Generalized Bi-Schrödinger Flows and Vortex Filament on Symmetric Lie Algebras

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Abstract

The theory of the vortex filament in three-dimensional fluid dynamics, consisting mainly of the models up to the third-order approximation (refer to [4, 23, 24]), is an attractive subject in both physics and mathematics. Many efforts have been devoted to the extension of the theory to higher-dimensional symmetric Lie algebras. However, such a generalization known in literature is strongly restricted by the integrable method. In this article, we endeavor to establish the third-order models of the vortex filament on symmetric Lie algebras in a purely geometric way by generalized bi-Schrödinger flows. Our generalization overcomes the limitation of integrability and creates successfully the desired models on Hermitian or para-Hermitian symmetric Lie algebras. Combining the result in this article with what have been known in literature for the leading-order and the second-order models, we actually exhibit the basic models and the related theory of the vortex filament on symmetric Lie algebras up to the third-order approximation.

Keywords: Vortex filament, Bi-Schrödinger flow, Kähler structure, Para-Kähler structure
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1 Introduction

Vortex rings in fluid physics are usually regarded as invariant states in the 3-dimensional inviscid incompressible fluid in which the vortex lines are endowed with curvature that enables the rings to propagate. The study of vortex rings can be traced back to Helmholtz in 1858 (see [29, 30]) and Kelvin in 1867 (see [31]). Since then, the theory of vortex filament dynamics in inviscid fluid has become an attractive subject in both physics and mathematics, and has been studied extensively and deeply by many physicists and mathematicians (see [4, 11, 27, 23, 38]). We should point out that, except for the hydrodynamic models in the classical mechanics, one also needs to characterize the motion behavior of vortex filament appearing in various physical models, such as kinematics of interfaces in crystal growth [32, 8], viscous fingering in a Hele-Shaw cell [44], charged fluid in a neutralizing background [48], superconductors, superfluid [6], etc.

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As an idealized model that essentially reveals some qualitative behaviors of a curved vortex filament, the so called “localized induction equation (LIE)” or in other words, “localized induction approximation (LIA)” was first introduced in literature [4]. The model in the Euclidean 3-space $\mathbb{R}^3$ is stated as follows. The filament curve $X = X(t, x)$, expressed by functions of the arclength $x$ and the time $t$, evolves according to

$$X_t = \kappa B,$$

where $\kappa$ is the curvature and $B$ the binormal vector at the position $X(t, x)$. This model is also called the binormal motion of space curves in $\mathbb{R}^3$, derived by Luigi Sante Da Rios, an Italian mathematician and physicist, in 1906 (see [11]). The distinguishing features of LIE (1) are its completely integrability and the fact that it is equivalent to the focus nonlinear Schrödinger equation (NLS$^+$)

$$i\varphi_t + \varphi_{xx} + 2|\varphi|^2\varphi = 0$$

by Hasimoto transform (see [27]). This produces the so-called LIE-NLS correspondence (see [34] for example), that is, if a curve $X$ evolves according to LIE (1), then the associated complex function $\varphi$ given by $\varphi(t, x) = \kappa(t, x) \exp \left( \int^x \tau(t, s) \, ds \right)$ evolves according to NLS$^+$, where $\tau$ stands for the torsion curvature. Magri [37] unveiled the Hamiltonian structure behind the integrability of LIE and created a recursion operator generating successfully an infinite sequence of commuting vector fields. Based on this, Langer and Perline in [33] constructed the counterparts for LIE. The resulting sequence of integrable vector fields is called the “localized induction hierarchy (LIH)”, the first three terms of which are

$$V^{(1)} = \kappa B,$$
$$V^{(2)} = \frac{1}{2}\kappa^2 T + \kappa_x N + \kappa \tau B,$$
$$V^{(3)} = \kappa^2 \tau T + (2\kappa_x \tau + \kappa \tau_x) N + \left( \kappa \tau^2 - \kappa xx - \frac{1}{2} \kappa^3 \right) B,$$

where $\{T, N, B\}$ is the Frenet frame along the filament curve. This further creates effective mathematical methods and techniques to explore and describe the property, hidden structure and symmetry of such vortex filaments. Meanwhile, physically speaking, some other models are related to particular adjustments of the velocity description of the vortex filament curve with respect to ratio $\varepsilon$ of the core radius to the ring radius (refer to [24] and the references therein for details). These models can be constructed by taking account of the effect from the higher-order corrections to that of curvature and torsion. For example, for the validity to $O(\varepsilon^2)$, a vortex filament with axial velocity is reducible to a summation of $V^{(1)}$ and $V^{(2)}$, which was written down explicitly by Fukumoto and Miyazaki in 1991 [23]. By omitting the first term and re-normalizing, the Fukumoto-Miyazaki’s model is stated (see §2 below for details) as

$$\frac{dX}{dt} = \frac{1}{2}\kappa^2 T + \kappa_x N + \kappa \tau B = X_{xxx} + \frac{3}{2}X_{xx} \times (X_x \times X_{xx}).$$

Hence the Fukumoto-Miyazaki model is the second-order (physical) correction model of the vortex filament in the Euclidean 3-space $\mathbb{R}^3$. According to this point of view, the LIE (1) is in fact the leading-order correction $O(\varepsilon)$-model of the vortex filament in the Euclidean 3-space $\mathbb{R}^3$.

In 2000, the third-order correction $O(\varepsilon^3)$-model of the vortex filament in the Euclidean 3-space $\mathbb{R}^3$ was derived by Fukumoto and Moffatt ([22, 24]) (see §2 below for more details) and is written

$$\frac{dX}{dt} = \lambda \left\{ \kappa B + \nu [\kappa^2 \tau T + (2\kappa_x \tau + \kappa \tau_x) N + (\kappa \tau^2 - \kappa xx) B] + \mu \kappa^3 B \right\}.$$
However, the Fukumoto-Moffatt’s model fails to be a summation of $V^{(1)}$ and $V^{(3)}$. Hence, surprisingly, the Fukumoto-Moffatt’s model of the vortex filament in the Euclidean 3-space $\mathbb{R}^3$ is in fact non-integrable and cannot be the fourth member in the localized induction hierarchy in general (see [24]). All these models constitute a complete mathematical treatment of the vortex filament up to the third-order approximation in Riemannian 3-manifolds. The similar timelike and spacelike models of vortex filament in the Minkowski 3-space $\mathbb{R}^{3,1}$ or Lorentzian 3-manifolds, which possess totally different geometric and dynamical properties by their timelike or spacelike stamp, are constructed and characterized in a series of works in [12, 14, 15, 17, 18, 19] (or see §2 for details). These models consist of interesting and indispensable supplements to the models of the vortex filament in $\mathbb{R}^3$.

On the other hand, in the past decades, many efforts have been devoted to the extension of the vortex filament to higher dimensional spaces. A successful generalization in literature seems to be the “localized induction (matrix) hierarchy” on a symmetric Lie algebra which depends crucially on the integrability (see, for example, [25, 34, 39, 47]). Fordy and Kulish constructed in [25] the matrix nonlinear Schrödinger equation associated to a Hermitian symmetric space, which also promoted the integrable matrix AKNS hierarchy on a Hermitian symmetric Lie algebra by the spectral method. They showed that the matrix nonlinear Schrödinger equation they constructed is in fact equivalent to a generalized Heisenberg ferromagnet. Langer and Perline applied the technique of Sym ([46]) and Pohlmeyer ([42]) to produce geometric realizations of the matrix nonlinear Schrödinger equation on Hermitian symmetric Lie algebras (see [34]). In this process, they illuminated the (integrable) “localized induction (matrix) hierarchy” for arclength-parameterized curves evolving in a Hermitian symmetric Lie algebra $\mathfrak{g}$. The first three of them are:

\[
\tilde{\gamma}_t = \tilde{\gamma}_x,
\]
\[
\tilde{\gamma}_t = [\tilde{\gamma}_x, \tilde{\gamma}_{xx}],
\]
\[
\tilde{\gamma}_t = \tilde{\gamma}_{xxx} + \frac{3}{2}[\tilde{\gamma}_{xx}, [\tilde{\gamma}_x, \tilde{\gamma}_{xx}]],
\]

where $[\cdot, \cdot]$ denotes the Lie bracket in $\mathfrak{g}$. We know that, corresponding to the three types of models of the vortex filament in $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$, there are three classes of important symmetric Lie algebras. The first class consists of the Hermitian symmetric Lie algebras of compact type, i.e. $\mathfrak{g} = u(n) \ (n \geq 2)$; the second class consists of the Hermitian symmetric Lie algebras of noncompact type, i.e. $\mathfrak{g} = u(k, n-k) \ (n \geq 2$ and $1 \leq k \leq n-1)$; and the third class consists of the para-Hermitian symmetric Lie algebras $\mathfrak{g} = gl(n, \mathbb{R}) \ (n \geq 2)$ (refer to §2 for more information). When $\mathfrak{g} = su(2)$, $su(1,1)$ and $sl(2)$, Eq.(4) returns respectively to the LIE (1) in $\mathbb{R}^3$, the timelike and spacelike LIE in $\mathbb{R}^{2,1}$, and meanwhile, Eq.(5) goes respectively back to the Fukumoto-Miyazaki’s model in $\mathbb{R}^3$, and the corresponding timelike and spacelike models in $\mathbb{R}^{2,1}$. We should point out that the Heisenberg ferromagnet constructed by Fordy and Kulish in [25] is exactly equivalent to model (4) of the localized induction hierarchy on the Hermitian symmetric Lie algebra $u(n) \ (n \geq 2)$.

The geometric study of vortex filament in literature seems very fascinating. It has been proved (refer to [21, 12, 14], for example) that the leading-order LIE model (1) in $\mathbb{R}^3$, the timelike LIE (see (7) below) and the spacelike LIE (see (8) below) in $\mathbb{R}^{2,1}$ of the vortex filament are geometrically equivalent respectively to the Schrödinger flow of maps from $\mathbb{R}$ to the 2-sphere $S^2 \hookrightarrow \mathbb{R}^3$, the hyperbolic 2-space $\mathbb{H}^2 \hookrightarrow \mathbb{R}^{2,1}$ and the de Sitter 2-space $S^{1,1} \hookrightarrow \mathbb{R}^{2,1}$ and are also respectively equivalent to the three typical 2nd-order integrable systems of the AKNS hierarchy,
i.e. the focus nonlinear Schrödinger equation (NLS$^+$) (in this case, also see [49]), the defocus nonlinear Schrödinger equation (NLS$^-$) $i\varphi_t + \varphi_{xx} - 2|\varphi|^2\varphi = 0$ and the nonlinear heat equation (NLH) $q_t = q_{xx} - 2qrr\varphi$, $r_t = -r_{xx} + 2r^2q$ (for the AKNS hierarchy, see [1, 2]). This indicates that the LIE-NLS correspondence is valid for all the three NLS-type equations and the LIE-type models. The second-order correction models of the vortex filament in $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$, i.e. the Fukumoto-Miyazaki’s model (2) in $\mathbb{R}^3$ and its sister timelike and spacelike models (see §2 for details) in $\mathbb{R}^{2,1}$, are proved to be equivalent to the geometric KdV flow of maps from $\mathbb{R}$ to $S^2 \hookrightarrow \mathbb{R}^3$, $\mathbb{H}^2 \hookrightarrow \mathbb{R}^{2,1}$ and $S^{1,1} \hookrightarrow \mathbb{R}^{2,1}$ respectively (see [17, 19]). This leads to that the four typical integrable equations: $r_t + r_{xxx} + 6rr_x = 0$ (KdV), $r_t + r_{xxx} + 6r^2r_x = 0$ (mKdV), $\varphi_t = \varphi_{xxx} - 6|\varphi|^2\varphi_x$ (cKdV$^-$) and $\varphi_t = \varphi_{xxx} + 6|\varphi|^2\varphi_x = 0$ (cKdV$^+$) existing in the 3rd-order integrable systems of the AKNS hierarchy are interpreted in terms of the geometric KdV flows in a unified way. Furthermore, it is also proved that the third-order correction models of the vortex filament in $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$, i.e. the non-integrable Fukumoto-Moffatt’s model (3) in $\mathbb{R}^3$ and its brothers, the timelike and spacelike non-integrable models in $\mathbb{R}^{2,1}$, are equivalent to the generalized bi-Schrödinger maps from $\mathbb{R}$ to $S^2 \hookrightarrow \mathbb{R}^3$, $\mathbb{H}^2 \hookrightarrow \mathbb{R}^{2,1}$ and $S^{1,1} \hookrightarrow \mathbb{R}^{2,1}$ respectively (see [18, 16]). We must emphasize that, besides the concepts of Schrödinger flows, the geometric KdV flows, the generalized bi-Schrödinger flows, the Kähler structure on $S^2$ (resp. $\mathbb{H}^2$) and the para-Kähler structure on $S^{1,1}$ play crucial roles in this aspect (see [14, 19, 16] for details).

Langer and Perline showed actually in [34] that the generalized LIE-NLS correspondence is still valid on a Hermitian symmetric Lie algebra $g$. It was proved by Terng and Uhlenbeck in [47] that the generalized LIE (4) on $u(n)$ is equivalent to the Schrödinger flow of maps from $\mathbb{R}$ to the Grassmannian manifold of compact type and by Chen in [9] that the generalized LIE on $u(k, n - k)$ or $gl(n)$ are respectively equivalent to the Schrödinger flow of maps from $\mathbb{R}$ to the Grassmannian manifold of noncompact type or the para-Grassmannian manifold, and are still equivalent respectively to the three typical 2nd-order matrix integrable systems of the matrix-AKNS hierarchy. Along this route, Ding and He in [15] showed that the third-order equation (5) of the localized induction (matrix) hierarchy on the symmetric Lie algebra $sl(2n, \mathbb{R})$ is equivalent to the geometric KdV flows from $\mathbb{R}$ to the para-Grassmannian manifold, and hence the (integrable) matrix KdV equation (see [35, 5]) is created exactly by a motion of Sym-Pohlmeyer curves in $sl(2n, \mathbb{R})$ with initial data being suitably restricted. Their result also signals that the third-order equation (5) in the localized induction (matrix) hierarchy on the symmetric Lie algebra $u(n)$ and $u(k, n - k)$ are respectively the geometric KdV flows from $\mathbb{R}$ to the Grassmannian manifold of compact type and of noncompact type. However, still absent, but very important is to have the general third-order non-integrable model of the vortex filament on symmetric Lie algebras that corresponds to the (non-integrable) Fukumoto-Moffatt’s model in the Euclidean 3-space $\mathbb{R}^3$, and the timelike and spacelike third-order correction models of the vortex filament in the Minkowski 3-space $\mathbb{R}^{2,1}$.

