THE COUPLING CONSTANTS AND MASSES OF THE STANDARD MODEL AS SYMMETRY NORMALIZATIONS

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Abstract

The numerical input for the quantitative consequences of the electroweak standard model, the hypercharge and isospin coupling constants and the Higgs field ground state mass value, are interpreted as normalizations of the symmetries involved. Using an additional statistical argument, a first order quantitative determination of the Weinberg angle as a normalization ratio gives the experimentally acceptable value $\tan^2 \theta = \frac{1}{3}$. 
Introduction

The standard model of the electroweak interactions\cite{1} derives its quantitative predictions with the experimental input of two independent coupling constants $g_1, g_2$ for the hypercharge $U(1)$ and isospin $SU(2)$ gauge fields resp. and the harmonic particle analysis fixing ground state value $M$ of the Higgs field. To determine those relevant scales and therewith the Weinberg ratio $\frac{g_1^2}{g_2^2} = \tan^2 \theta$ and the masses of the weak bosons, different strategies and model extensions are proposed in the literature, e.g. the imbedding of $U(1)$ and $SU(2)$ as subgroups of larger groups or substructures, etc.

In this paper the scales mentioned above are interpreted as symmetry normalizations which are determined by the representations of the relevant groups: $g_1^2, g_2^2$ as normalizations of $U(1)$ and $SU(2)$, the neutral weak boson mass as normalization of the spin group in the Lorentz group $O(3) \subset O(1,3)$ and the fine structure constant as normalization of the polarization group $O(2) \subset O(1,3)$. The ground state mass $M^2$ is related to the normalization of the electroweak $U(2)$.

In addition to such a qualitative interpretation of the electroweak scales a quantitative attempt is given to determine with such a strategy the value of the Weinberg angle. This proposal relies on the experimental observation that the electroweak group seems not to be a direct product, but the product $U(2) = U(1)_2 \circ SU(2)$ with the nontrivial hypercharge-isospin correlation\cite{2} \cite{3} $U(1)_2 \cap SU(2) = \{ \pm 1 \}$. With an additional statistical argument a first approximation of the normalization ratio for the defining representation gives the value $\tan^2 \theta = \frac{1}{3}$. 
Chapter 1
Masses and Coupling Constants

In this chapter, the coupling constants and masses for the standard model vector fields are given in a formulation adapted for an interpretation as group normalization constants.

1.1 Particle and Interaction Normalizations in the Standard Model

Obviously, coupling constants and masses play an important role in the standard model interactions\[1\]. They allow the experimental tests of the model by quantifying the transition from the interaction oriented field language to the ground state related particle language.

The standard model ground state properties are implemented by the complex 2-dimensional Higgs field $\varphi^{\alpha=1,2}(x)$ with a mass $M$ as ground state value, characterizing the minima of the potential $V(\varphi)$

$$V(\varphi) = \lambda_0(\varphi^*\varphi - M^2)^2, \quad \lambda_0 > 0$$

$$\langle \varphi(x) \rangle = \underline{\varphi} = \left( \begin{array}{c} 0 \\ M \end{array} \right), \quad M \neq 0 \quad (1.1)$$

Therewith a translation invariant internal reference system is fixed for the hyperisospin field symmetry group $U(2) = U(1_2) \circ SU(2)$ up to a remaining abelian particle symmetry with respect to the electromagnetic $U(1)_+ \cong U(1)$ subgroup

$$\varphi(x) \mapsto u(\alpha_0, \vec{\alpha}) \varphi(x), \quad u(\alpha_0, \vec{\alpha}) = e^{i\alpha_0 \frac{1_2 + \vec{\alpha}}{\sqrt{2}}} \in U(2)$$

$$\underline{\varphi} \mapsto u(\alpha_+) \underline{\varphi}, \quad u(\alpha_+) = e^{i\alpha_+ \frac{1_2 + \tau_3}{\sqrt{2}}} \in U(1)_+ \quad (1.2)$$

The electromagnetic $U(1)_+$ is spanned with the projector $\frac{1_2 + \tau_3}{2}$ connecting hypercharge $U(1_2) \cong U(1)$ and isospin $SU(2)$. With respect to the Killing form the $\sqrt{2}$-normalization is used in the $U(2)$ Lie algebra basis $\{i\frac{1_2}{\sqrt{2}}, i\frac{\tau_a}{\sqrt{2}}\}$ with the convenient 'double' trace $\text{tr} \frac{\tau_a}{\sqrt{2}} \frac{\tau_b}{\sqrt{2}} = \delta_{ab}$.

With regard to the internal symmetries the real 3-dimensional Goldstone manifold $U(2)/U(1)_+$ characterizes the transition from fields to particles. In contrast to the $U(2)$-symmetry for fields, particles can have only a nontrivial
electromagnetic $U(1)$-symmetry. All particles come as $SU(2)$-singlets - there is no mass degenerated nontrivial particle isomultiplet.

The kinetic terms for the Higgs field in connection with the vector gauge fields $B_j^{1,2,3}(x)$ for hypercharge $U(1)_{\mu}$ symmetry and $\vec{W}^j(x)$ for isospin $SU(2)$ symmetry are given in the Lagrangian

$$L(\varphi, B, W) = |\partial_j - i\frac{B_j + \vec{W}_j}{\sqrt{2}}\varphi|^2 - V(\varphi)$$
$$+ F^{jk}\partial_j B_k - \partial_j F^{jk} + g_1 F^{jk} F_{jk}$$
$$+ \tilde{F}^{jk}\partial_j \tilde{W}_k - \partial_j \tilde{F}_{jk} + g_2 \tilde{F}^{jk} \tilde{F}_{jk} - \frac{(\tilde{F}^{jk} \times \tilde{W}_j)\tilde{W}_k}{2}$$

(1.3)

The gauge fixing terms are omitted. For the vector fields a 1st order derivative formalism is used with gauge fields and curvatures as canonical pairs, $(B, F)$ and $(\tilde{W}, \tilde{F})$, connected in the derivative terms. The curvatures arise by variation in the equations of motion

$$g_1^2 F^{jk} = \partial^k B^j - \partial^j B^k,$$
$$g_2^2 \tilde{F}^{jk} = \partial^k \tilde{W}^j - \partial^j \tilde{W}^k + \tilde{W}^k \times \tilde{W}^j$$

(1.4)

where the coupling constants show up as curvature normalizations. $g_1^2$ and $g_2^2$ are called internal normalizations for hypercharge and isospin interaction resp.

A more familiar notation uses the coupling constants in the current gauge field coupling with the rescaling

$$B = g_1 \frac{\vec{B}}{\vec{W}}, \quad F = \frac{1}{g_1} F \quad \tilde{F} = \frac{1}{g_2} \tilde{F}$$

$$\Rightarrow J^k B_k + \tilde{J}^k \tilde{W}_k = g_1 J^k \frac{\vec{B}}{\vec{W}} + g_2 \tilde{J}^k \tilde{W}_k$$

(1.5)

which shows that the coupling constants are dynamically relevant only for the case of a nontrivial interaction. Only the squares $g^2$ occur, therefore $g = +\sqrt{g^2}$ can be defined as positive definite. Positive and negative electromagnetic charges arise by $U(1)_+\text{-representations with positive or negative winding numbers.}$

The hyperisospin transformation on the gauge sector is given by

$$B_j \rightarrow B_j + \partial_j \alpha_0,$$  
$$\tilde{W}_j \rightarrow O(\vec{\alpha})(\tilde{W}_j) + \partial_j \vec{\alpha},$$

$$F^{jk} \rightarrow F^{jk},$$

$$\tilde{F}^{jk} \rightarrow O(\vec{\alpha})(\tilde{F}^{jk}),$$

$$O(\vec{\alpha}) \in SO(3) \cong U(2)/U(1)$$

(1.6)

The projection of the interaction fields to the ground state gives the vector particle mass terms

$$L(\varphi, B, W) = \left| \left( \frac{B_j - W_j^3}{w_j^1 + i w_j^2} \right) \right|^2 \frac{m^2}{2} + \ldots$$

$$+ F^{jk}\partial_j B_k - \partial_j F^{jk} + g_1^2 F^{jk} F_{jk}$$

(1.7)

diagonalizable with the Weinberg rotation $O(\theta)$ defining the massive vector field $Z^j(x)$ and the massless electromagnetic one $A^j(x)$

$$O(\theta) = \left( \begin{array}{cc} \cos \theta & \tan \theta \\ -\sin \theta & \cos \theta \end{array} \right) \in SO(2) :$$

$$\begin{cases} \left( \frac{F_A}{F_Z} \right) = O(\theta) \left( \frac{g_1 F}{g_2 F} \right) = O(\theta) \left( \frac{F_A}{F_Z} \right), \\
\left( \frac{A}{Z} \right) = O(\theta) \left( \frac{g_1 B}{g_2 W} \right) = O(\theta) \left( \frac{A}{Z} \right) \end{cases}$$

