Stokes Phenomenon and Reflection Equations

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Abstract: In this paper, we study the Stokes phenomenon of the cyclotomic Knizhnik–Zamolodchikov equation associated to the symmetric pair \((\mathfrak{gl}_n, \mathfrak{so}_n)\), and prove that its two types of Stokes matrices satisfy the Yang–Baxter and reflection equations respectively. We briefly discuss its isomonodromy deformation, and its relations with cyclotomic associators, twists and quantum symmetric pairs.

1. Introduction

The Stokes phenomenon, in complex analysis, states that the asymptotic behavior of functions can differ in different angular sectors surrounding a singularity. In the papers [44–46], such jumps of the asymptotics of solutions of Knizhnik–Zamolodchikov (KZ) equations with irregular singularities, from one sector to another, are shown to encode various structures in representation theory, including the braid relation in quantum groups, \(\mathfrak{gl}_n\)-crystals, representations of quantum group \(U_q(\mathfrak{gl}_n)\) and the Gelfand-Tsetlin basis. More importantly, the relation with representation theory brings new insights into some open analysis problems themselves: motivated by the Gelfand-Tsetlin theory, we derive an explicit expression of Stokes matrices of meromorphic linear systems of ordinary differential equations of Poincaré rank 1 via the asymptotics of solutions of associated isomonodromy deformation equations [45]; motivated by the theory of canonical basis and representations of quantum groups at roots of unity, we give conjectural (and prove some special case of) algebraic characterizations of the WKB approximation and soliton solutions of meromorphic differential equations [45,46]. In this paper, we deepen the relation between Stokes phenomenon and representation theory, by providing a construction of the universal solutions of reflection equations via Stokes phenomenon. Before stating our main theorems, we first give a brief recall of the theory of reflection equations and KZ equations.

The theory of reflection equations was initiated by Cherednik [11] in the study of factorized scattering on the half line, and by Sklyanin in the investigation of quantum in-
Integrable models with boundary conditions [37]. Examples of explicit universal solutions of reflection equations can be found in Kulish, Sasaki and Schiewebert [28]. As the theory of quantum groups is governed by Yang–Baxter equations, the theory of quantum symmetric pair is closely related to reflection equations. In particular, Balagovic and Kolb [2], by generalizing the construction of Bao and Wang [3] for the quantum symmetric pair of type AIII, showed that any quantum symmetric pair in Letzter’s classification [30] gives a universal solution of the reflection equation, called a universal $K$-matrix.

A categorical framework for solutions of the reflection equation was proposed by T. tom Dieck and R. Haring-Oldenburg [39,40], and the universal $K$-matrix corresponds to twisted braiding on module categories over braided monoidal categories, see e.g., Enriquez [17], Brochier [8,9].

The Knizhnik–Zamolodchikov (KZ) equation was discovered in the study of conformal field theory [25]. It is a local system on the configuration space of points, whose monodromy has been studied by many people and is closely related to conformal field theory, quantum groups, representation theory of affine Lie algebras, hypergeometric functions, Hecke type algebras, geometry of cycles and so on. See e.g., Cherednik [14], Drinfeld [16], Kohno [26], Tsuchiya-Kanie [41], Varchenko [42], and the reference therein. The KZ equations have various of generalizations, like allowing more general $r$-matrix form [12] and the computation of its monodromy by Cherednik [13], allowing discretization, known as the quantum KZ equations by Frenkel and Reshetikhin [21], allowing irregular singularities and compatible dynamical equations, known as generalized KZ (gKZ) equations by Felder, Markov, Tarasov and Varchenko [20]. Another generalization is the cyclotomic KZ equation, following Leibman [29], Golubeva-Leksin [22], Enriquez-Etingof [18, Section 4.2], which is designed to incorporate various automorphisms on Lie algebras. When the automorphism is simply an involution, the relation between its monodromy and quantum symmetric pairs, has been studied by many authors, see e.g., Enriquez [17], De Commer-Neshveyev-Tuset-Yamashita [15], to some extent, generalizing the works of Drinfeld and Kohno.

In this paper, we introduce a cyclotomic KZ equation coupled with extra irregular singularities, called a generalized cyclotomic KZ (gcKZ) equation, see Definition 2.1. We then study the Stokes phenomenon of the gcKZ equation around irregular singularities. In the end, we show that the Stokes matrices of the gcKZ equation gives rise to universal solutions of reflection equations.

### 1.1. Stokes matrices, Yang–Baxter and reflection equations.

Throughout this paper, let us take the complex Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$, and take the negative transpose $\tau$ as an involution of $\mathfrak{g}$ with spectral decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$, where the fixed point Lie algebra is $\mathfrak{k} = \text{so}_n$. Let $\mathfrak{h}$ be the set of diagonal matrices, as a Cartan subalgebra of $\mathfrak{g}$. Let $\{e_i\}_{i \in I_{\pm}}$ be an orthonormal basis of $\pm 1$-eigenspaces $\mathfrak{t}$ and $\mathfrak{p}$ with respect to the Killing form of $\mathfrak{g}$. Set $\Omega_\mathfrak{t} = \sum_{i \in I_{\pm}} e_i \otimes e_i \in \mathfrak{t} \otimes \mathfrak{t}$, $\Omega_\mathfrak{p} = \sum_{i \in I_{\pm}} e_i \otimes e_i \in \mathfrak{p} \otimes \mathfrak{p}$, and denote $\Omega = \Omega_\mathfrak{t} + \Omega_\mathfrak{p} \in \mathfrak{g} \otimes \mathfrak{g}$. Furthermore, let us denote the Casimir element by $C_\mathfrak{t} = \sum_{i \in I_{\pm}} e_i e_i \in U(\mathfrak{t})$.

The involution $\tau$ extends to an automorphism of $U(\mathfrak{g})$. Let $V$ be a finite dimensional $U(\mathfrak{g}) \times \mathbb{Z}/2\mathbb{Z}$-module, with $1 \in \mathbb{Z}/2\mathbb{Z}$ encoding $\tau$, and $W$ be a finite dimensional $U(\mathfrak{t})$-module. We also denote by $\tau$ the action of $\tau$ on $V$. We consider respectively the two equations for a $V^{\otimes 2}$-valued function $Y(z)$ and a $W \otimes V$-valued function $F(z)$,

$$
\kappa \frac{dY}{dz} = \left( u^{(2)} + \frac{\Omega}{z} \right) \cdot Y,
$$

(1)
\[
\kappa \frac{dF}{dz} = \left( u^{(1)} + \frac{2\Omega_{\tau} + C_{\tau}^{(1)}}{z} \right) \cdot F,
\]

(2)

where \( \kappa \) is a purely imaginary number, \( u \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \) the set of real regular elements in \( \mathfrak{h} \), and \( u^{(1)} \) and \( C_{\tau}^{(1)} \) denote the action of \( u \) and \( C_{\tau} \) on the first component of \( V \) in \( V \otimes V \) and \( W \otimes V \) respectively.

The equation (1) has an irregular singularity at \( z = \infty \) of Poincaré rank 1. Since \( \frac{z}{\kappa} \) is purely imaginary, the Stokes rays of (1) lie on the imaginary axis, and the Stokes sectors are the right half plane \( \mathbb{H}_+ \) and the left half plane \( \mathbb{H}_- \). Then following the theory of meromorphic linear systems (see e.g., [5,6] or the Appendix A), it has a unique formal power series fundamental solution \( Y(z) \in \text{End}(V^{\otimes 2}) \) around \( z = \infty \), which under the Borel-Laplace transform gives rise to a canonical holomorphic solution \( Y_{+}(z) \) (resp. \( Y_{-}(z) \)) in \( \mathbb{H}_+ \) (resp. in \( \mathbb{H}_- \)). The discontinuity of the two solutions \( Y_{\pm} \) is measured by the Stokes matrices \( S_{\pm}(u), S_{\pm}(u) \in \text{End}(V^{\otimes 2}) \), which are determined by

\[
Y_{-}(z; u) = Y_{+}(z; u) \cdot e^{-\frac{z}{\kappa}[\Omega]} S(u), \quad Y_{+}(z; u) = Y_{-}(z; u) \cdot S_{-}(u) e^{\frac{z}{\kappa}[\Omega]}
\]

(3)

where \( [\Omega] \in \mathfrak{h} \otimes \mathfrak{h} \) is the projection of \( \Omega \in \mathfrak{g} \otimes \mathfrak{g} \) to \( \mathfrak{h} \otimes \mathfrak{h} \), and the first (resp. second) identity is understood to hold in \( \mathbb{H}_- \) (resp. \( \mathbb{H}_+ \)) after \( Y_{+} \) (resp. \( Y_{-} \)) has been analytically continued counterclockwise.

In the same way, we define the Stokes matrices \( K(u), K_{-}(u) \in \text{End}(W \otimes V) \) of equation (2) in the two Stokes sectors \( \mathbb{H}_{\pm} \).

**Theorem 1.1.** For any \( u \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \), the two Stokes matrices \( K(u), K_{-}(u) \in \text{End}(W \otimes V) \) and \( S(u) \in \text{End}(V \otimes V) \) satisfy the Yang–Baxter and the \( \tau \)-twisted reflection equations

\[
S^{12} S^{13} S^{23} = S^{23} S^{13} S^{12} \in \text{End}(V^{\otimes 3}),
\]

\[
K^{01} S^{21} K^{02} S^{21} = S^{21} K^{02} S^{12} K^{01} \in \text{End}(W \otimes V^{\otimes 2}).
\]

Here \( S_{\tau} := (\tau \otimes \text{id})S \) and we index \( W \) in \( W \otimes V^{\otimes 2} \) as the 0-th component, then \( S^{ij}, K^{ij} \) denote that the two components of \( S, K \) act respectively on the \( i \)-th and \( j \)-th components.

As in Sect. 2, the \( \tau \) twist in the reflection equation naturally comes from the action of \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) on the gcKZ equations with 2 variables.