The purpose of this paper is just to establish the third-order non-integrable models of the vortex filament on symmetric Lie algebras such that they are natural extensions of the Fukumoto-Moffatt’s model (3) in $\mathbb{R}^3$. Because of the non-integrability, we can not employ integrable methods to address the problem. In order to overcome the difficulty, we need to carefully make correct characterizations of the third-order Fukumoto-Moffatt’s (physical) correction model of the vortex filament motion in the Euclidean 3-space $\mathbb{R}^3$ in the way they fit symmetric Lie algebras. The main idea applied here is the Kähler or para-Kähler intrinsic characterization of the symmetric spaces and then we derive the model on a symmetric Lie algebra purely from the viewpoint of differential geometry. Recall that we have introduced the concept of the so-called
“generalized bi-Schrödinger flows” from a Riemannian manifold to Kähler and para-Kähler manifolds in [18]. However, this concept is not relevant to symmetric spaces in general. So our first task is to correctly modify the geometric concept of generalized bi-Schrödinger flows such that it is applicable to Hermitian and para-Hermitian symmetric spaces. A key observation we obtained is that the generalized bi-energy functional for maps \( u : \mathbb{R} \to (N, I, h) \) should be modified into

\[
E_{\alpha, \beta, \gamma}(u) = \alpha \int_{\mathbb{R}} |\nabla u|^2 \, dx + \beta \int_{\mathbb{R}} |d\nabla u|^2 \, dx + \gamma \int_{\mathbb{R}} \langle R(\nabla u, I_\mathbb{R} \nabla u) I_\mathbb{R} \nabla u, \nabla u \rangle \, dx,
\]

where \( \alpha, \beta \) and \( \gamma \) are real parameters, \( (N, I, h) \) is a Kähler (or para-Kähler) manifold with \( I \) being the complex (or para-complex) structure, \( R(\cdot, \cdot) \) is the curvature operator of \( N \) and \( \langle \cdot, \cdot \rangle \) stands for the inner product induced from the metric \( h \) on \( N \). Then, the generalized bi-Schrödinger flow in [18] is modified to be the Hamiltonian (or para-Hamiltonian) gradient flow of the above generalized bi-energy functional. Obviously, such a flow relies seriously on the intrinsic geometry of the symmetric Lie algebras, as we shall elucidate in §2 below.

By using this modification of the generalized bi-Schrödinger flows, we successfully create the third-order non-integrable model of the vortex filament on a symmetric Lie algebra \( g \) by the equation of the generalized bi-Schrödinger flows from \( \mathbb{R} \) to the corresponding symmetric space. Furthermore, by applying the geometric concept of PDEs with the given curvature representation which was proposed in the category of Yang-Mills theory in [20, 13], and with the aid of gauge transformations which can be regarded as the generalization of the Hasimoto transform in higher dimensions for non-integrable PDEs, we transform the third-order model of the vortex filament on a symmetric Lie algebra into a second to fourth order nonlinear Schrödinger-like differential-integral matrix equation. All the results obtained in this article coincide exactly with what we knew for the vortex filament in \( \mathbb{R}^3 \) and \( \mathbb{R}^{2,1} \) when the symmetric Lie algebra \( g \) goes back to \( su(2) \), \( su(1, 1) \) and \( sl(2) \). Therefore, combining our result with what have been known in literature for the leading and second order models, we actually exhibit the basic models and the related theory of the vortex filament on symmetric Lie algebras up to the third-order approximation. The main results are summarized in Theorems 1-3.

The article is organized as follows. In section 2, we give the preliminaries about symmetric Lie algebras, the symmetric Grassmannian or para-Grassmannian manifolds, the isospectral way to obtain the matrix AKNS systems and the geometric concept of generalized bi-Schrödinger maps from a Riemannian manifold to a Kähler or para-Kähler manifold. In section 3, we deduce the equation of the generalized bi-Schrödinger maps from \( \mathbb{R} \) to the three different types of the symmetric Grassmannian or para-Grassmannian manifolds. In section 4, the non-integrable third-order models of the vortex filament on three different types of symmetric Lie algebras are proved to be geometrical realizations of Sym-Pohlmeyer curves in their own symmetric Lie algebras and are transformed respectively to the second to fourth order nonlinear Schrödinger-like differential-integral matrix equations by gauge transformations. Conclusions and remarks are given in section 5.

2 Preliminaries

In this section, we will briefly mention the background of the vortex filament in three dimensional Euclidean and Minkowski spaces. We also recall some fundamental knowledge and facts about Hermitian or para-Hermitian symmetric spaces and symmetric Lie algebras. Besides, we need to elucidate the way of modifying the geometric concept on generalized bi-Schrödinger flows.
2.1 Three dimensional motions of vortex filaments

Suppose that we have a fluid in a Riemannian 3-manifold $M^3$ or a pseudo-Riemannian 3-manifold $M^{2,1}$ that evolves according to a given one-parameter family of diffeomorphism yielding the position of a fluid particle. Mathematically, the corresponding vortex filament $L$ is assumed to be parameterized by arclength $x$ and hence, we express a point (position) $X(t,x)$ as $X(t,x)$. At a point $X(t,x)$ on the filament $L$, we have the Frenet frame $\{T, N, B\}$, the curvature $k$ and the torsion $\tau$. The kinematics is now described by

$$v = v_T\varepsilon_1 T + v_1\varepsilon_2 N + v_2\varepsilon_3 B,$$

(6)

where $\varepsilon_i$ ($i = 1, 2, 3$) are respectively the sign of the tangent vector $T$, the normal vector $N$ and the binormal vector $B$ which are called the causal characters of $X(t,x) = X(t,x)$, $v = \frac{dX}{dt}(t,x)$ is the velocity of the filament, and $v_T$, $v_1$ and $v_2$ are given quantities according to the physical background (these quantities cannot be arbitrarily given, refer to [16]). The general theory of the vortex filament is to solve Eq.(6) (i.e. to determine $X(t,x)$) for given $v_T$, $v_1$ and $v_2$. It is obvious to see that Eq.(6) is not easy to solve, so it is difficult to understand the dynamical behavior of solutions to Eq.(6).

The so-called “localized induction equation” (LIE), or in other words, “localized induction approximation” (LIA), in $\mathbb{R}^3$ is an important idealized model of the vortex filament in approximation put forward by Arms and Hama (see [4]), which consists of solutions to the system (6) with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, assuming that $v_T = v_1 = 0$ and $v_2 = \kappa$ (refer to [4]). This model is exactly the model (1), as has been mentioned previously in the Introduction, which is the leading-order correction model of the vortex filament. The LIE in Minkowski (or Lorentzian) 3-space $\mathbb{R}^{2,1}$ consists of solutions to the system (6), assuming that $v_T = v_1 = 0$ and $v_2 = \varepsilon_2 \kappa$. Hence, the corresponding timelike LIE model in Minkowski 3-space $\mathbb{R}^{2,1}$ ($\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$) reads

$$\frac{dX}{dt} = \kappa B.$$

(7)

One of the spacelike LIE models in Minkowski 3-space $\mathbb{R}^{2,1}$ ($\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and $\varepsilon_3 = 1$) states

$$\frac{dX}{dt} = -\kappa B.$$

(8)

Another one is the case that $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_3 = -1$. One may verify directly that the system in this case is equivalent to the system (8), so we only consider in this article the case: $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and $\varepsilon_3 = 1$ when the filament is spacelike. Eqs.(7,8) are called respectively the binormal motions of timelike and spacelike curves in $\mathbb{R}^{2,1}$ in [14].

The second-order correction model of the vortex filament in $\mathbb{R}^3$ was deduced in 1991 by Fukumoto and Miyazaki (see [23]). This model consists of solutions to (6) with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, assuming that $v_T = \frac{1}{2} \kappa^2$, $v_1 = \lambda \kappa x$ and $v_2 = \kappa + \lambda \kappa \tau$ ($\lambda$ is a real parameter), i.e.,

$$\frac{dX}{dt} = \kappa B + \lambda \left(\frac{1}{2} \kappa^2 T + \kappa x N + \kappa \tau B\right),$$

$$= X_x \times X_{xx} + \lambda \left[X_{xxx} + \frac{3}{2} X_{xx} \times (X_x \times X_{xx})\right].$$

(9)
Omitting the first term and re-normalizing, we obtain the Fukumoto-Miyazaki’s model (2) indicated in the Introduction. Similar to the Fukumoto-Miyazaki’s model (2), the second-order timelike and spacelike correction models in the Minkowski 3-space \( \mathbb{R}^{2,1} \) are respectively deduced and characterized in [19] as follows:

\[
\frac{dX}{dt} = \frac{1}{2} \kappa^2 \mathbf{T} - \kappa_x \mathbf{N} - \kappa \tau \mathbf{B} = X_{xxx} + \frac{3}{2} X_{xx} \dot{X}_x \times (X_x \times X_{xx}) \tag{10}
\]

and

\[
\frac{dX}{dt} = \frac{1}{2} \kappa^2 \mathbf{T} + \kappa_x \mathbf{N} + \kappa \tau \mathbf{B} = X_{xxx} - \frac{3}{2} X_{xx} \dot{X}_x \times (X_x \times X_{xx}). \tag{11}
\]

The third-order correction model of the vortex filament in the Euclidean 3-space \( \mathbb{R}^3 \) was derived by Fukumoto and Moffatt in 2000 ([22, 24]). According to the present description, the Fukumoto-Moffatt’s model is just Eq. (6) with \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1 \), assuming that \( v_T = \lambda \nu \kappa^2 \), \( v_1 = \lambda \nu (2 \kappa_x \tau + \kappa \tau_x) \) and \( v_2 = \lambda \kappa + \lambda \nu (\kappa \tau^2 - \kappa_{xx}) + \lambda \mu \kappa^3 \) (\( \lambda, \nu \) and \( \mu \) are real parameters), which is explicitly expressed by Eq. (3) stated in the Introduction. Meanwhile, the third-order correction model in the Minkowski 3-space \( \mathbb{R}^{2,1} \) was proposed in [16], which consists of solutions to (6) adjusted by assuming (\( \lambda, \nu \) and \( \mu \) are also real parameters) that

\[ v_T = \lambda \nu \varepsilon_3 \kappa^2 \tau, \quad v_1 = \lambda \nu \varepsilon_2 \varepsilon_3 (2 \kappa_x \tau + \kappa \tau_x), \quad v_2 = \lambda \varepsilon_2 \kappa - \lambda \nu \varepsilon_3 (\varepsilon_2 \kappa_{xx} - \varepsilon_3 \kappa \tau^2) + \lambda \mu \kappa^3. \]

Hence, the timelike third-order model of the vortex filament in \( \mathbb{R}^{2,1} \) \( (\varepsilon_1 = -1 \text{ and } \varepsilon_2 = \varepsilon_3 = 1) \) reads

\[
X_t = \lambda \left\{ \kappa \mathbf{B} + \nu \left[ -\kappa^2 \tau \mathbf{T} + (2 \kappa_x \tau + \kappa \tau_x) \mathbf{N} - (\kappa_{xx} - \kappa \tau^2) \mathbf{B} \right] + \mu \kappa^3 \mathbf{B} \right\} \tag{12}
\]

and the spacelike one \( (\varepsilon_1 = 1, \varepsilon_2 = -1 \text{ and } \varepsilon_3 = 1) \) is

\[
X_t = \lambda \left\{ -\kappa \mathbf{B} + \nu \left[ \kappa^2 \tau \mathbf{T} + (2 \kappa_x \tau + \kappa \tau_x) \mathbf{N} + (\kappa_{xx} + \kappa \tau^2) \mathbf{B} \right] + \mu \kappa^3 \mathbf{B} \right\}. \tag{13}
\]

Models (1), (2) and (3) are respectively the leading order, the second order and the third order correction equations of the vortex filament in the Euclidean 3-space \( \mathbb{R}^3 \). These models characterize the phenomenon of the vortex filament in a 3-dimensional Riemannian fluid manifold. The exploitation of dynamical and geometric properties of these models constitutes the complete theory of the vortex filament in \( \mathbb{R}^3 \) up to the third-order approximation. The models (7), (10) and (12) for the timelike vortex filament and the models (8), (11) and (13) for the spacelike vortex filament in \( \mathbb{R}^{2,1} \) provide the basic differential equations of the theory of the vortex filament in a 3-dimensional Lorentzian fluid manifold up to the third-order approximation.

### 2.2 Symmetric Lie algebras and matrix-AKNS hierarchy

A so-called symmetric Lie algebra \( \mathfrak{g} \) is a Lie algebra that has a decomposition as a vector space sum: \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \) satisfying the (bracket) symmetric conditions: \( \{ \mathfrak{k}, \mathfrak{k} \} \subset \mathfrak{k}, \{ \mathfrak{m}, \mathfrak{m} \} \subset \mathfrak{k} \) and \( \{ \mathfrak{k}, \mathfrak{m} \} \subset \mathfrak{m} \) (see [5, 25, 28, 34]). In such a symmetric Lie algebra there is an element denoted by \( \sigma_3 \) in \( \mathfrak{k} \) such that \( \mathfrak{k} = \text{Ker}(ad_{\sigma_3}) = \{ \chi \in \mathfrak{g} | [\chi, \sigma_3] = 0 \} \). In this section, let’s first recall the three typical classes of symmetric Lie algebras. Then we briefly review the matrix AKNS hierarchy on a symmetric Lie algebra.
The first class of symmetric Lie algebras consists of Hermitian symmetric Lie algebras \( u(n) \) \((n \geq 2)\) with index \( k \) \((1 \leq k < n)\) of compact type. In fact, for any given \( 1 \leq k < n \), let
\[
\sigma_3 = \frac{\sqrt{-1}}{2} \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix},
\]
and we see that \( u(n) \) is decomposable as \( u(n) = k \oplus m \) satisfy the symmetric conditions: \([k, k] \subset k, [m, m] \subset k, [k, m] \subset m\), where
\[
k = \text{Kernel}(ad_{\sigma_3}) = \left\{ \begin{pmatrix} A_{k \times k} & 0 \\ 0 & B_{(n-k) \times (n-k)} \end{pmatrix} \in u(n) \right\}
\]
and
\[
m = \left\{ \begin{pmatrix} 0 & U_{k \times (n-k)} \\ -U^*_{k \times (n-k)} & 0 \end{pmatrix} \in u(n) \right\},
\]
where \( U^*_{k \times (n-k)} \) stand for the transposed conjugate matrix of \( U_{k \times (n-k)} \). This indicates that \( u(n) \) is a symmetric Lie algebra depending on \( k \) \((1 \leq k < n)\). We should point out that when \( n = 2m \) is even and \( k = m \) the half of \( n \), the Hermitian symmetric Lie algebras \( u(2m) \) with index \( m \) may be replaced exactly by \( su(2m) \) with index \( m \), since in this case, \( \sigma_3 \) given by (14) belongs to \( su(2m) \) itself.

The second class consists of Hermitian symmetric Lie algebras \( u(k, n - k) \) with index \( k \) \((1 \leq k < n)\) of noncompact type. In this case, let \( \sigma_3 \) be also given by (14) and we see that \( u(k, n - k) \) is decomposable as \( u(k, n - k) = k \oplus m \) satisfy the symmetric conditions, where
\[
k = \text{Kernel}(ad_{\sigma_3}) = \left\{ \begin{pmatrix} A_{k \times k} & 0 \\ 0 & B_{(n-k) \times (n-k)} \end{pmatrix} \in u(k, n - k) \right\}
\]
and
\[
m = \left\{ \begin{pmatrix} 0 & U_{k \times (n-k)} \\ U^*_{k \times (n-k)} & 0 \end{pmatrix} \in u(k, n - k) \right\}.
\]
Similarly, when \( n = 2m \) is even and \( k = m \) the half of \( n \), the Hermitian symmetric Lie algebras \( u(m, m) \) with index \( m \) is replaced just by \( su(m, m) \) with index \( m \). This is because \( \sigma_3 \) given by (14) belongs to \( su(m, m) \) in this case.

