Quantum Phase from the Twin Paradox

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Abstract. The modern concept of spacetime usually emerges from the consideration of moving clocks on the assumption that world-lines are continuous. In this paper we start with the assumption that natural clocks are digital and that events are discrete. By taking different continuum limits we show that the phase of non-relativistic quantum mechanics and the odd metric of spacetime both emerge from the consideration of discrete clocks in relative motion. From this perspective, the continuum limit that manifests itself in ‘spacetime’ is an infinite mass limit. The continuum limit that gives rise to the Schrödinger equation retains a finite mass as a beat frequency superimposed on the ‘Zitterbewegung’ at the Compton frequency. We illustrate this in a simple model in which a Poisson process drives a relativistic clock that gives rise to a Feynman path integral, where the phase is a manifestation of the twin paradox. The example shows that the non-Euclidean character of spacetime and the wave-particle duality of quantum mechanics share a common origin. They both emerge from the necessity that clocks age at rates that are path dependent.

1. Introduction

Historically and pedagogically, quantum mechanics and special relativity appear as two separate revolutions in physics. They were effectively merged by Dirac in his famous equation and it is customary to think of the merger as producing a relativistic form of Schrödinger’s equation

\[ \partial U / \partial t = D \partial^2 U / \partial x^2 \]

\[ \partial \Psi / \partial t = i D \partial^2 \Psi / \partial x^2 \]

\[ \partial^2 U / \partial t^2 = c^2 \partial^2 U / \partial z^2 + a^2 U \]

\[ \partial^2 \Psi / \partial t^2 = c^2 \partial^2 \Psi / \partial z^2 + (im)^2 \Psi \]

\[ \partial U / \partial t = c \sigma_z \partial U / \partial z + a \sigma_x U \]

\[ \partial \Psi / \partial t = c \sigma_z \partial \Psi / \partial z + i m \sigma_x \Psi \]

Table 1. The statistical mechanics underneath the Classical phenomenological equations is well accepted. The formal similarity with the Quantum equations is rendered mysterious by the presence of \( i \).

| 'Path' picture | Classical | Quantum |
|---------------|-----------|---------|
| Diffusion     | Wiener/Kac| Feynman |
| Schrödinger   | \( \partial U / \partial t = D \partial^2 U / \partial x^2 \) | \( \partial \Psi / \partial t = i D \partial^2 \Psi / \partial x^2 \) |
| Telegraph     | \( \partial^2 U / \partial t^2 = c^2 \partial^2 U / \partial z^2 + a^2 U \) | \( \partial^2 \Psi / \partial t^2 = c^2 \partial^2 \Psi / \partial z^2 + (im)^2 \Psi \) |
| K. G.         | Telegraph | Dirac   |
|               | \( \partial U / \partial t = c \sigma_z \partial U / \partial z + a \sigma_x U \) | \( \partial \Psi / \partial t = c \sigma_z \partial \Psi / \partial z + i m \sigma_x \Psi \) |
| Telegraph     | Decays   | Oscillates |
| Dirac         | \( e^{-at} \) | \( e^{iat} \) |
| Clock         | Thermodynamic | Deterministic |

In this talk we reconsider the relationship between quantum mechanics and relativity. Specifically, we question whether the two theories ‘live’ separately in the continuum and have
ultimately to be merged into a comprehensive theory, as is commonly supposed, or whether they both emerge as separate continuum limits of a discrete theory.

To motivate an emergent picture, there are interesting similarities between the two theories in their relationships with their predecessors. Both theories mathematically employ a formal analytic continuation to convert mathematical expressions from within classical mechanics to expressions in the modified theory.

In Table (1) we see two sets of partial differential equations. The left-most set are phenomenological equations describing diffusive processes. All of these equations have well known statistical bases. The right hand column contains equations thought to be fundamental in the quantum context. They are obtained from the phenomenological equation by a formal analytic continuation that can be accomplished by the replacement of $t$ by $it$. This replacement, while giving the quantum equations, does not directly yield a physically sensible statistical mechanics from which the equations emerge.

