EXOTIC GROUP $C^*$-ALGEBRAS IN NONCOMMUTATIVE DUALITY

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Abstract. We show that for a locally compact group $G$ there is a one-to-one correspondence between $G$-invariant weak*-closed subspaces $E$ of the Fourier-Stieltjes algebra $B(G)$ containing $B_r(G)$ and quotients $C^*_E(G)$ of $C^*(G)$ which are intermediate between $C^*(G)$ and the reduced group algebra $C^*_r(G)$. We show that the canonical comultiplication on $C^*(G)$ descends to a coaction or a comultiplication on $C^*_E(G)$ if and only if $E$ is an ideal or subalgebra, respectively. When $\alpha$ is an action of $G$ on a $C^*$-algebra $B$, we define "$E$-crossed products" $B \rtimes_{\alpha,E} G$ lying between the full crossed product and the reduced one, and we conjecture that these "intermediate crossed products" satisfy an "exotic" version of crossed-product duality involving $C^*_E(G)$.

1. Introduction

It has long been known that for a locally compact group $G$ there are many $C^*$-algebras between the full group $C^*$-algebra $C^*(G)$ and the reduced algebra $C^*_r(G)$ (see [Eym64]). However, little study has been made regarding the extent to which these intermediate algebras can be called group $C^*$-algebras.

This paper is inspired by recent work of Brown and Guentner [BG], which studies such intermediate algebras for discrete groups, and [Oka], which shows that in fact there can be a continuum of such intermediate algebras. We shall consider a general locally compact group $G$, and show that by elementary harmonic analysis there is a one-to-one correspondence between $G$-invariant weak*-closed subspaces $E$ of the Fourier-Stieltjes algebra $B(G)$ containing $B_r(G)$ and quotients $C^*_E(G)$ of $C^*(G)$ which are intermediate between $C^*(G)$ and the reduced group algebra $C^*_r(G)$.

We are primarily interested in the following results:

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• $E$ is an ideal if and only if there is a coaction $C^*_E(G) \to M(C^*_E(G) \otimes C^*(G))$.
• $E$ is a subalgebra if and only if there is a comultiplication $C^*_E(G) \to M(C^*_E(G) \otimes C^*_E(G))$.

(See Propositions 3.13 and 3.16 for more precise statements.) These $C^*$-algebras can be used to describe various properties of $G$, e.g., if $G$ is discrete and $E = B(G) \cap c_0(G)$, then $G$ has the Haagerup property if and only if $C^*_E(G) = C^*(G)$ (see [BG, Corollary 3.4]). Brown and Guentner also prove that (again, in the discrete case) $C^*_E(G)$ is a compact quantum group, because it carries a comultiplication, and this caught our attention since it makes a connection with noncommutative crossed-product duality.

If we have a $C^*$-dynamical system $(B, G, \alpha)$, one can form the full crossed product $B \rtimes_{\alpha} G$ or the reduced crossed product $B \rtimes_{\alpha, r} G$. We show in Section 6 that for $E$ as above there is an “$E$-crossed product” $B \rtimes_{\alpha, E} G$, and we speculate that these “intermediate” crossed products satisfy an “exotic” version of crossed-product duality involving $C^*_E(G)$.

After a short section on preliminaries, in Section 3 we prove the above-mentioned results concerning the existence of a coaction or comultiplication on $C^*_E(G)$.

In Section 4 we briefly explore the analogue for arbitrary locally compact groups of the construction used in [BG], where for discrete groups they construct group $C^*$-algebras starting with ideals of $\ell^\infty(G)$.

In Section 5 we specialize (for the only time in this paper) to the discrete case, showing that a quotient $C^*_E(G)$ is a group $C^*$-algebra if and only if it is topologically graded in the sense of [Exe97].

Finally, in Section 6 we outline a possible application of our exotic group algebras to noncommutative crossed-product duality.

After this paper was circulated in preprint form, we learned that Buss and Echterhoff [BE] have given counterexamples to Conjecture 6.12 and have proven Conjecture 6.14.

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2. Preliminaries

All ideals of $C^*$-algebras will be closed and two-sided. If $A$ and $B$ are $C^*$-algebras, then $A \otimes B$ will denote the minimal tensor product.

For one of our examples we will need the following elementary fact, which is surely folklore.
Lemma 2.1. Let $A$ be a $C^*$-algebra, and let $I$ and $J$ be ideals of $A$. Let $\phi : A \to A/I$ and $\psi : A \to A/J$ be the quotient maps, and define
\[ \pi = \phi \oplus \psi : A \to (A/I) \oplus (A/J). \]
Then $\pi$ is surjective if and only if $A = I + J$.

Proof. First assume that $\pi$ is surjective, and let $a \in A$. Choose $b \in A$ such that
\[ \pi(b) = (\phi(a), 0), \]
i.e., $\phi(b) = \phi(a)$ and $\psi(b) = 0$. Then $a - b \in I, b \in J$, and $a = (a - b) + b$.
Conversely, assume that $A = I + J$, and let $a \in A$. Choose $b \in I$ and $c \in J$ such that $a = b + c$. Then $\psi(c) = 0$, and $\phi(c) = \phi(a)$ since $a - c \in I$. Thus
\[ \pi(c) = (\phi(a), 0). \]
It follows that $\pi(A) \supset (A/I) \oplus \{0\}$, and similarly $\pi(A) \supset \{0\} \oplus (A/J)$, and hence $\pi$ is onto. \qed

A point of notation: for a homomorphism between $C^*$-algebras, or for a bounded linear functional on a $C^*$-algebra, we use a bar to denote the unique strictly continuous extension to the multiplier algebra.

We adopt the conventions of [EKQR06] for actions and coactions of a locally compact group $G$ on a $C^*$-algebra $A$. In particular, we use full coactions $\delta : A \to M(A \otimes C^*(G))$, which are nondegenerate injective homomorphisms satisfying the coaction-nondegeneracy property
\[ \text{span}\{\delta(A)(1 \otimes C^*(G))\} = A \otimes C^*(G) \] (2.1)
and the coaction identity
\[ \delta \otimes \text{id} \circ \delta = \text{id} \otimes \delta_G \circ \delta, \] (2.2)
where $\delta_G$ is the canonical coaction on $C^*(G)$, determined by $\delta_G(x) = x \otimes x$ for $x \in G$ (and where $G$ is identified with its canonical image in $M(C^*(G))$). Recall that $\delta$ gives rise to a right $B(G)$-module structure on $A^*$ given by
\[ \omega \cdot f = \omega \otimes f \circ \delta \quad \text{for } \omega \in A^* \text{ and } f \in B(G), \]
and also to a left $B(G)$-module structure on $A$ given by
\[ f \cdot a = \text{id} \otimes f \circ \delta(a) \quad \text{for } f \in B(G) \text{ and } a \in A, \]
and that moreover
\[ (\omega \cdot f)(a) = \omega(f \cdot a) \quad \text{for all } \omega \in A^*, f \in B(G), \text{ and } a \in A. \]

Further recall that $1_G \cdot a = a$ for all $a \in A$, where $1_G$ is the constant function with value $1$. In fact, suppose we have a homomorphism $\delta : A \to M(A \otimes C^*(G))$ satisfying all the conditions of a coaction except...
Perhaps injectivity. Then $\delta$ is in fact a coaction, because injectivity follows automatically, by the following folklore trick:

**Lemma 2.2.** Let $\delta : A \to M(A \otimes C^*(G))$ be a homomorphism satisfying (2.1) and (2.2). Then for all $a \in A$ we have
\[
id \otimes 1_G \circ \delta(a) = a,
\]
where $1_G \in B(G)$ is the constant function with value 1. In particular, $\delta$ is injective and hence a coaction.