The third class consists of para-Hermitian symmetric Lie algebras \( gl(n, \mathbb{R}) \) \((n \geq 2)\) with index \( k \) \((1 \leq k < n)\). In this case, for any given \( 1 \leq k < n \), setting
\[
\sigma_3 = \frac{1}{2} \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix},
\]
we see that \( gl(n, \mathbb{R}) = k \oplus m \) is a symmetric Lie algebra, where
\[
k = \text{Kernel}(ad_{\sigma_3}) = \left\{ \begin{pmatrix} A_{k \times k} & 0 \\ 0 & B_{(n-k) \times (n-k)} \end{pmatrix} \in gl(n, \mathbb{R}) \right\}
\]
and
\[
m = \left\{ \begin{pmatrix} 0 & U^+_{k \times (n-k)} \\ U^-_{k \times (n-k)} & 0 \end{pmatrix} \in sl(n, \mathbb{R}) \right\} \forall U^+_{k \times (n-k)} \text{ and } U^-_{(n-k) \times k} \}.
\]
When \( n = 2m \) and \( k = m \), the para-Hermitian symmetric Lie algebra \( gl(2m, \mathbb{R}) \) with index \( m \) is replaced by the symmetric Lie algebra \( sl(2m, \mathbb{R}) \) with index \( m \) (see [15]). It should also be pointed out that para-Hermitian symmetric Lie algebras relate to the geometry of the para-complex structure and the para-Kähler structure, which has been studied since the first papers of Rashevskii [43] and Libermann [36] until now. For developments on this subject, referred to works [3, 10] and the references therein.

We should mention that, for a given symmetric Lie algebra \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \) belonging to the above three classes and a matrix \( E \in \mathfrak{g} \), we usually denote by \( E^{(\text{diag})} \) (resp. \( E^{(\text{off-diag})} \)) its \( \mathfrak{k} \)-part (resp. \( \mathfrak{m} \)-part) and is called the diagonal (resp. off-diagonal) part of \( E \).

It is also well-known that the above three classes of symmetric Lie algebras relate to three different types of symmetric spaces, which may be presented by adjoint orbits (refer to, for example, [7, 28]). The first one consists of the complex compact (Kähler) Grassmannian manifolds:

\[
G_{n,k} = \{ E^{-1} \sigma_3 E \mid \forall E \in U(n) \} := U(n)/U(k) \times U(n-k),
\]

where \( \sigma_3 \) is given by (14). The second one consists of the complex noncompact (Kähler) Grassmannian manifolds:

\[
G^k_n = \{ E^{-1} \sigma_3 E \mid \forall E \in U(k, n-k) \} := U(k, n-k)/U(k) \times U(n-k),
\]

where \( \sigma_3 \) is still given by (14). The third one consists of the (para-Kähler) para-Grassmannian manifolds:

\[
\tilde{G}_{n,k} = \{ E^{-1} \sigma_3 E \mid \forall E \in GL(n, \mathbb{R}) \} := GL(n, \mathbb{R})/GL(k, \mathbb{R}) \times GL(n-k, \mathbb{R}),
\]

where \( \sigma_3 \) is given by (15). We point out that, as mentioned above, we especially have

\[
G_{2m,m} = SU(2m)/SU(m) \times SU(m),
\]

\[
G^m_{2m} = SU(m, m)/SU(m) \times SU(m)
\]

and

\[
\tilde{G}_{2m,m} = SL(2m, \mathbb{R})/SL(m, \mathbb{R}) \times SL(m, \mathbb{R}).
\]

For \( M = G_{n,k}, G^k_n \) and \( \tilde{G}_{n,k} \), the Kähler structure \( J_u \) (resp. the para-Kähler structure \( K_u \)) at the point \( u \in M \) is given by (see [47, 9, 15])

\[
J_u \text{ (resp. } K_u) = [u, \cdot] : T_uM \to T_uM.
\]

Next, in order to describe the matrix AKNS hierarchy, we consider an overdetermined linear differential system as follows

\[
\varphi_x = (\lambda \sigma_3 + Q) \varphi, \quad \varphi_t = V \varphi.
\]

This system involves two independent variables \( x \) (position) and \( t \) (time), and a spectral parameter \( \lambda \). The eigenfunction \( \varphi \) takes values in a Lie group \( \mathbf{G} \), while \( (\lambda \sigma_3 + Q) \) and \( V \) take values in the Lie algebra \( \mathfrak{g} \) of \( \mathbf{G} \). For our purpose, \( \mathfrak{g} \) is required to be a symmetric Lie algebra, and \( \sigma_3, Q \) are suitably chosen for this structure. In such a symmetric Lie algebra, \( \sigma_3 \) is chosen as indicated above, \( Q \) is required to be an \( \mathfrak{m} \)-potential (i.e. \( Q(t, x) \in \mathfrak{m} \) for all \( t \)) and \( V \) a function of \( Q \) and its derivatives. One notes that the system (20) gives the zero curvature condition

\[
Q_t = V_x - [\lambda \sigma_3 + Q, V].
\]
So, in finding $V$ in terms of $Q$ and its derivatives such that the integrability condition (21) is satisfied, we invoke a polynomial ansatz to $V$ as $V = \sum_{i=0}^{N} P^{(i)} \lambda^i$, where $\forall i$, $P^{(i)}$ is a function of $Q$ and its derivatives independent of $\lambda$. The strategy here is to carry out $V$ for a given integer $N$ and finally, to obtain a nonlinear integrable PDE for the $m$-potential $Q$. For details, refer to [25, 5, 34].

It is shown in [25] that the above process for $N = 2$ on the Hermitian symmetric Lie algebra $u(n)$, which is the so-called second-order isospectral flow of the matrix Schrödinger equation on $u(n)$, leads to the matrix nonlinear Schrödinger equation:

$$iQ_t + Q_{xx} + 2QQ^*Q = 0.$$  

To obtain the matrix KdV equation, we work on the Lie algebra $g = sl(2n, \mathbb{R})$. By a simple argument, the above process for $N = 3$ (i.e, the third-order isospectral flow) on $sl(2n, \mathbb{R})$ gives the ansatz expression for $V = Q_{xx} - 2QQ^*Q + [Q, [Q, Q]] + 2[Q, [\sigma_3, Q]] + 2\lambda + Q^2 + \sigma_3 \lambda^3$, which leads to the following equation for an $m$-potential $Q = \left( \begin{array}{c} 0 \\ U^+ \\ U^- \\ 0 \end{array} \right)$:

$$Q_t = Q_{xxx} - 2(Q^3)_{xx} - [Q, [Q, Q]].$$  

(22)

Eq.(22) is rewritten as the following (noncommutative) coupled matrix KdV equation:

$$\begin{cases}
U_t^+ = U_{xxx}^+ - 3U^+U^-U_x^+ - 3U_x^+U^-U^+,
U_t^- = U_{xxx}^- - 3U^-U^+U_x^- - 3U_x^-U^+U^-.
\end{cases}$$  

(23)

Generally, $U^+$ and $U^-$ in the coupled matrix KdV equation (23) are a pair of $k \times n$ and $n \times k$ matrix. Here we only deal with Eq.(23) with $U^+$ and $U^-$ being $n \times n$ matrices. Now by the reduction: $U = U^+$ and $U^- = I_n$, we obtain the matrix KdV equation

$$U_t = U_{xxx} - 3UU_x - 3U_xU$$

from Eq.(23). By a different reduction: $U^-(t,x) = U^+(t,x) = U(t,x)$, Eq.(23) leads to the modified matrix KdV equation:

$$U_t = U_{xxx} - 3U^2U_x - 3U_xU^2.$$  

If one wants to obtain the fourth order integrable matrix Schrödinger equation on $u(n)$, then by the same process, for $N = 4$ (i.e, the fourth-order isospectral flow) with a somewhat long computation, he has

$$iQ_t + q_{xxxx} + 4qq_{xx}\lambda^2 q + 2qq_xq^*q + 4qq^*q_{xx} + 2q_xq^*q + 4qq_x^*q + 6q_xq^*q_x + 6qq^*qq^*q = 0,$$  

where

$$Q = \left( \begin{array}{cc} 0 & q \\ -q^* & 0 \end{array} \right)$$

is an $m$-potential of $u(n)$ with $q^*$ being the transposed conjugate matrix of the complex $k \times (n - k)$-matrix $q$. 

10
2.3 Modification of generalized bi-Schrödinger flows

We know that the geometric concepts of Schrödinger flows and geometric KdV flows introduced recently in literature (see, for example, [21, 47, 45, 19]) are of independent research interest and enrich the content of research on LIE models. In particular, it is worthy of note that to make use of the para-Kähler structure provides a new perspective of utilizing the concepts of Schrödinger flows and geometric KdV flows. The results obtained in [14, 19, 15] illuminate that the nonlinear Schrödinger equations have the Kähler structure rather than the para-Kähler structure, while the nonlinear coupled heat equation and the KdV equation have the para-Kähler structure rather than the Kähler structure. The recognition that the KdV equation possesses the para-Kähler structure seems very appealing. As we shall see below, the Kähler structure on the Grassmannian manifolds $G_{n,k}$ and $G_k^{*n}$, and the para-Kähler structure on the para-Grassmannian manifold $\tilde{G}_k^{*n}$ described above will still play crucial roles in establishing the non-integral third-order model of the vortex filament on a symmetric Lie algebra.

Let $(M, g)$ be a Riemannian manifold and $(N, h)$ a (pseudo) Riemannian manifold. Recall that the energy functional and the bi-energy functional of a map $u : M \to N$ are defined respectively by

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 dv_g$$

and

$$E_2(u) = \frac{1}{2} \int_M \|(d + d^*)^2 u\|^2,$$

where $d^*$ denotes the co-differential operator. It is well-known that harmonic maps (resp. bi-harmonic maps) are defined by critical points of the energy functional (resp. the bi-energy functional). If the target manifold $(N, h)$ is a Kähler manifold with a compatible complex structure $J$, then in [18] generalized bi-Schrödinger flow of map $u$ from the Riemannian manifold $(M, g)$ to the Kähler manifold $(N, J, h)$ is defined by the Hamiltonian gradient flow of the generalized bi-energy functional

$$E^*_{\alpha, \beta, \gamma}(u) = \left\{ \alpha E(u) + \beta E_2(u) + \gamma \int_M \|\nabla u\|^4 \right\}.$$  \hspace{1cm} (25)

In other words, $u$ satisfies the equation

$$u_t = J_u \nabla E^*_{\alpha, \beta, \gamma}(u).$$  \hspace{1cm} (26)

Let $(N, K, h)$ be a para-Kähler manifold with $K$ as its para-Kähler structure. Similarly, a map $u = u(t, x) : [0, T) \times M \to N$, where $0 < T \leq \infty$, is called a generalized bi-Schrödinger flow of map from the Riemannian manifold $(M, g)$ to the para-Kähler manifold $(N, K, h)$ if $u$ fulfills the following para-Hamiltonian gradient flow of the functional $E^*_{\alpha, \beta, \gamma}(u)$

$$u_t = K_u \nabla E^*_{\alpha, \beta, \gamma}(u).$$

We point out that, when $\alpha \neq 0$ and $\beta = \gamma = 0$, generalized bi-Schrödinger maps reduce to Schrödinger flows (see [21, 47]), or in other words, the equation of Schrödinger flows is just the Hamiltonian gradient flow of the energy functional $E(u)$.

In order for the generalized bi-Schrödinger flows to be applicable in establishing the desired third-order models of the vortex filament on symmetric Lie algebras, we need to modify the generalized bi-energy functional (25). Our key observation comes from the following facts. For a map $u$ from $\mathbb{R} \to S^2 \hookrightarrow \mathbb{R}^3$, $\mathbb{R}^2 \leftrightarrow \mathbb{R}^{2,1}$ and $S^{1,1} \hookrightarrow \mathbb{R}^{2,1}$, we have respectively the identities:

$$R(u_x, J_u u_x) J_u u_x = |u_x|^2 u_x,$$

$$R(u_x, J_u u_x) J_u u_x = -|u_x|^2 u_x.$$
and
\[ R(u_x, K_u u_x)K_u u_x = |u_x|^2 u_x, \]
where \( R(\cdot, \cdot) \) stands for the curvature operator on the target manifold \( M = S^2, \mathbb{H}^2 \) or \( S^{1,1} \) (refer to [19]). Hence, the third term \( \int_M \|\nabla u\|^4 \) in (25) for the three different cases can be unified by
\[ \int_M \langle R(u_x, J_u u_x)J_u u_x, u_x \rangle, \]
where \( \langle \cdot, \cdot \rangle \) and \( J_u \) denote respectively the inner product induced from the standard (pseudo) Riemannian metric and the complex (or para-complex) structure on \( M = S^2, \mathbb{H}^2 \) or \( S^{1,1} \).

Based on this observation, we see that the generalized bi-energy functional (25) of smooth maps \( u \) from a Riemannian manifold \( (M, g) \) to a Kähler manifold \( (N, J, h) \) might be modified as follows. Let \( \{e_1, e_2, \cdots, e_n\} \) be a local frame of \( (M, g) \), and the metric \( g \) in the frame is expressed by \( g = (g_{ij}) \) and \( (g^{ij}) \) its inverse. Then we define
\[ E_{\alpha, \beta, \gamma}(u) = \left\{ \alpha E(u) + \beta E_2(u) + \gamma \int_M \sum_{i,j,k,l=1}^n g^{ij} g^{kl} \langle R(\nabla_{e_i} u, J_u \nabla_{e_j} u)J_u \nabla_{e_k} u, \nabla_{e_l} u \rangle \right\}, \tag{27} \]
where \( R(\cdot, \cdot) \) is the curvature operator on the target manifold \( N \) and \( \nabla_{e_k} \) the covariant derivative on the pull-back bundle \( u^{-1}TN \) induced from the Levi-Civita connection on \( N \). We should point out that the density function
\[ \sum_{i,j,k,l=1}^n g^{ij} g^{kl} \langle R(\nabla_{e_i} u, J_u \nabla_{e_j} u)J_u \nabla_{e_k} u, \nabla_{e_l} u \rangle \]
of the \( \gamma \)-term in (27) is independent of the choice of a local frame and hence the definition of the generalized bi-energy function (27) is of significance.

If the target manifold is a para-Kähler manifold denoted by \( (N, K, h) \), similarly we should define the corresponding functional as follows
\[ E_{\alpha, \beta, \gamma}(u) = \left\{ \alpha E(u) + \beta E_2(u) + \gamma \int_M \sum_{i,j,k,l=1}^n g^{ij} g^{kl} \langle R(\nabla_{e_i} u, K_u \nabla_{e_j} u)K_u \nabla_{e_k} u, \nabla_{e_l} u \rangle \right\}. \]

**Definition 1** Let \( (N, J, h) \) (resp. \( (N, K, h) \)) be a Kähler (resp. para-Kähler) manifold with \( J \) (resp. \( K \)) as its Kähler (resp. para-Kähler) structure. A map \( u = u(t, x) : [0, T) \times M \to N, \) where \( 0 < T \leq \infty, \) is called a generalized bi-Schrödinger flow from the Riemannian manifold \( (M, g) \) to \( (N, J, h) \) (resp. \( (N, K, h) \)) if \( u \) satisfies the following Hamiltonian gradient flow
\[ u_t = J_u \nabla E_{\alpha, \beta, \gamma}(u) \quad (\text{resp.} u_t = K_u \nabla E_{\alpha, \beta, \gamma}(u)), \tag{28} \]
where \( E_{\alpha, \beta, \gamma}(u) \) is the generalized bi-energy functional defined by (27).

