On Dwork cohomology and algebraic $\mathcal{D}$-modules

Francesco Baldassarri    Andrea D’Agnolo

Abstract. After works by Katz, Monsky, and Adolphson-Sperber, a comparison theorem between relative de Rham cohomology and Dwork cohomology is established in a paper by Dimca-Maaref-Sabbah-Saito in the framework of algebraic $\mathcal{D}$-modules. We propose here an alternative proof of this result. The use of Fourier transform techniques makes our approach more functorial.

2000 Mathematics Subject Classification: 32S40, 14F10

1. Review of algebraic $\mathcal{D}$-modules

For the reader’s convenience, we recall here the notions and results from the theory of algebraic $\mathcal{D}$-modules that we need. Our references were [4, 3, 6, 7].

1.1. Basic operations.

Let $X$ be a smooth algebraic variety over a field of characteristic zero, and let $\mathcal{O}_X$ and $\mathcal{D}_X$ be its structure sheaf and the sheaf of differential operators, respectively. Let $\operatorname{Mod}(\mathcal{D}_X)$ be the abelian category of left $\mathcal{D}_X$-modules, $\mathcal{D}^b(\mathcal{D}_X)$ its bounded derived category, and $\mathcal{D}^b_{\text{qc}}(\mathcal{D}_X)$ the full triangulated subcategory of $\mathcal{D}^b(\mathcal{D}_X)$ whose objects have quasi-coherent cohomologies.

Let $f: X \rightarrow Y$ be a morphism of smooth algebraic varieties, and denote by $\mathcal{D}_{X\rightarrow Y}$ and $\mathcal{D}_{Y\leftarrow X}$ the transfer bimodules. We use the following notation for the operations of tensor product, inverse image, and direct image for $\mathcal{D}$-modules

\[
\begin{align*}
\otimes &: \mathcal{D}^b_{\text{qc}}(\mathcal{D}_X) \times \mathcal{D}^b_{\text{qc}}(\mathcal{D}_X) \to \mathcal{D}^b_{\text{qc}}(\mathcal{D}_X), \quad (\mathcal{M}, \mathcal{M}') \mapsto \mathcal{M} \otimes^L_{\mathcal{O}_X} \mathcal{M}', \\
f^* &: \mathcal{D}^b_{\text{qc}}(\mathcal{D}_Y) \to \mathcal{D}^b_{\text{qc}}(\mathcal{D}_X), \quad \mathcal{N} \mapsto \mathcal{D}_{X\rightarrow Y} \otimes^L_{\mathcal{D}_Y} f^{-1} \mathcal{N}, \\
f_+ &: \mathcal{D}^b_{\text{qc}}(\mathcal{D}_X) \to \mathcal{D}^b_{\text{qc}}(\mathcal{D}_Y), \quad \mathcal{M} \mapsto Rf_*(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{M}).
\end{align*}
\]

1. About the tensor product, note that $\mathcal{M} \otimes^L_{\mathcal{O}_X} \mathcal{M'} \simeq (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes^L_{\mathcal{D}_X} \mathcal{M}' \simeq \mathcal{M} \otimes^L_{\mathcal{D}_X} (\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{M}')$, where $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ (resp. $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{M}'$) is given the natural structure of left-right (resp. left-left) $\mathcal{D}_X$-bimodule, and $\otimes^L_{\mathcal{D}_X}$ always uses up the “trivial” $\mathcal{D}_X$-module structure.
If \( f : X \to Y \) and \( g : Y \to Z \) are morphisms of smooth algebraic varieties, then there are natural functorial isomorphisms
\[
\begin{align*}
    f^* g^* & \simeq (g \circ f)^*, \\
    g_+ f_+ & \simeq (g \circ f)_+.
\end{align*}
\] (1.1) (1.2)

For \( \mathcal{N}, \mathcal{N}' \in \text{D}_{qc}^b(\mathcal{D} Y) \), there is a natural isomorphism
\[
f^* (\mathcal{N} \otimes \mathcal{N}') \simeq f^* \mathcal{N} \otimes f^* \mathcal{N}'.
\] (1.3)

