BIRATIONAL SUPERRIGID CYCLIC TRIPLE SPACES

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ABSTRACT. We prove the birational superrigidity and the nonrationality of a cyclic triple cover of $\mathbb{P}^2$ branched over a nodal hypersurface of degree $3n$ for $n \geq 2$. In particular, the obtained result solves the problem of the birational superrigidity of smooth cyclic triple spaces. We also consider certain relevant problems.

1. Introduction.

The problem of the rationality of an algebraic variety is one of the most interesting problems in algebraic geometry. Global holomorphic differential forms are natural birational invariants of a smooth algebraic variety that solve the problem of the rationality of algebraic curves and surfaces (see [205], [100]). However, even in three-dimensional case there are nonrational varieties that are very close to being rational. In particular, available discrete invariants does not solve the rationality problem for higher-dimensional algebraic varieties. For example, there are nonrational unirational 3-folds (see [103], [48]), which imply that the Lüroth problem in dimension 3 has a negative answer. Unfortunately, there are no known simple way of proving the nonrationality of higher-dimensional rationally connected varieties (see [115], [104], [117]).

There are few known methods of proving the nonrationality of rationally connected varieties. The finiteness of the group of birational automorphisms of a smooth quartic 3-fold is proved in [103], which implies its nonrationality. The nonrationality of a smooth cubic 3-fold is proved in [48] through the study of its intermediate Jacobian. The birational invariance of the torsion subgroup of a group $H^3(Z)$ is used in [6] to prove the nonrationality of certain unirational conic bundles. The nonrationality of a wide class of rationally connected varieties is proved in [114] by means of the reduction into the positive characteristic (see [115], [104], [117]).

Every methods of proving the nonrationality of an algebraic variety has advantages and disadvantages. For example, the method of the intermediate Jacobian can be applied only to 3-folds, and except a single csee (see [182], [183], [199], [184], [185], [47]) only to 3-folds fibered into conics (see [191], [192], [10], [193]). On the other hand, in the three-dimensional case the method of the intermediate Jacobian can be often when all other methods simply can not be used. The degeneration method (see [10], [193], [45], [12], [38], [40]) shows that sometimes the Griffiths component of the intermediate Jacobian is the most subtle three-dimensional birational invariant. For example, an important case of the rationality criterion of a three-dimensional conic bundle (see [96], [97], [98], [99]) is proved in [172] using the intermediate Jacobian method. However, there are nonrational 3-folds whose group $H^3(Z)$ is trivial (see [169]). In many interesting cases, for example, for smooth complete intersections, the group $H^3(Z)$ has no torsion and, therefore, the method of [6] can not be applied (see [49], [144]). The method of [114] works in any

The author is very grateful to M.Grinenko, V.Iskovskikh, S.Kudryavtsev, J.Park, Yu.Prokhorov, A.Pukhlikov and V.Shokurov for fruitful conversations.

1 All varieties are assumed to be projective, normal and defined over $\mathbb{C}$.
dimension, but it proves the nonrationality of a very general element of an appropriate family. The technique of [103] also works in any dimension (see [156]), but in general it can be applied only to varieties that stand too far from the rational ones. For example, it is hard to believe that one can use the technique of [103] to get an example of a smooth deformation of a nonrational variety into a rational one (see [193]). The latter example is expected to exist in dimension greater than 3 (see [189], [190], [86], [87]).

Let us consider the following notion, which is implicitly introduced in the paper [103], but historically it goes back to the classical papers [138], [69], [70], but its modern form is considered relatively recently (see [54], [163]). Note that the class of terminal singularities is a higher-dimensional generalization of smooth points of algebraic surfaces that is closed with respect to the good birational maps (see [113]). The $\mathbb{Q}$-factoriality simply means that a multiple of every Weil divisor on a variety is a Cartier divisor. In particular, every smooth variety has terminal $\mathbb{Q}$-factorial singularities.

**Definition 1.** A terminal $\mathbb{Q}$-factorial Fano variety $V$ with $\text{Pic}(V) \cong \mathbb{Z}$ is birationally superrigid if the following 3 conditions hold:

1. the variety $V$ is not birational to a fibration, whose generic fiber is a smooth variety of Kodaira dimension $-\infty$;
2. the variety $V$ is not birational to a $\mathbb{Q}$-factorial terminal Fano variety with Picard group $\mathbb{Z}$ that is not biregular to $V$;
3. $\text{Bir}(V) = \text{Aut}(V)$.

The paper [103] contains an implicit proof that every smooth quartic 3-fold in $\mathbb{P}^4$ is birationally superrigid (see [53]). The technique of [103] can be applied to certain Fano 3-folds with non-trivial group of birational automorphisms (see [95]). Therefore one can consider the following weakened version of the birational superrigidity.

**Definition 2.** A terminal $\mathbb{Q}$-factorial Fano variety $V$ with $\text{Pic}(V) \cong \mathbb{Z}$ is called birationally rigid if the first two conditions of Definition 1 are satisfied.

Birationally rigid varieties are nonrational. In particular, there are no birationally rigid del Pezzo surfaces defined over an algebraically closed field. However, there are birationally rigid del Pezzo surfaces over an algebraically non-closed field (see [100]). Namely, the results of [130] and [131] imply the birational superrigidity of smooth del Pezzo surfaces of degree 1 and the birational rigidity of smooth del Pezzo surfaces of degree 2 and 3 that are defined over a perfect algebraically non-closed field and have Picard group $\mathbb{Z}$. In particular, minimal smooth cubic surfaces in $\mathbb{P}^3$ are birationally equivalent if and only if they are projectively equivalent (see [132]).

The birational rigidity and superrigidity can be defined for a fibration into Fano varieties as well (see [72], [103]). To be precise, the birational rigidity and superrigidity can be defined for Mori fibrations (see [53]). Today the birational rigidity is proved for many smooth 3-folds (see [95], [168], [150], [54]), for many smooth varieties whose dimension is greater than 3 (see [169], [145], [146], [151], [27], [153], [151], [165], [174], [158], [159], [160], [72], [83], [162]), and for many singular varieties (see [147], [149], [56], [54], [133], [161], [12], [11]). For some birationally nonrigid algebraic varieties it is possible to find all Mori fibrations birational to them (see [77], [55], [80], [81]). Unfortunately, despite the obvious success in this area of algebraic geometry there are many still unsolved relevant classical problems such as finding the generators of the group $\text{Bir}(\mathbb{P}^3)$ or finding

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2 For every fibration $\tau : Y \to Z$ we assume that $\dim(Y) > \dim(Z) \neq 0$ and $\tau_*(\mathcal{O}_Y) = \mathcal{O}_Z$. 

the generators of the group of birational automorphisms of a smooth cubic 3-fold. The
solution of the latter problem is announced in the classical paper [70], but the proof
contains many gaps.

In the given paper we will prove the following result.

**Theorem 3.** Let \( \pi : X \to \mathbb{P}^{2n} \) be a cyclic triple cover\(^3\) such that \( \pi \) is branched over a
hypersurface \( S \subset \mathbb{P}^{2n} \) of degree \( 3n \), \( n \geq 2 \) and the hypersurface \( S \) has at most ordinary
double points. Then \( X \) is a terminal \( \mathbb{Q} \)-factorial Fano variety with \( \text{Pic}(X) \cong \mathbb{Z} \) such that \( X \) is birationally superrigid, the group \( \text{Bir}(X) \) is finite and for sufficiently general
hypersurface \( S \subset \mathbb{P}^{2n} \) it is isomorphic to \( \mathbb{Z}_3 \). In particular, the variety \( X \) is nonrational.

**Remark 4.** Under the conditions of Theorem 3, the variety \( X \) can be considered as a hy-
persurface in the weighted projective space \( \mathbb{P}(1^{2n+1}, n) \) of degree \( 3n \) given by the equation
\[
y^3 = f_{3n}(x_0, \ldots, x_{2n}) \subset \mathbb{P}(1^{2n+1}, n) \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_{2n}, y]),
\]
where \( f_{3n} \) is a homogenous polynomial of degree \( 3n \) (see [135], [74], [178], [179], [181]), and \( \pi : X \to \mathbb{P}^{2n} \) is a restriction of the natural projection \( \mathbb{P}(1^{2n+1}, n) \to \mathbb{P}^{2n} \) induced by the
embedding of the graded algebras \( \mathbb{C}[x_0, \ldots, x_{2n}] \subset \mathbb{C}[x_0, \ldots, x_{2n}, y] \). Moreover, the
hypersurface \( S \subset \mathbb{P}^{2n} \) is given by the equation \( f_{3n}(x_0, \ldots, x_{2n}) = 0 \).

**Remark 5.** Consider a cyclic triple cover \( \pi : X \to \mathbb{P}^k \) such that \( \pi \) is branched over a nodal
hypersurface \( S \subset \mathbb{P}^k \) of degree \( 3n \) and \( k \geq 3 \). Then \( X \) is not birationally superrigid in the case when \( k < 2n \), because it has pencils of varieties of Kodaira dimension \( -\infty \). On the
other hand, the Kodaira dimension of the variety \( X \) is non-negative when \( k > 2n \) and the
variety \( X \) is not even uniruled in this case. Therefore, all birationally superrigid smooth
cyclic triple covers are described by Theorem 3.

**Corollary 6.** Let \( f(x_0, \ldots, x_{2n}) \) be a homogeneous polynomial of degree \( 3n \) such that
\[
f(x_0, \ldots, x_{2n}) = 0 \subset \mathbb{P}^{2n} \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_{2n}])
\]
is a nodal or smooth hypersurface. Then \( \mathbb{C}(\nu_1, \ldots, \nu_{2n}) \sqrt[3]{f(1, \nu_1, \ldots, \nu_{2n})} \) is a purely tran-
scendental extension of the field \( \mathbb{C} \) if and only if the equality \( n = 1 \) holds.

**Example 7.** Let \( X \) be a hypersurface in \( \mathbb{P}(1^{2n+1}, n) \) of degree \( 3n \) whose equation is
\[
y^3 = \sum_{i=0}^{2n} x_i^{3n} \subset \mathbb{P}(1^{2n+1}, n) \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_{2n}, y]),
\]
and \( n \geq 2 \). Then the projection \( \pi : X \to \mathbb{P}^{2n} \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_{2n}]) \) is a cyclic triple
cover branched over a smooth hypersurface \( \sum_{i=0}^{2n} x_i^{3n} = 0 \), the variety \( X \) is birationally
superrigid by Theorem 3 and
\[
\text{Bir}(X) = \text{Aut}(X) \cong \mathbb{Z}_3 \oplus \text{Aut}(\sum_{i=0}^{2n} x_i^{3n} = 0) \cong \mathbb{Z}_3 \oplus (\mathbb{Z}_3 \rtimes S_{2n+1}),
\]
where \( S_{2n+1} \) is a symmetric group (see [198], [170], [171], [123]). Hence \( X \) is nonrational
and \( \mathbb{C}(\nu_1, \ldots, \nu_{2n}) \sqrt[3]{1 + \sum_{i=1}^{2n} \nu_i^{3n}} \) is not a purely transcendental extension of \( \mathbb{C} \).

\(^3\)\text{A finite morphism of degree 3 that induces the cyclic extension of the fields of rational functions.}
Example 8. Let $X$ be a hypersurface $\mathbb{P}(1^{2n+1}, n)$ of degree $3n$ whose equation is
\[ y^3 = \sum_{i=1}^{n} a_i(x_0, \ldots, x_{2n})x_i \subset \mathbb{P}(1^{2n+1}, n) \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_{2n}, y]), \]
where $a_i$ is a sufficiently general homogeneous polynomial of degree $3n - 1$. Then the natural projection $\pi : X \to \mathbb{P}^{2n}$ is a cyclic triple cover such that $\pi$ is branched over a nodal hypersurface $S \subset \mathbb{P}^{2n}$ of degree $3n$, which is given by the equation
\[ \sum_{i=1}^{n} a_i x_i = 0 \subset \mathbb{P}^{2n} \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_{2n}]) \]
and which has $(3n - 1)^n$ ordinary double points. The variety $X$ is birationally superrigid and nonrational for $n \geq 2$ by Theorem 3, and the group $\text{Bir}(X)$ is finite.

Example 9. Let $X$ be a hypersurface in $\mathbb{P}(1^{2n+1}, n)$ of degree $3n$ whose equation is
\[ y^3 = \sum_{i=1}^{n} a_i(x_0, \ldots, x_{2n})b_i(x_0, \ldots, x_{2n}) \subset \mathbb{P}(1^{2n+1}, n) \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_{2n}, y]), \]
where $a_i$ and $b_i$ are sufficiently general homogeneous polynomials of degree $2n$ and $n$ respectively. Then the natural projection $\pi : X \to \mathbb{P}^{2n}$ is a cyclic triple cover branched over a nodal hypersurface $S \subset \mathbb{P}^{2n}$ of degree $3n$ having $2^n n^{2n}$ ordinary double points, the variety $X$ is birationally superrigid and nonrational for $n \geq 2$ by Theorem 3, and the group $\text{Bir}(X)$ is finite.

The main reason why the variety $X$ in Theorem 3 is birationally superrigid is the following: the anticanonical degree $(-K_X)^{\dim(X)} = 3$ of the variety $X$ is very small and the singularities of the variety $X$ are relatively mild. Roughly speaking, a Fano variety must become more rational when the anticanonical degree getting bigger and the singularities getting worse. This general principle may not necessary be true in certain extremely singular cases (see [25]). However, it follows from the classification that a smooth Fano 3-fold is rational if its degree is bigger than 24 (see [104]). Singular Fano 3-folds are not classified even in the case when their anticanonical divisors are Cartier divisors (see [20], [142], [107]), but many examples affirm the intuition in the singular case as well (see [56], [35], [10], [40], [42]). Therefore the nonrationality of the variety $X$ in Theorem 3 is very natural.

Due to natural reasons, it makes sense to consider birational superrigidity and birational rigidity only for Mori fibrations (see [53]). In particular, in the case of Fano varieties we must assume that for a given Fano variety its singularities are $\mathbb{Q}$-factorial and its rank of the Picard group is 1. Many examples suggest that a Fano variety may not be birationally rigid if its degree is not sufficiently small. Moreover, it is intuitively clear that the quantitative characteristics of singularities (number of isolated singular points or anticanonical degree of the corresponding subvarieties of singular points) is important only to provide the $\mathbb{Q}$-factoriality condition (see [55], [133], [12], [10], [41]). On the other hand, the qualitative characteristics of singularities (multiplicity and analytical local type) can crucially influence the birational geometry of a Fano variety (see [54], [75]).

Unfortunately, all existent proofs of the birational rigidity or birational superrigidity of a Fano variety crucially depend on the projective geometry of the given variety related to the anticanonical map. It is natural to expect that some claims on birational rigidity can be proven without implicit usage of the properties of the anticanonical ring. For example, we expect that the following is true (cf. [158], [72]).
Conjecture 10. Let $X$ be a smooth Fan variety of dimension $k$ such that $\text{Pic}(X) \cong \mathbb{Z}$ and $(-K_X)^k \leq 2(k - 1)$. Then $X$ is birationally rigid.

It should be pointed out that Conjecture 10 is proved only in dimension 3 through the explicit classification of smooth Fano 3-folds (see [104]). It is very possible that the proof of Conjecture 10 can be extremely hard. On the other hand, it is very natural to expect that the following weakened version of the Conjecture 10 can be proved relatively soon using methods of [54], [62], [112].

Conjecture 11. Let $X$ be a smooth Fano variety of dimension $k$ such that $\text{Pic}(X) \cong \mathbb{Z}$ and $(-K_X)^k = 1$. Then $X$ is birationally superrigid.

Remark 12. It is well known that any statement on birational rigidity remains true over any field of definition of the considered varieties with a single exception. Namely, the characteristic of the field of definition must be zero in order to use the Kawamata–Viehweg vanishing theorem (see [111], [195]). However, in the case of algebraic surfaces it is enough to assume that the field of definition is just perfect (see [130], [131]). Moreover, one can consider equivariant version of any statement on birational rigidity when the acting group is finite (see [93], [76], [100]). The latter can be used in classification of all nonconjugate finite subgroups of corresponding groups of birational automorphisms (see [102]).

It should be pointed out that the nonrationality and the non-ruledness of a cyclic triple cover of $\mathbb{P}^{3n}$ branched over a very general smooth hypersurface of degree 3 with $n \geq 2$ are implied by Theorem 5.13 in [115] that claims the following.

Theorem 13. Let $\xi : V \to \mathbb{P}^k$ be a cyclic cover of prime degree $p \geq 2$ branched over a very general hypersurface $F \subset \mathbb{P}^k$ of degree $pd$ such that $k \geq 3$ and $d > \frac{k+1}{p}$. Then $V$ is nonruled and, in particular, the variety $V$ is nonrational.

In the conditions and notations of Theorem 13, it is natural to ask how many singular points can $X$ have. The singular points of the variety $X$ are in one-to-one correspondence with ordinary double points of the hypersurface $S \subset \mathbb{P}^{3n}$ of degree 3. Therefore, the best known bound is due to [194]. Namely, the number of singular points of $X$ does not exceed the Arnold number $A_{2n}(3n)$, where $A_{2n}(3n)$ is a number of points $(a_1, \ldots, a_{2n}) \subset \mathbb{Z}^{2n}$ such that the inequalities

$$3n^2 - 3n + 2 \leq \sum_{i=1}^{2n} a_i \leq 3n^2$$

hold and $a_i \in (0, 3n)$. In particular, the number of singular points of the variety $X$ does not exceed 320, 115788 and 85578174 when $n = 2, 3$ and 4 respectively. However, this bound seems not to be sharp for $n \gg 0$ (see [12], [175], [19], [177], [8], [106], [196]).

Remark 14. It is well known that the variety $X$ in Theorem 13 is a rationally connected variety (see [120], [122], [122], [115]). Namely, there is an irreducible rational curve on the variety $X$ passing through any two sufficiently general points of $X$.