A map \( u = u(t, x) : [0, T) \times M \to N \) is called a bi-Schrödinger flow from the Riemannian manifold \( (M, g) \) to \( (N, J, h) \) if \( u \) satisfies the following Hamiltonian gradient flow
\[ u_t = J_u \nabla E_{0,1, \gamma}(u) \quad (\text{resp.} u_t = K_u \nabla E_{0,1, \gamma}(u)). \tag{29} \]
We remark that without the contribution of the $\gamma$-term appearing in the generalized bi-energy functional $E_{\alpha,\beta,\gamma}(u)$, one cannot obtain completely the desired model on a symmetric Lie algebra, as we shall see in §3. This gives the reason why the $\gamma$-term which relies seriously on the intrinsic geometry of the target manifold in the functional $E_{\alpha,\beta,\gamma}(u)$ is indispensable. However, one also notes that the concept of generalized bi-Schrödinger maps is mainly based on the bi-energy and hence bi-Schrödinger maps. Therefore, the bi-energy plays an essential role in this respect. Furthermore, comparing $E_{\alpha,\beta,\gamma}(u)$ (27) with $E'_{\alpha,\beta,\gamma}(u)$ (25) in the case that the start manifold is $\mathbb{R}$ and the target manifold is to be $\mathbb{H}^2$, there is an additional sign in the third term of the generalized bi-energy functional $E_{\alpha,\beta,\gamma}(u)$. The so-called mysterious question why the parameters $\beta$ and $\gamma$ have to satisfy the different conditions arises from the geometric explanation of the second to fourth order integrable equations in the AKNS hierarchy by generalized bi-Schrödinger flows in [16] becomes clear now. By using the present concept of generalized bi-Schrödinger flows, the parameters $\beta$ and $\gamma$ do indeed satisfy the same condition: $\gamma = -\frac{\beta}{8}$ (one may refer to §3 of [16] for details). Hence, the modification looks very natural and effective. In the following context, generalized bi-Schrödinger flows are always referred to in the present sense, unless otherwise specified.

**Examples:**

a) The equation of generalized bi-Schrödinger flows from $\mathbb{R}$ to $S^2 \hookrightarrow \mathbb{R}^3$ is just the one displayed in [18, 16], i.e. for $s = (s_1, s_2, s_3) \in \mathbb{R}^3$,

$$s_t = s \times (-\alpha s_{xx} + \beta s_{xxxx} - (4\gamma - 2\beta)(|s_x|^2 s_x)_x), \quad |s|^2 := s_1^2 + s_2^2 + s_3^2 = 1. \quad (30)$$

b) The equation of generalized bi-Schrödinger flows from $\mathbb{R}$ to $\mathbb{H}^2 \hookrightarrow \mathbb{R}^{2,1}$ is the one displayed in [18, 16] with $\gamma$ changed by $-\gamma$, i.e. for $s = (s_1, s_2, s_3) \in \mathbb{R}^{2,1}$,

$$s_t = s \times (-\alpha s_{xx} + \beta s_{xxxx} + (4\gamma - 2\beta)(|s_x|^2 s_x)_x), \quad |s|^2 := s_1^2 + s_2^2 - s_3^2 = -1. \quad (31)$$

c) The equation of generalized bi-Schrödinger flows from $\mathbb{R}$ to $S^{1,1} \hookrightarrow \mathbb{R}^{2,1}$ is the same one as displayed in [16], i.e.

$$s_t = s \times (-\alpha s_{xx} + \beta s_{xxxx} - (4\gamma - 2\beta)(|s_x|^2 s_x)_x), \quad |s|^2 := s_1^2 + s_2^2 - s_3^2 = 1. \quad (32)$$

It is proved in [18, 16] that the equations (30), (31) and (32) of the generalized bi-Schrödinger flows are respectively equivalent to the non-integrable Fukumoto-Moffatt’s model (3) in $\mathbb{R}^3$, the timelike third-order model (12) and the spacelike third-order model (13) of the vortex filament in the Minkowski space $\mathbb{R}^{2,1}$.

### 3 Generalized bi-Schrödinger flows into symmetric spaces

In this section, we shall use our new generalized bi-Schrödinger flows introduced above to deduce the equations of the generalized bi-Schrödinger flows to the compact Kähler Grassmannian manifolds $G_{n,k}$, the noncompact Kähler Grassmannian manifolds $G^h_k$ and the para-Kähler Grassmannian manifolds $\tilde{G}_{n,k}$ respectively. These models return respectively to the non-integrable Fukumoto-Moffatt’s model (3) in $\mathbb{R}^3$, the timelike third-order model (12) and the spacelike third-order model (13) of the vortex filament in $\mathbb{R}^{2,1}$ when the target manifold $M$ goes back to $S^2$, $\mathbb{H}^2$ and $S^{1,1}$ correspondingly. Therefore, they are naturally the Fukumoto-Moffatt’s counterparts of the vortex filament on symmetric Lie algebras.
First of all, we note that \( \langle A, B \rangle = -\text{tr}(AB) \) is the bi-invariant inner product on \( u(n) \) (refer to, for example, [47]), \( \langle A, B \rangle = \text{tr}(AB) \) is the bi-invariant inner product on \( u(k, n - k) \) (refer to, for example, [28]) and \( \langle A, B \rangle = \text{tr}(AB) \) is the bi-invariant pseudo inner product on \( gl(n, \mathbb{R}) \). Now for a map \( u \) from \( \mathbb{R} \) to the Kähler Grassmannian manifolds \( G_{n,k} = \{ E^{-1} \sigma_3 E \mid \forall E \in U(n) \} \) of compact type, or to the Kähler Grassmannian manifolds \( G_n^k = \{ E^{-1} \sigma_3 E \mid \forall E \in U(k, n - k) \} \) of noncompact type and to the para-Kähler Grassmannian manifolds \( \tilde{G}_{n,k} = \{ E^{-1} \sigma_3 E \mid \forall E \in GL(n, \mathbb{R}) \} \), we see that the \( \gamma \)-term of the generalized bi-energy functional \( E_{\alpha, \beta, \gamma}(u) \) given by (27) becomes

\[
\int_{\mathbb{R}} \langle R(u_x, J_u u_x)J_u u_x, u_x \rangle \quad \text{resp} \quad \int_{\mathbb{R}} \langle R(u_x, K_u u_x)K_u u_x, u_x \rangle,
\]

where \( x \) stands for the standard coordinate on \( \mathbb{R} \) and \( u_x = \nabla_{\partial x} u \). Now we have

**Theorem 1** The equation of the generalized bi-Schrödinger flows (28) from \( \mathbb{R} \) into the symmetric space \( M \) reads

\[
\varphi_t = \left[ \varphi, -\alpha \varphi_{xx} + \beta \varphi_{xxxx} + (4\gamma - 2\beta) \left( \varphi_x \varphi^{-1} \varphi_{x} \varphi^{-1} \right)_x \right],
\]

(33)

where \( M \) is one of \( G_{n,k}, G_n^k \) or \( \tilde{G}_{n,k} \), this corresponds respectively to the Hermitian symmetric Lie algebra \( u(n) \) with index \( k \) of compact type, the Hermitian symmetric Lie algebra \( u(k, n - k) \) with index \( k \) of noncompact type and the para-Hermitian symmetric Lie algebra \( gl(n, \mathbb{R}) \) with index \( k \).

**Proof:** We will reach the conclusion case by case as follows.

1) Let \( M = G_{n,k} = \{ E^{-1} \sigma_3 E \mid \forall E \in U(n) \} \), the Grassmannian manifolds of compact type, and \( \varphi = E^{-1}(t, x) \sigma_3 E(t, x) \) be a map from the line \( \mathbb{R} \) to \( M = \{ E^{-1} \sigma_3 E \mid \forall E \in U(n) \} \), where \( \sigma_3 \) is given by (14). Without loss of generality, we may assume that \( E \) satisfies: \( E_x = PE \) for some \( P \in \mathfrak{m} \), where \( \mathfrak{m} \) fits the symmetric condition: \( u(n) = \mathfrak{k} \oplus \mathfrak{m} \). In fact, if \( E \) does not meet the requirement, we may make a transform:

\[
E \to \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} E
\]

(this is because the form \( E^{-1} \sigma_3 E \) is invariant up to the transform) such that by suitably choosing \( A \) and \( B \) (through solving a linear differential system of \( A \) and \( B \)) \( P \) can be modified so that the new \( P \) satisfies \( P \in \mathfrak{m} \). Based on this fact, we have

\[
\varphi^{-1} = 4\varphi^* = -4\varphi, \quad \Rightarrow \quad (\varphi^{-1})^2 = -4I,
\]

(34)

\[
\varphi_x = E^{-1}[\sigma_3, P]E, \quad \Rightarrow \quad [\varphi, \varphi_x] = -E^{-1}PE,
\]

(35)

\[
\varphi_x \varphi^{-1} = -2E^{-1}PE, \quad \varphi^{-1} \varphi_x = 2E^{-1}PE.
\]

(36)

Combining (35) with (36), we have

\[
[\varphi, \varphi_x] = J_\varphi \varphi_x = \frac{1}{2} \varphi_x \varphi^{-1}
\]

(37)

and

\[
\varphi_x^2 = E^{-1}[\sigma_3, P]^2 E = E^{-1}P^2 E = \frac{1}{4} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1}.
\]

(38)
Now we rewrite the generalized bi-energy $E_{\alpha,\beta,\gamma}(\varphi)$ as

$$E_{\alpha,\beta,\gamma}(u) = \left\{ \alpha E(u) + \beta E_2(u) + \gamma \int_{\mathbb{R}} (R(u_x, J_u u_x) J_u u_x, J_u u_x) \right\} := \alpha E(\varphi) + \beta E_2(\varphi) + \gamma \tilde{E}(\varphi).$$

Hence the calculation of the gradient of the generalized bi-energy $E_{\alpha,\beta,\gamma}(\varphi)$ reduces to calculating the gradient of $E(\varphi)$, $E_2(\varphi)$ and $\tilde{E}(\varphi)$ respectively. Firstly, we come to calculate explicitly the gradient of the energy functional $E(\varphi)$. It is well known (see, for example, [47]) that

$$\nabla E = \tau(\varphi) = -\nabla_{\varphi_x} \varphi_x,$$

where $\tau(\varphi)$ stands for the tension field of $\varphi$. One may verify directly by using (35) and (36) that

$$\nabla_{\varphi_x} \varphi_x = \varphi_{xx} + E^{-1}[P, [\sigma_3, P]]E \quad \text{(taking the tangent part)}$$

$$= \varphi_{xx} - \varphi_x \varphi^{-1} \varphi_x. \quad (40)$$

Next, by using the identities (36,34) and the fact (40) (with (39)), we see that the bi-energy functional $E_2(\varphi)$ can be re-written as

$$E_2(\varphi) = \frac{1}{2} \int_{\mathbb{R}} \| (d + d^*)^2 \varphi \|^2 = \frac{1}{2} \int_{\mathbb{R}} (\varphi_{xx}, \varphi_{xx})$$

$$= \frac{1}{2} \int_{\mathbb{R}} \left( \langle \varphi_{xx}, \varphi_{xx} \rangle - 2 \langle \varphi_{xx}, \varphi_x \varphi^{-1} \varphi_x \rangle + \langle \varphi_x \varphi^{-1} \varphi_x, \varphi_x \varphi^{-1} \varphi_x \rangle \right)$$

$$= \frac{1}{2} \int_{\mathbb{R}} \langle \varphi_{xx}, \varphi_{xx} \rangle - \int_{\mathbb{R}} \langle \varphi_{xx}, \varphi_x \varphi^{-1} \varphi_x \rangle + \frac{1}{2} \int_{\mathbb{R}} \langle \varphi_x \varphi^{-1} \varphi_x, \varphi_x \varphi^{-1} \varphi_x \rangle$$

$$:= E_{21}(\varphi) - E_{22}(\varphi) + E_{23}(\varphi), \quad (41)$$

where

$$E_{21}(\varphi) = \frac{1}{2} \int_{\mathbb{R}} \langle \varphi_{xx}, \varphi_{xx} \rangle,$$

$$E_{22}(\varphi) = \int_{\mathbb{R}} \langle \varphi_{xx}, \varphi_x \varphi^{-1} \varphi_x \rangle,$$

$$E_{23}(\varphi) = \frac{1}{2} \int_{\mathbb{R}} \langle \varphi_x \varphi^{-1} \varphi_x, \varphi_x \varphi^{-1} \varphi_x \rangle.$$

The calculation of the gradient of the bi-energy functional $E_2(\varphi)$ comes down to calculating the gradients of $E_{21}(\varphi)$, $E_{22}(\varphi)$ and $E_{23}(\varphi)$ respectively.

It is easy to see that the variation of the functional $E_{21}(\varphi)$ is

$$\lim_{\varepsilon \to 0} \frac{E_{21}(\varphi + \varepsilon h) - E_{21}(\varphi)}{\varepsilon} = \int_{\mathbb{R}} \langle \varphi_{xx}, h_{xx} \rangle = \int_{\mathbb{R}} \langle \varphi_{xxx}, h \rangle.$$ 

This indicates that

$$\nabla E_{21} = \varphi_{xxx} + k_1 \quad (42)$$
for some \( k \)-valued smooth function \( \hat{k}_1 \). To calculate the gradient of \( E_{22}(\varphi) \), we reexpress it as follows,

\[
E_{22}(\varphi) = \int_\mathbb{R} \langle \varphi_{xx}, \varphi_x \varphi^{-1} \varphi_x \rangle
\]

\[
= -\int_\mathbb{R} \langle \varphi_{xx}, \varphi^{-1} \varphi_x \varphi_x \rangle
\]

\[
= \frac{1}{4} \int_\mathbb{R} \text{tr} (\varphi_x \varphi^{-1} \varphi_x \varphi_x \varphi^{-1}).
\]

(43)

Here we have used the formulae (36) and (38) as well as the invariant inner product on \( u(n) \) mentioned above. Therefore

\[
\lim_{\varepsilon \to 0} \frac{E_{22}(\varphi + \varepsilon h) - E_{22}(\varphi)}{\varepsilon} = \frac{1}{4} \int_\mathbb{R} \text{tr} \left( h_{xx} \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x + \varphi_{xx} \varphi^{-1} \varphi_x \varphi^{-1} h_x \varphi^{-1} + \varphi_{xx} \varphi^{-1} h_x \varphi^{-1} \varphi_x \varphi^{-1}
\right.
\]

\[
- (\varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_{xx} \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} + \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} + \varphi_{xx} \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} h)
\]

\[
= \frac{1}{4} \int_\mathbb{R} \text{tr} \left( \left( (\varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1})_{xx} - (\varphi^{-1} \varphi_{xx} \varphi^{-1} \varphi_x \varphi^{-1})_x - (\varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_{xx} \varphi^{-1} \varphi_x \varphi^{-1})_x
\right.
\]

\[
- (\varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_{xx} \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1})_x \right) h.
\]

Hence we obtain

\[
\nabla E_{22} = -\frac{1}{4} \left( (\varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1})_{xx} - (\varphi^{-1} \varphi_{xx} \varphi^{-1} \varphi_x \varphi^{-1})_x - (\varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_{xx} \varphi^{-1} \varphi_x \varphi^{-1})_x \right) + \hat{k}_2
\]

\[
= (\varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_{xx} \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1})_x + \hat{k}_2
\]

(44)

for some \( k \)-valued smooth function \( \hat{k}_2 \).

