Global Existence and Uniform Boundedness in a Fully Parabolic Keller–Segel System with Non-monotonic Signal-dependent Motility

Yamin Xiao∗, Jie Jiang†

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Abstract

This paper is concerned with global solvability of a fully parabolic system of Keller–Segel-type involving non-monotonic signal-dependent motility. First, we prove global existence of classical solutions to our problem with generic positive motility function under a certain smallness assumption at infinity, which however permits the motility function to be arbitrarily large within a finite region. Then uniform-in-time boundedness of classical solutions is established whenever the motility function has strictly positive lower and upper bounds in any dimension $N \geq 1$, or decays at a certain slow rate at infinity for $N \geq 2$. Our results remove the crucial non-increasing requirement on the motility function in some recent work [10,13,16] and hence allow for both chemo-attractive and chemo-repulsive effect, or their co-existence in applications. The key ingredient of our proof lies in an important improvement of the comparison method developed in [10,16,18].

Keywords: Classical solutions, global existence, boundedness, Keller–Segel models, comparison.

1 Introduction

In this paper, we consider global existence of classical solutions to an initial-Neumann boundary value problem of the following Keller–Segel system:

\begin{align*}
  u_t &= \Delta (u^{\gamma}(v)), \quad (x,t) \in \Omega \times (0,\infty), \quad (1.1a) \\
  \tau v_t - \Delta v + v &= u, \quad (x,t) \in \Omega \times (0,\infty), \quad (1.1b) \\
  \nabla (u^{\gamma}(v)) \cdot \nu - \nabla v \cdot \nu &= 0, \quad (x,t) \in \partial \Omega \times (0,\infty), \quad (1.1c) \\
  u(x,0) &= u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega. \quad (1.1d)
\end{align*}

∗The Graduate School of China Academy of Engineering Physics, Beijing 100088, P.R. China, xiaoyamin20@gscaep.ac.cn.
†Innovation Academy for Precision Measurement Science and Technology, CAS, Wuhan 430071, HuBei Province, P.R. China, jiang@apm.ac.cn.
Here, \( u \) and \( v \) denote the density of cells and the signal concentration, respectively. \( \tau > 0 \) is a positive constant. \( \Omega \subset \mathbb{R}^N \) with \( N \geq 1 \) is a bounded domain with smooth boundary. The above system features a signal-dependent motility function \( \gamma(\cdot) \), which satisfies

\[
\gamma \in C^3((0, \infty)), \quad \gamma > 0 \text{ in } (0, \infty). \tag{A0}
\]

This PDE system was originally proposed by Keller and Segel in their seminal work [8], which models the oriented movement of cells due to a local sensing mechanism for chemotaxis. The sign of \( \gamma' \) corresponds to different chemotaxis phenomena: cells are attracted by signals when \( \gamma' < 0 \), and are repelled when \( \gamma' > 0 \).

Recently, an extended three-component chemotaxis system of (1.1a)-(1.1b) has been developed in [21], where a third variable \( n \) denoting the nutrient level is introduced. The full system reads as

\[
\begin{align*}
\partial_t u &= \Delta (u\gamma(v)) + \theta uf(n), & (x,t) \in \Omega \times (0, \infty), \tag{1.2a} \\
\tau \partial_t v &= \Delta v - v + u, & (x,t) \in \Omega \times (0, \infty), \tag{1.2b} \\
\partial_t n &= \Delta n - \theta uf(n), & (x,t) \in \Omega \times (0, \infty). \tag{1.2c}
\end{align*}
\]

In their work, \( \gamma \) is assumed to be a positive non-increasing function, which stands for a repressive effect of the signal on cellular motility. By numerical and experimental analyses, it is shown that the system can foster spatially periodic patterns in a growing bacteria population merely under this motility control. We note that setting \( \theta = 0 \) in (1.2) cancels the coupling and hence we recover system (1.1a)-(1.1b) for \( (u,v) \).

The mathematical analysis for the above Keller–Segel systems with signal-dependent motility has attracted a lot of research interest in recent years. We refer the readers to [16] for a detailed review on this topic where the non-increasing monotonicity of \( \gamma \) is mostly assumed. For general \( \gamma \) satisfying (A0), \( \gamma' \leq 0 \) and \( \lim_{s \to \infty} \gamma(s) < 1/\tau \), global existence of unique classical solution to problem (1.1) was established in [10] when \( N = 2 \), and in [13] when \( N \geq 3 \).

In contrast, existence theories without the monotonicity requirement are largely missing in the literature. On the one hand, if \( \gamma \) has strictly positive bounds from above and below, and moreover \( |\gamma'| \) is bounded, Tao and Winkler [23] prove that there exists a unique globally bounded classical solution for \( N = 2 \), and global weak solution is constructed for \( N \geq 3 \) provided that \( \Omega \) is convex. If the motility is permitted to approach zero at infinity, global existence of (very) weak solutions is established if \( \gamma \) either satisfies \( \sup_{s \geq 0} \gamma(s) < 1/\tau \) in [18], or exhibits an algebraic decay at infinity [5].

On the other hand, it is worth noting that for the parabolic-elliptic version of (1.1) with \( \tau = 0 \), existence of classical solution is verified simply under the assumption (A0) and that \( \sup_{s \in [a, \infty)} \gamma(s) < \infty \) for all \( a > 0 \) (or equivalently, \( \limsup_{s \to \infty} \gamma(s) < \infty \)), without the need of non-increasing property. However, a corresponding theory for the fully parabolic case is still open. Thus, there arises a natural question that whether the monotonicity requirement is removable or replaceable to ensure the existence of global classical solutions for the fully parabolic system.

The first task of this work is to give an affirmative answer to the above question. More precisely, we establish the following existence result.
Theorem 1.1. Let \( N \geq 1 \). Suppose that \( \gamma \) satisfies assumptions (A0), and that the initial condition \((u_0, v_0)\) satisfies
\[
(u_0, v_0) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega), \quad u_0 \geq 0, \; v_0 > 0 \quad \text{in} \; \bar{\Omega}, \quad u_0 \neq 0. \tag{1.3}
\]
Moreover, we assume
\[
\gamma_\infty := \limsup_{s \to \infty} \gamma(s) < 1/\tau. \tag{A1}
\]
Then problem (1.1) has a unique global non-negative classical solution \((u, v) \in C^0(\bar{\Omega} \times [0, \infty); \mathbb{R}^2) \cap C^{2,1}(\bar{\Omega} \times (0, \infty); \mathbb{R}^2)\).

Remark 1.1. Note the above result now permits sign-changing of \( \gamma' \) and thus applies for both chemo-attraction and chemo-repulsion.

In [18], global existence of weak solutions was proved where sign-changing of \( \gamma' \) is also permitted. However, a crucial condition \( \sup_{s \in (0, \infty)} \gamma(s) < 1/\tau \) is assumed there which requires uniform smallness of \( \gamma \) on \((0, \infty)\). In contrast, a much weakened assumption (A1) in the current work is proposed where smallness is merely presumed near infinity and hence \( \gamma \) can be arbitrarily large within any finite interval.

Our second task is to study the dynamic of global classical solutions. More precisely, we are interested in the boundedness of global solutions to problem (1.1). If \( \gamma \) has strictly positive lower and upper bounds, i.e.,
\[
\text{there are generic constants } \gamma_* \gamma^* > 0 \text{ such that } \gamma_* \leq \gamma(\cdot) \leq \gamma^* \text{ in } (0, \infty), \tag{A2}
\]
there are limited evidences indicating that the system fosters bounded solutions. In this direction, the first result is the one we mentioned before by Tao and Winkler in [23], proving that when \( N = 2 \), problem (1.1) permits a unique globally bounded classical solution on a convex domain provided that \( \gamma \) satisfies (A2) and \(|\gamma'|\) is also bounded. The other result in higher dimension \( N \geq 3 \) is provided in [15, Remark 1.3] for the parabolic-elliptic simplification of system (1.1) \((\tau = 0)\), establishing the uniform boundedness when \( \gamma \) is non-increasing and has strictly positive lower and upper bounds.

The second main result of this paper considers the case when \( \gamma \) simply satisfies (A0) and (A2). We establish global existence of uniformly bounded classical solutions in any dimension. Here and after, we use the short notation \( \| \cdot \|_p \) for the norm \( \| \cdot \|_{L^p(\Omega)} \) with \( p \in [1, \infty] \).

Theorem 1.2. Let \( N \geq 1 \). Suppose that \( \gamma \) satisfies assumptions (A0) and (A2), and that the initial condition \((u_0, v_0)\) satisfies (1.3). Then problem (1.1) has a unique global non-negative classical solution that is uniformly-in-time bounded, i.e.,
\[
\sup_{t \geq 0} \{ \|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_{W^{1,\infty}} \} < \infty. \tag{1.4}
\]

Remark 1.2. Note that we do not need assumption (A1) here. Our result improves the 2D boundedness result in [23, Theorem 1.1] by establishing boundedness in any dimension and removing the boundedness assumption on \(|\gamma'|\), as well as the convexity of domain as requested there.
Remark 1.3. In view of the time-independent positive lower bound for $v$ that are also independent of the choice of $\gamma$, see (2.2) below, assumption (A2) can be slightly weakened as there are generic constants $\gamma_{*}, \gamma^* > 0$ such that $\gamma_{*} \leq \gamma(\cdot) \leq \gamma^*$ in $[v_{*}, \infty)$.

