Necessary Optimality Conditions for Fractional Action-Like Integrals of Variational Calculus with Riemann-Liouville Derivatives of Order \((\alpha, \beta)\)∗

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Abstract

We derive Euler-Lagrange type equations for fractional action-like integrals of the calculus of variations which depend on the Riemann-Liouville derivatives of order \((\alpha, \beta)\), \(\alpha > 0, \beta > 0\), recently introduced by J. Cresson and S. Darses. Some interesting consequences are obtained and discussed.

Mathematics Subject Classification 2000: 49K05, 49S05, 70H33, 26A33.

Keywords: fractional action-like variational approach, fractional Euler-Lagrange equations, fractional constants of motion.

1 Introduction

Fractional derivatives and integrals play a leading role in the understanding of complex, classical or quantum, conservative or nonconservative, dynamical systems with holonomic as well as with nonholonomic constraints. Many applications of fractional differentiation and integration can be found in turbulence

∗This is a preprint of an article accepted for publication (28-02-2007) in Math. Methods Appl. Sci., Wiley, http://www.interscience.wiley.com
and fluid dynamics, chaotic dynamics, solid state physics, chemistry, stochastic
dynamical systems, plasma physics and controlled thermonuclear fusion, kinetic
theory, quantization, field theory, nonlinear control theory, image processing,
nonlinear biological systems, material sciences, rheological properties of rocks,
scaling phenomena, astrophysics, etc. (see e.g. [3, 19, 22]). Its origin goes
back more than three centuries, when in 1695 L'Hopital made some remarks
to Leibniz about the mathematical meaning of a fractional derivative of order
$1/2$. Leibniz’s response was “an apparent paradox, from which one day useful
consequences will be drawn”. In these words fractional calculus was born.
It was primarily a study reserved to the best minds in mathematics, including
J. Fourier, N. H. Abel, J. Liouville, B. Riemann, H. Holmgren, etc., who
contributed strongly to the fractional analysis program. A deep research was
carried out by Liouville from 1832 to 1837, where he succeeded in defining the
first fractional integration operator. Later on, further developments lead to the
construction of the well-known Riemann-Liouville fractional integral operator,
which plays nowadays a leading and important role in the analysis of complex
dynamical systems. There exist today many different forms of fractional integral
operators, ranging from divided-difference types to infinite-sum types, including
the Grunwald-Letnikov fractional derivative, the Caputo fractional derivative,
etc. One can say, however, that the Riemann-Liouville operator is still one of
the most frequently used for fractional integration.

The fractional calculus was considered a theoretical mathematical field with
no physical applications for more than three centuries. Last decades have shown
the contrary. Several fields of application of fractional differentiation and frac-
tional integration are already well established, some others have just started.
One of the areas that recently emerged within the fractional framework and
which is being subject to strong research is the Calculus of Variations (CoV)
and respective Euler-Lagrange type equations. In 1996 F. Riewe formulated
the CoV problem with fractional derivatives and obtained the respective Euler-
Lagrange equations (ELe’s), combining both conservative and non-conservative
cases [17]. In 2001 another approach was developed by M. Klimek by consider-
ing fractional problems of the CoV but with symmetric fractional derivatives.
Correspondent ELe’s were obtained, using both Lagrangian and Hamiltonian
formalisms [12]. In 2002 O. Agrawal extended Klimek variational problems by
considering right and left fractional derivatives in the Riemann-Liouville sense
[1]. In 2004 Agrawal ELe’s were used by D. Baleanu and T. Avkar to investigate
problems with Lagrangians which are linear in the velocities [2]. In all the above
mentioned studies, ELe’s depend on left and right fractional derivatives, even
when the problem depends only on one type of them. To avoid this situation, in
2005 M. Klimek studied problems depending on symmetric derivatives, proving
that in this case ELe’s include only the derivatives that appear in the formu-
lation of the problem [13]. One major problem with all the above mentioned
approaches is the presence of a non-local fractional differential operator with an
adjoint which is not its symmetric. Other difficulties arise due to a complicated
Leibniz rule and the non-existence of a fractional analogue to the chain rule.

The physical reasons for the appearance of fractional equations are, in gen-

eral, long-range dissipation and non-conservatism. For this reason, it seems of interest to study the fractional Hamiltonian of nonconservative dynamical systems. With this in mind, the first author has proposed in 2005 a novel approach, entitled Fractional Action-Like Variational Approach (FALVA), to model nonconservative dynamical systems [6, 7]. The Euler-Lagrange equations proved to be similar to the classical ones, with no fractional derivatives appearing, but with the presence of a fractional generalized external force acting on the system. The momentum, the Hamiltonian and Hamilton’s equations were proved to depend on the fractional order of integration and to vary inversely to time. In the present work we extend FALVA formalism and previous results by considering Riemann-Liouville fractional derivatives of order \((\alpha, \beta)\). Respective Euler-Lagrange equations are obtained using a Taylor-Riemann series expansion; the concept of fractional constant of motion is introduced and illustrated.

