We study the short-time and medium-time behavior of the survival probability of decaying states in the framework of the $N$-level Friedrichs model. The degenerated and nearly degenerated systems are discussed in detail. We show that in these systems decay can be considerably slowed down or even stopped by appropriate choice of initial conditions. We analyze the behavior of the multilevel system within the so-called Zeno era. We examine and compare two different definitions of the Zeno time. We demonstrate that the Zeno era can be considerably enlarged by proper choice of the system parameters.

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I. INTRODUCTION

Since the very beginning of the quantum mechanics, the measurement process has been a most fundamental issue which is still not completely understood. The main characteristic feature of the quantum measurement is that the measurement changes the dynamical evolution. This is the main difference of quantum measurement compared to its classical analogue. In this framework, Misra and Sudarshan pointed out [1] that repeated measurements can prevent an unstable system from decaying (the quantum Zeno effect, QZE).

The QZE has been discussed for many physical systems [2, 3, 4, 5, 6]. Later it has been found [7, 8] that under some conditions the repeated observations could not only slow down but, in contrary, speed up the decay of the quantum state (the quantum anti-Zeno effect). The anti-Zeno effect has been further analyzed in [9, 10, 11].

While there exist experiments [12, 13] demonstrating the perturbed evolution of a coherent dynamics, the demonstration of the QZE for an unstable system with exponential decay, as originally proposed in [1], has long been an open question. Only recently, both Zeno and anti-Zeno effects have been observed experimentally [14].

Zeno effect was recently proposed as a tool to prepare and maintain the purity of the initial state of quantum systems [15], to protect it against decoherence [16, 17], and for the purification of quantum states [18]. Multilevel systems are necessary in order to have decoherence-free subspaces, as in this case several qubits are required [19]. Zeno effect for multilevel systems has already been discussed [20, 21]. However, only non-degenerated systems have been considered so far, although in actual applications we have quantum systems with (nearly) identical elements (atoms, ions, quantum dots etc.). Therefore, the study of the Zeno (and anti-Zeno) effects in (nearly) degenerated systems is necessary. This is the object of this work in the frame of the Friedrichs model [22], which is appropriate for the discussion of decay and unstable states. The original Friedrichs model contains two discrete energy levels, a ground state and an excited state, coupled with the continuum, which is bounded from below. The time dependence of the survival probability of excited states has been extensively studied in the literature [10, 11, 23, 24, 25, 26].

For physically motivated values of the parameters of the model the decay is approximately exponential with a short non-exponential initial era and a non-exponential long tail.

The analytical structure of the multilevel Friedrichs model has also been widely discussed [27, 28, 29, 31, 32, 33]. However, these discussions are mainly concentrated on the non-degenerated systems, and only in a few papers the time dependence of the survival probability for $N$-level models has been discussed [34, 35].

The degenerated multilevel Friedrichs model is presented in Section II. In Section III we discuss the exponential decay era and analyze few cases when decay can be substantially slowed down or even stopped. This discussion is extensively used in the paper. In Section IV we analyze the Zeno effect and different Zeno times. We show that in almost completely degenerate systems, the Zeno era can be enlarged. In Section V we analyze the anti-Zeno effect in the almost completely degenerate systems.
The Hamiltonian of the $N$-level Friedrichs model is:

$$H = H_0 + \lambda V,$$

where

$$H_0 = \sum_{k=1}^{N} \omega_k |k\rangle\langle k| + \int_{0}^{\infty} d\omega \omega |\omega\rangle\langle \omega|,$$

$$V = \sum_{k=1}^{N} \int_{0}^{\infty} d\omega \hat{f}_k(\omega) (|k\rangle\langle \omega| + |\omega\rangle\langle k|).$$

Here $|k\rangle$ represent states of the discrete spectrum with the energy $\omega_k$, $\omega_k > 0$. Degeneracy is reflected in $\omega_k$. The vectors $|\omega\rangle$ represent states of the continuous spectrum with the energy $\omega$, $\hat{f}_k(\omega)$ are the form factors for the transitions between the discrete and the continuous spectrum, and $\lambda$ is the coupling parameter. The vacuum energy is chosen to be zero. The states $|k\rangle$ and $|\omega\rangle$ form a complete orthonormal basis:

$$\langle k|k\rangle = \delta_{kk}, \quad \langle \omega|\omega\rangle = \delta(\omega - \omega'), \quad \langle \omega|k\rangle = 0,$$

$$k, k' = 1 \ldots N,$$

$$\sum_{k=1}^{N} |k\rangle\langle k| + \int_{0}^{\infty} d\omega |\omega\rangle\langle \omega| = I,$$

where $\delta_{kk'}$ is the Kronecker symbol, $\delta(\omega - \omega')$ is Dirac’s delta function and $I$ is the unity operator. The Hamiltonian $H_0$ has continuous spectrum over the interval $[0, \infty)$ and discrete spectrum $\omega_1, \ldots, \omega_k$ embedded in the continuous spectrum. As the interaction $\lambda V$ is switched on, the discrete energy levels of $H_0$ become resonances of $H$ as in the case of the one-level Friedrichs model [22]. As a result, the total evolution normally leads to the decay of any initial state in point spectrum eigenspace:

$$|\Phi\rangle = \sum_k \alpha_k |k\rangle, \quad \langle \Phi|\Phi\rangle = 1.$$

