(A_{ij})$, $1 \leq i, j \leq h$-the Cartan matrix and $b_{ij}$-the matrix of scalar products of the simple roots, $A_{ij} = 2b_{ij}/b_{ii}$. We denote by $L$ the lattice with the basis $\{\delta_1, \cdots, \delta_h\}, L \cong \mathbb{Z}^h$. We will construct an associative $L$-graded algebra which is an elliptic deformation of the universal enveloping algebra of the following Lie algebra. Let $\mathfrak{G}$ be the Kac-Moody algebra with the root system $\Gamma$, $\mathfrak{G} = N_+ \oplus f \oplus N_-$-the Cartan decomposition. Let $\{g_\gamma, \gamma \in \Gamma^+\}$ be a basis in $N_+$ and $\varphi_{\gamma_1, \gamma_2}$-the structure constant $[g_{\gamma_1}, g_{\gamma_2}] = \varphi_{\gamma_1, \gamma_2} \cdot g_{\gamma_1 + \gamma_2}$. We denote by $n$ the dominant weight of the $\mathfrak{G}$, $n_i = \langle n, \delta_i \rangle \geq 0$. Our Lie algebra $L(\mathcal{E}, n) = L_0 \oplus (\oplus L_\gamma), \gamma \in \Gamma_+$. We assign to each positive root $\gamma$ a line bundle $\xi(\gamma)$ on $\mathcal{E}$ such that $\xi(\gamma_1) \otimes \xi(\gamma_2) = \xi(\gamma_1 + \gamma_2)$ if $\gamma_1 + \gamma_2$ is a root and $\dim H^0(\xi(\delta_i)) = n_i$. The space $L_0$ has a basis $\{t_1, t_2, \cdots, t_h\}$ and it is an abelian Lie algebra; $L_\gamma \cong H^0(\xi(\gamma))$. The bracket $L_{\gamma_1} \otimes L_{\gamma_2} \rightarrow L_{\gamma_1 + \gamma_2}$ is: $[f_1, f_2] = g_{\gamma_1, \gamma_2} \cdot f_1 \cdot f_2$, where $f_1 \cdot f_2$ is just a product of sections. The bracket $[t_i, f_\gamma] = \frac{1}{n_i} \langle \delta_i, \gamma \rangle f_\gamma$, where $f_\gamma \in L_\gamma$, $1 \leq i \leq h$ and $\langle \cdot, \cdot \rangle$-the scalar product on the root space. Let $K$ be the field of meromorphic functions in variables $t_1, t_2, \cdots, t_h$ and $U = K \otimes C[t_1, \cdots, t_h] U(L(\mathcal{E}, n))$. This means that we add to $U(L(\mathcal{E}, n))$ the “arbitrary” functions in $t_1, \cdots, t_h$.

Now we construct the $L$-graded algebra $Q_{n, \Gamma}(\mathcal{E}, \tau)$ which is a graded deformation of $U$. $Q_{n, \Gamma}(\mathcal{E}, \tau) = K \oplus (\oplus l F_l), l$ is the sum $\sum l_i \delta_i, l_i \geq 0$. The relation between $K$ and $F_l$ is: $f(t_1, \cdots, t_h) \cdot u = u \cdot f(t_1 + \frac{2}{n_1} \langle \delta_1, l \rangle \tau, t_2 + \frac{2}{n_2} \langle \delta_2, l \rangle \tau, \cdots, t_h + \frac{2}{n_h} \langle \delta_h, l \rangle \tau), u \in F_l$. The space $F_l$ is a free $K$-module, $F_l = K \otimes R_l$ and $R_l$ is a space of meromorphic sections of line bundle $\xi^l(\xi) = S^{l_1} \xi(\delta_1) \boxtimes S^{l_2} \xi(\delta_2) \boxtimes \cdots \boxtimes S^{l_h} \xi(\delta_h)$ which is a bundle over $S^{l_1} \mathcal{E} \times S^{l_2} \mathcal{E} \times \cdots \times S^{l_h} \mathcal{E}$, $\boxtimes$ is outer tensor product. Elements from $R_l$ we will write as functions in $l_1 + l_2 + \cdots + l_h$ variables: $f(u_{1,1}, u_{2,1}, \cdots, u_{1,1}; u_{1,2}, u_{2,2}, \cdots, u_{2,2}; \cdots; u_{1,h}, \cdots, u_{l_h,h})$ which is symmetric with respect to the first group of variables, the second, $\cdots$. Elements from $F_l = K \otimes R_l$ are functions in variables $t_1, \cdots, t_h, u_{11}, \cdots, u_{l_h,h}$ and they are $\theta$-functions in the variables.
\{u_{ij}\} \) (this is the meaning of the symbol ̃). These functions have to satisfy additional properties.

