Saturated Models in Mathematical Fuzzy Logic*

Guillermo Badia  
Department of Knowledge-Based Mathematical Systems  
Johannes Kepler University Linz  
Linz, Austria  
guillebadia89@gmail.com  

Carles Noguera  
Institute of Information Theory and Automation  
Czech Academy of Sciences  
Prague, Czech Republic  
noguera@utia.cas.cz  

Abstract—This paper considers the problem of building saturated models for first-order graded logics. We define types as pairs of sets of formulas in one free variable which express properties that an element is expected, respectively, to satisfy and to falsify. We show, by means of an elementary chains construction, that each model can be elementarily extended to a saturated model where as many types as possible are realized. In order to prove this theorem we obtain, as by-products, some results on tableaux (understood as pairs of sets of formulas) and their consistency and satisfiability, and a generalization of the Tarski–Vaught theorem on unions of elementary chains.  

Index Terms—mathematical fuzzy logic, first-order graded logics, uninorms, residuated lattices, logic UL, types, saturated models, elementary chains

I. INTRODUCTION

Mathematical fuzzy logic studies graded logics as particular kinds of many-valued inference systems in several formalisms, including first-order predicate languages. Models of such first-order graded logics are variations of classical structures in which predicates are evaluated over wide classes of algebras of truth degrees, beyond the classical two-valued Boolean algebra. Such models are relevant for recent computer science developments in which they are studied as weighted structures (see e.g. [23]).

The study of models of first-order fuzzy logics is based on the corresponding strong completeness theorems [10], [21] and has already addressed several crucial topics such as: characterization of completeness properties with respect to models based on particular classes of algebras [7], models of logics with evaluated syntax [26], [27], study of mappings and diagrams [13], ultraproduct constructions [14], [15], characterization of elementary equivalence in terms of elementary mappings [16], characterization of elementary classes as those closed under elementary equivalence and ultraproducts [15], Löwenheim–Skolem theorems [17], and back-and-forth systems for elementary equivalence [18]. A related stream of research is that of continuous model theory [2], [6].

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Another important item in the classical agenda is that of saturated models, that is, the construction of structures rich in elements satisfying many expressible properties. In continuous model theory the construction of such models is well known (cf. [1]). However, the problem has not yet been addressed in mathematical fuzzy logic, but only formulated in [15], where Dellunde suggested that saturated models of fuzzy logics could be built as an application of ultraproduct constructions. This idea followed the classical tradition found in [5]. However, in other classical standard references such as [22], [24], [28] the construction of saturated structures is obtained by other methods. The goal of the present paper is to show the existence of saturated models for first-order graded logics by means of an elementary construction.

The paper is organized as follows: after this introduction, Section II presents the necessary preliminaries we need by recalling several semantical notions from mathematical fuzzy logic, namely, the algebraic counterpart of extensions of the uninorm logic UL, fuzzy first-order models based on such algebras, and some basic model-theoretic notions. Section III introduces the notion of tableaux (necessary for our treatment of types) as pairs of sets of formulas and proves that each consistent tableau has a model. Section IV defines types as pairs of sets of formulas in one free variable (roughly speaking, expressing the properties that an element should satisfy and falsify) and contains the main results of the paper: a fuzzy version of the Tarski-Vaught theorem for unions of elementary chains and the existence theorem for saturated models. Finally, Section V ends the paper with some concluding remarks.

II. PRELIMINARIES

In this section we introduce the object of our study, fuzzy first-order models, and several necessary related notions for the development of the paper. For comprehensive information on the subject, one may consult the handbook of Mathematical Fuzzy Logic [8] (e.g. Chapters 1 and 2).

We choose, as the underlying propositional basis for the first-order setting, the class of residuated uninorm-based logics [25]. This class contains most of the well-studied particular systems of fuzzy logic that can be found in the literature and has been recently proposed as a suitable framework for reasoning with graded predicates in [11], while it retains important properties, such as associativity and commutativity.
of the residuated conjunction, that will be used to obtain the results of this paper.

The algebraic semantics of such logics is based on UL-algebras, that is, algebraic structures in the language \( \mathcal{L} = \{ \land, \lor, \& \rightarrow, \top, \bot, \top \} \) of the form \( A = \langle A, \land^A, \lor^A, \&^A, \rightarrow^A, \top^A, \bot^A, \top^A, \bot^A \rangle \) such that

- \( \langle A, \land^A, \lor^A, \&^A, \rightarrow^A, \top^A, \bot^A \rangle \) is a bounded lattice,
- \( \langle A, \&^A, \top^A \rangle \) is a commutative monoid,
- for each \( a, b, c \in A \), we have:

\[
A \text{ is called a UL-chain if its underlying lattice is linearly ordered. Standard UL-chains are those define over the real unit interval } [0, 1] \text{ with its usual order; in that case the operation } \&^A \text{ is a residuated uninorm, that is, a left-continuous binary associative commutative monotonous operation with a neutral element } \top^A \text{ (which need not coincide with the value } 1).}
\]

