CZF does not have the Existence Property

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Abstract

Constructive theories usually have interesting metamathematical properties where explicit witnesses can be extracted from proofs of existential sentences. For relational theories, probably the most natural of these is the existence property, EP, sometimes referred to as the set existence property. This states that whenever $(\exists x)\phi(x)$ is provable, there is a formula $\chi(x)$ such that $(\exists! x)\phi(x) \land \chi(x)$ is provable. It has been known since the 80’s that EP holds for some intuitionistic set theories and yet fails for IZF. Despite this, it has remained open until now whether EP holds for the most well known constructive set theory, CZF. In this paper we show that EP fails for CZF.

1 Introduction

1.1 Existence Properties

Constructive theories are known for having metamathematical properties that are often not shared by stronger classical theories such as ZFC. The principles below are amongst the most well known of these properties.

Constructive mathematicians choose to interpret disjunctions and existential quantifiers more strictly than classical mathematicians. For the constructive mathematician, in order to know the disjunction $\phi \lor \psi$, one must either know $\phi$ or know $\psi$. They therefore often expect their formal theories to have the following property:

**Definition 1.1.** A theory, $T$ has the disjunction property (DP) if whenever $T \vdash \phi \lor \psi$, either $T \vdash \phi$ or $T \vdash \psi$.

In order to know $(\exists x)\phi(x)$, the constructive mathematician must be able to “construct” some witness $a$ such that one knows $\phi(a)$. We certainly know what it means to construct an element of $\omega$: we must be able to write down an actual natural number. We also know what it means to construct a function $\mathbb{N} \to \mathbb{N}$: we must be able to able to find (a number encoding) an algorithm whose graph is that function. Hence the constructive mathematician expects their formal theories to have the following properties. In the definitions below we assume that $T$ has a constant $\omega$ such that $T$ proves that $\omega$ is the natural numbers and
for each $n$ a constant $\bar{n}$ such that $T$ proves $\bar{0}$ is empty and $\bar{n} + 1$ is the successor of $\bar{n}$. For any theory that could “reasonably” be called a set theory, there will be at least a conservative extension with this property.

**Definition 1.2.** $T$ has the *numerical existence property* (NEP) if whenever $T \vdash (\exists x \in \omega) \phi(x)$, there is some natural $n$ such that $T \vdash \phi(\bar{n})$.

**Definition 1.3.** $T$ is closed under *Church’s Rule* (CR) if whenever $T \vdash (\forall x \in \omega)(\exists y \in \omega) \phi(x, y)$, there is some natural $e$ such that $T \vdash (\forall x \in \omega) \phi(x, \{e\}(x))$ (where $\{e\}(x)$ denotes the result of applying the $e$th recursive function to $x$).

What it means to construct mathematical objects in general is less clear, but a common interpretation of this is that they should at least be definable, in the sense below.

**Definition 1.4.** $T$ has the *existence property* (EP) if whenever $T \vdash (\exists x) \phi(x)$, there is some formula $\chi(x)$ that only has free variable $x$ such that $T \vdash (\exists ! x) \phi(x) \land \chi(x)$.

### 1.2 IZF, CST, and CZF

If one wants a theory with some of the metamathematical properties appearing in section 1.1 but has no other objections to classical mathematics, one may be satisfied with the theory IZF, which can be regarded as “ZF without excluded middle.”

**Definition 1.5.** IZF is the theory with (intuitionistic logic and) the following axioms:

1. Extensionality
2. Separation
3. Pairing
4. Union
5. Infinity
6. Power Set
7. $\in$-induction
8. Collection
Collection is the following schema:

\[(\forall x \in a)(\exists y)\phi(x, y) \rightarrow (\exists z)(\forall x \in a)(\exists y \in z)\phi(x, y)\]

Compare this with the schema (equivalent in ZF) Replacement:

\[(\forall x \in a)(\exists! y)\phi(x, y) \rightarrow (\exists z)(\forall x \in a)(\exists y \in z)\phi(x, y)\]

**Definition 1.6.** \(\text{IZF}_R\) is the set theory with the axioms of \(\text{IZF}\) except that it has Replacement instead of Collection.

\(\text{IZF}\) is extremely powerful. In fact Friedman showed in [6] that it has the same consistency strength as \(\text{ZF}\). On the other hand, \(\text{IZF}\) has most of the existence properties we saw earlier.

Often one may be doing mathematics constructively for philosophical reasons. One may be an intuitionist: one believes mathematical objects only exist if they can be “mentally constructed.” One may be a predicativist: one believes that a mathematical object cannot be constructed until it is defined predicatively - that is without quantifiers whose range includes the object being constructed. In this case one needs to ensure that the axioms of the set theory are constructively justified. There are (at least) two ways to go about this:

1. Directly justify each axiom as “true” with philosophical reasoning
2. Find another theory that already has a strong constructive foundation and interpret your set theory into it

Myhill in [19] took the first approach, introducing the following theories. Both of these are over a three sorted language with sorts for numbers, sets, and partial functions.

**Definition 1.7.** \(\text{CST}^-\) is the theory with (intuitionistic logic and) the following axioms:

1. Extensionality (for sets)
2. Bounded Separation (that is, separation for formulae where every quantifier is bounded)
3. Pairing
4. Union
5. Exponentiation (that is, given any sets \(A\) and \(B\) there is a set containing precisely the functions \(f : A \rightarrow B\))
6. Replacement
7. Axioms of Heyting Arithmetic for the number sort

**Definition 1.8.** \(\text{CST}\) is the theory \(\text{CST}^-\) together with relativised dependent choices \(\text{RDC}\).
In particular Myhill rejected the power set axiom in favour of the weaker exponentiation axiom because of the more predicative nature of exponentiation. He chose bounded separation over full separation for the same reason.

CZF arose via the second approach in [1] where Aczel showed that set theory can be interpreted into the predicative Martin-Löf type theory. Aczel also dropped the three sorted approach of CST and defined the following theories over the same language as ZF.

**Definition 1.9.** CZF is the theory with (intuitionistic logic and) the following axioms

1. Extensionality
2. Bounded Separation
3. Pairing
4. Union
5. Strong Infinity
6. Subset Collection: the schema
   \[(\exists c)(\forall u)((\forall x \in a)(\exists y \in b)\psi(x, y, u) \rightarrow (\exists d \in c)((\forall x \in a)(\exists y \in d)\psi(x, y, u) \land (\forall y \in d)(\exists x \in a)\psi(x, y, u)))\]
7. \(\in\)-induction
8. Strong Collection: the schema
   \[(\forall x \in a)(\exists y)\phi(x, y) \rightarrow (\exists b)((\forall x \in a)(\exists y \in b)\phi(x, y) \land (\forall y \in b)(\exists x \in a)\phi(x, y))\]

Subset collection implies exponentiation and is implied by power set (see [2]) and can be seen as an “artefact” of the interpretation of set theory into type theory. As an alternative to subset collection, one may instead assume the equivalent fullness axiom. Given sets \(A\) and \(B\), define \(\text{mv}(A, B)\) to be the class of multivalued relations as

\[\text{mv}(A, B) := \{ R \subseteq A \times B \mid (\forall a \in A)(\exists b \in B)(a, b) \in R\}\]

The fullness axiom can then be stated as follows

\[(\forall A, B)(\exists C \subseteq \text{mv}(A, B))(\forall R \in \text{mv}(A, B) \rightarrow (\exists S \in C)(S \subseteq R)\]

(For a more detailed discussion of the fullness axiom see [2].)

One can see that the fullness axiom asserts the existence of sets for which there is no apparent definition. We will prove that for the case \(A = \mathbb{N}^{\mathbb{N}}, B = \mathbb{N}\), there is no definable \(C\).
CZF is stronger than CST in two respects: replacement has been strengthened to strong collection and exponentiation has been strengthened to subset collection.

CZF is regarded today as the standard set theory for formalising constructive mathematics. This is because it is constructively valid because of its interpretation into type theory and yet can be used to prove mathematically interesting results that do not hold in weaker theories. For example, in [15] Lubarsky and Rathjen showed that the theory CZF that has only exponentiation in place of subset collection does not prove that the Dedekind reals form a set.

1.3 Existence Properties of these Set Theories

The properties DP, NEP, and CR work extremely well as characterisations of constructive formal theories. None can hold for consistent recursively axiomatisable theories that have excluded middle, but on the other hand they hold for a rich variety of constructive theories.

In [18] Friedman and Myhill showed that IZF (that is, IZF with replacement instead of collection), has the existence property. In [19], Myhill showed the set theory CST also has EP and also that both CST and CST have DP and NEP, leaving open whether CST has EP. In [8] Friedman and Ščedrov showed that IZF + RDC has EP, establishing that even set theories with choice principles can have EP.

Beeson then developed q-realizability, allowing him to show in [3] that NEP, DP, and CR hold for IZF and IZF + RDC. Rathjen developed realizability with truth based partly on Beeson’s methods to show in [20] and [23] that DP, NEP, CR and other properties hold for a wide variety of intuitionistic set theories including CZF, CZF + REA, IZF, IZF + REA with any combination of the axioms MP, ACω, DC, RDC and PAx.

One can see that EP does not work so well as a characterisation of constructive theories as the other properties we have seen. As remarked in [20] EP can hold for classical theories, even extensions of ZFC. On the other hand, Friedman and Ščedrov showed in [21] that IZF does not have EP.

Friedman and Ščedrov’s proof that EP fails for IZF makes use of full separation and collection. Since IZF does have EP, it might seem reasonable to think that collection is responsible for the failure of EP and the use of full separation is only incidental. However due to recent work by Rathjen, this turns out not to be the case. Set theories with collection but only bounded separation can have EP.

In [24] Rathjen defined the following two variations on EP,

**Definition 1.10.** 1. T has the weak existence property, wEP, if whenever

\[ T \vdash (\exists x)\phi(x) \]
there is some formula $\chi(x)$ having at most the free variable $x$ such that

$$
T \vdash (\exists x)\chi(x)
$$

$$
T \vdash (\forall x)(\chi(x) \rightarrow (\exists u)u \in x)
$$

$$
T \vdash (\forall x)(\chi(x) \rightarrow (\forall u \in x)\phi(u))
$$

2. $T$ has the uniform weak existence property, uwEP, if whenever

$$
T \vdash (\forall u)(\exists x)\phi(u, x)
$$

there is some formula $\chi(u, x)$ having at most the free variables $u, x$ such that

$$
T \vdash (\forall u)(\exists x)\chi(u, x)
$$

$$
T \vdash (\forall u)(\forall x)(\chi(u, x) \rightarrow (\exists z)z \in x)
$$

$$
T \vdash (\forall u)(\forall x)(\chi(u, x) \rightarrow (\forall z \in x)\phi(u, z))
$$

As remarked in [24], by analysing Friedman and Ščedrov’s proof in [9] one can see that IZF doesn’t even have wEP. On the other hand any extension of ZF has uwEP - consider $V_\alpha$ where $\alpha$ is the least ordinal such that $V_\alpha$ contains a witness.

In [24], Rathjen refers to the theories CZF$^-$, CZF$\xi$ and CZF$\rho$. CZF$^-$ is CZF without subset collection. CZF$\xi$ is CZF$^-$ with the exponentiation axiom. CZF$\rho$ is CZF together with the power set axiom. All three of these theories have strong collection, and yet Rathjen shows in [24] that all three have uwEP (and hence wEP). In that paper he refers to a paper in preparation where he will show by using this result together with ordinal analysis that these three theories in fact have EP.

