Notes on the q-Analogues of the Natural Transforms and 
Some Further Applications

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Abstract

As an extension to the Laplace and Sumudu transforms the classical Natural 
transform was proposed to solve certain fluid flow problems. In this paper, we in-
vestigate q-analogues of the q-Natural transform of some special functions. We derive 
the q-analogues of the q-integral transform and further apply to some general spe-
cial functions such as: the exponential functions, the q-trigonometric functions, the 
q-hyperbolic functions and the Heaviside Function. Some further results involving 
convolutions and differentiations are also obtained.

Keywords: q-hyperbolic function; q-trigonometric function, q-Natural transform; 
Heaviside Function; Natural transform.

1 Introduction

The subject of fractional calculus (integrals and derivatives of any real or complex 
order) has gained noticeable importance and popularity due to mainly its demon-
strated applications in many seemingly diverse fields of science and engineering.
Much of the theory of fractional calculus is based upon the familiar Riemann-
Liouville fractional derivatives and integrals. Recently, there was a significant in-
crease of activity in the area of the q-calculus due to applications of the q-calculus 
in mathematics, statistics and physics.

Jackson in [12] presented a precise definition of the so-called q-Jackson integral 
and developed a q-calculus in a systematic way. Some remarkable integral trans-
forms have different q-analogues in the theory of q-calculus. Among those q-integrals 
we recall here are the q-Laplace integral transform [1, 15, 17], q-Sumudu integral
transform \cite{2} \cite{3}, Weyl fractional \( q \)-integral operator \cite{18}, \( q \)-Wavelet integral transform \cite{9}, \( q \)-Mellin integral transform \cite{9}, and few others. In this paper, we give some \( q \)-analogues of some recently investigated transform named as Natural transform and obtain some desired \( q \)-properties.

In the following section, we present some notations and terminologies from the \( q \)-calculus. In Section 3, we recall the definition and properties of the Natural transform. In Section 4, we derive the definition of the first \( q \)-analogue of the Natural transform and apply to some special functions. Sections 5-7 are devoted to some applications of the first \( q \)-analogue of the \( q \)-Natural transform to Heaviside Functions, convolutions and differentiations. The remaining two sections are investigating the second \( q \)-analogue of the \( q \)-Natural transform of some elementary functions and some applications.

2 Definitions and Preliminaries

We present some usual notions and notations used in the \( q \)-calculus see \cite{10} \cite{12} \cite{13}. Throughout this paper, we assume \( q \) to be a fixed number satisfying \( 0 < q < 1 \).

The \( q \)-calculus beings with the definition of the \( q \)-analogue \( d_q f (x) \) of the differential of functions,

\[
d_q f (x) = f (qx) - f (x).
\] (1)

Having said this, we immediately get the \( q \)-analogue of the derivative of \( f (x) \), called its \( q \)-derivative,

\[
(D_q f) (x) := \frac{d_q f (x)}{d_q x} := \frac{f (x) - f (qx)}{(1-q) x}, \quad \text{if} \; x \neq 0,
\] (2)

\( (D_q f) (0) = \dot{f} (0) \), provided \( \dot{f} (0) \) exists. If \( f \) is differentiable, then \( (D_q f) (x) \) tends to \( \dot{f} (0) \) as \( q \) tends to 1.

Notice that the \( q \)-derivative satisfies the following \( q \)-analogue of Leibnitz rule,

\[
D_q (f (x) g (x)) = g (x) D_q f (x) + f (qx) D_q g (x).
\] (3)

The \( q \)-Jackson integrals from 0 to \( x \) and from \( x \) to \( \infty \) are defined in \cite{9} \cite{12} by

\[
\int_0^x f (x) d_q x = (1-q) x \sum_0^\infty f (x q^k) q^k,
\] (4)

\[
\int_0^\infty f (x) d_q x = (1-q) x \sum_{-\infty}^\infty f (q^k) q^k,
\] (5)

provided the sum converges absolutely.
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The \(q\)-Jackson integral in a generic interval \([a, b]\) is given by in [12] as

\[
\int_a^b f(x) \, dq x = \int_0^b f(x) \, dq x - \int_0^a f(x) \, dq x.
\]  

The improper integral is defined in the way that

\[
\int_0^\infty f(x) \, dq x = (1 - q) \sum_{-\infty}^\infty f \left( \frac{q^k}{A} \right) \frac{q^k}{A}
\]  

and, for \(n \in \mathbb{Z}\), we have

\[
\int_0^\infty q^n f(x) \, dq x = \int_0^\infty f(x) \, dq x.
\]

