On the equipartition theorem and black holes nongaussian entropies

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Abstract

In this Letter we have shown that, from the standard thermodynamic functions, the mathematical form of an equipartition theorem may be related to the algebraic expression of a particular entropy initially chosen to describe the black hole thermodynamics. Namely, we have different equipartition expressions for distinct statistics. To this end, four different mathematical expressions for the entropy have been selected to demonstrate our objective. Furthermore, a possible phase transition is observed in the heat capacity behavior of the Tsallis and Cirto entropy model.

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I. INTRODUCTION

Black holes are among the most interesting and charming objects that constitutes the main targets of quantum gravity analysis. The origin of a black hole, a collapsing star, for example, makes us to see that the geometry of short-distance fluctuations can be augmented to macroscopic measures. This connection between high-energy and low-energy physics can bring serious questions about these systems dynamics at the Planck scale dimensions, which has consequences for the low-energy experiments.

The pioneering papers of Hawking and Bekenstein on black holes physics [1, 2] lead to a profound connection between gravity and thermodynamics. We can mention, for example, that quantities as entropy and temperature can be associated with the black hole horizon.

The aim of this paper is to show that the algebraic expression of an equipartition theorem will depend firstly on the entropy initially chosen to describe the black hole thermodynamics. To this end, we will use thermodynamic functions, which are normally used in the black holes physics, that are the entropy and the temperature. Throughout the paper we use $\hbar = c = k_B = 1$. In the context of the usual Schwarzschild black hole entropy, $S_{BH} = 4\pi G M^2$, where $G$ and $M$ are the gravitational constant and the black hole mass respectively, it will be assumed that the number $N$ of degrees of freedom (DF) of the horizon satisfies the standard equipartition law [3],

$$M = \frac{1}{2} N T,$$

where $T$ is the temperature of black hole. To better organize our paper, we will follow a sequence such that in the second section we have explained our procedure and consequently derived the usual equipartition theorem. In section three we have presented the Tsallis and Cirto black hole entropy model [4, 5] and the respective equipartition law. In section four the modified Rényi entropy [6–8] was introduced and the corresponding equipartition theorem was derived. In section five the equipartition theorem in the black hole horizon for the Sharma-Mittal entropy [9, 10] is derived. The conclusions and final remarks were given in the last section.

II. BLACK HOLE MASS ENTROPY

The thermodynamics of black holes is defined, in principle, on the basis of the concept of the entropy. As we have mentioned in the introduction, the Schwarzschild black hole entropy is written as

$$S_{BH} = 4\pi G M^2.$$  

(2)

The temperature is given by

$$\frac{1}{T} = \frac{\partial S(M)}{\partial M},$$

(3)
and using Eq. (2) we have that
\[ \frac{1}{T} = \frac{\partial S_{BH}(M)}{\partial M} = 8\pi G M. \] (4)

The number \( N \) of DF on the boundary of the black hole can be given by assuming the relation \[11\]
\[ N = 4S, \] (5)
where \( S \) is an specific entropy describing the black hole. So, using Eq. (5) in our initial case, we have
\[ N = 16\pi GM^2. \] (6)

Combining Eqs. (4) and (6) and making some algebra then we can derive the usual equipartition theorem
\[ M = \frac{1}{2} NT. \] (7)

To establish the physical coherence a standard test is to calculate the heat capacity of the model. The sign of the heat capacity can support us in determining the stability of black holes. Namely, a positive heat capacity is meaningful. On the other hand, a negative heat capacity in such system violates the laws of thermodynamics, namely, it shows a thermodynamical unstablleness.

The heat capacity can be computed from the expression
\[ C = -\frac{[S'_{BH}(M)]^2}{S''_{BH}(M)}, \] (8)
where the prime means a derivative relative to \( M \). So, substituting the entropy of Eq. (2) into Eq. (8) we have that
\[ C_{BH} = -8\pi GM^2, \] (9)
which means, as well known, that a black hole is thermally unstable. The negative heat capacity in this regime means that a slight drop in black hole temperature will cause an additional drop as energy is absorbed. The mechanism will continue forever and the black hole will keep feeding on the surrounding heat bath. On the other hand, a slight rise in black hole temperature from the equilibrium value will cause the black hole to radiate some net energy. In this way, it will increase its temperature still further. It will, eventually, conduct to the explosive vanishing of the black hole altogether.

