A TWISTED BASS-HELLER-SWAN DECOMPOSITION FOR THE NON-CONNECTIVE K-THEORY OF ADDITIVE CATEGORIES

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Abstract. We prove a twisted Bass-Heller-Swan decomposition for both the connective and the non-connective K-theory spectrum of additive categories.

INTRODUCTION

Statement of the main results. Let \( A \) be a (small) additive category together with an automorphism \( \Phi : A \xrightarrow{\sim} A \) of additive categories. Let \( A_\Phi[t, t^{-1}] \) be the associated twisted finite Laurent category (see Definition 1.1) and denote by \( A_\Phi[t] \) and \( A_\Phi[t^{-1}] \) the obvious additive subcategories of \( A_\Phi[t, t^{-1}] \) (see Definition 1.2). Denote by \( K_\infty(A) \) the non-connective K-theory spectrum of the additive category \( A \). Denote by \( \text{TK}^\infty(\Phi^{-1}) \) the mapping torus of the map of spectra \( K^\infty(A) \rightarrow K^\infty(\Phi^{-1}) \). Define \( NK_\infty(A_\Phi[t, t^{-1}]) \) to be the homotopy fiber of the map of spectra \( \text{ev}_{\Phi^0} : K^\infty(A_\Phi[t, t^{-1}]) \rightarrow K^\infty(A) \) induced by the functor of additive categories \( \text{ev}_{\Phi^0} : A_\Phi[t, t^{-1}] \rightarrow A \) obtained by evaluating at \( t = 0 \). There is a certain Nil-category \( \text{Nil}(A, \Phi) \) for which its non-connective K-theory \( K_\text{Nil}^\infty(A, \Phi) \) is a certain delooping of the connective K-theory \( K(\text{Nil}(A, \Phi)) \), where \text{Idem} stands for idempotent completion.

The main theorem of this paper is

**Theorem 0.1** (The Bass-Heller-Swan decomposition for non-connective K-theory of additive categories). Let \( A \) be an additive category. Let \( \Phi : A \rightarrow A \) be an automorphism of additive categories.

(i) There exists a weak homotopy equivalences of spectra, natural in \( A \),

\[
a^\infty \vee b^\infty \vee b^\infty : \text{TK}^\infty(\Phi^{-1}) \vee NK^\infty(A_\Phi[t]) \vee NK^\infty(A_\Phi[t^{-1}]) \xrightarrow{\sim} K^\infty(A_\Phi[t, t^{-1}]);
\]

(ii) There exist weak homotopy equivalence of spectra, natural in \( (A, \Phi) \),

\[
\Omega NK^\infty(A_\Phi[t]) \xrightarrow{\sim} \text{E}^\infty(A, \Phi);
\]

\[
K^\infty(A) \vee \text{E}^\infty(A, \Phi) \xrightarrow{\sim} K^\infty(\text{Nil}(A, \Phi)).
\]

Next we state what we get after applying homotopy groups.

**Remark 0.2** (Wang sequence). We obtain for all \( n \in \mathbb{Z} \) a natural splitting

\[
K_n(A_\Phi[t, t^{-1}]) \xrightarrow{\sim} C_n(A_\Phi[t, t^{-1}]) \oplus NK_n(A) \oplus NK_n(A),
\]

if we define \( C_n(A_\Phi[t, t^{-1}]) \) to be the cokernel of the split injective homomorphism \( K_n(b^\infty_+) \oplus K_n(b^\infty_-) : NK_n(A) \oplus NK_n(A) \rightarrow K_n(A_\Phi[t, t^{-1}]) \), and get a long exact...
Theorem 0.4
(The Bass-Heller-Swan decomposition for connective
induces isomorphisms on
\( A \) of the twisted group ring
\( R \) of the idempotent completion yields a map on the connected
monoid of stable isomorphism classes of objects in \( A \). The composition is given by matrix multiplication. The
(categorical) direct sum of \( m \) and \( n \) is \( m + n \) and on morphisms given by taking block matrices.

Then \( \mathcal{R} \) is a skeleton of the category of finitely generated free right \( R \)-modules, \( \mathcal{R}[t, t^{-1}] \) is a skeleton for the category of finitely generated free modules over the group ring \( R[t, t^{-1}] \), Theorem 0.1 (i) reduces for \( \mathcal{A} = \mathcal{R} \) to the classical Bass-Heller Swan isomorphism

\[
K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_{n}(R) \xrightarrow{\sim} K_{n}(R[t, t^{-1}]) \quad \text{for } n \in \mathbb{Z},
\]

and Theorem 0.1 (ii) reduces for \( \mathcal{A} = \mathcal{R} \) and \( n \geq 0 \) to the classical isomorphism

\[
K_n(\text{Nil}(R)) \xrightarrow{\sim} K_n(R) \oplus NK_{n+1}(R).
\]

If \( R \) comes with a ring automorphism \( \phi: R \to R \) and we equip \( \mathcal{R} \) with the induced automorphism \( \Phi: \mathcal{R} \xrightarrow{\sim} \mathcal{R} \), then \( \mathcal{R}[t, t^{-1}] \) is equivalent to the category of finitely generated free modules over the twisted group ring \( R[\phi, t, t^{-1}] \). Hence Theorem 0.1 (i) provides, after applying \( \pi_n \), a twisted Bass-Heller-Swan decomposition of the twisted group ring \( R[\phi, t, t^{-1}] \).

There is also a version for the connective \( K \)-theory spectrum \( K \).

Theorem 0.4 (The Bass-Heller-Swan decomposition for connective \( K \)-theory of additive categories). Let \( \mathcal{A} \) be an additive category which is idempotent complete. Let \( \Phi: \mathcal{A} \to \mathcal{A} \) be an automorphism of additive categories.

(i) Then there is weak equivalences of spectra, natural in \( (\mathcal{A}, \Phi) \),

\[
a \vee b, a \in \mathcal{A}, b \in \mathcal{B}: T_{K(\Phi^{-1})} \vee NK(\mathcal{A}[t]) \vee NK(\mathcal{A}[t^{-1}]) \xrightarrow{\sim} K(\mathcal{A}[t, t^{-1}]);
\]

(ii) There exist weak homotopy equivalence of spectra, natural in \( (\mathcal{A}, \Phi) \),

\[
\Omega NK(\mathcal{A}[t]) \xrightarrow{\sim} E(\mathcal{A}, \Phi);
\]

\[
K(\mathcal{A}) \vee E(\mathcal{A}, \Phi) \xrightarrow{\sim} K(\text{Nil}(\mathcal{A}; \Phi)).
\]

We emphasize that for the connective version some care is necessary concerning the interpretation after applying \( \pi_n \) in the case \( n = 0 \), since in contrast to the non-connective \( K \)-theory spectrum the passage from an additive category to its idempotent completion does change the zeroth \( K \)-group and the assumption that \( \mathcal{A} \) is idempotent complete does not imply that \( \mathcal{A}[t] \), \( \mathcal{A}[t^{-1}] \), or \( \mathcal{A}[t, t^{-1}] \) is idempotent complete. At least the canonical inclusion of an additive category in its idempotent completion yields a map on the connected \( K \)-theory spectra which induces isomorphisms on \( \pi_n \) for \( n \geq 1 \).

Recall that \( K_0(\mathcal{A}) \) is obtained as the Grothendieck construction of the abelian monoid of stable isomorphism classes of objects in \( \mathcal{A} \) under direct sum. (Two objects \( A_0 \) and \( A_1 \) are stably isomorphic if there exists an object \( B \) such that \( A_0 \oplus B \) and \( A_1 \oplus B \) are isomorphic.) We get for the connective version in degree 0

\[
\pi_0(\text{NK}(\mathcal{A}[t])) = \pi_0(\text{NK}(\mathcal{A}[t^{-1}]))) = 0,
\]
since $i_+ : A \to A_{\Phi}[t^{\pm 1}]$ is bijective on objects and $\text{ev}_0^* i_+ = \text{id}_A$, and therefore $\pi_0(\text{ev}_0^* i_+ : K_0(A_{\Phi}[t^{\pm 1}]) \to K_0(A)$ is bijective. The Wang sequence associated to Theorem 0.3 agrees with the one in Remark 0.2 in degree $n \geq 1$ and ends in degree zero by
\[
\cdots \xrightarrow{\partial_1} K_0(A) \xrightarrow{K_0(\Phi) - \text{id}} K_0(A) \xrightarrow{K_0(i_0)} K_0(A_{\Phi}[t, t^{-1}]) \to 0.
\]

Relation to other work. We start by giving a (incomplete) list of previous work on the Bass-Heller-Swan decomposition. In [4], Bass-Heller-Swan proved a decomposition of $K_1(R[t, t^{-1}])$ for regular rings $R$. Original sources for the Bass-Heller-Swan decomposition of $K_1(R[t, t^{-1}])$ for an arbitrary ring $R$ are Bass [3, Chapter XII] and Swan [22, Chapter 16]. Bass used the decomposition to define negative $K$-groups and to extend the Bass-Heller-Swan decomposition in this range. Ranicki [18, Chapter 10] extended this decomposition of middle and lower $K$-groups to additive categories. Farrell-Hsiang [6] gave a decomposition of $K_1$ of twisted group rings. More treatments of the classical Bass-Heller-Swan decomposition can be found e.g., in [19, Theorem 3.2.22 on page 149, Theorem 3.3.3 on page 155, Theorem 5.3.30 on page 295] and [21, Theorem 9.8 on page 207].

Grayson [7] proved a Bass-Heller-Swan decomposition on the level of higher algebraic $K$-groups, restricting to the case of a ring. In later work [8] he generalized this result to the case of a twisted group ring. The connective $K$-theory of generalized Laurent extensions of rings is treated in Waldhausen [25, 26]. Hüttemann-Klein-Vogell-Waldhausen-Williams [9] proved a Bass-Heller-Swan decomposition for connective algebraic $K$-theory of spaces on the spectrum level; Klein-Williams [10] identified the relative terms with the $K$-theory spectrum of homotopy-nilpotent endomorphisms.

In a companion paper [13] to the present work we will, building on Bass’s approach, develop a non-connective delooping machine for functors from additive categories to spectra which are $n$-contracting, which roughly speaking means that the (untwisted) Bass-Heller-Swan map is bijective on $\pi_i$ for $i \geq n + 1$ and its reduced version is split injective on $\pi_i$ for $i \leq n$. It will come with a universal property. This will enable us to make sense of $K^\infty(A)$ and $K^\infty(\text{Nil}(A; \Phi))$ and to deduce Theorem 0.4 from Theorem 0.3. This delooping machine is of interest in its own right since it is rather elementary and comes with a universal property.

A definition of $K^\infty(A)$ for an additive category $A$ has also been given by Pedersen-Weibel using controlled topology in [15]. It can be identified with our approach using the universal property. It is also obvious from the construction of our approach that $K_n(A)$ agrees with the original definition of Bass using contracting functors. Notice that we cannot define $K^\infty(\text{Nil}(A; \Phi))$ using Pedersen-Weibel [15] since we use a different exact structure than the one coming from split exact sequences. There is a definition of negative $K$-groups for exact categories presented by Schlichting [20] which we has not yet been identified with our approach for $\text{Nil}(A, \Phi)$. The problem in our context with the construction of Schlichting [20] is that it is not clear to us whether the analog of the Approximation Theorem and Fibration Theorems of Waldhausen make sense also in Schlichting’s setting.

The result of this paper will play a key role in a forthcoming paper by the same authors [14] where an explicit splitting on spectrum level of the relative Farrell-Jones assembly map from the family of finite subgroups to the family of virtually subgroups is given and the involution on the relative term is analyzed. Such a splitting, but without identifying the relative term, has already been constructed by Bartels [2] using controlled topology.
We try to keep the presentation of the present paper as self-contained as possible, relying just on some fundamental results in algebraic $K$-theory \cite{5, 23, 27}, the companion paper \cite{13}, and some very basic stable homotopy theory and category theory. While Quillen’s setting for algebraic $K$-theory is very well adapted to proving the Bass-Heller-Swan decomposition for rings, it is not for the more general setup of additive categories, as the necessary localization sequences are not available. The remedy is to pass to the category of chain complexes over $\mathcal{A}$, which is a Waldhausen category, i.e., categories with weak equivalences and cofibrations in the sense of Waldhausen \cite{27}, and to use Waldhausen’s approach to algebraic $K$-theory. Thus it is not surprising that our proof of Theorem 0.4 follows the same global pattern as the one given in the non-linear setting by Hüttemann-Klein-Vogell-Waldhausen-Williams \cite{9} and Klein-Williams \cite{10}. Some extra work is necessary to pass back from the category of homotopy-nilpotent endomorphisms of chain complexes over $\mathcal{A}$ (“Waldhausen setting”) to the category of nilpotent endomorphisms in $\mathcal{A}$ (“Quillen setting”). Such a reduction was carried out by Ranicki \cite{18, Chapter 9} on the level of path-components.

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1. Preliminaries about additive categories

In this section we present some basics about additive categories

1.1. The twisted finite Laurent category. Let $\mathcal{A}$ be an additive category. Let $\Phi: \mathcal{A} \to \mathcal{A}$ be an automorphism of additive categories.

Definition 1.1 (Twisted finite Laurent category $\mathcal{A}_\Phi[t, t^{-1}]$). Define the $\Phi$-twisted finite Laurent category $\mathcal{A}_\Phi[t, t^{-1}]$ as follows. It has the same objects as $\mathcal{A}$. Given two objects $A$ and $B$, a morphism $f: A \to B$ in $\mathcal{A}_\Phi[t, t^{-1}]$ is a formal sum $f = \sum_{i \in \mathbb{Z}} f_i \cdot t^i$, where $f_i: \Phi^i(A) \to B$ is a morphism in $\mathcal{A}$ from $\Phi^i(A)$ to $B$ and only finitely many of the morphisms $f_i$ are non-trivial. If $g = \sum_{j \in \mathbb{Z}} g_j \cdot t^j$ is a morphism
in $\mathcal{A}_R[t, t^{-1}]$ from $B \to C$, we define the composite $g \circ f : A \to C$ by

$$g \circ f := \sum_{k \in \mathbb{Z}} \left( \sum_{i, j \in \mathbb{Z}, i+j=k} g_j \circ \Phi^i(f_i) \right) \cdot t^k.$$ 

The direct sum and the structure of an abelian group on the set of morphisms from $A$ to $B$ in $\mathcal{A}_R[t, t^{-1}]$ is defined in the obvious way using the corresponding structures in $\mathcal{A}$.

So the decisive relation is for a morphism $f : A \to B$ in $\mathcal{A}$

$$(\text{id}_{\Phi(B)} \cdot t) \circ (f \cdot t^0) = \Phi(f) \cdot t.$$

We have already explained in Example [0.3] that for a ring $R$ the passage from $\mathcal{R}$

to $\mathcal{R}_\Phi[t, t^{-1}]$ corresponds to the passage of finitely generated free modules over $R$
to finitely generated free modules over the twisted group ring $R_\Phi[t, t^{-1}]$.

**Definition 1.2**: ($\mathcal{A}_R[t]$ and $\mathcal{A}_R[t^{-1}]$). Let $\mathcal{A}_R[t]$ and $\mathcal{A}_R[t^{-1}]$ respectively be the additive subcategory of $\mathcal{A}_R[t, t^{-1}]$ whose set of objects is the set of objects in $\mathcal{A}$

and whose morphisms from $A$ to $B$ are the finite formal Laurent series $\sum_{i \in \mathbb{Z}} f_i \cdot t^i$ with $f_i = 0$ for $i < 0$ and $i > 0$ respectively.

In the setting of Example [0.3] the additive subcategories $\mathcal{R}_\Phi[t]$ and $\mathcal{R}_\Phi[t^{-1}]$ of $\mathcal{R}_\Phi[t, t^{-1}]$ correspond to the category of finitely generated free modules over the subrings $R_\Phi[t]$ and $R_\Phi[t^{-1}]$ of $R_\Phi[t, t^{-1}]$.

1.2. **Idempotent completion.** Given an additive category $\mathcal{A}$, its idempotent completion $\text{Idem}(\mathcal{A})$ is defined to be the following additive category. Objects are morphisms $p : A \to A$ in $\mathcal{A}$ satisfying $p \circ p = p$. A morphism $f$ from $p_1 : A_1 \to A_1$ to $p_2 : A_2 \to A_2$ in $\mathcal{A}$

satisfying $p_2 \circ f \circ p_1 = f$. If $\mathcal{A}$ has the structure of an additive category with a model for finite or countable the direct sums, $\text{Idem}(\mathcal{A})$ inherits such a structure. Obviously a functor of additive categories $F : \mathcal{A} \to \mathcal{B}$ induces a functor $\text{Idem}(F) : \text{Idem}(\mathcal{A}) \to \text{Idem}(\mathcal{B})$ of additive categories.

There is a obvious embedding

$$\eta(\mathcal{A}) : A \to \text{Idem}(\mathcal{A})$$

sending an objects $A$ to $\text{id}_A : A \to A$ and a morphism $f : A \to B$ to the morphisms given by $f$ again. An additive category $\mathcal{A}$ is called idempotent complete if $\eta(\mathcal{A}) : A \to \text{Idem}(\mathcal{A})$ is an equivalence of additive categories, or, equivalently, if for every idempotent $p : A \to A$ in $\mathcal{A}$ there exists objects $B$ and $C$ and an isomorphism $f : A \cong B \oplus C$ in $\mathcal{A}$ such that $f \circ p \circ f^{-1} : B \oplus C \to B \oplus C$ is given by

$$\begin{pmatrix} \text{id}_B & 0 \\ 0 & 0 \end{pmatrix}.$$

The idempotent completion $\text{Idem}(\mathcal{A})$ is idempotent complete.

Given a ring $R$, then $\text{Idem}(\mathcal{R})$ of the additive category $\mathcal{R}$ defined in Example [0.3]
is a skeleton of the additive category of finitely generated projective $R$-modules.

**Theorem 1.3** (Passage to the idempotent completion). Let $\mathcal{A}$ be an additive category and $\eta(\mathcal{A}) : A \to \text{Idem}(\mathcal{A})$ its canonical embedding in to its idempotent completion.

(i) The map of connective spectra $K(\eta(\mathcal{A})) : K(\mathcal{A}) \to K(\text{Idem}(\mathcal{A}))$ induces an isomorphism on $\pi_n$ for $n \geq 1$ and an injection for $n = 0$;

(ii) The map of non-connective spectra $K^\infty(\eta(\mathcal{A})) : K^\infty(\mathcal{A}) \to K^\infty(\text{Idem}(\mathcal{A}))$
is a weak homotopy equivalence.

**Proof.** (1) This is proved in [23, Theorem A.9.1].

(11) This follows from assertion (11) and [13 Corollary 3.7].
1.3. **Infinite direct sums.** If \( A \) is an additive category, there is a functorial way of adjoining countable direct sums to \( A \): Let \( A^f \) and \( A^\kappa \) be the additive categories, and \( i_f: A \to A^f \) and \( i^\kappa: A \to A^\kappa \) be the embeddings of additive categories constructed in [1] Lemma 9.2, where \( \kappa \) stands for the cardinality of the integers. Then by construction \( i_f \) is an equivalence of additive categories, \( A^f \) is a full additive subcategory of \( A^\kappa \) and in \( A^\kappa \) there exists a categorical model for the direct sum of countably many object. All this data depends functorially on \( A \). In the sequel we will identify \( A \) with its image in \( A^\kappa \) by the embedding \( i^\kappa \circ i_f \).

In the specific situation of Example 0.3 the category \( R^\kappa \) corresponds to the category of countably generated free \( R \)-modules and the embedding of \( R \) into \( R^\kappa \) corresponds to the passage from finitely generated free \( R \)-module to countably generated free \( R \)-modules. Notice that a countable direct sum of countably generated free \( R \)-modules is again a countably generated free \( R \)-module.

1.4. **Induction.** Define functors of additive categories

\[
\begin{align*}
(1.4) \quad & i_0: A \to A_{\Phi}[t, t^{-1}]; \\
(1.5) \quad & i_{\pm}: A \to A_{\Phi}[t^{\pm 1}]; \\
(1.6) \quad & j_{\pm}: A_{\Phi}[t^{\pm 1}] \to A_{\Phi}[t, t^{-1}]; \\
(1.7) \quad & ev_{\pm}: A_{\Phi}[t^{\pm 1}] \to A,
\end{align*}
\]

as follows. The functors \( i_0, i_+ \) and \( i_- \) send a morphism \( f: A \to B \) in \( A \) to the morphism \( f \cdot t^0: A \to B \). The functors \( j_{\pm} \) are just the inclusions. The functor \( ev_0: A_{\Phi}[t^{\pm 1}] \to A \) is given by evaluation at \( t^0 \), i.e., it sends a morphism \( \sum_{i \geq 0} f_i \cdot t^i \) in \( A_{\Phi}[t] \) or \( \sum_{i < 0} f_i \cdot t^i \) in \( A_{\Phi}[t^{-1}] \) respectively to \( f_0 \). Notice that \( ev_0 \circ i_\pm = j_\pm \circ i_\pm \) is the identity \( id_A \) and \( i_0 = j_+ \circ i_+ = j_- \circ i_- \).

1.5. **Restriction.** In the setting of Example 0.3 the additive subcategories \( R_{\Phi}[t] \) and \( R_{\Phi}[t^{-1}] \) of \( R_{\Phi}[t, t^{-1}] \) correspond to the categories of finitely generated free modules over the subrings \( R_{\Phi}[t] \) and \( R_{\Phi}[t^{-1}] \) of \( R_{\Phi}[t, t^{-1}] \), respectively, and the functors \( i_0, i_+ \) and \( i_- \) corresponds to induction. If we allow countably generated modules, it is well known that all the three functors have right adjoints. Thus the following result is not surprising.

**Lemma 1.8.** Suppose that \( A \) has all countable direct sums. Then the induction functors \( i_{\pm} \) and \( i_0 \) from above have additive right adjoints

\[
\begin{align*}
i^0: A_{\Phi}[t, t^{-1}] \to A, \\
i^\pm: A_{\Phi}[t^{\pm 1}] \to A.
\end{align*}
\]

In other words, for any object \( A \in A \) and any objects \( B_+ \in A_{\Phi}[t] \), \( B_- \in A_{\Phi}[t^{-1}] \) and \( B \in A_{\Phi}[t, t^{-1}] \) there are natural group isomorphisms

\[
\begin{align*}
\text{adj: } \text{mor}_{A_{\Phi}[t, t^{-1}]}(i_0 A, B) \xrightarrow{\cong} \text{mor}_A(A, i^0 B); \\
\text{adj: } \text{mor}_{A_{\Phi}[t]}(i_+ A, B_+) \xrightarrow{\cong} \text{mor}_A(A, i^+ B_+); \\
\text{adj: } \text{mor}_{A_{\Phi}[t^{-1}]}(i_- A, B_-) \xrightarrow{\cong} \text{mor}_A(A, i^- B_-).
\end{align*}
\]

**Proof.** The functor \( i^0 \) sends an object \( A \) of \( A[t, t^{-1}] \) to the infinite countable direct sum \( \bigoplus_{k=\infty}^{\infty} \Phi^{-k}(A) \). A morphism in \( A_{\Phi}[t, t^{-1}] \) of the shape \( f \cdot t^0: A \to B \) for a morphism \( f: A \to B \) in \( A \) is sent to

\[
\bigoplus_{k=\infty}^{\infty} \Phi^{-k}(f): \bigoplus_{k=\infty}^{\infty} \Phi^{-k}(A) \to \bigoplus_{k=\infty}^{\infty} \Phi^{-k}(A).
\]
A morphism in \( \mathcal{A}_\Phi[t,t^{-1}] \) of the shape \( \text{id}_A \cdot t: \Phi^{-1}(A) \to A \) is sent to the shift automorphism

\[
\text{sh}: \bigoplus_{k=-\infty}^{\infty} \Phi^{-k}(\Phi^{-1}(A)) \to \bigoplus_{k=-\infty}^{\infty} \Phi^{-k}(A)
\]

which sends the \( k \)-th summand \( \Phi^{-k}(\Phi^{-1}(A)) = \Phi^{-(k+1)}(A) \) of the source identically to the \( (k+1) \)-summand of the target. Since any morphism in \( \mathcal{A}_\Phi[t,t^{-1}] \) is a finite sum of such morphisms, this specifies the desired functor \( i^0: \mathcal{A}_\Phi[t,t^{-1}] \to \mathcal{A}^c \).

