Numerical radii of accretive matrices

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ABSTRACT
The numerical radius of a matrix is a scalar quantity that has many applications in the study of matrix analysis. Due to the difficulty in computing the numerical radius, inequalities bounding it have received considerable attention in the literature. In this article, we present many new inequalities for the numerical radius of accretive matrices. The importance of this study is the presence of a new approach that treats a specific class of matrices, namely the accretive ones. While some of these inequalities can be considered as refinements of other existing ones, others present new insight to some known results for positive matrices.

1. Introduction
Let $\mathcal{M}_n$ be the algebra of all complex $n \times n$ matrices. For $A \in \mathcal{M}_n$, the numerical radius $w(A)$ and the operator norm $\|A\|$ of $A$ are defined, respectively, by

$$w(A) = \max\{ |\langle Ax, x \rangle| : x \in \mathbb{C}^n, \|x\| = 1 \}$$

and

$$\|A\| = \max\{ |\langle Ax, y \rangle| : x, y \in \mathbb{C}^n, \|x\| = \|y\| = 1 \}.$$ 

It is well known that $w(\cdot)$ defines a norm on $\mathcal{M}_n$ that is equivalent to the operator norm, via the relation [1, p. 114.]

$$\frac{1}{2} \|A\| \leq w(A) \leq \|A\|, \quad A \in \mathcal{M}_n. \quad (1)$$

Interest in bounding the numerical radius has grown due to the fact that computing the operator norm is much easier than that of the numerical radius.
The numerical range of $A \in \mathcal{M}_n$ is defined by the set $$W(A) = \{(Ax, x) : x \in \mathbb{C}^n, \|x\| = 1\}.$$ If $W(A) \subset (0, \infty)$, we say that $A$ is positive, and we simply write $A > 0$. It is well known that when $A > 0$, we have $w(A) = \|A\|$. A more general class of matrices than that of positive ones is the so called accretive matrices. A matrix $A \in \mathcal{M}_n$ is said to be accretive when $\Re A > 0$. Noticing that $\Re A > 0 \iff W(A) \subset (0, \infty) \times (-\infty, \infty) \subset \mathbb{C}$, where $\Re A = (A + A^*)/2$ is the real part of $A$. It is clear that when $A$ is positive, it is necessarily accretive.

The main goal of this article is to present many new relations for $w(A)$ when $A$ is accretive. Some of these new forms present a new direction in this study, while others can be looked at as refinements of some known results, in a new setting.

When talking about accretive matrices, we need to introduce sectorial matrices. A matrix $A \in \mathcal{M}_n$ is said to be sectorial if, for some $0 \leq \alpha < \pi/2$, we have $W(A) \subset S_\alpha := \{z \in \mathbb{C} : |\Im z| \leq \tan \alpha \Re z\}$. The smallest such $\alpha$ will be called the sectorial index of $A$. When $W(A) \subset S_\alpha$, we will write $A \in S_\alpha$. Further, in the sequel, it will be implicitly understood that the notions of $S_\alpha$ and $S_\alpha$ are defined only when $0 \leq \alpha < \pi/2$.

In [2], Drury defined the matrix geometric mean of two accretive matrices $A$ and $B$ by the equation

$$A \# B = \left(\frac{2}{\pi} \int_0^\infty (tA + t^{-1}B)^{-1} \frac{dt}{t}\right)^{-1}. \tag{2}$$

In that reference, several properties for $A \# B$ were shown, with an emphasis on the connection with the geometric mean of two positive matrices. Later, in [3], the definition (2) was extended to the weighted geometric mean $A \#_t B$, $0 < t < 1$.

Recently, in [4] the matrix mean of two accretive matrices $A, B \in \mathcal{M}_n$ has been defined by

$$A \sigma_f B = \int_0^1 (A!_t B) \, d\nu_f(s), \tag{3}$$

where $A!_t B = ((1 - s)A^{-1} + sB^{-1})^{-1}$ is the weighted harmonic mean of $A, B$, the function $f : (0, \infty) \rightarrow (0, \infty)$ is an operator monotone function with $f(1) = 1$ and $\nu_f$ is a probability measure characterizing $\sigma_f$. In [4], detailed discussion was presented to emphasize that (2) and its extension in [3] follow from (3) by letting $f(x) = x^t, 0 < t < 1$. We encourage the reader to refer to [4] as an essential reference to this article, with a full survey to needed background and a full explanation of the motivation.

