1 Introduction

In [4], following up work of Hitchin [9], the author found it useful to express Nahm’s equations, for a matrix group, in terms of the motion of a particle in a Riemannian symmetric space, subject to a potential field. This point of view lead readily to an elementary existence theorem for solutions of Nahm’s equation, corresponding to particle paths with prescribed end points. The original motivation for this article is the question of formulating an analogous theory for the Nahm equations associated to the infinite-dimensional Lie group of area-preserving diffeomorphisms of a surface—in the spirit of [5]. We will see that this can be done, and that a form of the appropriate existence theorem holds—essentially a special case of a result of Chen. However the main focus of the article is not on existence proofs but on the various formulations of the problem, and connections between them. In these developments, one finds that the natural context is rather more general than the original question, so we will start out of a different tack, and return to Nahm’s equations in Section 5.

Consider the following set-up in Euclidean space $\mathbb{R}^3$, in which we take co-ordinates $(x_1, x_2, z)$—thinking of $z$ as the vertical direction. (We will use the notation $\partial_1 = \partial_{x_1}, \partial_2 = \partial_{x_2}$.) Suppose we have a strictly positive function $H(x_1, x_2)$. This defines a domain

$$\Omega_H = \{(x_1, x_2, z) : 0 < z < H(x_1, x_2)\},$$

whose boundary has two components $\{z = 0\}$ and $\{z = H\}$. We consider the Dirichlet problem for the standard Laplacian: to find a harmonic function $\theta$ on $\Omega_H$ with $\theta = 0$ on $\{z = 0\}$ and $\theta = 1$ on $\{z = H\}$. To set up this problem precisely, let us assume that the data $H$ is $\mathbb{Z}^2$-periodic on $\mathbb{R}^2$, and seek a $\mathbb{Z}^2$-periodic solution $\theta$. Now we have a unique solution $\theta$ to our Dirichlet problem. Consider the flux of the gradient of $\theta$ through the boundary $\{z = H\}$. This defines another function $\rho$ on $\mathbb{R}^2$. To be precise, if $\iota_H$ is the obvious map from $\mathbb{R}^2$ to the boundary $\{z = H\}$ then the flux is defined by

$$\iota_H^*(\ast d\theta) = \rho dx_1 dx_2.$$  \hspace{1cm} (1)
Explicitly

\[ \rho = \partial_z \theta - (\partial_1 \theta \partial_1 H + \partial_2 \theta \partial_2 H), \]

with the right hand side evaluated at \((x_1, x_2, H(x_2, x_2))\). By the maximum principle, \(\rho\) is a positive function, since the normal derivative of \(\theta\) is positive in the positive \(z\) direction on \(\{z = H\}\). We consider the following free boundary problem: given a positive periodic function \(\rho\) does it arise from some periodic \(H\), and is \(H\) unique?

One can gain some physical intuition for this question by supposing that that the lower half-space \(\{z \leq 0\}\) represents a body with an infinite specific heat capacity fixed at temperature 0 and \(\Omega_H\) corresponds to a layer of ice covering this body. We choose units so that the melting temperature of the ice is 1. Sunlight shines vertically downwards onto the upper surface \(\{z = H\}\) of the ice, but with a variable intensity so that heat is transmitted to the surface according to the density function \(\rho\). We suppose that the surface of the ice is sprinkled by rain, which will instantly freeze if the surface temperature of the ice is less than 1. We also suppose that a wind blows across the surface, instantly removing any surface water. Then we see that the solution to our free boundary problem represents a static physical state, in which the upper surface of the ice is just at freezing point, the lower surface is at the imposed sub-freezing temperature and the heat generated by the given sunlight flows through the ice without changing the temperature. Physical intuition suggests that there should indeed be a unique solution.

We can express the free-boundary problem considered above as a special case of another question. Suppose now that we have a pair of periodic functions \(H_0, H_1\) on \(\mathbb{R}^2\) with \(H_0 < H_1\). Then we have a domain \(\Omega_{H_0,H_1} = \{H_0(x_1, x_2) < z < H_1(x_1, x_2)\}\), with two boundary components. Let \(\theta\) be the harmonic function in this domain equal to 0, 1 on \(\{z = H_0\}, \{z = H_1\}\) respectively. Then we obtain a pair of flux-functions \(\rho_0, \rho_1\) as before. By Gauss’ Theorem, these satisfy a constraint

\[ \int_{[0,1]^2} \rho_0 \, d^2x = \int_{[0,1]^2} \rho_1 \, d^2x, \tag{2} \]

since \([0,1]^2\) is a fundamental domain for the \(\mathbb{Z}^2\)-action. Obviously, if we replace \(H_0, H_1\) by \(H_0 + c, H_1 + c\) for any constant \(c\) we get the same pair \(\rho_0, \rho_1\). We ask: given \(\rho_0, \rho_1\) satisfying the integral constraint (2), is there a corresponding pair \((H_0, H_1)\), and if so is the solution unique up to the addition of a constant? A positive answer to this question implies a positive answer to the previous one, by a simple reflection argument. (Given \(\rho\), as in the first problem, take \(\rho_0 = \rho_1 = \rho/2\). Then uniqueness implies that the solution has reflection symmetry about \(\theta = 1/2\) and we get a solution to the first problem by changing \(\theta\) to \(2\theta - 1\).)

Of course we can also imagine a physical problem corresponding to this second question: for example a layer of ice in the region \(\Omega_{H_0,H_1}\). We can now vary the problem by supposing that in place of ice we have a horizontally stratified material in which heat can only flow in the horizontal directions. Thus the steady-state condition, for a temperature distribution \(\theta(x_1, x_2, z)\) is

\[ (\partial_1^2 + \partial_2^2)\theta = 0. \tag{3} \]
We define flux-functions $\rho_0, \rho_1$ by pulling back the 2-form
$$\partial_1 \theta \, dx_2 dz - \partial_2 \theta \, dx_1 dz,$$
and the integral constraint (2) still holds. So we ask: given $\rho_0, \rho_1$ satisfying (2), is there a pair $H_0, H_1$ and a function $\theta$ on $\Omega_{H_0, H_1}$, equal to 0, 1 on the two boundary components, which has these fluxes, and is the solution essentially unique? (In this case one has to relax the condition on the domain to $H_0 \leq H_1$.)

