A class of higher order Painlevé systems arising from integrable hierarchies of type $A$

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Abstract
A relationship between Painlevé systems and infinite-dimensional integrable hierarchies is studied. We derive a class of higher order Painlevé systems from Drinfeld-Sokolov (DS) hierarchies of type $A$ by similarity reductions. This result allows us to understand some properties of Painlevé systems, Hamiltonian representations, affine Weyl group symmetries and Lax forms.

Key Words and Phrases: Affine Lie algebra, Integrable systems, Painlevé equations.
2000 Mathematics Subject Classification: 34M55, 17B80, 37K10.

1 Introduction
The connection between the second Painlevé equation and the KdV equation was clarified by Ablowitz and Segur [2]. Since their result, a relationship between (higher order) Painlevé systems and infinite-dimensional integrable hierarchies has been studied. In a recent work [7], a class of fourth order Painlevé systems was derived from the DS hierarchies of type $A$ by similarity reductions. In this article, we give its development, namely, we derive a class of higher order Painlevé systems.

The DS hierarchies are extensions of the KdV hierarchy for the affine Lie algebras [5, 8]. They are characterized by the Heisenberg subalgebras of the affine Lie algebras. And the isomorphism classes of the Heisenberg subalgebras are in one-to-one correspondence with the conjugacy classes of the finite Weyl group [14]. Thus we can classify the DS hierarchies of type $A_n^{(1)}$ in terms of the partitions of the natural number $n + 1$. By means of
this viewpoint, we list the known connections between Painlevé systems and integrable hierarchies of type $A$ in Table 1 and 2.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Partition & (2) & (1, 1) & (3) & (2, 1) & (1, 1, 1) & (4) & (2, 2) \\
\hline
Painlevé eq. & $P_{II}$ & $P_{IV}$ & $P_{V}$ & $P_{VI}$ & $P_{V}$ & $P_{VI}$ & $P_{VI}$ \\
\hline
Ref. & [2] & [11] & [1] & [10] & [12] & [1] & [7] \\
\hline
\end{tabular}
\caption{Painlevé equations and DS hierarchy}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Partition & (3, 1) & (4, 1) & (2, 2, 1) & (3, 3) & $(n + 1)$ for $n \geq 4$ \\
\hline
Painlevé sys. & $P(A_4)$ & $P(A_5)$ & $P(A_5^*)$ & $P(A_5^*)$ & $P(A_n)$ \\
\hline
Order of sys. & 4 & 4 & 4 & 4 & $n$ for $n$ even \hline
& $n - 1$ for $n$ odd \\
\hline
Ref. & [7] & [7] & [7] & [7] & [1, 15] \\
\hline
\end{tabular}
\caption{Higher order Painlevé systems and DS hierarchy}
\end{table}

Here the symbol $P(A_n)$ stands for the higher order Painlevé system of type $A_n^{(1)}$ [15], or equivalently, the $(n + 1)$-periodic Darboux chain [1]. The symbol $P(A_5^*)$ stands for the fourth order Painlevé system with the coupled sixth Painlevé Hamiltonian [7]; we describe its explicit formula below.

In this article, we consider a higher order generalization of the above facts. The obtained results are listed in Table 3.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
Partition & $(2n - 1, 1)$ & $(2n, 1)$ & $(n, n, 1)$ & $(n + 1, n + 1)$ \\
\hline
Painlevé sys. & $P(A_{2n})$ & $P(A_{2n+1})$ & $P(A_{2n+1}^*)$ & $P(A_{2n+1}^*)$ \\
\hline
Order of sys. & $2n$ & $2n$ & $2n$ & $2n$ \\
\hline
Ref. & Sec4 & Sec5 & Sec6 & Sec3 \\
\hline
\end{tabular}
\caption{The result obtained in this article}
\end{table}

The Painlevé system $P(A_{2n+1}^*)$ is a Hamiltonian system
\[
\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (i = 1, \ldots, n),
\]
with a coupled Painlevé VI Hamiltonian
\[
t(t - 1)H = \sum_{i=1}^{n} H_{VI} \left[ \sum_{j=0}^{n} \alpha_{2j+1} - \alpha_{2i-1} - \eta_{i} \sum_{j=0}^{i-1} \alpha_{2j} \sum_{j=i}^{n} \alpha_{2j} \alpha_{2i-1} \eta ; q, p_i \right] \\
+ \sum_{1 \leq i < j \leq n} (q_i - 1)(q_j - t) \{(q_i p_i + \alpha_{2i-1})p_j + p_i (p_j q_j + \alpha_{2j-1})\},
\]
where
\[
H_{VI}[\kappa_0, \kappa_1, \kappa_t, \kappa; q, p] = q(q - 1)(q - t)p^2 - \kappa_0(q - 1)(q - t)p \\
- \kappa_1(q - t)p - (\kappa_t - 1)q(q - 1)p + \kappa q.
\]
Here the parameters $\alpha_0, \ldots, \alpha_{2n+1}$ satisfy the relation $\sum_{i=0}^{2n+1} \alpha_i = 1$. This system admits the affine Weyl group symmetry of type $A^{(1)}_{2n+1}$ and has a Lax form associated with the loop algebra $\mathfrak{sl}_{2n+2}[z, z^{-1}]$; we discuss their details in Section 3. Note that $P(A^*_n)$ is equivalent to the sixth Painlevé equation.

**Remark 1.1.** The regular conjugacy classes of $W(A_n)$ correspond to the partitions $(p, \ldots, p)$ and $(p, \ldots, p, 1)$; cf. [4, 6]. Therefore any hierarchy in Table 1, 2 and 3 is associated with the regular conjugacy class of $W(A_n)$.

**Remark 1.2.** The DS hierarchy for the partition $(n + 1, n + 1)$ is equivalent to the $(n + 1, n + 1)$-periodic reduction of the two-component KP hierarchy; cf. [3, 20].

**Remark 1.3.** The higher order Painlevé system of type $D^{(1)}_{2n+2}$ was proposed by Sasano [18]. It is expressed as a Hamiltonian system of $2n$-th order with a coupled sixth Painlevé Hamiltonian, as well as $P(A^*_{2n+1})$. The relationship between those two systems is not clarified.

**Remark 1.4.** Recently, the system $P(A^*_{2n+1})$ is derived independently by Tsuda [19] via a similarity reduction of the UC hierarchy. According to it, $P(A^*_{2n+1})$ is given as the monodromy preserving deformation of a Fuchsian differential equation with a spectral type

$$(1^{n+1}), \ (1^{n+1}), \ (n, 1), \ (n, 1).$$

Furthermore, $P(A^*_5)$ appears in the classification of the fourth order isomonodromy equations [17].

This article is organized as follows. In Section 2 we first recall the affine Lie algebra of type $A^{(1)}_n$. We next formulate the DS hierarchies of type $A^{(1)}_n$ and their similarity reductions. In Section 3 we derive the Painlevé system $P(A^*_{2n+1})$ from the hierarchy for the partition $(n + 1, n + 1)$. We also discuss a group of symmetries and a Lax form for $P(A^*_{2n+1})$. In Section 4, 5 and 6 we derive $P(A_{2n})$, $P(A_{2n+1})$ and $P(A^*_{2n+1})$ from the hierarchy for a partition $(2n - 1, 1), (2n, 1)$ and $(n, n, 1)$, respectively.

## 2 DS hierarchy

In this section, we first recall the affine Lie algebra of type $A^{(1)}_n$, following the notation in [9, 7]. We next formulate the DS hierarchies of type $A^{(1)}_n$ and their similarity reductions. The Lax forms of the similarity reductions are also proposed in a framework of the loop algebra $\mathfrak{sl}_{n+1}[z, z^{-1}]$. 
2.1 Affine Lie algebra

The affine Lie algebra \( \hat{\mathfrak{g}} = \mathfrak{g}(A_1^n) \) is a Kac-Moody Lie algebra whose generalized Cartan matrix \( A = [a_{ij}]_{i,j=0}^n \) is defined by

\[
\begin{align*}
  a_{i,i} &= 2 \quad (i = 0, \ldots, n), \\
  a_{i,i+1} &= a_{i,0} = a_{i+1,i} = a_{0,n} = -1 \quad (i = 0, \ldots, n-1), \\
  a_{i,j} &= 0 \quad (\text{otherwise}).
\end{align*}
\]

It is generated by the Chevalley generators \( e_i, f_i, \alpha_i^\vee (i = 0, \ldots, n) \) and the scaling element \( d \) with the fundamental relations

\[
\begin{align*}
  [\alpha_i^\vee, \alpha_j^\vee] &= 0, \quad [\alpha_i^\vee, e_j] = a_{i,j}e_j, \quad [\alpha_i^\vee, f_j] = -a_{i,j}f_j, \quad [e_i, f_j] = \delta_{i,j}\alpha_i^\vee, \\
  [d, \alpha_i^\vee] &= 0, \quad [d, e_i] = \delta_{i,0}e_i, \quad [d, f_i] = -\delta_{i,0}f_i, \\
  (\text{ad} e_i)^{1-a_{i,j}}(e_j) &= 0, \quad (\text{ad} f_i)^{1-a_{i,j}}(f_j) = 0 \quad (i \neq j),
\end{align*}
\]

for \( i, j = 0, \ldots, n \). The canonical central element of \( \hat{\mathfrak{g}} \) is given by

\[
K = \alpha_0^\vee + \alpha_1^\vee + \ldots + \alpha_n^\vee.
\]

The normalized invariant form is given by the conditions

\[
\begin{align*}
  (\alpha_i^\vee | \alpha_j^\vee) &= a_{i,j}, \quad (e_i | f_j) = \delta_{i,j}, \quad (\alpha_i^\vee | e_j) = (\alpha_i^\vee | f_j) = 0, \\
  (d | d) &= 0, \quad (d | \alpha_j^\vee) = \delta_{0,j}, \quad (d | e_j) = (d | f_j) = 0,
\end{align*}
\]

for \( i, j = 0, \ldots, n \). We set

\[
e_{i,j} = \text{ad} e_i \text{ad} e_{i+1} \ldots \text{ad} e_{i+j-1}(e_{i+j}), \quad f_{i,j} = \text{ad} f_i \text{ad} f_{i+j-1} \ldots \text{ad} f_{i+1}(f_i),
\]

where \( e_{i+n+1} = e_i \) and \( f_{i+n+1} = f_i \).

The Cartan subalgebra of \( \hat{\mathfrak{g}} \) is defined by

\[
\mathfrak{h} = \mathbb{C} \alpha_0^\vee \oplus \mathbb{C} \alpha_1^\vee \oplus \ldots \oplus \mathbb{C} \alpha_n^\vee \oplus \mathbb{C} d.
\]

Let \( \mathfrak{n}_+ \) and \( \mathfrak{n}_- \) be the subalgebras of \( \hat{\mathfrak{g}} \) generated by \( e_i \) and \( f_i \) \( (i = 0, \ldots, n) \), respectively. Then the Borel subalgebra \( \mathfrak{b}_+ \) of \( \hat{\mathfrak{g}} \) is given by \( \mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+ \).

