Fluctuations in the Hawking-Turok model

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(April 1998)

Scalar and gravity wave subcurvature fluctuations are calculated for a background approximating the Hawking and Turok open universe model. The gravity wave cosmic microwave background contribution is finite and it appears that a normalizable scalar supercurvature mode is possible in some regions of parameter space.

I. INTRODUCTION

The Hartle-Hawking wavefunction [1] has been suggested to describe ‘tunneling from nothing’ and has recently been proposed as a way to produce an open universe [2]. The interpretation of this mechanism, the most likely value of \( \Omega \) and the singular nature of the background are all under study and criticism at the moment (with varying interpretations, counterexamples and extensions in [3–9]). In the following, the subcurvature scalar and (‘electric parity’) gravity wave fluctuations are calculated for a model of this sort, using the approximate form for the background given in the papers above. The ‘electric parity’ gravity waves contribute to the cosmic microwave background (CMB) while the ‘magnetic parity’ ones do not.

It will be assumed that the singularity does not cause a difficulty aside from imposing boundary conditions on basis functions to be regular in its presence. This requirement is discussed in [7]. The time dependence will be taken to be the same, at ‘tunneling time’, as that of the Bunch-Davies vacuum, for reasons described in section two below. It will be assumed that the value of \( \Omega \) is appropriate.

One reason this calculation might be of interest is that for a Bunch-Davies vacuum in an open universe, the contribution of gravity waves to the cosmic microwave background is divergent [10]. In the model considered here, the subcurvature gravity wave contribution is found to be finite. The background metric and field are described in the rest of this section. In the next section the scalar perturbations are considered, followed by a section on supercurvature modes and then gravity waves.

The instanton describes several regions of spacetime (see [2] for further details). The Euclidean (tunneling) part of the wavefunction is described by a metric

\[
ds^2 = d\sigma^2 + b^2(\sigma)(d\psi^2 + \sin^2 \psi d\Omega_2^2)\tag{1}
\]

At the ‘tunneling time’ \( \psi = \pi/2 \), one can continue this to the Lorentzian region by

\[
\psi = \pi/2 + i\tau\tag{2}
\]

The metric then becomes

\[
ds^2 = d\sigma^2 + b^2(\sigma)(-d\tau^2 + \cosh^2 \tau d\Omega_2^2)\tag{3}
\]

As \( \sigma \) is a spatial coordinate, this spacetime is not of FRW form. Here, the parameter \( \sigma \) runs between \( \sigma_f \geq \sigma \geq 0 \), where \( b(0) = b(\sigma_f) = 0 \).

The open universe is the forward light cone of the point \( \sigma = 0 \) with \( \sigma = it, \tau = i\pi/2 + \chi \) and \( a(t) = -ib(it) \). The metric in this region is

\[
ds^2 = -dt^2 + a(t)^2(d\chi^2 + \sinh^2 \chi d\Omega_2^2)\tag{4}
\]

There is also a closed universe obtained by continuation at the maximum radius \( \sigma = \sigma_{max} \) with

\[
ds^2 = -dT^2 + b^2(T)(d\psi^2 + \sin^2 \psi d\Omega_2^2)\tag{5}
\]

That is \( 0 \leq \sigma \leq \sigma_{max} \) for the Euclidean region and \( \sigma = \sigma_{max} + iT \) in the Lorentzian region.
The action is
\[ S_E = \int d^4 x \sqrt{g} (-V(\phi) + \frac{1}{16} F^2) + \frac{1}{8\pi G} \int d^4 x \sqrt{h} K . \] (6)

The boundary term, \( \frac{1}{8\pi G} \int d^3 x \sqrt{h} K = -\frac{1}{8\pi G} (b^3)' \int d\Omega^3 \), comes in as there is a small area excised around \( \sigma = \sigma_f \), and the equations of motion have been used.

We will use a linear potential here, \( V(\phi) = V_\phi \phi \). The equations of motion in terms of \( \sigma \) are
\[ \phi''(\sigma) + \frac{3V'_{\phi}(\sigma)}{V_{\phi}(\sigma)} \phi' = V_{\phi} \] (7)
\[(b'(\sigma))^2 - 1 = \frac{3(b(\sigma)^2 - (\phi'(\sigma))^2)}{4} - V_F(\phi(\sigma)) \]

Here \( V_F = V(\phi) + F^2/48 \), and \( F \) is a constant [11].

The instanton background solving these equations is, to leading order \[2\],

\[ b_R(\sigma) \sim \sigma \quad \phi_R(\sigma) \sim \phi_0 + O(\sigma^2) \quad \sigma \sim 0 \]
\[ b_L(\sigma) \sim C(\sigma_f - \sigma)^{1/3} \quad \phi_L(\sigma) \sim -\sqrt{\frac{A}{3\kappa}} \ln(\sigma_f - \sigma) \quad \sigma \sim \sigma_f \] (8)

with \( \kappa = 8\pi G \).