Moreover, characterization of the operator monotone function for an accretive matrix was given in [4]: let $A \in \mathcal{M}_n$ be accretive and $f : (0, \infty) \rightarrow (0, \infty)$ be an operator
monotone function with \( f(1) = 1 \), then

\[
f(A) = \int_0^1 ((1 - s)I + sA^{-1})^{-1} \, d\nu_f(s),
\]

where \( \nu_f \) is probability measure satisfying \( f(x) = \int_0^1 ((1 - s) + sx^{-1})^{-1} \, d\nu_f(s) \).

In [5], the logarithmic mean of accretive matrices \( A, B \) was defined by

\[
\mathcal{L}(A, B) = \int_0^1 A_{s,t} B \, dt,
\]

where \( A_{s,t} B \) is the weighted geometric mean of the accretive matrices \( A, B \) defined in [3]. The Heinz mean is defined in [6] as

\[
H_t(A, B) = \frac{A_{s,t} B + A_{s,1-t} B}{2}, \quad 0 \leq t \leq 1.
\]

It should be mentioned that the idea of matrix means for accretive matrices was first given by Drury in [2], who defined \( A_{s,1/2} B \). However, this idea has received a considerable attention, as one can see in [3, 4, 6–11].

When \( A \in \mathcal{S}_0 \), then \( A \) is positive, and we have \( w(A) = \|A\| \). Our first simple observation will be that when \( A \in \mathcal{S}_\alpha \), we have

\[
\cos \alpha \|A\| \leq w(A) \leq \|A\|.
\]

Notice that this new inequality is better than (1) when \( 0 \leq \alpha < \pi/3 \). Many extensions of some numerical radius inequalities will be shown for accretive and sectorial matrices, including power inequalities and submultiplicative behaviour.

Another set of new inequalities for accretive matrices is the treatment of \( w(f(A)) \) and \( w(A\sigma B) \), where \( f \) is an operator monotone function and \( \sigma \) is a matrix mean. Such inequalities have not been treated in the literature due to the fact that when \( A, B \) are positive, \( f(A) \) and \( A\sigma B \) are positive, and hence their numerical radius and operator norm coincide. So, when \( A, B \) are accretive, this presents a new direction.

Many other results will be presented, such as subadditivity of the numerical radius, relations among \( w(A) \) and \( w(\Re A) \), and many others.

For our purpose, we will need the following notation.

\[
m = \{ f(x) \text{ where } f : (0, \infty) \to (0, \infty) \text{ is an operator monotone function with } f(1) = 1 \}.
\]

2. Some preliminary discussion

In this part of the paper, we discuss some needed results and terminologies related to accretive matrices.

**Lemma 2.1:** [4] Let \( A, B \in \mathcal{S}_\alpha \). If \( f \in m \), then

\[
\Re(A\sigma_f B) \leq \sec^2 \alpha \, (\Re A) \, \sigma_f (\Re B).
\]
Lemma 2.2 ([12]): Let $A, B \in \mathcal{M}_n$ be two positive matrices. Then, for any non-negative operator monotone function $f$ on $[0, \infty)$,

$$|||f(A + B)||| \leq |||f(A) + f(B)|||.$$  \hspace{1cm} (8)

Lemma 2.3 ([13]): Let $A, B \in \mathcal{M}_n$ be positive. If $f \in \mathfrak{m}$, then

$$|||A\sigma_f B||| \leq |||A|||\sigma_f |||B|||.$$  \hspace{1cm} (9)

for any unitarily invariant norm $||| \cdot |||$ on $\mathcal{M}_n$.

Lemma 2.4 ([4]): Let $A \in S_\alpha$. If $f \in \mathfrak{m}$, then

$$f(\Re A) \leq \Re f(A) \leq \sec^2 \alpha f(\Re A).$$  \hspace{1cm} (10)

Lemma 2.5 ([4]): Let $A \in S_\alpha$. If $f \in \mathfrak{m}$, then

$$f(\|\Re A\|) \leq \|\Re f(A)\| \leq \sec^2 \alpha f(\|\Re A\|).$$

Lemma 2.6 ([7]): Let $A \in S_\alpha$ and $t \in [-1, 0]$. Then

$$\Re A^t \leq \Re^t A \leq \cos^2 \alpha \Re A^t$$  \hspace{1cm} (11)

A reverse of Lemma 2.6 is as follows.

