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A Gaussian correlation inequality
for plurisubharmonic functions

F. Barthe and D. Cordero-Erausquin

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Abstract
A positive correlation inequality is established for circular-invariant plurisubharmonic functions, with respect to complex Gaussian measures. The main ingredients of the proofs are the Ornstein-Uhlenbeck semigroup, and another natural semigroup associated to the Gaussian $\overline{\partial}$-Laplacian.

1 Introduction

The motivation of the present work comes from a Gaussian moment inequality in $\mathbb{C}^n$ due to Arias de Reyna [1]. We will show that his result is in fact a very particular case of a new correlation inequality, which can be seen as the complex analogue of the following correlation inequality for convex function in $\mathbb{R}^n$ due to Hu [6]: if $\mu$ is a centered Gaussian measure on $\mathbb{R}^n$ and if $f, g : \mathbb{R}^n \to \mathbb{R}$ are convex functions in $L^2(\mu)$ and $f$ is even, then

$$\int f g \, d\mu \geq \int f \, d\mu \int g \, d\mu.$$ 

We will say that a function on $\mathbb{C}^n$ is circular-symmetric if it is invariant under the action of $S^1$ (i.e. multiplication by complex numbers of modulus one); in other words a function $f$ defined on $\mathbb{C}^n$ is circular-symmetric if $f(e^{i\theta}w) = f(w) \quad \forall \theta \in \mathbb{R}, \quad \forall w \in \mathbb{C}^n$.

A function $u : \mathbb{C}^n \to [-\infty, +\infty)$ is plurisubharmonic (psh) if it is upper semi-continuous and for all $a, b \in \mathbb{C}^n$ the function $z \in \mathbb{C} \mapsto u(a + zb)$ is subharmonic. Classically, a twice continuously differentiable function $u : \mathbb{C}^n \to \mathbb{R}$ is psh if and only if for all $w, z \in \mathbb{C}^n$

$$\sum_{j,k} \partial^2_{z_j z_k} u(z) w_j \overline{w_k} \geq 0,$$

where $\partial_{z_j} = \frac{1}{2} (\partial_x - i \partial_y)$, $\partial_{\overline{z}_k} = \frac{1}{2} (\partial_x + i \partial_y)$, $z = x + iy$ with $x, y \in \mathbb{R}^n$. The later condition means that the complex Hessian $D^2 \! C u$ is pointwise Hermitian semi-definite positive. We refer e.g. to the textbook [5] for more details.

We consider the standard complex Gaussian measure $\gamma$ on $\mathbb{C}^n$,

$$d\gamma(w) = d\gamma_n(w) = \pi^{-n} e^{-|w|^2} \overline{w} \, d\ell(w) = \pi^{-n} e^{-|w|^2} \, d\ell(w),$$

where $\ell$ denotes the Lebesgue measure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ and $w \cdot w' = \sum w_j w'_j$ for $w, w' \in \mathbb{C}^n$. For convenience, let us introduce the following class of $L^2(\gamma)$ functions with controlled growth at infinity:

$$\mathcal{G} := \left\{ f : \mathbb{C}^n \to \mathbb{C} ; \ f \in L^1_{loc}(\lambda) \text{ and } \exists c, C > 0 \text{ such that } |f(w)| \leq e^{c|w|^2 - \epsilon}, \ \forall |w| \geq C \right\}.$$

In particular any function (locally $L^2$) dominated by a polynomial function on $\mathbb{R}^{2n}$ belongs to $\mathcal{G}$. Our main result reads as follows:
Theorem 1 (Correlation for psh functions). Let \( f, g : \mathbb{C}^n \to [-\infty, \infty) \) be two plurisubharmonic functions belonging to \( \mathcal{G} \). If \( f \) is circular-symmetric, then

\[
\int f g \, d\gamma \geq \int f \, d\gamma \int g \, d\gamma.
\]

One can extend the result by approximation to more general psh functions in \( L^2(\gamma) \). The inequality also extends to arbitrary centered complex Gaussian measure, which are images of \( \gamma \) by \( \mathbb{C} \)-linear maps. Indeed composing a psh function with a \( \mathbb{C} \)-linear map gives another psh function.

Let us give some direct consequences of this theorem. First, we see that when \( f = |F|^\alpha \) is psh, and if \( F \) is homogeneous, then \( f \) is also circular-symmetric. This argument can also be used for products of the form

\[
f := |F_1|^{\alpha_1} \ldots |F_k|^{\alpha_k}
\]

where the \( F_j \) are holomorphic and the \( \alpha_j \)'s are nonnegative real numbers, so that \( f \) is log-psh, in the sense that

\[
\log f(w) = \sum_{\ell=1}^{k} \alpha_{\ell} \log |F_\ell(w)|
\]

is psh. This implies that \( f \) is also psh, and if the holomorphic functions \( F_j \) are homogeneous then \( f \) is also circular-symmetric.

