is simply called the rotation set of $M$ where
\[
\rho(F) = \left\{ \rho \in \mathbb{R}^2 \mid \exists n_i \to \infty, z_i \in \pi^{-1}(M) : \lim_{i \to \infty} (F^{n_i}(z_i) - z_i)/n = \rho \right\},
\]
where $\pi : \mathbb{T}^2 \to \mathbb{T}$ denotes the canonical projection. In case $M = \mathbb{T}^2$, the set $\rho(F) = \rho_{\mathbb{T}^2}(F)$ is simply called the rotation set of $F$. It takes a central place in the classification of torus homeomorphisms, since a wealth of dynamical information can be obtained from the shape of $\rho(F)$ (see, for example, \cite{4}–\cite{6} and references therein). A crucial fact in this context is that $\rho(F)$ is always compact and convex \cite{7}. Concerning the rotational behaviour on minimal subsets, it is known that if $\rho(F)$ has non-empty interior, then for every vector $\rho \in \text{int}(\rho(F))$ there exists a minimal set $M_\rho \subseteq \mathbb{T}^2$ with $\rho_{M_\rho}(F) = \{\rho\}$ \cite{8}. Further, if $M$ is minimal, then $\rho_{M}(F)$ is always compact and connected \cite{9}, and examples in \cite{9} show that it can be a line segment of the form $\{0\} \times [a,b]$ with $a < b$.

The aim of this note is to explore more complex rotational behaviour on minimal sets. The bottomline is that apparently no restrictions exist for the associated rotation sets, besides compactness and connectedness. We demonstrate this by means of three types of examples, which are actually all realised by the same torus homeomorphism. Denote by $\text{Homeo}_0(\mathbb{T}^2)$ the set of homeomorphisms of $\mathbb{T}^2$ homotopic to the identity.

**Theorem 1.1.** There exists $f \in \text{Homeo}_0(\mathbb{T}^2)$ with lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that

(a) for an open set $V \subseteq \mathbb{R}^2$ and all $v \in V$ there is a minimal set $M_v$ such that $\rho_{M_v}(F)$ is a line segment of positive length contained in $v + \mathbb{R}v^\perp$;

(b) for some minimal set $M$, the associated rotation set $\rho_{M}(F)$ is plane-separating;

(c) for some minimal set $M'$, the associated rotation set $\rho_{M'}(F)$ has non-empty interior.

The proof of Theorem 1.1 can roughly be outlined as follows. The homeomorphism $f$ is chosen such that it has a rotational horseshoe with three symbols and the topology depicted in Figure 1.1. This construction essentially goes back to \cite{10}, where it is implemented in much greater generality to show that every rational polygon can occur as the rotation set of a torus homeomorphism. For our purposes, the important fact is that in this situation we obtain an invariant set $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(\pi(D))$, where $D \subseteq \mathbb{R}^2$ is a topological disk that projects injectively to $\mathbb{T}^2$, and a symbolic coding $h : \Lambda \to \{0,1,2\}^\mathbb{Z}$ such that $h \circ f = \sigma \circ h$. Moreover, given $z \in \Lambda$, the entry $h(z)_0$ determines whether a lift $\tilde{z} \in D$ of $z \in \pi(D)$ remains
Figure 1.1. Geometry of a rotational horseshoe: The horseshoe is located in the topological disk $\pi(D) \subseteq \mathbb{T}^2$ shown on the left. The Markov partition is given by the preimages of the connected components of $\pi(D) \cap f(\pi(D))$.

The situation for the lift is depicted on the right, the displacement vectors are $v_0 = (0,0)$, $v_1 = (1,0)$ and $v_2 = (0,1)$.

in $D$, moves to $D + (1,0)$ or to $D + (0,1)$. Consequently, if we let $v_0 = (0,0)$, $v_1 = (1,0)$ and $v_2 = (0,1)$, then the displacement vector $F^n(z) - z$ differs from the vector $\sum_{i=0}^{n-1} v_{h(z)}$, only by an error term that is bounded uniformly in $n \in \mathbb{N}$ and $z \in \Lambda$. Asymptotically, this means that rotation vectors and sets are completely determined by the coding, and the rotational behaviour on minimal sets can be studied on a purely symbolic level. The crucial issue on the technical side then is to construct suitable almost periodic sequences that produce the desired rotation sets. To that end, we work within the class of irregular Toeplitz sequences, which have been used previously to produce a number of interesting examples in topological and symbolic dynamics [11, 12, 13, 14]. In certain aspects, our construction is reminiscent of these more classical ones.

It is well-known that a dynamical situation like the one in Figure 1.1 is stable under perturbations. Hence, our construction immediately yields an open set of torus homeomorphisms that satisfy the assertions of Theorem 1.1. Moreover, it is known that the existence of rotational horseshoes is $C^0$-generic within an open and dense subset of $\text{Homeo}_0(\mathbb{T}^2)$, see [9]. In order to give a precise statement in our context, we denote by $\mathcal{F}$ the set of those $f \in \text{Homeo}_0(\mathbb{T}^2)$ whose rotation sets have non-empty interior. Then $\mathcal{F}$ is open in the $C^0$-topology [8], and we have

**Theorem 1.2.** The set of $f \in \text{Homeo}_0(\mathbb{T}^2)$ which satisfy the assertions of Theorem 1.1 form an open and dense subset of $\mathcal{F}$.

In fact, we believe that this set is equal to $\mathcal{F}$ and, that arbitrary continua in the interior of $\rho(F)$ can be realised. This leads to the following

**Conjecture 1.3.** Given $f \in \text{Homeo}_0(\mathbb{T}^2)$ with $\text{int}(\rho(F)) \neq \emptyset$ and any continuum $C \subseteq \text{int}(\rho(F))$, there exists a minimal set $M_C$ such that $\rho_{M_C}(F) = C$.

