\textbf{N-order Darboux transformation and a spectral problem on semiaxis}

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\textbf{Abstract.} N-order Darboux transformation operator is defined on the basis of a general notion of transformation operators. Factorisation properties of this operator are studied. The Darboux transformation operator technique is applied to construct and investigate potentials with bound states at arbitrary energies for the spectral problem on semiaxis.

1. Introduction

In the inverse quantum scattering theory one usually uses the integral transformation operators (see, e.g., \cite{1}). An essential element in their construction is the Gelfand-Levitan-Marchenko equation. The differential transformation operators, such as the Darboux transformation operators (\cite{2}), have no such a prevalence. It was established in recent papers (\cite{3, 4}) that the integral transformation with a degenerate kernel is equivalent to a differential one. This possibility has been earlier noted by L. D. Faddeev \cite{5} and was recently discussed in (\cite{6}). Many properties of Darboux transformations are studied in the book by V.B. Matveev and M.A. Salle \cite{7}. They define this transformation on the basis of covariance property of the Schrödinger equation with respect to a transformation of the wave function and potential energy. We confine ourselves with other definition \cite{3, 4, 8}. Our definition is based on a general notion of transformation operators introduced and investigated by Delsart and Lions \cite{9} (see also \cite{10}). In terms of these notions Darboux \cite{2} investigated differential first order transformation operators for the Sturm-Liouville problem. This is the reason, in our opinion, to call any differential transformation operator \textit{Darboux transformation operator}.

In recent years one has investigated such transformations in connection with the spectral properties of the Schrödinger operator (\cite{11}) and as a source of new exactly solvable potentials (\cite{12, 13}) and reflectionless potentials with an infinite discrete spectrum (\cite{14}). In the supersymmetric quantum mechanics the N-order differential transformation operators are used in the construction of higher derivative supercharge operators (\cite{15}) and lead to new peculiarities in the supersymmetry breakdown (\cite{16}). In this connection the investigation of the properties of the N-order differential transformation operators is of interest.

In this paper we define the N-order Darboux transformation operator and give an improved formulation and a new proof of the theorem about the factorisation of this operator by the first order Darboux transformation operators which represent a dressing chain. It is established that the ordinary Darboux transformation
operator provide a one-to-one correspondence between the solution spaces of the input and output Schrödinger equations. We then apply this technique to construct and investigate potentials with given discrete spectrum for the spectral problem on semiaxis. We obtain an exactly solvable equation with arbitrarily disposed discrete spectrum levels. We give for the latter equation discrete spectrum eigenfunctions, Jost solution, and Jost function. The expression for the Jost function shows that the potential so obtained pertains to the Bargman type potentials. Other applications of these transformations may be found in [3, 4, 8], [11]-[15].

2. N-order Darboux transformation operator

Let us consider the one dimensional Schrödinger equation

\[ h_0 \psi(x) = E \psi(x), \quad h_0 = -D^2 + V_0(x), \quad D = -d/dx, \quad x \in [0, \infty) \]  

where \( V_0(x) \) is a sufficiently smooth real valued function. Let \( T_0 \) be the functional (if necessary topological) space of the solutions of the equation \( (1) \).

**Definition 1.** Let us call the N-order linearly differential operator \( L^{(N)} \) with the coefficient at \( D^N \) equal to unity, acting from \( T_0 \) to

\[ T_{N1} = \{ \varphi : \varphi = L^{(N)} \psi, \ \psi \in T_0 \}, \]

an N-order Darboux transformation operator for the Hamiltonian \( h_0 \) if it satisfies the following operator equation:

\[ L^{(N)} h_0 - h_0 L^{(N)} = A_N(x) L^{(N)}, \]

where \( A^{(N)}(x) \) is a sufficiently smooth function. If \( A^{(N)}(x) \equiv 0 \), the operator \( L^{(N)} \) is called trivial.

**Remark 1.** Every trivial transformation operator is a symmetry operator for equation \( (1) \).

It follows from Definition 1 that the function \( \varphi = L^{(N)} \psi \) satisfies the Schrödinger equation with the potential \( V_N(x) = V_0(x) + A_N(x) \) (transformed Schrödinger equation) and the space \( T_{N1} \subset T_N \), where \( T_N \) is a functional space of the solutions of the latter equation.