(1.8)
The Weinberg angle $\theta$ and the electromagnetic $\mathbf{U}(1)_+$-normalization (coupling constant) $e^2$ are related to the $\mathbf{U}(1)_2$ and $\mathbf{SU}(2)$ normalizations (coupling constants) $g_{1,2}^2$ by two conditions: The massive field $Z$ has to occur in the combination of the $M^2$-proportional term in the Lagrangian and the massless field $A$ has to be normalized with respect to the remaining gauge invariance

$$Z_j \sim B_j - W_j^3, \quad \Rightarrow \begin{align*}
\tan \theta &= \frac{g_1}{g_2} \\
O(\theta) &= \frac{1}{\sqrt{g_1^2 + g_2^2}} \left( \begin{array}{cc}
g_2 & g_1 \\
g_1 & -g_2 \end{array} \right)
\end{align*} \quad (1.9)$$

These two relations for $\theta$ and $e^2$ define the electroweak rectangular triangle$^3$ with the $\mathbf{U}(1)_+$-hypotenuse $\frac{1}{e}$ and the $\mathbf{U}(1)_2$ and $\mathbf{SU}(2)$ sides $\frac{1}{g_1}$ and $\frac{1}{g_2}$ resp. The $Z$-coupling constant $\frac{1}{g_2}$ is determined as the height of this triangle

$$eg_Z = g_1g_2 \Rightarrow g_Z^2 = g_1^2 + g_2^2 \quad (1.10)$$

The orthogonal Weinberg transformation $O(\theta)$ for the rescaled fields $B, F$ etc. is a special linear transformation $\mathcal{S}(\theta)$ for the unscaled fields $B, F$ etc., contragredient (inverse transposed, denoted by $-1T$) to each other for the canonical pairs

$$\mathcal{S}(\theta) = \left( \begin{array}{cc}
\cos^2 \theta & -\sin^2 \theta \\
\sin^2 \theta & \cos^2 \theta 
\end{array} \right) \in \mathbf{SL}(\mathbb{R}^2) \quad \Rightarrow \begin{align*}
\left( \begin{array}{c}
F_A \\
F_Z
\end{array} \right) &= \mathcal{S}(\theta) \left( \begin{array}{c}
F \\\nF_3
\end{array} \right) \\
\left( \begin{array}{c}
A \\
Z
\end{array} \right) &= \mathcal{S}(\theta) \left( \begin{array}{c}
B \\
W_3
\end{array} \right)
\end{align*} \quad (1.11)$$

The transformation $\mathcal{S}(\theta)$ shows the ground state induced direction $\mathbf{1}_2 + \tau_3$ in the first line as $(1, 1)$, i.e. $F + F_3$, the contragredient transformation $\tilde{\mathcal{S}}(\theta)$ involves the complementary 'broken' direction $\mathbf{1}_2 - \tau_3$ in the 2nd line $(-1, 1)$, i.e. $-W_3 + B$.

The normalizations (coupling constants) of the $\mathbf{U}(1)_2$ and $\mathbf{SU}(2)$ interaction fields on the one side and ground state projected fields on the other side are related to each other as follows

$$\mathcal{S}(\theta) \left( \begin{array}{c}
\frac{1}{g_1^2} \\
0 \\
\frac{1}{g_2^2}
\end{array} \right) \mathcal{S}(\theta)^T = \left( \begin{array}{cc}
1 & 0 \\
0 & \frac{1}{g_2^2}
\end{array} \right) \quad \text{for} \quad \tan^2 \theta = \frac{g_2^2}{g_1^2} \quad (1.12)$$

The Weinberg rotation from interaction fields to particle fields leads to the kinetic terms for the vector fields

$$\mathbf{L}(B, W) = F_A^{jk} \partial_j A_k - \partial_j A_k + e^2 F_A^{jk} F_A^{jkl}$$

$$+ \frac{M^2}{g_2^2} F_Z^{jk} \partial_j Z_k \frac{Z_k}{2} + g_2^2 F_Z^{jk} F_Z^{jkl} + \frac{M^2}{g_2^2} Z_j Z^l + \sum_{a=1,2} \left( \frac{F_a^{jk} \partial_j W_{ak} - \partial_j W_{ak}}{2} + g_2^2 F_a^{jk} F_{a,jk} + \frac{M^2}{g_2^2} W_{aj} W_j \right) \quad (1.13)$$

The particle masses $\mu$ arise from the products of the coefficients for the curvature $\frac{F^2}{4}$ and the vector field $\frac{Z^2 W^2}{2}$ whereas the quotients of the coefficients define constants $\ell^2$ and $\frac{1}{\ell^2}$, called external normalizations related to the
particles

\[ \mathbf{L}(B, W) = F^j_{jk} \partial_t A_k - \frac{1}{2} \partial_t A_j + e^2 F^j_{jk} \]

\[ + F^j_{Zk} \partial_t Z_k - \frac{1}{2} \partial_t Z_j + \mu Z \left( \frac{F^j_{Zk} F_{jk}}{4} + \frac{1}{\ell_2^2} Z_j Z^j \right) \]

\[ + \sum_{a=1,2} \left( F^j_{a k} \partial_t W_{ak} - \frac{1}{2} \partial_t W_{a j} \right) + \mu_W \left( \frac{F^j_{a k} F_{jk}}{4} + \frac{1}{\ell_4} W_{ak} W_{a} \right) \right) \] (1.14)

with \[
\begin{align*}
\mu_Z^2 &= (g_1^2 + g_2^2) M^2 = g_2^2 M^2, \quad \ell_4^2 = \frac{g_1^2 + g_2^2}{M^2}, \quad \ell_4^2 = \frac{g_2^2}{M^2}
\end{align*}
\]

Planck’s constant \( h \) and the maximal action velocity \( c \) are used as natural units.

The currents \( J^k(x) \) for hypercharge \( \mathbf{U}(1) \) and \( \bar{J}^k(x) \) for isospin \( \mathbf{SU}(2) \) have to be transformed as the curvatures

\[
\left( \begin{array}{c}
J_A \\
J_Z
\end{array} \right) = S(\theta) \left( \begin{array}{c}
J \\
J_3
\end{array} \right) = \left( -\sin^2 \theta J + \cos^2 \theta J_3 \right)
\]

(1.15) to obtain the interaction in the asymptotic particle basis

\[
- \mathbf{L}(J, B, W) = J^k B_k + \bar{J}^k \bar{W}_k = J^k A_k + J^k_Z Z_k + \sum_{a=1,2} J^k_a W_{ak}
\]

\[
= (J^k + J^k_3) A_k + (-\sin^2 \theta J^k + \cos^2 \theta J^k_3) Z_k + \sum_{a=1,2} J^k_a W_{ak}
\]

(1.16)

The standard model cannot determine the dimensionless values \( c^2 \sim \frac{1}{45}, \tan^2 \theta \sim 0.3 \) and the ratio of \( M \sim 123 \text{GeV} \) to any reference mass. Those values are taken from the experiments with the energy momentum dependent Weinberg angle at the \( Z \)-mass. Using dual rectangular triangles

\[
\mathcal{T} = \left( a_1, a_2; c/h \right): \quad \frac{a_1^2}{a_1} + \frac{a_2^2}{a_2} = c^2
\]

\[
\mathcal{T} = \left( \frac{a_1}{a_2}, \frac{1}{a_1}, \frac{1}{c} \right): \quad \frac{a_1}{a_2} + \frac{1}{a_1} = \frac{1}{c^2}
\]

(1.17)

one has for the coupling constants \( \mathcal{G} \), the masses \( \mathcal{M} \) and the normalization constants \( \mathcal{L}^2 \)

\[
\text{couplings: } \mathcal{G} = (g_1, g_2, g_Z | e) \sim (\frac{1}{29}, \frac{1}{10}, \frac{1}{14})
\]

\[
\text{masses: } \mathcal{M} = (\mu_1, \mu_W, \mu_Z | \mu_e) \sim (43.4, 80.2, 91.2 | 38.2) \text{GeV}
\]

\[
\text{normalizations: } \mathcal{L}^2 = (\ell_1^2, \ell_2^2, \ell_3^2 | \ell_e)
\]

\[
\mathcal{M} = M^2, \quad \mathcal{L}^2 = \frac{1}{M} \mathcal{G}
\]

(1.18)

\( \mu_e \) and \( \mu_1 \) are no particle masses in contrast to the \( W \)- and \( Z \)-masses \( \mu_W \) and \( \mu_Z \) resp.

