PARABOLIC OMORI-YAU MAXIMUM PRINCIPLE FOR MEAN CURVATURE FLOW AND SOME APPLICATIONS

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Abstract. We derive a parabolic version of Omori-Yau maximum principle for a proper mean curvature flow when the ambient space has lower bound on $\ell$-sectional curvature. We apply this to show that the image of Gauss map is preserved under a proper mean curvature flow in euclidean spaces with uniform bounded second fundamental forms. This generalizes the result of Wang [12] for compact immersions. We also prove a Omori-Yau maximum principle for properly immersed self-shrinkers, which improves a result in [2].

1. Introduction

Let $(M, g)$ be a Riemannian manifold and let $u : M \to \mathbb{R}$ be a twice differentiable function. If $M$ is compact, $u$ is maximized at some point $x \in M$. At this point, basic advanced calculus implies

$$u(x) = \sup u, \quad \nabla^M u(x) = 0, \quad \Delta^M u(x) \leq 0.$$ 

Here $\nabla^M$ and $\Delta^M$ are respectively the gradient and Laplace operator with respect to the metric $g$. When $M$ is noncompact, a bounded function might not attain a maximum. In this situation, Omori [9] and later Yau [13] provide some noncompact versions of maximum principles. We recall the statement in [13]:

Theorem 1.1. Let $(M, g)$ be a complete noncompact Riemannian manifold with bounded below Ricci curvature. Let $u : M \to \mathbb{R}$ be a bounded above twice differentiable function. Then there is a sequence $\{x_i\}$ in $M$ such that

$$u(x_i) \to \sup u, \quad |\nabla u|(x_i) \to 0, \quad \lim_{i \to \infty} \Delta^M u(x_i) \leq 0.$$ 

Maximum principles of this form are called Omori-Yau maximum principles. The assumption on the lower bound on Ricci curvature in Theorem 1.1 has been weaken in (e.g.) [3], [10]. On the other hand, various Omori-Yau type maximum principles have been proved for other elliptic operators and on solitons in geometric flows, such as Ricci soliton [2] and self-shrinkers in mean curvature flows [4]. The Omori-Yau maximum principles are powerful tools in studying noncompact manifolds and have a lot of geometric applications. We refer the reader to the book [1] and the reference therein for more information.

In this paper, we derive the following parabolic version of Omori-Yau maximum principle for mean curvature flow.

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Theorem 1.2 (Parabolic Omori-Yau Maximum Principle). Let \( n \geq 2 \) and \( m \geq 1 \). Let \( (\overline{M}^{n+m}, \overline{g}) \) be a \( n+m \)-dimensional noncompact complete Riemannian manifold such that the \((n-1)\)-sectional curvature of \( \overline{M} \) is bounded below by \(-C\) for some positive constant \( C \). Let \( M^n \) be a \( n \)-dimensional noncompact manifold and let \( F : M^n \times [0, T] \to \overline{M} \) be a proper mean curvature flow. Let \( u : M \times [0, T] \to \mathbb{R} \) be a continuous function which satisfies

1. \( \sup_{(x,t) \in M \times [0, T]} u > \sup_{x \in M} u(\cdot, 0) \),
2. \( u \) is twice differentiable in \( M \times (0, T) \), and
3. (sublinear growth condition) There are \( B > 0 \), \( \alpha \in [0, 1) \) and some \( y_0 \in \overline{M} \) so that

\[
(1.1) \quad u(x, t) \leq B(1 + d_{\overline{M}}(y_0, F(x, t))^\alpha), \quad \forall (x, t) \in M \times [0, T].
\]

Then there is a sequence of points \((x_i, t_i) \in M \times (0, T)\) so that

\[
(1.2) \quad u(x_i, t_i) \to \sup u, \quad |\nabla M^a u(x_i, t_i)| \to 0, \quad \liminf_{i \to \infty} \left( \frac{\partial}{\partial t} - \Delta M^a \right) u(x_i, t_i) \geq 0.
\]

We remark that the above theorem makes no assumption on the curvature of the immersion \( F_t \). See section 2 for the definition of \( \ell \)-sectional curvature.

With this parabolic Omori-Yau maximum principle, we derive the following results.