**Proof.** First of all,
\[
A = \overline{\text{span}} \left\{ (\text{id} \otimes g)(\delta(a)(1 \otimes c)) : g \in B(G), a \in A, c \in C^*(G) \right\}
\]
\[
= \overline{\text{span}} \left\{ \text{id} \otimes c \cdot g \circ \delta(a) : g \in B(G), a \in A, c \in C^*(G) \right\}
\]
\[
= \overline{\text{span}} \left\{ \text{id} \otimes f \circ \delta(a) : f \in B(G), a \in A \right\}.
\]
Now the following computation suffices: for all $a \in A$ and $f \in B(G)$ we have
\[
id \otimes 1_G \circ \delta(\text{id} \otimes f \circ \delta(a))
\]
\[
= \text{id} \otimes 1_G \circ \text{id} \otimes \text{id} \otimes f \circ (\delta \otimes \text{id}) \circ \delta(a)
\]
\[
= \text{id} \otimes 1_G \otimes f \circ (\text{id} \otimes \delta_G) \circ \delta(a)
\]
\[
= \text{id} \otimes f \circ \delta(a)
\]
\[
= \text{id} \otimes f \circ \delta(a)
\]

3. **Exotic Quotients of $C^*(G)$**

Let $G$ be a locally compact group. We are interested in certain quotients $C^*_E(G)$ (see Definition 3.2 for this notation). We will always assume that ideals of $C^*$-algebras are closed and two-sided. Let $B(G)$ denote the Fourier-Stieltjes algebra, which we identify with the dual of $C^*(G)$. We give $B(G)$ the usual $C^*(G)$-bimodule structure: for $a,b \in C^*(G)$ and $f \in B(G)$ we define
\[
\langle b, a \cdot f \rangle = \langle ba, f \rangle \quad \text{and} \quad \langle b, f \cdot a \rangle = \langle ab, f \rangle.
\]
This bimodule structure extends to an $M(C^*(G))$-bimodule structure, because for $m \in M(C^*(G))$ and $f \in B(G)$ the linear functionals $a \mapsto \langle am, f \rangle$ and $a \mapsto \langle ma, f \rangle$ on $C^*(G)$ are bounded. Regarding $G$ as canonically embedded in $M(C^*(G))$, the associated $G$-bimodule structure on $B(G)$ is given by
\[
(x \cdot f)(y) = f(yx) \quad \text{and} \quad (f \cdot x)(y) = f(xy)
\]
for $x,y \in G$ and $f \in B(G)$. 
A quotient $C^*(G)/I$ is uniquely determined by the annihilator $E = I^\perp$ in $B(G)$, which is a weak*-closed subspace. We find it convenient to work in terms of $E$ rather than $I$, keeping in mind that we will have $I = ^\perp E$, the preannihilator in $C^*(G)$. First we record the following well-known property:

**Lemma 3.1.** For any weak*-closed subspace $E$ of $B(G)$, the following are equivalent:

1. $^\perp E$ is an ideal;
2. $E$ is a $C^*(G)$-subbimodule;
3. $E$ is $G$-invariant.

*Proof.* (1)$\iff$(2) follows from, e.g., [Ped79, Theorem 3.10.8], and (2)$\iff$(3) follows by integration. □

**Definition 3.2.** If $E$ is a weak*-closed $G$-invariant subspace of $B(G)$, let $C^*_E(G)$ denote the quotient $C^*(G)/^\perp E$.

Note that the above definition makes sense, by Lemma 3.1.

**Example 3.3.** Of course we have $C^*(G) = C^*_{B(G)}(G)$.

Also,

$$C^*_r(G) = C^*_{B_r(G)}(G),$$

where $B_r(G)$ is the regular Fourier-Stieltjes algebra of $G$, because if $\lambda : C^*(G) \to C^*_r(G)$ denotes the regular representation of $G$ then

$$(\ker \lambda)^\perp = B_r(G).$$

Recall for later use that the intersection $C_c(G) \cap B(G)$ is norm-dense in the Fourier algebra $A(G)$ (for the norm of functionals on $C^*(G)$), and is weak*-dense in $B_r(G)$ [Eym64].

**Remark 3.4.** If $E$ is a weak*-closed $G$-invariant subspace of $B(G)$, and $q : C^*(G) \to C^*_E(G)$ is the quotient map, then the dual map $q^* : C_*^*(G)^* \to C^*(G)^* = B(G)$ is an isometric isomorphism onto $E$, and we identify $E = C^*_E(G)^*$ and regard $q^*$ as an inclusion map.

Inspired in part by [BG], we pause here to give another construction of the quotients $C^*_E(G)$:

1. Start with a $G$-invariant, but not necessarily weak*-closed, subspace $E$ of $B(G)$.
(2) Call a representation \(U\) of \(G\) on a Hilbert space \(H\) an \(E\)-representation if there is a dense subspace \(H_0\) of \(H\) such that the matrix coefficients
\[
x \mapsto \langle U_x \xi, \eta \rangle
\]
are in \(E\) for all \(\xi, \eta \in H_0\).

(3) Define a \(C^*\)-seminorm \(\| \cdot \|_E\) on \(C^c(G)\) by
\[
\| f \|_E = \sup \{ \| U(f) \| : U \text{ is an } E\text{-representation of } G \}.
\]

The following lemma is presumably well-known, but we include a proof for the convenience of the reader.

**Lemma 3.5.** With the above notation, let \(I\) be the ideal of \(C^*(G)\) given by

\[
I = \{ a \in C^*(G) : \| a \|_E = 0 \}.
\]

Then:

1. \(I = ^{-}E\).
2. The weak*-closure \(\overline{E}\) of \(E\) in \(B(G)\) is \(G\)-invariant, and \(C^*_E(G) = C^*(G)/I\) is the Hausdorff completion of \(C^c(G)\) in the seminorm \(\| \cdot \|_E\).
3. If \(E\) is an ideal or a subalgebra of \(B(G)\), then so is \(\overline{E}\).

**Proof.** (1) To show that \(I \subset ^{-}E\), let \(a \in I\) and \(f \in E\). Since \(f \in B(G)\), we can choose a representation \(U\) of \(G\) on a Hilbert space \(H\) and vectors \(\xi, \eta \in H\) such that
\[
f(x) = \langle U_x \xi, \eta \rangle \quad \text{for } x \in G.
\]
Let \(K_0\) be the smallest \(G\)-invariant subspace of \(H\) containing both \(\xi\) and \(\eta\), and let \(K = \overline{K_0}\). Then \(K\) is a closed \(G\)-invariant subspace of \(H\), so determines a subrepresentation \(\rho\) of \(G\). For every \(\zeta, \kappa \in K_0\), the function \(x \mapsto \langle U_x \zeta, \kappa \rangle\) is in \(E\) because \(E\) is \(G\)-invariant. Thus \(\rho\) is an \(E\)-representation. We have
\[
|\langle a, f \rangle| = |\langle \rho(a) \xi, \eta \rangle| \\
\leq \|\rho(a)\| \|\xi\| \|\eta\| \\
\leq \|a\|_E \|\xi\| \|\eta\| \\
= 0.
\]
Thus \(a \in ^{-}E\).

