Resilient Distributed $H_{\infty}$ Estimation via Dynamic Rejection of Biasing Attacks

V. Ugrinovskii

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Abstract

We consider the distributed $H_{\infty}$ estimation problem with additional requirement of resilience to biasing attacks. An attack scenario is considered where an adversary misappropriates some of the observer nodes and injects biasing signals into observer dynamics. Using a dynamic modelling of biasing attack inputs, a novel distributed state estimation algorithm is proposed which involves feedback from a network of attack detection filters. We show that each observer in the network can be computed in real time and in a decentralized fashion. When these controlled observers are interconnected to form a network, they are shown to cooperatively produce an unbiased estimate the plant, despite some of the nodes are compromised.

1 Introduction

Problems of resilient control and estimation came into prominence after situations were discovered where an adversary was able to interfere with the control task by covertly injecting false information into the measurement data [6, 1, 8, 16]. Networked control systems are particularly vulnerable to data injection attacks, since operation of such systems depends on the integrity of communicated data.

The information shared between the nodes can also be utilized for monitoring integrity of the network. We demonstrate in this paper that the information routinely collected and shared within a distributed observer network can be used to detect compromised observers and neutralize their biasing effect. For this, we propose a novel distributed observer augmented with a network of attack detector filters; the latter filters provide feedback to the node observers which neutralizes rogue biasing inputs, if such inputs are present.

The model of misappropriation attack considered here is the same as in [2]: it captures essential features of the biasing attack described in [11]. It assumes that the adversary gains access to one or several nodes of the observer network and injects biasing inputs directly into the state estimation algorithm; also, cf. [10]. Similarly to [2], our approach to detecting such biasing behaviour is by tracking changes in the behaviour of the estimation errors caused by malicious biasing inputs. However different from [2], our method allows each observer node to compute its attack detection filter in decentralized fashion, without communicating with other nodes.

The decentralization of computations is accomplished in this paper using a decoupling technique which we have developed previously for the distributed observer design in [15]. The technique involves an initial ‘setup’ step which requires the network to compute certain auxiliary parameters that are then distributed among the nodes. Although this initial setup must be carried out centrally, it involves only the information about the communication network, and does not require knowledge of the plant observed. The auxiliary parameters computed at this setup step are then used for computing a collection of controlled node observers equipped with output feedback $H_{\infty}$ controllers; the latter can detect and cancel the attack using the same sensory data that are available for estimation of the state of the observed plant. Also, our method...
allows for the consideration of time-varying distributed filters. The robustness against uncertainties in the sensors and the plant model is guaranteed as well.

The feedback-controlled nature of the proposed distributed observer distinguishes this paper from the companion paper [14], where we used a similar decoupling technique. Here, the observers are computed jointly with the attack detection filters. This requirement of co-design did not arise in the attack detection problem considered in [2, 14], but arises in this paper since here the objective shifts from detecting and signalling a biasing attack to ensuring the distributed observer network is resilient to this kind of attacks.

Notation: \( \mathbb{R}^n \) denotes the real Euclidean \( n \)-dimensional vector space, with the norm \( \|x\| = (x'x)^{1/2} \); here the symbol \( ' \) denotes the transpose of a matrix or a vector. The symbol \( I \) denotes the identity matrix. For real symmetric \( n \times n \) matrices \( X \) and \( Y, Y > X \) (respectively, \( Y \geq X \)) means the matrix \( Y - X \) is positive definite (respectively, positive semidefinite). The notation \( L_2[0, \infty) \) refers to the Lebesgue space of \( \mathbb{R}^n \)-valued vector-functions \( z(t) \), defined on the time interval \([0, \infty)\), with the norm \( \|z\|_2 \equiv \left( \int_0^\infty \|z(t)\|^2\,dt \right)^{1/2} \) and the inner product \( \int_0^\infty z_1(t)z_2(t)\,dt \).

2 Biasing misappropriation attacks on distributed observers

A distributed observer problem consists in obtaining an estimate of the state of a time varying plant

\[
\dot{x} = A(t)x + B(t)w, \quad x(0) = x_0,
\]

which is subject to an unknown modeling disturbance \( w \). The estimate is to be obtained from a collection of measurements

\[
y_i = C_i(t)x + D_i(t)v_i, \quad i = 1, 2, \ldots, N,
\]

taken at \( N \) nodes of a sensor network, each perturbed by a measurement disturbance \( v_i \). In the distributed estimation setting a state estimate must be obtained at each network node without sending the data to a central data processing facility, and the nodes must obtain the same estimate of the plant. This is achieved by interconnecting the observers into a network. This way, the nodes can use the information which they receive from their neighbours to correct their plant state estimates until all nodes reach an agreement.