Theorem 2. Let \( F_1, \ldots, F_N \) be a family of homogeneous polynomial functions on \( \mathbb{C}^n \). Then for any \( \alpha_1, \ldots, \alpha_N \geq 0 \) and \( k \leq N-1 \) we have

\[
\int \prod_{j=1}^{N} |F_j|^{\alpha_j} \, d\gamma \geq \left( \int \prod_{j=1}^{k} |F_j|^{\alpha_j} \, d\gamma \right) \left( \int \prod_{j=k+1}^{N} |F_j|^{\alpha_j} \, d\gamma \right) \geq \prod_{j=1}^{N} |F_j|^{\alpha_j} \, d\gamma.
\]

A standard complex Gaussian vector in \( \mathbb{C}^n \) is a random vector taking values in \( \mathbb{C}^n \) according to the distribution \( \gamma = \gamma_n \). A random vector \( X = (X_1, \ldots, X_N) \in \mathbb{C}^N \) is a centered complex Gaussian vector if there is an \( n \), a standard complex Gaussian vector \( G \) in \( \mathbb{C}^n \) and a \( \mathbb{C} \)-linear map \( A : \mathbb{C}^n \to \mathbb{C}^N \) such that \( X = AG \). It turns out that the law for \( X \) is then characterized by its complex covariance matrix \((E(X_k X_l))_{1 \leq k, l \leq N}\). Denoting by \( a_1, \ldots, a_N \in \mathbb{C}^n \) the rows the matrix of \( A \) in the canonical basis, \( X_j = G \cdot a_j \). Applying the later theorem to the complex linear forms \( F_j(w) = w \cdot a_j \) yields the following result.

Theorem 3. Let \( (X_1, \ldots, X_N) \in \mathbb{C}^N \) be a centered complex Gaussian vector, and let \( \alpha_1, \ldots, \alpha_N \in \mathbb{R}^+ \). Then, for any \( k \leq N-1 \)

\[
E \prod_{j=1}^{N} |X_j|^{\alpha_j} \geq \left( E \prod_{j=1}^{k} |X_j|^{\alpha_j} \right) \left( E \prod_{j=k+1}^{N} |X_j|^{\alpha_j} \right) \tag{1}
\]

and in particular

\[
E \prod_{j=1}^{N} |X_j|^{\alpha_j} \geq \prod_{j=1}^{N} E |X_j|^{\alpha_j}. \tag{2}
\]

In other words, among centered complex Gaussian vectors \( (X_1, \ldots, X_N) \in \mathbb{C}^N \) with fixed diagonal covariance (i.e. \((E|X_j|^2)_{j \leq N}\) fixed) the expectation of \( \prod_{j=1}^{N} |X_j|^{\alpha_j} \) is minimal when the variables are independent.
Inequality (2) is an extension of an inequality of Arias de Reyna [1], who established the particular case where all the $a_j = 2p_j$ are even integer by rewriting the left hand side in terms of a permanent of a $2m$ matrix ($m = \sum p_j$) and using an inequality for permanents due to Lieb. Actually, Inequality (1) in the case where the $a_j$ are even integers is equivalent to Lieb’s permanent inequality, so in particular we are giving a new proof of this inequality.

In the next section we will introduce the tools that will be used in the proof, that is two semi-groups: the usual Ornstein-Uhlenbeck semi-group and another natural semi-group associated to the $\mathcal{D}$ operator (the generator of which could be called, depending from the context, Landau or magnetic Laplacian). In the last section we give the proof of our correlation inequality.

2 Semi-groups

To get the result, we will let the circular-symmetric psh function evolve along the Ornstein-Uhlenbeck semi-group on $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n \simeq \mathbb{R}^{2n}$ associated with the measure $\gamma$ and the scalar product $: w, w' : = \Re(w \cdot \overline{w'})$. We recall that its generator is given, for smooth functions $f$, writing $w = x + iy$, $x, y \in \mathbb{R}^n$, by

$$L^0_t f(w) := \frac{1}{4} \Delta f(w) - \frac{1}{2} (w, \nabla) f(w)$$

$$= \sum_{j=1}^{n} \left( \frac{1}{4} \partial^2_{x_j x_j} f(w) + \partial^2_{y_j y_j} f(w) \right) - \frac{1}{2} \left( x_j \partial_{x_j} f(w) + y_j \partial_{y_j} f(w) \right).$$

Note that the normalization differs slightly from the usual one on $\mathbb{R}^{2n}$ because our Gaussian measure has complex covariance $\text{Id}_n$ but real covariance equal to $\frac{1}{2} \text{Id}_{2n}$. Accordingly, the spectrum of $-L^0_t$ on $L^2(\gamma)$ is here $\mathbb{R} \cup \{0\}$. The Ornstein-Uhlenbeck semi-group $P^0_t = e^{tL^0}$ admits the representation, for suitable functions $f : \mathbb{C}^n \to \mathbb{C}$,

$$P^0_t f(z) = \int f(e^{-t/2} z + \sqrt{1-e^{-t}} w) d\gamma(w)$$

$$= \pi^{-n} (1 - e^{-t})^{-n} \int f(w) e^{-\frac{1}{2} \sum_{j=1}^{n} |w_j - e^{-t/2} z_j|^2} d\ell(w)$$

As usual in semi-group methods, it is convenient to work with a nice stable space of functions. Here, we can for instance consider

$G^\infty := \{ f \in C^\infty(\mathbb{C}^n) : f \text{ and all its derivatives belong to } G \}$.

