Inertial Krasnosel’skii–Mann Method in Banach Spaces

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Abstract: In this paper, we give a general inertial Krasnosel’skii–Mann algorithm for solving inclusion problems in Banach Spaces. First, we establish a weak convergence in real uniformly convex and $q$-uniformly smooth Banach spaces for finding fixed points of nonexpansive mappings. Then, a strong convergence is obtained for the inertial generalized forward-backward splitting method for the inclusion. Our results extend many recent and related results obtained in real Hilbert spaces.

Keywords: Krasnosel’skii–Mann; nonexpansive mapping; forward-backward splitting method; $q$-uniformly smooth; Banach spaces

1. Introduction

Let $X$ be a real Banach space and given a single and set-valued operators $A : X \to X$ and $B : X \to 2^X$, respectively. We consider the following inclusion problem:

\[
\text{find } \hat{x} \in X \text{ such that } 0 \in A\hat{x} + B\hat{x}. \tag{1}
\]

Such inclusion problems are quite general since it include as special cases various problems such as: non-smooth convex optimization problems, variational inequalities and convex-concave saddle-point problems, just to name a few. (see, e.g., [1–5]).

A known and popular method for solving problem (1) is the forward-backward splitting method [6,7], which is defined in the following manner: $x_1 \in X$ and

\[
x_{n+1} = J_B^{r_n} (x_n - r A x_n), \quad n \geq 1, \tag{2}
\]

where provided that $B$ is maximally monotone and $A$ is co-coercive (or equivalent assumptions) and $J_B^{r} := (I + rB)^{-1}, \quad r > 0$, is called the “resolvent of $B$”. The forward-backward splitting method (2) includes the proximal point algorithm, (see, e.g., [8–12]), and the gradient method (see, for example, [2,13]). It has been shown that (2) in general only converges weakly to a zero of (1) (see, for example, [3,6,14,15]).

The following method was introduced in [16] (see also [14]) for finding zero of (1) when $A = 0$ and $B$ is maximal monotone operator: $x_0, x_1 \in H$;

\[
\begin{cases}
y_n = x_n + \theta_n (x_n - x_{n-1}) \\
x_{n+1} = J_B^{r_n} (y_n), \quad n \geq 1.
\end{cases} \tag{3}
\]

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Alvarez and Attouch [16] established the weak convergence of \((3)\) under some appropriate conditions on \(\{\theta_k\}\) and \(\{r_n\}\). Several other modifications of \((2)\) with inertial extrapolation step have been considered in Hilbert spaces by many authors, see, for example, [17–21].

Based on the above mentioned results [19,22–26], our main contribution in this paper is the following. We extend the results of [17] concerning the inertial Krasnoselskii–Mann iteration for fixed point of nonexpansive mappings to uniformly convex and \(q\)-uniformly smooth Banach space. We also extend the forward-backward splitting method with inertial extrapolation step for solving \((1)\) from Hilbert spaces to Banach spaces. While the mentioned results establish only weak convergence, we also provide strong convergence analysis in Banach spaces.

The outline of the paper is as follows. We first recall some basic definitions and results in Section 2. Our algorithms are presented and analysed in Section 3. In Section 4 an infinite dimensional example is presented and final remarks and conclusions are given in Section 5.

2. Preliminaries

Let \(X\) be a real Banach space. The \textit{modulus of convexity} of \(X\) is defined as the function \(\delta : (0, 2] \rightarrow [0, 1],\)

\[
\delta(\varepsilon) = \inf \left\{ 1 - \frac{x + y}{2} : x, y \in X, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}.
\] (4)

\(X\) is said to be \textit{uniformly convex} if \(\delta(\varepsilon) > 0\) for all \(\varepsilon \in (0, 2].\)

The \textit{modulus of smoothness} of \(X\) is the function \(\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) defined by

\[
\rho(t) = \sup \left\{ \frac{\|x + ty\| + \|x - ty\|}{2} - 1 : x, y \in X, \|x\| = \|y\| = 1 \right\}.
\] (5)

We say \(X\) is uniformly smooth if \(\lim_{t \to 0} \rho(t)/t = 0.\) \(X\) is said to be \(q\)-uniformly smooth with \(1 < q \leq 2,\) if there exists a constant \(k_q > 0\) such that \(\rho(t) \leq k_q t^q\) for \(t > 0.\) If \(X\) is \(q\)-uniformly smooth, then it is uniformly smooth (see, e.g., [27]). Suppose that \(X^*\) is the dual space of \(X.\) The generalized duality mapping \(J_q(q > 1)\) of \(X\) is defined by \(J_q(x) := \{j_q(x) \in X^* : \langle x, j_q(x) \rangle = \|x\|^q, \|j_q(x)\| = \|x\|^{q-1}\}, \forall x \in X,\) where \(\langle \cdot, \cdot \rangle\) denotes the duality pairing between \(X\) and \(X^*.\) In particular, we call \(J_2 := J,\) the \textit{normalized duality mapping} on \(X.\) Furthermore, (see, e.g., [28] (p. 1128))

\[
j_q(x) = \|x\|^{q-2}j(x), \ x \neq 0.
\] (6)