Note that we have the triangular decomposition

\[
\hat{\mathfrak{g}} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{n}_- \oplus \mathfrak{b}_+.
\]

The isomorphism classes of the Heisenberg subalgebras of \( \hat{\mathfrak{g}} \) are in one-to-one correspondence with partitions of the natural number \( n + 1 \). Let
\( n = (n_1, \ldots, n_k) \) be a partition of \( n + 1 \). Then the corresponding Heisenberg subalgebra is defined by

\[
\mathfrak{s}_n = \mathcal{P}_{n_1-1} \oplus \ldots \oplus \mathcal{P}_{n_k-1} \oplus \mathcal{H}_{k-1} \oplus \mathbb{C}K,
\]

where \( \mathcal{P}_n \oplus \mathbb{C}K \) and \( \mathcal{H}_n \oplus \mathbb{C}K \) are isomorphic to the principal and homogeneous Heisenberg subalgebra of \( \mathfrak{g}(A_n^{(1)}) \), respectively [4].

The partition \( n \) determines a grading operator \( \vartheta_n \in \mathfrak{h} \), whose explicit formula is not given here (see Section 3 of [7]). The operator \( \vartheta_n \) defines a \( \mathbb{Z} \)-gradation of type \( s \) by

\[
\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k(s), \quad \mathfrak{g}_k(s) = \{ x \in \mathfrak{g} \mid [\vartheta_n, x] = kx \},
\]

where \( s = (s_0, \ldots, s_n) \) is a vector of non-negative integers given by

\[
(\vartheta_n|\alpha_i) = s_i \quad (i = 0, \ldots, n).
\]

Note that

\[
[\vartheta_n, e_i] = s_i e_i, \quad [\vartheta_n, f_i] = -s_i f_i \quad (i = 0, \ldots, n).
\]

The Heisenberg subalgebra \( \mathfrak{s}_n \) admits the gradation defined by \( \vartheta_n \).

### 2.2 DS hierarchy and similarity reduction

The positive part of the Heisenberg subalgebra \( \mathfrak{s}_n \) has a graded basis \( \{\Lambda_k\}_{k \in \mathbb{N}} \) satisfying

\[
[\Lambda_k, \Lambda_l] = 0, \quad [\vartheta_n, \Lambda_k] = d_k \Lambda_k \quad (k, l \in \mathbb{N}),
\]

where \( d_k \) is a positive integer. We assume that \( d_k \leq d_{k+1} \) for any \( k \in \mathbb{N} \). In this subsection, we formulate the DS hierarchy associated with \( \mathfrak{s}_n \) by using those \( \Lambda_k \).

Introducing time variables \( t_k \ (k \in \mathbb{N}) \), we consider the Sato equation for \( n_- \)-valued function \( W = W(t_1, t_2, \ldots) \)

\[
\partial_k - B_k = \exp(\text{ad}W)(\partial_k - \Lambda_k) \quad (k \in \mathbb{N}), \quad (2.1)
\]

where \( \partial_k = \partial/\partial t_k \) and \( B_k \) stands for the \( b_+ \)-component of \( \exp(\text{ad}W)(\Lambda_k) \).

The compatibility condition of (2.1) gives the DS hierarchy

\[
[\partial_k - B_k, \partial_l - B_l] = 0 \quad (k, l \in \mathbb{N}). \quad (2.2)
\]
We now require a similarity condition
\[
\vartheta_n - \rho - \sum_{k=1}^{\infty} d_k t_k \partial_k = \exp(\text{ad}W) \left( \vartheta_n - \rho - \sum_{k=1}^{\infty} d_k t_k \partial_k \right),
\]
with an element \( \rho \in \mathfrak{h} \) satisfying
\[
[\partial_k, \rho] = 0, \quad [\Lambda_k, \rho] = 0 \quad (k \in \mathbb{N}).
\]
Then the compatibility condition of (2.1) and (2.3) gives
\[
[\vartheta_n - M, \partial_k - B_k] = 0 \quad (k \in \mathbb{N}),
\]
where
\[
M = \rho + \sum_{k=1}^{\infty} d_k t_k B_k.
\]
We call the system (2.2) and (2.4) a similarity reduction of the DS hierarchy.

Note that \( M \) is the \( b_+ \)-component of \( \exp(\text{ad}W)(\rho + \sum_{k=1}^{\infty} d_k t_k \Lambda_k) \).

In the following section, we always assume that \( t_2 = 1 \) and \( t_k = 0 \) for any \( k \geq 3 \). Under this specialization, the similarity reduction is described as a system of ordinary differential equations
\[
[\vartheta_n - M, \partial_1 - B_1] = 0, \quad M = \rho + d_1 t_1 B_1 + d_2 B_2,
\]
from which the Painlevé system is derived.

2.3 Lax form

The affine Lie algebra \( \hat{\mathfrak{g}} \) can be identified with the loop algebra \( \mathfrak{sl}_{n+1}[z, z^{-1}] \) under the specialization \( K = 0 \). In this subsection, we propose a Lax form of the similarity reduction in a framework of \( \mathfrak{sl}_{n+1}[z, z^{-1}] \).

Let \( E_{i,j} \) be a \((n+1) \times (n+1)\) matrix with 1 on the \((i, j)\)-th entry and zeros elsewhere. For each partition \( \mathbf{n} \), we define the graded Chevalley generators of \( \mathfrak{sl}_{n+1}[z, z^{-1}] \) by
\[
e_0 = z^{s_0}E_{n+1,1}, \quad f_0 = z^{-s_0}E_{1,n+1}, \quad \alpha_0^\vee = E_{n+1,n+1} - E_{1,1},
\]
\[
e_i = z^{s_i}E_{i,i+1}, \quad f_i = z^{-s_i}E_{i+1,i}, \quad \alpha_i^\vee = E_{i,i} - E_{i+1,i+1} \quad (i = 1, \ldots, n),
\]
and the grading operator by \( \vartheta_\mathbf{n} = zd/dz \). Recall that \( \vartheta_\mathbf{n} \) implies the \( \mathbb{Z} \)-gradation of type \( \mathbf{s} = (s_0, \ldots, s_n) \). The Lie bracket is given by
\[
[z^k X, z^l Y] = z^{k+l}(XY - YX) \quad (k, l \in \mathbb{Z}; X, Y \in \mathfrak{sl}_{n+1}).
\]
In a framework of $\mathfrak{sl}_{n+1}[z, z^{-1}]$, the Sato equation (2.1) and the similarity condition (2.3) are described as
\[
\frac{\partial}{\partial t_k} \exp W = B_k \exp W - \exp W \Lambda_k \quad (k \in \mathbb{N}),
\]
\[
z \frac{d}{dz} \exp W = M \exp W - \exp W \left( \rho + \sum_{k=1}^{\infty} d_k t_k \Lambda_k \right).
\]
Under them, we consider a wave function
\[
\Psi = \exp W z^\rho \exp \left( \sum_{k=1}^{\infty} t_k \Lambda_k \right).
\]
Then we obtain a system of linear differential equations
\[
\frac{\partial \Psi}{\partial t_k} = B_k \Psi \quad (k \in \mathbb{N}), \quad z \frac{d \Psi}{dz} = M \Psi. \tag{2.6}
\]
The system (2.6) is the Lax form of the similarity reduction (2.2) and (2.4). In fact, the compatibility condition of (2.6) gives
\[
\frac{\partial B_k}{\partial t_l} - \frac{\partial B_l}{\partial t_k} = [B_l, B_k], \quad \frac{\partial M}{\partial t_k} - z \frac{dB_k}{dz} = [B_k, M] \quad (k, l \in \mathbb{N}).
\]

3 For the partition $(n + 1, n + 1)$

In this section, we derive the Painlevé system $P(A^{*}_{2n+1})$ from the similarity reduction (2.5) for the partition $(n + 1, n + 1)$. We also discuss a group of symmetries and a Lax form for $P(A^{*}_{2n+1})$. Indices of the Chevalley generators and variables are congruent modulo $2n + 2$ in this section.

3.1 Similarity reduction of the DS hierarchy

At first, we give an explicit formula of the Heisenberg subalgebra $\mathfrak{s}_{(n+1, n+1)}$ of $\hat{\mathfrak{g}} = g(A^{(1)}_{2n+1})$ following [8, 7, 13]. Let
\[
\Lambda_{2k-1} = \sum_{i=0}^{n} e_{2i+1, 2k-1}, \quad \Lambda_{2k} = \sum_{i=0}^{n} e_{2i+2, 2k-1},
\]
\[
\bar{\Lambda}_{2k-1} = \sum_{i=0}^{n} f_{2i+1, 2k-1}, \quad \bar{\Lambda}_{2k} = \sum_{i=0}^{n} f_{2i+2, 2k-1}.
\]
for \( k \in \mathbb{N} \). Then \( s_{(n+1,n+1)} \) is expressed as

\[
s_{(n+1,n+1)} = \bigoplus_{k \in \mathbb{N} \setminus (2n+2)\mathbb{N}} C\Lambda_k \oplus CK \oplus \bigoplus_{k \in \mathbb{N} \setminus (2n+2)\mathbb{N}} C\Lambda_k.
\]

The grading operator \( \vartheta_{(n+1,n+1)} \) is given by

\[
\vartheta_{(n+1,n+1)} = (n+1)d + \sum_{i=0}^{n} i(n-i+1)\alpha_{2i}^\vee + \sum_{i=0}^{n} \left(\frac{(2i+1)n - 2i^2}{2}\right)\alpha_{2i+1}^\vee.
\]

It implies a \( \mathbb{Z} \)-gradation of type \((1,0,\ldots,1,0)\), namely

\[
(\vartheta_{(n+1,n+1)}|\alpha_{2i}^\vee) = 1, \quad (\vartheta_{(n+1,n+1)}|\alpha_{2i+1}^\vee) = 0,
\]

for \( i = 0, \ldots, n \). Note that

\[
[\vartheta_{(n+1,n+1)}, \Lambda_{2k-1}] = k\Lambda_{2k-1}, \quad [\vartheta_{(n+1,n+1)}, \Lambda_{2k}] = k\Lambda_{2k} \quad (k \in \mathbb{N}).
\]

The similarity reduction (2.5) associated with \( s_{(n+1,n+1)} \) is described as

\[
[\vartheta_{(n+1,n+1)} - M, \partial_1 - B_1] = 0,
\]

where

\[
B_1 = \sum_{i=0}^{2n+1} u_i \alpha_i^\vee + \sum_{i=0}^{2n+1} x_i e_i + \Lambda_1, \quad M = \sum_{i=0}^{2n+1} \kappa_i \alpha_i^\vee + \sum_{i=0}^{2n+1} \varphi_i e_i + t_1 \Lambda_1 + \Lambda_2.
\]

Note that

\[
\Lambda_1 = \sum_{i=0}^{n} e_{2i+1,1}, \quad \Lambda_2 = \sum_{i=0}^{n} e_{2i+2,1}.
\]

In terms of those variables, the system (3.1) is expressed as

\[
\partial_1(\kappa_i) = 0, \quad \partial_1(\varphi_i) = (u|\alpha_i^\vee)\varphi_i + x_i(\kappa|\alpha_i^\vee),
\]

for \( i = 0, \ldots, 2n + 1 \) and

\[
(u|\alpha_{2i}^\vee + \alpha_{2i+1}^\vee) - x_{2i+1}\varphi_{2i} + x_{2i}\varphi_{2i+1} = 0, \quad t_1(u|\alpha_{2i+1}^\vee + \alpha_{2i+2}^\vee) - x_{2i+2}\varphi_{2i+1} + x_{2i+1}\varphi_{2i+2} + (\kappa|\alpha_{2i+1}^\vee + \alpha_{2i+2}^\vee) = 1, \quad (3.3)
\]

\[
t_1 x_{2i} - x_{2i+2} - \varphi_{2i} = 0, \quad x_{2i+1} - t_1 x_{2i+3} + \varphi_{2i+3} = 0,
\]

for \( i = 0, \ldots, n \), where

\[
u = \sum_{i=0}^{2n+1} u_i \alpha_i^\vee, \quad \kappa = \vartheta_{(n+1,n+1)} - \sum_{i=0}^{2n+1} \kappa_i \alpha_i^\vee.
\]

In the next subsection, we express the system (3.2) with (3.3) as a Hamiltonian system.
3.2 Hamiltonian system

The operators $B_1$ and $M$ are defined as the $\mathfrak{g}_+$-components of $\exp(\text{ad}W)(\Lambda_1)$ and $\exp(\text{ad}W)(\rho + t_1\Lambda_1 + \Lambda_2)$, respectively. The $\mathfrak{g}$-valued operator $\rho$ is given by

$$\rho = \rho_1 \sum_{i=0}^{n} \alpha_{2i+1},$$

where $\rho_1$ is independent of $t_1$. By using this fact, we derive a Hamiltonian system.

