Functional equations of Selberg and Ruelle zeta functions for non-unitary twists

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Abstract

We consider the dynamical zeta functions of Selberg and Ruelle associated with the geodesic flow on a compact odd-dimensional hyperbolic manifold. These dynamical zeta functions are defined for a complex variable \( s \) in some right-half plane of \( \mathbb{C} \). In [Spi18], it was proved that they admit a meromorphic continuation to the whole complex plane. In this paper, we establish functional equations for them, relating their values at \( s \) with those at \( -s \). We prove also a determinant representation of the zeta functions, using the regularized determinants of certain twisted differential operators.

Keywords — Dynamical zeta functions Functional equations Eta invariant Determinant formula.

1 Introduction

We consider the twisted Selberg and Ruelle zeta functions of compact hyperbolic manifolds \( X \) of odd dimension \( d \). Let \( G, K \) be either \( G = \text{SO}^0(d, 1), K = \text{SO}(d) \) or \( G = \text{Spin}(d, 1), K = \text{Spin}(d) \). Then, \( K \) is a maximal compact subgroup of \( G \). Let \( \tilde{X} := G/K \). \( \tilde{X} \) can be equipped with a \( G \)-invariant metric, which is unique up to scaling and is of constant negative curvature. If we normalize this metric such that it has constant negative curvature \(-1\), then \( \tilde{X} \) equipped with this metric is isometric to the \( d \)-dimensional hyperbolic space \( \mathbb{H}^d \). Let \( \Gamma \) be a discrete, torsion-free subgroup of \( G \) such that \( \Gamma \setminus G \) is compact. This means that \( \Gamma \) has no elements of finite order. \( \Gamma \) acts by isometries on \( \tilde{X} \) and \( X = \Gamma \setminus \tilde{X} \) is a compact oriented hyperbolic manifold of dimension \( d \). This is a case of a locally symmetric space of non-compact type of real rank 1. This means that in the Iwasawa decomposition \( G = KAN \) of \( G \), \( A \) is a multiplicative torus of dimension \( 1 \), i.e., \( A \cong \mathbb{R}^+ \).

For a given \( \gamma \in \Gamma \), we denote by \([\gamma]\) the \( \Gamma \)-conjugacy class of \( \gamma \). If \( \gamma \neq e \), then there is a unique closed geodesic \( c_\gamma \) associated with \([\gamma]\). We denote by \( l(\gamma) \) the length of \( c_\gamma \). The conjugacy class \([\gamma]\) is called prime if there exist no \( k \in \mathbb{N} \) with \( k > 1 \) and \( \gamma_0 \in \Gamma \) such that \( \gamma = \gamma_0^k \). The prime closed geodesics correspond to the prime conjugacy classes. These geodesics trace out their image exactly once. Let \( M := \text{Cent}_K(A) \) be the centralizer of \( A \) in \( K \). Since \( \Gamma \) is a cocompact subgroup of \( G \), every element \( \gamma \in \Gamma \setminus \{e\} \) is hyperbolic. By [Wal76, Lemma 6.5], there exist a \( g \in G \), a \( m_\gamma \in M \), and an \( a_\gamma \in A \), such that \( g^{-1} \gamma g = m_\gamma a_\gamma \).
The element $m_\gamma$ is determined up to conjugacy in $M$, and the element $a_\gamma$ is uniquely determined by $\gamma$.

Let $\mathfrak{g}, \mathfrak{a}$ be the Lie algebras of $G$ and $A$, respectively. Let $\Delta^+(\mathfrak{g}, \mathfrak{a})$ be the set of positive roots of $(\mathfrak{g}, \mathfrak{a})$. Then, $\Delta^+(\mathfrak{g}, \mathfrak{a})$ consists of a single root $\alpha$. Let $\mathfrak{g}_0$ be the corresponding root space. Let $\mathfrak{p}$ be the negative root space of $(\mathfrak{g}, \mathfrak{a})$. Let $S^k(\text{Ad}(m_\gamma a_\gamma))$ be the $k$-th symmetric power of the adjoint map $\text{Ad}(m_\gamma a_\gamma)$ restricted to $\mathfrak{p}$ and $\rho$ be defined as $\rho := \frac{1}{2} \dim(\mathfrak{g}_0)\alpha$. We define the twisted zeta functions associated with unitary irreducible representations $\sigma$ of $M$ and finite-dimensional representations $\chi$ of $\Gamma$. The twisted Selberg zeta function $Z(s; \sigma, \chi)$ is defined for $s \in \mathbb{C}$ by the infinite product

$$Z(s; \sigma, \chi) := \prod_{[\gamma] \neq e, \text{ prime}} \prod_{k=0}^{\infty} \det \left( \text{Id} - (\chi(\gamma) \otimes \sigma(\gamma) \otimes S^k(\text{Ad}(m_\gamma a_\gamma))_{|\mathfrak{p}}) \right)e^{-(s+|\rho|)l(\gamma)}.$$  

$Z(s; \sigma, \chi)$ converges absolutely and uniformly on compact subsets of some half-plane of $\mathbb{C}$ ([Spi18, Proposition 3.4]). The twisted Ruelle zeta function $R(s; \sigma, \chi)$ is defined for $s \in \mathbb{C}$ by the infinite product

$$R(s; \sigma, \chi) := \prod_{[\gamma] \neq e, \text{ prime}} \det(\text{Id} - (\chi(\gamma) \otimes \sigma(\gamma))e^{-s l(\gamma)}).$$  

$R(s; \sigma, \chi)$ converges absolutely and uniformly on compact subsets of some half-plane of $\mathbb{C}$ ([Spi18, Proposition 3.5]).

The connection of the Ruelle zeta function to spectral invariants such as the analytic torsion has been studied for example by Fried ([Fri86]), Bunke and Olbrich ([BO95]), Wotzke ([Wot08]), Müller ([Mü12]) for compact hyperbolic manifolds, under certain assumptions on the representation of the fundamental group of the manifold. For the case of a hyperbolic manifold of finite volume, we refer the reader to the work of Park ([Par09]) and Pfaff ([Pfa14], [Pfa15]). A far more advanced study of the dynamical zeta functions of locally symmetric manifolds of higher rank is due to Moscovici and Stanton in [MS91, Deitmar in [Dei95], Schen in [She18] and Moscovici, Stanton and Prahm in [MSP18]. Fedosova and Pohl in [FP20], studied the Selberg zeta function on hyperbolic surfaces for geometrically finite, non-elementary Fuchsian groups $\Gamma$ and finite-dimensional representations with non-expanding cusp monodromy. They use transfer operators techniques to prove the meromorphic continuation of the zeta function.

For hyperbolic manifolds of odd dimension, Fried in [Fri86] considered an orthogonal representation $g: \Gamma \to O(m)$ of $\Gamma$. Using the Selberg trace formula for the heat operator $e^{-t \Delta_j}$, where $\Delta_j$ is the Hodge Laplacian on $j$-forms on $X$ with values in the flat vector bundle $E_\phi$ associated with $g$, he proved the meromorphic continuation of the zeta functions to the whole complex plane $\mathbb{C}$, as well as functional equations for the Selberg zeta function ([Fri86, p. 531-532]). Under the assumption that $g$ is acyclic, i.e., the cohomology with coefficients in the local system defined by $\rho$ vanish for all $j$, the Ruelle zeta function, defined for $\text{Re}(s) > d - 1$ by

$$R(s; g) := \prod_{[\gamma] \neq e, \text{ prime}} \det(\text{Id} - g(\gamma)e^{-s l(\gamma)}),$$
admits a meromorphic extension to \( \mathbb{C} \). In addition, it is regular at \( s = 0 \) and
\[
|R(0; \varrho)| = T_X(\varrho)^2,
\]
where \( T_X(\varrho) \) is the Ray-Singer analytic torsion defined as in \([RS71]\).

For unitary representations \( \chi \), the dynamical zeta functions have been studied by Bunke and Olbrich in \([BO95]\) for all locally symmetric spaces of real rank 1. They proved that the zeta functions admit a meromorphic continuation to the whole complex plane and they satisfy functional equations.

Wotzke in \([Wot08]\) extended this result for representations of \( \Gamma \), which are not necessary unitary, but very special ones. In particular, he considered a compact odd-dimensional hyperbolic manifold and a finite-dimensional irreducible representation \( \tau : G \to \text{GL}(V) \) of \( G \), such that \( \tau \neq \tau_\vartheta \), where \( \tau_\vartheta = \tau \circ \theta \) and \( \theta \) denotes the Cartan involution of \( G \). Under these assumptions, he proved that the Ruelle zeta function admits a meromorphic continuation to the whole complex plane. In addition, it is regular at \( s = 0 \) and
\[
|R(0; \tau)| = T_X(\tau)^2.
\]

Wotzke’s method is based on the fact that if one considers the restrictions \( \tau|_K \) and \( \tau|_\Gamma \) of \( \tau \) to \( K \) and \( \Gamma \), respectively, there is an isomorphism between the locally homogeneous vector bundle \( E_\tau \) over \( X \) associated with \( \tau|_K \) and the flat vector bundle \( E_{\text{flat}} \) over \( X \) associated with \( \tau|_\Gamma \). By \([MM63, \text{Lemma 3.1}]\), a Hermitian fiber metric in \( E_\tau \) descends to a fiber metric in \( E_{\text{flat}} \). Therefore, all tools from harmonic analysis on locally symmetric spaces are available.

In our case the representation of \( \Gamma \) is arbitrary. Hence, these tools are no longer applicable. In \([Spi18]\), it is proved that the twisted Selberg and Ruelle zeta functions associated with an arbitrary finite-dimensional representation of \( \Gamma \) admit a meromorphic continuation to the whole complex plane. In the present paper, we extend the results of \([BO95]\) to the case of the non-unitary twist \( \chi \). We prove functional equations for the twisted dynamical zeta functions, relating their values at \( s \) with those at \(-s\). Moreover, we prove a determinant formula, which expresses the zeta function in terms of regularized determinants of certain twisted Laplace-type operators. We state here our main results. In the functional equation (1.1) below, \( P_{\sigma}(s) \) denotes the Plancherel polynomial associated with \( \sigma \in \hat{M} \) (see Section 2).

**Theorem A.** The Selberg zeta function \( Z(s; \sigma, \chi) \) satisfies the following functional equation
\[
\frac{Z(s; \sigma, \chi)}{Z(-s; \sigma, \chi)} = \exp\left( -4\pi \dim(V_{\chi}) \operatorname{Vol}(X) \int_0^s P_{\sigma}(r) dr \right). \tag{1.1}
\]

Let \( M' := \text{Norm}_K(A) \) be the normalizer of \( A \) in \( K \). We define the restricted Weyl group by the quotient \( W_A := M'/M \). Then, \( W_A \) has order 2. \( W_A \) acts on \( \hat{M} \) by \( (w\sigma)(m) := \sigma(m_{w}^{-1}mm_{w}) \), where \( w \) is a non-trivial element in \( W_A \), \( m_{w} \) is a representative of \( w \) in \( M' \) and \( m \in M \). In this paper, we distinguish always the case of \( \sigma \in \hat{M} \) being non-Weyl invariant, i.e., \( w\sigma \neq \sigma \). In that case, we define the super Ruelle zeta function by
\[
R^s(s; \sigma, \chi) := \frac{R(s; \sigma, \chi)}{R(s; w\sigma, \chi)}.
\]
The twisted Dirac operator associated with an arbitrary representation of $\Gamma$, acting on smooth sections of twisted vector bundles is defined in [Spi18, Section 6] (see also Section 4 in the present paper). As in the case of the twisted Bochner-Laplace operator ([Spi18]), this operator acts locally as the identity operator on smooth sections of the flat vector bundle. In Section 4, we define the eta function of the twisted Dirac operator and further its eta invariant. In addition, we prove a formula (Lemma 4.6), which generalizes the usual integral representation of the eta function to the case of non-unitary twists. In the right-hand side of the functional equation (1.2) below, $\tilde{D}_\chi^\natural(\sigma)$ is a twisted Dirac operator (see p. 30) and the term $\eta(\tilde{D}_\chi^\natural(\sigma))$ is defined in terms of the eta function (Definition 4.5, Section 4) at zero of twisted Dirac operators (for more details see p. 30-31).

**Theorem B.** The super Ruelle zeta function associated with a non-Weyl invariant representation $\sigma \in \hat{M}$ satisfies the functional equation

$$R^s(s; \sigma, \chi)R^s(-s; \sigma, \chi) = e^{2\pi i \eta(\tilde{D}_\chi^\natural(\sigma))}.$$  

(1.2)

The determinant formula for the twisted Ruelle zeta function (Proposition 7.9) is a direct consequence of the determinant formula for the twisted Selberg zeta function (Theorem 7.8). We denote by $\sigma_p$ the standard representation of $M$ in $\Lambda^p \mathbb{R}^{d-1} \otimes \mathbb{C}$. Let $|\rho|$ be the norm of $\rho$, induced by the normalized Killing form (see (2.1), Section 2). We consider the operators $A_\chi^\natural(\sigma_p \otimes \sigma) + (s + |\rho| - p)^2$, acting on smooth sections of graded twisted vector bundles (see [Spi18, p. 174–175], Section 4 and Section 7 in the present paper). These operators are elliptic, differential operators of second order, which are not self adjoint. Since the vector bundles we consider are $\mathbb{Z}_2$-graded, we consider graded regularized determinants of these operators (see Section 7).

**Proposition A.** The Ruelle zeta function has the representation

$$R(s; \sigma, \chi) = \prod_{p=0}^{d-1} \det_{gr}(A_\chi^\natural(\sigma_p \otimes \sigma) + (s + |\rho| - p)^2)^{(-1)^p} \exp\left( (-1)^{\frac{d+1}{2}} \pi (d + 1) \dim(V_\sigma) \dim(V_\chi) \frac{\text{Vol}(X)}{\text{Vol}(S^d)} s \right).$$

This paper is organized as follows. In Section 2, we introduce the basic setup. In Section 3, we define the twisted dynamical zeta functions. Section 4 is devoted to the study of the twisted Dirac operator and the definition of the associated eta function. The functional equations for the Selberg zeta function are derived in Section 5. In Section 6, we prove the functional equations for the Ruelle and super Ruelle zeta function. Finally, in Section 7 we prove the determinant formula for the Ruelle zeta function.

## 2 Preliminaries

### 2.1 Representation theory of Lie groups

We introduce here our algebraic setting and fix some notation. For further details we refer the reader to [Spi18 Section 2]. Let $n \in \mathbb{N}_{>0}$. Let $d = 2n + 1$ be
an odd integer. We consider the universal coverings \( G = \text{Spin}(d, 1) \) and \( K = \text{Spin}(d) \) of \( \text{SO}(d) \), respectively. We set \( \tilde{X} := G/K \). Let \( \mathfrak{g} \) be the Lie algebra of \( G \) and \( K \), respectively. We consider the following complexification \( \mathfrak{g}_C := \mathfrak{g} \oplus i\mathfrak{g} \). Let \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) be the Cartan decomposition of \( \mathfrak{g} \) with respect to the Cartan involution of \( G \). There exists a canonical isomorphism \( T_{eK}\tilde{X} \cong \mathfrak{p} \).

Let \( \mathfrak{a} \) be a maximal abelian subalgebra of \( \mathfrak{p} \). Let \( A \) be the subgroup of \( G \) with Lie algebra \( \mathfrak{a} \). Let \( M := \text{Centr}_K(A) \) be the centralizer of \( A \) in \( K \). Then, \( M = \text{Spin}(d-1) \). Let \( \mathfrak{m} \) be the Lie algebra of \( M \) and \( \mathfrak{m}_C := \mathfrak{m} \oplus \mathfrak{i} \mathfrak{m} \) the complexification of \( \mathfrak{m} \). Let \( \mathfrak{b}, \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{m} \) and \( \mathfrak{g} \), respectively. Let \( B(X,Y) \) be the Killing form on \( \mathfrak{g} \times \mathfrak{g} \) defined by \( B(X,Y) = \text{Tr}(\text{ad}(X) \circ \text{ad}(Y)) \). It is a symmetric bilinear form. We consider the symmetric bilinear form

\[
(Y_1, Y_2) := \frac{1}{2(d-1)}B(Y_1, Y_2), \quad Y_1, Y_2 \in \mathfrak{g}.
\] (2.1)

The restriction of \((\cdot, \cdot)\) to \( \mathfrak{p} \) defines an inner product on \( \mathfrak{p} \) and therefore induces a \( G \)-invariant Riemannian metric on \( \tilde{X} \), which has constant curvature \(-1\). \( \tilde{X} \) with this metric is isometric to the \( d \)-dimensional real hyperbolic space \( \mathbb{H}^d \). Let \( \Gamma_1 \) be a torsion free cocompact discrete subgroup of \( \text{SO}(d,1) \). We assume that \( \Gamma_1 \) can be lifted to a subgroup \( \Gamma \) of \( G \). Then, \( X := \Gamma \backslash \tilde{X} \) is a compact hyperbolic manifold of odd dimension \( d \).