For \( N = 1 \) equation \( (2) \) defines the well known Darboux transformation with the transformation operator of the form

\[ L^{(1)} = L = -u'_\alpha / u_\alpha + D, \quad A_1(x) = -2 (\log u_\alpha)'', \]

where the prime denotes the derivative with respect to \( x \). The function \( u_\alpha \) called a transformation function is defined by the initial Hamiltonian \( h_0 \): \( h_0 u_\alpha = \alpha u_\alpha \), \( \alpha \in \mathbb{R} \), \( \text{Im} u_\alpha = 0 \). It is clear from \( (3) \) that \( \ker L = \text{span} \{ u_\alpha \} \), where ”span” stands for the linear hull over the complex number field \( \mathbb{C} \).
If $\bar{u}_\alpha$ is chosen such that $W(u_\alpha, \bar{u}_\alpha) = 1$ where $W$ stands for the Wronskian of the functions $u_\alpha$ and $\bar{u}_\alpha$, then we have $L\bar{u}_\alpha = u^{-1}_\alpha = v_\alpha$ and $h_1v_\alpha = \alpha v_\alpha$, $h_1 = h_0 + A_1(x)$. It is not difficult to convince ourselves that

$$\lim_{E \to \alpha} R^{-1}(E)L\psi_E(x) = \bar{v}_\alpha(x), \quad R(E) = E - \alpha$$

under the condition $\psi_E(x) \to u_\alpha(x)$ as $E \to \alpha$. In this case we have $h_1\bar{v}_\alpha = \alpha \bar{v}_\alpha$ and $W(v_\alpha, \bar{v}_\alpha) = W(u_\alpha, \bar{u}_\alpha) = 1$. Therefore we can always define on the space $T_0$ a linear operator $\hat{L}$ by putting

$$\hat{L}\psi_E = R^{-1/2}(E)L\psi_E, \quad \forall E \neq \alpha, \quad \hat{L}u_\alpha = L\bar{v}_\alpha = v_\alpha = u^{-1}_\alpha$$

and

$$\hat{L}\bar{v}_\alpha = L\bar{v}_\alpha = u_\alpha = v^{-1}_\alpha, \quad \hat{L}v_\alpha = \bar{u}_\alpha.$$  

The operator $\hat{L}$ maps every solution of the Schrödinger equation with the Hamiltonian $h_0$ to the solution of the same equation with the Hamiltonian $h_1$ and $W(\varphi_E, \bar{\varphi}_E) = W(\psi_E, \bar{\psi}_E)$, $\forall \psi_E, \bar{\psi}_E \in T_0$.

It follows from (2) that for the real valued function $A_N(x)$ the operator $L_1(N)$ where $L_1(N)$ is the symmetry operator for equation (1) and consequently it is an $N$-order polynomial with respect to $h_0$.

We will first consider three lemmas which are necessary for the proof of the main theorem.

**Lemma 1.** Operator $L \equiv L^{(1)}$ is the Darboux transformation operator if and only if $L^+L = h_0 - \alpha$, $\alpha \in \mathbb{R}$.

This lemma can easily be proved by direct calculations.

Since we have $\ker L^+ = \text{span} \{v_\alpha = u^{-1}_\alpha\}$, we can define an operator $\hat{L}^+$ in the space $T_1$ by putting

$$\hat{L}^+\varphi_E = R^{-1/2}(E)L^+\varphi_E, \quad \forall E \neq \alpha$$

and

$$\hat{L}^+\bar{v}_\alpha = L^+\bar{v}_\alpha = u_\alpha = v^{-1}_\alpha, \quad \hat{L}^+v_\alpha = \bar{u}_\alpha.$$  

The operators $\hat{L}$ and $\hat{L}^+$ assure a one-to-one correspondence between the spaces $T_0$ and $T_1$. Moreover, we have

$$T_0 = T_{01} \cup \text{span} \{\bar{u}_\alpha\}, \quad T_1 = T_{11} \cup \text{span} \{\bar{v}_\alpha\},$$

$$T_{01} = \{\psi: \psi = L^+\varphi, \varphi \in T_1\}.$$

**Lemma 2.** Operator $L \equiv L^{(2)}$ can always be presented in the form $L = L_2L_1$, where $L_1 = -u_1'/u_1 + D$ and $L_2 = -v'/v + D$ are the first order Darboux transformation operators, $u_1$ is the transformation function satisfying equation (4) with the eigenvalue $C_1$, $v$ is the transformation function for the iterated Darboux transformation satisfying the Schrödinger equation with an intermediate potential $V_1$ obtained after the Darboux transformation with the operator $L_1$ and corresponding to the
eigenvalue $C_2$. If $C_1$ and $C_2 \in \mathbb{R}$, then they are arbitrary and the functions $u_1$ and $v$ are real valued. If $C_1$ and $C_2 \in \mathbb{C}$, then $C_2 = \overline{C_1}$ (overline signifies the complex conjugation) and $v = L_1 \overline{u_1}$. The potential difference is a real valued function.