It is natural to extend these questions to a general compact oriented Riemannian manifold $X$ (which would be the flat torus $\mathbb{R}^2/\mathbb{Z}^2$ in the discussion above). Write $d\mu$ for the Riemannian volume form on $X$. We fix a real parameter $\epsilon \geq 0$ and define a map $*\epsilon$ from $T^*(X \times \mathbb{R})$ to $\Lambda^n T^*(X \times \mathbb{R})$ by
$$*\epsilon dz = \epsilon d\mu, \quad *\epsilon \alpha = (*_X \alpha) dz,$$
for $\alpha \in T^* X$, where $*_X$ is the usual Hodge $*$-operator on $X$. Then for a function $\theta$ on a domain in $X \times \mathbb{R}$
$$d *\epsilon d\theta = (\Delta_\epsilon \theta) dz d\mu,$$
where
$$\Delta_\epsilon \theta = (-\epsilon \partial_z^2 + \Delta_X) \theta,$$
with $\Delta_X$ the standard Laplace operator on $X$. (We use the sign convention that $\Delta_X$ is a positive operator, so when $\epsilon = 1$ our $\Delta_\epsilon$ is the standard Laplace operator on $X \times \mathbb{R}$.) If $\theta$ is defined on a domain $\Omega_{H_0, H_1}$, as above, we define the flux $\rho_i$ on the boundary $\{z = H_i\}$ by pulling back $*\epsilon d\theta$ just as before. We consider a pair of functions $\rho_0, \rho_1 > 0$ with
$$\int_X \rho_0 \, d\mu = \int_X \rho_1 \, d\mu = \int_X d\mu$$
and we ask

**Question 1** Is there a pair $H_0 \leq H_1$ and a function $\theta$ on the set $\Omega_{H_0, H_1} \subset X \times \mathbb{R}$ with $\theta = 0, 1$ on the hypersurfaces $\{z = H_0\}, \{z = H_1\}$, with fluxes $\rho_i$ and with $\Delta_\epsilon \theta = 0$? If so, is the solution essentially unique?

For any $\epsilon > 0$ the equation $\Delta_\epsilon \theta = 0$ can be transformed into the standard Laplace equation on the product, by rescaling the $z$ variable. When $\epsilon = 0$ the equation has a very different character: it is not elliptic and we obviously do not have automatic interior regularity with respect to $z$.

## 2 An infinite-dimensional Riemannian manifold

We now start in a different direction. Given our compact Riemannian manifold $X$ we let $\mathcal{H}$ be the set of functions $\phi$ on $X$ such that $1 - \Delta_X \phi > 0$. We make
\( \mathcal{H} \) into a Riemannian manifold, defining the norm of a tangent vector \( \delta \phi \) at a point \( \phi \) by
\[
\| \delta \phi \|_{\phi}^2 = \int_X (\delta \phi)^2 \left( 1 - \Delta_X \phi \right) d\mu.
\]
Thus a path \( \phi(t) \) in \( \mathcal{H} \), parametrised by \( t \in [0,1] \) say, is simply a function on \( X \times [0,1] \) and the “energy” of the path is
\[
\frac{1}{2} \int_0^1 \int_X \left( \frac{\partial \phi}{\partial t} \right)^2 \left( 1 - \Delta_X \phi \right) d\mu \, dt.
\]
When \( X \) is 2-dimensional and orientable, this definition coincides with the metric on the space of “Kahler potentials” discussed by Mabuchi \[11\], Semmes \[12\] and the author \[6\]. The general context in those references is a compact Kahler manifold: here we are considering a different extension of the 2-dimensional case, and we will see that some new features emerge. The account below follows the approach in \[6\] closely.

It is straightforward to find the Euler-Lagrange equations associated to the energy (3). These are
\[
\ddot{\phi} = \frac{[\nabla_X \dot{\phi}]^2}{1 - \Delta_X \phi}.
\]
These equations define the geodesics in \( \mathcal{H} \). We can read off the Levi-Civita connection of the metric from this geodesic equation, as follows. Let \( \phi(t) \) be any path in \( \mathcal{H} \) and \( \psi(t) \) be another function on \( X \times [0,1] \), which we regard as a vector field along the path \( \phi(t) \). Then the covariant derivative of \( \phi \) along the path is given by
\[
D_t \phi = \frac{d\phi}{dt} + (W_t, \nabla_X \psi),
\]
where
\[
W_t = \frac{-1}{1 - \Delta_X \phi} \nabla_X \dot{\phi}
\]
and \( (, ) \) is the Riemannian inner product on tangent vectors to \( X \). (We write \( \nabla_X \), or sometimes just \( \nabla \), for the gradient operator on \( X \), so \( W_t \) is a vector field on \( X \).) This has an important consequence for the holonomy group of the manifold \( \mathcal{H} \). Observe that the tangent space to \( \mathcal{H} \) at a point \( \phi \) is the space of functions on \( X \) endowed with the standard \( L^2 \) inner product associated to the measure
\[
d\mu_{\phi} = (1 - \Delta \phi) d\mu_0.
\]
So, in a general way, the parallel transport along a path from \( \phi_0 \) to \( \phi_1 \) should be an isometry from \( L^2(X, d\mu_{\phi_0}) \) to \( L^2(X, d\mu_{\phi_1}) \). (Here we are ignoring the distinction between, for example, smooth functions and \( L^2 \) functions.) What we see from equation (5) is that this isometry is induced by a diffeomorphism \( f : X \to X \) with
\[
f^*(d\mu_{\phi_1}) = d\mu_{\phi_0}.
\]
The diffeomorphism is obtained by integrating the time-dependent vector field $W_t$ and equation (6) follows from the identity

$$L_{W_t} \, d\mu_\phi = d \ast \left( \frac{1}{1 - \Delta_X} d\phi \ast X \, d\mu_\phi \right) = \Delta \phi = -\frac{d}{dt} \mu_\phi.$$

(Here $L$ denotes the Lie derivative on $X$.) We conclude that the holonomy group of $\mathcal{H}$ is contained in the group $\mathcal{G}$ of volume-preserving diffeomorphisms of $(X, d\mu_0)$, regarded as a subgroup of the orthogonal group of $L^2(X, d\mu_0)$. (This can also be expressed by saying that there is an obvious principal $\mathcal{G}$-bundle over $\mathcal{H}$ with the tangent bundle as an associated vector bundle, and the Levi-Civita connection is induced by a connection on this $\mathcal{G}$-bundle.)

We now move on to discuss the curvature tensor of $\mathcal{H}$. Let $\phi$ be a point of $\mathcal{H}$ and let $\alpha, \beta$ be tangent vectors to $\mathcal{H}$ at $\phi$—so $\alpha$ and $\beta$ are just functions on $X$. The curvature $R_{\alpha, \beta}$ should be a linear map from tangent vectors to tangent vectors: that is from functions on $X$ to functions on $X$. The discussion of the holonomy above tells us that this map must have the form

$$R_{\alpha, \beta}(\psi) = (\nu_{\alpha, \beta}, \nabla \psi),$$

for some vector field $\nu_{\alpha, \beta}$ on $X$, determined by $\phi, \alpha, \beta$. Moreover we know that we must have

$$L_{\nu_{\alpha, \beta}}(d\mu_\phi) = 0.$$

To identify this vector field we introduce some notation. For vector fields $v, w$ on $X$ we write $v \times w$ for the exterior product: a section of the bundle $\Lambda^2 TX$. We define a differential operator

$$\text{curl} : \Gamma(\Lambda^2 TX) \to \Gamma(TX),$$

to be the composite of the standard identification:

$$\Lambda^2 TX \cong \Lambda^{n-2} T^* X,$$

(using the Riemannian volume form $d\mu$), the exterior derivative

$$d : \Gamma(\Lambda^{n-2} T^* X) \to \Lambda^{n-1} T^* X,$$

and the standard identification

$$\Lambda^{n-1} T^* X \cong TX,$$

(using the volume form $d\mu$ again). Then we have

**Theorem 1** The curvature of $\mathcal{H}$ is given by (7) and the vector field

$$\nu_{\alpha, \beta} = \frac{1}{1 - \Delta \phi} \text{curl} \left( \frac{1}{1 - \Delta \phi} \nabla \alpha \times \nabla \beta \right).$$

