Green functions and dimensional reduction of quantum fields on product manifolds

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Abstract
We discuss Euclidean Green functions on product manifolds $\mathcal{P} = \mathcal{N} \times \mathcal{M}$. We show that if $\mathcal{M}$ is compact and $\mathcal{N}$ is not compact then the Euclidean field on $\mathcal{P}$ can be approximated by its zero mode which is a Euclidean field on $\mathcal{N}$. We estimate the remainder of this approximation. We show that for large distances on $\mathcal{N}$ the remainder is small. If $\mathcal{P} = \mathbb{R}^{D-1} \times S^\beta$, where $S^\beta$ is a circle of radius $\beta$, then the result reduces to the well-known approximation of the $D$-dimensional finite temperature quantum field theory by $(D - 1)$-dimensional one in the high-temperature limit. Analytic continuation of Euclidean fields is discussed briefly.

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1. Introduction

The aim of the Kaluza–Klein program [1] is a unification of interactions by means of an extension of the number of dimensions $D$. Then, $D - 4$ dimensions are supposed to be unobservable. In this paper, we ask the question of whether a higher dimensional quantum field theory can be approximated by a lower dimensional one (not necessarily from the point of view of the Kaluza–Klein program). We have shown in our earlier paper [2] (see also [3]) that such an approximation applies near the bifurcate Killing horizon when $D$-dimensional quantum field theory can be approximated by a two-dimensional one. We are interested in the dimensional reduction from the point of view of Green functions, i.e., correlation functions of quantum fields. We examine the question of whether the compact dimensions are negligible. In terms of the Green functions this means that Green functions in higher dimensions should be approximated by those in lower dimensions. Such a property cannot be true at arbitrarily small distances (except of some brane-type models [4, 5]) because the singularity of the Green function depends on the dimension. So, the approximation can make sense only above a certain length scale. If the manifold $\mathcal{P}$ is the product manifold $\mathcal{P} = \mathcal{N} \times \mathcal{M}$ then we expect...
that the Green functions on \( \mathcal{P} \) can be approximated by those on \( \mathcal{N} \) if the distances on \( \mathcal{N} \) are big in comparison to the size of \( \mathcal{M} \). In the conventional approach to Kaluza–Klein theories [1], the Fourier expansion of functions on the compact manifold \( \mathcal{M} \) leads to large masses which make (by a formal argument) propagators of the massive Kaluza–Klein particles negligible, realizing in this way the dimensional reduction. We are interested to see how this decoupling works in the configuration space.

In order to study the problem in a rigorous way we work in the Euclidean (Riemannian instead of pseudo-Riemannian) formulation of quantum field theory. Although the Euclidean approach to fields on a manifold is not as complete as in the flat space there are already some crucial results concerning the analytic continuation and construction of quantum fields [6, 7]. The quantum fields are determined by the Green functions. We discuss in this paper only the two-point function which is sufficient for a construction of free fields. The analytic continuation of interacting Euclidean fields can be performed if \( \mathcal{P} \) and the interaction have an additional reflection symmetry [6, 7].

In section 2 we define the Green functions. The Green functions are expanded in eigenfunctions in section 3. We distinguish the contribution of the zero mode which determines the dimensional reduction. In section 4 we discuss a special case of two-dimensional manifold \( \mathcal{N} \). We estimate the correction to the lower dimensional approximation in section 5. In section 6 we construct quantum fields from the Green functions. We discuss a possible extension of the results in section 7.

2. Warped metric on a product manifold

We consider a manifold in the form of a product \( \mathcal{P} = \mathcal{N} \times \mathcal{M} \) where \( \mathcal{N} \) has \( D-d \) dimensions and \( \mathcal{M} \) is a \( d \)-dimensional manifold. We assume that a metric on \( \mathcal{P} \) can be expressed in the warped form [8]

\[
d s^2 = \sigma_{AB} \, dX^A \, dX^B = g_{ab}(x) \, dx^a \, dx^b + v^2(x) \, h_{jk}(y) \, dy^j \, dy^k,
\]

where the coordinates on \( \mathcal{P} \) are denoted by the capital \( X = (x, y) \), those on \( \mathcal{N} \) by \( x \) and the coordinates on \( \mathcal{M} \) are denoted by \( y \). If \( v = 1 \) then the metric on the product manifold is just a product of the metrics.

