Power Sum Decompositions of Elementary Symmetric Polynomials

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Abstract

We bound the tensor ranks of elementary symmetric polynomials, and we give explicit decompositions into powers of linear forms. The bound is attained when the degree is odd.

1 Introduction

Given a form $F \in \mathbb{C}[x_1, \ldots, x_n]$ of degree $d$, the symmetric tensor rank (rank in short) of $F$ is the least integer $s$ such that $f = \sum_{i=1}^{s} L_i^d$, where the $L_i$’s are linear forms. For a generic form, the rank is known for any $n$ and $d$ by a work of Alexander and Hirschowitz [1], and a simple proof is proposed by Chandler [4].

On the contrary, only a few cases is known for that of specific form [2, 3, 8, 9]. To describe one of them, we define an index-membership function $\delta$; for an integer set $I$ and an integer $i$, define $\delta(I, i) = -1$ if $i \in I$, or 1 otherwise. A decomposition of the monic square-free monomial $\sigma_{n,n}$ in $n$ variables was given by Fischer [6]. He showed that

$$2^{n-1} n! \cdot \sigma_{n,n} = \sum_{I \subset [n] \setminus \{1\}} (-1)^{|I|}(x_1 + \delta(I,2)x_2 + \cdots + \delta(I,n)x_n)^n,$$

(1.1)

where $[n]$ denotes the set $\{1, 2, \ldots, n\}$, which will be used though this paper. For a general monomial, such a decomposition is given in [2, Corollary 3.8] so that (1.1) is a special case.

We extend the decomposition (1.1) for general elementary symmetric polynomials $\sigma_{d,n}$. As an example, we have

$$24 \sigma_{3,5}(a, b, c, d, e) = abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde$$

$$= 3(a + b + c + d + e)^3 - (a + b + c + d + e)^3 - (a + b + c + d + e)^3$$

$$- (a + b - c + d + e)^3 - (a + b + c - d + e)^3 - (a + b + c + d - e)^3,$$

and it gives an upper bound $\text{rank}(\sigma_{3,5}) \leq 6$. 


In general, such a decomposition provides an upper bound

$$\text{rank}(\sigma_{d,n}) \leq \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{n}{i}$$

for the rank of $\sigma_{d,n}$ (Corollary 2.2, Corollary 4.4).

This paper is organized as follows. In section 2 we present a power sum decomposition of $\sigma_{d,n}$ for odd degree case. Theorem 2.1 gives an analogue of (1.1).

In Section 3 we observe a structure of the catalecticants of $\sigma_{d,n}$. Lemma 3.2 says that each catalecticant matrix is essentially full rank, in the sense that it can be refined to a full rank matrix after removing all zero rows and zero columns. Theorem 3.4 tell us that the lower bound derived by Lemma 3.2 matches the number of components in the decomposition given in Section 2, hence we get the rank of $\sigma_{d,n}$ for odd $d$.

Section 4 discusses the even degree case. A power sum decomposition of $\sigma_{d,n}$ can be obtained from that of $\sigma_{d+1,n}$ (Theorem 4.1), or it can be derived directly (Remark 4.2). By Corollary 4.4 we see that upper and lower bounds are not the same, and we shortly explain why it seems to be hard to improve these bounds.

## 2 An upper bound

Through this paper, our main object is the elementary symmetric polynomial $\sigma_{d,n}$. It is defined by the sum of all square-free monomials. In symbols,

$$\sigma_{d,n} = \sum_{I \subset [n], |I|=d} \prod_{i \in I} x_i.$$ 

In this section we only focus on the odd degree case. Let $d = 2k + 1$ be odd, and consider the monomial case $n = d$. The first step is to write (1.1) into a symmetric way. Let $L_I = \delta(I, 1)x_1 + \delta(I, 2)x_2 + \cdots + \delta(I, n)x_n$ for some $I \subset [n]$, recalling that $\delta(I, i) = \pm 1$ and takes $-1$ if and only if $i \in I$. When $|I| \geq k+1$, we use $-(L_I)^n = -(L_{\bar{I}})^n$ instead of $(L_I)^n$ in the expression (1.1). Now all the linear terms appearing in (1.1) have less than half as many possible minus signs, and the coefficient of $x_1$ can be $-1$. After consideration of the power of $-1$, we get

$$2^{n-1}n! \cdot \sigma_{n,n} = \sum_{|I| \leq k} (-1)^{|I|}(\delta(I, 1)x_1 + \delta(I, 2)x_2 + \cdots + \delta(I, n)x_n)^n. \quad (2.1)$$

Using this symmetric representation as the initial step, we can prove following theorem using induction.
Theorem 2.1. Let \( d = 2k+1 \) be odd, and \( n \geq d \). Then the elementary symmetric polynomial \( \sigma_{d,n} \) admits the power sum decomposition

\[
2^{d-1}d! \cdot \sigma_{d,n} = \sum_{I \subset [n], |I| \leq k} (-1)^{|I|} \binom{n-k-|I|-1}{k-|I|}\left(\delta(I,1)x_1 + \delta(I,2)x_2 + \cdots + \delta(I,n)x_n\right)^d
\]

where \( \delta(I,i) = -1 \) if \( i \in I \), or \( 1 \) otherwise.

