Spin and Statistics in Nonrelativistic Quantum Mechanics: The Spin-Zero Case

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It is proved from stated assumptions of nonrelativistic quantum mechanics based on the Schrödinger equation that identical spin-zero particles must obey symmetric statistics.

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I. INTRODUCTION

The connection between spin and statistics (CSS) appears in nonrelativistic quantum mechanics as a constraint on the solutions of the Schrödinger equation. Practitioners usually introduce coordinate and spin variables of individual particles as if the identities of individual particles were observable. For integer spin they then constrain the wave function to be symmetric under exchange of the labels of any pair. For half-integer spin, the wave function is made antisymmetric. Equivalently, the total spin \( S \) and the relative orbital angular momentum \( \ell \) of each pair are constrained to be both even or both odd. Alternatively, the machinery of Fock space is introduced with creation and annihilation operators obeying commutation relations designed to achieve the same end \[ \ldots \]. To date, the available experimental information confirms the CSS but one may still ask whether quantum mechanics requires that all particles must obey it.

Pauli’s original proof of the CSS depended upon the somewhat hazardous assumptions of local relativistic quantum field theory. That was followed by similar proofs based on more modern versions of quantum field theory \[ \ldots \]. Feynman, in his lectures \[ \ldots \], expressed the dissatisfaction with that kind of proof by saying, “. . . An explanation has been worked out by Pauli from complicated arguments of quantum field theory and relativity . . . we have not been able to find a way of reproducing his arguments on an elementary level. This probably means we do not have a complete understanding of the fundamental principle involved.” Duck and Sudarshan \[ \ldots \], in their recent extensive critical review of the relevant theoretical and experimental literature, found that Feynman’s challenge had not been answered satisfactorily.

Messiah and Greenberg \[ \ldots \] investigated the status of the CSS within nonrelativistic quantum mechanics by rigorously analyzing the consequences of the particles’ being identical for the behavior of wave functions under permutations. They found that the minimal assumptions of quantum mechanics that they used permit symmetric, antisymmetric, and intermediate statistics for identical particles of all spins.

Leinaas and Myrheim \[ \ldots \] introduced a new approach by taking seriously the assumption that the quantum mechanical variables representing physical observables should stand in one-to-one correspondence with those observables. For two identical spinless particles with no other observables the configuration space consists of the unordered pairs of vectors \( \{ r_1, r_2 \} \), for which the subscripts label points in space, not particles, and \( \{ r_2, r_1 \} \) represents the same point in the six-dimensional configuration space as does \( \{ r_1, r_2 \} \). For particles with non-zero spins, the spin variables must also be included, but that complication will not necessary for present purposes. Leinaas and Myrheim were able, by analyzing phases introduced by parallel transport in the so-defined space of \( \{ r_1, r_2 \} \), to eliminate intermediate statistics, but not to choose between symmetric and antisymmetric statistics for particles of any spin. Their work was later extended by Berry and Robbins \[ \ldots \], who gave it a rigorous mathematical foundation for all spins and showed exactly how different allowable assumptions about parallel transport relate to geometrical properties of the configuration space, including the geometrical Berry phase. The outcome for the connection between spin and statistics was, however, the same. Particles of any spin can have symmetric or antisymmetric statistics; neither is excluded by the physical assumptions that they used.

This paper is limited to the case of spinless particles. I follow Leinaas and Myrheim by identifying \( \{ r_2, r_1 \} \) with \( \{ r_1, r_2 \} \), but I additionally assume that the wave function \( \Psi(\{ r_1, r_2 \}) \) must be a continuous function of \( r_1 \) and \( r_2 \) because of the second derivatives in the Schrödinger equation. Those assumptions lead unambiguously to the result that identical spinless particles must be bosons, not fermions. The method used here is elementary, involving only the properties of rotation and angular momentum. Parallel transport and its connection with global geometric properties

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of the configuration space are not needed.

II. ASSUMPTIONS

Assumption 1. The only independent dynamical variables are the positions and momenta of the particles. There are no spins or other internal variables. Then the wave function for a single particle has one component and is scalar under rotation. In other words, \( \Psi(\mathbf{r}) = \Psi(\mathbf{R}^{-1}\mathbf{r}) \) for any rotation \( \mathbf{R} \).

Assumption 2. For any two spinless particles, the wave functions are products of scalar, one-component single-particle wave functions or linear combinations of such products. For identical spinless particles, the configuration space is that of Leinaas and Myrheim, consisting of the unordered pairs \( \{\mathbf{r}_1, \mathbf{r}_2\} \), and the wave functions are scalar, one-component functions \( \Psi(\{\mathbf{r}_1, \mathbf{r}_2\}) \), where the subscripts label points in space, not particles. This assumption is, as mentioned above, motivated by the principle of quantum mechanics that the dynamical variables should be observable, at least in principle.