In [12], the author studies the gauss map along the mean curvature flow in the euclidean space. He shows that if the image of the gauss map stays inside a geodesic submanifold in the Grassmanians, the same is also true along the flow when the initial immersion is compact. As a first application, we extend Wang’s theorem to the noncompact situation.

Theorem 1.3. Let \( F_0 : M^n \to \mathbb{R}^{n+m} \) be a proper immersion and let \( F : M^n \times [0, T] \to \mathbb{R}^{n+m} \) be a mean curvature flow of \( F_0 \) with uniformly bounded second fundamental form. Let \( \Sigma \) be a compact totally geodesic submanifold of the Grassmanians of \( n \)-planes in \( \mathbb{R}^{n+m} \).

If the image of the Gauss map \( \gamma \) satisfies \( \gamma(\cdot, 0) \subset \Sigma \), then \( \gamma(\cdot, t) \subset \Sigma \) for all \( t \in [0, T] \).

As a corollary, we have the following:

Corollary 1.1. Let \( F_0 : M^n \to \mathbb{R}^{2n} \) be a proper Lagrangian immersion and let \( F : M \times [0, T] \to \mathbb{R}^{2n} \) be a mean curvature flow with uniformly bounded second fundamental form. Then \( F_t \) is Lagrangian for all \( t \in [0, T] \).

The above result is well-known when \( M \) is compact [11], [12]. Various forms of Corollary 1.1 are known to the experts (see remark 2 below).

The second application is to derive a Omori-Yau maximum principle for the \( \mathcal{L} \)-operator of a proper self-shrinker. The \( \mathcal{L} \) operator is introduced in [5] when the authors study the entropy stability of a self-shrinker. Since then it proves to be an important operator in mean curvature flow. Using Theorem 1.2 we prove

Theorem 1.4. Let \( \tilde{F} : M^n \to \mathbb{R}^{n+m} \) be a properly immersed self-shrinker and let \( f : M^n \to \mathbb{R} \) be a twice differentiable function so that

\[
(1.3) \quad f(x) \leq C(1 + |\tilde{F}(x)|^\alpha)
\]

\( f(x) \leq C(1 + |\tilde{F}(x)|^\alpha) \)
for some $C > 0$ and $\alpha \in [0,1)$. Then there exists a sequence \( \{x_i\} \) in \( M \) so that
\[
(1.4) \quad f(x_i) \to \sup_M f, \quad |\nabla f|(x_i) \to 0, \quad \limsup_{i \to \infty} Lf(x_i) \leq 0.
\]

The above theorem is a generalization of Theorem 5 in [2] since we assume weaker conditions on \( f \).

In section 2, we prove the parabolic Omori-Yau maximum principle. In section 3 we prove Theorem 1.3 and in section 4 we prove Theorem 1.4. The author would like to thank Jingyi Chen for the discussion on Omori-Yau maximum principle and Kwok Kun Kwong for suggesting the work of Li and Wang [7].

2. PROOF OF THE PARABOLIC OMORI-YAU MAXIMUM PRINCIPLE

Let \((\overline{M}^{n+m}, g)\) be an \( n + m \) dimensional complete noncompact Riemannian manifold. Let \( F: M \times [0, T] \to \overline{M} \), where \( M \) is an \( n \)-dimensional noncompact manifold, be a family of immersions \( \{F(\cdot, t): M \to \overline{M}\} \) which satisfies the mean curvature flow equation
\[
(2.1) \quad \frac{\partial F}{\partial t}(x, t) = \vec{H}(x, t).
\]

Here \( \vec{H}(x, t) \) is the mean curvature vector given by
\[
(2.2) \quad \vec{H} = \text{tr}A
\]
and \( A(X,Y) = (\nabla_X Y)^\perp \) is the second fundamental form of the immersion \( F(\cdot, t) \).

Next we recall the definition of \( \ell \)-sectional curvature in [7]. Let \( \overline{M}^N \) be an \( N \)-dimensional Riemannian manifold. Let \( p \in \overline{M}, 1 \leq \ell \leq N - 1 \). Consider a pair \( \{w, V\} \), where \( w \in T_p\overline{M} \) and \( V \subset T_p\overline{M} \) is a \( \ell \)-dimensional subspace so that \( w \) is perpendicular to \( V \).