Decay is described by the survival probability $p(t)$

$$p(t) \equiv |\langle \Phi|e^{-iHt}|\Phi\rangle|^2 = |A(t)|^2,$$

where $A(t)$ is the survival amplitude. The survival amplitude can be calculated as

$$A(t) \equiv \langle \Phi|\Phi(t)\rangle = \sum_{k,m=1}^{N} \alpha_k \alpha_m^* \langle k|m\rangle_t = \sum_{k,m=1}^{N} \alpha_k \alpha_m^* A_{km}(t),$$

where the partial survival amplitudes $A_{km}(t)$ are written as

$$A_{km}(t) = -\sum_{j} r_{km}^j e^{-i\gamma_j t} + \frac{1}{2\pi i} \int_{C} d\omega e^{-i\omega t} \hat{G}_{km}(\omega).$$

Here $\hat{G}_{km}(\omega)$ are the matrix elements of the partial resolvent $\hat{G}(\omega)$, which is defined as:

$$\hat{G}_{km}(\omega) = (\omega_k - \omega) \delta_{km} - \lambda^2 \int_{0}^{\infty} d\omega' \frac{\hat{f}_k(\omega') \hat{f}_m(\omega')}{\omega' - \omega + i0^+}.$$
structures mainly) for identical (A) and different (B) form factors. We analyze the behavior of the system in the exponential era (i.e. the poles $\epsilon_k$ are small, and consider the expansion in the vicinity of $\epsilon = 0$, the partial resolvent can be found explicitly: $G_{km}(x) = \Lambda \hat{G}(x)$, we find

$$G^{-1}_{km}(x) = (x_k - x)\delta_{km} - \lambda^2 \int_0^\infty dx' \frac{\hat{f}(x') f_m(x')}{x' - x + i0},$$

and

$$A_{km}(t) = - \sum_j r^j_{km} e^{-i\Lambda x_j t} + \frac{1}{2\pi i} \int_C dx e^{-i\Lambda x t} G_{km}(x).$$

### III. THE EXPONENTIAL ERA

We concentrate mainly on the model close to the completely degenerate one while the model without degeneracy has been discussed previously. We analyze the behavior of the system in the exponential era (i.e. the poles structures mainly) for identical (A) and different (B) form factors.

We suppose that the form factors can be expressed as

$$f_k(x) = p^{k-1} f(x) + \varepsilon q_k(x), \quad p \text{ is a constant.}$$

We assume that $\varepsilon$ is small, and consider the expansion in the vicinity of $\varepsilon = 0$. This choice is motivated by the expected similarity of the form factors for the degenerate levels. Then the integrals in Eq. (10) can be written as

$$\int_0^\infty dx' \frac{\hat{f}(x') f_m(x')}{x' - x + i0} = W_{km}(x) + \varepsilon R_{km}(x),$$

where

$$W_{km}(x) = p^{k+m-2} W(x) = p^{k+m-2} \int_0^\infty dx' \frac{f^2(x')}{x' - x + i0},$$

$$R_{km}(x) = \int_0^\infty dx' \frac{f(x')(p^{k-1} q_m(x') + p^{m-1} q_k(x'))}{x' - x + i0} + \varepsilon \int_0^\infty dx' \frac{q_k(x') q_m(x')}{x' - x + i0}.$$

The matrix elements of the inverse partial resolvent (10) are:

$$G^{-1}_{km}(x) = (x_k - x)\delta_{km} - \lambda^2 p^{k+m-2} W(x) - \lambda^2 \varepsilon R_{km}(x).$$

As both matrix elements $W(x)$ and $R(x)$ are complex, the resolvent $G(x)$ is also a complex function.

#### A. Identical form factors

For $\varepsilon = 0$, the partial resolvent can be found explicitly:

$$G_{km}(x) = \frac{\delta_{km}}{x_m - x} + \frac{\lambda^2 p^{k+m-2} W(x)}{(x_m - x)(x_k - x)(1 - \lambda^2 W(x) \sum_i p^{2(i-1)} (x_k - x_i)^{-1})},$$

as well as the determinant

$$\det G^{-1}(x) = (1 - \lambda^2 W(x) \sum_i p^{2(i-1)} (x_k - x_i)^{-1}) \prod_k (x_k - x).$$
When the energy levels $x_k$ are well separated, each of them becomes a resonance $z_k$ for nonzero $\lambda^2$:

$$z_k = x_k - \lambda^2 p^{2(k-1)} W(x_k) + O(\lambda^4).$$

This case is discussed in detail in [35].

As we focus on the model close to the completely degenerate one, we suppose that $p = 1$. First, we consider the case when all energy levels are degenerated: $x_k = \bar{x}$ for any $k$. Then we deduce the partial resolvent and the determinant:

$$G_{km}(x) = \frac{1}{x - \bar{x}} \left( \delta_{km} + \frac{\lambda^2 W(x)}{x - x - N\lambda^2 W(x)} \right),$$

$$\det G^{-1}(x) = (\bar{x} - x)^{N-1} (\bar{x} - x - N\lambda^2 W(x)).$$

The matrix elements [18] have two poles in the vicinity of $\bar{x}$: the real pole $z_1 = \bar{x}$ and the complex pole $z_2$ defined by the equation $\bar{x} - z_2 - N\lambda^2 W(z_2) = 0$. For small $\lambda^2$ we find

$$z_2 = \bar{x} - \lambda^2 N W(\bar{x}) + O(\lambda^4).$$

The residues of $G_{km}(x)$ at the poles are

$$\text{res}_{x=z_2} G_{km}(x) = -\delta_{km} + \frac{1}{N},$$

$$\text{res}_{x=z_1} G_{km}(x) = \frac{-1}{N(1 + \lambda^2 N W'(z_2))} = \frac{-1}{N(1 + \lambda^2 N W'(\bar{x}))} + O(\lambda^4),$$

where $W'(x)$ is the derivative of $W(x)$.

