Complete reducibility of subgroups of reductive algebraic groups over nonperfect fields III

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Abstract

Let $k$ be a nonperfect separably closed field. Let $G$ be a (possibly non-connected) reductive group defined over $k$. We study rationality problems for Serre’s notion of complete reducibility of subgroups of $G$. In our previous work, we constructed examples of subgroups $H$ of $G$ that are $G$-completely reducible but not $G$-completely reducible over $k$ (and vice versa). In this paper, we give a theoretical underpinning of those constructions. To illustrate our result, we present a new such example in a non-connected reductive group of type $D_4$ in characteristic 2. Then using Geometric Invariant Theory, we generalize the theoretical result above obtaining a new result on the structure of $G(k)$-(and $G$-) orbits in an arbitrary affine $G$-variety. We translate our result into the language of spherical buildings to give a new topological view. A problem on centralizers of completely reducible subgroups and a problem concerning the number of conjugacy classes are also considered.

Keywords: algebraic groups, geometric invariant theory, complete reducibility, rationality, spherical buildings

1 Introduction

Let $k$ be a field. We write $\overline{k}$ for an algebraic closure of $k$. Let $G$ be a (possibly non-connected) affine algebraic $k$-group: we regard $G$ as a $k$-defined algebraic group together with a choice of $k$-structure in the sense of Borel [9, AG. 11]. We say that $G$ is reductive if the unipotent radical $R_u(G)$ of $G$ is trivial. Throughout, $G$ is always a (possibly non-connected) reductive $k$-group. In this paper, we continue our study of rationality problems for complete reducibility of subgroups of $G$ [26, Sec. 3]. In particular if $H$ is not contained in any proper $k$-defined $R$-parabolic subgroup of $G$, $H$ is called $G$-irreducible over $k$ ($G$-ir over $k$ for short).

Definition 1.1. A subgroup $H$ of $G$ is called $G$-completely reducible over $k$ ($G$-cr over $k$ for short) if whenever $H$ is contained in a $k$-defined $R$-parabolic subgroup $P$ of $G$, then $H$ is contained in a $k$-defined $R$-Levi subgroup of $P$. In particular if $H$ is not contained in any proper $k$-defined $R$-parabolic subgroup of $G$, $H$ is called $G$-irreducible over $k$ ($G$-ir over $k$ for short).

We define $R$-parabolic subgroups and $R$-Levi subgroups in the next section (Definition 2.2). These concepts are essential to extend the notion of complete reducibility (initially defined only for subgroups of connected $G$ [26, Sec. 3]) to subgroups of non-connected $G$ [3, 4, Sec. 6]. We
defined complete reducibility for a possibly non-$k$-defined subgroup of $G$. This is because for a subgroup $H$ of $G$, some closely related important subgroups of $G$ are not necessarily $k$-defined even if $H$ is $k$-defined. For example, centralizers or normalizers of $k$-subgroups of $G$ are not necessarily $k$-defined; see [37] Thm. 1.2 and Thm. 1.7 for such examples. If $G$ is connected and $H$ is a subgroup of $G(k)$, our notion of complete reducibility agrees with the usual one of Serre.

**Definition 1.2.** Let $k'$ be an algebraic extension of $k$. We say that a subgroup $H$ of $G$ is $G$-completely reducible over $k'$ ($G$-cr over $k'$ for short) if whenever $H$ is contained in a $k'$-defined $R$-parabolic subgroup $P$ of $G$, $H$ is contained in a $k'$-defined $R$-Levi subgroup of $P$. In particular, if $H$ is not contained in any proper $k'$-defined $R$-parabolic subgroup of $G$, $H$ is $G$-irreducible over $k'$ ($G$-ir over $k'$ for short). We simply say that $H$ is $G$-cr ($G$-ir for short) if $H$ is $G$-cr over $\overline{k}$ ($G$-ir over $\overline{k}$).

So far, most studies on complete reducibility is for complete reducibility over $\overline{k}$ only; see [19], [29], [30] for example. Not much is known on complete reducibility over $k$ (especially for nonperfect $k$) except a few theoretical results and important examples in [4] Sec. 5), [1], [37], [34]. In particular, in [36] Thm. 1.10, [35] Thm. 1.8, [37] Thm. 1.2, we found several examples of $k$-subgroups of $G$ that are $G$-cr over $k$ but not $G$-cr (and vice versa). The main result of this paper is to give a theoretical underpinning for our (possibly somewhat mysterious) construction of those examples. For an algebraic extension $k'$ of $k$ and an affine group $H$, we denote the set of $k'$-points of $H$ by $H(k')$. We write $h \cdot H$ for $hHh^{-1}$ where $h \in H$.

**Theorem 1.3.** Let $k$ be a nonperfect separably closed field. Suppose that a subgroup $H$ of $G$ is $G$-cr but not $G$-cr over $k$. Let $P$ be a minimal $k$-defined $R$-parabolic subgroup of $G$ containing $H$. Then there exists a unipotent element $u \in R_u(P)(\overline{k})$ such that $u \cdot H$ is $G$-cr over $k$.

**Theorem 1.4.** Let $k$ be a nonperfect separably closed field. Suppose that a subgroup $H$ of $G$ is $G$-cr over $k$ but not $G$-cr. Let $P$ be a minimal $k$-defined $R$-parabolic subgroup of $G$ containing $H$. Then

1. $H$ is $L$-ir over $k$ for some $k$-defined $R$-Levi subgroup $L$ of $P$.

2. Moreover, there exists an element $l \in L(\overline{k})$ such that $l \cdot H$ is not $G$-cr over $k$.

To illustrate our theoretical results (Theorems 1.3, 1.4) and ideas in the proofs, we present a new example of a $k$-subgroup of $G$ that is $G$-cr over $k$ but not $G$-cr (and vice versa).

**Theorem 1.5.** Let $k$ be a nonperfect separably closed field of characteristic 2. Let $\tilde{G}$ be a simple $k$-group of type $D_4$. Let $\sigma$ be a non-trivial element in the graph automorphism of $\tilde{G}$. Let $G := \tilde{G} \rtimes \langle \sigma \rangle$. Then there exists a $k$-subgroup $H$ of $G$ that is $G$-cr over $k$ but not $G$-cr (and vice versa).

A few comments are in order. First, the non-perfectness of $k$ is (almost) essential in Theorems 1.3, 1.4 and 1.5 in view of the following [5] Thm. 1.1:

**Proposition 1.6.** Let $G$ be connected. Let $H$ be a subgroup of $G$. Then $H$ is $G$-cr over $k$ if and only if $H$ is $G$-cr over $k_s$.

So in particular if $k$ is perfect and $G$ is connected, a subgroup of $G$ is $G$-cr over $k$ if and only if it is $G$-cr. The forward direction of Proposition 1.6 holds for non-connected $G$. The reverse direction depends on the recently proved center conjecture of Tits [26], [31], [21] in spherical buildings, but this method does not work for non-connected $G$; the set of $R$-parabolic
subgroups does not form a simplicial complex in the usual sense of Tits \cite{22} as we have shown in \cite[Thm. 1.12]{21}. In the following we assume that $k$ is separably closed. So every maximal $k$-torus of $G$ splits over $k$, thus $G$ is $k$-split. This simplifies arguments in many places. For the theory of complete reducibility over arbitrary $k$, see \cite{11, 12}.

Second, note that the $k$-definability of $H$ in Theorem \ref{thm:1.5} is important. Actually it is not difficult to find a $k$-subgroup with the desired property. For our construction to work, it is essential for $H$ to be nonseparable in $G$. We write $\text{Lie}(G)$ or $\mathfrak{g}$ for the Lie algebra of $G$. Recall \cite[Def. 1.1]{7}.

**Definition 1.7.** A subgroup $H$ of $G$ is nonseparable if the dimension of $\text{Lie}(C_G(H))$ is strictly smaller than the dimension of $\mathfrak{z}_g(H)$ (where $H$ acts on $\mathfrak{g}$ via the adjoint action). In other words, the scheme-theoretic centralizer of $H$ in $G$ (in the sense of \cite[Def. A.1.9]{13}) is not smooth.

We exhibit the importance of nonseparability of $H$ in the proof of Theorem \ref{thm:1.5}. Nonseparable $k$-subgroups of $G$ are hard to find, and only a handful examples are known \cite[Sec. 7, Thm. 1.8]{21}, \cite[Thm. 1.2]{35}, \cite[Thm. 1.2]{37}. Note that if characteristic of $k$ is very good for connected $G$, every subgroup of $G$ is separable \cite[Thm. 1.2]{7}. Thus, to find a nonseparable subgroup we are forced to work in small $p$ (at least for connected $G$). See \cite{14, 15} for more on separability.

Our second main result is a generalization of Theorems \ref{thm:1.3} and \ref{thm:1.4} using the language of Geometric Invariant Theory (GIT for short) \cite{22}. Let $V$ be a (possibly non-connected) affine $k$-variety. When $G$ acts on $V$ $k$-morphically, we say that $V$ is a $G$-variety. One of the main themes of GIT is to study the structure of $G$-orbits (and $G(k)$-orbits) in $V$ \cite[Prop. 2.16, Thm. 3.1]{18}, \cite[Thm. 9.3]{37}. Recently studies on completely reducibility (over $k$) via GIT have been very fruitful; GIT gives a very short and uniform proof for many results \cite, \cite[8, 1]{1}. This makes a striking contrast to traditional representation theoretic methods (which depend on long case-by-case analyses) \cite[29, 30].