For the opposite containment, suppose by way of contradiction that we can find \(a \in ^{-}E \setminus I\). Then \(\|a\|_E \neq 0\), so we can also choose an \(E\)-representation \(U\) of \(G\) on a Hilbert space \(H\) such that \(U(a) \neq 0\). Let \(H_0\) be a dense subspace of \(H\) such that for all \(\xi, \eta \in H_0\) the function
$x \mapsto \langle U_x \xi, \eta \rangle$ is in $E$. By density we can choose $\xi, \eta \in H_0$ such that $\langle U(a) \xi, \eta \rangle \neq 0$. Then $g(x) = \langle U_x \xi, \eta \rangle$ defines an element $g \in E$, and we have

$$\langle a, g \rangle = \langle U(a) \xi, \eta \rangle \neq 0,$$

which is a contradiction. Therefore $\perp E \subset I$, as desired.

(2) Since $I = \perp E$ we have $E = I \perp$, which is $G$-invariant because $I$ is an ideal, by Lemma 3.1. We have $I = \perp E$, so $C^*_E(G) = C^*(G)/I$ by Definition 3.2. Since $C_c(G)$ is dense in $C^*(G)$, the result now follows by the definition of $I$ in (3.1).

(3) This follows immediately from separate weak*-continuity of multiplication in $B(G)$. This is a well-known property of $B(G)$, but we include the brief proof here for completeness: the bimodule action of $B(G)$ on the enveloping algebra $W^*(G) = B(G)^*$, given by

$$\langle a \cdot f, g \rangle = \langle a, fg \rangle = \langle f \cdot a, g \rangle \quad \text{for } a \in W^*(G), f, g \in B(G),$$

leaves $C^*(G)$ invariant, because it satisfies the submultiplicativity condition $\|a \cdot f\| \leq \|a\| \|f\|$ on norms and leaves $C_c(G) \subset C^*(G)$ invariant. Thus, if $f_i \to 0$ weak* in $B(G)$ and $g \in B(G)$, then for all $a \in C^*(G)$ we have

$$\langle a, f_i g \rangle = \langle a \cdot g, f_i \rangle \to 0.$$

□

Corollary 3.6.

(1) A representation $U$ of $G$ is an $E$-representation if and only if, identifying $U$ with the corresponding representation of $C^*(G)$, we have $\ker U \supset \perp E$.

(2) A nondegenerate homomorphism $\tau : C^*(G) \to M(A)$, where $A$ is a $C^*$-algebra, factors through a homomorphism of $C^*_E(G)$ if and only if

$$\overline{\omega} \circ \tau \in \overline{E} \quad \text{for all } \omega \in A^*,$$

where again $\overline{E}$ denotes the weak*-closure of $E$.

Proof. This follows readily from Lemma 3.5. □

Remark 3.7. In light of Lemma 3.5 if we have a $G$-invariant subspace $E$ of $B(G)$ that is not necessarily weak*-closed, it makes sense to, and we shall, write $C^*_E(G)$ for $C^*_E(G)$. However, whenever convenient we can replace $E$ by its weak*-closure, giving the same quotient $C^*_E(G)$.

Observation 3.8. By Lemma 3.5, if $E$ is a $G$-invariant subspace of $B(G)$ then:

(1) $C^*_E(G) = C^*(G)$ if and only if $E$ is weak*-dense in $B(G)$.

(2) $C^*_E(G) = C^*_r(G)$ if and only if $E$ is weak*-dense in $B_r(G)$. 
We record an elementary consequence of our definitions:

**Lemma 3.9.** For a weak*-closed $G$-invariant subspace $E$ of $B(G)$, the following are equivalent:

1. $\perp E \subset \ker \lambda$;
2. $E \supset B_r(G)$;
3. $E \supset A(G)$;
4. $E \supset (C_c(G) \cap B(G))$;
5. there is a (unique) homomorphism $\rho : C^*_E(G) \to C^*_r(G)$ making the diagram commute.

**Definition 3.10.** For a weak*-closed $G$-invariant subspace $E$ of $B(G)$, we say the quotient $C^*_E(G)$ is a group $C^*$-algebra of $G$ if the above equivalent conditions (1)–(4) are satisfied. If $B_r(G) \subsetneq E \neq B(G)$ we say the group $C^*$-algebra is exotic.

We will see in Proposition 5.1 that if $G$ is discrete then a quotient $C^*_E(G)$ is a group $C^*$-algebra of $G$ if and only if it is topologically graded in Exel’s sense [Exe97, Definition 3.4].

We are especially interested in group $C^*$-algebras that carry a coaction or a comultiplication. We will need the following result, which is folklore among coaction cognoscenti:

**Lemma 3.11.** If $\delta : A \to M(A \otimes C^*(G))$ is a coaction of $G$ on a $C^*$-algebra $A$ and $I$ is an ideal of $A$, then the following are equivalent:

1. there is a coaction $\tilde{\delta}$ on $A/I$ making the diagram commute (where $q$ is the quotient map);
2. $I \subset \ker q \otimes \text{id} \circ \delta$.
3. $I^\perp$ is a $B(G)$-submodule of $A^*$.
Proof. This is well-known, but difficult to find in the literature, so we include the brief proof for the convenience of the reader. There exists a homomorphism \( \tilde{\delta} \) making the diagram (3.2) commute if and only if (2) holds, and in that case \( \tilde{\delta} \) will satisfy the coaction-nondegeneracy (2.1) and the coaction identity (2.2). By Lemma 2.2 this implies that \( \tilde{\delta} \) is a coaction. Thus (1) \( \iff \) (2), and (2) \( \iff \) (3) follows from a routine calculation using the fact that \( \{ \psi \otimes f : \psi \in (A/I)^*, f \in B(G) \} \) separates the elements of \( M(A/I \otimes C^*(G)) \). \( \Box \)

Recall that the multiplication in \( B(G) \) satisfies
\[
\langle a, fg \rangle = \langle \delta_G(a), f \otimes g \rangle \quad \text{for } a \in C^*(G) \text{ and } f, g \in B(G),
\]
where here we use the notation \( f \otimes g \) to denote the functional in \( (C^*(G) \otimes C^*(G))^* \) determined by
\[
\langle x \otimes y, f \otimes g \rangle = f(x)g(y) \quad \text{for } x, y \in G.
\]

Remark 3.12. Note that we need to explicitly state the above convention for \( f \otimes g \), since we are using the minimal tensor product: if \( G \) is a group for which the canonical surjection \( C^*(G) \otimes \max C^*(G) \to C^*(G) \otimes C^*(G) \) is noninjective\(^1\), then
\[
C^*(G) \otimes C^*(G) \neq C^*(G \times G)
\]
\[
(C^*(G) \otimes C^*(G))^* \neq B(G \times G),
\]
because \( C^*(G \times G) = C^*(G) \otimes \max C^*(G) \).

Corollary 3.13. Let \( E \) be a weak*-closed \( G \)-invariant subspace of \( B(G) \), and let \( q : C^*(G) \to C^*_E(G) \) be the quotient map. Then there is a coaction \( \delta^E_G \) of \( G \) on \( C^*_E(G) \) such that
\[
\delta^E_G(q(x)) = q(x) \otimes x \quad \text{for } x \in G
\]
if and only if \( E \) is an ideal of \( B(G) \).

Proof. Since \( E \) is the annihilator of \( \ker q \), this follows immediately from Lemma 3.11. \( \Box \)

Recall that in Definition 3.10 we called \( C^*_E(G) \) a group \( C^* \)-algebra if \( E \) is a weak*-closed \( G \)-invariant subspace of \( B(G) \) containing \( B_r(G) \); this latter property is automatic if \( E \) is an ideal (as long as it’s nonzero):

\[^1\text{e.g., any infinite simple group with property T — see [BO08, Theorem 6.4.14 and Remark 6.4.15]}\]
Lemma 3.14. Every nonzero norm-closed $G$-invariant ideal of $B(G)$ contains $A(G)$, and hence every nonzero weak*-closed $G$-invariant ideal of $B(G)$ contains $B_r(G)$.