Proof. Write \( F_{d,n} \) for the expression (2.2). Putting \( x_n = 0 \), we get

\[
F_{d,n}(x_1, \ldots, x_{n-1}, 0) = \sum_{I \subset [n-1], |I| \leq k} (-1)^{|I|} \binom{n-k-|I|-1}{k-|I|}\left(\delta(I,1)x_1 + \cdots + \delta(I,n-1)x_{n-1}\right)^d = F_{d,n-1}
\]

Recursively we have \( F_{d,n}(x_1, \ldots, x_d, 0, \ldots, 0) = F_{d,d} \). By (2.1) we obtain

\[
F_{d,d} = \sum_{I \subset [d], |I| \leq k} (-1)^{|I|} \binom{k-|I|}{k-|I|}\left(\delta(I,1)x_1 + \delta(I,2)x_2 + \cdots + \delta(I,d)x_d\right)^d = 2^{d-1}d! \cdot \sigma_{d,d}.
\]

It shows that \( F_{d,n} \) consists of square-free monomials only. Using the symmetry of \( F_{d,n} \) we conclude

\[
F_{d,n} = 2^{d-1}d! \cdot \sigma_{d,n}.
\]

Counting the number of summands, we get an upper bound for the rank of \( \sigma_{d,n} \).

Corollary 2.2. For \( d \) odd, the rank of \( \sigma_{d,n} \) is bounded by

\[
\text{rank}(\sigma_{d,n}) \leq \sum_{i=0}^{(d-1)/2} \binom{n}{i}.
\]

Another consequence is a summation identity, as done in [6]. For \( n \geq 2k + 1 \) the identity

\[
\sum_{i=0}^{k} (-1)^i \binom{n-k-1-i}{k-i} \binom{n}{i}(n-2i)^{2k+1} = \frac{2^{2k}n!}{(n-2k-1)!}
\]

(2.3)

can be obtained by choosing \( d = 2k+1 \) and all \( x_i = 1 \) in the equation in Theorem 2.2.
3 A lower bound

Let \( S = \mathbb{C}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}] \) be the ring of differential operations with constant coefficients. It naturally acts on \( R = \mathbb{C}[x_1, \ldots, x_n] \) by differentiation. For a form \( F \) in \( R \) its apolar ideal \( F^\perp \) is defined by the annihilator of \( F \) in \( R \).

**Theorem 3.1 (Apolarity Lemma).** For a degree \( d \) form \( F \in R_d \) there is a power sum decomposition

\[
F = \sum_{i=1}^{s} L_i^d, \quad L_i \text{ linear}
\]

if and only if there exists a set of \( s \) distinct points in \( \mathbb{P}(S_1) \) whose defining ideal is contained in \( F^\perp \).

The apolarity lemma plays a key role in computations of symmetric tensor rank, especially for lower bounds. Many (possibly all) known lower bounds, given in [3, 8, 10] for instance, are related to the apolar ideal or at least catalecticants.

The \( r \)-th catalecticant of a given form \( F \in R_d \) is a linear map \( \phi_r : S_r \rightarrow R_{d-r} \) given by \( \phi(g) = gF \). Using monomial basis, \( \phi_r \) can be written as an \( (n+d-r-1) \times (n+d-1) \) matrix \( M_r \). Studying catalecticants and apolar ideals are essentially same, by the relation \( F^\perp = \bigcup \ker \phi_r \) or equivalently \( \ker \phi_r = (F^\perp)_r \). Note that it implies \( \text{Hilb}(S/F^\perp, r) = \text{rank}(M_r) \).

For our case \( F = \sigma_{d,n} \), the matrix \( M_r \) has many zero columns and zero rows. A column or a row of \( M_r \) is nonzero if and only if its index is square-free. Removing all zero columns and zero rows, we get a \( \left(\begin{array}{c} n \\ d-r \end{array}\right) \times \left(\begin{array}{c} n \\ r \end{array}\right) \) submatrix \( \tilde{M}_r \) of \( M_r \), whose indices are square-free monomials of corresponding degrees. Collecting subindices of monomials, we regards the indices of rows and columns as a subset of \( [n] \). The matrix \( \tilde{M}_r \) is binary with \((\tilde{M}_r)_{I,J} = 1\) if and only if \( I \) and \( J \) are disjoint.

