UNIQUENESS OF ORDERS AND PARAMETERS IN MULTI-TERM TIME-FRACTIONAL DIFFUSION EQUATIONS BY INEXACT DATA

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ABSTRACT

As the most significant difference from parabolic equations, the asymptotic behavior of solutions to time-fractional evolution equations is dominated by the fractional orders, whose unique determination has been frequently investigated in literature. Unlike all the existing results, in this article we prove the uniqueness of orders and parameters (up to a multiplier for the latter) only by the inexact data satisfying certain conditions near \( t = 0 \) at a single point. Moreover, we discover special conditions on unknown initial values or source terms for the coincidence of observation data. As a byproduct, we can even conclude the uniqueness for initial values or source terms by the same data. The proof relies on the asymptotic expansion after taking the Laplace transform and the completeness of generalized eigenfunctions.

Keywords Multi-term time-fractional diffusion equation · Parameter inverse problem · Uniqueness · Inexact data

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1 Introduction

Let \( T \in \mathbb{R}_+ := (0, \infty) \) and \( \Omega \subset \mathbb{R}^d \) \((d \in \mathbb{N} = \{1, 2, \ldots\})\) be a bounded domain with a smooth boundary \( \partial \Omega \). For a constant \( m \in \mathbb{N} \), let \( \alpha_j, q_j \) \((j = 1, \ldots, m)\) be positive constants such that \( 1 > \alpha_1 > \alpha_2 > \cdots > \alpha_m > 0 \). We define an elliptic operator \( L : \mathcal{D}(L) := H^2(\Omega) \cap H^1_0(\Omega) \longrightarrow L^2(\Omega) \) by

\[
Lh(x) := -\text{div}(a(x)\nabla h(x)) + b(x) \cdot \nabla h(x) + c(x) h(x), \quad x \in \Omega,
\]
where \( \cdot \) and \( \nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}) \) refer to the inner product in \( \mathbb{R}^d \) and the gradient in \( x \), respectively. We emphasize that \( L \) is not necessarily symmetric. Here

\[
\alpha = (a_{ij})_{1 \leq i,j \leq d} \in C^1(\overline{\Omega}; \mathbb{R}^{d \times d}), \quad b = (b_1, \ldots, b_d) \in C^1(\overline{\Omega}; \mathbb{R}^d), \quad c \in C(\overline{\Omega})
\]

are real-valued \( x \)-dependent matrix-, vector- and scalar-valued functions, respectively. In particular, we assume that \( \alpha \) is symmetric and strictly positive-definite on \( \Omega \), i.e., there exists a constant \( \delta > 0 \) such that

\[
a_{ij}(x) = a_{ji}(x), \quad \alpha(x) \cdot \xi \geq \delta |\xi|^2, \quad \forall i,j = 1, \ldots, d, \quad \forall x \in \overline{\Omega}, \quad \forall \xi \in \mathbb{R}^d,
\]

where \( |\xi|^2 = \xi \cdot \xi \).

In this article, we are concerned with the following initial-boundary value problem for a multi-term time-fractional diffusion equation

\[
\begin{cases}
\sum_{j=1}^{m} q_j \partial_t^{\alpha_j} u(x,t) + L u(x,t) = \rho(t) f(x), & (x,t) \in \Omega \times (0,T), \\
u(x,0) = \alpha(x), & x \in \Omega, \\
u(x,t) = 0, & (x,t) \in \partial \Omega \times (0,T).
\end{cases}
\]

Here \( \partial_t^{\alpha_j} \) denotes the \( \alpha_j \)-th order Caputo derivative in time, which is traditionally defined by (e.g. Podlubny [27])

\[
\partial_t^{\alpha_j} h(t) := \frac{1}{\Gamma(1-\alpha_j)} \int_0^t \frac{h'(s)}{(t-s)^{\alpha_j}} ds, \quad h \in C^1[0, \infty).
\]

Here \( \Gamma(\cdot) \) stands for the Gamma function. Throughout this article, we basically assume \( \rho \in L^1(0,T) \) and for convenience, we understand \( \rho \in L^1(\mathbb{R}_+ \times \Omega) \) by identifying it with its zero extension in \( (T, \infty) \).

The governing equation in (1.3) is called a single-term time-fractional diffusion equation if \( m = 1 \), while is called a multi-term one if \( m \geq 2 \). In the last decades, they have gathered increasing population as typical nonlocal models for anomalous diffusion in heterogeneous media or fractals (e.g. [4, 8]). Especially, linear theories for the case of \( m = 1 \) have been gradually established in recent years, represented by [7, 15, 28]. Compared with classical parabolic equations, single-term time-fractional diffusion equations inherit the same time-analyticity and maximum principle (e.g. [26, 28]). Meanwhile, they also share quantitative similarities in the smoothing effect and the unique continuation property (e.g. [10, 28]). In contrast, they show essential difference from their integer counterpart in the short- and long-time asymptotic behaviors taking the form of \( t^{-\alpha} \) (e.g. [23]).

Though less studied, it reveals that the case of \( m \geq 2 \) in (1.3) also behaves similarly to that of \( m = 1 \). We refer to [19] for the well-posedness and the limited smoothing effect of solutions, and [24, 25] for weak and strong maximum principles, respectively. In [4, 16], the parameters \( q_j \) in (1.3) were generalized to be \( x \)-dependent at the cost of slightly weaker solution regularity. Remarkably, the short- and long-time asymptotic behaviors of solutions depend on the highest order \( \alpha_1 \) and the lowest one \( \alpha_m \), respectively (e.g. [16, 19]).

Since the orders of fractional derivatives are closely related to the heterogeneity and fractal dimensions of media, the determination of \( \alpha_j \) in (1.3) turns out to be significant in both theory and practice. For the single-term case, Cheng et al. [6] first proved the uniqueness of \( \alpha_1 \) by the data \( u|_{[x_0] \times (0,T)} \) with \( x_0 \in \partial \Omega \). Employing the above mentioned asymptotic behaviors, Hatano et al. [9] deduced inversion formulæ

\[
\alpha_1 = - \lim_{t \rightarrow \infty} \frac{t \partial_t u(x_0, t)}{u(x_0, t)} = \lim_{t \rightarrow 0} \frac{t \partial_t u(x_0, t)}{u(x_0, t) - a(x_0)},
\]

with \( x_0 \in \Omega \). For the multi-term case, Li and Yamamoto [21] showed the unique determination of \( m, \alpha_j, q_j \) with constants \( q_j > 0 \) by \( u|_{[x_0] \times (0,T)} \) with \( x_0 \in \partial \Omega \). For \( x \)-dependent \( q_j \geq 0 \), Li et al. [18] proved the unique determination of \( m, \alpha_j, q_j \) and the zeroth-order coefficient \( c \) using the Dirichlet-to-Neumann map, and Li et al. [16] uniquely identified \( \alpha_j \) by \( u|_{[x_0] \times (0,T)} \) with \( x_0 \in \Omega \). We refrain from providing a complete list, and refer to Li et al. [20] as a survey covering most literature on parameter inverse problems for (1.3) before 2019. We notice that due to the dimension of observation data, most literature dealt with the simultaneous identification of orders and unknown sources or coefficients in (1.3) (see [14, 22, 23, 30, 54, 55]). Only [17] obtained the stability of determining a single order by the data.
Throughout this article, by the notation $h(t) \sim t^\mu$ for a.e. $t \ll 1$ we mean that there exists a constant $C' \neq 0$ such that $\lim_{t \to 0^+} t^{-\mu} h(t) = C'$. In this case, we know $h(t) = O(t^{\mu'})$ but $h(t) \neq o(t^{\mu'})$ for any $\mu' > \mu$ a.e. as $t \to 0^+$. We define $\sim$ for a.e. $t \gg 1$ in the same manner.

In the sequel, the inner product of $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$. By $\sigma(L) = \{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$ we denote the set of all distinct eigenvalues of $L$. Throughout this article, for our arguments we assume

$$0 \notin \sigma(L).$$

(2.1)
For the sake of self-containedness, we collect necessary ingredients concerning the eigenvalue theory of \( L \), which can be found e.g. in Kato \cite{Kato}. For each \( n \in \mathbb{N} \), we take a circle \( \gamma_n \) centered at \( \lambda_n \) with sufficiently small radius such that \( \gamma_n \) does not enclose \( \lambda_{n'} \) with any \( n' \neq n \). Defining an operator

\[
P_n := -\frac{1}{2\pi \sqrt{-1}} \int_{\gamma_n} (L - z)^{-1} \, dz,
\]

we know that

\[
P_n P_{n'} = \begin{cases} P_n, & n' = n, \\ 0, & n' \neq n. \end{cases}
\]

We call \( P_n \) the eigenprojection of \( \lambda_n \) and any \( \varphi \in P_n L^2(\Omega) \) a generalized eigenfunction of \( \lambda_n \). Meanwhile, the space \( P_n L^2(\Omega) \) is finite dimensional and we set \( d_n := \dim P_n L^2(\Omega) < \infty \). We note that if \( L \) is symmetric, then there holds for all \( n \in \mathbb{N} \) that

\[
\lambda_n \in \mathbb{R}, \quad P_n L^2(\Omega) = \ker(L - \lambda_n), \quad d_n = \dim \ker(L - \lambda_n).
\]

Next, we introduce the eigennilpotent of \( \lambda_n \) as

\[
D_n := (L - \lambda_n)^{d_n} P_n.
\]

Then we can verify that \( P_n \varphi = \varphi \) for \( \varphi \in P_n L^2(\Omega) \) and

\[
P_n L^2(\Omega) \subset D(L), \quad LP_n L^2(\Omega) \subset P_n L^2(\Omega), \quad D_n P_n L^2(\Omega) \subset P_n L^2(\Omega), \quad D_n^{d_n} = (L - \lambda_n)^{d_n} P_n = 0.
\]

Using \( P_n \) and \( D_n \), we have the following Laurent expansion for the resolvent:

\[
(L - z)^{-1} P_n = \frac{P_n}{\lambda_n - z} + \sum_{k=1}^{d_n-1} \frac{(-1)^k D_n^k}{(\lambda_n - z)^{k+1}}, \quad z \notin \sigma(L).
\]

On the other hand, we recall the following resolvent estimate (see e.g. Tanabe \cite{Tanabe}): there exist constants \( z_0 > 0 \) and \( C > 0 \) such that

\[
\| (L + z)^{-1} h \|_{L^2(\Omega)} \leq \frac{C}{|z|} \| h \|_{L^2(\Omega)}, \quad \forall \, z > z_0, \quad \forall \, h \in L^2(\Omega).
\]

Finally, we introduce the fractional power \( L^\gamma \) of \( L \) along with its domain \( D(L^\gamma) \) for \( \gamma > 0 \). First for \( 0 < \gamma < 1 \), the fractional operator \( L^{\gamma} \) is defined by

\[
L^{\gamma} h := \frac{\sin \pi \gamma}{\pi} \int_{\mathbb{R}^+} z^{-\gamma}(L + z)^{-1} h \, dz, \quad h \in L^2(\Omega).
\]

Then there holds \( L^{-\gamma} P_n L^2(\Omega) \subset P_n L^2(\Omega) \) and hence \( L^{\gamma} P_n L^2(\Omega) \subset P_n L^2(\Omega) \) for all \( \gamma \geq 0 \). Then for \( 0 < \gamma < 1 \), we define \( L^{\gamma} \) as the inverse of \( L^{-\gamma} \). For \( \gamma \in \mathbb{R} \setminus \mathbb{N} \), we define \( L^\gamma = L^{-\gamma} \circ L^{\gamma} \), where \( \lfloor \cdot \rfloor := \max \{ n \in \mathbb{Z} \mid n \leq \gamma \} \) stands for the flooring function and \( \circ \) denotes the composite. Then we set \( D(L^\gamma) := \{ h \in L^2(\Omega) \mid L^\gamma h \in L^2(\Omega) \} \) for \( \gamma > 0 \).

