Embedded fracture model in numerical simulation of the fluid flow and geo-mechanics using Generalized Multiscale Finite Element Method

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Abstract. In this work, we consider a poroelasticity problem in fractured porous media. Mathematical model contains a coupled system of equations for pressure and displacements, for which we use an embedded fracture model. The fine grid approximation is constructed based on the finite volume approximation for the pressure in fractured media and finite element method for the displacements. Multiscale approximation is developed using a structured coarse grid and is based on the Generalized Multiscale Finite Element Method for pressures and displacements. The performance of the method is tested using a two-dimensional model problem with different number of the multiscale basis functions.

1. Introduction

Effective numerical simulations of the problems in fractured porous media is important for many real world applications, for example, in oil and gas reservoirs, geothermal fields and underground waste disposal [1, 2]. Mathematical models of the flow problems in fractured porous media are constructed based on the mixed dimensional formulations [3, 4]. Fine grid approximation depends on the mesh construction, where the discrete fracture model is used for conforming fracture and porous matrix grid [5, 6, 7], and in the case of separate independent construction of the fracture and porous matrix grids, an embedded fracture model is used [8, 9, 10]. Mixed dimensional formulation of the flow problem is similar to the dual continuum approaches [11].

For accurate numerical solution of the poroelasticity problem using embedded fracture model, we should use a fine grid which leads to the large discrete system of equations. To reduce the size of the discrete system, multiscale methods are used [12, 13, 14, 15]. In this paper, we develop a Generalized Multiscale Finite Element method for solution of the poroelasticity problems in fractured media with embedded fracture model. We construct multiscale basis functions for pressures and displacement via solution of the local spectral problems. We numerically...
investigate our method using a two-dimensional model problems in fractured and heterogeneous poroelastic media.

The paper is organized as follows. In Section 2, we present the mathematical model and fine grid approximation of the poroelasticity problem with embedded fracture model. In Section 3, we construct a multiscale coarse grid solver using Generalized Multiscale Finite Element Method. Numerical results for two-dimensional model poroelasticity problems are presented in Section 4.

2. Mathematical model and fine grid approximation

Let $\Omega$ is computation domain for the porous matrix and $\gamma$ is the lower dimensional fracture domain. We consider a mathematical model of flow and geo-mechanics in fractured poroelastic medium that described by a following system of equations for displacements, pressure in porous matrix and fractures

$$-\text{div}\, \sigma(u) + \alpha \text{grad} \, p = 0, \quad x \in \Omega,$$

$$c_m \frac{\partial p_m}{\partial t} - \text{div} \, (k_m \text{grad} \, p_m) + r_{mf} (p_m - p_f) = f_m, \quad x \in \Omega,$$

$$c_f \frac{\partial p_f}{\partial t} - \text{div} (k_f \text{grad} \, p_f) - r_{fm} (p_m - p_f) = f_f, \quad x \in \gamma,$$

with a linear relation between stress $\sigma$ and strain $\varepsilon$ tensors

$$\sigma(u) = \lambda \varepsilon(u) I + 2\mu \varepsilon(u), \quad \varepsilon(u) = 0.5 (\nabla u + (\nabla u)^T),$$

where $\lambda$, $\mu$ are the Lames parameters, $u$ is the displacements, $p$ is the pressure, $\alpha$ is the Biot coefficient, $f$ is the source term, $|k_\alpha| = \kappa_\alpha / \nu$, $\nu$ is the viscosity, $\kappa_\alpha$ is the permeability for $\alpha = m, f$, $r_{\alpha\beta} = \eta_{\alpha\beta} r$, $r$ is the transfer coefficient, $\eta_{\alpha\beta}$ is the geometric factors, $c_\alpha$ is the compressibility for $\alpha = m, f$. Note that, we neglect the gravitational forces, suppose that effect of the mechanics to the flow is relatively small and consider effect of the porous matrix pressure to the displacements.