**Definition 2.1.** The \( \ell \)-sectional curvature of \( \{w, V\} \) is given by
\[
(2.3) \quad K_{\overline{M}}^\ell(w, V) = \sum_{i=1}^{\ell} \langle R(w, e_i)w, e_i \rangle,
\]
where \( R \) is the Riemann Curvature tensor on \( \overline{M} \) and \( \{e_1, \ldots, e_\ell\} \) is any orthonormal basis of \( V \).

We say that \( \overline{M} \) has \( \ell \)-sectional curvature bounded from below by a constant \( C \) if
\[
K_{\overline{M}}^\ell(w, V) \geq \ell C
\]
for all pairs \( \{w, V\} \) at any point \( p \in M \). In [7], the authors prove the following comparison theorem for the distance function \( r \) on manifolds with lower bound on \( \ell \)-sectional curvatures.

**Theorem 2.1.** ([Theorem 1.2 in [7]]) Assume that \( \overline{M} \) has \( \ell \)-sectional curvature bounded from below by \( -C \) for some \( C > 0 \). Let \( p \in M \) and \( r(x) = d_{\overline{M}}(x, p) \). If \( x \) is not in the cut
locus of \( p \) and \( V \subset T_{x_0}M \) is perpendicular to \( \nabla r(x) \), then

\[ \sum_{i=1}^{\ell} \nabla^2 r(e_i, e_i) \leq \ell \sqrt{C \coth(\sqrt{C} r)}, \]

where \( \{e_1, \cdots, e_\ell\} \) is any orthonormal basis of \( V \).

Now we prove Theorem 1.2. We recall that \( F \) is assumed to be proper, and \( u \) satisfies condition (1)-(3) in the statement of Theorem 1.2.

**Proof of Theorem 1.2.** Adding a constant to \( u \) if necessary, we assume

\[ \sup_{x \in M} u(x, 0) = 0. \]

By condition (1) in Theorem 1.2 we have \( u(y, s) > 0 \) for some \( (y, s) \). Note that \( s > 0 \). Let \( y_0 \in \overline{M} \) and \( r(y) = d_\rho(y, y_0) \) be the distance to \( y_0 \) in \( \overline{M} \). Let \( \rho(x, t) = r(F(x, t)) \).

Note that \( u(y, s) - \epsilon \rho(y, s)^2 > 0 \) whenever \( \epsilon \) is small. Let \( (\bar{x}_i, s_i) \) be a sequence so that \( u(\bar{x}_i, s_i) \to \sup u \in (0, \infty) \). Let \( \{\epsilon_i\} \) be a sequence in \( (0, \epsilon) \) converging to 0 which satisfies

\[ \epsilon_i \rho(\bar{x}_i, s_i)^2 \leq \frac{1}{i}, \quad i = 1, 2, \cdots. \]

Define

\[ u_i(x, t) = u(x, t) - \epsilon_i \rho(x, t)^2. \]

Note that \( u_i(y, s) > 0 \) and \( u_i(\cdot, 0) \leq 0 \). Using condition (3) in Theorem 1.2 there is \( R > 0 \) so that \( u_i(x, t) \leq 0 \) when \( F(x, t) \notin B_R(y_0) \), the closed ball in \( \overline{M} \) centered at \( y_0 \) with radius \( R \). Since \( \overline{M} \) is complete, \( B_R(y_0) \) is a compact subset. Furthermore, \( F \) is proper and thus \( u_i \) attains a maximum at some \( (x_i, t_i) \in M \times (0, T] \). From the choice of \( (\bar{x}_i, s_i) \) and \( \epsilon_i \) in (2.5),

\[ u(x_i, t_i) \geq u_i(x_i, t_i) \geq u_i(\bar{x}_i, s_i) \geq u(\bar{x}_i, s_i) - \frac{1}{i}. \]

Thus we have

\[ u(x_i, t_i) \to \sup u. \]