A background solution for \( \phi, b \) for all \( \sigma_f \geq \sigma \geq 0 \) will be obtained by extending the background metric near both endpoints and connecting them near \( \sigma \sim \sigma_f \). Extend the background metric near \( \sigma_f \) as above. To go to higher order in \((\sigma_f - \sigma)\), write
\[ b_L(\sigma) = C(\sigma_f - \sigma)^{1/3}(1 + A(\sigma_f - \sigma)^{4/3}) \] (10)

Dimensionally, \( [C] \sim \sigma^{2/3}, [A] \sim \sigma^{-4/3} \). The power of the additional term has been chosen to cancel the curvature in the equations of motion (13). Then
\[ \partial_\sigma b_L(\sigma) = -\frac{C}{3} (\sigma_f - \sigma)^{-2/3}(1 + 5A(\sigma_f - \sigma)^{4/3}) \] (11)

so that the first equation of motion in (11) then gives
\[ \partial_\sigma \phi_{b,L} = \sqrt{\frac{2}{3\kappa}} \left( \frac{1}{(\sigma_f - \sigma)} - 3A(\sigma_f - \sigma)^{1/3} \right) - V_\phi (\sigma_f - \sigma)/2 + O((\sigma_f - \sigma)^{5/3}) \] (12)

which integrates to
\[ \phi_{b,L} = -\sqrt{\frac{2}{3\kappa}} \ln(\sigma_f - \sigma) + \frac{9A}{4} \sqrt{\frac{2}{3\kappa}} (\sigma_f - \sigma)^{4/3} + \frac{V_\phi}{4}(\sigma_f - \sigma)^2 + c_0 + O((\sigma_f - \sigma)^{8/3}) \] (13)

These are solutions to the second equation of motion (19), up to \( O(\ln(\sigma_f - \sigma)(\sigma_f - \sigma)^{2/3}) \), provided \( AC^2 = 9/14 \) and \( V_\phi \) small.

The background metric and field \( \phi_b \) are to be matched at equal values of the metric. The approximate solution near \( \sigma \sim \sigma_f \) is only valid very close to \( \sigma_f \), so the matching will be done very close to \( \sigma \sim \sigma_f \). Setting the scale factors and their first derivatives equal,
\[ b_R(\sigma_m) = H^{-1} \sin H\sigma_m = C(\sigma_f - \sigma_m - \delta)^{1/3} = b_L(\sigma_m + \delta) \]
\[ \partial_\delta b_R = \cos H\sigma_m = -\frac{4}{3} (\sigma_f - \sigma_m - \delta)^{-2/3} = \partial_\sigma b_L \] (14)

If, for example, \( H\sigma_m \sim \pi \), neglecting the second term in \( b_L(\sigma_f - \sigma_m - \delta)^{4/3} \), down by 1/14 gives
\[
\frac{C}{3(\sigma_f - \sigma_m - \delta)} = (\sigma_f - \sigma_m - \delta)^{2/3} \ll 1
\]

For the background field values, using the solution in [12] for \( \phi_{b,R} \),

\[
\phi_{b,R}(\sigma_m) = \phi_0 + \frac{V_\phi}{6H^2} \left( -4 \ln \cos \frac{H\sigma_m}{2} + \frac{1}{(\cos \frac{H\sigma_m}{2})^2} \right)
\]

\[
\phi_{b,L}(\sigma_m + \delta) = -\sqrt{\frac{2}{3\kappa}} \log(\sigma_f - \sigma_m - \delta) \]

\[
\phi_{b,R}(\sigma_m) = \frac{V_\phi \cos^3 H\sigma_m - 3 \cos H\sigma_m + 2}{3 \sin^3 H\sigma_m} = \frac{V_\phi 2 + \cos H\sigma_m}{6 \cos^3 \frac{H\sigma_m}{2}} \sin \frac{H\sigma_m}{2}
\]

\[
\phi_{b,L}(\sigma_m + \delta) = \sqrt{\frac{2}{3\kappa}} \left( \frac{1}{(\sigma_f - \sigma_m - \delta)} - 3A(\sigma_f - \sigma_m - \delta)^{1/3} + O((\sigma_f - \sigma_m - \delta)) \right)
\]

Setting the first two lines equal can be used to determine \( \phi_0 \), and setting the next two lines equal can be used to determine \( V_\phi \).

Both of these are good approximate solutions for a potential of the form

\[
V = V_\phi \phi
\]

with \( V_\phi \) small. The numerical solutions shown in [3] seem to have this approximate behavior as well.

**II. SCALAR PERTURBATIONS**

In open universe models where the fields tunnel from a false vacuum, the tunneling 'time' is on a proper Cauchy surface exterior (except for a point, \( \sigma = 0 \)) to the subsequent open universe. As pointed out in [13,14], although this fixed time surface is a proper Cauchy surface for quantization it is not a spatially homogeneous background. Therefore quantization in this background does not split up into tensor, vector and scalar in the same way as it does once the system is continued into the open universe. For the calculations here, the distinction into scalar/vector/tensor will be made on the basis of what the modes continue into in the open universe.