We note first of all that a similar statement is discussed in the paper by Andrianov et al. (1995) but we need here some details of the proof and so we give a complete proof of this lemma.

Consider a second order differential operator of the general form $L = a_0(x) + a_1(x) D + a_2(x) D^2$. Equation (4) reduces to a system of differential equations for the coefficients $a_i(x)$, $i = 0, 1, 2$ and $A(x) \equiv A_2(x)$. It follows from this system that $a_2 = \text{const}$ and without the loss of generality we put $a_2 = 1$. We find then $A = 2a'_1$. After excluding $a_0$ and $A$ we obtain a differential equation for the function $a_1$ which can readily be twice integrated with integration constants $2\alpha_1, \alpha_2 \in \mathbb{R}$. As a result we obtain the following differential equation for the function $a_1$:

$$a_1^2 V_0 + a_1^2 a'_1 - \frac{1}{2} a_1 a''_1 + \frac{1}{4} a_1^4 - \frac{1}{4} a_1^4 - \alpha_1 a_1^2 - \alpha_2 = 0 .$$

(4)

After introducing a new dependent variable $u_1$ with the help of the following relation

$$u'_1 / u_1 = \frac{1}{2} a'_1 / a_1 - \frac{1}{2} a_1 - \sqrt{\alpha_2} / a_1,$$

(5)

we rewrite equation (4) in the form

$$-D^2 u_1 + (V_0 - C_1) u_1 = 0,$$

where $C_1 = \alpha_1 - \sqrt{\alpha_2}$. We state consequently that the function $u_1$ is a solution of the initial Schrödinger equation. Since the solutions of this equation are supposed to be known, the function $u_1$ is determined. We solve now equation (5). For this purpose we introduce a new function $v$ by putting $a_1 = -[\ln(v u_1)]'$. Equation (5) reduces to the following equation for the function $v$:

$$-D^2 v + (V_1 - C_2) v = 0,$$

(6)

where $C_2 = \alpha_1 + \sqrt{\alpha_2}$ and $V_1 = V_0 - 2(\ln u_1)'$. Equation (5) shows that the function $v$ is a solution of the intermediate Schrödinger equation obtained by the Darboux transformation with the operator $L_1$ and the transformation function $u_1$. Taking into account the expressions for the functions $a_1$ and $a_0$

$$a_1 = -[\ln(v u_1)]', \quad a_0 = u'_1 v' / (u_1 v) - (\ln u_1)'',$

we obtain an expression for the operator $L$ formulated in the assertion of the lemma. For $C_2 \neq C_1$ the function $v$ is a transform of the function $u_2$: $v = L_1 u_2 = u_1^{-1} W(u_1, u_2)$ which is a proper function of the initial Hamiltonian: $h_0 u_2 = C_2 u_2$. The potential difference is given by the expression

$$A_2 = -2[\ln W(u_1, u_2)]'' .$$

(7)
For $C_2 = C_1$ we have $v = \beta_1 u_1^{-1} + \beta_2 \bar{v}$, where $\beta_1, \beta_2 \in \mathbb{R}$ and $u_1^{-1}, \bar{v}$ are linearly independent solutions of equation (6). The Wronskian $W$ should be replaced in this case by $\beta_1 + \beta_2 u_1 \bar{v}$.

**Corollary 1.** It follows from Lemmas 1 and 2 that

\[ L^+ L = (h_0 - C_1)(h_0 - C_2), \]
\[ LL^+ = (h_2 - C_1)(h_2 - C_2), \quad h_2 = h_0 + A_2. \]

**Remark 2.** For $C_1 = C_2 = C \in \mathbb{R}$ and $v = u_1^{-1}$ ($\beta_2 = 0$) the operator $L = -L^+ L_1 = C - h_0$ is a trivial transformation operator.