**Remark 1.2.** The twisted reflection equations [2, Equation (9.17) and Remark 9.7] are introduced to unify various reflection equations associated to different quantum symmetric pairs. The twist for the reflection equation of a given quantum symmetric pair is determined by the Dynkin data that characterises it in Araki’s classification [1], which (in type AI case) is related to \( \tau \) by an inner automorphism of \( \mathfrak{g} \).

There are various monodromy relations among \( S(u), K(u) \) and the Casimir elements. See e.g., Proposition 1.8. In particular, we have the relation \( K(u) = -(\text{id} \otimes \tau) K_{-}(u)^{-1} \in \text{End}(W \otimes V) \), which follows from the facts that \( \tau(u) = -u \) and \( (\text{id} \otimes \tau)(2\Omega_{\tau} + C_{\tau}^{(1)}) = 2\Omega_{\tau} + C_{\tau}^{(1)} \).

1.2. Variation of \( u \). Let us take a root space decomposition \( \mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in \Delta} \mathfrak{c} \mathfrak{e}_{\alpha} \). For any root \( \alpha \in \Delta \), set \( C_{\alpha} = e_{\alpha} e_{-\alpha} \), and \( C_{\tau, \alpha} = \frac{1}{2}(e_{\alpha} + \tau(e_{\alpha})) e_{-\alpha} \).
Theorem 1.3. As a function of \( u \), the two Stokes matrices \( S(u) \) and \( K(u) \) satisfy respectively

\[
\kappa d_\hbar S(u) = \frac{1}{2} \sum_{\alpha \in \Delta} \frac{d\alpha}{\alpha} \left[ C^{(1)}_\alpha + C^{(2)}_\alpha, S(u) \right],
\]

\[
\kappa d_\hbar K(u) = \sum_{\alpha \in \Delta} \frac{d\alpha}{\alpha} \left[ C^{(0)}_{\tau,\alpha} + C^{(1)}_{\tau,\alpha}, K(u) \right],
\]

where \( d_\hbar \) is the de Rham differential on \( \hbar \), and again we index \( W \) in \( W \otimes V \) as the 0-th component.

A priori the Stokes matrices are defined on the real part \( u \in \hbar_{\text{reg}}(\mathbb{R}) \), but one can extend them to other points by the continuation of solutions of the above differential equation.

For a local picture, the theorem implies that the type \( B \) braid group representation, as the monodromy representation \( \rho(u) \) of generalized cyclotomic KZ equations (see Sect. 2), for different \( u \) are equivalent. In a categorical setting, see e.g., [9, 15, 17, 39, 40] for various versions, the theorem implies that the \( \tau \)-braided module category \( \text{Rep}(\mathfrak{g}) \) over (the braided monoidal category) \( \text{Rep}(\mathfrak{g}) \), constructed from \( S(u) \) and \( K_+(u) \) with different \( u \) are equivalent.

For a global picture, it is interesting to study the monodromy of the equation (5) with respect to \( u \in \hbar_{\text{reg}} \). Based on the quantum algebra version of the above results in Sect. 1.3, its monodromy should be related to the braid group actions on quantum symmetric pairs in type AI [34], see [10, 33], as a cyclotomic version of the Drinfeld-Kohno theorem for Casimir equations [38].

### 1.3. Comodule algebras, cyclotomic associators and twists.

In this subsection, we present a quantum algebra counterpart of the above categorical construction. Let us first recall the definition of comodule algebras, see e.g., [27, Definition 2.7].

**Definition 1.4.** Let \((H, \Delta, R)\) be a quasi-triangular bialgebra with universal \( R \)-matrix \( R \), an algebra involution \( \tau : H \rightarrow H \) such that \((\tau \otimes \tau)(R) = R \). A right \( H \)-comodule algebra \( B \) with coaction \( \Delta_B : B \rightarrow B \otimes H \) is called quasi-triangular if there exists an invertible element \( K \in B \otimes H \) such that

\[
\begin{align*}
(\text{K1). } K \Delta_B(b) &= (\text{id} \otimes \tau) \Delta_B(b) K, \\
(\text{K2). } (\Delta_B \otimes \text{id})(K) &= R^{21}_\tau K^{02} R^{12}, \\
(\text{K3). } (\text{id} \otimes \Delta)(K) &= R^{12} K^{02} R^{12} K^{01}.
\end{align*}
\]

Here \( R_\tau := (\text{id} \otimes \tau)(R) \), we label the tensor components of \( B \otimes H \otimes H \) by 0, 1, 2. The element \( K \) is called a universal \( K \)-matrix for the \( H \)-comodule algebra \( B \).

Set \( \U = U(\mathfrak{g})[[h]] \), \( \U_\hat{t} = U(\mathfrak{t})[[h]] \), denote by \( \U \hat{\otimes} \U \) and \( \U_\hat{t} \hat{\otimes} \U \) the completed tensor product of \( \mathbb{C}[[h]] \)-modules. Now let us consider the equations (1) and (2), with \( \kappa = h^{-1} \) a formal parameter and \( u \) rescaled by \( h^{-1} \), but for functions \( Y_h(z) \) and \( F_h(z) \) valued in \( \U_\hat{t} \hat{\otimes} \U \) and \( \U \hat{\otimes} \U \) respectively. Similar to finite dimensional case, one can study the Stokes phenomenon of these equations, and introduce canonical solutions, Stokes matrices and so on. Then Theorem 1.1 carries directly to this setting.
Theorem 1.5. The two (quantum) Stokkes matrices 

\[ S_h(u) \in \mathcal{U}\hat{\otimes}\mathcal{U} \text{ of } \frac{dY_h}{dz} = \left( u^{(2)} + \frac{\Omega}{z} \right) \cdot Y_h, \]  

\[ K_h(u) \in \mathcal{U}\hat{\otimes}\mathcal{U} \text{ of } \frac{dF_h}{dz} = \left( u^{(1)} + \frac{2\Omega^{01}_t + C^{(1)}_t}{z} \right) \cdot F_h, \] 

satisfy the Yang–Baxter and \( \tau \)-twisted reflection equations. Here we index \( \mathcal{U}_t \) as the 0-th factor.

Furthermore let us introduce the two (quantum) connection matrices \( C_h(u) \) and \( T_h(u) \), as the monodromy from 0 to \( \infty \), of the equations in (6) and (7) respectively. See the Appendix A for the definition of connection matrices. Then we have

Theorem 1.6. For any \( u \in h_{\text{reg}}(\mathbb{R}) \), \( B(u) = (\mathcal{U}_t, T_h \Delta T_h^{-1}) \) is a \( H(u) \)-comodule algebra with the universal \( K \)-matrix \( K_h(u) \), over the bialgebra \( H(u) = (\mathcal{U}, C_h \Delta C_h^{-1}, S_h(u)) \). Here \( \Delta \) is the standard (cocommutative) coproduct.

Let us give a sketch of a proof of the theorem and explain its relation with the cyclotomic associators and twists [17, 22]. In particular, a precursor of Definition 1.4 is the notion of a quasi-reflection algebra (QRA) over a quasi-triangular quasi-bialgebra (QTQBA) [17, Definition 4.1]. These data satisfy in particular the same axioms with the notion of a quasi-reflection algebra (QRA) over a quasi-triangular quasi-bialgebra (QTQBA) [17, Definition 4.1]. These data satisfy in particular the same axioms with the notion of a quasi-reflection algebra (QRA) over a quasi-triangular quasi-bialgebra (QTQBA) [17, Definition 4.1].

Remark 1.7. To be more precise, Enriquez [17] considered the semidirect product \( U(g) \rtimes \mathbb{Z}/2\mathbb{Z}[h] \) with \( \mathbb{Z}/2\mathbb{Z} \) encoding the involution \( \tau \). His definition of a QRA over a QTQBA avoids the \( \tau \) twist, and doesn’t necessarily satisfy condition (K3) with the cyclotomic associator. Here we take \( U(g) \) instead of the semidirect product, thus the conditions (K1) and (K2) with the associator of a QRA over a QTQBA involve the \( \tau \) twist, see the convention from [15, Section 3.2]. Furthermore, condition (K3) is a ribbon \( \tau \)-braid relation [9] as explained in [15, Proposition 3.6]. By the same reason, in the following we have to take a \( \tau \)-twisted version of the notion of twists in [17, Section 4.2].

A \( \tau \)-twisted twist of \( H_{KZ} \) and \( B_{KZ} \) is a pair \( (F, G) \), where \( F \in \mathcal{U}\hat{\otimes}\mathcal{U} \) and \( G \in \mathcal{U}\hat{\otimes}\mathcal{U} \) are invertible elements. Under the twist, we get

\[ H_{KZ}^F = \left( \mathcal{U}, F \Delta F^{-1}, F^{21} e^{\pi i h \Omega} F^{-1}, F^{23} (id \otimes \Delta)(id \otimes \Delta)(F) \Phi_{KZ}(F^{12} (\Delta \otimes id)(F))^{-1} \right), \]

\[ B_{KZ}^{F,G} = \left( \mathcal{U}_t, G \Delta G^{-1}, \tau(G) e^{\pi i h \Omega(C^2_t)} G^{-1}, F^{23} (id \otimes \Delta)(G) \Psi_{KZ}(F^{12} (\Delta \otimes id)(G))^{-1} \right), \]

and \( H_{KZ}^F \) is a QRA over the QTQBA \( B_{KZ}^{F,G} \). Here \( \tau(G) \) acts on the (first) \( \mathcal{U} \) component of \( G \). When the associators equal to 1, \( H_{KZ}^F \) is a quasitriangular bialgebra, and \( B_{KZ}^{F,G} \) becomes a comodule algebra with the universal \( K \)-matrix \( \tau(G) e^{\pi i h \Omega(C^2_t)} G^{-1} \), that is the case of Definition 1.4. Actually, we can prove that the twist of \( (H_{KZ}, B_{KZ}) \), under the quantum connection matrices \( (F = C_h(u), G = T_h(u)) \), is \( (H(u), B(u)) \).
Proposition 1.8. 1. The quantum connection matrices \((C_h(u), T_h(u))\) are the twist killing the cyclotomic associators \((\Phi_{KZ}, \Psi_{KZ})\). In particular, \(B(u)\) is a reflection algebra over the quasi-triangular bialgebra \(H(u)\), twist equivalent to the QRA \(B_{KZ}\) over the QTQBA \(H_{KZ}\).

