Experimental Measurement of Multi-dimensional Entanglement via Equivalent Symmetric Projection

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We construct a linear optics measurement process to determine the entanglement measure, named I-concurrence, of a set of $4 \times 4$ dimensional two-photon entangled pure states produced in the optical parametric down conversion process. In our experiment, an equivalent symmetric projection for the two-fold copy of single subsystem (presented by L. Aolita and F. Mintert, Phys. Rev. Lett. 97, 050501 (2006)) can be realized by observing the one-side two-photon coincidence without any triggering detection on the other subsystem. Here, for the first time, we realize the measurement for entanglement contained in bi-photon pure states by taking advantage of the indistinguishability and the bunching effect of photons. Our method can determine the I-concurrence of generic high dimensional bipartite pure states produced in parametric down conversion process.

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I. INTRODUCTION

The characterization and quantification for quantum entanglement has become one of the most central issues in quantum information theory. Various approaches for characterization of entanglement in quantum states have been proposed. These are based on quantum state tomography [1, 2, 3], entanglement witnesses [4, 5, 6, 7, 8, 9], and quantum nonlocality [10, 11, 12, 13]. The most important approach for quantifying the amount of entanglement in any quantum state is entanglement measure. Up till now, several well defined entanglement measures have been established, e.g. concurrence [14] and entanglement of formation [15]. The experimental determination of entanglement measure is, however, a very difficult task, since many measures are complicated nonlinear functions of the density matrix of the quantum state. The situation is even worse for multipartite and multi-dimensional quantum systems. The most straightforward way is to reconstruct the quantum state fully through quantum state tomography [1, 2, 3]. However, this method has the disadvantage in being unscalable, and not all the state parameters are necessary for the determination of entanglement measures.

Recently, there are increasing interests in the entanglement measured by concurrence, originally defined for two-qubit entanglement and later generalized to multipartite and multi-dimensional quantum systems [16, 17, 18, 19]. It is one of the most fundamental entanglement measures and has been widely used in many fields of quantum information theory, e.g. the research of entanglement in quantum phase transitions [20]. One important property of concurrence is that it depends on a polynomial function of the elements of the density matrix. This makes it possible to observe concurrence through some appropriate observables with two-fold copy of quantum states [19]. In Ref. [21], Walborn, etc reported an experimental determination of concurrence for two-qubit pure states.

In this paper, we report an experimental determination of the generalized concurrence [16], i.e. I-concurrence, of $4 \times 4$ dimensional pure states produced by optical parametric down conversion (PDC) by using the polarization and time-energy mode. Different from the measurements on two-fold copy of quantum states, here, our strategy is to detect coincidence counts of high order optical PDC directly, which contains all the information about the amplitudes in low order optical PDC and corresponds to an equivalent symmetric projection for the two-fold copy of single subsystem. Our scheme is simpler than the two-fold copy measurement, moreover, can be generalized to the measurement of higher dimensional bipartite pure states produced in optical PDC.

The structure of this paper is as follows. In Sec. II we present a brief review for I-concurrence of bipartite pure states. In Sec. III quantum states produced in optical PDC are investigated. We find the I-concurrence of an entangled state produced in 1-order optical PDC can be measured by detecting 2-order optical PDC process, which corresponds to the implementation of an equivalent symmetric projection for two-fold copy of this state. Sec. IV

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II. A REVIEW FOR I-CONCURRENCE

For a pure state $|\psi\rangle$ of a $d_1 \times d_2$ quantum system, the $I$-concurrence is defined as \[ C = \sqrt{2(1 - Tr\rho_1^2)} \] (1)

where $\rho_1$ is the reduced density matrix of the 1st subsystem. The above generalized concurrence is simply related to the purity of the marginal density matrices. The maximum value of $I$-concurrence is $\sqrt{2(M-1)/M}$, where $M = \min(d_1, d_2)$. We note that $Tr\rho_1^2$ is a quadratic function of the elements of the density matrix $\rho_1$. Thus, one could always find an observable $\hat{A}$ on 2 copies of $\rho_1$, such that $Tr\rho_1^2 = Tr(\hat{A}\rho_1 \otimes \rho_1)$ \[ 23 \]. This allows to measure $C$ without quantum state tomography. Actually, it has also been shown that $Tr\rho_1^2 = 1 - 2Tr(P_+^1 \rho_1 \otimes \rho_1) = 2Tr(P_+^1 \rho_1 \otimes \rho_1) - 1$, where $P_+^1$ and $P_-^1$ are the projectors onto the symmetric and antisymmetric subspace of the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_1$, which describes the two-fold copy of the 1st subsystem \[ 19 \]. Therefore, the $I$-concurrence can be expressed as the expectation value of a Hermitian operator $\hat{A}$ on $\mathcal{H} \otimes \mathcal{H}$, i.e.

$$C = \sqrt{\langle \psi \otimes \langle \psi | \hat{A} | \psi \rangle \otimes | \psi \rangle}$$

where $\hat{A} = 4P_+^1 = 4(I - P_-^1)$. Thus, we can determine the $I$-concurrence by measuring one single factorizable observable $\hat{A}$ on two-fold copy of one subsystem.

