Matched Queues with Matching Batch Pair \((m, n)\)

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Abstract

In this paper, we discuss an interesting but challenging bilateral stochastically matching problem: A more general matched queue with matching batch pair \((m, n)\) and two types (i.e., types A and B) of impatient customers, where the arrivals of A- and B-customers are both Poisson processes, \(m\) A-customers and \(n\) B-customers are matched as a group which leaves the system immediately, and the customers’ impatient behavior is to guarantee the stability of the system. We show that this matched queue can be expressed as a novel bidirectional level-dependent quasi-birth-and-death (QBD) process. Based on this, we provide a detailed analysis for this matched queue, including the system stability, the average stationary queue lengths, and the average sojourn time of any A-customer or B-customer. We believe that the methodology and results developed in this paper can be applicable to dealing with more general matched queueing systems, which are widely encountered in various practical areas, for example, sharing economy, ridesharing platform, bilateral market, organ transplantation, taxi services, assembly systems, and so on.

Keywords: Queueing; Markov process; Matched Queue; impatient customer; RG-factorization.

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1 Introduction

In this paper, we consider an interesting but challenging bilateral stochastically matching problem: A more general matched queue with matching batch pair \((m, n)\) and two types of impatient customers, where \(m\) A-customers and \(n\) B-customers are matched as a group which leaves the system immediately; and the customers’ impatient behavior is to guarantee the stability of the system. We show that the matched queue with matching batch pair \((m, n)\) can be expressed as a novel bidirectional level-dependent QBD process. Based on this, we can conduct a detailed analysis for the matched queue with matching batch pair \((m, n)\), such as the system stability, the average stationary queue lengths, the average sojourn time of any A- or B-customer, and so on. To our best knowledge, this paper is the first to study such a bilateral matched queue with matching batch pair \((m, n)\), in which matching batch pair \((m, n)\) always makes the analysis of the matched queue more complicated and difficult. To deal with this problem, we develop some new theory of bidirectional QBD processes including the system stable conditions, the stationary probability vectors and the first passage times.

Such a matched queue with matching batch pair \((m, n)\) has been widely encountered in many different practical areas. Important examples include: Organ transplantation by Zenios [64], Boxma et al. [8], Stanford et al. [56], and Elalouf et al. [24]; taxi services by Giveen [26, 27], Kashyap [35, 36, 37], Bhat [7], Baik et al. [4], Shi and Lian [54], and Zhang et al. [65]; baggage claim by Browne et al. [12]; sharing economy by Cheng [16], Sutherland and Jarrahi [58], and Benjaafar and Hu [6]; assembly systems by Hopp and Simon [30], Som et al. [55], and Ramachandran and Delen [51]; health care by Pandey and Gangeshwer [49]; multimedia synchronization by Steinmetz [57] and Parthasarathy et al. [50]; and so on. Recently, an emerging hot research topic of matched queues has been focused on ridesharing platform. Many ridesharing companies spring up as a result of rapid development of mobile networks, smart phones and location technologies, for example, Uber in transportation, Airbnb in housing, Eatwith in eating, Rent the Runway in dressing, and so on. For details of these examples, readers may refer to, such as Azevedo and Weyl [3], Duenyas et al. [23], Hu and Zhou [31], Banerjee and Johari [5] and Braverman et al. [9]. Obviously, the matched queues play a key role in research and applications of ridesharing platforms.

The matched queues are a type of interesting and classic systems in early research of
queuing systems. However, the available methodologies and results are still fewer than those in other types of classic queueing systems, for example, processor-sharing queues by Yashkov [61] and Yashkov and Yashkov [62], retrial queues by Falin and Templeton [25] and Artalejo and Gómez-Corral [2], fluid queues by Chapter 7 in Li [42], and so forth. Although a matched queue seems simple due to having only a few random factors, its analysis is actually difficult and challenging due to the bidirectional state space \{\ldots, -2, -1, 0, 1, 2, \ldots\} whose discussion is still quite rare up to now. To this end, we summarize the main literature of matched queues from the following three aspects:

**Matching batch pair (1, 1).** Early research of matching queues first focused on some simple double-ended systems with matching batch pair (1, 1). Also, the matching queues with matching batch pair (1, 1) have attracted numerous researchers’ attention since a pioneering work by Kendall [38], and crucially, some effective methodologies and available results have been developed from multiple research perspectives, which are listed as follows.

The Markov processes: They were the first effective method employed by early research of matched queues. Almost at the same time, for a simple matched queue, Sasieni [52], Giveen [26] and Dobbie [22] established the Chapman-Kolmogorov forward differential-difference equations, in which the customers’ impatient behavior is also introduced to guarantee the stability of the system. Since then, the Markov process analysis of matched queues were further developed from two different research lines as follows:

(a) A finite state space. When two waiting rooms of the matched queue are both finite, the Markov process is established with a finite state space. In this case, Jain [32] and Kashyap [35, 36, 37] applied the supplementary variable method to deal with the matched queue with a Poisson arrival process and a renewal arrival process. Takahashi et al. (2000) considered the matched queue with a Poisson arrival process and a PH-renewal arrival process. In addition, Sharma and Nair (1991) used the matrix theory to analyze the transient behavior of a Markovian matched queue. Chai et al. [15] considered a batch matching queueing system with impatient servers and bounded rational customers, and each server serves the customers in batches with finite service capacity.

(b) An infinite state space. When two waiting rooms of the matched queue are both infinite, the Markov process is set up with a bidirectional infinite state space. In general, it is always difficult and challenging to analyze such a Markov process with a bidirectional infinite state space. Latouche [10] applied the matrix-geometric solution to analyze several
bilateral matching queues with paired input. Conolly et al. [17] applied the Laplace transform to discuss the time-dependent performance measures of the matched queue with state-dependent impatience. Di Crescenzo et al. [19, 20] discussed the transient and stationary probabilities of a time-nonhomogeneous matched queue with catastrophes and repairs. Diamant and Baron [21] analyzed a matching queue with priority and impatient customers.

The fluid and diffusion approximations: In a matched queue, if the arrivals of A- and B-customers are both general renewal processes, then the fluid and diffusion approximations become an effective (but approximative) method. Jain [34] applied diffusion approximation to discuss the \(G^X/G^Y/1\) matched queue. Di Crescenzo et al. [19, 20] discussed a matched queue by means of a jump-diffusion approximation. Liu et al. [47] discussed some diffusion models for the matched queues with renewal arrival processes. Büke and Chen [14] applied the fluid and diffusion approximations to study the probabilistic matching systems. Liu [46] used the diffusion approximation to analyze the matched queues with reneging in heavy traffic.

Other effective methods: Kim et al. [39] provided a simulation model to analyze a more general matched queue. Jain [33] proposed a sample path analysis for studying the matched queue with time-dependent rates. Afeche et al. (2014) applied the level-cross method to discuss the matched batch queue with abandonment.

Control of matched queues: Hlynka and Sheahan [29] analyzed the control rates in a matched queue with two Poisson inputs. Gurvich and Ward [28] discussed dynamic control of the matching queues. Büke and Chen [13] studied stabilizing admission control policies for the probabilistic matching systems. Lee et al. [41] discussed optimal control of a time-varying double-ended production queueing model.

Matching batch pair \((1,n)\). As a key extension and generalization, Xu et al. [60] first discussed a matched queue with matching batch pair \((1,n)\), in which for the two waiting rooms, one is finite while another is infinite. Under two Poisson inputs and a PH service time distribution, they applied the matrix-geometric solution to obtain the system stable condition, and to study the stationary queue lengths for the both classes of customers. Yuan [63] applied Markov chains of M/G/1 type to consider a matched queue with matching batch pair \((1,n)\) and under two Poisson inputs and a general service time distribution. Further, Li and Cao [43] discussed the matched queue with matching batch pair \((1,n)\) and under two batch Markovian arrival processes (BMAPs) and a general
Matching batch pair \((m, n)\). To our best knowledge, this paper is the first to study the matched queues with matching batch pair \((m, n)\), in which the two waiting rooms are both infinite. For this matched queue, we express it as a bidirectional level-dependent QBD process, and apply the RG-factorizations proposed by Li \[42\] to obtain the average stationary queue lengths, and the average sojourn time of any A- or B-customer. Note that Liu et al. \[45\] conducted a close work for studying the block-structure matched queue with matching batch pair \((1, 1)\) and under two Markovian arrival processes (MAPs). Different from Liu et al. \[45\] and since the matching batch pair \((m, n)\) always makes the block structure of the Markov process more complicated from the bidirectional infinity, to deal with the difficulty of analyzing the sojourn times, a new phase-type distribution of bidirectional infinity is set up by means of the RG-factorizations in this research.