For \( \mathcal{M} \in \text{D}_{qc}^b(\mathcal{D} X) \) and \( \mathcal{N} \in \text{D}_{qc}^b(\mathcal{D} Y) \), there is a projection formula\(^2\)
\[
f_+ (\mathcal{M} \otimes f^* \mathcal{N}) \simeq f_+ \mathcal{M} \otimes \mathcal{N}.
\] (1.4)

Consider a Cartesian square of smooth algebraic varieties
\[
\begin{array}{ccc}
X' & \xrightarrow{b'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{h} & Y.
\end{array}
\] (1.5)

For \( \mathcal{M} \in \text{D}_{qc}^b(\mathcal{D} X) \), there is a base change formula\(^3\)
\[
f_* h^* \mathcal{M}[d_X - d_X'] \simeq h^* f_* \mathcal{M}[d_Y' - d_Y],
\] (1.6)

where \( d_X \) denotes the dimension of \( X \).

### 1.2. Relative cohomology.

Let \( S \) be a closed subscheme of \( X \), and denote by \( \mathcal{I}_S \subset \mathcal{O}_X \) the corresponding ideal of \( \mathcal{O}_X \). For \( \mathcal{F} \in \text{Mod}(\mathcal{O}_X) \) one sets\(^3\)
\[
\Gamma_{[S]}(\mathcal{F}) = \lim_{\to} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_S^m, \mathcal{F}).
\]

We point out that \( \Gamma_{[S]}(\mathcal{F}) = \Gamma_{[S^{red}]}(\mathcal{F}) \). If \( \mathcal{M} \in \text{Mod}(\mathcal{D} X) \) one checks that \( \Gamma_{[S]}\mathcal{M} \) has a natural left \( \mathcal{D} X \)-module structure, and one considers the right derived functor
\[
R\Gamma_{[S]} : \text{D}^b_{qc}(\mathcal{D} X) \to \text{D}^b_{qc}(\mathcal{D} X).
\]

Let \( i : X \setminus S \to X \) be the open embedding, and \( \mathcal{M} \in \text{D}_{qc}^b(\mathcal{D} X) \). There is a distinguished triangle in \( \text{D}_{qc}^b(\mathcal{D} X) \)
\[
R\Gamma_{[S]} \mathcal{M} \to \mathcal{M} \to i_* i^* \mathcal{M} \overset{+1}{\to}.
\] (1.7)

---

\(^2\)In the appendix we recall the proofs of base change and projection formulae.

\(^3\)In other words, for any open subset \( \mathcal{V} \subset X \), \( \Gamma_{[S]}\mathcal{F}(\mathcal{V}) = \{ s \in \mathcal{F}(\mathcal{V}) : (\mathcal{I}_S|\mathcal{V})^m s = 0, \, m \gg 0 \} \). Recall that if \( \mathcal{F} \) is quasi-coherent, Hilbert’s Nullstellensatz implies that \( \Gamma_{[S]}\mathcal{F} \simeq \Gamma_{\mathcal{S}}\mathcal{F} \), the subsheaf of \( \mathcal{F} \) whose sections are supported in \( S \).
For $S, S' \subset X$ possibly singular closed subvarieties, and $\mathcal{M} \in \mathcal{D}_{qc}^b(\mathcal{D}_X)$, one has
\begin{align*}
R\Gamma_{[S]}\mathcal{M} & \simeq \mathcal{M} \otimes R\Gamma_{[S]}\mathcal{O}_X, \\
R\Gamma_{[S]}R\Gamma_{[S']}\mathcal{M} & \simeq R\Gamma_{[S \cap S']}\mathcal{M}.
\end{align*}
(1.8)