It is well known that (see, for example, [27]) \(X\) is uniformly smooth if and only if the duality mapping \(J_q\) is single-valued and norm-to-norm uniformly continuous on bounded subsets of \(X.\)

Let \(B : X \rightarrow 2^X.\) We denote the \textit{domain} of \(B\) by \(D(B) = \{x \in X : Bx \neq \emptyset\}\) and its \textit{range} by \(R(B) = \bigcup\{Bz : z \in D(B)\}.\) We say that \(B\) is \textit{accretive} if, for each \(x, y \in D(A),\) there exists \(j(x - y) \in J(x - y)\) such that (see, for example, [25])

\[
\langle u - v, j(x - y) \rangle \geq 0, u \in Bx, v \in By.
\] (7)

\(B\) is said to be \(m\)-\textit{accretive} if \(R(I + rB) = X\) for all \(r > 0.\) Given \(a > 0\) and \(q \in (1, \infty),\) we say that a single-valued accretive operator \(A\) is \(a\)-\textit{inverse strongly accretive} (a-isa, for short) of order \(q\) if, for each \(x, y \in D(A),\) there exists \(j_q(x - y) \in j_q(x - y)\) such that

\[
\langle Ax - Ay, j_q(x - y) \rangle \geq a\|u - v\|^q.
\] (8)

We say that \(A\) is \(a\)-\textit{strongly accretive} of order \(q\) if, for each \(x, y \in D(A),\) there exists \(j_q(x - y) \in j_q(x - y)\) such that

\[
\langle Ax - Ay, j_q(x - y) \rangle \geq a\|x - y\|^q.
\] (9)
Lemma 1 ([29] p. 33). Let $q > 1$ and $X$ be a real normed space with the generalized duality mapping $J_q$. Then, for any $x, y \in X$, we have
\[
\|x + y\|^{q} \leq \|x\|^{q} + q(y, j_q(x + y))
\] (11)
for all $j_q(x + y) \in I_q(x + y)$.

Lemma 2 ([28] Cor. 1'). Let $1 < q \leq 2$ and $X$ be a smooth Banach space. Then the following statements are equivalent:
(i) $X$ is $q$-uniformly smooth.
(ii) There is a constant $k_q > 0$ such that for all $x, y \in X$
\[
\|x + y\|^{q} \leq \|x\|^{q} + q(y, j_q(x)) + k_q\|y\|^{q}.
\] (12)

The best constant $k_q$ will be called the $q$-uniform smoothness coefficient of $X$.

Lemma 3 ([25] Lem. 3.1, 3.2). Let $X$ be a Banach space. Let $A : X \to X$ be an $\alpha$-isa of order $q$ and $B : X \to 2^X$ an $m$-accretive operator. Then we have
(i) For $r > 0$, $Fix(T_r^{A, B}) = (A + B)^{-1}(0)$.
(ii) For $0 < s \leq r$ and $x \in X$, \(\|x - T_r^{A, B}x\| \leq 2\|x - T_r^{A, B}x\|\).

Lemma 4 ([25] Lem. 3.3). Let $X$ be a uniformly convex and $q$-uniformly smooth Banach space for some $q \in (1, 2]$. Assume that $A$ is a single-valued $\alpha$-isa of order $q$ in $X$. Then, given $r > 0$, there exists a continuous, strictly increasing and convex function $\phi_q : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_q(0) = 0$ such that, for all $x, y \in B_r$, \[
\|T_r^{A, B}x - T_r^{A, B}y\|^{q} \leq \|(x - y) - r(aq - r^{q-1}k_q)\|Ax - Ay\|^{q} \leq \phi_q((I - f_B^r)(I - rA)x - (I - f_B^r)(I - rA)y),
\] (13)
where $k_q$ is the $q$-uniform smoothness coefficient of $X$.