We only describe the group isomorphism \( \text{mor}_{\mathcal{A}_\Phi[t,t^{-1}]}(i_0A,B) \to \text{mor}_A(A,i^0B) \). It sends a morphism \( \sum_{i=-\infty}^{\infty} f_i \cdot t: A \to B \) in \( \mathcal{A}_\Phi[t,t^{-1}] \) to the morphism \( A \to \bigoplus_{k=-\infty}^{\infty} \Phi^{-k}(B) \) in \( \mathcal{A} \), whose component to the \( k \)-th-summand \( \Phi^{-k}(B) \) is given by \( \Phi^{-k}(f_k) \). This makes sense since \( i_0A = A \) and \( f_i: \Phi^i(A) \to B \) is different from zero only for finitely many values of \( i \).

The functor \( i^\pm \) are defined analogously, e.g., \( i^+ \) and \( i^- \) send an object \( A \) in \( \mathcal{A}_\Phi[t] \) and \( \mathcal{A}_\Phi[t^{-1}] \) respectively to \( \bigoplus_{k=0}^{\infty} \Phi^{-k}(A) \) and \( \bigoplus_{k=-\infty}^{0} \Phi^{-k}(A) \) respectively.

If \( \mathcal{A} \) does not have countable coproducts, we can still embed \( \mathcal{A}_\Phi[t,t^{-1}] \) into \( \mathcal{A}_{\Phi^0}[t,t^{-1}] \) (noticing that by naturality the automorphism \( \Phi \) extends over \( \mathcal{A}^c \)). In this case we will denote by \( i^\pm \) and \( i^0 \) the restriction of the corresponding left adjoint on \( \mathcal{A}^c \) along the corresponding inclusion

\[
\mathcal{A}_\Phi[t^{\pm 1}] \to \mathcal{A}_{\Phi^0}[t^{\pm 1}], \quad \mathcal{A}_\Phi[t,t^{-1}] \to \mathcal{A}_{\Phi^0}[t,t^{-1}].
\]

2. Strategy of proof for Theorem 0.4 (i)

In this section we present the details of the formulation and then the basic strategy of proof of Theorem 0.4 (i).

In the sequel \( \text{K}(\mathcal{C}) \) denotes the connective \( K \)-theory spectrum of a Waldhausen category \( \mathcal{C} \), i.e., a category with cofibrations and weak equivalences \( \mathcal{C} \), in the sense of Waldhausen [27].

**Remark 2.1** (Exact categories as Waldhausen categories). Any additive (in fact, any exact) category has a canonical Waldhausen structure where the cofibrations are the admissible monomorphisms and the weak equivalences are the isomorphisms.

In the situation of Example 0.3 we get that \( \pi_n(\text{K}(\mathcal{R})) = K_n(\mathcal{R}) \) for \( n \geq 1 \), the map \( \mathbb{Z} \to K_0(\mathcal{R}) \) sending \( n \) to \([R^n]\) is surjective and even bijective if \( R^n \cong R^m \) implies \( m = n \), and \( \pi_n(\text{K}(\mathcal{R})) ) = 0 \) for \( n \leq -1 \). If we pass to the idempotent completion \( \text{Idem}(\mathcal{R}) \), then we obtain \( \pi_n(\text{K}(\text{Idem}(\mathcal{R}))) = K_n(\mathcal{R}) \) for \( n \geq 0 \), where \( K_0(\mathcal{R}) \) is the projective class group, and \( \pi_n(\text{K}(\text{Idem}(\mathcal{R}))) = 0 \) for \( n \leq -1 \).

2.1 The NK-terms and the maps a and b.

**Definition 2.2** (NK(\( \mathcal{A}_\Phi[t] \)) and NK(\( \mathcal{A}_\Phi[t^{-1}] \))). Define \( \text{NK}(\mathcal{A}_\Phi[t]) \) and \( \text{NK}(\mathcal{A}_\Phi[t^{-1}]) \) to be the homotopy fiber of the map of spectra \( \text{K}(\text{ev})^*_t: \text{K}(\mathcal{A}_\Phi[t^{\pm 1}]) \to \text{K}(\mathcal{A}) \).

Let \( b^\pm: \text{NK}(\mathcal{A}_\Phi[t^{\pm 1}]) \to \text{K}(\mathcal{A}_\Phi[t^{\pm 1}]) \) be the canonical map of spectra.

Let \( S: i_0 \circ \Phi^{-1} \to i_0 \) be the natural transformation of functors of additive categories \( \mathcal{A} \to \mathcal{A}_\Phi[t,t^{-1}] \) which is given on an object \( A \) in \( \mathcal{A} \) by the isomorphism \( \text{id}_A \cdot t: \Phi^{-1}(A) \to A \). It induces a (preferred) homotopy

\[
(2.3) \quad \text{K}(S): \text{K}(\mathcal{A}) \wedge I_+ \to \text{K}(\mathcal{A}_\Phi[t,t^{-1}])
\]
from \( K(i_0) \circ K(\Phi^{-1}) \) to \( K(i_0) \). Recall that the mapping torus of \( K(\Phi^{-1}) \) is by definition the pushout

\[
K(A) \vee K(A) = K(A) \wedge \partial I_+ \xrightarrow{n} K(A) \wedge I_+ \xrightarrow{K(\Phi^{-1}) \vee id_{K(A)}} K(A) \vee K(A) = \partial I_+ + n \rightarrow K(A) \wedge I_+ \xrightarrow{\downarrow} K(A) \wedge I_+ \xrightarrow{\downarrow} T K(\Phi^{-1})
\]

where the upper horizontal map \( n \) is given by the inclusion \( \partial I \rightarrow I \). Hence \( S \) yields a map of spectra

\[
a : T K(\Phi^{-1}) \rightarrow K(A \Phi[t,t^{-1}]).
\]

Thus we have explained all terms appearing Theorem 0.4 (i). Next we explain the strategy of its proof.

2.2. The twisted projective line. We define the twisted projective line to be the following additive category \( \mathcal{X} = \mathcal{X}(A, \Phi) \). Objects are triples \((A^+ , f, A^-)\) consisting of objects \( A^+ \) and \( A^- \) in \( A \) and an isomorphism \( f : A^+ \rightarrow A^- \) in \( A \Phi[t,t^{-1}] \). A morphism \((u^+, u^-) : (A^+, f, A^-) \rightarrow (B^+ , g, B^-) \) in \( \mathcal{X} \) consists of morphisms \( u^+ : A^+ \rightarrow B^+ \) in \( A \Phi[t] \) and a morphism \( u^- : A^- \rightarrow B^- \) in \( A \Phi[t^{-1}] \) such that the following diagram commutes in \( A \Phi[t,t^{-1}] \)

\[
\begin{array}{ccc}
A^+ & \xrightarrow{f} & A^- \\
\downarrow{u^+} & & \downarrow{u^-} \\
B^+ & \xrightarrow{g} & B^- \\
\end{array}
\]

Let

\[
k^\pm : \mathcal{X} \rightarrow A \Phi[t^\pm]
\]

be the functor sending \((A^+ , f, A^-)\) to \( A^\pm \).

The category \( \mathcal{X} \) is naturally an exact category by declaring a sequence to be exact if and only if becomes (split) exact both after applying \( k^+ \) and \( k^- \).

The proof of the next result is deferred to Section 5.

Theorem 2.5. Consider the following (not necessarily commutative) diagram of spectra

\[
\begin{array}{ccc}
K(\mathcal{X}) & \xrightarrow{K(k^-)} & K(A \Phi[t^{-1}]) \\
\downarrow{K(k^+)} & & \downarrow{K(j^-)} \\
K(A \Phi[t]) & \xrightarrow{K(j_+)} & K(A \Phi[t,t^{-1}])
\end{array}
\]

There is a natural equivalence of functors \( T : j_+ \circ k^+ \cong j_- \circ k^- \) which is given on an object \((A^+, f, A^-)\) by \( f \). It induces a preferred homotopy \( K(j_+) \circ K(k^+) \simeq K(j_-) \circ K(k^-) \).

If \( A \) is idempotent complete, then the diagram above is a weak homotopy pullback, i.e., the canonical map from \( K(\mathcal{X}) \) to the homotopy pullback of

\[
K(A \Phi[t]) \xrightarrow{K(j_+)} K(A \Phi[t,t^{-1}]) \xleftarrow{K(j_-)} K(A \Phi[t^{-1}])
\]

is a weak homotopy equivalence.

Let

\[
l_i : A \rightarrow \mathcal{X} \quad \text{for } i = 0, 1
\]
be the functor which sends an object $A$ to $(A, \text{id}, A)$ for $i = 0$ and to the object $(\Phi^{-1}(A), \text{id}_A, t, A)$ for $i = 1$, and a morphism $f: A \to B$ in $\mathcal{A}$ to the morphism $(i_+(f), i_-(f))$ for $i = 0$ and $(i_+(\Phi^{-1}(f)), i_-(f))$ for $i = 1$.

The proof of the next result is deferred to Section 9.

**Theorem 2.7.** Suppose that $\mathcal{A}$ is idempotent complete. Then the map of spectra

$$K(l_0) \lor K(l_1): K(\mathcal{A}) \lor K(\mathcal{A}) \cong K(\mathcal{X}).$$

is a weak homotopy equivalence.

### 2.3. Proof of Theorem 0.4 (ii)

In this subsection we finish the proof of Theorem 0.4 (ii) assuming that Theorem 2.5 and Theorem 2.7 are true.

There is a not necessarily commutative diagram

$$\begin{array}{ccc}
K(\mathcal{A}) \lor K(\mathcal{A}) & \xrightarrow{K(i_-) \lor K(i_-)} & K(A_\Phi[t^{-1}]) \\
K(i_+ \circ \Phi^{-1}) \lor K(i_+) & \downarrow & K(j_-) \\
K(A_\Phi[t]) & \xrightarrow{K(j_+)} & K(A_\Phi[t, t^{-1}])
\end{array}$$

The homotopy $K(S): K(\mathcal{A}) \land I_+ \to K(A_\Phi[t, t^{-1}])$ of (2.8) induces a preferred homotopy $K(j_+) \circ (K(i_+ \circ \Phi^{-1}) \lor K(i_+)) \simeq K(j_-) \circ (K(i_-) \lor K(i_-))$.

**Theorem 2.9.** Suppose that $\mathcal{A}$ is idempotent complete. With respect to this choice of homotopy, the diagram (2.8) is a weak homotopy pushout, i.e., the canonical map from the homotopy pushout of

$$K(\mathcal{A}_\Phi[t]) \leftarrow K(i_+ \circ \Phi^{-1}) \lor K(i_+) \xrightarrow{K(i_-) \lor K(i_-)} K(\mathcal{A}) \lor K(\mathcal{A}) \xrightarrow{K(i_-) \lor K(i_-)} K(A_\Phi[t^{-1}])$$

to $K(A_\Phi[t, t^{-1}])$ is a weak homotopy equivalence.

**Proof.** Combining Theorem 2.5 and Theorem 2.7 shows that the diagram of spectra (2.8) is a weak homotopy pullback. This implies that (2.8) is a weak homotopy pushout. The latter claim follows for commutative squares of spectra from [12, Lemma 2.6] and then follows easily for squares commuting up to a preferred homotopy.

Consider the following commutative diagram

$$\begin{array}{ccc}
K(\mathcal{A}) \lor N K(A_\Phi[t]) & \xrightarrow{m_1 \circ (K(\Phi^{-1}) \lor \text{id})} & K(\mathcal{A}) \lor K(\mathcal{A}) \xrightarrow{m_1 \circ (\text{id} \lor \text{id})} K(\mathcal{A}) \lor N K(A_\Phi[t]) \\
K(i_+ \lor b_{+}) & \downarrow & K(i_- \lor b_{-}) \\
K(A_\Phi[t]) & \xrightarrow{K(i_+ \circ \Phi^{-1}) \lor K(i_+)} K(\mathcal{A}) \lor K(\mathcal{A}) & \xrightarrow{K(i_- \lor K(i_-))} K(A_\Phi[t^{-1}])
\end{array}$$

where $m_1$ here and in the sequel denotes the inclusion of the first summand. Let $E_t$ and $E_{b_0}$ respectively be the homotopy pushout of the top and of the bottom row of the diagram (2.10) respectively. One easily checks using the fact that the composite $K(\mathcal{A}) \xrightarrow{K(i_+ \circ \Phi^{-1})} K(A_\Phi[t]) \lor K(A_\Phi[t^{-1}]) \xrightarrow{K(\text{id} \lor \text{id})} K(\mathcal{A})$ is the identity that all vertical arrows in the diagram (2.10) are weak equivalences. Hence the diagram (2.10) induces a weak homotopy equivalence $e: E_t \to E_{b_0}$.

Let $f: E_{b_0} \to K(A_\Phi[t, t^{-1}])$ be homotopy equivalence coming from (2.8) and Theorem 2.7.

Next we construct a weak homotopy equivalence

$$g: E_u \to T K(\Phi^{-1}) \lor N K(A_\Phi[t]) \lor N K(A_\Phi[t^{-1}]).$$
Consider the following not necessarily commutative diagram

\[
\begin{array}{ccc}
K(A) \oplus NK(A)[t] & \xrightarrow{m_1:(K[\varphi^{-1}] \oplus id)} & K(A) \oplus K(A) \xrightarrow{m_1:(id \oplus 1)} K(A) \oplus NK(A)[t] \\
\text{id} \oplus id & & \text{id} \\
K(A) \oplus NK(A)[t^{-1}] & \xrightarrow{m_1:(K[\varphi^{-1}] \oplus K(id_A))} & K(A) \oplus K(A) \oplus I_+ \\
\end{array}
\]

where \( m_0 \) comes from the inclusion \( \{0\} \to I \), and \( n \) comes from the inclusion \( \partial I \to I \). The left square commutes. The right square commutes up to a preferred homotopy coming from the standard homotopy from the inclusion \( \partial I \to I \) to the constant map \( \partial I \to I \) with value 0. Since the pushout of the lower row is \( T_{K[\varphi]} \oplus NK(A)[t^{-1}] \), we obtain a map \( g: E_n \to T_{K[\varphi]} \oplus NK(A)[t^{-1}] \). Since the horizontal right arrow in the diagram above is a cofibration and all vertical arrows are weak homotopy equivalences, the map \( g \) is a weak homotopy equivalence. One easily checks that it fits into the following commutative diagram

\[
\begin{array}{ccc}
E_n & \xrightarrow{g} & T_{K[\varphi]} \oplus NK(A)[t] \oplus NK(A)[t^{-1}] \\
\text{id} & & \text{id} \\
E_0 & \xrightarrow{f} & K(A)[t, t^{-1}] \\
\end{array}
\]

This finishes the proof of Theorem 3.1.1, i.e., that the right vertical arrow in the diagram above is a weak homotopy equivalence, provided that Theorem 2.5 and Theorem 2.7 are true.

3. Preliminaries about chain complexes

Consider an additive category \( \mathcal{A} \). The notions of chain complexes over \( \mathcal{A} \), chain maps, chain homotopies, chain contractions, and short exact sequence of chain complexes are defined in the obvious way.

We write all chain complexes homologically. If \( C \) is a chain complex in \( \mathcal{A} \), we denote its \( n \)-th object by \( C_n \) and its \( n \)-differential by \( c_n: C_n \to C_{n-1} \).

3.1. Mapping cylinders and mapping cones. Let \( f: C \to D \) be a chain map. Define its mapping cylinder \( \text{cyl}(f) \) to be the chain complex with \( n \)-th differential

\[
\begin{pmatrix}
-c_{n-1} & 0 & 0 \\
{id} & c_n & 0 \\
f_{n-1} & 0 & d_n
\end{pmatrix}
\]

There are obvious inclusions \( i_C: C \to \text{cyl}(f) \) and \( i_D: D \to \text{cyl}(f) \) and an obvious projection \( p_D: \text{cyl}(f) \to D \) such that \( p_D \circ i_C = f \), \( p_D \circ i_D = \text{id}_D \); and both \( p_D \) and \( i_D \) are chain homotopy equivalences. Define the mapping cone \( \text{cone}(f) \) of \( f \) to be the cokernel of \( i_C: C \to \text{cyl}(f) \). Hence the \( n \)-th differential of \( \text{cone}(f) \) is

\[
\begin{pmatrix}
-c_{n-1} & 0 & 0 \\
f_{n-1} & 0 & d_n
\end{pmatrix}
\]

We write \( \Sigma C := \text{cone}(\text{id}_C) \). Given a chain complex \( C \), define its suspension \( \Sigma C \) to be the cokernel of the obvious embedding \( C \to \text{cone}(C) \), i.e., to be the chain complex with \( n \)-th differential

\[
\begin{pmatrix}
-c_{n-1} & 0 \\
0 & 0
\end{pmatrix}
\]

C_{n-1} \to C_{n-2}.
We will call a chain complex elementary if it is the finite direct sum of chain complexes $el(X,d)$ for objects $X$ and integers $d$, where $el(X,d)$ is concentrated in dimension $d$ and $d+1$ and has as $(d+1)$-th differential $id_X: X \to X$. Notice that elementary chain complexes are contractible.

We call a chain complex $C$ concentrated in degrees $[a,b]$ if $C_n = 0$ for $n < a$ and for $n > b$. The minimal possible nonnegative number $b−a$ is the length of $C$. We call $C$ bounded if there are natural numbers $a,b$ such that $C$ is concentrated in degrees $[a,b]$. For an object $A$ of $\mathcal{A}$ we denote by $A[n]$ the chain complex concentrated in degrees $[n,n]$ whose single object is $A$.

We collect the following elementary statements about chain complexes.

**Lemma 3.1.** Let $f: C \to D$ be a chain map and $E$ be a chain complex.

(i) There are obvious short exact sequences of chain complexes

$$
0 \to C \xrightarrow{\delta(C)} cyl(f) \to cone(f) \to 0;
$$

$$
0 \to D \xrightarrow{\delta(D)} cyl(f) \to cone(C) \to 0;
$$

(ii) The natural projection $pr(D): cyl(f) \to D$ is the chain map given by $pr(D)_n = (0, f_n, id_{D_n}): C_n - \oplus C_n - \oplus D_n \to D_n$. Then $pr(D) \circ i(D) = id_D$ and there is a chain homotopy $h(D): id_{cyl(f)} \simeq i(D) \circ pr(D)$ given by

$h(D)_n = \begin{pmatrix}
0 & id_{C_n} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}: C_{n-1} \oplus C_n \oplus D_n \to C_{n-1} \oplus C_n \oplus D_{n+1}$.

(iii) Consider the following (not necessarily commutative) diagram of chain complexes

$$
\begin{array}{ccc}
C & \xrightarrow{f} & D \\
\downarrow{u} & & \downarrow{v} \\
C' & \xrightarrow{f'} & D'
\end{array}
$$

Consider a chain homotopy $h: v \circ f' \simeq f' \circ u$.

Then we obtain a chain map $g: cone(f) \to cone(f')$ by

$g_n = \begin{pmatrix}
u_{n-1} & 0 \\
h_{n-1} & v_n
\end{pmatrix}: C_{n-1} \oplus D_n \to C'_{n-1} \oplus D'_{n}$.

Conversely, a chain map $g: cone(f) \to cone(f')$ given by

$g_n = \begin{pmatrix}
u_{n-1} & w_n \\
h_{n-1} & v_n
\end{pmatrix}: C_{n-1} \oplus D_n \to C'_{n-1} \oplus D'_{n}$

yields such a diagram and homotopy;

(iv) Let $f: C \to D$, $u: C \to E$, and $v: D \to E$ be chain maps and let $h: v \circ f \simeq u$ be a chain homotopy. Then we obtain a chain map $F: cyl(f) \to E$ by

$F_n := (h_{n-1}, u_n, v_n): C_{n-1} \oplus C_n \oplus D_n \to E_n$

such that the composite of $F$ with the canonical inclusions of $C$ and $D$ into $cyl(f)$ are $u$ and $v$.

The converse is also true, i.e., a chain map $F$ yields chain maps $u, v$ and a chain homotopy $h: v \circ f \simeq u$;

(v) A chain map is a chain homotopy equivalence if and only if its mapping cone is contractible.
(vi) Let \( 0 \to C \xrightarrow{i} D \xrightarrow{p} E \to 0 \) be an exact sequence of chain complexes. Suppose that \( E \) is contractible. Then there exists a chain map \( s: E \to D \) with \( p \circ s = \text{id}_C \). In particular we get a chain isomorphism
\[
i \oplus s: C \oplus E \xrightarrow{p} D;
\]

(vii) Consider the following commutative diagram of chain complexes
\[
\begin{array}{cccccc}
0 & \to & C & \xrightarrow{f} & D & \xrightarrow{g} & E & \to & 0 \\
& & f & \downarrow & & g & \downarrow & & \\
0 & \to & C' & \xrightarrow{h} & D' & \xrightarrow{E} & 0
\end{array}
\]
If two of the chain maps \( f, g \) and \( h \) are chain homotopy equivalences, then all three are;

(viii) Let \( f: C \to D \) be a map of bounded chain complexes in an additive category \( A \). Then the following statements are equivalent:
(a) \( f \) is a chain homotopy equivalence;
(b) There are elementary chain complexes \( E, E' \) in \( A \) and a commutative diagram
\[
\begin{array}{cccc}
C & \xrightarrow{f} & C \oplus E & \\
\downarrow & & \downarrow & \\
D & \xrightarrow{p} & D \oplus E'
\end{array}
\]
where the horizontal maps are the canonical inclusion and projection and the right vertical arrow is a chain isomorphism.
(ix) Let \( C \) be a chain complex concentrated in degrees \( [a,b] \) (where \( a < b + 1 \)) such that the last differential \( c_{a+1} \) is split surjective. Then, for any split \( \gamma \) of \( c_{a+1} \) there is a short exact sequence
\[
0 \to \text{el}(C_a, a) \xrightarrow{i} C \oplus \text{el}(C_a, a+1) \xrightarrow{p} D \to 0
\]
with a chain complex \( D \) concentrated in degrees \( [a,b+1] \). It is uniquely split and natural in \((C, \gamma)\).

Proof. (i) This is obvious.
(ii) This follows from a direct calculation.
(iii) This is obvious.
(iv) This is obvious.
(v) See for instance [11, Lemma 11.5 a) on page 214].
(vi) For each \( n \) we there exists a morphism \( t_n: E_n \to D_n \) with \( p_n \circ t_n = \text{id}_{D_n} \). Let \( \gamma \) be a chain contraction for \( E \). Define \( s_n: E_n \to D_n \) by \( d_{n+1} \circ \gamma_n \circ t_n + t_n \circ \gamma_{n-1} \circ e_n \). Then the collection \( s = (s_n) \) is a chain map \( s \circ E \to D \) with \( p \circ s = \text{id}_E \).
(vii) The commutative diagram induces a short exact sequence of chain complexes
\[
0 \to \text{cone}(f) \to \text{cone}(g) \to \text{cone}(h) \to 0
\]
with a chain complex \( D \) concentrated in degrees \( [a,b+1] \). It is uniquely split and natural in \((C, \gamma)\).
If $D$ and $E$ are contractible, we get from assertion (vi) a short exact sequence
$$0 \to E \to D \to C \to 0$$
and conclude from the previous case that $C$ is contractible. The implication $(b) \implies (a)$ is obvious, it remains to prove the implication $(a) \implies (b)$. We have the exact sequences $0 \to C \to \text{cyl}(f) \to \text{cone}(f) \to 0$ and $0 \to D \to \text{cyl}(f) \to \text{cone}(C) \to 0$. The chain complexes $\text{cone}(f)$ and $\text{cone}(C)$ are contractible by assertion (v). Because of assertion (vi) it suffices to show for a bounded contractible chain complex $C$ that there are elementary chain complexes $X$ and $X'$ together with chain isomorphisms $C \oplus X' \xrightarrow{=} X$. We use induction over the length of $C$. The induction beginning $d = 1$ is obvious since then $C$ looks like
$$\cdots \to 0 \to C_{n+1} \xrightarrow{c_{n+1}} C_n \to 0 \to \cdots$$
and $c_{n+1}$ is an isomorphism. The induction step from $(d - 1)$ to $d \geq 2$ is done as follows.