By comparison, the replacement of $t$ by $it$ takes a four dimensional Euclidean metric to Minkowski’s spacetime, giving us the transition from classical mechanics to special relativity.

We shall explore the idea that classical special relativity and quantum propagation are not independent physical constraints to be married, but rather that they emerge as different continuum limits of simple clocks Fig.[1(B)].

We shall follow a hint from the last row of Table 1. Diffusive systems are examples of thermodynamic clocks. They measure time in terms of an approach to an equilibrium state. Quantum systems on the other hand are deterministic clocks that measure time via a periodic process. The clock frequency is ultimately determined by a ‘mass’ parameter. We shall exploit this feature by building a set of minimalist classical discrete clocks as models of ‘point’ particles. We shall see that by starting with discrete clocks, the odd metric of spacetime and quantum propagation are manifestations of the continuum limit. This is missed in conventional formulations that start in the continuum.

When we speak of classical digital clocks we mean that we can interpret everything about the clocks in a classical way and classical mechanics can be seen as a particular continuum limit.

The first clock we shall consider is a simple digital, deterministic, two-photon clock. The second clock, ‘Minkowski’, is a continuum limit of the two-photon clock. The concept of ‘Spacetime’ arises from the consideration of this clock. The third clock, ‘Poisson’, is a Minkowski clock driven by a Poisson process in a non-relativistic limit. It gives us Feynman’s path integral directly in a context where the origin of phase is transparent. The phase is a remnant of the relativistic feature that objects, moving on different paths in spacetime, age at different rates.
Feynman’s phase is a remnant of the twin paradox in special relativity.

**Figure 2.** Two featureless photons confined to a box. They cross at $x = 0$ every 2 units of time giving an audible event. They move with speed $c = 1$. Points where paths change direction are considered silent events. Silent events are part of the spacetime geometry of the paths but are not part of the audible world line event sequence.

2. The Two-Photon Clock
Consider two featureless photons confined to a box Fig.[2]. We consider events of two types, audible and silent. Audible events line up along the $t$ axis and may be regarded as ticks of a clock. The set of these audible ticks is a digital analog of the clock’s worldline. Direction changes in the paths of the two photons are called silent events because they are important to make the clock tick, but are not observed directly.

To use the two-photon clock as a clock we have to count events. To handle boosts, we have to keep track of at least two components, Fig.[3]. This is done through the use of a two-component state variable that alternates sign and switches on and off with direction. The state variable $s_k$

**Figure 3.** A two-photon clock from two different frames. The area between the paths between audible events is fixed, as is the speed $c$. This determines the shape of the rectangular areas and the position of the ticks on the space-time diagram. The change in the lengths of the sides of the rectangles with changing relative velocities requires a two-component characteristic function.
Figure 4. We use the right-hand boundary of the chain of rectangles to count events. A 2-component column vector cycles between the 4 possible states of the boundary. Each state gives rise to a density that changes sign with every two events.

is a characteristic function for orientation and it takes on one of four values:

\[ s_k \in S = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \} \]  

(1)

If the two-photon clock is in state \( s_k \) at time \( k \) then the subsequent state, after the next event is

\[ s_{k+1} = Ts_k \]  

(2)

where \( T \) is the transfer matrix:

\[ T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]  

(3)

Since \( s_k = T^ks_0 \) the power of the transfer matrix corresponds to the displacement in time! Fig.[4] shows the relation between \( s \) and the path boundary. The two components of \( s \), made necessary by the need to be able to handle boosts, suggests that we can visualize the cycling of \( s \) by adding an extra dimension. Fig.[5] shows a visualization of the state variable with the extra dimension. The graph of the state function forms a four position spiral.

The digital clock in its current state would function well as a classical relativistic clock for any process that does not need time resolution on scales comparable with, or below, the digital period of the two-component clock. However special relativity needs analog reference frame clocks that are precise on all scales and we seek to find a relation between the two-photon clock and conventional analog time.