Lemma 5 ([26] Lem. 3.1). Let \(\{a_n\}\) and \(\{c_n\}\) be sequences of nonnegative real numbers such that
\[
a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \quad n \geq 1,
\] (14)
where \(\{\delta_n\}\) is a sequence in $(0, 1)$ and \(\{b_n\}\) is a real sequence. Assume \(\sum_{n=1}^{\infty} c_n < \infty\). Then the following results hold:
(i) If $b_n \leq \delta_n M$ for some $M \geq 0$, then \(\{a_n\}\) is a real sequence.
(ii) If \(\sum_{n=1}^{\infty} \delta_n = \infty\) and $\limsup_{n \to \infty} b_n / \delta_n \leq 0$, then $\lim_{n \to \infty} a_n = 0$.

Lemma 6 (Maingé [30]). Let \(\{\varphi_n\}, \{\delta_n\}\) and \(\{\theta_n\}\) be sequences in $[0, +\infty)$ such that
\[
\varphi_{n+1} \leq \varphi_n + \theta_n(\varphi_n - \varphi_{n-1}) + \delta_n, \quad \forall n \geq 1, \sum_{n=1}^{\infty} \delta_n < +\infty,
\]
and there exists a real number $\theta$ with $0 \leq \theta_n \leq \theta < 1$ for all $n \in \mathbb{N}$. Then the following hold:
(i) \(\sum_{n=1}^{\infty} |\varphi_n - \varphi_{n-1}| < +\infty\), where $[t]_+ := \max\{t, 0\}$;
Algorithm 1

Inertial Krasnoselskii–Mann iteration.

Algorithm 1 is easily implemented. Furthermore, observe that by the assumption that $\{\epsilon_n\} \subset [0, 1], \sum_{n=1}^{\infty} \sigma_n = \infty$;

(a) $\limsup \sigma_n \leq 0$;

(b) $\gamma_n \geq 0 (n \geq 1), \sum_{n=1}^{\infty} \gamma_n < \infty$.

Then, $a_n \to 0$ as $n \to \infty$.

Notations: $x_n \to x, n \to \infty$ means $\{x_n\}$ converges weakly to $x$ and $x_n \to x, n \to \infty$ means $\{x_n\}$ converges strongly to $x$.

3. The Algorithm

In this section, we introduce our method and give the convergence analysis. Recall that $\ell_1$ is the space of all sequences whose series is absolutely convergent.

Let $E$ be a uniformly convex Banach space and $T : E \to E$ a nonexpansive mapping and $\text{Fix}(T) \neq \emptyset$.

Remark 1. Observe that since the value of $\|x_n - x_{n-1}\|$ is a priori known before $\theta_n$, then Step (2) in Algorithm 1 is easily implemented. Furthermore, observe that by the assumption that $\{\epsilon_n\}_{n=1}^{\infty} \subset \ell_1$, we have that $\sum_{n=0}^{\infty} \epsilon_n \|x_n - x_{n-1}\| < \infty$ and $\sum_{n=0}^{\infty} \epsilon_n \|x_n - x_{n-1}\|^q < \infty$.

Algorithm 1 Inertial Krasnoselskii–Mann iteration.

1: Choose sequence $\{\epsilon_n\} \subset \ell_1$ and pick $\theta \in [0, 1)$. Select $x_0, x_1 \in E$ and set $n := 1$.

2: Given the iterations $x_n, x_{n-1}$, choose $\theta_n$ such that $0 \leq \theta_n \leq \theta$, where

$$\theta_n = \min \left\{ \theta, \frac{\epsilon_n}{\|x_k - x_{k-1}\|^q}, \frac{\epsilon_n}{\|x_k - x_{k-1}\|^q} \right\}, \quad x_k \neq x_{k-1}$$

otherwise.

3: Compute

$$w_n = x_n + \theta_n (x_n - x_{n-1}),$$

and

$$x_{n+1} = (1 - \lambda_n) w_n + \lambda_n T w_n.$$
Theorem 1. Suppose $T : E \to E$ is a nonexpansive mapping and $\text{Fix}(T) \neq \emptyset$. Assume that $0 < a \leq \lambda_n \leq b < 1$. Then $\{x_n\}$ generated by Algorithm 1 converges weakly to a point in $F(T)$.