Let \( G = KAN \) be the standard Iwasawa decomposition of \( G \). Let \( \Delta^+(\mathfrak{g}, \mathfrak{a}) \) be the set of positive roots of \( (\mathfrak{g}, \mathfrak{a}) \). Then, \( \Delta^+(\mathfrak{g}, \mathfrak{a}) \) consists of a single root \( \alpha \). Let \( \mathfrak{g}_\alpha \) be the corresponding root space. We define

\[
\rho := \frac{1}{2} \dim(\mathfrak{g}_\alpha) \alpha.
\] (2.2)

Let \( \Delta^+(\mathfrak{m}_C, \mathfrak{b}) \) be the set of the positive roots of the system \((\mathfrak{m}_C, \mathfrak{b})\). We define

\[
\rho_m := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{m}_C, \mathfrak{b})} \alpha.
\] (2.3)

Let \( \Delta^+(\mathfrak{g}_C, \mathfrak{h}) \) be the set of the positive roots of the system \((\mathfrak{g}_C, \mathfrak{h})\). We define

\[
\rho_\mathfrak{g} := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}_C, \mathfrak{h})} \alpha.
\] (2.4)

Let \( \nu_\sigma \) be the highest weight of \( \sigma \). We have

\[
\nu_\sigma = (\nu_1, \ldots, \nu_{n-1}, \nu_n),
\] (2.5)

where \( \nu_1 \geq \ldots \geq \nu_{n-1} \geq |\nu_n| \) and \( \nu_i, i = 1, \ldots, n \) are all half integers ([BO95, p. 20]).

Let \( W_A \) be the restricted Weyl group defined by \( W_A := M'/M \), where \( M' = \text{Norm}_K(A) \) is the normalizer of \( A \) in \( K \). \( W_A \) has order 2. Let \( \widetilde{M} \) be the set of the equivalent classes of irreducible unitary representations of \( M \). For a non-trivial element \( w \) of \( W_A \), the action of \( W_A \) on \( \widetilde{M} \) is defined by

\[
(w\sigma)(m) := \sigma(m_w^{-1}mm_w), \quad m \in M, \sigma \in \widetilde{M},
\]

where \( m_w \) is a representative of \( w \) in \( M' \). We distinguish the following two cases for this action:
• case (a): \( \sigma \) is invariant under the action of the restricted Weyl group \( W_A \);
• case (b): \( \sigma \) is not invariant under the action of the restricted Weyl group \( W_A \).

2.2 The principal series representation

Let \( P = MAN \) be the standard parabolic subgroup of \( G \). Let \( \rho \) be defined by (2.2). Let \(|\rho|\) be its norm, induced by the symmetric bilinear form, defined by (2.1). For \((\sigma, V_\sigma) \in \hat{M}\), we define the space \( H_{\sigma, \lambda} \) of continuous functions on \( G \) by

\[ H_{\sigma, \lambda} := \{ f \in C(G, V_\sigma) : f(gman) = e^{-i(\lambda + |\rho|) \log a - (\mu_\sigma + \nu_\sigma + \rho) \cdot (m \cdot g)} , \forall g \in G, \forall man \in P \}, \]

where \( \lambda \in \mathbb{C} \), with norm

\[ \|f\|_c = \int_K \|f(k)\|^2 dk. \quad (2.6) \]

The principal series representation are defined by

\[ \pi_{\sigma, \lambda} := \text{Ind}_{G}^{P}(\sigma \otimes e^{i\lambda} \otimes \text{Id}), \quad (2.7) \]

with representation space the Hilbert space, obtained by completion of \( \mathcal{H}_{\sigma, \lambda} \) with respect to the norm \( \|\cdot\|_c \) in (2.6). For \( f \in \mathcal{H}_{\sigma, \lambda} \), the action of \( G \) on \( f \) is given by \( \pi_{\sigma, \lambda}(g)f(g') = f(g^{-1}g') \). Let \( a_C^* \) be the space of the linear functionals on \( a_C \). In the definition of the space \( \mathcal{H}_{\sigma, \lambda} \), \( \lambda \) is a complex number. \( a_C^* \) is identified with \( \mathbb{C} \), using the positive root. If \( \lambda \in \mathbb{R} \), then the representation \( \pi_{\sigma, \lambda} \) is unitary.

2.3 Plancherel measure

Let \( \mu_{PL}(\pi_{\sigma, \lambda}) \) be the Plancherel measure, viewed as a measure on the set of the principal series representations \( \pi_{\sigma, \lambda} \). Since rank(\( G \)) > rank(\( K \)), by classical result of Harish-Chandra ([HC66]), the set of the discrete series representations of \( G \) is empty. By [Kna86] Theorem 13.2, there exists a constant \( c(n) \in \mathbb{R}, c(n) \neq 0 \), such that

\[ d\mu_{PL}(\pi_{\sigma, \lambda}) = -c(n)Q_\sigma(i\lambda) d\lambda, \]

where \( Q_\sigma(i\lambda) \) is the Plancherel polynomial given by

\[ Q_\sigma(i\lambda) = \prod_{\alpha \in \Delta^+(gC, h)} \frac{\langle i\lambda + \nu_\sigma + \mu_m, \alpha \rangle}{\langle \rho_\sigma, \alpha \rangle}, \quad (2.8) \]

where \( \langle \cdot, \cdot \rangle \) is defined by (2.1). Let \( z = i\lambda \in \mathbb{C} \). By [Pla12] eq. (2.33), (2.34)],

\[ c(n) = \frac{(-1)^{n+1}}{2\text{Vol}(S^d)}, \quad (2.9) \]

where \( \text{Vol}(S^d) \) denotes the volume of the \( d \)-dimensional Euclidean unit sphere. By [Mia79] p. 264-265], \( Q_\sigma(z) \) is an even polynomial of \( z \) and hence \( Q_\sigma(z) = Q_\sigma(-z) \). We set

\[ P_\sigma(z) = -c(n)Q_\sigma(z). \]

Remark: We note that in the definition of the Plancherel measure in [Spi18] 2.6], the constant \( c(n) \) is missing.
3 The dynamical zeta functions of Ruelle and Selberg

In this section we define the twisted Selberg and Ruelle zeta functions associated with the geodesic flow on the unit sphere bundle $S(X)$ of $X$. Here, we identify $S(X) = \Gamma \backslash G/M$. For more details about this identification, we refer the reader to [BO95 Subsection 1.1.1 and Section 3.1]. There exists a 1-1 correspondence between the closed geodesics on a manifold $X$ with negative sectional curvature and the non-trivial conjugacy classes of the fundamental group $\pi_1(X)$ ([GKM68]). The hyperbolic elements of $\Gamma$ are the semisimple elements of this group. Since $\Gamma$ is a cocompact and torsion free, every element $\gamma \in \Gamma - \{e\}$ is hyperbolic. Let $[\gamma]$ be the conjugacy class of $\gamma$, defined by $[\gamma] := \{\gamma' \gamma (\gamma')^{-1} : \gamma' \in \Gamma\}$. We denote by $c_\gamma$ the closed geodesic on $X$ associated with the hyperbolic conjugacy class $[\gamma]$, and by $l(\gamma)$ the length of $c_\gamma$.

A primitive element $\gamma_0 \in \Gamma$ corresponds to a prime closed geodesic on $X$. This is a geodesic of minimal length. Hence, if a hyperbolic element $\gamma$ in $\Gamma$ is generated by a primitive element $\gamma_0$, then there exists an $n_\Gamma(\gamma) \in \mathbb{N}$ such that $\gamma = \gamma_0^{n_\Gamma(\gamma)}$ and the corresponding closed geodesic is of length $l(\gamma) = n_\Gamma(\gamma)l(\gamma_0)$.

We define the dynamical zeta functions for an arbitrary finite-dimensional representation $\chi$ of $\Gamma$.

**Definition 3.1.** Let $\chi : \Gamma \to \text{GL}(V_\chi)$ be a finite-dimensional representation of $\Gamma$ and $\sigma \in \hat{M}$. The twisted Selberg zeta function $Z(s; \sigma, \chi)$ for $X$ is defined by the infinite product

$$Z(s; \sigma, \chi) := \prod_{[\gamma] \neq e \atop \gamma \text{ prime}} \prod_{k=0}^{\infty} \det \left( \text{Id} - (\chi(\gamma) \otimes \sigma(m, a)) \otimes S^k(\text{Ad}(m, a)) \right) e^{-s(l(\gamma))},$$

(3.1)

where $s \in \mathbb{C}$, $\mathfrak{p}$ is the negative root space of $(\mathfrak{g}, \mathfrak{a})$ and $S^k(\text{Ad}(m, a)) \otimes \mathfrak{p}$ denotes the $k$-th symmetric power of the adjoint map $\text{Ad}(m, a)$ restricted to $\mathfrak{p}$.

By [Spi18 Proposition 3.4], there exists a positive constant $c$, such that the infinite product in (3.1) converges absolutely and uniformly on compact subsets of the half-plane $\text{Re}(s) > c$.

**Definition 3.2.** Let $\chi : \Gamma \to \text{GL}(V_\chi)$ be a finite-dimensional representation of $\Gamma$ and $\sigma \in \hat{M}$. The twisted Ruelle zeta function $R(s; \sigma, \chi)$ for $X$ is defined by the infinite product

$$R(s; \sigma, \chi) := \prod_{[\gamma] \neq e \atop \gamma \text{ prime}} \det \left( \text{Id} - (\chi(\gamma) \otimes \sigma(m, a)) e^{-s(l(\gamma))} \right),$$

(3.2)

where $s \in \mathbb{C}$.

By [Spi18 Proposition 3.5], there exists a positive constant $r$, such that the infinite product in (3.2) converges absolutely and uniformly on compact subsets of the half-plane $\text{Re}(s) > r$. We consider now the case in which $\sigma$ is not invariant.
under the action of the restricted Weyl group $W_A$. This is case (b) (Section 2). We recall from [Spi18, Section 3] that in case (b), for a non-trivial element $w$ of $W_A$, we define in addition the symmetrized zeta function

$$S(s; \sigma, \chi) := Z(s; \sigma, \chi)Z(s; w\sigma, \chi);$$

the super zeta function

$$Z^\ast(s; \sigma, \chi) := \frac{Z(s; \sigma, \chi)}{Z(s; w\sigma, \chi)};$$

and the super Ruelle zeta function

$$R^\ast(s; \sigma, \chi) := \frac{R(s; \sigma, \chi)}{R(s; w\sigma, \chi)}.$$

We put

$$L(\gamma; \sigma, \chi) := \frac{\text{tr}(\chi(\gamma) \otimes \sigma(m_\gamma))e^{-\rho|\gamma|}}{\det(1 - \text{Ad}(m_\gamma)a_\gamma)}.$$  

By [Spi18] Lemma 3.9, the logarithmic derivatives $L(s), L_S(s), L^\ast(s)$ of the Selberg, symmetrized and super zeta function, respectively, are given by

$$L(s) := \frac{d}{ds} \log(Z(s; \sigma, \chi)) = \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_{\Gamma}(\gamma)} L(\gamma; \sigma, \chi)e^{-s\gamma};$$

$$L_S(s) := \frac{d}{ds} \log(S(s; \sigma, \chi)) = \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_{\Gamma}(\gamma)} L(\gamma; \sigma, \chi + w\sigma, \chi)e^{-s\gamma};$$

$$L^\ast(s) := \frac{d}{ds} \log(Z^\ast(s; \sigma, \chi)) = \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_{\Gamma}(\gamma)} L(\gamma; \sigma - w\sigma, \chi)e^{-s\gamma}.$$

4 The eta function of the twisted Dirac operator

Let $\chi: \Gamma \to \text{GL}(V_\chi)$ be a finite-dimensional representation of $\Gamma$ and $\sigma \in \tilde{M}$. In this section, we define the eta function associated with the twisted Dirac operator, which will be denoted by $D^\chi_\gamma(\sigma)$. This operator is associated with the representations $\sigma$ and $\chi$. It is an elliptic first order differential operator, but no longer self-adjoint. We recall from [Spi18] Section 6 the definition of $D^\chi_\gamma(\sigma)$.

Let $s$ be the spin representation of $\text{Spin}(d)$ with representation space $S$. Let $R(K), R(M)$ be the representation rings over $\text{Z}$ of $K$ and $M$, respectively. Let $i^*: R(K) \to R(M)$ be the pullback of the embedding $i: M \hookrightarrow K$. By [BO95] Proposition 1.1, (3)], there exists a unique element $\tau(\sigma) \in \tilde{K}$ and a splitting $s \otimes \tau(\sigma) = \tau^+(\sigma) \otimes \tau^-(\sigma)$, where $\tau^+(\sigma), \tau^-(\sigma) \in R(K)$, such that $\sigma + w\sigma = i^*(\tau^+(\sigma) - \tau^-(\sigma))$. Let $V_{\tau(\sigma)}$ be the representation space of $\tau(\sigma)$. We define the representation

$$\tau_s(\sigma) := s \otimes \tau(\sigma)$$

of $K$ with representation space $V_{\tau_s(\sigma)} = S \otimes V_{\tau(\sigma)}$. We consider the homogeneous vector bundle $E_{\tau(\sigma)}$ defined by

$$E_{\tau(\sigma)} := G \times_{\tau(\sigma)} V_{\tau(\sigma)} \to \tilde{X}.$$
Let \( \nabla^{\tau(\sigma)} \) be the canonical connection in \( \tilde{E}_{\tau(\sigma)} \). Let \( E_s \) be the spinor bundle over \( \tilde{X} \) associated with \( s \) and equipped with the spin connection \( \nabla \). The vector bundle \( \tilde{E}_{\tau_s(\sigma)} := \tilde{E}_{\tau(\sigma)} \otimes E_s \) over \( \tilde{X} \) carries a connection \( \nabla^{\tau_s(\sigma)} \), defined by the formula

\[
\nabla^{\tau_s(\sigma)} := \nabla^{\tau(\sigma)} \otimes 1 + 1 \otimes \nabla.
\]

Let \( \cdot : p \otimes S \to S \) be the Clifford multiplication on \( p \otimes S \). We extend the Clifford multiplication by requiring that it acts on \( V_{\tau_s(\sigma)} = S \otimes V_{\tau(\sigma)} \) by

\[
e \cdot (\phi \otimes \psi) = (e \cdot \phi) \otimes \psi, \quad e \in \text{Cl}(p), \phi \in S, \psi \in V_{\tau(\sigma)}.
\]

Let \( (e_1, \ldots, e_d) \) be a local orthonormal frame field over an open set \( U \subset \tilde{X} \). The Dirac operator \( \tilde{D}(\sigma) \) acting on \( C^\infty(\tilde{X}, \tilde{E}_{\tau_s(\sigma)}) \) is defined by

\[
\tilde{D}(\sigma)f = \sum_{i=1}^d e_i \cdot \nabla^{\tau_s(\sigma)} f.
\]

Let \( E_{\tau_s(\sigma)} := \Gamma \backslash \tilde{E}_{\tau_s(\sigma)} \) be the locally homogeneous vector bundle over \( X \). Let \( \chi : \Gamma \to \text{GL}(V_{\chi}) \) be an arbitrary finite-dimensional representation of \( \Gamma \). Let \( E_{\chi} \) be the associated flat vector bundle over \( X \), equipped with a flat connection \( \nabla^\chi \). We consider the product vector bundle \( E_{\tau_s(\sigma)} \otimes E_{\chi} \) over \( X \) and equip this bundle with the product connection \( \nabla^{E_{\tau_s(\sigma)} \otimes E_{\chi}} \) defined by

\[
\nabla^{E_{\tau_s(\sigma)} \otimes E_{\chi}} := \nabla^{\tau_s(\sigma)} \otimes 1 + 1 \otimes \nabla^\chi.
\]

We define the Clifford multiplication on \( V_{\tau_s(\sigma)} \otimes V_{\chi} \) by requiring that it acts only on \( V_{\tau_s(\sigma)} \), i.e.,

\[
e \cdot (w \otimes v) = (e \cdot w) \otimes v, \quad e \in \text{Cl}(p), w \in V_{\tau_s(\sigma)}, v \in V_{\chi}.
\]

We consider an open subset \( U \) of \( X \) such that \( E_{\chi}|_U \) is trivial. Let also \( (v_j), j = 1, \ldots, m \), be any base of flat sections of \( E_{\chi}|_U \), where \( m = \text{rank}(E_{\chi}) \), and \( \phi_j \in C^\infty(U, E_{\tau_s(\sigma)}|_U) \). Then,

\[
E_{\tau_s(\sigma)} \otimes E_{\chi}|_U \cong \bigoplus_{j=1}^m E_{\tau_s(\sigma)}|_U,
\]

and for each \( \phi \in C^\infty(U, E_{\tau_s(\sigma)} \otimes E_{\chi}|_U) \),

\[
\phi = \sum_{j=1}^m \phi_j \otimes v_j.
\]

Then,

\[
\nabla^{E_{\tau_s(\sigma)} \otimes E_{\chi}} \phi = \sum_{j=1}^m \nabla^{E_{\tau_s(\sigma)}} \phi_j \otimes v_j.
\]

The definition above is independent of the choice of the base of flat sections of \( E_{\chi}|_U \), since the transition maps comparing flat sections are constant. The twisted Dirac operator \( D^1_{\chi}(\sigma) \) associated with \( \nabla^{E_{\tau_s(\sigma)} \otimes E_{\chi}} \) is locally defined by

\[
D^1_{\chi}(\sigma) \phi := \sum_{i=1}^d e_i \cdot \nabla^{E_{\tau_s(\sigma)} \otimes E_{\chi}} \phi.
\]
The local definition of the twisted Dirac operator is independent of the choice of the orthonormal frame field. We consider the pullbacks $\tilde{E}_{\tau}(\sigma), \tilde{E}_{\chi}$ to $\tilde{X}$ of $E_{\tau}(\sigma), E_{\chi}$, respectively. Then, $\tilde{E}_{\chi} \cong \tilde{X} \times V_{\chi}$ and

$$C^\infty(\tilde{X}, \tilde{E}_{\tau(\sigma)} \otimes \tilde{E}_{\chi}) \cong C^\infty(\tilde{X}, \tilde{E}_{\tau(\sigma)}) \otimes V_{\chi}.$$  

With respect to this isomorphism, it follows from the definition of the twisted Dirac operator $D^\chi_{\tau}(\sigma)$ that the lift $\tilde{D}^\chi_{\lambda}(\sigma)$ of $D^\chi_{\tau}(\sigma)$ to $\tilde{X}$ is of the form

$$\tilde{D}^\chi_{\lambda}(\sigma) = \tilde{D}(\sigma) \otimes Id_{V_{\chi}}.$$  

We recall here the definition of the associated operator $A^\chi_{\lambda}(\sigma)$ acting on smooth sections of twisted vector bundles ([Spi18, p. 174-175]). Following the proof of Proposition 1.1 in [BO95, p. 22] (see also [Pfa12, Proposition 2.3]), there exist unique integers $m_\tau(\sigma) \in \{-1, 0, 1\}$, which are equal to zero except for finitely many $\tau \in \hat{K}$, such that,

- if $\sigma$ is Weyl invariant, $\sigma = \sum_{\tau \in \hat{K}} m_\tau(\sigma) \lambda^*(\tau)$;
- if $\sigma$ is non-Weyl invariant, $\sigma + w \sigma = \sum_{\tau \in \hat{K}} m_\tau(\sigma) \lambda^*(\tau)$.

