AN OPTIMIZATION PROBLEM FOR NONLINEAR STEKLOV EIGENVALUES WITH A BOUNDARY POTENTIAL

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Abstract. In this paper, we analyze an optimization problem for the first (nonlinear) Steklov eigenvalue plus a boundary potential with respect to the potential function which is assumed to be uniformly bounded and with fixed $L^1$-norm.

1. Introduction

In recent years a great deal of attention has been put on optimal design problems for eigenvalues (both linear and nonlinear) due to many interesting applications. For a comprehensive description of the current developments in the field in the case of linear eigenvalues and very interesting open problems, we refer to [12]. In the nonlinear setting, we refer to the recent research papers [3, 4, 5, 7, 8, 11] and references therein.

To be precise, the eigenvalue problem that we are interested in is the following

\begin{equation}
\begin{aligned}
-\Delta_p u + |u|^{p-2}u = 0 & \quad \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial \mathbf{n}} + \sigma \phi|u|^{p-2}u = \lambda|u|^{p-2}u & \quad \text{in } \partial \Omega.
\end{aligned}
\end{equation}

Here $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $\Delta_p u$ is the usual $p$-Laplace operator defined as $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$, $\mathbf{n}$ denotes the outer unit normal vector to $\partial \Omega$, $\phi \in L^\infty(\partial \Omega)$ is a nonnegative boundary potential and $\sigma > 0$ is a real parameter.

Under these hypotheses, the functional associated to (1.1) is trivially coercive, that is

\begin{equation}
I(u, \phi) = \int_{\Omega} |\nabla u|^p + |u|^p dx + \sigma \int_{\partial \Omega} \phi |u|^p d\mathcal{H}^{n-1} \geq \|u\|_{W^{1,p}(\Omega)}^p.
\end{equation}

This functional is associated to (1.1) in the sense that eigenvalues $\lambda$ of (1.1) are critical values of $I$ restricted to the manifold $\|u\|_{L^p(\partial \Omega)} = 1$. See [9].

In particular, it is easy to see that the minimum value of $I$

\begin{equation}
\lambda(\sigma, \phi) = \inf \left\{ I(u, \phi) : u \in W^{1,p}(\Omega), \|u\|_{L^p(\partial \Omega)} = 1 \right\}
\end{equation}

is the first (lowest) eigenvalue of (1.1). Therefore, the existence of the first eigenvalue and the corresponding eigenfunction $u$ follows from the compact embedding $W^{1,p}(\Omega) \subset L^p(\partial \Omega)$.

In this work, we are interested in the minimization problem for $\lambda(\sigma, \phi)$ with respect to different configurations for the boundary potential $\phi$. That is, given certain class of admissible potentials $\mathcal{A}$, we look for the minimum possible value of $\lambda(\sigma, \phi)$ when $\phi \in \mathcal{A}$.

2010 Mathematics Subject Classification. 35J70.

Key words and phrases. Eigenvalue optimization, Steklov eigenvalues, $p$-Laplacian.
This study complements the ones started in [7]. In that paper, the authors analyzed the Steklov problem but with an interior potential and show the connections of that problem with the one considered in [11].

In this opportunity, we consider the class of uniformly bounded potentials, i.e.

$$\mathcal{A} = \{ \phi \in L^\infty(\partial \Omega) : 0 \leq \phi \leq 1 \}.$$  

Observe that $\mathcal{A}$ is the closure of the characteristic functions in the weak* topology.

Clearly, the minimization problem in the whole class $\mathcal{A}$ has no sense since the infimum is realized with $\phi \equiv 0$. The relevant problem here is to consider the minimization among those potentials in $\mathcal{A}$ that has fixed $L^1$--norm. That is

$$\Lambda(\sigma, a) = \inf \left\{ \lambda(\sigma, \phi) : \phi \in \mathcal{A}, \int_{\partial \Omega} \phi dH^{n-1} = a \right\}$$

The first result in this paper is the existence of an optimal potential for $\Lambda(\sigma, a)$ and, moreover, it is shown that this optimal potential can be taken as the characteristic function a sub-level set $D_\sigma$ of the corresponding eigenfunction. See [1, 2] for related results.