Note that for $f \in G$, we can define $P^0_t f$ by (3) and then we have that $(t, z) \mapsto P^0_t f(z)$ is smooth on $(0, \infty) \times \mathbb{C}^n$, with $P^0_t f \in G^\infty$ and $\partial f P^0_t f = L^0_t P^0_t f$ for every $t > 0$, and $P^0_t f \to f$ in $L^2(\gamma)$ as $t \to 0$. We refer to [3] for details. Let us also mention here that with formula (3) it is readily checked that properties like convexity, subharmonicity, pluri-subharmonicity are preserved along $P^0_t$.

Another natural operator will be used. Indeed, since pluri-subharmonicity is characterized through a “$\partial_{\overline{z}}$ operation”, we shall also use the following differential operator on smooth functions $f$ on $\mathbb{C}^n$:

$$L f = \sum_{j=1}^{n} \left( \partial^2_{\overline{z}_j f} - \overline{\partial f} \right) = \sum_{j=1}^{n} e^{\frac{|z|^2}{2}} \partial_{\overline{z}_j} \left( e^{-|z|^2} \partial f \right).$$

Note that $L f = 0$ if (and only if, see below) $f$ is holomorphic. Formally $L = -\overline{\partial} \partial$ on $L^2(\gamma)$ equipped with the Hermitian structure $(f, g) = \int f \overline{g} d\gamma$. More precisely, denoting for a differentiable function

$$\partial f = (\partial_{\overline{z}_1} f, \ldots, \partial_{\overline{z}_n} f) \quad \text{and} \quad \partial_{\overline{z}} f := (\partial_{z_1} f, \ldots, \partial_{z_n} f)$$

we have the following standard fact.
Fact 4 (Integration by parts). For regular enough functions \( f, g : \mathbb{C}^n \to \mathbb{C} \), for instance for functions in \( \mathcal{G}^\infty \), we have

\[
\int (Lf) \overline{g} \, d\gamma = - \int \partial_x f \cdot \overline{\partial_x g} \, d\gamma
\]

and in particular \( \int (Lf) \overline{f} \, d\gamma = - \int |\partial_x f|^2 \, d\gamma \leq 0 \). We can also write

\[
\int (Lf) \overline{g} \, d\gamma = - \int \partial_x f \cdot \partial_x \overline{g} \, d\gamma = \int f(Lg) \, d\gamma
\]

where \( Lf := \sum_{j=1}^n \left( \partial_{z_j}^2 f - z_j \partial_{\overline{z}_j} f \right) \).

Indeed, it suffices to sum over \( j \) the equations

\[
\int \left[ e^{\left| z \right|^2} \partial_z (e^{-\left| z \right|^2} \partial_{\overline{z}} f) \right] \overline{g} \, d\gamma = \pi^{-n} \int \partial_z (e^{-\left| z \right|^2} \partial_{\overline{z}} f) \overline{g} \, d\lambda(z) = - \int \partial_{\overline{z}} f \partial_z \overline{g} \, d\gamma = - \int \partial_{\overline{z}} f \partial_{\overline{z}} \overline{g} \, d\gamma.
\]

The assumption that \( f, g \in \mathcal{G}^\infty \) guarantees that the boundary terms (at infinity) in the integration by parts vanish.

As a consequence, we see that the Gaussian measure \( \gamma \) is invariant for \( L \), and actually that \( L \) is a symmetric nonpositive operator on \( L^2(\gamma) \) with the above-mentioned Hermitian structure. The kernel of \( L \) is the Bargmann space \( \mathcal{H}_0 \) formed by the holomorphic functions on \( \mathbb{C}^n \) that belong to \( L^2(\gamma) \).

We want to work with the semi-group \( P_t = e^{tL} \) which is also Hermitian (formally):

\[
\int (P_t f) \overline{g} \, d\gamma = \int \overline{T_t g} \, d\gamma
\]

Although we will not explicitly use it, let us discuss a bit the (well known) spectral analysis of \( L \) on the complex Hilbert space \( L^2(\gamma) \). This analysis is indeed fairly standard using the ideas introduced by Landau. Following for instance the presentation given in [4, Section 4], consider the “annihilation” operators \( a_j = \partial_x \) and their adjoints, the “creation” operators \( b_j := a_j^* = \overline{\partial_x} - \partial_{\overline{z}_j} \). Then \( L = - \sum_{j \leq n} b_j \circ a_j \), with \( [a_j, b_j] = 1 \), and all these operators commute for distinct indices \( j \). Plainly, if a function \( f \) and a scalar \( \lambda \in \mathbb{C} \) satisfy \(-L f = \lambda f\), then \(-L(a_j f) = (\lambda - 1)a_j f \) and \(-L(b_j f) = (\lambda + 1)b_j f \). This implies that the spectrum of \(-L\) is \( \mathbb{N} \) and that the eigenspace associated to the eigenvalue \( k \in \mathbb{N} \) is given by the sum of the spaces \( b^m \mathcal{H}_0 \) with \( m = (m_1, \ldots, m_n) \in \mathbb{N}^n \), \( |m| := \sum_{j \leq n} m_j = k \) and the convention \( b^m := b_1^{m_1} \circ \cdots \circ b_n^{m_n} \). Moreover, if we introduce the classical projection \( \Pi_0 : L^2(\gamma) \to \mathcal{H}_0 \) onto holomorphic functions

\[
\Pi_0 f(z) := \int f(w) e^{\overline{z} w} \, d\gamma(w) = \int f(z + w) e^{-\overline{w} z} \, d\gamma(w),
\]

then the projector \( \Pi_k \) on the \( k \)-eigenspace can be expressed in terms of \( \Pi_0 \) and the creation and annihilation operators. This allows to compute the reproducing kernel of \( \Pi_k \), in terms of classical families of orthogonal polynomials. Next, one can sum over \( k \) and obtain the kernel \( K_t(z, w) \) for \( e^{tL} = \sum_k e^{t \Pi_k} \). Only the formula for \( K_1 \) will be useful in the sequel and we shall actually check below that this suggested formula is indeed the kernel of \( e^{tL} \).