Let the state \( x \) and the disturbance \( w \) be vectors in \( \mathbb{R}^n, \mathbb{R}^m \) respectively, and each measurement \( y_i \) be a vector in \( \mathbb{R}^{n_i} \). The disturbances \( w \) and \( v_i \) \( \in \mathbb{R}^{m_i} \) will be assumed to be \( L_2^2 \) integrable signals defined on the interval \([0, \infty)\). The initial state \( x_0 \) is also assumed to be unknown. A typical distributed estimation problem involves a network of filters connected over a graph with vertices in the set \( \{1, \ldots, N\} \), each representing a node of the network:

\[
\dot{\hat{x}}_i = A(t)\hat{x}_i + L_i(t)(y_i - C_i(t)x_0) + \sum_{j \in \mathbb{N}_i} K_{ij}(t)(c_{ij} - W_{ij}\hat{x}_j), \quad \hat{x}_i(0) = \xi_i.
\]

Each observer \( \xi_i \) produces an estimate \( \hat{x}_i(t) \) of the plant state \( x(t) \). For this, it uses its measurement \( y_i \) and the information received from the neighbours; the latter information is communicated over noisy communication channels in the form of \( p_{ij} \)-dimensional signals

\[
c_{ij} = W_{ij}\hat{x}_j + H_{ij}v_{ij}, \quad j \in \mathbb{N}_i.
\]

Since the plant is time-varying, the observer gains \( L_i, K_{ij} \) in (3) are allowed to be time-varying.

The signals \( c_{ij} \) complement the local measurements \( y_i \) at node \( i \) and assist in obtaining a high fidelity estimate of the plant. Each such signal contains information about the neighbour’s estimate \( \hat{x}_j \) of the plant state \( x \). That is, the observers \( \xi_i \) are coupled via the signals \( c_{ij} \), forming a distributed observer network. Such a coupling between the observer nodes is essential in situations where the plant is not detectable from local measurements at some of the nodes, and these nodes require additional information which can only be obtained from their neighbours. The matrix \( W_{ij} \) determines the part of the vector \( \hat{x}_j \) which node \( j \) shares with node \( i \). Since this information is usually delivered over noisy communication links, a disturbance \( v_{ij} \) is included in (4) which is also assumed to be an \( L_2^2 \) integrable signal.
The task of distributed estimation using the observer network (3) is to ensure that each estimate \( \hat{x}_i(t) \) converges to \( x(t) \) as \( t \to \infty \) in some sense, with some robustness against disturbances in the plant model, measurements and interconnection channels. A large body of literature is dedicated to the question as to how the observers (3) can be constructed which achieve this objective; e.g. [7, 3, 5, 4]. However, the dependency on information sharing leaves the distributed observers vulnerable to attacks seeking to disrupt the estimation task. A scenario of such attacks usually considers an injection of false signals into sensor measurements or communicated data [8]. Here we follow [2] and consider a different scenario where the adversary substitutes one or several observers (3) with their biased versions

\[
\dot{\hat{x}}_i = A(t)\hat{x}_i + L_i(t)(y_i(t) - C_i(t)\hat{x}_i) + \sum_{j \in N_i} K_{ij}(t)(c_{ij} - W_{ij}\hat{x}_i) + F_i f_i, \quad \hat{x}_i(0) = \xi_i, \quad (5)
\]

where the Laplace transform of \( f_i(t) \), \( f_i(s) \), is such that \( \sup_{\omega} |\omega f_i(j\omega)|^2 < \infty \) and \( f_i \in L_2[0, \infty) \).

Obviously, biasing inputs with rational Laplace transforms which have no more than one pole at the origin and the rest of the poles located in the open left half-plane of the complex plane have this property. We will use the notation \( \mathcal{F} \) for the set of such inputs. It includes biasing attack inputs introduced in [11] consisting of a steady-state component and an exponentially vanishing component generated by a low pass filter.

The following lemma characterizes the properties of biasing inputs of this class. Its proof is given in the journal version of [2]. Let \( G_i(s) \) be a proper transfer function for which the system in Fig. 1 is stable, and \( \hat{f}_i \) be an output of that system.