Now we consider the derivatives of \( u \) at \( (x_i, t_i) \). If \( F(x_i, t_i) \) is not in the cut locus of \( y_0 \), then \( \rho \) is differentiable at \( (x_i, t_i) \). Then so is \( u_i \) and we have

\[ \nabla^{M_{t_i}} u_i = 0 \quad \text{and} \quad \left( \frac{\partial}{\partial t} - \Delta^{M_{t_i}} \right) u_i \geq 0 \quad \text{at} \quad (x_i, t_i). \]

(The inequality holds since \( t_i > 0 \).) The first equality implies

\[ \nabla^{M_{t_i}} u = \epsilon_i \nabla^{M_{t_i}} \rho^2 = 2 \epsilon_i \rho(\nabla r)^\top \]

at \( (x_i, t_i) \), where \((\cdot)^\top\) denotes the projection onto \( T_{x_i}M_{t_i} \). Let \( \{e_1, \cdots, e_n\} \) be any orthonormal basis at \( T_{x_i}M_{t_i} \) with respect to \( g_{t_i} \). Then

\[ \Delta^{M_{t_i}} \rho^2 = 2 \sum_{i=1}^{n} |\nabla^{M_{t_i}} r(e_i)|^2 + 2 \rho \sum_{i=1}^{n} \nabla^2 r(e_i, e_i) + 2 \rho \bar{g}(\nabla r, \vec{H}). \]
Next we use the lower bound on \((n-1)\)-sectional curvature of \(\overline{M}\) to obtain the following lemma.

**Lemma 2.1.** There is \(C_1 = C_1(n, C) > 0\) so that

\[
\sum_{i=1}^{n} \nabla^2 r(e_i, e_i) \leq C_1 \rho. \tag{2.9}
\]

**Proof of lemma.** We consider two cases. First, if \(\gamma'\) is perpendicular to \(T_{x_i} M_t\), write

\[
\sum_{i=1}^{n} \nabla^2 r(e_i, e_i) = \frac{1}{n-1} \sum_{j=1}^{n-1} \sum_{i \neq j} \nabla^2 r(e_i, e_i).
\]

Since \(\overline{M}\) has \((n-1)\)-sectional curvature bounded from below by \(-C\), we apply Theorem 2.1 to the plane \(V\) spanned by \(\{e_1, \cdots, e_n\} \setminus \{e_i\}\) for each \(i\). Thus

\[
\sum_{i=1}^{n} \nabla^2 r(e_i, e_i) \leq \frac{n}{n-1} \sum_{j=1}^{n-1} \sqrt{C} \rho \coth(\sqrt{C} \rho)
\]

\[
= n \sqrt{C} \rho \coth(\sqrt{C} \rho). \tag{2.10}
\]

Second, if \(\gamma'\) is not perpendicular to \(T_{x_i} M_t\), since the right hand side of (2.9) is independent of the orthonormal basis chosen, we can assume that \(e_1\) is parallel to the projection of \(\gamma'\) onto \(T_{x_i} M_t\). Write

\[
e_1 = e_1^+ + a \gamma',
\]

where \(e_1^+\) lies in the orthogonal complement of \(\gamma'\) and \(a = \langle e_1, \gamma' \rangle\). By a direct calculation,

\[
\nabla^2 r(e_1, e_1) = (\nabla_{e_1} \nabla r)(e_1)
\]

\[
= e_1 \langle \gamma', e_1 \rangle - \langle \gamma', \nabla_{e_1} e_1 \rangle
\]

\[
= \langle \nabla_{e_1} \gamma', e_1 \rangle
\]

\[
= \langle \nabla_{e_1^+ + a \gamma'} \gamma', e_1^+ + a \gamma' \rangle
\]

\[
= \langle \nabla_{e_1^+} \gamma', e_1^+ \rangle + a \langle \nabla_{e_1^+} \gamma', \gamma' \rangle
\]

\[
= \nabla^2 r(e_1^+, e_1^+). \tag{2.11}
\]