Based on the above analysis, we summarize the main contributions of this paper as follows:

(1) We describe a more general matched queue with matching batch pair \((m, n)\) and impatient customers, where \(m\) A-customers and \(n\) B-customers are matched as a group which leaves the system immediately, and the customers’ impatient behavior is also taken into account to guarantee the stability of the system.

(2) We express the matched queue with matching batch pair \((m, n)\) as a bidirectional level-dependent QBD process, and apply the RG-factorizations developed by Li \[42\] to obtain the stationary probability vector of bidirectional infinity, which is used for analyzing the average stationary queue lengths.

(3) We study the average sojourn time of any A- or B-customer by means of the total probability formula, and provide an upper bound of this average sojourn time by using a new phase-type distribution of bidirectional infinity, whose computation is established in terms of the RG-factorizations given in Li \[42\].

The structure of this paper is organized as follows. Section 2 describes a more general matched queue with matching batch pair \((m, n)\) and impatient customers. Section 3 expresses this matched queue as a new bidirectional level-dependent QBD process. Based on this, the system stable condition is given. Section 4 studies the stationary probability vector of the bidirectional QBD process, and computes the average stationary queue lengths.
of the A- and B-customers, respectively. Section 5 discusses the average sojourn time of any A- or B-customer, and provides a better upper bound of this average sojourn time by using a new phase-type distribution of bidirectional infinity. Finally, some concluding remarks are given in Section 6.

2 Model Description

In this section, we describe a more general matched queue with matching batch pair \((m,n)\) and impatient customers, and also introduce operational mechanism, system parameters and basic notation.

In the matched queue with matching batch pair \((m,n)\), \(m\) A-customers and \(n\) B-customers are matched as a group which leaves the system immediately once the matching is successful, and also the customers’ impatient behavior is used to guarantee the stability of the system. Figure 1 provides a physical illustration for such a matched queue.

![Figure 1: A physical illustration of the matched queue](image)

Now, we provide a more detailed description for the matched queue as follows:

(1) **Arrival processes.** The A- and B-customers arrive at the queueing system according to the Poisson processes with rates \(\lambda_1\) and \(\lambda_2\), respectively.

(2) **Matched processes.** Once \(m\) A-customers and \(n\) B-customers match as a group, which immediately leaves the queueing system for \(m,n > 0\). Their matching process follows a First-Come-First-Match discipline. We assume that two waiting spaces of A- and B-customers are both infinite.

(3) **Impatient behavior.** If an A-customer (resp. a B-customer) stays at the queue-
ing system for a long time, then she will have some impatient behavior. We assume that
the impatient time of an A-customer (resp. a B-customer) is exponentially distributed
with impatient rate $\theta_1$ (resp. $\theta_2$) for $\theta_1, \theta_2 > 0$.

We assume that all the random variables defined above are independent of each other.

**Remark 1**

(a) The customers’ impatient behavior given in Assumption (3) is used to
ensure the stability of the matched queue.

(b) The matching discipline given in Assumption (2) indicates that over $m$ A-customers
and over $n$ B-customers cannot simultaneously exist in their respective waiting spaces.

### 3 A Bidirectional QBD Process

In this section, we use a new bidirectional level-dependent QBD process to express the
matched queue with matching batch pair $(m, n)$ and impatient customers, and obtain a
sufficient condition under which this matched queue is stable.

We denote by $N_1(t)$ and $N_2(t)$ the numbers of A- and B-customers in the matched
queue at time $t \geq 0$, respectively. Then the matched queue with matching batch pair
$(m, n)$ is related to a two-dimensional Markov process $\{(N_1(t), N_2(t)), t \geq 0\}$. Note
that once $m$ A-customers and $n$ B-customers match as a group, which immediately leaves
the queueing system, thus over $m$ A-customers and over $n$ B-customers cannot simultane-
ously exist in their waiting spaces. Based on this, the state space of the Markov process
$\{(N_1(t), N_2(t)), t \geq 0\}$ is given by

$$
\Omega = \{(i, j) : 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\} \cup \{(i, j) : i \geq m, 0 \leq j \leq n - 1\}
\cup \{(i, j) : 0 \leq i \leq m - 1, j \geq n\}.
$$

In this case, we write

Level 0 $= \{(0, 0), (0, 1), \ldots, (0, n - 1); (1, 0), (1, 1), \ldots, (1, n - 1);$

$\ldots; (m - 1, 0), (m - 1, 1), \ldots, (m - 1, n - 1)\},$

for $k \geq 1,$

Level $k = \{(km, 0), (km, 1), \ldots, (km, n - 1); (km + 1, 0), (km + 1, 1), \ldots, (km + 1, n - 1);$

$\ldots; (km + (m - 1), 0), (km + (m - 1), 1), \ldots, (km + (m - 1), n - 1)\},$
and $l \leq -1$,

Level $l = \{(0, (-l) n), (1, (-l) n), \ldots, (m - 1, (-l) n); (0, (-l) n + 1), (1, (-l) n + 1), \ldots, (m - 1, (-l) n + 1); \\
\ldots; (0, (-l) n + n - 1), (1, (-l) n + n - 1), \ldots, (m - 1, (-l) n + n - 1)\}$.

Therefore, we have

$$\Omega = \bigcup_{k=-\infty}^{\infty} \text{Level } k.$$  

**Example one:** As an illustrated example, we take $m = 2$ and $n = 3$. In this case, the state transition relations of Markov process \{$(N_1(t), N_2(t)), t \geq 0$\} is depicted in Figure 2. Also, we observe that each levels is a state set formed by many states in a rectangle (i.e., multiple state lines).

From Levels $k$ for $-\infty < k < \infty$ or Figure 2, it is easy to see that the Markov process \{$(N_1(t), N_2(t)), t \geq 0$\} is a new bidirectional level-dependent QBD process whose infinitesimal generator is given by

$$Q = \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots \\
B_0^{(-3)} & B_1^{(-3)} & B_2^{(-3)} & \\
B_0^{(-2)} & B_1^{(-2)} & B_2^{(-2)} & \\
B_0^{(-1)} & B_1^{(-1)} & B_2^{(-1)} & \\
B_0^{(0)} & A_1^{(0)} & A_2^{(0)} & A_0^{(0)} \\
A_1^{(1)} & A_2^{(1)} & A_0^{(1)} & \cdots \\
A_1^{(2)} & A_2^{(2)} & A_0^{(2)} & \cdots \\
A_1^{(3)} & A_2^{(3)} & A_0^{(3)} & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix}, \quad (1)$$

where

$$A_0^{(k)} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
\lambda_1 \\
\cdots \\
\lambda_1 \\
\vdots \\
(m-1)n \\
\lambda_1 \\
\lambda_1 \\
\vdots \\
0 \cdots 0
\end{pmatrix}_{mn \times mn}, \quad k \geq 0,$$
Figure 2: The state transition relations of the bilateral QBD process

\[
A_2^{(k)} = \begin{pmatrix}
0 & \cdots & km\theta_1 \\
\cdots & \ddots & \cdots \\
\lambda_2 & 0 & \cdots \\
0 & \cdots & 0 \\
\lambda_2 & 0 & \cdots \\
\end{pmatrix}^{nn} \begin{pmatrix}
0 & \cdots & km\theta_1 \\
\cdots & \ddots & \cdots \\
\lambda_2 & 0 & \cdots \\
0 & \cdots & 0 \\
\lambda_2 & 0 & \cdots \\
\end{pmatrix}^{nn} 
\]

, \ k \geq 1,
\[ A_1^{(k)} = \begin{pmatrix} A_{1,1} & A_{2,1} \\ A_{3,1} & A_{1,2} & A_{2,2} \\ & \ddots & \ddots & \ddots \\ & & A_{3,m-2} & A_{1,m-1} & A_{2,m-1} \\ & & & A_{3,m-1} & A_{1,m} \end{pmatrix}_{mn \times mn}, \quad k \geq 0, \]

\[ A_{1,i} = \begin{pmatrix} a_{1,i}^{(1)} & \lambda_2 \\ \theta_2 & a_{1,i}^{(2)} & \lambda_2 \\ & \ddots & \ddots & \ddots \\ & & (n-2) \theta_2 & a_{1,i}^{(n-1)} & \lambda_2 \\ & & & (n-1) \theta_2 & a_{1,i}^{(n)} \end{pmatrix}_{n \times n}, \quad 1 \leq i \leq m, \]

\[ a_{1,i}^{(r)} = - (\lambda_1 + \lambda_2 + (r-1) \theta_2 + (km + i - 1) \theta_1), \quad 1 \leq r \leq n, \]

\[ A_{2,i} = \begin{pmatrix} \lambda_1 \\ \lambda_1 \\ & \ddots \\ & & \lambda_1 \end{pmatrix}_{n \times n}, \quad 1 \leq i \leq m-1, \]

\[ A_{3,i} = \begin{pmatrix} (km + i) \theta_1 \\ & \ddots \\ & & (km + i) \theta_1 \end{pmatrix}_{n \times n}, \quad 1 \leq i \leq m-1, \]

\[ B_0^{(0)} = \begin{pmatrix} 0 \\ \vdots \\ \lambda_2 \end{pmatrix}_{n \times n}, \quad (n-1)m \]

\[ B_0^{(0)} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots \\ \lambda_2 \end{pmatrix}_{n \times n}, \quad (n-1)m \]
\[
B_2^{(-1)} = \begin{pmatrix}
\lambda_1 \\
\lambda_1 \\
\vdots \\
0
\end{pmatrix}^m, \quad (n - 1)^m, \quad l \leq -1,
\]

\[
B_0^{(l)} = \begin{pmatrix}
0 & \lambda_1 \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}^m, \quad (n - 1)^m, \quad l \leq -2,
\]

\[
B_2^{(l)} = \begin{pmatrix}
0 & \lambda_1 \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}^m, \quad (n - 1)^m, \quad l \leq -2,
\]

\[
B_2^{(l)} = \begin{pmatrix}
0 & \lambda_1 \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}^m, \quad (n - 1)^m, \quad l \leq -2,
\]
Now, we discuss the stability of the bidirectional QBD process $Q$. Note that such a bidirectional QBD process was first introduced in Latouche \[40\] and was further developed in Li and Cao \[44\].