The internal interaction normalizations are \( \mathcal{G}^2 \) and \( \frac{1}{\mathcal{G}^2} \), the external particle normalizations \( \mathcal{L}^2 \) and \( \frac{1}{\mathcal{L}^2} \), related to each other by the ground state mass value \( M \).

The Feynman propagators for the particle vector fields are characterized by the mass as momentum space pole and the residue as the coupling constant
involved

\[ \langle TZ^k Z^j \rangle(x) = -\frac{i}{\pi} \int \frac{d^4 q e^{iqx}}{(2\pi)^4} \left( \frac{g^j g^i}{\mu_2^2} - \frac{g^k g^j}{\mu_Z^2} \right) \frac{g_Z^2}{q^2 + i\alpha - \mu_Z^2}, \quad g_Z^2 = \frac{\mu_2^2}{M_Z^2} = \mu Z \ell_Z^2 \]

\[ \langle TA^k A_j \rangle(x) = -\frac{i}{\pi} \int \frac{d^4 q e^{iqx}}{(2\pi)^4} \left( \frac{g^j g^i}{\mu_2^2} + \text{gauge dep.} \right), \quad g^2 = \frac{\mu_2^2}{M^2} = \mu e \ell_e^2 \]

\[ \langle TW_a W_b \rangle(x) = -\delta_{ab} \frac{i}{\pi} \int \frac{d^4 q e^{iqx}}{(2\pi)^4} \left( \frac{g^j g^i}{\mu_W^2} - \frac{g^k g^j}{\mu_W^2} \right) \frac{g_W^2}{q^2 + i\alpha - \mu_W^2}, \quad g_W^2 = \frac{\mu_2^2}{M_W^2} = \mu W \ell_W^2 \]

(1.19)

1.2 The Quantum Mechanical Oscillators

The quantum mechanical oscillators can be used as a simple model to illustrate the interplay of masses and coupling constants.

The isotropic Bose oscillator in \( n \) dimensions has as Hamiltonian with mass \( m \) and spring constant \( k \)

\[ H_n = \frac{\vec{p}^2}{2m} + k\vec{q}^2, \quad (\vec{p}, \vec{q}) = (p_a, q_a)_{a=1} \] (1.20)

to be compared with the field theoretical expression \( g^2 \frac{\vec{p}^2}{m} + M^2 \frac{\vec{q}^2}{2} \) e.g. for the massive \( W_{1,2} \)-fields. The quantum mechanical analogue of the coupling constants \( g \) for relativistic fields and the ground state mass \( M^2 \) is the inverse inert mass \( \frac{1}{m} \) for mass points and the spring constant \( k \) resp. The characteristic frequeny \( \mu \) and the intrinsic length \( \ell \) of the oscillator are given with the product and the quotient of inverse mass and spring constant

\[ H_n = \frac{\mu^2 \vec{p}^2 + k \vec{q}^2}{2}, \quad \mu^2 = \frac{k}{m}, \quad \ell^4 = \frac{1}{mk} \] (1.21)

\( \hbar \) is used as natural unit. The mass of a particle, e.g. the \( Z \) or \( W \) mass \( \mu_{Z,W} \), has its analogue in the mass point frequency \( \mu \), not in the inert mass \( m \).

The time dependent quantization of the harmonic oscillator with the shorthand notation \([a, b](t - s) = [a(s), b(t)]\) is a \( U(1) \cong SO(2) \) representation

\[ D_{ab}(t|\mu) = \left( \begin{array}{cc} [i p_a, q_b] & [q_a, q_b] \\ [p_a, -i p_b] & [q_a, -i p_b] \end{array} \right) (t) \delta_{ab} D_1(t|\mu) \]

\[ D_1(t|\mu) = \left( \begin{array}{cc} \cos t \mu & i t^2 \sin t \mu \\ \frac{1}{t^2} \sin t \mu & \cos t \mu \end{array} \right) e^{it\mu N(\ell^2)}, \quad N(\ell^2) = \left( \begin{array}{cc} 0 & \ell^2 \\ \ell^2 & 0 \end{array} \right) \] (1.22)

The harmonic oscillator mass arises in the residue \( \frac{1}{m} \) of the integral representation for the time dependent quantization. One has with the principal value \( (\mu^2)_P \) integration contour in the energy plane

\[ [q_a, q_b](t) = \delta_{ab} i \ell^2 \sin t \mu = -\delta_{ab} \frac{it \mu}{\pi} \int dE e^{itE} \frac{1}{E^2 - (\mu^2)_P}, \quad \frac{1}{m} = \frac{\mu^2}{k} = \mu \ell^2 \] (1.23)

The characteristic length \( \ell \) can be absorbed in the 'free' oscillator by the rescaling

\[ \vec{q} = \ell \vec{q}, \quad \vec{p} = \frac{1}{\ell^2} \vec{p} \] (1.24)
i.e. it has no physical relevance. If, however, a nontrivial rotation $O(n)$-invariant $q^2$-dependent potential $V$ is added, e.g. $(q^2)^2$ for an anharmonic oscillator or $-k\frac{q^2}{2} - \frac{1}{\sqrt{q^2}}$ for the Kepler dynamics, $\ell^2$ is relevant for the coupling constant

$$H_n + V = \mu \ell^2 \frac{\vec{p}^2 + \frac{1}{2} \vec{q}^2}{2} + V(q^2) = \mu \frac{\vec{\rho}^2 + \vec{q}^2}{2} + V(\ell^2 \vec{q}^2)$$ (1.25)

The oscillator length $\ell$ normalizes the 'real' and 'imaginary' combination of the creation operator $u$ and the annihilation operator $u^*$

$$\vec{q} = \ell \frac{u + u^*}{\sqrt{2}}, \quad \vec{p} = i \ell \frac{u - u^*}{\sqrt{2}}$$ (1.26)

In the complex creation-annihilation formulation the characteristic length does not show up in the $U(n)$-symmetric Hamiltonian

$$H_n = \mu \frac{(u^* u)}{2}, \quad [u^*_a, u_b](t) = \delta_{ab} e^{it\mu}$$ (1.27)

The quantum mechanical meaning of the particle normalization $\ell^2$ is given with the Fock ground state of the harmonic oscillator. For the Fock space with the ground state values

$$n = 1 : \langle (u^* u)^N \rangle = 1, \quad N = 0, 1, \ldots$$
$$n \geq 1 : \langle (u^*_a u_a) \cdots (u^*_N u_N) \rangle = \delta_{a_1 b_1} \cdots \delta_{a_N b_N}$$ (1.28)

the time dependent Fock values for position and momenta with the shorthand notation $\langle \{a, b\}(t - s) = \{a(s), b(t)\}$ are the product of the particle normalization matrix $N(\ell^2)$ with a time representation

$$\begin{pmatrix}
\langle \{p_a, q_b\} \rangle \\
\langle \{p_a, p_b\} \rangle \\
\langle \{p_a, q_b\} \rangle \\
\langle \{p_a, p_b\} \rangle
\end{pmatrix}(t) = \delta_{ab} \begin{pmatrix}
\frac{i \sin t\mu}{\ell^2} \\
\frac{\cos t\mu}{\ell^2} \\
\ell^2 \cos t\mu \\
\ell^2 \sin t\mu
\end{pmatrix} = \delta_{ab} N(\ell^2) D_1(t|\mu)$$ (1.29)

$N(\ell^2)$ contains the dual normalizations of positions and momenta, i.e. $\ell^2$ and $\frac{1}{\ell^2}$ resp.