2 Fractional Euler-Lagrange Equations

We begin by recalling the standard definitions of Riemann-Liouville fractional integrals and fractional derivatives.

**Definition 2.1** (Left Riemann-Liouville fractional integral). If \(f(t) \in C(a, b)\), and \(\alpha > 0\), then the left Riemann-Liouville fractional integral of order \(\alpha\) is defined and denoted by

\[
I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} f(\tau) (t - \tau)^{\alpha - 1} d\tau .
\]

**Definition 2.2** (Right Riemann-Liouville fractional integral). If \(f(t) \in C(a, b)\), and \(\alpha > 0\), then the right Riemann-Liouville fractional integral of order \(\alpha\) is defined and denoted by

\[
I_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{b} f(\tau) (\tau - t)^{\alpha - 1} d\tau .
\]

**Definition 2.3** (Left and Right Riemann-Liouville fractional derivatives). If \(f(t) \in C^1(a, b)\), and \(\alpha > 0\), then the left and right Riemann-Liouville fractional derivatives of order \(\alpha\), denoted respectively by \(D_{a+}^{\alpha}\) and \(D_{b-}^{\alpha}\), are defined by

\[
D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_{a}^{t} f(\tau) (t - \tau)^{n-\alpha-1} d\tau ,
\]

\[
D_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \left( -\frac{d}{dt} \right)^n \int_{t}^{b} f(\tau) (\tau - t)^{n-\alpha-1} d\tau ,
\]

where \(n\) is such that \(n - 1 \leq \alpha < n\).
The classical integrals, and derivatives of order one, are obtained setting $\alpha = 1$. For an introduction to the fractional calculus we refer the reader to [15, 16, 18].

In this work we consider the fractional derivative operator of order $(\alpha, \beta)$ recently introduced by J. Cresson and S. Darses [5] (see also [4]):

**Definition 2.4.** Given $a, b \in \mathbb{R}$, $a < b$, and $\gamma \in \mathbb{C}$, the fractional derivative operator of order $(\alpha, \beta)$, $\alpha, \beta > 0$, is defined by

$$D^{\alpha, \beta}_{\gamma} = \frac{1}{2} \left[ D^{\alpha}_{a+} - D^{\beta}_{b-} \right] + \frac{i \gamma}{2} \left[ D^{\alpha}_{a+} + D^{\beta}_{b-} \right]$$

where $i = \sqrt{-1}$.

**Remark 2.1.** The operator $D^{\alpha, \beta}_{\gamma}$ extends the classical Riemann-Liouville fractional derivatives: for $\gamma = -i$ we have $D^{\alpha, \beta}_{\gamma} = D^{\alpha}_{a+}$; for $\gamma = i$ we obtain $D^{\alpha, \beta}_{\gamma} = -D^{\beta}_{b-}$.

The following Lemma is useful: we make use of (1) to prove the fractional Euler-Lagrange equations associated to our problem (cf. proof of Theorem 2.2).

**Lemma 2.1** ([4]). If $f, g \in C^1$ with $f(a) = f(b) = 0$ or $g(a) = g(b) = 0$, then

$$\int_a^b D^{\alpha, \beta}_{\gamma} f(t) g(t) \, dt = - \int_a^b f(t) D^{\beta, \alpha}_{\gamma} g(t) \, dt.$$  (1)

We are now in conditions to formulate the $(\alpha, \beta)$ fractional action-like variational problem.

**Definition 2.5.** Consider a smooth manifold $M$ and let $L$ be a smooth Lagrangian function $L : \mathbb{C}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $d \geq 1$. For any piecewise smooth path $q : [a, b] \to M$ satisfying fixed boundary conditions $q(a) = q_a$ and $q(b) = q_b$, we define the following fractional action integral:

$$S^{\alpha, \beta}_{\gamma,(a,b)}[q] = \frac{1}{\Gamma(\alpha)} \int_a^b L \left( D^{\alpha, \beta}_{\gamma} q(\tau), q(\tau), \tau \right) (t - \tau)^{\alpha-1} \, d\tau.$$  (2)

The $(\alpha, \beta)$ fractional action-like variational problem consists in finding an admissible $q(\cdot)$ which minimizes (2).