Now we recall the well-posedness results concerning the forward problems \cite{Kato}.

**Lemma 2.1.** Assume \( \mathbf{(1.1)} - \mathbf{(1.2)} \) and \( \mathbf{(2.1)} \). Let \( u \) satisfy \( \mathbf{(1.3)} \) with \( f \equiv 0 \) in \( \Omega \) and \( a \in D(L^\gamma) \) with \( \gamma \geq 0 \). Then for any fixed constant \( \eta \in (0, 1) \), the problem \( \mathbf{(1.3)} \) admits a unique solution \( u \in C([0, \infty); D(L^\gamma)) \cap C(\mathbb{R}_+; D(L^{\gamma + \eta})) \). Moreover, there exists a constant \( C > 0 \) depending on \( \alpha_j, q_j \) \( (j = 1, \ldots, m) \), \( \Omega \), \( L \), \( \gamma \) such that

\[
\| u(\cdot, t) \|_{D(L^{\gamma + \eta})} \leq C t^{-\alpha_1 \eta \epsilon_1} C_{\gamma} \| a \|_{D(L^\gamma)}, \quad \forall \, t > 0.
\]

The above lemma asserts that the solution norm \( \| u(\cdot, t) \|_{D(L^{\gamma + \eta})} \) increases at most exponentially as \( t \to \infty \), which guarantees the existence of its Laplace transform \( \tilde{u}(\cdot, p) \) for \( p > 1 \). Lemma 2.1 slightly improves Li, Huang and Yamamoto \cite{Li} [Theorem 2.3] in the sense of more general choice of \( a \) and the key estimate \( \mathbf{(2.5)} \). Especially, in \cite{Li} [Theorem 2.3] the estimate \( \mathbf{(2.5)} \) only held for \( t \in (0, T) \) with fixed \( T > 0 \). However, a close scrutiny into the proof reveals that \( T \) can be chosen arbitrarily. On the other hand, it was proved in \cite{Li} that if \( b = 0 \) and \( c \geq 0 \) on \( \Omega \), then the solution \( u(\cdot, t) \) obeys the decay rate \( t^{-\alpha} \) as \( t \to \infty \). Nevertheless, instead of the sharp decay estimate, here \( \mathbf{(2.5)} \) is enough for applying the Laplace transform, which plays an essential role in the proof (see Section 4).
3 Main Results and Examples

Now we are well prepared to state the main results of this article. To this end, we introduce another initial-boundary value problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
\sum_{j=1}^{m'} r_j \partial_j^\beta_j + L \\
\end{array} \right\} v(x, t) = \rho(t)g(x), \quad (x, t) \in \Omega \times (0, T), \\
v(x, 0) = b(x), \quad x \in \Omega, \\
v(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T),
\end{aligned}
\]

where \( m' \in \mathbb{N} \) is a constant and \( \beta_j, r_j \ (j = 1, \ldots, m') \) are positive constants satisfying \( 1 > \beta_1 > \beta_2 > \cdots > \beta_{m'} > 0 \). We notice that (1.3) and (3.1) share the same elliptic part \( L \) and the temporal component \( \rho \) of the source term. Nevertheless, their initial values \( a, b \) and spatial components \( f, g \) of source terms differ from each other, which are basically assumed also to be unknown.

Let us state our first main result on the unique determination of orders and parameters with inexact data for homogeneous problems, i.e., \( f = g \equiv 0 \) in (1.3) and (3.1).

**Theorem 3.1.** Assume (1.1), (1.2) and (2.1). Let \( u, v \) satisfy (1.3) and (3.1) respectively with \( f = g \equiv 0 \) in \( \Omega \), where \( a, b \in \mathcal{D}(L^\gamma) \) with \( \gamma > 1 + d/4 \). Fix \( \tau > 0 \) (\( \tau \ll 1 \)), pick \( x_0 \in \Omega \) such that \( La(x_0) \neq 0, Lb(x_0) \neq 0 \) and denote \( \kappa := La(x_0)/Lb(x_0) \).

(i) If there exist constants \( C > 0 \) and \( \nu > \min\{\alpha_1, \beta_1\} \) such that

\[
|u(x_0, t) - v(x_0, t)| \leq C t^\nu, \quad 0 \leq t \leq \tau,
\]

then

\[
\alpha_1 = \beta_1, \quad \frac{\kappa_1}{r_1} = \kappa.
\]

(ii) If further \( a, b \in \mathcal{D}(L^{1+\gamma}) \) and (3.2) is satisfied with \( \nu > 2 \min\{\alpha_1, \beta_1\} \), then (3.2) implies

\[
m = m', \quad \alpha_j = \beta_j, \quad \frac{\kappa_j}{r_j} = \kappa, \quad j = 1, \ldots, m.
\]

**Remark 3.2.** (1) According to the Sobolev embedding theorem (see Adams [1]), we see that \( a, b \in \mathcal{D}(L^\gamma) \) with \( \gamma > 1 + d/4 \) implies \( a, b \in C^2(\overline{\Omega}) \) and thus \( La(x_0), Lb(x_0) \) are well-defined. On the other hand, taking \( t = 0 \) in (3.2), we find the hidden assumption on the initial value

\[
a(x_0) = b(x_0).
\]

(2) Unlike most results on the uniqueness for inverse problems, we only need the inexact data satisfying the short-time asymptotic behavior (3.2) instead of their coincidence. Moreover, we even do not need \( a = b \) as was assumed in most literature. Therefore, we minimize the assumptions on initial values to \( La(x_0) \neq 0, Lb(x_0) \neq 0 \) and (3.5).

(3) The choices of the order \( \nu \) in (3.2) in Theorem 3.1(i)–(ii) turn out to be the minimum necessary condition used in the proof. If there is an a priori upper bound \( \overline{\nu} \) of \( \min\{\alpha_1, \beta_1\} \), then safer requirements of \( \nu \) would be \( \nu > \overline{\nu} \) in (i) and \( \nu > 2\overline{\nu} \) in (ii). If there is no a priori information on \( \alpha_1, \beta_1 \), then we should require \( \nu \geq 1 \) in (i) and \( \nu \geq 2 \) in (ii).

(4) Indeed, by the arguments in [19], one can prove

\[
|u(x_0, t) - a(x_0)| = O(t^{\alpha_1}), \quad |v(x_0, t) - b(x_0)| = O(t^{\beta_1}) \quad \text{as } t \to 0,
\]

which, together with (3.5), implies

\[
|u(x_0, t) - v(x_0, t)| \leq |u(x_0, t) - a(x_0)| + |v(x_0, t) - b(x_0)| = O(t^{\min\{\alpha_1, \beta_1\}}) \quad \text{as } t \to 0.
\]

However, (3.2) does not follow automatically from (3.6).
Since the key assumption (3.2) is satisfied if
\[ \exists \tau > 0 \text{ such that } u(x_0, t) = v(x_0, t), \quad 0 \leq t \leq \tau, \] (3.7)

Theorem 3.1 definitely covers the uniqueness with usual exact data. On the contrary, by an argument of the reverse proposition, we can immediately obtain the following distinguishability property: if there exists a constant \( \nu < \min\{\alpha_1, \beta_1\} \) such that
\[ |u(x_0, t) - v(x_0, t)| \geq C t^\nu \quad \text{near } t = 0, \]
then there should be \( \alpha_1 \neq \beta_1 \).

In general, one can at most identify the parameters \( q_j \) up to a multiplier \( \kappa \), but cannot further assert \( q_j = r_j \) (\( j = 1, \ldots, m \)) from (3.2). This is demonstrated by the following counterexample.

**Example 3.3.** In (1.3) and (3.1), let us simply take
\[ \Omega = (0, \pi), \quad f = g \equiv 0 \text{ in } \Omega, \quad m = m' = 1, \quad \alpha_1 = \beta_1, \quad q_1 = 4, \quad r_1 = 1, \quad L = -\partial_x^2. \]

We pick any \( x_0 \in (0, \pi) \setminus \{\pi/2\} \) and select
\[ a(x) = \frac{\sin 2x}{2 \cos x_0}, \quad b(x) = \sin x. \]

Then it is readily seen that
\[ a(x_0) = b(x_0) = \sin x_0, \quad q_1 \frac{r_1}{r_1} = 4 = \frac{\alpha_{1\nu}(x_0)}{b_{1\nu}(x_0)}. \]

However, employing the Mittag-Leffler function
\[ E_{\alpha_1,1}(z) := \sum_{\ell=0}^\infty \frac{z^\ell}{\Gamma(\alpha_1 \ell + 1)}, \]
we can easily obtain the explicit solutions
\[ u(x, t) = E_{\alpha_1,1}(-t^{\alpha_1}) \frac{\sin 2x}{2 \cos x_0}, \quad v(x, t) = E_{\alpha_1,1}(-t^{\alpha_1}) \sin x \]
and hence \( u(x_0, t) = E_{\alpha_1,1}(-t^{\alpha_1}) \sin x_0 = v(x_0, t) \). This means that even the exact data for all \( t \geq 0 \) fails to guarantee \( q_1 = r_1 \).

Similarly to Theorem 3.1, we can obtain the same uniqueness with inexact data (3.2) for inhomogeneous problems, i.e., \( a = b = 0 \) in (1.3) and (3.1).

**Theorem 3.4.** Assume (1.1), (1.2) and (3.1). Let \( u, v \) satisfy (1.3) and (3.1) respectively with \( a = b \equiv 0 \in \Omega \), where \( f, g \in \mathcal{D}(L^\gamma) \) with \( \gamma > d/4 \) and there exists a constant \( \mu > -1 \) such that
\[ \rho(t) \sim t^\mu \quad \text{for a.e. } t \ll 1. \]

Fix \( \tau > 0 (\tau \ll 1) \), pick \( x_0 \in \Omega \) such that \( f(x_0) \neq 0, g(x_0) \neq 0 \) and denote \( \kappa := f(x_0)/g(x_0) \).