Boundedness results are also available in the literature if $\gamma$ is non-increasing and tending to zero at infinity, i.e.,

$$\gamma' \leq 0, \quad \text{and } \lim_{s \to \infty} \gamma(s) = 0.$$  \hfill (A3)

Thus the motility function has no strictly positive lower bound and the above asymptotic vanishing property leads to a possible degenerate problem. Under the circumstances, dynamic of global solutions is shown to be closely related to the decay rate of $\gamma$ at infinity. Exponential decay rate is verified to be critical if $N = 2$ in a sense that the solution can become unbounded at time infinity when $\gamma(s) = e^{-s}$ with some large mass initial data $[7, 9, 10]$, whereas it is always uniformly bounded when $\gamma$ decays slower than a negative exponential function, i.e., $\gamma$ satisfies

$$\liminf_{s \to \infty} e^{\chi s} \gamma(s) > 0$$ \hfill (1.5)

for all $\chi > 0$ [11, 16], e.g., $\gamma(s) = s^{-k_1} \log^{-k_2}(1 + s)$ with any $k_1, k_2 > 0$, or $\gamma(s) = e^{-s^\alpha}$ with any $\alpha \in (0, 1)$. If $N \geq 3$, the global classical solution is proved to be uniformly-in-time bounded in [14–16], provided that $\gamma$ exhibits an algebraic decay at infinity. More precisely, the following algebraic growth assumption is requested on $1/\gamma$:

there are $k \geq l \geq 0$ such that $\liminf_{s \to \infty} s^k \gamma(s) > 0$ and $\limsup_{s \to \infty} s^l \gamma(s) < \infty.$ \hfill (A4)

A specific example satisfying (A4) is $\gamma(s) = s^{-k}$. For the parabolic-elliptic version of system (1.1) ($\tau = 0$), it is shown in [15] that the global solution is bounded if we require further that

$$k < \frac{N}{N - 2} \quad \text{and} \quad k - l < \frac{2}{N - 2}.$$ \hfill (1.6)

See also [1, 24] for the specific case $\gamma(s) = s^{-k}$ with $k < \frac{2}{N - 2}$. For the fully parabolic case $\tau > 0$, however, an additional technical assumption below is also needed in [16]:

there is $b_0 \in (0, 1]$ such that, for any $s \geq s_0 > 0$,

$$s\gamma(s) + (b_0 - 1) \int_1^s \gamma(\eta)d\eta \leq K_0(s_0),$$ \hfill (A5)

where $K_0(s_0) > 0$ depends only on $\gamma, b_0$, and $s_0$.

One another aim of the present paper is to revisit the boundedness result in [11, 16] for $N \geq 2$. Thanks to an improvement of the comparison argument, we are now able to remove the non-increasing monotonicity and the above technical assumption (A5) on $\gamma$ under the same decay rate assumption as in [11, 16].

Now we rephrase the boundedness result for problem (1.1) with possible decaying motility as follows.
Theorem 1.3. Let $N = 2$. Suppose that $\gamma$ satisfies assumptions (A0) and (A1). Moreover, there is $\chi > 0$ such that (1.5) is satisfied. If the initial condition $(u_0,v_0)$ satisfies (1.3) with
\[ \|u_0\|_1 < \frac{4\pi(1 - \tau\gamma_{\infty})}{\chi}, \]  
then the global classical solution $(u,v)$ to (1.1) is uniformly-in-time bounded.

In particular, if (1.5) is satisfied for all $\chi > 0$, then the global classical solution to (1.1) is uniformly-in-time bounded for any given initial data satisfying (1.3).

Remark 1.4. If $\lim_{s \to \infty} \gamma(s) = 0$, then $\gamma_{\infty} = 0$ and our result ensures boundedness of classical solutions if $\|u_0\|_1 < 4\pi/\chi$, or (1.5) is satisfied for all $\chi > 0$, which improves previous results in [11, 16] by removing the monotonicity assumption. In this case, the decay rate (1.5) is optimal since infinite-time blowup occurs if $\gamma(s) = e^{-\chi s}$ with some initial data of total mass exceeding $4\pi/\chi$, see [7, 9, 10].

Theorem 1.4. Let $N \geq 3$. Suppose that $\gamma$ satisfies assumptions (A0) and (A1). Moreover, assumption (A4) is fulfilled with some $k \geq l \geq 0$ satisfying (1.6). Then, for any initial condition $(u_0,v_0)$ satisfying (1.3), the global classical solution $(u,v)$ to (1.1) is uniformly-in-time bounded in the sense that it satisfies (1.4).

Remark 1.5. In Theorem 1.4, assumption (A1) is only explicitly needed in the case $l = 0$ and $0 < k < \frac{2}{N-2}$; see the proof of Lemma 5.3.

Remark 1.6. If $\gamma(s) = s^{-k}$, we note that (1.1a)-(1.1b) can be written as
\[ \begin{cases} u_t = \nabla \cdot (\gamma(v)\nabla u + u\gamma'(v)\nabla v) = \nabla \cdot \left(\gamma(v)(\nabla u - ku\nabla \log v)\right) \\ \tau v_t - \Delta v + v = u. \end{cases} \]  
System (1.8) resembles the logarithmic Keller–Segel system and they share the same steady states. For the latter system, it was conjectured in [12] that $k_c = \frac{N}{N-2}$ is the threshold number that separates boundedness and blowup of classical solutions.

According to Theorem 1.4, we achieve boundedness of global classical solutions to (1.8) in the sub-critical case $k < k_c$, whereas for the logarithmic Keller–Segel system, boundedness results in the sub-critical region are still far from satisfaction.

The main idea of our proof relies on a significant improvement of the comparison argument of a systematic approach developed recently in [9, 10, 13–16, 18] for systems (1.1) and (1.2). Since the approach is quite different from the conventional energy method (e.g., [1, 3, 7, 23]), let us outline its procedure here. The key ingredient lies in an introduction of an auxiliary function $w(x,t)$, which satisfies an elliptic problem:
\[ \begin{cases} w - \Delta w = u, & (x,t) \in \Omega \times (0,\infty), \\ \nabla w \cdot \nu = 0, & (x,t) \in \partial \Omega \times (0,\infty). \end{cases} \]  
By taking an inverse operator $\mathcal{A} = (I - \Delta)^{-1}$ on both sides of (1.1a) with $\Delta$ being the Laplacian operator with homogeneous Neumann boundary condition, one obtains the following key identity:
\[ w_t + u\gamma(v) = \mathcal{A}^{-1}[u\gamma(v)]. \]  

(1.9)
This identity is uncovered in [10] and has since then been used efficiently to investigate the global existence and boundedness of classical solutions to (1.1) and its variants (1.2), see [5,9–11,15,16,18–20].

Substituting \( u = w - \Delta w \) into the key identity, we transform the problem to a thorough study of the following quasilinear parabolic equation involving a non-local term:

\[
wt - \gamma(v)\Delta w + \gamma(v)w = A^{-1}[u\gamma(v)].
\] (1.10)

The proof for global existence and boundedness mainly consists of four steps as we sketch below.

- **Step I**: estimates of \( w \) and \( v \) in \( L^\infty(0,T;L^\infty(\Omega)) \) for arbitrary \( T > 0 \), with a possible dependence on \( T \). For this purpose, a three-step comparison argument is developed.
  
  (a) Firstly, the above key identity (1.9) allows us to derive a (time-dependent) upper bound of \( w \) via comparison principle of elliptic equations; see Lemma 2.3 below;
  
  (b) Secondly, an application of comparison principle for heat equations yields to an upper control of \( v \) by \( w \) and thus gives a (time-dependent) upper bound of \( v \);
  
  (c) Thirdly, we use comparison principle of elliptic as well as heat equations again to prove a reverse control. Hence we establish a two-sided control of \( v \) by \( w \).

- **Step II**: time-independent upper bounds for \( w \) and \( v \).
  
  (a) If \( N \leq 2 \), the uniform-in-time upper bounds are derived via a simple way relying on a crucial time-independent duality estimate and Sobolev embeddings.
  
  (b) When \( N \geq 3 \), the proof is more tricky. Thanks to the two-sided control and the monotonicity of \( \gamma \), we can replace \( \gamma(v) \) by \( \gamma(Cw) \) in (1.10) with some generic positive constant \( C \). Then a delicate Moser-Alikakos iteration is applied to derive the uniform-in-time upper bounds for \( w \) and \( v \).

- **Step III**: Hölder estimate for \( v \) in \( C^{\alpha}(\bar{\Omega} \times [0,T]) \) with arbitrary \( T > 0 \), with a possible dependence upon \( T \) of both the estimate and the exponent \( \alpha \in (0,1) \). We establish a local energy estimate which implies that \( w \) is bounded in \( C^{\alpha}(\bar{\Omega} \times [0,T]) \) by the classical result in [17]. The Hölder estimate for \( v \) then follows by standard Schauder’s theory.

- **Step IV**: estimates of \( (u,v) \) in \( L^\infty(0,T;L^\infty(\Omega) \times W^{1,\infty}(\Omega)) \), still possibly depending on \( T \). The above Hölder continuity enables us to regard \( -\gamma(v(t))\Delta \) as a generator of an evolution operator. Thus we can apply the abstract semigroup theory in [2] to deduce the high-order estimates for \( w \). Finally, we use energy method to get the \( W^{1,\infty} \)-boundedness of \( v \), and \( L^\infty \)-boundedness of \( u \), respectively.

For global existence of classical solutions, the proof goes along Step I-(a, b), Step III and Step IV. For uniform-in-time boundedness, the proof consists of Step I-(b), Step II-(a), Step III and Step IV when \( N \leq 2 \), and it follows Step I-(b, c), Step II-(b), Step III and Step IV when \( N \geq 3 \).

