Oversampling and transmission of hidden information

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Abstract

The lack of uniqueness arising by oversampling of Fourier coefficients is shown to provide a way of transmitting hidden information. A basic encoding/decoding system, developed on the basis of such a possibility, is discussed. The system is devised with the double purpose of: (a) enabling the transmission of an arbitrary signal and (b) allowing for the transmission of a hidden embedded code.

1 Introduction

In a previous publication [1] the oversampling problem of signal representation by Discrete Fourier Transform (DFT) has been addressed from the Frame Theory [2,3,4] point of view. Within this structure the signal reconstruction appears as a tight frame superposition. As remarked in [1], the implicit redundancy of frames is the cause of noise reduction in signal reconstruction and also the reason that the representation is not unique. In this communication we show that the lack of uniqueness can be used for dissembling information.

We propose a basic encoding/decoding procedure, which enables the transmission of hidden information when transmitting some arbitrary signal. Although we shall restrict our considerations to redundancy arising by DFT oversampling, the proposal for transmitting hidden information could be equivalently implemented using any other redundant transformation. In particular, recently introduced techniques for the construction of tight frames [5,6,7,8] should be relevant to development of more sophisticated systems based on the principles proposed.
here. Nevertheless, we feel it is appropriate to present the central ideas of this contribution by making use of the oversampled DFT. Among a number of important reasons leading to this choice we would like to point out the following: i) The Fast Fourier Transform (FFT), fast implementation of DFT, is still without doubt the most popular and widely used signal processing tool. ii) Oversampling FFT, by padding with zeros, is a common procedure for noise reduction and smoothing, hence the importance of stressing that it also leaves room for the transmission of hidden information. iii) All readers, regardless of their area of expertise, can be assumed to be familiar with the corresponding mathematical background. Consequently, we shall introduce the proposed technique within the framework arising by oversampling DFT coefficients. The effect of additive noise in the transmission channel is also considered. In order to discriminate the level of additive zero-mean noise in which the system can safely operate, two possible situations are analysed.

The letter is organised as follows: In Section 2 the lack of uniqueness inherent to oversampling, as well as its relevance in relation to transmission of hidden information, are discussed. A basic DFT based encoding/decoding system is proposed in Section 3, and some examples to illustrate its performance in the presence of additive zero-mean noisy are given in Section 4. The conclusions are drawn in Section 5.

2 Oversampling and lack of uniqueness

Let us represent a signal \( f(t) \), which is defined for \( t \in [-T, T] \), through its Discrete Fourier expansion, i.e.

\[
f(t) = \frac{1}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi t}{T}}.
\]

(1)
Since for $t \in [-T, T]$ the complex exponentials in (1) constitute an orthonormal basis, the coefficients $c_n$ in (1) are obtained as:

$$c_n = \frac{1}{\sqrt{2T}} \int_{-T}^{T} f(t) e^{-i\frac{an\pi}{T}} dt. \quad (2)$$

Let us consider now the re-scaling operation: $t \rightarrow at$, with $a$ a positive real number less than 1, and construct the functions $\frac{\chi_T(t)}{\sqrt{2T}} e^{\frac{an\pi}{T}}$, with $\chi_T(t)$ defined as: $\chi_T(t) = 1$ if $t \in [-T, T]$ and zero otherwise. The new functions $\frac{\chi_T(t)}{\sqrt{2T}} e^{\frac{an\pi}{T}}$ are no longer a basis but a tight frame for the space of time limited signal with time-width $T$ (the corresponding frame-bound being $a^{-1}$ [1]). This has a remarkable consequence, namely the coefficients $c_n$ of the linear expansion:

$$f(t) = \frac{\chi_T(t)}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} c_n e^{\frac{an\pi}{T}} \quad (3)$$

are not unique. There exist infinitely many different sets of coefficients $c_n$ which can reproduce an identical signal $f$ by the above linear superposition. A particular set of coefficients $c_n$ is obtained as:

$$c_n = \frac{a}{\sqrt{2T}} \int_{-T}^{T} f(t) e^{-i\frac{an\pi}{T}} dt. \quad (4)$$

Out of all possible sets of coefficients, the ones given by the above equation constitute the coefficients of minimum 2-norm [2, 3].