**Description of \( F_I \).** (a) function \( f \) has a pole of order \( \leq 1 \) on the divisors \( u_{\alpha,j} - u_{\beta,i} - t_j + t_i = 0 \) if \( \langle \delta_i, \delta_j \rangle \neq 0 \) and \( i \neq j \), and holomorphic outside the union of these divisors. (b) functions from \( F_I \) are zero on the following submanifolds: \( \{ M_{i,j} \} \). They are labeled by the pairs of simple roots \( \delta_i, \delta_j \), such that \( 2\langle \delta_i, \delta_j \rangle / \langle \delta_i, \delta_i \rangle = p_{i,j} < 0, p_{ij} \in \mathbb{Z} \). Recall, that \( p_{ij} \) is an element of Cartan matrix. The manifold \( M_{i,j} \) consists of such \( \{ t, u_{i,j} \} \), that \( u_{\alpha,i} = u_{\alpha_2,i} + \langle \delta_1, \delta_i \rangle \cdot t, u_{\alpha_3,i} = u_{\alpha_1,i} + \langle \delta_1, \delta_i \rangle \cdot t, \cdots u_{\alpha_{p_{ij}+1},i} = u_{\alpha_{p_{ij}+1}+1} + \langle \delta_i, \delta_i \rangle \cdot t \). A formula for the product in the algebra \( Q_{n,\Gamma}(E, \tau) \) is: \( f \in F_I \) and depends on \( \{ t, \cdots, t_h, u_{i,j} \} \), \( g \in F_I' \) and depends on \( \{ t, \cdots, t_h, u_{i,j}' \} \), \( l = \{ l_1, \cdots, l_h \} \), \( l' = \{ l'_1, \cdots, l'_h \} \)

\[
\begin{align*}
    f \ast g(t_1, \cdots, t_h, u_{i_1,1}, \cdots, u_{i_l,1}, u_{i_1,1}', \cdots, u_{i_l,1}', \cdots, u_{i_h,1}, \cdots, u_{i_{l'},1}, \cdots, u_{i_h,1}) &=
    \frac{1}{l_1!l_1'! \cdots l_h!l_h'} Symm \left( \prod_{\substack{i \leq j \leq h \\
i \leq \alpha \leq j \\
i \leq \beta \leq j'}} \frac{\theta(u_{i,\alpha} - u_{i,\beta} - t_i + t_j - \langle \delta_i, \delta_j \rangle \tau)}{\theta(u_{i,\alpha} - u_{i,\beta} - t_i + t_j)} \right) \times \\
    & \times f(t_1, \cdots, t_h, u_{\gamma,\delta}) \cdot g(t_1, \cdots, t_h, u_{i_1,1}' - \frac{2}{n_1} \langle \delta_i, \delta_1 \rangle \tau, \cdots, u_{i_l,1}' - \frac{2}{n_1} \langle \delta_i, \delta_1 \rangle \tau; \cdots \\
    & u_{i_h,1}' - \frac{2}{n_h} \langle \delta_i, \delta_h \rangle \tau, \cdots, u_{i_l',1}' - \frac{2}{n_h} \langle \delta_i, \delta_h \rangle \tau) 
\end{align*}
\]

The symmetrization here has a usual sense: the expression in the bracket \([ \cdot ]\) has not the desirable symmetry properties, so we symmetrize to get them.

**Main properties of algebras \( Q_{n,\Gamma}(E, \tau) \).**

(a) If \( \Gamma \) is a root system of \( A_1 \) then \( Q_{n,\Gamma}(E, \tau) \cong Q_{n_1}(E, \frac{2}{n_1} \tau) \otimes K \), where \( K \) is the ring of functions in \( t_1 \).