Let’s now treat \( E_{23}(\varphi) \). One sees that

\[
E_{23}(\varphi) = \frac{1}{2} \int_\mathbb{R} \langle \varphi_x \varphi^{-1} \varphi_x, \varphi_x \varphi^{-1} \varphi_x \rangle
\]

\[
= -\frac{1}{2} \int_\mathbb{R} \text{tr} (\varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x)
\]

\[
= \frac{1}{8} \int_\mathbb{R} \text{tr} (\varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1}).
\]

(45)

Here we have used the formula (38). Before we calculate the gradient of \( E_{23}(\varphi) \), let’s treat the \( \gamma \)-term

\[
\tilde{E}(\varphi) = \int_\mathbb{R} (R(\varphi_x, J_\varphi \varphi_x) J_\varphi \varphi_x, \varphi_x)
\]

of the generalized bi-energy functional \( E_{a, \beta, \gamma}(\varphi) \). We know that (refer to [40])

\[
R(\varphi_x, J_\varphi \varphi_x) J_\varphi \varphi_x = [J_\varphi \varphi_x, [\varphi_x, J_\varphi \varphi_x]].
\]
Hence, by using (34,37) and (38), we obtain
\[ R(\varphi x, J_\varphi \varphi x) J_\varphi \varphi x = \frac{1}{4} [\varphi x \varphi^{-1}, [\varphi x, \varphi x \varphi^{-1}]] = \frac{1}{4} [\varphi x \varphi^{-1}, \varphi x^2 \varphi^{-1} - \varphi x \varphi^{-1} \varphi x] \]
\[ = \frac{1}{4} (2\varphi x \varphi^{-1} \varphi x^2 \varphi^{-1} - \varphi x \varphi^{-1} \varphi x - \varphi x^2 \varphi^{-1} \varphi x) \]
\[ = \frac{1}{4} (2\varphi x \varphi^{-1} \left( \frac{1}{4} \varphi x \varphi^{-1} \varphi x \right) \varphi^{-1} - \varphi x \varphi^{-1} \varphi x - \varphi x \varphi^{-1} \varphi x)
\[ - \left( \frac{1}{4} \varphi x \varphi^{-1} \varphi x \right) \varphi^{-1} \varphi x \]
\[ = -\varphi x \varphi^{-1} \varphi x \varphi^{-1} \varphi x \]

and
\[ \tilde{E}(\varphi) = \int_R (R(\varphi x, J_\varphi \varphi x) J_\varphi \varphi x, \varphi x) \]
\[ = -\int_R \text{tr} \left(-\varphi x \varphi^{-1} \varphi x \varphi^{-1} \varphi x \cdot \varphi x\right) \]
\[ = \int_R \text{tr} \left(-\varphi x \varphi^{-1} \varphi x \varphi^{-1} \varphi x \cdot \left( \frac{1}{4} \varphi^{-1} \right)\right) \]
\[ = \frac{1}{4} \int_R \text{tr} \left(\varphi x \varphi^{-1} \varphi x \varphi^{-1} \varphi x \varphi^{-1} \varphi x\right) \]
\[ = 2\tilde{E}_{23}(\varphi). \quad (46) \]

It is direct to see that the variation of \( \tilde{E}(\varphi) \) is
\[ \lim_{\varepsilon \to 0} \frac{\tilde{E}(\varphi + \varepsilon h) - \tilde{E}(\varphi)}{\varepsilon} \]
\[ = \frac{1}{4} \lim_{\varepsilon \to 0} \int_R \text{tr} \left(\left((\varphi + \varepsilon h)x(\varphi + \varepsilon h)^{-1}\right)^4 - (\varphi x \varphi^{-1})^4\right) \]
\[ = \frac{1}{4} \int_R \text{tr} \left(\{\varphi x \varphi^{-1} \varphi x \varphi^{-1}, \{\varphi x \varphi^{-1}, h x \varphi^{-1} - \varphi x \varphi^{-1} h \varphi^{-1}\}\}\right) \]
\[ = \frac{1}{4} \int_R \text{tr} \left(-4(\varphi x \varphi^{-1} \varphi x \varphi^{-1} x) h \varphi^{-1} - 4(\varphi x \varphi^{-1})^4 h \varphi^{-1}\right) \]
\[ = \int_R (\varphi x \varphi^{-1} \varphi x \varphi^{-1} x) + (\varphi x \varphi^{-1})^4, h \varphi^{-1}). \quad (47) \]

Hence
\[ \nabla \tilde{E} = \varphi^{-1} \left((\varphi x \varphi^{-1} \varphi x \varphi^{-1} \varphi x \varphi^{-1})_x + (\varphi x \varphi^{-1})^4 + \hat{k}_3\right) \quad (48) \]
and
\[ \nabla \tilde{E}_{23} = \frac{1}{2} \left((\varphi^{-1} \varphi x \varphi^{-1} \varphi x \varphi^{-1} \varphi x \varphi^{-1})_x + \hat{k}_3\right) \quad (49) \]
for some \( k \)-valued smooth function \( \hat{k}_3 \).
Combining (42,44) with (48) and (49), we see that the equation of the generalized bi-Schrödinger flow of maps from $\mathbb{R}$ to the Kähler Grassmannian manifolds $G_{n,k}$ of compact type is

$$\varphi_t = \left[ \varphi, \alpha \nabla E + \beta \nabla E_2 + \gamma \nabla E_3 \right]$$

$$= \left[ \varphi, \alpha \nabla E + \beta \nabla (E_21 - E_22) + (\beta + 2\gamma) \nabla E_23 \right]$$

$$= \left[ \varphi, -\alpha (\varphi_{xx} - \varphi_x \varphi^{-1} \varphi_x) + \beta \left( \varphi_{xxxx} + \hat{k}_1 - (\varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1}) x + \hat{k}_2 \right) + \frac{1}{2}(\beta + 2\gamma) \left( (\varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1}) x + \hat{k}_3 \right) \right]$$

$$= \left[ \varphi, -\alpha \varphi_{xx} + \beta \varphi_{xxxx} + \left( \gamma - \frac{\beta}{2} \right) \varphi^{-1} (\varphi_x \varphi^{-1} \varphi_x \varphi^{-1}) x \varphi^{-1} \right]$$

$$= \left[ \varphi, -\alpha \varphi_{xx} + \beta \varphi_{xxxx} + (4\gamma - 2\beta) (\varphi_x \varphi^{-1} \varphi_x \varphi^{-1}) x \right],$$

which is just Eq.(33). Here we have used the identity (34) and the symmetric relation: $[k,k] = 0$. This completes the proof of the theorem in the case $M = G_{n,k}$.

2) For $M = G_{n,k}^\alpha = \{ E^{-1} \sigma_3 E \mid \forall E \in U(k, n-k) \}$, i.e. the Kähler Grassmannian manifolds of noncompact type, one may verify that the formulas (34 ~ 38) are still valid in the present case. One notes that the metric used for the Kähler Grassmannian manifolds $G_{n,k}^\alpha$ of noncompact type, i.e. in the case of the symmetric Lie algebra $u(k, n-k)$, is given by $\langle A, B \rangle = \text{tr}(AB), \forall A, B \in u(k, n-k)$. Following the line of proof of Theorem 1, we may similarly have

$$\nabla E = \tau(\varphi) = -\nabla_{\varphi_x} \varphi_x = -\varphi_{xx} + \varphi_x \varphi^{-1} \varphi_x,$$

$$E_{22}(\varphi) = -\frac{1}{4} \int_{\mathbb{R}} \text{tr} \left( \varphi_{xx} \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \right)$$

and

$$\tilde{E}(\varphi) = \int_{\mathbb{R}} \langle R(\varphi_x, J_{\varphi_x}) J_{\varphi_x}, \varphi_x \rangle$$

$$= -\frac{1}{4} \int_{\mathbb{R}} \text{tr} \left( \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \right)$$

$$= 2E_{23}(\varphi).$$

Here we have used the invariant (pseudo) inner product on $u(k, n-k)$. Therefore,

$$\nabla E_{21} = \varphi_{xxxx} + \hat{k}_1 \quad (50)$$

for some $k$-valued smooth function $\hat{k}_1$,

$$\nabla E_{22} = (\varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1}) x + \hat{k}_2 \quad (51)$$

for some $k$-valued smooth function $\hat{k}_2$ and

$$\nabla E_{23} = \frac{1}{2} \left( (\varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1}) x + \hat{k}_3 \right) \quad (52)$$

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for some \( k \)-valued smooth function \( \hat{k}_3 \). By using the formulae (50,51) and (52), we see that the equation of the generalized bi-Schrödinger flow of maps from \( \mathbb{R} \) to the Kähler Grassmannian manifolds \( G_n^k \) of noncompact type is

\[
\varphi_t = \left[ \varphi, \alpha \nabla E + \beta \nabla (E_{21} - E_{22}) + (\beta + 2\gamma) \nabla E_{23} \right] = \left[ \varphi, -\alpha (\varphi_{xx} - \varphi_x \varphi^{-1} \varphi_x) + \beta \left( \varphi_{xxxx} + \hat{k}_1 - (\varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1})_x + \hat{k}_2 \right) \\
+ \frac{1}{2} (\beta + 2\gamma) \left( (\varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x)_x \varphi^{-1} \right) \right] = \left[ \varphi, -\alpha \varphi_{xx} + \beta \varphi_{xxxx} + \left( \gamma - \frac{\beta}{2} \right) (\varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x)_x \varphi^{-1} \right] = \left[ \varphi, -\alpha \varphi_{xx} + \beta \varphi_{xxxx} + (4\gamma - 2\beta) (\varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x)_x \right],
\]

which also has the same form as Eq. (33).

3) For the case \( M = G_{n,k} = \{ E^{-1} \sigma_3 E \mid \forall E \in GL(n, \mathbb{R}) \} \), i.e. the para-Kähler Grassmannian manifolds, using the invariant metric on \( gl(n) \) or \( sl(2n) \), one may verify that the formulas (34-38) become

\[
\varphi^{-1} = 4\varphi, \quad \implies (\varphi^{-1})^2 = 4I, \quad (53) \\
\varphi_x = E^{-1} [\sigma_3, P] E, \quad \implies [\varphi, \varphi_x] = E^{-1} P E, \quad (54) \\
\varphi_x \varphi^{-1} = -2E^{-1} P E, \quad \varphi^{-1} \varphi_x = 2E^{-1} P E. \quad (55)
\]

Combining (54) with (55), we have

\[
[\varphi, \varphi_x] = J_{\varphi_x} \varphi_x = -\frac{1}{2} \varphi_x \varphi^{-1} \quad (56)
\]

and

\[
\varphi^2 = E^{-1} [\sigma_3, P]^2 E = -E^{-1} P^2 E = -\frac{1}{4} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1}. \quad (57)
\]

As in the case of \( u(n) \), we similarly have that

\[
\nabla E = \tau(\varphi) = -\nabla_{\varphi_x} \varphi_x = -\varphi_{xx} + \varphi_x \varphi^{-1} \varphi_x,
\]

\[
E_{22}(\varphi) = -\frac{1}{4} \int_{\mathbb{R}} \text{tr} \left( \varphi_{xx} \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \right),
\]

and

\[
\tilde{E}(\varphi) = \int_{\mathbb{R}} \{ R(\varphi_x, J_{\varphi_x} \varphi_x) J_{\varphi_x} \varphi_x, \varphi_x \} \\
= \frac{1}{4} \int_{\mathbb{R}} \text{tr} \left( \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \right).
\]
Here we have used the invariant (pseudo) inner product on $gl(n, \mathbb{R})$. Hence,

\[
\nabla E_{21} = \varphi_{xxxx} + \hat{k}_1
\]

for some $k$-valued smooth function $\hat{k}_1$,

\[
\nabla E_{22} = - (\varphi^{-1} \varphi^{-1} \varphi^{-1} \varphi^{-1} x)_x + \hat{k}_2
\]

for some $k$-valued smooth function $\hat{k}_2$ and

\[
\nabla E_{23} = \frac{1}{2} \left( - (\varphi^{-1} \varphi^{-1} \varphi^{-1} \varphi^{-1} x)_x + \hat{k}_3 \right)
\]

for some $k$-valued smooth function $\hat{k}_3$. By definition and the formulae (58,59) and (60), the equation of the generalized bi-Schrödinger flow of maps from $\mathbb{R}$ to $\hat{G}_{n,k}$ is

\[
\varphi_t = \left[ \varphi, \alpha \nabla E + \beta \nabla (E_{21} - E_{22}) + (\beta + 2\gamma) \nabla E_{23} \right]
\]

\[
= \left[ \varphi, \alpha \nabla E + \beta \nabla (E_{21} - E_{22}) + (\beta + 2\gamma) \nabla E_{23} \right]
\]

\[
= \left[ \varphi, -\alpha (\varphi_{xx} - \varphi^{-1} \varphi_x)_x + \beta \left( \varphi_{xxxx} + \hat{k}_1 + (\varphi^{-1} \varphi^{-1} \varphi^{-1} \varphi^{-1} x)_x + \hat{k}_2 \right)
\]

\[
\quad + \frac{1}{2}(\beta + 2\gamma) \left( -(\varphi^{-1} \varphi^{-1} \varphi^{-1} \varphi^{-1} x)_x + \hat{k}_3 \right) \right]
\]

\[
= \left[ \varphi, -\alpha \varphi_{xx} + \beta \varphi_{xxxx} + \left( -\gamma + \frac{\beta}{2} \right) \varphi^{-1} (\varphi^{-1} \varphi^{-1} \varphi^{-1} \varphi^{-1} \varphi_x)_x \varphi^{-1} \right]
\]

which is still written as Eq.(33). Here we have used the identity (53) in computation of the last equality. The proof of Theorem 1 is completed. \( \Box \)

**Corollary 1** For $g = su(2)$, or equivalently $M = SU(2)/SU(1) \times SU(1) = \{ E^{-1} \sigma_3 E \mid E \in SU(2) \} \cong S^2$, Eq.(33) in Theorem 1 reduces exactly to the equation (30) of the generalized bi-Schrödinger flows from $\mathbb{R}$ to $S^2 \hookrightarrow \mathbb{R}^3$.

**Proof:** In fact, for $E = \left( \begin{array}{cc} \hat{a} & -b \\ \hat{b} & a \end{array} \right) \in SU(2)$ (i.e. $|a|^2 + |b|^2 = 1$), one sees that

\[
\varphi = \left( \begin{array}{cc} a & b \\ -b & \hat{a} \end{array} \right) \left( \begin{array}{cc} i & 0 \\ 0 & -\frac{i}{2} \end{array} \right) \left( \begin{array}{cc} \hat{a} & -b \\ \hat{b} & a \end{array} \right) = \frac{1}{2} \left( \begin{array}{ccc} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{array} \right)
\]

with $x_1 = |a|^2 - |b|^2$, $x_2 = -i(ab - \hat{a}\hat{b})$, $x_3 = -(ab + \hat{a}\hat{b})$ and hence $x_1^2 + x_2^2 + x_3^2 = 1$. Therefore, the correspondence between

\[
M = SU(2)/SU(1) \times SU(1) = \{ E^{-1} \sigma_3 E \mid E \in SU(2) \}
\]
and $S^2 \to \mathbb{R}^3$ reads

$$\varphi \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in S^2 \to \mathbb{R}^3.$$ 

In this case, one may verify directly by using (35) and (36) that

$$\varphi_x \varphi_x^{-1} \varphi_x \varphi_x^{-1} = \begin{pmatrix} -(x_{1x})^2 - (x_{2x})^2 - (x_{3x})^2 \\ 0 \\ -(x_{1x})^2 - (x_{2x})^2 - (x_{3x})^2 \end{pmatrix}.$$ 

Substituting this expression into Eq. (33), one sees easily that Eq. (33) in this case is exactly the equation (30) of the generalized bi-Schrödinger flows from $\mathbb{R}$ to the 2-sphere $S^2 \to \mathbb{R}^3$ given in §2 (also refer to [17] or [16]).

**Corollary 2** For $g = su(1, 1)$, or equivalently $M = SU(1, 1)/SU(1) \times SU(1) = \{ E^{-1} \sigma_3 E \mid E \in SU(1, 1) \} \cong \mathbb{H}^2 \to \mathbb{R}^{2,1}$, Eq. (33) reduces exactly to the equation (31) of the generalized bi-Schrödinger flows from $\mathbb{R}$ to $\mathbb{H}^2 \to \mathbb{R}^{2,1}$.