We further split into two situations. If \(e_1^+ = 0\), then the above shows \(\nabla^2 r(e_1, e_1) = 0\). Using Theorem 2.1 we conclude

\[
\sum_{i=1}^{n} \nabla^2 r(e_i, e_i) = \sum_{i=2}^{n} \nabla^2 r(e_i, e_i)
\]

\[
\leq (n-1) \sqrt{C} \rho \coth(\sqrt{C} \rho). \tag{2.12}
\]
If \(e_1^\perp \neq 0\), write \(b = \|e_1^\perp\|\) and \(f_1 = b^{-1}e_1^\perp\). Then \(\{f_1, e_2, \ldots, e_n\}\) is an orthonormal basis of a \(n\)-dimensional plane in \(T_{F(x_i,t_i)}M\) orthogonal to \(\gamma\). Using (2.11),

\[
\sum_{i=1}^{n} \nabla^2 r(e_i, e_i) = \nabla^2 r(e_1^\perp, e_1^\perp) + \sum_{i=2}^{n} \nabla^2 r(e_i, e_i)
\]

\[
= b^2 \nabla^2 r(f_1, f_1) + \sum_{i=2}^{n} \nabla^2 r(e_i, e_i)
\]

\[
= b^2 \left( \nabla^2 r(f_1, f_1) + \sum_{i=2}^{n} \nabla^2 r(e_i, e_i) \right) + (1 - b^2) \sum_{i=2}^{n} \nabla^2 r(e_i, e_i).
\]

Now we apply Theorem 2.1 again (note that the first term can be dealt with as in (2.10))

\[
\sum_{i=1}^{n} \nabla^2 r(e_i, e_i) \leq b^2 n \sqrt{C} \rho \coth(\sqrt{C} \rho) + (1 - b^2) (n - 1) \sqrt{C} \rho \coth(\sqrt{C} \rho)
\]

(2.13)

Summarizing (2.10), (2.12) and (2.13), we have

\[
\sum_{i=1}^{n} \nabla^2 r(e_i, e_i) \leq n \sqrt{C} \rho \coth(\sqrt{C} \rho) \leq C_1 \rho
\]

for some \(C_1 = C_1(n, C) > 0\). Thus the lemma is proved.

Using Lemma 2.1, (2.8) and \(\frac{\partial \rho^2}{\partial t} = 2 \rho \bar{g}(\nabla r, \bar{H})\),

\[
\left( \frac{\partial}{\partial t} - \Delta^M_{t_i} \right) \rho^2 = -2 \sum_{i=1}^{n} |\nabla^{M_{t_i}}_{e_i} r|^2 - 2 \rho \sum_{i=1}^{n} \nabla^2 r(e_i, e_i)
\]

\[
\geq -2n - 2C_1 \rho
\]

(2.14)

(2.7) and (2.14) imply that at \((x_i, t_i)\) we have respectively

\[
|\nabla u| \leq 2 \epsilon_i \rho
\]

(2.15)

and

\[
\left( \frac{\partial}{\partial t} - \Delta^M_{t_i} \right) u \geq -2 \epsilon_i (n + C_1 \rho).
\]

(2.16)

Note

\[
u(x_i, t_i) - \epsilon_i \rho(x_i, t_i)^2 = u_i(x_i, t_i) \geq u_i(y, s) > 0.
\]

This implies

\[
\rho(x_i, t_i)^2 \leq u(x_i, t_i) \epsilon_i^{-1}.
\]

Using the sub-linear growth condition (3) of \(u\) and Young’s inequality, we have

\[
\rho(x_i, t_i)^2 \leq B \epsilon_i^{-1} + B \epsilon_i^{-1} \rho(x_i, t_i)^\alpha
\]

\[
\leq B \epsilon_i^{-1} + \frac{1}{2} \rho(x_i, t_i)^2 + \frac{1}{2} (B \epsilon_i^{-1})^{\frac{2}{2-\alpha}}.
\]
Thus we get
\[ \rho(x_i, t_i) \epsilon_i \leq \sqrt{2B \sqrt{\epsilon_i}} + B^{\frac{1}{2} - \alpha} \epsilon_i^\frac{1}{2} - \alpha. \]
Together with (2.15), (2.16) and that \( \epsilon_i \to 0 \),
\[ |\nabla u|(x_i, t_i) \to 0, \quad \liminf_{i \to \infty} \left( \frac{\partial}{\partial t} - \Delta_{M_{t_i}} \right) u(x_i, t_i) \geq 0. \]
This proves the theorem if \( \rho \) is smooth at \((x_i, t_i)\) for all \( i \). When \( \rho \) is not differentiable at some \((x_i, t_i)\), one applies the Calabi’s trick by considering \( r \epsilon(y) = d_{\bar{g}}(y, y_\epsilon) \) instead of \( r \), where \( y_\epsilon \) is a point close to \( y_0 \). The method is standard and thus is skipped. □

