Analytical approximations of two and three dimensional time-fractional telegraphic equation by reduced differential transform method

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Abstract

In this article, an analytical solution based on the series expansion method is proposed to solve the time-fractional telegraph equation (TFTE) in two and three dimensions using a recent and reliable semi-approximate method, namely the reduced differential transformation method (RDTM) subjected to the appropriate initial condition. Using RDTM, it is possible to find exact solution or a closed approximate solution of a differential equation. The accuracy, efficiency, and convergence of the method are demonstrated through the four numerical examples.

Keywords:
Two and three dimensional TFTEs
Fractional calculus
Reduced differential transform method (RDTM)
Analytical solutions

1. Introduction

Several real phenomena emerging in engineering and science fields can be demonstrated successfully by developing models using the fractional calculus theory. Fractional differential theory has gained much more attention as the fractional order system response ultimately converges to the integer order equations. The applications of the fractional differentiation for the mathematical modeling of real world physical problems such as the earthquake modeling, the traffic flow model with fractional derivatives, measurement of viscoelastic material properties, etc., have been widespread in this modern era. Before the nineteenth century, no analytical solution method was available for such type of equations even for the linear fractional differential equations. Recently, Keskin and Oturanc [1] developed the reduced differential transform...
method (RDTM) for the fractional differential equations and showed that RDTM is the easily usable semi analytical method and gives the exact solution for both the linear and nonlinear differential equations.

Let us assume that \( u(x,y,t) \) and \( i(x,y,t) \) denote the electric voltage and the current in a double conductor, then the time-fractional telegraphic equations (TFTEs) in the two dimensional (2D) are given as

\[
\begin{align*}
\frac{\partial^p u}{\partial x^p} + 2p \frac{\partial^q u}{\partial x^q} + q^2 u = \frac{\partial^p i}{\partial x^p} + 2p \frac{\partial^q i}{\partial x^q} + f_1(x, y, t),
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^p i}{\partial x^p} + 2p \frac{\partial^q i}{\partial x^q} + q^2 i = \frac{\partial^p v}{\partial x^p} + 2p \frac{\partial^q v}{\partial x^q} + f_2(x, y, t),
\end{align*}
\]

where \( \Omega = [a,b] \times [c,d] \times [t>0] \). The initial conditions are assumed to be

\[
\begin{align*}
&u(x,y,0) = \phi_1(x, y), \\
&u_t(x,y,0) = \phi_2(x, y), \\
&i(x,y,0) = \chi_1(x, y), \\
&i_t(x,y,0) = \chi_2(x, y)
\end{align*}
\]

Similarly, the three dimensional (3D) time-fractional order telegraphic equation (TFTE) can be given as

\[
\begin{align*}
\frac{\partial^p u}{\partial x^p} + 2p \frac{\partial^q u}{\partial x^q} + q^2 u = \frac{\partial^p i}{\partial x^p} + 2p \frac{\partial^q i}{\partial x^q} + f_1(x, y, z, t),
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^p i}{\partial x^p} + 2p \frac{\partial^q i}{\partial x^q} + q^2 i = \frac{\partial^p v}{\partial x^p} + 2p \frac{\partial^q v}{\partial x^q} + f_2(x, y, z, t),
\end{align*}
\]

where \( \Omega = [a,b] \times [c,d] \times [e,f] \times [t>0] \), with initial conditions

\[
\begin{align*}
&u(x,y,z,0) = \xi_1(x, y, z), \\
&u_t(x,y,z,0) = \xi_2(x, y, z), \\
&i(x,y,z,0) = \psi_1(x, y, z), \\
&i_t(x,y,z,0) = \psi_2(x, y, z)
\end{align*}
\]

In Eqs. (1) and (3) \( p \) and \( q \) denote constants. For \( p > 0, q = 0 \), (1) and (3) represent time-fractional order damped wave equations in two and three dimensions respectively.