**Lemma 1**  
(i) Consider a class of signals \( f_i(t) \) that admit the decomposition (6). Then for all such signals \( f_i(t) \) it holds that

\[
\int_0^\infty \| f_i - \hat{f}_i \|^2 dt < \infty. \quad (7)
\]

(ii) If in addition, \( G_i \) is selected so that

\[
\lim_{s \to 0} \| (I + \frac{1}{s} G_i(s))^{-1} \| = 0,
\]

then \( \lim_{t \to \infty} \| f_i(t) - \hat{f}_i(t) \| = 0 \) for all inputs \( f_i \in \mathcal{F} \).

According to Lemma 1, biasing inputs that have the form (6) can be ‘tracked’ using a system shown in Fig. 1. Of course, in reality it is not possible to track covert attack inputs. Nonetheless, the model in Fig. 1 allows us to associate the class of biasing attack inputs with the minimal realization of the strictly proper transfer function \( \frac{1}{s} G_i(s) \), of the form

\[
\dot{\epsilon}_i = \Omega_i \epsilon_i + \Gamma_i \nu_i, \quad \epsilon_i(0) = 0, \quad (9)
\]

\[
\hat{f}_i = \Upsilon_i \epsilon_i, \quad (10)
\]

Figure 1: An auxiliary ‘input tracking’ representation of a biasing attack input introduced in [2].
where \( \nu_i = \hat{f}_i - f_i \) is an \( L_2 \)-integrable input, according to (7). Clearly, each signal \( \nu_i \) corresponds to a certain unknown biasing input \( f_i \); it represents a mismatch error between the attack input \( f_i \) and the output \( \hat{f}_i \) of the system (9). In the sequel, this error will be regarded as an additional \( L_2 \)-integrable disturbance which will arise when we replace \( f_i \) with \( \hat{f}_i \) in the derivation of our attack detection and resilient estimation algorithms.

Apart from ensuring stability of the system in Fig. 1, according to Lemma 1 the proper transfer function \( G_i(s) \) can be selected arbitrarily.

### 3 Problem Formulation

In this paper we are concerned with the design of resilient version of the distributed observer (3). Our approach is to augment each node observer with additional dynamic feedback controllers to suppress the attack inputs. To accomplish this task, we introduce the following controlled modification of the observers (3), (5):

\[
\dot{x}_i = A(t)x_i + L_i^r(t)(y_i(t) - C_i(t)x_i) + \sum_{j \in N_i} K_{ij}^r(t)(c_{ij} - W_{ij}\hat{x}_i) + F_i f_i + u_i, \tag{10}
\]

\[
\dot{\hat{x}}_i(0) = \xi_i,
\]

The superscript \( r \) is to emphasize that the gains \( L_i^r(t), K_{ij}^r(t) \) are to be different from the gains \( L_i(t), K_{ij}(t) \) of the original observer (5). Also, \( u_i \) denotes the control input.

We propose the following observer-based feedback structure for generating the controls \( u_i \):

\[
u_i = -F_i \varphi_i, \tag{11}\]

Here \( \varphi_i \) denotes an output of a filter

\[
\hat{\mu}_i = \hat{\omega}_i(t)\mu_i + L_{d,i}(t)(\zeta_i - W_{d,i}\mu_i) + \sum_{j \in N_i} K_{d,ij}(t)(\zeta_{ij} - W_{d,ij}(\hat{\mu}_j - \mu_j)), \tag{12}\]

\[
\varphi_i = C_{d,i}(t)\mu_i, \quad \mu_i(0) = \mu_{i,0};
\]

where \( \hat{\omega}_i(t), L_{d,i}(t), K_{d,ij}(t), W_{d,i}, W_{d,ij}, C_{d,i} \) are matrix coefficients to be found. Each filter (12) is governed by the innovation signals \( \zeta_i, \zeta_{ij} \):

\[
\zeta_i = y_i - C_i(t)\hat{x}_i, \tag{13}\]

\[
\zeta_{ij} = c_{ij} - W_{ij}\hat{x}_j. \tag{14}\]

The filter (12) must generate \( \varphi_i \) so that when node \( i \) is under attack, the signal \( u_i \) counters the biasing input \( f_i \). Also, at the nodes which are not attacked directly, \( u_i \) must not interfere with the state observer. This requires the output \( \varphi_i \) of the filter (12) to track the biasing signal \( f_i \), turning (12) into an attack detector.