Lemma 2.7 ([7]): Let $A \in S_\alpha$ and $t \in [0, 1]$. Then

$$\cos^2 \alpha \Re A^t \leq \Re^t A \leq \Re A^t$$  \hspace{1cm} (12)

It is well known that for any matrix $A \in \mathcal{M}_n$, $|||\Re A||| \leq |||A|||$, for any unitarily invariant norm $||| \cdot |||$ on $\mathcal{M}_n$. The following lemma presents a reversed version of this inequality for sectorial matrices.

Lemma 2.8 ([14]): Let $A \in S_\alpha$ and let $||| \cdot |||$ be any unitarily invariant norm on $\mathcal{M}_n$. Then

$$\cos \alpha \ |||A||| \leq \ |||\Re(A)||| \leq |||A|||.$$  \hspace{1cm} (13)

Lemma 2.9 ([15]): Let $A \in \mathcal{M}_n$. Then

$$w(\Re A) \leq w(A).$$  \hspace{1cm} (13)

Lemma 2.10 ([6]): Let $A, B \in S_\alpha$. Then for $t \in (0, 1)$,

$$\cos^3 \alpha \ |||A^t B||| \leq |||\mathcal{H}_t(A, B)||| \leq \frac{\sec^3 \alpha}{2} \ |||A + B|||,$$  \hspace{1cm} (14)

for any unitarily invariant norm on $\mathcal{M}_n$.

Lemma 2.11 ([2]): Let $A \in S_\alpha$ and $t \in (0, 1)$. Then $W(A^t) \subset S_{t\alpha}$.

Also note that $W(A^{-1}) \subset S_{\alpha}$. This follows because $W(A^{-1}) \subset S_{\alpha}$ when $W(A) \subset S_{\alpha}$.
Lemma 2.12 ([16]): Let $A, B \in \mathcal{M}_n$ be positive. Then

$$w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \frac{1}{2} \|A + B\|. \tag{15}$$

The following two lemmas are well known.

Lemma 2.13: Let $A, B \in \mathcal{M}_n$. Then

$$\left\|\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right\| = \max(\|A\|, \|B\|). \tag{16}$$

Lemma 2.14: Let $A \in \mathcal{M}_n$ be invertible. Then

$$\|A\|^{-1} \leq \|A^{-1}\|. \tag{17}$$

Lemma 2.15 ([17]): Let $A, B, C, D \in \mathcal{M}_n$ be positive. Then $A \leq C$ and $B \leq D$ imply $A \sigma B \leq C \sigma D$, for any matrix mean $\sigma$.

3. Main results

Now we are ready to present our results. We will present our results in three subsections. In the first subsection, we present inequalities for the numerical radii of accretive matrices that extend some well known inequalities for the numerical radius. However, in the second subsection, we present a new type of numerical radius inequalities that has never been tickled in the literature. The last subsection treats inequalities for the numerical radius and it’s connection to matrix means.

3.1. Accretive versions of some known numerical radius inequalities

First, we have the simple accretive version of (1).

Proposition 3.1: Let $A \in S_\alpha$. Then

$$\cos \alpha \|A\| \leq w(A) \leq \|A\|. \tag{18}$$

Proof: Noting that $w(\Re A) = \|\Re A\|$, since $\Re A > 0$, Lemma 2.8 implies

$$\cos \alpha \|A\| \leq \|\Re A\| = w(\Re A) \leq w(A) \leq \|A\|. \nabla$$

Remark 3.1: Notice that when $0 < \alpha < \pi / 3$, $\cos \alpha > \frac{1}{2}$. This means that, for such $\alpha$,

$$\frac{1}{2} \|A\| < \cos \alpha \|A\| \leq w(A) \leq \|A\|,$$

providing a considerable refinement of the left inequality in (1).

Corollary 3.1: Let $A \in S_\alpha$. Then for $t \in (-1, 1)$,

$$\cos(t\alpha) \|A^t\| \leq w(A^t) \leq \|A^t\|. \tag{19}$$
\textbf{Proof:} Proposition 3.1, Lemmas 2.8 and 2.11, imply the desired result. \hfill \blacksquare

While \(w(\mathfrak{M}A) \leq w(A)\) for any matrix \(A\), a reversed version can be found via sectorial matrices, as follows.