The sum of time ordered quantization and Fock values is the quantum mechanical analogue of the Feynman propagator, given for the position with the oscillator frequency as energy pole and the inverse mass as residue

$$\langle Tqq \rangle(t) = \langle \{q, q\}(t) - \epsilon(t)[q, q](t) \rangle = \ell^2 (\cos t\mu - i\epsilon(t) \sin t\mu) = \ell^2 e^{-i|t|\mu} = -\frac{i}{\pi} \int dE e^{itE} \frac{1}{E^2 + i\omega - \mu^2}, \quad \frac{1}{m} = \frac{\mu^2}{k} = \mu \ell^2$$ (1.30)
1.3 Fock Normalization for Particles

The particle expansion of the vector fields in the standard model

\[
Z^j(x) = \sum_{n=1,2,3} \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{\mu_Z}{q_0}} \Lambda(\frac{q}{\mu_Z})^j_n \ell_Z \frac{e^{i\vec{q} \cdot \vec{u}_n(q)} + e^{-i\vec{q} \cdot \vec{u}_n(q)}}{\sqrt{2}}
\]

\[
W^j_a(x) = \sum_{n=1,2,3} \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{\mu_W}{q_0}} \Lambda(\frac{q}{\mu_W})^j_n \ell_W \frac{e^{i\vec{q} \cdot \vec{u}_n(q)} + e^{-i\vec{q} \cdot \vec{u}_n(q)}}{\sqrt{2}}, \quad a = 1, 2
\]

\[
A^j(x) = \sum_{n=0,1,2,3} \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{\mu_s}{q_0}} H(\frac{q}{q_0})^j_n \ell_e \left( \begin{array}{c}
\cdots
\end{array} \right)
\]

uses the energy \( q_0 = \sqrt{\mu^2 + \vec{q}^2} \) for \( \mu = \mu_Z, \mu_W, 0 \) and the boost \( \Lambda(\frac{q}{\mu}) \) with positive mass \( \mu > 0 \) for \( SO^+(1,3)/SO(3) \) to transform to a rest system and \( H(\frac{q}{q_0}), q_0 = |\vec{q}| \), i.e. with vanishing mass \( \mu = 0 \), for \( SO^+(1,3)/SO(2) \) to transform to a rest system with a polarization direction. The two nonparticle components (\( \cdots \)) (Coulomb and gauge degree of freedom for \( n = 0, 3 \)) for the massless electromagnetic field are not given explicitly.

The quantization and Fock values for the creation and annihilation operators

\[
[u^*(\vec{p}), u(\vec{q})] = \delta(\vec{q} - \vec{p}) = \langle \{u^*(\vec{p}), u(\vec{q})\}\rangle
\]

(1.32) exhibit the external normalizations \( \ell_Z^2, \ell_W^2 \) for the massive particles involved

\[
\langle Z^k, Z^j \rangle(x) = \sum_{n,m=1,2,3} \int \frac{d^3q}{(2\pi)^3} \frac{\mu_Z}{q_0} \Lambda(\frac{q}{\mu_Z})^k_m \ell_Z^2 \delta^{mn} \cos x_0 q_0 \Lambda(\frac{q}{\mu_Z})^j_n
\]

\[
\langle W^k_a, W^j_b \rangle(x) = \sum_{n,m=1,2,3} \int \frac{d^3q}{(2\pi)^3} \frac{\mu_W}{q_0} \Lambda(\frac{q}{\mu_W})^k_m \ell_W^2 \delta^{mn} \delta_{ab} \cos x_0 q_0 \Lambda(\frac{q}{\mu_W})^j_n
\]

(1.33)

The Fock values for the two massless particle degrees of freedom (photons) \( m, n = 1, 2 \) in the electromagnetic field

\[
\langle A^k, A^j \rangle(x) = \sum_{n,m=0,1,2,3} \int \frac{d^3q}{(2\pi)^3} \frac{\mu_e}{q_0} H(\frac{q}{q_0})^k_m \ell_e^2 \left( \begin{array}{c}
\cdots
\end{array} \right) \frac{\mu_e}{\mu_e} H(\frac{q}{q_0})^j_n
\]

(1.34) have as external particle normalization \( \ell_e^2 = \frac{e^2}{\mu_e} \).
Chapter 2

Normalization of Symmetries

In this chapter, the mathematical structure for symmetry normalizations (coupling constants and particle masses) is given together with its application to the standard model vector fields.

2.1 Dual Normalizations

A nondegenerate quadratic form \( \zeta \), i.e. a symmetric real bilinear or complex sesquilinear form of a finite dimensional vector space \( V \) with scalars \( \mathbb{K} = \mathbb{R} \) and \( \mathbb{K} = \mathbb{C} \) resp.

\[
\zeta(\ ,
) : V \times V \longrightarrow \mathbb{K}, \quad \begin{cases} 
\zeta(v, w) = \zeta(w, v), \text{ bilinear} \\
\zeta(v, w) = \zeta(w, v), \text{ sesquilinear}
\end{cases}
\tag{2.1}
\]

induces an (anti)linear isomorphism to the dual space \( V^T \), containing the linear \( V \)-forms

\[
\zeta : V \longrightarrow V^T, v \longmapsto \zeta(v) = \zeta(v, \ ), \quad \begin{cases} 
\zeta(v + w) = \zeta(v) + \zeta(w) \\
\zeta(\alpha v) = \alpha \zeta(v), \ \alpha \in \mathbb{R}, \text{ bilinear} \\
\zeta(\alpha v) = \overline{\alpha} \zeta(v), \ \alpha \in \mathbb{C}, \text{ sesquilinear}
\end{cases}
\tag{2.2}
\]

and therewith the inverse quadratic form \( \tilde{\zeta} \) for the linear forms

\[
\tilde{\zeta}(\ ,
) : V^T \times V^T \longrightarrow \mathbb{K}, \quad \tilde{\zeta}(\omega, \theta) = \zeta(\zeta^{-1}(\omega), \zeta^{-1}(\theta))
\tag{2.3}
\]

With the isomorphism \( \zeta \), the quadratic form can be expressed by the dual product of vectors \( V \) and linear forms \( V^T \)

\[
\text{dual product} : \quad V^T \times V \longrightarrow \mathbb{K}, \quad (\omega, v) \longmapsto \omega(v) = \langle \omega, v \rangle \\
\zeta(\ ,
) : V \times V \longrightarrow \mathbb{K}, \quad \zeta(v, w) = \langle \zeta(v), w \rangle \tag{2.4}
\]

\[
\tilde{\zeta}(\ ,
) : \ V^T \times V^T \longrightarrow \mathbb{K}, \quad \tilde{\zeta}(\omega, \theta) = \langle \omega, \zeta^{-1}(\theta) \rangle 
\]

Nondegenerate quadratic forms are characterized by a signature \((n_+, n_-)\), i.e. by their real orthogonal invariance groups \( \text{O}(n_+, n_-) \) for real vector spaces \( V \cong \mathbb{R}^n \), \( n = n_+ + n_- \), or unitary ones \( \text{U}(n_+, n_-) \) for complex vector spaces \( V \cong \mathbb{C}^n \).
\[ V \cong \mathbb{C}^n. \] If \( \{e^a\}_a \) and \( \{\bar{e}_a\}_a \) are dual bases of \( V \) and \( V^T \), i.e. with the dual product \( \langle \bar{e}_a, e^b \rangle = \delta^b_a \), their squares build the matrices \( \zeta(e^a, e^b) = \zeta^{ab} \) and \( \bar{\zeta}(\bar{e}_a, \bar{e}_b) = \bar{\zeta}_{ab} \).

A quadratic form of a complex vector space defines a \( U(n+, n-) \)-conjugation

\[ U(n+, n-) : \quad \zeta(v) = v^* = \zeta(v, ) \quad \zeta^{-1}(\omega) = \omega^* = \bar{\zeta}(\omega, ) \quad (2.5) \]

Only real, not complex quadratic forms are expressable by power two tensors, written in, but independent of bases

\[ O(n+, n-) : \quad \bar{\zeta} = \zeta_{ab} e^a \otimes e^b \in V \otimes V \]
\[ \zeta = \zeta^{ab} \bar{e}_a \otimes \bar{e}_b \in V^T \otimes V^T \quad (2.6) \]

Orthogonal bases of vector spaces with definite bilinear forms, i.e. of scalar products with \( n_- = 0 \), define dual normalizations \( \ell^2 \) and \( \frac{1}{\ell^2} \) for the compact symmetries acting on the vector spaces \( V \) and \( V^T \)

\[ \text{for } O(n), U(n) : \]
\[ \begin{cases} \zeta(e^a, e^b) = \ell^2 \delta^{ab} \\ \bar{\zeta}(\bar{e}_a, \bar{e}_b) = \frac{1}{\ell^2} \delta_{ab} \end{cases}, \quad \ell^2 > 0 \quad (2.7) \]

\[ \text{for } O(n) : \]
\[ \begin{cases} \bar{\zeta} = \frac{1}{\ell^2} \delta_{ab} e^a \otimes e^b \in V \otimes V \\ \zeta = \ell^2 \delta^{ab} \bar{e}_a \otimes \bar{e}_b \in V^T \otimes V^T \end{cases} \]

\[ \sigma(, ) : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{R}, \]
\[ \sigma(q, q') = \sigma(q', q) \]
\[ \sigma(q, q) > 0 \iff q \neq 0 \]

\[ \bar{\sigma}(, ) : \mathbb{S}^T \times \mathbb{S}^T \rightarrow \mathbb{R}, \]
\[ \bar{\sigma}(p, p') = \bar{\sigma}(p', p) \]
\[ \bar{\sigma}(p, p) > 0 \iff p \neq 0 \quad (2.8) \]

