Nonlinear variations in axisymmetric accretion

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ABSTRACT
Stationary solutions of an inviscid and rotational accretion process have been subjected to a time-dependent radial perturbation, whose equation includes nonlinearity to any arbitrary order. Regardless of the order of nonlinearity, the equation of the perturbation bears a form that is remarkably similar to the metric equation of an analogue acoustic black hole. Casting the perturbation as a standing wave and maintaining nonlinearity in it up to the second order, bring out the time-dependence of the perturbation in the form of a Liénard system. A dynamical systems analysis of this Liénard system reveals a saddle point in real time, with the implication that instabilities will develop in the accreting system when the perturbation is extended into the nonlinear regime. The instability of initial sub-critical states may also adversely affect the non-perturbative drive of the flow towards a final and stable critical state.

Key words: accretion, accretion discs – black hole physics – hydrodynamics – instabilities – methods: analytical

1 INTRODUCTION
Compressible fluids, possessing angular momentum, will execute rotational motion when drawn into the gravitational potential well of an accretor (Frank et al. 2002). Devising self-consistent and effective mathematical models to capture the underlying physics of such phenomena is a matter of abiding interest to the researcher in astrophysical flows. This is more so true, when the accretor in question is a black hole. Astrophysical black holes make their heavy presence felt only by dint of their strong gravitation. Due to their event horizons, no direct spectral information about black holes will ever be forthcoming. In this situation, observational evidence for black holes can only be obtained indirectly from the way their strong gravitational fields influence any proximate astrophysical fluid.

To keep matters mathematically simple, a common assumption about rotational accretion is that the fluid flow in question is non-self-gravitating (in contrast to, say, a star), and that the motion is axisymmetric. Nevertheless, the mathematical problem remains complex enough, involving nonlinear partial differential equations of fluid dynamics (Landau & Lifshitz 1987). One of the principal topics in studies of axisymmetric accretion is the manner in which angular momentum is transported, enabling a fluid element to move towards the accretor in a spiral trajectory. The viscosity of the fluid affords the formation of this type of a differentially rotating hydrodynamic flow (Lynden-Bell 1969; Shakura & Sunyaev 1973, Lynden-Bell & Pringle 1974, Pringle 1981, Frank et al. 2002), although the exact nature of the viscous transport is a question that has not yet received its final answer (Pringle 1981, Papaloizou & Lin 1995, Frank et al. 2002). This, notwithstanding the effort that has already gone into comprehending the role of viscosity in axisymmetric accretion from multiple perspectives (Shakura & Sunyaev 1973, Novikov & Thorne 1973, Lynden-Bell & Pringle 1974, Eardley & Lightman 1975, Shapiro et al. 1976, Shakura & Sunyaev 1976, Ichimaru 1977, Katz 1977, Begelman 1978, Piran 1978, Li & Thomson 1980, Pringle 1981, Begelman & Rees 1982, Rees et al. 1982, Matsumoto et al. 1984, Muchmore & Czerny 1986, Abramowicz et al. 1988, Eggum et al. 1988, Björnsson & Svensson 1991, Luo & Liang 1994, Narayan & Yi 1994, Abramowicz et al. 1995, Chen et al. 1995, Chakrabarti & Titarchuk 1995, Kato et al. 1996, Mann & Titarchuk 1996, Chakrabarti 1996, Chen et al. 1997, Peitz & Appel 1999, Balbus & Hawley 1998, Frank et al. 2002, Afshordi & Paczynski 2003, Chakrabarti & Das 2004, Becker & Subramanian 2005, Islam & Balbus 2008, Umurhan et al. 2008, Das 2007, Lanzafame 2008, Sharma 2008, Subramanian et al. 2008, Bhattacharjee et al. 2009).

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In a different approach to understanding the role of angular momentum, another model, that of the sub-Keplerian inviscid flow with a low angular momentum, has been formulated and has become well-established by now (Abramowicz & Zurek 1981; Fukue 1987; Chakrabarti 1989; Nakayama & Fukue 1989; Chakrabarti 1990; Kafatos & Yang 1994; Yang & Kafatos 1995; Pares 1996; Molteni et al. 1996; Chakrabarti 1996b; Lu et al. 1997; Das 2002; Das et al. 2003; Roy 2003; Barai et al. 2004; Das 2004; Abrahamed et al. 2006; Chaudhury et al. 2006; Ray & Bhattacharjee 2007a; Goswami et al. 2007; Das et al. 2007; Roy & Ray 2009; Nag et al. 2012). In hydrodynamic problems it is not uncommon to model real fluid flows as inviscid (Chandrasekhar 1981), and as regards rotational accretion, it provides a suitable model for describing the flow in the innermost region of the accretion disc, very close to the event horizon of a black hole. Large radial velocities in this region imply that the time scale of the dynamic infall process is much smaller than the viscous time scale. Even on large length scales the radial velocity has high values due to the fact that the angular momentum of the flow continues to be low. So when it comes to modelling these particular features of axisymmetric accretion, the inviscid rotational flow certainly has its utility (Itumenschev & Beloborodov 1997; Beloborodov & Illarionov 2001; Proga & Begelman 2003).

In feasible models of accretion flows, the boundary conditions require that at large distances from the accretor, the flow is very much subsonic, or very close to the accretor, the flow ought to become highly supersonic, a fact that is especially true if the accretor is a black hole (Novikov & Thorne 1973; Liang & Thomson 1980). In other words, in the intermediate region, the bulk flow attains the speed of acoustic propagation in the fluid (broadly speaking, the sonic velocity), and becomes transonic in character. Such critical features have been investigated in the stationary phase portrait of the flow, and it is known by now that these critical conditions are the consequences of a coupled first-order dynamical system that can be crafted out of the equations governing the stationary flow (Muchotrzeb-Czerny 1986; Ray & Bhattacharjee 2002; Afshordi & Paczynski 2003; Chaudhury et al. 2006; Ray & Bhattacharjee 2007a; Mandal et al. 2007; Goswami et al. 2007; Roy & Ray 2007; 2009; Bhattacharjee et al. 2009; Nag et al. 2012).

Open critical solutions, connecting the outer boundary of the flow to the surface of an accretor, must pass through saddle points in the phase portrait of the stationary solutions, and particularly where the accretor is a black hole, the phase portrait will have multiple saddle points. This is tantamount to saying that an inflow process that is made to traverse all these saddle points, must be multi-critical (Liang & Thomson 1980; Abramowicz & Zurek 1981; Muchotrzeb & Paczynski 1982; Muchotrzeb 1983; Muchotrzeb-Czerny 1986; Fukue 1987; Abramowicz & Kato 1989; Chakrabarti 1989, 1990; Kafatos & Yang 1994; Yang & Kafatos 1995; Pares 1996; Lu et al. 1997; Peitz & App 1997; Caditz & Tsuruta 1998; Das 2002; Barai et al. 2004; Das 2004; Abrahamed et al. 2006; Das et al. 2007), a fact that is made possible physically by the occurrence of standing shocks in the flow, as it proceeds from an outer saddle point to the next inner one (Fukue 1987; Chakrabarti 1989; Nakayama & Fukue 1989; Chakrabarti 1990; Kafatos & Yang 1994; Yang & Kafatos 1995; Lu et al. 1997; Caditz & Tsuruta 1998; Das 2002; Das et al. 2003; Chakrabarti & Das 2004; Abrahamed et al. 2006; Das 2007; Das et al. 2007; Fukumura & Kazanas 2007; Lanzafame Nagakura & Yamada 2008, 2009).