The geometrical meaning of Theorem 13 has the same nature as the Noether theorem that claims that the group $\text{Bir}(X)$ is generated by the Cremona involution and projective automorphisms (see [138], [92], [53]). The Noether theorem is related to many interesting problems. For example, the Noether theorem is related to the problem of birational classification of plane elliptic pencils. Originally it was considered in [11], but later the
ideas of [11] were put into proper and correct form in the paper [60] that proves that any plane elliptic pencil can be birationally transformed into a special plane elliptic pencil, so-called Halphen pencil (see §5.6 in [57]), which was studied in [83]. A similar problem can be considered for the variety \( X \) in Theorem 3. Namely, we prove the following result.

**Theorem 15.** Under the conditions of Theorem 3, the variety \( X \) is not birational to any elliptic fibration.

Birational transformations into elliptic fibrations were used in [14], [15], [84] in the proof of the potential density\(^5\) of rational points on smooth Fano 3-folds, where the following result was proved.

**Theorem 16.** Rational points are potentially dense on all smooth Fano 3-folds with a possible exception of a double cover of \( \mathbb{P}^3 \) ramified in a smooth sextic surface.

The existence of a possible exception in Theorem 16 is explained by the following result proved in [28]: a smooth sextic double solid is the only smooth Fano 3-fold that is not birationally isomorphic to an elliptic fibration (see [104]). It should be pointed out that a double cover of \( \mathbb{P}^3 \) branched over a sextic having one ordinary double point can be birationally transformed into an elliptic fibration in a unique way (see [30]) and rational points on such 3-fold are potentially dense (see [12]).

**Remark 17.** Let \( \pi : X \to \mathbb{P}^4 \) be a cyclic triple cover such that \( \pi \) is branched over a hypersurface \( S \subset \mathbb{P}^4 \) of degree 6, \( n \geq 2 \), and \( S \) has one ordinary singular point \( O \in S \) of multiplicity 3. Then the projection \( \gamma : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3 \) from \( O \) induces a rational map \( \gamma \circ \pi \) such that the normalization of the generic fiber of \( \gamma \circ \pi \) is an elliptic curve. In particular, the variety \( X \) does not satisfy the conditions of Theorem 3. Namely, \( S \) is not nodal.

The nodality condition in Theorems 3 and 15 is rather natural. Indeed, ordinary double points are the simplest singularities of algebraic varieties and the geometry of nodal varieties is related to many interesting problems (see [186], [46], [73], [200], [108], [16], [141], [58], [64], [59]). However, we can consider a wider class of singularities in the problems similar to the claim of Theorems 3. The proofs of Theorems 3 and 15 together with the inequality for global log canonical thresholds (see [31], [43], [63]) give a proof of the following simple generalization of Theorems 3 and 15.

**Theorem 18.** Let \( \pi : X \to \mathbb{P}^{2n} \) be a cyclic triple cover such that \( \pi \) is branched over a hypersurface \( S \subset \mathbb{P}^{2n} \) of degree \( 3n, n \geq 2 \), and the only singularities of \( S \) are ordinary double and triple points. Namely, the multiplicity of any singular point of \( S \) does not exceed 3 and the projectivization of the tangent cone to the hypersurface \( S \) at this point is smooth. Then \( X \) is a Fano variety with \( \mathbb{Q} \)-factorial terminal singularities, \( \text{Pic}(X) \cong \mathbb{Z} \), the variety \( X \) is birationally superrigid, and the group \( \text{Bir}(X) \) is finite. Moreover, the only way to birationally transform \( X \) into an elliptic fibration is by means of the construction in Remark 17, which implies \( n = 2 \) and \( S \) has a triple point.

Therefore, it follows from Theorem 18 that the methods of [14], [15], [84] can not be used to prove the potential density of rational points on the variety \( X \) in Theorem 18 in the case when the variety \( X \) is defined over a number field, with a single exception of a cyclic triple cover of \( \mathbb{P}^4 \) branched over a hypersurface of degree 6 having at least one triple point. It should be pointed out that the geometrical unirationality of a variety defined

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\(^5\)The set of rational points of a variety \( V \) defined over a number field \( \mathbb{F} \) is called potentially dense if for a finite extension of fields \( \mathbb{K}/\mathbb{F} \) the set of \( \mathbb{K} \)-rational points of the variety \( V \) is Zariski dense.
over a number field implies the potential density of rational points. Therefore, if rational points are not potentially dense on some of the considered cyclic triple covers, then it is rationally connected but not unirational! On the other hand, as of today there is no known example of a rationally connected variety that is not unirational (cf. Conjecture 4.1.6 in [117]). Therefore, it is natural to expect that the methods of [14], [15] and [84] can be applied to prove potential density of rational points of a cyclic triple cover of $\mathbb{P}^4$ which is defined over a number field and branched over a hypersurface of degree 6 having at least one singular point of multiplicity 3. We will prove the this statement in the general case only. Namely, we will prove the following result using the method of [14], [15], [84].

**Theorem 19.** Let $\pi : X \to \mathbb{P}^4$ be a cyclic triple cover branched over a sufficiently general\(^6\) hypersurface $S \subset \mathbb{P}^4$ of degree 6 such that $S$ is defined over a number field and $S$ has an ordinary triple point. Then rational points are potentially dense on $X$.

Actually, our methods can be used to prove the following result. Let us remind that canonical singularities are higher-dimensional generalization of Du Val singularities (see [113]).

**Theorem 20.** Under the conditions of Theorem 19 or Theorem 18, let $\rho : X \to V$ be a birational map such that $V$ is a Fano variety with canonical singularities. Then $\rho$ is an isomorphism.

The claim of Theorem 20 is a generalization of one of the claims of Theorem 3. However, we think that Theorem 20 has certain importance. For example, the similar claim for smooth minimal cubic surfaces defined over an algebraically non-closed field (see [28]) generalizes the classical birational classification (see [132]) in the following way: a smooth minimal cubic surface in $\mathbb{P}^3$ is birational to a cubic surface in $\mathbb{P}^3$ with Du Val singularities if and only if they are projectively equivalent. Moreover, the expanded version of the latter claim (see [28]) gives a description of all finite subgroups of the group of birational automorphisms of a smooth minimal cubic surface (see [36]), which answers Question 1.10 in the book [132]. The latter problem was originally solved in [109] by group-theoretic methods using the explicit description of the group of birational automorphisms of a smooth minimal cubic surface obtained in [131] and [132].

**Remark 21.** The claims similar to Theorems 15 and 20 are proved for many algebraic varieties (see [28], [27], [29], [30], [105], [32], [33], [34], [37], [39], [42]).

Double covers of projective spaces are generalizations of hyperelliptic curves, triple covers of projective spaces are generalizations of trigonal curves. However, triple covers are not necessary Galois covers. The study of discrete invariants of cyclic covers of $\mathbb{P}^2$ goes back to [52], [202], [203], which was continued in the papers [105], [128], [167], [188], [20] and [124]. Certain questions related to triple covers of algebraic surfaces were considered in [187], [178], [179]. The topological questions related to covers of projective spaces were considered in [127] and [75]. Structural results related to triple covers were obtained in [135], [74], [139], [21], [180], [181], [65]. Some results of sporadic nature were obtained in [197], [140], [129]. In the framework of birational geometry triple covers of projective spaces were considered in [125] and [126]. The nonrationality of general cyclic covers of projective spaces were considered in [115] (see Theorem 13).
2. **Movable log pairs.**

In this section we will consider properties of so-called movable log pairs that were introduced in [2]. Movable log pairs were used implicitly in [138], [69], [70], [103].

**Definition 22.** A movable log pair \((X,M_X)\) is pair consisting of a variety \(X\) and a movable boundary \(M_X\), where \(M_X = \sum_{i=1}^{n} a_i M_i\) is a formal finite linear combination of linear systems \(M_i\) on variety \(X\) such that the base locus of every \(M_i\) has codimension at least 2 in \(X\) and \(a_i \in \mathbb{Q}_{\geq 0}\).

It is clear that every movable log pair can be considered as a usual log pair with an effective boundary whose components does not have multiplicities greater than 1 by replacing every linear system either by its general element or by the appropriate weighted sum of its general elements. In particular, for a given movable log pair \((X,M_X)\) we may consider movable boundary \(M_X\) as an effective divisor. Thus the numerical intersection of the movable boundary \(M_X\) with curves on the variety \(X\) is well defined in the case when the variety \(X\) is \(\mathbb{Q}\)-factorial. Hence we can consider the formal sum \(K_X + M_X\) as a log canonical divisor of the movable log pair \((X,M_X)\). In the rest of this section we assume that all log canonical divisors are \(\mathbb{Q}\)-Cartier divisors.

**Remark 23.** For a movable log pair \((X,M_X)\) the self-intersection \(M_X^2\) can be considered as a well-defined effective codimension-two cycle in the case when the singularities of the variety \(X\) are \(\mathbb{Q}\)-factorial.

The image of a movable boundary under a birational map is naturally well defined, because base loci of the components of a movable boundary do not contain divisors.

**Definition 24.** Movable log pairs \((X,M_X)\) and \((Y,M_Y)\) are called birationally equivalent if there is a birational map \(\rho : X \dashrightarrow Y\) such that \(M_Y = \rho(M_X)\).

The standard notions such as discrepancies, terminality, canonicity, log terminality and log canonicity can be defined for movable log pairs in a similar way as they are defined for usual log pairs (see [113]).

**Definition 25.** A movable log pair \((X,M_X)\) has canonical (terminal respectively) singularities if for every birational morphism \(f : W \rightarrow X\) there is an equivalence

\[
K_W + f^{-1}(M_X) \sim_{Q} f^*(K_X + M_X) + \sum_{i=1}^{n} a(X,M_X,E_i)E_i
\]

such that every rational number \(a(X,M_X,E_i)\) is non-negative (positive respectively), where \(E_i\) is an \(f\)-exceptional divisor. The rational number \(a(X,B_X,E_i)\) is called a discrepancy of the movable log pair \((X,B_X)\) in the \(f\)-exceptional divisor \(E_i\).

**Example 26.** Let \(X\) be a 3-fold and \(\mathcal{M}\) be a linear system on \(X\) such that the base locus of the linear system \(\mathcal{M}\) has codimension at least 2. Then the log pair \((X,\mathcal{M})\) has terminal singularities if and only if the linear system \(\mathcal{M}\) has only isolated simple base points, which are smooth points of the 3-fold \(X\).

**Remark 27.** The application of Log Minimal Model Program (see [113]) to a movable log pair having canonical or terminal singularities preserves the canonicity or terminality respectively.
Singularities of a movable log pair coincide with the singularities of the variety outside of the base loci of the components of the movable boundary. Therefore the existence of a resolution of singularities (see \[90\]) implies that every movable log pair is birationally equivalent to a log pair with canonical or terminal singularities.

**Definition 28.** A proper irreducible subvariety \(Y \subset X\) is called a center of canonical singularities of a movable log pair \((X, M_X)\) if there is a birational morphism \(f : W \to X\) and an \(f\)-exceptional divisor \(E_1 \subset W\) such that

\[
K_W + f^{-1}(M_X) \sim_\mathbb{Q} f^*(K_X + M_X) + \sum_{i=1}^{k} a(X, M_X, E_i)E_i,
\]

where \(a(X, M_X, E_i) \in \mathbb{Q}\), \(E_i\) is an \(f\)-exceptional divisor, \(a(X, M_X, E_i) \leq 0\), \(f(E_i) = Y\).

**Definition 29.** The set \(\text{CS}(X, M_X)\) is a set of all centers of canonical singularities of a movable log pair \((X, M_X)\) and \(\text{CS}(X, M_X)\) is a set-theoretic union of all centers of canonical singularities of the movable log pair \((X, M_X)\).

In particular, a log pair \((X, M_X)\) is terminal if and only if \(\text{CS}(X, M_X) = \emptyset\).

**Remark 30.** Let \((X, M_X)\) be a log pair with terminal singularities. Then the singularities of the log pair \((X, \epsilon M_X)\) are terminal for any small enough rational number \(\epsilon > 1\).

**Remark 31.** Let \((X, M_X)\) be a movable log pair and \(Z \subset X\) be a proper irreducible subvariety such that \(X\) is smooth at the generic point of the subvariety \(Z\). Then elementary properties of blow ups imply

\[
Z \in \text{CS}(X, M_X) \Rightarrow \text{mult}_Z(M_X) \geq 1
\]

and in the case \(\text{codim}(Z \subset X) = 2\) we have \(\text{mult}_Z(M_X) \geq 1 \Rightarrow Z \in \text{CS}(X, M_X)\).

**Remark 32.** Let \((X, M_X)\) be a movable log pair, \(H\) be a general hyperplane section of the variety \(X\), and \(Z \in \text{CS}(X, M_X)\) such that \(\dim(Z) \geq 1\). Then \(Z \cap H \in \text{CS}(H, M_X|_H)\).

**Definition 33.** For a movable log pair \((X, M_X)\) consider any birationally equivalent movable log pair \((W, M_W)\) such that its singularities are canonical. Let \(m\) be a natural number such that \(m(K_W + M_W)\) is a Cartier divisor. The Kodaira dimension \(\kappa(X, M_X)\) of the log pair \((X, M_X)\) is the maximal dimension of the image \(\phi_{|nm(K_W + M_W)|}(W)\) for \(n \gg 0\) in the case when \(|n(K_W + M_W)|\) is not empty for some \(n\). In the case when complete linear systems \(|n(K_W + M_W)|\) are empty for all \(n \gg 0\) we simply put \(\kappa(X, M_X) = -\infty\).

**Lemma 34.** The Kodaira dimension of a movable log pair is well-defined, namely, it does not depend on the choice of the birationally equivalent movable log pair having canonical singularities in Definition 29.

**Proof.** Let \((X, M_X)\) and \((Y, M_Y)\) be movable log pairs having canonical singularities such that we have \( M_X = \rho(M_Y)\) for some birational map \(\rho : Y \dasharrow X\). Take positive integer \(m\) such that \(m(K_X + M_X)\) and \(m(K_Y + M_Y)\) are Cartier divisors. To conclude the claim it is enough to show that \(\phi_{|nm(K_X + M_X)|}(X) = \phi_{|nm(K_Y + M_Y)|}(Y)\) for \(n \gg 0\) or

\[
|nm(K_X + M_X)| = |nm(K_Y + M_Y)| = \emptyset \text{ for all } n \in \mathbb{N}.
\]

Let us consider a commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{f} & X \\
\downarrow{g} & & \downarrow{\rho} \\
& Y \\
\end{array}
\]
such that $W$ is smooth, $g : W \to X$ and $f : W \to Y$ are birational morphisms. Then

$$K_W + M_W \sim_Q g^*(K_X + M_X) + \Sigma_X \sim_Q f^*(K_Y + M_Y) + \Sigma_Y,$$

where $M_W = g^{-1}(M_X)$, $\Sigma_X$ and $\Sigma_Y$ are exceptional divisors of $g$ and $f$ respectively.

The canonicity of log pairs $(X, M_X)$ and $(Y, M_Y)$ implies the effectiveness of the exceptional divisors $\Sigma_X$ and $\Sigma_Y$. However, the effectiveness of $\Sigma_X$ and $\Sigma_Y$ implies that

$$\dim(|km(K_W + M_W)|) = \dim(|g^*(km(K_X + M_X))|) = \dim(|f^*(km(K_Y + M_Y))|)$$

for $k \gg 0$ if they are not empty and

$$\phi|km(K_W + M_W)| = \phi|g^*(km(K_X + M_X))| = \phi|f^*(km(K_Y + M_Y))|,$$

which implies the claim. \qed

By definition, the Kodaira dimension of a movable log pair is a birational invariant and a non-decreasing function of the coefficients of the movable boundary.

**Definition 35.** For a given movable log pair $(X, M_X)$, a movable log pair $(V, M_V)$ is called a canonical model of $(X, M_X)$ if $M_V = \psi(M_X)$ for a birational map $\psi : X \dasharrow V$, the divisor $K_V + M_V$ is ample, and singularities of $(V, M_V)$ are canonical.

The given definition of a canonical model of a movable log pair coincide with the classical definition of a canonical model in the case of empty boundary (see [113]). The existence of the canonical model of a movable log pair implies that its Kodaira dimension equals to the dimension of the variety.

**Lemma 36.** A canonical model of a movable log pair is unique if it exists.

**Proof.** Let $(X, M_X)$ and $(V, M_V)$ be canonical models such that $M_X = \rho(M_V)$ for a birational map $\rho : V \dasharrow X$. Consider a commutative diagram

```
\begin{array}{ccc}
W & \xrightarrow{g} & X \\
\downarrow{f} & & \downarrow{\rho} \\
Y & \xleftarrow{\rho^{-1}} & \end{array}
```

such that $W$ is smooth, $g : W \to X$ and $f : W \to Y$ are birational morphisms. Then

$$K_W + M_W \sim_Q g^*(K_X + M_X) + \Sigma_X \sim_Q f^*(K_Y + M_Y) + \Sigma_Y,$$

where $M_W = g^{-1}(M_X)$, $\Sigma_X$ and $\Sigma_Y$ are exceptional divisors of $g$ and $f$ respectively. Then

$$K_W + M_W \sim_Q g^*(K_X + M_X) + \Sigma_X \sim_Q f^*(K_Y + M_Y) + \Sigma_Y,$$

where $M_W = g^{-1}(M_X) = f^{-1}(M_Y)$, and $\Sigma_X$ and $\Sigma_Y$ are the exceptional divisors of birational morphisms $g$ and $f$ respectively. The canonicity of the singularities of the movable log pairs $(X, M_X)$ and $(V, M_V)$ implies that $\Sigma_X$ and $\Sigma_V$ are effective.