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2. Rotational horseshoes and the symbolic computation of rotation sets

Rotational horseshoes. We say that $R \subset \mathbb{T}^2$ is a (topological) rectangle if it is homeomorphic to the unit square $[0,1]^2$. Given an invariant set $C \subset \mathbb{T}^2$ of $f \in \text{Homeo}_0(\mathbb{T}^2)$, we say that a family of pairwise disjoint rectangles $\mathcal{R} = \{R_0, \ldots, R_N\}$ is a partition of $C$ if $C \subset \bigcup_{i=0}^N R_i$. In this case, we let $\Sigma := \{0, \ldots, N\}^\mathbb{Z}$ and denote by $\mathcal{S}$ the set of those sequences $\omega \in \Sigma$ for which there exists $x \in C$ with $f^n(x) \in R_{\omega(i)}$ for all $i \in \mathbb{Z}$. Then $\mathcal{S}$ is compact and invariant under the shift $\sigma$ on $\mathcal{S}$. If it happens, as in the classical horseshoe construction, that for every sequence $\xi \in \mathcal{S}$ there is a unique $x \in C$ with $f^n(x) \in R_{\omega(i)}$ for all $i \in \mathbb{Z}$, then the map $h_{\mathcal{R}} : \mathcal{S} \to C$ taking $\xi$ to the corresponding $x$ is a conjugacy from $\sigma^\mathcal{S}$ to $f_C$. This happens to be the case for any zero-dimensional hyperbolic set $C$ in $\mathbb{T}^2$ with local product structure. In fact, in this situation the partition $\mathcal{R}$ can be chosen such that it is a Markov partition, that is, $\mathcal{S}$ is a subshift of finite type (see [9, 15]). If $\mathcal{S} = \Sigma$, we call $C$ a horseshoe and say it is rotational if in addition the following two properties are satisfied: (i) there exists a bounded topological disk $D \subset \mathbb{R}^2$ such that $\pi(D) \subset \mathbb{T}^2$ is a topological disk and $\bigcup_{i=0}^N R_i \subset \pi(D)$; (ii) for each $i = 0, \ldots, N$ there exists a unique vector $v_i$ such that if $z \in D \cap \pi^{-1}(C)$ and $\pi(z) \in R_i$, then $f(z) \in D + v_i$. In other words, in a rotational horseshoe the symbolic coding determines to which copy of $\Sigma$ a point is mapped by $D$. As mentioned before, this allows to compute rotation sets and rotation vectors on a purely symbolic level.

More precisely, given a finite word $w = w_1 \ldots w_m$, let $|w| = m$ be the length of $w$ and $\psi(w) = \sum_{j=1}^m v_{w_j}$. Further, for a closed and $\sigma$-invariant set $M \subset \mathcal{S}$ we define

$$ \rho_M = \left\{ \lim_{n \to \infty} \frac{\psi(w^{(n)})}{|w^{(n)}|} \mid w^{(n)} \text{ is a subword of some } \omega^{(n)} \in M \text{ and } |w^{(n)}| \geq n \right\}. $$

The following lemma provides the crucial estimate that allows to translate these symbolic to dynamical rotation sets. Given $\omega \in \Sigma$, we let $\omega_{[1,n]} = \omega(1)\omega(2)\ldots\omega(n)$.

**Lemma 2.1** ([9], Proposition 2.1). There exists $r > 0$ so that for any $z \in C$ we have $F^{n+1}(z) - z \in B_r(\psi(h_{\mathcal{R}}^{-1}(z)_{[1,n]})))$.

As a direct consequence, we obtain

**Corollary 2.2.** $\rho_M = \rho_{\mathcal{R}(\mathcal{S}(M))}(F)$.

A sequence $\omega \in \Sigma$ is almost periodic if any finite subword occurs infinitely often and the time between two occurrences is uniformly bounded. It is well-known that $\omega$ is almost periodic if and only if $\mathcal{O}_x(\omega)$ is minimal. Moreover, in this case $\mathcal{O}_x(\omega)$ coincides with the set of those sequences $\xi \in \Sigma$ which have exactly the same subwords as $\omega$ [10]. Together with Corollary 2.2, this yields the following statement.

**Proposition 2.3.** Given an almost periodic sequence $\omega \in \Sigma$, the set $M = \mathcal{O}_f(h(\omega))$ is minimal with respect to $f$ and we have

$$ \rho_M(F) = \left\{ \lim_{n \to \infty} \frac{\psi(w^{(n)})}{|w^{(n)}|} \mid w^{(n)} \text{ is a subword of } \omega, |w^{(n)}| \geq n \right\}. $$

For constructing suitable almost periodic sequences, it is convenient to work only in the one-sided shift space. Due to the following folklore lemma, this is sufficient.

**Lemma 2.4.** Suppose $\omega^+ \in \Sigma^+$ is almost periodic and $\omega$ is any sequence in $\Sigma$ whose right side coincides with $\omega^+$. Then $\mathcal{O}_x(\omega)$ is minimal and coincides with the set of sequences that have exactly the same finite subwords as $\omega^+$. 
A particular case of almost periodic sequences are Toeplitz sequences. A sequence \( \omega^+ \in \Sigma^+ \) (\( \xi \in \Sigma \)) is called a \textit{Toeplitz sequence} if for every \( j \in \mathbb{N} \) (\( j \in \mathbb{Z} \)) there exists \( p \in \mathbb{N} \) so that \( \omega^+_{j+n} = \omega^+_j \) for all \( n \in \mathbb{N} \) (\( \omega^+_{j+np} = \omega^+_j \) for all \( n \in \mathbb{Z} \)). In other words, every entry of a Toeplitz sequence occurs periodically. However, since the periods depend on the position, the sequence itself need not be periodic. In fact, aperiodicity is often included in the definition, and we will follow this convention here.

3. Realisation of rotation sets by Toeplitz sequences

3.1 Preliminary notions. We fix \( f \in \text{Homeo}_0(T^2) \) such that \( f \) has a rotational horseshoe \( C \) with three symbols and displacement vectors \( v_0 = (0,0) \), \( v_1 = (1,0) \) and \( v_2 = (0,1) \), as in Figure 1. Thus, there exists a bounded topological disk \( D \subseteq \mathbb{R}^2 \) and a partition \( \mathcal{R} = \{ R_0, R_1, R_2 \} \) of \( C \) with \( \bigcup_{i=0}^2 R_i \subseteq \pi(D) \) such that \( F(\pi^{-1}(R_i) \cap D) \subseteq D + v_i \). As before, we denote by \( h_R \) the conjugacy between the shift \( \sigma \) on \( \Sigma := \{0,1,2\}^\mathbb{Z} \) and \( f \bigcirc \). As we will see in Section 4, the family of such maps is open and dense in the set \( F \subseteq \text{Homeo}_0(T^2) \) of torus homeomorphisms with non-empty interior rotation sets. According to Corollary 2.2 and Proposition 2.3, our aim is to construct almost periodic sequences whose associated rotation sets are line segments of positive length, separate the plane or have non-empty interior. To that end, we first introduce a general block structure which produces Toeplitz sequences through an inductive construction.

A general block structure. Suppose \((b_n)_{n \in \mathbb{N}} \) and \((d_n)_{n \in \mathbb{N}} \) are sequences of positive integers, with \( d_{n+1} \) a multiple of \( d_n \) for all \( n \in \mathbb{N} \). Let \( a_1 \in \mathbb{N} \). Slightly abusing notation, we denote by \([k,l]\) the interval of all integers \( i \) with \( k \leq i \leq l \), similarly for open and half-open intervals.

Then we recursively define

- \( a_{n+1} = (b_n d_n + 1)a_n \)
- \( A_n = [1, a_n] + d_n a_n \mathbb{N} \)
- \( B_n = \bigcup_{i=1}^n A_n \)
- \( C_n = B_n \backslash B_{n-1} \)

We call the maximal intervals in \( A_n \) blocks of level \( n \). If such a block is not equal to the first block \([1, a_1] \), we call it a repeated block. The following facts are easy to check.

