STRICHARTZ ESTIMATES FOR THE MAGNETIC SCHRÖDINGER EQUATION

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Abstract. We prove global, scale invariant Strichartz estimates for the linear magnetic Schrödinger equation with small time dependent magnetic field. This is done by constructing an appropriate parametrix. As an application, we show a global regularity type result for Schrödinger maps in dimensions $n \geq 6$.

1. Introduction

In this paper, we investigate the global behavior of certain quantum dynamical systems in the presence of magnetic field. To describe the relevant equations, introduce the magnetic Laplace operator

$$\Delta A = \sum_{j=1}^{n} (i\partial_j + A_j)^2$$

The magnetic Schrödinger equation is

$$\begin{cases} 
  u_t - i(\Delta A + V)u = F, \\
  u(0, x) = f(x) \in L^2(\mathbb{R}^n)
\end{cases}$$

(1)

In the physically important case of a real valued $\vec{A}$ and $V$, one has conservation of charge, $\|u(t, \cdot)\|_{L^2} = \|u(0, \cdot)\|_{L^2}$. More generally, by a result of Leinfelder and Simader, [19] if $A \in L^4_{\text{loc}}(\mathbb{R}^n)$ and $\text{div}(A) \in L^2_{\text{loc}}(\mathbb{R}^n)$, $V$ is relatively bounded with bound less than one with respect to $\Delta$, one has that the operator $\Delta A + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. In particular, the spectrum is real and one can define functional calculus.

In this paper, we shall be concerned mainly with the case of time dependent vector potentials $\vec{A}$, which are small and real-valued. This is dictated by certain partial differential equations, appearing naturally in geometry and physics. More specifically, we have in mind the Schrödinger map equation in Hodge gauge [5], [23], the Ishimori system, [17], [22], the Maxwell-Schrödinger system, [11],[24], [31] and several other models, related to the Landau-Lifshitz theory of electromagnetism.

In the case of magnetic-free field ($\vec{A} = 0$), great progress has been made to address the question for global/local existence and uniqueness for solutions of (1), [3], [30]. In particular, when $V$ is small and $n \geq 3$, one can use the standard Strichartz estimates to show by a perturbation argument that the corresponding equation has an unique global solution under reasonable assumptions on the right hand-side and the data $f$. In the same spirit, one can obtain local well-posedness results for large $V$.

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In the magnetics-free case, the Strichartz estimates are well-known and play a fundamental role in proving the existence and uniqueness results alluded to above. Introduce
\[
\|u\|_{L^q_t L^r_x} := \left( \int_0^\infty \left( \int_{\mathbb{R}^n} |u(t,x)|^r \, dx \right)^{\frac{q}{r}} \, dt \right)^{1/q}
\]
for every pair \(q, r \geq 1\) and similarly the mixed Lebesgue spaces \(L^q_t L^r_x \cdots L^r_x\).

We say that a pair of indices \((q, r)\) is Strichartz admissible if \(2 \leq q, r \leq \infty, 2/q + n/r = n/2\) and \((q, r, n) \neq (2, \infty, 2)\). Then, by a classical result of Strichartz, [29], later improved by Ginibre-Velo, [10] and finally Keel-Tao, [14], we have
\[
\begin{align*}
(2) & \quad \|e^{it\Delta} f\|_{L^q_t L^r_x} \leq C \|f\|_{L^2} \\
(3) & \quad \| \int_0^t e^{i(t-s)\Delta} F(s, \cdot) ds \|_{L^q_t L^r_x} \leq C \|F\|_{L^{q'}_t L^{r'}_x},
\end{align*}
\]
where \((\tilde{q}, \tilde{r})\) is another Strichartz admissible pair and \(q' = q/(q-1)\).

Clearly (2) and (3) are equivalent to \(\|u\|_{L^q_t L^r_x} \leq C \|f\|_{L^2} + \|F\|_{L^{q'}_t L^{r'}_x}\), whenever \(u\) is a solution to the free Schrödinger equation with initial data \(f\) and forcing term \(F\).

Another equivalent formulation is that there exists a constant \(C\), so that for all test functions \(\psi\):
\[
\|\psi\|_{L^q_t L^r_x} \leq C (\|\psi(0, \cdot)\|_{L^2} + \|\partial_t \pm i\Delta\psi\|_{L^{q'}_t L^{r'}_x}).
\]
In the sequel, we will make extensive use of all these points of view.

For the case of small (but non-zero) potential \(\vec{A}\), we have \(\Delta_{\vec{A}} = -\Delta + 2i\vec{A} \cdot \nabla + (idiv(\vec{A}) + \sum_j A_j^2)\). = \(-\Delta + 2i\vec{A} \cdot \nabla + \text{small potential}\), we can effectively treat the the magnetic Schrödinger equation in the form
\[
\begin{align*}
(4) & \quad \partial_t u - i\Delta u + \vec{A}(t,x) \cdot \nabla u = F(t,x) \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \\
& \quad u(0, x) = f(x)
\end{align*}
\]
where the terms in the form \((idiv(\vec{A}) + \sum_j A_j^2)u\) are subsumed in the right hand side.

Next, we explain the relevance of the magnetic Strichartz estimates to the Cauchy problem for
\[
(5) \quad \partial_t u - i\Delta u + \vec{A}(u) \cdot \nabla u = F(u).
\]
If \(\vec{A} = 0\), we can clearly use (2), (3) to set up an iteration scheme for the semilinear problem (5) in a ball \(B = B(0, R)\) in the “Strichartz space” \(\cap_{(q,r)} L^q((0,T) L^r\) to solve for arbitrary \(L^2\) data, provided one can show \(\|F(u)\|_{L^1(0,T) L^2} \leq T^\delta M(\|u\|_{L^2(0,T) L^2})\) for some bounded function \(M\). Choosing \(R \sim \|f\|_{L^2}\) and \(T : T^\delta M(R) << R\) makes such scheme successful to show that the solution exists for some time \(T = T(\|f\|_{L^2})\). There are of course, remaining unresolved by this approach, including globality of such solutions\(^1\), smoothness of the solution etc.

Clearly, to study (5) with \(\vec{A} \neq 0\), one cannot use the standard Strichartz estimates, by the obvious derivative loss. One of the goals of this paper is to derive global scale invariant Strichartz estimates under appropriate smallness assumptions on the vector potential \(\vec{A}\).

\(^1\)Note that such approach usually guarantees the global existence of small solutions.
The pioneering work of Barcelo-Ruiz-Vega, [1] has addressed some of these issues\(^2\), but was restricted to (essentially) radial vector potentials \(\vec{A}\). To the best of our knowledge, the results in Theorem 1 below are the first global estimates of such type for Schrödinger equations, that work for general non-radial potentials \(\vec{A}\).

Let us explain the general scheme for applying such Strichartz estimates to concrete quasi-linear PDE’s. Suppose, we have such estimates for the linear gradient Schrödinger equation (4), provided \(\|\vec{A}\|_{Y_T} \leq \varepsilon\), for some concrete Banach space \(Y_T\) appearing in Section 1.1. We apply the magnetic Strichartz estimates to the nonlinear equations of the type (5) as follows. For small initial data \(f\), run an iteration scheme in the ball \(B_X(0, R)\) in an appropriate Strichartz space\(^3\) \(X\), see Section 9 for precise definitions. This is possible if

\begin{itemize}
  \item one can ensure \textit{a priori} the smallness condition \(\|\vec{A}(u)\|_{Y_T} \leq \varepsilon\) for all functions \(u\), that are solutions to (4) satisfying \(\|u\|_X < R\) and for all times \(T \leq T_0 = T_0(R) \leq \infty\).
  \item \(\|F(u)\|_{L^1(0,T)L^2} \leq R/2\), whenever \(u\) is a function with \(\|u\|_X < R\) and for all times \(T \leq T_0 = T_0(R) \leq \infty\).
\end{itemize}

\section{1.1. Strichartz estimates for the magnetic Schrödinger operator.}
The existence and uniqueness problem for (4) has been studied extensively by many authors in the mathematics literature. We should first point to the pioneering work of Doi, [8], [9], who has devised a method to obtain solutions via energy estimates. The approach then relies on cleverly exploiting the properties of pseudodifferential operators of order zero to obtain \textit{a priori} control of \(\|u(t, \cdot)\|_{L^2}\) in terms of \(\|f\|_{L^2}\) and \(\|F\|_{L^1_tL^2}\).

We also mention the far reaching generalization of Doi’s results, due to Kenig-Ponce-Vega, [16]\(^4\). The authors have considered more general equations and were able to derive \textit{a priori} estimates for the \(L^2\) norms of the solution as well as the validity of a local smoothing effect, phenomenon well-known for the potential free case.

Note that (4) has the important scaling invariance \(u \rightarrow u^\lambda(t,x) = u(\lambda^2 t, \lambda x), \ A \rightarrow A^\lambda(t,x) = \lambda A(\lambda^2 t, \lambda x), \ F \rightarrow F^\lambda(t,x) = \lambda^2 F(\lambda^2 t, \lambda x)\). That is, whenever \((u,A,F)\) satisfy (4), so does \((u^\lambda,A^\lambda,F^\lambda)\) with initial data \(f^\lambda(x) = f(\lambda x)\).

We describe the space \(Y\) of vector potentials \(\vec{A}\), so that the corresponding magnetic Schrödinger operator satisfies the Strichartz estimates. Let \(SU(\mathbb{R}^n)\) be the special unitary
group acting on \( \mathbb{R}^n \) and \( x(t) : \mathbb{R}_+^1 \to \mathbb{R}^n \) be arbitrary measurable function. Define
\[
\|A\|_{Y_0} := \|\nabla A\|_{L^1 L^\infty} + \|A\|_{L^2 L^\infty} + \left( \sum_l 2^{2l(1+h)} \|A_l\|_{L^1 L^{n/h}}^2 \right)^{1/2}
\]
\[
\|A\|_{Y_1} := \sum_k 2^{k(n-1)/2} \|A_k\|_{L^\infty L^2_k}
\]
\[
\|A\|_{Y_2} := \sum_k 2^{k(n-1)/2} \sup_{t \in \mathbb{S}(n)} \sup_x \|A_k(t, x + U z)\|_{L^\infty L^2_{t z_1}} + \sum_k 2^{k(n-5)/2} \sup_{t \in \mathbb{S}(n), x(t)} \|(|\partial^2 A_k| + |\partial_t A_k|)(t, x(t) + U z)\|_{L^\infty L^2_{t z_1}}
\]
\[
\|A\|_{Y_3} := \sum_k 2^{k(n+3)/2} \sup_{t \in \mathbb{S}(n)} \sup_x \|A_k(t, x + U z)\|_{L^1 L^2_{t z_1}} + \sum_k 2^{k(n-1)/2} \sup_{t \in \mathbb{S}(n), x(t)} \|(|\partial^2 A_k| + |\partial_t A_k|)(t, x(t) + U z)\|_{L^1 L^2_{t z_1}}.
\]
In the case \( n \geq 4 \), we can replace \( Y_1 \) by a bigger space \( \tilde{Y}_1 \) (with the smaller norm)
\[
\|A\|_{\tilde{Y}_1} = \sum_k 2^{k(n-1)/p_0} \sup_{t \in \mathbb{S}(n), x} \|A_k(t, x + U z)\|_{L^\infty L^{p_0}_{t z_1}},
\]
for some \( p_0 : p_0 < (n - 1)/2 \).

**Theorem 1.** Let \( n \geq 2 \). Then, there exists an \( \varepsilon = \varepsilon(n) > 0 \), so that whenever \( \tilde{A} : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n \) is a real-valued vector potential with \( \|\tilde{A}\|_{Y_0 \cap Y_1 \cap Y_2 \cap Y_3} \leq \varepsilon \), (which can be relaxed to \( \|\tilde{A}\|_{Y_0 \cap \tilde{Y}_1 \cap Y_2 \cap Y_3} \leq \varepsilon \), if \( n \geq 4 \) (4) has an unique global solution, whenever the initial data \( f \in L^2(\mathbb{R}^n) \) and the forcing term \( F \in L^1 L^2 \). In addition, there exists a constant \( C = C(n) \), so that the a priori estimate
\[
\sup_{q, r - \text{Str.}} \|u\|_{L^q L^r} \leq C(\|f\|_{L^2} + \|F\|_{L^1 L^2}).
\]
holds true\(^5\). Moreover, for every \( \psi \in \mathcal{S} \)
\[
\sup_{q, r - \text{Strichartz admissible}} \|\psi\|_{L^q L^r} \leq C(\|\psi(0, \cdot)\|_{L^2} + \|(\partial_t - i\Delta + \tilde{A} \cdot \nabla)\psi\|_{L^1 L^2}).
\]
One also has the \( L^2 \) Besov space version
\[
\sum_k \|\psi_k\|_{L^q L^r}^2)^{1/2} \leq C(\|\psi(0, \cdot)\|_{L^2} + \|(\partial_t - i\Delta + \tilde{A} \cdot \nabla)\psi\|_{L^1 L^2}).
\]

**Remark**
- Note that for all \( Y_j, j = 0, 1, 2, 3 \) we have that \( \|\tilde{A}\|_{Y_j} = \|\tilde{A}\|_{Y_j} \), that is the spaces are scale invariant with respect to the natural scaling \( A \to A^\lambda \).
- In the case \( n = 1 \), the theorem holds as well. Our results are however far from optimal, as shown recently by Burq-Planchon, [2]. It seems that in the one dimensional case one only needs to require \( \sup_t \|A(t, \cdot)\|_{L^1(\mathbb{R}^1)} < \infty \), if \( A \) is a real-valued potential.

\(^5\)For the two dimensional case, the constant \( C \) does also depend on \( (q, r) \). More specifically the constant blows up as \( (q, r) \to (2, \infty) \).
• If \( A : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{C} \) is complex valued, and satisfies
  \[
  \sup_{t,x} \left| \int_{-\infty}^{x} A(t, y) dy \right| < \varepsilon \quad \text{and} \quad \left\| \int_{-\infty}^{x} (\partial_t - i\partial_y^2) A(t, y) dy \right\|_{L^1_t L^\infty_y} < \varepsilon ,
  \]
  one has the results of Theorem 1. This is shown in [27], together with some applications and uniqueness issues, consult [27] for more details. Note that by recent examples on ill-posedness for derivative Schrödinger equations in \( \mathbb{R}^1 \) (due to M. Christ, [6]), some smallness assumptions are necessary even for a local well-posedness.

1.2. Some corollaries. We present some corollaries of Theorem 1. Observe that by Bernstein inequality (Lemma 1), one can bound
  \[
  \| A \|_{Y_0 \cap Y_1 \cap Y_2 \cap Y_3} \leq C_n \left( \sum_k 2^{k(n-1)} \| A_k \|_{L^\infty_t L^1} + 2^{k(n-3)} \| \partial_t A_k \|_{L^\infty_t L^1} \right) +
  \]
  \[
  + C_n \left( \sum_k 2^{k(n+1)} \| A_k \|_{L^1_t L^1} + 2^{k(n-1)} \| \partial_t A_k \|_{L^1_t L^1} \right).
  \]

We thus have

**Corollary 1.** There exists a small positive \( \varepsilon > 0 \), so that whenever a real valued vector potential \( \vec{A} \) satisfies

\[
\sum_k 2^{k(n-1)} \| A_k \|_{L^\infty_t L^1} + 2^{k(n-3)} \| \partial_t A_k \|_{L^\infty_t L^1} + 2^{k(n+1)} \| A_k \|_{L^1_t L^1} + 2^{k(n-1)} \| \partial_t A_k \|_{L^1_t L^1} \leq \varepsilon ,
\]

the conclusions (6), (7) and (8) hold true.

For the case of time independent magnetic potential \( \vec{A} \), we can formulate the following immediate corollary of Theorem 1.

**Corollary 2.** Let \( n \geq 2 \). Then there exists an \( \varepsilon = \varepsilon(n) \), so that the magnetic Schrödinger equation (4) has an unique global solution, provided \( \vec{A} = \vec{A}(x) \) is real-valued vector function and

\[
\sum_k 2^{k(n+1)} \| A_k \|_{L^1} < \infty
\]

\[
\sum_k 2^{k(n-1)} \| A_k \|_{L^1_x} \leq \varepsilon .
\]

Moreover, the solution satisfies

\[
\sup_{q,r-Str.} \| u \|_{L^q(0,T) L^r} \leq C(T, n)(\| f \|_{L^2} + \| F \|_{L^1_t L^2}).
\]

Corollary 2 follows easily by just applying Corollary 1 with a vector potential of the form \( \vec{A}(t, x) := \vec{A}(x) \chi(t/\delta) \) for some appropriate smooth cutoff function \( \chi \) and a small \( \delta \). This will produce a solution in a small time interval, say \( (0, \delta/2) \), which is iterated and so on. The smallness of the potential \( \vec{A} \) is achieved by the smallness of \( \delta \) (used to satisfy the requirements \( A \in Y_0 \cap Y_2 \cap Y_3 \) and by Sobolev embedding and the condition \( \sum_k 2^{k(n-1)} \| A_k \|_{L^1_x} \leq \varepsilon \) in the case \( A \in Y_1 \). Such a proof provides an upper bound \( C(t, n) \sim C(T/\varepsilon) \), which is not optimal in general.
1.3. Strichartz estimates with derivatives. The Strichartz estimates described in Theorem 1 can be of course extended to control the norms of the solution $u$ in Besov type norms involving derivatives. One way to do that is considering the Littlewood-Paley reduction of the equation to a fixed frequency $k$, applying the regular Strichartz estimates (either (7) or (8) with appropriate $p_1, p_2, q_1, q_2$), then multiplying by the corresponding power of $2^{ks}$ and square summing in $k$. The result is

**Theorem 2.** Let $n \geq 2$ and $\tilde{A}$ satisfies the assumptions in Theorem 1. Then there exists a constant $C = C(n)$ ($C = C(n, q, r)$, if $n = 2$), so that for every $s > 0$, initial data $f \in H^s$ and forcing term $F \in L_t^1 H^s$, the global solution $u$ of (4) satisfies

$$
(\sum_k 2^{2ks} \|u_k(t)\|_{L_t^2 L_x^2}^2)^{1/2} \leq C(\|f\|_{H^s} + \|F\|_{L_t^1 H^s}) + C\|\nabla u\|_{L_t^2 L_x^2} \left(\sum_k 2^{2ks} \|A_k\|_{L_t^2 L_x^{2n/(n-2)}}^2\right)^{1/2},
$$

for every Strichartz admissible pair $(q, r)$.

1.4. Strichartz estimates in $L_t^2 L_x^{2(n-1)/(n-3)} L_x^2$. We present an extension of Theorem 1, which allows us to control a larger set of norms.

**Proposition 1.** Let $n \geq 4$ and $\tilde{A}$ satisfies the assumptions in Theorem 1. Then there exists a constant $C_n$, so that the solutions of (4), satisfy

$$
(\sum_k \sup_{U \in SU(R^n), x(t)} \|u_k(t, x(t) + U z)\|_{L_t^2 L_x^{2(n-1)/(n-3)} L_x^2}^2)^{1/2} \leq C(\|f\|_{L^2} + \|F\|_{L_t^1 L_x^2}).
$$

For $n = 3$, take any $(q, r) : 1/q + 1/r = 1/2$ and $q > 2$. Then there exists a constant $C_q$ (which may blow up as $q \to 2$), so that

$$
(\sum_k \sup_{U, x(t)} \|u_k(t, x(t) + U z)\|_{L_t^1 L_x^q L_x^2}^2)^{1/2} \leq C_q(\|f\|_{L^2} + \|F\|_{L_t^1 L_x^2}).
$$

For $n = 2$,

$$
(\sum_k \sup_{U, x(t)} \|u_k(t, x(t) + U z)\|_{L_t^1 L_x^2 L_x^2}^2)^{1/2} \leq C(\|f\|_{L^2} + \|F\|_{L_t^1 L_x^2}).
$$

We also have a generalization of Proposition 1 to the setting of Theorem 2, that is involving derivatives. Namely, say when $n \geq 4$, one has the a priori estimate

$$
(\sum_k \sup_{U \in SU(R^n), x(t)} 2^{2ks} \|u_k(t, x(t) + U z)\|_{L_t^2 L_x^{2(n-1)/(n-3)} L_x^2}^2)^{1/2} \leq C(\|f\|_{H^s} + \|F\|_{L_t^1 H^s}) + C\|\nabla u\|_{L_t^2 L_x^n} \left(\sum_k 2^{2ks} \|A_k\|_{L_t^2 L_x^{2n/(n-2)}}^2\right)^{1/2}.
$$

Such results are needed to connect the “solutions space” with the space of admissible vector potentials $Y$. More precisely, in the applications, we have $A = A(u)$, where the relation is usually in the form $A = Q(u, \bar{u})$, where $Q$ is a bilinear (or multilinear) form acting on the solution and its conjugate. For example, in the Schrödinger map case (see Section 1.5 below), we have schematically $A = |\nabla|^{-1}(u\bar{u})$ and for the Maxwell-Schrödinger system (see Section 1.6 below) we have $A = \Box^{-1}(u\nabla u)$.

Ignoring the derivatives (and the invertibility of $\Box$ in the MS case) for a second, we see that by Hölder’s inequality, to have estimates of the form

$$
\|A(u)\|_{Y_2 Y_3} \leq C\|u\|_{H^s}^2
$$
where $X$ is the solutions space, we must rely on mixed Lebesgue estimates like the one in Theorem 2. Moreover, $X$ must be intersected with a space given by one of the norms involved in the left hand side of (10).

1.5. Applications to Schrödinger maps. In this section, we present a global regularity type result for the so-called Modified Schrödinger map system (MSM), which was derived in [23]. According to Theorem 2.2, [23], the Schrödinger map problem, with with target $S^{n-1}$, was shown to be equivalent (modulo a Lorentz type gauge transformation), to a overdetermined system of Schrödinger equations with attached consistency conditions.

We will not discuss here, whether the MSM and the Schrödinger map problem are equivalent, and how the (properties of the) solutions to one relate to the solutions of the other, with the acknowledgement that these are by no means unimportant or irrelevant issues. We will concentrate instead on the question of existence of solutions for MSM, which is mathematically more tractable.

Consider the MSM, which takes the form

$$\frac{\partial u_j}{\partial t} = i\Delta u_j - 2\sum_{k=1}^{n}a_k \frac{\partial u_j}{\partial x_k} - \left(\sum_{k=1}^{n}a_k^2\right) u_j +$$

$$+ 2\sum_{k=1}^{n}Im(u_j u_k)u_j - ia_0 u_j \quad j = 1, \ldots, n;$$

where

$$a_k = \sum_{l=1}^{n} \frac{\partial \kappa_{lk}}{\partial x_l};$$

$$d \kappa = 0;$$

$$\Delta \kappa_{kj} = -4 \Im(u_k \overline{u_j}) \quad j = 0, 1, \ldots, n;$$

$$k = 1, \ldots, n;$$

$$\Delta a_0 = -4 \sum_{j=1}^{n} \sum_{k=1}^{n} \left[ \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} Re(u_k \overline{u_j}) - \frac{1}{2} \left( \frac{\partial}{\partial x_k} \right)^2 u_j \overline{u_j} \right].$$

In short, we will consider the following system of Schrödinger equations

$$\begin{cases}
\partial_t u - i\Delta u + A(u) \cdot \nabla u = N(u) \\
u(0, x) = f,
\end{cases}$$

where $A$ is a real-valued vector potential ($A = (a_1, \ldots, a_n)$ in MSM),

$$A(u) = \partial^{-1}Q_1(u, \bar{u})$$

$$a_0(u) = \partial^2 \Delta^{-1}Q_2(u, \bar{u})$$

$$N(u) = Q_3(u, u, \bar{u}) + Q_4(A, A, u) + Q_5(a_0, u).$$

Here, $Q_1, \ldots, Q_5$ are multilinear forms of their arguments, i.e.

$$Q_j(u^1, \ldots, u^r) = \sum_{j \in \mathbb{N}_0^k} c_{j_1, \ldots, j_k} u^{i_1} \cdots u^{i_r}$$

for some constants $c$. We have also used the notation $\partial^s$ to denote a multiplier type operator, whose smooth symbol satisfies $|s(\xi)| \sim |\xi|^s$.