(b) Suppose that \( n = (n_1, \cdots, n_h) \) is not beside the origin, it means that all \( n_i > 1 \). Then the algebra \( Q_{n,\Gamma}(E, \tau) \) is generated by \( t_1, \cdots, t_h \) and \( F_{\delta_1}, F_{\delta_2}, \cdots, F_{\delta_h} \).

(c) Consider an algebra \( A(I, \Gamma) \), \( I \) is a set of indices; \( A(I, \Gamma) \) is generated by \( \{ e_{\alpha,i}; u_{\alpha,i} \}, \alpha \in I, 1 \leq i \leq h \), the relations are:

\[
e_{\alpha,i}e_{\beta,j} = -e^{2\pi i (u_{\beta,j} - u_{\alpha,i} - t_j + t_i)} \frac{\theta(u_{\alpha,i} - u_{\beta,j} - t_i + t_j - \langle \delta_i, \delta_j \rangle \tau)}{\theta(u_{\beta,j} - u_{\alpha,i} - t_j + t_i - \langle \delta_i, \delta_j \rangle \tau)} e_{\beta,j} \cdot e_{\alpha,i}
\]
\[ e_{\alpha,i}u_{\beta,j} = (u_{\beta,j} - \frac{2}{n_j}\langle \delta_i, \delta_j \rangle \tau)e_{\alpha,i}; \text{ here } \alpha \neq \beta \text{ when } i = j \]

\[ e_{\alpha,i}u_{\alpha,i} = (u_{\alpha,i} + 2(2 - \frac{4}{n_i})\tau)e_{\alpha,i}; e_{\alpha,i}t_j = (t_j - \frac{2}{n_j}\langle \delta_i, \delta_j \rangle \tau)e_{\alpha,i} \]

There is a homomorphism \( Q_{n,\Gamma}(E, \tau) \to A(I, \Gamma), f \in F_{\delta_i} \) goes to \( \sum_{\alpha} f(u_{\alpha,i})e_{\alpha,i} \)

VI. Serre relations and classical limits.

Algebra \( Q_{n,\Gamma}(E, \tau) \) is generated by \( \{t_i\} \) and \( \{F_{\delta_i}\} \) only in the case when \( \tau \) is not a point of finite order. If \( \tau \) is a point of finite order, the subalgebra \( \tilde{Q}_{n,\Gamma}(E, \tau) \) generated by \( \{t_i\} \) and \( \{f_{\delta_i}\} \) is smaller. For example, suppose \( \tau^N = 0 \) on \( E, \tau \neq 0 \). In this case functions from \( \tilde{Q}_{u,\Gamma}(E, \tau) \) have additional zeros. Namely, \( f(t_1, \ldots, t_h; u_{\gamma,\delta}) \) is zero if \( u_{\alpha_1,i} = u_{\alpha_2,i} + \langle \delta_i, \delta_i \rangle \tau, u_{\alpha_2,i} = u_{\alpha_3,i} + \langle \delta_i, \delta_i \rangle \tau, \ldots, u_{\alpha_{N-1},i} = u_{\alpha_N,i} + \langle \delta_i, \delta_i \rangle \tau \) for each \( i, \alpha_1, \ldots, \alpha_N \).

It is natural to conjecture that \( \tilde{Q}_{n,\Gamma}(E, \tau) \) coincides with the space of functions with this property. All these zero conditions (see also point (b) from the description of \( F_1 \)) are a sort of Serre relations in the universal enveloping of the nilpotent part in universal enveloping of \( q \)-deformed Kac-Moody.

To make it clear consider the following construction. Let \( L \) be a lattice and \( D = \oplus D_l, l \in L \) is \( L \)-graded associative algebra; \( \delta_1, \ldots, \delta_h \) basis in \( L \) and the space \( D_l, l = l_1^1 \delta_1 + l_2^2 \delta_2 + \cdots + l_k^k \delta_k \) is a space of functions \( f \) in variables \( u_{1,1}, u_{2,1}, \ldots, u_{l_1^1,1}; u_{1,2}, u_{2,2}, \ldots, u_{l_2^2,2}; \ldots; u_{1,h}, u_{2,h}, \ldots, u_{l_h^h,h}; \) which are symmetric with respect to the first group of \( l_1 \) variables, with respect to the second group of \( l_2 \) variables, \( \cdots \). The product is given by our standard formula:

\[ f \ast g(u_{1,1}, \ldots, u_{l_1^1,1}, u_{1,1}', \ldots, u_{l_1^1,1}'; \ldots; u_{1,h}, \ldots, u_{l_h^h,h}, u_{1,h}', \ldots, u_{l_h^h,h}') = Symm\left( \prod_{1 \leq i,j \leq h} \lambda_{i,j}(u_{\alpha,i}, u_{\beta,j}') \right) \times f(u_{\gamma,\delta}) \cdot g(u_{\gamma,\delta}'; \beta, \delta) \]