(F1) Given \( k < k' \), any block of level \( k' \) starts and ends with a block of level \( k \).

(F2) If two blocks of levels \( k \) and \( k' \) are disjoint and \( k \leq k' \), then the interval between the blocks has length \( \geq (d_k - 1)a_k \).

(F3) The asymptotic density of \( B_n \) is at most \( \delta_n = \sum_{j=1}^n \frac{1}{d_j} \).

(F4) If \( J \) is an interval of integers whose length is a multiple of \( a_n d_n \), then \( \frac{1}{M} |J \cap B_n| \leq \delta_n \).

Consequently, given \( M \in \mathbb{N} \) and any interval \( J' \) of length \( \geq a_n d_n / M \) we have \( \frac{1}{M} |J' \cap B_n| \leq M \delta_n \). Here \( |J| \) denotes the cardinality of a set \( J \subseteq \mathbb{N} \).

(F5) If a sequence \( \omega = (a_i)_{i \in \mathbb{N}} \) is chosen so that for all \( n \in \mathbb{N} \), \( j \in [1, a_n] \) and \( k \in \mathbb{N} \) it satisfies \( a_{j+k} = a_{j+k(a_n d_n)} = a_j \), then \( \omega \) is Toeplitz.

We let \( \delta_\infty = \lim_{n \to \infty} \delta_n = \sup_n \delta_n \).

3.2 Line segments. We first need to specify the open set \( V \subseteq \mathbb{R}^2 \) in Theorem 1.1. In principle, we could take the whole interior of the simplex \( \Delta \) spanned by the vectors \( v_0, v_1 \) and \( v_2 \) defined above. However, for the sake of convenience we let \( \alpha = \alpha(v) := \left\langle v_1, \frac{v_2}{||v_2||} \right\rangle \), \( \beta = \beta(v) := \left\langle v_2, \frac{v_1}{||v_1||} \right\rangle \) and define \( V \) as the subset of vectors in \( \Delta \) for which \( ||v|| \leq \min(\alpha, \beta) \), which will simplify our construction below to some extent.

Given \( \omega^+ \in \Sigma^+ \), we denote by \( \mathcal{M}(\omega^+) \) the omega-limit set of a sequence \( \omega \in \Sigma \) whose right side coincides with \( \omega^+ \). According to Proposition 2.3, \( \mathcal{M}(\omega) \) is a minimal set, and the subwords of sequences in \( \mathcal{M}(\omega) \) are exactly the subwords of \( \omega^+ \). Given \( v \in V \), our aim is now to construct a one-sided sequence \( \omega_v = (\omega_v(j))_{j \in \mathbb{N}} \) such that \( \rho_{\mathcal{M}(\omega_v)} \) is a line segment of positive length contained in \( v + \mathbb{R} \omega^+ \). To that end, we use the above general block structure with the following specifications. We let \( b_n = 1 \) and \( d_n = 2^{n+1} \) for some integer \( t \) such that \( \delta_\infty \leq \frac{\max(||v||)}{10 \max(\alpha, \beta)} \). We start the construction with an integer \( a_1 \geq 2M/||v|| + 1 \).
where $M = \|v\| + \max\{\alpha, \beta\}$. Further, we let $D(l, j) = \left(\sum_{i=1}^{l} v_{\omega_i(i)} \frac{v_i}{\|v_i\|} - (j - l + 1)\|v\|\right)$. The sequence $\omega_v$ will be constructed by induction on the sets $C_n$. To that end, we first define $\omega_v$ on $[0, a_n] \cap C_n$ and then extend it to the whole of $C_n$ by $a_n d_n$-periodic repetition. On $[0, a_n]$, we choose the entries $\omega_v(j)$ by induction on $j$ according to the following rules.

(I) If $n$ is odd, we let $i = 1$, if $n$ is even we let $i = 2$.

(II) If neither of $j, j+1, \ldots, j + K$ intersects a block of level $< n$, then we choose $\omega_v(j) \in \{0, i\}$ such that $D(1, n)$ is contained in the interval $[0, M]$. If this is true for both possible choices 0 and $i$, we let $\omega_v(j) = 0$.

(III) If $B = [m + 1, m + a_k] \subseteq [0, a_n]$ is a block of level $k < n$ which is not contained in a larger block of level $< n$, then we choose $\omega_v(m + K + 1), \ldots, \omega_v(m)$ such that $D(1, j) \in [-M, M]$ for all $j = m - K + 1, \ldots, m$ and $D(1, m) \in [-D(1, a_k), M - D(1, a_k)]$. In order to make this choice unique, we require in addition that $D(1, j)$ always takes the smallest value which is possible under these conditions. This means we put 0 whenever possible, and $i$ only when necessary.

In order to see that these rules are consistent, note that if $\omega_v(j) = 0$, then $D(1, j) = D(1, j - 1) - \|v\|$, if $\omega_v(j) = 1$ then $D(1, j) = D(1, j - 1) + \alpha$ and if $\omega_v(j) = 2$ then $D(1, j) = D(1, j - 1) + \beta$. In each step, we therefore have the choice to either increase or decrease the value of $D(1, j)$. Thus, if $D(1, j - 1) \in [0, M]$, then due to the choice of $M$ we can always choose $\omega_v(j)$ in such a way that $D(1, j) \in [0, M]$ as well. Since rule (III) ensures that $D(1, j - 1) \in [0, M]$ whenever $j - 1$ is the end of a block of level $< n$, it is possible to follow rule (II) whenever it applies. If $j = m - K + 1$, where $m + 1$ is the starting point of a block of level $< n$ and $D(1, j - 1) \in [0, M]$, then choosing $\omega_v(i) = 0$ for all $i = j, \ldots, m$ would yield $D(v) \leq M - K\|v\| \leq -M$. Thus, by replacing some of the zeros with $i$'s, it is also possible to meet the requirements of rule (III). Note here that due to the choice of $K = a_1 - 1$ and the spacing of the blocks, the integers $j, \ldots, m$ are not contained in any repeated block of level $< n$. Altogether, this implies that the above algorithm yields a well-defined sequence $\omega_v$. Furthermore, by construction we obtain that $|D(1, j)| \in [0, M]$ whenever $j$ is not contained in a repeated block.

In order to ensure that $\rho(\omega_v) \subseteq v + \mathbb{R}v^+$, we need to show that

$$\lim_{n \to \infty} \frac{1}{n} \max \{|D(i, j)| \mid |j - i| \leq n\} = 0.$$ 

Since the $a_n$ grow super-exponentially, this will be a direct implication of the following.

**Proposition 3.1.** If $0 < j - i \leq a_n$, then $|D(i, j)| \leq 2nM + 1$.

For the proof, we need to introduce some further notation. We say that $j \in \mathbb{N}$ has depth $d$, and write $\text{depth}(j) = d$, if $d$ is the maximal integer such that $j \in B_d$ and $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_d$ is a nested sequence of blocks with $\min B_i < \min B_{i+1}$ and $\max B_i > \max B_{i+1}$ for all $i = 1, \ldots, d - 1$. Note that the nested sequence could be given by only one block $B_1 = [1, a_n]$, but it always exists since every integer is contained in some initial block. For the same reason, $B_1$ will always be an initial block and $B_d$ is the largest repeated block that contains $j$. Note also that the level of the blocks is decreasing, and if $j \in [1, a_n]$ then depth$(j) \leq n$. Moreover, if $n$ is the smallest integer such that $j \in [1, a_n]$, then $B_1$ is equal to $[1, a_n]$.

**Lemma 3.2.** We have $|D(1, j)| \leq M \text{depth}(j)$ for all $j \in \mathbb{N}$. In particular $|D(1, j)| \leq Mn$ for all $j \in [1, a_n]$.

**Proof.** We prove the lemma for all $j \in [1, a_n]$ by induction on $n$. The statement holds for $j \in [1, a_1]$, since on this interval we apply rule II to all $j$ and consequently $D(1, j) \in [0, M]$. Assume that the estimate holds for all $j \in [1, a_n]$ and let $j' \in [1, a_{n+1}]$. If depth$(j') = 1$, the statement holds by construction. Note here that if $\omega_v(j')$ is chosen according to rule II, then $|D(1, j')| \in [0, M]$, whereas if we apply rule III then $|D(1, j')| \in [-M, M]$.

Now, assume that depth$(j') = d$ and the block $B_2 = [m + 1, m + a_k] \subseteq [1, a_{n+1}]$ is of level $k \leq n$. Then $m$ is not contained in any block and thus $|D(1, m)| \leq M$ again by construction.
Further, we have $\text{depth}(j' - m) \leq \text{depth}(j') - 1$, and consequently
\[
|D(m + 1, j')| = |D(1, j' - m)| \leq M(\text{depth}(j') - 1).
\]
Note that here $\omega_v(m + 1), \ldots, \omega_v(m + a_k) = \omega_v(1), \ldots, \omega_v(a_k)$. Together, we obtain
\[
|D(1, j')| \leq |D(1, m)| + |D(m + 1, j')| \leq C \text{depth}(j')
\]
as required. \hfill \Box

Proof of Proposition 3.1. Fix $k > 0$. We proceed again by induction on $n$. Assume that the statement holds for all $i, j \in [1, a_1], l \leq n$, and suppose that $i, j \in [1, a_{n+1}]$ with $|D(i, j)| > 2Mk + 1$. We have to show that $j - i > a_k$.

If $i, j$ are both contained in a repeated block $B = [m + 1, m + a_p]$ of level $p \leq n$, then $|D(i, j)| = |D(i - m, j - m)|$ and the induction statement applies. Thus, we may assume that this does not happen. Due to Lemma 3.7 we have
\[
|D(i, j)| \leq |D(1, j)| + |D(1, i - 1)| \leq M(\text{depth}(j) + \text{depth}(i - 1)),
\]
so that either $j$ or $i - 1$ has depth bigger than $k$. We distinguish three cases.

First, if both have depth bigger or equal to $k$, then as they cannot both be contained in a single repeated block, they have to be contained in disjoints blocks of level bigger than or equal to $k$. However, as two such blocks are at least $(d_k - 1)a_k$ apart, the statement follows.

Secondly, assume that $d = \text{depth}(i - 1) > k$ and $\text{depth}(j) \leq k$. Let $B_1 \supseteq \cdots \supseteq B_d$ be a nested sequence of blocks as in the definition of $\text{depth}(i - 1)$, with $B_1 = [1, a_{n+1}]$ and $B_2 = [m + 1, m + a_p]$. Since $\text{depth}(i - 1) > k$, we have $p \geq k$. In the case $i - 1 \notin [m + a_p - a_k + 1, m + a_p]$, we have $j - i > a_k$ as required. Otherwise, we have that
\[
|D(i, m + a_p)| = |D(i - m - a_p + a_k, a_k)| = |D(1, a_k)| + |D(1, i - m - a_p + a_k - 1)| \leq Mk + 1,
\]
using $D(1, a_k) \in [0, M]$ and Lemma 3.7. Consequently, we obtain
\[
|D(i, j)| \leq |D(i, m + a_p) + D(m + a_s + 1, j)| = |D(i, m + a_p) - D(1, m + a_s) + D(1, j)| \leq Mk + 1 + M \text{depth}(j) \leq 2Mk + 1,
\]
contradicting our assumption. Finally, the case $\text{depth}(i - 1) \leq k$ and $\text{depth}(j) > k$ can be treated in an analogous way. \hfill \Box

As mentioned above, Proposition 3.1 implies that $\rho_{\mathcal{M}(\omega_i)} \subseteq v + Ru^\perp$, and if we let $M_i = h_R^{-1}(\mathcal{M}(\omega_i))$, then according to Corollary 2.2 the same will be true for the rotation set $\rho_{M_i}(F)$. It remains to show that $\rho_{\mathcal{M}(\omega_i)}$ is a segment of positive length. To that end, we note that for all $n \in \mathbb{N}$ and $j \in [1, a_n] \cap C_n$ we have $\omega_v(j) \in \{0, i\}$, where $i = 1$ if $n$ is odd and $i = 2$ if $n$ is even. In the first case, \((F4)\) implies that the fraction of $2'$s in the interval $[1, a_n]$ is bounded by $\delta = \frac{1}{10\max\{\alpha, \beta\}}$. At the same time, the requirement that $D(1, a_n) \in [0, M]$ implies that a proportion of $\|v\|/\max\{\alpha, \beta\}$ symbols of $[1, a_n]$ must be non-zero. This yields that the frequency of $1$'s in $[1, a_n]$ is greater than $96$. For even $n$, we obtain exactly the opposite estimates for the frequencies of $1$'s and $2$'s. In the limit $n \to \infty$, this yields the existence of two distinct vectors in $\rho_{\mathcal{M}(\omega_i)}$. This completes the proof of Theorem 1.1(a).

3.3. Plane separating continua. For the construction we make again use of the general block structure presented above, this time with the following specifications.

(i) We choose $(d_n)_{n \in \mathbb{N}}$ so that all $d_n$ are even and $\delta_\infty \leq \frac{1}{32}$.

(ii) We choose integers $K \geq 17$ and $L \geq 64$ and let $a_1 = (3L + 4)K$ and $b_n = (3L + 4)K$ for all $n \in \mathbb{N}$. The sequence $(a_n)_{n \in \mathbb{N}}$ is then defined inductively by $a_{n+1} = (b_n d_n + 1)a_n$, according to the general block structure introduced above.