**Proof:** For $E = \begin{pmatrix} \bar{a} & -b \\ b & a \end{pmatrix} \in SU(2)$ (i.e. $|a|^2 - |b|^2 = 1$), one sees that

$$\varphi = \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \bar{a} & -b \\ -b & a \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i x_3 & x_1 + i x_2 \\ x_1 - i x_1 & -i x_3 \end{pmatrix}$$

with $x_3 = |a|^2 + |b|^2$, $x_2 = -(ab + \bar{a}b)$, $x_1 = -i(ab - \bar{a}b)$ and hence $x_1^2 + x_2^2 - x_3^2 = -1$. Therefore, the correspondence between $M = SU(1, 1)/SU(1) \times SU(1) = \{ E^{-1} \sigma_3 E \mid E \in SU(2) \}$ and $\mathbb{H}^2 \to \mathbb{R}^{2,1}$ reads

$$\varphi \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{H}^2 \to \mathbb{R}^{2,1}.$$ 

In this case, one may verify directly by using (35) and (36) that

$$\varphi_x \varphi_x^{-1} \varphi_x \varphi_x^{-1} = \begin{pmatrix} (x_{1x})^2 + (x_{2x})^2 - (x_{3x})^2 \\ 0 \\ (x_{1x})^2 + (x_{2x})^2 - (x_{3x})^2 \end{pmatrix}.$$ 

Substituting this expression into Eq. (33), one sees easily that there is a sign difference in $\gamma$. This is because of our new definition of the generalized bi-energy functional. By noting this point, we see that Eq. (33) in this case is exactly the equation (31) of the generalized bi-Schrödinger flows from $\mathbb{R}$ to the hyperbolic plane $\mathbb{H}^2 \to \mathbb{R}^{2,1}$ given in §2 (refer to [16]).

**Corollary 3** For $g = sl(2, \mathbb{R})$, or equivalently $M = SL(2, \mathbb{R})/SL(1, \mathbb{R}) \times SL(1, \mathbb{R}) \cong S^{1,1}$, Eq. (33) reduces exactly to the equation (32) of the generalized bi-Schrödinger flows from $\mathbb{R}$ to the de Sitter 2-space $S^{1,1} \to \mathbb{R}^{2,1}$.

**Proof:** Since a matrix $E \in SL(2, \mathbb{R})$ is explicitly expressed by

$$E = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$
with \( ad - bc = 1 \), one sees that
\[
\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & d & -b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 - x_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_1 \\ x_2 - x_3 \end{pmatrix}
\]
with \( x_1 = ad + bc, x_2 = -(ab - cd), x_3 = -(ab + cd) \) and hence \( x_1^2 + x_2^2 - x_3^2 = 1 \). Therefore, the correspondence between \( M = SL(2, \mathbb{R})/SL(1, \mathbb{R}) \times SL(1, \mathbb{R}) = \{ E^{-1} \sigma_3 E \mid E \in SL(2, \mathbb{R}) \} \) and \( S^{1,1} \hookrightarrow \mathbb{R}^{2,1} \) reads
\[
\varphi \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in S^{1,1} \hookrightarrow \mathbb{R}^{2,1}.
\]
In this case, one may verify directly by using (35) and (36) that
\[
\varphi_x \varphi^{-1} \varphi_x \varphi^{-1} = \begin{pmatrix} -(x_{1x})^2 - (x_{2x})^2 + (x_{3x})^2 & 0 \\ 0 & -(x_{1x})^2 - (x_{2x})^2 + (x_{3x})^2 \end{pmatrix}.
\]
Substituting this expression into Eq.(33), one sees easily that Eq.(33) in this case is just the equation (32) of the generalized bi-Schrödinger flows from \( \mathbb{R} \) to the de Sitter 2-space \( S^{1,1} \hookrightarrow \mathbb{R}^{2,1} \) given in §2 (also see [16]). □

From Corollaries 1, 2 and 3 we see that Eq.(33) on the Hermitian symmetric Lie algebra \( u(n) \) with index \( k \) of compact type, and the Hermitian symmetric Lie algebra \( u(k, n - k) \) with index \( k \) of noncompact type, and the para-Hermitian symmetric Lie algebra \( \mathfrak{g}l(n, \mathbb{R}) \) with index \( k \), is exactly the desired non-integrable third-order models of the vortex filament on its corresponding symmetric Lie algebra. This is because when \( n = 2 \) and \( k = 1 \), these models are equivalent respectively to the Fukumoto-Moffatt’s model (3) in \( \mathbb{R}^3 \), the timelike third-order corrected model (12) and spacelike third-order corrected model (13) of the vortex filament in the Minkowski 3-space \( \mathbb{R}^{2,1} \).

In the next section, like the Fukumoto-Moffatt’s model (3) in \( \mathbb{R}^3 \) and models (12) and (13) in \( \mathbb{R}^{2,1} \), we shall pursue whether Eq.(33) on the Hermitian or para-Hermitian symmetric Lie algebras admits the property expressed as a motion of moving curves in the corresponding Lie algebra, and whether Eq.(33) on the Hermitian or para-Hermitian symmetric Lie algebras can be transformed equivalently into a second to fourth order nonlinear Schrödinger-like matrix equation.

### 4 Sym-Pohlmeyer curves moving in symmetric Lie algebras

In this section, we shall apply the technique of Sym ([46]) and Pohlmeyer ([42]) to produce geometric realizations of Eq.(33) in its corresponding symmetric Lie algebra \( \mathfrak{g} \). Then, by applying the geometric concept of PDEs with the given curvature representation proposed in the category of Yang-Mills theory by the first author and his collaborators in [20, 13] with the aid of gauge transformations, we transform Eq.(33) equivalently into a second to fourth order nonlinear Schrödinger-like matrix equation. All these results return to what are known for the Fukumoto-Moffatt’s models (3), (12) and (13) of the vortex filament when \( \mathfrak{g} \) goes back to \( su(2), su(1, 1) \) and \( sl(2, \mathbb{R}) \) respectively.

It should be mentioned that curves evolving in a symmetric Lie algebra \( \mathfrak{g} \) described in [42, 46, 34] constitute a very special subclass of the unit speed curves in \( \mathfrak{g} \cong \mathbb{R}^{\dim(\mathfrak{g})} \) (one notes here that \( \mathbb{R}^{\dim(\mathfrak{g})} \) is regarded as the Euclidean space in the case that \( \mathfrak{g} \) is in the first class
or the pseudo-Euclidean space otherwise). It is shown in [34] that the moving equation (4): 
\[ \gamma_t = [\gamma_x, \gamma_{xx}] \] (i.e. the leading-order model) in the symmetric Lie algebra \( u(n) \) is a geometric realization of the nonlinear Schrödinger matrix equation. Meanwhile, the moving equation (5): 
\[ \gamma_t = \gamma_{xxx} + \frac{2}{3} [\gamma_{xx}, [\gamma_x, \gamma_{xx}]] \] (i.e. the second-order model) in the symmetric Lie algebra \( sl(2n, \mathbb{R}) \) is a geometric realization of the KdV type matrix equation (see [15]). Now the following theorem shows that the third-order Eq.(33) also admits a geometric realization in its corresponding symmetric Lie algebra.

**Theorem 2** The following moving equation in the Hermitian (or para-Hermitian) symmetric Lie algebra \( g = u(n) \) with index \( k \), or \( u(k, n-k) \) with index \( k \) or \( gl(n, \mathbb{R}) \) with index \( k \),

\[
\gamma_t = -a[\gamma_x, \gamma_{xx}] + 2\beta \left( [\gamma_x, \gamma_{xxxx}] - [\gamma_{xx}, \gamma_{xxx}] \right) + (4\gamma - 2\beta) [\gamma_x, \gamma_{xx}] \gamma_{x}^{-1} \gamma_x \gamma_{xx}^{-1} \gamma_{xxx}^{-1}
\]  
(61)

is equivalent to Eq.(33) given in Theorem 1, where \( \gamma \in g \) with \( \gamma_x \in \{ E^{-1} \sigma_3 E | E \in g \} \).

**Remark 1** We point out that there are lots of moving Sym-Pohlmeyer curves in \( g \) meeting the requirement of (61). In fact, for a solution \( \varphi(t, x) = E^{-1}(t, x)\sigma_3 E(t, x) \) to Eq.(33) on \( g = k \oplus m \) with the fixed index, letting

\[
\tilde{\gamma} = \int_0^x \varphi(t, s) ds = \int_0^x E^{-1}(t, s) \sigma_3 E(t, s) ds,
\]

we see that \( \tilde{\gamma} \) solves Eq.(61) and satisfies \( (\tilde{\gamma})_x = \varphi(t, x) \in \{ E^{-1} \sigma_3 E | E \in g \} \). Theorem 2 indicates that the moving equation (61) is exactly the counterpart of the Fukumoto-Moffatt’s model on the symmetric Lie algebra \( g \), which is what we pursue to find in this article.

**Proof of Theorem 2** It is direct to verify, by taking the derivative with respect to \( x \), that if \( \tilde{\gamma} \) satisfies Eq.(61) with \( \tilde{\gamma}_x \in \{ E^{-1} \sigma_3 E | E \in g \} \), then \( \varphi = \tilde{\gamma}_x \) satisfies Eq.(33). Conversely, if \( \varphi(t, x) = E^{-1}(t, x)\sigma_3 E(t, x) \) satisfies Eq.(33), then it is easy to see that \( \tilde{\gamma} = \int_0^x \varphi(t, s) ds \) solves Eq.(61) and satisfies \( (\tilde{\gamma})_x = \varphi(t, x) \in \{ E^{-1} \sigma_3 E | E \in g \} \). □

It was proved by Hasimoto transform that the Fukumoto-Moffatt’s model (3) (resp. (12, 13)) is equivalent to a second to fourth order nonlinear Schrödinger-like equation (refer to [24, 16]). In what follows, we shall show that Eq.(61) (or equivalently Eq.(33)) is also equivalent to a second to fourth order nonlinear Schrödinger-like matrix equation. However, since we work on symmetric Lie algebras, the Hasimoto transform is not applicable here. It is widely recognized in literature that gauge transformation for a 1 + 1 integrable equation with zero curvature representation is equivalent to the Hasimoto transform. Therefore, the gauge transformation is a good candidate in higher dimensions that generalizes the Hasimoto transform. But in our process we need to throw away the integrability of the equations we deal with. Thanks to the works by the first author and his collaborators in [20, 13] where contrary to the concept of PDEs with zero curvature representation (i.e. integrable PDEs), the geometric concept of PDEs with the given curvature was proposed in the category of Yang-Mills theory. Thus, we may still apply the gauge transformation to equations with a given curvature representation that generalizes the Hasimoto transform for these non-integrable PDEs.

Recall that by using the language in Yang-Mills theory, the generalized LIE (4) on the Hermitian symmetric Lie algebra \( u(n) \) with the index \( k \) can be expressed by an equation with zero curvature representation. More precisely, we first rewrite the integrable Eq.(4) as

\[
\varphi_t = [\varphi, \varphi_{xx}],
\]
(62)
where $\varphi = \tilde{\gamma}_x = E^{-1}\sigma_3 E$ for some $E \in U(n)$. Then we define a connection on the trivial bundle $\mathbb{R}^2 \times U(n)$ by
\[
A = \lambda \varphi dx + (\lambda^2 \varphi + \lambda [\varphi, \varphi_x]) dt.
\]
One may directly verify that $F_A = dA + A \wedge A = \lambda (-\varphi_t + [\varphi, \varphi_{xx}])$ and hence Eq.(62) is equivalent to the following zero curvature representation:
\[
F_A = dA + A \wedge A = 0.
\]
For a given differential equation, if there exists a connection 1-form $A$ and a given curvature 2-form $K$ such that the equation is equivalent to the following formula:
\[
F_A = dA + A \wedge A = K,
\]
then it is called an equation with the given curvature representation. We should point out that, roughly speaking, equations with the given non-zero curvature representation are non-integrable in general.

By using the above idea of the given curvature representation with the aid of gauge transformations, we may obtain the following result.

**Theorem 3** On the symmetric Lie algebra $g = k \oplus m = u(n)$ with index $k$, $u(k, n-k)$ with index $k$ or $gl(n, \mathbb{R})$ with index $k$, Eq.(33) is equivalent to the following second to fourth order nonlinear Schrödinger-like matrix equation by the gauge transformation:
\[
2 \rho_t \sigma_3 = \alpha \left\{ - P_{xx} + 2 P^3 \right\} + \beta \left\{ P_{xxxx} - [P, [P, P_x]]_x - P \left((P^2)_{xx} - 3 P^3_x\right) - ((P^2)_{xx} - 3 P^3_x) P - 2(P^3)_{xx} + 6 P^5 \right\} - 2(8\gamma + \beta) \left\{ - (P^3)_{xx} + 2 P^5 \right\} + P \left( \int_0^x (PPP_x P + PP_x PP) ds \right) + \left( \int_0^x (PPP_x P + PP_x PP) ds \right) P, \tag{63}
\]
where $P \in m$ is an unknown $m$-valued-matrix function.

**Proof**: 1) The case $g = u(n)$ with index $k$ ($1 \leq k \leq n - 1$). First of all, we rewrite Eq.(33) as follows:
\[
\varphi_t = \left[ \varphi, -\alpha \varphi_{xx} + \beta \varphi_{xxxx} + 4(4\gamma - 2\beta) (\varphi_x^2)_x \right], \tag{64}
\]
\[\text{since } \varphi_x \varphi^{-1} \varphi_x \varphi^{-1} = 4\varphi_x^2 \text{ (see (38))}. \]
Next we come to construct a connection 1-form $A$ and a curvature 2-form $K$ on the trivial bundle $\mathbb{R}^2 \times U(n)$ such that Eq.(64) can be expressed by the given curvature representation
\[
F_A = dA + A \wedge A = K. \tag{65}
\]
In fact, for a given solution $\varphi = E^{-1}\sigma_3 E$ to Eq.(64) on $g = u(n)$ with index $k$, where the matrix $E \in U(n)$ is assumed to satisfy $E_x = P E$ with $P \in m$ as indicated in the proof of Theorem 1, we define
\[
A = \lambda \varphi dx + \left\{ \lambda^4 P_0 + \lambda^3 P_1 + \lambda^2 P_2 \right\}
\]
\[ + \lambda \left( [\varphi, -\alpha \varphi_x + \beta \varphi_{xxx} + 4(4\gamma - 2\beta)\varphi^3_x] - \beta[\varphi_x, \varphi_{xx}] \right) \right) dt, \] (66)

where \( P_0, P_1 \) and \( P_2 \) independent of \( \lambda \) are functions of \( P \) and its derivatives which will be determined later. Here the choice of the coefficient of \( \lambda \) in (66) is to guarantee the validity of Eq.(64), which is enlightened by the definition of the connection for the generalized LIE (62) described above. It can be directly verified that

\[
F_A = dA + A \wedge A \\
= \left\{ \lambda \left( -\varphi_t + [\varphi, -\alpha \varphi_x + \beta \varphi_{xxx} + 4(4\gamma - 2\beta)\varphi^3_x] - \beta[\varphi_x, \varphi_{xx}] \right) \right. \\
+ \lambda^2 \left( (P_2)_x + \left[ \varphi, [\varphi, -\alpha \varphi_x + \beta \varphi_{xxx} + 4(4\gamma - 2\beta)\varphi^3_x] - \beta[\varphi_x, \varphi_{xx}] \right] \right) \\
+ \lambda^3 \left( ((P_1)_x + [\varphi, P_2]) + \lambda^2 \left( (P_0)_x + [\varphi, P_1] \right) \right) \right. \right) dx \wedge dt. \] (67)

By using the identities: \( \varphi_{xxx} = -\varphi_{xx} \varphi - 3(\varphi^2_x)x \) and hence

\[
[\varphi, [\varphi, -\alpha \varphi_x + \beta \varphi_{xxx} + 4(4\gamma - 2\beta)\varphi^3_x] - \beta[\varphi_x, \varphi_{xx}]] = \alpha \varphi_x - \beta \varphi_{xxx} + 6\beta(\varphi^2_x \varphi)_x - 2(8\gamma + \beta)\varphi^3_x,
\]

which is deduced from the identities (34-37) and the relation \( E_x = PE \) with \( P \in m \), we see from (67) that in order for the two coefficients of \( \lambda^2 \) in both sides of (65) to be identically equal,

\[
P_2 = -\alpha \varphi + \beta(\varphi_{xx} - 6\varphi^2_x \varphi)
\]

and

\[
K = \left( -\lambda^2 2(8\gamma + \beta)\varphi^3_x \right) dx \wedge dt \] (68)

is chosen to be the curvature 2-form. Furthermore, by the vanish of the coefficients of \( \lambda^3 \) and \( \lambda^4 \) in the left-hand-side of (65) we obtain

\[
P_1 = -\beta[\varphi_x], \quad P_0 = -\beta \varphi
\]

from (67). To summarize, if one defines

\[
A = \lambda \varphi dx + \left\{ -\lambda^2 \beta \varphi - \lambda^3 \beta [\varphi_x] + \lambda^2 \left( -\alpha \varphi + \beta(\varphi_{xx} - 6\varphi^2_x \varphi) \right) \right. \\
+ \lambda \left( [\varphi, -\alpha \varphi_x + \beta \varphi_{xxx} + 4(4\gamma - 2\beta)\varphi^3_x] - \beta[\varphi_x, \varphi_{xx}] \right) \right. \right) dt, \] (69)

and \( K \) is given by (68), then Eq.(64) (or equivalently Eq.(33)) is equivalent to (65). In other words, Eq.(64) in this case possesses the given curvature representation Eq.(65) with \( A \) and \( K \) given by (69) and (68) respectively.