Let \( \text{FM}_\mathcal{L} \) denote the set of propositional formulas written in the language of UL-algebras with a denumerable set of variables and let \( \text{FM}_\mathcal{L} \) be the absolutely free algebra defined on such set. Given a UL-algebra \( A \), we say that an \( A \)-evaluation is a homomorphism from \( \text{FM}_\mathcal{L} \) to \( A \). The logic of all UL-algebras is defined by establishing, for each \( \Gamma \cup \{ \} \subseteq \text{FM}_\mathcal{L} \), \( \Gamma \models \phi \) if and only if, for each UL-algebra \( A \) and each \( A \)-evaluation \( e \), we have \( e(\phi) \geq \top^A \), whenever \( e(\psi) \geq \top^A \) for each \( \psi \in \Gamma \). The logic UL is, hence, defined as preservation of truth over all UL-algebras, where the notion of truth is given by the set of designated elements, or filter, \( \mathcal{F}^A = \{ a \in A \mid a \geq \top^A \} \). The standard completeness theorem of UL proves that the logic is also complete with respect to its intended semantics: the class of UL-chains defined over \([0, 1]\) by residuated uninorms (the standard UL-chains); this justifies the name of UL (uninorm logic).

Most well-known propositional fuzzy logics can be obtained by extending UL with additional axioms and rules (in a possibly expanded language). Important examples are Gödel–Dummett logic \( G \) and Łukasiewicz logic \( L \).

A predicate language \( \mathcal{P} \) is a triple \((\mathcal{P}, F, ar)\), where \( \mathcal{P} \) is a non-empty set of predicate symbols, \( F \) is a set of function symbols, and \( ar \) is a function assigning to each symbol a natural number called the arity of the symbol. Let us further fix a denumerable set \( V \) whose elements are called object variables. The sets of \( \mathcal{P} \)-terms, atomic \( \mathcal{P} \)-formulas, and \((\mathcal{L}, \mathcal{P})\)-formulas are defined as in classical logic. A \( \mathcal{P} \)-structure \( \mathfrak{M} \) is a pair \((A, M)\) where \( A \) is a UL-chain and \( M = \langle M, (P_M)_{P \in \mathcal{P}}, (F_M)_{F \in \mathcal{F}} \rangle \), where \( M \) is a non-empty domain; \( P_M \) is a function \( M^n \to A \), for each \( n \)-ary predicate symbol \( P \in \mathcal{P} \); and \( F_M \) is a function \( M^n \to M \) for each \( n \)-ary function symbol \( F \in \mathcal{F} \). An \( \mathfrak{M} \)-evaluation of the object variables is a mapping \( v : V \to M \); by \( v[x \to a] \) we denote the \( \mathfrak{M} \)-evaluation of the object variables where \( v[x \to a](x) = a \) and \( v[x \to a](y) = v(y) \) for each object variable \( y \neq x \). We define the values of the terms and the truth values of the formulas as (where for \( \circ \) stands for any \( n \)-ary connective in \( \mathcal{L} \)):

\[
\| x \|_\mathfrak{M} = v(x),
\| F(t_1, \ldots, t_n) \|_\mathfrak{M} = F_M((\| t_1 \|_\mathfrak{M}, \ldots, \| t_n \|_\mathfrak{M}))
\| P(t_1, \ldots, t_n) \|_\mathfrak{M} = P_M((\| t_1 \|_\mathfrak{M}, \ldots, \| t_n \|_\mathfrak{M}))
\| \circ(\varphi_1, \ldots, \varphi_n) \|_\mathfrak{M} = \circ^A((\| \varphi_1 \|_\mathfrak{M}, \ldots, \| \varphi_n \|_\mathfrak{M}))
\| (\exists x)\varphi \|_\mathfrak{M} = \sup \{ \| \varphi \|_\mathfrak{M} | m \in M \}
\| (\forall x)\varphi \|_\mathfrak{M} = \inf \{ \| \varphi \|_\mathfrak{M} | m \in M \}.
\]

If the infimum or supremum does not exist, the corresponding value is undefined. We say that \( \mathfrak{M} \) is a safe if \( \| \varphi \|_\mathfrak{M} \) is defined for each \( \mathcal{P} \)-formula \( \varphi \) and each \( \mathfrak{M} \)-evaluation \( v \). Formulas without free variables are called sentences and a set of sentences is called a theory. Observe that if \( \varphi \) is a sentence, then its value does not depend on a particular \( \mathfrak{M} \)-evaluation; we denote its value as \( \| \varphi \|_\mathfrak{M} \). If \( \varphi \) has free variables among \( \{ x_1, \ldots, x_n \} \) we will denote it as \( \varphi(x_1, \ldots, x_n) \); then the value of the formula under a certain evaluation \( v \) depends only on the values given to the free variables; if \( v(x_i) = d_i \in \mathcal{M} \) we denote \( \| \varphi \|_v \) as \( \| \varphi(d_1, \ldots, d_n) \|_\mathfrak{M} \). We say that \( \mathfrak{M} \) is a model of a theory \( T \), in symbols \( \mathfrak{M} \models T \), if it is safe and for each \( \varphi \in T \), \( \| \varphi \|_\mathfrak{M} \geq \top^A \).