CZF$\rho$, which has EP, is simply IZF with bounded separation in place of full separation, so the use of full separation in Friedman and Ščedrov’s proof must be essential. Furthermore, CZF lies between CZF$\xi$ and CZF$\rho$, two theories both satisfying EP and uwEP.

However, due to problems defining witnesses for the fullness axiom, these proofs do not apply to CZF itself. Rathjen goes so far as to conjecture in [24] that CZF does not even have wEP. In this paper we prove that this conjecture is correct. CZF does not have wEP, and the fullness axiom is responsible.

1.4 Pcas

When defining realizability, one usually starts with a partial combinatory algebra (pca).

Definition 1.11. A pca, $A$ is a set $A$ together with a partial binary operation, $\cdot$ referred to as application, and distinguished elements, $s$ and $k$ such that,
1. \( s \neq k \)
2. for all \( a, b \in \mathcal{A} \), \( kab \simeq a \)
3. for all \( a, b \in \mathcal{A} \), \( sa \downarrow, sab \downarrow \)
4. for all \( a, b, c \in \mathcal{A} \), \( sabc \simeq ac(bc) \)

Recall the following from, for example [27] or [3].

**Definition 1.12.** Given a pca, \( \mathcal{A} \), we define terms over \( \mathcal{A} \) inductively as follows

1. There is a countable supply of free variables, \( x_i \), each of which is a term.
2. Each element, \( a \) of \( \mathcal{A} \) is a term.
3. If \( s \) and \( t \) are terms, then the ordered pair, \( \langle s, t \rangle \) is also a term. We write this as \( (s.t) \).

We say that a term is **closed** if it contains no free variables.

**Definition 1.13.** We define inductively what it means for a closed term, \( s \), to denote \( a \in \mathcal{A} \)

1. If \( a' \in \mathcal{A} \), then \( a' \) denotes \( a \) if and only if \( a = a' \)
2. \( (s'.s'') \) denotes \( a \) if and only if there are \( a', a'' \in \mathcal{A} \) such that \( s' \) denotes \( a' \), \( s'' \) denotes \( a'' \), and \( a'.a'' \simeq a \).

If \( t \) is a closed term and there is an \( a \in \mathcal{A} \) such that \( t \) denotes \( a \), we write \( t \downarrow \).

If \( t(x_1,\ldots,x_n) \) is an open term with free variables amongst \( x_1,\ldots,x_n \), we write \( t \downarrow \) to mean that for every \( a_1,\ldots,a_n \in \mathcal{A} \), \( t(a_1,\ldots,a_n) \downarrow \).

**Proposition 1.14.** For any term, \( t(x_1,\ldots,x_n) \), over \( \mathcal{A} \) with free variables \( x_1,\ldots,x_n \) there is a term \( t^* \) such that for all \( a_1,\ldots,a_n \in \mathcal{A} \), \( ta_1\ldots a_{n-1} \downarrow \) and

\[
t^*a_1\ldots a_n \simeq t(a_1,\ldots,a_n)
\]

We will write \( t^* \) as \( (\lambda x_1,\ldots,x_n)t(x_1,\ldots,x_n) \).

**Proposition 1.15.** For any \( \mathcal{A} \) there are \( y, y' \in \mathcal{A} \) such that for all \( f \in \mathcal{A} \),

1. \( yf \simeq f(yf) \)
2. \( y'f \downarrow \) and for all \( e \in \mathcal{A} \), \( (y'f)e \simeq f(y'f)e \)

One can use this to construct pairing and projection operators that we will refer to as \( p, p_0 \) and \( p_1 \). We will write \( (e)_i \) to mean \( p_ie \) for \( i = 0,1 \). One can further define numerals that we will denote \( n \) for each \( n \in \omega \). All recursive functions can then be represented. See chapter 6 of [3] or chapter 1 of [27] for details.
1.5 The Model $V(\mathcal{A})$

Realizability is one of the main tools in the study of intuitionistic theories and was used for many well known results including those mentioned in section 1.3.

This variant of realizability has its roots in [13], where Kriesel and Troelstra adapted Kleene’s realizability from [12] to work with second order arithmetic. This was later adapted and used by Friedman in [7], by Myhill in [18] and [19] and by Beeson in [4] and [9]. In [16] and [17], McCarty adapted Beeson’s definition to work for set theories with extensionality. The definition below is the variation introduced by Rathjen in [22], where bounded quantifiers are kept separate in the definition.

We start by constructing the class $V(\mathcal{A})$ inductively as follows

$$
V(\mathcal{A})_{\alpha+1} = P(|\mathcal{A}| \times V(\mathcal{A})_{\alpha})
$$

$$
V(\mathcal{A})_{\lambda} = \bigcup_{\beta < \lambda} V(\mathcal{A})_{\beta}
$$

$$
V(\mathcal{A}) = \bigcup_{\alpha \in \text{On}} V(\mathcal{A})_{\alpha}
$$

We then introduce a relation $\vdash$ between $\mathcal{A}$ and formulae with parameters over $V(\mathcal{A})$. The first two lines of the definition are defined simultaneously by $\in$-induction and the remaining lines allow one to inductively define realizability for any sentence, $\phi$.

$$
e \vdash a \in b \text{ iff } (\exists \langle e,0, c \rangle \in b) (e)_1 \vdash a = c$$

$$
e \vdash a = b \text{ iff } (\forall \langle f,c \rangle \in a)(e)_0 f \vdash c \in b \land (\forall \langle f,c \rangle \in b)(e)_1 f \vdash c \in a$$

$$
e \vdash \phi \land \psi \text{ iff } (e)_0 \vdash \phi \land (e)_1 \vdash \psi$$

$$
e \vdash \phi \lor \psi \text{ iff } ((e)_0 = 0 \land (e)_1 \vdash \phi) \lor ((e)_0 = 1 \land (e)_1 \vdash \psi)$$

$$
e \vdash \phi \rightarrow \psi \text{ iff } f \vdash \phi \text{ implies } e.f \vdash \psi$$

$$
e \vdash (\exists x \in a) \phi(x) \text{ iff } \exists \langle(e)_0,b \rangle \in a)(e)_1 \vdash \phi(b)$$

$$
e \vdash (\forall x \in a) \phi(x) \text{ iff } (\forall \langle f,b \rangle \in a) e.f \vdash \phi(b)$$

$$
e \vdash (\exists x) \phi(x) \text{ iff } (\exists x \in V(\mathcal{A})) e \vdash \phi(a)$$

$$
e \vdash (\forall x) \phi(x) \text{ iff } (\forall a \in V(\mathcal{A})) e \vdash \phi(a)$$

$$
e \vdash \neg \phi \text{ iff } \nexists f \text{ such that } f \vdash \phi$$

If $\phi$ has free variables amongst $x_1,\ldots,x_n$, we write $e \vdash \phi$ to mean $e \vdash (\forall x_1,\ldots,x_n) \phi$ (the universal closure of $\phi$).

We write $V(\mathcal{A}) \models \phi$ to mean that there is $e \in \mathcal{A}$ such that $e \vdash \phi$.

This structure has been defined so that we get soundness for IZF in the following sense.

**Theorem 1.16.** Suppose that $\phi$ is a theorem of IZF. Then, $V(\mathcal{A}) \models \phi$. 

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Recall from chapter 5 of [16] that \( V(A) \) has certain standard representations of the naturals and Baire space.

Define

\[
\pi = \{ (m, \bar{m}) \mid m < n \} \\
\omega = \{ (n, \bar{n}) \mid n \in \omega \}
\]

Then \( V(A) \) has a realizer for the statement that \( \omega \) is the set of natural numbers.

Suppose that \( f \in A \) satisfies that for all \( n \in \omega \), there exists \( m \in \omega \) such that \( f\bar{n} = \bar{m} \). Then write

\[
\mathcal{F} = \{ (n, (\pi, \bar{f}n)) \mid n \in \omega \}
\]

(where we write \((, )\) for \( V(A) \)'s internal notion of ordered pair).

There is a realizer in \( V(A) \) for the statement that the set of functions from \( \omega \) to \( \omega \) is precisely

\[
\{ (f, \bar{f}) \mid (\forall n \in \omega) (\exists m \in \omega) f\bar{n} = \bar{m} \}
\]

2 Outline of the Proof

We will show that wEP fails for CZF. We do this by first showing that for any pca, \( A \), we can construct three realizability models, \( V(A) \), \( V_0^{ip}(A) \), and \( V_0^{T}(A) \). \( V(A) \) is the usual realizability model of IZF from section 1.5. \( V_0^{ip}(A) \) and \( V_0^{T}(A) \) are based on injectively presented sets and symmetric sets respectively and will be described in detail below.

The heart of the proof is that \( V_0^{ip}(A) \) and \( V_0^{T}(A) \) are essentially different models and yet both can be embedded into \( V(A) \) in such a way that realizability is preserved by the embedding. \( V_0^{ip}(A) \) and \( V_0^{T}(A) \) must both provide witnesses of existential statements that are still valid in \( V(A) \). Definability will imply that these witnesses are realizably equal in \( V(A) \).

The final step of the proof is to construct a particular pca, \( \mathcal{T} \), based on term models and normal filter \( \Gamma \) to show wEP fails. The cause of this failure will be a simple instance of the fullness axiom.

3 The Model \( V_0^{T}(A) \)

3.1 Definitions

A standard technique for showing the independence of choice principles in classical set theories is by using symmetric models. These can be seen as boolean valued models where every element is “symmetric.” See, for example [3, 11, 10], or [14] for a detailed description. We will construct a realizability model, \( V_0^{T}(A) \) based on the same ideas.

We start by defining the model \( V_1(A) \), as follows.
\[
V_1(A)_{\alpha+1} = P(2 \times |A| \times V_1(A)_\alpha)
\]
\[
V_1(A)_\lambda = \bigcup_{\beta<\lambda} V_1(A)_\beta
\]
\[
V_1(A) = \bigcup_{\alpha \in \text{On}} V_1(A)_\alpha
\]

One can think of \(V_1(A)\) as things from \(V(A)\) with an extra label from 2. Hence given any element of \(V_1(A)\) we can think of it as an element of \(V(A)\) by ignoring this extra label. Explicitly, we define this recursively as follows. Given \(a \in V_1(A)\),
\[
a^\circ := \{\langle 0, a^0 \rangle, \langle s, e, b \rangle \in a \}
\]

We write \(e \models_{1} \phi\) to mean that \(e \models \phi\) in \(V(A)\) when each parameter, \(a\), in \(\phi\) has been replaced by \(a^\circ\).

**Definition 3.1.** Let \(A\) be a pca. We say that \(\alpha\) is an **automorphism** of \(A\) if it is a bijection \(A \to A\) that such that both \(\alpha\) and \(\alpha^{-1}\) preserve application and fix \(s\) and \(k\).

Given an automorphism, \(\alpha\), of \(A\), we can lift this inductively to \(V_1(A)\) as follows:
\[
\alpha(a) = \{\langle 0, \alpha(e), \alpha(b) \rangle \mid \langle 0, e, b \rangle \in a \} \cup \{\langle 1, \alpha(e), \alpha(b) \rangle \mid \langle 1, e, b \rangle \in a \}
\]

So this is simply the natural action of the automorphism group on \(V_1(A)\).