Let us now find the time evolution of state $|\Phi\rangle$ [3]. In our case, there exist only two poles $z_1$ and $z_2$ in a vicinity of the real axis. The survival amplitude $A(t)$ is

$$A(t) = e^{-ix\Lambda t} \left( 1 - \frac{1}{N} \left( 1 - \frac{e^{-i(z_2-z_1)\Lambda t}}{1 + \lambda^2 N W'(z_2)} \right) |\hat{\alpha}|^2 \right) + \frac{\lambda^2 |\hat{\alpha}|^2}{2\pi i} \int_C dx W(x) e^{-ix\Lambda t} \frac{1}{(\bar{x} - x - N\lambda^2 W(x))(\bar{x} - x)},$$

where we have introduced

$$\hat{\alpha} = \sum_k \alpha_k.$$

We observe that for almost all initial states, the oscillations of the survival amplitude may not vanish with time. Indeed, if $\hat{\alpha} = 0$ then there is no decay at all and the survival probability is $p(t) = |A(t)|^2 = 1$. The survival probability decays to zero if and only if the initial state is $\alpha_k = e^{i\phi}/\sqrt{N}$ for any $k$ and some real $\phi$. For such states one has $|\hat{\alpha}|^2 = N$. For an arbitrary initial state, the decay is incomplete and

$$\lim_{t \to \infty} p(t) = \left( 1 - \frac{|\hat{\alpha}|^2}{N} \right)^2 \neq 0.$$ 

We note that the decay defined by Eq. (22) is oscillating [33]. The frequency of oscillations is

$$F_{osc} = \text{Re}(z_2 - \bar{x})\Lambda/2\pi \sim \lambda^2 N\Lambda.$$

The decay rate is

$$\gamma = \text{Im}(z_2)\Lambda \sim \lambda^2 N\bar{x}\Lambda,$$

which is considerably smaller than $F_{osc}$ for the physically motivated form factors. This oscillating decay is presented in Fig. 1. However, if the parameters of the quantum system are such that the system experiences complete decay (i.e. $\lim_{t \to \infty} p(t) = 0$) then there are no oscillations in the exponential term in (22). The integral term is negligible in the exponential era, therefore the system decays without oscillations, that is also shown in Fig. 1.

Let us now discuss the situation when the system is not completely degenerate, but it is close to the degenerate one. Namely, when we consider the case when one energy level differs from others: $x_k = \bar{x}$ for $k = 1 \ldots N-1$, $x_N = \bar{x} + \Delta$. If the form factors are identical, the determinant [17] takes the following form:

$$\det G^{-1}(x) = (\bar{x} - x)^{N-1}(x_N - x) - \lambda^2 W(x)((N-1)(\bar{x} - x)^{N-2}(x_N - x) + (\bar{x} - x)^{N-1}).$$
FIG. 1: The survival probability for the completely degenerated system. From above, the curves correspond to $N = 3$, $|\alpha|^2 = 0.2, 1.0, 3.0$. For comparison, the decay of the one-level system is also shown (the dashed line). The parameters of the model are selected for the hydrogen atom: $\Lambda = 8.498 \times 10^{18} \text{s}^{-1}$, $\omega = 1.55 \times 10^{16} \text{s}^{-1}$, $\lambda^2 = 6.43 \times 10^{-9}$.

It has three different roots: the root $z_1 = \bar{x}$ with the multiplicity $N - 2$ and the roots $z_{2,3}$ defined by

$$z_{2,3} = \bar{x} + \frac{\Delta - \lambda^2 N W(z) \mp \sqrt{(\Delta - \lambda^2 N W(z))^2 + 4\lambda^2 \Delta (N - 1) W(z)}}{2}.$$

Expression (23) gives the values for the roots for any $\Delta$ and $\lambda^2$. However, the limit when both these parameters go to zero is irregular. We will show that the pole and resolvent structure depends on the order, in which the limits are taken.

When $\lambda^2 \text{Re} W(\bar{x}) \ll \Delta$, we find

$$z_2 = \bar{x} - \lambda^2 (N - 1) W(\bar{x}) + O(\lambda^4), \quad z_3 = x_N - \lambda^2 W(x_N) + O(\lambda^4).$$

The root $z_2$ corresponds to the root (20) (the multiplicity is less by 1), and the root $z_3$ in this case is well-separated from $z_{1,2}$. Therefore the structure of the residues of the partial resolvent is composed of two independent blocks: one block coincides with Eq. (21) for the multiplicity $N - 1$, and the second block corresponds to the non-degenerated level (i.e. $\text{res}_{x=x_N} G_{NN}(x) = -1$, and all other residues are equal to 0).
Again, the root $z_2$ corresponds to the root $z_{20}$ and its imaginary part does not disappear when $\Delta \to 0$. The third root $z_3$ becomes real for identical energies, and the corresponding decay rate

$$\gamma_3 = -\frac{2\pi A \lambda^2 |\bar{\bar{x}}|^2 \Delta^2}{\lambda^2 W(\bar{\bar{x}})^2} \frac{N - 1}{N^3} \to 0 \quad \text{when} \quad \Delta \to 0.$$ 

The residues of the resolvent $G(x)$ at poles of the determinant are

$$\text{res}_{x=z_1} G(x) = \begin{pmatrix} -I^{(N-1)} + \frac{1}{N-1} P^{(N-1)} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{res}_{x=z_2} G(x) = \frac{1}{N} P^{(N)},$$

$$\text{res}_{x=z_3} G(x) = \left( \frac{1}{N(N-1)} P^{(N-1)} - \frac{1}{N-1} \right),$$

where $I^{(m)}$ is the $m \times m$ unit matrix and the $m \times m$ matrix $P^{(m)}$ is defined as $P^{(m)}_{ik} = 1$. The survival amplitude $A(t)$ is

$$A(t) \approx e^{-i\bar{\bar{x}}\Lambda t} \left( \sum_{k=1}^{N-1} |\alpha_k|^2 - \frac{1}{N-1} \right) \left( \sum_{k=1}^{N-1} |\alpha_k|^2 + \frac{1}{N} e^{-i(z_2-\bar{\bar{x}})\Lambda t} |\hat{\alpha}|^2 \right) + e^{-i(z_3-\bar{\bar{x}})\Lambda t} \left( \frac{1}{N-1} \sum_{k=1}^{N-1} |\alpha_k|^2 + |\alpha_N|^2 - \frac{1}{N} |\hat{\alpha}|^2 \right).$$