III. I-CONCURRENCE FOR TWO-PHOTON STATES PRODUCED IN OPTICAL PDC

Experimentally, entangled two-photon state can be produced through optical PDC. When we consider different degrees of freedom (DOF) of photons, such as polarization, time-energy, etc., high dimensional entangled pure states can be constructed by using appropriate linear optical methods. In the Schmidt decomposition, the high dimensional degrees of freedom (DOF) of photons, such as polarization, time-energy, can be constructed by using appropriate linear optical methods. In the Schimdt decomposition, the high dimensional two-photon entangled pure states. Sec. V contains conclusions and some discussions.

A bipartite pure state can be represented as:

$$|\Psi\rangle = \sum_{i<j} \lambda_i \lambda_j a_i^\dagger a_j^\dagger b_i^\dagger b_j^\dagger |\text{vac}\rangle$$

where $\lambda_i$ are the Schmidt coefficients. The maximum value of the $I$-concurrence is $\sqrt{2(M-1)/M}$, where $M = \min(d_1, d_2)$. We note that $Tr\rho_1^2$ is a quadratic function of the elements of the density matrix $\rho_1$. Thus, one could always find an observable $\hat{A}$ on 2 copies of $\rho_1$, such that $Tr\rho_1^2 = Tr(\hat{A}\rho_1 \otimes \rho_1)$ \[ 23 \]. This allows to measure $C$ without quantum state tomography. Actually, it has also been shown that $Tr\rho_1^2 = 1 - 2Tr(P_+^1 \rho_1 \otimes \rho_1) = 2Tr(P_+^1 \rho_1 \otimes \rho_1) - 1$, where $P_+^1$ and $P_-^1$ are the projectors onto the symmetric and antisymmetric subspace of the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_1$, which describes the two-fold copy of the 1st subsystem \[ 19 \]. Therefore, the $I$-concurrence can be expressed as the expectation value of a Hermitian operator $\hat{A}$ on $\mathcal{H} \otimes \mathcal{H}$, i.e.

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where $\hat{A} = 4P_+^1 = 4(I - P_-^1)$. Thus, we can determine the $I$-concurrence by measuring one single factorizable observable $\hat{A}$ on two-fold copy of one subsystem.
where the first item indicates that $a_i^\dagger a_j^\dagger b_i^\dagger b_j^\dagger |\text{vac}\rangle$ and $a_j^\dagger a_i^\dagger b_i^\dagger b_j^\dagger |\text{vac}\rangle$ are not distinguishable, and the second item will cause the bunching effect of identical photons. So, we may deduce that: $\langle \Psi_2 | \langle \Psi_2 | 4P_1^\dagger \rangle \Psi_2 \rangle = \langle \Psi_4 | \Psi_4 \rangle = 4(\sum_{i<j} \lambda_i \lambda_j + \sum_i \lambda_i^2) = 2(1 + \sum_i \lambda_i^2)$.

Eq.(4) suggests we can determine the entanglement of state $|\Psi_4\rangle$ by probing the inner product of the state $|\Psi_4\rangle$ in the full wavefunction $|\Psi\rangle$ in Eq.(3). (Here, we omit multi-photon ($\geq 6$) components due to their tiny probability.) When mode A passes through the beamsplitter (see Fig.1), $|\Psi_4\rangle$ is transformed into the superposition of two orthogonal components: $\frac{1}{\sqrt{2}}(|\Psi_e\rangle + |\Psi_s\rangle)$, where $|\Psi_e\rangle$ is the component of wavefunction giving rise to coincidence counts between $A_1$ and $A_2$ and $|\Psi_s\rangle$ is not. The form of $|\Psi_e\rangle$ is as follows:

$$|\Psi_e\rangle = \sum_{i<j} \sqrt{\lambda_i \lambda_j} \sum_{i,A_1} a_i^\dagger A_1 a_j^\dagger A_2 b_i^\dagger A_1 b_j^\dagger A_2 |\text{vac}\rangle$$
$$+ \sum_i \lambda_i a_i^\dagger A_1 a_i^\dagger A_2 b_i^\dagger A_1 b_i^\dagger A_2 |\text{vac}\rangle,$$