2. We have the monodromy relations \(S_h = C_{h}^{21} e^{\pi i h \Omega} C_{h}^{-1} \) and \(K_h = \tau^{(1)}(T_h) e^{\pi i h (\Omega + C_t^{(1)})} T_h^{-1}\).

Remark 1.9. The two sides of any identity in (2) computes the monodromy along a semicircle around 0 (in anti-clockwise direction) and the monodromy along a semicircle around \(\infty\) (in the clockwise direction) respectively, which by homotopy are the same. It interprets geometrically the twists in the universal \(R\) and \(K\) matrices. Furthermore, the relation \(K_h = \tau^{(1)}(T_h) e^{\pi i h (\Omega + C_t^{(1)})} T_h^{-1}\) implies \(\tau^{(1)}(\Delta_B(b)) K_h = K_{h} \Delta_B(b)\) for any \(b \in B(u)\), which is the axiom \((K1)\) of a universal \(K\)-matrix.

The proof of the fact that \(C_h(u)\) and \(T_h(u)\) are the twist killing respectively the associators \(\Phi_{KZ}\) and \(\Psi_{KZ}\) is standard, i.e., uses the monodromy relation between certain asymptotics zones of gcKZ equations for \(n = 3\) and \(n = 2\) respectively. So we skip the proof. Here we just remark that the connection matrices between regular singularities in the gcKZ equation are the associators \(\Phi_{KZ}\) and \(\Psi_{KZ}\). However, if we switch to the asymptotics zones near the irregular singularities, then the irregular singularities dominate, thus the connection matrices (the KZ associators) between regular singularities in the asymptotics zones are trivial. This proposition is a cyclotomic analog of the construction due to Toledano Laredo [38], where he proved that the connection matrix \(C_h\) (of gKZ equation for \(n = 2\)) is a Drinfeld twist killing the KZ associator \(\Phi_{KZ}\).

Remark 1.10. The automorphism \(\tau\) on \(U(g)\)-Mod in the obvious way: if \(V\) is a \(U(g)\)-module, then \(\tau(V) = V\) as a vector space, while the module structure is given by \(x \cdot_\tau v = \tau(x)v\). In particular, the result in this section defines a ribbon \(\tau\)-braided right module category \(\text{Rep}(\mathfrak{t})\) over the braided monoidal category \(\text{Rep}(g)\) in the sense of Brochier [9].

The theory of quantum symmetric pairs was developed by Noumi, Sugitani and Dijkhuizen for classical Lie algebras, see e.g., [34–36], and developed by Letzter [30, 31] for all semisimple Lie algebras via the Drinfeld-Jimbo presentation of quantized enveloping algebras [23]. As before, we focus on the case \(g = \mathfrak{gl}_n\) with the involution \(\tau\), and \(\mathfrak{t} = \mathfrak{so}_n\) the corresponding fixed Lie subalgebra. A coideal subalgebra \(B\) of \(U_h(g)\) was introduced by Noumi [34], see also [31, Section 7], and the pair \((U_h(g), B)\) is a quantum symmetric pair of type AI (here \(U_h(g)\) is the topological \(\mathbb{C}[[h]]\)-version of \(U_q(g)\), and we take the topological version of the quantum symmetric pair). In this paper, we use Stokes phenomenon to give a transcendental construction of the pair \((H(u), B(u))\) with a universal \(K\)-matrix \(K_h(u)\), for any \(u \in \mathfrak{h}_{\text{reg}}(\mathbb{R})\). We expect that by a cohomological argument, for any fixed \(u\) the transcendental construction \((H(u), B(u))\) and algebraic one \((U_h(g), B)\) are equivalent to each other.

Remark 1.11. The Drinfeld-Kohno theorem [16, 26] states that the monodromy representation of KZ equations and the “algebraic” representations of braid group \(B_n\) from universal \(R\)-matrices are equivalent. The equivalence between \((H(u), B(u))\) and \((U_h(g), B)\) will provide a Drinfeld-Kohno type theorem for type B braid group representation, coming from gcKZ equations and the universal \(K\)-matrix of algebraic quantum symmetric pairs respectively. Note that the monodromy representation of cyclotomic KZ equations and gcKZ equations are equivalent by the twist equivalence in Proposition 1.8, it will be
closely related to the conjecture (for the type AI case) studied by Commer, Neshveyev, Tuset and Yamashita [15, Conjecture 4.1].

In this paper we use the Stokes phenomenon to construct a family of universal $K$-matrices parameterized by $h_{\text{reg}}(\mathbb{R})$. On the one hand, as pointed by Etingof to us, it motivates an interesting question to study a dynamical analog of $K$-matrices, via the Tannakian duality for the fibre functors $F_d : \text{Rep}(U_h(g)) \to A$-bimodules, where $A$ is an algebra of appropriate functions on the Cartan subalgebra $h$, as in the dynamical $R$-matrices. On the other hand, a theory of canonical bases and Schur duality for quantum symmetric pairs was set up by Bao and Wang [3,4]. Note that the WKB approximation of quantum Stokes matrices $S_h(u)$ is proposed to be completely characterized by the crystal structures in [45]. It is thus interesting to study the further relation between the WKB approximation of quantum Stokes matrices $S_h(u)$ and $K_h(u)$ (as $h \to \infty$) and the canonical bases of quantum groups and quantum symmetric pairs respectively. In the end, we remark that the focus of this paper is on the gcKZ equation associated to the symmetric pair of Type AI. See Remark 2.3 for a discussion on the generalization to general types.

2. Monodromy of Generalized Cyclotomic Knizhnik–Zamolodchikov Equations

This section computes the monodromy of gcKZ equations and proves Theorem 1.1. In particular, Sect. 2.1 and 2.2 introduce the gcKZ equation and its monodromy representation. Section 2.3 introduces the canonical solutions of gcKZ equations. In the end, Sect. 2.4 computes the monodromy of gcKZ equations and proves Theorem 1.1.

2.1. Generalized cyclotomic KZ equations associated to the symmetric pair $(\mathfrak{gl}_n, \mathfrak{so}_n)$.

Let $\tau$ be the involution on $\mathfrak{g} = \mathfrak{gl}_n$ fixing the Lie subalgebra $\mathfrak{k} = \mathfrak{so}_n$. Let $V$ be a finite dimensional $U(\mathfrak{g}) \rtimes \mathbb{Z}/2\mathbb{Z}$-module, and $W$ be a finite dimensional $U(\mathfrak{k})$-module.

**Definition 2.1.** The gcKZ equation, for a function $F(z_1, \ldots, z_n)$ of $n$ complex variables with values in $W \otimes V^\otimes n$, is

$$\kappa \frac{\partial F}{\partial z_i} = \left( u^{(i)} + \frac{2\Omega^0_{ij} + C^{(i)}_{\mathfrak{k}}}{z_i} + \sum_{j \neq i, j=1}^{n} \frac{\Omega^i_j}{z_i - z_j} + \sum_{j \neq i, j=1}^{n} \frac{2\Omega^i_j - \Omega^j_i}{z_i + z_j} \right) F, \text{ for } i = 1, \ldots, n. \quad (8)$$

Here $\Omega^{ij}$ or $\Omega^{ij}_{\mathfrak{k}}$ means $\Omega$ or $\Omega_{\mathfrak{k}}$ acting on the $i$-th and $j$-th factors of $W \otimes V^\otimes n$ (we index $W$ as the 0-th factor), $u^{(i)}$ and $C^{(i)}_{\mathfrak{k}}$ mean $u$ and $C_{\mathfrak{k}}$ acting on the $i$-th $V$ factor of $W \otimes V^\otimes n$.

We assume henceforth that $\kappa$ is a purely imaginary number and $u \in h_{\text{reg}}(\mathbb{R})$.

**Proposition 2.2.** The gcKZ equation is a compatible system of differential equations over the configuration space $X_n = (\mathbb{C}^\times)^n \setminus \{ z_i = \pm z_j \}$.

**Proof.** Note that when $u$ vanishes, it reduces to the cyclotomic KZ equation (see e.g., [15, Definition 3.3]), which is compatible (see e.g., [15, Lemma 3.4]). So it is necessary to check for different $i$ and $j$

$$[u^{(i)}, \Omega^{ij}_{\mathfrak{k}}] - [\Omega^{ij}_{\mathfrak{k}}, u^{(j)}] = 0.$$
Without loss of generality, we assume $i = 1$ and $j = 2$. The first identity follows from $[u^{(1)} + u^{(2)}, \Omega] = 0$. By $(\text{id} \otimes \tau)(2\Omega - \Omega) = \Omega$, the second identity is equivalent to $(\text{id} \otimes \tau)[u^{(1)} - \tau(u)^{(2)}, \Omega] = 0$, which follows from the relation $\tau(u) = -u$ and the first identity.

\[ [u^{(i)}, 2\Omega^{ij} - \Omega^{ij}] + [2\Omega^{ij} - \Omega^{ij}, u^{(j)}] = 0. \]

\textbf{Remark 2.3.} The cyclotomic KZ equation is defined for the symmetric pairs of any types. This paper only focuses on the symmetric pair $(\mathfrak{gl}_n, \mathfrak{so}_n)$ of type $A_l$, simply because in this case there is an obvious way to add the irregular singularities, parameterized by a regular Cartan element $u$, to the cyclotomic KZ equation as in (8), such that the resulting gcKZ equation is still compatible. However, for the symmetric pair $(\mathfrak{gl}_n, \mathfrak{sp}_n)$ of type $A_{2l}$, the associated differential equation (8) (i.e., replacing $\ell = \mathfrak{so}_n$ by $\ell = \mathfrak{sp}_n$) is not compatible any more. Thus extra effort is needed for generalizing the gcKZ equation with irregular singularities to other types.