Stationary rotational flows may thus be realised in mathematical models by ingeniously tackling the complications that angular momentum in the flow presents. For all that, one does not go too far in deriving insight about the dynamic and the nonlinear aspects of the fluid flow problem that astrophysical accretion really is. Mathematically, all of fluid dynamics is a nonlinear problem, and accretion flows can be no exception to this rule. So it should be a worthwhile exercise to understand the effects of nonlinearity in an accretion process. In making a case for nonlinearity, one just needs to look at the well-known nonlinear problem of the realizability of solutions passing through saddle points in a stationary phase plot. The very existence of these critical solutions is threatened by even an infinitesimal deviation from making a case for nonlinearity, one just needs to look at the well-known nonlinear problem of the realisability of solutions passing through an infinitude of intermediate sub-critical states. To accomplish perturbative evolution of the accreting system, it is feasible to suggest that the initial condition of the evolution is globally sub-critical, with gravity subsequently driving the solution to a critical state, sweeping through an infinitude of intermediate sub-critical states. To accomplish
a smooth temporal convergence to a stable critical state, the stability of the sub-critical states is imperative. To investigate this aspect under relatively simple nonlinear conditions, all orders of nonlinearity beyond the second order have been truncated in the equation of the perturbation. Following this, the spatial dependence of the perturbation has been integrated out with the help of well-defined boundary conditions on globally sub-critical flows. After this, only the time-dependent part of the perturbation is extracted, and, very intriguingly, it acquires the mathematical appearance of a Liénard system (Strogatz 1994; Jordan & Smith 1999). Application of the common analytical tools of dynamical systems to study the equilibrium features of this Liénard system, shows the existence of a saddle point in real time. This unequivocally implies that the stationary background solutions will be unstable in time, if the perturbation is extended into the nonlinear regime.

To summarise the import of this work, conservative momentum balance and continuity conditions, as appropriate for an inviscid axisymmetric flow, have been subjected to time-dependent radial perturbations. On allowing nonlinearity to play a part, an instability is seen to develop in this hydrodynamic system. And finally, it will not be out of place to mention here that the analytical methods of this work are very nearly identical to those of a similar study on spherically symmetric accretion (Sen & Ray 2012). Considering the great difference in the respective geometries of spherically symmetric accretion and axisymmetric accretion, as well as major differences in much of the respective physics, the close similarity of the mathematical treatment in both the cases is, indeed, a matter of deep curiosity. The entire mathematical treatment so described, and all its attendant physical conclusions, have been presented in what follows.

2 THE CONDITIONS FOR INVISCID AXISYMMETRIC ACCRETION

In models of accretion discs, imposing the condition of hydrostatic equilibrium along the vertical direction (Matsumoto et al 1984; Frank et al. 2002) and also performing a vertical integration, will effectively collapse the vertical geometry of the flow on the equatorial plane of the disc. The equatorial flow is described by two coupled fields, the radial drift velocity, \( v \), and the surface density, \( \Sigma \), of which, the latter is defined by vertically integrating the volume density, \( \rho \), over the disc thickness, \( H \). This gives 

\[
\rho \Sigma \sim K \frac{\pi r^2}{\Omega v},
\]

where \( \Sigma \) has been defined as the surface density, \( \Omega \) is the local angular velocity of the flow, and the torque, \( \Gamma \), defined as

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( \Sigma vr \right) = 0.
\]

The axisymmetric accretion flow is driven by the gravitational field of a centrally located non-rotating black hole, but the structure of the neighbouring geometry, through which the flow takes place, is still set according to the Newtonian construct of space and time, by taking recourse to what is known as a pseudo-Newtonian potential (Paczyński & Witt 1980; Nowak & Wagoner 1991; Artemova et al. 1996). This effectively substitutes the properties of the Schwarzschild space-time geometry with a potential function, and to a large extent the analytical results pertaining to the rotational flow remain unaffected by the choice of a particular pseudo-Newtonian potential. Involving such a potential, \( \Phi \), the height of the disc, under hydrostatic equilibrium in the vertical direction, is given as

\[
H = \gamma^{-1/2} r(a/v_K),
\]

where \( a \) is the local speed of sound, \( v_K \) is the local Keplerian velocity, defined, respectively, by

\[
a^2 = \frac{\gamma P}{\rho} \quad \text{and} \quad v_K^2 = r \Phi'(r),
\]

where the prime indicates a derivative with respect to \( r \). The pressure, \( P \), as it has been introduced in the definition of \( a \), is expressed in terms of the volume density, \( \rho \), by a polytropic equation of state, \( P = k \rho^\gamma \), with \( \gamma \) being the polytropic exponent. In consequence of this definition of \( P \), the speed of sound may also be noted from

\[
d^2 = \frac{\partial P}{\partial \rho} = \gamma k \rho^{\gamma - 1}.
\]

It can be shown mathematically, by going back to the first law of thermodynamics (Chandrasekhar 1939), that \( \gamma \) varies between unity (the isothermal limit) and \( c_p/c_v \), which is the ratio of the two coefficients of specific heat capacity of a gas (corresponding simply to the conserved adiabatic limit), i.e. \( 1 \leq \gamma \leq c_p/c_v \). So the polytropic prescription is of a much more general scope than the simple conserved adiabatic case, and is suited well for the study of open systems like astrophysical flows.

Now, using the relationship between \( a \) and \( \rho \), the disc height can be explicitly written in terms of the standard fluid flow variables as

\[
H = (\gamma k)^{1/2} \frac{\rho^{(\gamma - 1)/2} r^{1/2}}{\sqrt{\gamma \Phi}}.
\]

With the foregoing result, the continuity condition, as given by equation (1), can be recast as

\[
\frac{\partial}{\partial r} (\rho^{(\gamma + 1)/2}) + \frac{\sqrt{\Phi'}}{r^{3/2}} \frac{\partial}{\partial r} \left[ \rho^{(\gamma + 1)/2} v r^{3/2} \right] = 0,
\]

which is one of the two required mathematical conditions governing the dynamics of the coupled fields, \( v \) and \( \rho \), in the radial direction.

