Bjorken flow in the general frame and its attractor

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ABSTRACT: It has been recently claimed that the first order hydrodynamics can be stable and casual in a general frame other than the usually used Landau or Eckart frames. We investigate the implications of the general frame approach for Bjorken flow. We show that certain transport coefficients that are introduced in this approach act as regulators similar to the Muller-Israel-Stewart parameters. The stable first-order hydro in the general frame approach gives rise to a non-linear equation of motion whose solutions decay to an attractor at late-times. We find an analytical approximation form for the attractor and show that its early-time behavior is consistent with the stability and casualty conditions proposed by Bemfica, Disconzi, Noronha, and Kovtun.
1 Introduction

Relativistic Hydrodynamics (RH) offers an effective approach to understanding different many-body systems and in particular, the quark-gluon plasma (QGP) produced in heavy-ion collisions (HICs) \[1-4\]. The word ”effective” here means that the infinite microscopic degrees of freedom (DoF) are replaced by a few macroscopic variables such as the energy density, pressure and fluids velocity [4]. One can regard RH as an extended version of relativistic thermodynamics which takes the out of equilibrium properties of the system into account. It is built upon two assumptions: the existence of stable local thermodynamic states in equilibrium and the possibility of a gradient expansion around such states out of the equilibrium. Therefore in equilibrium, the conserved charges are written in terms of a chosen set of hydrodynamical variables that completely determine the local thermal state. In the zeroth-order, no derivative term appears and the resulting conservation laws describe the evolution of a perfect fluid. To incorporate the out of equilibrium effects one adds derivatives of the hydrodynamical variables up to an arbitrary order of truncation [4]. If the fluid is not far from the equilibrium one anticipates that truncation of the gradient expansion at first-order might be a good approximation.

A set of hydrodynamical variables that are commonly used in RH includes the temperature, fluid velocity, and chemical potential. Although these quantities are well-defined in equilibrium, they lose their uniqueness when the equilibrium is disturbed: they can be varied, or redefined, without harming the constitutive relations. Different definitions of the hydrodynamical variables are often referred to as choosing a hydrodynamic frame [4]. Traditionally one exhausts the frame choice freedom before proceeding any further calculation. Two common frame choices are Landau [5] and Eckart [6] frames. However, it is known for years that first-order RH in both of these frames gives rise to the propagation of superluminal fluctuations [7-9]. To remedy this stability and causality (SC) problem, Muller, Israel, and Stewart (MIS)
suggested that the RH’s DoF have to be expanded to include new dynamical variables such as the viscous stress tensor [10–12]. In the MIS framework, the dynamics of the new variables is obtained using the second law of thermodynamics. These variables evolve such that they relax to their first-order forms. For the MIS framework to work, certain parameters need to satisfy non-trivial conditions [13, 14]. The existence of such conditions may have been a motivation for further inspection of the SC problem. One could ask if a formulation of the first-order hydrodynamics exists that resolves the SC problem without the introduction of new DoFs? The MIS dynamical variables are ad-hoc in the sense that they do not stem from the gradient expansion. Recently it has been claimed that such a formulation is possible [15, 16]. In this approach, one starts with the most general frame (GF) and looks for the conditions on the transport coefficients that ensure stability and casualty.

In practice, one would solve the RH equations to obtain the evolution of hydrodynamical variables in terms of spacetime coordinates. Unsurprisingly solving RH equations for explaining experimental data demands numerical methods. However, there are a few simple setups that give rise to an analytical solution [17–19]. The simplest setup in the QGP context is Bjorken flow in which every quantity is a function of a single variable, i.e. the proper time. The original Bjorken flow for the perfect fluid can be generalized to the first-order in the Landau frame. Such a generalization gives rise to a first-order linear equation that has an exact solution [20]. However, the generalization of Bjorken flow to the MIS framework at first-order leads to a non-linear second-order differential equation [21, 22]. The non-linearity of the MIS equation of motion has crucial implications. Unlike linear differential equations, the non-linear equations with different initial conditions may decay to a common attractor, and therefore lose the initial information [21–27]. Another crucial point is the late-time expansion solution is a divergent but asymptotic series. In the present paper, we show that the same trend is obtained using the GF approach.