The \(q\)-integration by parts is defined for functions \(f\) and \(g\) by

\[
\int_a^b g(x) \, D_q f(x) \, dq x = f(b) \, g(b) - f(a) \, g(a) - \int_a^b f(qx) \, D_q g(x) \, dq x.
\]

For \(x \in \mathbb{C}\), the \(q\)-shifted factorials are defined by

\[
(x; q)_0 = 1; (x, q)_t = (x; q)_\infty; (x, q)_n = \prod_{k=0}^{n-1} \left( 1 - xq^k \right)
\]

and

\[
(x; q)_\infty = \prod_{k=0}^{\infty} \left( 1 - xq^k \right),
\]

\(n = 0, 1, 2, \ldots\).

The \(q\)-analogue of \(x\) and \(\infty\) is defined by

\[
[x] = \frac{1 - q^x}{1 - q} \quad \text{and} \quad [\infty] = \frac{1}{1 - q}.
\]

The important \(q\)-analogues of the exponential function of first and second kinds are respectively given as:

\[
E_q(x) = \sum_{n=0}^{\infty} q^{\frac{a(n-1)}{2}} \frac{x^n}{[n]_q} \quad (t \in \mathbb{C}),
\]

and

\[
e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q} \quad (|t| < 1),
\]

where \([n]_q) = [n]_q [n - 1]_q \ldots [2]_q [1]_q, [n] = \frac{1 - q^n}{1 - q} = q^{n-1} + \ldots + q + 1.
However, due to the product expansions, \((e_q(x))^{-1} = E_q(-x)\) (not \(e_q(-x)\)), which explains the need of both \(q\)-analogues of the exponential function.

The \(q\)-derivative of \(E_q(x)\) is \(D_q E_q(x) = E_q(qx)\), whereas, the \(q\)-derivative of \(e_q(x)\) is \(D_q e_q(x) = e_q(x), e_q(0) = 1\).

The gamma and beta functions satisfy the \(q\)-integral representations

\[
\Gamma_q(t) = \int_0^1 x^{t-1} E_q(-qx) \, dqx \\
\text{and} \\
B_q(t; s) = \int_0^1 x^{t-1} (1 - qx)^{s-1} \, dqx, (t, s > 0)
\]

that satisfy \(B_q(t; s) = \Gamma_q(s) \Gamma_q(t) / \Gamma_q(s+t)\) and \(\Gamma_q(t+1) = [t]_q \Gamma_q(t)\). Due to (12) and (13), the \(q\)-analogues of sine and cosine functions of the second and first kinds are respectively given as:

\[
\sin_q(at) = \sum_0^\infty (-1)^n (a t^{2n+1})_q \frac{(2n+1)_q!}{(2n+1)_{q(n+1)}} a^{2n+1} t^{2n}; \\
\cos_q(at) = \sum_0^\infty (-1)^n (a t^{2n+1})_q \frac{(2n+1)_q!}{(2n+1)_{q(n+1)}} a^{2n+1} t^{2n}
\]

3 The Natural Transform

The Natural transform of a function \(f(x)\) on \(0 < x < \infty\) then it was proposed by Khan and Khan [14] as an extension to the Laplace and Sumudu transforms to solve some fluid flow problems.

Later, Silambarasan and Belgacem [16] have derived certain electric field solutions of the Maxwell’s equation in conducting media. In [4], the author applied the Natural transform to some ordinary differential equations and some space of Boehmians. Further investigation of the Natural transform can be obtained from [4] and [7].

The Natural transform of a function \(f(t), 0 < t < \infty\) is defined over the set \(A\), where

\[
A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{t/\tau}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}
\]

by ([13], (1))

\[
(Nf)(u; v) = \int_0^\infty f(ut) \exp(-vt) \, dt \quad (u, v > 0).
\]
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Provided the integral on the right of (16) exists it is easy to see that
\[(Nf)(u;1) = (Sf)(u) \quad \text{and} \quad (Nf)(1,v) = (Lf)(v)\]  \hspace{1cm} (17)
where \(Sf\) and \(Lf\) are respectively the Sumudu and Laplace transforms of \(f\).

Moreover, the Natural-Laplace and Natural-Sumudu dualities are given in [4, 2, 3] as
\[(Nf)(u;v) = \frac{1}{u} \int_0^\infty f(t) \exp\left(-\frac{vt}{u}\right) dt\]  \hspace{1cm} (18)
and
\[(Nf)(u;v) = \frac{1}{v} \int_0^\infty f\left(\frac{ut}{v}\right) \exp(-t) dt,\]  \hspace{1cm} (19)
respectively.

