Structural Identification of Fractional-Order Dynamical Network with Different Orders

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Abstract: Topology structure and system parameters have a great influence on the dynamical behavior of dynamical networks. However, they are sometimes unknown or uncertain in advance. How to effectively identify them has been investigated in various network models, from integer-order networks to fractional-order networks with the same order. In the real world, many systems consist of subsystems with different fractional orders. Therefore, the structure identification of a dynamical network with different fractional orders is investigated in this paper. Through designing proper adaptive controllers and parameter updating laws, two network estimators are well constructed. One is for identifying only the unknown topology structure. The other is for identifying both the unknown topology structure and system parameters. Based on the Lyapunov function method and the stability theory of fractional-order dynamical systems, the theoretical results are analytically proved. The effectiveness is verified by three numerical examples as well. In addition, the designed estimators have a good performance in monitoring switching topology. From the practical viewpoint, the designed estimators can be used to monitor the change of current and voltage in the fractional-order circuit systems.

Keywords: structure identification; fractional-order networks; different orders

1. Introduction

Topology structure and system parameters, as we know, play a great role in many studies on dynamical networks. In pinning control, as a specific example, how to choose the pinned nodes is usually based on the node degree calculated from the topology structure. In practical applications, however, the topology structure and system parameters of dynamical networks may be unknown or partially unknown. In cellular networks, for instance, the interactions among the protein-DNAs, described by the topological structure, are sometimes unknown. Since the DNA interactions have a great effect on cellular processes, how to effectively identify them is an interesting and important issue and deserves deep study. Thus far, researchers have performed much research on the structure identification issue of dynamical networks [1–15]. In [15], Waarde et al. addressed the problem of identifying the graph structure of a dynamical network using measured input/output data. In [9], Ding et al. investigated the topology identification problem for power systems. In [10], Farajollahi et al. investigated the topology identification problem for distribution systems. It is noted that the studies on structure identification of dynamical networks have potential applications in real systems.

Recently, the fractional-order calculus, as an extension of the integer-order calculus, has received increasing attention since it has some good properties (such as the infinite memory and genetic characteristics) and more degrees of freedom compared with the integer-order calculus [16,17]. That is, the fractional dynamical networks can more precisely describe the characteristics of physical systems. Nowadays, many valuable results about the fractional-order networks have been obtained [18–25]. In [18], Si et al. gave...
the sufficient condition of structure identification in uncertain fractional dynamical networks, and further discussed the effects of coupling strength and fractional order on the identification process. In [21], Du investigated the parameters identification issue of fractional dynamical networks by achieving the modified function projective synchronization between the drive-response networks.

It is worth noting that the networks in the aforementioned literature [18–28] are formed by many coupled subsystems with the same fractional order. In fact, complex networks may consist of subsystems with different fractional orders in the real world. In [29], Kaczorek used fractional-order current and voltage equations with different orders to describe the linear circuits. In [30], Datsko et al. studied fractional reaction-diffusion systems, in which the derivative order of activator and inhibitor variables are different. Naturally, how to identify the unknown topology structure of a dynamical network with different fractional orders is a challenging issue and deserves further study.

Motivated by the above discussions, this paper investigates the structure identification of a fractional-order dynamical network with different orders. The main contribution is to design network estimators based on the adaptive feedback control method and parameter updating laws in two cases: (i) assume that only the topology is unknown and to be identified; (ii) assume that both the topology and system parameters are unknown.

The rest of this paper is organized as follows. In Section 2, the fractional-order complex network model is introduced, and some necessary definitions, assumptions and lemmas are given. In Section 3, two corresponding effective estimators are designed. In Section 4, numerical examples are performed to verify the correctness and effectiveness of the theoretical results. Finally, the conclusions are drawn in Section 5.

2. Model Description and Preliminaries

Throughout this paper, the fractional-order calculus is Caputo calculus. In the following, some basic definitions and lemmas about Caputo calculus are presented.

Definition 1 ([16]). The fractional integral of order \( \alpha \) for function \( f(t) \) is defined as:

\[
D^{-\alpha}_{t_0} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \tau)^{\alpha-1} f(\tau) d\tau,
\]

where \( t \geq t_0, \alpha > 0, \) and \( \Gamma(\cdot) \) is the Gamma function.

Definition 2 ([16]). The \( \alpha \)-th-order Caputo fractional derivative of the given function \( f(t) \) is defined as:

\[
C D^{\alpha}_{t_0} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t - \tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau,
\]

where \( m-1 < \alpha < m, m \in \mathbb{Z}^+ \).