Proof. Let $E$ be the ideal. It suffices to show that $E \cap A(G)$ is norm dense in $A(G)$. There exist $t \in G$ and $f \in E$ such that $f(t) \neq 0$. By [Eym64, Lemma 3.2] there exists $g \in A(G) \cap C_c(G)$ such that $g(t) \neq 0$, and then $fg \in E \cap C_c(G)$ is nonzero at $t$. By $G$-invariance of $E$, for all $x \in G$ there exists $f \in E$ such that $f(x) \neq 0$. Then for any $y \neq x$ we can find $g \in A(G) \cap C_c(G)$ such that $g(x) \neq 0$ and $g(y) = 0$, and so $fg \in E$ is nonzero at $x$ and zero at $y$. Thus $E \cap A(G)$ is an ideal of $A(G)$ that is nowhere vanishing on $G$ and separates points, so by [Eym64, Corollary 3.38] $E \cap A(G)$ is norm dense in $A(G)$, so we are done. \hfill \square

Recall that a comultiplication on a $C^*$-algebra $A$ is a homomorphism (which we do not in general require to be injective) $\Delta : A \to M(A \otimes A)$ satisfying the co-associativity property
\[ \Delta \otimes \text{id} \circ \Delta = \text{id} \otimes \Delta \circ \Delta \]
and the nondegeneracy properties
\[ \overline{\text{span}}\{\Delta(A)(1 \otimes A)\} = A \otimes A = \overline{\text{span}}\{(A \otimes 1)\Delta(A)\}. \]
A $C^*$-algebra with a comultiplication is called a $C^*$-bialgebra (see [Kaw08] for this terminology). A comultiplication $\Delta$ on $A$ is used to make the dual space $A^*$ into a Banach algebra in the standard way:
\[ \omega \psi := \omega \otimes \psi \circ \Delta \quad \text{for} \quad \omega, \psi \in A^*. \]

The following is another folklore result, proved similarly to Lemma 3.11:

Lemma 3.15. If $\Delta : A \to M(A \otimes A)$ is a comultiplication on a $C^*$-algebra $A$ and $I$ is an ideal of $A$, then the following are equivalent:

1. there is a comultiplication $\tilde{\Delta}$ on $A/I$ making the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{\Delta} & M(A \otimes A) \\
\downarrow q & & \downarrow q \otimes q \\
A/I & \xrightarrow{\tilde{\Delta}} & M(A/I \otimes A/I)
\end{array}
\]
commute (where $q$ is the quotient map);
2. $I \subset \ker q \otimes q \circ \Delta$.
3. $I^\perp$ is a subalgebra of $A^*$.

We apply this to the canonical comultiplication $\delta_G$ on $C^*(G)$:
Proposition 3.16. Let $E$ be a weak*-closed $G$-invariant subspace of $B(G)$, and let $q : C^*(G) \to C^*_E(G)$ be the quotient map. Then the following are equivalent:

1. there is a comultiplication $\Delta$ making the diagram

\[
\begin{array}{ccc}
C^*(G) & \xrightarrow{\delta_G} & M(C^*(G) \otimes C^*(G)) \\
\downarrow q & & \downarrow \overline{q \otimes q} \\
C^*_E(G) & \xrightarrow{\Delta} & M(C^*_E(G) \otimes C^*_E(G))
\end{array}
\]

commute;

2. $\perp E \subset \ker q \otimes q \circ \delta_G$;

3. $E$ is a subalgebra of $B(G)$.

Remark 3.17. Proposition 3.16 tells us that if $E$ is a weak*-closed $G$-invariant subalgebra of $B(G)$, then the group algebra $C^*_E(G)$ is a $C^*$-bialgebra. However, this probably does not make $C^*_E(G)$ a locally compact quantum group, since this would require an antipode. It might be difficult to investigate the general question of whether there exists some antipode on $C^*_E(G)$ that is compatible with the comultiplication; it seems more reasonable to ask whether the quotient map $q : C^*(G) \to C^*_E(G)$ takes the canonical antipode on $C^*(G)$ to an antipode on $C^*_E(G)$. This requires $E$ to be closed under inverse i.e., if $f \in E$ then so is the function $f^\vee$ defined by $f^\vee(x) = f(x^{-1})$. Now, $f^\vee(x) = \overline{f^*(x)}$ where $f^*$ is defined by $f^*(a) = f(a^*)$ for $a \in C^*(G)$. Since $f \in E$ if and only if $f^* \in E$, we see that $E$ is invariant under $f \mapsto f^\vee$ if and only if it is invariant under complex conjugation. In all our examples (in particular Section 4) $E$ has this property. Note that $C^*_E(G)$ always has a Haar weight, since we can compose the canonical Haar weight on $C^*_r(G)$ with the quotient map $C^*_r(G) \to C^*_E(G)$. However, this Haar weight on $C^*_E(G)$ is faithful if and only if $E = B_r(G)$.

Remark 3.18. By Lemma 3.5, if $E$ is a $G$-invariant ideal of $B(G)$ and $I = \perp E$, then $E$ is also a $G$-invariant ideal, so by Proposition 3.13 there is a coaction $\delta^E_G$ of $G$ on $C^*_E(G)$ such that

\[
\delta^E_G(q(x)) = q(x) \otimes x \quad \text{for } x \in G,
\]

where $q : C^*(G) \to C^*_E(G)$ is the quotient map.

Similarly, if $E$ is a $G$-invariant subalgebra of $B(G)$ then $\overline{E}$ is also a $G$-invariant subalgebra, so by Proposition 3.16 there is a comultiplication $\Delta$ on $C^*_E(G)$ such that

\[
\Delta(q(x)) = q(x) \otimes q(x) \quad \text{for } x \in G.
\]
Example 3.19. Note that if the quotient \( C^*_E(G) \) is a group \( C^* \)-algebra, then the quotient map \( q : C^*(G) \to C^*_E(G) \) is faithful on \( C_c(G) \), and so by Lemma 3.5 \( C^*_E(G) \) is the completion of \( C_c(G) \) in the associated norm \( \| \cdot \|_E \). However, \( q \) being faithful on \( C_c(G) \) is not sufficient for \( C^*_E(G) \) to be a group \( C^* \)-algebra. The simplest example of this is in [FD88, Exercise XI.38] (which we modify only slightly): let \( 0 \leq a < b < 2\pi \), and define a surjection

\[
q(n)(t) = e^{int}.
\]

Then the unitaries \( q(n) \) are linearly independent, so \( q \) is faithful on \( c_c(\mathbb{Z}) \), but \( q(C^*(\mathbb{Z})) \) is not a group \( C^* \)-algebra because \( \ker q \) is a non-trivial ideal of \( C^*(\mathbb{Z}) \) and \( \mathbb{Z} \) is amenable, so that \( \ker \lambda = \{0\} \).

Example 3.20. The paper [EQ99] shows how to construct exotic group \( C^* \)-algebras \( C^*_E(G) \) (see also [KS, Remark 9.6] for similar exotic quantum groups) with no coaction: let

\[
q = \lambda \oplus 1_G,
\]

where \( 1_G \) denotes the trivial 1-dimensional representation of \( G \). The quotient \( C^*_E(G) \) is a group \( C^* \)-algebra since \( \ker q = \ker \lambda \cap \ker 1_G \). On the other hand, we have

\[
E = (\ker q)^\perp = B_r(G) + \mathbb{C}1_G,
\]

which is not an ideal of \( B(G) \) unless it is all of \( B(G) \), i.e., unless \( q \) is faithful; as remarked in [EQ99], this behavior would be quite bizarre, and in fact we do not know of any discrete nonamenable group with this property.