We recall the following algebro-geometric characterization for complete reducibility (over $k$) via GIT (\cite[Prop. 2.16, Thm. 3.1]{18} and \cite[Thm. 9.3]{37}). This turns problems on complete reducibility into problems on the structure of $G(k)$-(or $G$-) orbits. Let $H$ be a subgroup of $G$ such that $H = \langle h_1, \ldots, h_n \rangle$ for some $n \in \mathbb{N}$ and $h := (h_1, \ldots, h_n) \in G^n$. Suppose that $G$ (and $G(k)$) acts on $h$ via simultaneous conjugation.

**Proposition 1.8.** $H$ is $G$-cr if and only if $G \cdot h$ is Zariski closed in $G^n$. Moreover, $H$ is $G$-cr over $k$ if and only if $G(k) \cdot h$ is cocharacter closed over $k$.

The definition of a cocharacter closed orbit is given in the next section (Definition \ref{def:2.7}). Using Proposition \ref{prop:1.8} and various techniques from GIT we can sometimes generalize results on complete reducibility (over $k$) to obtain new results on GIT where $G$ (or $G(k)$) acts on an arbitrary affine $G$-variety rather than on some tuple of $G$; see \cite[8, 11] for example. We follow the same philosophy here and generalize Theorems \ref{thm:1.3} and \ref{thm:1.4}.

**Theorem 1.9.** Let $k$ be nonperfect. Suppose that there exists $v \in V$ such that $G \cdot v$ is Zariski closed but $G(k) \cdot v$ is not cocharacter closed over $k$. Let $\Delta_{v,k}$ be the set of $k$-cocharacters of $G$ destabilizing $v$ over $k$. Pick $\lambda \in \Delta_{v,k}$ such that $P_\lambda$ is minimal among $R$-parabolic subgroups $P_\mu$ for $\mu \in \Delta_{v,k}$. Then there exists a unipotent element $u \in R_u(P_\lambda)(\overline{k})$ such that $G(k) \cdot (u \cdot v)$ is cocharacter closed over $k$.

**Theorem 1.10.** Let $k$ be nonperfect. Suppose that there exists $v \in V$ such that $G(k) \cdot v$ is cocharacter closed over $k$ but $G \cdot v$ is not Zariski closed. Let $\Delta_{v,k}$ be the set of $k$-cocharacters of $G$ destabilizing $v$ over $k$. Pick $\lambda \in \Delta_{v,k}$ such that $P_\lambda$ is minimal among $R$-parabolic subgroups $P_\mu$ for $\mu \in \Delta_{v,k}$. Then
1. There exists $\xi \in \Delta_{v,k}$ such that $P_\lambda = P_\xi$ and $\xi(k)$ fixes $v$.

2. Any $k$-defined cocharacter of $L_\xi$ destabilizing $v$ over $k$ is central in $L_\xi$.

3. Moreover, there exists an element $l \in L_\xi(k)$ such that $G(k) \cdot (l \cdot v)$ is not cocharacter closed over $k$.

Roughly speaking, we say that $v \in V$ is destabilized over $k$ by a $k$-cocharacter $\lambda$ of $G$ if $v \in V$ is taken outside of $G(k) \cdot v$ by taking a limit of $v$ along $\lambda$ in the sense of GIT 
[13], [22]; see Definition [28] for the precise definition. Note that if $k$ is perfect, Theorems [1.9] and [1.10] have no content: in that case $G(k) \cdot v$ is cocharacter closed if and only if $G \cdot v$ is Zariski closed [1 Cor. 7.2, Prop. 7.4].

To complement the paper we also investigate the structure of centralizers of completely reducible subgroups of $G$. In particular we ask [34] Open Problem 1.4:

**Open Problem 1.11.** Suppose that a $k$-subgroup $H$ of $G$ is $G$-cr over $k$. Is $C_G(H)$ $G$-cr over $k$?

We have some partial answer [34] Thm. 1.5]:

**Proposition 1.12.** Let $G$ be connected. Suppose that a $k$-subgroup $H$ of $G$ is $G$-cr over $k$. If $C_G(H)$ is reductive, then it is $G$-cr over $k$.

We need the connectedness assumption in Proposition [1.12] since it depends on the center conjecture of Tits. If $k = \overline{k}$ (or more generally if $k$ is perfect), the answer to Open Problem [1.11] is “yes” by [1] Cor. 3.17] (and Proposition [1.9]). A trouble arises for nonperfect $k$ since $C_G(H)$ is not necessarily reductive even if $H$ is $G$-cr over $k$ [37] Rem. 3.11]. This does not happen if $k = \overline{k}$ by [1 Prop. 3.12] that depends on a deep result of Richardson [24] Thm. A].

In general, if $k$ is nonperfect, even if a $k$-subgroup of $G$ is $G$-cr over $k$, it is not necessarily reductive [37] Prop. 1.10]. This pathology happens because the classical construction of Borel-Tits [11 Prop. 3.1] fails over nonperfect $k$; see [37 Sec. 3.2]. This does not happen if $k = \overline{k}$; a $G$-cr subgroup is always reductive [26] Prop. 4.1].

Here is our third main result in this paper. Let $G$ be connected. Fix a maximal $k$-torus $T$ of $G$. We write $w_{0,G}$ for an automorphism of $G$ which normalizes $T$ and induces $-1$ on $\Psi^+(G)$ (the set of positive roots of $G$). It is known that $w_{0,G} = w_{0,G}$ for $G$ of not type $A_2$, $D_{2n+1}$, or $E_6$, and $w_{0,G} = w_{0,G}\sigma_G$ for $G$ of type $A_n$, $D_{2n+1}$, or $E_6$ where $w_{0,G}$ is the longest element of the Weyl group of $G$ and $\sigma_G$ is a suitable graph automorphism of $G$ (cf. [19] Proof of Thm. 4.1]).

**Theorem 1.13.** Let $G$ be connected. Suppose that a semisimple $k$-subgroup $H$ of $G$ is $G$-cr over $k$. Let $P$ be a minimal $k$-parabolic subgroup containing $HC_G(H)$, and $L$ be a $k$-Levi subgroup of $P$. If the automorphism $w_{0,L}$ of $L$ extends to an automorphism of $G$ (in particular if $L$ is not of type $A_n$, $D_{2n+1}$, or $E_6$), then $C_G(H)$ is $G$-cr over $k$.

Here is the structure of the paper. In Section 2, we set out the notation and show some preliminary results. Then in Section 3, we prove our first main result (Theorems [1.3] and [1.4]). In Section 4, we present the $D_4$ example (Theorem [1.5]). In Section 5, we generalize Theorems [1.3] and [1.4] and prove our second main result (Theorems [1.9] and [1.10]). In Section 6, we translate Theorems [1.3] and [1.4] into the language of spherical buildings, and prove Theorems [6.3] and [6.5]. This gives a new topological perspective for the rationality problems for complete reducibility and GIT. Then in Section 7, we attack Open Problem [1.11] and prove Theorems [1.13] in a purely combinatorial way. In Section 8, we consider a problem on the number of conjugacy classes and prove Theorem [8.3]. We note that nonseparability comes into play in a crucial way in the proof of Theorem [8.1].
2 Preliminaries

Throughout, we denote by $k$ a separably closed field. Our references for algebraic groups are [9, 10, 13, 17, and 23].

Let $H$ be a (possibly non-connected) affine algebraic group. We write $H^o$ for the identity component of $H$. It is clear that if $H$ is $k$-defined, $H^o$ is $k$-defined. We write $[H,H]$ for the derived group of $H$. A reductive group $G$ is called simple as an algebraic group if $G$ is connected and all proper normal subgroups of $G$ are finite. We write $X_k(G)$ and $Y_k(G)$ ($X(G)$ and $Y(G)$) for the set of $k$-characters and $k$-cocharacters ($\bar{k}$-characters and $\bar{k}$-cocharacters) of $G$ respectively. For $\bar{k}$-characters and $\bar{k}$-cocharacters of $G$ we simply say characters and cocharacters of $G$.

Fix a maximal $k$-torus $T$ of $G$ (such a $T$ exists by [3 Cor. 18.8]). Then $T$ splits over $k$ since $k$ is separably closed. Let $\Psi(G,T)$ denote the set of roots of $G$ with respect to $T$. We sometimes write $\Psi(G)$ for $\Psi(G,T)$. Let $\zeta \in \Psi(G)$. We write $U_{\zeta}$ for the corresponding root subgroup of $G$. We define $G_{\zeta} := \langle U_{\zeta}, U_{-\zeta} \rangle$. Let $\zeta, \xi \in \Psi(G)$. Let $\xi^\vee$ be the coroot corresponding to $\xi$. Then $\zeta \circ \xi^\vee : \bar{k}^* \to \bar{k}^*$ is a $k$-homomorphism such that $(\zeta \circ \xi^\vee)(a) = a^n$ for some $n \in \mathbb{Z}$. Let $s_\xi$ denote the reflection corresponding to $\xi$ in the Weyl group of $G$. Each $s_\xi$ acts on the set of roots $\Psi(G)$ by the following formula [23, Lem. 7.1.8]: $s_\xi \cdot \zeta = \zeta - (\zeta, \xi^\vee) \xi$. By [12, Prop. 6.4.2, Lem. 7.2.1] we can choose $k$-homomorphisms $\epsilon_\xi : \bar{k} \to U_{\zeta}$ so that $n_{\xi\xi^\vee}(a)n_{\xi}^{-1} = \epsilon_{\epsilon_{\xi^\vee}(a)}$ where $n_{\xi} = \epsilon_{\xi(1)}\epsilon_{-\xi(-1)}\epsilon(1)$.

