A VERSION OF THE VOLUME CONJECTURE

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Abstract. We propose a version of the volume conjecture that would relate a certain limit of the colored Jones polynomials of a knot to the volume function defined by a representation of the fundamental group of the knot complement to the special linear group of degree two over complex numbers. We also confirm the conjecture for the figure-eight knot and torus knots. This version is different from S. Gukov’s because of a choice of polarization.

1. Introduction

For a knot $K$ in the three-sphere $S^3$, one can define the colored Jones polynomial $J_N(K; t)$ as the quantum invariant corresponding to the $N$-dimensional irreducible representation of the Lie algebra $sl(2; \mathbb{C})$ [3] (see also [8]).

The volume conjecture [6, 14] states that the limit of $\log(J_N(K; \exp(2\pi \sqrt{-1}))) / N$ would determine the simplicial volume of the knot complement $S^3 \setminus K$. In [15], Y. Yokota and the author proved that for the figure-eight knot $E$ and a complex number $r$ the limit $\log(J_N(E; \exp(2\pi r \sqrt{-1}))) / N$ also exists and defines the volume for the three-manifold obtained from $S^3$ by certain Dehn surgery if $r$ is close to $1$.

In this paper we will show that a similar phenomenon appears for torus knots, which are not hyperbolic. We also propose a version of the volume conjecture which relates the limit of $\log(J_N(K; \exp(2\pi r \sqrt{-1}))) / N$ to the volume function corresponding to a representation of $\pi_1(S^3 \setminus K)$ to $SL(2; \mathbb{C})$ such that ratio of the eigenvalues of its image of the meridian is $\exp(2\pi r \sqrt{-1})$. Our version of generalization of the volume conjecture is different from S. Gukov’s [4] due to a choice of polarization.

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Conjecture 2.1 (Parameterized Volume Conjecture). There exists an open subset $O_K$ of $\mathbb{C}$ such that for any $u \in O_K$ the following limit exists:

\[
\lim_{N \to \infty} \frac{\log J_N(K; \exp((u+2\pi\sqrt{-1})/N))}{N}.
\]

Moreover the function of $u$

\[
H(K; u) := (u+2\pi\sqrt{-1}) \lim_{N \to \infty} \frac{\log J_N(E; \exp((u+2\pi\sqrt{-1})/N))}{N}
\]

is analytic on $O_K$. If we put

\[
v_K(u) := 2 \frac{dH(K; u)}{du} - 2\pi\sqrt{-1},
\]

then the following formula holds:

\[
V(K; u) = \text{Im}(H(K; u)) - \pi \text{Re}(u) - \frac{1}{2} \text{Re}(u) \text{Im}(v_K(u)),
\]

where $V(K; u)$ is the volume function corresponding to the representation from $\pi_1(S^3 \setminus K)$ to $SL(2; \mathbb{C})$ sending the meridian and the longitude to the elements the ratios of whose eigenvalues are $\exp(u)$ and $\exp(v_K(u))$ respectively [11 § 4.5].

Remark 2.2. If $u$ is parameterized by a real number $t$, then $V(K; u(t))$ satisfies the following differential equation from Schl"afli's formula:

\[
\frac{dV(K; u(t))}{dt} = -\frac{1}{2} \left( \text{Re}(u(t)) \frac{d\text{Im}(v_K(u(t)))}{dt} - \text{Re}(v_K(u(t))) \frac{d\text{Im}(u(t))}{dt} \right).
\]

See [10 § 5] and [11 § 4.5]. Note that we use the same convention for the meridian/longitude pair as in [10], which is different from that in [11] and [11]. Note also that the right hand side of the equation in the last line of Page 62 of [11] should be multiplied by four (see [11 (5.6)]).

Remark 2.3. If there exist coprime integers $p$ and $q$ satisfying $pu+qv_K(u) = 2\pi\sqrt{-1}$, then $u$ would define the $(p, q)$-Dehn surgery along $K$. If this is a hyperbolic manifold, $V(K; u)$ is its hyperbolic volume.

Remark 2.4. The open set $O_K$ may not contain $0$. Therefore Conjecture 2.1 is not a generalization of the volume conjecture. (Recall that the volume conjecture states that when $u = 0$, then the limit 2.1 gives the simplicial volume of the knot complement.) In fact the case where $K$ is a torus knot, the limit 2.1 is not continuous at 0 [11, 12]. See also [11 Proposition B.2].

