Communicating with a wave packet using quantum superarrival

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An analytical treatment of a propagating wave packet incident on a transient barrier reveals a counterintuitive quantum mechanical effect in which, for a particular time interval, the time-varying transmission probability exceeds (‘superarrival’) that for the free propagation of the wave packet. It is found that the speed with which the information about the barrier perturbation propagates across the wave packet can exceed the group velocity of the wave packet. An interesting implication of this effect regarding information transfer is analyzed by showing one-to-one correspondence between the strength of the barrier and the magnitude of ‘superarrival’.

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A. Introduction.— A number of interesting phenomena have been uncovered in recent years using the dynamics of quantum wave packets. Among these, a class of novel quantum effects as in the revival of wave packets [1] and quantum transients [2] are worth mentioning. In particular, for propagating wave packets, appropriate changes in the boundary conditions for suitable potentials can give rise to curious dynamical features [3–5]. One such striking effect, which we call ‘quantum superarrival’ is analytically demonstrated in this paper by considering a Gaussian wave packet which is incident on a time-dependent potential barrier. For the purpose of the analytical treatment given in this paper, the form of this barrier is chosen such that it corresponds to a transient parabolic barrier acting over a small time interval during which the peak of the propagating wave packet crosses the maximum of the parabolic barrier. In this case, we find that there exists an interval of time during which there is an enhancement (‘superarrival’) of the time-evolving transmission probability as compared to the case of a wave packet freely propagating in the absence of any potential barrier.

In the usual studies, the transmission/reflected probabilities for the scattering of wave packets by potential barriers are calculated after a complete time-evolution when the asymptotic values have been attained. In the present work, based on the analytical solution of the relevant time-dependent Schrödinger equation, a phenomenon is displayed which occurs during the time evolution of such a probability that is found to have the following salient features. While the effect of barrier perturbation resulting in ‘superarrival’ is discernible by measuring the transmission probability, it becomes more pronounced with the increase of the rate at which the barrier perturbation occurs (in the case considered, it is the strength of the barrier). Further, it is shown that the effect of barrier perturbation propagates across the wave packet at a speed that depends upon the strength of the barrier, thereby leading to a new concept of what we call ‘information velocity’.

In particular, for appropriate choices of the relevant parameters, it is shown that this information velocity can be higher than the group velocity of the incident wave packet, thereby illustrating that the information content of a wave packet does not always propagate with the group velocity of a wave packet. Here a local change in the potential affects a wave packet globally through its time evolution where the wave function plays the role of a carrier through which the information about the barrier perturbation is transmitted. Interestingly, by exploiting this feature of ‘superarrival’, it is possible to develop a scheme for communication whose basic idea is indicated in this paper. For this, we proceed by first delineating the relevant details of the analytical treatment that leads to the phenomenon of ‘superarrival’ in the example we consider in this paper.

B. Superarrival: the phenomenon.— We begin our analysis by considering a Gaussian wave packet peaked at $q_0$

$$\psi(x, t_0) = \left( \frac{2m}{\pi \alpha_0^2} \right)^{1/4} e^{-m[x-q_0]^2/\alpha_0^2} e^{i\phi_0[x-q_0]/\hbar}$$

which is incident on a time-dependent barrier given by

$$V(x, t) = -\frac{1}{2} m ke^{-g(t-t_0)^2} x^2,$$

where

$$g(t) = \frac{1}{t-t_0}.$$
that corresponds to the appearance of a parabolic barrier during a small time interval. This is achieved by choosing a Gaussian form for the time window, with the parameters $t_b$ and $g$ indicating the peak time and inverse width of the window. $k$ determines the barrier strength.