It further from (18) and (19) can be easily observed that
\[(Nf)(u;v) = \frac{1}{u} (Lf)\left(\frac{u}{v}\right) \quad \text{and} \quad (Nf)(u;v) = \frac{1}{v} (Sf)\left(\frac{u}{v}\right).\]  \hspace{1cm} (20)

Some values of the Natural transform of some known functions we mention here are [4, p.731]

(i) \(N(a)(u;v) = \frac{1}{v}\), where \(a\) is a constant.

(ii) \(N(\delta)(u;v) = \frac{1}{v}\), where \(\delta\) is the delta function.

(iii) \(N(e^{at})(u;v) = \frac{1}{v - au}\), \(a\) is a constant.

(iv) The scaling property is written in two ways as
\[(Nf(kt))(u;v) = \frac{1}{k} (Nf)(ku;v) \quad \text{and} \quad (Nf(kt))(u;v) = \frac{1}{k} (Nf)\left(\frac{u}{k};v\right).\]

4 The q-Analogue of the q-Natural Transform of First Kind

Hahn [11] and later Ucar and Albayrak [17] defined the q-analogue of first and second types of the well-known Laplace transform by means of the q-integrals
\[L_q (f(t);s) = \frac{1}{1-q} \int_0^{\frac{1}{q}} E_q (qst) f(t) dq_t\]  \hspace{1cm} (21)
and
\[qL (f(t);s) = \frac{1}{1-q} \int_0^{\infty} e_q (-st) f(t) dq_t.\]  \hspace{1cm} (22)

The q-analogues of the Sumudu transform of first and second types are defined by [2, 3]
\[S_q (f(t);s) = \frac{1}{(1-q)s} \int_0^{s} E_q \left(\frac{qt}{s}\right) f(t) dq_t\]  \hspace{1cm} (23)
and

\[ qS(f(t); s) = \frac{1}{1-q} \int_0^\infty e_q \left( -\frac{t}{s} \right) f(t) \, dq \, t \tag{24} \]

where \( s \in (-\tau_1, \tau_2) \) and \( f \) is a function belongs to the set \( A \),

\[ A = \left\{ f(t) \left| \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\tau_j}, t \in (-1)^j \times [0, \infty) \right. \right\} . \]

Now, we are in a position to demonstrate our results as follows. The \( q \)-analogue of the Natural transform of first kind as

\[ (N_q f)(u; v) := (N_q f; (u, v)) := \frac{1}{u} \int_0^\infty E_q \left( -\frac{vt}{u} \right) \, dq \, t, \tag{25} \]

provided the function \( f(t) \) is defined on \( A \) and, \( u \) and \( v \) are the transform variables.

The series representation of (25) can be written as:

\[ (N_q f)(u; v) = \frac{1}{(1-q)} \sum_{k \in \mathbb{Z}} q^k f(q^k) E_q \left( -\frac{q^{k+1}v}{u} \right), \tag{26} \]

and by (10), (26) can be put into the form

\[ (N_q f)(u; v) = \frac{(q; q)_\infty}{(1-q)} \sum_{k \in \mathbb{Z}} \frac{q^k f(q^k)}{(-\frac{v}{u}; q)_{k+1}} . \tag{27} \]

We derive now some values of \( N_q \) of some special functions.

**Theorem 1.** Let \( \alpha \in \mathbb{R} \), then we have

\[ (N_q t^\alpha)(u; v) = \frac{u^\alpha}{v^{\alpha+1}} \Gamma (\alpha + 1) . \tag{28} \]

**Proof.** By setting the variables, we get

\[
(N_q t^\alpha)(u; v) = \frac{u^\alpha}{v^{\alpha+1}} \int_0^\infty t^\alpha E_q (-qt) \, dq \, t
\]

\[
= \frac{u^\alpha}{v^{\alpha+1}} \Gamma_q (\alpha + 1) .
\]

The theorem hence follows.

A direct corollary of (28) can be

\[ (N_q t^n)(u; v) = \frac{u^n}{v^{n+1}} \left( [n]_q \right) !. \tag{29} \]

**Lemma 2.** Let \( a \) be a positive real number. Then, we have

\[ (N_q e_q(at))(u; v) = \frac{1}{v - au}, \text{ } au < v. \tag{30} \]
Proof. Using the second kind $q$-analogue of the $q$-exponential function, we write

$$
(N_q e_q(at)) (u; v) = \frac{1}{u} \int_0^\infty e_q(at) E_q \left( -\frac{v}{u} t \right) d_q t. \tag{31}
$$

In series representation (31) can be written as

$$
(N_q e_q(at)) (u; v) = \sum_{n=0}^\infty \frac{a^n}{[n]_q!} t^n E_q \left( -\frac{v}{u} t \right) d_q t. \tag{32}
$$

By (29), (32) gives a geometric series expansion and hence,

$$
(N_q e_q(at)) (u; v) = \frac{1}{v} \sum_{n=0}^\infty \left( \frac{a u}{v} \right)^n = \frac{1}{v - a u}, \quad au < v.
$$

This completes the proof of the lemma.

