Performance Analysis of a Cooperative Wireless Network with Adaptive Relays: A Network-Level Study

Ioannis Dimitriou, Nikolaos Pappas

Abstract

In this work, we investigate the stability region, the throughput performance, and the queueing delay of an asymmetric relay-assisted cooperative random access wireless network with multipacket reception (MPR) capabilities. We consider a network of \( N \) saturated source users that transmit packets to a common destination node with the cooperation of two relay nodes. The relays are equipped with infinite capacity buffers, and assist the users by forwarding the packets that failed to reach the destination. Moreover, the relays have also packets of their own to transmit to the destination. We assume random access of the medium and slotted time. With the relay-assisted cooperation, the packets originated by each user can be transmitted to the destination through multiple relaying paths formed by the relays. We assume that the relays employ an adaptive retransmission control mechanism. In particular, a relay node is aware of the status of the other relay, and accordingly adapts its transmission probability. Such a protocol is towards self-aware networks and leads to substantial performance gains in terms of delay. We investigate the stability region and the throughput performance for the full MPR model. Moreover, for the two-user, two-relay case we derive the generating function of the stationary joint queue-length distribution at the relays by solving a Riemann-Hilbert boundary value problem. Finally, for the symmetric case we obtain explicit expressions for the average queueing delay in a relay node without the need of solving a boundary value problem. Numerical examples are presented providing insights on the system performance.

Index Terms

Adaptive transmission, Stability region, Queueing analysis, Boundary value problem

I. Dimitriou is with the Department of Mathematics, University of Patras, Patra, Peloponnese, Greece. (e-mail: idimit@math.upatras.gr).
N. Pappas is with the Department of Science and Technology, Linköping University, Norrköping SE-60174, Sweden. (e-mail: nikolaos.pappas@liu.se).
I. INTRODUCTION

Over the past few decades, wireless communications and networking have witnessed an unprecedented growth. The growing demands require high data rates, considerably large coverage areas and high reliability. Relay-assisted wireless networks have been proposed as a candidate solution to fulfill these requirements [1], since relays can decrease the delay and can also provide increased reliability and higher energy efficiency [2], [3]. A relay-based cooperative wireless system operates as follows: There is a number of source users that transmit packets to a common destination node, and a number of relay nodes that assist the source users by storing and retransmitting the packets that failed to reach the destination; e.g., [4], [5], [6], [7]. A cooperation strategy among sources and relays specifies which of the relays will cooperate with the sources. This problem gives rise to the usage of a cooperative space diversity protocol [8], where each user has a number of “partners” (i.e., relays) that are responsible for retransmitting its failed packets.

A. Related work

Cooperative relaying is mostly considered at the physical layer, and is based on information-theoretic considerations. The classical relay channel was first examined in [9] and later in [2]. Recently cooperative communications have received renewed attention, as a powerful technique to combat fading and attenuation in wireless networks; e.g., [8], [10]. Most of the research has concentrated on information-theoretic studies. Recent works [4], [5], [7], [11] shown that similar gains can be achieved by network-layer cooperation. By network-layer cooperation, relaying is assumed to take place at the protocol level avoiding physical layer considerations.

In addition, random access recently re-gained interest due to the increased number of communicating devices in 5G networks, and the need for massive uncoordinated access [12]. Random access and alternatives schemes and their effect on the operation of LTE and LTE-A are presented in [12], [13], [14]. In [15], the effect of random access in Cloud-Radio Access Network is considered.

The characterization of the stable throughput region, i.e. the stability region, which gives the set of arrival rates such that there exist transmission probabilities under which the system is stable, is a meaningful metric to measure the impact of bursty traffic and the concept of interacting nodes in a network; e.g., [16], [17], [18].

Except throughput, delay is another important metric, which recently received considerable attention due to the rapid growth on supporting real-time applications, which in turn require delay-based guarantees. However, due to the interdependence among queues, the characterization of the delay even in small networks with random access is a rather difficult task, even for the non-cooperative collision channel model [19]. In [19], queueing delay was studied with the aid of the theory of boundary value problems. The traditional collision model is accurate for modeling wire-line communication, however, it is not an appropriate model for probabilistic reception in wireless multiple access
networks. For the non-cooperative multipacket reception (MPR) model, delay analysis was performed in [20], based on the assumption of a symmetric network. Recently, the authors in [21] generalized the model in [20], [19] by considering time-varying links between nodes where the channel state information was modeled according to a Gilbert-Elliot model. The study of queueing systems using the theory of boundary value problems was initiated in [22], and a concrete methodological approach was given in [23], [24]. The vast majority of queueing models are analyzed with the aid of the theory of boundary value problems referring to continuous time models, e.g., [25], [26], [27], [28], [29], [30], [31], [32]. On the contrary, there are very few works on the analysis of discrete time models [19], [21], [18], [33], [34]. This is mainly due to the complex boundary behavior of the underlying random walk, which reflects the interdependence and coupling among the queues.

B. Contribution

Our contribution is summarized as follows. We consider a cooperative wireless network with $N$ saturated source users, two relay nodes with adaptive transmission control, and a common destination. Our primary interest is to investigate the stability conditions, throughput performance, and provide expressions for the queueing delay experienced at the buffers of relay nodes. The time is slotted, corresponding to the duration of a transmission of a packet, and the sources/relay nodes access the medium in a random access manner. The sources transmit packets to the destination with the cooperation of the two relays. If a transmission of a user’s packet to the destination fails, the relays store it in their queues and try to forward it to the destination at a subsequent time slot. Moreover, the relays have also external bursty arrivals that are stored in their infinite capacity queues. We consider MPR capabilities at the destination node.

We assume that there is no coordination among relays and sources, but the destination node can sense both of them. Here, we assume that the destination node gives “priority” to the sources when it senses that they will transmit. If it senses that all of the sources will remain silent then it switches to the relays. The relays are accessing the wireless channel randomly and employ a state-dependent transmission protocol. More precisely, a relay adapts its transmission characteristics based on the status of the other relay in order to exploit its idle slots, and to increase its transmission efficiency, which in turn leads towards self-aware networks [35]. More specifically, we assume that each relay node is aware of the state of the other one. Note that this feature is common in cognitive radios [17], [35]. To the best of our knowledge this variation of random access has not been reported in the literature. The contribution of this work has two main parts focused on the stable throughput region, and the detailed analysis of the queueing delay at relay nodes.

1) Stability analysis and throughput performance: We provide the throughput analysis of the general two-user network with MPR capabilities and the symmetric $N$-user network under random access. The performance characterization for
symmetric users can provide insights on scalability of the network. In addition, we provide the stability conditions for the queues at the relays.

2) Delay Analysis: The second part of the contribution of this work is the delay analysis. Except its practical implications, our work is also theoretically oriented. To the best of our knowledge there is no other work in the related literature that deals with the detailed delay analysis of an asymmetric random access cooperative wireless system with adaptive transmissions and MPR capabilities.

To enhance the readability of our work we consider the case of \( N = 2 \) source users, and focus on a subclass of MPR models, called the “capture” channel, under which at most one packet can be successfully decoded by the receiver of the node \( D \), even if more than one nodes transmit. We need to mention, that the assumption of two users is not restrictive, and our analysis can be extended to the general case of \( N \) users. Moreover, our analysis remains valid even for the case of general MPR model. However, in both cases some important technical requirements must be further taken into account, which in turn will worse the readability of the paper. Besides, our aim here is to focus on the fundamental problem of characterizing the delay in a cooperative wireless network with two relay nodes, and our model and analysis can serve as a building block for the more general case.

Our system is modeled as a two-dimensional discrete time Markov chain, and we show that the generating function of the stationary joint relay queue length distribution by solving a fundamental functional equation with the aid of a Riemann-Hilbert boundary value problem. Furthermore, each relay node employs an adaptive transmission policy, under which it adapts its transmission probabilities based on the status of the queue of the other relay. Moreover, the kernel of this functional equation has never been treated in the related literature. More precisely,

- Based on a relation among the values of the transmission probabilities we distinguish the analysis in two cases, which are different both in the modeling, and in the technical point of view. In particular, the analysis leads to the formulation of two boundary value problems \[36\] (i.e., a Dirichlet, and a Riemann-Hilbert problem), the solution of which will provide the generating function of the stationary joint distribution of the queue size for the relays. This is the key element for obtaining expressions for the average delay at each user node. To our best knowledge, it is the first time in the related literature on cooperative networks with MPR capabilities, where such an analysis is performed.

- Furthermore, for the two-user, two-relay symmetric system, we provide explicit expression for the average queueing delay, without the need of solving a boundary value problem.

Concluding, the analytical results in this work, to the best of our knowledge, have not been reported in the literature.

The rest of the paper is organized as follows. In Section II we describe the system model in detail. Section III is devoted to the investigation of the throughput and the stability conditions for the asymmetric MPR model of \( N = 2 \), while in
Section IV we generalize our previous results for the general case of $N$ users with MPR capabilities. In Section V we focus on the delay analysis for the general asymmetric two-user with two-relays network. The fundamental functional equation is derived, and some preparatory results in view of the resolution of the functional equation are obtained. We formulate and solve two boundary value problems, the solution of which provide the generating function of the stationary joint queue length distribution of relay nodes. The basic performance metrics are obtained, and important hints regarding their numerical evaluation are also given. In Section VI we obtain explicit expressions for the average delay at each relay node for the symmetrical system without solving a boundary value problem. Finally, numerical examples that shows insights in the system performance are given in Section VII.

II. MODEL DESCRIPTION AND NOTATION

In this work, we consider a network consisting of $N$ saturated users-sources, two relays, and one destination node. In this section, we will describe the case of $N = 2$ saturated users assisted by two relays as depicted in Fig. 1. We focus on the two-user scenario in order to facilitate the presentation and the description of the cooperation protocol.

A. Network Model

We consider a network of $N = 2$ saturated source users, i.e. sources 1 and 2, two relay nodes, denoted by $R_1$ and $R_2$, and a common destination node $D$ depicted in Fig. 1. The sources transmit packets to the node $D$ with the cooperation of the relays. The packets have equal length and the time is divided into slots corresponding to the transmission time of a packet.

We assume that the relays and the destination have multipacket reception (MPR) capabilities and the success probabilities for the transmissions will be provided in Section II-C. MPR is a more suitable model than the collision channel since it can capture better the wireless transmissions. The source-users have random access to the medium with no coordination among them. At the beginning of a slot, the source user $P_k$ attempts to transmit a packet with a probability $t_k$, $k = 1, 2$, i.e., with probability $\bar{t}_k = 1 - t_k$ remains silent. The nodes are assume to have priority over the relays. More specifically, the sources and the relays transmit in different channels, however, the destination node can overhear both of the channels. However, the destination node gives “priority” to the sources if it senses that they will transmit. If the destination senses no activity from the sources, it switches to the relays. We assume that this sensing time is negligible. If a packet transmission from a source to the destination fails and at the same time if at least one of the relays will be able to decode this packet, then will store it in its queue with a probability, and it will forward it to the destination at a subsequent time slot. The queues at the relays are assumed to have infinite size.

\textsuperscript{1}In this section we will present the system model for the case of two users. However, in Section IV we consider the case where there are $n$-symmetric users.
Fig. 1. An instance of the two-relay cooperative wireless network with two users. In addition, the relays $R_1$ and $R_2$ have their own traffic $\hat{\lambda}_1$ and $\hat{\lambda}_2$ respectively, and they are assisting the users $P_1$ and $P_2$ by forwarding part of users’ packets to the destination $D$. The relays are assumed to have infinite capacity buffers. The case of pure relays can be obtained by replacing $\hat{\lambda}_1 = \hat{\lambda}_2 = 0$.

B. Description of Relay Cooperation

If a transmission of a user’s packet to the destination fails, the relays overhear the wireless transmission, they can store it in their queues with a probability, and try to forward it to the destination at a subsequent time slot. In case that both relays receive the same packet from a user, they choose randomly which will store the packet. In particular, we define the probability $p_{a_{i,j}}$ that a transmitted packet from the $i$-th source will be stored at the queue of $j$-th relay if the relay is able to decode it. This probability captures two scenarios, (i) the partial cooperation of a relay, which was introduced in [37] and (ii) when both relays receive the failed packet from node $i$, then the first one will keep in its queue with probability $p_{a_{i,1}}$ and with probability $p_{a_{i,2}} = 1 - p_{a_{i,1}}$ will be stored in the queue of the second relay. We would like to emphasize that in case that only one relay, i.e. the first relay, will receive correctly a failed packet, then it will store it in its queue with probability $p_{a_{i,1}}$. This probability, controls the amount of the cooperation that this relay provides. However, in this work we assume that if only one relay receives successfully a packet that fails to reach the destination, then this packet will be stored in its queue. When both relays decode correctly a failed packet, if we assume that $p_{a_{i,1}} + p_{a_{i,2}} = 1$, then the packet will enter one queue only, either the first or the second one. If we will assume that $p_{a_{i,1}} + p_{a_{i,2}} < 1$, then there is a probability that the failed packet will not be accepted in the queues of the relays and it has to be retransmitted in a future timeslot by its source.

Let $N_{i,n}$ be the number of packets in the buffer of relay node $R_i$, $i = 1, 2$, at the beginning of the $n$th slot. Moreover, during the time interval $(n, n + 1]$ (i.e., during a time slot) the relay $R_i$, $i = 1, 2$ generates also packets of its own (i.e., exogenous traffic). Let $\{A_{i,n}\}_{n \in \mathbb{N}}$ be a sequence of i.i.d. random variables where $A_{i,n}$ represents the number of packets which arrive at $R_i$ in the interval $(n, n + 1]$, with $E(A_{i,n}) = \tilde{\lambda}_i < \infty$. The network with pure relays can be obtained by replacing $\hat{\lambda}_1 = \hat{\lambda}_2 = 0$. 
In case node $D$ senses no activity from the source users at the beginning of a slot, it switches to the channel of relay nodes. If there are stored packets in the buffers of the relays, they will also attempt to transmit a packet to the node $D$ with a probability.