We assume that the pairing and projection elements and numerals that appear in the definition of realizability over \(V(A)\) are defined using \(s\) and \(k\) and therefore fixed by any automorphism. Hence we get

**Proposition 3.2.** Suppose that \(\alpha\) is an automorphism of \(A\) and
\[
e \models \phi
\]
Then if \(\phi^\circ\) is the result of replacing any parameters \(c\) in \(\phi\) by \(\alpha(c)\), we have
\[
\alpha(e) \models \phi^\circ
\]

Recall that normal filters are defined as follows

**Definition 3.3.** Let \(G\) be a group. Then a set of subgroups, \(\Gamma\), is a **normal filter** on \(G\) if

1. \(G \in \Gamma\)
2. \(H \in \Gamma\) and \(H\) is a subgroup of \(H'\) implies that \(H' \in \Gamma\)
3. \( H, H' \in \Gamma \) implies that \( H \cap H' \in \Gamma \)

4. \( H \in \Gamma \) and \( g \in G \) implies \( gHg^{-1} \in \Gamma \)

**Definition 3.4.** Given a normal filter, \( \Gamma \), we define the class \( V^0_1(A) \subseteq V_1(A) \), of partly symmetric sets inductively as follows.

Given \( a \in V_1(A) \), we say \( a \in V^0_1(A) \) if \( \text{Stab}_G(a) \in \Gamma \) and for every \((0, e, b) \in a\) we have \( b \in V^0_1(A) \).

In other words, \( a \), has a “large” stabiliser and every element that has been labelled with a 0 is also partly symmetric. Note that this property is preserved by automorphisms, and one can easily show the following.

**Proposition 3.5.** If \( a \in V_1(A) \) and \( \alpha \in G \) is such that \( \alpha(a) = a \), then \( \alpha(a^\circ) = a^\circ \).

In particular if we take an element of \( V^0_1(A) \), then it still has \( \text{Stab}_G(a) \in \Gamma \) when we consider it as an element of \( V(A) \).

We can now define realizability on \( V^0_1(A) \) as follows:

\[
e \models_0 a \in b \text{ iff } (\exists (0, (e)0, c) \in b) (e)_1 \models_0 a = c
\]

\[
e \models_0 a = b \text{ iff } (\forall (0, f, c) \in a)(e)_0 f \models_0 c \in b \wedge
(\forall (0, f, c) \in b)(e)_1 f \models_0 c \in a \wedge e \models_1 a = b
\]

\[
e \models_0 \phi \wedge \psi \text{ iff } (e)_0 \models_0 \phi \wedge (e)_1 \models_0 \psi
\]

\[
e \models_0 \phi \vee \psi \text{ iff } ((e)_0 = 0 \wedge (e)_1 \models_0 \phi) \vee ((e)_0 = 1 \wedge (e)_1 \models_0 \psi)
\]

\[
e \models_0 ^0 \phi \rightarrow \psi \text{ iff } f \models_0 ^0 \phi \text{ implies } e.f \models_0 ^0 \psi \text{ and } e \models_1 ^0 \phi \rightarrow \psi
\]

\[
e \models_0 (\exists x \in a) \phi(x) \text{ iff } (\exists (0, (e)0, b) \in a)(e)_1 \models_0 \phi(b)
\]

\[
e \models_0 (\forall x \in a) \phi(x) \text{ iff } (\forall (0, f, b) \in a)e.f \models_0 \phi(b) \text{ and } e \models_1 (\forall x \in a) \phi(x)
\]

\[
e \models_0 (\exists x) \phi(x) \text{ iff } (\exists a \in V^0_0(A))e \models_0 \phi(a)
\]

\[
e \models_0 (\forall x) \phi(x) \text{ iff } (\forall a \in V^0_1(A))e \models_0 \phi(a) \wedge (\forall a \in V_1(A))e \models_1 \phi(a)
\]

\[
e \models_0 \neg \phi \text{ iff } \#f f \models_1 \phi
\]

We write \( V^0_0(A) \models_0 \phi \) to mean that there is some \( e \in A \) such that \( e \models_0 \phi \).

We clearly have the following proposition.

**Proposition 3.6.** Suppose that \( \alpha \) is an automorphism and

\[
e \models_0 \phi
\]

Then, writing \( \phi^\alpha \) for the formula obtained by replacing any parameters, \( a \), in \( \phi \) by \( \alpha(a) \),

\[
\alpha(e) \models_0 \phi^\alpha
\]
The definition above can be seen as a combination of realizability and Kripke models of intuitionistic logic. See for example [26] for a description of Kripke models. Like in [9], the poset used in this model would have just two elements. On this basis, one should not be surprised by the following proposition.

**Proposition 3.7.** Suppose that \( e \vDash_0 \phi \). Then also \( e \vDash_1 \phi \).

*Proof.* We show this by induction on formulae, \( \phi \). One can see that the definition of \( \vDash_0 \) has been carefully chosen so that we can perform the induction at =, universal quantifiers, implication, and negation. One can check that the induction holds at conjunction, disjunction, \( \in \), and existential quantifiers. \( \square \)

### 3.2 Soundness Theorems

We now need to show soundness for intuitionistic logic and the axioms of CZF.

Throughout the soundness theorems, the following proposition is useful.

**Proposition 3.8.**

1. To show \( e \vDash_0 (\forall x_1) \ldots (\forall x_n) \phi(x_1, \ldots, x_n) \) it is sufficient to show that for all \( a_1, \ldots, a_n \in V_0^1(A) \),

\[
e \vDash_0 \phi(a_1, \ldots, a_n)
\]

and for all \( a_1, \ldots, a_n \in V_1(A) \),

\[
e \vDash_1 \phi(a_1, \ldots, a_n)
\]

2. To show \( e \vDash_0 \phi_1 \rightarrow (\phi_2 \rightarrow (\ldots \rightarrow (\phi_n \rightarrow \psi) \ldots) \), it is sufficient to show that for any \( e_1, \ldots, e_{n-1} \in A \), \( ee_1 \ldots e_{n-1} \downarrow \) and that whenever \( e_i \vDash_0 \phi_i \) for each \( i = 1, \ldots, n \) we have

\[
ee_1 \ldots e_n \vDash_0 \psi
\]

and whenever \( e_i \vDash_1 \phi_i \) for each \( i = 1, \ldots, n \) we have

\[
ee_1 \ldots e_n \vDash_1 \psi
\]

*Proof.* Both parts can be proved by induction on \( n \). \( \square \)

### 3.3 First Order Logic

**Proposition 3.9.** \( V_0^1(A) \) satisfies soundness for first order logic.

Explicitly this means that for every axiom \( \phi \) of the intuitionistic predicate calculus, \( V_0^1(A) \models \phi \), and for every inference rule, \( \frac{\phi_1, \ldots, \phi_n}{\psi} \), if \( V_0^1(A) \models \phi_i \) for \( i = 1, \ldots, n \), then \( V_0^1(A) \models \psi \).

The reader may wish to compare the following with the proofs for soundness of intuitionistic logic in realizability and Kripke models in, for example, [26].
3.3.1 Axioms

The axioms of intuitionistic predicate calculus are as follows.

1. $\phi \rightarrow (\psi \rightarrow \phi)$
2. $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$
3. $\phi \rightarrow (\psi \rightarrow \phi \land \psi)$
4. $\phi \land \psi \rightarrow \phi$
5. $\phi \land \psi \rightarrow \psi$
6. $\phi \rightarrow \phi \lor \psi$
7. $\psi \rightarrow \phi \lor \psi$
8. $(\phi \lor \psi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$
9. $(\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \neg \psi) \rightarrow \neg \phi)$
10. $\phi \rightarrow (\neg \phi \rightarrow \psi)$
11. $(\forall x)\phi(x) \rightarrow \phi(y)$, where $y$ is free for $x$ in $\phi(x)$
12. $\phi(y) \rightarrow (\exists x)\phi(x)$, where $y$ is free for $x$ in $\phi(x)$

As an example we will prove 2 and 11. These demonstrate the main ideas that are used for the remaining axioms.

2 We claim that

$s \vDash_0 (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$

By proposition 3.8, it is enough to show that whenever $e \vDash_0 \phi \rightarrow (\psi \rightarrow \chi)$
$f \vDash_0 \phi \rightarrow \psi$ and $g \vDash_0 \phi$ we have

$sefg \vDash_0 \chi$

and whenever $e \vDash_1 \phi \rightarrow (\psi \rightarrow \chi)$ $f \vDash_1 \phi \rightarrow \psi$ and $g \vDash_1 \phi$ we have

$sefg \vDash_1 \chi$

However, $sefg = eg(fg)$, so one can easily check that this is the case. (We also have that by definition $sef \downarrow$ for all $e$ and $f$.)
Let $I := \text{skk}$ be the identity. Then we claim

$I \models_0 (\forall x)\phi(x) \to \phi(y)$

What we actually mean is the universal closure of this axiom. Without loss of generality we can assume the universal closure is (ignoring any additional parameters) the following:

$I \models_0 (\forall y)((\forall x)\phi(x) \to \phi(y))$

Expanding this out, this means that for $b \in V^I_0(A)$,

$I \models_0 (\forall x)\phi(x) \to \phi(b)$

and for $b \in V(A)$

$I \models_1 (\forall x)\phi(x) \to \phi(b)$

So suppose that $e \models_0 (\forall x)\phi(x)$ and $b \in V^I_0(A)$. Then in particular,

$e \models_0 \phi(b)$

If $e \models_1 (\forall x)\phi(x)$, then

$e \models_1 \phi(b)$

So we have shown

$I \models_0 (\forall x)\phi(x) \to \phi(b)$

However, we can similarly show that for $b \in V(A)$

$I \models_1 (\forall x)\phi(x) \to \phi(b)$

So we can deduce

$I \models_0 (\forall y)((\forall x)\phi(x) \to \phi(y))$

as required.

### 3.3.2 Inference Rules

The inference rules of IPL are

1. \[ \frac{\phi \phi \to \psi}{\psi} \]

2. \[ \frac{\psi \to \phi(x)}{\psi \to (\forall x)\phi(x)} \] where $x \notin FV(\psi)$

3. \[ \frac{\phi(x) \to \psi}{(\exists x)\phi(x) \to \psi} \] where $x \notin FV(\psi)$
1 (Modus Ponens) Note first that we can assume that

\[ e \Vdash_0 (\forall x_1,\ldots,x_n) \phi(x_1,\ldots,x_n) \]
\[ f \Vdash_0 (\forall x_1,\ldots,x_n) \phi(x_1,\ldots,x_n) \Rightarrow \psi(x_1,\ldots,x_n) \]

where the free variables for \( \phi \) and \( \psi \) are amongst \( x_1,\ldots,x_n \).

