Variance Allocation and Shapley Value

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Abstract

Motivated by the problem of utility allocation in a portfolio under a Markowitz mean-variance choice paradigm, we propose an allocation criterion for the variance of the sum of \( n \) possibly dependent random variables. This criterion, the Shapley value, requires to translate the problem into a cooperative game. The Shapley value has nice properties, but, in general, is computationally demanding. The main result of this paper shows that in our particular case the Shapley value has a very simple form that can be easily computed. The same criterion is used also to allocate the standard deviation of the sum of \( n \) random variables and a conjecture about the relation of the values in the two games is formulated.

Keywords: Shapley value; core; variance game; covariance matrix; computational complexity.

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1 Introduction

In the mean-variance model of Markowitz (1952) preferences for prospects are represented by a linear combination of the mean and the variance of the prospect:

\[ U_\theta[X] = E[X] - \theta \text{Var}[X], \]  

(1.1)

for some \( \theta > 0 \) (see, e.g., Steinbach, 2001).

If an investor is dealing with a portfolio whose returns are \( X_1, \ldots, X_n \), it may be important for her to allocate to each asset of the portfolio its contribution to the total utility score \( U_\theta[\sum_{i=1}^n X_i] \). Given that returns are typically correlated, allocating to asset \( i \) a contribution equal to \( E[X_i] - \theta \text{Var}[X_i] \) would not solve the problem. A similar problem is considered and solved (under some conditions) in the Capital Asset Pricing Model of Sharpe (1964) and Lintner (1965) (see, e.g., Fama and French, 2004). In this paper we use tools of cooperative game theory to solve the problem. In particular we resort to the Shapley value (Shapley, 1953), a standard solution concept for cooperative games, which has nice properties and has been extensively used in a variety of fields. First we define a cooperative game where the players are the assets in the portfolio and the worth of each coalition, i.e., of every subportfolio, is the utility score of this subportfolio. Then we compute the Shapley value of each component of the portfolio.

In general the Shapley value is used to allocate costs or gains among different agents who contribute to a joint project. The basic idea is that the allocation has to be fair for each player. In order to achieve this fairness each player is allocated a value that corresponds to her average marginal contribution to the worth of a coalition, the average being suitably taken over all possible coalitions.

Different solution concepts exist, based on different principles. For instance the core, which embodies an idea of stability, since it is the set of all allocations such that no coalition has an incentive to deviate from the grand coalition. We refer the reader to Peleg and Sudhölter (2007) or Maschler, Solan, and Zamir (2013) for a nice treatment of cooperative games and their solution concepts.

One of the main drawbacks of the Shapley value is its computational complexity as the number of players grows. This is due to the fact that its expression is an average over \( n! \) marginal contributions. In our problem, though, the expression of the Shapley value is extremely easy to compute and the complexity for the computation of the whole vector of Shapley values is quadratic in \( n \). We will explain how this result fits into a more general result of Conitzer and Sandholm (2004) about the Shapley value of decomposable games.
1.1 Related literature

We are not the first to propose game-theoretic tools for the analysis of cost allocations related to risks.

In an innovative paper in actuarial science, Lemaire (1984) proposes to use tools from cooperative game theory to allocate operating costs among different lines of an insurance company. The novelty of his paper is to solve a complicated accounting problem by computing a suitable solution of a cooperative game. Whereas the accounting problem is typically quite cumbersome, once the translation in game-theoretic language is performed, the solution is elegant and easy to interpret. Lemaire’s paper gave rise to a whole literature on cost allocation in insurance. In this subsequent literature the attention is focused on the allocation of costs when dealing with a portfolio of risks. As Denault (2001) points out, “the problem of allocation is interesting and non-trivial because the sum of the risk capitals of each constituent is usually larger than the risk capital of the firm taken as a whole, something called the diversification effect. This decrease of total costs, or ‘rebate,’ needs to be shared fairly between the constituents.” His goal is to provide an allocation criterion that is based on fairness. Starting from the axiomatic definition of coherent risk measures provided by Artzner, Delbaen, Eber, and Heath (1999), he proposes a set of axioms for the coherence of risk capital allocation principles. He ends up with an allocation that corresponds to the Aumann-Shapley value of nonatomic cooperative games (see Aumann and Shapley, 1974). Tsanakas and Barnett (2003) and Tsanakas (2004) propose the Aumann-Shapley value as an allocation mechanism when the risk measure is given by a distortion premium principle. Tsanakas (2009) does the same when convex risk measures are used. Abbasi and Hosseinifard (2013) use the Shapley value to allocate capital in the tail conditional expectation model.