\textbf{Corollary 3.2:} Let \(A \in \mathcal{S}_\alpha\). Then

\[ w(A) \leq \sec \alpha \ w(\mathfrak{M}A). \]  \hfill (20)

\textbf{Proof:} Let \(A \in \mathcal{S}_\alpha\). Then \(w(\mathfrak{M}A) = \| \mathfrak{M}A \|\), since \(\mathfrak{M}A > 0\). Proposition 3.1 implies

\[ w(A) \leq \| A \| \leq \sec \alpha \ \| \mathfrak{M}A \| = \sec \alpha \ w(\mathfrak{M}A). \] \hfill \blacksquare

Notice that when \(A\) is positive, then \(\alpha = 0\) and the above inequality becomes \(w(A) \leq w(\mathfrak{M}A)\), which then implies \(w(A) = w(\mathfrak{M}A)\), as well known for positive matrices.

In the next results, we present accretive versions of the well known power inequality \[1\]

\[ w(A^k) \leq w^k(A), A \in \mathcal{M}_n, k = 1, 2, \ldots \]  \hfill (21)

It should be noted that in (21), only positive integer powers are treated. Now we add the interval \((0, 1)\) to these powers. The significance of these results is the observation that when \(A\) is positive, \(w(A^t) = \| A^t \|\) for any \(t \in (0, 1)\). For such powers, we find no version of (21) in the literature. Now we have one that reads as follows.

\textbf{Theorem 3.1:} Let \(A \in \mathcal{S}_\alpha\). Then, for \(t \in (0, 1)\),

\[ \cos(t\alpha) \ \cos^t \alpha \ w^t(A) \leq w(A^t) \leq \sec(t\alpha) \ \sec^{2t} \alpha \ w^t(A). \]  \hfill (22)

\textbf{Proof:} Let \(t \in (0, 1)\). Then

\[
w(A^t) \leq \| A^t \| \leq \cos(t\alpha) \ \| \mathfrak{M}A^t \| \quad \text{(by Lemma 2.8)}
\]
\[
\leq \cos(t\alpha) \ \sec^{2t} \alpha \ \| \mathfrak{M}A \| \quad \text{(by Lemma 2.7)}
\]
\[
= \sec(t\alpha) \ \sec^{2t} \alpha \ \| \mathfrak{M}A \|^t
\]
\[
= \sec(t\alpha) \ \sec^{2t} \alpha \ w^t(\mathfrak{M}A)
\]
\[
\leq \sec(t\alpha) \ \sec^{2t} \alpha \ w^t(A) \quad \text{(by Lemma 2.9)}.
\]

Thus, we have shown the second inequality. To show the first inequality, we have

\[
w(A^t) \geq \cos(t\alpha) \ \| A^t \| \geq \cos(t\alpha) \ \| \mathfrak{M}A^t \| \quad \text{(by Lemma 2.8)}
\]
\[
\geq \cos(t\alpha) \ \| \mathfrak{M}A \|^t \quad \text{(by Lemma 2.7)}
\]
\[
= \cos(t\alpha) \ \| \mathfrak{M}A \|^t
\]
\[
\geq \cos(t\alpha) \ \cos^t \alpha \ \| A \|^t \quad \text{(by Lemma 2.8)}
\]
\[
\geq \cos(t\alpha) \ \cos^t \alpha \ w^t(A) \quad \text{(by (1))}.
\]

This completes the proof. \hfill \blacksquare
When $A$ is positive, then $\alpha = 0$, and we obtain the well known equality $\|A^t\| = \|A\|^t$. On the other hand, a negative-power version of (21) can be stated as follows.

**Theorem 3.2:** Let $A \in S_\alpha$. Then, for $t \in [0, 1]$,
\[
\cos(t\alpha) \cos^{2t} \alpha w^{-t}(A) \leq w(A^{-t}). \tag{23}
\]

**Proof:** For $t \in [0, 1]$, we have
\[
w(A^{-t}) \geq \cos(t\alpha) \|A^{-t}\| \geq \cos(t\alpha) \|\Re A^{-t}\| \quad \text{(by Lemma 2.8)}
\]
\[
\geq \cos(t\alpha) \cos^{2t} \alpha \|\Re \|A\|^{-t}\| \quad \text{(by Lemma 2.6)}
\]
\[
\geq \cos(t\alpha) \cos^{2t} \alpha \|\Re A\|^{-t} \quad \text{(by Lemma 2.14)}
\]
\[
= \cos(t\alpha) \cos^{2t} \alpha w^{-t}(\Re A)
\]
\[
\geq \cos(t\alpha) \cos^{2t} \alpha w^{-t}(A) \quad \text{(by Lemma 2.9),}
\]
completing the proof.

