Halo Orbits around $L_1$ and $L_2$ in the Photogravitational Sun-Earth System with Oblateness

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Abstract The Photogravitational Restricted Three Body Problem with oblateness has been studied to obtain halo orbits around the Lagrangian points $L_1$ and $L_2$ of the Sun-Earth system in which the Sun is taken as radiating and the Earth as an oblate spheroid. The halo orbits corresponding to fourth and fifth order approximations around $L_1$ and $L_2$ for actual oblateness of the Earth and for different radiation pressures for the Sun are displayed graphically. The time period of halo orbits around $L_1$ decreases with increase in oblateness and increases with increase in radiation pressure. A reverse effect is observed due to increase in oblateness and radiation pressure on time period of orbits around $L_2$. It is also observed that halo orbits around $L_1$ shifts towards the source of radiation due to increase in both radiation pressure and oblateness. However, halo orbits around $L_2$ shifts towards the source of radiation due to increase in radiation but recedes with increase in oblateness.

Keywords Restricted Three Body Problem, Photogravitational Sun-Earth System, Oblateness, Halo orbits

1 Introduction

Restricted Three Body Problem (RTBP) deals with the motion of an infinitesimal body which moves under the gravitational influence of two massive bodies called the primaries. The infinitesimal body is called the secondary body. The only force acting on this system is the gravitational attraction force between the primaries. The mass of the secondary body is negligible compared to the primary masses and it does not influence the motion of the primaries. RTBP is very useful for describing the motion of planets, asteroids, comets and satellites (Plummer 1919; Winter 1941; Brouwer and Clemence 1961; Danby 1964; Pollard 1966; Murray and Dermot 1999). It plays an important role in space dynamics, celestial dynamics and analytic dynamics. It has applications in the fields of mathematics, theoretical physics and quantum physics. In Circular Restricted Three Body Problem (CRTBP), the primaries move in a circular path around their common centre of mass. This is a particular case of RTBP (Moulton 1914; McCuskey 1963; Szebehely 1967; Roy 2005; Fitzpatrick 2012; Vallado 2013). Most of the celestial bodies are radiating and hence study of RTBP incorporating radiation, usually called photogravitational RTBP, is pertinent. The solar radiation pressure force changes with the distance in a similar law as the gravitational attraction force but acts in an opposite direction to it. This reduces the effective mass of the Sun (Poynting 1903; Robertson and Russell 1937; Schuerman 1980; Simmons et al. 1985; Abouelmagd 2013; Pathak et al. 2016). In the case of planar CRTBP, there exist five equilibrium points known as Libration points or Lagrangian points. Among these, three points, denoted by $L_1, L_2$ and $L_3$ are collinear with $L_1$ lying between the primaries. The remaining Lagrangian points $L_4$ and $L_5$ lie opposite sides of the joining the primaries. The three dimensional periodic orbits around Lagrangian points are called halo orbits. Halo orbits were introduced by Farquhar (1968). He discovered the trajectories around the Earth-Moon $L_2$ which could be used to place a communication satellite that would continuously link between the Earth and the Moon. Other researchers (Breakwell and Brown 1979; Howell and V. Breakwell 1984; Howell 1984) have studied halo orbit families for
the Earth-Moon system. ISEE-3 was the first halo orbit mission. A third order approximation was introduced by Richardson (1980) to represent halo orbits in the Sun-Earth system. Tiwary and Kushvah (2015) have computed a first guess of halo orbits up to fourth order approximation using the Lindstedt-Poincaré method in the photogravitational RTBP with oblateness. In this paper we have computed halo orbits around the Lagrangian points $L_1$ and $L_2$ in the Sun-Earth system considering the Sun as radiating body and the Earth as oblate spheroid. The fourth order approximations to solutions given by (Tiwary and Kushvah 2015) have been improved incorporating fifth order approximation. The comparison of the orbits obtained by fourth and fifth order approximations are shown graphically.

Equations of motion of an infinitesimal body in a synodic system are described in Section 2. Equations of motion of infinitesimal body with oblateness and solar radiation pressure are given as (McCuskey 1963; Sharma 1987; Tiwary and Kushvah 2015)

\[
\begin{align*}
\dot{x} - 2n\dot{y} &= \frac{\partial \Omega^*}{\partial x}, \\
\dot{y} + 2n\dot{x} &= \frac{\partial \Omega^*}{\partial y}, \\
\ddot{z} &= \frac{\partial \Omega^*}{\partial z},
\end{align*}
\]

where

\[
\Omega^* = n^2\frac{x^2 + y^2}{2} + \frac{(1 - \mu)q}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3},
\]

and

\[
\begin{align*}
q &= \left(1 - \frac{F_p}{F_g}\right), \quad n = \sqrt{\left(1 + \frac{3}{2}A_2\right)}, \quad A_2 = \frac{R_2^2 - R_p^2}{5R^2}, \\
r_1 &= \sqrt{(x + \mu)^2 + y^2 + z^2}, \\
r_2 &= \sqrt{(x + \mu - 1)^2 + y^2 + z^2}
\end{align*}
\]

are the distances of the infinitesimal body from the bigger and smaller primaries, respectively.

### 3 Computation of Halo orbits

Lindstedt-Poincaré method (Koon et al. 2011) is used to compute the halo orbits around the libration points $L_1$ and $L_2$. It is used for solving non-linear ordinary differential equation when the regular perturbation method fails by removing secular terms and thereby converting to weakly non-linear equation with finite oscillatory solutions.

#### 3.1 Equations of motion near $L_1$ and $L_2$

To obtain the halo orbits around the Lagrangian point the origin is shifted to the location of the Lagrangian point. Then the new coordinates are given by (Koon et al. 2011)

\[
\begin{align*}
X &= \frac{1}{\gamma}(x + \mu - 1 \pm \gamma), \\
Y &= \frac{1}{\gamma}y, \\
Z &= \frac{1}{\gamma}z,
\end{align*}
\]

where $\gamma$ is the distance between the Lagrangian point and the smaller primary. In (7), upper sign corresponds to $L_1$ and lower sign corresponds to $L_2$. The variables $X,Y$ and $Z$ are normalized so that the distance between the Lagrangian point and the smaller primary is 1. Using the above transformation in the equations of...
motion (1)-(3), we obtain

\[
\gamma \dot{X} - 2n\dot{Y} = \frac{1}{\gamma} \frac{\partial \Omega}{\partial X},
\]

\[
\gamma \dot{Y} + 2n\dot{X} = \frac{1}{\gamma} \frac{\partial \Omega}{\partial Y},
\]

\[
\gamma \ddot{Z} = \frac{1}{\gamma} \frac{\partial \Omega}{\partial Z},
\]

where

\[
\Omega = \frac{n^2}{2} \left( (\gamma X + 1 - \mu) \gamma + (\gamma Y)^2 + (\gamma Z)^2 \right) + \frac{(1 - \mu)q}{R_1} + \frac{\mu A_2}{2R_2} + \frac{\mu A_2}{2R_2},
\]

and

\[
R_1 = \sqrt{(\gamma X + 1 - \mu) \gamma + (\gamma Y)^2 + (\gamma Z)^2},
\]

\[
R_2 = \sqrt{(\gamma X + 1 - \mu) \gamma + (\gamma Y)^2 + (\gamma Z)^2}.
\]

Expanding the nonlinear terms \(\frac{(1 - \mu)q}{R_1} + \frac{\mu A_2}{2R_2}\) of (13) using Legendre polynomials, equations of motion can be written as (Koon et al. 2011; Tiwary and Kushvah 2015)

\[
\ddot{X} - 2n\dot{Y} - (n^2 + 2C_2)\dot{X} = \frac{\partial}{\partial X} \sum_{k \geq 3} C_k \rho^k P_k \left( \frac{X}{\rho} \right),
\]

\[
\ddot{Y} + 2n\dot{X} + (C_2 - n^2)\dot{Y} = \frac{\partial}{\partial Y} \sum_{k \geq 3} C_k \rho^k P_k \left( \frac{X}{\rho} \right),
\]

\[
\ddot{Z} + C_2\dot{Z} = \frac{\partial}{\partial Z} \sum_{k \geq 3} C_k \rho^k P_k \left( \frac{X}{\rho} \right).
\]

In above equations, the left hand side contains the linear terms and the right hand side contains the non-linear terms. The coefficients \(C_k\) are given by

\[
C_k = \frac{1}{\gamma^3} \left[ \frac{(-1)^k q (1 - \mu) \gamma^{k+1}}{k! \gamma^{k+1}} + (\pm 1)^k \left( \mu + \frac{3\mu A_2}{2\gamma^2} \right) \right],
\]

for \(k \geq 1\). Considering only linear terms in equations (14)-(16), the solution of the linearized equations is

\[
X(t) = A_1 e^{\omega t} + A_2 e^{-\omega t} + A_3 \cos \lambda t + A_4 \sin \lambda t,
\]

\[
Y(t) = -k_1 A_1 e^{\omega t} + k_1 A_2 e^{-\omega t} - k_2 A_3 \sin \lambda t + k_2 A_4 \cos \lambda t,
\]

\[
Z(t) = A_5 \cos \sqrt{C_2} t + A_6 \sin \sqrt{C_2} t,
\]

where \(A_1, A_2, A_3, A_4, A_5\) and \(A_6\) are arbitrary constants,

\[
\alpha = \sqrt{-\left(2n^2 - C_2\right) + \sqrt{9C_2^2 - 8n^2C_2}},
\]

\[
\lambda = \frac{2n^2 - C_2 + \sqrt{9C_2^2 - 8n^2C_2}}{2},
\]

\[
k_1 = \frac{(2C_2 + n^2) - \alpha^2}{2n\alpha},
\]

\[
k_2 = \frac{(2C_2 + n^2) + \lambda^2}{2n\lambda}.
\]