It is interesting to notice that the Shapley value has been employed in various problems that involve probability models. For instance it has been implicitly used in reliability theory. Barlow and Proschan (1975) define an importance index of system components, whose \( j \)-th coordinate indicates the probability that the failure of component \( j \) causes the whole system to fail. Marichal and Mathonet (2013) point out that this index is actually a Shapley value. Cooperative game theory tools have been used in queueing theory, see, e.g., Anily and Haviv (2010), among others, and in inventory, see, e.g., Müller, Scarsini, and Shaked (2002) and Montrucchio and Scarsini (2007). Lipovetsky (2006) and Mishra (2016) have used the Shapley value in regression analysis. We refer the reader to Moretti and Patrone (2008) for a nice survey of possible applications of the Shapley value in various fields.

Several authors have considered computational issues related to the Shapley value and have pro-
posed efficient algorithms in some special cases, which typically involve voting games or games with a graph structure. Among them Deng and Papadimitriou (1994), Conitzer and Sandholm (2004), Ieong and Shoham (2005, 2006), Fatima, Wooldridge, and Jennings (2008, 2010), Azari Soufiani, Chickering, Charles, and Parkes (2014). The interested reader should consult the book by Chalkiadakis, Elkind, and Wooldridge (2011) for a nice survey of computational aspect of cooperative game theory.

1.2 Organization of the paper

In Section 2 we introduce some fundamental concepts of cooperative game theory and some important solution concepts. In Section 3 variance games are defined and analyzed. Section 4 deals with standard-deviation games and proposes a conjecture about the comparison between the two classes of games. Section 5 deals with some computational aspects of the Shapley value and explains why the complexity of its calculation in the variance game is polynomial.

2 Cooperative games

We start introducing some basic concepts in cooperative game theory. Given a set of players $N = \{1, \ldots, n\}$, a cooperative game is a pair $\langle N, \nu \rangle$, where $\nu : 2^N \to \mathbb{R}$ is such that $\nu(\emptyset) = 0$. Any subset $J \subset N$ is called a coalition. The set $N$ is called the grand coalition. The function $\nu$ is called the characteristic function of the game. Given that the set of players $N$ is fixed, for the sake of simplicity, we will just call game its characteristic function. So, if $\nu$ represents utilities, then $\nu(J)$ is the utility that the coalition $J$ can achieve by itself. If $\nu$ represents costs, then $\nu(J)$ the cost that the coalition must pay if it acts by itself. We call $\mathcal{G}(N)$ the class of all games on $N$.

2.1 Core and anticiore

A game $\xi$ is called additive if for all $I, J \subset N$ such that $J \cap J = \emptyset$, we have

$$\xi(I \cup J) = \xi(I) + \xi(J).$$

We call $\mathcal{M}(N)$ the class of additive games on $N$.  

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The core (anticore) of the game $\nu$ is defined as the set of all vectors $x = (x_1, \ldots, x_n)$ such that

\begin{align*}
\nu(J) &\leq (\geq) \sum_{i \in J} x_i \quad \text{for all } J \subset N, \\
\nu(N) &= \sum_{i \in M} x_i.
\end{align*}

(2.1)

(2.2)

A vector $x \in \mathbb{R}^n$ such that (2.2) holds represents a possible allocation among players of what achieved by the grand coalition. If $\nu$ represents a utility for coalitions, then the core is the set of all possible allocations that are stable, that is, no possible coalition has an incentive to deviate. Stability is given by (2.1), which tells that no coalition by itself can achieve more than what allocated to it. If the function $\nu$ represents costs, the set of stable allocations is given by the anticore. The core (anticore) of a game can be empty.

A game $\nu$ is called supermodular (submodular) if for all $I, J \subset N$

$$\nu(I \cup J) + \nu(I \cap J) \geq (\leq) \nu(I) + \nu(J).$$

It is well known (Shapley, 1971/72) that the core of a supermodular game and the anticore of a submodular game are non-empty.