All the results that we obtain for (11) cover the MSM system, which is our main motivation.
Theorem 3. (Global regularity of MSM in high dimensions) Let $n \geq 6$, $s_0 = n/2 - 1$ and $s \geq (n + 1)/2$. Then, there exists $\varepsilon > 0$, so that whenever $g \in H^s$, with $\|g\|_{H^s} \leq 1$, the solution to (11) with initial data $f = \varepsilon g$ exists globally and satisfies
\[
\sup_t \|u(t, \cdot)\|_{H^s} \leq C\varepsilon
\]
for some constant $C$ depending only on the dimension and $s$.

For $n = 5$, there is an appropriate Besov spaces analogue.

1.6. Maxwell-Schrödinger system. Another system of nonlinear PDE’s, for which the Strichartz estimates of Theorem 1 are applicable is the Maxwell-Schrödinger system. That is
\[
\begin{aligned}
i\partial_t u + \Delta_A u &= g(|u|^2)u \\
\Box A &= P \text{Im}(\bar{u} \nabla_A u),
\end{aligned}
\]
where $P$ is the Leray projection onto the divergence free vector fields and $g(|u|^2)$ is either the Hartree interaction $g(|u|^2) := \int G(x - y)|u|^2(y)dy$ or simply $g(|u|^2) = |u|^2$.

This system was studied by Tsutsumi, [31], where he constructs the wave operator on a class of small scattered states. In particular, Tsutsumi showed global existence for a particular class of small data.

Recently, Nakamura and Wada, [24] have considered the MS system (12) as well. They have obtained local well-posedness with data $u_0 \in H^{5/3}(\mathbb{R}^3)$ by using energy estimates approach. For related results and recent developments for (12), one might consult the recent work of Ginibre and Velo, [11].

We do not state any precise results here for the Maxwell-Schrödinger system, but it is clear that a variant of Proposition 1 for solutions of the linear wave equation (among other things) will be required. These issues will be addressed in a forthcoming paper, [28].

A short outline of the paper is as follows. In Section 2, we give some definitions from harmonic analysis as well as some facts from the abstract Strichartz estimates theory due to Keel and Tao, [14]. In Section 3, we give some classical energy estimates and Littlewood-Paley reductions, which reduce the problem to the existence of parametrix construction. In Section 4, we motivate and construct the parametrix and then we prove some of its main properties. In Section 5, we describe an important angular decomposition for the phase of the parametrix and as a corollary we show the crucial pointwise estimates, which are used throughout in the sequel. In Section 6, we show that the parametrix satisfies $L^q_t L^r_x$ estimates. In Section 7, we show that the parametrix almost satisfy the magnetic Schrödinger equation. In Section 8, we prove the Strichartz estimates stated in Proposition 1. In Section 9, we show the global regularity for the modified Schrödinger maps. Some of the technical lemmas used in Sections 4 and 7 are formulated and proved in the Appendix.

2. Preliminaries

2.1. Fourier transform and Littlewood-Paley projections. Define the Fourier transform by
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx
\]

\footnote{Here $G$ is the Green function in $n$ dimension.}
and hence 

\[ f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \]

Introduce a positive, smooth and even function \( \chi : \mathbb{R}^1 \to \mathbb{R}^1 \), supported in \( \{ \xi : |\xi| \leq 2 \} \) and \( \chi(\xi) = 1 \) for all \( |\xi| \leq 1 \). Define \( \varphi(\xi) = \chi(\xi) - \chi(2\xi) \), which is supported in the annulus \( 1/2 \leq |\xi| \leq 2 \). Clearly \( \sum_{k \in \mathbb{Z}} \varphi(2^{-k}\xi) = 1 \) for all \( \xi \neq 0 \).

The \( k^{th} \) Littlewood-Paley projection is defined as a multiplier type operator by

\[ \widehat{P_k f}(\xi) = \varphi(2^{-k}\xi) \hat{f}(\xi). \]

Note that the kernel of \( P_k \) is integrable, smooth and real valued for every \( k \). In particular, it commutes with differential operators.

Also of interest will be the properties of products under the action of \( P_k \). We have that for any two (Schwartz) functions \( f, g \)

\[ P_k(fg) = \sum_{l \geq k-2} P_k(f_l g_{l-2 \leq l \leq l+2}) + \text{symmetric term} + \]

\[ + P_k(f_{l \leq k-4} g_{k-1 \leq l \leq k+1}) + \text{symmetric term} + \]

\[ = f_{l \leq k-4} g_k + [P_k, f_{l \leq k-4}] g_{k-1 \leq l \leq k+1} + \text{symmetric terms} + \]

\[ + \sum_{l \geq k-2} P_k(f_l g_{l-2 \leq l \leq l+2}) + \text{symmetric term}. \]

**Lemma 1.** *(Bernstein inequality)* Let \( f \) be Fourier supported in a rectangle \( Q \subset \mathbb{R}^n \). Then for every \( 1 \leq p \leq q \leq \infty \), one has

\[ \|f\|_{L^q} \leq C_n |Q|^{1/p-1/q} \|f\|_{L^p} \]

If \( Q = \xi : |\xi| \sim 2^k \), one can extend to mixed \( L^p L^q \) spaces. Suppose \( p_1 > p_2 \geq r \). Then

\[ \|f\|_{L^{p_1}_{x_1}, \ldots, x_n L^{q_1}_{x_1}} \leq C_n 2^{k(n-1)(1/p_2-1/p_1)} \|f\|_{L^{p_2}_{x_2}, \ldots, x_n L^{q_2}_{x_1}}. \]

**Proof.** The first statement is standard.

For the second statement, it is equivalent to the boundedness of \( P_k : L^{p_2}_{x_2}, \ldots, x_n L^{q_2}_{x_1} \to L^{p_1}_{x_2}, \ldots, x_n L^{q_1}_{x_1} \) with bound \( C_n 2^{k(n-1)(1/p_2-1/p_1)} \). One can rescale to the case \( k = 0 \), since these estimates are scale invariant. Since \( P_0 \) has integrable kernel, we have

\[ P_0 : L^q_{x_2}, \ldots, x_n L^r_{x_1} \to L^q_{x_2}, \ldots, x_n L^r_{x_1} \]

for every \( 1 \leq q, r \leq \infty \).

On the other hand, an application of the Bernstein inequality in the \((n-1)\) variables \( x_2, \ldots, x_n \)

\[ \|P_0 u\|_{L^{p_2}_{x_2}, \ldots, x_n L^{q_2}_{x_1}} \leq \|P_0 u\|_{L^{p_1}_{x_2}, \ldots, x_n L^{q_1}_{x_1}} \leq \|P_0 u\|_{L^{p_1}_{x_1} L^{q_1}_{x_1}} \leq \|u\|_{L^{p_1}_{x_2}, \ldots, x_n L^{q_1}_{x_1}}. \]

A complex interpolation between the last estimate and \( P_0 : L^q L^1 \to L^q L^1 \) (for every \( 1 \leq q \leq \infty \)) yields \( P_0 : L^p L^1 \to L^p L^1 \), whenever \( 1 \leq \tilde{p} \leq p \leq \infty \). Interpolation between the last estimate and \( p_0 : L^p L^\infty \to L^p L^\infty \) yields \( P_0 : L^p_{x_2}, \ldots, x_n L^{r_1}_{x_1} \to L^p_{x_2}, \ldots, x_n L^{r_1}_{x_1} \) for every \( r \leq \tilde{p} \leq p \leq \infty \).

This is the second statement of Lemma 1 for an appropriate choice of \( p, \tilde{p} \).

We also need the following technical lemma in the sequel

**Lemma 2.** Let \( \{a_i\}, \{b_i\} \) are two sequences and \( h > 0 \). Then

\[ \left( \sum_{k} 2^{2hk} \left( \sum_{l \geq k-2} 2^{-hl} a_l b_l \right)^2 \right)^{1/2} \leq C_h \|a\|_{L^\infty} \|b\|_{L^2}. \]
Proof. Fix the sequence \( \{a_l\} \) and consider the linear operator (mapping sequence into a sequence)
\[
(Tb)_k := 2^{\varepsilon(k-l)} \sum_{l \geq k-2} a_l b_l
\]
We will show that \( T : l^1 \to l^1 \) and \( T : l^\infty \to l^\infty \). Indeed,
\[
\|Tb\|_{l^1} \leq \sum_l |a_l| |b_l| \sum_{k \leq l+2} 2^{\varepsilon(k-l)} \lesssim \varepsilon^{-1} \|a\|_{l^\infty} \|b\|_{l^1},
\]
\[
\|Tb\|_{l^\infty} \leq \sup_k |a_k| \sup_l |b_l| \sum_{l \geq k-2} 2^{\varepsilon(k-l)} \lesssim \varepsilon^{-1} \|a\|_{l^\infty} \|b\|_{l^\infty}.
\]
It follows that for \( 1 \leq p \leq \infty : T : l^p \to l^p \) with norm no bigger than \( C\varepsilon \|a\|_{l^\infty} \), hence the statement of the lemma. \( \square \)

2.2. Keel-Tao theory. It is well-known that decay and energy estimates imply Strichartz estimates in the context of various dispersive equations. We would like to state an abstract result due to M. Keel and T. Tao, [14], which proved out to be very useful in this context. Let us recall, that this method in conjunction with the Hausdorff-Young inequality was used by Ginibre and Velo in their proof of the Strichartz estimates for the linear Schrödinger equation away from the endpoint.

The abstract version of Keel and Tao has the (somewhat) unexpected consequence that the endpoint Strichartz estimate follows only from decay and energy estimates.

Let \( H \) be a Hilbert space, and \((X, dx)\) be a measure space and \( U(t) : H \to L^2(X) \) be a bounded operator. Suppose that \( U \) satisfies
\[
\|U(t) f\|_{L^2} \leq C \|f\|_H
\]
\[
\|U(t) s f\|_{L^\infty} \leq C|t - s|^{-\sigma} \|f\|_{L^1}.
\]
Suppose also that \((q, r)\) are \( \sigma \) admissible, that is \((q, r) : q, r \geq 2; 1/q + \sigma/r = \sigma/2 \) and \((q, r, \sigma) \neq (2, \infty, 1)\).

Proposition 2. (Keel-Tao, [14]) Let \((q, r)\) and \((\tilde{q}, \tilde{r})\) be both \( \sigma \) admissible and \( U(t) \) obeys (13) and (14). Then
\[
\|U(t) f\|_{L^q_t L^r_x} \leq C \|f\|_{L^2}
\]
\[
\left\| \int_0^t U(t) (U(s)^*) F(s, \cdot) ds \right\|_{L^q_t L^r_x} \leq C \|F\|_{L^q_t L^r_x}.
\]

Remark Note that by the Ginibre-Velo original argument (see also [14]), the Strichartz estimates (15), (16) follows only assuming the energy bound (13) and the “modified decay bound”
\[
\|U(t) s f\|_{L^q_x} \leq C \|t - s|^{-(2/p-1)\sigma} \|f\|_{L^p},
\]
for any \( \sigma : (1 - 2/p)\sigma > 1 \). Note that (17) follows by interpolation between (13) and (14), and so it is in general easier to establish. In fact, we use the modified decay bound, instead of the \( L^1 \to L^\infty \) decay bound (14) in order to reduce the smoothness assumptions\(^7\) on our

\(^7\)We show (14) under the condition \( A \in Y_1 \) and then interpolate with (13) to obtain (17) under the less restrictive condition \( A \in Y_1 \).
vector potentials, see Section 6.

We need an extension of Proposition 2, which follows from the same proof as in [14].

**Proposition 3.** Let $W(t)$ is an operator defined on all Schwartz functions on $\mathbb{R}^n$ and it satisfies

\[
\|W(t)f\|_{L^2_{x_2,...,x_n}L^2_{x_1}} = \|W(t)f\|_{L^2_{x_1}} \leq C\|f\|_{L^2_{x_1}}
\]

\[
\|W(t)W(s)^*f\|_{L^\infty_{x_2,...,x_n}L^2_{x_1}} \leq C|t-s|^{-\sigma}\|f\|_{L^2_{x_2,...,x_n}L^2_{x_1}}.
\]

Then

\[
(18) \quad \|W(t)f\|_{L^q_{t,x}L^r_{x_2,...,x_n}L^2_{x_1}} \leq C\|f\|_{L^2},
\]

for all $\sigma$ admissible pairs $(q,r)$. The usual averaging argument then implies the “retarded estimate”

\[
\left\| \int_0^t W(t)W(s)^*F(s,\cdot)ds \right\|_{L^q_tL^r_{x_2,...,x_n}L^2_{x_1}} \leq C\|F\|_{L^1L^2}.
\]

Indeed, for the “retarded estimate”, assume (18) to get

\[
\left\| \int_0^t W(t)W(s)^*F(s,\cdot)ds \right\|_{L^q_tL^r_{x_2,...,x_n}L^2_{x_1}} \leq \left\| \int_0^t \|W(t)W(s)^*F(s,\cdot)\|_{L^r_{x_2,...,x_n}L^2_{x_1}} ds \right\|_{L_t^q} \lesssim \int_0^\infty \|W(s)^*F(s,\cdot)\|_{L^r_{x_2,...,x_n}L^2_{x_1}} ds \lesssim \int_0^\infty \|W(s)^*F(s,\cdot)\|_{L^2} ds \leq C\|F\|_{L^1L^2},
\]

where we have used (18) in the form $\|W(t)G(s,\cdot)\|_{L^q_tL^r_{x_2,...,x_n}L^2_{x_1}} \lesssim \|G(s,\cdot)\|_{L^2}$ as well as the energy estimate $\|W(s)^*F\|_{L^2} \leq C\|F\|_{L^2}$.

3. Proof of Theorem 1

3.1. Energy estimates and Littlewood-Paley reductions. To start our argument, we shall need the following $L^2$ existence result, due to Kenig-Ponce-Vega, [16], which generalizes an earlier work of Doi, [8], [9]. We state it here only in the particular case of interest to us, namely Schrödinger equation with first order perturbations.

**Proposition 4.** (Kenig-Ponce-Vega, [16]) For the equation

\[
\begin{align*}
\partial_t u - i\Delta u + \vec{b}(t,x) \cdot \nabla u &= F \\
u(x,0) &= f
\end{align*}
\]

there is an unique global solution, provided $\vec{b} \in C^N$, $|\text{Im } \vec{b}| \leq x^{-m}$ for some large integers $N, m$. Moreover the solutions are smooth, provided $f, F$ are smooth and for every $T > 0$

\[
\|u\|_{L^\infty(0,T)L^2} \leq C_T(\|f\|_{L^2} + \|F\|_{L^1(0,T)L^2}).
\]

In the case, when $\vec{b}$ is real valued and $\|\vec{b}\|_{L^1L^\infty} < 1/2$, one can derive a priori estimates for $\|u\|_{L^\infty(0,T)L^2}$ that are $T$ independent.
This is very standard energy estimate. Indeed, multiply both sides by $\bar{u}$, integrate in the spatial variable and take real part\textsuperscript{8}. We obtain
\[
\partial_t \int |u|^2 dx + \int \vec{b} \cdot \nabla |u|^2 dx = \frac{1}{2} \int (F\bar{u} + Fu) dx
\]
Integrate by parts and then integrate in $(0, T)$ to get
\[
\|u(T, \cdot)\|_{L^2}^2 - \|f\|_{L^2}^2 = \int_0^T \int \text{div}(\vec{b}) |u|^2 dx dt + \int_0^T (F\bar{u} + Fu) dx dt \leq \|
abla \vec{b}\|_{L^1} \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^2}^2 + \|F\|_{L^1} \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^2}.
\]
whence since $\|
abla \vec{b}\|_{L^1} < 1/2$,
\[
\|u(T, \cdot)\|_{L^2} \leq C(\|f\|_{L^2} + \|F\|_{L^1}).
\]
Thus, we have shown the following

**Proposition 5.** Let $f, F$ be smooth functions. Let also $\vec{A}$ be a smooth, real-valued potential with $\|
abla \vec{A}\|_{L^1} < 1/2$. Then the Schrödinger equation
\[
\begin{aligned}
\partial_t u - i\Delta u + \vec{A}(t, x) \cdot \nabla u &= F \\
u(x, 0) &= f
\end{aligned}
\]
has an unique global solution and moreover there exists an absolute constant $C$, so that for every $T > 0$
\begin{equation}
\|u(T, \cdot)\|_{L^2} \leq C(\|f\|_{L^2} + \|F\|_{L^1}).
\end{equation}

We may restate Proposition 5 in a slightly different manner. Namely, the linear operators $U_A(t, s) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where $U_A(t, s)f$ is the unique solution $u$ of
\[
\begin{aligned}
\partial_t u - i\Delta u + \vec{A}(t, x) \cdot \nabla u &= 0 \\
u(s, x) &= f
\end{aligned}
\]
amre well-defined. Moreover, by uniqueness and since the equation is time reversible, we can define $U_A(s, t) := U_A(t, s)^{-1}$, which is the solution operator to the same equation with data at time $t$ backwards in time to $s$.

The Duhamel’s formula may be used to write the unique solution to (4) as
\begin{equation}
\begin{aligned}
u(t, x) &= U_A(t, 0)f + \int_0^t U_A(t, s)F(s, \cdot)ds.
\end{aligned}
\end{equation}
Next, we take a Littlewood-Paley projections of (4). We get the equations
\begin{equation}
\begin{aligned}
\partial_t u_k - i\Delta u_k + \vec{A}_{\leq k-4} \cdot \nabla u_k &= F_k + E_k \\
u_k(0) &= f_k
\end{aligned}
\end{equation}
\textsuperscript{8}By the smoothness of the solutions all the operations are justified.
where $E^k$ is the error term
\begin{equation}
E^k = |P_k, \tilde{A}_{k-4} \nabla u_{k-1} + \sum_{l \geq k-2} P_k(\tilde{A}_{l} \cdot \nabla u_{l-2} \leq l+2) + \\
+ \sum_{l \geq k-2} P_k(\tilde{A}_{l-2} \leq l+2 \cdot \nabla u_{l}) + P_k(\tilde{A}_{k-1} \leq k+1 \cdot \nabla u_{k-4}).
\end{equation}

Denote
\begin{align*}
\mathcal{L} \psi &:= \psi_t - i\Delta \psi + A \cdot \nabla \psi, \\
\mathcal{L}^k \psi &:= \psi_t - i\Delta \psi + A_{k-4} \cdot \nabla \psi.
\end{align*}

Our next observation is a naive Strichartz estimate for $L$, which will be the starting point in a continuity argument later on.

**Proposition 6.** For a fixed integer $k_0$, there exists a time $T_0 = T_0(k_0) \leq \infty$, so that whenever $0 < T < T_0$, and for every $\psi \in \mathcal{S}$
\begin{equation}
\|P_{<k_0} \psi\|_{L^q(0,T) L^r} \leq C(T, k_0)(\|\psi(0, \cdot)\|_{L^2} + \|\mathcal{L} \psi\|_{L^1 L^2}).
\end{equation}

Moreover, $C(T, k_0)$ depends on $T$ in a continuous way.

**Proof.** The proof is based on the standard Strichartz estimates for the linear Schrödinger equation, as we treat the term $\tilde{A} \nabla \psi$ as a perturbation. By Hölder’s inequality and since $\|\nabla P_{<k_0} \psi\|_{L^p} \lesssim 2^{k_0} \|\psi\|_{L^p}$
\begin{align*}
\|P_{<k_0} \psi\|_{L^q(0,T) L^r} &\leq C(\|\psi(0)\|_{L^2} + \|\tilde{A} \nabla P_{<k_0} \psi\|_{L^1(0,T) L^2} + \|\mathcal{L} P_{<k_0} \psi\|_{L^1 L^2}) \\
&\leq C_1(\|\psi(0)\|_{L^2} + \sqrt{T} 2^{k_0} \|\tilde{A}\|_{L^2 L^\infty} \|\psi\|_{L^\infty(0,T) L^2}) + \\
&+ C_1(\|P_{<k_0} \mathcal{L} \psi\|_{L^1 L^2} + \|P_{<k_0, \tilde{A}} \nabla \psi\|_{L^1 L^2}) \\
&\leq C_1(\|\psi(0)\|_{L^2} + \|\mathcal{L} \psi\|_{L^1 L^2}) + \\
&+ C_1(\sqrt{T} 2^{k_0} \|\tilde{A}\|_{L^2 L^\infty} + \|\nabla \tilde{A}\|_{L^1 L^\infty}) \|\psi\|_{L^\infty(0,T) L^2} \\
&\leq C_1(\|\psi(0)\|_{L^2} + \|\mathcal{L} \psi\|_{L^1 L^2}) + C_1 \varepsilon(\sqrt{T} 2^{k_0} + 1) \|\psi\|_{L^\infty(0,T) L^2}.
\end{align*}

According to (19), the last expression is controlled by
\begin{equation}
C_1(1 + C \varepsilon(\sqrt{T} 2^{k_0} + 1)) (\|\psi(0)\|_{L^2} + \|\mathcal{L} \psi\|_{L^1 L^2}).
\end{equation}

Hence, (23) holds with $C(T, k_0) = C_1(1 + C \varepsilon(\sqrt{T} 2^{k_0} + 1))$. \hfill \Box

Fix $k_0$ and a small $\varepsilon = \varepsilon(n)$ (to be chosen later). Set $0 < T^* \leq \infty$ to be the maximum time, so that for all $0 < T < T^*$ and for all $\psi \in \mathcal{S}$:
\begin{equation}
\|P_{<k_0} \psi\|_{L^q(0,T) L^r} \leq \varepsilon^{-1}(\|\psi(0, \cdot)\|_{L^2} + \|\mathcal{L} \psi\|_{L^1 L^2}).
\end{equation}

If $T^* = \infty$, (24) holds for the fixed $k_0$.
We will show that $T^* < \infty$ leads to a contradiction, provided $\varepsilon = \varepsilon_n$ was chosen suitably small. We need to consider separately the homogeneous and inhomogeneous problems.
3.2. **Homogeneous problem for $L$.** Consider

$$\begin{align*}
\mathcal{L} g &= 0 \\
g(0, x) &= f(x),
\end{align*}$$

where $g$ is constructed as $g(t, x) = U_{A}(t, 0)f$. We will show that there exists a constant $C$, depending on the dimension $n$, but not on $T, k_0$, or $f$, so that for any $T < T^*$ and all $k \leq k_0$

$$\left(\sum_k \|P_{<k_0} P_k g\|_{L^2(0,T)}^2\right)^{1/2} \leq C \|f\|_{L^2}.$$  

(25)

An elementary computation shows that $g_k$ solves

$$\begin{align*}
\mathcal{L} g_k &= -[P_k, \bar{A}] \nabla g, \\
g_k(0, x) &= f_k.
\end{align*}$$

(26)

3.3. **A priori $L^2$ estimates for (26).** First, we show an a priori estimate for $(\sum_k \|g_k\|_{L^\infty L^2}^2)^{1/2}$ which generalizes (19).