Then for any \( a_1,\ldots,a_n \in V_0^A \),

\[ e \Vdash_0 \phi(a_1,\ldots,a_n) \]
\[ f \Vdash_0 \phi(a_1,\ldots,a_n) \Rightarrow \psi(a_1,\ldots,a_n) \]

and hence \( e.f \Vdash_0 \psi(a_1,\ldots,a_n) \). Similarly, for any \( a_1,\ldots,a_n \in V_1(A) \), \( e.f \Vdash_1 \psi(a_1,\ldots,a_n) \). So we have shown

\[ e.f \Vdash_0 (\forall x_1,\ldots,x_n) \psi(x_1,\ldots,x_n) \]

2 We show that if \( e \Vdash_0 \psi \Rightarrow \phi(x) \), then

\[ e \Vdash_0 \psi \Rightarrow (\forall x) \phi(x) \]

So suppose that \( e \Vdash_0 \psi \Rightarrow \phi(x) \). Then, more explicitly (ignoring any additional free variables) this is

\[ e \Vdash_0 (\forall x)(\psi \Rightarrow \phi(x)) \]

In particular, if \( a \in V_0^A \), then

\[ e \Vdash_0 \psi \Rightarrow \phi(a) \]

and if \( a \in V(A) \) then

\[ e \Vdash_1 \psi \Rightarrow \phi(a) \]

Now suppose that \( f \Vdash_0 \psi \). We need to show that for any \( a \in V_0^A \), \( e.f \Vdash_0 \phi(a) \) and for any \( a \in V_1(A) \), \( e.f \Vdash_1 \phi(a) \). But this is clear from the above, so we can deduce

\[ e.f \Vdash_0 (\forall x) \phi(x) \]

We can similarly show that if \( f \Vdash_1 \psi \), then

\[ e.f \Vdash_1 (\forall x) \phi(x) \]

and so

\[ e \Vdash_0 \psi \Rightarrow (\forall x) \phi(x) \]
We claim that if \( e \vdash_0 \phi(x) \rightarrow \psi \), then \( e \vdash_0 (\exists x)\phi(x) \rightarrow \psi \). First note as before, that what we actually assume is that

\[
e \vdash_0 (\forall x)(\phi(x) \rightarrow \psi)
\]

Now suppose that \( f \vdash_0 (\exists x)\phi(x) \). Then there is \( a \in V^0_0(A) \) such that \( f \vdash_0 \phi(a) \). But we know from the above that

\[
e \vdash_0 \phi(a) \rightarrow \psi
\]

And so,

\[
e.f \vdash_0 \psi
\]

We similarly know that if \( f \vdash_1 (\exists x)\phi(x) \), then \( e.f \vdash_1 \psi \). So we can deduce

\[
e \vdash_0 (\exists x)\phi(x) \rightarrow \psi
\]

### 3.3.3 Axioms of Equality

In the following, we will need to work inductively on the definition of \( V^0_0(A) \), so it will be useful to have a notion of rank that we can induct on.

**Definition 3.10.** The rank, \( \text{rank}(a) \) of \( a \in V^1_1(A) \) is defined inductively as follows:

\[
\text{rank}(a) = \bigcup_{(s,c,b) \in a} (\text{rank}(b) + 1)
\]

**Proposition 3.11.** Suppose that \( a,b \in V^1_1(A) \). If \( V^1_1(A) \models a = b \), then \( \text{rank}(a) = \text{rank}(b) \).

**Proof.** We show by induction that for any \( \alpha \), for any \( a,b \), if \( V^1_1(A) \models a = b \) and \( \text{rank}(a) = \alpha \), then \( \text{rank}(b) = \alpha \).

Let \( \langle e, c \rangle \in a \). Then there is \( \langle e', c' \rangle \in b \) such that \( V^1_1(A) \models e = c \). Then \( \text{rank}(c) \in \alpha \) and so we may assume by induction that \( \text{rank}(c) = \text{rank}(c') \), and so

\[
\text{rank}(c) + 1 = \text{rank}(c') + 1 \\
\leq \text{rank}(b)
\]

Hence \( \text{rank}(a) \subseteq \text{rank}(b) \). If \( \langle e, c \rangle \in b \), then there is some \( \langle e', c' \rangle \in a \) such that \( V^1_1(A) \models e = c' \). Then we must also have \( V^1_1(A) \models c' = c \) and we know that \( \text{rank}(c') \in \alpha \). So \( \text{rank}(c) = \text{rank}(c') \). By the same reasoning as above \( \text{rank}(b) \subseteq \text{rank}(a) \) and so \( \text{rank}(a) = \text{rank}(b) \). 

**Proposition 3.12.** One can construct realizers \( \mathbf{i}_r, \mathbf{i}_s, \mathbf{i}_i, \mathbf{i}_0, \mathbf{i}_1 \) such that

1. \( \mathbf{i}_r \vdash_0 (\forall x)x = x \)
2. \( \mathbf{i}_s \vdash_0 (\forall x,y)x = y \rightarrow y = x \)
3. $i_\iota \vdash_0 (\forall x, y, z) (x = y \rightarrow (y = z \rightarrow x = z))$

4. $i_0 \vdash_0 (\forall x, y, z) (x = y \rightarrow (y \in z \rightarrow x \in z))$

5. $i_1 \vdash_0 (\forall x, y, z) (x = y \rightarrow (z \in x \rightarrow z \in y))$

Furthermore, for each formula (without parameters), $\phi(x, z_1, \ldots, z_n)$, there is $i_\phi$ such that

$$i_\phi \vdash_0 x = y \rightarrow (\phi(x, z_1, \ldots, z_n) \rightarrow \phi(y, z_1, \ldots, z_n))$$

Proof. We take these realizers from the proof of theorem 6.3 in [16] and check that they still work in this context.

Define $i_r$ from the fixed point theorem so that

$$(i_r)_0 f = f$$

$$(i_r)_0 f_1 = i_r$$

$$(i_r)_1 f_0 = f$$

$$(i_r)_1 f_1 = i_r$$

In order to show $i_r \vdash_0 (\forall x) x = x$, by proposition 3.8, what we need to show is

1. for every $a \in V_0^G(A)$, $i_r \vdash_0 a = a$

2. for every $a \in V_1(A)$, $i_r \vdash_1 a = a$

However note that the second of these conditions is basically the same as the statement $i_r \vdash a = a$ in $V(A)$. Hence we only have to check the first condition.

Furthermore, since we already know that for every $a \in V_0^G(A)$, $i_r \vdash_1 a = a$, all we have to check is the following:

$$\forall (0, f, b) \in a (i_r)_0 \vdash_0 b \in a$$

and

$$\forall (0, f, b) \in a (i_r)_1 \vdash_0 b \in a$$

We show by induction that these conditions hold for every $a \in V_0^G(A)$.

Suppose that $(0, f, b) \in a$. Then since this has been labelled with 0, we know that $b$ is also partly symmetric. Also $b$ is of strictly lower rank, so we can apply induction here and the above arguments to get

$$i_r \vdash_0 b = b$$

However, recall that we defined $i_r$ using the fixed point theorem so that for all $f$,
\[(i_r)_0 f = f\]
\[(i_r)_1 f = i_r\]

(and the same equations for \((i_r)_1\)).

Hence \(i_r \models_0 a = a\) as required.

The proof that \(i_s\) works as required is trivial and still holds here.

\(i_0, i_1\) are also the same as in [10] and the proofs that they are as required can be similarly adapted to this context.

The \(i_\phi\) are constructed by induction on the construction of \(\phi\). We will explicitly show how to do this for unbounded universal quantifiers and implication since these contain the main ideas for the rest of the induction.

We first show how to construct \(i_{\phi \rightarrow \psi}\).

Suppose that \(a, b, c \in V_0(A), e \models_0 a = b\) and \(f \models_0 \phi(a, c) \rightarrow \psi(a, c)\).

Suppose further that \(g \models_0 \phi(b, c)\).

Then
\[i_\phi(i_s e)g \models_0 \phi(a, c)\]
and so
\[f(i_\phi(i_s e)g) \models_0 \psi(a, c)\]
and finally
\[i_\psi(f(i_\phi(i_s e)g)) \models_0 \psi(b, c)\]

Hence we can apply similar reasoning for \(\models_1\) and for \(a, b, c \in V_1(A)\) and use proposition \([3.8]\) to show that we can take \(i_{\phi \rightarrow \psi}\) to be
\[i_{\phi \rightarrow \psi} := (\lambda x, y, z).i_\psi x(y(i_\phi i_s x z))\]

For unbounded universal quantifiers, we show that we can take \(i_{(\forall z)\phi(x, z)} := i_{\phi(x, z)}\). Suppose that
\[i_{\phi(x, z)} \models_0 (\forall z)(x = y \rightarrow (\phi(x, z) \rightarrow \phi(y, z)))\]
and suppose that for \(a, b \in V_0(A), e \models_0 a = b\) and
\[f \models_0 (\forall z)\phi(a, z)\]
Then for all \(c \in V_0(A),\)
\[f \models_0 \phi(a, c)\]
and so
\[i_{\phi(x, z)} e f \models_0 \phi(b, c)\]
One can check the corresponding case for \(c \in V_1(A)\) to get
\[i_{\phi(x, z)} e f \models_0 (\forall z)\phi(b, z)\]
as required.
Proposition 3.13. Bounded and unbounded quantifiers agree. That is, we can
find realizers for the following statements.

1. $\forall x \in a \phi(x) \rightarrow (\forall x)(x \in a \rightarrow \phi(x))$
2. $(\forall x)(x \in a \rightarrow \phi(x)) \rightarrow (\forall x \in a)\phi(x)$
3. $(\exists x \in a)\phi \rightarrow (\exists x)(x \in a \land \phi(x))$
4. $(\exists x)(x \in a \land \phi(x)) \rightarrow (\exists x \in a)\phi(x)$

Proof. The proof of theorem 4.3 from [22] can easily be adapted by applying
proposition 3.8 where necessary.

The following help illustrate the relation between realizability in $V_0^Γ(A)$ and
$V(A)$.

Definition 3.14. We say that $a \in V_0^Γ(A)$ is (completely) symmetric if every
element of $a$ is of the form

$$\langle 0, e, b \rangle$$

where $b$ is completely symmetric. (This is an inductive definition).

Proposition 3.15. Suppose that $\phi$ is a bounded formula, all of whose parameters
are completely symmetric. Then

$$e \vDash_0 \phi \iff e \vDash_1 \phi$$

Proof. When all parameters are completely symmetric the two definitions of
realizability agree for everything except unbounded quantifiers.

We now move on to the proof of soundness for the axioms of set theory. To
make things easier, we assume a background universe of $\text{ZFC}$, and show the
soundness of $\text{IZF}$.

Theorem 3.16. $V_0^Γ(A)$ satisfies the axioms of $\text{IZF}$.

We first deal with what are sometimes referred to as “set existence axioms.”
That is, axioms of the form

$$(\forall z_1, \ldots, z_n)(\forall x)(\exists y)\phi(x, y, z_1, \ldots, z_n)$$

where the free variables of $\phi$ are amongst $x, y, z_1, \ldots, z_n$. For these axioms we
can apply proposition 3.8 to show that it is sufficient to find $e$ such that for
every $a, c_1, \ldots, c_n \in V_1(A)$, there is $b \in V_1(A)$ such that

$$e \vDash_1 \phi(a, b, c_1, \ldots, c_n)$$

and for every $a, c_1, \ldots, c_n \in V_1^Γ(A)$ there is $b \in V_1^Γ(A)$ such that

$$e \vDash_0 \phi(a, b, c_1, \ldots, c_n)$$

However, the first of these statements follows from the soundness theorem
for $V(A)$. Hence we only have to check the second of these conditions.
Separation  By the above reasoning, what we need to show is the following statement:

Suppose that \( e \) is the usual realizer for separation from [16] or [22], \( A \) is a partly symmetric set, and \( \phi(x) \) is a formula with partly symmetric parameters. Then there is a partly symmetric set, \( S \), such that

\[
e \models_0 ((\forall x \in A)\phi(x) \rightarrow x \in S) \land ((\forall x \in S)x \in A \land \phi(x))
\]

We construct this \( S \) as follows:

\[
S_0 = \{ \langle 0, pf_g, a \rangle \mid \langle 0, f, a \rangle \in A \land g \models_0 \phi(a) \}
\]

\[
S_1 = \{ \langle 1, pf_g, a \rangle \mid \langle s, f, a \rangle \in A \land g \models_1 \phi(a) \}
\]

\[
S = S_0 \cup S_1
\]

Suppose that \( H \) is the intersection of stabilisers of \( A \) and all the parameters of \( \phi \). Note that \( H \in \Gamma \).

