Selection criteria of the addendum modification coefficients of spur gear pairs with smaller number of pinion teeth

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Abstract. A design procedure for the optimum distribution of the addendum modification coefficients of spur gear pairs with smaller number of pinion teeth is presented for the case of a fixed centred distance. The geometrical, kinematics and load capacity criteria are considered in the design analysis. The geometric and kinematics criteria are used to prevent the negative phenomena of the generating and engagement processes. The relation between the contact pressure of meshing teeth and specific sliding are analysed in relation with addendum modification coefficients. A dynamic model is developed to simulate the load sharing characteristics through a mesh cycle. The specific phenomenon of contact tooth pairs alternation during mesh cycle is integrated in this dynamic load modelling. A comparative study is included, which shows the effects of the distribution factor of the addendum modification coefficients on the contact surface characteristics of the gear pairs.

1. Introduction
The gear pairs with smaller number of pinion teeth came into use for the possibility of a larger gear ratio. There have been several researches on the specific aspects of gear with smaller number of teeth, such as geometric relationships [1, 4, 9] load capacity [7], and manufacturing modelling [6]. Roth and Kollenrott [9] have significantly contributed to the design and manufacturing of gears with small number of teeth and involute profile for tooth module \( m_n < 1[mm] \). Ishibashi et. al. [7] investigated the design, load capacity and efficiency of helical gears with two to four pinion teeth for tooth module \( m_n > 1[mm] \) and tooth geometry according to the standard rack. Chen and Tsay [6] developed a mathematical model of the modified helical gears with small number of teeth in order to improve their manufacture. Other reported studies focused on the analysis of the specific geometric relationships of helical gears [4] and spur gears [1] with small number of pinion teeth.

Addendum modification coefficients of the pinion and driven gear have the advantage of improving the load meshing performances. Reported studies [2, 5, 8] investigated different algorithms for the optimal selection of the addendum modification coefficients by considering only the principle of equalized sliding coefficients [8] or the influence of these coefficients on the efficiency of gear pairs [5]. Arikan [2] analyzed the maximum possible contact ratios for spur gears with small number of teeth in relation with the addendum modification coefficients.

The problem of the selection of these coefficients is an optimizing one and geometrical, kinematics and contact load criteria are needed to be taken into account.
In the case of gear pairs with smaller number of pinion teeth there are some constraints due to the negative phenomena of generating process and specific features of engagement. A supplementary compulsion occurs in the case of a fixed center distance, when the sum of the addendum modification coefficients results as a constant.

In this study, by considering the spur gear pairs with a constant gear ratio and center distance, the specific gear performance variables are calculated in order to select the addendum modification coefficients for the pinion and the gear by using geometrical and contact load capacity criteria. In the analysis of these spur gears, the involute gear teeth are geometrically defined by the standard basic rack.

2. Geometrical Design Criterion
The optimum employment of cylindrical gears with small number of teeth and involute profile requires to solve some problems of geometrical computations. Thus, at the generating process, the decrease of the number of teeth may be accompanied by the negative phenomena: tooth undercutting and tooth top sharpening. These phenomena reduces the transverse contact ratio and the load capacity.

The geometrical criteria are used to prevent the negative phenomena of the generating and engagement processes of helical gears with small number of teeth. Thus, in the design stage, the following condition must be accomplished

\[ x_{\min} \leq x \leq x_{\max} \]  

where \( x_{\min} \) represents the minimum value of the addendum modification coefficient to avoid tooth undercutting and \( x_{\max} \) corresponds to the maximum value of the addendum modification coefficient to avoid tooth top sharpening.

In the generating process of the involute flank profile it is possible to appear the involute undercutting, especially in the case of the gears with smaller number of teeth. The undercutting of gear teeth reduces the load capacity of the tooth and transverse contact ratio.

For design procedure, the addendum modification coefficient \( x_{\min} \) to avoid undercutting results in the following form

\[ x_{\min} = h_{ao}^{*} - \frac{z \cdot \sin \alpha_{o}^{2}}{2} - \rho_{o}^{*}(1 - \sin \alpha_{o}) \]  

where \( z \) represents teeth number, \( h_{ao}^{*} \) - addendum coefficient of basic rack, \( \rho_{o}^{*} \) - radii of rack – type tool tip, \( \rho_{o} = \rho_{o} / m \), \( m \) - tooth module.

The condition to avoid the sharp tooth tip can be expressed as

\[ s_{a} \geq \delta \cdot m \]  

where \( s_{a} \) is the top land thickness. The recommended value \( \delta = 0.25 \) for quenched and tempered steel gears is considered in this investigation. At boundary, the above condition take the form

\[ F(x, \Delta_{a}) = s_{a} - \delta \cdot m \]  

where

\[ s_{a} = d_{a}(\text{inv} \alpha_{o} - \text{inv} \alpha_{a} + \frac{0.5\pi + 2 \tan \alpha_{o}}{z}) \]  

\[ d_{a} = m \cdot z + 2m(h_{ao}^{*} + x - \Delta_{a}) \]
For helical gears with a small number of teeth, when \( x \geq x_{\text{min}} \) but Eq. (3) is not carried out, it is necessary to reduce the whole addendum with a value \( \Delta_m \) to prevent tooth top sharpening. In the design stage it is considered a value \( x \geq x_{\text{min}} \) to prevent involute tooth undercutting and the coefficient \( \Delta_a \) is computed on basis of Eq.(4) by using a numerical method. The outside diameter \( d_a \) of the pinion can be calculated from Eq.(6).

