MAXIMAL ERGODIC INEQUALITIES FOR BANACH FUNCTION SPACES

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Abstract. We analyse the Transfer Principle, which is used to generate weak type maximal inequalities for ergodic operators, and extend it to the general case of \( \sigma \)-compact locally compact Hausdorff groups acting measure-preservingly on \( \sigma \)-finite measure spaces. We show how the techniques developed here generate various weak type maximal inequalities on different Banach function spaces, and how the properties of these function spaces influence the weak type inequalities that can be obtained. Finally, we demonstrate how the techniques developed imply almost sure pointwise convergence of a wide class of ergodic averages.

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1. Introduction

Pointwise ergodic theorems have had an illustrious history spanning over 80 years since G.D. Birkhoff first proved the foundational result in 1931. The proof of
his ergodic theorem has been so refined that one can give an elementary, leisurely demonstration in about two pages \[13\]. To work with more general ergodic averages however, it seems one must still rely on a different approach. Indeed, one of the main techniques that has been developed to prove pointwise ergodic theorems, is supple enough to deal with pointwise convergence phenomena for a great variety of different ergodic averages, for different groups, function spaces and averages.

This is the technique of maximal operators. The idea is that once one estimates the behaviour of these maximal operators, proving the ergodic theorems becomes quite simple. (We explain the proof strategy in Section \[5\] in the form of a three-step programme). This is how the pointwise ergodic theorems are proved in \[10\] and \[22\]. It is also how results on entropy and information are obtained in \[20\] Chapter \[6\].

In fact, Wiener \[28\] developed a method, later greatly embellished by Calderón \[3\], for computing the requisite properties of the maximal operators, a method that is the central theme of this work: the Transfer Principle. It is our goal to extend the scope of this Principle and hence the scope of the maximal operator technique in proving pointwise ergodic theorems.

Broadly speaking, a dynamical system consists of three elements: a measure space \((\Omega, \mu)\), a topological group \(G\), and an action \(\alpha\), continuous in some sense, that binds them together by mapping \(G\) into the group of invertible measure-preserving transformations of \(\Omega\). The Transfer Principle refers to a body of techniques that allow one to transform certain types of operators acting on function spaces over \(G\) to corresponding transferred operators acting on function spaces over \(\Omega\), in such a way that many essential properties of the operator are preserved.

We have three aims: firstly, to extend the Transfer Principle to a larger class of groups, measure spaces and operators, secondly to broaden the reach of the techniques used to determine the weak type of the transferred operator, and finally to outline how these results can be used to derive a number of new pointwise ergodic theorems.

We define the Transfer Principle in quite a general setting (Definition \[2.1\]). If the operator to be transferred - call it \(T\) - is linear, then \(G\) and \(\Omega\) need only be \(\sigma\)-finite (Definition \[2.10\]) and if it is sublinear, then it must have separable and metrisable range (Definition \[2.9\]). The determination of the weak type of the transferred operator - call it \(T^\#\) - rests on results requiring \(\Omega\) to be countably generated and resonant as defined just before Proposition \[3.7\]. This fulfills the first aim.

Computing the weak type of a transferred operator \(T^\#\) is achieved with Corollary \[4.3\]. A noteworthy feature of this result is that it shows that the most important factors determining the weak type of \(T^\#\) are the fundamental functions associated with the function spaces defining the weak type of \(T\). In practice, however, one must most often deal with sequences of transferable operators. To compute the weak type of the transfer of the limit, one requires a more delicate analysis. We present our results in Proposition \[4.4\] and Theorems \[4.7\] \[4.8\] and \[4.13\]. Of these, Theorems \[4.7\] \[4.8\] and \[4.13\] are extensions of \[3\] Theorem \[1\] to the case of general amenable locally compact groups acting on measure spaces \(\Omega\) that may possibly have unbounded measure. By comparison, Theorem \[4.13\] applies to vastly more general types of rearrangement invariant function spaces. The price for this generality is the not too onerous restriction that \(\Omega\) have finite measure.
In this way, the problem of computing the weak type of the transferred operator is reduced to computations involving certain well-behaved real-valued functions. In particular, these results allow us to estimate the weak type of the maximal operator associated with ergodic averages over a wide class of rearrangement invariant Banach function spaces. This completes the second aim.

Finally, we address the third aim by proving pointwise ergodic theorems, transferring information obtained using Fourier analysis on the group to properties of the ergodic averages, and information on the function space on which they act, that is encoded in the fundamental function. The main results are Theorem 5.5 and Corollaries 5.6 and 5.7.

The importance of the Transfer Principle in ergodic theory has long been appreciated - see the excellent overview given in [1]. Apart from Calderón’s seminal paper [3], this principle is treated in some detail in the monograph [4] and employed extensively in [22]. In [13] the author makes use of Orlicz spaces to prove results about the pointwise convergence of ergodic averages along certain subsets of the natural numbers. In [8] the Transfer Principle of Coifman and Weiss is extended to weighted Orlicz spaces for group actions that are uniformly bounded in a sense determined by the space.

Definition 1.1. A dynamical system consists of a multiplicative locally compact group $G$ acting on a measure space $(\Omega, \mu)$ via an action of $G$ on $\Omega$ by $\alpha$. The action is measure-preserving in the sense that for any measurable subset $A \subseteq \Omega$ and $g \in G$, $\mu(\alpha_g^{-1}(A)) = \mu(A)$. Furthermore, the map $\tilde{\alpha} : G \times \Omega \to \Omega : t \times \omega \mapsto \alpha_t(\omega)$ is also measurable. The data is summarised by these four objects:

$$(\Omega, \mu, G, \alpha).$$

The condition that $\tilde{\alpha} : G \times \Omega \to \Omega : t \times \omega \mapsto \alpha_t(\omega)$ be measurable is equivalent to stating that if $f$ is a measurable function of $\Omega$, then the function $F(t, \omega) := f(\alpha_t(\omega))$ is measurable on $G \times \Omega$ because $F = f \circ \tilde{\alpha}$. With a slight abuse of notation, for any measurable function $f$ on $\Omega$ and $t \in G$ we can then define $\alpha_t(f)$ by setting $\alpha_t(f)(\omega) := f(\alpha_t(\omega))$ for a $\mu$-almost all $\omega \in \Omega$.

We shall denote the Haar measure on $G$ by the symbol $h$.

One final notational convention: if $A$ is a measurable subset of a measure space $(\Omega, \mu)$, we shall for brevity write $|A| := \mu(A)$. Likewise, if $K$ is a measurable subset of the locally compact group $G$, we shall denote the Haar measure of $K$ by $|K|$.

Let us briefly describe the organisation of the paper. In Section 2 we define the Transfer Principle and analyse it in some detail. This involves quite intricate measure-theoretic considerations, including the development of a theory of locally Bochner integrable functions in parallel with the classical theory of Bochner integrable functions.

In Section 3 we bring to mind some basic constructions and definitions in the theory of rearrangement invariant Banach function spaces. We emphasise how in the general theory a central role is played by the fundamental function of such spaces, and how a great deal of their structure and behaviour is reflected in this function. We also estimate some integrals that arise naturally for functions on product spaces (Proposition 3.7).

Section 4 contains the main results for estimating the weak type of the transferred operator, namely Corollary 4.3 and Theorems 4.8 and 4.13. This Section is based
on the work of the previous two sections and an extension of an inequality of Kolmogorov (Theorem 4.1).

The final section contains derivations of pointwise ergodic theorems. We show how properties of the function spaces and the transfer operators combine to determine a variety of ergodic theorems.

2. The Transfer Principle

2.1. Construction and measure theoretic considerations. The first order of business is to specify what the Transfer Principle is and to which operators the procedure applies. A word on terminology. If $T$ is a mapping between locally compact vector spaces, we say that $T$ has metrisable range if the relative topology on the range is metrisable. An operator $T$ whose domain is some linear subspace of the measurable functions on $(\Omega, \mu)$ and mapping into the measurable functions on a measure space $(\Omega_1, \mu_1)$ is said to be sublinear if for any $f$ and $g$ in the domain of $T$ and complex $\lambda$, we have $|T(f+g)| \leq |T(f)| + |T(g)|$ and $|T(\lambda f)| = |\lambda||T(f)|$.

Definition 2.1. (Transferable operators) Let $T$ be an operator having the class of locally integrable functions over $G$ as domain and having $C(G)$, the space of continuous functions on $G$ given the compact-open topology, as range. It is called transferable if it satisfies the following conditions:

1. $T$ is either a sublinear mapping with metrisable range or a continuous linear mapping
2. $T$ is semilocal: there exists an open neighbourhood $U$ of $1 \in G$ with compact closure such that if $\text{supp} f$ is contained in a set $V$, then $\text{supp} T(f)$ is contained in $UV$.
3. $T$ is translation invariant: for all $t \in G$ and $f \in L^\text{loc}(G)$,
   $$\tau_t \circ T(f) = T \circ \tau_t(f),$$
   where $\tau_t(f)$ is the function defined by $s \mapsto f(ts)$ for all locally integrable $f$ and $s \in G$.

Note that if $C(G)$ is metrisable under the compact-open topology, then $T$ automatically has metrisable range. For instance, this occurs when $G$ is second countable and metrisable. For then the compact-open topology on $G$ is the same as the topology of uniform convergence, as shown in [16]. Moreover, if $\{g_n\}$ is a countable dense subset of $G$ we can form the countable collection of closed balls $\{B(g_n, q) : n \in \mathbb{N}, q \in \mathbb{Q}\}$, and as any compact set can be covered by a finite number of these balls, we can see that $C(G)$ under the compact-open topology is a Fréchet space.

For a given transferable operator $T$, the remainder of this section is devoted to the construction of the transfer of $T$, denoted by $T^\#$, and some of its basic properties. It shall be defined as the composition of maps that arise naturally in the study of vector-valued measure theory, which we shall define and analyse. These constructions will allow us to handle the delicate measure theory on product spaces that shall appear.

We shall pay special attention to separability and countability properties of the space $(\Omega, \mu)$ and group $G$ and how they affect the transfer operator. As we will see in the sequel, if $\Omega$ and $G$ are $\sigma$-finite, the transfer operator will be well-defined. If $T$ is also metrisably valued (whether sublinear or linear) we will be able to write
down the construction of the Transfer Principle in an even more intuitively direct way, as given in Remark 2.13. Indeed, on a first reading, one may skip directly to this Remark as it is this construction that shall be used in the rest of the paper.

On the space of measurable functions over a measure space $(Ω, µ)$ we can define a family of seminorms $p_A(f) := \int_A |f| \, dµ$ as $A$ ranges over all subsets of $Ω$ of finite measure. Those measurable functions for which $p_A(f)$ is finite for all $A$ of finite measure form the locally convex vector space of \textit{locally integrable functions}. This space is called $L^\text{loc}(Ω)$ and is topologised by the family $\{p_A\}$.

If we have two measure spaces $(Ω_1, µ_1)$ and $(Ω_2, µ_2)$ then we can consider the family of seminorms $p_{A×B}$ defined on the set of measurable functions on $(Ω_1 × Ω_2, µ_1 × µ_2)$ by setting $p_{A×B}(f) = \int_{A×B} |f| \, dµ_1 × µ_2$ and define $L^{r-\text{loc}}(Ω_1 × Ω_2)$, the space of all \textit{rectangular locally integrable functions}, to consist of those measurable functions for which all such seminorms are finite. As with the locally integrable functions, we use the family $\{p_{A×B}\}$ to define the topology on $L^{r-\text{loc}}(Ω_1 × Ω_2)$.

In the category of locally convex spaces, it is possible to topologise the tensor product of two spaces in many ways. Two such tensor topologies can be distinguished: the projective tensor product used above and the injective tensor product. Here is one of two results on projective tensor products that we shall need.

**Proposition 2.2.** If $(Ω_1, Σ_1, µ_1)$ and $(Ω_2, Σ_2, µ_2)$ are two σ-finite measure spaces, then there is a continuous linear injection $ι_1 : L^{r-\text{loc}}(Ω_1 × Ω_2) → L^{\text{loc}}(Ω_1) ⊗_π L^{\text{loc}}(Ω_2)$.

**Proof.** The space $L^{\text{loc}}(Ω_1)$ may be recognised as the projective limit of the family of spaces $\{L^1(A)\}$, as $A$ ranges over all finitely measurable subsets in $Σ_1$ and mappings $r_{BA} : L^1(B) → L^1(A)$ where $B ⊇ A$ and $r_{BA}$ is the restriction map. Similarly for $L^{\text{loc}}(Ω_2)$. Hence by [12, Theorem 15.2], $L^{\text{loc}}(Ω_1) ⊗_π L^{\text{loc}}(Ω_2)$ is the projective limit of the family $L^1(A) ⊗_π L^1(B)$, as $A$ and $B$ range over all finitely measurable subsets of $Ω_1$ and $Ω_2$ respectively.

Now $L^1(A) ⊗_π L^1(B) ≃ L^1(A × B)$ as shown for example in [25]. As the restriction maps $r_{A×B} : L^{r-\text{loc}}(Ω_1 × Ω_2) → L^1(A × B)$ are well-defined and continuous, from the universal property of the projective limit (as described in [14, Chapter 3.4] or [27, Appendix I]) there is a unique continuous linear mapping $ι_1 : L^{r-\text{loc}}(Ω_1 × Ω_2) → L^{\text{loc}}(Ω_1) ⊗_π L^{\text{loc}}(Ω_2)$ such that the following diagrams commute for all finitely measurable $A ∈ Σ_1$ and $B ∈ Σ_2$.

$$
\begin{array}{ccc}
L^{r-\text{loc}}(Ω_1 × Ω_2) & \xrightarrow{ι_1} & L^{\text{loc}}(Ω_1) ⊗_π L^{\text{loc}}(Ω_2) \\
& & \downarrow r_A ⊗ r_B \\
& & L^1(A × B)
\end{array}
$$

It is clear from this diagram that $ι_1$ is injective: if $f$ and $g$ are distinct elements of $L^{r-\text{loc}}(Ω_1 × Ω_2)$, then there is some rectangle $A × B$ on which they are not equal a.e.; this means that $r_{A×B}(f - g) \neq 0$ and so $ι_1(f - g) \neq 0$. \□

We now note the result about the injective product that we shall employ, namely that if $X$ is locally compact Hausdorff and $E$ is a complete locally convex vector space then

$$
C(X, E) ≃ C(X) ⊗_π E
$$
where $C(X, E)$ and $C(X)$ denote respectively the continuous $E$-valued and $\mathbb{C}$-valued functions on $X$ equipped with the compact-open topology and $\otimes_\alpha$ denotes the completion of the injective tensor product of the two spaces. This result is proved in [12, Corollary 3, Section 16.6].

We now turn to some tensor constructions of functions that will be necessary when defining and working with the transfer operator. Recall first that given a $\mu$-a.e. finite measurable function $f$ on $(\Omega, \mu)$, the distribution function $s \mapsto m(f, s)$ is defined by

$$m(f, s) = \mu(\{\omega \in \Omega : |f(\omega)| > s\})$$

for all $s \geq 0$. Two measurable functions $f$ and $g$ are equimeasurable if we have $m(f, s) = m(g, s)$ for all $s \geq 0$.

**Definition 2.3.** Given a dynamical system $(G, \alpha, \Omega, \mu)$ as in Definition 1.1, let $f$ and $g$ be measurable functions on $G$ and $\Omega$ respectively. The $\alpha$-skew tensor $f \otimes_\alpha g$ is a measurable function on $G \times \Omega$ defined by

$$f \otimes_\alpha g(t, \omega) = f(t)g(\alpha_t(\omega)).$$

There is a strong link between the skew tensor product and the standard tensor product of two functions that will come in handy.

**Lemma 2.4.** Given $f$ and $g$ as above, the functions $f \otimes_\alpha g$ and $f \otimes g$ on $G \times \Omega$ are equimeasurable.

**Proof.** Let $\lambda \in \mathbb{R}^+$ be fixed and define the following sets:

$$E = \{(t, \omega) \in G \times \Omega : |f \otimes_\alpha g(t, \omega)| > \lambda\},
E' = \{(t, \omega) \in G \times \Omega : |f \otimes g(t, \omega)| > \lambda\}.$$

Moreover, for a fixed $t \in G$, we define

$$E_t = \{\omega \in \Omega : |g(\alpha_t(\omega))| > \lambda/|f(t)|\},
E'_t = \{\omega \in \Omega : |g(\omega)| > \lambda/|f(t)|\}.$$

Now because $\alpha_t(g)$ and $g$ are equimeasurable,

$$\mu(E_t) = m(\alpha_t(g), \lambda/|f(t)|) = m(g, \lambda/|f(t)|) = \mu(E'_t).$$

Furthermore, $h \times \mu(E) = \int_G \mu(E_t)dt = \int_G \mu(E'_t)dt = h \times \mu(E')$. Hence

$$m(f \otimes_\alpha g, \lambda) = m(f \otimes g, \lambda).$$

If $f$ is a measurable function on $\Omega$, we define

$$F := \otimes_{\alpha,G}(f) := \chi_G \otimes_\alpha f.$$  \hfill (2.1)

In other words, $F(t, \omega) = f(\alpha_t(\omega))$. This function is measurable on $G \times \Omega$. To see this, recall from Definition 1.1 that $\alpha : G \times \Omega \rightarrow \Omega : t \times \omega \mapsto \alpha_t(\omega)$ is measurable, which implies that $F = f \circ \alpha$ is measurable too.

We define the Banach space $L^{1+\infty}(\Omega)$ to be the set of a.e.-finite measurable functions on $(\Omega, \mu)$ that can be written in the form $f + g$, where $f \in L^1(\Omega)$ and $g \in L^\infty(\Omega)$. The norm on $L^{1+\infty}(\Omega)$ is given by

$$\|h\|_{L^{1+\infty}(\Omega)} = \inf\{\|f\|_{L^1} + \|g\|_{L^\infty} : f \in L^1(\Omega),\ g \in L^\infty(\Omega),\ h = f + g\}.$$
Such a space is a rearrangement invariant Banach function space, about which we shall have more to say in the next Section.

**Lemma 2.5.** If \( f \in L^{1+\infty}(\Omega) \), then \( F \) is rectangular-locally integrable on \( G \times \Omega \). Furthermore \( \otimes_{\alpha,G} \) is a continuous mapping from \( L^{1+\infty}(\Omega) \) to \( L^{r \text{ loc}}(G \times \Omega) \).

**Proof.** Let us write \( f = g_1 + g_2 \), where \( g_1 \in L^1(\Omega) \) and \( g_2 \in L^\infty(\Omega) \). Now for any subsets \( K \subset G \) and \( A \subset \Omega \) of finite measure, we must show that \( \int_{K \times A} |F| \, dh \times \mu \) is finite, where as per our convention, \( h \) denotes the Haar measure on \( G \).

Note that for any \( t \in G \), the measure-invariance of \( \alpha \) ensures that

\[
\int_{A} |f|(\alpha_t(\omega)) \, d\mu(\omega) = \int_{\alpha_{t^{-1}}(A)} |f|(\omega) \, d\mu(\omega)
\]

\[
\leq \int_{\alpha_{t^{-1}}(A)} |g_1|(\omega) \, d\mu(\omega) + \int_{\alpha_{t^{-1}}(A)} |g_2|(\omega) \, d\mu(\omega)
\]

\[
\leq \|g_1\|_1 + |A||g_2||_\infty.
\]

By Fubini’s theorem and the measurability of \( F \),

\[
\int_{K \times A} |F| \, dh \times \mu = \int_{K} \int_{A} |f(\alpha_t(\omega))| \, d\mu(\omega) dh(t)
\]

\[
\leq \int_{K} \|g_1\|_1 + |A||g_2||_\infty \, dh
\]

\[
= |K|(\|g_1\|_1 + |A||g_2||_\infty) < \infty.
\]

Hence \( F \) is rectangular locally integrable. Furthermore, as

\[
|K|(\|g_1\|_1 + |A||g_2||_\infty) < |K|(1 + |A|)(\|g_1\|_1 + \|g_2\|_\infty),
\]

we have

\[
\int_{K \times A} |F| \, dh \times \mu \leq |K|(1 + |A|) \|f\|_{1+\infty},
\]

which implies the continuity of \( \otimes_{\alpha,G} \). \( \square \)

In order to define and analyse the transfer operator construction, we require a modest extension of the theory of Bochner integrable functions to functions taking values not in a Banach space, but more general locally convex vector spaces. Let \((\Omega,\mu)\) be a \(\sigma\)-finite measure space and \(E\) a complete locally convex vector space whose topology is defined by the family \(\{p_\alpha\}_{\alpha \in A}\) of seminorms. Here the theory and proofs closely follow the standard treatments for Banach space-valued functions, such as [25] or [4]. A \(\mu\)-simple measurable function \(f : \Omega \to E\) is a function \(f = \sum_{i=1}^N \chi_{E_i} x_i\), where \(E_1, \ldots, E_N\) are \(\mu\)-measurable subsets of \(\Omega\) and \(x_1, \ldots, x_N \in E\). A function \(f : \Omega \to E\) is said to be \(\mu\)-measurable if there is a sequence of \(\mu\)-simple measurable functions \((f_n)\) that converges \(\mu\)-almost everywhere to \(f\).