We define a locally homogeneous vector bundle $E(\sigma)$ associated to $\sigma$ in the following way.

$$E(\sigma) = \bigoplus_{\tau \in \hat{K}, m_\tau(\sigma) \neq 0} E_{\tau},$$

where $E_{\tau}$ is the locally homogeneous vector bundle over $X$ associated with $\tau \in \hat{K}$. By the construction of the locally homogeneous vector bundles $E(\sigma)$ and $E_{\tau(\sigma)}$ over $X$ (see [BO95, Propos 1.1]), $E(\sigma) = E_{\tau(\sigma)}$ up to a $\mathbb{Z}_2$-grading.

Let $\tau \in \hat{K}$ and $E_{\tau}$ be the locally homogeneous vector bundle over $X$ associated with $\tau$. Let $\Delta_{\tau}$ be the Bochner-Laplace operator associated with $\tau$, acting on smooth sections of $E_{\tau}$. Let $\Lambda_{\tau}$ be the lift of $\Delta_{\tau}$ to $\tilde{X}$. As in [Spi18, eq. (5.23)], we set $\Lambda_{\tau} := \Lambda_{\tau} - \lambda_{\tau} \cdot Id$, where $\lambda_{\tau}$ is the Casimir eigenvalue of $\tau$ ([Spi18, eq. (5.4)]). Let $\Delta_{\tau, \chi}$ be the twisted Bochner-Laplace operator acting on smooth sections of the twisted vector bundle $E_{\tau} \otimes E_{\chi}$. This operator is defined in [Spi18, Section 4]. Then, the operator $A^\chi_{\tau, \chi}$ is induced by the twisted Bochner-Laplace operator $\Delta^\chi_{\tau, \chi}$ and acts on smooth sections of the twisted vector bundle $E_{\tau} \otimes E_{\chi}$.

The lift $\tilde{A}^\chi_{\tau, \chi}$ of $A^\chi_{\tau, \chi}$ to $\tilde{X}$ is given by

$$\tilde{A}^\chi_{\tau, \chi} = \tilde{\Lambda}_{\tau} \otimes Id_{V_{\chi}},$$

([Spi18, eq. (5.24)]). Let $\rho, \rho_{m}, \nu_\sigma$ be as in (2.2), (2.3) and (2.5), respectively. We define the number $c(\sigma)$ by

$$c(\sigma) := -|\rho|^2 - |\rho_m|^2 + |\nu_\sigma + \rho_m|^2,$$

and the operator $A^\chi_{\lambda}(\sigma)$ by

$$A^\chi_{\lambda}(\sigma) := \bigoplus_{m_\tau(\sigma) \neq 0} A^\chi_{\tau, \chi} + c(\sigma). \quad (4.1)$$
By \(\text{Spi18}, \text{eq. (6.8)}\), the Parthasarathy formula generalizes as
\[
D^2_{\chi}(\sigma)^2 = A^2_{\chi}(\sigma).
\]
The square \(D^2_{\chi}(\sigma)^2\) of the twisted Dirac operator acting on smooth sections of \(E_{\tau_\gamma(\sigma)} \otimes E_\chi\) is a second order elliptic differential operator but no longer self-adjoint. Nevertheless, for \(x \in X\) and \(\xi \in T^*_x X\), its principal symbol is given by
\[
\sigma_{D^2_{\chi}(\sigma)^2}(x, \xi) = \|\xi\|^2 \otimes \text{Id}_{(V_{\tau_\gamma(\sigma)} \otimes V_\chi)_x}.
\]
Hence it has nice spectral properties. By \(\text{Spi18}, \text{Lemma 8.6}\), its spectrum is discrete and contained in a translate of a positive cone \(C \subset C\). We consider now the corresponding heat operators \(e^{-tD^2_{\chi}(\sigma)^2}, D^2_{\chi}(\sigma)e^{-tD^2_{\chi}(\sigma)^2}\). Let \(dg\) a Haar measure on \(G\), normalized as in \(\text{Spi18}\, p.\, 157\). We equip the spaces \(\tilde{X}\) and \(X\) with the induced quotient measure. By \(\text{Spi18}\, p.\, 171-173\) and \(p.\, 181\), \(e^{-tD^2_{\chi}(\sigma)^2}, D^2_{\chi}(\sigma)e^{-tD^2_{\chi}(\sigma)^2}\) are well defined, trace class operators, acting on smooth sections of the vector bundle \(E_{\tau_\gamma(\sigma)} \otimes E_\chi\).

**Lemma 4.1.** The asymptotic expansion of the trace of the kernel \(K^{\tau_\gamma(\sigma),\chi}_t\) of the operator \(D^2_{\chi}(\sigma)e^{-tD^2_{\chi}(\sigma)^2}\) is given by
\[
\text{tr} K^{\tau_\gamma(\sigma),\chi}_t(x, x) \sim_{t \to 0^+} \dim(V_\chi)(a_0(\tilde{x})t^{1/2} + O(t^{3/2}, \tilde{x})),
\] (4.2)
where \(\tilde{x} \in \tilde{X}\) is the lift of \(x\) to \(\tilde{X}\), \(a_0(\tilde{x})\) is a \(C^\infty\)-function on \(\tilde{X}\) and \(O(t^{3/2}, \tilde{x})\) is uniform on \(\tilde{X}\).

**Proof.** Let \(K^{\tau_\gamma(\sigma)}_t\) be the kernel of the the operator \(D(\sigma)e^{-tD(\sigma)^2}\). Let \(x, y \in X\). Let \(\tilde{x}, \tilde{y} \in \tilde{X}\) be the lifts of \(x, y\) to \(\tilde{X}\). The action of \(\Gamma\) on \(\tilde{X}\) induces the following isomorphism of the fibres: \(R_\gamma: (\tilde{E}_{\tau_\gamma(\sigma)})_{\tilde{x}} \to (\tilde{E}_{\tau_\gamma(\sigma)})_{\tilde{y}}\). As in [Spi18, p. 180–181], the kernel \(K^{\tau_\gamma(\sigma),\chi}_t\) of the operator \(D^2_{\chi}(\sigma)e^{-tD^2_{\chi}(\sigma)^2}\) is given by
\[
K^{\tau_\gamma(\sigma),\chi}_t(x, y) = \sum_{\gamma \in \Gamma} K^{\tau_\gamma(\sigma)}_t(\tilde{x}, \gamma \tilde{y}) \circ (R_\gamma \otimes \chi(\gamma)).
\] (4.3)
The right-hand side of (4.3) can be written as
\[
K^{\tau_\gamma(\sigma)}_t(\tilde{x}, \tilde{y}) \otimes \text{Id}_{V_\chi} + \sum_{\gamma \in \Gamma, \gamma \neq e} K^{\tau_\gamma(\sigma)}_t(\tilde{x}, \gamma \tilde{y}) \circ (R_\gamma \otimes \chi(\gamma)).
\] (4.4)
Let \(|\cdot|\) denote the pointwise norm of the homomorphism \(K^{\tau_\gamma(\sigma)}_t \in \text{Hom}((\tilde{E}_{\tau_\gamma(\sigma)})_{\tilde{x}}, (\tilde{E}_{\tau_\gamma(\sigma)})_{\tilde{y}})\).
Let \(d(\cdot, \cdot)\) be the the geodesic distance on \(\tilde{X}\). Recall that the length \(l(\gamma)\) of the geodesic associated with \(\gamma\) is given by \(l(\gamma) := \inf\{d(\tilde{x}, \gamma \tilde{x}) : \tilde{x} \in \tilde{X}\}\). By [MHPB Prop. 3.2], for every \(T > 0\), there exists \(c > 0\) such that
\[
|K^{\tau_\gamma(\sigma)}_t(\tilde{x}, \tilde{y})| \leq c t^{-d/2} e^{-\frac{d^2(\tilde{x}, \tilde{y})}{4}}.
\]
for $0 < t \leq T$ and $\bar{x}, \bar{y} \in \bar{X}$. Hence,

$$
\sum_{\gamma \in \Gamma} \left|\text{tr} K_t^{\tau_0(\sigma)}(\bar{x}, \gamma \bar{x})\right| \leq c t^{-d/2} \sum_{\gamma \in \Gamma} e^{-\frac{d(\gamma^2)}{4 t}} \left|\gamma \right| \leq c t^{-d/2} \sum_{\gamma \in \Gamma} e^{-\frac{d(\gamma^2)}{4 t}}.
$$

By the normalization of the Haar measure on $G$ as in [Spi18, p. 157], there is a positive constant $C > 0$ such that for every $R > 0$

$$\text{Vol}(B(x_0, R)) \leq Ce^{2|\rho|R},$$

where $\rho$ is as in (2.2). $\Gamma$ is a cocompact lattice of $G$. This implies that, there exists a positive constant $C'$ such that

$$
\sharp\{[\gamma] : l(\gamma) < R\} \leq \sharp\{\gamma \in \Gamma : l(\gamma) \leq R\} \leq C'e^{2|\rho|R}
$$

(4.6). We define

$$
N(R) := \sharp\{[\gamma] \in C(\Gamma) : l(\gamma) \leq R\}, \quad R \geq 0,
$$

where $C(\Gamma)$ denotes the set of $\Gamma$-conjugacy classes. Since there exists a $c > 0$ such that $c \leq l(\gamma)$, for every $\gamma \neq e$, we have the following estimates.

$$
\sum_{\gamma \in \Gamma} e^{-\frac{(l(\gamma))^2}{t}} = \sum_{k=1}^{\infty} \sum_{\gamma \neq e, k \leq l(\gamma) < (k+1)e} e^{-\frac{(l(\gamma))^2}{t}},
$$

$$
\leq \sum_{k=1}^{\infty} N((k+1)c) e^{-\frac{2k^2}{4t}}.
$$

By (4.6), we get for $0 < t \leq T$,

$$
\sum_{\gamma \in \Gamma} e^{-\frac{(l(\gamma))^2}{t}} \leq C' \sum_{k=1}^{\infty} e^{2|\rho|(k+1)c} e^{-\frac{2k^2}{4t}} < \infty.
$$

(4.7)

By [Gan68, Lemma 5.1 and Corollary 5.2] (see also [Wol08, p. 28-30]), there exists a $c_0 > 0$ such that for every $\gamma \in \Gamma$ with $\gamma \neq e$, $l(\gamma) > 2c_0$. Then, by (4.5)

$$
\sum_{\gamma \in \Gamma, \gamma \neq e} \left|\text{tr} K_t^{\tau_0(\sigma)}(\bar{x}, \gamma \bar{x})\right| \leq c t^{-d/2} e^{-\frac{c_0^2}{4t}} \sum_{\gamma \in \Gamma} e^{-\frac{(l(\gamma))^2}{4t}},
$$

and $l(\gamma)^2 - c_0^2 > \frac{1}{2} l(\gamma)^2$. Hence, the series on the right-hand side converges and by (4.7) there exists a $C'' > 0$ such that for all $t$ with $0 < t \leq T$,

$$
\sum_{\gamma \in \Gamma, \gamma \neq e} \left|\text{tr} K_t^{\tau_0(\sigma)}(\bar{x}, \gamma \bar{x})\right| \leq C'' t^{-d/2} e^{-\frac{c_0^2}{4t}}.
$$

(4.8)
We use now Theorem 2.4 in [BF86] for the asymptotic expansion of the trace of the kernel \( K^{\tau,\chi}_{\tau,\chi} \) as \( t \to 0^+ \). This theorem is proved for compact manifolds. However, the proof of the existence of the asymptotic expansion is purely local. Hence, it holds for the non-compact case as well. Therefore, we have the following asymptotic expansion.

\[
\text{tr} \ K^{\tau,\chi}_{\tau,\chi}(\tilde{x}, \tilde{x}) \sim_{t \to 0^+} a_0(\tilde{x})t^{1/2} + O(t^{3/2}, \tilde{x}),
\]

(4.9)

where \( a_0(\tilde{x}) \) is a smooth function determined by the total symbol of \( D(\tau,\chi) \) and \( O(t^{3/2}, \tilde{x}) \) is uniform in \( \tilde{x} \in \tilde{X} \). Hence, by (4.3), (4.4), (4.8) and (4.9) we get

\[
\text{tr} \ K^{\tau,\chi}_{\tau,\chi}(x, x) \sim_{t \to 0^+} \dim(V_\chi)(a_0(\tilde{x})t^{1/2} + O(t^{3/2}, \tilde{x})).
\]

\[\square\]

**Remark:** We mention here that in Proposition 3.2 in [Mü198], the representation of \( K \) is irreducible. However, the irreducibility condition is not used in the proof. Hence, this result can be extended to the case of the non-irreducible representation \( \tau(\sigma) \) of \( K \).

We consider now the operator \( e^{-t(\tau(\sigma) + c(\gamma))} \) induced by each summand \( A^\tau_{\tau,\chi} + c(\gamma) \) in the definition (4.1) of \( A^\tau_{\tau,\chi} \).

**Lemma 4.2.** There exist coefficients \( c_j \) such that the asymptotic expansion of the trace of the kernel \( H^\tau(\tau,\chi)(x, y) \) of the operator \( e^{-t(\tau(\sigma) + c(\gamma))} \) is given by

\[
\text{tr} \ H^\tau(\tau,\chi)(x, x) \sim_{t \to 0^+} \dim(V_\chi) \sum_{j=0}^\infty c_j t^{\frac{j}{4}}.
\]

(4.10)

Proof. Let \( Q^\tau_t \) be the kernel associated with the operator \( e^{-t(\tau(\sigma) + c(\gamma))} \) (see [Sp18] p. 175-176]). Let \( x, y \in X \). Let \( \tilde{x}, \tilde{y} \in \tilde{X} \) be the lifts of \( x, y \) to \( \tilde{X} \), respectively. Let \( R_\gamma \) be the following isomorphism: \( R_\gamma : (\tilde{E}_\tau)_{\gamma} \to (\tilde{E}_\tau)_{\gamma} \). As in [Sp18] p. 175], the kernel \( H^\tau(\tau,\chi) \) of the operator \( e^{-t(\tau(\sigma) + c(\gamma))} \) is given by

\[
H^\tau(\tau,\chi)(x, y) = \sum_{\gamma \in \Gamma} e^{-tc(\gamma)}Q^\tau_t(\tilde{x}, \tilde{y}) \circ (R_\gamma \otimes \chi(\gamma)).
\]

(4.11)

Equivalently,

\[
H^\tau(\tau,\chi)(x, y) = e^{-tc(\gamma)}Q^\tau_t(\tilde{x}, \tilde{y}) \otimes \text{Id}_{V_\chi} + \sum_{\gamma \in \Gamma \setminus \chi(\gamma)} e^{-tc(\gamma)}Q^\tau_t(\tilde{x}, \tilde{y}) \circ (R_\gamma \otimes \chi(\gamma)).
\]

(4.11)

As in the proof of Lemma 4.1, one can prove that there exists a \( c > 0 \) such that for \( t \to 0^+ \),

\[
\sum_{\gamma \in \Gamma \setminus \chi(\gamma)} e^{-tc(\gamma)} |\text{tr} Q^\tau_t(\tilde{x}, \tilde{y})| \leq ct^{-d/2} e^{-c^2/4t} e^{-tc(\gamma)}.
\]

(4.12)

By [Gl95] Lemma 1.7.4] the trace of the kernel \( Q^\tau_t \), associated with the operator \( e^{-t(\tau(\sigma))} \), has the asymptotic expansion

\[
\text{tr} Q^\tau_t(\tilde{x}, \tilde{x}) \sim_{t \to 0^+} \sum_{j=0}^\infty c_j(\tilde{x}) t^{\frac{j}{4}}.
\]
where \( c_j(\tilde{x}) \) are smooth functions determined by the total symbol of the operator \( \tilde{\Delta}_\tau \). The proof of the existence of the asymptotic expansion is purely local. Therefore, it applies in our case as well. Since \( e^{-t\tilde{\Delta}_\tau} \) commutes with the action of \( G \), and \( G \) acts transitively on \( \tilde{X} \), it follows that \( Q^\tau_t(\tilde{x}, \tilde{x}) \) is independent of \( \tilde{x} \). Therefore, the coefficients \( c_j(\tilde{x}) \) are also independent of \( \tilde{x} \). Let \( \tilde{x}_0 = eK \in \tilde{X} \) be the base point. Then, we have

\[
\text{tr} \, Q^\tau_t(\tilde{x}_0, \tilde{x}_0) \sim _{t \to 0^+} \sum_{j=0}^{\infty} c_j t^{j-d/2}.
\]

(4.13)

Moreover, one can use the expansion in power series of the term \( e^{-tc(\sigma)} \). By (4.11), (4.12) and (4.13), we get

\[
\text{tr} \, H^\tau_{t,\chi}(x,x) \sim _{t \to 0^+} \dim(V_\chi) \sum_{j=0}^{\infty} c_j t^{j-d/2}
\]

We need the following definitions.