For instance, the second catalecticant of \( \sigma_{4,5} \) is given by

\[
M_2 = \begin{pmatrix}
11 & 12 & 13 & 14 & 15 & 22 & 23 & 24 & 33 & 34 & 35 & 44 & 45 & 55 \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
13 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
14 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
15 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
22 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
24 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
25 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
34 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
44 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
45 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
55 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Its rows and columns are indexed by 2-subsets of \( \{1, \ldots, 5\} \), and a row (resp. a column) indexed by \( ij \) corresponds to the monomial \( x_i x_j \) (resp. \( \frac{\partial^2}{\partial x_i \partial x_j} \)). Removing zero rows and zero columns indexed by \( 11, 22, \ldots, 55 \), we obtain

\[
\tilde{M}_2 = \begin{pmatrix}
12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \\
12 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
13 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
14 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
15 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
23 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
24 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
25 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
34 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
35 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
45 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

and one may check that it has full rank. Next lemma shows that it is true in general, including the case that the resulting \( \tilde{M}_r \) is not a square matrix.

**Lemma 3.2.** For all \( r, d, \) and \( n \) with \( r \leq d \leq n \), the matrix \( \tilde{M}_r \) is of full rank.

**Proof.** Since \( \tilde{M}_r = (\tilde{M}_{d-r})^T \), we may assume \( r \leq d - r \). Note that it forces \( \binom{n}{r} \geq \binom{n}{d-r} \) and \( 2r \leq n \). Our goal is to show \( \text{rank}(\tilde{M}_r) = \binom{n}{r} \).

Define an \( \binom{n}{r} \times \binom{n}{r} \) binary matrix \( D_r^n \) whose columns and rows are indexed by \( r \)-subsets of \([n]\), and \( (D_r^n)_{I,J} = 1 \) if and only if \( I \) and \( J \) are disjoint. In [7, Example 2.12], the matrix \( D_r^n \) is shown to be invertible when \( 2r \leq n \).

We claim that the row space of \( D_r^n \) is a subspace of the row space of \( \tilde{M}_r \). Pick a row vector \( v_I \) of \( D_r^n \) indexed by an \( r \)-subset \( I \). Consider a row vector

\[
w_I = \sum_{J \supset I} (\tilde{M}_r)_{I,J^*}
\]

where each summand is the row of \( \tilde{M}_r \) indexed by \( J \). By the symmetry of indices, \( w_I = c \cdot v_I \) for some integer \( c \). Since \( \tilde{M}_r \) has no zero row, \( c \neq 0 \) and we deduce the claim. \( \square \)

Since zero rows and zero columns do not contribute to the matrix rank, the following corollary is an immediate consequence.

**Corollary 3.3.** The Hilbert function of \( S/(\sigma_{d,n})^\perp \) is given by

\[
\text{Hilb}(S/(\sigma_{d,n})^\perp, r) = \begin{cases} 
\binom{n}{r} & \text{if } r \leq \lfloor d/2 \rfloor \\
\binom{n}{d-r} & \text{if } r > \lfloor d/2 \rfloor
\end{cases}
\]

It gives a lower bound \( \text{rank}(\sigma_{d,n}) \geq \max\{\text{Hilb}(S/(\sigma_{d,n})^\perp, r)\} = \binom{n}{\lfloor d/2 \rfloor} \) by apolarity lemma, but we can go further. For any nonzero linear form \( L \in S \), the number \( \dim \mathbb{C} S/((F^\perp : L) + (L)) \) gives a lower bound for the rank of \( F \) [3, Theorem 3.3].
Take $L = \frac{\partial}{\partial x_n}$. Then it is easy to see that
\[
((\sigma_{d,n})^\perp : L) + L = S \cdot (\sigma_{d-1,n-1})^\perp + L.
\]
Here, $(\sigma_{d-1,n-1})^\perp$ can be computed in either $S$ or $S' = \mathbb{C}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}] \subset S$. Since $S/L = S'$, we conclude
\[
\text{rank}(\sigma_{d,n}) \geq \dim \mathbb{C} S'/((\sigma_{d-1,n-1})^\perp).
\]