A function \(f : \Omega \to E\) is said to be \(\mu\)-weakly measurable if the scalar-valued function \(e^t f\) is \(\mu\)-measurable for every \(e^t \in E^\prime\) and Borel measurable if for every open subset \(O\) of \(E\), \(f^{-1}(O)\) is a measurable subset of \(\Omega\). Finally, \(f\) is \(\mu\)-essentially \((\text{separably/\text{metrisably}})\) valued if there is a \(\mu\)-measurable subset \(A\) of \(\Omega\) whose complement has measure 0, such that \(f(A)\) is contained in a \((\text{separably/\text{metrisable}})\) subspace of \(E\).
Theorem 2.6. (Pettis Measurability Theorem) For a $\sigma$-finite measure space $(\Omega, \mu)$ and dual pair $(E, E')$ under a topology $\xi$, the following are equivalent for a $\mu$-essentially metrisable valued function $f : \Omega \to E$:

1. $f$ is $\mu$-measurable
2. $f$ is $\mu$-weakly measurable and essentially separably valued
3. $f$ is Borel $\mu$-measurable and essentially separably valued.

The proof of this theorem is a straightforward adaptation of the proof of the Banach space-valued proof presented in [25].

We define the integral of a simple measurable function $s = \sum_{i=1}^{N} \chi_{E_i}x_i$ over a set $A \subseteq \Omega$ of finite measure by defining

$$\int_{A} s \, d\mu = \sum_{i=1}^{n} \mu(A \cap E_i)x_i.$$

We say that a measurable function $f : \Omega \to E$ is locally Bochner integrable if there exists a sequence of measurable simple functions $(f_n)$ converging almost everywhere to $f$ and satisfying

$$\lim_{n \to \infty} \int_{A} p_{\alpha}(f - f_n) = 0$$

for every finitely measurable subset $A \subseteq \Omega$ and every seminorm $p_{\alpha}$. In this case the Bochner integral of $f$ over $A$ is

$$\int_{A} f \, d\mu = \lim_{n \to \infty} \int_{A} f_n \, d\mu$$

and it can be shown as in the case of scalar functions that this limit exists and is independent of the particular choice of the sequence $(f_n)$.

The locally convex space of all locally Bochner integrable functions is called $L^{1}(\Omega, E) \simeq L^{1}(\Omega) \otimes_{\sigma} E$, which is proved in detail in [26] and [6]. From this fact, we can construct a similar, though weaker, relation between $L^{\loc}(\Omega, E)$ and $L^{\loc}(\Omega) \otimes_{\sigma} E$ where $E$ is a complete locally convex space, which is given in the following lemma.

Lemma 2.7. There is a naturally defined continuous injective (though in general not surjective) mapping $\iota_{2} : L^{\loc}(\Omega, E) \to L^{\loc}(\Omega) \otimes_{\sigma} E$.

Proof. We shall work once more with restriction maps as we did in Proposition 2.2.

Any locally convex space $E$ is the projective limit of a family of Banach spaces $E_{\alpha}$. For every $\alpha$, let $q_{\alpha} : E \to E_{\alpha}$ be the continuous linear mapping induced by the projective limit. Then we can define $q_{\alpha}^{\circ} : L^{\loc}(\Omega, E) \to L^{\loc}(\Omega, E_{\alpha})$ to be the mapping $f \mapsto q_{\alpha}^{\circ} f$.

For any subset $A \subseteq \Omega$ of finite measure, define $r_{A} : L^{\loc}(\Omega, E) \to L^{\loc}(A, E)$ to be the restriction map $f \mapsto f|_{A}$.

Now we can define maps $\pi_{A,\alpha} : L^{\loc}(\Omega, E) \to L^{1}(A, E_{\alpha})$ by setting $\pi_{A,\alpha} := r_{A} \circ q_{\alpha}^{\circ}$. On the other hand, there are the mappings $r_{A} \otimes q_{\alpha} : L^{\loc}(\Omega) \otimes_{\sigma} E \to L^{1}(A) \otimes_{\sigma} E_{\alpha}$. Identifying $L^{1}(A) \otimes_{\sigma} E_{\alpha}$ and $L^{1}(A, E_{\alpha})$, we see that $L^{\loc}(\Omega) \otimes_{\sigma} E$ is the projective limit of the family $\{L^{1}(A, E_{\alpha})\}$. By the universal property of the
projective limit, there must be a unique mapping $\iota_2 : L^{\text{loc}}(\Omega, E) \rightarrow L^{\text{loc}}(G) \otimes_\pi E$ such that the diagrams

$$
\begin{array}{c}
L^{\text{loc}}(\Omega, E) \\
\downarrow \pi_{A,\alpha} \\
L^1(A, E_\alpha)
\end{array}
\xrightarrow{\iota_2}
\begin{array}{c}
L^{\text{loc}}(\Omega_1) \otimes_\pi E \\
\downarrow r_A \otimes q_\alpha
\end{array}
$$

commute for all $\alpha$ and finitely measurable $A$. From these diagrams the injectivity of $\iota_2$ is clear.

There is a strong link between the spaces $L^{r-\text{loc}}(G \times \Omega)$, $L^{\text{loc}}(\Omega, L^{\text{loc}}(G))$ and $L^{\text{loc}}(G) \otimes_\pi L^{r-\text{loc}}(\Omega)$ that builds on Proposition 2.2 and Lemma 2.7. To establish this link, we need the following elementary lemma.

**Lemma 2.8.** If $(\Omega_1, \mu_1)$ and $(\Omega_2, \mu_2)$ are $\sigma$-finite measure spaces then the collection of rectangular simple functions with support of finite measure is dense in $L^{r-\text{loc}}(\Omega_1 \times \Omega_2)$.

**Proof.** Let $f \in L^{r-\text{loc}}(\Omega_1 \times \Omega_2)$ be a $[0, \infty]$-valued function. To prove the Lemma, it suffices to show that there is an increasing sequence $(f_n)$ of non-negative rectangular simple functions with support of finite measure such that $\lim_{n \to \infty} f_n(\omega_1, \omega_2) = f(\omega_1, \omega_2)$ $\mu_1 \times \mu_2$-a.e., because then by the Monotone Convergence Theorem

$$
\lim_{n \to \infty} \int_{A \times B} |f - f_n| \, d\mu_1 \times \mu_2 = 0
$$

for any $\mu_1 \times \mu_2$-finite rectangle $A \times B$. By [24] Theorem 1.17 there is an increasing sequence of simple functions $(g_n)$ converging pointwise a.e. to $f$. We may assume that each $g_n$ has $\mu_1 \times \mu_2$-finite support: as $\Omega_1 \times \Omega_2$ is $\sigma$-finite, there is an increasing sequence of finitely measurable sets $(A_n)$ whose union is all of $\Omega_1 \times \Omega_2$. Then the sequence $(g_n \chi_{A_n})$ has all the desired properties.

Let us observe how a simple function with finite-measured support can be approximated by a rectangular simple function of finite support. Fix an $\epsilon > 0$. If $g = \sum_{i=1}^m \chi_{E_i}c_i$ is a simple function with finite-measured support, the measurability of each $E_i$ in $\Omega_1 \times \Omega_2$ implies that there is a set $E'_i \subseteq E_i$ such that $E'_i$ is the union of finitely many rectangles and $\mu_1 \times \mu_2(E_i \setminus E'_i) < \epsilon/2^m$. Then $g' = \sum_{i=1}^m \chi_{E'_i}c_i$ is a rectangular simple function and $g'$ differs from $g$ on a set of measure at most $\epsilon$, and $g' \leq g$.

Starting from the sequence $(g_n)$, define $f_n$ to be the approximation of $g_n$ as described in the previous paragraph such that $\text{supp}(f_{n+1}) \supseteq \text{supp}(f_n)$ and $\text{supp}(g_n) \setminus \text{supp}(f_n) < 1/n$ for all $n \in \mathbb{N}$. Clearly,

$$
\lim_{n \to \infty} f_n(\omega_1, \omega_2) = \lim_{n \to \infty} g_n(\omega_1, \omega_2) = f(\omega_1, \omega_2)
$$

for a.e. $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, proving the Lemma.

Define $\iota_3 : L^{r-\text{loc}}(G \times \Omega) \rightarrow L^{\text{loc}}(\Omega, L^{\text{loc}}(G))$ by setting $f \mapsto F$ where $F(\omega) = f_\omega$ and where $f_\omega(t) := f(t, \omega)$. If $f$ is a rectangular simple function then $f_\omega$ is a simple
function on $G$ for every $\omega \in \Omega$ and so for any subsets $A \subset \Omega$ and $B \subset G$ of finite measure,

$$\int_A \int_B |f_\omega(t)| \, dt \, d\mu(\omega) < \infty,$$

which shows that $\iota_3(f)$ is a simple function in $L^{\text{loc}}(\Omega, L^{\text{loc}}(G))$. Furthermore, as by Lemma 2.8 $f \in L^{\text{loc}}(G \times \Omega)$ is the limit of rectangular simple functions, so is $F := \iota_3(f)$, implying the measurability of $F$. Denoting by $p_B$ the seminorm $g \mapsto \int_B |g| \, dh$ on $L^{\text{loc}}(G)$,

$$\int_A p_B(F(\omega)) \, d\mu(\omega) = \int_A \int_B |f| \, dt \, d\mu(\omega) < \infty,$$

proving the well-definedness and continuity of $\iota_3$. Clearly $\iota_3$ is injective. The mappings $\iota_1, \iota_2$ and $\iota_3$ are related by the following diagram.

The commutativity of this diagram can be checked for simple functions and tensors. As these collections are dense in their respective spaces, the continuity of the arrows yields the desired commutativity.

Now we are in a position to define the transfer operator. We do so first for the case of transferable operators that are sublinear with metrisable range, then for the linear case. Thereafter we show that the two definitions agree for linear transferable operators with metrisable range.

**Definition 2.9.** (Sublinear transfer operators with metrisable range) Let $T$ be a sublinear transferable operator on $L^{\text{loc}}(G)$ with metrisable range. We define the transfer operator $T^\#$ on $L^{1+\infty}(\Omega)$ as the composition

$$L^{1+\infty}(\Omega) \xrightarrow{\iota_3} L^{\text{loc}}(G \times \Omega) \xrightarrow{T} L^{\text{loc}}(\Omega, L^{\text{loc}}(G)) \xrightarrow{\overline{\epsilon}_1} L^{\text{loc}}(\Omega).$$

Here $\overline{T}$ is defined as $\overline{T}(f) = T \circ f$ for all $f \in L^{\text{loc}}(\Omega, L^{\text{loc}}(G))$, $\overline{\epsilon}_1 : C(G) \to \mathbb{C}$ is the evaluation map at $t = 1$ in $G$, and $\overline{\epsilon}_1(g) = \epsilon_1 \circ g$ for all $g \in L^{\text{loc}}(\Omega, C(G))$.

Fix an $f \in L^{\text{loc}}(\Omega, L^{\text{loc}}(G))$. To ensure the well-definedness of $\overline{T}$, note that there is a sequence of simple functions $(f_n)$ converging a.e. to $f$. Then $(\overline{T}(f_n))$ is a sequence of simple functions in $L^{\text{loc}}(\Omega, C(G))$ converging a.e. to $\overline{T}(f)$. To ensure the well-definedness of $\overline{\epsilon}_1$ we invoke the fact that $\overline{T}(f)$ has metrisable range. The Pettis Measurability Theorem 2.6 then implies that as $\overline{T}(f)$ is weakly measurable and $\overline{\epsilon}_1$ is a continuous linear functional on $C(G)$, $\overline{\epsilon}_1 \circ \overline{T}(f)$ is indeed measurable. That it is locally integrable is now easy to confirm.
Definition 2.10. (Linear transfer operator) Let $T$ be a linear transferable operator on $L^{\text{loc}}(G)$. We define the transfer operator $T^\#$ on $L^{1+\infty}(\Omega)$ as the composition

$$
L^{1+\infty}(\Omega) \xrightarrow{\otimes, \alpha, G} L^{\text{loc}}(G) \xrightarrow{\iota_1} L^{\text{loc}}(G) \otimes L^{\text{loc}}(G) \xrightarrow{I \otimes T} L^{\text{loc}}(\Omega) \otimes C(G) \xrightarrow{I \otimes \epsilon_1} L^{\text{loc}}(\Omega).
$$

As in Definition 2.9, $\epsilon_1 : C(G) \to \mathbb{C}$ is the evaluation map at $t = 1$ in $G$, so $I \otimes \epsilon_1$ maps $L^{\text{loc}}(\Omega) \otimes_C C(G)$ to $L^{\text{loc}}(G) \otimes \mathbb{C}$ which is naturally isomorphic to $L^{\text{loc}}(\Omega)$, which explains the slight abuse of notation in the diagram.

Lemma 2.11. Let $T$ be a linear and metrisably valued transferable operator. Then the two Definitions 2.9 and 2.10 agree.

Proof. The proof is encapsulated in the following diagram, which combines the two diagrams of Definitions 2.9 and 2.10.

$$
L^{\text{loc}}(\Omega, L^{\text{loc}}(G)) \xrightarrow{\iota_3} L^{\text{loc}}(\Omega, C(G)) \xrightarrow{\tilde{T}} L^{\text{loc}}(\Omega, C(G)) \xrightarrow{\iota_1} L^{\text{loc}}(\Omega) \otimes C(G) \xrightarrow{I \otimes T} L^{\text{loc}}(\Omega) \otimes_C C(G) \xrightarrow{I \otimes \epsilon_1} L^{\text{loc}}(\Omega).
$$

Its commutativity can be easily checked for simple functions in $L^{r-\text{loc}}(G \times \Omega)$, and so by continuity of all the arrows, the commutativity of the diagram follows. □

There is also a relation between the spaces $L^{\text{loc}}(\Omega, C(G))$ and $L^{\text{loc}}(\Omega) \otimes_C C(G) \simeq C(G, L^{\text{loc}}(\Omega))$ that is of interest.

Lemma 2.12. The space $L^{\text{loc}}(\Omega, C(G))$ may be continuously embedded in $C(G, L^{\text{loc}}(\Omega))$.

Proof. Consider the mapping $\iota_2 : L^{\text{loc}}(\Omega, C(G)) \to L^{\text{loc}}(\Omega) \otimes_C C(G)$ given in Lemma 2.7 the canonical injection $i : L^{\text{loc}}(\Omega) \otimes_C C(G) \to L^{\text{loc}}(\Omega) \otimes C(G)$ and the isomorphism $L^{\text{loc}}(\Omega) \otimes_C C(G) \simeq C(G, L^{\text{loc}}(\Omega))$. The composition of these maps gives us the required continuous linear injection of $L^{\text{loc}}(\Omega, C(G))$ into $C(G, L^{\text{loc}}(\Omega))$. □

Now that the well-definedness of the Transfer Principle has been established and it is clear what countability assumptions are used, let us give a simpler, working definition. We emphasise that this working definition depends on the above analysis for its correctness.

Remark 2.13. Let $T$ be a sublinear transferable operator with metrisable range. Starting with a function $f \in L^{1+\infty}(\Omega)$, define as before $F(t, \omega) := f(\alpha_t(\omega))$ and $F'(t, \omega) := (T(F_\omega))(t)$, where $F_\omega$ is the cross section of $F$ at $\omega \in \Omega$. Finally, we set $T^\#(f) := F'(1, \omega)$. Note that $F'$ as defined above is in $L^{\text{loc}}(\Omega, C(G))$. 
Obviously, the steps followed in this Remark are exactly the steps used in Definition \ref{defn}. As we will be working quite a bit with the functions \( F \) and \( F' \) given above, there is another point that must be made about them. Firstly \( F \) is measurable on \( G \times \Omega \) and indeed rectangular locally integrable as shown in Lemma \ref{lem:rect_loc_int}. For \( F' \), the situation is a little more tricky.

**Lemma 2.14.** Let \( f \) be an element of \( L^{\text{loc}}(\Omega, C(G)) \). Then the function \( f'(t, \omega) := f(\omega)(t) \) defined on \( (G \times \Omega, h \times \mu) \) is measurable.

Furthermore, if \( f \) has metrisable range then \( f' \) satisfies the following restricted Fubini theorem: for any \( A \subseteq \Omega \) of finite measure and compact \( K \subset G \),

\[
\int_A \int_K f'(t, \omega) \ dt \ d\mu = \int_K \int_A f'(t, \omega) \ dt \ d\mu.
\]  

**(2.2)**

**Proof.** The function \( f \) is measurable, which means by definition that there is a sequence \((f_n)\) of \( C(G)\)-valued simple functions converging \( \mu \)-a.e. to \( f \). For a \( C(G)\)-valued simple function \( g \), the function \( g'(t, \omega) := g(\omega)(t) \) is easily seen to be measurable.

Now for \( \mu \)-a.e. \( \omega \in \Omega \), \( f_n(\omega) \to f(\omega) \) in \( C(G) \). In particular, for any \( t \in G \), \( f_n(\omega)(t) \to f(\omega)(t) \). This is synonymous with the expression \( f'_n(t, \omega) \to f'(t, \omega) \), which implies the measurability of \( f' \).

We turn now to the Fubini-type result. Let \( M \) be the subspace (in the relative topology) of all elements in \( L^{\text{loc}}(\Omega, C(G)) \) with metrisable range. Consider the following diagram.

\[
\begin{array}{ccc}
M & \xrightarrow{\int_A} & C(G) \\
\downarrow{\int_K} & & \downarrow{\int_K} \\
L^{\text{loc}}(\Omega) & \xrightarrow{\int_A} & \mathbb{C}
\end{array}
\]

Here \( A \) is a subset of \( \Omega \) of finite measure and \( K \) is a compact subset of \( G \). The maps are naturally defined: \( \int_A : L^{\text{loc}}(\Omega, C(G)) \to C(G) \) sends an \( F \) to \( \int_A F(\omega) \ d\mu \) which is well-defined because of the local integrability of \( F \); \( \int_K : L^{\text{loc}}(\Omega, C(G)) \to L^{\text{loc}}(\Omega) \) sends an \( F \) to \( \tilde{F}(\omega) := \int_K F(\omega) \ d\mu \). The well-definedness of this map depends on the Pettis Measurability Theorem \ref{pettis} because the functional \( f \mapsto \int_K f \ dt \) is a continuous linear functional on \( C(G) \). The maps \( \int_A : L^{\text{loc}}(\Omega) \to \mathbb{C} \) and \( \int_K : C(G) \to \mathbb{C} \) are defined in the obvious way.

So the arrows of the diagram are all continuous linear mappings. It is easy to see that the diagram commutes for all simple functions in \( L^{\text{loc}}(\Omega, C(G)) \). As the simple functions form a dense subset, the commutativity of the diagram and the validity of \( (2.2) \) is proved.

Consequently, \( F' \) is a well-defined measurable function. However, it is not necessarily locally integrable or even rectangular locally integrable. But when working out the maximal inequalities of Section \ref{sec:max_ineq} we shall need only the Fubini result of equation \( (2.2) \).

All the different integrability conditions used in this section can be unified by the following concept. Let \( (\Omega, \Sigma, \mu) \) be a measure space and let \( \mathcal{A} \subset \Sigma \) be an algebra
of measurable sets. A measurable function \( f \) on \( \Omega \) is called \( \mathcal{A} \)-integrable if

\[
\int_A |f| \, d\mu
\]
is finite for every \( A \in \mathcal{A} \). On the product space \((\Omega_1 \times \Omega_2, \Sigma_1 \times \Sigma_2, \mu_1 \times \mu_2)\), when \( \mathcal{A} \) is the algebra of rectangles \( A \times B \) where \( A \) and \( B \) have finite measure, the \( \mathcal{A} \)-
integrable functions on \( \Omega_1 \times \Omega_2 \) are the rectangular locally integrable functions. On \( G \times \Omega \), where \( \mathcal{A} \) consists of all rectangles \( K \times A \) where \( K \subset G \) is compact and \( A \subset \Omega \) has finite measure, the functions considered in Lemma 2.14 are \( \mathcal{A} \)-measurable.