**Definition 4.3.** Let \( R_\theta := \{ \rho e^{i\theta} : \rho \in [0, \infty] \} \). The angle \( \theta \in [0, 2\pi) \) is a principal angle for the elliptic operator \( D^\chi_\lambda(\sigma) \) if

\[
\text{spec}(\sigma_{D^\chi_\lambda(\sigma)}(x,\xi)) \cap R_\theta = \emptyset, \quad \forall x \in X, \forall \xi \in T^*_x X, \xi \neq 0.
\]

**Definition 4.4.** Let \( I \subset [0, 2\pi) \). Let \( L_I \) be a solid angle defined by

\[
L_I := \{ \rho e^{i\theta} : \rho \in (0, \infty), \theta \in I \}.
\]

The angle \( \theta \) is an Agmon angle for the elliptic operator \( D^\chi_\lambda(\sigma) \), if it is a principal angle for \( D^\chi_\lambda(\sigma) \) and there exists an \( \varepsilon > 0 \) such that

\[
\text{spec}(D^\chi_\lambda(\sigma)) \cap L_{[\theta-\varepsilon,\theta+\varepsilon]} = \emptyset.
\]

We define here the eta function associated with non-self-adjoint operators with elliptic, self-adjoint principal symbol (see \[Gil84\]).

**Definition 4.5. Eta function of** \( D^\chi_\lambda(\sigma) \). The principal symbol of \( D^\chi_\lambda(\sigma) \) is self-adjoint. Hence, the angles \( \pm \pi/2 \) are principal angles for \( D^\chi_\lambda(\sigma) \). Since \( D^\chi_\lambda(\sigma) \) possesses a principal angle, it also possesses an Agmon angle (see \[BK08, Section 3.10\]). Let \( \theta \) be an Agmon angle for \( D^\chi_\lambda(\sigma) \). Let \( \text{spec}(D^\chi_\lambda(\sigma)) = \{ \lambda_k : k \in \mathbb{N} \} \) be the spectrum of \( D^\chi_\lambda(\sigma) \). It is a discrete subset of \( \mathbb{C} \).

By \[Mar88 \S 1.6\], the space \( L^2(X, E_{\tau_\sigma(\sigma)} \otimes E_\chi) \) of square integrable sections of \( E_{\tau_\sigma(\sigma)} \otimes E_\chi \) is the closure of the algebraic direct sum of finite dimensional \( D^\chi_\lambda(\sigma) \)-invariant subspaces

\[
L^2(X, E_{\tau_\sigma(\sigma)} \otimes E_\chi) = \bigoplus_{k \geq 1} \Lambda_k,
\]

(4.14)

such that the restriction of \( D^\chi_\lambda(\sigma) \) to \( \Lambda_k \) has a unique eigenvalue \( \lambda_k \) and \( \lim_{k \to \infty} |\lambda_k| = \infty \). In general, the sum (4.14) is not a sum of mutually orthogonal subspaces.
The spaces $\Lambda_k$ are called the root vectors if $D^k_\chi(\sigma)$ with eigenvalue $\lambda_k$. The algebraic multiplicity $m_k$ of the eigenvalue $\lambda_k$ is defined as the the dimension of the space $\Lambda_k$.

Denote by $\log_\theta \lambda_k$ the branch of the logarithm in $\mathbb{C}\setminus R_\theta$ with $\theta < \text{Im}(\log_\theta \lambda_k) < \theta + 2\pi$. Let $(\lambda_k) := e^{\log_\theta \lambda_k}$. For $\text{Re}(s) > 0$, we define the eta function $\eta_\theta(s, D^k_\chi(\sigma))$ of $D^k_\chi(\sigma)$ by

$$\eta_\theta(s, D^k_\chi(\sigma)) := \sum_{\text{Re}(\lambda_k) > 0} m_k(\lambda_k)^{-s} - \sum_{\text{Re}(\lambda_k) < 0} m_k(-\lambda_k)^{-s}.$$

Note that since the angles $\pm \pi/2$ are principal angles for $D^k_\chi(\sigma)$, there are at most finitely many eigenvalues of $D^k_\chi(\sigma)$ on or near the imaginary axis. Hence, the eigenvalues of $D^k_\chi(\sigma)$ that are purely imaginary do not contribute to the definition of the eta function.

It has been shown by Grubb and Seeley (GSS95 Theorem 2.7) that $\eta_\theta(s, D^k_\chi(\sigma))$ has a meromorphic continuation to the whole complex plane with isolated simple poles and that is regular at $s = 0$. Moreover, the number $\eta_\theta(0, D^k_\chi(\sigma))$ is independent of the Agmon angle $\theta$. Hence, we write $\eta(0, D^k_\chi(\sigma))$ instead of $\eta_\theta(0, D^k_\chi(\sigma))$. We give here a short description of the proof.

Let $\theta$ be an Agmon angle for $D^k_\chi(\sigma)$. We assume that $0$ is not an eigenvalue of $D^k_\chi(\sigma)$. To define the operator $D^k_\chi(\sigma)^{-s}$, one has to use the contour $\Gamma_{\theta, \rho_0}$, described as in [Shu87, p. 88]. There exists a $\rho_0 > 0$ such that

$$\text{spec}(D^k_\chi(\sigma)) \cap \{z \in \mathbb{C} : |z| \leq 2\rho_0\} = \emptyset.$$

We consider the contour $\Gamma_{\theta, \rho_0} \subset \mathbb{C}$, defined as $\Gamma_{\theta, \rho_0} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where $\Gamma_1 = \{re^{i\theta} : \infty > r \geq \rho_0\}, \Gamma_2 = \{\rho_0 e^{i\beta} : \theta \leq \beta \leq \theta - 2\pi\}, \Gamma_3 = \{re^{i(\beta + 2\pi)} : \rho_0 \leq r < \infty\}$. On $\Gamma_1$, $r$ runs from $\infty$ to $\rho_0$, $\Gamma_2$ is oriented counterclockwise, and on $\Gamma_3$, $r$ runs from $\rho_0$ to $\infty$. Then, for $\text{Re}(s) > 0$ we define

$$D^k_\chi(\sigma)^{-s} := \frac{i}{2\pi} \int_{\Gamma_{\theta, \rho_0}} \lambda^{-s}(D^k_\chi(\sigma) - \lambda \text{Id})^{-1} d\lambda.$$

Let $\Pi_\succ$ (resp. $\Pi_\prec$) be the pseudo-differential projection whose image contains the span of all generalized eigenvectors of $D^k_\chi(\sigma)$ corresponding to eigenvalues $\lambda$ with $\text{Re}(\lambda) > 0$ (resp. $\text{Re}(\lambda)<0$) (for more details see [BK07 Definition 6.16]). The zeta function $\zeta_\theta(s, P, D^k_\chi(\sigma))$ is define for $\text{Re}(s) > d$ by,

$$\zeta_\theta(s, P, D^k_\chi(\sigma)) := \text{Tr}(PD^k_\chi(\sigma)^{-s}),$$

where $P = \Pi_\succ, \Pi_\prec$ ([BK07, p. 38]). Then, Definition 4.5 can be read as

$$\eta_\theta(s, D^k_\chi(\sigma)) = \zeta_\theta(s, \Pi_\succ, D^k_\chi(\sigma)) - \zeta_\theta(s, \Pi_\prec, D^k_\chi(\sigma)).$$

The zeta function $\zeta_\theta(s, P, D^k_\chi(\sigma))$ is define for $\text{Re}(s) > d$ by,

$$\zeta_\theta(s, P, D^k_\chi(\sigma)) := \text{Tr}(PD^k_\chi(\sigma)^{-s}).$$

If we integrate by parts the integral above, the operator $(D^k_\chi(\sigma) - \lambda \text{Id})^{-k}$ will occur. By (GSS95 Theorem 2.7), for $k > d$, there exists an asymptotic expansion of the trace of the operator $P(D^k_\chi(\sigma) - \lambda \text{Id})^{-k}$ as $|\lambda| \to \infty$:

$$\text{Tr}(P(D^k_\chi(\sigma) - \lambda \text{Id})^{-k}) \sim \sum_{j=1}^{\infty} c_j \lambda^{d-j-k} + \sum_{l=1}^{\infty} (c_{l}^j \log \lambda + c_{l}^j) \lambda^{-k-l},$$

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We mention here that we can use the Lidskii’s theorem ([Sim05, Theorem 3.7], where the coefficients $c_j$ and $c_j’$ are determined from the symbols of $D^1_h(\sigma)$ and $P$, and the coefficients $c_j”$ are in general globally determined.

Let $m_+$, respectively $m_-$, denote the the number of the eigenvalues of $D^1_h(\sigma)$ on the positive, respectively negative, part of the imaginary axis. We define the eta invariant $\eta(D^1_h(\sigma))$ of the operator $D^1_h(\sigma)$ by

$$\eta(D^1_h(\sigma)) = \frac{\eta(0, D^1_h(\sigma)) + m_+ - m_-}{2}.$$  

Let now $\Pi_+$ be the projection on the span of the root spaces corresponding to eigenvalues $\lambda$ with $\text{Re}(\lambda^2_h) > 0$. We define the functions

$$\eta_0(s, D^2_h(\sigma)) := \sum_{\text{Re}(\lambda^2_h) > 0} m_k \lambda^s_k - \sum_{\text{Re}(\lambda^2_h) < 0} m_k \lambda^s_k$$

$$\eta_1(s, D^2_h(\sigma)) := \sum_{\text{Re}(\lambda^2_h) > 0} m_k \lambda^s_k - \sum_{\text{Re}(\lambda^2_h) < 0} m_k \lambda^s_k.$$  

By Definition 4.5, the eta function $\eta(s, D^1_h(\sigma))$ satisfies the equation

$$\eta(s, D^1_h(\sigma)) = \eta_0(s, D^1_h(\sigma)) + \eta_1(s, D^1_h(\sigma)).$$

Since the spectrum of $D^2_h(\sigma)^2$ is discrete and contained in a translate of a positive cone in $C$, there are only finitely many eigenvalues of $D^1_h(\sigma)$ with $\text{Re}(\lambda^2_h) \leq 0$.

**Lemma 4.6.** The eta function $\eta(s, D^1_h(\sigma))$ satisfies the equation

$$\eta(s, D^1_h(\sigma)) = \eta_0(s, D^1_h(\sigma)) + \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr}(\Pi_+ D^2_h(\sigma)e^{-tD^1_h(\sigma)^2}) t^{s-1} dt.$$  

(4.15)

**Proof.** Let $\lambda_k$ be an eigenvalue of $D^2_h(\sigma)$ such that $\text{Re}(\lambda^2_k) > 0$. We have

$$(\lambda^2_k)^{-\frac{s-1}{2}} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\lambda^2_k t} t^{s-1} dt.$$  

We mention here that we can use the Lidskii’s theorem ([Sim05, Theorem 3.7, p. 35]) to express the trace of the operator $D^1_h(\sigma)e^{-tD^1_h(\sigma)^2}$ in terms of its eigenvalues $\lambda_k$

$$\text{Tr}(D^1_h(\sigma)e^{-tD^1_h(\sigma)^2}) = \sum_{\lambda_k \neq 0} m_k \lambda_k e^{-t\lambda^2_k}.$$  

Taking the sum over the eigenvalues $\lambda_k$ of $D^1_h(\sigma)$, counting also their algebraic multiplicities, we have

$$\text{Tr}(\Pi_+ D^2_h(\sigma)(D^2_h(\sigma)^2)^{-\frac{s-1}{2}}) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr}(\Pi_+ D^2_h(\sigma)e^{-tD^1_h(\sigma)^2}) t^{s-1} dt.$$  

(4.16)
We use now Weyl’s law for the non-self-adjoint operator $D$ which is a holomorphic function for $\text{Re}(s) > 0$.

To prove the convergence of the above integral, we first observe that

$$
\text{Tr}(\Pi_t D^2(\sigma)(D^2(\sigma)^2)^{-1}) = \int_0^1 \text{Tr}(\Pi_t D^2(\sigma)e^{-tD^2(\sigma)^2})t^{-\frac{1}{2}}dt + \int_1^\infty \text{Tr}(\Pi_t D^2(\sigma)e^{-tD^2(\sigma)^2})t^{-\frac{1}{2}}dt.
$$

(4.17)

For the first integral in the right-hand side of (4.17), we use the asymptotic expansion of the trace of the operator $D^2(\sigma)e^{-tD^2(\sigma)^2}$. By Lemma 4.1, we have

$$
\int_0^1 \text{Tr}(\Pi_t D^2(\sigma)e^{-tD^2(\sigma)^2})t^{-\frac{1}{2}}dt = \int_0^1 \dim(V_{\chi}(a_0^2t^{1/2} + O(t^{3/2})])t^{-\frac{1}{2}}dt
$$

$$
= \dim(V_{\chi})a_0^2 \frac{2}{s+2} + \int_0^1 O(t^{3/2})t^{-\frac{1}{2}}dt,
$$

(4.18)

which is a holomorphic function for $\text{Re}(s) > -2$. Here,

$$
a_0 = \int_X a_0(x) d\mu(x),
$$

where $a_0(x)$ is a smooth function, and $\mu(x)$ is the volume measure determined by the Riemannian metric on $X$.