When $A$ is positive, then $\alpha = 0$, and we obtain the well known inequality $\|A\|^{-t} \leq \|A^{-t}\|$, for $t \in [0, 1]$. $\blacksquare$

In particular, we have the following interesting inverse relation. It should be noted that in general, we have no relation between $w^{-1}(A)$ and $w(A^{-1})$. Now we have the following accretive version.

**Corollary 3.3:** Let $A \in S_\alpha$. Then
\[
\cos^3 \alpha w^{-1}(A) \leq w(A^{-1}). \tag{24}
\]

**Proof:** Let $t = 1$ in (23). $\quad \blacksquare$

In the following result, we present a new submultiplicative inequality for the numerical radius. Recall that for general $A, B \in M_n$, one has $w(AB) \leq 4w(A)w(B)$. When $A$ and $B$ commute, the factor 4 can be reduced to 2, while it can be reduced to 1 when $A$ and $B$ are normal [1, p.114.]. The following result presents a new bound, that has its significance when $0 < \alpha < \pi / 3$.

**Theorem 3.3:** Let $A, B \in S_\alpha$. Then
\[
w(AB) \leq \sec^2 \alpha w(A)w(B). \tag{25}
\]

**Proof:** We have
\[
w(AB) \leq \|AB\| \quad \text{(by (1))}
\]
\[
\leq \|A\| \|B\| \leq \sec^2 \alpha \|\Re A\| \|\Re B\| \quad \text{(by Lemma 2.8)}
\]
\[ \sec^2 \alpha \cdot w(\Re A)w(\Re B) \leq \sec^2 \alpha \cdot w(A)w(B) \quad \text{(by Lemma 2.9)}, \]

which completes the proof.

When \( A, B \) are positive, then \( \alpha = 0 \), and we obtain the well known inequality \( w(AB) \leq w(A)w(B) \).

### 3.2. The numerical radius and operator monotone functions

A new type of numerical radius inequalities is discussed then, where relations for \( w(f(A)) \) and \( f(w(A)) \) are found. However, for such inequalities to be studied, we prove first that when \( A \in \mathcal{S}_\alpha \) and \( f \in \mathcal{m} \), then \( f(A) \in \mathcal{S}_\alpha \). This follows from the following.

**Proposition 3.2:** Let \( A, B \in \mathcal{S}_\alpha \) and let \( f \in \mathcal{m} \). Then \( A\sigma f B \in \mathcal{S}_\alpha \).

**Proof:** Let \( A, B \in \mathcal{S}_\alpha \) and notice that [4, Definition 4.1]

\[ A\sigma f B = \int_0^1 (A)_s B \, dv_f(s) \quad \text{(see (3))} \]

for some positive measure \( v_f(s) \) on \([0,1]\). Then for any unit vector \( x \in \mathbb{C} \), we have

\[ \langle A\sigma f Bx, x \rangle = \int_0^1 ((A)_s Bx, x) \, dv_f(s) \]

\[ = \int_0^1 h(s) \, dv_f(s) \quad \text{(where } h(s) = ((A)_s Bx, x)) \]

\[ = c + id, \]

where

\[ c = \Re \int_0^1 h(s) \, dv_f(s), \quad d = \Im \int_0^1 h(s) \, dv_f(s). \]

We notice that for each \( s \in [0,1] \), \( h(s) \in \mathcal{S}_\alpha \) since \( A, B \in \mathcal{S}_\alpha \). This is due to the fact that \( \mathcal{S}_\alpha \) is invariant under inversion and addition. To show that \( A\sigma f B \in \mathcal{S}_\alpha \), we need to show that \( \langle A\sigma f Bx, x \rangle \in \mathcal{S}_\alpha \), or \( |d| \leq \tan(\alpha)c \). In fact, we have

\[ |d| = \left| \Im \int_0^1 h(s) \, dv_f(s) \right| \]

\[ \leq \int_0^1 |\Im h(s)| \, dv_f(s) \]

\[ \leq \int_0^1 \tan(\alpha) \Re h(s) \, dv_f(s) \quad \text{(since } h(s) \in \mathcal{S}_\alpha) \]

\[ = \tan(\alpha)c. \]

This shows that \( A\sigma f B \in \mathcal{S}_\alpha \) and completes the proof. \[ \blacksquare \]
Noting (4), we have

\[ I \sigma f A = f(A), f \in \mathfrak{m}. \]

Then Proposition 3.2 implies the following.

**Corollary 3.4:** Let \( A \in S_\alpha \) and \( f \in \mathfrak{m} \). Then \( f(A) \in S_\alpha \).

As a special case, we have the following.

