Application of Unscented Kalman Filter in Satellite Orbit Simulation

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ABSTRACT A new estimate method is proposed, which takes advantage of the unscented transform method, thus the true mean and covariance are approximated more accurately. The new method can be applied to nonlinear systems without the linearization process necessary for the EKF, and it does not demand a Gaussian distribution of noise and what's more, its ease of implementation and more accurate estimation features enables it to demonstrate its good performance in the experiment of satellite orbit simulation. Numerical experiments show that the application of the unscented Kalman filter is more effective than the EKF.

KEYWORDS EKF; unscented transform; unscented Kalman filter (UKF); orbit simulation

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Introduction

As is well known, the Kalman filter (KF) is always used to deal with the system whose dynamics and observation models are linear, and the extended Kalman filter (EKF) is the most widely used estimator for nonlinear systems. In the EKF the kalman filter is usually applied to nonlinear systems by simply linearizing all the nonlinear models so that the traditional Kalman filter equations can be applied. However, as people have found, the use of the EKF in practice has two well-known drawbacks. The first one is that linearization can produce highly unstable filters if the assumptions of local linearity is violated. The second one is that the derivation of the Jacobian matrices is nontrivial in most applications and often leads to significant implementation difficulties.

In this paper a new estimator, UKF, is introduced which yields performance equivalent to the Kalman filter for linear systems, yet generalizes elegantly to nonlinear systems without the linearization steps required by the EKF.

1 Unscented Kalman filter

The UKF approximates the statistical distribution of the state random variable in a more accurate way. The Gaussian random variable that represents the state distribution is determined by a specific sample point set which contains information on the mean and covariance of the Gaussian random variable. Therefore, no matter how nonlinear the system is, the posteriori mean and covariance with accuracy up to the third order in terms of Taylor series expansion can be derived through nonlinear propagation. The specific minimal sample point (Sigma point) set is obtained using the unscented transform.

1.1 Unscented transform

The unscented transform was proposed by Ju-
lier and Uhlmann to calculate statistical properties of random variables after nonlinear propagation. Consider the problem: propagate a random variable $x$, whose dimension is $d_x$, through a nonlinear function $y = f(x)$. Assuming that $x$ has mean value $\mu$ and covariance $P_x$. To compute the statistical properties of $y$, the following formulas can be used

$$
\begin{align*}
&\chi_0 = x \\
&\chi_i = x + (\sqrt{(d_x + \lambda)P_x}), i = 1, \cdots, d_x \\
&\chi_i = x - (\sqrt{(d_x + \lambda)P_x}), i = d_x + 1, \cdots, 2d_x
\end{align*}
$$

(1)

where $\chi$ is a matrix consisting of $2d_x + 1$ vectors $\chi_i$ (called Sigma point) $i\lambda = \alpha^2 (d_x + \kappa) - d_x$ being a scaling parameter, and constant $\alpha$ determines the extension of these vectors around $x$ and usually set $1 \leq 2 \leq \alpha \leq 1$. $\kappa$ is another scaling factor, and often set as zero for state estimation problems; $(\sqrt{(d_x + \lambda)P_x})_i$ is the $i$-th column of the square root of the matrix. Here $\gamma = \sqrt{(d_x + \lambda)}$.

The acquired Sigma points are transformed or propagated through the nonlinear function

$$
y_i = f(\chi_i), i = 0, 1, \cdots, 2d_x
$$

(2)
to obtain the transformed vectors $y_i$. And then the mean value and covariance of $y$ are approximated using the weighted mean and covariance of the transformed vectors

$$
y \approx \sum_{i=0}^{2d_x} w^{(m)}_i y_i
$$

$$
P_y \approx \sum_{i=0}^{2d_x} w^{(c)}_i (y_i - y)(y_i - y)^T
$$

(4)

where weights $w_i$ are given as

$$
\begin{align*}
&w^{(m)}_i = \frac{\lambda}{d_x + \lambda} \\
&w^{(c)}_i = \frac{\lambda}{d_x + \lambda} + (1 - \alpha^2 + \beta) \\
&w^{(m)}_i = \frac{1}{2(d_x + \lambda)}, i = 1, 2, \cdots, 2d_x
\end{align*}
$$

(5)

And the parameter $\beta$ contains the a priori information of $x$ (for Gaussian distribution, $\beta = 2$).