We assume for simplicity that $C$ is concentrated in degrees $[0, d]$. Choose a chain contraction $\gamma$ for $C$. Now by part (ix), there is an isomorphism
$$\text{el}(C_0, 0) \oplus D \cong C \oplus \text{el}(C_0, 1)$$
where $D$ is concentrated in degrees $[1, d]$. Since the induction hypothesis applies to $D$, the claim follows.

Again we assume that $C$ is concentrated in degrees $[0, d]$. The splitting of the last differential induces a chain map $\Gamma: \text{el}(C_0, 0) \to C$.

The commutative diagram
$$\begin{array}{ccc}
C_0[0] & \xrightarrow{\text{id}} & \text{el}(C_0, 0) \\
\downarrow{\text{id}} & & \downarrow{i} \\
C_0[0] & \xrightarrow{\gamma} & C
\end{array}$$
induces a map
$$i: \text{el}(C_0, 0) \to \text{cone}(\Gamma)$$
on the vertical cones. Here the symbol “$\hookrightarrow$” denotes the inclusion into a direct summand. It follows that $i$ is also the inclusion into a direct summand, so it extends to a short exact sequence
$$0 \to \text{el}(C_0, 0) \xrightarrow{i} \text{cone}(\Gamma) \xrightarrow{p} D \to 0$$
in $\mathcal{A}$. But the $0$-th object of $\text{cone}(\Gamma)$ is just $C_0$, so $D$ concentrated in degrees $[1, d]$. Moreover the map $i$ is (uniquely) split on the $0$-th level; as the domain of $i$ is elementary, it follows $i$ has a (unique) splitting.

Finally, as $\text{el}(C_0, 0)$ is canonically contractible, the map $\Gamma$ is canonically null-homotopic. It follows that
$$\text{cone}(\Gamma) \cong \text{cone}(0): \text{el}(C_0, 0) \to C \cong \text{el}(C_0, 1) \oplus C.$$
Let
\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{k} & & \downarrow{l} \\
C & \xrightarrow{g} & D
\end{array}
\]
be a square in \( \mathbf{Ch}(A) \) which commutes up to a homotopy \( h : g \circ k \simeq l \circ f \). We call this square \textit{homotopy cartesian} if one of the following equivalent conditions hold:

(i) The induced map \( \text{cone}(f) \to \text{cone}(g) \) is a homotopy equivalence.
(ii) The induced map \( \text{cone}(k) \to \text{cone}(l) \) is a homotopy equivalence.
(iii) The induced sequence
\[
A \xrightarrow{(f,k)} B \oplus C \xrightarrow{g-l} D
\]
together with the null-homotopy induced by \( h \) is a fiber sequence.

We conclude from Lemma 3.1(v) and the fact that the mapping cones of the maps \( \text{cone}(f) \to \text{cone}(g) \), \( \text{cone}(k) \to \text{cone}(l) \) and \( \text{cone}(f,k) \to D \) are isomorphic that these three conditions above are indeed equivalent.

3.3. \textit{Detecting contractibility by restriction.} If \( S \) is a subring of \( R \) and \( C \) is a bounded \( R \)-chain complex, such that each \( R \)-module \( C_n \) of the shape \( R \otimes_S C'_n \), for some \( S \)-module \( C'_n \), and \( C \) considered as \( S \)-chain complex is contractible, then \( C \) is contractible as \( R \)-chain complex. We will later need the following version of this fact for \( A \subseteq \mathbf{A}_R[t, t^{-1}] \).

**Lemma 3.2.** Let \( f : C \to D \) be an \( \mathbf{A}_R[t] \)-chain map of bounded \( \mathbf{A}_R[t] \)-chain complexes. Then \( f \) is an \( \mathbf{A}_R[t] \)-chain homotopy equivalence if and only if its restriction \( i^+ f : i^+ C \to i^+ D \) is a \( \mathbf{A}_C[t] \)-chain homotopy equivalence.

The proof of this Lemma builds on the following result of category theory:

**Lemma 3.3.** Let \( i_+ : A \subseteq B ; i^+ \) be an adjunction between categories, such that the right adjoint \( i^+ \) is faithful. Then, for any two objects \( A \) of \( A \) and \( B \) of \( B \), the injection
\[
i^+ : B(i_+ A, B) \to A(i^+ i_+ A, i^+ B)
\]
has a splitting \( r \) which is natural in \( A \) and \( B \).

If \( A \) and \( B \) are additive and \( i^+ \) and \( i_+ \) are additive, then so is the splitting.

**Proof of Lemma 3.2** We conclude from the fact that \( i^+ \circ \text{cone}(f) = \text{cone}(i^+ f) \) and Lemma 5.3(v) that it suffices to show for a bounded \( \mathbf{A}_R[t] \)-chain complex \( C \) that \( C \) is contractible as \( \mathbf{A}_R[t] \)-chain complex if and only if \( i^+ C \) is contractible as \( \mathbf{A}_C[t] \)-chain complex.

We argue by induction on the length \( d \) of \( C \). The induction step \( d = 0 \) is trivial; the induction step from \( d - 1 \geq 0 \) to \( d \) is done as follows.

We assume for simplicity that \( C \) is concentrated in degrees \([0,d]\). Since \( i^+ C \) is contractible, there exists a morphism \( s_0 : i^+ C_0 \to i^+ C_1 \) in \( \mathbf{A}_C \) such that the composite \( i^+ c_1 \circ s_0 : i^+ C_0 \to i^+ C_1 \) is the identity. Let \( \gamma_0 := r(s_0) \). As it stands, this is a morphism in \( \mathbf{A}_R[t] \) but we can also consider it as a morphism in its full subcategory \( \mathbf{A}_\mathbb{Z}[t] \). Then, by naturality,
\[
c_1 \circ \gamma_0 = (c_1)_* r(s_0) = r((c_1)_* s_0) = r(i^+ c_1 \circ s_0) = r(id) = id.
\]

By Lemma 5.1(vi) it follows that there is an \( \mathbf{A}_R[t] \)-chain complex \( D \) and elementary \( \mathbf{A}_R[t] \)-chain complexes \( E \) and \( E' \) such that
\[
C \oplus E \cong D \oplus E'
\]
and $D$ are concentrated in degrees $[1,d]$. Since $i^+C$ is contractible, $i^+D$ is contractible by Lemma 3.1(vii). By the induction hypothesis $D$ is a contractible $A_Φ[t,t^{-1}]$-chain complex. Therefore $C$ is a contractible $A_Φ[t]$-chain complex, again by Lemma 3.1(vii). □

Proof of Lemma 3.3. Denote by $η_A: A → i^+i_+A$ and $ε_B: i_+i^+B → B$ the unit and the co-unit of the adjunction. The retraction sends a morphism $f: i^+i_+A → i^+B$ to the composite $r(f): i_+A \xrightarrow{i_+η_A} i_+i^+i_+A \xrightarrow{i_+f} i_+i^+B \xrightarrow{ε_B} B$.

This is clearly natural in $A$ and $B$. Moreover it is an elementary property of adjunctions that $i^+r(id_{i_+i_+A}) = id_{i_+i_+A}$. As $i^+$ was assumed to be faithful, we conclude $r(id) = id$.

If $f$ is of the form $i^+g$, then by naturality we have $r(i^+g) = r(g\ast id) = g\ast r(id) = g\ast id = g$

so $r$ is indeed a retraction. □

3.4. Finitely dominated chain complexes. Let $C$ be an $A^e$-chain complex. Recall that we view $A$ as a full additive subcategory of $A^e$. We call $C$ finitely dominated if there exists a bounded $A$-chain complex $D$ and $A^e$-chain maps $i: C → D$ and $r: D → C$ such that $r \circ i$ is $A^e$-chain homotopic to the identity. A proof of the next result can be found in see [17, Proposition 3.2 (ii)].

Lemma 3.4. Suppose that $A$ is idempotent complete. Then an $A^e$-chain complex is finitely dominated if and only if it is $A^e$-chain homotopy equivalent to a bounded $A$-chain complex.

3.5. Homotopy finite chain complexes. Let $B$ be an additive category with a full additive subcategory $A ⊆ B$. We call a $B$-chain complex $C$ homotopy $A$-finite if $C$ is $B$-chain homotopy equivalent to a bounded $A$-chain complex.

Lemma 3.5. Let $0 → C \xrightarrow{i} D \xrightarrow{p} E → 0$ be an exact sequence of $B$-chain complexes. Suppose that two of the three $B$-chain complexes $C$, $D$, and $E$ are $A$-homotopy finite.

Then all three are homotopy $A$-finite.

Proof. We begin with the case where $C$ and $D$ are homotopy $A$-finite. We have to show that $E$ is homotopy $A$-finite.

Choose bounded $A$-chain complexes $P$ and $Q$ together with $B$-chain homotopy equivalences $v: P → C$ and $w: Q → D$. Then there is a $A$-chain map $f: P → Q$ such that $w \circ f \simeq u \circ v$ holds as $B$-chain maps. From Lemma 3.1(iv) we obtain a $B$-chain map $F: cyl(f) → D$ satisfying $F \circ i = u \circ v$ for the canonical inclusion $i: P → cyl(f)$. We obtain a commutative diagram of $B$-chain complexes whose rows are short exact sequences

$$
\begin{array}{ccccccccc}
0 & \rightarrow & C & \xrightarrow{i} & D & \xrightarrow{p} & E & \rightarrow & 0 \\
& & v & & F & & T & & \\
0 & \rightarrow & P & \xrightarrow{i} & cyl(f) & \rightarrow & \text{cone}(f) & \rightarrow & 0 \\
\end{array}
$$

Since $v$ and $F$ are $B$-chain homotopy equivalences, $T$ is a $B$-chain homotopy equivalence by Lemma 3.1(vii). Since $P$ and $Q$ are bounded $A$-chain complexes, $\text{cone}(f)$ is a bounded $A$-chain complexes. This proves that $E$ is homotopy $A$-finite.

Next we deal with the second case, where $D$ and $E$ are homotopy $A$-finite. We have to show that $C$ is homotopy $A$-finite. The exact sequence $0 → C \xrightarrow{i} D \xrightarrow{p}$
$E \to 0$ induces an exact sequence of $\mathcal{B}$-chain complexes $0 \to D \to \text{cyl}(p) \to \Sigma C \to 0$ and $\text{cyl}(p)$ is $\mathcal{B}$-chain homotopy equivalent to $E$. Since $D$ and $\text{cyl}(p)$ are homotopy $\mathcal{A}$-finite, the first case applied to $0 \to D \to \text{cyl}(p) \to \Sigma C \to 0$ implies that $\Sigma C$ and hence $C$ are homotopy $\mathcal{A}$-finite.

If $C$ and $E$ are homotopy $\mathcal{A}$-finite, then $\text{cyl}(p)$ and $\Sigma C$ are homotopy $\mathcal{A}$-finite and by the second case applied to $0 \to D \to \text{cyl}(p) \to \Sigma C \to 0$ we conclude that $D$ is homotopy $\mathcal{A}$-finite. \hfill $\square$

### 3.6. Chain homotopy equivalences and cofibrations

The next lemma is well-known for cofibrations of spaces, see for instance [24, Proposition 5.2.5 on page 108].

**Lemma 3.6.** Let $j(D) \colon C \to D$ and $j(E) \colon C \to E$ be cofibrations of $\mathcal{A}$-chain complexes. Suppose that there exists an $\mathcal{A}$-chain homotopy equivalence $v \colon E \to D$ such that $v \circ j(E) = j(D)$.

Then there exists an $\mathcal{A}$-chain map $w \colon D \to E$ with $w \circ j(D) = j(E)$ together with chain homotopy $h : v \circ w \simeq \text{id}_D$ satisfying $h \circ j(D) = 0$.

*Proof.* Choose a chain map $w' \colon D \to E$ together with a chain homotopy $h' : w' \circ v \simeq \text{id}_E$. Since $j(D) \colon C \to D$ is a cofibration, i.e., for each $n$ there exists a morphism $\tau : E_n \to C_{n+1}$ with $\tau \circ j(E)_n = \text{id}_C_{n+1}$, we can choose for each $n$ a morphism $H'_{n+1} : D_n \to E_{n+1}$ satisfying $H'_{n+1} \circ j(D) = H'_{n+1} \circ j(E)_n$. Define a new chain map $w'' : D \to E$ by putting $w''_n = w'_n + H'_{n+1} \circ H'_{n+1} \circ d_n$. Then $w''$ is homotopic to $w'$ and hence still a chain homotopy inverse of $v$ and satisfies

$$w'' \circ j(D) = (w' + e \circ H' \circ d) \circ j(D) = w' \circ j(D) + e \circ H' \circ j(D) + H' \circ j(D) \circ c = w' \circ j(D) + e \circ h' \circ j(E) + h' \circ j(E) \circ c = w' \circ j(D) + e \circ h' \circ j(E) + h' \circ e \circ j(E) = w' \circ j(D) + (e \circ h' + h' \circ e) \circ j(E) = w' \circ j(D) + (\text{id}_D - w' \circ v) \circ j(E) = w' \circ j(D) + j(E) - w' \circ v \circ j(E) = w' \circ j(D) + j(E) - w' \circ j(D) = j(E).$$

Let $[D, E]_C$ be the set of chain homotopy classes relative $C$ of chain maps $f : D \to E$ satisfying $f \circ j(D) = j(C)$, where chain homotopy $h$ relative $C$ is a chain homotopy satisfying $h \circ j(D) = 0$. Define $[D, D]_C$ analogously. We obtain maps $w''_C : [D, D]_C \to [D, E]_C$ and $v_C : [D, E]_C \to [D, D]_C$ by taking composites. Fix a chain homotopy $h : v \circ w'' \simeq \text{id}_D$. Define a map $h'_2 : [D, D]_C \to [D, D]_C$ by sending the class of $f : D \to D$ to the class of $f + d \circ h + h \circ d$. This is well-defined since

$$ (d \circ h + h \circ d) \circ j(D) = (v \circ w'' - \text{id}) \circ j(D) = v \circ w'' \circ j(D) - j(D) = j(D) - j(D) = 0. $$

Next we prove

$$ h_2 \circ v_C \circ w'_C = \text{id}_{[D, D]_C}.$$
Consider a chain map \( f: D \to D \) with \( f \circ j(D) = j(D) \). Then \( h \circ w \circ v([f]) \) is the chain homotopy class relative \( C \) of \( v \circ w \circ f + d \circ h + h \circ d \). We compute
\[
\begin{align*}
v \circ w' \circ f + d \circ h + h \circ d - f &= (v \circ w' + d \circ h + h \circ d) \circ f - d \circ h \circ f - h \circ d \circ f - h \circ d \circ f - h \circ d - f \\
&= \text{id}_D \circ f + d \circ (h - h \circ f) + (h - h \circ f) \circ d - f \\
&= d \circ (h - h \circ f) + (h - h \circ f) \circ d.
\end{align*}
\]
This implies \((3.7)\) since \((h - h \circ f) \circ j(D) = h \circ j(D) - h \circ f \circ j(D) = h \circ j(D) - h \circ j(D) = 0\) holds. Obviously \( h \) is a bijection, an inverse is given by \((h)_2\). We conclude from \((3.7)\) that \( v_\ast : [D, E]_C \to [D, D]_C \) is surjective. Let \( w: D \to E \) be any chain map with \( w \circ j(E) = j(D) \) such that the class of \([w]\) is mapped under \( v_\ast \) to the class of the identity. \( \square \)

4. Some basic tools for connective \( K \)-theory

We collect some basic tools about connective \( K \)-theory of exact categories and Waldhausen categories.

4.1. The Gillet-Waldhausen Theorem. Throughout this subsection, let \( \mathcal{E} \) be an exact category. The Gillet-Waldhausen theorem compares the \( K \)-theory of the category \( \text{Ch}(\mathcal{E}) \) of (bounded) chain complexes over \( \mathcal{E} \). This is a slightly subtle question since on \( \text{Ch}(\mathcal{E}) \) there might be different notions of weak equivalences which give potentially different \( K \)-theories.

In this section we define a notion of a \textit{canonical Waldhausen structure} on \( \text{Ch}(\mathcal{E}) \) such that the following holds:

**Theorem 4.1** (Gillet-Waldhausen Theorem). The inclusion functor \( \mathcal{E} \to \text{Ch}(\mathcal{E}) \) which considers an object as a 0-dimensional chain complex induces a homotopy equivalence
\[
\mathbf{K}(\mathcal{E}) \cong \mathbf{K}(\text{Ch}(\mathcal{E})),
\]
provided \( \text{Ch}(\mathcal{E}) \) carries the canonical Waldhausen structure, see Definition 4.10.

**Definition 4.2** (Exact structure on chain complexes). A chain complex \( C \) in \( \mathcal{E} \) is called \textit{exact} if each differential factors as
\[
c_n: C_n \xrightarrow{p_n} Z_{n-1} \xrightarrow{i_n} C_{n-1}
\]
such that all the sequences
\[
Z_n \xrightarrow{i_n} C_n \xrightarrow{p_n} Z_{n-1}
\]
are exact.

Notice that if \( \mathcal{E} \) is abelian then this is just the usual notion of exactness in the sense that the chain complex has no homology.

**Definition 4.3** (Property (P)). We say that \( \mathcal{E} \) satisfies property (P) if any split surjection \( p: A \to B \) in \( \mathcal{E} \) is an admissible epimorphism, i.e., is part of an exact sequence \( K \to A \to B \).

In the case where \( \mathcal{E} \) satisfies property (P) the canonical Waldhausen structure agrees with the one naturally defined by the exact structure. In this case the Gillet-Waldhausen Theorem is well-known and can be found for instance in [23, 1.11.7].

We first discuss the choice of Waldhausen structure provided that property (P) may not hold. It is not a good idea to declare a chain map to be a weak equivalence by demanding that its mapping cone is exact. Namely, the following example shows...
that a contractible chain complex $C$ may not be exact and that a direct summand of an exact chain complex may not be exact either.

**Example 4.4.** Let $M$ be a module over some ring $R$ such that $M$ is not free but stably free, i.e., such that $M \oplus R^m$ is free for some $m$. Then, the chain complex

$$
R^m \xrightarrow{(0 \ 1)} M \oplus R^m \xrightarrow{(1 \ 0)} M \oplus R^m \xrightarrow{(0 \ 1)} R^m
$$

is a chain complex in the exact category $\mathcal{R}$ of finitely generated free $R$-modules. As such it is not exact, for the last map has no kernel in $\mathcal{R}$. On the other hand, it is chain contractible and acyclic as a chain complex of $R$-modules.

Moreover, if we take direct sum with the exact chain complex $R^m \xrightarrow{id} R^m$ to the middle degrees, then the resulting chain complex is exact in the category of free $R$-modules.

First we explain how property (P) ensures that this pathology does not arise.

**Definition 4.5 (Waldhausen structure for an exact category with Property (P)).** If $\mathcal{E}$ satisfies property (P), a chain map $f: C \to D$ is called

(i) a cofibration if it is degree-wise an admissible monomorphism;

(ii) a weak equivalence if its mapping cone is an exact chain complex.

For the following, we say that an exact functor $F: \mathcal{E} \to \mathcal{E}'$ between exact categories reflects exactness provided

$$
E_0 \to E_1 \to E_2 \text{ is exact in } \mathcal{E} \iff F(E_0) \to F(E_1) \to F(E_2) \text{ is exact in } \mathcal{E}'.
$$

We say that $F$ reflects admissible epimorphisms provided

$$
E_1 \to E_2 \text{ admissible epimorphism in } \mathcal{E} \iff F(E_1) \to F(E_2) \text{ admissible epimorphism in } \mathcal{E}'.
$$

**Lemma 4.6.** If an exact functor $F: \mathcal{E} \to \mathcal{E}'$ reflects exactness and admissible epimorphisms between categories with property (P), then $\text{Ch}(F): \text{Ch}(\mathcal{E}) \to \text{Ch}(\mathcal{E}')$ reflects weak equivalences.

**Proof.** It suffices to show that if $F(C)$ is exact in $\text{Ch}(\mathcal{E}')$ then so is $C$ in $\text{Ch}(\mathcal{E})$. The argument is by induction on the length of $C$. If it is has length at most 2, then the claim holds as $\mathcal{P}F$ reflects exactness.

For the inductive step, note that a chain complex $C$ concentrated in $[0, n]$ is exact if and only if $c_1: C_1 \to C_0$ is an admissible epimorphism with kernel $Z_1$ so that the induced chain complex

$$
\ldots \xrightarrow{c_2} C_2 \xrightarrow{c_2} Z_1 \to 0
$$

is exact. Thus if $F(C)$ is exact in $\text{Ch}(\mathcal{E}')$, then $F(c_1)$ is an admissible epimorphism in $\mathcal{E}'$. But $\mathcal{P}F$ reflects admissible epimorphisms, so $c_1$ is an admissible epimorphism in $\mathcal{E}$, with a kernel $Z_1$. Applying the inductive hypothesis to the chain complex concludes the proof.

**Lemma 4.8.** Suppose that $\mathcal{E}$ satisfies property (P). Then:

(i) With the above choice of cofibration and weak equivalence, $\text{Ch}(\mathcal{E})$ is a Waldhausen category that satisfies the saturation, extension, and cylinder axioms;

(ii) If the exact structure on $\mathcal{E}$ is the split-exact structure of the underlying additive category, then a chain map is a weak equivalence if and only if it is a chain homotopy equivalence.
Proof. In the case where $\mathcal{E}$ is an abelian category (so that weak equivalences are homology equivalences by the long exact homology sequence), the conclusion is well-known.

In the general case, denote by $i: \mathcal{E} \to \mathcal{E}'$ the Gabriel-Quillen embedding \cite[7.1]{iii}, where $\mathcal{E}'$ is the abelian category of contravariant left exact functors $\mathcal{E} \to \text{Ab}$ to the abelian category $\text{Ab}$ of abelian groups. (Note that surjections in $\mathcal{E}'$ are not the objectwise surjective transformations.) By \cite[7.16]{iii}, the image if $i$ is a full subcategory, and $i$ reflects exactness and admissible epimorphisms. Hence Lemma \ref{Lem:QuillenEmbedding} implies that $\text{Ch}(\mathcal{E})$ is a full Waldhausen subcategory of $\text{Ch}(\mathcal{E}')$ which is closed under taking mapping cylinders; in particular it is a Waldhausen category satisfying the three extra axioms.