**Corollary 3.5:** Let \( A \in S_\alpha \) and \( t \in (0,1) \). Then \( A^t \in S_\alpha \).

It should be noted that in [2], it is shown that if \( A \in S_\alpha \), then \( A^t \in S_{\alpha t}, t \in (0,1) \), a stronger version of Corollary 3.5.

Now we are ready to present the following new relation that allows switching the numerical radius and operator monotone functions.

**Theorem 3.4:** Let \( A \in S_\alpha \). If \( f \in \mathfrak{m} \), then

\[
\cos \alpha f(w(A)) \leq w(f(A)) \leq \sec^3 \alpha f(w(A)). \tag{26}
\]

**Proof:** First we note that for \( f \in \mathfrak{m} \) and every \( 0 \leq s \leq 1 \), one can get \( f(sx) \geq sf(x) \). Next we show the first inequality

\[
\cos \alpha f(w(A)) \leq f(\cos \alpha w(A)) \\
\leq f(w(\Re A)) \quad \text{(by (20))} \\
= f(\|\Re A\|) \\
\leq \|\Re f(A)\| \quad \text{(by Lemma 2.5)} \\
= w(\Re f(A)) \leq w(f(A)).
\]

Thus, we have shown the first inequality. To show the second inequality, noting Corollary 3.4, we have

\[
w(f(A)) \leq \|f(A)\| \leq \sec \alpha \|\Re f(A)\| \quad \text{(by Lemma 2.8)} \\
\leq \sec^3 \alpha f(\|\Re A\|) \quad \text{(by Lemma 2.5)} \\
= \sec^3 \alpha f(w(\Re A)) \\
\leq \sec^3 \alpha f(w(A)),
\]

where we have used the fact that \( f \) is monotone to obtain the last inequality. This completes the proof.

When \( A \) is positive, then \( \alpha = 0 \), and we obtain the well known inequality \( \|f(A)\| = f(\|A\|) \), when \( f \in \mathfrak{m} \). ■

**Proposition 3.3:** Let \( A, B \in S_\alpha \). If \( f \in \mathfrak{m} \), then for \( \lambda \in (0,1) \),

\[
w((1-\lambda)f(A) + \lambda f(B)) \leq \sec^3 \alpha f((1-\lambda)w(A) + \lambda w(B)). \tag{27}
\]
**Proof:** We have

\[
  w((1 - \lambda)f(A) + \lambda f(B)) \leq (1 - \lambda)w(f(A)) + \lambda w(f(B)) \\
  \leq \sec^3 \alpha \left( (1 - \lambda)w(f(A)) + \lambda w(f(B)) \right) \quad \text{(by (26))} \\
  \leq \sec^3 \alpha f((1 - \lambda)w(A) + \lambda w(B)),
\]

where we have used the fact that \( f \) is concave to obtain the last inequality. This completes the proof. \( \blacksquare \)

In the next result, we show a subadditivity behaviour of the numerical radius.

**Corollary 3.6:** Let \( A, B \in S_\alpha \). Then, for \( 0 < t < 1 \),

\[
  w(A^t + B^t) \leq 2^{1-t} \sec^3 \alpha \left( w(A) + w(B) \right)^t.
\]

**Proof:** In Proposition 3.3, let \( f(x) = x^t, \ t \in (0,1) \) and \( \lambda = \frac{1}{2} \). \( \blacksquare \)

On the other hand, a subadditive inequality for the numerical radius with operator monotone functions is shown next. This inequality is the numerical radius version of the celebrated result stating that

\[
  |||f(A + B)||| \leq |||f(A) + f(B)|||, \ A, B \geq 0, \ f \in m,
\]

shown in [12] for any unitarily invariant norm \( ||| \cdot ||| \) on \( \mathcal{M}_n \). Now we present the numerical radius version of this inequality, noting that the numerical radius is not a unitarily invariant norm.

