THE CLT FOR ROTATED ERGODIC SUMS
AND RELATED PROCESSES

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Abstract. Let \((\Omega, \mathcal{A}, P, \tau)\) be an ergodic dynamical system. The rotated ergodic sums of a function \(f\) on \(\Omega\) for \(\theta \in \mathbb{R}\) are \(S^\theta_n f := \sum_{k=0}^{n-1} e^{2\pi i k \theta} f \circ \tau^k, n \geq 1\).

Using Carleson’s theorem on Fourier series, Peligrad and Wu proved in [14] that \((S^\theta_n f)_{n \geq 1}\) satisfies the CLT for a.e. \(\theta\) when \((f \circ \tau^n)\) is a regular process.

Our aim is to extend this result and give a simple proof based on the Fejér-Lebesgue theorem. The results are expressed in the framework of processes generated by \(K\)-systems. We also consider the invariance principle for modified rotated sums. In a last section, we extend the method to \(\mathbb{Z}^d\)-dynamical systems.

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Introduction. Let \((\Omega, \mathcal{A}, P, \tau)\) be a dynamical system, i.e., a probability space \((\Omega, \mathcal{A}, P)\) and a measurable transformation \(\tau : \Omega \to \Omega\) which preserves \(P\). The
rotated ergodic sums of a function $f$ on $\Omega$ are defined for $\theta \in \mathbb{R}$ by

$$S^\theta_n f := \sum_{k=0}^{n-1} e^{2\pi i k \theta} f \circ \tau^k, \ n \geq 1. \quad (1)$$

Using Carleson’s theorem on Fourier series, Peligrad and Wu proved in [14] that $(S^\theta_n f)_{n \geq 1}$ satisfies the CLT for a.e. $\theta$ when $(f \circ \tau^n)$ is a regular process.

It is remarkable that this result does not require a rate of decorrelation or regularity of the function $f$ generating the stationary process $(f \circ \tau^n)$. Our aim is to give a simple proof based on the Fejér-Lebesgue theorem of this result and to extend it. The results are expressed in the framework of processes generated by $K$-systems. We also consider the invariance principle for modified rotated sums like in [13].

As an application, we prove a recurrence property for a class of rotated stationary processes. In the last section, the method is extended to $\mathbb{Z}^d$-dynamical systems. Other extensions will be presented in a further paper.

The proofs are based on the approximation of the ergodic sums by a martingale and lead to two separate questions: validity of a mean square approximation by a martingale, limiting behavior of the approximating martingale.

1. Preliminaries.

1.1. Rotated ergodic sums, $T$-filtrations. Let $T$ be a unitary operator on a Hilbert space $\mathcal{H}$. The spectral measure with respect to $T$ of a vector $f \in \mathcal{H}$ is the positive measure $\nu_f$ on the unit circle $T^1$ with Fourier coefficients $\hat{\nu}_f(n) = \langle T^n f, f \rangle, n \in \mathbb{Z}$. The results in this section are purely spectral, although later $T$ will be the Koopman operator induced by an automorphism.

We denote by $\varphi_f$ the density of the absolutely continuous part of the spectral measure $\nu_f$ with respect to the Lebesgue measure $\lambda$ on $T^1$. Let $K_n$ be the Fejér kernel. For the rotated ergodic sums $S^\theta_n f$ defined by (1), we have

$$\frac{1}{n} \|S^\theta_n f\|_2^2 = \int_0^1 \frac{1}{n} \left| \frac{\sin \pi n(t - \theta)}{\sin \pi(t - \theta)} \right|^2 \nu_f(dt) = (K_n * \nu_f)(\theta). \quad (2)$$

This formula implies by the Fejér-Lebesgue theorem (cf. [19, Ch. III, Th. 8.1]):

**Lemma 1.1.** For every $f \in L_2^0(\mathbb{P})$, we have, for a.e. $\theta$, $\lim_{n \to \infty} \frac{1}{n} \|S^\theta_n f\|_2^2 = \varphi_f(\theta)$.

It follows that the asymptotic variance of the rotated sums $\lim_{n \to \infty} \frac{1}{n} \|S^\theta_n f\|_2^2$ exists for a.e. $\theta$. Therefore a question is the behavior of the normalized ergodic sums in distribution, for a.e. $\theta$.

Remark that $\|S^\theta_n f\|_2 = o(\sqrt{n})$ for a.e. $\theta$, if the spectral measure is singular.

**T-filtrations**

**Definitions, notations 1.2.** Let $T$ be a unitary operator on a Hilbert space $\mathcal{H}$. A closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that $T^{-1}\mathcal{H}_0 \subset \mathcal{H}_0$ defines a Hilbertian $T$-filtration $\mathcal{F}$, i.e., an increasing family of closed subspaces $\mathcal{H}_n$ of $\mathcal{H}$ with $\mathcal{H}_n = T^n\mathcal{H}_0, \forall n \in \mathbb{Z}$.

The closed subspace generated by $\bigcup_{n \in \mathbb{Z}} \mathcal{H}_n$ is denoted by $\mathcal{H}_\infty$ and the intersection $\bigcap_{n \in \mathbb{Z}} \mathcal{H}_n$ by $\mathcal{H}_{-\infty}$. We say that an element $f \in \mathcal{H}$ is $\mathcal{F}$-regular (or simply regular), if $f$ belongs to $\mathcal{H}_\infty$ and is orthogonal to $\mathcal{H}_{-\infty}$.
Orthogonal decomposition

Let \( \Pi_n \) be the orthogonal projection on the subspace \( \mathcal{K}_n := \mathcal{H}_n \ominus \mathcal{H}_{n-1} \) of vectors in \( \mathcal{H}_n \) which are orthogonal to \( \mathcal{H}_{n-1} \). We have the orthogonal decomposition \( \mathcal{H}_\infty = \mathcal{H}_\infty \ominus_n \mathcal{K}_n \) and for \( f \in \mathcal{H}_\infty \):

\[
f = \Pi_{-\infty}f + \sum_{n \in \mathbb{Z}} \Pi_n f, \quad \|f\|^2 = \|\Pi_{-\infty}f\|^2 + \sum_{n \in \mathbb{Z}} \|\Pi_n f\|^2.
\] (3)

Note that \( \mathcal{K}_f = T\mathcal{K}_0 \) and, for every \( g \in \mathcal{K}_0 \), \( \Pi T^k g = \delta_{k,\ell} T^k g, \forall k, \ell \in \mathbb{Z} \), where \( \delta_{k,\ell} \) is the Kronecker symbol.

Let \( (\psi_j)_{j \in \mathcal{J}} \) be an orthonormal basis of \( \mathcal{K}_0 = \mathcal{H}_0 \ominus \mathcal{H}_{-1} \). It yields an orthonormal basis \( (T^k \psi_j)_{j \in \mathcal{J}, k \in \mathbb{Z}} \) of the subspace \( \oplus_n \mathcal{K}_n \) generated by the filtration. Let \( f \) be a \( \mathcal{F} \)-regular function in \( \mathcal{H}_\infty \). By restricting to the closed subspace generated by \( (T^n f)_{n \in \mathbb{Z}} \), we can assume that \( J \) is countable. Setting

\[
a_{j,n} := \langle f, T^n \psi_j \rangle,
\] (4)

we have \( \Pi_n f = \sum_{j \in J} a_{j,n} T^n \psi_j \) and \( \|f\|^2 = \sum_{n \in \mathbb{Z}} \|\Pi_n f\|^2 = \sum_{n \in \mathbb{Z}} \sum_{j \in J} |a_{j,n}|^2 \).

Notations 1.3. For each \( j \in \mathcal{J} \), let \( \gamma_j \) be an everywhere finite square integrable function with Fourier coefficient \( a_{j,n}, n \in \mathbb{Z} \), defined by (4), that is,

\[
\gamma_j(t) = \sum_{n \in \mathbb{Z}} a_{j,n} e^{2\pi i nt}.
\] (5)

A simple computation shows that \( |\gamma_j|^2 \) is the spectral density of the orthogonal projection \( f_j \) of \( f \) on the subspace generated by \( (T^k \psi_j)_{k \in \mathbb{Z}} \); hence

\[
\varphi_f = \sum_{j \in J} |\gamma_j|^2 \quad \text{and} \quad \int \sum_{j \in J} |\gamma_j(\theta)|^2 d\theta = \sum_{j \in J, k = -\infty} \sum_{|k|} |a_{j,k}|^2 = \int \varphi_f(\theta) d\theta = \|f\|^2. \quad (6)
\]

By (6), the set \( \Lambda_0 := \{ \theta \in \mathbb{T} : \sum_{j \in J} |\gamma_j(\theta)|^2 < \infty \} \) has full measure. For \( \theta \in \Lambda_0 \), we define \( M_\theta f \in \mathcal{K}_0 \) by:

\[
M_\theta f := \sum_{j} \gamma_j(\theta) \psi_j.
\]

The function \( \theta \to \|M_\theta f\|_{L^2}^2 = \sum_{j \in J} |\gamma_j(\theta)|^2 \) is a version of the spectral density \( \varphi_f \) of \( f \).

1.2. Approximation of the rotated ergodic sums. Here we show that, for almost every \( \theta \), the process with orthogonal increments \( (\sum_{k=0}^{n-1} e^{2\pi i k \theta} T^k M_\theta)_{n \geq 1} \) approximates the rotated process in mean square.

**Proposition 1.4.** If \( f \in \mathcal{H}_\infty \) is \( \mathcal{F} \)-regular, the set \( \Lambda(f) \subset \Lambda_0 \) of \( \theta \in \mathbb{T}^1 \), such that

\[
\frac{1}{n} \|\sum_{k=0}^{n-1} e^{2\pi i k \theta} T^k (f - M_\theta f)\|_{L^2}^2 \to 0,
\]

has full Lebesgue measure in \( \mathbb{T}^1 \).

**Proof.** The spectral density of \( f - M_\theta f \) is

\[
\varphi_{f-M_\theta f}(t) = \sum_{j \in J} |\gamma_j(t) - \gamma_j(\theta)|^2 = \sum_{j \in J} |\gamma_j(t)|^2 + \sum_{j \in J} |\gamma_j(\theta)|^2 - \sum_{j \in J} \gamma_j(t) \gamma_j(\theta) - \sum_{j \in J} \gamma_j(t) \gamma_j(\theta).
\]
We have (cf. Lemma 1.1):
\[
\frac{1}{n} \|S_n^\theta(f - M_\theta f)^2\|_2 = \int_0^1 \frac{1}{n} \left| \sum_{k=0}^{n-1} e^{2\pi i k(t-\theta)} \right|^2 \varphi_{f-M_\theta f}(t) \, dt
\]
\[
= \int_0^1 K_n(t-\theta) \varphi_{f-M_\theta f}(t) \, dt.
\]

Let \(\Lambda_0\) be the set of full measure of \(\theta\)'s such that the convergence given by the Fejér-Lebesgue theorem holds at \(\theta\) for the functions \(|\gamma_j|^2, \gamma_j, \forall j \in J, \) and \(\sum_{j \in J} |\gamma_j|^2\).