$(ii)$ If the exact structure is the split exact one, then any additive contravariant functor $\mathcal{E} \to \text{Ab}$ is automatically left exact. This implies that $\mathcal{E}'$ coincides with the abelian category of contravariant functors $\mathcal{E} \to \text{Ab}$ where the abelian structure on $\mathcal{E}'$ is given objectwise by the one on $\text{Ab}$, and the Gabriel-Quillen embedding $i$ is just the additive Yoneda embedding. Therefore the image of any object of $\mathcal{E}$ under $i$ is projective. So for a chain map $f$ in $\mathcal{E}$ its image $i(f)$ is exact (in other words, a homology equivalence) if and only if $i(f)$ is a chain homotopy equivalence in $\mathcal{E}'$. But this is equivalent to $f$ being a chain homotopy equivalence as the embedding $i$ is full. \qed

Lemma 4.9. \begin{itemize} \item[(i)] There is an endofunctor $\mathcal{P}$ on the category of exact categories such that $\mathcal{P}\mathcal{E}$ satisfies property (P), for any exact category $\mathcal{E}$; \item[(ii)] It comes with a natural full embedding $I: \mathcal{E} \to \mathcal{P}\mathcal{E}$ \item[(iii)] The embedding $I$ induces a homotopy equivalence on $K$-theory. \end{itemize}

Proof. By \cite[7.1]{iii}, the category $\text{Idem}(\mathcal{E})$ becomes an exact category if we call a sequence exact if it is a direct summand of an exact sequence of $\mathcal{E}$; moreover $\eta(\mathcal{E}) \subset \text{Idem}(\mathcal{E})$ is an exact subcategory.

Now let $\mathcal{PE} \subset \text{Idem}(\mathcal{E})$ be the full exact subcategory of all objects $A$ that are stably in $\mathcal{E}$, i.e., for which $A \oplus A'$ is isomorphic to an object of $\mathcal{E}$, for some $A' \in \mathcal{E}$. Then $\eta: \mathcal{E} \to \text{Idem}(\mathcal{E})$ factors through $I: \mathcal{E} \to \mathcal{PE}$. As $\eta$ reflects exactness, the same is true for $I$. Hence $I(\mathcal{E}) \subset \mathcal{PE}$ is an exact subcategory. It is clear that if $\mathcal{E}$ satisfies property (P) then $I$ is an equivalence of categories.

We claim that $\mathcal{PE}$ always satisfies property (P). In fact, suppose that $p: A \to B$ is a split surjection in $\mathcal{PE}$, with split $s$. Then $K = (A, 1 - s \circ p)$ is a kernel of $p$ in $\text{Idem}(\mathcal{E})$ and $K \to A \to B$ is exact. We need to show that $K \in \mathcal{PE}$.

To do that, let $A', B' \in \mathcal{E}$ such that $A \oplus A'$ and $B \oplus B'$ are isomorphic to objects in $\mathcal{E}$. Let $p' := p \oplus \text{id}: A \oplus A' \to B \oplus A' \oplus B'$.

$p$ and $p'$ obviously have isomorphic kernel $K$. As $p'$ is isomorphic to a split surjection in $\mathcal{E}$, we have $K \in \mathcal{PE}$.

$(iii)$ By definition $\mathcal{E} \subset \mathcal{PE}$ is strictly cofinal so the inclusion induces a homotopy equivalence on $K$-theory, by Waldhausen’s Cofinality Theorem, see \cite[1.5.9]{iii}. \qed

Now we proceed to define the canonical Waldhausen structure on an arbitrary exact category $\mathcal{E}$. 

Definition 4.10 (Canonical Waldhausen structure). The canonical Waldhausen structure on \( \text{Ch}(\mathcal{E}) \) is the Waldhausen structure which is induced by the embedding \( I : \text{Ch}(\mathcal{E}) \to \text{Ch}(\mathcal{PE}) \) from the Waldhausen structure on \( \text{Ch}(\mathcal{PE}) \).

Thus, a morphism \( f : C \to D \) in \( \text{Ch}(\mathcal{E}) \) is a canonical cofibration if and only if \( I(f)_n : I(C)_n \to I(D)_n \) is an admissible monomorphism in \( \mathcal{P}(\mathcal{E}) \) for each \( n \in \mathbb{Z} \), \( f \) is a canonical weak equivalence if and only if \( I(f) \) has an exact mapping cone in \( \mathcal{P}(\mathcal{E}) \).

Lemma 4.11. For any exact category \( \mathcal{E} \), the category \( \text{Ch}(\mathcal{E}) \), when endowed with its canonical Waldhausen structure, we have

(i) With the above choice of cofibration and weak equivalence, \( \text{Ch}(\mathcal{E}) \) is a Waldhausen category that satisfies the saturation, extension, and cylinder axioms;

(ii) If the exact structure on \( \mathcal{E} \) is the split-exact structure of the underlying additive category, then a chain map is a weak equivalence if and only if it is a chain homotopy equivalence;

(iii) The inclusion \( \text{Ch}(\mathcal{E}) \to \text{Ch}(\mathcal{PE}) \) induces a homotopy equivalence on \( K \)-theory.

Proof. (i) This follows from Lemma 4.8 (i) applied to \( \mathcal{PE} \) and Lemma 4.9 (i) since \( \text{Ch}(\mathcal{E}) \) is a full Waldhausen subcategory, closed under taking mapping cylinders, of \( \text{Ch}(\mathcal{PE}) \) with the Waldhausen structure defined in Definition 4.5.

(ii) If the exact structure on \( \mathcal{E} \) is the split exact one, then the same is true for \( \text{Idem}(\mathcal{E}) \) and hence for \( \mathcal{PE} \). Hence assertion (ii) follows from Lemma 4.8 (i) applied to \( \mathcal{PE} \).

(iii) This follows from Waldhausen’s Cofinality Theorem, see [27, 1.5.9], as \( \text{Ch}(\mathcal{E}) \) is strictly cofinal in \( \text{Ch}(\mathcal{PE}) \). □

Proof of the Gillet-Waldhausen Theorem. If \( \mathcal{E} \) satisfies property (P), then the Gillet-Waldhausen Theorem is proved in [29, 1.11]. In the general case there is a commutative diagram

\[
\begin{array}{ccc}
\text{K}(\mathcal{E}) & \longrightarrow & \text{K}(\text{Ch}(\mathcal{E})) \\
\downarrow^{\simeq} & & \downarrow^{\simeq} \\
\text{K}(\mathcal{PE}) & \longrightarrow & \text{K}(\text{Ch}(\mathcal{PE}))
\end{array}
\]

where the lower horizontal map is a homotopy equivalence since \( \mathcal{PE} \) satisfies property (P) by Lemma 4.9 (i) and the vertical maps are homotopy equivalence by Lemma 4.9 (iii) and Lemma 4.11 (iii). □

To identify the canonical weak equivalences on the exact categories we need to consider, we use the following generalization of Lemma 4.6:

Lemma 4.12. Let \( F : \mathcal{E} \to \mathcal{E}' \) be an exact functor between exact categories which reflects exactness and admissible epimorphisms. Then \( \text{Ch}(f) : \text{Ch}(\mathcal{E}) \to \text{Ch}(\mathcal{E}') \) reflects canonical weak equivalences.

Proof. Obviously \( \mathcal{PE} \) is exact. Let \( E_0 \to E_1 \to E_2 \) be a sequence in \( \mathcal{PE} \) which becomes exact after applying \( \mathcal{PF} \). For suitable \( Y, Z \in \mathcal{E} \), the direct sum

\[
(4.13) \quad (E_0 \to E_1 \to E_2) \oplus (Y \to Y \oplus Z \to Z)
\]
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is a sequence in $\mathcal{E}$ which becomes exact after applying $F$. As $F$ reflects exactness, (4.13) is an exact sequence in $\mathcal{E}$. This implies that the first summand on (4.13) is an exact sequence in $\mathcal{P}\mathcal{E}$.

This argument shows that $\mathcal{P}F$ reflects exactness. A similar argument shows that $\mathcal{P}F$ reflects admissible epimorphisms. Now apply Lemma 4.6. □

Example 4.14 (Twisted Nil category). Given an additive category $\mathcal{A}$ with automorphism $\Phi$, consider the twisted Nil category $\text{Nil}(\mathcal{A}, \Phi)$ from Section 7. It comes with an additive functor $F: \text{Nil}(\mathcal{A}, \Phi) \rightarrow \mathcal{A}$, sending $(\mathcal{A}, f)$ to its underlying object $\mathcal{A}$. This functor reflects exactness by definition, and it is not hard to see that it reflects admissible epimorphisms. Hence a chain map $\varphi$ in $\text{Nil}(\mathcal{A}, \Phi)$ is a canonical weak equivalence if and only if $F(\varphi)$ is one. By conclusion (ii) of Lemma 4.11 the latter statement is equivalent to $F(\varphi)$ being a chain homotopy equivalence.

Example 4.15 (Projective line). A similar statement holds for the twisted projective line category $\mathcal{X}$ from Section 2.2: By definition, the functor $F = (k^+, k^-): \mathcal{X} \rightarrow \mathcal{A} \Phi[1] \times \mathcal{A} \Phi[1]$ reflects exactness. It is not hard either to see that it reflects admissible epimorphisms.

Hence a chain map $f$ in $\mathcal{X}$ is a canonical weak equivalence if and only if both $k^+(f)$ and $k^-(f)$ are chain homotopy equivalences.

Unless specified otherwise, all the chain categories in the sequel will carry the canonical Waldhausen structure and often use Lemma 4.11 (ii) without mentioning this again.

4.2. The Fibration Theorem. In the sequel we use the definitions and notation of Waldhausen [27]. Suppose that $\mathcal{C}$ is a category with cofibrations and that $\mathcal{C}$ is equipped with two categories of weak equivalences, one finer than the other, $v\mathcal{C} \subseteq w\mathcal{C}$. Thus $\mathcal{C}$ becomes a Waldhausen category in two ways. Let $\mathcal{C}^w$ denote the subcategory with cofibrations of $\mathcal{C}$ given by the objects $C$ in $\mathcal{C}$ having the property that the map $A \rightarrow \text{pt}$ belongs to $w\mathcal{C}$. Then $\mathcal{C}^w$ inherits two Waldhausen structures if we put $v\mathcal{C}^w = \mathcal{C}^w \cap v\mathcal{C}$ and $w\mathcal{C}^w = \mathcal{C}^w \cap w\mathcal{C}$.

Theorem 4.16 (Fibration Theorem). Suppose that $\mathcal{C}$ has a cylinder functor, and the category of weak equivalences $w\mathcal{C}$ satisfies the cylinder axiom, saturation axiom, and extension axiom. Then:

(i) The square of path connected spaces

\[
\begin{array}{ccc}
|v\mathcal{C}| & \rightarrow & |w\mathcal{C}| \\
\downarrow & & \downarrow \\
|v\mathcal{S}| & \rightarrow & |w\mathcal{S}|
\end{array}
\]

is homotopy cartesian, and the upper right term is contractible;

(ii) We get a homotopy fibration of spectra

\[K(\mathcal{C}^w, v) \rightarrow K(\mathcal{C}, v) \rightarrow K(\mathcal{C}, w).\]

Proof. (i) This is proved in [27, Theorem 1.6.4].

(ii) The functor loop space $\Omega$ commutes with homotopy pullbacks and homotopy fibrations. The $K$-theory spectrum $K(\mathcal{C})$ is given by a sequence of maps

\[|w\mathcal{C}| \rightarrow \Omega|w\mathcal{S}| \rightarrow \Omega\Omega|w\mathcal{S}| \rightarrow \cdots\]

where all structure maps are weak equivalences possibly except the first one, see [27, page 330]. Hence assertion (ii) follows from assertion (i). □
4.3. The Approximation Theorem. The following result is taken from [27, Theorem 1.6.7].

**Theorem 4.17 (Approximation Theorem).** Let $C_0$ and $C_1$ be Waldhausen categories. Suppose that the weak equivalences in $C_0$ and $C_1$ satisfy the saturation axiom. Suppose further that $C_0$ has a cylinder functor and the weak equivalences in $C_0$ satisfy the cylinder axiom. Let $F: C_0 \to C_1$ be an exact functor. Suppose $F$ has the approximation property, i.e., satisfies the following two conditions:

(i) An arrow in $C_0$ is a weak equivalence in $C_0$ if and only if its image in $C_1$ is a weak equivalence in $C_1$;

(ii) Given any object $C_0$ in $C_0$ and any map $f: F(C_0) \to C_1$, there exist a cofibration $i: C_0 \to C_0'$ in $C_0$ and a weak equivalence $g: F(C_0') \to C_1$ in $C_1$ satisfying $f = g \circ F(i)$.

Then the map of spectra $|\pi|: |\pi|C_0 \to |\pi|C_1$ and the map of spectra $|\pi|: |\pi|C_0 \to |\pi|C_1$ are homotopy equivalences.

4.4. Cisinski’s version of the Approximation Theorem. The following result is a consequence of [5, Proposition 2.14].

**Theorem 4.18 (Cisinski’s Approximation Theorem).** Let $F: C_0 \to C_1$ be an exact functor of Waldhausen categories. Suppose for $k = 0, 1$ that $C_k$ satisfy the saturation axiom and any morphism $f: C \to C''$ in $C_k$ factorizes as $C \xrightarrow{i} C' \xrightarrow{w} C''$ for a cofibration $i$ and a weak equivalence $w$. Furthermore, we assume

(i) An arrow in $C_0$ is a weak equivalence in $C_0$ if and only if its image in $C_1$ is a weak equivalence in $C_1$;

(ii) Given any object $C_0$ in $C_0$ and any map $f: F(C_0) \to C_1$ in $C_1$, there exists a commutative diagram in $C_1$

\[
\begin{array}{ccc}
F(C_0) & \xrightarrow{f} & C_1 \\
F(w) \downarrow & \equiv & \downarrow v \\
F(D_0) & \xrightarrow{\sim} & D_1 \\
\end{array}
\]

for a morphism $w: C_0 \to D_0$ in $C_0$ and weak equivalences $v: C_1 \to D_1$ and $w: F(D_0) \to D_1$ in $C_1$.

Then the map of spectra $K(F): K(C_0) \xrightarrow{\sim} K(C_1)$ is a weak homotopy equivalence.

5. Proof of Theorem 2.5

This section is entirely devoted to the proof of Theorem 2.5. In the first step of the proof of Theorem 2.5 we replace the additive category $A_\Phi[t]$ by a larger exact category $\mathcal{Y}$ with equivalent $K$-theory. It is defined as follows: An object of $\mathcal{Y}$ is an object $(A, f, B)$ of $\mathcal{X}$. A morphism from $(A, f, B)$ to $(C, g, D)$ is a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\varphi \downarrow & & \downarrow \varphi \\
C & \xrightarrow{g} & D \\
\end{array}
\]

where $\varphi^+: A \to C$ is in $A_\Phi[t]$ and $\varphi: B \to D$ is in $A_\Phi[t, t^{-1}]$ (rather than $A_\Phi[t, t^{-1}]$ as for the definition of $\mathcal{X}$); the diagram is supposed to commute in $A[t, t^{-1}]$. The category $\mathcal{Y}$ is exact in the same way as $\mathcal{X}$ is.
Lemma 5.1. The functors
\[ u: \mathcal{A}[t] \to \mathcal{Y}, \quad A \mapsto (A, \text{id}, A) \]
and
\[ v: \mathcal{Y} \to \mathcal{A}[t], \quad (A^+, f, A^-) \mapsto A^+ \]
are exact. The composite \( v \circ u \) is the identity and the composite \( u \circ v \) is naturally isomorphic to the identity functor. In particular, they induce homotopy equivalences on \( \mathbf{K} \)-theory, homotopy inverse to each other.

Proof. It is clear that the functors are exact. Obviously \( v \circ u \) is the identity. The composite \( u \circ v \) is naturally isomorphic to the identity functor: the isomorphism in \( \mathcal{Y} \) at the object \((A^+, f, A^-)\) is given for an object \((A^+, f, A^-)\) by \((\text{id}, f): (A^+, f, A^-) \xrightarrow{\simeq} (A^+, \text{id}, A^+)\). This implies \( K(u) \circ K(v) \simeq \text{id} \). □

Denote by \( k': \mathcal{X} \to \mathcal{Y} \) the inclusion functor, and define
\[ j': \mathcal{Y} \to \text{Ch}(\mathcal{A}_\Phi[t, t^{-1}]), \quad (A^+, f, A^-) \mapsto A^- . \]

Then the square
\[
\begin{array}{ccc}
\text{Ch}(\mathcal{A}) & \xrightarrow{\text{Ch}(k^-)} & \text{Ch}(\mathcal{A}_\Phi[t^{-1}]) \\
\downarrow{\text{Ch}(k')} & & \downarrow{\text{Ch}(j^-)} \\
\text{Ch}(\mathcal{Y}) & \xrightarrow{\text{Ch}(j')} & \text{Ch}(\mathcal{A}_\Phi[t, t^{-1}])
\end{array}
\]
is strictly commutative, and we are going to show that it induces a homotopy pullback after applying \( K \). To show that the square is a homotopy pullback on \( \mathbf{K} \)-theory, we are going to show that the horizontal homotopy fibers of \( K(\text{Ch}(k^-)) \) and \( K(\text{Ch}(j')) \) agree.

Let \( w\text{Ch}(\mathcal{X}) \) be the subcategory of \( \text{Ch}(\mathcal{X}) \) consisting of all chain maps which become weak equivalences in \( \mathcal{A}_\Phi[t^{-1}] \), after applying \( \text{Ch}(k^-) \), and let \( \text{Ch}(\mathcal{X})^w \) be the full subcategory of \( \text{Ch}(\mathcal{X}) \) of all objects which are \( w \)-acyclic, i.e., the map from it to pt lies in \( w\text{Ch}(\mathcal{X}) \). In other words, an object \((C^+, f, C^-)\) belongs to \( \text{Ch}(\mathcal{X})^w \) if and only if \( C^- \) is contractible as an \( \mathcal{A}_\Phi[t, t^{-1}] \)-chain complex. Similarly, denote by \( w\text{Ch}(\mathcal{Y}) \) the subcategory of all morphisms \( f \) such that \( \text{Ch}(j')(f) \) is a chain homotopy equivalence in \( \text{Ch}(\mathcal{A}_\Phi[t, t^{-1}]) \), and adopt the notation \( \mathcal{Y}^w \) for the \( w \)-acyclic objects.

Lemma 5.2. The maps
\[ K(\text{Ch}(k^-)): K(\text{Ch}(\mathcal{X}), w) \to K(\text{Ch}(\mathcal{A}_\Phi[t^{-1}])) \]
\[ K(\text{Ch}(j')): K(\text{Ch}(\mathcal{Y}), w) \to K(\text{Ch}(\mathcal{A}_\Phi[t, t^{-1}])) \]
are homotopy equivalences.

Proof. We want to apply the Approximation Theorem \[4.17\]. We give the details only for \( K(\text{Ch}(k^-)) \), the analogous proof for \( K(\text{Ch}(j')) \) is left to the reader.

It suffices to verify the assumptions appearing in the Approximation Theorem \[4.17\]. The saturation and cylinder axioms are satisfied by Lemma \[4.11\]. The main task is to verify the conditions \([I]\) and \([II]\) appearing in the Approximation Theorem \[4.17\].

A morphism \( f \) in \( \text{Ch}(\mathcal{X}) \) is by definition in \( w\text{Ch}(\mathcal{X}) \) if and only if \( \text{Ch}(k^-)(f) \) is a chain homotopy equivalence in \( \text{Ch}(\mathcal{A}_\Phi[t^{-1}]) \). This takes care of condition \([I]\) for \( K(\text{Ch}(k^-)) \).
Finally we deal with condition (ii). Consider an object \((C^+, f, C^-)\) in \(\text{Ch}(\mathcal{C})\) and a morphism \(\varphi^- : C^- \to D^-\) in \(\text{Ch}(A_{\mathcal{C}}[t^{-1}])\). We will extend \(\varphi^-\) to a morphism \(\varphi = (\varphi^+, \varphi^-) : (C^+, f, C^-) \to (D^+, g, D^-)\) in \(\text{Ch}(\mathcal{C})\). Then, factoring \(\varphi = \mu \circ \psi\) into a cofibration \(\psi\) followed by a weak equivalence \(\mu\) (using the mapping cylinder), we can write \(g^- = \mu^- \circ \text{Ch}(k^-)(\psi)\) where \(\psi\) is a cofibration and \(\mu^-\) is a weak equivalence, as required in condition (ii).

It remains to construct the extension \(\varphi = (\varphi^+, \varphi^-)\).

Let \(m \in \mathbb{Z}\) such that \(D^-_m = 0\) for \(* > m\). Choosing \(K \gg 0\), let \(D^+\) be the following chain complex:

\[
\cdots \to 0 \to D^-_m \xrightarrow{\partial^K} D^-_{m-1} \xrightarrow{\partial^K} \cdots
\]

where \(\partial^K\) is the differential of \(D^-\). Notice that \(D^+\) is a chain complex in \(A_{\mathcal{C}}[t]\) provided \(K\) was chosen big enough. Let \(c^+_m\) be the differential of \(C^+_m\). Enlarging \(K\) if necessary, the following diagram provides a factorization of \(\varphi^- \circ f\) into an \(A_{\mathcal{C}}[t]\)-morphism \(\varphi^+ : C^+ \to D^+\), followed by the \(A_{\mathcal{C}}[t, t^{-1}]\)-isomorphism \(g : D^+ \to D^-\):

\[
\begin{array}{ccc}
C^+_m & \xrightarrow{\partial^K} & D^+_m \\
\downarrow{c^+_m} & & \downarrow{d^+_m} \\
C^+_{m-1} & \xrightarrow{\partial^K} & D^+_{m-1} \\
\downarrow{c^+_{m-1}} & & \downarrow{d^+_{m-1}} \\
& & \vdots
\end{array}
\]

\[
\begin{array}{ccc}
\cdots & \xrightarrow{\partial^K} & \cdots \\
\downarrow{\cdots} & & \downarrow{\cdots} \\
D^-_m & \xrightarrow{\partial^K} & D^-_{m-1} \\
\downarrow{\cdots} & & \downarrow{\cdots} \\
\cdots & \xrightarrow{\partial^K} & \cdots
\end{array}
\]

Hence \(\varphi = (\varphi^+, \varphi^-)\) is a morphism in \(\text{Ch}(\mathcal{C})\) as required. \(\square\)

**Theorem 5.3.** There are fibration sequences

\[
\begin{align*}
\text{K}(\text{Ch}(\mathcal{C})^w) & \to \text{K}(\mathcal{C}) \to \text{K}(A_{\mathcal{C}}[t^{-1}]); \\
\text{K}(\text{Ch}(\mathcal{D})^w) & \to \text{K}(\mathcal{D}) \to \text{K}(A_{\mathcal{D}}[t, t^{-1}]).
\end{align*}
\]

**Proof.** We give the details only for the first sequence, the analogous proof for the second one is left to the reader.

We apply the Fibration Theorem 4.16 in the case \(C = \text{Ch}(\mathcal{C})\), \(w\) as described above and \(v\) the structure of weak equivalences coming from chain homotopy equivalences. Thus we obtain homotopy fibration of spectra

\[
\text{K}(\text{Ch}(\mathcal{C})^w) \to \text{K}(\text{Ch}(\mathcal{C})) \to \text{K}(\text{Ch}(\mathcal{C}), w)
\]

Because of Lemma 5.2 we obtain a homotopy fibration

\[
\text{K}(\text{Ch}(\mathcal{C})^w) \to \text{K}(\text{Ch}(\mathcal{C})) \to \text{K}(\text{Ch}(A_{\mathcal{C}}[t^{-1}])))
\]

Now the claim follows from Theorem 4.1. \(\square\)

To finish the proof of Theorem 5.5 it is by Lemma 5.3 and Lemma 5.4 enough to show:

**Lemma 5.4.** The functor \(k'\) induces a homotopy equivalence

\[
\text{K}(\text{Ch}(\mathcal{C})^w) \simeq \text{K}(\text{Ch}(\mathcal{D})^w).
\]

**Proof.** Again we will use the Approximation Theorem 4.17. Let

\[
\begin{array}{ccc}
C^+ & \xrightarrow{f} & C^- \\
\downarrow{\varphi^+} & & \downarrow{\varphi^-} \\
D^+ & \xrightarrow{g} & D^-
\end{array}
\]
be a morphism in $\text{Ch}(X)^w$ which maps to a weak equivalence in $\text{Ch}(Y)^w$. Then $\varphi^+$ is a chain homotopy equivalence in $A_\Phi[t]$ and $\varphi^-$ is a chain homotopy equivalence in $A_\Phi[t,t^{-1}]$. By assumption, $C^-$ and $D^-$ are contractible in $A_\Phi[t^{-1}]$, so $\varphi^-$ has to be an equivalence in $A_\Phi[t,t^{-1}]$. It follows that the morphism given by Lemma 3.1 (viii) is a weak equivalence in $\text{Ch}(X)^w$ already. This takes care of condition [1].