**Theorem 3.** Let $a$ be a positive real number. Then, we have

$$
(N_q E_q(at)) (u; v) = \frac{1}{v} \sum_{n=0}^\infty q \frac{n(n-1)}{2} \left( \frac{au}{v} \right)^n. \tag{33}
$$

Proof. By the first kind $q$-analogue of the exponential function we indeed get

$$
(N_q E_q(at)) (u; v) = \sum_{n=0}^\infty \frac{a_n q^{n(n-1)} / 2}{[n]_q!} \frac{1}{u} \int_0^\infty t^n E_q \left( -\frac{v}{u} t \right) d_q t. \tag{34}
$$

The parity of (29) gives

$$
(N_q E_q(at)) (u; v) = \frac{1}{v} \sum_{n=0}^\infty q \frac{n(n-1)}{2} \left( \frac{au}{v} \right)^n.
$$

This completes the proof of the theorem.

The hyperbolic $q$-cosine and $q$-sine functions are given as

$$
\cosh^q t = \frac{e_q(t) + e_q(-t)}{2} \quad \text{and} \quad \sinh^q t = \frac{e_q(t) - e_q(-t)}{2}.
$$

Hence, as a corollary of Theorem 2, we have

$$
(N_q \cosh^q t) (u; v) = \frac{v}{v^2 + a^2 u^2}, \quad au < v. \tag{35}
$$

and

$$
(N_q \sinh^q t) (u; v) = \frac{au}{v^2 + a^2 u^2}, \quad au < v. \tag{36}
$$
**Theorem 4.** Let \( a \) be a positive real number. Then, we have

\[
(N_q \cos^q at) (u; v) = \frac{v}{v^2 - a^2u^2},
\]

provided \( au < v \).

**Proof.** On account of (15) and (29) we obtain

\[
(N_q \cos^q at) (u; v) = \sum_{n=0}^{\infty} \frac{a^{2n}}{u \left(\frac{n}{2}\right)_q} \int_0^{\infty} t^{2n} E_q \left(-q\frac{v}{u}t\right) dt
\]

\[
= \frac{1}{v} \sum_{n=0}^{\infty} \left(\frac{au}{v}\right)^{2n}.
\]

For \( au < v \), the geometric series converges to the sum

\[
(N_q \cos^q at) (u; v) = \frac{v}{v^2 - a^2u^2}, \quad au < v.
\]

This completes the proof of the theorem.

Similarly, by (15) and (29), we deduce that

\[
(N_q \sin^q at) (u; v) = \frac{au}{v^2 - a^2u^2}, \quad au < v.
\]

(38)

**5 \( N_q \) and \( q \)-Differentiation**

In this section of this paper we discuss some \( q \)-differentiation formulae. On account of (12) we derive the following differentiation result.

**Lemma 5.** Let \( u, v > 0 \), then we have

\[
D_q E_q \left(-q\frac{v}{u}t\right) = \frac{v}{u} \sum_{n=0}^{\infty} (-1)^{n+1} q^{\frac{(n+1)(n+2)}{2}} \frac{v^n}{u^n} t^n.
\]

(39)

**Proof.** By using the \( q \)-representation of \( E_q \) in (12) we write

\[
D_q E_q \left(-q\frac{v}{u}t\right) = D_q \sum_{n=0}^{\infty} (-1)^{n+1} \frac{q^{\frac{(n+1)(n+2)}{2}}}{\left[\frac{n}{2}\right]_q} \frac{v^n}{u^n} t^n
\]

\[
= \sum_{n=1}^{\infty} (-1)^{n+1} q^{\frac{(n+1)(n+2)}{2}} \frac{v^n}{u^n} t^{n-1} \frac{n(n+1)}{n!} u^n t^n
\]

\[
= \sum_{n=1}^{\infty} (-1)^{n+1} q^{\frac{(n+1)(n+2)}{2}} \frac{v^n}{u^n} t^n.
\]

Hence, it follows that

\[
D_q E_q \left(-q\frac{v}{u}t\right) = \frac{v}{u} \sum_{n=0}^{\infty} (-1)^{n+1} q^{\frac{(n+1)(n+2)}{2}} \frac{v^n}{u^n} t^n.
\]
This completes the proof of the theorem.