Let

$$W = - \sum_{i=0}^{2n+1} w_i f_i - \sum_{k=1}^{\infty} \sum_{i=0}^{2n+1} w_{i,k} f_{i,k}. $$

Then we obtain

$$u_{2i} = - \frac{1}{2} t_{1} w_{2i-1} w_{2i} + w_{2i-1,1}, \quad u_{2i+1} = \frac{1}{2} t_{1} w_{2i+1} w_{2i+2} + w_{2i+1,1}, \quad (3.4)$$

and

$$x_{2i} = - w_{2i-1}, \quad x_{2i+1} = w_{2i+2},$$

and

$$\kappa_{2i} = - \frac{1}{2} t_{1} w_{2i-1} w_{2i} + t_{1} w_{2i-1,1} + \frac{1}{2} t_{1} w_{2i} w_{2i+1} + w_{2i,1},$$

$$\kappa_{2i+1} = - \frac{1}{2} t_{1} w_{2i+1} w_{2i+1} + w_{2i,1} + \frac{1}{2} t_{1} w_{2i+1} w_{2i+2} + t_{1} w_{2i+1,1} + \rho_1, \quad (3.5)$$

$$\varphi_{2i} = - t_{1} w_{2i-1} w_{2i+1}, \quad \varphi_{2i+1} = - w_{2i} + t_{1} w_{2i+2},$$

for $i = 0, \ldots, n$. The equations (3.4) and (3.5) imply

Lemma 3.1. The $\mathfrak{g}_+$-valued functions $B_1$ and $M$ can be expressed in terms of the dependent variables $w_{2i+1}, \varphi_{2i+1} (i = 0, \ldots, n)$ as

$$u_{2i} - u_{2i+1} = - \sum_{j=0}^{n} \frac{t_{1}^{j+1}}{t_{1}^{j+1} - 1} w_{2i+1} \varphi_{2i+2j+1} + \frac{1}{t_{1}} (\rho_1 + \kappa_{2i} - \kappa_{2i+1}),$$

$$u_{2i+1} - u_{2i+2} = \sum_{j=0}^{n} \frac{t_{1}^{j}}{t_{1}^{j+1} - 1} w_{2i+1} \varphi_{2i+2j+3},$$

$$x_{2i} = - w_{2i-1}, \quad x_{2i+1} = \sum_{j=0}^{n} \frac{t_{1}^{j}}{t_{1}^{j+1} - 1} \varphi_{2i+2j+3},$$

$$\varphi_{2i} = - t_{1} w_{2i-1} + w_{2i+1},$$

for $i = 1, \ldots, n + 1$. Furthermore, those variables satisfy

$$\sum_{i=0}^{n} w_{2i+1} \varphi_{2i+1} = - \sum_{i=0}^{n} (\rho_1 + \kappa_{2i} - \kappa_{2i+1}).$$
Thanks to Lemma 3.1, we can express the system (3.2) with (3.3) as a system of ordinary differential equations in terms of the variables $w_{2i+1}, \varphi_{2i+1}$ ($i = 0, \ldots, n$). Note that those variables are taken from the $g_0(1, 0, \ldots, 1, 0)$-components of $W$ and $M$. We also remark that

$$w_{2i+1} = -\sum_{j=0}^{n} \frac{t_i^j}{t_i^{n+1} - 1} \varphi_{2i-2j} \quad (i = 0, \ldots, n).$$

Following [16], we define the Kostant-Kirillov structure for the operator $M$ by

$$\{\varphi_{2i}, \varphi_{2i+1}\} = -(n + 1), \quad \{\varphi_{2i+1}, \varphi_{2i+2}\} = -(n + 1)t_i \quad (i = 0, \ldots, n).$$

Then we arrive to

**Theorem 3.2.** In terms of the variables $w_{2i+1}, \varphi_{2i+1}$ ($i = 0, \ldots, n$) with the Poisson structure

$$\{\varphi_{2i+1}, w_{2j+1}\} = (n + 1)\delta_{i,j} \quad (i, j = 0, \ldots, n),$$

the similarity reduction (3.1) is expressed as the Hamiltonian system

$$\partial_1(w_{2i+1}) = \{H, w_{2i+1}\}, \quad \partial_1(\varphi_{2i+1}) = \{H, \varphi_{2i+1}\} \quad (i = 0, \ldots, n), \quad (3.6)$$

with the Hamiltonian

$$H = \sum_{i=0}^{n} \frac{n}{2(n + 1)^2 t_i} \left( w_{2i+1}\varphi_{2i+1} + 2\kappa_{2i} - 2\kappa_{2i+1} \right) w_{2i+1}\varphi_{2i+1}$$

$$- \sum_{i=0}^{n} \sum_{j=1}^{n} \frac{1}{(n + 1)^2 t_i} \left( w_{2i+2j+1}\varphi_{2i+2j+1} + \kappa_{2i+2j} - \kappa_{2i+2j+1} \right) w_{2i+1}\varphi_{2i+1}$$

$$- \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{t_i^j}{(n + 1)(t_i^{n+1} - 1)} \left( w_{2i+1}\varphi_{2i+1} + (\kappa |\varphi_{2i+1}) \right) w_{2i+1}\varphi_{2i+2j+3}, \quad (3.7)$$

and the relation

$$\sum_{i=0}^{n} w_{2i+1}\varphi_{2i+1} = - \sum_{i=0}^{n} (\rho_1 + \kappa_{2i} - \kappa_{2i+1}). \quad (3.8)$$
The system (3.6) with (3.7) and (3.8) can be rewritten into the Hamiltonian system in terms of the canonical coordinates. The equation (3.8) implies
\[ \sum_{i=0}^{n} d\varphi_{2i+1} \wedge dw_{2i+1} = \sum_{i=0}^{n-1} d\varphi_{2i+1} \wedge dw_{2i+1} = \sum_{i=0}^{n-1} d\frac{w_{2i+1}\varphi_{2i+1}}{w_{2n+1}} \wedge dw_{2n+1} \]
\[ = \sum_{i=0}^{n-1} d(w_{2n+1}\varphi_{2i+1}) \wedge d\frac{w_{2i+1}}{w_{2n+1}}. \]

Therefore we can take
\[ q_i = \frac{w_{2i-1}}{t_i^\alpha} \quad p_i = \frac{t_i^\alpha w_{2n+1}\varphi_{2i-1}}{n+1} \quad (i = 1, \ldots, n), \quad (3.9) \]
as canonical coordinates of a 2n-dimensional system with a Poisson structure
\[ \{p_i, q_j\} = \{p_i, w_{2n+1}\} = 0 \quad (i, j = 1, \ldots, n). \]

We denote the parameters by
\[ \alpha_i = \left(\frac{\kappa_1^{\alpha}}{n+1}\right) \quad (i = 0, \ldots, 2n+1), \quad \eta = \sum_{j=0}^{n} \frac{\rho_1 + \kappa_2 - \kappa_2 j + 1}{n+1}. \]

Via a transformation of the independent variable \( t = t_i^{-(n+1)} \), we obtain

**Corollary 3.3.** The variables \( q_i, p_i \ (i = 1, \ldots, n) \) defined by (3.9) satisfy the Painlevé system \( P(A_{2n+1}^*) \). Then the variable \( w_{2n+1} \) satisfies
\[ t(t-1) \frac{d}{dt} \log w_{2n+1} = -\sum_{i=1}^{n} \{(q_i - 1)(q_i - t)p_i + \alpha_{2i-1} q_i\} - \alpha_{2n+1} \]
\[ + \frac{nt + n + 2}{n+1} \eta + \sum_{i=0}^{n} \frac{n-2i}{2n+2} (\alpha_{2i-1} + \alpha_{2i})(t-1). \]

We remark that the parameter \( \eta \) satisfies
\[ \{\eta, q_i\} = \{\eta, p_i\} = 0 \quad (i = 1, \ldots, n), \quad \{\eta, w_{2n+1}\} = w_{2n+1}. \]

Thus the Poisson algebra generated by \( w_{2i+1}, \varphi_{2i+1} \ (i = 0, \ldots, n) \) is equivalent to one generated by \( q_i, p_i \ (i = 1, \ldots, n), \ w_{2n+1} \) and \( \eta \).
3.3 Affine Weyl group symmetry

The Painlevé system \( P(A_{2n+1}^*) \) admits the affine Weyl group symmetry of type \( A_{2n+1}^{(1)} \). In this section, we describe its action on the dependent variables and parameters.

Recall that the affine Weyl group of type \( A_{2n+1}^{(1)} \) is generated by the transformations \( r_i \) (\( i = 0, \ldots, 2n+1 \)) with the fundamental relations

\[
\begin{align*}
r_i^2 &= 1 \quad (i = 0, \ldots, 2n+1), \\
(r_ir_j)^{2-a_{ij}} &= 0 \quad (i, j = 0, \ldots, 2n+1; i \neq j).
\end{align*}
\]

where

\[
\begin{align*}
a_{i,i} &= 2 \quad (i = 0, \ldots, 2n+1), \\
a_{i,i+1} &= a_{2n+1,0} = a_{i+1,i} = a_{0,2n+1} = -1 \quad (i = 0, \ldots, 2n), \\
a_{i,j} &= 0 \quad (\text{otherwise}).
\end{align*}
\]

Under the Sato equation (2.1) and (2.3), we consider gauge transformations

\[
r_i(\exp W) = \exp(\gamma_if_i) \exp W \quad (i = 0, \ldots, 2n+1).
\]

They imply the transformations for the similarity reduction (3.1)

\[
r_i(\vartheta_{(n+1,n+1)} - M) = \exp(\text{ad} \gamma_if_i)(\vartheta_{(n+1,n+1)} - M) \quad (i = 0, \ldots, 2n+1).
\]

We look for the gauge parameters \( \gamma_0, \ldots, \gamma_{2n+1} \) such that \( r_i(\vartheta_{(n+1,n+1)} - M) \in \mathfrak{b}_+ \). Then we can show that

\[
\gamma_i = -\frac{\langle \kappa| \alpha_i^- \rangle}{\varphi_i} \quad (i = 0, \ldots, 2n+1).
\]

Such transformations give the group of symmetries for the Hamiltonian system.