In general, the initial time \( t_0 \) is often considers as \( t_0 = 0 \). Then, for the sake of simplicity, the \( D^{-\alpha}_{t_0} \) is denoted by \( D^{-\alpha} \) and \( C D^{\alpha}_{t_0} \) by \( D^\alpha \).

Lemma 1 ([31]). If \( x(t), z(t) \in C^1[t_0, b] \), and \( \alpha > 0, \beta > 0, \) then

\[
(1) \quad D^\alpha \tilde{D}^{-\beta} x(t) = \tilde{D}^{\alpha-\beta} x(t),
\]

\[
(2) \quad D^\alpha (x(t) \pm z(t)) = D^\alpha x(t) \pm D^\alpha z(t).
\]

Lemma 2 ([32]). Let \( x(t) \in \mathbb{R}^n \) be a differentiable vector function. Then, for any time instant \( t \geq 0 \)

\[
\frac{1}{2} D^\alpha \left( x^T(t) x(t) \right) \leq x^T(t) (D^\alpha x(t)),
\]
where $\alpha \in (0, 1)$ is the fractional order.

Consider the following fractional-order dynamical network

$$D^\alpha y_i(t) = f_i(y_i(t), \phi_i) + c \sum_{j=1}^{N} a_{ij} H y_j(t), \ i = 1, 2, \ldots, N,$$

where $0 < a_i < 1$, $y_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^T \in \mathbb{R}^n$ is the state vector of the $i$th node, $f_i$ is a continuous and differentiable vector function, $\phi_i \in \mathbb{R}^{m_i}$ is a parameter vector, $c > 0$ is the coupling strength, $H = \text{diag}(h_1, h_2, \ldots, h_n)$ is the inner coupling matrix, $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ is the zero-row-sum outer coupling matrix, defined as: if there is a connection from node $j$ to node $i$, then $a_{ij} \neq 0$ ($i \neq j$); otherwise, $a_{ij} = 0$.

In this paper, $f_i$ is assumed to be a linear function about $\phi_i$, namely

$$f_i(y_i(t), \phi_i) = g_i(y_i(t)) + G_i(y_i(t))\phi_i.$$  

**Assumption 1.** Suppose that there exists a positive definite matrix $\Delta$ such that

$$(\hat{y} - y)^T (f_i(\hat{y}, \phi_i) - f_i(y, \phi_i)) \leq (\hat{y} - y)^T \Delta (\hat{y} - y)$$

for any $\hat{y}, y \in \mathbb{R}^n$ and $i = 1, 2, \ldots, N$, where $\Delta = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$.

**Assumption 2.** Denote $G_i(y_i) = (G_{i1}^1(y_i), \ldots, G_{i1}^n(y_i))$. Suppose that $\{G_i^k(y_i)\}_{k=1}^{q_i}$ are linearly independent on the orbits $\{y_i(t)\}_{i=1}^{N}$ for $t > 0$.

**3. Main Results**

Firstly, assume that only the topology is unknown. The network estimator is designed as

$$D^\alpha \hat{y}_i(t) = f_i(\hat{y}_i(t), \phi_i) + c \sum_{j=1}^{N} \hat{a}_{ij}(t) H \hat{y}_j(t) + u_i(t), \ i = 1, 2, \ldots, N,$$

where $u_i(t)$ are controllers to be designed, $\hat{A}(t) = (\hat{a}_{ij}(t)) \in \mathbb{R}^{N \times N}$ is the estimation of the unknown matrix $A$.

Define $e_i(t) = \hat{y}_i(t) - y_i(t)$, then the error system is

$$D^\alpha e_i(t) = f_i(\hat{y}_i(t), \phi_i) - f_i(y_i(t), \phi_i) + c \sum_{j=1}^{N} a_{ij} H e_j(t) + c \sum_{j=1}^{N} (\hat{a}_{ij}(t) - a_{ij}) H \hat{y}_j(t) + u_i(t).$$

**Theorem 1.** Suppose that Assumptions 1 and 2 hold. Then the unknown matrix $A$ of network (1) is identified by the $\hat{A}(t)$ of network estimator (4) with the following adaptive controllers $u_i(t)$ and updated laws

$$u_i(t) = -w(t)e_i(t),$$

$$D^\alpha w(t) = \theta \sum_{i=1}^{N} e_i(t)^T e_i(t),$$

$$D^\alpha \hat{a}_{ij}(t) = -\mu_{ij} \hat{y}_j^T(t) He_i(t),$$

where $i, j = 1, 2, \ldots, N$, $\theta > 0, \mu_{ij} > 0$ are adaptive gains, and $\alpha_0 = \max_{1 \leq k \leq N} \{a_k\}$. 
Proof. Consider the following Lyapunov function:

\[
V(t) = \frac{1}{2} \sum_{i=1}^{N} D^{n_i-a_i}(e_i^T(t)e_i(t)) + \frac{1}{2\theta}(w(t) - \hat{\omega})^2
\]

+ \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{c}{\theta^{ij}} D^{n_i-a_i}((\tilde{a}_{ij}(\tau) - a_{ij})^2),
\]

where \(\hat{\omega}\) is a positive constant to be determined.