Using dual and orthonormal bases \( \{q_a, p_a\}_a \) one has the invariant tensors with the intrinsic oscillator length \( \ell \)

\[ \langle p_a, q_b \rangle = \delta_{ab}, \quad \begin{cases} \sigma(q_a, q_b) = \ell^2 \delta_{ab} \quad \Rightarrow \quad \sigma = \ell^2 p_a \otimes p_a \\ \bar{\sigma}(p_a, p_b) = \frac{1}{\ell^2} \delta_{ab} \quad \Rightarrow \quad \bar{\sigma} = \frac{1}{\ell^2} q_a \otimes q_a \end{cases} \quad (2.9) \]

The sum of the dual scalar products, multiplied with a constant \( \mu \) leads to the quantum mechanical Hamiltonian

\[ \mu^{\frac{\sigma + \bar{\sigma}}{2}} = \mu^{\frac{1}{2} \frac{\ell^2 p_a \otimes p_a + \ell^2 q_a \otimes q_a}{2}} \quad \Rightarrow \quad H_n = \mu^{\frac{1}{2} \frac{\ell^2 p_a p_a + \ell^2 q_a q_a}{2}} \quad (2.10) \]

\( \ell^2 \) and \( \frac{1}{\ell^2} \) are the dual particle normalizations of \( O(n) \) acting on positions and momenta.
2.3 Interaction Normalizations for the Standard Gauge Fields

With respect to the internal degrees of freedom the gauge field sector in the standard model is built with dual real 4-dimensional vector spaces for gauge vector fields and curvatures which are the direct sums of 1- and 3-dimensional subspaces

\[ \text{curvatures: } G = G_1 \oplus G_3 \cong \mathbb{R} \oplus \mathbb{R}^3 \text{ with basis } \{F, F_a\} \]
\[ \text{gauge vectors: } G^T = G^T_1 \oplus G^T_3 \cong \mathbb{R} \oplus \mathbb{R}^3 \text{ with basis } \{B, W_a\} \]
\[ \text{dual bases: } \langle B, F \rangle = 1, \quad \langle W_a, F \rangle = \delta_{ab} \]

The hyperisospin group \( U(2) = U(1_2) \circ SU(2) \) acts nontrivially only on the isospin subspaces \( G_3, G^T_3 \) as \( U(2)/U(1_2) \cong SO(3) \).

The squares of the coupling constants reflect a \( U(2) \)-invariant scalar product with dual normalizations \( g^2 \) and \( \frac{1}{g^2} \) of the internal groups acting on gauge vectors and curvatures

\[ g(\ , \ ) : G \times G \rightarrow \mathbb{R}, \quad g(F, F) = \frac{1}{g^2}, \quad g(F_a, F_b) = \delta_{ab} \frac{1}{g^2} \]
\[ \tilde{g}(\ , \ ) : G^T \times G^T \rightarrow \mathbb{R}, \quad \tilde{g}(B, B) = g^2_1, \quad \tilde{g}(W_a, W_b) = \delta_{ab} g^2_2 \]

The Lagrangian uses the gauge invariant metrical tensor \( \tilde{g} \) for the curvatures.

The distinction of the electromagnetic group \( U(1) \) in the ground state defines the basis transformation \( S(\theta) \in SL(\mathbb{R}^2) \) in the 2-dimensional curvature subspace spanned by \( \{F, F_3\} \) and to the contragredient transformation in the vector field space spanned by \( \{B, W_3\} \) (section 1.1)

\[ \Theta : G \rightarrow G, \quad \left( \begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_1,2 \end{array} \right) = \Theta \left( \begin{array}{c} F \\ F_1,2 \end{array} \right), \quad \Theta = \left( \begin{array}{cc} 1 & 1 \\ -\sin^2 \theta & \cos^2 \theta \\ 0 & 0 \end{array} \right) \in SL(\mathbb{R}^4) \]
\[ \tilde{\Theta} : G^T \rightarrow G^T, \quad \left( \begin{array}{c} A \\ Z \\ W_1,2 \end{array} \right) = \tilde{\Theta} \left( \begin{array}{c} B \\ W_1,2 \end{array} \right), \quad \tilde{\Theta} = \left( \begin{array}{cc} \cos^2 \theta & \sin^2 \theta \\ 0 & 1 \\ 0 & 0 \end{array} \right) \in SL(\mathbb{R}^4) \]

The internal normalizations read in the new basis

\[ \Theta \left( \begin{array}{c} 1 \\ \frac{1}{g^2_1} \\ 0 \\ \frac{1}{g^2_2} \end{array} \right) \tilde{\Theta}^T = \left( \begin{array}{cccc} \frac{1}{g^2_1} & 0 & 0 \\ 0 & \frac{1}{g^2_1} & 0 \\ 0 & 0 & \frac{1}{g^2_2} \end{array} \right) \]
\[ \tilde{g} = g^2_1 F^2 + g^2_2 \tilde{F}^2 = e^2 F_A^2 + g^2_2 F_Z^2 + g^2_2 \sum_{a=1,2} F_a^2 \]

2.4 Higgs Mass as Unit for the Goldstone Manifold

The algebraic framework for the ground state properties implementing complex Higgs vectors \( \varphi \) (section 1.1) uses a \( U(2) \)-invariant scalar product of the
complex 2-dimensional Higgs vector space \( \mathbb{H} \cong \mathbb{C}^2 \)

\[
\langle | \rangle : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}, \quad \begin{cases} 
\langle \varphi | \varphi' \rangle = \overline{\langle \varphi' | \varphi \rangle} \\
\langle \varphi | \varphi \rangle > 0 \iff \varphi \neq 0 \\
\varphi^\ast = \langle \varphi | \rangle \in \mathbb{H}^T \end{cases}
\] (2.15)

In the real 4-dimensional manifold \( \text{GL}(\mathbb{C}^2)_\mathbb{R}/U(2) \) of the classes of \( U(2) \)-equivalent bases with \( \text{GL}(\mathbb{C}^2)_\mathbb{R} \) taken as real 8-dimensional Lie group, the orthogonal bases \( \{\varphi^\alpha\}_{\alpha=1,2} \) with the diagonal matrix

\[
\langle \varphi^\alpha | \varphi^\beta \rangle = M^2 \delta^{\alpha \beta}, \quad \langle | \rangle \cong M^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\] (2.16)

define the normalization \( M^2 \) of the hyperisospin group \( U(2) \) acting on the Higgs space.

Any nontrivial vector \( \varphi \in \mathbb{H} \) fixes an orthogonal basis up to the electromagnetic \( U(1)_+ \)-direction. Therefore \( M \) can also be called the unit of the Goldstone manifold \( U(2)/U(1)_+ \) which relates fields and particles with respect to the internal degrees of freedom.

### 2.5 Induced Normalizations for Lie Algebras

If a real Lie algebra \( L \) acts on a real or complex vector space \( V \) by linear endomorphisms

\[
L \times V \rightarrow V, \quad l \circ v = \mathcal{D}(l)(v)
\] (2.17)
each vector \( v \in V \) induces via a quadratic form \( \zeta \) of \( V \) a symmetric bilinear form on the Lie algebra, definite for a scalar product

\[
\zeta_v(\ , \ ) : L \times L \rightarrow \mathbb{R}, \quad \begin{cases} 
\zeta_v(l, l') = \zeta(l \circ v, l' \circ v) = \zeta_v(l', l) \\
\text{if } \zeta(v, v) > 0 \text{ for } v \neq 0 \Rightarrow \zeta_v(l, l) \geq 0
\end{cases}
\] (2.18)
The induced bilinear Lie algebra form is trivial for the Lie subalgebra which acts trivially on the vector \( v \)

\[
\zeta_v(l, L) = \{0\} \quad \text{for} \quad l \in \text{FIX}_v L = \{l \in L \mid l \circ v = 0\}
\] (2.19)
\( \zeta_v \) is nondegenerate for the quotient vector space \( L/\text{FIX}_v L \) and a scalar product thereon for a scalar product \( \zeta \).