The Natural transform of the $q$-derivative $D_q f$ can be written as follows.

**Theorem 6.** Let $u, v > 0$, then we have

$$N_q (D_q f (t)) (u; v) = -f (0) + \frac{v}{u} N_q (f) (u; v).$$

**Proof.** Using the idea of $q$-integration by parts and the formula in (9) we write

$$N_q (D_q f (t)) (u; v) = \int_0^\infty D_q f (t) E_q \left(-\frac{v}{u} t\right) d_q t$$

$$= f (t) D_q E_q \left(-\frac{v}{u} t\right) \bigg|_0^\infty - \int_0^\infty f (qt) D_q E_q \left(-\frac{v}{u} t\right) d_q t.$$

The parity of Lemma 5 (Eq.4.39) gives

$$N_q (D_q f (t)) (u; v) = -f (0) - \int_0^\infty f (qt) \frac{v}{u} \sum_{n=0}^\infty \frac{(-1)^n}{q^n} \frac{(n+1)(n+2)}{[n]_q} \frac{v^n}{u^n} t^n d_q t.$$

Changing the variables $qt = y$, and $t^n = q^{-n} y^n$ imply

$$N_q (D_q f (t)) (u; v) = -f (0) + \frac{v}{u} \int_0^\infty f (y) \sum_{n=0}^\infty (-1)^n q^{\frac{n(n-1)}{2}} \frac{v^n}{u^n} y^n d_q y$$

$$= -f (0) + \frac{v}{u} \int_0^\infty f (t) \sum_{n=0}^\infty (-1)^n q^{\frac{n(n-1)}{2}} \frac{v^n}{u^n} t^n d_q t.$$

By virtue of (12), (41) yields

$$N_q (D_q f (t)) (u; v) = -f (0) + \frac{v}{u} \int_0^\infty f (t) E_q \left(-\frac{v}{u} t\right) d_q t.$$

Hence,

$$N_q (D_q f (t)) (u; v) = -f (0) + \frac{v}{u} N_q (f) (u; v).$$

This completes the proof of the theorem.

Now we extend Theorem 6 to $n$th derivatives.

**Theorem 7.** Let $u, v > 0$ and $n \in \mathbb{Z}^+$. Then, we have

$$N_q (D_q^n f (t)) (u; v) = \frac{v^n}{u^n} (N_q (f)) (u, v) - \sum_{i=0}^{n-1} \frac{u}{v}^{n-1-i} D_q^i f (0).$$
Proof. On account of Theorem 6, we can write

\[
N_q (D^2_q f(t))(u;v) = D_q f(0) + \frac{v}{u} N_q (D_q f)(u;v)
\]
\[
= -D_q f(0) + \frac{v}{u} \left(-f(0) + \frac{v}{u} (N_q f)(u;v)\right)
\]
\[
= -D_q f(0) - \frac{v}{u} f(0) + \frac{v^2}{u^2} (N_q f)(u;v).
\]

(43)

Proceeding as in (43) we obtain

\[
N_q (D^n_q f(t))(u;v) = \frac{v^2}{u^2} N_q (f)(u;v) - \sum_{i=0}^{n-1} \left(\frac{v}{u}\right)^{n-1-i} D^i_q f(0).
\]

This completes the proof of the theorem.

6 \(N_q\) of q-Convolutions

Let functions \(f\) and \(g\) be in the form \(f(t) = t^\alpha\) and \(g(t) = t^{\beta-1}\) for \(\alpha, \beta > 0\). We define the q-convolution of \(f\) and \(g\) as

\[
(f \ast g)_q(t) = \int_0^t f(\tau) g(t - q\tau) \, dq t
\]

(44)

Theorem 8. Let \(\alpha, \beta > 0\). Then, we have

\[
N_q \left((f \ast g)_q\right)(u;v) = u^{2} N_q (f^\alpha)(u,v) N_q \left(t^{\beta-1}\right)(u;v)
\]

(45)

Proof. By aid of (44) and (14) we get

\[
N_q \left((f \ast g)_q\right)(u;v) = \frac{B_q(\alpha + 1, \beta)}{u} \int_0^{\infty} t^{\alpha+\beta} E_q \left(-\frac{vt}{u}\right) \, dq t.
\]

Hence, by (28) we obtain

\[
N_q \left((f \ast g)_q\right)(u;v) = \frac{\Gamma_q(\alpha + 1) \Gamma_q(\beta)}{\Gamma_q} \frac{u^{\alpha+\beta+1}}{v^{\alpha+\beta+1}}
\]
\[
= \Gamma_q(\alpha + 1) \Gamma_q(\beta) \frac{u^{\alpha+\beta+1}}{v^{\alpha+\beta+1}}.
\]

(46)

Simple motivation on (46) gives

\[
N_q \left((f \ast g)_q\right)(u;v) = u^{2} \left(N_q t^\alpha\right)(u,v) \left(N_q t^{\beta-1}\right)(u;v).
\]

The proof is therefore completed.