Remark 2.5. S. Garoufalidis and T. Le proved that if $u$ is close enough to $-2\pi\sqrt{-1}$, then $J_N(K; \exp((u+2\pi\sqrt{-1})/N))$ converges to $1/\Delta(K; \exp(u+2\pi\sqrt{-1}))$, where $\Delta(K; t)$ is the Alexander polynomial of $K$ [2]. (See also [11 for the figure-eight knot.) In this case the right hand side of 2.1 vanishes and so we have $H(K; u) = 0$ and $v_K(u) = -2\pi\sqrt{-1}$. Therefore from 2.3 the volume function $V(K; u)$ vanishes. This corresponds to the case where $u$ defines an abelian representation, whose volume function is zero. See [3 Appendix B]. So we exclude such a case in Conjecture 2.1.

Note that the conjecture above is proved for the figure-eight knot by Yokota and the author [15]. In fact we proved [15 Corollary 2.4] that for the figure-eight knot $E$

\[
V(E; u) = \text{Im}(H(E; u)) - \pi \text{Re}(u) - \frac{1}{4} \text{Im}(u v_E(u)) - \frac{\pi}{2} \text{length}(\gamma),
\]
where \(\text{length}(\gamma)\) is the length of the geodesic \(\gamma\) added to complete the incomplete hyperbolic structure of \(S^3 \setminus E\) corresponding to \(u\). But since \(\text{length}(\gamma) = -\text{Im}(u v_E(u))/(2\pi)\) from [15] (34)], we have
\[
\begin{align*}
V(E; u) &= \text{Im}(H(E; u)) - \pi \text{Re}(u) - \frac{1}{4}\text{Im}(u v_E(u)) + \frac{1}{4}\text{Im}(u v_E(u)) \\
&= \text{Im}(H(E; u)) - \pi \text{Re}(u) - \frac{1}{2}\text{Re}(u)\text{Im}(v_E(u))
\end{align*}
\]
as required.

See also [13] (4.1). (The sign of \(\pi \text{Re}(u)\) in [13] (4.1)) should be negative because the author used a wrong definition of the function \(H\) in the old version of [15].)

Note also that Gukov uses a different polarization in his generalization of the volume conjecture [1] (5.12)]. It agrees with Conjecture 2.1 when \(\text{Re}(u) = 0\). The difference for \(\text{Re}(u) \neq 0\) can be explained by a choice of polarization. Details will be described in a forthcoming paper.

3. PROOF

In this section we prove Conjecture 2.1 for torus knots. (To be honest, 2.3] is proved only up to a constant, that is, we prove 2.4] instead.)

Let \(T(a, b)\) be the torus knot of type \((a, b)\), where \(a\) and \(b\) are coprime integers with \(a > 1\) and \(b > 1\). Then the author proved in [12] Theorem 1.1] the following theorem.

**Theorem 3.1** [12]. Suppose that \(|r| > 1/(ab)\), \(\text{Re} r > 0\) and \(\text{Im} r > 0\), then
\[
\lim_{N \to \infty} \frac{\log J_N (T(a, b); \exp(2\pi r \sqrt{-1}/N))}{N} = \left(1 - \frac{1}{2abr} - \frac{abr}{2} \right) \pi \sqrt{-1}.
\]

Therefore the functions \(H(T(p, q); u)\) and \(v_{T(a, b)}(u)\) in Conjecture 2.1 are defined as follows:
\[
H(T(p, q); u) = (u + 2\pi \sqrt{-1}) \left(1 - \frac{1}{2ab \left(1 + \frac{u}{2\pi \sqrt{-1}}\right)} - \frac{ab \left(1 + \frac{u}{2\pi \sqrt{-1}}\right)}{2}\right) \pi \sqrt{-1}
\]
and
\[
v_{T(a, b)}(u) = 2 \frac{d}{du} H(T(a, b); u) - 2\pi \sqrt{-1} = -ab(u + 2\pi \sqrt{-1})
\]
for \(u\) with \(|u + 2\pi \sqrt{-1}| > 2\pi/(ab)\), \(\text{Re}(u) < 0\) and \(\text{Im}(u) > -2\pi\).

So the volume function \(V(T(a, b); u)\) is
\[
V(T(a, b); u) = \text{Im} \left(H(T(a, b); u) - \pi \text{Re}(u) - \frac{1}{2}\text{Re}(u)\text{Im}(v_{T(a, b)}(u))\right)
\]
\[
= \frac{1}{4} ab \text{Im}(u^2) - ab\pi \text{Re}(u) + \pi \text{Re}(u)
\]
\[
- \pi \text{Re}(u) - \frac{1}{2}\text{Re}(u)(-ab \text{Im}(u) - 2ab\pi)
\]
\[
= 0.
\]

On the other hand the right hand side of 2.4] equals
\[
- \frac{1}{2} \left(\text{Re}(u(t))(-ab) \frac{d}{dt} \text{Im}(u(t)) + ab\text{Re}(u(t)) \frac{d}{dt} \text{Im}(u(t))\right) = 0,
\]
which proves (2.4).