(i) If there exist constants \( C > 0 \) and \( \nu > \min\{\alpha_1, \beta_1\} + \mu \) such that (3.2) is satisfied, then (3.3) holds.

(ii) If further \( f, g \in \mathcal{D}(L^{1+\gamma}) \) and (3.2) is satisfied with \( \nu > 2 \min\{\alpha_1, \beta_1\} + \mu \), then (3.2) implies (3.3).

The assumption (3.8) describes the short-time asymptotic behavior of the temporal component \( \rho \) of the source terms in (1.3) and (3.1), which turns out to be essential in proving Theorem 3.4. Indeed, since the inexact data (3.2) is only given near \( t = 0 \), the general assumption \( \rho \in L^1(\mathbb{R}_+) \) is too coarse to depict the local information. On the other hand, the solution behavior near \( t = 0 \) is also influenced by that of \( \rho \) in view of forward problems. Therefore, compared with Theorem 3.1, the difference \( \nu - \mu \) of orders instead of \( \nu \) itself appears in the conditions guaranteeing the uniqueness in Theorem 3.4. We notice that (3.8) indicates \( \rho \neq 0 \) near \( t = 0 \) and is not restrictive because it covers a wide choice of \( \rho \) behaving asymptotically like a power function for \( t \ll 1 \). However, it does exclude e.g.
\[ \rho(t) = \begin{cases} 0, & t = 0, \\ \exp(-1/t), & 0 < t < T, \end{cases} \]
which satisfies $\rho^{(i)}(0) = 0$ for all $i = 0, 1, \ldots$. With such functions, it is usually rather difficult to obtain uniqueness or stability for inverse problems. On the contrary, the ill-posedness is reduced if there exists $i = 0, 1, \ldots$ such that $\rho^{(i)}(0) \neq 0$, which falls within the framework of (3.8).

Similarly to Example 3.3 one can easily construct counterexample against $q_j = r_j$ ($j = 1, \ldots, m$) in the framework of Theorem 3.4. Owing to multiple unknown parameters, even usual exact data (3.7) do not yield the uniqueness for our inverse problem. Finally, we discuss the uniqueness and the non-uniqueness by (3.7).

**Theorem 3.5.** Assume that the elliptic operator $L$ is symmetric, i.e., $b \equiv 0$ on $\Gamma$. Let

$$\ker(L - \lambda_n) = \text{span}\{\varphi_{n,k} \mid k = 1, \ldots, d_n\},$$

where $\varphi_{n,k}$ ($k = 1, \ldots, d_n$) are the orthonormal eigenfunctions associated with $\lambda_n$. Define a countable set $\Sigma := \{\lambda_n/\lambda_n' \mid n, n' \in \mathbb{N}\} \subset \mathbb{R}$.

(1) Let $u, v$ satisfy (1.3) and (3.1) respectively with $f = g \equiv 0$ in $\Omega$. Under the same assumptions in Theorem 3.1 (3.7) holds if and only if

$$m = m', \quad \alpha_j = \beta_j, \quad \frac{q_j}{r_j} = \frac{La(x_0)}{Lb(x_0)} =: \kappa \in \Sigma, \quad j = 1, \ldots, m \tag{3.9}$$

and there exist nonempty sets $M_\kappa, M'_\kappa \subset \mathbb{N}$ satisfying

$$\{\lambda_n \in \sigma(L) \mid n \in M_\kappa\} = \{\kappa \lambda_n \mid \lambda_n \in \sigma(L), n \in M'_\kappa\} \tag{3.10}$$

and a bijection $\theta_\kappa : M_\kappa \rightarrow M'_\kappa$ such that

$$\begin{cases}
\begin{align*}
P_n a(x_0) &= P_{\theta_\kappa(n)} b(x_0), \\
P_n La(x_0) &= \kappa P_{\theta_\kappa(n)} Lb(x_0), &n \in M_\kappa, \\
P_n a(x_0) &= P_n La(x_0) = 0, &n \notin M_\kappa, \\
P_n b(x_0) &= P_n Lb(x_0) = 0, &n \notin M'_\kappa.
\end{align*}
\end{cases} \tag{3.11}$$

(2) Let $u, v$ satisfy (1.3) and (3.1) respectively with $a = b \equiv 0$ in $\Omega$. Under the same assumptions in Theorem 3.4 (3.7) holds if and only if

$$m = m', \quad \alpha_j = \beta_j, \quad \frac{q_j}{r_j} = \frac{f(x_0)}{g(x_0)} =: \kappa \in \Sigma, \quad j = 1, \ldots, m \tag{3.12}$$

and there exist the same sets $M_\kappa, M'_\kappa \subset \mathbb{N}$ and bijection $\theta_\kappa : M_\kappa \rightarrow M'_\kappa$ as that in (1) such that

$$\begin{cases}
\begin{align*}
P_n a(x_0) &= \kappa P_{\theta_\kappa(n)} g(x_0), &n \in M_\kappa, \\
P_n a(x_0) &= 0, &n \notin M_\kappa, \\
P_n g(x_0) &= 0, &n \notin M'_\kappa.
\end{align*}
\end{cases} \tag{3.13}$$

**Remark 3.6.** Theorem 3.5 characterizes the coincidence (3.2) of observation data via initial values $a, b$ or source terms $f, g$. From the definition of $\Sigma$, it is obvious that $1 \in \Sigma$. We discuss the cases of $\kappa = 1, \kappa \notin \Sigma$ and $\kappa \in \Sigma \setminus \{1\}$ separately.

(1) If $\kappa = 1$, i.e., $La(x_0) = Lb(x_0)$ or $f(x_0) = g(x_0)$, then $M_1 = M'_1 = \mathbb{N}$ and $\theta_1$ is an identity operator. In this case, it is readily seen that (3.11) and (3.13) are equivalent to

$$\begin{align*}
P_n a(x_0) &= P_n b(x_0), \\
P_n La(x_0) &= P_n Lb(x_0), &\forall n \in \mathbb{N}, \\
P_n f(x_0) &= P_n g(x_0), &\forall n \in \mathbb{N}, \tag{3.14}
\end{align*}$$

respectively.

(2) If $\kappa \notin \Sigma$, then Theorem 3.5 asserts that the observation data $u(x_0, t)$ and $v(x_0, t)$ ($0 \leq t \leq \tau$) cannot coincide.

(3) If $\kappa \in \Sigma \setminus \{1\}$, then it is indicated in (3.11) and (3.13) that the initial values $a, b$ or the source terms $f, g$ should satisfy a rather special relation. To illustrate this point, we provide the following example.
Example 3.7. Similarly to Example 3.3 in (1.3) and (3.1) we take
\[
\Omega = (0, \pi), \quad f = g \equiv 0 \text{ in } \Omega, \quad \kappa = \frac{La(x_0)}{Lb(x_0)} = 4, \quad L = -\partial_x^2.
\]
Then it is readily seen that
\[
\lambda_n = n^2, \quad \Sigma = \{(n/n')^2 \mid n, n' \in \mathbb{N}\} = \mathbb{Q}^2 \setminus \{0\}, \quad P_n f = (f, \varphi_n)\varphi_n, \quad f \in L^2(0, \pi),
\]
where \(\varphi(x) := \sqrt{2/\pi} \sin nx \quad (n \in \mathbb{N})\). Further, by the definitions of \(M_\kappa, M'_\kappa\) and \(\theta_\kappa\), it is straightforward to verify that \(M_4 = 2\mathbb{N}, M'_4 = \mathbb{N}\) and \(\theta_4(n) = n/2\). Therefore, the relation (3.11) can be rephrased as
\[
(a, \varphi_{2n})\varphi_{2n}(x_0) = (b, \varphi_n)\varphi(x_0), \quad (a, \varphi_{2n-1})\varphi_{2n-1}(x_0) = 0, \quad n \in \mathbb{N}.
\]
Especially, if \(x_0 \neq \pi\), i.e., \(x_0\) is not a zero of \(\varphi_n \quad (\forall n \in \mathbb{N})\), then \(a\) should take the special form of
\[
a(x) = \sum_{n=1}^{\infty} \frac{(b, \varphi_n)}{\varphi_{2n}(x_0)}\varphi_{2n}(x_0) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{(b, \varphi_n)}{\cos nx_0} \sin 2nx,
\]
which is a generalization of Example 3.3.

On the same direction of Theorem 3.5 and Example 3.7, it turns out that in a further special case, we can even obtain the uniqueness of \(a, b\) or \(f, g\) up to a bijection.

Corollary 3.8. Under the same assumptions in Theorem 3.5 further assume that all eigenvalues of \(L\) are simple and
\[
x_0 \not\in \omega := \bigcup_{n=1}^{\infty} \{x \in \Omega \mid \varphi_n(x) = 0\} \subset \Omega,
\]
where \(\varphi_n\) is the unique eigenfunction associated with \(\lambda_n\). Let \(\Sigma, \Omega, M_\kappa, M'_\kappa, \theta_\kappa\) be the same as that in Theorem 3.5.

(1) Let \(u, v\) satisfy (1.3) and (3.1) respectively with \(f = g \equiv 0\) in \(\Omega\). Then (3.7) holds if and only if (3.9) holds and
\[
a = \sum_{n \in M_\kappa} (a, \varphi_n)\varphi_n, \quad b = \sum_{n \in M'_\kappa} (b, \varphi_n)\varphi_n \quad \text{with} \quad (a, \varphi_n)\varphi_n(x_0) = (b, \varphi_{\theta_\kappa(n)})\varphi_{\theta_\kappa(n)}(x_0), \quad n \in M_\kappa.
\]
Especially if \(\kappa = 1\), i.e. \(La(x_0) = Lb(x_0)\), then (3.16) is equivalent to \(a \equiv b\) in \(\Omega\).

(2) Let \(u, v\) satisfy (1.3) and (3.1) respectively with \(a = b \equiv 0\) in \(\Omega\). Then (3.7) holds if and only if (3.12) holds and
\[
f = \sum_{n \in M_\kappa} (f, \varphi_n)\varphi_n, \quad g = \sum_{n \in M'_\kappa} (g, \varphi_n)\varphi_n \quad \text{with} \quad (f, \varphi_n)\varphi_n(x_0) = \kappa(g, \varphi_{\theta_\kappa(n)})\varphi_{\theta_\kappa(n)}(x_0), \quad n \in M_\kappa.
\]
Especially if \(\kappa = 1\), i.e. \(f(x_0) = g(x_0)\), then (3.17) is equivalent to \(f \equiv g\) in \(\Omega\).