Here \( f \in D_l, l = \sum l_i^i \delta_i \) and depends on \( U_{\gamma,\delta} \) and \( g \in D_{l'}, l' = \sum l_i'^i \delta_i \) and depends on \( u_{\gamma,\delta}' \) and \( f \times g \in D_{l+l'} \) and depends on \( u_{\gamma,\delta}, u_{\gamma,\delta}', \{ \lambda_{i,j} \} \) is an arbitrary set of function in two variables. Suppose for simplicity that the functions \( \lambda_{i,j} \) are regular and let \( K_{i,j} \) be a divisor of zeros of the function \( \lambda_{i,j} \). Let us denote by \( \tilde{D} \) the subalgebra in \( D \) generated by \( \oplus_i D_{\delta_i} \).
Proposition. Functions from $\tilde{D}$ have zeros on the set of $\{u_{\gamma,\delta}\}$ such that there exists the subsequence of points: $\{u_{1,i_1}, u_{2,i_1}, \ldots, u_{j_1,i_1}; u_{1,i_2}, u_{2,i_2}, \ldots, u_{j_2,i_2}; \ldots; u_{1,i_s}, \ldots, u_{i_s,i_s}\}$ such that $\{u_{1,i_1}, u_{2,i_1}\} \in K_{i_1,i_1}; \{u_{2,i_1}, u_{3,i_1}\} \in K_{i_1,i_1}; \ldots; \{u_{j_1-1,i_1}, u_{j_1,i_1}\} \in K_{i_1,i_1}; \{u_{i_1,i_1}, u_{1,i_2}\} \in K_{i_1,i_2}, \{u_{1,i_2}, u_{2,i_2}\} \in K_{i_2,i_2}, \ldots; \{u_{j_s-1,i_s}, u_{i_1,i_s}\} \in K_{i_s,i_s},$ and the last is $\{u_{j_s,i_s}, u_{1,i_1}\} \in K_{i_s,i_1}$. Algebra $Q_{n,\Gamma}(E, \tau)$ is a subalgebra in the twisted version of $D$ with some special $\{\lambda_{i,j}\}$ they have poles but it is not essential, the situation is the same. It is interesting that in this case these zero conditions are consequences of the conditions for $\{u_{1,i_1}, u_{2,i_1}, \ldots, u_{j_1,i_1}; u_{1,i_2}, u_{2,i_2}\}$.  

When $\tau \to 0$ algebra $Q_{n,\Gamma}(E, \tau)$ tends to commutative algebra. Therefore it is possible to define the quasi-classical limit of $Q_{n,\Gamma}(E, \tau)$, to find out symplectic leaves and so on. Another way to construct the $\tau \to 0$ limit is the following. Using the considerations of the point I we first have the braided tensor product $Q_{n,\Gamma} \otimes Q_{n,\Gamma}$ and the comultiplication map $Q_{n,\Gamma} \to Q_{n,\Gamma} \otimes Q_{n,\Gamma}$. The simplest definition of $Q_{n,\Gamma} \otimes Q_{n,\Gamma}$ can be done using algebra $A(I, \Gamma)$ from point V. Suppose I is infinite, then $Q_{n,\Gamma}(E, \tau) \to A(I, \Gamma)$ is imbedding. Decompose I into two infinite subsets $I_1 \cap I_2 = I$, so $A(I_j, \Gamma) \subset A(I, \Gamma)$ and two copies of $Q_{n,\Gamma}$ is contained in $A(I, \Gamma)$: one is $Q_{n,\Gamma}^{(1)} \subset A(I_1, \Gamma)$ and the second $Q_{n,\Gamma}^{(2)} \subset A(I_2, \Gamma)$. These $Q_{n,\Gamma}^{(1)}$ and $Q_{n,\Gamma}^{(2)}$ generate $Q_{n,\Gamma} \otimes Q_{n,\Gamma}$. Comultiplication is given by the familiar formula $f \in F_{\delta_i} \to f \otimes 1 + 1 \otimes f, t_j \to t_j \otimes 1 + 1 \otimes t_j$. When $\tau \to 0$ comultiplication degenerate into the non-trivial operator $Q_{n,\Gamma}(E, 0) \to Q_{n,\Gamma}(E, 0) \otimes Q_{n,\Gamma}(E, 0)$. Hence we get a Hopf algebra. Dual object is algebra $U(L, E)$-the universal enveloping of the Lie algebra from the beginning of point V. Informally it means that algebra $Q_{n,\Gamma}(E, 0)$ is more or less an algebra of functions on a group of currents from $E$ into maximal nilpotent subgroup into Kac-Moody algebra. Algebra $Q_{n,\Gamma}(E, \tau)$ for generic $\tau$ is generated by $\{t_i\}$ and the sum of $\{F_{\delta_i}\}$. When $\tau \to 0$, then the relations between $\{F_{\delta_i}\}$ tend to Serre relations in $U(L, E)$.