To ascertain the second condition necessary for determining the dynamics in the radial direction, one first has to look at the dynamics along the azimuthal coordinate. This gives the condition for the balance of the specific angular momentum of the flow as (Pringle 1981; Frank et al. 2002)

\[
\frac{\partial}{\partial t} (\Sigma r^2 \Omega) + \frac{1}{r} \frac{\partial}{\partial r} \left[ (\Sigma vr) r^2 \Omega \right] = \frac{1}{2\pi r} \left( \frac{\partial Q}{\partial r} \right),
\]

in which \( \Omega \) is the local angular velocity of the flow, and the torque, \( Q \), is to be read as

\[
Q = 2\pi r v \Sigma r^2 \frac{\partial \Omega}{\partial r},
\]

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with $\nu$ being the kinematic viscosity associated with the flow. Using the continuity condition, as equation (1) gives it, and going by the Shakura & Sunyaev (1973) prescription for the kinematic viscosity, $\nu = \alpha a H$, it would be easy to reduce equation (4) to the form (Frank et al. 2002; Narayan & Yi 1994).

$$\frac{1}{v} \frac{\partial}{\partial t} \left( r^2 \Omega \right) + \frac{\partial}{\partial r} \left( r^2 \Omega \right) = \frac{1}{\rho v r \Omega} \left[ \frac{\partial}{\partial r} \left( \frac{\partial \Omega}{\partial r} \right) \right].$$

(6)

with $\Omega_{K}$ being defined from $\nu_{K} = r \Omega_{K}$. In equation (6), the condition for an inviscid and axially symmetric rotation of the flow is achieved by putting $\alpha = 0$. This delivers the simplest possible integral solution, $r^2 \Omega = \lambda$, with the constant of the motion, $\lambda$, being interpreted physically as the globally constant specific angular momentum of the flow. This interpretation affords great convenience in setting down the second mathematical condition of the flow along the radial direction. This is the condition of the radial momentum balance (in essence the conservative Euler equation), in which the centrifugal term, arising due to the rotational motion of the flow, is fixed as $\lambda^2 / r^2$. After that, the radial momentum balance equation is written as

$$\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial r} \left( \nu \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial P}{\partial r} + \Phi'(r) - \frac{\lambda^2}{r^2} \right) = 0. \quad (7)$$

On specifying the functions, $\Phi(r)$ and $P$, equations (6) and (7) can give a complete hydrodynamic description of the axisymmetric flow in terms of the two fields, $v(r, t)$ and $\rho(r, t)$. From these dynamic variables, the steady solutions of the flow are obtained by making explicit time-dependence disappear, i.e. $\partial / \partial t \equiv 0$. The resulting differential equations of the inviscid rotational flow, involving full spatial derivatives only, can then be easily integrated to get the stationary global solutions (Chakrabarti 1989; Das 2002; Das et al. 2003). A noticeable feature of these stationary solutions is that they remain invariant under the transformation $v \rightarrow -v$, i.e. the mathematical problem of inflows ($v < 0$) and outflows ($v > 0$) is identical in the stationary state (Ray & Bhattacharjee 2007a). This invariance has some adverse implications for the critical flows in inviscid axisymmetric accretion processes. Critical solutions pass through saddle points in the stationary phase portrait of the flow (Ray & Bhattacharjee 2002; Chaudhury et al. 2006; Ray & Bhattacharjee 2007a), and it can be demonstrated that generating a stationary solution through a saddle point will be impossible by any physical means, because it calls for an infinite precision in the required outer boundary condition (Ray & Bhattacharjee 2002, 2007a). Nevertheless, criticality is not a matter of doubt in accretion processes (Liang & Thomson 1980). The key to resolving this paradox lies in considering explicit time-dependence in the flow, because of which, as one may note from equations (6) and (7), the invariance under the transformation, $v \rightarrow -v$, breaks down. Obviously then, a choice of inflows ($v < 0$) or outflows ($v > 0$) has to be made at the very beginning (at $t = 0$, as it were), and solutions generated thereafter will be free of all the difficulties associated with the presence of a saddle point in the stationary flow.

In the stationary phase portrait of the flow, criticality is obtained when the bulk flow velocity matches the speed with which an acoustic wave can propagate through the moving compressible fluid. In the case of an inviscid, polytropic, rotational flow, this speed is not exactly given by the sonic condition, but differs slightly from it by a constant numerical factor, $\sqrt{2(\gamma + 1)^{1/2}}$, due to the geometry of the flow (Ray 2003; Chaudhury et al. 2006; Ray & Bhattacharjee 2007a; Roy & Ray 2009, Nag et al. 2012). For the conserved and inviscid flow, the critical points are either saddle points, through which open solutions may pass, or they are centre-type points, around which the stationary solutions will form closed trajectories (Chaudhury et al. 2004). Depending on the global nature of the phase portrait, the solutions which pass through the saddle points can be either homoclinic or heteroclinic (Chakrabarti 1989; Das 2002; Das et al. 2003). The number of critical points that may be obtained, depends on the choice of the potential, $\Phi(r)$. For the simple Newtonian potential, only two critical points result (Ray 2003), one a centre-type point and the other a saddle point. On the other hand, in studies of axisymmetric accretion on to a black hole, it is an expedient practice to employ a pseudo-Newtonian potential to drive the flow. In that case, the number of critical points will exceed two, and going by the nature of critical points (Jordan & Smith 1999), it happens that there will be multiple saddle points. This can have a significant bearing on the way a fluid element may reach the accretor, after having started from an outer boundary point.

Imposing various boundary conditions on the stationary integral solutions, will result in multiple classes of flow (Chakrabarti 1989; Das 2002; Das et al. 2003). Of these, the one of enduring interest in rotational accretion obeys the boundary conditions, $v \rightarrow 0$ as $r \rightarrow \infty$ (the outer boundary condition) and $v > a$ at small values of $r$. The inner boundary condition naturally suggests itself when it comes to accretion onto a black hole, for which, the final infall across the event horizon must necessarily be highly supersonic (Novikov & Thorne 1973; Liang & Thomson 1980). Therefore, it is quite obvious that an open solution that starts with a very low velocity far away from the accretor, but has to allow a fluid element to reach the accretor with supersonic speeds, perforce has to pass through a saddle point (where criticality in the flow is attained). This implies that travelling from an outer boundary point, the first critical point that a physically relevant open inflow solution encounters is a saddle point. If, however, between this point and the surface of the accretor, there are other saddle points to be encountered, then what transpires is a multi-critical flow. In such a flow, a fluid element will reach the accretor after having travelled through more than one saddle point, and in between two successive saddle points, the flow is bound to suffer a shock (Chakrabarti 1989; Das 2002; Das et al. 2003). It is at the discontinuity of a shock, that a solution is demoted from its super-critical state to a sub-critical one, following which, the solution has to regain super-criticality by travelling through another saddle point (Chakrabarti 1982; Das 2002; Das et al. 2003). Evidently, multi-criticality and shocks are closely related in models of conserved, rotational inflows onto a black hole.