To analyze the matched queue with matching batch pair $(m,n)$ and impatient customers, it is worthwhile to note that a simple relation between Levels 0 and $-1$ is a key. From Levels 0 and $-1$, we can divide the bidirectional QBD process $Q$ into two unilateral QBD processes: $Q_A$ and $Q_B$. Based on this, the infinitesimal generators of the two
unilateral QBD processes $Q_A$ and $Q_B$ are respectively given by

$$Q_A = \begin{pmatrix}
  A_1^{(0)} & A_0^{(0)} \\
  A_1^{(1)} & A_0^{(1)} \\
  A_1^{(2)} & A_0^{(2)} \\
  A_1^{(3)} & A_0^{(3)} \\
  \vdots & \vdots & \vdots \\
  A_1^{(0)} & A_0^{(0)} \\
  A_1^{(1)} & A_0^{(1)} \\
  A_1^{(2)} & A_0^{(2)} \\
  A_1^{(3)} & A_0^{(3)} \\
  \vdots & \vdots & \vdots
\end{pmatrix}$$

and

$$Q_B = \begin{pmatrix}
  B_1^{(-1)} & B_0^{(-1)} \\
  B_2^{(-2)} & B_1^{(-2)} & B_0^{(-2)} \\
  B_2^{(-3)} & B_1^{(-3)} & B_0^{(-3)} \\
  B_2^{(-4)} & B_1^{(-4)} & B_0^{(-4)} \\
  \vdots & \vdots & \vdots
\end{pmatrix}$$

or

$$Q_B = \begin{pmatrix}
  \ldots & \ldots & \ldots & \ldots \\
  B_0^{(-4)} & B_1^{(-4)} & B_2^{(-4)} \\
  B_0^{(-3)} & B_1^{(-3)} & B_2^{(-3)} \\
  B_0^{(-2)} & B_1^{(-2)} & B_2^{(-2)} \\
  B_0^{(-1)} & B_1^{(-1)} & B_2^{(-1)} \\
  \ldots & \ldots & \ldots & \ldots
\end{pmatrix}.$$

In the remainder of this section, we study the stability of the matched queue with matching batch pair $(m, n)$ and impatient customers. It is easy to see that the customers’ impatient behavior plays a key role in the stable analysis.

The following theorem provides a sufficient condition under which the bidirectional QBD process is stable.

**Theorem 1** If $(\theta_1, \theta_2) > 0$, then the bidirectional QBD process $Q$ is irreducible and positive recurrent. Thus, the matched queue with matching batch pair $(m, n)$ and impatient customers is stable.

**Proof.** It is clear that the bidirectional QBD process $Q$ is irreducible through observing Figures 1 and 2, since this matched queue contains two Poisson inputs and two exponential impatient times.

From returning to Level 0, it is easy to see that the bidirectional QBD process $Q$ is positive recurrent if and only if the two unilateral QBD processes $Q_A$ and $Q_B$ are
both positive recurrent. Thus, our aim is to prove that if \((\theta_1, \theta_2) > 0\), then the two unilateral QBD processes \(Q_A\) and \(Q_B\) are both positive recurrent by means of the mean drift technique by Neuts [48] and Li [42].

For the QBD process \(Q_A\), let 
\[
A_k = \begin{pmatrix}
A_{1,1,1} & A_{2,1} & A_{2,2} \\
A_{3,1} & A_{1,2,1} & A_{2,2} \\
& \ddots & \ddots \ddots \\
& & A_{3,m-2} & A_{1,m-1,1} & A_{2,m-1} \\
A_0 & & & & A_{3,m-1} & A_{1,m,1}
\end{pmatrix}_{mn \times mn},
\]
where
\[
A_0 = \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_1
\end{pmatrix}_{n \times n},
A_{2,2,2} = \begin{pmatrix}
km \theta_1 \\
\vdots \\
km \theta_1
\end{pmatrix}_{n \times n},
\]
\[
A_{1,i,1} = \begin{pmatrix}
a^{(1)}_{1,i} & \lambda_2 & \lambda_2 & \vdots & \lambda_2 \\
\theta_2 & a^{(2)}_{1,i} & \lambda_2 & \vdots & \lambda_2 \\
& \ddots & \ddots & \ddots & \ddots \\
& & (n-2) \theta_2 & a^{(n-1)}_{1,i} & \lambda_2 \\
& & & (n-1) \theta_2 & a^{(n)}_{1,i}
\end{pmatrix}_{n \times n},
\]
\[
a^{(r)}_{1,i} = - (\lambda_1 + \lambda_2 + (r-1) \theta_2 + (km + i - 1) \theta_1),
\]
\[
A_{2,i} = \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_1
\end{pmatrix}_{n \times n},
\]
\[
A_{3,i} = \begin{pmatrix}
(km + i) \theta_1 \\
(km + i) \theta_1 \\
& \ddots & \ddots \\
& & (km + i) \theta_1
\end{pmatrix}_{n \times n},
\]

Let \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n; \ldots; \alpha_{(m-1)n+1}, \alpha_{(m-1)n+2}, \ldots, \alpha_{(m-1)n+n})\) be the stationary probability vector of the Markov process \(A_k\). Then
\[
\alpha A_k = 0, \ \alpha e = 1.
\]
Note that $\alpha > 0$, since the Markov process $A_k$ is irreducible.

Once the stationary probability vector $\alpha$ is obtained, we can compute the (upward and downward) mean drift rates of the QBD process $Q_A$. From Level $k$ to Level $k + 1$, the upward mean drift rate is given by

$$\alpha A_0^{(k)} e = \lambda_1 (\alpha_{(m-1)n+1} + \alpha_{(m-1)n+2} + \cdots + \alpha_{(m-1)n+n}) .$$

Similarly, from Level $k$ to Level $k - 1$, the downward mean drift rate is given by

$$\alpha A_2^{(k)} e = \lambda_2 (\alpha_n + \alpha_{2n} + \cdots + \alpha_{mn}) + km\theta_1 (\alpha_1 + \alpha_2 + \cdots + \alpha_n) .$$

Note that $k$ is a positive integer, $\lambda_1 > 0$ and $\theta_1 > 0$, it is easy to check that if $k > \max \{1, \lambda_1/m\theta_1\}$, then $\alpha A_0^{(k)} e < \alpha A_2^{(k)} e$. Therefore, the QBD process $Q_A$ is positive recurrent due to the fact that the mean drift rates: $\alpha A_0^{(k)} e < \alpha A_2^{(k)} e$ for a bigger positive integer $k$, this can hold because $k$ goes to infinity.