As another application we investigate the connection with the optimization problem considered in [6]. That is, given $E \subset \partial \Omega$, consider the equation

$$\begin{cases}
-\Delta_p u + |u|^{p-2} u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{in } E \\
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2} u \quad \text{in } \partial \Omega \setminus E
\end{cases}$$

whose first eigenvalue is given by

$$\lambda(\infty, E) := \inf \left\{ \|u\|_{W^{1,p}(\Omega)} : \|u\|_{L^p(\partial \Omega)} = 1, u = 0, H^{n-1} \text{ a.e. in } E \right\},$$

Associated to (1.5) we have the optimal configuration problem

$$\Lambda(\infty, a) = \inf \{ \lambda(\infty, E) : H^{n-1}(E) = a \}.$$  

Our second result shows that $\Lambda(\sigma, a) \to \Lambda(\infty, a)$ as $\sigma \to \infty$ and, moreover, the optimal configuration $\phi_\sigma = \chi_{D_\sigma}$ of $\Lambda(\sigma, a)$ converges (in the topology of $L^1$--convergence of the characteristic functions) to an optimal configuration of the limit problem $\Lambda(\infty, a)$.

The remaining of the paper is devoted to analyze qualitative properties of optimal configurations for $\Lambda(\sigma, a)$.

First, we consider the spherical symmetric case, that is when $\Omega$ is a ball, and in this simple case by means of symmetrization arguments we can give a full description of the optimal configurations.

Finally, we address the general problem and study the behavior of $\lambda(\sigma, \chi_D)$ for regular deformations of the set $D$. We employ the so--called method of Hadamard and prove differentiability of $\lambda(\sigma, \chi_D)$ with respect to regular deformations and provide a simple formula for the derivative of the eigenvalue. The main novelty of this formula is that it involves a $(n-2)$--dimensional integral along the boundary of $D$ relative to $\partial \Omega$. Up to our knowledge, this is the first time that this type of lower-dimensional integrals were observed in this type of computations.

We want to remark that the results in this work are new even in the linear setting, $p = 2$. 


2. PRELIMINARY REMARKS

A simple modification of the arguments in [10] shows that, given \( \phi \in A \) and \( \sigma > 0 \), the first eigenvalue \( \lambda(\sigma, \phi) \) is simple, i.e. any two eigenfunctions are multiple of each other. Therefore, there exists a unique nonnegative, normalized eigenfunction \( u \) (normalized means that \( \|u\|_{L^p(\partial \Omega)} = 1 \)).

The purpose of this very short section is to recall some regularity properties of this eigenfunction.

First, we note that by [15], there exists \( \alpha > 0 \) such that \( u \in C^{1,\alpha}_\text{loc}(\Omega) \). Now, by an usual argument, we have that \( |u| \) is an eigenfunction associated to \( \lambda(\sigma, \phi) \). Hence, the Harnack inequality, c.f. [15], implies that any first eigenfunction \( u \) has constant sign and, moreover, that \( u > 0 \) in \( \Omega \).

Next, by the results of [14], an eigenfunction of (1.1) is continuous up to the boundary. In fact, \( u \in C^\beta(\bar{\Omega}) \) for some \( \beta > 0 \).

Summing up, we have

**Proposition 2.1.** Given \( \phi \in A \) and \( \sigma > 0 \), there exists a unique nonnegative eigenfunction \( u \in W^{1,p}(\Omega) \) of (1.1) associated to \( \lambda(\sigma, \phi) \). Moreover, this eigenfunction \( u \) verifies that \( u \in C^{1,\alpha}_\text{loc}(\Omega) \cap C^\beta(\bar{\Omega}) \) for some \( \alpha, \beta > 0 \). Finally, \( u > 0 \) in \( \Omega \).

3. EXISTENCE OF OPTIMAL CONFIGURATIONS

In this section we first establish the existence of optimal configurations for \( \Lambda(\sigma, a) \). Then we analyze the limit \( \sigma \to \infty \) and show the convergence to the problem \( \Lambda(\infty, a) \).

Let us begin with the existence result.