According to Lemmas 1 and 2, the derivative of \(V(t)\) is calculated as

\[
D^{\theta_0}V(t) = \sum_{i=1}^{N} D^{\theta_i} \left( \frac{1}{2} e_i^T(t)e_i(t) \right) + \frac{1}{\theta} (w(t) - \hat{\omega})(D^{\theta_0}w(t))
\]

+ \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{c}{\theta^{ij}} (\tilde{a}_{ij}(t) - a_{ij})(D^{\theta_i}\tilde{a}_{ij}(t))
\]

\[
\leq \sum_{i=1}^{N} e_i^T(t)(D^{n_i}e_i(t)) + \frac{1}{\theta} (w(t) - \hat{\omega}) \left( \theta \sum_{i=1}^{N} e_i^T(t)e_i(t) \right)
\]

+ \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{c}{\theta^{ij}} (\tilde{a}_{ij}(t) - a_{ij})(-\mu_{ij}\hat{y}_i^T(t)He_i(t))
\]

\[
= \sum_{i=1}^{N} e_i^T(t)(f_i(\hat{y}_i(t),\phi_i) - f_i(y_i(t),\phi_i)) + c \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}e_i^T(t)He_i(t)
\]

+ (w(t) - \hat{\omega}) \sum_{i=1}^{N} e_i^T(t)e_i(t) - c \sum_{i=1}^{N} \sum_{j=1}^{N} (\tilde{a}_{ij}(t) - a_{ij})\hat{y}_i^T(t)He_i(t)
\]

\[
= \sum_{i=1}^{N} e_i^T(t)(f_i(\hat{y}_i(t),\phi_i) - f_i(y_i(t),\phi_i))
\]

+ c \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}e_i^T(t)He_i(t) - \hat{\omega} \sum_{i=1}^{N} e_i^T(t)e_i(t)
\]

\[
\leq \sum_{i=1}^{N} e_i^T(t)(\Delta - \hat{\omega}I_n)e_i(t) + c \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}e_i^T(t)He_i(t)
\]

\[
= e^T(t)(I_N \otimes \Delta - \hat{\omega}I_N \otimes I_n + c(A^s \otimes H))e(t),
\]

where \(e(t) = (e_1^T(t),e_2^T(t),\ldots,e_N^T(t))^T\), \(A^s = (A + A^T)/2\), \(I_n\) and \(I_N\) are \(n\)- and \(N\)-dimensional identity matrices respectively, ‘\(\otimes\)’ denotes the Kronecker product.

Let \(\hat{\delta} = \max_{1 \leq \mu \leq N} \{\delta_{\mu}\}\), \(\lambda\) be the largest eigenvalues of \(A^s \otimes H\). Choose \(\hat{\omega} = \hat{\delta} + c\lambda + 1\), we obtain

\[
D^{\theta_0}V(t) \leq -e^T(t)e(t).
\]

Based on the Lyapunov stability theory, we have \(e(t) \to 0\) as \(t \to \infty\). Then rewrite (5) as

\[
\sum_{j=1}^{N} (\tilde{a}_{ij}(t) - a_{ij})Hy_j(t) = 0, i = 1, 2, \ldots, N.
\]

By Assumption 2, it is clear that \(\tilde{a}_{ij}(t) \to a_{ij}\), i.e., the unknown \(A\) is identified. Thus the proof is completed. \(\Box\)
where $i = 1, 2, \ldots, N$, $\hat{A}(t) = (\hat{a}_{ij}(t)) \in \mathbb{R}^{N \times N}$ and $\phi_i(t)$ are the estimations of $A$ and $\phi_i$, respectively.