At the fixed time of tunneling, the background (\( b(\sigma) \), \( \phi_b \)) depends only on \( \sigma \). In particular, \( \phi_b \) is not a constant, unlike many simpler cases ('thin wall' for example), and so the scalar field and scalar metric perturbations become coupled. It is better in this case to use the 'gauge invariant gravitational potential' in the formalism of [13,14], in particular its open universe generalization detailed in section 7 of [12]. Equivalent descriptions using gauge fixing outside the open universe are detailed in [13].

In the forward light cone (inside the open universe) the gauge invariant potential \( \Phi \) obeys [12]

\[
\ddot{\Phi} + \left( \frac{\dot{a}}{a} - 2 \frac{\dot{b}}{b} \frac{\dot{\phi}_b}{\phi_b} \right) \dot{\Phi} + \left( \frac{4}{a^2} (-\nabla^2 - 4K) + 2 \frac{\dot{\phi}_b}{a} - 2 \frac{\dot{\phi}_b}{a} \frac{\phi_b}{\dot{\phi}_b} \right) \Phi = 0
\]

Continuing out of the light cone, and writing \( \Phi = -i \partial_\sigma \Phi, \frac{\dot{a}}{a} = -i \partial_\sigma b/b \) and \( b^2 = -a^2 \), this equation becomes

\[
\partial_\sigma^2 \Phi + \left( \frac{\partial_\sigma b}{b} - 2 \frac{\partial_\sigma^2 \phi_b}{\partial_\sigma \phi} \right) \partial_\sigma \Phi + \left( \frac{\nabla^2 - 4K}{b^2} + 2 \partial_\sigma \frac{\partial_\sigma b}{b} - 2 \partial_\sigma \frac{b}{b} \partial_\sigma \phi_b \right) \Phi = 0
\]

The gauge invariant gravitational potential can be expanded in terms of basis functions

\[
\Phi(\sigma, \tau, \theta, \phi) = \Phi_p(\sigma) Y_{p\ell m}(\tau, \theta, \phi)
\]

where the eigenfunctions of the three dimensional laplacian \( Y_{p\ell m} \) obey
\[ \nabla^2 Y_{p \ell m} = -(p^2 + 1) Y_{p \ell m} \] (21)

and for an open universe \( K = -1 \). The time dependence in \( Y_{p \ell m} \) implicitly corresponds to a particular definition of positive and negative frequency. The surface where analytic continuation from the Euclidean to Lorentzian signature is done defines a complex structure, as described in, for example, [10]. This method applies rigorously when the surface of analytic continuation is the only boundary of the space, so for the case here, due to the singularity in the instanton, this prescription is only a guide. In terms of the five-dimensional embedding space, as described in [17], at the surface of analytic continuation \( x^0 = \sin \sigma \sin(i\tau) \rightarrow ix^0 = \sin \sigma \sin \tau \). The Bunch Davies vacuum positive frequency wavefunctions are regular for \( x^0 > 0 \) [17]. For the tunneling prescription [16] the wavefunctions must be bounded for \( \tau_E = -i \tau < 0 \). As \( 0 \leq \sigma \leq \pi \), \( \sin \sigma \geq 0 \) and hence these two definitions of the split between positive and negative frequency coincide. One could also argue that since the \( O(3,1) \) symmetry persists in the presence of this instanton [6], the Bunch Davies vacuum is appropriate for the \( \tau \) dependence.

In view of these arguments, the Bunch Davies vacuum will be used for the \( \tau \) dependent part of the fluctuations, and thus the appropriate expression for \( Y_{p \ell m} \) can be found for example in [17]:

\[ Y_{p \ell m} \propto \frac{P_{p \ell m}(\cosh(\tau + \pi i/2))}{\sqrt{\sinh(\tau + \pi i/2)}} Y_{\ell m}(\theta, \phi) \] (22)

The \( Y_{p \ell m} \) are normalized identically to earlier calculations such as those in [13,14], as all the background \( b(\sigma), \phi_b(\sigma) \) dependence affects only \( \Phi_{p}(\sigma) \).

We will want to solve equation [14] for the gauge invariant gravitational potential \( \Phi \), in the two asymptotic regimes above. Then these will be matched at the boundary between the two backgrounds, given by equations [14,17]. The resulting basis function will then be normalized.