Linearized equations corresponding to equations (14)-(16) have two real roots which are equal in magnitude and opposite in sign. If the initial conditions are chosen arbitrarily, then these roots give rise to unbounded solutions. To avoid this, we take \(A_1 = A_2 = 0\) and \(A_3 = -A_X \cos \phi, A_4 = A_X \sin \phi, A_5 = A_Z \sin \psi\) and \(A_6 = A_Z \cos \psi\) and get the bounded solution in the following form: (Koon et al. 2011)

\[
X(t) = -A_X \cos(\lambda t + \phi),
\]

\[
Y(t) = k A_X \sin(\lambda t + \phi),
\]

\[
Z(t) = A_Z \sin(\sqrt{C_2} t + \psi),
\]

where \(A_X\) and \(A_Z\) are amplitudes; \(\lambda\) and \(\sqrt{C_2}\) are the frequencies; \(k = k_2\); \(\phi\) and \(\psi\) are phases of the in-plane and out of plane motions respectively. The ratio of \(\lambda\) and \(\sqrt{C_2}\) is irrational. This gives Lissajous(quasi periodic) orbits.

3.2 Lindstedt-Poincaré Method for the Halo Orbits

Halo orbits are important for spacecraft mission design. Many researchers have obtained the halo orbits up to third order approximation (Richardson 1980; Howell 1984; Breakwell and Brown 1979; Koon et al. 2011; Chidambarran and Sharma 2016; Pushparaj and Sharma 2016; Ghotekar and Sharma 2019). Tiwary and Kushvah (2015) have computed halo orbits up to fourth order approximation with the Sun as a radiating body and the Earth as an oblate spheroid using Lindstedt-Poincaré method. Here, we have computed halo orbits up to fifth order approximation with radiation pressure and oblateness using Lindstedt-Poincaré method. The non-linear terms in (14)-(16) change the frequency of the linearized system. Due to this secular terms appear in successive approximations. To change the frequency, we take a new independent variable \(\tau = \omega t\), where \(\omega\) is a frequency connection. Then the equations of motion.
(14)-(16) in terms of $\tau$ truncated at degree 5 are:

$$\begin{align*}
\omega^2 X'' - 2n\omega X' - (n^2 + 2C_2)X \\
= & \frac{3}{2} C_3(2X^2 - Y^2 - Z^2) + 2C_4X(2X^2 - 3Y^2 - 3Z^2) \\
+ & \frac{5}{8} C_5[8X^2\{X^2 - 3(Y^2 + Z^2)\} + 3(Y^2 + Z^2)^2] \\
+ & 3C_6[2X^3\{X^2 - 5(Y^2 + Z^2)\} + \frac{15}{4} X(Y^2 + Z^2)^2], \\
\text{(18)}
\end{align*}$$

$$\begin{align*}
\omega^2 Y'' + 2n\omega X' + (C_2 - n^2)Y \\
= & -3C_4XY - \frac{3}{2} C_4Y(4X^2 - Y^2 - Z^2) \\
- & \frac{5}{2} C_5XY(4X^2 - 3Y^2 - 3Z^2) \\
+ & \frac{15}{2} C_6[X^2Y\{-2X^2 + 3(Y^2 + Z^2)\} - \frac{1}{4} Y(Z^2 + Z^2)^2] \\
+ & \Delta Z, \\
\text{(19)}
\end{align*}$$

where $\Delta = \lambda^2 - C_2$ is the frequency correction term to obtain halo orbit and $\Delta = O(\epsilon^2)$.

The solutions of (18)-(20) are assumed in the perturbations form as (Thurman and Worfolk 1996):

$$\begin{align*}
X(\tau) = & \epsilon X_1(\tau) + \epsilon^2 X_2(\tau) + \epsilon^3 X_3(\tau) + \epsilon^4 X_4(\tau) + \epsilon^5 X_5(\tau) + \ldots, \\
Y(\tau) = & \epsilon Y_1(\tau) + \epsilon^2 Y_2(\tau) + \epsilon^3 Y_3(\tau) + \epsilon^4 Y_4(\tau) + \epsilon^5 Y_5(\tau) + \ldots, \\
Z(\tau) = & \epsilon Z_1(\tau) + \epsilon^2 Z_2(\tau) + \epsilon^3 Z_3(\tau) + \epsilon^4 Z_4(\tau) + \epsilon^5 Z_5(\tau) + \ldots, \\
\text{(21)-(23)}
\end{align*}$$

and let

$$\omega = 1 + \omega_1 + \epsilon^2 \omega_2 + \epsilon^3 \omega_3 + \epsilon^4 \omega_4 + \ldots \\
\text{(24)}$$

Substituting the solutions (21)-(24) into equations of motion (18)-(20) and equating the coefficients of the same order of $\epsilon, \epsilon^2, \epsilon^3$, and $\epsilon^4$, we obtain the first, second, third and fourth order equations, respectively (Thurman and Worfolk 1996; Tiwary and Kushvah 2015). For obtaining more accurate solutions of the equations we have collected the coefficients of $\epsilon^5$ and obtained the fifth order equations.

3.2.1 Fifth Order Equations

Collecting the coefficients of $\epsilon^5$ and incorporating all the solutions and conditions used up to fourth order approximations (Tiwary and Kushvah 2015), we get the fifth order equations as:

$$\begin{align*}
X_5'' - 2nY_5' - (n^2 + 2C_2)X_5 = & \gamma_{51} \\
Y_5'' + 2nX_5' + (C_2 - n^2)Y_5 = & \gamma_{52} \\
Z_5' + \lambda^2 Z_5 = & \left\{ \begin{array}{ll}
f_3, & p = 0,2 \\
f_4, & p = 1,3 \\
\end{array} \right.
\end{align*}$$

(25)-(27)

where

$$\begin{align*}
\gamma_{51} = & [v_4 + 2\lambda A_4 \omega_4(nk - \lambda)] \cos \tau_1 \\
+ & \gamma_8 \cos 3\tau_1 + \gamma_9 \cos 5\tau_1, \\
\gamma_{52} = & [v_5 + 2\lambda A_4 \omega_4(\lambda k - n)] \sin \tau_1 \\
+ & \beta_9 \sin 3\tau_1 + \beta_{10} \sin 5\tau_1, \\
f_3 = & [v_6 + 2\lambda A_4 \lambda^2 A_2] \sin \tau_1 + \delta_8 \sin 3\tau_1 + \delta_9 \sin 5\tau_1, \\
f_4 = [v_6 + 2\lambda A_4 \lambda^2 A_2] \cos \tau_1 + \delta_8 \cos 3\tau_1 + \delta_9 \cos 5\tau_1
\end{align*}$$

and the remaining coefficients are given in Appendix. In $f_3$, upper sign corresponds to $p = 0$ and lower sign corresponds to $p = 2$. Similarly, in $f_4$, upper sign corresponds to $p = 1$ and lower corresponds to $p = 3$.

The secular term can be removed from (27) if

$$v_6 + 2\lambda A_4 \lambda^2 A_2 = 0, \\
\text{(28)}$$

where the upper sign corresponds to $p = 0,1$ and the lower sign corresponds to $p = 2,3$.

To remove the secular terms from (25) and (26), we use a single condition from their particular solution (Thurman and Worfolk 1996; Tiwary and Kushvah 2015)

$$[v_4 + 2\lambda A_4 \omega_4(\lambda k - n)] - k[v_5 + 2\lambda A_4 \omega_4(\lambda k - n)] = 0.$$  

(29)

From equation (29), we get

$$\omega_4 = \frac{v_4 - kv_5}{2\lambda A_4(\lambda(k^2 + 1) - 2nk)^2}. \\
\text{(30)}$$

Using conditions (28) and (30) in equations (25)-(27), the equations of motion take the following form:

$$\begin{align*}
X_5'' - 2nY_5' - (n^2 + 2C_2)X_5 \\
= & k\beta_{11} \cos \tau_1 + \gamma_8 \cos 3\tau_1 + \gamma_9 \cos 5\tau_1, \\
Y_5'' + 2nX_5' + (C_2 - n^2)Y_5 \\
= & \beta_{11} \sin \tau_1 + \beta_9 \sin 3\tau_1 + \beta_{10} \sin 5\tau_1,
\end{align*}$$

(31)-(32)
\[ Z'' + \lambda^2 Z = \begin{cases} \delta_8 \sin 3\tau_1 + \delta_9 \sin 5\tau_1, & p = 0, 2, \\ \delta_8 \cos 3\tau_1 + \delta_9 \cos 5\tau_1, & p = 1, 3, \end{cases} \quad (33) \]

where \( \beta_{11} = v_5 + 2\lambda A_X \omega_4 (\lambda k - n) \). The solution of equations (31)-(33) is given by

\[ X_5(\tau) = \rho_{51} \cos 3\tau_1 + \rho_{52} \cos 5\tau_1, \quad (34) \]
\[ Y_5(\tau) = \rho_{51} \sin 3\tau_1 + \sigma_{53} \sin 5\tau_1, \quad (35) \]
\[ Z_5(\tau) = \begin{cases} k_{51} \sin 3\tau_1 + k_{52} \sin 5\tau_1, & p = 0, 2, \\ k_{51} \cos 3\tau_1 + k_{52} \cos 5\tau_1, & p = 1, 3, \end{cases} \quad (36) \]

where the coefficients are given in the Appendix.

