Heterotic horizons, Monge-Ampère equation and del Pezzo surfaces

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Abstract

Heterotic horizons preserving 4 supersymmetries have sections which are $T^2$ fibrations over 6-dimensional conformally balanced Hermitian manifolds. We give new examples of horizons with sections $S^3 \times S^3 \times T^2$ and $SU(3)$. We then examine the heterotic horizons which are $T^4$ fibrations over a Kähler 4-dimensional manifold. We prove that the solutions depend on 6 functions which are determined by a nonlinear differential system of 6 equations that include the Monge-Ampère equation. We show that this system has an explicit solution for the Kähler manifold $S^2 \times S^2$. We also demonstrate that there is an associated cohomological system which has solutions on del Pezzo surfaces. We raise the question of whether for every solution of the cohomological problem there is a solution of the differential system, and so a new heterotic horizon. The horizon sections have topologies which include $((k - 1)S^2 \times S^4 \# k(S^3 \times S^3)) \times T^2$ indicating the existence of exotic black holes. We also find an example of a horizon section which gives rise to two different near horizon geometries.
1 Introduction

In the past few years much work has been done to understand the topology and geometry of higher dimensional black holes following earlier work in four dimensions [1]-[7]. It has been realized that the four dimensional uniqueness theorems fail to hold in higher dimensions, and that the horizon sections can have many different topologies, including $S^n \times S^m$, $n \geq 0, m \geq 1$ [8]-[19].

More recently, the near horizon geometry of supersymmetric heterotic black holes has been investigated [20] utilizing the solution of the Killing spinor equations of heterotic supergravity in [21]. It was found that the heterotic horizons are either products $AdS_3 \times X$ for a suitable manifold $X$ or $(AdS_3 \times Y)/S^1$, where $AdS_3$ twists over a base space with a $U(1)$ connection. The heterotic horizons preserving 8 supersymmetries are isometric to $AdS_3 \times S^3 \times T^4$ or $AdS_3 \times S^3 \times K_3$ with constant dilaton and have horizon section $S^1 \times S^3 \times T^4$ or $S^1 \times S^3 \times K_3$, respectively.

In this paper, we shall give new heterotic horizons which preserve $N = 4$ supersymmetries. Moreover, we shall provide evidence that heterotic black holes can have increasingly involved horizon topologies which are distinct from those expected from lifting 4- and 5-dimensional black holes to 10 dimensions. It is known that $N = 4$ horizon sections $S$ are holomorphic $T^2$ fibrations over a complex conformally balanced 6-dimensional manifold $B$.

First, we shall reorganize the differential conditions which arise from supersymmetry in a form which is amenable to a cohomological analysis. Then we shall demonstrate that the differential system has solutions provided that certain conditions on the cohomology of $B$ are satisfied. We present explicit heterotic horizon solutions with $B = SU(3)/T^2$ equipped with the balanced Hermitian structure and $B = (S^3 \times S^3)/S^1 \times S^1$. The horizon sections are $S = SU(3)$ and $S = S^3 \times S^3 \times T^2$, and the spacetime is $M = (SL(2,\mathbb{R}) \times SU(3))/U(1)$ and $M = AdS_3 \times S^3 \times S^3 \times S^1$, respectively. In the former case, $AdS_3$ twists over $B$ with a $U(1)$ connection. In the latter case, the radius of $AdS_3$ is twice that of $S^3$. The half supersymmetric horizon $AdS_3 \times S^3 \times K_3$ is also of this type with $B = \mathbb{P}^1 \times K_3$ Kähler and so Hermitian and balanced.

To construct more examples of heterotic horizons, one can take $B$ to be a holomorphic $T^2$-fibration over a 4-dimensional Kähler manifold $X$. Such complex manifolds with skew-symmetric torsion and holonomy contained in $SU(3)$ have been considered before in [22, 23, 24, 25]. For such $B$, the horizon section $S$ is a $T^4$ fibration over $X$. We show that the resulting field equations and supersymmetry conditions lead to a non-linear system of 6 differential equations for 6 functions. The 6 functions include the dilaton, the deformation of the Kähler metric of $X$ within its Kähler class as well as the deformations of connections of holomorphic $U(1)$ bundles within their Chern classes. The deformation of the metric within its Kähler class leads to a complex Monge-Ampère equation similar to that which appears in Yau’s proof of the Calabi conjecture. The system also includes a conformally rescaled Hermitian-Einstein equation. However, the six differential equations do not separate. As in the general case, the differential equations give rise to some cohomological conditions for classes in $X$. These are necessary for the differential system to have a solution. We raise the question of whether for every solution of the cohomological

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The system we derive is different from that of [25] as in our case the torsion is closed on the horizon section $S$. 

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problem there is also a solution of the differential system.

The non-linear differential system of equations has solutions. One explicit solution is $X = \mathbb{P}^1 \times \mathbb{P}^1$, $B = \mathbb{P}^1 \times \mathbb{P}^1 \times T^2$ and $S = S^3 \times S^3 \times T^2$. The heterotic horizon spacetime is $M = (SL(2, \mathbb{R}) \times S^3 \times S^3 \times T^2)/S^3$. Another example is the half supersymmetric horizon $AdS_3 \times S^3 \times T^4$ with $X = \mathbb{P}^1 \times T^2$, and either $B = \mathbb{P}^1 \times T^2 \times T^2$ or $B = S^3 \times T^3$, and $S = S^3 \times T^5$. One of the consequences of the explicit examples we have constructed is that the same horizon section $S = S^3 \times S^3 \times T^2$ gives rise to two different near horizon geometries $M = AdS_3 \times S^3 \times S^3 \times S^1$ and $M = (SL(2, \mathbb{R}) \times S^3 \times S^3 \times T^2)/S^1$. As far as we know, this is the first time that such a phenomenon has been observed. As such it represents an additional difficulty in the classification of black holes in higher dimensions as well as in the investigation of their thermodynamical properties.

To give evidence that there are more heterotic horizons, we demonstrate that the associated cohomological conditions have solutions for $X$ a del Pezzo surface. Considering the del Pezzo surfaces as $\mathbb{P}^2$ blown up at $k < 9$ points, we find that if $k$ is odd the cohomological conditions are met provided that the Kähler class is identified with the anti-canonical class. If $k$ is even, then a different choice for a Kähler class has to be made. There are solutions for all del Pezzo surfaces. We also investigate the topology of the associated heterotic horizons. For this we compute the de Rham cohomology of $S$. We find that in some cases the horizon sections have the same de Rham cohomology as that of $((k-1)(S^2 \times S^4) \# k(S^3 \times S^3)) \times T^2$, and under an additional assumption they are diffeomorphic to them. We also point out that the cohomology of these black hole horizons, and in particular some intersection matrices, are related to exceptional groups. This is inherited from the relation of the intersection matrix of the second cohomology of del Pezzo surfaces to the Cartan matrix of the exceptional groups. Our cohomological conditions do not have a solution on $\mathbb{P}^2$ blown up at more than 9 points. However this restriction can be removed if the Bianchi identity of the 3-form field strength is modified either by adding point sources or by taking into account the heterotic anomaly.

We also explore the possibility of extending our near horizon geometries to full black hole solutions. One may expect that there is a continuous interpolation between a horizon section and a section of the asymptotic geometry of a black hole, ie the two sections are cobordant. We argue that in the presence of fermions, and in particular supersymmetry, the two sections must represent the same class in the spin co-bordism ring $\Omega^\text{spin}_*$. Similar tests have been proposed elsewhere [19] using oriented cobordism. We find that most of our horizons can be associated with asymptotically flat or $AdS$ black holes.

Since for all our solutions the 3-form field strength is closed, they can also be interpreted as solutions IIA, IIB and 11-dimensional supergravity. Lifting our heterotic solutions to 11-dimensional supergravity and then reducing them in a different direction, we show that new solutions can be constructed in IIA supergravity which have an $AdS_2$ component and RR fluxes. These can also be further T-dualized to find new solutions in IIB supergravity. Interpreting these new solutions as near horizon geometries, we provide evidence that both IIA and IIB supergravity admit black hole solutions with non-trivial topology and with all form fluxes non-vanishing.

In the cases that have been investigated so far, the near horizon supersymmetric solutions also arise as near brane geometries. For the 1/2 supersymmetric near horizon heterotic geometries this has been demonstrated in [20]. This is also the case for the 1/4
supersymmetric $AdS_3 \times S^3 \times S^3 \times S^1$ solution. It turns out that this is the near brane geometry of two 5-branes with a localized string superposition. It is likely that the rest of the solutions have a near brane interpretation.

This paper is organized as follows. In section two, we use the results of [20] to reconstruct the geometry of the horizon $M$ and that of the horizon section $S$ from geometric data on $B$ and those of the toric fibration over $B$. We also summarize and extend some results of [23] on the relation between Hermitian conformally balanced manifolds and toric fibrations. In section 3, we give the differential system needed to construct heterotic horizon sections as $T^2$ fibrations over a conformally balanced Hermitian 6-dimensional manifold. We present an explicit solution with horizon section $SU(3)$. In section 4, we construct heterotic horizon sections as $T^4$ fibrations over Kähler 4-dimensional manifolds $X$. We show that this leads to a differential system of 6 non-linear equations for 6 functions and has a cohomological analogue in the cohomology ring of $X$. We give an explicit example that solves the differential system. In section 5, we find solutions of the cohomological problem on del Pezzo surfaces. In section 6, we find brane configurations with near brane geometries similar to those that arise as near horizon geometries. In section 7, we explore our solutions in type II and 11-dimensional supergravities. In section 8, we use spin cobordism to provide evidence that our near horizon geometries can be extended to black hole solutions and in section 9, we give our conclusions. In appendix A, we compute the cohomology of horizon sections, and in appendix B we generalized the differential system that arises in the context of heterotic horizons.

2 Geometric conditions

Heterotic horizons which preserve 4 supersymmetries, and their associated sections, are fibrations over a 6-dimensional manifold $B^6$. In particular, the horizon spacetime $M$ is a $SL(2,\mathbb{R}) \times U(1)$ fibration while the section $S$ is a $T^2$ fibration. The fibre directions twist over $B^6$ with a non-trivial connection. In what follows, we shall reconstruct both $M$ and $S$ from geometric data given on $B^6$ and on the twisting of the fibration.

2.1 Reconstruction of the horizon sections

The geometry of a horizon spacetime $M$ is completely determined in terms of the geometric data on the horizon section $S$. Because of this, we first reconstruct $S$ as a $T^2$ fibration over $B^6$. Let $ds^2_{(6)}$ be the metric and $H_{(6)}$ be the torsion of $B^6$. The metric and torsion of $S$ can be written as

$$d\tilde{s}^2_{(8)} = k^{-2}h \otimes h + k^{-2}\ell \otimes \ell + ds^2_{(6)}, \quad \tilde{H}_{(8)} = k^{-2}h \wedge dh + k^{-2}\ell \wedge d\ell + H_{(6)},$$

respectively, where $h$ and $\ell$ are 1-forms on $S$ which are interpreted as principal bundle connections associated with the fibration, and $k^2$ is the constant length of $h$ and $\ell$. Both $d\tilde{s}^2_{(8)}$ and $\tilde{H}_{(8)}$ are invariant under the rotations of the $T^2$ fibre generated by the vector fields $\xi$ and $\eta$ dual to $h$ and $\ell$, respectively, relative to the metric\(^2\) $d\tilde{s}^2_{(8)}$. In particular,

\(^2\)One could write a more general metric on $S$ by introducing a non-trivial constant metric on the fibre directions. The choice we have made suffices for our purpose.
one has
\[ h = d\tau + \alpha_i e^i, \quad \ell = d\sigma + \beta_i e^i \] (2.2)
where \( 0 \leq \tau, \sigma \leq 2\pi \) are the coordinates of \( T^2 \), \( \alpha \) and \( \beta \) are connections on \( B^6 \) and \((e^i)\) is a local frame, \( ds^2_{(6)} = \delta_{ij} e^i e^j \). The components of the metric and torsion depend only on the coordinates of \( B^6 \).