### 2.2 Shapley value

As seen in Subsection 2.1, the core is a set of stable allocations. Its appeal is the stability of its allocations. Among its shortcomings we have the fact that it may be empty and, when it is not a singleton, it is not clear how to choose one single suitable allocation. We now introduce a different solution concepts that is obtained axiomatically and produces a single allocation.

Call player $i$ a dummy if for all $J \subset N$ we have

$$\nu(J \cup \{i\}) = \nu(J).$$

Call players $i, j$ symmetric if for all $J \subset N$ such that $i \not\in J$ and $j \not\in J$ we have

$$\nu(J \cup \{i\}) = \nu(J \cup \{j\}).$$

The Shapley value of $\nu$ is a function $\phi : \mathcal{G}(N) \to \mathbb{R}^n$ that satisfies the following properties:

1. Efficiency: $\sum_{i=1}^n \phi_i(\nu) = \nu(N)$. 

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2. **Symmetry**: If $i$ and $j$ are symmetric, then $\phi_i(\nu) = \phi_j(\nu)$.

3. **Dummy player**: If player $i$ is dummy, then $\phi_i(\nu) = 0$.

4. **Linearity**: for $\nu, \mu \in \mathcal{G}(N)$ and $\alpha, \beta \in \mathbb{R}$ we have

   $$\phi(\alpha \nu + \beta \mu) = \alpha \phi(\nu) + \beta \phi(\mu).$$

Shapley (1953) showed that the only function $\phi$ with these four properties has the following form

$$\phi_i(\nu) = \sum_{J \subset N \setminus \{i\}} \frac{|J|!(N - |J| - 1)!}{N!}(\nu(J \cup \{i\}) - \nu(J))$$

$$= \frac{1}{n!} \sum_{\psi \in \mathcal{P}(N)} (\nu(P^\psi(i) \cup \{i\}) - \nu(P^\psi(i))),$$

where $\mathcal{P}(N)$ is the set of all permutations of $N$, $P^\psi(i)$ is the set of players who precede $i$ in the order determined by permutation $\psi$, and $|J|$ is the cardinality of $J$.

In general the Shapley value of a game does not necessarily lie in its core. If the game is supermodular (submodular), the Shapley value lies in the core (anticore), it is actually its barycenter.

### 2.2.1 Shapley fusion property

Consider a game $\langle N, \nu \rangle$. For $J \subset N$ consider the new game $\langle N^J, \nu^J \rangle$ where all players in $J$ are fused into a single player. A game $\langle N, \nu \rangle$ such that, for all $J \subset N$, we have

$$\phi_J(\nu^J) = \sum_{i \in J} \phi_i(\nu).$$

is said to satisfy the **Shapley fusion property**.

In general the Shapley fusion property does not hold, as the following counterexample shows.

**Example 2.1.** Take $N\{1, 2, 3\}$ and, for every $i, j \in N$, with $i \neq j$,

$$\nu(\{i\}) = 0,$$

$$\nu(\{i, j\}) = 1$$

$$\nu(N) = 1.$$
By symmetry, $\phi_i(\nu) = 1/3$, but, for $J = \{2,3\}$, we have

$$\nu^J(\{1\}) = 0, \quad \nu^J(J) = 1, \quad \nu^J(\{1,J\}) = 1, \quad (2.5)$$

hence $\phi_J(\nu^J) = 1 \neq \phi_2(\nu) + \phi_3(\nu)$.

## 3 Variance games

We consider a random vector $\mathbf{X} := (X_1, \ldots, X_n)$ whose components can be seen for instance as the returns of $n$ securities in a portfolio. The return of the whole portfolio is then $\sum_{i=1}^n X_i$. If preferences are represented by the utility score $U_\theta$ defined in (1.1), we want to fairly allocate the utility $U_\theta[\sum_{i=1}^n X_i]$ of the portfolio to each of its components. To do this we have to take into account the correlation among the various returns. To achieve this goal we turn to cooperative game theory. We define a suitable cooperative game based on $U_\theta$ and we use the Shapley value of this game as the allocation criterion.