The problem of resilient estimation under consideration is now formally stated as the problem of constructing a network of filters (12) which, when interconnected with the modified state observers (10) via the feedback control (11), achieve the following properties

(i) In the absence of disturbances and when the system is not under attack, at every node \( i \), \( \|x(t) - \hat{x}_i\| \) and \( \varphi_i \) converge to 0 exponentially.

(ii) In the presence of uncertainties and/or attack,

\[
\int_0^{+\infty} \|\varphi_i - f_i\|^2 dt < +\infty \quad \forall i, \tag{15}\]

\[
\int_0^{+\infty} e^t P\dot{e} dt < +\infty;
\]
here $P = P' \geq 0$ is an $nN \times nN$ matrix and $e = [e'_1 \ldots e'_N]'$, where $e_i = x - \hat{x}_i$ denotes the estimation error of the observer (10) at node $i$.

The first condition in (15) formalizes the requirement for the filters (12) to track the corresponding attack inputs in the $L_2$ sense. Therefore, by monitoring the behaviour of the outputs $\varphi_i$, it will be possible to establish which node has been attacked. The second condition in (15) describes the desired resilience property of the observers. The matrix $P$ is considered to be given. The resilience of the modified observers (10), (11), (12) requires (15) to hold for any collection of admissible biasing inputs $f_i$ described in Section 2. The problem in this paper is to determine the characteristics $A_d(t)$, $L_{d,i}(t)$, $K_{d,ij}(t)$, $W_{d,i}$, $W_{d,ij}$, $C_{d,i}(t)$ of the filter (12) which guarantee that the above conditions (i) and (ii) hold.

4 Design of resilient distributed observers

4.1 Analysis of error dynamics

To construct suitable filters (12) consider the dynamics of the estimation errors of the controlled observers (10), (11). It is easy to see that these errors evolve according to

$$
\dot{e}_i = (A(t) - L'_i(t)C_i(t))e_i - \sum_{j \in N_i} K'_{ij}(t)W_{ij}e_i + \sum_{j \in N_i} K'_{ij}(t)W_{ij}e_j + B(t)w - L'_i(t)D_i(t)v_i - \sum_{j \in N_i} K'_{ij}(t)H_{ij}v_{ij} + F_i\varphi_i,
$$

(16)

$$
e_i(0) = x_0 - \xi_i.
$$

Noting that $f_i = \Upsilon_i e_i - \nu_i$, combine the dynamics of the systems (16) and (9) into an augmented system with $(e'_i, \epsilon'_i)'$ as a state vector:

$$
\dot{e}_i = (A(t) - L'_i(t)C_i(t))e_i - \sum_{j \in N_i} K'_{ij}(t)W_{ij}e_i + \sum_{j \in N_i} K'_{ij}(t)W_{ij}e_j - F_i\Upsilon_i e_i + F_i\varphi_i + B(t)w - L'_i(t)D_i(t)v_i - \sum_{j \in N_i} K'_{ij}(t)H_{ij}v_{ij} + F_i\nu_i,
$$

$$
\dot{\epsilon}_i = \Omega_i \epsilon_i + \Gamma_i \nu_i,
$$

$$
\dot{f}_i = \Upsilon_i e_i,
$$

(17)

$$
e_i(0) = x_0 - \xi_i, \quad \epsilon_i(0) = 0.
$$

Observe that the system (17) at node $i$ depends on the estimation errors at the neighboring nodes $j \in N_i$. Therefore, we propose a distributed observer of the form (12) to estimate the state of the extended system.
and its outputs $\hat{f}_i$ from the outputs (13), (14):

$$\dot{e}_i = (A(t) - L_i^* C_i(t) - \sum_{j \in N_i} K_{ij}^* W_{ij}) \hat{e}_i$$

$$+ \sum_{j \in N_i} K_{ij}^* W_{ij} \dot{e}_j + \bar{L}_i^*(\zeta_i - C_i(t) \dot{e}_i)$$

$$+ \sum_{j \in N_i} \bar{K}_{ij}^* (\zeta_{ij} - W_{ij} (\dot{e}_i - \dot{e}_j)),$$

$$\dot{\zeta}_i = \Omega_i \dot{e}_i + \bar{L}_i^* (\zeta_i - C_i(t) \dot{e}_i)$$

$$+ \sum_{j \in N_i} \bar{K}_{ij}^* (\zeta_{ij} - W_{ij} (\dot{e}_i - \dot{e}_j)),$$

$$\hat{e}_i(0) = 0, \quad \dot{e}_i(0) = 0.$$  \hfill (18)