Let

\[
\Delta_\mathcal{P} = \frac{1}{\sqrt{\sigma}} \partial_A \sigma^{AB} \sqrt{\sigma} \partial_B
\]

be the Laplace–Beltrami operator on \( \mathcal{P} \). We are interested in the calculation of the Green functions \( (\sigma = \det(\sigma_{AB})) \)

\[
(-\Delta_\mathcal{P} + m^2) \mathcal{G}^m = \frac{1}{\sqrt{\sigma}} \delta.
\]

In the metric (1) equation (3) reads

\[
(-v^{d-2} \sqrt{\sigma} (\Delta_\mathcal{M} - m^2) - \partial_a g^{ab} v^d \sqrt{\sigma} h_b) \mathcal{G}^m = h^{-\frac{d}{2}}_M \delta (X - X').
\]

A solution of equation (3) can be expressed by the fundamental solution of the diffusion equation

\[
\partial_t P_t = \frac{1}{2} \Delta_\mathcal{P} P_t
\]

with the initial condition \( P_0(X, X') = \sigma^{-\frac{d}{2}} \delta (X - X') \). Then

\[
\mathcal{G}^m = \frac{1}{2} \int_0^\infty \, dt \exp \left( -\frac{1}{2} m^2 t \right) P_t.
\]
If \( v = 1 \) then we may write equation (4) in the form (here \( h_M = \det(h_{jk}) \) and \( g = \det(g_{ab}) \))

\[
(-\Delta_M + m^2 - \Delta_N)G^M = h_M^{-\frac{1}{2}}g^{-\frac{1}{2}}\delta(X - X').
\]

(7)

In such a case from equation (5) we obtain a simple formula (in the sense of a product of semigroups)

\[
P^P_\tau = P^N_\tau P^M_\tau,
\]

(8)

where the upper index of the heat kernel denotes the manifold of its definition.

Hence

\[
G^M(X, X') = \frac{1}{2} \int_0^\infty d\tau \exp \left( -\frac{1}{2}m^2\tau \right) P^N_\tau(x, x')P^M_\tau(y, y').
\]

(9)

The formula (9) is useful if we have a reliable approximation for the heat kernels on \( N \) and \( M \).

We could conclude from equation (9) (using the Schwinger–DeWitt asymptotic expansion) that if \( X \) is close to \( X' \) then \( G \simeq \left( s_N^{-2}(x, x') + s_M^{-2}(y, y') \right)^{-\frac{D+1}{2}} \), where \( s_M \) denotes the geodesic distance on \( M \). We are interested in the behaviour of the Green functions when \( s_N \gg s_M \). For such a purpose the formula (9) does not seem useful. We apply eigenfunction expansions of the heat kernels in the following section.

The special case of \( D - d = 2 \) and \( v = 1 \) can be studied in more detail. We choose the isometric coordinates with \( g_{ab} = \delta_{a\bar{b}}a^2 \). In such a case equation (4) reads

\[
(a^2(x)(-\Delta_M + m^2) - \Delta_2)G^M = h_M^{-\frac{1}{2}}\delta(X - X'),
\]

(10)

where \( \Delta_2 \) is the Laplacian on \( R^2 \).

3. Eigenfunction expansions

We assume in this section that \( M \) is a compact manifold without a boundary. Then, \(-\Delta_M\) has a complete discrete set of orthonormal eigenfunctions \([9]\)

\[
-\Delta_M u_k = \epsilon_k u_k
\]

(11)

satisfying the completeness relation

\[
\sum_k \overline{u}_k(y)u_k(y') = h_M^{-\frac{1}{2}}\delta(y - y').
\]

(12)

Let us note that 1 is an eigenfunction (11) with the eigenvalue 0 (we use the normalization \( \int dy \sqrt{h_M} = 1 \)). Then (distinguishing the zero mode) we can expand the heat kernel in eigenfunctions

\[
P^M_\tau(y, y') = 1 + \sum_{k \neq 0} \exp \left( -\frac{1}{2}\epsilon_k \tau \right) \overline{u}_k(y)u_k(y').
\]

(13)

For the \( N \) part of \( P \) we consider the eigenvalue problem in \( L^2(dx) \) suggested by equation (4),

\[
A_k\phi^k_E = (v^{d-2}\sqrt{g}\omega_k^2 - \partial_a\partial^b v^{d}\sqrt{g}\partial_a\partial_b)\phi^k_E = E_k\phi^k_E,
\]

(14)

where

\[
\omega_k^2 = \epsilon_k + m^2.
\]

(15)

The eigenfunctions satisfy the completeness relation

\[
\sum_E \phi^E_E(x)\phi^E_E(x') = \delta(x - x'),
\]

3
where the sum must be replaced by an integral if the spectrum of $A_k$ in equation (14) is continuous.

We expand the Green function in eigenfunctions $u_k$ of the Laplace–Beltrami operator $\triangle_M$,

$$G^m(X, X') \equiv \sum_k g^m_k(X, X') = \sum_k g^m_k(x, x')u_k(y)u_k(y').$$  \hspace{1cm} (16)

Then, $g_k$ is expanded in the eigenfunctions (14),

$$g^m_k(x, x') = \sum_{E} E^{-1} \phi^k_E(x) \phi^k_E(x'),$$  \hspace{1cm} (17)

$g^m_k$ is a solution of the equation

$$A_k g^m_k(x, x') = \delta(x - x'),$$  \hspace{1cm} (18)