This result reproduces formula (22) when $\Delta \to 0$. In our assumptions, we have two time scales for exponential decay: a fast decay defined by $z_2$ with decay rate proportional to $\lambda^2 N$, and a slow decay defined by $z_3$ with decay rate proportional to $\Delta^2$. This slow decay is manifestation of non-degeneracy. The non-decaying subspaces of the system are now defined by two conditions: $|\hat{\alpha}| = 0$ and $\alpha_N = 0$.

### B. Different form factors

In the case of different form factors, $\varepsilon \neq 0$, one cannot find a general explicit expression for the matrix elements of the partial resolvent $G_{km}$ and its determinant. However, as the problem is the eigenvalue problem for a finite matrix, a general qualitative description is known (see e.g. theorem XII.2 in [41]). According to this theorem, for the system with identical energies $x_k \equiv \bar{\bar{x}}$ in the vicinity of $\bar{\bar{x}}$ there exist $N - 1$ roots of the determinant, additionally to the root $z_{20}$. Generally speaking, these roots give rise to exponentially decreasing terms in the survival amplitude $A(t)$. However, there may also exist real roots corresponding to bound states. These roots result in non-decaying behaviour of the survival probability. The existence and the number of such roots depend on specific properties of the form factors.

Having in mind this qualitative description we shall analyze the poles structure of $G(x)$ by perturbation expansion. In the first non-vanishing order of the perturbation expansion with respect to $\varepsilon$, we have for the roots of the determinant the following equation:

$$\det G_0^{-1}(x) - \varepsilon \lambda^2 \det G_0^{-1}(x) \tr (G_0(x) R(x)) = 0,$$

where the resolvent is defined by Eq. (16): $G_0(x) \equiv G|_{\varepsilon=0}(x)$. This equation gives us the following roots: $N - 2$ roots of the type of $z_1 = \bar{\bar{x}}$, the root $z_{20}$ and the new root $z_3$:

$$z_3 = \bar{\bar{x}} - \varepsilon^2 \lambda^2 \tr \left( (I - \frac{1}{N} P) Q(\bar{\bar{x}}) \right) + O(\varepsilon^3 \lambda^2),$$

where the $N \times N$ matrix $Q(x)$ is defined as

$$Q_{ik}(x) = \int dx' \frac{q_k(x')q_k(x')}{(x' - x + i0)}.$$
We note that the perturbation series for $z_3$ starts from the $\varepsilon^2$-term (see also a unified framework for noiseless quantum subsystems in [19]), while for one non-degenerated level $\Re(z_3 - \bar{x}) \sim \Delta$ and $\Im(z_3) \sim \Delta^2$. In the present case, the imaginary part of $z_3$ can be calculated as

$$\text{Im}(z_3) = 2\pi\varepsilon^2\lambda^2 \left( \sum_k q_k^2(\bar{x}) - \frac{1}{N} \left( \sum_k q_k(\bar{x}) \right)^2 \right) + O(\varepsilon^3\lambda^2).$$

One can see that $\text{Im}(z_3)$ can be equal to zero even for different form factors $q_k(x)$. In this case, the decay will be slower, its width will be proportional to $\varepsilon^3\lambda^2$.

When one considers next terms of the perturbation series with respect to $\varepsilon$, more resonances are split from the energy level $\bar{x}$. Generally speaking, each additional term in the perturbation series gives an additional resonance so $\text{Im}(z_k) \sim \varepsilon^{k-1}$ for $k = 3 \ldots N$, and all the roots become complex when the complete series is taken into account.

**IV. ZENO EFFECT AND ZENO TIME**

In order to estimate the short-time behaviour for the system [2], we will use the Taylor expansion of the survival probability. We shall assume here the existence of all necessary matrix elements, and denote the expectation values as $\langle \cdot \rangle = \langle \Phi | \cdot | \Phi \rangle$. Then, following the results of [41], we find

$$p(t) = \langle e^{-iHt} \rangle = 1 - t^2 \left( \langle H^2 \rangle - \langle H \rangle^2 \right) + t^4 \left( \frac{1}{4} \langle H^4 \rangle - \frac{1}{12} \langle H^2 \rangle^2 - \frac{1}{3} \langle H \rangle \langle H^3 \rangle \right) + O(t^6) = 1 - \frac{t^2}{r_a^2} + \frac{t^4}{r_b^4} + O(t^6).$$

The expressions for the times $t_a$ and $t_b$ can be deduced using the special structure of the potential $V$ [2]:

$$\frac{1}{\lambda^2 r_a^2} = \sum_k |\alpha_k|^2 x_k^2 - \left( \sum_k |\alpha_k|^2 x_k \right)^2 + \lambda^2 R_1,$$

$$\frac{1}{\lambda^4 r_b^4} = \frac{1}{4} \left( \sum_k |\alpha_k|^2 x_k^2 \right)^2 + \frac{1}{12} \sum_k |\alpha_k|^2 x_k^4 \sum_k |\alpha_k|^2 x_k - \frac{1}{3} \sum_k |\alpha_k|^2 x_k \sum_k |\alpha_k|^2 x_k^3$$

$$+ \lambda^2 \left( \frac{1}{12} R_1 \sum_k |\alpha_k|^2 x_k^2 + \frac{1}{12} R_3 - \frac{1}{3} R_2 \sum_k |\alpha_k|^2 x_k \right) + \lambda^4 \left( \frac{1}{12} R_4 + \frac{1}{4} R_1^2 \right).$$