We recall the notions of $R$-parabolic subgroups and $R$-Levi subgroups from [23 Sec. 2.1–2.3]. These notions are essential to define $G$-complete reducibility for subgroups of non-connected $G$ and also to translate results on complete reducibility into the language of GIT; see [3] and [4, Sec. 6].

Definition 2.1. Let $X$ be an affine $k$-variety. Let $\phi : \bar{k}^* \to X$ be a $k$-morphism of affine $k$-varieties. We say that $\lim_{a \to 0} \phi(a)$ exists if there exists a $k$-morphism $\hat{\phi} : \bar{k} \to X$ (necessarily unique) whose restriction to $\bar{k}^*$ is $\phi$. If this limit exists, we set $\lim_{a \to 0} \phi(a) = \hat{\phi}(0)$.

Definition 2.2. Let $\lambda \in Y(G)$. Define $P_\lambda := \{g \in G \mid \lim_{a \to 0} \lambda(a)g\lambda(a)^{-1}$ exists$\}$, $L_\lambda := \{g \in G \mid \lim_{a \to 0} \lambda(a)g\lambda(a)^{-1} = 1\}$, $R_\lambda(P_\lambda) := \{g \in G \mid \lim_{a \to 0} \lambda(a)g\lambda(a)^{-1} = 0\}$.

We call $P_\lambda$ an $R$-parabolic subgroup of $G$, $L_\lambda$ an $R$-Levi subgroup of $P_\lambda$. Note that $R_\lambda(P_\lambda)$ the unipotent radical of $P_\lambda$. If $\lambda$ is $k$-defined, $P_\lambda$, $L_\lambda$, and $R_\lambda(P_\lambda)$ are $k$-defined [23 Sec. 2.1-2.3]. Any $k$-defined parabolic subgroups and $k$-defined Levi subgroups of $G$ arise in this way since $k$ is separably closed. It is well known that $L_\lambda = C_G(\lambda(\bar{k}))$. Note that $k$-defined $R$-Levi subgroups of a $k$-defined $R$-parabolic subgroup $P$ of $G$ are $R_a(P)(k)$-conjugate [3 Lem. 2.5(iii)]. Let $M$ be a reductive $k$-subgroup of $G$. Then, there is a natural inclusion $Y_k(M) \subseteq Y_k(G)$ of $k$-cocharacter groups. Let $\lambda \in Y_k(M)$. We write $P_\lambda(G)$ or just $P_\lambda$ for the $R$-parabolic subgroup of $G$ corresponding to $\lambda$, and $P_\lambda(M)$ for the $R$-parabolic subgroup of $M$ corresponding to $\lambda$. It is clear that $P_\lambda(M) = P_\lambda(G) \cap M$ and $R_\lambda(P_\lambda(M)) = R_\lambda(P_\lambda(G)) \cap M$. If $G$ is connected, $R$-parabolic subgroups and $R$-Levi subgroups are parabolic subgroups and Levi subgroups in the usual sense [23 Prop. 8.4.5].

The next result is used repeatedly to reduce problems on $G$-complete reducibility to those on $L$-complete reducibility where $L$ is an $R$-Levi subgroup of $G$.

Proposition 2.3. Suppose that a subgroup $H$ of $G$ is contained in a $k$-defined $R$-Levi subgroup of $G$. Then $H$ is $G$-cr over $k$ if and only if it is $L$-cr over $k$.

Proof. This follows from Proposition [13 and 11 Thm. 5.4(ii)].
The next result shows how complete reducibility behaves under central isogenies.

**Definition 2.4.** Let $G_1$ and $G_2$ be reductive $k$-groups. A $k$-isogeny $f : G_1 \to G_2$ is central if $\ker df_1$ is central in $g_1$, where $\ker df_1$ is the differential of $f$ at the identity of $G_1$ and $g_1$ is the Lie algebra of $G_1$.

**Proposition 2.5.** Let $G_1$ and $G_2$ be reductive $k$-groups. Let $H_1$ and $H_2$ be subgroups of $G_1$ and $G_2$ be subgroups of $G_1$ and $G_2$ respectively. Let $f : G_1 \to G_2$ be a central $k$-isogeny.

1. Suppose that $H_1 < G_1^0$ and $f(H_1) < G_2^0$ (in particular if $H_1$ is connected). If $H_1$ is $G_1$-cr over $k$, then $f(H_1)$ is $G_2$-cr over $k$.
2. Suppose that $H_2 < G_2^0$ and $f^{-1}(H_2) < H_1^0$. If $H_2$ is $G_2$-cr over $k$, then $f^{-1}(H_2)$ is $G_1$-cr over $k$.

**Proof.** Proposition [16] and [1] Cor. 5.3 show that a subgroup $H$ of a reductive $G$ is $G$-cr over $k$ if and only if it is $G^r$-cr over $k$. Now the result follows from the connected case [37], Prop. 1.12.

**Remark 2.6.** In Proposition 2.5 if we know that a $k$-defined $R$-parabolic subgroup of $G_1$ always arises as the inverse image of a $k$-defined $R$-parabolic subgroup of $G_2$, then a similar argument as in the proof of [37], Prop. 1.12] goes through and we can omit the assumptions “$H_1 < G_1^0$” and $f(H_1) < G_2^0$” in Part 1 and “$H_2 < G_2^0$ and $f^{-1}(H_2) < H_1^0$” in Part 2. We do not know this is the case or not.

Now we recall some terminology from GIT [1] Def. 1.1, Sec. 2.4]. Let $V$ be a $G$-variety. Let $v \in V$.

**Definition 2.7.** We say that $G(k) \cdot v$ is cocharacter closed over $k$ if for every $\lambda \in Y_k(G)$ such that $v' := \lim_{a \to 0} \lambda(a) \cdot v$ exists, $v'$ is $G(k)$-conjugate to $v$. Moreover, we say that $G \cdot v$ is cocharacter closed if for every cocharacter $\lambda$ of $G$ such that $v' := \lim_{a \to 0} \lambda(a) \cdot v$ exists, $v'$ is $G$-conjugate to $v$.

Note that by the Hilbert-Mumford theorem [18], $G \cdot v$ is cocharacter closed if and only if it is Zariski closed.

**Definition 2.8.** Let $\lambda \in Y_k(G)$. We say that $\lambda$ destabilizes $v$ over $k$ if $\lim_{a \to 0} \lambda(a) \cdot v$ exists. Moreover if $v' := \lim_{a \to 0} \lambda(a) \cdot v$ exists and $v'$ is not $G(k)$-conjugate to $v$, we say that $\lambda$ properly destabilizes $v$ over $k$. Similarly, for $\lambda \in Y(G)$, if $\lim_{a \to 0} \lambda(a) \cdot v$ exists, we say that $\lambda$ destabilizes $v$. If $v' := \lim_{a \to 0} \lambda(a) \cdot v$ exists for $\lambda \in Y(G)$ and $v'$ is not $G$-conjugate to $v$, we say that $\lambda$ properly destabilizes $v$.

We use the following very useful results from GIT [8] Thm. 3.3] and [1] Cor. 5.1.

**Proposition 2.9.** Let $k$ be perfect. Suppose that $v' := \lim_{a \to 0} \lambda(a) \cdot v$ exists for $\lambda \in G$ and $v'$ is $G(k)$-conjugate to $v$. Then $v'$ is $R_a(P_{\lambda})(k)$-conjugate to $v$.

For nonperfect $k$, we do not know whether Proposition 2.9 still holds [1] Question 7.8]. It is known that if the centralizer of $v$ in $G$ is separable, it holds for nonperfect $k$ [1] Thm. 7.1].

**Proposition 2.10.** Suppose that $v' := \lim_{a \to 0} \lambda(a) \cdot v$ exists for $\lambda \in G$ and $v'$ is $G(k)$-conjugate to $v$. If $G(k) \cdot v$ is cocharacter closed over $k$, then $v'$ is $R_a(P_{\lambda})(k)$-conjugate to $v$. 


3 $G$-cr over $k$ vs $G$-cr

We prove theorems 1.3 and 1.4. Our proof works for both connected and non-connected $G$ in a uniform way.

Proof of Theorem 1.3. Since $H$ is not $G$-cr over $k$, there exists a proper $k$-defined $R$-parabolic subgroup $P$ of $G$ containing $H$. Let $P = P_\lambda$ be a minimal such $k$-defined $R$-parabolic subgroup where $\lambda \in Y_k(G)$. Since $H$ is $G$-cr and $R$-Levi subgroups of $P_\lambda$ are $R_u(P_\lambda)(\overline{k})$-conjugate by Cor. 6.7, there exists $u \in R_u(P_\lambda)(\overline{k})$ such that $H$ is contained in $u^{-1} \cdot L_\lambda$. Then $u \cdot H$ is contained in $L_\lambda$. Suppose that $u \cdot H$ is not $G$-cr over $k$. Then it is not $L_\lambda$-cr over $k$ by Proposition 2.3. So there exists a proper $k$-defined $R$-parabolic subgroup $P_\mu$ of $L_\lambda$ containing $u \cdot H$. Thus $u \cdot H$ is contained in a $k$-defined $R$-parabolic subgroup $Q := P_\mu \ltimes R_u(P_\lambda)$ of $G$. Then $H$ is contained in $u^{-1} \cdot Q$. Note that $u \in R_u(P_\lambda) < Q$. Thus $u^{-1} \cdot Q = Q$ and we have $H < Q$. It is clear that $Q$ is strictly contained in $P_\lambda$. This contradicts the minimality of $P_\lambda$. So we conclude that $u \cdot H$ is $G$-cr over $k$. ⊓⊔

Proof of Theorem 1.4. We start with Part 1. Let $P_\lambda$ be a minimal $k$-defined $R$-parabolic subgroup of $G$ containing $H$. Since $H$ is $G$-cr over $k$, there exists a $k$-defined $R$-Levi subgroup $L$ of $P_\lambda$ containing $H$. Since $k$-defined $R$-Levi subgroups of $P_\lambda$ are $R_u(P_\lambda)(k)$-conjugate, there exists $u \in R_u(P_\lambda)(k)$ such that $L = u \cdot L_\lambda$. Let $L_\mu := u \cdot L_\lambda$. Suppose that $H$ is not $L_\mu$-ir over $k$. So there exists a $k$-defined proper $R$-parabolic subgroup $P_\lambda$ of $L_\mu$ containing $H$. Then we have $H < P_\lambda < P_\mu \ltimes R_u(P_\lambda)$. Since $P_\mu \ltimes R_u(P_\lambda)$ is a $k$-defined $R$-parabolic subgroup of $G$ strictly contained in $P_\lambda$, this contradicts the minimality of $P_\lambda$.