In this work, we use the two-dimensional problem for illustration of the presented method and consider an implicit scheme for approximation of time with given time step $\tau$. Let $T_h = \bigcup \mathcal{N}_i$ be a fine scale finite element partition of the domain $\Omega$ and $\mathcal{E}_\gamma = \bigcup \mathcal{I}_l$ is the fracture mesh. $N_{fm}$ is the number of cells in $T_h$, $N_{f}$ is the number of cell for fracture mesh $\mathcal{E}_\gamma$.

For approximation of the flow problem, we use a finite volume approximation on the structured fine grids and obtain the following discrete system

$$c_m \frac{p_m,i - \bar{p}_m,i}{\tau} |s_i| + \sum_j T_{ij} (p_m,i - p_m,j) + \sum_l q_{il} (p_m,i - p_f,l) = f_m |s_i|, \quad \forall i = 1, N_{fm}^m,$$

$$c_f \frac{p_f,l - \bar{p}_f,l}{\tau} |s_l| + \sum_n W_{ln} (p_f,l - p_f,n) - \sum_i q_{il} (p_m,i - p_f,l) = f_f |s_l|, \quad \forall l = 1, N_{f}^f,$$

where $T_{ij} = k_m |E_{ij}| / \Delta_{ij}$ ($|E_{ij}|$ is the length of interface between cells $s_i$ and $s_j$, $\Delta_{ij}$ is the distance between mid point of cells $s_i$ and $s_j$), $W_{ln} = k_f / \Delta_{ln}$ ($\Delta_{ln}$ is the distance between points $l$ and $n$), $|s_i|$ and $|s_l|$ is the volume of the cells $s_i$ and $s_l$, $q_{il} = r$ if $s_i \cap s_l \neq \emptyset$ and equals zero otherwise. Here $(\bar{p}_m, \bar{p}_f)$ are solutions from the previous times step.

For displacement, we use Galerkin method with linear basis functions.

$$\int_\Omega \sigma(u) : \varepsilon(v) \, dx - \int_\Omega \alpha p_m I \cdot \varepsilon(v) \, dx = 0,$$
Therefore, we have a following computational algorithm for solution on the fine grid in the matrix form

- Solve pressure system for \( p = (p_m, p_f)^T \):
  \[
  \left( \frac{1}{\tau} M + A \right) p = F,
  \]
  
  where \( M = \begin{pmatrix} M_m & 0 \\ 0 & M_f \end{pmatrix}, \quad F = \begin{pmatrix} F_m + \frac{1}{\tau} M_m \tilde{p}_m \\ F_f + \frac{1}{\tau} M_f \tilde{p}_f \end{pmatrix}, \quad A = \begin{pmatrix} A_m + Q & -Q \\ -Q & A_f + Q \end{pmatrix}, \]

  \[ M_m = \{ m_{ij}^m \}, \quad m_{ij}^m = \begin{cases} c_{m|i|} & i = j, \\ 0 & i \neq j \end{cases}, \quad M_f = \{ m_{in}^f \}, \quad m_{in}^f = \begin{cases} c_{f|i|} & l = n, \\ 0 & l \neq n \end{cases}, \]

  and \( A_m, A_f \) are the transmissibility matrices, \( Q \) is the transfer term matrix between porous matrix and fractures, \( F_m = \{ f_i^m \}, \quad f_i^m = f_m|\Omega|, \quad F_f = \{ f_i^f \}, \quad f_i^f = f_f|\Omega| \).

- Solve displacements system for \( u = (u_x, u_y) \)
  \[
  D u = B,
  \]
  
  where \( D \) is the elasticity stiffness matrix

  \[
  D = \begin{pmatrix} D_{xx} & D_{xy} \\ D_{xy} & D_{yy} \end{pmatrix}, \quad B = \begin{pmatrix} B_x \\ B_y \end{pmatrix},
  \]

  and \( D_{xx} = \int_\Omega \sigma_x(\psi_i) : \varepsilon_x(\psi_j) \, dx, \quad D_{yy} = \int_\Omega \sigma_y(\psi_i) : \varepsilon_y(\psi_j) \, dx, \quad D_{xy} = \int_\Omega \sigma_{xy}(\psi_i) : \varepsilon_{xy}(\psi_j) \, dx, \quad B_x = \int_\Omega \alpha p_m \cdot \varepsilon_x(\psi_j) \, dx, \quad B_y = \int_\Omega \alpha p_m \cdot \varepsilon_y(\psi_j) \, dx \) with linear basis functions \( \psi_i \).