Observe that we allow arbitrary UL-chains and we do not focus in any kind of standard completeness properties.

Using the semantics just defined, the notion of semantical consequence is lifted from the propositional to the first-order level in the obvious way. Such first-order logics satisfy two important properties that we will use in the paper (see e.g. [9]), for each theory \( T \cup \{ \varphi, \psi, \chi \} \) (inductively defining for each formula \( \alpha : \alpha^0 = \top \), and for each natural \( n, \alpha^{n+1} = \alpha^n \& \alpha \)):

1. Local deduction theorem: \( T, \varphi \models \psi \) if, and only if, there is a natural number \( n \) such that \( T \models (\varphi \land \chi)^n \ra \psi \).
2. Proof by cases: If \( T, \varphi \models \chi \) and \( T, \psi \models \chi \), then \( T, \varphi \lor \psi \models \chi \).
3. Consequence compactness: If \( T \models \varphi \), then for some finite \( T_0 \subseteq T \), \( T_0 \models \varphi \).

Observe that alternatively we could have introduced calculi and a corresponding notion of deduction \( \vdash \) for these logics, but we prefer to keep the focus of the paper on the semantics.

III. Tableaux

A tableau is a pair \( \langle T, U \rangle \) such that \( T \) and \( U \) are sets of formulas. A tableau \( \langle T_0, U_0 \rangle \) is called a subtableau of \( \langle T, U \rangle \) if \( T_0 \subseteq T \) and \( U_0 \subseteq U \). \( \langle T, U \rangle \) is satisfied by a model \( \mathfrak{M} = \langle A, M \rangle \), if there is an \( \mathfrak{M} \)-evaluation \( v \) such that for each \( \varphi \in T \), \( \| \varphi \|_v \geq \top^A \), and for all \( \psi \in U \), \( \| \psi \|_v < \top^A \). Also, we write \( \langle T, U \rangle \models \varphi \) meaning that for any model and evaluation that satisfies \( \langle T, U \rangle \), the model and the evaluation must make \( \varphi \) true as well. A tableau \( \langle T, U \rangle \) is said to be consistent if there is no finite \( U_0 \subseteq U \) such that \( T \models \lor U_0 \). In the extreme case, we define \( \lor \emptyset \) as \( \bot \). Our choice of terminology here comes from [4], where such tableaux are introduced for the intuitionistic setting, where Boolean negation is also absent.
The intuitive idea is that in a semantic tableau as we go along we try to make everything on the left true while falsifying everything on the right.

Following [21], we say that a set of sentences $T$ is a $\exists$-Henkin theory if, whenever $T \models (\exists x)\varphi(x)$, there is a constant $c$ such that $T \models \varphi(c)$. $T$ is a Henkin theory if $T \not\models (\forall x)\varphi(x)$ implies that there is a constant $c$ such that $T \not\models \varphi(c)$. $T$ is doubly Henkin if it is both $\exists$-Henkin and Henkin. $T$ is a linear theory if for any pair of sentences $\varphi, \psi$ either $T \models \varphi \rightarrow \psi$ or $T \not\models \varphi$.

The following result shows that each consistent tableau has a model, which will be necessary in the next section.

**Theorem 1. (Model Existence Theorem)** Let $\langle T, U \rangle$ be a consistent tableau. Then there is a model that satisfies $\langle T, U \rangle$.

**Proof.** We will prove this for countable languages, though the generalization to arbitrary cardinals is straightforward. We start by adding a countable set $C$ of new constants to the language. We enumerate as $\varphi_0, \varphi_1, \varphi_2, \ldots$ all the formulas of the expanded language, and as $\langle \theta_0, \psi_0 \rangle, \langle \theta_1, \psi_1 \rangle, \langle \theta_2, \psi_2 \rangle, \ldots$ all pairs of such formulas. We modify the proofs of Theorems 4 and Lemma 2 from [21] by building two chains of theories $T_0 \subseteq \cdots \subseteq T_n \subseteq \cdots$ and $U_0 \subseteq \cdots \subseteq U_n \subseteq \cdots$ such that $\langle U_{i+1}, T_i \rangle$ is a consistent tableau (checking that at each stage we obtain a consistent tableau $\langle T_i, U_i \rangle$), plus $U_{i+1} \subseteq U_i$ is a linear doubly Henkin theory. Then, we will simply construct the canonical model as in Lemma 3 from [21]. We proceed by induction:

**Stage 0:** Define $T_0 = T$ and $U_0 = U$.

**Stage $s + 1 = 3i + 1$:** At this stage, we make sure that our final theory will be Henkin. To this end we follow the proof of Lemma 2 (1) from [21]. If $\varphi$ is not of the form $(\forall x)\chi(x)$, then define $T_{s+1} = T_s$ and $U_{s+1} = U_s$. Assume now that $\varphi = (\forall x)\chi(x)$. Then, we consider the following two cases:

(i) There is a finite $U' \subseteq U_s$ such that $T_s \models (\forall U') \chi(x)$. Then, we define $T_{s+1} = T_s \cup \{ (\forall x)\chi(x) \}$ and $U_{s+1} = U_s$.