Equations (4) and (6) permit to calculate the maximum value of the addendum modification coefficient \( x_{\text{max}} \) or the coefficient \( \Delta_a \) to avoid tooth top sharpening.

From the viewpoint point of the engagement process, the contact ratio \( \varepsilon \) must be always bigger than the unity, in order to preserve the continuity of engagement gears.

3. The Criterion of the Contact Surface Durability

The tooth surface durability depends basically on the tooth contact fatigue as pitting failure in the case of medium-hardness gears. This contact failure occurs on the region of contact where the contact pressure and the relative negative sliding are highest. Therefore, these parameters are analyzed in order to obtain the favourable condition for surface fatigue life of contacting teeth.

The Hertz contact pressure between the meshing teeth of the involute spur gear pair can be expressed as:

\[
p_H = Z_E \left[ W_n \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \right]^{1/2}
\]

(7)

where \( W_n \) is the normal load per unit length and \( \rho \) represents the radius of curvature at the point of contact. The subscript 1 is for pinion and 2 is for gear.

The elasticity factor \( Z_E \) is defined as

\[
Z_E = \sqrt{\frac{1}{1 - \nu_1^2 E_1 + \frac{1 - \nu_2^2}{E_2}}}
\]

(8)

where \( E \) represents the Young’s modul of elasticity and gear and \( \nu \) is the Poisson’s ratio.

Eq. (7) is valid at any point during the contact cycle and the contact pressure can be computed if the values of \( \rho_1 \) and \( \rho_2 \) are known.

The specific sliding for each point and each profile of contacting teeth are defined as follows:

\[
\xi_1 = \frac{\omega_1 \rho_1 - \omega_2 \rho_2}{\omega_1 \rho_1}
\]

\[
\xi_2 = \frac{\omega_2 \rho_2 - \omega_1 \rho_1}{\omega_2 \rho_2}
\]

(9)

where \( \omega_1, \omega_2 \) represent the angular velocity of the pinion and gear, respectively.

4. Dynamic Load Sharing

Dynamic load sharing becomes important to improve the evaluation accuracy of the tooth contact stress for a mesh cycle.

The dynamic model for a gear pair in mesh is shown in figure 1. The gear mesh interface is represented by the time-varying mesh stiffness \( k_i(t) \) and the viscous damper \( c \). The profile deviations should be included in the composite tooth profile error \( e_i(t) \).

The differential equations of motion can be expressed as:

\[
J_1 \ddot{\theta}_1 + c(\dot{\theta}_1 r_{b1} - \dot{\theta}_2 r_{b2})r_{b1} + k_i(t)(\theta_1 r_{b1} - \theta_2 r_{b2} + e_i(t))r_{b1} = T_1
\]

(10)
\[ J_2 \dot{\theta}_2 + c(\dot{\theta}_1 r_{b1} - \dot{\theta}_2 r_{b2}) r_{b2} - k_i(t)(\dot{\theta}_1 r_{b1} - \dot{\theta}_2 r_{b2} + e_i(t)) r_{b1} = -T_2 \] (11)

\[ F_{di}(t) = k_i(t)(r_{b1} \dot{\theta}_1 - r_{b2} \dot{\theta}_2 + e_i(t)) + c(r_{b1} \dot{\theta}_1 - r_{b2} \dot{\theta}_2) \] (12)

Figure 1. Dynamic model of a spur gear pair.

where \( \theta_1, \theta_2 \) are the rotation angle of the pinion and the driven gear, respectively. \( J_1 \) and \( J_2 \) are the mass moments of inertia of the gears. \( T_1 \) and \( T_2 \) denote the external torques applied on the gear system and \( r_{b1}, r_{b2} \) are the base circle radii of the gears.

The dynamic normal load between two meshing gear teeth is expressed as

The time-varying mesh stiffness is mainly caused by the following factors: (i) the variation of the single mesh stiffness along the line of action; (ii) the fluctuation of the total number of total pairs in contact during the engagement cycle. The effect of bending, shear and Hertzian contact deformation is taking into account in the analytical method to calculate the tooth deformation [3].

5. Analysis

For the given centre distance and gear ratio results the sum of addendum modification coefficients as \( x_x = x_1 + x_2 \). The distribution of the sum \( x_n \) between pinion and driven gear is influenced by the geometrical conditions, specific sliding and contact stress. The following performance characteristics should be taken into consideration for the selection of the addendum modification coefficients of the pinion and driven gear:

a) avoiding negative phenomena of generation and engagement process;
b) minimizing the design contact stress under dynamic condition;
c) ensuring the least values of negative specific slidings at the contacting zone with larger value of contact stress.