Supersymmetry restricts both the connection \( h \) and \( \ell \), and the geometry of the base space \( B^6 \) as follows. \( B^6 \) is a Hermitian manifold with Hermitian form \( \omega_{(6)} \) which is compatible with the metric connection, \( \hat{\nabla}_{(6)} \), with skew-symmetric torsion, \( H_{(6)} \), ie
\[ \hat{\nabla}_{(6)} \omega_{(6)} = 0 . \] (2.3)
This is equivalent to setting
\[ H_{(6)} = -i I_{(6)} d\omega_{(6)} , \quad dH_{(6)} \neq 0 , \] (2.4)
where \( I_{(6)} \) is the complex structure on \( B^6 \). Moreover, it is required that \( B^6 \) is conformally balanced. This means that the Lee form of \( B^6 \) is
\[ \theta_{\omega_{(6)}} = 2d\Phi , \] (2.5)
where \( \Phi \) is the dilaton that depends only on the coordinates of \( B^6 \). This summarizes the conditions on \( B^6 \).

The restriction on the twisting of the \( T^2 \) fibration over \( B^6 \) is made by putting appropriate conditions on the connections \( h \) and \( \ell \). In particular, \( N = 4 \) supersymmetry requires that
\[ dh^{2,0} = d\ell^{2,0} = 0 , \quad dh_{ij} \omega_{(6)}^{ij} = 0 , \quad d\ell_{ij} \omega_{(6)}^{ij} = -2k^2 . \] (2.6)
This means that the curvature of the torus fibration is \((1,1)\) with respect to the complex structure of \( B^6 \) and in addition one of the components of the connection is traceless while the other has constant trace.

The first two conditions on the curvature of the fibration can be solved by requiring that \( S \) is a holomorphic fibration. It is straightforward to see that the Hermitian form
\[ \omega_{(8)} = \frac{1}{k^2} h \wedge \ell + \omega_{(6)} , \] (2.7)
gives rise to an integrable complex structure on \( S \).

Collecting the above data, one finds that
\[ \hat{\nabla}_{(8)} \omega_{(8)} = 0 , \quad \hat{\nabla}_{(8)} h = \hat{\nabla}_{(8)} \ell = 0 , \] (2.8)
where \( \hat{\nabla}_{(8)} \) is the metric connection on \( S \) with skew-symmetric torsion \( H_{(8)} \). Consistency therefore requires that
\[ \hat{H}_{(8)} = -i I_{(8)} d\omega_{(8)} , \] (2.9)
which is equivalent to (2.1) and (2.4). So far the data imply that the holonomy of $\hat{\nabla}(8)$ is contained in $U(3)$, $\text{hol}(\hat{\nabla}(8)) \subseteq U(3)$.

Furthermore, supersymmetry requires that the holonomy of $\hat{\nabla}(8)$ must be contained in $SU(3)$. One way to enforce this is to require that the Ricci form of the $\hat{\nabla}(8)$ connection vanishes

$$\hat{\rho}(8) = 1\over 4 (\hat{R}(8))_{k\ell} i_{\frac{k}{\ell}} i_{\frac{k}{\ell}} e^k \wedge e^\ell = 0 \ ,$$

(2.10)

where $k, \ell$ are $S$ frame indices and $\hat{R}(8)$ is the curvature of the connection $\hat{\nabla}(8)$. This requirement gives several conditions. The only one which is independent from those that have already been stated is

$$(\hat{\rho}(6))_{ij} - d\ell_{ij} = 0 \ ,$$

(2.11)

where $\hat{\rho}(6)$ is the Ricci form of the $\hat{\nabla}(6)$ connection. If $B^6$ is simply connected, this and (2.8) are necessary and sufficient conditions for $\text{hol}(\hat{\nabla}(8)) \subseteq SU(3)$. In particular, this implies that there is a $(3,0)$ form $\chi$ such that

$$\hat{\nabla}(8)\chi = 0 \ , \ i_\xi \chi = i_\eta \chi = 0 \ ,$$

(2.12)

as stated in [20]. The geometries of $B^6$ and $S$ are summarized in table 1.

| Geometry                     | $B^6$ | $S$ |
|------------------------------|-------|-----|
| Hermitian                    | yes   | yes |
| Conformally balanced $\theta = 2d\Phi$ | yes   | no  |
| $\text{hol}(\nabla) \subseteq SU(3)$ | no    | yes |
| $\text{hol}(\nabla) \subseteq U(3)$ | yes   | no  |
| $dH_{(6)} = 0$               | no    | yes |

Table 1: The geometry of $B^6$ and $S$ is summarized. Observe that both geometries elegantly avoid the conditions of the no-go theorem of [29] and so compact examples exist.

The only remaining condition that needs to be satisfied in order to find a solution for both the Killing spinor and field equations of the theory is $dH_{(8)} = 0$. This leads to

$$d\tilde{H}(8) = k^{-2} dh \wedge dh + k^{-2} d\ell \wedge d\ell + dH_{(6)} = k^{-2} dh \wedge dh + k^{-2} d\ell \wedge d\ell - dI_{(6)} d\omega_{(6)} = 0 \ .$$

(2.13)

One can easily modify this condition if the heterotic anomaly is taken into account. But we shall not investigate this case here.

To summarize, if $S$ is taken to be a holomorphic torus fibration over a 6-dimensional Hermitian manifold, the conditions that must be satisfied to find a solution are

$$dh_{ij} \omega_{ij}^{(6)} = 0 \ , \ d\ell_{ij} \omega_{ij}^{(6)} = -2k^2 \ , \ \theta_{\omega(6)} = 2d\Phi \ ,$$

$$\hat{\rho}(6)_{ij} - d\ell_{ij} = 0 \ , \ k^{-2} dh \wedge dh + k^{-2} d\ell \wedge d\ell - dI_{(6)} d\omega_{(6)} = 0 \ .$$

(2.14)

In what follows, we investigate these conditions and give some explicit solutions.
2.2 Reconstruction of spacetime

The spacetime $M$ of heterotic horizons preserving 4 supersymmetries is a $SL(2, \mathbb{R}) \times U(1)$ fibration over $B^6$. The base space $B^6$ satisfies all the properties mentioned in the previous section for constructing $\mathcal{S}$. It remains to give the connection $\lambda$ of $M$. This is expressed in terms of the connections $h$ and $\ell$ of $\mathcal{S}$ as follows:

\[
\begin{align*}
\lambda^- &= \mathbf{e}^- , \\
\lambda^+ &= \mathbf{e}^+ - \frac{1}{2} k^2 u^2 e^- - uh , \\
\lambda^1 &= k^{-1} (h + k^2 u e^-) , \\
\lambda^6 &= k^{-1} \ell ,
\end{align*}
\]

(2.15)

where

\[
\begin{align*}
\mathbf{e}^- &= dr + rh , \\
\mathbf{e}^+ &= du .
\end{align*}
\]

(2.16)

The spacetime metric and torsion are given as

\[
\begin{align*}
ds^2 &\equiv 2\mathbf{e}^- \mathbf{e}^+ + (\lambda^1) + (\lambda^6)^2 + ds^2_{(6)} \\
H &\equiv C \mathcal{S}(\lambda) + H_{(6)} \\
&\equiv du \wedge dr \wedge h + r du \wedge dh + k^{-2} h \wedge dh + k^{-2} \ell \wedge d\ell - i I_{(6)} d\omega_{(6)} .
\end{align*}
\]

(2.17)

It is clear that given the geometric data on $\mathcal{S}$, the geometry of $M$ is completely described.

2.3 Toric fibrations

It is clear from the conditions that arise from supersymmetry as well as the examples that we shall investigate later that toric fibrations are central in the examination of near horizon geometries. Because of this, we shall derive some useful formulae for the analysis which shall follow. Toric fibrations in the context of manifolds with skew symmetric torsion and $SU$ holonomy have been investigated before [22, 23, 24, 25].

Suppose that $2n$-dimensional manifold $Y$ is a $T^{2(n-m)}$ fibration over $2m$-dimensional manifold $X$. Write the metric $ds^2$ and torsion $H$ on $Y$ as

\[
\begin{align*}
ds_{(2n)}^2 &= \delta_{ab} \lambda^a \lambda^b + ds_{(2m)}^2 , \\
H_{(2n)} &= \delta_{ab} \lambda^a \wedge d\lambda^b + H_{(2m)} ,
\end{align*}
\]

(2.18)

where $ds_{(2m)}^2$ and $H_{(2m)}$ are the metric and torsion on $X$.

Suppose now that in addition $X$ is a Hermitian manifold with complex structure $I_{(2m)}$ compatible with $ds_{(2m)}^2$. As is well known, the condition

\[
\hat{\nabla}_{(2m)} I_{(2m)} = 0 ,
\]

(2.19)

implies that

\[
H_{(2m)} = - i I_{(2m)} d\omega_{(2m)} ,
\]

(2.20)

where $\omega_{(2m)}$ is the Hermitian form of $X$. 
In turn, \( Y \) admits an almost Hermitian form \( \omega^{(2n)} \) compatible with \( ds^2_{(2n)} \) given by

\[
\omega^{(2n)} = - \sum_{k=1}^{n-m} \lambda^k \wedge \lambda^{n-m+k} + \omega_{(2m)} .
\]  

(2.21)

The associated almost complex structure \( I_{(2n)} \) is integrable provided that the curvatures

\[
\mathcal{F}^a = d\lambda^a
\]

are \((1,1)\)-forms on \( X \),

\[
(\mathcal{F}^a)^{2,0} = 0 .
\]

(2.22)

(2.23)

Furthermore, the connection with torsion, \( \hat{\nabla}_{(2n)} \), on \( Y \) is compatible with the complex structure, \( I_{(2n)} \), on \( Y \), ie

\[
\hat{\nabla}_{(2n)} I_{(2n)} = 0 ,
\]

(2.24)

and so

\[
H_{(2n)} = -i I_{(2n)} d\omega_{(2n)} .
\]

(2.25)

In addition\(^3\)

\[
\hat{\nabla}_{(2n)} \lambda^a = 0 .
\]

(2.26)

As a result, the holonomy of \( \hat{\nabla}_{(2n)} \) is contained in \( U(m) \),

\[
\text{hol}(\hat{\nabla}_{(2n)}) \subseteq U(m) .
\]

(2.27)

Now let us investigate the conditions under which the holonomy of \( \hat{\nabla}_{(2n)} \) reduces further to a subgroup of \( SU(m) \). For this, we express the curvature of the \( \hat{\nabla}_{(2n)} \) connection in terms of that of \( \hat{\nabla}_{(2n)} \) to find

\[
\hat{R}_{k\ell, i j} = \hat{R}_{k\ell, i j} - \delta_{ab} \mathcal{F}^a_{k\ell} \mathcal{F}^{bi} , \\
\hat{R}_{ab, i j} = \mathcal{F}^i_{ab} \mathcal{F}^k_{b i j} - \mathcal{F}^i_{ab} \mathcal{F}^k_{k i j} , \\
\hat{R}_{ak, i j} = \hat{\nabla}_{k} \mathcal{F}^i_{ab} ;
\]

(2.28)

A necessary condition\(^4\) for the holonomy of \( \hat{\nabla}_{(2n)} \) to reduce to \( SU(n) \) is that the Ricci form

\[
\hat{\rho}_{(2n)} = \frac{1}{4} \hat{R}_{k\ell, i j} \hat{\Omega}^i \hat{e}^k \wedge \hat{e}^\ell ,
\]

(2.29)

\[^3\]This has not been observed in [23]. As a result, the holonomy of the connection with skew-symmetric torsion of 6-dimensional manifolds which are \( T^2 \) fibrations is contained in \( SU(2) \) rather than \( SU(3) \).

\[^4\]It is also sufficient if the base space \( X \) is simply connected.
of $Y$ vanishes. This in turn gives

$$\left( \hat{\rho}_{(2m)} \right)_{k\ell} - \frac{1}{2} \delta_{ab} \mathcal{F}^a_{k\ell} \mathcal{F}^b_{ji} (I_{(2m)})^j_i = 0 ,$$
$$\hat{\nabla}_k \left( \mathcal{F}^a_{i} (I_{(2m)})^j_i \right) = 0 . \quad (2.30)$$

The first condition above can be simplified somewhat provided that $X$ is conformally balanced, i.e.

$$\theta_{\omega(2m)} = 2 \delta \Phi .$$

In particular after a bit of computation, one finds that

$$\hat{\rho}_{(2m)} = \frac{1}{4} \hat{R}_{k\ell, ij} (I_{(2m)})^j_i e^k \wedge e^\ell = \frac{1}{2} ddI \log \det g - 2ddI \Phi$$
$$= -i \partial \bar{\partial} \log \det g_{(2m)} + 4i \partial \bar{\partial} \Phi , \quad (2.31)$$

where $\det g_{(2m)}$ is the determinant of the Hermitian metric of $X$, i.e.

$$\det g_{(2m)} = \det (g_{(2m)_{\alpha \beta}}) .$$

Observe that the expression for the Ricci form of Hermitian conformally balanced manifolds is very similar to that of Kähler manifolds.