### 3.2.2 Final Approximation

Final approximation is obtained by removing \( \epsilon \) from all the equations. For that we take the mapping \( A_X \rightarrow \frac{A_X}{A_X} \) and \( A_Z \rightarrow \frac{A_Z}{A_Z} \). Combining the solutions component wise in (21)-(23), we get (Tiwary and Kushvah 2015)

\[ X(\tau) = (\rho_{20} + \rho_{40}) - A_X \cos \tau_1 \]
\[ + (\rho_{21} + \sigma_{22} + \rho_{41}) \cos 2\tau_1 \]
\[ + (\rho_{23} + \rho_{51}) \cos 3\tau_1 + \rho_{42} \cos 4\tau_1 + \rho_{52} \cos 5\tau_1, \quad (37) \]
\[ Y(\tau) = (k A_X + \sigma_{52} + \sigma_{51}) \sin \tau_1 \]
\[ + (\sigma_{21} + \sigma_{41} + \zeta \sigma_{22}) \sin 2\tau_1 \]
\[ + (\sigma_{31} + \sigma_{52}) \sin 3\tau_1 + \sigma_{42} \sin 4\tau_1 + \sigma_{53} \sin 5\tau_1, \quad (38) \]
\[ Z(\tau) = \begin{cases} f_5, & p = 0, 2, \\ f_6, & p = 1, 3, \end{cases} \quad (39) \]

where

\[ f_5 = (-1)^{\frac{1}{2}} (A_Z \sin \tau_1 + k_{21} \sin 2\tau_1 + k_{31} \sin 3\tau_1) \]
\[ + k_{41} \sin 2\tau_1 + k_{42} \sin 4\tau_1 + k_{51} \sin 3\tau_1 + k_{52} \sin 5\tau_1, \]
\[ f_6 = (-1)^{\frac{p-1}{2}} (A_Z \cos \tau_1 + k_{21} \cos 2\tau_1 + k_{22} + k_{32} \cos 3\tau_1) \]
\[ + k_{40} + k_{41} \cos 2\tau_1 + k_{42} \cos 4\tau_1 \]
\[ + k_{51} \cos 3\tau_1 + k_{52} \cos 5\tau_1. \]

Using equations (37)-(39), we can get the first guess of halo orbits.

### 4 Discussion

The halo orbits in the photogravitational Sun-Earth system with oblateness upto fourth order approximations using Lindstedt-Poincaré method are obtained by Tiwary and Kushvah (2015). Here, the first guess of the halo orbit in the same system is obtained upto fifth order approximation using Lindstedt-Poincaré method. Equations (37)-(39) are used with the amplitudes \( A_X = 206000 \) km and \( A_Z = 110000 \) km from the ISEE-3 mission.

EDITOR: PLACE FIGURE 1 HERE.
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The orbits are plotted for different values of phases. Fig.1 to Fig.4 show halo orbits around \( L_1 \) for different values of \( q \), mass reduction factor. Orbits coloured in blue represents fourth order orbits and red corresponds to fifth order orbits.

EDITOR: PLACE FIGURE 5 HERE.

The effects of radiation pressure on the position of halo orbits are given in Fig.5. Fig.5 shows the positions of halo orbits for \( q = 0.9995, 0.9945, 0.9895 \) and 0.9845 labeled as 1, 2, 3 and 4, respectively, with the actual oblateness of Earth. As radiation pressure increases, the halo orbits move towards the source of radiation. This agrees with conclusions of Eapen and Sharma (2014).

EDITOR: PLACE FIGURE 6 HERE.
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EDITOR: PLACE FIGURE 9 HERE.

Fig.6 – 9 represent halo orbits around \( L_2 \) corresponding to mass reduction factor 0.9995, 0.9945, 0.9895 and 0.9845, respectively, with oblateness \( A_2 = 2.4 \times 10^{-12} \), the oblateness of the Earth.

EDITOR: PLACE FIGURE 10 HERE.

Fig.10 shows the variation in position of halo orbits due to radiation pressure. Here, the orbits labeled as 1, 2, 3 and 4 correspond to \( q = 0.9995, 0.9945, 0.9895, 0.9845 \) and 0.9845, respectively, with oblateness \( A_2 = 2.4 \times 10^{-12} \). Halo orbits move towards the source of radiation with the increase in radiation pressure.

EDITOR: PLACE TABLE 1 HERE.
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Table1, Table2, Table3 and Table4 show the variation in coefficients, the position of Lagrangian points, \( \Delta \) and time period \( \tau \) due to variation in radiation pressure and oblateness. \( \Delta = \lambda^2 - C_2 \) is the frequency correction term to obtain the halo orbits. \( \tau \) is the time taken by the infinitesimal body to complete one rotation about
the Lagrangian point. Table1 shows the effect of radiation pressure on parameters of orbits around $L_1$. It can be observed that as the radiation pressure increases, that is, $q$ decreases, $L_1$ move towards the source of radiation, the Sun. Also, the time period of orbits increase with the increase in radiation pressure. Table2 represents the effect of oblateness on various parameters of orbits around $L_1$. With the increase in radiation pressure, orbits move towards the Sun and their time period is decreased. From Table3, it can be observed that due to increase in radiation pressure, the orbits around $L_2$ move towards the Sun and their time period is decreased. Effect of oblateness on position of orbits and time period can be observed from Table 4. Halo orbits around $L_2$ move away from the source of radiation and also time period of orbits increase due to increase in oblateness. Fig.11 represents the effect of oblateness on position of $L_1$. As oblateness increases, $L_1$ moves towards the source of radiation, the Sun. In Fig.12, the reverse effect of oblateness is observed on the position of $L_2$. That is, as oblateness increases, $L_2$ moves away from the Sun. Fig.13 and Fig.14 show the variation in position of $L_1$ and $L_2$ due to radiation pressure, respectively. With the increase in radiation pressure, $L_1$ and $L_2$ both move towards the Sun. The effect of radiation pressure and oblateness on time period is graphically shown in Fig.15 – 18. Time periods of halo orbits decrease with the increase in oblateness around $L_1$ while they increase with the increase in oblateness around $L_2$. With the increase in radiation pressure, time period of orbits around $L_1$ increases and decreases around $L_2$.

5 Conclusion

Photogravitational RTBP with oblateness, where the Sun is radiating and the Earth an oblate spheroid, is studied for halo orbits. We have improved the fourth order equations obtained by (Tiwary and Kushvah 2015) using Lindstedt-Poincaré method to fifth order and to obtain halo orbits around $L_1$ and $L_2$. The deviations of the orbits around $L_1$ and $L_2$ obtained from fourth order and fifth order equations are shown graphically. The variations in position and time of halo orbits around $L_1$ and $L_2$ due to radiation pressure and oblateness are studied. It is found that the halo orbits around $L_1$ shift towards the source of radiation (Sun) as the radiation pressure and oblateness increase. However, the time period of halo orbits around $L_1$ increases with the increase in radiation pressure but decreases with the increase in the oblateness. Halo orbits around $L_2$ approaches the source of radiation with increase in the radiation pressure but recedes from the source of radiation due to increase in the oblateness. The period of halo orbits around $L_2$ decreases with increase in the radiation pressure but increases with increase in oblateness.

Acknowledgements

One of the authors (DS) would like to thank Council of Scientific and Industrial Research (CSIR) for financial support through JRF (File No. 09/114(0218)/2019-EMR-I).

Compliance with Ethical Standards

Conflict of Interest: Author Dhwani Sheth has received Junior Research Fellowship (JRF) from CSIR (File No. 09/114(0218)/2019-EMR-I).