It has been observed that telegraphic equation is more suitable than ordinary diffusion equation in modeling reaction diffusion. The hyperbolic partial differential equations model the vibrations of structures (e.g. machines, buildings and beams) and they are the basis for fundamental equations of atomic physics. The telegraph equation is an important equation for modeling several relevant problems in engineering and science such as wave propagation, random walk theory, signal analysis etc. In recent years, from the literature it can be seen that much attention has been given to the development of analytical and numerical schemes for the one dimensional and two dimensional hyperbolic and non-fractional TFTEs. To the best of our knowledge till now no one has applied the RDTM to solve the time-fractional order telegraphic equations in two and three dimensions.

In this paper, we propose an analytical scheme namely the reduced differential transformation method based on series solution method to find analytical solutions of the time-fractional telegraph equation (TFTE) in two and three dimensions. The accuracy and efficiency of the proposed method are demonstrated by the four test examples. The main advantage of the method is that it solves the telegraph equation directly without using linearization, transformation, discretization or restrictive assumptions. Also, the RDTM scheme is very easy to implement for the multidimensional time-fractional order physical problems emerging in various fields of engineering and science.

### 2. Fractional calculus

In this section, we demonstrate some notations and definitions that will be used further in the study. Fractional calculus theory is almost more than two decades’ old in the literature. Several definitions of fractional integrals and derivatives have been proposed but the first major contribution to give proper definition is due to Liouville as follows.

**Definition 2.1.** A real function \( f(x), x > 0 \) is said to be in the space \( C_{\alpha, \mu} \) if there exists a real number \( q(\leq \mu) \), such that \( f(x) = x^q g(x) \), where \( g(x) \in C[0, \infty) \), and it is said to be in the space \( C_{\alpha}^{m} \) if \( f^{(m)} \in C_{\alpha, m} \) for \( m \in N \).

**Definition 2.2.** For a function \( f \), the Riemann–Liouville fractional integral operator \([23]\) of order \( \alpha \geq 0 \), is defined as

\[
\begin{align*}
J_{0}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0; x > 0, \\
J_{0}^{0} f(x) &= f(x)
\end{align*}
\]

The Riemann–Liouville derivative has certain disadvantages when trying to model real world problems with fractional differential equations. To overcome this discrepancy, Caputo and Mainardi\([24]\) proposed a modified fractional differentiation operator \( D_{\alpha}^{\alpha} \) in his work on the theory of viscoelasticity. The Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations.

**Definition 2.3.** The fractional derivative of \( f \) in the Caputo sense \([25]\) can be defined as

\[
\begin{align*}
D_{\alpha}^{\alpha} f(x) &= \frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \\
D_{\alpha}^{0} f(x) &= f(x)
\end{align*}
\]

for \( m-1 < \alpha \leq m, m \in N; x > 0, f \in C_{m-1}^{m} \).

The fundamental basic properties of the Caputo fractional derivative are given as.

**Lemma.** If \( m-1 < \alpha \leq m, m \in N \) and \( f \in C_{m}^{m}, \mu \geq -1, \) then

\[
\begin{align*}
D_{\alpha}^{m-\alpha} f(x) &= f(x), x > 0, \\
D_{\alpha}^{m+\alpha} f(x) &= \sum_{k=0}^{m} f^{(k)}(0^{+}) \frac{x^{k}}{k!}, x > 0,
\end{align*}
\]
In this study, the Caputo fractional derivative is taken since it allows traditional initial and boundary conditions to be included in the derivation of the problem. Some other properties of fractional derivative can be found in [25,26].

### 3. Reduced differential transform method (RDTM)

In this section, we introduce the basic definitions of the reduced differential transformations.

Consider a function of four variables $w(x,y,z,t)$, and assume that it can be represented as a product $w(x,y,z,t) = F(x,y)G(t)$. On extending the basis of the properties of the one-dimensional differential transformation [26,27], the function $w(x,y,z,t)$ can be represented as

$$w(x,y,z,t) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \sum_{i_3=0}^{\infty} \sum_{j=0}^{\infty} F(i_1,i_2,i_3) x^{i_1} y^{i_2} z^{i_3} t^j G(j) t^j,$$

where $W(i_1,i_2,i_3) = F(i_1,i_2,i_3) G(j)$ is called the spectrum of $w(x,y,z,t)$.