Assumption 3. The wave function \( \Psi(\{\mathbf{r}_1, \mathbf{r}_2\}) \) is a continuous function of the variables \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \). Continuity is required by the appearance of \( \nabla_1^2 \) and \( \nabla_2^2 \) in the Hamiltonian. It is in fact possible to avoid continuity by making a gauge transformation that results in a discontinuous vector potential in the Hamiltonian, but that will not change the physics and I will not consider it further.

Assumption 4. In extending the conclusions from two identical particles to many, I will make use of an assumption of asymptotic separability to be explained below.

III. TWO SPINLESS PARTICLES

The domain of the relative coordinate \( \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \) is only half of three-dimensional space because \( \mathbf{r} \) and \( -\mathbf{r} \) cannot both be in that domain since no observation can distinguish between them. For fixed \( r = |\mathbf{r}| \), Leinaas and Myrheim represent that domain graphically by a hemisphere of radius \( r \) whose base is a circle in the \( z = 0 \) plane but with only half of that circle included.

\[
\Psi(\{\mathbf{r}_1, \mathbf{r}_2\}) = \sum_{\ell m} a_{\ell m}(\mathbf{R}, r) Y_{\ell m}(\vartheta, \varphi),
\]

where \( \mathbf{R} \) is the center-of-mass coordinate and the domain of the relative coordinate \( \mathbf{r} \) is given by

\[
\begin{align*}
r &\geq 0, \\
0 &\leq \vartheta \leq \frac{\pi}{2}, \\
0 &\leq \varphi < \begin{cases} 2\pi, & \text{for } \vartheta < \pi/2, \\
\pi, & \text{for } \vartheta = \pi/2. \end{cases}
\end{align*}
\]

The domain of \( \mathbf{r} \) defined here differs technically from that of Leinaas and Myrheim in that the one defined here is simply connected. No two points \( \mathbf{r} \) and \( -\mathbf{r} \) need to be identified in that domain because \( -\mathbf{r} \) is never in the domain of \( \mathbf{r} \). The choice of the hemisphere based on the \( z \) axis is of course arbitrary. Any other hemisphere would represent the same physics.

The sets \( Y_{\ell m} \) for even and for odd \( \ell \) are separately complete on the hemisphere. It is proved in Appendix A below that even values and odd are superselected from each other; both cannot appear in one wave function. What remains is to eliminate one of them.

Consider a point \( \{\mathbf{r}_1, \mathbf{r}_2\} \) corresponding to

\[
\begin{align*}
\mathbf{r}_1 &= (x, y, \varepsilon), & \mathbf{r}_2 &= (x, -y, -\varepsilon), & \mathbf{r} &= (2x, 2y, 2\varepsilon).
\end{align*}
\]

For infinitesimal \( \varepsilon \), that point must be infinitesimally close to the point

\[
\begin{align*}
\mathbf{r}_1 &= (x, y, -\varepsilon), & \mathbf{r}_2 &= (x, -y, \varepsilon), & \mathbf{r} &= (-2x, -2y, 2\varepsilon).
\end{align*}
\]

The two points have the same \( r \) and \( \vartheta \), but differ in \( \varphi \) by \( \pi \). Continuity under changes in \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) requires

\[
\Psi(\mathbf{R}, r, \vartheta, \varphi) \rightarrow \Psi(\mathbf{R}, r, \vartheta, \varphi \pm \pi)
\]
as \( \vartheta \to \pi/2 \) from below.

\[
\Psi(R, r, \vartheta, \varphi) \to \sum_{\ell m} a_{\ell m}(R, r) Y_{\ell m}\left(\frac{\pi}{2}, \varphi\right)
\]

Eqs. (6) are consistent only if \( Y_{\ell m}\left(\frac{\pi}{2}, \varphi\right) \) vanishes for all odd values of \( m \). That is the case when \( \ell \) is even but not when \( \ell \) is odd. Therefore, \( \ell \) must be even and the two identical spinless particles must have symmetric statistics.