In the standard model (section 1.1) the vector fields obtain their masses in the mechanism just described: The Lie algebras for hypercharge and for isospin \( \log U(2) = \log U(1_2) \oplus \log SU(2) \) act on the Higgs vectors \( \mathbb{H} \cong \mathbb{C}^2 \)

\[
l^0 \circ \varphi^\alpha = \frac{i}{\sqrt{2}} \varphi^\alpha, \quad l^0 \in \log U(1) \cong \mathbb{R}
\]

\[
l^a \circ \varphi^\alpha = \frac{i}{\sqrt{2}} (\tau^a)_{\beta}^\alpha \varphi^\beta, \quad l^a \in \log SU(2) \cong \mathbb{R}^3
\] (2.20)
The electromagnetic Lie algebra \( \log U(1)_+ \) is the Lie algebra acting trivially on each nontrivial Higgs vector

\[
\mathbb{H} \ni \varphi \cong \begin{pmatrix} 0 \\ M \end{pmatrix} \neq 0, \quad \langle \varphi | \varphi \rangle = M^2
\]

\[
\text{FIX}_\varphi \log U(2) = \log U(1)_+ \cong i \mathbb{R} \frac{1+i\tau_3}{\sqrt{2}}
\] (2.21)
The scalar product induced on the Lie algebra $\log U(2)$ is nontrivial on the vector space $\log U(2)/U(1) \cong \mathbb{R}^3$ for the massive $Z$- and $W_{1,2}$-particle fields

$$\langle l^0 - l^3 | l^0 - l^3 \rangle = \langle i \frac{l^0 - l^3}{\sqrt{2}} | i \frac{l^0 - l^3}{\sqrt{2}} \rangle = \frac{M^2}{2}$$

$$\langle l^a | l^b \rangle = \langle i \frac{l^a}{\sqrt{2}} | i \frac{l^b}{\sqrt{2}} \rangle = \delta_{ab} \frac{M^2}{2}$$ for $a, b = 1, 2$ \hfill (2.22)

### 2.6 Particle Normalization

for the Standard Gauge Fields

The normalization of the Lorentz group $SO^+(1, 3)$-action shows up in the particle normalizations $ℓ^2$ (section 1.1). The distinction of the definite subgroups of the Lorentz group goes parallel with the Wigner classification of the representations of the Poincaré group: Via the massive particles the definite spin group $SO(3) \cong SU(2)/\mathbb{I}_2$ is normalized, via the massless ones the definite cirularity (polarization) group $SO(2) \cong U(1)$ inside the Lorentz group $SO^+(1, 3) \cong SL(\mathbb{C}^2)_R/\mathbb{I}_2$.

Similar to the internal symmetries with the Goldstone manifold $U(2)/U(1)_+$ characterizing the field particle transition, there arise for the external symmetries the Sylvester and Witt manifolds $SO^+(1, 3)/SO(3)$ and $SO^+(1, 3)/SO(2)$ resp. relevant for the harmonic particle analysis of the relativistic fields. In contrast to the fields, particles have no full Lorentz symmetry, massive particles keep the spin $SO(3)$ symmetry, massless ones only the circularity (polarization) $SO(2)$ symmetry.

To be more explicit: The Lagrangian for a massive vector field

$$L(Z, F_Z) = \frac{1}{2} F_Z^{jk} \ell^k_m \partial_m Z_j + \mu_Z (\ell_Z^{jk} F_Z^{jk} + Z^m Z_m) \frac{Z_1 Z_2}{2}$$. \hfill (2.23)

with the Clebsch-Gordan coefficients $\ell^k_m = \delta^k_m - \delta^m_k$ and dual normalization constant $\ell_Z^k$ and $\ell_Z^m$ give rise to the commutators of the canonical pair $(Z, F_Z)$

$$\begin{pmatrix}
  [F_Z^k, Z^l] \\
  [F_Z^m, Z^l] \\
  [Z^k, -i F_Z^l]
\end{pmatrix} (x) =
\begin{pmatrix}
  -i \ell_Z^k (\eta_{m+n} \frac{\partial^m}{\nu_2} - i \delta^m_{l+n} \frac{\partial_n}{\nu_2}) \\
  -i \ell_Z^m (\eta_{k+n} \frac{\partial^m}{\nu_2} - i \delta^m_{l+n} \frac{\partial_n}{\nu_2})
\end{pmatrix} \int \frac{d^3 q}{(2\pi)^3} \epsilon(q^0) \mu_Z \delta(q^2 - \mu_Z^2) (-\eta^k_j + q^k q^j\frac{\nu_2}{\mu_Z^2}) \hfill (2.24)
$$

The $Z$-commutators

$$[Z^k, Z^j] (x) = \int \frac{d^3 q}{(2\pi)^3} \frac{\mu_Z}{q_0} \Lambda(q, Z)(\frac{\mu_Z}{Z})^k_m [Z Z]^m_n (x_0) \Lambda(\frac{\mu_Z}{Z})^j_n \hfill (2.25)$$

involve the $SO(3)$ normalization $\ell_Z^2$

$$[Z Z] (x_0) = \ell_Z^2 \begin{pmatrix}
  0 & 0 \\
  0 & 1
\end{pmatrix} i \sin x_0 q_0, \quad \ell_Z^2 = \frac{\mu_Z^2}{Z} \hfill (2.26)$$

and the boost $\Lambda(\frac{q}{\mu_Z})$ for the transformation to a Sylvester basis

$$\Lambda(\frac{q}{\mu_Z}) \cong \frac{1}{\mu_Z} \begin{pmatrix}
  q_0 & q_0 + \frac{q_0 \mu_Z}{q_0 + \mu_Z} \\
  \frac{\mu_Z}{q_0} & \frac{\mu_Z}{q_0 + \mu_Z}
\end{pmatrix}, \quad q^2 = \mu_Z^2 \hfill (2.27)$$
The boosts represent $\text{SO}(3)$-classes of the Sylvester manifold $\text{SO}^+(1, 3)/\text{SO}(3)$. The Fock values of the anticommutators use the analogue structures with

$$\langle\{Z^k, Z^j\}\rangle(x) = \int \frac{d^4q}{(2\pi)^4} \frac{m}{q_0} \Lambda(\frac{q}{\mu^2})^k_l \langle\{Z Z\}^m_n\rangle(x_0) \Lambda(\frac{q}{\mu^2})^j_l$$

$$\langle\{Z Z\}\rangle(x_0) = \ell^2_e \left( \begin{array}{cc} 0 & 1_3 \end{array} \right) \cos x_0 q_0$$

(2.28)

For massive vector particles a rest system with a time direction has to be fixed up to space rotations $\text{SO}(3)$.

For the massless electromagnetic vector fields the Lagrangian contains in addition to the coupling constant $e^2$ also the gauge fixing terms with a scalar field $L(x)$ and gauge fixing parameter $\kappa$

$$L(A, F_A, L) = \frac{1}{2} F_{jk}^i (\partial_j A_k - \partial_k A_j + e^2 F_{jk}^i A_{ik}) + \kappa L^2$$

The curvature $F_A$ and the gauge fixing field $L$ are the canonical partners for the gauge field $A$. The nontrivial commutators are given by

$$\left[ [i F_A^k, A^l], [A^k, A^l] \right](x) = \left( \begin{array}{cc} -ie^{[k, l]} & \delta^l_j \delta^l_k \\ -i\ell_{\mu} e_{\mu} \epsilon^{l[j, k]} & -i\ell_{\mu} e_{\mu} \delta^{l[j, k]} \end{array} \right)[A^j, A^l](x)$$

$$= \int \frac{d^4q}{(2\pi)^4} \epsilon(q_0) \mu_e \left( \begin{array}{cc} q^0 e^{-i\frac{q^i q^j}{q_0}} & 0 \\ 0 & e^{i\frac{q^i q^j}{q_0}} \end{array} \right) \left( \begin{array}{cc} 0 & 0 \\ 0 & \ell^2_e - \frac{e^2}{2c^2} q^i q^j \end{array} \right)$$

(2.30)

In the space-time translations analysis the vector field components are expressed in a Witt basis

$$[A^k, A^l](x) = \int \frac{d^4q}{(2\pi)^4} \mu_e \epsilon(q_0) \{ AA \}_{kk}^l(x_0) \{ AA \}_{ll}^j(x_0)$$

(2.31)

The transversal components (photons) have the $\text{SO}(2)$-normalization $\ell^2_e$, the nondiagonal light fixing parameter $\alpha$ is combined from $e^2$ and the gauge fixing parameter $\kappa$

$$[AA](x_0) = \ell^2_e \left\{ \begin{array}{cc} 0 & 0 \\ 0 & 0 -\alpha \end{array} \right\} i \sin x_0 q_0 + i x_0 q_0 \beta \left( \begin{array}{cc} e^{-i\alpha q_0} & 0 \\ 0 & 0 \end{array} \right)$$

$$\ell^2_e = \frac{e^2}{\mu_e M^2}, \quad \alpha = -\frac{3e^2 + \kappa}{2c^2}, \quad \beta = -\frac{e^2 + \kappa}{2c^2}$$