An explicit formula for \( K_1 \) can be found in [2]: setting

\[
K_1(z, w) := \frac{1}{\pi^n (1 - e^{-t})^n} \exp \left( z \cdot w - \frac{e^{-t} |z - w|^2}{1 - e^{-t}} - |w|^2 \right),
\]

then

\[
P_t f(z) = \int f(w) K_t(z, w) \, d\gamma(w)
\]

\[
= (1 - e^{-t})^{-n} \int f(w) e^{\overline{z} w} \frac{e^{-t|z-w|^2}}{1 - e^{-t}} \, d\gamma(w)
\]
Next, let us note that by performing the change of variable \( w = z + \sqrt{1 - e^{-t}} \xi \) for fixed \( z \) we find

\[
P_t f(z) = \int f(z + \sqrt{1 - e^{-t}} \xi) e^{-\sqrt{1 - e^{-t}} z \xi} \, d\gamma(\xi).
\]  

(10)

On this formula, we see immediately that \( P_0 = \text{Id} \) and \( P_\infty = \Pi_0 \).

To avoid discussions regarding unbounded operators and existence of semi-groups, we will proceed in the opposite direction and use the previous formula to define \( P_t f \). Actually, to be fair, we should mention that later, in the proof of our result, we only need to work with smooth functions \( f \in G^\infty \); these functions provide nice initial data, ensuring existence and uniqueness of strong solutions for the semi-group equation. Nevertheless, we feel it is of independent interest to start from the integral formula (9) or (10) and derive from it the semi-group properties; we will first check that (9) solves (11) indeed. The drawback is that some properties that are obvious (formally) for \( e^{tL} f \) need to be checked thoroughly when using this kernel representation, in particular because the kernel (8) is not Markovian.

Formulas (9)-(10) make sense pointwise, for \( z \in \mathbb{C}^n \) fixed, as long as \( f \in L^2(\gamma) \); actually we have the pointwise estimate \( |P_t f(z)| \leq C_t(z) \| f \|_{L^2(\gamma)} \) for some constant \( C_t(z) > 0 \) depending on \( t \) and \( z \). In order to derive some properties of \( P_t f \), a stronger integrability condition (as \( f \in \mathcal{G} \)) will be assumed.

**Lemma 5.** Given \( f \in \mathcal{G} \), let us define \( P_t f \) using the formula (9) or (10). Then one has that \( (t, z) \rightarrow P_t f(z) \) is smooth on \( (0, \infty) \times \mathbb{C}^n \), with for any \( t > 0 \), \( P_t f \in \mathcal{G}_\infty \) and

\[
\partial_t P_t f = LP_t f.
\]

Moreover we have

\[
\|P_t f\|_{L^2(\gamma)} \leq \|f\|_{L^2(\gamma)}
\]

and \( P_t f \rightarrow f \) in \( L^2(\gamma) \) as \( t \rightarrow 0 \).

**Proof.** It is readily checked that, for any fixed \( w \in \mathbb{C}^n \) one has

\[
\partial_t K_t(\cdot, w) = L K_t(\cdot, w).
\]

Moreover, for \( T, R, k > 0 \) fixed, there exists constants \( c = c(T, R, k) \) such that for \( F = \partial K_t \), or \( F = K_t \) or else \( F \) being any of the first \( k \)th partial derivatives of \( K_t \), it holds that \( |F(z, w)| \leq C e^{-c|w|^2} \) for all \( w \in \mathbb{C}^n \), all \( t \in (0, T) \) and all \( |z| \leq R \). From this and the definition of \( \mathcal{G} \), we can call upon dominated convergence to conclude to the smoothness of \( (t, z) \rightarrow P_t f(z) \) and to the fact that \( P_t f \in \mathcal{G}_\infty \) with \( \partial_t P_t f = LP_t f \).

Regarding the contraction property, we want to avoid direct computations or spectral arguments, and so we make a detour and use some obvious but important properties of \( P_t \).