It is important to notice that even if $X$ is conformally balanced, $Y$ may not be. In particular, if $X$ is conformally balanced a sufficient condition for $Y$ to be conformally balanced is

$$\mathcal{F}^a_{ij} \omega^j_i (2m) = 0 . \quad (2.32)$$

We shall use this when we consider horizon sections which are $T^4$ fibrations over Kähler 4-dimensional manifolds.

## 3 Horizons with 4 supersymmetries

### 3.1 Forms and geometric conditions

To solve the conditions (2.14), we shall take $B^6$ to be a Hermitian conformally balanced manifold and the $T^2$-torus fibration to be holomorphic. Write $\mathcal{S} = P \boxtimes Q$, where $P$ and $Q$ are principal circle bundles over $B^6$ associated with the curvatures $dh$ and $d\ell$, respectively.

It is convenient to rewrite the conditions stated in (2.14) in form notation. After some straightforward computation, one finds that

$$dh \wedge \omega^2_{(6)} = 0 , \quad d\ell \wedge \omega^2_{(6)} = -\frac{k^2}{3} \omega^3_{(6)} , \quad d(e^{-2\Phi} \omega^2_{(6)}) = 0 ,$$
$$\hat{\rho}_{(6)} - d\ell = 0 , \quad k^{-2} dh \wedge dh + k^{-2} d\ell \wedge d\ell - di_{(6)} d\omega_{(6)} = 0 . \quad (3.1)$$

The first two conditions can be easily recognized as the Hermitian-Einstein conditions without and with cosmological constant, respectively, appropriately generalized for Hermitian conformally balanced manifolds. The third condition is the conformal balanced condition. The fourth condition identifies $Q$ with the canonical bundle $K$ over $B^6$. Moreover observe that for conformally balanced manifolds

$$\hat{\rho}_{(6)} = \frac{1}{2} ddI \log \det g_{(6)} - 2ddI \Phi = -i \partial \bar{\partial} \log \det g_{(6)} + 4i \partial \bar{\partial} \Phi , \quad (3.2)$$
where $\det g(6) = \det (g(6)_{\alpha \bar{\beta}})$ is the determinant of the Hermitian metric of $B^6$.

The conditions stated in (3.1) can be easily adapted to the cases for which $B^6$ admits some additional structure. Several possibilities are available. For example, one can assume that $B^6$ with respect to $\omega(6)$ is Kähler, conformally Kähler or balanced. In particular, the latter condition requires that the dilaton is constant. We shall demonstrate that the differential system (3.1) admits solutions. Another class of examples can be generated by taking $B^6$ to be a $T^2$ fibration over a Hermitian 4-manifold. This case will be investigated separately. The general case leads to a non-linear system that contains equations of Monge-Ampère type.

### 3.2 Cohomological conditions

All conditions in (3.1) lead to restrictions on the cohomology ring of $B^6$. To see this observe that the conformal balanced condition for $B^6$ implies that $e^{-2\Phi} \omega^2_{(6)}$ is closed. Denote the cohomology class of $e^{-2\Phi} \omega^2_{(6)}$ with $[e^{-2\Phi} \omega^2_{(6)}]$. Then (3.1) implies that the cohomology classes are restricted as

\[
\begin{align*}
    c_1(P) \wedge [e^{-2\Phi} \omega^2_{(6)}] &= 0 , \\
    c_1(Q) \wedge [e^{-2\Phi} \omega^2_{(6)}] &= -\frac{k^2}{6\pi} [e^{-2\Phi} \omega^3_{(6)}] , \\
    c_1(B^6) - c_1(Q) &= 0 , \\
    c_1(P) \wedge c_1(P) + c_1(Q) \wedge c_1(Q) &= 0 ,
\end{align*}
\]

where $c_1$ denotes the first Chern class of the appropriate circle bundle and $c_1(B^6) = \frac{1}{2\pi} \hat{\rho}(6)] = K$ is the first Chern class of the canonical bundle of $B^6$.

The conditions on the cohomology stated above are necessary for the existence of solutions. This means one should first seek manifolds $B^6$ and $T^2$-bundles over them that satisfy the cohomological conditions (3.3) and then try to solve the differential conditions (3.1). One can also show that some of the cohomological conditions are also sufficient to find solutions to some of the equations, e.g., solutions for the first equation in (3.1). However, it is not known in general that holomorphic $T^2$ fibrations over conformally balanced Hermitian manifolds that satisfy all the cohomological conditions (3.3) are also solutions of (3.1). We shall demonstrate though that (3.1) admits solutions by constructing explicit examples.

### 3.3 New solution with horizon section $SU(3)$

We shall demonstrate that $S = SU(3)$ is a solution of (3.1) with $B^6 = SU(3)/T^2$, where $T^2$ is identified with a maximal torus of $SU(3)$. The metric of $S$ is written

\[
d\tilde{s}_8^2 = \delta_{ij} e^i \wedge e^j
\]

for $i = 1, \ldots, 8$, and $e^i$ satisfy the Maurer-Cartan equations

\[
de^{i} = -\frac{1}{2} c^{ij}_{\lambda} e^j \wedge e^\lambda
\]
where $c$ are the structure constants of the Lie algebra $SU(3)$ with respect to a real basis. In particular, we work with a normalization with respect to which

$$
c_{126} = c_{135} = c_{368} = -c_{234} = -c_{258} = -c_{456} = -\frac{1}{\sqrt{2}},
$$

$$
c_{148} = -\sqrt{2},
$$

$$
c_{257} = c_{367} = -\sqrt{3},
$$

(3.6)

The hermitian form on $SU(3)$ is taken to be

$$
\omega^{(8)} = e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6 + e^7 \wedge e^8,
$$

(3.7)

and it is straightforward to check that the Maurer-Cartan equations imply that $\omega^{(8)}$ is integrable. This complex structure has been introduced on $SU(3)$ in [30] written in a different basis. The 3-form flux $H^{(8)} = -di_{I^{(8)}}d\omega^{(8)}$ is given by

$$
H^{(8)} = -c,
$$

(3.8)

which is covariantly constant with respect to the Levi-civita connection and hence closed, and $\hat{\nabla}^{(8)}$ is flat. In the conventions we have adopted, $SU(3)$ can be considered as a $T^2$ fibration over $B^6$, where the $T^2$ lies in the directions spanned by $e^7$ and $e^8$, so

$$
d\hat{s}^2_{(6)} = \delta_{ij} e^i \wedge e^j,
$$

$$
\omega_{(6)} = e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6
$$

(3.9)

for $i, j = 1, \ldots, 6$, $e^i = e^4$. Note that the Hermitian form $\omega_{(6)}$ is different from the Kirillov symplectic form put on regular co-adjoint orbits of semi-simple groups. In particular, $\omega_{(6)}$ defined above is not closed, however $SU(3)/T^2$ is balanced since

$$
d\omega^2_{(6)} = 0,
$$

(3.10)

and so the dilaton is constant. The connections $h, \ell$ are defined as

$$
h = \sqrt{6} e^8 - \sqrt{2} e^7,
$$

$$
\ell = -\sqrt{6} e^7 - \sqrt{2} e^8
$$

(3.11)

and so for these solutions, $k = 2\sqrt{2}$. One can use the Maurer-Cartan equations to verify that the associated curvatures $dh$ and $d\ell$ satisfy the conditions (3.1), with flux $H_{(6)} = -i_{I^{(6)}}d\omega_{(6)}$ given by

$$
H_{(6)} = \frac{1}{\sqrt{2}} \left( e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^5 \wedge e^6 - e^2 \wedge e^4 \wedge e^6 - e^3 \wedge e^4 \wedge e^5 \right).
$$

(3.12)

Note that $H_{(6)}$ is not closed. Thus $SU(3)$ satisfies all the properties of a heterotic horizon. The associated spacetime is $(SL(2, \mathbb{R}) \times SU(3))/U(1)$.

### 3.4 Horizon section $S^1 \times S^3 \times K_3$

The $S^1 \times S^3 \times K_3$ horizon section is a $T^2$ fibration over $B = S^2 \times K_3$. There are two ways of demonstrating this. One is to begin from $S^1 \times S^3 \times K_3$ and project down on
\( B = S^2 \times K_3 \) or alternatively reconstruct \( S^1 \times S^3 \times K_3 \) from \( S^2 \times K_3 \). It is convenient to do the former operation. Of course both procedures give to the same result. To begin introduce the left invariant 1-forms on the 3-spheres as \( \sigma^i \), and the 1-form \( \tau \) along \( S^1 \). The Maurer-Cartan equations give

\[
\begin{align*}
  d\sigma^3 &= \sigma^1 \wedge \sigma^2, & d\tau &= 0, &(3.13) \\
  \\
  ds^2 &= (\sigma^3)^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\tau)^2 + ds^2(K_3), &(3.14)
\end{align*}
\]

where \( ds^2(K_3) \) is the hyper-Kähler metric on \( K_3 \).

The Hermitian form on \( S^1 \times S^3 \times K_3 \) is

\[
\omega(8) = -\sigma^3 \wedge \tau - \sigma^1 \wedge \sigma^2 + \omega(K_3),
\]

where \( \omega(K_3) \) is a (1,1)-Kähler form on \( K_3 \). It is easy to see that the complex structure is integrable. Moreover, one finds that

\[
\tilde{H}(8) = \sigma^1 \wedge \sigma^2 \wedge \sigma^3.
\]

(3.16)

Since the connection with torsion on the group manifold \( S^1 \times S^3 \) is flat, the only contribution in the holonomy of \( \tilde{\nabla}(8) \) comes from the Levi-Civita connection of \( K_3 \) and so

\[
\text{hol}(\tilde{\nabla}(8)) = SU(2).
\]

Therefore

\[
\hat{\rho}(8) = 0.
\]

(3.17)

Observe though that \( S^3 \times S^3 \times T^2 \) is not (conformally) balanced.

To investigate the geometry of \( B = S^2 \times K_3 \) consider the 2-form

\[
\omega = -\sigma^1 \wedge \sigma^2 + \omega(K_3),
\]

(3.18)

on \( S^1 \times S^3 \times K_3 \). If the \( T^2 \) directions of the fibre are along \( \sigma^3 \) and \( \tau \), \( \omega \) descends to a Hermitian form on \( B \) as

\[
i_{\sigma^3} \omega = i_{\tau} \omega = 0,
\]

(3.19)

and

\[
\mathcal{L}_{\sigma^3} \omega = i_{\sigma^3} d\omega = 0,
\]

(3.20)

and similarly

\[
\mathcal{L}_{\tau} \omega = 0.
\]

(3.21)

So we set \( \omega_6 = \omega \). Since \( d\omega = 0 \), \( B = S^2 \times K_3 \) is Kähler and so balanced as required. The dilaton is constant.

The curvatures of the principal bundle connections are

\[
\begin{align*}
  d\ell &= \sigma^1 \wedge \sigma^2, \\
  dh &= 0.
\end{align*}
\]

(3.22)

In particular the connection which twists \( AdS_3 \) is flat and so the spacetime is a product \( AdS_3 \times S^3 \times K_3 \). Moreover \( d\ell \) is (1,1) and its trace is constant. Note in addition that \( \rho(6) \) does not vanish. It receives a contribution from \( S^2 \). However, this cancels the contribution of \( d\ell \) so that (3.17) is satisfied. The solution preserves 1/2 of the supersymmetry [20].
3.5 New solution with horizon section $S^3 \times S^3 \times T^2$

The $S^3 \times S^3 \times T^2$ horizon section is a $T^4$ fibration over $X = S^2 \times S^2$. As in the $S^1 \times S^3 \times K_3$ case investigated previously, we shall begin from $S^3 \times S^3 \times T^2$ and project down to $B^6$. To begin introduce the left invariant 1-forms on the two 3-spheres as $\sigma^i$ and $\rho^i$ which satisfy the Maurer-Cartan equations

$$d\sigma^3 = \sigma^1 \wedge \sigma^2, \quad d\rho^3 = \rho^1 \wedge \rho^2,$$

and cyclically in the indices 1,2,3, respectively. Then write the metric on $S$ as

$$ds^2_{(8)} = (\sigma^3)^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\rho^3)^2 + (\rho^1)^2 + (\rho^2)^2 + (\tau^1)^2 + (\tau^2)^2,$$

where

$$d\tau^1 = d\tau^2 = 0,$$

are the standard 1-forms on $T^2$.