Similarly, we discuss the stability of the QBD process $Q_B$. Let $B_l = B_0^{(l)} + B_1^{(l)} + B_2^{(l)}$ in Level $l$ for $l \leq -2$. Then

$$B_k = \begin{pmatrix}
B_{1,1,1} & B_{2,1} & & B_0 \\
B_{3,1} & B_{1,2,1} & B_{2,2} & \\
& \ddots & \ddots & \ddots \\
B_{3,n-2} & B_{1,n-1,1} & B_{2,n-1} & \\
B_{2,2,2} & & B_{3,n-1} & B_{1,n,1}
\end{pmatrix}_{mn \times mn},$$

where

$$B_0 = \begin{pmatrix}
\lambda_2 \\
& \ddots \\
& & \lambda_2
\end{pmatrix}_{m \times m}, \quad B_{2,2,2} = \begin{pmatrix}
-ln\theta_2 \\
& \ddots \\
& & -ln\theta_2
\end{pmatrix}_{m \times m},$$

$$B_{1,i,1} = \begin{pmatrix}
b_{1,i}^{(1)} (m-1) \theta_1 \\
\lambda_1 & b_{1,i}^{(2)} (m-2) \theta_1 \\
& \ddots & \ddots \\
& & \lambda_1 & b_{1,i}^{(m-1)} \theta_1 \\
& & & \lambda_1 & b_{1,i}^{(m)} \theta_1
\end{pmatrix}_{m \times m}, \quad 1 \leq i \leq n,$$

$$b_{1,i}^{(r)} = -(\lambda_1 + \lambda_2 + (m-r) \theta_1 + (-ln + n - i) \theta_2), \quad 1 \leq r \leq m,$$
\[
B_{2,i} = \begin{pmatrix}
(-ln + n - i) \theta_2 \\
(-ln + n - i) \theta_2 \\
\vdots \\
(-ln + n - i) \theta_2 \\
\end{pmatrix}_{m \times m}, 1 \leq i \leq n - 1,
\]

\[
B_{3,i} = \begin{pmatrix}
\lambda_2 \\
\lambda_2 \\
\vdots \\
\lambda_2 \\
\end{pmatrix}_{m \times m}, 1 \leq i \leq n - 1.
\]

Let \( \beta = (\beta_1, \beta_2, \ldots, \beta_m; \beta_{(n-1)m+1}, \beta_{(n-1)m+2}, \ldots, \beta_{(n-1)m+m}) \) be the stationary probability vector of the Markov process \( \mathbb{B}_l \). Then

\[
\beta \mathbb{B}_l = 0, \ \beta \mathbf{e} = 1.
\]

Now, we compute the (upward and downward) mean drift rates of the QBD process \( Q_B \).

From Level \( l \) to Level \( l - 1 \), the upward mean drift rate is given by

\[
\beta B_0^{(l)} \mathbf{e} = \lambda_2 (\beta_1 + \beta_2 + \cdots + \beta_m).
\]

Similarly, from Level \( l \) to Level \( l + 1 \), the downward mean drift rate is given by

\[
\beta B_2^{(l)} \mathbf{e} = \lambda_1 \left( \beta_1 + \beta_{m+1} + \cdots + \beta_{(n-1)m+1} \right)
- \ln \theta_2 \left( \beta_{(n-1)m+1} + \beta_{(n-1)m+2} + \cdots + \beta_{(n-1)m+m} \right).
\]

Note that \( l \) is a negative integer, \( \lambda_2 > 0 \) and \( \theta_2 > 0 \), it is easy to check that if \( l < - \max \{1, \lambda_2/n \theta_2\} \), then \( \beta B_0^{(l)} \mathbf{e} < \beta B_2^{(l)} \mathbf{e} \). This holds because \( l \) goes to negative infinity.

Therefore, the QBD process \( Q_B \) is positive recurrent.

Based on the above two analysis, the two QBD processes \( Q_A \) and \( Q_B \) are both positive recurrent. Thus the bidirectional QBD process \( Q \) is irreducible and positive recurrent. This further shows that the matched queue with matching batch pair \( (m, n) \) and impatient customers is stable. This completes the proof.

\[\blacksquare\]

4 The Stationary Queue Length

In this section, we first provide a bilateral matrix-product expression for the stationary probability vector of the bidirectional level-dependent QBD process by means of the RG-
factorizations given in Li [42]. Then we compute two average stationary queue lengths for the A- and B-customers, respectively.

We write
\[ p_{i,j}(t) = P\{N_1(t) = i, N_2(t) = j\}. \]
Since the bidirectional level-dependent QBD process is stable, we have
\[ \pi_{i,j} = \lim_{t \to +\infty} p_{i,j}(t). \]
For \( l = -1, -2, -3, \ldots \), we write
\[ \pi_l = (\pi_{0,0}, \pi_{0,1}, \ldots, \pi_{0,n-1}, \pi_{1,0}, \pi_{1,1}, \ldots, \pi_{1,n-1}, \ldots, \pi_{m-1,0}, \pi_{m-1,1}, \ldots, \pi_{m-1,n-1}); \]
for \( k = 0 \), we write
\[ \pi_0 = (\pi_{0,0}, \pi_{0,1}, \ldots, \pi_{0,n-1}, \pi_{1,0}, \pi_{1,1}, \ldots, \pi_{1,n-1}, \ldots, \pi_{m-1,0}, \pi_{m-1,1}, \ldots, \pi_{m-1,n-1}); \]
for \( k = 1, 2, 3, \ldots \), we write
\[ \pi_k = (\pi_{km,0}, \pi_{km,1}, \ldots, \pi_{km,n-1}, \pi_{km+1,0}, \pi_{km+1,1}, \ldots, \pi_{km+1,n-1}); \]
and
\[ \pi = (\ldots, \pi_{-2}, \pi_{-1}, \pi_0, \pi_1, \pi_2, \ldots). \]

To compute the stationary probability vector of the bidirectional level-dependent QBD process \( Q \), we first need to compute the stationary probability vectors of the two unilateral QBD processes \( Q_A \) and \( Q_B \). Then we use Levels 1, 0 and \(-1\) as three interaction boundary levels, which are used to further determine the stationary probability vectors from the interaction boundary levels to the bilateral internal levels.

Note that the two unilateral QBD processes \( Q_A \) and \( Q_B \) are level-dependent, thus we need to apply the RG-factorization given in Li [42] to calculate their stationary probability vectors. To this end, we need to introduce the \( U \)-, \( R \)- and \( G \)-measures for the two unilateral QBD processes \( Q_A \) and \( Q_B \), respectively. In fact, such a level-dependent QBD process was given a detailed analysis in Li and Cao [44].

For the unilateral QBD process \( Q_A \), we define the \( UL \)-type \( U \)-, \( R \)- and \( G \)-measures as
\[ U_k = A_1^{(k)} + A_0^{(k)} (-U_{k+1}^{(k)}) A_2^{(k+1)}, \quad k \geq 0, \]
\[ R_k = A_0^{(k)} (-U_{k+1}^{-1}), \quad k \geq 0, \]

and

\[ G_k = (-U_{k}^{-1}) A_2^{(k)}, \quad k \geq 1. \]

Obviously, it is well-known from Li and Cao \[44\] that the matrix sequence \( \{ R_k, k \geq 0 \} \) is the minimal nonnegative solution to the system of nonlinear matrix equations

\[ A_0^{(k)} + R_k A_1^{(k+1)} + R_k R_{k+1} A_2^{(k+2)} = 0, \quad k \geq 0; \quad (3) \]

and the matrix sequence \( \{ G_k, k \geq 1 \} \) is the minimal nonnegative solution to the system of nonlinear matrix equations

\[ A_0^{(k)} G_{k+1} + A_1^{(k)} G_k + A_2^{(k)} = 0, \quad k \geq 1. \quad (4) \]

Once the matrix sequence \( \{ R_k, k \geq 0 \} \) or \( \{ G_k, k \geq 1 \} \) is given, for \( k \geq 0 \) we have

\[ U_k = A_1^{(k)} + A_0^{(k)} (-U_{k+1}^{-1}) A_2^{(k+1)} \]
\[ = A_1^{(k)} + R_k A_2^{(k+1)} \]
\[ = A_1^{(k)} + A_0^{(k)} G_{k+1}. \]

For the unilateral QBD process \( Q_A \), by following Chapters 1 and 2 of Li \[42\] or Li and Cao \[44\], the UL-type RG-factorization is given by

\[ Q_A = (I - R_U) U_D (I - G_L), \quad (5) \]

where

\[ U_D = \text{diag} (U_0, U_1, U_2, U_3, \ldots), \]

\[ R_U = \begin{pmatrix}
0 & R_0 \\
0 & R_1 \\
0 & R_2 \\
0 & \ddots \\
0 & \ddots
\end{pmatrix}, \quad G_L = \begin{pmatrix}
0 & G_1 & 0 \\
G_2 & 0 & \ddots \\
g_3 & 0 & \ddots
\end{pmatrix}. \]

Let \( \pi_A = (\pi_0^A, \pi_1^A, \pi_2^A, \ldots) \) be the stationary probability vector of the unilateral QBD process \( Q_A \). Then from the UL-type RG-factorization given in Chapter 2 of Li \[42\], by using the \( R \)-measure \( \{ R_k : k \geq 0 \} \) we have

\[ \pi_k^A = \pi_0^A R_0 R_1 \cdots R_{k-1}, \quad k \geq 1. \quad (6) \]
By conducting a similar analysis to those in Equations (2) to (5), we can give the stationary probability vector, \( \pi_B = (\pi_0^B, \pi_1^B, \pi_2^B, \ldots) \) of the unilateral QBD process \( Q_B \). Here, we only provide the \( R \)-measure \( \{ R_l : l \leq -1 \} \), while the \( U \)-measure \( \{ U_l : l \leq -1 \} \) and \( G \)-measure \( \{ G_l : l \leq -2 \} \) is omitted for brevity.