We treat now the second integral in the right-hand side of (4.17). Since there are finitely many eigenvalues $\lambda_k$ with $|\lambda_k| < 1$, we have for $t \geq 1$,

$$
\left| \sum_{\text{Re}(\lambda_k^2) > 0} m_k \lambda_k e^{-t\lambda_k^2} \right| \leq C \sum_{\text{Re}(\lambda_k^2) > 0} m(\lambda_k^2)|\lambda_k^2|e^{-t\text{Re}(\lambda_k^2)},
$$

(4.19)

where $C$ is a positive constant. We recall here that the spectrum of $D^2(\sigma)^2$ is discrete and contained in a translate of a positive cone in $\mathbb{C}$. We set $c_0 := \frac{1}{2}\min\{\text{Re}(\lambda_k^2) : \text{Re}(\lambda_k^2) > 0\}$. Then, we have for $t \geq 1$,

$$
\sum_{\text{Re}(\lambda_k^2) > 0} m(\lambda_k^2)|\lambda_k^2|e^{-t\text{Re}(\lambda_k^2)} \leq e^{-ct/2} \sum_{\text{Re}(\lambda_k^2) > 0} m(\lambda_k^2)|\lambda_k^2|e^{-\text{Re}(\lambda_k^2)/2}.
$$

(4.20)

We use now Weyl’s law for the non-self-adjoint operator $D^2(\sigma)^2$ to estimate the last sum. Let $r$ be a positive constant. We define the counting function $N(r)$ by

$$
N(r) := \sum_{\lambda^2 \in \text{spec}(D^2(\sigma)^2) \leq r} m(\lambda^2).
$$

In [Mul11], the generalization of Weyl’s law for the non-self-adjoint case is proved. By [Mul11, Lemma 2.2], we have

$$
N(r) = \frac{\text{rank}(\text{rank}(E(\tau_\sigma(\Gamma) \otimes E_\chi))) \text{Vol}(X)}{(4\pi)^{d/2}2\Gamma(d/2 + 1)} r^{d/2} + o(r^{d/2}), \quad r \to \infty,
$$

where $\text{rank}(E(\tau_\sigma(\Gamma) \otimes E_\chi))$ denotes the rank of the product vector bundle $E(\tau_\sigma(\Gamma) \otimes E_\chi)$. Let $a > 0$ be the slope of the boundary of the cone, in which all the eigenvalues $\lambda_j^2$ of $D^2(\sigma)^2$ are contained. We have

$$
2\{j : |\text{Re}(\lambda_j^2)| \leq r\} \leq 2\{j : |\lambda_j^2| \leq \sqrt{1 + a^2r}\} \leq N(\sqrt{1 + a^2r}).
$$

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Hence, we have
\[
\sum_{\text{Re}(\lambda_i^2) > 0} m(\lambda_i^2)|\lambda_i^2|^2 e^{-\text{Re}(\lambda_i^2)/2} < \infty. \tag{4.21}
\]
By (4.19), (4.20) and (4.21), there exists a positive constant \(c\) such that
\[
\left| \sum_{\text{Re}(\lambda_i^2) > 0} m_k \lambda_k e^{-t\lambda_i^2} \right| \leq ce^{-ct/2}.
\]
Therefore,
\[
\int_1^\infty |\text{Tr}(\Pi_+ D^2_\lambda(\sigma) e^{-tD^2_\lambda(\sigma)^2})|^{\frac{t}{2}} dt \leq c\int_1^\infty e^{-ct} e^{\frac{t}{2}\text{Re}(\sigma)} - \frac{1}{2} dt < \infty. \tag{4.22}
\]
By (4.17), (4.18), and (4.22), it follows that for \(\text{Re}(s) > -2\),
\[
\eta(s, D^2_\lambda(\sigma)) = \eta_0(s, D^2_\lambda(\sigma)) + \frac{1}{\Gamma(\frac{t}{2})} \int_0^\infty \text{Tr}(\Pi_+ D^2_\lambda(\sigma) e^{-tD^2_\lambda(\sigma)^2})|^{\frac{t}{2}} dt.
\]

5 Functional equations for the Selberg zeta function

The meromorphic continuation of the Selberg and Ruelle zeta function associated with non-unitary representations of the subgroup \(\Gamma\) is proved in [Spi18]. Our aim is to derive the functional equations for the Selberg zeta function in case (a), and for the symmetrized and super zeta function in case (b). We recall here the formulas proved in [Spi18] (see (5.1) and (5.2) below), using the generalized resolvent identity and the trace formulas. More precisely, let \(N \in \mathbb{N}\) and \(i = 1, \ldots, N\). Let \(s_i\) be complex numbers with \(s_i^k \neq s_j^k\) for all \(i \neq j\). Let \(\text{Re}(s_i^k) \gg 0\). We have the following expressions
\[
R(s_i^k) = (A^2_\lambda(\sigma) + s_i^k)^{-1} = \int_0^\infty e^{-ts_i^k} e^{-tA^2_\lambda(\sigma)} dt;
\]
\[
D^2_\lambda(\sigma)R(s_i^k) = D^2_\lambda(\sigma)(D^2_\lambda(\sigma)^2 + s_i^k)^{-1} = \int_0^\infty e^{-ts_i^k} D^2_\lambda(\sigma) e^{-tD^2_\lambda(\sigma)^2} dt
\]
(recall that \(D^2_\lambda(\sigma)^2 = A^2_\lambda(\sigma)\)). We choose \(N\) such that \(N > \frac{d}{2} + 1\). Then, the operators \(D^2_\lambda(\sigma)R(s_i^k)\) and \(R(s_i^k)\) are trace class. By [Spi18 Lemma 7.5], for \(N > \frac{d}{2}\), the operator \(\prod_{i=1}^N R(s_i^k)\) is trace class, and for \(N > \frac{d}{2} + 1\), the operator \(D^2_\lambda(\sigma) \prod_{i=1}^N R(s_i^k)\) is trace class. We use the generalized resolvent identity as in Lemma 3.5 in [BO95]. In our case this lemma reads
\[
\prod_{i=1}^N R(s_i^k) = \frac{1}{s_i^k} \left( \prod_{j=1}^N \frac{1}{s_j^k - s_i^k} \right) R(s_i^k);
\]
\[
D^2_\lambda(\sigma) \prod_{i=1}^N R(s_i^k) = D^2_\lambda(\sigma) \sum_{i=1}^N \left( \prod_{j=1}^N \frac{1}{s_j^k - s_i^k} \right) R(s_i^k).
\]
Hence, if we take into account the integral representations of the resolvent operators above, we have

\[ \prod_{i=1}^{N} (A_{\chi}^\#(s) + s_i^2)^{-1} = \int_0^\infty \sum_{i=1}^{N} \left( \prod_{j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} e^{-tA_{\chi}^\#(s)} dt; \]

\[ D_{\chi}^\#(s) \prod_{i=1}^{N} (D_{\chi}^\#(s)^2 + s_i^2)^{-1} = \int_0^\infty \sum_{i=1}^{N} \left( \prod_{j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_{\chi}^\#(s) e^{-tD_{\chi}^\#(s)} dt. \]

We plug now the trace of the corresponding operators and use the trace formulas \cite[eq. (7.27) and (7.34)]{Spi18}.

Remark: In the trace formulas (7.27) and (7.34) in \cite{Spi18}, the traces of the operators are super traces, corresponding to the grading of the locally homogeneous vector bundle \( E(\sigma) \) over \( X \), defined as in Section 4. For more details about the grading of \( E(\sigma) \), we refer the reader to \cite[p. 27, 29]{BO95} and \cite[p. 175]{Spi18}). We denote the super trace of the corresponding operators by \( \text{Tr}_s \). By \cite[eq. (7.27) and (7.34)]{Spi18}, we have

- **case (a)**

\[
\begin{align*}
\text{Tr}_s \prod_{i=1}^{N} (A_{\chi}^\#(s) + s_i^2)^{-1} &= \sum_{i=1}^{N} \left( \prod_{j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi \dim(V_{\chi}) \text{Vol}(X) \text{P}_s(s_i)}{s_i} \\
&\quad + \sum_{i=1}^{N} \frac{1}{2s_i} \left( \prod_{j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) L(s_i). \tag{5.1}
\end{align*}
\]

- **case (b)**

\[
\begin{align*}
\text{Tr}_s \prod_{i=1}^{N} (A_{\chi}^\#(s) + s_i^2)^{-1} &= \sum_{i=1}^{N} \left( \prod_{j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi \dim(V_{\chi}) \text{Vol}(X) \text{P}_s(s_i)}{s_i} \\
&\quad + \sum_{i=1}^{N} \left( \prod_{j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} L_s(s_i). \tag{5.2}
\end{align*}
\]

We consider first case (a).

**Lemma 5.1.** The logarithmic derivative \( L(s) \) of the Selberg zeta function satisfies the following functional equation

\[ L(s) + L(-s) = -4\pi \dim(V_{\chi}) \text{Vol}(X) \text{P}_s(s). \tag{5.3} \]

**Proof.** We fix the complex numbers \( s_2, \ldots, s_N \in \mathbb{C} \) and let \( s_1 = s \in \mathbb{C} \) vary. The left-hand side of (5.1) is invariant under \( s \mapsto -s \). Hence, the right-hand
side of (5.1) is also invariant under \( s \mapsto -s \). We have
\[
\left( \prod_{j=2 \atop j \neq i}^{N} \frac{1}{s_{j}^{2} - s^{2}} \right) \frac{1}{2s} L(s) \mapsto \left( \prod_{j=2 \atop j \neq i}^{N} \frac{1}{s_{j}^{2} - s^{2}} \right) \frac{1}{-2s} L(-s).
\]

Since the Plancherel polynomial is an even polynomial of \( s \)
\[
\left( \prod_{j=2 \atop j \neq i}^{N} \frac{1}{s_{j}^{2} - s^{2}} \right) \frac{\pi}{s} \dim(V_{\chi}) \operatorname{Vol}(X) P_{\sigma}(s) \mapsto \left( \prod_{j=2 \atop j \neq i}^{N} \frac{1}{s_{j}^{2} - s^{2}} \right) \frac{\pi}{s} \dim(V_{\chi}) \operatorname{Vol}(X) P_{\sigma}(s).
\]

We subtract the resulting equation from (5.1). Hence,
\[
\left( \prod_{j=2 \atop j \neq i}^{N} \frac{1}{s_{j}^{2} - s^{2}} \right) \frac{1}{2s} (L(s) + L(-s)) = \left( \prod_{j=2 \atop j \neq i}^{N} \frac{1}{s_{j}^{2} - s^{2}} \right) \frac{2\pi}{s} \dim(V_{\chi}) \operatorname{Vol}(X) P_{\sigma}(s).
\]

We multiply the above equation by the function \( 2s \prod_{j=2}^{N} (s_{j}^{2} - s^{2}) \). Then, we get
\[
L(s) + L(-s) = -4\pi \dim(V_{\chi}) \operatorname{Vol}(X) P_{\sigma}(s).
\]

**Theorem 5.2.** The Selberg zeta function \( Z(s; \sigma, \chi) \) satisfies the following functional equation
\[
Z(s; \sigma, \chi) Z(-s; \sigma, \chi) = \exp \left( -4\pi \dim(V_{\chi}) \operatorname{Vol}(X) \int_{0}^{s} P_{\sigma}(r) dr \right).
\]

**Proof.** We integrate over \( s \) and exponentiate equation (5.3). The assertion follows.

We treat now case (b).

**Lemma 5.3.** The logarithmic derivative \( L_{S}(s) \) of the symmetrized zeta function satisfies the following functional equation
\[
L_{S}(s) + L_{S}(-s) = -8\pi \dim(V_{\chi}) \operatorname{Vol}(X) P_{\sigma}(s).
\]

**Proof.** We consider (5.2) at \( s \mapsto -s \). We subtract the resulting equation from (5.2). Using the same argument as in Lemma 5.1, we have
\[
\left( \prod_{j=2}^{N} \frac{1}{s_{j}^{2} - s^{2}} \right) \frac{1}{2s} (L_{S}(s) + L_{S}(-s)) = \left( \prod_{j=2}^{N} \frac{1}{s_{j}^{2} - s^{2}} \right) \frac{4\pi}{s} \dim(V_{\chi}) \operatorname{Vol}(X) P_{\sigma}(s).
\]

We multiply the above equation by the function
\[
2s \prod_{j=2}^{N} (s_{j}^{2} - s^{2}),
\]
and get equation (5.5).
Theorem 5.4. The symmetrized zeta function satisfies the following functional equation

\[ S(s; \sigma, \chi) = \exp \left( -8 \pi \dim(V_\chi) \Vol(X) \int_0^\infty P_\sigma(r) dr \right). \] (5.6)

Proof. We integrate over \( s \) and exponentiate equation (5.5). The assertion follows.

Theorem 5.5. The super zeta function satisfies the functional equation

\[ Z^*(s; \sigma, \chi) Z^*(-s; \sigma, \chi) = e^{2\pi i \eta(0, D_\chi^2(\sigma))}, \] (5.7)

where \( \eta(0, D_\chi^2(\sigma)) \) is the function \( \eta(s, D_\chi^2(\sigma)) \) as in (4.14) in Lemma 4.6, at \( s = 0 \). Furthermore,

\[ Z^*(0; \sigma, \chi) = e^{\pi i \eta(0, D_\chi^2(\sigma))}. \] (5.8)

Proof. By [Spi18, Proposition 7.7], we have

\[ \Tr(D_\chi^2(\sigma) \prod_{i=1}^N (D_\chi^2(\sigma)^2 + s_i^2)^{-1}) = \frac{-i}{2} \sum_{i=1}^N \left( \prod_{j \neq i} \frac{1}{s_j - s_i} \right) L^*(s_i). \]

Equivalently, for \( \Re(s_i^2) \gg 0 \), we get

\[ \sum_{i=1}^N \left( \prod_{j=1}^N \frac{1}{s_j^2 - s_i^2} \right) L^*(s_i) = 2i \Tr(D_\chi^2(\sigma) \prod_{i=1}^N (D_\chi^2(\sigma)^2 + s_i^2)^{-1}) = 2i \int_0^\infty \sum_{i=1}^N \left( \prod_{j \neq i} \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^2(\sigma) e^{-tD_\chi^2(\sigma)^2} dt, \]

Using the same argument as in the proof of Proposition 7.7 in [Spi18], we obtain

\[ \sum_{i=1}^N \left( \prod_{j=1}^N \frac{1}{s_j^2 - s_i^2} \right) L^*(s_i) = 2i \int_0^\infty \sum_{i=1}^N \left( \prod_{j \neq i} \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \Tr(D_\chi^2(\sigma) e^{-tD_\chi^2(\sigma)^2}) dt. \] (5.9)

We fix now \( s_2, \ldots, s_N \in \mathbb{C} \) and let \( s_1 = s \in \mathbb{C} \) vary. We choose \( s_i \) such that \( \Re(s_i^2) \to \infty \). In the right-hand side of (5.9), the integrals that include the exponentials \( e^{-ts_i^2} \) are

\[ \int_0^\infty e^{-ts_i^2} \Tr(D_\chi^2(\sigma) e^{-tD_\chi^2(\sigma)^2}) dt. \]

As \( t \to \infty \), \( \Tr(D_\chi^2(\sigma) e^{-tD_\chi^2(\sigma)^2}) \) and \( e^{-ts_i^2} \) decay exponentially. As \( t \to 0^+ \), we use the asymptotic expansion of the trace of the operator \( D_\chi^2(\sigma) e^{-tD_\chi^2(\sigma)} \). By Lemma 4.1, we have

\[ \Tr(D_\chi^2(\sigma) e^{-tD_\chi^2(\sigma)}) \sim_{t \to 0^+} \dim(V_\chi)(\alpha_0 t^{1/2} + O(t^{3/2})), \]
where
\[ \alpha_0 = \int_X a_0(x) d\mu(x), \]
\( a_0(x) \) is a smooth function and \( \mu(x) \) is the volume measure determined by the Riemannian metric on \( X \). Hence, each of the integrals
\[ \int_0^\infty e^{-ts^2} \text{Tr}(D^2_X(\sigma) e^{-t D^2_X(\sigma)^2}) dt, \]
is well defined. We multiply both sides of (5.9) by the finite product
\[ \prod_{j=2}^N s_j^2 - s^2. \]
Then, we obtain
\[ L^s(s) = 2i \int_0^\infty e^{-ts^2} \text{Tr}(D^2_X(\sigma) e^{-t D^2_X(\sigma)^2}) dt + \left( \prod_{j=2}^N s_j^2 - s^2 \right) \Xi(s_2, \ldots, s_N). \]
(5.10)
The term \( \Xi(s_2, \ldots, s_N) \) comes from the terms that correspond to the summands over \( i = 2, \ldots, N \) and hence it has a fixed value in \( \mathbb{C} \), since \( s_2, \ldots, s_N \) are fixed. The estimation of the sum on the right hand side of (3.14) on p. 163 of [Spi18] can be improved to show that the log of the Selberg zeta function is exponentially decreasing. Hence, following a similar consideration, by (3.7), \( L^s(s) \) decreases exponentially, as \( \text{Re}(s_i) \to \infty \). Therefore, the even polynomial arising from the term
\[ \left( \prod_{j=2}^N s_j^2 - s^2 \right) \Xi(s_2, \ldots, s_N) \]
vanishes. Hence, we have
\[ L^s(s) = 2i \int_0^\infty e^{-ts^2} \text{Tr}(\Pi_+ D^2_X(\sigma) e^{-t D^2_X(\sigma)^2}) dt. \]
Let \( \Pi_+ \) (resp. \( \Pi_- \)) be the projection on the span of the root spaces corresponding to eigenvalues \( \lambda \) of \( D^2_X(\sigma) \) with \( \text{Re}(\lambda^2) > 0 \) (resp. \( \text{Re}(\lambda^2) \leq 0 \)). Recall that there are only finitely many eigenvalues of \( D^2_X(\sigma) \) such that \( \text{Re}(\lambda^2) \leq 0 \). We write
\[ L^s(s) = 2i \int_0^\infty e^{-ts^2} \text{Tr}(\Pi_+ D^2_X(\sigma) e^{-t D^2_X(\sigma)^2}) dt \\
+ 2i \int_0^\infty e^{-ts^2} \text{Tr}(\Pi_- D^2_X(\sigma) e^{-t D^2_X(\sigma)^2}) dt. \]
(5.11)
We set
\[ I_+(s) := \int_0^\infty e^{-ts^2} \text{Tr}(\Pi_+ D^2_X(\sigma) e^{-t D^2_X(\sigma)^2}) dt \]
\[ I_-(s) := \int_0^\infty e^{-ts^2} \text{Tr}(\Pi_- D^2_X(\sigma) e^{-t D^2_X(\sigma)^2}) dt. \]
For \( s \in \mathbb{C} \) with \( \text{Re}(s) > 0 \), we consider the integral
\[
\int_s^\infty I_+(w)dw = \int_s^\infty \int_0^\infty e^{-tw^2} \text{Tr}(\Pi_+ D^2_\sigma e^{-tD_\sigma^2})dt dw
\]
\[
= \int_0^\infty \int_s^\infty e^{-tw^2} \text{Tr}(\Pi_+ D^2_\sigma e^{-tD_\sigma^2})dw dt.
\]

If we make the change of variables \( w \mapsto \frac{1}{\sqrt{t}} u \), we get
\[
\int_s^\infty I_+(w)dw = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{t}} e^{-u^2} \text{Tr}(\Pi_+ D^2_\sigma e^{-tD_\sigma^2})dudt.
\]

We use now the error function for a complex number \( z \in \mathbb{C} \),
\[
\Phi(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du.
\]
It holds
\[
\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du = 1 - \Phi(z).
\]
Hence,
\[
\int_s^\infty I_+(w)dw = \int_0^\infty \frac{\sqrt{\pi}}{2\sqrt{t}} \left( 1 - \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{ts}} e^{-u^2} du \right) \text{Tr}(\Pi_+ D^2_\sigma e^{-tD_\sigma^2})dt,
\]
and
\[
\int_{-s}^\infty I_+(w)dw = \int_0^\infty \frac{\sqrt{\pi}}{2\sqrt{t}} \left( 1 - \frac{2}{\sqrt{\pi}} \int_0^{-\sqrt{ts}} e^{-u^2} du \right) \text{Tr}(\Pi_+ D^2_\sigma e^{-tD_\sigma^2})dt.
\]
We add together (5.11) and (5.12) to get
\[
\int_s^\infty I_+(w)dw + \int_{-s}^\infty I_+(w)dw = \int_0^\infty \frac{\sqrt{\pi}}{\sqrt{t}} \text{Tr}(\Pi_+ D^2_\sigma e^{-tD_\sigma^2})dt.
\]

For \( s \in \mathbb{C} \) with \( \text{Re}(s) > 0 \), we consider now the integral \( \int_s^\infty I_-(w)dw \). Since there are only finitely many eigenvalues of \( D^2_\sigma \) with \( \text{Re}(\lambda^2) \leq 0 \), we can interchange the order of integration and write
\[
\int_s^\infty I_-(w)dw = \int_s^\infty \int_0^\infty e^{-tw^2} \text{Tr}(\Pi_- D^2_\sigma e^{-tD_\sigma^2})dt dw
\]
\[
= \sum_{\lambda, \text{Re}(\lambda^2) \leq 0} \lambda \int_s^\infty \int_0^\infty e^{-tw^2} e^{-t\lambda^2} dt dw
\]
\[
= \sum_{\lambda, \text{Re}(\lambda^2) \leq 0} \lambda \int_s^\infty \frac{1}{w^2 + \lambda^2} dw.
\]

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We substitute at (5.14) \( s \mapsto -s \) and add the resulting equation to (5.14). Then, by change of variables, \( w \mapsto w' = w/\lambda \), we obtain

\[
\int_s^\infty L_-(w)dw + \int_{-s}^\infty L_-(w)dw = \sum_{\text{Re}(\lambda) > 0} \pi - \sum_{\text{Re}(\lambda) < 0} \pi
\]

The sums over \( \lambda \) in the equation above are finite, because we sum over \( \lambda \) with \( \text{Re}(\lambda^2) \leq 0 \), and there are only finitely many eigenvalues such that \( \text{Re}(\lambda^2) \leq 0 \).