3. Minkowski’s Clock

Periodic digital clocks measure time by counting the number of clock events between external events that happen on longer time scales. Although we cannot in practice construct clocks with arbitrarily high frequency, we can imagine sequences of clocks with increasing frequency and extract from this an approximation of an analog clock, Fig.[6].
Figure 5. The two-photon clock digitizes a path boundary. Counting events with a two-component characteristic function suggests we include an extra dimension to visualize the event sequence. The path used by the clock is planar but the state function that is used for counting may be visualized with an extra dimension where it forms a four state spiral.

To this end, if $T_\nu$ is the transfer matrix for a high-frequency $N$-clock, synchronized with $T$, we must have an equivalence between the $N$-th power of $T_\nu$ and $T$ itself. Thus

$$T_\nu^N = T$$

(4)

giving $T_\nu$ as an $N$th root of $T$ or

$$T_\nu = \begin{pmatrix}
\cos \left( \frac{\pi}{2N} \right) & -\sin \left( \frac{\pi}{2N} \right) \\
\sin \left( \frac{\pi}{2N} \right) & \cos \left( \frac{\pi}{2N} \right)
\end{pmatrix} = \cos \left( \frac{\pi}{2N} \right) I_2 + \sin \left( \frac{\pi}{2N} \right) T,$$

(5)

Fig.[7].

Taking a continuum limit gives a transfer matrix representing a reference clock with arbitrarily high precision. That is we define $T_R$ as

$$T_R(t) = \lim_{N \to \infty} T_\nu^N = \begin{pmatrix}
\cos \left( \frac{\pi t}{2} \right) & -\sin \left( \frac{\pi t}{2} \right) \\
\sin \left( \frac{\pi t}{2} \right) & \cos \left( \frac{\pi t}{2} \right)
\end{pmatrix}
= \cos \left( \frac{\pi t}{2} \right) I_2 + T \sin \left( \frac{\pi t}{2} \right)$$

(6)

where $I_2$ is the $2 \times 2$ identity matrix. This clock agrees with the two-photon clock $T$ at the integers but uses a rotation matrix to interpolate between events. That is $T_R(n) = T^n$ for integer $n$.

Figure 6. We can imagine a sequence of two-photon clocks of ever higher frequency, synchronized with the initial clock in such a way that all the events of the initial low-frequency clock coincide with a subset of the events of the high-frequency clocks.
Figure 7. In the first frame the two-photon clocks ‘counts’ with the four-state indicator function. In the second frame the frequency is doubled. The eight states are just a ‘rotation’ of the original four. In the third frame the sampling frequency is 10 times the original. The approach to a spiral is evident.

The two-photon clock viewed from another inertial frame preserves the areas in the chain of areas and preserves the speed $c$. For the transfer matrix to accommodate the change in location of the audible events it has to take into account the relative velocity of the frame and the change in residence times in the two states Fig.[8,9]. For small time steps $\epsilon$ the modification of the Transfer matrix is:

$$T_M(\epsilon) = \begin{pmatrix} 1 + \frac{\pi}{2} v \epsilon & -\frac{\pi}{2} \epsilon \\ \frac{\pi}{2} \epsilon & 1 - \frac{\pi}{2} v \epsilon \end{pmatrix} = I_2 - \frac{\pi}{2} \epsilon (i\sigma_y + v\sigma_z)$$

(7)

where we have used the Pauli matrices to show the structure of the matrix. Comparing this to a small $t$ expansion of (5) we see that the only difference is the $\sigma_z$ term.