This confirms Conjecture (2.1) for the torus knot $T(a, b)$.

**Remark 3.2.** The volume function for a torus knot would be zero. See [3] Appendix B.

**Remark 3.3.** Note that the pair $(u, v_{T(a, b)}(u))$ satisfies the equality:

$$(−1) \times u + \left(\frac{-1}{ab}\right) \times v_{T(a, b)}(u) = 2π\sqrt{-1}.$$  

So the corresponding geometric object would be the generalized Dehn surgery along the torus knot $T(a, b)$ with invariant $(1, 1/(ab))$, or the $ab$-Dehn surgery with cone-angle $2\pi ab$. [4] Chapter 4, § 4.5] (see also Remark 2.4). It would be interesting that the $ab$-Dehn surgery along the torus knot $T(a, b)$ is reducible [11].

### 4. Comments

In [11] (5.29)], Gukov proposed the following conjecture.

**Conjecture 4.1.** [4] Let $K$ be a knot in the three-sphere. For $a \in \mathbb{C} \setminus \mathbb{Q}$, define the function $l(a)$ as follows:

$$(4.1) \quad l(a) := -\frac{d}{da} \left\{ a \lim_{N \to \infty} \frac{\log J_N(K; \exp(2\pi a\sqrt{-1}/N))}{N} \right\}.$$  

Then the pair $(\exp(l(a)), −\exp(\pi a \sqrt{-1}))$ is a zero of the $A$-polynomial of $K$ introduced in [1].

Using our parameterization $u := 2\pi a\sqrt{-1} − 2\pi \sqrt{-1}$, we have $l(a) = −v_R(u)/2 − π\sqrt{-1}$. Then Conjecture 4.1 states that the pair $(−\exp(−v_R(u)/2), \exp(u/2))$ is a zero of the $A$-polynomial.

In the case of the figure-eight knot, we can prove this. For the torus knot $T(a, b)$, either $−\exp(−v_R(u)/2), \exp(u/2))$ or $−\exp(v_R(u)/2), \exp(u/2))$ is a zero of the $A$-polynomial. (See, for example, [15] Example 4.1 for the $A$-polynomials of torus knots.) This would depend on how we choose a meridian/longitude pair.

**Remark 4.2.** If $|a|$ is small enough the right hand side of (4.1) vanishes [3]. This corresponds to the $(1-1)$-factor of the $A$-polynomial [11 2.5]. See Remark 2.5.

The function $H(K; u)$ defined by [22] is a kind of potential function introduced in [15] Theorem 3]. (To be more precise, $\Phi(u) = 4H(K; u) − 4\pi u\sqrt{-1}$, where $\Phi(u)$ is Neumann–Zagier’s potential function.) The following observation indicates a relation between $H(K; u)$ and the Alexander polynomial.

**Observation 4.3.** Let $\Delta(K; t)$ be the Alexander polynomial of a knot $K$. For the figure-eight knot and torus knots, the equations $H(K; u) = 0$ and $\Delta(K; \exp(u)) = 0$ share a root. Note that here we regard $H(K; u)$ as a function defined on the whole complex plane.

**Proof.** First note that $\Delta(E; t) = −t^2 + 3t − 1$ and $\Delta(T(a, b); t) = (t^{ab} − 1)(t − 1)/(t^a − 1)(t^b − 1)$.

For the figure-eight knot $E$, we have from [15]

$H(E; u) = \text{Li}_2(y^{-1} \exp(-u)) - \text{Li}_2(y \exp(-u)) + (\log(-y) + \pi\sqrt{-1}) u,$

where $y$ is a solution to the equation $y + y^{-1} = 2\cosh(u) − 1$. Put $u := \pm \text{arcosh}(3/2)$. Then $y = 1$ and so $H(E; u)$ vanishes. Note that we take the branch of log so that $\log(-1) = -\pi\sqrt{-1}$. It is easy to see that $\exp(\pm \text{arcosh}(3/2))$ are the roots of $\Delta(E; \exp(u)) = 0$.

For the torus knot $T(a, b)$ the zero of $H(T(a, b); u)$ is $2\pi\sqrt{-1}/(ab) − 2\pi\sqrt{-1}$. It is also a zero of $\Delta(T(a, b); \exp(u))$. □
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