The assumption (3.15) is a special case of the well-known rank condition, which is commonly used for observability of PDEs (e.g. [29]). In some special cases such as that in Example 3.7 the Lebesgue measure of \(\omega\) defined in (3.15) is zero. Therefore, the uniqueness claimed in Corollary 3.8 is generic, which is rather surprising because the single point observation can even almost determine the initial value or the spatial component of the source term uniquely.

4 Proofs of Main Results

For the proof of Theorem 3.1 we need the completeness of all the generalized eigenfunctions of \(L\).

Lemma 4.1. (i) For any \(h \in L^2(\Omega)\), there exists a sequence \(\{h_N\}_{N \in \mathbb{N}} \subset L^2(\Omega)\) such that
\[
h_N \in \sum_{n=1}^{N} P_n L^2(\Omega), \quad \lim_{N \to \infty} \|h - h_N\|_{L^2(\Omega)} = 0.
\]
(ii) For any $h \in \mathcal{D}(L^\gamma)$ with $\gamma \geq 0$, there exists a sequence $\{h_N\}_{N \in \mathbb{N}} \subset \mathcal{D}(L^\gamma)$ such that

$$h_N \in \sum_{n=1}^{N} P_n L^2(\Omega), \quad \lim_{N \to \infty} \| L^\gamma (h - h_N) \|_{L^2(\Omega)} = 0. $$

Proof. (i) It follows immediately from the density of the linear subspace spanned by all generalized eigenfunctions of $L$ in $L^2(\Omega)$ (see Agmon [2]).

(ii) Since $h \in \mathcal{D}(L^\gamma)$ implies $L^\gamma h \in L^2(\Omega)$, we can apply (i) to choose a sequence $\{\psi_N\}_{N \in \mathbb{N}} \subset L^2(\Omega)$ such that

$$\psi_N \in \sum_{n=1}^{N} P_n L^2(\Omega), \quad \lim_{N \to \infty} \| \psi_N - L^\gamma h \|_{L^2(\Omega)} = 0. $$

Owing to (2.1), we see that $L^{-\gamma} : L^2(\Omega) \to L^2(\Omega)$ exists and is bounded, indicating

$$\lim_{N \to \infty} \| L^\gamma (L^{-\gamma} \psi_N) - L^\gamma h \|_{L^2(\Omega)} = 0. $$

By (4.8), we see that $h_N := L^{-\gamma} \psi_N \in \sum_{n=1}^{N} P_n L^2(\Omega) \subset \mathcal{D}(L^\gamma)$ and thus

$$\lim_{N \to \infty} \| L^\gamma h_N - L^\gamma h \|_{L^2(\Omega)} = 0.$$  

The proof of Lemma 4.1 is complete.

Proof of Theorem 3.1 Let $f = g \equiv 0$ in $\Omega$ in (1.3) and (3.1). We divide the proof into 4 steps.

**Step 1** To begin with, we make some overall preparations. Since the assumption $\gamma - 1 > d/4$ implies the Sobolev embedding $\mathcal{D}(L^{\gamma-1}) \subset C(\Omega)$, it follows from Lemma 2.1 that $u(x_0, t)$ is well-defined for $t > 0$ and fulfills the estimate

$$|u(x_0, t)| \leq \| u(\cdot, t) \|_{C(\Omega)} \leq C \| u(\cdot, t) \|_{\mathcal{D}(L^{\gamma-1})} \leq C e^{Ct} \| a \|_{\mathcal{D}(L^{\gamma-1})}, \quad t > 0.$$  

The similar estimate also holds for $v(x_0, t)$, which, together with the key assumption [3.2], indicates

$$|u(x_0, t) - v(x_0, t)| \leq \begin{cases} C t^{\nu}, & 0 \leq t \leq \tau, \\ C e^{Ct}, & t > \tau. \end{cases}$$

Then for $p \gg 1$, we estimate the Laplace transform of $u(x_0, t) - v(x_0, t)$ from above as

$$\hat{u}(x_0, p) - \hat{v}(x_0, p) \leq \int_{\mathbb{R}^n} e^{-pt} |u(x_0, t) - v(x_0, t)| \, dt \leq C \int_{0}^{\tau} e^{-pt^{\nu}} \, dt + C \int_{\tau}^{\infty} e^{-(p-C)t} \, dt$$

$$\leq C \int_{\mathbb{R}^n} e^{-pt^{\nu}} \, dt + \frac{C e^{-(p-C)\tau}}{p-C} = \frac{C \Gamma(\nu + 1)}{p^{\nu+1}} + o(p^{-\nu-1}) \leq C p^{-\nu-1}.$$  

We can similarly obtain the lower estimate $\hat{u}(x_0, p) - \hat{v}(x_0, p) \geq -C p^{-\nu-1}$ for $p \gg 1$ and hence

$$\hat{u}(x_0, p) - \hat{v}(x_0, p) = O(p^{-\nu-1}) \quad \text{for} \quad p \gg 1.$$  

(4.1)

Now we shall represent $\hat{u}(x_0, p)$ and $\hat{v}(x_0, p)$ explicitly for $p \gg 1$. Performing the Laplace transform in (1.3) and using the formula

$$\hat{\partial_t^\alpha h}(p) = p^\alpha \hat{h}(p) - p^{\alpha-1} h(0),$$

we derive the boundary value problem for an elliptic equation for $\hat{u}(\cdot, p)$:

$$\begin{cases} \left( \sum_{j=1}^{m} q_j p^{\alpha_j} + L \right) \hat{u}(\cdot, p) = a \sum_{j=1}^{m} q_j p^{\alpha_j-1} \quad \text{in} \ \Omega, \\ \hat{u}(\cdot, p) = 0 \quad \text{on} \ \partial\Omega. \end{cases}$$
According to Lemma 4.1, there exist 

\[ a_N = \sum_{n=1}^{N} a_{N,n} \quad \text{and} \quad b_N = \sum_{n=1}^{N} b_{N,n} \quad (N \in \mathbb{N}) \]  

such that 

\[ a_{N,n}, b_{N,n} \in P_n L^2(\Omega), \quad \lim_{N \to \infty} \| L^\gamma (a - a_N) \|_{L^2(\Omega)} = \lim_{N \to \infty} \| L^\gamma (b - b_N) \|_{L^2(\Omega)} = 0. \]  

Especially, since \( \gamma > 1 + d/4 \), the Sobolev embedding implies 

\[ \lim_{N \to \infty} L a_N(x_0) = La(x_0), \quad \lim_{N \to \infty} L b_N(x_0) = Lb(x_0). \]  

Then for any fixed \( N \in \mathbb{N} \), we decompose \( a = a_N + (a - a_N) \) and employ the expansion (2.3) to rewrite (4.2) as 

\[ p \hat{u}(\cdot, p) = \frac{1}{p} \sum_{j=1}^{m} q_j \alpha_j \left( L + \sum_{j=1}^{m} q_j \alpha_j \right)^{-1} \left\{ \sum_{n=1}^{N} \alpha_j \right\} \]  

\[ = \frac{1}{p} \sum_{j=1}^{m} q_j \alpha_j \sum_{k=1}^{N-1} (L + \sum_{j=1}^{m} q_j \alpha_j)^{-1} \left\{ \sum_{n=1}^{N} \alpha_j \right\} \]  

\[ = \frac{1}{p} \sum_{j=1}^{m} q_j \alpha_j \sum_{k=1}^{N-1} \frac{(-1)^k D_n^{k} a_{N,n}}{\lambda_n + \sum_{j=1}^{m} q_j \alpha_j} + (a - a_N) - \left( L + \sum_{j=1}^{m} q_j \alpha_j \right)^{-1} \left( a - a_N \right) \]  

\[ = \frac{1}{p} \sum_{j=1}^{m} q_j \alpha_j \sum_{k=1}^{N-1} \frac{(-1)^k D_n^{k} a_{N,n}}{\lambda_n + \sum_{j=1}^{m} q_j \alpha_j} + (a - a_N) - \left( L + \sum_{j=1}^{m} q_j \alpha_j \right)^{-1} \left( a - a_N \right) \]  

\[ = a - \sum_{n=1}^{N} \frac{\lambda_n a_{N,n}}{\lambda_n + \sum_{j=1}^{m} q_j \alpha_j} + \sum_{j=1}^{m} q_j \alpha_j \sum_{n=1}^{N-1} \frac{(-1)^k D_n^{k} a_{N,n}}{\lambda_n + \sum_{j=1}^{m} q_j \alpha_j} \]  

\[ - \left( L + \sum_{j=1}^{m} q_j \alpha_j \right)^{-1} \left( a - a_N \right). \]  

Rewriting (4.3) in the same manner, we obtain 

\[ p \hat{v}(\cdot, p) = \frac{1}{p} \sum_{j=1}^{m'} r_j \beta_j \left( L + \sum_{j=1}^{m'} r_j \beta_j \right)^{-1} \]  

\[ = \frac{1}{p} \sum_{j=1}^{m'} r_j \beta_j \sum_{k=1}^{N-1} \frac{(-1)^k D_n^{k} b_{N,n}}{\lambda_n + \sum_{j=1}^{m'} r_j \beta_j} + (b - b_N) - \left( L + \sum_{j=1}^{m'} r_j \beta_j \right)^{-1} \left( b - b_N \right). \]  

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Taking $x = x_0$ in (4.6)–(4.7) and substituting them into (4.1), we notice (3.5) to arrive at

$$\sum_{i=1}^{3} (I^i_N(p) - J^i_N(p)) = O(p^{-\nu}) \quad \text{for } p \gg 1,$$

where

$$I^1_N(p) := \sum_{n=1}^{N} \frac{\lambda_n a_{N,n}(x_0)}{\lambda_n + \sum_{j=1}^{m} q_j p^{\alpha_j}} \quad J^1_N(p) := \sum_{n=1}^{N} \frac{\lambda_n b_{N,n}(x_0)}{\lambda_n + \sum_{j=1}^{m} r_j p^{\beta_j}},$$

$$I^2_N(p) := \sum_{j=1}^{m} q_j p^{\alpha_j} \sum_{n=1}^{N} \sum_{k=1}^{d_a-1} \frac{(-1)^{k+1} (D^n_k a_{N,n})(x_0)}{(\lambda_n + \sum_{j=1}^{m} q_j p^{\alpha_j})^{k+1}}, \quad J^2_N(p) := \sum_{j=1}^{m} r_j p^{\beta_j} \sum_{n=1}^{N} \sum_{k=1}^{d_a-1} \frac{(-1)^{k+1} (D^n_k b_{N,n})(x_0)}{(\lambda_n + \sum_{j=1}^{m} r_j p^{\beta_j})^{k+1}},$$

$$I^3_N(p) := \left( L + \sum_{j=1}^{m} q_j p^{\alpha_j} \right)^{-1} L(a - a_N) (x_0), \quad J^3_N(p) := \left( L + \sum_{j=1}^{m} r_j p^{\beta_j} \right)^{-1} L(b - b_N) (x_0).$$