The error system is

\[
D^N e_i(t) = f_i(\hat{y}_i(t), \phi_i) - f_i(y_i(t), \phi_i) + G_i(\hat{y}_i(t))(\phi_i(t) - \phi_i)
+ c \sum_{j=1}^N a_{ij} H e_j(t) + c \sum_{j=1}^N (\hat{a}_{ij}(t) - a_{ij}) H \hat{y}_j(t) + u_i(t).
\]  

\[V(t) = \frac{1}{2} \sum_{i=1}^N D^{\alpha_0} (e_i(t)) (e_i(t)) + \frac{1}{2\theta} (w(t) - \bar{w})^2
+ \sum_{i=1}^N \sum_{j=1}^N c \mu_{ij} D^{\alpha_0}((\hat{a}_{ij}(t) - a_{ij})^2)
+ \sum_{i=1}^N \frac{1}{\xi_i} D^{\alpha_0}((\phi_i(t) - \phi_i)^T (\phi_i(t) - \phi_i)),
\]

where $\bar{w}$ is a positive constant to be determined.

\[D^{\alpha_0} V(t) \leq \sum_{i=1}^N \sum_{j=1}^N c \xi_i D^{\alpha_0}((\hat{a}_{ij}(t) - a_{ij})^2)
+ \sum_{i=1}^N \sum_{j=1}^N c \xi_i (\phi_i(t) - \phi_i)^T (\phi_i(t) - \phi_i))
\]

\[= e^T(t)(I_N \otimes \Delta - \bar{w} I_N \otimes I_n + c (A^T \otimes H))e(t),
\]
Then similar to Theorem 1, we have $e(t) \to 0$ as $t \to \infty$. Rewrite (10) as

$$G_i(y_i(t))(\phi_i(t) - \phi_i) + c \sum_{j=1}^{N} (\hat{a}_{ij}(t) - a_{ij})H y_j(t) = 0.$$  

By Assumption 2, it is clear that $\hat{a}_{ij}(t) \to a_{ij}$ and $\phi_i(t) \to \phi_i$, i.e., the unknown structure is identified. Thus, the proof is completed. □

**Remark 1.** If choose $\alpha_1 = \alpha_2 = \cdots = \alpha_N = \alpha$, network (1) and network estimator (9) can be rewritten as follows

$$D^\alpha y_i(t) = f_i(y_i(t), \phi_i) + c \sum_{j=1}^{N} a_{ij}H y_j(t),$$

$$D^\alpha \hat{y}_i(t) = g_i(\hat{y}_i(t)) + G_i(\hat{y}_i(t))\phi_i(t) + c \sum_{j=1}^{N} \hat{a}_{ij}(t)H \hat{y}_j(t) + u_i(t).$$

The detailed discussions of this special case can be seen in Theorem 1 in Reference [18].

**Remark 2.** Recently, many valuable results about topology identification of integer-order networks have been obtained by various methods, such as the graph-theoretic approach [8], compressive sensing [9], mixed integer linear program (MILP) method [10], and so on. Compared with these results, we consider the structure identification problem of a fractional-order network. Especially, the fractional orders are nonidentical. How to generalize the mentioned results in integer-order networks for fractional-order network deserves further study in the future.

4. Numerical Examples

Consider a fractional-order dynamical network with six nodes, and choose the node dynamics as the fractional-order Chen system [33]

$$D^\alpha x_i = \begin{cases} 
\phi_{11}(x_{i2} - x_{i1}), \\
(\phi_{13} - \phi_{11})x_{i1} - x_{i1}x_{i3} + \phi_{13}x_{i2}, \\
x_{i1}x_{i2} - \phi_{i2}x_{i3}.
\end{cases} \quad i = 1, 2,$$

and the fractional-order Lorenz system [34]

$$D^\alpha x_i = \begin{cases} 
\phi_{11}(x_{i2} - x_{i1}), \\
\phi_{i3}x_{i1} - x_{i1}x_{i3} - x_{i2}, \\
x_{i1}x_{i2} - \phi_{i2}x_{i3}.
\end{cases} \quad i = 3, 4,$$

and the fractional-order Lü system [35]

$$D^\alpha x_i = \begin{cases} 
\phi_{11}(x_{i2} - x_{i1}), \\
-x_{i1}x_{i3} + \phi_{i2}x_{i2}, \\
x_{i1}x_{i2} - \phi_{i3}x_{i3}.
\end{cases} \quad i = 5, 6,$$

where $(a_1, a_2, \ldots, a_6) = (0.97, 0.98, 0.995, 0.998, 0.99, 0.99), \phi_1 = \phi_2 = (35, 3, 28)^T, \phi_3 = \phi_4 = (10, 8/3, 28)^T$ and $\phi_5 = \phi_6 = (35, 28, 3)^T.$
Choose $c = 0.1$, $H = \text{diag}(1, 1, 1)$ and

$$A = \begin{bmatrix}
-5 & 3 & 2 & 0 & 0 & 0 \\
0 & -4 & 1 & 3 & 0 & 0 \\
0 & 1 & -3 & 0 & 2 & 0 \\
0 & 1 & 1 & -2 & 0 & 0 \\
4 & 0 & 2 & 0 & -6 & 0 \\
0 & 2 & 0 & 2 & 0 & -4
\end{bmatrix}.$$ 