Having in mind the path-integral properties for quadratic Lagrangians (see below), the time-dependent wavefunction is obtained from the ansatz

$$\psi(x,t) = \left( \frac{2m}{\pi \alpha^2(t)} \right)^{1/4} e^{-[x-q(t)]^2} e^{i\phi(t)} e^{-[m\phi(t)]^2/2m[\phi(t) - \phi_0]},$$

(3)

It can be checked by direct substitution that Eq. (3) obeys the Schrödinger equation with the initial condition $\psi(0)=1$, provided $q(t)$ and $p(t)$ obey the classical equations of motion, i.e.,

$$\frac{\partial^2 q(t)}{\alpha(t)} = \omega^2(t) q(t)$$

and $\partial_t p(t) = m \partial_q^2 q(t)$, with $q_0 \equiv q(t_0)$ and $p_0 \equiv p(t_0)$; $\alpha(t)$ is a solution of the nonlinear equation

$$\frac{\partial^2 \alpha(t)}{\alpha(t)} - \omega^2(t) = \frac{4\hbar^2}{\alpha^4(t)}$$

(5)

which forms with the linear equation (4) an Ermakov pair (see Refs. therein). This means that $\alpha(t)$ can be expressed in terms of two linearly independent solutions of Eq. (4), the precise choice of a given function $\alpha(t)$ depending on two arbitrary constants (denoted $I$ and $c$ in Fig. 1). These are fixed so that initially $\alpha(t_0) = \alpha_0$ and $\alpha'(t_0) = 0$, as required so that Eq. (1) is consistent with Eq. (4). One then has $q(t) = \sqrt{2I} \alpha(t) \sin \phi(t)$, where $\phi(t)$ which appears in Eq. (4) is known in the context of Ermakov systems as the phase function; it is given by $\partial_t \phi(t) \equiv \hbar \alpha^{-2}(t)$. Note that Ermakov systems have often been employed in order to study the solutions of the classical time dependent harmonic oscillator. Besides transforming an ubiquitous nonlinear equation into a linear one, they offer several advantages: for example by construction $\alpha(t)$ is a positive definite quadratic form ensuring that $\psi(x,t)$ given by Eq. (4) is normalizable.

We consider situations in which an initial Gaussian wavefunction $\psi_0$ lies far on the negative axis and is launched at $t_0$ towards the right. The wavepacket spreads while travelling to the right (with the spread controlled by $\alpha(t)$). A detector placed at a point $x_T$ far beyond $q_0$ measures the time-dependent transmission probability by counting the transmitted particles arriving there up to various instants. At any instant before the asymptotic value of the reflection probability is attained, the time evolving transmission probability in the region $x_T \leq x < \infty$ is given by

$$T(x_T,t) = \int_{x_T}^{\infty} |\psi(x,t)|^2 \, dx$$

(6)

and then, it follows from Eq. (4) that

$$T(x_T,t) = \frac{1}{2} \text{erfc} \left[ \frac{\sqrt{2m}(x_T-q(t))}{\alpha(t)} \right].$$

(7)

We compute the transmission probabilities for various sets of parameters. In order to assess the influence of the appearance of the time-dependent barrier on the transmitted wavepacket, we set $t_b$ in Eq. (2) so that the maximum of the free Gaussian reaches $x=0$ when the barrier strength is the greatest, i.e $V(x,t_b) = -\frac{1}{2} m k x^2$ and $q(t_b) = 0$ where $f$ denotes the free case ($k=0$). Taking identical initial Gaussians in the free and barrier cases, we can appropriately choose the initial position and momentum parameters of the initial wavefunction so that $\psi(x,t)$ and $\psi_f(x,t)$ remain almost identical up to times slightly below $t=t_b$. At that point, the rising barrier perturbs the wavepacket, whereas in the free case the Gaussian keeps propagating with an average momentum $\frac{p_f(t)}{t_f} = \frac{p_0}{t_0}$. The transmission probabilities which are plotted (as a function of time) in Fig. 1(a) in the free case (black curve) and for barriers with increasing strength (as the colouring goes from blue to red). We denote the transmitted probability for the free and the barrier-perturbed cases as $T_f(t)$ and $T_b(t)$ respectively.
We observe that $T_k(t) > T_f(t)$ during the time interval $t_d < t < t_e$ (superarrival). (Here $t_k$ is the instant at which the perturbation starts, $t_e$ the instant when the free and the perturbed curves cross each other, and $t_d$ the time from which the curve corresponding to the perturbed case starts deviating from that in the free case, so that $t_e > t_d > t_k$).