The outputs of this observer

$$\varphi_i = \Upsilon_i \dot{e}_i,$$  \hfill (19)

will be constructed so that each output signal $\varphi_i$ approximates the attack input $f_i$ at the corresponding node $i$. This will allow it to be used for feedback to compensate the attack as well as for signalling the biasing attack. Note that the innovation signals (13), (14) can be written as

$$\zeta_i = C_i(t) e_i + D_i v_i,$$  \hfill (20)

$$\zeta_{ij} = -W_{ij} (e_i - e_j) + H_{ij} v_{ij}, \quad j \in N_i,$$  \hfill (21)

and can be regarded as outputs of the interconnected large-scale uncertain system comprised of systems (17).

### 4.2 The design algorithm

To present the procedure for constructing a resilient observer of the form (10), let us consider the error dynamics of the observer (18). Define $z_i = e_i - \hat{e}_i$, $\delta_i = e_i - \dot{e}_i$. Then it follows from (17), (18) that

$$\dot{z}_i = (A(t) - \bar{L}_i^* C_i(t) - \sum_{j \in N_i} \bar{K}_{ij}^* W_{ij}) z_i - F_i \Upsilon_i \delta_i$$

$$+ \sum_{j \in N_i} \bar{K}_{ij}^* (W_{ij} z_j - H_{ij} v_{ij}) + B w$$

$$- \bar{L}_i^* (t) D_i(t) v_i + F_i \nu_i,$$

$$\dot{\delta}_i = \Omega_i \delta_i - \bar{L}_i^* (t) C_i(t) z_i - \sum_{j \in N_i} \bar{K}_{ij}^* (t) W_{ij} z_i + \Gamma_i \nu_i$$

$$+ \sum_{j \in N_i} \bar{K}_{ij}^* (t) (W_{ij} z_j - H_{ij} v_{ij})$$

$$z_i(0) = x_0 - \xi_i, \quad \delta_i(0) = 0.$$  \hfill (22)

Here we used the notation

$$\bar{L}_i^* = L_i^* + \bar{L}_i^*, \quad \bar{K}_{ij}^* = K_{ij}^* + \bar{K}_{ij}^*.$$  \hfill (23)

Although the equations describing the evolution of $z_i$ and $\delta_i$ look identical to the equations describing dynamics of the detector errors in (2) [14], the distinction lies in how $\bar{L}_i^*$, $\bar{K}_{ij}^*$ are split to provide the gains for the state observer and the attack detection filter at node $i$. Contrast to (2) [14], in this paper the matrices $L_i^*$, $K_{ij}^*$ are not considered to be given; they are determined jointly with $\bar{L}_i^*$, $\bar{K}_{ij}^*$ using the following procedure.
1. First, the coefficients $\hat{L}_i(t)$, $\hat{K}_{ij}(t)$, $\hat{L}_i(t)$, $\hat{K}_{ij}(t)$ for each system (22) are derived, to stabilize the uncertain interconnected system comprised of the systems (22) in an $\mathcal{L}_2$ sense. Then with these coefficients, one has $\bar{U}_i(e_i - \hat{e}_i) \in \mathcal{L}_2[0, \infty)$.

2. The coefficients $\hat{L}_i(t)$, $\hat{K}_{ij}(t)$ for the controlled distributed plant observer (10) are computed in parallel with the previous step. Since with the parameters derived in the previous step, the signal $f_i - \varphi_i = \bar{U}_i(e_i - \hat{e}_i) - \nu_i$ is $\mathcal{L}_2$ integrable for every admissible attack input $f_i$, this will be accomplished by treating $f_i - \varphi_i$ as a disturbance perturbing the error dynamics (16). The coefficients $\hat{L}_i(t)$, $\hat{K}_{ij}(t)$ are computed to attenuate these disturbances, along with $w$, $\nu_i$, $\nu_{ij}$. Essentially, we redesign the original unbiased distributed plant observer (5) to make it is robust against attack tracking errors which will arise as a result of applying the attack cancelling control (11).

3. Finally, the remaining coefficients $\hat{L}_i(t)$, $\hat{K}_{ij}(t)$ of the attack detector (13) are obtained from (23), using the values $\hat{L}_i(t)$, $\hat{K}_{ij}(t)$ and $\hat{L}_i(t)$, $\hat{K}_{ij}(t)$ obtained in the previous steps.