The Hermitian form on $S^3 \times S^3 \times T^2$ is

$$\omega_{(8)} = \frac{1}{\sqrt{2}} \tau^1 \wedge (\sigma^3 + \rho^3) - \sigma^1 \wedge \sigma^2 - \rho^1 \wedge \rho^2 - \frac{1}{\sqrt{2}} \tau^2 \wedge (\sigma^3 - \rho^3),$$

leading to

$$\tilde{H}_{(8)} = \sigma^1 \wedge \sigma^2 \wedge \sigma^3 + \rho^1 \wedge \rho^2 \wedge \rho^3,$$

which is the expected 3-form field strength for group manifolds. It is easy to see using the Maurer-Cartan equations that the associated complex structure is integrable. Moreover $\hat{\nabla}_{(8)}$ is flat and so its holonomy is contained in $SU(3)$. Observe though that $S^3 \times S^3 \times T^2$ is not (conformally) balanced.

The $T^2$ fibration of the horizon section is chosen along the vector fields dual to $\tau^1$ and $\sigma^3 + \rho^3$. Moreover set

$$\omega = -\sigma^1 \wedge \sigma^2 - \rho^1 \wedge \rho^2 - \frac{1}{\sqrt{2}} \tau^2 \wedge (\sigma^3 - \rho^3).$$

After a straightforward calculation, it is easy to see that

$$i_{\sigma^3 + \rho^3} \omega = i_{\tau^1} \omega = 0,$$

and

$$\mathcal{L}_{\sigma^3 + \rho^3} \omega = i_{\sigma^3 + \rho^3} d\omega = 0,$$

and similarly

$$\mathcal{L}_{\tau^1} \omega = 0.$$
Therefore \( \omega \) descends to a Hermitian form on \( B^6 = (S^3 \times S^3)/S^1 \times S^1 \). So \( \omega = \omega_{(6)} \) and the metric on \( B^6 \) is
\[
\text{ds}^2_{(6)} = \frac{1}{2}(\sigma^3 - \rho^3)^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\rho^1)^2 + (\rho^2)^2 + (\tau)^2 .
\]
Observe that \( B \) is balanced
\[
d\omega^2_{(6)} = 0 ,
\]
as expected. Choosing
\[
h = \sqrt{2} \tau^1 , \quad \ell = \sigma^3 + \rho^3 ,
\]
so that \( k = \sqrt{2} \), one finds that
\[
dh = 0 , \quad d\ell = \sigma^1 \wedge \sigma^2 + \rho^1 \wedge \rho^2 .
\]
So \( h, \ell \) satisfy the properties for the curvatures of the \( T^2 \) fibration as required by supersymmetry. Since \( dh = 0 \), the spacetime is \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \). As we shall demonstrate later, this is the near horizon geometry of two 5-branes intersecting on a string with the string localized on the transverse space.

4 Horizons as torus fibrations over a 4-manifold

4.1 Geometric conditions

A large class of heterotic horizons can be found by taking \( B^6 \) to be a \( T^2 \) fibration over a 4-dimensional balanced Hermitian manifold \( X \). All 4-dimensional conformally balanced Hermitian manifolds are conformally Kähler. Since the conformal balanced condition is with respect to the dilaton \( \Phi \), one writes for the Hermitian form of \( X \)
\[
\omega_X = e^{2\Phi} \kappa , \quad d\kappa = 0 ,
\]
where \( \kappa \) is the Kähler form. Therefore \( S \) is a \( T^4 \) fibration over \( X \). As before we consider the two principal bundle connections \( \ell \) and \( h \), set \( h = h^1 \), and introduce two connections \( h^2 \) and \( h^3 \) along the two additional torus directions. It follows from the properties of torus fibrations in section 2 that \( B^6 \) is conformally balanced provided that the curvatures \( dh^2 \) and \( dh^3 \) are (1,1) and traceless with respect to \( \kappa \). Using this, we summarize the conditions required for a spacetime to preserve 4 supersymmetries as follows
\[
-\frac{k^2}{2} e^{2\Phi} \kappa^2 , \quad d\kappa = 0 ,
\]
\[
-k^{-2} dh^1 \wedge dh^1 + k^{-2} dh^2 \wedge dh^2 + k^{-2} dh^3 \wedge dh^3 + k^{-2} d\ell \wedge d\ell + 2i\partial \bar{\partial} e^{2\Phi} \wedge \kappa = 0 .
\]

As in the general case the above conditions lead to restrictions on the cohomology of \( X \). Here, in addition, the above differential system can be rewritten as a system of six equations for six functions. Before we do this, we shall first describe the conditions on the cohomology on \( X \).
4.2 Cohomological conditions

It is clear that $S = P_1 \otimes P_2 \otimes P_3 \otimes Q$, where $P_1, P_2, P_3, Q$ are principal circle bundles over $X$ and $P_1 = P$. The conditions (4.2) imply the cohomology classes are restricted as

$$c_1(P_1) \wedge [\kappa] = 0, \quad c_1(P_2) \wedge [\kappa] = 0, \quad c_1(P_3) \wedge [\kappa] = 0,$$

$$c_1(Q) \wedge [\kappa] = -\frac{k^2}{4\pi} [e^{2\Phi} \kappa^2], \quad c_1(X) - c_1(Q) = 0,$$

$$c_1(P_1) \wedge c_1(P_1) + c_1(P_2) \wedge c_1(P_2) + c_1(P_3) \wedge c_1(P_3) + c_1(Q) \wedge c_1(Q) = 0,$$  \hspace{1cm} (4.3)

where $c_1$ denotes the first Chern class of the appropriate circle bundle and $c_1(X)$ is the first Chern class of the canonical bundle of $X$. It is clear from the cohomology conditions above that $c_1(Q)$ must be identified with the canonical class of $X$. Moreover, the class $c_1(Q) \wedge [\kappa]$ must be a negative multiple of the volume class of $X$.

The above conditions can also be rewritten in terms of the intersection form. 

$$[\alpha] \cdot [\beta] = \int_X \alpha \wedge \beta.$$  \hspace{1cm} (4.4)

The only difference is that the ring product in $H^2(X)$ is replaced with the product as defined by the intersection form.

4.3 Differential system

To express the differential conditions as a system of six equations for six functions, we shall use the $\partial \bar{\partial}$-lemma and the cohomological conditions stated above. It is assumed that $X$ is chosen such that the conditions (4.3) have a solution. The procedure resembles the background field method used for quantum calculations in field theory. Thus one splits the fields, which are represented by the connections, metric, hermitian form and dilaton, into a background part and a fluctuation. Though here, the fluctuations are not restricted to be small. In particular, introduce fixed background fields $\tilde{h}^1, \tilde{h}^2, \tilde{h}^3, \tilde{\ell}, \tilde{\Phi}$ and $\tilde{\kappa}$ which satisfy the cohomological conditions (4.3). Then using the $\partial \bar{\partial}$-lemma, there are \((1,1)\)-forms $\alpha^1, \alpha^2, \alpha^3, \chi$ and $\psi$, and a function $f$, which depend only on the background data, such that

$$d\tilde{h}^1 \wedge \tilde{\kappa} = i\partial \bar{\partial} \alpha^1, \quad d\tilde{h}^2 \wedge \tilde{\kappa} = i\partial \bar{\partial} \alpha^2, \quad d\tilde{h}^3 \wedge \tilde{\kappa} = i\partial \bar{\partial} \alpha^3,$$

$$d\tilde{\ell} \wedge \tilde{\kappa} = -\frac{k^2}{2} e^{2\Phi} \kappa^2 + i\partial \bar{\partial} \chi, \quad i\partial \bar{\partial} \log \det(i\tilde{\kappa}) + i\partial \bar{\partial} f - d\tilde{\ell} = 0,$$

$$k^{-2}(d\tilde{h}^1 \wedge dh^1 + dh^2 \wedge dh^2 + dh^3 \wedge dh^3 + d\tilde{\ell} \wedge d\tilde{\ell} + i\partial \bar{\partial} \psi = 0.$$  \hspace{1cm} (4.5)

Since we are seeking solutions that preserve the chosen cohomological classes, and using again the $\partial \bar{\partial}$-lemma, there are functions $s^1, s^2, s^3, v$ and $w$ such that

$$dh^1 = \tilde{h}^1 + i\partial \bar{\partial} s^1, \quad dh^2 = \tilde{h}^2 + i\partial \bar{\partial} s^2, \quad dh^3 = \tilde{h}^3 + i\partial \bar{\partial} s^3,$$

$$d\ell = d\tilde{\ell} + i\partial \bar{\partial} v, \quad \kappa = \tilde{\kappa} - i\partial \bar{\partial} w, \quad \Phi = \tilde{\Phi} + \varphi.$$  \hspace{1cm} (4.6)

where we have also expressed the dilaton\(^5\) in terms of the background field and a fluctuation $\varphi$.

\(^5\)It is not necessary to split the dilaton in a background and a fluctuation but this was done for uniformity.
Substituting (4.6) into (4.2) and subtracting from the resulting expressions (4.5), one finds
\[
\begin{align*}
i\bar{\partial}\dd s^1 & \wedge \kappa - \bar{d}h^1 \wedge i\bar{\partial}\dd w + \partial\dd s^1 \wedge \partial\dd w = -i\bar{\partial}\dd\alpha^1 , \\
i\bar{\partial}\dd s^2 & \wedge \kappa - \bar{d}h^2 \wedge i\bar{\partial}\dd w + \partial\dd s^2 \wedge \partial\dd w = -i\bar{\partial}\dd\alpha^2 , \\
i\bar{\partial}\dd s^3 & \wedge \kappa - \bar{d}h^3 \wedge i\bar{\partial}\dd w + \partial\dd s^3 \wedge \partial\dd w = -i\bar{\partial}\dd\alpha^3 , \\
i\bar{\partial}\dd v & \wedge \kappa - d\ell \wedge i\bar{\partial}\dd w + \partial\dd v \wedge \partial\dd w = -\frac{k^2}{2} e^{2(\Phi+\varphi)}(\kappa - i\bar{\partial}\dd w)^2 + \frac{k^2}{2} e^{2\Phi} \kappa^2 - i\bar{\partial}\dd\chi ,
\end{align*}
\]
where \( c \) is a constant.

This is a non-linear system of six equations for the six functions \( s^1, s^2, s^3, v, w \) and \( \varphi \). It contains a Monge-Ampère type of equation. It is easy to see that each equation can be solved for one unknown function treating the remaining functions as sources. For example if \( v \) is a source, the Monge-Ampère equation is identical to the one solved by Yau for the proof of the Calabi conjecture. However, it is less clear that the full non-linear system has solutions. Therefore the question arises as to whether for every solution of the cohomological conditions (4.3), there is a smooth solution of (4.7).

There is also the possibility that (4.7) that does not have any solutions at all. This is not the case. We shall give explicit examples below which solve all the conditions. Moreover, we shall explore the conditions on the cohomology. We shall demonstrate that these have solutions for \( X \) a del Pezzo surface.

### 4.4 New solution with horizon section \( S^3 \times S^3 \times T^2 \)

The \( S^3 \times S^3 \times T^2 \) horizon section is a \( T^4 \) fibration over \( X = S^2 \times S^2 \). As in the \( S^1 \times S^3 \times K_3 \) case investigated previously, we shall begin from \( S^3 \times S^3 \times T^2 \) and project down to \( B \) and to \( X \). To begin introduce the left invariant 1-forms \( \sigma^i \) and \( \rho^i \) on the two 3-spheres which satisfy the Maurer-Cartan equations as in (3.23). Then write the metric on \( S \) as
\[
ds^2_{(8)} = (\sigma^3)^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\rho^3)^2 + (\rho^1)^2 + (\rho^2)^2 + (\tau^1)^2 + (\tau^2)^2 ,
\]
where
\[
d\tau^1 = d\tau^2 = 0 ,
\]
are the standard 1-forms on \( T^2 \).

The Hermitian form on \( S^3 \times S^3 \times T^2 \) is
\[
\omega_{(8)} = -\sigma^2 \wedge \rho^3 - \sigma^1 \wedge \sigma^2 - \rho^1 \wedge \rho^2 - \tau^1 \wedge \tau^2 ,
\]
leading to
\[
\tilde{H}_{(8)} = \sigma^1 \wedge \sigma^2 \wedge \sigma^3 + \rho^1 \wedge \rho^2 \wedge \rho^3 ,
\]
which is the expected 3-form field strength for group manifolds. It is easy to see using the Maurer-Cartan equations that the associated complex structure is integrable. Moreover \( \hat{\nabla}_{(8)} \) is flat and so its holonomy is contained in \( SU(3) \). Observe though that \( S^3 \times S^3 \times T^2 \) is not (conformally) balanced.