1 is an eigenfunction of $-\triangle_M$ with the eigenvalue 0. If we subtract the zero mode $G^m_0$ (corresponding to $u_0 = 1$) from $G^m$ then

$$G^m(X, X') - G^m_0(X, X') = \sum_{E, k > 0} E^{-1} \phi^k_E(x) \phi^k_E(x')u_k(y)u_k(y)$$  \hspace{1cm} (19)

with

$$G^m_0(X, X') = G^m(x, x'),$$  \hspace{1cm} (20)

where $G^m$ is a solution of the equation

$$(-\partial_ag^{ab}v^d \partial_b + m^2v^{d-2}g^{\frac{1}{2}})G^m = \delta(x - x').$$  \hspace{1cm} (21)

We investigate in this paper whether $G^m$ can be approximated by $G^m_0$, i.e., by the Green function $G^m_0$ on $\mathcal{N}$. Such an approximation cannot be true for small distances because if the metric tensor is a regular function then the singularity of the Green function depends on the dimension of the spacetime (see the discussion in section 7). We expect that the approximation makes sense for large distances in $\mathcal{N}$. It can be seen that a decay of eigenfunctions $\phi^k_E$ is sufficient for a disappearance of each term on the rhs of equation (19) at large $x$ (this is not a necessary condition as we show soon). The eigenfunctions $\phi^k_E$ are localized if the spectrum of the operators $A_k$ (14) is discrete. We can estimate the decay of eigenfunctions $\phi^k_E$ applying the eikonal (WKB) approximation to equation (14). This means that we write $\phi^k_E = \exp(-\omega_k W)$ assuming that $W$ is growing uniformly in each direction for large distances. Then, in the leading order we obtain for large $x$ the equation

$$1 = v^2 g^{ab} \partial_a W \partial_b W.$$  \hspace{1cm} (22)

We obtain an exponential localization (increasing with the eigenvalue $\epsilon_k$) of $\phi^k_E$ if $v^{-2}g^{ab}$ is uniformly growing to infinity for large distances. Let us note that $v^{-2}g^{ab}$ is the metric on $\mathcal{N}$ related to that of equation (1) by a conformal transformation. The growth of $g^{ab} = v^{-2}g^{ab}$ means that the volume element $\int dx \sqrt{g}$ is infinite. Such a property could be used to characterize the manifolds $\mathcal{P}$ whose Green function is dominated by the zero mode. Our rough arguments need a confirmation by a mathematical theory of the eigenvalue problems of second-order differential operators (see [10] for some partial results). The sum over eigenvalues $\epsilon_k$ (the rhs of equation (19)) will be discussed in section 5.

In the special case (10) when $v = 1$ and $D - d = 2$ the eigenvalue equation (14) reduces to the well-known problem of quantum mechanics

$$A\phi^k_E = (-\triangle_2 + \omega_k^2a^2(x))\phi^k_E = E_k\phi^k_E.$$  \hspace{1cm} (23)

Here, $\triangle_2$ denotes the two-dimensional Laplacian. In this special case we have simple criteria for the discreteness of the spectrum of $A$. If $a^2$ is growing uniformly in all directions then
the spectrum of \( A \) is discrete and the eigenfunctions are localized (see [11] for a precise formulation and proofs). The eikonal approximation reads

\[ \nabla W \nabla W = a^2(x). \]  

(24)

Hence, \( |\nabla W(x)| \) is growing like \( a(x) \). The decay of eigenfunctions derived from equation (24) is in agreement with exact results [11]. If the eigenfunctions \( \phi_k^E \) decay for large \( x \) then the volume of \( \mathcal{N} \), equal to \( \int dx a^2(x) \), is infinite (\( \mathcal{N} \) is not compact).

The localization of eigenfunctions \( \phi_k^E(x) \) is not necessary for a decrease of \( G^m - G^m_0 \). Let us consider the simplest case of a non-compact \( \mathcal{N} = \mathbb{R}^{d-d} \) with \( v = 1 \) and \( g_{ab} = \delta_{ab} \). Then, \( \mathcal{A}_k \) has a continuous spectrum, its eigenfunctions are not localized, but

\[ (G^m - G^m_0)(X, X') = \sum_{k \neq 0} \int_0^\infty d\tau \exp \left( -\frac{1}{2} a_k^2 \tau - \frac{1}{2} (x - x')^2 \right) (2\pi \tau)^{-\frac{D-d}{2}} \Psi_k(y) u_k(y'). \]

(25)

where

\[ g_k^m(x - x') = 2(2\pi)^{\nu-1} |x - x'|^\nu a_k^{-\nu} K_\nu(a_k |x - x'|), \]

with \( \nu = -\frac{D-d}{2} + 1 \), where \( K_\nu \) is the modified Bessel function of the third kind [12]. From the asymptotic expansion of \( K_\nu \) it follows that (for any \( m^2 \geq 0 \)) each term on the rhs of equation (25) is decaying exponentially for large \( |x - x'| \). The sum on the rhs of equation (26) will be estimated in section 5.