We point out that monotonicity of \( \gamma \) is not necessary in Step III and Step IV once we obtain the (time-dependent or time-independent) upper bounds of \( w \) and \( v \). However, the
non-increasing property plays a crucial role in comparison argument part Step I-(b,c), which restrict the previous studies [9,10,13–16] mainly focusing on the chemo-attraction case.

We stress again that the main contribution of the present work is to completely remove the monotonicity assumption on the motility function, which achieves a crucial technical improvement of the above mentioned method and an important progress in the existence/boundedness theory for the Keller–Segel system (1.1). Moreover, it has significance in physical applications, since now \( \gamma' \) is permitted to change its sign, both chemo-attraction and chemo-repulsion are included.

As we analyzed above, the cornerstone of our proofs is the derivation of \( L^\infty \)-bounds for \( w \) and \( v \) without monotonicity, which relies on an essential improvement and a careful revisit of the above mentioned steps. Firstly, we develop a variant version of the second comparison argument in Step I-(b) which completely removes the non-increasing assumption; see Lemma 3.1 below. Secondly, under the assumption of strictly lower and upper boundedness of \( \gamma \), we can directly apply an iteration argument on (1.9) together with comparison techniques to derive the uniform boundedness of \( w \) without the monotonicity assumption on \( \gamma \), which allows us to skip Step II. Lastly, we observe that under the assumptions (A1) and (A4) with some \( k \geq l \geq 0 \), monotonicity in Step I-(c) and Step II can also be dropped as well.

The paper is organized as follows. In Section 2, we recall some preliminary results. In Section 3, we provide the key variant proof for the upper control of \( v \) by \( w \), which removes the monotonic non-increasing property and enables us to prove Theorem 1.1. In Section 4, we use an iteration together with a comparison argument to derive the uniform upper bound for \( w \) directly and then prove Theorem 1.2. In the last section, we discuss the boundedness when \( \gamma \) satisfies some decay condition at infinity.

2 Preliminaries

In this section, we recall some basic results for problem (1.1). First, we have the local well-posedness. We refer the reader to [16, Proposition 2.1] for a detailed proof.

**Theorem 2.1.** Suppose that \( \gamma \) satisfies (A0) and \((u_0,v_0)\) satisfies (1.3). Then there exists \( T_{\text{max}} \in (0,\infty) \) such that problem (1.1) has a unique non-negative classical solution \((u,v)\in C(\bar{\Omega} \times [0,T_{\text{max}}); \mathbb{R}^2) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\text{max}}); \mathbb{R}^2)\). The solution \((u,v)\) satisfies the mass conservation

\[
\int_{\Omega} u(x,t) \, dx = \int_{\Omega} u_0(x) \, dx \quad \text{for all } t \in (0,T_{\text{max}}). \tag{2.1}
\]

Moreover, there is \( v_* > 0 \) depending only on \( \Omega, v_0, \) and \( \|u_0\|_1 \) such that

\[
v(x,t) \geq v_* , \quad (x,t) \in \bar{\Omega} \times [0,T_{\text{max}}). \tag{2.2}
\]

Finally, if \( T_{\text{max}} < \infty \), then

\[
\lim_{t \to T_{\text{max}}} \sup_{\Omega} \|u(\cdot,t)\|_\infty = \infty.
\]

The following result implies that under the assumptions (A0) and (A4), \( \gamma \) is controlled from the above and below by two algebraically decay functions. The proof is trivial and we omit the details here.
Lemma 2.1. Suppose that $\gamma$ satisfies (A0) and (A4). Then for any $a > 0$, there are constants $C_1, C_2 > 0$ depending on $a$ and $\gamma$ such that

$$C_1 s^{-k} \leq \gamma(s) \leq C_2 s^{-l}, \quad s \in [a, \infty).$$  \tag{2.3}$$

Next, we define the operator $\mathcal{A}$ on $L^2(\Omega)$ as

$$\text{dom}(\mathcal{A}) \triangleq \{ z \in H^2(\Omega) : \nabla z \cdot \nu = 0 \text{ on } \partial \Omega \}, \quad \mathcal{A}z \triangleq z - \Delta z, \quad z \in \text{dom}(\mathcal{A}).$$

We recall that $\mathcal{A}$ generates an analytic semigroup on $L^p(\Omega)$ and is invertible on $L^p(\Omega)$ for all $p \in (1, \infty)$, see e.g., [2]. Then we introduce $w(\cdot, t) \triangleq \mathcal{A}^{-1}[u(\cdot, t)] \geq 0$, for $t \in [0, T_{\text{max}})$, the non-negativity of $w$ being a consequence of that of $u$ and the comparison principle. Due to the time continuity of $u$,

$$w_0(x) \triangleq w(x, 0) = \mathcal{A}^{-1}[u_0(x)] \tag{2.4}$$

and it follows the regularity assumption (1.3) that $w_0 \in W^{2,p}(\Omega)$ with any $1 \leq p < \infty$.

We also need the following result given in [24, Lemma 3.3], which is similar to the celebrated Brezis-Merle inequality [4, Theorem 1], see [23, Lemma A.3] for related results.

Lemma 2.2. Assume that $N = 2$. For any $f \in L^1(\Omega)$ such that

$$\|f\|_1 = \Lambda > 0$$

and $0 < R < \frac{4\Lambda}{\pi}$, there is $C(\Lambda, R) > 0$ depending on $\Omega$, $\Lambda$, and $R$ such that the solution $z$ to

$$\begin{cases} -\Delta z + z = f, & x \in \Omega, \\ \nabla z \cdot \nu = 0, & x \in \partial \Omega, \end{cases} \tag{2.5}$$

satisfies

$$\int_{\Omega} e^{Rz} \, dx \leq C(\Lambda, R).$$

Last we recall the following key lemma. See [10, Lemma 5] for a detailed proof.

Lemma 2.3. Assume that $\gamma$ satisfies (A0). Then the following key identity holds

$$\partial_t w + u\gamma(v) = \mathcal{A}^{-1}[u\gamma(v)] \tag{2.6}$$

in $\Omega \times (0, T_{\text{max}})$. Moreover, if there is a constant $\gamma^* > 0$ such that $\gamma(v(x, t)) \leq \gamma^*$ for $(x, t) \in \Omega \times [0, T_{\text{max}})$, then there holds

$$w_* \leq w(x, t) \leq w_0(x)e^{\gamma^* t}, \quad \text{for } (x, t) \in \Omega \times [0, T_{\text{max}}) \tag{2.7}$$

with $w_*$ being a positive constant depending only on $\Omega$ and $\|u_0\|_1$. 

8
3 Global existence with non-monotonic motility

In this section, we aim to establish global existence of classical solutions to problem (1.1) under the assumptions of Theorem 1.1. To begin with, we provide an upper control of $v$ by $w$ under the assumption (A1), which enables us to consider problem (1.1) without monotonicity assumption on $\gamma$ in this work.

First, we notice that under the assumptions (A0) and (A1), there is a positive constant $\gamma^* > 0$ such that $\gamma(s) \leq \gamma^*$ for $s \in [v_*, \infty)$ with $v_*$ being the constant given in (2.2). Then it results from Lemma 2.3 that

$$w_* \leq w(x, t) \leq w_0(x) e^{\gamma^* t} \quad \text{for} \quad (x, t) \in \Omega \times [0, T_{\max}).$$

Lemma 3.1. Under the assumption of Theorem 1.1, for any $\varepsilon > 0$, there is a constant $C_\infty > 0$ depending only on $\Omega$, $\gamma$, $\tau$, $\varepsilon$ and the initial data such that for all $(x, t) \in \bar{\Omega} \times [0, T_{\max})$,

$$v(x, t) \leq \left( \frac{1}{1 - \tau \gamma_\infty} + \varepsilon \right) (w + C_\infty).$$

Proof. Recall that $\limsup_{s \to \infty} \gamma(s) = \gamma_\infty < 1/\tau$ and introduce

$$\varepsilon_0 = (1 - \tau \gamma_\infty) \left( 1 - \frac{1}{1 + \varepsilon(1 - \tau \gamma_\infty)} \right) \in (0, 1 - \tau \gamma_\infty).$$

Since $w, v$ are both non-negative, we may pick $C_\infty > 0$ sufficiently large such that

$$\gamma(\sigma v + w + C_\infty) \leq \gamma_\infty + \varepsilon_0 / \tau, \quad \forall \sigma > 0$$

as well as

$$v_0(x) \leq w_0(x) + C_\infty.$$ (3.4)