**Theorem 3.5:** Let \( A, B \in S_\alpha \). If \( f \in m \), then

\[
  w(f(A + B)) \leq \sec^3 \alpha w(f(A) + f(B)). \quad \text{(28)}
\]

**Proof:** We have the following chain of inequalities

\[
  w\left(f(A + B)\right) \leq \|f(A + B)\| \leq \sec \alpha \|\Re f(A + B)\| \quad \text{(by Lemma 2.8)} \\
  \leq \sec^3 \alpha \|f(\Re A + \Re B)\| \quad \text{(by Lemma 2.4)} \\
  \leq \sec^3 \alpha \|f(\Re A) + f(\Re B)\| \quad \text{(by Lemma 2.2)} \\
  \leq \sec^3 \alpha \|\Re(f(A) + f(B))\| \quad \text{(by Lemma 2.4)} \\
  = \sec^3 \alpha w(\Re f(A) + f(B))) \\
  \leq \sec^3 \alpha w(f(A) + f(B)) \quad \text{(by Lemma 2.9)}
\]

which completes the proof. \( \blacksquare \)

**Corollary 3.7:** Let \( A, B \in S_\alpha \). Then, for \( 0 < t < 1 \),

\[
  w((A + B)^t) \leq \sec^3 \alpha w(A^t + B^t). \quad \text{(29)}
\]
Proof: This is an immediate consequence of Theorem 3.5, by putting \( f(x) = x^t \), for \( 0 \leq t \leq 1 \).

When \( A, B \) are positive, then \( \alpha = 0 \), and we obtain the well-known inequality \( \| (A + B)^t \| \leq \| A^t + B^t \| \), for \( t \in [0,1] \).

Corollary 3.8: Let \( A, B \in S_\alpha \). Then, for \( 0 < t < 1 \),

\[
\cos^3 \alpha ((A + B)^t) \leq w(A^t + B^t) \leq 2^{1-t} \sec^3 \alpha (w(A) + w(B))^t.
\]

Proof: This follows from Corollary 3.6 and Corollary 3.7.

When \( A, B \in M_n \) are positive, then clearly \( w(A + B) \geq \max(w(A), w(B)) \). If either \( A \) or \( B \) is not positive, this inequality is not necessarily true. However, when \( A, B \) are sectorial, we have the following version.

Theorem 3.6: Let \( A, B \in S_\alpha \). Then

\[
\cos^2 \alpha \max(w(A), w(B)) \leq w(A + B).
\]

Proof: Let \( A, B \in S_\alpha \). Then

\[
w(A + B) \geq \cos \alpha \| A + B \| \quad \text{(by Proposition 3.1)}
\]

\[
\geq \cos \alpha \| \Re A + \Re B \| \quad \text{(by Lemma 2.8)}
\]

\[
= 2 \cos \alpha w \left( \begin{bmatrix} 0 & \Re A \\ \Re B & 0 \end{bmatrix} \right) \quad \text{(Lemma 2.12)}
\]

\[
\geq \cos \alpha \left\| \begin{bmatrix} 0 & \Re A \\ \Re B & 0 \end{bmatrix} \right\| \quad \text{(by (1)})
\]

\[
= \cos \alpha \max (\| \Re A \|, \| \Re B \|) \quad \text{(by Lemma 2.13)}
\]

\[
= \cos \alpha \max (w(\Re A), w(\Re B))
\]

\[
\geq \cos^2 \alpha \max (w(A), w(B)) \quad \text{(by 20)}
\]

completing the proof.

3.3. The numerical radius and matrix mean

In this part of the paper, we present another new type of numerical radius inequalities, where the numerical radius of matrix means is discussed. When \( A, B \in M_n \) are positive, then for any matrix mean \( \sigma \), one has

\[ A \sigma B > 0 \Rightarrow w(A \sigma B) = \| A \sigma B \|. \]

This makes the study of numerical radius inequalities of means of positive matrices trivial.

Now we have the following numerical radius action over the matrix mean of sectorial matrices.
**Theorem 3.7:** Let $A, B \in S_\alpha$. If $f \in \mathcal{M}$, then
\[
w(A \sigma_f B) \leq \sec^3 \alpha \left( w(A) \sigma_f w(B) \right).
\] (32)

**Proof:** Noting Proposition 3.2, we have
\[
w(A \sigma_f B) \leq \|A \sigma_f B\| \leq \sec \|\Re(A \sigma_f B)\| \tag{by Lemma 2.8}
\]
\[
\leq \sec^3 \alpha \|\Re(A) \sigma_f \Re(B)\| \tag{by Lemma 2.1}
\]
\[
\leq \sec^3 \alpha \left( \|\Re(A)\| \sigma_f \|\Re(B)\| \right) \tag{by Lemma 2.3}
\]
\[
= \sec^3 \alpha \left( w(\Re A) \sigma_f w(\Re B) \right)
\]
\[
\leq \sec^3 \alpha \left( w(A) \sigma_f w(B) \right) \tag{by Lemma 2.9 and Lemma 2.15},
\]
which completes the proof. ■