Proof. Take $z \in F(T)$. Then

$$
\|x_{n+1} - z\|^q \leq \left((1 - \lambda_n)\|w_n - z\| + \lambda_n\|Tw_n - z\|\right)^q
\leq \left(1 - \lambda_n\right)\|w_n - z\|^q + \lambda_n\|Tw_n - Tz\|^q
\leq \left(1 - \lambda_n\right)\|w_n - z\|^q + \lambda_n\|w_n - z\|^q
= \|w_n - z\|^q
$$

(15)

and

$$
\|w_n - z\|^q = \left\|x_n - z + \theta_n(x_n - x_{n-1})\right\|^q
\leq \left\|x_n - z\right\|^q + \theta_n\left(\left\|x_n - z\right\|^q - \left\|x_{n-1} - z\right\|^q\right) + c_q\theta_n\|x_n - x_{n-1}\|^q.
$$

(16)

Observe that

$$
q(x_n - x_{n-1}, j_q(x_n - z)) \leq \|x_n - z\|^q - \|x_{n-1} - z\|^q + c_q\|x_n - x_{n-1}\|^q.
$$

(17)

From (16) and (17), we have (noting that $\theta_n \leq \theta_n$)

$$
\|w_n - z\|^q \leq \left\|x_n - z\right\|^q + \theta_n\left(\left\|x_n - z\right\|^q - \left\|x_{n-1} - z\right\|^q\right) + c_q\theta_n\|x_n - x_{n-1}\|^q
\leq \left\|x_n - z\right\|^q + \theta_n\left(\left\|x_n - z\right\|^q - \left\|x_{n-1} - z\right\|^q\right) + 2c_q\theta_n\|x_n - x_{n-1}\|^q.
$$

(18)

It follows from (15) and (18) that

$$
\|x_{n+1} - z\|^q \leq \left\|x_n - z\right\|^q + \theta_n\left(\left\|x_n - z\right\|^q - \left\|x_{n-1} - z\right\|^q\right) + 2c_q\theta_n\|x_n - x_{n-1}\|^q.
$$

(19)

By Lemma 6, we deduce that $\{\|x_n - z\|\}$ is convergent. Thus, $\{x_n\}$ is bounded and $\sum_{n=1}^{\infty}\|x_{n+1} - z\|^q - \|x_n - z\|^q < \infty$.

We next show that $\lim_{n \to \infty} \|Tw_n - w_n\| = 0$. From the update of $x_{n+1}$ in Algorithm 1, we get

$$
\|x_{n+1} - z\|^q = \|w_n - z\|^q + (1 - \lambda_n)(\|w_n - z\| + \lambda_n\|Tw_n - z\|)^q
\leq \left(1 - \lambda_n\right)\|w_n - z\|^q + \lambda_n\|Tw_n - z\|^q
\leq \|w_n - z\|^q - \lambda_n\|Tw_n - w_n\|^q.
$$

(20)

Using (18) in (20), we get

$$
\|w_q(\lambda_n)\phi(\|Tw_n - w_n\|) \leq \|x_n - z\|^q - \|x_{n+1} - z\|^q + \theta_n\left(\left\|x_n - z\right\|^q - \left\|x_{n-1} - z\right\|^q\right) + 2c_q\theta_n\|x_n - x_{n-1}\|^q.
$$

(21)
Also,\[
\|w_n - x_n\|^q = \theta_n^q \|x_n - x_{n-1}\|^q \\
\leq \theta_n \|x_n - x_{n-1}\|^q \to 0, n \to \infty.
\] (22)

Since \(\lim_{n \to \infty} \|x_n - x_{n-1}\|^q = 0\) and \(\lim_{n \to \infty} \|x_n - z\|^q\) exists, we obtain from (21) that \(\lim_{n \to \infty} w_q(\lambda_n) \varphi(\|Tw_n - w_n\|) = 0\). Since \(\inf_{n \to \infty} \lambda_n (1 - \lambda_n) > 0\), we get \(\lim_{n \to \infty} \varphi(\|Tw_n - w_n\|) = 0\) and by the continuity of \(\varphi\), we get \(\lim_{n \to \infty} \|Tw_n - w_n\| = 0\).

Furthermore, since \(\{x_n\}\) is bounded, there exists \(\{x_{n_k}\} \subset \{x_n\}\) such that \(x_{n_k} \to p \in B\). By (22), we have that \(w_{n_k} \to p \in B\). Using the demiclosedness of \(I - T\) in Lemma 7, we get that \(p \in F(T)\).

By the results in [33], we have that \(\{x_n\}\) has exactly one weak limit point and hence \(\{x_n\}\) is weakly convergent. This ends the proof. \(\Box\)

Remark 2.
(a) We mention here that quasi-nonexpansiveness is a weaker sufficient condition for Theorem 1.
(b) It can also be shown in Theorem 1 that
\[
\|x_n - Tw_n\| \leq \|x_n - w_n\| + \|w_n - Tw_n\| + \|Tw_n - Tx_n\| \\
\leq 2\|x_n - w_n\| + \|w_n - Tw_n\| \to 0, n \to \infty.
\]

Therefore, Algorithm 1 preserves certain properties of the Krasnoselskii–Mann iteration.