**Corollary 1** The manifold $\mathcal{H}$ has non-positive sectional curvature.
The sectional curvature corresponding to a pair of tangent vectors \( \alpha, \beta \) at a point \( \phi \) is 

\[ K_{\alpha,\beta} = \langle R_{\alpha,\beta}(\alpha), \beta \rangle. \]

In our case this is

\[ K_{\alpha,\beta} = \int_X (\nu_{\alpha,\beta}, \nabla \alpha) \beta (1 - \Delta X \phi) d\mu. \]

Unwinding the algebraic identifications we used above, the integrand can be written in terms of differential forms as

\[ \frac{1}{1 - \Delta X \phi} d\alpha \wedge d \left( \frac{1}{1 - \Delta X \phi} \ast (d\alpha \wedge d\beta) \right) \beta (1 - \Delta X \phi). \]

So

\[ K_{\alpha,\beta} = \int_X d\alpha \wedge d \left( \frac{1}{1 - \Delta X \phi} \ast (d\alpha \wedge d\beta) \right) \beta. \]

Applying Stokes’ Theorem this is

\[ K_{\alpha,\beta} = -\int_X \frac{1}{1 - \Delta X \phi} d\alpha \wedge d\beta \wedge \ast (d\alpha \wedge d\beta) = -\int_X \frac{1}{1 - \Delta X \phi} |d\alpha \wedge d\beta|^2 d\mu \leq 0. \]

In the proof of Theorem 1 we will make use of two identities. For any pair of vector fields \( v, w \) and function \( f \)

\[ \text{curl} \ (v \times w) = [v, w] + (\text{div} \ v) w - (\text{div} \ v) w \] (8)

\[ \text{curl} \ (f (v \times w)) = f \text{curl} \ (v \times w) + (v, \nabla f) w - (w, \nabla f) w. \] (9)

We leave the verification as an exercise. (Considering geodesic coordinates we see that it suffices to treat the case of Euclidean space. Our notation has been chosen to agree with standard notation in the case of vector fields in \( \mathbb{R}^3 \).)

To calculate the curvature we consider a 2-parameter family \( \phi(s, t) \) in \( \mathcal{H} \), with a corresponding vector field \( \psi(s, t) \) along the family. Then we will compute the commutator \( (D_s D_t - D_t D_s) \psi(s, t) \). Evaluating at \( \phi = \phi(0, 0) \) this is \( R_{\alpha,\beta}(\psi) \) where \( \psi = \psi(0, 0), \alpha = \partial_s \phi, \beta = \partial_t \phi. \)

Now we write

\[ D_s = \frac{\partial}{\partial s} + W_s, \quad D_t = \frac{\partial}{\partial t} + W_t, \]

where the vector fields \( W_s, W_t \) are regarded as operators on the functions on \( X \). So \( D_s D_t - D_t D_s \) is the operator given by the vector field

\[ \nu = \frac{\partial W_s}{\partial t} - \frac{\partial W_t}{\partial s} - [W_s, W_t], \]

and \( \nu \) is exactly the vector field \( \nu_{\alpha,\beta} \) we need to identify. Recall that

\[ W_s = -\nabla \partial_s \phi \frac{1}{1 - \Delta \phi}, \quad W_t = -\nabla \partial_t \phi \frac{1}{1 - \Delta \phi}. \]
\[
\frac{\partial W_s}{\partial t} = -\frac{1}{1 - \Delta \phi} \nabla \left( \frac{\partial^2 \phi}{\partial s \partial t} \right) + \frac{1}{(1 - \Delta \phi)^2} \Delta \partial_s \phi \nabla \partial_t \phi.
\]
Evaluating at \( s = t = 0 \) where \( \partial_s \phi = \alpha, \partial_t \phi = \beta \) we have
\[
\frac{\partial W_s}{\partial t} - \frac{\partial W_t}{\partial s} = \frac{1}{(1 - \Delta \phi)^2} (\Delta \alpha \nabla \beta - \Delta \beta \nabla \alpha).
\]
Write \( g \) for the function \((1 - \Delta \phi)^{-1}\). Combining with the Lie bracket term we obtain
\[
\nu_{\alpha, \beta} = [g \nabla \alpha, g \nabla \beta] + g^2 (\Delta \alpha \nabla \beta - \Delta \beta \nabla \alpha).
\]
Now applying (8) we have
\[
[g \nabla \alpha, g \nabla \beta] = \text{curl} \ (g^2 \nabla \alpha \times \nabla \beta) + \text{div} \ (g \nabla \alpha) \nabla \beta - \text{div} \ (g \nabla \beta) \nabla \alpha.
\]
Applying (9) we have
\[
\text{curl} \ (g^2 \nabla \alpha \times \nabla \beta) = g \text{curl} \ (g \nabla \alpha \times \nabla \beta) + g (\nabla g, \nabla \alpha) \nabla \beta - (\nabla g, \nabla \beta) \nabla \alpha).
\]
Since
\[
\text{div} \ (g \nabla \alpha) = g \Delta \alpha - (\nabla g, \nabla \alpha), \quad \text{div} \ (g \nabla \beta) = g \Delta \beta - (\nabla g, \nabla \beta)
\]
we see that
\[
\nu_{\alpha, \beta} = g \text{curl} \ (g \nabla \alpha \times \nabla \beta),
\]
as required.

In the case when \( X \) has dimension 2—as discussed in [6], [11], [12]—the space \( H \) is formally a symmetric space. This is not true in general, since the curvature tensor is not preserved by the action of the group \( G \).

We define a functional on \( H \) by
\[
V(\phi) = \int_X \phi \, d\mu.
\]
This function is convex along geodesics in \( H \), since the geodesic equation implies \( \ddot{\phi} \geq 0 \). Now introduce a real parameter \( \epsilon \geq 0 \) and consider the functional on paths in \( H \):
\[
E = \int \frac{1}{2} |\dot{\phi}|^2 + \epsilon V(\phi) \, dt,
\]
corresponding to the motion of a particle in the potential \(-\epsilon V\). The Euler-Lagrange equations are
\[
\ddot{\phi} = \frac{|\nabla_X \phi|^2 + \epsilon}{1 - \Delta_X \phi}.
\]
3 Three equivalent problems

In this section we will explain that there are three equivalent formulations of
the same PDE problem associated to a compact Riemannian manifold $X$. We
have essentially encountered two of these already.