Take \(\theta \in \Lambda_0 \cap \Lambda'_0\). Let \(\varepsilon > 0\) and let \(J_0 = J_0(\varepsilon, \theta)\) be a finite subset of \(J\) such that \(\sum_{j \notin J_0} |\gamma_j(\theta)|^2 < \varepsilon\). Since
\[
\lim_n \int_0^1 K_n(t-\theta) \sum_{j \notin J_0} |\gamma_j(t)|^2 \, dt
\]
\[
= \lim_n \int_0^1 K_n(t-\theta) \sum_{j \in J} |\gamma_j(t)|^2 \, dt - \lim_n \int_0^1 K_n(t-\theta) \sum_{j \notin J_0} |\gamma_j(t)|^2 \, dt
\]
\[
= \sum_{j \notin J_0} |\gamma_j(\theta)|^2;
\]
we have
\[
\limsup_n \int_0^1 K_n(t-\theta) \varphi_{f-M_\theta f}(t) \, dt
\]
\[
\leq \lim_n \int_0^1 K_n(t-\theta) \sum_{j \notin J_0} \left[ |\gamma_j(t)|^2 + |\gamma_j(\theta)|^2 - \gamma_j(t)\gamma_j(\theta) - \gamma_j(t)\gamma_j(\theta) \right] \, dt
\]
\[
+ 2 \lim_n \int_0^1 K_n(t-\theta) \sum_{j \notin J_0} \left[ |\gamma_j(t)|^2 + |\gamma_j(\theta)|^2 \right] \, dt
\]
\[
= \sum_{j \notin J_0} \left[ |\gamma_j(\theta)|^2 + |\gamma_j(\theta)|^2 - \gamma_j(\theta)\gamma_j(\theta) - \gamma_j(\theta)\gamma_j(\theta) \right] + 4 \sum_{j \notin J_0} |\gamma_j(\theta)|^2 \leq 0 + 4\varepsilon.
\]

This shows that \(\Lambda_0 \cap \Lambda'_0 \subset \Lambda(f)\); hence \(\Lambda(f)\) has full measure. \(\square\)

**Corollary 1.5.** Let \(f_{-\infty}\) be the orthogonal projection on \(\mathcal{H}_{-\infty}\) of a vector \(f \in \mathcal{H}_{\infty}\). If the spectral measure of \(f_{-\infty}\) is singular, then for a.e. \(\theta\)
\[
\frac{1}{n} \| \sum_{k=0}^{n-1} e^{2\pi i k\theta} T^k (f - M_\theta (f - f_{-\infty})) \|_2^2 \to 0.
\]

**Proof.** By Lemma 1.1, \(\lim \frac{1}{n} \| \sum_{k=0}^{n-1} e^{2\pi i k\theta} T^k (f - f_{-\infty}) \|_2^2 = 0\) for a.e. \(\theta\). As \(f - f_{-\infty}\) is regular, Proposition 1.4 implies the result. \(\square\)

**Remark 1.6.** We may also deal with \(T\) isometry. For example, when \(\tau\) is an endomorphism (i.e., non-invertible measure preserving), the induced Koopman operator \(T\) is only an isometry. We may transfer some results which hold for automorphisms to endomorphisms in the following way (cf. Nagy-Foias [12, Proposition 6.2]). If \(U\) is the unitary dilation of \(T\) and \(h\) is a polynomial, by the construction of the dilation we obtain \(\|h(T)\| \leq \|h(U)\|\). In particular, Proposition 1.4 holds for an endomorphism and the associated decreasing filtration. The spectral measure that is used
in the proof is replaced by the spectral measure of \( f \) with respect to the unitary dilation. Analogous results hold for any commuting finite family of endomorphisms.

The following result will be used later for the computation of the variance for rotated processes.

**Lemma 1.7.** Let \( \mu \) be an atomless finite complex measure on \([\frac{-1}{2}, \frac{1}{2}]\) and denote by \((\hat{\mu}(n))\) its Fourier-Stieltjes coefficients. Let \( \theta_1, \theta_2 \in [\frac{-1}{2}, \frac{1}{2}] \) with \( \theta_1 \neq \theta_2 \). Then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i(k\theta_1 - \theta_2)} \hat{\mu}(k - p) = 0.
\]

**Proof.** It is enough to prove that \( \lim_{n \to \infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \hat{\mu}(\ell)e^{2\pi i\ell\theta_2} \sum_{k=0}^{\ell} e^{2\pi ik(\theta_1 - \theta_2)} = 0. \)

From the inequality \( |\sum_{\ell=0}^{n-1} e^{2\pi i\ell(\theta_1 - \theta_2)}| \leq \frac{C_{\theta_1, \theta_2}}{n} \) and Cauchy-Schwarz inequality we obtain

\[
\left| \frac{1}{n} \sum_{\ell=0}^{n-1} \hat{\mu}(\ell)e^{2\pi i\ell\theta_2} \sum_{k=0}^{\ell} e^{2\pi ik(\theta_1 - \theta_2)} \right|^2 \\
\leq \left( \frac{1}{n} \sum_{\ell=0}^{n-1} |\hat{\mu}(\ell)|^2 \right) \left( \frac{1}{n} \sum_{k=0}^{\ell} |\sum_{\ell=0}^{n-1} e^{2\pi ik(\theta_1 - \theta_2)}|^2 \right) \\
\leq \frac{C_{\theta_1, \theta_2}}{n} \sum_{\ell=0}^{n-1} |\hat{\mu}(\ell)|^2 \to C_{\theta_1, \theta_2} \sum_{x} |\mu(x)|^2 = 0.
\]

\[\square\]

For a unitary operator \( T \) and for \( f_1, f_2 \in L^2(\Omega) \), there is a complex spectral measure \( \mu_{1,2} \) on the torus (the cross spectral measure of \((f_1, f_2)\) with respect to \( T \)) such that \( \hat{\mu}(k) := \langle T^k f_1, f_2 \rangle = \int e^{-2\pi ik\cdot t} d\mu_{1,2}(t) \). The measure \( \mu_{1,2} \) is a linear combination of the spectral measures of \( f_1 \pm f_2 \) and \( f_1 \pm if_2 \) with respect to \( T \). If \( T \) is weakly mixing, then \( \mu_{1,2} \) is continuous for every \( f_1 \) and \( f_2 \).

**Proposition 1.8.** Let \( T \) be a unitary operator on \( \mathcal{H} \) and let \( f_1, f_2 \in \mathcal{H} \). Denote by \( \varphi_{f_i}, i = 1, 2 \) the densities of the absolutely continuous part of the spectral measures of \( f_i \) with respect to \( T \). Assume that \((f_1, f_2)\) has an atomless cross spectral measure with respect to \( T \). Then for a.e. \( (\theta_1, \theta_2) \in \mathbb{T}^2 \)

\[
\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \left( e^{2\pi ik\theta_1} T^k f_1 + e^{2\pi ik\theta_2} T^k f_2 \right) \right\|^2 = \varphi_{f_1}(\theta_1) + \varphi_{f_2}(\theta_2).
\]

**Proof.** We expand

\[
\frac{1}{n} \left\| \sum_{k=0}^{n-1} \left( e^{2\pi ik\theta_1} T^k f_1 + e^{2\pi ik\theta_2} T^k f_2 \right) \right\|^2 \\
= \frac{1}{n} \left\| \sum_{k=0}^{n-1} e^{2\pi ik\theta_1} T^k f_1 \right\|^2 + \frac{1}{n} \left\| \sum_{k=0}^{n-1} e^{2\pi ik\theta_2} T^k f_2 \right\|^2 \\
+ \frac{2}{n} \Re \left( \sum_{k=0}^{n-1} e^{2\pi ik\theta_1} T^k f_1 \sum_{k=0}^{n-1} e^{2\pi ik\theta_2} T^k f_2 \right).
\]

By the Fejér-Lebesgue theorem, for a.e. \( \theta_1, \theta_2 \) the first two terms tend respectively to \( \varphi_{f_1}(\theta_1) \) and \( \varphi_{f_2}(\theta_2) \). The inner product term tends to 0 by the previous lemma. \(\square\)
The results of this section were of spectral nature and valid for general stationary processes of second order. Now we apply them to regular processes, with the terminology of dynamical systems and specifically of \( K \)-systems.

2. CLT for rotated processes. Let \( (\Omega, \mathcal{A}, \mathbb{P}, \tau) \) be a dynamical system. If \( \mathcal{B} \subset \mathcal{A} \) is a sub \( \sigma \)-algebra, then \( L^2_0(\mathcal{B}) \) is the subspace of centered \( \mathcal{B} \)-measurable functions in \( L^2(\Omega, \mathcal{A}, \mathbb{P}) \). Recall (cf. Rohlin [15]) that any dynamical system has a largest zero entropy factor \( \mathcal{P}(\tau) \) (Pinsker factor) and a sub-\( \sigma \)-algebra \( \mathcal{A}_0 \) of \( \mathcal{A} \) with the following properties: (i) (increasing) \( \mathcal{A}_0 \subset \tau^{-1} \mathcal{A}_0 \); (ii) (generating) \( \mathcal{A}_\infty := \sigma (\cup_{n=-\infty}^{\infty} \tau^{-n} \mathcal{A}_0) = \mathcal{A} \); (iii) \( \mathcal{A}_{-\infty} := \cap_{n=-\infty}^{\infty} \tau^{-n} \mathcal{A}_0 = \mathcal{P}(\tau) \).

The increasing sequence \( \mathcal{A}_n := \tau^{-n} \mathcal{A}_0 \) defines a filtration with the corresponding Hilbertian \( T \)-filtration:

\[
L^2_0(\mathcal{P}(\tau)) = L^2_0(\mathcal{A}_{-\infty}) \subset ... \subset L^2_0(\mathcal{A}_{-1}) \subset L^2_0(\mathcal{A}_0) \subset L^2_0(\mathcal{A}_1) \subset ... \subset L^2_0(\mathcal{A}_\infty) = L^2_0(\mathcal{A}).
\]

Every function in \( L^2(\mathcal{A}) \) which is orthogonal to \( L^2(\mathcal{P}(\tau)) \) is regular for the associated Hilbertian \( T \)-filtration.

In particular, when the system is a \( K \)-system (equivalently has completely positive entropy, i.e., with a trivial Pinsker factor), there exists an increasing generating sub-\( \sigma \)-algebra \( \mathcal{A}_0 \) of \( \mathcal{A} \) such that \( \mathcal{A}_{-\infty} \) is trivial. Every function in \( L^2_0(\mathcal{P}) \) is then regular for the associated Hilbertian \( T \)-filtration. Therefore Proposition 1.4 applies to \( K \)-systems.

If the Pinsker factor is not trivial but has a singular spectral type, i.e., if the spectral measure \( \nu_f \) is singular for every \( f \in L^2_0(\mathcal{P}(\tau)) \), then Corollary 1.5 applies.

When \( \tau \) is an endomorphism we have a decreasing filtration \( \mathcal{A}_n = \tau^{-n} \mathcal{A}, n \geq 0 \). Here \( \mathcal{A}_{-\infty} = \mathcal{A} \) and \( \mathcal{A}_{-\infty} \) is replaced by \( \cap_{n=0}^{\infty} \mathcal{A}_n \). This induces a one-sided Hilbertian \( T \)-filtration. Every function in \( L^2(\mathcal{A}) \) orthogonal to \( L^2(\mathcal{A}_{-\infty}) \) is regular. If the system is exact, that is, \( \mathcal{A}_{-\infty} \) is trivial, every function in \( L^2_0(\mathcal{A}) \) is regular. Using the unitary dilation (see Remark 1.6) and reversing the filtration order, Proposition 1.4 holds for endomorphisms.

2.1. CLT. In this section we show a Central Limit Theorem (CLT) for “rotated” processes.

Let \( (\Omega, \mathcal{A}, \mathbb{P}, \tau) \) be a \( K \)-system. Let \( \mathcal{A}_0 \) be an increasing and generating sub-\( \sigma \)-algebra of \( \mathcal{A} \) as in the beginning of Section 2 and \( \mathcal{A}_n = \tau^{-n} \mathcal{A}_0 \). With the notations of Subsection 1.1, we put \( \mathcal{K}_0 := L^2_0(\mathcal{A}_0) \oplus L^2_0(\mathcal{A}_{-1}) \).

For \( \theta \in \mathbb{R} \), we consider the product system \( (T^1 \times \Omega, \lambda \otimes \mathbb{P}, \tau_\theta) \), where

\[
\tau_\theta : (x, \omega) \rightarrow (x + \theta \mod 1, \tau \omega).
\]

The Pinsker factor of \( \tau_\theta \) is isomorphic to the rotation by \( \theta \) on the circle. Let \( (\psi_j)_{j \in J} \) be an orthonormal family of real functions which is a basis of \( \mathcal{K}_0 \).