Apply (19) for the solutions of (26). We get

$$\|g_k\|_{L^\infty L^2} \leq C(\|f_k\|_{L^2} + \|[P_k, \bar{A}] \nabla g\|_{L^1 L^2}).$$

To tackle the term $\|[P_k, \bar{A}] \nabla g\|_{L^1 L^2}$, we split $\bar{A}$ in low and high frequencies portions respectively. We have

$$\sum_k \|P_k, \bar{A}\|_{L^1 L^2} \leq \sum_k \|P_k, \bar{A}_{<k-4}\|_{L^1 L^2} + \sum_k \|P_k, \bar{A}_{>k-4}\|_{L^1 L^2}.$$ 

Since $[P_k, \bar{A}_{<k-4}] \nabla g = [P_k, \bar{A}_{<k-4}] P_{k-2<,k+2} \nabla g$, we estimate this portion by

$$\sum_k \|P_k, \bar{A}_{<k-4}\|_{L^1 L^2} \leq \|\nabla \bar{A}\|_{L^1 L^2} \sum_k \|g_{k-2<,k+2}\|_{L^\infty L^2} \leq \varepsilon \sum_k \|g_k\|_{L^\infty L^2}.$$ 

For the high-frequency portion, we have

$$[P_k, \bar{A}_{>k-4}] \nabla g = -\bar{A}_{>k-4} \nabla g_k + P_k(\bar{A}_{>k-4} \nabla g).$$

The first term is trivially estimated by

$$\sum_k \|\bar{A}_{>k-4} \nabla g_k\|_{L^1 L^2} \leq \|\nabla \bar{A}\|_{L^1 L^2} \sum_k \|g_k\|_{L^\infty L^2}^2.$$ 

\(^9\)The constant $C$ may also depend on the Strichartz pair $(q, r)$ in dimension two, although we suppress that dependence.
The rest is estimated by
\[
\sum_k \left\| P_k(\tilde{A}_{> k-4} \nabla g) \right\|_{L^1 L^2}^2 \leq \sum_k \left\| P_k(\tilde{A}_{k-4 < k+4} \nabla g) \right\|_{L^1 L^2}^2 + \\
+ \sum_k \left\| P_k(\tilde{A}_{> k+4} \nabla g) \right\|_{L^1 L^2}^2 \lesssim \sum_k \left\| \nabla A_{k-4 < k+4} \right\|_{L^1 L^\infty} \|g\|_{L^\infty L^2}^2 + \\
+ \sum_k \left( \sum_{l \geq k+4} \left\| P_k(\tilde{A}_l \nabla g_{l-2 < l+2}) \right\|_{L^1 L^2} \right)^2 \lesssim \\
\lesssim \varepsilon^2 \|g\|_{L^\infty L^2}^2 + \sum_k \left( \sum_{l \geq k+4} 2^{hk} \left\| P_k(\tilde{A}_l \nabla g_{l-2 < l+2}) \right\|_{L^1 L^{2/(n+2h)}} \right)^2 \lesssim \\
\lesssim \varepsilon^2 \|g\|_{L^\infty L^2}^2 + \sum_k 2^{2hk} \left( \sum_{l \geq k+4} 2^l \left\| \tilde{A}_l \right\|_{L^1 L^{n/h}} \|g_{l-2 < l+2}\|_{L^\infty L^2} \right)^2 \lesssim \\
\lesssim \varepsilon^2 \|g\|_{L^\infty L^2}^2 + \sum_k 2^{2hk} \left( \sum_{l \geq k+4} 2^{-hl} \left\| \tilde{A}_l \right\|_{L^1 W^{n/h,1+h}} \|g_{l-2 < l+2}\|_{L^\infty L^2} \right)^2
\]
for some \( h : 0 < h < 1/4 \). By Lemma 2, we get that the second expression above is bounded by \( C \sup_t \|g_{l-2 < l+2}\|_{L^\infty L^2}^2 \sum_l \|\tilde{A}_l\|_{L^1 W^{n/h,1+h}}^2 \).

Altogether, we obtain the estimate
\[
\sum_k \|g_k\|_{L^\infty L^2}^2 \leq C \|f\|_{L^2}^2 + C \varepsilon^2 \sum_k \|g_k\|_{L^\infty L^2}^2,
\]
whence for \( \varepsilon \) small enough
\[
(\sum_k \|g_k\|_{L^\infty L^2}^2)^{1/2} \leq 2C \|f\|_{L^2},
\]
which generalizes (19), since \( \|g\|_{L^\infty L^2} \leq (\sum_k \|g_k\|_{L^\infty L^2}^2)^{1/2} \).

We also need the following crucial lemma, whose proof we postpone for Section 4.

**Lemma 3.** (existence of parametrix) Given a potential \( \left\| \tilde{A} \right\|_{Y_1 \cap Y_2 \cap Y_3} \leq \varepsilon \) and integer \( k \) and \( T > 0 \), and for every function \( f_k \in L^2(\mathbb{R}^n) \) with \( \text{supp} \, \hat{f}_k \subset \{ |\xi| \sim 2^k \} \), one can find a function \( v_k : [0,T) \times \mathbb{R}^n \to C \), so that \( \text{supp} \, \hat{v}_k \subset \{ |\xi| \sim 2^k \} \) and
\[
\|v_k(0,x) - f_k\|_{L^2} \leq C \varepsilon \|f_k\|_{L^2} \\
\|v_k\|_{L^\infty(0,T)L^2} \leq C \|f_k\|_{L^2} \\
\|L v_k\|_{L^1 L^2} \leq C \varepsilon \|f_k\|_{L^2},
\]
for some \( C \) independent of \( f, k, T \).
Assuming for the moment the validity of Lemma 3, we show (25). Take $v_k$ as in Lemma 3. Applying the a priori estimate (24) yields
\[
\|P_{<k_0}P_kg\|_{L^q(0,T)\Lr} \leq \|P_{<k_0}(P_kg - v_k)\|_{L^q(0,T)\Lr} + \|P_{<k_0}v_k\|_{L^q(0,T)\Lr} \\
\leq \varepsilon^{-1}(\|f_k - v_k(0, x)\|_{L^2} + \|\mathcal{L}P_kg\|_{L^1\Lr} + \|\mathcal{L}v_k\|_{L^1\Lr}) + C\|v_k\|_{L^q(0,T)\Lr} \\
\leq \varepsilon^{-1}(C\|f_k\|_{L^2} + \|[P_k, \mathcal{A}\nabla g]\|_{L^1\Lr}) + C\|f_k\|_{L^2}.
\]

For the proof of (27), we have already estimated
\[
\sum_k \left\| [P_k, \mathcal{A}\nabla g]\right\|_{L^1\Lr}^2 \leq C\varepsilon^2 \sum_k \|g_k\|_{L^\infty\Lr}^2,
\]
which by (27) is bounded by $C\|f\|_{L^2}^2$. Squaring and summing the estimates for $\|P_{<k_0}P_kg\|_{L^q(0,T)\Lr}$ yields
\[
\|P_{<k_0}g\|_{L^q(0,T)\Lr} \leq \left(\sum_k \|P_{<k_0}P_kg\|_{L^q(0,T)\Lr}^2\right)^{1/2} \leq C\|f\|_{L^2}.
\]
This proves (25). Rewriting (25) in terms of the operators $U$ yields
\[
\left(\sum_k \|P_{<k_0}P_k U(t, 0) f\|_{L^q(0,T)\Lr}^2\right)^{1/2} \leq C\|f\|_{L^2},
\]
or more generally since $U(t, s) = U(t, 0)U(0, s)$
\[
\left(\sum_k \|P_{<k_0}P_k U(t, s) f_s\|_{L^q(0,T)\Lr}^2\right)^{1/2} \leq C\|U(0, s)f_s\|_{L^2} \leq C\|f_s\|_{L^2}.
\]
which is sometimes more convenient to use. In particular, we obtain
\[
\|P_{<k_0}U(t, s)f_s\|_{L^q(0,T)\Lr} \leq C\|f_s\|_{L^2}.
\]

3.4. The inhomogeneous problem for $\mathcal{L}$. We derive estimates similar to (25) for the inhomogeneous problem associated with $\mathcal{L}$. Namely, we consider
\[
\begin{cases}
\mathcal{L}w = G \\
w(0, x) = 0,
\end{cases}
\]
We show that the solution $w$ (constructed by the Duhamel’s formula (20)) satisfy
\[
\left(\sum_k \|P_{<k_0}P_k w\|_{L^q(0,T)\Lr}^2\right)^{1/2} \leq C\|G\|_{L^1(0,T)\Lr},
\]
whenever $T < T^*$ and the constant $C$ is dependent only on $n$.
Our first step as in the homogeneous case is to project the inhomogeneous equation by the Littlewood-Paley operator $P_k$. We have that
\[
\mathcal{L}P_kw = P_k G - [P_k, \mathcal{A}\nabla w],
\]
whence exactly as in the homogeneous case (and by using the smallness assumptions on $\mathcal{A}$), we conclude
\[
\left(\sum_k \|P_kw\|_{L^\infty\Lr}^2\right)^{1/2} \leq C\left(\sum_k \|G_k\|_{L^1\Lr}^2\right)^{1/2} \leq C\|G\|_{L^1\Lr},
\]
\[
\left(\sum_k \|[P_k, \mathcal{A}\nabla w]\|_{L^1\Lr}^2\right)^{1/2} \leq C\varepsilon\left(\sum_l \|P_l w\|_{L^\infty\Lr}^2\right)^{1/2} \leq C\varepsilon\|G\|_{L^1\Lr}.
\]
By the Duhamel’s formula applied to (31) and (29)

\[
\left(\sum_{k} \left| P_{<k_0} P_k w \right|^2_{L^2(0,T) L^r} \right)^{1/2} = \left(\sum_{k} \left\| \int_0^t P_{<k_0} U(t, s)(P_k G(s, \cdot)) ds \right\|_{L^2(0,T) L^r}^2 \right)^{1/2} + \\
\left(\sum_{k} \left\| \int_0^t P_{<k_0} U(t, s)([P_k, \tilde{A}] \nabla w) ds \right\|_{L^2(0,T) L^r}^2 \right)^{1/2} \lesssim \\
\lesssim \left(\sum_{k} \left\| P_k G(s, \cdot) \right\|^2_{L^2} ds \right)^{1/2} + \left(\sum_{k} \left\| [P_k, \tilde{A}] \nabla w(s, \cdot) \right\|^2_{L^2} ds \right)^{1/2} \lesssim \\
\lesssim \left(\sum_{k} \|G_k\|^2_{L^1 L^2} \right)^{1/2} + \left(\sum_{k} \| [P_k, \tilde{A}] \nabla w\|^2_{L^1 L^2} \right)^{1/2} \lesssim \|G\|_{L^1 L^2}.
\]

In particular,

\[(32) \quad \| P_{<k_0} w \|_{L^2(0,T) L^r} \leq C \| G \|_{L^1 L^2}.
\]

Combining (29), (32) yields that the solution to $\mathcal{L}\psi = G$, $\psi(0, x) = f$, satisfies

\[
\| P_{<k_0} \psi \|_{L^2(0,T) L^r} \leq C_n(\| f \|_{L^2} + \| G \|_{L^1 L^2}),
\]

which is a contradiction with the maximality of $T^*$ (see (24)), provided $\varepsilon < 1/C_n$. Thus $T^* = \infty$ and one has the inequality

\[
\| P_{<k_0} \psi \|_{L^2 L^r} \leq C_n(\| \psi(0, \cdot) \|_{L^2} + \| \mathcal{L}\psi \|_{L^1 L^2}).
\]

for all $k_0$. Taking a limit $k_0 \to \infty$ establishes the Strichartz estimate (7). Moreover, we have established a more general Besov type estimate

\[
\left(\sum_{k} \| \psi_k \|^2_{L^2 L^r} \right)^{1/2} \leq C_n(\| \psi(0, \cdot) \|_{L^2} + \| \mathcal{L}\psi \|_{L^1 L^2}).
\]

4. THE CONSTRUCTION OF THE PARAMETRIX

In this section, we show the existence of approximate solution (in sense of Lemma 3) to the equation $\mathcal{L}g = 0$, $g(x, 0) = f_k$.

We first make some reductions. Our first reduction is that it will suffice to show that there exists $v_k$ with the properties as listed in Lemma 3, where $v_k$ satisfies $\| \mathcal{L}^k v_k \|_{L^1 L^2} \leq C \varepsilon \| f_k \|_{L^2}$ instead of $\| \mathcal{L} v_k \|_{L^1 L^2} \leq C \varepsilon \| f_k \|_{L^2}$. Indeed, suppose there is $v_k$ with $\| \mathcal{L}^k v_k \|_{L^1 L^2} \leq C \varepsilon \| f_k \|_{L^2}$. Then

\[
\| \mathcal{L} v_k \|_{L^1 L^2} \leq \| \mathcal{L}^k v_k \|_{L^1 L^2} + \| \tilde{A}_{k-4} \|_{L^1 L^\infty} \| \nabla v_k \|_{L^\infty L^2} \leq \\
\leq \| \mathcal{L}^k v_k \|_{L^1 L^2} + C \| \partial \tilde{A} \|_{L^1 L^\infty} \| v_k \|_{L^\infty L^2} \leq C \varepsilon \| f_k \|_{L^2}.
\]

Suppose that one has already a function $v_k$ as in Lemma 3 without $supp \tilde{v}_k \subset \{ |\xi| \sim 2^k \}$. Our claim is that $\tilde{P}_k v_k$ will satisfy all conditions in Lemma 3, with $\mathcal{L}$ replaced by $\mathcal{L}^k$ according to our first reduction.

By construction $supp \tilde{P}_k v_k \subset \{ |\xi| \sim 2^k \}$. Since $\tilde{P}_k f_k = f_k$,

\[
\left\| \tilde{P}_k v_k(0, x) - f_k \right\|_{L^2} = \left\| \tilde{P}_k (v_k(0, x) - f_k) \right\|_{L^2} \leq \left\| v_k(0, x) - f_k \right\|_{L^2} \leq \varepsilon \| f \|_{L^2}.
\]
Next,\[
\|\tilde{P}_k v_k\|_{L^q L^r}^q \leq \|v_k\|_{L^q L^r}^q \leq C\|f_k\|_{L^2}.
\]
Finally, since \(\mathcal{L}^k \tilde{P}_k v_k = \mathcal{L}^k v_k - [\tilde{P}_k, \tilde{A}^-] \nabla v_k\)

\[
\left\|\mathcal{L} \tilde{P}_k v_k\right\|_{L^1 L^2} \leq \left\|\mathcal{L} v_k\right\|_{L^1 L^2} + \left\|[\tilde{P}_k, \tilde{A}^-] \nabla v_k\right\|_{L^1 L^2} \leq C\varepsilon \|f_k\|_{L^2} + \left\|\partial \tilde{A}^-\right\|_{L^1 L^\infty} \|v_k\|_{L^\infty L^2} \leq C\varepsilon \|f_k\|_{L^2}.
\]

Next, since all our estimates will be scale invariant, we may rescale and assume without loss of generality \(k = 0\). Thus, matters are reduced to the following

**Lemma 4.** Let \(\varepsilon > 0\) and \(\|\tilde{A}\|_{Y_1 \cap Y_2 \cap Y_3} < \varepsilon\) be a potential with \(\text{supp } \hat{A}(t, \xi) \subset \{||\xi|| \lesssim 1\}\). Then for every \(T > 0\) and for every function \(f \in L^2(\mathbb{R}^n)\) with \(\text{supp } \hat{f} \subset \{||\xi|| \sim 1\}\), one can find a function \(v : [0, T) \times \mathbb{R}^n \rightarrow C\), so that

\[
\|v(0, x) - f\|_{L^2} \leq C\varepsilon \|f\|_{L^2} \tag{33}
\]

\[
\|v\|_{L^q(0, T) L^r} \leq C\|f\|_{L^2} \tag{34}
\]

\[
\left\|\partial_t v - i\Delta v + \tilde{A} \cdot \nabla v\right\|_{L^1 L^2} \leq C\varepsilon \|f\|_{L^2}, \tag{35}
\]

for some \(C\) independent on \(f, T, \varepsilon\).

**Proof.** (Lemma 4) We construct \(v\) in the form

\[
v(t, x) = \Lambda f(t, x) = \int e^{i\sigma(t, x, \xi)} e^{-4\pi^2 t|\xi|^2} e^{2\pi i (\xi, x)} \Omega(\xi) \hat{f}(\xi) d\xi,
\]

where the \(\Omega\) is a smooth cut-off of the annulus \(||\xi|| \sim 1\) and the phase correction \(\sigma\) is to be selected momentarily. We have

\[
\mathcal{L} v = \partial_t v - i\Delta v + \tilde{A} \cdot \nabla v =
\]

\[
= \int (i\partial_t \sigma + \Delta \sigma + 2\pi i (\nabla \sigma, \xi) + \tilde{A} \cdot \xi) + i[(\nabla \sigma)^2 + \tilde{A} \cdot \nabla \sigma]) \times
\]

\[
\times e^{i\sigma(t, x, \xi)} e^{-4\pi^2 t|\xi|^2} e^{2\pi i (\xi, x)} \Omega(\xi) \hat{f}(\xi) d\xi.
\]

We first comment on possible choices for \(\sigma\). Clearly, since we are trying to almost solve \(\mathcal{L} v = 0\), we should choose \(\sigma\) in a way, so that the main terms are resolved in the formula for \(\mathcal{L} v\). We see that since the potential \(\tilde{A}\) is supported in the low frequencies and is small, the main terms are those, that are either linear in \(\tilde{A}\) or linear in \(\nabla \sigma\). It seems then reasonable to choose \(\sigma\), so that

\[
\langle \nabla \sigma, \xi \rangle + \tilde{A} \cdot \xi = 0.
\]

However, it turns out that when \(||\xi|| / ||\eta|| \lesssim |\eta|^2\) (here \(\eta\) is the Fourier variable for \(\sigma\)), one has that \(\Delta \sigma\) is actually “bigger” compared to \(\langle \nabla \sigma, \xi \rangle\). We therefore modify our choice as follows.
Set $\sigma = \sigma^0 + \sigma^1$, where

$$
\sigma^0(t, x, \xi) = \sum_{k \leq -2} \int_{0}^{\infty} \tilde{A}_k(t, x + z\xi/|\xi|) \cdot \frac{\xi}{|\xi|} \chi(2^{2k}z)dz,
$$

$$
\sigma^1(t, x, \xi) = 2\pi i \sum_{k \leq -2} 2^{2k} \int_{0}^{\infty} \Delta^{-1} \tilde{A}_k(t, x + z\xi/|\xi|) \cdot \xi \chi'(2^{2k}z)dz.
$$

It is elementary to see that

$$
\langle \nabla \sigma^0, \xi \rangle + \tilde{A} \cdot \xi = -\sum_{k} 2^{2k} \int_{0}^{\infty} \langle \tilde{A}_k(t, x + z\xi/|\xi|), \xi \rangle \chi'(2^{2k}z)dz,
$$

whence

$$
\Delta \sigma^1 + 2\pi i \langle \nabla \sigma^0, \xi \rangle + \tilde{A} \cdot \xi = 0.
$$

Denote $\tilde{A}_k := 2^{2k} \Delta^{-1} \tilde{A}_k$. Then, we rewrite

$$
\sigma^1(t, x, \xi) = 2\pi i \sum_{k \leq -2} \int_{0}^{\infty} \tilde{A}_k(t, x + z\xi/|\xi|) \cdot \xi \chi'(2^{2k}z)dz.
$$

According to the choice of $\sigma^0, \sigma^1$, we get

$$
\mathcal{L}(\Lambda f)(x, t) = \int (i\partial_t(\sigma^0 + \sigma^1) + \Delta \sigma^0 + 2\pi i \langle \nabla \sigma^1, \xi \rangle + i[(\nabla \sigma)^2 + \langle \tilde{A}, \nabla \sigma \rangle]) \times e^{i\sigma(t, x, \xi)} e^{-\frac{4\pi^2 i t |\xi|^2}{\langle \tilde{A}, \nabla \sigma \rangle}} \Omega(\xi) \hat{f}(\xi) d\xi.
$$

Now every term in the formula for $\mathcal{L}(\Lambda f)$ except $\langle \nabla \sigma^1, \xi \rangle$ either has a two spatial derivatives or one time derivative acting on it (recall that in our scaling time derivatives are worth two spatial derivatives) or is quadratic in $\nabla \sigma$ (since $\nabla \sigma \sim A$).

However by our choice of $\sigma^1$, we have

$$
\langle \nabla \sigma^1, \xi \rangle = 2\pi i \sum_{k \leq -2} \int (\sum_{j,m=1}^{n} \partial_{m} \tilde{A}_k^j(x + z\xi/|\xi|) \xi_j \xi_m) \chi'(2^{2k}z)dz =
$$

$$
= 2\pi i \sum_{k \leq -2} \int \left( \frac{d}{dz} \tilde{A}_k(x + z\xi/|\xi|) \cdot \xi \right) |\xi| \chi'(2^{2k}z)dz =
$$

$$
= -2\pi i \sum_{k \leq -2} 2^{2k} \int \tilde{A}_k(x + z\xi/|\xi|) \cdot \xi |\xi| \chi''(2^{2k}z)dz.
$$

In this last expression, one has multiplication by $2^{2k}$, which behaves like two spatial derivatives on $\tilde{A}_k$.

Now that we have made our selection of $\sigma$, we go back to the proof of Lemma 4. First, expanding the exponential $e^{i\sigma}$ in Taylor series yields the representation

$$
\Lambda = \sum_{\alpha \geq 0} i^{\alpha} (\alpha!)^{-1} \Lambda^\alpha,
$$

where

$$
\Lambda^\alpha f(t, x) = \int (\sigma(t, x, \xi))^\alpha e^{-\frac{4\pi^2 i t |\xi|^2}{2}} e^{2\pi i \langle \xi, x \rangle} \Omega(\xi) \hat{f}(\xi) d\xi.
$$
and similar for the expression for $L(Af)$. It is clear now that in this formulation, it is convenient to think that $\sigma$ is in the form

$$\sigma(t, x, \xi) \sim \sum_{k \leq -2} \sum_{l \leq -2k} \int_0^\infty A_k(t, x + z\xi/|\xi|) \chi(2^k z) dz$$

for some $C_0^\infty$ function $\chi$ supported in $(-2, 2)$. This is done by subsuming the harmless terms $\xi/|\xi|$, $\xi$ and $\xi|\xi|$ in the multiplier $\Omega(\xi)$ and by considering the resulting expressions componentwise. This will be a good strategy for all estimates involving $L^2_x$ norms.\footnote{We will omit the interval of integration $(0, \infty)$ in the formula for $\sigma$. In other words, we will tacitly replace $\chi(2^k z)$ by $\chi_+(2^k z) := \chi(2^k z)\chi_{(0, \infty)}(z)$. This is not going to make any difference in the $L^2$ estimates, since no smoothness of the amplitude $\chi_+(2^k z)$ is needed.}

We will however need to show also decay estimates for $\Lambda^a f$, in which case, it is better to think of $\sigma$ in the form

$$\sigma(t, x, \xi) \sim \sum_{k \leq -2} \sum_{l \leq -2k} \int_0^\infty A_k(t, x + z\xi/|\xi|) \varphi(2^{-l} z) dz.$$ 

This is so since $\sum_{l \leq -2k} \varphi(2^{-l} z) = \chi(2^k z)$ and $\chi'(2^k z)$ which enters in $\sigma^1$ has essentially the same form as $\varphi(2^k z)$. Note that since $\varphi(2^{-l} z)$ has support away from the endpoints of the interval of integration, we can also write

$$\sigma(t, x, \xi) \sim \sum_{k \leq -2} \sum_{l \leq -2k} \int_0^\infty A_k(t, x + z\xi/|\xi|) \varphi(2^{-l} z) dz,$$

with some smooth compactly supported function $\varphi$ with $\text{supp} \varphi \subset (1/2, 4)$.