**Remark 3.** In the case $C_2 \neq C_1$, the intermediate transformation function can be expressed through the solution $u_2$ of the initial equation (1). In this case we obtain the known (14) expression for the transformation operator, which in the general case has the form

\[
L^{(N)} = L_N L_{N-1} ... L_1 = W^{-1}(u_1, ..., u_N) \left| \begin{array}{cccc}
  u_1 & u_2 & \cdots & 1 \\
  u_1' & u_2' & \cdots & D \\
  \cdots & \cdots & \cdots & \cdots \\
  u_1^{(N)} & u_2^{(N)} & \cdots & D^N
\end{array} \right|. \quad (8)
\]

In addition, the following factorizations are valid ([14, 3]):

\[
L^{(N)} L^{(N)} = P(h_0) = \prod_{i=1}^{N}(h_0 - C_i), \quad L^{(N)} L^{(N)+} = P(h_N)
\]

where $h_N = h_0 + A_N$, $A_N = -2 \log W(u_1, ..., u_N)$, $h_0 u_i = C_i u_i$ and all $C_i$ are different. If the coefficients of the polynomial $P(x)$ belong to the field $\mathbb{R}$, then the intermediate potentials (in the case where the polynomial $P(x)$ has complex zeros) can be complex valued, but the final potential is real valued if the transformation functions are chosen appropriately.

Let us denote by $C_1, \ldots, C_N$ the zeros of the polynomial $P(x)$ which can be of an arbitrary order.

**Lemma 3.** If $L^{(N)}$ is an $N$-order Darboux transformation operator with the transformation functions $u_i$ such that $h_0 u_i = C_i u_i$, then we have

\[
\ker L^{(N)} \cap \bigcup_{i=1}^{N} \ker (h_0 - C_i) \neq \emptyset.
\]

Consider one of the zeros of the polynomial $P(x)$, e.g., $C_1$. If $\ker L^{(N)} \cap \ker (h_0 - C_1) \neq \emptyset$, then the lemma is proved. Let we have

\[
\ker L^{(N)} \cap \ker (h_0 - C_1) = \emptyset. \quad (9)
\]
Consider \( v_1 = L^{(N)}u_1 \) and \( \tilde{v}_1 = L^{(N)}\tilde{u}_1 \), where \( u_1 \) and \( \tilde{u}_1 \) form a basis in the space \( \ker(h_0 - C_1) \). By virtue of the linearity of the operator \( L^{(N)} \) and assumption \( ([1]) \), the space span \( \{v_1, \tilde{v}_1\} \) cannot be a one dimensional space. Then we have

\[
\text{span}\ \{v_1, \tilde{v}_1\} = \ker(h_N - C_1) \subset \ker L^{(N)^+} .
\]

The operator \( L^{(N)^+} \), with the help of Proposition 2.1 given in the book by Berkovich [22], can be presented in the form \( L^{(N)^+} = L^{(N-2)^+}L_2^+L_1^+ \), where

\[
L_1^+ = \frac{d}{dx} \ln v_1 - D, \quad L_2^+ = \frac{d}{dx} \ln \frac{W(v_1, \tilde{v}_1)}{v_1} - D .
\]

Using expression \( ([10]) \), we obtain

\[
L^{(N)^+} = -L^{(N-2)^+}(h_N - C_1) .
\]

Here, \( L^{(N-2)^+} \) is an \((N-2)\)-order Darboux transformation operator which transforms the solutions of the Schrödinger equation with the Hamiltonian \( h_N \) to the solutions of the same equation with the Hamiltonian \( h_0 \). It follows from \( ([1]) \) that \( L^{(N)} = -L^{(N-2)}(h_0 - \overline{C}_1) \). When \( C_1 \in \mathbb{R} \), the latter expression contradicts to \( ([4]) \) and when \( C_1 \in \mathbb{C} \) it leads to the lemma’s assertion since \( \overline{C}_1 \) in this case is as well the zero of the polynomial \( P(x) \). \( \square \)

We can now formulate and prove the main theorem.

**Theorem 1.** The action of every nontrivial transformation operator \( L^{(N)} \) is equivalent to the resulting action of any chain of \( k \) first order Darboux transformation operators.

Using Lemmas 2 and 3 and the induction, we can always present an operator \( L^{(N)} \) in the form \( L^{(N)} = L_N L_{N-1} \ldots L_1 \) which corresponds to a chain of \( N \) first order Darboux transformations. If we are under the hypothesis of Remark 2, then some of the products of the operators from this chain are trivial transformation operators. In this case we have \( L^{(N)} = L^{(k)}P(h_0) \), where \( L^{(k)} = L_{t+k}L_{t+k-1} \ldots L_t \) with any \( t \) and \( P(x) \) is some polynomial. The transformation operators \( L^{(N)} \) and \( L^{(k)} \) give the same potential difference \( A_N(x) \), and the theorem is proved. \( \square \)

**Remark 4.** This theorem stresses the existence of trivial transformation operators which can appear in any transformation chain for an arbitrary initial potential \( V_0(x) \). There exist potentials (see \([7]\)) for which the chain of \( k \geq 2 \) first order operators gives a trivial transformation operator, while the product of \( k - 1 \) operators is a nontrivial transformation operator.