Here

$$R_1 = \sum_{ik} \alpha_i \alpha_k^* F_{ik}^0,$$

$$R_2 = \sum_{ik} \alpha_i \alpha_k^* \left( (x_i + x_k) F_{ik}^0 + F_{ik}^1 \right),$$

$$R_3 = \sum_{ik} \alpha_i \alpha_k^* \left( (x_i^2 + x_i x_k + x_k^2) F_{ik}^0 + (x_i + x_k) F_{ik}^1 + F_{ik}^2 \right),$$

$$R_4 = \sum_{ik} \alpha_i \alpha_k^* (|F_{ik}|^2)_{im},$$

where

$$F_{ik}^p = \int_0^\infty dx x^p f_i(x) f_k(x).$$

The probability that the state $|\Phi\rangle$ after $M$ equally spaced measurements during the time interval $[0, T]$ has not decayed, is $p_M(T) = p^M(T/M)$ [11]. We are interested in the limit $M \to \infty$ or, equally, the time interval between the measurements $\tau = T/M$ goes to zero:

$$\lim_{\tau \to 0} p_M(T) = \lim_{\tau \to 0} p(\tau)^{T/\tau} = \begin{cases} 1, & \text{when } p'(0) = 0, \\ e^{-cT}, & \text{when } p'(0) = -c, \\ 0, & \text{when } p'(0) = -\infty. \end{cases}$$
This limit corresponds to so-called continuously ongoing measurements during the entire time interval $[0, T]$. Obviously, this is an idealization. In practice we have a manifestation of the Zeno effect if the probability $p_M(T)$ increases as the time interval $\tau$ between measurements decreases. Formula (37) may be accepted as an approximation only for very short time intervals $\tau$.

As one refers in discussions about the Zeno effect on the Taylor expansion (30) of survival probability for short times, and specifically on the second term, we shall define the Zeno time $T_z$ as corresponding to the region where the second term dominates. Hence, one way is to define the Zeno time $t_Z$ (see paper [11]) as a boundary where the second and third terms have the same amplitude:

$$\frac{t_Z^2}{t_a} = \frac{t_Z^2}{t_b}, \quad \text{so} \quad t_Z = t_b^2/t_a.$$

(38)

Another way to discuss the Zeno and the anti-Zeno effects is the variable decay rate $\gamma(t)$ [37]. Namely, we represent the survival probability as $p(t) = e^{-2\gamma(t)t}$. Then we find

$$\gamma(t) = -\frac{1}{2t} \log p(t).$$

(39)

In terms of the $\gamma(t)$, we find that $p_M(T) = e^{-\gamma(T/M)T}$. Therefore, we have the Zeno region (deceleration of the decay) when $\gamma(\tau) < \gamma(T)$ and the anti-Zeno region (acceleration of the decay) when $\gamma(\tau) > \gamma(T)$. If the expansion (30) is valid, then $\gamma(\tau) \approx \tau/(2t_a^2)$ for $\tau \lesssim t_Z$, and we always have the Zeno region for short times. Then we can define the Zeno time $t_Z$ as the boundary between the Zeno and anti-Zeno regions: $\gamma(t_Z) = \gamma(T)$. Again, if the expansion (30) is valid and $T$ is within the exponential era, then $t_Z \approx 2t_a^2/t_d$, where $t_d$ is the time interval when the decay is almost exponential (the so-called “exponential era”).

Even if expansion (30) is not valid, we can still apply the same idea and use the short-time expansion of the survival probability. For example, for the form factor $\frac{t_Z}{t_a}$ one finds in the one-level Friedrichs model [11] that $p(t) \approx 1-(t/t_a)^{1.5}$ for small $t$, and $t_Z \approx 4t_a^3/t_d^2$. We compare two definitions of the Zeno time in Table 1 for different form factors in the frame of the one-level Friedrichs model [11]. One can see that $t_Z \approx Ct_Z \ll t_Z$ for physically motivated parameters. Hence the Zeno time $t_Z$ is even shorter than the previously estimated time $t_Z$ [11]. The time $t_Z$ gives an interesting connection between the decay time and the energy uncertainty of the initial state (as $t_a^{-2} = \langle H^2 \rangle - \langle H \rangle^2$ when these matrix elements exist).

Expressions (31) and (32) include many different parameters and can hardly be analyzed in the general case. We consider here three specific representative examples for the physically motivated weak coupling model [11] with

$$\lambda^2 \ll 1 \quad \text{and} \quad x_k = \omega_k/\Lambda \ll 1.$$

(40)

**Example A. The decay of one level**

We consider here the multilevel Friedrichs model. We do not introduce any assumption on the energy levels and the form factors. The initial condition was chosen so that the only one level $l$ is occupied: $\alpha_l = 1$, $\alpha_k = 0$ for $k \neq l$. Then expressions (31) and (32) become

$$\frac{1}{t_a} = \lambda^2 \Lambda^2 F_{ll}^0,$$

$$\frac{1}{t_b} = \lambda^2 \Lambda^4 \left( \frac{1}{12} x_l^2 F_{ll}^0 - \frac{1}{6} x_l F_{ll}^0 + \frac{1}{12} F_{ll}^0 \right) + \lambda^4 \Lambda^4 \left( \frac{(F_{ll}^0)^2}{4} \right) + \frac{1}{12} ((F_{ll}^0)^2).$$

It is not surprising that the expressions for $t_a$ and $t_b$ practically coincide with those for the one-level Friedrichs model [11]. Therefore, the Zeno time $t_Z$ is also the same:

$$t_Z \sim \frac{1}{\Lambda} \sqrt{\frac{12 F_{ll}^0}{F_{ll}^2}}.$$
The only difference in the last term for \( t_b^4 (\sum_{m} F_{lm}^0 F_{ml}^0) \) instead of \((F^0)^2\) for one-level model does not influence the results for a number of levels \( N \ll 1/\lambda^2\) and \( \lambda^2 \ll 1 \).