Let $n \in \mathbb{N}$ be a big and divisible enough number such that $n(K_W + M_W)$, $n(K_X + M_X)$ and $n(K_Y + M_Y)$ are Cartier divisors. Then the effectiveness of $\Sigma_X$ and $\Sigma_V$ implies

$$\phi|n(K_W + M_W)| = \phi|g^*(n(K_X + M_X))| = \phi|f^*(n(K_Y + M_Y))|$$

and $\rho$ is an isomorphism, because $K_X + M_X$ and $K_Y + M_Y$ are ample. \qed
In the case of empty movable boundary the claim of Lemma 36 about the uniqueness of a canonical modal of an algebraic variety is well known. The latter implies that all birational automorphisms of a canonical model are biregular. However, the absence of non-biregular birational automorphisms is also a property of a birationally superrigid variety (see Definition 1). We show later that Lemma 36 explains the geometrical nature of this phenomenon in the both cases. In the case of a birationally rigid varieties Lemma 36 is nothing but a veiled Noether–Fano–Iskovskikh inequality (see [156]).

3. Preliminary results.

Properties of movable log pairs (see Definition 22) reflects birational geometry of a given variety (see Lemma 36). Canonical and terminal singularities are most appropriate classes of singularities for movable log pairs (see Remark 27). Many geometrical problems can be translated into the language of movable log pairs. Movable log pair always can be considered as usual log pairs, and movable boundaries always can be considered as effective divisors. On the other hand, we can consider log pairs with both movable and fixed components (linear systems can have both movable and fixed parts). Moreover, we can consider log pairs with negative coefficients as well. We must consider such generalizations due to several reasons.

For a movable log pair \((X, M_X)\) and birational morphism \(f : V \to X\), the birationally equivalent log pair \((V, M_V)\) does not reflect the properties of the log pair \((X, M_X)\), but the log-pullback (see Definition 37) of the log pair \((X, M_X)\) reflects the properties of the log pair \((X, M_X)\). However, the log pull back \((V, M^V)\) of the movable log pair \((X, M_X)\) is not necessary a movable log pair and \(M^V\) is not necessary an effective divisor. This is the first reason to consider log pairs with both fixed and movable components and possibly negative coefficients.

Canonical singularities and centers of canonical singularities (see Definition 28) do not have good functorial properties when considered apart from the birational context, but log canonical singularities and centers of log canonical singularities (see Definition 38) have good functorial properties, and they role in the modern algebraic geometry is very important (see [113], [119], [116], [173], [136], [137], [158], [72]). Log canonical singularities and canonical singularities are related mostly through the log adjunction (see [32] and Theorem 49), but also through other ways (see [158]). However, log adjunction for movable log pair can lead to a non-movable log pair. This is another reason to consider log pairs with both fixed and movable components.

In this section we do not impose any restrictions on boundaries. In particular, boundaries may not be effective unless otherwise stated. For simplicity, we assume that log canonical divisors of all log pairs are \(\mathbb{Q}\)-Cartier divisors.

**Definition 37.** A log pair \((V, B^V)\) is called a log pull back of a log pair \((X, B_X)\) with respect to a birational morphism \(f : V \to X\) if we have

\[
B^V = f^{-1}(B_X) - \sum_{i=1}^{n} a(X, B_X, E_i)E_i
\]

such that the equivalence \(K_V + B^V \sim_{\mathbb{Q}} f^*(K_X + B_X)\) holds, where \(E_i\) is an \(f\)-exceptional divisor and \(a(X, B_X, E_i) \in \mathbb{Q}\). The rational number \(a(X, B_X, E_i)\) is called a discrepancy of the log pair \((X, B_X)\) in the \(f\)-exceptional divisor \(E_i\).
**Definition 38.** A proper irreducible subvariety $Y \subset X$ is called a center of log canonical singularities of the log pair $(X, B_X)$ if there are a birational morphism $f : V \to X$ and a not necessary $f$-exceptional divisor $E \subset V$ such that $E$ is contained in the effective part of the support of the divisor $[B^V]$ and $f(E) = Y$.

**Definition 39.** The set of centers of log canonical singularities of the log pair $(X, B_X)$ is denoted as $\text{LCS}(X, B_X)$. The set-theoretic union of all elements in $\text{LCS}(X, B_X)$ is called a locus of log canonical singularities of the log pair $(X, B_X)$, it is denoted as $\text{LCS}(X, B_X)$.

**Remark 40.** Let $H$ be a general hyperplane section of $X$ and $Z \in \text{LCS}(X, B_X)$ such that the inequality $\dim(Z) \geq 1$ holds. Then $Z \cap H \in \text{LCS}(H, B_X|_H)$.

Let $X$ be a variety and $B_X = \sum_{i=1}^n a_i B_i$ be a boundary on $X$, where $a_i$ is a rational number and $B_i$ is a prime divisor on $X$. Let $f : V \to X$ be a birational morphism such that $V$ is smooth and the union all $f$-exceptional divisors and $\cup_{i=1}^n f^{-1}(B_i)$ forms a divisor with simple normal crossing. The morphism $f$ is called is called a log resolution of the log pair $(X, B_X)$. Then the equivalence

$$K_Y + B^Y \sim_\mathbb{Q} f^*(K_X + B_X)$$

holds, where $(Y, B^Y)$ is a log pull back of the log pair $(X, B_X)$.

**Definition 41.** Let $\mathcal{I}(X, B_X) = f_* (\mathcal{O}_V([-B^V]))$. Then the subscheme $\mathcal{L}(X, B_X)$ associated to the ideal sheaf $\mathcal{I}(X, B_X)$ is called a log canonical singularity subscheme of the log pair $(X, B_X)$.

Note, that by definition we have $\text{Supp}(\mathcal{L}(X, B_X)) = \text{LCS}(X, B_X) \subset X$. The following result is the Shokurov vanishing theorem (see [173], [3]).

**Theorem 42.** Let $(X, B_X)$ be a log pair, and $H$ be a nef and big divisor on $X$ such that the boundary $B_X$ is effective, and $D = K_X + B_X + H$ is a Cartier divisor. Then the cohomology group $H^i(X, \mathcal{I}(X, B_X) \otimes D)$ vanishes for $i > 0$.

**Proof.** Let $f : W \to X$ be a log resolution of $(X, B_X)$. Then

$$R^i f_*(f^*(K_X + B_X + H) + [-B^W]) = 0$$

for $i > 0$ by the Kawamata-Viehweg vanishing (see [111], [195], [113]). The degeneration of the local-to-global spectral sequence and

$$R^0 f_*(f^*(K_X + B_X + H) + [-B^W]) = \mathcal{I}(X, B_X) \otimes D$$

imply that for all $i \geq 0$ we have

$$H^i(X, \mathcal{I}(X, B_X) \otimes D) = H^i(W, f^*(K_X + B_X + H) + [-B^W]),$$

but $H^i(W, f^*(K_X + B_X + H) + [-B^W]) = 0$ for $i > 0$ by the Kawamata-Viehweg vanishing. \hfill \Box

Consider the following two application of Theorem 42, which are special cases of a more general result in [31] (see [13], [63], [72]).

**Lemma 43.** Let $V = \mathbb{P}^1 \times \mathbb{P}^1$ and $B_V$ be an effective boundary on $V$ of bi-degree $(a, b)$ such that $a$ and $b \in \mathbb{Q} \cap [0, 1)$. Then $\text{LCS}(V, B_V) = \emptyset$. 

---

**Note:** The text contains mathematical definitions, theorems, and proofs that require a deep understanding of algebraic geometry. The notation used is standard in the field, and the results discussed are foundational in the study of log canonical singularities and vanishing theorems.
Proof. Let \( B_V = \sum_{i=1}^{k} a_i B_i \), where \( a_i \) is a positive rational number, and \( B_i \) is an irreducible reduced curve on the surface \( V \). Intersecting the boundary \( B_V \) with the rulings of \( V \) we get the inequality \( a_i < 1 \). Thus the set \( \mathrm{LCS}(V, B_V) \) does not contains curves on \( V \).

Suppose that the set \( \mathrm{LCS}(V, B_V) \) contains a point \( O \). Take a divisor \( H \in \text{Pic}(V) \otimes \mathbb{Q} \) of bi-degree \((1 - a, 1 - b)\). Then the divisor \( H \) is ample. Moreover, there is a divisor

\[
D \sim_{\mathbb{Q}} K_V + B_V + H
\]

such that \( D \) is a Cartier divisor and \( H^0(\mathcal{O}_V(D)) = 0 \). On the other hand, the map

\[
H^0(\mathcal{O}_V(D)) \to H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}(D))
\]

is surjective by Theorem 42, which is a contradiction. \( \square \)

Lemma 44. Let \( V \subset \mathbb{P}^n \) be a smooth hypersurface of degree \( k < n \), and \( B_V \) be an effective boundary on \( V \) such that \( B_V \equiv rH \), where \( r \in \mathbb{Q} \cap [0, 1) \), and \( H \) is a hyperplane section of the hypersurface \( V \subset \mathbb{P}^n \). Then \( \mathrm{LCS}(V, B_V) = \emptyset \).

Proof. Suppose that the set \( \mathrm{LCS}(V, B_V) \) contains a subvariety \( Z \subset V \). Then \( \dim(Z) = 0 \) by Theorem 2 in \([148]\) (see Lemma 3.18 in \([33]\)). Therefore the set \( \mathrm{LCS}(V, B_V) \) contains only closed points of the hypersurface \( V \). In particular, the support of the scheme \( \mathcal{L}(V, B_V) \) is zero-dimensional and \( H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}) \neq 0 \).

Note, that \( K_V + B_V + (1 - r)H \equiv (k - n)H \) and \( H^0(\mathcal{O}_V((k - n)H)) = 0 \), because the inequality \( k < n \) holds. However, Theorem 42 implies the surjectivity

\[
H^0(\mathcal{O}_V((k - n)H)) \to H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}((k - n)H)) \to 0,
\]

which is a contradiction, because \( H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}((k - n)H)) = H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}) \).

\( \square \)

Example 45. Let \( V \subset \mathbb{P}^n \) be a smooth hypersurface

\[
x_0^k = \sum_{i=1}^{n} x_i^k \subset \mathbb{P}^n \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_n]),
\]

and \( B_V = \frac{n - 1}{k} H \), where \( H \) is a hyperplane section of the hypersurface \( V \) that is cut by the equation \( x_0 = x_1 \). Then the hypersurface \( V \) is smooth and the set \( \mathrm{LCS}(V, B_V) \) consist of a single point \((1 : 1 : 0 : \ldots : 0) \in V \subset \mathbb{P}^n \).

The arguments of the proofs of Lemmas 43 and 44 can be applied in much more general situation. Namely, for a given Cartier divisor \( D \) on the variety \( X \), let us consider the exact sequence of sheaves

\[
0 \to \mathcal{I}(X, B_X) \otimes D \to \mathcal{O}_X(D) \to \mathcal{O}_{\mathcal{L}(X, B_X)}(D) \to 0,
\]

and the corresponding exact sequence of cohomology groups. Now Theorem 42 implies the following two connectedness results (see \([173]\)).

Theorem 46. Let \( (X, B_X) \) be a log pair, and let \( B_X \) be an effective boundary such that the divisor \( -(K_X + B_X) \) is nef and big. Then the locus \( \mathrm{LCS}(X, B_X) \) is connected.

Theorem 47. Let \( (X, B_X) \) be a log pair, \( B_X \) be an effective boundary, \( g : X \to Z \) be morphism with connected fibers such that \( -(K_X + B_X) \) is \( g \)-nef and \( g \)-big. Then \( \mathrm{LCS}(X, B_X) \) is connected in a neighborhood of each fiber of \( g \).

Similarly, one can prove the following result, which is Theorem 17.4 in \([119]\).
Theorem 48. Let $g : X \to Z$ be a morphism with connected fibers, $D = \sum_{i \in I} d_i D_i$ be a divisor on $X$, $h : V \to X$ be a resolution of singularities of the variety $X$ such that the union of all $h$-exceptional divisors and $\cup_{i \in I} h^{-1}(D_i)$ is a simple normal crossing divisor, the divisor $-(K_X + D)$ is $g$-nef and $g$-big, and the inequality $\text{codim}(g(D_i) \subset Z) \geq 2$ holds whenever $d_i < 0$. For any divisor $E \subset V$ let $a(E) \in \mathbb{Q}$ such that the equivalence

$$K_V \sim_{\mathbb{Q}} f^*(K_X + D) + \sum_{E \subset V} a(E)E$$

holds. Then $\cup_{a(E) \leq -1} E$ is connected in the neighborhood of every fiber of $g \circ h$.

Proof. Let $f = g \circ h$, $A = \sum_{a(E) > -1} E$, and $B = \sum_{a(E) \leq -1} E$. Then

$$[A] - [B] \sim_{\mathbb{Q}} K_V - h^*(K_X + D) + \{ -A \} + \{ B \}$$

and $R^1 f_* \mathcal{O}_V([A] - [B]) = 0$ by the Kawamata–Viehweg vanishing. Hence the map

$$f_* \mathcal{O}_V([A]) \to f_* \mathcal{O}_B([A])$$

is surjective. Every component of $[A]$ is either $h$-exceptional or a proper transform of a divisor $D_j$ with $d_j < 0$. Thus $h_*([A])$ is $g$-exceptional and $f_* \mathcal{O}_V([A]) = \mathcal{O}_Z$. So the map

$$\mathcal{O}_Z \to f_* \mathcal{O}_B([A])$$

is surjective, which implies the connectedness of $[B]$ in a neighborhood of every fiber of morphism $f$, because $[A]$ is effective and has no common component with $[B]$. \qed

We defined the notion of a center of canonical singularities in Definition 28 for a movable log pair. However, we did not use the movability of a boundary in Definitions 28 and we can consider centers of canonical singularities of any log pair.

Theorem 49. Let $(X, B_X)$ be a log pair, $Z$ be an element in $\mathcal{CS}(X, B_X)$, $H$ be an effective irreducible Cartier divisor on $X$ such that $Z \subset H$, $X$ and $H$ are smooth at the generic point of $Z$, $H$ is not a component of $B_X$, and $B_X$ is effective. Then $\mathbb{LCS}(H, B_X|_H) \neq \emptyset$.

Proof. Let $f : W \to X$ be a log resolution of $(X, B_X + H)$. Put $\hat{H} = f^{-1}(H)$. Then

$$K_W + \hat{H} \sim_{\mathbb{Q}} f^*(K_X + B_X + H) + \sum_{E \neq \hat{H}} a(X, B_X + H, E)E$$

and by assumption we have $\{ Z, H \} \subset \mathcal{LCS}(X, B_X + H)$. Therefore applying Theorem 48 to the log pullback of $(X, B_X + H)$ on $W$, we get $\hat{H} \cap E \neq \emptyset$ for some divisor $E \neq \hat{H}$ on the variety $W$ such that $f(E) = Z$ and $a(X, B_X, E) \leq -1$. Now the equivalences

$$K_{\hat{H}} \sim (K_W + \hat{H})|_{\hat{H}} \sim_{\mathbb{Q}} f|_{\hat{H}}^*(K_H + B_X|_H) + \sum_{E \neq \hat{H}} a(X, B_X + H, E)E|_{\hat{H}}$$

imply the claim. \qed

Corollary 50. Let $(X, M_X)$ be a movable log pair, $O$ be a smooth point of $X$, $H_i$ be a general hyperplane section of $X$ passing through the point $O$ for $i = 1, \ldots, k \leq \dim(X) - 2$ such that $O \in \mathcal{CS}(X, M_X)$, $M_X$ is effective, and $\dim(X) \geq 3$. Then $O \in \mathcal{LCS}(S, M_S)$, where $S = \cap_{i=1}^k H_i$ and $M_S = M_X|_S$.

It should be pointed out that Theorem 49 is a special case of a general phenomenon, which is known as log adjunction (see [119], [54]). In particular, simple modification of the proof of Theorem 49 implies the following result.
Lemma 55. Let \((X, M_X)\) be a movable log pair, \(O\) be an isolated hypersurface singular point of the variety \(X\), and \(H_i\) be a general hyperplane section of \(X\) passing through the point \(O\) for \(i = 1, \ldots, k \leq \dim(X) - 2\) such that \(O \in \mathbb{CS}(X, M_X)\), the boundary \(M_X\) is effective, and \(\dim(X) \geq 3\). Then \(O \in \mathbb{LCS}(S, M_S)\), where \(S = \cap_{i=1}^{k} H_i\) and \(M_S = M_X|_S\).

The following result is a Theorem 3.1 in [54], which gives the shortest proof of the main result of [103] modulo Theorem 49 (see [54]).

Theorem 52. Suppose that \(\dim(X) = 2\), the boundary \(B_X\) is effective and movable, and there is a smooth point \(O \in X\) such that \(O \in \mathbb{LCS}(X, (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_X)\), where \(\Delta_1\) and \(\Delta_2\) are smooth curves on \(X\) intersecting normally at \(O\), and \(a_1\) and \(a_2\) are arbitrary non-negative rational numbers. Then we have

\[
\text{mult}_O(B_X^2) \geq \begin{cases} 
4a_1a_2 & \text{if } a_1 \leq 1 \text{ or } a_2 \leq 1 \\
4(a_1 + a_2 - 1) & \text{if } a_1 > 1 \text{ and } a_2 > 1.
\end{cases}
\]

Most applications of Theorem 52 use the simplified version Theorem 53 (see Corollary 53) that involves only movable boundary. Moreover, Theorem 52 was created in order to be applied to movable log pairs. However, the proof of Theorem 52 in [54] is inductive by the number of blow ups required to obtain the appropriate negative discrepancy. It is easy to see that the inductive proof of Theorem 52 is much easier to apply when we have nonmovable components of the boundary. In certain sense the main difficulty in the proof of Theorem 52 is to find the right form of Theorem 52 which is suitable for the inductive proof. On the other hand, Theorem 52 with nontrivial nonmovable components of the boundary has nice higher-dimensional applications (see [27], [33]). More general approach to Theorem 52 was found in [72], where an analog of Theorem 52 was used to prove the generalization of the main inequality of [158]. Note, that Theorem 2.1 in [72] is a generalization of Theorem 52 in the case when the nonmovable part of the boundary consists of a single component. However, such weaken version of Theorem 52 is not suitable for some applications (see [27]).