Let the matrix sequence \( \{ R_l, l \leq -1 \} \) be the minimal nonnegative solution to the system of nonlinear matrix equations

\[
B_0^{(l)} + R_l B_1^{(l-1)} + R_l R_{l-1} B_2^{(l-2)} = 0, \quad l \leq -1, \tag{7}
\]

By using the \( R \)-measure \( \{ R_l : l \leq -1 \} \) we obtain

\[
\pi_l^B = \pi_{-1}^B R_{-1} R_{-2} \cdots R_{l+1}, \quad l \leq -2. \tag{8}
\]

The following theorem expresses the stationary probability vector \( \pi = (\ldots, \pi_{-2}, \pi_{-1}, \pi_0, \pi_1, \pi_2, \ldots) \) of the bidirectional level-dependent QBD process \( Q \) by means of the stationary probability vectors \( \pi_A = (\pi_2^A, \pi_3^A, \pi_4^A, \ldots) \) given in (6) and \( \pi_B = (\pi_{-2}^B, \pi_{-3}^B, \pi_{-4}^B, \ldots) \) given in (8).

**Theorem 2** *The stationary probability vector \( \pi \) of the bidirectional level-dependent QBD process \( Q \) is given by*

\[
\pi_k = c \tilde{\pi}_k, \tag{9}
\]

and

\[
\tilde{\pi}_k = \begin{cases} 
\tilde{\pi}_{-1} R_{-1} R_{-2} \cdots R_{k+1}, & k \leq -2, \\
\tilde{\pi}_{-1}, & \\
\tilde{\pi}_0, & \\
\tilde{\pi}_1, & \\
\tilde{\pi}_1 R_1 R_2 \cdots R_{k-1}, & k \geq 2,
\end{cases} \tag{10}
\]

where the three boundary vectors \( \tilde{\pi}_{-1}, \tilde{\pi}_0, \tilde{\pi}_1 \) are uniquely determined by the following system of linear equations

\[
\begin{aligned}
\tilde{\pi}_0 A_0^{(0)} + \tilde{\pi}_1 \left[ A_1^{(1)} + R_1 A_2^{(2)} \right] &= 0, \\
\tilde{\pi}_{-1} B_2^{(-1)} + \tilde{\pi}_0 A_1^{(0)} + \tilde{\pi}_1 A_2^{(1)} &= 0, \\
\tilde{\pi}_0 B_0^{(0)} + \tilde{\pi}_{-1} \left[ B_1^{(-1)} + R_{-1} B_2^{(-2)} \right] &= 0,
\end{aligned} \tag{11}
\]

and the positive constant \( c \) is uniquely given by

\[
c = \frac{1}{\sum_{k \leq -2} \tilde{\pi}_{-1} R_{-1} R_{-2} \cdots R_{k+1} e + \tilde{\pi}_{-1} e + \tilde{\pi}_0 e + \tilde{\pi}_1 e + \sum_{k=2}^{\infty} \tilde{\pi}_1 R_1 R_2 \cdots R_{k-1} e}. \tag{12}
\]
Proof. The proof is easy through checking whether $\pi$ satisfies the system of linear equations: $\pi Q = 0$ and $\pi e = 1$. To this end, we consider the following three different cases:

**Case one:** $k \geq 2$. In this case, we need to check that

$$\pi_{k-1}A_0^{(k-1)} + \pi_k A_1^{(k)} + \pi_{k+1} A_2^{(k+1)} = 0. \tag{13}$$

Since $\pi_{k-1} = c\pi_{k-1} = c\pi_1 R_1 R_2 \cdots R_{k-2}$, $\pi_k = c\pi_k = c\pi_1 R_1 R_2 \cdots R_{k-1}$ and $\pi_{k+1} = c\pi_{k+1} = c\pi_1 R_1 R_2 \cdots R_k$, we obtain

$$\pi_{k-1}A_0^{(k-1)} + \pi_k A_1^{(k)} + \pi_{k+1} A_2^{(k+1)}$$

$$= c\pi_1 R_1 R_2 \cdots R_{k-2}A_0^{(k-1)} + c\pi_1 R_1 R_2 \cdots R_{k-1} A_1^{(k)} + c\pi_1 R_1 R_2 \cdots R_k A_2^{(k+1)}$$

$$= c\pi_1 R_1 R_2 \cdots R_{k-2} \left( A_0^{(k-1)} + R_{k-1} A_1^{(k)} + R_{k-1} R_k A_2^{(k+1)} \right) = 0$$

by means of (8).

**Case two:** $k \leq -2$. In this case, we need to check that

$$\pi_{k+1} B_0^{(k+1)} + \pi_k B_1^{(k)} + \pi_{k-1} B_2^{(k-1)} = 0. \tag{14}$$

Note that $\pi_{k+1} = c\pi_{k+1} = c\pi_1 \mathbb{R}_{-1} \mathbb{R}_{-2} \cdots \mathbb{R}_{k+2}$, $\pi_k = c\pi_k = c\pi_1 \mathbb{R}_{-1} \mathbb{R}_{-2} \cdots \mathbb{R}_{k+1}$ and $\pi_{k-1} = c\pi_{k-1} = c\pi_1 \mathbb{R}_{-1} \mathbb{R}_{-2} \cdots \mathbb{R}_k$, we have

$$\pi_{k+1} B_0^{(k+1)} + \pi_k B_1^{(k)} + \pi_{k-1} B_2^{(k-1)}$$

$$= c\pi_1 \mathbb{R}_{-1} \mathbb{R}_{-2} \cdots \mathbb{R}_{k+2} B_0^{(k+1)} + c\pi_1 \mathbb{R}_{-1} \mathbb{R}_{-2} \cdots \mathbb{R}_{k+1} B_1^{(k)} + c\pi_1 \mathbb{R}_{-1} \mathbb{R}_{-2} \cdots \mathbb{R}_k B_2^{(k-1)}$$

$$= c\pi_1 \mathbb{R}_{-1} \mathbb{R}_{-2} \cdots \mathbb{R}_{k+2} \left( B_0^{(k+1)} + R_{k+1} B_1^{(k)} + R_{k+1} R_k B_2^{(k-1)} \right) = 0$$

in terms of (7).

**Case three:** $k = 1, 0, -1$. In this case, we obtain

$$\begin{cases} 
\pi_0 A_0^{(0)} + \pi_1 A_1^{(1)} + \pi_2 A_2^{(2)} = 0, \\
\pi_{-1} B_2^{(-1)} + \pi_0 A_1^{(0)} + \pi_1 A_2^{(1)} = 0, \\
\pi_{-2} B_2^{(-2)} + \pi_{-1} B_1^{(-1)} + \pi_0 B_0^{(0)} = 0.
\end{cases} \tag{15}$$

Note that $\pi_0 = c\pi_0, \pi_1 = c\pi_1, \pi_2 = c\pi_2 = c\pi_1 R_1, \pi_{-1} = c\pi_{-1}$ and $\pi_{-2} = c\pi_{-2} = c\pi_{-1} \mathbb{R}_{-1}$, we can give (11).

Note that $\pi e = 1$, we have

$$\sum_{-\infty < k < \infty} \pi_k e = 1 \tag{16}$$
by means of $\pi_k = c\tilde{\pi}_k = c\tilde{\pi}_1R_1R_2 \cdots R_{k-1}$ for $k \geq 2$, $\pi_1 = c\tilde{\pi}_1$, $\pi_0 = c\tilde{\pi}_0$, $\pi_{-1} = c\tilde{\pi}_{-1}$ and $\pi_k = c\tilde{\pi}_{k-1}R_{k-1} \cdots R_{l+1}$ for $l \leq -2$. Thus we have

$$1 = \sum_{-\infty < k < \infty} \pi_k e = \sum_{l \leq -2} \pi_l e + \pi_{-1} e + \pi_{0} e + \pi_{1} e + \sum_{k=2}^{\infty} \pi_k e$$

$$= \sum_{l \leq -2} c\tilde{\pi}_{l-1}R_{l-1} \cdots R_{l+1} e + c\tilde{\pi}_{-1} e + c\tilde{\pi}_0 e + c\tilde{\pi}_1 e + \sum_{k=2}^{\infty} c\tilde{\pi}_1 R_2 \cdots R_{k-1} e$$

$$= c \left( \sum_{l \leq -2} \tilde{\pi}_{l-1} R_{l-2} \cdots R_{l+1} e + \tilde{\pi}_{-1} e + \tilde{\pi}_0 e + \tilde{\pi}_1 e + \sum_{k=2}^{\infty} \tilde{\pi}_1 R_2 \cdots R_{k-1} e \right),$$

this gives the positive constant $c$ in (12). This completes the proof. ■

**Remark 2** As seen from Theorem 2, generally there is no explicit expression for the stationary probability vector of the bidirectional level-dependent QBD process, in which the impatient customers always make a level-dependent Markov process whose stationary probability vector computation is more complicated and difficult. Thus we cannot use an analytic expression to further discuss performance measures of the matched queue with matching batch pair $(m, n)$ and impatient customers.