**Theorem 3.1.** For any \( \sigma > 0 \) and \( 0 \leq a \leq H^{n-1}(\partial \Omega) \) there exist an optimal pair \( (u, \phi) \in W^{1,p}(\Omega) \times A \), which has the following properties

1. \( u \in C^{1,\alpha}_\text{loc}(\Omega) \cap C(\bar{\Omega}) \)
2. \( \phi = \chi_D \) where, for some \( s \), \( \{u < s\} \subset D \subset \{u \leq s\} \), \( H^{n-1}(D) = a \)

**Proof.** We consider a minimizing sequence \( \{\phi_k\}_{k \in \mathbb{N}} \subset A \) of (1.3) and their associated normalized eigenfunctions \( \{u_k\}_{k \in \mathbb{N}} \subset W^{1,p}(\Omega) \).

From the reflexivity of the Sobolev space \( W^{1,p}(\Omega) \), the compactness of the embeddings \( W^{1,p}(\Omega) \hookrightarrow L^p(\partial \Omega) \) and \( W^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \) and \( L^\infty(\partial \Omega) \) being a dual space, we obtain a subsequence (again denoted \( \{u_k, \phi_k\} \)) and \( u \in W^{1,p}(\Omega), \phi \in L^\infty(\partial \Omega) \) such that

\[
\begin{align*}
(3.1) & \quad u_k \rightharpoonup u \quad \text{in} \quad W^{1,p}(\Omega) \\
(3.2) & \quad u_k \to u \quad \text{in} \quad L^p(\partial \Omega) \\
(3.3) & \quad u_k \to u \quad \text{in} \quad L^p(\Omega) \\
(3.4) & \quad \phi_k \rightharpoonup \phi \quad \text{in} \quad L^\infty(\partial \Omega)
\end{align*}
\]

From the admissibility of \( \phi_k \) and (3.4), we get \( 0 \leq \phi \leq 1 \) and \( \int_{\partial \Omega} \phi dH^{n-1} = a \). Using (3.2), we get \( \|u\|_{L^p(\partial \Omega)} = 1 \). As a consequence of the lower semicontinuity of the norm \( \|\cdot\|_{W^{1,p}(\Omega)} \) with respect to weak convergence, we obtain

\[
\begin{align*}
(3.5) & \quad \int_{\Omega} |\nabla u|^p + |u|^p dx \leq \liminf_{k \to \infty} \int_{\Omega} |\nabla u_k|^p + |u_k|^p dx
\end{align*}
\]
Using (3.2), we can see that \(|u_k|^p \to |u|^p\) in \(L^1(\partial \Omega)\). Therefore, taking into account (3.3) we obtain
\[
\int_{\partial \Omega} \phi_k |u_k|^p d\mathcal{H}^{n-1} \to \int_{\partial \Omega} \phi |u|^p d\mathcal{H}^{n-1}
\]
From (3.5) and (3.6), we have \((u, \phi)\) is an optimal pair for (1.3).

By an elementary variation of the Bathtub Principle ([14, Pag. 28]), we can prove that the minimization problem
\[
\inf_{\int_{\partial \Omega} \phi d\mathcal{H}^{n-1}=a} \int_{\partial \Omega} \phi |u|^p d\mathcal{H}^{n-1},
\]
has a solution of the form \(\phi = \chi_D\), where \(\{u < s\} \subset D \subset \{u \leq s\}\) and \(\mathcal{H}^{n-1}(D) = a\) and therefore \((\chi_D, u)\) is an optimal pair for \(\Lambda(\sigma, a)\).

Now we prove a Lemma about the continuity of the eigenvalues and eigenfunctions with respect to the potential \(\phi\) in the weak * topology.

**Lemma 3.2.** Let \(\phi_j, \phi \in L^\infty(\partial \Omega)\) be such that \(\phi_j \rightharpoonup \phi\) in \(L^\infty(\partial \Omega)\). Let \(\lambda_j = \lambda(\sigma, \phi_j)\) and \(\lambda = \lambda(\sigma, \phi)\) the eigenvalues defined by (1.2) and let \(u_j, u \in W^{1,p}(\Omega)\) be the positive normalized eigenfunctions associated to \(\lambda_j\) and \(\lambda\) respectively.