Fist, we will use the semi-group property: for \( t, s > 0 \) that \( P_t \circ P_s = P_{t+s} \). This can be seen in two ways. One can invoke that for a smooth function \( F := P_{t} f \in \mathcal{G}_\infty \) then there is existence and uniqueness for the equation \( \partial_t F_t = LF_t \) with initial condition \( F_0 = F \) and we have seen that \( (P_{t+s} f) \) is a solution; from there one can conclude. Or else, in a more pedestrian way, one can check that

\[
\int K_t(z, w)K_s(\xi, z) \, d\ell(z) = K_{t+s}(\xi, w).
\]

For this, one may use that given \( z, \xi \in \mathbb{C}^n \) and \( c \in \mathbb{C} \) with \( \Re(c) > 0 \),

\[
\int e^{-(w+z)\xi - e^{-c}|w|^2} \, d\ell(w) = n^n e^{-n} e^{z \xi / c}.
\]

(12)

Next, recall that \( P_t \) is Hermitian, in the sense (7), on \( \mathcal{G} \subset L^2(\gamma) \); this can be seen directly from the integral formula since \( K_t(z, w) = \overline{K_t(w, z)} \). Finally, we will use that \( \|P_t f\|_{L^2(\gamma)} \) decreases for \( t \in (0, \infty) \). This is immediate from the non-positivity of \( L \) since for \( t > 0 \) we have \( P_t f \in \mathcal{G}_\infty \) and

\[
\frac{d}{dt} \int |P_t f|^2 \, d\gamma = 2 \int (LP_t f)\overline{P_t f} \, d\gamma = -2 \int \partial_t |P_t f|^2 \, d\gamma \leq 0.
\]
This establishes the desired continuity.

Using (10) we see that \( P_t f \) converge point-wise to \( f \) and that for \( t \in (0,1) \) we have \( |P_t f(z)| \leq C e^{\epsilon |z|} \) for some constant \( c, C > 0 \); so we can conclude by dominated convergence. For \( f \in \mathcal{G} \) and \( \epsilon > 0 \), introduce \( g \) smooth compactly supported such that \( \|f - g\|_{L^2(\gamma)} \leq \epsilon \) and let \( \delta > 0 \) be such that \( t \leq \delta \) ensures that \( \|P_t g - g\|_{L^2(\gamma)} \leq \epsilon \) holds. For \( t \leq \delta \),

\[
\|P_t f - f\|_{L^2(\gamma)} \leq \|P_t f - P_t g\|_{L^2(\gamma)} + \|P_t g - g\|_{L^2(\gamma)} + \|g - f\|_{L^2(\gamma)} \leq 2\|f - g\|_{L^2(\gamma)} + \epsilon \leq 3\epsilon.
\]

This establishes the desired continuity.

\[ \square \]

**Remark 6 (Contraction property).**

1. We observe that some results, which can be deduced from the spectral decomposition and Hilbertian analysis, may be derived in a soft way thanks to flexible semigroup techniques. We have proved that starting from formula (9) we have \( \|P_t f\|_{L^2(\gamma)} \leq \|f\|_{L^2(\gamma)} \) on the dense subspace \( \mathcal{G} \), which together with the pointwise estimate given just before the previous Lemma implies by density that

\[
\|P_t\|_{L^2(\gamma) \rightarrow L^2(\gamma)} \leq 1.
\]

Formally, by letting \( t \to \infty \) in \( \|P_t\|_{L^2(\gamma) \rightarrow L^2(\gamma)} \leq 1 \) we recover that

\[
\|\Pi_0\|_{L^2(\gamma) \rightarrow L^2(\gamma)} \leq 1.
\]

Actually, the convergence of \( P_t \) towards \( \Pi_0 \) can be quantified rigorously through Hörmander’s inequality,

\[
\|\varphi - \Pi_0 \varphi\|_{L^2(\gamma)}^2 \leq \|\partial \varphi\|_{L^2(\gamma)}^2 = \int (-L\varphi) \varphi \, d\gamma
\]

valid for any suitable \( \varphi \), for instance for \( \varphi \in \mathcal{G}^\infty \). Note that from formula (10), \( P_t \) acts as the identity on holomorphic functions, so \( \Pi_0 P_t = \Pi_0 \). A classical Grönwall type argument (using the previous Lemma to justify the computation of \( \frac{d}{dt} \int |P_t(f - \Pi_0 f)|^2 \, d\gamma \)) ensures that for \( f \in \mathcal{G} \) and \( t \geq 0 \)

\[
\|P_t f - \Pi_0 f\|_{L^2(\gamma)}^2 \leq e^{-2t} \|f - \Pi_0 f\|_{L^2(\gamma)}^2.
\]

2. In analogy with the Markovian case \( P^p_t \) we may wonder if \( P_t \) is also a contraction on some \( L^p(\gamma) \). However, for any \( p \neq 2 \) we have

\[
\|P_t\|_{L^p(\gamma) \rightarrow L^p(\gamma)} = +\infty,
\]

as it can be seen by taking, in dimension \( n = 1 \), for \( a \in \mathbb{R} \),

\[
f_a(w) := e^{aw + \varphi^2}, \quad w \in \mathbb{C}.
\]

Indeed, repeated applications of (12) with \( c = 1 \) show, setting \( s_t := \sqrt{1 - e^{-t}} \) and using (10), that \( P_t f_a(z) = e^{s_t^2 a z + (1 - s_t^2) \varphi^2} \) and that

\[
\frac{\|P_t f_a\|^p_{L^p(\gamma)}}{\|f_a\|^p_{L^p(\gamma)}} = C(t, p) e^{a s_t^2(p - p^2/2)}.
\]

The next result describes how derivatives and \( P_t \) commute, an important issue in semi-group methods.
Lemma 7 (Commutation relations). For any suitable \( f \), say \( f \in \mathcal{G}^\infty \), and \( t > 0 \) we have for every \( 1 \leq j \leq n \) and \( z \in \mathbb{C}^n \),

\[
\partial_{z_j}(P_t f(z)) = P_t(\partial_{z_j} f)(z) \quad \text{and} \quad \partial_{\Pi_t}(P_t f(z)) = e^{-t}P_t(\partial_{\Pi_t} f)(z).
\]