The Zeno time \( \tau_Z \) is more sensitive to the structure of the energy levels than \( t_z \). If the non-degenerated energy level \( x_1 \) is well separated from the others, then the survival probability in the exponential era coincides with the survival probability for the one-level Friedrichs model. In this case, the decay time \( t_d = 1/\text{Im}z_i \) and \( \tau_Z = t_a^2/t_d \) as for the one-level model. As we shall show later, the situation is quite different when the energy level is degenerated.

**Example B. The completely degenerate case**

In this case, all frequencies are identical, \( x_k = \bar{x} \) for any \( k \). Expressions (31,32) become

\[
\frac{1}{t_a^2} = \lambda^2 A^2 (F^0), \quad \frac{1}{t_b^4} = \lambda^2 A^4 \left( \frac{1}{12} \bar{x}^2 (F^0) - \frac{1}{6} \bar{x} (F^1) + \frac{1}{12} (F^2) \right) + \lambda^2 A^4 \left( \frac{(F^0)^2}{4} + \frac{1}{12} (F^0)^2 \right),
\]

where

\[
\langle F^p \rangle = \sum_{ik} \alpha_i \alpha_k^* F_{ik}^p.
\]

The matrices \( F^p \) are the Gramm matrices, which have the following property (32):

\[
\langle F^p \rangle \geq 0, \quad \langle F^p \rangle = 0 \quad \text{for some } |\Phi\rangle, \quad \text{iff the form factors } f_k(x) \text{ are linearly dependent: } \sum_k m_k f_k(x) \equiv 0.
\]

When all averages \( \langle \cdot \rangle \) are separated from zero, e.g. the form factors \( f_k(x) \) are linearly independent, it resembles Example A. The Zeno time is of the same order of magnitude

\[
t_Z \sim \frac{1}{\lambda} \sqrt{\frac{12 (F^0)}{(F^2)}}.
\]

However, in the case of linearly dependent form factors, the situation may change for special initial conditions \( |\Phi\rangle \). Assuming for the sake of simplicity that the form factors are identical, \( \bar{x} \ll 1, \lambda^2 \ll 1 \), we find that

\[
\frac{1}{t_a^2} = \lambda^2 A^2 F^0 |\hat{\alpha}|^2, \quad \frac{1}{t_b^4} \approx \frac{\lambda^2 A^4}{12} F^2 |\hat{\alpha}|^2, \quad t_Z \approx \frac{1}{\lambda} \sqrt{\frac{12 F^0}{F^2}}.
\]

We can now see that the Zeno time \( t_Z \) is independent of the initial conditions. However, both \( t_a \) and \( t_b \) increase to the infinity when \( \hat{\alpha} \) goes to zero. This means that the state \( |\Phi\rangle \) with \( \hat{\alpha} = 0 \) does not decay, while the relation between \( t_a \) and \( t_b \) is unchanged. The same is true when the form factors are not identical but linearly dependent: there also exists a non-decaying state \( |\Phi\rangle \).

The Zeno time \( \tau_Z \) essentially depends on the time of the observation \( T \). If \( T \) is considerably bigger than the decay time \( t_d \), then \( \gamma(T) \approx -\log (1 - |\hat{\alpha}|^2/N)/T \), and the Zeno time

\[
\tau_Z \approx \frac{-2 \log (1 - |\hat{\alpha}|^2/N)}{\lambda^2 A^2 F^0 |\hat{\alpha}|^2 T}.
\]

If the observation time \( T \sim Ct_d \) then for small \( |\hat{\alpha}|^2 \) we find \( \tau_Z \sim \bar{x}/A \). Hence both times \( t_Z \) and \( \tau_Z \) have for small \( |\hat{\alpha}|^2 \) finite values while the initial state \( |\Phi\rangle \) does not decay for \( |\hat{\alpha}|^2 = 0 \).

Therefore, both Zeno times \( t_Z \) and \( \tau_Z \) may not be relevant for the \( N \)-level model and should be modified. We would like to note that this problem does not appear for the one-level model analyzed in [11].

**Example C. \( N \)-level model with one different level**

In this case, the energy of one level differs from the others: \( x_k = \bar{x}, k = 1 \ldots N-1, x_N = \bar{x} + \Delta, \) and the form factors are identical: \( f_k(x) = f(x) \). Then we have for the time \( t_a \)

\[
\frac{1}{t_a^2} = \lambda^2 A^2 \left( |\alpha_N|^2 - |\alpha_N|^4 \right) + \lambda^2 A^2 F^0 |\hat{\alpha}|^2.
\]

The time \( t_b \) is first analyzed under conditions (40):

\[
\frac{1}{t_b^4} \approx \frac{\lambda^4 A^4}{12} \left( |\alpha_N|^2 - |\alpha_N|^4 \right) + \frac{\lambda^4 A^4}{12} F^2 |\hat{\alpha}|^2.
\]
Both these times and the Zeno times depend on the initial vector. One can easily see that if $|\hat{\alpha}|^2 \neq 0$, the Zeno time $t_Z$ has a maximum as the function of the energy difference $\Delta$. We can estimate the position and the value of this maximum:

$$t_Z(\Delta) \approx \sqrt{\lambda} \frac{36(|\alpha_N|^2 - |\alpha_N|^4)}{F^2 |\hat{\alpha}|^2} \frac{1}{\Delta \sqrt{\lambda}}.$$ (43)