Then \(\lambda_j \to \lambda\) and \(u_j \to u\) strongly in \(W^{1,p}(\Omega)\) as \(j \to \infty\).

**Proof.** First, define \(v \equiv \mathcal{H}^{n-1}(\partial \Omega)^{-1/p}\) and from (1.2) we get
\[
\lambda_j \leq I(v, \phi_j) = \frac{\|\Omega\|}{\mathcal{H}^{n-1}(\partial \Omega)} \leq C
\]
for every \(j \in \mathbb{N}\). Therefore, since \(\|u_j\|_{W^{1,p}(\Omega)} \leq \lambda_j\) (recall that the eigenfunctions \(u_j\) are normalized) it follows that \(\{u_j\}_{j \in \mathbb{N}}\) is bounded in \(W^{1,p}(\Omega)\).

From these, we obtain the existence of a function \(w \in W^{1,p}(\Omega)\) such that, for a subsequence,
\[
\begin{align*}
  u_j & \to w \quad \text{weakly in } W^{1,p}(\Omega) \\
  u_j & \to w \quad \text{strongly in } L^p(\Omega) \\
  u_j & \to w \quad \text{strongly in } L^p(\partial \Omega)
\end{align*}
\]
It then follows that \(w \geq 0\) and that \(\|w\|_{L^p(\partial \Omega)} = 1\).

Now, from the weakly sequentially lower semicontinuity it holds
\[
\lambda \leq I(w, \phi) \leq \liminf I(u_j, \phi) = \liminf I(u_j, \phi_j) + \sigma \int_{\partial \Omega} (\phi - \phi_j) |u_j|^p d\mathcal{H}^{n-1}.
\]
Since \(|u_j|^p \to |u|^p\) strongly in \(L^1(\partial \Omega)\), it easily follows that
\[
\lambda \leq \liminf \lambda_j.
\]

For the reverse inequality, we proceed in a similar fashion. In fact, from (1.2)
\[
\lambda_j \leq I(u, \phi_j).
\]
Therefore
\[
\limsup \lambda_j \leq \lim I(u, \phi_j) = I(u, \phi) = \lambda,
\]
so \(\lambda_j \to \lambda\).

Finally, from (3.7), one obtains that \(I(w, \phi) = \lambda\) and since there exists a unique nonnegative normalized eigenfunction associated to \(\lambda\) it follows that \(w = u\). Moreover, again from (3.7) it is easily seen that \(\|u_j\|_{W^{1,p}(\Omega)} \to \|u\|_{W^{1,p}(\Omega)}\) and so \(u_j \to u\).
strongly in $W^{1,p}(\Omega)$ and, since the limit is uniquely determined, the whole sequence $\{u_j\}_{j \in \mathbb{N}}$ is convergent.

The next Lemma, that was proved in [6] gives the strict monotonicity of the quantity $\Lambda(\infty, a)$ with respect to $a$ and will be helpful in showing the behavior of $\Lambda(\sigma, a)$ for $\sigma \to \infty$.

**Lemma 3.3** (Corollary 3.7, [6]). The function $\Lambda(\infty, \cdot)$ is strictly monotonic.

Now we are ready to prove the convergence of $\Lambda(\sigma, a)$ to $\Lambda(\infty, a)$ as $\sigma \to \infty$.

**Theorem 3.4.** If $\sigma_j$ is a sequence tending to $\infty$ and $(u_j, D_j)$ associated optimal pairs of (1.3), then there exists a subsequence (that we still call $\sigma_j$) and an optimal pair $(u, D)$ of the problem (1.3) such that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega)$, $\chi_{D_j} \to \chi_D$ in $L^\infty(\partial \Omega)$.