(ii) Otherwise, let $T_{s+1} = T_s$ and $U_{s+1} = U_s \cup \{ \chi(c) \}$ (where $c$ is the first unused constant from $U_s$ up to this stage).

We have to check that $\langle T_{s+1}, U_{s+1} \rangle$ is consistent in both cases. Suppose that (i) holds and that $T_s \cup \{ (\forall x)\chi(x) \} \models U'$ for some finite $U' \subseteq U_s$. By construction, we must have that $T_s \models (\forall U') \chi(x)$ for some finite $U' \subseteq U_s$. Take the finite set $U_s = U'_s \cup U''_s$; clearly we also have $T_s \models (\forall U'_s) \chi(x)$. Now, by the local deduction theorem, $T_s \models (\forall U'_s) \chi(x)$ for some $n$. On the other hand, $T_s \models \forall U'_s \rightarrow \forall U''_s$. Recall, that $\forall U'_s \chi(x) = (\forall U''_s \chi(x) \land \exists n)$ (this follows from the rules $\varphi \models \varphi \land \exists n$ and $\varphi, \psi \models \varphi \land \psi$). So, by proof by cases, we have that $T_s \models \forall U''_s \chi(x)$, which means that $T_s \models (\forall U''_s) \chi(x)$, a contradiction since by induction hypothesis $\langle T_s, U''_s \rangle$ is consistent. If (ii) holds, suppose that $\langle T_s, U_s \cup \{ \chi(c) \} \rangle$ is not consistent; then, $T_s \models (\forall U''_s) \chi(x)$ for some finite $U'' \subseteq U_s$. Quantifying the new constant $c$, we must have $T_s \models (\forall x)((\forall U''_s) \chi(x))$, so $T_s \models (\forall U''_s) \chi(x)$, which contradicts the fact that we are considering case (ii).

**Stage $s + 1 = 3i + 2$:** At this stage we make sure that we will eventually obtain an $\exists$-Henkin theory. If $\varphi_i$ is not of the form $(\exists x)\chi(x)$, then let $T_{s+1} = T_s$ and $U_{s+1} = U_s$. Otherwise, as in Lemma 2 (2) from [21], we have two cases to consider:

(i) There is a finite $U' \subseteq U_s$ such that $T_s \cup \{ \varphi_i \} \models U'$, then we define $T_{s+1} = T_s$ and $U_{s+1} = U_s$.

(ii) Otherwise, define $T_{s+1} = T_s \cup \{ \chi(c) \}$ (where $c$ is the first unused constant from $C$) and $U_{s+1} = U_s$.

Again, in both cases $\langle T_{s+1}, U_{s+1} \rangle$ is consistent (check the proof of Lemma 2 (2) from [21]).

**Stage $s + 1 = 3i + 3$:** At this stage we work to ensure that our final theory will be linear. So given the pair $\langle \theta_i, \psi_i \rangle$ proceed as in Lemma 2 (3) from [21]. That is, we start from the assumption that $\langle T_s, U_s \rangle$ is consistent and letting $U_{s+1} = U_s$ we look to add one of $\theta_i \rightarrow \psi_i$ or $\psi_i \rightarrow \theta_i$ to $T_s$ to obtain $T_{s+1}$ while making the resulting tableau $\langle T_{s+1}, U_{s+1} \rangle$ consistent. Note that if $T_s \cup \{ \theta_i \rightarrow \psi_i \} \models U_{s+1}$ and $T_s \cup \{ \psi_i \rightarrow \theta_i \} \models U''_{s+1}$, then $T_s \cup \{ \theta_i \rightarrow \psi_i \} \models U''_{s+1}$ and $T_s \cup \{ \psi_i \rightarrow \theta_i \} \models U''_{s+1} \cup \{ \psi_i \rightarrow \theta_i \} \models U''_{s+1} \cup \{ \psi_i \rightarrow \theta_i \}$ by proof by cases, and since $\varphi \models (\psi_i \rightarrow \theta_i)$, we obtain that $T_s \models (\forall U''_{s+1}) \chi(x)$, a contradiction.

We can already introduce the general notion of type with respect to a given table:

**Definition 1.** A pair of sets of formulas $\langle p, p' \rangle$ is a type of a tableau $\langle T, U \rangle$ if the tableau $\langle T \cup p, U \cup p' \rangle$ is satisfiable.