Appendix

$$v_4 = \begin{cases} v_{41}, & \text{when } p = 0, 2, \\ v_{42}, & \text{when } p = 1, 3. \end{cases}$$

$$v_{41} = \lambda \omega_2(2n\sigma_{32} - \lambda \omega_2 A_X) + \frac{3}{2} C_3(-2A_X(2\rho_{40} + \rho_{41}) + 2\rho_{31}(\rho_{21} + \rho_{22}) - k A_X \sigma_{41} - (\sigma_{21} + \sigma_{22})(\sigma_{31} + \sigma_{32}) - (-1)^2 A_Z k_{41} - k_{21} k_{31}) + \frac{3}{2} C_4(2A_X^2 \rho_{31} - 4A_X ((\rho_{20} + \rho_{21} + \rho_{22})^2 + \rho_{20}^2) + 2k A_X^2 (\sigma_{31} + \sigma_{32}) + 2A_X (\sigma_{21} + \sigma_{22})^2 - 4k A_X \rho_{20} (\sigma_{21} + \sigma_{22}) + k^2 A_X^2 \rho_{31} + 2A_X A_Z k_{31} + 2A_X k_{21}^2 - 4A_Z k_{21} \rho_{20} + A_Z^2 \rho_{31}) + \frac{5}{2} C_5(-2A_X^3 (3\rho_{20} + 2(\rho_{21} + \rho_{22})) - 3k A_X^3 (\sigma_{21} + \sigma_{22}) + 3k^2 A_X^2 \rho_{20} - 3A_X^2 A_Z k_{21} + 3A_X A_Z^2 \rho_{20} + \frac{3}{4} k^3 A_X^3 (\sigma_{21} + \sigma_{22}) + \frac{3}{4} k^2 A_X^2 A_Z k_{21} + \frac{3}{4} k A_X A_Z^2 (\sigma_{21} + \sigma_{22}) + \frac{3}{4} A_Z^3 k_{21}) + \frac{15}{32} C_6(-8A_X^5 + 8k^2 A_X^5 + 8A_X^3 A_Z^2 - 3k^4 A_X^5 - 6k^2 A_X^3 A_Z^2 - 3A_X A_Z^4),$$
\[ v_{12} = \lambda_2 (2n\sigma_{32} - \lambda_2 A_X) \]
\[ + \frac{3}{2} C_5 (-2A_X (2\rho_40 + \rho_{41}) + 2\rho_{31} (\rho_{21} - \rho_{22})) \]
\[ - kA_X \sigma_{41} - (\sigma_{21} - \sigma_{22}) (\sigma_{31} + \sigma_{32}) \]
\[ - \rho_{31} (\sigma_{21} - \sigma_{22}) + kA_X (2\rho_{40} - \rho_{41})) \]
\[ - \frac{3}{2} C_4 (A^4_A X) \sigma_{31} + \sigma_{32} - 16A_X \rho_0 \sigma_{21} - \sigma_{22}) \]
\[ + 8kA^3_A X \rho_{31} \]
\[ + 3k^3A^3_A X (2\rho_{20} - 2\rho_{21} - \rho_{22}) + (\rho_{21} - \rho_{22}) \]
\[ + 3k^3A^3_A X (\sigma_{31} - 3\sigma_{32}) - 6kA_X (\sigma_{21} - \sigma_{22}) \]
\[ + 2kA_X A_Z k_{31} - 2kA_X k_{21} + 2k^2_{21} + 2k^2_{22} \]
\[ - 4A_Z k_{22} (\sigma_{21} - \sigma_{22}) - A^2_Z \sigma_{31} + \sigma_{32}) \]
\[ + \frac{5}{8} C_5 (4A^4_A X) (\sigma_{21} - \sigma_{22}) \]
\[ + 3k^3A^3_A X (2\rho_{20} - 2\rho_{21} - \rho_{22}) - 9k^2A^3_A (\sigma_{21} - \sigma_{22}) \]
\[ - 12kA^4_A X \rho_0 - 6kA^2_A X A_Z k_{22} - 3A_A X A^2_Z (\sigma_{21} - \sigma_{22}) \]
\[ + 3kA_X A^2_Z (\sigma_{21} - \sigma_{22}) + \frac{15}{64} C_6 (-8kA^5_A X + 12k^3A^5_A X \]
\[ + 12kA^4_A X - 5k^5A^5_A X - 2k^3A^3_A X A^2_Z - kA_X A^3_Z) \]
\[ \frac{v_6}{v_6} = \begin{cases} 
 0, & \text{when } p = 0, \\
 1, & \text{when } p = 1, \\
 2, & \text{when } p = 2, \\
 3, & \text{when } p = 3. 
\end{cases} 
\]
\[ v_{61} = \omega_2^2 \lambda^2 A_Z - \frac{3}{2} C_3 (A_X (2k_{40} + k_{41}) + k_{32} (p_{21} - r_{22}) \\
+ r_{31} k_{21} + A_Z (2p_{40} + r_{41}) - \frac{3}{2} C_4 (A_X^2 k_{32} \\
- 4 A_X (r_{20} (k_{21} + 2k_{22}) + (p_{21} - r_{22})) (k_{21} + k_{22})) \\
- 2 A_X A_Z p_{31} \\
+ 2 A_Z \{2p_{20} (p_{20} + r_{21} - r_{22}) + (r_{21} - r_{22})^2 \} \\
+ \frac{1}{4} k^2 A_X^2 k_{32} - 4 A_X k_{22} (\sigma_{21} - \sigma_{22}) \\
- \frac{1}{2} k A_X A_Z (\sigma_{31} + \sigma_{32}) - \frac{3}{4} A_Z^2 k_{32} \\
- \frac{1}{2} A_Z (\sigma_{21} - \sigma_{22})^2 - \frac{3}{2} A_Z (k_{21}^2 + 2k_{21} k_{22} + 2k_{22}^2) \\
- \frac{5}{8} C_5 (4 A_X^3 (2k_{21} + 3k_{22}) \\
+ 12 A_X^2 A_Z (3p_{20} + 2 (p_{21} - r_{22}) \\
+ 9 A_X A_Z^2 (2k_{21} + 3k_{22}) \\
- 3 A_Z^2 (3p_{20} + 2 (p_{21} - r_{22}) + 3k^2 A_X^2 k_{22} \\
+ 6k A_X A_Z (\sigma_{21} - \sigma_{22}) - 3k^2 A_Z^2 A_Z (p_{20}) \\
+ \frac{15}{64} C_6 (40 A_X^4 A_Z + 12k^2 A_X^2 A_Z + 60 A_X^2 A_Z^2 \\
- k^4 A_X^4 A_Z - 2k^2 A_X^2 A_Z^2 - 5 A_Z^5), \\
\]

\[ v_{62} = -\omega_2^2 \lambda^2 A_Z - \frac{3}{2} C_3 (A_X (2k_{41} + k_{41} - k_{31} (p_{21} + r_{22}) \\
+ r_{31} k_{21} - A_Z (2p_{40} + r_{41}) \\
- \frac{3}{2} C_4 (A_X^2 k_{31} + 4 A_X p_{20} k_{21} - 2 A_X A_Z p_{31} \\
- 2 A_Z \{2p_{20} (p_{21} - r_{21} - r_{22}) + (r_{21} + r_{22})^2 \} \\
+ \frac{1}{4} k^2 A_X^2 k_{31} + k A_X k_{21} (\sigma_{21} + \sigma_{22}) \\
- \frac{1}{2} k A_X A_Z (\sigma_{31} - 3 \sigma_{32}) - \frac{3}{4} A_Z^2 k_{31} \\
+ \frac{1}{2} \frac{15}{64} C_6 (8 A_X^2 A_Z - 12k^2 A_X^2 A_Z - 12 A_X^2 A_Z^2 \\
+ 5k^4 A_X^4 A_Z + 10k^2 A_X^2 A_Z^2 + 5 A_Z^5), \\
\]

\[ v_{63} = -\omega_2^2 \lambda^2 A_Z - \frac{3}{2} C_3 (A_X (2k_{40} + k_{41}) - k_{32} (p_{21} - r_{22}) \\
+ r_{31} k_{21} - A_Z (2p_{40} + r_{41}) - \frac{3}{2} C_4 (A_X^2 k_{32} + 4 A_X (p_{20} (k_{21} + 2k_{22}) + (p_{21} - r_{22})) (k_{21} + k_{22}) \\
+ 2 A_X A_Z p_{31} \\
- 2 A_Z \{2p_{20} (p_{21} + r_{21} - r_{22}) + (r_{21} - r_{22})^2 \} \\
- \frac{1}{4} k^2 A_X^2 k_{32} + 4 A_X k_{22} (\sigma_{21} - \sigma_{22}) \\
+ \frac{1}{2} k A_X A_Z (\sigma_{31} + \sigma_{32}) + \frac{3}{4} A_Z^2 k_{32} \\
+ \frac{1}{2} A_Z (\sigma_{21} - \sigma_{22})^2 + \frac{3}{2} A_Z (k_{21}^2 + 2k_{21} k_{22} + 2k_{22}^2) \\
- \frac{5}{8} C_5 (4 A_X^3 (2k_{21} + 3k_{22}) \\
+ 12 A_X^2 A_Z (3p_{20} + 2 (p_{21} - r_{22}) \\
- 9 A_X A_Z^2 (2k_{21} + 3k_{22}) + 3 A_Z^2 (3p_{20} + 2 (p_{21} - r_{22}) \\
- 3k^2 A_X^2 k_{22} - 6 k A_X A_Z (\sigma_{21} - \sigma_{22}) + 3k^2 A_Z^2 A_Z (p_{20}) \\
+ 3k^2 A_Z^2 A_Z (p_{20}) + \frac{15}{64} C_6 (40 A_X^4 A_Z - 12k^2 A_X^2 A_Z \\
- 60 A_X^2 A_Z^2 + k^4 A_X^4 A_Z + 2k^2 A_X^2 A_Z^2 + 5 A_Z^5), \\
\]