Now we shall make a gauge transformation to transform Eq.(64) into its equivalent second to fourth order nonlinear Schrödinger-like matrix equation. Indeed, since the given solution
\[ \varphi = E^{-1} \sigma_3 E \] to Eq.(64) on \( g = u(n) \) with index \( k \) satisfies: \( E_x = P E \) with \( P \in m \) as indicated above, we make the following gauge transformation via \( G = E^{-1} \):

\[ A \mapsto \tilde{A} = G^{-1} dG + G^{-1} AG \tag{70} \]

on the connection \( A \) given by (69). It is well-known from the Yang-Mills theory that under the gauge transformation, the relation between the curvature \( F_A \) of \( A \) and the curvature \( F_{\tilde{A}} \) of \( \tilde{A} \) is:

\[ F_{\tilde{A}} = G^{-1} F_A G = G^{-1} KG. \tag{71} \]

By a direct calculation, we see from (70) that

\[ \tilde{A} = Ed(E^{-1}) + EAE^{-1} = -(dE)E^{-1} + EAE^{-1} \]

\[ = (\lambda \sigma_3 - P) dx + \left\{ -\lambda^4 \sigma_3 + \lambda^3 \beta P + \lambda^2 \left( -\alpha \sigma_3 - 2 \beta (P_x + P^2) \sigma_3 \right) + \lambda \left( \alpha P + \beta (-P_{xx} - 8 \beta^3 - [P, P_x]) - 4(4 \gamma - 2 \beta) P^3 \right) - E_t \right\} dt, \tag{72} \]

where \( E_t \) independent of \( \lambda \) is to be determined later. Here we have used the fact: for a matrix \( \lambda \in \mathfrak{g} \) and \( \lambda = \alpha \sigma_3 - \beta(P_x + P^2) \sigma_3 \).

Furthermore, by using (72) a direct computation shows that

\[ F_{\tilde{A}} = d\tilde{A} + \tilde{A} \wedge \tilde{A} \]

\[ = \left\{ P_t + \left( -\lambda^4 \beta \sigma_3 + \lambda^3 \beta P + \lambda^2 \left( -\alpha \sigma_3 - 2 \beta (P_x + P^2) \sigma_3 \right) + \lambda \left( \alpha P + \beta (-P_{xx} - 8 \beta^3 - [P, P_x]) - 4(4 \gamma - 2 \beta) P^3 \right) - E_t \right) \right\} dx \wedge dt \tag{73} \]

and

\[ \tilde{K} = G^{-1} F_A G = E \left( -\lambda^2 2(8 \gamma + \beta) \varphi_3^3 \right) E^{-1} dx \wedge dt = \left( 4(8 \gamma + \beta) \varphi_3^3 \right) dx \wedge dt. \tag{74} \]

Substituting (73) and (74) into (71), we see that the coefficients of \( \lambda^5 \), \( \lambda^4 \), \( \lambda^3 \) and \( \lambda^2 \) in both sides of (71) are equal to each other. The fact that the coefficient of \( \lambda \) in the left-hand-side of (71) is zero implies that

\[ [\sigma_3, E_t] = \alpha P_x - \beta (P_{xxx} - [P, [P, P_x]]) - 16 \gamma (P^3)_x \]

and hence

\[ E_t^{(\text{off-diag})} = 2 \alpha P_x \sigma_3 - 2 \beta (P_{xxx} - [P, [P, P_x]]) \sigma_3 - 32 \gamma (P^3)_x \sigma_3, \tag{75} \]

where \( E_t^{(\text{off-diag})} \) is the \( m \)-part or off-diagonal part of \( E_t \). Here we have used the fact: for a matrix \( X \in u(n) = k \oplus m \)

\[ [\sigma_3, X] \sigma_3 = \frac{1}{2} X^{(\text{off-diag})}. \]

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Meanwhile, the fact that the coefficient of $\lambda^0$ in the left-hand-side of (71) is zero implies that

$$ P_t - (E_t)_x + [P, E_t] = 0. \quad (76) $$

By taking the diagonal part of (76), we see that

$$ (E_t^{(\text{diag})})_x = [P, E_t^{(\text{off-diag})}] = [P, 2\alpha P_x \sigma_3 - 2\beta (P_{xxx} - [P, P_x]) \sigma_3 - 32\gamma (P^3)_x \sigma_3] $$

and hence

$$ E_t^{(\text{diag})} = 2\alpha P^2 \sigma_3 - 2\beta ((P^2)_x \sigma_3 - 3(P^2)_x \sigma_3) + 32\gamma (P^3)_x \sigma_3 $$

By taking the off-diagonal part of (76):

$$ P_t - (E_t^{(\text{off-diag})})_x + [P, E_t^{(\text{diag})}] = 0 $$

and using (75,77), we finally arrive at the following second to fourth order nonlinear Schrödinger matrix equation:

$$ 2P_t \sigma_3 = \alpha \left\{ P_{xx} + 2P^3 \right\} + \beta \left\{ P_{xxx} - [P, [P, P_x]]_x - P ((P^2)_x - 3P^3) \right\} $$

$$ - ((P^2)_x - 3P^3) P - 2(P^3)_{xx} + 6P^5 \right\} - 2(8\gamma + \beta) \left\{ (P^3)_{xx} + 2P^5 \right\} $$

$$ + P \left( \int_0^x (PPP_s P + PP_s PP) ds \right) + \left( \int_0^x (PPP_s P + PP_s PP) ds \right) P, $$

which is just Eq.(63). This proves that any solution to Eq.(64) is transformed into a solution to Eq.(63) by the gauge transformation. Conversely, it is easy to see that every step in the above proof is invertible and hence any solution to Eq.(63) is transformed into a solution to Eq.(64). The proof that Eq.(33) on $u(n)$ is equivalent to Eq.(63) is completed.

2) The case $g = u(k, n - k)$ with index $k$ ($1 \leq k \leq n - 1$). Since the identities (34-38) are the same as in the case $g = u(n)$ with index $k$, following exactly the line in the proof of 1), we see that Eq.(33) on $u(k, n - k)$ with the index $k$ is transformed into Eq.(63) and vice versa. This proves that Eq.(33) on $u(k, n - k)$ is equivalent to Eq.(63).

3) The case $g = gl(n, \mathbb{R})$ with index $k$ ($1 \leq k \leq n - 1$). First of all, we rewrite Eq.(33) in this case as follows:

$$ \varphi_t = \left[ \varphi - \alpha \varphi_{xx} + \beta \varphi_{xxxx} - 4(4\gamma - 2\beta)(\varphi_x^3)_x \right], \quad (78) $$

since $\varphi_x \varphi^{-1} \varphi_x \varphi^{-1} = -4\varphi_x^2$ (see (57)). Next we come to construct a connection 1-form $A$ and a curvature 2-form $K$ on the trivial bundle $\mathbb{R}^2 \times gl(n, \mathbb{R})$ such that Eq.(78) can be expressed as an equation with the given curvature representation. In fact, unlike in the proof of Theorem
we replace a solution of the form $\varphi = E^{-1}\sigma_3 E$ to Eq. (78) by $\varphi = E\sigma_3 E^{-1}$ with the matrix $E \in GL(n, \mathbb{R})$ being now assumed to satisfy $E_x = EP$ with $P \in \mathfrak{m}$. One notes that the new $P$ here equals $-P$ with $P$ used in the proof of Theorem 1. By using the relation $E_x = EP$ and hence the identity:

$$[\varphi, [\varphi, -\alpha \varphi_{xx} + \beta \varphi_{xxxx} - 4(4\gamma - 2\beta) (\varphi_x^3)_x]] - \beta [\varphi, [\varphi_x, \varphi_{xx}]]$$

$$= -\alpha \varphi_x + \beta (\varphi_{xxx} + 6(\varphi_x^2)_x \varphi) - 4(4\gamma - 2\beta) \varphi_x^3 - 4\beta \varphi_x^3,$$

we define a connection 1-form by

$$A = \lambda \varphi dx + \left\{ -\lambda^4 \beta \varphi + \lambda^3 \beta [\varphi, \varphi_x] + \lambda^2 (\alpha \varphi - \beta (\varphi_{xx} + 6 \varphi_x^2 \varphi))
\right.$$  

$$+ \lambda \left( [\varphi, -\alpha \varphi_x + \beta \varphi_{xxx} - 4(4\gamma - 2\beta) \varphi_x^3] - \beta [\varphi_x, \varphi_{xx}] \right) \right\} \, dt, \quad (79)$$

and

$$K = \left( -\lambda^2 2(8\gamma + \beta) \varphi_x^3 \right) dx \wedge dt \quad (80)$$

to be a given curvature 2-form. Then Eq. (78) (or equivalently Eq. (33) on $gl(n, \mathbb{R})$) admits the following given curvature representation:

$$F_A = dA + A \wedge A = K. \quad (81)$$

The key point here in making a gauge transformation for Eq. (81) is as follows. Since the given solution $\varphi = E\sigma_3 E^{-1}$ to Eq. (78) on $\mathfrak{g} = gl(n, \mathbb{R})$ with index $k$ is obtained by assuming that $E \in GL(n, \mathbb{R})$ satisfies $E_x = EP$ with $P \in \mathfrak{m}$. The gauge transformation via $G = E$ for the connection $A$ given by (79) is:

$$A \mapsto \tilde{A} = E^{-1} dE + E^{-1} AE. \quad (82)$$

Hence

$$\tilde{K} = F_{\tilde{A}} = E^{-1} F_A E = E^{-1} K E = \left( \lambda^2 4(8\gamma + \beta) P^3 \sigma_3 \right) dx \wedge dt.$$

It is also through a direct computation and by using (53-57) that $\tilde{A}$ given by (82) can be explicitly expressed as

$$\tilde{A} = (\lambda \sigma_3 + P) dx + \left\{ -\lambda^4 \beta \sigma_3 - \lambda^3 \beta P + \lambda^2 (\alpha \sigma_3 - 2\beta (P_x - P^2) \sigma_3)
\right.$$  

$$+ \lambda (\alpha P - \beta (P_{xx} - [P, P_x]) - 16\gamma P^3) - E_t \right\} dt, \quad (83)$$

where $E_t$ independent of $\lambda$ is to be determined later. Similarly, from the relation $F_{\tilde{A}} = d\tilde{A} + \tilde{A} \wedge \tilde{A} = \tilde{K}$, one may obtain

$$E_t^{(\text{off-diag})} = -2\alpha P_x \sigma_3 + 2\beta P_{xxx} \sigma_3 - 2\beta [P, [P, P_x]] \sigma_3 + 32\gamma (P^3) x \sigma_3$$
and
\[
\left( E^{(\text{diag})}_i \right)_x = 2\alpha(P^2)_x \sigma_3 - 2\beta((P^2)_{xx} - 3P^2_x)_x \sigma_3 + 6\beta(P^4)_x \sigma_3 - (32\gamma + 4\beta)((P^4)_x + P_2P_xP + PP_xP^2) \sigma_3.
\]

The above identities come from equalizing the coefficients of \( \lambda \) and the diagonal part of the constant terms in \( d\bar{A} + \bar{A} \wedge A = \bar{K} \) respectively. Based on the above two identities, the off-diagonal part of the constant term in \( d\bar{A} + \bar{A} \wedge A = \bar{K} \) implies
\[
-P_t + \left( E^{(\text{off-diag})}_i \right)_x + [P, E^{(\text{diag})}_i] = 0,
\]
which can be explicitly rewritten as
\[
2P_t \sigma_3 = \alpha \left\{ -P_{xx} + 2P^3 \right\} + \beta \left\{ P_{xxxx} - [P, [P, P_x]]_x - P((P^2)_{xx} - 3P^2_x) \right. \\
- ((P^2)_{xx} - 3P^2_x)P - 2(P^3)_{xx} + 6P^5 \right\} - 2(8\gamma + \beta) \left\{ -(P^3)_{xx} + 2P^5 \\
+ P \left( \int_0^x (PPP_xP + PP_xPP)ds \right) + \left( \int_0^x (PPP_xP + PP_xPP)ds \right) P \right\}.
\]

This equation is exactly Eq.(63) and hence any solution to Eq.(78) is transformed to a solution to Eq.(63) by the gauge transformation. One also notes that all the steps in the above proof are invertible, or in other words, Eq.(63) in this case is transformed to Eq.(78) by the gauge transformation. The proof that Eq.(78) on \( gl(n, \mathbb{R}) \) and Eq.(86) are equivalent to each other is finished. \( \square \)

The following proposition gives a detailed description of Eq.(63) on the three different types of symmetric Lie algebras.