Thus far, the properties of the accretion flow have been understood from a stationary perspective, which is relatively easy to follow. It is the dynamics of the flow (even a conservative one) that offers a problem of greater mathematical complexity. In comparatively simpler studies involving time-dependence (Ray 2003; Chaudhury et al. 2006), the inviscid and axisymmetric flow is found to be stable under the effect of linearised perturbations. However, this does not shed much light on the way the velocity and density fields in an accretion process
may evolve non-perturbatively in time. In such a mathematical problem, one is required to work with a coupled set of nonlinear partial differential equations, as implied by equations (3) and (4), but no analytical solution exists for these coupled, dynamic nonlinear equations.

Now, so far as generating the critical flow is concerned, the non-perturbative dynamic evolution of global $v(r, t)$ and $\rho(r, t)$ profiles is very crucial. It is the way in which the two fields evolve vis-à-vis each other that decides if the critical state would be achieved or not. The dynamic process is to be envisaged mathematically as one in which initially both the radial drift velocity and the density fields, $v(r, t)$ and $\rho(r, t)$, are sub-critical and nearly uniform for all values of $r$, in the absence of any driving force. Then with the introduction of a gravitational field (at $t = 0$), about whose centre, the fluid distribution may randomly possess some net angular momentum, the hydrodynamic fields, $v$ and $\rho$, start evolving in time. In the regions where the temporal growth of $v$ outpaces the temporal growth of $\rho$ (to which $a$ is connected), and gravity (going as $r^{-2}$) dominates the inhibitive centrifugal effects of rotation (going as $r^{-3}$), the infall process will become super-critical. Otherwise, it will continue to remain sub-critical. Under the approximation of a "pressureless" motion of a fluid in a gravitational field (Shu 1991), qualified support for the attainment of criticality was provided from a non-perturbative dynamic perspective (Ray & Bhattacharjee 2007a), guided by the physical criterion that the total specific mechanical energy at the end of the evolution will remain the same as what it was at the start of the evolution. A similar principle makes transonicity possible in spherically symmetric accretion (Bondi 1952; Garlick 1979; Ray & Bhattacharjee 2002; Roy & Ray 2007). Nevertheless, a completely analytical solution of the dynamic nonlinear problem continues to be elusive.

3 NONLINEARITY IN THE PERTURBATIVE ANALYSIS

Equations (3) and (4) are easy to integrate in their stationary limits, and the resulting velocity and density fields, derived from these two equations, have only spatial profiles, $v \equiv v_0(r)$ and $\rho \equiv \rho_0(r)$. A standard method of perturbative analysis, implemented originally in spherically symmetric accretion (Petterson et al. 1980), entails applying small time-dependent radial perturbations on the stationary profiles, $v_0(r)$ and $\rho_0(r)$, and then linearising the perturbed quantities. This, however, does not offer much insight into the time-dependent evolutionary aspects of the hydrodynamic flow. The next logical act, therefore, is to incorporate nonlinearity in the perturbative method. With the inclusion of nonlinearity in progressively higher orders, the perturbative analysis incrementally approaches the actual time-dependent evolution of the global solutions, after it has started with a given stationary profile at $t = 0$ (it makes physical sense to suggest that this initial profile is spatially sub-critical over the entire flow domain).

The prescription for the perturbation is $v(r, t) = v_0(r) + v'(r, t)$ and $\rho(r, t) = \rho_0(r) + \rho'(r, t)$, in which the primed quantities indicate a perturbation about a stationary background. It is now necessary to define a new variable, $f(r, t) = \rho^{(\gamma+1)/2}v^{3/2}/\sqrt{\Phi}$, following a similar mathematical procedure employed in some previous studies on inviscid, axisymmetric accretion (Ray 2003; Chaudhury et al. 2006; Ray & Bhattacharjee 2007a). It is easy to see that this variable emerges as a constant of the motion from the stationary limit of equation (3). This constant, $f_0$, can be identified closely with the matter flow rate, within a fixed geometrical factor, and in terms of $v_0$ and $\rho_0$, this constant is given as $f_0 = \rho_0^{(\gamma+1)/2}v_0^{3/2}/\sqrt{\Phi}$. On applying the perturbation scheme for $v$ and $\rho$, the perturbation in $f$, without losing anything of nonlinearity, is derived as

$$f_{\text{f}} = \frac{\zeta}{\beta^2} \frac{\rho'}{\rho_0} + \frac{v'}{v_0} + \frac{\zeta}{\beta^2} \frac{\rho'}{\rho_0} v', $$

(8)

in which,

$$\zeta = 1 + \frac{1}{1 \cdot 2} \left( \gamma - 1 \right) \frac{\rho'}{\rho_0} + \frac{1}{1 \cdot 2 \cdot 3} \left( \gamma - 1 \right) \left( \gamma - 3 \right) \left( \frac{\rho'}{\rho_0} \right)^2 + \cdots, $$

(9)

and $\beta = \sqrt{2(\gamma + 1)^{-1/2}}$. Equation (8) connects the perturbed quantities, $v'$, $\rho'$ and $f'$, to one another. To get a relation between only $\rho'$ and $f'$, one has to go back to equation (3), and apply the perturbation scheme on it. This will result in

$$\frac{\partial}{\partial t} \left( \frac{\zeta}{\beta^2} \frac{\rho'}{\rho_0} \right) = -v_0 \frac{\partial}{\partial r} \left( \frac{f'}{f_0} \right). $$

(10)

To obtain a similar relationship solely between $v'$ and $f'$, one needs to combine the conditions given in equations (8) and (10), to get

$$\frac{\partial v'}{\partial t} = \frac{1}{f} \left( \frac{\partial f'}{\partial t} + \frac{v}{\rho} \frac{\partial f'}{\partial r} \right). $$

(11)

It is worth stressing at this point that in equations (8), (10) and (11), all orders of nonlinearity have been maintained. Now returning to equation (3), and on taking the partial time derivative of $\rho$ in it, an alternative form of the perturbation on the continuity condition appears as

$$\frac{1}{\rho} \frac{\partial \rho'}{\partial t} = -\frac{\beta^2 v}{f} \left( \frac{\partial f'}{\partial r} \right). $$

(12)

Equations (10) and (12) are equivalent expressions of the same principle. However, the latter expression is more expedient than the former, when one takes the second-order partial time derivative of equation (7), and applies the perturbation scheme on it, to arrive finally at

$$\frac{\partial^2 v^2}{\partial t^2} + \frac{\partial}{\partial r} \left[ \frac{\partial v'}{\partial t} + \frac{v^2}{\rho} \frac{\partial \rho'}{\partial t} \right] = 0. $$

(13)
Now making use of equation (11), its second-order partial time derivative, and equation (12), one obtains a fully nonlinear equation of the perturbation from equation (13), running in a symmetric form as

$$\frac{\partial}{\partial t} \left( h^{tt} \frac{\partial f'}{\partial t} \right) + \frac{\partial}{\partial r} \left( h^{tr} \frac{\partial f'}{\partial r} \right) + \frac{\partial}{\partial r} \left( h^{rt} \frac{\partial f'}{\partial r} \right) + \frac{\partial}{\partial r} \left( h^{rr} \frac{\partial f'}{\partial r} \right) = 0,$$

in which,

$$h^{tt} = \frac{v}{r}, \quad h^{tr} = h^{rt} = \frac{v^2}{r}, \quad h^{rr} = \frac{v}{r} \left( v^2 - a^2 \right).$$