Denote $L[z] = \tau z_t - \Delta z + z = \tau z_t + A z$ and observe that $A^{-1}[u \gamma(v)] \geq 0$ by comparison principle. We infer from (1.1b), the definition of $w$, the key identity (2.6) and (3.3) that

$$\mathcal{L}[v] = u = w - \Delta w = \mathcal{L}[w] - \tau w_t$$

$$= \mathcal{L}[w] + \tau u \gamma(v) - \tau A^{-1}[u \gamma(v)]$$

$$\leq \mathcal{L}[w] + \tau u \left( \gamma(v) - \gamma(\sigma v + w + C_\infty) \right) + \tau u \gamma(\sigma v + w + C_\infty)$$

$$\leq \mathcal{L}[w] + \tau u \left( \gamma(v) - \gamma(\sigma v + w + C_\infty) \right) + (\tau \gamma_\infty + \varepsilon_0) u,$$ (3.5)

which entails that

$$(1 - \tau \gamma_\infty - \varepsilon_0) u$$

$$\leq \mathcal{L}[w] + \tau u \left( \gamma(v) - \gamma(\sigma v + w + C_\infty) \right)$$

$$= \mathcal{L}[w] + \left( \tau u \int_0^1 \gamma'(\theta v + (1 - \theta)(\sigma v + w + C_\infty)) d\theta \right) ((1 - \sigma)v + w - C_\infty)$$

$$\leq \mathcal{L}[w + C_\infty] + \left( \tau u \int_0^1 \gamma'((1 - \theta)(\sigma v + w + C_\infty)) d\theta \right) ((1 - \sigma)v + w - C_\infty)$$ (3.6)
Now, we let $1-\tau = 1-\tau_\infty - \varepsilon_0 \in (0,1)$ and notice that $(1-\tau_\infty - \varepsilon_0)u = \mathcal{L}[(1-\tau_\infty - \varepsilon_0)v] = \mathcal{L}[(1-\tau)v]$. It follows from (3.6) that

$$
\mathcal{L}[(1-\tau)v - w - C_\infty] \leq \left(\tau u \int_0^1 \gamma'(\theta v + (1-\theta)(\sigma v + w + C_\infty))d\theta \right)((1-\tau)v - w - C_\infty).
$$

(3.7)

Denote

$$
g(x,t) := \tau u \int_0^1 \gamma'(\theta v + (1-\theta)(\sigma v + w + C_\infty))d\theta
$$

which belongs to $C(\bar{\Omega} \times [0,T_{\text{max}}])$ according to Theorem 2.1 and let $h(x,t) := (1-\tau)v - w - C_\infty$.

The preceding inequality now reads as

$$
\tau \partial_t h - \Delta h + (1-g)h \leq 0.
$$

(3.8)

Moreover, we note that $\nabla h \cdot \nu = 0$ on $\partial\Omega$, and $h(x,0) = (1-\tau)v_0 - w_0 - C_\infty \leq v_0 - w_0 - C_\infty \leq 0$ due to (3.4). For any fixed $T \in (0,T_{\text{max}})$, since $1-g \in C_b(\bar{\Omega} \times [0,T])$, we may apply comparison principles of linear parabolic equations (see, e.g., [17, Theorem 2.2, Chapter 1]) to deduce that $h(x,t) \leq 0$ for all $(x,t) \in \bar{\Omega} \times [0,T]$. As a result, we get

$$(1-\tau)v(x,t) \leq w(x,t) + C_\infty, \quad \text{for } (x,t) \in \bar{\Omega} \times [0,T_{\text{max}}).$$

This completes the proof by substituting the definition of $\varepsilon_0$.

As a result, we obtain an upper control of $v$ by $w$ as well as a time-dependent upper bound of $v$.

**Corollary 3.1.** Under the assumption of Theorem 1.1, there is a constant $B > 0$ depending only on $\Omega$, $\gamma$, $\tau$ and the initial data such that

$$
v(x,t) \leq Bw(x,t), \quad \text{for } (x,t) \in \bar{\Omega} \times [0,T_{\text{max}})
$$

(3.9)

and

$$
v(x,t) \leq \frac{2(w_0 e^{\gamma t} + C_\infty)}{1 - \tau_\infty}, \quad \text{for } (x,t) \in \bar{\Omega} \times [0,T_{\text{max}}).$$

(3.10)

**Proof.** Fixing any $\varepsilon > 0$ in Lemma 3.1, then there is a positive time-independent constant $C > 0$ such that

$$
v(x,t) \leq Cw(x,t) + C \quad \text{for } (x,t) \in \bar{\Omega} \times [0,T_{\text{max}}).
$$

Thanks to the strictly positive lower bound given in (2.7), we infer that

$$
v(x,t) \leq C(w(x,t) + w(x,t)/w_*) = C(1 + 1/w_*)w(x,t) := Bw(x,t).
$$

(3.11)

The second upper bound estimate (3.10) simply follows from Lemma 3.1 and (3.1) by taking $\varepsilon = 1/(1 - \tau_\infty)$. This completes the proof. \qed
Next, we prove the Hölder continuity of \( w \) and \( v \). First, we recall that the auxiliary function \( w \) solves the initial-boundary value problem
\[
\begin{align*}
\partial_t w + u \gamma(v) &= A^{-1}[w \gamma(v)], \quad (t, x) \in (0, T_{\max}) \times \Omega, \\
-\Delta w + w &= u, \quad (t, x) \in (0, T_{\max}) \times \Omega, \\
\nabla w \cdot \nu &= 0, \quad (t, x) \in (0, T_{\max}) \times \partial \Omega, \\
w(x, 0) &= w_0(x), \quad x \in \Omega.
\end{align*}
\]
(3.12a) \hspace{1cm} (3.12b) \hspace{1cm} (3.12c) \hspace{1cm} (3.12d)

For any fixed \( T \in (0, T_{\max}) \), denote \( J_T = [0, T] \). Introducing \( w^* = \|w_0\|_{\infty}e^{\gamma^* T} \) being the upper bound of \( w \) on \( \hat{\Omega} \times J_T \) according to Lemma 2.3, the upper bound of \( v \) on \( \hat{\Omega} \times J_T \) then is \( v^* = 2(w^* + C_\infty)/(1 - \tau \gamma_\infty) \) due to (3.10). We note that with the upper and lower bounds of \( v \), under the assumptions of (A0) and (A1), one can find two positive constants \( \gamma^*_s(T), \gamma^* > 0 \) with \( \gamma^*_s \) depending on \( T \) and \( \gamma^* \) independent of \( T \) such that \( \gamma(v(x, t)) \in [\gamma^*_s(T), \gamma^*] \) for \( (x, t) \in \hat{\Omega} \times J_T \).

Now, we may proceed along the lines of [17, Chapter V, Section 7] to establish the following local energy estimate for \( w \); see also [16, Lemma 4.1].

**Lemma 3.2.** Let \( \delta \in (0, 1) \). Suppose that \( \gamma_s \leq \gamma(v(x, t)) \leq \gamma^* \) and \( 0 < w_s \leq w(x, t) \leq w^* \leq v^* \) for \( (x, t) \in \hat{\Omega} \times J_T \). There is \( C > 0 \) depending on \( \gamma_s, \gamma^* \) and \( w^* \) such that, if \( \vartheta \in C^\infty(\hat{\Omega} \times J_T) \), \( 0 \leq \vartheta \leq 1 \), \( \sigma \in \{-1, 1\} \), and \( h \in \mathbb{R} \) are such that
\[
\sigma w(x, t) - h \leq \delta, \quad (x, t) \in \text{supp } \vartheta,
\]
then
\[
\begin{align*}
\int_{\Omega} \vartheta^2 (\sigma w(t) - h)^2_+ \ dx + \frac{\gamma^*_s}{2} \int_{t_0}^t \int_{\Omega} \vartheta^2 |\nabla (\sigma w(s) - h)|^2 \ dx \ ds \\
\leq \int_{\Omega} \vartheta^2 (\sigma w(t_0) - h)^2_+ \ dx + C \int_{t_0}^t \int_{\Omega} (|\nabla \vartheta|^2 + \vartheta |\partial_t \vartheta|) (\sigma w(\tau) - h)^2_+ \ dx \ ds \\
+ C \int_{t_0}^t \int_{A_{h, \vartheta, \sigma}(s)} \vartheta \ dx \ ds
\end{align*}
\]  
(3.14)

for \( 0 \leq t_0 \leq t \leq T \), where
\[
A_{h, \vartheta, \sigma}(s) \triangleq \{ x \in \Omega : \sigma w(x, s) > h \} , \quad s \in [0, T].
\]

**Proof.** By (2.6),
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \vartheta^2 (\sigma w - h)^2_+ \ dx &= \sigma \int_{\Omega} \vartheta^2 (\sigma w - h)_+ \partial_t w dx + \int_{\Omega} (\sigma w - h)^2_+ \vartheta \partial_t \vartheta dx \\
&= -\sigma \int_{\Omega} \vartheta^2 (\sigma w - h)_+ u \gamma(v) dx + \sigma \int_{\Omega} \vartheta^2 (\sigma w - h)_+ A^{-1}[u \gamma(v)] dx \\
&\quad + \int_{\Omega} (\sigma w - h)^2_+ \vartheta \partial_t \vartheta dx.
\end{align*}
\]
(3.15)

By comparison principle of elliptic equations, we observe that \( A^{-1}[u \gamma(v)] \leq \gamma^* w^* \). Hence,
\[
\sigma \int_{\Omega} \vartheta^2 (\sigma w - h)_+ A^{-1}[u \gamma(v)] dx \leq \gamma^* w^* \int_{\Omega} \vartheta^2 (\sigma w - h)_+ dx.
\]
Either $\sigma = 1$ and it follows from the boundedness of $\gamma$, (3.12b) and the non-negativity of $u$ and $w$ that

$$
- \sigma \int_{\Omega} \nabla^2 (\sigma w - h) + w \gamma(v) \, dx \\
\leq -\gamma_* \int_{\Omega} \nabla^2 (w - h) + u \, dx \\
= -\gamma_* \int_{\Omega} \nabla^2 (w - h) + (w - \Delta w) \, dx \\
\leq -\gamma_* \int_{\Omega} \nabla \left[ \nabla^2 (w - h) \right] \cdot \nabla w \, dx \\
\leq -\gamma_* \int_{\Omega} \nabla^2 \nabla (w - h) + dx + 2\gamma_* \int_{\Omega} \nabla |\nabla \nabla (w - h) + \nabla (w - h) + dx .
$$