### 2.2. The effect of semilocality and translation invariance

Let \( T \) be a transferable operator with metrisable range. We now describe some approximation properties of \( T^# \) that will be useful in the next section. We start by extending some constructions that we have used earlier. Let \( K \) be any measurable subset of \( G \).

As in equation (2.1), we can define the operator \( \otimes_{\alpha,K} \) on the set of measurable functions on \( \Omega \) by setting \( F_K := \otimes_{\alpha,K} f := \chi_K \otimes f \) on \( G \times \Omega \) using Definition 2.3. Hence

\[
F_K(t, \omega) = \begin{cases} 
  f(\alpha(\omega)) & \text{if } t \in K \\
  0 & \text{if } t \notin K.
\end{cases}
\]  

(2.3)

In this notation, \( F_G = F \). We shall also use the notation \( F_K' = \tilde{T}(F_K) \) where \( \tilde{T} \) is as given in Definition 2.4. In particular, \( F' = F'_G \). By Lemma 2.14, the functions \( F_K' \) are measurable on \( G \times \Omega \).

**Lemma 2.15.** Let \( T \) be a transferable operator with metrisable range and let \( U \) be the open neighbourhood guaranteed by Definition 2.1(2). Let \( K \) and \( E \) be measurable subsets of \( G \) such that \( EU^{-1} \subseteq K \). For any \( f \in L^{1+\infty}(\Omega) \), and almost all \((t, \omega) \in E \times \Omega\), we have

\[
|F'(t, \omega)| \leq |F_K(t, \omega)|.
\]  

(2.4)

**Proof.** First note that \( F_{K'} = (\chi_G - \chi_K) \otimes f = \chi_G \otimes f - \chi_K \otimes f = F - F_K \). Consequently,

\[
|F'| = |\tilde{T}(F)| = |\tilde{T}(F - F_K + F_K)| \\
\leq |\tilde{T}(F_{K'})| + |\tilde{T}(F_K)| \\
= |F_{K'}| + |F'_K|.
\]

By the semilocality of \( T \), for almost every \( \omega \in \Omega \), the measurable map \( t \mapsto F_{K'}(t, \omega) \) has support in \( K^c U \). Because \( EU^{-1} \subseteq K \), it follows that \((K^c U) \cap E\) is empty since \( a \in (K^c U) \cap E \) implies that there is \( a b^{-1} \in K^c \cap (EU^{-1}) \), which is impossible if \( EU^{-1} \subseteq K \). Hence \( |F'(t, \omega)| \leq |F_K(t, \omega)| \) for almost all \( \omega \in \Omega \) and all \( t \in E \).

Translation invariant operators, a class that includes all convolution operators, are automatically equimeasurable-preserving, in a sense made precise by the following Lemma.

**Lemma 2.16.** Let \( T \) be a transferable operator with metrisable range. For any \( f \in L^{1+\infty}(\Omega) \), all \( s, t \in G \) and almost all \( \omega \in \Omega \),

\[
F^s(t, \alpha_s(\omega)) = F^t(ts, \omega).
\]

Moreover, for any \( t_1, t_2 \in G \), the mappings \( \omega \mapsto F^t(t_1, \omega) \) and \( \omega \mapsto F^t(t_2, \omega) \) are equimeasurable.
Proof. By definition of $F$, for $\mu$-a.e. $\omega$ and any $s,t \in G$, we have

$$F(t,\alpha_s(\omega)) = f \circ \alpha_t(\alpha_s(\omega)) = f \circ \alpha_t(\alpha_s(\omega)) = F(ts,\omega).$$

Let $\tau_t : G \to G$ be defined by $\tau_t(s) = ts$ for all $s,t \in G$ as in Definition 2.1. By definition of $F'$ and the translation-invariance of $T$, we have

$$F'(t,\alpha_s(\omega)) = \tilde{T} \circ F(t,\alpha_s(\omega)) = \tilde{T} \circ F(ts,\omega) = \tilde{T} \circ \tau_t \circ F(s,\omega) = \tau_t \circ \tilde{T} \circ F(s,\omega) = F'(ts,\omega).$$

Finally, let $s = t_1^{-1}t_2$ and $\lambda > 0$. Then as $F'(t_2,\omega) = F'(t_1,\alpha_s(\omega))$, we see that

$$\mu(\{\omega : |F'(t_2,\omega)| > \lambda\}) = \mu(\{\omega : |F'(t_1,\alpha_s(\omega))| > \lambda\}) = \mu(\{\alpha_s^{-1}(\omega) : |F'(t_1,\omega)| > \lambda\}),$$

proving the equimeasurability of the maps $\omega \mapsto |F'(t_1,\omega)|$ and $\omega \mapsto |F'(t_2,\omega)|$. \hfill \Box

2.3. Examples. One of the main sources of transferable operators in applications is convolution operators. The straightforward construction of ergodic averaging operators from convolution operators demonstrates the utility and ubiquity of the transfer operator construction.

Suppose that $O$ is a measure-preserving automorphism on the measure space $(\Omega, \mu)$. It induces an action $\alpha$ of $\mathbb{Z}$ on $\Omega$ via $\alpha(n) := O^n$. If $S$ is a finite subset of $\mathbb{Z}$, let $T_S$ be the convolution operator defined on the space of all locally integrable functions $f$ on $\mathbb{Z}$ by

$$T_S(f) := \frac{1}{|S|} \chi_S * f,$$

where of course $\chi_S$ is the characteristic function of $S$. Bearing in mind that the set of locally integrable functions on $\mathbb{Z}$ is precisely the set of all complex-valued functions, we see that $T_S$ is well-defined, linear and takes its values in $C(\mathbb{Z})$, which is metrisable. From the properties of convolution, it is clearly semi-local. Indeed, if $N = \max\{|s| : s \in S\}$ and $f$ is a function with support in $[-M,M]$, then $T_S(f)$ will have support in $[-M-N,M+N]$.

Let us now determine the transfer operator $T_S^\#$. Let $f \in L^{1+\infty}(\Omega)$. With the help of Definition 2.1 and Lemma 2.14 we compute:

$$\tilde{T}_S(F(t,\omega)) = \tilde{T}_S(f(\alpha_t(\omega))) = \frac{1}{|S|} \sum_{s \in S} f(\alpha_t(\omega)).$$

Hence

$$T_S^\#(f)(\omega) = \tilde{e}_0 \circ \tilde{T}_S(F)(\omega) = \frac{1}{|S|} \sum_{s \in S} f(\alpha_s(\omega)),$$

which is a locally integrable function on $\Omega$. (Note that we write $\tilde{e}_0$ above because 0 is the identity element of $\mathbb{Z}$).
3. Rearrangement invariant Banach function spaces

3.1. Basic definitions and constructions. We start by recalling the definition of a rearrangement invariant Banach function space (hereafter referred to as a r.i. BFS) over a resonant measure space \((\Omega, \mu)\). Our source for this material is mainly [2], and also [17].

Given an a.e. finite measurable function \(f\) on \((\Omega, \mu)\), we have already defined the distribution function \(s \mapsto m(f, s) = \mu(\{\omega \in \Omega : |f(\omega)| > s\}\). The decreasing rearrangement \(f^*(s)\) is defined as

\[
 f^*(s) = \sup\{t : m(f, t) \leq s\}.
\]

Note that if \(f\) and \(g\) are equimeasurable functions on \((\Omega, \mu)\) then \(f^*(t) = g^*(t)\) for all \(t \geq 0\). By [2, Definition 2.2.3], the space \((\Omega, \mu)\) is said to be resonant if for each measurable finite a.e. functions \(f\) and \(g\), the identity

\[
 \int_0^\infty f^*(t)g^*(t) \, dt = \sup \int_{\Omega} |f\tilde{g}| \, d\mu
\]

holds as \(\tilde{g}\) ranges over all functions equimeasurable with \(g\).

One also defines a primitive maximal operator \(f \mapsto f^{**}\) as

\[
 f^{**}(s) = \frac{1}{s} \int_0^s f^*(t) \, dt.
\]

We call \(f^{**}\) the double decreasing rearrangement of \(f\).

When the function norm \(\rho\) that defines the Banach function spaces has the property that \(\rho(f) = \rho(g)\) for all equimeasurable functions \(f\) and \(g\), the Banach space is called rearrangement invariant - see [2, Definitions 1.1, 4.1].

For any r.i. BFS \(X\) we define another r.i. BFS \(X'\), called the associate space, to be the subset of the a.e.-finite measurable functions \(f\) on \((\Omega, \mu)\) for which \(\|f\|_{X'}\) is finite, where

\[
 \|f\|_{X'} = \sup \{ |\int_\Omega f(\omega)g(\omega) \, d\mu(\omega)| : g \in X, \|g\|_X \leq 1\}.
\]

We shall also have need of another Banach space \(X_b \subseteq X\), which is the closure in \(X\) of the set of all simple functions in \(X\). This is not in general a r.i. BFS itself, but is useful for the role it plays in the duality theory.

Associated with any rearrangement invariant BFS \(X\), there is a fundamental function \(\varphi_X : [0, \infty) \to [0, \infty)\) defined by

\[
 \varphi_X(t) = \|\chi_E\|_X,
\]

where \(E\) is any subset of \(\Omega\) such that \(\mu(E) = t\). By the rearrangement invariance of \(X\), this function is well-defined. It is a quasi-concave function, as explained in [2, Definition 2.5.6]. Such functions are automatically subadditive and continuous on \((0, \infty)\). We denote by \(\varphi_X^*\) the associate fundamental function of \(\varphi_X\), where \(\varphi_X^* = \varphi_X'\). We shall often make use of the identity

\[
 \varphi_X(t)\varphi_X^*(t) = t
\]

for all \(t \geq 0\), as proved in [2, Theorem 2.5.2].

Recall that a Young’s function is a convex, nondecreasing function \(\Phi : [0, \infty) \to [0, \infty]\) for which \(\Phi(0) = 0\), \(\lim_{x \to \infty} \Phi(x) = \infty\) and which is neither identically zero nor infinite valued on all of \((0, \infty)\).
One class of spaces that we shall study is that of Orlicz spaces. The theory of these important spaces, which include the standard $L^p$-spaces, is developed in [21]. In [2] and [17], they are also studied in some depth. We bring to mind the most salient features of their construction. The Luxemburg norm $\| \cdot \|_{L(\Phi)}$ is defined by a Minkowski functional on the set of all finite a.e. measurable functions on $(\Omega, \mu)$ by the formula

$$\| f \|_{L(\Phi)} = \inf \{ k^{-1} : \int_\Omega \Phi(k|f|) \, d\mu \leq 1 \}.$$ 

The set of all $f$ for which $\| f \|_{L(\Phi)} < \infty$ is the Orlicz space $L(\Phi)$.

There is also another Young’s function, called the complementary Young’s function. This is the function $\Psi$ defined by

$$\Psi(x) = \sup_{y > 0} \{ xy - \Phi(y) \}.$$ 

Using this complementary Young’s function, it is possible to define another, equivalent norm on the Orlicz space $L(\Phi)$. To this end, define the Orlicz norm $\| \cdot \|_{L(\Phi)}$ on the space of measurable functions $f$ on $(\Omega, \mu)$ by setting

$$\| f \|_{L(\Phi)} = \sup \left\{ \int_0^\infty f^*(s) g^*(s) \, ds : \| g \|_{L(\Psi)} \leq 1 \right\}. \quad (3.1)$$

Now the Orlicz and Luxemburg norms on the Orlicz space $L(\Phi)$ are equivalent. In fact it is proved in [2, Theorem 4.8.14] that

$$\| f \|_{L(\Phi)} \leq \| f \|_{L(\Phi)} \leq 2 \| f \|_{L(\Phi)}. \quad (3.2)$$

Note that for an Orlicz space $L(\Phi)$ equipped with the Luxemburg norm, its fundamental function $\varphi$ is related to $\Phi$ by the equation

$$\varphi(t) = 1 / \Phi^{-1}(1/t) \quad (3.3)$$

for all $0 < t \leq |\Omega|$ as shown in [2] Lemma 4.8.17. In the sequel, given a Young’s function $\Phi$ we define the fundamental function associated to $\Phi$ to be the quasiconcave function defined by (3.3).

Given a r.i. BFS $X$ over $(\Omega, \mu)$, we canonically associate two other r.i. BFSs with $X$, apart from the Orlicz space. If $\varphi_X$ is the fundamental function associated with $X$, define the first space $M(X)$ to consist of all the measurable functions $f$ over $\Omega$ such that

$$\| f \|_{M(X)} = \sup_{s > 0} f^{**}(s) \varphi_X(s) < \infty.$$ 

The space $M(X)$ is a Banach space with the norm $\| \cdot \|_{M(X)}$. This is the largest r.i. BFS with fundamental function $\varphi_X$. In other words, if $Y$ is any other r.i. BFS with fundamental function $\varphi_Y$, then $Y$ is contractively embedded in $M(X)$. Note that as the quasiconcave function $\varphi_X$ is the only property of $X$ required to construct $M(X)$, we may just as well denote this space by $M(\varphi_X)$.

The second space is the associate of $M(X')$. We define $\Lambda(X)$ to consist of all measurable functions $f$ over $\Omega$ such that

$$\| f \|_{\Lambda(X)} = \sup \left\{ \int_0^\infty f^*(s) g^*(s) \, ds : \| g \|_{M(\varphi_X)} \leq 1 \right\} < \infty.$$ 

The set $\Lambda(X)$ is a Banach space with norm $\| \cdot \|_{\Lambda(X)}$.

As in the case of $M(X)$, we can just as well write $\Lambda(\varphi_X)$, as $\varphi_X$ is the only property of $X$ employed in the construction of $\Lambda(X)$. This is the smallest r.i. BFS
with fundamental function $\varphi_X$; if $Y$ is any other r.i. BFS with this fundamental function then there is a continuous injection of $\Lambda(X)$ into $Y$.

There is another function space, denoted $M^*(X)$, that can be constructed from a given r.i. BFS $X$. In general, this is not a Banach space and consists of all those finite a.e. measurable functions $f$ on $(\Omega, \mu)$ for which the norm $\| \cdot \|_{M^*(X)}$ defined by

$$\| f \|_{M^*(X)} = \sup_{s > 0} f^*(s) \varphi_X(s)$$

is finite. Again, note that the only property of $X$ required for this construction is its fundamental function $\varphi_X$. While $M^*(X)$ is not necessarily a Banach space, it is a quasi-Banach space, in that $\| f \|_{M^*(X)} = 0$ if and only if $f = 0$ a.e., $\| \lambda f \|_{M^*(X)} = |\lambda| \| f \|_{M^*(X)}$ for all complex $\lambda$ and

$$\| f + g \|_{M^*(X)} \leq 2(\| f \|_{M^*(X)} + \| g \|_{M^*(X)})$$

for all $f, g \in M^*(X)$. This space was introduced in [26] - see also [2, Ch. 4, exercise 21].

We provide a useful equivalent definition of the $M^*(X)$-norm.

**Lemma 3.1.** If $\varphi$ is a fundamental function, then $\sup_{t > 0} f^*(t) \varphi(t) = \sup_{s > 0} s \varphi(m(f, s))$.

**Proof.** We follow [11] Proposition 1.4.5.16, p46]. Given $s > 0$, pick $\epsilon \in (0, s)$. As $f^*(m(f, s) - \epsilon) > s$,

$$\sup_{t > 0} f^*(t) \varphi(t) \geq f^*(m(f, s) - \epsilon) \varphi(m(f, s) - \epsilon) > s \varphi(m(f, s) - \epsilon).$$

Because $\varphi$ is continuous on $(0, \infty)$, as $\epsilon \to 0$, we obtain

$$\sup_{t > 0} f^*(t) \varphi(t) \geq s \varphi(m(f, s))$$

for all $s > 0$, which proves that $\sup_{t > 0} f^*(t) \varphi(t) \geq \sup_{s > 0} s \varphi(m(f, s))$.

Conversely, given $t > 0$, if $f^*(t) > 0$ pick $\epsilon \in (0, f^*(t))$. Then $m(f, f^*(t) - \epsilon) > t$, meaning that

$$\sup_{s > 0} s \varphi(m(f, s)) \geq (f^*(t) - \epsilon) \varphi(m(f, f^*(t) - \epsilon)).$$

As $\varphi$ is nondecreasing, we have $(f^*(t) - \epsilon) \varphi(m(f, f^*(t) - \epsilon)) \geq (f^*(t) - \epsilon) \varphi(t)$.

Letting $\epsilon \to 0$, we obtain

$$\sup_{s > 0} s \varphi(m(f, s)) \geq f^*(t) \varphi(t)$$

for all $t > 0$. If $f^*(t) = 0$, this inequality is trivially satisfied.

Hence $\sup_{t > 0} f^*(t) \varphi(t) \leq \sup_{s > 0} s \varphi(m(f, s))$. \qed

Let us make some remarks on operators between function spaces, in particular on the weak type of an operator. There are two standard definitions of this concept. Let $X$ and $Y$ be rearrangement invariant BFSs. We say that a sublinear operator $T$ has Marcinkiewicz weak type $(X, Y)$ if $T$ maps $X$ into $M^*(Y)$ and that $T$ has Lorentz weak type $(X, Y)$ if it maps $\Lambda(X)$ into $M^*(Y)$. Clearly if $T$ is of Marcinkiewicz weak type $(X, Y)$ then it is of Lorentz weak type $(X, Y)$. In the sequel, we shall write ‘weak type’ for ‘Marcinkiewicz weak type.’

For an operator $T$ of weak type $(X, Y)$ one can write $\| T f \|_{M^*(Y)} \leq c \| f \|_X$. The smallest value of $c$ for which this equation holds is called the norm of $T$. 

As we shall mostly be working with \( \Lambda \)-, \( M \)- and Orlicz-spaces, let us fix some terminology for dealing with the weak types associated with these kinds of spaces.

**Definition 3.2. (Weak type)** Let \( \Phi_A \) and \( \Phi_B \) be Young’s functions with associated fundamental functions \( \varphi_A \) and \( \varphi_B \) respectively. We say that a sublinear operator \( T \) has \( \Lambda \)-, \( M \)- or \( L \)-weak type \((\varphi_A, \varphi_B)\) if it respectively maps \( \Lambda(\Phi_A), M(\Phi_A) \) or \( L(\Phi_A) \) into \( M^*(\varphi_B) \).

Bear in mind that if an operator is of \( M \)-weak type \((\varphi_A, \varphi_B)\), then it is automatically of \( L \)- and \( \Lambda \)-weak types \((\varphi_A, \varphi_B)\) too.

### 3.2. Comparison of fundamental functions.

It should be quite clear that a lot rests upon the analysis of the various fundamental functions associated with r.i. BFSs. Indeed, based on techniques for comparing quasiconcave functions developed in the sequel, we will derive our results on the weak type of the transferred operator.

The growth properties of a Young’s function have great bearing on the properties of the associated Orlicz space. The same holds more generally for the growth properties of a fundamental function and its associated BFSs. This part of the work will be concerned with translating some standard growth conditions on Young’s functions into conditions on fundamental functions. Then we shall analyse these conditions in terms of inequalities prominent in O’Neil’s work [17].

Recall from [21] that a Young’s function \( \Phi \) is said to satisfy the \( \Delta_2 \) condition (globally), denoted \( \Phi \in \Delta_2 \) (\( \Phi \in \Delta_2 \) (globally)) if

\[
\Phi(2x) \leq K \Phi(x), \quad x \geq x_0 \geq 0 \quad (x_0 = 0)
\]

for some absolute constant \( K > 0 \). The Young’s function \( \Phi \) satisfies the \( \nabla_2 \) condition (globally), denoted \( \Phi \in \nabla_2 \) (\( \Phi \in \nabla_2 \) (globally)) if

\[
\Phi(x) \leq \frac{1}{2\ell} \Phi(\ell x), \quad x \geq x_0 \geq 0 \quad (x_0 = 0)
\]

for some \( \ell > 1 \). A subclass of the Young’s functions satisfying the \( \Delta_2 \) condition, are those satisfying the so-called \( \Delta' \) condition. Specifically a Young’s function \( \Phi \) is satisfies the \( \Delta' \) condition (globally),

\[
\Phi(xy) \leq K \Phi(x)\Phi(y), \quad x, y \geq x_0 \geq 0 \quad (x_0 = 0)
\]

for some absolute constant \( K > 0 \). The class of Orlicz spaces associated with such Young’s functions embody many of the properties of the classical \( L^p \)-spaces.

We now state these definitions in terms of fundamental functions.