The following result is a special case of Theorem 0.1 in [71].

Corollary 53. Let \(H\) be a surface, \(O\) be a smooth point on \(H\), and \(M_H\) be an effective movable boundary on \(H\) such that \(O \in \mathbb{LCS}(H, M_H)\). Then the inequality \(\text{mult}_O(M_H^2) \geq 4\) holds and the equality \(\text{mult}_O(M_H^2) = 4\) implies \(\text{mult}_O(M_H) = 2\).

The following result is due to [156].

Theorem 54. Let \(X\) be a variety, \(M_X\) be an effective movable boundary on \(X\), and \(O\) be a smooth point of \(X\) such that \(O \in \mathbb{CS}(X, M_X)\) and \(\dim(X) \geq 3\). Then \(\text{mult}_O(M_X^2) \geq 4\) and the equality \(\text{mult}_O(M_X^2) = 4\) implies \(\text{mult}_O(M_X) = 2\) and \(\dim(X) = 3\).

Proof. The claim is implied by Corollaries 50 and 53. \(\square\)

The proof of Theorem 54 in [156] is elementary but technical, which is valid even over fields of positive characteristic. The proof in [156] and the proof in [54] does not explain the geometrical nature of Theorem 54 which is pointed out in [53] and requires the following well known result (see [119]).

Lemma 55. Let \(X\) be a smooth 3-fold, \(O\) be a point on \(X\), and \(M_X\) be an effective movable boundary on the variety \(X\) such that the singularities of the log pair \((X, M_X)\) are canonical, and \(O \in \mathbb{CS}(X, M_X)\). Then there is a birational morphism \(f : V \to X\) such that the 3-fold \(V\) has \(\mathbb{Q}\)-factorial terminal singularities, the morphism \(f\) contracts exactly one divisor \(E\), \(f(E) = O\), and \(K_V + M_V \sim_{\mathbb{Q}} f^*(K_X + M_X)\), where \(M_V = f^{-1}(M_X)\).
Proof. There are finitely many divisorial discrete valuations $\nu$ of the field of rational functions of $X$ whose center on $X$ is the point $O$ and whose discrepancy $a(X, M_X, \nu)$ is non-positive, because $(X, M_X)$ has canonical singularities. Therefore we may consider a birational morphism $g : W \to X$ such that $W$ is smooth, $g$ contracts $k$ divisors,

$$K_W + M_W \sim_{Q} g^*(K_X + M_X) + \sum_{i=1}^{k} a_i E_i,$$

movable log pair $(W, M_W)$ has canonical singularities, and the set $\mathcal{CS}(W, M_W)$ does not contain subvarieties of $\bigcup_{i=1}^{\infty} E_i$, where $M_W = g^{-1}(M_X)$, $g(E_i) = O$, and $a_i \in \mathbb{Q}$. Applying the relative Log Minimal Model Program (see [11]) to the log pair $(W, M_W)$ over $X$, we may assume that the 3-fold $W$ has terminal $\mathbb{Q}$-factorial singularities and

$$K_W + M_W \sim_{Q} g^*(K_X + M_X)$$

because the singularities of the movable log pair $(X, M_X)$ are canonical. Now applying the relative Minimal Model Program to the variety $W$ over $X$, we get the necessary 3-fold and birational morphism.

Remark 56. Let $X$ be a smooth variety, $O$ be a point of $X$, and $f : V \to X$ be a birational morphism such that $V$ has terminal $\mathbb{Q}$-factorial singularities, $f$ contracts a single exceptional divisor $E$, and $f(E) = O$. Then there is a movable log pair $(X, M_X)$ such that the boundary $M_X$ is effective, the singularities of the log pair $(X, M_X)$ are canonical, $K_V + M_V \sim_{Q} f^*(K_X + M_X)$, and $O \in \mathcal{CS}(X, M_X)$, where $M_V = f^{-1}(M_X)$.

The following result is conjectured in [53] and proved in [10].

Theorem 57. Let $X$ be a smooth 3-fold, $O$ be a point of $X$, and $f : V \to X$ be a birational morphism such that the singularities of $V$ are terminal and $\mathbb{Q}$-factorial, $f$ contracts a single divisor $E \subset V$, and $f(E) = O$. Then $f$ is a weighted blow up at the point $O$ with weights $(1, K, N)$ in suitable local coordinates on $X$, where $K$ and $N$ are coprime naturals.

Actually, Theorem 54 was proved in [53] modulo Theorem 57 in the following way, which explains the geometrical nature of Theorem 54.

Proposition 58. Let $X$ be a smooth 3-fold, $O$ be a point of $X$, and $M_X$ be an effective movable boundary on $X$, and $f : V \to X$ be a weighted blow up of $O$ with weights $(1, K, N)$ in suitable local coordinates on $X$ such that

$$K_V + M_V \sim_{Q} f^*(K_X + M_X),$$

where natural numbers $K$ and $N$ are coprime and $M_V = f^{-1}(M_X)$. Then

$$\text{mult}_O(M_X^2) \geq \frac{(K + N)^2}{KN} = 4 + \frac{(K - N)^2}{KN} \geq 4,$$

where $K = N$ implies that $f$ is a standard blow up of $O$ and $\text{mult}_O(M_X) = 2$.

Proof. Let $E \subset V$ be an $f$-exceptional divisor. Then

$$K_V \sim_{Q} f^*(K_X) + (N + K)E$$

and $M_V \sim_{Q} f^*(M_X) + mE$ for some $m \in \mathbb{Q}_{>0}$. Thus $m = K + N$. Now intersecting the effective cycle $M_X^2$ with a general hyperplane section of $X$ passing through $O$, we obtain the inequality

$$\text{mult}_O(M_X^2) \geq m^2E^3 = \frac{(K+N)^2}{KN}.$$

The following application of Theorem 49 is Theorem 3.10 in [54].
Theorem 59. Let $X$ be a variety, $O$ be an ordinary double point of $X$, and $B_X$ an effective boundary on the variety $X$ such that $O \in \mathcal{CS}(X, B_X)$, $B_X$ is a $\mathbb{Q}$-Cartier divisor, and $\dim(X) \geq 3$. Then $\text{mult}_O(B_X) \geq 1$, and $\text{mult}_O(B_X) = 1$ implies $\dim(X) = 3$, where the positive rational number $\text{mult}_O(B_X)$ is defined through the standard blow up of $O$.

Proof. We may assume that $X$ is a 3-fold due to Corollary 51. Let $f : W \to X$ be a blow up at the point $O$ and $E$ be an $f$-exceptional divisor. Then

$$K_W + B_W \sim \mathbb{Q} f^*(K_X + B_X) + (1 - \text{mult}_O(B_X))E,$$

where $B_W = f^{-1}(B_X)$. Suppose that the inequality $\text{mult}_O(B_X) < 1$ holds. Then there is a proper subvariety $Z \subset Q$ such that $Z \in \mathcal{CS}(W, B_W)$. Hence

$$\text{LCS}(E, B_W|_E) \neq \emptyset$$

by Theorem 49 which is impossible by Lemma 43 because $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. □

Proposition 60. Let $X$ be a variety, $B_X$ be an effective boundary on $X$, and $O$ be an isolated singular point on $X$ such that $X$ is locally given by the equation $y^3 = \sum_{i=1}^{\dim(X)} x_i^2$ in the neighborhood of $O$, the boundary $B_X$ is a $\mathbb{Q}$-Cartier divisor on $X$, $O \in \mathcal{CS}(X, B_X)$, and $\dim(X) \geq 4$. Then $\text{mult}_O(B_X) > 1$, where $\text{mult}_O(B_X) \in \mathbb{Q}$ is defined naturally by means of the standard blow up of the point $O$.

Proof. The claim is implied by Corollary 51 and Theorem 59. □

Theorem 61. Let $X$ be a variety of dimension $n \geq 4$, $B_X$ be an effective boundary on the variety $X$, $O$ be an ordinary triple point of the variety $X$ such that $O \in \mathcal{CS}(X, B_X)$, and the boundary $B_X$ is a $\mathbb{Q}$-Cartier divisor on $X$. Then the inequality $\text{mult}_O(B_X) \geq 1$ holds, and $\text{mult}_O(B_X) = 1$ implies $n = 4$, where the rational number $\text{mult}_O(B_X)$ is defined naturally through the standard blow up of the point $O$.

Proof. Let $f : W \to X$ be a blow up of the point $O$. Then

$$K_W + B_W \sim \mathbb{Q} f^*(K_X + B_X) + (n - 3 - \text{mult}_O(B_X))E,$$

where $B_W = f^{-1}(B_X)$ and $E = f^{-1}(O)$. Suppose that $\text{mult}_O(B_X) < n - 3$. Then there is a subvariety $Z \subset E$ such that

$$Z \in \mathcal{CS}(W, B_W - (n - 3 - \text{mult}_O(B_X))E) \subseteq \mathcal{CS}(W, B_W),$$

and the inequalities $n > 4$ and $\text{mult}_O(B_X) \leq 1$ imply that

$$\mathcal{CS}(W, B_W - (n - 3 - \text{mult}_O(B_X))E) \subseteq \mathcal{CS}(W, \lambda B_W)$$

for some positive rational number $\lambda < 1$. In particular, $\text{LCS}(E, B_W|_E) \neq \emptyset$ in the case when $\text{mult}_O(B_X) < 1$ by Theorem 49. Moreover, we have $\text{LCS}(E, \lambda B_W|_E) \neq \emptyset$ in the case when $\text{mult}_O(B_X) \leq 1$ and $n > 4$. Therefore, in both cases we proved the claim that contradicts to Lemma 44. □

It is easy to see that Theorems 59 and 61 are special cases of the following general result, which is left without a proof, because its proof is very similar to the proof of Theorem 61.

7Namely, the point $O$ is an isolated hypersurface singular point of $X$ such that the projectivization of the tangent cone to $X$ at the point $O$ is a smooth hypersurface in $\mathbb{P}^{n-1}$ of degree 3.
Theorem 62. Let $X$ be a variety of dimension $n$, $B_X$ be an effective boundary on the variety $X$, and $O$ be an ordinary singular point of multiplicity $k$ such that $O \in \mathbb{CS}(X, B_X)$, the inequality $n > k$ holds, and $B_X$ is a $\mathbb{Q}$-Cartier divisor. Then $\text{mult}_O(B_X) \geq 1$, and the equality $\text{mult}_O(B_X) = 1$ implies $n = k + 1$, where $\text{mult}_O(B_X) \in \mathbb{Q}$ is defined naturally through the standard blow up of the point $O$.

Corollary 63. Let $f : V \to X$ be a birational morphism, $O$ be an ordinary singular point of the variety $X$ of multiplicity $\dim(X) - 1$ such that $X$ and $V$ have terminal $\mathbb{Q}$-factorial singularities, the morphism $f$ contracts a single divisor $E$, and $f(E) = O$. Then $f$ is a standard blow up of the point $O$.

4. The Noether–Fano–Iskovskikh inequality.

In this chapter we consider the Noether–Fano–Iskovskikh inequality and give two generalization of this inequality. Let $X$ be a Fano variety with terminal $\mathbb{Q}$-factorial singularities such that $\text{Pic}(X) \cong \mathbb{Z}$. For example, we can always substitute $X$ by the variety that satisfies all conditions of Theorems 3 or 18 (see Lemma 58 and Remark 57). We assume that all movable boundaries are effective. The following result is due to [53], but its special cases can be found in [138, 169, 170, 130, 131, 103, 95, 168] and [169].

Theorem 64. Suppose that every movable log pair $(X, M_X)$ such that $K_X + M_X \sim_{\mathbb{Q}} 0$ has canonical singularities. Then the Fano variety $X$ is birationally superrigid.

Proof. Suppose that $X$ is not birationally superrigid. Let $\rho : X \dashrightarrow Y$ be a birational map such that the rational map $\rho$ is not biregular and either $Y$ is a Fano variety of Picard rank 1 with terminal $\mathbb{Q}$-factorial singularities or there is a fibration $\tau : Y \to Z$ whose generic fiber has Kodaira dimension $-\infty$. We may assume that

Suppose that we have a fibration $\tau : Y \to Z$ such that the generic fiber of $\tau$ is a variety of Kodaira dimension $-\infty$. Take a very ample divisor $H$ on $Z$ and some positive rational number $\mu$. Put $M_Y = \mu|\tau^*(H)|$ and $M_X = \mu\rho^{-1}(|\tau^*(H)|)$. Then we have

$$\kappa(X, M_X) = \kappa(Y, M_Y) = -\infty$$

by construction. Choose $\mu$ such that $M_X \sim_{\mathbb{Q}} -K_X$. Then the singularities of $(X, M_X)$ are not canonical, because otherwise $\kappa(X, M_X) = 0$. Therefore we get a contradiction with our initial assumption.

Suppose that $Y$ is a $\mathbb{Q}$-factorial terminal Fano variety of Picard rank 1. Take a positive rational number $\mu$. Let $M_Y = \frac{\mu}{n} - nK_Y$ and $M_X = \rho^{-1}(M_Y)$ for $n \gg 0$. Then

$$\kappa(X, M_X) = \kappa(Y, M_Y) = \begin{cases} 
\text{dim}(Y) & \text{for } \mu > 1 \\
0 & \text{for } \mu = 1 \\
-\infty & \text{for } \mu < 1
\end{cases}$$

by construction. Choose $\mu$ such that $M_X \sim_{\mathbb{Q}} -K_X$. Then the singularities of the movable log pair $(X, M_X)$ are canonical by assumption. Hence $\kappa(X, M_X) = 0$ and $\mu = 1$.

Let us consider a commutative diagram

\[ 
\begin{array}{ccc}
W & \xrightarrow{g} & Y \\
\downarrow f & & \\
X & \xleftarrow{\rho} & Y 
\end{array}
\]

\[ ^8\text{Namely, the point } O \text{ is an isolated hypersurface singular point on } X \text{ such that the projectivization of the tangent cone to } X \text{ at the point } O \text{ is a smooth hypersurface in } \mathbb{P}^{n-1}. \]
such that $W$ is smooth, $g : W \to X$ and $f : W \to Y$ are birational morphisms. Then
\[
\sum_{j=1}^{k} a(X, M_X, F_j)F_j \sim_{\mathbb{Q}} \sum_{i=1}^{l} a(Y, M_Y, G_i)G_i,
\]
where $G_i$ is a $g$-exceptional divisor and $F_j$ is an $f$-exceptional divisor. We may assume that $k = l$, because $X$ and $Y$ are $\mathbb{Q}$-factorial and have Picard rank 1. Every $a(X, M_X, F_j)$ is non-negative and every $a(Y, M_Y, G_i)$ is positive, because $(Y, M_Y)$ has terminal singularities by assumption. Thus it follows from Lemma 2.19 in [119] that
\[
\sum_{j=1}^{k} a(X, M_X, F_j)F_j = \sum_{i=1}^{l} a(Y, M_Y, G_i)G_i,
\]
which implies that the singularities of the log pair $(X, M_X)$ are terminal.

Now take $\mu > 1$ such that the singularities of the log pairs $(X, M_X)$ and $(Y, M_Y)$ are still terminal (see Remark 30). Then both log pairs $(X, M_X)$ and $(Y, M_Y)$ are canonical models. Thus $\rho$ is an isomorphism by Lemma 36, which contradicts to our assumption and concludes the proof. \hfill \Box

The roots of Theorem 64 can be found in [138], [69] and [70]. In two-dimensional case an analog of Theorem 64 is proved in [130], [131], in the three-dimensional case an analog of Theorem 63 is proved in [103], [95].

**Corollary 65.** Suppose that $X$ is not birationally superrigid. Then there is a movable log pair $(X, M_X)$ such that the divisor $-(K_X + M_X)$ is ample and $\mathcal{CS}(X, M_X) \neq \emptyset$.

The following two generalizations of Theorem 63 are due to [28].

**Theorem 66.** Let $\rho : V \dasharrow X$ be birational map such that there is a morphism $\tau : V \to Z$ whose generic fiber is a smooth elliptic curve, and let $D$ be a very ample divisor on the variety $Z$ and $\mathcal{D} = [\tau^*(D)]$. Put $\mathcal{M} = \rho(D)$ and $M_X = \gamma \mathcal{M}$, where $\gamma$ is a positive rational number such that $K_X + \gamma M_X \sim_{\mathbb{Q}} 0$. Then $\mathcal{CS}(X, M_X) \neq \emptyset$.

**Proof.** Suppose that the set $\mathcal{CS}(X, M_X)$ is empty. Then the singularities of the movable log pair $(X, M_X)$ are terminal. In particular, for some rational number $\epsilon > \gamma$ the movable log pair $(X, \epsilon \mathcal{M})$ is a canonical model (see Remark 30). In particular, the equality
\[
\kappa(X, \epsilon \mathcal{M}) = \dim(X)
\]
holds. On the other hand, the log pairs $(X, \epsilon \mathcal{M})$ and $(V, \epsilon \mathcal{D})$ are birationally equivalent and have the same Kodaira dimensions. However, by construction
\[
\kappa(V, \epsilon \mathcal{D}) \leq \dim(Z) = \dim(X) - 1,
\]
which is a contradiction. \hfill \Box

**Theorem 67.** Let $\rho : V \dasharrow X$ be a birational map such that $V$ is a Fano variety with canonical singularities. Put $\mathcal{D} = \lfloor -nK_V \rfloor$ for $n \gg 0$, $\mathcal{M} = \rho(\mathcal{D})$, and $M_X = \gamma \mathcal{M}$, where $\gamma \in \mathbb{Q}$ such that $K_X + \gamma M_X \sim_{\mathbb{Q}} 0$. Then either $\rho$ is not biregular or $\mathcal{CS}(X, M_X) \neq \emptyset$.