IV. DISCUSSION

It has been proved, under stated general assumptions of quantum mechanics, that two identical spinless particles with no internal degrees of freedom must have even relative orbital angular momentum, which implies that they are bosons, not fermions. This proof did not make use of relativity or of quantum field theory. The approach used here is based on the requirement that the point \( \{r_1, r_2\} \) in the configuration space for two identical spinless particles is the same point as \( \{r_2, r_1\} \). This approach was enabled to go beyond previous work departing from the same requirement and to find unambiguously that the two spinless particles are bosons by the introduction of the additional requirement that wave functions must be continuous under variations of the particle coordinates \( r_1 \) and \( r_2 \) because of the second derivative in the Hamiltonian and the Schrödinger equation, and by consideration of the relative orbital angular momentum. The continuity in \( r_1 \) and \( r_2 \) was used to relate the wave function at a relative coordinate \( r \) near the relative \( z=0 \) plane to the wave function at a rotated point. No question of multiple-valued wave functions arose because the domain of \( r \) is simply connected.

The extension of the spin-statistics connection to many particles has been given in a general way by Berry and Robbins [8]. Here, to complete the discussion of spinless particles, I give a simple heuristic proof that should apply to any theory that is asymptotically separable in the sense that moving all particles except two to a great distance is the same as removing them; the motion of the two remaining particles is unaffected by the presence or absence of other particles a great distance. For many particles, the configuration space consists of the unordered multiplets \( \{r_1, r_2, r_3, \ldots, r_N\} \), where the subscripts label points in space, not particles. Select any pair of \( r_j \) and consider a wave function \( \Psi(\{u, v, w\}) \), where \( u \) and \( v \) are the two selected \( r_j \), \( w \) stands for all the other \( r_j \), and \( s \) is a scale factor. For sufficiently large \( s \), it is assumed that the dynamics in the neighborhood of \( u \) and \( v \) is unaffected by the existence of the remaining particles. Then the relative orbital angular momentum \( \ell \) of the particles at those points must be even. Now reduce \( s \) continuously to \( s = 1 \). If the wave function is to be continuous, \( \ell \) values cannot jump so they must remain even. Then the relative angular momentum of each pair is even and the particles are bosons.

This paper addresses the motion of identical particles in all of three-dimensional space. Boundary conditions, even ones that confine two particles to two separate boxes, appear not to challenge the results because such boundary conditions are idealizations. In physical reality they can be replaced by sufficiently high potential barriers, whose existence is not precluded by the the assumptions used here. The same is true of the Aharonov-Bohm effect [10], where real flux lines have nonvanishing thickness and nonvanishing penetrability. The hypothetical case of the motion of charged particles near a Dirac monopole, where the multiply-connected domain of the individual particle coordinates is not merely an idealized limit, is not covered by the results found here, nor apparently in other treatments of the connection between spin and statistics.

The methods used here are not directly applicable in their present simple form to particles with spin because the spatial continuity condition alone is insufficient to determine the relative phase of \( \Psi(\varphi) \) and \( \Psi(\varphi \pm \pi) \) at \( \vartheta = (\pi/2) - \varepsilon \) when \( \Psi \) contains spinors in addition to its spatial variables.

Berry and Robbins, in what they call a perverse case, give a wave function for two identical spinless fermions which in the present notation would be

\[
\Psi(R, r, \vartheta, \varphi) = \frac{r}{R} \sum_{\ell m} a_{\ell m}(R, r) Y_{\ell m}\left(\frac{\pi}{2}, \varphi\right).
\]

Under the standard assumptions of quantum mechanics used here, this is not an admissible wave function for two spinless particles because it is a three-component vector, not a one-component scalar, under rotation.

The proof of the CSS in nonrelativistic theory differs fundamentally from the proofs in quantum field theory. In nonrelativistic theory, states with different numbers of identical particles are superselected from each other and live in spaces with different topologies. In quantum field theory states with different particle numbers are not superselected and the spatial variables are only labels and have the same topology for all states.
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APPENDIX A: SUPERSELECTION OF EVEN FROM ODD ANGULAR MOMENTUM

Superselection of even from odd $\ell$ is required by the condition that the rotation of two spinless particles through angle $\pi$ about an axis through their center of mass and perpendicular to their relative coordinate must restore their initial physical state. Consider an initial state $|r_0\rangle$ with $0 < \vartheta_0 < \pi/2$. For this purpose, the irrelevant center-of-mass coordinate $R$ is suppressed.

Define unit vectors $n_j$ and “body-fixed” angular momentum projections $K_j$ by

$$
n_3 = \frac{r_0}{r_0}, \quad n_1 = \frac{\hat{z} \times r_0}{|\hat{z} \times r_0|}, \quad n_2 = -n_3 \times n_1,
$$

where $K_j = n_j \cdot L$, $[K_i, K_j] = i\epsilon_{ijk}K_k$, $K_1^2 + K_2^2 + K_3^2 = L^2$, $|r_0\rangle = \sum_{\ell\mu} \alpha_{\ell\mu}(r_0) |\ell\mu\rangle_1$, $K_1|\ell, \mu\rangle_1 = \mu|\ell, \mu\rangle_1$, $L^2|\ell, \mu\rangle_1 = \ell(\ell + 1)|\ell, \mu\rangle_1$.