We notice as well that the operator \( L^{(k)^+} \) realizes the transformation from the solutions of the Schrödinger equation with the potential \( V_N \) to the solutions of equation \( ([1]) \) and consequently can be used for construction of the operator \( L^{(k)^-1} \). Moreover, if \( C_1, \ldots, C_q \) are different zeros of the polynomial \( P(h_0) = L^{(k)^+}L^{(k)} \), then for the space \( T_0 \) we can write the following decomposition:

\[
T_0 = T_{01} \cup \bigcup_{i=1}^{q} \text{span}\ \{\tilde{u}_i\} , \quad \ker(h_0 - C_i) = \text{span}\ \{u_i, \tilde{u}_i\}, \quad i = 1, \ldots, q .
\]
The functions $u_i$ are the transformation functions for the intermediate transformation operator $L^{(q)} = L_q L_{q-1} \ldots L_1$. A similar decomposition is valid for the space $T_N$.

3. Potentials with given discrete spectrum on semiaxis

We will now apply the Darboux transformation operator technique to generate and investigate one class of potentials with $N$ discrete energy levels disposed in a desirable manner. To construct such potentials, one usually uses integral transformation operators. For the spectral problem on full real axis, these potentials are known as $N$-soliton ones (see, e.g., [17]). The properties of the $N$-soliton solutions of the KdV equation were studied by Wadati and Toda [18]. The term 'N-soliton potential' was introduced by Its and Matveev [19]. Sukumar [20] and Berezovoi and Pashnev [21] give a simple prescription for getting $N$-soliton potentials by means of a chain of Darboux transformations. Recently ([4]) an analytic expressions for these potentials and for discrete spectrum eigenfunctions have been obtained.

Following the papers [20, 21, 3, 4], we consider the following solutions of the Schrödinger equation with a zero potential ($h_0 = -D^2$) as transformation functions:

$$u_k(x) = \cosh (a_k x + b_k), \quad u_{k+1}(x) = \sinh (a_{k+1} x + b_{k+1}),$$

$$h_0 u_i = -a_i^2 u_i, \quad 0 < a_1 < a_2 < \ldots.$$  \hspace{1cm} (13)

The resulting action of such a chain is equivalent to the action of the $N$-order transformation operator (8), and the resulting potential difference is given by (7) with the replacement $W(u_1, u_2) \rightarrow W(u_1, \ldots, u_N)$.

For arbitrary $N$, the following can be proved:

**Proposition.** The Wronskian of functions (12) is a linear combination of hyperbolic cosines:

$$W(u_1, \ldots, u_N) = 2^{1-N} \sum_{(\varepsilon_1, \ldots, \varepsilon_N)} \varepsilon_2 \varepsilon_4 \cdots \varepsilon_p \prod_{j>i}^{N} (\varepsilon_j a_j - \varepsilon_i a_i) \cosh[\sum_{l=1}^{N} \varepsilon_l (a_l x + b_l)],$$

where $\varepsilon_i = \pm 1$; the summation is over all different ordered sets $(\varepsilon_1, \ldots, \varepsilon_N)$ consisted of +1 and -1 (sets $(\varepsilon_1, \ldots, \varepsilon_N)$ and $(-\varepsilon_1, \ldots, -\varepsilon_N)$ are considered identical); the index at $\varepsilon_p$ is taken equal to $N$ or $N-1$ with $N$ being even or odd, respectively.

The proof can be broken down into two parts:

a) The following formula is proved by the induction method:

$$W(u_1, \ldots, u_N) = 2^{-N} \sum_{\varepsilon_i = \pm 1} \varepsilon_2 \varepsilon_4 \cdots \varepsilon_p \prod_{j>i}^{N} (\varepsilon_j a_j - \varepsilon_i a_i) \exp[\sum_{l=1}^{N} \varepsilon_l (a_l x + b_l)];$$

b) The number of different ordered sets $(\varepsilon_1, \ldots, \varepsilon_N)$ equals $2^N$. As for any set $(\varepsilon_1, \ldots, \varepsilon_N)$ there is a set with an opposite sign $(-\varepsilon_1, \ldots, -\varepsilon_N)$, we can detach
paired members in the sum, which are packed into the hyperbolic cosines, and in doing so, we obtain the above formula with the number of members $2^{N-1}$ and the factor $2^{1-N}$.