- **The “$\theta$ equation”**

This is the problem we set up in Section 1. We are given positive functions
$\rho_0, \rho_1$ on $X$, with

$$\int_X \rho_i \, d\mu = \int_X d\mu. \tag{12}$$

We seek a domain $\Omega_{H_0, H_1} \subset X \times \mathbb{R}$ defined by $H_0, H_1 : X \to \mathbb{R}$ and a
function $\theta$ on $\Omega_{H_0, H_1}$, equal to 0, 1 on the two boundary components, with
fluxes $\rho_0, \rho_1$ and satisfying the equation

$$\Delta_{\epsilon} \theta = 0.$$

- **The “$\Phi$ equation”**

Here we are given $\phi_0, \phi_1$ on $X$, with $1 - \Delta \phi_i > 0$. We seek a function
$\Phi$ on $X \times [0,1]$, equal to $\phi_0, \phi_1$ on the two boundary components, with
$1 - \Delta \Phi > 0$ for all $t$ and satisfying the nonlinear equation

$$\frac{\partial^2 \Phi}{\partial t^2} (1 - \Delta X \Phi) - |\nabla \left( \frac{\partial \Phi}{\partial t} \right)|^2 = \epsilon. \tag{13}$$

As we have explained in Section 2, this is the same as finding a path in the
space $\mathcal{H}$, with end points $\phi_0, \phi_1$, corresponding to the motion of a particle
in the potential $-\epsilon V$.

Now we introduce the third problem.

- **The “$U$ equation”**

We are given positive functions $\phi_0, \phi_1$, with $1 - \Delta \phi_i > 0$, as above. Define
a function $L$ on $X \times \mathbb{R}$ by

$$L(x, z) = \max(\phi_0(x) - \phi_1(x) + z, 0).$$

We seek a $C^1$ function $U(x, z)$ on $X \times \mathbb{R}$ with $U \geq L$ everywhere and
satisfying the equation

$$\Delta U = (1 - \Delta \phi_0) \tag{14}$$
on the open set $\Omega$ where $U > L$. 

8
The equivalence of these three problems (assuming suitable regularity for the solutions in each case) arises from elementary, but not completely obvious, transformations. We describe these now.

- \( \theta \)-equation \( \Rightarrow \phi \)-equation

Suppose we have a solution \( \theta \) on a domain \( \Omega_{H_0, H_1} \). Then \( \partial_z \theta = \frac{\partial \theta}{\partial z} \) is positive on the boundary components of \( \Omega_{H_0, H_1} \). The function \( \partial_z \theta \) satisfies the equation \( \Delta_z (\partial_z \theta) = 0 \) and it follows from this that \( \partial_z \theta \) is positive throughout the domain. This implies that, for any \( t \in [0,1] \), the set \( \theta^{-1}(t) \) is the graph of a smooth function \( h_t \) on \( X \). By definition \( h_0 = H_0 \) and \( h_1 = H_1 \). We also write this function as \( h(t,x) \) where convenient. For each fixed \( t \) we can define a function \( \rho_t \) on \( X \) by the flux of \(*_t d\theta\), just as before.

We claim that

\[
\frac{\partial \rho_t}{\partial t} = \Delta_X h_t
\]

We show this by direct calculation (there are more conceptual, geometric arguments). For simplicity we treat the case when the metric on \( X \) is locally Euclidean, so \( \Delta_X = -\sum \partial_i^2 \) where \( \partial_i = \frac{\partial}{\partial x_i} \), for local coordinates \( x_i \). The identity

\[
\theta(x, h_t(x)) = t
\]

implies that

\[
\partial_t \theta + \partial_z \theta \partial_i h = 0 \quad (16)
\]

and

\[
\partial_z \theta \partial_t h = 1. \quad (17)
\]

Now

\[
\Delta_X h_t = -\sum_i (\partial_i + (\partial_i h) \partial_z) \partial_i h,
\]

and this is

\[
\Delta_X h_t = -\sum_i (\partial_i - \frac{\partial \theta}{\partial_z} \partial_z) \left( -\frac{\partial_i \theta}{\partial_z} \right)
\]

which is

\[
-\sum_i \left( \frac{\partial_i^2 \theta}{\partial_i \theta} - 2 \frac{\partial_i \theta \partial_i \partial_z \theta}{(\partial_i \theta)^2} + \frac{\partial_i \theta \partial_i \partial_z \theta \partial_z \theta}{(\partial_z \theta)^3} \right).
\]

On the other hand the flux \( \rho_t \) is given by pulling back the differential form \(*_t d\theta\) on the product by the map \( x \mapsto (x, h_t(x)) \) and this gives

\[
\rho_t = c \partial_z \theta + \frac{1}{\partial_z \theta} \sum_i (\partial_i \theta)^2.
\]

So

\[
\frac{\partial \rho}{\partial t} = \frac{1}{\partial_z \theta} \partial_z \left( c \partial_z \theta + \sum_i (\partial_i \theta)^2 \right).
\]
This is

\[ \partial_t \rho_t = \epsilon \frac{\partial^2 \rho_t}{\partial z^2} + 2 \sum_i \frac{\partial_i \theta \partial_i \rho_t}{(\partial_z \theta)^2} - \sum_i (\partial_i \theta)^2 \frac{\partial_i \rho_t}{(\partial_z \theta)^3} \].

So we see that \( \partial_t \rho_t = -\Delta_X h_t \), since \( \epsilon \partial_z \partial_z \theta = -\sum_i \partial_i \partial_i \theta \).

Now the normalisation (13) implies that there is a function \( \phi_0 \) on \( X \) such that \( \rho_0 = 1 - \Delta_X \phi_0 \). For \( t > 0 \) we define \( \phi_t \) by

\[ \phi_t = \phi_0 + \int_0^t h_\tau d\tau. \]

We can also regard this family of functions as a single function \( \Phi \) on \( X \times [0,1] \). Then (15) implies that \( \rho_t = 1 - \Delta_X \phi_t \) for each \( t \). We have

\[ \frac{\partial^2 \Phi}{\partial t^2} = \partial_t h = \frac{1}{\partial_z \theta} \]

and

\[ 1 - \Delta_X \Phi = \epsilon \partial_z \theta + \frac{1}{\partial_z \theta} \sum_i (\partial_i \theta)^2. \]

So

\[ \frac{\partial^2 \Phi}{\partial t^2} (1 - \Delta_X \Phi) = \epsilon + \sum_i \left( \frac{\partial_i \theta}{\partial_z \theta} \right)^2. \]

Now since

\[ \partial_t \partial_t \Phi = \partial_t h_t = -\frac{1}{\partial_z \theta} \partial_z \theta \]

we can write the above as

\[ \frac{\partial^2 \Phi}{\partial t^2} (1 - \Delta_X \Phi) = \epsilon + |\nabla_X \frac{\partial \Phi}{\partial t}|^2 \]

as required.

- \( \Phi \)-equation \( \implies \) \( U \)-equation.