\section*{2.2. Braid groups in type $B$ and monodromy representation of gcKZ equations.}

There is a natural action of $G_{n,2} = (\mathbb{Z}/2\mathbb{Z})^n \times S_n$ on $X_n$, where the symmetric group $S_n$ acts by permutation of variables, and $i$-th $\mathbb{Z}/2\mathbb{Z}$ acts on $i$-th variable by sign permutation. The action of $\tau$ on $V$ induces an action of $(\mathbb{Z}/2\mathbb{Z})^n$ on $W \otimes V^{\otimes n}$ (which acts trivially on $W$), and $S_n$ acts on $W \otimes V^{\otimes n}$ by permuting the $V$ components. Then the gcKZ equation is $G_{n,2}$-equivariant with respect to these actions, which implies that it also induces a system on the quotient space $X_n/G_{n,2}$. Actually, by $(\text{id} \otimes \tau)(\Omega) = 2\Omega - \Omega$, the last two terms of the equation’s coefficients can be written in a more symmetric way as

\[ \sum_{k=1,2} \sum_{j \neq i, j=1}^{n} \frac{(\text{id} \otimes \tau^k(\Omega))^{ij}}{z_{i+1} + \tau^k(z_j)}. \]

Following [7], the fundamental group of $X_n/G_{n,2}$ is isomorphic to the braid group $B_n^1$ in type $B$ with generators $\sigma, b_1, \ldots, b_{n-1}$ and relations

\[ \sigma b_i = b_i \sigma, \quad i \geq 2, \tag{9} \]

\[ \sigma b_1 \sigma b_1 = b_1 \sigma b_1 \sigma, \tag{10} \]

\[ b_i b_j = b_j b_i, \quad |i - j| > 1, \tag{11} \]

\[ b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}. \tag{12} \]

Actually choose a base point $z = (z_1, \ldots, z_n)$ such that $z_i \in \mathbb{R}, 0 < z_1 < z_2 < \cdots < z_n$, then a homomorphism is given by $B_n^1 \to \pi_1(X_n/G_{n,2}); \sigma \mapsto \text{the path in Fig. 1}, b_i \mapsto \text{the path in Fig. 2}.$

The operators of holonomy along the paths $\sigma, b_1, \ldots, b_{n-1}$ gives a representation of $\pi_1(X_n/G_{n,2}) \cong B_n^1$. To be more precise, we fix a base point $z$, and denote by
in the rest of this section, we fix a $a$ and $-a$ and study the fundamental solutions of (8) with a prescribed asymptotics. To this end, let us consider the map

$$P : \mathbb{C}^n \times X_n \to X_n; \quad P(z, \xi_1, \ldots, \xi_n) = (z\xi_1, \ldots, z\xi_n).$$

Then the pull-back of the gcKZ equation (8) under $P$ becomes

$$\kappa \frac{\partial F}{\partial z} = \left( \sum_{i=1}^{n} \xi_i u^{(i)} + \frac{2 \sum_{0 \leq i < j \leq n} \Omega^i_{j} \Omega^j_i}{z} + \sum_{i=1}^{n} C^{(i)} \right) \cdot F, \quad i = 1, \ldots, n. \quad (13)$$

In the rest of this section, we fix a $u \in h_{\text{reg}}(\mathbb{R})$. Let us first assume that $\xi_1, \ldots, \xi_n$ are real, and for any $k = -1, 0, 1, \ldots, n - 1$, define the domain in $(\mathbb{R}^\times)^n$,

$$D_{-1} : = \{ \xi \in \mathbb{R}^n | \xi_1 < 0 < \xi_2 < \cdots < \xi_n \},$$

$$D_0 : = \{ \xi \in \mathbb{R}^n | 0 < \xi_1 < \xi_2 < \cdots < \xi_n \},$$

$$D_k : = \{ \xi \in \mathbb{R}^n | 0 < \xi_1 < \cdots < \xi_{k-1} < \xi_k+1 < \xi_k < \xi_{k+1} < \cdots < \xi_n \} \text{ for } k = 1, \ldots, n - 1.$$

For any fixed point $\xi \in D_k$, equation (13) becomes a meromorphic ordinary differential equation with an irregular singularity at $z = \infty$. There is a standard way to produce solutions of equation (13) with prescribed asymptotics at $z = \infty$.

Let us take a root space decomposition $g = h \oplus_{\alpha \in \Delta} \mathbb{C} e_{\alpha}$ with $(e_{\alpha}, e_{-\alpha}) = 1$, and for any root $\alpha \in \Delta$. In particular, $h$ is the space of diagonal matrices, and for each $\alpha$ the elements $e_{\alpha}, e_{-\alpha}$ and $\tau(e_{\alpha})$ are the generators of $g = gl_n$ taking the forms of $e_{ij}, e_{ji}$ and $-e_{ji}$ respectively. Set $\Omega_{\alpha} = e_{\alpha} \otimes e_{-\alpha}, \Omega_{\xi, \alpha} = \frac{1}{2}(e_{\alpha} + \tau(e_{\alpha})) \otimes e_{-\alpha}$ and recall that $C_{\alpha} = e_{\alpha} e_{-\alpha}, C_{\xi, \alpha} = \frac{1}{2}(e_{\alpha} + \tau(e_{\alpha})) e_{-\alpha}$. Then

**Lemma 2.5.** The element

$$Y(\xi) = -\sum_{\alpha \in \Delta} \frac{1}{\alpha(u)} \left( \sum_{1 \leq i < j \leq n} \left( \frac{\Omega^i_{j} \Omega^j_{i}}{\xi_i - \xi_j} + \frac{2 \Omega^i_{j, \alpha} - \Omega^i_{j, \alpha}}{\xi_i + \xi_j} \right) - \sum_{i=1}^{n} \frac{2 \Omega^0_{i, \alpha} + \frac{1}{2} C^{(i)}_{\xi, \alpha}}{\xi_i} \right) \quad (15)$$
satisfies
\[ [Y(\xi), \sum_{i=1}^{n} \xi_i u^{(i)}] = 2 \sum_{0 \leq i < j \leq n} \Omega^{ij}_\xi + \sum_{k=1}^{n} (C^{(i)}_\xi - C^{(i)}_0), \tag{16} \]
where \( C_0 := \frac{1}{2} \sum_{\alpha \in \Delta} C_\alpha \), and the right hand sides of (15) and (16) are seen as the image in the representation space \( \text{End}(W \otimes V^\otimes n) \).

**Proof.** Note that since \( \xi \in D_k \), \( Y(\xi) \) is well defined, i.e., the denominator in (15) can not be zero. To verify (16), one uses the identities (recall that \( \tau(u) = -u \))

\[
\sum_{k=1}^{n} \xi_k u^{(k)}, \Omega^{ij}_\alpha] = \sum_{k=1}^{n} \xi_k (\delta_{ki} - \delta_{kj}) \alpha(u) \Omega^{ij}_\alpha = (\xi_i - \xi_j) \alpha(u) \Omega^{ij}_\alpha,
\]

\[
\sum_{k=1}^{n} \xi_k u^{(k)}, \tau^{(j)}(\Omega^{ij}_\alpha)] = \sum_{k=1}^{n} \xi_k (1 - 2 \delta_{kj}) \tau^{(j)}([u^{(k)}, \Omega^{ij}_\alpha]) = (\xi_i + \xi_j) \alpha(u) \tau^{(j)}(\Omega^{ij}_\alpha)
\]

\[
\sum_{k=1}^{n} \xi_k u^{(k)}, -2 \Omega^{0i}_{\xi,\alpha} - \frac{1}{2} C^{(i)}_{\xi,\alpha} = \xi_i \alpha(u) (2 \Omega^{0i}_{\xi,\alpha} + C^{(i)}_{\xi,\alpha} - \frac{1}{2} C^{(i)}_\alpha),
\]

and the identities

\[
(id \otimes \tau)(\Omega) = 2 \Omega - \Omega, \quad \Omega = \sum_{\alpha \in \Delta} \Omega_{\xi,\alpha}, \quad C = \sum_{\alpha \in \Delta} C_{\xi,\alpha}.
\]

\[ \square \]

For any fixed \( u \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \), there exists an open dense subset \( D_u \subset D_k \), such that for any \( \xi \in D_u \) the image of the adjoint operator \( \text{ad}_{\sum_{i=1}^{n} \xi_i u^{(i)}} \) in \( \text{End}(W \otimes V^\otimes n) \) has the maximal dimension. Then the vector space \( \text{End}(W \otimes V^\otimes n) \) decomposes into the direct sum of the kernel space and the image space \( \text{Im} \) of the operator \( \text{ad}_{\sum_{i=1}^{n} \xi_i u^{(i)}} \), and \( \text{ad}_{\sum_{i=1}^{n} \xi_i u^{(i)}} \) is invertible on \( \text{Im} \). Since \( \left[ \sum_{i=1}^{n} \xi_i u^{(i)}, \sum_{i=1}^{n} C^{(i)}_0 \right] = 0 \), following Lemma 2.5 the projection of \( 2 \sum_{0 \leq i < j \leq n} \Omega^{ij}_\xi + \sum_{i=1}^{n} C^{(i)}_\xi \) to the centralizer of \( \sum_{i=1}^{n} \xi_i u^{(i)} \) is \( \sum_{i=1}^{n} C^{(i)}_0 \).