Remark. There is a way to generalize our construction. Algebra from the point IV is connected with maximal nilpotent subalgebra in Lie superalgebra $osp(1,2)$ which has a basis $l_1, l_2$ and a bracket $[l_1, l_2] = l_2, [l_2, l_1] = 0, \deg l_1 = 1, \deg l_2 = 2$. Changing the functions $\lambda_{i,j}$ we can cover many superalgebras. It will be a subject of another paper.
Let $N_{n,k}$ be the following moduli space. Consider the moduli space of indecomposable $k$-dimensional bundles of degree $n$. More precisely we need corresponding universal family of bundles $\xi_{n,k}(z)$, $z \in \mathbb{C} \to \mathbb{C}/\Gamma$, is a parameter on the elliptic curve $\mathbb{C}/\Gamma = \mathcal{E}$. Element from the space $N_{n,k}$ is a pair $(z, \beta)$, $z \in \mathcal{E}$ and $\beta \in \text{Ext}^1(\xi_{n,k}(z), \xi_{0,1})$. Define an algebra $\hat{Q}_{n,k}(\mathcal{E}, \tau)$ with generators $x_1, \ldots, x_n, z$. Elements $\{x_i\}$ obey the same relations as the generators of algebra $Q_{n,k}(\mathcal{E}, \tau)$ (see the beginning of introduction), $x_i z = (z + \tau') x_i$, $\tau' \in \mathbb{C}$.

Algebra $\hat{Q}_{n,k}(\mathcal{E}, \tau)$ consists of expressions $\sum x_1^{i_1} \cdots x_n^{i_n} f_{i_1, \ldots, i_n}(z)$, where $f$ are elliptic functions associated with $\mathcal{E}$. If $\tau$ and $\tau'$ go to zero ($\tau/\tau'$ is generic) we get the hamiltonian structure on $N_{n,k}$.

Let $\text{Mod}$ be a set of vector bundles on $\mathcal{E}$ defined up to isomorphism. Define an action of $\mathcal{E}$ on $\text{Mod}$ by the formula: $\beta \in \mathcal{E}$ acts as $T_\beta: \xi_{n_1,k_1}(\alpha_1) + \xi_{n_2,k_2}(\alpha_2) + \cdots + \xi_{n_p,k_p}(\alpha_p) \to \xi_{n_1,k_1}(\overline{\alpha_1}) + \cdots + \xi_{n_p,k_p}(\overline{\alpha_p})$

where $\overline{\alpha_i} = \alpha_i + \beta \cdot (n_i(k+1) - k_i \cdot n)$, $n = n_1 + n_2 + \cdots + n_p$, $k + 1 = k_1 + \cdots + k_p$.

There is a map

$$\theta: N_{n,k} \to \text{Mod} \to \text{Mod}/\mathcal{E}.$$  

Here $\text{Mod}/\mathcal{E}$ is a space of orbits of $\mathcal{E}$ acting on $\text{Mod}$ by $\beta \to T_\beta$.

**Proposition.** Symplectic leaves of the hamiltonian structure on $N_{n,k}$ are the fibers of map $\theta: N_{n,k} \to \text{Mod}/\mathcal{E}$.