If \( \mathcal{N} \) is a compact manifold without a boundary then the spectrum \( \lambda_n \) of the Laplace–Beltrami operator \( \Delta_N \) on \( \mathcal{N} \) is discrete

\[ -\Delta_N \psi_n = \lambda_n \psi_n. \]

(27)

In such a case

\[ P^N_\tau(x, x') = 1 + \sum_{n \neq 0} \exp \left( -\frac{1}{2} \lambda_n \tau \right) \bar{\Psi}_n(x) \psi_n(x'). \]

(28)

Hence, if \( v = 1 \) then from equation (9)

\[ (G^m - G^m_0)(X, X') = \sum_{k>0} g_k(x, x') \bar{\Psi}_k(y) u_k(y'), \]

(29)

where

\[ g_k^m(x, x') = \sum_n (\lambda_n + \epsilon_k + m^2)^{-1} \bar{\Psi}_n(x) \psi_n(x') \]

(30)

and

\[ G^m_0(x, x') = G^m(x, x') = \sum_n (\lambda_n + m^2)^{-1} \bar{\Psi}_n(x) \psi_n(x') \]

(31)

is the Green function (3) on \( \mathcal{N} \) (solving equation (21) for \( v = 1 \)).

However, if \( \mathcal{N} \) is compact then there is no reason to neglect the rhs of equation (29).
4. A two-dimensional manifold $\mathcal{N}$ with a Killing vector

In this section we discuss in more detail the product manifold $\mathcal{P} = \mathcal{N} \times \mathcal{M}$ ($\nu = 1$) when the two-dimensional manifold $\mathcal{N}$ has a symmetry generated by a Killing vector $K$. In an adapted system of coordinates such that $K = \partial_1$ the metric can be written in the form

$$\mathrm{d}s^2 = \mathrm{d}x_0^2 + a_1^2(x_0) \mathrm{d}x_1^2 + \sum_{jk} h_{jk}(y) \mathrm{d}y_j \mathrm{d}y_k. \quad (32)$$

In such a case the equation for the Green function reads

$$-\left(\partial_0 a_1 \partial_0 + a_1^{-1} \partial_1^2 + a_1 \Delta_{\mathcal{M}} - m^2 a_1\right)G^m = h_M^{-1} \delta. \quad (33)$$

We can write the metric in an equivalent form. Let

$$\hat{x}_0 = \int \mathrm{d}x_0(a_1(x_0))^{-1}. \quad (34)$$

Then,

$$\mathrm{d}s^2 = a_1^2(\hat{x}_0)(\mathrm{d}\hat{x}_0^2 + \mathrm{d}x_1^2) + \mathrm{d}s_M^2, \quad (35)$$

where $\mathrm{d}s_M^2$ is the metric on $\mathcal{M}$. In the new coordinates

$$(-\partial_0^2 - \partial_1^2 - a_1^2 \Delta_{\mathcal{M}} + m^2 a_1^2)G^m = h_M^{-1} \delta, \quad (36)$$

where $\delta = \delta(\hat{x} - \hat{x}')$ depends on $\hat{x}$ variables.

As an example, let $a_1(x_0) = x_0$ then

$$\hat{x}_0 = \ln(x_0). \quad (37)$$

Hence, equation (36) takes the form

$$(-\partial_0^2 - \partial_1^2 - \exp(2\hat{x}_0)(\Delta_{\mathcal{M}} - m^2))G^m = h_M^{-1} \delta. \quad (38)$$

In spite of $a$ vanishing at zero (in the original $x_0$ coordinate) it can be checked by means of a calculation of the curvature tensor $R$ that $R$ is a continuous function (if $\mathcal{M} = R^n$ then the formula for the curvature in Bianchi-type spacetimes derived in [13, 14] gives $R = 0$). We have discussed the model (38) in detail in [2]. It has been shown that the model serves as an approximation to the Green function on a spacetime with the bifurcate Killing horizon.

If $\mathcal{M} = R^d$ then equation (38) defines the Green function on the Euclidean version of the Rindler space.

An interesting class of models results from a choice of a metric which has a power-like singularity at $x_0$ when approaching $x_0 = 0$. We may choose $a_1(x_0)^2 = |x_0|^{2\gamma}$ (the curvature tends to infinity at the singularity if $\gamma \neq 1$). Then, in the coordinates (34) we have

$$a_1(x_0)^2 = |\hat{x}_0|^{2\gamma}. \quad (39)$$

in equations (35)–(36).