Here we suppose we have a solution \( \Phi(x,t) \) of the \( \Phi \) equation and we write \( \Phi(x,0) = \phi_0, \Phi(x,1) = \phi_1 \). We essentially take the Legendre transform in the \( t \)-variable. The discussion is slightly more complicated when \( \epsilon = 0 \), so for simplicity we treat the case when \( \epsilon > 0 \) and \( \partial_t^2 \Phi \) is strictly positive. Write \( H_1(x), H_2(x) \) for the derivatives \( \partial_t \Phi \) evaluated at \( (x,0), (x,1) \) respectively, so \( H_0 < H_1 \). We calculate first in the open set \( \Omega_{H_0,H_1} \). For each fixed \( x \in X \) and each \( z \) in the interval \( (H_0(x), H_1(x)) \) there is a \( t = t(x,z) \) such that \( z = \partial_t \Phi \). We set

\[ U(x,z) = \Phi(x,0) - \Phi(x,t) + zt. \]

This defines a function \( U \) in \( \Omega_{H_0,H_1} \). We define \( U \) outside this set by setting \( U(x,z) = 0 \) if \( z \leq H_0(x) \) and \( U(x,z) = L(x,z) = \phi_0 - \phi_1 - z \) if
$z \geq H_1(x)$. It follows from the definitions that $U$ is $C^1$, that $U \geq L$ and that the set where $U > L$ is exactly $\Omega_{H_0,H_1}$. We calculate on this set. Then $\partial_t U = t$ and
\[ \partial^2_z U = (\partial^2_t \Phi)^{-1}. \] (18)
Differentiating with respect to the parameters $x_i$ we have
\[ \partial_i U = \partial_i \phi_0 - \partial_t \Phi, \]
and
\[ \partial^2_i U = \partial^2_i \phi_0 - \partial^2_t \Phi - \frac{\partial t}{\partial x_i} \frac{\partial^2 \Phi}{\partial t \partial x_i}. \]
Differentiating the identity $z = \partial_i \Phi$ gives
\[ 0 = \frac{\partial^2 \Phi}{\partial t \partial x_i} + \frac{\partial^2 \Phi}{\partial t^2} \frac{\partial t}{\partial x_i}, \]
so we can write
\[ \partial^2_i U = \partial^2_i \phi_0 - \partial^2_t \Phi + \frac{1}{\partial^2_t \Phi} (\partial_t \partial_i \Phi)^2. \]
Summing over $i$ and using the formula (18) for $\partial^2_z U$ we obtain
\[ \epsilon \partial^2_z U - \Delta X U = \frac{1}{\partial^2_t \Phi} (\epsilon + |\nabla X \Phi|^2) - \Delta_X \phi_0 + 1, \]
and so
\[ \Delta_x U = 1 - \Delta_X \phi_0. \]

• $U$-equation $\implies$ $\theta$-equation

Now suppose we have a solution $U$ of $\Delta U = \rho_0$ in a domain $\Omega_{H_0,H_1}$, satisfying the appropriate boundary conditions, where $\rho_0 = 1 - \Delta_X \phi_0$. We set
\[ \theta = \frac{\partial U}{\partial z}. \]
Then $\Delta_x \theta = 0$ and $\theta = 0,1$ on the two boundary components. We have to check that the fluxes of $\epsilon \epsilon d\theta$ on the boundary components are $\rho_i = 1 - \Delta_X \phi_i$. Consider first the boundary component where $z = H_0$. The flux is
\[ \epsilon \partial_z \theta + \frac{|\nabla_X \theta|^2}{\partial_z \theta} = \epsilon \partial^2_z F + \frac{1}{\partial^2_t F} \sum (\partial_z \partial_i F)^2. \]
Now we have identities
\[ (\partial_i F)(x, H_0(x)) = 0, \quad (\partial_z F)(x, H_0(x)) = 0. \]
Differentiating the first of these with respect to $x_i$ we get

$$\partial^2_i F + \partial_i H_0 \partial_i \partial_z F = 0,$$

on the boundary. Differentiating the second gives

$$\partial_i \partial_z F + \partial_i H_0 \partial^2_z F = 0$$

on the boundary. Combining these we have

$$(\partial_z \partial_i F)^2 = (\partial^2_z F)(\partial^2_i F).$$

Hence the flux is

$$\epsilon \partial^2_z F + \sum_i \partial^2_i F = \rho_0.$$

The argument for the other boundary component $\{z = H_1(x)\}$ is similar.

4 Existence results and discussion

We have set up a class of PDE problems associated to any compact Riemannian manifold, and seen that these have three equivalent formulations. In this section we will make some remarks about existence results, and comparison with the free-boundary literature. This discussion is unfortunately rather incomplete, mainly due to the authors limited grasp of the background.

4.1 Monge-Ampère and the results of Chen

For a function $\Phi$ on $X \times (0, 1)$ write $q(\Phi)$ for the nonlinear differential operator

$$q(\Phi) = \partial^2_t \Phi (1 - \Delta_X \Phi) - |\nabla_X \frac{\partial}{\partial t} \Phi|^2.$$

So our “$\Phi$-equation” is $q(\Phi) = \epsilon$. When $X$ has dimension 1—a circle with local coordinate $x$—we can write $\Delta_X = -\partial_x^2$ and the equation is the real Monge-Ampère operator

$$q(\Phi) = \det \begin{pmatrix} \partial^2_t \Phi & \partial_t \partial_i \Phi \\ \partial_x \partial_t \Phi & 1 + \partial^2_x \Phi \end{pmatrix}.$$

When $X$ has dimension 2 the operator can be expressed as a complex Monge-Ampère operator. That is, we regard $X$ as a Riemann surface and identify the Laplace operator on $X$ with $i\overline{\partial}\partial$. We take the product with a circle, with coordinate $\alpha$, and let $\tau = t + i\alpha$ be a complex coordinate on the Riemann surface $S^1 \times (0, 1)$. Then, in differential form notation, our nonlinear operator is given by

$$(\omega_0 + i\overline{\partial}\partial \Phi)^2 = q(\Phi)\omega_0 d\tau d\overline{\tau},$$

where $\omega_0$ is the Riemannian area form of $X$ lifted to $X \times S^1 \times (0, 1)$. Our Dirichlet problem becomes a Dirichlet problem for $S^1$-invariant solutions of this
complex Monge-Ampère equation on $X \times S^1 \times (0, 1)$. This was studied by Chen \cite{Chen} and it follows from his results that, for any $\epsilon > 0$ there is a unique solution to our problem, and hence an affirmative answer to Question 1 in this case. (Chen does not state this result explicitly, but it follows from the continuity method developed in \cite{Chen}, Section 3, that for any strictly positive smooth function $\nu$ on $X \times [0, 1]$ there is a solution of the equation $q(\Phi) = \nu$ with prescribed boundary values $\phi_0, \phi_1$.)

It seems quite likely that the techniques used by Chen can be extended to the higher dimensional case. The foundation for this should be provided by a convexity property of the nonlinear operator which we will now derive. Let $A$ be a symmetric $(n+1) \times (n+1)$ matrix with entries $A_{ij}$, $0 \leq i, j \leq n$. Define

$$Q(A) = A_{00} \sum_{i=1}^{n} A_{ii} - \sum_{i=1}^{n} A_{i0}^2.$$ 

Thus $Q$ is a quadratic function on the vector space of symmetric $(n+1) \times (n+1)$ matrices.

**Lemma 1**

1. If $A > 0$ then $Q(A) > 0$ and if $A \geq 0$ then $Q(A) \geq 0$.

2. If $A, B$ are matrices with $Q(A) = Q(B) > 0$ and if the entries $A_{00}, B_{00}$ are positive then for each $s \in [0, 1]$

$$Q(sA + (1-s)B) \geq Q(A), \quad Q(A-B) < 0.$$

Moreover, if $A \neq B$ then strict inequality holds.