(2.32)

The transmutation to a Witt basis is performed by $H(\frac{q}{q_0})$ (Sylvester indices $k, j, \ldots = 0, 1, 2, 3$, Witt indices $k, j, \ldots = 0, 1, 2, 3$)

$$H(\frac{q}{q_0})^k_l = O(\frac{q}{q_0})^{k_j} \circ w^l_j \approx \frac{1}{q^2} \left( \begin{array}{cccc} \frac{q^0}{\sqrt{2}} & 0 & 0 & \frac{q^0}{\sqrt{2}} \\ \frac{q^j}{\sqrt{2}} & \frac{q^0 - (q^i q^i)^0}{\sqrt{2}(q^0 + q^0)} & q^j - \frac{q^j}{\sqrt{2}} \frac{q^0}{\sqrt{2}} & \frac{q^j}{\sqrt{2}} \\ \frac{q^j}{\sqrt{2}} & \frac{q^0 - (q^i q^i)^0}{\sqrt{2}(q^0 + q^0)} & \frac{q^j}{\sqrt{2}} - \frac{q^j}{\sqrt{2}} \frac{q^0}{\sqrt{2}} & \frac{q^j}{\sqrt{2}} \\ \frac{q^j}{\sqrt{2}} & \frac{q^0 - (q^i q^i)^0}{\sqrt{2}(q^0 + q^0)} & \frac{q^j}{\sqrt{2}} - \frac{q^j}{\sqrt{2}} \frac{q^0}{\sqrt{2}} & \frac{q^j}{\sqrt{2}} \end{array} \right), \quad q^2 = 0$$

$$H(\frac{q}{q_0})^0_0 \left( \begin{array}{cccc} 0 & 0 & 0 & 1_3 \\ 0 & 0 & 0 & 0 \end{array} \right) H(\frac{q}{q_0})^j_l = \left( \begin{array}{cc} -1 & 0 \\ 0 & 1_3 \end{array} \right)$$

(2.33)

For massive particles a rest system with one space axis, e.g. a 3rd direction, has to be fixed up to space plane rotations $\text{SO}(2)$.

The 1st and 2nd component have a particle structure with Fock value

$$\vec{k}, \vec{j} \in \{1, 2\} : \langle\{AA\}\rangle(x_0) \cong \ell^2_e 1_2 \cos x_0 q_0$$

(2.34)
Chapter 3

Determination of the Weinberg Angle

In this chapter a symmetry normalization related framework is proposed and conditions are given therein leading to a numerical determination of the Weinberg angle.

To recapitulate: The four coupling constants $g$ or masses $\mu$ in the electroweak triangle for the standard model

$$G = (g_1, g_2; g_Z|e) = \frac{1}{M} (\mu_1, \mu_W; \mu_Z|\mu_e) = M (\ell_1^2, \ell_2^2; \ell_Z^2|\ell_e^2) \sim \left( \frac{1}{2\pi}, \frac{1}{1\pi}; \frac{1}{1\pi} \right)$$

(3.1)

contain the interaction symmetry normalizations $g_1^2$ for hypercharge $U(1)_2$ and $g_2^2$ for isospin $SU(2)$. The particle normalizations $\ell_Z^2$ and $\ell_e^2$ are related to the stability groups $SO(3)$ and $SO(2)$ resp. of the neutral massive and massless particles resp. in the Lorentz group $SO^+(1,3)$. $M^2$ normalizes the internal group $U(2)$.

To compute the interaction and particle normalizations (for a first purely algebraic orientation without the interaction related running of the coupling constants), one can proceed in three steps: First, the ratio of the $U(1)_2$ and $SU(2)$ normalizations has to be understood, i.e. the Weinberg angle $\tan^2 \theta = \frac{g_1^2}{g_2^2}$. An attempt for this first step is given in this chapter. Therewith the form of the electroweak triangle and the ratios of electroweak masses, coupling constants and normalizations are determined. As a second step, the absolute value of one internal coupling constant, e.g. the fine structure constant $\frac{e^2}{4\pi}$, has to be given. Finally, the $U(2)$ unit $M^2$ has to be related to other masses, e.g. to the Planck mass or to the proton mass.

3.1 Invariant Lie Algebra Forms

Representations of Lie symmetries have their intrinsic metrical structure: Any representation of a Lie algebra $L$ on a vector space $V$, both finite dimensional, in the endomorphisms algebra with its natural Lie algebra structure

$$L \times V \rightarrow V, \ l \cdot v = D(l)(v)$$

(3.2)
gives rise to the \( V \)-associated \( L \)-invariant multilinear trace forms

\[
t^k_V : L \times \cdots \times L \to \mathbb{K}, \quad t^k_V(l_1, \ldots, l_k) = \text{tr} \mathcal{D}(l_1) \circ \cdots \circ \mathcal{D}(l_k)
\]

\[
l \cdot t^k_V(l_1, \ldots, l_k) = t^k_V([l, l_1], \ldots, l_k) + \cdots + t^k_V(l_1, \ldots, [l, l_k]) = 0 \tag{3.3}
\]

Any irreducible representation of a semisimple Lie algebra \( L \) on a complex vector space \( V \) gives an invariant bilinear Lie algebra form, unique up to a normalization. These associated bilinear forms are multiples \( \alpha t^2_V, \alpha \in \mathbb{K} \), of the ‘double’ trace

\[
t^2_V(\cdot, \cdot) : L \times L \to \mathbb{K}, \quad \begin{cases} t^2_V(l, m) = \text{tr} \mathcal{D}(l) \circ \mathcal{D}(m) = t^2_V(m, l) \\ t^2_V([k, l], m) + t^2_V(l, [k, m]) = 0 \end{cases} \tag{3.4}
\]

The bilinear form associated to the adjoint representation is the Killing form.

E.g. the real Lie algebras \( \log \text{SU}(n) \), especially the isospin Lie algebra \( \log \text{SU}(2) \), can be spanned with the \((n^2 - 1)\) generalized Pauli matrices as basis

- basis of \( \log \text{SU}(n) \): \( \{ l^a = \frac{i}{\sqrt{2}} \tau(n)^a \mid a = 1, \ldots, n^2 - 1 \} \)
- for \( \text{SU}(2) \): \( \{ \tau(2)^a = \tau^a \mid \text{Pauli matrices} \} \)
- for \( \text{SU}(3) \): \( \{ \tau(3)^a = \lambda^a \mid \text{Gell-Mann matrices} \} \)

The generalized Pauli matrices have the structure constants

\[
\tau(n)^a \tau(n)^b = \frac{2}{n} \delta^{ab} 1_n + (\delta^{abc} + \alpha^{abc}) \tau(n)^c
\]

totally symmetric \( \delta^{abc} \in \mathbb{R} \)

totally antisymmetric \( \alpha^{abc} \in \mathbb{R} \) \tag{3.6}

The defining representation of \( \log \text{SU}(n) \) on \( V \cong \mathbb{C}^n \) has as associated bilinear forms

\[
t^2_n(\cdot, \cdot) : \log \text{SU}(n) \times \log \text{SU}(n) \to \mathbb{R}, \quad t^2_n(l^a, l^b) = \alpha_n \text{tr} \frac{i}{\sqrt{2}} \tau(n)^a \frac{i}{\sqrt{2}} \tau(n)^b = -\alpha_n \delta^{ab}, \quad \alpha_n \in \mathbb{R} \tag{3.7}
\]

\( \text{SU}(n) \) has no structure which distinguishes any normalization \( \alpha_n \).

Since for a semisimple Lie algebra the derived Lie algebra \([L, L]\) coincides with \( L \), all nontrivial \( L \)-representations have to be traceless, i.e. there do not exist nontrivial invariant linear forms for semisimple Lie algebras.

This is different for abelian Lie algebras where the ‘single’ trace is a basis for the invariant linear forms

\[
L \times V \to V, \quad l \cdot v = \mathcal{D}(l)(v)
\]

\[
t^1_V : L \to \mathbb{K}, \quad \begin{cases} t^1_V(l) = \text{tr} \mathcal{D}(l) \\ t^1_V([k, l]) = 0 \end{cases} \tag{3.8}
\]

Therewith one obtains the invariant bilinear forms of abelian Lie algebras from the squared ‘single’ trace

\[
t^{1 \cdot 1}_V(\cdot, \cdot) : L \times L \to \mathbb{K}, \quad t^{1 \cdot 1}_V(l, m) = \text{tr} \mathcal{D}(l) \text{tr} \mathcal{D}(m) \tag{3.9}
\]
E.g. the real Lie algebras $\log U(1_2)$ with basis $\{l^0 = \frac{i}{\sqrt{2}} 1_2\}$, especially the hypercharge Lie algebra $\log U(1_2)$, have as invariant bilinear forms in the defining complex $n^2$-dimensional representation

$$t_n^{1,1}(l^0, l^0) = \beta_n (\text{tr} \frac{i}{\sqrt{2}} 1_2)^2 = -\frac{n^2}{2} \beta_n, \quad \beta_n \in \mathbb{R}$$

(3.10)

Again, $U(1_n)$ has no structure which distinguishes any normalization $\beta_n$.