**Theorem 3.4.** The Hamiltonian system (3.6) with (3.7) and (3.8) is invariant under the birational transformations defined by

\[
\begin{align*}
r_{2i}(w_{2j+1}) &= w_{2j+1}, \\
r_{2i}(\varphi_{2j+1}) &= \varphi_{2j+1} + \frac{\langle \kappa| \alpha_i^- \rangle}{(n+1)(t_1 w_{2i-1} - w_{2i+1})} \{t_1 w_{2i-1} - w_{2i+1}, \varphi_{2j+1}\}, \\
r_{2i+1}(w_{2j+1}) &= w_{2j+1} + \frac{\langle \kappa| \alpha_{i+1}^- \rangle}{(n+1)} \{\varphi_{2i+1}, w_{2j+1}\}, \\
r_{2i+1}(\varphi_{2j+1}) &= \varphi_{2j+1},
\end{align*}
\]

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for\ i,\ j = 0,\ldots,\ n \ and
\[
\begin{align*}
  r_i(\kappa_j) &= \kappa_j + (\kappa|\alpha_i^\vee) \quad (i, j = 0, \ldots, 2n + 1). \tag{3.11}
\end{align*}
\]

Furthermore, a group of symmetries \( \langle r_0, \ldots, r_{2n+1} \rangle \) is isomorphic to the affine Weyl group of type \( A_{2n+1}^{(1)} \).

**Corollary 3.5.** The Painlevé system \( P(A_{2n+1}^*) \) is invariant under the birational canonical transformations \( r_0, \ldots, r_{2n+1} \) defined by
\[
\begin{align*}
  r_0(q_j) &= q_j, \quad r_0(p_j) = p_j - \frac{\alpha_0}{q_1 - 1}\{p_j, q_1\}, \\
  r_{2i-1}(q_j) &= q_j + \frac{\alpha_{2i-1}}{p_i}\{p_i, q_j\}, \quad r_{2i-1}(p_j) = p_j \quad (i = 1, \ldots, n), \\
  r_{2i}(q_j) &= q_j, \quad r_{2i}(p_j) = p_j - \frac{\alpha_{2i}}{q_i - q_{i+1}}\{p_j, q_i - q_{i+1}\} \quad (i = 1, \ldots, n - 1), \\
  r_{2n}(q_j) &= q_j, \quad r_{2n}(p_j) = p_j - \frac{\alpha_{2n}}{q_n - t}\{p_j, q_n\}, \\
  r_{2n+1}(q_j) &= q_j + \sum_{j=1}^{2n+1} q_j p_j + \eta, \quad r_{2n+1}(p_j) = p_j - \frac{\alpha_{2n+1} p_j}{\sum_{j=1}^{2n+1} q_j p_j + \eta},
\end{align*}
\]

for \( j = 1, \ldots, n \) and
\[
\begin{align*}
  r_i(\alpha_j) &= \alpha_j - a_{i,j}\alpha_i, \quad r_i(\eta) = \eta + (-1)^i\alpha_i \quad (i, j = 0, \ldots, 2n + 1).
\end{align*}
\]

## 3.4 Lax form

In the previous subsection, the Painlevé system \( P(A_{2n+1}^*) \) has been derived. In this subsection, we give its Lax form in a framework of the loop algebra \( \mathfrak{sl}_{2n+2}[z, z^{-1}] \).

Under the specialization \( t_1 = t^{-1/(n+1)} \), \( t_2 = 1 \) and \( t_k = 0 \ (k \geq 3) \), the Lax form \( (2.10) \) is described as
\[
\begin{align*}
  t(t - 1)\frac{d\Psi}{dt} &= B\Psi, \\
  z\frac{d\Psi}{dz} &= M\Psi. \tag{3.12}
\end{align*}
\]

The matrix \( B \) is given by
\[
\begin{align*}
  B &= \sum_{i=1}^{2n+2} u'_i E_{i,i} + \sum_{i=1}^{n+1} x'_{2i-1} E_{2i-1,2i} + x'_{0} z E_{2n+2,1} + \sum_{i=1}^{n} x'_{2i} z E_{2i,2i+1} \\
  &\quad - \sum_{i=1}^{n} \frac{t - 1}{(n+1)t^{1/(n+1)}} z E_{2i-1,2i+1} - \frac{t - 1}{(n+1)t^{1/(n+1)}} z E_{2n+1,1}.
\end{align*}
\]
where

\[ u'_{2i-1} = -\sum_{j=1}^{i-1} q_j (q_j - 1)p_j - \sum_{j=i}^{n} q_i (q_j - t)p_j - \eta q_i \]

\[ + \frac{t - 1}{n + 1} \eta + \sum_{j=0}^{2n+2} \frac{n - 2j}{2n+2} (\alpha_{2i+2j-3} + \alpha_{2i+2j-2})(t - 1), \]

\[ u'_{2i} = \sum_{j=1}^{i} q_j (q_j - 1)p_j + \sum_{j=i+1}^{n} q_i (q_j - t)p_j + \eta q_i, \]

\[ u'_{2n+1} = -\sum_{j=1}^{n} t(q_j - 1)p_j - \frac{nt + 1}{n + 1} \eta - \sum_{j=0}^{2n+2} \frac{n - 2j}{2n+2} (\alpha_{2n+2j-1} + \alpha_{2n+2j})(t - 1), \]

\[ u'_{2n+2} = \sum_{j=1}^{n} (q_j - t)p_j + \eta, \]

and

\[ x'_{2i-1} = -\frac{i^{i/(n+1)}}{w'_{2n+1}} \left\{ \sum_{j=1}^{i} (q_j - 1)p_j + \sum_{j=i+1}^{n} (q_j - t)p_j + \eta \right\}, \]

\[ x'_{2n+1} = -\frac{1}{w'_{2n+1}} \left\{ \sum_{j=1}^{n} (q_j - t)p_j + \eta \right\}, \]

\[ x'_{0} = \frac{(t - 1)w_{2n+1}}{(n + 1)t^{(i+1)/(n+1)}}, \]

\[ x'_{2i} = \frac{(t - 1)w_{2n+1}}{(n + 1)t^{(i+1)/(n+1)}} q_i, \]

for \( i = 1, \ldots, n \). The matrix \( M \) is given by

\[ M = \sum_{i=1}^{2n+2} (\kappa_i - \kappa_{i-1})E_{i,i} + \sum_{i=1}^{n+1} \varphi_{2i-1} E_{2i-1,2i} + \varphi_0 z E_{2n+2,1} + \sum_{i=1}^{n} \varphi_{2i} z E_{2i,2i+1} \]

\[ + \sum_{i=1}^{n} \frac{1}{t^{i/(n+1)}} z E_{2i-1,2i+1} + \frac{1}{t^{i/(n+1)}} z E_{2n+1,1} + \sum_{i=1}^{n} z E_{2i,2i+2} + z E_{2n+2,2}, \]

where

\[ \varphi_{2i-1} = \frac{(n + 1)t^{i/(n+1)}}{w'_{2n+1}} p_i \quad (i = 1, \ldots, n), \]

\[ \varphi_{2n+1} = -\frac{n + 1}{w'_{2n+1}} \left( \sum_{j=1}^{n} q_j p_j + \eta \right), \]

\[ \varphi_0 = -\frac{w'_{2n+1}}{t^{(i+1)/(n+1)}} (1 - q_1), \]

\[ \varphi_{2i} = -\frac{w'_{2n+1}}{t^{(i+1)/(n+1)}} (q_i - q_{i+1}) \quad (i = 1, \ldots, n - 1), \quad \varphi_{2n} = -\frac{w'_{2n+1}}{t} (q_n - t). \]

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Proposition 3.6. The compatibility condition of the Lax form (3.12) gives the Painlevé system $P(A_{2n+1}^*)$.

Remark 3.7. The group of symmetries defined in Section 3.3 arises from the gauge transformations for the Lax form

$$r_i(\Psi) = \exp \left( -\frac{\kappa |\alpha_i^\vee|}{\varphi_i} f_i \right) \Psi \quad (i = 0, \ldots, 2n + 1).$$

4 For the partition $(2n - 1, 1)$

The Painlevé system $P(A_{2n})$ is a Hamiltonian system of $2n$-th order with a coupled Painlevé IV Hamiltonian

$$H = \sum_{i=1}^{2n} H_{IV} \left[ \alpha_{2i}, \sum_{j=1}^i \alpha_{2j-1}; q_i, p_i \right] + \sum_{1 \leq i < j \leq n} 2q_ip_ip_j,$$

where

$$H_{IV}[a, b; q, p] = qp(p - q - t) - aq - bp.$$

It is known that $P(A_{2n})$ is derived from the hierarchy for the partition $(2n + 1)$. In this section, we discuss its derivation from the hierarchy for a partition $(2n - 1, 1)$, from which we obtain a new Lax pair for $P(A_{2n})$.

4.1 Similarity reduction of the DS hierarchy

Let

$$\Lambda_{k+(2n-1)(l-1)} = \sum_{i=1}^{2n-1-k} e_{i,k-1+(2n-1)(l-1)} + \sum_{i=2n-k}^{2n-1} e_{i,k+(2n-1)(l-1)},$$

$$\tilde{\Lambda}_{k+(2n-1)(l-1)} = \sum_{i=1}^{2n-1-k} f_{i,k-1+(2n-1)(l-1)} + \sum_{i=2n-k}^{2n-1} f_{i,k+(2n-1)(l-1)},$$

for $k = 1, \ldots, 2n - 1$ and $l \in \mathbb{N}$. Then the Heisenberg subalgebra $\mathfrak{s}_{(2n-1,1)}$ of $\widehat{\mathfrak{g}} = \mathfrak{g}(A_{2n-1}^{(1)})$ is expressed as

$$\mathfrak{s}_{(2n-1,1)} = \bigoplus_{k \in \mathbb{N}} \mathbb{C}\Lambda_k \oplus \mathbb{C}K \oplus \bigoplus_{k \in \mathbb{N}} \mathbb{C}\Lambda_k.$$

The grading operator $\vartheta_{(2n-1,1)}$ is given by

$$\vartheta_{(2n-1,1)} = (2n - 1)d + \sum_{i=1}^{2n-1} \frac{i}{2} \left( 2n - i - \frac{1}{n} \right) \alpha_i^\vee.$$
It implies a $\mathbb{Z}$-gradation of type $(1, \ldots, 1, 0)$, namely

$$(\vartheta_{2n-1,1}|\alpha^\gamma_i) = 1 \quad (i = 0, \ldots, 2n - 2), \quad (\vartheta_{2n-1,1}|\alpha^\gamma_{2n-1}) = 0.$$ 

Note that

$$[\vartheta_{2n-1,1}, \Lambda_k] = k\Lambda_k \quad (k \in \mathbb{N}).$$

The similarity reduction (2.5) associated with $s_{(2n-1,1)}$ is described as

$$[\vartheta_{2n-1,1} - M, \partial_1 - B_1] = 0,$$  \hspace{1cm} (4.1)

where

$$B_1 = \sum_{i=0}^{2n-1} u_i\alpha_i^\gamma + x_0e_0 + x_{2n-1}e_{2n-1} + \Lambda_1,$$

$$M = \sum_{i=0}^{2n-1} \kappa_i\alpha_i^\gamma + \sum_{i=0}^{2n-1} \varphi_i e_i + \varphi_{0,1}e_{0,1} + \varphi_{2n-2,1}e_{2n-2,1} + \varphi_{2n-1,1}e_{2n-1,1} + 2\Lambda_2.$$