In particular, when $A, B \in \mathcal{M}_n$ are positive, then we may select $\alpha = 0$, and (32) implies the known inequality
\[
\|A \sigma_f B\| \leq \|A\| \sigma_f \|B\|.
\]

**Corollary 3.9:** Let $A, B \in S_\alpha$. Then, for $0 < t < 1$,
\[
w(A^t_\sharp B) \leq \sec^3 \alpha \ w^{1-t}(A)w^t(B).
\] (33)

**Proof:** By (32) and for $\sigma_f = \sharp_t$, where $t \in (1, 0)$
\[
w(A^t_\sharp B) \leq \sec^3 \alpha \left( w(A) \sharp_t w(B) \right)
\]
\[
= \sec^3 \alpha \ w^{1-t}(A)w^t(B),
\]
where the last inequality follows from the definition of $\sharp_t$. This completes the proof. ■

In a similar way, the logarithmic mean satisfies similar property, as follows.

**Theorem 3.8:** Let $A, B \in S_\alpha$. Then, for $0 < t < 1$,
\[
w(\mathcal{L}(A, B)) \leq \sec^3 \alpha \ \mathcal{L}(w(A), w(B)).
\] (34)

**Proof:** By definition of the logarithmic mean (5), we get
\[
w(\mathcal{L}(A, B)) = w\left( \int_0^1 A^t_\sharp B \, dt \right)
\]
\[
\leq \int_0^1 w(A^t_\sharp B) \, dt
\]
\[ \leq \sec^3 \alpha \int_0^1 w^{1-t}(A) w'(B) \, dt \quad \text{(by (33))} \]
\[ = \sec^3 \alpha \mathcal{L}(w(A), w(B)), \]
completing the proof.

The Heinz means follow the same theme too.

**Theorem 3.9:** Let \( A, B \in S_\alpha \). Then for \( t \in (0, 1) \),
\[ w(\mathcal{H}_t(A, B)) \leq \sec^3 \alpha \mathcal{H}_t(w(A), w(B)). \tag{35} \]

**Proof:** Compute
\[
w(\mathcal{H}_t(A, B)) = w\left( \frac{A^{\sharp t} B + A^{\sharp 1-t} B}{2} \right) \quad \text{(by (6))}
\]
\[ \leq \frac{w(A^{\sharp t} B) + w(A^{\sharp 1-t} B)}{2} \]
\[ \leq \frac{\sec^3 \alpha}{2} \left( w^{1-t}(A) w'(B) + w'(A) w^{1-t}(B) \right) \quad \text{(by (33))}
\]
\[ = \sec^3 \alpha \mathcal{H}_t(w(A), w(B)). \]

The proof is complete.

A Heinz-type inequality for the numerical radii of accretive matrices maybe stated as follows.

**Theorem 3.10:** Let \( A, B \in S_\alpha \). Then for \( t \in (0, 1) \),
\[ \cos^4 \alpha \ w(A^\sharp B) \leq w(\mathcal{H}_t(A, B)) \leq \frac{\sec^4 \alpha}{2} \ w(A + B). \tag{36} \]

**Proof:** We prove the first inequality.
\[ w(A^\sharp B) \leq \|A^\sharp B\| \quad \text{(by Proposition 3.1)} \]
\[ \leq \sec^3 \alpha \ \| \mathcal{H}_t(A, B) \| \quad \text{(by Lemma 2.10)} \]
\[ \leq \sec^4 \alpha \ w(\mathcal{H}_t(A, B)). \quad \text{(by Proposition 3.1)} \]

We now prove the second inequality.
\[ w(\mathcal{H}_t(A, B)) \leq \| \mathcal{H}_t(A, B) \| \quad \text{(by Proposition 3.1)} \]
\[ \leq \sec^3 \alpha \ \left\| \frac{A + B}{2} \right\| \quad \text{(by Lemma 2.10)} \]
\[ \leq \frac{\sec^4 \alpha}{2} \ w(A + B). \quad \text{(by Proposition 3.1)} \]

The proof is complete.
**Corollary 3.10:** Let $A, B \in S_\alpha$. Then for $t \in (0, 1)$,

$$\cos \alpha \, w^{1/2}(AB) \leq \mathcal{H}_t(w(A), w(B)). \quad (37)$$

**Proof:** By Theorem 3.3, we get

$$\cos \alpha \, w^{1/2}(AB) \leq \sqrt{w(A)w(B)} \leq \mathcal{H}_t(w(A), w(B)).$$

\[\blacksquare\]

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