Here for $J^3_N(p)$, it follows from the Sobolev embedding and the estimate (2.4) that

$$|J^3_N(p)| \leq \left\| L + \sum_{j=1}^{m} q_j p^{\alpha_j} \right\|^{-1} L(a - a_N) \left\| C(\Omega) \right\| \leq C \left\| L + \sum_{j=1}^{m} q_j p^{\alpha_j} \right\|^{-1} L(a - a_N) \left\| D(L^{-1}) \right\|$$

$$\leq C \left\| L + \sum_{j=1}^{m} q_j p^{\alpha_j} \right\|^{-1} L^\gamma(a - a_N) \left\| L^2(\Omega) \right\| \leq \frac{C}{\sum_{j=1}^{m} q_j p^{\alpha_j}} \| L^\gamma(a - a_N) \|_{L^2(\Omega)}$$

$$\leq C p^{-\alpha_1} \| a - a_N \|_{D(L^\gamma)}.$$

Similarly, for $J^3_N(p)$ we have

$$|J^3_N(p)| \leq C p^{-\beta_1} \| b - b_N \|_{D(L^\gamma)}.$$  

Finally, we note that the regularity of $a, b$ guarantees the uniform boundedness of

$$\sum_{n=1}^{N} \lambda_n a_{N,n}(x_0), \quad \sum_{n=1}^{N} \lambda_n b_{N,n}(x_0), \quad \sum_{n=1}^{N} \sum_{k=1}^{d_a-1} (-1)^{k+1} (D^n_k a_{N,n})(x_0), \quad \sum_{n=1}^{N} \sum_{k=1}^{d_a-1} (-1)^{k+1} (D^n_k b_{N,n})(x_0)$$

for all $N \in \mathbb{N}$. Especially, by the identity $L P_n = \lambda_n P_n + D_n$ we know that

$$\sum_{n=1}^{N} \lambda_n a_{N,n}(x_0) + \sum_{n=1}^{N} D_n a_{N,n}(x_0) = \sum_{n=1}^{N} L a_{N,n}(x_0) = L a_N(x_0),$$

$$\sum_{n=1}^{N} \lambda_n b_{N,n}(x_0) + \sum_{n=1}^{N} D_n b_{N,n}(x_0) = \sum_{n=1}^{N} L b_{N,n}(x_0) = L b_N(x_0).$$

**Step 2** Now we are well prepared to prove part (i). We starting with showing $\alpha_1 = \beta_1$ by contradiction.

First let us assume that $\alpha_1 < \beta_1$. Our strategy is first multiplying both sides of (4.8) by $p^{\alpha_1}$, then passing $N \to \infty$ and finally passing $p \to \infty$ to deduce a contradiction. To this end, we calculate $p^{\alpha_1} I^1_N(p), p^{\alpha_1} J^1_N(p), p^{\alpha_1} I^2_N(p)$ and $p^{\alpha_1} J^2_N(p)$ as

$$p^{\alpha_1} I^1_N(p) = \sum_{n=1}^{N} \frac{\lambda_n a_{N,n}(x_0)}{\lambda_n p^{\alpha_1} + \sum_{j=1}^{m} q_j p^{\alpha_j}} = \frac{1 + o(1)}{q_1} \sum_{n=1}^{N} \lambda_n a_{N,n}(x_0),$$

$$p^{\alpha_1} J^1_N(p) = \sum_{n=1}^{N} \frac{\lambda_n b_{N,n}(x_0)}{\lambda_n p^{\alpha_1} + \sum_{j=1}^{m} r_j p^{\beta_j}} = \frac{p^{\alpha_1 - \beta_1}(1 + o(1))}{r_1} \sum_{n=1}^{N} \lambda_n b_{N,n}(x_0) = O(p^{\alpha_1 - \beta_1}),$$

$$p^{\alpha_1} I^2_N(p) = \sum_{n=1}^{N} \frac{\lambda_n a_{N,n}(x_0)}{\lambda_n p^{\alpha_1} + \sum_{j=1}^{m} q_j p^{\alpha_j}} = \frac{1 + o(1)}{q_1} \sum_{n=1}^{N} \lambda_n a_{N,n}(x_0),$$

$$p^{\alpha_1} J^2_N(p) = \sum_{n=1}^{N} \frac{\lambda_n b_{N,n}(x_0)}{\lambda_n p^{\alpha_1} + \sum_{j=1}^{m} r_j p^{\beta_j}} = \frac{p^{\alpha_1 - \beta_1}(1 + o(1))}{r_1} \sum_{n=1}^{N} \lambda_n b_{N,n}(x_0) = O(p^{\alpha_1 - \beta_1}).$$
\[ p^{\alpha_1} f_N^2(p) = \sum_{j=1}^{m} q_j p^{\alpha_j - \alpha_1} \sum_{n=1}^{N} \left\{ \frac{D_n a_{N,n}(x_0)}{(\lambda_n p^{-\alpha_1} + \sum_{j=1}^{m} q_j p^{\alpha_j - \alpha_1})^2} + O(p^{-\alpha_1}) \right\} \]
\[ = \frac{1 + o(1)}{q_1} \sum_{n=1}^{N} D_n a_{N,n}(x_0) + O(p^{-\alpha_1}), \quad \text{(4.14)} \]
\[ p^{\alpha_1} f_N^2(p) = O(p^{\alpha_1 + \beta_1}) \sum_{n=1}^{N} O(p^{-2\beta_1}) \sum_{k=1}^{d_n-1} (-1)^{k+1} (D_n^k b_{N,n})(x_0) = O(p^{\alpha_1 - \beta_1}) \quad \text{(4.15)} \]

for \( p \gg 1 \). Substituting (4.12)–(4.15) and (4.9)–(4.10) into (4.8), we obtain

\[ O(p^{-\nu + \alpha_1}) = p^{\alpha_1} \sum_{i=1}^{3} (f_N^i(p) - J_N^i(p)) = \frac{1 + o(1)}{q_1} \left\{ \sum_{n=1}^{N} \lambda_n a_{N,n}(x_0) + \sum_{n=1}^{N} D_n a_{N,n}(x_0) \right\} + O(p^{\alpha_1 - \beta_1}) + O(p^{-\alpha_1}) \]
\[ + O(1) \| a - a_N \|_{D(L^\gamma)} + O(p^{\alpha_1 - \beta_1}) \| b - b_N \|_{D(L^\gamma)} \]
\[ = \frac{1 + o(1)}{q_1} L_N a_N(x_0) + O(1) \| a - a_N \|_{D(L^\gamma)} + o(1) \quad \text{for} \ p \gg 1, \]

where we used (4.11) and the uniform boundedness of \( \| b - b_N \|_{D(L^\gamma)} \) with respect to \( N \). By the assumption \( \nu > \min \{ \alpha_1, \beta_1 \} = \alpha_1 \), the above equation becomes

\[ \frac{1 + o(1)}{q_1} L_N a_N(x_0) + O(1) \| a - a_N \|_{D(L^\gamma)} = o(1) \quad \text{for} \ p \gg 1, \quad \text{(4.16)} \]

Then we pass \( N \to \infty \) in (4.16) and employ (4.4)–(4.5) to deduce

\[ \frac{1 + o(1)}{q_1} L a_N(x_0) = o(1) \quad \text{for} \ p \gg 1. \]

Consequently, passing \( p \to \infty \) yields \( L a(x_0) = 0 \), which contradicts with the assumption \( L a(x_0) \neq 0 \).

In a completely parallel manner, assuming \( \alpha_1 > \beta_1 \) results in a contradiction with the assumption \( L b(x_0) \neq 0 \), indicating the only possibility \( \alpha_1 = \beta_1 \).

Now it suffices to verify \( q_1 / r_1 = L a(x_0) / L b(x_0) \). Similarly to the treatment for (4.12) and (4.14), we substitute \( \alpha_1 = \beta_1 \) into \( p^{\alpha_1} f_N^1(p) \) and \( p^{\alpha_1} f_N^2(p) \) to see that now (4.13) and (4.15) become

\[ p^{\alpha_1} f_N^1(p) = \frac{1 + o(1)}{r_1} \sum_{n=1}^{N} \lambda_n b_{N,n}(x_0), \quad p^{\alpha_1} f_N^2(p) = \frac{1 + o(1)}{r_1} \sum_{n=1}^{N} D_n b_{N,n}(x_0) + O(p^{-\alpha_1}), \]

respectively. Substituting the above equalities, (4.12), (4.14), (4.9), (4.10) into (4.8) and repeating the same calculation as before, we arrive at

\[ (1 + o(1)) \left( \frac{L_N a_N(x_0)}{q_1} - \frac{L_N b_N(x_0)}{r_1} \right) + O(1) \left( \| a - a_N \|_{D(L^\gamma)} + q_1 \| b - b_N \|_{D(L^\gamma)} \right) = o(1) \quad \text{for} \ p \gg 1. \]

Again passing first \( N \to \infty \) and then \( p \to \infty \), we conclude

\[ \frac{L a(x_0)}{q_1} - \frac{L b(x_0)}{r_1} = 0 \]

or equivalently \( q_1 / r_1 = L a(x_0) / L b(x_0) \). This completes the proof of part (i).

**Step 3** From now on we proceed to the proof of part (ii). For later convenience, we set \( \kappa := L a(x_0) / L b(x_0) \). Notice that owing to the further assumption \( a, b \in D(L^{1+\gamma}) \), the sequences

\[ \sum_{n=1}^{N} \lambda_n^2 a_{N,n}(x_0), \quad \sum_{n=1}^{N} \lambda_n^2 b_{N,n}(x_0), \quad \sum_{n=1}^{N} \lambda_n (D_n a_{N,n})(x_0), \quad \sum_{n=1}^{N} \lambda_n (D_n b_{N,n})(x_0) \]
are also uniformly bounded for all \( N \in \mathbb{N} \).

In this step, we shall show by induction that for \( j = 1, 2, \ldots, \min\{m, m'\} \), there holds

\[
\alpha_j = \beta_j, \quad \frac{q_j}{r_j} = \kappa.
\]  

(4.17)

Since the assumption of \( \nu \) in part (ii) is stronger than that in part (i), the case of \( j = 1 \) was already shown in Step 2.