\[ \gamma_8 = \begin{cases} 
\gamma_{s_1}, & \text{when } p = 0, 2, \\
\gamma_{s_2}, & \text{when } p = 1, 3.
\end{cases} \\
\]

\[ \gamma_8 = 6 \lambda \omega_2 (3 \lambda p_{31} + n \sigma_{31}) + \frac{3}{2} C_5 (-2 A_X (p_{41} + p_{42}) \\
+ 4 \rho_{20} p_{31} + k A_X (\sigma_{41} - \sigma_{42}) + \sigma_{32} (\sigma_{21} + \sigma_{22}) \\
+ (1 - \frac{1}{2}) A_Z (k_{41} - k_{42})) \\
+ \frac{3}{2} C_4 (4 A_X^2 \rho_{31} - 2 A_X (p_{21} + p_{22}) (4 \rho_{20} + p_{21} + p_{22}) \\
+ 2k A_X (\sigma_{31} + \sigma_{32}) (2 \rho_{20} - (p_{21} + p_{22})) \\
- 2k^2 A_X^2 \rho_{31} + 4 A_Z k_{21} + 2 A_Z k_{21} (2 \rho_{20} - (p_{21} + p_{22})) \\
- 2k^2 A_X^2 \rho_{31} + 4 A_Z k_{21} + 2 A_Z k_{21} (2 \rho_{20} - (p_{21} + p_{22})) \\
+ 12 k A_X (\sigma_{21} + \sigma_{22}) - 12 k^2 A_X (2 \rho_{20} - (p_{21} + p_{22})) \\
+ 12 k A_X A_Z (k_{21} - 12 A_X A_Z (2 \rho_{20} - (p_{21} + p_{22})) \\
- 9k A_X A_Z (k_{21} - 12 A_X A_Z (2 \rho_{20} - (p_{21} + p_{22})) \\
- 9k A_X A_Z (k_{21} - 12 A_X A_Z (2 \rho_{20} - (p_{21} + p_{22})) \\
+ \frac{5}{16} C_5 (-8 A_X^2 (2p_{20} + 3 (p_{21} + p_{22})) \\
+ 12 k A_X (\sigma_{21} + \sigma_{22}) - 12 k^2 A_X (2 \rho_{20} - (p_{21} + p_{22})) \\
+ 12 k A_X A_Z (k_{21} - 12 A_X A_Z (2 \rho_{20} - (p_{21} + p_{22})) \\
- 9k A_X A_Z (k_{21} - 12 A_X A_Z (2 \rho_{20} - (p_{21} + p_{22})) \\
- 9k A_X A_Z (k_{21} - 12 A_X A_Z (2 \rho_{20} - (p_{21} + p_{22})) \\
+ \frac{15}{64} C_6 (-8 A_X^2 - 8k^2 A_X^2 - 8 A_X^2 A_Z + 9k A_X A_Z \\
+ 18k^2 A_X A_Z + 9 A_X A_Z) \\
\]
\[\gamma_{92} = \frac{3}{2} C_3(-2A_X \rho_{42} + 2 \rho_{31}(\rho_{21} - \rho_{22}) + k A_X \sigma_{42} + \sigma_{31}(\sigma_{21} - \sigma_{22}) - (-1)^{\frac{\gamma}{2}} A_Z k_{42} - k_{21} k_{31}) + \frac{3}{2} C_4(2 A_X^2 \rho_{31} - 2A_X(\rho_{21} - \rho_{22})(4 \rho_{20} + \rho_{21} - \rho_{22}) - 2k A_X^2 \sigma_{32} - A_X(\sigma_{21} - \sigma_{22})^2 + 2k A_X(\sigma_{21} - \sigma_{22})(2\rho_{20} - (\rho_{21} - \rho_{22})) - 2k^2 A_X^2 \rho_{31} + 4A_X A_Z k_{32} + A_X(k_{21}^2 + 4k_{21} k_{22}) - 2A_Z(k_{21}(2\rho_{20} + \rho_{21} - \rho_{22}) + 2k_{22}(\rho_{21} - \rho_{22})) - 2A_Z^2(\rho_{20} + 3(\rho_{21} - \rho_{22})) + 12k A_X^3(\sigma_{21} - \sigma_{22}) - 12k^2 A_X^2(2\rho_{20} - (\rho_{21} - \rho_{22})) - 12A_X A_Z(3k_{21} + 2k_{22}) + 12A_X A_Z^2(2\rho_{20} + 3(\rho_{21} - \rho_{22})) - 9k^3 A_X^3(\sigma_{21} - \sigma_{22}) + 3k^2 A_X^2 A_Z(k_{21} - 2k_{22}) - 3k A_X A_Z^2(2\rho_{21} - 2k_{22}) + 15 A_X^3(\sigma_{21} - \sigma_{22}) + 3A^2(3k_{21} + 2k_{22}) + \frac{15}{64} C_6(-8A_X^5 - 8k^2 A_X^5 + 40A_X^4 A_Z^2 + 9k^4 A_X^3 + 6k^2 A_X^3 A_Z^2 - 15A_X A_Z^4). \]

\[\beta_9 = \begin{cases} 
\beta_{91}, & \text{when } p = 0, 2, \\
\beta_{92}, & \text{when } p = 1, 3. 
\end{cases} \]

\[\beta_{91} = \frac{3}{2} C_3(-2A_X \rho_{42} + 2 \rho_{31}(\rho_{21} - \rho_{22}) + k A_X \sigma_{42} + \sigma_{31}(\sigma_{21} + \sigma_{22}) + (-1)^{\frac{\gamma}{2}} A_Z k_{42} + k_{21} k_{31}) + \frac{3}{2} C_4(2 A_X^2 \rho_{31} - 2A_X(\rho_{21} + \rho_{22})^2 - 2k A_X^2 \sigma_{31} - A_X(\sigma_{21} + \sigma_{22})^2 + 2k A_X(\sigma_{21} + \sigma_{22})(\rho_{21} + \rho_{22}) + k^2 A_X^2 \rho_{31} - 2A_X A_Z k_{31} - A_X k_{31}^2 + 2A_Z k_{21} - A_X k_{21}^2 + 2A_X A_Z^2(2\rho_{21} - 2k_{21}) + \frac{5}{16} C_5(-8A_X^3(\rho_{21} + \rho_{22}) + 12k A_X^3(\sigma_{21} + \sigma_{22})) - 12k^2 A_X^2(2\rho_{21} - 2k_{21}) - 3k A_X A_Z(3k_{21} + 2k_{22}) + \frac{5}{16} C_6(-8A_X^5 - 40k^2 A_X^5 + 40A_X^4 A_Z^2 - 15k^4 A_X^3 + 30k^2 A_X^3 A_Z^2 - 15A_X A_Z^4), \]

\[\gamma_9 = \begin{cases} 
\gamma_{91}, & \text{when } p = 0, 2, \\
\gamma_{92}, & \text{when } p = 1, 3. 
\end{cases} \]

\[\beta_{91} = 6\lambda\omega_2(3\lambda \rho_{31} + n \sigma_{31}) - \frac{3}{2} C_3(-A_X(\sigma_{21} + \sigma_{22}) + 2\rho_{20} + \sigma_{32}(\rho_{21} + \rho_{22}) + k A_X(\sigma_{21} - \sigma_{22}) - \frac{3}{8} C_4(4A_X^2(2\sigma_{31} + \sigma_{32})) - 8A_X(\sigma_{21} + \sigma_{22})(2\rho_{20} + \rho_{21} + \rho_{22}) + 4k A_X(\rho_{21} + \rho_{22})(4\rho_{20} - (\rho_{21} + \rho_{22})) - 3k^2 A_X^2(2\sigma_{31} - \sigma_{32}) - 3k A_X(\sigma_{21} + \sigma_{22})^2 - 4k A_X A_Z k_{31} - k A_X k_{21}^2 - 2A_Z k_{21}(\sigma_{21} + \sigma_{22}) - A_Z^2(2\sigma_{31} - \sigma_{32}) + \frac{5}{16} C_5(12A_X^3(\sigma_{21} + \sigma_{22}) - 12k A_X^3(\rho_{20} + \rho_{21} + \rho_{22}) + 3k^2 A_X^2(\sigma_{21} + \sigma_{22})) + 3k A_X A_Z^2(2\rho_{20} + 3(\rho_{21} + \rho_{22})) - 9k^3 A_X^3(\sigma_{21} + \sigma_{22}) - 6k A_X A_Z k_{21} - 3A_X A_Z^2(\sigma_{21} + \sigma_{22}) + 3k A_X A_Z^2(2\rho_{21} - 2k_{21}) + \frac{15}{128} C_6(-24k A_X^5 + 12k^3 A_X^3 A_Z^2 - 5k A_X A_Z^5 + 10k^3 A_X A_Z^5 + 5k A_X A_Z^5), 
\]

