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ROBUST ESTIMATION AND MOMENT SELECTION IN DYNAMIC FIXED-EFFECTS PANEL MODELS

By

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Robust estimation and moment selection in dynamic fixed-effects panel data models

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Abstract

This paper extends an existing outlier-robust estimator of linear dynamic panel data models with fixed effects, which is based on the median ratio of two consecutive pairs of first-differenced data. To improve its precision and robust properties, a general procedure based on many pairwise differences and their ratios is designed. The proposed two-step GMM estimator based on the corresponding moment equations relies on an innovative weighting scheme reflecting both the variance and bias of those moment equations, where the bias is assumed to stem from data contamination. To estimate the bias, the influence function is derived and evaluated. The asymptotic distribution as well as robust properties of the estimator are characterized; the latter are obtained both under contamination by independent additive outliers and the patches of additive outliers. The proposed estimator is additionally compared with existing methods by means of Monte Carlo simulations.

Keywords: dynamic panel data, fixed effects, generalized method of moments, influence function, pairwise differences, robust estimation

JEL codes: C13, C23

1 Introduction

Dynamic panel data models with fixed effects have proven to be very attractive models in empirical applications; see among others Harris et al. (2008) for an overview of the extensive literature. One reason for this and an important advantage of these models is that they allow to disentangle the persistent component due to the (time-invariant) unobserved heterogeneity.

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from the one based on the dynamic behavior. Despite the complexity of the data structure of dynamic panels, almost all literature focuses on the models assuming that data are free of influential observations or outliers. This is often not the case in reality, not even in relatively reliable macroeconomic data as documented in Zaman et al. (2001). This issue is even more important in the case of panel data, where erroneous observations can be masked by the complex structure of the data.

Despite its relevance, the study of robust techniques for panel data seems to be rather limited. Few contributions are available for static models (e.g., Bramati and Croux, 2007; Aquaro and Čížek, 2013) and even fewer for the dynamic setting. For example, Lucas et al. (2007) constructs the generalized method of moment estimator with a bounded influence function and Galvao (2011) proposes to estimate the dynamic panel model estimated using quantile regression techniques. Both these procedures focus on methods that are only locally robust. On the contrary, Dhaene and Zhu (2009) and recently Aquaro and Čížek (2014) propose median-based robust estimators that are both globally robust. These estimators are based on the median ratios of the first differences of the dependent variable and of the first- or higher-order differences of the lagged dependent variable. There are two main shortcomings of these methods. The first one concerns robustness: since both methods are based only on the first-differences of the dependent variable, they might be overly sensitive to innovation outliers and patches of outliers. The second shortcoming is complementarity: estimation based on the first differences is suitable in the case of weak time dependence and randomly occurring outliers, whereas using additional higher-order differences of lagged dependent variable is beneficial in the case of strong time dependence and innovation outliers or patches of outliers; but the data generating process and outlier structure are not known a priori.

Our aim is to extend these median-based estimators of Dhaene and Zhu (2009) and Aquaro and Čížek (2014) by means of the multiple pairwise difference transformation to obtain a globally robust estimator that addresses above mentioned concerns and exhibits as good finite-sample performance as the commonly used non-robust estimators – such as the one by Blundell and Bond (1998) – in data free of outlying and aberrant observations. The proposed method using higher-order differences of the dependent variable is not new (see Aquaro and Čížek, 2013), but presents two big challenges when applied in dynamic models. In particular, higher-order differences have not been previously used since (i) they can result in a substantial increase in bias in the presence of particular types of outliers and (ii) their number grows quadratically with the number of time periods, which can lead to additional biases due to weak identification or outliers.