Remark 1. Condition (1) in the above theorem is used solely to exclude the case that \( u_i \) is maximized at \((x_i, 0)\) for some \( x_i \in M \). The condition can be dropped if that does not happen (see the proof of Theorem 1.4).

3. Preservation of Gauss image

In this section we assume that \( F_0 : M^n \to \mathbb{R}^{n+m} \) is a proper immersion. Let \( F : M \times [0, T] \to \mathbb{R}^{n+m} \) be a mean curvature flow starting at \( F_0 \). We further assume that the second fundamental form are uniformly bounded: there is \( C_0 > 0 \) so that
\[ \|A(x, t)\| \leq C_0, \quad \text{for all } (x, t) \in M \times [0, T]. \]

Lemma 3.1. The mapping \( F \) is proper.

Proof. Let \( B_0(r) \) be the closed ball in \( \mathbb{R}^{n+m} \) centered at the origin with radius \( r \). Then by (2.1) and (3.1) we have
\[ |F(x, t) - F(x, 0)| = \left| \int_0^t \frac{\partial F}{\partial s}(x, s)ds \right| \]
\[ = \left| \int_0^t \tilde{H}(x, s)ds \right| \]
\[ \leq \sqrt{n} \int_0^t \|A(x, s)\|ds \]
\[ \leq C_0 \sqrt{n} T. \]
Thus if \((x, t) \in F^{-1}(B_0(r))\), then \( x \) is in \( F_0^{-1}(B_0(r + C_0 \sqrt{n} T)) \). Let \((x_n, t_n) \in F^{-1}(B_0(r))\). Since \( F_0 \) is proper, a subsequence of \( \{x_n\} \) converges to \( x \in M \). Since \([0, T]\) is compact, a subsequence of \((x_n, t_n)\) converges to \((x, t)\), which must be in \( F^{-1}(B_0(r)) \) since \( F \) is continuous. As \( r > 0 \) is arbitrary, \( F \) is proper. □

In particular, the parabolic Omori-Yau maximum principle (Theorem 1.2) can be applied in this case.

Let \( G(n, m) \) be the real Grassmanians of \( n \)-planes in \( \mathbb{R}^{n+m} \) and let
\[ \gamma : M \times [0, T] \to G(n, m), \quad x \mapsto F_* T_x M \]
be the Gauss map of \( F \).

Now we prove Theorem 1.3 which is a generalization of a Theorem of Wang [12] to the noncompact situation with bounded second fundamental form.
Proof of Theorem 1.3. Let \( d : G(n, m) \to \mathbb{R} \) be the distance to \( \Sigma \). That is \( d(\ell) = \inf_{L \in \Sigma} d(L, \ell) \). Since \( \gamma(\cdot, 0) \subset \Sigma \), we have \( d \circ \gamma = 0 \) when \( t = 0 \). Using chain rule and (3.1), as \( d\gamma = A \),

\[
d(\gamma(x, t)) = d(\gamma(x, t)) - d(\gamma(x, 0)) = \int_{0}^{t} \nabla d \circ d\gamma(x, s) ds \leq tC_0.
\]

Since \( \Sigma \subset G(n, m) \) is compact, there is \( \epsilon_0 > 0 \) so that the open set

\[ V = \{ \ell \in G(n, m) : d(\ell, \Sigma) < \sqrt{\epsilon_0} \} \]

lies in a small tubular neighborhood of \( \Sigma \) and the function \( d^2 \) is smooth on this neighborhood. Let \( T' = \epsilon_0/2C_0 \). Then the image of \( f := d^2 \circ \gamma \) lies in this tubular neighborhood if \( t \in [0, T'] \) and \( f \) is a smooth function on \( M \times [0, T'] \).