Recall the definition of \( \eta_0(0, D^\sharp_\chi(\sigma)) \) from Section 4:

\[
\eta_0(s, D^\sharp_\chi(\sigma)) := \sum_{\text{Re}(\lambda) > 0} \lambda^{-s} - \sum_{\text{Re}(\lambda) < 0} \lambda^{-s}.
\]

Then,

\[
\int_s^\infty L_-(w)dw + \int_{-s}^\infty L_-(w)dw = \pi \eta_0(0, D^\sharp_\chi(\sigma)). \tag{5.16}
\]

By Lemma 4.6, we have,

\[
\eta(s, D^\sharp_\chi(\sigma)) = \eta_0(s, D^\sharp_\chi(\sigma)) + \frac{1}{\Gamma(s+1)} \int_0^\infty \text{Tr}(\Pi_+ D^\sharp_\chi(\sigma)e^{-t(D^\sharp_\chi(\sigma))^2})t^{s+1-1}dt.
\]

Hence, by (5.10), (5.13) and (5.15) we get

\[
\log Z^s(s; \sigma, \chi) + \log Z^s(-s; \sigma, \chi) = \int_s^\infty L^s(w)dw + \int_{-s}^\infty L^s(w)dw \tag{5.17}
\]

\[
= 2i \int_s^\infty I_+(w)dw + 2i \int_{-s}^\infty I_+(w)dw
\]

\[
+ 2i \int_s^\infty I_-(w)dw + 2i \int_{-s}^\infty I_-(w)dw
\]

\[
= 2\pi i (\eta_1(0, D^\sharp_\chi(\sigma)) + \eta_0(0, D^\sharp_\chi(\sigma)))
\]

\[
= 2\pi i \eta_0(0, D^\sharp_\chi(\sigma)). \tag{5.18}
\]

We mention here that the logarithm of the super zeta function

\[
\log Z^s(s; \sigma, \chi) = \int_s^\infty L^s(w)dw, \tag{5.19}
\]

does depend on the choice of the path connecting \( s \) and infinity. However, in order to obtain the functional equations, we have to exponentiate the integral in the right-hand side of (5.18). Since the residues of the singularities of the super zeta functions are all integers ([Spi18, Theorem 7.8]), the exponential of this integral is independent of the choice of the path. Equation (5.7) follows by exponentiation of (5.17). Also, by (5.17),

\[
\log Z^s(s; \sigma, \chi) + \log Z^s(-s; \sigma, \chi) = 2\pi i \eta_0(0, D^\sharp_\chi(\sigma))
\]

For \( s = 0 \),

\[
2 \log Z^s(0; \sigma, \chi) = 2\pi i \eta_0(0, D^\sharp_\chi(\sigma))
\]
Equivalently,
\[ Z^*(0; \sigma, \chi) = e^{\pi i \eta(0, D^\chi_\sigma(\sigma))}. \]

We prove now the functional equation for the Selberg zeta function in case (b).

**Theorem 5.6.** The Selberg zeta function satisfies the following functional equation
\[ \frac{Z(s; \sigma, \chi)}{Z(-s; w\sigma, \chi)} = e^{\pi i \eta(0, D^\chi_\sigma(\sigma))} \exp \left( -4\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(r) dr \right). \] (5.20)

**Proof.** By (3.3) and (3.4), we have
\[ \frac{Z(s; \sigma, \chi)}{Z(-s; w\sigma, \chi)} = \prod_{\sigma' \in \hat{M}} \frac{Z^*(s; \sigma, \chi)}{Z^*(-s; w\sigma, \chi)} \]
\[ = \frac{S(s; \sigma, \chi) Z^*(s; \sigma, \chi)}{S(-s; w\sigma, \chi) Z^*(-s; \sigma, \chi)} \]
\[ = e^{\pi i \eta(0, D^\chi_\sigma(\sigma))} \exp \left( -4\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(r) dr \right), \]
where in the last equation we have employed Theorem 5.4 and 5.5. We mention here that we consider only the plus sign of the square root in the equation above since by the functional equations (5.8), the super zeta function at \( s = 0 \) is equal to \( e^{\pi i \eta(0, D^\chi_\sigma(\sigma))} \) only.

### 6 Functional equations for the Ruelle zeta function

To prove the functional equations for the Ruelle zeta function, we have to consider Selberg zeta functions associated with representations of \( M \), which are not irreducible. In such case, for a finite dimensional representation \( \sigma \) of \( M \), we have
\[ Z(s; \sigma, \chi) = \prod_{\sigma' \in \hat{M}} Z(s; \sigma', \chi)^{[\sigma : \sigma']}, \]
where \([\sigma : \sigma']\) is the multiplicity of \( \sigma' \) in \( \sigma \).

Recall the Iwasawa decomposition \( G = KAN \) of \( G \). Let \( \mathfrak{n} \) be the Lie algebra of \( N \). Let \( \mathfrak{n}_C \) be the complexification of \( \mathfrak{n} \). Let \( \nu_p \) be the representation of \( MA \) in \( \Lambda^p \mathfrak{n}_C \), given by the \( p \)-th exterior power of the adjoint representation
\[ \nu_p := \Lambda^p \text{Ad}_{\mathfrak{n}_C} : MA \to \text{GL}(\Lambda^p \mathfrak{n}_C), \quad p = 0, 1, \ldots, d - 1. \]

We consider the following identification \( \mathfrak{a}_C^* \cong \mathbb{C} \). Let \( \alpha > 0 \) be the unique positive root of \( (\mathfrak{g}, \mathfrak{a}) \). Let \( \lambda : A \to \mathbb{C}^\times \) be the character, defined by \( \lambda(a) = e^{\alpha \log a^*} \).

We consider also finite-dimensional, irreducible representations \( (\psi_p, V_{\psi_p}) \in \hat{M} \).
We define the set $J_p$ as the subset $J_p \subseteq \{ (\psi_p, \lambda) : \psi_p \in \hat{M}, \lambda \in \mathbb{C} \}$ such that the following decomposition holds.

$$A^p_{\mathfrak{n}_\mathbb{C}} = \sum_{(\psi_p, \lambda) \in J_p} V_{\psi_p} \otimes C_\lambda.$$ 

By Poincaré duality ([BO95, p. 122]), we have for $p < \frac{d-1}{2},$

$$J_{d-1-p} \subseteq \{ (\psi_p, 2|\rho| - \lambda) : \psi_p \in \hat{M}, \lambda \in \mathbb{C} \}. \quad (6.1)$$

Let $p_d = i p_d.$ Let $g_d$ be the subalgebra of $g_{\mathbb{C}}$ defined by $g_d = k \oplus p_d.$ Let $G_d$ be the Lie group that corresponds to $g_d.$ $G_d$ is the the compact real form of the complexification $G_{\mathbb{C}}$ of $G.$ Let $\mathfrak{a}_d$ be the subalgebra of $p_d$ defined by $\mathfrak{a}_d = i \mathfrak{a}.$ Let $A_d$ be the corresponding Lie group. We set $L := G_d/MA_d,$ which is a Kähler manifold ([BO95, p. 123]). For $\psi_p, \sigma \in \hat{M},$ we define

$$Q_{\psi_p \otimes \sigma}(s) := \sum_{\sigma' \in \hat{M}} [(\psi_p \otimes \sigma) : \sigma'] Q_{\sigma'}(s),$$

where $[(\psi_p \otimes \sigma) : \sigma']$ is the multiplicity of $\sigma'$ in $\psi_p \otimes \sigma.$

**Lemma 6.1.** Let $(\sigma, V_{\sigma}) \in \hat{M}.$ Let $Q_{\psi_p \otimes \sigma}(s), s \in \mathbb{C},$ be the Plancherel measure associated with the representation $\psi_p \otimes \sigma,$ $p = 0, \ldots, d - 1.$ Let $f(s)$ be the polynomial of $s$ given by

$$f(s) := (-1)^{\frac{d-1}{2}} Q_{\psi_{d-1} \otimes \sigma}(s)$$

$$+ \sum_{p=0}^{d-1} \sum_{(\psi_p, \lambda) \in J_p} (-1)^p [Q_{\psi_p \otimes \sigma}(s + |\rho| - \lambda) + Q_{\psi_p \otimes \sigma}(s - |\rho| + \lambda)]$$

$$= \sum_{p=0}^{d-1} \sum_{(\psi_p, \lambda) \in J_p} (-1)^p Q_{\psi_p \otimes \sigma}(s; \rho, \lambda). \quad (6.2)$$

Then,

$$f(s) = (d + 1) \dim(V_{\sigma}). \quad (6.3)$$

**Proof.** As in the proof of Theorem 4.4 in [BO95, p. 127], the polynomial $f(s)$ is given by

$$f(s) = \dim(V_{\sigma}) \chi(L).$$

The Euler characteristic $\chi(L)$ is also calculated in [BO95 p.127]:

$$\chi(L) = d + 1.$$

The assertion follows.

**Theorem 6.2.** The Ruelle zeta function satisfies the following functional equation

$$\frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} = \exp \left( (-1)^{\frac{d+1}{2}} \cdot 2\pi (d + 1) \dim(V_{\sigma}) \dim(V_{\chi}) \frac{\text{Vol}(X)}{\text{Vol}(S^d)} s \right). \quad (6.4)$$

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Proof. By [Spi18] Theorem 7.12, we have the following representation of the Ruelle zeta function.

\[ R(s; \sigma, \chi) = \prod_{p=0}^{d-1} \prod_{(\psi_p, \lambda) \in I_p}(Z(s + |\rho| - \lambda; \psi_p \otimes \sigma, \chi))^{(-1)^p}. \tag{6.5} \]

Using Poncaré duality, i.e., (6.1), and considering

\[ s + |\rho| - \lambda \mapsto s + |\rho| - (2|\rho| - \lambda) = s - |\rho| + \lambda, \]

we get

\[ R(s; \sigma, \chi) = Z(s; \psi_{d-1} \otimes \sigma, \chi)^{(-1)^{d-1}} \prod_{p=0}^{d-1} \prod_{(\psi_p, \lambda) \in I_p}(Z(s + |\rho| - \lambda; \psi_p \otimes \sigma, \chi)Z(s - |\rho| + \lambda; \psi_p \otimes \sigma, \chi))^{(-1)^p}. \]

Hence,

\[
\frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} = \left( \frac{Z(s; \psi_{d-1} \otimes \sigma, \chi)}{Z(-s; \psi_{d-1} \otimes \sigma, \chi)} \right)^{(-1)^{d-1}} \prod_{p=0}^{d-1} \prod_{(\psi_p, \lambda) \in I_p} \left( \frac{Z(s + |\rho| - \lambda; \psi_p \otimes \sigma, \chi)Z(s - |\rho| + \lambda; \psi_p \otimes \sigma, \chi)}{Z(-s + |\rho| - \lambda; \psi_p \otimes \sigma, \chi)Z(-s - |\rho| + \lambda; \psi_p \otimes \sigma, \chi)} \right)^{(-1)^p}.
\]

By Theorem 5.2, we get

\[
\frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} = \exp \left\{ -4\pi \dim(V_{\chi}) \Vol(X) \left( (-1)^{d-1} \int_0^s P_{\psi_{d-1} \otimes \sigma}(r)dr + \sum_{p=0}^{d-1} \sum_{(\psi_p, \lambda) \in I_p} (-1)^p \left( \int_0^{s+|\rho| - \lambda} P_{\psi_p \otimes \sigma}(r)dr + \int_0^{s-|\rho| + \lambda} P_{\psi_p \otimes \sigma}(r)dr \right) \right) \right\}. \tag{6.6}
\]

We set

\[ F(s) = (-1)^{d-1} \int_0^s Q_{\psi_{d-1} \otimes \sigma}(r)dr + \sum_{p=0}^{d-1} \sum_{(\psi_p, \lambda) \in I_p} (-1)^p \left( \int_0^{s+|\rho| - \lambda} Q_{\psi_p \otimes \sigma}(r)dr + \int_0^{s-|\rho| + \lambda} Q_{\psi_p \otimes \sigma}(r)dr \right). \]

Then,

\[
\frac{d}{ds} F(s) = f(s),
\]

where \( f(s) \) is as in (6.2). Using Lemma 6.1, we get

\[
\frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} = \exp \left( 4\pi c(n) \dim(V_{\chi}) \Vol(X) [(d + 1) \dim(V_{\sigma})s + C] \right). \tag{6.7}
\]

where \( c(n) \) is as in (2.9) and \( C \in \mathbb{R} \) is a real constant. On the other hand, if we set \( s = 0 \) in (6.7), we get \( 1 = \exp(4\pi c(n) \dim(V_{\chi}) \Vol(X)C) \), and hence \( C = 0 \). The assertion follows. \( \square \)
We examine now case (b). Let $\tau_p$ be the standard representation of $K$ in $\Lambda^p\mathbb{R}^d \otimes \mathbb{C}$. Let $(\sigma_p, V_{\sigma_p})$ be the standard representation of $M$ in $\Lambda^p\mathbb{R}^{d-1} \otimes \mathbb{C}$. Let $\alpha > 0$ be the unique positive root of $(g, a)$. Let $\lambda \equiv \lambda_p: A \to \mathbb{C}^\times$ be the character, defined by $\lambda \equiv \lambda_p(a) = e^{p\alpha \log a}$. Then, as a representation of $MA$, one has $\nu_p = \sigma_p \otimes \lambda_p$. We denote by $C_p \cong \mathbb{C}$ the representation space of $\lambda_p$.

Then, in the sense of $MA$-modules, we have

$$\Lambda^p u_C = \Lambda^p \mathbb{R}^{d-1} \otimes C_p.$$  \hfill (6.8)

Let $D_\chi^2(\sigma)$ be the twisted Dirac operator acting on $C^\infty(X, E_{\tau_1(\sigma)} \otimes E_\chi)$. We introduce here the twist $\tilde{D}_\chi^2(\sigma)$ of the Dirac operator $D_\chi^2(\sigma)$ acting on smooth sections of the vector bundle

$$V_{\tau_1(\sigma)} \otimes V_\chi \otimes \left(\sum_{k=0}^{d-1} (-1)^k (d - k) \Lambda^k T^* X\right).$$

The twisted Dirac operator $\tilde{D}_\chi^2(\sigma)$ is defined in a similar way as the Dirac operator $D_\chi^2(\sigma)$ in Section 4, equipping the bundle $\Lambda^k T^* X$ with the Levi-Civita connection of $X$.