$K_1$ generates the rotations of $|r_0\rangle$ about an axis perpendicular to $|r_0\rangle$. A rotation of $r_1$ and $r_2$ through angle $\pi$ around that axis must restore the initial state except for a possible phase factor $e^{i\delta}$. In terms of the relative $r$, that rotation appears as a rotation through angle $(\hat{z} - \vartheta_0)$ down to the $z = 0$ plane, followed by a rotation through $(\hat{z} + \vartheta_0)$ for a total angle of $\pi$ to restore $|r_0\rangle$

$$
e^{\pi K_1} |r_0\rangle = \sum_{\ell\mu} \alpha_{\ell\mu}(r_0) e^{i\pi \mu} |\ell\mu\rangle = e^{i\delta} \sum_{\ell\mu} \alpha_{\ell\mu}(r_0) |\ell\mu\rangle .
$$

In writing Eq.(A4), I have implicitly assumed that the angular momentum eigenfunctions are continuous under infinitesimal changes of $r_1$ and $r_2$, as in Section III above.

Eqs.(A2) constrain the values of $\mu$ to be integers or integers plus one-half. Then, Eq.(A4) requires that

$$
\mu = 2n + \delta ,
$$

where the values of $n$ are integers and $\delta$ is zero, one-half, one, or one-and-one-half. If states with different $\delta$ exist, they are superselected from each other and cannot appear in the same wave function. In other words, even $\mu$ are superselected from odd.

The state $|r_0\rangle$ is an eigenfunction of $K_3$ with eigenvalue equal to zero. From Eqs.(A2),

$$
|r_0\rangle = \sum_{\ell} \beta_{\ell} |\ell, 0\rangle_3 ,
$$

for some $\beta_{\ell}$, where $|\ell, 0\rangle_3$ is an eigenfunction of $L^2$ and $K_3$ with eigenvalues $\ell(\ell + 1)$ and zero, respectively. Then the values of $\ell$, and consequently of $\mu$, must be integers.

The eigenfunctions of $K_1$ are related to those of $K_3$ by a rotation through angle $(\pi/2)$ around the $n_2$ axis. Therefore, from Eq.(A4),

$$
|r_0\rangle = \sum_{\ell\mu} D_{\mu 0}^{\ell}(\mathcal{R}) \beta_{\ell} |\ell, \mu\rangle_1 .
$$

Here, $\mathcal{R}$ is a rotation through angle $(\pi/2)$ around the $n_2$ axis, which carries $n_3$ into $n_1$, and $D_{\mu 0}^{\ell}(\mathcal{R})$ is the rotation matrix for wave functions of angular momentum $\ell$. For the $\pi/2$ rotation, all the $D_{\mu 0}^{\ell}(\mathcal{R})$ vanish except for even values of $\ell$-$\mu$. Then, since even $\mu$ are superselected from odd, even $\ell$ must likewise be superselected from odd.
This proof has been given for a single $|r_0\rangle$, but continuity assures that one selection of even versus odd applies to all $|r_0\rangle$.

[1] H.J. Lipkin, *Quantum Mechanics: New Approaches to Selected Topics*, (North-Holland, Amsterdam, London, 1973).
[2] W. Pauli, Phys. Rev. **58**, 716 (1940).
[3] e.g., S. Weinberg, *The Quantum Theory of Fields*, (Cambridge Univ. Press, 1975) Vol.1, Chap. 5.
[4] R. P. Feynman, *The Feynman Lectures of Physics* (Addison-Wesley, 1965) Vol. 3
[5] I. Duck and E. C. G. Sudarshan, Am. J. Phys. **66**, 284 (1998).
[6] A. M. L. Messiah and O. W. Greenberg, Phys. Rev. B **248**,136 (1964).
[7] J. M. Leinaas and J. Myrheim, Nuovo Cim. B **37**, 1 (1977).
[8] M. V. Berry and J. M. Robbins, Proc. Roy. Soc. Lond. A **453**, 1771 (1997).
[9] M. V. Berry and J. M. Robbins, J. Phys. A: Math. Gen. **33**, L207 (2000).
[10] M. Peshkin and A. Tonomura, *The Aharonov-Bohm effect*, (Springer Verlag, 1989).