Going by the symmetry of equation (14), it can be recast in a compact form as

$$\partial_\mu \left( h^{\mu\nu} \partial_\nu f' \right) = 0,$$

with the Greek indices running from 0 to 1, under the equivalence that 0 stands for $t$, and 1 stands for $r$. At this stage it should be important to realise that equation (16), or equivalently, equation (14), is a nonlinear equation containing arbitrary orders of nonlinearity in the perturbative expansion. All of the nonlinearity is carried in the metric elements, $h^{\mu\nu}$. If one were to have worked with a linearised equation only, then $h^{\mu\nu}$ could be read from the symmetric matrix \cite{Chaudhury2006, RayBhattacharjee2007a},

$$h^{\mu\nu} = \frac{v_0}{f_0} \left( \begin{array}{cc} 1 & \frac{v_0}{v_0^2 - \beta^2 a_0^2} \\ \frac{v_0}{v_0^2 - \beta^2 a_0^2} & 0 \end{array} \right),$$

in which $\alpha_0 \equiv \alpha_0(r)$ is the steady state value of the local speed of sound. Now, in Lorentzian geometry the d’Alembertian for a scalar field in curved space is expressed in terms of the metric, $g_{\mu\nu}$, as

$$\Delta \varphi \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \varphi \right),$$

with $g^{\mu\nu}$ being the inverse of the matrix implied by $g_{\mu\nu}$. Comparing equations (16) and (18) with each other, one could look for an equivalence between $h^{\mu\nu}$ and $\sqrt{-g} g^{\mu\nu}$. What can easily be appreciated from this comparison is that equation (16) gives an expression for $f'$ which is of the type given by equation (18). In the linear order, the metrical part of equation (15), as equation (17) shows it, may then be extracted, and its inverse will incorporate the notion of the horizon of an acoustic black hole, when $v_0^2 = \beta^2 a_0^2$. This point of view has features that are similar to the metric of a wave equation obtained by setting up the velocity of an irrotational, inviscid and barotropic fluid flow as the gradient of a scalar potential, and then by imposing a perturbation on this scalar potential \cite{Visser1998, Barcelo2005}. In contrast to this approach of exploiting the conservative nature of the flow to craft a scalar potential, the derivation of equation (16) makes use of the continuity condition. The latter method is more robust because the continuity condition is based on matter conservation, which is a firmer conservation principle than that of energy conservation, on which the scalar-potential approach is founded. If, for instance, the fluid system under study were to have contained dissipative factors (as it is happens in models of axisymmetric accretion), then the potential-flow formalism would have been ineffective, but the present method of making use of the matter conservation principle, would still have delivered an equation of the perturbation.

However, the close correspondence between the physics of acoustic flows and many features of black hole physics is valid only as far as the linear ordering goes. When nonlinearity is to be accounted for, then instead of equation (17), it will be equations (15) which will define the elements, $h^{\mu\nu}$, depending on the order of nonlinearity that one wishes to retain (in principle one could go up to any arbitrary order). The first serious consequence of including nonlinearity is to compromise the argument in favour of an inflow solution attaining the critical status, because the description of $h^{\mu\nu}$, as stated in equation (17), will not suffice any longer. This view is in perfect conformity with a numerical study conducted by Mach & Maled \cite{Mach2008} for the case of spherically symmetric accretion, in which it was shown that if the perturbations were to become strong then the analogy between the “sonic horizon” and the event horizon of a black hole would not hold.

Finally, when discussing an axisymmetric flow, it has been possible to invoke the critical aspects of spherical geometry, because the mathematical methods employed in both the cases are nearly identical \cite{Sen2012}. While this unifying principle is worthy of a careful note in its own right, another very remarkable fact to have emerged in consequence of including nonlinearity in the perturbative analysis is that regardless of the order of nonlinearity that one may desire to go up to, the symmetric form of the Lorentzian metric equation will remain unchanged, as shown very clearly by equation (16). For the laboratory fluid problem of the hydraulic jump, a similar type of symmetry is known to exist, going up to the second order of nonlinearity \cite{RayBhattacharjee2007b}.

4 STANDING WAVES ON SUB-CRITICAL INFLOWS

All physically relevant stationary inflow solutions obey the outer boundary condition, $v(r) \rightarrow 0$ as $r \rightarrow \infty$. Among these solutions, a critical inflow will reach the accretor with a high super-critical speed, but somewhere along the way, this flow will also undergo a discontinuity due to a shock. So critical inflows are highly sub-critical at the outer boundary, highly super-critical near the accretor (the inner boundary), and have a discontinuity in the interim region \cite{Chakrabarti1989, Das2002, Das2003}. On the other hand, there is an entire class of inflow solutions which are globally sub-critical, conforming to the inner boundary condition, $v(r) \rightarrow 0$ as $r \rightarrow 0$. From the point of view of a gravity-driven evolution of an inflow solution to a critical state, the sub-critical flows have a great importance, because the initial state
of an evolution, as well as the intermediate states in the progression towards criticality, should realistically be sub-critical. So the stability of globally sub-critical solutions will have a significant bearing on how a critical solution will develop eventually. Imposing an Eulerian perturbation on these sub-critical inflows, their stability has been studied by now under a linearised regime (Ray 2003; Chaudhury et al. 2006), and the amplitude of the perturbation, fashioned to be a standing wave, was seen to maintain a constant profile in time. Further, the perturbation was also made to behave like a travelling wave under a WKB approximation, and it was found that although the relative amplitude of the perturbation grew locally when the bulk velocity decreased, the total energy in the wave remained fixed. In all of these respects one may then say that the solutions do not exhibit any obvious instability. However, it is never prudent to extend this argument too far, especially when one considers nonlinearity in the perturbative effects, as it rightly ought to be done in a fluid flow problem.

Now a nonlinear equation of the perturbation, that accommodates nonlinearity up to any desired order, has already been derived, as equation (14) shows. This equation can then be applied to study the stability of stationary sub-critical flows in a nonlinear regime. The perturbation is designed to behave like a standing wave about a globally sub-critical stationary solution, obeying the boundary condition that the spatial part of the perturbation vanishes at two radial points in the axisymmetric geometry — one at a great distance from the accretor (the outer boundary), and the other very close to it (the inner boundary). While the former boundary is a self-evident fact of the flow, there is a certain measure of difficulty in identifying the latter. The guiding principle behind the choice of the two boundaries is that the background stationary solution should be continuous in the interim region. For a globally continuous sub-critical solution, that connects the outer boundary to the accretor, the inner boundary is obviously the surface of the accretor itself. If, on the other hand, even a sub-critical inflow is also disrupted by a shock, then the inner boundary should be the standing front of the shock itself. It is conceivable that no part of the perturbation on the background flow may percolate through the shock front, and so, the discontinuous front itself may be set as a boundary for the perturbation. Such piecewise continuity of a stability analysis, on either side of a discontinuity, is not uncommon in studies on fluid flows (Ray & Bhattacharjee 2007b).