In this paper, we first generalize the results of Dhaene and Zhu (2009) for a generic $s$th difference transformation, $s \in \mathbb{N}$, and combine multiple pairwise differences by means of
the generalized method of moments (GMM). To account for the shortcomings of the current methods and to extend the analysis of Aquaro and Čížek (2014), we first analyze the robustness of the median-based moment conditions, derive their influence functions, and quantify the bias caused by data contamination. Subsequently, we use the maximum bias to create two-step GMM estimator, which weights the (median-based) moment conditions both by their variance and bias; this guarantees that imprecise or biased moment conditions get low weights in estimation. Finally, as the number of applicable moment conditions grows quadratically with the number of time periods, a suitable number of moment conditions for the underlying data generating process needs to be selected using a robust version of moment selection procedure of Hall et al. (2007).

The paper is organized as follows. In Section 2, the new estimator is introduced and its asymptotic distribution is presented. Its robust properties are studied in Section 3. The results of the Monte Carlo simulations are summarized in Section 4. The proofs are in the Appendix.

2 Median-based estimation of dynamic panel models

The dynamic panel data model (Section 2.1) and its median-based estimation (Section 2.2) will be now discussed. Later, the two-step GMM estimation procedure (Section 2.3) and the moment selection method (Section 2.4) will be introduced.

2.1 Dynamic panel data model

Consider the dynamic panel data model \((i = 1, \ldots, n; t = 1, \ldots, T; T \geq 3)\)

\[
y_{it} = \alpha y_{it-1} + \eta_i + \varepsilon_{it},
\]

(1)

where \(y_{it}\) is the response variable, \(\eta_i\) is the unobservable fixed effect, and \(\varepsilon_{it}\) represents the idiosyncratic error. To guarantee the stationarity of the data following the model, \(|\alpha| < 1\) is assumed. The time dimension \(T\) is assumed to be fixed. Consequently, fixed or stochastic effects \(\eta_i\) are nuisance parameters, which cannot be consistently estimated. We concentrate on the estimation of this simple dynamic model as the main difficulty lies in the estimation of the autoregressive parameter \(\alpha\) and the extension of the discussed estimators to a model including exogenous covariates is straightforward (see Dhaene and Zhu, 2009, Section 4.1).

As in Aquaro and Čížek (2014) and similarly to Han et al. (2014), we will consider model (1) under the following assumptions:
A.1 Errors $\varepsilon_{it}$ are assumed to be independent across $i = 1, \ldots, n$ and $t = 1, \ldots, T$ and to possess finite second moments. Errors $\{\varepsilon_{it}\}_{t=1}^{T}$ are also independent of fixed effects $\eta_i$.

A.2 The sequences $\{y_{it}\}_{t=1}^{T}$ are time stationary for all $i = 1, \ldots, n$. In particular, the first and second moments of $y_{it}$ conditional to $\eta_i$ do not depend of time.

A.3 Let $\varepsilon_{it} \sim N(0, \sigma^2_\varepsilon)$ for all $i = 1, \ldots, n$ and $t = 1, \ldots, T$.

First, note that no assumptions are made about the unobservable fixed effects $\eta_i$ except for Assumption A.1. The errors $\varepsilon_{it}$ are also not required to follow the same distribution across cross-sectional units $i$: although we derive the results under the normality of the errors, see Assumption A.3, the discussed estimators are consistent as long as the joint distributions of errors $\{\varepsilon_{it}\}_{t=1}^{T}$ are elliptically contoured (see Dhaene and Zhu, 2009, Section 4.2). The normal error distribution as a classical light-tailed distribution is imposed to obtain conservative characterization of the robustness to deviations from the baseline model (1), which naturally depends on non-contaminated error distribution. Finally, the stationarity Assumption A.2 is used not only by the discussed robust estimators, but also by frequently applied GMM estimators such as Blundell and Bond (1998) and it is implied by the assumptions of Han et al. (2014) for $|\alpha| < 1$.

2.2 Median-based moment conditions

To generalize the estimator by Dhaene and Zhu (2009), let $\Delta^s$ denote the $s$th difference operator, that is, $\Delta^sv_t := v_t - v_{t-s}$ (cf. Abrevaya, 2000; Aquaro and Čížek, 2013). Given model (1), it holds under stationarity for $s, q, p \in \mathbb{N}$ that

$$E(\Delta^s y_{it} | \Delta^p y_{it-q}) = r_j \Delta^p y_{it-q}, \quad (2)$$

where the triplet $j = (s, q, p)'$ and $r_j$ are independent of $i$ and $t$, $\max\{s, p+q\} < T$, and $r_j = \text{cov}(\Delta^s y_{it}, \Delta^p y_{it-q})/\text{var}(\Delta^p y_{it-q})$; see for example Bain and Engelhardt (1992, Theorem 5.4.6).