The organization of this paper is as follows. In Sec. 2 we present the solution to Bjorken flow in the GF approach for conformal uncharged fluid. We show that the equation of motion is non-linear and therefore there are certain terms in the GF approach that act as regulators: The exponential decay of transient modes which were previously reported in the MIS framework [28], are also observed in the GF approach. We present an analytical approximation for the attractor and show that SC constraints found in [15, 16] can be derived from an inspection of the attractor. We also discuss the entropy current and anisotropy induced by GF terms. We conclude the paper in Sec. 3. We use natural units in which $\hbar = c = 1$, and the metric convention is mostly plus. All equations are written in Milne coordinates, namely $x^\mu = (\tau, x, y, \eta)$ in which $\tau = \sqrt{t^2 - z^2}$ and $\eta = \frac{1}{2} \log \left( \frac{t + z}{t - z} \right)$. The line element of the flat spacetime in this coordinate system reads

$$ds^2 = -d\tau^2 + dx^2 + dy^2 + \tau^2 d\eta^2.$$
2 Conformal Bjorken flow

In this section, we present the solution to Bjorken flow in the GF for a conformal fluid. We show that the GF approach gives rise to similar results known from the MIS framework [22] without relying on any ad-hoc relation. However, it does not mean that the two frameworks exactly match in the frame-dependent transport coefficients. Also, we confirm the stability conditions found in [15, 16] by investigating the attractor.

Setup The energy momentum tensor for an uncharged conformal fluid in the GF reads [15]

\[ T^\mu\nu = T^{D-1} \left( pT + \frac{\pi_1 T}{T} + \frac{\pi_1 \nabla u^\lambda}{D-1} \right) (g^\mu\nu + D u^\mu u^\nu) \]

\[ + \theta_1 T^{D-1} \left[ \left( \frac{\mu}{T} + \frac{\nabla u^\lambda}{T} \right) u^\nu + (\mu \leftrightarrow \nu) \right] - \eta T^{D-1} \sigma^\mu\nu + O(\partial^2). \] (2.1)

For uncharged fluids parameters \( p, \pi_1, \) and \( \theta_1 \) are dimensionless constants and \( D \) is the spacetime dimension. Using \( Dp = Ts, \) \( p = pT^D, \)

we find \( s = DpT^{D-1} \) and \( \eta/s = \eta/(Dp). \) For simplicity we define

\[ \nu \equiv \frac{\eta}{Dp}, \quad \alpha \equiv \frac{\pi_1}{4Dp\nu}, \]

With the above parameterization, the stability requirement \( \pi_1 > 4\eta [15] \) is translated into \( \alpha > 1. \) Bjorken symmetries [17, 18] eliminate the heat flow and reduce the equation of motion \( \nabla_\mu T^{\mu\nu} = 0 \) to a single ordinary differential equation which in \( D = 4 \) reads

\[ 4\alpha \nu \dot{T} + \dot{T} \left[ T + 4\alpha \nu \left( \frac{2\dot{T}}{T} + \frac{7}{3\tau} \right) \right] + \frac{T}{3\tau} \left( T + \frac{4(\alpha - 1)\nu}{3\tau} \right) = 0. \] (2.2)

Here \( \dot{T} = \frac{dT}{d\tau} \) and \( \ddot{T} = \frac{d^2T}{d\tau^2}. \) The elimination of heat flow also happens in any other flow with at least three spacetime symmetries\(^1\). Therefore, one cannot investigate the parameter \( \theta_1 \) in such flows. If we compare Eq. (2.2) with Eq. 11 of [25], we observe that most terms exactly match. In particular, the comparison confirms that our parameter \( \nu \) coincides with \( C_\eta \) in MIS terminology. This is not surprising since the shear viscosity is invariant under frame transformation [4]. However, there are terms in the two equations that cannot be matched. The GF transport coefficient \( \pi_1 \) plays the role of both \( \tau_1 \) and \( \lambda_1 [21, 23, 26] \) in the MIS framework.