Now let us assume that for some \( \ell = 2, \ldots, \min\{m, m'\} \), (4.17) holds for \( j = 1, \ldots, \ell - 1 \). Then our aim is to show that (4.17) also holds for \( j = \ell \). Notice that now the terms \( J_N^1(p) \) and \( J_N^2(p) \) become

\[
J_N^1(p) = \sum_{n=1}^{N} \frac{\lambda_n b_{N,n}(x_0)}{\lambda_n + \kappa^{-1} \sum_{j=1}^{m} q_j p^{\alpha_j} + \sum_{j=1}^{m'} r_j p^{\beta_j}},
\]

\[
J_N^2(p) = \left( \kappa^{-1} \sum_{j=1}^{\ell-1} q_j p^{\alpha_j} + \sum_{j=\ell}^{m'} r_j p^{\beta_j} \right) \sum_{n=1}^{N} \frac{\lambda_n}{\lambda_n + \kappa^{-1} \sum_{j=1}^{\ell-1} q_j p^{\alpha_j} + \sum_{j=\ell}^{m'} r_j p^{\beta_j}} \frac{(-1)^{k+1}(D_n^{k,b_{N,n}})(x_0)}{(\lambda_n + \kappa^{-1} \sum_{j=1}^{\ell-1} q_j p^{\alpha_j} + \sum_{j=\ell}^{m'} r_j p^{\beta_j})^{k+1}}.
\]

(4.18)

Similarly to Step 2, we start with showing \( \alpha_\ell = \beta_\ell \) by contradiction.

Again let us first assume that \( \alpha_\ell < \beta_\ell \). Now our strategy is first multiplying both sides of (4.18) by \( p^{2\alpha_\ell - \beta_\ell} \), then passing \( N \to \infty \) and finally passing \( p \to 1 \) to deduce a contradiction.

We first calculate \( p^{2\alpha_\ell - \beta_\ell}(I_N^1(p) - J_N^1(p)) \). To this end, we rewrite the denominator in \( I_N^1(p) \) as

\[
\frac{1}{\lambda_n + \sum_{j=1}^{m} q_j p^{\alpha_j}} = \frac{1}{\lambda_n + \sum_{j=1}^{\ell-1} q_j p^{\alpha_j}} \left( 1 + \frac{\sum_{j=\ell}^{m} q_j p^{\alpha_j}}{\lambda_n + \sum_{j=1}^{\ell-1} q_j p^{\alpha_j}} \right)^{-1}
\]

\[
= 1 - \frac{\sum_{j=\ell}^{m} q_j p^{\alpha_j}}{\lambda_n + \sum_{j=1}^{\ell-1} q_j p^{\alpha_j}} + O \left( \frac{(\sum_{j=\ell}^{m} q_j p^{\alpha_j})^2}{(\lambda_n + \sum_{j=1}^{\ell-1} q_j p^{\alpha_j})^2} \right)
\]

\[
= \lambda_n p^{-\alpha_\ell} - \sum_{j=1}^{\ell-1} q_j p^{\alpha_j - \alpha_\ell} - \frac{p^{\alpha_\ell - 2\alpha_\ell}(q_\ell + o(1))}{(q_1 + o(1))^2} + O(p^{2\alpha_\ell - 3\alpha_\ell}) \quad \text{for } p \gg 1.
\]

(4.19)

For the denominator of the first term on the right-hand side, we further expand

\[
\frac{1}{\lambda_n p^{-\alpha_\ell} + \sum_{j=1}^{\ell-1} q_j p^{\alpha_j - \alpha_\ell}} = \frac{1}{\sum_{j=1}^{\ell-1} q_j p^{\alpha_j - \alpha_\ell}} \left( 1 + \frac{\lambda_n p^{-\alpha_\ell}}{\sum_{j=1}^{\ell-1} q_j p^{\alpha_j - \alpha_\ell}} \right)^{-1}
\]

\[
= \sum_{j=1}^{\ell-1} q_j p^{\alpha_j - \alpha_\ell} + O \left( \frac{\lambda_n p^{-\alpha_\ell}}{(\sum_{j=1}^{\ell-1} q_j p^{\alpha_j - \alpha_\ell})^2} \right)
\]

\[
= \sum_{j=1}^{\ell-1} q_j p^{\alpha_j - \alpha_\ell} + \lambda_n O(p^{-\alpha_\ell}) \quad \text{for } p \gg 1.
\]

(4.20)

Plugging the above expansion into (4.19), we obtain

\[
\frac{1}{\lambda_n + \sum_{j=1}^{m} q_j p^{\alpha_j}} = \frac{p^{-\alpha_\ell}}{\sum_{j=1}^{\ell-1} q_j p^{\alpha_j - \alpha_\ell}} - \frac{p^{\alpha_\ell - 2\alpha_\ell}(q_\ell + o(1))}{(q_1 + o(1))^2} + \lambda_n O(p^{-2\alpha_\ell}) + O(p^{2\alpha_\ell - 3\alpha_\ell}) \quad \text{for } p \gg 1.
\]

(4.21)

Treating the denominator in \( J_N^1(p) \) similarly and substituting them into \( p^{2\alpha_\ell - \beta_\ell}(I_N^1(p) - J_N^1(p)) \), we deduce

\[
p^{2\alpha_\ell - \beta_\ell}(I_N^1(p) - J_N^1(p)) = \sum_{n=1}^{N} \lambda_n \left( \frac{a_{N,n}(x_0)}{\sum_{j=1}^{\ell-1} q_j p^{\alpha_j - \alpha_\ell}} - \frac{b_{N,n}(x_0)}{\sum_{j=1}^{\ell-1} q_j p^{\alpha_j - \alpha_\ell}} \right)
\]

\[
- p^{\alpha_\ell - \beta_\ell}(q_\ell + o(1)) \sum_{n=1}^{N} \lambda_n a_{N,n}(x_0) (q_1 + o(1))^2 + (r_\ell + o(1)) \sum_{n=1}^{N} \lambda_n b_{N,n}(x_0) (q_1 + o(1))^2
\]
where we utilized the assumption $\alpha_1 < \beta_1$ and the uniform boundedness of all involved sequences with respect to $N$.

Next we calculate $p^{2\alpha_1 - \beta_1} (I_N^2(p) - J_N^2(p))$. We further decompose $I_N^2(p)$ as

\[
I_N^2(p) = \sum_{j=1}^{m} q_j p^{\alpha_1} \sum_{n=1}^{N} \left\{ \frac{D_n a_{N,n}(x_0)}{(\lambda_n + \sum_{j=1}^{m} q_j p^{\alpha_j})^2} + \frac{D_n b_{N,n}(x_0)}{(\lambda_n + \sum_{j=1}^{m} q_j p^{\alpha_j})^2} \right\}
\]

Plugging the expansion (4.20) into the above equality and treating $J_N^2(p)$ in the same manner, we deduce

\[
p^{2\alpha_1 - \beta_1} (I_N^2(p) - J_N^2(p)) = p^{\alpha_1 - \beta_1} \sum_{n=1}^{N} \left( \frac{D_n a_{N,n}(x_0)}{\sum_{j=1}^{m} q_j p^{\alpha_j - \alpha_1}} - \frac{D_n b_{N,n}(x_0)}{\sum_{j=1}^{m} q_j p^{\alpha_j - \beta_1}} \right)
\]

\[
- p^{\alpha_1 - \beta_1} (q_l + o(1)) \sum_{n=1}^{N} \frac{D_n a_{N,n}(x_0)}{(q_l + o(1))^2} + (r_l + o(1)) \sum_{n=1}^{N} \frac{D_n b_{N,n}(x_0)}{(r_l + o(1))^2}
\]

\[
+ O(p^{\beta_1}) \sum_{n=1}^{N} \lambda_n D_n(\kappa - b_{N,n})a_{N,n}(x_0) + O(p^{2\alpha_1 - \beta_1}) \sum_{n=1}^{N} D_n a_{N,n}(x_0)
\]

\[
+ O(p^{\beta_1}) \sum_{n=1}^{N} D_n b_{N,n}(x_0)
\]

\[
= p^{\alpha_1 - \beta_1} \sum_{n=1}^{N} \left\{ D_n a_{N,n}(x_0) - \kappa b_{N,n}\right\} + \sum_{n=1}^{N} \frac{r_l + o(1)}{r_l} \sum_{n=1}^{N} \lambda_n b_{N,n}(x_0)
\]

\[
+ o(1) \quad \text{for } p \gg 1.
\]
Again passing first (4.17) holds for all \( j \).

Notice that now a contradiction.

Step 4 At last, it remains to show \( \alpha \) identically parallel manner, we can exclude the possibility of \( \alpha \beta = \beta \).

Consequently, passing \( p \to \infty \) yields \( Lb(x_0) = 0 \), which contradicts with the assumption \( Lb(x_0) \neq 0 \). In an identically parallel manner, we can exclude the possibility of \( \alpha \beta > \beta \) and conclude \( \alpha \beta = \beta \).

Now it suffices to verify \( q_\ell/r_\ell = \kappa \). Similarly to the argument at the end of Step 2, it follows immediately from \( \alpha \beta = \beta \) that now (4.21) and (4.22) become

\[
p^{2\alpha-\beta}(I_N^1(p) - J_N^1(p)) = p^{\alpha-\beta}(1 + o(1)) \sum_{n=1}^{N} \lambda_n(a_{N,n} - \kappa b_{N,n})(x_0) - \frac{q_\ell + o(1)}{q_\ell^2} \sum_{n=1}^{N} \lambda_n a_{N,n}(x_0) + \frac{r_\ell + o(1)}{r_\ell^2} Lb(x_0) = o(1)
\]

Converting the denominators in (O.28) – (O.35) again to deduce

\[
\frac{r_\ell + o(1)}{r_\ell^2} Lb(x_0) = o(1)
\]

for \( p \gg 1 \). By the definition of \( \kappa \), we pass \( N \to \infty \) in (4.23) and employ (4.4) – (4.5) again to deduce

\[
r_\ell + o(1) \sum_{n=1}^{N} \lambda_n b_{N,n}(x_0) + o(1),
\]

Consequently, passing \( p \to \infty \) yields \( Lb(x_0) = 0 \), which contradicts with the assumption \( Lb(x_0) \neq 0 \). In an identically parallel manner, we can exclude the possibility of \( \alpha \beta > \beta \) and conclude \( \alpha \beta = \beta \).

Now it suffices to verify \( q_\ell/r_\ell = \kappa \). Similarly to the argument at the end of Step 2, it follows immediately from \( \alpha \beta = \beta \) that now (4.21) and (4.22) become

\[
p^{2\alpha-\beta}(I_N^1(p) - J_N^1(p)) = p^{\alpha-\beta}(1 + o(1)) \sum_{n=1}^{N} \lambda_n(a_{N,n} - \kappa b_{N,n})(x_0) - \frac{q_\ell + o(1)}{q_\ell^2} \sum_{n=1}^{N} \lambda_n a_{N,n}(x_0) + \frac{r_\ell + o(1)}{r_\ell^2} Lb(x_0) = o(1)
\]

Repeating the same calculation as before, we arrive at

\[
o(1) = O(p^{\alpha-\alpha j}) \left( L(a_N - \kappa b_N)(x_0) + \|a - a_N\|_{\mathcal{D}(L^\gamma)} + \|b - b_N\|_{\mathcal{D}(L^\gamma)} \right)
\]

\[
- \frac{q_\ell + o(1)}{q_\ell^2} L_{\alpha N}(x_0) + \frac{r_\ell + o(1)}{r_\ell^2} L_{\alpha N}(x_0)
\]

for \( p \gg 1 \).