We apply the eigenfunction expansion (23) of section 3 to the case when $a_1$ depends only on $x_0$. We consider equation (36) (we omit the hat over $x_0$). We expand the Green function in a complete set of orthonormal eigenfunctions of the one-dimensional quantum-mechanical problem

$$(-\partial_0^2 + \omega_k^2 a_1(x_0)^2)\phi_n^k = \lambda_n(k) \phi_n^k, \quad (40)$$

where $\omega_k$ is defined in equation (15) and

$$\sum_n \phi_n^k(x_0)\phi_n^k(x_0') = \delta(x_0 - x_0'). \quad (41)$$
Then, a solution of the equation for the Green function (10) has an expansion
\[ G_m(X, X') = \pi^{-1} \sum_{k,n} \bar{u}_k(y) u_k(y') \phi^k_n(x_0) \lambda_n(k)^{-1} \exp(-\lambda_n(k)|x_1 - x'_1|). \] (42)

The formula (42) follows from equations (16)–(17) if we write
\[ \phi^k_E(x_0, x_1) = \exp(i p_1 x_1) \phi^k_n(x_0) \]
with \( E_k = p_1^2 + \lambda_n(k)^2 \) and
\[ \sum_E = \int dp_1 \sum_n. \]

Then, the integral over \( p_1 \) in equation (17) leads to equation (42).

The eigenfunctions \( \phi^k \) are decaying exponentially if \( a_1 \) is growing at infinity, as can be seen, e.g., from the WKB approximation (see [11] for rigorous results)
\[ \phi^k_n(x) \simeq \exp \left( -\omega_k \int dx a_1(x) \right). \] (43)

Hence, each term in the expansion (42) is decaying exponentially. Note that according to equations (22), (24) and (43) the terms with larger eigenvalues \( \epsilon_k \) are decaying faster than the ones with the lower eigenvalue.

5. The correction to the contribution of the zero mode

We expect that in general (for a non-compact manifold \( \mathcal{N} \)) the difference \( G^m - G^m_0 \) is negligible for large distances on \( \mathcal{N} \). First, we must estimate the Green functions \( g^m_k \) (18) of the second-order differential operators \( A_k \) (depending on \( \epsilon_k \)) on \( \mathcal{N} \) for large distances. We write
\[ g^m_k(x, x') = \exp(-\omega_k W(x, x')). \] (44)

Assuming that \( W \) is growing uniformly in each direction we obtain in the leading order for large distances equation (22) for \( W \). We recognize equation (22) as an equation for a geodesic distance \( s_N(x, x') \) on the manifold \( \mathcal{N} \) with the metric \( v^{-2} g_{ab} \) [15]. Hence, the geodesic distance \( W(x, x') = s_N(x, x') \) is the solution of equation (22) which is symmetric under the exchange of the points and satisfies the boundary condition \( W(x, x) = 0 \). We insert the approximate solutions \( g^m_k \) (44) (for some rigorous results on the large distance behaviour of Green functions of second-order differential operators see [10], [16]) into the sum (19) over eigenvalues and eigenfunctions of \( -\Delta \mathcal{M} \). We approximate the sum over large eigenvalues by an integral (Weyl approximation) assuming \( |u_k(y)| \leq C \). Then
\[ |(G^m - G^m_0)(X, X')| \leq A_1 \sum_{k < \Lambda} \exp(-\omega_k s_N^2(x, x')) |u_k(y)| |u_k(y')| \\
+ A_2 \int_{|k| > \Lambda} dk \exp(-\omega_k s_N^2(x, x')) \]
where in the integral over \( k \) we set \( \omega_k = \sqrt{k^2 + m^2} \). The integral over \( k \) is decreasing exponentially as a function of \( s_N \).

We make the estimate precise in the simple model of \( \mathcal{N} = R^{D-d} \) (equation (25)). We apply Weyl theory [17] saying that for large eigenvalues the sum over eigenvalues of the Laplace–Beltrami operator on a \( d \)-dimensional compact manifold \( \mathcal{M} \) can be approximated by a \( d \)-dimensional integral with \( \epsilon_k \simeq k^2 \). If additionally we assume \( |u_k| \leq C \) then in equation (25),
\[ |(G^m - G^m_0)(X, X')| \leq A_1 \sum_{k < \Lambda} |c_k(x - x')||u_k(y)||u_k(y')| + R_\Lambda(x - x'), \] (46)
where from the Weyl approximation
\[
R_A(x - x') = C^2 \int_0^\infty dr \int_{|k| > \Lambda} dk \exp \left( -\frac{1}{2} (k^2 + m^2) \tau - \frac{1}{2\tau} (x - x')^2 \right) \left( 2\pi \tau \right)^{-\frac{n-1}{2}} 
\]
\[
= A \int_0^\infty d\tau \tau^{-\frac{n}{2}} \Gamma \left( \frac{d}{2} \right) \left( \frac{\Lambda^2}{\tau} \right) \exp \left( -\frac{1}{2} m^2 \tau - \frac{1}{2\tau} (x - x')^2 \right) \left( 2\pi \tau \right)^{-\frac{n-1}{2}},
\]
(47)
where \( \Gamma \) denotes the incomplete gamma function [12]. From the asymptotic expansion of \( \Gamma \) we obtain that
\[
R_A(x - x') \simeq \exp(-\sqrt{\Lambda^2 + m^2}|x - x'|)
\]
(48)
for large \(|x - x'|\). Hence, we can conclude that \( G^m - G_0^m \) for large \(|x - x'|\) is decreasing as \( \exp(-\sqrt{m^2 + \epsilon|x - x'|}) \), where \( \epsilon \) is the lowest nonzero eigenvalue of \(-\Delta_M\). The decay of \(|G^m - G_0^m(X, X')|\) for large distances is determined by the first term on the rhs of equation (46).

