The Landau problem and noncommutative quantum mechanics

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The conditions under which noncommutative quantum mechanics and the Landau problem are equivalent theories is explored. If the potential in noncommutative quantum mechanics is chosen as \( V = \Omega \hat{N} \) with \( \hat{N} \) defined in the text, then for the value \( \hat{\theta} = 0.22 \times 10^{-11} \) cm\(^2\) (that measures the noncommutative effects of the space), the Landau problem and noncommutative quantum mechanics are equivalent theories in the lowest Landau level. For other systems one can find different values for \( \hat{\theta} \) and, therefore, the possible bounds for \( \hat{\theta} \) should be searched in a physical independent scenario. This last fact could explain the different bounds for \( \hat{\theta} \) found in the literature.

The presence of magnetic fields in the string effective action suggest a noncommutative structure for the spacetime. This fact and the possibility of compactifying string theory via the noncommutative torus \( \$ \) have stimulated an important amount of work during the last three years \( \$ \).

A noncommutative space is an intriguing and revolutionary possibility that could have important consequences in our conception of the quantum structure of nature. A noncommutative space is related to the fundamental new commutation relation

\[ [x, y] \sim \alpha \theta, \quad (1) \]

where we will define \( \alpha \theta \) as \( \hat{\theta} \) that is a parameter with dimensions (length\(^2\)).

In a previous paper \( \$ \) the noncommutative quantum mechanics for a two dimensional central field was studied and an explicit realization of the Seiberg-Witten map, at quantum mechanical level, was found. In this letter we would like to argue that the usual Landau problem also can be understood in terms of noncommutative quantum mechanics. Additionally, we will show that the experimental value for the magnetic field in the QHE \( \$ \) implies

\[ \hat{\theta} \sim 0.22 \times 10^{-11} \text{ cm}^2, \quad (2) \]

\textit{i.e.} the characteristic magnetic length can be derived from noncommutative quantum mechanics.

Let us start with the Moyal product for potential term in the Schrödinger equation in a noncommutative plane

\[ V(x) \star \psi(x) = V(x - \frac{1}{2} \hat{\theta}) \psi(x), \quad (3) \]

where \( \star \) denotes the Moyal product and \( \hat{\theta}_{ij} = \theta^{ij} j_k p_k \).

Then for a two dimensional central field \( \H \) \( V(|x|^2) \), \( \$ \) yields

\[ V(|x|^2) \star \psi(x) = V(\hat{\theta}) \psi(x), \quad (4) \]

where the aleph (\( \H \)) operator is defined as

\[ \hat{\theta} = \frac{\theta^2}{4} p_x^2 + x^2 + \frac{\theta^2}{4} p_y^2 + y^2 - \theta L_z \]

\[ = \hat{H}_{HO} - \theta \hat{L}_z. \quad (5) \]

In the last expression \( \hat{H}_{HO} \) is the hamiltonian for a two dimensional harmonic oscillator with mass \( 2/\theta^2 \), frequency \( \omega = \theta \) and \( L_z \) is the z-component of the angular momentum defined as \( L_z = xp_y - yp_x \).

From \( \H \) one see that \( \H \) has (length\(^2\)) and, furthermore, the dimensions of \( \theta \) are time/mass. This last fact imply that in \( \H \) one must choose \( \alpha = \hbar \) in \( \H \), i.e.

\[ \hat{\theta} = \hbar \theta, \quad (7) \]

Thus, from \( \H \) one can think that \( \hat{\theta} \) effectively measure the noncommutative effects of the space.

The eigenstates and the eigenvalues of the \( \H \) operator were computed in \( \H \) and the result is

\[ \hat{\theta} jm >= \theta [2j + 1 - 2m] jm >, \quad (8) \]

with the selection rules

\[ j = 0, 1, 2, 1, 3, 2, ..., \]

\[ m = j, j - 1, j - 2, ..., -j. \quad (9) \]

Thus, the hamiltonian for noncommutative quantum mechanics in a central field becomes

\[ \hat{H} = \frac{1}{2M} p_x^2 + V(\hat{\theta}). \quad (10) \]

If we choose the potential

\[ \begin{array}{l}
\end{array} \]
being $\Omega$ an appropriate constant, then the hamiltonian can be written as
\[
H = \frac{1}{2M} + \frac{\Omega \theta^2}{4} (p_x^2 + p_y^2) + \Omega (x^2 + y^2) - \Omega \theta L_z.
\] (12)

Our next step is to consider the two-dimensional Landau problem in the symmetric gauge, whose Hamiltonian for a particle with mass $\mu$ is given by
\[
\hat{H}_{\text{Landau}} = \frac{1}{2\mu} (p_x^2 + p_y^2) + \frac{e^2 \hbar^2}{8\mu} (x^2 + y^2) - \frac{eH_0}{2\mu} L_z.
\] (13)

Now we note from (12) and (13) that these two problems are equivalent. This equivalence means that the magnetic field must be strong enough in order to confine the particles in the plane $x-y$. After this identification, the following relations are satisfied:
\[
\frac{1}{2M} + \frac{\Omega \theta^2}{4} = \frac{1}{2\mu} \quad \Omega \theta = \frac{eH_0}{2\mu}.
\] (14) (15) (16)

These equations are consistent if and only if $M = \infty$. Furthermore, the Hamiltonians (12) and (13) describes the same noncommutative system in the lowest Landau level, i.e. in the strong regime of the magnetic field.

Using (13) and (16) one find that
\[
\tilde{\theta} = \frac{4\hbar}{eH_0},
\] (17)

In the QHE experiments ([3], the magnetic fields are typically about 12T. In this way we get
\[
\tilde{\theta} = 0.22 \times 10^{-11} \text{ cm}^2.
\] (18)

For this value of $\tilde{\theta}$ one cannot distinguish between noncommutative quantum mechanics [4] and the usual Landau problem. This route was explored by Bellisard [5] and other authors [1, 11] using different points of view (for other bounds for $\tilde{\theta}$ see [1] and references therein). Qualitatively one could also observe noncommutative effects in the Aharonov-Bohm experiments as was proposed in [10].

Finally we would like to point out the following: 1) the magnetic field in equation (17) cannot be arbitrarily small due to the identification we have done assuming the existence of the lowest Landau level and 2) the value for $\tilde{\theta}$ is strongly model dependent. Therefore, if spacetime is physically realized as a noncommutative structure, then the bounds for $\tilde{\theta}$ should have a universal character and should be independent of a particular physical scenario. One possibility could be to explore Lorentz violation invariance as in [11].

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