**Proof.** We consider $E \subset \partial \Omega$ closed such that $H^{n-1}(E) = a$ and $v \in W^{1,p}(\Omega)$, $\|v\|_{L^p(\partial \Omega)} = 1$ such that $v = 0$ in $E$. Therefore

$$\|u_j\|_{W^{1,p}(\Omega)}^p \leq I(u_j, \chi_{D_j}) = \Lambda(\sigma_j, a) \leq \lambda(\sigma_j, \chi_E) \leq I(v, \chi_E) = \|v\|_{W^{1,p}(\Omega)}^p$$

Hence, the sequence $u_j$ is bounded in $W^{1,p}(\Omega)$. Therefore we can assume that there exists $u_\infty \in W^{1,p}(\Omega)$ and $\phi_\infty \in L^\infty(\partial \Omega)$ such that

$$\begin{align*}
(3.8) & \quad u_j \rightharpoonup u_\infty \text{ in } W^{1,p}(\Omega) \\
(3.9) & \quad u_j \to u_\infty \text{ in } L^p(\Omega) \\
(3.10) & \quad u_j \to u_\infty \text{ in } L^p(\partial \Omega) \\
(3.11) & \quad \chi_{D_j} \to \phi_\infty \text{ in } L^\infty(\partial \Omega)
\end{align*}$$

From (3.10) and (3.11) we have that $\|u_\infty\|_{L^p(\partial \Omega)} = 1$, $\int_{\partial \Omega} \phi_\infty dH^{n-1} = a$ and $0 \leq \phi_\infty \leq 1$. The rest of the proof follows in a completely analogous way, using Lemma 3.3 to [7] Theorem 1.2.

4. **Symmetry**

Throughout this section we assume that $\Omega$ is the unit ball $B(0, 1)$. The goal of the section is to show that there exists an optimal pair $(u, \chi_D)$ of the problem (1.1) with $D$ a spherical cup in $S^{n-1} = \partial \Omega$. A key tool is played by the spherical symmetrization.

The spherical symmetrization of a set $A \subset \mathbb{R}^n$ with respect to an axis given by a unit vector $e$ is defined as follows: Given $r > 0$ we consider $s_r > 0$ such that $H^{n-1}(A \cap \partial B(0, r)) = H^{n-1}(B(re, s_r) \cap \partial B(0, r))$. We note that the sets $A \cap \partial B(0, r)$ are $H^{n-1}$-measurable for almost every $r \geq 0$. Now we put:

$$A^* = \bigcup_{0 \leq r \leq 1} B(re, s_r) \cap \partial B(0, r)$$

The set $A^*$ is well defined and measurable whence $A$ is a measurable set. If $u \geq 0$ is a measurable function, we define its symmetrized function $u^*$ so that satisfies the relation $\{u^* \geq t\} = \{u \geq t\}^*$. We refer to [13] for an exhaustive study of this symmetrization. In particular, we need the following known results:

**Theorem 4.1.** Let $0 \leq u \in W^{1,p}(\Omega)$ and let $u^*$ be its symmetrized function. Then

1. $u^* \in W^{1,p}(\Omega)$
(2) $u^*$ and $u$ are equi-measurable, i.e. they have the same distribution function. Hence for every continuous increasing function $\Phi$: $\int_\Omega \Phi(u^*)dx = \int_\Omega \Phi(u)dx$

(3) $\int_\Omega uvdx \leq \int_\Omega u^*v^*dx$, for every measurable positive function $v$.

(4) In a similar way $u$ and $u^*$ are equimeasurable with respect to the Hausdorff measure on boundary of balls. Therefore, the two previous items holds with $\partial \Omega$ and $dH^{n-1}$ instead of $\Omega$ and $dx$, respectively.

(5) $\int_\Omega |\nabla u|^pdx \leq \int_\Omega |\nabla u|^pdx$.

With these preliminaries, we can now prove the main result of the section.

**Theorem 4.2.** Let $\Omega = B(0,1)$. Then there exists an optimal pair $(u, \chi_E)$ of the problem (1.1) with $E$ a spherical cup in $\partial \Omega$.

**Proof.** Let $(u, \chi_D)$ be an optimal pair. We define $E := ((D^c)^*)^c$. Since $(D^c)^*$ is a spherical cup it follows that $E$ is also a spherical cup.