Let us stress the cause for the coefficients in the tight frame expansion not to be unique. The reason being that, for $a < 1$ with the restriction $t \in [-T, T]$ the exponentials $\frac{1}{\sqrt{2T}} e^{\frac{an\pi}{T}}$ are not linearly independent i.e., we can have the situation:

$$\frac{1}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} c'_n e^{\frac{an\pi}{T}} = 0 \quad \text{for} \quad \sum_{n=-\infty}^{\infty} |c'_n|^2 \neq 0$$

or, taking inner products both sides with $\frac{1}{\sqrt{2T}} e^{\frac{an\pi}{T}}$,

$$\frac{1}{2T} \sum_{n=-\infty}^{\infty} c'_n \int_{-T}^{T} e^{-i\frac{an\pi}{T}} e^{\frac{an\pi}{T}} dt = 0 \quad \text{for} \quad \sum_{n=-\infty}^{\infty} |c'_n|^2 \neq 0,$$
which can be recast in the fashion:

\[ G\vec{c} = 0 \quad \text{for} \quad ||\vec{c}||^2 = \sum_{n=-\infty}^{\infty} |c'_n|^2 \neq 0, \]

with \( G \) a matrix of elements:

\[ g_{m,n} = \frac{1}{2T} \int_{-T}^{T} e^{-i\frac{am\pi}{T}} e^{i\frac{an\pi}{T}} dt = \frac{\sin a(m-n)\pi}{a(m-n)\pi} \]

and \( \vec{c} \) a vector of components \( c'_n \).

Notice that all vectors \( \vec{c} \) satisfying \( G\vec{c} = 0 \) belong, by definition, to Null(\( G \)), the Null space of \( G \). All such vectors satisfy:

\[ f(t) = \frac{\chi_T(t)}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} c_n e^{i\frac{an\pi}{T}} + \frac{\chi_T(t)}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} c'_n e^{i\frac{an\pi}{T}} = \frac{\chi_T(t)}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} c''_n e^{i\frac{an\pi}{T}}, \]

where we have defined \( c''_n = c_n + c'_n \) with \( c_n \) as in (4) and \( c'_n \) the components of an arbitrary vector \( \vec{c} \in \text{Null}(G) \). Vectors \( \vec{c} \) and \( \vec{c}' \) will hereafter be referred to as signal coefficients and hidden code coefficients respectively. The fact that all coefficients \( \vec{c}' = \vec{c} + \vec{c} \) reproduce an identical signal as coefficients \( \vec{c} \) provides us with the foundation to construct an encoding/decoding scheme for transmitting hidden information.

3 The encoding-decoding system

Let us assume that, in addition to transmitting an arbitrary signal \( f \), we wish to transmit a hidden code \( \vec{h} \) consisting of \( K \) numbers. For practical implementation we give to the oversampling parameter \( a \) a positive value, less than one, and consider that \( G \) is an \( M \times M \) matrix of elements as given in (5). We select \( K \) eigenvectors of \( G \) corresponding to the zero eigenvalues, which are assumed to be orthonormal, and construct a vector \( \vec{c}' \in \text{Null}(G) \) as follows:

\[ \vec{c}' = UB_s\vec{h} \]
where $U$ is an $M \times K$ matrix, the columns of which are the $K$ selected eigenvectors and $B_s$ is a $K \times K$ unitary random matrix. Note: the subindex $s$ indicates that the random generator used for constructing the matrix is initialized at state $s$. Such a state is needed to be known at the decoding stage.

**Encoding process**

Consider that the signal $f$ to be transmitted is given as an $N$-dimension data vector and proceed as follows:

- Compute the signal coefficients $\vec{c}$ as in (4).
  
  Note that this calculation can be carried out with Fast Fourier Transform (FFT) by adding $\frac{N(1-a)}{2a}$ zeros at the beginning and at the end of the data vector $f$, so as to obtain the required vector $\vec{c}$ of dimension $M = \frac{N}{a}$.

- Compute the hidden code coefficients $\vec{c}'$ as prescribed in (7).

- Transmit the coefficients $\vec{c}'' = \vec{c} + \vec{c}'$ to the receiver.

**Decoding process**

- Use the received vector $\vec{c}''$ for recovering the signal $f$.
  
  In practice this can be computed by Inverse Fast Fourier Transform (IFFT) on the received vector $\vec{c}''$.

- Use the signal $f$ to compute the signal coefficients $\vec{c}$ as in (4), which can be accomplished by applying FFT on the signal $f$ recovered in the previous step.