The mathematical treatment involving nonlinearity will be confined to the second order only (the lowest order of nonlinearity). Even simplified so, the entire procedure will still carry much of the complications associated with a nonlinear problem. The restriction of not going beyond the second order of nonlinearity implies that $h^{\mu\nu}$ in equations (13) will contain primed quantities in their first power only. Taken together with equation (14), this will preserve all the terms which are nonlinear in the second order. So, performing the necessary expansion of $v = v_0 + \nu'$, $\rho = \rho_0 + \rho'$ and $f = f_0 + f'$ in equations (15) up to the first order only, and defining a new set of metric elements, $q^\mu\nu = f_0 h^{\mu\nu}$, one obtains

$$\partial_\mu \left( q^{\mu\nu} \partial_\nu f' \right) = 0,$$

(19)

in which $\mu$ and $\nu$ are to be read just as in equation (16). In the preceding expression, the elements, $q^{\mu\nu}$, carry all the three perturbed quantities, $\rho'$, $\nu'$ and $f'$. The next process to perform is to substitute both $\rho'$ and $\nu'$ in terms of $f'$, since equation (19) is over $f'$ only. To make this substitution possible, first one has to make use of equation (6) to close $\nu'$ in terms of $\rho'$ and $f'$ in all $q^{\mu\nu}$. While doing so, the product term of $\rho'$ and $\nu'$ in equation (6) is to be ignored, because including it will raise equation (19) to the third order of nonlinearity. By the same token, one also has to take only $\xi = 1$ from equation (9). Once $\nu'$ has been eliminated in this manner, one has to write $\rho'$ in terms of $f'$. This can be done by invoking equation (10), with the reasoning that if $\rho'$ and $f'$ are both separable functions of space and time, with the time part being oscillatory (all of which are standard mathematical prescriptions in perturbative analysis), then

$$\frac{1}{\beta^2 \rho_0} = \sigma(r) \frac{f'}{f_0},$$

(20)

with $\sigma$ being a function of $r$ only (which extends a crucial advantage in simplifying much of the calculations to follow). The exact functional form of $\sigma(r)$ will be determined by the way the spatial part of $f'$ is set up. It is known that $\sigma(r)$ is indeed a real function, going as $\sigma(r) = v_0 (v_0^2 - \beta a_0)^{-1}$, when the spatial part of $f'$ is set down as a power series in the WKB approximation (Ray 2003; Chaudhury et al. 2006). In any case, since $\rho'$, $\nu'$ and $f'$ are all real fluctuations, $\sigma$ cannot be anything but real.

Following all of these algebraic details, the elements, $q^{\mu\nu}$, in equation (19), can finally be expressed entirely in terms of $f'$ as

$$q^{tt} = v_0 \left( 1 + \epsilon \xi^{\nu} \frac{f'}{f_0} \right), \quad q^{tr} = v_0 \left( 1 + \epsilon \xi^{\nu} \frac{f'}{f_0} \right), \quad q^{rr} = v_0 \left( 1 + \epsilon \xi^{\nu} \frac{f'}{f_0} \right), \quad q^{tr} = v_0 \left( 1 + \epsilon \xi^{\nu} \frac{f'}{f_0} \right),$$

(21)

in all of which, $\epsilon$ has been introduced as a nonlinear “switch” parameter to keep track of all the nonlinear terms. When $\epsilon = 0$, only linearity remains. In fact, in this limit one converges to the familiar linear result implied by equation (17). In the opposite extreme, when $\epsilon = 1$, in addition to the linear effects, the lowest order of nonlinearity (the second order) becomes activated in equation (19). It may be noted that equations (21) also contain the factors, $\xi^{\mu\nu}$, all of which are to be read as

$$\xi^{tt} = -\sigma, \quad \xi^{tr} = \xi^{tt} = 1 - 2\sigma, \quad \xi^{rr} = 2 - \sigma \left[ 3 + \left( \frac{\gamma - 3}{\gamma + 1} \right) \frac{\beta^2 a_0^2}{v_0} \right].$$

(22)

Taking equations (19), (21) and (22) together, one gets a nonlinear equation of the perturbation, completed up to the second order, without the loss of any relevant term.

To render equation (19), along with all $q^{\mu\nu}$ and $\xi^{\mu\nu}$, into a workable form, it will first have to be written explicitly, and then divided throughout by $v_0$. While doing so, the symmetry lent by $\xi^{tt} = \xi^{rr}$ is also to be exploited. The desirable form of the equation of the perturbation should be such that its leading term would be a second-order partial time derivative of $f'$, with unity as its coefficient. To arrive
at this form, an intermediate step will involve a division by \(1 + \epsilon \xi^{tt}(f' / f_0)\), which, binomially, is the equivalent of a multiplication by \(1 - \epsilon \xi^{tt}(f' / f_0)\), with a truncation applied thereafter. This is dictated by the simple principle that to keep only the second-order nonlinear terms, it will suffice to retain just those terms which carry \(\epsilon\) in its first power. The result of this entire exercise is

\[
\frac{\partial^2 f}{\partial t^2} + 2 \frac{\partial}{\partial r} \left( v_0 \frac{\partial f}{\partial t} \right) + \frac{1}{v_0} \frac{\partial}{\partial r} \left( v_0 (v_0^2 - \beta^2 a_0^2) \frac{\partial f}{\partial t} \right) + \frac{\epsilon}{f_0} \left( \xi^{tt} \left( \frac{\partial f}{\partial t} \right)^2 + \frac{\partial}{\partial r} \left( \xi^{tt} v_0 \frac{\partial f^2}{\partial t^2} \right) - \frac{v_0}{2} \frac{\partial \xi^{tt}}{\partial t} \frac{\partial f^2}{\partial t} \right) \right] = 0,
\]

(23)
in which, if one were to set \(\epsilon = 0\), then what would remain would be the linear solution discussed in detail in some earlier works (e.g., Ray 2003; Chaudhury et al. 2006). To progress further, a solution of \(f'(r, t)\), separable in space and time, is to be applied. This will bear the form, \(f'(r, t) = R(r)\phi(t)\). Using this separable solution in equation (23), then multiplying the resulting expression throughout by \(v_0 R\), and then performing some algebraic simplifications by partial integrations, will finally lead to

\[
\ddot{\phi} v_0 R^2 + \frac{d}{dr} \left( v_0 R^2 \right)^2 + \frac{\epsilon}{f_0} \left[ \dot{\phi}^2 \xi^{rr} v_0 R^3 + \dot{\phi} \phi \left( \frac{d}{dr} \left( \xi^{tt} v_0^2 R^3 \right) + \xi^{tt} v_0^2 \frac{d^2 R^3}{dr^2} - \xi^{tt} R \frac{d}{dr} \left( v_0 R^2 \right) \right) \right] \right] + \phi^2 \left( v_0 \left( v_0^2 - \beta^2 a_0^2 \right) \frac{dR}{dr} \left( \xi^{tt} R^2 \right) - \xi^{tt} v_0^2 R \left( \xi^{tt} \frac{dR}{dr} \right)^2 - \frac{d}{dr} \left[ \xi^{tt} v_0^2 \left( v_0^2 - \beta^2 a_0^2 \right) \frac{dR^3}{dr^2} \right] + \frac{d}{dr} \left( \xi^{rr} v_0^2 \frac{dR^3}{dr^2} \right) \right] \right) = 0,
\]