First, we shall investigate the geometry of \( B = S^2 \times S^2 \times T^2 \) and then of \( X \). For this, consider the 2-form

\[
\omega = -\sigma^1 \wedge \sigma^2 - \rho^1 \wedge \rho^2 - \tau^1 \wedge \tau^2 , \tag{4.12}
\]
on \( S^3 \times S^3 \times T^2 \). If the fibre directions are along \( \sigma^3 \) and \( \rho^3 \), \( \omega \) descends to a Hermitian form on \( B \) as

\[
i_{\sigma^3} \omega = i_{\rho^3} \omega = 0 , \tag{4.13}
\]
and

\[
\mathcal{L}_{\sigma^3} \omega = i_{\sigma^3} d\omega = 0 , \tag{4.14}
\]
and similarly

\[
\mathcal{L}_{\rho^3} \omega = 0 . \tag{4.15}
\]
So we set \( \omega_{(6)} = \omega \). Since \( d\omega = 0 \), \( B = S^2 \times S^2 \times T^2 \) is Kähler and so balanced as required. The dilaton is constant.

The canonical bundle of \( S^2 \times S^2 \times T^2 \) is not trivial. However, it becomes trivial after pulling it back on \( S^3 \times S^3 \times T^2 \). To see this consider the (3,0) form

\[
\chi = \frac{1}{2\sqrt{2}} (\sigma^1 + i\sigma^2) \wedge (\rho^1 + i\rho^2) \wedge (\tau^1 + i\tau^2) , \tag{4.16}
\]
on \( S^3 \times S^3 \times T^2 \). Clearly

\[
i_{\sigma^3} \chi = i_{\rho^3} \chi = 0 , \tag{4.17}
\]
but

\[
\mathcal{L}_{\sigma^3 + \rho^3} \chi = 2i\chi , \quad \mathcal{L}_{\sigma^3 - \rho^3} \chi = 0 . \tag{4.18}
\]
Since \( \chi \) is transformed up to a phase in the \( \sigma^3 + \rho^3 \) direction, the trivial bundle with section \( \chi \) over \( S^3 \times S^3 \times T^2 \) is projected down to a non-trivial bundle over \( S^2 \times S^2 \times T^2 \).

Furthermore, one can set

\[
h = -(\sigma^3 - \rho^3) , \quad \ell = (\sigma^3 + \rho^3) . \tag{4.19}
\]
Then

\[
dh = -(\sigma^1 \wedge \sigma^2 - \rho^1 \wedge \rho^2) \tag{4.20}
\]
which is (1,1) and traceless and

\[
d\ell = (\sigma^1 \wedge \sigma^2 + \rho^1 \wedge \rho^2) \tag{4.21}
\]
which is (1,1) but not traceless. The trace of \( d\ell \) is constant as required.

Since \( B = S^2 \times S^2 \times T^2 \), the 4-dimensional Kähler manifold \( X \) is \( X = S^2 \times S^2 \). The projection from \( B \) to \( X \) is along the trivial \( T^2 \) fibration. The Kähler form on \( X \) is \( \kappa = -\sigma^1 \wedge \sigma^2 - \rho^1 \wedge \rho^2 \). Clearly, \( S^3 \times S^3 \times T^2 \) is a \( T^4 \) fibration over \( S^2 \times S^2 \) with principal bundle connection \( \ell = \ell , h^1 = h , h^2 = \tau^1 \) and \( h^3 = \tau^2 \). The spacetime is isomorphic to \( (AdS_3 \times S^3 \times S^3)/S^1 \times T^2 \).
4.5 Horizon section $S^3 \times T^5$

To describe the geometry, we introduce the left invariant 1-forms $\sigma$ on $S^3$ as in the previous example. We also consider the 1-forms $\tau^3, \tau^4$ and $\tau^5$ spanning the $T^3$ directions that replace the second $S^3$. So now
\[d\tau^3 = d\tau^4 = d\tau^5 = 0\ . \tag{4.22}\]

The analysis is identical to the one presented in the previous example. The only difference is that the 1-forms $\tau$ which replace the $\rho$‘s are closed instead of satisfying (3.23). Thus one finds that $B = S^2 \times T^4$, which is Kähler, and $X = S^2 \times T^2$.

The principal bundle connections are $\ell = \sigma^3, h^1 = \tau^1, h^2 = \tau^2$ and $h^3 = \tau^3$. The curvatures of the principal bundle connections are
\[d\ell = \sigma^1 \wedge \sigma^2, \quad dh^1 = dh^2 = dh^3 = 0\ . \tag{4.23}\]

Clearly all the curvatures are (1,1) and $d\ell$ has constant trace. The spacetime is $AdS_3 \times S^3 \times T^4$ and preserves 1/2 of supersymmetry [20]. Since $dh = 0$, $h = h^1$, $AdS_3$ does not twist over $B$.

5 Solutions of cohomology conditions

We have demonstrated that the differential system (4.7) admits solutions. Now we shall provide evidence that there may be a large class of solutions to (4.7) by proving that there are many manifolds that satisfy the cohomological conditions (4.3). We shall take $X$ to be $\mathbb{P}^2$ blown up at $k$ points\(^6\). This surface has been used before in the context of manifolds with skew-symmetric torsion [24]. If $k < 9$, then these manifolds are del Pezzo surfaces $dP_{9-k}$. These have found applications in the context of mysterious duality [27], which relates the U-duality [26] 1/2 BPS states of M-theory toroidal compactifications to rational curves in del Pezzo surfaces, and supergravity [28].

Appropriately choosing the points that $\mathbb{P}^2$ is blown up, the cohomology $H^2(X, \mathbb{Z})$ is generated by the hypersurface class $H$ and the exceptional divisors $E_i, i = 1, \ldots, k$ [36]. The intersection form is
\[H \cdot H = 1, \quad H \cdot E_i = 0, \quad E_i \cdot E_j = -\delta_{ij}\ . \tag{5.1}\]

The anti-canonical class, $-K$, of $X$ is
\[-K = 3H - E_1 - E_2 - \cdots - E_k\ . \tag{5.2}\]

There is another basis in the cohomology which consists of the anti-canonical class $-K$ and
\[\alpha_i = E_i - E_{i+1}, \quad i = 1, \ldots, k - 1,\]

\(^6\)Throughout this section $k$ is the number of points that $\mathbb{P}^2$ is blown up and it should not be confused with the normalization factor that appears in the definition of metric and fluxes in previous sections, see eg (2.1).
\[ \alpha_k = H - E_1 - E_2 - E_3. \]  

(5.3)

The anti-canonical class is orthogonal to the rest of the generators. The intersection matrix in the \((\alpha_i), i = 1, \ldots, k,\) basis is

\[ K \cdot \alpha_i = 0, \quad \alpha_i \cdot \alpha_j = -A_{ij}, \quad i, j = 1, \ldots, k, \]  

(5.4)

where \((A_{ij})\) is the Cartan matrix of exceptional Lie algebras \(E_k\), see table 1. As a result, the intersection matrix can be represented with the associated Dynkin diagram.

| \(k\) | \(E_k\) |
|-------|--------|
| 1     | \(A_1\) |
| 2     | \(A_1 \oplus A_1\) |
| 3     | \(A_2 \oplus A_1\) |
| 4     | \(A_4\) |
| 5     | \(D_5\) |
| 6     | \(E_6\) |
| 7     | \(E_7\) |
| 8     | \(E_8\) |
| \(k > 9\) | \(E_k\) |

Table 2: The intersection matrix of a del Pezzo surface, \(k < 9\), is given by the Cartan matrix of exceptional Lie algebras. The intersection matrix of of \(\mathbb{P}^2\) blown up at \(k > 9\) in general position are also given by the Cartan matrix of exceptional algebras \(E_k\).

The \((4.3)\) conditions require that

\[ c_1(Q) = K. \]  

(5.5)

To solve the remaining conditions in \((4.3)\), one has to choose appropriately the Chern classes of the bundles \(P_1, P_2\) and \(P_3\), and \([\kappa]\). We shall consider the cases for which \(X\) is a del Pezzo surface and for which \(k > 9\) separately.

### 5.1 del Pezzo

#### 5.1.1 \(k < 9\) odd

There are two cases to consider here depending on the choice of Kähler class. Not all classes in \(H^2(X, \mathbb{R})\) can be represented by a Kähler form. Those that can obey the Nakai-Moishezon criteria. These will be stated later. In this case, the most straightforward choice is \([\kappa] = -K\). Observe that \(c_1(Q) \cdot [\kappa] = -K^2 = k - 9 < 0\), ie \(c_1(Q) \wedge [\kappa]\) is a negative multiple of the volume class of \(X\) as required. Setting

\[ c_1(P_s) = n_s H - \sum_i m_{si} E_i, \quad s = 1, 2, 3, \]  

(5.6)
we find that the condition \( c_1(P_s) \cdot [\kappa] = 0 \) implies that

\[
3n_s - \sum_i m_{si} = 0 .
\]  

(5.7)

Moreover the cohomological condition which arises from the closure of \( H_{(8)} \) reads

\[
9 - k + \sum_s (n_s^2 - \sum_i (m_{si})^2) = 9 - k - \sum_s (8n_s^2 + 2 \sum_{i<j} m_{si}m_{sj}) = 0 .
\]  

(5.8)

It is clear that for the above choice of Kähler class, there are solutions only if \( k \) is odd. Moreover, it is straightforward to show that for \( k = 3, 5, 7 \), (5.7) and (5.8) imply that \( |m_{si}| \leq 2 \) for all \( s, i \). So there are only finitely many solutions. Furthermore one also finds, for \( k = 3, 5, 7 \), that at most one of the \( m_{si} \) can take the value \( \pm 2 \), the remaining \( m_{si} \) must be \( \pm 1 \) or 0. One can then enumerate all possible solutions.

\( k = 1 \)

The solution for \( k = 1 \) is unique, modulo overall sign changes in the \( c_1(P_s) \) and permutations of the \( P_s \). The Chern class can be chosen, without loss of generality, as

\[
c_1(P_1) = H - 3E_1 , \quad c_1(P_2) = c_1(P_3) = 0 .
\]  

(5.9)

\( k = 3 \)

There are seven distinct types of solutions up to permutations in the \( P_s \) and overall changes of sign in \( c_1(P_s) \). In particular, one finds

\[
c_1(P_1) = E_{a_1} - E_{a_2} , \quad c_1(P_2) = E_{b_1} - E_{b_2} , \quad c_1(P_3) = E_{f_1} - E_{f_2} ,
\]

\[
c_1(P_1) = H - E_1 - E_2 - E_3 , \quad c_1(P_2) = E_{a_1} - E_{a_2} , \quad c_1(P_3) = E_{b_1} - E_{b_2} ,
\]

\[
c_1(P_1) = \pm c_1(P_2) = H - E_1 - E_2 - E_3 , \quad c_1(P_3) = E_{a_1} - E_{a_2} ,
\]

\[
c_1(P_1) = \pm c_1(P_2) = \pm c_1(P_3) = H - E_1 - E_2 - E_3 ,
\]

\[
c_1(P_1) = -2E_{a_1} + E_{a_2} + E_{a_3} , \quad c_1(P_2) = c_1(P_3) = 0 ,
\]

\[
c_1(P_1) = H - 2E_{a_1} - E_{a_2} , \quad c_1(P_2) = E_{b_1} - E_{b_2} , \quad c_1(P_3) = 0 ,
\]

\[
c_1(P_1) = H - 2E_{a_1} - E_{a_2} , \quad c_1(P_2) = H - E_1 - E_2 - E_3 , \quad c_1(P_3) = 0 ,
\]  

(5.10)

where \( \{a_i\} \) are distinct, \( \{b_i\} \) are distinct, \( \{f_i\} \) are distinct taking values in the set \( \{1, 2, \ldots, k\} \), \( k = 3 \). Though we do allow for \( a_i = b_j \) and so on.