Proof. We use (10). The first equality is obvious. For the second one, setting \( s_t = \sqrt{1 - e^{-t}} \), we have

\[
\partial_{\Pi_t}(P_t f)(z) = P_t(\partial_{\Pi_t} f)(z) - s_t \int f(z + s_t \xi) e^{-s_t |\xi|^2} \xi_j d\gamma(\xi),
\]

and

\[
\pi^{-n} \int f(z + s_t \xi) e^{-s_t |\xi|^2} \xi_j e^{-\xi \cdot \bar{\xi}} d\ell(\xi) = -\pi^{-n} \int f(z + s_t \xi) e^{-s_t |\xi|^2} \partial_{\gamma_j}(e^{-\xi \cdot \bar{\xi}}) d\ell(\xi) = s_t \int (\partial_{\Pi_t} f)(z + s_t \xi) e^{-s_t |\xi|^2} d\gamma(\xi) = s_t P_t(\partial_{\Pi_t} f)(z).
\]

\[\square\]

Now, and for the rest of this section, we focus on the case of circular-symmetric functions. Given \( \theta \in \mathbb{R} \) and a function \( f \) we denote \( f_\theta \) the function \( f_\theta(w) = f(e^{i\theta}w) \). Note that

\[
P_t(f_\theta) = (P_t f)_\theta.
\]

Recall that a function \( f \) is said to be circular-symmetric if \( f_\theta = f \) for every \( \theta \). It is worth noting that a holomorphic function on \( \mathbb{C}^n \) is necessarily constant when circular-symmetric. Indeed if \( h : \mathbb{C} \to \mathbb{C} \) has both properties then invariance and the Cauchy formula give \( h(1) = \int_0^{2\pi} h(e^{i\theta}) d\theta/(2\pi) = h(0) \); next if \( f : \mathbb{C}^n \to \mathbb{C} \) is holomorphic and circular symmetric, then for any \( (z_1, \ldots, z_n) \in \mathbb{C}^n \), the function \( h : z \in \mathbb{C} \mapsto f(z z_1, \ldots, z z_n) \) is also holomorphic and circular-symmetric, hence \( f(z_1, \ldots, z_n) = h(1) = h(0) = f(0) \). Accordingly if \( f \in L^2(\gamma) \) is circular-symmetric then \( \Pi_0 f = \int f \ d\gamma \) since the gaussian density is also circular-symmetric. Actually, much more can be said, as we shall see below.

Let us first investigate the relation between \( L \) and \( L^{ou} \). Note that one can write

\[
L f = L^{ou} f + \frac{i}{2} \sum_{j=1}^n \left( y_j \partial_{z_j} f - x_j \partial_{\bar{z}_j} f \right).
\]

So we have \( L^{ou} = \Re(L) = \frac{L + \bar{L}}{2} \) where \( \Re(f) = \sum_{j=1}^n \left( \partial_{z_j}^2 f - z_j \partial_{\bar{z}_j} f \right) \) has a kernel formed by the anti-holomorphic functions. The operators \( L \) and \( \bar{L} \) are Hermitian symmetric, whereas \( L^{ou} \) is symmetric for the real and the Hermitian product, and preserves the subspace of real valued functions. As we said, its spectrum is \(-\frac{1}{2} \mathbb{N}_0\), as can be seen also from the formula \( L^{ou} = \frac{L + \bar{L}}{2} \).

Let us illustrate this on two examples, obtained by applying the creation operator \( h_1 = \partial_{\bar{z}_1} \) to the holomorphic functions \( z \mapsto 1 \), and \( z \mapsto z_1 \). The function \( z \mapsto \bar{z}_1 \) is an eigenfunction for \( L \) with eigenvalue \(-1\), for \( \bar{L} \) with eigenvalue \(0\), and for \( L^{ou} \) with eigenvalue \(-1/2\). The function \( z \mapsto |z_1|^2 - 1 \) is an eigenfunction for \( L \) with eigenvalue \(-1\), for \( \bar{L} \) with eigenvalue \(-1\), and for \( L^{ou} \) with eigenvalue \(-1\).

The special role played by circular-symmetric functions is due to the fact that these operators, and the associated semi-groups, coincide for them.

Lemma 8 (Action of \( L \) and \( P_t \) on circular-symmetric functions). If \( f \) is a smooth circular-symmetric function, then we have

\[Lf = \bar{L}f = L^{ou}f.
\]

In particular we have, when \( f, g \in \mathcal{G}^\infty \) and \( f \) is circular-symmetric,

\[\int (Lf) g \ d\gamma = \int f Lg \ d\gamma = -\int \partial_{z_j} f \cdot \partial_{\bar{z}_j} g \ d\gamma.\]
Also, if $f \in L^2(\gamma)$ is circular-symmetric then we have

$$P_t f = P_t^{ou} f$$

for every $t \geq 0$.