Or $\sigma = -1$ and we infer that

$$
- \sigma \int_{\Omega} \nabla^2 (\sigma w - h) + w \gamma(v) \, dx \\
\leq \gamma_* \int_{\Omega} \nabla^2 (-w - h) + u \, dx \\
\leq \gamma_* w^* \int_{\Omega} \nabla^2 (-w - h) + dx \\
+ \gamma_* \int_{\Omega} \nabla \left[ \nabla^2 (-w - h) \right] \cdot \nabla w \, dx \\
\leq -\gamma_* \int_{\Omega} \nabla^2 \nabla (-w - h) + dx \\
+ 2\gamma_* \int_{\Omega} \nabla |\nabla \nabla (-w - h) + \nabla (-w - h) + dx \\
+ C(\gamma^*, w^*) \int_{\Omega} \nabla^2 (-w - h) + dx .
$$

Inserting the above estimates in (3.15) and using Young's inequality lead us to

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \nabla^2 (\sigma w - h) + dx \\
\leq -\gamma_* \int_{\Omega} \nabla^2 |\nabla (\sigma w - h) + |^2 \, dx \\
+ \frac{\gamma_*}{2} \int_{\Omega} \nabla^2 |\nabla (\sigma w - h) + |^2 \, dx + C(\gamma^*, \gamma^*) \int_{\Omega} |\nabla \nabla (\sigma w - h) + |^2 \, dx \\
+ C(\gamma^*, w^*) \int_{\Omega} \nabla^2 (\sigma w - h) + dx + \int_{\Omega} (\sigma w - h) + \nabla \nabla |\nabla \nabla (\sigma w - h) + dx \\
\leq -\frac{\gamma_*}{2} \int_{\Omega} \nabla^2 |\nabla (\sigma w - h) + |^2 \, dx \\
+ C(\gamma^*, \gamma^*) \int_{\Omega} \left( |\nabla \nabla |^2 + \nabla \nabla (\sigma w - h) + \right) (\sigma w - h) + dx + C(\gamma^*, w^*) \int_{\Omega} \nabla^2 (\sigma w - h) + dx .
$$

We now use (3.13) to estimate from above the last term by

$$
C(\gamma^*, w^*) \int_{\Omega} \nabla^2 (\sigma w - h) + dx \leq C(\gamma^*, w^*) \delta \int_{A_{h, \sigma}} \nabla^2 \, dx \leq C(\gamma^*, w^*) \int_{A_{h, \sigma}} \nabla \, dx .
$$
and integrate the above differential inequality over \((t_0, t)\) to complete the proof. \(\square\)

We are now in a position to apply \([17, \text{Chapter II, Theorem 8.2}]\) to obtain a Hölder estimate for \(w\), the proof of which was given in \([16, \text{Corollary 4.2}]\).

**Corollary 3.2.** Under the assumptions of Lemma 3.2, there is \(\alpha \in (0, 1)\) depending on \(\gamma_s, \gamma^*, w^*, \delta, N\) and the initial data such that \(w \in C^{\alpha}(\bar{\Omega} \times J_T)\).

Next, we use standard Schauder’s estimate to derive Hölder continuity for \(v\).

**Lemma 3.3.** The function \(v\) belongs to \(C^{\alpha, 1+\alpha}(\bar{\Omega} \times J_T)\) with the exponent \(\alpha\) given in Corollary 3.2.

**Proof.** Set \(r \triangleq A^{-1}[v]\) and \(r_0 \triangleq A^{-1}[v_0]\). In view of the regularity \(v_0 \in W^{1, \infty}(\Omega)\), there holds that \(r_0 \in C^{2+\alpha}(\bar{\Omega})\) with \(\alpha \in (0, 1)\) being the exponent given in Corollary 3.2. Besides, we infer from (1.1b) that \(r\) is a solution to

\[
\begin{cases}
\tau \rho_t - \Delta r + r = w, & (x, t) \in \Omega \times (0, T_{\text{max}}) \\
\nabla r \cdot v = 0, & (x, t) \in \partial \Omega \times (0, T_{\text{max}}) \\
r(x, 0) = r_0(x) & x \in \Omega,
\end{cases}
\]

so that Schauder’s theory of heat equations, along with Corollary 3.2, ensures that \(r\) belongs \(C^{2+\alpha, 1+\alpha}(\bar{\Omega} \times J_T)\). As a result, we obtain that \(v = r - \Delta r \in C^{\alpha, 1+\alpha}(\bar{\Omega} \times J_T)\). \(\square\)

**Remark 3.1.** We remark that the exponent \(\alpha\) and the Hölder estimates of \(v\) obtained above depend on \(\gamma_s, \gamma^*, w^*, \Omega\) and the initial data. As a result, in this section, they depend on \(T\) in view of the dependence of \(\gamma_s\) and \(w^*\) on time.

**Proof of Theorem 1.1.** With the Hölder continuity of \(v\) at hand, we can regard \(\gamma(v)\Delta\) as a generator of evolution operators. Then according to \([16, \text{Lemma 4.4 and Proposition 4.5}]\), we have for any \(q \in (1, \infty)\)

\[
\sup_{0 \leq t \leq T} (\|u(\cdot, t)\|_q + \|v(\cdot, t)\|_{W^{1, \infty}}) \leq C(q, T). \tag{3.16}
\]

Note that the results mentioned above in \([16]\) hold for the extended system (1.2), which covers our case by simply taking \(f = n \equiv 0\).

Next we write the first equation as

\[
u_t = \nabla \cdot (\gamma(v(x, t)) \nabla u) + \nabla \cdot (w \gamma'(v) \nabla v).
\]

Note that \(\gamma(v(x, t)) \geq \gamma_s\) and \(w \gamma'(v) \nabla v\) is bounded in \(L^\infty(0, T; L^q(\Omega))\) with any \(q \in (1, \infty)\) by (3.16). This allows us to apply \([22, \text{Lemma A.1}]\) by taking \(m = 1\), fixing any \(q_1 > N + 2\), \(q_2 > \frac{N + 2}{2}\) and choosing \(p_0\) sufficiently large. Thus, we finally deduce that

\[
\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{\infty} \leq C(T).
\]

According to Theorem 2.1, we deduce that \(T_{\text{max}} = \infty\) and this finishes the proof of Theorem 1.1.
4 Uniform-in-time boundedness with non-vanishing motility

In this section, we study uniform-in-time boundedness of classical solution to problem (1.1) with non-vanishing motility. More precisely, we assume within this section that there are two positive constants $\gamma_s$ and $\gamma^*$ such that

$$0 < \gamma_s \leq \gamma(\cdot) \leq \gamma^* \text{ on } (0, \infty).$$

(4.1)

If in particular $\gamma$ is non-increasing on $(0, \infty)$, i.e., $\gamma'(\cdot) \leq 0$ on $(0, \infty)$, then one has $\gamma^* = \gamma(v_*)$. First, we prove uniform-in-time $L^p$-estimates for $w$ with any $1 \leq p < \infty$.

Lemma 4.1. Under the assumption of (A0) and (4.1), for any $1 \leq p < \infty$ there holds

$$\|w(\cdot, t)\|_p \leq C(p) \quad \forall \ t \in [0, T_{\text{max}}),$$

(4.2)

where $C(p) > 0$ is a constant depending only on $\Omega$, $\tau$, $\gamma$, $p$ and the initial data.

Proof. Thanks to the upper and lower bounds of $\gamma$, the non-negativity of $u$, the fact $w - \Delta w = u$, and the comparison principle of elliptic equations, we deduce from the key identity that

$$w_t - \gamma_s \Delta w + \gamma_s w = w_t + \gamma_s u \leq w_t + \gamma(v) u = A^{-1}[u \gamma(v)] \leq A^{-1}[\gamma^* u] = \gamma^* w.$$

(4.3)

A multiplication of (4.3) by $p w^{p-1}$ with any $p \geq 2$ and an integration over $\Omega$ give rise to

$$\frac{d}{dt} \int_\Omega w^p \, dx + \frac{p-1}{4p} \gamma_s \int_\Omega |\nabla w|^{2p/2} \, dx + p \gamma_s \int_\Omega w \, dx \leq p \gamma^* \int_\Omega w \, dx.$$

(4.4)

Recall the Gagliardo-Nirenberg inequality

$$\|z\| \leq C \|\nabla z\|^{\alpha} \|z\|^{1-\alpha} + C \|z\|_1$$

(4.5)

with

$$\alpha = \frac{N}{N+2} \in (0, 1).$$

(4.6)

Letting $z = w^{p/2}$, it follows by the above inequality and Young’s inequality that

$$p \gamma^* \int_\Omega w^p \, dx \leq C p \gamma^* \left( \int_\Omega |\nabla w|^{p/2} \, dx \right)^\alpha \left( \int_\Omega w^{p/2} \, dx \right)^{2(1-\alpha)} + C p \gamma^* \left( \int_\Omega w^{p/2} \, dx \right)^2.$$

$$\leq \frac{\gamma_s}{16} \int_\Omega |\nabla w|^{p/2} \, dx + C(N, p, \gamma_s, \gamma^*) \left( \int_\Omega w^{p/2} \, dx \right)^2.$$

Since $p \geq 2$, we notice that $\frac{p-1}{4p} \geq \frac{1}{8}$. Thus, we arrive at

$$\frac{d}{dt} \int_\Omega w^p \, dx + \frac{\gamma_s}{16} \int_\Omega |\nabla w|^{p/2} \, dx + p \gamma_s \int_\Omega w^p \, dx \leq C(N, p, \gamma_s, \gamma^*) \left( \int_\Omega w^{p/2} \, dx \right)^2.$$

(4.7)

Solving the above ODI with $p = p_k = 2^k$, $(k = 1, 2, 3, \ldots)$ successively in view of the fact $\int_\Omega w \, dx = \int_\Omega u \, dx = \|u_0\|_1$ yields to the $L^p$-estimates for $w$ with any $p \geq 2$ and the remainder case $1 \leq p < 2$ follows by the obvious embedding $L^2 \hookrightarrow L^p$ on bounded domains. This completes the proof.
Now we are ready to prove the uniform-in-time bound of \( w \).