5. Pointwise estimates on $\sigma$

To start us off, we will need an additional angular decomposition for $\sigma$, which we describe next. Note that in this section, we completely ignore the $t$ dependence, since it is irrelevant in that setting. This is so, because in the spaces of interest to us (i.e. the Strichartz space and its dual) the $L^r_x$ norm always comes first.

To put ourselves in the desired framework, let us fix a family of unit vectors

$$\{\theta_j^m\}_{j \in [1, c2^{m(n-1)}], m \in [-2, \infty)}$$

with the property: For fixed $m$, the family of balls $\{B(\theta_j^m, 2^{-m})\} =: \{B_j^m\}$ forms a covering of $S^{n-1}$ with every ball in the family intersecting at most a fixed number (depending only on the dimension $n$) of other balls in the family.

In other words for fixed $m$, $\theta_j^m$ forms a $2^{-m}$ net over $S^{n-1}$ and the distance between any two $\theta_j^m, \theta_s^m$ is bounded below by $c2^{-m}$.

For the locally finite covering $\{B(\theta_j^m, 2^{-m})\}$ of the annulus $\{\xi : 1 - 2^{-m-5} \leq |\xi| \leq 1 + 2^{-m-5}\}$, we find a smooth partition of unity subordinated to the covering. That is, there exists a family $\{\psi_{j, m}\} \subset C_0^\infty$ with

$$\sum_j \psi_{j, m}(2^m (\xi/|\xi| - \theta_j^m)) = 1.$$ 

Moreover, the functions $\psi_{j, m}$ satisfy uniform bound on their derivatives, i.e. $\sup_{j, m} |\partial^\gamma \psi_{j, m}(x)| \leq C_n^\gamma$.
Fix $k, l : l > -k$ For any function $H_k$, expand $H_k(x + z\xi/|\xi|)$ in Taylor series around $x + z\theta_j^{l+k}$, when $|\xi/|\xi| - \theta_j^{l+k} \leq 2^{-l-k}$. We get

$$H_k(x + z\xi/|\xi|)\psi_{j,l+k}(2^{l+k}(\xi/|\xi| - \theta_j^{l+k})) =$$

$$= \sum_{\gamma \geq 0} \frac{1}{\gamma!} \partial^\gamma H_k(x + z\theta_j^{l+k})z|\gamma(\xi/|\xi| - \theta_j^{l+k})^\gamma\psi_{j,l+k}(2^{l+k}(\xi/|\xi| - \theta_j^{l+k})) =$$

$$= \sum_{\gamma \geq 0} \frac{z|\gamma|}{2!|\gamma|!} H_k'(x + z\theta_j^{l+k})\psi_{j,l+k}(2^{l+k}(\xi/|\xi| - \theta_j^{l+k}))$$

where $\psi_{j,m}(\mu) = \psi_{j,m}(\mu)|\gamma|$. We will also adopt the convention of naming a function $H_k^\gamma$, whenever $H_k^\gamma$ has the same Fourier support properties as $H_k$ and $\|H_k^\gamma\|_Z \leq C|\gamma|\|H_k\|_Z$ for all Banach spaces $Z$ used throughout the paper. Using the formula for $H_k(x + z\xi/|\xi|)\psi_{j,l+k}(2^{l+k}(\xi/|\xi| - \theta_j^{l+k}))$, we arrive at

$$\int H_k(x + z\xi/|\xi|)\psi_{j,l+k}(2^{l+k}(\xi/|\xi| - \theta_j^{l+k}))\varphi(2^{-l}z)dz =$$

$$(38) \quad \sum_{\gamma \geq 0} (\gamma!)^{-1}(\int H_k'(x + z\theta_j^{l+k})\varphi(2^{-l}z)dz)\psi_{j,l+k}(2^{l+k}(\xi/|\xi| - \theta_j^{l+k}))$$

If $l \leq -k$, the above decomposition trivializes in the sense that we can write

$$\int H_k(x + z\xi/|\xi|)\varphi(2^{-l}z)dz =$$

$$= \sum_{\gamma \geq 0} (\gamma!)^{-1}(\int H_k'(x + z\theta_j)\varphi(2^{-l}z)dz)(2^{l+k}(\xi/|\xi| - e_1))^\gamma,$$

that is, just the vector $e_1$ would suffice in that situation.

For the $L^2$ estimates, we use the decomposition

$$(39) \quad \int H_k(x + z\xi/|\xi|)\psi_{j,-k}(2^{-k}(\xi/|\xi| - \theta_j^{-k}))\chi(2^{2k}z)dz =$$

$$= \sum_{\gamma \geq 0} (\gamma!)^{-1}(\int H_k'(x + z\theta_j^{-k})\chi(2^{2k}z)dz)\psi_{j,-k}(2^{-k}(\xi/|\xi| - \theta_j^{-k})),$$

which is derived similar to (38).

We have the following pointwise bound on functions in the form $\int H_k(x + \theta_j^{l+k})\varphi(2^{-l}z)dz$.

**Lemma 5.** Let $k$ be fixed integer and $\varphi$ be a fixed Schwartz function with supp $\varphi \subset (1/2, 4)$. Then

$$(40) \quad \sup_x \sum_{l=-k}^{l} \sum_{j} \int |H_k(x + z\theta_j^{l+k})|\varphi(2^{-l}z)dz \leq C_n 2^{k(n-1)}\|H_k\|_{L^2}.$$

In the case $l \leq -k$, we trivially have

$$\sup_x \sum_{l=-k}^{l} \sum_{j} \int |H_k(x + z\theta_j^{l+k})|\varphi(2^{-l}z)dz \leq C 2^{-k}\|H_k\|_{L^\infty} \leq C_n 2^{k(n-1)}\|H_k\|_{L^2}.$$

**Proof.** We concentrate on the case $l > -k$, since the other inequality is obvious.

Represent $H_k(x) = 2^k \int \varsigma(2^{2k}|x - y|^2)H_k(y)dy$ with some suitable Schwartz function $\varsigma : \mathbb{R}^3 \to \mathbb{R}$. Clearly

$$2^{2k}|x + z\theta_j^{l+k} - y|^2 = 2^{2k}(|x - y| - z)^2 + 2(2^{2k})(|x - y|)(1 - (\theta_j^{l+k}, (y - x)/|y - x|).$$
We thus have
\[ |s(2^k|z - y|^2)| \leq \frac{C_n}{(1 + 2^k|x-y|^2)^N} \]
where \( \gamma \) is the angle between the unit vectors \( \theta_{j+k} \) and \( (y-x)/|y-x| \) and \( N \) is arbitrary integer. It follows that
\[
\sum_{l>\frac{1}{k}} \sum_{j} \int |H_k(x+z\theta_j)|\varphi(2^{-l}z)dz \leq \sum_{l>\frac{1}{k}} \sum_{j} \int \frac{C_n 2^{kn} |H_k(y)|\varphi(2^{-l}z)dz}{(1 + 2^k|x-y|^2)^N} dydz = \sum_{l>\frac{1}{k}} \sum_{j} \int \frac{C_n 2^{kn} |H_k(x+r\theta)|r^{-1}\varphi(2^{-l}z)dz}{(1 + 2^k|r-z|^2)^N} drd\theta dz
\]
The main term in the expression above is when the integration is over \( r \sim 2^l, |z-r| \leq 2^{-k} \) and \( |\theta_{j+k} - \theta| \leq 2^{-l-} \), with the corresponding decay away from this set. We estimate by
\[
C \sum_{l>\frac{1}{k}} \sum_{j} 2^{k(n-1)} \int_{r\sim 2^l} \int_{|\theta_{j+k} - \theta| \leq 2^{-l-k}} |H_k(x+r\theta)||r|^{n-1} dr
\]
Clearly the summation in \( j \) is sum of integrals over (almost) disjoint subsets of \( S^n \) and as a result it gives the integration over the whole \( S^n \) (since \( \{\theta_{j+k}\} \) were chosen to be a \( 2^{-l-k} \) net of \( S^n \)).

We get
\[
\sum_{l>\frac{1}{k}} \sum_{j} \int |H_k(x+z\theta_{j+k})|\varphi(2^{-l}z)dz \lesssim 2^{k(n-1)} \int |H_k(x+y)|dy \leq 2^{k(n-1)}\|H_k\|_{L^1}.
\]

6. \( L^dL^r \) estimates for the parametrix

We show that the parametrix is close to the initial data at \( t = 0 \) and stays in the Strichartz space \( L^dL^r \). Taking into account that \( \Lambda^0f = e^{it\Delta}f \), it is clear that (33), (34) will follow from
\[
(41) \quad \|\Lambda^\alpha f\|_{L^dL^r} \leq C_n^\alpha \|A\|_{L^1} \|f\|_{L^2}.
\]
The case \( \alpha = 0 \) corresponds to the case of free solutions, which are in \( L^dL^r \) by the standard Strichartz estimates. We prove (41) by showing that \( \Lambda^\alpha \) satisfies appropriate decay and energy estimates.

We will show that for a fixed \( s, t \)
\[
(42) \quad \|\Lambda^\alpha f(t, \cdot)\|_{L^2} \leq C_n^\alpha \left( \sum_{k} \sup_{x \in \mathbb{R}^n, \theta \in S^{n-1}} \int_{x} |A_k(t, x + z\theta)|dz \right)^\alpha \|f\|_{L^2}
\]
\[
(43) \quad |\Lambda^\alpha(t)\Lambda^\alpha(s)^* f(x)| \leq C_n^{2\alpha} \left( \sum_k 2^{k(n-1)}\|A_k\|_{L^\infty L^1} \right)^{2\alpha} |t-s|^{-n/2}\|f\|_{L^1},
\]
whence by the abstract Strichartz estimates of Keel and Tao, [14](see also Proposition 2), one gets (41).
Note that by the Bernstein inequality
\[
\sum_k \sup_{x \in \mathbb{R}^n, \theta \in S^{n-1}} \int |A_k(t, x + z\theta)|dz = \sum_k \sup_{U \subseteq SU(n), x} \|A_k(t, x + Uz)\|_{L_t^\infty L_x^{2\alpha}} \lesssim \sum_k 2^{k(n-1)}\|A_k\|_{L_x^\infty} = \|A\|_{Y_1}
\]
and therefore (42) and (43) hold for \(A \in Y_1\).

If however \(n \geq 3\), we have by complex multilinear interpolation between (42) and (43) that the “modified decay estimate” (see (17))
\[
\|A^\alpha(t)\Lambda^\alpha(s)^*f(x)\|_{L^{p'}} \leq C_n 2^{k(n-1)/p} \sup_{U \subseteq SU(n), x} \|A_k(t, x + Uz)\|_{L_t^\infty L_x^{2\alpha}} 2^\alpha \times |t - s|^{-n/(2p)}\|f\|_{L^p} = C_n \|A\|_{Y_1}^{2\alpha} \|f\|_{L^p},
\]
which suffices for (41). Thus, (42) and (43) imply (41).

6.1. Energy estimates: Proof of (42). It is more convenient to show \((\Lambda^\alpha)^* : L^2 \to L^2\), which is equivalent to (42). Clearly
\[
(\Lambda^\alpha)^* f(x, t) = \int e^{4\pi^2t|\xi|^2} e^{2\pi i(\xi, x-y)} \Omega(\xi) f(y) \overline{\sigma^\alpha(t, y, \xi)} dyd\xi.
\]
Having in mind the specific form of \(\sigma\), matters reduce to

**Lemma 6.** Let \(\alpha\) be an positive integer and \(k_1, k_2, \ldots, k_\alpha\) are integers. Let \(\{F^\mu_{k_\nu}\}\) be a collection of functions with supp\(\overline{F^\mu_{k_\nu}} \subseteq \{\xi : |\xi| \sim 2^{k_\nu}\}\). Then for the multilinear operator
\[
\Xi^{k_1, \ldots, k_\alpha}_{F_1, \ldots, F_\alpha} f(t, x) = \int e^{4\pi^2t|\xi|^2} e^{2\pi i(\xi, x-y)} \Omega(\xi) f(y) \prod_{\mu=1}^{\alpha} \int F^\mu_{k_\nu}(t, y + z\xi/|\xi|) \chi(2^{k_\nu}z) dz dy,
\]
there is the estimate
\[
\left\|\Xi^{k_1, \ldots, k_\alpha}_{F_1, \ldots, F_\alpha} f(t, \cdot)\right\|_{L_x^2} \leq C_n \|f\|_{L^2} \prod_{\mu=1}^{\alpha} \sup_{x, \theta} \int |F^\mu_{k_\nu}(t, x + z\theta)|dz.
\]

The Lemma is applied to \((\Lambda^\alpha)^*\) in an obvious way. That is, write
\(
\sigma = \sum_{k \leq -2} A_k(t, x + z\xi/|\xi|) \chi(2^{2k}z) dz
\)

\(
(\Lambda^\alpha)^* f(t, x) = \sum_{k_1, \ldots, k_\alpha} \int e^{4\pi^2t|\xi|^2} e^{2\pi i(\xi, x-y)} \Omega(\xi) f(y) \prod_{\mu=1}^{\alpha} A^\mu_{k_\nu}(t, y + z\xi/|\xi|) \chi(2^{2k_\nu}z) dz dyd\xi = \sum_{k_1, \ldots, k_\alpha} \Xi^{k_1, \ldots, k_\alpha}_{A, \ldots, A} f(t, x)
\)

It follows that
\[
\|\Lambda^\alpha f\|_{L^2} \leq \sum_{k_1, \ldots, k_\alpha} \left\|\Xi^{k_1, \ldots, k_\alpha}_{A, \ldots, A} f\right\|_{L^2} \leq C_n \|f\|_{L^2} \left(\sum_{k} \sup_{x, \theta} \int |A_k(t, x + z\theta)|dz\right)^\alpha,
\]
as claimed.
Proof. (Lemma 6) Let us first present to the proof in the case $\alpha = 1$, since the proof in the general case follows similar scheme, with somewhat cumbersome notations.

The basic idea is to “pretend” that $\int F_{k}(t, y + z\xi/|\xi|)\chi(2^{2k}z)dz$ is independent of $\xi$. We show that this is almost true, modulo the angular decomposition, that we have described earlier.

For the given $k$, introduce the partition of unity
$$\sum_{j} \psi_{j,-k}(2^{-k}(\xi/|\xi| - \theta_{j}^{-k})) = 1$$

Next, expand $F_{k}(y + z\xi/|\xi|)$ around $y + z\theta_{j}^{-k}$. We get
$$F_{k}(t, y + z\xi/|\xi|)\chi(2^{2k}z)\psi_{j,-k}(2^{-k}(\xi/|\xi| - \theta_{j}^{-k})) =$$
$$= \sum_{\gamma \geq 0} (\gamma!)^{-1} F_{k}^\gamma(t, y + z\theta_{j}^{-k})\chi(2^{2k}z)\psi_{j,-k}(2^{-k}(\xi/|\xi| - \theta_{j}^{-k}))$$

We drop the $\gamma$’s, since in the end, we always add up with the help of $(\gamma!)^{-1}$. We have

(45) $\Xi f(t, x) = e^{-it\Delta} \sum_{j} P^{j}[f(\cdot) \int F_{k}(t, \cdot + z\theta_{j}^{-k})\chi(2^{2k}z)dz]$, where $\hat{P}^{j}g(\xi) = \hat{g}(\xi)\psi_{j,-k}(2^{-k}(\xi/|\xi| - \theta_{j}^{-k}))$. Note that the function $y \mapsto \int F_{k}(y + z\theta_{j}^{-k})\chi(2^{2k}z)dz$ has a Fourier transform supported in $\{\xi : |\xi| \sim 2^{k}\}$. It follows

(46) $P^{j}[f(\cdot) \int F_{k}(t, \cdot + z\theta_{j}^{-k})\chi(2^{2k}z)dz] = P^{j}[\int F_{k}(t, \cdot + z\theta_{j}^{-k})\chi(2^{2k}z)dz \hat{P}^{j}(f)]$, for some $\hat{P}^{j} : \hat{P}^{j}P^{j} = P^{j}$ and $\hat{P}^{j}$ has multiplier with support inside $|\xi/|\xi| - \theta_{j}^{-k}| \leq 2^{2k+1}$.

This implies

$$\|\Xi f(t, x)\|_{L^2}^{2} \lesssim \left( \sum_{j} \left\| P^{j}[f(\cdot) \int F_{k}(t, \cdot + z\theta_{j}^{-k})\chi(2^{2k}z)dz]\right\|_{L^2}^{2} \right)^{1/2} =$$

$$= \left( \sum_{j} \left\| P^{j}[\int F_{k}(t, \cdot + z\theta_{j}^{-k})\chi(2^{2k}z)dz] \hat{P}^{j}f\right]\right\|_{L^2}^{2} \right)^{1/2} \lesssim$$

$$\lesssim \left( \sum_{j} \left\| \int F_{k}(t, \cdot + z\theta_{j}^{-k})\chi(2^{2k}z)dz \hat{P}^{j}f\right]\right\|_{L^2}^{2} \right)^{1/2} \lesssim$$

$$\lesssim \left( \sum_{j} \left\| \int F_{k}(t, \cdot + z\theta_{j}^{-k})\chi(2^{2k}z)dz\right\|_{L^\infty}^{2} \left\| \hat{P}^{j}f\right\|_{L^2}^{2} \right)^{1/2} \lesssim$$

$$\leq C_{n} \sup_{y, \theta} \left\| \int F_{k}(t, y + z\theta)\chi(2^{2k}z)dz\right\|_{L^2} \left\| f\right\|_{L^2}.$$}

Note that the previous calculation requires slightly augmenting $P^{j}$ to $\hat{P}^{j}$ for an additional constant $C_{n}$ coming on the account of the extra overlap of the supports of different $\hat{P}^{j}$.

Unfortunately, we have to be extra careful for the case of $\alpha > 1$, because the constants (following this argument) are estimated by $\alpha^\alpha$, whereas we need (and can manage) constants
of magnitude $C_n^\alpha$. This problem occurs, when too many of the frequencies $k_1, \ldots, k_\alpha$ are equal.

Thus, we start our considerations for $\alpha > 1$ by ordering these frequencies. Without loss of generality let us assume $k_1 = \ldots = k_{s_1} < k_{s_1+1} = \ldots = k_{s_2} < \ldots k_{s_{m-1}+1} = \ldots = k_{s_m} = k_\alpha$, where we have set $s_0 = 0$ for convenience.

Since for every $1 \leq r \leq m$, and $\mu \in [s_{r-1} + 1, s_r]$

$$
\sum_{j_r} \psi_{j_r, -k_\mu}(2^{-k_\mu}(\xi/|\xi| - \theta_{j_r}^{-k_\mu})) = 1,
$$

we expand $F_{k_\mu}^\mu(t, y + z\xi/|\xi|)$ around $y + z\theta_{j_r}^{-k_\mu}$. Just as in the case $\alpha = 1$, this allows us to write

$$
(47) \quad \Xi_{F_1, \ldots, F_\alpha}^k f(t, x) = e^{-i\Delta t} \sum_{j_1, \ldots, j_m} \sum_{\gamma_1, \ldots, \gamma_\alpha} P^{j_1} \ldots P^{j_m}[f(\cdot) \prod_{\mu=1}^\alpha g_\mu^\gamma(t, \cdot)]/(\gamma_1! \ldots \gamma_\alpha!),
$$

where for $\mu \in [s_{r-1} + 1, s_r]$

$$
\tilde{P^{j_r}} g(\xi) = \hat{\tilde{\psi}}(\xi) \psi_{j_r, -k_\mu}(2^{-k_\mu}(\xi/|\xi| - \theta_{j_r}^{-k_\mu})),
$$

$$
g_\mu^\gamma(t, y) = \int F_{k_\mu}^\mu(t, y + z\theta_{j_r}^{-k_\mu}) \chi(2^{k_\mu}z) dz.
$$

As always we drop the $\gamma$’s and concentrate on the case $\gamma_1 = \ldots = \gamma_\alpha = 0$.