\(^1\)Gubser flow [18] is an example.
Hydrodynamic expansion  Let us begin our investigation with the ordinary hydrodynamic expansion that is valid after some initial proper time $\tau_0$. Inspired by the ideal Bjorken flow \[17\] and its MIS generalization \[1\] we introduce the following Ansatz
\[
T = \frac{T_0}{\tau_0} \Pi(\rho), \quad \rho \equiv \left( \frac{\tau_0}{\tau} \right)^{1/3},
\] (2.3)
in which $T_0$ is a dimensionless normalization constant. Without loss of generality, we assume $T_0 = 1$ in what follows. Plugging the above formulation, which is appropriate only for the near equilibrium treatment, into (2.2) gives rise to
\[
\Pi'(\rho) - \frac{\Pi(\rho)}{\rho} - \frac{4\nu(\alpha - 1)}{3} \rho^2 = \frac{4\alpha \nu \rho^4}{3(\Pi(\rho))} \left[ \Pi''(\rho) - \Pi'(\rho) \left( \frac{3}{\rho} - \frac{2\Pi'(\rho)}{\Pi(\rho)} \right) \right].
\] (2.4)

One should bear in mind that small $\rho$ is equivalent to the late times and vice versa. As the Eq. (2.4) suggests, the transport coefficient $\pi_1$ (or equivalently $\alpha$) is the source of non-linearity in the equation of motion. The equation reduces to the linear equation of motion in the Landau frame \[20\] with $\alpha = 0$. This nonlinearity has significant consequences which are discussed above:

I. The canonical entropy reads \[15\]
\[
s^\mu = \frac{4p \Pi(\rho)^3}{\tau_0^3} \left( 1 + \frac{4\alpha \nu \rho^3}{\Pi(\rho)} \left( 1 - \rho \frac{\Pi'(\rho)}{\Pi(\rho)} \right), 0 \right) = s(\rho) \left( 1 + \frac{4\alpha \nu \rho^3}{\Pi(\rho)} \left( 1 - \rho \frac{\Pi'(\rho)}{\Pi(\rho)} \right) \right),
\] (2.5)
that gives rise to the following relation for final produced entropy \[18\]
\[
dS = \frac{4\pi R^2 \Pi(\rho)^3}{\tau_0^2} \left( \rho^3 + \frac{4\alpha \nu}{\Pi(\rho)} \left( 1 - \rho \frac{\Pi'(\rho)}{\Pi(\rho)} \right) \right).
\] (2.6)

In the Landau frame, the $\alpha$-dependent term in (2.5) does not appear and the entropy current of the uncharged fluid has a similar form to the perfect fluid’s case, i.e. $su^\mu$. As Eq. (2.5) suggests this extra term tends to zero for $\alpha \nu \rho^3 \ll 1$. This is consistent with the impression that $\pi_1$ should be similar to the relaxation time in the MIS framework. Eq. (2.6) also suggests that $\Pi(\rho)$ may be close to $\rho$, i.e. the ideal Bjorken flow, close to the freezeout.

II. The pressure anisotropy is given by
\[
\frac{P_L}{P_T} = 16w^3 \nu [(\alpha - 1) \Pi(\rho) - \alpha \rho \Pi'(\rho)] + 3\Pi(\rho)^2
\]
\[
\frac{P_T}{8w^3 \nu [(2\alpha + 1) \Pi(\rho) - 2\alpha \rho \Pi'(\rho)] + \Pi(\rho)^2}.
\] (2.7)

Here $P_L = T_{\eta\eta}/\tau^2$ and $P_T = T_{xx} = T_{yy}$ \[1\]. Eq. (2.2) has an exact solution that is independent of any initial or boundary condition
\[
\Pi_{\text{exact}}(\rho) = \frac{2\nu}{3} (16\alpha - 1) \rho^3.
\] (2.8)
The occurrence of such solutions is an aspect of non-linear differential equations and thus cannot appear in the Landau frame. However, a solution that is independent of any initial or boundary conditions may not be physical. In our case, the unphysical nature of the exact solution is confirmed by the observation that it gives rise to a negative value for $P_L$ that is

$$P_L = -\frac{16p^4\nu^4}{9\tau_0^4}(16\alpha - 1)^3.$$ 

Also, the exact solution violates the second law of thermodynamics

$$\nabla\mu s^\mu = -\frac{64p^4\nu^3(16\alpha - 1)^2(4\alpha - 1)}{27\tau_0^4} < 0.$$ 