Due to the interference among the relays, we consider the following opportunistic access policy: If both relays are non empty, $R_i$, $i = 1, 2$, transmits a packet with probability $\alpha_i$. If $R_1$ (resp. $R_2$) is the only non-empty, it adapts its transmission probability. More specifically, it transmits a packet with a probability $\alpha_i^* > \alpha_i$, in order to utilize the idle slot of the neighbor relay node\footnote{We consider the general case for $\alpha_i^*$ instead of assuming directly $\alpha_i^* = 1$. This can handle cases where the node cannot transmit with probability one even if the other node is silent, e.g., when the nodes are subject to energy limitations. It is outside of the scope of this work to consider specific cases and we intent to keep the proposed analysis general.}. Note that in such a case, a relay node is aware about the state of its neighbor\footnote{In such a shared access network, it is practical to assume a minimum exchanging information of one bit between the nodes.}

C. Physical Layer Model

The MPR channel model used in this work is a generalized form of the packet erasure model. In wireless networks, a transmission over a link is successful with a probability. We denote $P_s(i, k, A)$ the success probability of the link between nodes $i$ and $k$ when the set of active transmitters are in $A$. For example, $P_s(1, R_1, \{1, 2\})$ denotes the success probability for the link between the first source and the first relay when both sources are transmitting. The probability that the transmission fails is denoted by $\overline{P}_s(1, R_1, \{1, 2\})$. In order to take also into account the interference among the relays, we have to distinguish the success probabilities when a relay transmits and the other is active or inactive (i.e., it is empty). Thus, when $i \in \{R_1, R_2\}$, the success probability of the link between relay node $i$ and node $D$ when relay node $i$ is the only non empty is denoted by $P_s^*(i, D, \{i\})$. In this work we distinguish this case, in order to have more general results that can capture scenarios that one relay can increase its transmission power when the other relay is empty, thus silent, in order to achieve a higher success probability. Thus, $P_s^*(R_1, D, \{R_1\}) \geq P_s(R_1, D, \{R_1\})$. The probabilities of successful packet reception can be obtained using the common assumption in wireless networks that a packet can be decoded correctly by the receiver if the received SINR (Signal-to-Interference-plus-Noise-Ratio) exceeds a certain threshold. The SINR depends on the modulation scheme, the target bit error rate and the number of bits in the packet\footnote{In such a shared access network, it is practical to assume a minimum exchanging information of one bit between the nodes.} and the expressions for the success probabilities can be found in several papers, i.e for the case of Raleigh fading refer in \cite{4}. On the other hand, if source user $k = 1, 2$, is the only that transmits, $P_s(k, D, \{k\})$ denotes the probability that its packet is successfully decoded by the destination, while with probability $\overline{P}_s(k, D, \{k\}) = 1 - P_s(k, D, \{k\})$ this transmission fails.
TABLE I

| Symbol  | Explanation                                                                 |
|---------|-----------------------------------------------------------------------------|
| $N_{i,n}$ | The number of packets in relay node $R_i$ at the beginning of slot $n$       |
| $A_{i,n}$ | The number of packets arriving during $(n, n+1]$ in relay node $R_i$, $i = 1, 2$ |
| $\lambda_i$ | The expected number of external arrivals in relay node $R_i$, $i = 1, 2$, during a slot |
| $t_k$   | Transmission probability of source $k$, $k = 1, 2$                         |
| $a_i$   | Transmission probability of relay node $R_i$, $i = 1, 2$, when both users are active (i.e., non-empty) |
| $a^*_i$ | Transmission probability of relay node $R_i$, $i = 1, 2$, when it is the only active (i.e., non-empty) node |
| $P_s(k, m, A)$ | Success probability of the link between node $k$ and $m$ when the set of the transmitted nodes are in $A$ |
| $P^*_s(R_i, D, \{R_i\})$ | Success probability of relay node $R_i$, $i = 1, 2$, when it is the only active (i.e., non-empty) node |
| $P_{s,k}(k, R_i, \{1, 2\})$ | Success probability of the link between source $k$ and $R_i$ when both sources transmit, but source $k$ fails to directly reach node $D$, $k = 1, 2$, $i = 1, 2$. |

We now provide the service rates $\mu_1$, $\mu_2$ seen at relay nodes. For the first relay we have

$$
\mu_1 = \tilde{t}_1 \tilde{t}_2 \left[ \Pr(N_2 = 0) \alpha^*_1 P^*_s(R_1, D, \{R_1\}) + \Pr(N_2 > 0) \alpha_1 \left( \alpha_2 P_s(R_1, D, \{R_1, R_2\}) + \alpha_2 P_s(R_1, D, \{R_2\}) \right) \right].
$$

(1)

Similarly we have the service rate at the second relay

$$
\mu_2 = \tilde{t}_1 \tilde{t}_2 \left[ \Pr(N_1 = 0) \alpha^*_2 P^*_s(R_2, D, \{R_2\}) + \Pr(N_1 > 0) \alpha_2 \left( \alpha_1 P_s(R_2, D, \{R_1, R_2\}) + \alpha_1 P_s(R_2, D, \{R_2\}) \right) \right].
$$

(2)

Note that the success probability $P_s(R_1, D, \{R_1, R_2\})$ (resp. $P_s(R_2, D, \{R_1, R_2\})$) refers to the case where a submitted packet from relay $R_1$ (resp. $R_2$) is successfully decoded by node $D$, and includes both the case where only a packet from $R_1$ (resp. $R_2$) is decoded, both the case where both relays have successful transmissions (i.e. MPR case).

We define the following two variables in order to simplify the presentation in the analysis

$$
\Delta_1 = P_s(R_1, D, \{R_1, R_2\}) - P_s(R_1, D, \{R_1\}),
$$

(3)

and

$$
\Delta_2 = P_s(R_2, D, \{R_1, R_2\}) - P_s(R_2, D, \{R_2\}).
$$

(4)

These variables can be seen as an indication regarding the MPR capability for each user. If $\Delta_1 \to 0$, then the interference caused by the other user is negligible.
III. THROUGHPUT AND STABILITY ANALYSIS FOR THE TWO-USER CASE – GENERAL MPR CASE

In this section, we provide the analysis for the two-user case under the MPR channel model. More specifically, we provide the throughput analysis for the two users and in addition we derive the stability conditions for the queues at the relays.

Based on the definition in [18], a queue is said to be stable if \( \lim_{n \to \infty} Pr[N_{i,n} < x] = F(x) \) and \( \lim_{x \to \infty} F(x) = 1 \). Loynes’ theorem [39] states that if the arrival and service processes of a queue are strictly jointly stationary and the average arrival rate is less than the average service rate, then the queue is stable. If the average arrival rate is greater than the average service rate, then the queue is unstable and the value of \( N_{i,n} \) approaches infinity almost surely. The stability region of the system is defined as the set of arrival rate vectors \( \lambda = (\lambda_1, \lambda_2) \), for which the queues in the system are stable.

We start the analysis by deriving the throughput per user which allow us to calculate the endogenous arrivals at the relays.

A. Throughput per user

Here we will consider the throughput per (source) user when both queues of the relays are stable. Conditions for stability are given in a subsequent subsection. When the queues at the relays are not stable the throughput per user can be obtained using the approach in [4].

The throughput per user \( T_k \), is the direct throughput when the transmission to the destination is successful plus the throughput contributed by the relays (if they can decode the transmission) in case of a failed transmission to the destination. Thus, the throughput seen by the first user is given by

\[
T_1 = T_{1,D} + T_{1,R},
\]

(5)

where

\[
T_{1,D} = t_1t_2P_s(1, D, \{1\}) + t_1t_2P_s(1, D, \{1, 2\}),
\]

(6)

and

\[
T_{1,R} = t_1(1 - t_2)P_s(1, D, \{1\})P_s(1, R_1, \{1\})P_s(1, R_2, \{1\}) + t_1t_2P_s(1, D, \{1, 2\})P_s(1, R_1, \{1, 2\})P_s(1, R_2, \{1, 2\}) + t_1(1 - t_2)P_s(1, D, \{1\})P_s(1, R_1, \{1\})P_s(1, R_2, \{1\}) + t_1t_2P_s(1, D, \{1, 2\})P_s(1, R_1, \{1, 2\})P_s(1, R_2, \{1, 2\}) + t_1(1 - t_2)P_s(1, D, \{1\})P_s(1, R_1, \{1\})P_s(1, R_2, \{1\}) + t_1t_2P_s(1, D, \{1, 2\})P_s(1, R_1, \{1, 2\})P_s(1, R_2, \{1, 2\}).
\]

Similarly we can obtain the throughput for the second user.
The aggregate or network-wide throughput of the network when the queues at the relays are both stable is

\[ T_{\text{aggr}} = T_1 + T_2 + \lambda_1 + \lambda_2. \]  

(7)

**B. Endogenous arrivals at the relays**

Here, we will derive the internal (or endogenous) arrival rate from the users to each relay. We would like to mention that the relays have also their own traffic (exogenous) denoted by \( \hat{\lambda}_i \) for the \( i \)-th relay.

A packet from a user can enter a queue at one relay if the transmission to the destination fails and at the same time at least one relay decodes correctly the packet. In the two-user case with MPR capabilities, in a relay up to two packets can enter the queue.

Here we will derive the endogenous arrival from user 1 to the first relay denoted by \( \lambda_{1,1} \). The \( \lambda_{1,1} \) is also the probability that a transmitted packet by the first user will enter the queue at the first relay. So, the term \( \lambda_{i,j} \) denotes the endogenous arrival probability from \( i \)-th user, \( i = 1, 2 \), to the queue at the \( j \)-th relay, \( j = 1, 2 \).

The endogenous arrival rate \( \lambda_{1,1} \) is given by

\[
\lambda_{1,1} = t_1 (1 - t_2) \bar{P}_s(1, D, \{1\}) P_s(1, R_1, \{1\}) \bar{P}_s(1, R_2, \{1\}) + t_1 t_2 \bar{P}_s(1, D, \{1, 2\}) P_s(1, R_1, \{1, 2\}) \bar{P}_s(1, R_2, \{1, 2\}) + \]
\[ + t_1 (1 - t_2) \bar{P}_s(1, D, \{1\}) P_s(1, R_1, \{1\}) P_s(1, R_2, \{1\}) p_{a_{1,1}} + t_1 t_2 \bar{P}_s(1, D, \{1, 2\}) P_s(1, R_1, \{1, 2\}) P_s(1, R_2, \{1, 2\}) p_{a_{1,1}}. \]

The endogenous arrival rate \( \lambda_{1,2} \) is given by

\[
\lambda_{1,2} = t_1 (1 - t_2) \bar{P}_s(1, D, \{1\}) P_s(1, R_1, \{1\}) P_s(1, R_2, \{1\}) + t_1 t_2 \bar{P}_s(1, D, \{1, 2\}) P_s(1, R_1, \{1, 2\}) P_s(1, R_2, \{1, 2\}) + \]
\[ + t_1 (1 - t_2) \bar{P}_s(1, D, \{1\}) P_s(1, R_1, \{1\}) P_s(1, R_2, \{1\}) p_{a_{1,2}} + t_1 t_2 \bar{P}_s(1, D, \{1, 2\}) P_s(1, R_1, \{1, 2\}) P_s(1, R_2, \{1, 2\}) p_{a_{1,2}}. \]

Similarly we can define \( \lambda_{2,1} \), and \( \lambda_{2,2} \). Note that \( T_{1,R} = \lambda_{1,1} + \lambda_{1,2} \), which is the relayed throughput for the first user defined in the previous subsection.

The average arrival rate at the relay \( i \) is given by

\[
\lambda_i = \hat{\lambda}_i + \lambda_{1,i} + \lambda_{2,i}. \]  

(8)

Recall that \( p_{a_{1,1}} \) denotes the probability that the transmitted by the first source packet which is correctly received by both relays and failed to reach the destination will enter the queue at the first relay. The term \( p_{a_{1,2}} = 1 - p_{a_{1,1}} \) denotes the probability that the packet will enter the queue at the second relay. Thus, a packet can enter only one queue so we avoid wasting resources by transmitting the same packet twice.
C. Stability conditions for the queues at the relays

We now proceed with the investigation of the stability conditions, based on the concept of stochastic dominant systems developed in [17], [18]. The stability region of the system is defined as the set of arrival rate vectors \( \lambda = (\lambda_1, \lambda_2) \), for which the queues of the relay nodes are stable. Here, we will derive the stability analysis for the total average arrival rate at each relay, \( \lambda_i \).

The next theorem provides the stability criteria for the two-user general MPR case.

**Theorem III.1.** The stability region \( \mathcal{R} \) is given by \( \mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \) where

\[
\mathcal{R}_1 = \left\{ (\lambda_1, \lambda_2) : \lambda_1 < \bar{t}_1 \bar{t}_2 \alpha_1^* P_s^*(R_1, D, \{R_1\}) - \frac{\lambda_2 [\alpha_1^* P_s^*(R_1, D, \{R_1\}) - \alpha_1 [\alpha_2 \Delta_1 + P_s(R_1, D, \{R_1\})]]}{\alpha_2 [\alpha_1 \Delta_2 + P_s(R_2, D, \{R_2\})]}, \right.
\]

\[
\lambda_2 < \bar{t}_1 \bar{t}_2 \alpha_2 \left[ \alpha_1 \Delta_2 + P_s(R_2, D, \{R_2\}) \right] \right\} \text{, (9)}
\]

and

\[
\mathcal{R}_2 = \left\{ (\lambda_1, \lambda_2) : \lambda_2 < \bar{t}_1 \bar{t}_2 \alpha_2^* P_s^*(R_2, D, \{R_2\}) - \frac{\lambda_1 [\alpha_2^* P_s^*(R_2, D, \{R_2\}) - \alpha_2 [\alpha_2 \Delta_1 + P_s(R_2, D, \{R_2\})]]}{\alpha_2 [\alpha_1 \Delta_2 + P_s(R_1, D, \{R_1\})]}, \right.
\]

\[
\lambda_1 < \bar{t}_1 \bar{t}_2 \alpha_1 \left[ \alpha_2 \Delta_1 + P_s(R_1, D, \{R_1\}) \right] \right\}. \text{ (10)}
\]

**Proof.** The average service rates of the first and second relay are given by (1) and (2), respectively.

Since the average service rate of each relay depends on the queue size of the other relay, the stability region cannot be computed directly. Thus, we apply the stochastic dominance technique introduced in [17], i.e. we construct hypothetical dominant systems, in which the relay with the empty queue transmits dummy packets, while the non-empty relay transmits according to its traffic.

In the first dominant system, the first relay transmit dummy packets and the second relay behaves as in the original system. All the rest operational aspects remain unaltered in the dominant system. Thus, in this dominant system, the first queue never empties, hence the service rate for the second relay is

\[
\mu_2 = \bar{t}_1 \bar{t}_2 \alpha_2 \left[ \alpha_1 P_s(R_2, D, \{R_1, R_2\}) + \bar{r}_1 P_s(R_2, D, \{R_2\}) \right]. \text{ (11)}
\]

Which can be rewritten as

\[
\mu_2 = \bar{t}_1 \bar{t}_2 \alpha_2 \left[ \alpha_1 \Delta_2 + P_s(R_2, D, \{R_2\}) \right]. \text{ (12)}
\]

Then, we can obtain stability conditions for the second relay by applying Loynes’ criterion [39]. The queue at the second source is stable if and only if \( \lambda_2 < \mu_2 \), that is \( \lambda_2 < \bar{t}_1 \bar{t}_2 \alpha_2 \left[ \alpha_1 \Delta_2 + P_s(R_2, D, \{R_2\}) \right] \). Then we can obtain the
probability that the second relay is empty by applying Little’s theorem, i.e.

\[
\Pr(N_2 = 0) = 1 - \frac{\lambda_2}{\bar{t}_1 \bar{t}_2 \alpha_2 \alpha_1 [\alpha_1 \Delta_2 + P_s(R_2, D, \{R_2\})]}. \quad (13)
\]

After replacing (13) into (1) we obtain

\[
\mu_1 = \bar{t}_1 \bar{t}_2 \alpha_1^* P_s^*(R_1, D, \{R_1\}) - \frac{\lambda_2 \alpha_1^* P_s^*(R_1, D, \{R_1\})}{\alpha_2 [\alpha_1 \Delta_2 + P_s(R_2, D, \{R_2\})]} + \frac{\lambda_2 \alpha_1 [\alpha_2 \Delta_1 + P_s(R_1, D, \{R_1\})]}{\alpha_2 [\alpha_1 \Delta_2 + P_s(R_2, D, \{R_2\})]} . \quad (14)
\]

Thus, after applying Loynes’ criterion, the stability condition for the first relay in the first dominant system is

\[
\lambda_1 < \bar{t}_1 \bar{t}_2 \alpha_1^* P_s^*(R_1, D, \{R_1\}) - \frac{\lambda_2 [\alpha_1^* P_s^*(R_1, D, \{R_1\}) - \alpha_1 [\alpha_2 \Delta_1 + P_s(R_1, D, \{R_1\})]]}{\alpha_2 [\alpha_1 \Delta_2 + P_s(R_2, D, \{R_2\})]} . \quad (15)
\]

The stability region \( \mathcal{R}_1 \) obtained from the first dominant system is given by

\[
\mathcal{R}_1 = \left\{ (\lambda_1, \lambda_2) : \lambda_1 < \bar{t}_1 \bar{t}_2 \alpha_1^* P_s^*(R_1, D, \{R_1\}) - \frac{\lambda_2 [\alpha_1^* P_s^*(R_1, D, \{R_1\}) - \alpha_1 [\alpha_2 \Delta_1 + P_s(R_1, D, \{R_1\})]]}{\alpha_2 [\alpha_1 \Delta_2 + P_s(R_2, D, \{R_2\})]} , \right. \\
\left. \lambda_2 < \bar{t}_1 \bar{t}_2 \alpha_2 [\alpha_2 \Delta_1 + P_s(R_2, D, \{R_2\})] \right\}. \quad (16)
\]

Similarly, we construct a second dominant system where the second relay transmits a dummy packet when it is empty and the first relay behaves as in the original system. All other operational aspects remain unaltered in the dominant system. Following the same steps as in the first dominant system, we obtain the stability region, \( \mathcal{R}_2 \), of the second dominant system.

\[
\mathcal{R}_2 = \left\{ (\lambda_1, \lambda_2) : \lambda_2 < \bar{t}_1 \bar{t}_2 \alpha_2^* P_s^*(R_2, D, \{R_2\}) - \frac{\lambda_1 [\alpha_2^* P_s^*(R_2, D, \{R_2\}) - \alpha_2 [\alpha_1 \Delta_2 + P_s(R_2, D, \{R_2\})]]}{\alpha_1 [\alpha_2 \Delta_1 + P_s(R_1, D, \{R_1\})]} , \right. \\
\left. \lambda_1 < \bar{t}_1 \bar{t}_2 \alpha_2 [\alpha_2 \Delta_1 + P_s(R_1, D, \{R_1\})] \right\}. \quad (17)
\]

An important observation made in [17] is that the stability conditions obtained by the stochastic dominance technique are not only sufficient but also necessary for the stability of the original system.