It remains to check condition [B]. Suppose now that (5.5) is now a morphism in $\text{Ch}(Y)^w$, with $(C^+, f, C^-)$ in $\text{Ch}(X)^w$. We have to factor this morphism through a map in $\text{Ch}(X)^w$ (which we may then replace by a cofibration using the mapping cylinder) and a weak equivalence in $\text{Ch}(Y)^w$.

Notice that the morphism $\varphi^-$ is a chain homotopy equivalence in $A_\Phi[t,t^{-1}]$, as both $C^-$ and $D^-$ are contractible in that category, by assumption. We conclude from Lemma 3.1 [viii] that there is a chain isomorphism of the shape

$$\begin{pmatrix} \varphi^- & y \\ x & z \end{pmatrix} : C^- \oplus E \cong D^- \oplus E'$$

where $E$ and $E'$ are elementary chain complexes in $A_\Phi[t,t^{-1}]$, or even in $A$ since both categories have the same objects.

For large enough $K > 0$, the commutative diagram

$$\begin{array}{ccc}
C^+ & \cong & C^- \\
\left( \begin{array}{cc}
\varphi^+ \\
\overline{tK : x \circ f} \\
\end{array} \right) & \left( \begin{array}{cc}
g^{-1} \circ \varphi^- \\
\overline{tK : x} \\
g^{-1} \circ y \\
\end{array} \right)^{-1} & \left( \begin{array}{cc}
0 \\
1 \end{array} \right) \\
C^- \oplus E' & \cong & C^- \oplus E \\
(1 & 0) & g \\
D^+ \oplus E' & \to & D^- \\
(1 & 0) & g & (\varphi^- & y) \\
D^+ & \to & D^- \\
\end{array}\right.$$

provides the desired factorization of (5.5). □

This finishes the proof of proof of Theorem 2.5.

6. PROOF OF THEOREM 2.7

Notation 6.1 (Truncation for objects). Let $A$ and $B$ be objects in $A$. Define for $a, b \in \mathbb{Z} \cap \{-\infty, \infty\}$ an object in $A^\infty$ by

$$A[a, b] = \bigoplus_{k=a} b \Phi^{-k}(A)$$

where $A[a, b]$ is defined to be zero if $a > b$ holds.

Given a morphism $f: A \to B$ in $A_\Phi[t, t^{-1}]$ and $a_0, b_0, a_1, b_1$ in $\mathbb{Z} \cap \{-\infty, \infty\}$, define the $A^\infty$ morphism $f[\cdot]$ in $A$ to be the composite

$$f[\cdot] \circ A[a_0, b_0] = A[-\infty, \infty] \circ t^0P \circ \overline{t^0f} \circ \overline{t^0}B = B[-\infty, \infty] \circ B[a_1, b_1],$$

where $i$ is the obvious inclusion and $p$ the obvious projection.

The morphism $f[\cdot] : A[-\infty, \infty] \to B[-\infty, \infty]$ agrees with $\overline{t^0f}$ for a morphism $f: A \to B$ in $A_\Phi[t, t^{-1}]$. If $f$ belongs to $A_\Phi[t^{-1}]$, we abbreviate $(j_+ f)[\cdot]$ by $f[\cdot]$ again.

Notice that $(g \circ f)[\cdot]$ is in general not equal to $g[\cdot] \circ f[\cdot]$ and $\text{id}[\cdot]$ is in general not the identity.

Notation 6.2 (Truncation for chain complexes). If $C^+$ is an $A_\Phi[t]$-chain complex and $a \in \{-\infty\} \cap \mathbb{Z}$, then we obtain an $A^\infty$-chain complex $C^+[a, \infty]$ by defining the $n$-chain object to be $C_n^+[a, \infty]$ and the $n$-th differential to be $c_n[] : C_n^+[a, \infty] \to C_{n-1}^+[a, \infty]$ if $c_n$ is the differential of $C^+$. (One has to check that $c_n[] \circ c_{n+1}[] = 0$.)
A chain map \( f : C^+ \to D^+ \) of \( A_\mathbb{F}[t] \)-chain complexes induces a \( A^c \)-chain map denoted by \( f[: C^+_{\mathbb{F}, a, \infty} \to D^+_{\mathbb{F}, b, \infty} \) provided that \( b \leq a \).

If \( C^- \) is an \( A_\mathbb{F}[t^{-1}] \)-chain complex and \( a \in \{\infty\} \mathbb{H}_\mathbb{Z} \), define the \( A^c \)-chain complex \( C^-_{\mathbb{F}, a, \infty} \) analogously. A chain map \( f : C^- \to D^- \) of \( A_\mathbb{F}[t^{-1}] \)-chain complexes induces a \( A^c \)-chain map denoted by \( f[: C^-_{\mathbb{F}, a, \infty} \to D^-_{\mathbb{F}, b, \infty} \) provided that \( b \geq a \).

Notice that Notation \([6.2]\) (in contrast to Notation \([6.1]\)) does in general not make sense if we replace \( \infty \) or \(-\infty \) by an integer.

**Definition 6.3** (Global section functor). The global section functor

\[ \Gamma : \text{Ch}(A) \to \text{Ch}(A^c) \]

sends an object \((C^+, f, C^-)\) to the \( A^c \)-chain complex

\[ \Sigma^{-1} \text{cone} \left( C^+[0, \infty] \oplus C^-[-\infty, 0] \xrightarrow{(-f[1, \text{id}])} C^-[-\infty, \infty] \right) . \]

A morphism \((\varphi^+, \varphi^-) : (C^+, f, C^-) \to (D^+, g, D^-)\) of \( \text{Ch}(A) \) is sent to the morphism in \( \text{Ch}(A^c) \) obtained by applying Lemma \([3.1](iii)\) to the commutative diagram (using the trivial homotopy)

\[
\begin{array}{ccc}
C^+[0, \infty] \oplus C^-[-\infty, 0] & \xrightarrow{(-f[1, \text{id}])} & C^-[-\infty, \infty] \\
(\varphi^+[1, \varphi^-]) & \downarrow & \varphi^-[1] \\
D^+[0, \infty] \oplus D^-[-\infty, 0] & \xrightarrow{(-g[1, \text{id}])} & D^-[-\infty, \infty]
\end{array}
\]

**Remark 6.4** (Comparison with global sections for modules). Consider the special case of modules over a ring \( R \) with automorphism \( \phi : R \to R \). The global section functor assigns to a triple \((M^+, f, M^-)\) consisting of a finitely generated free \( R_\phi[t]\)-module \( M^- \), a finitely generated free \( R_\phi[t^{-1}]\)-module \( M^- \) and an \( R_\phi[t, t^{-1}]\)-isomorphism \( f : j_+M^+ := R_\phi[t, t^{-1}] \otimes_{R_\phi[t]} M^+ \xrightarrow{\cong} j_-M^- := R_\phi[t, t^{-1}] \otimes_{R_\phi[t^{-1}]} M^- \) the finitely generated projective \( R \)-module

\[ \Gamma(f) := \{(a^+, a^-) \in M^+ \oplus M^- \mid f \circ j_+(a) = j_-(b)\} \]

where \( i^\pm M^\pm \) is the restriction to an \( R \)-module and \( j^\pm : M^\pm \to R_\phi[t, t^\pm 1] \otimes_{R_\phi[t^\pm 1]} M^\pm \) is the obvious map. This can be rewritten as the kernel of the \( R \)-homomorphism

\[ i^+M^+ \oplus i^-M^- \xrightarrow{(-f j_+, j_-)} i^0j_-M^- , \]

where \( i^0 \) is again restriction to \( R \). In the case of modules over rings, global sections and its derived functors can be used to compute the \( K \)-theory of the projective line \([10] \) Theorem 3.1 in Section 8.3 on page 59]. In our situation, the above kernel might not exist since \( A \) is not necessarily abelian, but we can replace it by the mapping cone construction.

Such an idea and a similar strategy of proof has been used by Huettemann-Klein-Vogell-Waldhausen-Williams \([9]\).

Let \( \text{Ch}^{bf}(A) \subset \text{Ch}(A^c) \) be the full subcategory of homotopy bounded chain complexes, i.e., chain complexes over \( A^c \) which are homotopy equivalent to a bounded chain complex over \( A \). This is a strictly smaller category if \( A \) does not have countable coproducts.

If \( A \) already has countable coproducts, then the categories \( A \) and \( A^c \) are equivalent. In this case we will think of truncation and global sections as taking values in \( A \).
Lemma 6.5. 

(i) The functor $\Gamma$ is Waldhausen exact (for the canonical Waldhausen structures).

(ii) Suppose that $\mathcal{A}$ is idempotent complete. Then for any object $(C^+, f, C^-)$ of $\text{Ch}(\mathcal{A})$, the chain complex $\Gamma(C^+, f, C^-) \in \text{Ch}(\mathcal{A}^\kappa)$ is chain homotopy equivalent to an object in $\text{Ch}(\mathcal{A})$. Thus, $\Gamma$ defines a Waldhausen exact functor $\Gamma: \text{Ch}(\mathcal{A}) \to \text{Ch}^h(\mathcal{A})$.

Proof. (ii) We showed in Section 4.1 that the functors $k^\pm: \text{Ch}(\mathcal{A}) \to \text{Ch}(\mathcal{A}_R[t^\pm1])$ are Waldhausen exact. The restriction functors from $\text{Ch}(\mathcal{A}_R[t])$, $\text{Ch}(\mathcal{A}_R[t^{-1}])$ and $\text{Ch}(\mathcal{A}_R[t, t^{-1}])$ to $\text{Ch}(\mathcal{A}^\kappa)$ are defined on the level of additive categories and hence are Waldhausen exact. Taking cones and suspensions is also Waldhausen exact. The isomorphism of $\text{Idem}(\mathcal{A}^\kappa)$ defines a Waldhausen exact functor $\Gamma(\mathcal{A}^\kappa) \to \text{Ch}^h(\mathcal{A})$.

(i) The following diagram of $\mathcal{A}^\kappa$-chain complexes has exact rows

\[
\begin{array}{ccccccccc}
0 & \to & C^+[-\infty,0] & \xrightarrow{0} & C^+[0,\infty] \oplus C^-[0,\infty] & \xrightarrow{(\text{id},0)} & C^+[0,\infty] & \to & 0 \\
0 & \to & C^-[-\infty,0] & \xrightarrow{\text{id}} & C^-[-\infty,\infty] & \xrightarrow{(-f[,\text{id}])} & C^-[-\infty,\infty] & \xrightarrow{\text{id}]} & 0 \\
\end{array}
\]

We conclude from Subsection 3.2

(6.6) $\Gamma(C^+, f, C^-) \simeq \Sigma^{-1} \text{cone}(-f[]): C^+[0,\infty] \to C^-[1,\infty]).$

Write $f_n = \sum_{k \in \mathbb{Z}} f_{n,k} \cdot t^k$. Now choose a natural number $N$ such that we have $f_{n,k} = 0$ for all $k \leq -N$ and all $n$. Then the composite

$C^+[N,\infty] \xrightarrow{f[1]} C^-[1,\infty] \xrightarrow{f^{-1}[1]} C^+[N,\infty]$

is the identity map since the following diagram commutes

\[
\begin{array}{ccccccccc}
C^+[N,\infty] & \xrightarrow{\text{id}} & C^+[-\infty,\infty] & \xrightarrow{\text{id}} & C^+[-\infty,\infty] & \xrightarrow{\text{id}} & C^+[N,\infty]
\end{array}
\]

Hence in $\text{Idem}(\mathcal{A}^\kappa)$, the chain complex $C^-[1,\infty]$ splits as

$C^-[1,\infty] \cong C^+[N,\infty] \oplus R.$

We argue that $R$ is actually isomorphic to an object of $\mathcal{A}$. In fact, denote by $r: C^-[1,\infty] \to R$ the projection and by $i: R \to C^-[1,\infty]$ the inclusion. For a large enough natural number $M \gg N$, the composite

$C^-[M,\infty] \xrightarrow{f^{-1}[1]} C^+[N,\infty] \xrightarrow{f[1]} C^-[M,\infty]$

is the identity, which shows that the restriction of $r$ onto $C^-[M,\infty]$ is zero. Hence $r$ factors as

$C^-[1,\infty] \xrightarrow{\text{in}} C^-[1, M] \xrightarrow{i'r} R.$

The isomorphism of $\text{Idem}(\mathcal{A}^\kappa)$-chain complexes

$(C^-[1,\infty], ir) \xrightarrow{\text{pir}} (C^-[1, M], pir')$

(with inverse $ir'$) now shows that $R = (C^-[1,\infty], ir)$ is isomorphic to a chain complex in $\text{Ch}(\text{Idem}(\mathcal{A}))$. Since $\mathcal{A}$ is idempotent complete, we can assume that
$R$ is an $A$-chain complex. Since $A^c$ is a full subcategory of $\Idem(A^c)$, we obtain a exact sequence of $A^c$-chain complexes such that $C^+[0, N]$ and $R$ are $A$-chain complexes

$$
\begin{array}{ccccccc}
0 & \longrightarrow & C^+[N, \infty] & \longrightarrow & C^+[0, \infty] & \longrightarrow & C^+[0, N] & \longrightarrow & 0 \\
\downarrow{\text{id}} & & \downarrow{-f[1]} & & \downarrow{g} & & \\
0 & \longrightarrow & C^+[N, \infty] & \longrightarrow & C^+[-1, \infty] & \longrightarrow & R & \longrightarrow & 0
\end{array}
$$

where $g$ is the induced map on the quotients. It shows that $\Sigma^{-1}\cone(-f[1]) \simeq \Sigma^{-1}\cone(g)$ which lies in $\Ch(A)$. Hence $\Gamma(C^+, f, C^-)$ belongs to $\Ch^hf((A)$ because of (6.6).

Recall that the automorphism $\Phi: A \to A$ extends to an automorphism, denoted by the same letter, $\Phi: A_\Phi[t, t^{-1}] \to A_\Phi[t, t^{-1}]$ by sending a morphism $\sum_{k=-\infty}^{\infty} g_k \cdot t^k: A \to B$ to $\sum_{k=-\infty}^{\infty} \Phi(g_k) \cdot t^k: \Phi(A) \to \Phi(B)$. It induces automorphisms $\Phi: A_\Phi[t^{\pm 1}] \to A_\Phi[t^{\pm 1}]$. In particular we get for an $A_\Phi[t]$-chain complex $C$ a new $A_\Phi[t]$-chain complex $\Phi^{-1}(C)$, an $A_\Phi[t]$-chain map $t: \Phi^{-1}(C) \to C$ and a $A_\Phi[t, t^{-1}]$-chain map $t: \Phi^{-1}(j_+C) = j_+\Phi^{-1}(C) \to j_+C$.

Denote by $s$: $\Ch(\mathcal{X}) \to \Ch(\mathcal{X})$ the functor which sends the object $(C^+, f, C^-)$ to $(\Phi^{-1}(C^+), f \circ t, C^-)$ and the morphism $(\varphi^+, \varphi^-)$ to $(\Phi^{-1}(\varphi^+), \varphi^-)$. This is well-defined since $j_+\Phi^{-1}(\varphi^+) = \Phi(j_+\varphi^+) = t^{-1} \circ j_+(\varphi^+) \circ t$ holds in $A_\Phi[t, t^{-1}]$. Put

$$
l_i := s^i \circ l_0: \Ch(A) \to \Ch(\mathcal{X});
$$

$$
\Gamma_i := \Gamma \circ s^{-i}: \Ch(\mathcal{X}) \to \Ch^hf(\Idem(A)).
$$

Denote by $(\Ch(\mathcal{X}), w_1)$ the Waldhausen category with underlying category the category with bounded chain complexes over $\mathcal{X}$ and the usual cofibrations, but with a new category of weak equivalences, namely, the one consisting of those chain maps that become a weak equivalence after applying $\Gamma_i$. Notice that $w_0 \cap w_1$ contains all chain homotopy equivalences so that $\Ch(\mathcal{X})$ is a Waldhausen subcategory of $(\Ch(\mathcal{X}), w_0 \cap w_1)$.

**Lemma 6.7.** The map induced by inclusion of Waldhausen categories

$$
\mathbf{K}(\Ch(\mathcal{X})) \to \mathbf{K}(\Ch(\mathcal{X}), w_0 \cap w_1)
$$

is a homotopy equivalence.

**Proof.** Let $\nu$ be the standard structure of weak equivalences in $\Ch(\mathcal{X})$. We will show that if an object $(C^+, f, C^-)$ of $\Ch(\mathcal{X})$ is $(w_0 \cap w_1)$-acyclic, then $C^+$ is contractible in $A_\Phi[t]$ and $C^-$ is contractible in $A_\Phi[t^{-1}]$, so that $(C^+, f, C^-)$ is $\nu$-acyclic. This statement implies that $\mathbf{K}(\Ch(\mathcal{X}), w_0 \cap w_1, v)$ is contractible, from which the Lemma follows by the Fibration Theorem 4.14.

First we want to show that the $A^c$-chain complex $C^-[-\infty, 0]$ is contractible. The following diagram of $A^c$-chain complexes commutes

$$
\begin{array}{ccc}
C^+[-1, \infty] & \oplus & C^-[-\infty, 0] \\
\downarrow{\text{id} \oplus \text{id}} & & \downarrow{(-f[,], \text{id} \oplus \text{id})} \\
C^+[-1, \infty] & \oplus & C^-[-\infty, 0]
\end{array}
$$

There is an obvious identification of $C[-1, \infty]$ with $\Phi(C)[0, \infty]$. Under this identification the map $(-f[,], \text{id} \oplus \text{id}): C[-1, \infty] \oplus C^-[-\infty, 0] \to C^-[-\infty, \infty]$ becomes the map $(-f \circ t^{-1}, \text{id} \oplus \text{id}): \Phi(C)[0, \infty] \oplus C^-[-\infty, 0] \to C^-[-\infty, \infty]$. Hence the mapping cone of the lower horizontal arrow agrees with the mapping cone appearing
in the definition of $\Gamma(C^+, f, C^-)$. The mapping cone of the horizontal map is the mapping cone appearing in the definition of $\Gamma_0(C^+, f, C^-)$. Since $(C^+, f, C^-)$ of $\text{Ch}(X)$ is $(w_0, w_1)$-acyclic by assumption, the mapping cone of both the upper horizontal and the lower horizontal arrow are contractible. We conclude from Lemma 3.1 (vii) that

$$\text{id}[]: C^+[0, \infty] \rightarrow C^+[-1, \infty]$$

is an $\mathcal{A}^\circ$-chain equivalence.

If we apply $\Phi^n$ for $n \in \mathbb{Z}, n \geq 0$ to the inclusion above and use the obvious identifications $\Phi^n(C^+[0, \infty]) = C^+[-n, \infty]$ and $\Phi^n(C^+[-1, \infty]) = C^+[-n - 1, \infty]$, we conclude that also the inclusion $\text{id}[]: C^+[-n + 1, \infty] \rightarrow C^+[-n, \infty]$ is an $\mathcal{A}^\circ$-chain equivalence. Hence the inclusion $\text{id}[]: C^+[0, \infty] \rightarrow C^+[-n, \infty]$ is a $\mathcal{A}$-chain homotopy equivalence for every $n \in \mathbb{Z}, n \geq 0$.

Next we want to show that $\text{id}[]: C^+[0, \infty] \rightarrow C^+[-\infty, \infty]$ is a $\mathcal{A}$-chain homotopy equivalence. Since the inclusion $\text{id}[]: C^+[0, \infty] \rightarrow C^+[-n, \infty]$ is levelwise split injective, the canonical projection from its mapping cone to $C^+[-n, \infty]/C^+[0, \infty]$ is a $\mathcal{A}$-chain homotopy equivalence by Lemma 3.1 (vii). Since $\text{id}[]: C^+[0, \infty] \rightarrow C^+[-n, \infty]$ is a $\mathcal{A}$-chain homotopy equivalence, Lemma 3.1 (v) implies that the $\mathcal{A}$-chain complex $C^+[-n, \infty]/C^+[0, \infty]$ is contractible.

Because of Lemma 3.1 (v) and (vii) we can find for $n \in \mathbb{Z}, n \geq 0$ chain contractions $\gamma_r[-n]$ for $C^+[-n, \infty]/C^+[0, \infty]$. Since $\gamma_r[-n]$ and $\gamma_r[-n - 1]$ are compatible with the inclusion $C^+[-n, \infty]/C^+[0, \infty] \rightarrow C^+[-n - 1, \infty]/C^+[0, \infty]$.

By inspecting the definitions of the various chain modules as direct sums, one sees that the $\mathcal{A}$-chain complex $C^+[-\infty, \infty]/C^+[0, \infty]$ is the colimit $\text{colim}_{n \rightarrow \infty} C^+[-n, \infty]$ within the category of $\mathcal{A}$-chain complexes. Hence $C^+[-\infty, \infty]/C^+[0, \infty]$ inherits a chain contraction from the various chain contractions $\gamma_r[-n]$. We conclude from Lemma 3.1 (v) and (vii) that $\text{id}[]: C^+[0, \infty] \rightarrow C^+[-\infty, \infty]$ is a $\mathcal{A}$-chain homotopy equivalence.

The following diagram commutes

$$\begin{array}{ccc}
C^+[0, \infty] \oplus C^-[-\infty, 0] & \xrightarrow{(-f|, \text{id}[])} & C^-[-\infty, \infty] \\
\text{id}[] \oplus \text{id}[] & \downarrow & \\
C^+[-\infty, \infty] \oplus C^-[-\infty, 0] & \xrightarrow{(-f, \text{id}[])} & \\
\end{array}$$

Since the vertical and the horizontal arrow are $\mathcal{A}$-chain homotopy equivalences, $(-f, \text{id}[]): C^+[-\infty, \infty] \oplus C^-[-\infty, 0] \rightarrow C^-[-\infty, \infty]$ is a $\mathcal{A}$-chain homotopy equivalence. Since $f: C^+[-\infty, \infty] \rightarrow C^-[-\infty, \infty]$ is a $\mathcal{A}$-chain homotopy equivalence, we conclude from Lemma 3.1 (vii) that $C^-[-\infty, 0]$ is contractible.

Analogously one proves that the $\mathcal{A}^\circ$-chain complex $C^+[0, \infty]$ is contractible. Namely, choose a $\mathcal{A}_f[t, t^{-1}]$-chain homotopy inverse $f^{-1}$ of $f$ and consider the triple $(C^-, f^{-1}, C^+)$ and conclude from the assumption that the object $(C^+, f, C^-)$ of $\text{Ch}(X)$ is $(w_0, w_1)$-acyclic that the mapping cones of the $\mathcal{A}^\circ$-chain maps

$$\begin{align*}
(-f^{-1}[], \text{id}[]): & \quad C^-[-\infty, 0] \oplus C^+[0, \infty] \rightarrow C^+[-\infty, \infty]; \\
(-f^{-1}[], \text{id}[]): & \quad C^-[-\infty, 1] \oplus C^+[0, \infty] \rightarrow C^+[-\infty, \infty],
\end{align*}$$

are contractible.

Finally we conclude from Lemma 3.3 that $C^+$ is contractible in $\mathcal{A}_f[t]$ and $C^-$ is contractible in $\mathcal{A}_f[t^{-1}]$. This finishes the proof of Lemma 3.7.