**Lemma 2.6.** For any fixed \( u \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \) and \( \xi \in D_u \subset D_k \), the ordinary differential equation (13) has a unique formal fundamental solution taking the form

\[
\hat{F}(z; \xi) = \hat{H}(z; \xi) e^{\xi (\sum_{i=1}^{n} \xi_i u^{(i)}) \frac{1}{2} \sum_{i=1}^{n} C^{(i)}_0}, \quad \text{for } \hat{H} = 1 + h_1 z^{-1} + h_2 z^{-1} + \cdots, \tag{17}
\]

where each \( h_i(\xi) \in \text{End}(W \otimes V^\otimes n) \).

**Proof.** Plugging \( \hat{F} \) in (17) into the equation (13) gives rise to the equation for the formal sum \( \hat{H} = 1 + h_1 z^{-1} + h_2 z^{-1} + \cdots, \)

\[
\kappa \frac{d\hat{H}}{dz} + \hat{H} \left( \sum_{i=1}^{n} \xi_i u^{(i)} + \sum_{i=1}^{n} C^{(i)}_0 \right) = \left( \sum_{i=1}^{n} \xi_i u^{(i)} + \sum_{0 \leq i < j \leq n} \Omega^{ij}_\xi + \sum_{i=1}^{n} C^{(i)}_\xi \right) \cdot \hat{H}. \tag{18}
\]
Comparing the coefficients of $z^{-1}$ on the both sides of (18), we see that $h_1(\xi)$ satisfies

$$[h_1(\xi), \sum_{i=1}^{n} \xi_i u^{(i)}] = 2 \sum_{0 \leq i < j \leq n} \Omega_{ij} + \sum_{k=1}^{n} (C_{ij} - C_{0j}).$$

(19)

Comparing the coefficients of $z^{-i}$, we see that $h_j(\xi)$, for all $j > 1$, satisfies

$$[h_j(\xi), \sum_{i=1}^{n} \xi_i u^{(i)}] = (\kappa (j - 1) + 2 \sum_{0 \leq i < j \leq n} \Omega_{ij} + \sum_{k=1}^{n} C_{ij}) \cdot h_{j-1}(\xi) - h_{j-1}(\xi) \cdot C_{0j}.$$  

(20)

All $h_j(\xi)$ are then uniquely determined in a recursive way: as $j = 1$, by Lemma 2.5, any solution $h_1(\xi)$ of (19) can be written as a sum $X(\xi) + X_1(\xi)$, where $X_1(\xi)$ is an element satisfying $[X_1(\xi), \sum_{i=1}^{n} \xi_i u^{(i)}] = 0$. Different chosen $X_1(\xi)$ will make the right hand side of (20) for $j = 2$ different. While since $\kappa$ is purely imaginary, there exists a unique $X_1(\xi)$ such that the right hand side of (20) for $j = 2$ lives in the image space $\text{Im}$ of the operator $\text{ad}_{\sum_{i=1}^{n} \xi_i u^{(i)}}$. That is there is a unique $h_1(\xi)$ such that the relation (20) for $j = 2$ is well posed. Note that for $\xi \in D_u$ the operator $\text{ad}_{\sum_{i=1}^{n} \xi_i u^{(i)}}$ is invertible on $\text{Im}$, we can continue and check that each $h_j(\xi)$ is determined by (20) up to some term $X_j(\xi)$ in the centralizer of $\sum_{i=1}^{n} \xi_i u^{(i)}$, while the term $X_j(\xi)$ is further fixed by requiring the right hand side of the relation (20) for $j + 1$ is in the space $\text{Im}$. It proves the existence and uniqueness of the formal fundamental solution.

Although $\hat{H}$ is a formal power series whose radius of convergence is in general zero, it is known that (see e.g., [5, 32]) its Borel resummation (Borel-Laplace transform) gives a holomorphic function in each Stokes sector around $z = \infty$. In particular, by our assumption on $\kappa$ and $u$, the irregular term $\sum_{i=1}^{n} \frac{1}{\kappa} \xi_i u^{(i)}$ is purely imaginary, thus the only two Stokes sectors of equation (13) are the right and left half planes $\mathbb{H}_+$. The Borel resummation of $\hat{H}$ in $\mathbb{H}_+$ can be used to construct a unique holomorphic function $H_k : \mathbb{H}_+ \to \text{End}(W \otimes V^{\otimes n})$ defined in a bigger sector $\mathbb{H} = \{\rho e^{i\theta} \mid \rho > 0, -\pi < \theta < \pi\}$, with the prescribed asymptotics $\hat{H}$ as $z \mapsto \infty$ within $\mathbb{H}_+$ (see Appendix A for more details). Let us choose the branch of $\log(z)$, which is real on the positive real axis, with a cut along the nonnegative imaginary axis $i\mathbb{R}_{\geq 0}$. Thus we have

**Lemma 2.7.** The function $F_{D_k} := H_k e^{\frac{i}{\kappa} (\sum \xi_i u^{(i)})} \frac{1}{\kappa} \sum_{i=1}^{n} C_{0i}^{(i)}$ is the unique holomorphic solution of (13) on $\mathbb{H}_+$, with the prescribed asymptotics $F_{D_k} e^{-\frac{i}{\kappa} (\sum \xi_i u^{(i)})} z^{-\frac{1}{\kappa} \sum_{i=1}^{n} C_{0i}^{(i)}} \sim \hat{H}(z)$ within $\mathbb{H}_+$.

Thus, for fixed $u$ and any fixed $\xi \in D_u$, we have found a solution of equation (13). For fixed $u$ let us now consider the variation of $\xi$ in $D_u \cap D_k \subset D_k$. By the compatibility, a solution of (13) and (14) will take the form $F_{D_k}(z; \xi) G_k(\xi)$ for $G_k(\xi)$ being a function of $\xi$. The following proposition provides such a choice of $G_k(\xi)$.

**Proposition 2.8.** For any fixed $u \in \mathfrak{h}_{\text{reg}}(\mathbb{R})$, the function $F_k(z, \xi) : \mathbb{H}_+ \times (D_u \cap D_k) \to \text{End}(W \otimes V^{\otimes n})$, given by

$$F_k(z, \xi) := H_k e^{\frac{1}{\kappa} (\sum z \xi_i u^{(i)})} \frac{1}{\kappa} (\sum C_{0i}^{(i)}) \cdot \prod_{i=1}^{n} (\xi_i) \frac{C_{0i}^{(i)}}{\kappa} \prod_{1 \leq i < j \leq n} \left( \frac{\xi_i - \xi_j}{\xi_i + \xi_j} \right)^{\frac{\nu_{ij}}{2}} T_k,$$

(21)
satisfies the equations (13) and (14), where the constants $T_0 = 1$ and $T_k := e^{\frac{\pi i}{k} \Omega_i^{k+1}}$ for $k = 1, \ldots, n - 1$, and $T_{-1} := e^{\frac{\pi i}{\kappa} C_0^{(1)}}$.

**Proof.**

Set

$$\nabla_z = \kappa \frac{d}{dz} \left( \sum_{i=1}^n \xi_i u^{(i)} + \frac{2 \sum_{0 \leq i < j \leq n} \Omega_{ij}^{(i)} + \sum_{i=1}^n C_{\xi}^{(i)}}{z} \right),$$

$$\Phi = \sum_{i=1}^n \left( z u^{(i)} + \frac{2 \Omega_{ij}^{(i)} + C_{\xi}^{(i)}}{\xi_i} + \sum_{j \neq i} \frac{\Omega_{ij}^{(i)}}{\xi_i - \xi_j} + \sum_{j \neq i} \frac{2 \Omega_{ij}^{(i)} - \Omega_{ij}^{(i)}}{\xi_i + \xi_j} \right) d\xi_i.$$  

From the compatibility of the equations (13) and (14), we have $\nabla_z (\kappa d_\xi F_k - \Phi F_k) = 0$, where $d_\xi$ denotes the exterior differentiation with respect to parameters $\xi_i$s. It implies that there exists a 1-form $B_k(\xi)$ independent of $z$ such that $\kappa d_\xi F_k = \Phi F_k = F_k B_k$, or equivalently

$$\kappa d_\xi F_k \cdot F_k^{-1} = \Phi = F_k B_k F_k^{-1}. \quad \text{(22)}$$

To show $B_k = 0$, we compare the expansion of the both sides of this equation at $z = \infty$.

Firstly, the formal sum $\hat{H} = 1 + h_{1z}^{-1} + h_{2z}^{-2} + \cdots$ in (17) satisfies

$$\kappa \frac{dH}{dz} + H \cdot \left( \sum_{i=1}^n \xi_i u^{(i)} + \sum_{i=1}^n C_{\xi}^{(i)} \right) = \left( \sum_{i=1}^n \xi_i u^{(i)} + \frac{2 \sum_{i=1}^n C_{\xi}^{(i)}}{z} \right) \cdot H.$$  

Comparing the coefficients of $z^{-1}$, we have seen that $h_{1}(\xi)$ satisfies the identity (19). Now differentiating (21) gives

$$\kappa d_\xi F_k \cdot F_k^{-1} = \kappa d_\xi H_k \cdot H_k^{-1} + H_k \sum_{i=1}^n \left( u^{(i)} z + \frac{C_{\xi}^{(i)}}{\xi_i} + \frac{[\Omega]^{ij}}{\xi_i - \xi_j} - \frac{[\Omega]^{ij}}{\xi_i + \xi_j} \right) d\xi_i \cdot H_k^{-1}.$$  

Then the asymptotics $H_k \sim \hat{H} = 1 + h_{1z}^{-1} + \cdots$ in $\hat{H}_+$ and the identity (19), we get

$$d_\xi F_k \cdot F_k^{-1} = \Phi + O(z^{-1}) \text{ for } z \to \infty \text{ within } \hat{H}_+. \quad \text{(23)}$$

It thus follows from the identity (22) that $F_k B_k F_k^{-1} = O(z^{-1})$ for $z \to \infty$ within $\hat{H}_+$.