Now taking \(T := T_r^{A,B}\) in Algorithm 1, we obtain the following result for inclusion problem (1).

Theorem 2. Let \(E\) be a uniformly convex and \(q\)-uniformly smooth Banach Space. Suppose that \(A : E \to E\) is \(\alpha\)-isa of order \(q\) and \(B : E \to 2^E\) an \(m\)-accretive operator. Assume that the solution set \(S\) of inclusion problem (1) nonempty. Let \(r \in (0, \left(\frac{\alpha q}{c_q}\right)^{\frac{1}{q+1}})\). Then the sequence \(\{x_n\}\) generated by Algorithm 1 with \(T := T_r^{A,B}\) converges weakly to a point in \(S\).

Proof. By Lemma 3 (i) and Lemma 4, we have that \(\text{Fix}(T_r^{A,B}) = (A + B)^{-1}(0) = S\) and \(T_r^{A,B}\) is nonexpansive. Therefore, by Theorem 1, we have that \(\{x_n\}\) converges weakly to a point in \(S\) and the desired result is obtained. \(\Box\)

We give two instances of strong convergence of the relaxed forward–backward Algorithm 1.

Theorem 3. Let \(E\) be a uniformly convex and \(q\)-uniformly smooth Banach Space. Assume that the solution set \(S\) of inclusion problem (1) nonempty and \(\{\lambda_n\} \subseteq (0,1)\) is such that \(\sum_{n=1}^{\infty} \lambda_n = \infty\). Let \(r \in \left(0, \left(\frac{\alpha q}{c_q}\right)^{\frac{1}{q+1}}\right)\).

Suppose that one of the following holds:

(i) \(A\) is \(\alpha\)-isa of order \(q\), \(B\) is \(\beta\)-strongly accretive of order \(q\), and \(r \in \left(0, \left(\frac{\alpha q}{c_q}\right)^{\frac{1}{q+1}}\right)\).

(ii) \(\beta \leq L\), \(A\) is \(\beta\)-strongly accretive and \(L\)-Lipschitz on \(E\) with \(r \in \left(0, \frac{2\beta}{L^2}\right)\).

Then \(\{x_n\}\) generated by Algorithm 1 with \(T := T_r^{A,B}\) converges strongly to a unique point in \(S\).

Proof. We first show that the inclusion problem (1) has a unique solution by showing that in each of the cases above \(T_r^{A,B}\) is a contraction map on \(E\).
(i) For all \( x, y \in E \), we have
\[
\|(I - rA)x - (I - rA)y\|^q = \|x - y - r(Ax - Ay)\|^q \\
\leq \|x - y\|^q + c_q r^q \|Ax - Ay\|^q - rq\langle Ax - Ay, j_q(x - y) \rangle \\
\leq \|x - y\|^q - \tau.Aq - r(1 - \|c_q \|A - Ay\|^q \\
\leq \|x - y\|^q.
\]
Therefore, \( I - rA \) is a nonexpansive mapping. Let \( x, y, u, v \in E \). Since \( B \) is \( \beta \)-strongly accretive of order \( q \), we have that
\[
(u, v) \in (f^p x, f^p y) \Leftrightarrow (x - u, y - v) \in Bu \times Bv \Rightarrow \langle (x - u) - (y - v), j_q(u - v) \rangle \geq \beta\|u - v\|^q \\
\Leftrightarrow (x - y, j_q(u - v)) \geq (\beta + 1)\|u - v\|^q.
\]
Hence,
\[
(\beta + 1){\|f^p x - f^p y\|^q} \leq \langle x - y, j_q(f^p x - f^p y) \rangle \\
\leq \|x - y\||j_q(f^p x - f^p y)|| \\
= \|x - y\||f^p x - f^p y||^{q-1}.
\]
Therefore, \( ||f^p x - f^p y|| \leq \frac{1}{\beta + 1} \|x - y\| \). So,
\[
||f^p (I - rA)x - f^p (I - rA)y|| \leq \frac{1}{\beta + 1} \|x - y\| = \tau \|x - y\|.
\]
(ii) Observe that \( r(\beta q - c_q r^{q-1} L^q) \in (0, 1) \) and define \( \tau := \left[1 - r(\beta q - c_q r^{q-1} L^q)\right]^\frac{1}{q} \). Then for all \( x, y \in E \),
\[
||f^p (I - rA)x - f^p (I - rA)y||^q \leq \|(I - rA)x - (I - rA)y\|^q \\
= \|x - y - r(Ax - Ay)\|^q \\
\leq \|x - y\|^q - rq\langle Ax - Ay, j_q(x - y) \rangle + c_q r^q \|Ax - Ay\|^q \\
\leq \|x - y\|^q - rq \beta \|x - y\|^q + c_q r^q L^q \|x - y\|^q \\
= (1 - r(\beta q - c_q r^{q-1} L^q)) \|x - y\|^q.
\]
Therefore, in both cases (i) and (ii), \( T_r^{A,B} \) is a contraction map on \( E \) with constant \( \tau \).
Each of these cases in (i) and (ii) above implies that the inclusion problem (1) has a unique solution \( x^* \in S \). Consequently, using the update of \( x_{n+1} \) in Algorithm 1 with \( T = T_r^{A,B} \), we get
\[
\|x_{n+1} - x^*\| \leq (1 - \lambda_n) \|w_n - x^*\| + \lambda_n \tau \|w_n - x^*\| \\
= (1 - \lambda_n (1 - \tau)) \|w_n - x^*\| \\
\leq (1 - \lambda_n (1 - \tau)) \|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\| \\
= (1 - \lambda_n (1 - \tau)) \|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|.
\]
Observe that by the update of \( \theta_n \) in Algorithm 1, we have \( \sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty \), using Lemma 9, we get that \( x_n \to x^* \), \( n \to \infty \), and the proof is complete. \( \square \)