We observe that the Zeno time is increased by the factor $\sim 1/\sqrt{\lambda}$ with respect to the one-level model. We note that even if we consider the exact expression for $t_b$, the same increase of the Zeno time takes place. We illustrate this increase on Fig. 2a using exact expression (32) for the time $t_b$. We use the parameters of the model associated with the hydrogen atom. One can see that for this system the maximum (43) cannot be reached. However, for small $\Delta$ we can find

$$t_Z(\Delta) \approx t_Z(0) \left(1 + \frac{|\alpha_N|^2 - |\alpha_N|^4}{2 \Lambda^2 F^0 |\hat{\alpha}|^2} \Delta^2 \right) > t_Z(0),$$ (44)

therefore some increase of the Zeno time always takes place.

![Graph](image)

**FIG. 2:** The Zeno time $t_Z$ (Fig. 2a) and the Zeno time $\tau_Z$ (Fig. 2b) as a function of the energy difference $\Delta$ for two-level Friedriths model. From above, the curves correspond to the initial condition $|\Phi\rangle$: (2a) $(\alpha_1, \alpha_2) = (1, -0.6), (1, 1), (1, 0.1)$ and $(1, 0$); (2b) $(\alpha_1, \alpha_2) = (1, 0.05), (1, 0.55), (1, 0), \text{and} (1, -0.6)$. The observation time $T = 5/\text{Im}(z_2 - \bar{x})$. The parameters of the model are the same as in Fig. 1.

For the Zeno time $\tau_Z$, the dependence $\tau_Z$ on $\Delta$ can hardly be analyzed analytically. Nonetheless, for big values of $\Delta$ and $N = 2$ we find

$$\tau_Z \approx -\Lambda \text{Im}(z_2 - \bar{x}) - \log (|\hat{\alpha}|^2/2)/T,$$

$$\frac{\Lambda^2 \Delta^2}{\Lambda^2 \Delta^2 (|\alpha_2|^2 - |\alpha_2|^4) + \lambda^2 \Lambda^2 F^0 |\hat{\alpha}|^2},$$

hence $\tau_Z$ decreases when $\Delta \to \infty$. We plot in Fig. 2b the time $\tau_Z$ for the same model as for the time $t_Z$. We can see that the ratio $t_Z/\tau_Z \approx \bar{x}$ is about the same as for one-level model. The increase of the Zeno time also takes place.
except for the cases when initial vector $|\Phi\rangle$ is close to the state which decays without oscillations, i.e. $|\hat{\alpha}|^2 = 2$. This differs from the behaviour of the Zeno time $t_Z$ where the only states, for which the increase does not take place, are ones with one populated level: $|\Phi\rangle = (1, 0)$ and $|\Phi\rangle = (0, 1)$. This difference is due to the fact that while $t_Z$ is defined only by the short time expansion, the time $\tau_Z$ depends on the behaviour of the system in the exponential era. This dependence is also responsible for the oscillations seen in Fig. 2b.

V. ANTI-ZENO EFFECT

In the previous section we have seen that for very short times $t \lesssim t_Z(\tau_Z)$, the Zeno effect takes place. We will not discuss here the transition region $t \sim t_Z(\tau_Z)$, where the Zeno region changes to the anti-Zeno one, but concentrate instead on the longer times $t \gtrsim t_Z(\tau_Z)$. We neglect the integral term in Eq. (8), as its magnitude is proportional to $\lambda^2$, and is much smaller than effects which we discuss. We shall analyze the same examples as in the previous section.

Example A. The decay of one level

The only level $l$ is initially occupied: $\alpha_l = 1$, $\alpha_k = 0$ for $k \neq l$. The survival probability is here

$$p(t) \sim |\sum_j r_{j l}^\dagger e^{-i\Lambda x_j t}|^2,$$

and it formally coincides with the survival probability for the one-level Friedrichs model. However, the partial resolvent $G(x)$ may be different, so the pole structure and the residues may also be different. If the non-degenerated energy level $x_l$ is well separated from the others, then the survival probability coincides with one for one-level Friedrichs model. In this case, the anti-Zeno region, defined by the value of $|r_{kl}^\dagger|$ has already been analyzed [11, 37]. If the energy level is degenerate, then the situation is different as it will be discussed in Examples B and C.

Example B. The completely degenerate case

All energy levels are identical: $x_k = \bar{x}$ for any $k$. Starting from Eq. (22), we find for the survival probability:

$$p(t) \approx \left|1 - \frac{\hat{\alpha}^2}{N} + \frac{\hat{\alpha}^2}{N} e^{-i(\bar{x} - \bar{x})t}\right|^2 \approx \left|1 - \frac{\hat{\alpha}^2}{N} \right|^2 e^{-t/td} e^{i\Lambda t/td},$$

where $s = \text{Re}W(\bar{x})/\text{Im}W(\bar{x})$. In the last expression we assume that $|r_{kl}^\dagger| = 1$. Then the decay rate $\gamma(t)$ for short time can be expressed as

$$\gamma(t) = |\hat{\alpha}|^2 \left(\Lambda^2 \text{Im}W(\bar{x}) + \frac{1}{2} (\Lambda^2 \lambda^4 (\text{Re}^2 W(\bar{x}) - \text{Im}^2 W(\bar{x}))) \right) = \frac{|\hat{\alpha}|^2}{N} \left(\frac{1}{t_d} - \frac{t}{2t_d^2} (1 - s^2)\right).$$

Expansion (46) can be used for the time interval $\tau_Z \lesssim t \ll t_d$. The decay rate (46) should be compared with the decay rate for the exponential era:

$$\gamma(t) \sim -\log(1 - |\hat{\alpha}|^2/N)/t \quad \text{for} \quad |\hat{\alpha}|^2 \neq N.$$

If $|\hat{\alpha}|^2 = N$, the system completely decays without oscillations with asymptotic decay rate $\gamma = N\Lambda^2\text{Im}W(\bar{x})$ and mimics the one-level model. Otherwise, we always have the anti-Zeno region as the variable decay rate within the exponential era goes to zero when the time of the observation increases.