Let us consider a function \( F \in L^2(T^1 \times \Omega) \) orthogonal to the functions depending only on \( \omega \),

\[
F(x, \omega) = \sum_{\ell \in \mathbb{Z} \setminus \{0\}} e^{2\pi i \ell x} f_\ell(\omega). \tag{8}
\]

We investigate the question of the CLT for \( \sum_{k=0}^{n-1} F(x+k\theta, \tau^k \omega) \) for a.e. \( \theta \). When \( F(x, \omega) = e^{2\pi i \ell x} f(\omega) \), we will obtain the CLT for “rotated ergodic sums” of Peligrad and Wu mentioned in the introduction.
The approximation martingale

The asymptotic variance

The Fourier series

The spectral density of \( \tilde{\varphi} \) is well defined for \( \ell \) in the set of \( \mathcal{K}_0 \) of \( L^2_\mathbb{G}(\mathbb{A}) \) to decompose the components \( f_\ell \) of \( F \). The \( L^2 \)-norm of \( F \) is

\[
\|F\|_2^2 = \sum_{\ell \neq 0} \|f_\ell\|_2^2 = \sum_{\ell \neq 0} \sum_j |(f_\ell, T^k \psi_j)|^2 < \infty.
\]

For each \( j \in J \), let \( \gamma_j, \ell \) be an everywhere finite square integrable function with Fourier series \( \gamma_j, \ell (t) = \sum_n (f_\ell, T^n \psi_j) e^{2\pi i k t} \) (cf. (5)). Let \( f_\ell (x, \omega) := e^{2\pi i k x} f_\ell (\omega) \). The spectral density of \( f_\ell \) for \( \tau_\theta \) is \( \varphi_{f_\ell} (t + \ell \theta) = \sum_j |\gamma_j, \ell (t + \ell \theta)|^2 \) and we have

\[
\sum_j \int |\gamma_j, \ell (t + \ell \theta)|^2 dt = \sum_j \int |\gamma_j, \ell (t)|^2 dt = \sum_j \sum_k |(f_\ell, T^k \psi_j)|^2.
\]

The spectral density of \( F \) in the dynamical system \((\mathbb{T}^1 \times \Omega, \tau_\theta)\) is

\[
\varphi_F (t) = \sum_{\ell \neq 0} \sum_j |\gamma_j, \ell (t + \ell \theta)|^2.
\]

The above formula defines for every \( \theta \) a positive function in \( L^1(\mathbb{T}^1) \) with integral \( \int_{\mathbb{T}^1} \varphi_F \, dt = \|F\|_2^2 \). The variance at time \( n \) is

\[
\frac{1}{n} \int_{\mathbb{T}^1 \times \Omega} |\sum_{k=0}^{n-1} F(x + k \theta, \tau^k \omega)|^2 \, dx \, P(\omega) = \sum_{\ell \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \int_{\Omega} |\sum_{k=0}^{n-1} e^{2\pi i k \theta} f_\ell (\tau^k \omega)|^2 \, P(\omega) = \sum_{\ell \in \mathbb{Z} \setminus \{0\}} (K_n * \varphi_{f_\ell}) (\ell \theta).
\]

The asymptotic variance \( \sigma_\theta^2 (F) \) for the action of \( \tau_\theta \) exists whenever the limit

\[
\lim_{n \to \infty} \sum_{\ell \in \mathbb{Z} \setminus \{0\}} (K_n * \varphi_{f_\ell}) (\ell \theta)
\]

exists and is finite.

**Definition of the approximating martingale** The functions \( \sum_j \gamma_j, \ell (\ell \theta) \psi_j \) are in \( \mathcal{K}_0 \). Let \( M_\theta f \) (or simply \( M_\theta \)) be defined by:

\[
M_\theta (x, \omega) := \sum_{\ell \neq 0} e^{2\pi i k x} \sum_j \gamma_j, \ell (\ell \theta) \psi_j.
\]

We have

\[
\int_{\mathbb{T}^1 \times \Omega} |M_\theta (x, \omega)|^2 \, dx \, P(\omega) = \sum_{\ell \neq 0} \sum_j |\gamma_j, \ell (\ell \theta)|^2 = \sum_{\ell \neq 0} \varphi_{f_\ell} (\ell \theta).
\]

Since \( \sum_{\ell \neq 0} \sum_{j \in J} |\gamma_j, \ell (\ell \theta)|^2 \, d\theta = \sum_{\ell \neq 0} \sum_{j \in J} \int |\gamma_j, \ell (\ell \theta)|^2 \, d\theta = \|F\|_2^2 < \infty \), \( M_\theta \) is well defined for \( \theta \) in the set of full measure

\[
\Lambda_0 (F) := \{ \theta \in \mathbb{T} : \sum_{\ell \neq 0} \sum_{j \in J} |\gamma_j, \ell (\ell \theta)|^2 < \infty \}.
\]

Denote by \( \mathcal{B} \) the Borel \( \sigma \)-algebra of \( \mathbb{T} \). With respect to the filtration \((\mathcal{B} \times \mathcal{A}_n)\), \((M_\theta (x + k \theta, \tau^n \omega))_{n \in \mathbb{Z}}\) is a \( \tau_\theta \)-stationary ergodic sequence of differences of martingale, with variance given by (10). Notice that \( M_\theta \) is real valued if \( F \) is a real valued function. The following two propositions are the main steps in the proof of the CLT for rotated sums.
Proposition 2.2. Suppose

Let

Proposition 2.1. Let $F$ be a real function in $L^2(T^1 \times \Omega)$ represented as in (8). For $\theta \in \Lambda_0(F)$, the asymptotic distribution with respect to the measure $\lambda \otimes P$ on $T^1 \times \Omega$ of $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} M_\theta(x + k\theta, \tau^k \omega)$ is the normal law with variance $\sum_{\ell \neq 0} \varphi_{f_\ell}(\ell \theta)$.

Proof. The result follows from Billingsley-Ibragimov theorem on stationary ergodic martingale differences (cf. [9]).

Proposition 2.2. Suppose $\sum_\ell \|f_\ell\|_2 < \infty$.

1) Then we have

$$\sum_\ell \sup_n (K_n * \varphi_{f_\ell})(\ell \theta) < \infty, \text{ for a.e. } \theta. \quad (11)$$

2) The asymptotic variance $\sigma_0^2(F) = \sum_\ell \varphi_{f_\ell}(\ell \theta)$ exists for a.e. $\theta$. The set $\Lambda(F)$ of elements $\theta$ in $\Lambda_0$ such that

$$\delta_0^\ell(F) := \frac{1}{n} \int_{T^1 \times \Omega} \left| \sum_{j=0}^{n-1} [F(x + k\theta, \tau^k \omega) - M_\theta(x + k\theta, \tau^k \omega)] \right|^2 dx \ P(d\omega) \to 0 \quad (12)$$

has full measure.

Proof. 1) By the maximal inequality for the convolution by the Fejér kernels $K_n$ (see Zygmund [19, Th. 8.1] and Garsia [7, p. 7]), there is $C$ such that, for every $\varphi \in L^1(T^1)$, $\lambda(\sup_n |(K_n * \varphi)| > \alpha) \leq \frac{C}{\alpha} \|\varphi\|_1$.

Recall that $\|\varphi_{f_\ell}\|_{L^1(\Omega)} = \|f_\ell\|_2$. Using the invariance of the Lebesgue measure on $T^1$ by the map $\theta \to k\theta \mod 1$, $k \neq 0$, and the maximal inequality, we obtain (implicitly we restrict the sum to indices $\ell$ such that $\|f_\ell\|_2 \neq 0$) for $s > 0$ the following maximal inequality which implies (11):

$$\lambda(\theta : \sum_\ell \sup_n (K_n * \varphi_{f_\ell})(\ell \theta) \geq s \sum_\ell \|f_\ell\|_2) \leq \sum_\ell \lambda(\theta : \sup_n (K_n * \varphi_{f_\ell})(\ell \theta) \geq s \|f_\ell\|_2) \leq \sum_\ell \lambda(\theta : \sup_n (K_n * \varphi_{f_\ell})(\theta) \geq s \|f_\ell\|_2) \leq \frac{C}{s} \sum_\ell \|f_\ell\|_2 \|\varphi_{f_\ell}\|_1 \leq \frac{C}{s} \sum_\ell \|f_\ell\|_2.$$

2) The existence of the asymptotic variance $\sigma_0^2(F)$ for a.e. $\theta$ follows from (11). We have

$$\int_{T^1 \times \Omega} \left| \sum_{j=0}^{n-1} [F(x + k\theta, \tau^k \omega) - M_\theta(x + k\theta, \tau^k \omega)] \right|^2 dx \ P(d\omega)$$

$$= \int_{T^1 \times \Omega} \left| \sum_{k=0}^{n-1} \sum_{\ell \neq 0} e^{2\pi i \ell (x + k\theta)} (f_\ell(\tau^k \omega) - \sum_j \gamma_{j,\ell}(\ell \theta) \psi_j(\tau^k \omega)) \right|^2 dx \ P(d\omega)$$

$$= \int_{T^1 \times \Omega} \left| \sum_{\ell \neq 0} e^{2\pi i \ell x} \sum_{k=0}^{n-1} e^{2\pi ik\ell \theta} (f_\ell(\tau^k \omega) - \sum_j \gamma_{j,\ell}(\ell \theta) \psi_j(\tau^k \omega)) \right|^2 dx \ P(d\omega)$$

$$= \sum_{\ell \neq 0} \int_{\Omega} \left| \sum_{k=0}^{n-1} e^{2\pi ik\ell \theta} (f_\ell(\tau^k \omega) - \sum_j \gamma_{j,\ell}(\ell \theta) \psi_j(\tau^k \omega)) \right|^2 P(d\omega).$$
hence (cf. (7)):

\[
\delta_n^\theta(F) = \frac{1}{n} \sum_{\ell \neq 0} \| \delta_n^\theta (f_\ell - M_{\ell \theta, \ell} f_\ell) \|_2^2 \\
= \sum_{\ell \neq 0} \int_0^1 \left| \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i k (t - \ell \theta)} \varphi_{f_\ell - M_{\ell \theta, \ell} f_\ell}(t) \right|^2 dt \\
= \sum_{\ell \neq 0} \int_0^1 K_n(t - \ell \theta) \varphi_{f_\ell - M_{\ell \theta, \ell} f_\ell}(t) dt,
\]  

(13)

where the spectral density (for \( \tau \)) of \( f_\ell - M_{\ell \theta, \ell} \) is

\[
\varphi_{f_\ell - M_{\ell \theta, \ell}}(t) = \sum_{j \in J} |\gamma_j(\ell)(t) - \gamma_j(\ell \theta)|^2.
\]

From Proposition 1.4, for a.e. \( \theta \) and for each \( \ell \), \( \lim\limits_{n \to \infty} \int_0^1 K_n(t - \ell \theta) \varphi_{f_\ell - M_{\ell \theta, \ell} f_\ell}(t) dt = 0 \). By (11), for a.e. \( \theta \) we can permute the limit and the sum in (13):

\[
\lim_{n \to \infty} \sum_{\ell \neq 0} \int_0^1 K_n(t - \ell \theta) \varphi_{f_\ell - M_{\ell \theta, \ell} f_\ell}(t) dt = 0.
\]

\[\square\]

**Theorem 2.3.** Let \((\Omega, \mathcal{A}, P, \tau)\) be a K-system. Let \( F \) be a real function in \( L^2(T_1 \times \Omega) \) orthogonal to the functions depending only on \( \omega \) with Fourier expansion (8) such that \( \sum \| f_\ell \|_2^2 < +\infty \). Then for a.e. \( \theta \) we have with respect to the measure \( dx \otimes P(d\omega) \)

\[
\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} F(x + k\theta, \tau^k \omega) \xrightarrow{dist, n \to \infty} \mathcal{N}(0, \sigma_\theta^2(F)),
\]

with asymptotic variance \( \sigma_\theta^2(F) = \sum_{\ell} \varphi_{f_\ell}(\theta) \).