4.2.1 Stabilization of the detector error dynamics (22)

This step is identical to the corresponding step in (14). Introduce the following notation:

\[
\begin{align*}
A_i(t) & = \begin{bmatrix} A(t) & -F_i \bar{U}_i \\ 0 & \Omega_i \end{bmatrix}, & B_i & = \begin{bmatrix} B(t) \\ F_i \end{bmatrix}, \\
C_i(t) & = \begin{bmatrix} C_i(t) \\ W_{ij} \\ \vdots \\ W_{ij} \\ 0 \end{bmatrix}, & \hat{L}_i = \begin{bmatrix} \hat{L}_i^r & \hat{L}_i^r & \cdots & \hat{L}_i^r \\ \hat{K}_{ij} & \hat{K}_{ij} & \cdots & \hat{K}_{ij} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{K}_{ij} & \hat{K}_{ij} & \cdots & \hat{K}_{ij} \\ \hat{K}_{ij} & \hat{K}_{ij} & \cdots & \hat{K}_{ij} \end{bmatrix}, \\
D_i(t) & = \begin{bmatrix} D_i(t) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & H_{ij} & \cdots & 0 & Z_{ij}^{1/2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_{ij} & 0 & \cdots & Z_{ij}^{1/2} \end{bmatrix}.
\end{align*}
\]

It is assumed that $E_i(t) = D_i(t)D_i^*(t) > 0$ for all $t$.

Also, following (15), introduce a collection of positive definite $(n + n_{f_j}) \times (n + n_{f_j})$ block-diagonal matrices $R_i, X_i, i = 1, \ldots, N$, partitioned as

\[
R_i = \begin{bmatrix} R_i & 0 \\ 0 & \hat{R}_i \end{bmatrix}, & X_i = \begin{bmatrix} X_i \\ 0 \\ \hat{X}_i \end{bmatrix},
\]

with $n \times n$ matrices $R_i, X_i$ and $n_{f_j} \times n_{f_j}$ matrices $\hat{R}_i, \hat{X}_i$. Also, define the block matrix $\Phi = [\Phi_{ij}]_{i,j=1}^N$,

\[
\Phi_{ij} = \begin{cases} \Delta_i, & i = j, \\ -W_{ij}U_{ij}^{-1}W_{ij}, & i \neq j, j \in N_i, \\ 0, & i \neq j, j \notin N_i, \end{cases}
\]

\[
U_{ij} = H_{ij}H_{ij}^* + Z_{ij}, & \Delta_i = \sum_{j \in N_i} W_{ij}U_{ij}^{-1}Z_{ij}U_{ij}^{-1}W_{ij},
\]

$Z_{ij} i = 1, \ldots, N, j \in N_i$, are square $p_{ij} \times p_{ij}$ positive definite matrices. Also, let $R = \text{diag}[R_1, \ldots, R_N]$, $\Delta = \text{diag}[\Delta_1, \ldots, \Delta_N]$.

**Lemma 2** (cf. (13) (15)) Suppose there exists a constant $\gamma > 0$ and symmetric matrices $R_i \geq 0$, $\hat{R}_i \geq 0$, $Z_{ij} > 0$, $j \in N_i, i = 1, \ldots, N$, such that
(i) the following linear matrix inequalities are satisfied
\[ R + \gamma^2(\Phi + \Phi' - \Delta) > 0, \quad \hat{R}_i > \hat{Y}_i'\hat{Y}_i; \]  
(ii) each differential Riccati equation
\[ \dot{Y}_i = A_i Y_i + Y_i A_i' - Y_i (C_i' E_i^{-1} C_i - \frac{1}{\gamma^2} R_i) Y_i + B_i B_i', \]  
has a positive definite symmetric bounded solution \( Y_i(t) \) on the interval \([0, \infty)\), i.e., for all \( t \geq 0 \), \( \alpha_1 I < Y_i(t) = Y_i'(t) < \alpha_2 I \), for some \( \alpha_{1,2} > 0 \).