The calculation in Wang [12] shows that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) f \leq C|A_i|^2 f,
\]

where \( C > 0 \) depends on \( \epsilon_0 \) and \( \Sigma \). Together with (3.1) this shows that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) f \leq C_1 f
\]

for some positive constant \( C_1 \).

Let \( g = e^{-(C_1+1)t} f \). Then \( g \) is bounded, nonnegative and \( g(\cdot, 0) \equiv 0 \). On the other hand,

\[
\left( \frac{\partial}{\partial t} - \Delta \right) g = -(C_1 + 1)g + e^{-(C_1+1)t} \left( \frac{\partial}{\partial t} - \Delta \right) f \leq -g.
\]

If \( g \) is positive at some point, Theorem 1.2 implies the existence of a sequence \((x_i, t_i)\) so that

\[
g(x_i, t_i) \to \sup g, \quad \limsup_{i \to \infty} \left( \frac{\partial}{\partial t} - \Delta \right) g(x_i, t_i) \geq 0.
\]

Take \( i \to \infty \) in (3.4) gives \( 0 \leq -\sup g \), which contradicts that \( g \) is positive somewhere. Thus \( g \) and so \( f \) is identically zero. This is the same as saying that \( \gamma(x, t) \in \Sigma \) for all \((x, t) \in [0, T']\). Note that \( T' \) depends only on \( C_0 \), so we can repeat the same argument finitely many time to conclude that \( \gamma(x, t) \in \Sigma \) for all \((x, t) \in M \times [0, T]\).

\[ \square \]

Proof of Corollary 1.1. An immersion is Lagrangian if and only if its Gauss map has image in the Lagrangian Grassmanians \( LG(n) \), which is a totally geodesic submanifold of \( G(n, n) \). The Corollary follows immediately from Theorem 1.3. \[ \square \]

Remark 2. Various forms of Corollary 1.1 are known to the experts. In [8], the author comments that the argument used in [11] can be generalized to the complete noncompact case, if one assumes the following volume growth condition:

\[
Vol(L_0 \cap B_R(0)) \leq C_0 R^n, \quad \text{for some } C_0 > 0.
\]

The above condition is needed to apply the non-compact maximum principle in [6].
4. OMORI-YAU MAXIMUM PRINCIPLE FOR SELF-SHRINKERS

In this section, we improve Theorem 5 in [2] using Theorem 1.2. The proof is more intuitive in the sense that we use essentially the fact that a self-shrinker is a self-similar solution to the mean curvature flow (possibly after reparametrization on each time slice).

First we recall some facts about self-shrinker. A self-shrinker to the mean curvature flow is an immersion $\tilde{F} : M^n \rightarrow \mathbb{R}^{n+m}$ which satisfies

\begin{equation}
\tilde{F}^\perp = -\frac{1}{2} \tilde{H}.
\end{equation}

Fix $T_0 \in (-1, 0)$. Let $\phi_t : M \rightarrow M$ be a family of diffeomorphisms on $M$ so that

\begin{equation}
\phi_{T_0} = \text{Id}_M, \quad \frac{\partial}{\partial t} (\tilde{F}(\phi_t(x))) = \frac{1}{2(-t)} \tilde{F}^\top(\phi_t(x)), \quad \forall t \in [-1, T_0].
\end{equation}

Let

\begin{equation}
F(x, t) = \sqrt{-t} \tilde{F}(\phi_t(x)), \quad (x, t) \in M \times [-1, T_0].
\end{equation}

Then $F$ satisfies the MCF equation since by (4.1),

\[
\frac{\partial F}{\partial t}(x, t) = \frac{\partial}{\partial t}(\sqrt{-t} \tilde{F}(\phi_t(x))) \\
= -\frac{1}{2\sqrt{-t}} \tilde{F}(\phi_t(x)) + \sqrt{-t} \frac{\partial}{\partial t}(\tilde{F}(\phi_t(x))) \\
= -\frac{1}{2\sqrt{-t}} \tilde{F}(\phi_t(x)) + \frac{1}{2\sqrt{-t}} \tilde{F}^\top(\phi_t(x)) \\
= \frac{1}{\sqrt{-t}} \tilde{H}_F(\phi_t(x)) \\
= \tilde{H}_F(x, t).
\]