Next, Equation (2) implies that the variables $\Delta^s y_{it} - r_j \Delta^p y_{it-q}$ and $\Delta^p y_{it-q}$ are uncorrelated, and by Assumption A.3, that they are independent and symmetrically distributed around zero. Thus, it follows that $E[\text{sgn}(\Delta^s y_{it} - r_j \Delta^p y_{it-q}) \text{sgn}(\Delta^p y_{it-q})] = 0$, which can be rewritten more conveniently as

$$E \left[ \text{sgn} \left( \frac{\Delta^s y_{it}}{\Delta^p y_{it-q}} - r_j \right) \right] = 0. \quad (3)$$
This facilitates the estimation of \( r_j \) by the sample analog of this condition:

\[
\hat{r}_{nj} = \text{med} \left\{ \frac{\Delta^s y_{it}}{\Delta^p y_{it-q}}; \ t = p + q + 1, \ldots, T; \ i = 1, \ldots, n \right\}.
\]  

(4)

To relate this estimator to the autoregressive coefficient \( \alpha \), Aquaro and Čížek (2014) derived under Assumption A.1–A.2 that the correlation coefficient \( r_j \) satisfies the moment condition

\[
g_j(\alpha) = 2(1 - \alpha^p)r_j - \alpha^q + \alpha^{q+p} + \alpha^{|s-q|} - \alpha^{|s-p-q|} = 0.
\]

(5)

By setting \( s, q, \) and \( p \) in (3) and (5) all equal to one, Dhaene and Zhu (2009)’s estimator is obtained. Then \( \alpha \in (-1, 1) \) is identified by \( g_{111}(\alpha) = (1 - \alpha)(2r_{111} + 1 - \alpha) = 0 \), where \( g_{111}(\hat{\alpha}) \) depends on data only via the median \( r_{111} \). The Dhaene and Zhu (DZ) estimator \( \hat{\alpha}_n \) therefore simply equals to \( 2\hat{r}_{n111} + 1 \) and it was proved to be consistent and asymptotically normal. Aquaro and Čížek (2014)’s estimator (AC-DZ) of \( \alpha \) uses \( s = q = 1 \) and \( p \) being odd, \( p < T - 1 \). Although cases \( s > 1 \) are mentioned there, they are not used due to their robustness properties: while they seem reliable robust to sequences of outliers grouped in several consecutive time periods, they can lead to large biases in the presence of randomly occurring outliers.

2.3 Two-step GMM estimation

To increase the precision and robustness of the estimation, we propose to extend the (AC-)DZ estimator by allowing for multiple differences with \( s = q \geq 1 \) and \( p \geq 1 \); the moment conditions (5) do not allow distinguishing outlying and regular observations for \( s \neq q \) as shown in Aquaro and Čížek (2014). It is interesting to note that, for \( s = q \), (5) simplifies after dividing by \( 1 - \alpha^p \) to

\[
g_j(\alpha) = 2r_j + 1 - \alpha^s = 0.
\]

(6)

The full set of moment conditions in (6) can be then written as

\[
g(\alpha) = \{g_j(\alpha)\}_{j \in \mathcal{J}} \text{ and a fixed finite set } \mathcal{J} \text{ contains all triplets } j = (s, q, p)’ \text{ that are considered in estimation. The DZ estimator corresponds then to the special case } \mathcal{J} = \{(1, 1, 1)’\}. \text{ The AC-DZ relies on a set } \mathcal{J} = \{(1, 1, p)’ : 1 \leq p < T - 1 \text{ odd}\}. \text{ Here we consider all combinations with } s = q \text{ odd and } p \text{ odd, } \mathcal{J} \subseteq \mathcal{J}_o = \{(s, s, p)’ : s \in \mathbb{N} \text{ odd, } p \in \mathbb{N} \text{ odd, } 1 \leq s + p < T\}, \text{ as the single moment conditions do not identify uniquely } \alpha \text{ for even values of } s \text{ or } p, \text{ which can then negatively affect the bias caused by contamination. (More specifically,}
if $s$ is even and $\alpha$ denotes a solution of (6), then $-\alpha$ solves (6) as well; for $p$ even, a similar argument holds for $r_j$.