Let $f : X \to Y$ be a morphism of smooth varieties, $Z \subset Y$ a possibly singular closed subvariety, and set $S = f^{-1}(Z) \subset X$. Then there is an isomorphism
\[ f_+ R\Gamma_{[S]} \mathcal{M} \simeq R\Gamma_{[Z]} f_+ \mathcal{M}. \]
(1.10)

Let $Y$ be a closed smooth subvariety of $X$ of codimension $d$, and denote by $j : Y \to X$ the embedding. Recall that Kashiwara's equivalence states that the functors $M \mapsto j_* M\big([-d]\big)$ and $N \mapsto j_+ N$ establish an equivalence between the category $\text{Mod}_{qc}(\mathcal{D}_Y)$ of quasi-coherent $\mathcal{D}_Y$-modules, and the full abelian subcategory of $\text{Mod}_{qc}(\mathcal{D}_X)$ whose objects $\mathcal{M}$ satisfy $\Gamma_Y \mathcal{M} \simeq \ldots$. This extends to derived categories. In particular, the functor $j_+: D_{qc}^b(\mathcal{D}_Y) \to D_{qc}^b(\mathcal{D}_X)$ is fully faithful, (1.11) and for $\mathcal{M} \in D_{qc}^b(\mathcal{D}_X)$ one has
\[ R\Gamma_Y \mathcal{M} \simeq j_+ j^* \mathcal{M}\big([-d]\big). \]
(1.12)

1.3. Fourier-Laplace transform.

To $\varphi \in \Gamma(X; \mathcal{O}_X)$ one associates the $\mathcal{D}_X$-module
\[ D_X e^\varphi = \mathcal{D}_X/\mathcal{I}_\varphi, \quad \mathcal{I}_\varphi(V) = \{ P \in \mathcal{D}_X(V) : P e^\varphi = 0 \}, \quad \forall V \subset X \text{ open}. \]

For $f : X \to Y$ a morphism of smooth algebraic varieties, and $\psi \in \Gamma(Y; \mathcal{O}_Y)$, one has
\[ f^* \mathcal{D}_Y e^\psi \simeq D_X e^{\psi \circ f}. \]
(1.13)

Let us denote by $\mathbb{A}_X^1$ the trivial line bundle on $X$, and by $t \in \Gamma(\mathbb{A}_X^1; \mathcal{O}_{\mathbb{A}_X^1})$ its fiber coordinate. Let $\pi : \tilde{V} \to X$ be a vector bundle of finite rank, $\hat{\pi} : \hat{V} \to X$ be the dual bundle, $\gamma_V : \hat{V} \times_X \tilde{V} \to \mathbb{A}_X^1$ be the natural pairing, and $V \xrightarrow{p_2} \hat{V} \times_X \tilde{V} \xrightarrow{p_2} \hat{V}$ be the natural projections. The Fourier-Laplace transform for $\mathcal{D}$-modules is the functor
\[ F_V : D_{qc}^b(\mathcal{D}_V) \to D_{qc}^b(\mathcal{D}_V) \]
\[ \mathcal{N} \mapsto p_2^* (p_1^* \mathcal{N} \otimes \gamma_V^* \mathcal{L}_1). \]

The Fourier-Laplace transform is involutive, in the sense that (cf [10, Lemma 7.1 and Appendix 7.5])
\[ F_V \circ F_V \simeq (-\text{id}_V)^*. \]
(1.14)

\text{\textsuperscript{4}}Equivalently, $D_X e^\varphi$ is the sheaf $\mathcal{O}_X$ with the $\mathcal{D}_X$-module structure given by the flat connection $1 \mapsto d\varphi$. 
Let \( f : V \to W \) be a morphism of vector bundles over \( X \), and denote by \( t^f : \check{W} \to \check{V} \) the transpose of \( f \). Then for any \( N \in D^b_{qc}(\mathcal{D}_V) \) and \( P \in D^b_{qc}(\mathcal{D}_W) \) there are natural isomorphisms\(^5\)

\[
\mathcal{F}_W f^* N \simeq (t^f)^* \mathcal{F}_V N, \quad (1.15)
\]

\[
\mathcal{F}_V f^* P \simeq (t^f)_+ \mathcal{F}_W P. \quad (1.16)
\]