Further analysis shows that, using the Sigma point transformation, the statistical properties of the random variable after nonlinear transformation can be approximated with higher accuracy than that using simple linearization used in the Kalman filter

$$
y_{\text{LIN}} = f(x)
$$

$$
(P_y)_{\text{LIN}} = A_x P_x A_x^T
$$

(6)

The difference between the mean of random variable through UT and that through nonlinear transformation only lies in the terms higher than the third order. The covariance through UT only differs from that through nonlinear transform in moments higher than the fourth order.

### 1.2 UKF Implementation

By applying the UT to the recursive estimation, the UKF method is derived. It must be pointed out that the implementation of the UKF needs no explicit calculation of Jacobians or Hessians, which are always non-trivial burden of computation for EKF in such case as perturbed orbit computation that involves variational equation as far as state transition matrix is concerned. Furthermore, the overall number of computations is of the same order as that of the EKF. The process of UKF is as follows.

1) Initialization.

$$
\hat{x}_0 = E[\chi_0] \\
P_0 = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T]
$$

2) As for $k \in (1, \cdots, \infty)$, compute Sigma points.

$$
\chi_{i-1} = \left[\hat{x}_{i-1} + \sqrt{(d_x + \lambda)P_{x,i-1}} \chi_{i-1} - \sqrt{(d_x + \lambda)P_{x,i-1}}\right]
$$

3) Prediction.

$$
\hat{x}_{i,k-1} = F[\chi_{i,k-1} , u_{i,k-1}]
$$

$$
P_{i,k-1} = \sum_{i=0}^{2d_x} W^{(m)}_i \chi_{i,k-1}
$$

$$
X_{i,k-1} = H[\chi_{i,k-1}]
$$

4) Correction.

$$
P_{i,k} = \sum_{i=0}^{2d_x} W^{(c)}_i [\chi_{i,k} - \hat{x}_i][\chi_{i,k} - \hat{x}_i]^T + R^w
$$

$$
\hat{y}_i = \sum_{i=0}^{2d_x} W^{(m)}_i \chi_{i,k} - \hat{x}_i
$$

$$
P_{y,i,k} = \sum_{i=0}^{2d_x} W^{(c)}_i [\chi_{i,k} - \hat{x}_i][\chi_{i,k} - \hat{x}_i]^T
$$

$$
K_i = \frac{P_{y,i,k}}{P_{y,i,k} + \beta}
$$

$$
\hat{x}_i = \hat{x}_i + K_i(y_i - \hat{y}_i)
$$

$$
P_i = P_i - K_i P_{y,i,k} K_i^T
$$
where \( \lambda \) is the composite scaling factor; \( d_i \) is the dimension of the state vector; \( R^p \) is the process noise covariance matrix; \( R^m \) is the measurement noise covariance matrix; \( W_i \) is weights calculated in Eq. (5).

## 2 Numerical experiment

Following a common mode of orbit determination of a satellite in the low Earth orbit, observations of horizontal angle, elevation angle and distance are made. To be simple, only two degree zonal harmonic perturbation and atmospheric resistance perturbation are taken into account. The true ephemeris is acquired with the DE algorithm, and the observations are acquired with simulation based on the true ephemeris and the known coordinates of observation station. Provided that the J2000 is the spatial reference system and the TT is the time reference datum, the state-space model of satellite dynamics can be established as follows.