**Lemma 4.2.** Under the assumptions of Theorem 1.2, there is a positive constant \( C \) depending only on \( \Omega, \tau, \gamma \) and the initial data such that

\[
\| w(\cdot, t) \|_{\infty} \leq C, \quad \forall \ t \in [0, T_{\text{max}}).
\]

**Proof.** The proof is based on a simple comparison argument. Let \( z \) be the solution of the following linear equation:

\[
\begin{aligned}
&z_t - \gamma^* \Delta z + \gamma^* z = \gamma^* w \\
&\nabla z \cdot \nu = 0 \\
&z(x, 0) = w_0(x)
\end{aligned}
\quad (x, t) \in \Omega \times (0, T_{\text{max}})
\quad (x, t) \in \partial \Omega \times (0, T_{\text{max}})
\quad x \in \Omega.
\]

Then we have the following representation formula

\[
z(\cdot, t) = e^{\gamma^*(\Delta-1)t}w_0 + \gamma^* \int_0^t e^{\gamma^*(\Delta-1)(t-s)}w(\cdot, s)ds.
\]

It follows from above and Lemma 4.1 that

\[
\| z(\cdot, t) \|_{\infty} \leq e^{\gamma^*(\Delta-1)t}w_0 + \gamma^* \int_0^t e^{\gamma^*(\Delta-1)(t-s)}w(s)ds
\]

\[
\leq C\| w_0 \|_{\infty} + C\gamma^* \int_0^t e^{-\gamma^*(t-s)}(t-s)^{\frac{N}{2p}}\| w(s) \|_p ds
\]

\[
\leq C(\| w_0 \|_{\infty}, \gamma^*, \gamma^*, p, N, \Omega)
\]

provided that \( p > N/2 \), since

\[
\int_0^\infty e^{-\gamma^*s}s^{-\frac{N}{2p}}ds < \infty.
\]

Then by comparison principle of heat equations, we have \( 0 \leq w \leq z \leq C(\| w_0 \|_{\infty}, \gamma^*, \gamma^*, N, \Omega) \) in view of (4.3). This completes the proof.

**Proof of Theorem 1.2.** Fix any \( T \in (0, T_{\text{max}}) \). Although we do not have the upper bound of \( v \) at the present stage, by Lemma 3.2, the strictly positive lower and upper bounds assumption (4.1) together with the uniform-in-time upper bound for \( w \) still allows us to establish the local energy estimate (3.14) for \( w \), where the constant \( C > 0 \) depend on \( \gamma^*, \gamma^* \) and \( \sup \| w \|_{\infty} \), and thus are both independent of \( T \). As a result, according to Corollary 3.2 and Lemma 3.3, there is \( \alpha > 0 \) independent of \( T > 0 \) such that \( w \) and \( v \) belong to Hölder spaces \( C^\alpha(\Omega \times J_T) \) and \( C^{\alpha,1+\alpha}(\Omega \times J_T) \), respectively. Moreover, their Hölder estimates are independent of \( T \) as well.

Finally, by the results in [16, Lemma 6.7 and Proposition 6.8], there is \( C(q) > 0 \) independent of \( T \) and \( T_{\text{max}} \) such that \( \sup_{0 \leq t < T_{\text{max}}} (\| w(\cdot, t) \|_q + \| v(\cdot, t) \|_{W^{1,q}}) \leq C(q) \) for any \( q \in (1, \infty) \). Then it follows by [22, Lemma A.1] that \( \sup_{0 \leq t < T_{\text{max}}} \| u(\cdot, t) \|_{\infty} < \infty \). This completes the proof according to Theorem 2.1.
5 Uniform-in-time boundedness with decaying motility

In this section, we consider problem (1.1) with a decaying motility function that may vanish at infinity. Under the circumstances, uniform boundedness of classical solution is related to the decay rate of $\gamma$ at infinity.

5.1 The case $N = 2$

In this part, we first consider the two-dimensional case and we assume that $\gamma$ satisfies (A0), (A1) and (1.5) with some $\chi > 0$. According to Theorem 1.1, classical solution exists globally. One also observes due to (A0) and (A1) that $\gamma$ has an upper bound on $[v_+, \infty)$ denoted by $\gamma^*$. Then the proof for Theorem 1.3 can be carried out in a similar manner as done in [11, Section 3.1]. We report it here for reader’s convenience.

**Lemma 5.1.** Assume (A0), (A1) and (1.5) with some $\chi > 0$. Suppose that $\|u_0\|_1 < 4\pi(1 - \tau\gamma_\infty)/\chi$. There is $C(\chi) > 0$ depends only on $\Omega, \gamma, \tau$ and the initial data such that

$$\sup_{t>0} \left( \|\nabla w\|^2 + \|w\|^2 + \int_t^{t+1} \int_\Omega u^2 \gamma(v) dx ds \right) \leq C(\chi).$$

(5.1)

**Proof.** We multiply the key identity (2.6) by $u$ and integrate over $\Omega$. Recalling that $w = \mathcal{A}^{-1}[u]$ and that $\mathcal{A}^{-1}$ is self-adjoint, we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \|\nabla w\|_2^2 + \|w\|_2^2 \right) + \int_\Omega u^2 \gamma(v) dx = \int_\Omega u \mathcal{A}^{-1}[u \gamma(v)] dx$$

$$= \int_\Omega u \gamma(v) \mathcal{A}^{-1}[u] dx$$

$$= \int_\Omega \gamma(v) uw dx.$$

Hence, by the upper bound $\gamma^*$ for $\gamma$,

$$\frac{1}{2} \frac{d}{dt} \left( \|\nabla w\|_2^2 + \|w\|_2^2 \right) + \int_\Omega u^2 \gamma(v) dx \leq \gamma^* \int_\Omega uw dx.$$

Also, note that

$$\|\nabla w\|^2_2 + \|w\|^2_2 = \int_\Omega wu dx.$$

Combining the above inequalities and using Young’s inequality, we arrive at

$$\frac{d}{dt} \left( \|\nabla w\|_2^2 + \|w\|_2^2 \right) + \|\nabla w\|^2_2 + \|w\|^2_2 + 2 \int_\Omega u^2 \gamma(v) dx$$

$$= (2\gamma^* + 1) \int_\Omega wu dx$$

$$\leq \int_\Omega u^2 \gamma(v) dx + \frac{(2\gamma^* + 1)^2}{4} \int_\Omega \frac{w^2}{\gamma(v)} dx.$$

We thus obtain

$$\frac{d}{dt} \left( \|\nabla w\|_2^2 + \|w\|_2^2 \right) + \|\nabla w\|^2_2 + \|w\|^2_2 + \int_\Omega u^2 \gamma(v) dx \leq C \int_\Omega \frac{w^2}{\gamma(v)} dx$$

(5.2)
with \( C > 0 \) independent of time.

We now deduce from the assumption (1.5) that there exist \( b > 0 \) and \( s_\gamma > v_* \) depending on \( \chi \) such that \( e^{\chi s_\gamma} \gamma(s) \geq 1/b \) for all \( s \geq s_\gamma \). Besides, owing to the continuity of \( \gamma \), there holds \( \gamma(s) \geq \gamma(s_\gamma, v_*) \) for \( s \in [v_*, s_\gamma] \); we end up with

\[
\frac{1}{\gamma(s)} \leq \max \left\{ b, \frac{1}{\gamma(s_\gamma, v_*)} \right\} e^{\chi s_\gamma}, \quad s \geq v_*.
\] (5.3)

Now, for \( \varepsilon > 0 \), we infer from (5.3), (3.2) and the elementary inequality \( e^{\varepsilon s} \geq \varepsilon^2 s^2, s > 0 \), that

\[
\int_{\Omega} \frac{w^2}{\gamma(v)} \, dx \leq C(\chi) \int_{\Omega} w^2 e^{\chi w} \, dx \leq C(\chi) \int_{\Omega} \frac{w^2 e^{\chi(1 - \tau \gamma_\infty) + \varepsilon)(w + C_\infty)}{\chi^2 \varepsilon^2} \, dx \leq C(\chi) e^{\chi C_\infty (1 - \tau \gamma_\infty + \varepsilon)} \int_{\Omega} e^{\chi(1 - \tau \gamma_\infty + 2\varepsilon w)} \, dx.
\] (5.4)

Since \( \|u\|_1 = \|u_0\|_1 < 4\pi (1 - \tau \gamma_\infty)/\chi \) by (2.1), we may choose \( \varepsilon_\chi > 0 \) such that

\[
\chi(\frac{1}{1 - \tau \gamma_\infty} + 2\varepsilon_\chi) < \frac{4\pi}{\|u_0\|_1} \quad \text{say} \quad \varepsilon_\chi \equiv \frac{\pi}{\chi\|u_0\|_1} - \frac{1}{4(1 - \tau \gamma_\infty)}
\] (5.5)

and deduce from Lemma 2.2 that

\[
\int_{\Omega} e^{\chi(1 - \tau \gamma_\infty + 2\varepsilon_\chi) w} \, dx \leq C(\|u_0\|_1, \chi(\frac{1}{1 - \tau \gamma_\infty} + 2\varepsilon_\chi)).
\] (5.6)