If we take the limit of the transfer matrix raised to the power $t/\epsilon$ as $\epsilon$ goes to zero we get

$$T_M(t) = \begin{pmatrix} \cos\left(\frac{\pi t}{2\gamma}\right) - v\gamma \sin\left(\frac{\pi t}{2\gamma}\right) & -\gamma \sin\left(\frac{\pi t}{2\gamma}\right) \\ \gamma \sin\left(\frac{\pi t}{2\gamma}\right) & \cos\left(\frac{\pi t}{2\gamma}\right) + v\gamma \sin\left(\frac{\pi t}{2\gamma}\right) \end{pmatrix}$$

$$= \cos\left(\frac{\pi t}{2\gamma}\right) I_2 - \sin\left(\frac{\pi t}{2\gamma}\right) \gamma(v\sigma_z + i\sigma_y)$$

(8)

Figure 8. From a moving frame the two-photon clock is a chain of oriented areas. The areas and orientations of the areas do not change between inertial frames. However the boundaries of the areas change to accommodate the boost. It is for this reason that the two-photon clock requires at least two components. In the figure, the area boundaries spend a greater proportion of time in the $(1,1)$ direction than the $(-1,1)$ direction.
Figure 9. The boost alters the percentage of time the boundary paths spend in the different states. If $-v$ is the velocity of the frame with respect to the clock then the transfer matrix readjusts the residency times.

Algebraically, the transfer matrix consists of two parts. The even part is a 'scalar', the $2 \times 2$ identity matrix $I_2$. The odd part is a vector quantity $M = \gamma (v \sigma_z + i \sigma_y)$ that includes the time dilation factor $\gamma = \frac{1}{\sqrt{1 - v^2}}$ as normalization. Notice that $M \cdot M = -I_2$ while $\sigma_z \cdot \sigma_z = I_2$ and $(i \sigma_y) \cdot (i \sigma_y) = -I_2$. The clock 'recognizes' the odd signature of spacetime through the transfer matrix. The vector $M$ 'points' in the direction of the proper time axis of the moving frame. If we regard $\sigma_z$ as the unit vector in space and $T = i \sigma_y$ as the unit vector in time, Minkowski’s clock stores both the tick rate information needed to be a clock and direction information needed by special relativity. By comparison, a Newtonian clock that measures absolute time could simply be represented by a complex number.

$T_M$ agrees with the two-photon clock at the 'dilated' integers, $\{0, \gamma, 2\gamma, \ldots\}$ as required by special relativity.

4. Minkowski’s Clock in 3-D Space
The 3-D transfer matrix functions in exactly the same way as in the previous version, except it codes for a reference frame that moves at constant velocity in a three dimensional space. The (1,1) and (2,2) blocks code for the Doppler shifts of the clock away from and towards the reference frame, as did the (1,1) and (2,2) elements in the two dimensional version. The result is that the vector in the odd part of (8) is embedded in four dimensions by $\sigma_z v \rightarrow \sigma_z \bigotimes (v_x \sigma_x + v_y \sigma_y + v_z \sigma_z)$ and $i \sigma_y \rightarrow i \sigma_y \bigotimes I_2$. The result is

$$T_M = \cos \left( \frac{\pi t}{2\gamma} \right) I_4 - \sin \left( \frac{\pi t}{2\gamma} \right) \gamma (v_x \alpha_x + v_y \alpha_y + v_z \alpha_z + i \beta)$$

where the $\alpha$'s and $\beta$ anticommute and have unit norm. Notice again the implication of the odd spacetime signature and the natural setting of 'spacetime algebra'. We started with the simple digital two-photon clock in a two-dimensional Euclidean space. The demands of special relativity for the constancy of the speed of light have forced the clock, in a continuum limit, to recognize Minkowski space. As we shall shortly see, the continuum limit is in fact an infinite mass limit and by implication conventional 'spacetime' also emerges as an infinite mass limit!

5. The Poisson Clock
In the previous sections there was no mention of quantum mechanics. We simply built a classical relativistic digital clock and found that the odd signature of spacetime emerged in a continuum limit. Interpretation was not an issue. We could in principle build a two-photon clock with audible tick rates limited only by technology. Provided the audible ticks were indeed audible, we would not find evidence of quantum behaviour in the (discrete) worldline of the clock.