3. Coarse grid approximation using GMsFEM

For coarse grid approximation of the poroelasticity problems in fractured porous media, we use the Generalized Multiscale Finite Element Method (GMsFEM). GMsFEM contains the following steps: (1) construction of the coarse and fine meshes; (2) generation of the projection matrix using local multiscale basis functions; (3) construction of the coarse grid system using projection matrix; (4) solution of the coarse scale problem and reconstruction of the fine grid solution.

For construction of the multiscale basis functions for pressures and displacements, we solve a spectral problem in the local domain \( \omega_i \)

- pressures, \( \phi_i^{\omega_i} = (\phi_i^{\omega_i m}, \phi_i^{\omega_i f}) \)
  \[
  A^{\omega_i} \phi_i^{\omega_i} = \lambda_{p,j} S^{\omega_i} \phi_i^{\omega_i},
  \]

- displacements, \( \Phi_i^{\omega_i} = (\Phi_i^{\omega_i x}, \Phi_i^{\omega_i y}) \)
  \[
  D^{\omega_i} \Phi_i^{\omega_i} = \lambda_{u,j} G^{\omega_i} \Phi_i^{\omega_i},
  \]

where \( D^{\omega_i} \) and \( A^{\omega_i} \) are the restrictions of the global matrices \( D \) and \( A \) to the local domain \( \omega_i \). Here

\[
S^{\omega_i} = \begin{pmatrix} S_m^{\omega_i} & 0 \\ 0 & S_f^{\omega_i} \end{pmatrix}, \quad G^{\omega_i} = \begin{pmatrix} G_x^{\omega_i} & 0 \\ 0 & G_y^{\omega_i} \end{pmatrix},
\]
\[ S_{m}^{\omega_{i}} = \{ s_{ij}^{\omega_{i},m} \}, \quad s_{ij}^{\omega_{i},m} = \begin{cases} k_{m} |s_{i}| & i = j, \\ 0 & i \neq j, \end{cases} \quad S_{f}^{\omega_{i}} = \{ s_{ln}^{\omega_{i},f} \}, \quad s_{ln}^{\omega_{i},f} = \begin{cases} k_{f} |t_{l}| & l = n, \\ 0 & l \neq n, \end{cases} \]

and \( G_{x}^{\omega_{i}} = G_{y}^{\omega_{i}} = [g_{ij}] = \int_{\omega} (\lambda + 2\mu) \psi_{j} dx. \)

We form the multiscale spaces for pressures and displacements using eigenvectors \( \{ \phi_{1}^{\omega_{i}}, \phi_{2}^{\omega_{i}}, \ldots, \phi_{L_{p}}^{\omega_{i}} \} \) and \( \{ \Phi_{1}^{\omega_{i}}, \Phi_{2}^{\omega_{i}}, \ldots, \Phi_{L_{u}}^{\omega_{i}} \} \) corresponding to the first smallest \( L_{p} \) and \( L_{u} \) eigenvalues, where \( \lambda_{p,1} \leq \lambda_{p,2} \leq \ldots \leq \lambda_{p,L_{p}} \) and \( \lambda_{u,1} \leq \lambda_{u,2} \leq \ldots \leq \lambda_{u,L_{u}} \). Furthermore, for obtaining conforming basis functions we use linear partition of unity functions \( \chi^{\omega_{i}} \). We construct transition matrices \( R_{u} \) and \( R_{p} \) from a fine grid to a coarse grid and use it for reducing the dimension of the problem