In similar way, we extend the q-convolution to functions of power series form.
Theorem 9. Let \( f(t) = \sum_{0}^{\infty} a_{i} t^{\alpha i} \) and \( g(t) = t^{\beta-1} \). Then, we have

\[
N_{q} \left( (f * g)_{q} \right)(u; v) = u^{2} (N_{q}f)(u; v) (N_{q}g)(u; v).
\]

**Proof.** Under the hypothesis of the theorem and Theorem 8 we write

\[
N_{q} \left( (f * g)_{q} \right)(u; v) = \sum_{0}^{\infty} a_{i} N_{q} \left( \left( t^{\alpha i} * t^{\beta-1} \right)_{q} \right)(u; v)
\]

\[
= \sum_{0}^{\infty} a_{i} \left( N_{q}t^{\alpha i} \right)(u; v) \left( N_{q}t^{\beta-1} \right)(u; v)
\]

\[
= u^{2} (N_{q}f)(u; v) (N_{q}g)(u; v).
\]

Hence, the proof of the theorem is completed.

7 \( N_{q} \) and Heaviside Functions

The Heaviside function is defined by

\[
N_{q}(\hat{u}(t - a)) = \begin{cases} 1 & , t \geq a \\ 0 & , 0 \leq t < a \end{cases}
\]

where \( a \) is a real number. In this part of the paper we merely establish the following theorem.

**Theorem 10.** If \( \hat{u} \) denotes the heaviside function and \( u, v > 0 \). Then, we have

\[
N_{q}(\hat{u}(t - a))(u; v) = \frac{1}{v} E_{q} \left( -\frac{v}{u}a \right).
\]

**Proof.** By (67) we have

\[
N_{q}(\hat{u}(t - a))(u; v) = \frac{1}{u} \int_{a}^{\infty} E_{q} \left( -\frac{v}{u}t \right) d_{q}t.
\]

On account of (21) we get

\[
N_{q}(\hat{u}(t - a))(u; v) = \frac{1}{v} - \frac{1}{u} \int_{a}^{\infty} \sum_{0}^{\infty} \frac{q^{n(\alpha-1)}}{[n]_{q}!} \left( -\frac{v}{u}t \right)^{n} d_{q}t
\]

\[
= \frac{1}{v} - \frac{1}{u} \sum_{0}^{\infty} (-1)^{n} \frac{q^{n(n-1)}}{[n]_{q}!} \frac{v^{n}}{u^{n}} \int_{a}^{\infty} t^{n} d_{q}t
\]
Integrating together with simple calculation reveal

\[
N_q(\hat{u}(t-a))(u;v) = \frac{1}{v} - \frac{1}{u} \sum_0^\infty \frac{q^{\frac{n(n-1)}{2}} v^n a^{n+1}}{[n]_q! u^n [n+1]_q} \nonumber
\]

\[
= \frac{1}{v} + \frac{1}{u} \sum_0^\infty (-1)^{n+1} \frac{q^{\frac{n+1}{2}} v^n a^{n+1}}{[n+1]_q! u^n [n+1]_q} \nonumber
\]

\[
= \frac{1}{v} + \frac{1}{u} \sum_0^\infty (-1)^{n+1} \frac{q^{\frac{n+1}{2}} v^n a^{n+1}}{[n+1]_q! u^n [n+1]_q} \nonumber
\]

This can be written as

\[
N_q(\hat{u}(t-a))(u;v) = \frac{1}{v} + \frac{1}{u} \sum_1^\infty (-1)^m \frac{q^{m(m-1)} v^m a^m}{[m]_q! u^m a^m} \nonumber
\]

Starting the summation from 0 gives

\[
N_q(\hat{u}(t-a))(u;v) = \frac{1}{v} \sum_1^\infty (-1)^m \frac{q^{m(m-1)} v^m a^m}{[m]_q! u^m a^m} \nonumber
\]

\[
= \frac{1}{v} E_q \left( \frac{-v}{u} \right) \nonumber
\]