Let \( \alpha \in H \), and \( \langle 0, pf_g, a \rangle \in S_0 \). Then \( \langle 0, f, a \rangle \in A \) and \( g \models_0 \phi(a) \). Since \( \alpha \in H \), we know that \( \langle 0, \alpha(f), \alpha(a) \rangle \in A \) and \( \alpha(g) \models_0 \phi(\alpha(a)) \). Hence we also have \( \langle 0, \alpha(pf_g), \alpha(a) \rangle \in S_0 \). One can show the same result for \( S_1 \) and hence get \( \text{Stab}_G(S) \in \Gamma \). Note further that if \( \langle 0, pf_g, a \rangle \in S \) then also \( \langle 0, f, a \rangle \in A \) and so \( a \) is partly symmetric. We can now deduce that \( S \) is partly symmetric.

One can easily check that the usual realizer does still work for \( S \).

**Power Set**  As before, note that we only have to check power set for partly symmetric sets. Hence let \( A \in V_0^\Gamma(A) \).

Let

\[
P_0 = \{ \langle 0, e, b \rangle \mid b \in V_0^\Gamma(A), e \models_0 b \subseteq A \}
\]

\[
P_1 = \{ \langle 1, e, b \rangle \mid b \in V_1(A), e \models_1 b \subseteq A \}
\]

\[
P = P_0 \cup P_1
\]

Note that by the arguments in [16], both \( P_0 \) and \( P_1 \) are sets. Further note that if \( e \models_0 b \subseteq A \) and \( \alpha \in \text{Stab}_G(A) \) then \( \alpha(e) \models_0 \alpha(b) \subseteq A \), and similarly if \( e \models_1 b \subseteq A \), and we have ensured that any elements of \( P \) labelled with 0 are partly symmetric. Hence \( P \) is partly symmetric.

One can easily show that the realizer in [16] still works here.

**Union**  We assume that we are given a set \( A \in V_0^\Gamma(A) \) and construct a set to show the union axiom. Since we already have full separation, we only have to construct a \( U \) such that we have a realizer for \((\forall x \in A)(\forall y \in x)y \in U \).

Let

\[
U = \{ \langle 0, 0, b \rangle \mid \langle 0, e, c \rangle \in A, \langle 0, f, b \rangle \in c \} \cup \{ \langle 1, 0, b \rangle \mid \langle s, e, c \rangle \in A, \langle s', f, b \rangle \in c \}
\]

Note that

\[
(\langle k(k(pf_0, i, r)) \rangle_0 \models_0 (\forall x \in A)(\forall y \in x)y \in U)
\]

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Pair Given \( a, b \in V_0^\Gamma(A) \), consider the set

\[
P = \{ \langle 0, 0, a \rangle, \langle 0, 1, b \rangle \}
\]

We can easily see that

\[
e \vDash_0 (\forall x)(x \in P \leftrightarrow (x = a \lor x = b))
\]

Infinity We check that the proof in [22] still holds here. We use the same \( \bar{\omega} \) as in section 1.5. We write \( \perp_v \) for the formula \( (\forall x \in v) \bot \), and write \( SC(x, y) \) for \( y = x \cup \{x\} \) (expressed as a bounded formula).

Note first that we can apply proposition 3.15 and the soundness theorem in [22] to reduce the problem to finding a realizer for

\[
(\forall v)((\bot_v \lor (\exists u \in \bar{\omega})SC(u, v)) \rightarrow v \in \bar{\omega})
\]

Since we can clearly find a realizer to show that the empty set is in \( \bar{\omega} \), this is reduced to finding a realizer for

\[
(\forall v)(\exists u \in \bar{\omega})SC(u, v) \rightarrow v \in \bar{\omega}
\]

Hence we assume that there is \( a \in V_0^\Gamma(A) \) with \( e \vDash_0 (\exists u \in \bar{\omega})SC(u, a) \). So there must be some \( n \) such that \( (e)_0 = n \) and \( (e)_1 \vDash_0 SC(\pi, a) \).

One can clearly find a realizer for \( SC(\pi, n + 1) \) and hence a realizer, using the soundness of extensionality (once we have checked this) for \( SC(u, v) \land SC(u, v') \rightarrow v = v' \). We can use these to construct a realizer for \( a \in \bar{\omega} \), as required.

Collection Assume

\[
e \vDash_0 (\forall x \in A)(\exists y)\phi(x, y)
\]

where \( \phi \) is a formula with all parameters partly symmetric.

By collection in the background universe, we can find a \( C_0 \) such that whenever \( \langle 0, f, a \rangle \in A \), there is \( \langle 0, 0, c \rangle \in C_0 \) such that \( c \) is partly symmetric and \( e.f \vDash_0 \phi(a, c) \). Note that \( C'_0 := C_0 \cap \{0\} \times \{0\} \times V_0^\Gamma(A) \) still has this property, but is an element of \( V_1(A) \) such that for every \( \langle 0, g, c \rangle \in C'_0 \), \( c \) is partly symmetric.

Similarly, there is a \( C'_1 \), such that every element of \( C'_1 \) is of the form \( \langle 1, 0, c \rangle \) with \( c \in V_1(A) \) and whenever \( \langle s, f, a \rangle \in A \), there is \( \langle 1, 0, c \rangle \in C'_1 \) such that \( e.f \vDash_1 \phi(a, c) \).

Let \( C = C'_0 \cup C'_1 \), and let \( C' \) be the closure of \( C \) under all automorphisms in \( G \). Note that \( C' \in V_1^\Gamma(A) \) and this set together with the usual realizer from [16] is enough to show the soundness of collection.

Extensionality One can check that the realizers for the formula

\[
((\forall x \in a)x \in b) \land ((\forall x \in b)x \in a)
\]

in fact are already realizers for \( a = b \), so we can use the identity to show extensionality (in this form).
Suppose that
\[ e \vdash_0 (\forall y)((\forall x \in y)\phi(x) \rightarrow \phi(y)) \]

Let \( e' = (\lambda x, y).e.x \) and let \( f \) be given by the fixed point theorem so that for all \( g \)
\[ f.g \simeq e'.f.g \]

Note that we know
\[ e \vdash_1 (\forall y)((\forall x \in y)\phi(x) \rightarrow \phi(y)) \]

and so by the usual proof we have that for all \( a \in V_1(A) \), and all \( g \in A \), \( f.g \vdash_1 \phi(a) \). We claim that for all \( a \in V_1^\alpha(A) \), and all \( g \in A \), \( f.g \downarrow \) and \( f.g \vdash_0 \phi(a) \).

So suppose that \( a \in V_1^\alpha(A) \). Then for every \( \langle 0, g, b \rangle \in a \), we know by induction in the background universe (since \( b \) must be partly symmetric and of strictly lower rank than \( a \)) that \( f.g \downarrow \) and \( f.g \vdash_0 \phi(b) \). We also know from the above that \( f \vdash_1 (\forall x \in a)\phi(x) \). Hence \( f \vdash_0 (\forall x \in a)\phi(x) \). Thus we have for any \( g \in A \), \( e'fg \simeq ef \) (is defined and) realizes \( \phi(a) \). But \( e'fg \simeq fg \) and so \( f.g \vdash_0 \phi(a) \) as required.

\[ \square \]

**Remark 3.17.** Note that when we proved the axiom of infinity we used the same standard representation \( \overline{\omega} \) as for \( V(A) \). Note further that if \( f \in A \) is such that for all \( n \in \omega \) there is \( m \in \omega \) with \( fn = m \), then the \( \overline{\tau} \) from section [1.3.9] is completely symmetric and hence we have the same standard representations of the naturals and Baire space as we did before.

### 4 The Model \( V_0^{ip}(A) \)

We say that \( a \in V(A) \) is **injectively presented** if for any \( \langle e, b \rangle, \langle e', b' \rangle \in a \), if \( e = e' \) then \( b = b' \).

Define \( V_0^{ip}(A) \) inductively as follows
\[
V_0^{ip}(A)_{\alpha+1} = \{ X \subseteq |A| \times V_0^{ip}(A)_{\alpha} \mid X \text{ is injectively presented} \}
\]
\[
V_0^{ip}(A)_{\lambda} = \bigcup_{\beta < \lambda} V_0^{ip}(A)_{\beta}
\]
\[
V_0^{ip}(A) = \bigcup_{\alpha \in \text{On}} V_0^{ip}(A)_{\alpha}
\]

We define realizability on \( V_0^{ip}(A) \) as follows. We write \( \vdash_1 \) for realizability at \( V(A) \).
\[
\begin{align*}
e \models_0 a \in b & \iff (\exists (e)_0, c) \in b)(e)_1 \models_0 a = c \\
e \models_0 a = b & \iff (\forall (f, c) \in a)(e)_0, f \models_0 c \in b \land \forall (f, c) \in b (e)_1 f \models_0 c \in a \\
e \models_0 \phi \land \psi & \iff (e)_0 \models_0 \phi \land (e)_1 \models_0 \psi \\
e \models_0 \phi \lor \psi & \iff ((e)_0 = 0 \lor (e)_1 \models_0 \phi) \lor ((e)_0 = 1 \land (e)_1 \models_0 \psi) \\
e \models_0 \phi \rightarrow \psi & \iff f \models_0 \phi \text{ implies } e.f \models_0 \psi \text{ and } e \models_1 \phi \rightarrow \psi \\
e \models_0 (\exists x \in a) \phi(x) & \iff (\exists ((e)_0, b) \in a)(e)_1 \models_0 \phi(b) \\
e \models_0 (\forall x \in a) \phi(x) & \iff (\forall (f, b)) e.f \models_0 \phi(b) \text{ and } \\
e \models_0 (\exists x) \phi(x) & \iff (\exists a \in V_0^\text{ip}(A)) e \models_0 \phi(a) \\
e \models_0 (\forall x) \phi(x) & \iff (\forall a \in V_0^\text{ip}(A)) e \models_0 \phi(a) \land (\forall a \in V(A)) e \models_1 \phi(a) \\
e \models_0 \neg \phi & \iff \# f \models f \models_1 \phi
\end{align*}
\]

We write \(V_0^\text{ip}(A) \models \phi\) to mean that there is some \(e \in A\) such that \(e \models_0 \phi\).

**Remark 4.1.** This is a much simpler embedding than that of \(V_0^\Gamma(A)\). We have not needed to alter the definition of realizability for bounded universal quantification and equality in order to ensure realizability is preserved. Hence, realizability for bounded formulae is identical in \(V_0^\text{ip}(A)\) and \(V(A)\).

**Proposition 4.2.** \(V_0^\text{ip}(A)\) is sound with respect to the intuitionistic predicate calculus and axioms of equality.

**Proof.** This follows by exactly the same proof as for \(V_0^\Gamma(A)\). \(\square\)

It remains to check that when we show the soundness of the axioms of CZF, we can assume the sets we construct are injectively presented. Since we will require choice in the background universe for this proof, we work over a background universe of ZFC.

**Theorem 4.3.** \(V_0^\text{ip}(A)\) is sound with respect to the axioms of CZF.

**Extensionality** This is the same as for \(V(A)\).

**Bounded Separation** Given \(A \in V_0^\text{ip}(A)\) and a bounded formula, \(\phi\), consider the set

\[
S = \{ (pef, a) \mid (e, a) \in A, f \models_0 \phi(a) \}
\]

Note that this is injectively presented, since \(A\) is, and since realizability for bounded formulae is identical in \(V_0^\text{ip}(A)\) and \(V(A)\), we can see that this can be used to show the soundness of bounded separation.
Pair  Given \( a, b \in V(A) \), consider
\[
P = \{ \langle 0, a \rangle, \langle 1, b \rangle \}
\]
This is clearly injectively presented, and we can easily use this to show the soundness of pair.