Secondly, by the definition of $F_k$ we have

$$G_k(\xi)^{-1} H_k^{-1} F_k B_k F_k^{-1} H_k G_k(\xi) = z^{\frac{1}{\kappa}} \left( \sum_{i=1}^n C_{\xi}^{(i)} \right) e^{\frac{\pi i}{\kappa} \sum_{i=1}^n \xi_i u^{(i)}} B_k e^{-\frac{\pi i}{\kappa} \sum_{i=1}^n \xi_i u^{(i)}} z^{-\frac{1}{\kappa}} \left( \sum_{i=1}^n C_{\xi}^{(i)} \right),$$

where

$$G_k(\xi) = \prod_{i=1}^n \xi_i \prod_{1 \leq i < j \leq n} \left( \frac{\xi_i - \xi_j}{\xi_i + \xi_j} \right) \frac{1}{\kappa \Omega_{ij}^{(i)}} \cdot T_k.$$  

By the asymptotics of $H_k$ and $F_k B_k F_k^{-1}$, it implies

$$z^{\frac{1}{\kappa}} \left( \sum_{i=1}^n C_{\xi}^{(i)} \right) e^{\frac{\pi i}{\kappa} \sum_{i=1}^n \xi_i u^{(i)}} B_k e^{-\frac{\pi i}{\kappa} \sum_{i=1}^n \xi_i u^{(i)}} z^{-\frac{1}{\kappa}} \left( \sum_{i=1}^n C_{\xi}^{(i)} \right) = O(z^{-1}) \text{ for } z \to \infty \text{ within } \hat{H}_+. \quad \text{(24)}$$
Since the exponentials dominate, \( B_k \) must commute with \( e^{-\frac{z}{\kappa} \sum_i \xi_i u^{(i)}} \), otherwise some entries of
\[
e^{-\frac{z}{\kappa} \sum_i \xi_i u^{(i)}} B_k e^{-\frac{z}{\kappa} \sum_i \xi_i u^{(i)}} \in \text{End}(W \otimes V^\otimes n)
\]
would grow exponentially for the opening of the sector \( \mathbb{H}_+ \) is larger than \( \pi \), which would contradict with the asymptotics (24). Furthermore, under our assumption of \( \kappa \) is purely imaginary, one checks that if \( z^{-\frac{1}{\kappa} \sum_i C_0^{(i)} B_k z^{-\frac{1}{\kappa} \sum_i C_0^{(i)}}} = O(z^{-1}) \), then \( B_k \) must be zero. It proves that \( d_{\xi} F_k(z, \xi) = \Phi F_k(z, \xi) \).

Note that for any \( u \) the solution \( F_k(z, \xi) \) given in Proposition 2.8 can be analytically continued from \( \mathbb{H}_+ \times (D_u \cap D_k) \) to the domain \( \mathbb{H}_+ \times D_k \). Therefore, switching back to the coordinates \( z_i = z \xi_i \) for \( i = 1, \ldots, n \), Proposition 2.8 enables us to construct a unique (therefore canonical) solution \( F_k(z_1, \ldots, z_n) \) of the equation (2.1) on each domain \( P(\mathbb{H}_+ \times D_k) \subset X_n \), with the prescribed asymptotics.

**Definition 2.9.** For any \( k = -1, 0, \ldots, n - 1 \), we call \( F_k(z_1, \ldots, z_n) \) the canonical solution of (8).

2.4. Monodromy representations. To compute the monodromy of equation (8), we take an infinite base point \( z = (z_1, \ldots, z_n) \) in \( D_0 \), where \( z_i \ll z_{i+1} \) for any \( i \) (See e.g., [19] Section 8.4 for the discussion on the monodromy representation with respect to an infinite base point). Then the induced braid group representation is
\[
\pi_1(X_n/G_{n,2}) \rightarrow \text{End}(W \otimes V^\otimes n); \quad \sigma \mapsto \tau^{(1)}(F_{-1}(z)F_0(z)^{-1}),
\]
\[
b_i \mapsto s_i(F_i(z) \cdot F_0(z)^{-1}),
\]
where \( \sigma \) and \( b'_i \)'s are the generators of \( B_n^1 \) as in Sect. 2.2, and the image of each generator is the ratio of the corresponding canonical solutions \( F_0 \) and \( F_i \) (after \( F_0 \) has been analytic continued to \( D_i \) along the path \( \tau \) or \( b_i \)).

**Theorem 2.10.** The monodromy representation of the gcKZ_n equation, with respect to the infinite base point, is given by
\[
\pi_1(X_n/G_{n,2}) \cong B_n \rightarrow \text{End}(W \otimes V^\otimes n); \quad \sigma \mapsto \tau^{(1)}(K(u)), \quad b_i \mapsto s_i(S(u)^{i,i+1}),
\]
where \( S(u) \) and \( K(u) \) are the Stokes matrices of equation (1) and (2) (associated to \( V \) and \( W \)) respectively.

**Proof.** To get the monodromy along the path \( b_i \), we need to compute the ratio of the canonical solutions \( F_0 \) and \( F_i \). To this end, we need the factorization property of the gcKZ_n equation.

**The case of** \( i \geq 1 \). Let us study the factorization problem in the asymptotic zone
\[
\frac{z_i - z_{i+1}}{z_i} \left| \frac{z_i - z_{i+1}}{z_i} \right|, \quad \frac{z_i - z_{i+1}}{z_{i+1}} \left| \frac{z_i - z_{i+1}}{z_{i+1}} \right|, \quad \frac{z_i - z_{i+1}}{z_{i+1}} \leq \frac{z_i}{z_{i+1}} \right|, \quad |z_i - z_{i+1}| \ll 1, \quad \forall \; j \neq i, i + 1.
\]

Let us introduce the new coordinates \( \omega = z_i, t = z_{i+1} - z_i, z_1, \ldots, z_{i-1}, z_{i+2}, \ldots, z_n \). Then for any fixed \( t \), the gcKZ_n equation (8) becomes an equation for a function \( I \) of \( n - 1 \) complex variables \( \omega, z_1, \ldots, z_{i-1}, z_{i+2}, \ldots, z_n \),
\[
\kappa \frac{\partial I}{\partial z_j} = \left( u^{(j)} + \frac{2s_0^0(t) + c^{(j)}}{z_j} \sum_{k \neq j} \frac{\Omega^{jk} + \sum_{k \neq j} 2\Omega^{jk} - \Omega^{jk}}{z_j + z_k} \right) \cdot I, \quad j \neq i, i + 1.
\]
\[
\kappa \frac{\partial I}{\partial \omega} = \left( u^{(i)} + u^{(i+1)} + \frac{2\Omega^0_t}{\omega} + \frac{2\Omega_{t,i}^{0,i+1} + C_{t}^{(i)}}{\omega + t} + \frac{2\Omega_{t,i+1}^{i+1} - \Omega_{t,i+1}^{i,i+1}}{\omega + \frac{1}{2}t} + M \right) \cdot I,
\]

where \(z_{i+1} = t + z_i\), and

\[
M = \sum_{k \neq i,i+1} \frac{\Omega^i_k}{\omega - z_k} + \sum_{k \neq i,i+1} \frac{2\Omega^0_t - \Omega^{i,k}}{\omega + z_k} + \sum_{k \neq i,i+1} \frac{\Omega^{i+1,k}}{\omega + t - z_k} + \sum_{k \neq i,i+1} \frac{2\Omega^{i+1,k} - \Omega^{i+1,k}}{\omega + t + z_k}.
\]

As in Sect. 2.3, let us take the new coordinates \(\omega = z\xi_i, t = z\xi_{i+1} - z\xi_i\) and \(z_j = z\xi_j\) for \(j = 1, \ldots, i-1, i+1, \ldots, n\). Then the system of equations (26) and (27) can be rewritten as a system of a differential equation with respect to \(z\) and \(n - 2\) differential equations with respect to variables \(\xi_1, \ldots, \xi_i, \xi_{i+2}, \ldots, \xi_n\). Here we restrict \(\xi_j\) for \(j \neq i + 1\) to be real, but allow the parameter \(\xi_{i+1}\) determined by the parameter \(t\) to be arbitrary complex number. Since \(\xi_{i+1}\) does not appear in the irregular part, the differential equation with respect to \(z\) still has two Stokes sectors \(\mathbb{H}_\pm\). And applying the same proof of Proposition 2.8, one shows that for any fixed \(u \in b_{\text{reg}}(\mathbb{R})\) and fixed constant \(t\), the system (26) and (27) has a unique solution \(I_i : \hat{\mathbb{H}}_+ \times D_i(n - 1) \to \text{End}(W \otimes V^\otimes n)\), with the following asymptotics as \(z \to \infty\) within \(\hat{\mathbb{H}}_+\) (in terms of \(z\) and \(\xi_j\))

\[
I_i \cdot N_i \sim 1.
\]

Here

\[
N_i = e^{-\frac{1}{\kappa} \left( \sum_{j \neq i+1} z\xi_j u^{(j)} \right)} e^{-\frac{z\xi_i + z\xi_{i+1}}{2}} \left( \prod_{1 \leq j < k \leq n} \frac{z\xi_j - z\xi_k}{z\xi_j + z\xi_k} \right)^{\frac{1}{\kappa} \left[ \Omega_{i,i+1} \right]} \prod_{j=1}^{n} \left( \frac{z\xi_j}{z\xi_j + \xi_j} \right)^{\frac{n-j}{\kappa} \left[ c_{i}^{(j)} \right]}.\]

and \(D_i(n - 1) \subset \mathbb{R}^{n-1}\) is the projection of \(D_k \subset \mathbb{R}^n\) by forgetting the \(i + 1\) component.