For part 2, let $L$ be a minimal $k$-defined $R$-Levi subgroup containing $H$. Since $H$ is not $L$-cr, there exists a proper $R$-parabolic subgroup of $L$ containing $H$. Let $P_\lambda$ be a minimal such $R$-parabolic subgroup of $L$ containing $H$. Since an $R$-parabolic subgroup of $L$ is $L(\overline{k})$-conjugate to a $k$-defined $R$-parabolic subgroup of $L$, there exists $l \in L(\overline{k})$ such that $l \cdot P_\lambda$ is a $k$-defined $R$-parabolic subgroup of $L$. Then $l \cdot H < l \cdot P_\lambda$. Suppose that $l \cdot H$ is $G$-cr over $k$. Then $l \cdot H$ is $L$-cr over $k$ by Proposition 2.3, so there exist a $k$-defined $R$-Levi subgroup $M$ of $l \cdot P_\lambda$ containing $l \cdot H$. Note that $l \cdot H$ is not $M$-cr since $H$ is not $G$-cr. Then there exists a proper $R$-parabolic subgroup $P_M$ of $M$ containing $l \cdot H$. Thus $P_\mu := P_M \ltimes R_u(l \cdot P_\lambda)$ is an $R$-parabolic subgroup of $L$ containing $l \cdot H$. Then $l^{-1} \cdot P_\mu$ is an $R$-parabolic subgroup of $L$ containing $H$. It is clear that $l^{-1} \cdot P_\mu$ is a proper subgroup of $P_\lambda$. This contradicts the minimality of $P_\lambda$. Thus $l \cdot H$ is not $G$-cr over $k$. ⊓⊔

Remark 3.1. Although Theorems 1.3 and 1.4 (and ideas in the proofs) explain necessary conditions to have examples of a subgroup $H$ of $G$ that is $G$-cr over $k$ but not $G$-cr (or vice versa), it is still a difficult problem to find concrete such examples with a $k$-defined $H$. In the next section, we use the converse of Theorems 1.3 and 1.4, we start with some subgroup of $G$ and conjugate it by $u$ (or $l$) as in the proof of Theorems 1.3 and 1.4 to obtain a subgroup with the desired property. For our construction to work, $u$ (or $l$) needs to be chosen very carefully and the choice is closely related to the nonseparability of $H$. We show all details in the next section. The same idea was used in [7, 37, 35, and 36].

4 The $D_4$ example

In this section we prove Theorem 1.5. We use the triality of $D_4$ in an essential way.

Let $\tilde{G}$ be a simple algebraic group of type $D_4$ defined over a nonperfect field of characteristic 2. Fix a maximal $k$-torus of $\tilde{G}$ and a $k$-defined Borel subgroup of $\tilde{G}$. let $\Psi(\tilde{G}) = \Psi(\tilde{G}, T)$ be
the set of roots corresponding to $T$, and $\Psi(\tilde{G})^+ = \Psi(\tilde{G}, B, T)$ be the set of positive roots of $\tilde{G}$ corresponding to $T$ and $B$. The following Dynkin diagram defines the set of simple roots of $\tilde{G}$.

Let $G := \tilde{G} \rtimes \langle \sigma \rangle$ where $\sigma$ is the non-trivial element of the graph automorphism group of $\tilde{G}$.

(normalizing $T$ and $B$) as the diagram defines; we have $\sigma \cdot \alpha = \gamma$, $\sigma \cdot \gamma = \delta$, $\sigma \cdot \delta = \alpha$, and $\beta$ is fixed by $\sigma$. We label $\Psi(\tilde{G})^+$ in the following. The corresponding negative roots are defined accordingly. Note that Roots 1, 2, 3, 4 correspond to $\alpha$, $\gamma$, $\delta$, $\beta$ respectively.

\[
\begin{array}{cccccccccccc}
1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 3 & 0 & 0 & 4 & 0 & 1 & 0 & 5 & 1 & 0 & 0 & 6 & 1 & 0 \\
7 & 0 & 1 & 1 & 8 & 1 & 1 & 0 & 9 & 1 & 1 & 1 & 10 & 0 & 1 & 1 & 11 & 1 & 1 & 1 & 12 & 1 & 2 & 1
\end{array}
\]

Define $\lambda := (\alpha + 2\beta + \gamma + \delta)^\vee = \alpha^\vee + 2\beta^\vee + \gamma^\vee + \delta^\vee$. Then

\[
P_{\lambda} = \langle T, \sigma, U_\zeta \mid \zeta \in \Psi(\tilde{G})^+ \cup \{-1, -2, -3\} \rangle,
\]

\[
L_{\lambda} = \langle T, \sigma, U_\zeta \mid \zeta \in \{\pm 1, \pm 2, \pm 3\} \rangle,
\]

\[
R_{n}(P_{\lambda}) = \langle U_\zeta \mid \zeta \in \Psi(\tilde{G})^+ \}\{1, 2, 3\} \rangle.
\]

Let $a \in k\backslash k^2$. Let $v(\sqrt{a}) := \epsilon_6(\sqrt{a})\epsilon_9(\sqrt{a}) \in R_{n}(P_{\lambda})(k)$. Define

\[
H := v(\sqrt{a}) \cdot ((n_\alpha \sigma), (\alpha + \gamma)^\vee(k^\vee)).
\]

Here is our first main result in this section.

**Proposition 4.1.** $H$ is $k$-defined. Moreover, $H$ is $G$-cr but not $G$-cr over $k$.

**Proof.** First, we have $(n_\alpha \sigma) \cdot (\beta + \delta) = (n_\alpha \sigma) \cdot 6 = 9$, $(n_\alpha \sigma) \cdot 9 = 6$. Using this and the commutation relations [17] Lem. 32.5 and Prop. 33.3], we obtain

\[
v(\sqrt{a}) \cdot (n_\alpha \sigma) = (n_\alpha \sigma)\epsilon_{12}(a).
\]

An easy computation shows that $v(\sqrt{a})$ commutes with $(\alpha + \gamma)^\vee(k^\vee)$. Now it is clear that $H$ is $k$-defined.

Now we show that $H$ is $G$-cr. It is sufficient to show that $H' := v(\sqrt{a})^{-1} \cdot H = \langle n_\alpha \sigma, (\alpha + \gamma)^\vee(k^\vee) \rangle$ is $G$-conjugate to $H$. Since $H'$ is contained in $L_{\lambda}$, by Proposition 2.3 it is enough to show that $H'$ is $L_{\lambda}$-cr. We actually show that $H'$ is $L_{\lambda}$-ir. Note that $L_{\lambda} = A_1 \times A_1 \times A_1 = L_\alpha \times L_\gamma \times L_\delta$. We have

\[
(n_\alpha \sigma) \cdot (\alpha + \gamma)^\vee(k^\vee) = (\gamma + \delta)^\vee(k^\vee), (n_\alpha \sigma)^3 = n_\alpha n_\gamma n_\delta.
\]

Thus $H'$ contains $(\alpha + \gamma)^\vee(k^\vee)$, $(\gamma + \delta)^\vee(k^\vee)$, and $n_\alpha n_\gamma n_\delta$. Now it is clear that $H'$ is $L_{\lambda}$-ir.

Next, we show that $H$ is not $G$-cr over $k$. Suppose the contrary. Clearly $H$ is contained in a $k$-defined $R$-parabolic subgroup $P_{\lambda}$. Then there exists a $k$-defined $R$-Levi subgroup of $P_{\lambda}$.
containing $H$. Then by [8 Lem. 2.5(iii)] there exists $u \in R_u(P_\lambda)(k)$ such that $H$ is contained in $u \cdot L_\lambda$. Thus $n_\alpha \sigma \varepsilon_{12}(a) < u \cdot L_\lambda$. So $u^{-1} \cdot (n_\alpha \sigma \varepsilon_{12}(a)) < L_\lambda$. By [28 Prop. 8.2.1], we set

$$u := \prod_{\zeta \in \Psi(R_u(P_\lambda))} \varepsilon_\zeta(x_\zeta).$$

We compute how $n_\alpha \sigma$ acts $\Psi(R_u(P_\lambda))$. Using the labelling of the positive roots above, we have $\Psi(R_u(P_\lambda)) = \{4, \cdots, 12\}$. We compute how $n_\alpha \sigma$ acts on $\Psi(R_u(P_\lambda))$:

$$n_\alpha \sigma = (4 \ 5 \ 8 \ 11 \ 10 \ 7)(6 \ 9)(12). \quad (4.1)$$

Using this and the commutation relations,

$$u^{-1} \cdot (n_\alpha \sigma \varepsilon_{12}(a)) = n_\alpha \sigma \varepsilon_7(x_4 + x_7) \varepsilon_{10}(x_7 + x_{10}) \varepsilon_9(x_6 + x_9) \varepsilon_{11}(x_{10} + x_{11})$$

$$\varepsilon_6(x_6 + x_9) \varepsilon_8(x_8 + x_{11}) \varepsilon_4(x_4 + x_5) \varepsilon_5(x_5 + x_8)$$

$$\varepsilon_{12}(x_5 x_{10} + x_5 x_{11} + x_7 x_8 + x_7 x_{11} + x_8 x_{10} + x_9^2 + a).$$