### 3.2 Relative Normalization of Hypercharge and Isospin

In the electroweak sector of the standard model the internal group comes as product group $U(2) = U(1_2) \circ SU(2)$, but apparently not as the direct product group $U(1) \times SU(2)$. The correlation of hypercharge and isospin via the common subgroup $U(1_2) \cap SU(2) = \{\pm 1_2\}$ can be seen on the colourless sector of the standard model[2, 3] where fields with integer and half integer isospin $T$ have always integer and half integer hypercharge $Y$ resp., e.g. the left handed electron fields $(T, Y) = (\frac{1}{2}, -\frac{1}{2})$, the right handed electron fields $(0, -1)$, the $SU(2)$ gauge fields $(1, 0)$ etc.

The situation is more complicated for the colour nontrivial sector, e.g. for the quark fields, where in addition to the $SU(2)$-center $\{\sqrt{1/2} = \pm 1_2\}$ (two-ality) also the $SU(3)$-center $\{\sqrt{1/3}\}$ (tri-ality) has to be taken into account. Triality is correlated to third integer hypercharges: E.g., the left handed quarks as isospin doublets $d_T = 2$ and colour triplets $d_C = 3$ have hypercharge $Y = \frac{1}{d_T d_C}$. This relation has been discussed to some extent in[4, 5]. I will consider in this paper only the colour trivial electroweak sector.

The invariant bilinear forms for the defining representation of the real $n^2$-dimensional Lie algebra $\log U(n)$ allow two real normalization constants $\beta_n$ and $\alpha_n$ for the Lie subalgebras $\log U(1_n)$ (abelian) and $\log SU(n)$ (simple) resp.

$$g_n(\ , \ ) : \log U(n) \times \log U(n) \longrightarrow \mathbb{R}, \quad g_n(l, m) \left\{ \begin{array}{l}
t_n^{1,1}(l, m) + t_n^{2,2}(l, m) \\
\beta_n \text{tr} l \text{tr} m + \alpha_n \text{tr} l \circ m
\end{array} \right.$$  

(3.11)

These bilinear forms read in the basis with Pauli matrices

$$g_n(\ , \ ) \cong -\left( \begin{array}{c|c}
\beta_n n^2 + \alpha_n n & 0 \\
0 & \alpha_n 1_{n^2-1}
\end{array} \right)$$  

(3.12)

The real 2-dimensional space of the invariant bilinear forms of $\log U(n)$ (coefficients $\alpha_n$, $\beta_n$) can be decomposed into the bilinear forms with symmetric and antisymmetric ’double’ trace, called Fierz symmetrical and antisymmetri-
cal forms with normalizations $\alpha_n^\pm$

$$g_n^\pm(, ) : \log U(n) \times \log U(n) \rightarrow \mathbb{R}; \quad \left\{ \begin{array}{ll}
g_n^+(l, m) = \alpha_n^+(\text{tr } l \text{ tr } m) \\
g_n^-(l, m) = \alpha_n^-(\text{tr } l \text{ tr } m)
\end{array} \right. \quad \alpha_n^\pm \in \mathbb{R}$$

(3.13)

Only the Fierz symmetrical forms $g_n^+$ are definite. The indefinite Fierz antisymmetrical forms have signature $(1, n^2 - 1)$. The invariance groups of the quadratic forms $g_n^+$ and $g_n^-$ are the compact group $O(n^2)$ and the noncompact group $O(1, n^2 - 1)$ with $SO(n^2 - 1)$ embedding $U(n)/U(1_n)$.

For the hyperisospin group $U(2)$ one obtains as Fierz (anti)symmetrical quadratic forms

$$g_2^+(, ) \cong -\alpha_2^+(\begin{pmatrix} 3 & 0 \\ 0 & 1_3 \end{pmatrix}), \quad g_2^-(, ) \cong -\alpha_2^-(\begin{pmatrix} 1 & 0 \\ 0 & -1_3 \end{pmatrix})$$

(3.14)

If one can give an argument in favour of the definite Fierz symmetrical form $g_2^+$ as the physically relevant hyperisospin form (next section), the ratio of the normalizations of abelian $U(1_n)$-symmetry and simple $SU(n)$-symmetry is fixed

$$U(2) : g_2^+(, ) \cong \left( \begin{array}{cc} \frac{1}{2} & 0 \\ 0 & g_2^+ \end{array} \right) \Rightarrow \tan^2 \theta = \frac{g_2^+}{g_2^-} = \frac{1}{3} \quad (3.15)$$

and in the general case

$$U(n) : g_n^+(, ) \cong \left( \begin{array}{cc} \frac{1}{2} & 0 \\ 0 & g_n^+ \end{array} \right) \Rightarrow \tan^2 \theta = \frac{g_n^+}{g_n^-} = \frac{1}{(n+1)} \quad (3.16)$$

### 3.3 Fierz (Anti)Symmetrical Forms

Statistical structures can determine the Fierz (anti)symmetry of the bilinear forms for the $U(2)$-normalization: The endomorphisms of a finite dimensional vector space $V$ are isomorphic to the tensor product $V \otimes V^T$ of space and dual space, the invariant linear trace form is the dual product

$$\text{tr} : V \otimes V^T \rightarrow \mathbb{K}, \quad \left\{ \begin{array}{l}
\text{tr } v \otimes \omega = \langle \omega, v \rangle \\
\text{tr } [f, g] = 0
\end{array} \right. \quad \text{dual bases } tr e^a \otimes \hat{e}_b = \langle \hat{e}_b, e^a \rangle = \delta_b^a \quad (3.17)$$

The extension of the trace from $V \otimes V^T$ as 'double trace' for invariant linear forms for the power two tensors $(V \otimes V^T)^2$ can be done in two ways, symmetrically or antisymmetrically

$$\text{tr}^2_\pm : (V \otimes V) \otimes (V \otimes V)^T \rightarrow \mathbb{K}$$

(3.18)

$$\text{tr}^2_\pm (v_1 \otimes v_2) \otimes (\omega_1 \otimes \omega_2) = \langle \omega_1, v_1 \rangle \langle \omega_2, v_2 \rangle \pm \langle \omega_1, v_2 \rangle \langle \omega_2, v_1 \rangle$$

These two possibilities are the appropriate nontrivial forms for the symmetric $\vee$ or antisymmetrical $\wedge$ 'square' of the vector space $V$

$$\text{tr}^2_\pm : (V \vee V) \otimes (V \vee V)^T \rightarrow \mathbb{K}$$

$$\text{tr}^2_\pm : (V \wedge V) \otimes (V \wedge V)^T \rightarrow \mathbb{K} \quad (3.19)$$
The gauge interactions of the standard model act on vector spaces $V$ with $U(2)$ representations. Those vector spaces carry in the quantum structure either Fermi or Bose statistics, characterized by $[v, u] = 0$ or $\{v, u\} = 0$ resp. for $v, u \in V$. Therefore only the symmetrical product $V \vee V$ or the antisymmetrical product $V \wedge V$ is nontrivial, e.g. for the Higgs vectors $\varphi \in \mathbb{H} \cong \mathbb{C}^2$ with Bose statistic

$$2\varphi^A \wedge \varphi^B = [\varphi^A, \varphi^B] = 0, \text{ nontrivial } 2\varphi^A \vee \varphi^B = \{\varphi^A, \varphi^B\} \quad (3.20)$$

In a quantum structure where the hyperisospin $U(2)$-interactions get their normalization via a vector space acted on only with the defining $U(2)$ representation, the associated bilinear form for the Lie algebra of $U(2)$ has to be either Fierz symmetrical or antisymmetrical. The definite forms $g_2^+$ require symmetrical combinations of the underlying 2-dimensional $U(2)$ representations.

Obviously only the normalization ratio (mixing angle) for $\log U(1_2)$ (hypercharge) and $\log SU(2)$ as Lie subalgebras of $\log U(2)$ is fixed by such a statistical argument, not the absolute normalizations.
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