In the remainder of this section, we compute two average stationary queue lengths for the A- and B-customers, respectively.

Note that the matched queue with matching batch pair $(m, n)$ is stable for $(\theta_1, \theta_2) > 0$, we denote by $Q^{(1)}$ and $Q^{(2)}$ the stationary queue lengths of the A- and B-customers, respectively. By using Theorem 2 we provide the average stationary queue lengths of the A- and B-customers, respectively, as follows:

(a) The average stationary queue length of the A-customers is given by

$$E[Q^{(1)}] = \sum_{k=1}^{m-1} \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \pi_{km+i,j} + \sum_{-\infty < l \leq 0} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \pi_{i,(-l)n+j}.$$  

(b) The average stationary queue length of the B-customers is given by

$$E[Q^{(2)}] = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} \pi_{km+i,j} + \sum_{-\infty < l \leq -1} \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} \pi_{i,(-l)n+j}.$$  

By using the efficient algorithms given in Bright and Taylor [10, 11] (e.g., see Liu et al. [15] for more details), we conduct numerical examples to analyze how the average
stationary queue lengths of A- and B-customers are influenced by two key parameters: $\theta_1$ and $\theta_2$. To this end, we take the system parameters: $\lambda_1 = 1$, $\lambda_2 = 2$, $m = 2$ and $n = 3$ for the purpose of illustration.

Figure 4 shows that the average stationary queue length $E \left[ Q^{(1)} \right]$ decreases as $\theta_1$ increases, while it increases as $\theta_2$ increases.

From Figure 5, it is easy seen that the average stationary queue length $E \left[ Q^{(2)} \right]$ increases as $\theta_1$ increases, while it decreases as $\theta_2$ increases.

The two numerical results are intuitive. As $\theta_1$ increases, more and more A-customers quickly leave the system so that $E \left[ Q^{(1)} \right]$ decreases. On the other hand, as $\theta_2$ increases, more and more B-customers quickly leave the system so that the probability that an A-customer can match a B-customer will become smaller and smaller. Thus, $E \left[ Q^{(1)} \right]$
increases as $\theta_2$ increases.

5 The Sojourn Time

In this section, we analyze the average sojourn time of any arriving A- or B-customer in the matched queue with matching batch pair $(m, n)$ and impatient customers.

In this stable matched queue, we denote by $W$ the sojourn time of any arriving A-customer. To compute the average sojourn time, we need to consider three different cases as follows:

(a) An arriving A-customer observes the system state $(i, j)$ for $0 \leq i \leq m - 1$ and $j \geq n$;

(b) an arriving A-customer observes the system state $(i, j)$ for $0 \leq i \leq m - 1$ and $0 \leq j \leq n - 1$; and

(c) an arriving A-customer observes the system state $(i, j)$ for $i \geq m$ and $0 \leq j \leq n - 1$.

Case a: An arriving A-customer observes the system state $(i, j)$ for $0 \leq i \leq m - 1$ and $j \geq n$.

In this case, if $i = m - 1$, then the arriving A-customer makes the number of A-customers become $m$. In this case, the $m$ A-customers can match $n$ B-customers as a group, which leaves this system immediately. Thus the average sojourn time is given by

$$E[W] = 0.$$  \hspace{1cm} (17)

If $0 \leq i \leq m - 2$, then the arriving A-customer still needs to wait for $m - (i + 1)$ arrivals of A-customers such that the number of A-customers is $m$. Thus the $m$ A-customers can match $n$ B-customers as a group, which leaves this system immediately.

Let $X_k$ be the $k$th interarrival time of the Poisson process with arrival rate $\lambda_1$, and $Z$ be the exponential impatient time of A-customer with impatient rate $\theta_1$. Then

$$W = \min \left\{ Z, \sum_{k=1}^{m-(i+1)} X_k \right\}.$$  

This gives

$$E[W] = E[Z] P \left\{ Z \leq \sum_{k=1}^{m-(i+1)} X_k \right\} + E \left[ \sum_{k=1}^{m-(i+1)} X_k \right] P \left\{ Z > \sum_{k=1}^{m-(i+1)} X_k \right\}.$$  

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Note that $\sum_{k=1}^{m-(i+1)} X_k$ is an Erlang-$(m - i - 1)$ distribution, we have

$$P\left\{ \sum_{k=1}^{m-(i+1)} X_k > u \right\} = e^{-\lambda_1 u} \sum_{k=0}^{m-(i+1)-1} \frac{(\lambda_1 u)^k}{k!}$$

this gives

$$\gamma_{m-i-1} = P\left\{ Z \leq \sum_{k=1}^{m-(i+1)} X_k \right\} = \int_0^{+\infty} P\left\{ \sum_{k=1}^{m-(i+1)} X_k > u \right\} dP\{Z < u\}$$

$$= \theta_1 \int_0^{+\infty} e^{-(\lambda_1+\theta_1)u} \sum_{k=0}^{m-(i+1)-1} \frac{(\lambda_1 u)^k}{k!} du.$$ 

Thus, we obtain

$$E[W] = 1 - \theta_1^{-1} \gamma_{m-i-1} + \frac{m-i-1}{\lambda_1} (1 - \gamma_{m-i-1}). \quad (18)$$

**Case b:** An arriving A-customer observes the system state $(i, j)$ for $0 \leq i \leq m - 1$ and $0 \leq j \leq n - 1$

In this case, if $i = m - 1$, then the arriving A-customer makes the number of A-customers become $m$. However, they still need to wait for $n - j$ arrivals of B-customers such that the number of B-customers is $n$. Thus the $n$ B-customers can match $m$ A-customers as a group, which leaves this system immediately.

Let $Y_k$ be the $k$th interarrival time of the Poisson process with arrival rate $\lambda_2$. Then

$$W = \min\left\{ Z, \sum_{k=1}^{n-j} Y_k \right\}.$$

Let

$$\delta_{n-j} = P\left\{ Z \leq \sum_{k=1}^{n-j} Y_k \right\} = \theta_1 \int_0^{+\infty} e^{-(\lambda_2+\theta_1)u} \sum_{k=0}^{n-j-1} \frac{(\lambda_2 u)^k}{k!} du.$$

Then

$$E[W] = \frac{1}{\theta_1} \delta_{n-j} + \frac{n-j}{\lambda_2} (1 - \delta_{n-j}). \quad (19)$$

If $0 \leq i \leq m - 2$, then we still need to not only wait for $m-(i+1)$ arrivals of A-customers such that the number of A-customers is $m$, but also wait for $n - j$ arrivals of B-customers such that the number of B-customers is $n$. In this case, the $n$ B-customers can match $m$ A-customers as a group, which leaves this system immediately. Thus we obtain

$$W = \min\left\{ Z, \max\left\{ \sum_{k=1}^{m-(i+1)} X_k, \sum_{k=1}^{n-j} Y_k \right\} \right\}.$$
Let
\[ \omega(m - i - 1, n - j) = P \left\{ Z \leq \max \left\{ \sum_{k=1}^{m-(i+1)} X_k, \sum_{k=1}^{n-j} Y_k \right\} \right\}. \]

Then
\[ E[W] = \frac{1}{\theta_1} \omega(m - i - 1, n - j) + E \left[ \max \left\{ \sum_{k=1}^{m-(i+1)} X_k, \sum_{k=1}^{n-j} Y_k \right\} \right] [1 - \omega(m - i - 1, n - j)]. \]

Let
\[ \psi(m - i - 1, n - j) = P \left\{ \sum_{k=1}^{m-(i+1)} X_k \leq \sum_{k=1}^{n-j} Y_k \right\}. \]

Then
\[ E \left[ \max \left\{ \sum_{k=1}^{m-(i+1)} X_k, \sum_{k=1}^{n-j} Y_k \right\} \right] = \psi(m - i - 1, n - j) \frac{n-j}{\lambda_2}
+ [1 - \psi(m - i - 1, n - j)] \frac{m - i - 1}{\lambda_1}. \]

Thus we obtain
\[ E[W] = \frac{1}{\theta_1} \omega(m - i - 1, n - j) + [1 - \omega(m - i - 1, n - j)] \]
\[ \times \left\{ \psi(m - i - 1, n - j) \frac{n-j}{\lambda_2} + [1 - \psi(m - i - 1, n - j)] \frac{m - i - 1}{\lambda_1} \right\}. \quad (20) \]

Case c: An arriving A-customer observes the system state \((i, j)\) for \(i \geq m\) and \(0 \leq j \leq n - 1\). Since \(i \geq m\), there exists a unique positive integer \(h\) such that \(i = hm + f\), where \(0 \leq f \leq m - 1\).