Thus if $u^{-1} \cdot (n_\alpha \sigma \varepsilon_{12+2\beta+\gamma+\delta}(a)) < L_\lambda$ we must have

$$x_4 = x_5 = x_7 = x_8 = x_{10} = x_{11}, \quad x_6 = x_9,$$

$$x_5 x_{10} + x_5 x_{11} + x_7 x_8 + x_7 x_{11} + x_8 x_{10} + x_9^2 + a = 0.$$

Set $x_4 = y$. Then we have $y^2 + x_9^2 + a = 0$. Thus $(y + x_9)^2 = a$. This is impossible since $y, x_9 \in k$ and $a \notin k^2$. We are done. \qed

**Remark 4.2.** From the computations above we see that the curve $C(x) := \{\varepsilon_6(x)\varepsilon_9(x) \mid x \in \mathcal{E}\}$ is not contained in $C_G(H)$, but the corresponding element in $\text{Lie}(G)$, that is, $\varepsilon_6 + \varepsilon_9$ is contained in $\mathfrak{c}_G(H)$. Then the argument in the proof of [30 Prop. 3.3] shows that $\text{Dim}(C_G(H))$ is strictly smaller than $\text{Dim}(\mathfrak{c}_G(H))$. So $H$ is non-separable in $G$.

Now we move on to the second main result in this section. We use the same $G$, $\lambda$, and $\alpha$ as above. We also use the same labelling of the roots of $G$. Let $v(\sqrt{a}) := \varepsilon_{-6}(\sqrt{a}) \varepsilon_{-9}(\sqrt{a})$. Let

$$K := v(\sqrt{a}) \cdot (n_\alpha \sigma, (\alpha + \gamma)^\vee(\mathcal{K})) = (n_\alpha \sigma \varepsilon_{-12}(a), (\alpha + \gamma)^\vee(\mathcal{K})).$$

Define

$$H := (K, \varepsilon_{11}(1)).$$

**Proposition 4.3.** $H$ is $k$-defined. Moreover, $H$ is $G$-ir over $k$ but not $G$-cr.

**Proof.** $H$ is clearly $k$-defined. First, we show that $H$ is $G$-ir over $k$. Note that

$$v(\sqrt{a})^{-1} \cdot H = (n_\alpha \sigma, (\alpha + \gamma)^\vee(\mathcal{K})), \varepsilon_{11}(1) \varepsilon_2(\sqrt{a}).$$

Thus we see that $v(\sqrt{a})^{-1} \cdot H$ is contained in $P_\lambda$. So $H$ is contained in $v(\sqrt{a}) \cdot P_\lambda$.

**Lemma 4.4.** $v(\sqrt{a}) \cdot P_\lambda$ is the unique proper $R$-parabolic subgroup of $G$ containing $H$.

**Proof.** Suppose that $P_\mu$ is a proper $R$-parabolic subgroup containing $v(\sqrt{a})^{-1} \cdot H$. In the proof of Proposition 4.1 we have shown that $M := (n_\alpha \sigma, (\alpha + \gamma)^\vee(\mathcal{K}))$ is $G$-cr. Then there exists a $R$-Levi subgroup $L$ of $P_\mu$ containing $M$ since $M$ is contained in $P_\mu$. Since $R$-Levi subgroups of $P_\mu$ are $R_u(P_\mu)$-conjugate by [8 Lem. 2.5(iii)], without loss, we set $L := L_\mu$. Then $M < L_\mu = C_G(\mu(\mathcal{K}))$, so $\mu(\mathcal{K})$ centralizes $M$. Recall that by [28 Thm. 13.4.2], $C_{R_u(P_\lambda)}(M)^\circ \times C_{L_\lambda}(M)^\circ \times C_{R_\lambda}(P_\lambda)^\circ$ is an open set of $C_G(M)^\circ$ where $P_\lambda^\circ$ is the opposite of $P_\lambda$ containing $L_\lambda$. [9]
Lemma 4.5. $C_G(M)^o = G_{12}$.

Proof. First of all, from Equation 4.11 we see that $G_{12}$ is contained in $C_G(n_o \sigma)$. Since $(\alpha + 2\beta + \gamma + \delta, (\alpha + \gamma)^\vee) = 0$, $G_{21}$ is also contained in $C_G((\alpha + \gamma)^\vee(k^r))$. So $G_{12}$ is contained in $C_G(M)$. Set $u := \prod_{i \in \Psi(R_u(P_\chi))} \epsilon_i(x_i)$ for some $x_i \in k$. Using Equation 4.11 and the commutation relations, we obtain

$$(n_o \sigma) \cdot u = \epsilon_4(x_7)\epsilon_5(x_4)\epsilon_6(x_9)\epsilon_7(x_10)\epsilon_8(x_5)\epsilon_9(x_6)\epsilon_{10}(x_{11})\epsilon_{12}(x_5 x_{10} + x_6 x_9 + x_{12}).$$

So, if $u \in C_{R_u(P_\chi)}(n_o \sigma)$ we must have $x_4 = x_5 = x_7 = x_8 = x_{10} = x_{11}, x_6 = x_9$. But $((\alpha + \gamma)^\vee, \alpha + \beta) = 2$, so $x_5 = 0$ for $u \in C_{R_u(P_\chi)}(M)$. Then

$$(n_o \sigma) \cdot u = \epsilon_6(x_6)\epsilon_9(x_6)\epsilon_{12}(x_6^2 + x_{12}).$$

So we must have $x_6^2 = 0$ if $u \in C_{R_u(P_\chi)}(M)$. Thus we conclude that $C_{R_u(P_\chi)}(M) = U_{12}$. Similarly, we can show that $C_{R_u(P_\chi)}(P_\lambda) = U_{-12}$. Now we show that $C_L(M) < G_{12}$. In the proof of Proposition 4.1 we have shown that $(\gamma + \delta)^\vee(k^r)$ is contained in $M$. So $C_{L_\chi}(M)$ is contained in $C_{L_\chi}((\alpha + \gamma)^\vee(k^r), (\gamma + \delta)^\vee(k^r))$. A direct computation shows that $C_{L_\chi}((\alpha + \gamma)^\vee(k^r), (\gamma + \delta)^\vee(k^r)) = T$ and $C_T(n_o \sigma) = (\alpha + 2\beta + \gamma + \delta)^\vee(k^r) < G_{12}$. We are done.

Since $\lambda(k^r)$ centralizes $M$, Lemma 4.5 yields $\mu(k^r) < G_{12}$. Then we can set $\mu := g \cdot (\alpha + 2\beta + \gamma + \delta)^\vee$ for some $g \in G_{12}$. By the Bruhat decomposition, $g$ is of one of the following forms:

1. $g = (\alpha + 2\beta + \gamma + \delta)^\vee(s)\epsilon_{12}(x_1)$,
2. $g = \epsilon_{12}(x_1)\epsilon_{12}(\alpha + 2\beta + \gamma + \delta)^\vee(s)\epsilon_{12}(x_2)$

for some $x_1, x_2 \in k$, $s \in k^r$.

We rule out the second case. Suppose $g$ is of the second form. Note that $\epsilon_{11}(1)\epsilon_2(\sqrt{\alpha}) \in v(\sqrt{\alpha})^{-1} \cdot H < P_\mu$. But $P_\mu = P_{g \cdot (\alpha + 2\beta + \gamma + \delta)^\vee} = g \cdot P_\mu$. So it is enough to show that $g^{-1} \cdot (\epsilon_{11}(1)\epsilon_2(\sqrt{\alpha})) \notin P_{g \cdot (\alpha + 2\beta + \gamma + \delta)^\vee}$. Since $U_{12}$ and $(\alpha + 2\beta + \gamma + \delta)^\vee(k^r)$ is contained in $P_{g \cdot (\alpha + 2\beta + \gamma + \delta)^\vee}$ we can assume $g = n_{12}$. We have

$$n_{12} = n_\alpha n_\beta n_an_\gamma n_\beta n_\alpha n_\delta n_\alpha n_\gamma n_\beta n_\delta (\text{the longest element in the Weyl group of } D_4).$$

Using this, we can compute how $n_{12}$ acts on each root subgroup of $G$. In particular $n_{12}^{-1} \cdot U_{11} = U_{-12}$ and $n_{12}^{-1} \cdot U_2 = U_{-2}$. Thus

$$n_{12}^{-1} \cdot (\epsilon_{11}(1)\epsilon_2(\sqrt{\alpha})) = \epsilon_{-12}(1)\epsilon_{-2}(\sqrt{\alpha}) \notin P_{(\alpha + 2\beta + \gamma + \delta)^\vee}.$$ 

So $g$ must be of the first form. Then $g \in P_\lambda$. Thus $P_\mu = P_{g \cdot \lambda} = g \cdot P_\lambda = P_\lambda$. We are done.

Lemma 4.6. $v(\sqrt{\alpha}) \cdot P_\lambda$ is not k-defined.