- Compute vector $\vec{c}'$ through $\vec{c}' = \vec{c}'' - \vec{c}$.

- Recover the hidden code $\vec{h}$ by noticing that:
(a) For constructing the matrix $U$ one can use all eigenvectors of the matrix $G$ corresponding to eigenvalues less than a previously specified tolerance parameter. Matrix $U$ is unitary, i.e., $U^{-1} = U^*$ and then we have: $B_s \vec{h} = U^* \vec{c}$, where $U^*$ indicates the transpose conjugate of matrix $U$.

(b) The dimension of matrix $B_s$ can be determined from the number of non-zero components of vector $U^* \vec{c}$. Thereby, state $s$ allows the reproduction of the random matrix $B_s$. Since this is also a unitary matrix $B_s^{-1} = B_s^*$.

Hence the vector $\vec{h}$ is obtained as:

$$\vec{h} = B_s^* U^* \vec{c}.$$ 

Remarks: In order to be able to implement the above described decoding process the receiver should know:

i) The transmitted vector $\vec{c}''$.

ii) The oversampling parameter $a$, yet this parameter might be actually estimated in some situations by counting the beginning/ending zeros apparent in the recovered signal.

iii) The state $s$ that was used for generating the unitary matrix $B_s$. Note: the restriction of unitariness is imposed in order to avoid amplification of errors in the inversion process. Such a condition is achieved by orthogonalization of a well posed random matrix.

With the knowledge of i) ii) and iii) the receiver should be able to reproduce both the signal $f$ and the hidden code $\vec{h}$.
4 Example

Consider that we wish to transmit the chirp signal of Figure 1. In order to have an acceptable representation of this signal we need $N = 200$ non-zero Fourier coefficients in the non-oversampled case (corresponding to considering $a = 1$). If instead we consider $a = 0.5$, we duplicate the number of coefficients used to represent the same signal, but it allows us to additionally transmit a hidden code. In our example we will send a code $\vec{h}$ consisting of $K = 12$ numbers, the first 5 digits of which are shown in the first column of Table I.

**Case 1:** If transmitted through a noise-free channel, the proposed encoding/decoding system is, of course, capable of transmitting a great deal of hidden information with accuracy limited only by machine precision of the calculations involved. The absolute value of the coefficients $\vec{c}$ conveying the information on the chirp signal are plotted in Figure 2a while the hidden code coefficients $\vec{c}'$ are plotted in Figure 2b. The absolute value of the transmitted coefficients $\vec{c}'' = \vec{c} + \vec{c}'$ are those of Figure 2c. For the sake of avoiding a notorious distortion of coefficients $\vec{c}$ we have diminished the magnitude of coefficients $\vec{c}'$ by multiplying by an appropriate scaling factor, which in the absence of noise can be arbitrarily small. As expected, our decoding system is capable of reconstructing the hidden code up to the precision of the numerical calculation, although for space limitation reasons we have shown only 5 digits (see the second column of Table I).

**Case 2:** Now let us illustrate the effect of adding zero mean random Gaussian noise to the transmitted coefficients. As would be expected, the quality of the recovery of our signal and hidden code depends on the variance of the noise ($\sigma^2$) relative to the size of the signal and
hidden code coefficients respectively. For this case and the following one $\sigma^2$ was fixed such that the signal to noise ratio (in terms of the signal coefficients) was 40 dB. This allows a reasonably good recovery of the signal with a small amount of noise distortion.

Now that the noise is fixed relative to the signal, we consider rescaling the hidden code coefficients $\vec{c}'$ by a constant to achieve the desired accuracy in the recovery of the code. We define the variance ratio as

$$\rho = \frac{\sigma^2}{||\vec{c}'||/\text{Dim}(\vec{c}')},$$

where $\text{Dim}(\vec{c}')$ denotes the dimension of vector $\vec{c}'$. In order to recover 4 significant digits of the code, we need a variance ratio of about $10^{-5}$ (see third column of Table I). To achieve such a ratio the magnitude of $\vec{c}'$ has to be increased, thereby the absolute value of the transmitted coefficients look as in Figure 2c. It is important to remark that, although the need to magnify the coefficients $\vec{c}'$ makes them “visible” during the transmission, this has no effect whatsoever on the signal reconstruction. Let us recall that $\vec{c}'$ is by definition in $\text{Null}(G)$, so these coefficients cannot affect the signal in any way. Hence in this case the hidden code coefficients $\vec{c}'$ actually play a double role. On one hand they cover the coefficients $\vec{c}$ conveying the information for recovering the signal $f$ and on the other hand they convey the information containing the hidden code.