The indistinguishability argument [17] applies to our problem as well. Based on the construction of the dominant system, it is easy to see that the queue sizes in the dominant system are always greater than those in the original system, provided they are both initialized to the same value and the arrivals are identical in both systems. Therefore, given \( \lambda_2 < \mu_2 \), if for some \( \lambda_1 \), the queue at the first relay is stable in the dominant system, then the corresponding queue
in the original system must be stable. Conversely, if for some $\lambda_1$ in the dominant system, the queue at the first relay saturates, then it will not transmit dummy packets, and as long as the first relay has a packet to transmit, the behavior of the dominant system is identical to that of the original system since dummy packet transmissions are eliminated as we approach the stability boundary. Therefore, the original and the dominant systems are indistinguishable at the boundary points.

The stability region obtained in Theorem III.1 is depicted in Fig. 2. To simplify presentation, we denote the points $A_1, A_2, B_1, B_2$ with the following expressions

$$A_1 = t_1 t_2 \alpha_1^* P_s^*(R_1, D, \{R_1\})$$
$$A_2 = t_1 t_2 \alpha_1 [\alpha_2 \Delta_1 + P_s(R_1, D, \{R_1\})]$$
$$B_1 = t_1 t_2 \alpha_2^* P_s^*(R_2, D, \{R_2\})$$
$$B_2 = t_1 t_2 \alpha_2 [\alpha_1 \Delta_2 + P_s(R_2, D, \{R_2\})].$$

**Remark 1.** The stability region is a convex polyhedron when the following condition holds

$$\frac{\alpha_1(P_s(R_1, D, \{R_1\} + \alpha_2 \Delta_1))}{\alpha_1^* P_s^*(R_1, D, \{R_1\})} + \frac{\alpha_2(P_s(R_2, D, \{R_2\} + \alpha_1 \Delta_2))}{\alpha_2^* P_s^*(R_2, D, \{R_2\})} \geq 1.$$

In the previous condition, when equality holds, the region becomes a triangle and coincides with the case of time-sharing of the channel between the relays. Convexity is an important property since it corresponds to the case when parallel concurrent transmissions are preferable to a time-sharing scheme. Additionally, convexity of the stability region implies that if two rate pairs are stable, then any rate pair lying on the line segment joining those two rate pairs is also stable.

**Remark 2.** The case of pure relays can be obtained easily by replacing $\widehat{\lambda}_1 = \widehat{\lambda}_2 = 0$. 

---

Fig. 2. The stability region described in Theorem III.1.
Remark 3. The network without relay’s assistance can be obtained by \( p_{a_{1,2}} = p_{a_{1,1}} = 0 \). In this case, we have a network with saturated users and also two users with bursty traffic that transmit packets only when the saturated users are silent.

Remark 4. One can connect the endogenous arrivals from the users to the relays with the stability conditions, obtained in Theorem III.1, by replacing the relevant expressions of \( \hat{\lambda}_1 \) and \( \hat{\lambda}_2 \) into \( \lambda_1 \) and \( \lambda_2 \).

Remark 5. A slightly different scenario is captured by the case where the relays can transmit in a different channel than the users and the destination can hear both channels at the same time. The receivers at the relays are operating at the same channels where the users are transmitting. In this case, we can have a full duplex operation at the relays on different bands. Thus, we have the following average service rate for the first relay

\[
\mu_1 = \Pr(N_2 = 0)\alpha_1^* P_s(R_1, D, \{R_1\}) + \Pr(N_2 > 0)\alpha_1 (\alpha_2 P_s(R_1, D, \{R_1, R_2\}) + \bar{\alpha}_2 P_s(R_1, D, \{R_1\})).
\] (18)

Similarly we have the service rate at the second relay

\[
\mu_2 = \Pr(N_1 = 0)\alpha_2^* P_s(R_2, D, \{R_2\}) + \Pr(N_1 > 0)\alpha_2 (\alpha_1 P_s(R_2, D, \{R_1, R_2\}) + \bar{\alpha}_1 P_s(R_2, D, \{R_2\})).
\] (19)

The stability analysis for this case can be trivially obtained by the presented analysis thus, it is omitted. However, this scenario has applicability in nowadays relay-assisted networks.

IV. Throughput and Stability Analysis – The Symmetric \(N\)-User Case for the General MPR Case

Here we will generalize the analysis provided in the previous section for the \(N\)-user case. However, due to presentation clarity we will focus on the symmetric user case. The users attempt to transmit with probability \( t \). The success probability from a user to the destination is the same for all the users, thus, in order to characterize it we just need the number of active users, i.e. the interference. This probability is denoted by \( P_s(D, i) \) to capture the case that \( i \) users are attempting transmission (including the user we intend to study its performance), similarly we define \( P_s(R_j, i), j = 1, 2 \).

A. Endogenous arrivals at the relays and throughput performance

The direct throughput of a user to the destination in the case of \( N \) symmetric users is given by

\[
T_D = \sum_{i=1}^{N} t^i (1-t)^{N-i} P_s(D, i)
\] (20)
We will derive the endogenous arrivals at the first and the second relay respectively in order to calculate the relayed throughput in the network. For the symmetric $N$-user case we denote the endogenous arrivals from the users at the first (second) relay as $\lambda_{1,u}$ ($\lambda_{2,u}$).

We need to characterize the average number of packet arrivals from the users at each relay. Thus, we define as $r_{k,1}$ the probability that $k$ packets will arrive in a timeslot at the first relay. Similarly, we define $r_{k,2}$. Then, the average endogenous arrival rate at the $j$-th relay is given by

$$\lambda_{j,u} = \sum_{k=1}^{N} kr_{k,j}, \quad j = 1, 2. \tag{21}$$

The probability $r_{k,1}$ where $1 \leq k \leq N$ is given by

$$r_{k,1} = \sum_{i=k}^{N} \sum_{l=0}^{k} \binom{N}{i} \binom{k}{l} t^{i} \tau^{N-i} \left[ (P_s(R_1, i))^k \left( \mathbb{P}_s(D, i) \right)^k \left( \mathbb{P}_s(R_2, i) \right)^{k-l} \left( \mathbb{P}_s(R_2, i) \right)^{l} \right] p_{a_1}^{l} \left[ 1 - P_s(R_1, i) \mathbb{P}_s(D, i) \right]^{i-k}. \tag{22}$$

Similarly, we obtain $r_{k,2}$ for the second relay. Note that the network-wide relayed throughput when both relays are stable is given by $\lambda_{1,u} + \lambda_{2,u}$. Thus, the aggregate or network-wide throughput of the network when both relays are stable is given by

$$T_{aggr} = NT_D + \lambda_{1,u} + \lambda_{2,u} + \hat{\lambda}_1 + \hat{\lambda}_2 \tag{23}$$

Recall that the total arrival rate at relay $i$ is $\lambda_i = \lambda_{i,u} + \hat{\lambda}_i$, consisting of the endogenous arrivals from the users and the external traffic.

Below provide the stability conditions at the relays.

B. Stability conditions for the queues at the relays

The service rates at the at the first relay is given by

$$\mu_1 = t^N \left[ \Pr(N_2 = 0)\alpha_1^s P_s(R_1, D, \{R_1\}) + \Pr(N_2 > 0)\alpha_1 (\alpha_2 P_s(R_1, D, \{R_1, R_2\}) + \bar{P}_s(R_1, D, \{R_1\})) \right]. \tag{24}$$

Similarly, we have the service rate at the second relay is given by

$$\mu_2 = t^N \left[ \Pr(N_1 = 0)\alpha_2^s P_s(R_2, D, \{R_2\}) + \Pr(N_1 > 0)\alpha_2 (\alpha_1 P_s(R_2, D, \{R_1, R_2\}) + \bar{P}_s(R_2, D, \{R_2\})) \right]. \tag{25}$$

Following the same methodology as in the proof of Theorem III.1 we obtain the stability conditions for the symmetric $N$-user case. The stability conditions are given by $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ where
\[ \mathcal{R}_1 = \left\{ (\lambda_1, \lambda_2) : \lambda_1 < t^N \alpha_1^* P_s^*(R_1, D, \{R_1\}) - \frac{\lambda_2 \left[ \alpha_1^* P_s^*(R_1, D, \{R_1\}) - \alpha_1 \left[ \alpha_2 \Delta_1 + P_s(R_1, D, \{R_1\}) \right] \right]}{\alpha_2 \left[ \alpha_1 \Delta_1 + P_s(R_2, D, \{R_2\}) \right]}, \right. \\
\left. \lambda_2 < t^N \alpha_2 \left[ \alpha_1 \Delta_1 + P_s(R_2, D, \{R_2\}) \right] \right\}. \quad (26) \]

and

\[ \mathcal{R}_2 = \left\{ (\lambda_1, \lambda_2) : \lambda_2 < t^N \alpha_2^* P_s^*(R_2, D, \{R_2\}) - \frac{\lambda_1 \left[ \alpha_2^* P_s^*(R_2, D, \{R_2\}) - \alpha_2 \left[ \alpha_1 \Delta_2 + P_s(R_2, D, \{R_2\}) \right] \right]}{\alpha_1 \left[ \alpha_2 \Delta_1 + P_s(R_1, D, \{R_1\}) \right]}, \right. \\
\left. \lambda_1 < t^N \alpha_1 \left[ \alpha_2 \Delta_1 + P_s(R_1, D, \{R_1\}) \right] \right\}. \quad (27) \]

V. DELAY ANALYSIS: THE TWO-USER CASE

This section is devoted to the analysis of the queueing delay experienced at the relays. Our aim is to obtain the generating function of the joint stationary distribution of the number of packets at relay nodes. In the following we consider the case of \( N = 2 \) users, and focus on a subclass of MPR models, called the “capture” channel, under which at most one packet can be successfully decoded by the receiver of the node \( D \), even if more than one nodes transmit.

In order to proceed, we have to provide some more information regarding the success probabilities of a transmission between nodes that were defined in subsection II-C. More precisely, we have to take into account the number as well as the type of nodes that transmit (i.e., source or relay node). This is due to several reasons, such as the fact that generally the channel quality between relay nodes and destination node is usually better than between sources and destination, as well as due to the wireless interference, since the channel quality is severely affected by the the number of nodes that attempt a transmission. Moreover, it is crucial to take into account the possibility that a failed packet can be successfully decoded by both relays, as well as the ability of the “smart” relay nodes to be aware of the status of the others, which in turn leads to self-aware networks. With that in mind we consider the following cases:

1) Both sources transmit

a) When both sources failed to transmit directly to the node \( D \), the failed packet of source \( k \) is success-
fully decoded by relay \( R_i \) with probability \( P_s(k, R_i, \{1, 2\}) \), \( k = 1, 2 \), \( i = 1, 2 \), where with proba-
bility \( \overline{P}_s(0, R_i, \{1, 2\}) \), the relay \( R_i \) failed to decode both packets. Note also that
\( \overline{P}_s(1, R_i, \{1, 2\}) = \overline{P}_s(0, R_i, \{1, 2\}) + P_s(2, R_i, \{1, 2\}) \), \( \overline{P}_s(2, R_i, \{1, 2\}) = \overline{P}_s(0, R_i, \{1, 2\}) + P_s(1, R_i, \{1, 2\}) \), \( i = 1, 2 \). Due
to the total probability law we have

\[ (\overline{P}_s(0, R_1, \{1, 2\}) + P_s(1, R_1, \{1, 2\}) + P_s(2, R_1, \{1, 2\})) (\overline{P}_s(0, R_2, \{1, 2\}) + P_s(1, R_2, \{1, 2\}) + P_s(2, R_2, \{1, 2\})) = \frac{1}{1 + \overline{P}_s(0, R_i, \{1, 2\})} \]
b) When source 1 (resp. source 2) is the only that succeeds to transmit a packet at node \( D \), i.e., its transmission was successfully decoded by node \( D \), then with probability \( P_{s,2}(2, R_i, \{1, 2\}) \) (resp. \( P_{s,1}(1, R_i, \{1, 2\}) \)), \( i = 1, 2 \), the failed packet of source 2 (resp. source 1) is successfully decoded by the relay \( R_i \). On the contrary, with probability \( \overline{P}_{s,2}(2, R_i, \{1, 2\}) \) (resp. \( \overline{P}_{s,1}(1, R_i, \{1, 2\}) \)), the relay \( R_i \) failed to decode the packet from source 2 (resp. source 1), and thus, it is considered lost. Due to the total probability law we have,

\[
(P_{s,2}(2, R_1, \{1, 2\}) + \overline{P}_{s,2}(2, R_1, \{1, 2\}))(P_{s,2}(2, R_2, \{1, 2\}) + \overline{P}_{s,2}(2, R_2, \{1, 2\})) = 1,
\]

\[
(P_{s,1}(1, R_1, \{1, 2\}) + \overline{P}_{s,1}(1, R_1, \{1, 2\}))(P_{s,1}(1, R_2, \{1, 2\}) + \overline{P}_{s,1}(1, R_2, \{1, 2\})) = 1.
\]

2) Only one source transmit, say source \( k \), and the other remains silent. When source \( k \) fails to transmit directly to node \( D \), its failed packet is successfully decoded by relay \( R_i \) with probability \( P_s(k, R_i, \{k\}) \), \( k = 1, 2 \), \( i = 1, 2 \), where with probability \( \overline{P}_s(0, R_i, \{k\}) \), the relay \( R_i \) fails to decode the packet. Due to the total probability law we have

\[
(\overline{P}_s(0, R_1, \{k\}) + P_s(k, R_1, \{k\}))(\overline{P}_s(0, R_2, \{k\}) + P_s(k, R_2, \{k\})) = 1.
\]

Note that the cases (1, b) and (2) refer to the case where only one source cooperate with a relay. However, we have to distinguish it in two cases because in the former one, there is an interaction among sources since both of them transmit, while in the latter one, only one source transmit and the other remains silent (i.e., there is no interaction). Such an interaction, plays a crucial role on the values of the success probabilities. In wireless systems, the feature of interference and interaction among transmitting nodes is of great importance and have to be taken into account.