III. Assuming that the r.h.s of Eq. (2.2) is highly suppressed in late times, we may solve the l.h.s to find

$$\Pi_{\text{l.h.s}}(\rho) = C_1\rho + \frac{2\nu(1 - \alpha)}{3} \rho^3.$$ (2.9)

On the other hand, if we put $\alpha = 0$ in Eq. (2.4), the Landau frame solution reported in (2.2) is reproduced

$$\Pi_{\text{Landau}}(\rho) = \left(1 + \frac{2\nu}{3}\right) \rho - \frac{2\nu}{3} \rho^3.$$ (2.10)

The above solution has a zero at $\rho_0 = \sqrt{1 + \frac{3}{2\nu}}$. This might not worry us if we assume the hydrodynamical evolution is only valid at $\rho \leq 1$. A much more significant point is that the Landau solution has a maximum at

$$\rho^* = \sqrt{\frac{1}{3} + \frac{1}{2\nu}}.$$ (2.11)

The maximum can appear before $\rho = 1$ under certain conditions which lead to a reheating effect [20]. A reheating cannot occur for the solution given in Eq. (2.9) if $C_1 > 0$ and $\alpha > 1$.

To find the hydrodynamic expansion, we utilize a power series around the ideal Bjorken flow and plug it into the Eq. (2.4)

$$\Pi_N(\rho) = C\rho + \frac{\nu\rho^3}{3} \sum_{n=0}^{N} P_n(\alpha) \left(\frac{\nu\rho^2}{3}\right)^n,$$ (2.12)

in which $C$ is a normalization constant and the coefficient of each power in (2.12) is a polynomial in $\alpha$

$$P_n(\alpha) = \sum_{m=0}^{n} c_m \alpha^m.$$
The leading power of $\alpha$ in $P_n$ has the following form

$$c_n = -\frac{2}{C \pi^8} (n+1)!.$$  \tag{2.13}

The above relation shows that (2.12) is a divergent but asymptotic series in $\rho \to 0$ with the optimal truncation roughly occurring around $N < \frac{3C}{8\nu \alpha}$. The factorial form that appears in (2.13) also suggests that the series in (2.12) is Borel resumable. A careful Borel resummation of the aforementioned series requires a thorough inspection of the analytical structure of the Borel transform $[22]$. Such an investigation is not discussed in the present work but we mention that the first real pole of the Borel transformed series occurs at $\frac{3C}{8\nu \alpha}$. When compared with the MIS results $[22, 25, 26]$, this trend of poles is consistent with the interpretation of $\tau_1$ as relaxation time. That being said, a naive resummation using (2.13) gives rise to a very good approximation of the solution at very late times

$$\Pi_{\text{Borel}}(\rho) = C \rho \left(1 - \frac{1}{\alpha} \exp \left(-\frac{3C}{8\nu \rho^2}\right) \Gamma \left[0, -\frac{3C}{8\nu \rho^2}\right]\right).$$  \tag{2.14}

The above resummation has a constant imaginary part which is anticipated by the non-sign-alternating form in (2.13). The appearance of an imaginary part shows that the resumed solution to Eq. (2.4) should be obtained using a trans-series $[22]$. For $C = 1$ the optimal truncation occurs at $N = 2$, i.e. up to $O(w^7)$ which reads

$$T(\tau) = \frac{T_0}{\tau_0} \left(\frac{\tau}{\tau_0}\right)^{1/3} \left[1 - \frac{2\nu}{3} \left(\frac{\tau}{\tau_0}\right)^{2/3} - \frac{16\nu^2}{9} \left(\frac{\tau}{\tau_0}\right)^{4/3} - \frac{256\nu^2 \nu^3}{27} \left(\frac{\tau}{\tau_0}\right)^{2}\right],$$  \tag{2.15}