We note that $\chi_E = 1 - (\chi_D)^*$, therefore it is easy to show, from (c) in Theorem 4.1 that

$$\int_{\partial \Omega} \chi_E |u^*|^p dH^{n-1} \leq \int_{\partial \Omega} \chi_D |u|^p dH^{n-1}.$$  

We note that $\int_{\partial \Omega} |u^*|^p dH^{n-1} = 1$, so $u^*$ is an admissible function in (1.2). Moreover,

$$\int_\Omega |\nabla u^*|^p + |u^*|^p dx + \sigma \int_{\partial \Omega} \chi_E |u^*|^p dH^{n-1} \leq \int_\Omega |\nabla u|^p + |u|^p dx + \sigma \int_{\partial \Omega} \chi_D |u|^p dH^{n-1}.$$  

Consequently, $(u^*, \chi_E)$ is an optimal pair. \qed

5. **DERIVATIVE OF EIGENVALUES**

Henceforth we put $\Gamma := \partial \Omega$. In this section we compute derivatives of the eigenvalues $\lambda(\sigma, \chi_D)$ with respect to perturbations of the set $D$. We also assume that the set $D \subset \Gamma$ is the closure of a regular relatively open set.

For this purpose, we introduce the vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ supported on a narrow neighborhood of $\Gamma$ with $V \cdot n = 0$, where $n$ is the outer normal vector. We consider the flow

$$\begin{cases}
\frac{d}{dt} \Psi_t(x) = V(\Psi_t(x)) \\
\Psi_0(x) = x
\end{cases}$$  

(5.1)

We note that the condition $V \cdot n = 0$ implies that $\Psi_t(\Gamma) = \Gamma$. From (5.1), it follows the asymptotic expansions

$$D \Psi_t = I + tDV + o(t),$$  

(5.2)

$$(D \Psi_t)^{-1} = I - tDV + o(t),$$  

(5.3)

$$J \Psi_t = 1 + t\text{div}V + o(t).$$  

(5.4)

Here $D \Psi_t$ and $J \Psi_t$ denote the differential matrix of $\Psi_t$ and its Jacobian, respectively. See [12].

In order to try with surface integrals, we need the following formulas whose proofs can be found in [12]. The tangential Jacobian of $\Psi_t$ is given by

$$J_{\Gamma} \Psi_t(x) = |(D \Psi_t(x))^{-1} n| J \Psi_t(x) = 1 + t\text{div}_{\Gamma}V + o(t) \quad x \in \Gamma$$  

where $\text{div}_{\Gamma}V$ is the tangential divergence operator defined by

$$\text{div}_{\Gamma}V = \text{div}V - n^T DV n.$$
The main result here is the following

**Theorem 5.1.** Let $\sigma > 0$ be fixed and $D \subset \Gamma$ be the closure of a smooth relatively open set. Let $u \in W^{1,p}(\Omega)$ be the nonnegative normalized eigenfunction for $\lambda(\sigma, \chi_{\Omega})$.

Then, the function $\lambda(t) := \lambda(\sigma, \chi_{D_t})$ where $D_t = \Psi_t(D)$ is differentiable at $t = 0$ and

$$
\lambda'(0) = -\sigma \int_{\partial D} |u|_p^p (n_\Gamma \cdot V) dH^{n-2}
$$

where $n_\Gamma$ denotes the unit normal vector exterior to $\partial D$ relative to the tangent space of $\Gamma$.

**Remark 5.2.** Observe that the results of Lemma 5.3 immediately imply the continuity of $\lambda(t)$ at $t = 0$ and also that the associated eigenfunctions $u_t$ strongly converge to the associated eigenfunction $u$ of $\lambda(0)$ in $W^{1,p}(\Omega)$.