**Definition 3.3.** A fundamental function \( \varphi \) is said to satisfy the \( \Delta_2 \) condition (globally), denoted \( \varphi \in \Delta_2 \) (\( \varphi \in \Delta_2 \) (globally)) if

\[
\varphi(Kx) \geq 2\varphi(x), \quad x \geq x_0 \geq 0 \quad (x_0 = 0)
\]

(3.4)

for some absolute constant \( K > 0 \).

A fundamental function \( \varphi \) is said to satisfy the \( \nabla_2 \) condition (globally), denoted \( \varphi \in \nabla_2 \) (\( \varphi \in \nabla_2 \) (globally)) if

\[
\varphi(x) \geq \frac{2}{\ell} \varphi(\ell x), \quad x \geq x_0 \geq 0 \quad (x_0 = 0)
\]

(3.5)

for some absolute constant \( \ell > 1 \).
Suppose that \( \varphi \) satisfies the \( \Delta_2 \) condition. Note that as \( \varphi \) is nondecreasing, if \( K < 1 \), then \( \varphi(Kx) < \varphi(x) \), so \( 2\varphi(x) < \varphi(x) \) for every \( x \geq x_0 \) - a contradiction as \( \varphi(x) = 0 \) if and only if \( x = 0 \). Hence \( K \geq 1 \). But then \( \varphi(Kx) \leq K\varphi(x) \), which follows from the nonincreasing behaviour of \( x \mapsto \varphi(x) / x \). Hence \( 2\varphi(x) \leq \varphi(x) \) for every \( x \geq x_0 \), so \( 1 \geq 2/\ell \), which implies that \( \ell \geq 2 \).

Likewise, we can provide a crude bound for \( \ell \) in the case that \( \varphi \) satisfies the \( \nabla_2 \) condition. As \( \ell > 1 \), \( \varphi(\ell x) \geq \varphi(x) \geq 2\ell \varphi(\ell x) \) for every \( x \geq x_0 \), so \( 1 \geq 2/\ell \), which implies that \( \ell \geq 2 \).

Note that the same reasoning applies to the selection of \( K \) and \( \ell \) for Young’s functions respectively satisfying the conditions \( \Delta_2 \) and \( \nabla_2 \). Indeed, let \( \Phi \in \Delta_2 \). By [21, Corollary I.3.2], we may write
\[
\Phi(x) = \int_0^x \phi(t) \, dt
\]
for some nondecreasing left continuous function \( \phi : [0, \infty) \to [0, \infty] \) for which \( \phi(0) = 0 \). Hence
\[
\Phi(2x) = \int_0^x \phi(t) \, dt + \int_x^{2x} \phi(t) \, dt 
\geq \int_0^x \phi(t) \, dt + \int_x^x \phi(t) \, dt 
= 2\Phi(x)
\]
on account of the fact that \( \phi \) is nondecreasing. Therefore \( 2\Phi(x) \leq \Phi(2x) \leq K\Phi(x) \), implying that \( K \geq 2 \).

If \( \Phi \in \nabla_2 \) and \( \Phi(x) \leq (1/2\ell)\Phi(\ell x) \) for some \( \ell < 2 \), note that by the convexity of \( \Phi \) and the identity \( \Phi(0) = 0 \),
\[
\Phi(\ell x) = \Phi((1-\ell/2)0 + (\ell/2)2x) \leq \frac{\ell}{2} \Phi(2x).
\]
Consequently, \( \Phi(x) \leq (1/2\ell)\Phi(\ell x) \leq (1/2\ell)(\ell/2)\Phi(2x) = (1/4)\Phi(2x) \). So \( \Phi \) satisfies the \( \nabla_2 \) condition with \( \ell = 2 \).

**Proposition 3.4.** Suppose that the quasiconcave function \( \varphi \) satisfies the \( \Delta_2 \) condition globally. Then there exist constants \( 1 \geq \epsilon > 0 \) and \( A > 0 \) such that for all \( y \geq 1 \), \( 0 < p < \epsilon \) and \( x \in \mathbb{R}^+ \),
\[
\varphi(yx) \geq Ay^p \varphi(x).
\]

Furthermore, if \( \varphi \) satisfies the \( \nabla_2 \) condition globally, then there exist constants \( 1 > \epsilon \geq 0 \) and \( B > 0 \) such that for all \( y \geq 1 \), \( \epsilon \leq p < 1 \) and \( x \in \mathbb{R}^+ \), it holds that
\[
\varphi(yx) \leq By^p \varphi(x).
\]

**Proof.** Suppose that \( \varphi \) satisfies the \( \Delta_2 \) condition. As noted in the remarks preceding this Proposition, we can choose a \( K \geq 2 \) such that (4.4) holds. Set \( \epsilon = \ln 2 / \ln K \) and let \( y \geq 1 \). By iterating the definition of the \( \Delta_2 \) condition, we see that for any \( n \in \mathbb{N} \) and \( x \geq 0 \),
\[
2^n \varphi(x) \leq \varphi(K^n x).
\]
There is a natural number \( n \) such that \( K^{n-1} \leq y < K^n \). Define \( \lambda \) such that \( y/K^n = \lambda \). Then \( 1/K \leq \lambda < 1 \), which allows the following computation:

\[
\varphi(y x) = \varphi(\lambda K^n x) \geq \lambda \varphi(K^n x) \geq \lambda^2 \varphi(x).
\]

(3.6)

Now \( y/\lambda = K^n \) implies that \( n = \ln y - \ln \lambda / \ln K \) and

\[
2^n = e^{n \ln 2} = e^{\ln K \cdot (\ln 2 - \ln \lambda / \ln K)} = y^{\ln 2 / \ln K} \lambda^{-\ln 2 / \ln K} = \lambda^2 n = y^{\ln 2 / \ln K} \lambda^{-\ln 2 / \ln K}.
\]

Setting \( A = (1/K)^{1-\ln 2 / \ln K} \), we see that

\[
\lambda^{2n} \geq Ay^p.
\]

(3.7)

Combining (3.6) and (3.7), for any \( 0 < p < \epsilon \) and \( y \geq 1 \), we have

\[
\varphi(y x) \geq Ay^p \varphi(x).
\]

Now suppose that \( \varphi \) satisfies the \( \nabla_2 \) condition, that is, inequality (3.5) with \( \ell \geq 2 \). Define \( \epsilon = 1 - \ln 2 / \ln \ell \) and let \( y \geq 1 \). There is a natural number \( n \) such that \( \ell^{n-1} \leq y < \ell^n \). Set \( \lambda = y/\ell^n \). Then \( 1/\ell \leq \lambda < 1 \) and

\[
\varphi(y x) = \varphi(\lambda \ell^n x) \leq \varphi(\ell^n x) \leq \ell^n / 2n \varphi(x).
\]

(3.8)

Now \( n = \ln y - \ln \lambda / \ln \ell \), so

\[
\frac{\ell^n}{2n} = e^{n \ln \ell - n \ln 2} = e^{\ln y - \ln \lambda - (\ln y - \ln \lambda) \ln 2 / \ln K} = e^{(1 - \ln 2 / \ln K) \ln y / \lambda} = y^{1 - \ln 2 / \ln K} \lambda^{\ln 2 / \ln K} - 1.
\]

(3.9)

Because \( 0 < \ln 2 / \ln \ell \leq 1 \) and \( 1/\ell \leq \lambda < 1 \), we have that \( \lambda^{\ln 2 / \ln \ell} \leq \ell^{1-\ln 2 / \ln \ell} \). Hence for any \( 1 > p \geq 1 - \ln 2 / \ln \ell = \epsilon \), from (3.9) we have

\[
\frac{\ell^n}{2n} \leq By^p,
\]

where \( B = \ell^{1-\ln 2 / \ln \ell} \). Substituting this into (3.8) yields the result.

The inequalities \( \varphi(y x) \geq Ay^p \varphi(x) \) and \( \varphi(y x) \leq By^p \varphi(x) \) obtained in the Proposition above are instances of tail growth conditions: they are valid for \( y \) large enough. A larger class of fundamental functions satisfying such conditions is provided in [26], which we now recall.

**Definition 3.5.** We define two classes of fundamental functions as follows.

1. \( \varphi \in U \) if for some \( 0 < \alpha < 1 \), there are positive constants \( A \) and \( \delta \) such that

\[
\varphi(ts) \leq At^\alpha \varphi(s) \text{ if } t \geq \delta.
\]

The \( U \)-index of \( \varphi \), denoted \( \rho_U^\varphi \), is the infimum of all \( \alpha \) for which the above inequality obtains.
Lemma 3.6. Let \( \varphi \) be a fundamental function.

(1) \( \varphi \in U \) if and only if \( A\varphi(uv) \geq v^\alpha \varphi(u) \) for some constants \( A, \delta > 0 \) and \( u > 0, v \leq \delta \).

(2) \( \varphi \in L \) if and only if \( A\varphi(uv) \leq v^\alpha \varphi(u) \) for some constants \( A, \delta > 0 \) and \( u > 0, v \leq \delta \).

Moreover, by adjusting \( A \) if necessary, we may always take \( \delta = 1 \).

Proof. Suppose \( \varphi \in U \) so that \( \varphi(ts) \leq A t^\alpha \varphi(s) \) if \( t \geq \delta \). By the change of variables \( u = ts \) and \( v = 1/t \), we get \( A\varphi(uv) \geq v^\alpha \varphi(u) \) for \( u > 0, v \leq 1/\delta \). Part (2) is treated with the same change of variables.

For the last part, suppose \( \varphi \in U \) so that \( \varphi(ts) \leq A t^\alpha \varphi(s) \) if \( t \geq \delta \). If \( \delta \leq 1 \), then this inequality is certainly true for all \( t \geq 1 \). If \( \delta > 1 \), then for all \( t \geq 1, s > 0 \),

\[
\varphi(ts) \leq \varphi(\delta ts) \leq A(\delta t)^\alpha \varphi(s)
\]

because \( \varphi \) is nondecreasing. Setting \( A' = A\delta^\alpha \), we have proven that \( \varphi(ts) \leq A't^\alpha \varphi(s) \) if \( t \geq 1 \). The same reasoning can be used for the case \( \varphi \in L \). \( \square \)

We have thus shown that \( U \) and \( L \) defined above are identical to the classes as defined in [26]. Furthermore, the \( U \)- and \( L \)-indices of \( \varphi \) are equal to the fundamental indices. We will prove this in Section 5. We pause to note that for a r.i. BFS \( X \) with fundamental function \( \varphi \), on setting \( M(t,X) = \sup_{s>0} \varphi(st)/\varphi(s) \), Zippin [29] defined the fundamental indices as

\[
\beta_X = \lim_{t \to 0^+} \frac{\ln M(t,X)}{\ln t}, \quad \beta_X = \lim_{t \to \infty} \frac{\ln M(t,X)}{\ln t}.
\]

These indices were then further discussed and analyzed in [26].

Proposition 3.4 shows that if \( \varphi \) satisfies the \( \Delta_2 \) condition, then \( \varphi \in L \) and if \( \varphi \) satisfies the \( \nabla_2 \) condition then \( \varphi \in U \).

3.3. Estimates of integrals and function norms. When working with maximal inequalities, there are certain integrals that we will need to estimate. The following Proposition covers the cases that we will need.

First, some terminology, following [14]: consider a measure space \((\Omega, \Sigma, \mu)\) and a countable collection \( D \subset \Sigma \) of measurable subsets of \( \mu \)-finite measure. The \( \sigma \)-algebra \( \sigma(D) \) generated by \( D \) is contained in \( \Sigma \). If for any \( F \in \Sigma \) there is a \( D \in \sigma(D) \) such that \( F \Delta D \) has null measure, where \( F \Delta D \) denotes the symmetric difference between \( D \) and \( F \), we say that \((\Omega, \Sigma, \mu)\) is countably generated modulo null sets, or just countably generated. We call \( D \) the generators of \( \Sigma \). Moreover, we may assume that \( D \) is an algebra, for if \( D \) is countable, so is the algebra it generates. If \( D \) is an algebra of sets that generates \( \Sigma \) in the above sense, it is easy to see that if \( F \subset \Omega \) is any \( \mu \)-finite subset and \( \epsilon > 0 \), then there is a \( D \in D \) such that \( \mu(D \Delta F) < \epsilon \) and \( |\mu(D) - \mu(F)| < \epsilon \).
Proposition 3.7. Let \((\Omega_1, \mu_1)\) and \((\Omega_2, \mu_2)\) be resonant spaces with \(\Omega_2\) countably generated. Let \(\Phi_A, \Phi_B\) and \(\Phi_C\) be Young’s functions and \(\varphi_A, \varphi_B\) and \(\varphi_C\) be their respective associated fundamental functions satisfying

\[
\theta \varphi_A(st) \geq \varphi_B(s) \varphi_C(t)
\]

(3.10)

for all \(s, t > 0\) and some \(\theta > 0\). Let \(f\) be a measurable function on \(\Omega_1 \times \Omega_2\) and \(E \subset \Omega_1\) a subset of finite measure.

1) If \(f \in M(\Phi_A)\), then

\[
\varphi_C(|E|) \int_E \|f\|_{M(\Phi_B)} \, d\mu_1(\omega_1) \leq 4e^3 \theta \|f\|_{M(\Phi_A)}.
\]

2) If \(f \in \Lambda(\Phi_A)\) and \(\lim_{t \to 0} \varphi_B^*(t) = 0\), then

\[
\frac{\varphi_C(|E|)}{|E|} \int_E \|f\|_{\Lambda(\Phi_B)} \, d\mu_1(\omega_1) \leq 6 \theta \|f\|_{\Lambda(\Phi_A)}.
\]

3) If \(f \in L(\Phi_A)\) and \(\lim_{t \to 0} \varphi_B^*(t) = 0\), then

\[
\frac{\varphi_C(|E|)}{|E|} \int_E \|f\|_{L(\Phi_B)} \, d\mu_1(\omega_1) \leq \theta \|f\|_{L(\Phi_A)}.
\]

As the proof of this Proposition relies heavily on [17, Theorem 8.18], it is worth mentioning that the condition on the fundamental functions given there, namely \(\Phi_A^{-1}(st) \Phi_B^{-1}(t) \leq \theta t \Phi_C^{-1}(s)\), can with the help of (3.3) and the identity \(\varphi_B(t) \varphi_B^*(t) = t\) be written in the equivalent form

\[
\theta \varphi_A(st) \geq \varphi_B(t) \varphi_C(s).
\]

Proof. Let \(D\) be a countable algebra that generates \((\Omega_2, \mu_2)\).

Suppose \(f \in M(\Phi_A)\). For any subset \(\Delta \subset \Omega_2\) of finite measure, define \(h_\Delta\) by

\[
h_\Delta(\omega_1) = \frac{1}{\varphi_B^*(|\Delta|)} \int_\Delta |f(\omega_1, \omega_2)| \, d\mu_2(\omega_2).
\]

Thus \(h_\Delta(\omega_1) = \int_{\Omega_2} |f|/\varphi_B^*(|\Delta|) \, d\mu_2\). Note that (3.10) can be written in the form

\[
\theta \varphi_A(st) \geq \varphi_B^*(s) \varphi_C(t)
\]

because for any fundamental function \(\varphi\), \((\varphi^*)^* = \varphi\). We apply [17, Theorem 8.18, part 1°] to conclude that \(h_\Delta \in M(\Phi_C)\), with \(\|h_\Delta\|_{M(\Phi_C)} \leq 4e^3 \theta \|f\|_{M(\Phi_A)}\). We also used the obvious fact that \(\|\chi_\Delta/\varphi_B^*(|\Delta|)\|_{\Lambda(\varphi_B^*)} = 1\).

Now define

\[
\hat{h} = \sup_{\Delta \in D} h_\Delta.
\]

As \(\hat{h}\) is the supremum of a countable number of functions, it is itself a measurable function.

For any \(\Delta \in D\) and \(\mu_1\)-almost every \(\omega_1 \in \Omega_1\),

\[
\tilde{h}(\omega_1) = \frac{1}{\varphi_B^*(|\Delta|)} \int_\Delta |f_{\omega_1}| \, d\mu_2 = \frac{1}{|\Delta|} \int_\Delta |f_{\omega_1}| \, d\mu_2 \varphi_B(|\Delta|)
\]

\[
\leq f_{\omega_1}^*(|\Delta|) \varphi_B(|\Delta|) \leq \|f_{\omega_1}\|_{M(\Phi_B)},
\]

by definition of the norm \(\|\cdot\|_{M(\Phi_B)}\). Hence \(\tilde{h}(\omega_1) \leq \|f_{\omega_1}\|_{M(\Phi_B)}\) \(\mu_1\) a.e.
On the other hand for any fixed $\epsilon > 0$, there is a $t > 0$ such that $f^{**}_{\omega_1}(t)\varphi_B(t) > \|f_{\omega_1}\|_{M(\Phi_B)} - \epsilon$. As $(\Omega_2, \mu_2)$ is a resonant space, by [2 Proposition 2.3.3], there is a subset $F$ such that $|F| = t$ and

$$\frac{1}{|F|} \int_F |f_{\omega_1}| \, d\mu_2 > f^{**}_{\omega_1}(t) - \epsilon/\varphi_B(t).$$

Hence

$$\frac{1}{\varphi_B(|F|)} \int_F |f_{\omega_1}| \, d\mu_2 > f^{**}_{\omega_1}(t)\varphi_B(t) - \epsilon.$$ 

Because $\mathcal{D}$ is dense in the Borel $\sigma$-algebra, there is a $\Delta \in \mathcal{D}$ such that

$$\left| \frac{1}{\varphi_B(|\Delta|)} \int_{\Delta} |f_{\omega_1}| \, d\mu_2 - \frac{1}{\varphi_B(|F|)} \int_F |f_{\omega_1}| \, d\mu_2 \right| < \epsilon.$$

Therefore

$$h_{\Delta}(\omega_1) = \frac{1}{\varphi_B(|\Delta|)} \int_{\Delta} |f_{\omega_1}| \, d\mu_2 > \frac{1}{\varphi_B(|F|)} \int_F |f_{\omega_1}| \, d\mu_2 - \epsilon > f^{**}_{\omega_1}(t)\varphi_B(t) - 2\epsilon > \|f_{\omega_1}\|_{M(\Phi_B)} - 3\epsilon,$$

whence

$$\tilde{h}(\omega_1) > \|f_{\omega_1}\|_{M(\Phi_B)} - 3\epsilon.$$ 

As $\epsilon > 0$ was arbitrary, it is clear that $\tilde{h}(\omega_1) \geq \|f_{\omega_1}\|_{M(\Phi_B)}$. So we have proved that $\tilde{h}(\omega_1) = \|f_{\omega_1}\|_{M(\Phi_B)}$ for almost all $\omega_1 \in \Omega_1$. Also, $\|\tilde{h}\|_{M(\Phi_C)} \leq 4e^3\theta\|f\|_{M(\Phi_A)}$ because as we have already shown, $\|h_{\Delta}\|_{M(\Phi_C)} \leq 4e^3\theta\|f\|_{M(\Phi_A)}$ for all $\Delta \in \mathcal{D}$. Combining these two facts yields part 1) of the Proposition.

For the second part, we shall follow a similar strategy to that of the first part. Consider the space $M(\varphi_B^*)_b$ over $\Omega_2$, which is the closure of the space of all simple functions in $M(\varphi_B^*)$ whose support has finite measure. The condition $\lim_{t \to 0} \varphi_B^*(t) = 0$ means that by [2 Theorem 2.5.5], $M(\varphi_B^*)_b$ is separable and that $(M(\varphi_B^*)_b)^* = \Lambda(\Phi_B)$. Let $\mathcal{D}$ be a countable dense subset of the unit ball of $M(\varphi_B^*)_b$. By the above remarks, this is a norming set for $\Lambda(\Phi_B)$, in that for any $g \in \Lambda(\Phi_B)$, we have

$$\|g\|_{\Lambda(\Phi_B)} = \sup_{\delta \in \mathcal{D}} \int_{\Omega_2} |g(\omega_2)\delta(\omega_2)| \, d\mu_1(\omega_2).$$

Now for each $\delta \in \mathcal{D}$, define the functions

$$h_\delta(\omega_1) = \int_{\Omega_2} |f(\omega_1, \omega_2)\delta(\omega_2)| \, d\mu_2(\omega_2)$$

$$\tilde{h}(\omega_1) = \sup_{\delta \in \mathcal{D}} h_\delta(\omega_1).$$

Note that as $\tilde{h}$ is the supremum of a countable number of measurable functions, it is itself measurable.