If both relays transmit simultaneously, with probability \( P_s(R_i, D, \{R_1, R_2\}) \), the packet transmitted from \( R_i \) is successfully received by node \( D \), while with probability \( \overline{P}_s(R_i, D, \{R_1, R_2\}) = 1 - \sum_{i=1,2} P_s(R_i, D, \{R_1, R_2\}) \), both of them failed to be received by the node \( D \), and have to be retransmitted in a later time slot. Recall also the success probabilities \( P_s^*(R_i, D, \{R_i\}) \), \( P_s(R_i, D, \{R_i\}) \) of \( R_i \) when the other relay node is active (i.e., non-empty), and inactive (i.e., empty) respectively. We assume that \( P_s^*(R_i, D, \{R_i\}) > P_s(R_i, D, \{R_i\}) > P_s(R_i, D, \{R_1, R_2\}) \). Denote the counter probabilities \( \overline{P}_s^*(R_i, D, \{R_i\}) = 1 - P_s^*(R_i, D, \{R_i\}) \), \( \overline{P}_s(R_i, D, \{R_i\}) = 1 - P_s(R_i, D, \{R_i\}) \), \( i = 1, 2 \).

In the following we proceed with the derivation of a fundamental functional equation, the solution of which, will provide the generating function of the stationary joint queue length distribution at relay nodes. The solution of this functional equation is the key element for obtaining expressions for the queueing delay at relay nodes.
A. Functional equation and preparatory results

Clearly, \( Y_n = (N_{1,n}, N_{2,n}) \) is a discrete time Markov chain with state space \( S = \{(k_1, k_2) : k_1, k_2 = 0, 1, 2, \ldots \} \). The queues of both relay nodes evolve as:

\[
N_{i,n+1} = [N_{i,n} + F_{i,n}]^+ + A_{i,n}, \quad i = 1, 2,
\]

where \( F_{i,n} \) is either the number of arrivals (in this case \( F_{i,n} \) equals 0 or 1) at relay \( R_i \) at time slot \( n \) (in case both source users transmit simultaneously, and the unsuccessful packet is stored in \( R_i \), or only a single source user transmits, but its transmission was unsuccessful), or the number of departures (in this case \( F_{i,n} \) equals 0 or \(-1\)) from \( R_i \) (this is because when the sources do not transmit, and \( R_i \) attempts to transmit a packet at node \( D \)) at time slot \( n \). Recall that the relays have their own traffic, and \( A_{i,n} \) represents the number of arrivals (of such generated traffic) in the the time interval \((n, n+1]\). Let \( H(x, y) \) be the generating function of the joint stationary queue process and \( Z(x, y) \) the generating function of the joint distribution of the number of arriving packets in any slot (i.e., self-generated traffic of the relays), viz.

\[
H(x, y) = \lim_{n \to \infty} E(x^{N_{1,n}} y^{N_{2,n}}), \quad |x| \leq 1, |y| \leq 1,
\]

\[
Z(x, y) = \lim_{n \to \infty} E(x^{A_{1,n}} y^{A_{2,n}}), \quad |x| \leq 1, |y| \leq 1.
\]

In the following we assume for sake of convenience only a particular distribution for the self-generated arrival processes at both relays, namely the geometric distribution\(^4\) [19]. We also assume that both arrival processes are independent. More precisely we assume hereon that

\[
Z(x, y) = [(1 + \lambda_1 (1 - x))(1 + \lambda_2 (1 - y))]^{-1}.
\]

Then, by exploiting (28), and using (52) (see Appendix A), we obtain after lengthy calculations

\[
R(x, y) H(x, y) = A(x, y) H(x, 0) + B(x, y) H(0, y) + C(x, y) H(0, 0),
\]

where,

\[
R(x, y) = Z^{-1}(x, y) - 1 + \tilde{l}_1 \tilde{l}_2 [\alpha_1 \tilde{\alpha}_2 (1 - \frac{1}{x}) + \alpha_2 \tilde{\alpha}_1 (1 - \frac{1}{y})] + (1 - x) L_1 + (1 - y) L_2 + (1 - xy) L_3,
\]

\(^4\)Note that such a distribution is natural in radio-packet networks.
and

\[ L_1 = t_1 \tilde{t}_2 \bar{P}_s(1, D, \{1\}) \bar{P}_s(1, R_2, \{1\}) P_s(1, R_1, \{1\}) + t_2 \tilde{t}_1 \bar{P}_s(2, D, \{2\}) \bar{P}_s(2, R_2, \{2\}) + t_1 \tilde{t}_2 \bar{P}_s(0, D, \{1, 2\}) \bar{P}_s(0, R_2, \{1, 2\}) (P_s(1, R_1, \{1, 2\}) + P_s(2, R_1, \{1, 2\})) + P_s(1, D, \{1, 2\}) \bar{P}_{s,2}(2, R_2, \{1, 2\}) + P_s(2, D, \{1, 2\}) \bar{P}_{s,1}(1, R_2, \{1, 2\}) P_s(1, R_1, \{1, 2\}) \],

\[ L_2 = t_1 \tilde{t}_2 \bar{P}_s(1, D, \{1\}) P_s(1, R_2, \{1\}) \bar{P}_s(1, R_1, \{1\}) + t_2 \tilde{t}_1 \bar{P}_s(2, D, \{2\}) P_s(2, R_2, \{2\}) \bar{P}_s(2, R_1, \{2\}) + t_1 \tilde{t}_2 \bar{P}_s(0, D, \{1, 2\}) \bar{P}_s(0, R_1, \{1, 2\}) (P_s(1, R_2, \{1, 2\}) + P_s(2, R_2, \{1, 2\})) + P_s(1, D, \{1, 2\}) \bar{P}_{s,2}(2, R_1, \{1, 2\}) + P_s(2, D, \{1, 2\}) \bar{P}_{s,1}(1, R_1, \{1, 2\}) P_s(1, R_2, \{1, 2\}) \],

\[ L_3 = t_1 \tilde{t}_2 \bar{P}_s(1, D, \{1\}) P_s(1, R_2, \{1\}) \bar{P}_s(1, R_1, \{1\}) + t_2 \tilde{t}_1 \bar{P}_s(2, D, \{2\}) P_s(2, R_2, \{2\}) \bar{P}_s(2, R_1, \{2\}) + t_1 \tilde{t}_2 \bar{P}_s(0, D, \{1, 2\}) \bar{P}_s(0, R_1, \{1, 2\}) (P_s(1, R_2, \{1, 2\}) + P_s(2, R_2, \{1, 2\})) + P_s(1, D, \{1, 2\}) \bar{P}_{s,2}(2, R_1, \{1, 2\}) + P_s(2, D, \{1, 2\}) \bar{P}_{s,1}(1, R_1, \{1, 2\}) P_s(1, R_2, \{1, 2\}) \],

\[
A(x, y) = \tilde{t}_1 \tilde{t}_2 [d_1(1 - \frac{1}{x}) + \alpha_2 \bar{\alpha}_1 (1 - \frac{1}{y})],
\]

\[
B(x, y) = \tilde{t}_1 \tilde{t}_2 [d_2(1 - \frac{1}{y}) + \alpha_1 \bar{\alpha}_2 (1 - \frac{1}{x})],
\]

\[
C(x, y) = \tilde{t}_1 \tilde{t}_2 [d_1(\frac{1}{x} - 1) + d_2(\frac{1}{y} - 1)],
\]

\[
\bar{\alpha}_i = \bar{\alpha}_i P_s(R_i, D, \{R_i\}) + \alpha_i P_s(R_i, D, \{R_1, R_2\}), i = 1, 2,
\]

\[
d_1 = \alpha_1 \bar{\alpha}_2 - \alpha_1 \bar{\alpha}_1 P_s^*(R_1, D, \{R_1\}),
\]

\[
d_2 = \alpha_2 \bar{\alpha}_1 - \alpha_2 \bar{\alpha}_2 P_s^*(R_2, D, \{R_2\}).
\]

**Remark 6.** Note that \( L_i, i = 1, 2, 3, \) has a clear probabilistic interpretation. Indeed, \( L_1 \) (resp. \( L_2 \)) is the probability that a (failed) transmitted source packet will be decoded and stored at relay \( R_i \). Moreover, \( L_3 \) is the probability that a failed transmitted source packet will be decoded and stored at both relays.

Some interesting relations can be obtained directly from (29). Taking \( y = 1 \), dividing by \( x - 1 \) and taking \( x \to 1 \) in (29) and vice versa yield the following “conservation of flow” relations:

\[
\lambda_1 = \tilde{t}_1 \tilde{t}_2 \{\alpha_1 \bar{\alpha}_2 (1 - H(0, 1)) - d_1 (H(1, 0) - H(0, 0))\},
\]

\[
\lambda_2 = \tilde{t}_1 \tilde{t}_2 \{\alpha_2 \bar{\alpha}_1 (1 - H(1, 0)) - d_2 (H(0, 1) - H(0, 0))\},
\]

where for \( i = 1, 2 \),

\[
\lambda_i = \tilde{\lambda}_i + \lambda_{1,i} + \lambda_{2,i},
\]
where now,

\[
\lambda_{1,1} = \left(1 + \frac{\alpha_1 + \alpha_2}{\alpha_1 P_s^*(R_1, D, \{1\}) + \alpha_2 P_s^*(R_2, D, \{1\})}\right) H(0, 0, 1) + d_2 \frac{\alpha_1}{\alpha_1 P_s^*(R_1, D, \{1\})} H(0, 0, 1) + d_2 \frac{\alpha_2}{\alpha_2 P_s^*(R_2, D, \{1\})} H(0, 0, 1)
\]

\[
\lambda_{1,2} = \left(1 + \frac{\alpha_1 + \alpha_2}{\alpha_1 P_s^*(R_1, D, \{1\}) + \alpha_2 P_s^*(R_2, D, \{1\})}\right) H(0, 0, 1) + d_2 \frac{\alpha_1}{\alpha_1 P_s^*(R_1, D, \{1\})} H(0, 0, 1) + d_2 \frac{\alpha_2}{\alpha_2 P_s^*(R_2, D, \{1\})} H(0, 0, 1)
\]

\[
\lambda_{2,1} = \left(1 + \frac{\alpha_1 + \alpha_2}{\alpha_1 P_s^*(R_1, D, \{1\}) + \alpha_2 P_s^*(R_2, D, \{1\})}\right) H(0, 0, 1) + d_2 \frac{\alpha_1}{\alpha_1 P_s^*(R_1, D, \{1\})} H(0, 0, 1) + d_2 \frac{\alpha_2}{\alpha_2 P_s^*(R_2, D, \{1\})} H(0, 0, 1)
\]

\[
\lambda_{2,2} = \left(1 + \frac{\alpha_1 + \alpha_2}{\alpha_1 P_s^*(R_1, D, \{1\}) + \alpha_2 P_s^*(R_2, D, \{1\})}\right) H(0, 0, 1) + d_2 \frac{\alpha_1}{\alpha_1 P_s^*(R_1, D, \{1\})} H(0, 0, 1) + d_2 \frac{\alpha_2}{\alpha_2 P_s^*(R_2, D, \{1\})} H(0, 0, 1)
\]

From (30), (31) we realize that the analysis is distinguished in two cases:

1) For \( \alpha_1 \neq \alpha_2 \), \( \frac{\alpha_1 + \alpha_2}{\alpha_1 P_s^*(R_1, D, \{1\}) + \alpha_2 P_s^*(R_2, D, \{1\})} = 1 \), (30), (31) yield

\[
H(0, 0) = 1 - \frac{\lambda_1}{t_1 t_2 (\alpha_1^2 + \alpha_2^2)} - \frac{\lambda_2}{t_1 t_2 (\alpha_1^2 + \alpha_2^2)} = 1 - \rho.
\]

2) For \( \alpha_1 = \alpha_2 \), \( \frac{\alpha_1 + \alpha_2}{\alpha_1 P_s^*(R_1, D, \{1\}) + \alpha_2 P_s^*(R_2, D, \{1\})} = 1 \), (30), (31) yield

\[
H(0, 0) = \frac{d_2 \lambda_1 + d_2 \alpha_2 (\alpha_1^2 + \alpha_2^2) - d_1 \alpha_2}{t_1 t_2 (\alpha_1^2 + \alpha_2^2)} H(0, 0) + d_2 \frac{\alpha_1}{\alpha_1 P_s^*(R_1, D, \{1\})} H(0, 0) + d_2 \frac{\alpha_2}{\alpha_2 P_s^*(R_2, D, \{1\})} H(0, 0).
\]

Our primary interest in the following is to obtain expressions for the queueing delay at relay nodes. The key element for doing this is to solve the functional equation (29) and obtain \( H(x, y) \). In order to this, we have to obtain the boundary functions \( H(x, 0) \), \( H(0, y) \) and the term \( H(0, 0) \). The basic tool for obtaining these functions is the theory of boundary value problems [23], [24]. Since we are dealing with a quite technical approach we summarized in the following the basic steps.

**Step 1.** From the functional equation (29), we prove that \( H(x, 0) \) and \( H(0, y) \) satisfy certain boundary value problems of Riemann-Hilbert-Carleman type [24], i.e., with boundary conditions on closed curves. These curves are studied in Lemma [V.3.1. The proof of this lemma (Appendix [A]) requires the investigation of the kernel \( R(x, y) \) (see
subsection V-B. All the required results are given in Lemmas V.1, V.2 (the proof of Lemma V.1 is given in the Appendix A). Note that based on the values of the parameters the unit disc may lie inside the region bounded by these contours. Clearly, the functions \( H(x,0), H(0,y) \) are analytic inside the unit disc, but they might have poles in the region bounded by the unit disc and these closed curves. The position of these poles (if exist) are investigated in the Appendix A. With that in mind, the boundary functions admit analytic continuations in the whole interiors of the curves above; see also Chapter 3 in [24]. Then, we have to obtain the precise boundary conditions on these curves. This is done in subsubsections V-C1, V-C2; see (35), (40) respectively.

**Step 2.** Next, we conformally transform these problems into boundary value problems of Riemann-Hilbert type on the unit disc; see [23]. This conversion is motivated by the fact that the latter problems are more usual and by far more treated in the literature. It is done using conformal mappings in subsubsections V-C1, V-C2; see (43).

**Step 3.** Finally we solve these new problems and we deduce an explicit integral representation of the unknown boundary functions. This will conclude subsubsections V-C1, V-C2; see (37), (42) respectively.