The potential calculated according to (7) is regular.

Let $\tilde{u}_i$ be the second linearly independent solution of equation (13) with the eigenvalue $E = -a_i^2$ and such that $W(\tilde{u}_i, \tilde{u}_i) = 1$. The action of operator $\tilde{S}$ to $\tilde{u}_i$ gives the following result:

$$\tilde{v}_i(x) = L^{(N)} \tilde{u}_i = W^{(i)} (u_1, \ldots, u_N) W^{-1} (u_1, \ldots, u_N),$$

where $W^{(i)} (u_1, \ldots, u_N)$ is an $(N-1)$-order Wronskian constructed from the functions $u_1, \ldots, u_N$ except for the function $u_i$. Moreover, the limit

$$\lim_{\psi_E \to \tilde{u}_i} (E + a_i^2)^{-1} L^{(N)} \psi_E (x) = v_i(x)$$

exists and gives the second linearly independent solution of the Schrödinger equation with the potential $V_N(x) = A_N(x)$ corresponding to the eigenvalue $E = -a_i^2$ and such that $W(v_i, \tilde{v}_i) = 1$.

Functions (14) with the transformation functions chosen according to (12), are square integrable on full real axis. Parameters $b_i$ realize the isospectral deformation of the potential $V_N$, and for the sake of simplicity we put all $b_i = 0$. In this case, for even $N$, the parity of functions (14) coincides with the parity of their number $i$ and for odd $N$ it is opposite to this parity. Hence, the potential $V_N$ considered on semiaxis for $N = 2k$ and $N = 2k + 1$ has $k$ discrete energy levels. For even $N$, just $N/2$ functions (14) with even numbers are the discrete spectrum eigenfunctions of the Hamiltonian $h_N$, and for odd $N$, we should take $(N-1)/2$ functions with odd numbers. The discrete spectrum eigenfunctions normalized to unity have the form

$$\varphi_i(x) = \left[a_i \prod_{j=1, j \neq i}^N \left|a_i^2 - a_j^2\right|\right]^{1/2} \tilde{v}_i(x).$$

They have the proper values equal to $-a_i^2$.

Let us construct now the Jost solution and the Jost function for the potential $V_N$. One can find their definition, for instance, in the first book of the Ref. [1].

Let $f^{(0)} (k) = \exp (ikx), E = k^2$, be the Jost solution for the initial potential. Then the solution $\psi^{(0)} = \frac{1}{k} \sin (kx)$ of the same equation, regular at $x = 0$, is expressed with the help of the Jost solution as follows:

$$\psi^{(0)} = \frac{1}{2k^2} \left[f^{(0)} (k) - f^{(0)} (-k)\right].$$

We need as well the second solution of the same equation corresponding to $E = k^2$:

$$\tilde{\psi}^{(0)} = \cos (kx) = \frac{1}{2} \left[f^{(0)} (k) + f^{(0)} (-k)\right].$$
It can readily be seen that the behavior of the functions $L^{(N)}\psi^{(0)}(0)$ and $L^{(N)}\tilde{\psi}(0)$ as $x \to 0$ depends on the parity of $N$. For an even $N$, we have

$$L^{(N)}\psi^{(0)} \approx (-1)^{N/2} \prod_{i=1}^{N/2} \left( k^2 + a_{2i}^2 \right) x,$$

and for an odd $N$ we obtain

$$L^{(N)}\psi^{(0)} \approx (-1)^{(N+1)/2} \prod_{i=0}^{(N-1)/2} \left( k^2 + a_{2i+1}^2 \right) x.$$

It is easily seen from these relations that the solution regular at the origin (i.e., the solution with the asymptotic $x^{-1}\psi^{(N)} \to 1$ as $x \to 0$) has the form

$$\psi^{(N)} = (-1)^{(N+1)/2} \prod_{i=0}^{(N-1)/2} \left( k^2 + a_{2i+1}^2 \right) L^{(N)}\tilde{\psi}(0)$$

(18)

for an odd $N$ and

$$\psi^{(N)} = (-1)^{N/2} \prod_{i=1}^{N/2} \left( k^2 + a_{2i}^2 \right) L^{(N)}\psi^{(0)}$$

(19)

for an even $N$.