By [14 Theorem 8.18, part 3)], $\|h_\delta\|_{L(\Phi_C)} \leq 6\theta\|f\|_{\Lambda(\Phi_A)}\|\delta\|_{M(\varphi_B^*)} \leq 6\theta\|f\|_{\Lambda(\Phi_A)}$. Hence $\|\tilde{h}\|_{L(\Phi_C)} \leq 6\theta\|f\|_{\Lambda(\Phi_A)}$. 
On the other hand, for each \( \omega_1 \in \Omega_1 \),

\[
\tilde{h}(\omega_1) = \sup_{\delta \in \mathcal{D}} \int_{\Omega_2} |f(\omega_1, \omega_2)\delta(\omega_2)| \, d\mu_2(\omega_2) = \|f_{\omega_1}\|_{L(\Phi_B)}
\]

where the last equality is true on account of \( \mathcal{D} \) being a norming subset of \( M(\varphi_B^*)_b \) for \( L(\Phi_B) \).

Hence if \( E \subset \Omega_1 \) is any set of finite measure, then by Hölder’s inequality

\[
\frac{\varphi_C(|E|)}{|E|} \int_E \|f_{\omega_1}\|_{L(\Phi_B)} \, d\mu_1(\omega_1) \leq \frac{\varphi_C(|E|)}{|E|} \|\tilde{h}\|_{L(\Phi_C)} \|\chi_E\|_{L(\Phi_C)} = \|\tilde{h}\|_{L(\Phi_C)} \leq 6\theta \|f\|_{L(\Phi_A)},
\]

proving part 2).

For the third part, let \( \Psi_B \) denote the Young’s function complementary to \( \Phi_B \) and note that because \( L(\Phi_B) \) is an Orlicz space with the Luxemburg norm, its associate space is the Orlicz space \( L(\Psi_B) \) under the Orlicz norm and with fundamental function \( \varphi_B^* \).

The proof now proceeds as for the second part. Because \( \lim_{t \to \infty} \varphi_B^*(t) = 0 \), by \cite{2} Theorem 2.5.5, \( L(\Psi_B) \) is separable and that \( (L(\Psi_B))^* = L(\Phi_B) \). Let \( \mathcal{D} \) be a countable dense subset of the unit ball of \( L(\Psi_B) \) and define as before the functions \( h_\delta \) and \( \tilde{h} \). By \cite{17} Theorem 8.18, part 2”,

\[
\|h_\delta\|_{L(\Phi_C)} \leq \theta \|f\|_{L(\Phi_A)} \|\delta\|_{L(\Phi_B)} \leq \theta \|f\|_{L(\Phi_A)} \|\tilde{h}\|_{L(\Phi_C)} \leq \theta \|f\|_{L(\Phi_A)},
\]

where \( \|\cdot\|_{L(\Phi_B)} \) and \( \|\cdot\|_{L(\Phi_B)} \) denote the Luxemburg and Orlicz norms respectively and we used the fact that by \( \text{(3.2)} \), \( \|\delta\|_{L(\Phi_B)} \leq \|\tilde{h}\|_{L(\Phi_B)} \leq 1 \). Hence \( \|\tilde{h}\|_{L(\Phi_C)} \leq \theta \|f\|_{L(\Phi_A)} \).

For each \( \omega_1 \in \Omega_1 \),

\[
\tilde{h}(\omega_1) = \sup_{\delta \in \mathcal{D}} \int_{\Omega_2} |f(\omega_1, \omega_2)\delta(\omega_2)| \, d\mu_2(\omega_2) = \|f_{\omega_1}\|_{L(\Phi_B)}
\]

where the last equality is true on account of \( \mathcal{D} \) being a norming subset of \( L(\Psi_B) \) for \( L(\Phi_B) \).

For a subset \( E \subset \Omega_1 \) of finite measure, Hölder’s inequality reveals that

\[
\frac{\varphi_C(|E|)}{|E|} \int_E \|f_{\omega_1}\|_{L(\Phi_B)} \, d\mu_1(\omega_1) \leq \frac{\varphi_C(|E|)}{|E|} \|\tilde{h}\|_{L(\Phi_C)} \|\chi_E\|_{L(\Phi_C)} \leq \theta \|f\|_{L(\Phi_A)},
\]

proving part 3). \hfill \Box

4. The weak type of the transfer operator

This Section is devoted to calculating the type of the transfer operator \( T^{\#} \) from information on the type of \( T \).

4.1. Kolmogorov’s inequality for r.i. BFSs. The following theorem will be useful in determining the weak type of an operator. It is an extension of Kolmogorov’s criterion as found in \cite{5}.
Let \( (Ω, μ) \) be a measure space and let \( T \) be an operator on some class of measurable functions on \( Ω \) that maps into the set of measurable functions on \( Ω \). Suppose that \( T \) is of weak type \((X, Y)\) for r.i. BFSs \( X \) and \( Y \) and has norm \( c \). Let \( ϕ \) be the fundamental function of the space \( Y \). If \( 0 < σ < 1 \) and \( A \) is any subset of \( Ω \) of finite measure, then for any \( f \in X \) we have

\[
\int_A |Tf|^σ \, dμ(x) \leq \frac{c^σ}{1 - σ} [ϕ^*[|A|]]^σ |A|^{1-σ} ||f||_X^σ.
\]

(4.1)

Conversely, if \( T \) satisfies this inequality for some \( c \) and \( 0 < σ < 1 \), and for each \( f \in X \) and each \( A \subset Ω \) with finite measure, then \( T \) is of weak type \((X, Y)\).

Proof. Suppose \( 0 < σ < 1 \). As \( t \mapsto ϕ(t)/t \) is nondecreasing, if \( s \leq t \), we have the implications

\[
\frac{ϕ(s)}{s} ≥ \frac{ϕ(t)}{t} \Rightarrow \frac{ϕ(t)}{ϕ(s)} ≤ \frac{t}{s} \Rightarrow \frac{ϕ^σ(t)}{ϕ^σ(s)} ≤ \left(\frac{t}{s}\right)^{σ-1} \frac{ϕ^σ(s)}{ϕ^σ(t)}.
\]

Note that \( χ_{[1,∞)}(t/s) = χ_{(0,t]}(s) \) for all \( s, t \in \mathbb{R}^+ \). On the multiplicative group of the positive reals, we now compute, using the convolution of the functions \([Tf]^σ* x^σ ϕ^σ(x)\) and \( x^{σ-1} χ_{[1,∞)}(x)\) at \( t \in \mathbb{R}^+ \):

\[
\frac{ϕ^σ(t)}{t} \int_0^t [(Tf)^σ* x^σ ϕ^σ(x)] \, ds = \int_0^t [(Tf)^σ* x^σ ϕ^σ(x)] \left(\frac{t}{s}\right)^{σ-1} \, ds
\]

\[
= \int_0^∞ [(Tf)^σ* x^σ ϕ^σ(x)] \left(\frac{t}{s}\right)^{σ-1} χ_{(0,t]}(s) \, ds
\]

\[
= \int_0^∞ [(Tf)^σ* x^σ ϕ^σ(x)] \left(\frac{t}{s}\right)^{σ-1} χ_{[1,∞)}(t/s) \, ds
\]

\[
= [(Tf)^σ* x^σ ϕ^σ(x) * x^{σ-1} χ_{[1,∞)}(x)](t).
\]

With this in hand, we exploit the inequality

\[
|[(Tf)^σ* x^σ ϕ^σ(x) * x^{σ-1} χ_{[1,∞)}(x)](t)| ≤ ||[(Tf)^σ* x^σ ϕ^σ(x)]||_∞ ||x^{σ-1} χ_{[1,∞)}(x)||_1.
\]

As \( ||x^{σ-1} χ_{[1,∞)}(x)||_1 = \int_1^∞ s^{σ-2} \, ds = \frac{1}{1 - σ} s^{σ-1} \bigg|_1^∞ = \frac{1}{1 - σ} \), we have

\[
\sup_{t>0} \frac{ϕ^σ(t)}{t} \int_0^t [(Tf)^σ* x^σ ϕ^σ(x)] \, ds ≤ \frac{1}{1 - σ} \sup_{t>0} [(Tf)^σ* x^σ ϕ^σ(t)] = \frac{1}{1 - σ} [(sup (Tf)^σ* x^σ ϕ^σ(t))] = (4.2)
\]

Here we used the fact that \( [(f)^σ* = ||f||^σ \) for all \( 0 < σ < ∞ \) (see Prop 2.1.7 p41 of [2]). We thus have the following inequality.

\[
\frac{ϕ^σ(t)}{t} \int_0^t [(Tf)^σ* x^σ ϕ^σ(x)] \, ds ≤ \frac{1}{1 - σ} ||Tf||_{M^σ(Y)}^σ
\]

\[
≤ \frac{c^σ}{1 - σ} ||f||_X^σ.
\]
by our hypothesis. It is obvious that \( \int_A |Tf|^{\sigma} \, d\mu \leq \int_0^{|A|} [(Tf)^{\sigma}]^* (s) \, ds \), and so we obtain
\[
\int_A |Tf|^{\sigma} \, d\mu \leq \frac{c^\sigma |A|}{1 - \sigma} \frac{\varphi\sigma(|A|)}{\varphi\sigma(|Tf|)} \|f\|_X^{\sigma} = \frac{c^\sigma |A|}{1 - \sigma} \varphi\sigma(|A|)^{\sigma} \|f\|_X^{\sigma}.
\]
To get the last equality, we used the identity \( \varphi\sigma(|A|) \varphi\sigma(|A|) = |A| \).

To prove the converse, suppose that \( T \) satisfies (4.1) for some \( 0 < \sigma < 1 \) and fix \( \lambda > 0 \). Consider a set \( K \subset \{ \omega : |Tf(\omega)| > \lambda \} \) of finite measure. By hypothesis,
\[
|K| \leq \int_K \frac{|Tf|^\sigma}{\lambda^\sigma} \, d\mu \leq \frac{1}{\lambda^\sigma} \frac{c^\sigma}{1 - \sigma} \varphi\sigma(|K|) |K|^{1-\sigma} \|f\|_X^{\sigma}.
\]
Consequently, the following computations are valid:
\[
\frac{|K|^\sigma}{\varphi\sigma(|K|)^\sigma} \leq \frac{1}{\lambda^\sigma} \frac{c^\sigma}{1 - \sigma} \|f\|_X^{\sigma};
\]
\[
\varphi(|K|) \leq \frac{1}{\lambda} \frac{c}{(1 - \sigma)^{1/\sigma}} \|f\|_X;
\]
\[
\lambda \varphi(|Tf|, \lambda) \leq \frac{c}{(1 - \sigma)^{1/\sigma}} \|f\|_X;
\]
\[
\|Tf\|_{M^* (Y)} \leq \frac{c}{(1 - \sigma)^{1/\sigma}} \|f\|_X;
\]
where in the last line we used the identity \( \|Tf\|_{M^* (Y)} = \sup_{\lambda > 0} \lambda \varphi_Y (m(|Tf|, \lambda)) \) of Lemma 3.1. This proves the converse. \( \square \)

Note that the proof remains correct even if \( \| \cdot \|_X \) is a seminorm. We will need this fact in the proof of Corollary 4.3.

4.2. Computation of the weak type of \( T^\# \) for general \( \Omega \). In the rest of the paper, we shall work with dynamical systems \((G, \Omega, \mu, \alpha)\) as given in Definition 1.1, where \((\Omega, \mu)\) is a \( \sigma \)-finite and resonant measure space, and \( G \) is a \( \sigma \)-finite locally compact group. We shall also often require that \( G \) be amenable in order to obtain the most powerful results.

The first set of results we prove, hold for general locally compact groups, with minimal restrictions on the spaces involved. However the price paid for achieving such generality, is that most of these results only apply to single operators, not sequences. The one exception to this rule, is transfers of convolution operators. Results that apply to sequences of more general classes of operators, can be produced by adding the restrictions that the group be amenable, and also either restricting the growth rates of the associated fundamental functions, or requiring \((\Omega, \mu)\) to be a finite measure space (as we shall do in the next subsection). Each of these categories of results will be considered in turn.

**Lemma 4.2.** Let \( X, Y \) be r.i. BFSs over \( G \) with fundamental functions \( \varphi_X \) and \( \varphi_Y \) respectively and let \( T \) be a transferable operator of weak type \((X, Y)\) with metrisable range. Let \( U \) be the open neighbourhood satisfying the conditions in Definition 2.1(2). Then for any subset \( A \subset \Omega \) of finite measure, and \( 0 < \sigma < 1 \), there is a
compact neighbourhood \( \tilde{K} \) of the identity such that

\[
\frac{1}{|A|} \int_A |T^\# f|^{\sigma} (\omega) d\mu \leq \frac{2c^\sigma}{1-\sigma} [\varphi_Y(|\tilde{K}|)]^{-\sigma} \left( \frac{1}{|A|} \int_A \|(F_{KU^{-1}}) \omega\|_X d\mu \right)^{\sigma},
\]

where for each \( \omega \in \Omega \) the cross section \( (F_{KU^{-1}})\omega \) is the measurable function defined on \( G \) by \( t \mapsto F_{KU^{-1}}(t, \omega) \).

**Proof.** By the Fubini-type Lemma 2.14 and Lemma 2.15, which we employ by identifying \( E \) with \( K \) and \( K \) with \( KU^{-1} \), we have:

\[
\int_{\tilde{K}} \int_A |F'(t, \omega)|^{\sigma} d\mu dt = \int_{\tilde{K}} \int_A |F'(t, \omega)|^{\sigma} dt d\mu \leq \int_{\tilde{K}} \int_A |F'_{KU^{-1}}(t, \omega)|^{\sigma} dt d\mu.
\]

As \( T \) is of weak type \( (X, Y) \), using Kolmogorov’s criterion \( [1,1] \) we have that

\[
\int_{\tilde{K}} |F'_{KU^{-1}}(t, \omega)|^{\sigma} dt \leq \frac{c^\sigma}{1-\sigma} [\varphi_Y(|\tilde{K}|)]^{-\sigma} |\tilde{K}|^{1-\sigma} \left( \int_{\tilde{K}} \|(F_{KU^{-1}})\omega\|_X d\mu \right)^{\sigma}.
\]

From Jensen’s inequality and the identity \( \varphi_Y(t) = t, \)

\[
\int_{\tilde{K}} \int_A |F'(t, \omega)|^{\sigma} d\mu dt \leq \frac{c^\sigma}{1-\sigma} [\varphi_Y(|\tilde{K}|)]^{-\sigma} |\tilde{K}|^{1-\sigma} \left( \int_{\tilde{K}} \|(F_{KU^{-1}})\omega\|_X d\mu \right)^{\sigma}.
\]

We rewrite this as

\[
\frac{1}{|K|} \left( \frac{1}{|A|} \int_A |F'(t, \omega)|^{\sigma} d\mu \right) \leq \frac{c^\sigma}{1-\sigma} [\varphi_Y(|\tilde{K}|)]^{-\sigma} \left( \frac{1}{|A|} \int_A \|(F_{KU^{-1}})\omega\|_X d\mu \right)^{\sigma}.
\]

As \( t \mapsto \frac{1}{|A|} \int_A |F'(t, \omega)|^{\sigma} d\mu \) is continuous, and as \( |F'(1, \omega)| = |T^\# f|(\omega) \) by definition, from the Lebesgue differentiation Theorem we obtain

\[
\frac{1}{|A|} \int_A |T^\# f|^{\sigma} (\omega) d\mu = \lim_{\tilde{K} \rightarrow (1)} \frac{1}{|K|} \left( \frac{1}{|A|} \int_A |F'(t, \omega)|^{\sigma} d\mu \right) dt.
\]

Hence for some \( K \) small enough,

\[
\frac{1}{|A|} \int_A |T^\# f|^{\sigma} (\omega) d\mu \leq \frac{2}{|K|} \left( \frac{1}{|A|} \int_{\tilde{K}} |F'(t, \omega)|^{\sigma} d\mu \right) dt
\]

\[
\leq \frac{2c^\sigma}{1-\sigma} [\varphi_Y(|\tilde{K}|)]^{-\sigma} \left( \frac{1}{|A|} \int_A \|(F_{KU^{-1}})\omega\|_X d\mu \right)^{\sigma}.
\]

\[\square\]

**Corollary 4.3.** Let \( (\Omega, \mu, G, \alpha) \) be a dynamical system with \( (\Omega, \mu) \) countably generated and resonant and let \( T \) be a transferable operator of weak type \( (X, Y) \) and suppose that \( \Phi_A \) and \( \Phi_B \) are Young’s functions with respective associated fundamental functions \( \varphi_A \) and \( \varphi_B \) satisfying

\[
\theta \varphi_A(st) \geq \varphi_X(s) \varphi_B(t)
\]

for some \( \theta > 0 \) and all \( s, t > 0 \).
1) If $X$ is an Orlicz space and $\lim_{t \to 0} \varphi_X^*(t) = 0$, then $T^#$ is of weak type $(L(\Phi_A), \varphi_B)$.
2) If $X$ is an $M$-space then $T^#$ is of weak type $(M(\varphi_A), \varphi_B)$.
3) If $X$ is a $\Lambda$-space and $\lim_{t \to 0} \varphi_X^*(t) = 0$, then $T^#$ is of weak type $(\Lambda(\varphi_A), \varphi_B)$.

Proof. We prove part 3). Let $A \subset \Omega$ have finite measure and let $U$ be the open neighbourhood guaranteed by Definition 2.1(2). Then for any $0 < \sigma < 1$, by Lemma 4.2 there is a compact neighbourhood of the identity $K \subset G$ such that
\[
\frac{1}{|A|} \int_A |T^# f|^{\sigma}(\omega) d\mu \leq \frac{2c_0}{1 - \sigma} [\varphi_Y(|K|)]^{-\sigma} \left( \frac{1}{|A|} \left\| (F_{KU^{-1}})_{\omega} \right\|_{\Lambda(\Phi_A)} \right)^{\sigma}.
\]
From Proposition 3.7 part 2),
\[
\varphi_B(|A|) \int_A \left\| (F_{KU^{-1}})_{\omega} \right\|_{\Lambda(\Phi_A)} d\mu(\omega) \leq 6\theta \|F_{KU^{-1}}\|_{\Lambda(\Phi_A)}.
\]
Combining these last two inequalities yields
\[
\frac{1}{|A|} \int_A |T^# f|^{\sigma}(\omega) d\mu \leq \frac{2c_0}{1 - \sigma} [\varphi_Y(|K|)]^{-\sigma} \max(1, |KU^{-1}|) \|f\|_{\Lambda(\Phi_A)}.
\]

A simple calculation shows that $\varphi_A(st) \leq \varphi_A(s) \max(1, t)$. Now note that the condition in [17] Theorem 8.15] on the Young’s functions may be rephrased as $\varphi_A(st) \leq \theta \varphi_A(s) \varphi_B(t)$ for all $s, t > 0$. Hence, that theorem is applicable here. Together with Lemma 2.4 we may conclude that $\|F_{KU^{-1}}\|_{\Lambda(\Phi_A)} \leq \max(1, |KU^{-1}|) \|f\|_{\Lambda(\Phi_A)}$ and so
\[
\frac{1}{|A|} \int_A |T^# f|^{\sigma}(\omega) d\mu \leq \frac{2c_0}{1 - \sigma} [\varphi_Y(|K|)]^{-\sigma} \max(1, |KU^{-1}|) \|f\|_{\Lambda(\Phi_A)}
\]
where $c_0 = 2^{1/\sigma} 6\theta c \varphi_Y(|K|)^{-1} \max(1, |KU^{-1}|)$.

Therefore
\[
\int_A |T^# f|^{\sigma}(\omega) d\mu \leq \frac{c_0}{1 - \sigma} [\varphi_B(|A|)]^{-\sigma} \|f\|_{\Lambda(\Phi_A)}^{\sigma},
\]
and so by Theorem 4.8, $T^#$ is of weak type $(\Lambda(\Phi_A), \varphi_B)$.

We remark again that in contrast to Theorem 4.8, Corollary 4.3 applies only to single transferable operators, not sequences. The obstacle in this regard is that norm bound obtained, namely $c_0$, depends on the set $U$, for which the measure may grow without bound in the case of sequences. Only in the case of transfers of convolution operators are we able to handle sequences in this generality, as is demonstrated by the next Proposition.

**Proposition 4.4.** Let $(T_n)$ be a sequence of operators given by (5.1). Suppose there are Young’s functions $\Phi_A, \Phi_B, \Phi_C, \Phi_D$ and $\Phi_E$ with associated fundamental functions $\varphi_A, \ldots, \varphi_E$ satisfying
\[
\varphi_C(t) \varphi_B^*(s) \leq \theta_1 \varphi_A(st),
\]
\[
\varphi_A(st) \leq \theta_2 \varphi_D(s) \varphi_E(t)
\]
for all $s, t > 0$.