\[\gamma_{92} = \frac{3}{2} C_3(-2A_X \rho_{42} + 2 \rho_{31}(\rho_{21} - \rho_{22}) + k A_X \sigma_{42} + \sigma_{31}(\sigma_{21} - \sigma_{22}) - (\frac{\gamma}{2})^{-1} A_Z k_{42} - k_{21} k_{31}) + \frac{3}{2} C_4(2 A_X^2 \rho_{31} - 2A_X(\rho_{21} - \rho_{22})^2 - 2k A_X^2 \sigma_{31} - A_X(\sigma_{21} - \sigma_{22})^2 + 2k A_X(\sigma_{21} - \sigma_{22})(\rho_{21} - \rho_{22}) + k^2 A_X^2 \rho_{31} + 2A_X A_Z k_{31} + A_X k_{31}^2 - 2A_Z k_{21}(\rho_{21} - \rho_{22}) - A_Z^2(\rho_{20} + 3(\rho_{21} - \rho_{22})), \]

\[\gamma_9 = \begin{cases} 
\gamma_{91}, & \text{when } p = 0, 2, \\
\gamma_{92}, & \text{when } p = 1, 3. 
\end{cases} \]
\[
\beta_{92} = 6\lambda\omega_2(3\lambda\sigma_{31} + n\rho_{31}) - \frac{3}{2}C_3(-A_X(\sigma_{41} + \sigma_{42}) + 2\rho_{20}\sigma_{31} + \sigma_{32}(\rho_{21} - \rho_{22}) + kA_X(\rho_{41} - \rho_{42})) - \frac{3}{8}C_4(4A_X^2(2\sigma_{31} + \sigma_{32}) - 8A_X(\sigma_{21} - \sigma_{22})(2\rho_{20} + \rho_{21} - \rho_{22}) + 4kA_X(\rho_{21} - \rho_{22})(4\rho_{20} - (\rho_{21} - \rho_{22})) - 3k^2A_X^2(2\sigma_{31} + \sigma_{32}) - 3kA_X(\sigma_{21} - \sigma_{22})^2 - kA_X(4k_{21}k_{22} - k^2)) - 2A_Z(k_{21} + 2k_{22})(\sigma_{21} - \sigma_{22}) - A_Z^2(2\sigma_{31} + \sigma_{32})) + \frac{5}{16}C_5(12A_X^3(\sigma_{21} - \sigma_{22}) - 12kA_X^3(2\rho_{20} + \rho_{21} - \rho_{22}) + 3k^3A_X^3(-2\rho_{20} + 3(\rho_{21} - \rho_{22})) - 9k^2A_X^3(\sigma_{21} - \sigma_{22}) - 6kA_X^2A_Z(k_{21} + 2k_{22}) - 9A_XA_Z^2(\sigma_{21} - \sigma_{22}) + 3kA_XA_Z^2(2\rho_{20} + \rho_{21} - \rho_{22})) + \frac{15}{128}C_6(-24kA_X^5 + 12k^3A_X^5 + 36kA_X^3A_Z^2 + 5k^5A_X^3 - 2k^3A_X^3A_Z^2 - 3kA_XA_Z^4).
\]

\[
\beta_{102} = -\frac{3}{2}C_3(-A_X\sigma_{42} + \sigma_{31}(\rho_{21} - \rho_{22}) + \rho_{31}(\sigma_{21} - \sigma_{22}) + kA_X\rho_{42}) - \frac{3}{8}C_4(4A_X^2\sigma_{31} - 8A_X(\sigma_{21} - \sigma_{22})(\rho_{21} + \rho_{22}) + 4kA_X(\rho_{21} + \rho_{22})(4\rho_{20} - (\rho_{21} + \rho_{22})) + 3kA_X(\sigma_{21} + \sigma_{22})^2 + 2kA_XA_Zk_{31} + kA_Xk_{21}^2 + 2A_Zk_{21}(\sigma_{21} + \sigma_{22}) + A_Z^2(\sigma_{21} + \sigma_{22})) + \frac{5}{16}C_5(12A_X^3(\sigma_{21} + \sigma_{22}) - 12kA_X^3(\rho_{21} + \rho_{22}) + 3k^3A_X^3(\rho_{21} + \rho_{22}) + 9k^2A_X^3(\sigma_{21} + \sigma_{22}) + 6kA_X^2A_Zk_{21} + 3A_XA_Z^2(\sigma_{21} + \sigma_{22}) + 3kA_XA_Z^2(\rho_{21} + \rho_{22}))) + \frac{15}{128}C_6(-24kA_X^5 + 12k^3A_X^5 + 12k^2A_X^4A_Z + 12A_X^3A_Z^2 + 5k^4A_X^3A_Z + 10k^2A_X^2A_Z^2 + 5A_Z^5) + \frac{\Delta}{\epsilon}k_{31},
\]

\[
\delta_8 = \begin{cases} 
\beta_{101}, & \text{when } p = 0, \\
\beta_{102}, & \text{when } p = 1, \\
\beta_{103}, & \text{when } p = 2, \\
\beta_{104}, & \text{when } p = 3.
\end{cases}
\]

\[
\delta_{80} = 18\lambda\omega_2^2\lambda^3k_{31} - \frac{3}{2}C_3(-A_X(k_{41} + k_{42}) + A_Z(\rho_{41} - \rho_{42}) + 2\rho_{20}k_{31}) - \frac{3}{8}C_4(8A_X^2k_{31} - 8A_Xk_{21}(2\rho_{20} + \rho_{21} + \rho_{22}) + 4kA_X(\rho_{21} + \rho_{22})(4\rho_{20} - (\rho_{21} + \rho_{22})) + 2kA_Xk_{21}(\sigma_{21} + \sigma_{22}) + kA_Xk_{31}^2 + 2A_Zk_{31}(\sigma_{21} + \sigma_{22}) + A_Z^2(\sigma_{21} + \sigma_{22})),
\]

\[
\delta_{81} = \frac{5}{16}C_5(12A_X^3k_{21} + 12A_X^3A_Z(2\rho_{20} + \rho_{21} + \rho_{22}) + 9A_XA_Z^2k_{31} + 3A_X^2(2\rho_{20} - 3(\rho_{21} + \rho_{22})) + 3k^2A_X^2k_{21} + 6kA_X^2A_Z(\sigma_{21} + \sigma_{22}) + 3k^2A_X^2A_Z(2\rho_{20} - 3(\rho_{21} + \rho_{22}))),
\]

\[
\delta_{82} = \frac{15}{128}C_6(-24A_X^5A_Z + 12k^2A_X^4A_Z + 12A_X^3A_Z^2 + 5k^4A_X^3A_Z + 10k^2A_X^2A_Z^2 + 5A_Z^5) + \frac{\Delta}{\epsilon}k_{31},
\]

\[
\delta_{83} = \frac{15}{128}C_6(-24A_X^5A_Z + 12k^2A_X^4A_Z + 12A_X^3A_Z^2 + 5k^4A_X^3A_Z + 10k^2A_X^2A_Z^2 + 5A_Z^5) + \frac{\Delta}{\epsilon}k_{31},
\]