However, these quotients \( C^*_E(G) \) are \( C^* \)-bialgebras, because \( B_r(G) + \mathbb{C}1_G \) is a subalgebra of \( B(G) \). Thus, these quotients give examples of exotic group \( C^* \)-bialgebras that are different from those in [BG, Proposition 4.4 and Remark 4.5]. It is interesting to note that these quotients of \( C^*(G) \) are of a decidedly elementary variety: by Lemma 2.1 we have

\[
C^*_E(G) = C^*_r(G) \oplus \mathbb{C},
\]

because \( C^*(G) = \ker \lambda + \ker 1_G \) since \( G \) is nonamenable. To see this latter implication, recall that if \( G \) is nonamenable then \( 1_G \) is not weakly contained in \( \lambda \), so \( \ker 1_G \not\subset \ker \lambda \), and hence \( C^*(G) = \ker \lambda + \ker 1_G \) since \( \ker 1_G \) is a maximal ideal.

Valette has a similar example in [Val84, Theorem 3.6] where he shows that if \( N \) is a closed normal subgroup of \( G \) that has property \( (T) \), then \( C^*(G) \) is the direct sum of \( C^*(G/N) \) and a complementary ideal.
For a different source of exotic group $C^*$-bialgebras, see Example \[3.22\].

**Example 3.21.** We can also find examples of group $C^*$-algebras with no comultiplication: modify the preceding example by taking
\[ q = \lambda \oplus \gamma, \]
where $\gamma$ is a nontrivial character of $G$ (assuming that $G$ has such characters). Then
\[ (\ker q)^\perp = B_r(G) + C\gamma, \]
which is not a subalgebra of $B(G)$ when $G$ is nonamenable.

**Example 3.22.** Let $G$ be a locally compact group for which the canonical surjection
\[ (3.3) \quad C^*(G) \otimes_{\max} C^*(G) \to C^*(G) \otimes C^*(G) \]
is not injective, where in the second tensor product we use the minimal $C^*$-tensor norm as usual (see Remark \[3.12\]). Let $I$ denote the kernel of this map. Since the algebraic product $B(G) \odot B(G)$ is weak*-dense in $(C^*(G) \otimes C^*(G))^*$, the annihilator $E = I^\perp$ is the weak*-closed span of functions of the form
\[ (x, y) \mapsto f(x)g(y) \quad \text{for } f, g \in B(G). \]
This is clearly a subalgebra, but not an ideal, because it contains 1. Also, $E \supset B_r(G \times G)$ because the surjection (3.3) can be followed by
\[ C^*(G) \otimes C^*(G) \to C_r^*(G) \otimes C_r^*(G) \cong C_r^*(G \times G). \]
Thus the canonical coaction $\delta_{G \times G}$ of $G \times G$ on $C^*(G \times G)$ descends to a comultiplication on the group $C^*$-algebra $C_r^*(G \times G) \cong C^*(G) \otimes C^*(G)$, but not to a coaction of $G \times G$.

4. **Classical ideals**

We continue to let $G$ be an arbitrary locally compact group.

We will apply the theory of the preceding sections to group $C^*$-algebras $C_r^*(G)$ with $E$ of the form
\[ E = D \cap B(G), \]
where $D$ is some familiar $G$-invariant set of functions on $G$.

**Notation 4.1.** If $D$ is a $G$-invariant set of functions on $G$, we write $\|f\|_D = \|f\|_{D \cap B(G)}$, and similarly $C^*_D(G) = C^*_D(G \cap B(G))$. 
So, for instance, we can consider $C^*_c(G)$, $C^*_{c_0}(G)$, and $C^*_{L^p(G)}(G)$. In each of these cases the intersection $E = D \cap B(G)$ is a $G$-invariant ideal of $B(G)$, so by Remark 3.18 and Lemma 3.14 these quotients are all group $C^*$-algebras carrying coactions of $G$, and hence by Proposition 3.16 they carry comultiplications. In the case that $G$ is discrete, $C^*_c(G)$, $C^*_0(G)$, and $\ell^p(G)$ could be regarded as classical ideals of $\ell^\infty(G)$; this is the context of Brown and Guentner’s “new completions of discrete groups” [BG].

We have

$$C^*_c(G) = C^*_{A(G)}(G) = C^*_r(G),$$

because $C_c(G) \cap B(G)$ is norm dense in $A(G)$, and hence weak*-dense in $B_r(G)$. However, the quotients $C^*_{c_0}(G)$ and $C^*_{L^p(G)}(G)$ are more mysterious. Nevertheless, we have the following (which, for the case of discrete $G$, is [BG, Proposition 2.11]):

**Proposition 4.2.** For all $p \leq 2$ we have $C^*_{L^p(G)}(G) = C^*_r(G)$.

**Proof.** Since $L^p(G) \cap B(G)$ consists of bounded functions, for $p \leq 2$ we have

$$C_c(G) \cap B(G) \subset L^p(G) \cap B(G) \subset L^2(G) \cap B(G).$$

Now, if $U$ is a representation of $G$ having a cyclic vector $\xi$ such that the function $x \mapsto \langle U_x \xi, \xi \rangle$ is in $L^2(G)$, then $U$ is contained in $\lambda$ (see, e.g., [Car76]), and consequently $L^2(G) \cap B(G) \subset A(G)$. Thus

$$B_r(G) = \overline{C_c(G) \cap B(G)}^{\text{weak}^*} \subset \overline{L^p(G) \cap B(G)}^{\text{weak}^*} \subset \overline{L^2(G) \cap B(G)}^{\text{weak}^*} \subset \overline{A(G)}^{\text{weak}^*} = B_r(G),$$

and the result follows. \(\square\)

**Remark 4.3.** (1) The proof of Proposition 4.2 is much easier when $G$ is discrete, because then for $\xi \in \ell^2(G)$ we have

$$\xi(x) = \langle \lambda_x \chi(e), \overline{\xi} \rangle,$$

so $\ell^2(G) \subset A(G)$.

(2) In general, $C^*_0(G) \cap B(G) \subset B_r(G)$, and the containment can be proper (for perhaps the earliest result along these lines, see [Men16]). When $G$ is discrete, this phenomenon occurs...
precisely when \( G \) is a-T-menable but nonamenable, by the result of [BG] mentioned in the introduction.

(3) Using the method outlined in this section, if we start with a \( G \)-invariant ideal \( D \) of \( L^\infty(G) \) and put \( E = D \cap B(G)^{\text{weak}^*} \), we get many weak*-closed ideals of \( B(G) \), but probably not all. For example, if we let \( z_F \) be the supremum in the universal enveloping von Neumann algebra \( W^*(G) = C^*(G)^{**} \) of the support projections of finite dimensional representations of \( G \), then it follows from [Wal75, Proposition 1, Theorem 2, Proposition 8] that \( (1-z_F) \cdot B(G) \) is an ideal of \( B(G) \) and \( z_F \cdot B(G) = AP(G) \cap B(G) \) is a subalgebra. It seems unlikely that for all locally compact groups \( G \) the ideal \( (1-z_F) \cdot B(G) \) arises as an intersection \( D \cap B(G) \) for an ideal \( D \) of \( L^\infty(G) \).

5. Graded algebras

In this short section we impose the condition that the group \( G \) is discrete. We made this a separate section for the purpose of clarity — here the assumptions on \( G \) are different from everywhere else in this paper. [Exe97, Definition 3.1] and [FD88, VIII.16.11–12] define \( G \)-graded \( C^* \)-algebras as certain quotients of Fell-bundle algebras. When the fibres of the Fell bundle are 1-dimensional, each one consists of scalar multiplies of a unitary. When these unitaries can be chosen to form a representation of \( G \), the \( C^* \)-algebra is a quotient \( C^*_E(G) \).