Strong Collection  Suppose that
\[
e \models_0 (\forall x \in A)(\exists y \in C) \phi(x, y)
\]
For each \( \langle f, a \rangle \in A \), we can assume by choice in the background universe that we have chosen a \( c_f \in V_0^A \) such that \( e.f \models_0 \phi(a, c_f) \) (and hence also \( e.f \models_{-1} \phi(a, c_f) \)).
Let
\[
C = \{ \langle f, c_f \rangle \mid \langle f, a \rangle \in A \}
\]
This is clearly injectively presented (since \( A \) is).
Note that
\[
(\lambda x).p x(e.x) \models_0 (\forall x \in A)(\exists y \in C) \phi(x, y)
\]
and in fact we can use exactly the same realizer again in
\[
(\lambda x).p x(e.x) \models_0 (\forall y \in C)(\exists x \in A) \phi(x, y)
\]
(since every element of \( C \) is of the form \( \langle f, c_f \rangle \) where \( \langle f, x \rangle \in A \) and \( e.f \models_0 \phi(x, c_f) \)). So we get soundness for strong collection.

Subset Collection  Suppose we are given sets \( A, B \in V^p_0(A) \). Suppose further that \( e \in A \) is such that for all \( \langle f, a \rangle \in A \), \( e.f \downarrow \) and there is \( \langle e.f, b \rangle \in B \) for some \( b \). In this case we can define
\[
\overline{c} := \{ \langle f, b \rangle \mid \exists a \langle f, a \rangle \in A, \langle e.f, b \rangle \in B \}
\]
(Clearly \( \overline{c} \in V^p_0(A) \)).
Now let
\[
D := \{ \langle e, \overline{c} \rangle \mid e \in A, \overline{c} \text{ is defined} \}
\]
Clearly \( D \in V^p_0(A) \). We shall show that we can use \( D \) to show the soundness of subset collection.
Suppose that \( u \in V(A) \) is such that
\[
e \models_{-1} (\forall x \in A)(\exists y \in B) \phi(x, y, u)
\]
Let
\[
e' := (\lambda x).e x_0
\]
Note that for every \( \langle f, a \rangle \in A \), we have \( e'.f \downarrow \) and there is (a unique) \( b \) with \( \langle e'.f, b \rangle \in B \), and so \( \langle e', \overline{c} \rangle \in D \). Furthermore \( (e.f)_1 \models_{-1} \phi(a, b, u) \), and so we can find a realizer for
\[
(\forall x \in A)(\exists y \in \overline{c}) \phi(x, y, u) \land (\forall y \in \overline{c})(\exists x \in A) \phi(x, y, u)
\]
We can do exactly same if
\[ e \Vdash_0 (\forall x \in A)(\exists y \in B)\phi(x, y, u) \]

Hence this does give a proof of the soundness of subset collection.

**Union** Suppose we have been given \( A \in V_0^{ip}(A) \). We want to find an injectively presented set that we can use to show the union axiom. So let
\[ U = \{ \langle pe, f, c \rangle \mid \langle f, b \rangle \in a, \langle e, c \rangle \in b \} \]

Then we see that
\[ (\lambda x, y)(p(pyx)i_x) \Vdash_0 (\forall x \in a)(\forall y \in x)y \in U \]

**Infinity** We note that the \( \exists \) given in section 1.5 is injectively presented, and since no other sets need to be constructed in the proof of infinity, this means we can use the same proof as usual here (see eg [22]).

\( \varepsilon \text{-Induction} \) The same proof as for \( V_0^T(A) \) still holds here. \( \Box \)

5 The Pca \( \mathcal{T} \)

5.1 Definition

We will define a term model based on combinatory logic. This is similar to the model \( NT \) that appears in chapter 6 of [3].

We start by adding constants \( \xi_i \) and \( \zeta_F \) to the language of combinatory logic.

**Definition 5.1.** The set, \( C \) of terms is defined inductively as follows

1. constants \( s \) and \( k \) are terms
2. free variables \( x_i \) for each \( i \in \omega \) are terms
3. for each \( i > 0 \), the constant \( \xi_i \) is a term (we will call these atoms)
4. for each bijection \( F : \omega_{>0} \rightarrow \omega_{>0} \), the constant \( \zeta_F \) is a term
5. if \( L \) and \( M \) are terms then \( L.M \) is a term

We will consider \( C \) as a term rewriting system. We have in particular the two standard reduction rules

\[ sxyz \rightarrow xz(yz) \]
\[ kxy \rightarrow x \]
In addition to these, we add a new reduction rule. In the below let $\overline{n}$ be $n$ encoded using $s$ and $k$ in the usual way. Then we define $\zeta$-reduction as follows:

$$\zeta F t \rightarrow \overline{n}$$

where $t$ is a closed term and $n$ is either maximal such that $n = F(m)$ where $\xi_m$ occurs in $t$ or $n = 0$ and no $\xi_m$ occurs in $t$.

Note that this term rewriting system is ambiguous. That is, there are terms that can be reduced in two incompatible ways. For example, the term $\zeta(\lambda x. x(kk\xi_1))$ can reduce either to 1 or to 0 depending on whether the subterm $kk\xi_1$ is reduced before or after $\zeta$-reduction. However, we still have a notion of normal form (when no reduction rule can be applied to a term) and leftmost innermost reduction, as defined below.

**Definition 5.2.** We define a sequence of partial operators, $\text{RED}_n$ for each $n$ as follows:

For $n = 0$, define $\text{RED}_0$ as follows:

1. if $t$ is a normal form, $\text{RED}_0(t) = t$
2. for $t = krs$ where $r$ and $s$ are normal forms, $\text{RED}_0(krs) = r$
3. for $t = \zeta F r$ where $r$ is a normal form, $\text{RED}_0(\zeta F r) = \overline{n}$ where $n$ is maximal such that $\xi_{F-1(n)}$ occurs in $r$ or 0 if no $\xi_i$ occurs in $r$

If $\text{RED}_n$ has been already been defined, then we define $\text{RED}_{n+1}$ as follows:

1. if $\text{RED}_n(t) \downarrow$, then $\text{RED}_{n+1}(t) = \text{RED}_n(t)$
2. for $t = sru$, where $r$, $s$, and $u$ are normal forms, $\text{RED}_{n+1}(sru) \simeq \text{RED}_n(\text{RED}_n(ru) \text{RED}_n(su))$
3. if $t = rs$ and neither of previous cases apply, then

$$\text{RED}_{n+1}(rs) \simeq \text{RED}_n(\text{RED}_n(r) \text{RED}_n(s))$$

We then define $\text{RED}$ as

$$\text{RED} = \bigcup_{n \in \omega} \text{RED}_n$$

Note that if $\text{RED}(t)$ is defined, then it is a normal form.

We now define our pca, $\mathcal{T}$

**Definition 5.3.** Let $\mathcal{T}$ be the set of normal forms of $\mathcal{C}$ together with the following application:

$$s.t := \text{RED}(s.t)$$

(undefined if $\text{RED}(s.t)$ is undefined)
Note that since this is a pca we can consider the notion of terms over \( T \) (ie definition 1.12) as well as terms in the sense of definition 5.1. Fortunately we are free to switch between thinking of terms and terms over \( T \) by the following proposition. (Note that this proposition is a characteristic of inside first reduction and is not shared by some similar structures: see Remark 6.1.4 in [3].)

**Proposition 5.4.** Suppose that \( t \) is a closed term over \( T \) (in the sense of definition 1.12) and write \( t^* \) for the corresponding term (in the sense of definition 5.1). Then \( \text{RED}(t^*) \) is defined if and only if \( t \) denotes, and in this case we have \( \text{RED}(t^*) = t \).

*Proof.* This essentially appears as parts (i) and (ii) of lemma 6.1.1 in chapter 6 of [3]. We simply note that the proof still holds in this setting where we also have \( \zeta \)-reduction. \( \square \)

**Proposition 5.5.** \( T \) is a pca.

*Proof.* Note firstly that \( s \) and \( k \) are normal terms and hence elements of \( T \).

If \( r \) and \( s \) are normal forms, then so are \( kr \) and \( srs \). Hence \( kr \downarrow \), \( sr \downarrow \), and \( srs \downarrow \). Also \( \text{RED}(krs) = r \), so \( krs = r \).

It remains only to check that for all \( r, s, t \), \( srs \simeq rt(st) \). However this is clear from the definition. (In fact the left hand side is defined at stage \( n + 1 \) if and only if the right hand side is defined at stage \( n \).) \( \square \)

### 5.2 Preservation of Atoms

The non trivial structure of \( V^F_0(T) \) will rely on the \( \xi_i \), and the rich supply of automorphisms arising from permutations of them. We will want to ensure therefore that under suitable conditions the atoms aren’t eliminated by the realizability structure. In this section, we will aim towards a lemma that will enable us to show this.

**Definition 5.6.** For any pca, \( \mathcal{A} \), one may consider the following classes of elements

1. \( f \in \mathcal{A} \) is *type 1* if for every \( n \in \omega \), \( f \updownarrow \), and there is some \( m \in \omega \) such that \( f \cdot n = m \)
2. \( e \in \mathcal{A} \) is *type 2* if for every type 1 \( f \), \( e \cdot f \downarrow \) and \( e \cdot f \) is type 1
3. \( e \in \mathcal{A} \) is a *type 2 identity* if it is type 2 and for all \( f \) type 1 and for all \( n \in \omega \), \( e \cdot f \cdot n = f \cdot n \)

We will now show that being able to decide whether a term is defined or not is equivalent to the halting problem.

**Proposition 5.7.** Suppose that \( t(x) = t_1(x)t_2(x) \), \( l \in \omega \), and \( r \) is a normal form. If \( \text{RED}_l((\lambda x).t(x)r) \downarrow \), then \( l > 0 \) and \( \text{RED}_{l-1}(t(r)) \downarrow \).
Proof. Note that from the definition of lambda terms over a pca (see [3] or [27]) we know that
\[(\lambda x).t(x) := s((\lambda x).t_1(x))(\lambda x).t_2(x)\]

Note firstly that \((\lambda x).t(x)r = s(\lambda x).t_1(x)(\lambda x).t_2(x)r\) and hence we can only have \(\text{RED}_l((\lambda x).t(x)r) \downarrow\) for \(l > 0\). Furthermore,

\[\text{RED}_l(s(\lambda x).t_1(x)(\lambda x).t_2(x)r) \simeq \text{RED}_{l-1}(\text{RED}_{l-1}((\lambda x).t_1(x)r) \text{RED}_{l-1}((\lambda x).t_2(x)r))\]

Since we are assuming that \(\text{RED}_l((\lambda x).t(x)r) \downarrow\), we know in particular that \(\text{RED}_{l-1}((\lambda x).t_1(x)r) \downarrow\) and \(\text{RED}_{l-1}((\lambda x).t_2(x)r) \downarrow\), and hence

\[\text{RED}_{l-1}(\text{RED}_{l-1}((\lambda x).t_1(x)r) \text{RED}_{l-1}((\lambda x).t_2(x)r)) = \text{RED}_{l-1}(t_1(r)t_2(r)) = \text{RED}_{l-1}(t(r))\]

and in particular \(\text{RED}_{l-1}(t(r)) \downarrow\). \(\square\)

**Proposition 5.8.** For any \(m, n \in \omega\), there is a closed normal form \(t_m\) and a normal form \(t'_m(x)\) with free variable \(x\) such that for all \(r \in T\)

1. \(\text{RED}(t_m r) \downarrow\) if and only if the \(m\)th Turing machine halts on input \(m\), and if this occurs \(\text{RED}(t_m r) = I\) (I := skk)
2. \(\text{RED}(t'_m(n)r) \downarrow\) if and only if the \(m\)th Turing machine halts on input \(m\), and if this occurs \(\text{RED}(t_m(n)r) = F(n)\)
3. \(t_m\) contains no \(\xi_i\) and \(t'_m\) contains \(\xi_i\) for \(i = n\) only

**Proof.** By representability of computable functions in pcas (see eg [27] or [3]), one can construct \(u_m\) such that for every \(k \in \omega\), and every \(v \in T\)

\[u_{m,k} = \begin{cases} 
  kI & \text{if the } m\text{th Turing machine halts by stage } k \\
  (\lambda z).(z \underline{k} + 1) & \text{if the } m\text{th Turing machine does not halt by stage } k 
\end{cases}\]

Then, following the construction in the fixed point theorem,

\[w := (\lambda x).(\lambda y).u_m(yx)\]
\[v := ww\]

\[= (\lambda y).u_m(ww)\]
Then if the $m$th Turing machine halts at stage $k$,

$\begin{align*}
v\mathbf{0} & \simeq u_m\mathbf{0}(ww) \\
& \simeq u_m\mathbf{0}v \\
& \simeq v_1 \\
& \vdots \\
& \simeq v_k \\
& \simeq u_mk(ww) \\
& \simeq (kI)(ww) \\
& \simeq I
\end{align*}$

In particular $v\mathbf{0} \downarrow$.