If \(f = m - 1\), then the arriving A-customer makes the number of A-customers become \((h + 1)m\). We still need to wait for \(hn + n - j\) arrivals of B-customers such that the number of B-customers becomes \((h + 1)n\). In this case, the \((h + 1)n\) B-customers can match \((h + 1)m\) A-customers as \(h + 1\) groups, which leave this system immediately. Thus we obtain
\[ W = \min \left\{ Z, \sum_{k=1}^{hn+n-j} Y_k \right\}. \]

This gives
\[ E[W] = \frac{1}{\theta_1} \delta_{hn+n-j} + \frac{hn + n - j}{\lambda_2} (1 - \delta_{hn+n-j}). \quad (21) \]
If $0 \leq f \leq m - 2$, then we still need to not only wait for $m - (f + 1)$ arrivals of A-customers such that the number of A-customers becomes $(h + 1)m$, but also wait for $hn + n - j$ arrivals of B-customers such that the number of B-customers becomes $(h + 1)n$. In this case, the $(h + 1)n$ B-customers can match $(h + 1)m$ A-customers to form $h + 1$ groups, which leave this system immediately. Thus we obtain

$$W = \min \left\{ Z, \max \left\{ \sum_{k=1}^{m-f-1} X_k, \sum_{k=1}^{hn+n-j} Y_k \right\} \right\}.$$ 

This gives

$$E[W] = \frac{1}{\theta_1} \omega (m - f - 1, hn + n - j) + [1 - \omega (m - f - 1, hn + n - j)]$$

$$\times \left\{ \psi (m - f - 1, hn + n - j) \frac{hn + n - j}{\lambda_2} + [1 - \psi (m - f - 1, hn + n - j)] \frac{m - f - 1}{\lambda_1} \right\}. \quad (22)$$

In what follows we provide an average stationary sojourn time of any arriving A-customer by means of Equations (17) to (22) and the stationary probability vector of this system. We write that for $l = -1, -2, -3, ...$,

$$\Phi_l = (\phi_{1,(-l)n}, \phi_{2,(-l)n}, \ldots, \phi_{m-1,(-l)n}; \phi_{1,(-l)n+1}, \phi_{2,(-l)n+1}, \ldots, \phi_{m-1,(-l)n+1}; \ldots;$$

$$\phi_{1,(l)n+n-1}; \phi_{2,(l)n+n-1}, \ldots, \phi_{m-1,(l)n+n-1}),$$

for $k = 0$,

$$\Phi_0 = (\phi_{1,0}, \phi_{1,1}, \ldots, \phi_{1,n-1}; \phi_{2,0}, \phi_{2,1}, \ldots, \phi_{2,n-1}; \ldots; \phi_{m-1,0}, \phi_{m-1,1}, \ldots, \phi_{m-1,n-1}),$$

for $k = 1, 2, 3, ...$,

$$\Phi_k = (\phi_{km,0}, \phi_{km,1}, \ldots, \phi_{km,n-1}; \phi_{km+1,0}, \phi_{km+1,1}, \ldots, \phi_{km+1,n-1}; \ldots;$$

$$\phi_{km+m-1,0}, \phi_{km+m-1,1}, \ldots, \phi_{km+m-1,n-1}),$$

and

$$\Phi = (\ldots, \Phi_{-2}, \Phi_{-1}, \Phi_0, \Phi_1, \Phi_2, \ldots). \quad (23)$$

For $1 \leq i \leq m - 1$ and $0 \leq j \leq n - 1$; $1 \leq i \leq m - 1$ and $j \geq n$; and $i \geq m$ and $0 \leq j \leq n - 1$, we write

$$\phi_{i,j} = \frac{1}{1 - \sum_{\infty < l \leq 0} \sum_{j=0}^{n-1} \Phi_{0,(-l)n+j}} \pi_{i,j}.$$
It is clear that $\Phi_e = 1$.

When the matched queue with matching batch pair $(m,n)$ and impatient customers is stable, we assume that an arriving A-customer enters State $(i-1,j)$ with probability $\phi_{i,j}$ for $i \geq 1$ at time 0. Once the A-customer enters the matched queue, the number of A-customers becomes $i$. In this case, we obtain

$$E[W] = \sum_{i=1}^{m-1} \sum_{j=n}^{\infty} \phi_{i,j} E[W | (N_1(0), N_2(0)) = (i,j)]$$  

Case a

$$+ \sum_{i=1}^{m-1} \sum_{j=0}^{n-1} \phi_{i,j} E[W | (N_1(0), N_2(0)) = (i,j)]$$  

Case b

$$+ \sum_{i=m}^{\infty} \sum_{j=1}^{\infty} \phi_{i,j} E[W | (N_1(0), N_2(0)) = (i,j)],$$  

Case c (24)

where $E[W | (N_1(0), N_2(0)) = (i,j)]$ is given in Case a by (17) and (18); Case b by (19) and (20); and Case c by (21) and (22).

In the remainder of this section, we provide a better upper bound of the average sojourn time $E[W]$ given in (24) by means of a new PH distribution of bidirectional infinite levels.

We write a first passage time of the Markov process $Q$ (or $\{(N_1(t), N_2(t)), t \geq 0\}$) as

$$\xi = \inf \{t : N_1(t) = 0\},$$

that is, $\xi$ is such a first passage time that the waiting room of A-customers becomes empty for the first time. It is easy to see that $E[W] \leq E[\xi]$, since it is possible that the waiting room of A-customers still contains some customers at the time that the arriving A-customer leaves the system. Note that the arriving A-customer enters State $(i-1,j)$ with probability $\phi_{i,j}$ for $i \geq 1$ at time 0. It is also possible that the waiting room of A-customers is empty at the time that the arriving A-customer leaves the system. Thus $E[\xi]$ can be a better upper bound of the average sojourn time $E[W]$.

Now, we compute the average first passage time $E[\xi]$. To do this, we take all the states: $(0,k)$ for $k \geq 0$, as an absorbing state $\Delta$. Therefore, from the Markov process $Q$, we can set up an absorbing Markov process whose infinitesimal generator is given by

$$Q = \begin{pmatrix} 0 & 0 \\ T^0 & T \end{pmatrix},$$

where the first row and the first column of the matrix $Q$ are related to the absorbing state $\Delta$. To write the matrix $T$, we need to check the levels of the absorbing Markov process $Q$.
as follows: For $k \geq 1$,

$$\text{Level } k = \text{Level } k,$$

$$\text{Level } 0 = \{(1, 0), (1, 1), \ldots, (1, n - 1); (2, 0), (2, 1), \ldots, (2, n - 1); \ldots; (m - 1, 0), (m - 1, 1), \ldots, (m - 1, n - 1)\},$$

and $l \leq -1$,

$$\text{Level } l = \{(1, (-l) n), (2, (-l) n), \ldots, (m - 1, (-l) n); (1, (-l) n + 1), (2, (-l) n + 1), \ldots, (m - 1, (-l) n + 1); \ldots; (1, (-l) n + n - 1), (2, (-l) n + n - 1), \ldots, (m - 1, (-l) n + n - 1)\}.$$

Thus the state space of the absorbing Markov process $Q$ is given by

$$\tilde{\Omega} = \{\Delta\} \cup \left\{ \bigcup_{-\infty < k < \infty} \text{Level } k \right\}$$

$$= \{\Delta\} \cup \left\{ \bigcup_{k=1}^{\infty} \text{Level } k \right\} \cup \left\{ \bigcup_{-\infty < l \leq 0} \text{Level } l \right\}.$$

By using the levels: Level $k$ for $k \geq 1$ and $\text{Level } l$ for $-\infty < l \leq 0$, we obtain

$$T = \begin{pmatrix}
\ddots & \ddots & \ddots \\
\overline{B}_0^{(-3)} & \overline{B}_1^{(-3)} & \overline{B}_2^{(-3)} \\
\overline{B}_0^{(-2)} & \overline{B}_1^{(-2)} & \overline{B}_2^{(-2)} \\
\overline{B}_0^{(-1)} & \overline{B}_1^{(-1)} & \overline{B}_2^{(-1)} \\
\overline{B}_0^{(0)} & \overline{A}_0^{(0)} & \overline{A}_0^{(1)} \\
\overline{A}_0^{(1)} & \overline{A}_0^{(2)} & \overline{A}_0^{(2)} \\
\overline{A}_2^{(3)} & \overline{A}_1^{(3)} & \overline{A}_0^{(3)} \\
\vdots & \ddots & \ddots
\end{pmatrix},$$

where $\overline{A}_0^{(0)}$, $\overline{A}_1^{(0)}$, $\overline{A}_2^{(1)}$ and $\overline{B}_i^{(-l)}$ for $i = 0, 1, 2$ and $l \leq 0$, can be written easily with details omitted here.