**Theorem 6.3.** The super Ruelle zeta function associated with a non-Weyl invariant representation $\sigma \in M$ satisfies the functional equation

$$R^s(s; \sigma, \chi)R^s(-s; \sigma, \chi) = e^{2\pi \eta(\tilde{D}_\chi^2(\sigma))},$$  \hfill (6.9)

where $\eta(\tilde{D}_\chi^2(\sigma))$ denotes the eta invariant of the twisted Dirac operator $\tilde{D}_\chi^2(\sigma)$. Moreover, the following equation holds

$$\frac{R(s; \sigma, \chi)}{R(-s; w\sigma, \chi)} = e^{\pi \eta(\tilde{D}_\chi^2(\sigma))} \exp \left((-1)^{d-1} \frac{1}{2} + 12\pi (d+1) \dim(V_\sigma) \dim(V_\chi) \frac{\text{Vol}(X)}{\text{Vol}(S^d)} s\right).$$  \hfill (6.10)

**Proof.** By [BO95, p. 23], we have

$$\sigma_p = i^s ((-1)^0 \tau_0 + (-1)^1 \tau_{p-1} + \ldots + (-1)^{p-1} (\tau_1 - \text{Id})),$$

$$p = 1, 2, \ldots d - 1, \quad s^+ + s^- = i^s(s), \quad \text{otherwise}.$$

If we take the alternating sum of $\sigma_p$ over $p$ we get

$$\sum_{p=0}^{d-1} (-1)^p \sigma_p = i^s \left(\sum_{k=0}^{d-1} (-1)^k (d - k) \tau_k\right).$$  \hfill (6.11)

By the definition of the super Ruelle zeta function, we have

$$R^s(s; \sigma, \chi)R^s(-s; \sigma, \chi) = \frac{R(s; \sigma, \chi)}{R(-s; w\sigma, \chi)} \frac{R(-s; \sigma, \chi)}{R(s; \sigma, \chi)}.$$  \hfill (6.12)

We use now the representation (6.5) of the Ruelle zeta function. By the Poincaré duality we obtain

$$R(s; \sigma, \chi) \overset{\text{def}}{=} Z(s; \sigma \oplus \sigma, \chi)^{(-1)^d} \prod_{p=0}^{d-1} (Z(s + |\rho| - \lambda; \sigma_p \otimes \sigma, \chi)Z(s - |\rho| + \lambda; \sigma_p \otimes \sigma, \chi))^{(-1)^p}.$$  \hfill (6.13)
If we substitute this expression in the right-hand side of (6.12), we have

\[ R^*(s; \sigma, \chi) R^*(-s; \sigma, \chi) = \frac{Z(s; \sigma_{-1} \otimes \sigma, \chi)}{Z(-s; \sigma_{-1} \otimes w\sigma, \chi)} \cdot \frac{1}{\prod_{p=0}^{d-3} (Z(s + |\rho| - \lambda; \sigma_p \otimes \sigma, \chi) Z(s - |\rho| + \lambda; \sigma_p \otimes \sigma, \chi)^{(-1)^p}} \cdot \frac{Z(-s; \sigma_{-1} \otimes \sigma, \chi)^{(-1)^d}}{Z(s; \sigma_{-1} \otimes w\sigma, \chi)} \]

By Theorem 5.6, we get

\[ R^*(s; \sigma, \chi) R^*(-s; \sigma, \chi) = (e^{2i\pi\eta(0, D_\chi^1(\sigma \otimes \sigma_{d-1/2}))} (s; \sigma, \chi)^{(-1)^d} \prod_{p=0}^{d-3} (e^{2i\pi\eta(0, D_\chi^p(\sigma \otimes \sigma_p))}) (s; \sigma, \chi)) \]

where we used the fact that the Plancherel polynomial is an even function. Finally, we have

\[ R^*(s; \sigma, \chi) R^*(-s; \sigma, \chi) = e^{2i\pi \sum_{p=0}^{d-1} (-1)^p \eta(0, D_\chi^p(\sigma \otimes \sigma_p))} \]

where \( \eta(D_\chi^\sigma(\sigma)) = \sum_{p=0}^{d-1} (-1)^p \eta(0, D_\chi^p(\sigma \otimes \sigma_p)) \). For the functional equations (6.10), we have

\[ \frac{R(s; \sigma, \chi)^2}{R(-s; w\sigma, \chi)^2} = \frac{R(s; \sigma, \chi)}{R(-s; w\sigma, \chi)} \cdot \frac{R(-s; \sigma, \chi)}{R(s; w\sigma, \chi)} \cdot \frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} \cdot \frac{R(s; w\sigma, \chi)}{R(-s; w\sigma, \chi)} \]

By (6.4), we get

\[ \frac{R(s; w\sigma, \chi)}{R(-s; w\sigma, \chi)} = \exp \left( (-1)^\frac{d-1}{2} + 2\pi(d+1) \dim(V_\sigma) \dim(V^\chi) \frac{\Vol(X)}{\Vol(S^d)} s \right). \]

Hence,

\[ \frac{R(s; \sigma, \chi)^2}{R(-s; w\sigma, \chi)^2} = e^{2i\pi\eta(D_\chi^2(\sigma))} \exp 2 \left( (-1)^\frac{d-1}{2} + 2\pi(d+1) \dim(V_\sigma) \dim(V^\chi) \frac{\Vol(X)}{\Vol(S^d)} s \right). \]

By (6.13), we have

\[ \log(R^*(s; \sigma, \chi)) + \log(R^*(-s; \sigma, \chi)) = 2i\pi \eta(D_\chi^2(\sigma)) \]
For $s = 0$,

$$2 \log(R^s(0; \sigma, \chi)) = 2i\pi\eta(D^\sharp_{\chi}(\sigma))$$

Equivalently,

$$R^s(0; \sigma, \chi) = \exp(i\pi\eta(D^\sharp_{\chi}(\sigma))). \quad (6.15)$$

Hence, by (6.14)

$$\frac{R(s; \sigma, \chi)}{R(-s; w\sigma, \chi)} = e^{i\pi\eta(D^\sharp_{\chi}(\sigma))} \exp \left( (-1)^{\frac{d-1}{2}+1}2\pi(d+1) \dim(V^\chi) \dim(V^\sigma) \frac{\text{Vol}(X)}{\text{Vol}(S^d)} s \right).$$

### 7 The determinant formula

We recall here the notion of the graded regularized determinant of an elliptic differential operator. Let $E = E^+ \oplus E^-$ be a $\mathbb{Z}_2$-graded vector bundle over a compact Riemannian manifold $X$. Let $P : C^\infty(X, E) \to C^\infty(X, E)$ be an elliptic differential operator, which is bounded from below. We assume that $P$ preserves the grading, i.e., we assume that with respect to the decomposition

$$C^\infty(X, E) = C^\infty(X, E^+) \oplus C^\infty(X, E^-),$$

$P$ takes the form,

$$P = \begin{pmatrix} P^+ & 0 \\ 0 & P^- \end{pmatrix}.$$

Then, the graded determinant $\det_{gr}(P)$ of $P$ is defined by

$$\det_{gr}(P) = \frac{\det(P^+)}{\det(P^-)},$$

where $\det(P^+)$ and $\det(P^-)$ denote the regularized determinants of the operators $P^+$ and $P^-$, respectively (see [BK08, Definition 3.6]).

As it is mentioned before, $E(\sigma)$ is a $\mathbb{Z}_2$-graded locally homogeneous vector bundle over $X$ ([BO95, p. 27, 29], [Spi18, p. 175]). Moreover, $A^\sharp_{\chi}(\sigma)$ acting on $C^\infty(X, E(\sigma) \otimes E^\chi)$ preserves the grading ([Spi18, p. 175]). Hence, we consider the super trace $\text{Tr}_{\sigma}(e^{-tA^\sharp_{\chi}(\sigma)})$ in Definition 7.1 of the xi function $\xi(z, s; \sigma)$ and the generalized zeta function $\zeta(z, s; \sigma)$ below. In addition, the regularized determinants of the operators $A^\sharp_{\chi}(\sigma) + s^2$ in Theorem 7.8 and Proposition 7.9 below are graded regularized determinants, i.e., we consider

$$\det_{gr}(A^\sharp_{\chi}(\sigma) + s^2) = \frac{\det(A^\sharp_{\chi,+}(\sigma) + s^2)}{\det(A^\sharp_{\chi,-}(\sigma) + s^2)}.$$

**Remark:** If $\theta$ is an Agmon angle for $A^\sharp_{\chi,+}(\sigma) + s^2$ and $A^\sharp_{\chi,-}(\sigma) + s^2$, then the corresponding regularized determinants do not depend on the choice of the Agmon angle $\theta$, since the operators have a self-adjoint principal symbol ([BK08], [Spi18]).
Section 3.10]). Hence, there is no notion of $\theta$ in our definition.

By Lemma 4.2, there exist coefficients $a_j \in \mathbb{C}$, $j = 0, 1, \ldots$ such that

$$\text{Tr}(e^{-tA^\sharp_{\chi, z}}(\sigma)) \sim_{t \to 0^+} \dim(V_\chi) \sum_{j=0}^{\infty} a_j t^{\frac{j}{2d}}. \quad (7.1)$$

**Definition 7.1.** Let $C \in \mathbb{R}$ be such that $\Re(\lambda) > C$ for all $\lambda \in \text{spec}(A^\sharp_{\chi}(\sigma))$. For $\Re(z) > d/2$ and $\Re(s) > -C$, we define the xi function associated to the operator $A^\sharp_{\chi}(\sigma)$ by

$$\xi(z, s; \sigma) := \int_0^\infty e^{-ts} \text{Tr}_s(e^{-tA^\sharp_{\chi}(\sigma)}) t^{z-1} dt, \quad (7.2)$$

and the generalized zeta function by

$$\zeta(z, s; \sigma) := \frac{1}{\Gamma(z)} \int_0^\infty e^{-ts} \text{Tr}_s(e^{-tA^\sharp_{\chi}(\sigma)}) t^{z-1} dt. \quad (7.3)$$

We define the theta function $\theta(t)$ associated with the operator $e^{-tA^\sharp_{\chi}(\sigma)}$ by

$$\theta(t) := \text{Tr}_s(e^{-tA^\sharp_{\chi}(\sigma)})$$

We set

$$\theta_1(t) := \text{Tr}(e^{-tA^\sharp_{\chi,+}(\sigma)});$$
$$\theta_2(t) := \text{Tr}(e^{-tA^\sharp_{\chi,-}(\sigma)}).$$

Then,

$$\theta(t) = \theta_1(t) - \theta_2(t).$$

We set

$$\xi_1(z, s; \sigma) := \int_0^\infty e^{-ts} \text{Tr}(e^{-tA^\sharp_{\chi,+}(\sigma)}) t^{z-1} dt;$$
$$\xi_2(z, s; \sigma) := \int_0^\infty e^{-ts} \text{Tr}(e^{-tA^\sharp_{\chi,-}(\sigma)}) t^{z-1} dt.$$

Then, $\xi(z, s; \sigma)$ is just the Mellin transform of $e^{-ts}\theta(t)$ and moreover

$$\xi(z, s; \sigma) = \xi_1(z, s; \sigma) - \xi_2(z, s; \sigma).$$

We study here $\xi_1(z, s; \sigma)$. One can proceed similarly for $\xi_2(z, s; \sigma)$. For $\lambda_j \in \text{spec}(A^\sharp_{\chi,+}(\sigma))$, let $L^+_j$ be the horizontal half line going from $-\infty$ to $-\lambda_j$. We define $U^+_L$ to be the complement of all the half lines $L^+_j$ in $\mathbb{C}$. We denote by $\mathbb{N}_0$ the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}.$

**Lemma 7.2.** For $s \in U^+_L$, $\xi_1(z, s; \sigma)$ admits a meromorphic continuation to $z \in \mathbb{C}$. Furthermore, it has simple poles at $k_r = -\left(\frac{d-r}{d}\right)$, where $r \in \mathbb{N}_0.$
Proof. We denote by \( m(\lambda) \) the algebraic multiplicity of the eigenvalue \( \lambda \in \text{spec}(A_{\chi,+}^2(\sigma)) \). We consider the following ordering \( \text{Re}(\lambda_1) \leq \text{Re}(\lambda_2) \leq \text{Re}(\lambda_3) \leq \ldots \) of the real parts of the eigenvalues of \( A_{\chi,+}^2(\sigma) \). We observe that for every positive real number \( c \), there are only finitely many eigenvalues \( \lambda_j \) such that \( \text{Re}(\lambda_j) \leq c \). Hence, there exists a positive integer \( N \geq 1 \), such that \( \text{Re}(\lambda_j) > c \), for every \( j > N \). Then, we have

\[
\left| \sum_{j=1}^{\infty} m(\lambda_j) e^{-t\lambda_j} - \sum_{j=1}^{N} m(\lambda_j) e^{-t\lambda_j} \right| = \left| \sum_{j=N+1}^{\infty} m(\lambda_j) e^{-t\lambda_j} \right| \leq \sum_{j=N+1}^{\infty} m(\lambda_j) e^{-t\text{Re}(\lambda_j)}. \tag{7.4}
\]

Then, we have for \( t \geq 1 \)

\[
\sum_{j=N+1}^{\infty} m(\lambda_j) e^{-t\text{Re}(\lambda_j)} \leq e^{-tc/2} \sum_{j=N+1}^{\infty} m(\lambda_j) e^{-\text{Re}(\lambda_j)/2}. \tag{7.5}
\]

To estimate the last sum, we will use Weyl’s law for the non-self-adjoint operator \( A_{\chi,+}^2(\sigma) \). Given a positive constant \( k \), we define the counting function \( N(k) \) by

\[
N(k) := \sum_{\substack{\lambda \in \text{spec}(A_{\chi,+}^2(\sigma)) \\ |\lambda| \leq k}} m(\lambda).
\]

In [MüI11], the generalization of Weyl’s law for the non-self-adjoint case is proved. By [MüI11, Lemma 2.2], we have

\[
N(k) = \frac{\text{rank}(E(\sigma) \otimes E_{\chi}) \text{Vol}(X)}{(4\pi)^{d/2} \Gamma(d/2 + 1)} k^{d/2} + o(k^{d/2}), \quad k \to \infty, \tag{7.6}
\]

where \( \text{rank}(E(\sigma) \otimes E_{\chi}) \) denotes the rank of the product vector bundle \( E(\sigma) \otimes E_{\chi} \). Let \( a > 0 \) be the slope of the boundary of the cone, in which all the eigenvalues \( \lambda_j \) of \( A_{\chi,+}^2(\sigma) \) are contained. We have

\[
\sharp\{ j : |\text{Re}(\lambda_j)| \leq \lambda \} \leq \sharp\{ j : |\lambda_j| \leq \sqrt{1 + a^2 \lambda} \} \leq N(\sqrt{1 + a^2 \lambda}).
\]

By (7.6), similarly to (4.21), we get

\[
\sum_{j=N+1}^{\infty} m(\lambda_j) e^{-\text{Re}(\lambda_j)/2} < \infty. \tag{7.7}
\]

where \( C_1 \) is a positive constant.

Hence, by (7.4), (7.5), (7.7) and the definition of \( \theta_1(t) \), we have that given a positive number \( c > 0 \), there exist a positive integer \( N \) and a \( K > 0 \) such that

\[
\left| \theta_1(t) - \sum_{j=1}^{N} m(\lambda_j) e^{-t\lambda_j} \right| \leq Ke^{-ct}, \quad t \geq 1. \tag{7.8a}
\]
Furthermore, by the asymptotic expansion of the trace of the operator $e^{-t A^t_{X,\epsilon}(\sigma)}$ (7.1), we have that for every positive integer $N$,

$$\theta_1(t) - \sum_{j=0}^{N} a_j t^{j/2} = O(t^{N+1/2}), \quad t \to 0. \quad (7.8b)$$

By (7.8a) and (7.8b), $\theta_1(t)$ satisfies the assumptions as in [JL93] AS 1, AS 2, p. 16. Hence, we can apply [JL93, Theorem 1.5] for $p = \frac{d}{2}$ and obtain the meromorphic continuation of $\xi_1(z,s;\sigma)$ to $z \in \mathbb{C}$. The simple poles are located at $k_r = -(\frac{d}{2} + n)$, where $r \in \mathbb{N}_0$.

Let now $\lambda_j \in \text{spec}(A^t_{X,\epsilon}(\sigma))$ and $L_j$ be the horizontal half line going from $-\infty$ to $-\lambda_j$. We define $U_{L_j}^-$ to be the complement of all the half lines $L_j$ in $\mathbb{C}$. Then, similarly, for $\xi_2(z,s;\sigma)$, we have the following lemma.

**Lemma 7.3.** For $s \in U_{L}^+$, $\xi_2(z,s;\sigma)$ admits a meromorphic continuation to $z \in \mathbb{C}$. Furthermore, it has simple poles at $k_{j,n} = -(\frac{d}{2} + n)$, where $n \in \mathbb{N}_0$.