By the same means, the behavior of the function $L^{(N)}f^{(0)}(k)$ as $x \to \infty$,

$$L^{(N)}f^{(0)}(k) \approx \prod_{j=1}^{N} (-a_j + i k) \exp (i k x),$$

permits one to write the Jost solution for the potential $V_N$

$$f^{(N)}(k) = \prod_{j=1}^{N} (-a_j + i k)^{-1} L^{(N)}f^{(0)}(k).$$

(20)

With the help of relations (16), (17), and (20), equations (18) and (19) can be rewritten as follows:

$$\psi^{(N)} = \frac{i}{2k} \left[ F^{(N)}(k) f^{(N)}(-k) - F^{(N)}(-k) f^{(N)}(k) \right],$$

(21)

where

$$F^{(N)}(k) = \prod_{j=N/2}^{N} \frac{k - i a_{2j-1}}{k + i a_{2j}}$$

(22)
is the Jost function for the potential $V_N$; $\gamma = 1/2$ when $N$ is odd and $\gamma = 1$ when $N$ is even; $a_0 = 0$. Equation (22) clearly shows that the potential $V_N$ is of the Bargman type.

4. Examples

Consider now simplest particular cases. For $N = 1$ the potential $V_N$ is the well known one soliton potential

$$V_1 = -2a_1^2 \text{sech}^2 (a_1 x).$$

This potential, for the spectral problem on full real axis, has one discrete energy level and, when being considered on semiaxis, it has no discrete spectrum at all. Using the transformation operator $L^{(1)}$, we easily obtain its Jost solution:

$$f^{(1)} (k) = \frac{a_1 \tanh (a_1 x) - i k}{a_1 - i k} \exp (i k x).$$

According to formula (22), we obtain the Jost function

$$F^{(1)} (k) = k (k + i a_1)^{-1}.$$

The module and phase of this function, $F^{(1)} (k) = F_1 e^{-i \delta}$, define the asymptotic of the regular solution of the Schrödinger equation at $x \to \infty$:

$$\psi^{(1)} = \frac{i}{2k} \left[ \frac{k}{k + i a_1} f^{(1)} (k) - \frac{-k}{-k + i a_1} f^{(1)} (-k) \right],$$

$$\psi^{(1)} \to \frac{F_1}{k} \sin (k x + \delta).$$

It is interesting to note that the choice $b_1 \neq 0$ corresponds to the translation of the origin along the $x$ axis by the value $\Delta = -b_1/a_1$. Choosing values of $b_1$ and $a_1$, we can give an arbitrary value to the quantity $\Delta$. Nevertheless, the potential $V_1$ not necessary has a discrete spectrum. This means that the absence on semiaxis and the presence on full real axis of a discrete spectrum are due to the asymptotic behavior of the potential $V_1$.

For $N = 2$ we have the two soliton potential

$$V_2 = \frac{8 (a_1^2 - a_2^2) [a_2^2 \cosh (a_1 x) + a_1^2 \sinh (a_2 x)]}{[(a_2 + a_1) \cosh [(a_2 - a_1) x] + (a_2 - a_1) \cosh [(a_2 + a_1) x]]^2}$$

having a single discrete spectrum level equal to $-a_1^2$ with the wave function

$$\varphi_0 = \frac{2 \sqrt{a_1 (a_2^2 - a_1^2) \sinh (a_2 x)}}{(a_2 + a_1) \cosh [(a_2 - a_1) x] + (a_2 - a_1) \cosh [(a_2 + a_1) x]}.$$

Its Jost function has the form:

$$F^{(2)} (k) = \frac{k - i a_1}{k + i a_2}.$$
5. Discussion and concluding remarks

We have shown that every $N$-order Darboux transformation operator defined in terms of a general notion of transformation operators may be presented as a chain of $k (\leq N)$ usual Darboux transformation operators (so called a dressing chain [1]). It is worth stressing that this chain may have ill defined elements. There are two possibilities for such elements. In the first case we may obtain complex-valued potentials for some intermediate Hamiltonians. Corresponding eigenvalue problem can not be treated as Schrödinger equation and no quantum mechanical interpretation exists for eigenfunctions. In the second case some intermediate potentials may have poles in the interval for the variable $x$ where the initial Schrödinger equation is defined. In this case several spectral problems arise. In general, the spectrum of such problems has no common points with the initial problem. Nevertheless, we may obtain solution to these auxiliary spectral problems (if necessary) with the help of the Darboux transformation operator method. Such a situation is studied in detail in [23]. In the case of the absence of ill defined elements in the chain of transformations we have a completely reducible chain. Conception of reducibility of chains of Darboux transformations with respect to complex-valued intermediate potentials has been introduced by Andrianov et. al. [14]. With regard to these remarks we state that the study of $N$-order Darboux transformation operators may be reduced to the study of different chains of the first order Darboux transformations.

