Introduction

A 2-category is a generalization of an ordinary category in which there are morphisms between morphisms. More generally, a strict $\infty$-category is an object in which there are notions of n-morphisms between n-1-morphisms for all integers $n$. In applications, we often consider (weak) $(\infty, n)$-categories, in which associativity and identity hold only up to isomorphism at the next level, and the morphisms below level $n+1$ are invertible. This is because $\infty$-categories can be studied using the techniques of homotopy theory. For instance, Kan complexes (i.e. spaces) are models for $\infty$-groupoids (higher categories where all morphisms are invertible up to homotopy).

Traditionally, a stack is defined to be a sheaf of groupoids on a small Grothendieck site $\mathcal{C}$ in which we can glue objects together along compatible families of isomorphisms. That is, a sheaf of groupoids which satisfies the effective descent condition (see [5]). A higher stack a presheaf of $(\infty, n)$-categories which satisfies glueing conditions involving higher morphisms.

A systematic theory of higher stacks with values in $\infty$-groupoids was developed by Jardine, which was useful in algebraic K-theory and cohomology theory (see [18]). Lurie developed the theory of $(\infty, 1)$-stacks (i.e. stacks valued in $\infty$-groupoids) in [13] internally to a quasi-category. Lurie’s approach has the advantage of being more conceptual and general, whereas Jardine’s approach is closer to the classical geometric language of the Grothendieck school.

Generalizing Jardine’s work in a different direction, Hirschowitz and Simpson developed a theory of $(\infty, n)$-stacks (stacks with values in $(\infty, n - 1)$-categories) for all $n \in \mathbb{N}$ in [7]. They used the iterated Segal construction to construct the homotopy theory of $(\infty, n)$-categories.
The main example of higher stacks that they produce are \((\infty, 2)\)-stacks of the form
\[ S_{Seg}L(M), \]
where \(M\) is a presheaf of model categories, \(S_{Seg}\) is the sectionwise fibrant replacement in the model structure for Segal precategories and \(L\) is simplicial localization in the sense of [4]. Hirschowitz and Simpson give a sufficient condition for \(S_{Seg}LM\) to be a higher stack ([7, Theorem 19.4]). Intuitively, this can be interpreted as a statement about how objects glue together along weak equivalences, since the object \(S_{Seg}LM\) acts as a higher categorical approximation to \(M\).

However, the object \(S_{Seg}L(M)\) is difficult to work with, and consequently the proof of the descent result is complicated. The purpose of this paper is to give a simpler account of the theory of \((\infty, 2)\)-stacks in the sense of [7], using quasi-category theory and the local Joyal model structure of [15]. These theories of higher stacks are equivalent because quasi-categories and Segal categories are both models of higher category theory. The theory based on quasi-categories is simpler because many quasi-categorical constructions are much more tractable than their Segal category analogues (compare the homotopy coherent nerve to \(S_{Seg}L\), for instance).

The paper is organized as follows. In the first section, we explain the background on the Joyal model structure, and various quasi-categorical results we need for the rest of the paper. The most important are a description of the homotopy coherent nerve functor \(B\) and its homotopy-theoretic significance, as well as a characterization of Joyal equivalences as DK-equivalences. The latter is essential for proof of the main descent theorem in the second section.

In the second section, we first show that local Joyal equivalences induce local weak equivalences of mapping space presheaves. This is an important auxiliary result needed to prove later descent theorem. Once this result is in hand, we apply the general argument of [7, Theorem 10.2] to produce its quasi-categorical analogue (2.15). We then easily deduce sufficient conditions for \(B(M^o)\) to satisfy descent (2.18 and 2.19), where \(M\) is a presheaf of simplicial model categories, and for a simplicial model category \(M\), \(M^o\) denotes the full subsimplicial category (in each simplicial degree), consisting of cofibrant-fibrant objects.

In the third section, we will apply 2.19 to construct the \((\infty, 2)\)-stack of simplicial \(R\)-module spectra, where \(R\) is a sheaf of rings on a site. The
model structure on simplicial $\mathcal{R}$-module spectra is Quillen equivalent to a model structure on unbounded chain complexes in which the weak equivalences are quasi-isomorphisms. That is, we have constructed a higher stack of unbounded complexes, generalizing the main example of a higher stack found in [7].

In the fourth and final section, we relate the more conceptual definition of descent in terms of homotopy to the intuition that higher stacks in Simpson’s sense should be presheaves of categories in which we can glue objects together along weak equivalences. The formal statement is [11,10].

Notational and Terminological Conventions

Given a category $C$, we write $BC$ for the nerve of a category. We write $\text{Iso}(C)$ for the subcategory of $C$ whose morphisms are isomorphisms of $C$. Given a category $C$ and $B \in \text{Ob}(C)$, we write $C/B$ for the usual slice category over $B$. Let $\text{sSet}, \text{sCat}$ denote the categories of simplicial sets and simplicial categories, respectively. Let $\text{Cat}$ denote the category of small categories.

Given 2 simplicial sets $X,Y$, write $X^Y$ for the internal hom in simplicial sets. We call a map of simplicial sets which has the right lifting property with respect to the horns $\Lambda^n_i \subseteq \Delta^n, 0 < i \leq n$ a right fibration. A map which has the left lifting property with respect to right fibrations is called a right anodyne map. A map which has the right lifting property with respect to the horns $\Lambda^n_i \subseteq \Delta^n, 0 < i < n$ is called an inner fibration. A map which has the left lifting property with respect to inner horns is called inner anodyne.

The path category functor $P : \text{sSet} \rightarrow \text{Cat}$ is the left adjoint to the nerve functor. We write $\pi(X)$ for the fundamental groupoid of a quasi-category $X$, which is defined to be the groupoid completion of $P(X)$.

We call a simplicially enriched category a simplicial category. Given a simplicial category $C$, and objects $x,y$ of $C$, we write $\text{hom}_C(x,y)$ for the simplicial set of morphisms between $x$ and $y$. Given a simplicial category $C$, we write $\pi_0(C)$ for its fundamental category. That is, the category whose objects are objects of $C$, such that for each $x,y \in C$, $\text{hom}_{\pi_0(C)}(x,y) = \pi_0(\text{hom}_C)(x,y)$. A fibrant simplicial category is a simplicial category which is fibrant in the Bergner model structure; i.e. all of its simplicial homs are Kan complexes. The Bergner model structure and the constructions for simplicial categories above can be found in [1].
1 Preliminaries on the Joyal model structure

The Joyal model structure on simplicial sets, whose existence is asserted in [13, Theorem 2.2.5.1] and [12, Theorem 6.12] is one of the main models for the homotopy theory of $(\infty, 1)$-categories. The fibrant objects of this model structure are the quasi-categories: simplicial sets $X$ such that the map $X \to *$ is an inner fibration. The cofibrations are the monomorphisms. The weak equivalence are called Joyal equivalences and can be described as follows. Given two simplicial sets $K$ and $X$, $\tau_0(K, X)$ will denote Joyal’s set, which is defined to be the set of isomorphism classes of objects in $P(X^K)$. The Joyal equivalences are defined to be maps $f : A \to B$ such that, for each quasi-category $X$, the map $\tau_0(B, X) \to \tau_0(A, X)$ is a bijection. The fibrations of this model structure are called quasi-fibrations. The trivial fibrations are the trivial Kan fibrations.

For a quasi-category $X$, let $J(X)$ denotes its maximal Kan subcomplex. Write $I = B\pi(\Delta^1)$.

The following is taken from [11, Section 1]:

**Theorem 1.1.** A quasi-category $X$ is a Kan complex iff $P(X)$ is a groupoid. Thus, $J(X)$ can be constructed by taking the maximal subcomplex of $X$ whose 1-simplices are invertible in $P(X)$. Furthermore, a 1-simplex $s : \Delta^1 \to X$ is invertible in $P(X)$ iff it extends to a map $I \to X$.

**Definition 1.2.** Given two simplicial sets $S$ and $T$, their join, denoted $S*T$, is a simplicial set whose n-simplices are described by the formula

$$(S*T)_n = S_n \cup T_n \cup i+j=n-1 (S_i \times T_j).$$

The ith degeneracy $d_i : (S*T)_n \to (S*T)_{n-1}$ is defined on the factors $S_n$ and $T_n$ using the degeneracy maps of $S_n$ and $T_n$. For $(\sigma, \sigma') \in S_i \times T_j$, we have the formula

$$d_k(\sigma, \sigma') = \begin{cases} (d_k \sigma, \sigma') & \text{if } k \leq i, i \neq 0 \\ (\sigma, d_{k-i-1}(\sigma')) & \text{if } k > i, j \neq 0 \end{cases}$$

**Definition 1.3.** Given a map of simplicial sets $p : K \to S$, there is a simplicial set $S/p$ such that
\[(S/p)_n = \text{hom}_p(\Delta^n \ast K, S),\]

where \(\text{hom}_p\) means simplicial set maps \(\phi\) such that \(\phi|_K = p\). We call this the slice over \(p\).

There is a natural map \(S/p \to S\) which is induced in simplicial degree \(n\) by \(\Delta^n \subseteq \Delta^n \ast K\). We call this the projection map.

**Definition 1.4.** Given a simplicial set \(X\) and \(x, y \in X\), the mapping space between \(x\) and \(y\) is defined to be the pullback

\[
\begin{array}{ccc}
\text{Map}_X(x, y) & \longrightarrow & X/y \\
\downarrow & & \downarrow q \\
* & \longrightarrow & X
\end{array}
\]

where \(q\) is the projection map.

In [13], Lurie writes \(\text{Hom}^R(x, y)\) for these mapping spaces and calls them right mapping spaces. The mapping spaces in a quasi-category are always Kan complexes.

**Definition 1.5.** Let \(f : X \to Y\) be a map of quasi-categories. We say that \(f\) is fully faithful iff for each \(x, y \in X\), \(\text{Map}_X(x, y) \to \text{Map}_Y(f(x), f(y))\) is a weak equivalence of Kan complexes. We say that \(f\) is essentially surjective iff \(P(f)\) is essentially surjective.

**Lemma 1.6.** A morphism \(f : X \to Y\) of quasi-categories is essentially surjective iff \(\pi_0 J(f)\) is surjective.

**Proof.** By [12] a 1-simplex \(s : \Delta^1 \to X\) is invertible iff it extends to a map \(B\pi\Delta^1 \to X\). Thus, there are bijections (natural in \(X\))

\[
\pi_0 \text{Iso}(P(X)) \cong \pi_I(*, X) \cong \pi_{\Delta^1}(*, J(X)) \cong \pi_0 J(X),
\]

where \(\pi_I\) denotes the homotopy classes of maps with respect to \(I = B\pi\Delta^1\) and \(\pi_{\Delta^1}\) those with respect to \(\Delta^1\).

The mapping space construction is important because of the following result:
Theorem 1.7. (see [13, Theorem 2.2.5.1], [3, Theorem 8.1]). Suppose that $f : X \to Y$ is a map of quasi-categories. Then $f$ is a Joyal equivalence iff it is fully faithful and essentially surjective.

Definition 1.8. Suppose that $\Omega^*$ is a cosimplicial object in a category $C$. Then there is a pair of adjoint functors associated to $\Omega^*$

$$ | |_{\Omega^*} : \text{sSet} \rightleftarrows C : \text{Sing}_{\Omega^*}.$$ 

The left adjoint is given by

$$|S|_{\Omega^*} = \lim_{\Delta^n \to S} \Omega^n $$

and the right adjoint is given by $\text{Sing}_{\Omega^*}(S)_n = \text{hom}(\Omega^n, S)$. The right adjoint is known as the singular functor associated to $\Omega^*$.

For each $n \in \mathbb{N}$ there is a simplicial category $\Phi^n$ such that:

1. The objects $\Phi^n$ are the objects in the set $\{0, 1, \ldots, n\}$.
2. $\text{hom}_{\Phi^n}(i, j)$ can be identified with the nerve of the poset $\mathcal{P}_n[i, j]$ of subsets of the interval $[i, j]$ which contains the endpoints. That is, $\text{hom}_{\Phi^n}(i, j) \cong (\Delta^1)^{i-j-1}$.
3. Composition is induced by union of posets.

These $\Phi^n$ glue together to give a cosimplicial object $\Phi$. The singular functor associated to $\Phi$ is called the homotopy coherent nerve, and is denoted $\mathcal{B}$. We write $\mathcal{C}$ for its left adjoint.

The homotopy coherent nerve is significant because of the following theorem, which relates the Bergner model structure (see [1]) to the Joyal model structure:

Theorem 1.9. There is a Quillen equivalence

$$ \mathcal{C} : \text{sSet} \rightleftarrows \text{sCat} : \mathcal{B}$$

between the Joyal model structure and the Bergner model structure.

In addition, the homotopy coherent nerve is important because if $M$ is a simplicial model category, then $\mathcal{B}(M^\circ)$ captures the essential homotopy-theoretic properties of $M$, as is immediate from the following result:
Theorem 1.10. Suppose that $M$ is a fibrant simplicial category. Then:

1. There is a natural isomorphism $\pi_0 \mathcal{C} \cong P$. In particular, we have equivalences of categories $\pi_0 M \simeq \pi_0 \mathcal{CB} \cong P \mathcal{B}(M)$.

2. For each $x, y \in M$ there is a zig-zag of weak equivalences

$$\text{Map}_{\mathcal{B}(M)}(x, y) \leftarrow S_{x,y} \rightarrow \text{hom}_{M}(x, y)$$

Proof. A simplicial set is a colimit of its non-degenerate simplices and we have isomorphisms (natural in ordinal numbers $n$)

$$P(\Delta^n) \cong [n] \cong \pi_0 \mathcal{C}(\Delta^n).$$

$\pi_0 \mathcal{C}$ and $P$ are both left adjoints and thus preserve colimits. Thus, we have

$$P \cong \pi_0 \mathcal{C}.$$ 

The remainder of 1. now follows from 1.9.

Statement 2. is the result of combining [13, 2.2.2.7, 2.2.2.10 and 2.2.2.13] and the fact that mapping spaces in a quasi-category are Kan complexes.

\[\square\]

2 DK equivalences and Descent

This chapter is devoted to proving descent results for presheaves of quasi-categories. Descent in our context is a variation (in fact a generalization of) of what is referred to as hyperdescent in the recent literature (see [18] or [13, Section 6.5.4]), which is a strictly stronger condition than what is typically called descent. Hyperdescent is in fact what we call injective descent; that is, descent with respect to the Jardine model structure.

The first part of the section is devoted to showing that local Joyal equivalences induce local weak equivalences of mapping space presheaves, and the formation of mapping space presheaves preserves the property of satisfying descent. This is an important ingredient in the proof of 2.15. Once this is in place, we prove the main descent results (2.10, 2.15 and 2.19).

A Grothendieck site is a category-theoretic generalization of the concept of topological space. It is specified by choosing a category $C$ and a
collection of coverings (sets of morphisms \(\{U_i \to U\}_{i \in I}\)), subject to various axioms. Grothendieck sites were invented to discuss cohomology theories in algebraic geometry, to define topologies that are finer than the usual Zariski topology on schemes. A good overview of this concept is found in [14, Chapter 3].

Throughout the rest of the paper, we fix a small Grothendieck site \(\mathcal{C}\). We will write \(s\text{Pre}(\mathcal{C})\) for the simplicial presheaves on \(\mathcal{C}\). We write \(s\text{Sh}(\mathcal{C})\) for the simplicial sheaves on \(\mathcal{C}\). We will identify simplicial sets with constant simplicial presheaves. Given a simplicial set and a simplicial presheaf \(K\), we will write \(X^K\) for the simplicial presheaf \(U \mapsto X(U)^K\). We denote sheafification by

\[
L^2 : s\text{Pre}(\mathcal{C}) \to s\text{Sh}(\mathcal{C}).
\]

Recall that the existence of the Jardine model structure on \(s\text{Pre}(\mathcal{C})\), in which the weak equivalences are local weak equivalence and the cofibrations are monomorphisms (i.e. in the case our topos has enough points, the weak equivalences are maps which induce weak equivalences on stalks). We will refer to this as the injective model structure in the paper. We call the fibrations injective fibrations.

There is also a model structure on \(s\text{Pre}(\mathcal{C})\), called the local Joyal model structure, in which the weak equivalences are local Joyal equivalences the cofibrations are the monomorphisms (see [15, Theorem 3.3] or [17]). The weak equivalences are called local Joyal equivalences and the fibrations are called quasi-injective fibrations.

The technique of Boolean localization is essential to the study of local model structures; overviews of this technique are given in [15, Section 2] and [10, Chapter 3]. A Boolean localization

\[
p = (p^*, p_*): \text{Sh}(\mathcal{B}) \to \text{Sh}(\mathcal{C})
\]

is a surjective geometric morphism with \(\mathcal{B}\) a complete Boolean algebra equipped with the canonical site. The Boolean localization allows us to reason about local weak equivalences by replacing them with sectionwise weak equivalences, thereby reducing many statements about local equivalences to the classical setting. We will fix a Boolean localization for \(\mathcal{C}\), denoted \(p\), and write

\[
p^* : s\text{Sh}(\mathcal{C}) \rightleftarrows s\text{Sh}(\mathcal{B}) : p_*
\]
for the adjoint pair obtained by applying the left and right adjoint parts of \( p \) sectionwise to a simplicial object in sheaves (simplicial sheaf). Note that all Grothendieck topoi have a Boolean localization by a theorem of Barr ([14, pg. 515]).

We write \( \mathcal{L}_{\text{Joyal}}, \mathcal{L}_{\text{inj}} \) for the fibrant replacement functors in the local Joyal and injective model structures on \( \text{sPre}(\mathfrak{C}) \), respectively. We write \( \mathcal{S}_{\text{Joyal}}, \mathcal{S}_{\text{inj}} \) for the sectionwise Joyal and standard fibrant replacement functors, respectively. We say that a presheaf of Kan complexes \( X \) satisfies \textbf{injective descent} iff \( X \to \mathcal{L}_{\text{inj}}(X) \) is a sectionwise weak equivalence. We say that a presheaf of quasi-categories \( X \) satisfies \textbf{quasi-injective descent} iff \( X \to \mathcal{L}_{\text{Joyal}}(X) \) is a sectionwise Joyal equivalence.

The following is [16, Theorem 4.7]:

**Theorem 2.1.** Suppose that \( X \) is a presheaf of quasi-categories. Then \( X \) satisfies quasi-injective descent iff for each \( n \in \mathbb{N} \) \( J(X^{\Delta^n}) \) satisfies injective descent. In particular, if \( X \) is a presheaf of Kan complexes, then it satisfies quasi-injective descent iff it satisfies injective descent.

**Definition 2.2.** Suppose that \( X, Y \) are simplicial presheaves. We define their \textbf{join}, \( X \star Y \), to be the simplicial presheaf obtained by applying the usual join operation sectionwise. Suppose that \( f : K \to X \) is a map of simplicial presheaves with \( K \) constant. Then we can form a simplicial presheaf \( X_{/f} \) such that \( X_{/f}(U) = X(U)_{/f} \).

Suppose that \( X \) is a presheaf of quasi-categories and let \( x, y : \ast \to X \) be global sections. Then the mapping space presheaf \( \text{Map}_X(x, y) \) is defined to be the pullback

\[
\begin{array}{ccc}
\text{Map}_X(x, y) & \longrightarrow & X_{/y} \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & X
\end{array}
\]

**Lemma 2.3.** Let \( A, B \) be presheaves of simplicial sets. Then

\[
p^\ast L^2(A \ast B) \cong L^2(p^\ast L^2 A \ast p^\ast L^2 B).
\]

**Proof.** Follows from the standard properties of Boolean localization found in [15, Lemma 2.6] and the formula in [12].
Lemma 2.4. If $K$ is finite simplicial set, $X$ is a simplicial presheaf and $f : K \to X$ is a map of simplicial presheaves, then

$$p^* L^2(X/f) \cong p^* L^2(X)/p^* L^2(f).$$

Proof. The n-simplices of $X/f$ can be described as a pullback

\[
\begin{array}{ccc}
(X/f)_n & \longrightarrow & \hom(K \Delta^n, X) \\
\downarrow & & \downarrow \\
* & \longrightarrow & \hom(K, X)
\end{array}
\]

Since $K$ is finite, the standard properties of Boolean localization found in [15, Lemma 2.6] imply that we have a pullback diagram

\[
\begin{array}{ccc}
p^* L^2(X/f)_n & \longrightarrow & \hom(K \Delta^n, p^* L^2 X) \\
\downarrow & & \downarrow \\
* & \longrightarrow & \hom(\Delta^n, p^* L^2 X)
\end{array}
\]

as required. \qed

Lemma 2.5. Suppose that $f : X \to Y$ is a local Joyal equivalence and $A$ is a simplicial presheaf. Then $X \ast A \to Y \ast A$ is a local Joyal equivalence.

Proof. The join operation preserves Joyal equivalences by [13, Corollary 4.2.1.3]. Thus, it suffices to show that $S_{Joyal}(X) \ast S_{Joyal}(A) \to S_{Joyal}(Y) \ast S_{Joyal}(A)$ is a local Joyal equivalence. Thus, we have reduced to the case that $A, X, Y$ are presheaves of quasi-categories. By [13, Corollary 4.2.1.3] and [15, Corollary 3.11] the map

$$p^* L^2(f) \ast id : p^* L^2 X \ast p^* L^2 A \to p^* L^2 Y \ast p^* L^2 A$$

is a sectionwise Joyal equivalence. Thus $L^2(p^* L^2(f) \ast id) \cong p^* L^2(f \ast id)$ is a local Joyal equivalence by [15, Corollary 3.2]. The result follows from that fact that $p^* L^2$ reflects local Joyal equivalences. \qed

Lemma 2.6. Suppose that $f : X \to Y$ is a presheaf of quasi-categories and $s : K \to X$ is a map with $K$ a constant simplicial presheaf with $K$ finite. Then $X_{f/s} \to Y_{f/s}$ is a local Joyal equivalence.
Proof. If \( p : K \to X, q : X \to Y \) are maps such that \( q \) is a Joyal equivalence, then \( X/p \to Y_{q\circ p} \) is a Joyal equivalence (this follows from the discussion of [13, pg. 241]). Combining this with [15, Corollary 3.11], we thus have a Joyal equivalence \( p^*L^2(X)_{/p^*L^2(f)} \to p^*L^2(Y)_{/p^*L^2(f)} \). This is naturally isomorphic to \( p^*L^2(X/s) \to p^*L^2(X_{/f \circ s}) \) by 2.4 But \( p^*L^2 \) reflects local Joyal equivalences, so the result follows.

Corollary 2.7. Let \( X \) is quasi-injective fibrant simplicial presheaf. Let \( s : K \to X \) be a map of simplicial presheaves with \( K \) constant. Then \( \forall x, y \in X(U), U \in \text{Ob}(\mathcal{C}) \), \( \text{Map}_X|U(x,y) \) satisfies injective descent.

Proof. First, we will show that \( \text{Map}_X|U(x,y) \) is quasi-injective fibrant. One wants to solve a lifting problem

\[
\begin{array}{c}
A \\
\downarrow \\
B
\end{array}
\quad \rightarrow 
\begin{array}{c}
\text{Map}_X|U(x,y) \\
\downarrow \\
B
\end{array}
\]

where \( A \to B \) is a local Joyal equivalence and a monomorphism. By adjunction, this is equivalent to a lifting problem

\[
\begin{array}{c}
(A \ast \Delta^0) \coprod_A B \\
\downarrow \\
B
\end{array}
\quad \rightarrow 
\begin{array}{c}
X \\
\downarrow \\
B
\end{array}
\]

where \( m|_B \) factors through the inclusion \( \ast|_U \to X \). But the vertical map is a local Joyal equivalence and a monomorphism, so the lift exists.

Now, \( \text{Map}_X|U(x,y) \) satisfies quasi-injective descent. Because the mapping spaces in a quasi-category are Kan complexes, \( J\text{Map}_X|U(x,y) = \text{Map}_X|U(x,y) \) and \( \text{Map}_X|U(x,y) \) satisfies injective descent by 2.1.

Lemma 2.8. Suppose that \( X \) is a presheaf of quasi-categories and let \( i : X \to \mathcal{L}_{Joyal}(X) \) be a quasi-injective fibrant replacement. Then \( i \) is essentially surjective in each section iff the \( \forall U \in \text{Ob}(\mathcal{C}), x, y \in X(U), \text{Map}_X|U(x,y) \) satisfies injective descent.

Proof. Let \( x, y \in X(U) \). Note that we have a diagram

\[
\begin{array}{c}
J((X|U)/y) \\
\downarrow \\
J((\mathcal{L}_{Joyal}(X)|U)/i(y))
\end{array}
\quad \rightarrow 
\begin{array}{c}
J(X|U) \\
\downarrow \\
J(\mathcal{L}_{Joyal}(X)|U)/i(x)
\end{array}
\]

\[
\begin{array}{c}
J(X|U) \\
\downarrow \\
X
\end{array}
\quad \leftarrow 
\begin{array}{c}
\text{Map}_X|U(x,y) \\
\downarrow \\
\ast
\end{array}
\]

11
The left vertical map is a local weak equivalence by 2.6 and the fact that $J$ sends local Joyal equivalences to local weak equivalences ([16, Lemma 4.5]). Since left fibrations of quasi-categories are quasi-fibrations ([12, Proposition 4.10]) and $J$ sends quasi-fibrations to Kan fibrations ([12, Proposition 4.27]) the left horizontal maps in the diagrams are sectionwise Kan fibrations. Thus, the pullback of each of the two rows are both homotopy Kan fibrations for the injective model structure by [10, Lemma 5.20]. The properness of the injective model structure and [6, Lemma II.8.19] implies that we have a local weak equivalence

\[ J\text{Map}_X\vert_U(x, y) \to J\text{Map}_{\mathcal{L}_{\text{Joyal}}}(x)\vert_U(i(x), i(y)) \]

The functor $J$ is the identity on Kan complexes, so we have a local weak equivalence

\[ \text{Map}_X\vert_U(x, y) \to \text{Map}_{\mathcal{L}_{\text{Joyal}}}(x)\vert_U(i(x), i(y)) \]

The object $\text{Map}_{\mathcal{L}_{\text{Joyal}}}(x)\vert_U(i(x), i(y))$ satisfies injective descent by 2.7. Thus, $\text{Map}_X\vert_U(x, y) \to \text{Map}_{\mathcal{L}_{\text{Joyal}}}(x)\vert_U(i(x), i(y))$ is a sectionwise weak equivalence iff $\text{Map}_X\vert_U(x, y)$ satisfies injective descent.

\begin{lemma}
The fibrant replacement $i : X \to \mathcal{L}_{\text{Joyal}}(X)$ is essentially surjective in sections iff $\pi_0 J(X) \to \pi_0 \mathcal{L}_{\text{inj}}(JX)$ is a surjection.
\end{lemma}

\begin{proof}
By [1.6], it suffices to show that $\pi_0 (JX) \to \pi_0 (\mathcal{L}_{\text{Joyal}}(X))$ is a surjection. One has a diagram

\[ J(X) \xrightarrow{} \mathcal{L}_{\text{inj}}(JX) \]

\[ J\mathcal{L}_{\text{Joyal}}(X) \]

Since $J$ sends local Joyal equivalences of presheaves of quasi-categories to local weak equivalences by [16, Lemma 4.5], we can produce a lifting

\[ J(X) \xrightarrow{} \mathcal{L}_{\text{inj}}(JX) \]

\[ J\mathcal{L}_{\text{Joyal}}(X) \]

The diagonal map is a local weak equivalence. Since $J\mathcal{L}_{\text{Joyal}}(X)$ satisfies injective descent by 2.1 the diagonal map is a bijection on path components. In particular, essential surjectivity is equivalent to the stated condition. \qed

12
Combining 2.8, 2.9, 1.7 and 2.10 we have:

**Corollary 2.10.** Let $X$ be a presheaf of quasi-categories. Then $X$ satisfies quasi-injective descent iff

1. For each $x, y \in X(U), U \in \text{Ob}(\mathcal{C})$, $\text{Map}_{X|_U}(x, y)$ satisfies injective descent.

2. $JX$ satisfies injective descent.

Recall that there is a **global injective model structure** on $s\text{Pre}(\mathcal{C})$ in which the cofibrations and weak equivalences are, respectively sectionwise cofibrations and sectionwise weak equivalences for the Joyal model structure.

**Definition 2.11.** Let $X$ be a presheaf of quasi-categories. Then $X$ is said to satisfy **effective descent** with respect to a covering sieve $R$ of an object $U$ if and only if $[*|_U, X]_q \to [*_R, X]_q$ is surjective, where $[\cdot]_q$ denotes maps in the homotopy category of the global injective Joyal model structure. $X$ satisfies effective descent if and only if it satisfies effective descent with respect to each covering sieve of each object.

**Remark 2.12.** Note that the effective descent condition is invariant under sectionwise Joyal equivalence.

**Lemma 2.13.** Suppose that $X$ is fibrant for the global injective Joyal model structure on $s\text{Pre}(\mathcal{C})$. Then $X$ satisfies effective descent with respect to covering $R$ of $U$ if and only if it has the right lifting property with respect to $*_R \to *|_U$.

**Proof.** Sufficiency is obvious. We prove necessity.

Suppose that we have a diagram

$$
\begin{array}{ccc}
*|_R & \rightarrow^s & X \\
\downarrow & & \downarrow \\
*|_U & \rightarrow^t & X
\end{array}
$$

By hypothesis, we can choose $t : *|_U \to X$ and a homotopy $h : I \times *|_R \to X$ between $s$ and $t|_R$. We can find a lift

$$
\begin{array}{ccc}
I|_R \times *|_R & \rightarrow^{(h,t)} & X \\
\downarrow & \phi & \downarrow \\
I|_U & \rightarrow & X
\end{array}
$$

13
Lemma 2.14. Suppose that \( X \) is a presheaf of quasi-categories, fibrant for the global injective model structure. Then the effective descent condition for \( X \) is equivalent to the following: for each \( U \in \text{Ob}(\mathcal{C}) \) and covering sieve \( R \) of \( U \), the map 
\[
\pi_0 JX(U) \to \pi_0 (\varprojlim_{V \in R} JX(V))
\]
is surjective.

Proof. We have bijections (natural in \( R \))
\[
\pi_I(\ast|_R, X) = \pi_{\Delta^I}(\ast|_R, JX) = \pi_0 \text{hom}(\ast|_R, JX),
\]
where the first follows from 1.1.

Here \( \text{hom} \) denotes the simplicial hom for the injective model structure, given by \( \text{hom}(A, B)_n = \text{hom}(\Delta^n \times A, B) \). Now, we have a bijection (natural in coverings \( R \))
\[
\text{hom}(\ast|_R, JX) \cong \varprojlim_{V \in R} JX(V)
\]
as required. \( \Box \)

The following theorem is a local Joyal analogue of [7, Theorem 10.2]; the proofs are quite similar.

Theorem 2.15. Let \( X \) be a presheaf of quasi-categories. Then \( X \) is satisfies quasi-injective descent iff

1. For each \( U \in \text{Ob}(\mathcal{C}), x, y \in X(U) \) \( \text{Map}_{X|_U}(x, y) \) satisfies injective descent.

2. \( X \) satisfies the effective descent condition with respect to all coverings \( R \) of objects \( U \).

Proof. Suppose that \( X \) satisfies quasi-injective descent. Then Condition (1) is satisfied by 2.7. To show that the effective descent condition is satisfied, note that the map \( X \to \mathcal{L}_{\text{Joyal}}(X) \) is a fibrant replacement for the global injective model structure since \( X \) satisfies quasi-injective descent. Given a
covering $R$ of an object $U$, $\ast|_R \to \ast|_U$ is a local Joyal equivalence and we can solve lifting problems

$$
\begin{array}{ccc}
\ast|_R & \longrightarrow & \mathcal{L}_{\text{Joyal}}(X) \\
\downarrow & & \downarrow \\
\ast|_U & & \\
\end{array}
$$

Thus, the result follows from 2.12 and 2.13.

Now, assume the two conditions in the statement of the theorem hold. By 1.7, 2.8 and the assumption on mapping space presheaves, it suffices to show that $X \to \mathcal{L}_{\text{Joyal}}(X)$ is essentially surjective in sections. Factor $X \to B \xrightarrow{g} \mathcal{L}_{\text{Joyal}}(X)$, where $g$ is an fibration for the global injective model structure and $X \to B$ is a sectionwise Joyal equivalence. It now suffices to show that $g$ is essentially surjective. Let $a : \ast|_U \to \mathcal{L}_{\text{Joyal}}(X)$ be a point. Form the pullback

$$
\begin{array}{ccc}
C & \longrightarrow & B \\
\downarrow & & \downarrow g \\
\ast|_U & \longrightarrow & \mathcal{L}_{\text{Joyal}}(X) \\
\end{array}
$$

The map $g$ is a sectionwise quasi-fibration and a local Joyal equivalence, and hence a local trivial fibration ([15, Definition 2.4]) by [15, Lemma 3.15]. Thus, $k : C \to \ast|_U$ is a local trivial fibration. In particular, this means that there exists a covering $R$ of $U$ such that $k|_R : C|_R \to \ast|_R$ is surjective. Let $S \subseteq \mathcal{L}_{\text{Joyal}}(X)$ be the full subpresheaf of quasi-categories (i.e. in sections see [13, 1.2.11]) such that

$$\text{Ob}(S) = \text{im}(\text{Ob}(g))$$

One can factor the map $g$ as $B \xrightarrow{g'} S \subseteq \mathcal{L}_{\text{Joyal}}(X)$. The map $g'$ is a fibration for the global injective model structure, since $g'$ can be expressed as a pullback

$$
\begin{array}{ccc}
B & \longrightarrow & B \\
\downarrow & & \downarrow g \\
S & \longrightarrow & \mathcal{L}_{\text{Joyal}}(X') \\
\end{array}
$$

$g'$ is essentially surjective by construction, and is fully faithful by [13, Remark 1.2.2.4] and the fact that $g$ is fully faithful. Thus, it is a sectionwise
Joyal equivalence by 1.7. Thus it is a trivial quasi-injective fibration, since trivial fibrations for the local Joyal and global injective quasi-category model structure coincide. \( k|_R : C|_R \rightarrow *|_R \) is isomorphic to the pullback

\[
\begin{array}{ccc}
C|_R & \xrightarrow{k|_R} & *|_R \\
\downarrow{l|_R} & & \downarrow{a|_R} \\
B & \xrightarrow{g} & S
\end{array}
\]

so that \( k|_R : C|_R \rightarrow *|_R \) is a trivial quasi-injective fibration. Thus, it has a section \( s : *|_R \rightarrow C|_R \). The morphism

\[ l|_R \circ s : *|_R \rightarrow B \]

extends to a map

\[ r : *|_U \rightarrow B \]

by the effective descent hypothesis, 2.13 and the fact that \( B \) is global quasi-injective fibrant replacement of \( X \). Now, there exists a lifting

\[
\begin{array}{ccc}
(*|_U \coprod *|_U) \coprod (*|_R \coprod *|_R)(I|_R) & \xrightarrow{(g \circ r, a|_R \circ t)} & \mathcal{L}_\text{Joyal}(X) \\
\downarrow & & \downarrow \\
I|_U & \rightarrow &
\end{array}
\]

where \( t : I|_R \rightarrow *|_R \) is the terminal map in sections (the vertical map is a local Joyal equivalence). Thus, the image of the point \( r \) is isomorphic to \( a \) in \( P(\mathcal{L}_\text{Joyal}(X)) \) by 1.1. Since \( a, U \) are arbitrary, we have proven that \( g \) is essentially surjective in sections.

**Definition 2.16.** We say that a site \( \mathcal{E} \) with fiber products has **sufficient coproducts compatible with fiber products** if there exists some regular cardinal \( \alpha \) such that the following conditions hold

1. \( \mathcal{E} \) has \( \alpha \)-bounded coproducts.
2. The sieves generated by covering families of size less than \( \alpha \) form a cofinal set in the collection of covering sieves and fiber products commute with \( \alpha \)-bounded coproducts.
Throughout the rest of the paper, we will assume that $C$ has fiber products, and satisfies the properties of 2.16.

**Theorem 2.17.** Let $X$ be a presheaf of quasi-categories such that $X(i) = \ast$, where $i$ is the initial object of $C$. Suppose that $X$ satisfies the following three conditions

1. For each $U \in \text{Ob}(C), x, y \in X(U)$ $\text{Map}_{X|U}(x, y)$ satisfies injective descent.

2. Any restriction map $X(U) \to X(V)$ is essentially surjective for each morphism $V \to U$ that generates a cover of $U$.

3. Let $\alpha$ be as in 2.16. For any $\alpha$-bounded collection of elements $\mathcal{V}$, the map induced by restriction $X(\coprod_{V \in \mathcal{V}} V) \to \prod_{V \in \mathcal{V}} X(V)$ is essentially surjective.

Then $X$ satisfies quasi-injective descent.

**Proof.** These conditions are invariant under sectionwise Joyal equivalence. Thus, it suffices to assume that $X$ is global injective fibrant. By 2.15 we want to verify the effective descent condition for each covering $R$ of an object $U$. Let $S$ be a subcovering of $R$ generated by an $\alpha$-bounded set of elements. Since $C$ admits pullbacks, the inclusion of subcategories $S \subseteq R$ is a cofinal functor. Thus, we have isomorphisms

$$\pi_0 \lim_{V \in S} JX(V) \cong \pi_0 \lim_{V \in R} JX(V)$$

so that effective descent with respect to the covering $S$ implies effective descent with respect to $R$ by 2.14.

Thus, it suffices to verify the effective descent condition for coverings generated by an $\alpha$-bounded set of elements. Let $\mathcal{V}$ be the set of elements generating such a cover $S$. Let $W = \coprod_{V \in \mathcal{V}} V$ and let $T$ be the sieve generated by the natural inclusions $\{V \to W : V \in \mathcal{V}\}$ (this is a sieve by condition (2) of 2.16). The facts that $X(i) = \ast$ and that $J$ is a right adjoint imply that we have a natural isomorphism

$$\lim_{V \in \mathcal{V}} JX(V) \cong \prod_{V \in \mathcal{V}} (JX(V)) \cong J \prod_{V \in \mathcal{V}} (X(V)).$$
Thus, \[2.14\] implies that \(X\) satisfying effective descent with respect to the covering \(T\) is equivalent to Condition (3) for the collection \(V\). \([*|_{U}, X]_{q} \rightarrow [*|_{S}, X]\) factors as

\[[*|_{U}, X]_{q} \rightarrow [*|_{W}, X] \rightarrow [*|_{S}, X]\]

and the second map is surjective by the preceding paragraph.

Thus, to check that \(X\) satisfies effective descent, it suffices to show that it satisfies the effective descent condition with respect to covers generated by a single morphism \(W \rightarrow U\). By \[2.14\] it suffices to check that \(\pi_{0}JX(U) \rightarrow \pi_{0}JX(W)\) is surjective, which is true by Condition (2).

**Corollary 2.18.** Suppose that \(M\) is a presheaf of fibrant simplicial categories such that \(M(i) = \ast\). Then \(\mathcal{B}(M)\) satisfies quasi-injective descent if

1. For each \(U \in \text{Ob} \mathcal{C}\), \(x, y \in M(U)\), \(\text{hom}_{M(U)}(x, y)\) satisfies injective descent.

2. \(\pi_{0}M(U) \rightarrow \pi_{0}M(V)\) is essentially surjective for each morphism \(V \rightarrow U\) that generates a cover of \(U\).

3. \(\pi_{0}M(\coprod_{i \in I} U_{i}) \rightarrow \pi_{0}(\prod_{i \in I} M(U_{i}))\) is essentially surjective for each \(\alpha\)-bounded set \(I\).

**Proof.** We verify conditions (1)-(3) of \[2.17\]

The zig-zag of weak equivalences explained in \[1.10\] is natural in simplicial sets. Thus, we have a zig-zag of sectionwise weak equivalences of simplicial presheaves

\(\text{hom}_{M(U)}(x, y) \leftarrow S \rightarrow \text{Map}_{\mathcal{B}(M)(U)}(x, y)\),

so that condition (1) above implies condition (1) of \[2.17\]

By \[1.10\] we have natural maps

\(P\mathcal{B}(M) \cong \pi_{0}\mathcal{C}\mathcal{B}(M) \cong \pi_{0}(M)\),

which are equivalences of categories in each section. Thus, condition (2) and (3) above imply that \(\mathcal{B}(M)\) satisfies condition (2) and (3) of \[2.17\]

We call a subsimplicial category \(M'\) of a simplicial model category full iff it is full in each simplicial degree. Given a full subsimplicial category \(M'\) of a simplicial model category \(M\), write \((M')^{\circ}\) for the full subsimplicial category consisting of cofibrant-fibrant objects of \(M'\).
We have the following theorem, which allows us to produce examples of higher stacks using presheaves of simplicial model categories:

**Theorem 2.19.** Suppose that $M$ is a presheaf of simplicial model categories on a site which satisfies the hypotheses of 2.16. Suppose that $M' \subseteq M$ is a full subpresheaf (i.e. in each simplicial degree) such that

1. For each morphism $\phi : V \to U$ of $\mathcal{C}$ generating a covering, restriction $r_{\phi} : M_0(U) \to M_0(V)$ is both a left and right quillen functor. Furthermore, if $\eta_{\phi}$ is left adjoint of restriction $r_{\phi} : M_0(U) \to M_0(V)$, then then the unit $\text{id} \to r_{\phi} \eta_{\phi}$ is a weak equivalence in $M(V)$.

2. $M'(U)$ is small for each $U \in \text{Ob}(\mathcal{C})$.

3. For each $x \in M'(U)$, $M'(U)$ contains a fibrant replacement of $x$.

4. Let $\phi : V \to U$ be a morphism generating a cover. Then if $\eta_{\phi}$ is the left adjoint of restriction, $M'_0(V) \subseteq M_0(V) \xrightarrow{\eta_{\phi}} M_0(U)$ has the same essential image as $M'_0(U) \subseteq M_0(U)$.

5. $\text{Map}_{M'_|U}(x, y)$ satisfies injective descent for each $x, y \in M'(U)$.

6. $\pi_0 M'(\coprod_{i \in I} U_i) \to \pi_0(\prod_{i \in I} M'(U_i))$ is essentially surjective for each $\alpha$-bounded set $I$.

Then $\mathfrak{B}((M')^\circ)$ satisfies quasi-injective descent.

**Proof.** It suffices to verify condition (2) of 2.18 applies to $M'$. Let $V \to U$ be a morphism generating a cover and an object $x \in (M')^\circ(V)$. Let $\eta_{\phi} : M_0(V) \to M_0(U)$ be the left adjoint of the restriction $r_{\phi}$. Choose a $z \cong \eta_{\phi}(x)$ and a fibrant replacement $z \to x'$ in $(M')_0(U)$. Then there is a weak equivalence $x \to r_{\phi}(\eta_{\phi}(x)) \to r_{\phi}(x')$. in $(M')^\circ(V)$. But $\pi_0((M')^\circ)(V)$ is a full subcategory of the homotopy category of $M(V)$, as required.

**Remark 2.20.** In constructing stacks, we don’t take them to be $\mathfrak{B}(M^\circ)$ for a simplicial model category $M$, because simplicial model categories are large categories. Instead, we replace $M$ with $M' \subseteq M$, where $M'(U)$ is a full, small subcategory of $M(U)$ for each object $U$. In this case, $\pi_0((M')^\circ)$ gets identified
with a full subcategory of $\text{Ho}(M)$. This is in contrast to recent literature, in which the size issue is dealt with by using Grothendieck universes. We have not taken this route, as Grothendieck universes tend to obscure cardinal arithmetic. Cardinality tricks are essential to proving ‘bounded cofibration arguments’, which appear in the construction of many local model structures (see [10]).

**Remark 2.21.** Condition 2, 3 and 4 in 2.19 are redundant if we are willing to take $M = M'$ and ignore size issues. Thus, 2.19 can be regarded as a generalization of [7, Theorem 19.4].

**Remark 2.22.** In practice, one produces $M'(U)$ in 2.19 by choosing the objects to be representatives of isomorphism classes of objects that satisfy a cardinality bound. Condition 3 in 2.19 holds, in effect, when $M$ is a presheaf of cofibrantly generated model categories and the cardinality bound is sufficiently high (see the argument of [9, Theorem 4.8]). Condition 4 holds when the left adjoint of restriction preserves the cardinality bound. This is illustrated in the next section.

### 3 The Higher Stacks of Unbounded Chain Complexes

In this section, we establish the existence of the higher stack of simplicial $\mathcal{R}$-module spectra (3.7). The first part of the section is devoted to reviewing the model structure for simplicial $\mathcal{R}$-module spectra and discussing its relationship to the derived category. In the second part, we apply 2.19 to show that this is a stack.

Throughout this section, we fix a sheaf of rings $\mathcal{R}$ on $\mathcal{C}$. Let $\text{sSh}_\mathcal{R}$ denote the category of sheaves of simplicial abelian groups with $\mathcal{R}$-module structure. Let $\text{Ch}_{\mathcal{R},+}$, $\text{Ch}_\mathcal{R}$ denote, respectively the categories of non-negatively graded and unbounded complexes of sheaves of $\mathcal{R}$-modules.

Given a chain complex $C$, we write $C[n]$ for the chain complex defined by the formula

$$C[n]_m = C_{n+m}.$$ 

There is a functor $\tau : \text{Ch}_\mathcal{R} \to \text{Ch}_{\mathcal{R},+}$, the ‘intelligent’ truncation functor, defined by

$$\tau(C)_n = \begin{cases} 
\ker(\partial : C_0 \to C_{-1}) & \text{if } n = 0 \\
C_n & \text{if } n > 0
\end{cases}$$
Theorem 3.1. (see [10, Theorem 8.6]). There is a cofibrantly generated, proper, simplicial model structure on $s\text{Sh}_R$ in which the weak equivalences are the local weak equivalences and the fibrations are the injective fibrations. The simplicial hom is the usual simplicial hom for the injective model structure. The Dold-Kan correspondence

$$N : s\text{Sh}_R \rightleftharpoons \text{Ch}_{R,+} : \Gamma$$

and the above model structure induce a model structure on $\text{Ch}_{R,+}$ in which the weak equivalences are the quasi-isomorphisms.

Given a simplicial $R$-module $X$ and a simplicial presheaf $K$, we write $X \otimes K$ for the simplicial sheaf defined as the sheafification of

$$U \mapsto X(U) \otimes R(K)(U),$$

where $R$ is the free $R$-module functor.

Definition 3.2. A Simplicial $R$-module spectrum $A$ consists of simplicial $R$-modules $A^n, n \geq 0$, together with simplicial $R$-module homomorphisms $\sigma : S^1 \otimes A^n \to A^{n+1}$, called bonding maps. A map of $f : A \to B$ of maps $f : A^n \to B^n$ such that

$$A^n \otimes S^1 \overset{f}{\longrightarrow} B^n \otimes S^1 \quad \sigma \downarrow \quad \sigma \downarrow$$

$$A^{n+1} \longrightarrow B^{n+1}$$

commutes.

We write $\text{Spt}(s\text{Sh}_R)$ for the category of simplicial $R$-module spectra. Given a simplicial $R$-module spectrum $X$ and a simplicial presheaf $K$, we write $X \otimes K$ for the simplicial $R$-module spectrum that is the sheafification of

$$(X \otimes K)^n = X^n \otimes K$$

and whose bonding maps come from those of $X$.

Given $X \in \text{Spt}(s\text{Sh}_R)$, there is a presheaf of stable homotopy groups $\pi_n^s(X)$, defined by

$$U \mapsto \pi_n^s(X(U))$$

(see [10] pg. 377). We say that a map of simplicial $R$-module spectra $f : A \to B$ is a local stable equivalence iff $L^2\pi_n^s(f)$ is an isomorphism for each $n \geq 0$. 

21
Theorem 3.3. There is a cofibrantly generated, proper simplicial model structure on $Spt(sSh_R)$ in which the weak equivalences are the local stable equivalences and the cofibrations are maps $f : A \to B$ such that:

1. $A^0 \to B^0$ is a cofibration in the model structure of 3.1.
2. $(S^1 \otimes B^n) \cup (S^1 \otimes A^n) A^{n+1} \to B^{n+1}$ is a cofibration for the model structure of 3.1 for all $n \geq 0$.

The simplicial hom is given by $\text{hom}(A, B)_n = \text{hom}(A \otimes \Delta^n, B)$.

Theorem 3.4. (see [8, Theorems 2.6 and 3.6]). There is a model structure on $Ch_R$ in which:

1. The weak equivalences are the quasi-isomorphisms.
2. A map $f$ is a cofibration iff for each $n \in \mathbb{N}$, $\tau(f[n])$ is a cofibration for the model structure of 3.1.

Moreover there is a Quillen equivalence

$$\mathcal{X} : Spt(sSh_R) \leftrightarrows Ch_R : Z.$$  

As a consequence of the preceding theorem

$$\text{Ho}(Spt(sSh_R))$$

is equivalent to the usual derived category of $\mathcal{R}$-modules.

The right adjoint $Z$ in the Quillen equivalence of 3.4 is given by $Z(A)^n = \Gamma(\tau(A[n]))$. The bonding map is a composite

$$S^1 \otimes \Gamma(\tau(A[n])) \to \bar{W}(\tau(A[n])) \cong \Gamma(\tau(A[n]))[-1] \xrightarrow{\Gamma(\sigma)} \Gamma(\tau(A[n+1]))$$

where $\sigma : \tau(A[n])[-1] \to \tau(A[n+1])$ is the obvious map and $\bar{W}$ is the ‘simplicial loop group functor’ ([6, V.7.7]) applied sectionwise.

If $\beta$ is an uncountable cardinal, we call a spectrum of simplicial $\mathcal{R}$-modules $A$ $\beta$-bounded iff each $A^n$ is $\beta$-bounded.

Let $\beta = 2^\gamma + 1$, where $\gamma > |\text{Mor}(\mathcal{C})|$, $\alpha$, with $\alpha$ as in 2.16. Write

$$Spt_{\mathcal{R}, \beta}$$

for the full subcategory of $Spt(sSh_R)$ consisting of objects are a set of representatives of the isomorphism classes of $\beta$-bounded simplicial $\mathcal{R}$-module spectra.
**Definition 3.5.** We define a presheaf of fibrant simplicial categories $SPT_\mathcal{R}$, such that the objects of $SPT_\mathcal{R}(U)$ are the objects in the image of the restriction of sites map $\text{Spt}_{\mathcal{R},\beta} \to \text{Spt}(s\text{Sh}_{\mathcal{R}|U})$ and such that $\text{hom}_{SPT_\mathcal{R}(U)}(A, B) = \text{hom}_{s\text{Sh}_{\mathcal{R}|U}}(A, B)$.

**Lemma 3.6.** *Restriction of sites* 

$$r_\phi : \text{Spt}(s\text{Sh}_{\mathcal{R}}) \to \text{Spt}(s\text{Sh}_{\mathcal{R}|U})$$

is a left and right Quillen adjunction. If $\eta_\phi$ is the left adjoint of $r_\phi$, then the unit $r_\phi \eta_\phi \to \text{id}$ is an isomorphism.

**Proof.** The generating cofibrations of the model structure of 3.1 are of the form $\mathcal{R}(f)$, where $f$ is a generating cofibration for the injective model structure on $s\text{Sh}(\mathcal{C})$ and $\mathcal{R}$ is the free $\mathcal{R}$-module functor. Thus, restriction of sites preserves cofibrations in the model structure of 3.1. It follows that it preserves cofibrations in the model structure of 3.3. Restriction preserves weak equivalences, so we have a left Quillen functor.

Restriction is a right Quillen functor for the model structure of 3.1 since it preserves injective fibrations. Its left adjoint is a left Quillen functor, and thus by definition the left adjoint of $r_\phi$ preserves cofibrations. $r_\phi$ preserves fibrations and trivial fibrations, as required.

The final statement is trivial. 

**Theorem 3.7.** $\mathcal{B}(SPT_\mathcal{R})$ satisfies quasi-injective descent.

**Proof.** We use 2.19. Condition 1 is 3.6. Condition 2 and 6 are trivial. Condition 3 follows from the argument of [9, Theorem 4.8] and the fact that the model structure for simplicial $\mathcal{R}$-module spectra is cofibrantly generated. By taking $\beta$ large enough, sheafification preserves $\beta$-bounded objects. Thus, so does the left adjoint of restriction, and Condition 4 holds. Condition 5 follows from an easy adjointness argument.

**Example 3.8.** The big Zariski, etale and flat sites of a (not necessarily Noetherian) scheme $X$ satisfy the hypotheses of 2.16 (note that by bounding the size of the schemes over $X$, we can take these to be small sites). Thus, 3.7 applies to the usual geometric contexts.
4 An Application: Glueing Chain Complexes

An intuitively appealing definition of higher stacks is a presheaf of categories in which we can glue objects together along some notion of weak equivalence. The purpose of this section is to demonstrate the relationship between the more abstract definition of higher stacks in terms of descent and the intuitive notion. That is, if \( \mathcal{B}(M^\circ) \) satisfies descent for a presheaf of simplicial model categories \( M \), then \( M \) satisfies a generalization of the effective descent condition involving weak equivalences (4.10).

Suppose that \( R \subseteq \text{hom}(-, U) \) is a covering sieve. Following Giraud (see [5]), an effective descent datum on the sieve \( R \) for a sheaf of groupoids \( H \) consists of:

1. Objects \( x_\phi \in H(V) \), one for each morphism \( \phi : V \to U \) in \( R \).
2. Morphisms \( x_\phi \xrightarrow{\alpha} \alpha^*(x_\psi) \) in \( H(V) \), one for each diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\alpha} & W \\
\phi \downarrow & & \downarrow \\
U, & & \\
\end{array}
\]

such that the diagram

\[
\begin{array}{ccc}
x_\phi & \xrightarrow{\beta_*} & \alpha^*(x_\psi) \\
(\alpha \circ \alpha)^* & \downarrow & \downarrow \alpha^* \beta_* \\
(\beta \circ \alpha)^*(x_\zeta) & = & \alpha^* \beta^*(x_\zeta)
\end{array}
\]

commutes for each composable pair of morphisms

\[
\begin{array}{ccc}
V & \xrightarrow{\alpha} & W & \xrightarrow{\beta} & W' \\
\phi \downarrow & \psi \downarrow & \downarrow \zeta & \downarrow & \\
U & \xrightarrow{\zeta} & & & \\
\end{array}
\]

The following discussion is taken from [10, pg. 276-277].

Let \( R \subseteq \mathcal{C}/U \) be a covering sieve of the object \( U \). Then there is a functor \( s : R \to \text{Pre}(\mathcal{C}) \), which associates to each object the representable functor
hom(−, U). If we consider the presheaf of groupoids $E_R$ given by

$$U \mapsto \int s(U),$$

where $\int$ is the Grothendieck construction, then the universal property of the Grothendieck construction implies that effective descent data for $H$ is equivalent to a morphism of presheaves of groupoids $E_R \to H$.

We have the following internal description of a presheaf $E_R$ associated to $R$. Its presheaf of objects is $\text{Ob}(E_R) = \coprod_{\phi : V \to U} \text{hom}(−, V)$, where the coproduct runs over all morphisms in $R$ whose target is $U$, and its presheaf of morphisms is

$$\text{Mor}(E_R) = \coprod_{W \to V \to U} \text{hom}(−, W)$$

where the coproduct runs over all composable $W \to V \to U$ in $R$. The source and target maps

$$s, t \coprod_{V \to W \to U} \text{hom}(−, V) \to \coprod_{W' \to U} \text{hom}(−, W').$$

are, respectively, the map that ‘forgets’ the map $f$ on each factor and the map given by composition with $f$ on each factor.

If $R \subseteq S$ is a refinement, then there is a natural map $E_R \to E_S$. If $\tilde{U}$ denotes the effective descent datum associated to the (trivial) covering that contains the identity, then the map $E_R \to \tilde{U}$ induces a map of groupoids

$$\Upsilon_{R,H} : H(U) \cong \text{hom}(\tilde{U}, H) \to \text{hom}(E_R, H).$$

We say that $H$ satisfies effective descent with respect to $R$ if this map is essentially surjective. Traditionally, a sheaf of groupoids $H$ is said to be a stack iff $H$ satisfies effective descent with respect to all covers.

Recall that there is a global projective model structure on $sPre(\mathcal{C})$ in which the weak equivalences are the sectionwise weak equivalences and the fibrations are sectionwise Kan fibration. A projective cofibrant simplicial presheaf is one which is cofibrant for this model structure. This is a simplicial model structure, with simplicial hom the same for the injective (Jardine) model structure.

**Lemma 4.1.** Suppose that $X$ is a simplicial presheaf such that:
1. $X_n$ is a coproduct of representables.

2. $X_n$ can be written as a coproduct of its non-degenerate and degenerate simplices.

The $X$ is global projective cofibrant.

Proof. See the proof of [2, Lemma 2.7].