Consider a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ with finite second moments and define, for every $J \subset N$

$$S_J := \sum_{i \in J} X_i. \quad (3.1)$$

For each $J \subset N$, define

$$\gamma(J) := U_\theta[S_J] = \mathbb{E}[S_J] - \theta \mathbb{V}[S_J]. \quad (3.2)$$

The expression (3.2) defines a cooperative game $\langle N, \gamma \rangle$. This game is a linear combination of two other games:

$$\gamma(\cdot) = \varepsilon(\cdot) - \theta \nu(\cdot), \quad (3.3)$$

where

$$\varepsilon(J) := \mathbb{E}[S_J] \quad \text{and} \quad \nu(J) := \mathbb{V}[S_J]. \quad (3.4)$$

Given Property 4 of the Shapley value, we have

$$\phi(\gamma) = \phi(\varepsilon) - \theta \phi(\nu).$$

Since the expectation is a linear operator, the game $\varepsilon$ is additive and therefore we have

$$\phi_i(\varepsilon) = \mathbb{E}[X_i].$$
Therefore the problem of finding the Shapley value of $\gamma$ reduces to finding the Shapley value of $\nu$, which we call a variance game.

### 3.1 Main result

There next result shows that the Shapley value of the game $\nu$ has a very intuitive simple form in terms of the covariance matrix of $X$.

**Theorem 3.1.** For $\nu$ defined as in (3.4) we have

$$
\phi_i(\nu) = \text{Cov}[X_i, S_N].
$$

(3.5)

**Proof.** From (3.4), for $i \notin J$, we have

$$
\nu(J \cup \{i\}) - \nu(J) = \text{Var} \left( \sum_{j \in J \cup \{i\}} X_j \right) - \text{Var} \left( \sum_{j \in J} X_j \right)
$$

$$
= \sum_{j, \ell \in J \cup \{i\}} \text{Cov}[X_j, X_\ell] - \sum_{j \in J} \text{Cov}[X_j, X_i]
$$

$$
= \text{Var}[X_i] + 2 \sum_{j \in J} \text{Cov}[X_i, X_j].
$$

Therefore from (2.4) it follows that

$$
\phi_i(\nu) = \frac{1}{n!} \sum_{\psi \in \mathcal{P}(N)} \left( \text{Var}[X_i] + 2 \sum_{j \in P^\psi(i)} \text{Cov}[X_i, X_j] \right)
$$

$$
= \text{Var}[X_i] + \frac{2}{n!} \sum_{j \in N \setminus \{i\}} \sum_{\psi \in P^\psi(i)} \text{Cov}[X_i, X_j]
$$

$$
= \text{Var}[X_i] + \sum_{j \in N \setminus \{i\}} \text{Cov}[X_i, X_j]
$$

$$
= \sum_{j=1}^n \text{Cov}[X_i, X_j]
$$

$$
= \text{Cov}[X_i, S_N].
$$

The allocation in (3.5) is similar (although not equal) to the one obtained by Wang (2002, Theorem 3.2) for multinormally distributed risks, when the exponential tilting model is used. Notice the allocation in (3.5) is similar (although not equal) to the one obtained by Wang (2002, Theorem 3.2) for multinormally distributed risks, when the exponential tilting model is used. Notice
that Theorem 3.1 does not make any parametric assumption on the distribution of $X$.

**Remark 3.2.** An immediate corollary of Theorem 3.1 is that the Shapley fusion property holds for variance games.

### 3.2 Additional properties

We examine now some interesting properties of the variance allocation through the Shapley value.

**Example 3.3.** The Shapley value of the variance game can assume negative values. Consider the case $X = (X_1, X_2)$ with $X_2 = -2X_1$ and $\text{Var}[X_1] = 1$. Then

\[ \text{Cov}[X] = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \]

hence $\phi_1(\nu) = -1$ and $\phi_2(\nu) = 2$. The idea is that, if a random variable contributes to hedge a risk, then it is “rewarded” with a negative Shapley value.

The following property says that if perfect hedging can be achieved then the Shapley value is identically zero, no matter what the individual variances are.

**Proposition 3.4.** If $\text{Var}[S_N] = 0$, then $\phi(\nu) = 0$.

**Proof.** Let $\text{Var}[S_N] = 0$, that is

\[ \sum_{i \in N} \sum_{j \in N} \text{Cov}[X_i, X_j] = 0. \tag{3.6} \]

Then $S_N$ is almost surely a constant, which, without any loss of generality, we can assume to be zero. Hence for each $i \in N$

\[ X_i = - \sum_{j \in N \setminus \{i\}} X_j, \]

which implies

\[ \text{Var}[X_i] = \sum_{j \in N \setminus \{i\}} \sum_{\ell \in N \setminus \{i\}} \text{Cov}[X_j, X_{\ell}]. \tag{3.7} \]

Plugging (3.7) into (3.6) we obtain

\[ \sum_{j \in N} \text{Cov}[X_i, X_j] = 0 \quad \text{for all } i \in N, \]

which, by Theorem 3.1, gives the desired result. \qed
As the following example shows, it is not possible to apply the result of Proposition 3.4 to a subvector of the vector \( X \).