Let $S_n(T, U)$ be the collection of all complete $n$-types (that is, pairs $\langle p, q \rangle$ in $n$-many free variables such that for any $\phi$, either $\phi \in p$ or $\phi \in q$) of the tableau $\langle T, U \rangle$. This is the space of prime filter-ideal pairs of the $n$-Lindebaum algebra of our logic with the quotient algebra constructed by the relation $\phi \equiv \psi$ iff $\langle T, U \rangle \models \phi \leftrightarrow \psi$.

Given formulas $\phi$ and $\theta$, we define $\langle \langle \sigma, \theta \rangle \rangle = \langle \langle p, p' \rangle \rangle \in S_n(T, U) \mid \sigma \in p, \theta \in p' \rangle$. Consider now the collection $B = \{ \langle \langle \phi, \psi \rangle \rangle \mid \phi, \psi \text{ are formulas} \}$. Intuitively, this simply contains all the sets of pairs of theories such that $\phi$ is expected to be true while $\psi$ is expected to fail, for any two formulas $\phi, \psi$. $B$ is the base for a topology on $S_n(T, U)$ since given $\langle \langle \phi, \psi \rangle \rangle, \langle \langle \phi', \psi' \rangle \rangle \in B$, we have that $\langle \langle \phi \land \phi', \psi \lor \psi' \rangle \rangle \subseteq \langle \langle \phi, \psi \rangle \rangle \land \langle \langle \phi', \psi' \rangle \rangle$. Then, there is a topology on $S_n(T, U)$ such that every open set of $T$ is just the union of a collection of sets from $B$. A topological space is said to be strongly $S$-closed if every family of open sets with the finite intersection property has a non-empty intersection [19]. Moreover, we will say that a space is almost strongly $S$-closed if every family of basic open sets with the finite intersection property has a non-empty intersection.

**Corollary 2.** (Tableaux almost strong $S$-closedness) Let $\langle T, U \rangle$ be a tableau. If every $\langle T_0, U_0 \rangle$, with $|T_0|, |U_0|$ finite and $T_0 \subseteq T$ and $U_0 \subseteq U$, is satisfiable, then $\langle T, U \rangle$ is satisfied in some model.
Proof. It suffices to show that \( (T, U) \) is consistent. Suppose otherwise, that is, there is a finite \( U_0 \subseteq U \) such that \( T \models \bigvee U_0 \). But then for some finite \( T_0 \subseteq T \), \( T_0 \models \bigvee U_0 \). Moreover, this implies that \( (T_0, \{\bigvee U_0\}) \) cannot be satisfiable, but this is a contradiction with the fact that \( (T_0, U_0) \) has a model. \( \square \)

IV. Main Results

Let us start by recalling the notion of (elementary) substructure (see e.g. [17]). \( (A, M) \) is a substructure of \( (B, N) \) if the following conditions are satisfied:

1) \( M \subseteq N \).
2) For each \( n \)-ary function symbol \( F \in F \), and elements \( d_1, \ldots, d_n \in M \),

\[ F_M(d_1, \ldots, d_n) = F_N(d_1, \ldots, d_n) \]

3) \( A \) is a subalgebra of \( B \).
4) For every quantifier-free formula \( \varphi(x_1, \ldots, x_n) \), and \( d_1, \ldots, d_n \in M \),

\[ \|\varphi(d_1, \ldots, d_n)\|_M^A = \|\varphi(d_1, \ldots, d_n)\|_N^B \]

Moreover, \( (A, M) \) is an elementary substructure of \( (B, N) \) if condition 4 holds for arbitrary formulas. In this case, we also say that \( (B, N) \) is an elementary extension of \( (A, M) \). When instead of subalgebra and subset we have a pair of injections \( (g, f) \) satisfying the corresponding conditions above, we have an embedding.

A sequence \( \{ (A_i, M_i) \, | \, i < \gamma \} \) of models is called a chain when for all \( i < j < \gamma \) we have that \( (A_i, M_i) \) is a substructure of \( (A_j, M_j) \). If, moreover, these substructures are elementary, we speak of an elementary chain. The union of the chain \( \{ (A_i, M_i) \, | \, i < \gamma \} \) is the structure \( \langle A, M \rangle \) where \( A \) is the classical union model of the classical chain of algebras \( \{ (A_i, M_i) \, | \, i < \gamma \} \) while \( M \) is defined by taking as its domain \( \bigcup_{i<\gamma} M_i \), interpreting the constants of the language as they were interpreted in each \( M_i \) and similarly with the relational symbols of the language. Let us note that since all the classes of algebras under consideration are classically \( \forall_1 \)-axiomatizable, \( A \) will always be an algebra of the appropriate sort. Observe as well that \( M \) is well defined given that \( \{ (A_i, M_i) \, | \, i < \gamma \} \) is a chain.