Suppose further that there are measurable functions $\ell_0$ and $\ell_1$ on $G$ such that $\sup |k_n(s^{-1})| = \ell_0(s) \ell_1(s)$, $\ell_0 \in L(\Phi_B)$ and $\ell_1 \in L(\Phi_E)$. Then the operator $T^#$
defined by $T^#f(\omega) = \sup_{n \in \mathbb{N}} |T^#_n f(\omega)|$ is a sublinear operator mapping $L(\Phi_D)$ into $L(\Phi_C)$.

Proof. For each $N \in \mathbb{N}$, let $S_N f(\omega) := \max_{1 \leq n \leq N} |T^#_n f(\omega)|$. By [17, Theorem 8.18],

$$\|S_N f\|_{L(\Phi_C)} \leq \|\max_{1 \leq n \leq N} |k_n(s^{-1})f(\alpha_s(\omega))|ds\|_{L(\Phi_C)}$$

$$\leq \theta \|\ell_1\|_{L(\Phi_B)} \|\ell_0 \otimes f\|_{L(\Phi_A)} \text{ (by Lemma 2.4)}$$

$$\leq \theta_1 \theta_2 \ell_1 \|\ell_0\|_{L(\Phi_E)} \|f\|_{L(\Phi_D)}$$

where the final inequality follows from [17, Theorem 8.15]. \qed

Under some mild restrictions on the fundamental functions of the associated spaces, one can obtain further results applicable to sequences. We start this phase of the analysis by defining some sets that shall be used extensively in the sequel.

**Definition 4.5.** For $f \in L^{1+\infty}(\Omega)$, let $K$ be a measurable subset of $G$ and $\lambda > 0$. Using the notation of Subsection 2.2, we define the following sets:

$$E = \{ \omega : |F'(1, \omega)| > \lambda \} \subset \Omega$$

$$E' = \{ (t, \omega) : |F'_{KU} - 1| > \lambda \} \in G \times \Omega.$$

Also, for each $t \in G$, we define

$$E'_t = \{ \omega : |F'_{KU} - 1| > \lambda \} \subset \Omega$$

and for each $\omega \in \Omega$,

$$E'_\omega := \{ t : |F'_{KU} - 1| > \lambda \} \subset G.$$

These sets are all measurable by Lemma 2.14 and so by Fubini’s theorem,

$$|E| = \int_\Omega |E_\omega| d\mu(\omega) = \int_G |E'| dt.$$

The next lemma is the key technical ingredient. It is in this Lemma that the semilocality and equimeasurability-preserving properties of transferable operators are used.

**Lemma 4.6.** Let $(\Omega, \mu, G, \alpha)$ be a dynamical system with $(\Omega, \mu)$ resonant, and let $T$ be a transferable operator of weak type $(X,Y)$ with norm majorised by $c$.

Let $f \in X$, and $\lambda > 0$, with $E, E', E'_t$ and $E'_\omega$ the sets given in Definition 4.5. Then we have

$$|E| \geq |K| |E'|$$

$$\varphi_Y(E'_\omega) \leq \frac{c}{\lambda} \|F'_{KU} - 1\|_X. \quad (4.5)$$

Proof. For all $t \in K$ and almost all $\omega \in \Omega$, we have by Lemmas 2.15 and 2.16 that

$$|F'_{KU} - 1| \geq |F'(1, \alpha_t(\omega))| = |F'(1, \alpha_t(\omega))|.$$
This implies that $|E| \geq |E|$, and hence that

$$|E| = \int_G |E| \, dt \geq \int_K |E| \, dt \geq \int_K |E| \, dt = |K||E|.$$  

By Theorem 4.1 and the hypothesis on the weak type of $T$,

$$|E_\omega| \leq \int_{E_\omega} \frac{|T((F_{KU^{-1}})_\omega)|^\sigma}{\lambda^\sigma} \, d\mu$$

$$\leq \frac{1}{\lambda^\sigma} \frac{e^\sigma}{1-\sigma} \|\phi_Y(\|E_\omega\|)\|X \leq \frac{1}{\lambda^\sigma} \frac{e^\sigma}{1-\sigma} \|F_{KU^{-1}}\|X,$$  

whence

$$\phi_Y(\|E_\omega\|) \leq \frac{1}{\lambda(1-\sigma)^{1/\sigma}} \|F_{KU^{-1}}\|X \leq \frac{ec}{\lambda} \|F_{KU^{-1}}\|X.$$  

(In the final inequality we used the elementary fact that

$$\sup_{0<\sigma<1} (1-\sigma)^{1/\sigma} = \lim_{\sigma \to 0} (1-\sigma)^{1/\sigma} = 1/e.$$  

\[\square\]

**Theorem 4.7.** Let $(\Omega, \mu, G, \alpha)$ be a dynamical system where the group $G$ is amenable. Suppose that we are given fundamental functions $\phi_A, \phi_W$ and $\phi_Z$ satisfying

$$\phi_A(t)\phi_X(s) \leq \phi_Y(\theta_1 st), \quad \text{and} \quad \phi_Z(st) \leq \theta_2 \phi_A(s)\phi_W(t) \quad \text{for all } t > 0, s > 0$$

with $\phi_Z \in \mathcal{U}$. Let $(T_n)$ be a sequence of transferable operators of weak type $(X, Y)$, and suppose that in addition and $T := \sup_n |T_n|$ is of weak type $(X, Y)$. Then the operator $T^\#: := \sup_n |T_n^\#|$ is of $\Lambda$-weak type $(\phi_W, \phi_Z)$.

**Proof.** Firstly observe that since $\phi_Z \in \mathcal{U}$, we must have that $\lim_{t \to \infty} \phi_Z(t) = \infty$. In fact more generally if all we had was that $\phi_Z \in \Delta_2$, we would still have that $\lim_{t \to \infty} \phi_Z(t) = \infty$. To see this simply let $x = 1$ in the first part of Proposition 3.4 and note what happens when $y \to \infty$. Since by hypothesis we have that $\phi_Z(t) \leq \theta_2 \phi_A(t)\phi_W(1)$ for all $t > 0$, we then also have that $\lim_{t \to \infty} \phi_A(t) = \infty$.

Next let $c$ be the bound of the operator $T$. For any $n \in \mathbb{N}$ and any $g \in X(G)$ we will then have that $\|T(g)\|_{M^*(Y)} \leq c\|g\|_X$. Let $n \in \mathbb{N}$ be given and let $E, E'$, $\overline{E}$ and $\overline{E}_\omega$ be the sets specified in Definition 4.6 for the operator $T_n$, a compact neighbourhood $K \subset G$ of $1 \in G$, and the function $f = \chi_A$, where $A \subset \Omega$ is measurable set with finite measure. Let $U$ be the set specified in part (2) of Definition 2.1.

It is easy to verify that $(\chi_{KU^{-1}} \otimes_\alpha \chi_A)(t, \omega) = \chi_{KU^{-1}}(t) \chi_A(\alpha_t(\omega))$ is a characteristic function on $G \times \Omega$. So for some measurable subset $\overline{E} \subset G \times \Omega$, we have
that $\chi_{\tilde{E}} = \chi_{KU^{-1} \otimes_A \chi_A}$. So in this specific case $(F_{KU^{-1}})_\omega = (\chi_{KU^{-1} \otimes_A \chi_A})_\omega = (\chi_{\tilde{E}})_\omega = \chi_{\tilde{E}_\omega}$. By Lemma 2.4, we moreover have that

$$
|\tilde{E}| = \int_{G \times \Omega} \chi_{KU^{-1} \otimes_A \chi_A} (dt \times d\mu(\omega))
= \int_{G \times \Omega} \chi_{KU^{-1} \otimes_A \chi_A} (dt \times d\mu(\omega))
= \int_G \int_{\Omega} \chi_{KU^{-1}(t)\chi_A}(\omega) \, d\omega \, dt
= |KU^{-1}| |A|.
$$

From inequality (4.5), we therefore have that

$$
\varphi_Y(|E_\omega|) \leq \frac{ce}{\lambda} \|(F_{KU^{-1}})_\omega\| X = \frac{ce}{\lambda} \|\chi_{E_\omega}\| X = \frac{ce}{\lambda} \varphi_X(|E_\omega|).
$$

Assume for the moment that $\varphi_A$ and $\varphi_Y$ each are invertible in the usual sense with $\frac{ce}{\lambda}$ in the domain of $\varphi_A^{-1}$. It then follows from the above inequality and the inequality $\varphi_A(t)\varphi_X(s) \leq \varphi_Y(\theta_1 st)$, that

$$
\varphi_Y(|E_\omega|) \leq \varphi_A(\varphi_A^{-1}(\frac{ce}{\lambda}))\varphi_X(|E_\omega|) \leq \varphi_Y(\theta_1 \varphi_A^{-1}(\frac{ce}{\lambda}))|E_\omega|,
$$

and hence that

$$
|E_\omega| \leq \theta_1 \varphi_A^{-1}(\frac{ce}{\lambda})|E_\omega|.
$$

Integrating over $\Omega$ now yields the conclusion that

$$
|K| |E| \leq |E| = \int_\Omega |E_\omega| \, d\mu \leq \theta_1 \varphi_A^{-1}(\frac{ce}{\lambda}) \int_\Omega |E_\omega| \, d\mu = \theta_1 \varphi_A^{-1}(\frac{ce}{\lambda}) |E| = \frac{ce}{\lambda} \varphi_X(|E|),
$$

and hence that

$$
|E| \leq \frac{ce}{\lambda} \varphi_X(|E|),
$$

On the one hand, $|KU^{-1}|/|K| \geq 1$. On the other hand, by definition of $U$, $\overline{U}$ is compact and so by [7, (cf. 19, Corollary 4.14)],

$$
\inf \left\{ \frac{|\overline{UK}|}{|K|} : K \in \mathcal{C}(G), |K| > 0 \right\} \leq 1,
$$

where $\mathcal{C}(G)$ is the collection of all compact subsets of $G$. Consequently, the fact that Haar measure is preserved under taking inverses yields that

$$
1 \leq \frac{|KU^{-1}|}{|K|} = \frac{|UK^{-1}|}{|K^{-1}|} \leq \frac{|\overline{UK}^{-1}|}{|K^{-1}|}
$$

and so

$$
\inf \left\{ \frac{|KU^{-1}|}{|K|} : K \in \mathcal{C}(G), |K| > 0 \right\} = 1.
$$

The inequality (4.6) then clearly yields the conclusion that

$$
|E| \leq \frac{ce}{\lambda} \varphi_X(|E|).
$$

But then

$$
\varphi_Z(|E|) \leq \varphi_Z(\theta_1 \varphi_A^{-1}(\frac{ce}{\lambda}) |A|) \leq \varphi_Z(\varphi_A^{-1}(\frac{ce}{\lambda}) \varphi_W(\theta_1 |A|)) \leq (\theta_1 + 1) \frac{ce}{\lambda} \varphi_W(|A|).
$$

(In the last inequality we used the fact that $\varphi_W(\theta_1 u) \leq (\theta_1 + 1) \varphi_W(u)$ for all $u > 0$, which in turn follows fairly directly from the facts that $\varphi_W$ is non-decreasing and...
\( \varphi_w(s) \) non-increasing.) Keeping in mind that \( |E| = m(T_N^f, \lambda) \), it is clear that we have proved that for all \( \lambda > 0 \),
\[
\lambda \varphi_Z(m(T_n^f(\chi_A), \lambda)) \leq (\theta_1 + 1)\theta_2 c e \varphi_W(|A|).
\]

We proceed to deal with the case where possibly \( \varphi_A \) and \( \varphi_Y \) are constant on some part of \([0, \infty)\), and/or \( \frac{e}{\lambda} \) lies outside of the range of \( \varphi_A \). With \( \varphi(0+) = \lim_{t \to 0} \varphi_A(t) \), we pick \( 1 > \epsilon > 0 \) so that \( \epsilon \varphi(0+) < \frac{e}{\lambda} \), and replace \( \varphi_A \) with the function \( \varphi_{A, \epsilon} \) assigned the value \( \varphi(0+) \) at 0, with \( \varphi_{A, \epsilon}(t) = \epsilon^{1/(1+t)} \varphi_A(t) \) for all \( t > 0 \). The function \( t \to \epsilon^{1/(1+t)} \) is a strictly increasing function mapping \([0, \infty)\) onto \((\epsilon, 1)\). Thus since by hypothesis \( \lim_{t \to \infty} \varphi_A(t) = \infty \), \( \varphi_{A, \epsilon} \) is then a strictly increasing function mapping \([0, \infty)\) onto \([\epsilon, 1)\). In addition for all \( t > 0 \) we have that \( \epsilon \varphi_A(t) \leq \varphi_{A, \epsilon}(t) \leq \varphi_A(t) \) for all \( t > 0 \). Given \( t > 0 \), the value \( \varphi_{A, \epsilon}(t) \) then clearly increases to \( \varphi_A(t) \) as \( \epsilon \) increases to 1. The functions \( \varphi_{X, \epsilon} \) and \( \varphi_{Y, \epsilon} \), are similarly defined. Observe that since \( \frac{1}{1+t} + \frac{1}{1+s} > \frac{1}{1+t+s} \) for all \( s, t > 0 \) and since \( 1 > \epsilon > 0 \), it is clear that \( \epsilon^{1/(1+t)} \epsilon^{1/(1+s)} < \epsilon^{1/(1+t+s)} \). Thus the inequalities
\[
\varphi_A(t) \varphi_X(s) \leq \varphi_Y(\theta_1 st), \quad \text{and} \quad \varphi_Z(st) \leq \theta_2 \varphi_A(s) \varphi_W(t)
\]
become
\[
\varphi_{A, \epsilon}(t) \varphi_{X, \epsilon}(s) \leq \varphi_{Y, \epsilon}(\theta_1 st) \quad \text{and} \quad \varphi_{Z}(st) \leq \theta_2 \epsilon \varphi_{A, \epsilon}(s) \varphi_W(t)
\]

Since in addition
\[
\varphi_{Y, \epsilon}(\overline{|E_w|}) < \varphi_Y(\overline{|E_w|}) \leq \frac{ce}{\lambda} \varphi_X(\overline{|E_w|}) < \frac{ce}{\epsilon \lambda} \varphi_{X, \epsilon}(\overline{|E_w|})
\]
with \( \epsilon \varphi_A(0+) < \frac{e}{\lambda} \), we may now argue as before to obtain the conclusion that
\[
\lambda \varphi_Z(m(T_n^f(\chi_A), \lambda)) \leq (\theta_1 + 1)\theta_2 ce \varphi_W(|A|).
\]

On letting \( \epsilon \) increase to 1, we have as before that
\[
\lambda \varphi_Z(m(T_n^f(\chi_A), \lambda)) \leq (\theta_1 + 1)\theta_2 c e \varphi_W(|A|).
\]
Notice that since \( \Lambda_W(\Omega) \) contractively embeds into \( L^{1+\infty}(\Omega) \), it clearly follows from Lemma 2.2 and the diagram in Definition 2.10 that \( T_n^f \) maps sequences converging in \( \Lambda_W(\Omega) \), onto sequences converging in measure on subsets of \( \Omega \) of finite measure. It therefore follows from [26, Theorem 2.5] that
\[
\sup_{t>0} \varphi_Z(t)(T_n^f(f))^*(t) = \sup_{\lambda>0} \lambda \varphi_Z(m(T_n^f(f), \lambda)) \leq (\theta_1 + 1)\theta_2 C(Z)e\|f\|_{\Lambda(W)}
\]
where \( C(Z) \) is a constant depending only on \( \varphi_Z \), and not on \( T_n^f \). Using this fact one can easily show that the operator \( T^f := \sup_n |T_n^f| \) is of \( \Lambda \)-weak type \((\varphi_W, \varphi_Z)\). \( \square \)

The above theorem of course applies in particular to operators with weak type \((p, p)\). However it does not guarantee a transfer of \( L \)-weak type. For such a result relatively strong restrictions need to be placed on the the fundamental function \( \varphi_X \) as is shown in the following result.

**Theorem 4.8.** Let \((\Omega, \mu, G, \alpha)\) be a dynamical system where the group \( G \) is amenable. Suppose that we are given fundamental functions \( \varphi_A, \varphi_W \) and \( \varphi_Z \) satisfying
\[
\varphi_A(t) \varphi_X(s) \leq \varphi_Y(\theta_1 st), \quad \text{and} \quad \varphi_Z(st) \leq \theta_2 \varphi_A(s) \varphi_W(t)
\]
for all \( t > 0, s > 0 \).
with \( \lim_{t \to \infty} \varphi_A(t) = \infty \) and \( \lim_{t \to 0} \varphi_W(t) = 0 \). Suppose also that \( \varphi_X \) is the fundamental function of the Orlicz space \( L_X \) equipped with the Luxemburg norm, where the associated Young’s function \( \Phi_X \) belongs to the class \( \Delta' \) globally. Let \( (T_n) \) be a sequence of transferable operators of \( L \)-weak type \((\varphi_X, \varphi_Y)\), and suppose that in addition and \( T := \sup_n |T_n| \) is of \( L \)-weak type \((\varphi_X, \varphi_Y)\). Then the operator \( T^\# := \sup_n |T_n^\#| \) is of \( L \)-weak type \((\varphi_X, \varphi_Z)\).

**Proof.** Let \( c \) be the bound of the operator \( T \). For any \( n \in \mathbb{N} \) and any \( g \in X(G) \) we will then have that \( \|T(g)\|_{M^r(Y)} \leq c\|g\|_X \). Using the fact that \( \Phi_X \) satisfies the \( \Delta' \) (and hence also the \( \Delta_2 \)) condition, it is an easy exercise to see that \( \Phi_X(t) > 0 \) for all \( t > 0 \), and hence that \( \Phi_X \) has a ‘proper’ inverse. For such Young’s functions it is also known that \( \int_G \Phi_X(|g|(t))dt < \infty \) for all \( g \in L_X(G) \), and similarly that \( \int_G \Phi_X(|f|(\omega))d\mu(\omega) < \infty \) for all \( f \in L_X(\Omega) \). Using the fact that \( \Phi_X(st) \leq c_0 \Phi_X(s)\Phi_X(t) \) for all \( s \) and \( t \), it is easy to see that for any \( g \in L_X(G) \), we have that

\[
\int_G \Phi_X \left( \frac{|g|(s)/|\varphi_X(c_0 \int_G \Phi_X(|g|)(t) dt)}{\varphi_X(c_0 \int_G \Phi_X(|g|)(t) dt)} \right) ds \\
\leq c_0 \Phi_X \left( \frac{1}{|\varphi_X(c_0 \int_G \Phi_X(|g|)(t) dt)} \right) \int_G \Phi_X(|g|(s)) ds \\
= c_0 \Phi_X \left( \frac{\Phi_X^{-1}(1/c_0 \int_G \Phi_X(|g|)(t) dt)}{\Phi_X^{-1}(1/c_0 \int_G \Phi_X(|g|)(t) dt)} \right) \int_G \Phi_X(|g|(s)) ds \\
= 1.
\]

It follows that

\[
\|g\|_X \leq \varphi_X \left( c_0 \int_G \Phi_X(|g|)(t) dt \right) \text{ for all } g \in L_X(G).
\]

Now let \( E, \mathcal{E}, \mathcal{E}' \) and \( \mathcal{E}_\omega \) be the sets specified in Definition \( \text{(1.9)} \) for the operator \( T_n \) and the function \( f \in L_X(\Omega) \). Let \( U \) be the set specified in part (2) of Definition \( \text{(2.1)} \). For some compact neighbourhood \( K \subset G \) of \( 1 \in G \), we then have by Lemma \( \text{(4.6)} \) that

\[
\varphi_Y(|\mathcal{E}_\omega|) \leq \frac{\lambda}{\varphi_X} \|(F_{KU^{-1}})\omega\|_X \leq \frac{\lambda}{\varphi_X} \left( c_0 \int_G \Phi_X(|(F_{KU^{-1}})\omega|(t)) dt \right)
\]

On arguing as in the proof of Theorem \( \text{(4.7)} \) it is clear that we may assume that both \( \varphi_A \) and \( \varphi_Y \) have ‘proper’ inverses, which we may then use to conclude from the above that

\[
\varphi_Y(|\mathcal{E}_\omega|) \leq \varphi_Y \left( \theta_1 c_0 \varphi_A^{-1} \left( \frac{\lambda e}{\lambda} \right) \int_G \Phi_X(|(F_{KU^{-1}})\omega|(t)) dt \right),
\]

and hence that

\[
|\mathcal{E}_\omega| \leq \theta_1 c_0 \varphi_A^{-1} \left( \frac{\lambda e}{\lambda} \right) \int_G \Phi_X(|(F_{KU^{-1}})\omega|(t)) dt. \tag{4.7}
\]