To see the first item, observe that we can change basis in $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ to reduce to the case when the block $A_{ij}, 1 \leq i, j \leq n$ is diagonal, with entries $b_i$ say. Then if $A \geq 0$ we have $A_{00}b_i \geq A_{i0}^2$, and so

$$Q(A) = \sum_{i=1}^{n} A_{00} \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} A_{i0}^2 \geq 0,$$

with strict inequality if $A > 0$.

For the second item, we just have to observe that $Q$ is induced from a a quadratic form of Lorentzian signature on $\mathbb{R}^{n+2}$ by the linear map

$$\pi : A \mapsto (A_{00}, \sum_{i=1}^{n} A_{ii}, A_{0i}).$$

The hypotheses imply that $\pi(A)$ and $\pi(B)$ are in the same component of a hyperboloid defined by this Lorentzian form and the statements follow immediately from elementary geometry of Lorentz space.

Using this Lemma we can deduce the uniqueness of the solution to our Dirichlet problem, in any dimension.

**Proposition 1** If $\phi_0, \phi_1 \in \mathcal{H}$ then there is at most one solution $\Phi$ of the equation $Q(\Phi) = \epsilon$ on $X \times [0, 1]$ with $1 - \Delta X \Phi > 0$ for all $t$ and with $\Phi(x, 0) = \phi_0(x), \Phi(x, 1) = \phi_1(x)$.  

13
We show that the functional $E(\Phi)$ given by (10) is convex with respect to the obvious linear structure. Thus we consider a 1-parameter family $\Phi_s = \Phi + s\psi$, with the fixed end points. We have

$$\frac{d}{ds}E(\Phi_s) = \int_0^1 \int_X 2\Phi_s \dot{\psi}(1 - \Delta X \Phi_s) - \dot{\Phi}^2 \Delta X \psi.$$ 

Integrating by parts (just as in the derivation of the geodesic equation) we obtain

$$\frac{d}{ds}E(\Phi_s) = \int_0^1 \int_X (q(\Phi_s) - \epsilon) \psi \, d\mu.$$ 

Suppose that $\Phi_0, \Phi_1$ are two different solutions, so when $s = 0, 1$ the term $q(\Phi_s) - \epsilon$ in the above expression vanishes pointwise. Item (2) in the lemma above implies that for $s \in (0, 1)$ we have $q(\Phi_s) - \epsilon \geq 0$, with strict inequality somewhere. This means that $E(\Phi_1) > E(\Phi_0)$. Interchanging the roles of $\Phi_0, \Phi_1$ we obtain the reverse inequality, and hence a contradiction.

One can also prove this uniqueness using the maximum principle. Note too that the uniqueness is what one would expect, formally, from the negative curvature of the space $\mathcal{H}$ and the convexity of the functional $V$.

4.2 Comparison with the free-boundary literature

The author is not at all competent to make this comparison properly. Suffice it to say, first, that the problem we are considering is very close to those which have been studied extensively in the applied literature. For example, in the $\theta$-formulation, the condition of prescribing the pull-back of the flux on the free boundary is the same as that in the classical problem of the “porous dam” ([1] Chapter 8, [8] Chapter 4.4), but with the difference that in that case $\rho$ is constant and there are additional boundary conditions on other boundary components.

Second, the constructions we have introduced in Section 3 above all appear in this literature. The transformation from $\theta$ to $\Phi$ taking the harmonic function $\theta$ as a new independent variable is called in ([3], Chapter 5) the “isothermal migration method”. The transformation from the formulation in terms of $\theta$ to that in terms of $U$ is known as the Baiocchi transformation [1], [8], [3]. The transformation of the free boundary problem for a linear equation to a nonlinear Dirichlet problem is used in [10] to derive fundamental regularity results.

An important feature of the $U$-formulation is that it admits a variational description. Recall that we are given a function $L = \max(\phi_0 - \phi_1 + z, 0)$ on $X \times \mathbb{R}$ and we seek a $C^1$ function $U$ with $U \geq L$ satisfying the equation $\Delta_x U = \rho_0$ on the set where $U > L$. This can be formulated as follows. We fix a large positive $M$ and consider the functional

$$\mathcal{E}_M(U) = \int \frac{1}{2} |\nabla_x U|^2 + \epsilon |\partial_z U|^2 - \rho_0 U \, d\mu dz,$$

over the space of functions satisfying the constraint $U \geq L$, where the integral is taken over $X \times [-M, M]$ in $X \times \mathbb{R}$ (which, a posteriori, should contain the set
Ω_{H_0,H_1} on which \( U > L \). Then the solution minimises \( E_M \) over all functions \( U \geq L \). This can be used to give another proof of the uniqueness of the solution to our problem. It seems likely that it could also be made the basis of an existence proof, following standard techniques in the free boundary literature. Now recall that our \( \Phi \)-formulation was based on a variational principle, with Lagrangian (10). To relate the two, we consider any function \( \Phi \) on \( X \times [0,1] \) with \( \partial_t \partial_t \Phi \geq 0 \) and define \( U \) by the recipe of Section 3. We suppose that \( -M < \partial_t \Phi(x,0) \) and \( \partial_t \Phi(x,1) < M \) for all \( x \in X \). Then we have

**Proposition 2** The functional \( E_M(U) \) is

\[
E(\Phi) + M \int_X (1-\Delta_X \phi_0)(\phi_0-\phi_1) + \frac{1}{2} |\nabla(\phi_1-\phi_0)|^2 d\mu + \left( \frac{M^2}{2} + \epsilon M \right) \int_X \mu - \epsilon \int_X \phi_1 d\mu
\]

Thus if we fix \( M \) and the end points \( \phi_0, \phi_1 \) the two functionals differ by a constant. The central step in the proof is the fact that the integrals

\[
\int_0^1 \int_X \partial_t^2 \Phi |\nabla_X \Phi|^2 d\mu dt
\]

\[
\int_0^1 \int_X \Delta_X \Phi (\partial_t \Phi)^2 d\mu dt
\]

are equal modulo boundary terms. We leave the full calculation as an exercise for the reader.

### 4.3 The degenerate case

So far, in this section, we have discussed the case when \( \epsilon > 0 \). In that case the equations we are studying are elliptic. The degenerate case, when \( \epsilon = 0 \), is much more delicate. In fact Chen’s main concern in [2] was to obtain results about this case, taking the limit as \( \epsilon \) tends to 0. Chen shows that the Dirichlet problem for \( \Phi \), with \( \epsilon = 0 \), has a \( C^{1,1} \) solution but the question of smoothness is open. The formulation of the problem in terms of the function \( U \) has particular advantages here, because the problem is set-up as a family of elliptic problems, and the issue becomes one of smooth dependence on parameters. (This is related to another approach, involving families of holomorphic maps, discussed in [12], [7].) We can express the central question as follows. Suppose we have a smooth function \( \lambda \) on a compact Riemannian manifold \( X \) and fix a smooth positive function \( \rho \). Let \( J \) be the functional

\[
J(u) = \int_X \frac{1}{2} |\nabla u|^2 - \rho u.
\]

For each \( z \in \mathbb{R} \) we set \( \lambda_z = \max(\lambda, z) \) and minimise the functional \( J \) over the set of functions \( u \geq \lambda_z \). Suppose we know that there is a minimiser \( u_z \) which is smooth on the open set \( \Omega_z \subset X \) where \( u_z > \lambda_z \). Let \( \Omega = \{ (x,z) : x \in \Omega_z \} \subset X \times \mathbb{R} \).