THEOREM 3. (Tarski-Vaught theorem on unions of elementary chains) Let \( A = \langle A, M \rangle \) be the union of an elementary chain \( \{ (A_i, M_i) \, | \, i < \gamma \} \). Then, for each sequence \( \tau \) of elements of \( M \) and every formula \( \varphi(\tau) \), \( \|\varphi(\tau)\|_M^A = \|\varphi(\tau)\|_M^{A_i} \).

Moreover, the union \( A = \langle A, M \rangle \) is a safe structure.

Proof. We proceed by induction on the complexity of \( \varphi \). When \( \varphi \) is atomic, the result follows by definition of \( A \). For any \( n \)-ary connective \( \circ \),

\[ \circ^A(\|\varphi(\tau)\|_M^{A_i}, \ldots, \|\varphi(\tau)\|_M^{A_i}) = \circ^A(\|\varphi(\tau)\|_M^A, \ldots, \|\varphi(\tau)\|_M^A) = \]

\[ \|\circ(\varphi(\tau), \ldots, \varphi(\tau))\|_M^A \]

where the second equality follows by the induction hypothesis and the definition of \( A \).

Let \( \varphi = (\exists x)\psi \) (the case of \( \varphi = (\forall x)\psi \) is analogous). Consider \( \|\psi(\tau, x)\|_M^A \) for \( b \in M^n \). Take \( j > i \) sufficiently large such that \( b \in M_j^n \). By induction hypothesis, \( \|\psi(\tau, b)\|_M^{A_i} = \|\psi(\tau, b)\|_M^{A_j} \). By the elementarity of the chain, \( \|\exists x\psi(\tau)\|_M^{A_i} = \|\exists x\psi(\tau)\|_M^{A_j} \). Hence, \( \|\exists x\psi(\tau, b)\|_M^{A_i} \leq A \)

\[ \|\exists x\psi(\tau)\|_M^{A_i} \]

Then \( \|\exists x\psi(\tau)\|_M^{A_i} \) is an upper bound for

\[ \{ \|\exists x\psi(\tau)\|_M^{A_i} \mid b \in M^n \} \]

in \( A \). Moreover, suppose that \( u \) is another such upper bound in \( A \). This means that we can find \( j \geq i \) such that \( u \in A_j \).

Let \( \psi(\tau, x) \parallel A_j \).

This establishes as well that the union is this chain of models is a safe structure, and hence, a model. \( \square \)

A structure \( \langle A, M \rangle \) is said to be exhaustive if every element of \( A \) is the value of some formula for some tuple of objects from \( M \). In the rest of the paper, we will assume that all models are exhaustive. For that purpose we need to make sure that our constructions always give us back exhaustive models. Clearly, the model obtained in the model existence theorem is exhaustive.

COROLLARY 4. The union of an elementary chain of exhaustive models is itself exhaustive.

Proof. Suppose that \( x \in A \), then \( x \in A_i \) for some \( i \), so \( x = \|\varphi(\tau)\|_M^{A_i} \) for some sequence \( \bar{a} \) of elements of \( M_i \) and formula \( \varphi \), but then \( x = \|\varphi(\tau)\|_M^{A_i} \) by Theorem 3. \( \square \)

Given a model \( \mathfrak{M} = \langle A, M \rangle \) and a collection \( D \subseteq M \), we denote by \( \text{Th}_D(\mathfrak{M}) \) the theory of \( \mathfrak{M} \) relative to \( D \), that is, the collection of all sentences \( \varphi \) in a language obtained by augmenting with a list of constants to denote the elements from \( D \) such that \( \|\varphi\|_M^A \geq \mathcal{T}^D \). On the other hand, \( \overline{T}_D(\mathfrak{M}) \) will simply denote the set-theoretic complement of \( \text{Th}_D(\mathfrak{M}) \).

We are finally ready to define the intended notion of type with respect to a model \( \mathfrak{M} \) (observe that it is a particular case of Definition 1 when the tableau is \( (\text{Th}_D(\mathfrak{M}), \overline{T}_D(\mathfrak{M})) \)).

DEFINITION 2. Let \( \mathfrak{M} = \langle A, M \rangle \) be a model. If \( \langle p, p' \rangle \) is a pair of sets of formulas in some variable \( x \) and parameters over some \( D \subseteq M \), we will call \( \langle p, p' \rangle \) a type of \( \langle A, M \rangle \) in \( D \) if the tableau \( \langle \text{Th}_D(\mathfrak{M}) \cup p, \overline{T}_D(\mathfrak{M}) \cup p' \rangle \) is satisfiable (consistent). We will denote the set of all such types by \( S(\langle A, M \rangle)(D) \).
The following definition captures the notion of a model realizing as many types as possible (under a certain cardinal restriction).

**Definition 3.** For any cardinal \( \kappa \), a model \( \mathfrak{M} \) is said to be \( \kappa \)-saturated if for any \( D \subseteq M \) such that \( |D| < \kappa \), any type in \( S^{\mathfrak{M}}(D) \) is satisfiable in \( \mathfrak{M} \).