Let $R_D$ denote the reduced differential transform operator and $R_D^{-1}$ the inverse reduced differential transform operator. The basic definition and operation of the RDTM method is described below.

**Definition 2.1.** If $w(x,y,z,t)$ is analytic and continuously differentiable with respect to space variables $x,y$ and time variable $t$ in the domain of interest, then the spectrum function [28,29]

$$R_D|w(x,y,z,t)| = \mathcal{W}_k(x,y,z) = \frac{1}{\Gamma(k+1)} \sum_{j=0}^{\infty} \frac{\partial^k}{\partial t^k} w(x,y,z,t) t^j$$

is the reduced transformed function of $w(x,y,z,t)$.

In this article, lowercase $w(x,y,z,t)$ represents the original function while uppercase $\mathcal{W}_k(x,y,z)$ stands for the reduced transformed function. The differential inverse reduced transform of $\mathcal{W}_k(x,y,z)$ is defined as

$$R_D^{-1}[\mathcal{W}_k(x,y,z)] = w(x,y,z,t) = \sum_{k=0}^{\infty} \mathcal{W}_k(x,y,z) (t - t_0)^k$$

Combining Eqs. (9) and (10), we get

$$w(x,y,z,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} \int_0^t \frac{\partial^k}{\partial t^k} w(x,y,z,t) \frac{dt}{t^{k+1}} (t - t_0)^k$$

When $t = 0$, Eq. (11) reduces to

$$w(x,y,z,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} \int_0^t \frac{\partial^k}{\partial t^k} w(x,y,z,t) \frac{dt}{t^{k+1}} t^k$$

From the Eq. (11), it can be seen that the concept of the reduced differential transform is derived from the power series expansion of the function.

**Definition 2.2.** If $w(x,y,z,t) = R_D^{-1}[U_k(x,y,z)], u(x,y,z,t) = R_D^{-1}[V_k(x,y,z)],$ and the convolution $\otimes$ denotes the reduced differential transform version of the multiplication, then the fundamental operations of the reduced differential transform are shown in the Table 1.

In Table 1, $\Gamma$ represents the Gama function, which is defined as

$$\Gamma(y) := \int_0^\infty e^{-tt^{-1}} dt, \gamma \in \mathbb{C}$$

### 4. RDTM for two dimensional TFTE

Applying the RDTM to the two dimensional TFTE (1), we have the following relation

Now applying the method to the initial conditions (2), we get

$$U_0(x,y) = \phi_1(x,y), \quad U_1(x,y) = \phi_2(x,y), \quad I_0(x,y) = \chi_1(x,y), \quad I_1(x,y) = \chi_2(x,y),$$

From above two equations we get the values of $U_0(x,y), \chi_1(x,y), k = 2, 3, 4, \ldots$ etc. Using the differential inverse reduced transform of $U_0(x,y), \chi_1(x,y), k = 0, 1, 2, 3, \ldots$, we get the approximate solution for $u(x,y,t)$ and $i(x,y,t)$ as
5. RDTM for three dimensional TFTE

Applying the RDTM to the three dimensional TFTE (4), we have the following relation