For the background \( b_R(\sigma) = H^{-1} \sin H \sigma \), it is convenient to use the conformal coordinate \( u \), defined by

\[ \tanh u = \cos H \sigma \), \( \sin H \sigma = \frac{1}{\cosh u} \), \( \tan \frac{H \sigma}{2} = e^{-u} \] (23)

so \( \partial_\sigma = -H \cosh u \partial_u \). The two independent solutions to [19] in this case were found in [12]

\[ \Phi_{R,+p}(u) = (e^{-u})^{1+ip}(1 - \frac{1 - ip}{1 + ip} e^{-2u}) \] (24)

\[ \Phi_{R,-p}(u) = (e^{-u})^{1-ip}(1 + \frac{1 + ip}{1 - ip} e^{-2u}) = \Phi_{R,+p}(u)^* \]

For the background with \( b_L = C(\sigma_f - \sigma)^{1/3}(1 + A(\sigma_f - \sigma)^{4/3}) \), and

\[ \partial_\sigma \phi_b, L = \sqrt{\frac{2}{3 \kappa}} \frac{1}{(\sigma_f - \sigma)} - 3A(\sigma_f - \sigma)^{1/3} - V_\phi(\sigma_f - \sigma)/2 + O((\sigma_f - \sigma)^{5/3}) \] (25)

The terms needed for the equation for \( \Phi_p \) are

\[ \frac{\partial_\sigma b}{b} \frac{\partial_\sigma^2 \phi_b}{\partial_\sigma^2} = \frac{-1}{3(\sigma_f - \sigma)^{2/3}} (1 + 4A(\sigma_f - \sigma)^{4/3} - \sqrt{3 \kappa} V_\phi(\sigma_f - \sigma)^2)) \] (26)

and

\[ \frac{\partial_\sigma b}{b} \partial_\sigma^2 \phi_b = \frac{-1}{3(\sigma_f - \sigma)^{2/3}} - \frac{8A}{3(\sigma_f - \sigma)^{2/3}} - \sqrt{6 \kappa} V_\phi + O((\sigma_f - \sigma)^{2/3}) \]. (27)

The equation of motion for \( \Phi \), to leading order and assuming \( \sigma_f - \sigma, \sqrt{6 \kappa} V_\phi \) small then becomes

\[ \partial_\sigma^2 \Phi_{L,p} - \frac{7}{3(\sigma_f - \sigma)} \partial_\sigma \Phi_{L,p} + \frac{p^2 + 9}{C^2(\sigma_f - \sigma)^{2/3}} \Phi_{L,p} = 0 \] (28)

This has solutions (via mathematica)

\[ \Phi(\sigma) = c_1 \frac{J_1[3 \sqrt{\frac{p^2 + 9}{C^2}(\sigma_f - \sigma)^{2/3}}]}{(\sigma_f - \sigma)^{2/3}} + c_2 \frac{K_1[3i \sqrt{\frac{p^2 + 9}{C^2}(\sigma_f - \sigma)^{2/3}}]}{(\sigma_f - \sigma)^{2/3}} \] (29)
In terms of the coordinate \( X = \frac{3}{4}(\sigma_f - \sigma)^{2/3} \) the solution is

\[
\Phi_{L,p}(X) = c_1 \frac{J_1[X \sqrt{p^2 + 9}]}{X} + c_2 \frac{K_1[X \sqrt{p^2 + 9}]}{X}
\]  

(30)

The asymptotics of the Bessel functions for small argument are

\[
J_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu + 1)}, \quad K_\nu(z) \sim \frac{1}{2} \Gamma(\nu) \left( \frac{2}{z} \right)^\nu
\]

(31)

and so as \( \sigma \rightarrow \sigma_f \),

\[
\Phi_{L,p}(X) \sim c_1 \frac{\sqrt{p^2 + 9}}{C} + c_2 \frac{C}{\sqrt{p^2 + 9}} X^{-2}.
\]

(32)

The measure in the normalization, equation [44], goes as \( X^3 \), and so the norm diverges for \( c_2 \neq 0 \). Thus we take \( c_2 \rightarrow 0 \), as mentioned in [7]. As \( J_\nu(ze^{m\pi i}) = e^{m\pi i\nu}J_\nu(z) \), taking the opposite sign in the argument of the Bessel function (the square root of \( p^2 + 9 \)) is not linearly independent and consequently there is only one basis function normalizable at the boundary.

The gauge invariant gravitational potentials \( \{ \Phi_{L,p}, \Phi_{R,\pm p} \} \) are now matched at

\[
\sigma_L = \sigma_m + \delta, \quad \sigma_R = \sigma_m
\]

(33)

with the backgrounds for \( b, \phi_b \) given in equations (14, 17). The gravitational potentials \( \Phi_{L,p}(\sigma + \delta) \) and \( \Phi_R(\sigma) \) must be matched at the same value of \( |p| \) so that the \( \tau \) dependence agrees. We have

\[
\Phi_{L,p}(\sigma_m + \delta) = \alpha_p \Phi_{R,p}(\sigma_m) + \beta_p \Phi_{R,-p}(\sigma_m)
\]

(34)

\[
\partial_\sigma \Phi_{L,p}(\sigma) |_{\sigma = \sigma_m + \delta} = \alpha_p \partial_\sigma \Phi_{R,p}(\sigma) |_{\sigma = \sigma_m + \delta} + \beta_p \partial_\sigma \Phi_{R,-p}(\sigma) |_{\sigma = \sigma_m + \delta}
\]