**Proof.** Suppose that $\mathcal{CS}(X, M_X) = \emptyset$. Then the singularities of the log pair $(X, M_X)$ are terminal. In particular, we have $\kappa(X, M_X) = 0$, which implies $\gamma = \frac{1}{n}$. Thus for some rational $\epsilon > \gamma$ the log pair $(X, \epsilon \mathcal{M})$ is canonical model. On the other hand, the movable log pair $(V, \epsilon \mathcal{D})$ is a canonical model as well. Hence $\rho$ is biregular by Lemma 36. \hfill \Box
5. Birational superrigidity.

In this section we prove Theorem \ref{thm:superrigidity}. Let $\pi : X \to \mathbb{P}^{2n}$ be a cyclic triple cover branched over a hypersurface $S \subset \mathbb{P}^{2n}$ of degree $3n$ such that the only singularities of the hypersurface $S$ are ordinary double points, and $n \geq 2$. Then $X$ is a Fano variety, the singularities of the variety $X$ are terminal Gorenstein singularities, and $K_X \sim \pi^*(O_{\mathbb{P}^{2n}}(-1))$.

The variety $X$ is a hypersurface
\[ y^3 = f_{3n}(x_0, \ldots, x_{2n}) \subset \mathbb{P}(1^{2n+1}, n) \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_{2n}, y]), \]
where $f_{3n}$ is a homogeneous polynomial of degree $3n$. The triple cover $\pi : X \to \mathbb{P}^{2n}$ is a restriction of the natural projection $\mathbb{P}(1^{2n+1}, n) \to \mathbb{P}^{2n}$ that is induced by the embedding of the graded algebras $\mathbb{C}[x_0, \ldots, x_{2n}] \subset \mathbb{C}[x_0, \ldots, x_{2n}, y]$. Moreover, the equation of the hypersurface $S \subset \mathbb{P}^{2n}$ is $f_{3n}(x_0, \ldots, x_{2n}) = 0$.

The variety $X$ is $\mathbb{Q}$-factorial, but this must be proved. We prove a stronger statement following the arguments in \cite{22, 23}. In fact, the $\mathbb{Q}$-factoriality of the variety $X$ must follow from the Lefschetz theorem (see \cite{17, 21, 52}), because $X$ has isolated singularities.

**Lemma 68.** The groups $\text{Cl}(X)$ and $\text{Pic}(X)$ are generated by the divisor $K_X$.

**Proof.** Let $D$ be a Weil divisor on $X$. We must show that $D \sim rK_X$ for some $r \in \mathbb{Z}$.

Let $H$ be a general divisor in $|-kK_X|$ for $k \gg 0$. Then $H$ is a smooth weighted complete intersection in $\mathbb{P}(1^{2n+1}, n)$ and $\dim(H) \geq 3$. In particular, $\text{Pic}(H)$ is generated by the divisor $K_{X|H}$ by Theorem 3.13 of chapter XI in \cite{52} (see \cite{13}, Lemma 3.2.2 in \cite{61}, or Lemma 3.5 in \cite{56}). Hence there is $r \in \mathbb{Z}$ such that $D|_H \sim rK_{X|H}$.

Let $\Delta = D - rK_X$. Then the sequence of sheaves
\[ 0 \to O_X(\Delta) \otimes O_X(-H) \to O_X(\Delta) \to O_H \to 0 \]
is exact, because $O_X(\Delta)$ is locally free in the neighborhood of $H$. Thus the sequence
\[ 0 \to H^0(O_X(\Delta)) \to H^0(O_H) \to H^1(O_X(\Delta) \otimes O_X(-H)) \]
is exact. The sheaf $O_X(\Delta)$ is reflexive (see \cite{85}). So there is an exact sequence of sheaves
\[ 0 \to O_X(\Delta) \to \mathcal{E} \to \mathcal{F} \to 0, \]
where $\mathcal{E}$ is a locally free sheaf, and $\mathcal{F}$ has no torsion. Hence the sequence
\[ H^0(\mathcal{F} \otimes O_X(-H)) \to H^1(O_X(\Delta - H)) \to H^1(\mathcal{E} \otimes O_X(-H)) \]
is exact. However, the group $H^0(\mathcal{F} \otimes O_X(-H))$ is trivial because $\mathcal{F}$ has no torsion, and the group $H^1(\mathcal{E} \otimes O_X(-H))$ is trivial by the lemma of Enriques–Severi–Zariski (see \cite{201}), because the variety $X$ is normal. Therefore
\[ H^1(O_X(\Delta) \otimes O_X(-H)) = 0 \]
and $H^0(O_X(\Delta)) = \mathbb{C}$. Similarly, we have $H^0(O_X(-\Delta)) = \mathbb{C}$. Thus $D \sim rK_X$.

Suppose that the Fano variety $X$ is not birationally superrigid. Let us show that this assumption leads to a contradiction. It is follows from Corollary \ref{cor:log_pairs} that there is a movable log pair $(X, M_X)$ such that the set $\mathcal{CS}(X, M_X)$ is not empty, the boundary $M_X$ is effective, and the divisor $-(K_X + M_X)$ is ample. The latter simply means that $M_X \sim_{\mathbb{Q}} -rK_X$ for a positive rational number $r < 1$. Let $Z \subset X$ be an element of the set $\mathcal{CS}(X, M_X)$.

**Lemma 69.** The subvariety $Z \subset X$ is not a smooth point of the variety $X$. 


Proof. Suppose that $Z$ is a smooth point of $X$. Let $H_1, \ldots, H_{2n-2}$ be sufficiently general divisors in the linear system $|\pi^*(O_{\mathbb{P}^{2n}}(1))|$ such that each $H_i$ passes through $Z$. Then

$$3 > M_X^2 \cdot H_1 \cdots H_{2n-2} \geq \text{mult}_Z(M_X^2) \text{mult}_Z(H_1) \cdots \text{mult}_Z(H_{2n-2}) > 4,$$

because $\text{mult}_Z(M_X^2) > 4$ by Theorem 54, which is a contradiction. \hfill $\square$

**Lemma 70.** The subvariety $Z \subset X$ is not a singular point of the variety $X$.

Proof. Suppose that $Z \subset X$ is a singular point of the variety $X$. Then $\pi(Z)$ is a singular point of the hypersurface $S \subset \mathbb{P}^{2n}$. Let $\alpha : V \to X$ be a usual blow up $Z$ and $G \subset V$ be an $\alpha$-exceptional divisor. Then $V$ is smooth and $G$ is a quadric of dimension $2n-1$ having a single singular point $O \in G$. Namely, the variety $G \subset V$ is a quadric cone with the vertex $O \in V$.

Let $M_V = \alpha^{-1}(M_X)$, and $\text{mult}_Z(M_X)$ be a rational number such that the equivalence

$$M_V \sim_{\mathbb{Q}} \alpha^*(M_X) - \text{mult}_Z(M_X)G$$

holds. Then $\text{mult}_Z(M_X) > 1$ by Proposition 60.

Put $H = \alpha^*(-K_X)$ and consider the linear system $|H - G|$. By construction, the rational map $\phi_{|H - G|}$ that is given by the linear system $|H - G|$ is the rational map

$$\gamma \circ \pi \circ \alpha : V \dashrightarrow \mathbb{P}^{2n-1},$$

where $\gamma : \mathbb{P}^{2n} \dashrightarrow \mathbb{P}^{2n-1}$ is a projection from the point $\pi(Z)$. The base locus of the linear system $|H - G|$ is not empty. Namely, its base locus consists of the vertex $O$ of the quadric cone $G$. Moreover, blowing up the point $O$ we resolve the indeterminacy of the rational map $\phi_{|H - G|}$, and the proper transform of the quadric cone $G$ is contracted by $\phi_{|H - G|}$ into the smooth quadric of dimension $2n-2$.

It should be pointed out that instead of blowing up the points $Z$ and $O$ we can resolve the indeterminacy of the rational map $\gamma \circ \pi : X \dashrightarrow \mathbb{P}^{2n-1}$ by a single weighted blow up $\beta : U \to X \subset \mathbb{P}(1^{2n+1}, n)$

of the point $Z$ with weights $(2, 3^{2n})$ in the corresponding local coordinates. The weighted blow up $\beta : U \to X$ can be described as a composition of three rational maps: the blow up $\alpha$, the blow up of the point $O$, and the consequent contraction of the proper transform of the quadric cone $G$. The exceptional divisor of $\beta$ is isomorphic to $\mathbb{P}^{2n-1}$, and it is a section of the fibration $\gamma \circ \pi \circ \beta : U \to \mathbb{P}^{2n-1}$. However, the variety $U$ is singular, namely, the variety $U$ has log terminal quotient singularities (see [164]) of type $1/3(1, 1)$ along the image of the quadric cone $G$ on the variety $U$.

Let $C$ be a general curve that is contained in the fibers of $\phi_{|H - G|}$. Then $C$ is irreducible and reduced, the curve $\pi \circ \alpha(C)$ is a line passing through the point $\pi(Z)$. Moreover, we have $C \cdot G = 2$, $C \cdot (H - G) = 1$, and $O \in C$. Intersecting the boundary $M_V$ with the curve $C$, we obtain the inequalities

$$1 > 3 - 2\text{mult}_Z(M_X) > M_V \cdot C \geq \text{mult}_O(M_V),$$

which imply $\text{mult}_Z(M_X) \leq \frac{3}{2}$ and $\text{mult}_O(M_V) < 1$. The equivalence

$$K_V + M_V \sim_{\mathbb{Q}} \alpha^*(K_X + M_X) + (2n - 2 - \text{mult}_Z(M_X))G$$

and $\text{mult}_Z(M_X) \leq \frac{3}{2}$ imply the existence of a proper subvariety $Y \subset G$ such that

$$Y \in \text{CS}(V, M_V - (2n - 2 - \text{mult}_Z(M_X))G).$$

The dimension of $Y$ does not exceed $2n - 2$, $\text{mult}_V(M_V) > 1$, and $Y \in \text{CS}(V, M_V)$. 21
Let $\dim(Y) = 2n - 2$. In the case when $O \in Y$ we have
\[ 1 > \text{mult}_O(M_V) \geq \text{mult}_Y(M_V) > 1, \]
which is impossible. Thus $O \notin Y$. Let $L$ be a general ruling of the cone $G$. Then
\[ \frac{3}{2} \geq \text{mult}_Z(M_X) = M_V \cdot L \geq \text{mult}_Y(M_V)L \cdot Y, \]
where $L \cdot Y$ is an intersection on $G$. Hence $L \cdot Y = 1$ and $Y$ is a hyperplane section of the quadric cone $G$ under the natural embedding $G \subset \mathbb{P}^{2n}$. Note, that we have
\[ Y \in \mathcal{LCS}(V, M_V - (2n - 3 - \text{mult}_Z(M_X))G), \]
and we can apply Theorem 52 to the log pair $(V, M_V - (2n - 3 - \text{mult}_Z(M_X))G)$ and to the subvariety $Y \subset V$ of codimension 2. This gives the inequality
\[ \text{mult}_Y(M^2_V) \geq (2n - 2 - \text{mult}_Z(M_X)) \geq 2, \]
because $\text{mult}_Z(M_X) \leq \frac{3}{2}$ and $n \geq 2$. Let $H_1, \ldots, H_{2n-2}$ be sufficiently general divisors in the linear system $|H - G|$. Then the inequalities
\[ 1 > 3 - 2\text{mult}_Z^2(M_X) > H_1 \cdot H_2 \cdots H_{2n-2} \cdot M^2_V \geq \text{mult}_Y(M^2_V)(H - G)^{2n-2} \cdot Y \geq 2 \]
hold, which is a contradiction.

Therefore $\dim(Y) < 2n - 2$. The inequality $\text{mult}_O(M_V) < 1$ implies that $O$ is not contained in $Y$. Let $P$ be a general point $P \in Y$. Then $\text{mult}_P(M^2_V) > 4$ by Theorem 54.

Let $D \subset |H - G|$ be a linear subsystem consisting of divisors that passes through the point $P$. The base locus of the linear system $D$ consists of 2 curves. The first one is a ruling $L_P$ of a quadric cone $G$ such that $P \in L_P$. The second one is a possibly reducible curve $C_P$ such that $\pi \circ \alpha(C_P) \subset \mathbb{P}^{2n}$ is a line that passes through the point $\pi(Z)$.

The line $\pi \circ \alpha(C_P)$ gives a point in the projectivization of the tangent cone of the hypersurface $S$ at the point $\pi(Z)$ that corresponds to the image of the point $\zeta(P)$, where $\zeta$ is a projection of the cone $G$ to its base. Note that the base of the cone $G$ is canonically isomorphic to the projectivization of the tangent cone to $S$ at $\pi(Z)$.

Let $D_1, \ldots, D_{2n-2}$ be general divisors in $D$, and $T$ be a one-cycle $H_1 \cdots H_{2n-3} \cdot M^2_V$ on the variety $V$. Then $T$ is an effective and $\text{mult}_P(T) > 4$. Unfortunately, we are unable to intersect properly the cycle $T$ with the remaining divisor $H_{2n-2}$, because $H_{2n-2}$ may contain components of the effective one-dimensional cycle $T$. Namely, $H_{2n-2}$ may contain either the curve $L_P$ or components of the the curve $C_P$ in the case if some of them are contained in $\text{Supp}(T)$.

Suppose that the curve $C_P$ is irreducible. Then
\[ T = \mu L_P + \lambda C_P + \Gamma, \]
where $\mu$ and $\lambda$ are nonnegative rational numbers, and $\Gamma$ is an effective one-dimensional cycle whose support does not contain the curves $L_P \cup C_P$. Then
\[ \text{mult}_P(\Gamma) > 4 - \text{mult}_P(L_P)\mu - \text{mult}_P(C_P)\lambda = 4 - \mu - \text{mult}_P(C_P)\lambda \geq 4 - \mu - 3\lambda, \]
because $\text{mult}_P(C_P) \leq 3$, which is implied by the fact that $C_P$ is a triple cover of a line that is blown up in a possible singular point. Intersecting the effective cycle $\Gamma$ with the divisor $H_{2n-2}$, we obtain the inequalities
\[ 3 - 2\text{mult}_Z^2(M_X) - \mu > \Gamma \cdot H_{2n-2} \geq \text{mult}_P(\Gamma) > 4 - \mu - 3\lambda, \]

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because $C_P \cdot H_{2n-2} = 0$. Therefore $\lambda > 1$. Intersecting the cycle $T$ with a sufficiently general divisor $H$ of the free linear system $|\alpha^*(-K_X)|$, we immediately obtain a contradiction, because $H \cdot C_P = 3$ and $H \cdot T < 3$.

Hence the curve $C_P$ is reducible. However, the triple cover $\pi$ is cyclic, which implies

$$C_P = C_1 + C_2 + C_3,$$

where $C_i$ is a nonsingular rational curve such that $\pi \circ \alpha(C_P)$ is a line, the restriction morphism $\pi \circ \alpha|_{C_i}$ is an isomorphism, $-K_X \cdot \alpha(C_i) = 1$, and $C_i \neq C_j$ if $i \neq j$. Put

$$T = \mu L_P + \sum_{i=1}^{3} \lambda_i C_i + \Gamma,$$

where $\mu$ and $\lambda_i$ are nonnegative rational numbers, and $\Gamma$ is an effective one-dimensional cycle whose support does not contain the curves $L_P$ and $C_i$. As in the case of the irreducible curve $C_P$ we can intersect properly the cycle $\Gamma$ with the divisor $H_{2n-2}$, which immediately implies the inequality $\sum_{i=1}^{3} \lambda_i > 1$. Intersecting the cycle $T$ with a general divisor $H$ in $|\alpha^*(-K_X)|$, we obtain a contradiction, because $H \cdot C_i = 1$ and $H \cdot T < 3$. □

**Lemma 71.** The inequality $\text{codim}(Z \subset X) > 2$ is impossible.

*Proof.* Suppose that $\text{codim}(Z \subset X) > 2$. Then $\dim(Z) \neq 0$ by Lemmas 69 and 70 and

$$\text{mult}_Z(M_X^2) \geq 4$$

by Theorem 52. Let $O$ be a general point on $Z$, and let $H_1, \ldots, H_{2n-2}$ be sufficiently general divisors in $|−K_X|$ such that each $H_i$ passes through the point $O$. Then

$$3 > M_X^2 \cdot H_1 \cdots H_{2n-2} \geq \text{mult}_Z(M_X^2) \geq 4,$$

which is impossible. □

Therefore we proved that $\text{codim}(Z \subset X) = 2$.

**Lemma 72.** The inequality $K_X^{2n-2} \cdot Z \leq 2$ holds.

*Proof.* The inequality $K_X^{2n-2} \cdot Z \leq 2$ easily follows from the equality $K_X^{2n} = 3$, the amenability of the divisor $−(K_X + M_X)$, and the inequality $\text{mult}_Z(M_X) \geq 1$. □

**Lemma 73.** The equality $n = 2$ holds, namely, $\dim(X) = 4$.

*Proof.* Suppose that $n > 2$. Let $V$ be a general divisor in $|−K_X|$. Then $V$ is a smooth hypersurface of degree $3n$ and of dimension $2n-1 \geq 5$ in $\mathbb{P}(1^{2n}, n)$. Hence the cohomology group $H_{4n-6}(V, \mathbb{C})$ is one-dimensional (see [170], Theorem 7.2 in [91], and §4 in [61]).

Let us show that the subvariety

$$Y = Z \cap V \subset V$$

of dimension $2n-3$ can not generate the group $H_{4n-6}(V, \mathbb{C})$. Let $Y \equiv \lambda D^2$ in $H_{4n-6}(V, \mathbb{C})$ for some $\lambda \in \mathbb{C}$, where $D = −K_X|_V$. The image $\pi(Z) \subset \mathbb{P}^{2n}$ is either a linear subspace of dimension $2n-2$ or a quadric of dimension $2n-2$ by Lemma 72. In particular, applying the Lefschetz theorem to a smooth hyperplane section of $S$, we see that $\pi(Z) \not\subset S$.