Then the probability of the two-photon counts after the BS is: $P_{A_1 A_2} = \frac{1}{4} \eta_{A_1} \eta_{A_2} |\eta|^2 \langle \Psi_e | \Psi_e \rangle = \frac{1}{2} \eta_{A_1} \eta_{A_2} |\eta|^2 \langle \Psi_4 | \Psi_4 \rangle = P_{A_1} P_{A_2} (\sum_i \lambda_i^2 + 1)$, where $P_{A_1} = \frac{1}{4} \eta_{A_1} \eta_{A_2} |\eta|$ is the single photon counts probability \cite{24}. $\eta_{A_1}$ and $\eta_{A_2}$ are photon collection efficiencies including the effect of photon coupling losses and the detector efficiency. Here, we find that the probability of the two-photon coincidence counts $P_{A_1 A_2}$ is always larger than the product of single photon counts probabilities $P_{A_1}$ and $P_{A_2}$. The reason relies on the indistinguishability and the bunching effect of photons in $|\Psi_4\rangle$. By defining $K = Tr\rho_1^2 = \frac{\sum_i \lambda_i^2}{P_{A_1} P_{A_2}} - 1$, the $I$-concurrence of a bipartite pure state is

$$C = \sqrt{2 - 2K}.$$

IV. EXPERIMENT

In our experiment, polarization and time-energy DOF are used to realize a 4 dimensional Hilbert space. Different polarization states are produced with two type-I PDC \cite{24} and time-energy DOF are introduced by using a 52.4 mm quartz crystal (QC) to induce different time delays for two polarization components of the pump beam. This is shown in Fig.2. The half wave plate (HWP) before the QC rotates the pump beam polarization. After the QC, the pump beam state is:

$$|P\rangle = \cos \theta_1 |HT_1\rangle + \sin \theta_1 |VT_2\rangle,$$
where $\theta_1$ is the angle of HWP$_1$ and the time delay $\Delta T = |T_1 - T_2| = 1.68$ ps. The pulse laser beam with a pulse width of $\tau_p = 150$ fs and repetition rate of $f = 76$ MHz from a Ti: Sapphire ultra-fast laser (Coherent D-900) is frequency doubled to 390 nm, which serves as the pump beam to two degenerated noncollinear type-I cut BBOs with mutually orthogonally optical axes. Each $|T_i\rangle$ ($i = 1, 2$) pulse generates a two-photon entangled state if we adjust the angle $\theta_2$ of HWP$_2$ [25]. Superposition of the two entangled polarization states with different $T_i$ is a two-photon four-dimensional state:

$$|\Psi_2\rangle = \cos 2\theta_1 (\cos 2\theta_2 |VT_1\rangle |VT_1\rangle + \sin 2\theta_2 |HT_1\rangle |HT_1\rangle)$$

$$+ \sin 2\theta_1 (\sin 2\theta_2 |VT_2\rangle |VT_2\rangle - \cos 2\theta_2 |HT_2\rangle |HT_2\rangle),$$

(9)

and the four bases are \{|HT_1\rangle, |VT_1\rangle, |HT_2\rangle, |VT_2\rangle\} (If $\theta_1$ or $\theta_2 = 0$, it reduces to 2 x 2 dimensional entangled state). To enhance the purity of two-photon states, we make down-converted photons pass through the interference filter, compensation crystal (CC) and enter into the single mode fiber. The interference filter is centered at 780 nm and its bandwidth is 3 nm, which corresponds to $\tau = 676$ fs for the correlation time of down converted photons. In our experiment, the two-photon coincidence window is $\Delta t = 3$ ns. The visibility for two-photon state and four-photon state are more than 96% and 95% respectively [26], indicating the high purity of the photon state.

**FIG. 2:** (color online) Experimental settings. The pseudo-twofold copy of the two-photon state is generated from the second order PDC.

It is worth mentioning the time scales in the experiment. $\Delta T > \tau_p$, $\tau$ guarantees good time separation of the two pulses so that the orthogonality condition $\langle P_i | T_i | P_j | T_j \rangle = \delta_{P_i, P_j} \delta_{ij} (P_i, P_j \in \{H, V\}; i, j \in \{1, 2\})$ holds. $\tau$, $\Delta T \ll \Delta t$ makes the time separation undetectable through the photon coincidence counts, so that quantum coherence of $|\Psi_2\rangle$ can be observed.