Observe that for all $\iota$ we have $\Gamma_{\iota-1} \circ l_{\iota} \simeq s$. In fact, $\Gamma_{\iota-1} \circ l_{\iota}(C) = \Gamma \circ s \circ l_0(C)$ is, up to a suspension, the cone of the chain isomorphism

$$\text{id}[] \oplus \text{id}[]: C[1, \infty] \oplus C[-\infty, 0] \xrightarrow{\cong} C[-\infty, \infty]$$
and therefore contractible. In particular, \( l_i \) induces a functor

\[
l_i : \mathbf{K}(\text{Ch}(A)) \rightarrow \mathbf{K}(\text{Ch}(X^{w_{i-1}}, w_i)),
\]

where \( \text{Ch}(X^{w_{i-1}}, w_i) \) is the full Waldhausen subcategory of \( (\text{Ch}(X), w_i) \) of those \( X \)-chain complexes which are \( w_{i-1} \)-acyclic.

**Lemma 6.8.** Suppose that \( A \) is idempotent complete. For any \( i \), the maps

\[
\mathbf{K}(\text{Ch}(A)) \rightarrow \mathbf{K}(\text{Ch}(X^{w_{i-1}}, w_i)), \quad \text{and}
\]

\[
\mathbf{K}(\text{Ch}(A)) \rightarrow \mathbf{K}(\text{Ch}(X), w_i)
\]

induced by \( l_i \) are chain homotopy equivalences.

**Proof.** Since \( s^i : (X, w_0) \rightarrow (X, w_i) \) is an isomorphism of Waldhausen categories, it suffices to treat the case \( i = 0 \). We have defined a functor

\[
l_0 : \text{Ch}(A) \rightarrow \text{Ch}(X)
\]

which sends an object \( C \) to the object \((i_+ C, \text{id}_{1C}, i_- C)\). Given an \( A \)-chain complex \( C \), we define a natural chain homotopy equivalence of \( A^n \)-chain complexes, natural in \( C \)

\[
T(C) : C \rightarrow \Gamma_0 \circ l_0(C) = \Sigma^{-1} \text{cone}(\langle - \text{id}[\cdot], \text{id}[\cdot] \rangle : C[0, \infty] \oplus C[-\infty, 0] \rightarrow \Gamma C[\infty, \infty]),
\]

by the short exact sequence of \( A^n \)-chain complexes

\[
0 \rightarrow C = C[0, 0] \rightarrow \frac{\text{id}[\cdot]}{[\text{id}[\cdot]]} \rightarrow C[0, \infty] \oplus C[-\infty, 0] \rightarrow [\text{id}[\cdot], \text{id}[\cdot]] \rightarrow \Gamma C[\infty, \infty] \rightarrow 0.
\]

using Lemma 3.1(iii) The chain map \( T(C) \) is a chain homotopy equivalence by Subsection 3.2. Thus we obtain a natural equivalence of functors from \( \text{Ch}(A) \rightarrow \text{Ch}^{hf}(A) \) from the inclusion \( i : \text{Ch}(A) \rightarrow \text{Ch}^{hf}(A) \), which is an equivalence on \( K \)-theory by the Approximation Theorem 4.17, to \( \Gamma \circ l_0 \). Hence we obtain a commutative diagram of spectra

\[
\begin{array}{c}
\mathbf{K}(\text{Ch}(A)) \\
\xrightarrow{\mathbf{K}(l_0)} \\
\xrightarrow{\mathbf{K}(z)} \\
\mathbf{K}(\text{Ch}(X^{w_{i-1}}, w_0)) \\
\xrightarrow{\mathbf{K}(\iota)} \\
\xrightarrow{\mathbf{K}(\Gamma)} \\
\mathbf{K}(\text{Ch}^{hf}(A))
\end{array}
\]

where \( z \) denotes the inclusion. We conclude that both

\[
\mathbf{K}(l_0) : \mathbf{K}(\text{Ch}(A)) \rightarrow \mathbf{K}(\text{Ch}(X^{w_{i-1}}, w_0))
\]

and

\[
\mathbf{K}(l_0) : \mathbf{K}(\text{Ch}(A)) \rightarrow \mathbf{K}(\text{Ch}(X), w_0)
\]

have homotopy left inverses.

Now consider an object \((C^+, f, C^-)\) in \( \text{Ch}(X) \). In the sequel we abbreviate \( \Gamma(f) := \Gamma(C^+, f, C^-) \). Since \( \Gamma(f) \) is the \( A^n \)-chain complex

\[
\Gamma(f) := \Sigma^{-1} \text{cone}(\langle -f[\cdot], \text{id}[\cdot] \rangle : C^+[0, \infty] \oplus C^-[-\infty, 0] \rightarrow C^-[-\infty, \infty])
\]

we obtain from Lemma 3.1(iii) the following (not necessarily) commutative diagram of \( A^n \)-chain complexes which commutes up to a preferred chain homotopy \( h : f \circ \mathbf{K}(l_0) \rightarrow \mathbf{K}(l_0) \circ \mathbf{K}(z) \rightarrow \mathbf{K}(\Gamma) \):
id[\circ \varphi^+ \simeq \text{id}[\circ \varphi^- \circ \text{id]}

\begin{align*}
\begin{array}{c}
\Gamma(f) \quad \text{id} \\
\varphi^+ \\
C^+[0, \infty] \quad C^+[-\infty, 0] \\
\text{id}[ \\
C^+[-\infty, \infty] \quad \simeq f[-\infty, \infty] \quad C^+[-\infty, \infty]
\end{array}
\end{align*}

¿From the adjunctions appearing in Lemma 1.8, now applied for $A^-\kappa$ instead of $A$, we obtain an $A^-\kappa\Phi[t]$-chain map

$$
\psi^+ : i_+ \Gamma(f) \rightarrow C^+
$$

from $\varphi^+$, an $A^-\kappa[t^{-1}]$-chain map

$$
\psi^- : i_- \Gamma(f) \rightarrow C^-
$$

from $\varphi^-$, and a homotopy of $A^-\kappa[t, t^{-1}]$-chain maps

$$
H : f \circ j_+ \psi^+ \simeq j_- \psi^-.
$$

Let $D$ be the mapping cylinder of the $A^-\kappa[t]$-chain map $\psi^+ : i_+ \Gamma(f) \rightarrow C^+$. Denote by $u : i_+ \Gamma(f) \rightarrow D$ and $v : C^+ \rightarrow D$ the canonical inclusions and by $p : D \rightarrow C^+$ the canonical projection. Notice that $j_+ D$ can be identified with the mapping cylinder of $j_+ \psi^+ : i_0 \Gamma(f) \rightarrow j_+ C^+$. Because of Lemma 3.1(v) we obtain from $H$ a $A^-\kappa[t, t^{-1}]$-chain map $f' : j_+ D \rightarrow j_- C^-$ so that the following diagram of $A^-\kappa[t, t^{-1}]$-chain complexes commutes (strictly)

\begin{align*}
\begin{array}{c}
i_0 \Gamma(f) \quad \text{id} \\
j_+ u \\
j_+ D \\
j_+ v \\
j_+ C^+ \quad f' \quad j_- C^-
\end{array}
\end{align*}

Thus we get morphisms

(6.10) $(u, \psi^-) : (i_+ \Gamma(f), \text{id}, i_+ \Gamma(f)) \rightarrow (D, f', C^-);

(6.11) $(v, \text{id}) : (C^+, f, C^-) \rightarrow (D, f', C^-),

in $\text{Ch}(\mathcal{X}(A^-)) = \text{Ch}(\mathcal{X}^\kappa)$.

Both maps are $w_0$-equivalences, i.e., they become weak equivalences after applying $\Gamma$. For (6.11) this is clear as $(v, \text{id})$ is a weak equivalence in $\text{Ch}(\mathcal{X})$ and $\Gamma$ is Waldhausen exact.

Abbreviating $\Gamma(f') := \Gamma(D, f', C^-)$, the following diagram of $A^-\kappa$-chain complexes is commutative

\begin{align*}
\begin{array}{c}
\Gamma(f') \quad \Gamma(f) \\
\text{\Gamma}(\text{\Gamma}(\text{f})) \quad \text{\Gamma}(\text{\Gamma}(\text{f})) \\
\text{\Gamma}(\text{\Gamma}(\text{f})) \quad \text{\Gamma}(\text{\Gamma}(\text{f}))
\end{array}
\end{align*}

where the $A^-\kappa$-chain homotopy equivalence $T(\Gamma(f))$ has been defined in (6.9). It follows that $\Gamma$ applied to (6.10) is also a weak equivalence.
Summarizing, we get three exact functors of Waldhausen categories
\[(\mathrm{Ch}(\mathcal{X}), w_0) \to (\mathrm{Ch}^{\bf{hf}}(\mathcal{X}), w_0),\]
where \(w_0\) on \(\mathrm{Ch}^{\bf{hf}}(\mathcal{X})\) is formed by those chain maps in \(\mathrm{Ch}(\mathcal{X}^\infty)\) which become \(\mathcal{A}^\infty\)-chain equivalences after applying \(\Gamma\): \(\mathrm{Ch}(\mathcal{X}^\infty) \to \mathrm{Ch}(\mathcal{X}^\infty)\). The first one is the inclusion \(\iota\). The second one is the composite of the functors \(\Gamma\): \((\mathrm{Ch}(\mathcal{X}), w_0) \to \mathrm{Ch}^{\bf{hf}}(\mathcal{A})\) and \(l_0\): \(\mathrm{Ch}^{\bf{hf}}(\mathcal{A}) \to (\mathrm{Ch}^{\bf{hf}}(\mathcal{X}), w_0)\). The third one \(F\) sends \((C^+, f, C^-)\) to \((D, f', C^-)\). The maps \((6.11)\) and \((6.11)\) are natural \(w_0\)-weak equivalences
\[(6.12)\]
\[\alpha: l_0 \xrightarrow{\cong} F;\]
\[(6.13)\]
\[\beta: l_0 \circ \Gamma \xrightarrow{\cong} F;\]
coming from the morphisms \((u, \psi^-)\) of \((6.10)\) and \((v, \text{id})\) of \((6.11)\). This implies that
\[K(\Gamma) \circ K(l_0), K(\iota): K(\mathrm{Ch}(\mathcal{X}), w_0) \to K(\mathrm{Ch}^{\bf{hf}}(\mathcal{X}), w_0)\]
are homotopic. The following diagram commutes and has homotopy equivalences as vertical arrows
\[
\begin{array}{ccc}
K(\mathrm{Ch}^{\bf{hf}}(\mathcal{A})) & \xrightarrow{K(l_0)} & K(\mathrm{Ch}^{\bf{hf}}(\mathcal{X}), w_0) \\
K(\iota) & \cong & \cong \\
K(\mathrm{Ch}(\mathcal{A})) & \xrightarrow{K(l_0)} & K(\mathrm{Ch}(\mathcal{X}), w_0)
\end{array}
\]
This implies that the composite of \(K(l_0): K(\mathrm{Ch}(\mathcal{A})) \to K(\mathrm{Ch}(\mathcal{X}), w_0)\) and \(K(\iota)^{-1} \circ K(\Gamma): K(\mathrm{Ch}(\mathcal{X}), w_0) \to K(\mathrm{Ch}(\mathcal{A}))\) is homotopic to the identity, in other words, \(K(l_0): \mathrm{Ch}(\mathcal{X}) \to K(\mathrm{Ch}(\mathcal{X}), w_0)\) has a homotopy right inverse. Since we already know that it has a homotopy left inverse,
\[K(l_0): K(\mathrm{Ch}(\mathcal{A})) \to K(\mathrm{Ch}(\mathcal{X}), w_0)\]
is a homotopy equivalence.

It remains to show that
\[K(l_0): K(\mathrm{Ch}(\mathcal{A})) \to K(\mathrm{Ch}(\mathcal{X}^{w-1}), w_0)\]
is a homotopy equivalence. We know already that it has a homotopy left inverse. We claim that a right homotopy inverse is given by the composite \(K(\Gamma \circ z): K(\mathrm{Ch}(\mathcal{X}^{w-1}), w_0) \to K(\mathrm{Ch}^{\bf{hf}}(\mathcal{X}^{w-1}), w_0)\) with a homotopy inverse of the homotopy equivalence \(K(\iota): K(\mathrm{Ch}(\mathcal{X}^{w-1}), w_0) \to K(\mathrm{Ch}^{\bf{hf}}(\mathcal{X}^{w-1}), w_0)\). This follows by the same argument as above using the fact that the morphisms \((u, \psi^-)\) of \((6.10)\) and \((v, \text{id})\) of \((6.11)\) are morphisms between \(w^{-1}\)-acyclic objects. \(\square\)

Finally, we can finish the proof of Theorem 2.7.

**Proof of Theorem 2.7** By the Fibration Theorem 4.16 there is a fibration sequence
\[K(\mathrm{Ch}(\mathcal{X}^{w_0}), w_1) \to K(\mathrm{Ch}(\mathcal{X}), w_0 \cap w_1) \to K(\mathrm{Ch}(\mathcal{X}), w_0))\]
where by Theorem 4.1 and Lemma 6.7 the middle term agrees with \(K(\mathcal{X})\). By Lemma 6.3 \(K(\mathcal{A})\) is homotopy equivalent to both the left-hand and the right-hand side, using the functors \(l_1\) and \(l_0\) respectively. The fibration sequence splits as \(l_0\) factors through the middle term. \(\square\)
7. Strategy of proof for Theorem 0.4(ii)

In this section we present the details of the formulation and then the basic strategy of proof of Theorem 0.4(ii).

Definition 7.1 (Nilpotent morphisms and Nil-categories). Let \( \mathcal{A} \) be an additive category and \( \Phi \) be an automorphism of \( \mathcal{A} \).

(i) A morphism \( f: \Phi(A) \to A \) of \( \mathcal{A} \) is called \( \Phi \)-nilpotent if for some \( n \geq 1 \), the \( n \)-fold composite

\[
f^{(n)} := f \circ \Phi(f) \circ \cdots \circ \Phi^{n-1}(f): \Phi^n(A) \to A.
\]

is trivial;

(ii) The category \( \text{Nil}(\mathcal{A}, \Phi) \) has as objects pairs \((A, \phi)\) where \( \phi: \Phi(A) \to A \) is a \( \Phi \)-nilpotent morphism in \( \mathcal{A} \). A morphism from \((A, \phi)\) to \((B, \mu)\) is a morphism \( u: A \to B \) in \( \mathcal{A} \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\Phi(A) & \xrightarrow{\phi} & A \\
\Phi(A) & \xrightarrow{\Phi(u)} & B \\
\end{array}
\]

The category \( \text{Nil}(\mathcal{A}, \Phi) \) inherits the structure of an exact category from \( \mathcal{A} \); a sequence in \( \text{Nil}(\mathcal{A}, \Phi) \) is declared to be exact if the underlying sequence in \( \mathcal{A} \) is (split) exact.

There is a functor

\[
\chi: \text{Nil}(\mathcal{A}, \Phi) \to \text{Ch}(\mathcal{A}_\Phi[[t^{-1}]]
\]

sending \( \phi: \Phi(A) \to A \) to the 1-dimensional chain complex \( A \xrightarrow{\cdot^{-1, -\phi}} \Phi(A) \). (See section 8 for more details.) Using the Gillet-Waldhausen theorem 4.1, this leads to a map

\[
K(\chi): K(\text{Nil}(\mathcal{A}, \Phi)) \to K(\mathcal{A}_\Phi[[t]]).
\]

The key ingredient in the proof Theorem 0.4(ii) is the following theorem whose proof is deferred to Section 8.

Theorem 7.2 (Fiber sequence for the Nil). Suppose that \( \mathcal{A} \) is idempotent complete.

The following is a homotopy fiber sequence, natural in \((\mathcal{A}, \Phi)\):

\[
\begin{array}{ccc}
K(\text{Nil}(\mathcal{A}, \Phi)) & \xrightarrow{K(\chi)} & K(\mathcal{A}_\Phi[[t^{-1}]]) \\
& \xrightarrow{K(j^{-1})} & K(\mathcal{A}_\Phi[[t, t^{-1}]])
\end{array}
\]

Next we explain how Theorem 0.4(ii) follows from Theorem 7.2. Define spectra

\[
\begin{align*}
E_0(\mathcal{A}, \Phi) & := \text{hofib}(K(i_+): K(\mathcal{A}) \to K(\mathcal{A}_\Phi[[t]])); \\
E_1(\mathcal{A}, \Phi) & := \text{hofib}(K(i_+ \circ \Phi^{-1}) \vee K(i_+): K(\mathcal{A}) \vee K(\mathcal{A}) \to K(\mathcal{A}_\Phi[[t]])); \\
E_2(\mathcal{A}, \Phi) & := \text{hofib}(K(\mathcal{A}_\Phi[[t^{-1}]]) \xrightarrow{K(j^{-1})} K(\mathcal{A}_\Phi[[t, t^{-1}]]));
\end{align*}
\]

The inclusion to the second summand \( K(\mathcal{A}) \to K(\mathcal{A}) \vee K(\mathcal{A}) \) induces a map of spectra

\[
w: E_0 \to E_1
\]

and the projection onto the first summand \( K(\mathcal{A}) \vee K(\mathcal{A}) \to K(\mathcal{A}) \) induces a map of spectra

\[
x: E_1 \to K(\mathcal{A}),
\]

such that the following is a fibration sequence of spectra

\[
E_0(\mathcal{A}, \Phi) \xrightarrow{w} E_1(\mathcal{A}, \Phi) \xrightarrow{x} K(\mathcal{A}).
\]
From the diagram \( \overset{2.8}{\square} \) we obtain a weak equivalence of spectra, natural in \((A, \Phi)\).

\[
y : E_1(A, \Phi) \xrightarrow{\simeq} E_2(A, \Phi).
\]

Let

\[
z : K(\text{Nil}(A, \Phi)) \to E_2(A, \Phi)
\]

be the in \((A, \Phi)\) natural weak homotopy equivalence associated to the homotopy fiber sequence of Theorem 7.2. Define \(E(A, \Phi)\) to be the homotopy pullback

\[
E(A, \Phi) \xrightarrow{\simeq} K(\text{Nil}(A, \Phi))
\]

\[
x \downarrow \simeq \quad \downarrow z
\]

\[
E_0(A, \Phi) \xrightarrow{\gamma} E_2(A, \Phi)
\]

It follows from the sequence \( \overset{2.8}{\square} \) that

\[
E(A, \Phi) \xrightarrow{\simeq} K(\text{Nil}(A, \Phi)) \xrightarrow{\gamma \circ y^{-1} \circ z} K(A)
\]

is a fibration sequence of spectra.

Now the inclusion \( i : A \to \text{Nil}(A, \Phi) \) sending \( A \) to \((A, 0)\) induces a map of spectra

\[
K(i) : K(A) \to K(\text{Nil}(A, \Phi)).
\]

**Lemma 7.4.** In the homotopy category, the following composite agrees with \(-K(\Phi^{-1})\):

\[
K(A) \xrightarrow{K(i)} K(\text{Nil}(A, \Phi)) \xrightarrow{z} E_2(A, \Phi) \xrightarrow{y^{-1}} E_1(A, \Phi) \xrightarrow{\simeq} K(A).
\]

**Proof.** As in section 5 but interchanging the roles of \( t \) and \( t^{-1} \) we denote by \( \text{Ch}(A_\Phi[t^{-1}])^w \subset \text{Ch}(A_\Phi[t^{-1}]) \) the full Waldhausen subcategory of chain complexes which are contractible over \( A_\Phi[t, t^{-1}] \), and by \( \text{Ch}(X)^w \subset \text{Ch}(X) \) the full Waldhausen subcategory or complexes whose plus-part is contractible. Then the results of that section imply \( E_2(A, \Phi) \simeq K(\text{Ch}(A_\Phi[t^{-1}])^w) \); moreover \( E_1(A, \Phi) \simeq K(\text{Ch}(X)^w) \) if we use the equivalence

\[
K(l_1) \vee K(l_0) : K(A) \vee K(A) \xrightarrow{\simeq} K(X)
\]

from Theorem 2.7.

Under these identifications, the composite \( z \circ K(i) \) corresponds to the map induced by the functor

\[
F_1 : A \to \text{Ch}(A_\Phi[t^{-1}])^w, \quad A \mapsto \text{cone}(\Phi(A) \xrightarrow{t^{-1}} A).
\]

But this is the image of the functor

\[
F_2 : A \to \text{Ch}(X)^w, \quad A \mapsto \text{cone}(l_1(\Phi^{-1}(A)) \xrightarrow{(t^{-1}, \text{id})} l_0(A))
\]

under the projection \( X \to A_\Phi[t^{-1}] \). Applying the Additivity theorem to the cylinder-cone-sequence shows that in \( K(X) \), we have

\[
K(F_2) \simeq K(l_0) - K(l_1 \circ \Phi^{-1}),
\]

which is the image under \( K(l_1) \vee K(l_0) \) of the map

\[
(\to K(\Phi^{-1}), \text{id}) : K(A) \to K(A) \vee K(A).
\]

Now project to the first variable. \( \square \)
Hence the fibration sequence splits and we obtain a weak equivalence
\[ K(i) \vee K(y \rightarrow w) : K(A) \vee E(A, \Phi) \xrightarrow{\sim} K(\text{Nil}(A, \Phi)). \]
As \( E_0(A, \Phi) \) is the homotopy fiber of \( K(i) : K(A) \to K(A\Phi[t]) \) and \( NK(A\Phi[t]) \) is the homotopy fiber of \( K(\text{ev}_{+}) : K(A\Phi[t]) \to K(A) \) and \( K(\text{ev}_{+}) \circ K(i) \) is the identity, we obtain a weak equivalence of spectra, natural in \((A, \Phi)\)
\[
u : E_0 \to \Omega NK(A\Phi[t]).
\]
Thus we obtain a weak equivalence of spectra, natural in \((A, \Phi)\),
\[ u : E_0 \to \Omega NK(A\Phi[t]). \]
Now Theorem 8.1 (ii) follows from (i) and (iii), provided that Theorem 8.2 holds.

8. On the Nil-category

Lemma 5.1 and Theorem 5.3 imply that there is homotopy fiber sequence
\[ K(\text{Ch}(A\Phi[t^{-1}])^w) \to K(A\Phi[t^{-1}]) \to K(A\Phi[t, t^{-1}]) \]
where \( \text{Ch}(A\Phi[t^{-1}])^w \) denotes the category of bounded chain complexes over \( A\Phi[t^{-1}] \) which are contractible as chain complexes over \( A\Phi[t, t^{-1}] \). The main goal of this section is to see that the first term of this sequence can be described in terms of the \( K \)-theory of the twisted Nil-category \( \text{Nil}(A, \Phi) \).

If \((A, \phi)\) is an object of \( \text{Nil}(A, \Phi) \), then there is an associated chain complex
\[ \Phi(A) \xrightarrow{\phi} A \]
over \( A\Phi[t^{-1}] \), concentrated in dimension 0 and 1. It is contractible over \( A\Phi[t, t^{-1}] \) since
\[ t^{-1} - i_{-}\phi = (1 - \phi \cdot t) \circ t^{-1} \]
and both \( 1 - \phi \cdot t \) and \( t^{-1} \) are invertible in \( A\Phi[t, t^{-1}] \): inverses are \( \sum_{i=0}^{n-1} (\phi \cdot t)^i \) and \( t \) respectively, where \( n \) is such that \( \phi(n) = 0 \). This induces a functor of Waldhausen categories
\[ \chi : \text{Nil}(A, \Phi) \to \text{Ch}(A\Phi[t^{-1}])^w. \]

The goal of this section is to prove the following result:

Theorem 8.1. Suppose that \( A \) is idempotent complete. Then the induced map on connective \( K \)-theory
\[ K(\chi) : K(\text{Nil}(A, \Phi)) \to K(\text{Ch}(A\Phi[t^{-1}])^w) \]
is a homotopy equivalence.

Notice that Theorem 7.2 follows from Theorem 8.1.

8.1. The characteristic sequence. The first step in the proof of Theorem 8.1 is to relate chain complexes over \( A\Phi[t^{-1}] \) with chain complexes over \( A \) equipped with an endomorphism. This relation is a consequence of the characteristic sequence, which we recall now.