Before we begin the proof of the main result below, we need to recall the notion of the elementary diagram of a structure. Given a model \( \langle A, M \rangle \), the elementary diagram of \( \langle A, M \rangle \), in symbols \( \text{El diag}(A, M) \), we will denote the theory of \( \langle A, M \rangle \) relative to the whole of \( M \). In a nutshell, \( \text{El diag}(A, M) = \text{Th}_M(A, M) \). This notion has been studied in detail in [13, 16, 21] and we refer the reader to those papers for further information. On the other hand, \( \text{El diag}(A, M) \) will denote the set-theoretic complement of \( \text{El diag}(A, M) \). The important fact for our purposes is that, there is a canonical model (those models given by the model existence theorem) constructed from \( \langle \text{El diag}(A, M), \text{El diag}(A, M) \rangle \) such that we can build an embedding from \( \langle A, M \rangle \) into the new canonical model.

We can observe that in the above definition it suffices to consider types in one free variable. Indeed, the more general case of finitely many variables, say, \( x_0, \ldots, x_n \), can be reduced to the one variable case by a standard argument. Suppose that \( \text{(Th}_D(\mathfrak{M}) \cup p, \text{Th}_D(\mathfrak{M}) \cup p') \) is satisfiable in some model \( \langle B, N \rangle \) obtained by the model existence theorem by a sequence \( e_0, \ldots, e_n \in N \). Thus, the type of \( e_0 \) with parameters over \( D \) is realized in \( \mathfrak{M} = \langle A, M \rangle \) by an element \( e_0' \). But then we can also realize in \( \langle A, M \rangle \) the type \( \langle T, U \rangle \) where

\[
T = \{ \varphi(x, e'_0) \mid \{ B, N \} \models \varphi(e_1, e_0) \}
\]

\[
U = \{ \psi(x, e'_0) \mid \{ B, N \} \not\models \psi(e_1, e_0) \}
\]

since it is satisfied in \( \langle B, N \rangle \) by interpreting \( e_0' \) as \( e_0 \). Keep going this way until we finally realize the type of an element \( e_n' \) with parameters in \( D \cup \{ e'_0, \ldots, e'_{n-1} \} \).

Given a collection of theories \( \Psi \) of our language and a theory \( T \), following Convention 3.22 from [10], we will write \( T \models \Psi \) if there is \( S \in \Psi \) such that \( T \models \varphi \) for each \( \varphi \in S \).

**Theorem 5.** For each cardinal \( \kappa \), each model can be elementarily extended to a \( \kappa^+ \)-saturated model.

**Proof.** Let \( \mathfrak{M} = \langle A, M \rangle \) be a model. Observe that, indeed,

\[
\{ D \subseteq M \mid |D| \leq \kappa \} \subseteq \{ M \}^\kappa.
\]

This means, together with that fact that \( |S^{\mathfrak{M}}(D)| \leq 2^\kappa \), that we can list all types in \( S^{\mathfrak{M}}(D) \) for \( D \subseteq M, |D| \leq \kappa \) as \( \{ \langle p, \alpha \rangle \mid \alpha < |M|^\kappa \} \).

We can find a model \( \langle A', M' \rangle \) that realizes all types in \( S^{\mathfrak{M}}(D) \) for any \( D \subseteq M, |D| \leq \kappa \). We will use the union of elementary chains construction, defining a sequence of models \( \{ \langle A_\alpha, M_\alpha \rangle \mid \alpha < |M|^\kappa \} \) which is an elementary chain, and where \( \langle A_\alpha, M_\alpha \rangle \) realizes \( \langle p_\alpha, \alpha \rangle \).

The goal is to build the model \( \bigcup_{\alpha < |M|^\kappa} \langle A_\alpha, M_\alpha \rangle \), which will be our \( \langle A', M' \rangle \).

We let

(i) \( \mathfrak{M}_0 = \langle A_0, M_0 \rangle = \langle A, M \rangle \)

(ii) \( \mathfrak{M}_\alpha = \langle A_\alpha, M_\alpha \rangle = \bigcup_{\beta < \alpha} \langle A_\beta, M_\beta \rangle \) when \( \alpha \) is a limit ordinal.

(iii) \( \mathfrak{M}_{\alpha+1} = \langle A_{\alpha+1}, M_{\alpha+1} \rangle \) is a elementary extension of \( \langle A_\alpha, M_\alpha \rangle \) which realizes \( \langle p_\alpha, \alpha \rangle \). We build \( \langle A_{\alpha+1}, M_{\alpha+1} \rangle \) using Lemma 3.24 [10], the construction of canonical models from that paper and our tableaux almost strong S-closedness. In what follows we will use the notation and definitions from [10]. We start by showing that

\[
\text{El diag}(A_\alpha, M_\alpha) \cup p_\alpha \not\models \{ X \},
\]

where \( X \) is an arbitrary finite subset of \( \text{El diag}(A_\alpha, M_\alpha) \cup p_\alpha \). Observe that the set of theories \( \{ X \} \) is trivially deductively directed in the sense of Definition 3.21 from [10]. Using the canonical model construction and Lemma 3.24 [10] we can then provide a model for the tableau \( \langle \text{El diag}(A_\alpha, M_\alpha) \cup p_\alpha, X \rangle \) for each such \( X \). Hence, an application of tableaux almost strong S-closedness provides us with a model of \( \langle \text{El diag}(A_\alpha, M_\alpha) \cup p_\alpha, \text{El diag}(A_\alpha, M_\alpha) \cup p_\alpha' \rangle \).