The subvariety $\pi^{-1}(\pi(Z))$ splits into three irreducible subvarieties, which are conjugated by the action of the group $\mathbb{Z}_3$ on the variety $X$ that interchanges the fibers of the triple cover $\pi$. Therefore $\lambda = \frac{2}{3}$, where $\alpha = K_X^{2n-2} \cdot Z = 1$ or 2 by Lemma 72. The equality

$$\alpha = Y \cdot D^{2n-3} = \lambda^{2-n} D \cdot Y^{n-2}$$

leads to a contradiction with the equality $\alpha = 1$ or 2.
applied Proposition 5 in [158] or Proposition 4.4 in [72] to \(S\) implies but

The surface Lemma 74.

When we vary the point Proof. In the smooth case the claim is trivial due to the Lefschetz theorem.

Let \(V \subset X\) be a general divisor in the linear system \(| - K_X|\). Then the induced morphism \(\tau = \pi|_V : V \to \mathbb{P}^3\) is a cyclic triple cover branched over a smooth hypersurface \(F = S \cap \pi(V) \subset \mathbb{P}^3\) of degree 6. Put \(M_V = M_X|_V\) and \(C = Z \cap V\). Then the boundary \(M_V\) is movable and effective, the curve \(C\) is smooth and rational, the curve \(\tau(C)\) is either a line or a conic, and the restriction morphism \(\tau|_C\) is birational. Moreover, the inequality \(\text{mult}_C(M_V) \geq 1\) and the equivalence \(M_V \sim Q \tau H\) hold, where \(H \sim \tau^*(O_{\mathbb{P}^3}(1))\) and \(r \in \mathbb{Q} \cap (0, 1)\).

Suppose that \(\tau(C) \subset F\). Let us show that the latter assumption leads to a contradiction. Let \(O\) be a point on \(C\). Put \(P = \tau(O) \in \tau(C)\). Let \(T \subset \mathbb{P}^3\) be a hyperplane that tangents the hypersurface \(F\) at the point \(P\). Then the curve \(Y = T \cap F\) is singular at the point \(P\). In the case when the multiplicity of the curve \(Y\) in the point \(P\) is 2, let \(L\) be a line in \(T\) that passes through the point \(O\) in the direction corresponding to any point in the projectivization of the tangent cone to the curve \(T\) at the point \(P\). In the case when the multiplicity of the curve \(Y\) in the point \(P\) is greater than 2, let \(L\) be any line in \(T\) that passes through the point \(P\). By construction, the line \(L\) tangents \(F\) such that the multiplicity of the tangency is at least 3.

Let \(L = \tau^{-1}(L)\). Then \(\text{mult}_O(L) = 3\). Intersecting the curve \(L\) with a movable boundary \(M_V\), we immediately obtain the following: at least one of the irreducible components of the curve \(L\) is contained in the base locus of one of the component of the movable boundary \(M_V\). However, the latter is impossible in the case when the line \(L\) spans at least a divisor in \(\mathbb{P}^3\) when we vary the point \(O\) on \(C\). Let us show that the line \(L\) always spans at least a divisor in \(\mathbb{P}^3\) when we vary the point \(O\) on the curve \(C\), which concludes the proof.

It should be pointed out that the hyperplane \(T\) tangents the hypersurface \(F\) in finite number of points. This is easily implied either by the Zak theorem on the finiteness of the Gauss map (see [73, 22, 201]) or by Theorem 2 in [148] (see Lemma 3.18 in [33]).

Suppose that \(\tau(C)\) is a line. Then \(\tau(C) \subset Y \subset T\), and \(T\) spans a pencil of hyperplanes in \(\mathbb{P}^3\) that pass through \(\tau(C)\) when we vary \(O\) on \(C\). Put \(Y = \tau(C) \cup R\). In the case when the point \(O\) is sufficiently general, the curve \(R\) is smooth and intersects \(\tau(C)\) transversally by the Bertini theorem. In particular, we always can choose the line \(L\) different from the line \(\tau(C)\). Therefore different choices of the sufficiently general point \(O\) on \(C\) give different line \(L\). Hence the line \(L\) spans a divisor when we vary the point \(O\) on \(C\).

So \(\tau(C)\) is a conic. Then \(\tau(C) \not\subset Y\) when \(O\) is a general point on \(O\). On the other hand, the hyperplane \(T\) tangents \(\tau(C)\) in \(P\). Therefore, the hyperplane \(T\) intersects the conic \(\tau(C)\) just by the point \(P\) if the point \(O\) is general on \(O\). However, the line \(L\) passes through the point \(P\), and \(L\) is contained in the hyperplane \(T\). Thus the different choices of the sufficiently general point \(O\) give different line \(L\). Hence the line \(L\) spans a divisor when we vary the point \(O\) on the curve \(C\), which concludes the proof. □
Lemma 75. The surface $\pi(Z)$ is not a plane in $\mathbb{P}^4$.

Proof. Suppose that $\pi(Z)$ is a two-dimensional linear subspace of $\mathbb{P}^4$. Let us consider the reduction to a smooth 3-fold to get a contradiction as in the proof of Lemma 74.

Let $V \subset X$ be a general divisor in $|-K_X|$, and
$$\tau = \pi|_V : V \rightarrow \mathbb{P}^3$$
be an induced cyclic triple cover branched over a smooth hypersurface $F = S \cap \pi(V) \subset \mathbb{P}^3$ of degree 6. Put $M_V = M_X|_V$ and $C = Z \cap V$. Then $M_V$ is an effective movable boundary, the curve $\tau(C)$ is a line, the morphism $\tau|_C$ is biregular, the curve $\tau(C)$ is not contained in the hypersurface $F$, $\text{mult}_C(M_V) \geq 1$, and $M_V \sim_{\mathbb{Q}} rH$, where $H \sim \tau^*(O_{\mathbb{P}^3}(1))$ and $r$ is a positive rational number such that $r < 1$. The variety $V$ is a smooth Calabi-Yau variety, namely, the rational equivalence $K_V \sim 0$ holds.

Let $\mathcal{D} \subset |\tau^*(O_{\mathbb{P}^3}(1))|$ be a pencil consisting of surfaces passing through $C$. The base locus of the pencil $\mathcal{D}$ consists of the curve $C$ and 2 different curves $\hat{C}$ and $\check{C}$ such that
$$\tau(C) = \tau(\check{C}) = \tau(\hat{C}).$$

The curves $C$, $\hat{C}$ and $\check{C}$ are conjugated by the action of the group $\mathbb{Z}_3$ on $V$ that interchanges the fibers of the cyclic triple cover $\tau$.

Let $f : U \rightarrow V$ be a blow up of $C$ and $E = f^{-1}(C)$. Put $\mathcal{P} = f^{-1}(\mathcal{D})$. Then
$$\mathcal{P} \sim D - E,$$
where $D = (\tau \circ f)^*(O_{\mathbb{P}^3}(1))$. On the other hand, the base locus of the pencil $\mathcal{P}$ consists of proper transforms of the curves $\check{C}$ and $\hat{C}$ on the variety $U$. In particular, the proper transforms of the curves $\check{C}$ and $\hat{C}$ on $U$ are the only curves on the variety $U$ that has negative intersection with the divisor $D - E$. Therefore the divisor $2D - E$ is numerically effective on the variety $U$. In particular, the inequality
$$(2D - E) \cdot M_V^2 \geq 0$$
holds, where $M_U$ is a proper transform of the movable boundary $M_V$ on the variety $U$.

Now we calculate the intersection $(2D - E) \cdot M_U^2 \geq 0$ implicitly. Firstly, the equalities
$$D^3 = 3, D^2 \cdot E = 0, D \cdot E^2 = -1$$
hold. Secondly, the equalities
$$E^3 = -\deg(N_{C/V}) = K_V \cdot C + 2 - 2g(C) = 2$$
holds (see [91]). Thirdly, the equivalence $M_U \sim_{\mathbb{Q}} rD - \text{mult}_C(M_V)E$ holds. Hence
$$(2D - E) \cdot M_U^2 = 6r^2 - 2\text{mult}_C(M_V) - 2r\text{mult}_C(M_V) - 2\text{mult}_C^2(M_V),$$
which implies $(2D - E) \cdot M_U^2 < 0$, because $r < 1$ and $\text{mult}_C(M_V) \geq 1$.

\[\square\]

Lemma 76. The surface $\pi(Z)$ is not a quadric in $\mathbb{P}^4$.

Proof. Suppose that $\pi(Z)$ is an irreducible two-dimensional quadric in $\mathbb{P}^4$. Then we can obtain the contradiction in the same way as in the proof of Lemma 75. Let us show the small modifications that must be done to the proof of Lemma 75. We use the notation of the proof of Lemma 75.

Firstly, the curve $\tau(C)$ is a conic. Secondly, the base locus of the linear system $|2D - E|$ is contained in the union $\check{C} \cup \hat{C}$, because $|2D - E|$ contains proper transforms of quadric cones in $\mathbb{P}^3$ over the conic $\tau(C)$. However, the intersections of the proper transforms of the curves $\check{C}$ and $\hat{C}$ on $U$ with $2D - E$ are non-negative. In particular, the divisor $2D - E$ is
numerically effective as in the proof of Lemma 79. Thus the inequality $(2D - E) \cdot M_U^2 \geq 0$ holds. Thirdly, the equality $D \cdot E^2 = -2$ holds, but $E^3 = 2$. Fourthly, the equality

$$(2D - E) \cdot M_U^2 = 6r - 4\text{mult}_C(M_V) - 4r\text{mult}_C(M_V) - 2\text{mult}_C(M_V)$$

holds, which implies $(2D - E) \cdot M_U^2 < 0$, because $r < 1$ and $\text{mult}_C(M_V) \geq 1$. □

Therefore, Theorem 3 is proved.

6. The absence of elliptic structures.

In this section we prove Theorem 15. Let $\pi : X \to \mathbb{P}^{2n}$ be a cyclic triple cover branched over a hypersurface $S \subset \mathbb{P}^{2n}$ of degree $3n$ such that the only singularities of the hypersurface $S$ are ordinary double points, and $n \geq 2$. Then $X$ is a Fano variety, the singularities of the variety $X$ are terminal and $\mathbb{Q}$-factorial (see Lemma 68), and the equivalence

$$K_X \sim \pi^*(\mathcal{O}_{\mathbb{P}^{2n}}(-1))$$

holds. Suppose that there are birational map $\rho : \hat{X} \dasharrow X$ and morphism $\nu : \hat{X} \to W$ such that the generic fiber of $\nu$ is a smooth elliptic curve. Let us show that the latter assumption leads to a contradiction.

Let $D$ be a very ample divisor $D$ on $W$. Put $D = |\nu^*(D)|$, $M = \rho(D)$, and $M_X = \gamma M$, where $\gamma \in \mathbb{Q}$ such that $\gamma M_X \sim Q - K_X$. Then $\text{CS}(X, M_X) \neq \emptyset$ by Theorem 66.

Remark 77. It follows from the proof of Theorem 3 that the singularities of the movable log pair $(X, M_X)$ are canonical (see 64).

The claim of Theorem 15 is a limit of the claim of Theorem 3. Therefore we can repeat almost all steps of the proof of Theorem 3 under slightly weaker conditions. However, we must modify the proof the proof of Theorem 3 using the following property of $(X, M_X)$.

Remark 78. The linear system $M$ is not composed from a pencil.\(^9\)

Let $Z \subset X$ be an element of the set $\text{CS}(X, M_X)$.

Proposition 79. The equality $\text{codim}(Z \subset X) = 2$ holds.

Proof. The claim is implied by the proofs of Lemmas 69, 70, 71. \hfill □

Proposition 80. The equality $\text{mult}_Z(M_X) = 1$ holds.

Proof. The claim follows from Proposition 79 and Remarks 77 and 81. \hfill □

Lemma 81. The inequality $K_X^{2n-2} \cdot Z \leq 2$ holds.

Proof. The inequality

$$K_X^{2n-2} \cdot Z \leq 3$$

follows from $K_X^{2n} = 3$, $M_X \sim Q - K_X$ and $\text{mult}_Z(M_X) = 1$.

Suppose that $K_X^{2n-2} \cdot Z = 3$. Let us show that $K_X^{2n-2} \cdot Z = 3$ leads to a contradiction.

Taking an intersection of the cycle $M_X^2$ with $2n - 2$ sufficiently general divisors in the linear system $| - K_X|$, we see that

$$\text{Supp}(M_X^2) = Z,$$

where the equality $\text{Supp}(M_X^2) = Z$ does not depend on the choice of two different divisors in the linear system $M$ in the definition of the cycle $M_X^2$.

\(^9\)Namely, the inequality $\dim(\psi_M(X)) > 1$ holds.
Let \( P \in X \setminus Z \) be a sufficiently general point, and \( \mathcal{D} \subset \mathcal{M} \) be a linear system consisting of divisors passing through the point \( P \). Then the base locus of the linear system \( \mathcal{D} \) has codimension at least 2 in \( X \), because \( \mathcal{M} \) does not composed from a pencil. Thus
\[
D_1 \cap D_2 = Z
\]
in a set-theoretic sense, where \( D_1 \) and \( D_2 \) are sufficiently general divisors in the linear system \( \mathcal{D} \). Indeed, the divisors \( D_1 \) and \( D_2 \) are contained in the linear system \( \mathcal{M} \) and we have \( \text{Supp}(M_X^2) = Z \). On the other hand, by definition \( P \in D_1 \cap D_2 \) and \( P \notin Z \), which is a contradiction. \( \square \)

It should be pointed out that the proof of Lemma 76 requires that the following properties of the subvariety \( Z \) hold: \( \text{codim}(Z \subset X) = 2 \) and \( K_X^{2n-2} \cdot Z \leq 2 \).

**Corollary 82.** The equality \( n = 2 \) holds, namely, we have \( \dim(X) = 4 \).

We must reprove Lemmas 74 and 75 under the new conditions. We prove them using the canonicity of the movable log pair \( (X, M_X) \) and the fact that \( \mathcal{M} \) is not composed from a pencil. However, the proof of Lemma 76 is valid under the new conditions once we have the claims of Lemmas 74 and 75.

**Corollary 83.** The case \( \pi(Z) \not\subset S \) and \( K_X^2 \cdot Z = 2 \) is impossible.

Hence we must get rid of the following three cases:

- \( \pi(Z) \not\subset S \) and \( K_X^2 \cdot Z = 1 \);
- \( \pi(Z) \subset S \) and \( K_X^2 \cdot Z = 1 \);
- \( \pi(Z) \subset S \) and \( K_X^2 \cdot Z = 2 \).

**Lemma 84.** The case \( \pi(Z) \not\subset S \) and \( K_X^2 \cdot Z = 1 \) is impossible.

*Proof.* Suppose that \( \pi(Z) \not\subset S \) and \( K_X^2 \cdot Z = 1 \). Let us show that this assumption leads to a contradiction. The surface \( \pi(Z) \) is a two-dimensional linear subspace of \( \mathbb{P}^4 \), which is not contained in the hypersurface \( S \). The triple cover \( \pi \) is cyclic, which implies the existence of two different surfaces \( \tilde{Z} \) and \( \hat{Z} \) such that
\[
\pi(Z) = \pi(\tilde{Z}) = \pi(\hat{Z}),
\]
and three surfaces \( Z, \tilde{Z} \) and \( \hat{Z} \) are conjugate under the action of the group \( \mathbb{Z}_3 \) on the variety \( X \), that interchanges the fibers of \( \pi \).

Let \( V \subset X \) be a sufficiently general divisor in the linear system \( | - K_X| \) and
\[
\tau = \pi|_V : V \to \mathbb{P}^3
\]
be an induced cyclic triple cover. Then the triple cover \( \tau \) is branched over a smooth hypersurface \( F = S \cap \pi(V) \subset \mathbb{P}^3 \) of degree 6.

Let \( \mathcal{H} = \mathcal{M}|_V \) and \( M_V = M_X|_V = \gamma \mathcal{H} \). Then the base locus of the linear system \( \mathcal{H} \) has codimension at least 2 in \( V \), the equivalence
\[
M_V \sim_{\mathbb{Q}} \tau^*(\mathcal{O}_{\mathbb{P}^3}(1))
\]
holds. Moreover, the generality in the choice of \( V \) implies that \( \mathcal{H} \) is not composed from a pencil. Let \( C = Z \cap V \), \( \tilde{C} = \tilde{Z} \cap V \), and \( \hat{C} = \hat{Z} \cap V \). Then \( \text{mult}_C(M_V) = 1 \).

Let \( f : U \to V \) be a blow up of a smooth curve \( C \), and \( E \) be an exceptional divisor of the blow up \( f \). Put \( D = f^{-1}(\mathcal{H}) \) and \( M_U = f^{-1}(M_V) = \gamma D \). Then
\[
M_U \sim_{\mathbb{Q}} D - E,
\]
where $D = (\tau \circ f)\ast(O_{\mathbb{P}^3}(1))$. However, the base locus of the pencil $|D - E|$ consists of proper transforms of the curves $\tilde{C}$ and $\hat{C}$ on the variety $U$. Moreover, the equalities

$$(D - E) \cdot \tilde{C} = (D - E) \cdot \hat{C} = -1$$

holds. Therefore the proper transforms of the curves $\tilde{C}$ and $\hat{C}$ on the variety $U$ are the only curves on $U$ that have non-positive intersection with $2D - E$. In particular, the divisor $2D - E$ is numerically effective and the inequality $(2D - E) \cdot M^2_U \geq 0$ holds.

The intersection $(2D - E) \cdot M^2_U$ can be easily calculated (see the proof of Lemma 75), namely, the equalities

$$(2D - E) \cdot M^2_U = 6 - 2\text{mult}_C(M_V) - 2\text{mult}_C(M_V) - 2\text{mult}_C^2(M_V) = 0$$

hold. Thus $\text{Supp}(M^2_U)$ is contained in the curves $\tilde{C}$ and $\hat{C}$. This is simply means that for any two different divisors $H_1$ and $H_2$ in the linear system $D$, the intersection $H_1 \cap H_2$ is contained in the union $\tilde{C} \cup \hat{C}$ in a set-theoretic sense.