Using

\[
\partial_\sigma = -H \cosh u \partial_u = -\sqrt{\frac{3}{2}} \frac{1}{X^{1/2}} \partial_X
\]

(35)

it is also convenient to define

\[
\tanh \tilde{u} = \cos H \sigma_m, \quad \tilde{X} = \frac{3}{2}(\sigma_f - \sigma_m - \delta)^{2/3}.
\]

(36)

By the matching conditions of the background metric [44], we also have that

\[
H \cosh \tilde{u} = \frac{1}{C} \sqrt{\frac{3}{2}} \frac{1}{X^{1/2}}
\]

(37)

The matching conditions become

\[
-\frac{2}{3} \frac{J_1(X \sqrt{p^2 + 9})}{X} \frac{1}{\cosh \tilde{u}} \partial_X \frac{2}{3} \frac{J_1(X \sqrt{p^2 + 9})}{X} X = \tilde{X}
\]

(38)

and so, with

\[
\alpha_p = \frac{\partial_\sigma \Phi_{R,-p} \Phi_{L,p} - \Phi_{R,-p} \Phi_{L,p}}{\partial_\sigma \Phi_{R,-p} \Phi_{R,p} - \Phi_{R,-p} \Phi_{R,p}}
\]

\[
\beta_p = \frac{-\partial_\sigma \Phi_{R,p} \Phi_{L,p} + \Phi_{R,p} \Phi_{L,p}}{\partial_\sigma \Phi_{R,-p} \Phi_{R,p} - \Phi_{R,-p} \Phi_{R,p}}
\]

(39)

we get

\[
\text{(39)}
\]
\[ \alpha_p = e^{\tilde{\alpha} (1+i\tilde{p})} \left( -1 + ip - \frac{e^{-2\tilde{u}}}{3} \frac{1+i\tilde{p}}{1-i\tilde{p}} \right) \frac{J_1(\tilde{X} \sqrt{\frac{\tilde{p}^2+9}{2}})}{X} - C(1 + \frac{1+i\tilde{p}}{1-i\tilde{p}} \frac{e^{-2\tilde{u}}}{3}) \frac{\partial \tilde{X}}{\tilde{X}} J_1(\tilde{X} \sqrt{\frac{\tilde{p}^2+9}{2}}) \]
\[ = e^{\tilde{\alpha} (1+i\tilde{p})} \left( -1 + ip - 2 \tanh \tilde{u} \frac{e^{-2\tilde{u}}}{3} \frac{1+i\tilde{p}}{1-i\tilde{p}} \right) \frac{J_1(y) - (1 - \frac{1+i\tilde{p}}{1-i\tilde{p}} \frac{e^{-2\tilde{u}}}{3}) \sqrt{\frac{p^2+9}{2} [J_0(y) - J_2(y)]}{3ip(1 + \frac{e^{-2\tilde{u}}}{3})^2} \]
\[ y = -\frac{\sqrt{p^2+9}}{2 \tanh \tilde{u}} \]

The second line comes from using the matching conditions equation (44), i.e. that \( C = -2 \tanh \tilde{u} \tilde{X} \). Also,
\[ \beta_p = \alpha_p^* \]

which could be seen as well from noting that \( \phi_L \) is real.

Thus the unnormalized gauge invariant gravitational potential \( \Phi_p \) is
\[ \Phi_p = \Phi_L(\sigma), \quad \sigma_f > \sigma > \sigma_m + \delta \]
\[ = \alpha_p \Phi_{R,+} + \alpha_p^* \Phi_{R,-}, \quad \sigma_m > \sigma \geq 0 \]

The measure for normalizing \( \Phi_p \) comes from relating \( \Phi_p \) to the scalar field fluctuation. The Klein Gordon normalization for the scalar field translates into
\[ \int d\sigma \frac{1}{b(p^2+4)} \frac{2}{\tilde{\kappa} \partial \tilde{\sigma} \tilde{\phi}_b} \Phi_p \Phi_{p'} = \delta(p-p') N_p^2 \]

Thus
\[ (p^2 + 4) \delta(p-p') N_p^2 = \int_0^{\sigma_f} d\sigma \frac{1}{b(\sigma)} \left( \frac{2}{\kappa \partial \sigma \phi_{b}} \right)^2 \Phi_p(\sigma) \Phi_{p'}(\sigma) \]
\[ = \int_0^{\sigma_f} d\sigma \frac{1}{b(\sigma)} \left( \frac{2}{\kappa \partial \sigma \phi_{b}} \right)^2 4 \mathcal{R}(\alpha_p \Phi_{R,+}(\sigma)) \mathcal{R}(\alpha_{p'} \Phi_{R,+}(\sigma)) \]
\[ + \int_{\sigma_m+\delta}^{\sigma_f} d\sigma \frac{1}{b(\sigma)} \left( \frac{2}{\kappa \partial \sigma \phi_{b}} \right)^2 \Phi_{L,p}(\sigma) \Phi_{L,p'}(\sigma) \]