The following can be regarded as a special case of [Exe97, Theorem 3.3]:

**Proposition 5.1.** Let \( E \) be a weak*-closed \( G \)-invariant subspace of \( B(G) \), and let \( q : C^*(G) \to C^*_E(G) \) be the quotient map. Then the following are equivalent:

1. \( C^*_E(G) \) is a group \( C^* \)-algebra in the sense of Definition 3.10;
2. there is a bounded linear functional \( \omega \) on \( C^*_E(G) \) such that
   \[
   \omega(q(x)) = \begin{cases} 
   1 & \text{if } x = e \\
   0 & \text{if } x \neq e;
   \end{cases}
   \]
3. \( E \) contains the canonical trace \( \text{tr} \) on \( C^*(G) \);
4. \( E \supset B_r(G) \);

\[\text{Exe97, FD88}\] would require the images of the fibres to be linearly independent.
(5) there is a (unique) homomorphism $\rho : C^*_E(G) \to C^*_r(G)$ making the diagram

\[
\begin{array}{ccc}
C^*_r(G) & \xrightarrow{\delta} & M(C^*_r(G) \otimes C^*_r(G)) \\
\downarrow{q} & & \downarrow{q \otimes \text{id}} \\
C^*_E(G) & \xrightarrow{\delta_E} & M(C^*_E(G) \otimes C^*_r(G))
\end{array}
\]

commute.

Proof. Assuming (2), the composition $\omega \circ q$ coincides with $\text{tr}$, so $\text{tr} \in E$, and conversely if $\text{tr} \in E$ then we get a suitable $\omega$. Thus (2) $\Leftrightarrow$ (3).

For the rest, just note that $B_r(G) = (\ker \lambda) ^\perp$ is the weak*-closed $G$-invariant subspace generated by $\text{tr} = \chi_{\{e\}}$, and appeal to Lemma 3.9.

Remark 5.2. Condition (2) in Proposition 5.1 is precisely what Exel’s [Exe97, Definition 3.4] would require to say that $C^*_E(G)$ is topologically graded.

6. Exotic coactions

We return to the context of an arbitrary locally compact group $G$.

The coactions appearing in noncommutative crossed-product duality come in a variety of flavors: reduced vs. full (see [EKQR06, Appendix] or [HQRW11], for example), and, among the full ones, a spectrum with normal and maximal coactions at the extremes (see [EKQ04], for example). In this concluding section we briefly propose a new program in crossed-product duality: “exotic coactions”, involving the exotic group $C^*$-algebras $C^*_E(G)$ in the sense of Definition 3.10. From now until Proposition 6.16 we are concerned with nonzero $G$-invariant weak*-closed ideals $E$ of $B(G)$.

By Lemmas 3.9 and 3.14 the quotient $C^*_E(G) = C^*_r(G) / ^\perp E$ is a group $C^*$-algebra. By Proposition 3.13, there is a coaction $\delta^E_G$ of $G$ on $C^*_E(G)$ making the diagram

\[
\begin{array}{ccc}
C^*_r(G) & \xrightarrow{\delta} & M(C^*_r(G) \otimes C^*_r(G)) \\
\downarrow{q} & & \downarrow{q \otimes \text{id}} \\
C^*_E(G) & \xrightarrow{\delta^E_G} & M(C^*_E(G) \otimes C^*_r(G))
\end{array}
\]
commute, where $q$ is the quotient map, and by Proposition 3.16 there is a quotient comultiplication $\Delta$ on $C^*_E(G)$. Recall that we defined the exotic group $C^*$-algebras to be the ones strictly between the two extremes $C^*(G)$ and $C^*_r(G)$, corresponding to $E = B(G)$ and $E = B_r(G)$, respectively.

On one level, we could try to study coactions of Hopf $C^*$-algebras associated to the locally compact group $G$ other than $C^*(G)$ and $C^*_r(G)$. However, there is an inconvenient subtlety here (see Remark 3.17). However, there is a deeper level to this program, relating more directly to crossed-product duality. At the deepest level, we aim for a characterization of all coactions of $G$ in terms of the quotients $C^*_E(G)$. We hasten to emphasize that at this time some of the following is speculative, and is intended merely to outline a program of study.

From now on, the unadorned term “coaction” will refer to a full coaction of $G$ on a $C^*$-algebra $A$.

Let $\psi : (A^m, \delta^m) \to (A, \delta)$ be the maximalization of $\delta$, so that $\delta^m$ is a maximal coaction, $\psi : A^m \to A$ is an equivariant surjection, and the crossed-product surjection

$$
\psi \times G : A^m \times_{\delta^m} G \to A \rtimes \delta G
$$

(for the existence of which, see [EKQR06, Lemma A.46], for example) is an isomorphism. Since $\delta^m$ is maximal, the canonical surjection

$$
\Phi : A^m \times_{\delta^m} G \times_{\delta^m} G \to A^m \otimes K(L^2(G))
$$

is an isomorphism (this is “full-crossed-product duality”). Blurring the distinction between $A^m \times_{\delta^m} G$ and the isomorphic crossed product $A \rtimes_{\delta} G$, and recalling that $\psi \times G : A^m \times_{\delta^m} G \to A \rtimes \delta G$ is $\delta^m - \delta$ equivariant, we can regard $\Phi$ as an isomorphism

$$
A \rtimes \delta G \times_{\delta} G \xrightarrow{\Phi} A^m \otimes K(L^2(G)).
$$

We have a surjection

$$
\psi \otimes \text{id} : A^m \otimes K(L^2(G)) \to A \otimes K(L^2(G)),
$$

whose kernel is $(\ker \psi) \otimes K(L^2(G))$ since $K(L^2(G))$ is nuclear. Let $K_\delta$ be the inverse image under $\Phi$ of this kernel, giving an ideal of $A \rtimes \delta G \rtimes_{\delta} G$ and an isomorphism $\Phi_\delta$ making the diagram

\[
\begin{array}{ccc}
A \rtimes \delta G \rtimes_{\delta} G & \xrightarrow{\Phi} & A^m \otimes K(L^2(G)) \\
\downarrow & & \downarrow \psi \otimes \text{id} \\
(A \rtimes \delta G \rtimes_{\delta} G) / K_\delta & \xrightarrow{\cong} & A \otimes K(L^2(G))
\end{array}
\]

(6.1)
commute, where \( Q \) is the quotient map. Adapting the techniques of [EQ02, Theorem 3.7], it is not hard to see that \( K_\delta \) is contained in the kernel of the regular representation \( \Lambda : A \rtimes_\delta G \rtimes_\delta G \to A \rtimes_\delta G \rtimes_{\delta,r} G \).

If \( \delta \) is maximal, then diagram 6.1 collapses to a single row. On the other hand, if \( \delta \) is normal, then \( Q \) is the regular representation \( \Lambda \) and in particular

\[
(A \rtimes_\delta G \rtimes_\delta G)/K_\delta = A \rtimes_\delta G \rtimes_{\delta,r} G.
\]

(In this case the isomorphism \( \Phi_\delta \) is “reduced-crossed-product duality”.)