Lastly, recall the $\mathcal{L}$ operator defined in [5]:

\begin{equation}
\mathcal{L} f = \Delta f - \frac{1}{2} \langle \nabla f, \tilde{F}^\top \rangle.
\end{equation}

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Recall $T_0 \in (-1, 0)$. Let $u : M \times [-1, T_0] \rightarrow \mathbb{R}$ be given by

\begin{equation}
u(x, t) = f(\phi_t(x)), \quad \forall (x, t) \in M \times [-1, T_0].
\end{equation}

Then

\[u(x, t) \leq C(1 + |\tilde{F}(\phi_t(x)|^\alpha) \leq C(-T_0)^{-\alpha/2}|F(x, t)|^\alpha.
\]

Thus we can apply Theorem 1.2 (The condition that $u(\cdot, 0) \equiv 0$ in Theorem 1.2 is used only to exclude the case $t_i = -1$. But since

\[u_i(x, t) = f(\phi_t(x)) - \epsilon_i |\sqrt{-t} \tilde{F}(\phi_t(x))|^2,
\]
in order that \( u_i \) is maximized at \( (x_i, t_i) \) we must have \( t_i = T_0 \). In particular \( t_i \neq -1 \).
Thus there is a sequence \( (x_i, T_0) \) so that
\[
u(x_i, T_0) \to \sup u,
\quad |\nabla^{M_{T_0}} u(x_i, T_0)| \to 0,
\quad \liminf_{i \to \infty} \left( \frac{\partial}{\partial t} - \Delta^{M_{T_0}} \right) u(x_i, T_0) \geq 0.
\]
Using \( \phi_{T_0} = \text{Id} \) and the definition of \( u \), the first condition gives
\[
(4.6) f(x_i) \to \sup f.
\]
Since \( \nabla^{M_{T_0}} = \frac{1}{\sqrt{-T_0}} \nabla^M \), the second condition gives
\[
(4.7) |\nabla^M f(x_i)| \to 0.
\]
Lastly,
\[
(4.8) \frac{\partial u}{\partial t}(x_i, T_0) = \frac{\partial f}{\partial t}(\phi_{T_0}(x)) \bigg|_{t=T_0} = \frac{1}{2(-T_0)} \langle \nabla f(x_i), \tilde{F}^\top(x_i) \rangle
\]
and
\[
\Delta^{M_{T_0}} u(x_i, T_0) = \Delta^{M_{T_0}} f(x_i) = \frac{1}{-T_0} \Delta^M f(x_i).
\]
Thus
\[
\left( \frac{\partial}{\partial t} - \Delta^{M_{T_0}} \right) u(x_i, T_0) = \frac{1}{T_0} \mathcal{L} f(x_i)
\]
and the result follows. \( \square \)

Remark 3. Note that the above theorem is stronger than Theorem 5 in [2], where they assume that \( f \) is bounded above (which corresponds to our case when \( \alpha = 0 \)).

Remark 4. Our growth condition on \( f \) is optimal: the function \( f(x) = \sqrt{|x|^2 + 1} \) defined on \( \mathbb{R}^n \) (as a self-shrinker) has linear growth, but the gradient of \( f \)
\[
\nabla f = \frac{x}{\sqrt{|x|^2 + 1}}
\]
does not tend to 0 as \( f(x) \to \sup f = \infty \).

Remark 5. In Theorem 4 of [2], the authors also derive a Omori-Yau maximum principle on a properly immersed self-shrinker for the Laplace operator. There they assume \( u : M \to \mathbb{R} \) satisfies the growth condition
\[
\lim_{x \to \infty} \frac{u(x)}{\log \left( \sqrt{|\tilde{F}(x)|^2 + 4} - 1 \right)} = 0.
\]
We remark that the condition can be weaken to
\[
\lim_{x \to \infty} \frac{u(x)}{|\tilde{F}(x)| + 1} = 0,
\]
since the Laplacian of the function \( |\tilde{F}|^2 \) satisfies better estimates: \( \Delta |\tilde{F}|^2 \leq 2n \). Thus one can argue as in p.79 in [1] to conclude.
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