\( k = 5 \)

There are seven distinct types of solutions up to permutations in the \( P_s \) and overall changes of sign in \( c_1(P_s) \). In all cases, one can take, without loss of generality \( c_1(P_3) = 0 \), and the remaining Chern classes \( c_1(P_1), c_1(P_2) \) are
\[ c_1(P_1) = -H - E_{a_1} + E_{a_2} + E_{a_3} + E_{a_4} + E_{a_5}, \quad c_1(P_2) = 0, \]
\[ c_1(P_1) = -E_{a_1} - E_{a_2} + E_{a_3}, \quad c_1(P_2) = 0, \]
\[ c_1(P_1) = -H + E_{a_1} + E_{a_2} + E_{a_3}, \quad c_1(P_2) = E_{b_1} - E_{b_2}, \]
\[ c_1(P_1) = E_{a_1} - E_{a_2}, \quad c_1(P_2) = E_{b_1} - E_{b_2}, \]
\[ c_1(P_1) = -H + E_{a_1} + E_{a_2} + E_{a_3}, \quad c_1(P_2) = -H + E_{b_1} + E_{b_2} + E_{b_3}, \]
\[ c_1(P_1) = -H + E_{a_1} + E_{a_2} + E_{a_3}, \quad c_1(P_2) = 0, \]
\[ c_1(P_1) = 2H - 2E_{a_1} - E_{a_2} - E_{a_3} - E_{a_4} - E_{a_5}, \quad c_1(P_2) = 0. \]  

(5.11)

\[ k = 7 \]

There are three distinct types of solutions again up to permutations in the \( P_s \) and overall changes of sign in \( c_1(P_s) \). One can take, without loss of generality, \( c_1(P_2) = c_1(P_3) = 0 \). The remaining Chern class \( c_1(P_1) \) is

\[ c_1(P_1) = -2H + E_{a_1} + E_{a_2} + E_{a_3} + E_{a_4} + E_{a_5}, \]
\[ c_1(P_1) = -H + E_{a_1} + E_{a_2} + E_{a_3}, \]
\[ c_1(P_1) = E_{a_1} - E_{a_2}. \]  

(5.12)

### 5.1.2 \( k < 9 \) even

As we have mentioned, if the Kähler class is identified with the anti-canonical class \(-K\), one cannot solve the cohomological conditions when \( k \) is even. However, this problem can be circumvented by choosing another Kähler class. For this write

\[ [\kappa] = pH - \sum_i q_i E_i. \]  

(5.13)

The requirement that \([\kappa]\) is represented by a Kähler class restricts the components \( p, q_i \). In particular according to Nakai-Moishezon criteria, \([\kappa]\) must satisfy

* \([\kappa] \cdot [\kappa] > 0.\)
* \([\kappa] \cdot D > 0 \) for any irreducible curve \( D \) with negative self intersection.
* \([\kappa] \cdot C > 0 \) for an ample divisor \( C \).

The divisor class \( nH - E_1 - E_2 - \cdots - E_k \) is ample for sufficiently large \( n \). On the blow up of distinct points on a smooth cubic in \( \mathbb{P}^2 \), the irreducible curves with negative intersection are \( E_i, H - E_i - E_j, i \neq j \) for \( k \leq 9 \). The above conditions lead to the restrictions

\[ p^2 > \sum_{i=1}^k q_i^2, \quad p > q_i + q_j, \quad q_i > 0, \quad i = 1, \ldots, k. \]  

(5.14)
Writing
\[ c_1(P_s) = n_s H - \sum_i m_{si} E_i , \] (5.15)
the conditions \( c_1(P_s) \cdot [\kappa] = 0 \) imply
\[ n_s p - \sum_i m_{si} q_i = 0 . \] (5.16)
In addition, \(-c_1(Q) \cdot [\kappa] > 0\) gives
\[ 3p - \sum_{i=1}^k q_i > 0 . \] (5.17)
Furthermore, the condition associated with the closure of \( H(8) \) gives
\[ 9 - k + \sum_s n_s^2 - \sum_s \sum_i (m_{si})^2 = 0 . \] (5.18)
Examples of solutions to the cohomological conditions are as follows
\[ k = 2 \]
\[ [\kappa] = 4H - E_1 - E_2 , \quad c_1(P_1) = H - 2E_1 - 2E_2 , \quad c_1(P_2) = c_1(P_3) = 0 . \] (5.19)
\[ k = 4 \]
\[ [\kappa] = 4H - E_1 - E_2 - E_3 - E_4 , \quad c_1(P_1) = H - E_1 - E_2 - E_3 - E_4 , \]
\[ c_1(P_2) = E_1 - E_2 , \quad c_1(P_3) = 0 . \] (5.20)
\[ k = 6 \]
\[ [\kappa] = 8H - 3E_1 - 3E_2 - 3E_3 - 3E_4 - 3E_5 - 3E_6 , \quad c_1(P_2) = c_1(P_3) = 0 , \]
\[ c_1(P_1) = 3H - 2E_1 - 2E_2 - E_3 - E_4 - E_5 - E_6 . \] (5.21)
\[ k = 8 \]
\[ [\kappa] = 17H - 6 \sum_{i=1}^8 E_i , \quad c_1(P_2) = c_1(P_3) = 0 , \]
\[ c_1(P_1) = 6H - 3E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 2E_7 - 2E_8 . \] (5.22)
A direct observation reveals that $c_1(P_s)^2 \leq 0$ as required for classes represented by anti-self-dual 2-forms. Moreover in all cases $-c_1(Q) \cdot [k] > 0$.

It is clear that there are many solutions of the cohomological conditions on del Pezzo surfaces. So it is likely that the differential system (4.2) admits solutions as well. This will give a new geometry on del Pezzo surfaces as it will be different from the Einstein metrics of [31]. This is because if the Einstein metric is chosen as a solution of the differential system (4.2), then $d\ell$ will be proportional to the Kähler form and so the dilaton has to be constant. As a result, it is not possible to solve the equation which arises from $dH = 0$ as the wedge product of harmonic forms may not be harmonic.

5.2 $k > 9$

The cohomological system does not have solutions for $k > 9$ for any choice of Kähler class. This is because now $K^2 = 9 - k < 0$ and $c_1^2(P_s) \leq 0$ since the latter are represented by anti-self-dual forms. Therefore the cohomological condition associated with $dH_{(8)} = 0$ cannot be satisfied. This can be confirmed using Nakai-Moishezon criteria for choosing the Kähler class and the rest of the conditions of the cohomological system. It is worth noting that for $k \geq 10$, apart from $E_i, H - E_i - E_j, i \neq j$, $-K$ is also an irreducible curve with negative self-intersection. Then (5.17) arises as part of the Nakai-Moishezon criteria.

To include solutions with $k > 9$, the cohomological condition associated with $d\tilde{H}_{(8)} = 0$ must be modified. One option is to include contributions from the anomaly cancelation which schematically leads to

$$d\tilde{H}_{(8)} = -\frac{\alpha'}{4}(\text{tr}R^2 - \text{tr}F^2).$$  \hspace{1cm} (5.23)

Alternatively, one can add point anti-5-brane sources as

$$d\tilde{H}_{(8)} = n\delta(NS5).$$  \hspace{1cm} (5.24)

Choosing these appropriately, one can cancel the negative contributions arising from the canonical class and the rest of the Chern classes of the circle bundles.

5.3 Topology of heterotic horizons

In all cases, the horizon spacetime is contractible to the horizon section $S$. In turn the horizon sections are $T^2$ fibrations over $B^6$. We shall focus on the case for which $B^6$ is a $T^2$-fibration over a Kähler 4-manifold $X$. Then we shall adapt the calculation for a del Pezzo surface $X = dP_{9-k}$. We shall compute the cohomology in three different scenarios depending on the number of non-trivial line bundles which appear in the construction of $S$ from $X$.

First suppose that $S = T^2 \times B$, and $B$ is a non-trivial $T^2$ fibration over $X$. To simplify the computations, let us focus on de Rham cohomology. To compute the de Rham cohomology of $S$, it suffices to find the cohomology of $B$. Since $B$ is a non-trivial fibration over $X$, the Chern classes $c_1(Q) = b_1$ and $c_1(P) = b_2$ can be chosen
as the first two basis elements in $H^2(X, \mathbb{R})$, i.e. $H^2(X, \mathbb{R}) = \mathbb{R}\langle b_1, \ldots, b_m \rangle$. Moreover let $H^2(T^2, \mathbb{R}) = \mathbb{R}\langle \theta_1, \theta_2, \theta_1 \wedge \theta_2 \rangle$. Using the spectral sequence for a fibration

$$E_2^{p,q} = H^p(X, H^q(T^2, \mathbb{R})) , \quad p = 0, \ldots, 4 , \quad q = 0, \ldots, 2 .$$

To find the cohomology of $B$, it suffices to calculate the action of the $d_2$ differential. This has been done in appendix A. It turns out that

$$H^0(B, \mathbb{R}) = H^6(B, \mathbb{R}) = \mathbb{R} , \quad H^2(B, \mathbb{R}) = H^4(B, \mathbb{R}) = \mathbb{R}^{m-2} , \quad H^3(B, \mathbb{R}) = \mathbb{R}^{2m-2} .$$

We can also give the intersection matrices of the cohomology of $B$. For this suppose that the intersection matrix $A$ of $X$ is

$$\int_X b_e \wedge b_f = A_{ef} , \quad e,f = 1, \ldots, m .$$

A basis in the cohomology of $B^6$ is

$$H^2(B, \mathbb{R}) = \mathbb{R}\langle b_a \rangle , \quad a \neq 1, 2 ,$$

$$H^3(B, \mathbb{R}) = \mathbb{R}\langle \theta_r \wedge b_a, \theta_1 \wedge b_1 + \theta_2 \wedge b_2, \theta_1 \wedge b_2 + \theta_2 \wedge b_1 \rangle .$$

Observe that one of the generators of $H^3(B, \mathbb{R})$ is the Chern-Simons form. It is straightforward to compute the integrals

$$\int_B \omega_{(6)} \wedge b_a \wedge b_b = -A_{ab} ,$$

$$\int_B \theta_r \wedge b_a \wedge \theta_s \wedge b_b = -\epsilon_{rs} A_{ab} ,$$

$$\int_B \theta_r \wedge b_a \wedge (\theta_1 \wedge b_1 + \theta_2 \wedge b_2) = -\epsilon_{r1} A_{a1} - \epsilon_{r2} A_{a2} ,$$

$$\int_B \theta_r \wedge b_a \wedge (\theta_1 \wedge b_2 + \theta_2 \wedge b_1) = -\epsilon_{r1} A_{a2} - \epsilon_{r2} A_{a1} ,$$

$$\int_B (\theta_1 \wedge b_1 + \theta_2 \wedge b_2) \wedge (\theta_1 \wedge b_2 + \theta_2 \wedge b_1) = -A_{11} + A_{22} .$$

Apart from the first, the rest give the intersection matrix of $B$. Note that even though $d\omega_{(6)} \neq 0$, the first integral does not depend on the representatives of the classes.

Now suppose that $X = dP_{3-k}$. For the solutions for which $[\kappa] = -b_1 = -K$, one can choose, up to an appropriate rescaling, $\alpha = -K$ and $\beta = b_2$ as $K \cdot b_2 = 0$ and $b_2^2 \neq 0$ to satisfy the assumptions stated in appendix A to calculate the cohomology. In the case that $[\kappa] \neq -K$, one chooses $\alpha = [\kappa]$. Moreover, there is always a class with the properties of $\beta$. Clearly, the intersection matrices of $B^6$ inherit the exceptional structure of the cohomology of del Pezzo surfaces.

Moreover, it turns out that if $X = dP_{3-k}$, $B^6$ has the same de Rham cohomology as $(k-1)(S^2 \times S^4) \# k(S^3 \times S^3)$. In fact under some additional conditions, it is diffeomorphic to it [24]. The black holes that can arise from such horizons have non-trivial topology.
Next suppose that $S = S^1 \times Y$ and $Y$ is a non-trivial $T^3$ fibration over $X$. In such case, the calculation explained in appendix A reveals that

$$
H^0(Y, \mathbb{R}) = H^7(Y, \mathbb{R}) = \mathbb{R}, \quad H^2(Y, \mathbb{R}) = H^5(Y, \mathbb{R}) = \mathbb{R}^{m-3},
H^3(Y, \mathbb{R}) = H^4(Y, \mathbb{R}) = \mathbb{R}^{3m-4}.
$$

(5.30)

Furthermore, if $S$ is a non-trivial fibration over $X$, then the calculation in appendix A gives

$$
H^0(S, \mathbb{R}) = H^8(S, \mathbb{R}) = \mathbb{R}, \quad H^2(S, \mathbb{R}) = H^6(S, \mathbb{R}) = \mathbb{R}^{m-4},
H^3(S, \mathbb{R}) = H^5(S, \mathbb{R}) = \mathbb{R}^{4m-7}, \quad H^4(S, \mathbb{R}) = \mathbb{R}^{6m-8}.
$$

(5.31)

It is clear that in all cases the cohomology of the horizon section is non-trivial indicating the existence of exotic heterotic black holes.