**Proof.** The result follows from the CLT for the martingale defined by (9) (Proposition 2.1) and the approximation (12) valid for a.e. \( \theta \), as shown in Proposition 2.2.

\[\square\]

**Corollary 2.4.** Let \( f \) be a real function in \( L^2_\mathbb{P}(\mathbb{P}) \). For a.e. \( \theta \), the CLT holds for the rotated sums:

\[
\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi i k \theta} f(\tau^k \omega) \xrightarrow{dist, n \to \infty} \mathcal{N}(0, \Gamma_\theta)
\]

with respect to the measure \( P(d\omega) \), with covariance matrix

\[
\Gamma_\theta = \begin{pmatrix}
\frac{1}{2} \varphi_f(\theta) & 0 \\
0 & \frac{1}{2} \varphi_f(\theta)
\end{pmatrix}.
\]

**Proof.** Theorem 2.3 implies the CLT for the sums \( e^{2\pi i x} \sum_{k=0}^{n-1} e^{2\pi i k \theta} f(\tau^k \omega) \) with respect to \( dx \otimes P(d\omega) \). An easy computation gives the covariance matrix for the (complex valued) martingale \( M_\theta \) associated to \( e^{2\pi i x} f(\omega) \). Using the lemma below, this implies the result.

\[\square\]
Lemma 2.5. The asymptotic distribution of \( \frac{1}{\sqrt{n}} e^{2\pi i x} \sum_{k=0}^{n-1} e^{2\pi i k \theta} f(\tau^k \omega) \) with respect to \( \lambda \otimes \mathbb{P} \) is, for a.e. \( \theta \in T^1 \), the same as that of \( \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi i k \theta} f(\tau^k \omega) \) with respect to \( \mathbb{P} \).

Proof. Let \( Z_n^\theta(\omega) := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi i k \theta} f(\tau^k \omega) \). By Theorem 2.3 for a.e. \( \theta \), \( e^{2\pi i x} Z_n^\theta(\omega) \) satisfies the CLT with respect to \( \lambda \otimes \mathbb{P} \). Fix \( \theta \) in this set of full measure. By Theorem 4 of Eagleson [5] (see also [18]), if we replace \( \lambda \otimes \mathbb{P} \) by an absolutely continuous probability measure with respect to \( \lambda \otimes \mathbb{P} \), the asymptotic distribution of \( e^{2\pi i x} Z_n^\theta(\omega) \) is unchanged (i.e., the limit is mixing).

For \( \varepsilon > 0 \) let \( J_\varepsilon \) be an interval neighborhood of 0 of length \( \varepsilon \). Let \( \zeta_\varepsilon := \varepsilon^{-1} 1_{J_\varepsilon} \). We consider the probability measure \( \zeta_\varepsilon(x) \lambda(dx) \otimes \mathbb{P}(d\omega) \).

If \( \Phi \) is a Lipschitz function defined on \( \mathbb{C} \) with Lipschitz constant \( C \), then for \( x \in J_\varepsilon \) we have \( \|\Phi(e^{2\pi i x} Y) - \Phi(Y)\| \leq C \varepsilon \) for every \( Y \in \mathbb{C} \). Hence using the Cauchy-Schwarz inequality, we have

\[
| \int (\Phi(e^{2\pi i x} Z_n^\theta) - \Phi(Z_n^\theta)) \zeta_\varepsilon(x) dx \, d\mathbb{P}(d\omega) | \leq C \varepsilon \left( \int |Z_n^\theta|^2 \, d\mathbb{P} \right)^{\frac{1}{2}}.
\]

Since \( \sup_n \int |Z_n^\theta|^2 \, d\mathbb{P} < +\infty \), it shows that:

\[
\lim_{\varepsilon \to 0} \sup_n \left| \int (\Phi(e^{2\pi i x} Z_n^\theta) - \Phi(Z_n^\theta)) \zeta_\varepsilon(x) dx \, \mathbb{P}(d\omega) \right| = 0.
\]

We obtain the result by applying what precedes to the family \( \Phi_t(Y) = e^{it \Re(e^{2\pi i x} Y)} \), for \( t \) and \( Y \in \mathbb{C} \).

\( \square \)

Remarks 2.6. 1) The CLT for the sums \( \sum_{k=0}^{n-1} e^{2\pi i k \theta} f(\tau^k \omega) \) under the measure \( d\theta \otimes \mathbb{P}(d\omega) \) follows from Corollary 2.4.

2) A variant of the proof of Corollary 2.4 consists in using a CLT for martingales with the weights \( \cos(2\pi k \theta) \). For this, we use the ergodic theorem to prove the convergence of the conditional variance and apply Brown’s Central Limit Theorem for martingale which extends Billingsley-Ibragimov theorem or Corollary 3.4 in [9].

Extensions. In the definition (1) of the rotated sums, it is natural to replace \( e^{2\pi i k \theta} \) for \( \theta \) in the circle by \( e^{2\pi i (\ell, \theta)} \), where \( \theta \) belongs to the \( d \)-dimensional torus. This leads to the following construction. For \( \theta = (\theta_1, ..., \theta_d) \in \mathbb{R}^d \), let us consider the system defined on \( T^d \times \Omega \), by \( \tau_\theta : (x, \omega) \to (x + \theta \mod 1, \tau_\omega) \). We denote by \( \lambda \) the Lebesgue measure on \( T^d \).

Let \( F \) be a function in \( L^2(T^d \times \Omega) \) orthogonal to the functions depending only on \( \omega \), i.e.,

\[
F(x, \omega) = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} e^{2\pi i (\ell, x)} f_\ell(\omega).
\]

Its \( L^2 \)-norm is

\[
\|F\|_2^2 = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \|f_\ell\|_2^2 = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \sum_k |(f_\ell, T^k \psi_j)|^2 < \infty.
\]

Observe that the push forward of the Lebesgue measure on \( T^d \) by the map \( \theta \in T^d \to (\ell, \theta) \in T^1 \) is the Lebesgue measure on \( T^1 \) if \( \ell \in \mathbb{Z}^d \setminus \{0\} \). Therefore the previous method applies. Now the martingale \( M_\theta \) is defined by

\[
M_\theta(x, \omega) := \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} e^{2\pi i (\ell, x)} M_{(\ell, \theta), x}(\omega), \quad \text{where} \quad M_{(\ell, \theta), x} := \sum_j \gamma_{\ell, j}(\ell, \theta) \psi_j.
\]
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Its variance is

\[ \int_{\mathbb{T}^d \times \Omega} |M_\theta(x,\omega)|^2 \, d\mathbb{P}(d\omega) = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \sum_{j \in J} |\gamma_{j, \ell}(\langle \ell, \theta \rangle)|^2 < +\infty, \text{ for a.e. } \theta. \quad (15) \]

The same proof as in Theorem 2.3 shows:

**Theorem 2.7.** Let \((\Omega, \mathcal{A}, \mathbb{P}, \tau)\) be a K-system. Let \(F\) be a real function in \(L^2(\mathbb{T}^d \times \Omega)\) orthogonal to the functions depending only on \(\omega\) with Fourier expansion (14) satisfying \(\sum \|f_\ell\|_2 < +\infty\). Then for a.e. \(\theta \in \mathbb{T}^d\) the CLT holds for \(\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} F(x + k\theta, \tau^k \omega)\) with respect to the measure \(\lambda \otimes \mathbb{P}\).

Remark that the proof applies if we replace \(\mathbb{T}^d\) by any compact abelian connected group \(G\), since for any character on \(G\) the map \(g \in G \rightarrow \chi(g) \in \mathbb{T}^1\) is a surjective homomorphism and the push forward of the Haar measure on \(G\) is the Lebesgue measure on \(\mathbb{T}^1\).

Now, as in Corollary 2.4, we consider the distribution with respect to the measure \(\mathbb{P}\). Note that, with the condition \(\sum \|f_\ell\|_2 < +\infty\), the function \(\omega \rightarrow F(\theta, \omega) := \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} e^{2\pi i \langle \ell, \theta \rangle} f_\ell(\omega)\) is in \(L^2(\Omega, \mathbb{P})\) for every \(\theta\).

**Theorem 2.8.** Let \((\Omega, \mathcal{A}, \mathbb{P}, \tau)\) be a K-system. Let \(F(\theta, \omega)\) with Fourier expansion (14). If \(\sum \|f_\ell\|_2 < +\infty\), then for a.e. \(\theta \in \mathbb{T}^d\) the CLT holds for \(\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} F(k\theta, \tau^k \omega)\) with respect to the measure \(\mathbb{P}\) and \(\lim_n \frac{1}{n} \| \sum_{k=0}^{n-1} F(k\theta, \tau^k \cdot) \|^2 = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \| \varphi_\ell(\langle \ell, \theta \rangle) \|^2\).

**Proof.** It follows from the hypothesis (cf. proof of Proposition 2.2):

\[ \lambda(\theta) : \sup_{n} \frac{1}{\sqrt{n}} \left( \sum_{\ell \in \mathbb{Z}^d} \| K_n * \varphi_\ell \|_2 \right)^{\frac{1}{2}} \geq s \sum_{\ell} \| f_\ell \|_2^{\frac{3}{2}} \]

\[ \leq s \lambda \left( \sup_{n} \frac{1}{\sqrt{n}} \| K_n * \varphi_\ell \|_2 \right)^{\frac{1}{2}} \geq s \sum_{\ell} \| f_\ell \|_2^{\frac{3}{2}} \]

\[ \leq C \frac{s}{\sqrt{2}} \sum_{\ell} \| f_\ell \|_2^{\frac{3}{2}} < +\infty. \quad (16) \]

With the notation \(S_{\ell,n}(\omega) := \sum_{k=0}^{n-1} e^{2\pi i \langle \ell, k\theta \rangle} f_\ell(\tau^k \omega)\), using the triangle inequality this implies:

\[ \lambda(\theta) : \sup_{n} \frac{1}{\sqrt{n}} \left( \sum_{\ell \in \mathbb{Z}^d} \| K_n * \varphi_\ell \|_2 \right)^{\frac{1}{2}} \geq s \sum_{\ell \in \mathbb{Z}^d} \| f_\ell \|_2^{\frac{3}{2}} \]

\[ \leq \lambda \left( \sup_{n} \frac{1}{\sqrt{n}} \| S_{\ell,n} \|_2 \right)^{\frac{1}{2}} \geq s \sum_{\ell} \| f_\ell \|_2^{\frac{3}{2}} \]

\[ \leq s \lambda \left( \sup_{n} \| K_n * \varphi_\ell \|_{(\langle \ell, \theta \rangle)} \right) \geq s \sum_{\ell} \| f_\ell \|_2^{\frac{3}{2}} \]

\[ \leq C \frac{s}{\sqrt{2}} \sum_{\ell} \| f_\ell \|_2^{\frac{3}{2}} < +\infty. \quad (16) \]

If \(L\) is a finite set of indices in \(\mathbb{Z}^d\), we have by the preceding inequalities:

\[ \lambda(\theta) : \sup_{n} \frac{1}{\sqrt{n}} \left( \sum_{k=0}^{n-1} \sum_{\ell \in \mathbb{Z}^d \setminus L} e^{2\pi i \langle \ell, k\theta \rangle} f_\ell \circ \tau^k \|_2 \right)^{\frac{1}{2}} \geq s \sum_{\ell \in \mathbb{Z}^d \setminus L} \| f_\ell \|_2^{\frac{3}{2}} \]

\[ \leq C \frac{s}{\sqrt{2}} \sum_{\ell \in \mathbb{Z}^d \setminus L} \| f_\ell \|_2^{\frac{3}{2}}. \]
Therefore, for an increasing sequence of finite sets \((L_p)\) in \(\mathbb{Z}^d\) with union \(\mathbb{Z}^d \setminus \{0\}\) and with

\[
Z_n^\theta, p := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \sum_{\ell \in L_p} e^{2\pi i \langle \ell, k \theta \rangle} f_k \circ \tau^k,
\]

we have \(\lim_p \sup_n \|Z_n^\theta - Z_n^\theta, p\|_2 = 0\) for a.e. \(\theta\).