With the ultimate goal (which at this time remains elusive — see Conjectures 6.12 and 6.14) of achieving an “\( E \)-crossed-product duality”, intermediate between full- and reduced-crossed-product dualities, below we will propose tentative definitions of “\( E \)-crossed-product duality” and “\( E \)-crossed products” \( B \rtimes_{\alpha,E} G \) by actions \( \alpha : G \to \text{Aut} B \), and we will prove that they have the following properties:

1. A coaction satisfies \( B(G) \)-crossed-product duality if and only if it is maximal.
2. A coaction satisfies \( B_r(G) \)-crossed-product duality if and only if it is normal.
3. \( B \rtimes_{\alpha,B(G)} G = B \rtimes_{\alpha} G \).
4. \( B \rtimes_{\alpha,B_r(G)} G = B \rtimes_{\alpha,r} G \).
5. The dual coaction \( \hat{\alpha} \) on the full crossed product \( B \rtimes_\alpha G \) satisfies \( B(G) \)-crossed-product duality.
6. The dual coaction \( \hat{\alpha}^n \) on the reduced crossed product \( B \rtimes_{\alpha,r} G \) satisfies \( B_r(G) \)-crossed-product duality.
7. In general, \( B \rtimes_{\alpha,E} G \) is a quotient of \( B \rtimes_\alpha G \) by an ideal contained in the kernel of the regular representation \( \Lambda : B \rtimes_\alpha G \to B \rtimes_{\alpha,r} G \).
8. There is a dual coaction \( \hat{\alpha}_E \) of \( G \) on \( B \rtimes_{\alpha,E} G \).

**Definition 6.1.** Define an ideal \( J_{\alpha,E} \) of the crossed product \( B \rtimes_\alpha G \) by

\[
J_{\alpha,E} = \ker \text{id} \otimes q \circ \hat{\alpha},
\]

and define the \( E \)-crossed product by

\[
B \rtimes_{\alpha,E} G = (B \rtimes_\alpha G)/J_{\alpha,E}.
\]

\(^3\)This is a convenient place to correct a slip in the last paragraph of the proof of [EQ02, Theorem 3.7]: “contains” should be replaced by “is contained in” (both times).
Note that the above properties (1)–(7) are obviously satisfied (because $\hat{\alpha}$ is maximal and $\hat{\alpha}^n$ is normal), and we now verify that (8) holds as well:

**Theorem 6.2.** Let $E$ be a nonzero weak*-closed $G$-invariant ideal of $B(G)$, and let $Q : B \rtimes_{\alpha} G \to B \rtimes_{\alpha,E} G$ be the quotient map. Then there is a coaction $\hat{\alpha}_E$ making the diagram

$$
\begin{array}{ccc}
B \rtimes_{\alpha} G & \longrightarrow & M((B \rtimes_{\alpha} G) \otimes C^*(G)) \\
Q & & Q \otimes \text{id} \\
B \rtimes_{\alpha,E} G & \longrightarrow & M((B \rtimes_{\alpha,E} G) \otimes C^*(G)) 
\end{array}
$$

commute.

**Proof.** By Lemma 3.13, we must show that

$$\text{ker } Q \otimes \text{id} \circ \hat{\alpha}.$$

Let $a \in J_{\alpha,E}$, $\omega \in (B \rtimes_{\alpha,E} G)^*$, and $g \in B(G)$. Then

$$\omega \otimes g \circ Q \otimes \text{id} \circ \hat{\alpha}(a) = Q^* \omega \otimes g \circ \hat{\alpha}(a) = Q^* \omega \circ \text{id} \otimes g \circ \hat{\alpha}(a) = Q^* \omega(g \cdot a).$$

Now, since $Q^* \omega \in J^\perp_{\alpha,E}$, it suffices to show that $g \cdot a \in J_{\alpha,E}$. For $h \in E$ we have

$$h \cdot (g \cdot a) = (hg) \cdot a = (gh) \cdot a = g \cdot (h \cdot a) = 0,$$

because $h \cdot a = 0$ by Lemma 6.3 below. □

**Lemma 6.3.** With the above notation, we have:

1. $J_{\alpha,E} = \{a \in B \rtimes_{\alpha} G : E \cdot a = \{0\}\}$, and
2. $J^\perp_{\alpha,E} = \text{span}\{(B \rtimes_{\alpha} G)^* \cdot E\}$, where the closure is in the weak*-topology.

**Proof.** (1) For $a \in B \rtimes_{\alpha} G$, we have

$$a \in J_{\alpha,E} \iff \text{id} \otimes q \circ \hat{\alpha}(a) = 0 \iff \omega \otimes h \circ \text{id} \otimes q \circ \hat{\alpha}(a) = 0$$

for all $\omega \in (B \rtimes_{\alpha,E} G)^*$ and $h \in C^*_E(G)^*$

$$\omega \otimes q^* h \circ \hat{\alpha}(a) = 0$$

for all $\omega \in (B \rtimes_{\alpha,E} G)^*$ and $h \in C^*_E(G)^*$.
\[ \omega \otimes g \circ \hat{\alpha}(a) = 0 \]
for all \( \omega \in (B \rtimes_{\alpha,E} G)^* \) and \( g \in E \)
\[ \omega \circ \text{id} \otimes g \circ \hat{\alpha}(a) = 0 \]
for all \( \omega \in (B \rtimes_{\alpha,E} G)^* \) and \( g \in E \)
\[ \omega(g \cdot a) = 0 \] for all \( \omega \in (B \rtimes_{\alpha,E} G)^* \) and \( g \in E \)
\[ g \cdot a = 0 \] for all \( g \in E \).

(2) If \( a \in J_{\alpha,E}, \omega \in (B \rtimes_{\alpha} G)^* \), and \( f \in E \),
\[ (\omega \cdot f)(a) = \omega(f \cdot a) = 0, \]
so \( \omega \cdot f \in J_{\alpha,E}^\perp \), and hence the left-hand side contains the right.

For the opposite containment, it suffices to show that
\[ J_{\alpha,E} \supset J_{\alpha,E}^\perp \cdot E. \]
If \( a \in J_{\alpha,E}^\perp \cdot E \), then for all \( \omega \in (B \rtimes_{\alpha} G)^* \) and \( f \in E \) we have
\[ 0 = (\omega \cdot f)(a) = \omega(f \cdot a), \]
so \( f \cdot a = 0 \), and therefore \( a \in J_{\alpha,E} \).

**Remark 6.4.** We could define a covariant representation \((\pi, U)\) of the action \((B, \alpha)\) to be an \(E\)-representation if the representation \(U\) of \(G\) is an \(E\)-representation, and we could define an ideal \(\tilde{J}_{\alpha,E}\) of \(B \rtimes_{\alpha} G\) by
\[ \tilde{J}_{\alpha,E} = \{ a : \pi \times U(a) = 0 \} \]
for every \(E\)-representation \((\pi, U)\). It follows from Corollary 3.6 that \((\pi, U)\) is an \(E\)-representation in the above sense if and only if
\[ \overline{\omega} \circ U \in E \] for all \( \omega \in (\pi \times U(B \rtimes_{\alpha} G))^* \),
where \(i_G : C^*(G) \to \text{M}(B \rtimes_{\alpha} G)\) is the canonical nondegenerate homomorphism, and consequently
\[ \tilde{J}_{\alpha,E} = \{ \omega \in (B \rtimes_{\alpha} G)^* : \overline{\omega} \circ i_G \in E \}. \]

In the following lemma we show one containment that always holds between (6.2) and the ideal of Definition 6.1, after which we explain why these ideals do not coincide in general.