Let $P \in U \setminus (\tilde{C} \cup \hat{C})$ be a sufficiently general point, and $P \subset D$ be a linear subsystem of divisors passing through the point $P$. Then the linear system $P$ does not have base components, because $D$ is not composed from a pencil. Let $D_1$ and $D_2$ be two sufficiently general divisors in the linear system $P$. Then in a set-theoretic sense

$$P \in D_1 \cap D_2 \subset \tilde{C} \cup \hat{C},$$

because $D_i \in D$. The obtained contradiction concludes the proof. \hfill $\Box$

**Lemma 85.** The case $\pi(Z) \subset S$ and $K^2_X \cdot Z = 1$ is impossible.

**Proof.** Suppose that $\pi(Z) \subset S$ and $K^2_X \cdot Z = 1$. Then $\pi(Z)$ is a two-dimensional linear subspace of $\mathbb{P}^4$, which is contained in the hypersurface $S$. The Lefschetz theorem implies that the hypersurface $S$ is singular.

We use the reduction to a smooth 3-fold as in the proof of Lemma 84. Moreover, let us use the notations of the proof of Lemma 84, which can be used in this case with the only difference that the surfaces $Z, \tilde{Z}, \tilde{Z}$ are coincide under the new conditions, because the surface $Z$ is invariant under the action of $\mathbb{Z}_3$ on the variety $X$ that interchanges the fibers of $\pi$.

All steps of the proof of Lemma 84 remains valid under new conditions except the very last one. Namely, the numerical effectivity of the divisor $2D - E$ is not clear, but it can be proved analyzing the class of the divisor $E\vert_E$ in the Picard group of the $f$-exceptional surface $E \cong \mathbb{F}_k$. However, we prove the numerical effectivity of the divisor $2D - E$ using more geometric ideas.

Let us consider the pencil $|D - E|$ on the variety $U$. The base locus of $|D - E|$ consists of a curve $C \subset E$ such that $C$ is a section of the projection $f\vert_E : E \to C$. It should be pointed out that the curve $C \subset E$ is an infinitesimal analog of the curve $\tilde{C}$ in the proof of Lemma 84. Moreover, blowing up the curve $C$, we can obtain an infinitesimal analog of the curve $\tilde{C}$ in the proof of Lemma 84, but we do not need this.

Let $Y$ be a general surface $Y$ in the pencil $|D - E|$. Then $Y$ is singular. Let us describe the singularities of the surface $Y$. The surface

$$\tau \circ f(Y) \subset \mathbb{P}^3$$

is a plane passing through the line $\tau(C) \subset F$, where $F$ is a ramification surface of the cyclic triple cover $\tau : V \to \mathbb{P}^3$. In particular, the curve

$$\tau \circ f(Y) \cap F$$

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is reducible, it consists of two irreducible components: the line \( \tau(C) \) and a plane quintic curve \( R \). Moreover, the quintic \( R \) is smooth by the Bertini theorem, and \( R \) intersects the line \( \tau(C) \) transversally in 5 points. On the other hand, the morphism

\[
\tau|_{f(Y)} : f(Y) \to \tau \circ f(Y) \cong \mathbb{P}^2
\]

is a cyclic triple cover branched over a curve \( \tau(C) \cup R \). Therefore the singularities of the surface \( f(Y) \) are 5 singular points of type \( A_2 \) contained in \( C \). The birational morphism

\[
f|_Y : Y \to f(Y)
\]

partially resolves the singularities of \( f(Y) \). Namely, the surface \( Y \) has 5 ordinary double points, and each of them dominates the corresponding singular point of the surface \( f(Y) \).

Let \( M_Y = M_U|_Y \). Then the boundary \( M_Y \) may not be movable, because it may contain a multiple of the curve \( C \) as a fixed component. Hence we can put

\[
M_Y = \alpha C + \Gamma,
\]

where \( \alpha \in \mathbb{Q}_{>0} \) and \( \Gamma \) is a movable boundary on the surface \( Y \). On the other hand, we have \( M_Y \sim_{\mathbb{Q}} 2C \). Moreover, it follows from the subadjunction formula (see [119]) that

\[
C^2 = -3 + \deg(\text{Diff}_C(0)) = -3 + \frac{5}{2} < 0,
\]

which implies \( \alpha = 2 \) and \( \Gamma = \emptyset \). So for any general divisors \( D \in D \) and \( H \in |D - E| \), the intersection \( D \cap H \) is contained in the curve \( C \) in a set-theoretic sense.

It should be pointed out that we used the following properties of the boundary \( M_X \) in the above arguments: \( M \) does not have fixed components, the equivalence \( M_X \sim_{\mathbb{Q}} -K_X \) holds, and \( \text{mult}_Z(M_X) = 1 \). In particular, we did not use the fact that \( M \) is not composed from a pencil. So we can repeat the previous arguments to any linear subsystem \( B \subset M \) such that \( B \) does not have fixed components. Indeed, the equivalence \( \gamma B \sim_{\mathbb{Q}} -K_X \) and the inequality \( \text{mult}_Z(\gamma B) \geq 1 \) are obvious, and the proof of Theorem [19] implies the canonicity of the log pair \( (X, \gamma B) \), which implies the equality \( \text{mult}_Z(\gamma B) = 1 \). Therefore for any sufficiently general divisor \( B \) in any linear system \( B \subset M \) with no fixed components and a sufficiently general divisor \( H \in |D - E| \), the intersection \( B \cap f(H) \) is contained in the curve \( C \) in a set-theoretic sense.

Let \( P \) be a sufficiently general point in \( X \setminus C \), and \( B \subset D \) be a linear subsystem of divisors passing through the point \( P \). Then \( B \) does not have fixed components, because the linear system \( M \) is not composed from a pencil. Let \( B \) be a general divisor in the linear system \( B \), and \( H \) be a general divisor in \( |D - E| \). Then the intersection \( B \cap f(H) \) contains the point \( P \notin C \). Thus \( B \cap f(H) \) is not contained in the curve \( C \) in a set-theoretic sense, which is a contradiction.

\[ \square \]

**Lemma 86.** The case \( \pi(Z) \subset S \) and \( K_X^2 \cdot Z = 2 \) is impossible.

**Proof.** Suppose that \( \pi(Z) \subset S \) and \( K_X^2 \cdot Z = 2 \). Let us show that this assumption leads to a contradiction. The surface \( \pi(Z) \) is a two-dimensional quadric in \( \mathbb{P}^4 \), which is contained in the sextic \( S \), which implies that \( Z \) is invariant under the action of the group \( \mathbb{Z}_3 \) on the variety \( X \) that interchanges the fibers of \( \pi \), because \( \pi(Z) \subset S \). The Lefschetz theorem implies that the hypersurface \( S \) is singular.

We reduce the problem to a smooth 3-fold. Let \( V \subset X \) be a sufficiently general divisor in the linear system \( |-K_X| \), and \( \tau = \pi|_V : V \to \mathbb{P}^3 \) be the induced cyclic triple cover branched over the smooth hypersurface

\[
F = S \cap \pi(V) \subset \mathbb{P}^3
\]
of degree 6. Put \( M_V = M_X|_V \). Then \( M_V \) is a movable boundary, the equivalence
\[
M_V \sim_{\mathbb{Q}} \tau^*(\mathcal{O}_{\mathbb{P}^3}(1))
\]
holds, and \( \text{mult}_C(M_V) = 1 \), where \( C = Z \cap V \). The curve \( \tau(C) \subset F \) is a smooth conic.

Let \( f : U \to V \) be a blow up of \( C \), and \( E = f^{-1}(C) \). Put \( M_U = f^{-1}(M_V) \). Then
\[
M_U \sim_{\mathbb{Q}} D - E,
\]
where \( D = (\tau \circ f)^*(\mathcal{O}_{\mathbb{P}^3}(1)) \). In the case when the divisor \( 2D - E \) is numerically effective, the explicit calculation of the intersection \( (2D - E) \cdot M_U^2 \geq 0 \) leads to a contradiction in the same way as in the proof of Lemma 68. So we may assume that \( 2D - E \) is not numerically effective.

The base locus of \( 2D - E \) is contained in \( E \), because \( (2D - E) \cdot s_{\infty} \geq 0 \), where \( s_{\infty} \) is an exceptional section of the ruled surface \( E \equiv \mathbb{P}k \), which concludes the proof. The curve \( C \) is smooth and \( C \cong \mathbb{P}^1 \). Hence
\[
\mathcal{N}_{C/V} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)
\]
for integers \( a \) and \( b \) such that \( b \geq a \). Then \( k = b - a \) and the equalities
\[
a + b = \deg(\mathcal{N}_{C/V}) = 2g(C) - 2 - K_V \cdot C = -2
\]
and \( E^3 = -\deg(\mathcal{N}_{C/V}) = 2 \) holds. On the other hand, the smooth curve \( C \) is contained in the smooth surface \( \tilde{F} = \tau^{-1}(F) \). Thus the sequence of sheaves
\[
0 \to \mathcal{N}_{C/F} \to \mathcal{N}_{C/V} \to \mathcal{N}_{\tilde{F}/V} \to 0
\]
is exact, where \( \mathcal{N}_{C/F} \cong \mathcal{O}_{\mathbb{P}^1}(-6) \), because \( C^2 = -6 \) on the surface \( \tilde{F} \cong F \) by the adjunction formula. Hence \( a \geq -6 \). Let \( l \subset E \) be a fiber of the projection \( f|_E \). Then
\[
-E|_E \sim s_{\infty} + rl
\]
for \( r = \frac{2 + k}{2} \), because the equalities
\[
2 = E^3 = (s_{\infty} + rl)^2 = -k + 2r,
\]
holds. So we have
\[
(2D - E) \cdot s_{\infty} = 4 - E \cdot s_{\infty} = 4 + (s_{\infty} + \frac{2 + k}{2}l) \cdot s_{\infty} = 4 - k + \frac{2 + k}{2} = \frac{10 - k}{2} = 6 + a \geq 0,
\]
which concludes the proof. \( \square \)

Therefore Theorem 13 is proved.

7. The proof of Theorems 18 and 20

In this section we prove Theorems 18 and 20. Let \( \pi : X \to \mathbb{P}^{2n} \) be a cyclic triple cover branched over a hypersurface \( S \subset \mathbb{P}^{2n} \) of degree \( 3n \) such that \( n \geq 2 \), and the only singularities of \( S \) are isolated ordinary double and triple points. Namely, the projectivization of the tangent cone to the hypersurface \( S \) at any singular point \( P \) of \( S \) is a smooth hypersurface in \( \mathbb{P}^{2n-1} \) of degree \( \text{mult}_P(S) \leq 3 \).

Remark 87. The proof of Lemma 68 implies that the groups \( \text{Pic}(X) \) and \( \text{Cl}(X) \) are generated by the divisor \(-K_X\), because the singularities of \( X \) are isolated.
Hence $X$ is a Fano variety with terminal $\mathbb{Q}$-factorial singularities. We must prove the following three results:

- the variety $X$ is birationally superrigid;
- the variety $X$ is not birationally equivalent to any Fano variety with canonical singularities that is not biregular to $X$;
- if the variety $X$ is birational to an elliptic fibration, then $n = 2$, the hypersurface $S$ has a triple point $O$ such that the elliptic fibration is induced by the projection $\gamma : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ from the point $O$.

Suppose that at least one of the above three claims is not true. Then there is a linear system $\mathcal{M}$ on the variety $X$ that satisfies the following properties:

- the linear system $\mathcal{M}$ does not have fixed components;
- the set $\mathcal{CS}(X, \frac{1}{d}\mathcal{M})$ is not empty, where $d \in \mathbb{N}$ such that $\mathcal{M} \sim -dK_X$;
- in the case when $n = 2$, for any point $O \in S$ such that $\text{mult}_O(S) = 3$,
  - the linear system $\mathcal{M}$ is not contained in the fibers of the rational map $\gamma \circ \pi$, where $\gamma : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ is a projection from the point $O$.

The existence of $\mathcal{M}$ follows from Theorems 66 and 67 and the proof of Theorem 64.

Let us show that the linear system $\mathcal{M}$ does not exist. Let $Z \subset X$ be a subvariety such that $Z \in \mathcal{CS}(X, \frac{1}{7}\mathcal{M})$. Then the proof of Theorems 3 and 15 implies that $Z$ is a singular point of the variety $X$ such that $O = \pi(Z)$ is an ordinary triple point of $S$.

Remark 88. The point $Z$ is an ordinary triple point of the variety $X$.

Let $\alpha : V \to X$ be a blow up of the point $O$, and $E = \alpha^{-1}(O)$. Then $E$ is a smooth hypersurface of degree $3$ in $\mathbb{P}^{2n}$, and $E|_E \sim H$, where $H$ is a hyperplane section of the hypersurface $E \subset \mathbb{P}^{2n}$. Moreover, the linear system

$$|\alpha^*(-K_X) - E|$$

is free and gives a morphism $\psi : V \to \mathbb{P}^{2n-1}$ such that $\psi = \gamma \circ \pi \circ \alpha$, where $\gamma : \mathbb{P}^{2n} \dashrightarrow \mathbb{P}^{2n-1}$ is a projection from the point $O$. Let $\text{mult}_Z(\mathcal{M})$ be an integer number such that

$$\mathcal{D} \sim \alpha^*(-dK_X) - \text{mult}_Z(\mathcal{M})E,$$

where $\mathcal{D}$ is a proper transform of the linear system $\mathcal{M}$ on the variety $V$. Let $C \subset V$ be a sufficiently general curve in a fiber of $\psi$. Then

$$\mathcal{D} \cdot C = 3(d - \text{mult}_Z(\mathcal{M})) \geq 0,$$

and the equality $\mathcal{D} \cdot C = 0$ implies that $\mathcal{D}$ is contained in the fibers of $\psi$. On the other hand, the inequality $\text{mult}_Z(\mathcal{M}) > d$ holds when $n > 2$ and the inequality $\text{mult}_Z(\mathcal{M}) \geq d$ holds when $n = 2$ by Theorem 61. Hence $n = 2$ and the linear system $\mathcal{M}$ is contained in the fibers of the rational map $\gamma \circ \pi$, which contradicts to one of the properties of the linear system $\mathcal{M}$. Thus both Theorems 13 and 20 are proved.

8. Potential density.

In this section we prove Theorem 19. Let $\pi : X \to \mathbb{P}^4$ be a cyclic triple cover branched over a hypersurface $S \subset \mathbb{P}^4$ of degree $6$ such that the hypersurface $S$ is defined over a number field $\mathbb{F}$. Suppose that the hypersurface $S$ has an ordinary triple point $O$, and the hypersurface $S$ is smooth outside of the point $O$. Thus the equality $\text{mult}_O(S) = 3$ holds, and the projectivization of the tangent cone to the hypersurface $S$ at the point $O$ is a smooth cubic surface in $\mathbb{P}^3$. The point $O$ is defined over the field $\mathbb{F}$. 

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The variety \( X \) can be considered as a hypersurface
\[
y^3 = x_0^3 f_3(x_1, \ldots, x_4) + x_0^2 f_4(x_1, \ldots, x_4) + x_0 f_5(x_1, \ldots, x_4) + f_6(x_1, \ldots, x_4)
\]
in the weighted projective space \( \mathbb{P}(1^5, 2) \cong \text{Proj}(\mathbb{F}[x_0, \ldots, x_4, y]) \), where \( f_i \) is a homogeneous polynomial of degree \( i \). The cyclic triple cover \( \pi : X \rightarrow \mathbb{P}^4 \) is a restriction to the hypersurface \( X \) of the natural projection \( \mathbb{P}(1^5, 2) \rightarrow \mathbb{P}^4 \) that is induced by the natural embedding of the graded algebras \( \mathbb{F}[x_0, \ldots, x_4] \subseteq \mathbb{F}[x_0, \ldots, x_4, y] \). Moreover, the hypersurface \( S \subseteq \mathbb{P}^4 \) is given by the equation
\[
x_0^3 f_3(x_1, \ldots, x_4) + x_0^2 f_4(x_1, \ldots, x_4) + x_0 f_5(x_1, \ldots, x_4) + f_6(x_1, \ldots, x_4) = 0,
\]
where the coordinates of the singular point \( O \) are \( (1 : 0 : \cdots : 0) \).

**Remark 89.** The equation \( f_3(x_1, \ldots, x_4) = 0 \) defines a smooth cubic surface in \( \mathbb{P}^3 \), which is a projectivization of the tangent cone to \( S \) at the point \( O \). In particular, \( f_3 \) is irreducible.

Suppose that \( X \) satisfies the following generality conditions:

1. \( f_4 \) is not divisible by \( f_3 \);
2. \( f_3^2 - 3f_4f_6 \) and \( f_3^2 f_2^2 - 4f_3^3 f_6 - 4f_3 f_5 f_6 + 18f_3 f_4 f_5 f_6 - 27 f_3^2 f_6^2 \) are coprime.

**Remark 90.** The required generality conditions are satisfied in the case when the polynomial \( f_i \) are chosen sufficiently general. The geometrical meaning of the generality conditions are the following:

1. a sufficiently general line \( L \) in \( \mathbb{P}^4 \) that passes through the point \( O \) and that is contained in the tangent cone to the hypersurface \( S \) at the point \( O \) intersects the hypersurface \( S \) in two points that are different from \( O \);
2. there is at most one-dimensional family of curves \( C \subseteq X \) such that the singular point \( P = \pi^{-1}(O) \) of the variety \( X \) is contained in the curve \( C \) and \( -K_X \cdot C = 1 \).

We use the methods of [14], [84], [15] to prove the following result implying Theorem 19.