Now, if we prove that \(Z_n^\theta \xrightarrow{\text{distr}} N(0, \sigma_\theta)\) for every \(p\), then we may use Theorem 3.2 in Billingsley [1] to conclude that \(Z_n^\theta \xrightarrow{\text{distr}} N(0, \sigma(\theta))\), with \(\sigma(\theta) := \lim_p \sigma_p(\theta) \in [0, +\infty]\).

Let \(Y_\ell = S^\theta_{\ell, n}(\omega)\) and let \(\gamma_\ell \in \mathbb{C}\) be any family of complex numbers. We have

\[
|e^{i\text{Re}(t \sum_{\ell \in L_p} \gamma_\ell Y_\ell)} - e^{i\text{Re}(t \sum_{\ell \in L_p} Y_\ell)}| \leq \sum_{\ell \in L_p} |e^{i\text{Re}(t \gamma_\ell Y_\ell)} - e^{i\text{Re}(t Y_\ell)}| \leq |t| \sum_{\ell \in L_p} |\gamma_\ell - 1||Y_\ell|.
\]

Therefore we can use the argument of Lemma 2.5. We obtain the CLT for \(Z_n^\theta, p\) for a fixed \(p\) with respect to \(\mathbb{P}\) with the same variance for the limit law as for the process \(\frac{1}{\sqrt{n}} \sum_{\ell \in L_p} e^{2\pi i \langle \ell, x \rangle} \sum_{k=0}^{n-1} e^{2\pi i \langle \ell, k \theta \rangle} f_k \circ \tau^k\) with respect to \(\lambda \times \mathbb{P}\).

Hence we obtain \(Z_n^\theta, p \xrightarrow{\mathbb{P}} N(0, \sigma_p(\theta))\), with for a.e. \(\theta\) (cf. (15))

\[
\sigma_\theta^2(\theta) = \sum_{\ell, \rho \in \mathbb{Z}^d \setminus \{0\}} |\gamma_{j, \ell}(\langle \ell, \theta \rangle)|^2.
\]

Since \(\sigma^2(\theta) = \lim_p \sigma^2_p(\theta) = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \sum_{j, \rho} |\gamma_{j, \ell}(\langle \ell, \theta \rangle)|^2\), we obtain the result. The convergence of the variance will be proved below.

**Corollary 2.9.** Let \(h\) be a continuous function on \(\mathbb{T}^d\) with \(\theta\) integral such that \(\sum_{\ell \in \mathbb{Z}^d} |\hat{h}(\ell)|^2 < \infty\). Let \(f\) be in \(L^2(\Omega)\). Then, for a.e. \(\theta \in \mathbb{T}^d\), the CLT holds for \(\sum_{k=0}^{n-1} h(k \theta) f \circ \tau^k\) with respect to \(\mathbb{P}(d\omega)\) and \(\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} h(k \theta) T^k f^2 = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} |\hat{h}(\ell)|^2 \varphi_f(\langle \ell, \theta \rangle)\).

**Proof.** We apply Theorem 2.8 to \(F(\theta, \omega) = h(\theta)f(\omega) = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} e^{2\pi i \langle \ell, \theta \rangle} \hat{h}(\ell) f(\omega)\).

**Remark 2.10.** 1) The previous results extend to vector valued functions.

2) Let \(P(\tau)\) be the Pinsker factor of a dynamical systems \((\Omega, A, \mathbb{P}, \tau)\). The previous results extend to functions \(F\) in \(L^2(\mathbb{T}^d \times \Omega)\) with \(f_\ell\) orthogonal to \(L^2(\mathbb{P}(\tau))\) for every \(\ell \in \mathbb{Z}^d \setminus \{0\}\), under the condition \(\sum \|f_\ell\|^2 < +\infty\).

They are valid for dynamical systems such that \(\mathbb{P}(\tau)\) has a singular spectrum (with a possibly degenerated limit law, see Corollary 1.5).

**Computation of the variance** Here we give a proof of the convergence of the variance as stated in Theorem 2.8. For this result we need only to assume that \(\tau\) is weakly mixing in order that the cross spectral measures with respect to the unitary operator \(T\) induced by \(\tau\) are without atoms.

**Theorem 2.11.** If \(F\) with Fourier series \(F(x, \omega) = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} e^{2\pi i \langle \ell, x \rangle} f_\ell(\omega)\) is such that \(\sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \|f_\ell\|^2 < +\infty\), then for a.e. \(\theta \in \mathbb{T}^d\)

\[
\lim_{n \to \infty} \frac{1}{n} \|\sum_{k=0}^{n-1} F(k \theta, \tau^k \omega)\|_{L^2(\mathbb{P}(d\omega))}^2 = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \varphi_{f_\ell}(\langle \ell, \theta \rangle).
\]
Proof. From Proposition 1.8 applied with distinct $\theta_1 = \ell \theta$ and $\theta_2 = \tilde{\ell} \theta$, we obtain the result when $F$ is trigonometric polynomial. We consider now the general case.

We put as above $S_{\ell,n}^\theta(\omega) := \sum_{k=0}^{n-1} e^{2\pi i k/\ell \theta} f_k(\tau^k \omega)$. The assumption of the theorem and Inequality (16) in Theorem 2.8 imply

$$
\sum_{\ell \neq 0} \sup_n \left\| \frac{1}{\sqrt{n}} S_{\ell,n}^\theta \right\| < +\infty, \text{for a.e. } \theta. \tag{17}
$$

It follows

$$
\frac{1}{n} \left\| \sum_{k=0}^{n-1} F(\ell \theta, \tau^k \cdot) \right\|^2 = \left\| \lim_{L \to \infty} \frac{1}{\sqrt{n}} S_{\ell,n}^\theta \right\|^2 = \left\| \sum_{\ell \neq 0} \frac{1}{\sqrt{n}} S_{\ell,n}^\theta \right\|^2.
$$

So, actually we need to compute $\lim_n \lim_L \left\| \sum_{\ell \neq 0} \frac{1}{\sqrt{n}} S_{\ell,n}^\theta \right\|^2$. After expanding the square of the norm, we have to compute the following two limits:

$$
\lim_n \sum_{\ell \neq 0} \frac{1}{n} \left\| S_{\ell,n}^\theta \right\|^2 \text{ and } \Re \left\{ \lim_n \sum_{\ell \neq 0} \left( \frac{1}{\sqrt{n}} S_{\ell,n}^\theta, \frac{1}{\sqrt{n}} S_{\tilde{\ell},n}^\theta \right) \right\}. \tag{18}
$$

By (17) $\sum_{\ell \neq 0} \sup_n \frac{1}{n} \left\| S_{\ell,n}^\theta \right\|^2 < +\infty$ and we may use Lebesgue dominated convergence theorem to permute $\lim_n$ with the sum in the first expression of (18). This expression converges to the desired limit of the theorem. For the second term, by Lemma 1.7 we have that $\lim_n \left( \frac{1}{\sqrt{n}} S_{\ell,n}^\theta, \frac{1}{\sqrt{n}} S_{\tilde{\ell},n}^\theta \right) = 0$ for $\ell \neq \tilde{\ell}$ and for every $\theta \in \mathbb{T}^d$ with rationally independent components. Since

$$
\sum_{\ell \neq \tilde{\ell}} \sup_n \left| \left( \frac{1}{\sqrt{n}} S_{\ell,n}^\theta, \frac{1}{\sqrt{n}} S_{\tilde{\ell},n}^\theta \right) \right| \leq \left( \sum_{\ell} \sup_n \left\| \frac{1}{\sqrt{n}} S_{\ell,n}^\theta \right\|^2 \right)^{1/2},
$$

we may use (17) and Lebesgue’s dominated convergence theorem to invert $\lim_n$ with the sum in the second expression of (18) to conclude that the limit for $n$ tending to $\infty$ of the mixed terms is $0$. \hfill \Box

2.2. Invariance principle for a.e. $\theta$ for modified sums. Let $(\Omega, \mathbb{P}, \tau)$ be a $K$-system and $(T^n f)$ a process generated by $f \in L^2_0(\Omega, \mathbb{P})$ where $T f = f \circ \tau$. Let us consider for a parameter $\alpha$ in $[0, 1]$ the modified rotated sums

$$
S_{n,\alpha}^\theta f = \sum_{k=0}^{n-1} e^{2\pi i k/\ell \theta} \left( 1 - \left( \frac{k}{n} \right)^\alpha \right) T^k f
$$

and the interpolated normalized continuous time process defined for $t \in [0, 1]$ (with the convention $\sum_{k=0}^{-1} = 0$) by

$$
W_{n,t,\alpha}^\theta f = \frac{t^\alpha}{\sqrt{n}} \sum_{k=0}^{[nt]-1} e^{2\pi i k/\ell \theta} \left( 1 - \left( \frac{k}{nt} \right)^\alpha \right) T^k f.
$$

For $\alpha \in [0, 1]$, we have $\lim_n \frac{1}{n} \sum_{k=1}^{n-1} \left( 1 - \left( \frac{k}{n} \right)^\alpha \right)^2 = \frac{2\alpha^2}{(1+\alpha)(1+2\alpha)}$. A computation analogous to that in [13] gives the following result:

Proposition 2.12. Let $\alpha$ be in $[0, 1]$. We have for a.e. $\theta$

$$
\| \sup_{0 \leq t \leq 1} |W_{n,t,\alpha}^\theta f - W_{n,t,\alpha}^\theta \mathcal{M}_\theta f|_2 \to 0. \tag{19}
$$
Proof. The inequalities \[ \sum_{j=k}^{n} \int_{j}^{j+1} \frac{dt}{t^{1-\alpha}} \leq \sum_{j=k}^{n} \frac{1}{j^{1-\alpha}} \leq \sum_{j=k}^{n} \int_{j}^{j+1} \frac{dt}{t^{1-\alpha}} \] imply

\[ \sum_{j=k}^{n} \frac{1}{j^{1-\alpha}} = \frac{1}{\alpha} (n^\alpha - k^\alpha) + \rho(k,n), \]

with \( 0 \leq \rho(k,n) \leq \int_{k-1}^{n} \frac{dt}{t^{1-\alpha}} - \int_{k}^{n} \frac{dt}{t^{1-\alpha}} = \frac{k^\alpha - (k-1)^\alpha}{\alpha}. \) (20)

We have the following relation,

\[ \frac{1}{n^{\frac{\alpha}{2}+\alpha}} \sum_{k=1}^{[nt]} S_k^\theta f k^{1-\alpha} = \frac{1}{n^{\frac{\alpha}{2}+\alpha}} \sum_{k=1}^{[nt]} e^{2\pi i(k-1)/\theta} \left( \frac{1}{k^{1-\alpha}} + \ldots + \frac{1}{[nt]^{1-\alpha}} \right) T^{k-1} f \]

\[ = \frac{1}{\alpha} \frac{1}{n^{\frac{\alpha}{2}+\alpha}} \sum_{k=1}^{[nt]} e^{2\pi i(k-1)/\theta} \left( [nt]^{\alpha} - k^\alpha + \rho(k,[nt]) T^{k-1} f \right) \]

\[ = \frac{1}{\alpha} \left( \frac{[nt]}{nt} \right)^\alpha \frac{l^{\alpha}}{n^{\frac{\alpha}{2}+\alpha}} \sum_{k=1}^{[nt]} e^{2\pi i(k-1)/\theta} (1 - (\frac{k}{[nt]})^\alpha) T^{k-1} f \]

\[ + \frac{1}{n^{\frac{\alpha}{2}+\alpha}} \sum_{k=1}^{[nt]} e^{2\pi i(k-1)/\theta} \rho(k,[nt]) T^{k-1} f; \]

hence:

\[ W_{n,t,\alpha}^\theta f \]

\[ = \left( \frac{nt}{[nt]} \right)^\alpha \frac{\alpha}{n^{\frac{\alpha}{2}+\alpha}} \sum_{k=1}^{[nt]} S_k^\theta f k^{1-\alpha} - \left( \frac{nt}{[nt]} \right)^\alpha \frac{\alpha}{n^{\frac{\alpha}{2}+\alpha}} \sum_{k=1}^{[nt]} e^{2\pi i(k-1)/\theta} \rho(k,[nt]) T^{k-1} f. \]