We repeat the estimates in the model (25) without any reference to the Weyl approximation in the simplest case of \( M = S^\beta \) where \( S^\beta \) is the circle of radius \( \beta \) (then the spectrum of \(-\Delta_M\) is known). The method of an explicit sum over eigenvalues is simple if \( m = 0 \). Then, from equations (9) and (17),
\[
G^0(X, X') = (2\pi \beta)^{-1} \int_0^\infty dr (2\pi \tau)^{-\frac{n-1}{2}} \exp \left( -\frac{1}{2\tau} (x - x')^2 \right) 
\]
\[
\times \sum_k \exp \left( -\frac{\tau}{2} \left( \frac{k}{\beta} \right)^2 \right) \exp \left( i\frac{k}{\beta} (y - y') \right) 
\]
\[
= \int_0^\infty d\tau (2\pi \tau)^{-\frac{n}{2}} \exp \left( -\frac{1}{2\tau} (x - x')^2 \right) \sum_k \exp \left( -\frac{1}{2\tau} (y - y' - 2\pi \beta k)^2 \right),
\]
(49)
Performing the integral over \( \tau \) we obtain
\[
G^0(X, X') = (2\pi)^{-D+2} \Gamma \left( \frac{D}{2} - 1 \right) \sum_k ((x - x')^2 + (y - y' - 2\pi \beta k)^2)^{-\frac{n}{2}+1}.
\]
(50)
In order to perform the sum we apply the formula
\[
\sum_k (\sigma^2 + (y - y' - 2\pi \beta k)^2)^{-1} 
\]
\[
= \frac{1}{4\beta \sigma} \left( \coth \left( \frac{1}{2\beta} \sigma + i(y - y') \right) + \coth \left( \frac{1}{2\beta} (\sigma - i(y - y')) \right) \right).
\]
(51)
The formula (51) can be applied directly to equation (50) with \( \sigma^2 = (x - x')^2 \) if \( D = 4 \). When \( D \) is even and bigger than 4 then we differentiate equation (51) over \( \sigma^2 \) and subsequently apply to the sum (50). If \( D = 2n + 1 \) is odd then we have to use differentiation \( n - 1 \) times together with an integration in order to perform the sum in equation (50).

We show this procedure for \( D = 3 \). This dimension is relevant for the model (25) of section 3 and in models of section 4. We use the integral
\[
\int_0^\infty dr (r^2 + a^2)^{-1} = \pi a^{-1}
\]
(52)
in order to represent the Green function in $D = 3$ dimensions in the form
\[
G^0(X, X') = 2^{-\frac{D}{2}} \pi^{-\frac{D}{2}} \beta^{-1} \int_0^\infty dr \frac{1}{4\sigma} \left( \coth \left( \frac{1}{2\beta}(\sigma + i(y - y')) \right) + \coth \left( \frac{1}{2\beta}(\sigma - i(y - y')) \right) \right),
\]
(53)
where $\sigma^2 = r^2 + (x_0 - x'_0)^2 + (x_1 - x'_1)^2$. Then,
\[
\begin{align*}
(G^0 - G^0_0)(X, X') &= 2^{-\frac{D}{2}} \pi^{-\frac{D}{2}} \beta^{-1} \int_0^\infty dr \frac{1}{4\sigma} \left( \coth \left( \frac{1}{2\beta}(\sigma + i(y - y')) \right) - 1 \\
&\quad + \coth \left( \frac{1}{2\beta}(\sigma - i(y - y')) - 1 \right),
\end{align*}
\]
(54)
where $G^0_0 = G^0$ is defined in equation (26) with $D - d = 2$ (for $m = 0$ and $D - d = 2$ the integral (26) is divergent at large $\tau$; this is the well-known infrared problem for massless scalar fields, it can be avoided by a choice of test functions for the smeared out fields with no support at the zero momentum, in such a case the integral (26) is defined as a logarithm of the distance).

Applying the formula
\[
\coth v - 1 = 2 \exp(-2v)(1 - \exp(-2v))^{-1}
\]
we bound the integral on the rhs of equation (54) by
\[
|(G^0 - G^0_0)(X, X')| \leq A \int_0^\infty dr \sigma^{-1} \exp(-2\sigma) = AK_0 \left( \frac{1}{\beta} \sqrt{(x_0 - x'_0)^2 + (x_1 - x'_1)^2} \right)
\]
for large $(x_0 - x'_0)^2 + (x_1 - x'_1)^2$, where $K_v$ denotes the modified Bessel function of the third kind [12], which is exponentially decreasing for large arguments.

We conclude from equation (54) that the logarithmic two-dimensional propagator well approximates the three-dimensional propagator for distances $(x_0 - x'_0)^2 + (x_1 - x'_1)^2 \gg \beta^2$.

Applying equations (50)–(51) we could confirm for any dimension $D - d$ the result following from equations (46)–(48) that the difference $G^0 - G^0_0$ is exponentially small for large distances (the length scale is $\beta$ which is equal to the radius of the circle $S^\beta$ or in other words it is the square root of the inverse of the lowest nonzero eigenvalue of $-\Delta_{M'}$). For $D > 3$ the Green function $G$ would be approximated by $G^0_0 \simeq |x - x'|^{-D+3}$ (by $\log |x - x'|$ in $D = 3$). The high-temperature limit of interacting field theories is discussed in [18, 19].

6. Quantum free fields on the product manifold

We introduce now a free Euclidean field as a random field with the two-point correlation function equal to the Green function (see [7]). The Green function (17) defines the Gaussian Euclidean field
\[
\Phi(x, y) \equiv \sum_k \Phi_k = \sum_k \chi_k(x) u_k(y),
\]
(55)
where
\[
(\chi_k(x) \chi_r(x')) = \delta_{k,r} g_k^m(x, x'),
\]
(56)
g_k^m as a Green function of the second-order differential operator is non-negative. This is a positive definite bilinear form. Hence, it defines the Gaussian Euclidean field $\phi_k$ on $\mathcal{N}$. In the example (25) of $\mathcal{N} = R^{D-d}$ we have $g_k^m = (-\Delta + \omega_k^2)^{-1}$. Then, $\phi_k^m$ is the Euclidean free field on $R^{D-d}$ with the mass $\omega_k$ ($\phi_0^m$ has the mass $m$). In an analytic continuation of the
model (25) to the Minkowski space $\chi_k$ becomes the free quantum field with the mass $\omega_k$ on the $(D - d)$-dimensional Minkowski space.

The models of section 4 have an expansion

$$\Phi(x_0, x_1, y) = \sum_k \Phi_k = \int dp_1 \exp(ip_1 x_1) \sum_{k,n} a_k(p_1, n) \phi_k(x_0) u_k(y)$$

$$= \Phi_0(x_0, x_1) + \sum_{k>0} \Phi_k(x_0, x_1, y), \quad (57)$$

where

$$\langle \Phi_k(p_1, n) a_k(p'_1, n') \rangle = \delta(p_1 - p'_1) \delta_{mn} \delta_{kk'} (p_1^2 + \lambda_n(k)^2)^{-1}. \quad (58)$$

We have

$$\langle \Phi_0(x_0, x_1) \Phi_0(x'_0, x'_1) \rangle = G^m_{00}(x, x') = \int_0^\infty d\tau \exp \left(-\frac{1}{2} m^2 \tau - \frac{1}{2\tau} (x - x')^2\right) (2\pi \tau)^{-1} \quad (59)$$

and for $k > 0$

$$\langle \Phi_k(x_0, x_1, y) \Phi_k(x'_0, x'_1, y') \rangle = \frac{1}{\pi} \sum_n \lambda_n(k)^{-1} \exp(-\lambda_n(k)|x_1 - x'_1|) \tilde{\phi}_k^* (x_0) \phi_k(x'_0) u_k(y)u_k(y'). \quad (60)$$

We can continue analytically the Green functions (60) of the model (57) into imaginary values of $x_1$ (then $x_1$ plays the role of time; the analytic continuation follows from the positivity of the Green function (60) under a reflection of $x_1$ [7])

$$G^m_{00}(x, i x', y, i y') = \frac{1}{\pi} \sum_{k,n} \lambda_n(k)^{-1} \exp(i\lambda_n(k)(x_1 - x'_1)) \tilde{\phi}_k^* (x_0) \phi_k(x'_0) u_k(y)u_k(y'). \quad (61)$$

Such Green functions result from the quantization in the Fock space

$$\Phi(x_0, x_1, y) = \sum_{k,n} \exp(-i\lambda_n(k)x_1) a_k(n) \phi_k^*(x_0) u_k(y)$$

$$+ \sum_{k,n} \exp(i\lambda_n(k)x_1) a_k^*(n) \phi_k(x_0) \bar{u}_k(y), \quad (62)$$

where

$$\left[ a_k(n), a_k^*(n') \right] = \frac{1}{\pi} \delta_{nm} \delta_{kk'} \lambda_n(k)^{-1} \quad (63)$$

are the creation and annihilation operators.

In order to define a quantum field from the Euclidean field (60) with another time we would need the reflection positivity in another direction. Various analytic continuations are possible on the same manifold $\mathcal{N}$. As an example, if $\mathcal{N}$ is the hyperbolic space then the Euclidean field on $\mathcal{N}$ can be analytically continued either to a quantum field on the DeSitter space or to the one defined on the anti-de Sitter space [7]. The model (38) of section 4 ($a_1 \simeq x_0^2$) with $\mathcal{M} = R^d$ as the Euclidean version of the Rindler space has an analytic continuation to the Rindler space or to the Milne space. The model is also related by a conformal transformation to the direct product of the real line and the hyperbolic space (the models and their analytic continuations are discussed in [20] and in the references cited there).