**Proposition 91.** Under the generality conditions, the rational points on \( X \) are potentially dense\(^\text{10}\), namely, the set of all \( \mathbb{K} \)-points of the variety \( X \) is Zariski dense in \( X \) for a finite extension of fields \( \mathbb{F} \subseteq \mathbb{K} \).

There are two ways of looking at the potential density of rational points. The optimistic point of view is the following: the potential density of rational points reflects the measure of how close a given variety to being rational. For example, the geometrical rationality obviously implies the potential density of rational points. From this point of view the claim of Theorem 19 is very natural, as well as the fact that we are unable to prove the potential density of rational points on many many rationally connected nonrational varieties. For example, it is unknown whether rational points are potentially dense on the generic quintic hypersurface in \( \mathbb{P}^5 \) or not (see [145], [27], [72]). The pessimistic point of view considers the potential density of rational points as a much weaker birational invariant. In particular, there is the following conjecture (see [84]).

**Conjecture 92.** Let \( V \) be a smooth variety such that \( V \) is defined over a number field, and \( -K_V \) is numerically effective. Then rational points on \( V \) are potentially dense.

Therefore from the point of view of Conjecture 92 the claim of Proposition 91 is just an illustration of a general principle. It is known that Conjecture 92 holds the following algebraic varieties: abelian varieties (see [88]), smooth Fano 3-folds except a double cover

\(^{10}\)To be precise we must say that \( \mathbb{F} \)-points are potentially dense on the variety \( X \).
of $\mathbb{P}^3$ ramified in a smooth sextic (see [14], [84]), smooth Enriques surfaces (see [13]), smooth elliptic K3 surfaces (see [15]), smooth K3 surfaces with an infinite group of automorphisms (see [15]), some symmetric products (see [89]). Therefore rational points are potentially dense on many varieties that are not rationally connected. However, it is unknown whether rational points are potentially dense on a generic double cover of $\mathbb{P}^2$ branched over a smooth quartic curve or not (see [14]).

Example 93. Let $C$ be a smooth connected curve such that the curve $C$ is defined over a number field, and $g(C) \geq 2$. Then rational points are not potentially dense on $C \times \mathbb{P}^k$ by the Faltings theorem (see [66], [67]).

It is natural to expect that the potential density of rational points reflects such birational properties of an algebraic variety as rational connectedness. However, even in the case of a smooth conic bundle $\zeta : V \to \mathbb{P}^n$ with sufficiently general and big discriminant it is not known whether rational points are potentially dense on $V$ or not in the case $n \geq 2$, but it is known that the potential density of rational points on $V$ is implied by the Schinzel conjecture for $\zeta : V \to \mathbb{P}^n$ (see [50]). The variety $V$ is nonrational (see [168], [169]) and it is expected that $V$ is not unirational. Perhaps, the potential density of rational points can be used to obtain an example of a rationally connected variety that is not unirational.

An example in [51] implies the following generalization of Conjecture 92.

Conjecture 94. Let $V$ be a smooth variety such that $V$ is defined over a number field, and the divisor $-K_V$ is not numerically effective. Then rational points on $V$ are potentially dense if there is no unramified finite morphism $f : U \to V$ such that there is a dominant rational map $g : U \dashrightarrow Z$, where $Z$ is a variety of general type of dimension $\dim(Z) > 0$.

It should be pointed out that both Conjectures 92 and 94 are logical negation of the following weak Lang conjecture, which is proved only for curves and subvarieties of abelian varieties (see [60], [67], [68]).

Conjecture 95. Let $V$ be a smooth variety of general type such that the variety $V$ is defined over a number field. Then rational points on $V$ are not potentially dense.

The claim of Theorem 19 must remain valid without any generality conditions. Moreover, in non-general case the proof of the potential density of rational points must be easier than in general case. The same can be said about the singularities. Namely, the proof of the potential density of rational points must become easier when the singularities become worse. However, there are exceptional extreme cases.

Example 96. Let $\chi : Y \to \mathbb{P}^4$ be a cyclic triple cover branched over a hypersurface $G$ of degree 6 such that $G$ is a union of 6 different hyperplanes defined over a number field $\mathbb{F}$ and passing through some two-dimensional linear subspace $\Pi \subset \mathbb{P}^4$. Then $Y$ is birational to the product $C \times \mathbb{P}^3$, where $C$ is a cyclic triple cover of $\mathbb{P}^1$ branched over 6 points, which are defined over the field $\mathbb{F}$. Then rational points on the variety $Y$ are not potentially dense, because $g(C) = 4$ (see Example 93).

Let us prove Proposition 91. The following result is due to [134].

Theorem 97. Let $\mathbb{F}$ be a number field. Then there is $n(\mathbb{F}) \in \mathbb{N}$ such that $n(\mathbb{F})$ depends only on the field $\mathbb{F}$, and the order of any torsion $\mathbb{F}$-point on any elliptic curve $C$ is bounded by $n(\mathbb{F})$, where $C$ is defined over $\mathbb{F}$.
Let $P = \pi^{-1}(O)$. Then $P$ is an ordinary triple point on $X$. Let $\alpha : U \to V$ be a blow up of $P$, and $E$ be an exceptional divisor of $\alpha$. Then $-K_U \sim \alpha^*(-K_X) - E$, the linear system $|-K_U|$ has no base points. Let

$$\psi : U \to \mathbb{P}^3$$

be a morphism that is given by $|-K_U|$. Then $\psi$ is an elliptic fibration such that $E$ is a three-section of $\psi$, and $\psi = \gamma \circ \pi$, where $\gamma : \mathbb{P}^4 \dasharrow \mathbb{P}^3$ is a projection from the point $O$.

**Remark 98.** The variety $E$ is a smooth cubic hypersurface in $\mathbb{P}^4$. The cubic $E$ is not rational over $\mathbb{C}$ (see [43]), but $E$ is unirational over $\mathbb{C}$ (see [132]). In particular, rational points on the variety $E$ are potentially dense.

Let $D$ be an intersection of two general divisors in $|−K_U|$. Then $D$ is a smooth elliptic surface. The restriction $\tau = \psi|_D : D \to \mathbb{P}^1$ is a canonical morphism of the surface $D$, namely, the equivalence $\tau^* = \psi^* |_{\mathbb{P}^4} (1)$ holds. The curve $Z = E \cap D$ is a smooth elliptic curve, and the restriction $\tau|_Z : Z \to \mathbb{P}^1$ is a cyclic triple cover branched over three points.

**Remark 99.** The proper transform on the variety $V$ of every irreducible component of any reducible fiber of the fibration $\tau$ is a rational curve whose intersection with the anticanonical divisor $−K_X$ is equal to 1. The generality conditions implies that there is no more than one-dimensional family of such curves on $V$. On the other hand, the generality in the choice of the surface $D$ in the fibers of $\psi$ and the equality $\dim(D \subset U) = 2$ imply that all fibers of the fibration $\tau$ are irreducible.

Let $F_1$, $F_2$, $F_3$ be fibers of $\tau$ that pass through the ramification points of the triple cover $\tau|_Z$. Then $F_i \neq F_j$ if $i \neq j$, because $X$ satisfies the generality conditions and the cubic 3-fold $E$ is smooth.

**Remark 100.** The surface $\pi \circ \alpha(D) = \Pi \subset \mathbb{P}^4$ is a sufficiently general two-dimensional linear subspace passing through the point $O$. The curve $\pi \circ \alpha(F_i) \subset \Pi$ is one of three curves that are cut on the plane $\Pi$ by the equation $f_3 = 0$. The line $\pi \circ \alpha(F_i)$ is different from the lines that are cut on $\Pi$ by the equation $f_4 = 0$. Indeed, the plane $\Pi$ is sufficiently general, but the polynomial $f_4$ is not divisible by the polynomial $f_3$ by assumption. Therefore the fiber $F_i$ is smooth in the point of intersection with the curve $Z$.

The restriction morphism $\alpha|_D$ contracts the elliptic curve $Z$ into the point $P$. The self-intersection of the curve $Z$ on the surface $D$ is $−3$. The restriction $\tau|_{\alpha(D)}$ is a cyclic triple cover of the plane $\Pi$ branched over a singular curve $\Pi \cap S$ of degree 6 whose singularities consist of the point, which is an ordinary triple point on the curve $\Pi \cap S$.

Let $H \subset D$ be a curve that is cut on the surface $D$ by a sufficiently general divisor in the linear system $|\alpha^*(-K_X)|$. The curve $H$ is smooth, the curve $H$ is a three-section of the elliptic fibration $\tau$, the equality $g(H) = 4$ holds. Moreover, the curve $\pi \circ \alpha(H) \subset \Pi$ is a line. Let $C_b$ be a fiber of the elliptic fibration $\tau : D \to \mathbb{P}^1$ over a point $b \in \mathbb{P}^1$. Then

$$H^2 = 3, H \cdot Z = C_b^2 = 0, Z^2 = −3, Z \cdot C_b = H \cdot C_b = 0$$

on the surface $D$.

**Lemma 101.** For a very general $\mathbb{C}$-point $b \in \mathbb{P}^1$ the equivalence

$$3np − nH|_{C_b} \not\sim 0$$

holds in $\text{Pic}(C_b)$ for every $n \in \mathbb{N}$, where $p$ is one of the points of $Z \cap C_b$. 

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**Proof.** Let \( T = Z \times_{P^1} D \) be a fiber product and

\[
\chi : T \rightarrow D
\]

be an induced morphism. Then \( \chi : T \rightarrow D \) is a cyclic triple cover branched over the curves \( F_1, F_2, F_3 \). In particular, the surface \( T \) is singular if and only if some fiber \( F_i \) is singular. However, the possible singularities of the surface \( T \) are easy to calculate in the case when we know the type of the singular fiber \( F_i \) of the elliptic fibration \( \tau \) (see \([9]\)).

The surface \( T \) is normal, and there is a well defined intersection form of Weil divisors on the surface \( T \) (see \([166]\)).

The fibration \( \tau \) induces an elliptic fibration \( \eta : T \rightarrow Z \) such that \( \eta \) is a Jacobian fibration of the fibration \( \tau \). Indeed, the curve \( \chi^{-1}(Z) \) splits into three irreducible components, which are interchanged by the action of the group \( \mathbb{Z}_3 \) on the surface \( T \) that interchanges the fibers of \( \chi \). Let \( \tilde{Z} \) be a component of the reducible curve \( \chi^{-1}(Z) \). Then \( \tilde{Z} \) is a section of the fibration \( \eta \), and \( \chi|_{\tilde{Z}} \) is an isomorphism.

Let \( \tilde{H} = \chi^{-1}(H) \) and \( L \) be a fiber of the fibration \( \eta \). Then the equalities

\[
\tilde{H}^2 = 9, \tilde{H} \cdot \tilde{Z} = L^2 = 0, \tilde{Z} \cdot L = 1, \tilde{H} \cdot L = 3
\]

hold on the surface \( T \). The curve \( \tilde{Z} \) is smooth and \( \tilde{Z} \subset T \setminus \text{Sing}(T) \), because the point of intersection \( F_i \cap Z \) is smooth on the fiber \( F_i \) (see Remark \([100]\)).

The self-intersection \( \tilde{Z}^2 \) on \( T \) can be calculated via the adjunction formula, namely, we have \( \tilde{Z}^2 = -9 \), because \( K_T \equiv 9L \).

It should be pointed out that in the case when the curve \( Z \) passes through the singular points of the surface \( T \) the self-intersection \( \tilde{Z}^2 \) can be calculated using the sub-adjunction formula with an appropriate different (see \([119]\)), which can be explicitly calculated for every type of singular point.

For every \( n \in \mathbb{N} \) we have

\[
3np - nH|_{C_b} \sim 0 \iff (3n\tilde{Z} - n\tilde{H})|_{L_a} \sim 0 \Rightarrow 3n\tilde{Z} - n\tilde{H} \equiv \Sigma,
\]

where \( C_b \) is a fiber of \( \tau \) over a very general \( \mathbb{C} \)-point \( b \in \mathbb{P}^1 \), \( p \) is one of the intersection points \( Z \cap C_b \), \( L_a \) is a fiber of \( \eta \) over a very general \( \mathbb{C} \)-point \( a \in Z \), and \( \Sigma \) is a divisor on the surface \( T \) such that \( \text{Supp}(\Sigma) \) is a union of the fibers of the elliptic fibration \( \eta \).

Note, that all fibers of \( \eta \) are irreducible, because all fibers of \( \tau \) are irreducible.

Suppose that the claim of the lemma is not true. Then the curves \( \tilde{Z}, \tilde{H}, L \) are linearly dependent in the group \( \text{Div}(T) \otimes \mathbb{Q} / \equiv \). However, the determinant of the matrix

\[
\begin{pmatrix}
\tilde{Z}^2 & \tilde{H} \cdot \tilde{Z} & L \cdot \tilde{Z} \\
\tilde{Z} \cdot \tilde{H} & \tilde{H}^2 & L \cdot \tilde{H} \\
L \cdot \tilde{Z} & L \cdot \tilde{H} & L^2
\end{pmatrix}
= 
\begin{pmatrix}
-9 & 0 & 1 \\
0 & 9 & 3 \\
1 & 3 & 0
\end{pmatrix}
\]

is \( 72 \neq 0 \), which contradicts to the linear dependence of the curves \( \tilde{Z}, \tilde{H}, L \). \qed

Now let us go from the surface \( D \) back to the variety \( U \). The generality in the choice of the surface \( D \) and Lemma \([101]\) imply that

\[
3np + \alpha^*(nK_X)|_{L_p} \not\sim 0
\]

in \( \text{Pic}(L_p) \) for a very general \( \mathbb{C} \)-point \( p \in E \) and all \( n \in \mathbb{N} \), where \( L_p \) is a fiber of the fibration \( \psi : U \rightarrow \mathbb{P}^3 \) over the point \( p \).

For every \( n \in \mathbb{N} \) let \( \Phi_n \subseteq E \) be a subset that is defined by the condition

\[
p \in \Phi_n \iff 3np \sim \alpha^*(-nK_X)|_{L_p}
\]

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in Pic($L_p$), where $L_p$ is a fiber of the elliptic fibration $\psi$ over the point $\psi(p)$ such that the fiber $L_p$ is smooth in a scheme-theoretic sense. Let $\Phi_n \subseteq E$ be a closure of the set $\Phi_n$ in the Zariski topology. Then $\Phi_n \neq E$ for every $n \in \mathbb{N}$.

**Remark 102.** The set $\Phi_n \cup \bigcup_{i=1}^{n-1} \Phi_i$ does not contain $\mathbb{F}$-points of the divisor $E$ for all natural numbers $n > n(\mathbb{F})$ by Theorem 97.

The rational points are potentially dense on the divisor $E$ (see Remark 98). Thus we can substitute $\mathbb{F}$ by its finite extension and assume that $\mathbb{F}$-points of $E$ are Zariski dense.

Take an $\mathbb{F}$-point

$$q \in E \setminus \left( \Delta \cup \bigcup_{i=1}^{n(\mathbb{F})} \Phi_i \right),$$

where $\Delta$ is a Zariski closed subset of the divisor $E$ consisting of points that are contained in the singular fibers of the elliptic fibration $\psi$. Let as before $L_q$ be a fiber of $\psi$ over the point $\psi(q)$. Then $L_q$ and $\psi(q)$ are defined over $\mathbb{F}$. Moreover, the curve $L_q$ is smooth.

By construction, the divisor $3q + \alpha^*(K_X)|_{L_q}$ is defined over the field $\mathbb{F}$ and it is not a torsion in Pic($L_q$). Therefore for any $n \in \mathbb{N}$ there is a unique $\mathbb{F}$-point $q_n \in L_b$ such that

$$q_n + (3n - 1)q + \alpha^*(nK_X)|_{L_q} \sim 0$$

in Pic($L_q$) by the Riemann–Roch theorem.

We have $q_i \neq q_j$ if $i \neq j$. Hence the curve $L_q$ is contained in the closure of all $\mathbb{F}$-points of the variety $U$ in the Zariski topology for every $\mathbb{F}$-point $q$ in a Zariski dense subset of the divisor $E$. Thus rational points are dense on the varieties $U$ and $X$. It should be pointed out that at certain point we substitute the field $\mathbb{F}$ by some its finite extension in order to get the density of $\mathbb{F}$-points on the divisor $E$. Hence Proposition 91 is proved.

During the proof of Proposition 91 we noticed that the surface $T$ is smooth if and only if each fiber $F_i$ of the elliptic fibration $\tau$ is smooth. It is natural to expect that the this is true for a sufficiently general $X$. Indeed, the smoothness of the fiber $F_i$ is implied by the fact that the line $\pi \circ \alpha(F_i)$ intersects the ramification hypersurface $S$ in three different points, one of which is the point $O$. The latter condition can be easily expressed in terms of the discriminant of the corresponding equation. Namely, it is enough to require that the polynomials $f_4$ and $f_5^2 - 4f_4f_6$ are not divisible by the irreducible polynomial $f_3$.

Suppose that the polynomials $f_4$ and $f_5^2 - 4f_4f_6$ are not divisible by the irreducible polynomial $f_3$. Then the divisor $E$ is a three-section of the elliptic fibration $\psi$ such that there is a smooth fiber $C$ of $\psi$ passing through one of the ramification points of the restriction triple cover $\psi|_E$. In the notations of the papers 13 and 14 such multi-section is called saliently ramified.

Let $C_b$ be a fiber of $\psi$ over a very general point $b \in \mathbb{P}^3$, $p_1$ and $p_2$ be two different points of $C_b \cap E$. Then $p_1 - p_2$ is not a torsion divisor on $C_b$. Indeed, otherwise the torsion divisor $p_1 - p_2$ goes to a trivial divisor on $C$ when we $C_b \to C$. This arguments can easily be put in algebraic form (see 13). Now we can prove the potential density of the rational points on $X$ in the same way as in the proof of Proposition 91, the only difference is the following: we must generate $\mathbb{F}$-points in the fibers of $\psi$ acting by the Jacobian fibration of $\psi$ without the usage the Riemann–Roch theorem (see 13).

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