If \( X \) is viewed as a zero-dimensional vector bundle over itself, the projection \( \pi : V \to X \) and the zero-section \( \check{i} : X \to \check{V} \) are transpose to each other. Hence (1.16) gives for \( M \in D^b_{qc}(\mathcal{D}_X) \) and \( Q \in D^b_{qc}(\mathcal{D}_\check{V}) \) the isomorphisms\(^6\)

\[
\check{i}_* M \simeq \mathcal{F}_V \pi^* M, \quad (1.17)
\]

\[
\check{i}^* Q \simeq \pi_* \mathcal{F}_\check{V} Q. \quad (1.18)
\]

## 2. Dwork cohomology

Let \( s : X \to \check{V} \) be a section of the vector bundle \( \hat{\pi} : \check{V} \to X \) of rank \( r \), and set \( \check{s} = \text{id}_V \times_X s : V \to V \times_X \check{V} \). Recall that \( \gamma_V : V \times_X \check{V} \to \mathbb{A}^1_X \) denotes the pairing, and let \( F \in \Gamma(V; \mathcal{O}_V) \) be the function

\[ F = t \circ \gamma_V \circ \check{s}. \]

Let us denote by \( S \) the reduced zero locus of \( s \), which is a possibly singular closed subvariety of \( X \), and by \( j : S \to X \) the embedding. The geometric framework is thus summarized in the commutative diagram with Cartesian squares

\[
\begin{array}{ccc}
X & \xrightarrow{i} & \check{V} \\
\downarrow{j_1} & & \downarrow{j_2} \\
S & \xleftarrow{\check{s}} & V \times_X \check{V} \\
\downarrow{p_2} & & \downarrow{p_1} \\
V & \xleftarrow{\pi} & \check{V}
\end{array}
\]

Then \( j_1 = j_2 = j \) on \( S = \check{S}^{\text{red}} \). Generalizing previous results of \([9, 13, 2]\), Theorem 0.2 of \([5]\) gives the following link between relative cohomology and Dwork cohomology

**Theorem 2.1.** For \( \mathcal{M} \in D^b_{qc}(\mathcal{D}_X) \) there is an isomorphism

\[
R\Gamma_{[S]} \mathcal{M}[r] \simeq \pi_+ (\pi^* \mathcal{M} \otimes \mathcal{D}_V e^F).
\]

Our aim here is to provide a more natural proof of this result.

**Proof.** For \( \mathcal{M} = \mathcal{O}_X \), the statement reads

\[
R\Gamma_{[S]} \mathcal{O}_X[r] \simeq \pi_+ \mathcal{D}_V e^F. \quad (2.1)
\]

\(^5\)See the appendix for a proof.

\(^6\)Note that isomorphism (1.17) is the content of \([5, \text{Lemma 2.3}]\), of which we have thus provided a more natural proof.
For a general \( M \in \mathbf{D}_{qc}^{b}(\mathcal{D}_X) \), there are isomorphisms
\[
R\Gamma_{[S]}M \cong M \otimes R\Gamma_{[S]}\mathcal{O}_X \quad \text{by (1.3),}
\]
and
\[
\pi^+(\pi^* M \otimes \mathcal{D}_Ve^F) \cong M \otimes \pi^+ \mathcal{D}_Ve^F \quad \text{by (1.4).}
\]
It is thus sufficient to prove (2.1). Setting \( L = \gamma^*_V \mathcal{D}_{\Lambda^k_V} e^t \), there is a chain of isomorphisms
\[
\mathcal{D}_Ve^F \cong \tilde{s}^* \mathcal{L} \quad \text{by (1.13)}
\]
\[
\cong \tilde{s}^* \mathcal{L} \otimes \mathcal{O}_V \quad \text{by (1.2)}
\]
\[
\cong p_{1+} \tilde{s}^* (\mathcal{L} \otimes \mathcal{O}_V) \quad \text{by (1.4)}
\]
\[
\cong p_{1+} (\mathcal{L} \otimes \tilde{s}^* \mathcal{O}_X) \quad \text{by (1.6)}
\]
\[
\cong p_{1+} (\mathcal{L} \otimes p^*_2 \mathcal{S} \mathcal{O}_X) \quad \text{by (1.4)}
\]
\[
= \mathcal{F}_V s^* \mathcal{O}_X.
\]
Hence we have
\[
\pi^+ \mathcal{D}_Ve^F \cong \pi^+ \mathcal{F}_V s^* \mathcal{O}_X
\]
\[
\cong \tilde{v}^* s^* \mathcal{O}_X \quad \text{by (1.13)},
\]
and to prove (2.1) we are left to establish an isomorphism
\[
\tilde{v}^* s^* \mathcal{O}_X \cong R\Gamma_{[S]} \mathcal{O}_X[r].
\] (2.2)