Recall that \( A^\circ \) is obtained from \( A \) by adjoining countable direct sums. We have defined induction and restriction functors \( i_- : A \to A\Phi[t^{-1}] \), \( i_+ : A\Phi[t^{-1}] \to A^\circ \) and \( i_- : A^\circ \to A\Phi[t^{-1}] \) in Subsections 1.4 and 1.5. Consider an object \( A \in A\Phi[t^{-1}] \). Let
\[ e : i_--i_- A \to A \]
be the morphism in \( A\Phi[t^{-1}] \) which is the adjoint of \( i_- A \to i_- A \) under the adjunction of Lemma 1.8 but now for \( A^\circ \) instead of \( A \). We have the morphism \( \text{id}_A \cdot t^{-1} : \Phi(A) \to A \) in \( A\Phi[t^{-1}] \). Applying the composite \( i_- i_- \) yields a morphism \( i_- i_- (\text{id}_A \cdot t^{-1}) : i_- i_- \Phi(A) \to i_- i_- A \) in \( A\Phi[t^{-1}] \). We also have the morphism
id_{i^{-1}A} \cdot t^{-1}; i_{-i}^{-} \Phi(A) \to i_{-i}^{-} A. We will abbreviate id_{i^{-1}A} \cdot t^{-1} and id_A \cdot t^{-1} by t^{-1}. The difference of the two morphisms above yields the morphism in $A_0^e[t^{-1}]$

$$t^{-1} - i_{-i}^{-} t^{-1} := id_{i^{-1}A} \cdot t^{-1} - i_{-i}^{-} (id_A \cdot t^{-1}): i_{-i}^{-} \Phi(A) \to i_{-i}^{-} A.$$  

Lemma 8.2. Let $A$ be an object in $A_0^e[t^{-1}]$. Then the so called characteristic sequence in $A_0^e[t^{-1}]$

$$0 \to i_{-i}^{-} \Phi(A) \xrightarrow{t^{-1} - i_{-i}^{-} t^{-1}} i_{-i}^{-} A \xrightarrow{\varepsilon} A \to 0$$

in $A_0^e[t^{-1}]$ is (split) exact and natural in $A$.

Proof. We have $i_{-i}^{-} \Phi(A) = \bigoplus_{i^{-1}} \Phi^{-i}(A)$ and $i_{-i}^{-} A = \bigoplus_{i^{-1}} \Phi^{-i}A$. Under this identification the short sequence under consideration becomes the sequence in $A_0^e[t]$

$$0 \to \bigoplus_{i^{-1}} \Phi^{-i}(A) \xrightarrow{\bigoplus_{i^{-1}} i_{-i}^{-} \Phi^{-i}(A)} \bigoplus_{i^{-1}} \Phi^{-i}(A) \xrightarrow{\bigoplus_{i^{-1}} \varepsilon} A \to 0.$$  

If we view this as a 2-dimensional chain complex, we obtain an $A_0^e[t]$-chain contraction by

$$\begin{pmatrix}
0 & -t^{-1} & -t^{-2} & -t^{-3} & \cdots \\
0 & 0 & -t^{-1} & -t^{-2} & \cdots \\
0 & 0 & 0 & -id & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \xrightarrow{\begin{pmatrix}
id \\
0 \\
0 \\
0
\end{pmatrix}} \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \xrightarrow{\begin{pmatrix}
id \\
0 \\
0 \\
0
\end{pmatrix}} A$$

\[ \square \]

Given an $A_0^e[t^{-1}]$-chain complex $C$, we obtain from Lemma 8.2 that $C$ can be resolved by an in $C$ natural short exact sequence

$$0 \to i_{-i}^{-} \Phi(C) \xrightarrow{t^{-1} - i_{-i}^{-} t^{-1}} i_{-i}^{-} C \xrightarrow{\varepsilon} C \to 0$$

of $A_0^e[t^{-1}]$-chain complexes.

Lemma 8.4. Consider a morphism $\phi: \Phi(A) \to A$ in $A^e$. Then we obtain a (split) exact and in $\Phi$-natural exact sequence of $A^e$-modules

$$0 \to i_{-i}^{-} \Phi(A) \xrightarrow{i_{-i}^{-} (t^{-1} - i_{-i}^{-} \phi)} i_{-i}^{-} A \xrightarrow{\varepsilon} A \to 0.$$  

Proof. The proof is analogous to the one of Lemma 8.2, the role of $t^{-1}$ is now played by $\phi$. Namely, we have $i_{-i}^{-} \Phi(A) = \bigoplus_{i^{-1}} \Phi^{-i}A$ and $i_{-i}^{-} A = \bigoplus_{i^{-1}} \Phi^{-i}A$. 

Under this identification the short sequence under consideration becomes the sequence in $\mathcal{A}^c$

$$0 \to \bigoplus_{i=-\infty}^{-1} \Phi^{-i}(A) \xrightarrow{0} \bigoplus_{i=-\infty}^{0} \Phi^{-i}(A)$$

where $\phi^{(k)} := \phi \circ \Phi^1(\phi) \circ \cdots \circ \Phi^{(k-1)}(\phi): \Phi^k(A) \to A$. If we view this as a 2-dimensional chain complex, we obtain a $\bigoplus_{i=-\infty}^{-1} \Phi^{-i}(A)$ weak equivalence in $\mathcal{A}^c$.

$$\begin{pmatrix}
0 & -\Phi(\phi) & -\Phi(\phi^2) & -\Phi(\phi^3) & \cdots \\
0 & 0 & -\Phi^2(\phi) & -\Phi^2(\phi^2) & \cdots \\
0 & 0 & 0 & -\Phi^3(\phi) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \xrightarrow{\text{id}} \begin{pmatrix}
0 \\
0 \\
0 \\
\vdots
\end{pmatrix}
$$

\[ \phi \circ \Phi^1(\phi) \circ \cdots \circ \Phi^{(k-1)}(\phi): \Phi^k(A) \to A \]

Notation 8.5 (End(Ch(A), $\Phi$)).
Denote by $\text{End}(\text{Ch}(A), \Phi)$ the Waldhausen category of $\Phi$-twisted endomorphisms of $\mathcal{A}$-chain complexes. An object is a pair $(C, \phi)$, where $C$ is a chain complex in $\mathcal{A}$, and $\phi: \Phi(C) \to C$ is a chain map. A morphism $u: (C, \phi) \to (D, \psi)$ is an $\mathcal{A}$-chain map $u: C \to D$ such that $u \circ \phi = \psi \circ \Phi(u)$. It is a cofibration or weak equivalence respectively if the underlying $\mathcal{A}$-chain map $u$ has this property.

Define functors of Waldhausen categories

\[ \chi_{\mathcal{A}^c}: \text{End}(\text{Ch}(\mathcal{A}^c), \Phi) \to \text{Ch}(\mathcal{A}^c_0[t^{-1}]), \]

\[ (C, \phi) \mapsto \text{cone}(i_- \Phi(C) \xrightarrow{t^{-1} - i \phi} i_- C); \]

\[ \chi_{\mathcal{A}^c}: \text{Ch}(\mathcal{A}^c_0[t^{-1}]) \to \text{End}(\text{Ch}(\mathcal{A}^c), \Phi), \quad D \mapsto (i^D, i^D_{-t^{-1}}). \]

Lemma 8.8. The functors $\chi_{\mathcal{A}^c}$ and $N_{\mathcal{A}^c}$ are inverse equivalences of Waldhausen categories. In particular they induce inverse equivalences on $K$-theory.

Proof. We obtain from the sequence (8.3) using Subsection 5.2.4 for any object $D$ in $\text{Ch}(\mathcal{A}^c_0[t^{-1}])$ a weak equivalence in $\text{Ch}(\mathcal{A}^c_0[t^{-1}])$

\[ T(D): \chi_{\mathcal{A}^c} \circ N_{\mathcal{A}^c} (D) = \text{cone}(i_- \phi \Phi(D) \xrightarrow{t^{-1} - i \phi} i_- i^D) \xrightarrow{\approx} (D), \]

and thus a natural weak equivalence $T: \chi_{\mathcal{A}^c} \circ N_{\mathcal{A}^c} \xrightarrow{\approx} \text{id}$.

It remains to construct a natural weak equivalence $S: N_{\mathcal{A}^c} \circ \chi_{\mathcal{A}^c} \xrightarrow{\approx} \text{id}$. Consider an object $(C, \phi)$ in $\text{End}(\text{Ch}(\mathcal{A}^c))$. From Lemma 8.4.1 we obtain the short exact sequence in $\text{Ch}(\mathcal{A}^c)$

\[ 0 \to i^- \phi C \xrightarrow{i^D(t^{-1} - i \phi)} i^- i_- C \xrightarrow{\approx} C \to 0 \]
Lemma 8.11. For $A(i) =$ proof.

D is homotopy nilpotent. By assumption complex $A$ it is $M$ be a large enough natural number so that the collection of maps $A$ is homotopy nilpotent and hence its cone $M$ is homotopy nilpotent. Let $\Phi$ be a null-homotopy for the bounded $A$-chain complex and, regarded as $A$-chain complex, and that $\chi_{A^c}(\phi) \simeq (C, \phi).$

This finishes the proof of Lemma 8.8. □

Definition 8.9 ((Homotopy) $\Phi$-nilpotent). A morphism $f: \Phi^n(A) \to A$ of $A$ is called $\Phi$-nilpotent if for some $n \geq 1$ the $n$-fold composite

\[ f^{(n)} := f \circ \Phi(f) \circ \cdots \circ \Phi^{n-1}(f) : \Phi^n(A) \to A \]

is trivial;

An $A$-chain map $f: \Phi^n(C) \to C$ is called homotopy $\Phi$-nilpotent if $f^{(n)}$ is $A$-chain homotopic to the trivial chain map.

Definition 8.10 (Homotopy finite). A chain complex in $A^c$ is homotopy finite if it is $A^c$-chain homotopy equivalent to a bounded chain complex in $A$.

In the sequel we will view $A^c_\Phi[t]$ as a subcategory $A_\Phi[t]^c$ in the obvious way.

Lemma 8.11. For $(C, \phi) \in \text{End}(\text{Ch}(A^c, \Phi))$, the following are equivalent:

(i) $C$ is homotopy finite and $\phi$ is homotopy nilpotent;
(ii) $\chi_{A^c}(C, \phi)$ is $A^c_\Phi[t^{-1}]$-chain homotopy equivalent to a bounded $A_\Phi[t^{-1}]$-chain complex and, regarded as $A^c_\Phi[t, t^{-1}]$-chain complex, is contractible.

Proof. (i) $\implies$ (ii) If we apply Lemma 8.16 below to the morphism $(0, 0) \to (C, \phi)$, we obtain a zigzag of weak equivalences $(E, \mu) \xrightarrow{\sim} (F, \sigma) \xleftarrow{\sim} (C, \phi)$ in $\text{End}(\text{Ch}(A^c, \Phi))$ such that $E$ is a bounded $A$-chain complex. This implies by Subsection 8.2 that $\chi(C, \phi)$ is $A^c_\Phi[t^{-1}]$-chain homotopy equivalent to $\chi(E, \sigma)$. Since $E$ is a bounded $A$-chain complex, $\chi_{A}(E, \sigma)$ is a bounded $A_\Phi[t^{-1}]$-chain complex.

Over $A^c_\Phi[t, t^{-1}]$ we may split $i_\phi - t^{-1} = (1 - \phi \cdot t) \circ t^{-1}$ where $t^{-1}$ is an isomorphism and $1 - \phi \cdot t$ is a homotopy equivalence if $\phi$ is homotopy nilpotent (with homotopy inverse $\sum_{i \geq 0}(\phi \cdot t^i)$). In this case $i_\phi - t^{-1}$ is a chain homotopy equivalence and hence its cone $\chi_{A^c}(C, \phi)$ is $A^c_\Phi[t, t^{-1}]$-contractible by Lemma 8.11 (v).

(ii) $\implies$ (i) Let $D' = \chi(C, \phi)$. Lemma 8.8 implies that $N_{A^c}(D') \simeq (C, \phi)$ holds in $\text{End}(\text{Ch}(A^c, \Phi))$. It suffices to show that the $A^c$-chain complex $i^{-1}D'$ is chain homotopy equivalent to a bounded $A$-chain complex and that $i^{-1}t^{-1}: i^{-1}D \to i^{-1}D$ is homotopy nilpotent. By assumption $D'$ is $A^c_\Phi[t^{-1}]$-chain homotopy equivalent to a bounded $A_\Phi[t^{-1}]$-chain complex $D$. Hence it remains to show for the bounded $A_\Phi[t^{-1}]$-chain complex $D$ which is contractible over $A_\Phi[t, t^{-1}]$ that the $A^c$-chain complex $i^{-1}D$ is $A^c$-chain homotopy equivalent to a bounded $A$-chain complex and $i^{-1}t^{-1}: i^{-1}D \to i^{-1}D$ is homotopy $\Phi$-nilpotent.

Let $H$ be a null-homotopy for the bounded $A^c_\Phi[t, t^{-1}]$-chain complex $j_\cdot D$. Let $M$ be a large enough natural number so that the collection of maps

\[ H_n[\cdot] : D_n[-\infty, -M] \to D_{n+1}[-\infty, 0] \]

(introduced in Notation 6.1) is a null-homotopy for the inclusion $D[-\infty, -M] \to D[-\infty, 0]$. Since the $A^c$-chain map $(i^{-1}t^{-1})^{(M)} : i^{-1}D = D[-\infty, 0] \to i^{-1}D =$
$D_{[-\infty,0]}$ factors through the inclusion $D_{[-\infty,-M]} \to D_{[-\infty,0]}$, it is $\Phi$-homotopy nilpotent. Moreover the exact sequence

$$0 \to D_{[-\infty, -M]} \to D_{[-\infty,0]} \to D_{[-\infty,0]} / D_{[-\infty, -M]} \to 0$$

shows using Subsection 8.2 that we obtain $\mathcal{A}^e$-chain homotopy equivalences

$$D_{[-\infty,0]} / D_{[-\infty, -M]} \simeq \text{cone}(D_{[-\infty, -M]} \to D_{[-\infty,0]}) \simeq D_{[-\infty,0]} \oplus \Sigma D_{[-\infty, -M]}.$$  

We conclude that $D_{[-\infty,0]}$ is a homotopy retract of the bounded $\mathcal{A}$-chain complex $D_{[-\infty,0]} / D_{[-\infty, -M]}$. As $\mathcal{A}$ was assumed to be idempotent complete, we conclude from Lemma 8.14 that $i^{-1}D = D_{[-\infty,0]}$ is $\mathcal{A}$-chain homotopy equivalent to a bounded $\mathcal{A}$-chain complex.

\begin{notation}
(\text{HNil}(\text{Ch}^{bf}(\mathcal{A}), \Phi) \text{ and } \text{Ch}^{bf}(\mathcal{A}_\Phi[t^{-1}])^w).
\end{notation}

Let $\text{HNil}(\text{Ch}^{bf}(\mathcal{A}), \Phi)$ be the full Waldhausen subcategory of $\text{End}(\text{Ch}(\mathcal{A}^e, \Phi))$ consisting of objects $(C, \phi)$ for which the $\mathcal{A}^e$-chain complex $C$ is homotopy finite and $\phi$ is homotopy $\Phi$-nilpotent.

Let $\text{Ch}^{bf}(\mathcal{A}_\Phi[t^{-1}])^w$ be the full Waldhausen subcategory of $\text{Ch}(\mathcal{A}_\Phi[t^{-1}])$ consisting of those objects $C$ which are $\mathcal{A}_\Phi[t^{-1}]$-chain homotopy equivalent to a bounded $\mathcal{A}_\Phi[t^{-1}]$-chain complex and become contractible over $\mathcal{A}_\Phi[t, t^{-1}]$.

\begin{lemma}
Consider a morphism $u: C \to D$ in the Waldhausen category $\text{Ch}^{bf}(\mathcal{A}_\Phi[t^{-1}])^w$.

Then the following assertions are equivalent

(i) The morphism $u: (C, \phi) \to (D, \psi)$ is a cofibration in $\text{Ch}^{bf}(\mathcal{A}_\Phi[t^{-1}])^w$;

(ii) The underlying chain map $u: C \to D$ is a cofibration in $\text{Ch}(\mathcal{A}_\Phi[t^{-1}]^e)$, the $\mathcal{A}_\Phi[t^{-1}]^e$-chain complex $D/C$ is $\mathcal{A}_\Phi[t^{-1}]$-chain homotopy equivalent to a bounded $\mathcal{A}_\Phi[t^{-1}]$-chain complex, and $D/C$ regarded as $\mathcal{A}[t, t^{-1}]^e$-chain complex is contractible;

(iii) The underlying chain map $u: C \to D$ is a cofibration in $\text{Ch}(\mathcal{A}_\Phi[t^{-1}])$;

(iv) The morphism $u_n: C_n \to D_n$ in $\mathcal{A}_\Phi[t^{-1}]$ is split injective for all $n \in \mathbb{Z}$.

\begin{proof}
The equivalences (i) $\iff$ (ii) and (iii) $\iff$ (iv) are true by definition. The implication (iii) $\implies$ (ii) is obviously true. The implication (iii) $\implies$ (i) follows from Lemma 8.1 (vii) and Lemma 8.6.
\end{proof}

\begin{lemma}
Consider a morphism $u: (C, \phi) \to (D, \psi)$ in the Waldhausen category $\text{HNil}(\text{Ch}^{bf}(\mathcal{A}), \Phi)$.

Then the following assertions are equivalent

(i) The morphism $u: (C, \phi) \to (D, \psi)$ is a cofibration in $\text{HNil}(\text{Ch}^{bf}(\mathcal{A}), \Phi)$;

(ii) The underlying chain map $u: C \to D$ is a cofibration in $\text{Ch}(\mathcal{A}^e)$, the $\mathcal{A}^e$-chain complex $D/C$ is homotopy finite and the induced $\mathcal{A}^e$-chain map $\psi: D/C \to D/C$ is homotopy nilpotent;

(iii) The underlying chain map $u: C \to D$ is a cofibration in $\text{Ch}(\mathcal{A}^e)$;

(iv) The morphism $u_n: C_n \to D_n$ in $\mathcal{A}^e$ is split injective for all $n \in \mathbb{Z}$.

\begin{proof}
The equivalences (i) $\iff$ (ii) and (iii) $\iff$ (iv) are true by definition. Obviously (ii) $\implies$ (iii) holds. Hence it remains to prove (iii) $\implies$ (i).

We have to show that $D/C$ is homotopy finite and $\psi$ is homotopy nilpotent.

Because of Lemma 8.11 it suffices to prove that $\chi_{\mathcal{A}^e}(D/C)$ is $\mathcal{A}_\Phi[t^{-1}]$-chain homotopy equivalent to a bounded $\mathcal{A}_\Phi[t^{-1}]$-chain complex and contractible over $\mathcal{A}_\Phi[t, t^{-1}]$ provided that both $\chi_{\mathcal{A}^e}(C)$ and $\chi_{\mathcal{A}^e}(D)$ are $\mathcal{A}_\Phi[t^{-1}]$-chain homotopy equivalent to a bounded $\mathcal{A}_\Phi[t^{-1}]$-chain complex and contractible over $\mathcal{A}_\Phi[t, t^{-1}]$.
Since \( u: C \to D \) is a cofibration in \( \text{Ch}(A^c) \) with quotient \( D/C \), we obtain a cofibration \( \chi_{A^c}(u): \chi_{A^c}(C) \to \chi_{A^c}(D) \) with quotient \( \chi_{A^c}(D/C) \) in \( \text{Ch}(A^c_\Phi[t]) \). Now the claim follows from Lemma 6.1(vii) and Lemma 6.9.

\[ \textbf{Proposition 8.15.} \text{ The functor } \chi_{A^c} \text{ appearing in Lemma 8.8 induces an equivalence of Waldhausen categories } \]

\[ \chi(A): \text{HNil}(\text{Ch}^\text{hf}(A), \Phi) \cong \text{Ch}^\text{hf}(A, [t^{-1}])^w, \]

In particular we obtain a homotopy equivalence

\[ K(\chi(A)): K(\text{HNil}(\text{Ch}^\text{hf}(A), \Phi)) \cong K(\text{Ch}^\text{hf}(A, [t^{-1}])^w), \]

\[ \textbf{Proof.} \text{ We conclude from Lemma 8.7 and Lemma 8.13 that the functor } \chi_A \text{ defined in (8.6) restricts to a functor of Waldhausen categories } \]

\[ \chi(A): \text{HNil}(\text{Ch}^\text{hf}(A), \Phi) \cong \text{Ch}^\text{hf}(A, [t^{-1}])^w. \]

Because of Lemma 8.8, Lemma 8.11, Lemma 8.13 and Lemma 8.14, an inverse up to natural equivalence of Waldhausen categories is given by the restriction of the functor \( N_{A^c} \).

\[ \square \]

### 8.2. Chain complexes with (nilpotent) endomorphisms

The next two lemmas will play a key role at the end of the proof of Lemma 8.24 when we want to apply Cisinski’s Approximation Theorem 4.18. In the first lemma we want to replace an \( A^c \)-chain complex which is \( A^c \)-chain homotopy equivalent to a bounded \( A \)-chain complex by a bounded \( A \)-chain complex, in the second one we want to replace a homotopy nilpotent endomorphism by a nilpotent endomorphism.

\[ \textbf{Lemma 8.16.} \text{ Let } C \text{ be bounded } A \text{-chain complex. Let } D \text{ be an } A^c \text{-chain complex which is homotopy equivalent to a bounded } A \text{-chain complex. Let } u: (C, \phi) \to (D, \psi) \text{ be a morphism in End}(\text{Ch}(A^c), \Phi). \]

Then there exists a commutative diagram in \( \text{End}(\text{Ch}(A^c), \Phi) \)

\[
\begin{array}{ccc}
(C, \phi) & \xrightarrow{u} & (D, \psi) \\
\downarrow & & \downarrow \\
(E, \mu) & \xrightarrow{\simeq} & (F, \sigma)
\end{array}
\]

where the arrows labelled by \( \simeq \) are weak equivalences, the vertical arrows are cofibrations, and \( E \) is a bounded \( A \)-chain complex as well.

\[ \textbf{Proof.} \text{ We begin with two reductions.} \]

\[ \textbf{Reduction 1:} \text{ It is enough to consider the special case where } u: C \to D \text{ is a cofibration.} \]

In fact, put \( D' = \text{cyl}(u) \). Let \( u': C \to D' \) be the canonical inclusion and \( p: D' \to D \) be the canonical projection. Since \( \psi \circ u = u \circ \phi \) holds, we conclude from naturality of the mapping cylinder construction that there exists a chain map \( \psi': \Phi(D') \to D' \) such that the following is a sequence in \( \text{End}(A^c, \Phi) \)

\[
\begin{array}{ccc}
(C, \phi) & \xrightarrow{u'} & (D', \psi') \\
\downarrow & & \downarrow \\
(E, \mu) & \xrightarrow{\simeq} & (F', \sigma')
\end{array}
\]

Applying now the special case \( (u' \text{ is a cofibration}) \), we obtain the left square in the following diagram, where \( E \) is a bounded \( A \)-chain complex:

\[
\begin{array}{ccc}
(C, \phi) & \xrightarrow{u'} & (D', \psi') \\
\downarrow & & \downarrow \\
(E, \mu) & \xrightarrow{\simeq} & (F', \sigma')
\end{array}
\]

\[
\begin{array}{ccc}
(D, \psi) & \xrightarrow{p} & (D, \psi) \\
\downarrow & & \downarrow \\
(G, \tau) & \xrightarrow{\simeq} & (G, \tau)
\end{array}
\]

\[ \square \]
The right square is obtained by applying the pushout construction to the pair \((p, y)\). All vertical maps are cofibrations and all maps marked with \(\simeq\) are chain homotopy equivalences. Now the outer square is the desired diagram in \(\text{End}(\text{Ch}(A^\kappa))\).