As an eigenfunction of a self-adjoint operator, the rescaled \( \Phi_p \) is orthogonal to any eigenfunction with a different value of \( p^2 \). Thus one can read off \( N_p^2 \) from the behavior at the endpoints (although there is a jump in \( \sigma \) in the integrand, continuity in the argument is ensured by the matching conditions). On the other hand, as \( \sigma \to 0 \) (or \( u \to \infty \)),
\[ \Phi_p \sim \alpha_p e^{-u(1+ip)} + \alpha_p^* e^{-u(1-ip)} \]

and
\[ \partial \sigma \phi_{R} \sim \frac{V_{\phi}}{2H} e^{-u} . \]

The asymptotics of the inner product in equation (44) thus give
\[ \int du \left( \frac{4}{\kappa} \right)^2 \frac{1}{p^2+4} \left( \frac{H}{V_{\phi}} \right)^2 \left( \alpha_p e^{-ipu} + \alpha_p^* e^{ipu} \right) \left( \alpha_{p'} e^{-ip'\bar{u}} + \alpha_{p'}^* e^{ip'\bar{u}} \right) \]
\[ \sim \pi \left( \frac{4}{\kappa} \right)^2 \frac{1}{p^2+4} \left( \frac{H}{V_{\phi}} \right)^2 \left[ (\alpha_p \alpha_{p'} + \alpha_p^* \alpha_{p'}^*) \delta(p+p') + (\alpha_p^* \alpha_{p'} + \alpha_p \alpha_{p'}^*) \delta(p-p') \right] \]

This is symmetric under \( p \to -p \) just as the basis functions are, so restricting to \( p > 0 \) the inner product becomes
\[ \alpha_p \alpha_p^* \frac{4}{\kappa} \frac{1}{p^2+4} \frac{8 \pi}{p^2+4} \left( \frac{H}{V_{\phi}} \right)^2 = N_p^2 \]

This gauge invariant gravitational potential can then be continued into the forward light cone for the open universe, and then used to compute a power spectrum.
\[
(\Phi(\chi = 0, t)\Phi(\chi, t)) = \int_0^\infty dp \frac{\sin p\chi}{\sinh \chi} P_\Phi(p, t) .
\]

As the basis functions used here are the same as those in earlier calculations \[12,18,19\], expressions for \(P_\Phi\) can be used to complete the calculation. The normalization \(N_p^2\) found here corresponds to taking \(b_+ = \frac{1}{|\alpha_p|}\) and \(b_- = 0\) in expression (5.9) of \[19\], and not summing over both \(\pm p\). As a result,

\[
P_{\Phi(p, t \to \infty)} = \frac{(GV_\phi)^2}{p (p^2 + 1) 2 \sinh \pi p 9H^2} \frac{4 |e^{ip/2} \alpha_p|}{1 + ip |\alpha_p|} e^{-\pi p/2} \]

Defining \(\alpha_p = A_p e^{i\lambda_p}\) with \(A_p, \lambda_p\) real,

\[
P_{\Phi} = \frac{4}{9H^2} (GV_\phi)^2 \frac{e^{ip\pi} - e^{-ip\pi} - 2 \frac{p^2 - 1}{p^2 + 1} \cos 2\lambda_p - 2ip(1 + p^2) \sin 2\lambda_p}{p(1 + p^2)(2 \sinh \pi p)}
\]

We see that just as for the subcurvature modes in the presence of a bubble wall, for \(p \geq 1\) the last 3 terms are bounded and become irrelevant.

### III. SUPERCURVATURE MODES

Supercurvature modes may appear in a field’s expansion in an open universes \[20,17,21\]. At first it was unclear whether they appeared in the vacuum expansion of a quantum field, but calculations \[17\] (see also \[21\]) of the wightman showed that they were required in the sum over modes in order to produce the Bunch Davies vacuum. Their effects can be large, and thus in some cases they constrain models of open inflation \[22\]. A supercurvature mode will have \(p\) imaginary. Take \(-ip = \Lambda > 0\), without loss of generality. In this case, the asymptotics of the basis function at \(u \sim \infty\), equation \[10\] requires \(\alpha_{i\Lambda} = 0\). Normalizability at \(\sigma \sim \sigma_f\) does not seem to constrain the the value of \(p\) as the basis function is bounded there independent of \(p\).