Note that

$$\Lambda_1 = \sum_{i=1}^{2n-2} e_i + e_{2n-1,1}, \quad \Lambda_2 = \sum_{i=1}^{2n-3} e_{i,1} + e_{2n-2,2} + e_{2n-1,2}.$$

### 4.2 Hamiltonian system

The operators $B_1$ and $M$ are defined as the $\mathfrak{b}_+$-components of $\exp(\text{ad}W)(\Lambda_1)$ and $\exp(\text{ad}W)(\rho + t_1\Lambda_1 + 2\Lambda_2)$, respectively. The $\mathfrak{h}$-valued operator $\rho$ is given by

$$\rho = \rho_1 \sum_{i=1}^{2n-1} i\alpha_i^\gamma,$$

where $\rho_1$ is independent of $t_1$. The $\mathfrak{n}_-$-valued function $W$ is described as

$$W = -\sum_{i=0}^{2n+1} w_i f_i - \sum_{k=1}^{\infty} \sum_{i=0}^{2n+1} w_{i,k} f_{i,k}.$$

In a similar manner given in Section 3.2, we obtain
Lemma 4.1. The $\mathfrak{b}_+$-valued functions $B_1$ and $M$ can be expressed in terms of the dependent variables $w, w_{2n-1}$ and $\varphi_i$ ($i = 0, \ldots, 2n - 1$) as

$$
\begin{align*}
&u_0 - u_1 = \sum_{j=1}^{n-1} \frac{1}{2} \varphi_{2j} - w_0 w_{2n-1} - \frac{n-1}{2} t_1, \\
u_{2i-1} - u_{2i} &= - \sum_{j=1}^{i} \frac{1}{2} \varphi_{2j-1} - \sum_{j=i+1}^{n-1} \frac{1}{2} \varphi_{2j} + w_0 w_{2n-1} + \frac{n}{2} t_1, \\
u_{2i} - u_{2i+1} &= \sum_{j=1}^{i} \frac{1}{2} \varphi_{2j-1} + \sum_{j=i+1}^{n-1} \frac{1}{2} \varphi_{2j} - w_0 w_{2n-1} - \frac{n-1}{2} t_1, \\
u_{2n-1} - u_0 &= w_0 w_{2n-1}, \quad x_0 = -w_{2n-1}, \quad x_{2n-1} = w_0,
\end{align*}
$$

for $i = 1, \ldots, n - 1$ and

$$
\begin{align*}
\varphi_{0,1} &= -2w_{2n-1}, \quad \varphi_{2n-2,1} = 2w_0, \\
\varphi_{2n-1,1} &= -\sum_{i=1}^{2n-2} \varphi_i + 2w_0 w_{2n-1} + (2n - 1)t_1.
\end{align*}
$$

Furthermore, we have a relation

$$
\begin{align*}
u_0 \left\{ \varphi_0 - \sum_{i=1}^{2n-2} w_{2n-1} \varphi_i + 2w_0 w_{2n-1}^2 + (2n-1)t_1 w_{2n-1} \right\} - w_{2n-1} \varphi_{2n-1} &= (2n-1)\rho_1 + \kappa_0 - \kappa_{2n-1}.
\end{align*}
$$

(4.2)

Thanks to Lemma 4.1, we can express the system (4.1) as a system of ordinary differential equations in terms of the variables $w, w_{2n-1}$ and $\varphi_i$ ($i = 0, \ldots, 2n - 1$).

The Kostant-Kirillov structure for the operator $M$ is defined by

$$
\begin{align*}
&\{ \varphi_1, \varphi_{i+1} \} = -2(2n-1) \quad (i = 1, \ldots, 2n - 3), \\
&\{ \varphi_{2n-2}, \varphi_{2n-1,1} \} = -2(2n-1), \quad \{ \varphi_{2n-2,1}, \varphi_{0} \} = -2(2n-1), \\
&\{ \varphi_{2n-1,1}, \varphi_{0,1} \} = -2(2n-1), \quad \{ \varphi_{2n-1,1}, \varphi_1 \} = -2(2n-1), \\
&\{ \varphi_{2n-2}, \varphi_{2n-1} \} = -(2n-1)\varphi_{2n-2,1}, \quad \{ \varphi_{2n-1}, \varphi_{0} \} = -(2n-1)\varphi_{2n-1,1}, \\
&\{ \varphi_0, \varphi_1 \} = -(2n-1)\varphi_{0,1}.
\end{align*}
$$

It is equivalent to

$$
\{ \mu_i, \lambda_j \} = (2n-1)\delta_{i,j} \quad (i, j = 1, \ldots, n + 1),
$$

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via a transformation of dependent variables

\[ \lambda_1 = w_{2n-1}, \quad \mu_1 = \varphi_{2n-1}, \]

\[ \lambda_i = -\sum_{j=1}^{i-1} \frac{1}{2} \varphi_{2n-2j} + w_0 w_{2n-1}, \quad \mu_i = \varphi_{2n-2i+1}, \]

\[ \lambda_{n+1} = w_0, \quad \mu_{n+1} = \varphi_0 - \sum_{i=1}^{2n-2} w_{2n-1} \varphi_i + 2w_0 w_{2n-1}^2 + (2n-1)t_1 w_{2n-1}, \]

for \( i = 2, \ldots, n \). Then those variables satisfy a Hamiltonian system; we do not give its explicit formula.

On the other hand, the equation (4.2) implies

\[ \sum_{i=1}^{n+1} d\mu_i \wedge d\lambda_i = \sum_{i=1}^{n} d\mu_i \wedge d\lambda_i + d\lambda_{n+1} \]

\[ = d\mu_1 \wedge d(\lambda_{n+1} \lambda_1) + \sum_{i=2}^{n} d\mu_i \wedge d\lambda_i. \]

Therefore we can take

\[ q_1 = \sqrt{\frac{2}{2n-1}} \lambda_{n+1} \lambda_1, \quad p_1 = \frac{\mu_1}{\sqrt{2(2n-1)}} \lambda_{n+1}, \]

\[ q_i = \sqrt{\frac{2}{2n-1}} \lambda_i, \quad p_i = \frac{\mu_i}{\sqrt{2(2n-1)}} \quad (i = 2, \ldots, n), \]

as canonical coordinates of a 2\(n\)-dimensional system with a Poisson structure

\[ \{ p_i, q_j \} = \delta_{i,j} \quad (i, j = 1, \ldots, n). \]

Denote the parameters by

\[ \alpha_0 = \rho_1 + \frac{1 - \kappa_0 + \kappa_1}{2n-1}, \quad \alpha_1 = -\rho_1 - \frac{\kappa_0 - \kappa_{2n-1}}{2n-1}, \]

\[ \alpha_i = \frac{(\kappa_1 \alpha_{2n-i+1})}{2n-1} \quad (i = 2, \ldots, 2n-1) \]

where

\[ \kappa = \vartheta_{(2n-1,1)} - \sum_{i=0}^{2n-1} \kappa_i \alpha_i^\vee. \]

Via a transformation of the independent variable \( t = \sqrt{\frac{2n-1}{2}} t_1 \), we arrive to
Theorem 4.2. The variables \( q_i, p_i \) \((i = 1, \ldots, n)\) defined by (4.3) satisfy the Painlevé system \( P(A_{2n}) \). Then the variable \( \lambda_{n+1} \) satisfies

\[
\frac{d}{dt} \log \lambda_{n+1} = \sum_{j=1}^{n} p_j - \frac{n}{2n-1} t.
\]

4.3 Lax form

In this subsection, we derive a Lax form for the Painlevé system \( P(A_{2n}) \) in a framework of \( sl_{2n}[z, z^{-1}] \).

Under the specialization \( t_1 = \sqrt{\frac{2}{2n-1}} t, t_2 = 1 \) and \( t_k = 0 \) \((k \geq 3)\), the Lax form (2.6) is described as

\[
\frac{d\Psi}{dt} = B\Psi, \quad z\frac{d\Psi}{dz} = M\Psi.
\]

(4.4)

The matrix \( B \) is given by

\[
B = \sum_{i=1}^{2n} u'_i E_{i,i} + x'_{2n-1} E_{2n-2,n}
\]

\[
+ x'_0 z E_{2n,1} + \sum_{i=1}^{2n-2} \sqrt{\frac{2}{2n-1}} z E_{i,i+1} + \sqrt{\frac{2}{2n-1}} z E_{2n-1,1},
\]

where

\[
u'_1 = q_n + \frac{n-1}{2n-1} t, \quad \nu'_i = \sum_{j=1}^{i} p_{n-j+1} - q_{n-i+1} - \frac{n}{2n-1} t,
\]

\[
u'_{2i} = -\sum_{j=1}^{i} p_{n-j+1} + q_{n-i} + \frac{n-1}{2n-1} t, \quad \nu'_2 = -q_1,
\]

\[
x'_{2n-1} = \sqrt{\frac{2}{2n-1}} \lambda_{n+1}, \quad x'_0 = \frac{-q_1}{\lambda_{n+1}},
\]

for \( i = 1, \ldots, n-1 \). The matrix \( M \) is given by

\[
M = \sum_{i=1}^{2n-1} (\kappa_i - \kappa_{i-1}) E_{i,i} + (\kappa_0 - \kappa_{2n-1}) E_{2n,2n} + \varphi_{2n-1} E_{2n-1,2n}
\]

\[
+ \varphi_0 z E_{2n,1} + \sum_{i=1}^{2n-2} \varphi_{i} z E_{i,i+1} + \varphi_{2n-2,1} z E_{2n-2,2n} + \varphi_{2n-1,1} z E_{2n-1,1}
\]

\[
+ \varphi_{0,1} z E_{2n,2} + \sum_{i=1}^{2n-3} 2z^2 E_{i,i+2} + 2z^2 E_{2n-2,1} + 2z^2 E_{2n-1,2},
\]

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where
\[ \varphi_{2n-1} = \sqrt{2(2n-1)\lambda_{n+1}} p_1, \quad \varphi_0 = \frac{2n-1}{\lambda_{n+1}} \left( \sum_{j=1}^{n} q_j p_j - q_1 q_n - t q_1 - \alpha_1 \right), \]
\[ \varphi_{2i-1} = \sqrt{2(2n-1)} p_{n-i+1}, \quad \varphi_{2i} = \sqrt{2(2n-1)} (q_{n-i} - q_{n-i+1}), \]
\[ \varphi_{2n-2,1} = 2 \lambda_{n+1}, \quad \varphi_{2n-1,1} = -\sqrt{2(2n-1)} \left( \sum_{j=2}^{n} p_j - q_n - t \right), \]
\[ \varphi_{0,1} = -\sqrt{2(2n-1)} q_1. \]

for \( i = 1, \ldots, n-1. \)

**Proposition 4.3.** The compatibility condition of the Lax form (4.4) gives the Painlevé system \( P(A_{2n}) \).

5 For the partition \((2n, 1)\)

The Painlevé system \( P(A_{2n+1}) \) is a Hamiltonian system of \( 2n \)-th order with a coupled Painlevé V Hamiltonian

\[ tH = \sum_{i=1}^{n} H_V \left[ \alpha_{2i}, \sum_{j=1}^{i} \alpha_{2j-1}, \sum_{j=1}^{n+1} \alpha_{2j-1}; q_i, p_i \right] + \sum_{1 \leq i < j \leq n} 2q_i p_i (q_j - 1) p_j, \]

where
\[ H_V[a, b; c; q, p] = q(q-1)p(p+t) + atq + b p - cq p. \]

It is known that \( P(A_{2n+1}) \) is derived from the hierarchy for the partition \((2n + 2)\). In this section, we discuss its derivation from the hierarchy for a partition \((2n, 1)\), from which we obtain a new Lax pair for \( P(A_{2n+1}) \).