Again passing first \( N \to \infty \) and then \( p \to \infty \), we conclude

\[
- \frac{q_\ell}{q_\ell^2} L_{\alpha N}(x_0) + \frac{r_\ell}{r_\ell^2} Lb(x_0) = 0
\]

or equivalently \( q_\ell/r_\ell = \kappa \). This completes the proof of (4.17) for \( j = \ell \). By the inductive assumption, eventually (4.17) holds for all \( j = 1, 2, \ldots, \min\{m, m'\} \).

Step 4 At last, it remains to show \( m = m' \) again by contradiction, that is, we assume \( m < m' \) without loss of generality. As those in the previous steps, our strategy is again multiplying both sides of (4.8) by \( p^{2\alpha_1-\beta_{m+1}} \) to deduce a contradiction.

Notice that now \( J_N^1(p) \) and \( J_N^1(p) \) becomes (4.18) with \( \ell = m + 1 \). Treating the denominators in \( I_N^1(p) \) and \( J_N^1(p) \) similarly as that in (4.20) yields

\[
\frac{1}{\lambda_n + \sum_{j=1}^{m} q_j p^{\alpha_j}} = \frac{p^{-\alpha_1}}{\sum_{j=1}^{m} q_j p^{\alpha_j}} + \lambda_n O(p^{-2\alpha_1}),
\]
where

\[ \frac{1}{\lambda_n + \sum_{j=1}^{m} r_j p^{\beta_j}} = \frac{\kappa p^{-\alpha_1}}{\sum_{j=1}^{m} q_j p^{\beta_j}} - \frac{p^{2\alpha_1 - \beta_{m+1}} (r_{m+1} + o(1))}{(r_1 + o(1))^2} + \lambda_n O(p^{-2\alpha_1}) + O(p^{2\beta_{m+1} - 3\alpha_1}) \]

for \( p \gg 1 \). Then

\[ p^{2\alpha_1 - \beta_{m+1}} (I_N^2(p) - J_N^2(p)) = \frac{p^{\alpha_1 - \beta_{m+1}} (1 + o(1))}{q_1} \sum_{n=1}^{N} \lambda_n (a_{N,n} - \kappa b_{N,n})(x_0) \]

\[ + \frac{r_{m+1} + o(1)}{r_1} \sum_{n=1}^{N} \lambda_n b_{N,n}(x_0) + o(1) \quad \text{for} \ p \gg 1. \]  

(4.24)

Treating \( I_N^1(p) \) and \( J_N^1(p) \) similarly as that in Step 3 and expanding the involved denominators as above, we have

\[ p^{2\alpha_1 - \beta_{m+1}} (I_N^1(p) - J_N^1(p)) = \frac{p^{\alpha_1 - \beta_{m+1}} (1 + o(1))}{q_1} \sum_{n=1}^{N} D_n (a_{N,n} - \kappa b_{N,n})(x_0) \]

\[ + \frac{r_{m+1} + o(1)}{r_1} \sum_{n=1}^{N} D_n b_{N,n}(x_0) + o(1) \quad \text{for} \ p \gg 1. \]  

(4.25)

Collecting (4.24), (4.25), (4.9), (4.10) and repeating the same calculation used in Step 3, we reach (4.23) with \( \ell = m + 1 \). Performing the same limiting process as before, we conclude again the contradiction \( Lb(x_0) = 0 \). This completes the proof of \( m = m' \) and thus finalizes the proof of part (ii).

Now we turn to the proof of Theorem 3.4 which relies on the following key lemma concerning the temporal component \( \rho(t) \) of the inhomogeneous terms in (1.3) and (3.1).

Lemma 4.2. Let \( \mu \in L^1(\mathbb{R}_+) \) satisfy (3.8). Then its Laplace transform \( \tilde{\mu}(p) \) satisfies \( \tilde{\mu}(p) \sim p^{-\mu-1} \) for \( p \gg 1 \).

Proof. According to (3.8), there exist constants \( \varepsilon > 0 \) and \( C_1, C_2 \in \mathbb{R} \) satisfying \( \varepsilon \ll 1 \), \( C_1 < C_2 \) and \( C_1 C_2 > 0 \) such that

\[ C_1 t^\mu \leq \rho(t) \leq C_2 t^\mu \quad \text{a.e.} \ t \in (0, \varepsilon). \]  

(4.26)

By the definition of the Laplace transform, we divide \( \tilde{\mu}(p) \) as

\[ \tilde{\mu}(p) = \left( \int_{\varepsilon}^{\infty} + \int_{0}^{\varepsilon} \right) e^{-pt} \rho(t) \, dt =: I_1(p) + I_2(p), \]

where

\[ |I_2(p)| \leq \int_{\varepsilon}^{\infty} e^{-pt} |\rho(t)| \, dt \leq e^{-\varepsilon p} \|\rho\|_{L^1(\mathbb{R}_+)} = o(p^{-\mu-1}) \quad \text{for} \ p \gg 1. \]

For \( I_1(p) \), it follows from (4.26) and

\[ \int_{0}^{\varepsilon} e^{-pt} t^\mu \, dt = \frac{1}{p^{\mu+1}} \int_{0}^{\varepsilon} e^{-p s} s^\mu \, ds \sim p^{-\mu-1} \quad \text{for} \ p \gg 1 \]

that \( I_1(p) \sim p^{-\mu-1} \) for \( p \gg 1 \), which completes the proof.

Proof of Theorem 3.4 Let \( a = b \equiv 0 \) in \( \Omega \) in (1.3) and (3.1). Similarly to Step 1 in the proof of Theorem 3.1 it is not difficult to conclude that the difference of \( \tilde{u}(x_0, p) \) and \( \tilde{v}(x_0, p) \) still satisfies (4.1) for \( p \gg 1 \).

Next, we take the Laplace transform in (1.3) and (3.1) and substitute \( x = x_0 \) to deduce

\[ \tilde{u}(x_0, p) - \tilde{v}(x_0, p) = \tilde{\mu}(p) (w(x_0, p) - y(x_0, p)) \quad \text{for} \ p \gg 1, \]

where

\[ w(\cdot, p) := \left( L + \sum_{j=1}^{m} q_j p^{\beta_j} \right)^{-1} f, \quad y(\cdot, p) := \left( L + \sum_{j=1}^{m'} r_j p^{\beta_j} \right)^{-1} g. \]  

(4.27)
Then we employ \((4.1)\) and Lemma \(4.2\) to obtain
\[
w(x_0, p) - y(x_0, p) = O(p^{\mu - \nu}) \quad \text{for } p \gg 1.
\] (4.28)

Arguing similarly as before, we utilize Lemma \(4.1\) to pick two sequences \(f_n = \sum_{n=1}^{N} f_{N,n}\) and \(g_n = \sum_{n=1}^{N} g_{N,n}\) \((N \in \mathbb{N})\) such that
\[
f_{N,n}, g_{N,n} \in P_nL^2(\Omega), \quad \lim_{N \to \infty} \|L^\gamma (f - f_N)\|_{L^2(\Omega)} = \lim_{N \to \infty} \|L^\gamma (g - g_N)\|_{L^2(\Omega)} = 0.
\]

Then for any fixed \(N \in \mathbb{N}\), we decompose \(f = f_N + (f - f_N)\) and employ the expansion \(2.3\) to rewrite \((4.27)\) as
\[
w(\cdot, p) = \left( L + \sum_{j=1}^{m} q_j p^{\alpha_j} \right)^{-1} \left\{ \sum_{n=1}^{N} f_{N,n} + (f - f_N) \right\}
\]
\[
= \sum_{n=1}^{N} \left( L + \sum_{j=1}^{m} q_j p^{\alpha_j} \right)^{-1} P_n f_{N,n} + \left( L + \sum_{j=1}^{m} q_j p^{\alpha_j} \right)^{-1} (f - f_N)
\]
\[
= \sum_{n=1}^{N} \frac{f_{N,n}}{\lambda_n + \sum_{j=1}^{m} q_j p^{\alpha_j}} + \sum_{n=1}^{N} \sum_{k=1}^{d_n-1} \frac{(-1)^k D_{n}^{\alpha_j} f_{N,n}}{\lambda_n + \sum_{j=1}^{m} q_j p^{\alpha_j}} + \left( L + \sum_{j=1}^{m} q_j p^{\alpha_j} \right)^{-1} (f - f_N)
\]
and similarly
\[
y(\cdot, p) = \sum_{n=1}^{N} \frac{g_{N,n}}{\lambda_n + \sum_{j=1}^{m} r_j p^{\beta_j}} + \sum_{n=1}^{N} \sum_{k=1}^{d_n-1} \frac{(-1)^k D_{n}^{\alpha_j} g_{N,n}}{\lambda_n + \sum_{j=1}^{m} q_j p^{\alpha_j}} + \left( L + \sum_{j=1}^{m} r_j p^{\beta_j} \right)^{-1} (g - g_N).
\]

Then taking \(x = x_0\) in the above expressions and substituting them into \((4.28)\) yield
\[
\sum_{i=1}^{3} (I^i_N(p) - J^i_N(p)) = O(p^{\mu - \nu}) \quad \text{for } p \gg 1,
\]
where
\[
I^1_N(p) := \sum_{n=1}^{N} \frac{f_{N,n}(x_0)}{\lambda_n + \sum_{j=1}^{m} q_j p^{\alpha_j}}, \quad J^1_N(p) := \sum_{n=1}^{N} \frac{g_{N,n}(x_0)}{\lambda_n + \sum_{j=1}^{m} r_j p^{\beta_j}},
\]
\[
I^2_N(p) := \sum_{j=1}^{m} q_j p^{\alpha_j} \sum_{n=1}^{N} \sum_{k=1}^{d_n-1} \frac{(-1)^k (D_{n}^{\alpha_j} f_{N,n})(x_0)}{\lambda_n + \sum_{j=1}^{m} q_j p^{\alpha_j}}(x_0), \quad J^2_N(p) := \sum_{j=1}^{m} r_j p^{\beta_j} \sum_{n=1}^{N} \sum_{k=1}^{d_n-1} \frac{(-1)^k (D_{n}^{\alpha_j} g_{N,n})(x_0)}{\lambda_n + \sum_{j=1}^{m} q_j p^{\alpha_j}}(x_0),
\]
\[
I^3_N(p) := \left( L + \sum_{j=1}^{m} q_j p^{\alpha_j} \right)^{-1} (f - f_N)(x_0), \quad J^3_N(p) := \left( L + \sum_{j=1}^{m} r_j p^{\beta_j} \right)^{-1} (g - g_N)(x_0).
\]

The remaining part of the proof turns out to be essentially the same as that of Theorem \(3.1\) and we refrain from repeating the details.