Here we have recovered the normalization constant $T_0$. Eq. (2.15) resembles the one that is found using MIS approach $[1]$. At the risk of repeating ourselves, we emphasize that a match between $\alpha$ and MIS transport coefficients cannot be found by simple comparison. Assuming that the higher-order terms match we end up with

$$\alpha = \frac{C_\pi}{8C_\eta} (1 - C_\lambda),$$

which for the $\mathcal{N} = 4$ SYM parameters $[22]$ gives rise to $\alpha \approx 0.3$ that is not in the stable range for GF $[15]$. Also the above formula cannot transform Eq. (2.2) to the MIS equations presented in $[25]$. A numerical investigation of the solutions is possible by assuming two initial conditions, i.e. $\Pi(0)$ and $\Pi'(0)$, from the solution (2.9). The numerical results represented in Fig. [1] confirm the analysis given in the previous lines. The parameters used for the numerical results are

$$\nu = \frac{1}{4\pi}, \quad \alpha = \frac{\pi}{3}, \quad \tau_0 = 0.6 \text{ fm}/c, \quad T(\tau_0) = 350 \text{ MeV}.$$  \tag{2.16}
The attractor  To understand the early time behavior of (2.2), we work with following parameters \[22\]
\[ w = T\tau, \quad f = \frac{\dot{w}}{w}, \quad (2.17) \]
The dimensionless function $f$ can be written as \[25\]
\[ f(w) = 1 + \frac{d\log T}{d\log \tau}. \quad (2.18) \]

For a physical system, one can argue that $f(w)$ must be smaller than unity. Otherwise, the energy in the system grows with time. Using
\[ \dot{w} = \frac{f(w)w}{\tau}, \quad \ddot{w} = \frac{w}{\tau^2} (wf'(w)f(w) + f(w)^2 - f(w)), \]
Eq. (2.2) is transformed into the following first order ODE
\[ 4\nu f f' + 12\nu f^2 - \frac{56}{3} f + w f + \frac{64\nu}{9} - \frac{4\nu}{9} - \frac{2w}{3} = 0. \quad (2.19) \]
Here $f'(w)$ denotes derivative with respect to $w$. Although this equation is quite similar to Eq. 14 of \[25\], there is no transformation that can exactly reproduces the latter. Again we conclude that some parts of this equation which depend on $\alpha$ are not frame-invariant.

To gain insight into the behavior of the solutions of (2.19) we examine the equation around $w = 0$. Such an examination gives rise to
\[ f(w) = \frac{7\alpha + \sqrt{\alpha(3 + \alpha)}}{9\alpha} + \mathcal{O}(w). \quad (2.20) \]

We employ the above equation to consider different initial conditions\[3\] for (2.19) and numerically solve it. An illustration of such a computation is represented in Fig. 2.

**Figure 1.** (color online). A numerical inspection of the hydrodynamic expansion (2.12) with parameters given in (2.16): (a) The error for different truncation, and (b) the temperature evolution. The asymptotic nature of the hydrodynamic expansion is manifest.
As this figure suggests the solutions with different initial conditions decay to an attractor in late times. The solid line in the aforementioned figure is the analytical attractor that will be presented in the subsequent lines. The Eq. (2.20) also gives rise to the stability condition for $\alpha$: $f(0)$ is equal to 1 for $\alpha = 1$ and blows up as $\alpha$ tends to zero. On the other hand, it has a finite lower bound when $\alpha$ tends to infinity

$$f(0) > \frac{8}{9}.$$ 

Therefore if the system is to be physical $\alpha$ must be greater than unity. This is in agreement with results of [15] and [16]. To find the attractor we need to understand