By (1.11), this follows from the chain of isomorphisms
\[
\tilde{i} + \tilde{v}^* s^* \mathcal{O}_X \cong \tilde{i} + \tilde{v}^* s^* \mathcal{O}_V
\]
\[
\cong R\Gamma_{[i(X)]} R\Gamma_{[s(X)]} \mathcal{O}_V[2r] \quad \text{by (1.12)}
\]
\[
\cong R\Gamma_{[i(S)]} R\Gamma_{[i(X)]} \mathcal{O}_V[2r] \quad \text{by (1.9)}
\]
\[
\cong R\Gamma_{[i(S)]} \tilde{i}^* \mathcal{O}_X[r] \quad \text{by (1.12)}
\]
\[
\cong \tilde{i}^* R\Gamma_{[S]} \mathcal{O}_X[r] \quad \text{by (1.10)}.
\]

\[\square\]

**Remark 2.2.** Kashiwara’s equivalence allows one\(^7\) to develop the theory of algebraic \( \mathcal{D} \)-modules on possibly singular varieties, so that the formulae stated in the previous section still hold. In this framework, (2.2) is obtained by
\[
\tilde{v}^* s^* \mathcal{O}_X \cong j^* j^* \mathcal{O}_X \quad \text{by (1.6)}
\]
\[
\cong R\Gamma_{[S]} \mathcal{O}_X[r] \quad \text{by (1.12)}.
\]

\(^7\)This is done for example in [3]. For \( S \) a singular closed subvariety of a smooth variety \( X \), the idea is to define \( \text{Mod}(\mathcal{D}_S) \) as the full abelian subcategory of \( \text{Mod}(\mathcal{D}_X) \) whose objects \( M \) satisfy \( \Gamma_{[S]} M \rightarrow M \).
A. Appendix

A.1. Base change and projection formulae.

The base change formula \(1.6\) is proved in \([4, \text{Theorem VI.8.4}]\) for \(h\) a locally closed embedding\(^8\). Let us recall how to deal with the general case.

**Proof of (1.6).** The Cartesian square (1.5) splits into the two Cartesian squares

\[
\begin{array}{ccc}
X' & \xrightarrow{(f',h')} & Y' \times X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{(id_Y, h)} & Y' \times Y \\
\end{array}
\]

where \(p_2\) and \(p'_2\) are the natural projections. Since \((id_Y, h)\) is a closed embedding, by \([4]\) the base change formula holds for the Cartesian square on the left hand side. We are thus left to prove the base change formula for the Cartesian square on the right hand side. For \(\mathcal{M} \in D^{bc}_{qc}(\mathcal{D}_X)\), one has the chain of isomorphisms

\[
(id_Y \times f)_+ p_2^* \mathcal{M} \simeq (id_Y \times f)_+ (\mathcal{O}_Y \boxtimes \mathcal{M}) \\
\simeq \mathcal{O}_Y \boxtimes f_+ \mathcal{M} \\
\simeq p_2^* f_+ \mathcal{M},
\]

where \(\boxtimes\) denotes the exterior tensor product. \(\square\)

Let us also recall, following \([3]\), how projection formula is deduced from base change formula.