The following theorem shows that the first passage time $\xi$ is of bidirectional infinite phase type, and thus provides a method to compute the average first passage time $E[\xi]$. The proof is easy and is omitted here.
Theorem 3 If the arriving A-customer enters State \((i - 1, j)\) with probability \(\phi_{i,j}\) for \(i \geq 1\) at time 0, then the first passage time \(\xi\) is of bidirectional infinite phase type with an irreducible representation \((\Phi, T)\), and

\[
E[\xi] = -\Phi T^{-1}e,
\]

where \(\Phi\) is given in (23).

When the PH distribution is unilateral infinite, Chapter 8 of Li [42] provides a detailed discussion for computing the inverse matrix \(T^{-1}\) by means of the RG-factorizations given in Li [42].

When the PH distribution is bidirectional infinite, to our best knowledge, this paper is the first to deal with the inverse matrix \(T^{-1}\) of bidirectional infinite. To this end, we write

\[
T = \begin{pmatrix}
T_{1,1} & T_{1,2} \\
T_{2,1} & T_{2,2}
\end{pmatrix},
\]

where

\[
T_{1,1} = \begin{pmatrix}
\cdots & \cdots & \cdots \\
\tilde{B}_0^{-2} & \tilde{B}_1^{-2} & \tilde{B}_2^{-2} \\
\tilde{B}_0^{-1} & \tilde{B}_1^{-1} & \tilde{B}_2^{-1} \\
\tilde{B}_0^{0} & \tilde{B}_1^{0} & \tilde{B}_2^{0}
\end{pmatrix},
\]

\[
T_{1,2} = \begin{pmatrix}
A_0^{(1)} \\
A_1^{(2)} \\
A_2^{(3)} \\
\cdots
\end{pmatrix},
\]

\[
T_{2,1} = \begin{pmatrix}
\tilde{A}_2^{(1)} \\
\tilde{A}_2^{(0)} \\
\tilde{A}_2^{(0)} \\
\cdots
\end{pmatrix},
\]

\[
T_{2,2} = \begin{pmatrix}
A_0^{(1)} & A_0^{(2)} & A_0^{(3)} \\
A_1^{(2)} & A_1^{(3)} \\
A_2^{(3)} & \cdots \\
\cdots & \cdots & \cdots
\end{pmatrix}.
\]

Note that the Markov chain \(Q\) is irreducible, thus the submatrix \(T_{2,2}\) must be invertible.

Based on this, it is easy to check that

\[
T^{-1} = \begin{pmatrix}
T_{1,1}^{-1} & T_{1,2}^{-1} - T_{1,1}^{-1} T_{1,2} T_{2,2}^{-1} \\
-T_{2,1}^{-1} T_{1,1}^{-1} T_{1,2}^{-1} & T_{2,2}^{-1} + T_{2,1}^{-1} T_{1,1}^{-1} T_{1,2} T_{2,2}^{-1}
\end{pmatrix},
\]

where

\[
T_{1,1}^{-1} = T_{1,1} - T_{1,2} T_{2,2}^{-1} T_{2,1}.
\]
It is easy to see that the inverse matrix $T^{-1}$ of bidirectional infinite can be expressed by means of the inverse matrix $T_{2,2}^{-1}$ of unilateral infinite. This relation plays a key role in setting up the PH distribution of bidirectional infinite.

In what follows, we simply discuss the inverse matrix $T_{2,2}^{-1}$ of unilateral infinite by means of the RG-factorizations, e.g., see Chapter 1 of Li [42] or Li and Cao [44] for more details.

For the QBD process $T_{2,2}$ of unilateral infinite, we define the UL-type $U$, $R$- and $G$-measures as

$$U_k = A_1^{(k)} + A_0^{(k)} (-U_{k+1}^{-1}) A_2^{(k+1)}, \quad k \geq 1,$$

$$R_k = A_0^{(k)} (-U_{k+1}^{-1}), \quad k \geq 1,$$

and

$$G_k = (-U_{k}^{-1}) A_2^{(k)}, \quad k \geq 2.$$

Note that the matrix sequence $\{R_k, k \geq 1\}$ is the minimal nonnegative solution to the system of nonlinear matrix equations

$$A_0^{(k)} + R_k A_1^{(k+1)} + R_k R_{k+1} A_2^{(k+2)} = 0, \quad k \geq 1;$$

and the matrix sequence $\{G_k, k \geq 2\}$ is the minimal nonnegative solution to the system of nonlinear matrix equations

$$A_0^{(k)} G_{k+1} A_2^{(k)} + A_1^{(k)} G_k + A_2^{(k)} = 0, \quad k \geq 2.$$

Based on this, the UL-type RG-factorization of the matrix $T_{2,2}$ of unilateral infinite is given by

$$T_{2,2} = (I - R_U) U_D (I - G_L),$$

where

$$I - R_U = \begin{pmatrix} I & -R_1 \\ & I & -R_2 \\ & & I & -R_3 \\ & & & \ddots \end{pmatrix},$$

and
$$U_D = \text{diag} (U_1, U_2, U_3, \ldots),$$

$$I - G_L = \begin{pmatrix}
  I & & & \\
  -G_2 & I & & \\
  & -G_3 & I & \\
  & & -G_4 & I \\
  & & & \ddots \\
\end{pmatrix}.$$ 

By using R-measure \(\{R_k : k \geq 1\}\) and the G-measure \(\{G_k : k \geq 2\}\), we write

$$X^{(k)}_k = I,$$

$$X^{(k)}_{k+l} = R_k R_{k+1} \cdots R_{k+l-1}, \quad k \geq 1, \quad l \geq 1,$$

$$Y^{(k)}_k = I,$$

$$Y^{(k)}_{k-l} = G_k G_{k-1} \cdots G_{k-l+1}, \quad k > l \geq 1.$$ 

From the UL-type RG-factorization (28), it is easy to check that the three matrices \(I - R_U, U_D\) and \(I - G_L\) are all invertible, and

$$(I - R_U)^{-1} = \begin{pmatrix}
  X^{(1)}_1 & X^{(1)}_2 & X^{(1)}_3 & X^{(1)}_4 & \cdots \\
  X^{(2)}_2 & X^{(2)}_3 & X^{(2)}_4 & \cdots \\
  X^{(3)}_3 & X^{(3)}_4 & \cdots \\
  X^{(4)}_4 & \cdots \\
  \vdots & & & & \ddots 
\end{pmatrix},$$

$$(I - G_L)^{-1} = \begin{pmatrix}
  Y^{(1)}_1 & Y^{(2)}_1 & Y^{(3)}_1 & Y^{(4)}_1 & \cdots \\
  Y^{(1)}_2 & Y^{(2)}_2 & Y^{(3)}_2 & Y^{(4)}_2 & \cdots \\
  Y^{(1)}_3 & Y^{(2)}_3 & Y^{(3)}_3 & Y^{(4)}_3 & \cdots \\
  Y^{(1)}_4 & Y^{(2)}_4 & Y^{(3)}_4 & Y^{(4)}_4 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}.$$ 

We obtain

$$T^{-1}_{2,2} = (I - G_L)^{-1} U_D^{-1} (I - R_U)^{-1}. \quad (29)$$ 

By using (26), (27) and (29), we obtain the inverse matrix \(T^{-1}\) of bidirectional infinite. Therefore, we can compute the average first passage time \(E[\xi]\), and \(E[W] \leq E[\xi]\).
Remark 3 From Theorem 1 for the system stability, Theorem 2 for the stationary probability vector, and Theorem 3 for the first passage time, we establish some new theory of the bidirectional level-dependent QBD processes, which will be necessary and useful in the study of stochastic models in practice.

6 Concluding Remarks

In this paper, we discuss an interesting but challenging bilateral stochastically matching problem: A more general matched queue with matching batch pair \((m, n)\) and impatient customers. We show that the matched queue with matching batch pair \((m, n)\) can be expressed as a novel bidirectional level-dependent QBD process. Based on this, we provide a detailed analysis for this matched queue, including the system stability, the two average stationary queue lengths, and the sojourn time of any arriving A- or B-customer. We believe that the methodology and results developed in this paper can be applicable to analyze more general matched queueing systems, which are widely encountered in various practical areas, for example, sharing economy, ridesharing platform, bilateral market, organ transplantation, taxi services, assembly systems, and so on.

Along these lines, we will continue our future research on the following directions:

- Considering probability matched system with matching pair \((1, 1)\).
- Studying probability matched system with matching batch pair \((m, n)\).
- Discussing some Phase-type and MAP factors in the study of matched queues.
- Analyzing fluid and diffusion approximations for matched system with matching batch pair \((m, n)\).
- Developing stochastic optimization, and Markov decision processes in the study of matched queues.

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