**Remark 2.2.** We consider two time variables: the intrinsic time $\tau$ and the observer time $t$. This multi-time characteristic is important in applications and is the main ingredient of the theory being developed by C. Udriste [21].

**Remark 2.3.** The fractional action integral (2) is a generalization of the FALVA action integral of [6, 7, 8]. Theorem 2.2 gives a necessary optimality condition for $q(\cdot)$ to be a solution of the $(\alpha, \beta)$ fractional action-like variational problem.
Theorem 2.2 \(((\alpha, \beta) \text{ fractional Euler-Lagrange equations})\). If \(q : [a, b] \rightarrow M\) is a minimizer of the \((\alpha, \beta) \text{ fractional action-like variational problem (cf. Definition 2.5)}\), then

\[
\frac{\partial L}{\partial q} \left(D_{\gamma,\alpha}^{\beta,\beta} q(\tau), q(\tau), \tau\right) - D_{\gamma,\alpha}^{\beta,\beta} \frac{\partial L}{\partial q} \left(D_{\gamma,\alpha}^{\beta,\beta} q(\tau), q(\tau), \tau\right)
= \frac{1 - \alpha}{t - \tau} \frac{\partial L}{\partial q} \left(D_{\gamma,\alpha}^{\beta,\beta} q(\tau), q(\tau), \tau\right) \quad (3)
\]

where \(D_{\gamma,\alpha}^{\beta,\beta}\) represents the fractional derivative with respect to time \(\tau\).

Remark 2.4. Theorem 2.2 is an extension of the Euler-Lagrange equations derived in FALVA: one just needs to choose \(\beta = 1\) in (3) to obtain the Euler-Lagrange equations of [6, 7].

Proof. We perform a small perturbation of the generalized coordinates as \(q \rightarrow q + \varepsilon h, \varepsilon \ll 1\). As a result, \(D_{\gamma,\alpha}^{\beta,\beta}(q + \varepsilon h) = D_{\gamma,\alpha}^{\beta,\beta}q + \varepsilon D_{\gamma,\alpha}^{\beta,\beta}h\) and

\[
S_{\gamma,(a,b)}^{\alpha,\beta}[q + \varepsilon h] = \frac{1}{\Gamma(\alpha)} \int_a^b \left(L \left(D_{\gamma,\alpha}^{\beta,\beta}q + \varepsilon D_{\gamma,\alpha}^{\beta,\beta}h, q + \varepsilon h, \tau\right) (t - \tau)^{\alpha - 1}\right) \, d\tau
\]

which, doing a Taylor expansion of \(L \left(D_{\gamma,\alpha}^{\beta,\beta}q + \varepsilon D_{\gamma,\alpha}^{\beta,\beta}h, q + \varepsilon h, \tau\right)\) in \(\varepsilon\) around zero, and integrating by parts, imply that

\[
S_{\gamma,(a,b)}^{\alpha,\beta}[q + \varepsilon h] = S_{\gamma,(a,b)}^{\alpha,\beta}[q]
- \frac{\varepsilon}{\Gamma(\alpha)} \int_a^b \left[ -\frac{\partial L}{\partial q} \left(D_{\gamma,\alpha}^{\beta,\beta}q(\tau), q(\tau), \tau\right) (t - \tau)^{\alpha - 1} h(\tau)\right.
+ (t - \tau)^{\alpha - 1} \frac{\partial L}{\partial q} \left(D_{\gamma,\alpha}^{\beta,\beta}q(\tau), q(\tau), \tau\right) D_{\gamma,\alpha}^{\beta,\beta}h(\tau)\]
+ (t - \tau)^{\alpha - 1} \frac{\partial L}{\partial q} \left(D_{\gamma,\alpha}^{\beta,\beta}q(\tau), q(\tau), \tau\right) D_{\gamma,\alpha}^{\beta,\beta}h(\tau) \bigg] \, d\tau + O(\varepsilon) .
\]

Making use of Lemma 2.1 and the least action principle we arrive to (3).

Definition 2.6. A path \(q : [a, b] \rightarrow M\) satisfying equation (3) is said to be a fractional extremal associated to the Lagrangian \(L\).

Definition 2.7. The right hand side of (3),

\[
F_{\gamma,\tau}^{\alpha,\beta} = \frac{\alpha - 1}{T} \frac{\partial L}{\partial q} \left(D_{\gamma,\alpha}^{\beta,\beta}q(\tau), q(\tau), \tau\right) , \quad (4)
\]

\(T = \tau - t\), defines the fractional decaying friction force.