Next observe that since \( \Phi_X \) is zero at zero, it is an exercise to see that

\[
\Phi_X(|F_{KU^{-1}}|(t, \omega)) = \Phi_X(X_{KU^{-1}}(t)|f|(\alpha(t))) = \chi_{KU^{-1}}(t) \Phi_X(|f|(\alpha(t))).
\]
Recall that (as noted earlier) we also have that \( \Phi_X(\lambda) \in L_1(\Omega) \) since \( f \in L_X(\Omega) \). So if we integrate \( \int_G \Phi_X((|F_{KU^{-1}}|)_\omega)(t) \, dt \) over \( \Omega \), we may use Lemma 2.4 to conclude that

\[
\int \int_G \Phi_X((|F_{KU^{-1}}|)(t, \omega) \, dt \, d\mu(\omega) = \int \int_G (\chi_{KU^{-1}} \otimes_{\alpha} \Phi_X(|f|))(t, \omega) \, dt \, d\mu(\omega)
\]

\[
= \int \int_G (\chi_{KU^{-1}} \otimes \Phi_X(|f|))(t, \omega) \, dt \, d\mu(\omega)
\]

\[
= |KU^{-1}| \int \Phi_X(|f|) \, d\mu(\omega).
\]

Integrating equation (4.7) over \( \Omega \) will therefore yield

\[
|K||E| \leq \theta_1 c_0 \varphi_{\lambda}^{-1} \left( \frac{c e}{\lambda} \right) |KU^{-1}| \int \Phi_X(|f|) \, d\mu(\omega).
\]

As in the proof of Theorem 4.7 inf\{\( |KU^{-1}|/|K| : K \in \mathcal{C}(G), |K| > 0 \}\} = 1. Hence the above inequality yields the fact that

\[
|E| \leq \theta_1 c_0 \varphi_{\lambda}^{-1} \left( \frac{c e}{\lambda} \right) \int \Phi_X(|f|) \, d\mu(\omega).
\]

Once again arguing as the proof of Theorem 4.7 we may now conclude from this inequality that

\[
\lambda \varphi_Z(|E|) \leq \theta_2 ce \varphi_W(\theta_1 c_0 \int \Phi_X(|f|) \, d\mu(\omega)).
\]

Taking the supremum over \( \lambda > 0 \), now yields the fact that

\[
\|T_n^\#(f)\|_{M^*(Z)} = \sup_{\lambda > 0} \lambda \varphi_Z(m(T_n^\#(\chi_A), \lambda)) \leq \theta_2 ce \varphi_Z(\theta_1 c_0 \int \Phi_X(|f|) \, d\mu(\omega)).
\]

This inequality ensures that \( T_n^\#(f_m) \) converges to \( T_n^\#(f) \) whenever \( \int \Phi_X(|f_m - f|) \, d\mu(\omega) \to 0 \) as \( m \to \infty \). But since \( \Phi_X \in \Delta_2 \), it follows from [21] Theorem 3.4.12 that the convergence \( \int \Phi_X(|f_m - f|) \, d\mu(\omega) \to 0 \) is equivalent to norm convergence of \( (f_m) \) to \( f \) in \( L_X(\Omega) \). It therefore follows that \( T_n^\# \) is a bounded linear operator from \( L_X(\Omega) \) to \( M^*(\varphi_Z) \).

We proceed to estimate the norm of \( T_n^\# \). For any \( f \in L_X(\Omega) \) with \( \|f\|_X \leq 1 \), we have that \( \int \Phi_X(|f|) \, d\mu(\omega) \leq \|f\|_X \leq 1 \) [2] Lemma 4.8.8. For such \( f \) it therefore follows that

\[
\|T_n^\#(f)\|_{M^*(Z)} \leq \theta_1 ce \varphi_Z(\theta_1 c_0 \int \Phi_X(|f|) \, d\mu(\omega)) \leq \theta_2 ce \varphi_Z(\theta_1 c_0).
\]

This in turn leads to the conclusion that \( \|T_n^\#\| \leq \theta_2 ce \varphi_Z(\theta_1 c_0) \). Since this holds for every \( n \in \mathbb{N} \) and since the upper estimate for the norm of \( T_n^\# \) does not depend on \( n \), one can now use this fact to show that the operator \( T^\# := \sup_n |T_n^\#| \) is of \( L \)-weak type \((\varphi_W, \varphi_Z)\) with norm \( \|T^\#\| \leq \theta_2 ce \varphi_Z(\theta_1 c_0) \). \( \square \)

If in the above theorem we take all fundamental functions to be \( t^{1/p} \) where \( 1 < p < \infty \), we obtain the following easy corollary. Note that this recovers Calderón’s result for transferable operators of weak type \((p, p)\). However it does go further than the result of Calderón, in that it is valid for actions of arbitrary amenable locally compact groups.
Corollary 4.9. Let \((\Omega, \mu, G, \alpha)\) be a dynamical system where \((\Omega, \mu)\) is resonant and the group \(G\) is amenable. Let \((T_n)\) be a sequence of transferable operators of weak type \((p, p)\) where \(1 \leq p \leq \infty\), and \(T := \sup_n |T_n|\) is also of weak type \((p, p)\). Then the operator \(T^# := \sup_n |T_n^#|\) has the same weak type as \(T\).

4.3. Computation of the weak type of \(T^#\) for \(\Omega\) of finite measure. In the case where \(\Omega\) is of finite measure, it is however possible to deal with sequences of transferable operators as in Theorem 4.8. We achieve the most general result for sequences of transferable operators in Theorem 4.13. To prove it requires a fairly technical analysis of fundamental functions, which we undertake in the next three Lemmas.

We recall the definition of the Boyd indices of a r.i. BFS \(X\) which is defined by a function norm \(\rho\). First, by the Luxemburg Representation Theorem [2, Theorem 2.4.10] there is a (not necessarily unique) r.i. BFS \(X\) over the positive reals with Lebesgue measure, defined by a function norm \(\rho\) which is related to \(\rho\) by the formula

\[
\rho(f^*) = \rho(f)
\]

for every \(f \in X\). Now for each \(t \in \mathbb{R}^+\) we define the dilation operator \(E_t\) by

\[
(E_t g)(s) = g(st)
\]

for all \(s \in \mathbb{R}^+\) and \(g\) a measurable and finite a.e. function on \([0, \infty)\). Let \(h_X(t)\) denote the operator norm of \(E_{1/t}\): that is, \(h_X(t) = \|E_{1/t}\|_{\mathcal{B}(X)}\) for \(t > 0\). Define \(h_X(0) = 0\). In [2, Section 3.5], the authors thoroughly develop the basics of the theory, including the fact that \(h_X\) is submultiplicative. Note that it is also quasiconcave, for by [2, Proposition 3.5.11], \(h_X\) is nondecreasing and \(h_X(t)/t = h_X(1/t)\), which is nonincreasing. Furthermore, \(h_X(t) > 0\) for \(t > 0\), which is a consequence of the next Lemma.

Lemma 4.10. If \(X\) is a r.i. BFS then for all \(s, t > 0\),

\[
\varphi_X(st) \leq h_X(t) \varphi_X(s)
\]

\[
\varphi_X(st) \geq h_X^*(t) \varphi_X(s).
\]

Proof. Consider the function space \(\overline{X}\) over the positive reals given by the Luxemburg Representation Theorem mentioned above and fix \(s, t \in \mathbb{R}^+\). Note that

\[
\varphi_X(st) = \|\chi_{(0, st)}\|_{\overline{X}}
\]

\[
= \|E_{1/t} \chi_{(0, s)}\|_{\overline{X}}
\]

\[
\leq \|E_{1/t}\|_{\mathcal{B}(\overline{X})} \|\chi_{(0, s)}\|_{\overline{X}}
\]

\[
= h_X(t) \varphi_X(s).
\]

From this inequality, we immediately deduce that \(\varphi_X^*(st) \geq h_X^*(t) \varphi_X^*(s)\). As \(\varphi_X^* = \varphi_X\), this can be written as \(\varphi_X^*(st) \geq h_X^*(t) \varphi_X^*(s)\). Equivalently, \(\varphi_X(st) \geq h_X^*(t) \varphi_X(s)\). \(\square\)

Note that as \(h_X\) is submultiplicative, \(h_X^*\) is super multiplicative, in that \(h_X^*(st) \geq h_X^*(s) h_X^*(t)\). So we have shown that any fundamental function \(\varphi_X\) can be bounded above by a submultiplicative and below by a supermultiplicative function (up to a constant factor), in that

\[
\varphi(1) h_X^*(t) \leq \varphi(t) \leq \varphi(1) h_X(t).
\]
The upper and lower Boyd indices of $X$, denoted respectively by $\underline{\alpha}_X$ and $\overline{\alpha}_X$ are given by

$$\underline{\alpha}_X = \lim_{t \to 0^+} \frac{\ln h_X(t)}{\ln t}, \quad \overline{\alpha}_X = \lim_{t \to \infty} \frac{\ln h_X(t)}{\ln t}.$$ 

Recall from Definition 3.5.20 the definitions of the $\mathcal{L}$- and $\mathcal{U}$-indices of a fundamental function. Note that the fundamental indices (defined at the end of subsection 3.2) of a fundamental function are always defined, even if the $\mathcal{L}$- or $\mathcal{U}$-indices are not. The relation between the Boyd and fundamental indices given in the next Lemma, is well known - see for example [2, Chapter 3, exercise 14].

**Lemma 4.11.** Let $X$ be a r.i. BFS with fundamental function $\varphi$. Then

$$0 \leq \underline{\alpha}_X \leq \overline{\alpha}_X \leq \underline{\beta}_X \leq \overline{\beta}_X \leq 1.$$ 

Moreover, $\varphi \in \mathcal{U}$ if and only if $\overline{\beta}_X < 1$ and in this case $\overline{\beta}_X = \rho_\varphi^\mathcal{U}$, and $\varphi \in \mathcal{L}$ if and only if $\overline{\beta}_X > 0$ and in this case $\overline{\beta}_X = \rho_\varphi^\mathcal{L}$.

**Proof.** From Lemma 4.10 it is easy to see that $M(t, X) \leq h_X(t)$. Hence if $t > 1$, $\ln M(t, X)/\ln t \leq \ln h_X(t)/\ln t$, so $\overline{\beta}_X \leq \overline{\alpha}_X$. On the other hand, if $t < 1$, $\ln M(t, X)/\ln t \geq \ln h_X(t)/\ln t$, so $\overline{\alpha}_X \geq \underline{\alpha}_X$. The fact that $M(t, X)$ is nondecreasing shows that $\overline{\alpha}_X \leq \overline{\beta}_X$.

The rest of the Lemma follows directly from the definition of the $\mathcal{U}$- and $\mathcal{L}$-indices.

**Lemma 4.12.** For any r.i.BFS $X$ and any $t > 0$, $h_X(t) \geq h_X^*(t)$.

**Proof.** From [2, Proposition 3.5.11], $h_X^*(t) = \theta h_X(1/t)$, which implies

$$h_X(1/t) = h_X^*(t)/t.$$ 

Note also that the submultiplicativity of $h_X$ and the fact that $h_X(1) = 1$ implies that

$$1 = h_X(1) \leq h_X(t)h_X(1/t),$$ 

which means that $h_X(1/t) \geq 1/h_X(t)$. Substituting the above into the identity $1 = h_X^*(t)h_X^*(t)/t$, we see that

$$1 = h_X^*(t)h_X(1/t) \geq h_X^*(t)\frac{1}{h_X(t)},$$

hence proving the result.

**Theorem 4.13.** Let $(\Omega, \mu, G, \alpha)$ be a dynamical system where $(\Omega, \mu)$ is countably generated, resonant and of finite measure, and the group $G$ is amenable. Let $T_n$ be a sequence of transferable operators of weak type $(X,Y)$ where the pair $(X,Y)$ are either $L_r$, $L_\sigma$- or $M$-type r.i.BFSs’ respectively corresponding to the fundamental functions $\varphi_X$ and $\varphi_Y$. Let $\varphi_B$ be another fundamental function such that the relation

$$\varphi_Y(st) \geq \theta \varphi_B(s)\varphi_X(t)$$

holds for all $s,t > 0$. If $T := \sup_n |T_n|$ is of weak type $(X,Y)$, then the operator $T^\# := \sup_n T_n^\#$ has weak type $(Y, L^1)$.

Note that a special case of this is where the operators $T_n$ are of weak type $(X,X)$, for by Lemma 4.10 $\varphi_X(st) \geq h_X(s)\varphi_X(t)$. 

Proof. As in the proof of Theorem 4.8 for each \( N \in \mathbb{N} \) we define the transferable operator \( T_N = \sup_{\alpha \leq N} |T_\alpha| \). Let \( f \in X \) and \( \lambda > 0 \), with \( E, \mathcal{F}, \mathcal{F}^* \) and \( \mathcal{F}_\omega \) the sets specified in Definition 4.6 for the function \( f \) and operator \( T_N \). Let \( U \) be the neighbourhood of \( 1 \in G \) specified in part (2) of Definition 2.1. Note that \( \varphi_Y \) is just the fundamental function of the Köthe dual \( Y' \) of \( Y \). So for any compact subset \( K \) of \( G \), we may apply Hölder’s inequality for r.i.BFS’s to see that

\[
\int \chi_K(t) \chi_{\mathcal{F}}(t, \omega) \, dt \leq \|\chi_K\|_{Y'} \|\chi_{\mathcal{F}}\|_Y = \varphi_Y(|K|) \varphi_Y(\mathcal{F}_\omega).
\]

But by Lemma 4.10 \( \varphi_Y(|K|) \geq h_{\mathcal{F}_\omega}(|K|) \varphi_Y(1) \), from which it follows that

\[
\frac{\varphi_Y(|K|)}{\varphi_Y(|K|)} \leq \frac{h_{\mathcal{F}_\omega}(|K|)}{h_{\mathcal{F}_\omega}(|K|)} \varphi_Y(1).
\]

Thus the first centered inequality becomes

\[
\frac{h_{\mathcal{F}_\omega}(|K|)}{|K|} \left( \int_K \chi_{\mathcal{F}}(t, \omega) \, dt \right) \leq \varphi_Y(\mathcal{F}_\omega).
\]

Now set \( c_1 := \varphi_Y(1) \) and integrate over \( \Omega \) to get

\[
\int_{\Omega} \varphi_Y(\mathcal{F}_\omega) \, d\mu(\omega) = c_1 h_{\mathcal{F}_\omega}(|K|) \frac{1}{|K|} \int_{K} \int_{\Omega} \chi_{\mathcal{F}}(t, \omega) \, dt \, d\mu(\omega)
\]

\[
= c_1 h_{\mathcal{F}_\omega}(|K|) \frac{1}{|K|} \int_{K} \int_{\Omega} |\mathcal{F}'| \, dt
\]

\[
\geq c_1 h_{\mathcal{F}_\omega}(|K|) \frac{|K|}{|E|} \text{ (as } |\mathcal{F}'| \geq |E|)
\]

\[
= c_1 h_{\mathcal{F}_\omega}(|K|) |E|.
\]

This completes the first part of the proof.

For the second part, we once again start from (4.8). Let \( A \subset \Omega \) be any subset of finite measure. Integrating (4.8) over \( A \), we get

\[
\int_A \varphi_Y(\mathcal{F}_\omega) \, d\mu(\omega) \leq \frac{ce}{\lambda} \int_A \|F_{KU^{-1}}(t, \omega)\|_X \, d\mu(\omega)
\]

\[
= \frac{ce}{\lambda} \varphi_B(|A|) \frac{\varphi_X(|A|)}{|A|} \int_A \|F_{KU^{-1}}(t, \omega)\|_X \, d\mu(\omega)
\]

where we used the identity \( \varphi_B(|A|) \varphi_X(|A|) = |A| \).

Since the measure of \( \Omega \) is finite, we may of course set \( A = \Omega \). Thus by Proposition 3.7 combined with the hypothesis that \( \varphi_Y(st) \geq \theta \varphi_B(s) \varphi_X(t) \) for all \( s, t > 0 \), we have that

\[
\frac{\varphi_B(|\Omega|)}{|\Omega|} \int_{\Omega} \|F_{KU^{-1}}(t, \omega)\|_X \, d\mu(\omega) \leq c_2 \|\chi_{KU^{-1}} \otimes \alpha f\|_Y
\]

for some constant \( c_2 \), and hence that

\[
\int_{\Omega} \varphi_Y(\mathcal{F}_\omega) \, d\mu(\omega) \leq \frac{c_2 ce}{\lambda} \varphi_B(|\Omega|) \|\chi_{KU^{-1}} \otimes \alpha f\|_Y
\]

\[
= \frac{c_2 ce}{\lambda} \varphi_B(|\Omega|) \|\chi_{KU^{-1}} \otimes f\|_Y
\]

\[
\leq \frac{c_2 ce}{\lambda} \varphi_B(|\Omega|) \|h_Y(|KU^{-1}|)\|_Y.
\]
(Here the second last line follows from Lemma 2.4 and the last line from Theorem 8.15 applied to the inequality \( \varphi_Y(st) \leq h_Y(s)\varphi_Y(t) \) in Lemma 4.10)

By inequality we now have that\
\[
c_1 h_Y^c(|K|)|E| \leq \int_\Omega \varphi_Y(|\mathcal{E}_\omega|) \, d\mu(\omega) \leq \frac{c_2 c_\lambda}{\lambda} \varphi_B^c(|\Omega|) h_Y(|K U^{-1}|) \|f\|_Y.
\]

Hence we may conclude from Lemma 4.12 that\
\[
|E| \leq \frac{c_2 c_\lambda}{c_1} \varphi_B^c(|\Omega|) \frac{h_Y(|K U^{-1}|)}{h_Y^c(|K|)} \|f\|_Y \leq \frac{c_2 c_\lambda}{c_1} \varphi_B^c(|\Omega|) \frac{h_Y(|K U^{-1}|)}{h_Y(|K|)} \|f\|_Y.
\]

Now since \( \frac{h_Y(t)}{t} \) is nonincreasing, it is clear that \( 1 \leq \frac{h_Y(|K U^{-1}|)}{h_Y(|K|)} \leq \frac{|K U^{-1}|}{|K|} \). As in the proof of Theorem 4.7, \( \inf \{ |K U^{-1}|/|K| : K \in \mathscr{C}(G), |K| > 0 \} = 1 \). Given that \( |E| = m(T_N^# f, \lambda) \), it therefore follows that\
\[
\lambda m(T_N^# f, \lambda) = \lambda |E| \leq \frac{c_2 c_\lambda \varphi_B^c(|\Omega|)}{c_1} \|f\|_Y.
\]

We have proved that \( T_N^# \) is of weak type \((Y, L^1)\) with constant \( c_2 c_\lambda \varphi_B^c(|\Omega|)/c_1 \). Since \( T_N^# \) is increasing with limit \( T^# \) as \( N \to \infty \), we may reason as in Theorem 4.8 to see that \( T^# \) is also of weak type \((Y, L^1)\).
\[ \square \]

5. Applications of the Transfer Principle

The three main results of the previous Section, Corollary 4.3, Theorems 4.8 and 4.13 are powerful enough to yield a great many maximal inequalities. We can of course use these maximal inequalities to derive a variety of pointwise convergence theorems. This is illustrated in the final part of this work by Theorem 5.5 and Corollaries 5.6 and 5.7.

In the last part of this work, we turn to the derivation of pointwise ergodic theorems. Henceforth, in the dynamical system \((\Omega, \mu, G, \alpha)\), not only will \((\Omega, \mu)\) be countably generated and resonant, but \(G\) will be an abelian, additive, second countable locally compact Hausdorff group with identity element 0, and \(X\) will be a r.i. BFS over the countably generated and resonant measure space \((\Omega, \mu)\). We shall work with transfer operators generated by sequences of convolution operators. That is, we consider a sequence \((T_n)\) of operators on \(L^{1\infty}(G)\) given by\
\[
T_n(f) = k_n * f,
\]
where \(k_n \in L^1(G)\) is bounded and has bounded support, and \(f\) is locally integrable. It is easy to see that the operators \(T_n\) are semilocal. As \(k_n * f\) is a continuous function and \(C(G)\) is metrisable, we see that these operators satisfy the definition of linear transferable operators with metrisable range given in Definition 2.1. We also define \(Tf := \sup_n |T_n(f)|\). Using the Transfer Principle and given information about the functions \(k_n\) and the space \(X\), we show that the transferred operators \(T_n^#\) satisfy a pointwise convergence theorem: that is, \(T_n^# f(\omega)\) converges a.e. for all \(f \in X\) as \(n\) tends to infinity.