One can look for a supercurvature mode in the case where \(H\sigma_m - \pi\) small, i.e. where equation (13) holds. Then \(\tilde{X}/C = \frac{1}{2}\) and the equation \[44\] for \(\alpha_{p = i\Lambda}\) becomes

\[
\alpha_{i\Lambda} \propto (-1 - \Lambda + \frac{e^{-2\tilde{a}}}{3} \frac{1 - \Lambda}{1 + \Lambda} (3 + \Lambda)) J_1(\sqrt{\frac{9 - \Lambda^2}{2}}) - \frac{1 - \Lambda e^{-2\tilde{a}}}{1 + \Lambda} \sqrt{9 - \Lambda^2} (J_0(\sqrt{\frac{9 - \Lambda^2}{2}}) - J_2(\sqrt{\frac{9 - \Lambda^2}{2}})) - 4 J_1(\sqrt{\frac{9 - \Lambda^2}{2}}) \\
\equiv 0
\]

The \(\tilde{X}\) dependence thus falls out and we need to solve

\[
0 = (-1 - \Lambda + \frac{e^{-2\tilde{a}}}{3} \frac{1 - \Lambda}{1 + \Lambda} (3 + \Lambda)) J_1(\sqrt{\frac{9 - \Lambda^2}{2}}) - (1 - \Lambda e^{-2\tilde{a}}) \frac{1 - \Lambda}{1 + \Lambda} \sqrt{9 - \Lambda^2} (J_0(\sqrt{\frac{9 - \Lambda^2}{2}}) - J_2(\sqrt{\frac{9 - \Lambda^2}{2}})) - 4 J_1(\sqrt{\frac{9 - \Lambda^2}{2}})
\]

This is satisfied for a range of \(e^{-2\tilde{a}}\) values (corresponding to a choice of model), as can be seen by plotting the right hand side. The approximations in equation \[17\] require \(e^{-2\tilde{a}}\) to be large enough that \(\tanh \tilde{u} \sim -1 - 2e^{2\tilde{a}} \sim -1\). Taking \(e^{2\tilde{a}} = .01\), this normalizability equation for the supercurvature mode has a solution for \(\Lambda \sim .97\). Increasing the value of \(e^{-2\tilde{a}}\) increases the required value of \(\Lambda\). Thus it seems that a normalizable supercurvature mode may be present. The interpretation for \(\Lambda > 1\), where the eigenvalue of the spatial laplacian \(- (p^2 + 1)\) changes sign, seems unclear.

### IV. GRAVITY WAVES

For gravity waves, again the \(\tau, \theta, \phi\) dependence of the gravity waves in this instanton background is the same as the earlier calculations in \[23,24,15\]. The gravity waves are perturbations on the background metric
\[ g_{\mu\nu} = g_{\mu\nu}^b + \hat{h}_{\mu\nu} \]  

(56)

where \( \hat{h} \) is transverse and traceless. Expanding \( \hat{h}_{ij} \) in terms of tensor eigenfunctions \( T^{p,\ell,m}_{ij}(\tau, \theta, \phi) \) of the \( \tau, \theta, \phi \) laplacian, one gets an expansion in terms of modes

\[ n(p)T^{p,\ell,m}_{ij}(\tau, \theta, \phi)T_p(\sigma) \]  

(57)

The instanton behavior appears in the \( \sigma \) dependence, and \( T_p \) has the equation of motion

\[ \partial^2_\sigma T_p + 3 \frac{\partial_x b}{b} \partial_x T_p + \frac{p^2 + 1}{b^2} T_p = 0. \]  

(58)

The normalization \[15\] is proportional to

\[ \int d\sigma b^3(\sigma)\partial_x(T_p)\partial_x(T^*_p) = (p^2 + 1) \int d\sigma b(\sigma)T_p T^*_p \]  

(59)

For \( b_R = \sim H^{-1} \sin H \sigma = \frac{1}{2 \cosh u} \), the fluctuations are

\[ T_{R,+p}(u) = (\tanh u - ip) \cosh ue^{ipu} \]  

\[ T_{R,-p}(u) = (\tanh u + ip) \cosh ue^{-ipu} \]  

(60)

while for \( b_L = C(\sigma_f - \sigma)^{1/3} = C\sqrt{\frac{2}{3}} X \),

\[ T_{L,p}(\sigma) = c_1 J_0[\frac{\sqrt{1 + p^2}}{C} X] + c_2 K_0[i\frac{\sqrt{1 + p^2}}{C} X] \]  

(61)

Since

\[ J_0(x) \sim 1 + \frac{x^2}{4} + O(x^2) \]  

\[ K_0(x) \sim (k_1 - \ln x) + (k_2 - \frac{\ln x}{4})x^2 + O(x^3) \]  

(62)

with \( k_1, k_2 \) constants, one can see using eqn.\((69)\) that normalizability requires \( c_2 = 0 \). Thus,

\[ T_{L,p}(X) = J_0(\frac{\sqrt{p^2 + 1}}{C} X) \]  

(63)