8 The q-Analogue of the q-Natural Transform of Second Kind

The q-analogue of the Natural transform of the second type is defined over the set A

\[
A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| |Me^{t/\tau_j}, t \in (-1)^j \times [0, \infty) \right\} \nonumber
\]

as

\[
(N_q f)(u;v) = \frac{1}{u} \int_0^\infty f(t) e_q \left( \frac{-v}{u} t \right) d_q t. \tag{49} \nonumber
\]

The q-analogue of the gamma function of the second kind is defined as

\[
\gamma_q(t) = \int_0^\infty x^{t-1} e_q(-x) d_q x, \tag{40} \nonumber
\]

and, hence, it follows that

\[
\gamma_q(1) = 1, \quad \gamma_q(t+1) = q^{-t} [t]_q \gamma_q(t) \quad \text{and} \quad \gamma_q(n) = q^{\frac{n(n-1)}{2}} \Gamma_q(n), \tag{41} \nonumber
\]

\Gamma_q\) being the q-analogue of gamma function of first kind.

We aim to derive certain results similar to that we have obtained in the previous sections.
**Lemma 11.** Let $\alpha > -1$, then we have

\[
(N^q t^\alpha)(u; v) = \frac{u^\alpha}{v^{\alpha+1}} \gamma_q (\alpha + 1).
\]  

In particular,

\[
(N^q t^n)(u; v) = \frac{u^n}{v^{n+1}} q^{-\frac{n(n-1)}{2}} ([n_q])!
\]

**Proof.** Let $\alpha > -1$, then by change of variables we have

\[
(N^q t^\alpha)(u; v) = \frac{1}{u} \int_0^\infty t^\alpha e_q \left( -\frac{vt}{u} \right) d_q t
\]

\[
= \frac{u^\alpha}{v^{\alpha+1}} \int_0^\infty t^\alpha e_q (-t) d_q t
\]

On aid of (40), we get

\[
(N^q t^\alpha)(u; v) = \frac{u^\alpha}{v^{\alpha+1}} \gamma_q (\alpha + 1).
\]

Proof of the second part of the theorem follows from (41).

Hence, we completed the proof of the theorem.

**Theorem 12.** Let $a \in \mathbb{R}, a > 0$, then we have

\[
(N^q e_q(at))(u; v) = \frac{1}{uv} \sum_{n=0}^\infty a^n u^n v^n q^{-\frac{n(n-1)}{2}}.
\]

**Proof.** By (13) we write

\[
(N^q e_q(at))(u; v) = \frac{1}{u} \int_0^\infty e_q(at) e_q \left( -\frac{vt}{u} \right) d_q t
\]

\[
= \frac{1}{u} \sum_{n=0}^\infty \frac{a^n}{([n]_q)!} \int_0^\infty t^n e_q \left( -\frac{vt}{u} \right) d_q t.
\]

By aid of Theorem 11, the above equation yields

\[
(N^q e_q(at))(u; v) = \frac{1}{uv} \sum_{n=0}^\infty a^n u^n v^n q^{-\frac{n(n-1)}{2}}.
\]

This completes the proof of the theorem.

**Theorem 13.** Let $a > 0, a \in \mathbb{R}$, then we have

\[
(N^q E_q(at))(u; v) = \frac{1}{u(v - au)}, \; au < v.
\]
Proof. After some calculations and by using Theorem 11, we obtain

\[
\begin{align*}
(N^q e_q(\alpha t)) (u; v) &= \frac{1}{u} \int_0^\infty E_q(\alpha t) e_q\left(-\frac{v}{u}t\right) dt \\
&= \frac{1}{u} \sum_{n=0}^\infty q^{n(n-1)} \frac{a^n}{[n]_q!} \int_0^\infty t^n e_q\left(-\frac{v}{u}t\right) dt \\
&= \frac{1}{uv} \sum_{n=0}^\infty a^n \frac{u^n}{v^n}.
\end{align*}
\]

Since the above series determine a geometric series, we get

\[
(N^q e_q(\alpha t)) (u; v) = \frac{1}{uv} \frac{1}{1 - \frac{au}{v}} = \frac{1}{u(v-au)}, au < v.
\]

Hence the theorem is proved.

The \(N^q\) transform of \(\cos^q\) and \(\sin^q\) is given as follows.