\textbf{Proof of Theorem \(3.3\)} We only provide the proof for part (1) because that of part (2) is mostly parallel. Then throughout this proof, we set \(f = g \equiv 0\) in \(\Omega\) in \(1.3\) and \(3.1\). Notice that thanks to the symmetry of \(L\), we have \(2.2\) and
\[
h = \sum_{n=1}^{\infty} P_n h, \quad \forall \ h \in L^2(\Omega), \quad (L - z)^{-1} P_n = \frac{P_n}{\lambda - z}, \quad z \notin \sigma(L).
\] (4.29)
First we assume (3.7) and show the “only if” part. According to Theorem 3.1, we have already reached (3.4). Together with (4.29), we can rewrite (4.2) and (4.3) as
\[
\hat{u}(\cdot, p) = \frac{1}{p} \sum_{j=1}^{m} q_j p^{\alpha_j} \sum_{n=1}^{\infty} \frac{P_n a}{\kappa \lambda_n + z} \quad \text{for } p \gg 1.
\]
(4.30)

On the other hand, owing to the time-analyticity of the solutions to (1.3) and (3.1), we see that (3.7) implies \( u(x_0, \cdot) = v(x_0, \cdot) \) in \( \mathbb{R}_+ \). Therefore, taking the Laplace transform results in \( \hat{u}(x_0, p) = \hat{v}(x_0, p) \) for \( p \gg 1 \) and hence
\[
z \sum_{n=1}^{\infty} \frac{P_n a(x_0)}{\lambda_n + z} = z \sum_{n=1}^{\infty} \frac{P_n b(x_0)}{\kappa \lambda_n + z} \quad \text{for } z \gg 1, \quad z := \sum_{j=1}^{m} q_j p^{\alpha_j}.
\]
Then we can analytically extend both sides of the above equality in \( z \in \mathbb{C} \) to obtain
\[
\sum_{n=1}^{\infty} \frac{P_n a(x_0)}{\lambda_n + z} = \sum_{n=1}^{\infty} \frac{P_n b(x_0)}{\kappa \lambda_n + z} = \sum_{n=1}^{\infty} \frac{P_n L a(x_0)}{\lambda_n + z} = \sum_{n=1}^{\infty} \frac{P_n L b(x_0)}{\kappa \lambda_n + z}, \quad z \in \mathbb{C} \setminus \Lambda,
\]
(4.31)
where
\[
\Lambda := (-\sigma(L)) \cup (-\kappa \sigma(L)), \quad \eta \sigma(L) := \{ \eta \lambda_n | \lambda_n \in \sigma(L) \}, \quad \eta \in \mathbb{R}.
\]
Here we used (3.5) and the fact that \( \lambda_n L = LP_n \).

Now we concentrate on \( \kappa \) and consider the cases of \( \kappa = 1 \) and \( \kappa \neq 1 \) separately.

Case 1 If \( \kappa = 1 \), then \( \Lambda = -\sigma(L) \) and for each \( n \in \mathbb{N} \), we can take a sufficiently small circle \( \gamma_n \) enclosing \( -\lambda_n \) and excluding \( -\sigma(L) \setminus \{ -\lambda_n \} \). Integrating (4.31) on \( \gamma_n \) with respect to \( z \), we employ Cauchy’s integral theorem to conclude (3.14). In view of Remark 3.6, this completes the proof of (3.11) for \( \kappa = 1 \).

Case 2 Now let us assume \( \kappa \neq 1 \). To obtain (3.9), it suffices to show \( \kappa \in \Sigma \) by contradiction. If \( \kappa \notin \Sigma \), then it follows from the definition of \( \Sigma \) that \( \sigma(L) \cap \kappa \sigma(L) = \emptyset \). Thus for each \( n \in \mathbb{N} \), we can take the same circle \( \gamma_n \) as that in Case 1 enclosing \( -\lambda_n \) and excluding \( \Lambda \setminus \{ -\lambda_n \} \). Then integrating (4.31) on \( \gamma_n \) again yields \( P_n L a(x_0) = 0 \) (\( \forall n \in \mathbb{N} \)) and hence
\[
L a(x_0) = \sum_{n=1}^{\infty} P_n L a(x_0) = 0,
\]
which contradicts with the assumption \( L a(x_0) \neq 0 \). This verifies \( \kappa \in \Sigma \) and also (3.9). Now we turn to demonstrating (3.11) for \( \kappa \neq 1 \). Again by the definition of \( \Sigma \), we know that \( \sigma(L) \cap \kappa \sigma(L) \neq \emptyset \) and the index sets \( M_\kappa, M'_\kappa \) defined by (3.10) are obviously nonempty. Meanwhile, by the strict monotonicity of \( \sigma(L) \), it is readily seen that for each \( n \in M_\kappa \), there exists a unique \( n' \in M'_\kappa \) such that \( \lambda_n = \kappa \lambda_{n'} \) and vise versa. Then the bijection \( \theta_\kappa : M_\kappa \rightarrow M'_\kappa \) can be defined by \( \theta_\kappa(n) = n' \).

Next, for each \( n \notin M_\kappa \), we have \( \lambda_n \notin \kappa \sigma(L) \). Then we can take the same circle \( \gamma_n \) as before and integrating (4.31) on \( \gamma_n \) to conclude
\[
P_n a(x_0) = P_n L a(x_0) = 0, \quad \forall n \notin M_\kappa.
\]
Similarly, we have
\[
P_n b(x_0) = P_n L b(x_0) = 0, \quad \forall n \notin M'_\kappa.
\]
Finally, for each \( n \in M_\kappa \), we can take a circle \( \gamma_n \) enclosing \( \lambda_n = \kappa \lambda_{\theta_\kappa(n)} \) and excluding \( \Lambda \setminus \{ -\lambda_n \} \). Then again integrating (4.31) on \( \gamma_n \) gives
\[
P_n a(x_0) = P_{\theta_\kappa(n)} a(x_0), \quad P_n L a(x_0) = \kappa P_{\theta_\kappa(n)} L b(x_0), \quad \forall n \in M_\kappa.
\]
Collecting the above three cases completes the proof of (3.11) for \( \kappa \neq 1 \).
Now it remains to show the “if” part. Taking $x = x_0$ in (4.30) and employing conditions (3.10)–(3.11), we obtain
\[
\hat{u}(x_0, p) = \frac{z}{p} \sum_{n=1}^{\infty} \frac{P_n(a(x_0))}{\lambda_n + z}
= \frac{z}{p} \sum_{n \in M_k} \frac{P_n(a(x_0))}{\lambda_n + z}
= \frac{z}{p} \sum_{n \in M_k'} \frac{P_n(b(x_0))}{\kappa \lambda_n + z}
= \frac{z}{p} \sum_{n=1}^{\infty} \frac{P_n(b(x_0))}{\kappa \lambda_n + z}
= \hat{v}(x_0, p) \quad \text{for } p > 1,
\]
where we abbreviated $z := \sum_{j=1}^{m} q_jp^{\alpha_j}$. Then the coincidence of $u$ and $v$ at $\{x_0\} \times \mathbb{R}_+$ follows immediately from the uniqueness of the Laplace transform. \hfill \Box

**Proof of Corollary 3.8** Similarly to the proof of Theorem 3.3, we only deal with part (1) and omit the proof of part (2). Then it suffices to show the equivalence of (3.11) and (3.16).

By the assumption (3.15) on the choice of $x_0$, there holds $\varphi_n(x_0) \neq 0$ for all $n \in \mathbb{N}$. On the other hand, since all eigenvalues of $L$ are assumed to be simple, it reveals that
\[
P_n h = (h, \varphi_n) \varphi_n, \quad P_n L h = \lambda_n (h, \varphi_n) \varphi_n, \quad \forall h \in D(L).
\]
Therefore, noticing that $\lambda_n = \kappa \lambda_{\theta, n}(n)$ for $n \in M_k$, we immediately see that (3.11) is equivalent to
\[
\begin{cases}
(a, \varphi_n(x_0)) = (b, \varphi_{\theta, n}(n)) \varphi_{\theta, n}(n)(x_0), & n \in M_k, \\
(a, \varphi_n) = 0, & n \notin M_k, \\
(b, \varphi_n) = 0, & n \notin M_k'.
\end{cases}
\]
Substituting the above relation into the Fourier expansions of $a, b$, we immediately obtain (3.16). The opposite side can also be verified similarly.

Especially if $\kappa = 1$, it turns out that (3.11) simply equals to
\[
(a, \varphi_n) = (b, \varphi_n), \quad \forall n \in \mathbb{N}
\]
or equivalently $a = b$ in $\Omega$. The proof of Corollary 3.8 is completed. \hfill \Box

5 **Concluding Remarks**

For the observation data, we only use the short-time inexact data near $t = 0$ satisfying (3.2). On the other hand, after taking Laplace transform of the solutions with respect to $t$, we only take advantage of the asymptotic behavior of $\hat{u}(x_0, p) - \hat{v}(x_0, p)$ for $p > 1$. This reflects the dominating effect of fractional orders and corresponding parameters to the solution. Simultaneously, it also suggests that the asymptotic behavior of solutions possesses abundant information about the equations.

The starting point of the proofs relies on the Laplace transforms of solutions, whose existence was guaranteed by the key estimate (2.5) in Lemma 2.1. Nevertheless, especially for $\sigma(L) \subset \mathbb{R}_+$, the estimate (2.5) is not at all sharp for large $t$. The sharp decay rate with a non-symmetric $L$ keeps open even in the single-term case of (1.3), which deserves further investigation.

By reviewing the proof, we immediately see that our arguments also work for multi-term time-fractional wave equations, that is, (1.3) and (3.1) with
\[
2 > \alpha_1 > \alpha_2 > \cdots > \alpha_m > 0, \quad 2 > \beta_1 > \beta_2 > \cdots > \beta_m' > 0,
\partial_t u = 0 \quad \text{if } \alpha_1 > 1, \quad \partial_t v = 0 \quad \text{if } \beta_1 > 1 \quad \text{in } \Omega \times \{0\}.
\]

However, since the forward theory for multi-term time-fractional wave equations (especially with a non-symmetric elliptic part) is not well established for the moment, we postpone this generalization as a future work. Meanwhile, other possible future topics include the case of $x$-dependent parameters $q_j(x), r_j(x)$. 

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Finally, though the single point observation is sufficient for the uniqueness of Problem \[1.1\] theoretically, it seems meaningful to develop numerical methods with multiple observation points in order to improve the accuracy. If the rank condition (see \[29\]) is also satisfied in this case, it is even possible to uniquely identify the initial value \(a\) or the spatial component \(f\) of the source term in \[1.3\], whose numerical reconstruction may also be interesting.

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