Proof. Suppose the contrary. Then $(v(\sqrt{\alpha}) \cdot P_\lambda)^o$ is k-defined. Since $P_\rho^\circ$ is k-defined, $v(\sqrt{\alpha}) \cdot P_\lambda$ is $G^o(k)$-conjugate to $P_\lambda^o$ by [1] Thm. 20.9. Thus we can put $g v(\sqrt{\alpha}) \cdot P_\lambda^o$ for some $g \in G(k)^o$. So $g v(\sqrt{\alpha}) \in P_\lambda^o$ since parabolic subgroups are self-normalizing. Then $g = p v(\sqrt{\alpha})^{-1}$ for some $p \in P_\lambda^o$. Thus $g$ is a k-point of $P_\rho^o R_u(P_\lambda^o)$. Then by the rational version of the Bruhat decomposition [3] Thm. 21.15], there exists a unique $p' \in P_\rho^o(k)$ and a unique $u' \in R_u(P_\lambda^o)(k)$ such that $g = p'u'$. This is a contradiction since $v(\sqrt{\alpha}) \notin R_u(P_\lambda^o)(k)$.

Now Lemmas 4.4 and 4.6 show that $H$ is G-ir over $k$. 

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Lemma 4.7. $H$ is not $G$-cr.

Proof. We had $C_G(M)^\circ = G_{12}$. Then $C_G(v(\sqrt{a})^{-1} \cdot H)^\circ < G_{12}$ since $M < v(\sqrt{a})^{-1} \cdot H$. Using the commutation relations, we see that $U_{12} < C_G(v(\sqrt{a})^{-1} \cdot H)$. Note that $v(\sqrt{a})^{-1} \cdot H$ contains $h := \epsilon_1(1)\epsilon_2(\sqrt{a})$ that does not commute with any non-trivial element of $U_{12}$. Also, since $(\alpha + \beta + \gamma + \delta, \lambda) = 4$, $h$ does not commute with any non-trivial element of $(\alpha + 2\beta + \gamma + \delta)^\circ(\sqrt{k})$. Thus we conclude that $C_G(v(\sqrt{a})^{-1} \cdot H)^\circ = U_{12}$. So $C_G(H)^\circ = v(\sqrt{a}) \cdot U_{12}$ which is unipotent. Then by the classical result of Borel-Tits [11, Prop. 3.1], we see that $C_G(H)^\circ$ is not $G$-cr. Since $C_G(H)^\circ$ is a normal subgroup of $C_G(H)$, by [8, Ex. 5.20], $C_G(H)$ is not $G$-cr. Then by [4, Cor. 3.17], $H$ is not $G$-cr.

\[\Box\]

Remark 4.8. Now we have a collection of examples of subgroups of $G$ that are $G$-cr over $k$ but not $G$-cr (or vice versa) for connected $G$ of type $G_2$, $E_6$, $E_7$, $E_8$, and for non-connected $G$ of type $A_2$, $A_4$, $D_4$ ([2, 37, 39, and 36]) all in characteristic 2. It would be interesting to find such examples in characteristic 3.

In general, combining [11, Thm. 1.5] and Proposition 1.3 we have

Proposition 4.9. Let $k$ be nonperfect. Suppose that $H$ is separable in $G$. If a $k$-subgroup $H$ of $G$ is $G$-cr, then it is $G$-cr over $k$.

Our examples of subgroups $H$ of $G$ in [37] (and the $D_4$ example in this paper) for the other direction ($G$-cr over $k$ but not $G$-cr) are all nonseparable. So it is natural to conjecture that the other direction holds if $H$ is separable. However, there exists a separable such subgroup in $G = PGL_2$:

Example 4.10. Let $k$ be a nonperfect field of characteristic $p = 2$. Let $a \in k \setminus k^2$. Let $G = PGL_2$. We write $\tilde{A}$ for the image in $PGL_2$ of $A \in GL_2$. Set $u = \begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix} \in G(k)$. Let $U := \langle u \rangle$. Then $U$ is unipotent, so by the classical result of Borel-Tits [11, Prop. 3.1] $U$ is not $G$-cr. However $U$ is not contained in any proper $k$-parabolic subgroup of $G$ since there is no nontrivial $k$-defined flag of $\mathbb{P}^1_k$ stabilized by $U$. So $U$ is $G$-ir over $k$, hence $G$-cr over $k$. Also the argument in [11, Ex. 7.6] shows that $U$ is separable in $G$.

The element $u$ in the example above is one of $k$-nonplongeable unipotent elements in [33].

Definition 4.11. A unipotent element $u$ of $G$ is $k$-nonplongeable unipotent if $u$ is not contained in the unipotent radical of any $k$-defined $R$-parabolic subgroup of $G$. In particular, if $u$ is not contained in any $k$-defined $R$-parabolic subgroup of $G$, $u$ is $k$-anisotropic unipotent.

Note that our definition of $k$-nonplongeability (and $k$-anisotropy) extends original Tits’ definition to non-connected $G$. Now we ask

Question 4.12. Let $k$ be nonperfect. Let $H$ be a $k$-subgroup of $G$. Suppose that every unipotent element of $G(k)$ is $k$-plongeable and $H$ is separable in $G$. Then if $H$ is $G$-cr over $k$, it is $G$-cr?

Remark 4.13. If every unipotent element of $G^\circ(k)$ is $k$-plongeable (in particular this holds if $[k : k^p] \leq p$ and $G^\circ$ is simply-connected by a deep result of Gille [14]) and if a connected $k$-subgroup $H$ of $G^\circ$ is $G^\circ$-cr over $k$, then $H$ is pseudo-reductive [37, Thm. 1.9]; see [33, Def. 1.1.1] for the definition of pseudo-reductivity. The proof of [37, Thm. 1.9] depends on the center conjecture. Question 4.12 is closely related to the so-called “strengthened Hilbert-Mumford theorem” in GIT; see [11, Sec. 5]. We believe that our $D_4$ example and examples in [37] give a clue to attack this problem.
5 Geometric Invariant Theory

In this section, we generalize Theorems 1.3 and 1.4 via GIT, and obtain new results (Theorems 1.9 and 1.10) concerning $G$ and $G(k)$-orbits in an arbitrary $G$-variety. The proof of Theorems 1.9 and 1.10 give a new GIT-theoretic proof to Theorems 1.3 and 1.4 using the geometric characterization of complete reducibility (Proposition 1.8). Let $H$ be a subgroup of $G$ such that $H = \langle h_1, \ldots, h_n \rangle$ for some $n \in \mathbb{N}$. Let $h := (h_1, \ldots, h_n) \in G^n$. Suppose that $G$ (and $G(k)$) acts on $h$ via simultaneous conjugation.

Proposition 5.1. Suppose that $G \cdot h$ is Zariski closed but $G(k) \cdot h$ is not cocharacter closed over $k$. Let $P$ be a minimal $k$-defined $R$-parabolic subgroup of $G$ containing $H$. Then there exists a unipotent element $u \in R_u(P)(k)$ such that $G(k) \cdot (u \cdot h) = G(k) \cdot (uh_1u^{-1}, \ldots, uh_Nu^{-1})$ is cocharacter closed over $k$.

Proof. This is a translation of Theorem 1.3 using Proposition 1.8. Alternatively, this follows from 1.9 since this is a special case of Theorem 1.9.

Proposition 5.2. Let $h$ and $H$ as in the last proposition. Suppose that $G(k) \cdot h$ is cocharacter closed over $k$ but $G \cdot h$ is not Zariski closed. Let $L$ be a minimal $k$-defined $R$-Levi subgroup of $G$ containing $H$. Then there exists an element $l \in L(k)$ such that $G(k) \cdot (l \cdot h)$ is not cocharacter closed over $k$.

Proof. This is a translation of Theorem 1.4 via Proposition 1.8. Also this is a special case of Theorem 1.10.

Now it is easy to see that Theorems 1.9 and 1.10 are natural generalizations of Theorems 1.3 and 1.4 via GIT.

Proof of Theorem 1.10. Let $\lambda$, $P_\lambda$, and $v$ as in the hypothesis. Let $v' := \lim_{a \to 0} \lambda(a) \cdot v$. Since $G \cdot v$ is Zariski closed, there exists $g \in G(k)$ such that $v' := g \cdot v$. Then, by Proposition 2.9 there exists $u \in R_u(P_\lambda)(k)$ such that $v' := u \cdot v$ since $k$ is perfect. Our goal is to show that $G(k) \cdot v'$ is cocharacter closed over $k$. Suppose the contrary. Then by Propositions 2.3 and 1.8 $L_\lambda(k) \cdot v'$ is not cocharacter closed over $k$. Then there exists a $k$-defined cocharacter $\mu$ of $L_\lambda$ properly destabilizing $v'$. So $\mu(k)$ is not central in $L_\lambda$. Then $Q := P_\mu(L_\lambda) \ltimes R_u(P_\lambda)$ is a $k$-defined $R$-parabolic subgroup of $G$ properly contained in $P_\lambda$. Set $Q := Q_\xi$ for some $\xi \in Y_k(G)$. We want to show that $\xi$ destabilizes $v$ over $k$. By [1 Lem. 6.2], we can set $\xi := m \lambda + \mu$ for some large $m \in \mathbb{N}$. Note that $\lambda$ and $\mu$ clearly commute since $\mu$ is a cocharacter of $L_\lambda = C_G(\lambda(k))$. So, if both $\lim_{a \to 0} \lambda(a) \cdot v'$ and $\lim_{a \to 0} \mu(a) \cdot v'$ exist, we are done thanks to [1 Lem. 2.7]. We have already shown that the second limit exists. Note that $\lim_{a \to 0} \lambda(a) \cdot u = 1$ since $u \in R_u(P_\lambda)$. So we have

$$\lim_{a \to 0} \lambda(a) \cdot v' = \lim_{a \to 0} \lambda(a) \cdot (u \cdot v) = \lim_{a \to 0} \lambda(a) \cdot v.$$ 

The last limit exists by our assumption, so the first limit exists. We are done.