We measure the single photon counting rate of $A_1$, $A_2$ and the coincidence counting rate between $A_1$ and $A_2$ as $N_{A_1}$, $N_{A_2}$ and $N_{A_1 A_2}$, respectively. Then $P_{A_1 A_2} = N_{A_1 A_2} / f$ and $K = f N_{A_1 A_2} / N_{A_1} N_{A_2} - 1$.

**FIG. 3:** (color online) Single photon Counts and two-photon coincidence counts with $\theta_1 = 22.5^\circ$.

Fig. 3 is the experimental photon counts with $\theta_1 = 22.5^\circ$. Single photon counts of $A_1$ and $A_2$ are indicated by black square and green circle points in Fig. 3(a). Fig. 3(b) shows coincidence counts between $A_1$ and $A_2$. The entanglement measurement result with different angles of the two HWP $\theta_1$ and $\theta_2$ are shown in Fig. 4. The green open circle points are the data of $K$ and the green dotted curves show the theoretical values with function of $K(\theta_1, \theta_2) = (\cos^4 2\theta_1 + \sin^4 2\theta_1)(\cos^4 2\theta_2 + \sin^4 2\theta_2)$. The solid red square points refer to relative $I$-concurrence and the theoretical values of $C(\theta_1, \theta_2) = \sqrt{2 - 2K(\theta_1, \theta_2)}$ are illustrated with red solid curves. Fig. 4(a) shows the experimental results for
FIG. 4: (color online) Plot of entanglement measurement results with different angles $\theta_1$ and $\theta_2$. The green square points and dashed curves are the experimental data and theoretic value of $K$. Red circle points and curves are measured relative concurrences and their theoretical values. The black dashed curves show the sub-concurrence on polarization DOF after the time-energy DOF is traced out.

$\theta_1 = 0^\circ$, which corresponds to the case of $2 \times 2$ entangled state. When $\theta_2 = 22.5^\circ$, it becomes the maximally entangled (Bell) state and the $I$-concurrence reaches 1.03±0.09. While $\theta_2 = 0^\circ$ or $45^\circ$, it becomes a product state with minimal entanglement. Fig.4(b), (c), and (d) depict the $4 \times 4$ dimensional entangled state with $\theta_1 = 7.5^\circ$, $15^\circ$, and $22.5^\circ$, respectively. The $4 \times 4$ maximally entangled state can be achieved when both angles of $\theta_1$ and $\theta_2$ are set to $22.5^\circ$. The measured $I$-concurrence for this state is 1.24±0.09, whereas the theoretical value is $\sqrt{6}/2$. When $\theta_2 = 0^\circ$ or $45^\circ$, the states are reduced to $2 \times 2$ dimension again. The experimental results agree with the theoretical values well within the experimental errors from photon counts variance. Moreover, the experimental data shows the $I$-concurrence $C$ is always no less than the sub-concurrence $C_{12} = 2((\cos^2 2\theta_1 - \sin^2 2\theta_1) \cos 2\theta_2 \sin 2\theta_2)$, the polarization-dependent concurrence when time-energy DOF is traced out. $C_{12}$ is plotted as the black dashed curve in Figs.4(b), (c), and (d).

V. DISCUSSIONS AND CONCLUSIONS

In our experiment, $K$ is always a little less than the theoretical value of $Tr\rho_A^2$. It is likely because there are other DOFs besides the polarization and time-energy DOF involved in our experiment. It may be the frequency DOF, despite the narrow frequency filters used to improve the purity of the two-photon state [27]. When these additional DOFs are present, the photon state will be that of a higher dimensional system. Generally, $K$ less than the theoretical value for the maximally entangled states indicates there are other dimensions not under consideration. Therefore, our scheme could act as an effective method to detect additional DOF whether it is entangled with the main DOF or not.

In conclusion, we experimentally determine the entanglement measure of two-partite pure photon state with an equivalent symmetric projection measurement for the two-fold copy of single subsystem. We find the $I$-concurrence of entangled states produced in 1-order optical PDC can be obtained by measuring entangled states produced in 2-order optical PDC. Our method, for the first time, takes advantage of the indistinguishability and the bunching effect of photons to measure the entanglement of bi-photon pure states, which is suitable for application in optical PDC process to determine the entanglement of high-dimensional bipartite pure states composed of other DOF of photons.
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