The above statement has been illustrated by the application of a completely reducible chain of transformations to create a potential with arbitrarily disposed discrete spectrum levels for the spectral problem on semiaxis. A closed formula for an $N$-order Wronskian has been given. To obtain the potential it is sufficient to take the second logarithmic derivative of this Wronskian. With the help of the $N$-order Darboux transformation operator the Jost solution and the Jost function have been obtained. Our analysis shows that the presence of a single discrete level and its absence when the same potential is considered on full real axis and on semiaxis is due to the asymptotic behaviour of the potential.

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References

[1] Chadan K and Sabatier PC 1989 Inverse problem in quantum scattering theory 2nd ed. (New-York: Springer-Verlag); Marchenko V A 1986 Sturm-Liouville operators and applications. (Basel: Berkhauser); Zakhariev B N and Suzko A A 1990 Potentials and quantum scattering. Direct and inverse problems (Heidelberg: Springer-Verlag)
[2] Darboux G 1889 *Leçons sur la théorie générale des surfaces et les application géométriques du calcul infinitésimal. Deuxièm partie.* (Paris: Gauthier-Villars et fils); Ince E L 1926 *Ordinary differential equations* (New York: Dover)

[3] Bagrov V G and Samsonov B F 1995 *Teor. Mat. Fiz.* **104** 356

[4] Samsonov B F 1995 *J. Phys. A: Math. and Gen.* **28** 6989

[5] Faddeev L D 1959 *Usp. Mat. Nauk.* **14** 57

[6] Schnizer W A and Leeb H 1994 *J. Phys. A: Math. Gen.* **27** 2605

[7] Matveev V and Salle M 1991 *Darboux transformations and solitons.* (New York: Springer)

[8] Bagrov V G and Samsonov B F 1997 *Fiz. Elem. Chastits Atom. Yadra* **28** 951; *Zh. Éksp. Teor. Fiz.* 1996 **109** 1105

[9] Delsart J 1938 *Comp. Rend. Acad. Sci. Paris.* **206**. 178; 1956 *Colloq. Int. Nancy* 29; Lions J L 1956 *Colloq. Intern. Nancy* 125; Delsart J and Lions J L *Comment. Math. Helv.* **32** 113

[10] Levitan B M 1973 *Generalized translation operators* (Moskow: Nauka)

[11] Veselov A P and Shabat A B 1993 *Func. Anal. Appl.* **27** 1

[12] Dudov S Yu Eleonsky V M and Kulagin N E 1994 *Int. J. Bifurc. Chaos.* **4** 47

[13] Degasperis A and Shabat A 1994 *Teor. Mat. Fiz.* **100** 230

[14] Andrianov A A Ioffe M V and Spiridonov V P 1993 *Phys. Lett. A.* **174** 273; Andrianov A A Ioffe M V and Nishnianidze D N 1995 *Teor. Mat. Fiz.* **104** 463; Andrianov A A Ioffe M V Cannata F and Dedonder J-P 1995 *Int. J. Mod. Phys. A.* **10** 2683

[15] Samsonov B F 1996 *Mod. Phys. Lett. A.* **11** 1563

[16] Crum M M 1955 *Quart. J. Math.* **6**, 121; Krein M G 1957 *Dokl. Akad. Nauk. SSSR* **113** 970
[17] Newell A C *Solitons in mathematics and physics* (Arizona: SIAM)

[18] Wadati M and Toda M 1972 *J. Phys. Soc. Jap.* **32** 1403

[19] Its A R and Matveev V B 1975 *Teor. Mat. Fiz.* **23**, 51

[20] Sukumar C V 1987 *J. Phys. A: Math. and. Gen.* **20** 2461

[21] Berezovoi V P and Pashnev A I 1989 *Teor. Mat. Fiz.* **78** 289

[22] Berkovich L M 1989 *Factorization and transformations of the ordinary differential equations.* (Saratov: Saratov University Press)

[23] Bagrov V G Ovcharov I N and Samsonov B F 1995 *J. Moscow Phys. Soc.* **5** 191