\[
\begin{aligned}
\frac{1}{\beta_1}\frac{\partial^2 u}{\partial x_1^2} &+ 2p \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_1 \partial x_3} + \frac{\partial^3 u}{\partial x_1 \partial y \partial z} + \frac{\partial^3 u}{\partial x_1 \partial z^2} + \frac{\partial^3 u}{\partial x_2 \partial y \partial z} + \frac{\partial^3 u}{\partial x_2 \partial z^2} + \frac{\partial^3 u}{\partial x_3 \partial y \partial z} + \frac{\partial^3 u}{\partial x_3 \partial z^2} = \sum_{k=0}^{\infty} U_k(x, y, z) \beta_k + \sum_{k=0}^{\infty} I_k(x, y, z) \beta_k + ... \\
\frac{1}{\beta_2}\frac{\partial^2 u}{\partial x_2^2} &+ 2q \frac{\partial^2 u}{\partial x_2 \partial x_3} + \frac{\partial^2 u}{\partial x_2 \partial x_1} + \frac{\partial^3 u}{\partial x_2 \partial y \partial z} + \frac{\partial^3 u}{\partial x_2 \partial z^2} + \frac{\partial^3 u}{\partial x_3 \partial y \partial z} + \frac{\partial^3 u}{\partial x_3 \partial z^2} + \frac{\partial^3 u}{\partial x_1 \partial y \partial z} + \frac{\partial^3 u}{\partial x_1 \partial z^2} = \sum_{k=0}^{\infty} U_k(x, y, z) \beta_k + \sum_{k=0}^{\infty} I_k(x, y, z) \beta_k + ... \\
\frac{1}{\beta_3}\frac{\partial^2 u}{\partial x_3^2} &+ 2r \frac{\partial^2 u}{\partial x_3 \partial x_1} + \frac{\partial^2 u}{\partial x_3 \partial x_2} + \frac{\partial^3 u}{\partial x_3 \partial y \partial z} + \frac{\partial^3 u}{\partial x_3 \partial z^2} + \frac{\partial^3 u}{\partial x_1 \partial y \partial z} + \frac{\partial^3 u}{\partial x_1 \partial z^2} + \frac{\partial^3 u}{\partial x_2 \partial y \partial z} + \frac{\partial^3 u}{\partial x_2 \partial z^2} = \sum_{k=0}^{\infty} U_k(x, y, z) \beta_k + \sum_{k=0}^{\infty} I_k(x, y, z) \beta_k + ...
\end{aligned}
\]

Now applying the method to the initial conditions (4), we get

\[
\begin{aligned}
U_0(x, y, z) &= \xi_1(x, y, z), \\
U_1(x, y, z) &= \xi_2(x, y, z), \\
I_0(x, y, z) &= \psi_1(x, y, z), \\
I_1(x, y, z) &= \psi_2(x, y, z).
\end{aligned}
\]

Applying the same procedure as in the case of 2D TFTE, we get the approximate solution for \(u(x, y, z, t)\) and \(i(x, y, z, t)\) as

\[
\begin{aligned}
u(x, y, z, t) &= \sum_{k=0}^{\infty} U_k(x, y, z) t^{\beta_k} = U_0(x, y, z) + U_1(x, y, z) t^{\beta_1} + U_2(x, y, z) t^{2\beta_1} + ..., \\
i(x, y, z, t) &= \sum_{k=0}^{\infty} I_k(x, y, z) t^{\beta_k} = I_0(x, y, z) + I_1(x, y, z) t^{\beta_1} + I_2(x, y, z) t^{2\beta_1} + ...
\end{aligned}
\]

Using the RDTM to the initial conditions (21), we have

\[
U_0(x, y, z) = e^{x+y}, \quad U_1(x, y, z) = -3e^{x+y}.
\]

From Eq. (23) into Eq. (22), we get the following \(U_k(x, y)\) values successively

\[
U_k(x, y) = \frac{(-3)^k}{\Gamma(\lambda + k)} \Gamma\left(\frac{\lambda + 1}{\lambda}\right) e^{x+y}, \quad k \geq 2.
\]

where \(\alpha = 1/\lambda, \lambda > 0\). Using the differential inverse reduced transform of \(U_k(x, y)\), we get

\[
u(x, y, z, t) = \sum_{k=0}^{\infty} U_k(x, y) t^{k/\lambda} = U_0(x, y) + U_1(x, y) t^{1/\lambda} + U_2(x, y) t^{2/\lambda} + U_3(x, y) t^{3/\lambda} + ... \\
i(x, y, z, t) = \sum_{k=0}^{\infty} I_k(x, y) t^{k/\lambda} = I_0(x, y) + I_1(x, y) t^{1/\lambda} + I_2(x, y) t^{2/\lambda} + I_3(x, y) t^{3/\lambda} + ... \\
\]

6. Numerical examples

In this section, we describe the method explained in the Sections 4 and 5 by taking four examples of both linear and nonlinear 2D and 3D TFTEs to validate the efficiency and reliability of the RDTM scheme.