**Theorem 3.4.** For $d$ odd, we have
\[
\text{rank}(\sigma_{d,n}) = \sum_{r=0}^{(d-1)/2} \binom{n}{r}.
\]

**Proof.** We have a lower bound
\[
\text{rank}(\sigma_{d,n}) \geq \dim \mathbb{C} S'/((\sigma_{d-1,n-1})^\perp) = \sum_{r} \text{Hilb}(S'/((\sigma_{d-1,n-1})^\perp), r).
\]
By Corollary 3.3, we get
\[
\sum_{r} \text{Hilb}(S'/((\sigma_{d-1,n-1})^\perp), r) = \sum_{r=0}^{k} \binom{n-1}{r} + \sum_{r=k+1}^{n-1} \binom{n-1}{d-r}
\]
\[
= \binom{n-1}{0} + \sum_{i=r}^{k} \left( \binom{n-1}{r} + \binom{n-1}{r-1} \right) = 1 + \sum_{r=1}^{k} \binom{n}{r}
\]
and it is coincide to the upper bound given in Corollary 2.2. \qed

**Remark 3.5.** One may ask for a power sum decomposition of a form over real field. The smallest number of required real linear forms is called the real rank of given form. Since all the coefficients in (2.2) are real, in fact rational, Theorem 3.4 also holds for the real rank.

### 4 Even degree

For even $n = d = 2k$, The equation (1.1) can be written in symmetric format as
\[
2^{n-1}n! \cdot \sigma_{n,n} = \sum_{I \subset [n], |I| < k} (-1)^{|I|}(\delta(I,1)x_1 + \delta(I,2)x_2 + \cdots + \delta(I,n)x_n)^n
\]
\[
+ \sum_{I \subset [n], |I| = k} (-1)^k \frac{2^k}{2} (\delta(I,1)x_1 + \delta(I,2)x_2 + \cdots + \delta(I,n)x_n)^n.
\]
However it is hard to generalize this expression to elementary symmetric polynomials directly, while it is in fact possible (Remark 4.2). Instead, we can easily get a power sum decomposition using Theorem 2.1.
Theorem 4.1. Let \( d = 2k \) be even, and \( n > d \). Then the elementary symmetric polynomial \( \sigma_{d,n} \) admits the power sum decomposition
\[
2^d(n - d)d! \cdot \sigma_{d,n} = \sum_{I \subset [n], |I| \leq k} (-1)^{|I|} \binom{n - k - |I| - 1}{k - |I|} (n - 2|I|)(\delta(I, 1)x_1 + \delta(I, 2)x_2 + \cdots + \delta(I, n)x_n)^d \tag{4.1}
\]
where \( \delta(I, i) = -1 \) if \( i \in I \), or 1 otherwise.

Proof. Note that
\[
\left(\frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}\right) \sigma_{d+1,n} = (n - d)\sigma_{d,n}.
\]
Applying (2.2) to the left side, we get the required decomposition. \( \square \)

Remark 4.2. After knowing the decomposition (4.1), Theorem 4.1 can be proven inductively as the proof of Theorem 2.1.

Remark 4.3. If we take \( x_i = 1 \) for all \( i \) in (4.1) then we get (2.3).

Since the number of summands are not equal to the lower bound given by Corollary 3.3 unless \( n = d \), we do not know whether the expression (4.1) is minimal.

Corollary 4.4. For \( d \) even, the rank of \( \sigma_{d,n} \) is bounded by
\[
\left(\sum_{r=0}^{d/2} \binom{n}{r}\right) - \binom{n - 1}{d/2} \leq \text{rank}(\sigma_{d,n}) \leq \sum_{r=0}^{d/2} \binom{n}{r}.
\]

We close this section by giving a difficulty of an improvement of Corollary 4.4. If \( d = 4 \) and \( n = 5 \), for instance, then the bound is \( 10 \leq \text{rank}(\sigma_{4,5}) \leq 16 \). It is easy to see that \( \text{rank}(\sigma_{4,5}) \geq 11 \) by observing \( \text{Hilb}(S/(\sigma_{4,5})^\perp, 2) = 10 \) and \( x_1^2, \ldots, x_5 \in (\sigma_{4,5})^\perp \). Both lower bounds given in [8] and [10] still give 11. The following proposition suggests that it will be tough to find a decomposition with less than 15 pure powers, even if there exists. It is checked by brute force using Macaulay2 [5].

Proposition 4.5. Let \( I \) be the defining ideal of 15 points among 16 points of the form \((\pm 1, \pm 1, \pm 1, \pm 1, \pm 1) \in \mathbb{P}^4\). Then \( I \) is not contained in \((\sigma_{4,5})^\perp\).

It means that \( \sigma_{4,5} \) can not be written as a linear combination of 15 (or less) polynomials of the form \((\pm x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5)^4\).

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