Now consider the case of algebra $Q_{n,k}(\mathcal{E}, \tau)$ without the additional variable $z$. In the introduction we described the symplectic leaves of the hamiltonian structure on $\mathbb{C}P^{n-1}$. This $\mathbb{C}P^{n-1}$ is a moduli space of exact sequences $0 \to \xi_{0,1} \to ? \to \xi_{n,k}(z) \to 0$ where $z$ is fixed. There is a birational map

$$\mathbb{C}P^{n-1} \to \text{Mod}_{n,k}^s(z).$$

Here $\text{Mod}_{n,k}^s(z)$ is the moduli space of stable $k + 1$-dimensional bundles of degree $n$ with fixed determinant. The determinant have to be equal to the determinant of $\xi_{0,1} \oplus \xi_{n,k}(z)$.
Due to §1 we know that $\text{Mod}^*_{n,k}(z)$ is isomorphic to $\mathbb{C}P^{c-1}$, where $c$ is the maximal common divisor of $n$ and $k + 1$. It is possible to draw the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{C}^n & \xrightarrow{\bar{\theta}} & \mathbb{C}^c \\
P_1 \downarrow & & \downarrow P_2 \\
\mathbb{C}P^{n-1} & \longrightarrow & \text{Mod}_{n,k}(z) \cong \mathbb{C}P^{c-1}
\end{array}
$$

Here $P_1$ and $P_2$ are natural projections. They are not defined in the origins. The map $\bar{\theta}$ is regular and can be written as a set $\{\theta_1, \ldots, \theta_c\}$ of polynomials of degree $n/c$. It is reasonable to think about $\mathbb{C}^c \setminus \{0\}$ as a space of determinant bundle over $\text{Mod}_{n,k}(z)$. So, the meaning of our diagram is that the rational map in the bottom is covered by the regular map of determinant bundles.

**Propositions.** The symplectic leaves of hamiltonian structure on $\mathbb{C}^n$ (it corresponds to the classical limit of $Q_{n,k}(\mathcal{E}, \tau)$) are the intersections of the fibers of the map $\mathbb{C}^n \rightarrow \text{Mod} \rightarrow \text{Mod}/\mathcal{E}$ and $\mathbb{C}^n \rightarrow \mathbb{C}^c$. In particular, $\bar{\theta}$ is hamiltonian map, if the hamiltonian structure on $\mathbb{C}^c$ is trivial.

Using this proposition and results from §1 we can describe all symplectic leaves of the classical limit of $Q_{n,k}(\mathcal{E}, \tau)$.

The similar results are true in the case of Borel bundles on $\mathcal{E}$. Consider for example the $sl_r$-case. So, let $M_{n_1, \ldots, n_{r-1}}$ be a moduli space of objects: $\Gamma$-dimensional bundle $\nu$ with filtration $\nu_0 \subset \nu_1 \subset \cdots \subset \nu_{r-1}$ and the isomorphisms: $\nu_0 \cong \xi_{0,1}(z_1), \nu_1/\nu_0 \cong \xi_{n_1,1}(z_2), \ldots, \nu_{r-1}/\nu_{r-2} \cong \xi_{n_{r-1},1}(z_r)$. Suppose also that the determinant of $\nu$ is fixed. The classical limit of the algebras $Q_{n,\Gamma}(\mathcal{E}, \tau)$, where $n = (n_1, n_2, \ldots, n_{r-1})$ and $\Gamma$ is a root system of $sl_r$ gives us a hamiltonian structure on $M_{n_1, \ldots, n_{r-1}}$. As in $Q_{n,k}(\mathcal{E}, \tau)$ case it is a hamiltonian map

$$
M_{n_1, \ldots, n_{r-1}} \rightarrow \mathbb{C}^h.
$$

Here $\mathbb{C}^h$ has a trivial hamiltonian structure and $h$ is a maximal common divisor of $r$ and $n_1 + 2n_2 + \cdots + (r-1)n_{r-1}$. It is also possible to define a map $M_{n_1, \ldots, n_{r-1}} \rightarrow \text{Mod} \rightarrow \text{Mod}/\mathcal{E}$ and description of symplectic leaves is the same as in $Q_{n,k}$-case.
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