III. TSALLIS-CIRTO ENTROPY

Our second case will be the Tsallis and Cirto entropy model\[4\]. In few words, we can say that entropies that generalize the Boltzmann-Gibbs one are necessary to retrieve the
thermodynamics extensivity of nonstandard scenarios. These authors have proposed a modified form of $S_{BH}$ such that

$$S_\delta = (S_{BH})^\delta, \quad \text{(10)}$$

where $\delta$ denotes the non-additivity parameter. We can see that in the limit $\delta = 1$ $S_{BH}$ is recovered. The cosmological implications of the modified form of $S_{BH}$, Eq. (10), can be found, for example, in reference [12]. For the Schwarzschild solution we can write Eq. (10) as

$$S_\delta = (4\pi G)^\delta M^{2\delta}. \quad \text{(11)}$$

Making use of Eqs. (3) and (5) we can derive, respectively, the Hawking temperature $T$ and the number $N$ of DF as

$$T = \frac{1}{2 \delta (4\pi G)^\delta M^{2\delta-1}}, \quad \text{(12)}$$

and

$$N = 4 (4\pi G)^\delta M^{2\delta}. \quad \text{(13)}$$

Combining Eqs. (12) and (13) and after some algebra then we can obtain an equipartition theorem for the black hole mass in the Tsallis and Cirto model as

$$M = \frac{1}{2} \delta NT. \quad \text{(14)}$$

From Eq. (14) we can observe the appearance of an extra term $\delta$ in the usual equipartition theorem, Eq. (7). When we make $\delta = 1$ we recover the usual equipartition law.

The heat capacity, using Eqs. (11) and (8) is

$$C_\delta = \frac{2\delta (4\pi G)^\delta}{1 - 2\delta} M^{2\delta}. \quad \text{(15)}$$

When we make $\delta = 1$ in (15) we recover the heat capacity, Eq. (9). Here it is important to mention that the heat capacity in the Tsallis and Cirto model, Eq. (15), presents a divergence for $\delta = 1/2$. This result may indicate a possible phase transition between a thermally stable state and a thermally unstable state. In Fig. 1, the heat capacity, Eq. (15), has been plotted as a function of the $\delta$ parameter. We can observe that for $\delta = 1/2$ the heat capacity diverges.
FIG. 1. Heat capacity, Eq. (15), as a function of $\delta$. In this Equation we make, for example, $G = 1$ and $M = 2$.

IV. MODIFIED RÉNYI ENTROPY

The third example that will be analyzed is the modified Rényi entropy. This model was suggested by Biró and Czinner and they considered the Schwarzschild black hole entropy as a nonextensive Tsallis entropy \[13, 14\]. Next they wrote the Rényi entropy as a function of Tsallis entropy. The final form is given by

$$S_R = \frac{1}{\lambda} \ln(1 + \lambda S_{BH}), \quad (16)$$

where $\lambda$ is a constant parameter. If we take the limit $\lambda \to 0$ in Eq. (16) then we have $S_R = S_{BH}$. Substituting the Schwarzschild solution into Eq. (16) we have

$$S_R = \frac{1}{\lambda} \ln \left(1 + 4\pi \lambda GM^2\right). \quad (17)$$

Using Eqs. (3) and (5) we obtain, respectively, the black hole temperature and the number $N$ of DF as

$$T = \frac{1 + 4\pi \lambda GM^2}{8\pi GM}, \quad (18)$$
and

\[ N = \frac{4}{\lambda} \ln \left( 1 + 4\pi \lambda GM^2 \right). \tag{19} \]

Using Eqs. (18) and (19) and after some algebra, we can derive a mathematical expression for the equipartition theorem compatible with Rényi entropy as

\[ M = \frac{1}{8\pi GT} e^{\frac{\lambda N}{4}}. \tag{20} \]

Also doing some algebra, i.e., writing initially \( M \) as \( \frac{M^2}{M} \), we can derive an equation for the equipartition theorem in the Rényi modified entropy model in another form as

\[ M = \frac{e^{\frac{\lambda N}{4}} - 1}{4\pi GT \left[ 1 - \sqrt{1 - \frac{\lambda^2}{4\pi^2 GT^2}} \right]} . \tag{21} \]

It is straightforward to show that if we take the limit \( \lambda \to 0 \) in Eq. (21) then we recover the usual equipartition law, \( M = \frac{1}{2} NT \).