Now let us consider the equation for a \(\text{End}(W \otimes V^\otimes n)\) valued function \(Y(z)\),

\[
\kappa \frac{dY}{dz} = \left( u^{(i+1)} + \frac{\Omega_{i,i+1}}{z} \right) \cdot Y.
\]

It has a unique solution \(Y_\pm(z)\) defined in the sectors \(\hat{\mathbb{H}}_\pm\) and at the same time having the prescribed asymptotics

\[
Y_\pm(z) e^{-z\left[ u^{(i+1)} \right]} z^{-\left[ \Omega_{i,i+1} \right]} \sim 1 \quad \text{as} \quad z \to \infty \quad \text{within} \quad \hat{\mathbb{H}}_\pm.
\]

**Lemma 2.11.** In their common domain, we have

\[
F_0(z_1, \ldots, z_n) = I_i(z_1, \ldots, z_{i-1}, z_{i+2}, \ldots, z_n; z_i, z_{i+1}) \cdot Y_+(z_{i+1} - z_i),
\]

\[
F_i(z_1, \ldots, z_n) = I_i(z_1, \ldots, z_{i-1}, z_{i+2}, \ldots, z_n; z_i, z_{i+1}) \cdot Y_-(z_{i+1} - z_i) \cdot T_i.
\]
Proof. Note that the product functions on the right hand sides of the above identities also satisfy the gcKZ\(_n\) equation. Thus, this lemma can be proved by comparing the asymptotics of both hand sides of the identities as \(z \to \infty\) within \(\hat{\mathbb{H}}_+\) (here \(z_k = z\xi_k\) for all \(k = 1, \ldots, n\)) and at the same time as \(z_1, \ldots, z_n\) in the asymptotic zone (25).

Recall that as a solution of (26) and (27), \(I_i\) is uniquely determined by the prescribed asymptotics (28). Although a priori the asymptotics of \(I_i\) only holds for fixed \(t = z\xi_i + z\xi_{i+1}\), one can prove that it has the same asymptotics as \(\xi_i + z\xi_{i+1}\) and \(z \to \infty\). To see this, one checks that the general terms, of the unique formal solution \(I_i\) (whose Borel resummation produces \(I_i\)) of (26) and (27), have an uniform bound with respect to \(\xi_i/\xi_{i+1} \in \mathbb{R} \setminus [0, 1]\). It is a standard procedure that we skip here. See e.g., [46] for an argument for a similar situation.

Since \(u^{(i)} + u^{(i+1)}\) and \(C_0^{(i)} + C_0^{(i+1)} + [\Omega]^{i,i+1}\) commute with the coefficient of the equation (29), we have

\[
Y_+ (z\xi_i + z\xi_{i+1}) \cdot e^{-\frac{z\xi_i}{\kappa} (u^{(i)} + u^{(i+1)})} = Y_+ (z\xi_i + z\xi_{i+1}) \cdot e^{-\frac{z\xi_i}{\kappa} (u^{(i)} + u^{(i+1)})},
\]

\[
Y_+ (z\xi_i + z\xi_{i+1}) \cdot (z\xi_i) \cdot e^{-\frac{z\xi_i}{\kappa} (u^{(i)} + u^{(i+1)})} = (z\xi_i) \cdot e^{-\frac{z\xi_i}{\kappa} (u^{(i)} + u^{(i+1)})} \cdot Y_+ (z\xi_i + z\xi_{i+1}).
\]

Using these identities, we get as \(\xi_i/\xi_{i+1} \to 1\) and \(z \to \infty\) within \(\hat{\mathbb{H}}_+\),

\[
I_i \cdot Y_+ \cdot e^{-\frac{1}{\kappa} (\sum_i z\xi_i u^{(i)})} \cdot \prod_{i=1}^n (z\xi_i) \cdot \frac{e^{(i)\xi_i}}{\kappa} \prod_{1 \leq i < j \leq n} \left( \frac{\xi_i - \xi_j}{\xi_i + \xi_j} \right) \frac{\Omega^{i,j}}{\kappa} \sim 1.
\]

The uniqueness of the solution with the prescribed asymptotics then guarantees that \(I_i Y_+\) must equal to \(F_0\). In the component of \(\xi_i < \xi_{i+1}\) a similar argument shows that \(I_i Y_- = F_i\).

Therefore using the factorization properties of \(F_0\) and \(F_i\), see [46] for analog to the KZ equation with three variables, the analytic continuation of \(F_0\) along the path \(b_i\) amounts to the continuation of \(I_i \cdot Y_+ (z_i, z_{i+1})\) along \(b_i\). Note that \(I_i\) is defined for all \(t = z_i + z_{i+1} \in \mathbb{C}\) and therefore does not contribute monodromy from the continuation along \(b_i\). Thus, after the analytic continuation of \(F_0\), we have

\[
F_0^{-1} \cdot F_i = Y_+ (z_i + z_{i+1})^{-1} Y_- (z_i + z_{i+1}) \cdot T_i = S^{i,i+1}.
\]

Here the second identity uses the definition (3) of the Stokes matrix \(S\) (the continuation of \(Y_+ (z_i + z_{i+1})\) along \(b_i\) is same as the continuation of \(Y_+ (z)\) from \(\mathbb{H}_+\) to \(\mathbb{H}_-\) in a counterclockwise direction). It proves that the monodromy of gcKZ equation along \(b_i\) is given by \(s_i (S(u))^{i,i+1}\).

The case of \(i = -1\). In a similar way, by studying the factorization problem in the asymptotic zone \(|\frac{z_i}{z_{i+1}}| \ll 1\) for any \(i > 1\), we can get that

\[
F_0 = U (z_2, \ldots, z_n; z_1) \cdot F_+ (z_1) \quad \text{and} \quad F_{-1} = U (z_2, \ldots, z_n; z_1) \cdot F_- (z_1) \cdot T_{-1},
\]

where \(F_\pm\) are the canonical solutions of equation (2) defined in \(\hat{\mathbb{H}}_\pm\), i.e., \(\kappa \frac{dF_\pm}{dz_1} = (u^{(1)} + \Omega^{1,2}) \cdot F_\pm\), with the prescribed asymptotics, and for any \(z_1 \in \mathbb{C}\), \(U (z_2, \ldots, z_n; z_1)\) is a function of \(z_2, \ldots, z_n\). By the same argument, it follows that the monodromy along \(\sigma\) is given by \(\tau^{(1)} (K(u))\).

As a corollary, we have
Theorem 2.12. For any \( u \), the Stokes matrices \( K(u) \) and \( S(u) \) satisfy the Yang–Baxter and the \( \tau \)-twisted reflection equations.

Proof. It follows from Theorem 2.10 and the braid relation in type \( B \), as well as the fact \((\tau \otimes \tau)(S) = S\).

\[ \square \]

Remark 2.13. Following Felder-Markov-Tarasov-Varchenko \([20, \text{Theorem 3.1}]\), the gKZ equation has solutions taking the form of (confluent) hypergeometric integrals over twisted cycles (see e.g., \([42]\)). One can study the asymptotics of these solutions at the irregular singularity, and try to get an integral expression of the Stokes matrices. For example, in the \( \text{gl}_2 \) case, the gKZ equation, with two variables and valued in a two-dimensional vector space, reduces to a confluent hypergeometric equation, and the integral solutions in \([20]\) are related to the integral representations of confluent hypergeometric functions. The Stokes phenomenon of these solutions amounts to the different asymptotics of confluent hypergeometric function \( F_1(a, b; z) \) as \( z \to \infty \) from two different sectors. The comparison of the different asymptotics gives the Stokes matrices. More details can be found in the explicit computation of Stokes matrices in rank two \([6, \text{Proposition 8}]\). One should compare it to the \( \text{gl}_2 \) example of KZ equation with three variables, and the hypergeometric function \( F_1(a, b, c; z) \), see e.g., \([42, \text{Section 1.1}]\).

2.5. Variation of \( u \). In this subsection, we will study the derivations of the Stokes matrices \( S(u) \) and \( K(u) \) with respect to the parameter \( u \), i.e., give a proof of Theorem 1.3. It needs the following lemma.

Proposition 2.14. The canonical solutions \( F_k \), for any \( k = -1, 0, 1 \ldots, n - 1 \), satisfy

\[ \kappa d_\eta F = \left( \sum_{i=1}^{n} z_i du^{(i)} + \sum_{\alpha \in \Delta} \frac{d\alpha}{\alpha} \Delta^{(n+1)}(C_{\xi,\alpha}) \right) F - F \left( \sum_{\alpha \in \Delta} \frac{d\alpha}{\alpha}(C_{\xi,\alpha}^{(0)} + \sum_{i=1}^{n} C_{\alpha}^{(i)}) \right). \]

Here \( d_\eta \) is the de Rham differential on \( \eta(\mathbb{R}) \), \( \Delta \) is the coproduct on \( \mathfrak{g} \), and \( \Delta^{(n)} : U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes n} \) is the iterated coproduct recursively defined by \( \Delta^{(1)} = \text{id} \), and \( \Delta^{(n)} = (\Delta \otimes \text{id}^{\otimes (n-2)}) \circ \Delta^{(n-1)} \) for \( n \geq 2 \).