**Example 3.5.** It is possible to have \( \text{Var}[S_J] = 0 \) for some \( J \subset N \), without having \( \phi_j(\nu) = 0 \) for all \( j \in J \). For instance, let \( X = (X_1, X_2, X_3, X_4) \) be such that

\[
-X_1 = X_2 = X_3 = X_4,
\]

with \( \text{Var}[X_1] = 1 \). Then

\[
\text{Cov}[X] = \begin{pmatrix}
1 & -1 & -1 & -1 \\
-1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1
\end{pmatrix},
\]

\( \text{Var}[X_1 + X_2] = 0 \) and \( \phi_1(\nu) = -2 \) and \( \phi_2(\nu) = 2 \).

This is due to the fact that the Shapley value is computed globally, looking at the marginal contributions of a random variable to the variance of all possible subvectors of the vector \( X \). On the other hand, what is true is that, if \( \text{Var}[S_J] = 0 \), then \( \sum_{j \in J} \phi_j = 0 \).

**Example 3.6.** Even if the Shapley value has the symmetry property, is possible to have \( X_i \) and \( X_j \) exchangeable (or even i.i.d.) without necessarily having \( \phi_i(\nu) = \phi_j(\nu) \). For instance, consider \( X = (X_1, X_2, X_3, X_4) \) such that \( X_2 \) and \( X_3 \) are i.i.d. and

\[
X_1 = X_2, \quad X_4 = -X_3.
\]

Let \( \text{Var}[X_1] = 1 \). Then

\[
\text{Cov}[X] = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & -1 & 1
\end{pmatrix},
\]

Therefore \( \phi_2(\nu) = 2 \) and \( \phi_3(\nu) = 0 \).

Again, this is due to the global property of the Shapley value. Two exchangeable random variables can have very different relations with the other components of \( X \), therefore their Shapley value can differ.

Finally, we look at supermodularity (submodularity) of the variance game, which, as mentioned before, has important implications for the nonemptiness of its core (anticore).
Proposition 3.7. (a) If \( \text{Cov}[X_i, X_j] \geq 0 \) for all \( i, j \in N \), then the game \( \nu \) is supermodular.

(b) If \( \text{Cov}[X_i, X_j] \leq 0 \) for all \( i, j \in N \), then the game \( \nu \) is submodular.

Proof. (a) If \( \text{Cov}[X_i, X_j] \geq 0 \), then we have

\[
\nu(I \cup J) + \nu(I \cap J) = \text{Var}[S_{I \cup J}] + \text{Var}[S_{I \cap J}]
\]

\[
= \sum_{i \in I \cup J} \sum_{j \in I \cup J} \text{Cov}[X_i, X_j] + \sum_{i \in I \cap J} \sum_{j \in I \cap J} \text{Cov}[X_i, X_j]
\]

\[
= \sum_{i \in I} \sum_{j \in I} \text{Cov}[X_i, X_j] + \sum_{i \in J} \sum_{j \in J} \text{Cov}[X_i, X_j] + 2 \sum_{i \in I \setminus J} \sum_{j \in J \setminus I} \text{Cov}[X_i, X_j]
\]

\[
\geq \sum_{i \in I} \sum_{j \in I} \text{Cov}[X_i, X_j] + \sum_{i \in J} \sum_{j \in J} \text{Cov}[X_i, X_j]
\]

\[
= \text{Var}[S_I] + \text{Var}[S_J]
\]

\[
= \nu(I) + \nu(J).
\]

(b) If \( \text{Cov}[X_i, X_j] \leq 0 \), then the inequality goes in the opposite direction.

\[\square\]

4 Standard deviation games

Given a random vector \( \mathbf{X} = (X_1, \ldots, X_n) \) we can define a standard deviation game \( \lambda \) on \( N = \{1, \ldots, n\} \) as follows:

\[
\lambda(J) = \sqrt{\text{Var}[S_J]},
\]

where \( S_J \) is defined as in (3.1). Computing the Shapley value for this game is much more difficult than for the variance game. We will examine the relation between these two types of games.