**Reduction 2:** It is enough to construct

(i) a zig-zag in \(\text{End}(A^\kappa, \Phi)\)

\[
(E, \mu) \xrightarrow{j} (C, \phi) \xrightarrow{u} (D, \psi),
\]

where \(E\) is a bounded \(A\)-chain complex and \(j\) is a cofibration;

(ii) a chain homotopy equivalence \(v: E \to D\) such that \(v \circ j = u\);

(iii) a chain homotopy

\[
H: \psi \circ v \simeq v \circ \Phi(\mu): \Phi(E) \to D
\]

which is stationary over \(\Phi(C)\) (i.e., \(H \circ \Phi(j) = 0\)).

In fact, suppose we are given this data. By Lemma 3.1 (iv), there exists a chain map

\[
\rho = F(\mu, \psi, H): \Phi(\text{cyl}(v)) \to \text{cyl}(v)
\]

such that the inclusions of the top and bottom end into the mapping cylinder give rise to a zig-zag

\[
(E, \mu) \xrightarrow{i(E)} (\text{cyl}(v), \rho) \xrightarrow{i(D)} (D, \psi).
\]

Explicitly,

\[
\rho_n: \Phi(E)_{n-1} \oplus \Phi(E)_n \oplus \Phi(D)_n \xrightarrow{egin{pmatrix}
\mu_{n-1} & 0 & 0 \\
0 & \mu_n & 0 \\
H_n & 0 & \psi_n
\end{pmatrix}} E_{n-1} \oplus E_n \oplus D_n.
\]

Thus we get a (non-commutative) diagram in \(\text{End}(\text{Ch}(A^\kappa))\):

\[
(C, \phi) \xrightarrow{u} (D, \psi)
\]

\[
\begin{array}{c}
E, \mu \simeq j \simeq i(D) \\
(C, \phi) \xrightarrow{u} (D, \psi)
\end{array}
\]

In order to obtain a commutative diagram, we have to pass to a quotient of \(\text{cyl}(v)\) as follows. Let \(k: \text{cyl}(u) \to \text{cyl}(v)\) be the obvious cofibration induced by \(j(C): C \to E\). Let \(i(C): C \to \text{cyl}(u)\) and \(pr: \text{cyl}(u) \to D\) be the canonical inclusion and projection, respectively. Define an \(A^\kappa\)-chain complex \(F\) by the pushout of \(A^\kappa\)-chain complexes

\[
\begin{array}{c}
\text{cyl}(u) \xrightarrow{pr} D \\
k \simeq F
\end{array}
\]

All arrows are chain homotopy equivalences: This is always true for the projection \(pr\) in the mapping cylinder and follows for \(pr\) from Subsection 3.2 since \(k\) is a cofibration. The morphism \(k\) is a chain homotopy equivalence, as both domain and target are canonically homotopy equivalent to \(D\).

As \(u: (C, \phi) \to (D, \psi)\) is a morphism in \(\text{End}(\text{Ch}(A^\kappa)), \Phi\), we obtain by naturality an induced map \(\rho': \Phi(\text{cyl}(u)) \to \text{cyl}(u)\) for which the projection \(pr\) becomes a morphism in the endomorphism category. Moreover \(\rho\) restricts to \(\rho'\) under \(k\). This follows from the explicit form of \(\rho\) as displayed above, together with the fact that \(\mu\) restricts to \(\phi\) and \(H\) restricts to 0 by assumption.
Hence we may define \((F, \sigma)\) to be the pushout in \(\End(\Ch(A^{\omega}), \Phi)\) in the right square of the following diagram:

\[
\begin{array}{ccc}
(C, \phi) & \xrightarrow{i(C)} & (\cyl(u), \rho') \\
\downarrow j & \approx & \downarrow k \\
(E, \mu) & \xrightarrow{w} & (\cyl(v), \rho) \\
\end{array}
\]

The outer square provides then the conclusion of the Lemma.

This finishes the proof of Reduction 2. We are left to show that the hypotheses of Reduction 2 are satisfied.

Choose a bounded \(A\)-chain complex \(D'\) together with an \(A^{\omega}\)-chain homotopy equivalence \(f: D' \to D\). Choose a homotopy inverse \(f^{-1}: D \to D'\). Consider \(f^{-1} \circ u: C \to D'\). Then we can choose a homotopy equivalence \(f \circ (f^{-1} \circ u) \simeq u\). Let \(E = \cyl(f^{-1} \circ u)\) and write \(e\) for its differential. Notice that \(E\) is a bounded \(A\)-chain complex. We obtain from Lemma 3.1 (iv) an \(A^{\omega}\)-chain map \(v: E \to D\) such that \(v \circ j(C) = u\) and \(v \circ j(D') = f\), where \(j := j(C): C \to E\) and \(j(D'): D' \to \cyl(f^{-1} \circ u)\) denote the canonical inclusions.

Since \(j(D')\) and \(f\) are chain homotopy equivalences, the same is true for \(v\). From Lemma 3.1 we obtain a chain map \(w: D \to E\) with \(w \circ u = j(C)\) and a chain homotopy \(h: v \circ w \simeq \id_D\) satisfying

\[
(8.18) \quad h \circ u = 0.
\]

Define \(\mu: \Phi(E) \to E\) to be \(w \circ \psi \circ \Phi(v)\). Since

\[
\begin{align*}
\mu \circ \Phi(j(C)) &= w \circ \psi \circ \Phi(v) \circ \Phi(j(C)) \\
&= w \circ \psi \circ \Phi(u) \\
&= w \circ u \circ \phi \\
&= j(C) \circ \phi,
\end{align*}
\]

we obtain a morphism \(j(C): (C, \phi) \to (E, \mu)\) in \(\End(\Ch(A), \Phi)\).

Consider the (not necessarily commutative) diagram of \(A^{\omega}\)-chain complexes

\[
\begin{array}{ccc}
\Phi(E) & \xrightarrow{\mu} & E \\
\Phi(v) & \simeq & v \\
\Phi(D) & \xrightarrow{\psi} & D
\end{array}
\]

It commutes up to the chain homotopy

\[
H := h \circ \psi \circ \Phi(v).
\]

Then \(H\) is stationary over \(\Phi(C)\): In fact, we compute

\[
(8.19) \quad H \circ \Phi(j(C)) = h \circ \psi \circ \Phi(v) \circ \Phi(j(C))
\]

\[
= h \circ \psi \circ \Phi(u)
\]

\[
= h \circ u \circ \phi
\]

\[
= 0
\]

This concludes the proof that the hypotheses of Reduction 2 are satisfied, and thus the proof of the Lemma.

\[\square\]

**Lemma 8.20.** Let \(C\) and \(D\) be a bounded \(A\)-chain complexes. Consider a morphism \(u: (C, \phi) \to (D, \psi)\) in \(\End(\Ch(A), \Phi)\). Suppose that \(\phi\) is \(\Phi\)-nilpotent and \(\psi\) is homotopy \(\Phi\)-nilpotent.
Then there exists a commutative diagram in $\text{End}(\text{Ch}(A), \Phi)$

\[
\begin{array}{ccc}
(C, \phi) & \xrightarrow{u} & (D, \psi) \\
\downarrow & & \downarrow \cong \\
(E, \mu) & \xrightarrow{\cong} & (F, \sigma)
\end{array}
\]

where the arrows labelled by $\cong$ are weak equivalences, the vertical arrows are cofibrations, and $\mu$ is $\Phi$-nilpotent.

**Proof.** The same argument as in the proof of Lemma 8.16 shows that we can make the following two reductions.

**Reduction 1:** It is enough to consider the special case where $u: \Phi(D) \to D$ is a cofibration.

**Reduction 2:** It is enough to construct (i) a zig-zag in $\text{End}(A, \Phi)$

\[
\begin{array}{ccc}
(E, \mu) & \xleftarrow{j} & (C, \phi) \xrightarrow{u} (D, \psi)
\end{array}
\]

where $\mu$ is $\Phi$-nilpotent and $j$ is a cofibration, (ii) a chain homotopy equivalence $v': D \to E$ satisfying $v' \circ u = j$, and (iii) a chain homotopy

\[H': \mu \circ \Phi(v') \simeq v' \circ \psi: \Phi(E) \to D\]

which is stationary over $\Phi(C)$ (i.e., $H' \circ \Phi(u) = 0$).

We now proceed to show that the assumptions of Reduction 2 are fulfilled. As a first step we prove that we can choose an integer $n$ satisfying

\[(8.21) \quad \phi^{(n)} = 0,\]

and a chain homotopy

\[h(\psi): \psi^{(n)} \simeq 0\]

satisfying

\[(8.22) \quad h(\psi) \circ \Phi^n(u) = 0.\]

Let $E$ be the cokernel of the cofibration $u: C \to D$. Let $\overline{\psi}: E \to E$ be the $A$-chain map induced by $\psi$. Because of Lemma 8.14 we can choose an integer $m$ such that $\phi^{(m)} = 0$ and there exists a nullhomotopy $H: \psi^{(m)} \simeq 0$. Since $u: C \to D$ is a cofibration, we can assume $D_k = C_k \oplus E_k$ and that the differential of $D$ looks like

\[d_k = \begin{pmatrix} c_k & x_k \\ 0 & e_k \end{pmatrix}: D_k = C_k \oplus E_k \to D_{k-1} = C_{k-1} \oplus E_{k-1},\]

if $c$ and $e$ denote the differentials of $C$ and $E$, and $\psi^{(m)}$ looks like

\[\psi_k^{(m)} = \begin{pmatrix} 0 & y_k \\ 0 & \overline{\psi}_k \end{pmatrix}: \Phi^m(D_k) = \Phi^m(C_k) \oplus \Phi^m(E_k) \to D_k = C_k \oplus E_k.\]

Define a homotopy

\[H_k = \begin{pmatrix} 0 & 0 \\ 0 & \overline{H}_k \end{pmatrix}: \Phi^m(D_k) = \Phi^m(C_k) \oplus \Phi^m(E_k) \to D_{k+1} = C_{k+1} \oplus E_{k+1}.\]
We have
\[ d_{k+1} \circ H_k + H_{k-1} \circ \Phi^m(d_k) = \left( \begin{array}{c} e_{k+1} \\ x_{k+1} \\ 0 \\ e_{k+1} \end{array} \right) \circ \left( \begin{array}{c} 0 \\ 0 \\ \overline{H}_k \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ \Phi^m(e_k) \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ e_{k+1} \circ \overline{H}_k \\ 0 \end{array} \right) \circ \left( \begin{array}{c} x_{k+1} \\ \overline{H}_k + \overline{H}_{k-1} \circ \Phi^m(e_k) \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ x_{k+1} \circ \overline{H}_k \\ 0 \end{array} \right). \]
Hence, if we put \( z_k = y_k - x_{k+1} \circ \overline{H}_k \) and define \( \omega : \Phi^m(D) \to D \) by
\[ \omega_k = \left( \begin{array}{c} 0 \\ z_k \\ 0 \\ 0 \end{array} \right) : \Phi^m(D_k) = \Phi^m(C_k) \oplus \Phi^m(E_k) \to D_k = C_k \oplus E_k, \]
then \( \omega \) is a chain map and \( H \) is a chain homotopy \( \psi(m) \simeq \omega \).

It is easy to verify that if \( f \simeq f' : C \to D \) are two chain maps which are homotopic via a chain homotopy \( H \), and \( g \simeq g' : D \to E \) are homotopic via \( K \), then \( g \circ f \simeq g' \circ f' \) via the chain homotopy \( g \circ H + K \circ f' \).

In our situation
\[ \psi(2m) = \psi(m) \circ \Phi^m(\psi(m)) \simeq \omega \circ \Phi^m(\omega) = 0 \]
via the homotopy \( h(\psi) := \psi(m) \circ \Phi^m(H) + H \circ \Phi^m(\omega) \). Since both \( \Phi^m(H) \) and \( \Phi^m(\omega) \) are zero when restricted along \( u : C \to D \), the same is true for \( h(\psi) \). This establishes \([8,22]\), with \( n = 2m \).

We get from Lemma \([8,112]\), \( \Phi^m \)-chain homotopies
\[ h(C) : id_{\text{cyl}(\phi)} \simeq l(C) \circ \text{pr}(C); \]
\[ h(D) : id_{\text{cyl}(\psi)} \simeq l(D) \circ \text{pr(D)}. \]
satisfying
\[ \text{pr}(C) \circ h(C) = 0; \]
\[ \text{pr}(D) \circ h(D) = 0; \]
\[ h(D) \circ \overline{\pi} = \overline{\pi} \circ h(C), \]
where \( l(C) : C \to \text{cyl}(\phi) \) and \( l(D) : C \to \text{cyl}(\psi) \) are the canonical inclusions, \( \text{pr}(C) : \text{cyl}(\phi) \to C \) and \( \text{pr}(D) : \text{cyl}(\psi) \to C \) the canonical projections, and \( \overline{\pi} \) is the chain map \( \text{cyl}(\phi) \to \text{cyl}(\psi) \) given by \( \overline{\pi}_n = \Phi(u_{n-1}) \oplus \Phi(u_n) \oplus u_n : \Phi(C_{n-1}) \oplus \Phi(C_n) \oplus C_n \to \Phi(D_{n-1}) \oplus \Phi(D_n) \oplus D_n \). Denote by \( C' \) and \( D' \) the iterated mapping cylinders.

\[ C' := \Phi^{n-2}(\text{cyl}(\phi)) \cup_{\Phi^{n-3}(C)} \Phi^{n-3}(\text{cyl}(\phi)) \cup_{\Phi^{n-4}(C)} \cdots \cup_{\Phi(C)} \text{cyl}(\phi); \]
\[ D' := \Phi^{n-2}(\text{cyl}(\psi)) \cup_{\Phi^{n-3}(D)} \Phi^{n-3}(\text{cyl}(\psi)) \cup_{\Phi^{n-4}(D)} \cdots \cup_{\Phi(D)} \text{cyl}(\psi). \]

Denote by
\[ i(C) : C \to C'; \]
\[ i(D) : D \to D'; \]
\[ i(\Phi^{n-1}(C)) : \Phi^{n-1}(C) \to C'; \]
\[ i(\Phi^{n-1}(D)) : \Phi^{n-1}(D) \to D'. \]
the obvious inclusions. The various chain maps \( \Phi^{i}(\text{pr}(C)) \) and \( \Phi^{i}(\text{pr}(D)) \) fit together to projections
\[ p(C) : C' \to C; \]
\[ p(D) : D' \to D. \]
We have
\[ p(C) \circ i(C) = \mathrm{id}_C; \]
\[ p(D) \circ i(D) = \mathrm{id}_D; \]
\[ p(C) \circ i(\Phi^{n-1}(C)) = \phi^{(n-1)}; \]
\[ p(D) \circ i(\Phi^{n-1}(D)) = \psi^{(n-1)}. \]

Since \( \psi \circ \Phi(u) = u \circ \phi \), the various maps \( \Phi^i(\Pi) \) fit together to a cofibration \( u': C' \to D' \)
satisfying
\[ u' \circ i(C) = i(D) \circ u; \]
\[ p(D) \circ u' = u \circ p(C). \]

The various chain homotopies \( \Phi^i(h(C)) \) and \( \Phi^i(h(D)) \) fit together to chain homotopies\(^{[8.21]}\)
\[ g(C): \mathrm{id}_{C'} \simeq i(C) \circ p(C); \]
\[ g(D): \mathrm{id}_{D'} \simeq i(D) \circ p(D), \]
satisfying
\[ p(C) \circ g(C) = 0; \]
\[ p(D) \circ g(D) = 0; \]
\[ u' \circ g(C) = g(D) \circ u. \]

Next we define a chain maps \( \phi': \Phi(C') \to C' \) and \( \psi': \Phi(D') \to D' \). The morphism \( \psi' \) is constructed as follows: For \( i < n - 2 \), on \( \Phi(\Phi^i(\mathrm{cyl}(\psi))) \) it is given by the inclusion
\[ \Phi(\Phi^i(\mathrm{cyl}(H))) = \Phi^{i+1}(\mathrm{cyl}(H)) \to D'. \]

It remains to define \( \psi' \) on \( \Phi(\Phi^{n-2}(\mathrm{cyl}(\psi))) \). Consider the following (not necessarily commutative) diagram of \( A \)-chain complexes
\[ \begin{array}{ccc}
\Phi^n(C) & \xrightarrow{\Phi^{n-1}(\phi)} & \Phi^n(D) \\
\Phi^{-1}(C) & \xrightarrow{i(\Phi^{-1}(D))} & C' \\
\Phi^{-1}(D) & \xrightarrow{i(\Phi^{-1}(D))} & D'
\end{array} \]
where \( i(\Phi^{-1}(C)) \) and \( i(\Phi^{-1}(D)) \) are the obvious inclusions. Using \( \Phi^i(\Pi) \) and \( \Phi^{i+1}(\Pi) \), we obtain explicit chain homotopies of chain maps \( \Phi^n(C) \to C' \) and \( \Phi(D) \to D' \)
\[ k(C) = g(C) \circ i(\Phi^{-1}(C)) \circ \Phi^{n-1}(\phi); \]
\[ k(D) = g(D) \circ i(\Phi^{-1}(D)) \circ \Phi^{n-1}(\psi) \]

satisfying because of \( \Phi^{-1}(\Pi) \)
\[ \begin{array}{c}
k(D) \circ \Phi^n(u) = u' \circ k(C); \\
p(C) \circ k(C) = 0. \end{array} \]

We obtain from Lemma \( \Phi^{n-1}(\mathrm{cyl}(\phi)) \to C' \) and \( \Phi^{n-1}(\mathrm{cyl}(\psi)) \to D' \) which will be declared to be the restrictions of \( \phi' \) and \( \psi' \) to \( \Phi^{n-1}(\mathrm{cyl}(\phi)) \) and \( \Phi^{n-1}(\mathrm{cyl}(\psi)) \to D' \). This finishes the construction of the chain maps\(^{[8.21]}\)
\[ \phi': \Phi(C') \to C'; \]
\[ \psi': \Phi(D') \to D'. \]
One easily checks

\[
\begin{align*}
(\phi')^{(n)} &= 0; \\
(\psi')^{(n)} &= 0; \\
\psi' &\circ \phi' = \psi' \circ \Phi(u'); \\
p(C) &\circ \phi' = \phi \circ \Phi(p(C)); \\
p(D) &\circ \psi' \circ \Phi(i(D)) = \psi.
\end{align*}
\]

We define \((E, \mu)\) by the pushout in \(\text{End}(\text{Ch}(\mathcal{A}, \Phi))\)

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\left( C', \phi' \right) \quad \xrightarrow{u'} \quad \left( D', \psi' \right) \\
p(C) \quad \xrightarrow{=} \quad \cong \quad \xrightarrow{p(C)} \\
\left( C, \phi \right) \quad \xrightarrow{j} \quad \left( E, \mu \right)
\end{array}
\end{array}
\end{array}
\]

Since \(\psi'\) is \(\Phi\)-nilpotent and \(p(C)\) and hence \(p(C)\) are split surjective, also \(\mu\) is \(\Phi\)-nilpotent.

Letting \(v' := p(C) \circ i(D)\), we obtain an explicit chain homotopy of chain maps \(\Phi(D) \to E\)

\[
H' := p(C) \circ g(D) \circ \psi' \circ \Phi(i(D)): \mu \circ \Phi(v') \simeq v' \circ \psi.
\]

The following computation shows that \(H'\) is stationary over \(\Phi(C)\):

\[
\begin{align*}
H' \circ \Phi(u) &= p(C) \circ g(D) \circ \psi' \circ \Phi(i(D)) \circ \Phi(u) \\
&= p(C) \circ g(D) \circ \psi' \circ \Phi(u') \circ \Phi(i(C)) \\
&= p(C) \circ g(D) \circ \psi' \circ \phi' \circ \Phi(i(C)) \\
&= j \circ p(C) \circ g(C) \circ \phi' \circ \Phi(i(C)) \\
&= j \circ 0 \circ \phi' \circ \Phi(i(C)) \\
&= 0,
\end{align*}
\]

This completes the verification of the assumptions of Reduction 2, and therefore completes the proof of the Lemma. \(\square\)

8.3. Comparing the finiteness conditions. By Lemma \([\text{II}](i)\), the Waldhausen categories \(\text{Ch}(\mathcal{A}_\Phi[t^{-1}])\) and \(\text{Ch}(\mathcal{A}_\Phi[t^{-1}])^w\) and hence also \(\text{Ch}(\mathcal{A}_\Phi[t^{-1}])^w\) and \(\text{Ch}^{\text{hf}}(\mathcal{A}_\Phi[t^{-1}])^w\) satisfy the saturation and the cylinder axiom. Cisinski’s Approximation Theorem \([\text{I}].18\) implies

**Lemma 8.23.** The inclusion

\[
\text{Ch}(\mathcal{A}_\Phi[t^{-1}])^w \to \text{Ch}^{\text{hf}}(\mathcal{A}_\Phi[t^{-1}])^w
\]

induces an equivalence on \(K\)-theory.

The goal of this section is to show the following Lemma \([\text{8}.24]\) since then Theorem \([\text{8}.3]\) will directly follow from Proposition \([\text{8}.15]\), Lemma \([\text{8}.23]\) and Lemma \([\text{8}.24]\).

**Lemma 8.24.** The inclusion

\[
\text{Nil}(\mathcal{A}, \Phi) \to \text{HNil}(\text{Ch}^{\text{hf}}(\mathcal{A}), \Phi)
\]

induces an equivalence on \(K\)-theory.
Proof. The inclusion \( \text{Nil}(A, \Phi) \rightarrow \text{HNil}(\text{Ch}^{bf}(A), \Phi) \) can be split into a sequence of inclusions
\[
\text{Nil}(A, \Phi) \xrightarrow{I_1} \text{Nil}(\text{Ch}(A), \Phi) \xrightarrow{I_2} \text{HNil}(\text{Ch}(A), \Phi) \xrightarrow{I_3} \text{HNil}(\text{Ch}^{bf}(A), \Phi).
\]
We will show that each of the three inclusions \( I_1, I_2 \) and \( I_3 \) induce homotopy equivalence on \( K \)-theory.

The morphism \( I_1 \) induces a homotopy equivalence on \( K \)-theory because of the Gillet-Waldhausen Theorem [4.1] and Example [4.14] using the identity of Waldhausen categories \( \text{Nil}(\text{Ch}(A), \Phi) = \text{Ch}(\text{Nil}(A, \Phi)) \).

The maps \( I_2 \) and \( I_3 \) induce equivalences on \( K \)-theory by Cisinski’s Approximation Theorem [4.18]. We have to check the various assumptions appearing in Theorem [4.18]. The categories \( \text{Ch}(A^k, \Phi) \), \( \text{Ch}(A_{k-1}[t^{-1}]^*) \) and \( \text{Nil}(\text{Ch}(A), \Phi) = \text{Ch}(\text{Nil}(A, \Phi)) \) satisfy the saturation axiom and the cylinder axiom because of Lemma [4.11(3)]. We conclude that \( \text{End}(\text{Ch}(A^k, \Phi)) \) and hence by from Lemma [8.14] the full Waldhausen subcategories \( \text{HNil}(\text{Ch}(A), \Phi) \) and \( \text{HNil}(\text{Ch}^{bf}(A), \Phi) \) satisfy the saturation axiom and the cylinder axiom. The inclusion functors \( I_2 \) and \( I_3 \) reflect weak equivalences by Lemma [4.12]. The second approximation property appearing in Theorem [4.18] was shown to hold in Lemma [8.16] and Lemma [8.20]. This finishes the proof of Lemma [8.21] and hence also the proof of Theorem [8.7].

9. PASSING TO NON-CONNECTIVE ALGEBRAIC \( K \)-THEORY

The details of the definitions of the non-connective versions of the \( K \)-groups appearing in Theorem [7.1] and the argument how Theorem [7.1] can be deduced from the connective version, are presented in [13] Section 6.

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