Proof. Let us prove it for any fix \( i \in \{-1, \ldots, n-1\} \). Let us use the setting in Proposition 2.8, i.e., consider the pull-back of the gcKZ equation under the map \( P : \mathbb{C}^\times \times X_n \to X_n \). Let us introduce the differential operators

\[ D_z := \kappa dz - \left( i \left( \sum_{i=1}^{n} \xi_i u^{(i)} + \frac{2 \sum_{i<j} \xi_i j \Omega^{(i)}_t + \sum_i C_{(i)}^{(i)}}{z} \right) - r \left( \sum_{i=1}^{n} \xi_i u^{(i)} + \sum_i C^{(i)}_0 \right) \right) dz, \]
\[ D_u := \kappa d_u - \left( \ell \left( \sum_{i=1}^{n} \xi_i d u^{(i)} + \sum_{\alpha \in \Delta} \frac{d \alpha}{\alpha} \Delta^{(n+1)} C_{t, \alpha} \right) \right) \]

\[ -r \left( \sum_{i=1}^{n} \xi_i d u^{(i)} + \sum_{\alpha \in \Delta} \frac{d \alpha}{\alpha} \left( C_{t, \alpha}^{(0)} + \sum_{i=1}^{n} C_{\alpha}^{(i)} \right) \right), \]

where \( l \) and \( r \) denote the left and right multiplication. First one checks that

\[ [D_u, D_z] = 0 \quad \text{and} \quad D_u^2 = 0, \quad \text{for any} \quad i, j = 1, \ldots, n. \]

Since the function \( F_k \) is a solution, we get \( D_z H_k = 0 \), see Proposition 2.8 for the definition of the function \( H_k \). It implies that \( D_z D_u H_k = D_u D_z H_k = 0 \). Thus to prove \( D_u H_k = 0 \), by the uniqueness argument we only need to show that \( D_u H_k \), as a function of \( z \), tends to 0 as \( z \to \infty \) in the sector \( \mathbb{H}_+ \). It can be seen as follows: plugging the expansion \( H_k \sim 1 + h_1 z^{-1} + O(z^{-2}) \) in \( D_u H_k \), we see that the limit of \( D_u H_k \) as \( z \to \infty \) within \( \mathbb{H}_+ \) is given by

\[ \left[ -\sum_{i=1}^{n} \xi_i d u^{(i)}, h_1 \right] - \sum_{\alpha \in \Delta} \frac{d \alpha}{\alpha} \Delta^{(n+1)} C_{t, \alpha}^{(0)} + \sum_{\alpha \in \Delta} \frac{d \alpha}{\alpha} \left( C_{t, \alpha}^{(0)} + \sum_{i=1}^{n} C_{\alpha}^{(i)} \right). \]

Then given the identity (19) satisfied by \( h_1 \), one checks that the above expression vanishes. \( \square \)

**Proof of Theorem 1.3.** From the definition of Stokes matrices and Proposition 2.14, recall that Theorem 2.10 (in particular the asymptotics in (30)) gives

\[ F_0^{-1} \cdot F_i = S^{i,i+1}. \]  (31)

Following Proposition 2.14, we have that both \( F_0 \) and \( F_i \) satisfy \( d_0 F = AF - BF \), where

\[ A = \sum_{i=1}^{n} z_i d u^{(i)} + \sum_{\alpha \in \Delta} \frac{d \alpha}{\alpha} \Delta^{(n+1)} C_{t, \alpha}^{(0)} \quad \text{and} \quad B = \sum_{\alpha \in \Delta} \frac{d \alpha}{\alpha} \left( C_{t, \alpha}^{(0)} + \sum_{i=1}^{n} C_{\alpha}^{(i)} \right). \]

Then taking the derivatives of both sides of (31) with respect to \( u \) gives

\[ d_0 S^{i,i+1} = -F_0^{-1} \cdot d_0 F_0 \cdot F_0^{-1} \cdot F_i + F_0^{-1} \cdot d_0 F_i \]
\[ = -F_0^{-1} \cdot AF_i + B F_0^{-1} \cdot F_i + F_0^{-1} AF_i - F_0^{-1} \cdot F_i B \]
\[ = [B, F_0^{-1} \cdot F_i] = [B, S^{i,i+1}]. \]

Taking respectively \( i = 1 \) and \( i = 0 \) in the above identity give rise to the expressions (4) and (4) in Theorem 1.3. \( \square \)

The equations (4) and (5), controlling the variation of \( u \), are equivalent to the isomonodromy deformation equations of the differential equations (1) and (2) respectively. We refer the reader to the theory of the isomonodromy deformation of linear systems of meromorphic ordinary differential equations by Jimbo, Miwa and Ueno [24]. Theorem 1.3 then states that the isomonodromy deformation amounts to gauge transformations of universal solutions of Yang–Baxter and reflection equations. In the end, we remark that it is interesting to study the asymptotics of the Stokes matrices \( S(u, \kappa) \) and \( K(u, \kappa) \) of the equations (1) and (2), as \( \kappa \to 0 \). Following [45,46], the leading order in the
asymptotics of \( S(u, \kappa) \), as \( \kappa \to 0 \), is related to the theory of canonical bases of quantum group \( U_q(\mathfrak{gl}_n) \). It is interesting to see if the leading order in the asymptotics of \( K(u, \kappa) \) is related to the theory of canonical bases for quantum symmetric pairs introduced by Bao and Wang in [4].

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**Appendices**

**A Canonical Solutions, Stokes Matrices and Connection Matrices**

Let us consider the linear system of meromorphic ordinary differential equations

\[
\frac{dF}{dz} = \left( u + \frac{A}{z} \right) \cdot F,
\]

where \( F(z) \in \text{GL}_n \), \( u = \text{diag}(u_1, \ldots, u_n) \) is a diagonal matrix, and \( A \in \mathfrak{gl}_n \). We divide \( \{1, \ldots, n\} \) into subsets \( \{I_l\}_{l=1,\ldots,k} \) such that \( u_i = u_j \) if \( i, j \in I_l \) for some \( l \), and \( u_i \neq u_j \) otherwise. We then assume that for any \( l = 1, \ldots, k \), no two eigenvalues of the submatrix of \( A \) formed by the rows and columns from the index set \( I_l \) differ by a non-zero integer. The equation has an irregular singularity at \( z = \infty \), and has a unique formal fundamental solution around \( \infty \) taking the form (see e.g., [5, Chapter 3])

\[
\hat{F}(z) = \hat{H}(z)z^{[A]}e^{zu}, \quad \text{for} \quad \hat{H}(z) = 1 + H_1z^{-1} + H_2z^{-2} + \cdots.
\]

Here \([A]\) takes the projection of \( A \) to the centralizer of \( u \) in \( \mathfrak{gl}_n \). In particular, if \( u \) has distinct diagonal elements, \([A]\) takes the diagonal part of \( A \). Although the radius of convergence of \( \hat{H}(z) \) is in general zero, its Borel-Laplace transform (see [5, Chapter 5]) produces different holomorphic functions \( H_i \), with the prescribed asymptotics \( \hat{H} \) in certain different sectors \( \tilde{\text{Sect}}_i \) of the complex plane, whose union covers a full neighborhood of the singularity \( z = \infty \). See e.g., [5, Chapter 8]. In this way, one gets canonical fundamental holomorphic solutions \( F_i(z) \) in each \( \tilde{\text{Sect}}_i \). The mismatch of two sectoral solutions on the overlap of the corresponding sectors, known as the Stokes phenomenon, may be measured in terms of the transition matrix relating the two fundamental solutions. In what follows, we give more details and introduce some necessary notations for the present paper.
Definition A.1. The anti-Stokes rays of the equation (32) are the directions along which 
\( e^{(u_i - u_j)z} \) decays most rapidly as \( z \mapsto \infty \) for some \( u_i \neq u_j \). The Stokes sectors are the open regions of \( \mathbb{C} \) bounded by two adjacent anti-Stokes rays.

Given an initial Stokes sector \( \text{Sect}_0 \), we label the anti-Stokes rays \( d_1, d_2, \ldots, d_{2l} \) going in a positive sense and starting on the positive edge of \( \text{Sect}_0 \). We denote by \( \text{Sect}_i \) the Stokes sector bounded by two adjacent anti-Stokes rays \( d_i \) and \( d_{i+1} \). Here indices are taken modulo \( 2l \), i.e., \( \text{Sect}_0 = \text{Sect}(d_{2l}, d_1) \). The following result can be found in e.g., [5, 6, 32].

Theorem A.2. On each \( \text{Sect}_i \), there is a unique (therefore canonical) holomorphic function \( H_i : \text{Sect}_i \to \text{GL}_n \) such that the function

\[
F_i(z) = H_i(z)e^{zu z^{[A]}}
\]
satisfies equation (32), and at the same time \( H_i(z) \) can be analytically continued to a bigger sector \( \hat{\text{Sect}}_i := (d_i - \frac{\pi}{2}, d_{i+1} + \frac{\pi}{2}) \) and is asymptotic to \( \hat{H} \) as \( z \to \infty \) within \( \hat{\text{Sect}}_i \).

Definition A.3. The Stokes matrices of the equation (32) (with respect to to \( \text{Sect}_0 \)) are the \( n \times n \) matrices \( S(A, u) \), \( S^-(A, u) \) determined by

\[
F_i(z) = F_0(z) \cdot e^{-\pi i [A]} S(A, u), \quad F_0(z) = F_i(z e^{2 \pi i}) \cdot e^{-\pi i [A]} S^-(A, u)
\]

where the first (resp. second) identity is understood to hold in \( \text{Sect}_i \) (resp. \( \text{Sect}_0 \)) after \( F_0 \) (resp. \( F_i \)) has been analytically continued counterclockwise.

Now if no two eigenvalues of \( A \) are differed by a non-zero integer, then the following fact is well-known (see e.g [43, Chapter 2]).

Lemma A.4. There is a unique holomorphic fundamental solution \( F_0(z) \in \text{GL}_n \) of the system (32) on a neighbourhood of \( \infty \) slit along the anti-Stokes ray \( d_0 \), such that \( F_0 \cdot z^A \to 1 \) as \( z \to 0 \).

Definition A.5. The connection matrix \( C(A, u) \in \text{GL}_n \) of the system (32) (with respect to \( \text{Sect}_0 \)) is determined by \( F_0(z) = F_+(z) \cdot C(u, A) \).

In a global picture, the connection matrix is related to the Stokes matrices by the following monodromy relation, which follows from the fact that a simple negative loop (i.e., in clockwise direction) around 0 is a simple positive loop around \( \infty \):

\[
C(A, u) e^{2 \pi i A} C(A, u)^{-1} = S^-(A, u) S(A, u)
\]

We remark that the Stokes matrices \( S(A, u) \), \( S^-(A, u) \) will in general depend on the irregular data \( u \) in (32). Such dependence was studied by many authors, see e.g., [24].

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