**Theorem 7.4.** For $s \in U_{L}^+$, $\xi_1(z,s;\sigma)$ is holomorphic at $z = 0$.

**Proof.** By [JL93] Theorem 1.6, $\xi_1(z,s;\sigma)$ has Laurent expansion at $z = 0$

$$\xi_1(z,s;\sigma) = R_{-1}(0,s;\sigma)z^{-1} + R_0(0,s;\sigma) + R_1(0,s;\sigma)z + \ldots,$$

where $R_{-1}(0,s;\sigma)$ is a polynomial of degree $\leq d/2$. In our case the pole at $z = 0$ is simple since the number $n(0')$ is zero. For justification, see the definition of $n(0')$ in [JL93] p. 17 and notice that the asymptotic expansion (7.1) corresponds to the so called special case, i.e., the coefficients in (7.1) are constants and there are no logarithmic terms (see [JL93] p. 15–16). By the proof of Theorem 1.6 in [JL93], the polynomial $R_{-1}(0,s;\sigma)$, can be explicitly given

$$R_{-1}(0,s;\sigma) = \sum_{j/2 - d/2 + k = 0} (-1)^k \frac{s^k}{k!} a_j^+, \quad (7.9)$$

where $a_j^+$ are the coefficients in the short time asymptotic expansion (7.1) of the heat operator, with $j/2 - d/2 < 0$. By [CH93] Lemma 1.7.4 (d), the coefficients $a_j^+$ vanish for $j$ odd. Hence, by (7.9) and the fact that $d = \dim X$ is odd, $R_{-1}(0,s;\sigma) = 0$. The assertion follows.

Similarly, for $\xi_2(z,s;\sigma)$, we have the following theorem.

**Theorem 7.5.** For $s \in U_{L}^-$, $\xi_2(z,s;\sigma)$ is holomorphic at $z = 0$.

We set $U_L := U_{L,+} \cup U_{L,-}$.

**Corollary 7.6.** For $s \in U_L$, the generalized zeta function $\zeta(z,s;\sigma)$ is holomorphic at $z = 0$. In addition,

$$\frac{d}{dz} \zeta(z,s;\sigma) \bigg|_{z=0} = \xi(0,s;\sigma). \quad (7.10)$$
Proof. The generalized zeta function is by definition the xi function divided by $\Gamma(z)$. We use
$$\frac{1}{\Gamma(z)} = z + \gamma z^2 + O(z^3),$$
where $\gamma$ is the Euler constant. We have
$$\frac{d}{dz} \zeta(z, s) \bigg|_{z=0} = \frac{d}{dz} \left( \frac{1}{\Gamma(z)} \xi_1(z, s; \sigma) - \frac{1}{\Gamma(z)} \xi_1(z, s; \sigma) \right) \bigg|_{z=0} = \xi_1(0, s; \sigma) - \xi_2(0, s; \sigma) = \xi(0, s; \sigma).$$

Definition 7.7. The graded regularized determinant of the operator $A^\sharp_\chi(\sigma) + s^2$ is defined by
$$\det_{gr}(A^\sharp_\chi(\sigma) + s^2) := \exp \left(- \frac{d}{dz} \zeta(z, s^2; \sigma) \bigg|_{z=0} \right).$$ (7.11)

By (7.10), we get
$$\det_{gr}(A^\sharp_\chi(\sigma) + s^2) = \exp(-\xi(0, s^2; \sigma)).$$

Equivalently,
$$\log(\det_{gr}(A^\sharp_\chi(\sigma) + s^2)) = -\xi(0, s^2; \sigma) = - (\xi_1(0, s^2; \sigma) - \xi_2(0, s^2; \sigma)).$$ (7.12)

Theorem 7.8. Let $\det_{gr}(A^\sharp_\chi(\sigma) + s^2)$ be the regularized determinant associated with the operator $A^\sharp_\chi(\sigma) + s^2$. Then,

1. case(a) the Selberg zeta function has the representation
$$Z(s; \sigma, \chi) = \det_{gr}(A^\sharp_\chi(\sigma) + s^2) \exp \left(- 2\pi \dim(V_\chi) \Vol(X) \int_0^s P(\sigma(t)) dt \right).$$ (7.13)

2. case(b) the symmetrized zeta function has the representation
$$S(s; \sigma, \chi) = \det_{gr}(A^\sharp_\chi(\sigma) + s^2) \exp \left(- 4\pi \dim(V_\chi) \Vol(X) \int_0^s P(\sigma(t)) dt \right).$$ (7.14)

Proof. By [Spi18, Lemma 7.1], [Spi18, eq. (7.9)] and arguing as in the proof of Proposition 7.7 in [Spi18] p. 187-188, we obtain
$$\Tr_s \prod_{i=1}^N (A^\sharp_\chi(\sigma) + s_i^2)^{-1} = \int_0^\infty \sum_{i=1}^N \left( \prod_{j=1}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-\lambda s_i^2} \Tr_s(e^{-tA^\sharp_\chi(\sigma)}) dt.$$
By [Spi18, Lemma 7.3] and (7.1), the integral in the right-hand side of the equation above is absolutely convergent. We have

\[
\int_0^\infty \sum_{i=1}^N \left( \prod_{\substack{j=1 \atop j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}_s(e^{-tA_k^L(\sigma)}) dt = \int_0^\infty \sum_{i=1}^N \left( \prod_{\substack{j=1 \atop j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i t} \left( - \frac{d}{ds_i} e^{-ts_i^2} \right) \text{Tr}_s(e^{-tA_k^L(\sigma)}) dt. \tag{7.15}
\]

The right-hand side of (7.15) gives

\[
\int_0^\infty \sum_{i=1}^N \left( \prod_{\substack{j=1 \atop j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i t} \left( - \frac{d}{ds_i} e^{-ts_i^2} \right) \text{Tr}_s(e^{-tA_k^L(\sigma)}) dt = \lim_{z \to 0} \int_0^\infty \sum_{i=1}^N \left( \prod_{\substack{j=1 \atop j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \left( - \frac{1}{z-1} \frac{d}{ds_i} e^{-ts_i^2} \right) \text{Tr}_s(e^{-tA_k^L(\sigma)}) dt
\]

\[
= \sum_{i=1}^N \left( \prod_{\substack{j=1 \atop j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \left( - \xi(0, s_i^2; \sigma) \right)
\]

\[
= \sum_{i=1}^N \left( \prod_{\substack{j=1 \atop j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \left( \log(\det_{gr}(A_k^L(\sigma) + s_i^2)) \right),
\]

where in the last equation we used (7.12). Therefore, (7.15) becomes

\[
\int_0^\infty \sum_{i=1}^N \left( \prod_{\substack{j=1 \atop j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}_s(e^{-tA_k^L(\sigma)}) dt = \sum_{i=1}^N \left( \prod_{\substack{j=1 \atop j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \left( \log(\det_{gr}(A_k^L(\sigma) + s_i^2)) \right). \tag{7.16}
\]

We treat here the case (b). One can proceed similarly for the case (a). The left-hand side of (7.16) can be developed more. Recall that $L(\gamma; \sigma, \chi)$ is given by (3.5). We insert the right-hand side of the trace formula [Spi18, eq. (5.39), Theorem 5.5] for the operator $e^{-tA_k^L(\sigma)}$. As before, we consider the super trace.
of the operator $e^{-tA^{\tau}_{\chi}(\sigma)}$.

$$\text{Tr}_s (e^{-tA^{\tau}_{\chi}(\sigma)}) = 2 \dim(V_{\chi}) \int_{\mathbb{R}} e^{-t\lambda^2} P_{\sigma}(i\lambda) d\lambda$$

$$+ \sum_{[\gamma] \not\equiv e} \frac{l(\gamma)}{n_T(\gamma)} L(\gamma; \sigma + w\sigma, \chi) e^{-l(\gamma)^2/4t} \left(\frac{4\pi t}{(4\pi t)^{1/2}}\right).$$

Then, the left-hand side of (7.16) becomes

$$\int_0^{\infty} \sum_{i=1}^N \left( \prod_{j=1}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}_s (e^{-tA^{\tau}_{\chi}(\sigma)}) dt = \int_0^{\infty} \sum_{i=1}^N \left( \prod_{j=1}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \left(2 \dim(V_{\chi}) \int_{\mathbb{R}} e^{-t\lambda^2} P_{\sigma}(i\lambda) d\lambda \right.$$

$$\left. + \sum_{[\gamma] \not\equiv e} \frac{l(\gamma)}{n_T(\gamma)} L(\gamma; \sigma + w\sigma, \chi) e^{-l(\gamma)^2/4t} \right) dt.$$
Hence, by (7.16), we get

\[
\sum_{i=1}^{N} \left( \prod_{j=1, j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \frac{d}{ds_i} \log(\det_{\text{gr}}(A_{\chi}^1(\sigma) + s_i^2)) = \frac{2\pi}{s_i} \text{dim}(V_{\chi}) \text{Vol}(X) P_\sigma(s_i)
\]

\[
+ \sum_{i=1}^{N} \left( \prod_{j=1, j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_{\Gamma}(\gamma)} L(\gamma; \sigma + w\sigma, \chi) e^{-s_i l(\gamma)}.
\]

(7.17)

We fix now the variables \(s_2, \ldots, s_N \in \mathbb{C}\) and let the variable \(s_1 = s \in \mathbb{C}\) vary. We multiply both sides of (7.17) by

\[
2s \prod_{j=2}^{N} (s_j^2 - s^2).
\]

Then, we get

\[
\frac{d}{ds} \log(\det_{\text{gr}}(A_{\chi}^1(\sigma) + s^2)) = 4\pi \text{dim}(V_{\chi}) \text{Vol}(X) P_\sigma(s) + \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_{\Gamma}(\gamma)} L(\gamma; \sigma + w\sigma, \chi) e^{-sl(\gamma)} + K'(s),
\]

(7.18)

where \(K'(s)\) is a certain odd polynomial, which is of the from

\[
K'(s) = 2s \prod_{j=2}^{N} (s_j^2 - s^2) \tilde{Q}(s_2, \ldots, s_N).
\]

The term \(\tilde{Q}(s_2, \ldots, s_N)\) comes from the terms that correspond to the summands over \(i = 2, \ldots, N\) and hence it has a fixed value in \(\mathbb{C}\), since \(s_2, \ldots, s_N\) are fixed.

Next, we can substitute the term in (7.18), that comes from the hyperbolic contribution of the trace formula, with the logarithmic derivative of the symmetrized zeta function. By (3.6), we have

\[
\frac{d}{ds} \log(\det_{\text{gr}}(A_{\chi}^1(\sigma) + s^2)) = 4\pi \text{dim}(V_{\chi}) \text{Vol}(X) P_\sigma(s) + L_S(s) + K'(s).
\]

We integrate with respect to \(s\) and get

\[
\log(\det_{\text{gr}}(A_{\chi}^1(\sigma) + s^2)) = 4\pi \text{dim}(V_{\chi}) \text{Vol}(X) \int_0^s P_\sigma(t) dt + \log S(s; \sigma, \chi) + K(s).
\]
Hence,
\[
\log S(s; \sigma, \chi) = \log(\det_{gr}(A^2_{\chi}(\sigma) + s^2)) - K(s) - 4\pi \dim(V_{\chi}) \Vol(X) \int_0^s P_\sigma(t) dt.
\]

(7.19)

We want to show that \(K(s) = 0\). To this end, we study the asymptotic behaviour of all terms in (7.19), as \(s \to \infty\). By Lemma 4.2, there exist coefficients \(c_j\) such that
\[
\Tr(e^{-tA^2_{\chi}(\sigma)}) \sim_{t \to 0} \sum_{j=0}^\infty c_j t^j.
\]

We use the asymptotic formula (13) from [QHS93]. By [Gil95, Lemma 1.7.4 (d)], \(c_j\) vanish for \(j\) odd. Hence, in our case, since \(d\) is odd, the first sum over the integers in the right-hand side of the asymptotic formula (13) in [QHS93] does not contribute. We have
\[
-\log \det_{gr}(A^2_{\chi}(\sigma) + s^2) \sim_{s \to \infty} \sum_{k=0}^\infty c_{2k} \Gamma(2k) s^{d-2k}.
\]

(7.20)

The estimation of the sum on the right hand side of (3.14) on p. 163 of [Spi18] can be improved to show that the log of the Selberg zeta function is exponentially decreasing. Therefore the log of the symmetrized zeta function also decreases exponentially. Hence, as \(s \to \infty\), the left-hand side of (7.19) goes to zero. On the other hand, as \(s \to \infty\), the right-hand side of (7.20) contains only odd powers of \(s\), the term that involves the Plancherel polynomial in the left hand side of (7.19) is an odd polynomial, and \(K(s)\) is an even polynomial. Hence, \(K(s)\) vanish identically. Moreover, we conclude
\[
-4\pi \dim(V_{\chi}) \Vol(X) \int_0^s P_\sigma(t) dt = \sum_{k=0}^{(d-1)/2} c_{2k} \Gamma(2k) s^{d-2k}
\]

and \(c_{2k} = 0\) for \(2k > d\).

We prove now a determinant formula for the Ruelle zeta function. Recall form Section 6 the standard representation \(\sigma_p\) of \(M\) in \(A^p \mathbb{R}^{d-1} \otimes \mathbb{C}\). Let \(\sigma, \sigma' \in \hat{M}\). We denote by \([\sigma_p \otimes \sigma : \sigma']\) the multiplicity of \(\sigma'\) in \(\sigma_p \otimes \sigma\). We distinguish again two cases for \(\sigma' \in \hat{M}\).

- **Case (a):** \(\sigma'\) is invariant under the action of the restricted Weyl group \(W_A\). Then, \(i^* (\tau) = \sigma'\), where \(\tau \in R(K)\).
- **Case (b):** \(\sigma'\) is not invariant under the action of the restricted Weyl group \(W_A\). Then, \(i^* (\tau) = \sigma' + w\sigma'\), where \(\tau \in R(K)\).

We define the operator
\[
A^2_{\chi}(\sigma_p \otimes \sigma) := \bigoplus_{[\sigma] \in \hat{M}/W_A} \bigoplus_{i=1}^{[\sigma_p \otimes \sigma : \sigma_i]} A^2_{\chi}(\sigma_i),
\]

and
acting on the space $C^\infty(X, E'(\sigma') \otimes E'_\chi)$, where $E(\sigma')$ is the vector bundle over $X$, constructed as in [Spi18, p. 175].

Let $\sigma$ be the unique positive root of $(g, a)$. Let $H \in a$ such that $\alpha(H) = 1$. Recall that the character $\lambda \equiv \lambda_p$ of $A$ is defined by $\lambda \equiv \lambda_p(a) = e^{\rho(a) \log a}$. Then, we can identify $\lambda$ with $p$.

**Proposition 7.9.** The Ruelle zeta function has the representation

- **case (a)**

$$R(s; \sigma, \chi) = \prod_{p=0}^{d-1} \det_{gr}(A^*_\chi(\sigma_p \otimes \sigma) + (s + |p| - p)^2)^{(-1)^p} \exp \left( (-1)^{\frac{d+1}{2}} + 1 \pi (d+1) \dim(V_\sigma) \frac{\text{Vol}(X)}{\text{Vol}(S^d)^s} \right).$$

(7.21)

- **case (b)**

$$R(s; \sigma, \chi) = \prod_{p=0}^{d-1} \det_{gr}(A^*_\chi(\sigma_p \otimes \sigma) + (s + |p| - p)^2)^{(-1)^p} \exp \left( (-1)^{\frac{d+1}{2}} + 1 \pi (d+1) \dim(V_\sigma) \frac{\text{Vol}(X)}{\text{Vol}(S^d)^s} \right).$$

(7.22)

**Proof.** We prove the assertion for case (b). One can proceed similarly for case (a). By [Spi18 Theorem 7.12], we have the expression of the Ruelle zeta function as a product of Selberg zeta functions. Then, we see

$$R(s; \sigma, \chi) = \prod_{p=0}^{d-1} Z(s + |p| - p; \sigma_p \otimes \sigma, \chi)^{(-1)^p} \prod_{p=0}^{d-1} Z(s + |p| - p; \sigma_p \otimes w \sigma, \chi)^{(-1)^p}$$

$$= \prod_{p=0}^{d-1} S(s + |p| - p; \sigma_p \otimes \sigma, \chi)^{(-1)^p}.$$

Hence, if we equip the determinant formula for the symmetrized zeta function (Theorem 7.8.2), we have

$$R(s; \sigma, \chi) = \prod_{p=0}^{d-1} \det_{gr}(A^*_\chi(\sigma_p \otimes \sigma) + (s + |p| - p)^2)^{(-1)^p} \exp \left( \sum_{p=0}^{d-1} (-1)^p (4\pi \dim(V_\chi) \text{Vol}(X)) \int_0^{s+|p|/2} P_{\sigma_p \otimes \sigma}(t) \, dt \right).$$

(7.23)

On the other hand,

$$\sum_{p=0}^{d-1} (-1)^p \int_0^{s+|p|/2} Q_{\sigma_p \otimes \sigma}(t) \, dt = \int_0^s f(t) \, dt,$$

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where \( f(t) \) is defined by (6.2). Therefore, as in the proof of Theorem 6.2,
\[
\sum_{p=0}^{d-1} (-1)^p \int_0^{s+|\rho|-p} Q_{\sigma_p \otimes \sigma} (t) dt = (d+1) \dim(V_\sigma) s.
\] (7.24)

Hence, (7.23) becomes by (7.24)
\[
R(s; \sigma, \chi) R(s; \sigma, \chi) = \prod_{p=0}^{d-1} \det_{\text{gr}} \left( A_\chi^2 (\sigma_p \otimes \sigma) + (s + |\rho| - p)^2 \right)^{(-1)^p} \exp \left( 4\pi c(n)(d+1) \dim(V_\chi) \dim(V_\sigma) \text{Vol}(X)s \right),
\]
where \( c(n) \) is as in (2.9). The assertion follows.

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