Then the network of systems (22) with the coefficients \( \hat{L}_{ij}^r, \hat{K}_{ij}^r, \hat{L}_{ij}', \hat{K}_{ij}' \), obtained by partitioning the matrices
\[ L_i'(t) = Y_i(t) C_i(t)' E_i^{-1}(t). \]  
according to (24), guarantees that the noise- and attack-free network is exponentially stable, and in the presence of disturbances or an attack it holds that
\[ \sum_{i=1}^{N} \int_0^\infty \| \dot{z}_i - \varphi_i \|^2 dt \leq \gamma^2 \sum_{i=1}^{N} \left( (x_0 - \xi_i)' X_i^{-1}(x_0 - \xi_i) + \int_0^\infty (\| w \|^2 + \| v_i \|^2 + \| \nu_i \|^2 + \sum_{j \in N_{ij}} \| v_{ij} \|^2)^2 dt \right). \]  

The proof of Lemma 2 is omitted for brevity, it uses a completion of squares argument to establish that \( V = \sum_{i=1}^{N} |z_i' \delta_i| Y_i^{-1}(t) |z_i' \delta_i| \) is a Lyapunov function for the large-scale system comprised of systems (22). Also, it trivially follows from (30) that each signal
\[ \eta_{ij} = -W_{ij} z_j, \quad j \in N_i, \]  
is \( L_2 \)-integrable; each such signal \( \eta_{ij} \) connects the system (22) at node \( j \) with the analogous system at node \( i, j \in N_i \). It then follows from Lemma 2 condition (ii), that each detector ensures the following decentralized disturbance attenuation performance:
\[ \int_0^\infty (\| z_i \|^2 R_i + \| \hat{\delta}_i \|^2 \hat{R}_i) dt \leq \gamma^2 \left( (x_0 - \xi_i)' X_i^{-1}(x_0 - \xi_i) + \int_0^\infty (\| w \|^2 + \| v_i \|^2 + \sum_{j \in N_{ij}} (\| \eta_{ij} \|^2 + \| v_{ij} \|^2)^2 dt \right). \]  

This explains the role of the matrices \( Z_{ij} \) as weights on the contribution of the interconnection signals \( \eta_{ij} \) in the individual performance of each detector component (18).

According to Lemma 2 each node computes the matrix \( L_i' \) and its components \( \hat{L}_{ij}^r, \hat{K}_{ij}^r(t), \hat{L}_{ij}', \hat{K}_{ij}'(t) \) independently from other nodes. For this, the respective Riccati differential equation (28) must be solved on-line; this allows the matrix \( L_i' \) to be computed and partitioned according to (24) in real time. Unlike (2), the nodes do not need to communicate to solve these Riccati equations. To setup these equations, the matrices \( R_i \) must be determined first from the LMI (27). Even though this step must be performed centrally, it does not require the knowledge of the parameters of the system observed; only the matrices \( W_{ij} \) and \( H_{ij} \) are required which characterize the communication network. Compared with (2), this reduces substantially the amount of information that the node must agree upon in advance. As long as the matrices \( W_{ij} \), \( H_{ij} \) and \( Z_{ij} \) and the disturbance attenuation parameter \( \gamma^2 \) do not change, the same matrices \( R_i \) and \( X_i \) can be utilized even when the plant changes substantially. In the case of such an event, each node must only update its Riccati equation (28); it can do so without communicating with its neighbours.
4.2.2 Design of the resilient distributed plant observer (10)

Now that we able to guarantee that $f_i - \varphi_i = (f_i - \varphi_i) - \nu_i \in \mathcal{L}_2$, the large-scale system comprised of the error dynamics (10) of the observer (10), (11) can be stabilized in an $H_{\infty}$ sense, while attenuating this disturbance. The coefficients $L^r_i, K^r_{ij}$ which accomplish this task can be computed in parallel with the coefficients of the attack detector, using the same approach based on the results of [15]. To present this step of our algorithm, introduce the notation

$$
B_{1,i} = \begin{bmatrix} B(t) & F_i \end{bmatrix}, \quad C_{1,i}(t) = \begin{bmatrix} C_i(t) & W_{ij} & \cdots & W_{ijN} \end{bmatrix}^T,
$$

$$
L_{1,i} = \begin{bmatrix} L^r_i & K^r_{ij} & \cdots & K^r_{ijN} \end{bmatrix}.
$$

(33)

**Lemma 3 (see [15])** Suppose there exists a constant $\gamma > 0$ and symmetric matrices $\bar{R}_i \geq 0, \bar{X}_i > 0, Z_{ij} > 0, j \in \mathbb{N}_i, i = 1, \ldots, N$, such that

(i) the following linear matrix inequality is satisfied

$$
\bar{R} + \gamma^2 (\Phi + \Phi^T - \Delta) > P,
$$

(34)

where $\bar{R} = \text{diag}[\bar{R}_1, \ldots, \bar{R}_N]$, and $\Phi, \Delta$ are the matrices defined in [25], [26] which are the same as in Lemma 2.