Remark 2.5. The fractional decaying friction force satisfy the asymptotic property \(\lim_{T \to \infty} F_{\gamma,\tau}^{\alpha,\beta} = 0\).
3 The Fractional Hamiltonian Formalism

We now consider a more general class of fractional optimal control problems:

\[ S_{\alpha,\beta}^{\alpha,\beta}(q, u) = \frac{1}{\Gamma(\alpha)} \int_{a}^{b} L(u(\tau), q(\tau), (t-\tau)^{\alpha-1} d\tau \rightarrow \min, \]

\[ D_{\gamma}^{\alpha,\beta} q(\tau) = \varphi(u(\tau), q(\tau), \tau), \]

\[ a, b \in \mathbb{R}, a < b. \]

In the particular case where \( \varphi(u, q, \tau) = u \), it reduces to (2). To obtain a necessary optimality condition for problem (5), we introduce the augmented action integral (cf. e.g. [9]):

\[ S_{\gamma, (a,b)}^{\alpha,\beta} [q, u, p^{\alpha,\beta}] = \frac{1}{\Gamma(\alpha)} \int_{a}^{b} \left[ H^{\alpha,\beta}(u(\tau), q(\tau), p^{\alpha,\beta}(\tau), \tau) \right. \]

\[ - p^{\alpha,\beta}(\tau) D_{\gamma}^{\alpha,\beta} q(\tau) d\tau \]

where \( p^{\alpha,\beta} \) is the fractional Lagrange multiplier and the fractional Hamiltonian \( H^{\alpha,\beta} \) is defined by

\[ H^{\alpha,\beta}(u, q, p^{\alpha,\beta}, \tau) = L(u, q, \tau)(t-\tau)^{\alpha-1} + p^{\alpha,\beta}\varphi(u, q, \tau). \]

**Theorem 3.1.** If \((q, u)\) is a minimizer of problem (5), then there exists a covector function \( p^{\alpha,\beta} \) such that the following conditions hold:

- the fractional Hamiltonian system

\[ \begin{align*}
D_{\gamma}^{\alpha,\beta} q(\tau) &= \frac{\partial H^{\alpha,\beta}}{\partial p^{\alpha,\beta}} (u(\tau), q(\tau), p^{\alpha,\beta}(\tau), \tau), \\
D_{-\gamma}^{\beta,\alpha} q^{\alpha,\beta}(\tau) &= - \frac{\partial H^{\alpha,\beta}}{\partial q} (u(\tau), q(\tau), p^{\alpha,\beta}(\tau), \tau); \tag{7}
\end{align*} \]

- the fractional stationary condition

\[ \frac{\partial H^{\alpha,\beta}}{\partial u} (u(\tau), q(\tau), p^{\alpha,\beta}(\tau), \tau) = 0. \tag{8} \]

**Remark 3.1.** For the fractional problem of the calculus of variations (2), one has

\[ H^{\alpha,\beta}(u, q, p^{\alpha,\beta}, \tau) = L(u, q, \tau)(t-\tau)^{\alpha-1} + p^{\alpha,\beta}\varphi(u, q, \tau). \]

The fractional stationary condition (5) reduces to

\[ p^{\alpha,\beta}(\tau) = - \frac{\partial L}{\partial u}(u(\tau), q(\tau), \tau)(t-\tau)^{\alpha-1}; \tag{9} \]

the first equation in (7) to

\[ u(\tau) = D_{\gamma}^{\alpha,\beta} q(\tau); \tag{10} \]
while the second equation in (7) takes the form

\[ D_{\beta,\alpha}^{\gamma,\tau} p^{\alpha,\beta} (\tau) = - \frac{\partial L}{\partial q}(u(\tau), q(\tau), \tau) (t-\tau)^{\alpha-1}. \] (11)

Using (9) and (10) in (11) we arrive to:

\[ D_{\beta,\alpha}^{\gamma,\tau} \left[ \frac{\partial L}{\partial u}(D_{\gamma,\tau}^{\alpha,\beta} q(\tau), q(\tau), \tau) (t-\tau)^{\alpha-1} \right] = \frac{\partial L}{\partial q}(D_{\gamma,\tau}^{\alpha,\beta} q(\tau), q(\tau), \tau) (t-\tau)^{\alpha-1}. \] (12)

Simple calculations show that (12) is equivalent to (3), that is, Theorem 3.1 is a generalization of Theorem 2.2 to the fractional optimal control problem (5).