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{(color online). Decay of solutions of (2.19) with different initial conditions from (2.20) to an attractor in late-times. The solid line is the approximate analytic attractor represented in (2.27).}
\end{figure}

the late-time behavior of Eq. (2.19). As mentioned before $\alpha$ is the source for non-linearity and therefore we assume the following form [22] to learn how the non-linearity decays at late-times

$$f(w) = \frac{2}{3} + \frac{4\nu}{9w} + \epsilon \delta f(w).$$

Plugging above into Eq. (2.19) and solve it around $w \to \infty$ gives rise to

$$\delta f \sim \exp\left(-\frac{3w}{8\alpha \nu}\right) (3w + 2\nu)^{1+1/(4\alpha)},$$

which can be compared with Eq. 11 of [22]. For the above perturbation to be suppressed at late-times, $\alpha$ must be positive. Putting $\alpha$ to zero and choosing the Landau frame destroys the regulator that is required for suppression of transient modes [28], and it is the source of the SC problem in the aforementioned frame. Also, the coefficient in the exponential and the power must be related to the analytical structure of
the Borel transformation of the asymptotic series at large times similar to [22]. The asymptotic series for large \( w \) is equivalent to a series in small \( v \) defined as

\[
v = \frac{2}{3} \sqrt{\frac{w}{3}}.
\]

Around \( v = 0 \) the asymptotic series read

\[
f_N(v) = \sum_{n=0}^{N} \lambda_n v^{2n},
\]

with

\[
\lambda_0 = \frac{2}{3}, \quad \lambda_1 = 1, \quad \lambda_n = 3\alpha \left(14\lambda_{n-1} - \frac{3(7 - n)}{2} \sum_{m=0}^{n} \lambda_m \lambda_{n-m-1}\right), \quad \text{for } n > 1.
\]

As in \((2.12)\), \( \lambda_n \) is polynomial in \( \alpha \) whose leading power is

\[
\lambda_n = (6\alpha)^n(n + 1)! + \cdots .
\]

The above form shows that the series is divergent, but asymptotic and Borel re-summable. Let the Borel transformed series be

\[
f_B(\xi) = \sum_{n=0}^{\infty} \frac{\lambda_n}{n!} \left[\frac{2\nu \xi^3}{3}\right]^n
\]

Our computations up to 600 orders in \( \xi \) and with different values of \( \alpha \) confirm that the first real positive pole of \((2.25)\) occurs at \( 3/(8\alpha \nu) \). As an example, for the following set of parameters, we find \( \xi_0 = 1.12495 \), which is in quite good agreement with the beforementioned analytical relation

\[
\nu = \frac{1}{4\pi}, \quad \alpha = \frac{4\pi}{3}.
\]

We also find the power using Eq. 17 of \([22]\) which for \(-\gamma \) gives rise to 1.03312. The latter exhibits an error of 2.5% in comparison to the analytical value \( 1+1/(4\alpha) \). Using this information one can reproduce the results of \([22, 25, 26]\) for the GF approach. We find the analytical form for the attractor by assuming \( \epsilon(w)/\delta = f'/f \) in which \( \delta \) is an arbitrary small number. Plugging this form into Eq. \((2.19)\) and expanding it in terms of \( \epsilon \) in the leading order gives rise to two solutions for \( f(w) \). We then expand each solution around \( w \to \infty \) and compare the results with late-time expansion given in \((2.22)\). By this comparison, we find the approximate attractor to be

\[
f(w) = \frac{2}{3} - \frac{w}{24\alpha \nu} + \frac{8\alpha \nu + \sqrt{192\alpha \nu^2 + (3w - 8\alpha \nu)^2}}{27\alpha \nu}.
\]
A numerical representation of the late-time asymptotic series of (2.22) and the approximate attractor of (2.27) for parameters given in (2.16) is depicted in Fig. 3. The late-time expansion has a small error for \( w > w_0 \sim 0.5 \). The optimal truncation is of order \( 1/w^4 \), and then the approximation gets worse. The value of \( w_0 \) increases with \( \alpha \).

Finally, we write the solution for early times as a series around \( w = 0 \)

\[
f(w) = f(0) + \sum_{n=0}^{\infty} a_n \left( \frac{3w}{\alpha \nu} \right)^n,
\]

with

\[
a_0 = -\frac{1}{84}, \quad a_n = \frac{a_{n-1} + 6(7+n) \sum_{m=0}^{n-1} (a_m a_{n-(m+1)})}{56 - 12(7+n)f(0)}.
\]

The numerical results for the set of parameters given in (2.16) are represented in Fig. 4. The early time expansion (2.28) has a marginal error up to \( w_0 \sim 2 \), whose value increases with \( \alpha_0 \). As the recursive relation (2.29) and numerical computation of coefficients up to very high orders suggest the early time expansion has a finite radius of convergence.