The next example shows that the Shapley fusion property does not hold for standard deviation games.

Example 4.1. Consider the following covariance matrix

\[
\Sigma = \begin{bmatrix}
1 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 9
\end{bmatrix}.
\]
The corresponding standard deviation game is

\[
\begin{align*}
\lambda(\{1\}) &= 1, & \lambda(\{2\}) &= 2, & \lambda(\{3\}) &= 3, \\
\lambda(\{1, 2\}) &= \sqrt{5}, & \lambda(\{1, 3\}) &= \sqrt{10}, & \lambda(\{2, 3\}) &= \sqrt{13}, \\
\lambda(\{1, 2, 3\}) &= \sqrt{14}.
\end{align*}
\]

Therefore the Shapley value of the above game is

\[
\begin{align*}
\phi_1(\lambda) &= \frac{1}{6} \left( 2\sqrt{14} + \sqrt{10} + \sqrt{5} - 3 - 2\sqrt{13} \right), \\
\phi_2(\lambda) &= \frac{1}{6} \left( 2\sqrt{14} + \sqrt{13} + \sqrt{5} - 2\sqrt{10} \right), \\
\phi_3(\lambda) &= \frac{1}{6} \left( 2\sqrt{14} + \sqrt{13} + \sqrt{10} + 3 - 2\sqrt{5} \right).
\end{align*}
\]

For \( S = \{2, 3\} \) the covariance matrix becomes

\[
\Sigma^S = \begin{bmatrix} 1 & 0 \\ 0 & 13 \end{bmatrix}
\]

and the corresponding games is

\[
\begin{align*}
\lambda^S(\{1\}) &= 1, & \lambda^S(S) &= \sqrt{13}, & \lambda^S(\{1, S\}) &= \sqrt{14}.
\end{align*}
\]

The Shapley value of the above game is

\[
\begin{align*}
\phi_1(\lambda^S) &= \frac{1}{2} \left( 1 + \sqrt{14} - \sqrt{13} \right), \\
\phi_2(\lambda^S) &= \frac{1}{2} \left( \sqrt{14} + \sqrt{13} - 1 \right).
\end{align*}
\]

We have

\[
\phi_2(\lambda^S) \neq \phi_2(\lambda) + \phi_3(\lambda).
\]
4.1 A conjecture

Given two vectors \( x, y \in \mathbb{R}^n \) we say that \( x \) is majorized by \( y \) (\( x \prec y \)) if

\[
\sum_{i=k}^{n} x(i) \leq \sum_{i=k}^{n} y(i) \quad \text{for all } k \in \{1, \ldots, n - 1\},
\]

\[
\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,
\]

where \( x(1) \leq x(2) \leq \cdots \leq x(n) \) is the increasing rearrangement of \( x \). The reader is referred to Marshall, Olkin, and Arnold (2011) for properties of majorization.

The following proposition shows that, for \( n = 2 \), the normalized Shapley value of the variance game majorizes the corresponding normalized Shapley value of the standard deviation game.

**Proposition 4.2.** Consider a covariance matrix

\[
\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}
\]

and the corresponding variance game \( \nu \) and standard deviation game \( \lambda \). Then

\[
\frac{1}{\phi_1(\lambda) + \phi_2(\lambda)} \phi(\lambda) < \frac{1}{\phi_1(\nu) + \phi_2(\nu)} \phi(\nu),
\]

**Proof.** Assume, without any loss of generality, that \( \sigma_1 \leq \sigma_2 \). We need to show that

\[
\frac{\phi_1(\lambda)}{\phi_1(\lambda) + \phi_2(\lambda)} \geq \frac{\phi_1(\nu)}{\phi_1(\nu) + \phi_2(\nu)},
\]

that is

\[
\frac{\sigma_1 + \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_{12} - \sigma_2}}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_{12}}} \geq \frac{\sigma_1^2 + \sigma_{12}}{\sigma_1^2 + \sigma_2^2 + 2\sigma_{12}}.
\]

After simple algebra, this corresponds to

\[
(\sigma_1 - \sigma_2)\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho \sigma_1 \sigma_2} + \sigma_2^2 + \rho \sigma_1 \sigma_2 \geq 0, \tag{4.1}
\]

where \( \sigma_{12} = \rho \sigma_1 \sigma_2 \).
For $\rho = -1$, expression (4.1) becomes

$$-\sigma_1^2 + \sigma_1 \sigma_2 \geq 0,$$

and is therefore true. Since the right hand side of (4.1) is increasing in $\rho$, we have the result. \square

We conjecture the above result to be true for all $n \in \mathbb{N}$.