Since all equations in (7) have to be satisfied simultaneously, the parameter $\alpha$ is estimated by the GMM procedure:

$$\hat{\alpha}_n = \arg\min_{c \in (-1, 1)} g_n(c)' A_n g_n(c),$$

(8)

where $g_n(c) = (g_{nj}(c))_{j \in J}$ is the sample analog of $g(\alpha)$ and corresponds to (6) with $r_j$ being replaced by $\tilde{r}_{nj}$ defined in (4). The weighting matrix $A_n$ has to be positive definite. A simple choice used by Aquaro and Čížek (2014) is proportional to the number of observations available for the estimation of each moment equation: $A_n = A = \text{diag}\{(T - p - s)/T\}$.

The estimator defined in equation (8) will be referred to as the pairwise-difference DZ (PD-DZ) estimator. Its asymptotic distribution has been derived by Aquaro and Čížek (2014, Theorem 1) for a fixed number $T$ of time periods and is presented here for the triplet sets such that $J \subset J_o$.

A.4 Assume that $A_n \to A$ in probability as $n \to \infty$ and $A$ is positive definite.

**Theorem 1.** Suppose that Assumptions A.1–A.4 hold. Let $(1, 1, 1)' \in J \subset J_o$ and $d = \partial g(\alpha)/\partial \alpha$, where $\alpha$ represents the true parameter value. Then for a fixed $T$ and $n \to \infty$, $\hat{\alpha}_n$ is consistent and asymptotically normal,

$$\sqrt{n}(\hat{\alpha}_n - \alpha) \to N(0, (d' Ad)^{-1} d' AVAd(d' Ad)^{-1}),$$

(9)

where $d = \partial g(\alpha)/\partial \alpha = \{-s\alpha^{s-1}\}_{j=(s,s,p)' \in J}$ and $V$ is has a typical element with indices $j = (s, s, p) \in J$, $j' = (s', s', p') \in J$ defined by

$$\frac{\pi^2 \sqrt{[1 - \alpha^s - \frac{1}{4}(1 - \alpha^s)^2(1 - \alpha^p)^2]\{1 - \alpha^{s'} - \frac{1}{4}(1 - \alpha^{s'})^2(1 - \alpha^{p'})^2\}}}{\sqrt{T - s - p}[T - s']}[\sum_{t=s+p+1}^{T} \text{sgn}(\Delta_s y_{it} - r_j \Delta^p y_{it-s}) \text{sgn}(\Delta^p y_{it-s}) \sum_{t=s'+p'+1}^{T} \text{sgn}(\Delta^s y_{it} - r_{j'} \Delta^{p'} y_{it-s'}) \text{sgn}(\Delta^{p'} y_{it-s'})].$$

Although not done in Aquaro and Čížek (2014) due to robustness considerations and a large number of moment conditions, the traditional choice of the GMM weighting matrix $A_n$ equals the inverse of the variance matrix $V_n$ of the moment conditions $g_n(\alpha)$. If $V_n$ converges to $V$ (under usual regularity conditions), the choice $A_n = V_n^{-1}$ minimizes in the limit the asymptotic variance of the GMM estimator, which then equals $(d' V^{-1} d)^{-1}$.