(ii) each differential Riccati equation

$$
\dot{Y}_i = AY_i + Y_i A^T - Y_i (C_{i1} E_{i1}^{-1} C_{1i1} - \frac{1}{\gamma^2} \bar{R}_i) Y_i + B_{1,i} B_{1,i}^T, \quad Y_i(0) = \bar{X}_i^{-1},
$$

(35)

has a positive definite symmetric bounded solution $Y_i(t)$ on the interval $[0, \infty)$, i.e., for all $t \geq 0$, $\alpha_1 I < Y_i(t) = Y_i(t) < \alpha_2 I$, for some $\alpha_{1,2} > 0$.

Then the network of systems (16), with the coefficients $L^r_i, K^r_{ij}$, obtained by partitioning the matrices

$$
L^r_{1,i}(t) = Y_i(t) C_{1,i}(t)' E_{i1}^{-1}(t).
$$

(36)

according to [33], guarantees that the noise- and attack-free network of error dynamics (16) is exponentially stable, and in the presence of disturbances or an attack it holds that

$$
\int_0^\infty e^T P e dt \leq \gamma^2 \sum_{i=1}^N \left( (x_0 - \xi_i)' X_i^{-1} (x_0 - \xi_i) + \int_0^\infty (\|w\|^2 + \|v_i\|^2 + \|\varphi_i - f_i\|^2 + \sum_{j \in \mathbb{N}_i} \|v_{ij}\|^2) dt \right).
$$

(37)

The proof of the lemma is analogous to the proof of the corresponding result in [15].

4.2.3 The main result

The main result of this paper follows from the properties of the observer errors (16) and the properties of the errors of the attack detection filters [13].

\[^1\text{Performance tuning of the algorithm may require one to choose different matrices } Z_{ij} \text{ in this step of the algorithm. In this case, the matrices } \Phi, \Delta, \text{ and } E_{i1} \text{ will also need to be updated, and will not be the same as in Lemma 2. However, this does not have any effect on the statement of Lemma 4.} \]
Theorem 1 Suppose the conditions of Lemmas 2 and 3 hold. Let the coefficients  \( \hat{L}_i, \hat{K}_{ij} \) of the detectors (18) be obtained as described in Lemma 2, and let the coefficients  \( \bar{L}_i, \bar{K}_{ij} \) be obtained using the matrices  \( \hat{L}_i, \hat{K}_{ij}, \bar{L}_i, \bar{K}_{ij} \) from Lemmas 2 and 3, as

\[
\hat{L}_i = \bar{L}_i - L_i, \quad \hat{K}_{ij} = \bar{K}_{ij} - K_{ij}.
\]

Then, the network of state observers (10), augmented with the network of attack detectors (18) produces state estimates \( \hat{x}_i \) which have the following convergence properties.

(i) In the absence of disturbances and biasing attacks, \( \|x - \hat{x}_i\| \to 0 \) exponentially as \( t \to \infty \).

(ii) When the plant and/or the network is subject to \( L_2 \)-integrable disturbances and/or admissible biasing attacks, the estimates \( \hat{x}_i \) converge to \( x \) in the \( L_2 \) sense, and the resilient performance of this observer network is characterized by the condition

\[
\int_0^\infty e^t P e dt \leq \gamma^2 \sum_{i=1}^N \left( (x_0 - \xi_i)^t (\hat{X}_i^{-1} + 2\gamma^2 X^{-1}_i) (x_0 - \xi_i) \\
+ (1 + 2\gamma^2) \int_0^\infty (\|w\|^2 + \|v_i\|^2 + \sum_{j \in N_{ij}} \|v_{ij}\|^2) dt \right) \\
+ 2\gamma^2 (1 + \gamma^2) \sum_{i=1}^N \int_0^\infty \|v_i\|^2 dt.
\]

Also, the outputs \( \varphi_i \) of the distributed attack detector network (18) track the attack inputs in the \( L_2 \) sense.

5 Conclusion

We have proposed a novel class of distributed observers for robust estimation of a linear plant, which are resilient to biasing misappropriation attacks. The observers involve feedback from an additional network of attack detection filters, which can also signal the attack. The design method is based on the methodology of distributed and decentralized \( H_\infty \) filtering which is combined with a decoupling technique to obtain observers which attenuate benign disturbances, while sensing and compensating biasing inputs.

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