Next, observe that since $e^{-i\Delta t}$ is an isometry on $L^2$, we can dispose of it immediately. We have

$$
\left\| \Xi_{F_1, \ldots, F_\alpha}^k f \right\|_{L^2}^2 < \sum_{j_1, \ldots, j_m} \left\| P^{j_1} \ldots P^{j_m}[f(\cdot) \prod_{\mu=1}^\alpha g_\mu^\gamma(t, \cdot)] \right\|_{L^2}^2.
$$

For technical reasons, it is more convenient to replace $P^{j_r}$ by “rough” versions of the same. Namely, introduce the Fourier restriction operators $Q^{j_r}$, which act via

$$
\tilde{Q^{j_r}} g(\xi) = \hat{\tilde{\psi}}(\xi) 1_{\{\xi/|\xi| - \theta_{j_r}^{-k_\mu} \leq 2^{k_\mu}\}} (\xi/|\xi| - \theta_{j_r}^{-k_\mu}).
$$

More generally, for any $a > 0$,

$$
\tilde{Q^{j_r}} g(\xi) = \hat{\tilde{\psi}}(\xi) 1_{\{\xi/|\xi| - \theta_{j_r}^{-k_\mu} \leq a\}} (\xi/|\xi| - \theta_{j_r}^{-k_\mu}).
$$

Since the Fourier supports of the multipliers of $P^{j_r}$ are in $\{\xi : |\xi/|\xi| - \theta_{j_r}^{-k_\mu} \leq 2^{k_\mu}\}$, we have

$$
(48) \quad \left\| \Xi_{F_1, \ldots, F_\alpha}^k f \right\|_{L^2}^2 < \sum_{j_1, \ldots, j_m} \left\| Q^{j_1} \ldots Q^{j_m}[f(\cdot) \prod_{\mu=1}^\alpha g_\mu(t, \cdot)] \right\|_{L^2}^2.
$$

Fix $j_1$. Clearly the summation in any $j_r$ runs only on $j_r : |\theta_{j_r}^{-k_\mu} - \theta_{j_1}^{-k_{s_1}}| \leq 2^{k_{s_1}} + 2^{k_\alpha}$, since otherwise $Q^{j_1} Q^{j_r} = 0$. Thus,

$$
(49) \quad \left\| \Xi_{F_1, \ldots, F_\alpha}^k f \right\|_{L^2}^2 < \sum_{j_1} \sum_{j_2, \ldots, j_m} \left\| Q^{j_1} [f(\cdot) \prod_{\mu=1}^\alpha g_\mu(t, \cdot)] \right\|_{L^2}^2.
$$
Note that since \( \text{supp} \, \hat{g}_\mu \subset \{ \xi : |\xi| \leq 2^{k_\mu+1} \} \), we have that \( \text{supp} \, g_1 \ldots g_{s_1} \subset \{ \xi : |\xi| \leq 2s_12^{k_{s_1}} \} \). It follows that

\[
Q^{j_1}[f(\cdot) \prod_{\mu=1}^{\alpha} \mu g_\mu(t, \cdot)] = Q^{j_1}[(g_1 \ldots g_{s_1})(f(\cdot) \prod_{\mu=s_1+1}^{\alpha} \mu g_\mu(t, \cdot))] = Q^{j_1}(g_1 \ldots g_{s_1}Q^{j_1}_{2^{k_{s_1}+2s_12^{k_{s_1}}}}(f(\cdot) \prod_{\mu=s_1+1}^{\alpha} \mu g_\mu(t, \cdot)))
\]

Plugging this back in (49) and taking into account \( \|Q^{j_1}\|_{L^2 \rightarrow L^2} = 1 \) yields

\[
\left\| \frac{k_1 \ldots k_\alpha}{f_1 \ldots f_\alpha} f \right\|_{L^2}^2 \leq \sum_{j_1} \sum_{j_2, \ldots, j_m : |\theta_{j_\nu} - k_{s_\nu}| \leq 2^{k_{s_\nu}+2^{k_{s_\nu}}} \theta_{j_1} - k_{s_1}| \leq 2^{k_{s_1}+2^{k_{s_1}}}} \left\| g_1 \ldots g_{s_1} Q^{j_1}_{2^{k_{s_1}+2s_12^{k_{s_1}}}}(f(\cdot) \prod_{\mu=s_1+1}^{\alpha} \mu g_\mu(t, \cdot)) \right\|_{L^2}^2 \leq \sum_{j_1} \sum_{j_2, \ldots, j_m : |\theta_{j_\nu} - k_{s_\nu}| \leq 2^{k_{s_\nu}+2^{k_{s_\nu}}} \theta_{j_1} - k_{s_1}| \leq 2^{k_{s_1}+2^{k_{s_1}}}} \left\| g_1 \ldots g_{s_1} \right\|_{L^\infty}^2 \left\| Q^{j_1}_{2^{k_{s_1}+2s_12^{k_{s_1}}}}(f(\cdot) \prod_{\mu=s_1+1}^{\alpha} \mu g_\mu(t, \cdot)) \right\|_{L^2}^2 \leq \prod_{\mu=1}^{s_1} \sup_{y, \theta} \left| \int F^\mu_{k_\mu}(t, y + z\theta) \chi(2^{k_\nu} z) dz \right|^2 \times 
\]

\[
\sum_{j_1} \sum_{j_2, \ldots, j_m : |\theta_{j_\nu} - k_{s_\nu}| \leq 2^{k_{s_\nu}+2^{k_{s_\nu}}} \theta_{j_1} - k_{s_1}| \leq 2^{k_{s_1}+2^{k_{s_1}}}} \left\| Q^{j_1}_{2^{k_{s_1}+2s_12^{k_{s_1}}}}(f(\cdot) \prod_{\mu=s_1+1}^{\alpha} \mu g_\mu(t, \cdot)) \right\|_{L^2}^2.
\]

It is now time to reintroduce the \( Q^{j_r} \) multipliers. Since \( |\theta_{j_r} - k_{s_1}| \leq 2^{k_{s_1}+2^{k_{s_\nu}}} \), we conclude that for every \( r : 1 \leq r \leq m \), \( Q^{j_r}_{2^{k_{s_1}+2s_12^{k_{s_1}}}} = Q^{j_r}_{2^{k_{s_1}+2s_12^{k_{s_1}}} \cap 2^{k_{s_1}+2s_12^{k_{s_1}}}} \) and we have

\[
\sum_{j_1} \sum_{j_2, \ldots, j_m : |\theta_{j_\nu} - k_{s_\nu}| \leq 2^{k_{s_\nu}+2^{k_{s_\nu}}} \theta_{j_1} - k_{s_1}| \leq 2^{k_{s_1}+2^{k_{s_1}}}} \left\| Q^{j_1}_{2^{k_{s_1}+2s_12^{k_{s_1}}}}(f(\cdot) \prod_{\mu=s_1+1}^{\alpha} \mu g_\mu(t, \cdot)) \right\|_{L^2}^2 \leq \sum_{j_2, \ldots, j_m} \sum_{j_1} \left\| Q^{j_1}_{2^{k_{s_1}+2s_12^{k_{s_1}}}} \cdot Q^{j_2}_{2^{k_{s_1}+(2s_1+2^{k_{s_1}})}} \ldots Q^{j_m}_{2^{k_{s_1}+(2s_1+2^{k_{s_1}})}}(f(\cdot) \prod_{\mu=s_1+1}^{\alpha} \mu g_\mu(t, \cdot)) \right\|_{L^2}^2.
\]

Note that

\[
(51) \sum_{j_1} \left\| Q^{j_1}_{2^{k_{s_1}+2s_12^{k_{s_1}}}} g \right\|_{L^2}^2 \leq C_n \left( \frac{2^{k_{s_1}+2s_12^{k_{s_1}}}}{2^{k_{s_1}}} \right)^n \|g\|_{L^2}^2 = C_n(1 + 2s_1)^n \|g\|_{L^2}^2,
\]

because of the extra overlap created by passing from \( Q^{j_1} \) to \( Q^{j_1}_{2^{k_{s_1}+2s_12^{k_{s_1}}}} \). It follows that

\[
\left\| \frac{k_1 \ldots k_\alpha}{f_1 \ldots f_\alpha} f \right\|_{L^2}^2 \leq C_n \prod_{\mu=1}^{s_1} \sup_{y, \theta} \left| \int F^\mu_{k_\mu}(t, y + z\theta) \chi(2^{k_\nu} z) dz \right|^2 (1 + 2s_1)^n \times 
\]

\[
\sum_{j_2, \ldots, j_m} \left\| Q^{j_2}_{2^{k_{s_1}+(2s_1+2^{k_{s_1}})}} \ldots Q^{j_m}_{2^{k_{s_1}+(2s_1+2^{k_{s_1}})}}(f(\cdot) \prod_{\mu=s_1+1}^{\alpha} \mu g_\mu(t, \cdot)) \right\|_{L^2}^2.
\]
This is very similar to (48), except that the sum in \( j_1 \) is taken care of and \( Q^{j_2} \ldots Q^{j_m} \) gets replaced by the slightly larger \( Q^{j_2} \ldots Q^{j_m} Q^{k_2 + (2s_1 + 2)k_3 + \ldots + (2s_m + 2)k_m} \). Continuing in this fashion yields the estimate
\[
\left\| \sum_{F_1, \ldots, F_n} f \right\|_{L^2_x}^2 \leq C_n \prod_{\mu=1}^n \left( \sum_{y, \theta} |F_{k, \mu}(t, y + z\theta)\chi(2^{2k\mu} z)dz|^2 \right)^{\frac{1}{2}} \times \prod_{r=2}^m \left( \frac{2^{2k_r} + (2s_1 + 2)2^{k_3} + (2(s_2 - s_1) + 2)2^{k_2} + \ldots + (2(s_r - s_{r-1}) + 2)2^{k_r}}{2^{2k_r}} \right)^{n-r}.
\]

Note that for every \( 1 \leq i \leq r \), there is \( k_{r-i} \geq r-i \), since there are \( r-i \) strict inequalities in the chain \( k_{s_i} < \ldots < k_{s_r} \). We therefore need an estimate for
\[
G = \prod_{r=1}^m \left( 1 + \sum_{i=1}^r (2(s_i - s_{i-1}) + 2)^{2i-r} \right)^n
\]

For \( \ln(G) \), note that since \( \ln(1 + x) \leq x \) and \( m \leq \alpha \), we can estimate
\[
\ln(G) \leq n \sum_{r=1}^m \sum_{i=1}^r (2(s_i - s_{i-1}) + 2)^{2i-r} \leq 4n(m + \sum_{i=1}^m (s_i - s_{i-1})) = 4n(m + \alpha) \leq 8n\alpha.
\]

It follows that \( G \leq e^{8n\alpha} \) and
\[
\left\| \sum_{F_1, \ldots, F_n} f \right\|_{L^2_x}^2 \leq C_n^\alpha \prod_{\mu=1}^n \left( \sum_{y, \theta} |F_{k, \mu}(t, y + z\theta)\chi(2^{2k\mu} z)dz|^2 \right)^{\frac{n}{2}}.
\]

\( \square \)

6.2. Dispersive estimates: Proof of (43). For the dispersive estimate, write
\[
\Lambda^\alpha(t)\Lambda^\alpha(s)^* f = \int \sigma^\alpha(t, x, \xi)\sigma^\alpha(s, y, \xi) e^{-4\pi^2i(t-s)|\xi|^2} e^{2\pi i(\xi, x-y)} \Omega^2(\xi) f(y)dyd\xi
\]
Clearly for \( \alpha = 0 \), we have (43) by the decay estimates for the free solution. Consider the case \( \alpha = 1 \) for simplicity, the general case to be addressed momentarily.

Expand \( \sigma(t, x, \xi) \) to get
\[
\Lambda^1(t)\Lambda^1(s)^* f = \sum_{k_1 \leq -2} \sum_{l_1 \leq -2} \sum_{k_1 \geq 0} (\gamma_1)^{-1} \sum_{j_1} \left( \int A_{k_1}^{\gamma_1}(t, x + z\theta_{j_1}^{l_1 + k_1}) \varphi^{\gamma_1}(2^{-l_1} z)dz \right) \times \int \psi_{j_1, l_1 + k_1}^{\gamma_1}(2^{l_1 + k_1}(\xi/|\xi| - \theta_{j_1}^{l_1 + k_1})) e^{-4\pi^2i(t-s)|\xi|^2} e^{2\pi i(\xi, x-y)} \Omega^2(\xi) f(y)dyd\xi = \sum_{k_1 \leq -2} \sum_{l_1 \leq -2} \sum_{k_1 \geq 0} (\gamma_1)^{-1} \sum_{j_1} \left( \int A_{k_1}^{\gamma_1}(t, x + z\theta_{j_1}^{l_1 + k_1}) \varphi^{\gamma_1}(2^{-l_1} z)dz \right) \Gamma_{j_1, l_1 + k_1}^{\gamma_1},
\]

\[
\Gamma_{j_1, l_1 + k_1}^{\gamma_1} f(x) = \int \sigma^\gamma(s, y, \xi) e^{-4\pi^2i(t-s)|\xi|^2} e^{2\pi i(\xi, x-y)} \psi_{j_1, l_1 + k_1}^{\gamma_1}(2^{l_1 + k}(\xi/|\xi| - \theta_{j_1}^{l_1 + k})) \Omega^2(\xi) f(y)dyd\xi
\]

\(^{11}\)In the case \( l_1 \leq -k_1 \) the summation in \( j_1 \) collapses to a single term and \( \theta_{j_1}^{l_1 + k_1} = e_1 \) as pointed out in the previous section.
It is easy to see that
\[
\|\Lambda^1(t)\Lambda^1(s)^* f\|_{L_\infty} \lesssim \\
\lesssim \sup_x \left( \sum_{k_1 \leq -2} \sum_{l_2 \leq -2k_1} \sum (\gamma_1!)^{-1} \sum_j A_{k_1}^\gamma (t, x + z \theta_{j_1}^{l_1+k_1}) |\varphi^\gamma (2^{-l_1} z) dz| \times \\
\times \sup_{j_1, l_1, k_1} \|\Gamma_{j_1, l_1+k_1} f\|_{L_\infty} \right).
\]

By the pointwise estimates of Lemma 5 and \( A \in Y_1 \), the last expression is bounded by \( C\|A\|^2_{Y_1} |t - s|^{-n/2} \), provided one can show \( \|\Gamma_{j_1, l_1+k_1} f\|_{L_\infty} \leq C\|A\|_{Y_1} |t - s|^{-n/2} \|f\|_{L^1} \).

More generally, it is easy to see that by iterating the argument above, we have
\[
\Lambda^\alpha(t)\Lambda^\alpha(s)^* f(x) = \\
= \prod_{\mu=1}^\alpha \left( \sum_{\mu \leq -2} \sum_{l_\mu \leq -2k_\mu} \sum (\gamma_\mu!)^{-1} \sum_j A_{k_\mu}^\gamma (t, x + z \theta_{j_\mu}^{l_\mu+k_\mu}) |\varphi^\gamma (2^{-l_\mu} z) dz| \times \\
\times \sup_{j_\mu, l_\mu, k_\mu} \|\Gamma_{j_\mu, l_\mu+k_\mu} f\|_{L_\infty} \right).
\]

Thus,
\[
\|\Lambda^\alpha(t)\Lambda^\alpha(s)^* f\|_{L_\infty} \leq \\
\leq \prod_{\mu=1}^\alpha \sup_x \left( \sum_{\mu \leq -2} \sum_{l_\mu \leq -2k_\mu} \sum (\gamma_\mu!)^{-1} \sum_j A_{k_\mu}^\gamma (t, x + z \theta_{j_\mu}^{l_\mu+k_\mu}) |\varphi^\gamma (2^{-l_\mu} z) dz| \times \\
\times \sup_{j_\mu, l_\mu, k_\mu} \|\Gamma_{j_\mu, l_\mu+k_\mu} f\|_{L_\infty} \right).
\]

By Lemma 5, we have
\[
\|\Lambda^\alpha(t)\Lambda^\alpha(s)^* f\|_{L_\infty} \leq C_n^{2\alpha} \|A\|^2_{Y_1} |t - s|^{-n/2} \|f\|_{L^1},
\]
provided one can show
\[
(52) \sup_{j_\mu, l_\mu, k_\mu} \|\Gamma_{j_\mu, l_\mu+k_\mu} f\|_{L_\infty} \leq C_n^\alpha \|A\|^\alpha_{Y_1} |t - s|^{-n/2} \|f\|_{L^1}.
\]

Note that \( \Gamma_{j_\mu, l_\mu+k_\mu} f \) looks somewhat like \( \Lambda(s)^* f \), with the important difference that it has the “multiplier” \( \prod_{\mu=1}^\alpha \psi_{j_\mu, l_\mu+k_\mu} (2^{l_\mu+k_\mu} (|\xi| - \theta_{j_\mu}^{l_\mu+k_\mu})) \) in its definition. Dualizing (52) leads us to showing that
\[
[(\Gamma_{j_1, l_1+k_1} \ldots j_\alpha, l_\alpha+k_\alpha)^* f](x) = \\
= \int \sigma^\alpha(s, x, \xi) e^{4\pi^2 i (t-s) |\xi|^2} e^{2\pi i (\xi, x-y)} \times \\
\times \prod_{\mu=1}^\alpha \psi_{j_\mu, l_\mu+k_\mu} (2^{l_\mu+k_\mu} (|\xi| - \theta_{j_\mu}^{l_\mu+k_\mu})) \Omega^2(\xi) f(y) dy d\xi.
\]
By the Krein-Milman theorem, span[δα] allows us to estimate and the required dispersive estimates hold.

\[ [(Γ_{j_1, l_1 + k_1}^{α_1}, ..., j_α, l_α + k_α)^α f](x) = \prod_{ν=1}^α \left( \sum_{k_ν \leq -2 l_ν \leq -2 k_ν} \sum (γ_ν ν)^{-1} \sum_{j_ν} \int A_κ^α γ_ν(s, x + zθ_{j_ν}^k + k_ν)ϕ^α k_ν (2^{-l_ν} z) dz \right) \times \times \int e^{4π^2 (t-s) |ξ|^2} e^{2πi (ξ, x-y)} \prod_{ν=1}^α ψ_{j_ν, l_ν + k_ν}^ν (2^{l_ν + k_ν} (ξ/|ξ| - θ_{j_ν}^k)) \times \times \prod_{μ=1}^α ψ_{j_μ, l_μ + k_μ}^γ μ (2^{l_μ + k_μ} (ξ/|ξ| - θ_{j_μ}^k)) 2^μ (ξ) f(y) dy dξ. \]

By the Krein-Milman theorem, span[δα] is w* dense in \( M(\mathbb{R}^n) \supset L^1(\mathbb{R}^n) \). This, together with Lemma 5 allows us to estimate

\[ \| (Γ_{j_1, l_1 + k_1}^{α_1}, ..., j_α, l_α + k_α)^α f \|_{L^∞_x} \leq C_n^α (\| A \|_{Y_1})^α \times \sup_{x, b \in \mathbb{R}^n; m_1, ..., m_2, α \geq 0; θ_1, ..., θ_2, α ∈ S^{α-1}} | \int e^{4π^2 i (t-s) |ξ|^2} e^{2πi (ξ, x-y)} \prod_{μ=1}^α ψ_{j_μ, l_μ + k_μ}^γ μ (2^{l_μ + k_μ} (ξ/|ξ| - θ_μ)) 2^μ (ξ) dξ |. \]

The oscillatory integral above is bounded by \( C_n^α |t-s|^{-n/2} \) by Lemma 10 in the Appendix, and the required dispersive estimates hold.

7. The Parametrix Almost Satisfies the Equation

In this section, we show that the parametrix satisfies (35). We have several types of terms that arise according to (36).

First, we take on the terms

\[ \int (i\partial_t (σ^0 + σ^1) + Δσ^0 + 2πi \langle \nabla σ^1, ξ \rangle) e^{iσ(t, x, ξ)} e^{4π^2 i (t-s) |ξ|^2} e^{2πi (ξ, x)} 2^μ (ξ) dξ. \]

These are all terms linear in either \( σ^0 \) or \( σ^1 \) with either a time derivative or two spatial derivative acting on them (recall from the expression (37) that \( \langle \nabla σ^1, ξ \rangle \) is also of this form).

7.1. Main result.

Lemma 7. Let \( Ξ_{F_1, ..., F_α}^{k_1, ..., k_α} \) be as in Lemma 6, i.e.

\[ Ξ_{F_1, ..., F_α}^{k_1, ..., k_α} f(t, x) = \prod_{μ=1}^α (\int F_κ^μ(t, x + zξ/|ξ|) 2^{2k_μ} z dz) e^{-4π^2 i (t-s) |ξ|^2} e^{2πi (ξ, x)} 2^μ (ξ) f(ξ) dξ. \]
Then there exists a constant $C_n$, so that

$$
\left\| \Xi_{k_1, \ldots, k_n} f \right\|_{L^1 L^2} \leq C_n^\alpha \| f \|_{L^2} \times
\prod_{\mu=1}^{\alpha-1} \left( 2k_\mu^{(n-1)/2} \sup_{U \in SU(n)} \sup_{t} \sup_{x} \| F_{k_\mu}^\mu(t, x + U z) \|_{L^2_2 \ldots L^1_1} + \right)
\times \left( 2k_\mu^{(n+3)/2} \sup_{U \in SU(n)} \| \sum_{\alpha} \alpha^\mu \langle \nabla \sigma^1 \rangle \right) \times
\times \left( 2k_\mu^{(n-1)/2} \sup_{U \in SU(n)} \| |F_{k_\mu}^\mu(t, x + U z) \|_{L^2_2 \ldots L^1_1} + \right)
\times \left( 2k_\mu^{(n-5)/2} \sup_{U \in SU(n)} \| \sum_{\alpha} \alpha^\mu \langle \nabla \sigma^1 \rangle \right).
$$

In a different form,

$$
\left\| \Xi_{k_1, \ldots, k_n} f \right\|_{L^1 L^2} \leq C_n^\alpha \| f \|_{L^2} \times
\prod_{\mu=1}^{\alpha-1} \left( 2k_\mu^{(n-1)/2} \sup_{U \in SU(n)} \sup_{t} \sup_{x} \| F_{k_\mu}^\mu(t, x + U z) \|_{L^\infty L^1_2 \ldots L^1_1} + \right)
\times \left( 2k_\mu^{(n+3)/2} \sup_{U \in SU(n), v \rightarrow x(t)} \| \sum_{\alpha} \alpha^\mu \langle \nabla \sigma^1 \rangle \right) \times
\times \left( 2k_\mu^{(n-1)/2} \sup_{U \in SU(n), v \rightarrow x(t)} \| |F_{k_\mu}^\mu(t, x(t) + U z) \|_{L^2_2 \ldots L^1_1} + \right)
\times \left( 2k_\mu^{(n-5)/2} \sup_{U \in SU(n), v \rightarrow x(t)} \| \sum_{\alpha} \alpha^\mu \langle \nabla \sigma^1 \rangle \right),
$$

where $\sup_{t \rightarrow x(t)}$ is taken over all measurable functions $x(\cdot) : \mathbb{R}^1 \rightarrow \mathbb{R}^n$.

7.2. Application of Lemma 7 to various terms of $\| f \|_{L^1 L^2}$. Assuming the validity of Lemma 7, one can easily handle the first type of terms.