**Proof of Theorem 5.1.** We will follow the same line that [8] Theorem 1.1. Let $u \in W^{1,p}(\Omega)$. We call $\overline{u} = u \circ \Psi_t$, then the following asymptotic expansions hold

$$
\int_\Omega |\nabla \overline{u}|^p + |\overline{u}|^p dx = \int_\Omega (|D\Psi_t \nabla u|^p + |u|^p) J\Psi_t^{-1} dx
$$

$$
= \int_\Omega (|(I + tDV + o(t))\nabla u|^p + |u|^p)(1 - t\text{div} V + o(t)) dx
$$

$$
= \int_\Omega |\nabla u|^p + |u|^p dx - t(\text{div} V(|\nabla u|^p + |u|^p) - p|\nabla u|^{p-2} (\nabla u)^t DV \nabla u) dx + o(t)
$$

$$
\int_\Gamma \chi_{\Omega_t} |\overline{u}|^p dH^{n-1} = \int_\Gamma \chi_D |u|^p J_T \Psi_t^{-1} dH^{n-1}
$$

$$
= \int_\Gamma \chi_D |u|^p (1 - t\text{div}_T V) dH^{n-1} + o(t)
$$

$$
\int_\Gamma |\overline{u}|^p dH^{n-1} = \int_\Gamma |u|^p J_T \Psi_t^{-1} dH^{n-1}
$$

$$
= \int_\Gamma |u|^p (1 - t\text{div}_T V) dH^{n-1} + o(t)
$$

From (5.5) and (5.6), we obtain

$$
I(\overline{u}, \chi_{\Omega_t}) = F(u) - tG(u) + o(t)
$$

where

$$
F(u) = \int_\Omega |\nabla u|^p + |u|^p dx + \sigma \int_\Gamma \chi_D |u|^p dH^{n-1}
$$

and

$$
G(u) = \int_\Omega \text{div} V(|\nabla u|^p + |u|^p) - p|\nabla u|^{p-2} (\nabla u)^t DV \nabla u dx + \sigma \int_\Gamma \chi_D |u|^p \text{div}_T V dH^{n-1}
$$
Now, take \( u \) to be a normalized eigenfunction associated to \( \lambda(0) \). Then we have

\[
\lambda(t) \leq \frac{I(\bar{u}, \chi_D)}{\int_{\Gamma} |\bar{u}|^p \, d\mathcal{H}^{n-1}} = \frac{F(u) - tG(u) + o(t)}{\int_{\Gamma} |u|^p \, d\mathcal{H}^{n-1} - t \int_{\Gamma} |u|^p \text{div} V \, d\mathcal{H}^{n-1} + o(t)}
\]

\[
= \frac{F(u)}{\int_{\Gamma} |u|^p \, d\mathcal{H}^{n-1}} + t \left( \frac{F(u) \int_{\Gamma} |u|^p \text{div} V \, d\mathcal{H}^{n-1}}{\left( \int_{\Gamma} |u|^p \, d\mathcal{H}^{n-1} \right)^2} - \frac{G(u)}{\int_{\Gamma} |u|^p \, d\mathcal{H}^{n-1}} \right) + o(t)
\]

\[
= \lambda(0) + t \left( \lambda(0) \int_{\Gamma} |u|^p \text{div} V \, d\mathcal{H}^{n-1} - G(u) \right) + o(t)
\]

Therefore

\[
\lambda(t) - \lambda(0) \leq t \left( \lambda(0) \int_{\Gamma} |u|^p \text{div} V \, d\mathcal{H}^{n-1} - G(u) \right) + o(t)
\]

Now, take \( u_\epsilon \in W^{1,p}(\Omega) \) a normalized eigenfunction associated to \( \lambda(t) \) and denote by \( \bar{u}_\epsilon = u_\epsilon \circ \Psi_\epsilon \). So

\[
\lambda(0) \leq \frac{I(\bar{u}_\epsilon, \chi_D)}{\int_{\Gamma} |\bar{u}_\epsilon|^p \, d\mathcal{H}^{n-1}} = \frac{F(u_\epsilon) + tG(u_\epsilon) + o(t)}{\int_{\Gamma} |u_\epsilon|^p \, d\mathcal{H}^{n-1} + t \int_{\Gamma} |u_\epsilon|^p \text{div} V \, d\mathcal{H}^{n-1} + o(t)}
\]

\[
= \frac{F(u_\epsilon)}{\int_{\Gamma} |u_\epsilon|^p \, d\mathcal{H}^{n-1}} - t \left( \frac{F(u_\epsilon) \int_{\Gamma} |u_\epsilon|^p \text{div} V \, d\mathcal{H}^{n-1}}{\left( \int_{\Gamma} |u_\epsilon|^p \, d\mathcal{H}^{n-1} \right)^2} - \frac{G(u_\epsilon)}{\int_{\Gamma} |u_\epsilon|^p \, d\mathcal{H}^{n-1}} \right) + o(t)
\]

\[
= \lambda(t) - t \left( \lambda(t) \int_{\Gamma} |u_\epsilon|^p \text{div} V \, d\mathcal{H}^{n-1} - G(u_\epsilon) \right) + o(t)
\]