Proof of Theorem 1.10. We start with part 1. Let $\lambda$, $P_\lambda$, $v$ as in the hypothesis. Let $v' := \lim_{a \to 0} \lambda(a) \cdot v$. Since $G(k) \cdot v$ is cocharacter closed over $k$, $v'$ is $G(k)$-conjugate to $v$. Then $v'$ is $R_u(P_\lambda)(k)$-conjugate to $v$ by Proposition 2.10 since $G(k)$ is cocharacter closed over $k$. So set $v' := u \cdot v$ for some $u \in R_u(P_\lambda)(k)$. Then it follows that $v$ is centralized by $u^{-1} \cdot \lambda$ by [8 Lem. 2.12]. Let $\xi := u^{-1} \cdot \lambda$. Then $\xi$ is a $k$-defined cocharacter of $G$ since $u$ is a $k$-point of $G$. Thus $L_\xi$ is a $k$-defined $R$-Levi subgroup of $G$. We have already shown that $\xi$ fixes $v$. It is clear that $P_\xi = P_\xi$ since $P_\xi = P_{u^{-1} \cdot \lambda} = u^{-1} \cdot P_\lambda = P_\lambda$.

For part 2, suppose that there exist a non-central $k$-defined cocharacter $\mu$ of $L_\xi$ destabilizing $v$ over $k$. Let $Q := P_\mu(L_\xi) \ltimes R_u(P_\xi)$. By the similar argument to that in the proof of
Theorem 1.9. We find a $k$-defined cocharacter of $L_\xi$ destabilizing $v$ over $k$ since both $\xi$ and $\lambda$ destabilize $v$ over $k$. Since $Q$ is a $k$-defined $R$-parabolic subgroup strictly contained in $P_\xi = P_\lambda$, this contradicts the minimality of $P_\lambda$. So we are done.

For part 3, let $L := L_\xi$. Since $G \cdot v$ is not Zariski closed, $L \cdot v$ is not Zariski closed by Propositions 2.3 and 1.8. Let $\Delta_{L,v,\overline{k}}$ be the set of $\overline{k}$-defined cocharacters of $L$ destabilizing $v$. Then by the Hilbert-Mumford theorem [18], $\Delta_{L,v,\overline{k}}$ is non-empty. Pick $\zeta \in \Delta_{L,v,\overline{k}}$ such that $P_\zeta(L)$ be a minimal $R$-parabolic subgroup of $L$ among $P_\mu$ for $\mu \in \Delta_{L,v,\overline{k}}$. Since an $R$-parabolic subgroup of $L$ is $L(\overline{k})$-conjugate to a $k$-defined $R$-parabolic subgroup of $L$, there exists $l \in L(\overline{k})$ such that $l \cdot P_\zeta(L)$ is a $k$-defined $R$-parabolic subgroup of $L$. Then there exists a $k$-defined cocharacter $\eta$ of $L$ such that $P_\eta(L) = l \cdot P_\zeta(L)$ since $k$ is separably closed. Then $P_\eta(L) = P_l(\zeta)(L)$. It is clear that $\eta$ is $P_\eta(L)$-conjugate to $l \cdot \zeta$. So set $\eta := s \cdot (l \cdot \zeta)$ for some $s \in P_\eta(L)$. Then $s = mu$ for some $m \in L_\eta(L)$ and $u \in R_u(P_\eta(L))$. Since $\eta$ fixes $m$, we have $\eta = u \cdot l \cdot \zeta$. Then

$$\lim_{a \to 0} \eta(a) \cdot (l \cdot v) = \lim_{a \to 0} (u \cdot l \cdot \zeta(a)) \cdot (l \cdot v) = ul \cdot (\lim_{a \to 0} \zeta(a) \cdot v).$$

We have assumed that the last limit exists, so the first limit also exists. Thus $\eta$ destabilizes $l \cdot v$ over $k$. Suppose that $G(k) \cdot (l \cdot v)$ is cocharacter closed over $k$. Our goal is to obtain a contradiction. Now $L(k) \cdot (l \cdot v)$ is cocharacter closed over $k$ by Propositions 2.3 and 1.8. Then by [3] Lem. 2.12 there exists $w \in R_u(P_\eta(L))(k)$ such that $w^{-1} \cdot \eta(\overline{k})$ fixes $l \cdot v$. Since $C := C_L(w^{-1} \cdot \eta(\overline{k}))$ is a $k$-defined $R$-Levi subgroup (since $w$ is a $k$-point in $L$ and $\eta$ is $k$-defined), by Propositions 2.3 and 1.8 again, $C(k) \cdot (l \cdot v)$ is not cocharacter closed over $k$ since $G(k) \cdot (l \cdot v)$ is not cocharacter closed over $k$. Then there exists a $k$-defined cocharacter $\tau$ of $C$ properly destabilizing $l \cdot v$ over $k$. In particular, $\tau(\overline{k})$ does not fix $l \cdot v$. Thus $P_\tau(C)$ is a proper $k$-defined $R$-parabolic subgroup of $C$. Define $Q := P_\tau(C) \ltimes R_u(P_\eta(L))$. Then by the similar argument to that of the proof of part 2, we find a $k$-defined cocharacter $\alpha$ of $C$ such that $Q = Q_\alpha$ and $\alpha$ destabilizes $l \cdot v$ over $k$ since both $\tau$ and $\eta$ destabilizes $l \cdot v$ over $k$. Since $Q$ is a $k$-defined $R$-parabolic subgroup of $L$ strictly contained in $P_\eta(L)$, this contradicts the minimality of $P_L$. We are done.

\[\square\]

Remark 5.3. Our proofs of Theorems 1.9 and 1.10 are very similar to those of Theorems 1.3 and 1.4. We believe many results on complete reducibility (over $k$) can be generalized in a similar way. Conversely, more results on GIT can be used to study complete reducibility; see [1, 8] for more on this.

6 Tits’ spherical buildings

In this section, we translate Theorems 1.3 and 1.4 into the language of spherical buildings [22]. This gives a new topological view to the rationality problems for complete reducibility and GIT. We assume that $G$ is connected in this section.

Definition 6.1. We write $\Delta_k(G)$ (or $\Delta(G)$) for the spherical building corresponding to the set of $k$-parabolic subgroups of $G$ (or parabolic subgroups of $G$ respectively). It is clear that $\Delta_k(G)$ is a subbuilding of $\Delta(G)$.

Definition 6.2. Let $H$ be a subgroup of $G$, we write $\Delta_k(G)^H$ (or $\Delta(G)^H$) for the subcomplex of $\Delta_k(G)$ (or $\Delta(G)$) corresponding to the set of $k$-parabolic subgroups (or parabolic subgroups) of $G$ containing $H$. We call $\Delta_k(G)^H$ (or $\Delta(G)^H$) the fixed point subcomplex of $\Delta_k(G)$ (or $\Delta(G)$) with respect to $H$.
We recall Serre’s characterization of complete reducibility (and complete reducibility over \(k\)) in terms of a topological property of the corresponding fixed point subcomplexes of \(\Delta(G)\) (and \(\Delta_k(G)\)) \cite[Sec. 2.2]{Serre}:

**Proposition 6.3.** Let \(H\) be a subgroup of \(G\). Then

1. \(H\) is \(G\)-cr over \(k\) if and only if \(\Delta_k(G)^H\) is not contractible.
2. \(H\) is \(G\)-cr if and only if \(\Delta(G)^H\) is not contractible.

We are ready to state our main results in this section.

**Theorem 6.4.** Let \(k\) be a nonperfect separably closed field. Let \(H\) be a subgroup of \(G\). Suppose that \(\Delta_k(G)^H\) is contractible but \(\Delta(G)^H\) is not contractible. Let \(P\) be a simplex \(s\) in \(\Delta_k(G)^H\) of minimal dimension. Let \(P\) be the \(k\)-parabolic subgroup of \(G\) corresponding to \(s\). Then there exists a unipotent element \(u \in R_u(P)(\overline{k})\) such that \(\Delta_k(G)^u\) is not contractible.

**Proof.** This follows from Theorem 1.3 and Proposition 6.3.

**Theorem 6.5.** Let \(k\) be a nonperfect separably closed field. Let \(H\) be a subgroup of \(G\). Suppose that \(\Delta_k(G)^H\) is not contractible but \(\Delta(G)^H\) is contractible. Let \(s\) be a simplex of \(\Delta_k(G)^H\) of minimal dimension. Let \(P\) be the \(k\)-parabolic subgroup corresponding to \(s\). Then the following hold:

1. There exists a \(k\)-Levi subgroup of \(P\) such that \(\Delta_k(L)^H = \{\emptyset\}\).
2. There exists an element \(l \in L(\overline{k})\) such that \(\Delta_k(G)^l\) is contractible.

**Proof.** This follows from Theorem 1.4 and Proposition 6.3.

**Remark 6.6.** Our proofs of Theorems 6.4 and 6.5 are translations of Theorems 1.3 and 1.4. It would be very interesting to find purely building-theoretic (geometric-topological) proofs of Theorems 6.4 and 6.5.

## 7 Centralizers and normalizers of \(G\)-cr subgroups

In this section, we assume \(G\) is connected. Our proof of Theorem 1.13 is purely combinatorial and just depends on the structure of the set of roots of \(G\) and the corresponding root subgroups of \(G\).