**Conjecture 4.3.** For any $n \times n$ covariance matrix $\Sigma$, if $\nu$ is the corresponding variance game and $\lambda$ the corresponding standard deviation game, then

$$\frac{1}{\sum_{i=1}^n \phi_i(\lambda)} \phi(\lambda) \prec \frac{1}{\sum_{i=1}^n \phi_i(\nu)} \phi(\nu).$$  \hspace{1cm} (4.2)

We have verified the conjecture numerically when the matrix $\Sigma$ is diagonal. The program that verifies the conjecture was written in C and is based on the following consideration. Let

$$S^{n-1} = \left\{(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}_+^n : \sum_{i=1}^n \sigma_i^2 = 1\right\},$$

$$D_n = \left\{(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}_+^n : \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_{n-1} \leq \sigma_n \right\},$$

$$M_n = S^{n-1} \cap D_n.$$  

Because of the normalization factors in both sides of (4.2), if the conjecture holds for each diagonal matrix $\Sigma = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$ then it holds also for the diagonal matrix $\Sigma' = \text{diag}(\alpha \sigma_{\psi(1)}^2, \ldots, \alpha \sigma_{\psi(n)}^2)$, with $\alpha > 0$ and $\psi$ any permutation of $1, \ldots, n$. Therefore, in order to verify the conjecture for any diagonal covariance matrix, it suffices to verify it for each $(\sigma_1, \sigma_2, \ldots, \sigma_n) \in M_n$.

The procedure works as follows. For a given number of players $n$ we extract $N$ independent normally distributed random vectors $Z_j = (Z_{1,j}, Z_{2,j}, \ldots, Z_{n,j}) \sim \mathcal{N}(0, I_n)$ where $I_n$ denotes the $n \times n$ identity matrix. It is well known (Muller, 1959) that $X_j = (X_{1,j}, X_{2,j}, \ldots, X_{n,j})$, where $X_{i,j} = Z_{i,j}/||Z_j||$, is uniformly distributed on $S^{n-1}$. Therefore $|X_j| := (|X_{1,j}|, |X_{2,j}|, \ldots, |X_{n,j}|)$ is uniformly distributed on the intersection of $S^{n-1}$ and the nonnegative orthant of $\mathbb{R}^n$. Call $\sigma_j = (\sigma_{1,j}, \sigma_{2,j}, \ldots, \sigma_{n,j})$ the nondecreasing rearrangement of $X_j$. Then the points $\{\sigma_j\}_{j=1}^N$ are independently uniformly distributed on the set $M_n$. The procedure checks for each point $\{\sigma_j\}_{j=1}^N$ whether the $n - 1$ inequalities given by the majorization conditions in (4.2) hold for the diagonal covariance matrix $\Sigma_j = \text{diag}(\sigma_{1,j}^2, \sigma_{2,j}^2, \ldots, \sigma_{n,j}^2)$. Conditions were tested when $n = 3, 4, 5$ and $N = 10^9$.
5 Computational aspects

Theorem 3.1 shows that the Shapley value of the variance game can be easily computed in polynomial time. For each $i \in \{1, \ldots, n\}$ the value $\phi_i(\nu)$ is just the sum of $n$ known covariances. We now want to frame this result in a more general framework concerning computational complexity of the Shapley value in suitable classes of games.

In a general coalition formation problem there are two main sources of computational complexity: the computation of each single coalition’s value and how to distribute this value among the participants of each coalition. The former appears when each coalition has to solve a hard optimization problem in order to compute its value. The latter depends on the characteristic function of the game and on the solution concept.