Expand $e^{i\sigma(t, x, \xi)}$ in powers of $\sigma$ as in the previous section yields

$$
\int \left( i \partial_t (\sigma^0 + \sigma^1) + \Delta \sigma^0 + 2\pi i \langle \nabla \sigma^1, \xi \rangle \right) e^{i\sigma(t, x, \xi)} e^{-4\pi i \tau |\xi|^2} e^{2\pi i \langle \xi, x \rangle} \Omega(\xi) \hat{f}(\xi) d\xi =
$$

$$
= \sum_{\alpha \geq 1} \frac{i^{\alpha-1}}{(\alpha - 1)!} \int \sigma(t, x, \xi)^{\alpha-1} (i \partial_t (\sigma^0 + \sigma^1) + \Delta \sigma^0 + 2\pi i \langle \nabla \sigma^1, \xi \rangle) \times
$$

$$
\times e^{-4\pi i \tau |\xi|^2} e^{2\pi i \langle \xi, x \rangle} \Omega(\xi) \hat{f}(\xi) d\xi = \sum_{\alpha \geq 1} \frac{i^{\alpha-1}}{(\alpha - 1)!} \sum_{k_1, \ldots, k_n}
$$

$$
= \int \prod_{\mu=1}^{\alpha-1} \left( \int A_{k_\mu}(t, x + z \xi | |\xi|) \chi(2k_\mu z) dz \right) \left( \int A_{k_\mu}(t, x + z \xi | |\xi|) \chi(2k_\mu z) dz \right) \times
$$

$$
\times e^{-4\pi i \tau |\xi|^2} e^{2\pi i \langle \xi, x \rangle} \Omega(\xi) \hat{f}(\xi) d\xi =
$$

$$
= \sum_{\alpha \geq 1} \frac{i^{\alpha-1}}{(\alpha - 1)!} \sum_{k_1, \ldots, k_n}
$$
where we have denoted by $\int \partial^2 A_{k_\alpha}(t, x + z \xi / |\xi|) \chi(2^{2k}z) dz$ all the terms $P_{k_\alpha}(i \partial_t (\sigma^0 + \sigma^1) + \Delta \sigma^0 + 2\pi i \langle \nabla \sigma^1, \xi \rangle)$. Thus, an application of Lemma 7 yields

$$\left\| \int (i \partial_t (\sigma^0 + \sigma^1) + \Delta \sigma^0 + 2\pi i \langle \nabla \sigma^1, \xi \rangle)e^{i\alpha(t,x,\xi)}e^{-4\pi^2it|\xi|^2}e^{2\pi i(\xi,x)}\Omega(\xi)\hat{f}(\xi)d\xi \right\|_{L^1L^2} \leq C_n\|f\|_{L^2} \exp(C_n(\sum_k 2^{k(n-1)/2} \sup_{U \in SU(n)} \sup_x \|A_k(t, x + Uz)\|_{L^1_{t}L^2_{x2},...,z_{n}L^1_{z1}} + 2^{k(n+3)/2} \sup_{U \in SU(n), t \rightarrow x(t)} \|A_k(t, x(t) + Uz)\|_{L^1_{t}L^1_{z1}}) \times \sum_k 2^{(k-1)/2} \sup_{U \in SU(n), x(t)} \|(|\partial^2 A_k| + |\partial_t A_k|)(t, x(t) + Uz)\|_{L^1_{t}L^2_{x2},...,z_{n}L^1_{z1}} + 2^{k(n-5)/2} \sup_{U \in SU(n), x(t)} \|(|\partial^2 A_k| + |\partial_t A_k|)(t, x(t) + Uz)\|_{L^\infty L^2_{x2},...,z_{n}L^1_{z1}}).$$

Next, we show how to use Lemma 7 to control terms in the form

$$\int (\nabla \sigma)^2 e^{i\alpha(t,x,\xi)}e^{-4\pi^2it|\xi|^2}e^{2\pi i(\xi,x)}\Omega(\xi)\hat{f}(\xi)d\xi.$$

in $L^1L^2$ norm. We expand the same way as before $e^{i\alpha} = \sum_{\alpha \geq 2} i^{\alpha-2}/(\alpha - 2)! \sigma^{\alpha-2}$ to get

$$\int (\nabla \sigma)^2 e^{i\alpha(t,x,\xi)}e^{-4\pi^2it|\xi|^2}e^{2\pi i(\xi,x)}\Omega(\xi)\hat{f}(\xi)d\xi = \sum_{\alpha \geq 2} i^{\alpha-2}/(\alpha - 2)! \sum_{k_1,...,k_\alpha} \prod_{\mu=1}^{\alpha-2} \int A_{k_\mu}(t, x + z\xi/|\xi|) \chi(2^{2k_\nu}z) dz \times (\int \partial A_{k_{\alpha-1}}(t, x + z\xi/|\xi|) \chi(2^{2k_{\alpha-1}}z) dz)(\int \partial A_{k_\alpha}(t, x + z\xi/|\xi|) \chi(2^{2k_\alpha}z) dz) \times e^{-4\pi^2it|\xi|^2}e^{2\pi i(\xi,x)}\Omega(\xi)\hat{f}(\xi)d\xi = \sum_{\alpha \geq 2} i^{\alpha-2}/(\alpha - 2)! \sum_{k_1,...,k_\alpha} \zeta_{k_1,...,k_\alpha} A_{k_1,...,k_\alpha, \partial A, \partial A}.$$
By the symmetry in the last entries $\partial A_{k_{\alpha-1}}$ and $\partial A_{k_{\alpha}}$, we will without loss of generality assume $k_{\alpha-1} \leq k_{\alpha}$. Applying Lemma 7 yields

$$
\left\| \int (\nabla \sigma)^2 e^{i\sigma(t,x,\xi)} e^{-4\pi^2 it|\xi|^2} e^{2\pi i(\xi, x)} \Omega(\xi) \tilde{f}(\xi) d\xi \right\|_{L^1 L^2} \leq
$$

$$
\leq C_n \|f\|_{L^2} \exp(C_n \sum_{k} 2^{k(n-1)/2} \sup_{U \in SU(n)} \sup_{x} \|A_k(t,x+Uz)\|_{L^\infty L^2_2,\ldots,n L^1_1} +
$$

$$
+2^{k(n+3)/2} \sup_{U \in SU(n), t \rightarrow x(t)} \|A_k(t,x(t)+Uz)\|_{L^1 L^2_2,\ldots,n L^1_1}) \times
$$

$$
\sum_{k_{\alpha-1} \leq k_{\alpha}} \sum_{k} 2^{k_{\alpha-1}(n-1)/2} \sup_{U \in SU(n)} \sup_{x} \|\partial A_{k_{\alpha-1}}(t,x+Uz)\|_{L^\infty L^2_2,\ldots,n L^1_1} +
$$

$$
+2^{k_{\alpha-1}(n+3)/2} \sup_{U \in SU(n), t \rightarrow x(t)} \|\partial A_{k_{\alpha-1}}(t,x(t)+Uz)\|_{L^1 L^2_2,\ldots,n L^1_1}) \times
$$

$$
\times (2^{k_{\alpha-1}(n-1)/2} \sup_{U \in SU(n)} \sup_{x} \|\partial A_{k_{\alpha}}(t,x+Uz)\|_{L^1 L^2_2,\ldots,n L^1_1} +
$$

$$
+2^{k_{\alpha-1}(n-5)/2} \sup_{U \in SU(n), t \rightarrow x(t)} \|\partial A_{k_{\alpha}}(t,x(t)+Uz)\|_{L^\infty L^2_2,\ldots,n L^1_1})).
$$

We simply now replace (at the expense of a constant) $\partial A_{k_{\alpha-1}}$ by $2^{k_{\alpha-1}} A_{k_{\alpha-1}}$, which is in turn smaller than $2^{k_{\alpha}} A_{k_{\alpha-1}}$ in all the norms above involving $A_{k_{\alpha-1}}$. We get an estimate

$$
\left\| \int (\nabla \sigma)^2 e^{i\sigma(t,x,\xi)} e^{-4\pi^2 it|\xi|^2} e^{2\pi i(\xi, x)} \Omega(\xi) \tilde{f}(\xi) d\xi \right\|_{L^1 L^2} \leq
$$

$$
\leq C_n \|f\|_{L^2} \exp(C_n \sum_{k} 2^{k(n-1)/2} \sup_{U, x} \|A_k(t,x+Uz)\|_{L^\infty L^2_2,\ldots,n L^1_1} +
$$

$$
+2^{k(n+3)/2} \sup_{U, x} \|A_k(t,x(t)+Uz)\|_{L^1 L^2_2,\ldots,n L^1_1}) \times
$$

$$
\sum_{k} 2^{k(n-1)/2} \sup_{U, x} \|A_k(t,x+Uz)\|_{L^\infty L^2_2,\ldots,n L^1_1} +
$$

$$
\sum_{k} 2^{k(n+3)/2} \sup_{U, x} \|A_k(t,x(t)+Uz)\|_{L^1 L^2_2,\ldots,n L^1_1}) \times
$$

$$
\times (\sum_{k} 2^{k(n+1)/2} \sup_{U, x} \|\partial A_k(t,x+Uz)\|_{L^1 L^2_2,\ldots,n L^1_1} +
$$

$$
+2^{k(n-3)/2} \sup_{U, x} \|\partial A_k(t,x(t)+Uz)\|_{L^\infty L^2_2,\ldots,n L^1_1})).
$$

Next, using the fact that $\|A\|_{L^2 L^\infty} \leq \varepsilon$, one sees that to control the term

$$
\int (i\langle \tilde{A}, \nabla \sigma \rangle) e^{i\sigma(t,x,\xi)} e^{-4\pi^2 it|\xi|^2} e^{2\pi i(\xi, x)} \Omega(\xi) \tilde{f}(\xi) d\xi.
$$

in $L^1 L^2$, we need to control $\| \int \nabla \sigma(t,x,\xi) e^{i\sigma(t,x,\xi)} e^{-4\pi^2 it|\xi|^2} e^{2\pi i(\xi, x)} \Omega(\xi) \tilde{f}(\xi) d\xi \|_{L^1 L^2}$. For the proof of that, we can proceed by interpolation between the $L^1 L^2$ estimates of Lemma 7 and the $L^\infty L^2$ estimates of Lemma 6. An even easier way is the following. By inspection of the proof of Lemma 7, one sees that $L^2_1 L^2_2$ estimate is much easier to prove.
(since $L_t^2$ norm commutes with $L_x^2$ norm) and one gets
\[
\| \int \nabla \sigma(t,x,\xi) e^{i\sigma(t,x,\xi)} e^{-4\pi^2|\xi|^2} e^{2\pi i (\xi, x)} \Omega(\xi) \tilde{f}(\xi) d\xi \|_{L_t^2 L_x^2} \leq \\
\leq C_n \| f \|_{L^2} \exp(C_n \sum_k 2^{k(n-1)/2} \sup_{U,x} \| A_k(t,x+Uz) \|_{L_t^\infty L_x^{L_x^2}\ldots L_x^{L_x^n} L_t^1}) \times \\
\times \sum_k 2^{k(n-1)/2} \sup_{U,x} \| \partial A_k(t,x+Uz) \|_{L_t^2 L_x^{L_x^2}\ldots L_x^{L_x^n} L_t^1}.
\]
Note that by the convexity of the norms
\[
2^{k(n-1)/2} \sup_{U,x} \| \partial A_k(t,x+Uz) \|_{L_t^2 L_x^{L_x^2}\ldots L_x^{L_x^n} L_t^1} \leq \\
\leq (2^{k(n-1)/2} \sup_{U,x} \| A_k(t,x+Uz) \|_{L_t^\infty L_x^{L_x^2}\ldots L_x^{L_x^n} L_t^1})^{1/2} \times \\
\times (2^{k(n-1)/2} \sup_{U,x} \| \partial^2 A_k(t,x+Uz) \|_{L_t^1 L_x^{L_x^2}\ldots L_x^{L_x^n} L_t^1})^{1/2},
\]
whence
\[
\sum_k 2^{k(n-1)/2} \sup_{U,x} \| \partial A_k(t,x+Uz) \|_{L_t^2 L_x^{L_x^2}\ldots L_x^{L_x^n} L_t^1} \leq \\
\leq (\sum_k 2^{k(n-1)/2} \sup_{U,x} \| A_k(t,x+Uz) \|_{L_t^\infty L_x^{L_x^2}\ldots L_x^{L_x^n} L_t^1})^{1/2} \times \\
\times (\sum_k 2^{k(n-1)/2} \sup_{U,x} \| \partial^2 A_k(t,x+Uz) \|_{L_t^1 L_x^{L_x^2}\ldots L_x^{L_x^n} L_t^1})^{1/2} \leq \| A \|_{Y_2 \cap Y_3}.
\]
Thus, it remains to prove Lemma 7.

### 7.3. Proof of Lemma 7

To outline the main ideas, we start the proof with the simpler case $\alpha = 1$.

#### 7.3.1. The case $\alpha = 1$

We follow the method of Lemma 6. According to (45), (46),
\[
\Xi_{F_k}^{k_1} f(t,x) = e^{-\Delta t} \sum_j P^j \left[ (\int F_k(t, \cdot + z\Theta_j^{-k}) \chi(2^{2k}z) dz) \tilde{P}^j(f) \right],
\]
where the summation in $j$ is over the family $\{\Theta_j^{-k}\}$. Taking $L_x^2$ norms and taking into account the (almost) orthogonality of the $j$ sum, we have
\[
\| \Xi_{F_k}^{k_1} f(t,\cdot) \|_{L_x^2} \leq C_n \left( \sum_j \left\| (\int F_k(t, \cdot + z\Theta_j^{-k}) \chi(2^{2k}z) dz) \tilde{P}^j(f) \right\|_{L_t^2}^2 \right)^{1/2} \leq \\
\leq C_n \left( \sum_j \sup_x \| h_j(t,x) \|_{L_x^2} \left\| \tilde{P}^j f \right\|_{L_t^2}^2 \right)^{1/2},
\]
where \( h_j(t, x) = \int F_k(t, x + z \theta_j^{-k}) \chi(2^{2k}z)dz \). Take \( L^1_t \) norm. We will show

\[
\left\| \left( \sum_j \sup_x |h_j(t, x)|^2 \left\| \tilde{P}^j f \right\|_{L^2} \right)^{1/2} \right\|_{L^1_t} \leq C_n 2^{k(n-1)/2} \sup_{U \in SU(n)} \left\| \sup_x \|F_k(t, x + U z)\|_{L^2_{x_2}, \ldots, x_n L^1_{x_1}} \right\| \|f\|_{L^2}.
\]

This follows by complex interpolation between

\[
\left\| \sum_j \sup_x |h_j(t, x)| \left\| \tilde{P}^j f \right\|_{L^2} \right\|_{L^1_t} \leq \sup_{U \in SU(n)} \left\| \sup_x \|F_k(t, x + U z)\|_{L^2_{x_2}, \ldots, x_n L^1_{x_1}} \right\| \sum_j \left\| \tilde{P}^j f \right\|_{L^2}
\]

and

\[
\left\| \sup_j \sup_x |h_j(t, x)| \left\| \tilde{P}^j f \right\|_{L^2} \right\|_{L^1_t} \leq C_n 2^{k(n-1)} \|F_k\|_{L^1_t L^1_z} \sup_j \left\| \tilde{P}^j f \right\|_{L^2},
\]

because \( \|F_k\|_{L^1_t L^1_z} = \sup_{U \in SU(n)} \| \sup_x \|F_k(t, x + U z)\|_{L^2_{x}} \|_{L^1_t} \).

The \( l^1_j \) estimate, (54) is straightforward, since \( \tilde{P}^j f \) is indendent of \( t \). We have

\[
\left\| \sum_j \sup_x |h_j(t, x)| \left\| \tilde{P}^j f \right\|_{L^2} \right\|_{L^1_t} = \sum_j \left\| \sup_x |h_j(t, x)| \left\| \tilde{P}^j f \right\|_{L^2} \right\|_{L^1_t} \leq \sup_j \left\| \sup_x |h_j(t, x)| \left\| \tilde{P}^j f \right\|_{L^2} \right\|_{L^1_t} \sum_j \left\| \tilde{P}^j f \right\|_{L^2}
\]

It remains to observe that

\[
\sup_j \left\| \sup_x |h_j(t, x)| \right\|_{L^1_t} \leq \sup_{\theta} \left\| \sup_x \left\| F_k(t, x + z \theta) \right\|_{L^1_t} \right\| = \sup_{U \in SU(n)} \left\| \sup_x \|F_k(t, x + U z)\|_{L^2_{x_2}, \ldots, x_n L^1_{x_1}} \right\|_{L^1_t}.
\]

For the \( l^\infty_j \) estimate, (55), write

\[
\left\| \sup_j \sup_x |h_j(t, x)| \left\| \tilde{P}^j f \right\|_{L^2} \right\|_{L^1_t} = \left\| \sup_j \sup_x |h_j(t, x)| \left\| \tilde{P}^j f \right\|_{L^2} \right\|_{L^1_t}.
\]

Note that

\[
\sup \sup_x |h_j(t, x)| \leq \sup_{\theta} \sup_{t \leq -2k} \sum_{l \leq 2k} \int |F_k(t, x + z \theta)\| \varphi(2^{-l}z)dz \leq \sum_{l \leq -2k} \sup_{\theta} \sup_x \int |F_k(t, x + z \theta)\| \varphi(2^{-l}z)dz.
\]
Since for every fixed $\theta$, $2^{(l+k)(n-1)} \int_{\theta_0 \in S^{n-1} : |\theta_0 - \theta| < 2^{-l-k}} d\theta_0 \gtrsim 1$, write
\[
\int |F_k(t, x + z\theta)| \varphi(2^{-l}z) dz \leq \leq C_n 2^{(l+k)(n-1)} \int_{\theta_0 \in S^{n-1} : |\theta_0 - \theta| < 2^{-l-k}} |F_k(t, x + z\theta)| \varphi(2^{-l}z) dz d\theta_0
\]
Expand out as before $F_k(t, x + z\theta)$ around $x + z\theta_0$. We get
\[
\sup_x \sup_\theta \int |F_k(t, x + z\theta)| \varphi(2^{-l}z) dz \leq \leq C_n 2^{(l+k)(n-1)} \sum_{\gamma \geq 0} (\gamma !)^{-1} \int_{\theta_0 \in S^{n-1} : |\theta_0 - \theta| < 2^{-l-k}} |F_k^\gamma(t, x + z\theta_0)| \varphi^\gamma(2^{-l}z) dz d\theta_0 \leq \leq C_n 2^{(l+k)(n-1)} \sum_{\gamma \geq 0} (\gamma !)^{-1} \int_{\theta_0 \in S^{n-1}} |F_k^\gamma(t, x + z\theta_0)| |z|^{-1} \varphi^\gamma(2^{-l}z) dz d\theta_0.
\]
Summing the last inequalities in $l$ implies
\[
|\sup_x \sup_j h_j (t, x)| \leq C_n 2^{(n-1)} \sum_{\gamma \geq 0} (\gamma !)^{-1} \sum_l \int_{\theta_0 \in S^{n-1}} |F_k^\gamma(t, x + z\theta_0)| |z|^{-1} \varphi^\gamma(2^{-l}z) dz d\theta_0 \leq C_n 2^{(n-1)} \|F_k(t, \cdot)\|_{L^1_x}
\]
Taking $L^1_x$ norm yields
\[
\left\| \sup_x \sup_j h_j(t, x) \right\|_{L^1_x} \leq C_n 2^{(n-1)} \|F_k\|_{L^1_x},
\]
as required.

7.3.2. The case $\alpha > 1$. The strategy here is to start “peeling off” the functions $\int F^\mu_{k^\alpha}(t, x + z\xi/|\xi|) \chi(2^{2k^\alpha}z) dz$ in a way similar to Lemma 6. Recall that the method presented in Lemma 6 starts the “peeling” argument with the terms with the lowest frequency $k_{\min} = \min(k_1, \ldots, k_\alpha)$.

We have here the extra complication of having to take $L^1_t$. Moreover, we must measure the last term $\int F^\mu_{k^\alpha}(t, x + z\xi/|\xi|) \chi(2^{2k^\alpha}z) dz$ in the $L^1_t$ norm, while all the other terms should be measured in $L^\infty$.

As we have alluded to above, the order of the frequencies is not insignificant. The case $\alpha = 1$, considered in the previous section roughly corresponds to the case when the last frequency $k_\alpha$ is maximal, i.e. $k_\alpha = \max(k_1, \ldots, k_\alpha)$. We consider this case, and then we indicate the necessary changes when $k_\alpha < \max(k_1, \ldots, k_\alpha)$.

**SubCase 1:** $k_\alpha = \max(k_1, \ldots, k_\alpha)$.

By the symmetry of the terms $1, \ldots, (\alpha - 1)$, assume without loss of generality $k_1 \leq \ldots \leq k_{\alpha-1}$. In fact, following Lemma 6, let $k_1 = \ldots = k_{s_1} < k_{s_1+1} = \ldots = k_{s_2} < \ldots k_{s_{m-1}+1} = \ldots = k_{s_m} = k_\alpha$.

According to (47), we can write $\Xi^{k_1, \ldots, k_\alpha}_{r_1, \ldots, r_\alpha}$ as a free solution (with time depending data). Recall also (48) and (49), which give an estimate for its $L^2$ norm (for a fixed time $t$). Take
into account (50), to get
\[ \| \Xi_{F_1,\ldots,F_n} \|_{L^2}^2 \leq \]
\[ \leq C_n \left( \sum_{j_1 \leq \cdots \leq j_m} \sum_{\theta_{j_1}^{-k_{s_1}} \cdots \theta_{j_1}^{-k_{s_m}}} \left\| g_{j_1}^{s_1} \ldots g_{j_1}^{s_2} Q_{2^{k_{s_1}}+2s_12^{k_{s_1}}} (f) \prod_{\mu=s_1+1}^\alpha g_{j_1}^{r(\mu)} (t, \cdot) \right\|_{L^2_x}^2 \right)^{1/2}, \]
where for \( \mu \in [s_{r-1}+1, s_r] \), we have introduced
\[ g_{j_1}^{r(\mu)} (t, x) = \int F_{k_\mu}^\mu (t, x + z \theta_{j_1}^{-k_{\mu}}) \chi(2^{2k_{\mu}}z) dz. \]
Furthermore, by support considerations (as discussed in the proof of Lemma 6),
\[ Q_{2^{k_{s_1}}+2s_12^{k_{s_1}}} = Q_{2^{k_{s_1}}+2s_12^{k_{s_1}}} \ldots Q_{2^{k_{s_r}}+(2s_1+2)2^{k_{s_1}}}. \]
We conclude
\[ \| \Xi_{F_1,\ldots,F_n} \|_{L^2}^2 \leq C_n \left( \sum_{j_1 \leq \cdots \leq j_m} \left\| g_{j_1}^{s_1} \ldots g_{j_1}^{s_2} Q_{2^{k_{s_1}}+2s_12^{k_{s_1}}} (f) \prod_{\mu=s_1+1}^\alpha g_{j_1}^{r(\mu)} (t, \cdot) \right\|_{L^2_x}^2 \right)^{1/2}. \]
(57)

Take \( L^1_t \) norm. Our claim is that
\[ \| \Xi_{F_1,\ldots,F_n} \|_{L^1_t L^2_x} \leq C_n (1 + 2s_1)^n \prod_{\mu=1}^{s_1} 2^{k_{\mu}(n-1)/2} \sup_t \sup_{U \in SU(n)} \sup_x \| F_{k_\mu}^\mu (t, x + U z) \|_{L^2_{t,x}} \]
\[ \times (\sum_{j_1 \leq \cdots \leq j_m} \left\| Q_{2^{k_{s_1}}+2s_12^{k_{s_1}}} \ldots Q_{2^{k_{s_m}}+(2s_1+2)2^{k_{s_1}}} (f) \prod_{\mu=s_1+1}^\alpha g_{j_1}^{r(\mu)} (t, \cdot) \right\|_{L^2_x}^2 )^{1/2} \| L^1_t. \]
On account of (57), this follows by a complex multilinear interpolation between the \( l^1_j \) estimate
\[ \| \sum_{j_1 \leq \cdots \leq j_m} \left\| g_{j_1}^{s_1} \ldots g_{j_1}^{s_2} Q_{2^{k_{s_1}}+2s_12^{k_{s_1}}} \ldots Q_{2^{k_{s_m}}+(2s_1+2)2^{k_{s_1}}} (f) \prod_{\mu=s_1+1}^\alpha g_{j_1}^{r(\mu)} (t, \cdot) \right\|_{L^2_x} \| L^1_t \leq \]
\[ \leq \prod_{\mu=1}^{s_1} (\sup_{t, x} \int | F_{k_\mu}^\mu (t, x + z \theta) |dz) \times \]
\[ \times \| \sum_{j_1 \leq \cdots \leq j_m} \left\| Q_{2^{k_{s_1}}+2s_12^{k_{s_1}}} \ldots Q_{2^{k_{s_m}}+(2s_1+2)2^{k_{s_1}}} (f) \prod_{\mu=s_1+1}^\alpha g_{j_1}^{r(\mu)} (t, \cdot) \right\|_{L^2_x} \| L^1_t. \]
and the \( l^\infty_j \) estimate
\[ \| \sup_{j_1 \leq \cdots \leq j_m} \left\| g_{j_1}^{s_1} \ldots g_{j_1}^{s_2} Q_{2^{k_{s_1}}+2s_12^{k_{s_1}}} \ldots Q_{2^{k_{s_m}}+(2s_1+2)2^{k_{s_1}}} (f) \prod_{\mu=s_1+1}^\alpha g_{j_1}^{r(\mu)} (t, \cdot) \right\|_{L^2_x} \| L^1_t \leq \]
\[ \leq C_n \prod_{\mu=1}^{s_1} 2^{k_{\mu}(n-1)} \| F_{k_\mu}^\mu \|_{L^\infty_t L^1_x} \times \]
\[ \times \| \sup_{j_1 \leq \cdots \leq j_m} \left\| Q_{2^{k_{s_1}}+2s_12^{k_{s_1}}} \ldots Q_{2^{k_{s_m}}+(2s_1+2)2^{k_{s_1}}} (f) \prod_{\mu=s_1+1}^\alpha g_{j_1}^{r(\mu)} (t, \cdot) \right\|_{L^2_x} \| L^1_t. \]
Note that after the complex interpolation, the sum in $j_1$ disappears (at the expense of a constant $C_n(1 + 2s_1)^n$), since we have estimated by (51).