This last inequality together with (5.11) give us

\[
t \left( \lambda(t) \int_{\Gamma} |u_\epsilon|^p \text{div} V \, d\mathcal{H}^{n-1} - G(u_\epsilon) \right) + o(t) \leq \lambda(t) - \lambda(0)
\]

\[
\leq t \left( \lambda(0) \int_{\Gamma} |u|^p \text{div} V \, d\mathcal{H}^{n-1} - G(u) \right) + o(t)
\]

So, by Remark [5.2] one gets

\[
\lambda'(0) = \left( \lambda(0) \int_{\Gamma} |u|^p \text{div} V \, d\mathcal{H}^{n-1} - G(u) \right).
\]

It remains to further simplify the expression for \( \lambda'(0) \). Let

\[
G(u) = \int_{\Omega} \text{div} V(|\nabla u|^p + |u|^p) - p|\nabla u|^{p-2} (\nabla u)^t D V \nabla u \, dx
\]

\[
+ \sigma \int_{\Gamma} \chi_D |u|^p \text{div} V \, d\mathcal{H}^{n-1}
\]

\[
= I_1 + I_2
\]
Now using $V \cdot \nabla u$ as test function in the equation $-\Delta p u + |u|^{p-2} u = 0$ and the boundary condition in \ref{eq:boundary} we obtain:

\begin{align*}
I_1 &= -p \int_{\Gamma} \nabla u|^{p-2} \frac{\partial u}{\partial n} \cdot \nabla u \, d\mathcal{H}^{n-1} \\
&= -p \int_{\Gamma} \lambda(0) |u|^{p-2} u (V \cdot \nabla u) - \sigma \chi_{\mathcal{D}} |u|^{p-2} u (V \cdot \nabla u) \, d\mathcal{H}^{n-1} \\
&= -\lambda(0) \int_{\Gamma} \nabla(|u|^p) \cdot V \, d\mathcal{H}^{n-1} + \sigma \int_{\Gamma} \chi_{\mathcal{D}} \nabla(|u|^p) \cdot V \, d\mathcal{H}^{n-1} \\
&= \lambda(0) \int_{\Gamma} |u|^p \text{div}_V \, d\mathcal{H}^{n-1} - \sigma \int_{\mathcal{D}} |u|^p \text{div}_V \, d\mathcal{H}^{n-1} + \sigma \int_{\partial \mathcal{D}} |u|^p V \cdot \mathbf{n}_\Gamma \, d\mathcal{H}^{n-2} \\
&= \lambda(0) \int_{\Gamma} |u|^p \text{div}_V \, d\mathcal{H}^{n-1} + \sigma \int_{\partial \mathcal{D}} |u|^p V \cdot \mathbf{n}_\Gamma \, d\mathcal{H}^{n-2} - I_2
\end{align*}

So,

\begin{align*}
G(u) &= \lambda(0) \int_{\Gamma} |u|^p \text{div}_V \, d\mathcal{H}^{n-1} + \sigma \int_{\partial \mathcal{D}} |u|^p V \cdot \mathbf{n}_\Gamma \, d\mathcal{H}^{n-2}
\end{align*}

and therefore

$$\lambda'(0) = -\sigma \int_{\partial \mathcal{D}} |u|^p V \cdot \mathbf{n}_\Gamma \, d\mathcal{H}^{n-2}$$

This completes the proof of the Theorem. \hfill \Box

\section*{Acknowledgements}

This work was partially supported by Universidad de Buenos Aires under grant UBACYT 20020100100400 and by CONICET (Argentina) PIP 5478/1438.

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