Specific characteristics of the analyzed gear pairs are shown in Table 1. Additionally, the design parameters are chosen as: \( \rho_o^* = 0.38 \); nominal value \( F_n/b = 80 \) [N/mm]; pinion speed, \( n_1 = 1800 \) rpm.
Table 1. Specifications of the helical gear pairs.

| Gear pairs | GP1 | GP2 | GP3 | GP4 |
|------------|-----|-----|-----|-----|
| Number of teeth \( z_1 / z_2 \) | 8 / 52 | 8 / 52 | 12 / 78 | 12 / 78 |
| Tooth module, \( m \) [mm] | 3 | 3 | 2 | 2 |
| Addendum modification coefficient, \( x_1 \) | 0.532* | 0.65 | 0.298* | 0.65 |
| Addendum modification coefficient, \( x_2 \) | -0.532 | -0.65 | -0.298 | -0.65 |
| Coefficient \( A_a \) | 0.13 | 0.19 | 0.0 | 0.085 |
| Face width, \( b \) [mm] | 20 | 20 | 20 | 20 |
| Transverse contact ratio, \( \varepsilon_\alpha \) | 1.287 | 1.209 | 1.529 | 1.349 |
| Section \( AB \),[mm] | 1.53 | 2.06 | 2.54 | 1.84 |
| Section \( AC \),[mm] | 3.86 | 1.98 | 3.86 | 2.93 |
| Section \( AE \),[mm] | 9.03 | 7.96 | 11.39 | 10.74 |
| Center distance \( a \) [mm] | 90 | 90 | 90 | 90 |
| Material | steel/steel | steel/steel | steel/steel | steel/steel |

A computer program was developed for simulating the dynamic characteristics of spur gear pairs. The equations of motion are solved by the fourth-order Runge-Kutta method. Computer analysis of dynamic characteristics includes gear pairs with different combination of the addendum modification coefficients. The dynamic factor \( c_{dl} \) of the single tooth pair is defined the ratio of the single dynamic load \( F_{dl} \) to the static load \( F_n \). Static load is the steady state force resulting from driving torque. The static factor \( c_s \) of the single tooth pair 1 is defined the ratio of the single static load \( F_s \) to the static load \( F_n \). If the effect of tooth errors is neglected, the tooth load sharing ratio \( c_{dl} \) depends on the mesh stiffness only.

The variation of the dynamic factor \( c_{dl} \) and static factor \( c_s \) on the path of contact are presented in figures 2 and 5 as a function of the addendum modification coefficients.

Figures 3 and 6 present the variation of the contact pressures \( p_{Hd} \) and \( p_{H_s} \) on the line of action, where \( p_{Hd} \) is computed under dynamic condition and \( p_{H_s} \) is computed under static condition. The variation of specific slidings \( \xi_1 \) and \( \xi_2 \) is presented in figures 4 and 7.

Referring to these figures, the following mesh points were used to represent the successive positions of contact point of a tooth as it passes through the zone of loading: the initial point of engagement, A; the lowest point of single-tooth contact, B; the highest point of single-tooth contact, D; and the final point of engagement, E. Section AB and DE are double-tooth contact zone and section BD is the single tooth - contact zone.

For \( x_1^* = x_1_{\text{min}} \) the maximum of both, contact pressure and negative specific slidings occur at the initial point of engagement. Under these conditions the contact fatigue life of the gear pair is considered to be governed by the pitting on the pinion teeth. The limitations of the contact pressure and negative specific slidings is essentially an attempt to prevent surface failure. This desideratum is obtained for an addendum modification coefficient \( x_1 \) a little larger than \( x_{\text{min}} \) and a such solution must be used. These situation are comparatively presented in figures 3, 4 and figures 6, 7.
Figure 2. Variation of the static and dynamic load factors.

Figure 3. Variation of the contact pressure under static and dynamic condition

Figure 4. Variation of the specific slidings.

A such of increase of the addendum modification coefficient $x_i$ causes a slight decrease in the contact ratio of the gear pair, which involves an slightly increase in the dynamic load. Therefore, the increase trend of the coefficient $x_i$ is limited.
6. Conclusion
Distributing the sum of addendum modification coefficients is crucial for a better meshing performance and a specific detailed analysis should be carried out for each gear-pair. In order to select the best values of the addendum modification coefficients of the pinion and driven gear, the geometrical and surface contact characteristics are considered as design criteria for spur gear pairs with smaller number of pinion teeth. The geometrical design procedure permits to establish the limit values of the addendum modification coefficients in order to avoid negative phenomena of the
generating process. From the viewpoint of the contact failure, it is necessary to analyze the variation of the contact stress and specific slidings for a meshing cycle in relation with the amount of addendum modification coefficients. The analysis of these contact characteristics under dynamic condition permitted to establish the unfavourable meshing regions for surface durability. The meshing conditions at initial point of contact zone should be improved by using an addendum modification coefficient a little larger than the limit value $x_{\text{min}}$. Such a selection has a favourable effect on both, the pressure contact and negative specific sliding and, therefore, the surface durability of the contacting teeth should be improved.

7. References

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