However, recognizing that conventional spacetime arrives as a continuum limit of a discrete clock, the natural question is “What happens between the ticks of a clock?” In the previous cases, we had a prescription for the photon paths between ticks and we used the same deterministic prescription down to arbitrarily small scales. Here we remove the determinism, not only between successive audible events but between a first and last observation many scales above
the tick frequency of the clock. The result is a familiar one, but in a context that demonstrates the intimate link between special relativity and quantum mechanics.

**Figure 10.** The Poisson clock is a stochastic version of the two-photon clock. The event sequence is determined by a Poisson process of fixed rate. The counting of events is implemented in the same way as for the two-photon clock. If we ultimately hear ticks only on scales very much larger than the natural ‘wavelength’, there are many possible discrete worldlines between these ticks.

Referring to Fig. [10] let \( w \) be the random variable generated by applying \( T \) to the clock state at every Poisson event. Thus if \( n(t) \) is the Poisson random variable, and the process starts out in \( s_1 \) we have

\[
w(t) = T^{n(t)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

The expected value of \( w(t) \) is then

\[
<w(t)> = e^{-\alpha t} \sum_{k=0}^{\infty} \frac{(\alpha t)^k}{k!} T^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-\alpha t} (I_2 \cos \alpha t + T \sin \alpha t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-\alpha t} e^{T\alpha t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

(10)

The prefactor \( e^{-\alpha t} \) signals an exponential decay superimposed on the Poisson clock. We thus define the random variable

\[
P(t) = e^{\alpha t} T^{n(t)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

(12)

giving us a normalized ‘Poisson clock’

\[
<w(t)> = e^{T\alpha t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (I_2 \cos \alpha t + T \sin \alpha t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

(13)

whose expected length does not change with time.
From an inertial frame moving at speed $-v$ with respect to the frame of the clock, the matrix $e^{T\alpha t}$ becomes

$$
e^{T\alpha t} \rightarrow \left( \begin{array}{cc} \cos \frac{\alpha t}{\gamma} + v\gamma \sin \frac{\alpha t}{\gamma} & -\gamma \sin \frac{\alpha t}{\gamma} \\ \gamma \sin \frac{\alpha t}{\gamma} & \cos \frac{\alpha t}{\gamma} - v\gamma \sin \frac{\alpha t}{\gamma} \end{array} \right)
$$

$$= I_2 \cos \left( \frac{\alpha t}{\gamma} \right) + \left[ \gamma (v\sigma_z + T) \right] \sin \left( \frac{\alpha t}{\gamma} \right)
$$

$$= e^{T_M\alpha t}$$

(14)

where $T_M = [\gamma(v\sigma_z + T)]$ is the transfer matrix in the moving frame.

Notice that $\gamma$ has two roles. It is the time dilation factor $\gamma = \frac{1}{\sqrt{1-v^2}}$ adjusting the frequency of the moving clock and it is the normalization factor that makes the vector $T_M$ a unit vector with negative norm. $T_M$ provides the directional information for the proper time axis and is responsible for the odd metric signature of Minkowski space for the clock.

In the non-relativistic limit, the unit vector $T_M$ may be approximated by $T$ and to first order in $v^2$ we have

$$<\hat{w}(t)> \approx e^{T\alpha t} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \approx e^{\alpha T} e^{\alpha T z^2} \left( \begin{array}{c} 1 \\ 0 \end{array} \right).$$

The term $e^{\alpha T}$ is a high frequency signal that is independent of the speed and direction of the reference frame. The remaining term is the relative phase of the moving clock.