Now suppose that the $m$th Turing machine never halts. We show by induction on $l$ that for all $l \in \omega$ and for all $k \in \omega$,

$\text{RED}_l(v_k) \uparrow$

Assume that for all $k \in \omega$ and for all $l' < l$ the statement above holds and assume for a contradiction that $\text{RED}_l(v_k) \downarrow$. Note that

$\text{RED}_l(v_k) = \text{RED}_l(((\lambda y).u_m y(ww))k)$

and so by proposition 5.7 we know in particular that $\text{RED}_{l-1}(u_mk(ww)) \downarrow$. But in this case

$\begin{align*}
\text{RED}_{l-1}(u_mk(ww)) &= \text{RED}_{l-2}(\text{RED}_{l-2}(u_mk)\text{RED}_{l-2}(ww)) \\
&= \text{RED}_{l-2}((\lambda z).(zk + 1)v) \\
&= \text{RED}_{l-3}(v_k + 1)
\end{align*}$

and so in particular $\text{RED}_{l-3}(v_k + 1) \downarrow$ giving a contradiction as required.

Finally, let $t_m = s(kv)(k\mathbf{0})$. Then, for all $r$, $t_m r \simeq v\mathbf{0}$, by the basic properties of $s$ and $k$.

For parts 2 and 3, let $t_m$ be as above and let $t'_m(x) = s(st_m(kx))(k\xi_n)$. Note that part 2 follows from the basic properties of $s$ and $k$ and that part 3 is clear from the definition of $t_m$.

**Lemma 5.9** (Preservation of Atoms). Let $e$ be a type 2 identity in $\mathcal{T}$. Then for any $n$, there is some type 1 $f$ in $\mathcal{T}$ such that $\text{RED}(e.f)$ contains the atom $\xi_n$ as a subterm and furthermore, only contains $\xi_i$ such that $i = n$.

**Proof.** We assume that this is not the case and derive a contradiction.

We will define a (computable) family $f_m(x)$ of normal forms with one free variable such that for each $F$, $f_m(\xi_F)$ is type 1 in $\mathcal{T}$.
Let $g_m \in T$ be such that for all $l \in \omega$, $g_m^l = k(k0)$ if the $m$th Turing machine with input $m$ has not halted by stage $l$ and $g_m^l = I$ if the $m$th Turing machine has halted by stage $l$. We can do this using the representability of primitive recursive functions in PCAS.

Then let $t'_m$ be as in proposition 5.8. Define

$$f_m(x) := s(s(g_m(k(t'_m(x))))I)$$

Note that this is in normal form, and that if the $m$th Turing machine has not halted by stage $l$ then for any $\zeta$,

$$\text{RED} (f_m(\zeta))^l \simeq \text{RED} (\text{RED} (s(g_m(k(t'_m(\zeta))))I))^l$$

$$\simeq \text{RED} (\text{RED} (g_m(t'_m(\zeta))^l)$$

$$\simeq \text{RED} (\text{RED} (k(k0)t'_m(\zeta))^l)$$

$$\simeq \text{RED} (k0)^l$$

In particular, see that $f_m(\zeta)^l \downarrow$ even if the $m$th Turing machine never halts on input $m$. If the $m$th Turing machine on input $m$ has halted by stage $l$, then

$$\text{RED} (f_m(\zeta))^l \simeq \text{RED} (\text{RED} (s(g_m(k(t'_m(\zeta))))I))^l$$

$$\simeq \text{RED} (\text{RED} (g_m(t'_m(\zeta))^l)$$

$$\simeq \text{RED} (\text{RED} (k(k0)t'_m(\zeta))^l)$$

$$\simeq \text{RED} (k0)^l$$

$$\simeq F(n)$$

Hence for any $m \in \omega$ and any $\zeta$, $f_m(\zeta)$ is type 1 in the sense we defined earlier.

We therefore know that $e.f_m(\zeta) \simeq \text{RED}(e.f_m(\zeta)) \downarrow$ and by hypothesis $\text{RED}(e.f)$ cannot contain $\xi_n$. For convenience, in the below we will assume that $F$ is chosen such that $\zeta$ does not occur anywhere in $e$.

Note that we can carry out an algorithm to find $\text{RED}(e.f_m(\zeta))$ from $m$. (Only finitely many $\xi_i$’s and $\zeta_G$’s occur in $e$ and $f_m(\zeta)$, so we can give these terms Gödel numbers and the $\zeta$-rule does not cause a problem here because we only need to know $G(k)$ where $\zeta_G$ occurs in $e$ or $G = F$ and $\xi_k$ occurs in $e$ or $k = n$ and this is only finitely much information).

Furthermore, note that when we carry out this algorithm we can check whether or not we ever need to evaluate $\text{RED}(t'_m(\zeta)\xi)$ for some $r$. If we did need to evaluate this, then in particular $\text{RED}(t'_m(\zeta)\xi) \downarrow$ and so the $m$th Turing machine must halt on input $m$. On the other hand, if we did not need to evaluate $\text{RED}(t'_m(\zeta)\xi)$, then $\zeta$ was never used in the $\zeta$-rule because it only ever occurs as a subterm of the normal form $t'_m(\zeta)$. Furthermore, by hypothesis $t'_m(\zeta)$ cannot occur as a subterm of $\text{RED}(e.f_m(\zeta))$, because otherwise $e.f_m(\zeta)$ would contain $\xi_n$. 
Hence if we choose $F'$ such that $F'(n) \neq F(n)$ then

$$\text{RED}(e.f_m(\zeta_F)) = \text{RED}(e.f_m(\zeta_{F'}))$$

But note that this means $f_m(\zeta_F)$ and $f_m(\zeta_{F'})$ must have the same value on every $l$. This can only happen if they are both identically zero and hence the $m$th Turing machine does not halt on input $m$.

Therefore we could use such an algorithm to solve the halting problem and we derive our contradiction. □

5.3 Automorphisms of $T$

Suppose that $\pi : \omega_{>0} \to \omega_{>0}$ is a permutation. Then $\pi$ induces an automorphism $\alpha : T \to T$ as follows.

1. $\alpha(\xi_n) = \xi_{\pi(n)}$
2. $\alpha(\zeta_F) = \zeta_{F\circ \pi^{-1}}$
3. $\alpha(s) = s$
4. $\alpha(k) = k$
5. $\alpha(s.t) = \alpha(s)\alpha(t)$

Note that we have chosen the action of $\alpha$ on the $\zeta_F$ so that it is compatible with the $\zeta$-rule and the action of $\alpha$ on the $\xi_n$. $\alpha$ is clearly therefore an automorphism of $T$.

6 A Useful Lemma

Before we move onto the proof itself, we prove a lemma that is true in general for any pca $A$. Informally, what this says is the property of being injectively presented can be inherited “up to realizability” across sets that are realizably equal.

Lemma 6.1. There is some $e \in A$ such that for any $a, b \in V(A)$, if $f \in A$ is such that $f \vdash a = b$ and $a$ is injectively presented, and if $\langle g, c \rangle, \langle g, c' \rangle \in b$, then

$$efg \vdash c = c'$$

Proof. Since $f \vdash a = b$, there must be $\langle((f)_1 g)_0, d\rangle, \langle((f)_1 g)_0, d'\rangle \in a$ such that

$$((f)_1 g)_1 \vdash c = d$$
$$((f)_1 g)_1 \vdash c' = d'$$

Since $a$ is injectively presented, we know in fact that $d = d'$ and so

$$i_1((f)_1 g)_1(i_1((f)_1 g)_1) \vdash c = c'$$

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Hence we can take
\[ e := (\lambda x, y).i((x)_{1}y)_{1}(i_{x}(x)_{1}y)_{1} \]

\[ \square \]

7 Failure of the Existence Property

We will show that the existence property fails for CZF in the following instance.

**Theorem 7.1.** There is no formula with one free variable \( \chi(x) \) such that

\[ \text{CZF} \vdash (\exists x)\chi(x) \]

and

\[ \text{CZF} \vdash \chi(x) \rightarrow x \subseteq \text{mv}(\mathbb{N}, \mathbb{N}) \land (\forall R \in \text{mv}(\mathbb{N}, \mathbb{N}))(\exists S \in x)S \subseteq R \]

This will immediately give the following corollary.

**Corollary 7.2.** CZF does not have wEP.

**Proof.** We know that

\[ \text{CZF} \vdash (\exists x)(x \subseteq \text{mv}(\mathbb{N}, \mathbb{N}) \land (\forall R \in \text{mv}(\mathbb{N}, \mathbb{N}))(\exists S \in x)S \subseteq R) \]

Suppose that there is some \( \psi(x) \) such that

\[ \text{CZF} \vdash (\exists x)\psi(x) \]

\[ \text{CZF} \vdash (\forall x)\psi(x) \rightarrow (\exists z)z \in x \]

\[ \text{CZF} \vdash (\forall x)\psi(x) \rightarrow (\forall z \in x) \]

\[ (z \subseteq \text{mv}(\mathbb{N}, \mathbb{N}) \land (\forall R \in \text{mv}(\mathbb{N}, \mathbb{N}))(\exists S \in z)S \subseteq R) \]

Then by taking \( \chi(w) \) to be \( \forall x\psi(x) \rightarrow w = \bigcup x \), we would get

\[ \text{CZF} \vdash (\exists w)\chi(w) \]

and

\[ \text{CZF} \vdash \chi(w) \rightarrow w \subseteq \text{mv}(\mathbb{N}, \mathbb{N}) \land (\forall R \in \text{mv}(\mathbb{N}, \mathbb{N}))(\exists S \in w)S \subseteq R \]

contradicting the theorem. \( \square \)

**Proof of theorem 7.1** Assume that there is such a \( \chi(x) \).

Let \( T \) be the pca from section 5 and let \( G \) be the group of all automorphisms obtained from permutations of \( \omega \), as in section 5.3. Let \( \Gamma \) be the normal filter generated by \( \{ \text{Stab}_G(\xi_n) \mid n \in \omega \} \). Hence if \( G \) acts on some class \( X \), then for \( x \in X \), \( \text{Stab}_G(x) \in \Gamma \) means that \( x \) “has finite support relative to the \( \xi_n \).”