We next present a complexity bound for Algorithm 1 in this result.
Theorem 4. Suppose that either of condition (i) or (ii) in Theorem 3 is satisfied and let $x^* \in S$ be the unique solution of the inclusion problem (1). Let $\lambda_n = \lambda$ and $\epsilon_n = \epsilon$ be constant. Then, given $\rho \in (0, \lambda(1 - \tau))$, for any

$$n \geq \bar{n} := \left\lfloor \log_{1-\rho} \left( \frac{\epsilon}{\|x_0 - x^*\|} \left( \frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho} \right) \right) \right\rfloor,$$

assuming $\bar{n} \geq 0$, it holds that

$$\|x_n - x^*\| \leq \epsilon \left( \frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho} + 1 \right),$$

(24)

where

(i) $\tau := \frac{1}{\rho r + 1}$ if $A$ is $\alpha$-isa of order $q$, $B$ is $\beta$-strongly accretive of order $q$, and $r \in (0, 2\alpha)$ and

(ii) $\tau := \sqrt{1 - r(2\beta - rL^2)}$ if $B \leq L$, $A$ is $\beta$-strongly accretive and $L$-Lipschitz on $E$ with $r \in \left(0, \frac{2\beta}{L^2}\right)$.

Proof. From the proof of Theorem 3, for any $n \geq 1$ we get

$$\|x_{n+1} - x^*\| \leq (1 - \lambda(1 - \tau))(\|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|) \leq (1 - \lambda(1 - \tau))(\|x_n - x^*\| + \epsilon).$$

(25)

Without the loss of generality, we assume that for every $n < \bar{n}$ we have

$$\|x_n - x^*\| \geq \epsilon \left( \frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho} \right).$$

(26)

Concatenating (25) and (26) we obtain, for every $k < \bar{k}$,

$$\|x_{n+1} - x^*\| \leq (1 - \lambda(1 - \tau)) \left( 1 + \frac{\lambda(1 - \tau) - \rho}{1 - \lambda(1 - \tau)} \right) \|x_n - x^*\| = (1 - \rho) \|x_n - x^*\|.$$

(27)

Therefore, by the definition of $\bar{n}$, it holds that

$$\|x_{\bar{n}} - x^*\| \leq (1 - \rho)^\bar{n} \|x_0 - x^*\| \leq \epsilon \left( \frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho} \right).$$

For any $n > \bar{n}$ there are two possibilities. If

$$\|x_{n-1} - x^*\| \leq \epsilon \left( \frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho} \right),$$

then, by (25) and recalling that $(1 - \lambda(1 - \tau)) \leq 1$, we obtain that $x_n$ satisfies (24). Otherwise, if

$$\epsilon \left( \frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho} \right) \leq \|x_{n-1} - x^*\| \leq \epsilon \left( \frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho} + 1 \right),$$

then

$$\|x_n - x^*\| \leq (1 - \rho) \|x_{n-1} - x^*\| \leq \|x_{n-1} - x^*\|,$$

and the desired result holds. \(\square\)

Remark 3. We observe that, in contradiction with the assumptions of Theorem 2, in Theorem 4 the summability of $\{\epsilon_n\}$ is not required. However if one wants a good bound in (24) then a small value of $\epsilon$ must be set, but, in this case, small values of $\theta_n$ are allowed.
To summarize and emphasize the novelty and major advantages of our proposed scheme, we list next several relations to recent works.