The proof of the $l_j^1$ estimate is immediate, by just pulling out

\[ \prod_{j=1}^{s_1} \sup_{t, x} |g_j^1(t, x) \cdots g_j^{s_1}(t, x)| \leq \prod_{j=1}^{s_1} \sup_{t, x} \int |F_{k_{j}}^\mu(t, x + z\theta)| dz. \]

For the $l_j^\infty$ estimate, estimate by

\[
\left\| \sup_{j_1, j_2, \ldots, j_m} \left\| g_{j_1} \cdots g_{j_{s_1}} Q_{2k_{s_1}+2s_12k_{s_1}} \cdots Q_{2k_{s_m}+(2s_1+2)2k_{s_1}} (f(\cdot) \prod_{\mu=s_1+1}^\alpha g_{j_{\mu}}^{(\mu)}(t, \cdot)) \right\|_{L_2^k} \right\|_{L_1^T} \leq \\
\leq \prod_{\mu=1}^{s_1} \sup_{t, j, x} |g_{j_{\mu}}^1(t, x)| \times \\
\times \left\| \sup_{j_1, j_2, \ldots, j_m} \left\| Q_{2k_{s_1}+2s_12k_{s_1}} \cdots Q_{2k_{s_m}+(2s_1+2)2k_{s_1}} (f(\cdot) \prod_{\mu=s_1+1}^\alpha g_{j_{\mu}}^{(\mu)}(t, \cdot)) \right\|_{L_2^k} \right\|_{L_1^T}.
\]

By (56) however,

\[ \sup_{t} \sup_{j, x} |g_{j_{\mu}}^1(t, x)| \leq C_n 2^{k_{s_1}(n-1)} \sup_t \left\| F_{k_{\mu}}^\mu(t, \cdot) \right\|_{L_2^k}, \]

as required.

This shows the main step in the argument. We continue in this fashion (and as in Lemma 6, we keep incurring constants coming from the increased overlap of the supports of the operators $Q^j$) until we reach the maximal frequency $k_n$. In this final step, we finally have to take the $L_1^T$ norm on $\int F_{k_{\mu}}(t, x + z\theta)\chi(2^{2k_{s_1}}z)dz$. We have

\[
\left\| \prod_{\mu=1}^{k_{s_m-1}} F_{1, \ldots, F_{\alpha}}^{k_{s_m}} \right\|_{L_1^TL_2^k} \leq \\
\leq C_n^\alpha \prod_{\mu=1}^{k_{s_m-1}} 2^{k_{s_1}(n-1)/2} \sup_t \sup_{U \in SU(n)} \sup_x \left\| F_{k_{\mu}}^\mu(t, x + Uz) \right\|_{L_2^k} \times \\
\times \left\| \left( \sum_{j_m} \left\| \prod_{\mu=s_1+1}^\alpha g_{j_{\mu}}^{(\mu)}(t, \cdot)Q_{2k_{s_m}+(2s_1+2)2k_{s_1}} \cdots (2(s_1-s_{s_1})+2)2k_{s_m} \left[f(\cdot)\right]_{L_2^k}^{2/2} \right\|_{L_1^T} \right\|_{L_1^T}.
\]

Observe that since $p(\alpha, m) = 2^{k_{s_m}} + (2s_1 + 2)^{2k_{s_1}} + \cdots (2(s_1-s_{s_1})+2)2^{k_{s_m}} \leq 4\alpha 2^{k_{s_m}}$, we have

\[ \left( \sum_{j_m} \left\| Q_{p(\alpha, m)}^{j_m} g \right\|_{L_2^k}^{2} \right)^{1/2} \leq C_n(p(\alpha, m)/2^{k_{s_m}})^n \left\| g \right\|_{L_2^k} \leq C_n\alpha n \left\| g \right\|_{L_2^k}. \]

It remains to show

\[
\left( \sum_{j_m} \left\| \prod_{\mu=s_1+1}^\alpha g_{j_{\mu}}^{(\mu)}(t, \cdot)Q_{p(\alpha, m)}^{j_m} \left[f(\cdot)\right]_{L_2^k}^{2} \right\|_{L_1^T} \right)^{1/2} \leq C_n \prod_{\mu=s_1+1}^{\alpha-1} \sup_{t, \theta, x} \int |F_{k_{\mu}}^\mu(t, x + z\theta)| dz \times \\
\times 2^{k_{s_1}(n-1)/2} \sup_{U \in SU(n)} \sup_x \left\| F_{k_{\mu}}(t, x + Uz) \right\|_{L_2^k} \left\| \left[f(\cdot)\right]_{L_2^k} \right\|_{L_1^T}. 
\]
This follows as in (53), by interpolating between
\[
\| \sum_{j_m} \| \prod_{\mu=s_{m-1}+1}^{\alpha} g_{\mu}^j(t, \cdot) Q_{p(\alpha,m)}^j[f(\cdot)] \|_{L^2} \|_{L^1_t} \leq \\
\leq C_n \left( \prod_{\mu=s_{m-1}+1}^{\alpha-1} \sup_{t, \theta, x} \int |F_{k_\mu}^\mu(t, x + z\theta)| dz \sum_{j_m} \| Q_{p(\alpha,m)}^j[f(\cdot)] \|_{L^2} \times \\
\times \sup_{\theta} \| \sup_{x} \| \int |F_{k_\mu}^\alpha(t, x + z\theta)| dz \|_{L^1_t}\right)
\]

and
\[
\| \sup_{j_m} \| \prod_{\mu=s_{m-1}+1}^{\alpha} g_{\mu}^j(t, \cdot) Q_{p(\alpha,m)}^j[f(\cdot)] \|_{L^2} \|_{L^1_t} \leq \\
\leq C_n \left( \prod_{\mu=s_{m-1}+1}^{\alpha-1} 2^{k_\mu(n-1)} \| F_{k_\mu}^\mu \|_{L^\infty_t L^2_x} \right) \sup_{j_m} \| Q_{p(\alpha,m)}^j[f(\cdot)] \|_{L^2} \times \\
\times 2^{k_\alpha(n-1)} \| F_{k_\alpha}^\alpha \|_{L^1_t L^1_x}).
\]

We omit the details, as they are exactly as in the proof of (53). This completes the proof of Subcase I.

**Subcase II:** \( k_\alpha < \max(k_1, \ldots, k_\alpha) \)

Let for some \( 1 \leq r_0 < \alpha \), we have \( k_{r_0} = \max(k_1, \ldots, k_\alpha) \). The estimates proved in Subcase I yield
\[
\left\| \mathcal{Z}^{k_1, \ldots, k_\alpha}_{F_1, \ldots, F_\alpha} \right\|_{L^1_t L^2_x} \leq C_n \left( \prod_{\mu=1, \mu \neq r_0}^{\alpha-1} 2^{k_\mu(n-1)/2} \sup_{t \in SU(n)} \sup_{x} \| F_{k_\mu}^\mu(t, x + U z) \|_{L^2_{t, \ldots, x} L^1_{x_1}} \times \\
\times 2^{k_{r_0}(n-1)/2} \sup_{t \in SU(n)} \sup_{x} \| F_{k_{r_0}}^\alpha(t, x + U z) \|_{L^2_{t, \ldots, x} L^1_{x_1}} \times \\
\times 2^{k_{r_0}(n-1)/2} \sup_{t \in SU(n)} \sup_{x} \| F_{k_{r_0}}^{r_0}(t, x + U z) \|_{L^2_{t, \ldots, x} L^1_{x_1}} \| f \|_{L^2_t}
\]

Since \( k_\alpha < \max(k_1, \ldots, k_\alpha) = k_{r_0} \), we have \( 2^{k_\alpha(n-1)/2} 2^{k_{r_0}(n-1)/2} \leq 2^{k_{r_0}(n-5)/2} 2^{k_{r_0}(n+3)/2} \). We get an estimate
\[
\left\| \mathcal{Z}^{k_1, \ldots, k_\alpha}_{F_1, \ldots, F_\alpha} \right\|_{L^1_t L^2_x} \leq C_n \left( \prod_{\mu=1, \mu \neq r_0}^{\alpha-1} 2^{k_\mu(n-1)/2} \sup_{t \in SU(n)} \sup_{x} \| F_{k_\mu}^\mu(t, x + U z) \|_{L^2_{t, \ldots, x} L^1_{x_1}} \times \\
\times 2^{k_{r_0}(n-5)/2} \sup_{t \in SU(n)} \sup_{x} \| F_{k_{r_0}}^\alpha(t, x + U z) \|_{L^2_{t, \ldots, x} L^1_{x_1}} \times \\
\times 2^{k_{r_0}(n+3)/2} \sup_{t \in SU(n)} \sup_{x} \| F_{k_{r_0}}^{r_0}(t, x + U z) \|_{L^2_{t, \ldots, x} L^1_{x_1}} \| f \|_{L^2_t}
\]

This finishes the Proof of Lemma 7.
8. Strichartz estimates in $L^2_t L^2_{x_2,...,x_n} L^2_{x_1}$

In this section, we establish Proposition 1. We concentrate on the case $n \geq 4$, since this is what we use anyway. On the other hand, minor changes are needed in the proofs for $n = 2$ and $n = 3$.

Let us show first Proposition 1 for the case $A = 0$, that is for free solutions.

**Lemma 8.** Let $n \geq 4$. Then

$$\sup_{U \in SU(R^n), x(t)} \left\| e^{it\Delta} f(x(t) + U x) \right\|_{L^2_t L^{2(n-1)/(n-3)}_{x_2,...,x_n} L^2_{x_1}} \leq C_n \| f \|_{L^2} \tag{58}$$

$$\sup_{U \in SU(R^n), x(t)} \left\| \int_0^t e^{i(t-s)\Delta} F(s, (x(s) + U x)ds) \right\|_{L^2_t L^{2(n-1)/(n-3)}_{x_2,...,x_n} L^2_{x_1}} \leq C_n \| F \|_{L^1_t L^2} \tag{59}$$

**Proof.** For a fixed $U \in SU(R^n)$ and fixed measurable function $x(t)$, denote $W(t)f(y) := (e^{it\Delta} f)(x(t) + U y)$. According to Proposition 3, we need to verify that $W(t) : L^2 \to L^2$ and $W(t)W(s)^* : L^1_{y_2,...,y_n} L^2_{y_1} \to L^\infty_{y_2,...,y_n} L^2_{y_1}$ with norm no larger than $C_n |t - s|^{(n-1)/2}$.

The $L^2$ boundedness is obvious since,

$$\| W(t)f \|_{L^2} = \| (e^{it\Delta} f)(x(t) + U y) \|_{L^2_y} = \| (e^{it\Delta} f)(x(t) + y) \|_{L^2_y} = \| e^{it\Delta} f \|_{L^2_y} = \| f \|_{L^2}.$$

For the dispersive estimates, note that

$$W(t)W(s)^* f(y) = (e^{i(t-s)\Delta} f)(y + U^* x(s) - U^* x(t)).$$

It suffices to verify the decay estimate for a family of extreme points, whose convex span is $u^*$ dense in the unit ball of $L^1_{y_2,...,y_n} L^2_{y_1}$. Since the $\delta$ functions provide such a set in $L^1_X$ for any measure space $X$, it will suffice to take $f(y) = \delta(\bar{y} - b) g(y_1)$, where $\bar{y} = (y_2, \ldots, y_n)$, $b \in R^{n-1}$ and $g \in L^2(R^1)$. Fix $s$ and $t$ and denote $z = U^* x(s) - U^* x(t)$, which is a fixed vector in $R^n$.

Compute $W(t)W(s)^* f$

$$W(t)W(s)^* f(y) = (e^{i(t-s)\Delta} g)(y_1 + z_1)\left[ e^{i(t-s)\Delta_{n-1}} \delta(\cdot - \bar{y} + \bar{z}) \right].$$

Immediately,

$$\| W(t)W(s)^* f(y) \|_{L^2_{y_1}} = \| e^{i(t-s)\Delta_{n-1}} \delta(\cdot - \bar{y} + \bar{z}) \| g \|_{L^2_{y_1}}$$

and thus

$$\| W(t)W(s)^* f \|_{L^\infty_{y_2,...,y_n} L^2_{y_1}} \leq C_n |t - s|^{-(n-1)/2} \| g \|_{L^2_{y_1}} = C_n |t - s|^{-(n-1)/2} \| f \|_{L^1_{y_2,...,y_n} L^2_{y_1}}.$$

Note that the constant $C_n$ is independent of $g$, and $z \in R^n$ and depends only on the dimension $n$. \hfill \Box

For the case $A \neq 0$, we proceed as in the proof of Theorem 1. We first establish a "naive" Strichartz estimate similar to Proposition 6. Then, we show that to extend the naive Strichartz estimate indefinitely in time (under the assumption that $A$ is small), we need an estimate of the parametrix, similar to Lemma 3 in the appropriate norms.

We start with the naive Strichartz estimate. This is done exactly the same way as Proposition 6, given that we already have Lemma 8.
Proposition 7. For a fixed integer $k_0$, there exists a time $T_0 = T_0(k_0) \leq \infty$, so that whenever $0 < T < T_0$, $\psi \in S$

\begin{equation}
\sup_{U \in SU(R^n), x(t)} \| (P_{<k_0} \psi)(t, x(t) + Uz) \|_{L^2(0, T) L^{2(n-1)/(n-3)}_{t z_2, \ldots, z_n} L^2_{z_1}} \leq C(T, k_0) (\| \psi(0, \cdot) \|_{L^2} + \| \mathcal{L} \psi \|_{L^1 L^2}).
\end{equation}

Moreover, $C(T, k_0)$ depends on $T$ in a continuous way.

Fix $k_0$ and a small $\varepsilon$. Set as before $0 < T^* \leq \infty$ to be the maximum time, so that for every $0 < T < T^*$, one has

\begin{equation}
\sup_{U \in SU(R^n), x(t)} \| (P_{<k_0} \psi)(t, x(t) + Uz) \|_{L^2(0, T) L^{2(n-1)/(n-3)}_{t z_2, \ldots, z_n} L^2_{z_1}} \leq \varepsilon^{-1} (\| \psi(0, \cdot) \|_{L^2} + \| \mathcal{L} \psi \|_{L^1 L^2}),
\end{equation}

for all Schwartz functions $\psi$.

The goal would be again to show that for small enough $\varepsilon$, we have that $T^* = \infty$. This is reduced in a standard way (recall that $P_{<k_0}$ has an integrable kernel and is therefore bounded on $L^2(0, T) L^{2(n-1)/(n-3)}_{t z_2, \ldots, z_n} L^2_{z_1}$) to the following estimate for the parametrix, constructed in Section 4. This needs to be compared to Lemma 3.

Lemma 9. Given $\| \hat{A} \|_{Y_1 \cap Y_2 \cap Y_3} \leq \varepsilon$, integer $k$ and $T > 0$, and for every function $f_k \in L^2(R^n)$ with $\text{supp} \hat{f}_k \subset \{ |\xi| \sim 2^k \}$, one can find a function $v_k : [0, T) \times R^n \rightarrow C$, so that for every $\hat{v}_k \subset \{ |\xi| \sim 2^k \}$ and

\begin{align*}
\| v_k(0, x) - f_k \|_{L^2} & \leq C\varepsilon \| f_k \|_{L^2} \\
\sup_{U \in SU(n), x(t)} \| v_k(t, x(t) + Uz) \|_{L^2 L^{2(n-1)/(n-3)}_{t z_2, \ldots, z_n} L^2_{z_1}} & \leq C\varepsilon \| f_k \|_{L^2} \\
\| \mathcal{L} v_k \|_{L^1 L^2} & \leq C\varepsilon \| f_k \|_{L^2},
\end{align*}

for some $C$ independent of $f, k, T$.

It remains to prove Lemma 9.

Proof. (9) We reduce as in Section 4 to the case, when $k = 0$ and $\text{supp} \hat{A} \subset \{ |\xi| \ll 1 \}$ and without the condition $\text{supp} \hat{v} \subset \{ |\xi| \sim 1 \}$. We again use the function $v$ constructed in Section 4.

Since we have already verified $v(0, x) - f \in L^2 \leq C\varepsilon \| f \|_{L^2}$ and $\| \mathcal{L} v \|_{L^1 L^2} \leq C\varepsilon \| f \|_{L^2}$ under the conditions imposed in Lemma 9, it remains to check

\begin{equation}
\sup_{U \in SU(n), x(t)} \| v(t, x(t) + Uz) \|_{L^2(0, T) L^{2(n-1)/(n-3)}_{t z_2, \ldots, z_n} L^2_{z_1}} \leq C\| f \|_{L^2}.
\end{equation}

for

\begin{equation}
v(t, x) = \Lambda f(t, x) = \int e^{i\sigma(t, x, \xi)} e^{-4\pi^2 |\xi|^2} e^{2\pi i \langle \xi, x \rangle} \Omega(\xi) \hat{f}(\xi) d\xi.
\end{equation}

Expanding $e^{i\sigma} = \sum_{\alpha} (i^n \sigma^{(\alpha)})/(\alpha !)$ reduces matters to showing

\begin{equation}
\sup_{U \in SU(n), x(t)} \| \Lambda^\alpha f(t, x(t) + Ux) \|_{L^2(0, T) L^{2(n-1)/(n-3)}_{t z_2, \ldots, z_n} L^2_{z_1}} \leq C_n^\alpha \varepsilon^\alpha \| f \|_{L^2}.
\end{equation}

Fix $U \in SU(n)$ and a measurable function $x(t)$. Set

\begin{equation}
W^\alpha(t) f(x) = [\Lambda^\alpha f](t, x(t) + Ux) = \int \sigma^\alpha(t, x(t) + Ux, \xi) e^{-4\pi^2 |\xi|^2} e^{2\pi i \langle \xi, x(t) + Ux \rangle} \Omega(\xi) \hat{f}(\xi) d\xi.
\end{equation}
By Proposition 3, (61) follows from the energy estimate

\[ \| W^\alpha(t)f \|_{L^2} \leq C_n^\alpha \left( \sum_k \sup_{U,z} \| A_k(t, x + U z) \|_{L^\infty t L^1 x} \right)^\alpha \| f \|_{L^2} \]

and the decay estimate

\[ \| W^\alpha(t)W^\alpha(s)^* f \|_{L^\infty_{t^2 \cdots x_n} L^2_{x_1}} \leq C_n^\alpha \left( \sum_k 2^{k(n-1)/2} \| A_k \|_{L^\infty t L^1 x} \right)^2 \| f \|_{L^\infty_{t^2 \cdots x_n} L^2_{x_1}}. \]

By interpolation between the last two estimates, we obtain the “modified decay estimate”

\[ \| W^\alpha(t)W^\alpha(s)^* f \|_{L^\infty_{t^2 \cdots x_n} L^2_{x_1}} \leq C_n^\alpha \left( \sum_k 2^{k(n-1)/p_0} \sup_{U,z} \| A_k(t, x + U z) \|_{L^\infty t L^1 x} \right)^2 \| f \|_{L^\infty_{t^2 \cdots x_n} L^2_{x_1}} = C_n^\alpha \| f \|_{L^\infty_{t^2 \cdots x_n} L^2_{x_1}}, \]

which implies (61), as long as \( p_0 < (n-1)/2 \), according to the Remark before Proposition 3.

8.1. Energy estimates for \( W^\alpha \). Observe that the case \( \alpha = 0 \) is simply the energy estimate in Lemma 8, while for the general case observe\(^{12}\)

\[ W^\alpha(t)f(x) = [L^\alpha f](t, x(t) + Ux) = \int e^\alpha(t, x(t) + Ux, U^*\xi)e^{-4\pi it|\xi|^2}e^{2\pi i(t, x(t) + Ux)\xi} \Omega(\xi) d\xi f(Uy)dy. \]

By a simple translational invariance, \( \| f(U\cdot) \|_{L^2} = \| f \|_{L^2} \) and \( \| A(t, x(t) + U\cdot) \|_{Y_j} = \| A \|_{Y_j} \), we have that by (41), the energy estimate \( \| W^\alpha(t)f \|_{L^2} \leq C_n^\alpha \| f \|_{L^2} \) is satisfied for every \( \alpha > 0 \).

8.2. Decay estimates for \( W^\alpha \). The case \( \alpha = 0 \) is the decay estimate in Lemma 8.

For \( \alpha \geq 1 \), we are following the approach of the dispersive estimates in Section 6. Write

\[ W^\alpha(t)W^\alpha(s)^* f(x) = \prod_{\mu=1}^\alpha \sum_{k_\mu \leq -2 l_\mu \leq -2k_\mu} \sum_{\gamma_\mu} (\gamma_\mu!)^{-1} \sum_{j_\mu} A_{k_\mu j_\mu}(t, x(t) + Ux + z \theta_{j_\mu}^{l_\mu + k_\mu} \varphi^{\gamma_\mu}(2^{-j_\mu} z) dz) \times \]

\[ \times H_{j_1, l_1; \cdots; j_\alpha, l_\alpha + k_\alpha} f(x), \]

where

\[ H_{j_1, l_1; \cdots; j_\alpha, l_\alpha + k_\alpha} f(x) = \int \sigma^\alpha(s, x(s) + Uy, \xi)e^{-4\pi^2 i(t-s) |\xi|^2}e^{2\pi i(t, x(t) - x(s) + Ux - Uy) x} \times \]

\[ \times \prod_{\mu=1}^\alpha \varphi^{\gamma_\mu}_{j_\mu, l_\mu + k_\mu} (2^{-l_\mu - k_\mu} (\xi/|\xi| - \theta_{j_\mu}^{l_\mu + k_\mu})) \Omega^2(\xi) f(y) dy d\xi. \]

\(^{12}\)Recall that \( \Omega \) is a radial function.
Hence by Lemma 5, we conclude
\[
\|W^\alpha(t)W^\alpha(s)\|_{L^\infty_{x_2,...,x_n}L^2_{x_1}} \leq \sup_x \prod_{\mu=1}^\alpha \left( \sum_{k_\mu \leq -2l_\mu \leq -2k_\mu} \sum (\gamma_\mu!)^{-1} \sum_{j_\mu} \int A_{k_\mu}^\gamma(s, t, x) + U x + z\theta_{j_\mu}^{\mu+k_\mu} \varphi^\gamma(2^{-l_\mu} z) dz \times \right. \\
\times \left. \sup_{j_1,l_1,...,j_n,l_n,k_\alpha} \left\| H_{j_1,l_1+1}^{\alpha,...,n} f \right\|_{L^\infty_{x_2,...,x_n}L^2_{x_1}} \right) \leq C_n(\|A\|_{Y_1})^\alpha \sup_{j_1,l_1,...,j_n,l_n,k_\alpha} \left\| H_{j_1,l_1+1}^{\alpha,...,n} f \right\|_{L^\infty_{x_2,...,x_n}L^2_{x_1}}.
\]
Dualizing the needed estimate for \( H_{j_1,l_1+1}^{\alpha,...,n} f \mid_{L^\infty_{x_2,...,x_n}L^2_{x_1}} \), reduces matters to showing that
\[
(H_{j_1,l_1+1}^{\alpha,...,n})^* f(x) = \int \sigma^\alpha(s, x, s) + U x, \xi \right) e^{4\pi^2i(t-s)} e^{2\pi i(x - x(t) + U x - U y)} \times \\
\times \prod_{\mu=1}^\alpha \psi_{j_\mu,l_\mu+k_\mu}^\gamma(2^{l_\mu+k_\mu} (\xi / |\xi| - \theta_{j_\mu}^{\mu+k_\mu})) \Omega^2(\xi) f(y) dy d\xi
\]
is a mapping \( L^1_{x_2,...,x_n}L^2_{x_1} \rightarrow L^\infty_{x_2,...,x_n}L^2_{x_1} \) with norm no bigger than \( C_n\|A\|_{Y_1} |t-s|^{(n-1)/2} \).