**Proof of (1.4).** Consider the diagram with commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \delta_X & & \downarrow \delta_Y \\
X \times X & \xrightarrow{f''} & X \times Y \\
\end{array}
\]

where \(\delta_X\) and \(\delta_Y\) are the diagonal embeddings, \(\delta_f\) is the graph embedding, \(f' = f \times id_Y\), and \(f'' = id_X \times f\). Then there is a chain of isomorphisms

\[
f_+ (\mathcal{M} \otimes f^* \mathcal{N}) \simeq f_+ \delta_X^* (\mathcal{M} \boxtimes f^* \mathcal{N}) \\
\simeq f_+ \delta_X^* f''^* (\mathcal{M} \boxtimes \mathcal{N}) \\
\simeq \delta_Y^* f'_+ (\mathcal{M} \boxtimes \mathcal{N}) \quad \text{by (1.6)} \\
\simeq \delta_Y^* (f_+ \mathcal{M} \boxtimes \mathcal{N}) \\
\simeq f_+ \mathcal{M} \otimes \mathcal{N}.
\]

\(\square\)

---

\(^8\)In the language of Gauss-Manin connections, the base change formula is stated in \([1, \S\ 3.2.6]\) for \(h\) flat.
A.2. Fourier-Laplace transform.

The formulae stated in section 1.3 for the Fourier-Laplace transform of algebraic $\mathcal{D}$-modules have their analogues for the Fourier-Deligne transform of $\ell$-adic sheaves (see [12] or [11, §III.13]), and for the Fourier-Sato transform of conic abelian sheaves (see [8]). Apart from [10], we do not have specific references for the algebraic $\mathcal{D}$-module case. We thus provide here some proofs.

**Proof of (1.15) and (1.16).** The following arguments are parallel to those in the proof of [12, Théorème 1.2.2.4] or [8, Proposition 3.7.14]. Consider the diagram with Cartesian squares

\[
\begin{array}{c}
\begin{array}{ccc}
\tilde{V} & \xrightarrow{\tilde{f}} & \tilde{W} \\
p_2 & & q_2 \\
V \times_X \tilde{V} & \xleftarrow{\alpha} & V \times_X \tilde{W} \\
p_1 & & q_1 \\
V & \xrightarrow{\beta} & W \times_X \tilde{W} \\
r_1 & & r_2 \\
V & \xrightarrow{f} & W,
\end{array}
\end{array}
\]

where the morphisms $p_i, q_i, r_i, \text{ for } i = 1, 2$ are the natural projections. Note that $\gamma_V \circ \alpha = \gamma_W \circ \beta$. The isomorphism (1.15) is obtained via the following chain of isomorphisms\footnote{Note that these arguments still apply if one replaces $\mathcal{L}_1 = \mathcal{D}_{\Lambda_X} e^t$ with an arbitrary quasi-coherent $\mathcal{D}_{\Lambda_X}$-module. On the other hand, in order to prove (1.16) we will use the fact that the Fourier transform is involutive.}, where we set $\mathcal{L}_1 = \mathcal{D}_{\Lambda_X} e^t$.