Gathering (5.2), (5.4) (with \( \varepsilon = \varepsilon_\chi \)), and (5.6) gives

\[
\frac{d}{dt} \left( \|\nabla w\|_2^2 + \|w\|_2^2 \right) + \|\nabla w\|_2^2 + \|w\|_2^2 \leq \int_{\Omega} u^2 \gamma(v) \, dx \leq C(\chi),
\]

from which Lemma 5.1 follows after integration with respect to time. \( \square \)

**Proposition 5.1.** Assume (A0), (A1) and (1.5) with some \( \chi > 0 \) and that \( \|u_0\|_1 < 4\pi (1 - \tau \gamma_\infty)/\chi \). There is \( C(\chi) > 0 \) such that

\[
\sup_{t \geq 0} \left( \|w(t)\|_\infty + \|v(t)\|_\infty \right) \leq C(\chi).
\] (5.7)

**Proof.** Let \( p \in (1, 2) \) and \( \varepsilon > 0 \) to be specified later. Since \( u = A[w] \), we infer from the Sobolev embedding theorem, Hölder’s inequality, (5.3) and (3.2) that

\[
\|w\|_\infty \leq C(p) \|w\|_{W^{2, p}} \leq C(p) \|u\|_p
\]

\[
\leq C(p) \left( \int_{\Omega} u^2 \gamma(v) \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (\gamma(v))^{-\frac{p}{2-p}} \, dx \right)^{\frac{2-p}{2p}}
\]

\[
\leq C(p, \chi) \left( \int_{\Omega} u^2 \gamma(v) \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} e^{\frac{2p}{2-p} \gamma(v)} \, dx \right)^{\frac{2-p}{2p}}
\]

\[
\leq C(p, \chi, \varepsilon) \left( \int_{\Omega} u^2 \gamma(v) \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} e^{\frac{2p}{2-p} \left( \frac{1}{1 - \tau \gamma_\infty} + \varepsilon \right) w} \, dx \right)^{\frac{2-p}{2p}}.
\] (5.8)
Choosing $p = p_\chi \triangleq \frac{2+4\varepsilon_\chi (1-\tau \gamma_\infty)}{2+4\varepsilon_\chi (1-\tau \gamma_\infty)} \in (1, 2)$ with $\varepsilon_\chi$ defined in (5.5), we observe that

$$\frac{\chi p_\chi}{(2-p_\chi)} \left( \frac{1}{1-\tau \gamma_\infty} + \varepsilon_\chi \right) = \chi \left( \frac{1}{1-\tau \gamma_\infty} + 2\varepsilon_\chi \right)$$

and we deduce from (5.6) and (5.8) (with $\varepsilon = \varepsilon_\chi$ and $p = p_\chi$) that

$$\|w\|_\infty \leq C(\chi) \left( \int_\Omega u^2 \gamma(v) \, dx \right)^{\frac{1}{2}}.$$

Combining the above estimate with Lemma 5.1, we obtain, for $t \geq 0$,

$$\int_t^{t+1} \|w(\cdot, s)\|_\infty \, ds \leq \left( \int_t^{t+1} \|w(\cdot, s)\|_\infty^2 \, ds \right)^{\frac{1}{2}} \leq C(\chi) \left( \int_t^{t+1} \int_\Omega u^2 \gamma(v) \, dx \, ds \right)^{\frac{1}{2}} \leq C(\chi). \tag{5.9}$$

Now, observing that the key identity (2.6), the non-negativity of $u$ and $\gamma$, and the elliptic comparison principle imply that

$$\partial_t w \leq \partial_t w + u \gamma(v) = A^{-1} [u \gamma(v)] \leq \gamma^* A^{-1} [u] = \gamma^* w,$$

we realize that $w(x, t + 1) \leq e^{\gamma^* (t+1-s)} w(x, s)$ for all $(x, s) \in \Omega \times (t, t+1)$. Consequently,

$$\|w(\cdot, t + 1)\|_\infty e^{\gamma^* (s-t-1)} \leq \|w(\cdot, s)\|_\infty, \quad s \in [t, t+1],$$

and it follows from (5.9) and the above inequality after integration with respect to $s$ over $[t, t+1]$ that

$$\|w(\cdot, t + 1)\|_\infty \frac{1 - e^{-\gamma^*}}{\gamma^*} \leq C(\chi).$$

Since we already know that $\|w(\cdot, t)\|_\infty \leq C$ for $t \in [0, 1]$ by Lemma 2.3, we have thus proved that

$$\|w(\cdot, t)\|_\infty \leq C(\chi) \quad \text{for } t \geq 0.$$

The claimed boundedness of $v$ now readily follows in view of (3.9) and the non-negativity of $v$.

**Proof of Theorem 1.3.** With Proposition 5.1 at hand, $\gamma$ has strictly positive time-independent upper and lower bounds on $\Omega \times [0, \infty)$, the result then directly follows from Theorem 1.2. \qed

### 5.2 The case $N \geq 3$

In this part, we assume that $N \geq 3$ and $\gamma$ satisfies (A0), (A1), (A4) and (1.6). We show that under the above assumptions, the technical condition (A5) and the non-increasing property of $\gamma$ in [16, Theorem1.4] are both removable.

Note that if $\gamma$ satisfies (A4) with $k = l = 0$, there are time-independent positive constants $\gamma_s$ and $\gamma^*$ such that $\gamma_s \leq \gamma(s) \leq \gamma^*$ for $s \in [v_*, \infty)$ according to Lemma 2.1, then the assertion of Theorem 1.4 follows by Theorem 1.2 in this special case. In the sequel, we aim to provide a unified approach to prove Theorem 1.4 for all cases under the assumption (1.6).

To begin with, we need to establish the following reverse control.
Lemma 5.2. Suppose that $\gamma$ satisfies (A0) and there is $C_l > 0$ such that $\gamma(s) \leq C_l s^{-l}$ for $s \geq v_*$ with some $l > 0$. Then there is a positive time-independent constant $A > 0$ such that

$$Aw(x, t) \leq v(x, t), \quad \text{for } (x, t) \in \Omega \times [0, T_{\max}).$$ \hfill (5.10)

Proof. We recall that by the key identity (2.6)

$$L[v] = u = w - \Delta w = L[w] - \tau w_t
= L[w] + \tau u \gamma(v) - \tau A^{-1}[u \gamma(v)].$$ \hfill (5.11)

Since $\gamma(s) \leq C_l s^{-l}$ for $s \geq v_*$ with some $l > 0$, we infer that

$$u \gamma(v) \leq C_l u w^{-l}$$

$$= C_l (\tau v_t - \Delta v + v) u^{-l}$$

$$= C_l \left( \tau \partial_t \Gamma(v) - \Delta \Gamma(v) + \Gamma(v) - lv^{-l-1}|\nabla v|^2 + v^{-l-1} - \Gamma(v) \right)$$

$$= C_l \left( L[\Gamma(v)] - lv^{-l-1}|\nabla v|^2 + v^{-l-1} - \Gamma(v) \right),$$ \hfill (5.12)

where

$$\Gamma(s) \triangleq \int_1^s \eta^{-l} d\eta = \begin{cases} \frac{s^{1-l}-1}{1-l}, & \text{if } l \neq 1, \\ \log s, & \text{if } l = 1. \end{cases}$$ \hfill (5.13)

Denote $s_+ = \max\{s, 0\}$. We observe that for any $s \geq a > 0$,

$$s^{1-l} - \Gamma(s) = \begin{cases} \frac{1-l s^{1-l}}{1-l}, & \text{if } l \neq 1, \\ 1 - \log s \leq (1 - \log a)_+ & \text{if } l = 1. \end{cases}$$ \hfill (5.14)

Let

$$C(a, l) := \begin{cases} \frac{1-l a^{1-l}}{1-l}, & \text{if } l > 1, \\ \frac{1}{1-l}, & \text{if } l < 1, \\ (1 - \log a)_+ & \text{if } l = 1. \end{cases}$$ \hfill (5.15)

Then, there is $C' = C(v_*, l) \geq 0$ such that

$$v^{1-l} - \Gamma(v) \leq C' \text{ for } (x, t) \in \Omega \times [0, T_{\max}),$$ \hfill (5.16)

which together with the non-negativity of $lv^{-l-1}|\nabla v|^2 \geq 0$, and (5.12) implies that

$$u \gamma(v) \leq C_l L[\Gamma(v) + C']$$ \hfill (5.17)

and hence by comparison principle that

$$A^{-1}[u \gamma(v)] \leq C_l A^{-1}[L[\Gamma(v) + C']] = C_l L[A^{-1}[\Gamma(v) + C']].$$

It follows from (5.11), the non-negativity of $u \gamma(v)$ and the above that

$$L[v] \geq L[w] - C_l \tau L[A^{-1}[\Gamma(v) + C']],$$ \hfill (5.18)
which according to comparison principle of heat equations, gives rise to
\[ w \leq v + C_l \tau A^{-1}[\Gamma(v)] + C_0, \quad \text{for } (x, t) \in \Omega \times [0, T_{\max}) \]  
(5.19)
with a generic positive constant \( C_0 \geq C' \) ensuring that
\[ w_0 \leq v_0 + C_l \tau A^{-1}[\Gamma(v_0)] + C_0 \quad \text{in } \bar{\Omega}. \]

Next, noticing that assumption (A1) is satisfied since \( \gamma(s) \leq C_l s^{-l} \) with \( l > 0 \), by Lemma 2.3, \( v \leq Bw \) for \( (x, t) \in \Omega \times [0, T_{\max}). \) Since \( \Gamma(s) \) is monotone increasing, we infer that
\[ \Gamma(v) \leq \Gamma(Bw) = \begin{cases} \frac{B^{1-l}w^{1-l} - 1}{1-l} = B^{1-l}\Gamma(w) + \frac{B^{1-l} - 1}{1-l} \log B + \log w & \text{if } l \neq 1, \\ 1 - lw & \text{if } l = 1. \end{cases} \]
We note that for \( l > 1 \), there holds \( \Gamma(Bw) \leq \frac{1}{l^l} \). Then it follows from (5.19) and comparison principle of elliptic equations that when \( l > 1 \),
\[ w \leq v + C_l \tau A^{-1}[\Gamma(Bw)] + C_0 \leq v + C'_0 \]
with \( C'_0 = \frac{C_l l}{l^l} + C_0 \).