The heat capacity, using Eqs. (17) and (8) is

\[ C_R = \frac{8\pi GM^2}{4\pi \lambda GM^2 - 1}. \tag{22} \]

When we make \( \lambda = 0 \) in (22) we recover the heat capacity, Eq. (9). Here it is important to mention that Eq. (22) has already been obtained in ref. [7].

V. THE SHARMA-MITTL AL ENTROPY

Rényi and Tsallis entropies are formulations where the probability distributions are substituted by power-law distributions and the results are the so-called generalized entropies. The Sharma-Mittal (SM) entropy is a combined generalization of both Tsallis and Rényi entropies. The SM entropy leads us to relevant results in the cosmological scenario such as the description of the current accelerated Universe by using conveniently the vacuum energy. For more reference see [15] and references therein.

Although it is common procedure to use NE entropies to analyze BH thermodynamics properties, it is very different to consider the SM entropy for it. Hence, it is completely new to investigate its equipartition law the way we do here.

We are interested here in the implications of the SM entropy in the equipartition law, which was not taken in consideration until now. Considering the Schwarzschild black hole entropy as Tsallis entropy then the SM entropy is written as

\[ S_{SM} = \frac{1}{R} \left[ (1 + \delta S_{BH})^{\frac{R}{\delta}} - 1 \right], \tag{23} \]

where \( R \) and \( \delta \) are free parameters. At the limits, \( R \to 0 \) and \( R \to \delta \), the Rényi and Tsallis entropy, respectively, can be recovered. Substituting the Schwarzschild solution into Eq. (23) we have
Using Eq. (3) we obtain the black hole SM temperature

\[ T = \frac{1}{8\pi GM (1 + 4\pi GM^2)^\frac{R}{2} - 1} \]  

(25)

As we have proceed in the last sections, a modified equipartition theorem for the SM statistics can be written as

\[ M = \frac{2}{\delta} \left[ \left( 1 + \frac{R}{4N} \right)^\frac{\delta}{\pi} - 1 \right] \left[ \left( 1 + \frac{R}{4N} \right)^\frac{2}{\pi} \right]^{\frac{R}{2} - 1} T. \]  

(26)

It is clear that when \( \delta = R \) we obtain the standard equipartition law.

The heat capacity, using Eqs. (24) and (8) is

\[ C_{SM} = -\frac{8\pi GM^2 (1 + 4\pi GM^2)^\frac{R}{2}}{1 + 8\pi RGM^2 - 4\pi GM^2 \delta}. \]  

(27)

When we make \( R = \delta \) in (27) we recover the heat capacity, Eq. (9). It is important to comment here that Eq. (27) has already been obtained in ref. [16].

VI. CONCLUSIONS

In this letter we have shown that the algebraic form of the equipartition theorem may depend on the particular form of the entropy initially chosen to describe the black hole thermodynamics. Four cases have been studied in this paper, which are the usual black hole entropy, the Tsallis-Cirto entropy, the modified Rényi entropy and the Sharma-Mittal entropy. For each case, the Hawking temperature and the DF number are different and the result is that equipartition theorems corresponding to each model will also be logically different. Hence, we should pay attention to the fact that the algebraic form of an equipartition theorem is very sensible to an entropy formula initially chosen to depict the black hole thermodynamics. Therefore, important results, which are Eqs. (14), (21) and (26) describe an equipartition theorem for black holes for different entropies models. Finally, we would like to mention that it is possible to obtain a black hole thermally stable state in the Tsallis and Cirto entropy model for \( \delta < 1/2 \) in Eq. (15).

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