(iii) Note that due to the choice of $b_n$ we have $a_{n+1} \geq 8a_n b_n$ for all $n \in \mathbb{N}$, which implies in particular that $\sum_{j=1}^{n-1} a_j d_j \leq a_n d_n/2$. 

Then we construct $\omega = (\omega_n)_{n \in \mathbb{N}}$ inductively on the sets $\mathcal{A}_n$ as follows. Suppose $\omega_j$ is defined for all $j \in [1, a_n]$, and hence for all $j \in \mathcal{A}_n$ (recall $\omega_{j+k_0a_n} = \omega_j$ for all $j \in [1, a_n]$ and $k \in \mathbb{N}$). We extend the definition to $[1, a_{n+1}]$, and thus to $\mathcal{A}_{n+1}$, as follows. Let

$$p_n = (L + 1)Ka_n d_n - a_n d_n + 1 + \sum_{j=1}^{n} a_j d_j/2,$$

(3.2)

$$q_n = (L + 1)Ka_n d_n + a_n d_n - \sum_{j=1}^{n} a_j d_j/2 - 1.$$

Then divide $[1, a_{n+1}]$ into the following seven intervals (see Figure 3.1).

- $I^0_1 = [1, (LKd_n + 1)a_n]$
- $I^1_1 = [(LKd_n + 1)a_n + 1, p_n - 1]$
- $I^2_1 = [p_n, q_n]$
- $I^1_2 = [q_n + 1, (L + 2)Kd_n a_n]$
- $I^0_2 = [(L + 2)Kd_n a_n + 1, ((2L + 2)Kd_n + 1)a_n]$
- $I^2_2 = [((2L + 2)Kd_n + 1)a_n + 1, (2L + 4)Kd_n a_n]$
- $I^0_3 = [(2L + 4)Kd_n a_n + 1, a_{n+1}]$

Due to the choice of $p_n$ and $q_n$, the following properties are easy to verify.

- (PQ1) The intervals $I^0_1$, $I^2_2$ and $I^0_3$ all have the same length $(LKd_n + 1)a_n$ and start and end with a block of level $n$ (and thus with blocks of all levels $k \leq n$).
- (PQ2) The length of $I^1_1$ and $I^2_2$ is between $(K - 1)a_n d_n$ and $K a_n d_n$.
  (Note here that due to (iii) we have $\sum_{j=1}^{n} a_j d_j/2 \leq a_n d_n/4$.)
- (PQ3) The length of $I^2_2$ is between $a_n d_n/2$ and $a_n d_n$.
- (PQ4) The interval $I^3_2$ is concentric around a block $B_n$ of level $n$.
- (PQ5) The distance of $p_n$ and $q_n$ to any block $B_k$ of level $k \leq n$ is at least $a_k d_k/4$.
  (In order to check this for $p_n$, note that for each $k \leq n$ a block of level $k$ starts at $(L + 1)Ka_n d_n - a_n d_n + 1 + \sum_{j=k+1}^{n} a_j d_j/2$ and $\sum_{j=1}^{k} a_j d_j/2 \leq 3a_k d_k/4$. A similar comment applies to $q_n$.)
- (PQ6) The intervals $I^*$ and $I^2_2$ both have length $(2Kd_n - 1)a_n$.

We define

$$\omega_j = \begin{cases} 
0 & \text{if } j \in I_0 \setminus B_n \\
1 & \text{if } j \in I_1 \setminus B_n \\
2 & \text{if } j \in I_2 \setminus B_n
\end{cases}$$

(3.3)

for all $j \in [1, a_n] \setminus B_n$ and $\omega_{j+k_0a_n} = \omega_j$ for all $j \in [1, a_n]$ and $k \in \mathbb{N}$. By induction on $n \in \mathbb{N}$ this yields a sequence $\omega = (\omega_j)_{j \in \mathbb{N}}$, which follows our general block structure introduced above and is, in particular, Toeplitz.

Recall that $v_0 = (0, 0)$, $v_1 = (1, 0)$ and $v_2 = (0, 1)$ are the integer vectors associated to the partition and for every interval $J \subseteq \mathbb{N}$ we write $\rho(J) = \frac{\psi(J)}{|J|}$, where $\psi(J) = \sum_{j \in J} v_{\omega_j}$.

Lemma 3.3. $\rho([1, a_{n+1}]) \in B^2_{\omega}(v_0)$ for all $n \in \mathbb{N}_0$. 

![Figure 3.1. The configuration of intervals.](image-url)
Proof. With the above notions, we have that \( \{ j \in [1, a_{n+1}] : \omega_j \neq 0 \} \subseteq ([1, a_{n+1}] \cap B_n) \cup I^1 \cup I^2 \). By (F4), we have \( |[1, a_{n+1}] \cap B_n| \leq \delta_\infty a_{n+1} \leq \frac{1}{\delta_\infty} a_{n+1} \). Further, we have that \( |I^1 \cup I^2| \leq \frac{1}{\delta_\infty} a_{n+1} \leq \frac{1}{\delta_\infty} a_{n+1} \). The statement follows.

\( \square \)

Lemma 3.4. Suppose \( J = [1, m] \) or \( J = [m, a_{n+1}] \) for some \( n \in \mathbb{N}_0 \) and \( m \in (1, a_{n+1}) \). Then \( \rho(J) \in B_k(v_0) \).

Proof. We proceed by induction on \( n \). Assume that the statement holds for \( n \in \mathbb{N}_0 \). Let \( J = [1, m] \subset [1, a_{n+1}] \) (the proof for \( J = [m, a_{n+1}] \) is similar). We distinguish several cases.

**Case 1:** \( J \subseteq I_1^0 \). Let

\[ J \cap B_n = B_1 \cup \cdots \cup B_j \cup B', \]

where \( B' = B_{j+1} \cap J \) and \( B_i, i = 1, \ldots, j + 1 \), are full blocks of level \( \leq n \). By the previous lemma \( \rho(B_i) \in B_k(v_0) \) for \( i = 1, \ldots, j \). Due to the induction hypothesis \( \rho(B_i) \in B_k(v_0) \).

**Case 2:** \( J \subseteq I_1^0 \cup I^* \) but \( J \not\subseteq I_1^0 \). By (F4) we have that \( |I_1^0 \cap B_n| \leq \frac{1}{\delta_\infty} |I_1^0| \), and hence \( \rho(I_1^0) = B_k(v_0) \). Further \( |J \cap I^*| \leq |I^*| \leq \frac{1}{\delta_\infty} |I_1^0| \). Thus \( \rho(J) \in B_k(v_0) \).