**Remark 4.**

1. Our result in Theorem 1 extends the results in [17,26,30,34,35] from Hilbert spaces to uniformly convex and \( q \)-uniformly smooth Banach spaces. Furthermore, when \( \theta_n = 0 \) in Algorithm 1, Theorem 1 reduces to the results in [33] and other related papers.

2. Our Theorem 2 extends the results in [16,19,21,22,24,36] from Hilbert spaces to uniformly convex and \( q \)-uniformly smooth Banach spaces.

3. Shehu in [37] obtained a nonasymptotic \( O(1/n) \) convergence rate result for a Krasnoselski–Mann iteration with inertial extrapolation step in real Hilbert spaces under the stringent condition of Boţ et al. [17] (Theorem 5). In this paper, we obtain the results for Krasnoselski–Mann iteration with inertial extrapolation step under easy assumptions and give some complexity results in uniformly convex Banach spaces.

4. Themelis and Patrinos in [38] study a Newton-type generalization of the classical Krasnoselski–Mann iteration in Hilbert spaces and obtained superlinear convergence when the direction satisfies Dennis-More condition in Hilbert spaces. However, Themelis and Patrinos in [38] do not consider Krasnoselski–Mann iteration in Hilbert spaces and obtained superlinear convergence when the direction satisfies Dennis-More condition.

5. In [39], Phon-on et al. established inertial S-iteration in Banach spaces and obtained convergence under boundedness of some generated sequence. In this paper, the boundedness assumption of any generated sequence is dispensed with in our results. Therefore, our results improve on the results of this paper.

4. Numerical Illustration

In this section, we present two numerical examples in order to illustrate the behaviour of our proposed method. For the first example we are concerned with the split convex feasibility problem (SCFP) (Censor and Elfving [40]) in an infinite-dimensional Hilbert space. Let \( H_1 \) and \( H_2 \) be two real Hilbert spaces and \( T : H_1 \to H_2 \) a bounded and linear operator and \( T^* \) its adjoint. Let \( C \subseteq H_1 \) and \( Q \subseteq H_2 \) be nonempty, closed and convex sets. The split convex feasibility problem is formulated as follows:

\[
\text{find a point } x \in C \text{ such that } Tx \in Q.
\]

So, if we take \( Ax := \nabla \left( \frac{1}{2} \left< Tx - P_Q Tx \right, \right| \right) = T^* (I - P_Q) Tx \), where \( P_Q \) is the metric projection onto \( Q \), \( \nabla \) is the gradient and \( B = \partial i_C \) is the characteristic function of the set \( C \). So, the SCFP has an inclusion structure as in (1). It can be seen that \( A \) is Lipschitz continuous with constant \( L = \| T \|^2 \) and \( B \) is maximal monotone, see e.g., [41].

**Example 1.** Let \( H_1 = L_2([0, 2\pi]) \) and norm \( \| x \| := \left( \int_0^{2\pi} |x(t)|^2 dt \right)^{1/2} \) and inner product \( \langle x, y \rangle := \int_0^{2\pi} x(t)y(t)dt \), \( \forall x, y \in H \). Consider the half-space

\[
C := \{ x \in L_2([0, 2\pi]) \mid \langle 1, x \rangle \leq 1 \} = \left\{ x \in L_2([0, 2\pi]) \mid \int_0^{2\pi} x(t)dt \leq 1 \right\}
\]

(28)

where \( 1 \equiv 1 \in L_2([0, 2\pi]) \). In addition, let the closed ball centered at \( \sin \in L_2([0, 2\pi]) \) with radius 4.

\[
Q := \{ x \in L_2([0, 2\pi]) \mid \| x - \sin \|_2^2 \leq 16 \} = \left\{ x \in L_2([0, 2\pi]) \mid \int_0^{2\pi} |x(t) - \sin(t)|^2 dt \leq 16 \right\}.
\]

(29)
Consider the mapping $T : L^2([0,2\pi]) \to L^2([0,2\pi])$ such that $(Tx)(s) = x(s), \ \forall x \in L^2([0,2\pi])$. Then $(T^*x)(s) = x(s)$ and $\|T\| = 1$. So, we wish to solve the following problem:

$$\text{find } x^* \in C \text{ such that } Tx^* \in Q. \quad (30)$$

Observe since $(Tx)(s) = x(s), \ \forall x \in L^2([0,2\pi])$, (30) reduces to the well-known convex feasibility problem of the form.

$$\text{find } x^* \in C \cap Q. \quad (31)$$

Moreover, the solution set of (30) is nonempty since clearly $x(t) = 0$ is a solution. As explained before, we define $Ax := \nabla \left( \frac{1}{2} \|Tx - P_QTx\|^2 \right) = T^*(I - P_Q)Tx$ and $B = \partial C$ and translate (30) to an inclusion formulation as in (1).