It remains to consider the case \( 0 < l \leq 1 \). Recall that \( w - \Delta w = u \) and observe that
\[ uw^{-l} = \Gamma(w) - \Delta \Gamma(w) - lw^{-l-1}|\nabla w|^2 + (w^{1-l} - \Gamma(w)) \]
\[ = A[\Gamma(w)] - lw^{-l-1}|\nabla w|^2 + (w^{1-l} - \Gamma(w)). \]
(5.20)
When \( l \in (0, 1) \), there holds that
\[ w^{1-l} - \Gamma(w) = \frac{1 - lw^{1-l}}{1-l} = 1 - l\Gamma(w). \]
Therefore, we infer from the above identity, (5.20), the non-negativity of \( uw^{-l} \) and \( lw^{-l-1}|\nabla w|^2 \) that
\[ ll\Gamma(w) \leq A[\Gamma(w)] + 1. \]
(5.21)
Hence, it follows by comparison principle of elliptic equations that
\[ A^{-1}[\Gamma(w)] \leq \frac{1}{l}(\Gamma(w) + 1) \]
(5.22)
Together with (5.19) and Young’s inequality yielding that
\[ w \leq v + C_l \tau A^{-1}[\Gamma(Bw)] + C_0 \]
\[ = v + C_l \tau A^{-1}[B^{1-l}\Gamma(w) + \frac{B^{1-l} - 1}{1-l}] + C_0 \]
\[ \leq v + \frac{C_l \tau B^{1-l}}{l}\Gamma(w) + \frac{C_l \tau B^{1-l}}{1-l} + C_0 \]
\[ = v + \frac{C_l \tau B^{1-l}w^{1-l}}{l(1-l)} + C_0 \]
\[ \leq v + \frac{1}{2} w + C'_0. \]
Thus, for $l \in (0, 1)$ we deduce that

$$w \leq 2(v + C_0').$$

Last, when $l = 1$, we recall that $\Gamma(w) = \log w$ and $w^{1-l} - \Gamma(w) = 1 - \log w$. We further infer from (5.20) that

$$\Gamma(w) \leq A[\Gamma(w)] + 1.$$ 

Thus,

$$A^{-1}[\Gamma(w)] \leq \Gamma(w) + 1$$

due to an application of the comparison principle. Finally, we deduce from (5.19) that

$$w \leq v + C_0'$$

Since $\log w \leq w^\delta$ for any $\delta > 0$, we infer by Young’s inequality again that

$$\frac{1}{2}w \leq v + C_0''.$$ 

In summary, we establish that

$$w \leq C(v + 1) = C(v + v/v_*) = C(1 + 1/v_*)v := \frac{1}{A}v.$$ 

This completes the proof. \qed

In view of Lemma 2.3 and Lemma 5.2, we have the following two-sided control.

**Corollary 5.1.** Suppose that $\gamma$ satisfies (A4) with some $k \geq l > 0$. Then there are positive time-independent constants $A, B > 0$ such that

$$Aw(x,t) \leq v(x,t) \leq Bw(x,t), \quad \text{for} \ (x,t) \in \Omega \times [0,T_{\text{max}}).$$

(5.24)

Now, we prove the following key inequality for $w$ when $\gamma$ satisfies (A4).

**Lemma 5.3.** Suppose that $\gamma$ satisfies (A0), (A1) and (A4), there exist time-independent positive constants $C_1, C_2$ and $C_3$, such that for $(x,t) \in \Omega \times [0,T_{\text{max}})$

$$w_t + C_1B^{-k}w^{-k}u \leq C_2A^{-l}(\Gamma(w) + C_3)$$

(5.25)

where $\Gamma(\cdot)$ is defined by (5.13).

**Proof.** Firstly, we consider the case $k = l = 0$. According to Lemma 2.1, there are two time-independent positive constants $C_1, C_2$ such that

$$0 < C_1 \leq \gamma(s) \leq C_2 \quad \text{for} \ s \in [v_*, \infty).$$
It follows from the key identity, the non-negativity of \( u \), and the comparison principle that

\[
w_t + C_1 u \leq w_t + u \gamma(v) = A^{-1}[u \gamma(v)] \leq C_2 A^{-1}[u] = C_2 w = C_2 \Gamma(w) + C_2.
\]

Secondly, we consider the case \( l = 0 \) and \( k > 0 \). By Lemma 2.1 again, there are two generic positive constants \( C_1, C_2 > 0 \) depending on \( \Omega, \gamma \) and the initial data such that

\[
C_1 s^{-k} \leq \gamma(s) \leq C_2 \quad \text{for } s \in [v_*, \infty).
\]

Since \( \gamma \) also satisfies (A1) (note that (A1) is only explicitly needed here for Theorem 1.4), we infer by the upper control \( v \leq Bw \) that

\[
w_t + C_1 B^{-k} w^{-k} u \leq w_t + \gamma(v) u \leq C_2 A^{-1}[u] = C_2 w = C_2 \Gamma(w) + C_2.
\]

Lastly, we consider the case \( k \geq l > 0 \). Thanks to Lemma 2.1 again, there holds

\[
C_1 s^{-k} \leq \gamma(s) \leq C_2 s^{-l} \quad \text{for } s \in [v_*, \infty),
\]

which together with the two-sided control in Corollary 5.1 yields that

\[
C_1 B^{-k} w^{-k} \leq C_1 v^{-k} \leq \gamma(v) \leq C_2 v^{-l} \leq C_2 A^{-1} w^{-l}
\]

(5.27)

Then an application of the comparison principle to the key identity gives rise to

\[
w_t + C_1 B^{-k} w^{-k} u \leq w_t + u \gamma(v) = A^{-1}[u \gamma(v)] \leq C_2 A^{-1} A^{-1}[uw^{-l}].
\]

(5.28)

Finally, by (5.14) and (5.15) there is \( C'' = C(w_*, l) \geq 0 \) such that

\[
w^{1-l} - \Gamma(w) \leq C''.
\]

Thus, it follows from (5.20) that

\[
A^{-1}[uw^{-l}] \leq \Gamma(w) + C''.
\]

(5.29)

This completes the proof.

Next, we establish the following energy inequality.

**Lemma 5.4.** There are time-independent positive constants \( \lambda_0 > 0 \) and \( C_0 > 0 \) such that, for any \( p > 1 + k \),

\[
\frac{d}{dt} \|w\|_p^p + \frac{\lambda_0 p(p - k - 1)}{(p - k)^2} \|\nabla w\|_2^{p-k} + \lambda_0 p \|w\|_{p-k}^{p-k} \leq C_0 p \int_\Omega (w^{p-1} \Gamma(w) + w^{p-1}) \, dx.
\]
Proof. Multiplying the inequality (5.25) by $w^{p-1}$ for some $p > 1 + k$, we obtain that
\[
\frac{1}{p} \frac{d}{dt} \|w\|_p^p + C_1 B^{-k} \int_\Omega uw^{p-1} \, dx \leq C_2 A^{-1} \int_\Omega w^{p-1}(\Gamma(w) + C_3) \, dx.
\]
Recalling that $w - \Delta w = u$, we observe that
\[
\int_\Omega w^{p-k-1} u \, dx = \int_\Omega w^{p-k-1} (w - \Delta w) \, dx = \|w\|_{p-k}^{p-k} + (p - k - 1) \int_\Omega w^{p-k-2} |\nabla w|^2 \, dx
= \|w\|_{p-k}^{p-k} + \frac{4(p - k - 1)}{(p - k)^2} \|\nabla w^\frac{k}{2}\|_2^2.
\]
The assertion follows by collecting the above estimates and this completes the proof. □

Proof of Theorem 1.4. We point out that energy inequality established for $w$ in Lemma 5.4 is exactly the same as the one in [15, Lemma 4.3] for $v$ (see also [16, Lemma 6.3] for the same energy inequality established for an auxiliary function $\tilde{S}$). Thus, starting from this energy inequality, we can proceed in the same manner by delicate Moser-Alikakos iteration argument to prove that under the same assumption (1.6) as done in [15,16], there is a time-independent constant $C > 0$ such that
\[
\sup_{0 \leq t \leq T_{\max}} \|w(\cdot, t)\|_\infty \leq C.
\]
The uniform-in-time $L^\infty$-bound for $v$ follows as well by Corollary 3.1. As a result, $\gamma$ is bounded from above and below by time-independent positive constants on $\Omega \times [0, T_{\max})$ and we may apply Theorem 1.2 to conclude Theorem 1.3 now. □

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