\[
R_{u} = \{ \chi^{\omega_{i}} \Phi_{1}^{\omega_{i}}, \chi^{\omega_{i}} \Phi_{2}^{\omega_{i}}, \ldots, \chi^{\omega_{i}} \Phi_{L_{u}}^{\omega_{i}}, \chi^{\omega_{u}N_{c}} \Phi_{1}^{\omega_{u}N_{c}}, \chi^{\omega_{u}N_{c}} \Phi_{2}^{\omega_{u}N_{c}}, \ldots, \chi^{\omega_{u}N_{c}} \Phi_{L_{u}}^{\omega_{u}N_{c}} \} \\
R_{p} = \{ \chi^{\omega_{i}} \phi_{1}^{\omega_{i}}, \chi^{\omega_{i}} \phi_{2}^{\omega_{i}}, \ldots, \chi^{\omega_{i}} \phi_{L_{p}}^{\omega_{i}}, \chi^{\omega_{u}N_{c}} \phi_{1}^{\omega_{u}N_{c}}, \chi^{\omega_{u}N_{c}} \phi_{2}^{\omega_{u}N_{c}}, \ldots, \chi^{\omega_{u}N_{c}} \phi_{L_{p}}^{\omega_{u}N_{c}} \}. 
\]

where \( \chi^{\omega_{i}} \) is linear partition of unity functions, \( L_{p} \) and \( L_{u} \) are the number of basis functions for pressure and displacements, \( N_{c} \) is the number of vertices of a coarse grid.

Then the system of equations can be translated into a coarse grid, and we have following computational algorithm in the matrix form

- Solve pressure system for \( p_{c} = (p_{c,m}, p_{c,f})^{T} \):

\[
\left( \frac{1}{\tau} M_{c} + A_{c} \right) p_{c} = F_{c}, \quad (6)
\]

where \( M_{c} = R_{p} M R_{p}^{T}, \quad A_{c} = R_{p} A R_{p}^{T} \) and \( F_{c} = R_{p} F \).

- Solve displacements system for \( u = (u_{x}, u_{y}) \)

\[
D_{c} u_{c} = B_{c}, \quad (7)
\]

where \( D_{c} = R_{u} D R_{u}^{T} \) and \( B_{c} = R_{u} B \).

After obtaining of a coarse-scale solution, we can reconstruct fine-scale solution \( u_{ms} = R_{u}^{T} u_{c} \) and \( p_{ms} = R_{p}^{T} p_{c} \). Next, we present numerical results for test problems and compare multiscale solutions for pressures and displacements \( (p_{ms} \text{ and } u_{ms}) \) with the reference fine grid solutions \( (p \text{ and } u) \).

4. Numerical results

![Figure 1](image.png)

**Figure 1.** Fracture distribution, fine grid, heterogeneous permeabilities \( K \) and elasticity parameter \( E \) (from left to right).
In this section, we consider poroelasticity problem in fractured porous media and consider two test problems. We set parameters of model problem $E = 10$, $\nu = 0.3$, $k_m = 10^{-5}$, $k_f = 1$ for Test 1. For Test 2, we use a heterogeneous coefficients, where elasticity modulus and heterogeneous permeability are presented in Figure 1. We set $\alpha = 1, \sigma = k_m$, $c_m = 0.1$ and $c_f = 0.01$. The calculation is performed by $T_{\text{max}} = 100$ with time step $\tau = 10$. Computational grids and fracture distribution are presented in Figure 1. Fine grid contains 40401 vertices and 40000 cells. For study of the presented multiscale method, we consider two coarse grids $10 \times 10$ and $20 \times 20$.

In Figures 2 and 3, distribution of pressure and displacement along $X$ and $Y$ directions at final time for 8 multiscale basis functions in homogeneous and heterogeneous media are presented. In

**Figure 2.** Distribution of pressure, displacement along $X$ and $Y$ directions at the last moment of time for 8 multiscale basis functions for Test 1.