The remaining cases \( J \subseteq I_1^0 \cup I^* \cup I_2^0 \), \( J \subseteq I_1^0 \cup I^* \cup I_2^0 \cup I_2^1 \) and \( J \subseteq [1, a_{n+1}] \) can be treated by similar arguments. \( \square \)

Lemma 3.5. Suppose that \( J \subseteq I_j^i \) is an interval of length \( \geq a_n d_n/2 \), where \( i = 0, 1, 2 \) and \( j = 1, 2 \) or \( i = 0 \) and \( j = 3 \). Then \( \rho(J) \in B_k(v_i) \).

Proof. By (F4), we obtain that \( |J \cap B_n|/|J| \leq \frac{1}{\delta_\infty} \). Since all free positions in \( J \) are filled by \( \delta_i \)'s in the \( (n+1) \)st step of the construction, this implies the statement. \( \square \)

Corollary 3.6. For \( i = 0, 1, 2 \), \( j = 1, 2 \) or \( i = 0 \) and \( j = 3 \) we have \( \rho(I_j^i) \in B_k(v_i) \).

Lemma 3.7. Let \( J \subseteq I \subseteq \mathbb{N} \) be intervals and assume that \( J \) has one endpoint in common with \( I \).

(a) If \( I = I_1^j \), \( j = 1, 2, 3 \), then \( \rho(J) \in B_k(v_0) \).

(b) If \( I = I_2^j \), \( j = 1, 2 \), then \( \rho(J) \in B_k(v_1) \).

(c) If \( I = I_2^j \), \( j = 1, 2 \), then \( \rho(J) \in B_k(v_2) \).

(d) If \( I = I^* \), then \( \rho(J) \in B_k(v_1) \).

Proof. (a) If \( J \subseteq I_1^i \) starts with \( 1 \) or \( J \subseteq I_1^i \) ends with \( a_{n+1} \), then the statement is contained in Lemma 3.4. However, by construction the configurations of symbols in the intervals \( I_1^0, I_2^0 \) and \( I_2^1 \) are identical. In order to see this, note that since these have the same length and all start and end with a block of level \( n \) by (PQ1), the configuration of the blocks is identical, and all positions not contained in previous blocks are filled by \( 0 \)'s. Hence, by symmetry the statements for \( I_1^0 \) and \( I_2^0 \) extend to the other intervals.

(b) and (c) The proofs of all cases of (b) and (c) are similar. Hence, we consider only one of them. The crucial observation is the fact that all endpoints of these intervals have distance \( \geq a_k d_k/4 \) to any block of level \( k \). For the points \( p_n \) and \( q_n \), this is true by construction, see (PQ5). For endpoints of \( I_1^0, I_2^0 \) and \( I_2^1 \) this holds since the adjacent intervals \( I_2^j, j = 1, 2, 3 \), start and end with blocks of level \( n \). Hence, the nearest block of any level \( k \leq n \) in one of the considered intervals can appear at distance \( (d_k - 1)a_k \) to the boundary points.

Assume that \( I = I_2^1 \) and \( J = [p_n, k] \). If \( J \) does not intersect any blocks of level \( k \leq n \), then \( \rho(J) = v_2 \). Otherwise, let \( m \) be the largest integer such that \( J \cap B \neq \emptyset \) for some \( m \)-block \( B \). Then by (PQ5) the length of \( J \) is at least \( a_m d_m/4 \), and due to (F4) we obtain that \( |J \cap B_n|/|J| \leq 4 \delta_\infty \leq \frac{1}{8} \), which implies \( \rho(J) \in B_k(v_2) \). As mentioned, the other cases are analogous.
Proof. We have that \( \rho(I_j^1) \in B_{\frac{1}{16}}(v_1) \) by Corollary 3.6 and \( |I_j^2|/|I_j^1| \leq \frac{1}{16} \).

□

Proposition 3.8. Let \( T = \{ \lambda v_i + (1 - \lambda)v_j : i, j \in \{0, 1, 2\}, \lambda \in [0, 1]\} \) and \( S = B_{\frac{1}{16}}(T) \). Then \( \rho(J) \in S \) for every \( J = [a, b] \subset \mathbb{N} \).

Proof. Let \( n \) be the smallest integer such that \( J \) is contained in a block of level \( n + 1 \). We prove the statement by induction on \( n \) and may thus assume that \( J \) is not entirely contained in any block of level \( k \leq n \). Moreover, since the structure inside all blocks of level \( n + 1 \) is the same, we may assume without loss of generality that \( J \subseteq [1, a_{n+1}] \). We distinguish several cases.

Case 1. Suppose that \( J \) intersects both \( I^* \) and \( I_2^0 \). In this case \( J \) contains \( I_0^0 \), and by Corollary 3.6 we have \( \rho(I_0^0) \in B_{\frac{1}{16}}(v_0) \). Moreover, \( |I^* \cup I_0^0|/|I_0^0| \leq \frac{1}{16} \leq 1/16 \). If \( J \) also intersects the intervals \( I_1^0 \) and \( I_3^0 \), say \( J = J \cap I_1^0 \) and \( J'' = J \cap I_3^0 \), then \( \rho(J'), \rho(J'') \in B_{\frac{1}{16}}(v_0) \) by Lemma 3.7(a). Consequently, \( J \subseteq B_{\frac{1}{16}}(v_0) \).

Case 2. Suppose that \( J \) intersects exactly two of the five intervals \( I_1^0, I^*, I_0^0, I_2^0, I_3^0 \). Since all the subcases are similar, we only treat one and assume \( J \) intersects \( I_1^0 \) and \( I^* \). Let \( J' = J \cap I_1^0 \) and \( J'' = J \cap I^* \). Then \( \rho(J') \in B_{\frac{1}{16}}(v_0) \) by Lemma 3.7(a), whereas \( \rho(J'') \in B_{\frac{1}{16}}(v_1) \) by Lemma 3.7(d). Consequently, \( \rho(J) \) is a convex combination of a vector in \( B_{\frac{1}{16}}(v_0) \) and a vector in \( B_{\frac{1}{16}}(v_1) \), and therefore belongs to \( S \).

Case 3. Suppose \( J \subseteq I^* \) intersects at least two of the intervals \( I_1^0, I_2^0, I_3^0 \). Let \( J = J' \cup J'' \cup J''' \), where \( J' = J \cap I_1^0 \), \( J'' = J \cap I_2^0 \), and \( J''' = J \cap I_3^0 \). Then \( \rho(J'), \rho(J''), \rho(J''') \in B_{\frac{1}{16}}(v_1) \) by Lemma 3.7(b), and \( \rho(J'') \in B_{\frac{1}{16}}(v_2) \) by Lemma 3.7(c). Hence, we obtain again that \( \rho(J) \in S \).