Starting at the origin, if the clock is moved at constant speed $v$ along the $x$-axis then at time $t$ in the rest frame of the origin, the expected relative phase of the clock is

$$\exp[\alpha T \frac{v^2}{2}] = \exp[\alpha T \frac{x^2}{2t}]$$

(15)

The only algebraic property of $T$ we need in the relative phase $\exp[\alpha T \frac{x^2}{2t}]$ is that $T^2 = -1$. We can then replace the column vectors by complex numbers and $T$ by the unit imaginary $i$. If we replace $\alpha$ by the real constant $\alpha = \frac{m}{\hbar}$ the expected phase is

$$\exp[i \frac{m x^2}{2\hbar t}]$$

(16)

As a unimodular complex number, this clock ‘runs’ at a relative rate appropriate to a clock moving at speed $v$ with respect to the origin, provided $v << 1$.

To obtain a spacetime density of phase we average $\exp[i \frac{m x^2}{2\hbar t}]$ assuming all speeds between $\pm 1$ are equally likely, the result in the non-relativistic limit is the density

$$K(0, 0; x, t) = \left( \frac{2\pi i \hbar t}{m} \right)^{-\frac{1}{2}} \exp \left( \frac{i m x^2}{2\hbar t} \right).$$

(17)

This is just Feynman’s free particle kernel. It appears here as the relative phase density due to an ensemble of relativistic clock paths! The path integral follows in the usual way by constructing polygonal paths using the above density as the phase of the individual segments.[1][2]

We pause here to notice a few things about the argument at this point. First of all, the replacement of $\alpha$ by $m/\hbar$ prior to equation (16) tells us that in the clock picture, what we
Figure 11. Spacetime and Quantum propagation emerge from a discrete clock via two different continuum limits. The classical limit produces a smooth world-line of audible events, but loses the geometric feature by which a particle can tell the time (Zitterbewegung). This corresponds to scaling up from deterministic clock behaviour. The quantum limit does not assume intermediate events are audible so world-lines between audible events are non-unique. This implicates the path-integral and leaves mass as a fundamental frequency that is observable through phase.

call mass is just the fundamental frequency of the clock. Referring to Fig. [6], this means that our continuum limit for Minkowski’s clock was really an infinite mass limit. Thus the usual formulations of Relativity employing spacetime frames assume an infinite mass limit for the frames themselves! Association of these frames with particles whose mass is provided by dynamics decouples the direct association between mass and clock frequency, losing the implication of wave propagation that Poisson’s clock demonstrates.

Minkowski’s clock was classical by construction. Poisson’s clock just adds a stochastic element (essentially Wiener paths). We might wonder, ‘Is this the source of the appearance of wave propagation?’ (cf. Stochastic mechanics and diffusion based models.) It is true that the Schrödinger form results from an ensemble average of stochastic paths and the non-relativistic approximation. On the other hand, it is ultimately the existence of phase that gives us wave propagation. This is ‘hidden’ in Minkowski’s clock and originates in the determination of spacetime events through spacetime areas! Poisson’s clock scales this feature up by not fixing audible events between observations (cf. superfluids.) By comparison, Minkowski’s clock scales this down, ultimately hiding any stochastic element underneath a continuum limit.

Figure[11] illustrates the different limits involved in the emergence of special relativity and quantum propagation from two-photon clocks. Minkowski space involves the scaling down of the wavelength of the two-photon clock in order to reach a continuum limit. This is an infinite frequency limit which, from the Poisson clock perspective, is an infinite mass limit. Quantum Mechanics is not apparent in this limit in agreement with the Quantum Zeno effect. Adding mass as a background feature through dynamics, as is customary in special relativity, completes the classical picture of particles ‘living’ classically in a spacetime ‘container’.

The quantum limit does not force the scale of observation to go to zero with the continuum limit. Instead, the audible events of Minkowski’s clock are assumed unobserved and are left variable except at an initial and final event. In this picture, the fundamental frequency (Compton) survives through the phase of the beats of the expected characteristic function of oriented area (Feynman’s kernel, equation (17)).
6. Discussion
With respect to the subject of this conference, this work has a number of points to offer. Ultimately, if we are looking for emergent quantum mechanics, we have to address the question ‘Emergent from what?’ The beautiful experiments of Couder’s group[3] and Bush[4] show an intriguing physical picture of wave-particle duality. The theoretical framework of Grössing, Fussy, Mesa Pascasio and Schwabl (see articles in this volume and[5]) takes this picture and replaces the underlying wave by a structured vacuum. In both cases the quantum behaviour arises from the phase of waves that are literally and metaphorically ‘underneath’ the particle.