**B. Analysis of the kernel**

In the following we focus on the kernel \( R(x,y) \), and provide some important properties for the following analysis. To the best of our knowledge, this type of kernel has never been treated in the related literature. Clearly,

\[
R(x,y) = a(x)y^2 + b(x)y + c(x) = \hat{a}(y)x^2 + \hat{b}(y)x + \hat{c}(y),
\]

where

\[
a(x) = -x[\hat{\lambda}_2(1 + \hat{\lambda}_1(1 - x)) + L_2 + L_3x],
\]

\[
b(x) = x[\hat{\lambda} + \hat{\lambda}_1\hat{\lambda}_2 + \tilde{t}_1\tilde{t}_2(\alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1) + L_1 + L_2 + L_3] - \tilde{t}_1\tilde{t}_2\alpha_1\hat{\alpha}_2 - [\hat{\lambda}_1(1 + \hat{\lambda}_2) + L_1]x^2,
\]

\[
c(x) = -\tilde{t}_1\tilde{t}_2\alpha_2\hat{\alpha}_1x,
\]

\[
\hat{a}(y) = -y[\hat{\lambda}_1(1 + \hat{\lambda}_2(1 - y)) + L_1 + L_3y],
\]

\[
\hat{b}(y) = y[\hat{\lambda} + \hat{\lambda}_1\hat{\lambda}_2 + \tilde{t}_1\tilde{t}_2(\alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1) + L_1 + L_2 + L_3] - \tilde{t}_1\tilde{t}_2\alpha_2\hat{\alpha}_1 - [\hat{\lambda}_2(1 + \hat{\lambda}_1) + L_2]y^2,
\]

\[
\hat{c}(y) = -\tilde{t}_1\tilde{t}_2\alpha_1\hat{\alpha}_2y.
\]

The roots of \( R(x,y) = 0 \) are \( X_{\pm}(y) = \frac{-b(y) \pm \sqrt{D_y(y)}}{2a(y)} \), \( Y_{\pm}(x) = \frac{-b(x) \pm \sqrt{D_x(x)}}{2a(x)} \), where \( D_y(y) = \hat{b}(y)^2 - 4\hat{a}(y)\hat{c}(y) \), \( D_x(x) = b(x)^2 - 4a(x)c(x) \).

**Lemma V.1.** For \(|y| = 1, y \neq 1\), the kernel equation \( R(x,y) = 0 \) has exactly one root \( x = X_0(y) \) such that \(|X_0(y)| < 1\). For \( \lambda_1 < \tilde{t}_1\tilde{t}_2\alpha_1\hat{\alpha}_2 \), \( X_0(1) = 1 \). Similarly, we can prove that \( R(x,y) = 0 \) has exactly one root \( y = Y_0(x) \), such that
\[ |Y_0(x)| \leq 1, \text{ for } |x| = 1. \]

**Proof.** See Appendix A .

Using simple algebraic arguments we can prove the following lemma, which provides information about the location of the branch points of the two-valued functions \( Y(x), X(y) \).

**Lemma V.2.** The algebraic function \( Y(x) \), defined by \( R(x, Y(x)) = 0 \), has four real branch points \( 0 < x_1 < x_2 \leq 1 < x_3 < x_4 < \frac{1+\lambda_1}{\lambda_1} \). Moreover, \( D_x(x) > 0 \), \( x \in (x_1, x_2) \cup (x_3, x_4) \) and \( D_x(x) > 0, x \in (-\infty, x_1) \cup (x_2, x_3) \cup (x_4, \infty) \).

Similarly, \( X(y) \), defined by \( R(X(y), y) = 0 \), has four real branch points \( 0 \leq y_1 < y_2 \leq 1 < y_3 < y_4 < \frac{1+\lambda_2}{\lambda_2} \), and \( D_x(y) < 0, y \in (y_1, y_2) \cup (y_3, y_4) < \text{ and } D_x(y) > 0, y \in (-\infty, y_1) \cup (y_2, y_3) \cup (y_4, \infty) \).

To ensure the continuity of the two valued function \( Y(x) \) (resp. \( X(y) \)) we consider the following cut planes: \( \tilde{C}_x = \mathbb{C}_x - ([x_1, x_2] \cup [x_3, x_4], \tilde{C}_y = \mathbb{C}_y - ([y_1, y_2] \cup [y_3, y_4], \) where \( \mathbb{C}_x, \mathbb{C}_y \) the complex planes of \( x, y \), respectively. In \( \tilde{C}_x \) (resp. \( \tilde{C}_y \)), denote by \( Y_0(x) \) (resp. \( X_0(y) \)) the zero of \( R(x, Y(x)) = 0 \) (resp. \( R(X(y), y) = 0 \)) with the smallest modulus, and \( Y_1(x) \) (resp. \( X_1(y) \)) the other one.

Define the image contours, \( \mathcal{L} = Y_0\left[\frac{x_1}{y}, \frac{x_2}{y}\right], \mathcal{L}_{\text{ext}} = Y_0\left[\frac{x_3}{y}, \frac{x_4}{y}\right], \mathcal{M} = X_0\left[\frac{y_1}{y}, \frac{y_2}{y}\right], \mathcal{M}_{\text{ext}} = X_0\left[\frac{y_3}{y}, \frac{y_4}{y}\right] \), where \( [u, v] \) stands for the contour traversed from \( u \) to \( v \) along the upper edge of the slit \( [u, v] \) and then back to \( u \) along the lower edge of the slit. The following lemma shows that the mappings \( Y(x), X(y) \), for \( x \in [x_1, x_2] \), \( y \in [y_1, y_2] \) respectively, give rise to the smooth and closed contours \( \mathcal{L}, \mathcal{M} \) respectively:

**Lemma V.3.**
1) For \( y \in [y_1, y_2] \), the algebraic function \( X(y) \) lies on a closed contour \( \mathcal{M} \), which is symmetric with respect to the real line and defined by

\[ |y|^2 = \frac{\Re(\zeta(y))}{\frac{\partial}{\partial(\zeta(y))}}, |y|^2 \leq \frac{\Re(\zeta(y))}{\frac{\partial}{\partial(\zeta(y))}}, \]

where, \( \zeta(\delta) = \frac{k_2(\delta) - \sqrt{k_2^2(\delta) - 4 \gamma_1 \gamma_2 \alpha_1 \alpha_2 \delta}}{2k_1(\delta)}, \)

\[ k_1(\delta) := \lambda_2(1 + \lambda_1) + L_2 + 2\delta(L_3 - \lambda_1 \lambda_2), \]

\[ k_2(\delta) := (1 + 2\delta)(L_1 + \lambda_1(1 + \lambda_2)) + \lambda_2 + L_2 + L_3 + \bar{\lambda}_1 \bar{\lambda}_2(\alpha_1 \alpha_2 + \alpha_2 \alpha_1). \]

Set \( \beta_0 := \sqrt{\frac{\Re(\zeta(y_2))}{\frac{\partial}{\partial(\zeta(y_2))}}, \beta_1 := -\sqrt{\frac{\Re(\zeta(y_1))}{\frac{\partial}{\partial(\zeta(y_1))}}} \) the extreme right and left point of \( \mathcal{M} \), respectively.

2) For \( x \in [x_1, x_2] \), the algebraic function \( Y(x) \) lies on a closed contour \( \mathcal{L} \), which is symmetric with respect to the real line and defined by

\[ |x|^2 = \frac{\Re(\delta(x))}{\frac{\partial}{\partial(\delta(x))}}, |x|^2 \leq \frac{\Re(\delta(x))}{\frac{\partial}{\partial(\delta(x))}}, \]

where, \( \delta(x) = \frac{\tilde{\lambda}_2(1 + \tilde{\lambda}_1) + \tilde{L}_2 + 2\delta(\tilde{L}_3 - \tilde{\lambda}_1 \tilde{\lambda}_2)}{2\tilde{k}_1(\delta)}, \)

\[ \tilde{k}_1(\delta) := \tilde{\lambda}_2(1 + \tilde{\lambda}_1) + \tilde{L}_2 + 2\delta(\tilde{L}_3 - \tilde{\lambda}_1 \tilde{\lambda}_2), \]

\[ \tilde{k}_2(\delta) := (1 + 2\delta)(\tilde{L}_1 + \tilde{\lambda}_1(1 + \tilde{\lambda}_2)) + \tilde{\lambda}_2 + \tilde{L}_2 + \tilde{L}_3 + \tilde{\bar{\lambda}}_1 \tilde{\bar{\lambda}}_2(\alpha_1 \alpha_2 + \alpha_2 \alpha_1). \]
Therefore, for $y \in D_y = \{ x \in C : |y| \leq 1, |X_0(y)| \leq 1 \}$,

$$\alpha_2 \alpha_1 (1 - \rho) C(x_0(y), y) = 0.$$  \hfill (33)

For $y \in D_y - [y_1, y_2]$ both $H(x_0(y), 0)$, and $H(0, y)$ are analytic and thus, by means of analytic continuation, we can also consider (33) for $y \in [y_1, y_2]$, or equivalently, for $x \in M$

$$\alpha_2 \alpha_1 H(x, 0) + d_2 H(0, y) + \frac{\alpha_2 \alpha_1 (1 - \rho) C(x_0(x), y) A(x_0(y), y)}{A(x_0(y), y)} = 0.$$ \hfill (34)

Then, multiplying both sides of (34) by the imaginary complex number $i$, and noticing that $H(0, Y_0(x))$ is real for $x \in M$, since $Y_0(x) \in [y_1, y_2]$, we have

$$\text{Re}(iH(x, 0)) = \text{Re}(-i \frac{C(x_0(x), y)}{A(x_0(x), y)}(1 - \rho), x \in M.$$ \hfill (35)

Clearly, some analytic continuation considerations must be made in order to be everything well defined. To do this, we have to check for poles of $H(x, 0)$ in $S := G_M \cap \mathbb{D}_x^*$, where $G_U$ be the interior domain bounded by $U$, and $D_x = \{x : |x| < 1\}, \mathbb{D}_x = \{x : |x| \leq 1\}, \mathbb{D}_x^* = \{x : |x| > 1\}$. These poles, if exist, they coincide with the zeros of $A(x, y_0(x))$ in $S_x$; see Appendix [A]. Note that equation (35) is defined on $M$. In order to solve (35) we must firstly conformally transform the problem from $M$ to the unit circle. Let the conformal mapping, $z = \gamma(x) : G_M \to \mathbb{C}$, and its inverse $x = \gamma_0(z) : \mathbb{C} \to G_M$. 

Proof. See Appendix [A]
Then, we have the following problem: Find a function $\tilde{T}(z) = H(\gamma_0(z), 0)$ regular for $z \in G_C$, and continuous for $z \in C \cup G_C$ such that, $Re(i\tilde{T}(z)) = w(\gamma_0(z))$, $z \in C$. In case $H(x, 0)$ has no poles in $S$, the solution of the Dirichlet problem with boundary condition (35) is:

$$H(x, 0) = -\frac{1-\rho}{2\pi} \int_{|t|=1} f(t) \frac{t+\gamma(x)}{t-\gamma(x)} dt + C, \quad x \in \mathcal{M}, \quad (36)$$

where $f(t) = Re(-i\frac{C(\gamma_0(t), y_0(\gamma_0(t))))}{A(\gamma_0(t), y_0(\gamma_0(t)))}$, $C$ a constant to be defined by setting $x = 0 \in G_M$ in (36) and using the fact that $H(0, 0) = 1 - \rho$, $\gamma(0) = 0$ (In case $H(x, 0)$ has a pole, say $\bar{x}$, we still have a Dirichlet problem for the function $(x - \bar{x})H(x, 0)$)

Following the discussion above,

$$C = (1 - \rho)(1 + \frac{1}{2\pi} \int_{|t|=1} f(t) \frac{d^x}{dt^x}),$$

Setting $t = e^{i\phi}$, $\gamma_0(e^{i\phi}) = \rho(\psi(\phi))e^{i\psi(\phi)}$, we obtain after some algebra,

$$f(e^{i\phi}) = \frac{d_1 \alpha_2 \sin(\psi(\phi))(1-Y_0(\gamma_0(e^{i\phi}))^{-1})}{\rho(\psi(\phi))k(\phi)},$$

which is an odd function of $\phi$, and

$$k(\phi) = [\alpha_2 \tilde{\alpha}_1 (1 - Y_0^{-1}(\gamma_0(e^{i\phi}))) + d_1 (1 - \frac{\cos(\psi(\phi))}{\rho(\psi(\phi))})]^2 + (\frac{\sin(\psi(\phi))}{\rho(\psi(\phi))})^2.$$

Thus, $C = 1 - \rho$. Substituting in (36) we obtain after simple calculations an integral representation of $H(x, 0)$ on a real interval, i.e.,

$$H(x, 0) = (1 - \rho)\{1 + \frac{2\gamma(x)i}{\pi} \int_0^\pi \frac{f(e^{i\phi})\sin(\phi)}{1-2\gamma(x)\cos(\phi)-\gamma(x)^2} d\phi\}, \quad x \in G_M. \quad (37)$$

Similarly, we can determine $H(0, y)$ by solving another Dirichlet boundary value problem on the closed contour $\mathcal{L}$. Then, using the fundamental functional equation (29) we uniquely obtain $H(x, y)$.

2) A homogeneous Riemann-Hilbert boundary value problem: In case $\frac{\alpha_1 \tilde{\alpha}_2}{\alpha_1 P^*_+ (R_1, D, \{R_1\})} + \frac{\alpha_2 \tilde{\alpha}_1}{\alpha_2 P^*_+ (R_2, D, \{R_2\})} \neq 1$, consider the following transformation:

$$G(x) := H(x, 0) + \frac{\alpha_1 P^*_+ (R_1, D, \{R_1\}) d_2 H(0, 0)}{d_1 d_2 - \alpha_1 \alpha_2 \tilde{\alpha}_2 \tilde{\alpha}_1}, \quad L(y) := H(0, y) + \frac{\alpha_2 P^*_+ (R_2, D, \{R_2\}) d_1 H(0, 0)}{d_1 d_2 - \alpha_1 \alpha_2 \tilde{\alpha}_2 \tilde{\alpha}_1}.$$

Then, for $y \in D_y$,

$$A(X_0(y), y)G(X_0(y)) = -B(X_0(y), y)L(y). \quad (38)$$
For \( y \in D_y - [y_1, y_2] \) both \( G(X_0(y)) \), \( L(y) \) are analytic and the right-hand side can be analytically continued up to the slit \([y_1, y_2]\) or equivalently, for \( x \in \mathcal{M} \)

\[
A(x, Y_0(x))G(x) = -B(x, Y_0(x))L(Y_0(x)).
\]  

(39)

Clearly, \( G(x) \) is holomorphic for \( D_x \), continuous for \( \bar{D}_x \). However, \( G(x) \) might has poles in \( S_x \), based on the values of the system parameters. These poles (if exist) coincide with the zeros of \( A(x, Y_0(x)) \) in \( S_x \); see Appendix [A]. For \( y \in [y_1, y_2] \), let \( X_0(y) = x \in \mathcal{M} \) and realize that \( Y_0(X_0(y)) = y \) so that \( y = Y_0(x) \). Taking into account the possible poles of \( G(x) \), and noticing that \( L(Y_0(x)) \) is real for \( x \in \mathcal{M} \), since \( Y_0(x) \in [y_1, y_2] \), we have

\[
\text{re}[iU(x)\tilde{G}(x)] = 0, \ x \in \mathcal{M},
\]

(40)

\[ U(x) = \frac{A(x, Y_0(x))}{(x - \bar{x})^2 B(x, Y_0(x))}, \ \tilde{G}(x) = (x - \bar{x})^r G(x), \]

where \( r = 0, 1 \), whether \( \bar{x} \) is zero or not of \( A(x, Y_0(x)) \) in \( S_x \). Thus, \( \tilde{G}(x) \) is regular for \( x \in G_M \), continuous for \( x \in M \cup G_M \), and \( U(x) \) is a non-vanishing function on \( M \). As usual, we must firstly conformally transform the problem (40) from \( M \) to the unit circle, using the mapping \( z = \gamma(x) : G_M \rightarrow G_C \), and its inverse given by \( x = \gamma_0(z) : G_C \rightarrow G_M \).