**Proposition 1** Eq.(63) given in Theorem 3 on the Hermitian symmetric Lie algebra \( u(n) \) with index \( k \) of compact type, the Hermitian symmetric Lie algebra \( u(k, n-k) \) with index \( k \) of non-compact type and the para-Hermitian symmetric Lie algebra \( gl(n, \mathbb{R}) \) with index \( k \) are respectively given by
\[
iq_t + \alpha \left\{ -q_{xx} + 2qq^*q \right\} + \beta \left\{ q_{xxxx} + 4q_{xx}q^*q + 2qq^*q_{xx} \\
+ 2q_xq^*q + 6q_xq^*q_x + 2qq^*_xq + 6qq^*q^*_xq \right\} - 2(8\gamma + \beta) \left\{ (qq^*)_xx + 2qq^*q^*_xq \\
+ q \left( \int_0^x q^*(qq^*)_xq^*ds \right) + \left( \int_0^x q(q^*)_xq^*ds \right) q \right\} = 0,
\]
where \( q \) is a \( k \times (n-k) \) complex-matrix-valued unknown function and \( q^* \) stands for the transposed conjugate matrix of \( q \).
\[
iq_t + \alpha \left\{ -q_{xx} + 2qq^*q \right\} + \beta \left\{ q_{xxxx} - 4q_{xx}q^*q - 2qq^*_xq - 4qq^*q_{xx} \\
- 2q_xq^*_x - 6q_xq^*_xq_x + 6qq^*_xq_x + 6qq^*_xq^*_xq \right\} - 2(8\gamma + \beta) \left\{ -(qq^*)_xx + 2qq^*q^*_xq \\
- 2q_xq^*_x - 6q_xq^*_xq_x - 2qq^*_xq_x + 6qq^*q^*_xq \right\} - 2(8\gamma + \beta) \left\{ -((qq^*)_xx + 2qq^*q^*_xq). \right.
\]

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\[ + q \left( \int \sigma^* \varepsilon \sigma \, ds \right) + \left( \int r(q) \, ds \right) q \right) = 0, \tag{85} \]

where \( q \) is a \( k \times (n-k) \) complex-matrix-valued unknown function, and

\[
\begin{aligned}
q_t &= \alpha \left\{ q_{xx} - 2q_{r}r \right\} + \beta \left\{ -q_{xxx} + 4q_{xx}r_q + 2q_{rr}r_q \right\} q \\
&\quad + 6q_{r}r_q + 2q_{rr}r_q - 6q_{rr}r_q + 2(8\gamma + \beta) \left\{ -q_{rr} - 2q_{rr}r_q \right\} q \\
\end{aligned}
\]

\[
\begin{aligned}
r_t &= \alpha \left\{ -r_{xx} + 2r_qr \right\} + \beta \left\{ r_{xxx} - 4r_{xx}r_q - 2r_{rr}r_q - 4q_{rr}r_q - 2r_{xx}r_q \right\} q \\
&\quad - 6r_{xx}r_q + 2r_{rr}r_q + 6q_{rr}r_q + 2(8\gamma + \beta) \left\{ (r_q)_{xx} - 2q_{rr}r_q \right\} q \\
\end{aligned}
\]

\( \left( \begin{array}{cc} 0 & q \\ -q^* & 0 \end{array} \right) \),

where \( q \) is a \( k \times (n-k) \) complex matrix and \( q^* \) stands for the transposed conjugate matrix of \( q \), and then by the identities (34-38) and a long direct computation, we see that Eq.(78) is equivalent to

\[
\begin{pmatrix}
0 & -i_q t \\
-i_q t^* & 0
\end{pmatrix}
= \alpha \begin{pmatrix}
0 & -q_{xx} - 2q_{q}q \\
q_{xx}^* + 2q_{q}^* q & 0
\end{pmatrix} + \beta \begin{pmatrix}
0 & A \\
-A^* & 0
\end{pmatrix}
\]

\[
-2(8\gamma + \beta) \begin{pmatrix}
0 & B \\
-B^* & 0
\end{pmatrix},
\]

where

\[
A = q_{xxx} + 4q_{xx}q^* q + 2q_{q_{xx}}q + 4q_{xx}q^* q + 2q_{xx}q^* q + 6q_{x}q^* q + 2q_{xx}q^* q + 6q_{x}^* q^* q\]

and

\[
B = (q_{q^* q})_{xx} + 2q_{q^* q}^* q + q \left( \int r(q) \, ds \right) q.
\]

Therefore, Eq.(63) in this case can be explicitly expressed by the following second to fourth order nonlinear Schrödinger matrix equation:

\[
i_q t + \alpha \left\{ -q_{xx} - 2q_{q}q \right\} + \beta \left\{ q_{xxx} + 4q_{xx}q^* q + 2q_{xx}q^* q + 4q_{xx}q^* q \\
+ 2q_{xx}q^* q + 6q_{x}q^* q + 2q_{xx}q^* q + 6q_{x}^* q^* q \right\} - 2(8\gamma + \beta) \left\{ (q_{q^* q})_{xx} + 2q_{q^* q}^* q \right\}
\]
\[ + q \left( \int_0^x q^* (q q^*)_s ds \right) + \left( \int_0^x q^* q^*_s q^* ds \right) q = 0, \]

which is just Eq.\( (84) \).

2) In the case \( g = k \oplus m = u(k, n - k) \) with index \( k \), we now write \( P \in m \subset u(k, n - k) \) as

\[ P = \begin{pmatrix} 0 & q^* \\ q & 0 \end{pmatrix}, \]

where \( q \) is a \( k \times (n - k) \) complex matrix and \( q^* \) stands for the transposed conjugate matrix of \( q \). Following the same computation as above, we see that Eq.\( (63) \) on \( u(k, n - k) \) is explicitly expressed by Eq.\( (85) \).

3) In the case \( g = k \oplus m = gl(n, \mathbb{R}) \) with index \( k \), we now write \( P \in m \subset gl(n, \mathbb{R}) \) as

\[ P = \begin{pmatrix} 0 & q^* \\ r & 0 \end{pmatrix}, \]

where \( q \) is a \( k \times (n - k) \) unknown real-valued matrix and \( r \) is an \( (n - k) \times k \) unknown real-valued matrix. By using the identities \( (53-57) \) and a long computation, one may verify directly that Eq.\( (63) \) is explicitly expressed by Eq.\( (86) \). The details are omitted here. The proof of the Proposition is completed. \( \square \)

**Remark 2** Unexpectedly, from Theorem 3 or Proposition 1 Eqs.\( (84, 85) \) and \( (86) \) are differential-integral equations in general. A peculiar feature of them is that these equations reduce to the second to fourth order completely integrable nonlinear Schrödinger AKNS-matrix differential equations when \( \gamma = -\frac{2}{3} \), and especially when \( \alpha = 0 \) and \( \gamma = -\frac{2}{3} \), Eq.\( (84) \) goes back to the fourth order integrable matrix Schrödinger equation \( (24) \) on \( u(n) \) given in \( \S 2 \). Hence the three typical second to fourth order integrable nonlinear Schrödinger AKNS-matrix equations also have geometric realizations of Sym-Pohlmeyer curves satisfying Eq.\( (61) \) with \( \gamma = -\frac{\beta}{8} \) in the symmetric Lie algebra \( g = k \oplus m = u(n), u(k, n - k) \) and \( gl(n, \mathbb{R}) \) respectively.

Another distinguishing feature of Eqs.\( (84, 85) \) and \( (86) \) is that these differential-integral equations reduce to partial differential equations when \( n = 2 \).

**Corollary 4** On \( g = su(2) \), Eq.\( (84) \) reduces exactly to

\[ i q_{t} - \alpha \left\{ q_{xx} + 2 |q|^2 q \right\} + \beta \left\{ q_{xxxx} + 6 \left[ |q|^2 q_{xx} + (q_{x})^2 q \right \right\}
\]

\[ + \left[ 6|q|^4 + 2(|q|^2)_{xx} \right] q \right\} - 2 (8 \gamma + \beta) \left\{ (|q|^2 q)_{xx} + 3|q|^4 q \right\} = 0, \tag{87} \]

where \( q \) is an unknown complex valued function. This equation is just the second to fourth order equation that equivalent to the Fukumoto-Maffott’s model \( (3) \) or in other words the generalized Schrödinger flows from \( \mathbb{R} \) to the 2-sphere \( S^2 \rightarrow R^3 \) (refer to \([24, 18]\)).

**Proof:** In this case, \( q^* = \bar{q} \) with \( q \) being an unknown complex function of \( t \) and \( x \). One may verify that

\[ q_{xxxx} + 4 q_{xx} q^* q + 2 q q^* q_{xx} + 2 q q^* q_{x} + 6 q_{x} q^* q_{x} + 2 q q^* q_{x} + 6 q q^* q q^* q \]
\[ q_{xxxx} + 6 \left[ |q|^2 q_{xx} + (q_x)^2 q \right] + [6|q|^4 + 2(|q|^2)_{xx}] q \]

and (up to a term \( c(t)q \) with \( c(t) \) depending only on \( t \) which can be concealed by the transform of the form: \( q \mapsto q \exp \left( i \int_0^t c(s) ds \right) \)

\[ q \left( \int_{x} q^*(qq^*)_x q ds \right) + \left( \int_{x} q^*(q^*q)_x q^* ds \right) q = |q|^4 q. \]

Substituting the above into (84), we see that Eq.(84) becomes exactly Eq.(87). \( \square \)

Similarly, we have

**Corollary 5** On \( g = su(1,1) \), Eq.(85) reduces exactly to

\[
\begin{align*}
  iq_t - \alpha \left( q_{xx} - 2|q|^2 q \right) + \beta \left( q_{xxxx} - 6 \left[ |q|^2 q_{xx} + (q_x)^2 q \right] \right) \\
  + \left[ 6|q|^4 + 2(|q|^2)_{xx} \right] q - 2(8\gamma + \beta) \left\{ (|q|^2 q)_{xx} - 3|q|^4 q \right\} = 0,
\end{align*}
\]

where \( q \) is an unknown complex valued function. This equation is just the second to fourth order equation that equivalent to the timelike third-order corrected model (12) of the vortex filament in \( \mathbb{R}^{2,1} \) or in other words the generalized Schrödinger flows from \( \mathbb{R} \) to the hyperbolic 2-space \( \mathbb{H}^2 \hookrightarrow R^{2,1} \) (see [18, 16]).

**Corollary 6** On \( g = sl(2,\mathbb{R}) \), Eq.(33) reduces exactly to

\[
\begin{align*}
  q_t &= \alpha(q_{xx} - 2q^2 r) + \beta \left( -q_{xxxx} + 6q_x^2 r + 4qq_x r_x + 8qr q_{xx} + 2q^2 r_{xx} - 6q^3 r^2 \right) \\
  &\quad - 2(8\gamma + \beta) \left( qr q)_{xx} - 3q^2 r^2 \right), \\
  r_t &= -\alpha(r_{xx} - 2q^2 r) + \beta \left( r_{xxxx} - 6q^2 r_x^2 - 4qr q_x r_x - 8qr r_{xx} - 2r^2 q_{xx} + 6q^2 r^3 \right) \\
  &\quad + 2(8\gamma + \beta) \left( (qr)_{xx} - 3q^2 y^3 \right),
\end{align*}
\]

where \( q \) and \( r \) are unknown real functions. This equation is just the second to fourth order equation that equivalent to the spacelike third-order corrected model (13) of the vortex filament in \( \mathbb{R}^{2,1} \) or equivalently the generalized Schrödinger flows from \( \mathbb{R} \) to the de Sitter 2-space \( \mathbb{S}^{1,1} \hookrightarrow R^{2,1} \) (refer to [16]).

From Proposition 1 and Corollaries 4, 5 and 6, we see that our exploitation of the third-order models of the vortex filament on Hermitian or para-Hermitian symmetric Lie algebras \( g \) returns exactly to what we knew in literature for the Fukumoto-Moffatt’s models (3) in the Euclidean 3-space \( \mathbb{R}^3 \) or the models (12) and (13) of the vortex filament in the Minkowski 3-space \( \mathbb{R}^{2,1} \) when \( g = su(2) \), or \( su(1,1) \), or \( sl(2,\mathbb{R}) \).
5 Conclusions and remarks

In this article, by modifying the concept of generalized bi-Schrödinger maps into Kähler or para-Kähler manifolds, we have established the third-order non-integrable model of the vortex filament on a Hermitian or para-Hermitian symmetric Lie algebra in a unified way. Combining it with what are known in literature for the leading order and the second order models, we may list all the basic models of the vortex filament on a symmetric Lie algebra $g$ up to the third-order approximation, which are moving equations of Sym-Pohlmeyer’s curves in the symmetric Lie algebra $g$:

\begin{align}
\tilde{\gamma}_t &= \{\tilde{\gamma}_x, \tilde{\gamma}_{xx}\}, \\
\tilde{\gamma}_t &= \tilde{\gamma}_{xxx} + \frac{3}{2}[\tilde{\gamma}_{xx}, \{\tilde{\gamma}_x, \tilde{\gamma}_{xx}\}], \\
\tilde{\gamma}_t &= \beta \left( \tilde{\gamma}_x, \tilde{\gamma}_{xxxx} - \{\tilde{\gamma}_{xx}, \tilde{\gamma}_{xxx}\} \right) + (4\gamma - 2\beta) \left[ \tilde{\gamma}_x, \tilde{\gamma}_{xx} \tilde{\gamma}_x^{-1} \tilde{\gamma}_{xx} \tilde{\gamma}_x^{-1} \tilde{\gamma}_{xx} \right],
\end{align}

where $\{\cdot, \cdot\}$ denotes the Lie bracket on $g$. Comparing these equations with the models existing in the localized induction (matrix) hierarchy, we see that Eq. (90) (resp. Eq. (91)) coincides exactly with Eq. (4) (resp. Eq. (5)). However, Eq. (92) is not the fourth order equation in the localized induction (matrix) hierarchy in general, unless the two parameters $\beta$ and $\gamma$ satisfy the condition: $\gamma = -\frac{4}{9}$ as mentioned in Remark 2. We should point out that the discovery of the third-order correction model (92) is based on exploiting the intrinsic geometry of the Hermitian or para-Hermitian symmetric Lie algebras and the corresponding symmetric spaces, especially their Kähler or para-Kähler structures. The application of generalized bi-Schrödinger flows overcomes the limitation of the methods in the theory of integrable systems and creates the model in a purely geometric way. Furthermore, by using the concept of equations with a given curvature representation with the aid of gauge transformations, the models (92) of the vortex filament on the three types of symmetric Lie algebras are transformed to three types of fourth order nonlinear Schrödinger-like matrix differential-integral equations. These three types of equations reduce to the three typical 4th order integrable nonlinear Schrödinger matrix differential equations in the AKNS matrix hierarchy when the parameters $\beta$ and $\gamma$ satisfy the relation: $\gamma = -\frac{4}{9}$. It is very appealing and delighting that when the symmetric Lie algebra $g$ goes back to $su(2)$, $su(1, 1)$ and $sl(2, \mathbb{R})$ respectively, the models (90,91) and (92) on $g$ reduce respectively to the LIE-type models (1,7) and (8), the Fukumoto-Miyazaki-type models (2,10) and (11), the Fukumoto-Moffatt-type models (3,12) and (13) in $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$, and all the theory we know about Eqs. (90,91) and (92) are reduced to the corresponding theory of the vortex filament in $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$.

This indicates that there exists a complete hidden theory of the vortex filament on symmetric Lie algebras, which consists of motions of arclength-parameterized Sym-Pohlmeyer curves evolving in the (Hermitian or para-Hermitian) symmetric Lie algebras we deal with. Models (90,91) and (92) provide the basic equations of the vortex filament on the symmetric Lie fluid algebra up to the third-order approximation. The theory of the vortex filament in the Euclidean 3-space $\mathbb{R}^3$ or in the Minkowski 3-space $\mathbb{R}^{2,1}$ can be understood uniformly on the level of symmetric Lie algebras.

The study of the vortex filament in the Euclidean and Minkowski 3-space or on symmetric Lie algebras is always significant. We believe that the model (92) and its related equations (33) and (63) as well as the geometric concept of generalized bi-Schrödinger flows will have much more deeper applications in both mathematics and physics. However, there are some questions still
unclear at the present time. For example, 1) The para-Kähler structure in geometry seems to be peculiar beautiful and mysterious to us, which has not been well understood up to now. How to characterize the dynamical properties of the third-order correction models \((92)\) in terms of Kähler or para-Kähler geometry? 2) Except the Hermitian symmetric spaces \(G_{m,k}\) of compact type (i.e., \(A\) III), there are \(C\) I, \(D\) III and \(BD\) I-types of symmetric spaces (and also two exceptional Hermitian symmetric spaces \(E\) III and \(E\) VII) (refer to [28, 25]). Can we establish a similar result by using the generalized bi-Schrödinger flow of maps from \(\mathbb{R}\) to these symmetric spaces? How about the other Hermitian symmetric spaces of noncompact type and the other para-Hermitian symmetric spaces? These questions deserve study in the future.

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