Example 6.1. Consider the 2D linear TFTE

\[
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + u &= \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \\
\end{aligned}
\]

subject to the initial conditions

\[
\begin{aligned}
u(x, y, 0) &= e^{x+y}, \\
i(x, y, 0) &= -3e^{x+y}.
\end{aligned}
\]

Applying the RDTM to Eq. (20), we obtain the following recurrence relation

\[
\begin{aligned}
\frac{1}{\beta_1}\frac{U_{k+2}(x, y)}{\Gamma(\alpha + 1)} &= \frac{1}{\beta_2}\frac{U_{k+1}(x, y)}{\Gamma(\alpha + 1)} + \frac{1}{\beta_3}\frac{U_k(x, y)}{\Gamma(\alpha + 1)} \\
&= \frac{\partial^2}{\partial x^2} U_k(x, y) + \frac{\partial^2}{\partial y^2} U_k(x, y) - U_k(x, y).
\end{aligned}
\]

Example 6.2. Consider the following 3D linear TFTE

\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + u &= \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \\
\end{aligned}
\]

subject to initial conditions
Applying the RDTM to Eq. (27), we obtain the following recurrence relation

\[
\frac{I(ka + 2a + 1)}{I(ka + 1)}U_{k+2}(x, y, z) + 2 \frac{I(ka + a + 1)}{I(ka + 1)}U_{k+1}(x, y, z)
\]

\[-\frac{\partial}{\partial x^2} U_k(x, y, z) + \frac{\partial}{\partial y^2} U_k(x, y, z) + \frac{\partial}{\partial z^2} U_k(x, y, z) - U_k(x, y, z).\]

(29)

Using the RDTM to the initial conditions (28), we have

\[
U_0(x, y, z) = \sinh(x)\sinh(y)\sinh(z);
\]

\[
U_1(x, y, z) = -\sinh(x)\sinh(y)\sinh(z).\]

(30)

From Eq. (30) into Eq. (29), we get the following \(U_k(x,y,z)\) values

\[
U_k(x, y, z) = \frac{(-1)^k}{I(\frac{n}{a} + 1)} I\left(\frac{n}{k} + 1\right) \sinh(x) \sinh(y) \sinh(z); k \geq 2.\]

(31)

Using the differential inverse reduced transform of \(U_k(x,y,z)\), we get

\[
u(x, y, z, t) = \sum_{k=0}^{\infty} U_k(x, y, z) t^{\frac{a}{2}}
\]

\[
= U_0(x, y, z) + U_1(x, y, z) t^{\frac{a}{2}} + U_2(x, y, z) t^{\frac{3a}{2}} + U_3(x, y, z) t^{\frac{5a}{2}} + \ldots
\]

\[
= \sinh(x) \sinh(y) \sinh(z) \left[ 1 - (-1) t^{\frac{a}{2}} + I(\frac{n}{a} + 1) \left( \frac{(-1)^n t^{\frac{3a}{2}}}{I(\frac{3a}{2})} + \frac{(-1)^n t^{\frac{5a}{2}}}{I(\frac{5a}{2})} + \ldots \right) \right].\]

(32)

Eq. (32) represents the solution of the TFTE (27). When \(\lambda = 1\), i.e. \(a = 1\), we get

\[
u(x, y, z, t) = e^{-t} \sinh(x) \sinh(y) \sinh(z),\]

(33)

which is the closed form solution of the non-fractional form of the TFTE (27).