**Case 3:** Let us finally discuss the limitation of the proposed system if one does not want the hidden code coefficients $\vec{c}'$ to dominate the value of the transmitted ones $\vec{c}''$. With $\sigma^2$ fixed as described in the previous case, the variance ratio was increased to 0.002 (by scaling $\vec{c}'$ appropriately) and the first digit of the code was still safely recovered (see the fourth column of Table I). The absolute value of the transmitted coefficients in this case are shown in Figure 2d. Note that the transmitted coefficient are dominated now by the signal coefficients.
5 Conclusions

Oversampling DFT has been considered as providing a means for transmitting hidden information. A basic encoding/decoding system has been discussed. The proposed scheme aims at transmitting an arbitrary signal and, simultaneously, embedding a hidden code. It is important to stress once again that the purpose was to discuss in the simplest possible way the possibility of using redundant transformations for embedding a hidden code while transmitting an arbitrary signal. To such an end the idea has being presented in the context of DFT oversampling. However, many other redundant transformations could have been considered by an equivalent treatment. Furthermore, different ways of embedding a hidden code in the redundant coefficients could be envisaged. We feel then confident that the simple scheme we have introduced here will stimulate further research in the subject.

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References

[1] L. Rebollo-Neira, A. G. Constantinides, “Power Spectrum Estimation from values of noisy autocorrelations.”, Signal Processing, Vol 50, 223-231, (1996).

[2] R. M. Young, “An introduction to Nonharmonic Fourier Series”, Academic Press, New York, (1980).

[3] I. Daubechies, “Ten Lectures on Wavelets”, CBMAS-NSF, SIAM, Philadelphia, (1992).

[4] O. Christensen, “An Introduction to Frames and Riesz Bases”, Birkhäuser, Berlin, (2002)

[5] J. J. Benedetto, S. Li, “The theory of multiresolution analysis frames and applications to filter banks”. Applied and Computational Harmonic Analysis, Vol 5 (4), pp 384-417, (1998).

[6] C. Chi, W He, “Compactly supported tight frames associated with refinable functions”, Applied and Computational Harmonic Analysis, Vol 8 (3), pp 293-319, (2000).

[7] I. W. Selesnick, “Smooth wavelet tight frmaes with zero moments”, Applied and Computational Harmonic Analysis, Vol 10 (2), pp 163-181, (2001).

[8] A. Petukhov, “Explicit Construction of Framelets”, Applied and Computational Harmonic Analysis, Vol 11 (2), pp 313-273 (2001).
Figure Captions:

Figure 1: Chirp signal to be transmitted.

Figures 2a-2e, from top to bottom. 2a depicts the absolute value of the signal coefficients used in all three cases, 2b the hidden code coefficients scaled as used in Case 1, and 2c the absolute value of the transmitted coefficients in Case 1. Figure 2d represents the absolute value of the transmitted coefficients in Case 2 and 2e the absolute value of the transmitted coefficients in Case 3.
| code    | Case 1 $\rho = 0$ | Case 2 $\rho = 10^{-5}$ | Case 3 $\rho = 0.002$ |
|---------|-------------------|--------------------------|------------------------|
| 3.1492  | 3.1492            | 3.1496                   | 3.2286                 |
| 2.1271  | 2.1271            | 2.1270                   | 2.1157                 |
| 5.1312  | 5.1312            | 5.1316                   | 5.2206                 |
| 1.2835  | 1.2835            | 1.2836                   | 1.2939                 |
| 7.7976  | 7.7976            | 7.7979                   | 7.8660                 |
| 3.7160  | 3.7160            | 3.7164                   | 3.7999                 |
| 8.4139  | 8.4139            | 8.4140                   | 8.4360                 |
| 1.9791  | 1.9791            | 1.9791                   | 1.9885                 |
| 0.5863  | 0.5863            | 0.5868                   | 0.6785                 |
| 5.8321  | 5.8321            | 5.8317                   | 5.7570                 |
| 8.1032  | 8.1032            | 8.1032                   | 8.1124                 |
| 6.4908  | 6.4908            | 6.4907                   | 6.4718                 |

Table 1: Recovered code with given variance ratio.
Figure 1:
Figure 2: