Time-dependent Perpendicular Transport of Energetic Particles for Different Turbulence Configurations and Parallel Transport Models

J. Lasuik and A. Shalchi

Department of Physics and Astronomy, University of Manitoba, Winnipeg, MB R3T 2N2, Canada; andreasm4@yahoo.com

Received 2017 July 11; revised 2017 August 15; accepted 2017 August 16; published 2017 September 14

Abstract

Recently, a new theory for the transport of energetic particles across a mean magnetic field was presented. Compared to other nonlinear theories the new approach has the advantage that it provides a full time-dependent description of the transport. Furthermore, a diffusion approximation is no longer part of that theory. The purpose of this paper is to combine this new approach with a time-dependent model for parallel transport and different turbulence configurations in order to explore the parameter regimes for which we get ballistic transport, compound subdiffusion, and normal Markovian diffusion.

Key words: diffusion – magnetic fields – turbulence

1. Introduction

The transport of electrically charged particles in a turbulent magnetized plasma is a topic of great interest in modern physics. Particularly in astrophysics, scientists are keen on understanding the motion of cosmic rays and solar energetic particles through the universe (see, e.g., Schlickeiser 2002 and Zank 2014 for reviews). If such particles would only interact with a constant magnetic field, their trajectory would be a perfect helix. However, real particles experience scattering due to the interaction with turbulent electric and magnetic fields. Therefore, one finds different transport processes such as parallel diffusion or stochastic acceleration. In particular, the motion of energetic particles across the mean magnetic field was the subject of numerous theoretical studies (see, e.g., Shalchi 2009 for a review).

A simple analytical description of perpendicular transport is provided by quasi-linear theory (see Jokipii 1966 for the original presentation of this approach) where it is assumed that perpendicular diffusion is caused by particles following magnetic field lines that themselves behave diffusively. Characteristic of this type of transport is that the corresponding perpendicular mean free path does not depend on particle energy nor does it depend on other particle properties. It is entirely controlled by magnetic field parameters. This type of transport is often referred to as the field line random walk (FLRW) limit. More realistic descriptions were developed later. It was shown, for instance, that parallel diffusion suppresses perpendicular transport to a subdiffusive level. Analytical descriptions of this type are usually called compound subdiffusion (see, e.g., Kota & Jokipii 2000; Webb et al. 2006).

Comprehensive numerical studies of perpendicular transport have been performed showing that one can indeed find subdiffusive transport if a turbulence configuration without any transverse structure in considered (see, e.g., Qin et al. 2002a). However, if there is transverse complexity of the turbulence, diffusion is restored (see, e.g., Qin et al. 2002b). Matthaeus et al. (2003) and Qin et al. (2002b), therefore, distinguished between first diffusion (meaning quasi-linear transport) and second diffusion. This process must be described by some type of nonlinear interaction between particles and magnetic fields and it must also be related to the transverse structure of the turbulence. It has to be emphasized that the recovery of diffusion due to collisions was described in the famous work of Rechester & Rosenbluth (1978), but Coulomb collisions, albeit relevant in laboratory plasmas, should not be important in astrophysical scenarios such as the solar wind or the interstellar medium.

Different attempts to describe second diffusion have been presented in the past, such as the pioneering work of Matthaeus et al. (2003). A few years later the so-called unified nonlinear transport (UNLT) theory has been derived (see Shalchi 2010), which contains the field line diffusion theory of Matthaeus et al. (1995), quasi-linear theory, and a Rechester and Rosenbluth type of diffusion as special limits (see Shalchi 2015 for a detailed discussion of this matter). However, the aforementioned theories rely on a diffusion approximation together with a late time limit. Therefore, such theories do not describe the early ballistic motion of the particles nor do they explain the subdiffusive regime and the recovery of diffusion. In Shalchi (2017), a time-dependent version of UNLT theory was developed. Although this description still relies on approximations and assumptions such as Corrsin’s independence hypothesis (see Corrsin 1959), one is now able to describe the transport as a full time-dependent process. Furthermore, this theory provides a simple condition that needs to be satisfied in order to find second diffusion, namely, \( \langle (\Delta x)^2 \rangle \gg 2t_i^2 \). Here, we have used the mean square displacement of possible particle orbits as well as a characteristic length scale of the turbulence in the perpendicular direction. Before this condition is satisfied, perpendicular transport is either ballistic, quasi-linear, or subdiffusive.

The purpose of this paper is to present a detailed analytical description of time-dependent perpendicular transport. We employ a general model for the transport of particles in the parallel direction containing ballistic and diffusive regimes. Furthermore, we employ different analytical models for magnetic turbulence such as the slab model, a noisy slab model, a Gaussian model, the two-dimensional model, as well as a two-component turbulence model. In all cases, we compute the running perpendicular diffusion coefficient as a function of time to explore the different transport regimes.
2. Full Time-dependent Description of Perpendicular Transport

A time-dependent theory for perpendicular transport was developed in Shalchi (2017) based on ideas discussed in Matthaeus et al. (2003) and Shalchi (2010). By using guiding center coordinates instead of particle coordinates, Corrsin’s independence hypothesis, and by assuming that the averages over particles properties can be written as a product of parallel and perpendicular correlation functions, the following equation was derived:

$$\langle V_z(t) V_z(0) \rangle = \frac{1}{B_0^2} \int \frac{d^3k}{(2\pi)^3} P_{xx}(k, t) \xi(k_{||}, t) e^{-\frac{1}{2}(\Delta k^2)^2},$$

(1)

where we used the $x$-component of the guiding center velocity $V_z(t)$, the mean magnetic field $B_0$, the $xx$-component of the magnetic correlation tensor $P_{xx}$ (see below for some examples), and the mean square displacement in the $x$-direction ($\langle \Delta x^2 \rangle$).

Equation (1) is valid for axisymmetric and dynamical turbulence. However, in this paper, we only consider magnetostatic turbulence where by definition $P_{xx}(k, t) = P_{xx}(k)$. Furthermore, Equation (1) is based on the assumption that $\delta B_z \ll B_0$. If one considers isotropic turbulence, for instance, one has to ensure that the turbulent magnetic field is not too strong. To generalize Equation (1) to allow turbulence that is not axisymmetric would be straightforward. In this case, Equation (1) would be replaced by a set of four coupled ordinary differential equations.

In Equation (1), we also used $\xi(k_{||}, t_1 = t_2 = 0)$ with the parallel correlation function

$$\xi(k_{||}, t_1, t_2) = k_{||}^{-2} \left\{ \left( \frac{d}{dt} e^{i(k_{||} t)} \right) \left( \frac{d}{dt} e^{-i(k_{||} t)} \right) \right\} = k_{||}^{-2} \frac{d}{dt_1} \frac{d}{dt_2} (e^{i(k_{||} t_1)} - e^{-i(k_{||} t_2)}).$$

(2)

This function was explored in Shalchi et al. (2011) based on the cosmic-ray Fokker–Planck equation

$$\frac{\partial \xi}{\partial t} + v \mu \frac{\partial \xi}{\partial z} = \frac{\partial}{\partial \mu} \left[ D_{\mu\mu}(\mu) \frac{\partial \xi}{\partial \mu} \right]$$

(3)

with an isotropic pitch-angle scattering coefficient $D_{\mu\mu} = (1 - \mu^2)D$. A detailed discussion of the analytical form of $D_{\mu\mu}$ and the validity of the isotropic regime can be found in Shalchi et al. (2009). The solution of Equation (3) describes the parallel motion of energetic particles while they experience pitch-angle scattering. As explained in Shalchi (2006), the solution describes the parallel motion as a ballistic motion at early times and then the solutions become diffusive. This is exactly what one observes in test-particle simulations performed in the past (see, e.g., Qin et al. 2002a, 2002b). A general solution of the two-dimensional Fokker–Planck equation is difficult to find. Within a two-dimensional subspace approximation, it was derived in Shalchi et al. (2011) that

$$\xi(k_{||}, t = 0) = \frac{v^2}{3}$$

(4)

thus, Equation (1) becomes

$$\langle V_z(t) V_z(0) \rangle = \frac{v^2}{3B_0^2} \int d^3k P_{xx}(k) = \frac{v^2}{3B_0^2} \frac{8B_0^2}{\omega_{\perp}^2 - \omega_+ e^{\omega_- t} - \omega_- e^{\omega_+ t}}.$$
where we have also used $\langle (\Delta x)^2 \rangle_{x=0} = 0$. After integrating this formula over time, and using Equation (7), we obtain for the running diffusion coefficient

$$
d_{\perp}(t) = \frac{v^2}{3} \frac{\delta B_0^2}{B_0^2} t.
$$

If we integrate again, we find

$$
\langle (\Delta x)^2 \rangle = \frac{v^2}{3} \frac{\delta B_0^4}{B_0^4} t^2.
$$

The motion obtained here corresponds to a ballistic motion where particles move unperturbed in the parallel direction while they follow ballistic magnetic field lines. We expect that for a given turbulence model, the running diffusion coefficient can be approximated by Equation (11) if early enough times are considered.

### 3.2. Slab Turbulence

For slab turbulence, corresponding to turbulence without any transverse structure, we have by definition

$$
P_{nm}(k) = g_{nm}(k) \delta(k_j),
$$

for $n, m = x, y$. Because of the solenoidal constraint, all other components of this tensor are zero. Due to the Dirac delta function, Equation (1) becomes

$$
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{8\pi}{B_0^2} \int_0^\infty dk g_{nm}^\perp(k) \xi(k_j, t).
$$

Here, we have used the one-dimensional spectrum of the slab modes $g_{nm}^\perp(k)$. If we combine this form with the TGK formula (7) and model (4), we find after straightforward algebra

$$
d_{\perp}(t) = \frac{4v^2}{3B_0^4} \int_0^\infty dk g_{nm}^\perp(k) \frac{1}{\omega_- - \omega_+} (e^{\omega_- t} - e^{\omega_+ t}),
$$

which was originally derived in Shalchi et al. (2011). In that paper, a more detailed discussion can be found as well as a visualization of the running perpendicular diffusion coefficient as a function of time (see Figure 3 of Shalchi et al. 2011). For $t \to \infty$, we only find a contribution to the $k_{||}$-integral of the smallest possible values of $\omega_- + \omega_-$, meaning $\omega_- = 0$ and $\omega_+ = -\kappa_\perp k_||^2$. Furthermore, we derive from Equation (5)

$$
\omega_- = 2\sqrt{D^2 - (vk_||)^2}/3 \approx 2D = v/\lambda_{\perp}.
$$

Hereby, Equation (15) becomes

$$
d_{\perp}(t) = \frac{4v\kappa_\perp}{B_0^2} \int_0^\infty dk g_{nm}^\perp(k) e^{-\kappa_\perp k_||^2 t},
$$

where we have also used $\lambda_{\perp} = 3\kappa_\perp/v$. This formula is in agreement with the result originally obtained in Shalchi & Döring (2007). As discussed there, the formula describes correctly compound subdiffusion as usually obtained for slab turbulence (see, e.g., Kót & Jokipii 2000; Webb et al. 2006). In the Appendix, we briefly show how quasi-linear theory can be recovered from Equation (14) as well.

### 3.3. Diffusion Approximation

Particularly in nonlinear treatments of particle transport, it is often assumed that perpendicular transport is diffusive for all times. In our notation this means that we set

$$
\langle (\Delta x)^2 \rangle = 2\kappa_\perp t \forall t
$$

in Equation (1). We like to emphasize that this is only an approximation. In reality one expects a ballistic motion and thereafter there could be a subdiffusive regime (see, e.g., Sections 3.2 and 4). Eventually diffusion is recovered if there is transverse structure (see, e.g., Shalchi 2017 and Section 4). The purpose of this paper is to explore the different transport regimes. Here, we employ the diffusion approximation only to restore previous equations for the perpendicular diffusion coefficient. With approximation (18) we derive from Equation (1)

$$
\langle V_t(t) V_t(0) \rangle = \frac{v^2}{3B_0^2} \int d^3k P_{xx}(k) \frac{1}{\omega_- - \omega_+} \times [\omega_- e^{\omega_- t} - \omega_+ e^{\omega_+ t}] e^{-\kappa_\perp k_||^2 t}.
$$

If we integrate this equation over time, and after employing the TGK formula (7) for $t \to \infty$, we obtain

$$
\kappa_\perp = \frac{v^2}{3B_0^2} \int d^3k P_{xx}(k) \times \frac{\kappa_\perp k_||^2}{(\kappa_\perp k_||^2)^2 - (\omega_- + \omega_+)\kappa_\perp k_||^2 + \omega_- \omega_+}.
$$

Now, we use Equation (5) to derive $\omega_- + \omega_+ = -2D = -v/\lambda_{\perp}$ and $\omega_- \omega_+ = (vk_||)^2/3$ to find

$$
\kappa_\perp = \frac{v^2}{3B_0^2} \int d^3k \frac{P_{xx}(k)}{\kappa_\perp k_||^2 + v/\lambda_{\perp} + F(k_|| k_\perp)},
$$

where we have used the function

$$
F(k_|| k_\perp) = (vk_||)^2/(3\kappa_\perp k_\perp^2).
$$

This result agrees with the integral equation provided by UNLT theory (see Shalchi 2010), apart from a factor of 4/3 in front of the first term in the denominator of Equation (21). The reason for the small discrepancy is that Equation (4) itself is an approximation. One can easily repeat the calculations performed above for the more general case of dynamical turbulence. In this case, one would obtain the formula originally derived in Shalchi (2011).

### 4. Time-dependent Transport in Turbulence with Transverse Complexity

In this section, we study particle transport in different turbulence models with transverse structure. This is important because this effect is essential in order to restore diffusion. As examples we consider the noisy slab model, a Gaussian correlation model, pure two-dimensional turbulence, and a two-component turbulence model consisting of slab and two-dimensional modes.
4.1. Noisy Slab Turbulence

As a first example for turbulence with transverse structure, we consider the noisy slab model originally proposed in Shalchi (2015) as a model with minimal transverse complexity. Within this model the magnetic correlation tensor has the components

$$ P_{nm}(k) = \frac{2k_n}{k_l} g^{slab}(k_l) \Theta(1 - k_n k_m) \left( \delta_{nm} \right) \frac{f_l}{k^2_l}, \quad (23) $$

where we have used the Heaviside step function $\Theta(x)$ and the perpendicular correlation scale of the turbulence $k_l$. This form can be understood as broadened slab turbulence. This model can be recovered in the limit $k_l \rightarrow \infty$.

For the noisy slab model Equation (1) becomes

$$ \langle V_s(t) V_s(0) \rangle = \frac{4\pi v^2 k_l}{3B_0^2} \int_0^\infty dk_l g^{slab}(k_l) $$

$$ \times \int_0^\infty d^2 \langle \Delta x \rangle \frac{1}{\omega_+ - \omega_-} [\omega_+ e^{i\omega t} - \omega_- e^{i\omega t}] $$

$$ \times \int_{k_l}^{\infty} dk_l e^{-\frac{1}{2}(\Delta x)^2 k_l^2}. \quad (24) $$

The perpendicular wavenumber integral therein can be expressed by an error function, and we derive

$$ \frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle $$

$$ = \frac{8\pi v^2 k_l}{3B_0^2} \sqrt{\frac{\pi}{2}} \text{Erf} \left( \sqrt{\frac{(\Delta x)^2}{2k^2_l}} \right) $$

$$ \times \int_0^\infty dk_l g^{slab}(k_l) \frac{1}{\omega_+ - \omega_-} [\omega_+ e^{i\omega t} - \omega_- e^{i\omega t}]. \quad (25) $$

For a numerical evaluation of Equation (25), it is convenient to employ the integral transformation $x = \xi_k k_l$ and to use the Kubo number $K = (\xi k^2 c^2)(\ell k^2 B_0)$, the dimensionless time $\tau = k_l t $, as well as $\sigma = ((\Delta x)^2)/k^2_l$.

For the spectrum $g^{slab}(k_l)$, we use the Bieber et al. (1994) model

$$ g^{slab}(k_l) = \frac{1}{2\pi} C(s) \delta B^2 k_l \left[ (\xi k^2) \right]^{-1/2}. \quad (26) $$

Here, we have used the normalization function

$$ C(s) = \frac{\Gamma(\frac{s}{2})}{2\sqrt{\pi} \Gamma(\frac{s-1}{2})} \quad (27) $$

with the inertial range spectral index $s$ and gamma functions. For the spectral index in the inertial range we use $s = 5/3$ throughout the whole paper as motivated by the famous work of Kolmogorov (1941). The parameter $\ell_0$ is the bendover scale in the parallel direction. After the integral transformation

**Figure 1.** Running diffusion ratio $D_r = (\ell_0^2 t_{ball})/(\ell_0^2 k_s)$ vs. time $\tau = k_l^2 t/k^2_0$ for the noisy slab model as obtained by solving Equation (28) numerically for a Kubo number of $K = 0.2$. We have shown the results obtained for different values of the parallel mean free path, namely, $\lambda_0/\ell_0 = 0.01$ (dotted line), $\lambda_0/\ell_0 = 0.1$ (dashed–dotted line), $\lambda_0/\ell_0 = 1$ (dashed line), as well as $\lambda_0/\ell_0 = 10$ (solid line). Note that the dotted, dashed–dotted, and dashed lines are in coincidence. The dot represents the result obtained by employing diffusive UNLT theory.

**Figure 2.** As in Figure 1, but here we have used $K = 0.7$ for the Kubo number. Note that the dotted and dashed–dotted lines are in coincidence.

$z = k_l \xi_0$. Equation (25) can be written as

$$ \frac{d^2}{dt^2} \sigma = 12 - \pi C(s) K^2 k_l^2 \text{Erf} \left( \frac{\sigma}{2} \right) $$

$$ \times \int_0^\infty dz \left( 1 + z^2 \right)^{-1/2} $$

$$ \times \frac{1}{\Omega^+ - \Omega^-} \left[ \Omega^+ e^{i\omega,\tau} - \Omega^- e^{i\omega,-\tau} \right]. \quad (28) $$

where we have also used

$$ \Omega_{\pm} = \frac{\ell_0^2}{\xi_{||} \omega_{||}} \pm \frac{\ell_0^2}{\xi_{\perp} \omega_{\perp}} \pm \frac{\ell_0^2}{\xi_{\perp} \omega_{\perp}} $$

$$ \times \frac{1}{\sqrt{\lambda^2 - \frac{4}{3}}} \quad (29) $$

Differential Equation (28) can be solved numerically. The corresponding running diffusion ratio $D_r = (\ell_0^2 t_{ball})/(\ell_0^2 k_s)$ is shown in Figures 1 and 2 for two different Kubo numbers. For the initial conditions we have set $\sigma(0) = 0$ and $(d\sigma)/(d\tau)(0) = 0$ corresponding to a ballistic motion. This is

---

\^ We would like to point out that there is a typo in Shalchi (2017). In that paper, the definition $D_r = (\ell_0^2 t_{ball})/(\ell_0^2 k_s)$ was used. The way parameter $D_r$ is defined in the current paper is correct.
used for all computations presented in this paper. In both considered cases, we find a subdiffusive motion directly after the ballistic regime. For a small Kubo number the subdiffusive regime persists for a long time, whereas for an intermediate Kubo number diffusion is restored earlier.

4.2. The Gaussian Correlation Model

In this subsection, we employ a Gaussian correlation model that is often used as an example (see, e.g., Neuer & Spatschek 2006). In this case, the components of the magnetic correlation tensor are given by

\[ P_{nm}(k) = \frac{\ell_1^2 \delta \ell_2^2}{2\pi^{3/2}} k_{1\perp} e^{-\frac{1}{2}(k_1^2) - \frac{1}{2}(k_2^2)} \left( \delta_{nm} - \frac{k_n k_m}{k_{1\perp}^2} \right) \]  

(30)

The parameters used in this model are the same as used above. If this model is combined with Equation (3), we derive

\[ \frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{2\ell_1^4 \delta \ell_2^2}{2\pi} \int_0^\infty dk_k \xi(k_\parallel, t) e^{-\frac{1}{2}(k_1^2)} \times \int_0^\infty dk_k k_\perp^3 e^{-\frac{1}{2}(k_1^2 + (\Delta x)^2)} k_\perp^2. \]  

(31)

The perpendicular wavenumber integral can be solved by

\[ \int_0^\infty dk_k k_\perp^3 e^{-\frac{1}{2}(k_1^2 + (\Delta x)^2)} k_\perp^2 = 2[\ell_1^2 + \langle (\Delta x)^2 \rangle]^{-2}. \]  

(32)

The corresponding Kubo number of the Gaussian model as obtained by solving Equation (34) numerically for a Kubo number of \( K = 0.2 \). We have shown the results obtained for different values of the parallel mean free path, namely, \( \lambda_1/\ell_1 = 0.01 \) (dotted line), \( \lambda_1/\ell_1 = 0.1 \) (dashed–dotted line), \( \lambda_1/\ell_1 = 1 \) (dashed line), as well as \( \lambda_1/\ell_1 = 10 \) (solid line). Note that the dotted and dashed–dotted lines are in coincidence. The dots represent the corresponding solutions of UNLT theory within the diffusion approximation.

4.3. Two-dimensional Turbulence

Here, we employ the so-called two-dimensional model where we have by definition

\[ P_{nm}^{2D}(k) = g^{2D}(k_\perp) \delta(k_\parallel) \left( \delta_{nm} - \frac{k_n k_m}{k_{1\perp}^2} \right) \]  

(35)

\[ g^{2D}(k_\perp) = \frac{2D(s, q)}{\pi} \delta \ell_2^2 \frac{(k_\perp \ell_1)^q}{[1 + (k_\perp \ell_1)^2]^{(s+q)/2}}. \]  

(36)

This spectrum contains a characteristic scale \( \ell_1 \) denoting the turnover from the energy range to the inertial range. In the inertial range, the spectrum scales like \( k_\perp^{-q} \), whereas in the energy range it scales like \( k_\perp^q \). The energy range spectral index \( q \) was discussed in detail in Matthaeus et al. (2007). In Equation (36), we have used the normalization function

\[ D(s, q) = \frac{\Gamma \left( \frac{s+q}{2} \right)}{2\Gamma \left( \frac{s-1}{2} \right) \Gamma \left( \frac{q+1}{2} \right)}. \]  

(37)
agreement with the conditions discussed in Matthaeus et al. (2007). The result is visualized in Figure 5. In this case, we find that diffusion is restored directly after the ballistic regime. A subdiffusive regime cannot be observed. We made some calculations for other values of the parameter $q$, but no qualitative difference was found.

### 4.4. Two-component Turbulence

Above, we have considered the slab model and the two-dimensional model as examples. It is often assumed that turbulence in the solar wind can be approximated by a two-component model in which we consider a superposition of slab and two-dimensional modes (see, e.g., Matthaeus et al. 1990, 1996; Zank & Matthaeus 1993; Oughton et al. 1994; Bieber et al. 1996; Dasso et al. 2005; Shaikh & Zank 2007; Hunana & Zank 2010 and Zank et al. 2017).

In this case, time-dependent UNLT provides the following differential equation:

\[
\frac{d^2}{d\tau^2} \langle (\Delta x)^2 \rangle = \frac{8\pi n^2}{3B_0^2} \int_0^\infty dk_i g^{2D}(k_i) \times e^{-\zeta(k_i)\tau} \langle (\Delta x)^2 \rangle_{k_i}^2,
\]

where we have combined Equations (4), (14), and (40). For the two spectra we employ Equations (26) and (36), respectively.

Using again the Kubo number $K = (\ell_0^2\delta B_s)/(\ell_0^2B_0)$, the dimensionless time $\tau = \kappa_i t/\ell_1^2$, $\sigma = \langle (\Delta x)^2 \rangle/\ell_1^2$, as well as the integral transformations $y = k_i \ell_1$ and $z = k_i \ell_1$ yields

\[
\frac{d^2}{d\tau^2} \sigma = 24\frac{\ell_0^2}{\lambda_s^2} K^2 \left( C(s) \frac{\delta B_{s\text{lab}}}{\delta B^2} \int_0^\infty dy \frac{y^q}{[1 + y^{2(q+q)/2}]^{5/2}} \times \frac{1}{(\Omega_+ - \Omega_-)} (\Omega_+ e^{\Omega_+\tau} - \Omega_- e^{\Omega_-\tau}) + D(s, q) \frac{\delta B_{s\text{lab}}}{\delta B^2} \int_0^\infty dz \frac{z^q}{[1 + z^{2(q+q)/2}]^{5/2}} \times e^{-3\xi^2/\lambda_s^2 - z^2/\lambda_s^2} \right),
\]

where we have used the total turbulent magnetic field $\delta B^2 = \delta B_{s\text{lab}}^2 + \delta B_{2D}^2 = 2\delta B_0^2$. For the energy range spectral index in the two-dimensional spectrum we again set $q = 3$. We would like to point out that the Kubo number is used here for convenience only and to make it easier to compare the results obtained for two-component turbulence with the other results.

Strictly speaking, the two-component model is a composition of models with Kubo numbers $K = 0$ and $K = \infty$.

In Figures 6 and 7, we visualize the numerical solution of Equation (44) for the case of an intermediate Kubo number of $K = 0.7$. Furthermore, we have considered two values of the slab fraction, namely, $\delta B_{s\text{lab}}/\delta B_0 = 0.2$ as originally suggested by Bieber et al. (1994, 1996), as well as the balanced case of $\delta B_{s\text{lab}}/\delta B_0 = 0.5$. In Figure 8, we show the results for a large Kubo number of $K = 7$ and a slab fraction of $\delta B_{s\text{lab}}/\delta B_0 = 0.2$. The case is relevant because it corresponds to a scale ratio of $\ell_1 = 0.1\ell_0$ used in previous work (see, e.g., Matthaeus et al. 2003).
The Astrophysical Journal, 847:9 (8pp), 2017 September 20

Lasuik & Shalchi

5. Summary and Conclusion

It is crucial in astrophysics, space science, and plasma physics to understand the motion of energetic particles across a mean magnetic field. Previous analytical theories were based on the diffusion approximation together with a late time limit. Therefore, such theories only provide formulas for the perpendicular diffusion coefficient or mean free path. Recently, a time-dependent version of the UNLT theory was presented in Shalchi (2017). This theory is no longer based on the diffusion approximation and allows one to describe the transport for early times as well.

In this paper, we combined the aforementioned theory with a more general parallel transport model and different turbulence models. We considered transport in slab, noisy slab, Gaussian, two-dimensional, and two-component turbulence. In all cases, we computed the running perpendicular diffusion coefficient as a function of time in order to explore ballistic, subdiffusive, and diffusive regimes.

For slab and noisy slab turbulence we find a subdiffusive regime directly after the ballistic regime. If transverse complexity is present, diffusion is restored as soon as the condition \((\Delta x)^2 \gg 2\sigma_0^2\) is satisfied. For all considered turbulence configurations we find a similar form of the running diffusion coefficient. In some cases, we find normal diffusion directly after the ballistic regime. This is in particular the case for large Kubo numbers and long parallel mean free paths. In some other cases, the subdiffusive regime persists for a very long time before diffusion is restored. All of our results are compared with the results obtained by using standard UNLT theory involving a diffusion approximation. We find that the diffusion approximation works well in the late time regime, but a small discrepancy can be found in some cases. This discrepancy could be relevant if analytical results are compared directly with test-particle simulations and the diffusion approximation itself can contribute to the mysterious factor \(a^2\) often used in nonlinear treatments of perpendicular diffusion (see, e.g., Matthaeus et al. 2003; Shalchi 2010). It needs to be emphasized that time-dependent UNLT theory also contains the effect of the implicit contribution of slab modes as described in Shalchi (2016).

In the past, authors discussed the possibility of nondiffusive transport in the literature (see, e.g., Zimbardo et al. 2006, 2012; Pommois et al. 2007; Shalchi & Kourakis 2007). The theory proposed in Shalchi (2017) and used in this paper allows us, in principle, to describe sub- and superdiffusive transport. Indeed, we found several cases where the transport is subdiffusive for a long time. However, in all considered cases, Markovian diffusion is restored in the late time limit if there is transverse complexity. If there is indeed nondiffusive transport in the late time limit, it must either be for an extreme turbulence model or a nondiffusive parallel transport model. Such cases could be explored in future work.

The theory used in this paper and originally developed in Shalchi (2017) is the most advanced analytical theory for perpendicular transport developed so far. It is able to describe perpendicular transport for arbitrary time and transport.

In all three cases, we observe that for a long parallel mean free path the diffusive regime comes directly after the ballistic regime. For shorter parallel mean free paths we find a ballistic regime, then subdiffusion, and at later times diffusion is recovered.

It is important to note here that the first contribution in Equation (44) corresponds to the usual slab result meaning that it behaves subdiffusively. However, the second term contains the mean square displacement \(\sigma\). Therefore, we expect an implicit contribution of the slab modes. This effect was already studied in Shalchi (2016). In that paper, however, diffusion theory was extended by a subdiffusive slab contribution. The equation used here is more general because the parameter \(\sigma\) is calculated without any assumption concerning diffusivity.

In Figure 6, but here we used \(\delta B_{\text{lab}}^2/\delta B^2 = 0.50\) for the slab fraction.

In Figure 8, but here we used \(K = 7\) for the Kubo number.

Figure 6. Running diffusion ratio \(D_t = (\langle x^2 \rangle_d) / (\langle x^2 \rangle_{K0})\) vs. time \(\tau = \kappa t / l_t^2\) for slab/2D composite turbulence as obtained by solving Equation (44) numerically for a Kubo number of \(K = 0.7\). In this plot, the slab fraction is \(\delta B_{\text{lab}}^2/\delta B^2 = 0.20\). We have shown the results obtained for different values of the parallel mean free path, namely, \(\lambda_p/l_t = 0.01\) (dotted line), \(\lambda_p/l_t = 0.1\) (dotted line), \(\lambda_p/l_t = 1\) (dashed line), as well as \(\lambda_p/l_t = 10\) (solid line). The dots represent the results obtained by employing diffusive UNLT theory.

Figure 7. As in Figure 6, but here we used \(\delta B_{\text{lab}}^2/\delta B^2 = 0.50\) for the slab fraction.

Figure 8. As in Figure 6, but here we used \(K = 7\) for the Kubo number.
behavior, including ballistic motion and compound subdiffusion, but it is also a theory that is no longer based on the diffusion approximation. The theory is still tractable because the perpendicular diffusion coefficient can easily be computed by solving an ordinary differential equation numerically (see, e.g., Equation (1)). One would, therefore, expect that there will be a variety of applications in astrophysics and space science.

Support by the Natural Sciences and Engineering Research Council (NSERC) of Canada is acknowledged.

Appendix
Quasi-linear Perpendicular Diffusion

The question arises how quasi-linear theory can be recovered from the time-dependent approach used in this paper. We assume slab turbulence as it was originally done in the work of Jokipii (1966). Therefore, we start our investigations with Equation (14). In order to obtain the quasi-linear limit, we have to consider the formal limit \( \lambda_i \to \infty \) and Equation (4) becomes

\[
\xi(k_i, t) = \frac{v^2}{3} \cos \left( \frac{1}{\sqrt{3}}vk_i t \right). \tag{45}
\]

Using this in Equation (14) yields

\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{8\pi v^2}{3 B_0} \int_0^\infty dk_i g_{\text{slab}}(k_i) \cos \left( \frac{1}{\sqrt{3}}vk_i t \right). \tag{46}
\]

After integrating this formula over time and employing Equation (6), we derive

\[
\kappa_\perp = \frac{4\pi v^2}{3 B_0} \int_0^\infty dk_i g_{\text{slab}}(k_i) \int_0^\infty dt \cos \left( \frac{1}{\sqrt{3}}vk_i t \right). \tag{47}
\]

With the relations (see, e.g., Zwillinger 2012)

\[
\int_0^\infty dt \cos (\alpha t) = \pi \delta(\alpha) \tag{48}
\]

and

\[
\delta(\alpha \alpha) = \frac{1}{|\alpha|} \delta(\alpha), \tag{49}
\]

we finally obtain for the perpendicular mean free path

\[
\lambda_\perp = \frac{3}{v} \kappa_\perp = \frac{2\sqrt{3} \pi^2}{B_0} \quad (g_{\text{slab}}(k_i) = 0), \tag{50}
\]

which, apart from a factor \( 2/\sqrt{3} \approx 1.15 \), agrees with the well-known quasi-linear formula (see, e.g., Shalchi 2005). The reason for this small disagreement is the fact that Equation (4) itself is an approximation.

References

Bieber, J. W., Matthaeus, W. H., Smith, C. W., et al. 1994, ApJ, 420, 294
Bieber, J. W., Wanner, W., & Matthaeus, W. H. 1996, JGR, 101, 2511
Corson, S. 1959, in Atmospheric Diffusion and Air Pollution, Advances in Geophysics, Vol. 6, ed. F. Frenkiel & P. Sheppard (New York: Academic), 161
Dasso, S., Milano, L., Matthaeus, W. H., & Smith, C. 2005, ApJ, 635, L181
Gradsteyn, I. S., & Ryzhik, I. M. 2000, Table of Integrals, Series, and Products (New York: Academic Press)
Green, M. S. 1951, JChPh, 19, 1036
Hunana, P., & Zank, G. P. 2010, ApJ, 718, 148
Jokipii, J. R. 1966, ApJ, 146, 480
Kolmogorov, A. N. 1941, DoSSR, 30, 301
Kota, J., & Jokipii, J. R. 2000, ApJ, 531, 1067
Kubo, R. 1957, JPSJ, 12, 570
Matthaeus, M. W., Gray, P. C., Pontius, D. H., Jr., & Bieber, J. W. 1995, PhRvL, 75, 2136
Matthaeus, W. H., Bieber, J. W., Ruffolo, D., Chuychai, P., & Minnie, J. 2007, ApJ, 667, 956
Matthaeus, W. H., Ghosh, S., Oughton, S., & Roberts, D. 1996, JGR, 101, 7619
Matthaeus, W. H., Goldstein, M. L., & Aaron, R. D. 1990, JGR, 95, 20673
Matthaeus, W. H., Qin, G., Bieber, J. W., & Zank, G. P. 2003, ApJL, 590, L53
Neuer, M., & Spatschek, K. H. 2006, PhRvE, 73, 26404
Oughton, S., Priest, E., & Matthaeus, W. H. 1994, JFM, 280, 95
Pommois, P., Zimbardo, G., & Veltri, P. 2007, PhPL, 14, 012311
Qin, G., Matthaeus, W. H., & Bieber, J. W. 2002a, GeoRL, 29, 1048
Qin, G., Matthaeus, W. H., & Bieber, J. W. 2002b, ApJL, 578, L117
Rechester, A. B., & Rosenbluth, M. N. 1978, PhRvL, 40, 38
Schlickeiser, R. 2002, Cosmic Ray Astrophysics (Berlin: Springer)
Shalchi, D., & Zank, G. P. 2007, ApJL, 656, L17
Shalchi, A. 2005, JGR, 110, A09103
Shalchi, A. 2006, A&A, 448, 809
Shalchi, A. 2009, Nonlinear Cosmic Ray Diffusion Theories, Vol. 362 (Berlin: Springer)
Shalchi, A. 2010, ApJL, 720, L127
Shalchi, A. 2011, PPCF, 53, 074010
Shalchi, A. 2015, PhPL, 22, 010704
Shalchi, A. 2016, ApJ, 830, 130
Shalchi, A. 2017, PhPi, 24, 050702
Shalchi, A., & Döring, H. 2007, JPhG, 34, 859
Shalchi, A., & Kourakis, I. 2007, A&A, 470, 405
Shalchi, A., Skoda, T., Tautz, R. C., & Schlickeiser, R. 2009, A&A, 507, 589
Shalchi, A., Tautz, R. C., & Rempel, T. J. 2011, PPCF, 53, 105016
Shalchi, A., & Weinhorst, B. 2009, AdSpR, 43, 1429
Taylor, G. I. 1922, Proceedings of the London Mathematical Society, 20, 196
Webb, G. M., Zank, G. P., Kaghashvili, E. K., & le Roux, J. A. 2006, ApJ, 651, 211
Zank, G. P. 2014, Transport Processes in Space Physics and Astrophysics, Vol. 877 (New York: Springer)
Zank, G. P., Adhikari, L., Hunana, P., et al. 2017, ApJ, 835, 147
Zank, G. P., & Matthaeus, W. H. 1993, PhFIA, 5, 257
Zimbardo, G., Perri, S., Pommois, P., & Veltri, P. 2012, AdSpR, 49, 1633
Zimbardo, G., Pommois, P., & Veltri, P. 2006, ApJL, 639, L91
Zwillinger, D. 2012, Standard Mathematical Tables and Formulae (Boca Raton, FL: CRC Press)