Since \( \frac{nt}{[nt]} \leq 2 \) for \( \frac{1}{n} \leq t \leq 1 \), putting \( \theta(n,t) = (\frac{nt}{[nt]})^\alpha \alpha \), we have from the previous relation:

\[ \sup_{\frac{1}{n} \leq t \leq 1} |W_{n,t,\alpha}^\theta f| \]

\[ \leq \sup_{\frac{1}{n} \leq t \leq 1} \left[ \theta(n,t) \frac{1}{n^{\frac{\alpha}{2}+\alpha}} \sum_{k=1}^{[nt]} \frac{|S_k^\theta f| k^{1-\alpha}}{k^{1-\alpha}} \right] + \sup_{\frac{1}{n} \leq t \leq 1} \left[ \frac{1}{n^{\frac{\alpha}{2}+\alpha}} \sum_{k=1}^{[nt]} \rho(k,[nt]) |T^{k-1} |f| \right] \]

\[ \leq \sup_{\frac{1}{n} \leq t \leq 1} \left[ \frac{2}{n^{\frac{\alpha}{2}+\alpha}} \sum_{k=1}^{n} \frac{|S_k^\theta f| k^{1-\alpha}}{k^{1-\alpha}} \right] + \sup_{\frac{1}{n} \leq t \leq 1} \left[ \frac{2}{n^{\frac{\alpha}{2}+\alpha}} \sum_{k=1}^{n} \frac{[k^\alpha - (k-1)^\alpha] T^{k-1} |f|}{k^{1-\alpha}} \right] \]

\[ \leq \frac{2}{n^{\frac{\alpha}{2}+\alpha}} \sum_{k=1}^{n} \frac{|S_k^\theta f| k^{1-\alpha}}{k^{1-\alpha}} + \frac{2}{n^{\frac{\alpha}{2}+\alpha}} \sum_{k=1}^{n} \frac{[k^\alpha - (k-1)^\alpha] T^{k-1} |f|}{k^{1-\alpha}}. \]
We use the previous inequality with \( f - M_\theta f \) instead of \( f \). By the triangle inequality, Cauchy-Schwarz inequality and (20) it follows:

\[
\| \sup_{0 \leq t \leq 1} |W_{n,t,\alpha}^\theta f - W_{n,t,\alpha}^\theta M_\theta f| \|_2 \leq \frac{1}{n^{2+\alpha}} \sum_{k=1}^{n} \| S_k^\theta f - S_k^\theta M_\theta f \|_2 + \frac{n^{\alpha}}{n^{2+\alpha}}(\|f\|_2 + \|M_\theta f\|_2)
\]

As \( \| S_k^\theta f - S_k^\theta M_\theta f \|_2 = o(k) \) for a.e. \( \theta \), this implies (19).

**Theorem 2.13.** For a.e. \( \theta \in \mathbb{T} \), the real and imaginary parts of the process \((W_{n,t,\alpha}^\theta)\) are asymptotically independent and satisfy the invariance principle with convergence to \( \frac{1}{2} \varphi_f(\theta) Z_\alpha(t) \), where \( Z_\alpha(t) \) is a Gaussian continuous process with covariance for \( s \leq t \),

\[
\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)} \frac{s^{2-\alpha}}{(1-\alpha)(2-\alpha)} \left[ t^{1-\alpha} - \frac{1}{3} s^{1-\alpha} \right].
\]

**Proof.** By Proposition 2.12 it is enough to prove the statement for the process \((W_{n,t,\alpha}^\theta M_\theta f)\). We use the Cramér-Wold device [3] to identify the 2-d limit. Then the proof goes along the same lines as in Peligrad-Peligrad [13]. Since the system is weakly mixing the factor \( \frac{1}{2} \varphi_f(\theta) \) comes from the following limit, for a.e. \( \theta \)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \cos^2(2\pi k\theta) T^k|M_\theta f|^2 = \frac{1}{2} \varphi_f(\theta), \text{ a.e. and in } L^1(\mathbb{P})
\]

and the same with \( \sin^2(2\pi k\theta) \). The asymptotic independence follows from the fact that for a.e. \( \theta \) the asymptotic correlation between the real and imaginary parts of the process is zero a.e. and in \( L^1(\mathbb{P}) \)-norm.

The above result is an a.e. result, but for modified sums. In [14] an invariance principle for the usual rotated sums is shown for the product measure. We present below their result (and include a proof for the sake of completeness) which is valid also in our setting. The key step is the following lemma, where Carleson’s result on Fourier series is used:

**Lemma 2.14.** For every regular function \( f \), the family \( \left( \frac{1}{n} \sup_{1 \leq k \leq n} |S_k^\theta f|^2 \right)_{n \geq 1} \) is uniformly integrable for the product measure \( d\theta \times \mathbb{P}(d\omega) \).

**Proof.** For every \( m \geq 1 \), we have:

\[
S_k^\theta f = \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^\ell f = \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^\ell \sum_{j=-\infty}^{\infty} \Pi_j f = \sum_{m=-m}^{m} \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^\ell \Pi_j f + \sum_{j \notin [-m,m]} \Pi_j f.
\]
First, let us consider the second term in (21). By Carleson-Hunt’s inequality, there is a constant $C$ such that:

$$\int_{\Omega} \max_{1 \leq k \leq n} \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^{\ell} \left| \sum_{j \in [-m,m]} \Pi_j f(\omega) \right|^2 d\theta \leq C \sum_{l=0}^{n-1} |T^l \sum_{j \in [-m,m]} \Pi_j f(\omega)|^2;$$

hence:

$$\int_{\Omega} \int_{\mathbb{T}} \frac{1}{n} \max_{1 \leq k \leq n} \left| \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^{\ell} \sum_{j \in [-m,m]} \Pi_j f(\omega) \right|^2 d\theta \, \mathbb{P}(d\omega)$$

$$\leq C \frac{1}{n} \int_{\Omega} \int_{\mathbb{T}} \left( \sum_{j \in [-m,m]} T^j \Pi_j f(\omega) \right)^2 \mathbb{P}(d\omega) = C \sum_{j \in [-m,m]} \|\Pi_j f\|^2 \to 0 \quad (n \to \infty). \tag{22}$$

In particular, we obtain that $\frac{1}{n} \max_{1 \leq k \leq n} |\sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^{\ell} \sum_{j \in [-m,m]} \Pi_j f(\omega)|^2$ is uniformly integrable for the product measure $d\theta \times \mathbb{P}(d\omega)$. Now we consider the first term in (21). For each $j \in [-m, m]$ and $\theta \in \mathbb{T}$, the sequence $M_j^\theta(k, \omega) := \sum_{\ell=0}^{k-1} e^{2\pi i \ell \theta} T^\ell \Pi_j f(\omega)$ is a square integrable martingale with respect to $(\mathcal{A}_{j+k-1})_{k \geq 1}$. We conclude that $(M_j^\theta(k, \omega))_{k \geq 1}$ is a square integrable martingale with respect to $(\mathcal{B}(T) \otimes \mathcal{A}_{j+k-1})_{k \geq 1}$. Proposition 1(a) in Dedecker and Rio [4] yields that $(\frac{1}{n} \max_{1 \leq k \leq n} |M_j^\theta(k, \omega)|^2)_{n \geq 1}$ is uniformly integrable for $d\theta \times \mathbb{P}(d\omega)$.

As a finite sum of uniformly integrable sequences is a uniformly integrable sequence, the sequence $\sum_{j \in [-m,m]} \frac{1}{n} \max_{1 \leq k \leq n} |M_j^\theta(k, \omega)|^2$ is uniformly integrable. This observation and (22) imply the assertion for $\frac{1}{n} \max_{1 \leq k \leq n} |S_k^{\theta}|^2$.

As a consequence of standard results from Billingsley [1] now it follows:

**Theorem 2.15 ([14]).** Under the product measure $d\theta \times \mathbb{P}(d\omega)$, the process $\left(\frac{1}{\sqrt{n}}S_{\lfloor nt \rfloor}^{\theta}\right)$ is tight in $D(0,1)$ and

$$\left(\Re\left\{\frac{1}{\sqrt{n}}S_{\lfloor nt \rfloor}^{\theta}\right\}, \Im\left\{\frac{1}{\sqrt{n}}S_{\lfloor nt \rfloor}^{\theta}\right\}\right) \Rightarrow \sqrt{\frac{1}{2}} \varphi_f(U)(W'(t), W''(t)), $$

where $U$ is a random variable uniformly distributed on $[0, 2\pi]$ and $W'(t)$ and $W''(t)$ are two independent standard Brownian motions independent of $U$.

### 2.3. Application to recurrence of rotated stationary walks in $\mathbb{C}$.

Let us consider an ergodic dynamical system $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ and a measurable function $\varphi$ on $\Omega$ with values in $\mathbb{R}^d$, $d \geq 1$. The skew product $\tau_\varphi : (\omega, y) \to (\tau \omega, y + \varphi(\omega))$ leaves invariant the $\sigma$-finite measure $\mathbb{P} \times dy$. It is known that it is either dissipative or conservative with respect to this measure. The latter case occurs if and only if the “stationary walk” (or “cocycle”) $S_n \varphi(\omega) := \sum_{k=0}^{n-1} \varphi(\tau^k \omega)$ is recurrent, i.e., for a.e. $\omega$, $S_{n_\omega} \varphi(\omega)$ belongs to an arbitrary neighborhood of 0 for an infinite sequence of times $n_k(\omega)$.

For $d = 1$, when $\varphi$ is integrable, the condition $\int \varphi \, d\mathbb{P} = 0$ is necessary and sufficient for recurrence. The construction of 2-dimensional stationary walks deserves attention, since, as shown by the i.i.d. case, the dimension $d = 2$ is critical for the recurrence property. We use the following sufficient recurrence criterion:

**Recurrence criterion** (K. Schmidt [17]) Let $B(\eta)$ be the ball of radius $\eta > 0$ centered at the origin in $\mathbb{R}^2$. If there exists $\delta > 0$ such that, for every $\eta > 0$, we have $\lim_{n} \mathbb{P}(n^{-1/2}S_n \varphi \in B(\eta)) \geq \delta \eta^2$, then $(S_n \varphi)$ is recurrent in $\mathbb{R}^2$. 

We apply the preceding results to rotated 2-dimensional walks, by modifying the increments of the cocycle at each step by a rotation.

**Theorem 2.16.** Let \((\Omega, \mathbb{P}, \tau)\) be a K-system or an exact system. Let \(\varphi \in L^2(\mathbb{P})\) with values in \(\mathbb{C}\). For a.e. \(\theta \in \mathbb{R}\), the process \(\sum_{k=0}^{n-1} e^{2\pi i k \theta} \varphi(\tau^k \omega)\) is recurrent and the map \(\tau_{\theta, \varphi} : (\omega, z) \rightarrow (\tau \omega, e^{2\pi i \theta} z + \varphi(\omega))\) from \(\Omega \times \mathbb{C}\) into itself is conservative.

**Proof.** We consider \((T^1 \times \Omega, \lambda \otimes \mathbb{P}, \tau_0)\), where \(\tau_0 : (x, \omega) \rightarrow (x + \theta \mod 1, \tau \omega)\). Let \(F(x, \omega) := e^{2\pi i x} \varphi(\omega)\).