\[
\delta_{84} = \frac{15}{128}C_6(-24A_X^5A_Z + 12k^2A_X^4A_Z + 12A_X^3A_Z^2 + 5k^4A_X^3A_Z + 10k^2A_X^2A_Z^2 + 5A_Z^5) + \frac{\Delta}{\epsilon}k_{31}.
\]
\[ \delta_{s1} = 18 \omega_2 \lambda^2 k_{32} - \frac{3}{2} C_3 (-A_X (k_{41} + k_{42}) + A_Z (\rho_{41} + \rho_{42})) + 2 \rho_{31} k_{22} + 2 \rho_{20} k_{32} - \frac{3}{8} C_4 (8 A_X^2 k_{32}) - 8 A_X (2 \rho_{20} k_{21} + (\rho_{21} - \rho_{22}) (k_{21} + 2 k_{22})) - 16 A_X A_Z (\rho_{31} + 4 A_X (\rho_{21} - \rho_{22}) (4 \rho_{20} + \rho_{21} - \rho_{22})) - 2 k^2 A_X^2 k_{32} - 2 k A_X (k_{21} - 2 k_{22}) (\sigma_{21} - \sigma_{22}) + 2 k A_X A_Z \sigma_{32} - 6 A_Z^2 k_{32} + A_Z (\sigma_{21} - \sigma_{22})^2 - 3 A_Z k_{21} (k_{21} + 4 k_{22})) - \frac{5}{16} C_5 (-4 A_X^3 (3 k_{21} + 2 k_{22})) + 12 A_X^3 A_Z (2 \rho_{20} + 3 (\rho_{21} - \rho_{22})) + 9 A_X A_Z^2 (2 k_{21} + 2 k_{22}) - 3 A_Z^2 (2 \rho_{20} + 3 (\rho_{21} - \rho_{22})) + 3 k^2 A_X^3 (k_{21} - 2 k_{22}) - 6 A_X^2 A_Z (\sigma_{21} - \sigma_{22}) + 3 k^2 A_X^2 A_Z (2 \rho_{20} - (\rho_{21} - \rho_{22})) + \frac{15}{128} C_6 (-40 A_X^4 A_Z - 12 k^2 A_X^4 A_Z + 60 A_X^2 A_Z^3 + 3 k^4 A_X^4 A_Z + 2 k^2 A_X^2 A_Z^2 - A_Z^2) + \frac{3}{2} C_3 (-A_X (k_{41} + k_{42}) - A_Z (\rho_{41} + \rho_{42}) - 2 \rho_{20} k_{31}) - \frac{3}{8} C_4 (-8 A_X^2 k_{31} + 8 A_X k_{21} (2 \rho_{20} + \rho_{21} + \rho_{22})) - 4 A_Z (\rho_{21} + \rho_{22}) (4 \rho_{20} - \rho_{21} - \rho_{22}) + 2 k^2 A_X^2 k_{31} + 2 k A_X k_{21} (\sigma_{21} - \sigma_{22}) + 2 k A_X A_Z (2 \sigma_{21} - \sigma_{22}) + 6 A_Z^2 k_{31} + A_Z (\sigma_{21} + \sigma_{22})^2 - 3 A_Z k_{21} (k_{21} + 2 k_{22})) - \frac{5}{16} C_5 (12 A_X^3 k_{21} - 12 A_X^2 A_Z (2 \rho_{20} + \rho_{21} + \rho_{22})) - 9 A_X A_Z^2 k_{21} - 3 A_Z^2 (2 \rho_{20} - 3 (\rho_{21} + \rho_{22})) - 3 k^2 A_X^3 k_{21} + 6 k^2 A_X^2 A_Z (\sigma_{21} + \sigma_{22}) - 3 k^2 A_X^2 A_Z (2 \rho_{20} - (\rho_{21} + \rho_{22})) + \frac{15}{128} C_6 (24 A_X^4 A_Z - 12 k^2 A_X^4 A_Z - 12 A_X^2 A_Z^3 - 5 k^4 A_X^4 A_Z - 10 k^2 A_X^2 A_Z^2 + 5 A_Z^2) - \frac{3}{2} C_3 (-A_X k_{42} + k_{31} (\rho_{21} + \rho_{22}) + \rho_{31} k_{21} + A_Z \rho_{42}) - \frac{3}{8} C_4 (4 A_X^2 k_{31} - 8 A_X k_{21} (\rho_{21} + \rho_{22}) - 8 A_X A_Z \rho_{31} + 4 A_Z (\rho_{21} + \rho_{22})^2 + k^2 A_Z^2 k_{31} + 2 k A_X k_{21} (\sigma_{21} + \sigma_{22}) + 2 k A_X A_Z \sigma_{31} + 3 A_Z^2 k_{31} + A_Z (\sigma_{21} + \sigma_{22})^2 + 3 A_Z k_{21} (k_{21} + 2 k_{22})) - \frac{5}{16} C_5 (-4 A_X^3 k_{21} + 12 A_X^2 A_Z (\rho_{21} + \rho_{22}) - 9 A_X A_Z^2 k_{21} + 3 A_Z^2 (\rho_{21} + \rho_{22}) - 3 k^2 A_X^3 k_{21} + 6 k^2 A_X^2 A_Z (\sigma_{21} + \sigma_{22}) + 3 k^2 A_X^2 A_Z (\rho_{21} + \rho_{22})) + \frac{15}{128} C_6 (-8 A_X^4 A_Z - 12 k^2 A_X^4 A_Z - 12 A_X^2 A_Z^3 - k^4 A_X^4 A_Z - 2 k^2 A_X^2 A_Z^2 - A_Z^2), \]

\[ \delta_{s8} = -18 \omega_2 \lambda^2 k_{31} \]

\[ \delta_{s9} = \begin{cases} 
\delta_{s0}, & \text{when } p = 0, \\
\delta_{s1}, & \text{when } p = 1, \\
\delta_{s2}, & \text{when } p = 2, \\
\delta_{s3}, & \text{when } p = 3.
\end{cases} \]
\[
\delta_{91} = \frac{3}{2} C_3 (-A_X k_{42} + k_{32}(\rho_{21} - \rho_{22}) + \rho_{31} k_{21} + A_Z \rho_{42}) \\
- \frac{3}{8} C_4 (4 A_X^2 k_{32} - 8 A_X k_{21}(\rho_{21} - \rho_{22}) - 8 A_X A_Z \rho_{31} \\
\quad + 4 A_Z (\rho_{21} - \rho_{22})^2 + k^2 A_X^2 k_{32} \\
\quad + 2 k A_X k_{21}(\sigma_{21} - \sigma_{22}) + 2 k A_X A_Z \sigma_{31} - 3 A_Z^2 k_{32} \\
\quad + A_Z (\sigma_{21} - \sigma_{22})^2 - 3 A_Z k_{21}^2) \\
- \frac{5}{16} C_5 (-4 A_X^3 k_{21} + 12 A_X^2 A_Z (\rho_{21} - \rho_{22}) \\
\quad + 9 A_X A_Z^2 k_{21} - 3 A_Z^3 (\rho_{21} - \rho_{22}) - 3 k^2 A_X^3 k_{21} \\
\quad - 6 k A_X^2 A_Z (\sigma_{21} - \sigma_{22}) + 3 k^2 A_X^3 A_Z (\rho_{21} - \rho_{22}) \\
\quad + \frac{15}{128} C_6 (-8 A_X^4 A_Z - 12 k^2 A_X^3 A_Z + 12 A_X^2 A_Z \\
\quad \quad - k^4 A_X A_Z + 2 k^2 A_X^3 A_X^4 - A_Z^5),
\]

\[
\delta_{92} = \frac{3}{2} C_3 (-A_X k_{42} - k_{31}(\rho_{21} + \rho_{22}) - \rho_{31} k_{21} - A_Z \rho_{42}) \\
- \frac{3}{8} C_4 (-4 A_X^2 k_{31} + 8 A_X k_{21}(\rho_{21} + \rho_{22}) \\
\quad + 8 A_X A_Z \rho_{31} - 4 A_Z (\rho_{21} + \rho_{22})^2 - k^2 A_X^2 k_{31} \\
\quad - 2 k A_X k_{21}(\sigma_{21} + \sigma_{22}) - 2 k A_X A_Z \sigma_{31} - 3 A_Z^2 k_{31} \\
\quad - A_Z (\sigma_{21} + \sigma_{22})^2 - 3 A_Z k_{21}^2) \\
- \frac{5}{16} C_5 (4 A_X^3 k_{21} - 12 A_X^2 A_Z (\rho_{21} + \rho_{22}) \\
\quad + 9 A_X A_Z^2 k_{21} - 3 A_Z^3 (\rho_{21} + \rho_{22}) + 3 k^2 A_X^3 k_{21} \\
\quad - 6 k A_X^2 A_Z (\sigma_{21} + \sigma_{22}) - 3 k^2 A_X^3 A_Z (\rho_{21} + \rho_{22}) \\
\quad + \frac{15}{128} C_6 (8 A_X^4 A_Z + 12 k^2 A_X^3 A_Z + 12 A_X^2 A_Z \\
\quad \quad + k^4 A_X A_Z + 2 k^2 A_X^3 A_X^4 + A_Z^5),
\]

\[
\delta_{93} = \frac{3}{2} C_3 (-A_X k_{42} - k_{32}(\rho_{21} - \rho_{22}) - \rho_{31} k_{21} - A_Z \rho_{42}) \\
- \frac{3}{8} C_4 (-4 A_X^2 k_{32} + 8 A_X k_{21}(\rho_{21} - \rho_{22}) \\
\quad + 8 A_X A_Z \rho_{31} - 4 A_Z (\rho_{21} - \rho_{22})^2 - k^2 A_X^2 k_{32} \\
\quad - 2 k A_X k_{21}(\sigma_{21} - \sigma_{22}) - 2 k A_X A_Z \sigma_{31} + 3 A_Z^2 k_{32} \\
\quad - A_Z (\sigma_{21} - \sigma_{22})^2 + 3 A_Z k_{21}^2) \\
- \frac{5}{16} C_5 (4 A_X^3 k_{21} - 12 A_X^2 A_Z (\rho_{21} - \rho_{22}) \\
\quad - 9 A_X A_Z^2 k_{21} + 3 A_Z^3 (\rho_{21} - \rho_{22}) + 3 k^2 A_X^3 k_{21} \\
\quad + 6 k A_X^2 A_Z (\sigma_{21} - \sigma_{22}) - 3 k^2 A_X^3 A_Z (\rho_{21} - \rho_{22}) \\
\quad + \frac{15}{128} C_6 (8 A_X^4 A_Z + 12 k^2 A_X^3 A_Z - 12 A_X^2 A_Z \\
\quad \quad + k^4 A_X A_Z - 2 k^2 A_X^3 A_X^4 + A_Z^5).}
\]

\[
\rho_{51} = \frac{6 n \lambda \beta_0 - (9 \lambda^2 + n^2 - C_2) \gamma_8}{(n^2 - 9 \lambda^2)^2 + C_2 (n^2 - 2 C_2 + 9 \lambda^2)},
\]

\[
\rho_{52} = \frac{10 n \lambda \beta_{10} - (25 \lambda^2 + n^2 - C_2) \gamma_9}{(n^2 - 25 \lambda^2)^2 + C_2 (n^2 - 2 C_2 + 25 \lambda^2)},
\]

\[
\sigma_{51} = \frac{k_3}{2 \lambda n},
\]

\[
\sigma_{52} = \frac{6 n \lambda \gamma_8 - (9 \lambda^2 + n^2 + 2 C_2) \beta_9}{(n^2 - 9 \lambda^2)^2 + C_2 (n^2 - 2 C_2 + 9 \lambda^2)},
\]

\[
\sigma_{53} = \frac{10 n \lambda \gamma_9 - (25 \lambda^2 + n^2 + 2 C_2) \beta_{10}}{(n^2 - 25 \lambda^2)^2 + C_2 (n^2 - 2 C_2 + 25 \lambda^2)},
\]

\[
k_{51} = \frac{- \delta_8}{8 \lambda^2},
\]

\[
k_{52} = \frac{- \delta_9}{24 \lambda^2}.
\]