**Remark 3.2.** Let us define the Poisson bracket of two dynamical quantities \( f \) and \( g \), with respect to coordinates \( q \) and fractional-momenta \( p^{\alpha,\beta} \), by

\[ \{ f, g \} = \frac{\partial f}{\partial p^{\alpha,\beta}} \cdot \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \cdot \frac{\partial g}{\partial p^{\alpha,\beta}}. \]

The fractional Hamiltonian system (7) can be written in the following form:

\[ \begin{cases} D_{\alpha,\beta}^{\gamma,\tau} q(\tau) = \{ \mathcal{H}^{\alpha,\beta}, q \}, \\ D_{\beta,\alpha}^{\gamma,\tau} p^{\alpha,\beta} (\tau) = \{ \mathcal{H}^{\alpha,\beta}, p^{\alpha,\beta} \}. \end{cases} \]

**Proof.** Theorem 3.1 is proved applying the \((\alpha, \beta)\) fractional Euler-Lagrange equations to the augmented action integral (6), i.e. applying (3) to

\[ S_{\gamma, (a,b)}^{\alpha,\beta} [q, u, p^{\alpha,\beta}] = \frac{1}{\Gamma (\alpha)} \int_{a}^{b} \left[ \frac{\mathcal{H}^{\alpha,\beta} - p^{\alpha,\beta}(\tau) D_{\gamma,\tau}^{\alpha,\beta} q(\tau)}{(t-\tau)^{\alpha-1}} \right] (t-\tau)^{\alpha-1} d\tau, \]

where \( \mathcal{H}^{\alpha,\beta} = \mathcal{H}^{\alpha,\beta} (u(\tau), q(\tau), p^{\alpha,\beta}(\tau), \tau) \). The Euler-Lagrange equation with respect to \( q \) gives the second equation of the fractional Hamiltonian system (7); the Euler-Lagrange equation with respect to \( u \) gives the fractional stationary condition (8); and, finally, the Euler-Lagrange equation with respect to \( p^{\alpha,\beta} \) gives the first equation of (7). \( \square \)

In classical mechanics, constants of motion are derived from the first integrals of the Euler-Lagrange equations. In the fractional case it is necessary to change the definition of constant of motion in a proper way \[9, 10, 11\]. Here, in order to account the presence of the fractional decaying friction force \( F_{\gamma,\tau}^{\alpha,\beta} \) (4), we propose the following definition of fractional constant of motion.

**Definition 3.1.** We say that a function \( C \) of \( \tau \) is a fractional constant of motion if and only if \( D_{\gamma,\tau}^{\alpha,\beta} C = 0 \).
Corollary 3.2. If $L$ and $\varphi$ do not depend on $q$, then it follows from the fractional Hamiltonian system (cf. Theorem 3.1) that $D^{\beta}_{-\gamma,\tau} p^{a,\beta} (\tau) = 0$, i.e., $p^{a,\beta}$ is a fractional constant of motion.

Definition 3.2 (fractional momentum of order $(\alpha, \beta)$). Associated with an $(\alpha, \beta)$ fractional action-like variational problem (2) we define the fractional momentum by (9)-(10):

$$p^{\alpha,\beta} (\tau) = -\frac{\partial L}{\partial q} (D^{\alpha,\beta} q (\tau), q (\tau), \tau) (t - \tau)^{\alpha-1}.$$ 

Corollary 3.3. For the fractional problem of the calculus of variations (2) with $\frac{\partial L}{\partial q} = 0$, the fractional momentum of order $(\alpha, \beta)$ is a fractional constant of motion.

4 Conclusions

We generalize previous results of the fractional variational calculus by using the action-like variational approach together with Riemann-Liouville fractional derivatives of order $(\alpha, \beta)$. We claim that the $(\alpha, \beta)$ fractional action-like variational problem here introduced offers a better mathematical model for weak dissipative and nonconservative dynamical systems. Both Lagrangian and Hamiltonian approaches are considered. We derive the $(\alpha, \beta)$ fractional Euler-Lagrange, Hamilton and Poisson equations. Standard Noetherian constants of motion are violated due to the presence of a fractional decaying friction term. To solve the problem we introduce a new notion of fractional constant of motion. The class of fractional Hamiltonian systems thus obtained has two parameters and is wider than the standard class of Hamiltonian dynamical systems. Our approach is different from the ones found in the literature: different from S. Muslih and D. Baleanu approach [14], where no fractional integral action is used and no decaying friction term considered; different from the recent approach [20] of A. Stanislavsky, where the generalization of the classical mechanics with fractional derivatives is based on the time-clock randomization of momenta and the coordinates are taken from the conventional phase space.

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