This work looks down in scale from the de Broglie wavelength to the Compton wavelength to ‘find’ the waves, or structured vacuum, that underlies the wave-particle duality we need for quantum mechanics to emerge.

The first result, Minkowski’s clock, gives us spacetime as it emerges from the consideration of an extremely simple two-photon clock. By seeing how spacetime results from a continuum limit, we get one extra feature that is not apparent in the usual invocation of spacetime. We see that from the perspective of actual clocks, spacetime is an infinite frequency, infinite mass object. While this is obvious looking at special relativity through the perspective of the uncertainty principle ($\Delta E \Delta t \geq \hbar$), the uncertainty principle is usually assumed (incorrectly) to be outside the domain of classical physics, and as a result the usual pedagogy for relativity misses the connection between mass and frequency. With that connection gone, quantum propagation has to be invoked in a separate theory.

The second result, Poisson’s clock, shows how the mechanism that enforces special relativity in a simple stochastic clock also gives rise to Feynman’s phase in the path integral. The phase itself is a manifestation of the twin paradox familiar from special relativity. Two different spacetime paths between two events give rise to two different ages for particles that traverse the paths. The ages are measured in terms of the fundamental frequency (Compton) of the particles and, on observation, the synchronicity of the various paths affect the likelihood of an observation taking place. The sum over paths still motivates paradoxical questions like ‘Did the clock move over one or all of the paths?’. However these questions are put into context by the realization that, from the perspective of actually building a clock, such questions are unanswerable even in the world of classical special relativity. Minkowski’s clock, which appears to bypass wave propagation, illustrates this. It is classical because the continuum limit confines spacetime paths to scales below a fundamental period which itself is going to zero in the continuum limit.

7. Conclusions
The view that spacetime and quantum propagation both emerge from the consideration of simple clocks is a picture that intersects many attempts to understand why quantum mechanics is both precise and inscrutable.

One bridge that connects clock arguments to other pictures is the feature of dimensionality. Classical physics, including relativity, interpolates a sequence of events of particle observation by smooth world-lines. However, relativistic physics implicitly connects the events by invariant spacetime areas. The control of events by areas is a progenitor of ‘wave-particle duality’ in quantum mechanics and Minkowski space in relativity. The important role of dimension in quantum mechanics is emphasized in Fractal Spacetime approaches[6]-[8] and is apparent from the path integral perspective[9, 10]. The connection to special relativity arises from Feynman’s chessboard model[11]-[12].

The light clock aspect of the model is similar to the Zig-Zag model of Penrose[13] with an emphasis on Clifford algebras as a geometric framework for the Dirac equation (see the papers of Hiley and de Gosson in this volume and[14, 15]). The clock model shares with deBroglie-Bohm pictures[16] the feature that it is ‘particle and wave’ as opposed to the ‘particle or wave’ of Copenhagen. Just as deBroglie-Bohm considers the particle and quantum potential as real
aspects of an object, so the clock model considers the events and the inter-event regions as part of the model.

Ultimately, the clock model suggests that at least part of the problem regarding the interpretation of quantum mechanics resides in the premature invocation of real numbers. Real numbers are a mathematical convenience that are not observable. Quantum mechanics tells us that the stability of Nature reflects a predisposition for discrete energy levels. Mathematical models that start in a spacetime continuum presuppose that a coherent discrete picture should emerge from a continuum description. The clock model (see also Kauffman’s paper in this volume) suggests that this may be putting the cart before the horse.

Thus it is from the clock model that we can see classical special relativity and quantum propagation as two different continuum limits of the same discrete model. After the limit is taken, the connection between the two pictures is severed, leaving only $i$ as the ghost of discrete physics that haunts both theories.

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