Again, as in Section 6, we expand
\[
\sigma^\alpha(s, x, s) + U x, \xi = \prod_{\nu=1}^\alpha (\sum_{k_\nu \leq -2l_\nu \leq -2k_\nu} \sum (\gamma_\nu!)^{-1} \int A_{k_\nu}^\gamma s, t, x) + U x + z\theta_{j_\nu}^{\nu+k_\nu} \varphi^\gamma(2^{-l_\nu} z) dz
\]
and estimate by Lemma 5. We get
\[
\left\| (H_{j_1,l_1+1}^{\alpha,...,n})^* f \right\|_{L^\infty_{x_2,...,x_n}L^2_{x_1}} \leq C_n\|A\|_{Y_1} \times \\
\times \int e^{4\pi^2i(t-s)} e^{2\pi i(x - x(t) + U x - U y)} \prod_{\mu=1}^\alpha \psi_{j_\mu,l_\mu+k_\mu}^\gamma(2^{l_\mu+k_\mu} (\xi / |\xi| - \theta_{j_\mu}^{\mu+k_\mu})) \times \\
\times \prod_{\nu=1}^\alpha \psi_{j_\nu,l_\nu+k_\nu}^\gamma(2^{l_\nu+k_\nu} (\xi / |\xi| - \theta_{j_\nu}^{\nu+k_\nu})) \Omega^2(\xi) f(y) dy d\xi
\]
By the Krein-Milman theorem, the linear span of \( \{ \delta_\delta(x_2, \ldots, x_n) g(x_1) : g \in L^2(\mathbb{R}^1) \} \) is \( w^* \)
dense in \( L^1_{x_2,...,x_n}L^2_{x_1} \). Thus, it will suffice to verify the estimate
\[
\sup_{b, \tilde{b} \in \mathbb{R}^{n-1}, z \in \mathbb{R}^n, \tilde{b}_1, \ldots, \tilde{b}_2 \in \mathbb{S}^{n-1}} \left\| \int e^{4\pi^2i(t-s)} e^{2\pi i(x - x(t) + U(x_1, \bar{z}) - U(y_1, b))} \times \\
\prod_{\mu=1}^{2n} \psi^{\mu}(2^{m_\mu} (\xi / |\xi| - \tilde{\theta}_\mu)) \Omega^2(\xi) g(y_1) dy_1 d\xi \right\|_{L^2_{x_2,\ldots,x_n}} \leq C_n\|t-s|^{(n-1)/2}\|g\|_{L^2(\mathbb{R}^1)}.
\]
Clearly, by rotational invariance, we can assume \( U = Id \). But then the expression above is equal to
\[
\int e^{4\pi^2i(t-s)} e^{2\pi i(\xi_1(x_1 + x_1 - y_1))} K(\xi_1) g(y_1) dy_1 = e^{-i(t-s)\partial^2} K(\partial_1/2i\pi) K(\xi_1)(z_1 + x_1),
\]
where
\[ K(\xi_1) = \int \int e^{4\pi^2i(t-s)|\xi|^2} e^{2\pi i(\xi_1 \cdot \bar{\xi} + b)} \prod_{\mu=1}^{2\alpha} \psi_\mu(2^{m_\mu} (|\xi| - \tilde{\theta}_\mu)) \Omega^2(\xi) d\xi_2 \ldots d\xi_n. \]

By Lemma 10 (see its second statement), we have
\[ \sup_{\xi_1} |K(\xi_1)| \leq C_n^\alpha |t - s|^{- (n - 1)/2}. \]

We get
\[ \sup_{b, \tilde{\theta} \in R^{n-1}, z \in R^n, \tilde{\theta}_1, \ldots, \tilde{\theta}_{2\alpha}} \| e^{-i(t-s)\partial_2^2} K(\partial_1/2i\pi)[G(\cdot)] \|_{L^2} \lesssim \| g \|_{L^2} \sup_{\xi_1} |K(\xi_1)| \leq C_n^\alpha |t - s|^{- (n - 1)/2} \| g \|_{L^2}, \]
as required.

9. Global regularity for Schrödinger maps

In this section, we sketch the proof of Theorem 3. As it was discussed earlier, it will suffice to show that \( A \) stays small in the space of vector potentials \( Y \), given the \( a \) \( priori \) information that \( u \) is small in (a portion of) the solution space to be described below. Let \( X^\alpha \) be the completion of all Schwartz functions in the norm
\[ \| u \|_{\tilde{X}^\alpha} = (\sum_k 2^{2ak} \sup_{q, r - \text{Str.}} \| u_k \|_{L^q L^r}^2 + 2^{2ak} \sup_{U \in SU(n), x(t)} \| u_k(t, x(t) + U x) \|_{L^2 L^{2(n-1)/(n-3)} L^{5/2}}^2)^{1/2} \]

Let \( X^s := \tilde{X}^s \cap \tilde{X}^0 \). We will generally measure the solution in \( X^s \), but moreover, we will show it is small in \( \tilde{X}^{s_0} \). Note that since \( s > s_0 \), \( \| u \|_{\tilde{X}^{s_0}} \lesssim \| u \|_{X^s} \).

The space \( Y \) of acceptable vector potentials is on the level of smoothness of \( \tilde{X}^{s_0} \). Fix \( \delta > 0 \), so that \( s > (n+1)/2 + \delta \). We will only assume that \( \| f \|_{\dot{H}^{(n+1)/2+\delta}} \| f \|_{\dot{H}^{(n-5)/2-\delta}} \ll 1 \). Clearly\(^\text{13} \) \( \| f \|_{\dot{H}^{s_0}} \leq (\| f \|_{\dot{H}^{(n+1)/2+\delta}} \| f \|_{\dot{H}^{(n-5)/2-\delta}})^{1/2} \ll 1 \).

Under these assumptions, it will suffice to check

- (\( A = A(u) \) is controlled by \( \| u \|_{\tilde{X}^{s_0}} \) and \( \| u \|_{X^s} \))
  \begin{align}
  \| A(u) \|_Y &\leq C_n \| u \|_{\tilde{X}^{(n+1)/2+\delta}} \| u \|_{\tilde{X}^{(n-5)/2-\delta}}, \\
  \sum_k 2^{2ks} \| A_k \|_{L^2 L^{2n/(n-2)}}^{1/2} &\leq C_n \| u \|_{\tilde{X}^s} \| u \|_{\tilde{X}^{s_0}} \quad \text{for every} \quad s \geq 0.
  \end{align}

- (\( N(u) \) is controlled by \( \| u \|_{\tilde{X}^{s_0}} \) and \( \| u \|_{X^s} \))
  \begin{align}
  \| \partial^s N(u) \|_{L^1 L^2} &\leq C_n \| u \|_{\tilde{X}^s} (\| u \|_{\tilde{X}^{s_0}}^2 + \| u \|_{X^{s_0}}^4)
  \end{align}
Le ts us first show how Theorem 3 follows from (62), (63), (64).

\(^{13}\)Note that in the formulation of the theorem, we have asked for a lot more, namely \( \| f \|_{\dot{H}^s} = \varepsilon \| g \|_{\dot{H}^s} \ll 1 \) for all \( s \in [0, (n+1)/2 + \delta] \).
To that end, we know that the Strichartz estimates hold for at least for some time $T$, so that $\|A\|_{Y_T} \leq \varepsilon$. Fix one such $T$. We have for every $s \geq 0$, by (63) and (64)
\[
\|u\|_{X_T^s} \leq C_n(\|f\|_{\dot{H}^s} + \|\partial^s N(u)\|_{L^1_T L^2_x}) + C_n\|\nabla u\|_{L^2_T L^6_x} \left( \sum_k 2^{2k\delta} \|A_k\|_{L^2_T L^{2n/(n-2)}_x} \right)^{1/2} \\
\leq C_n\|f\|_{\dot{H}^s} + C_n\|u\|_{X_T^s} \left( \|u\|_{X_T^{s_0}}^2 + \|u\|_{X_T^{s_0}}^4 \right) + C_n\|u\|_{\dot{X}_T^s} \left( \sum_k 2^{2k\delta} \|A_k\|_{L^2_T L^{2n/(n-2)}_x} \right)^{1/2} \\
\leq C_n(\|f\|_{\dot{H}^s} + \|u\|_{\dot{X}_T^{s_0}}(\|u\|_{X_T^{s_0}}^2 + \|u\|_{X_T^{s_0}}^4)).
\]

In particular, for $s = s_0$ and the smallness of $\|f\|_{\dot{H}^{s_0}}$, it follows that
\[
\|u\|_{\dot{X}_T^{s_0}} \leq C_n\|f\|_{\dot{H}^{s_0}}.
\]

This means that $\|u\|_{\dot{X}_T^{s_0}}$ is small (independently of $T$), which in turn implies that
\[
\|u\|_{\dot{X}_T^{s}} \leq C_n\|f\|_{\dot{H}^{s}},
\]
for every $s \geq 0$. But how far can we really push that? Recall (62), which gives us a control of $A$ back in terms of $u$. Namely, since
\[
\|A(u)\|_{Y_T} \leq C_n\|u\|_{\dot{X}_T^{(n+1)/2-\delta}} \|u\|_{\dot{X}_T^{(n-5)/2-\delta}} \leq C_n\|f\|_{\dot{H}^{(n+1)/2-\delta}} \|f\|_{\dot{H}^{(n-5)/2-\delta}} < < 1.
\]

This implies that $\|A\|_{Y_T}$ is small and one could apply back the Strichartz estimates, which means that $T$ could be taken to be $\infty$. Theorem 3 follows.

9.1. **Proof of** (62), (63). We will not give the full details of (62), (63), since these are standard Besov type estimates for products. Let us for example consider the estimate for $\|A\|_{Y_3}$. First, it is not hard to see that the terms containing $\partial_tA$ one uses the equation (11), to write it like
\[
\partial_tA = \partial_t Q_1(u, \bar{u}) \sim \dot{Q}_1(u, u_t) \sim \ddot{Q}_1(u, \Delta u) + \dot{Q}_1(u, N(u)).
\]

Thus, everything is reduced to the terms containing $\partial^2A$ and $N(u)$, the latter being easy to treat.

So, we concentrate on the terms involving $\partial^2A$. For those, take into account $\partial^2A_k \sim 2^{2k}A_k$ and $A \sim \partial^{-1}Q(u, \bar{u})$, to conclude
\[
\|A\|_{Y_3} \sim \sum_k 2^{(n+3)/2} \sup_{U \in SU(n), x(t)} \|A_k(t, x(t) + Ux)\|_{L^1_T L^{2^2} x_2 \ldots x_n L^1_{x_1}} \sim \\
\sim \sum_k 2^{(n+1)/2} \sup_{U \in SU(n), x(t)} \||uv\|_k(t, x(t) + Ux)\|_{L^1_T L^{2^2} x_2 \ldots x_n L^1_{x_1}}.
\]

Following Lemma 3.1 and Lemma 3.2 in [25], we have to split $|uv|_k$ in two types of terms - high low interactions $u \sim k v < k + 5$ and high-high interactions $\sum_{l > k - 5} (uv_{l-2} < l+2)_k$.

\[\text{Here } v \text{ might be either } u \text{ or } \bar{u}.\]
The high-low interactions are more difficult to handle in this context\textsuperscript{15}, so let us concentrate on these. We have by Cauchy-Schwartz and Bernstein inequalities
\[
2^{(n+1)/2} \sup_{U \in SU(n), x(t)} \| u_k v_{k-m}(t, x(t) + U x) \|_{L^1_t L^2_{x1}} \leq \\
\leq 2^{(n+1)/2} \sup_{U, x(t)} \| u_k(t, x(t) + U x) \|_{L^2_t L^{2(n-1)/(n-3)}_{x1}} \| v_{k-m}(t, x(t) + U x) \|_{L^2_t L^{n-1} L^2_{x1}} \\
\leq C_n 2^{-\delta m} \sup_{U, x(t)} 2^{(n+1)/(2+\delta)} \| u_k(t, x(t) + U x) \|_{L^2_t L^{2(n-1)/(n-3)}_{x1}} \times \\
\sup_{U, x(t)} 2^{(k-m)(5/2-\delta)} \| v_{k-m}(t, x(t) + U x) \|_{L^2_t L^{2(n-1)/(n-3)}_{x1}}
\]

Note that at this stage to apply the Bernstein inequality in the variables \(x_2, \ldots, x_n\) \(\| v_{k-m} \|_{L^{2(n-1)/(n-3)}_{x1}} \leq C_n 2^{(k-m)(5/2-\delta)} \| v_{k-m} \|_{L^{2(n-1)/(n-3)}_{x1}}\), we needed \(n-1 > 2(n-1)/(n-3)\), which is the dimensional restriction \(n > 5\).

Summing in \(m > -5\) yields
\[
\sum_k 2^{k(n+1)/2} \sup_{U \in SU(n), x(t)} \| u_k v_{k-m}(t, x(t) + U x) \|_{L^1_t L^2_{x1}} \leq \\
\leq (\sum_k 2^{k((n+1)+2\delta)} \sup_{U, x(t)} \| u_k(t, x(t) + U x) \|_{L^2_t L^{2(n-1)/(n-3)}_{x1}}) \times \sup_{U, x(t)} 2^{(k-(5-2\delta))} \| v_{k}(t, x(t) + U x) \|_{L^2_t L^{2(n-1)/(n-3)}_{x1}} \leq \\
\leq C_n \| u \|_{\dot{X}^{(n+1)/2+\delta}} \| v \|_{\dot{X}^{(n-1)/2-\delta}}.
\]

The proof of (63) is in fact very similar and boils down to the same Besov space estimates for products.

9.2. Proof of (64). The estimates for the nonlinearities are the easiest ones. It basically suffice to apply the Kato-Ponce type-estimates
\[
\| \partial^s (uv) \|_{L^r} \leq C \| \partial^s u \|_{L^p} \| v \|_{L^q} + C \| \partial^s v \|_{L^p} \| u \|_{L^q}
\]

whenever \(1/r = 1/p + 1/q\). We omit the details.

10. Appendix

10.1. Decay estimates for the free Schrödinger equation with initial data Fourier supported in a small cap.

\textbf{Lemma 10.} Let \(k_1, \ldots, k_\mu\) be positive integers. Let also \(\{\theta_j\} \in S^{n-1}\) and \(\psi_j\) be smooth cutoff functions adapted to \(\{\xi| \leq 1\}\), whose smoothness bounds are uniform in \(j\). Then there exists a constant \(C\) depending only on the dimension, so that
\[
(65) \quad \sup_{\theta_1, \ldots, \theta_\mu \in S^{n-1}, k_\mu} \sup_{x} \left| \int e^{-4\pi^2 t|\xi|^2} e^{2\pi i \xi \cdot x} \prod_{j=1}^\mu \psi_j(2k_j(\xi/|\xi| - \theta_j)) \varphi(\xi) d\xi \right| \leq C \| \theta \|_{-n/2}.
\]

\textsuperscript{15}For the high-high interactions, one can actually split the \((n+1)/2\) derivatives between the two terms and get a better more balanced estimate.
Also, if one fixes $\xi_1$ and integrates with respect to $\xi_2, \ldots, \xi_n$,

$$\sup_{\theta_1, k_1, \ldots, \theta_n, \xi_2, \ldots, \xi_n} \sup_{x, \xi_1} |\int e^{-4\pi^2it|\xi|^2} e^{2\pi i \xi \cdot x} \prod_{j=1}^\mu \psi_j(2^{k_j}(\xi/|\xi| - \theta_j)) \varphi(\xi) d\xi_2 \ldots d\xi_n| \leq C_n|t|^{-(n-1)/2}.$$  

Proof. We prove only (65), since the second statement in Lemma 10 requires only a slight adjustment of the argument.

This a standard stationary phase argument, except that we have to keep track of the derivatives that may pile up from the cutoffs $\psi_j(2^{k_j}(\xi/|\xi| - \theta_j))$. Fix $k_1, \ldots, k_\mu$ and let $k = \max(k_1, \ldots, k_\mu)$ and let $k = k_0$. If $2^k \geq \sqrt{t}$, we pass to polar coordinates and estimate by

$$C \int_{|\theta - \theta_\mu| \leq 2^{-k}} |\int e^{-4\pi^2it\rho^2} e^{2\pi i \rho \cdot (\theta, x)} \varphi(\rho) d\rho| d\theta \leq C 2^{-(k-1)|t|^{-1/2}} \leq C|t|^{-n/2},$$

where we have used the decay estimate by $Ct^{-1/2}$ for the 1D Schrödinger equation.

Thus assume $2^k \leq \sqrt{t}$. We (smoothly) split the region of integration, according to the size of the derivative of the phase. If $| - 8\pi^2 s \xi + 2\pi x | \leq \sqrt{t}$, we estimate by absolute values and obtain the desired estimate by the volume of the $\xi$ support, which is $C_n|t|^{-n/2}$.

It remains to show that

$$\int e^{-4\pi^2it|\xi|^2} e^{2\pi i \xi \cdot x} \varphi(2^{-m}t^{-1/2}( - 8\pi^2 t \xi + 2\pi x)) \prod_{j=1}^\mu \psi_j(2^{k_j}(\xi/|\xi| - \theta_j)) \varphi(\xi) d\xi$$

$$\leq C 2^{-m} C_n^m |t|^{-n/2},$$

Now that we have $2^{k_j} \leq 2^k \leq \sqrt{t}$. Then, the argument goes as in the classical estimate, that is after $n + 1$ integration by parts with the phase function $\varphi(\xi) = i(-4\pi^2 t|\xi|^2 + 2\pi \langle \xi, x \rangle)$, (at each step one gains at least a factor of $2^{-m}|t|^{-1/2}$ and loses a factor of $C_n \sqrt{|t|}$ at the most), we put in absolute values. Taking into account the volume of the support $\leq C_n 2^{m} t^{-n/2}$, we estimate by

$$D_n 2^m 2^{-m} |t|^{-n/2} \leq C_n^m 2^{-m} |t|^{-n/2},$$

whence summation by $m \geq 0$ yields the result. \hfill \Box

10.2. Estimates on the error term. In this section we give an estimate on the error terms $E_k$, defined in (22).

Lemma 11. Let $\|A\|_{Y_0} \leq \varepsilon$ and $s \geq 0$. Let also $1 = 1/p_1 + 1/p_2; 1/2 = 1/q_1 + 1/q_2$. Then

$$\sum_k 2^{2ks} \|E_k\|_{L^1 L^2}^2 \lesssim \varepsilon^2 \sum_k 2^{2ks} \|u_k\|_{L^{\infty} L^2}^2 + \|u\|_{L^p_{L^1 K_1}}^2 \sum_k 2^{2ks+1} \|A_k\|_{L^p_{L^2 K} L^q_{L^2}}^2,$$

$$\sum_k 2^{2ks} \|E_k\|_{L^1 L^2}^2 \lesssim \varepsilon^2 \sum_k 2^{2ks} \|u_k\|_{L^{\infty} L^2}^2 + \|\nabla u\|_{L^p_{L^1 K_1}}^2 \sum_k 2^{2ks} \|A_k\|_{L^p_{L^2 K} L^q_{L^2}}^2.$$

In particular, from (66), with $s = 0, q_1 = 2, p_1 = \infty, p_2 = 1, q_2 = \infty$, and since $\|A_k\|_{L^1 L^\infty} \leq 2^{kh} \|A_k\|_{L^1 L^\infty}$

$$\sum_k \|E_k\|_{L^1 L^2}^2 \lesssim \varepsilon^2 \sum_k \|u_k\|_{L^\infty L^2}^2$$

(68)
Proof. We estimate on a term-by-term basis in formula (22). For the first term, we have
\[ \sum_k 2^{2ks} \left\| P_k \left( A_{k-4} \nabla u_{k-4} \right) \right\|_{L^1 L^2}^2 \lesssim \left\| \nabla A_t \right\|_{L^1 L^\infty}^2 \sum_k 2^{2ks} \left\| u_k \right\|_{L^\infty L^2}^2. \]

The second and third terms in (22) are treated in a similar fashion, so we concentrate on the second one. For any positive \( h \leq 1 \), we have
\[ \sum_k 2^{2ks} \left( \sum_{l \geq k-2} \left\| P_k \left( A_{l-2} \nabla u_{l-2} \right) \right\|_{L^1 L^2} \right)^2 \lesssim \sum_k 2^{2ks} \left( \sum_{l \geq k-2} 2^{lh} \left\| P_k \left( A_{l-2} \nabla u_{l-2} \right) \right\|_{L^1 L^{2n/(n+h)}} \right)^2 \lesssim \sum_k 2^{2k(s+h)} \left( \sum_{l \geq k-2} 2^l \left\| A_{l-2} \right\|_{L^1 L^{2n/h}} \left\| u_{l-2} \right\|_{L^\infty L^2} \right)^2 \lesssim \sum_k 2^{2k(s+h)} \left( \sum_{l \geq k-2} 2^{-hl} \left( 2^{l+h} \left\| A_{l-2} \right\|_{L^1 L^{n/h}} \right) \left\| u_{l-2} \right\|_{L^2 L^{2n/(n-2)}} \right)^2 \]

One obtains by Lemma 2 an estimate by
\[ \sum_l 2^{2l(1+h)} \left\| A_l \right\|_{L^1 L^{n/h}}^2 \sum_l 2^{2ls} \left\| u_l \right\|_{L^\infty L^2}^2. \]

The last fourth term in (22) can be bounded in two ways.
\[ \sum_k 2^{2ks} \left\| A_{k-1} \nabla u_{k-4} \right\|_{L^1 L^2}^2 \lesssim \left\| u \right\|_{L^p L^q}^2 \sum_k 2^{2k(s+1)} \left\| A_k \right\|_{L^p L^q}^2, \]
\[ \sum_k 2^{2ks} \left\| A_{k-1} \nabla u_{k-4} \right\|_{L^1 L^2}^2 \lesssim \left\| \nabla u \right\|_{L^p L^q}^2 \sum_k 2^{2ks} \left\| A_k \right\|_{L^p L^q}^2. \]

\[ \square \]

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