**Proof of Theorem 7.1.** Suppose that \(H\) is \(G\)-cr over \(k\) but \(C_G(H)\) is not. Then by \cite[Prop. 3.3]{Serre} (this depends on the recently proved center conjecture of Tits), there exists a proper \(k\)-parabolic subgroup of \(G\) containing \(HC_G(H)\). Let \(P\) be a minimal such \(k\)-parabolic subgroup. Let \(P = P_\lambda\) for some \(\lambda \in Y_k(G)\). Since \(H\) is \(G\)-cr over \(k\), there exists a \(k\)-Levi subgroup \(L\) of \(P_\lambda\) containing \(H\). Since \(k\)-Levi subgroups of \(P\) are \(R_u(P)(k)\)-conjugate by \cite[Lem. 2.5(iii)]{Serre}, we may assume \(L = L_\lambda\). Then \(\lambda(\overline{k}) < C_G(H)\) since \(L_\lambda = C_G(\lambda(\overline{k}))\).

Suppose that there exists \(1 \neq u \in R_u(P_\lambda) \cap C_G(H)\). Define \(w := \overline{w_{0,L} \circ \overline{w_{0,G}}}\). Then

\[
    w \cdot (R_u(P_\lambda)) = \overline{w_{0,L}} \cdot (\overline{w_{0,G}} \cdot R_u(P_\lambda)) = \overline{w_{0,L}} \cdot (R_u(P_{-\lambda})) = R_u(P_{-\lambda}).
\]

Note that for any root \(i \in \Psi(L)\) we have

\[
    w \cdot i = \overline{w_{0,L}} \cdot (\overline{w_{0,G}} \cdot i) = \overline{w_{0,L}} \cdot (-i) = i.
\]
Then $w$ centralizes $[L, L]$ and $H$ as well since $H$ is contained in $[L, L]$ (we have assumed that $H$ is semisimple). Thus

\[ w \cdot (C_G(H)) = C_G(w \cdot H) = C_G(H). \]

So we have $1 \neq w \cdot u \in R_u(P_{-\lambda}) \cap C_G(H)$. This is a contradiction since $C_G(H) < P_\lambda$. So $R_u(P_{-\lambda}) \cap C_G(H) = 1$. Thus $HC_G(H) < L$. Since $C_G(H)$ is not $G$-cr over $k$, it is not $L$-cr over $k$ by Proposition 2.3. Then there exists a proper $k$-parabolic subgroup $P_L$ of $L$ containing $HC_G(H)$ by [37] Prop 3.3. Then $P_L \ltimes R_u(P_{-\lambda})$ is a $k$-parabolic subgroup of $G$ containing $HC_G(H)$ and strictly contained in $P_\lambda$. This contradicts the minimality of $P_\lambda$. Hence $C_G(H)$ is $G$-cr over $k$.

**Remark 7.1.** The condition on $\varpi_{0, L}$ in the hypothesis of Theorem 7.2 is necessary for our proof to work since otherwise the operation of $\varpi_{0, L}$ on $R_u(P_{-\lambda})$ might not make sense. Consider the following case: $G = A_3$, $\Psi(G) = \{ \pm \alpha, \pm \beta, \pm \gamma, \pm(\alpha + \beta), \pm(\beta + \gamma), \pm(\alpha + \beta + \gamma) \}$, $L = L_{\alpha \beta \gamma}$. Then $\varpi_{0, L} = n_\alpha n_\beta n_\gamma$, where $n_\alpha$, $n_\beta$, and $n_\gamma$ are reflections corresponding to $\alpha$, $\beta$, and $\gamma$ respectively. We see that $\beta + \gamma \in R_u(P_{-\lambda})$ but $\gamma$ cannot be applied to $\beta + \gamma$.

Replacing $C_G(H)$ with $N_G(H)$ in the proof of Theorem 7.2 we obtain

**Theorem 7.2.** Let $G$ be connected. Suppose that a semisimple $k$-subgroup $H$ of $G$ is $G$-cr over $k$. Let $P$ be a minimal $k$-parabolic subgroup containing $N_G(H)$, and $L$ be a $k$-Levi subgroup of $P$. If the automorphism $\varpi_{0, L}$ of $L$ extends to an automorphism of $G$, then $N_G(H)$ is $G$-cr over $k$.

For the converse of Theorem 7.2 we have

**Proposition 7.3.** Let $G$ be connected. Let $H$ be a (not necessarily $k$-defined) subgroup of $G$. Suppose that $N_G(H)$ is $G$-cr over $k$. If $N_G(H)$ is defined over $k$, then $H$ is $G$-cr over $k$.

**Proof.** This is a consequence of [37] Prop. 3.7. Note that $N_G(H)$ needs to be $k$-defined to apply [37] Prop. 3.7.

## 8 On the number of conjugacy classes

Let $n$ be a natural number. Let $G$ be a (possibly non-connected) reductive groups defined over an algebraically closed field $k$ of characteristic $p$. Suppose that $M$ is a (possibly non-connected) reductive subgroup of $G$. Using Richardson’s beautiful tangent space argument [23], Slowdowy [27] showed that if $G$ and $M$ are connected and $p$ is very good for $G$, $G \cdot (m_1, \cdots, m_n) \cap M^n$ is a finite union of $M$-conjugacy classes where $m_i \in M$ and $G$ acts on $(m_1, \cdots, m_n)$ via simultaneous conjugation. Moreover Guralnick showed that the same result holds for non-connected $G$ and $M$ for any $n$ if $n = 1$ [15] Thm. 1.2.

The following Theorem 8.1 shows the two things: even if $p$ is very good for $G^o$, 1. Slowdowy’s result fails if $G$ and $M$ are non-connected, 2. Guralnick’s result fails if $n > 1$. Theorem 8.1 works because if $G$ is non-connected of type $A_2$, there exists a nonseparable subgroup of $G$ even if $p$ is very good for $G^o$.

**Theorem 8.1.** Let $k$ be an algebraically closed field of characteristic 2. Let $G = SL_3 \rtimes \langle \alpha \rangle$ be defined over $k$ where $\alpha$ is a non-trivial graph automorphism of $SL_3$. Then there exists a (non-connected) reductive subgroup $M$ of $G$ and a pair $(m_1, m_2) \in M^2$ such that $G \cdot (m_1, m_2) \cap M^2$ is an infinite union of $M$-conjugacy classes.
Proof. Let $G$ and $\sigma$ be as in the hypothesis. Fix a maximal torus $T$ of $G$ and a Borel subgroup $B$ of $G^\circ$ containing $T$. Let $\Psi(G)^+ = \{\alpha, \beta, \alpha + \beta\}$ be the set of positive roots of $G$ with respect to $T$ and $B$. Then $\sigma \cdot \alpha = \beta$ and $\sigma \cdot \beta$. In the following we use the commutation relations \cite{17} Sec. 33.3 repeatedly.

Let $\lambda := (\alpha + \beta)^+ \in Y(G)$. Then $P_\lambda = \langle T, U_\alpha, U_\beta, U_{\alpha+\beta}, \sigma \rangle$, and $R_u(P_\lambda) = \langle U_\alpha, U_\beta, U_{\alpha+\beta} \rangle$. Let $\epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_{\alpha+\beta}(z)$ be an arbitrary element in $R_u(P_\lambda)$. For $x, y, z \in k$, we have

$$\sigma \cdot (\epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_{\alpha+\beta}(z)) = \epsilon_\beta(x)\epsilon_\alpha(y)\epsilon_{\alpha+\beta}(z) = \epsilon_\alpha(y)\epsilon_\beta(x)\epsilon_{\alpha+\beta}(xy + z).$$

Thus $\epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_{\alpha+\beta}(z)$ is not in $C_G(\sigma)$ unless $x = y = 0$. In particular, a curve $\{\epsilon_\alpha(x)\epsilon_\beta(x) \mid x \in k\}$ is not in $C_G(H)$. On the other hand, we have

$$\sigma \cdot (\epsilon_\alpha + \epsilon_\beta) = e_\alpha + e_\beta.$$  

The same argument as in Remark \[3.2\] shows that $\langle \sigma \rangle$ is non-separable in $G$.

Now let $v(x) := \epsilon_\alpha(x)\epsilon_\beta(x)$ for $x \in k$. Clearly $v(x) \in R_u(P_\lambda)$. Let $M := \langle \sigma, G_{\alpha+\beta} \rangle$. Then $M$ is (non-connected) reductive. We have $Z(R_u(P_\lambda)) = U_{\alpha+\beta}$. Let $(m_1, m_2) := (\sigma, \epsilon_{\alpha+\beta}(1))$. Define

$$(m_1, m_2)_x := v(x) \cdot (m_1, m_2) = (\sigma\epsilon_{\alpha+\beta}(x^2), \epsilon_{\alpha+\beta}(1)).$$

We see that $(m_1, m_2)_x \in M^2$. Note that $\sigma$ centralizes $(m_1, m_2)$. Since $M^\circ = SL_2$, a direct computation shows that if $a \neq b$, $(m_1, m_2)_a$ is not $M$-conjugate to $(m_1, m_2)_b$. \qed

Remark 8.2. The same example was used to answer negatively to a question of Külshammer on representations of finite groups \cite[Thm. 1.14]{35}. We believe that the conjugacy class problem in this section is closely related to Külshammer’s question. See \cite{6} and \cite{20} for more on this.

Remark 8.3. In the proof of Theorem \[5.1\] $p = 2$ is crucial. Even if $p = 3$ is bad for our $G$, if $p = 3$ our argument breaks down (an easy computation shows that $\langle \sigma \rangle$ becomes separable in $G$).

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