### 6 Near brane geometries for localized brane intersections

In all known cases, black hole horizons also arise as near brane geometries. It is likely that our heterotic horizon solutions can also be interpreted as the near brane geometry of a brane configuration. Since the only fluxes that are switched on are the dilaton and the 3-form field strength, it is expected that they correspond to the near brane geometry of a configuration of 5-branes and fundamental strings. In particular, we shall show that the $AdS_3 \times S^3 \times S^3 \times S^1$ solution is the near brane geometry of two 5-branes intersecting on a string with the latter localized in all transverse directions. The metric, 3-form field strength and dilaton for this configuration can be written as

$$
d s^2 = h_{-1}(x, y)(-d t^2 + d \sigma^2) + h_5(x)dx^2 + h'_5(y)dy^2,
H = d t \wedge d \sigma \wedge dh_{-1} + *x dh_5 + *y dh'_5,
$$

$$
e^{2 \Phi} = h_{-1}^{-1}h_5h'_5,
$$

(6.1)

where $dx^2$ and $dy^2$ is the Euclidean metric on the transverse spaces $\mathbb{R}^4$ of the corresponding 5-brane and all the Hodge duals have been taken with the flat metric. If

$$
h_5 = 1 + \frac{q_5}{|x|^2}, \quad h'_5 = 1 + \frac{q'_5}{|y|^2},
$$

(6.2)

| 5brane | 0 | 1 | 2 | 3 | 4 | 5 |
|--------|---|---|---|---|---|---|
| 5brane | 0 | 1 | 6 | 7 | 8 | 9 |
| string  | 0 | 1 |

Table 3: The worldvolume directions of the branes are given.
then $dH = 0$. To determine $h_s$, one has to solve the field equation for the 2-form gauge potential which gives

$$h_5\delta^{ab}\partial_a h_s + h_5\delta^{a'b'}\partial_{a'} \partial_{b'} h_s = 0 \ .$$

(6.3)

A solution for $h_s$ is

$$h_s = 1 + \frac{q_1}{|x|^2} + \frac{q_1'}{|y|^2} + \frac{s}{|y|^2|x|^2} \ .$$

(6.4)

see also [32, 33]. The geometry near the common intersection is recovered in the limit $|y|^2, |x|^2 \to 0$ with the ratio $|y|/|x|$ fixed. In this limit, the last term in the above equation dominates. Evaluating the metric near this limit after a change of co-ordinates one finds $AdS_3 \times S^3 \times S^3 \times S^1$ with constant dilaton. Moreover

$$H = 2d\text{vol}(AdS_3) + d\text{vol}(S^3) + d\text{vol}(S^3) \ .$$

(6.5)

This is the near horizon example which we have found in section 3.5.

7 Near horizon geometries for type II black holes

The solutions we have found have closed 3-form field strength and so they are solutions of type IIA and IIB supergravities. As solutions to IIA supergravity, they can be lifted to 11 dimensions. In particular the general form of the solution in (2.17) lifted to 11 dimensions reads

$$ds^2_{(11)} = e^{-\frac{2\Phi}{3}} ds^2_{(10)} + e^{\frac{4\Phi}{3}} dy^2$$

$$F = dy \wedge H \ .$$

(7.1)

Reducing the solution back to IIA along the non-trivial fibre direction $h$, we find that

$$ds^2_{(A)} = e^{-\Phi}[-k^2 r^2 du^2 + 2dudr + k^{-2} \ell \wedge \ell + ds^2_{(6)} + e^{4\Phi} dy^2]$$

$$F_2 = k^{-2}(dh + dr \wedge du) , \quad H = -k^{-2} dy \wedge dh ,$$

$$F_4 = dy \wedge [k^{-2} \ell \wedge d\ell + H_{(6)}] , \quad e^{2\Phi_{(A)}} = e^{-\Phi} \ .$$

(7.2)

One may notice that the metric $e^\Phi ds^2_{(10)}$ has an $AdS_2$ spanned by the coordinates $r, u$. Therefore if the dilaton is constant, and this is the case for many of our solutions, the resulting spacetime can be interpreted as the near horizon geometry of IIA black holes. The solution preserves all 4 supersymmetries as the reduction is done along $h$ and $dh$ is $(1,1)$ and traceless. This is because the spinorial Lie derivative of all the Killing spinors along the vector field associated to $h$ vanishes [34].

The solution can be further be T-dualized to IIB along the direction of $\ell$. In this case, the metric of the IIB background is

$$ds^2_{(B)} = e^{-\Phi}[-k^2 r^2 du^2 + 2dudr + ds^2_{(6)} + e^{4\Phi} dy^2] + k^2 e^\Phi dx^2 \ .$$

(7.3)

Again for constant dilaton, the metric has an $AdS_2$ component. Therefore such spacetimes can be interpreted as the near horizon geometries of IIB black holes. However in this
case, all 4 supersymmetries of the IIA background will be broken after T-duality to IIB as \(d\ell\) although \((1,1)\) is not traceless. The spinorial Lie derivative along the vector field associated to \(\ell\) on the Killing spinor will not vanish.

Supersymmetry under T-duality can be preserved provided that \(B^6\) satisfies additional properties and the T-duality operation is taken along another direction. In particular if \(B^6\) is a \(T^2\)-fibration over a Kähler manifold, then the two additional fibre directions are associated with curvatures \(dh^r, r = 2, 3,\) which are \((1,1)\) and traceless. In this case, after T-duality along any of these two directions one obtains a IIB background with an \(AdS_2\) factor preserving 4 supersymmetries. This is because again the spinorial Lie derivative of the IIA Killing spinor vanishes when is taken along one of the two fibre directions. So even though IIB supergravity does not have a 1-form gauge potential, there are supersymmetric black hole near horizon geometries.

8 From horizons to black holes

Having found a near horizon geometry, it is natural to ask whether this can be extended to a full black hole geometry. It is not expected that all near horizon geometries will give rise to black hole solutions. In the absence of a solution to the field equations, some qualitative tests have been devised. One such test is based on the expectation that there is a continuous interpolation of the horizon section to the compact section of the asymptotic geometry via a Cauchy surface which lies outside the horizon. Such a test suggests that the horizon and asymptotic sections are cobordant. The cobordism mostly used for this is the oriented cobordism ring \(\Omega_*\), see [19]. However, it seems to us that in the presence of spinors which must be defined at both the horizon and the asymptotic region of spacetime, and in particular supersymmetry, the most relevant equivalence is that of spin cobordism ring \(\Omega_*^{\text{spin}}\). \(\Omega_n^{\text{spin}}, n \leq 8,\) has been computed in [35], see also references within. Here we shall use that

\[
\Omega_1^{\text{spin}} = \Omega_2^{\text{spin}} = \mathbb{Z}_2, \quad \Omega_4^{\text{spin}} = \mathbb{Z}, \quad \Omega_5^{\text{spin}} = \Omega_6^{\text{spin}} = \Omega_7^{\text{spin}} = 0, \quad \Omega_8^{\text{spin}} = \mathbb{Z} \oplus \mathbb{Z} \quad (8.1)
\]

Moreover \(\Omega_1^{\text{spin}}\) is generated by the circle with the periodic spin structure, and \(\Omega_3^{\text{spin}}\) is generated by \(\mathbb{P}^2(\mathbb{H})\), ie the space of quaternionic lines in \(\mathbb{H}^3\), and a manifold \(L^8\) satisfying the relation \(4L^8 = K_3 \times K_3\). Clearly heterotic horizons lie in \(\Omega_8^{\text{spin}}\).

These data indicate that all the solutions\(^7\) we have found, including a large class which may arise from del Pezzo surfaces, can be the near horizon geometries of asymptotically flat or asymptotically AdS black holes. For such black holes the asymptotic section is a sphere \(S^8\) and so it represents the trivial class in \(\Omega_8^{\text{spin}}\). This is also the case for most Kaluza-Klein black holes. On the other hand all our explicit horizon sections, apart from \(SU(3)\), are products \(S^3 \times \mathbb{Z}\). Since \(S^3\) represents the trivial class, because \(\Omega_3^{\text{spin}}\) vanishes, one can consider \(D^4 \times \mathbb{Z}\) which has boundary \(S^3 \times \mathbb{Z}\) and so \(S^3 \times \mathbb{Z}\) also represents the trivial class in \(\Omega_8^{\text{spin}}\). Therefore all such horizon sections can be associated with asymptotically flat or AdS black holes. The same argument applies for the examples based on del Pezzo.

\(^7\)All our horizon sections admit metrics with positive scalar curvature and so satisfy the positive Yamabe type condition of [37].
surfaces since for most of them the horizon section is $S = S^1 \times M^7$. Since $\Omega^\text{spin}_7 = 0$, one can again argue that $S^1 \times M^7$ represents the trivial class in $\Omega^\text{spin}_8$.

It may appear that the cobordism equivalence between the horizon and asymptotic sections do not impose much restriction. This is mostly the case but not always. One of the near horizon geometries that arises in heterotic theory has section $K_3 \times K_3$ [20]. As we have mentioned this represents a non-trivial class in $\Omega^\text{spin}_8$. Therefore it is not in the same cobordism class as $S^8$ and so it cannot be the near horizon geometry of an asymptotically flat or AdS black hole. Moreover, it cannot be the near horizon geometry of most Kaluza-Klein black holes which arise from lifting a 4- or 5-dimensional black hole to 10 dimensions. This is because the asymptotic section of such Kaluza-Klein black holes is expected to be either products of spheres or $S \times H$, where again $S$ is a product of spheres and $H$ is a special holonomy manifold, ie $H$ is either $K_3$ or a 6-dimensional Calabi-Yau. However such spaces represent the trivial class in $\Omega^\text{spin}_8$ and so cannot be cobordant to $K_3 \times K_3$.

9 Concluding Remarks

We have constructed explicit examples of near horizon geometries of heterotic supergravity which preserve 4 supersymmetries. Amongst the horizon sections we have found are $SU(3)$ and $S^3 \times S^3 \times T^2$. The near horizon geometry in the $SU(3)$ case is $(SL(2, \mathbb{R}) \times SU(3))/U(1)$ and $SL(2, \mathbb{R})$ is twisted with respect to a $U(1)$ connection. The horizon section $S^3 \times S^3 \times T^2$ gives rise to two different near horizon geometries. One near horizon geometry is $AdS_3 \times S^3 \times S^3 \times S^1$. But there is also the possibility that the near horizon geometry is $(AdS_3 \times S^3 \times S^3)/S^1 \times T^2$. Therefore a horizon section does not determine the near horizon geometry uniquely. To our knowledge, it is the first time that such a possibility has been observed.

We have also demonstrated that a large class of solutions can arise provided that the horizon section is chosen to be a $T^4$ fibration over 4-dimensional a Kähler manifold $X$. The resulting differential system contains 6 equations for 6 unknown functions which include the Monge-Ampère equation and a conformally rescaled Hermitian-Einstein equation. We have shown that the non-linear system has solutions for $X = \mathbb{P}^1 \times \mathbb{P}^1$. We have also given a set of conditions in the cohomology of $X$ which are necessary for the existence of solutions. We have found many solutions of this cohomological system when $X$ is a del Pezzo surface. We have also raised the question of whether for every solution of the cohomological system there is a solution of the differential equations and so a new heterotic horizon.

We have investigated the topology of the horizons we have found by computing their de Rham cohomology. Those horizons that are associated with del Pezzo surfaces exhibit cohomology with intersection matrices which are closely related to exceptional groups. Of course this is inherited from the intersection matrix of del Pezzo surfaces which is the Cartan matrix of exceptional Lie algebras. The cohomological properties of our horizons point to a relation to U-duality invariant brane configurations but we have not been able to make this more precise.

There is not an apriori reason to believe that a near horizon solution is associated with a black hole. However, there are some tests that one can do. One of them is to
argue that the asymptotic section of a black hole is cobordant to the horizon section. In the presence of spinors, this means that both sections are spin cobordant. We have found that all our horizon sections which preserve 4 supersymmetries are in the trivial class and so they may be near horizons of asymptotically flat or AdS black holes.