The duration of the anti-Zeno region depends both on the initial vector $|\Phi\rangle$ and the level of degeneracy $N$. If the initial vector $|\Phi\rangle$ is such that $|\hat{\alpha}|^2$ is independent of the degeneracy $N$, the decay rate for short times is not sensitive to $N$, and the beginning of the anti-Zeno region is rather stable with respect to $N$, see Fig. 3. However, the duration strongly depends on $N$. We should note that in this case system decays to different survival probabilities $p_\infty = (1 - |\hat{\alpha}|^2/N)^2$. If we fix the final amplitude $p_\infty$, then $|\hat{\alpha}|^2 = N(1 - \sqrt{p_\infty})$, and the anti-Zeno region is independent of $N$ in the units of the decay time $t_d$, see Eq. (46). Hence, the duration and the position of the anti-Zeno region is proportional to $N$.

Example C. N-level model with one different level

In this case, all energy levels are identical except one: $x_k = \bar{x}$, $k = 1 \ldots N - 1$; $x_N = \bar{x} + \Delta$, and all form factors are identical: $f_k(x) = f(x)$. The survival probability can be calculated from Eq. (27). We consider the simplest non-degenerate case with $N = 2$. The more general situation $N \geq 3$ does not add anything essentially new. For the survival amplitude we have

$$p(t) = \frac{1}{2} e^{-i(\bar{x} - \bar{x})t} |\hat{\alpha}|^2 + e^{-i(z_2 - \bar{x})t} (1 - \frac{1}{2} |\hat{\alpha}|^2)^2.$$

(47)
The decay rate for short times is

$$\gamma(t) = \Lambda C_1 - \frac{\Lambda^2 |\hat{\alpha}|^2 C_2}{4} t + O(t^2), \quad (48)$$

where

$$C_1 = -\frac{|\hat{\alpha}|^2}{2} \text{Im} \Delta z_2 - (1 - \frac{|\hat{\alpha}|^2}{2}) \text{Im} \Delta z_3 > 0, \quad C_2 = (1 + \frac{|\hat{\alpha}|^2}{2})(\text{Im}(z_2 - z_3))^2 - (1 - \frac{|\hat{\alpha}|^2}{2})(\text{Re}(z_2 - z_3))^2.$$ 

For the case $\Delta \ll \lambda^2$ and for long times we find $\gamma(t) \sim -\text{Im} \Delta z_3$. Therefore, for the initial vector $|\Phi\rangle$ such that $|\hat{\alpha}|^2 \neq 0$, the decay rate for short times is bigger than $-\text{Im} \Delta z_3$. In this case a wide stable anti-Zeno region of the size $\sim \text{Im} \Delta z_2$ appears (see Fig. 4). When the $\Delta$ becomes bigger, the position and the size of the anti-Zeno region considerably changes.

The origin of the anti-Zeno behaviour for Example A and Examples B, C is very different. For Example A, as well for the one-level Friedrichs model, the anti-Zeno behaviour is due to the fact the the residues $r_{ll}^j$ are not equal to 1: $|r_{ll}^j - 1| \sim \lambda^2$. While this is also true for Examples B and C, this difference is much smaller than the non-exponentiality due to the degeneracy (or nearly degeneracy) of the energy levels. This non-exponentiality is the origin of the anti-Zeno behaviour in Examples B and C.

VI. CONCLUDING REMARKS

We have considered the temporal behaviour of the survival probability of excited states in the $N$-level Friedrichs model for degenerate and nearly degenerate situations. For various initial conditions we have determined the duration

FIG. 3: The probability $p_M(\tau)$ for the completely degenerated system as a function of the duration $\tau$ between measurements. From above, the curves correspond to degeneracy $N = 20, 10, 4, 3$, and $|\hat{\alpha}|^2 = 0.9$. The observation time $T = 3/\text{Im}(z_2 - \bar{x})$. The parameters of the model are the same as in Fig. 1.
FIG. 4: The probability $p_M(t)$ for the model with two close levels as a function of the duration $\tau$ between measurements. From above, the curves correspond to energy splitting $\Delta = 10^{-12}, 10^{-9}, 10^{-8}$, and $|\hat{\alpha}|^2 = 0.05$. The observation time $T = 2/\text{Im}(z_2 - \bar{x})$. The parameters of the model are the same as in Fig. 1.

of the Zeno and anti-Zeno eras and analyzed the behaviour on the intermediate exponential era where we have found a rich variety of behaviour from pure exponential decay to exponentially decaying oscillations. For initial states belonging to the nondecaying subspaces, these oscillations stabilize without decaying to zero. The experimental implementation of this result should be exploited as a mean of suppression of decoherence.

In the short-time scale, our analysis has shown also a possibility for considerable slowing down of the decay due to the Zeno effect in the nearly degenerate system for a special class of initial conditions. If these systems and conditions are realizable experimentally (in atoms, ions, quantum dots etc.), one has a new possibility for efficient suppression of decoherence in quantum computation and communication using the Zeno effect. We have analyzed and compared two different definitions of the Zeno time. This allows for a better estimation of the time intervals where the Zeno effect may be really helpful in quantum computations.

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