Then, the problem in (40) is reduced to the following: Find a function \( F(z) := \tilde{G}(\gamma_0(z)) \), regular in \( G_C \), continuous in \( G_C \cup C \) such that, \( \text{re}[iU(\gamma_0(z))F(z)] = 0, \ z \in C \).

A crucial step in the solution of the boundary value problem is the determination of the index \( \chi = \frac{1}{\pi} [\text{arg}\{U(x)\}]_{x \in M} \), where \( [\text{arg}\{U(x)\}]_{x \in M} \), denotes the variation of the argument of the function \( U(x) \) as \( x \) moves along the closed contour \( \mathcal{M} \) in the positive direction, provided that \( U(x) \neq 0, \ x \in \mathcal{M} \). Following the lines in [24] we have,

**Lemma V.4.** 1) If \( \lambda_2 < \lambda_2^* \), then \( \chi = 0 \) is equivalent to

\[
\frac{dA(x, Y_0(x))}{dx} \bigg|_{x=1} < 0 \iff \lambda_1 < \bar{t}_1 \bar{t}_2 [\alpha_1^* P_s^*(R_1, D, \{R_1\}) + \frac{d_2 \lambda_2}{\bar{t}_1 \bar{t}_2 \alpha_1}],
\]

\[
\frac{dB(X_0(y), y)}{dy} \bigg|_{y=1} < 0 \iff \lambda_2 < \bar{t}_1 \bar{t}_2 [\alpha_2^* P_s^*(R_2, D, \{R_2\}) + \frac{d_3 \lambda_1}{\bar{t}_1 \bar{t}_2 \alpha_2}].
\]

2) If \( \lambda_2 \geq \lambda_2^* \), \( \chi = 0 \) is equivalent to

\[
\frac{dB(X_0(y), y)}{dy} \bigg|_{y=1} < 0 \iff \lambda_2 < \bar{t}_1 \bar{t}_2 [\alpha_2^* P_s^*(R_2, D, \{R_2\}) + \frac{d_2 \lambda_1}{\bar{t}_1 \bar{t}_2 \alpha_2}],
\]

Thus, under stability conditions (see Lemma III.1) the problem defined in (40) has a unique solution given by,

\[
H(x, 0) = K(x - \bar{x})^{-r} \exp\left[\frac{1}{2\pi} \int_{|t|=1} \frac{\log|J(t)|}{t - \gamma(x)} dt\right] - \frac{\alpha_1^* P_s^*(R_1, D, \{R_1\}) d_2 H(0, 0)}{d_1 d_2 - \bar{t}_1 \bar{t}_2 \alpha_1 \alpha_2}, \ x \in G_M,
\]

(41)

where \( K \) is a constant and \( J(t) = \frac{U_1(t)}{U_1(1)}, \ U_1(t) = U(\gamma_0(t)), \ |t| = 1 \). Setting \( x = 0 \) in (41) we derive a relation between \( K \) and \( H(0, 0) \). Then, for \( x = 1 \in G_M \), and using the first in (32) we can obtain \( K \) and \( H(0, 0) \). Substituting back in
we split the interval \( [0, \pi] \). The integral equation has to be solved numerically by an iterative procedure. For the numerical evaluation of the integrals points given by their angles \( \psi \), i.e.,

\[
(41) \quad \text{we obtain for } x \in G_M, \\
H(x, 0) = \frac{\lambda_1 d_2 + \alpha \alpha_d (t, \ell_2 \alpha_2 P^*(R_2, D, \{R_2\}) - \lambda_2)}{\lambda_1 d_2 + \alpha \alpha_d (t, \ell_2 \alpha_2 P^*(R_2, D, \{R_2\}) - \lambda_2)} \left( (\bar{x} - x)^r \exp \left[ \frac{\gamma(x) - \gamma(1)}{2\pi} \int_{|t|=1} \log \left| \frac{J(t)}{t - \gamma(x)} \right| dt \right] 
+ \frac{\alpha \alpha_d P^*(R_2, D, \{R_2\})}{\alpha \alpha_d (t, \ell_2 \alpha_2 P^*(R_2, D, \{R_2\}))} \exp \left[ \frac{-\gamma(1)}{2\pi} \int_{|t|=1} \log \left| \frac{J(t)}{t - \gamma(1)} \right| dt \right] \right).
\]

Similarly, we can determine \( H(0, y) \) by solving another Riemann-Hilbert boundary value problem on the closed contour \( \mathcal{L} \). Then, using the fundamental functional equation (29) we uniquely obtain \( H(x, y) \).

D. Construction of the conformal mappings and numerical issues

The construction of the conformal mapping \( \gamma(x) \) is not a trivial task. However, we can construct the inverse of it in order to obtain expressions for the expected value of the queue lengths in each relay node. To proceed, we need a representation of \( \mathcal{M} \) in polar coordinates, i.e., \( \mathcal{M} = \{ x : x = \rho(\phi) \exp(i\phi), \phi \in [0, 2\pi] \} \). This procedure is described in detail in (43).

In the following we summarize the basic steps: Since \( 0 \in G_M \), for each \( x \in \mathcal{M} \), a relation between its absolute value and its real part is given by \( |x|^2 = m(Re(x)) \) (see Lemma \[V.3\]). Given the angle \( \phi \) of some point on \( \mathcal{M} \), the real part of this point, say \( \delta(\phi) \), is zero of \( \delta - \cos(\phi) \sqrt{m(\delta)} \), \( \phi \in [0, 2\pi] \). Since \( \mathcal{M} \) is a smooth, egg-shaped contour, the solution is unique. Clearly, \( \rho(\phi) = \frac{\delta(\phi)}{\cos(\phi)} \), and the parametrization of \( \mathcal{M} \) in polar coordinates is fully specified.

Then, the mapping from \( z \in G_C \) to \( x \in G_M \), where \( z = e^{i\phi} \) and \( x = \rho(\psi(\phi))e^{i\psi(\phi)} \), satisfying \( \gamma_0(0) = 0 \), \( \gamma_0(z) = \overline{\gamma_0(z)} \) is uniquely determined by (see [23], Section I.4.4),

\[
\gamma_0(z) = z \exp \left[ \frac{1}{2\pi} \int_0^{2\pi} \log \left( \rho(\psi(\omega)) \right) \frac{d\omega}{\omega - \phi} + \frac{z}{\omega - \phi} \right], |z| < 1,
\psi(\phi) = \phi - \frac{1}{2\pi} \int_0^{2\pi} \log \left( \rho(\psi(\omega)) \right) \cot \left( \frac{\omega - \phi}{2} \right) d\omega, 0 \leq \phi \leq 2\pi,
\]

i.e., \( \psi(.) \) is uniquely determined as the solution of a Theodorsen integral equation with \( \psi(\phi) = 2\pi - \psi(2\pi - \phi) \). This integral equation has to be solved numerically by an iterative procedure. For the numerical evaluation of the integrals we split the interval \( [0, 2\pi] \) into \( M \) parts of length \( 2\pi/M \), by taking \( M \) points \( \phi_k = \frac{2k\pi}{M}, k = 0, 1, ..., M - 1 \). For the \( M \) points given by their angles \( \{ \phi_0, ..., \phi_{M-1} \} \) we should solve the second in (43) to obtain the corresponding points \( \{ \psi(\phi_0), ..., \psi(\phi_{M-1}) \} \), iteratively from,

\[
\psi_0(\phi_k) = \phi_k, \\
\psi_{n+1}(\phi_k) = \phi_k - \frac{1}{2\pi} \int_0^{2\pi} \log \left\{ \frac{\delta(\psi_n(\omega))}{\cos(\psi_n(\omega))} \right\} \cot \left( \frac{1}{2} (\omega - \phi_k) \right) d\omega,
\]

where \( \lim_{n \to \infty} \psi_{n+1}(\phi) = \psi(\phi) \), and \( \delta(\psi_n(\omega)) \) is determined by,

\[
\delta(\psi_n(\omega)) = \cos(\psi_n(\omega)) \sqrt{m(\delta(\psi_n(\omega)))},
\]
using the Newton-Raphson root finding method. For each step, the integral in \(44\) is numerically determined by again using the trapezium rule with \(M\) parts of equal length \(2\pi/M\). For the iteration, we have used the following stopping criterion \(\max_{k \in \{0,1,\ldots,M-1\}} |\psi_{n+1}(\phi_k) - \psi_n(\phi_k)| < 10^{-6}\).

Having obtained \(\psi(\phi)\) numerically, the values of the conformal mapping \(\gamma_0(z)\), \(|z| \leq 1\), can be calculated by applying the Plemelj-Sokhotski formula to the first in \(43\)

\[
\gamma_0(e^{i\phi}) = e^{i\psi(\phi)} \frac{\delta(\psi(\phi))}{\cos(\psi(\phi))} = \delta(\psi(\phi))[1 + i \tan(\psi(\phi))], \quad 0 \leq \phi \leq 2\pi.
\]

We further need to find \(\gamma(1), \gamma'(1)\). To do this, one needs to use the Newton’s method and solve \(\gamma_0(z_0) = 1\), in \([0,1]\), i.e., \(z_0\) is the zero in \([0,1]\) of \(\gamma_0(z) = 1\). Then, \(\gamma(1) = z_0\). Moreover, using the first in \(43\)

\[
\gamma'(1) = (\gamma_0(z_0))^{-1} = \left\{ \frac{1}{\gamma(1)} + \frac{1}{2\pi i} \int_0^{2\pi} \log(\rho(\psi(\omega))) \frac{2e^{i\omega}}{(e^{i\omega} - \gamma(1))^2} d\omega \right\}^{-1}, \quad (45)
\]

which can be obtained numerically by using the Trapezoidal rule for the integral on the right-hand side of \(45\).

Clearly, the numerical computation of the exact conformal mappings is generally time consuming. Since \(\mathcal{M}, \mathcal{L}\) are close to ellipses, alternatively, we can approximate them by conformal mappings that map the interior of ellipses to \(G_C\). In particular, we can approximate the contour \(\mathcal{M}\) by ellipse \(\mathcal{E}\) with semi-axes \(\rho(0), \rho(\pi/2)\). Then, \(\epsilon(x)\) maps \(G_{\mathcal{E}}\) to \(G_C\), where

\[
\epsilon(x) = \sqrt{k}sn\left(\frac{2Q}{\pi} \sin^{-1}\left(\frac{x}{\sqrt{\rho^2(0) - \rho^2(\pi/2)}}; k^2\right)\right), \quad k = 16q \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^8, \quad q = \left(\frac{\rho(0)-\rho(\pi/2)}{\rho(0)+\rho(\pi/2)}\right)^2, \quad Q = \int_0^1 dt \sqrt{(1+t^2)(1-k^2 t^2)},
\]

where \(sn(w; l)\) is the Jacobian elliptic function. Our approximation for \(\gamma(x)\) is \(\epsilon(x), \ x \in \mathcal{M} \cup G_{\mathcal{M}}\).

\[ \text{E. Performance metrics} \]

In the following we derive formulas for the expected number of packets and the average delay at each user node in steady state, say \(E_i\) and \(D_i\), \(i = 1, 2\), respectively. Denote by \(H_1(x,y), H_2(x,y)\) the derivatives of \(H(x,y)\) with respect to \(x\) and \(y\) respectively. Then, \(E_i = H_i(1,1)\), and using Little’s law \(D_i = H_i(1,1)/\lambda_i, \ i = 1, 2\). Using the functional equation \(29, 30\) and \(31\) we arrive after simple calculations in

\[
E_1 = \frac{\lambda_x d_1 H_1(1,0)}{t_{12} \alpha_1 \alpha_2}, \quad E_2 = \frac{\lambda_x d_2 H_2(0,1)}{t_{12} \alpha_1 \alpha_2}. \quad (46)
\]
We only focus on $E_1$, $D_1$ (similarly we can obtain $E_2$, $D_2$). Note that $H_1(1,0)$ can be obtained using either (42) or (37). For the general case $\frac{\alpha_1 \tilde{\alpha}_2}{\alpha_1 P^*(R_1, D_1, \{R_1\})} + \frac{\alpha_2 \tilde{\alpha}_1}{\alpha_2 P^*(R_2, D_1, \{R_2\})} \neq 1$, and using (42),

$$H_1(1,0) = \frac{\lambda_1 d_1 + \alpha_1 \tilde{\alpha}_2 t}{\alpha_1 P^*(R_1, D_1, \{R_1\})} + \frac{\alpha_2 \tilde{\alpha}_1 t}{\alpha_2 P^*(R_2, D_1, \{R_2\})} \lambda - \lambda_2) \gamma' \left( \frac{1}{2 \pi t} \int_{|t|=1} \log \{J(t)\} dt. \right. \tag{47}$$

Substituting (47) in (46) we obtain $E_1$, and dividing with $\lambda_1$, the average delay $D_1$. Note that the calculation of (46) requires the evaluation of integrals (43), (45), (47) using the trapezoid rule, and $\gamma(1), \gamma'(1)$, as described above.

**VI. Explicit expressions for the symmetrical model**

In the following we consider the symmetrical model and obtain exact expressions for the average delay without solving a boundary value problem.

As symmetrical, we mean the case where $\tilde{\lambda}_k = \tilde{\lambda}, \tilde{P}_s(k, D, \{k\}) = 1 - P_s(k, D, \{k\}) = 1 - q = \tilde{q}, P_s(k, D, \{k\}) = \tilde{q}, t_k = t, k = 1, 2, \alpha_i^* = \alpha^*, \alpha_i = \alpha, P_s(r_i, D, \{R_i\}) = \tilde{s}, P_s(R_i, D, \{R_1, R_2\}) = s_{1,2}, P_s(r_i, D, \{R_i\}) = \tilde{s}, P_s(k, R_i, \{k\}) = P_s(1, R_i, \{1, 2\}) + P_s(2, R_i, \{1, 2\}) = P_s(k, R_i, \{1, 2\}) = r, k = 1, 2, i = 1, 2$. As a result, $d_1 = d_2 = d$.