**Fig. 1** $4^{th}$ and $5^{th}$ order halo orbits around $L_1$ corresponding to $A_2 = 2.4 \times 10^{-12}$, $q = 0.9995$
Table 1  Effect of radiation pressure on different parameters of orbits around $L_1$ when $A_2 = 2.4 \times 10^{-12}$

| $q$ | $A_2$ | $2.4 \times 10^{-12}$ | $3.5 \times 10^{-12}$ | $4 \times 10^{-12}$ |
|-----|-------|------------------------|----------------------|---------------------|
|     | $\gamma$ | 0.009966562831474 | 0.010822806024997 | 0.010822806024997 |
|     | $L_1$ | 0.990030433536519 | 0.98975190565718 | 0.98975190565718 |
|     | $C_2$ | 0.963443359558607 | 0.93213970575904 | 0.93213970575904 |
|     | $C_3$ | 0.302436775959918 | 0.2972624703974638 | 0.2972624703974638 |
|     | $C_4$ | 0.30334155218881 | 0.298315954578718 | 0.298315954578718 |
|     | $C_5$ | 0.303340367603850 | 0.298305329754523 | 0.298305329754523 |
|     | $C_6$ | 0.303341732214042 | 0.2983054372628669 | 0.2983054372628669 |
|     | $\lambda$ | 0.87480924217118 | 0.2983159543754524 | 0.2983159543754524 |
|     | $\Delta$ | 0.29225189059122 | 0.291658764200673 | 0.291658764200673 |
|     | $k$ | 3.23801097192164 | 3.21264343931957 | 3.21264343931957 |
|     | $\tau$ | 3.01027546921636 | 3.028304497912615 | 3.028304497912615 |

Table 2  Effect of oblateness on different parameters of orbits around $L_1$ when $q = 0.9995$

| $A_2$ | $2.4 \times 10^{-12}$ | $3.5 \times 10^{-12}$ | $4 \times 10^{-12}$ |
|-------|------------------------|----------------------|---------------------|
| $\gamma$ | 0.010022806024997 | 0.010822806024997 | 0.010822806024997 |
| $L_1$ | 0.989974190567518 | 0.989974190567518 | 0.989974190567518 |
| $C_2$ | 0.4013217360998824 | 0.4013217360998824 | 0.4013217360998824 |
| $C_3$ | 0.2972624703974638 | 0.297264704093073 | 0.297264704093073 |
| $C_4$ | 0.2983159543754524 | 0.2983159543754524 | 0.2983159543754524 |
| $C_5$ | 0.2983054372628669 | 0.2983054372628669 | 0.2983054372628669 |
| $C_6$ | 0.2983054372628669 | 0.2983054372628669 | 0.2983054372628669 |
| $\lambda$ | 2.078419527003888 | 2.078419527003888 | 2.078419527003888 |
| $\Delta$ | 0.291658764200673 | 0.291658764200673 | 0.291658764200673 |
| $k$ | 3.21264343931957 | 3.21264343931957 | 3.21264343931957 |
| $\tau$ | 3.0283049479712615 | 3.0283049479712615 | 3.0283049479712615 |

Table 3  Effect of radiation pressure on different parameters of orbits around $L_2$ when $A_2 = 2.4 \times 10^{-12}$

| $q$ | $A_2$ | $2.4 \times 10^{-12}$ | $3.5 \times 10^{-12}$ | $4 \times 10^{-12}$ |
|-----|-------|------------------------|----------------------|---------------------|
| $\gamma$ | 0.009966562831474 | 0.009966562831474 | 0.009966562831474 | 0.009966562831474 |
| $L_2$ | 0.009966562831474 | 0.009966562831474 | 0.009966562831474 | 0.009966562831474 |
| $C_2$ | 3.942590937851239 | 3.99327144921635 | 4.51388113295898 | 5.08772984797185 |
| $C_3$ | 2.98348088154385 | -0.0329693790758554 | -3.55564417985380 | -4.313098527601873 |
| $C_4$ | 2.973863414322752 | 0.302304928754649 | 3.546861815969391 | 4.12458572165059 |
| $C_5$ | 2.973863414322752 | 3.21264343931957 | 3.21264343931957 | 3.21264343931957 |
| $C_6$ | 2.973863414322752 | 3.21264343931957 | 3.21264343931957 | 3.21264343931957 |
| $\lambda$ | 2.057993351632755 | 2.06959687116124 | 2.19321760935948 | 2.3212498886311 |
| $\Delta$ | 0.302304928754649 | 0.302304928754649 | 0.302304928754649 | 0.302304928754649 |
| $k$ | 3.186540491593625 | 3.20569040467006 | 3.38249069868882 | 3.56774774450672 |
| $\tau$ | 3.05315281375049 | 3.03542752850363 | 2.864825235043783 | 2.706576818596858 | 2.56064201507853 |
| \( A_2 \) | 0 | \( 2.4 \times 10^{-12} \) | \( 3 \times 10^{-12} \) | \( 3.5 \times 10^{-12} \) | \( 4 \times 10^{-12} \) |
|---|---|---|---|---|---|
| \( \gamma \) | 0.009978343518616 | 0.009978343639533 | 0.009978343669762 | 0.009978343694953 | 0.009978343720144 |
| \( L_2 \) | 1.00997534008281 | 1.009975340129198 | 1.009975340159427 | 1.009975340184618 | 1.009975340209809 |
| \( C_2 \) | 3.993273145236837 | 3.993273144291635 | 3.993273144055562 | 3.993273143858681 | 3.993273143661800 |
| \( C_3 \) | -3.032693791243736 | -3.032693790758545 | -3.032693790637474 | -3.032693790536429 | -3.032693790435384 |
| \( C_4 \) | 3.023203497879979 | 3.023203497285466 | 3.023203497137065 | 3.023203497013244 | 3.023203496889423 |
| \( C_5 \) | -3.023109736060348 | -3.023109735463630 | -3.02310973514677 | -3.023109735190398 | -3.023109735066117 |
| \( C_6 \) | 3.023108809716084 | 3.023108809119334 | 3.02310880970372 | 3.02310880846086 | 3.023108808721798 |
| \( \lambda \) | 2.069950687346945 | 2.069950687116124 | 2.069950687058475 | 2.069950687010396 | 2.069950686962317 |
| \( \Delta \) | 0.291422702811252 | 0.291422702800878 | 0.291422702798289 | 0.291422702796127 | 0.291422702793967 |
| \( k \) | 3.205690410801889 | 3.205690410467006 | 3.205690410383364 | 3.205690410313609 | 3.205690410243853 |
| \( \tau \) | 3.035427532446555 | 3.035427532785036 | 3.035427532860575 | 3.035427532940079 | 3.035427533010583 |
Fig. 2 4th and 5th order halo orbits around $L_1$ corresponding to $A_2 = 2.4 \times 10^{-12}, q = 0.9945$

Fig. 3 4th and 5th order halo orbits around $L_1$ corresponding to $A_2 = 2.4 \times 10^{-12}, q = 0.9895$

Fig. 4 4th and 5th order halo orbits around $L_1$ corresponding to $A_2 = 2.4 \times 10^{-12}, q = 0.9845$

Fig. 5 Effect of radiation pressure on the position of halo orbits around $L_1$

Fig. 6 4th and 5th order halo orbits around $L_2$ corresponding to $A_2 = 2.4 \times 10^{-12}, q = 0.9995$

Fig. 7 4th and 5th order halo orbits around $L_2$ corresponding to $A_2 = 2.4 \times 10^{-12}, q = 0.9945$
Fig. 8 4th and 5th order halo orbits around $L_2$ corresponding to $A_2 = 2.4 \times 10^{-12}, q = 0.9895$

Fig. 9 4th and 5th order halo orbits around $L_2$ corresponding to $A_2 = 2.4 \times 10^{-12}, q = 0.9845$

Fig. 10 Effect of radiation pressure on the position of halo orbits around $L_2$

Fig. 11 Effect of oblateness on the position of $L_1$

Fig. 12 Effect of oblateness on the position of $L_2$

Fig. 13 Effect of radiation pressure on the position of $L_1$

Fig. 14 Effect of radiation pressure on the position of $L_2$
Fig. 15  Effect of oblateness on time period of halo orbits around $L_1$

Fig. 16  Effect of oblateness on time period of halo orbits around $L_2$

Fig. 17  Effect of radiation pressure on time period of halo orbits around $L_1$

Fig. 18  Effect of radiation pressure on time period of halo orbits around $L_2$
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This manuscript was prepared with the AAS IMpX macros v5.2.