In our setting, coalitions do not face any hard optimization problem to compute their values and among the solution concepts we use the Shapley value as an allocation criterion. Thus, the interesting question that we address is why the Shapley value of the variance game can be easily computed in an efficient way, whereas a similar method cannot be used for the standard deviation game. The reason lies in the form of the characteristic function of the two games. The characteristic function of the variance game is easily decomposable in a sum of distinct easy functions, whereas the characteristic function of the standard deviation game is not decomposable due to the presence of the square root. Therefore, to the best of our knowledge, all algorithms to compute exactly the Shapley value of the standard deviation game are non-polynomial.

Conitzer and Sandholm (2004) prove that the Shapley value is efficiently computable if the characteristic function of the game can be decomposed in a specific form. In the following we show that the characteristic function of the variance game respects their decomposition requirements. We first introduce the definition of decomposition of a characteristic function.

**Definition 5.1** (Conitzer and Sandholm (2004, Definition 4)). The vector of characteristic functions $(\nu_1, \nu_2, \ldots, \nu_T)$, with each $\nu_t : 2^N \rightarrow \mathbb{R}$, is a decomposition over $T$ issues of characteristic functions $\nu : 2^N \rightarrow \mathbb{R}$ if for any $J \subseteq N$, $\nu(J) = \sum_{t=1}^{T} \nu_t(J)$.

The decomposition of the original characteristic function is particularly convenient if each $\nu_t$ restricts its focus on a subset of agents.

**Definition 5.2** (Conitzer and Sandholm (2004, Definition 5)). We say that $\nu_t$ concerns only $C_t \subseteq N$ if $\nu_t(J_1) = \nu_t(J_2)$ whenever $C_t \cap J_1 = C_t \cap J_2$. In this case, we only need to define $\nu_t$ over $2^{C_t}$.

This representation shrinks the number of values from $2^{|N|}$ to $\sum_{t=1}^{T} 2^{|C_t|}$, exponentially fewer than
the original representation. Notice that when $|C_t|$ is bounded by a small constant, the number of values is linear in $T$.

Now, the characteristic function of the variance game is represented by

$$
\nu(J) = \text{Var} \left[ \sum_{i \in J} X_i \right] = \sum_{i \in J} \sum_{j \in J} \text{Cov}[X_i, X_j].
$$

Thus, for each set $J$, $\nu$ can be decomposed into $\left| J \right|(\left| J \right| + 1)/2$ terms, considering that $\text{Cov}[X_i, X_j] = \text{Cov}[X_j, X_i]$. Notice that for each set $J' \subseteq J$ all the characteristic functions in $J'$ are also present in $J$. We can represent $T$ as the set of all pairs $(i, j) \in N \times N$ with $i \leq j$. Consequently, $|T| = |N|(\left| N \right| + 1)/2 < |N|^2$.

For each $(i, j) \in T$ we have

$$
\nu_{(i,j)}(J) = \begin{cases} 
\text{Var}[X_i] & \text{if } J = (i, i), \\
2 \text{Cov}(X_i, X_j) & \text{if } J = (i, j) \text{ and } i \neq j, \\
0 & \text{otherwise}.
\end{cases}
$$

Therefore, by Definition 5.2 we know that each $\nu_t \in T$ concerns at most 2 players, thus $|C_t| \leq 2$ for all $t \in T$. We can then apply the following theorem:

**Theorem 5.3** (Conitzer and Sandholm (2004, Theorem 1)). Suppose we are given a characteristic function with a decomposition $\nu = \sum_{t=1}^{T} \nu_t$, represented as follows. For each $t$ with $1 \leq t \leq T$ we are given $C_t \subseteq N$, so that each $\nu_t$ concerns only $C_t$. Each $\nu_t$ is flatly represented over $2^{C_t}$, that is, for each $t$ with $1 \leq t \leq T$, we are given $\nu_t(J_t)$ explicitly for each $J_t \subseteq C_t$. Then (assuming that table lookups for the $\nu_t(J_t)$, as well computations of factorials, multiplications and subtractions take constant time), we can compute the Shapley value of $\nu$ for any given agent in time $O(\sum_{t=1}^{T} 2^{|C_t|})$, or less precisely $O(T \cdot 2^{\max_{t \leq T} |C_t|})$. This holds whether or not the characteristic function is increasing, and whether or not it is superadditive.

This confirms the outcome of our Theorem 3.1, that is, the Shapley value of the variance game is computable in polynomial time.

Similar computational aspects of the Shapley value based on decomposition ideas have been studied by Ieong and Shoham (2005, 2006).
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