**Proof.** Writing $w = x + iy$, $x, y \in \mathbb{R}^n$, the symmetry rewrites as $f((\cos(\theta)x - \sin(\theta)y) + i(\cos(\theta)y + i \sin(\theta)x)) = f(x + iy)$. Taking the derivative at $\theta = 0$ we find

$$\sum_{j=1}^{n} \left( - y_j \partial_x f(x + iy) + x_j \partial_y f(x + iy) \right) = 0,$$

and this for every $x, y \in \mathbb{R}^n$. This implies in view of (14) that $L f = L^{ou} f = \mathcal{L} f$. Next, for any smooth function $g$ we have, using that $L f = \mathcal{L} f$ and (6),

$$\int (L f) g \, d\gamma = \int (\mathcal{L} f) g \, d\gamma = \int f \mathcal{L} g \, d\gamma = - \int \partial_x f \cdot \partial_x g \, d\gamma.$$

Although it is formally trivial that equality of $L$ and $L^{ou}$ on circular-symmetric functions implies equality of the semi-groups $P_t$ and $P_t^{ou}$, a bit more should be said since we defined the semi-group using the explicit formula (9). And it is anyway instructive to compute the kernels on circular-symmetric functions. Denote by $K_t^{ou}$ the kernel of the Ornstein-Uhlenbeck semi-group that we recalled above: $K_t^{ou}(z, w) = \pi^{-n}(1 - e^{-t})^{-n} e^{-\frac{1}{2} |w - e^{-t/2}z|^2}$. So we have, setting $c_t := e^{-t/2}$ and $s_t := \sqrt{1 - e^{-t}}$,

$$K_t^{ou}(z, w) = \pi^{-n} s_t^{-2n} e^{-s_t^{-2} |w|^2 - s_t^{-2} c_t^2 |z|^2} e^{s_t^{-2} c_t(w \cdot \overline{w} \cdot \overline{z})}$$

and

$$K_t(z, w) = \pi^{-n} s_t^{-2n} e^{-s_t^{-2} |w|^2 - s_t^{-2} c_t^2 |z|^2} e^{s_t^{-2} c_t^2(w \cdot \overline{w} \cdot \overline{z})}$$

Note that only the last exponentials differ in these two formulas. When $f$ is circular-symmetric, in order to check that $P_t f = P_t^{ou} f$ it suffices to check that for fixed $w, z, t$ one has

$$\frac{1}{2\pi} \int_0^{2\pi} K_t(z, e^{i\theta} w) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} K_t^{ou}(z, e^{i\theta} w) \, d\theta.$$

Observe that for $a, b \in \mathbb{C}$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} e^{a e^{i\theta} + b e^{-i\theta}} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \sum_{p, q \in \mathbb{N}} \frac{a^p b^q}{p! q!} e^{i(p - q)\theta} \, d\theta = \sum_{n \geq 0} \frac{(ab)^n}{(n!)^2} = B(ab)$$

with $B(x) := \sum_{n \geq 0} \frac{x^n}{(n!)^2}$. Therefore, we have

$$\frac{1}{2\pi} \int_0^{2\pi} K_t(z, e^{i\theta} w) \, d\theta = \pi^{-n} s_t^{-2n} e^{-s_t^{-2} |w|^2 - s_t^{-2} c_t^2 |z|^2} B(s_t^{-4} c_t^2 |w \cdot \overline{w} \cdot \overline{z}|^2)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} K_t^{ou}(z, e^{i\theta} w) \, d\theta,$$

as wanted. \hfill \square

### 3 Proof of Theorem 1

First, let us note that we can assume that $g$ is smooth, and actually that $g \in \mathcal{G}^{\infty}$. Indeed, if $g \in \mathcal{G}$ then $P_t^{ou} g \in \mathcal{G}^{\infty}$ and we mentioned that $P_t^{ou} g$ converges to $g$ in $L^2(\gamma)$ and therefore also in $L^1(\gamma)$, as $t \to 0$. Consequently, if we know the conclusion for a function in $\mathcal{G}^{\infty}$, then

$$\int f P_t^{ou} g \, d\gamma \geq \int f \, d\gamma \int P_t^{ou} g \, d\gamma$$

8
Since \( P \) which is enough, since a convex function with a bounded limit at zero. Since \( P \) the continuity at zero). Since \( P \) going on? along the Ornstein-Uhlenbeck semi-group exactly as in the case of convex functions, so what is going on?

As in the proof of the correlation for convex functions [6], we will compute some kind of second derivative in \( t \) for integrals involving \( P_{t}^{\alpha} f \); recall that \( \partial_{k} P_{t}^{\alpha} f = L_{t} P_{t}^{\alpha} f \). The novelty is that, along the way, we will also use \( P_{t} f \) which satisfies \( \partial_{k} P_{t} f = L_{t} P_{t} f \) (Lemma 5).