**Question 2** In this situation, does \( u_z \) vary smoothly with \( z \) in \( \Omega \)?

The interesting case here seems to be when \( z \) is a critical value of \( g \).
5 Relation with Nahm’s equations

We recall that Nahm’s equations are a system of ODE for three functions $T_1, T_2, T_3$ taking values in a fixed Lie algebra:

$$\frac{dT_i}{dt} = [T_j, T_k],$$

where $i, j, k$ run over cyclic permutations of $1, 2, 3$. To simplify notation, let us fix on the Lie algebra $u(n)$. It is equivalent (at least in the finite-dimensional case) to introduce a fourth function $T_0$ and consider the equations

$$\frac{dT_i}{dt} + [T_0, T_i] = [T_j, T_k],$$

with the action of the “gauge group” of $U(n)$-valued functions $u(t)$:

$$T_i \mapsto uT_i u^{-1}, T_0 \mapsto u T_0 u^{-1} - \frac{du}{dt} u^{-1}$$

which preserved solutions to (20). (That is, using the gauge group we can transform $T_0$ to 0.) The equations imply that

$$\frac{d}{dt}(T_2 + iT_3) = [T_0 + iT_1, T_2 + iT_3],$$

so $T_2 + iT_3$ moves in a single adjoint orbit in the Lie algebra of $GL(n, \mathbb{C})$. Conversely if we fix some $B$ in this complex Lie algebra, introduce a function $g(t)$ taking values in $GL(n, \mathbb{C})$ and define skew-Hermitian matrices $T_i(t)$ by

$$T_0 + iT_1 = \frac{dg}{ds} g^{-1},$$

$$T_2 + iT_3 = gBg^{-1},$$

then two of the three Nahm equations are satisfied identically. The remaining equation can be expressed in terms of the function $h(t) = g^*(t)g(t)$, taking values in the space $\mathcal{H}$ of positive definite Hermitian matrices, which we can also regard as the quotient space $GL(n, \mathbb{C})/U(n)$. This equation for $h(t)$ is a second order ODE which is the Euler-Lagrange equation for the Lagrangian

$$E(h) = \int |\frac{dh}{dt}|^2 + V_B(h) dt.$$ 

Here $|\cdot|$ denotes the standard Riemannian metric on $\mathcal{H}$. The function $V$ on $\mathcal{H}$ is

$$V_B(h) = \text{Tr}(hBh^{-1}B^*).$$

If $g$ is any element of $GL(n, \mathbb{C})$ with $g^* = h$ then

$$V_B(h) = |gBg^{-1}|^2.$$
so $V_B$ is determined by the norm of matrices in the adjoint orbit of $B$. (See [4] for details of the manipulations involved in all the above.) The result in [4], mentioned in the introduction to this article, is that for any two points $h_0, h_1 \in \mathcal{H}$ there is a unique solution $h(t)$ to the Euler-Lagrange equations for $t \in [0, 1]$ with $h(0) = h_0, h(1) = h_1$.

These constructions go over immediately to the case when $U(n)$ is replaced by any compact Lie group and $GL(n, \mathbb{C})$ by the complexified group. We want to extend them to the situation where $U(n)$ is replaced by the group $\mathcal{G}$ of Hamiltonian diffeomorphisms of a surface $\Sigma$ with a fixed area form (or more precisely, the extension of this group given by a choice of Hamiltonian). The essential difficulty is that this group does not have a complexification. However, as explained in [6], [11], [12], the space $\mathcal{H}$ of Kahler potentials behaves formally like the quotient space $\mathcal{G}^c/\mathcal{G}$ for a fictitious group $\mathcal{G}^c$. Thus the problem we have formulated in Section 2 can be viewed as an analogue of the desired kind provided that our potential function $V$ can be seen as an analogue of $V_B$ in the finite-dimensional case.

If we have a path $\phi_t$ in $\mathcal{H}$ with $\phi_0 = 0$ and a function $\beta : \Sigma \to \mathbb{C}$ we can write down a differential equation for a one-parameter family $\beta_t$ which corresponds, formally, to the adjoint action of the complexified group $\mathcal{G}^c$, with the initial condition $\beta_0 = \beta$. The equation has the shape

$$\frac{\partial \beta_t}{\partial t} = \nabla \phi \bar{\phi} \beta.$$ 

The problem is that this evolution equation will not have solutions, even for a short time, in general. But if we suspend for a moment our assumption that we are working over a compact Riemann surface and suppose that $\beta$ is a holomorphic function then there is a trivial solution $\beta_t = \beta$. So, formally, the functional $V_\beta$ on $\mathcal{H}$ is given by the $L^2$ norm of $\beta$ with respect to the measure $d\mu_\phi$:

$$\int (1 - \Delta_X \phi)|\beta|^2.$$ 

Even if this integral is divergent, the variation with respect to compactly supported variations in $\phi$ is well-defined, and this is what corresponds to the gradient of $V_B$ appearing in the equations of motion. Moreover, we can integrate by parts to get another formal representation of a functional with the same variation

$$- \int \phi \Delta_X |\beta|^2 = \int \phi |\nabla \beta|^2.$$ 

Now take the compact Riemann surface $\Sigma$ to be a 2-torus, and identify the space $\mathcal{H}$ with periodic Kahler potentials on the universal cover $\mathbb{C}$. On this cover the identity function $\beta$ is holomorphic, and we see from the above that the formal expression

$$V_\beta = \int_{\mathbb{C}} \phi.$$ 

17
is analogous to the function $V_B$ in the finite-dimensional case. Of course the integrand is periodic and so the integral will be divergent but we can return to the compact surface $\Sigma$ and consider the well-defined functional

$$V_\beta(\phi) = \int_\Sigma \phi$$

which will generate the same equations of motion. So we see that, modulo some blurring of the distinction between $\Sigma$ and its universal cover, the functional we have been considering is indeed analogous to that in the finite-dimensional case.

Using the transformation from the $\Phi$ equation to the $\theta$ equation, we obtain a relation between Nahm’s equations for the Hamiltonian diffeomorphisms of a surface and harmonic functions on $\mathbb{R}^3$. This can be seen in other ways. Most directly, we consider three one-parameter families of functions $h_i(t)$ on a surface $\Sigma$ with an area form which satisfy:

$$\frac{dh_i}{dt} = \{h_j, h_k\}, \quad (22)$$

where $\{ , \}$ is the Poisson bracket. We think of these as a one-parameter family of maps $h_t : \Sigma \to \mathbb{R}^3$, and assume for simplicity that these are embeddings, with disjoint images. Then it is a simple exercise to show that the equations (22) imply that the images $h_t(\Sigma)$ are the level sets of a harmonic function on a domain in $\mathbb{R}^3$. From another point of view, the geometric structure defined by a solution to the $\Phi$ equation is an $S^1$ invariant Kahler metric $\Omega = \omega_0 + i\partial \bar{\partial} \Phi$ on $\Sigma \times S^1 \times (0, 1)$ with volume form

$$\Omega^2 = drd\beta d\bar{r}d\bar{\beta}.$$ 

Since $drd\beta$ is an $S^1$-invariant holomorphic 2-form, what we have is an $S^1$ invariant hyperkahler structure. Then the relation with harmonic functions appears as the Gibbons-Hawking construction for hyperkahler metrics.