5.1 Similarity reduction of the DS hierarchy

Let
\[ \Lambda_{k+2n(l-1)} = \sum_{i=1}^{2n-k} e_{i,k-1+2n(l-1)} + \sum_{i=2n-k+1}^{2n} e_{i,k+2n(l-1)}, \]
\[ \tilde{\Lambda}_{k+2n(l-1)} = \sum_{i=1}^{2n-k} f_{i,k-1+2n(l-1)} + \sum_{i=2n-k+1}^{2n} f_{i,k+2n(l-1)}, \]
for \( k = 1, \ldots, 2n \) and \( l \in \mathbb{N} \). Then the Heisenberg subalgebra \( \mathfrak{s}_{(2n,1)} \) of \( \hat{\mathfrak{g}} = \mathfrak{g}(A^{(1)}_{2n}) \) is expressed as

\[
\mathfrak{s}_{(2n,1)} = \bigoplus_{k \in \mathbb{N}} \mathbb{C}\tilde{\Lambda}_k \oplus \mathbb{C}K \oplus \bigoplus_{k \in \mathbb{N}} \mathbb{C}\Lambda_k.
\]

The grading operator \( \vartheta_{(2n,1)} \) is given by

\[
\vartheta_{(2n,1)} = 2nd + \sum_{i=1}^{2n} i \left( 2n + 1 - i - \frac{2}{2n + 1} \right) \alpha_i^\vee.
\]

It implies a \( \mathbb{Z} \)-gradation of type \((1, \ldots, 1, 0)\), namely

\[
(\vartheta_{(2n,1)} | \alpha_i^\vee) = 1 \ (i = 0, \ldots, 2n - 1), \quad (\vartheta_{(2n,1)} | \alpha_{2n}^\vee) = 0.
\]

Note that

\[
[\vartheta_{(2n,1)}, \Lambda_k] = k\Lambda_k \quad (k \in \mathbb{N}).
\]

The similarity reduction (2.5) associated with \( \mathfrak{s}_{(2n,1)} \) is described as

\[
[\vartheta_{(2n,1)} - M, \partial_1 - B_1] = 0,
\]

(5.1)

where

\[
B_1 = \sum_{i=0}^{2n} u_i \alpha_i^\vee + x_0 e_0 + x_{2n} e_{2n} + \Lambda_1,
\]

\[
M = \sum_{i=0}^{2n} \kappa_i \alpha_i^\vee + \sum_{i=0}^{2n} \varphi_i e_i + \varphi_{0,1} e_{0,1} + \varphi_{2n-1,1} e_{2n-1,1} + \varphi_{2n,1} e_{2n,1} + 2\Lambda_2.
\]

Note that

\[
\Lambda_1 = \sum_{i=1}^{2n-1} e_i + e_{2n,1}, \quad \Lambda_2 = \sum_{i=1}^{2n-2} e_{i,1} + e_{2n-1,2} + e_{2n,2}.
\]

### 5.2 Hamiltonian system

The operators \( B_1 \) and \( M \) are defined as the \( \mathfrak{h}_+ \)-components of \( \exp(\text{ad}W)(\Lambda_1) \) and \( \exp(\text{ad}W)(\rho + t_1 \Lambda_1 + 2\Lambda_2) \), respectively. The \( \mathfrak{h} \)-valued operator \( \rho \) is given by

\[
\rho = \rho_1 \sum_{i=1}^{2n} i\alpha_i^\vee,
\]

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Lemma 5.1. In a similar manner given in Section 3.2, we obtain 

\[ W = - \sum_{i=0}^{2n+1} w_i f_i - \sum_{k=1}^{\infty} \sum_{i=0}^{2n+1} w_{i,k} f_{i,k}. \]

In a similar manner given in Section [5.2] we obtain

**Lemma 5.1.** The \( b_\pm \)-valued functions \( B_1 \) and \( M \) can be expressed in terms of the dependent variables \( w_0, \varphi_0, \varphi_i \) (\( i = 2, \ldots, 2n - 1 \)) and \( w_{2n}, \varphi_{2n} \) as

\[
\begin{align*}
    u_0 - u_1 &= \sum_{j=1}^{n-1} \sum_{k=1}^{j} \frac{1}{2nt_1} \varphi_{2j+1} \varphi_{2k} - \sum_{j=1}^{n-1} \frac{1}{nt_1} w_0 w_{2n} \varphi_{2j} - \frac{1}{nt_1} w_{2n} \varphi_{2n}, \\
    u_{2i-1} - u_{2i} &= - \sum_{j=1}^{n-1} \sum_{k=1}^{j} \frac{1}{2nt_1} \varphi_{2j+1} \varphi_{2k} + \sum_{j=1}^{n-1} \frac{1}{nt_1} w_0 w_{2n} \varphi_{2j} + \frac{1}{nt_1} w_{2n} \varphi_{2n}, \\
    u_{2i} - u_{2i+1} &= \sum_{j=1}^{n-1} \sum_{k=1}^{j} \frac{1}{2nt_1} \varphi_{2j+1} \varphi_{2k} - \sum_{j=1}^{n-1} \frac{1}{nt_1} w_0 w_{2n} \varphi_{2j} - \frac{1}{nt_1} w_{2n} \varphi_{2n}, \\
    u_{2n-1} - u_{2n} &= - \sum_{j=1}^{n-1} \sum_{k=1}^{j} \frac{1}{2nt_1} \varphi_{2j+1} \varphi_{2k} + \sum_{j=1}^{n-1} \frac{1}{nt_1} w_0 w_{2n} \varphi_{2j} + \frac{1}{nt_1} w_{2n} \varphi_{2n}, \\
    u_{2n} - u_0 &= w_0 w_{2n},
\end{align*}
\]

for \( i = 1, \ldots, n - 1 \),

\[ x_0 = -w_{2n}, \quad x_{2n} = w_0, \]

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and

\[ \varphi_1 = -\sum_{j=1}^{n-1} \varphi_{2j+1} + 2w_0w_{2n} + nt_1, \]

\[ \varphi_{0,1} = -2w_{2n}, \quad \varphi_{2n-1,1} = 2w_0, \quad \varphi_{2n,1} = -\sum_{j=1}^{n-1} \varphi_{2j} + nt_1. \]

Furthermore, we have a relation

\[ w_0(\varphi_0 + nt_1w_{2n}) - w_{2n} \left( \varphi_{2n} + \sum_{i=1}^{n-1} w_{2n}\varphi_{2i} \right) = 2np_1 + \kappa_0 - \kappa_{2n}. \quad (5.2) \]

Thanks to Lemma 5.1 we can express the system (5.1) as a system of ordinary differential equations in terms of the variables \( w_0, \varphi_0, \varphi_i (i = 2, \ldots, 2n - 1) \) and \( w_{2n}, \varphi_{2n} \).

The Kostant-Kirillov structure for the operator \( M \) is defined by

\[ \{ \varphi_i, \varphi_{i+1} \} = -4n \quad (i = 1, \ldots, 2n - 2), \quad \{ \varphi_{2n-1}, \varphi_{2n,1} \} = -4n, \]

\[ \{ \varphi_2n-1, \varphi_0 \} = -4n, \quad \{ \varphi_{2n}, \varphi_{0,1} \} = -4n, \quad \{ \varphi_{2n,1}, \varphi_1 \} = -4n, \]

\[ \{ \varphi_{2n-1}, \varphi_{2n} \} = -2n\varphi_{2n-1,1}, \quad \{ \varphi_{2n}, \varphi_0 \} = -2n\varphi_{2n,1}, \quad \{ \varphi_0, \varphi_1 \} = -2n\varphi_{0,1}. \]

It is equivalent to

\[ \{ \mu_i, \lambda_j \} = 2n\delta_{i,j} \quad (i, j = 1, \ldots, n + 1), \]

via a transformation of dependent variables

\[ \lambda_i = \sum_{j=1}^{i} \varphi_{2j}, \quad \mu_i = \frac{1}{2}\varphi_{2i+1} \quad (i = 1, \ldots, n - 1), \]

\[ \lambda_n = \varphi_{2n} + \sum_{i=1}^{n-1} w_0\varphi_{2i}, \quad \mu_n = -w_{2n}, \]

\[ \lambda_{n+1} = w_0, \quad \mu_{n+1} = \varphi_0 + nt_1w_{2n}. \]

Then those variables satisfy a Hamiltonian system; we do not give its explicit formula.

On the other hand, the equation (5.2) implies

\[ \sum_{i=1}^{n+1} d\mu_i \wedge d\lambda_i = \sum_{i=1}^{n} d\mu_i \wedge d\lambda_i - \frac{\lambda_n\mu_n}{\lambda_{n+1}} \wedge d\lambda_{n+1}, \]

\[ = \sum_{i=1}^{n-1} d\mu_i \wedge d\lambda_i + d(\lambda_{n+1}\mu_n) \wedge d\frac{\lambda_n}{\lambda_{n+1}}. \]
Therefore we can take
\[
q_i = \frac{\lambda_i}{nt_1}, \quad p_i = \frac{t_1\mu_i}{2} \quad (i = 1, \ldots, n - 1),
\]
\[
q_n = \frac{\lambda_n}{nt_1\lambda_{n+1}}, \quad p_n = \frac{t_1\lambda_{n+1}\mu_n}{2},
\]
as canonical coordinates of a 2n-dimensional system with a Poisson structure
\[
\{p_i, q_j\} = \delta_{i,j} \quad (i, j = 1, \ldots, n).
\] (5.3)

Denote the parameters by
\[
\alpha_i = \left(\kappa_{i+1}^\vee\right)^{2n} \quad (i = 0, \ldots, 2n - 1),
\]
\[
\alpha_{2n} = -\rho_1 - \frac{\kappa_0 - \kappa_{2n}}{2n}, \quad \alpha_{2n+1} = \rho_1 + \frac{1 - \kappa_0 + \kappa_1}{2n},
\]
where
\[
\kappa = \vartheta_{(2n,1)} - \sum_{i=0}^{2n} \kappa_i\alpha_i^\vee.
\]

Via a transformation of the independent variable \(t = -\frac{2}{3}t_1^2\), we arrive to

**Theorem 5.2.** The variables \(q_i, p_i \ (i = 1, \ldots, n)\) defined by (5.3) satisfy the Painlevé system \(P(A_{2n+1})\). Then the variable \(\lambda_{n+1}\) satisfies
\[
t \frac{d}{dt} \log \lambda_{n+1} = -\sum_{i=1}^{n} q_i p_i - tq_n + \frac{n + 1}{2} t - \frac{1}{4} \sum_{j=0}^{n} (\alpha_{2j} - \alpha_{2j+1}).
\]

### 5.3 Lax form

In this subsection, we derive a Lax form for the Painlevé system \(P(A_{2n+1})\) in a framework of \(\mathfrak{sl}_{2n+1}[z, z^{-1}]\).