**Lemma 4.2.** $B(E_R)$ is global projective cofibrant.

Proof. First, it follows from the internal description of $E_R$ that the $n$-simplices of $B(E_R)$ are

$$
\coprod_{V_1 \rightarrow V_2 \rightarrow \cdots \rightarrow V_{n+1} \rightarrow X} S_{f_1, \ldots, f_n, \phi}
$$

where $S_{f_1, \ldots, f_n, \phi}$ is the subpresheaf of

$$
\hom(-, V_1) \times \hom(-, V_2) \times \cdots \times \hom(-, V_{n+1})
$$

such that $S_{f_1, \ldots, f_n, \phi}(W)$ consists of $(a_1, a_2, \ldots, a_n)$ such that $a_{i+1} = (f_i \circ a_i)$. That is, it is isomorphic to $\hom(-, V_1)$. Furthermore, the non-degenerate $n$-simplices are precisely those corresponding to summands where no $f_i$ is an identity map. The result now follows from [4,1].

**Lemma 4.3.** Suppose that $X$ satisfies injective descent. Let $R$ be a covering of an object $U \in \text{Ob}(C)$. Then $\pi_0 \hom(B(\bar{U}), X) \rightarrow \pi_0 \hom(B(E_R), X)$ is surjective.

Proof. By the preceding lemma, the map

$$
\pi_0 \hom(B(E_R), X) \rightarrow \pi_0 \hom(B(E_R), L_{inj}(X))
$$

can be identified with the map $[B(E_R), X]_{GProj} \rightarrow [B(E_R), L_{inj}(X)]_{GProj}$ induced by $X \rightarrow L_{inj}(X)$, where $[,]_{GProj}$ denotes maps in the homotopy category of the global projective model structure, and is thus a bijection. Now, consider the diagram

$$
\pi_0 \hom(B(\bar{U}), X) \rightarrow \pi_0 \hom(B(\bar{U}), L_{inj}(X)) \rightarrow \pi_0 \hom(B(E_R), X) \rightarrow \pi_0 \hom(B(E_R), L_{inj}(X))
$$
The top horizontal arrow can be identified with $\pi_0X(U) \to \pi_0(L_{inj}(X))$, and is thus a bijection. The right vertical map surjective, since $B(E_R) \to B(U)$ is a local weak equivalence by [10, Lemma 9.29] and $L_{inj}(X)$ is injective fibrant. Thus, we conclude that the left vertical map is surjective, as required.

**Lemma 4.4.** There is an isomorphism $\mathcal{B}(A) \cong B(A)$, natural in discrete simplicial categories $A$.

**Proof.** The functor $\pi_0$ is left adjoint to the functor which regards a category as a discrete simplicial category. Thus, we have isomorphisms

$$\mathcal{B}(A)_n \cong \text{hom}(\mathcal{C}(\Delta^n), A) \cong \text{hom}(\pi_0\mathcal{C}(\Delta^n), A) \cong \text{hom}([n], A) \cong B(A)_n.$$

**Lemma 4.5.** Suppose that $A$ is a discrete simplicial category and $M$ is a simplicial category. Then any map $\mathcal{B}(A) \to \mathcal{B}(M)$ factors as

$$\mathcal{B}(A) \to \mathcal{B}(M_0) \to \mathcal{B}(M).$$

**Proof.** By Yoneda, the map

$$\mathcal{B}(A)_n = \text{hom}(\mathcal{C}(\Delta^n), A) \to \text{hom}(\mathcal{C}(\Delta^n), M) = \mathcal{B}(M)_n$$

is induced by a map $A \to M$, which factors through $A \to M_0 \to M$. These factorizations are compatible with degeneracy and face maps, as required.

Recall that given a simplicial category $M$, we can an object of $M_0$ an equivalence iff it represents an isomorphism in $\pi_0M$.

**Lemma 4.6.** Suppose that $M$ is a fibrant simplicial category and $\phi : [1] \to M_0$ be an equivalence of $M$. Then $\mathcal{B}(\phi) : B([1]) \to \mathcal{B}(M)$ factors through $J\mathcal{B}(M)$.

**Proof.** We have a natural equivalence $P\mathcal{B} \simeq \pi_0$, so that $P\mathcal{B}(\phi)$ represents an isomorphism in $P\mathcal{B}(M)$. The result follows from the description of $J$ in [1,1].

If $M$ is a fibrant simplicial category, then write $M_{eq}$ for the subcategory of $M_0$ consisting of equivalences of $M$. If $M$ is a presheaf of fibrant simplicial categories, then we write $M_{eq}$ for the presheaf defined by $U \mapsto M(U)_{eq}$.
Lemma 4.7. Suppose that $M$ is a presheaf of fibrant simplicial categories such that $\mathcal{B}(M)$ satisfies quasi-injective descent. Let $R$ be a covering of an object $U$. Let $\Upsilon_{R,M_{eq}} : M_{eq}(U) \to \text{hom}(E_R, M_{eq})$ be the map of $\mathcal{B}$. Then if $\sigma \in \text{hom}(E_R, M_{eq})$, there exists a morphism

$$\Upsilon_{R,M_{eq}}(\sigma') \to \sigma$$

in $\text{hom}(E_R, M_{eq})$ for some object $\sigma'$.

Proof. Consider the map

$$\mathcal{B}(\sigma) : B(E_R) \to \mathcal{B}(M).$$

By 4.6, we can regard $\mathcal{B}(\sigma)$ as a map $\gamma : B(E_R) \to JB(M)$. By 4.3 and the fact that $JB(M)$ satisfies injective descent, we can find a map

$$B(E_R) \times \Delta^1 \xrightarrow{f} JB(M)$$

such that $f|_{B(E_R) \times \{0\}} = B(\Upsilon_R(\sigma'))$ and $f|_{B(E_R) \times \{1\}} = \gamma$. Since $B(E_R) \times \Delta^1 \cong B(E_R \times [1])$, 4.5 and 4.4 imply that $f$ factors as

$$B(E_R \times [1]) \xrightarrow{f'} B(M_{eq}) \to JB(M)$$

and the map $f'$ gives a natural transformation $E_R \times [1] \to M_{eq}$ between $\sigma$ and $\Upsilon_R(\sigma')$.

Suppose that $M$ is a presheaf of model categories. Regarding $M$ as a functor to the category of big categories, let $\int M$ denote the Grothendieck construction. Write $\text{Sect}(\mathcal{C}^{op}, \int M)$ for the sections of the projection map $\int M \to \mathcal{C}^{op}$.

Remark 4.8. It is easy to see that there is an isomorphism of categories $\text{Sect}(\mathcal{C}^{op}, \int M) \cong \text{hom}(E_R, M)$.

The following is [7, Theorem 17.1].

Theorem 4.9. Suppose that $M$ is a presheaf of model categories such that each restriction map is a left Quillen functor and for each $U \in \text{Ob}(\mathcal{C})$, $M(U)$ is cofibrantly generated. Then there exists a model structure on $\text{Sect}(\mathcal{C}^{op}, \int M)$ such that
1. The weak equivalences are maps $X \to Y$ such that $X(U) \to Y(U)$ is a weak equivalence in $M(U)$ for each $U \in \text{Ob}(\mathcal{C})$.

2. The fibrations are maps $X \to Y$ such that $X(U) \to Y(U)$ is a fibration in $M(U)$ for each $U \in \text{Ob}(\mathcal{C})$.

**Theorem 4.10.** Suppose that $M$ is a presheaf of simplicial model categories such that $\mathfrak{B}(M^\circ)$ satisfies descent and $M$ satisfies the hypotheses of 4.9. Let $M_{we} \subseteq M_0$ be such that $M_{we}(U)$ is the subcategory of $M_0(U)$ whose morphisms are the weak equivalences of $M$. Then for each

$$\sigma \in \text{hom}(E_R, M_{we})$$

where $R$ is a covering of some object $U$, then there exists a zig-zag

$$\Upsilon_{R,M_{we}}(\sigma'') \to \sigma' \leftarrow \sigma$$

in $\text{hom}(E_R, M_{we})$.

**Proof.** Using cofibrant and fibrant replacement in the model structure of 4.9, we can produce a zig-zag

$$\sigma'' \to \sigma' \leftarrow \sigma,$$

where $\sigma''$ lies in $M^\circ$. But $M_{eq}^\circ = M_{we}^\circ$, since $M$ is a simplicial model category. The result now follows from 4.7.

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