Consider

\[
\alpha(t) := \int (P_{t}^{\alpha} f) g \, d\gamma = \int (P_{t} f) g \, d\gamma \in \mathbb{R}.
\]

The function \( \alpha \) is, by construction, smooth on \((0, \infty)\) and continuous on \([0, \infty)\) (see Lemma 5 for the continuity at zero). Since \( P_{t}^{\alpha} f \) tends to the constant \( \int f \, d\gamma \) when \( t \to \infty \), we have

\[
\alpha(t) \to \int f \, d\gamma.
\]

In order to conclude, it suffices to show that \( \alpha \) decreases. Actually we will prove that \( \alpha \) is convex; which is enough, since a convex function with a bounded limit at \( +\infty \) cannot increase. It holds

\[
\alpha'(t) = \int L^{\alpha}(P_{t}^{\alpha} f) g \, d\gamma = \int L P_{t} f g \, d\gamma. \tag{15}
\]

Since \( P_{t} f \) is also circular-symmetric, we can invoke Lemma 8 and write

\[
\alpha'(t) = -\int \partial_{z} P_{t} f \cdot \partial_{\overline{z}} g \, d\gamma.
\]

Next, using the first commutation relation from Lemma 7 we get

\[
\alpha'(t) = -\sum_{j=1}^{n} \int P_{t}(\partial_{z_{j}} f) \partial_{\overline{z}_{j}} g \, d\gamma.
\]

We stress that \( \partial_{z} f \) is no longer circular-symmetric, so we cannot exchange \( P_{t} \) and \( P_{t}^{\alpha} \). The second derivative of \( \alpha \) is, using Fact 4,

\[
\alpha''(t) = -\sum_{j=1}^{n} \int L(P_{t}(\partial_{z_{j}} f)) \partial_{\overline{z}_{j}} g \, d\gamma = \sum_{j=1}^{n} \int \partial_{\overline{z}_{j}}(P_{t}(\partial_{z_{j}} f)) \cdot \partial_{z_{j}}(\partial_{\overline{z}_{j}} g) \, d\gamma = \sum_{j,k=1}^{n} \int \partial_{\overline{z}_{j}} P_{t}(\partial_{z_{j}} f) \partial_{z_{k}}^{2} \overline{g} \, d\gamma.
\]

Using the commutation relation from Lemma 7 we can write

\[
\alpha''(t) = \sum_{j,k=1}^{n} \int \partial_{\overline{z}_{j}}^{2} P_{t}(f) \partial_{z_{k}}^{2} \overline{g} \, d\gamma = \int \text{Tr} \left( (D_{z}^{2} P_{t} f)(z) (D_{\overline{z}}^{2} g)(z) \right) \, d\gamma
\]

where for a \( C^{2} \) function \( h \) on \( \mathbb{C}^{n} \) the notation \( D_{z}^{2} h(z) \) refers to the Hermitian matrix \( \left( \partial_{z_{j}} \overline{\partial_{z_{k}}} h(z) \right)_{j,k \leq n} \).

Since \( P_{t} f = P_{t}^{\alpha} f \) and \( g \) are psh, the corresponding matrices are nonnegative Hermitian matrices, which means that the trace of their product is still nonnegative. This shows that \( \alpha'' \geq 0 \) and finishes the proof of the theorem.

\[\square\]

We would like to conclude with a discussion of the differences between the real case and the complex case. After all, we are computing second derivatives of the same object

\[
\alpha(t) = \int (P_{t}^{\alpha} f) g \, d\gamma
\]

along the Ornstein-Uhlenbeck semi-group exactly as in the case of convex functions, so what is going on?
In both cases we prove that \( \alpha \) decreases by showing that \( \alpha' \leq 0 \) using the next derivative somehow, but we compute these derivatives differently. The argument for convex function goes as follows. A direct computation and usual commutation properties show that, if \( \int \nabla f \, d\gamma = 0 \), which is the case when \( f \) is even, then

\[
\alpha'(t) = -e^{-t/2} \int_t^\infty \left( \int \text{Tr} \left( (D^2 P_t^{ou} f)(z) \ (D^2 g)(z) \right) d\gamma(z) \right) e^{s/2} ds
\]

where \( D^2 \) refers to the usual (real) Hessian on \( \mathbb{R}^{2n} \); from this we conclude to the correlation for convex functions. On the other hand, we have proved, when \( f \) is circular-symmetric, that

\[
\alpha'(t) = -\int_t^\infty \left( \int \text{Tr} \left( (D^2 P_t^{ou} f)(z) \ (D^2 g)(z) \right) d\gamma(z) \right) ds.
\]

Note that we have used here that \( \alpha'(t) \) tends to 0 when \( t \to +\infty \); this follows from the fact that \( \alpha \) is convex and has a finite limit at \( +\infty \), and can also be seen from (15) since \( P_t^{ou} f \) tends to a constant when \( t \to +\infty \). It is because we wanted to work with complex derivatives that we aimed at inserting \( L \) in place of \( L^{ou} \); recall that \( \partial_z^2 f \) need not be circular-symmetric even when \( f \) is, although the second derivatives are again circular-symmetric.

Finally, let us observe that if we consider in dimension 1 the circular-symmetric psh functions \( f(w) = |w|^{1/3} \) and \( g(w) = |w|^4 \) on \( \mathbb{C} \simeq \mathbb{R}^2 \), then, as we already mentioned,

\[
\text{Tr} \left( (D^2 f)(D^2 g) \right) = \Delta f \Delta g \geq 0,
\]

but a direct computation shows that

\[
\text{Tr} \left( (D^2 f)(D^2 g) \right) = \frac{4}{3} |z|^{1/3} \leq 0 \quad \forall z \in \mathbb{C}.
\]

Of course, this discrepancy cannot hold at all times for \( P_t^{ou} f \) in place of \( f \) (and moreover \( f \) is not smooth at zero, although this is not really an issue). But it suggests that the two formulas above for \( \alpha'(t) \) are indeed quite different.

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