**Lemma 6.5.** With the above notation, we have
\[ \tilde{J}_{\alpha,E} \subset J_{\alpha,E}. \]

**Proof.** If \( \omega \in (B \rtimes_{\alpha} G)^* \) and \( f \in E \), then
\[ \overline{\omega} \cdot f \circ \overline{i_G} = \overline{\omega} \otimes \overline{f} \circ \overline{\hat{\alpha}} \circ \overline{i_G} \]
\[ = \overline{\omega} \otimes \overline{f} \circ \overline{i_G} \otimes \text{id} \circ \delta_G \]
\[
\begin{align*}
\omega \circ i_G & \otimes f \circ \delta_G \\
= (\omega \circ i_G) f,
\end{align*}
\]
which is in \(E\) because \(f \in E\) and \(E\) is an ideal of \(B(G)\). Thus \(\omega \cdot f \in \tilde{J}_{\alpha,E}^\perp\). \(\square\)

**Example 6.6.** To see that the inclusion of Lemma 6.5 can be proper, consider the extreme case \(E = B_r(G)\), so that \(B \rtimes_{\alpha,E} G = B \rtimes_{\alpha,r} G\). In this case \(J_{\alpha,E}\) is the kernel of the regular representation \(\Lambda : B \rtimes_{\alpha} G \to B \rtimes_{\alpha,r} G\). On the other hand, \(\tilde{J}_{\alpha,E}\) comprises the elements that are killed by every representation \(\pi \times U\) for which \(U\) is weakly contained in the regular representation \(\lambda\) of \(G\). [QS92, Example 5.3] gives an example of an action \((B,\alpha)\) having a covariant representation \((\pi,U)\) for which \(U\) is weakly contained in \(\lambda\) but \(\pi \times U\) is not weakly contained in \(\Lambda\). Thus \(\ker \pi \times U\) contains \(\tilde{J}_{\alpha,E}\) and \(J_{\alpha,E}\) has an element not contained in \(\ker \pi \times U\), so \(\tilde{J}_{\alpha,E}\) is properly contained in \(J_{\alpha,E}\) in this case.

**Definition 6.7.** We say that \(G\) is \(E\)-amenable if there are positive definite functions \(h_n\) in \(E\) such that \(h_n \to 1\) uniformly on compact sets.

**Lemma 6.8.** If \(G\) is \(E\)-amenable and \((A,G,\alpha)\) is an action, then \(J_{\alpha,E} = \{0\}\), so
\[
A \rtimes_{\alpha} G \cong A \rtimes_{\alpha,E} G.
\]

**Proof.** By Lemma 6.3, we have \(h_n \cdot a = 0\) for all \(a \in J_{\alpha,E}\). Since \(h_n \to 1\) uniformly on compact sets, it follows that \(h_n \cdot a \to a\) in norm. To see this, note that since the \(h_n\) are positive definite and \(h_n \to 1\), the sequence \(\{h_n\}\) is bounded in \(B(G)\), and certainly for \(f \in C_c(G)\) we have
\[
h_n \cdot (fa) = (h_n f)a \to f a
\]
in norm, because the pointwise products \(h_n f\) converge to \(f\) uniformly and hence in the inductive limit topology since \(\text{supp} f\) is compact. Therefore \(J_{\alpha,E} = \{0\}\). \(\square\)

**Remark 6.9.** In [BG] Section 5, Brown and Guentner study actions of a discrete group \(G\) on a unital abelian \(C^*\)-algebra \(C(X)\), and introduce the concept of a \(D\)-amenable action, where \(D\) is a \(G\)-invariant ideal of \(\ell^\infty(G)\). In particular, if \(G\) is \(D\)-amenable then every action of \(G\) is \(D\)-amenable. They show that if the action is \(D\)-amenable then \(\tilde{J}_{\alpha,E} = \{0\}\), i.e.,
\[
C_D^*(X \rtimes G) \cong C(X) \rtimes_{\alpha} G.
\]
Here we have used the notation of [BG]: $C^*_E(X \rtimes G)$ denotes the quotient of the crossed product $C(X) \rtimes \alpha G$ by the ideal $\tilde{J}_{\alpha,E}$ (although Brown and Guentner give a different, albeit equivalent, definition).

**Question 6.10.** With the above notation, form a weak*-closed $G$-invariant ideal $E$ of $B(G)$ by taking the weak*-closure of $D \cap B(G)$. Then is the stronger statement $J_{\alpha,E} = \{0\}$ true? (One easily checks it for $E = B_r(G)$, and it is trivial for $E = B(G)$.)

Note that the techniques of [BG] rely heavily on the fact that they are using ideals of $\ell^\infty(G)$, whereas our methods require ideals of $B(G)$.

**Definition 6.11.** A coaction $(A, \delta)$ satisfies $E$–crossed-product duality if

$$K_{\delta} = J_{\delta,E},$$

where $K_{\delta}$ is the ideal from (6.1) and $J_{\delta,E}$ is the ideal associated to the dual action $\hat{\delta}$ in Definition 6.1.

Thus $(A, \delta)$ satisfies $E$–crossed-product duality precisely when we have an isomorphism $\Phi_E$ making the diagram

$$
\begin{array}{ccc}
A \rtimes_{\delta} G \rtimes_{\delta} G & \xrightarrow{\Phi} & A \otimes K(L^2(G)) \\
Q \downarrow & \cong & \Phi_E \\
A \rtimes_{\delta} G \rtimes_{\hat{\delta},E} G & \end{array}
$$

commute, where $Q$ is the quotient map.

**Conjecture 6.12.** Every coaction satisfies $E$–crossed-product duality for some $E$.

**Observation 6.13.** If $E$ is an ideal of $B(G)$, then every group $C^*$-algebra $C^*_E(G)$ is an $E$-crossed product:

$$C^*_E(G) = \mathbb{C} \rtimes_{\iota,E} G,$$

where $\iota$ is the trivial action of $G$ on $\mathbb{C}$, because the kernel of the quotient map $C^*_E(G) \to C^*_E(G)$ is $\perp E$. This generalizes the extreme cases

1. $C^*_E(G) = \mathbb{C} \rtimes_{\iota} G$;
2. $C^*_E(G) = \mathbb{C} \rtimes_{\iota,r} G$.

**Conjecture 6.14.** If $(B, \alpha)$ is an action, then the dual coaction $\hat{\alpha}_E$ on the $E$-crossed product $B \rtimes_{\alpha,E} G$ satisfies $E$–crossed-product duality.

**Remark 6.15.** In particular, by Observation 6.13 Conjecture 6.14 would imply as a special case that the canonical coaction $\delta^E_G$ on the group algebra $C^*_E(G)$ satisfies $E$–crossed-product duality.
For our final result, we only require that $E$ be a weak*-closed $G$-invariant subalgebra of $B(G)$ (but not necessarily an ideal). By Proposition 3.16, $C_E^*(G)$ carries a comultiplication $\Delta$ that is a quotient of the canonical comultiplication $\delta_G$ on $C^*(G)$.

Techniques similar to those used in the proof of Theorem 6.2, taking $g \in E$ rather than $g \in B(G)$, can be used to show:

**Proposition 6.16.** Let $E$ be a weak*-closed $G$-invariant subalgebra of $B(G)$, and let $(B, \alpha)$ be an action. Then there is a coaction $\Delta_\alpha$ of the $C^*$-bialgebra $C_E^*(G)$ making the diagram

\[
\begin{array}{ccc}
B \rtimes_\alpha G & \xrightarrow{\Delta_\alpha} & M((B \rtimes_\alpha G) \otimes C^*(G)) \\
\downarrow & & \downarrow \\
B \rtimes_{\alpha, E} G & \xrightarrow{\Delta_\alpha} & M((B \rtimes_{\alpha, E} G) \otimes C_E^*(G))
\end{array}
\]

commute, where we use notation from Theorem 6.2.

We close with a rather vague query:

**Question 6.17.** What are the relationships among $E$-crossed products, $E$-coactions, and coactions of the $C^*$-bialgebra $C_E^*(G)$?

We hope to investigate this question, together with Conjectures 6.12 and 6.14 in future research.

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