(24)
in which the overdots indicate full derivatives in time. Quite evidently, equation (24) is a second-order nonlinear differential equation in both space and time. The way forward now is to integrate all spatial dependence out of equation (24), and then study the nonlinear features of the time-dependent part. The integration over the spatial part will necessitate invoking two boundary conditions. One of these (the outer boundary condition) is situated very far from the accretor, while the other one (the inner boundary condition) could either be very close to the accretor, or at a standing shock front where the background solution becomes discontinuous. At both of these boundary points, the perturbation will have a vanishing amplitude in time, while the background solution will maintain a continuity in the interim region. The boundary conditions will ensure that all the “surface” terms of the integrals in equation (24) will vanish (which explains the tedious mathematical exercise to extract several “surface” terms). So after carrying out the required integration on equation (24), over the entire region trapped between the two specified boundaries, all that will remain is the purely time-dependent part, having the form

\[
\ddot{\phi} + \epsilon \left( A\phi + B\dot{\phi} \right) \dot{\phi} + C\phi + \epsilon D\phi^2 = 0,
\]

(25)
in which the constants, \(A, B, C\) and \(D\) are to be read as

\[
A = \frac{1}{f_0} \left( \int v_0 R^2 \, dr \right)^{-1} \left( \int \xi^{tt} v_0^2 \frac{dR^3}{dr} - \xi^{tt} R \frac{d}{dr} \left( v_0 R^2 \right)^2 \right) \, dr,
\]

\[
B = \frac{1}{f_0} \left( \int v_0 R^2 \, dr \right)^{-1} \int \xi^{tt} v_0 R^3 \, dr,
\]

\[
C = -\left( \int v_0 R^2 \, dr \right)^{-1} \int v_0 \left( v_0^2 - \beta^2 a_0^2 \right) \left( \frac{dR}{dr} \right)^2 \, dr,
\]

\[
D = \frac{1}{f_0} \left( \int v_0 R^2 \, dr \right)^{-1} \left( \int v_0 \left( v_0^2 - \beta^2 a_0^2 \right) \frac{dR}{dr} \left( \xi^{tt} R^2 \right) - \xi^{tt} v_0^2 R \left( \frac{dR}{dr} \right)^2 \right) \, dr,
\]

(26)
respectively. The form in which equation (25) has been abstracted is that of a general Liénard system (Strogatz 1994; Jordan & Smith 1999). All the terms of equation (25), which carry the parameter, \(\epsilon\), have arisen in consequence of nonlinearity. When one sets \(\epsilon = 0\), one readily regains the linear results presented earlier (Ray 2003). However, to go beyond linearity, and to appreciate the role of nonlinearity in the perturbation, one now has to understand the Liénard system that equation (25) has brought forth.

5 EQUILIBRIUM AND INSTABILITY IN THE LIÉNARD SYSTEM

The mathematical form of a Liénard system is like a damped nonlinear oscillator equation, going as (Strogatz 1994; Jordan & Smith 1999)

\[
\ddot{\phi} + \epsilon H(\phi, \dot{\phi}) \dot{\phi} + V(\phi) = 0,
\]

(27)
in which, \(H\) is a nonlinear damping coefficient (the retention of the parameter, \(\epsilon\), alongside \(H\), attests to the nonlinearity), and \(V\) is the “potential” of the system (with the prime on it indicating its derivative with respect to \(\phi\)). In the present study,
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\[ \mathcal{H}(\phi, \dot{\phi}) = A\phi + B\dot{\phi}, \]  
and

\[ V(\phi) = C\phi^2 + \epsilon D\phi^3, \]

with the constant coefficients, \( A, B, C \) and \( D \) having to be read from equations (26).

To investigate the properties of the equilibrium points resulting from equation (27), it will be necessary to decompose this second-order differential equation into a coupled first-order dynamical system. To that end, on introducing a new variable, \( \psi \), equation (27) can be recast as (Jordan & Smith 1999)

\[
\begin{align*}
\dot{\phi} &= \psi \\
\dot{\psi} &= -\epsilon (A\phi + B\psi) \psi - (C\phi + \epsilon D\phi^2).
\end{align*}
\]

Equilibrium conditions are established with \( \dot{\phi} = \dot{\psi} = 0 \). For the dynamical system implied by equations (30), this will immediately lead to two equilibrium points on the \( \phi-\psi \) phase plane. Labelling the equilibrium points with a superscript, one can easily see that \( (\phi^*, \psi^*) = (0, 0) \) in one case, whereas in the other case, \( (\phi^*, \psi^*) = (-C/(\epsilon D), 0) \). In effect, both the equilibrium points lie on the line, \( \psi = 0 \), and correspond to the turning points of \( V(\phi) \). Higher orders of nonlinearity may simply have the effect of proliferating equilibrium points on the line, \( \psi = 0 \). For the present case of second-order nonlinearity, one of the equilibrium points is located at the origin of the \( \phi-\psi \) phase plane, while the location of the other will depend both on the sign and the magnitude of \( C/D \).

Having identified the position of the two equilibrium points, the next task would be to understand their stability. To do so, both equilibrium points are to be subjected to small perturbations, following which, a linear stability analysis will have to be carried out. The perturbation scheme on both \( \phi \) and \( \psi \) is \( \phi = \phi^* + \delta\phi \) and \( \psi = \psi^* + \delta\psi \). Applying this scheme on equation (30), and then linearising in \( \delta\phi \) and \( \delta\psi \), will lead to the coupled linear dynamical system,

\[
\begin{align*}
d\delta\phi/dt &= \delta\psi \\
d\delta\psi/dt &= -V''(\phi^*)\delta\phi - \epsilon\mathcal{H}(\phi^*, \psi^*)\delta\psi,
\end{align*}
\]

in which \( V''(\phi^*) = C + 2\epsilon D\phi^* \). Using solutions of the type, \( \delta\phi \sim \exp(\omega t) \) and \( \delta\psi \sim \exp(\omega t) \), in equations (31), the eigenvalues of the Jacobian matrix of the dynamical system follow as

\[
\omega = -\frac{\epsilon}{2} \pm \sqrt{\frac{\epsilon^2 H^2}{4} - V''(\phi^*)},
\]

with \( H \equiv \mathcal{H}(\phi^*, \psi^*) \) having to be evaluated at the equilibrium points.