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By the soundness theorems, there must be $C_{ip} \in V_{0}^{ip}(T)$ and $C^{\Gamma} \in V_{0}^{\Gamma}(T)$ such that

$$V_{0}^{ip}(T) \models \chi(C_{ip}) \quad V_{0}^{\Gamma}(T) \models \chi(C^{\Gamma})$$

Hence we must have that

$$V(T) \models \chi(C_{ip}) \land \chi(C^{\Gamma})$$

and so

$$V(T) \models C_{ip} = C^{\Gamma}$$

This allows to apply lemma [6.1] and deduce that there is some $e_0$ such that for any $\langle f, c \rangle, \langle f, c' \rangle \in C^{\Gamma}$,

$$e_0.f \upharpoonright_1 c = c'$$

In fact this is the only point where we need $C_{ip}$ and we can now derive a contradiction by examining $C^{\Gamma}$ carefully.

Recall that we can assume that the elements of $\mathbb{N}^\mathbb{N}$ are of the form $\langle f, (\lambda x).x \rangle$ as described in section [1.5].

Write $\zeta_1$ for $\zeta_{(\lambda x).x}$ and for each $N$, construct $R_N \in V_{0}^{\Gamma}(T)$,

$$R_N := \{ \langle 0, f, (\overline{f}, \overline{n}) \rangle \mid f \text{ is type } 1, n \leq N, \overline{n} = \zeta_1 f \} \cup$$

$$\{ \langle 0, f, (\overline{f}, \overline{n}) \rangle \mid f \text{ is type } 1, n > N, \zeta_1 f > N \}$$

(where we write $(\cdot)$ for $V_{0}^{\Gamma}(T)$’s internal notion of ordered pairs)

**Lemma 7.3.** We have constructed these $R_N$ so that the following hold:

1. $R_N \in V_{0}^{\Gamma}(T)$. In fact $\bigcap_{i=1}^{N} \text{Stab}_G(\xi_i) \subseteq \text{Stab}_G(R_N)$.

2. There is some $e_1 \in T$ such that for all $N$, $e_1 \upharpoonright_0 R_N \in \text{mv}(\mathbb{N}^\mathbb{N}, \mathbb{N})$.

3. Suppose that $\langle f, a \rangle \in R_N$ and $\xi_i$ occurs in $f$ only if $i \leq N$. Then $a = (\overline{f}, \overline{\overline{n}})$ where $n = \zeta_1 f$ (and $n \leq N$).

**Proof.** For 1, note that each set in the binary union in the definition of $R_N$ is preserved by elements of $\bigcap_{i=1}^{N} \text{Stab}_G(\xi_i)$.

For 2, note that each $R_N$ can be “represented” by $\zeta_1$. This can clearly be used to produce a realizer that these are multi valued functions.

Part 3 is clear from the definition. \qed

We will aim for our contradiction by first showing a lemma stating that any automorphism satisfying certain properties has to be the identity. This will use the key lemma from section [5.2] as well as the basic properties of $R_N$. We will then construct a non trivial automorphism satisfying these conditions. In this lemma we work over $V(T)$ rather than $V_{0}^{\Gamma}(A)$. 


Lemma 7.4. Suppose that \( a \in V(T) \), \( N < N' \in \mathbb{N} \), and \( e, f \in T \) are such that

1. For any \( \xi_i \) occurring in \( e \) or \( f \), \( i \leq N \)

2. \( \bigcap_{i=1}^{N} \text{Stab}_G(\xi_i) \subseteq \text{Stab}_G(a) \)

3. \( e \models_1 (\forall x \in a)x \in R_N \)

4. \( f \models_1 (\forall x \in \mathbb{N}^N)(\exists y \in a)(\exists z \in \mathbb{N})y = (x, z) \)

Then, whenever \( \alpha \in G \) fixes \( \xi_i \) for \( i \leq N \) and \( i > N' \), \( \alpha \) must also fix \( \xi_i \) for \( N < i \leq N' \) and hence \( \alpha \) must be the identity.

Proof. We first check that \( (\lambda x). (e(fx)_0)_0 \) is of type 2 identity.

Let \( g \) be type 1. Then \( (g,\overrightarrow{y}) \in \mathbb{N}^{\mathbb{N}} \). Therefore there is \( b \) such that \( \langle (fg)_0, b \rangle \in a \) and \( (fg)_1 \models_1 (\exists z \in \mathbb{N})b = (\overrightarrow{z}, z) \).

Let \( h := (e(fg)_0)_0 \). Then we know that there is some \( c \) such that \( \langle h, c \rangle \in R_N \) and \( (e(fg)_0)_1 \models_1 b = c \). By the basic properties of \( R_N \), we know that \( c \) must be of the form \( (\overrightarrow{h}, \overrightarrow{m}) \) for some \( m \in \mathbb{N} \). From above we know that \( V(T) \models (\exists z \in \omega) b = (\overrightarrow{z}, z) \). Hence \( g \) and \( h \) must have the same graphs as type 1 elements, and so \( (\lambda x). (e(fx)_0)_0 \) is of type 2 identity as required.

Now let \( \alpha \in G \) fix \( \xi_i \) for \( i \leq N \) and \( i > N' \). Suppose for a contradiction that there is some \( n \) with \( N < n < N' \) such that \( \alpha(\xi_n) \neq \xi_n \).

By applying lemma 5.3. we can find a first order \( g \) such that \( g \) only contains \( \xi_i \) for \( i = n \) and such that \( \xi_n \) does occur in \( (e(fg)_0)_0 \). Let \( b, h \), and \( c \) be as above, but for this particular \( g \).

Since \( e \) and \( f \) only contain \( \xi_i \) for \( i \leq N \), we know that \( h \) can only contain \( \xi_i \) for \( i \leq N \) or \( i = n \). Since we have guaranteed that \( h \) does contain \( \xi_n \), we know that \( \alpha_1 h = \overrightarrow{m} \). In particular \( n \leq N' \), so we know from the definition of \( R_{N'} \) that \( c \) must be of the form \( (\overrightarrow{h}, \overrightarrow{m}) \).

Since \( \alpha \) fixes \( \xi_i \) for \( i \leq N \) we know from our assumptions that \( \alpha \) also fixes \( a \). Therefore, since \( ((fg)_0, b) \in a \) we must also have \( \alpha((fg)_0, \alpha(b)) \in a \). Hence if \( h' := (e\alpha((fg)_0)_0 \) we know that there is some \( \xi \) with \( \langle h', \xi \rangle \in R_{N'} \) and \( V(T) \models \alpha(b) = \xi \).

Since \( \alpha \) fixes \( \xi_i \) for \( i \leq N \) we know that \( \xi_i \) can only occur in \( e \) and \( \alpha(f) \) for \( i \leq N' \). Since \( \alpha \) fixes \( \xi_i \) for \( i > N' \), we know that \( \alpha(\xi_n) \) must be amongst \( \xi_i \) for \( i \leq N' \). Hence \( h' \) only contains \( \xi_i \) for \( i \leq N' \). Furthermore neither \( \alpha(f) \) nor \( \alpha(g) \) contains \( \xi_n \) and from the assumption that \( \alpha(\xi_n) \neq \xi_n \) we also know that \( \xi_n \) does not occur in \( \alpha(g) \). Hence \( \xi_1 h' = \overrightarrow{m} \) for some \( m \leq N' \) with \( m \neq n \). Again from the definition of \( R_{N'} \), we know therefore that \( \xi \) is of the form \( (\overrightarrow{h}, \overrightarrow{m}) \).

But then since \( V(T) \models b = (\overrightarrow{h}, \overrightarrow{m}) \), we have that \( V(T) \models \alpha(b) = (\alpha(\overrightarrow{h}), \alpha(\overrightarrow{m})) \). Together with \( V(T) \models \alpha(b) = (\overrightarrow{h}, \overrightarrow{m}) \) this gives \( V(T) \models \alpha(\overrightarrow{m}) = \overrightarrow{m} \). In fact \( \alpha(\overrightarrow{m}) = \overrightarrow{m} \), so \( V(T) \models \overrightarrow{m} = \overrightarrow{m} \). But this is a contradiction since \( m \neq n \). \( \square \)
Since we know $V^\Gamma_0(T) \models (\forall x \in \text{mv}(\mathbb{N}^n, \mathbb{N}))(\exists y \in C)(y \subseteq x \land y \in \text{mv}(\mathbb{N}^n, \mathbb{N}))$, there must be $f, e_2, e_3 \in T$ and $c_n$ such that for all $n$,

$$
\langle 0, f, c_n \rangle \in C \\
e_2 \models^\Gamma_0 (\forall x \in c_n)(x \in R_n) \\
e_3 \models^\Gamma_0 (\forall x \in \mathbb{N}^N)(\exists y \in c_n)(\exists z \in \mathbb{N})y = (x, z)
$$

In particular we know that for all $n$, $\text{Stab}_G(c_n) \in \Gamma$ and hence by proposition 3.2. $\text{Stab}_G(c_n^i) \in \Gamma$. From now on we will work entirely over $V(T)$.

Recall that we chose $e_0$ so that for all $m$ and $n$,

$$e_0 \circ f \models^1 (\forall x \in c_0^m) c_n^o = c_n^o$$

and so by substitution we can use $e_0 \circ f$ and $e_2$ to construct $e_4$ such that for all $m$ and $n$

$$e_4 \models^1 (\forall x \in c_0^m)x \in R_n^o$$

Now let $N$ be large enough such that the list $\xi_1, \ldots, \xi_N$ includes any $\xi_n$ in a support of $c_0^m$, or appearing in $e_0, e_1, e_2, e_3$ or $e_4$.

Let $N' = N + 2$.

Note that we have

$$e_4 \models^1 (\forall x \in c_0^m)x \in R_{N'}^o$$

Let $\alpha$ be the automorphism that swaps round $\xi_{N+1}$ and $\xi_{N+2}$, fixing everything else. Then we know that $\alpha$ fixes $\xi_i$ for $i \leq N$ (and hence also fixes $c_0^m$ and any $\xi_i$ occurring in $e_3$ and $e_4$) and fixes $\xi_i$ for $i > N'$. However, clearly $\alpha$ does not fix $\xi_{N+1}$. Hence we can finally get a contradiction by applying lemma 7.3.

8 Conclusion

8.1 What the failure of EP means

We have shown that CZF does not have EP, or indeed even wEP. Since EP was described in the introduction as a property to be expected from constructive formal theories, one might ask if its failure indicates some weakness in CZF as a constructive theory. The short answer is no: CZF is still a sound foundation for constructive mathematics.

The main theorem of this paper shows essentially that CZF asserts the existence of mathematical objects that it does not know how to construct. However, CZF does have natural interpretations in which these objects can be constructed. One example is Aczel’s original interpretation of CZF into type theory in [1]. Here, the sets asserted in the fullness axiom are sets of those multivalued relations that arise from elements of a particular exponential type. Another (related) interpretation is Rathjen’s “formulas as classes” in [21], in which CZF is interpreted into CZF$_E$. In this example the full sets appear as
exponentials in the background universe. In [25] Rathjen and Tupailo showed using these techniques that CZF with a choice principle $\Pi_\Sigma - AC$ has a form of the existence property.

8.2 Further Work

In this paper we used the axioms of ZFC in several places, with what appears to be an essential use of choice in the soundness of strong collection in $V^\text{ip}_0(A)$. This means that the final result of the paper was only proved on the assumption that ZFC is consistent. We conjecture that in fact this assumption is unwarranted and with a more sophisticated construction the result can shown only on the assumption that CZF is consistent.

Choice principles tend to fail in $V^T_0(A)$ (in fact one can check that countable choice fails in $V^T_0(T)$), so it remains open whether CZF together with particular choice principles have EP.

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