Under the specialization \(t_1 = \frac{2}{\sqrt{-nt}}, \ t_2 = 1\) and \(t_k = 0 \ (k \geq 3)\), the Lax form (2.6) is described as
\[
t \frac{d\Psi}{dt} = B\Psi, \quad z \frac{d\Psi}{dz} = M\Psi.
\] (5.4)

The matrix \(B\) is given by
\[
B_1 = \sum_{i=1}^{2n+1} u'_i E_{i,i} + x'_{2n} E_{2n,2n+1} + x'_0 z E_{2n+1,1} + \sum_{i=1}^{2n-1} \frac{1}{\sqrt{-nt}} z E_{i,i+1} + \frac{1}{\sqrt{-nt}} z E_{2n,1}.
\]
where
\[
\begin{align*}
u_1' &= -\sum_{j=1}^{n} q_j p_j - \frac{n-1}{2n} t - \frac{1}{4} \sum_{j=0}^{n} (\alpha_{2j} - \alpha_{2j+1}), \\
u_2' &= \sum_{j=1}^{n} (q_j - 1) p_j - \frac{n-1}{2n} t + \frac{1}{4} \sum_{j=0}^{n} (\alpha_{2j} - \alpha_{2j+1}), \\
u_{2i+1}' &= -\sum_{j=1}^{n} (q_j - 1) p_j + \frac{n+1}{2n} t - \frac{1}{4} \sum_{j=0}^{n} (\alpha_{2j} - \alpha_{2j+1}), \\
u_{2i+2}' &= \sum_{j=1}^{n} (q_j - 1) p_j + t q_i - \frac{n-1}{2n} t + \frac{1}{4} \sum_{j=0}^{n} (\alpha_{2j} - \alpha_{2j+1}), \\
u_{2n+1}' &= \frac{1}{2} p_n,
\end{align*}
\]
for \(i = 1, \ldots, n-1\) and
\[
x_{2n} = \frac{\lambda_{n+1}}{\sqrt{-nt}}, \quad x_0 = \frac{1}{2\lambda_{n+1}} p_n.
\]

The matrix \(M\) is given by
\[
M = \sum_{i=1}^{2n} (\kappa_i - \kappa_{i-1}) E_{i,i} + (\kappa_0 - \kappa_{2n}) E_{2n+1,2n+1} + \varphi_{2n} E_{2n,2n+1} + \sum_{i=0}^{2n-1} \varphi_i z E_{i,i+1} + \varphi_{2n-1,1} z E_{2n-1,2n+1} + \varphi_{2n,1} z E_{2n,1} + \varphi_{0,1} z^2 E_{2n+1,2} + \sum_{i=1}^{2n-2} 2z^2 E_{i,i+2} + 2z^2 E_{2n-1,1} + 2z^2 E_{2n,2},
\]
where
\[
\begin{align*}
\varphi_0 &= -\frac{2n}{\lambda_{n+1}} (q_n - 1) p_n + \alpha_{2n}, \quad \varphi_1 = -2\sqrt{-nt} \left( \sum_{j=1}^{n} p_j + t \right), \\
\varphi_2 &= \frac{2n}{\sqrt{-nt}} q_1, \quad \varphi_{2i-1} = -2\sqrt{-nt} p_{i-1} \quad (i = 2, \ldots, n), \\
\varphi_{2i} &= \frac{2n}{\sqrt{-nt}} (q_i - q_{i-1}) \quad (i = 2, \ldots, n-1), \quad \varphi_{2n} = \frac{2n\lambda_{n+1}}{\sqrt{-nt}} (q_n - q_{n-1}), \\
\varphi_{2n-1,1} &= 2\lambda_{n+1}, \quad \varphi_{2n,1} = -\frac{2n}{\sqrt{-nt}} (q_{n-1} - 1), \quad \varphi_{0,1} = \frac{2\sqrt{-nt}}{\lambda_{n+1}} p_n.
\end{align*}
\]

**Proposition 5.3.** The compatibility condition of the Lax form (5.4) gives the Painlevé system \(P(A_{2n+1})\).
6 For the partition \((n, n, 1)\)

In Section 3, we have derived the Painlevé system \(P(A_{2n+1}^{*})\) from the hierarchy for the partition \((n + 1, n + 1)\). In this section, we discuss its derivation from the hierarchy for a partition \((n, n, 1)\), from which we obtain another Lax pair for \(P(A_{2n+1}^{*})\).

6.1 Similarity reduction of the DS hierarchy

Let

\[
\Lambda_{2k-1+2n(l-1)} = \sum_{i=1}^{n-k} e^{2i_1,2k-1+(2n+1)(l-1)} + \sum_{i=n-k+1}^{n} e^{2i_1,2k+(2n+1)(l-1)};
\]

\[
\Lambda_{2k+2n(l-1)} = \sum_{i=1}^{n-k} e^{2i_1,2k-1+(2n+1)(l-1)} + \sum_{i=n-k+1}^{n} e^{2i_1,2k+(2n+1)(l-1)};
\]

\[
\bar{\Lambda}_{2k-1+2n(l-1)} = \sum_{i=1}^{n-k} f^{2i_1,2k-1+(2n+1)(l-1)} + \sum_{i=n-k+1}^{n} f^{2i_1,2k+(2n+1)(l-1)};
\]

\[
\bar{\Lambda}_{2k+2n(l-1)} = \sum_{i=1}^{n-k} f^{2i_1,2k-1+(2n+1)(l-1)} + \sum_{i=n-k+1}^{n} f^{2i_1,2k+(2n+1)(l-1)};
\]

for \(k = 1, \ldots, n\) and \(l \in \mathbb{N}\). Then the Heisenberg subalgebra \(s_{(n,n,1)}\) of \(\hat{\mathfrak{g}} = \mathfrak{g}(A_{2n+1}^{(1)})\) is expressed as

\[
s_{(n,n,1)} = \bigoplus_{k \in \mathbb{N}} \mathbb{C}\bar{\Lambda}_{k} \oplus \mathbb{C}K \oplus \bigoplus_{k \in \mathbb{N}} \mathbb{C}\Lambda_{k}.
\]

The grading operator \(\vartheta_{(n,n,1)}\) is given by

\[
\vartheta_{(n,n,1)} = nd + \sum_{i=1}^{n} (n - i + 1) \left( i - \frac{n}{2n+1} \right) \alpha_{2i-1}^{\gamma} + \sum_{i=1}^{n} i \left( \frac{2n}{2n+1} - i \right) \alpha_{2i}^{\gamma}.
\]

It implies a \(\mathbb{Z}\)-gradation of type \((0, 1, 0, \ldots, 1, 0)\), namely

\[
(\vartheta_{(n,n,1)}|\alpha_{2i}^{\gamma}) = 0, \quad (\vartheta_{(n,n,1)}|\alpha_{2i-1}^{\gamma}) = 1, \quad (\vartheta_{(n,n,1)}|\alpha_{2k}^{\gamma}) = 0 (i = 1, \ldots, n).
\]

Note that

\[
[\vartheta_{(n,n,1)}, \Lambda_{2k-1}] = k\Lambda_{2k-1}, \quad [\vartheta_{(n,n,1)}, \Lambda_{2k}] = k\Lambda_{2k} \quad (k \in \mathbb{N}).
\]

The similarity reduction (2.5) associated with \(s_{(n,n,1)}\) is described as

\[
[\vartheta_{(n,n,1)} - M, \partial_{t} - B_{1}] = 0,
\]

(6.1)
where
\[
B_1 = \sum_{i=0}^{2n} u_i \alpha_i^\vee + \sum_{i=0}^{2n} x_i e_i + x_{2n-1,1} e_{2n-1,1} + x_{2n,1} e_{2n,1} + \Lambda_1,
\]
\[
M = \sum_{i=0}^{2n} \kappa_i \alpha_i^\vee + \sum_{i=0}^{2n} \varphi_i e_i + \varphi_{0,1} e_{0,1} + \varphi_{2n-1,1} e_{2n-1,1} + \varphi_{2n,1} e_{2n,1} + t_1 \Lambda_1 + \Lambda_2.
\]

Note that
\[
\Lambda_1 = \sum_{i=1}^{n-1} e_{2i-1,1} + e_{2n-1,2}, \quad \Lambda_2 = \sum_{i=1}^{n-1} e_{2i,1} + e_{2n,2}.
\]

### 6.2 Hamiltonian system

The operators $B_1$ and $M$ are defined as the $\mathfrak{b}_+$-components of $\exp(\text{ad} W)(\Lambda_1)$ and $\exp(\text{ad} W)(\rho + 2t_1 \Lambda_1 + 2\Lambda_2)$, respectively. The $\mathfrak{b}$-valued operator $\rho$ is given by
\[
\rho = \rho_1 \sum_{i=1}^{n} 2i(\alpha_{2i-1}^\vee + \alpha_{2i}^\vee) + \rho_2 \sum_{i=1}^{n} 2(n - i + 1)(\alpha_{2i-2}^\vee + \alpha_{2i-1}^\vee),
\]
where $\rho_1, \rho_2$ are independent of $t_1$. The $\mathfrak{n}_+$-valued function $W$ is described as
\[
W = - \sum_{i=0}^{2n+1} w_i f_i - \sum_{k=1}^{\infty} \sum_{i=0}^{2n+1} w_{i,k} f_{i,k}.
\]

In a similar manner given in Section 3.2 we obtain

**Lemma 6.1.** The $\mathfrak{b}_+$-valued functions $B_1$ and $M$ can be expressed in terms of the dependent variables $w_0, \varphi_0, w_{2i-1}, \varphi_{2i-1} (i = 1, \ldots, n)$ and $w_{2n}, \varphi_{2n}$ as
\[
u_{2i-2} - u_{2i-1} = \sum_{j=1}^{i-1} \frac{t_{i-1}^{n+i-j-1}}{t_1^{n-1}} w_{2i-1} \varphi_{2j-1} - \sum_{j=1}^{n} \frac{t_{i-1}^{j-i-1}}{t_1^{n-1}} w_{2i-1} \varphi_{2j-1} + \frac{t_{i-1}^{n-i}}{t_1^{n-1}} w_0 w_{2i-1} w_{2n} + \frac{1}{t_1} (\rho_1 + \kappa_{2i-2} - \kappa_{2i-1}),
\]
\[
u_{2i-1} - u_{2i} = \sum_{j=1}^{i} \frac{t_{i-1}^{n+i-j-1}}{t_1^{n-1}} w_{2i-1} \varphi_{2j-1} + \sum_{j=1}^{n} \frac{t_{i-1}^{j-i-1}}{t_1^{n-1}} w_{2i-1} \varphi_{2j-1} - \frac{t_{i-1}^{n-i}}{t_1^{n-1}} w_0 w_{2i-1} w_{2n},
\]
\[
u_{2n} - u_0 = \frac{1}{t_1} w_0 (\varphi_0 - w_1 w_{2n}).
\]
for $i = 1, \ldots, n$,

$$x_0 = \frac{1}{t_1}(\varphi_0 - w_1 w_{2n}),$$

$$x_{2i-1} = \sum_{j=1}^{i} \frac{t_1^{n-i+j-1}}{t_1^n - 1} \varphi_{2j-1} + \frac{t_1^{i-1}}{t_1^n - 1} \varphi_{2i-1} - \frac{t_1^{n-i}}{t_1^n - 1} w_0 w_{2n},$$

$$x_{2i} = -w_{2i-1}, \quad x_{2n-1} = \sum_{j=1}^{n} \frac{t_1^{j-1}}{t_1^n - 1} \varphi_{2j-1} - \frac{1}{t_1^n - 1} w_0 w_{2n},$$

$$x_{2n} = w_0 w_{2n-1}, \quad x_{2n-1,1} = -w_0, \quad x_{2n,1} = -w_{2n-1},$$

for $i = 1, \ldots, n - 1$ and

$$\varphi_{2i} = -t_1 w_{2i-1} + w_{2i+1} \quad (i = 1, \ldots, n - 1), \quad \varphi_{2n,1} = -t_1 w_{2n-1} + w_1.$$

Furthermore, we have a relation

$$\sum_{i=1}^{n} w_{2i-1} \varphi_{2i-1} + w_0(\varphi_0 - w_1 w_{2n}) = -\sum_{i=1}^{n} (\rho_i + \kappa_2 - \varphi_{2i-1}),$$

$$\sum_{i=1}^{n} w_{2i-1} \varphi_{2i-1} + w_{2n}(\varphi_{2n} - t_1 w_0 w_{2n-1}) = -\sum_{i=1}^{n} (\rho_i - \kappa_2 + \varphi_{2i}).$$

Thanks to Lemma 6.1, we can express the system (6.1) as a system of ordinary differential equations in terms of the variables $w_0, \varphi_0, w_{2i-1}, \varphi_{2i-1}$ ($i = 1, \ldots, n$) and $w_{2n}, \varphi_{2n}$. Note that

$$w_{2i-1} = -\sum_{j=1}^{i-1} \frac{t_1^{n-i+j-1}}{t_1^n - 1} \varphi_{2j} - \sum_{j=i}^{n-1} \frac{t_1^{n-i+j-1}}{t_1^n - 1} \varphi_{2j} - \frac{t_1^{i-1}}{t_1^n - 1} \varphi_{2n,1},$$

for $i = 1, \ldots, n$.