Since \(F\) satisfies the CLT by Theorem 2.3 (see Remark 2.10), the recurrence criterion implies that for a.e. \(\theta\), for a.e. \((x, \omega)\), the process \(\sum_{k=0}^{n-1} F(x + k \theta, \tau^k \omega)\) is recurrent in any neighborhood of 0. Hence for a.e. \(\theta\) the process \(\sum_{k=0}^{n-1} e^{2\pi i k \theta} \varphi(\tau^k \omega)\) is also recurrent.

The map \(\tau_{\theta, \varphi}\) defines a measure preserving transformation on \(\Omega \times \mathbb{C}\), where \(m\) is the Lebesgue measure on \(\mathbb{R}^2\). By iteration of this map, we get: 
\[ \tau^{n \theta}_{\varphi} = (\tau^n \omega, e^{2\pi i n \theta} z + \varphi_n(\theta, \omega)), \]
with \(\varphi_n(\theta, \omega) = \sum_{k=0}^{n-1} e^{2\pi i (n-1-k) \theta} \varphi(\tau^k \omega)\) which behaves in modulus like the rotated ergodic sums \(\sum_{k=0}^{n-1} e^{2\pi i k \theta} \varphi(\tau^k \omega)\).

This implies the conservativity of the map \(\tau_{\theta, \varphi}\) for the set (of full measure) of values of \(\theta\) such that \(\sum_{k=0}^{n-1} e^{-2\pi i k \theta} \varphi(\tau^k \omega)\) is recurrent. Indeed, suppose that, for such a \(\theta\), \(\tau_{\theta, \varphi}\) is not conservative on \(\Omega \times \mathbb{C}\) for the measure \(\mu := d\mathbb{P}(\omega) \otimes dm(z)\). Then there is a measurable subset \(D\) of positive measure in \(\Omega \times \mathbb{C}\) such that the sets \(\tau_{\theta, \varphi}^n D\), \(n \in \mathbb{Z}\), are pairwise disjoint. Let \(K\) be a compact disk in \(\mathbb{C}\) centered at the origin such that \((\Omega \times K) \cap D\) has a positive measure. Let \(V\) be a compact neighborhood of zero in \(\mathbb{C}\) and \(B_K := \Omega \times (K + V)\). We have \(\sum_{n=0}^{\infty} \mu(\tau_{\theta, \varphi}^n B_K \cap D) = \sum_{n \geq 0} \mu(B_K \cap \tau_{\theta, \varphi}^n D) \leq \mu(B_K) < +\infty\). Therefore, by the Borel-Cantelli Lemma, for a.e. \((\omega, z)\) there is \(N(\omega, z)\) such that \((\omega, z) \notin \tau_{\theta, \varphi}^{-n} B_K \cap D\) for \(n > N(\omega, z)\).

Hence, for a.e. \((\omega, z)\) in \(D\), \(\tau_{\theta, \varphi}^n(\omega, z)\) is not in \(B_K\) for \(n\) big enough. This contradicts that, for a.e. point of \((\Omega \times K) \cap D\), \(\tau_{\theta, \varphi}^n(\omega, z)\) belongs to \(B_K\) infinitely often by recurrence of \(e^{2\pi i n \theta} z + \varphi_n(\theta, \omega)\) in \(B_K\).

Another situation is provided by isometric extensions.

**Theorem 2.17.** Let \((\Omega, \mathcal{A}, \mathbb{P}, \tau)\) be a K-system or an exact system. Let \(\varphi\) be a measurable real function on \(\Omega\) and \(\varphi_k(\omega) := \varphi(\omega) + \varphi(\tau\omega) + \ldots + \varphi(\tau^{k-1}\omega)\). For a.e. \(\theta\) the asymptotic distribution of \(n^{-1/2} \sum_{k=0}^{n-1} e^{2\pi i k \theta} e^{2\pi i \varphi_k}\) with respect to \(\mathbb{P}\) is a normal law. For a.e. \(\theta\), for \(\mathbb{P}\)-a.e. \(\omega\) the process \(\sum_{k=0}^{n-1} e^{2\pi i k \theta} e^{2\pi i \varphi_k(\omega)}\) visits infinitely often every neighborhood of 0.

**Proof.** The isometric extension \(\tau_{\varphi} : (\omega, y) \rightarrow (\tau \omega, y + \varphi(\omega) \mod 1)\) defines a measure preserving dynamical system on \(\Omega \times T^1\). Its Pinsker factor \(\mathcal{P}(\tau_{\varphi})\) is invariant for the circle action on \(\Omega \times T^1\) for the second coordinate. The sets which are invariant by this action belong to the Pinsker factor of \(\tau\), hence are trivial. Therefore the circle action on \(\mathcal{P}(\tau_{\varphi})\) is ergodic. Since it has a discrete spectrum, the map \(\tau_{\varphi}\), which commutes with this action, has also a discrete (here singular) spectrum in restriction to \(\mathcal{P}(\tau_{\varphi})\).

Let \(F \in L^2(\mathbb{P} \times dy)\). By Corollary 2.4 and Remark 1.6, the CLT (with a limiting law which can be degenerated) is satisfied for a.e. \(\theta\) for the rotated ergodic sums \(S^\theta_n F := \sum_{k=0}^{n-1} e^{2\pi i k \theta} F \circ \tau_{\varphi}^k, n \geq 1\).
In particular consider the function \( F(\omega, y) := e^{2\pi iy} \) on \( \Omega \times T^1 \). Its ergodic sums for the action of \( \tau_\varphi \) read \( e^{2\pi iy} \sum_{k=0}^{n-1} e^{2\pi ik\theta} e^{2\pi i \varphi_k(\omega)} \). The result follows from the previous theorem.

**Example.** The map \( \tau : x \to 2x \mod 1 \) is exact for the Lebesgue measure on \( T^d \). For \( \varphi \) on the circle, let \( \varphi_k(x) := \varphi(x) + \varphi(2x) + \ldots + \varphi(2^{k-1}x) \). It is known that, if \( \varphi \) is Hölderian, the process \( (\sum_{n=0}^{n-1} e^{2\pi i \varphi_n}) \) is recurrent in \( C \) (cf. [8]). The previous theorem shows that for any measurable real function \( \varphi \) the process \( (\sum_{n=0}^{n-1} e^{2\pi ik\theta} e^{2\pi i \varphi_n})_{n \geq 1} \) is recurrent in \( C \) for a.e. \( \theta \).

3. **CLT for rotated \( Z^d \)-actions.** The above results for rotated ergodic sums rely on the duality between \( Z \), the group action corresponding to 1-dimensional discrete time dynamical systems, and the group of the circle. In higher dimension this suggests that the same method applies to multidimensional abelian actions. In this section, we show how to obtain a CLT for rotated ergodic sums of \( L^2 \)-functions on \( \mathbb{Z}^d \) dynamical systems \((d > 1)\) with \( K \)-property which allows to use martingale methods as in the previous section. For simplicity we take \( d = 2 \).

We write \( n = (n_1, n_2) \) for an element of \( \mathbb{Z}^2 \) and we put on \( \mathbb{Z}^2 \) the lexicographic order \( n \leq p \iff (n_1 < p_1) \ or (n_1 = p_1 \ and \ n_2 \leq p_2) \).

**Spectral properties** Part of the spectral considerations of section 1.1 extends to a unitary action of a discrete abelian group on a Hilbert space. For an action of \( \mathbb{Z}^d \), the Fourier analysis on \( T^1 \), used for the spectral properties of a single map, is replaced by the Fourier analysis on the \( d \)-dimensional torus. First let us recall some classical facts about Fourier series in several variables. For simplicity, we consider two variables.

Using the usual one-dimensional Fejér kernel, the two-dimensional Fejér kernel on \( T^2 \) corresponding to rectangles is defined as \( K_{N,P}(\theta_1, \theta_2) = K_N(\theta_1) \cdot K_P(\theta_2) \).

For an integrable function \( \varphi \) on \( T^2 \) with Fourier coefficients \((\hat{\varphi}_{k,\ell})_{k,\ell \in \mathbb{Z}} \), the Fejér kernel applied to \( \varphi \) gives

\[
K_{N,P}(\varphi)(\theta_1, \theta_2) := (K_{N,P} \ast \varphi)(\theta_1, \theta_2)
= \frac{1}{NP} \sum_{n=1}^{N} \sum_{p=1}^{P} \sum_{k=-n+1}^{n-1} \sum_{\ell=-p+1}^{p-1} \hat{\varphi}_{k,\ell} e^{2\pi i (k\theta_1 + \ell\theta_2)}.
\]

We have the following results (cf. Zygmund [19, Ch. XVII], Th. 2.14, Th. 3.1):

**Proposition 3.1.** (i) If \( \varphi \) is integrable, then \( \lim_{N \to \infty} K_{N,N}(\varphi) = \varphi \) a.e.

(ii) If \( \varphi \log^+ |\varphi| \) is integrable, then \( \lim_{\min(N,P) \to \infty} K_{N,P}(\varphi) = \varphi \) a.e.

**Rotated ergodic sums and approximation** Let \( \mathcal{H} \) be a Hilbert space. If \( T_1, T_2 \) are two commuting unitary operators on \( \mathcal{H} \), they define a \( \mathbb{Z}^2 \)-unitary action on \( \mathcal{H} \) by \( f \in \mathcal{H} \mapsto T_1^{k_1} T_2^{k_2} f, \ k \in \mathbb{Z}^2 \).

The rotated ergodic sum corresponding to a rectangle \( R_{N,L} = \{0,N\times[0,L]\} \) is, for \( f \) in \( \mathcal{H} \),

\[
S_{N,L}^{\theta_1,\theta_2} f = \sum_{k \in R_{N,L}} e^{2\pi i (k_1\theta_1 + k_2\theta_2)} T_1^{k_1} T_2^{k_2} f.
\]

For \( L = N \) we simply write \( R_N \) for \( R_{N,N} \) and \( S_{N}^{\theta_1,\theta_2} f \). The variance of the rotated 2-dimensional process (for squares), when it exists, is \( \lim_N \frac{1}{N^2} \|S_N^{0,\theta_2} f\|_2^2 \).
Suppose that $\mathcal{K}_0$ is a closed subspace of $\mathcal{H}$ such that the subspaces $T_1^{k_1}T_2^{k_2}\mathcal{K}_0$ are pairwise orthogonal. If $(\psi_j)_{j \in J}$ is an orthonormal basis of $\mathcal{K}_0$, the family $(T_1^{n_1}T_2^{n_2}\psi_j, j \in J, 1 \leq j \leq T^2)$ is an orthonormal basis of the closed subspace $\mathcal{H}_\infty$ generated by these subspaces.

**Notations 3.2.** For $f \in \mathcal{H}_\infty$, we set $a_{j,n} := (f, T_1^{n_1}T_2^{n_2}\psi_j)$. For $j \in J$, let $\gamma_j$ be an everywhere finite square integrable function on $T^2$ with Fourier coefficients $a_{j,n}$. The density of the spectral measure for the $\mathbb{Z}^2$-action is $\varphi_f = \sum_{j \in J} |\gamma_j|^2$.

Since
\[ \int_{T^2} \sum_{j \in J} |\gamma_j(\theta_1, \theta_2)|^2 \, d\theta_1 d\theta_2 = \sum_{j \in J} \sum_{n \in \mathbb{Z}^2} |a_{j,n}|^2 = \int \varphi_f(\theta_1, \theta_2) \, d\theta_1 d\theta_2 = \|f\|^2 < \infty, \]
the set $\Lambda_0 := \{ \theta \in T^2 : \sum_{j \in J} |\gamma_j(\theta)|^2 < \infty \}$ has full measure. For $\theta \in \Lambda_0$, let
\[ M_0 f := \sum_{j} \gamma_j(\theta) \psi_j \in \mathcal{K}_0. \] (23)

With a proof analogous to that of Proposition 1.4, Proposition 3.1 implies:

**Proposition 3.3.** For $f \in \mathcal{H}_\infty$, the set $\Lambda(f)$ of elements $\theta = (\theta_1, \theta_2)$ in $\Lambda_0$ such that
\[ \lim_N \frac{1}{N^2} \left\| \sum_{k \in \mathbb{Z}^2} e^{2\pi i (k_1 \theta_1 + k_2 \theta_2)} T_1^{k_1}T_2^{k_2} (f - M_0 f) \right\|^2_2 = 0. \]
has full measure in $T^2$.