![Figure 3](image3.png)

**Figure 3.** (color online). A numerical inspection of the late-time expansion (2.22) with parameters given in (2.16): (a) The error for different truncation, and (b) \( f(w) \) from numerical solution, different truncation of (2.22) and the attractor (2.27).

### 3 Concluding remarks

In the present work, we investigated the GF approach to stable hydrodynamics in first-order [15, 16] using Bjorken conformal flow as a toy model. We showed that this approach introduces the non-linearity required for the hydrodynamization that is traditionally incorporated using somehow ad-hoc MIS approach [3, 22]. The non-linearity seems to be a requirement for a theory that describes the transition from pre-equilibrium to equilibrium in a physical system. In hydrodynamics, such non-linearity gives rise to the appearance of attractors. In a simple sense, this means
that the solutions of non-linear equations lose the initial information and decay to a common function. Using previously introduced techniques in the MIS framework \[22, 25, 26\], we presented an approximate analytical form for the attractor in the general frame approach. Our results confirm that the two frameworks are equivalent in some sense, although they cannot be exactly matched. This is consistent with the frame-dependency of hydrodynamical variables such as the temperature. However, the emergence of attractors in the GF approach shows that the late-time hydrodynamical behavior is independent of the chosen regulator \[28\]. We also deduced the stability conditions from the investigation of the attractor and showed that the GF gradient expansion is a divergent but asymptotic one.

Although the present work can be extended in many ways, we suggest that two significant directions are the most crucial ones. The first one is to break the boost-invariance to reveal the role of the heat flow regulator, and the second one is to investigate the general frame’s transport coefficients from a microscopic point of view. The reader can find our comments on strategies for the first direction in Appendix A.

### A Comments on 1+1 self-similar flow

A minimal approach to boost-invariant breaking is introduced in the 1+1 Self-similar solution to ideal hydrodynamics \[19\], which is based on the assumption that although the pressure is required to be boost-invariant in the Bjorken flow, the temperature and entropy density can have \(\eta\) dependency. However, it is required that their \(\eta\) dependency cancel out such that the pressure remains boost-invariant. Such scaling is invalid for a conformal fluid and one needs to assume an equation of state of form

\[
\mathcal{E} = \kappa P, \tag{A.1}
\]
in which $\kappa$ is constant. To make the problem manageable, one may assume that the above equation gives rise to

$$\epsilon = \kappa p \quad \epsilon_i = \kappa \pi_i.$$  \hfill (A.2)

Also, it is required to assume that the temperature and entropy density have separable functionality in $\tau$ and $\eta$

$$T = F(\tau)T(\eta), \quad s = \frac{G(\tau)}{T(\eta)}, \quad p = \frac{T_s}{\kappa + 1},$$  \hfill (A.3)

and assume some appropriate forms for the transport coefficients. The heat flow in Milne coordinates reads

$$Q_\mu = \theta(\tau, \eta) \frac{T'(\eta)}{T(\eta)} (0,0,0,1).$$  \hfill (A.4)

Plugging above into the Euler equation gives rise to

$$\theta(\tau, \eta) \sim \frac{1}{\tau} \theta(\eta).$$  \hfill (A.5)

Assuming $\theta(\eta) = \theta_0$, the energy equation becomes separable in $\tau$ and $\eta$ with the following immediate result

$$T(\eta) \sim \exp \left( -\frac{\eta^2}{\theta_0(1 + \kappa)} \right).$$  \hfill (A.6)

To obtain the above form, we have assumed that $\eta = \nu G(\tau)$ with $\nu$ being constant and other coefficients to be boost invariant. What remains is one equation for two unknowns $F$ and $G$ that cannot be solved unless we assume a relation between them. From a phenomenological perspective, the Gaussian form appearing in (A.6) looks plausible. However, in the spirit of GF, it is maybe regarded as a hint on how the perturbations decay. We believe that a comprehensive investigation of boost invariance breakdown is crucial for a thorough understanding of the general frame approach.

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