Example 6.3. Consider the following 2D nonlinear TFTE

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial u}{\partial t} + u^2 - e^{2(x+y)} - 4t + e^{2(x+y)} - 2t
\]

(34)

under the initial conditions

\[
u(x, y, 0) = e^{x+y},
\]

\[
u_t(x, y, 0) = -2e^{x+y}.\]

(35)

Applying the RDTM technique to Eq. (34), we obtain the following iterative formula:

\[
\frac{I(ka + 2a + 1)}{I(ka + 1)}U_{k+2}(x, y, z) + 2 \frac{I(ka + a + 1)}{I(ka + 1)}U_{k+1}(x, y, z)
\]

\[-\frac{\partial^2}{\partial x^2} U_k(x, y, z) + \frac{\partial^2}{\partial y^2} U_k(x, y, z) + \frac{\partial}{\partial z^2} U_k(x, y, z) - U_k(x, y, z).\]

When \(\lambda = 1\), we get the exact solution of the non-fractional form of the TFTE (34) as

\[
u(x, y, t) = e^{2(x+y) - 2t}.\]

(40)

Example 6.4. Consider the 3D nonlinear TFTE given as

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2 \frac{\partial u}{\partial t} + u^2 - e^{2(x+y-z) - 4t} + e^{2(x+y-z) - 2t}
\]

(41)

subject to the initial conditions

\[
u(x, y, z, 0) = e^{x+y+z},
\]

\[
u_t(x, y, z, 0) = -e^{x+y+z}.\]

(42)

Applying the RDTM technique to Eq. (41), we obtain the following iterative formula:
\[
\frac{\Gamma(k + 2a + 1)}{\Gamma(k + 1)} U_k(x, y, z) - 2 \frac{\Gamma(k + a + 1)}{\Gamma(k + 1)} U_{k+1}(x, y, z) + \frac{\gamma^2}{\partial x^2} U_k(x, y, z) + \frac{\gamma^2}{\partial y^2} U_k(x, y, z) - \sum_{l=0}^{k} U_l(x, y, z) U_{k-l}(x, y, z) + \frac{\gamma^2}{\partial z^2} \left( -\frac{4}{k!} \right) \left( -\frac{2}{k!} \right) \right).
\] (43)

Using the RDTM to the initial conditions \((42)\), we get

\[
U_0(x, y, z) = e^{x-y-z}, \quad U_1(x, y, z) = -e^{x-y-z}.
\] (44)

Using Eq. \((44)\) in Eq. \((43)\), we get the following \(U_k(x, y, z)\) values successively

\[
U_k(x, y, z) = \frac{(-1)^k}{\Gamma\left(\frac{k}{2} + 1\right)} \left( \frac{k+1}{2} \right) e^{x-y-z}, \quad k \geq 2.
\] (45)

Using the differential inverse reduced transform of \(U_k(x, y, z)\), we get

\[
u(x, y, z, t) = \sum_{k=0}^{\infty} U_k(x, y, z) t^{k/2} = \sum_{k=0}^{\infty} U_k(x, y, z) t^{k/2}.
\]

\[
= U_0(x, y, z) + U_1(x, y, z) t^{1/2} + U_2(x, y, z) t^{1/2} + U_3(x, y, z) t^{1/2} + \ldots
\]

\[
= e^{x-y-z} \left[ 1 + \left( -1 \right)^{1/2} + \Gamma\left(\frac{1}{2}\right) \left( \frac{1}{\Gamma\left(\frac{3}{2}\right)} \right) + \frac{1}{2} \left( \frac{1}{\Gamma\left(\frac{5}{2}\right)} \frac{3}{2} \right) + \ldots \right].
\] (46)

When \(a = 1\), the exact solution of the non-fractional form of the nonlinear TFTE \((41)\) is obtained as

\[
u(x, y, z, t) = e^{x-y-z}.
\] (47)

7. Conclusions

In the present study, we have illustrated the reduced differential transform method for the analytical solution of two and three dimensional second order hyperbolic linear and nonlinear TFTEs. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. The effectiveness of the method is shown from the computational results. These results show that the RDTM technique is highly accurate, rapidly converge and is very easily implementable mathematical tool for the multidimensional physical problems emerging in various fields of engineering and sciences.

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