**C. Communication using superarrival.** — We have seen above that the transmission probability for the perturbed barrier exceeds that of the free case in a particular time interval. The detector during this time interval therefore records more number of particles than it would have in the free case. At this stage, a particularly relevant question arises as to when would Bob at the detector realize that Alice has perturbed the wave packet? Or, in other words, how fast does the information about barrier perturbation travel to the detector? Bob who records the growth of the transmission probability becomes aware of the perturbation (occurring from the instant $t_k$) at the instant $t_d$ when the transmission probability starts deviating from the free case. If Alice (located at the potential barrier) and Bob are separated by the distance $D$, one can define information velocity $v_I$ by

$$v_I(k) = \frac{D}{t_d - t_k}$$

(8)

We compute $v_I(k)$ and plot the function $v_I(k)/v_g$ (where $v_g$ refers to the group velocity of the wave packet in the free case) versus the strength of the barrier (Fig. 1(b)). Note that information of barrier perturbation travels from the barrier to the detector with a velocity which could exceed the group velocity of the wave packet.

Fig. 2 displays superarrivals with parameters chosen so as to enhance the ratio $v_I(k)/v_g$ (see Fig. 2(b)).

In order for Alice and Bob to use this information transfer as means of communication, we define the quantity $\eta$ which determines the magnitude of superarrival by

$$\eta(k) = \frac{I_k - I_f}{I_f}$$

(9)

where $I_k$ and $I_f$ are defined with respect to the time interval $\Delta t = t_e - t_d$ during which superarrival occurs, as follows:

$$I_k = \int_{\Delta t} T_k(t)dt; \quad I_f = \int_{\Delta t} T_f(t)dt$$

(10)

The magnitude of superarrival $\eta$ is a function of the barrier strength $k$. In Figs. 1(c) and 2(c) we plot $\eta$ versus $k$ for a couple of different sets of parameter values.

Now, suppose a particular functional relation between $k$ and $\eta(k)$ (say, curve 1(c)) is chosen as a key which is shared by Alice and Bob. Alice at the barrier receives a continuous inflow of particles whose wave function is given by the initial Gaussian. At first, she does nothing and Bob at the detector simply records the particle counts, thereby generating the curve $T_f$. She then introduces the barrier perturbation choosing random different values for the barrier strength $k$ corresponding to the different runs of the experiment. For a particular run, Bob has to decipher this specific value of $k$ chosen by Alice. Bob monitors the time evolution of $T_k$ through the detector counts, and by comparing with $T_f$, is able to obtain $\Delta_t$ and compute $\eta$ using it. Thus, he is able to decipher the value of $k$ chosen by Alice using the key $\eta(k)$. This whole procedure may be repeated as many times as required by Alice and Bob in order to exchange any required (classical) information between them.

The security of this protocol is checked by using the relation between $k$ and $v_I$ (the curve 1(b) in this case). The value of $v_I$ can be obtained by Bob by observing $t_d$, and using pre-determined (with Alice) values of the distance $D$ and the time $t_k$ at which Alice starts her perturbation. An attempt of eavesdropping would involve distortion of the wave packet thereby affecting the correspondence between the barrier strength and the information velocity. Bob checks whether his deciphered value of $k$ together with the measured $v_I$ lie on the curve 1(b). If the pair of values fall outside the curve, then Bob is able to detect disturbance by the eavesdropper, and asks Alice to reject that particular run of the experiment.
produced by the diffraction of the wave packet. It is analytically shown that there exists a short time interval over which the transmission probability of a Gaussian wave packet impinging on a transient parabolic potential is maximized at the time of the wave packet’s interaction with the parabola.

The important message encapsulated by the propagation of superarrivals is that every point $x'$ of the initial wavepacket is carried to the point $(x', t - t_0)$ by a single propagating wave packet on the barrier. Further work involving the time-dependent reflection and transmission of wave packets from various types of transient barriers is required for the purpose of experimental tests and applications of superarrivals. In particular, one may consider a variant of our example by introducing two qubits localized on opposite sides of the barrier, and thereby inducing an entanglement between them by their interaction with the reflected and the transmitted parts of a wave packet, in the context of which superarrivals can be employed in schemes aiming at the speed-up of entanglement generation.

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