Firstly, assume that only $A$ is unknown. Choose the adaptive gains $\theta = 1.6, \mu_{ij} = 16$, the initial values $w(0) = 0.1, \hat{a}_{ij}(0) = 1, \text{for } i, j = 1, 2, \ldots, 6$, the initial values of the state variable $y_i(0) = (1 + 0.2i, 2 + 0.2i, 3 + 0.2i)^T$ and $\hat{y}_i(0) = (6 + 0.2i, 5 + 0.2i, 4 + 0.2i)^T$. Figure 1 shows the evolution of $\hat{a}_{ij}(t)$. For a clearer view, Figure 2 shows the orbits of $\hat{a}_{24}(t), \hat{a}_{32}(t), \hat{a}_{35}(t)$ and $\hat{a}_{51}(t)$. Clearly, the unknown topology is effectively identified at about $t = 50$.

Secondly, assume that both $A$ and $\phi_i$ are unknown. Choose $\theta = 1.6, \mu_{ij} = 16, \xi_{ij} = 25$, $w(0) = 0.1, \hat{a}_{ij}(0) = 1, \varphi_1(0) = (34.2, 27)^T, \varphi_2(0) = (34.5, 2.5, 27.5)^T, \varphi_3(0) = (9, 2.27)^T$, $\varphi_4(0) = (9.5, 2, 27.5)^T, \varphi_5(0) = (34, 27)^T, \varphi_6(0) = (34.5, 27.5, 2.5)^T$, for $i, j = 1, 2, \ldots, 6$, $y_i(0) = (1 + 0.2i, 2 + 0.2i, 3 + 0.2i)^T$ and $\hat{y}_i(0) = (6 + 0.2i, 5 + 0.2i, 4 + 0.2i)^T$. Figures 3 and 4 show some orbits of $\hat{a}_{ij}(t)$ and $\varphi(t)$, respectively. Compared with Example 1, the topology identification needs more times. That is, the unknown system parameters have a great influence on the topology identification. How to improve the performance deserve further studies.

Finally, suppose that $a_{35} = 2$ switches to $a_{35} = 0$ at $t = 100$ in the first example. Figure 5 shows that $a_{35}$ is identified quickly. That is, the network estimator (4) has good performance in real-time monitoring. From the practical viewpoint, the designed estimator can be used to monitor the switching phenomenon in circuits.

Figure 1. The evolution of $\hat{a}_{ij}(t), i, j = 1, 2, \ldots, 6$. 

![Figure 1: The evolution of $\hat{a}_{ij}(t), i, j = 1, 2, \ldots, 6$.](image)
Figure 2. (Color online) The orbits of $\tilde{a}_{24}(t)$, $\tilde{a}_{32}(t)$, $\tilde{a}_{35}(t)$ and $\tilde{a}_{51}(t)$.

Figure 3. (Color online) The orbits of $\tilde{a}_{24}(t)$, $\tilde{a}_{32}(t)$, $\tilde{a}_{35}(t)$ and $\tilde{a}_{51}(t)$.
Figure 4. (Color online) The blue, red and yellow lines denote the first, second and third components of the state variables $\varphi_i(t)$ respectively.

Figure 5. (Color online) Identification of the switching topology.

5. Conclusions and Discussions

The structure identification problem of uncertain fractional-order complex networks with different orders is investigated. Two network estimators are designed via adaptive feedback control and parameter updating laws. The valuable results for the integer-order networks and the fractional-order networks with the same order are generalized to the networks with different fractional orders. In this paper, all the state variables are assumed to be observable. In real cases, the state variables may be partially observable [11]. Thus, how to identify the topology from partial observations for fractional-order networks, especially with different orders, deserves further studies. On the other hand, many other methods are introduced to address the issue of integer-order systems. Compressive sensing based on transient dynamics is used to identify the topology for power systems [9]. An MILP method is used for distributed systems [10]. We will adopt the mentioned methods to consider the issue of fractional-order networks in future studies.
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