Then, by exploiting the symmetry of the model we clearly have $H_1(1,1) = H_2(1,1), H_1(1,0) = H_2(0,1)$. Recall that $E_i = H_i(1,1)$ the expected number of packets in relay node $R_i$. Therefore, after simple calculations using (29) we obtain,

$$E_1 = \frac{\tilde{\lambda} + t + 2t\tilde{q}r + \tilde{t}^2 d H_1(1,0)}{2 \alpha \alpha - (\lambda + t + 2t\tilde{q}r)}. \tag{48}$$

Setting $x = y$ in (29), differentiating it with respect to $x$ at $x = 1$, and using (30) we obtain

$$E_1 + E_2 = 2E_1 = \frac{2(\tilde{\lambda} + t + 2t\tilde{q}) - \tilde{\lambda}^2 + 2H_1(1,0)\tilde{t}^2 (\alpha \tilde{\alpha} + d)}{2[\alpha \tilde{\alpha} - (\lambda + t + 2t\tilde{q})r]} \tag{49}.$$

Using (48), (49) we finally obtain

$$E_1 = E_2 = \frac{\tilde{\lambda} d + 2 \tilde{\lambda} \alpha \tilde{\alpha} + \lambda (\lambda + 2\tilde{q}r)(2\alpha \tilde{\alpha} - rd)}{2\alpha \tilde{\alpha} [\lambda \alpha \tilde{\alpha} - (\lambda + \alpha \tilde{\alpha} + \lambda \tilde{\lambda} r)]}. \tag{50}$$

Therefore, using Little’s law the average delay in a node is given by

$$D_1 = D_2 = \frac{\tilde{\lambda} d + 2 \tilde{\lambda} \alpha \tilde{\alpha} + t + 2t\tilde{q}r)(2\alpha \tilde{\alpha} - rd)}{2\alpha \tilde{\alpha} [\lambda \tilde{\lambda}]} \tag{51},$$

where $\lambda = \tilde{\lambda} + t( + 2t\tilde{q}r)$, and $\tilde{t}^2 \alpha \tilde{\alpha} - \lambda > 0$ due to the stability condition.
VII. NUMERICAL RESULTS

In this section we evaluate numerically the theoretical results obtained in the previous sections. We will focus on a symmetric users setup in order to simplify the presentation. In particular, we consider the case where $\alpha^* = 1$, $\alpha = 0.7$, and $t = 0.1$. The distance between the users and the destination is 110m, the distance between the users and the relays is 80m and between the destination and the relays the distance is 80m. The path-loss exponent is assumed to be four, we also consider Raleigh fading for the channel gain. The transmit power for both relays is $10mW$ and for the users is $1mW$. We consider two cases for the SINR threshold, $SINR_t = 0.2, 1$, when $SINR_t = 0.2$ the MPR capability is stronger thus, we can have more than two concurrent successful transmissions. We can compute the success probabilities using Equation (1) in [4], for $SINR_t = 0.2$ we obtain $P_s(D, 1) = 0.74, P_s(R_1, 1) = P_s(R_2, 1) = 0.92, P_s(R_1, D, \{R_1\}) = P_s(R_2, D, \{R_2\}) = 0.99, P_s(R_1, D, \{R_1, R_2\}) = P_s(R_2, D, \{R_1, R_2\}) = 0.83.$

For $SINR_t = 1$ we obtain $P_s(D, 1) = 0.23, P_s(R_1, 1) = P_s(R_2, 1) = 0.66, P_s(R_1, D, \{R_1\}) = P_s(R_2, D, \{R_2\}) = 0.96, P_s(R_1, D, \{R_1, R_2\}) = P_s(R_2, D, \{R_1, R_2\}) = 0.5.$

A. Stability

Here we present the effect of the number of users on the stability region. We consider the cases where the number of users is varying from 1 to 11, i.e. $N = 1, ..., 11$. In Fig. 3 we consider the case where $SINR_t = 0.2$, the outer curve in the plot represents the case where $N = 1$, the inner the case corresponds to $N = 11$. Since we have stronger MPR capabilities at the receivers we observe that the region for up to four users is convex thus, it the performance is better than a time division scheme as also described in Sections III and IV.

In Fig. 4 we consider the case where $SINR_t = 1$, the outer curve in the plot represents the case where $N = 1$ and the inner the case for $N = 11$. In this plot, we observe that above two users the region is not a convex set. Thus, a time division scheme would be preferable as also described in Sections III and IV.

In both cases, we observe that as the number of users is increasing, then the number of slots that the relays can transmit packets from their queues is decreasing. Thus, when $N = 11$, the stability region is approaching the empty set, which is an indication that the relays cannot sustain the traffic in their queues.

B. Throughput performance

We provide numerical evaluation regarding throughput per user and aggregate throughput for the case of pure relays, i.e. $\hat{\lambda}_1 = \hat{\lambda}_2 = 0$.

The throughput per user as the number of users increasing in the network is depicted in Fig. 5. The throughput per user is decreasing as the number of users is increasing, in addition we can observe for $SINR_t = 0.2$, the system
becomes unstable after 12 users, for $SINR_t = 1$ the system remains stable when the number of users is up to 6. The aggregate throughput is depicted in Fig. 5, the maximum aggregate throughput for $SINR_t = 0.2$ and $SINR_t = 1$ is achieved for twelve and six users respectively.
Fig. 5. Throughput per user versus the number of users.

Fig. 6. Aggregate throughput versus the number of users.

C. Average Delay and Stability Region for the capture model

In this part we will evaluate the average delay performance. The setup will be different from the previous two subsection due to the capture channel model assumed in the analysis.

a) Example 1. The symmetrical system: In the following we consider the symmetric system and we investigate the effect of the system parameters on the average delay. We assume that \( q = 0.5, \bar{s} = 0.8, \bar{\delta} = 0.9, s_{12} = 0.4 \). In Fig. 7 we can see the effect of \( r \) (i.e., the reception probability of blocked packet by a relay node) on the average delay for
increasing values of \( \hat{\lambda} \) (i.e., the average number of external packet arrivals at a relay node during a time slot) and \( \alpha \) (i.e., the transmission probability of a relay). As expected, the increase in \( \hat{\lambda} \) increases the expected delay, and that decrease becomes more apparent as \( \alpha \) takes small values and \( r \) increases.

![Graph 7](image7.png)

Fig. 7. The average delay vs \( \alpha \) and \( \hat{\lambda} \) for different values of \( r \).

Figure 8 illustrate how sensitive is the average delay as we increase the probability of a direct transmission (at the beginning of a slot) of a source. In particular, as \( t \) remains small, the increase in \( \hat{\lambda} \) from 0.1 to 0.15 will not affect the average delay. As \( t \) increases, the simultaneous increase in \( \hat{\lambda} \) will cause a rapid increase in the average delay, even when we set the transmission probability \( \alpha = \alpha^* \). This is expected, since at the beginning of a slot, source users have precedence over the relays.

![Graph 8](image8.png)

Fig. 8. Effect of \( \hat{\lambda} \) on average delay.
Similar observations can be deduced by Fig. 9, where we can observe the average delay as a function of $\alpha^*$ and $\hat{\lambda}$. Clearly, as $t$ increases from 0.3 to 0.4, the average delay increases rapidly, especially when, $\hat{\lambda}$ tends to 0.1.

**Fig. 9.** Effect of $t$ on average delay.

*b) Example 2. Stability region:* We now focus on the general model, and specifically on the case $\frac{\alpha_1\hat{\alpha}_2}{\alpha_1^*s_1/(1)} + \frac{\alpha_2\hat{\alpha}_1}{\alpha_2^*s_2/(1)} \neq 1$. We investigate the effect of parameters on the stability region, i.e., the set of arrival rate vectors $(\lambda_1, \lambda_2)$, for which the queues of the system are stable. In what follows, let $\alpha_1 = 0.7$, $\alpha_2 = 0.6$, $\alpha_2^* = 0.9$, $P_s^*(R_1, D, \{R_1\}) = P_s^*(R_2, D, \{R_2\}) = 0.9$, $P_s(R_1, D, \{R_1\}) = P_s(R_2, D, \{R_2\}) = 0.8$, $P_s(R_i, D, \{R_1, R_2\}) = 0.4$, $i = 1, 2$, $t_2 = 0.3$.

In Fig. 10 we observe the impact of $t_1$, i.e., the probability of transmission of a packet of source user 1 at the beginning of a slot, on the stability region. Note that this factor plays a crucial role in the performance of the system, since although the destination node hears both source users and the relays, but gives priority to the source users at this time slot. We can easily observe that the increase of $t_1$ from 0.2 to 0.4, will cause a deterioration of the stability region. Moreover, that increase will affect both relays, i.e., both adequate the values of $\hat{\lambda}_1$, and $\hat{\lambda}_2$ will be decreased in order to sustain stability.

In Fig. 11 we set $t_1 = 0.2$, and investigate the impact of the adaptive transmission control in relay node 1 on the stability region. In particular, first we assume that $\alpha_1^* = \alpha_1 = 0.7$, i.e., the relay 1 does not adapt its transmission probability when it senses relay 2 inactive. In such case, the stability region is the part in Fig. 11 colored in blue and yellow. When relay 1 adapts its transmission probability to $\alpha_1^* = 0.9$ (when it senses relay 2 inactive) the stability region changes and is given by the part of Fig. 11 colored in blue and red. Note that the increase of $\alpha_1^*$ will affect the relay 2, since a packet in relay 1 is more likely to be transmitted. Thus, we expect that adequate values of $\hat{\lambda}_2$ to be lower in order to ensure stability.
VIII. Conclusions and Future Work

In this work we obtained the stable throughput region, and investigated the delay analysis of a relay-assisted cooperative wireless network with MPR capabilities and adaptive transmission policy. By applying the stochastic dominance technique we obtained the stability region under general MPR both for the asymmetric network of two source-users, two-relay nodes, and for the symmetric model of $N$ users. In addition, we provided the aggregate throughput...
and the throughput per user in terms of closed form expressions.

We investigated the fundamental problem of delay analysis, and for the asymmetric network of two-users, two-relays we performed a detailed mathematical analysis, which led to the determination of the generating function of the stationary joint queue length distribution of relay nodes in terms of a solution of a Riemann-Hilbert boundary value problem. For the symmetrical model, closed form expressions for the expected delay at each relay node were also derived without the need of solving a boundary value problem. Extensive numerical results were obtained providing insights in the system performance.

In a future work we plan to investigate the stable throughput and delay for the case of a completely random access network, where there is no coordination between source users and relays. A challenging task will be the extension to the case of more than two relay nodes. The investigation of such a network is an open problem, and using our current work as a building block we plan to investigate the possibility of obtaining at least some bounds for the expected delay at relay nodes.

**APPENDIX**

The queue evolution equation (28) implies

\[
E(x^{N_{1,n}}y^{N_{2,n+1}}) = D(x, y) \left\{ t_1 t_2 [P(N_1, n = N_2, n = 0) + E(x^{N_{1,n}} 1_{\{N_{1,n}>0,N_{2,n}=0\}}) (1 + \alpha_1^* P_s(R_1, D, \{R_1\}) (\frac{1}{x} - 1)] \\
+ E(y^{N_{2,n}} 1_{\{N_{1,n}=0,N_{2,n}>0\}}) (1 + \alpha_2^* P_s(R_2, D, \{R_2\}) (\frac{1}{y} - 1)) + E(x^{N_{1,n}} y^{N_{2,n}} 1_{\{N_{1,n}>0,N_{2,n}>0\}}) (1 + \alpha_1 \alpha_2 (\frac{1}{x} - 1) \\
+ \alpha_2 \hat{\alpha}_1 (\frac{1}{y} - 1)) + E(x^{N_{1,n}} y^{N_{2,n}}) [t_1 t_2(1 + S_1(x, y)) + t_2 t_1(1 + S_2(x, y)) + t_2 t_1(1 + S_3(x, y))] \right\},
\]

(52)

where \(1_A\) denotes the indicator function of the event \(A\), and

\[
S_1(x, y) = \{P_s(0, D, \{1, 2\}) P_s(0, R_2, \{1, 2\}) (P_s(1, R_1, \{1, 2\}) + P_s(2, R_1, \{1, 2\})) \\
+ P_s(1, D, \{1, 2\}) P_s(2, R_2, \{1, 2\}) (P_s(1, R_1, \{1, 2\}) + P_s(2, R_1, \{1, 2\})) \},
\]

\[
S_2(x, y) = \{P_s(1, D, \{1, 2\}) P_s(2, R_2, \{1, 2\}) (P_s(1, R_1, \{1, 2\}) + P_s(2, R_1, \{1, 2\})) \}
\]

\[
S_3(x, y) = \{P_s(1, D, \{1, 2\}) P_s(2, R_2, \{1, 2\}) P_s(2, R_1, \{1, 2\}) \}
\]
Note that
\[
H(0, 0) = \lim_{n \to \infty} P(N_{1,n} = N_{2,n} = 0),
\]
\[
H(x, 0) - H(0, 0) = \lim_{n \to \infty} E(x^{N_{1,n}}1_{\{N_{1,n}>0, N_{2,n}=0\}}),
\]
\[
H(0, y) - H(0, 0) = \lim_{n \to \infty} E(y^{N_{2,n}}1_{\{N_{1,n}=0, N_{2,n}>0\}}),
\]
\[
H(x, y) - H(x, 0) - H(0, y) + H(0, 0) = \lim_{n \to \infty} E(x^{N_{1,n}}y^{N_{2,n}}1_{\{N_{1,n}>0, N_{2,n}>0\}}).
\]

Then, using (52) we obtain the functional equation (29).

It is easily seen that \(R(x, y) = \frac{xy - \Psi(x, y)}{xy D(x, y)}\), where \(\Psi(x, y) = D(x, y)[xy(1 + L_3(xy - 1)) + y(1 - x)(\hat{t}_1 \hat{t}_2 \alpha_1 \hat{\alpha}_2 - L_1 x) + x(1 - y)(\hat{t}_1 \hat{t}_2 \alpha_2 \hat{\alpha}_1 - L_2 y)]\), where for \(|x| \leq 1, |y| \leq 1\), \(\Psi(x, y)\) is a generating function of a proper probability distribution. Now, for \(|y| = 1, y \neq 1\) and \(|x| = 1\) it is clear that \(|\Psi(x, y)| < 1 = |xy|\). Thus, from Rouché’s theorem, \(xy - \Psi(x, y)\) has exactly one zero inside the unit circle. Therefore, \(R(x, y) = 0\) has exactly one root \(x = X_0(y)\), such that \(|x| < 1\). For \(y = 1\), \(R(x, 1) = 0\) implies
\[
(x - 1) \left( \lambda_1 - \frac{\hat{t}_1 \hat{t}_2 \alpha_1 \hat{\alpha}_2}{2} \right) = 0.
\]

Therefore, for \(y = 1\), and since \(\lambda_1 < \hat{t}_1 \hat{t}_2 \alpha_1 \hat{\alpha}_2\), the only root of \(R(x, 1) = 0\) for \(|x| \leq 1\), is \(x = 1\). \(\square\) We will prove the part related to \(M\). Similarly, we can also prove the other. For \(y \in [y_1, y_2]\), \(D_y(y)\) is negative, so \(X_0(y)\), \(X_1(y)\) are complex conjugates. It also follows that
\[
Re(X(y)) = \frac{-\beta(y)}{2\alpha(y)}.
\]
Therefore, \(|X(y)|^2 = \frac{\beta(y)}{\alpha(y)} = g(y)\). Clearly, \(g(y)\) is an increasing function for \(y \in [0, 1]\) and thus, \(|X(y)|^2 \leq g(y_2) = \beta_0\).

Using simple algebraic considerations we can prove that, \(X_0(y_1) := \beta_1 = -g(y_1)\) is the extreme left point of \(M\). Finally, 
\(\zeta(\delta)\) is derived by solving (53) for \(y\) with \(\delta = Re(X(y))\), and taking the solution such that \(y \in [0, 1]\). \(\square\) In the following, we focus on the location of the intersection points of \(R(x, y) = 0\), \(A(x, y) = 0\) (resp. \(B(x, y)\)). These points (if they exist) are potential singularities for the functions \(H(x, 0)\), \(H(0, y)\), and thus, their investigation is crucial regarding the analytic continuation of \(H(x, 0)\), \(H(0, y)\) outside the unit disk; see also Lemma 2.2.1 and Theorem 3.2.3 in [22] for alternative approaches.