Matching these two asymptotic values and their derivatives at the juncture given by equations \([34, 7]\),

\[ J_0(\frac{\sqrt{p^2 + 1}}{C} \tilde{X}) = \alpha_{h,p}(\tanh \tilde{u} - ip) \cosh \tilde{u} e^{ip\tilde{u}} + \beta_{h,p}(\tanh \tilde{u} + ip) \cosh \tilde{u} e^{-ip\tilde{u}} \]  

\[ -\sqrt{\frac{2}{X}} \partial_x J_0(\sqrt{\frac{p^2 + 1}{X}})|_{X = \tilde{X}} = -\alpha_{h,p} H \cosh^2 \tilde{u} (1 + p^2) e^{ip\tilde{u}} - \beta_{h,p} H \cosh^2 \tilde{u} (1 + p^2) e^{-ip\tilde{u}}. \]  

(64)

So,

\[ \alpha_{h,p} = \frac{(1 + p^2) \cosh^2 \tilde{u} J_0(\sqrt{\frac{p^2 + 1}{C}} X) - C \partial_x J_0(\sqrt{\frac{p^2 + 1}{C}} X)(\sinh \tilde{u} + ip \cosh \tilde{u}) e^{-ip\tilde{u}}}{-2ip \cosh^3 \tilde{u} (1 + p^2)} \]  

\[ = \frac{(1 + p^2) \cosh^2 \tilde{u} J_0(y_h) - \frac{\sqrt{p^2 + 1}}{2}[J_1(y_h) - J_1(y_h)](\sinh \tilde{u} + ip \cosh \tilde{u}) e^{-ip\tilde{u}}}{-2ip \cosh^3 \tilde{u} (1 + p^2)} \]  

\[ y_h = -\frac{\sqrt{p^2 + 1}}{2 \tanh \tilde{u}} \]  

and again

\[ \beta_{h,p} = \alpha_{h,p}^* . \]  

(66)
Equation 59 can be used to get the normalization dependence on $\alpha_p, \beta_p$, which is proportional to $|\alpha_p|^2$. The power spectrum can be taken over from the results in [23,24]. The two point function dependence on $T_{h,p}$ goes as

$$P_h(p) \sim H^2 \frac{1}{2p(p^2+1) \sinh \pi p} |e^{\pi p/2} \frac{\alpha_{h,p}}{\alpha_{h,p}} + e^{-\pi p/2} \frac{\alpha_{h,p}^*}{\alpha_{h,p}}|^2 = H^2 \frac{e^{\pi p} + e^{-\pi p} + 2 \frac{\alpha_{h,p}^2 + \alpha_{h,p}^{*2}}{\alpha_{h,p}}}{2p(p^2+1) \sinh \pi p}$$

(67)
times $p$ independent terms. Writing $\alpha_{h,p} = A_{h,p} e^{i\lambda_{h,p}}$, equation 66 implies that

$$\lambda_{h,p} = -\frac{\pi}{2} + O(p)$$

(68)

so that as $p \to 0$, equation (67) goes as

$$\frac{e^{\pi p} + e^{-\pi p} + 2 \cos \lambda_{h,p}}{2p(p^2+1) \sinh \pi p} \to_{p \to 0} \frac{2 + 2 \cos(-\pi + O(p))}{p^2} = \frac{O(p^2)}{p^2} = O(1)$$

(69)

which is finite. As the $p \to 0$ divergence in the pure Bunch Davies vacuum [10] does not appear, the gravity wave contribution to the CMB is finite.

One can also look for the possibility of normalizable supercurvature modes here, although they are not present in other cases which have been studied so far [23]. Setting $ip = \Lambda$ in equation (68), and using the approximation equation (15), it seems that one cannot have normalizability, i.e. $\alpha_{h,-i\Lambda} = 0$, for $0 \leq \Lambda \leq 1$.

V. CONCLUSION

Assuming the background model and resulting $\Omega$ value are consistent, subcurvature scalar and gravity wave fluctuations were found for a background approximating that described in [2]. The initial conditions were found by requiring regularity at the singular boundary along with the time dependence (on the surface of analytic continuation) corresponding to the Bunch Davies vacuum. Even though a bubble wall is not present, the gravity waves were found to be finite. Both scalar and gravity wave power spectra, for ‘momentum’ $p \geq 2$, go over to the conformal vacuum case (found for scalars in [23]). A normalizable scalar supercurvature mode appears possible for some values of the matching parameters and might have observable effects. The power spectra for $P_\Phi$ and $P_h$ can be used to calculate cosmic microwave background anisotropies. The the subcurvature scalar power spectrum $P_\Phi$ lies within the same envelope as subcurvature modes in other open universe models. Thus varying parameters in this background will change the subcurvature scalar contribution to the CMB in a bounded way and only affect the largest scales just as in the other open universe models. The gravity wave contribution to the CMB is finite. The corresponding gravity wave CMB spectrum may provide constraints on viable backgrounds, just as it does in some of the open universe models (see, e.g. [23]).

VI. ACKNOWLEDGEMENTS

This work was supported by an NSF Career Advancement Award, NSF-PHY-97-22787. I thank S. Carlip and M. White for discussions, and J. Garriga for comments on the draft.

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