**Theorem 14.** Let \(a > 0\), then we have

\[
(N^q \cos^q (\alpha t)) (u; v) = \frac{1}{u} \sum_{n=0}^\infty (-1)^n a^{2n} \frac{u^{2n}}{v^{2n}} q^{-2n \frac{(2n-1)}{2}}.
\] (46)

**Proof.** Using the definition of \(\cos^q\) we write

\[
\begin{align*}
(N^q \cos^q (\alpha t)) (u; v) &= \frac{1}{u} \int_0^\infty \cos^q (\alpha t) e_q\left(-\frac{v}{u}t\right) dt \\
&= \frac{1}{u} \sum_{n=0}^\infty (-1)^n a^{2n} \frac{u^{2n}}{[2n]_q!} \int_0^\infty t^{2n} e_q\left(-\frac{v}{u}t\right) dt \\
&= \frac{1}{u} \sum_{n=0}^\infty (-1)^n a^{2n} \frac{u^{2n}}{v^{2n} q^{-2n \frac{(2n-1)}{2}}}.
\end{align*}
\]

This completes the proof of the theorem.

The \(N^q\) transform of \(\sin^q(\alpha t)\) is given as follows:

**Theorem 15.** Let \(a > 0\), then we have

\[
(N^q \sin^q (\alpha t)) (u; v) = \frac{1}{uv} \sum_{n=0}^\infty (-1)^n a^{2n+1} \frac{u^{2n+1}}{v^{2n+1} q^{-2n \frac{(2n-1)}{2}}}.
\] (47)

Proof of Theorem 15 follows from similar proof to that of Theorem 14.
Notes on the $q$-analogues of the Natural transforms and some further applications

9 \( N^q \) of \( q-\)Differentiation

Before we start investigations, we first assert that
\[
D_q e_q \left( -\frac{v}{u} t \right) = -\frac{v}{u} e_q \left( -\frac{v}{u} t \right).
\]
For further details, we have
\[
D_q e_q \left( -\frac{v}{u} t \right) = \sum_{0}^{\infty} \frac{(-1)^n}{[n]_q} \frac{v^n}{u^n} t^n
= \sum_{1}^{\infty} \frac{(-1)^n}{[n-1]_q} \frac{v^n}{u^n} t^{n-1}
= \sum_{0}^{\infty} \frac{(-1)^{n+1}}{[n]_q} \frac{v^{n+1}}{u^{n+1}} t^n
= -\frac{v}{u} \sum_{0}^{\infty} \frac{(-1)^n}{[n]_q} \frac{v^n}{u^n} t^n
= -\frac{v}{u} e_q \left( -\frac{v}{u} t \right).
\]
This proves the above assertion.

Hence we prove the following theorem.

\textbf{Theorem 16.} Let \( u, v > 0 \), then we have
\[
(N^q D_q f (t)) (u; v) = -f (0) - \int_{0}^{\infty} f (qt) D_q e_q \left( -\frac{v}{u} t \right) d_q t.
\] \hspace{1cm} (49)

\textbf{Proof.} By (48), the above equation gives
\[
(N^q D_q f (t)) (u; v) = \int_{0}^{\infty} D_q f (t) e_q \left( -\frac{v}{u} t \right) d_q t
= -f (0) + \frac{v}{u} \int_{0}^{\infty} f (qt) e_q \left( -\frac{v}{u} t \right) d_q t
= -f (0) + \frac{v}{u} q^{-1} \int_{0}^{\infty} f (t) e_q \left( -\frac{v}{u} q^{-1} t \right) d_q t.
\]
By setting variables we have
\[
(N^q D_q f (t)) (u; v) = -f (0) + \frac{v}{u} q^{-1} (N^q f) (q^{-1} v; u).
\]
This completes the proof.
Now, we extend (49) to have
\[
(N^q D^2_q f)(u; v) = (N^q D_q (D_q f))(u; v)
\]
\[
= -D_q f(0) + \frac{v}{u} q^{-1} (N^q D_q f)(q^{-1}v; u)
\]
\[
= -D_q f(0) + \frac{v}{u} q^{-1} \left( -f(0) + \frac{v}{u} q^{-1} (N^q f)(q^{-2}v; u) \right)
\]
\[
= -D_q f(0) - \left( \frac{v}{u} \right) q^{-1} f(0) + \left( \frac{v}{u} \right) q^{-2} (N^q f)(q^{-2}v; u).
\]

Proceeding to \( n \)th derivatives, we get
\[
(N^q D^n_q f)(u; v) = \left( \frac{v}{u} \right)^n q^{-n} (N^q f)(q^{-n}v; u) - \sum_{i=0}^{n-1} \left( \frac{v}{u} \right)^{n-1-i} D^i_q f(0).
\]

This completes the proof of the theorem.

Competing interests
The authors declare that they have no competing interests.

Author’s contributions
All the authors jointly worked on deriving the results and approved the final manuscript.

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