State vector:

\[
y(t) = \left[ \begin{array}{c} r(t) \\ v(t) \end{array} \right]
\]

Equation of dynamic model is

\[
\dot{y}(t) = \left[ \begin{array}{c} v(t) \\ a_0(t) + a_1(t) + a_{DRG}(t) \end{array} \right] + w'(t)
\]

State prediction:

\[
y_{k+1} = y_k + \int_{t_k}^{t_{k+1}} \left[ \begin{array}{c} v(t) \\ a_0(t) + a_1(t) + a_{DRG}(t) \end{array} \right] dt + w_k
\]

where \( r(t) \) is the vector of position; \( v(t) \) is the vector of velocity; \( a_0(t) \) is the central gravitational acceleration; \( a_1(t) \) is the non-spherical gravitational perturbation acceleration of \( J_2 \); \( a_{DRG}(t) \) is atmospheric resistance; \( w'(t) \) is mechanical noise; \( w_k \) is state noise.

As observation is made under the topocentric coordinate system, the satellite coordinates of J2000 should be converted into those under local Horizon coordinate system. Disregarding minute corrections like aberration, atmospheric refraction, etc., the observation equations are set up, including azimuth \( A \), elevation angle \( E \) and distance \( \rho \) in the horizon coordinates being

\[
s = [s_E \ s_N \ s_Z]^T
\]

\[
A = \arctan \left( \frac{s_N}{s_E} \right) + \nu_A
\]

\[
E = \arctan \left( \frac{s_Z}{\sqrt{s_E^2 + s_N^2}} \right) + \nu_E
\]

\[
\rho = \sqrt{s_E^2 + s_N^2 + s_Z^2} + \nu_\rho
\]

where \( s_E \) is the coordinate component along eastern axis; \( s_N \) is the coordinate component along northern axis; \( s_Z \) is the coordinate component along zenith axis; \( \nu_A \) is the noise of the observed azimuth; \( \nu_E \) is the noise of the observed elevation angle; \( \nu_\rho \) is the noise of the observed distance.

The accurate estimation that is assumed as the true value is

\[
y_0 = \begin{bmatrix} -6345.000 \times 10^3 \\ -3723.000 \times 10^3 \\ -580.000 \times 10^3 \\ +2.169 \times 10^3 \\ -9.266 \times 10^3 \\ -1.079 \times 10^3 \end{bmatrix}
\]

\[
P_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \times 10^{-8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \times 10^{-8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \times 10^{-9} \end{bmatrix}
\]

The initial value for filtering is

\[
y_0^+ = \begin{bmatrix} -6345.000 \times 10^3 + 100 \\ -3723.000 \times 10^3 - 100 \\ -580.000 \times 10^3 + 100 \\ +2.169 \times 10^3 - 0.1 \\ -9.266 \times 10^3 + 0.1 \\ -1.079 \times 10^3 - 0.1 \end{bmatrix}
\]

\[
P_0^+ = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.01 & 0 \end{bmatrix}
\]

The initial epoch of observation is UTC19990301T00; 00; 00. The frequency is 1 Hz, and the error of observation is \( \sigma_\theta = 0.01^\circ \times \cos(E) \cdot \sigma_E = 0.01^\circ, \sigma_\rho = 1 \) m. For 100 sets of observations, the filtered results of UKF and EKF are shown in Fig. 1, Fig. 2.

As shown in Fig. 1, the accuracy of UKF is basically equivalent to that of EKF, which is due
to the weak nonlinearity for short observation interval between epochs.

To test the performance under strong nonlinearity, the observation interval is set as one period \( T = 38,048 \) s. And the filtered results with UKF and EKF for 10 \( T \) are given in Fig. 3, Fig. 4. Fig. 3, Fig. 4 show that EKF demonstrates great deviation and divergency, whereas UKF can lead to satisfactory results while maintaining certain accuracy.

### 3 Conclusions

This paper indicates that new state and observation of a dynamic system can be accurately predicted through the unscented transform. And the unscented Kalman filter has two prominent advantages. One is that it can predict the state of a nonlinear dynamic system and gain accuracy up to the 3rd order for Gaussian noises. The result is especially better for strongly nonlinear system. The other advantage is that it is easy to perform as no Jacobian or Hessian matrix of nonlinear system is needed.

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