We implement our algorithm with different starting point $x_0(t) = x_1(t), \ t \in [0,2\pi]$. We choose the stopping criterion $\|x_n - y_n\| < 10^{-5}$ and other parameters are chosen as $\varepsilon_n = 1/n^2, \lambda_n = 1/n, \theta = 0.5, \tau = 0.5$. To justify our algorithm’s name we compare it with the standard Krasnoselskii–Mann, which is the update of $x_{n+1}$ in Algorithm 1 with $w_n$ replaced by $x_n$ and $\lambda_n \in (0,1)$. The results for different starting points are presented in Table 1.

Recall the definition of the operator $T_{r}^{AB}$ (10) and following [41] (Example 23.4) and [42] we get the following results. For $z \in L_2([0,2\pi])$ we have

$$(I + \lambda_n B)^{-1}(z) = (I + \lambda_n \partial C)^{-1}(z) = \arg \min_{u \in L^2([0,2\pi])} \left\{ ic(u) + \frac{1}{2\lambda_n} \|u - z\|_2^2 \right\} = P_C(z). \quad (32)$$

Moreover, by [42]

$$P_C(z) = \begin{cases} 1 - \int_0^{2\pi} z(t)dt \frac{\pi}{2\pi}, & \int_0^{2\pi} z(t)dt > 1 \\ z, & \int_0^{2\pi} z(t)dt \leq 1. \end{cases}$$

For $w \in L_2([0,2\pi])$ we also have

$$P_Q(w) = \begin{cases} \sin + \frac{4}{\int_0^{2\pi} |w(t) - \sin(t)|^2 dt}, & \int_0^{2\pi} |w(t) - \sin(t)|^2 dt > 16 \\ w, & \int_0^{2\pi} |w(t) - \sin(t)|^2 dt \leq 16. \end{cases}$$

| Starting Points | CPU Time | Iterations |
|-----------------|----------|------------|
| $x_0 = x_1 = \frac{\pi}{2}, \ 0.054$ | 0.201 | 7 | 17 |
| $x_0 = x_1 = \frac{\pi}{2}, \ 0.056$ | 0.254 | 11 | 28 |
| $x_0 = x_1 = 2 \sin(5\pi) - 3 \cos(-2\pi), \ 0.0653$ | 0.103 | 4 | 15 |
| $x_0 = x_1 = \frac{\pi - \exp(-1)}{2}, \ 0.0732$ | 0.142 | 5 | 15 |
| $x_0 = x_1 = \frac{1}{2} \exp(-\frac{1}{2})$ | 0.103 | 0.243 | 9 | 14 |

Example 2. Take $E = L_p([0,2\pi]), 2 \leq p < \infty$. Then, $E$ is 2-uniformly smooth and uniformly convex and so $q = 2$ in Algorithm 1. Define $(Tx)(s) := \max\{ -x(s), 0 \} \ \forall x \in L_p([0,2\pi])$. Then, $T$ is nonexpansive and $F(T) = 0$. In the below numerical illustration we choose $p = 4, 10, 100$, starting points $x_0 = x_1 = 2 \sin(5\pi)$ and other parameters as in the previous example. Based on the example setting and Remark 4 we find Shehu’s
algorithm [37] (Equation (3)) most suitable for comparison with our Algorithm 1. The results are reported next in Table 2.

Table 2. Comparison of Algorithm 1 and Shehu’s Algorithm.

| p   | CPU Time | ∥x∥_L_p |
|-----|----------|---------|
|     | Algorithm 1 | Shehu’s Algorithm | Algorithm 1 | Shehu’s Algorithm |
| 4   | 1.1250    | 1.2813  | 11.7172 | 11.7172 |
| 10  | 0.9531    | 1.3906  | 3.8892  | 4.8892  |
| 1000| 0.0625    | 0.2500  | 2.1135  | 2.8135  |

5. Conclusions

In this paper, we give weak and strong convergence results for relaxed inertial forward-backward splitting method in uniformly convex and q-uniformly smooth Banach spaces under some appropriate conditions. Our results are new in Banach spaces, and generalize some existing results in the literature. In our future project, we will generalize our results in this paper to finding zero of maximal monotone operators in a more general Banach space.

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Abbreviations

The following abbreviations are used in this manuscript:

isa inverse strongly accretive
SCFP split convex feasibility problem

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