Case 4. Finally, suppose that \( J \) is contained in just one of the seven intervals of the decomposition of \([1, a_{n+1}]\), say \( J \subseteq I_j^1 \). Then

\[
J \cap B_n = B' \cup B_1 \cup \ldots \cup B_{m+1} \cup B''',
\]

where \( B' = J \cap B_1 \), \( B'' = J \cap B_{m+1} \), and the \( B_i \) with \( i = 0, \ldots, m + 1 \) are those maximal blocks contained in \( B_n \) which intersect \( J \), ordered in an increasing way. Since \( J \) is not entirely contained in one block, we can use Lemmas 3.3 and 3.4 to see that \( \rho(B'), \rho(B_1), \ldots, \rho(B_{m+1}), \rho(B''') \in B_{\frac{1}{16}}(v_1) \), and hence \( \rho(J \cap B) \in B_{\frac{1}{16}}(v_0) \). At the same time we have \( \rho(J \setminus B) = v_1 \), such that again \( \rho(J) \) is contained in \( S \).

□

Proposition 3.9. \( \rho(\partial(O(\omega_{\sigma}, \sigma))) \) separates the plane.

Proof. We have that

\[
\rho(\partial(O(\omega_{\sigma}, \sigma))) = \bigcap_{k \in \mathbb{N}} \bigcup_{n \geq k} \rho[J] \subseteq \mathbb{N}
\]

where \( k = \{ \rho(J) \mid J \subseteq \mathbb{N} \) is a finite interval with \( |J| = n \} \).

Given \( 0 \leq i < j \leq 2 \) we let \( S_{ij} = \{ \lambda v_i + (1 - \lambda)v_j : \lambda \in [0, 1]\} \). For \( n \in \mathbb{N} \), let \( J_1 = I_1^0 = [p_n, q_n] \) and choose an interval \( J_2 \subseteq I_2^0 \) which has the same length as \( J_1 \) and is concentric around a block of level \( n \). Since this also holds for \( J_1 \), we have that the configuration of blocks inside both intervals is the same. Since free positions in both intervals are both filled by 2’s, we have that \( \rho(J_1) = \rho(J_2) \in B_{\frac{1}{16}}(v_2) \).

Let \( M_n \in \mathbb{N} \) be such that \( J_2 = J_1 + M_n = \{ j + M_n \mid j \in J_1 \} \) and let \( \rho_i = \rho(J_1 + i) \) for \( i = 0, \ldots, M_n \). Then all the \( \rho_i \) belong to \( S' \), and the distance between \( \rho_i \) and \( \rho_{i+1} \) is at most \( 2/|J_1| \). Moreover, as \( i \) increases from 0 to \( M_n \), the interval \( J_1 + i \) will leave \( I_2^0 \) in order to enter \( I_3^0 \) and eventually enter \( I_4^0 \) until it stops at \( J_1 + M_n = J_2 \).

According to Lemmas 3.3 and 3.7 the corresponding vectors \( \rho_i \) always remain in \( S' \). Further, they start in \( B_{\frac{1}{16}}(v_2) \), then move to \( B_{\frac{1}{16}}(v_1) \) while remaining in \( S_{01} \), then move to \( B_{\frac{1}{16}}(v_2) \) while remaining in \( S_{02} \), and finally return to \( B_{\frac{1}{16}}(v_2) \) while remaining in \( S_{03} \). Note
here that \(|J_i| \geq a_n d_n/2\), such that Lemma 3.5 applies whenever \(J_i + i\) is entirely contained in one of the intervals of the decomposition, and otherwise we can always combine two of the statements of Lemma 3.7.

Since \(|J_i| \not	o \infty\) as \(n \to \infty\), it follows easily from these facts that the upper Hausdorff limit of the sequence of finite sets \(\{\rho^0_n, \ldots, \rho^k_n\} \subseteq k\) contains a continuum \(\mathcal{C}\) that separates the two connected components of the complement of \(S\). Since \(\mathcal{C} \subseteq \rho_{\mathcal{C}(\omega_{\phi}, \sigma)}\), this completes the proof. \(\Box\)

This shows Theorem 1.1(b).

3.4. Non-empty interior. It remains to construct \(\omega' \in \Sigma^+\) such that \(\rho_{M}(\omega')\) has non-empty interior. It turns out that in comparison with the previous cases this is quite easy. We use the same block construction as before, with \(b_n = 1\) for all \(n \in \mathbb{N}\) and \(d_n\) chosen such that \(\delta = \delta_{\infty} < 1/10\). Let \(\Delta_{\delta} = \{sv_1 + tv_2 \mid s, t > \delta, s + t < 1 - \delta\}\) and choose a sequence of vectors \(\rho_n \in \Delta_{\delta}\) such that the coordinates of \(\rho_n\) are integer multiples of \(1/a_n\) and \(\{\rho_n \mid n \in \mathbb{N}\}\) is dense in \(\Delta_{\delta}\). Then we simply define \(\omega'\) inductively on \([1, a_n]\) in such a way that \(\frac{1}{a_n} \sum_{i=1}^{n} v_{\omega'(i)} = \rho_n\) for all \(n \in \mathbb{N}\). This is possible, since in each stage of the construction we have \([1, a_n] \setminus B_{n-1} = a_n - [1, a_n] \cap B_n \geq (1 - \delta_n) a_n\). We thus obtain that \(\Delta_{\delta} \subseteq \bigcap_{n \in \mathbb{N}} \{\rho_k \mid k \geq n\} \subseteq \rho_{M}(\omega')\).

This proves Theorem 1.1(c) and thus completes the proof of Theorem 1.1.

4. Abundance.

Finally, in this section we want to prove that the phenomena given by Theorem 1.1 are abundant, in the sense that they occur for an open set in \(\text{Homeo}_0(T^2)\). Recall that we denote \(\mathcal{F}\) the set of those \(f \in \text{Homeo}_0(T^2)\) having non-empty interior rotation set. The result we want to prove is the following.

**Theorem 4.1.** The family \(\mathcal{G}\) in Theorem 1.1 contains an open and dense set of \(\mathcal{F}\).

This statement essentially follows from series of results on Axiom A diffeomorphisms which is already collected in [9]. We mainly follow that paper and keep the exposition brief. Recall that \(f \in \text{Homeo}_0(T^2)\) is an Axiom A diffeomorphism if the non-wandering set is hyperbolic and contains a dense set of periodic points. We call by \(\mathcal{F}_0 \subset \mathcal{F}\) the set of those axiom A maps having zero-dimensional (totally disconnected) non-wandering set. The elements of \(\mathcal{F}_0\) are called fitted Axiom A.

**Theorem 4.2 ([17]).** The set \(\mathcal{F}_0\) is dense in \(\mathcal{F}\).

**Theorem 4.3 ([15]).** For any \(f \in \mathcal{F}_0\) there is a \(C^0\)-neighborhood \(\mathcal{U}(f)\) of \(f\) so that for all \(g \in \mathcal{U}(f)\) there exists a semiconjugacy \(h\) between \(g\) and \(f\), that is, a continuous onto map \(h\) so that \(h \circ g = f \circ h\). Moreover, the semiconjugacy can be chosen in the homotopy class of the identity.