The plethora of near horizon solutions we have found suggest that there are many exotic supersymmetric black holes in heterotic supergravity. Some of them will be Kaluza-Klein black holes but there is a possibility that some will have a purely 10-dimensional origin. The existence of such black holes will give new insights into string theory and M-theory.

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Appendix A  Topology of horizon sections

To compute the cohomology of the total space of the torus fibration, we shall use the spectral sequences method explained in [38], see also [24]. Suppose that $S = T^2 \times B$, where $B$ is a non-trivial $T^2$ fibration over $X$. $X$ is simply connected. In such a case, the $E_2^{*,*}$ part of the spectral sequence defined in (5.25) is summarized in table 4.

| $\mathbb{R}\langle \theta_1 \rangle$ | $\mathbb{R}\langle \theta_2 \rangle$ | $\mathbb{R}\langle \theta_2 \wedge b_a \rangle$ | $\mathbb{R}\langle \theta_2 \wedge v \rangle$ |
|----------------|----------------|----------------|----------------|
| $\mathbb{R}\langle \theta_2 \rangle$ | $\mathbb{R}\langle \theta_2 \rangle$ | $\mathbb{R}\langle \theta_2 \wedge b_a \rangle$ | $\mathbb{R}\langle \theta_2 \wedge v \rangle$ |
| $\mathbb{R}$ | $\mathbb{R}$ | $\mathbb{R}$ | $\mathbb{R}$ |

Table 4: The entries are the elements of of $E_2^{*,*}$

To compute the action of the $d_2$ differential first observe that $d_2 : \mathbb{R}\langle \theta_2 \rangle \to \mathbb{R}\langle b_a \rangle$ with $d_2(\theta_2) = b_r$ which is clearly 1-1. Next $d_2 : \mathbb{R}\langle \theta_1 \rangle \to \mathbb{R}\langle \theta_2 \wedge b_a \rangle$ with $d_2(\theta_1) = \theta_1 \wedge b_a - \theta_2 \wedge b_a$ which is also 1-1. Moreover $d_2 : \mathbb{R}\langle \theta_1 \wedge b_a \rangle \to \mathbb{R}\langle v \rangle$ with $d_2(\theta_1 \wedge b_a) = b_r \wedge b_a$, where $v$ is the volume class of $X$. Suppose now that there are $\alpha, \beta \in H^2(X, \mathbb{R})$ such that

$$b_1 \wedge \alpha = v, \quad b_2 \wedge \beta = v, \quad b_1 \wedge \beta = b_2 \wedge \alpha = 0.$$ (A.1)

In such a case, $d_2 : \mathbb{R}\langle \theta_2 \wedge b_a \rangle \to \mathbb{R}\langle v \rangle$ is onto. Furthermore, $d_2 : \mathbb{R}\langle \theta_1 \wedge b_a \rangle \to \mathbb{R}\langle \theta_2 \wedge b_a \rangle$ with $d_2(\theta_1 \wedge b_a) = \theta_1 \wedge b_a - \theta_2 \wedge b_a$ is also onto. As a result, $E_3^{*,*}$ is given in table 5 and converges in the cohomology of the bundle space.

Therefore the cohomology of $B$ is given by

$$H^0(B, \mathbb{R}) = H^0(B, \mathbb{R}) = \mathbb{R}, \quad H^1(B, \mathbb{R}) = H^1(B, \mathbb{R}) = \mathbb{R}^{m-2}, \quad H^2(B, \mathbb{R}) = \mathbb{R}^{2m-2}. \quad (A.2)$$

Suppose that $S = S^1 \times Y$, where $Y$ is a non-trivial $T^3$ fibration over $X$. To find the cohomology of $Y$, the $E_2^{*,*}$ part of the the spectral sequence is given in table 6. Moreover observe that $d_2 : \mathbb{R}\langle \theta_2 \rangle \to \mathbb{R}\langle b_a \rangle$ with $d_2(\theta_2) = b_r$ which is clearly 1-1. Next
Table 5: The entries are the elements of of $E_3^{*,*}$

| $\mathbb{R}(\theta_{123})$ | $\mathbb{R}(\theta_{123} \wedge b_a)$ | $\mathbb{R}(\theta_{123} \wedge v)$ |
|-----------------------------|--------------------------------------|-------------------------------------|
| $\mathbb{R}(\theta_{rs})$  | $\mathbb{R}(\theta_{rs} \wedge b_a)$ | $\mathbb{R}(\theta_{rs} \wedge v)$ |
| $\mathbb{R}(\theta_{r})$   | $\mathbb{R}(\theta_{r} \wedge b_a)$  | $\mathbb{R}(\theta_{r} \wedge v)$  |
| $\mathbb{R}$ $0$           | $\mathbb{R}(b_a)$                    | $\mathbb{R}(v)$                    |

Table 6: The entries are the elements of $E_2^{*,*}$

$$d_2 : \mathbb{R}(\theta_{rs}) \rightarrow \mathbb{R}(\theta_r \wedge b_a)$$ with $d_2(\theta_{rs}) = \theta_r \wedge b_s - \theta_s \wedge b_r$ which is also 1-1. Moreover $d_2 : \mathbb{R}(\theta_r \wedge b_a) \rightarrow \mathbb{R}(v)$ with $d_2(\theta_r \wedge b_a) = b_r \wedge b_a$. Suppose now that there are $\alpha_s \in H^2(X, \mathbb{R})$ such that

$$b_r \wedge \alpha_s = \delta_{rs} v . \quad (A.3)$$

In such a case, $d_2 : \mathbb{R}(\theta_r \wedge b_a) \rightarrow \mathbb{R}(v)$ is onto. Furthermore, $d_2 : \mathbb{R}(\theta_{rs} \wedge b_a) \rightarrow \mathbb{R}(\theta_r \wedge v)$ with $d_2(\theta_{rs} \wedge b_a) = \theta_r \wedge b_s \wedge b_a - \theta_s \wedge b_r \wedge b_a$ is also onto. Next $d_2 : \mathbb{R}(\theta_{123}) \rightarrow \mathbb{R}(\theta_{rs} \wedge b_a)$ with $d_2(\theta_{123}) = \theta_{12} \wedge b_3 + \theta_{31} \wedge b_2 + \theta_{23} \wedge b_1$, and $d_2 : \mathbb{R}(\theta_{rs} \wedge b_a) \rightarrow \mathbb{R}(\theta_r \wedge v)$ with $d_2(\theta_{rs} \wedge b_a) = \theta_r \wedge b_s \wedge b_a - \theta_s \wedge b_r \wedge b_a$. The former map is 1-1 and the second is onto. As a result, $E_3^{*,*}$ is given in table 7 and converges in the cohomology of the bundle space.

Table 7: The entries are the elements of of $E_3^{*,*}$

| $0$ $0$ $\mathbb{R}^{m-3}$ $0$ $\mathbb{R}$ |
|-----------------|------------------|------------------|
| $0$ $0$ $\mathbb{R}^{m-4}$ $0$ $0$ |
| $0$ $0$ $\mathbb{R}^{m-4}$ $0$ $0$ |
| $\mathbb{R}$ $0$ $\mathbb{R}^{m-3}$ $0$ $0$ |

In particular, ones finds that

$$H^0(Y, \mathbb{R}) = H^7(Y, \mathbb{R}) = \mathbb{R} , \quad H^2(Y, \mathbb{R}) = H^5(Y, \mathbb{R}) = \mathbb{R}^{m-3} , \quad H^3(Y, \mathbb{R}) = H^4(Y, \mathbb{R}) = \mathbb{R}^{m-4} . \quad (A.4)$$

Suppose that $S$ is a non-trivial $T^4$ fibration over $X$. The $E_2^{*,*}$ of part of the spectral sequence is given in table 8.

A similar analysis to the one we have presented for the two similar cases above gives that the elements of $E_3^{*,*}$ are given in table 9.

In particular, one finds that

$$H^0(S, \mathbb{R}) = H^8(S, \mathbb{R}) = \mathbb{R} , \quad H^2(S, \mathbb{R}) = H^6(S, \mathbb{R}) = \mathbb{R}^{m-4} , \quad H^3(S, \mathbb{R}) = H^5(S, \mathbb{R}) = \mathbb{R}^{4m-7} , \quad H^4(S, \mathbb{R}) = \mathbb{R}^{6m-8} . \quad (A.5)$$
Table 8: The entries are the elements of $E^*_2$

| $\mathbb{R}\langle \theta_{1234} \rangle$ | 0 | $\mathbb{R}\langle \theta_{1234} \wedge b \rangle$ | 0 | $\mathbb{R}\langle \theta_{1234} \wedge v \rangle$ |
| $\mathbb{R}\langle \theta_{rst} \rangle$ | 0 | $\mathbb{R}\langle \theta_{rst} \wedge b \rangle$ | 0 | $\mathbb{R}\langle \theta_{rst} \wedge v \rangle$ |
| $\mathbb{R}\langle \theta_1 \rangle$ | 0 | $\mathbb{R}\langle \theta_1 \wedge b \rangle$ | 0 | $\mathbb{R}\langle \theta_1 \wedge v \rangle$ |
| $\mathbb{R}\langle v \rangle$ | 0 | $\mathbb{R}\langle b \rangle$ | 0 | $\mathbb{R}\langle v \rangle$ |

Table 9: The entries are the elements of $E^*_3$

| 0 | 0 | $\mathbb{R}^{m-4}$ | 0 | $\mathbb{R}$ |
| 0 | 0 | $\mathbb{R}^{4m-7}$ | 0 | 0 |
| 0 | 0 | $\mathbb{R}^{9m-8}$ | 0 | 0 |
| 0 | 0 | $\mathbb{R}^{4m-9}$ | 0 | 0 |
| $\mathbb{R}$ | 0 | $\mathbb{R}^{m-4}$ | 0 | 0 |

Appendix B A generalization of the differential system

The differential (4.2) and cohomological (4.3) systems can be easily generalized to holomorphic $T^{2n}$-fibrations $Y$ over 4-dimensional Kähler manifolds $X$. First consider a conformal scaling of the Kähler form $\kappa$ of $X$ as

$$\omega_X = e^{2\Phi} \kappa , \quad d\kappa = 0 . \quad (B.1)$$

Then introduce connections $h^r$ adapted to the $T^n$ fibration for which the curvature $dh^r$ is (1,1). The metric and 3-form field strength on $Y$ are

$$ds^2 = G_{rs} h^r h^s + e^{2\Phi} ds^2(X)$$
$$H = G_{rs} h^r \wedge dh^s - i_I d e^{2\Phi} \wedge \kappa \quad (B.2)$$

where $G_{rs}$ is a constant fibre metric and $I$ is the complex structure on $X$. The Hermitian form on $Y$ is

$$\omega(Y) = \phi_{rs} h^r \wedge h^s + e^{2\Phi} \kappa \quad (B.3)$$

where $\phi$ is a compatible constant Hermitian form which together with $G$ gives rise to a complex structure on the fibre. Requiring that the connection $\hat{\nabla}$ on $Y$ has holonomy contained in $SU(2) \subset SU(2 + n)$, one finds that

$$dh^r \wedge \kappa = \frac{v^r}{2} e^{2\Phi} \kappa^2$$
$$- i \bar{\partial} \partial \log \det(i\kappa) + 2i \bar{\partial} \partial \Phi + G_{rs} v^r dh^s = 0 ,$$
$$G_{rs} dh^r \wedge dh^s + 2i \bar{\partial} \partial e^{2\Phi} \wedge \kappa = 0 , \quad (B.4)$$

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where \(v^r\) are constants. One can rewrite the differential equations in terms of a non-linear system for \(2n + 2\) functions. The calculation is an adaptation of the one we have done already for heterotic horizons and we shall not repeat it here.

As for heterotic horizons, there is an associated cohomological system given by

\[
\begin{align*}
    c_1(P_r) \wedge \kappa &= \frac{1}{2} \left[ e^{2\Phi} \kappa^2 \right], \\
    c_1(X) + v_s c_1(P^s) &= 0, \\
    G_{rs} c_1(P^r) \wedge c_1(P^s) &= 0,
\end{align*}
\]

(B.5)

where \(c_1(P^r)\) denotes the first Chern class of the \(P^r\) circle bundle, \(c_1(X)\) is the first Chern class of the canonical bundle of \(X\) and \(v_s = G_{rs} v^r\). Since \(c_1(P^r), c_1(X) \in H^2(X, \mathbb{Z})\), it is required that \(G_{rs} v^r c_1(P^s) \in H^2(X, \mathbb{Z})\) imposing restrictions on the fibre metric and \(v\).

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