**K-systems** Now we consider two commuting automorphisms $\tau_1$ and $\tau_2$ of a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. They define a measure preserving action of $\mathbb{Z}^2$. We write $T_1 f$ (resp. $T_2 f$) for $f \circ \tau_1$ (resp. $f \circ \tau_2$). Recall that the entropy of such an action can be defined, as well as the Pinsker factor which is the largest factor with zero entropy. The action has a completely positive entropy if every non trivial factor has positive entropy (equivalently its Pinsker factor is trivial). The notion of $K$-system can be extended to a $\mathbb{Z}^2$-action in the following way:

For a sub $\sigma$-algebra $\mathcal{B}$ of $\mathcal{A}$, we denote by $\mathcal{B}^i$, $i = 1, 2$, the $\sigma$-algebra generated by $(\bigcup_{n=-N}^N \tau_i^n \mathcal{B}, N \geq 1)$.

A sub-$\sigma$-algebra $\mathcal{A}_0$ is increasing if $(n', p') \leq (n, p) \Rightarrow \tau_1^{n'} \tau_2^{p'} \mathcal{A}_0 \subset \tau_1^n \tau_2^p \mathcal{A}_0$. An increasing sub-$\sigma$-algebra $\mathcal{A}_0$ of $\mathcal{A}$ has the property $K$ if
(i) it is generating: the sub-$\sigma$-algebra is generated by $\cup_{n,p} \tau_1^n \tau_2^p \mathcal{A}_0$ is $\mathcal{A}$;
(ii) its remote past is trivial, i.e., $\bigcap_1 \tau_2^p \mathcal{A}_0 = \tau_1^{-1} \mathcal{A}_0$ and $\bigcap_1 \tau_1^p \mathcal{A}_0$ is trivial.

The system is a $K$-system if there exists an increasing sub $\sigma$-algebra with the property $K$. In that case, the system has completely positive entropy. Conversely B. Kamiński ([11]) proved the existence of a sub $\sigma$-algebra with the $K$-property, when the action has completely positive entropy.

**Examples.** 1) The simplest examples of $K$-systems for a $\mathbb{Z}^2$-action are the $\mathbb{Z}^2$-Bernoulli schemes (full 2-shift) defined as the shift action on the coordinates on the product $\Omega = I^{\mathbb{Z}^2}$ endowed with a product probability measure $p^\otimes \mathbb{Z}^2$, where $p$ is a probability vector on a finite space $I$.

2) For $K$-systems of algebraic origin see the book of K. Schmidt ([16]).
3) Examples are also provided by statistical mechanics (for instance, see [10] for examples of $\mathbb{Z}^2$-systems which are $K$ but not Bernoulli).

**Rotated ergodic sums and martingale approximation** Let $\mathcal{A}_0$ be a sub $\sigma$-algebra and $K_0$ be a subspace in $L^2(\mathcal{A}_0)$ of functions which satisfy

$$\mathbb{E}(M_0 f \circ \tau_{k_1}^{1} \tau_{k_2}^{2} | \tau_{k_1}^{-1} \tau_{k_2}^{-2} \mathcal{A}_0) = 0, \quad \forall k > k', \quad (24)$$

Let $f$ be in $L^2_0(\mathbb{P})$. Let $M_{(\theta_1, \theta_2)} \in K_0$ be associated to $f$ for a.e. $(\theta_1, \theta_2)$ by (23).

**Theorem 3.4.**

$$\frac{1}{n} \sum_{k \in \mathbb{R}^n} M_0(\tau_1^{k_1} \tau_2^{k_2} \omega) \xrightarrow{distr} \mathcal{N}(0, \Gamma(\theta))$$

with respect to $\mathbb{P}$, where $\Gamma(\theta)$ is the covariance matrix

$$\Gamma(\theta) = \left( \begin{array}{cc} \frac{1}{2} \varphi_f(\theta_1, \theta_2) & 0 \\ 0 & \frac{1}{2} \varphi_f(\theta_1, \theta_2) \end{array} \right).$$

**Proof.** Although we consider the 2-dimensional case, as long as we deal with finite sums, by ordering, we may consider them as 1-dimensional sums. The ergodic theorem for the ergodic means on squares implies the convergence of the conditional variance. The theorem then follows from Theorem 3.2 in [9] for martingales (cf. Remark 2.6).

Now we can apply the martingale approximation to the 2-dimensional rotated ergodic sums.

**Theorem 3.5.** Let $(\Omega, \mathcal{A}, \mathbb{P}, (\tau_1, \tau_2))$ be a 2-d $K$-system. Let $f$ be in $L^2_0(\mathbb{P})$ with spectral density $\varphi_f$. Then for Lebesgue-a.e. $\theta$ the asymptotic distribution (with respect to $\mathbb{P}$) of

$$\left( \frac{1}{n} \sum_{k \in \mathbb{R}^n} \cos(2\pi(k_1 \theta_1 + k_2 \theta_2)) f(\tau_1^{k_1} \tau_2^{k_2} \omega), \frac{1}{n} \sum_{k \in \mathbb{R}^n} \sin(2\pi(k_1 \theta_1 + k_2 \theta_2)) f(\tau_1^{k_1} \tau_2^{k_2} \omega) \right)$$

is the centered normal law in $\mathbb{R}^2$ with covariance matrix

$$\frac{1}{2} \left( \begin{array}{cc} \varphi_f(\theta_1, \theta_2) & 0 \\ 0 & \varphi_f(\theta_1, \theta_2) \end{array} \right).$$

If the spectral density $\varphi_f$ is in $L^1(\log L)$, then the result is valid for ergodic sums on rectangles.

**Proof.** By Proposition 3.3, the mean square approximation of the bi-dimensional process by a bi-dimensional array satisfying a martingale property holds. The result then follows from the previous theorem. The asymptotic covariance computation follows from the same argument as in the proof of Theorem 2.13.

If $\varphi_f$ is in $L^1(\log L)$, ii) of Proposition 3.1 allows to extend the result to ergodic sums on sequences of rectangles $R_{N,L}$ and take the limit when $\min\{N, L\} \to \infty$. 

**Examples of commuting endomorphisms** Let us consider the commuting (non invertible) endomorphisms on $\mathbb{T}^1$ given by $\tau_1 x = 2x \mod 1$ and $\tau_2 x = 3x \mod 1$ ($2$ and $3$ can be replaced by any pair of coprime integers $> 1$). They generate an action of the semi-group $\mathbb{N}^2$. The invertible extension of this action has a Lebesgue spectrum.
Let $J$ be the subset of all non-zero integers which are not divisible by 2 or 3. Every non-zero integer $n \in \mathbb{N}$ can be written in a unique way as $n = 2^{k_1}3^{k_2}j$, for some non-negative integers $k_1, k_2$ and $j \in J$.

We define $K_0$ as the closed subspace of $L_0^2(\mathbb{T}^1)$ generated by $e^{2\pi i j x}$ for $j \in J$. The subspaces $T_{1j}^kT_{2j}^kK_0$ are pairwise orthogonal and with the previous notations $\mathcal{H}_\infty = L_0^2(\mathbb{T}^1)$.

Let $f$ be a function if $L_0^2(\mathbb{T}^1)$ with Fourier series $\sum_{n \in \mathbb{Z}} \hat{f}_n e^{2\pi i n x}$. Let

$$\gamma_j(\theta_1, \theta_2) = \sum_{k_1, k_2 \geq 0} \hat{f}_{2^{k_1}3^{k_2}j} e^{2\pi i (k_1 \theta_1 + k_2 \theta_2)}.$$  

As $j \in J$ is not divisible by 2 and by 3, for any $f$ in $K_0$ and any $g$ in $L_0^2$, $T_{1j}^kT_{2j}^k f$ is orthogonal to $T_{1j}^kT_{2j}^k g$ if $k < k'$. Let $M_\theta f(x) := \sum_{j \in J} \gamma_j(\theta) e^{2\pi i j x}$. Let $\gamma_j(\theta) e^{2\pi i j x}$.

Let us mention that a Central Limit Theorem like Theorem 3.5 can be shown ([2]). The main step in the proof is a result analogous to Theorem 3.4 for the sums $\sum_{k \in \mathbb{Z}} M_\theta(\tau_{1k}^{k_1}\tau_{2k}^{k_2} x)$. This can be done by using the properties of the set of integers $(2^{k_1}3^{k_2}, (k_1, k_2) \in \mathbb{N}_0^2)$ as in [6].

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REFERENCES

[1] P. Billingsley, “Convergence of Probability Measures,” Second edition, Wiley Series in Probability and Statistics: Probability and Statistics, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1999.

[2] G. Cohen and J.-P. Conze, Central limit theorem for commutative semigroups of toral endomorphisms, preprint, (2012).

[3] H. Cramér and H. Wold, Some theorems on distribution functions, J. Lond. Math. Soc., 21 (1946), 290-294.

[4] J. Dedecker and E. Rio, On the functional central limit theorem for stationary processes, Ann. Inst. H. Poincaré Probab. Statist., 36 (2000), 1-34.

[5] G. K. Eagleson, Some simple conditions for limit theorems to be mixing, Teor. Verojatnost. i Primenen., 21 (1976), 653-660.

[6] K. Fukuyama and B. Petit, Le théorème limite central pour les suites de R. C. Baker, Ergodic Theory Dynam. Systems, 21 (2001), 479–492.

[7] A. M. Garsia, “Topics in Almost Everywhere Convergence,” Lectures in Advanced Mathematics, 4, Markham Publishing Co., Chicago, Ill., 1970.

[8] Y. Guivarc’h and J. Hardy, Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d’Anosov, Ann. Inst. H. Poincaré Probab. Statist., 24 (1988), 73–98.

[9] P. Hall and C. C. Heyde, “Martingale Limit Theory and its Application,” Probability and Mathematical Statistics, Academic Press, Inc., New York-London, 1980.

[10] C. Hoffman, A Markov random field which is $K$ but not Bernoulli, Israel J. Math., 112 (1999), 249–269.

[11] B. Kamiński, The theory of invariant partitions for $\mathbb{Z}^d$-actions, Bul. Acad. Pol. Sci. Sér. Sci. Math., 29 (1981), 349–362.

[12] B. Sz.-Nagy and C. Foiaş, “Harmonic Analysis of Operators on Hilbert Space,” Translated from the French and revised North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest, 1970.

[13] M. Peligrad and C. Peligrad, On the invariance principle under martingale approximation, Stoch. Dyn., 11 (2011), 95–105.
[14] M. Peligrad and W. B. Wu, *Central limit theorem for Fourier transforms of stationary processes*, Ann. Probab., 38 (2010), 2009–2022.

[15] V. A. Rohlin, *Lectures on the entropy of measure-preserving transformations*, Russian Mathematical Surveys, 22 (1967), 1–52.

[16] K. Schmidt, “Dynamical Systems of Algebraic Origin,” Progress in Mathematics, 128, Birkhäuser Verlag, Basel, 1995.

[17] K. Schmidt, *On joint recurrence*, C. R. Acad. Sci. Paris Sér. I Math., 327 (1998), 837–842.

[18] R. Zweimüller, *Mixing limit theorems for ergodic transformations*, J. Theor. Prob., 20 (2007) 1059–1071.

[19] A. Zygmund, “Trigonometric Series: Vols. I, II,” Second edition, Cambridge University Press, London-New York, 1968.

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