The last theorem implies in particular that given \(g \in \mathcal{U}(f)\) as above, we have \(\rho_{\mathcal{C}_g}(F) = \rho_{\mathcal{C}_g}(F)\) for any closed invariant set \(\mathcal{C}\) of \(g\). Thus if we prove that \(\mathcal{F}_0 \subset \mathcal{G}\), we automatically have that

\[
\bigcup_{f \in \mathcal{F}_0} \mathcal{U}(f) \subset \mathcal{G}
\]

by means of the last theorem, which proves Theorem 1.2. Thus, the remainder of this section is devoted to showing that \(\mathcal{F}_0 \subset \mathcal{G}\).

Recall that a **basic piece** \(\Lambda \subset T^2\) of a diffeomorphism \(f\) is a hyperbolic transitive set which is locally maximal. Given a set \(X \subset \mathbb{R}^2\) we denote its convex hull by \(\text{conv}(X)\). The proof of the following statement can be found in [9, Corollary 5.2].

**Theorem 4.4.** Every \(f \in \mathcal{F}_0\) has a basic piece \(\Lambda\) so that \(\text{Conv}(\rho_{\Lambda}(F))\) has non-empty interior.
We denote the basic set given in the last theorem by \( \Lambda_{\text{rot}} \). The following results is a mixture of [9, Lemma 5.2] and the proof of [9, Theorem 5.2]. There is only one new consideration which is not done in [9], which is the following.

In [9, Lemma 5.2] the assumption is that we have a basic piece whose rotation set is not a single point, and the conclusion is the existence of a heteroclinic relation for its lift between a fixed point and an integer translation of it. In our situation given by Theorem 4.4 we have a basic piece whose rotation set has at least two non-collinear vectors, and the conclusion we need is the existence of a fixed point of the lift that has heteroclinic relations with two non-collinear integer translations of itself. However, the proof for this case is completely analogous to that of [9, Lemma 5.2]. Then, applying the same arguments as in the proof of [9, Theorem 5.2], one easily obtains the following result.

**Theorem 4.5.** Suppose \( f \in \mathcal{F}_0 \) and let \( \Lambda_{\text{rot}} \) be the basic piece given by Theorem 4.4. Then there exists a positive integer \( n \) and an invariant set \( \Lambda \subset \Lambda_{\text{rot}} \) which admits a Markov partition \( \mathcal{R} = \{ R_0, R_1, R_2 \} \), so that \( \bigcup_{i=0}^2 R_i \) is contained in a topological disk \( D \), \( f|_{\Lambda} \) is conjugated via \( h_\mathcal{R} \) to the full shift on \( \{0,1,2\}^\mathbb{Z} \), and if we consider lifts \( \tilde{R}_0, \tilde{R}_1, \tilde{R}_2 \) of \( R_0, R_1, R_2 \) we have:

- \( F^n(\tilde{R}_0) \cap \tilde{R}_0 \neq \emptyset \),
- \( F^n(\tilde{R}_1) \cap \tilde{R}_1 + v \neq \emptyset \),
- \( F^n(\tilde{R}_2) \cap \tilde{R}_2 + w \neq \emptyset \),

where \( v, w \in \mathbb{Z}^2 \setminus \{0\} \) are non-collinear.

This implies that any element \( f \) in \( \mathcal{F}_0 \) has a power which already has very similar properties to the ones we used in the constructions in the previous sections. In fact, by considering an iterate \( F^n \) of \( f \) and performing a linear change of coordinates on \( \mathbb{T}^2 \), we may assume that \( v = v_1 \) and \( w = v_2 \) (see [10] for details). Therefore \( F^n \) has minimal sets with the rotation sets described in Theorem 4.4. However, since \( \rho_M(F^n) = n \rho_M(F) \), the same applies to \( f \) itself. This completes the proof of Theorem 4.4.

**References**

[1] J. Franks. Realizing rotation vectors for torus homeomorphisms. *Trans. Am. Math. Soc.*, 311(1):107–115, 1989.

[2] J. Llibre and R.S. MacKay. Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity. *Ergodic Theory Dynam. Syst.*, 11:115–128, 1991.

[3] J. Franks. The rotation set and periodic points for torus homeomorphisms. In N. Aoki et al., editor, *Dynamical systems and chaos. Vol. 1: Mathematics, engineering and economics*. Singapore, World Scientific, pages 41–48, 1995.

[4] T. Jäger. Linearisation of conservative toral homeomorphisms. *Invent. Math.*, 176(3):601–616, 2009.

[5] P. Davalos. On torus homeomorphisms whose rotation set is an interval. *Math. Z.*, 2012. To appear, arXiv:1111.2378.

[6] A. Koropecki and F. Armando Tal. Strictly toral dynamics. *Invent. Math.*, 2013. To appear, arXiv:1201.1168.

[7] M. Misiurewicz and K. Ziemian. Rotation sets for maps of tori. *J. Lond. Math. Soc.*, 40:490–506, 1989.

[8] M. Misiurewicz and K. Ziemian. Rotation sets and ergodic measures for torus homeomorphisms. *Fundam. Math.*, 137(1):45–52, 1991.

[9] A. Passaggi. Rational polygons as rotation sets of generic homeomorphisms of the two-torus. *J. Lond. Math. Soc.*, 8(1):235–254, 2014.

[10] J. Kwapisz. Every convex polygon with rational vertices is a rotation set. *Ergodic Theory Dynam. Syst.*, 12:333–339, 1992.

[11] K. Jacobs and M. Keane. 0-1-sequences of toepoldt type. *Z. Wahrsch. verw. Gebiete*, 13(2):123–131, 1969.

[12] N.G. Markley and M.E. Paul. Almost automorphic symbolic minimal sets without unique ergodicity. *Israel J. Math.*, 34(3):259–272, 1979.

[13] S. Williams. Toepoldt minimal flows which are not uniquely ergodic. *Z. Wahrsch. verw. Gebiete*, 67(1):95–107, 1984.

[14] T. Downarowicz. Survey of odometers and Toepoldt flows. *Cont. Math.*, 385:7–38, 2005.

[15] F. Béguin. Classification of Smale diffeomorphisms of surfaces: realisable geometric types. *Ann. Inst. Fourier*, 52(4):1135–1185, 2002.

[16] J. Auslander. *Minimal flows and their extensions*. Elsevier Science Ltd, 1988.

[17] S. Shub and D. Sullivan. Homology theory and dynamical systems. *Topology*, 14:109–132, 1975.

[18] Z. Nitek. On Semi-Stability for Diffeomorphisms. *Invent. Math.*, 14:83–122, 1971.