The Kostant-Kirillov structure for the operator $M$ is defined by

$$\{\varphi_{2i-1}, \varphi_2\} = -nt_1, \quad \{\varphi_2, \varphi_{2i+1}\} = -n \quad (i = 1, \ldots, n - 1),$$

$$\{\varphi_{2n-1}, \varphi_{2n,1}\} = -nt_1, \quad \{\varphi_{2n,1}, \varphi_1\} = -n,$$

$$\{\varphi_{2n-1,1}, \varphi_0\} = -nt_1, \quad \{\varphi_2, \varphi_{0,1}\} = -n,$$

$$\{\varphi_0, \varphi_1\} = -n \varphi_{0,1}, \quad \{\varphi_{2n-1}, \varphi_{2n}\} = -n \varphi_{2n-1,1}, \quad \{\varphi_{2n}, \varphi_0\} = -n \varphi_{2n,1}.$$

It is equivalent to

$$\{\mu_i, \lambda_j\} = n \delta_{i,j} \quad (i, j = 1, \ldots, n + 2),$$

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via a transformation of dependent variables

\[ \lambda_i = w_{2i-1}, \quad \mu_i = \varphi_{2i-1} \quad (i = 1, \ldots, n), \]
\[ \lambda_{n+1} = w_0, \quad \mu_{n+1} = -\varphi_0 + w_1w_{2n}, \]
\[ \lambda_{n+2} = w_{2n}, \quad \mu_{n+2} = -\varphi_{2n} + t_1w_0w_{2n-1}. \]

Then those variables satisfy a Hamiltonian system; we do not give its explicit formula.

On the other hand, the equation (6.2) implies

\[
\sum_{i=1}^{n+2} d\mu_i \wedge d\lambda_i = \sum_{i=1}^{n} d\mu_i \wedge d\lambda_i + \sum_{i=1}^{n} d\mu_{n+1} \wedge \frac{\lambda_i \mu_i}{\lambda_{n+2}} + \sum_{i=1}^{n} d\frac{\lambda_i \mu_i}{\lambda_{n+2}} \wedge d\lambda_{n+2},
\]

Therefore we can take

\[ q_i = \frac{\lambda_{n+2}}{t_i-1} \frac{1}{\mu_{n+1}} \lambda_i, \quad p_i = \frac{t_i^{-1}}{n\lambda_{n+2}} \mu_i \quad (i = 1, \ldots, n), \]

as canonical coordinates of a 2n-dimensional system with a Poisson structure

\[ \{p_i, q_j\} = \delta_{i,j} \quad (i, j = 1, \ldots, n). \]

Denote the parameters by

\[ \alpha_i = \frac{(\kappa_i \alpha_i^\vee)}{n} \quad (i = 0, \ldots, 2n-1), \quad \alpha_{2n} = \rho_1 - \rho_2 + \frac{\kappa_0 - \kappa_{2n}}{n}, \]
\[ \alpha_{2n+1} = -\rho_1 + \rho_2 + \frac{1 + \kappa_{2n-1} - \kappa_{2n}}{n}, \quad \eta = \sum_{j=1}^{n} \frac{\rho_1 + \kappa_{2j-2} - \kappa_{2j-1}}{n}, \]

where

\[ \kappa = \partial_{(n,n,1)} - \sum_{i=0}^{2n} \kappa_i \alpha_i^\vee. \]

Via a transformation of the independent variable \( t = t_1^{-n} \), we arrive to

**Theorem 6.2.** The variables \( q_i, p_i \ (i = 1, \ldots, n) \) defined by (6.3) satisfy the
Painlevé system $P(A_{2n+1}^*)$. Then the variables $\mu_{n+1}, \lambda_{n+2}$ satisfy

$$t(t-1) \frac{d}{dt} \log \mu_{n+1} = - \sum_{i=1}^{n} q_i \{ (q_i - 1)p_i + \alpha_{2i-1} \} - \alpha_{2n}t - \frac{t-n-1}{n} \eta$$

$$- \left\{ \frac{n+1}{2n} \alpha_0 + \sum_{j=1}^{n} \frac{n-2j+1}{2n} (\alpha_{2j-1} + \alpha_{2j}) \right\} (t-1),$$

$$t(t-1) \frac{d}{dt} \log \lambda_{n+2} = - \sum_{i=1}^{n} t(q_i - 1)p_i - t\eta.$$

### 6.3 Lax form

In this subsection, we derive a Lax form for the Painlevé system $P(A_{2n+1}^*)$ in a framework of $\mathfrak{sl}_{2n+1}[z, z^{-1}]$.

Under the specialization $t_1 = t^{-n}$, $t_2 = 1$ and $t_k = 0$ ($k \geq 3$), the Lax form (6.4) is described as

$$t(t-1) \frac{d \Psi}{dt} = B \Psi, \quad z \frac{d \Psi}{dz} = M \Psi. \quad (6.4)$$

The matrix $B$ is given by

$$B = \sum_{i=1}^{2n+1} u'_i E_{i,i} + x'_0 E_{2n+1,1} + \sum_{i=1}^{n} x'_{2i} E_{2i,2i+1} + x'_{2n,1} E_{2n,1} + \sum_{i=1}^{n} x'_{2i-1} zE_{2i-1,2i}$$

$$+ x'_{2n-1,1} zE_{2n-1,2n+1} - \sum_{i=1}^{n-1} \frac{t-1}{nt^{1/n}} zE_{2i-1,2i} - \frac{t-1}{nt^{1/n}} zE_{2n-1,1},$$

where

$$u'_{2i-1} = -q_i \left\{ \sum_{j=1}^{i-1} (q_j - 1)p_j + \sum_{j=i}^{n} (q_j - t)p_j + \eta \right\} + \frac{t-1}{n} \eta$$

$$- \left\{ \sum_{j=1}^{i-1} \frac{j}{n} (\alpha_{2j-1} + \alpha_{2j} - \frac{1}{n}) + \sum_{j=i}^{n-1} \frac{n-j}{n} (\alpha_{2j-1} + \alpha_{2j} - \frac{1}{n}) \right\} (t-1),$$

$$u'_{2i} = q_i \left\{ \sum_{j=1}^{i} (q_j - 1)p_j + \sum_{j=i+1}^{n} (q_j - t)p_j + \eta \right\},$$

$$u'_{2n+1} = -(t-1) \left( \sum_{j=1}^{n} q_j p_j + \eta \right).$$

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for $i = 1, \ldots, n$ and
\[
x_0' = \frac{(t - 1)\mu_{n+1}}{n}, \quad x_{2i}' = \frac{(t - 1)\mu_{n+1}}{n^{t/n}\lambda_{n+2}}q_i \quad (i = 1, \ldots, n - 1),
\]
\[
x_{2n}' = -\frac{t - 1}{\lambda_{n+2}}q_n \left( \sum_{j=1}^{n} q_j p_j + \eta \right), \quad x_{2n,1}' = \frac{(t - 1)\mu_{n+1}}{n\lambda_{n+2}}q_n,
\]
\[
x_{2i-1}' = -\frac{t(i-1)/n\lambda_{n+2}}{\mu_{n+1}} \left\{ \sum_{j=1}^{i} (q_j - 1)p_j + \sum_{j=i+1}^{n} (q_j - t)p_j + \eta \right\} \quad (i = 1, \ldots, n),
\]
\[
x_{2n-1,1}' = \frac{t - 1}{t^{1/n}\mu_{n+1}} \left( \sum_{j=1}^{n} q_j p_j + \eta \right).
\]

The matrix $M$ is given by
\[
M = \sum_{i=1}^{2n} (\kappa_i - \kappa_{i-1}) E_{i,i} + (\kappa_0 - \kappa_{2n}) E_{2n+1,1} + \varphi_0 E_{2n+1,1} + \sum_{i=1}^{n} \varphi_{2i} E_{2i,2i+1}
\]
\[
+ \varphi_{2n,1} E_{2n,1} + \sum_{i=1}^{n} \varphi_{2i-1} z\varphi_{2i} E_{2i-1,2i} + \varphi_{0,1} z\varphi_{2n+1} E_{2n+1,2} + \varphi_{2n-1,1} z\varphi_{2n} E_{2n-1,2n+1}
\]
\[
+ \sum_{i=1}^{n-1} t^{-1/n} z\varphi_{2i-1} E_{2i-1,2i+1} + t^{-1/n} z\varphi_{2n-1} E_{2n-1,1} + \sum_{i=1}^{n-1} z\varphi_{2i} E_{2i,2i+2} + z\varphi_{2n} E_{2n,2},
\]
where
\[
\varphi_0 = \mu_{n+1} (q_1 - 1), \quad \varphi_{2i} = -\frac{\mu_{n+1}}{t^{i/n}\lambda_{n+2}} (q_i - q_{i+1}) \quad (i = 1, \ldots, n - 1),
\]
\[
\varphi_{2n} = \frac{n}{t\lambda_{n+2}} \left\{ (q_n - t) \left( \sum_{j=1}^{n} q_j p_j + \eta \right) + \alpha_{2n} t \right\},
\]
\[
\varphi_{2n,1} = \frac{\mu_{n+1}}{\lambda_{n+2}} (q_1 - t q_n), \quad \varphi_{2i-1} = \frac{n t (i-1)/n\lambda_{n+2}}{\mu_{n+1}} p_i \quad (i = 1, \ldots, n),
\]
\[
\varphi_{0,1} = \lambda_{n+2}, \quad \varphi_{2n-1,1} = -\frac{n}{t^{1/n}\mu_{n+1}} \left( \sum_{j=1}^{n} q_j p_j + \eta \right).
\]

**Proposition 6.3.** The compatibility condition of the Lax form (6.4) gives the Painlevé system $P(A^*_{2n+1})$.

**Acknowledgement**

The author is grateful to Professors Laszlo Fehér, Kenta Fuji, Saburo Kakei, Hidetaka Sakai, Teruhisa Tsuda and Yasuhiko Yamada for valuable discus-
sions and advices.

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