A. Intersection points between \(R(x, y) = 0\), \(A(x, y) = 0\).

Let \(R(x, y) = 0\), \(x = X_{\pm}(y)\), \(y \in \tilde{C}_y\). We can easily show that the resultant in \(x\) of the two polynomials \(R(x, y)\) and \(A(x, y)\) is \(Res_x(R, A; y) = y(y - 1)Q(y)\), where
\[
Q(y) = -d_1[\lambda_2d_1 + a_2\hat{\alpha}_1(\hat{\lambda}_2 + \hat{\lambda}_1 + L_2)]y^2 + ya_2\hat{\alpha}_1[\lambda_1(\lambda_1 + \hat{\lambda}_2 + \hat{\lambda}_1 + L_2)\
-\hat{t}_1 \hat{t}_2 a_1^* P^*_1(R_1, D, \{R_1\})(a_2\hat{\alpha}_1 + d_1)] + \hat{t}_1 \hat{t}_2 a_1^* P^*_1(R_1, D, \{R_1\})(a_2\hat{\alpha}_1)^2.
\]
Note that \( Q(0) = t_1^2 a_3^1 R_1, D, \{ R_1 \} (a_2 \bar{a}_1)^2 > 0 \) and \( Q(1) = d_1[\lambda_1 \alpha_2 \bar{a}_1 - \lambda_2 d_1 - t_1^2 \alpha_2 \bar{a}_1 \alpha_1^* P_s(R_1, D, \{ R_1 \}) > 0 \), since \( d_1 < 0 \) and due to the stability condition.

Similarly, for \( R(x, y) = 0, y = Y_\pm(x), x \in \tilde{C}_x \), the resultant in \( y \) of the two polynomials \( R(x, y), A(x, y) \) is 
\[
Res_y(R, A; x) = x(x - 1)t_1^2 \alpha_2 \bar{a}_1 \tilde{Q}(x), \]
where,
\[
\tilde{Q}(x) = -[\alpha_2 \bar{a}_1 \lambda_1 + (\bar{\lambda}_1(1 + \tilde{\lambda}_2) + L_1)d_1]x^2 + x[(\bar{\lambda}_1(1 + \tilde{\lambda}_2) + \lambda_2 + L_1)d_1 + (\alpha_2 \bar{a}_1 + d_1)\alpha_1^* P_{s}(R_1, D, \{ R_1 \})t_1^2 - \alpha_1^* P_{s}(R_1, D, \{ R_1 \})t_1^2].
\]

Note also that \( \tilde{Q}(0) = -\alpha_1^* P_{s}(R_1, D, \{ R_1 \})t_1^2 t_2 > 0 \) since \( d_1 < 0 \) and \( \tilde{Q}(1) = t_1^2 \alpha_2 \bar{a}_1 \alpha_1^* P_{s}(R_1, D, \{ R_1 \}) - \lambda_1 \alpha_2 \bar{a}_1 + \lambda_2 d_1 > 0 \) due to the stability conditions (see Lemma [3, 1]). If \( \alpha_1^* \leq \min\{1, (\frac{\alpha_2 \bar{a}_1 \lambda_1 + (\bar{\lambda}_1(1 + \tilde{\lambda}_2) + L_1)\alpha_2 \bar{a}_1}{(\lambda_1(1 + \tilde{\lambda}_2) + L_1)P_{s}(R_1, D, \{ R_1 \})}) \}, \) then \( \lim_{x \to \infty} \tilde{Q}(x) = -\infty, \) and \( \tilde{Q}(x) = 0 \) has two roots of opposite sign, say \( x_\ast < 0 < 1 < x^\ast. \) If \( \frac{\alpha_2 \bar{a}_1 \lambda_1 + (\bar{\lambda}_1(1 + \tilde{\lambda}_2) + L_1)\alpha_2 \bar{a}_1}{(\lambda_1(1 + \tilde{\lambda}_2) + L_1)P_{s}(R_1, D, \{ R_1 \})} < \alpha_1^* \leq 1, \) then \( \lim_{x \to \infty} \tilde{Q}(x) = +\infty, \) and \( \tilde{Q}(x) = 0 \) has two positive roots, say \( 1 < \tilde{x}_\ast < x_3 < x_4 < \bar{x}_\ast, \) due to the stability conditions. In the former case we have to check if \( x^\ast \) is in \( S_x, \) while in the latter case if \( \tilde{x}_\ast \) is in \( S_x. \)

These zeros, if they lie in \( S_x \) such that \( |Y_0(x)| \leq 1, \) are poles of \( H(x, y). \) Denote by

\[
\tilde{x} = \begin{cases} 
  x^\ast, & \alpha_1^* \leq \min\{1, (\frac{\alpha_2 \bar{a}_1 \lambda_1 + (\bar{\lambda}_1(1 + \tilde{\lambda}_2) + L_1)\alpha_2 \bar{a}_1}{(\lambda_1(1 + \tilde{\lambda}_2) + L_1)P_{s}(R_1, D, \{ R_1 \})}) \}, \\
  \tilde{x}_\ast, & (\frac{\alpha_2 \bar{a}_1 \lambda_1 + (\bar{\lambda}_1(1 + \tilde{\lambda}_2) + L_1)\alpha_2 \bar{a}_1}{(\lambda_1(1 + \tilde{\lambda}_2) + L_1)P_{s}(R_1, D, \{ R_1 \})}) < \alpha_1^* \leq 1.
\end{cases}
\]

\[ B. \text{ Intersection points between } R(x, y) = 0, B(x, y) = 0. \]

Let \( y \in \tilde{C}_y \) and \( R(x, y) = 0, x = X_\pm(y). \) It is easily shown that the resultant in \( x \) of \( R(x, y), B(x, y) \) is
\[ Res_x(R, B, y) = y(y - 1)T(y), \]
where
\[
T(y) = -[\alpha_2 \bar{a}_1 \lambda_2 + (\tilde{\lambda}_2(1 + \tilde{\lambda}_1) + L_2)d_2]y^2 + y[(\tilde{\lambda}_2(1 + \tilde{\lambda}_1) + \lambda_2 + L_2)d_2 + (\alpha_2 \bar{a}_1 + d_2)\alpha_2 \lambda_2 P_{s}(R_2, D, \{ R_2 \})t_1^2 - \alpha_2 \lambda_2 P_{s}(R_2, D, \{ R_2 \})t_1^2].
\]

Note that \( T(0) = -\alpha_2 \lambda_2 P_{s}(R_2, D, \{ R_2 \})d_2^2 t_1^2 t_2 > 0, \)
\[
T(1) = t_1^2 \alpha_2 \bar{a}_1 \alpha_2 \lambda_1 P_{s}(R_2, D, \{ R_2 \}) - \lambda_2 \alpha_2 \lambda_2 + \lambda_1 d_2 > 0,
\]
since \( d_2 < 0 \) and due to the stability conditions. If \( \alpha_2^* < \min\{1, (\frac{\alpha_2 \bar{a}_1 \lambda_2 + (\tilde{\lambda}_2(1 + \tilde{\lambda}_1) + L_2)\alpha_2 \lambda_2}{(\lambda_2(1 + \lambda_1) + L_2)P_{s}(R_2, D, \{ R_2 \})}) \}, \) then \( \lim_{y \to \infty} T(y) = -\infty, \) and \( T(x) \) has two roots of opposite sign, say \( y_\ast, y^\ast \) such that \( y_\ast < 0 < 1 < y^\ast, \) which in turn implies that \( B(X_0(y), y) \neq 0, y \in [y_1, y_2] \subset (0, 1), \) or equivalently \( B(x, Y_0(x)) \neq 0, x \in M. \) When \( \frac{\alpha_2 \bar{a}_1 \lambda_2 + (\tilde{\lambda}_2(1 + \tilde{\lambda}_1) + L_2)\alpha_2 \lambda_2}{(\lambda_2(1 + \lambda_1) + L_2)P_{s}(R_2, D, \{ R_2 \})} < \alpha_2^* \leq 1, \)
\[
\lim_{y \to \infty} T(y) = +\infty, \] and \( T(y) \) has two positive roots, say \( \tilde{y}_\ast, \tilde{y}^\ast \) such that \( 1 < \tilde{y}_\ast < y_3 < y_4 < \tilde{y}^\ast, \) which in turn implies that \( B(X_0(y), y) \neq 0, y \in [y_1, y_2], \) i.e., \( B(x, Y_0(x)) \neq 0, x \in M. \)

References

[1] R. e. a. Pabst, “Relay-based deployment concepts for wireless and mobile broadband radio,” IEEE Communications Magazine, vol. 42, no. 9, pp. 80–89, 2004.
[2] T. Cover and A. Gamal, “Capacity theorems for the relay channel,” *IEEE Trans. Inform. Theory*, vol. 25, no. 5, pp. 572–584, 1979.

[3] A. Nosratinia, T. Hunter, and A. Hedayat, “Cooperative communication in wireless networks,” *IEEE Communications Magazine*, vol. 42, no. 10, pp. 74–80, 2004.

[4] N. Pappas, M. Kountouris, A. Ephremides, and A. Traganitis, “Relay-assisted multiple access with full-duplex multi-packet reception,” *IEEE Transactions on Wireless Communications*, vol. 14, pp. 3544–3558, July 2015.

[5] G. Papadimitriou, N. Pappas, A. Traganitis, and V. Angelakis, “Network-level performance evaluation of a two-relay cooperative random access wireless system,” *Computer Networks*, vol. 88, pp. 187–201, 2015.

[6] N. Pappas, A. Ephremides, and A. Traganitis, “Stability and performance issues of a relay assisted multiple access scheme with mpr capabilities,” *Computer Communications*, vol. 42, pp. 70–76, 2014.

[7] A. Sadek, K. Liu, and A. Ephremides, “Cognitive multiple access via cooperation: Protocol design and performance analysis,” *IEEE Trans. Infor. Th.*, vol. 53, no. 10, pp. 3677–3696, 2007.

[8] A. Sendonaris, E. Erkip, and B. Aazhang, “User cooperation diversity-part i: system description,” *IEEE Trans. Commun.*, vol. 51, pp. 1927–1938, 2003.

[9] E. C. van der Meulen, “Three-terminal communication channels,” *Adv. Appl. Probab.*, vol. 3, pp. 120–154, 1971.

[10] J. Laneman, D. Tse, and G. Wornell, “Cooperative diversity in wireless networks: efficient protocols and outage behavior,” *IEEE Trans. Inform. Theory*, vol. 50, pp. 3062–3080, 2004.

[11] B. Rong and A. Ephremides, “Cooperation above the physical layer: The case of a simple network,” in *IEEE International Symposium on Information Theory (ISIT)*, pp. 1789–1793, 2009.

[12] A. Laya, L. Alonso, and J. Alonso-Zarate, “Is the random access channel of lte and lte-a suitable for m2m communications? a survey of alternatives,” *IEEE Communications Surveys Tutorials*, vol. 16, pp. 4–16, First 2014.

[13] M. Koseoglu, “Lower bounds on the lte-a average random access delay under massive m2m arrivals,” *IEEE Transactions on Communications*, vol. 64, pp. 2104–2115, May 2016.

[14] J. B. Seo and V. C. M. Leung, “Performance modeling and stability of semi-persistent scheduling with initial random access in lte,” *IEEE Transactions on Wireless Communications*, vol. 11, pp. 4446–4456, December 2012.

[15] Z. Utkovski, O. Simeone, T. Dimitrova, and P. Popovski, “Random access in c-ran for user activity detection with limited-capacity fronthaul,” *IEEE Signal Processing Letters*, vol. 24, pp. 17–21, Jan 2017.

[16] W. Luo and A. Ephremides, “Stability of n interacting queues in random-access systems,” *IEEE Transactions on Information Theory*, vol. 45, pp. 1579–1587, Jul 1999.

[17] R. Rao and A. Ephremides, “On the stability of interacting queues in a multiple-access system,” *IEEE Transactions on Information Theory*, vol. 34, pp. 918–930, Sep 1988.

[18] W. Szpankowski, “Stability conditions for multidimensional queueing systems with computer applications,” *Operations Research*, vol. 36, no. 6, pp. 944–957, 1988.

[19] P. Nain, “Analysis of a two-node aloha-network with infinite capacity buffers,” in *Int. Seminar on Computer Networking and Performance Evaluation*, pp. 49–63, September 1985.

[20] V. Naware, G. Mergen, and L. Tong, “Stability and delay of finite-user slotted aloha with multipacket reception,” *IEEE Transactions on Information Theory*, vol. 51, pp. 2636–2656, July 2005.

[21] I. Dimitriou and N. Pappas, *Stability and Delay Analysis of an Adaptive Channel-Aware Random Access Wireless Network*, pp. 63–80. Cham: Springer International Publishing, 2017.
[22] G. Fayolle and R. Iasnogorodski, “Two coupled processors: The reduction to a riemann-hilbert problem,” Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, vol. 47, no. 3, pp. 325–351, 1979.

[23] J. Cohen and O. Boxma, Boundary value problems in queueing systems analysis. Amsterdam, Netherlands: North Holland Publishing Company, 1983.

[24] G. Fayolle, R. Iasnogorodski, and V. Malyshev, Random walks in the quarter-plane: Algebraic methods, boundary value problems, applications to queueing systems and analytic combinatorics. Springer-Verlag, Berlin, 2017.

[25] K. Avrachenkov, P. Nain, and U. Yechiali, “A retrial system with two input streams and two orbit queues,” Queueing Systems, vol. 77, no. 1, pp. 1–31, 2014.

[26] J. Cohen, “Boundary value problems in queueing theory,” Queueing Syst., vol. 3, pp. 97–128, 1988.

[27] I. Dimitriou, “A two class retrial system with coupled orbit queues,” Prob. Engin. Infor. Sc., vol. 31, no. 2, pp. 139–179, 2017.

[28] I. Dimitriou, “A queueing model with two types of retrial customers and paired services,” Ann. Oper. Res., vol. 238, no. 1, pp. 123–143, 2016.

[29] C. Fricker, F. Guillemin, P. Robert, and G. Thompson, “Analysis of an offloading scheme for data centers in the framework of fog computing,” ACM Trans. Model. Perform. Eval. Comput. Syst., vol. 1, pp. 16:1–16:18, Sept. 2016.

[30] F. Guillemin and D. Pinchon, “Analysis of generalized processor-sharing systems with two classes of customers and exponential services,” J. Appl. Probab., vol. 41, pp. 832–858, 09 2004.

[31] F. Guillemin, C. Knessl, and J. S. H. van Leeuwaarden, “Wireless three-hop networks with stealing ii: exact solutions through boundary value problems,” Queueing Systems, vol. 74, pp. 235–272, Jun 2013.

[32] J. Van Leeuwaarden and J. Resing, “A tandem queue with coupled processors: Computational issues,” Queueing Syst., vol. 50, pp. 29–52, 2005.

[33] W. Szpankowski, “Stability conditions for some distributed systems: buffered random access systems,” Advances in Applied Probability, vol. 26, no. 2, pp. 498–515, 1994.

[34] W. Szpankowski, “Bounds for queue lengths in a contention packet broadcast system,” IEEE Transactions on Communications, vol. 34, pp. 1132–1140, Nov 1986.

[35] Q. Mahmoud, Cognitive networks: towards self-aware networks. Wiley-Interscience, 2007.

[36] F. Gakhov, Boundary value problems. Pergamon Press, Oxford, 1966.

[37] N. Pappas, J. Jeon, A. Ephremides, and A. Traganitis, “Wireless network-level partial relay cooperation,” in 2012 IEEE International Symposium on Information Theory Proceedings, pp. 1122–1126, July 2012.

[38] D. Tse and P. Viswanath, Fundamentals of wireless communication. New York, NY, USA: Cambridge University Press, 2005.

[39] R. Loynes, “The stability of a queue with non-independent inter-arrival and service times,” Math. Proc. Camb. Philos. Soc., vol. 58, pp. 497–520, 1962.

[40] Z. Nehari, Conformal mapping. New York: McGraw-Hill, 1952.