In general, if we are able to define a quantum field with the Green function defined by the analytic continuation of $\tilde{\phi}_k(x_0)$ (18) (for every $k$) then we can define the quantum field on an analytic continuation of $\mathcal{P}$ by means of the expansion (55).
7. Discussion

We have investigated some aspects of quantum free fields on product manifolds \( \mathcal{P} = \mathcal{N} \times \mathcal{M} \). Such studies have a long history. It is an old argument that on a compact manifold \( \mathcal{M} \) the heavy modes (KK-modes) decouple from the zero mode (defining the dimensional reduction). In this paper, we have studied the problem in the configuration space. An approximation of quantum field theory by a lower dimensional one is not trivial because the theories in various dimensions have different short-distance behaviour. The dimensional reduction of the Kaluza–Klein type can make sense only above a certain length scale determined by the size of the compact manifold (or equivalently by the inverse of the square root of the lowest nonzero eigenvalue of \(-\Delta_{\mathcal{M}}\)). In section 5 we have estimated the difference \( G - \hat{G}_0 \), where \( \hat{G}_0 \) is the \((D-d)\)-dimensional Green function. We have shown in some models that there is no dimensional reduction for a non-compact \( \mathcal{M} \)[2]. Nevertheless, a study of Green functions with non-compact manifolds \( \mathcal{M} \) can lead to some unexpected results. The conventional behaviour \( |x - x'|^{-D+2} \) in \( D \) dimensions of the Green functions of quantum massless fields can fail on a manifold. The Green functions of the model of section 4 can be more singular than it would follow from the dimension-dependent index \( D - 2 \). A change of the short-distance behaviour has been shown in the brane model of Dvali et al[5]. Their model can be formulated in the framework of section 2 with \( D - d = 1 \). The equation for the Green function reads

\[
\left(-\partial_0^2 + a^2(x_0)(-\Delta + m^2)\right) G^m = \delta. \tag{64}
\]

The Fourier transform of \( G \) in \( D-1 \) variables satisfies an equation for the quantum-mechanical Green function with the potential \( V = p^2 a^2(x_0) \). We have studied solutions of the equations for Green functions in [21] by means of path integrals. The short-distance behaviour depends on the singularity of \( a^2(x_0) \) for \( x_0 \to 0 \). In the model of [5],

\[
a^2(x_0) = \alpha + \beta \delta(x_0). \tag{65}
\]

It follows from [5, 21] that \( G(0, y, 0, y') \simeq |y - y'|^{-D+1} \). This is the behaviour of the Green function in \( D - 1 \) dimensions. Such a behaviour of the Green function at short distances may be called a dimensional reduction in spite of the continuous spectrum of \( \Delta \) on \( \mathcal{M} \).

Let us still consider the models with a singular metric of section 4 in view of the results of [21]. The equation for the Green function reads

\[
\left(-\partial_0^2 - \partial_1^2 + |\tilde{x}_0|^2 \right)(-\Delta + m^2) G^m = \delta, \tag{66}
\]

where \( \tilde{x}_0 \) has been called \( \eta \) in [21]. Applying the methods and results of [21] we obtain

\[
G(0, x_1, 0; 0, x'_1, 0) \simeq |x_1 - x'_1|^\frac{n}{D-2} \tag{67}
\]

and

\[
G(0, 0, y; 0, 0, y') \simeq |y - y'|^{-D+2} \tag{68}
\]

for small distances. Hence, the Green functions in the \( x_1 \) direction are more singular (if \( \gamma > 0 \)) than those in \( R^D \), whereas the singularity in the \( y \) direction is the same as that in the \( D \)-dimensional free field theory on a flat space.

In the model of Dvali et al[5] the long-distance behaviour of massless fields is the same as that in \( D \) dimensions. The dimensional reduction discussed in the present paper concerns the long-distance behaviour. The short-distance behaviour can be different from the canonical one (with the index \( D - 2 \)) only if the metric is singular.

An analytic continuation of Green functions of the models of section 4 with a Killing vector on \( \mathcal{N} \) as discussed in section 6 follows from the general theory of a quantization of the symmetry generated by the Killing vector formulated in [7]. In the system of coordinates
adapted to this Killing vector the analytic continuation concerns the coordinate $x_1$. It gives the Hilbert space, the Hamiltonian and a unitary evolution. There can be more reflection symmetries on a manifold. The Riemannian manifolds of section 4 are invariant under the reflection $x_0 \to -x_0$. However, a reflection positivity (with respect to $x_0$), necessary for a definition of the quantum field (with $x_0$ as an imaginary time), remains unclear. The reflection positivity may have a physical meaning. Tunnelling between different topologies of spacetime may require a reflection invariant Riemannian manifold [22]. Models with a prior to the big bang evolution [23, 24] seem to require a Euclidean version which is invariant under a time reflection.

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