Once the eigenvalues have been determined, it is now a simple task to classify the stability of an equilibrium point by putting its coordinates in equation (32). The equilibrium point at the origin has the coordinates, \((0, 0)\). Using these coordinates in equation (32), one gets the two roots of the eigenvalues as \( \omega = \pm i\sqrt{C} \). It can be readily seen that if \( C > 0 \), then the eigenvalues will be purely imaginary quantities, and consequently, the equilibrium point at the origin of the \( \phi-\psi \) plane will be a centre-type point (Jordan & Smith 1999). And indeed, when the stationary inflow solution, about which the perturbation is constrained to behave like a standing wave, is sub-critical over the entire region of the spatial integration, then \( C > 0 \), because in this situation, \( v_0^2 < \beta^2 a_0^2 \) (Ray 2003). Therefore, the centre-type equilibrium point at the origin of the phase plane indicates that the standing wave will be purely oscillatory in time, with no change in its amplitude. This very conclusion was made in a previous study (Ray 2003) on the linearised analysis of the standing wave, and it could be arrived at equally correctly by setting \( \epsilon = 0 \) (the linear condition) in equation (32).

The centre-type point at the origin of the phase plane has confirmed the results known already. It is the second equilibrium point that offers some novelties. This equilibrium point is entirely an outcome of taking nonlinearity to its lowest order (the second order) in the standing wave. The coordinates of this equilibrium point in the phase plane are \((-C/(\epsilon D), 0)\), and using these coordinates in equation (32), the eigenvalues become specified as

\[
\omega = \frac{AC}{2D} \pm \sqrt{\left(\frac{AC}{2D}\right)^2 + C}.
\]

Noting as before, that \( C > 0 \), and that \( A, C \) and \( D \) are all real quantities, the inescapable conclusion is that the eigenvalues, \( \omega \), are real quantities, with opposite signs. In other words, the second equilibrium point is a saddle point (Jordan & Smith 1999), and as such its implications may be far-reaching when it comes to evolving a critical solution in the axisymmetric flow through the dynamic process.

To understand this fact, the first thing that has to be realised is that if the magnitude of the temporal part of the perturbation exceeds a certain critical value, i.e. if \( |\phi| > |C/D| \), then the perturbation will undergo a divergence in one of its modes, and the stationary, sub-critical background solution will become unstable under the influence of the perturbation. This is how it must be in the vicinity of a saddle point, and higher orders of nonlinearity (third order onwards) will not be able to make this effect disappear (Strogatz 1994; Jordan & Smith 1999). The best that one may hope for is that the instability may grow in time till it reaches a saturation level imposed by a higher order of nonlinearity, a feature that has a precedence in the laboratory fluid problem of the hydraulic jump (Volovik 2006; Ray & Bhattacharjee 2007b).
While all of this gives the perturbative perspective, the implications of the saddle point for the non-perturbative evolutionary dynamics are also noteworthy. It is evident that there can be no critical solution without gravity driving the infall process. So, from a dynamic point of view, gravity starts the evolution towards the critical state from an initial (and arguably nearly uniform) sub-critical state. If, however, during the evolution in real time, a saddle point is to be encountered, then there should be difficulties in reaching a stable and stationary critical end.

Under linearised conditions, the perturbation on globally sub-critical flows maintains a constant amplitude. Viewed in the phase portrait, this feature translates into closed phase trajectories around a centre-type point. Now, from dynamical systems theory, centre-type points are known to be “borderline” cases (Strogatz 1994; Jordan & Smith 1999). In such situations, the linearised treatment will show apparently stable behaviour but an instability may emerge immediately on accounting for nonlinearity (Strogatz 1994; Jordan & Smith 1999). This instability in the nonlinear analysis may be connected to the constant distribution of angular momentum in the inviscid accretion disc. To consider an analogous situation in an inviscid and incompressible Couette flow (which also has an axial symmetry, just like the inviscid accretion disc), if Rayleigh’s criterion for stability is invoked, then the stratification of angular momentum in the flow is stable if and only if it increases monotonically outwards, i.e. have a positive gradient (Chandrasekhar 1981). If, however, the gradient of the angular momentum is negative, then the rotational flow will be unstable. To carry this analogy over to the inviscid accretion disc, the gradient of the constant distribution of angular momentum is zero. So effectively this implies a borderline case between a stable positive gradient, and an unstable negative gradient. Such borderline cases may show the apparently stable features of a centre-type point in a linearised dynamical systems analysis, but in these situations it is always proper to draw a definitive conclusion regarding stability only from a nonlinear analysis (Strogatz 1994; Jordan & Smith 1999). And in the present study the nonlinear analysis does reveal an instability. One may also be careful to note that the Couette flow is mathematically modelled to have an infinite extent along the vertical direction (Chandrasekhar 1981), while the inviscid disc in this study has a finite thickness, and is modelled mathematically by taking vertically integrated quantities on the equatorial plane of the disc.

6 CONCLUDING REMARKS

The Liénard system derived in this work indicates that the number of equilibrium points will depend on the order of nonlinearity that may be retained in the equation of the perturbation. Additional equilibrium points, resulting from higher orders of nonlinearity, may temper the instability that has been found here. However, up to the second order at least, an instability in real time appears to be an undeniable fact, and it has been suggested that this instability, arising from the nonlinear order, could be connected to the gradient of the angular momentum. There are other disc models, in which angular momentum maintains a positive gradient, as for example the Keplerian accretion disc (Frank et al. 2002). Examining the stability of such configurations, may lead to a general understanding of the connection between the stability of rotational flows and the distribution of their angular momenta.

Now, real fluid flows are also influenced by viscosity. In fact, fluid flows are usually affected both by nonlinearity and viscosity, occasionally as competing effects. In models of accretion discs, viscous dissipation usually brings about stability, but in one of the proposed models of axisymmetric accretion, the quasi-viscous accretion disc, viscosity is seen to destabilise the flow (Bhattacharjee & Ray 2007; Bhattacharjee et al. 2009). In this model, viscosity, quantified by the $\alpha$ parameter (Shakura & Sunyaev 1973), is made to behave as a small first-order perturbative effect about a background inviscid flow. This instability, known as secular instability, is not without its precedence. Exactly this kind of instability is also seen to grow in Maclaurin spheroids on the introduction of a kinematic viscosity to a first order (Chandrasekhar 1987). So when it comes to stability, the role of viscosity is not as clearly defined as it is for enabling the infall process by an outward transport of the angular momentum. In any case, beyond simple hydrodynamics, the stability of accretion discs is more likely to be governed by phenomena like radiative processes, turbulence and magnetohydrodynamics (Balbus & Hawley 1998; Bhattacharjee et al. 2009). Stability of fluids is also studied by constraining a perturbation to behave like a travelling wave (Petterson et al. 1980; Cross 1986; Ray 2003; Ray & Bhattacharjee 2007). At times, one encounters the surprising situation of a fluid flow being stable under one type of perturbation, but unstable under the effect of another (Cross & Hohenberg 1993; Ray & Bhattacharjee 2007). With nonlinearity lending an additional aspect, these effects are worth a close look in future studies.

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