The development above is rather limited, since we have only been able formulate an analogue of our Nahm’s equation problem for a single function $\beta$. One can go further, and arrive at other interesting free boundary problems. Consider for example the case when the surface $\Sigma$ is the 2-sphere with the standard area form, and the orientation-reversing map $\sigma : \Sigma \to \Sigma$ given by reflection in the $x_1, x_2$ plane. Now consider maps $\beta : \Sigma \to \mathbb{C}$ with $\beta = \beta \circ \sigma$ which are diffeomorphisms on each hemisphere. Then the push-forward of the area form on the upper hemisphere defines a 2-form $\rho_\beta$ on $\mathbb{C}$ with support in a topological disc $\beta(\Sigma) \subset \mathbb{C}$. (The form $\rho_\beta$ will not usually be smooth, but will behave like $d^{-1/2}$ where $d$ is the distance to the boundary of $\beta(\Sigma)$.) Clearly the form $\rho_\beta$ determines $\beta$ up to the action of the $\sigma$-equivariant Hamiltonian diffeomorphisms of $\Sigma$. Suppose that $h$ is a $\sigma$-invariant function on $\Sigma$. We can regard this as an element of the Lie algebra of $\mathcal{G}^c$ and consider its action on $\beta$. This is given by $\Delta_C h$ where $h$ is thought of as a function on $\mathbb{C}$, vanishing outside $\beta(\Sigma)$. So a reasonable candidate for a model of the quotient of the space of maps $\beta$ by the action of $\mathcal{G}^c$ is given by the following. We consider 2-forms $\rho$ supported on
topological discs in $C$, with singularities at the boundary of the kind arising above, and impose the equivalence relation that $\rho_0 \sim \rho_1$ if there is a compactly supported harmonic function $F$ on $C$ with $\Delta F = \rho_0 - \rho_1$.

Now let $\theta(x_1, x_2, z)$ be a harmonic function on an open set $\Omega \subset \mathbb{R}^3$, with $\theta(x_1, x_2, z) = \theta(x_1, x_2, -z)$. Suppose that $\Omega$ is diffeomorphic to $S^2 \times (0, 1)$, that $\theta = 0$ on the inner boundary component $\Sigma_0$ and $\theta = 1$ on the outer boundary component $\Sigma_1$. Suppose also that the projections of $\Sigma_0, \Sigma_1$ to the $(x_1, x_2)$ plane are diffeomorphisms on each upper hemisphere, mapping to a pair of topological disc $D_0 \subset D_1$. Then the flux of $\nabla \theta$ on each boundary component pushes forward to define a pair of compactly supported 2-forms $\rho_0, \rho_1$ on $C$. These are equivalent in the sense above, since $\rho_0 - \rho_1 = \Delta_C F$ for the function

$$F(x_1, x_2) = \int z \frac{\partial \theta}{\partial z} dz,$$

where the integral is taken over the intersection of the vertical line through $(x_1, x_2, 0)$ with $\Omega$. Our hypotheses imply that $F \geq 0$, and $F$ is supported on the larger disc $D_1$.

The question we are lead to is the following

**Question 3** Suppose $D_0 \subset D_1$ are topological discs in $C$, that $\rho_i$ are 2-forms supported on $D_i$ and that there is a non-negative function $F$ on $C$, supported on $D_1$, with $\rho_0 - \rho_1 = \Delta_C F$ (where the Laplacian is defined in the distributional sense). Do $\rho_0, \rho_1$ arise from a unique harmonic function $\theta$ on a domain in $\mathbb{R}^3$, by the construction above?

(For simplicity we have not specified precisely what singularities should be allowed in the forms $\rho_i$; this specification should be a part of the question.)

Hitchin showed in [9] that Nahm’s equations form an integrable system. The root of this is the invariance of the conjugacy class given by (21), together with the family of similar statements that arise from the $SO(3)$ action on the set-up. In this vein, we can write down infinitely many conserved quantities for the solutions of our equation (11) on the Riemannian manifold $H$. Let $f_\lambda$ be an eigenfunction of the Laplacian $\Delta_X$, with eigenvalue $\lambda > 0$. Then we have

**Proposition 3** For any $\epsilon > 0$, if $\phi_t$ satisfies (11) then the quantity

$$\int_X \exp \left( \sqrt{\frac{\lambda}{\epsilon}} \phi \right) f_\lambda(1 - \Delta \phi) \, d\mu,$$

does not vary with $t$.

This becomes rather transparent in the $\theta$-formulation, using the fact that the function

$$K_\lambda(x, z) = f_\lambda(x) \exp \left( \sqrt{\frac{\lambda}{\epsilon}} z \right),$$

satisfies $\Delta_x K_\lambda = 0$.

The author is grateful to Professors Colin Atkinson, Xiuxiong Chen, Darryl Holm and John Ockendon for helpful discussions.
References

[1] C. Baiocchi and A. Capelo *Variational and Quasivariational inequalities* Wiley 1984

[2] X-X Chen *The space of Kahler metrics* Journal of Differential Geometry 56 2000 189-234

[3] J. Crank *Free and moving boundary problems* Oxford U.P. 1984

[4] S. K. Donaldson *Nahm’s equations and the classification of monopoles* Commun. Math. Phys. 1983

[5] S. K. Donaldson *Complex cobordism, Ashtekar’s equations and diffeomorphisms* In: Symplectic Geometry (Salamon ed.) Cambridge U.P. 1993 45-55

[6] S. K. Donaldson *Symmetric spaces, Kahler geometry and Hamiltonian dynamics* In: Northern California Symplectic Geometry seminar (Eliashberg et al eds.) American Math. Soc 1999 13-33

[7] S. K. Donaldson *Holomorphic discs and the complex Monge-Ampère equation* Jour. Symplectic Geometry 1 2002 171-196

[8] C. M. Elliot and J. R. Ockendon *Weak and variational methods for moving boundary problems* Pitman 1982

[9] N. J. Hitchin *On the construction of monopoles* Commun. Math. Phys. 89 1983 145-190

[10] D. Kinderlehrer and L. Nirenberg *Regularity in free boundary problems* Ann. del. Scuola Normale Sup. Pisa tome 4 No. 2 1979 373-391

[11] T. Mabuchi *Some symplectic geometry on Kahler manifolds, I* Osaka J. Math. 24 1987 227-252

[12] S. Semmes *Complex Monge-Ampère and symplectic manifolds* Amer. Jour. Math. 114 1992 495-550