Suppose, for a contradiction, that for each \( \chi \in X \),

\[
\text{El diag}(A_\alpha, M_\alpha) \cup p_\alpha \not\models \chi.
\]

Then take \( \psi \in X \). There are two possibilities. First suppose that \( \psi \in \text{El diag}(A_\alpha, M_\alpha) \). Since \( \text{El diag}(A_\alpha, M_\alpha) \cup p_\alpha \models \psi \), by the local deduction theorem, \( p_\alpha \models (\varphi \land \bar{1})^\alpha \rightarrow \psi \) where \( \varphi \) is some lattice conjunction of elements of \( \text{El diag}(A_\alpha, M_\alpha) \). Quantifying away the new constants (so only constants from the particular \( A \subseteq M \) remain), we obtain that \( p_\alpha \models (\forall \varphi \land \bar{1})^\alpha \rightarrow \psi \). But then, taking the model of \( \text{Th}_A(\mathfrak{M}) \cup p_\alpha, \text{Th}_A(\mathfrak{M}) \cup p_\alpha' \), we get a contradiction because \( (\forall \varphi \land \bar{1})^\alpha \rightarrow \psi \) would have to be in \( \text{Th}_A(\mathfrak{M}) \), which is in turn contained in \( \text{Th}_A(\mathfrak{M}_\alpha) \). But then \( (\forall \varphi \land \bar{1})^\alpha \models A_\alpha \) and hence, \( (\varphi \land \bar{1})^\alpha \models A_\alpha \) which leads to a contradiction. On the other hand, suppose that \( \psi \in p_\alpha' \). Similarly, we can obtain that \( \text{El diag}(A_\alpha, M_\alpha) \models (\forall \varphi \land \bar{1})^\alpha \rightarrow \psi \) where this time \( \varphi \) is a lattice conjunction of elements from \( p_\alpha \). Then \( (\forall \varphi \land \bar{1})^\alpha \rightarrow \psi \) would have to be in \( \text{Th}_A(\mathfrak{M}_\alpha) \), which in turn is contained in \( \text{Th}_A(\mathfrak{M}) \) which would be a contradiction given the model of \( \text{Th}_A(\mathfrak{M}) \cup p_\alpha, \text{Th}_A(\mathfrak{M}) \cup p_\alpha' \).

Next we build another elementary chain to get the \( \kappa^+ \)-saturated structure \( \langle D, O \rangle \). This time we put:

(i) \( \mathfrak{D}_0 = \langle A, M \rangle \)

(ii) \( \mathfrak{D}_\alpha = \bigcup_{\beta < \alpha} \langle D_\beta, O_\beta \rangle \) when \( \alpha \) is a limit ordinal.

(iii) \( \mathfrak{D}_{\alpha+1} = \langle D_{\alpha+1}, O_{\alpha+1} \rangle \) is a model that elementarily extends \( \langle D_\alpha, O_\alpha \rangle \) and realizes all types in \( S^{(D_\alpha, O_\alpha)}(A) \) for any \( A \subseteq M_\alpha, |A| \leq \kappa \).

Consider now \( \bigcup_{\alpha < \kappa^+} \langle D_\alpha, O_\alpha \rangle \), which will be our \( \langle D, O \rangle \).

Now suppose that \( A \subseteq N, |A| \leq \kappa \) and \( \langle p, p' \rangle \in S^{(D, O)}(A) \). By the regularity of the cardinal \( \kappa^+ \), we must have that
indeed $A \subseteq O_\alpha$ for some $\alpha < \kappa^+$. But, of course, since $\text{Th}_A(D, O) = \text{Th}_A(D_\alpha, O_\alpha)$ and $\text{Th}_A(D, O) = \text{Th}_A(D_\alpha, O_\alpha)(p, p') \in S(D_\alpha, O_\alpha)(A)$, so it is in fact realized in $\langle D_{\alpha+1}, O_{\alpha+1}\rangle$, and hence in $\langle D, O\rangle$. 

V. Conclusions

In this paper we have shown the existence of saturated models, that is, models realizing as many types as possible (given some cardinality restrictions). A complementary task would be that of building models realizing very few types, which in classical model theory is accomplished by means of the Omitting Types Theorem. Some work has already been started along these lines in the context of mathematical fuzzy logic in [3, 12, 26], that have focused on types with respect to a theory. In a forthcoming investigation we plan to extend these works by considering, in the fashion of the present paper, omission of types given by tableaux.

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