To achieve this goal, our strategy is the following three step programme.

1. Given the weak type of the operator \(T\), find the weak type of \(T^#\).

2. In the domain of \(T^#\) computed in step (1), identify a dense subset \(D\) for which the pointwise convergence of \((T_n^# f)\) can be verified for all \(f \in D\).
(3) Use an appropriate version of Banach’s Principle to extend the a.e. convergence of step (2) to the whole domain of $T^\#$.

To do step (1), we shall use results obtained earlier in this paper. For step (3), we prove the following variation on the theme of [2, Corollary 4.5.8] and [10, Theorem 1.1.1], which provides the final link between maximal inequalities and pointwise ergodic theorems.

**Proposition 5.1.** Let $X$ and $Y$ be r.i. BFSs over a measure space $(\Omega, \mu)$. Let $(T_n)$ be a sequence of linear operators on $X$ and define the maximal operator $T$ by $T(f) = \sup_n |T_n(f)|$. If

1. there is a dense subset $D \subseteq X$ such that for all $f \in D$, $(T_n(f)(\omega))$ converges for $\mu$-a.e. $\omega \in \Omega$,
2. $T$ is of weak-type $(X,Y)$,

then $(T_n(f)(\omega))$ converges for $\mu$-a.e. $\omega \in \Omega$ and all $f \in X$.

**Proof.** Define the oscillation $O_f$ of $f \in X$ as follows. For any $\omega \in \Omega$ set

$$O_f(\omega) = \limsup_{n,m \to \infty} |T_n(f)(\omega) - T_m(f)(\omega)|.$$ 

Clearly the linearity of the operators $T_n$ implies that $O_f(\omega) \leq O_g(\omega) + O_{f-g}(\omega)$.

For any $g \in D$ and $\delta > 0$, we have $\mu(\{\omega : O_g(\omega) > \delta\}) = 0$, due to the $\mu$-a.e. convergence of $(T_n)$ on $D$. So $O_g = 0$ $\mu$-a.e. Pick an $f \in X$. Now for any $\eta > 0$, there is a $g \in D$ such that $\|f - g\|_X < \eta$ and

$$\mu(\{\omega : O_f(\omega) > \delta\}) \leq \mu(\{\omega : O_{f-g}(\omega) > \delta\}).$$

Furthermore, by the definition of the oscillation, $O_f(\omega) \leq 2T(f)(\omega)$ a.e. Similarly for $O_{f-g}$. Hence

$$\mu(\{\omega : O_f(\omega) > \delta\}) \leq \mu(\{\omega : 2T(f-g)(\omega) > \delta\}) = m(2T(f-g), \delta).$$

As $T$ is of weak-type $(X,Y)$, $\|2T(f-g)\|_{M^\#(Y)} \leq 2\beta \|f-g\|_X < 2\beta \eta$ where $\beta$ depends only on $T$. Rewriting this using Lemma 3.1

$$\sup_{s>0} s \varphi_Y(m(2T(f-g),s)) \leq 2\beta \eta,$$

In particular, $\delta \varphi_Y(m(2T(f-g),\delta)) \leq 2\beta \eta$. Therefore

$$\varphi_Y(\mu(\{\omega : O_f(\omega) > \delta\})) \leq \frac{2\beta \eta}{\delta}.$$ 

As $\eta$ is arbitrary, $\varphi_Y(\mu(\{\omega : O_f(\omega) > \delta\})) = 0$. Because a fundamental function is 0 only at the origin, $\mu(\{\omega : O_f(\omega) > \delta\}) = 0$. Because $\delta$ is arbitrary, $O_f = 0$ $\mu$-a.e. which implies that $(T_n f)$ does indeed converge $\mu$-a.e. □

The fundamental result towards completing step (2) of the three-step programme is given in Proposition [5,3]. From this, many interesting pointwise ergodic theorems can be deduced, given further information on the nature of $X$. We start by constructing subsets $D$ of $X$ for whose elements a.e. convergence is easy to check. To this end, for any $f \in L^1(G)$ and $x \in X$, we define

$$\alpha_f(x) = \int_G \alpha_t(x) f(t) \, dt,$$

(5.2)
where the integral is a Bochner integral. Because the action of $G$ on $(\Omega, \mu)$ is measure-preserving, on any r.i. BFS the automorphism $\alpha_t$ is an isometry and so $\alpha_f(x) \in X$ too.

Note that the above equation actually gives a bounded bilinear mapping from $L^1(G) \times X$ into $X$, given by $(f, x) \mapsto \alpha_f(x)$.

**Definition 5.2.** Let $Y$ be a set of measurable functions on $(\Omega, \mu)$ and $\mathcal{L} \subseteq L^1(G)$. Define

$$D_X(Y, \mathcal{L}) = \{ \alpha_f(x) : f \in \mathcal{L}, \ x \in X \cap Y \},$$

which is a subset of $X$. In particular, if $\mathcal{F}_0$ consists of those integrable functions on $G$ with vanishing integral, we shall simply write $D_X$ for $D_X(L^\infty(\Omega), \mathcal{F}_0)$.

**Proposition 5.3.** Let $(\Omega, \mu, G, \alpha)$ be a dynamical system and $(T_n)$ a sequence of convolution operators given by (5.1). Suppose that the sequence $\{ \int_G k_n(t) \, dt \}$ converges and that $(k_n \ast \phi)$ converges weakly in $L^1$ for all $\phi \in L^1(G)$ with vanishing integral.

Then given a r.i. BFS $X$, the sequence $(T_n^\# f)$ converges a.e. for every $f \in D_X$.

**Proof.** We begin by describing $T_n^\#$ explicitly, using the construction of Remark 2.13. Let $f \in X$. For almost every $\omega \in \Omega$, the function $t \mapsto f(\alpha_t \omega)$ is locally integrable by Lemma 5.2, and so by definition of $T_n$ we have

$$T_n f(\alpha_{t} \omega) = \int_G k_n(-s) f(\alpha_{t-s} \omega) \, ds.$$

Because we have assumed that $k_n$ is bounded and has bounded support, the integral converges for any locally integrable $f$, and in particular for any $f \in X$. Setting $t = 0$, we obtain

$$T_n^\# f(\omega) = \int_G k_n(-s) f(\alpha_s \omega) \, ds. \tag{5.3}$$

We prove that for any $f \in D_X$, the sequence $(T_n^\# (f))$ converges a.e. By definition of $D_X$, there exists a $g \in L^\infty(\Omega) \cap X$ and $\psi \in L^1(G)$ with vanishing integral, such that $f = \alpha_\psi(g)$. We compute:

$$T_n^\# f(\omega) = \int_G k_n(-t) f(\alpha_t \omega) \, dt$$

$$= \int_G k_n(-t) \int_G g(\alpha_{t+s} \omega) \psi(s) \, ds \, dt$$

$$= \int_G g(\alpha_t \omega) \int_G k_n(s - t) \psi(s) \, ds \, dt.$$ 

As the inner integrals converge weakly in $L^1(G)$, and bearing in mind that $g \in L^\infty(\Omega)$, we have proved that $(T_n^\# (f))$ converges a.e. $\square$

In the light of Propositions 5.1 and 5.3 to complete the three-step programme and prove pointwise ergodic theorems, we indicate situations where we can use the space $D_X$ to construct dense subsets of $X$.

First, we must establish a lemma that will allow us to construct invariant subsets for a given dynamical system. Recall from [2, Definition 1.3.1] that a function norm on a r.i. BFS $X$ over a measure space $(\Omega, \mu)$ is absolutely continuous if for every sequence $(A_n)$ of measurable subsets of $\Omega$ such that $A_n \to \emptyset$ $\mu$-a.e. as $n \to \infty$, and any $f \in X$, we have $\| f \chi_{A_n} \|_X \to 0$ as $n \to \infty$. 

Lemma 5.4. Let $X$ be a r.i. BFS over a $\sigma$-finite measure space $(\Omega, \mu)$, $f$ be a function in $X$ and $\lambda > 0$. If $X$ has absolutely continuous norm then $m(f, \lambda) < \infty$. In particular, the conclusion holds if $X$ is reflexive.

Proof. Suppose to the contrary that there is a $\lambda > 0$ and $f \in X$ such that $m(f, \lambda) = \infty$. Set $\Lambda = \{\omega : |f(\omega)| > \lambda\}$. As $\Omega$ is $\sigma$-finite there is an increasing sequence of finite-measure sets $K_n$ such that $\cup K_n = \Lambda$. Set $C_n = \Lambda \setminus K_n$. Clearly $\mu(C_n) = \infty$.

Because $\lambda \chi_A \leq |f|, \lambda \chi_A$, and hence $\chi_A$ itself, are in $X$. So are the characteristic functions $\chi_{C_n}$ for all $n$.

But as $C_n \downarrow \emptyset$, the hypothesis that $X$ has absolutely continuous norm implies that
\[
\lim_{n \to \infty} \|\chi_{C_n} \|_X = \lim_{n \to \infty} \|\chi_A \chi_{C_n} \|_X = 0.
\] (5.4)

However, it is easy to compute that $\chi_{C_n}^*(t) = 1$ for all $t > 0$ and so $\varphi_X(1) \leq \sup_{t > 0} \chi_{C_n}^*(t) \varphi_X(t) = \|\chi_{C_n} \|_{M(X)} \leq \|\chi_{C_n} \|_X$.

This contradicts (5.4) and so we conclude that our hypothesis $m(f, \lambda) = \infty$ is false.

In the case that $X$ is reflexive, note that by [2, Corollary 1.4.4], its norm is absolutely continuous. □

Theorem 5.5. In the setup of Proposition 5.3, suppose that $X$ is reflexive and that $T^\#$ has weak type $(X,Y)$. Then $(T^\#_n f)$ converges a.e. for every $f \in X$.

Proof. By hypothesis, $T^\#$ is of weak type $(X,Y)$. The plan of the proof is to execute steps (2) and (3) of the three step programme: identify a dense subset $D$ of $X$ and use Proposition 5.3 to show the a.e. convergence of the sequence $(T^\#_n f)$ for $f \in D$, prove that $D$ is dense in $X$, and then finally invoke Proposition 5.1 to show that $(T^\#_n f)$ converges a.e. for all $f \in X$.

We consider the set $D = DX + F$, where $F$ is the subspace of all fixed points of the action $a$ in $X$. We have already seen in Proposition 5.3 that $(T^\#_n f)$ converges a.e. for all $f \in DX$. Similarly, if $g \in F$, it is easy to see from (5.3) that as $t \to g(\alpha_\xi) = g(\xi)$ for almost every $\xi \in \Omega$, $T^\#_n g(\xi) = g(\xi) \int_G k_{n}(s) \, ds$ a.e. From the assumption that the sequence $\left( \int_G k_n(s) \, ds \right)$ converges, it follows that $(T^\#_n g(\xi))$ converges a.e. for all $g \in F$. Therefore $(T^\#_n f)$ converges a.e. for all $f \in D$.

The next task is to demonstrate that $D$ is dense in $X$. By [2, Corollary 1.4.3], the associate space $X'$ is also the dual of $X$. Let $\ell \in X'$ be orthogonal to $D$. We must show that $\ell = 0 \mu$-a.e. We may assume that $\ell$ is real-valued, because if $\ell$ is orthogonal to $D$, so is $\overline{\ell}$.

For any $f = \alpha_\psi(g) \in DX \subseteq D$ we have
\[
0 = \int_\Omega \ell(\omega) f(\omega) \, d\mu(\omega) = \int_\Omega \ell(\omega) \int_G g(\alpha_\omega \psi(t)) \psi(t) \, dtd\mu(\omega) = \int_\Omega \int_G \ell(\omega) g(\alpha_\omega \psi(t)) \, dtd\mu(\omega) = \int_\Omega \int_G \ell(\alpha^{-1} \omega) g(\omega) \psi(t) \, dtd\mu(\omega) = \int_\Omega g(\omega) \int_G \psi(t) \ell(\alpha^{-1} \omega) \, dtd\mu(\omega).
\]
The absolute continuity of the norm of $X$ and \cite{2} Theorem 1.3.11 imply that $X_b = X$. Hence $L^\infty(\Omega) \cap X$ is dense in $X$. By definition of $D_X$, $\xi$ is an arbitrary element of $L^\infty(\Omega) \cap X$ and therefore

$$
\int_G \psi(t) \ell(\alpha_{-t}(\omega)) \, dt = 0.
$$

Now we use some basic ideas from spectral synthesis, as presented in \cite{9} Section 4.6 and \cite{23} Section 7.8. We write

$$
F_0^\perp = \{ \xi \in L^\infty(G) : \int_G \psi(t) \xi(t) \, dt = 0 \text{ for all } \psi \in F_0 \}. 
$$

Clearly $F_0^\perp$ is a translation-invariant subspace of $L^\infty(G)$ and $F_0$ is a closed ideal in $L^1(G)$. Furthermore,

$$
\nu(F_0) = \{ \xi \in \hat{G} : \hat{f}(\xi) = 0 \text{ for all } f \in F_0 \} = \{0\}.
$$

To see this, first note that for any $f \in L^1(G)$, $\int_G f(t) \, dt = \hat{f}(0)$, so the fact that each $f \in F_0$ has vanishing integral means that $0 \in \nu(F_0)$.

Now suppose there was a $\xi \in \hat{G}$, $\xi \neq 0$, such that $\xi \in \nu(F_0)$. Then $f \in F_0$ implies that $\hat{f}(\xi) = 0$. In other words, $\ker \xi \supseteq F_0 = \ker 0$.

But as $\overline{U}$ and $\overline{\xi}$ are linear functionals, this means that $\ker \overline{U} = \ker \overline{\xi}$, and that there is a non zero $\lambda \in \mathbb{C}$ such that $\overline{U} = \overline{\xi} \xi$.

Now take any two $f, g \in L^1(G)$ such that $\overline{U}(f), \overline{U}(g) \neq 0$. Then

$$
\lambda \overline{\xi} \overline{U}(f \star g) = \overline{U}(f \star g) = \overline{U}(f) \overline{U}(g) = \lambda^2 \overline{\xi} \overline{U}(f \star g),
$$

so $\lambda = 1$. Hence $\overline{U} = \overline{\xi}$ and $\xi = 0$, a contradiction.

Recall that for any translation-invariant subspace $\mathcal{M} \subseteq L^\infty(G)$, the spectrum $\sigma(\mathcal{M})$ is the set of all continuous characters in $\mathcal{M}$:

$$
\sigma(\mathcal{M}) = \mathcal{M} \cap \hat{G}.
$$

By \cite{9} Proposition 4.73, $\sigma(F_0^\perp) = \nu(F_0) = \{0\}$. By \cite{9} Proposition 4.75 b), $F_0^\perp$ is the linear span of the constant character and so consists of the constant functions.

From this analysis we conclude that $t \mapsto \ell(\alpha_{-t}(\omega)) \in F_0^\perp$ is a constant for a.e. $\omega \in \Omega$. We can write this fact as $\alpha_t(\ell) = \ell$ for all $t \in G$.

To complete the proof that $D$ is dense in $X$, we must show that $\ell = 0$ a.e. Fix a $\lambda > 0$ and define $\Lambda = \{ \omega : \ell(\omega) > \lambda \}$.

By the reflexivity of $X' = X^*$ and Lemma 5.4, $|\Lambda| \leq m(f, \lambda) < \infty$. This set is invariant under $\alpha$ (because $\alpha_t(\ell) = \ell$ for all $t \in G$) and of course $\chi_\Lambda \in X$. So a fortiori $\chi_\Lambda \in F$. As $\ell$ is orthogonal to $F$ as well, we have

$$
0 = \int_\Omega \chi_\Lambda \ell \, d\mu \geq \lambda |\Lambda|,
$$

which implies that $|\Lambda| = 0$. As this holds for all $\lambda > 0$, we have proved that $\ell \leq 0$ $\mu$-a.e. But now the same argument applied to the sets $\Lambda' = \{ \omega : \ell(\omega) < -\lambda \}$, where $\lambda$ is an arbitrary positive number, shows that $|\Lambda'| = 0$ for all $\lambda > 0$, and so $\ell = 0$ $\mu$-a.e.

Thus we have shown that any function in $X'$ orthogonal to $D$ must be the zero function, proving the density of $D$ in $X$. We have already proved that $(T_n^d(f))$ converges a.e. for all $f \in D$. Applying Proposition 5.1 the Theorem is proved.
If we have more information about the action \( \alpha \), we can relax the conditions on the space \( X \). In the next corollary, given additional assumptions about the group action, we shall not need the reflexivity of \( X \).

**Corollary 5.6.** In the setup of Theorem 5.5, if \( X \) has absolutely continuous norm and the fixed point space \( F \) is finite dimensional, then \( (T_n^f) \) converges a.e. for every \( f \in X \).

In particular, the result holds if the dynamical system is ergodic.

**Proof.** As \( F \) is finite dimensional, we can find a finite number of mutually disjoint subsets \( \{A_n\}_{n=1}^N \) such that \( F \) is spanned by \( \chi_{A_n} \) for \( 1 \leq n \leq N \). Indeed if \( f \in F \), then for every \( \lambda > 0 \), if we set \( E_\lambda = \{ \omega : |f(\omega)| > \lambda \} \), then \( \chi_{E_\lambda} \in F \). As \( F \) is finite dimensional, there can only be a finite number of different sets \( E_\lambda \), and so \( f \) is a linear combination of a finite number of characteristic functions, making \( f \) a simple function. Hence \( F \) is spanned by a collection of simple functions. As \( F \) is finite dimensional, such a collection must be finite.

By [2 Corollary 1.4.3], \( X^* = X^\ast \). Reasoning as in Theorem 5.5, we consider the set \( D = D_X + F \), where \( F \) is the subspace of all fixed points of the action \( \alpha \) in \( X \) and let \( \ell \in X' \) orthogonal to \( D \). To prove the density of \( D \) in \( X \), we must again show that \( \ell = 0 \) \( \mu \)-a.e.

Firstly, as shown in the proof of Theorem 5.5, \( \ell \) is invariant under the action \( \alpha \). Hence for any \( \lambda > 0 \), \( \{ \omega : |\ell(\omega)| > \lambda \} \) is an invariant subset of \( \Omega \). This implies that the set \( \{ \omega : |\ell(\omega)| > \lambda \} \cap A_n \) is invariant and is either empty or equal to \( A_n \). Hence \( \ell \) is invariant under the action \( \alpha \) if \( A_n \) or \( A_n^c \). From this we conclude that \( \ell \) is a (possibly trivial) linear combination of the characteristic functions \( \chi_{A_n} \) and \( \chi_{A_n^c} \) for \( 1 \leq n \leq N \). Note that as \( \chi_{A_n} \in F \), it is a linear combination of the functions \( \{\chi_{A_n}\}_{n=1}^N \), so \( \ell \) is really just a linear combination of the functions \( \chi_{A_n} \).

As \( \ell \) is orthogonal to \( F \), this implies that \( \ell = 0 \) a.e. Hence \( D \) is dense in \( X \). Applying Proposition 5.1, the Corollary is proved.

It is easy to prove that the action of \( G \) on \( (\Omega, \mu) \) is ergodic if and only if the only invariant function is the characteristic function \( \chi_\Omega \). In this case, \( F \) has dimension one if \( \Omega \) has finite measure, for then \( \chi_\Omega \in F \), and \( F = \{0\} \) if \( \Omega \) has infinite measure. This is because if there was a non-zero \( f \in F \), then as we have seen above, \( \{ \omega : |f(\omega)| > \lambda \} \) is an invariant subset of \( \Omega \) for each \( \lambda > 0 \), contradicting the ergodicity of the action.

Finally, let us mention another way to obtain a dense subset of a r.i. BFSs on which the pointwise convergence of the ergodic averages can readily be checked.

**Corollary 5.7.** In the setup of Theorem 5.8 if \( T^\# \) is also of weak type \((E,Z)\) for r.i. BFSs \( E \) and \( Z \) where \( E \) is reflexive and \( X \cap E \) is dense in \( X \), then \( (T_n^f) \) converges a.e. for every \( f \in X \).

**Proof.** By Theorem 5.8, \( (T_n^f) \) converges a.e. for all \( f \in E \). Hence we have a dense subset of \( X \), namely \( X \cap E \), on which the ergodic averages converge pointwise. Applying Proposition 5.1 finishes the proof.

If for example \( X = L^1(\Omega) \) and \( E = Z = L^p(\Omega) \) for some \( 1 < p < \infty \), this Corollary is applicable. A special case of this setup in given in [3 Theorem 3].
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