\[
(\tilde{f})^* \mathcal{F}_V \mathcal{N} = (\tilde{f})^* p_{2+}(p_1^* \mathcal{N} \otimes \gamma_V^* \mathcal{L}_1) \\
\cong r_{2+} \alpha^*(p_1^* \mathcal{N} \otimes \gamma_V^* \mathcal{L}_1) \quad \text{by (1.6)} \\
\cong r_{2+} (\alpha^* p_1^* \mathcal{N} \otimes \alpha^* \gamma_V^* \mathcal{L}_1) \quad \text{by (1.3)} \\
\cong r_{2+} (r_1^* \mathcal{N} \otimes \alpha^* \gamma_V^* \mathcal{L}_1) \quad \text{by (1.1)} \\
\cong q_{2+} (r_1^* \mathcal{N} \otimes \beta^* \gamma_W^* \mathcal{L}_1) \quad \text{by (1.1)} \\
\cong q_{2+} (r_1^* \mathcal{N} \otimes \beta^* \gamma_W^* \mathcal{L}_1) \quad \text{by (1.2)} \\
\cong q_{2+} (r_1^* \mathcal{N} \otimes \gamma_W^* \mathcal{L}_1) \quad \text{by (1.3)} \\
\cong q_{2+} (\beta^* q_1^* f_+ \mathcal{N} \otimes \gamma_W^* \mathcal{L}_1) \quad \text{by (1.6)} \\
\cong q_{2+} (\beta^* q_1^* f_+ \mathcal{N} \otimes \gamma_W^* \mathcal{L}_1) \quad \text{by (1.6)} \\
\cong q_{2+} (q_1^* f_+ \mathcal{N} \otimes \gamma_W^* \mathcal{L}_1) \quad \text{by (1.6)} \\
= \mathcal{F}_W f_+ \mathcal{N}.
\]

Applying the functor $\mathcal{F}_W$ to the isomorphism (1.15) with $\mathcal{N} = \mathcal{F}_V Q$, $Q \in \mathcal{D}_b^{qc}(\mathcal{D}_V)$, and using (1.14), we get

\[
f_+ \mathcal{F}_V Q \cong \mathcal{F}_W (\tilde{f})^* Q.
\]

The isomorphism (1.16) is obtained from the one above by interchanging the roles of $f$ and $\tilde{f}$. \qed
References

[1] Y. André and F. Baldassarri, *De Rham cohomology of differential modules on algebraic varieties*, Progress in Mathematics, 189, Birkhäuser, 2001.

[2] A. Adolphson and S. Sperber, *Dwork cohomology, de Rham cohomology, and hypergeometric functions*, Amer. J. Math. 122 (2000), no. 2, 319–348.

[3] J. Bernstein, *Lectures on algebraic D-modules at Berkeley*, 2001 (unpublished).

[4] A. Borel, *Algebraic D-modules*, Perspectives in Mathematics, 2. Academic Press, 1987.

[5] A. Dimca, F. Maaref, C. Sabbah, M. Saito, *Dwork cohomology and algebraic D-modules*, Math. Ann. 318 (2000), no. 1, 107–125.

[6] M. Kashiwara, *Algebraic study of systems of partial differential equations*, Mémoire Soc. Math. France (N.S.) (1995), no. 63, xiv+72, Kashiwara’s Master’s Thesis, Tokyo University 1970, translated from the Japanese by A. D’Agnolo and J.-P. Schneiders.

[7] ———, *D-modules and microlocal calculus* (translated from the 2000 Japanese original by M. Saito), Transl. of Math. Monographs 217, A.M.S. (2003).

[8] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften, 292, Springer, 1990.

[9] N. M. Katz, *On the differential equations satisfied by period matrices*, Inst. Hautes Études Sci. Publ. Math. 35 (1968) 223–258.

[10] N. M. Katz and G. Laumon, *Transformation de Fourier et majoration de sommes exponentielles*, Inst. Hautes Études Sci. Publ. Math. 62 (1985), 361–418.

[11] R. Kiehl and R. Weissauer, *Weil conjectures, perverse sheaves and l-adic Fourier transform*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 42. Springer, 2001.

[12] G. Laumon, *Transformation de Fourier, constantes d’équations fonctionnelles et conjecture de Weil*, Inst. Hautes Études Sci. Publ. Math. 65 (1987), 131–210.

[13] P. Monsky, *Finiteness of de Rham cohomology*, Amer. J. Math. 94 (1972), 237–245.

Dipartimento di Matematica Pura ed Applicata; Università di Padova; via G. Belzoni, 7; 35131 Padova; Italy

Email: baldassa@math.unipd.it, dagnolo@math.unipd.it