**Figure 3.** Distribution of pressure, displacement along $X$ and $Y$ directions at the last moment of time for 8 multiscale basis functions for Test 2.

| $L$ | $10 \times 10$ | $20 \times 20$ | $10 \times 10$ | $20 \times 20$ |
|-----|------|-------|------|-------|
|     | $e^u$ | $e^p$ | $e^u$ | $e^p$ |
| 1   | 4.850 | 1.840 | 1.794 | 0.896 |
| 2   | 1.933 | 1.332 | 0.535 | 0.550 |
| 4   | 1.120 | 0.648 | 0.372 | 0.292 |
| 8   | 0.395 | 0.488 | 0.136 | 0.195 |
| 12  | 0.289 | 0.330 | 0.077 | 0.144 |

**Table 1.** Relative errors in % for displacement and pressure with different numbers of multiscale basis functions for Test 1 (left) and Test 2 (right). $L_p = L_u = L$. 
Table 1, we present the relative $L_2$ errors for different number of the multiscale basis functions on the coarse grid $10 \times 10$ and $20 \times 20$ between multiscale and fine grid solutions. Presented results of the poroelasticity problem show that proposed multiscale method can provide good accuracy for a small number of multiscale basis functions for embedded fracture model.

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References
[1] I Yucel Akkutlu, Yalchin Efendiev, Maria Vasilyeva, and Yuhe Wang. Multiscale model reduction for shale gas transport in a coupled discrete fracture and dual-continuum porous media. *Journal of Natural Gas Science and Engineering*, 48:65–76, 2017.
[2] I Yucel Akkutlu, Ebrahim Fathi, et al. Multiscale gas transport in shales with local kerogen heterogeneities. *SPE journal*, 17(04):1–002, 2012.
[3] Vincent Martin, Jérôme Jaffré, and Jean E Roberts. Modeling fractures and barriers as interfaces for flow in porous media. *SIAM Journal on Scientific Computing*, 26(5):1667–1691, 2005.
[4] Luca Formaggia, Alessio Fumagalli, Anna Scotti, and Paolo Ruffo. A reduced model for darcy’s problem in networks of fractures. *ESAIM: Mathematical Modelling and Numerical Analysis*, 48(4):1089–1116, 2014.
[5] I Yucel Akkutlu, Yalchin Efendiev, Maria Vasilyeva, and Yuhe Wang. Multiscale model reduction for shale gas transport in poroelastic fractured media. *Journal of Computational Physics*, 353:356–376, 2018.
[6] Liyong Li, Seong H Lee, et al. Efficient field-scale simulation of black oil in a naturally fractured reservoir through discrete fracture networks and homogenized media. *SPE Reservoir Evaluation & Engineering*, 11(04):750–758, 2008.
[7] Mohammad Karimi-Fard, Luis J Durlofsky, Khalid Aziz, et al. An efficient discrete fracture model applicable for general purpose reservoir simulators. In *SPE Reservoir Simulation Symposium*. Society of Petroleum Engineers, 2003.
[8] Seong H Lee, MF Lough, and CL Jensen. Hierarchical modeling of flow in naturally fractured formations with multiple length scales. *Water resources research*, 37(3):443–455, 2001.
[9] Liyong Li, Seong Hee Lee, et al. Efficient field-scale simulation for black oil in a naturally fractured reservoir via discrete fracture networks and homogenized media. In *International oil & gas conference and exhibition in China*. Society of Petroleum Engineers, 2006.
[10] Matei Tene, MS Al Kobaisi, H Hajibeygi, et al. Algebraic multiscale solver for flow in heterogeneous fractured porous media. In *SPE Reservoir Simulation Symposium*. Society of Petroleum Engineers, 2015.
[11] GI Barenblatt, In P Zheltov, and IN Kochina. Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks [strata]. *Journal of applied mathematics and mechanics*, 24(5):1286–1303, 1960.
[12] Y. Efendiev, J. Galvis, and T. Hou. Generalized multiscale finite element methods. *Journal of Computational Physics*, 251:116–135, 2013.
[13] Y. Efendiev and T. Hou. Multiscale Finite Element Methods: Theory and Applications. 4, 2009.
[14] Patrick Jenny, Seong H Lee, and Hamdi A Tchelepi. Adaptive multiscale finite-volume method for multiphase flow and transport in porous media. *Multiscale Modeling & Simulation*, 3(1):50–64, 2005.
[15] Hadi Hajibeygi, Giuseppe Bonfigli, Marc Andre Hesse, and Patrick Jenny. Iterative multiscale finite-volume method. *Journal of Computational Physics*, 227(19):8604–8621, 2008.