However, we aim to account also the presence of outlying observations that can substan-
tially bias the estimates. Hence, we propose to minimize the mean squared error (MSE) of estimates instead of the asymptotic variance. First, let us denote the MSE of $g_n(\alpha)$ by $W_n$,

$$W_n = MSE\{g_n(\alpha)\} = Bias\{g_n(\alpha)\}Bias\{g_n(\alpha)\}' + Var\{g_n(\alpha)\} = b_n b'_n + V_n.$$ 

Given a weighting matrix $A_n$ and the asymptotic linearity of $\hat{\alpha}_n$ (see Aquaro and Čížek, 2014, the proof of Theorem 1)

$$\hat{\alpha}_n - \alpha = (d'A_n d)^{-1} d'A_n g_n(\alpha) + o_p(1) \quad (10)$$

as $n \to \infty$, it immediately follows that the MSE of $\hat{\alpha}_n$ equals

$$(d'A_n d)^{-1} d'A_n W_n A_n d(d'A_n d)^{-1} + o_p(1),$$

which is (asymptotically) minimized by choosing $A_n = W_n^{-1}$ (see Hansen, 1982, Theorem 3.2). Thus, the optimal weighting matrix is inversely proportional to the MSE matrix $W_n$ of the moment conditions, or alternatively, to the sum of the usual variance matrix and the squared-bias matrix of the moment conditions.

Next, to create a feasible procedure, both the variance and squared bias matrices have to be estimated because they depend on the data generating process and the amount and type of data contamination present in the data. The estimation thus proceeds in two steps: first, the (AC-)DZ estimator is applied to obtain an initial parameter estimates; then – after estimating the bias $b_n$ and variance $V_n$ of moment conditions – the GMM estimator with all applicable pairwise differences is evaluated using the estimate of the weighting matrix $A_n = [b_n b'_n + V_n]^{-1}$.

Whereas the estimate $\hat{V}_n$ of $V_n$ can be directly obtained from Theorem 1 using initial estimates of $r_j$ and $\alpha$, the estimating $b_n$ by $\hat{b}_n$ requires first studying the biases of median-based moment conditions and constructing a feasible estimate thereof in Section 3. Using estimates $\hat{V}_n$ and $\hat{b}_n$ to construct $\hat{W}_n = \hat{b}_n \hat{b}'_n + \hat{V}_n$ and $\hat{A}_n = \hat{W}_n^{-1}$ then leads to the proposed second-step GMM estimator

$$\hat{\alpha}_n = \arg \min_{c \in (-1,1)} g_n(c)' \hat{A}_n g_n(c) = \arg \min_{c \in (-1,1)} g_n(c)' [\hat{b}_n \hat{b}'_n + \hat{V}_n]^{-1} g_n(c). \quad (11)$$

### 2.4 Robust moment selection

The proposed two-step GMM estimator is based on the moment conditions (6), and given that we consider only odd $s$ and $p$, their number equals approximately $T(T - 1)/8$ and grows quadratically with the number of time periods. Although the extra moment conditions based on higher-order differences might improve precision of estimation for larger values of $|\alpha|$, their
usefulness is rather limited if \( \alpha \) is close to zero. At the same time, a large number of moment conditions might increase estimation bias due to outliers. More specifically, Aquaro and Čížek (2014) showed for \( \alpha \) close to 0 that the original moment condition of the DZ estimator \( s = q = p = 1 \) is least sensitive to random outliers, for instance; including higher-order moment conditions then just increases bias, does not improve the variance, and is thus harmful.

To account for this, we propose to select the moment conditions used in estimation by a robust analog of the moment selection criterion of Hall et al. (2007). They propose the so-called relevant moment selection criterion (RMSC) that – for a given set of moment conditions defined by triplets \( \mathcal{J} \) in our case – equals

\[
RMSC(\mathcal{J}) = \ln(|\hat{V}_{n,\mathcal{J}}|) + \kappa(|\mathcal{J}|, n).
\]

Matrix \( \hat{V}_{n,\mathcal{J}} \) represents an estimate of the variance matrix \( V_{\mathcal{J}} \) of moment conditions (7) defined by triplets \( \mathcal{J} \) and \( \kappa(\cdot, \cdot) \) is a penalty term depending on the number \(|\mathcal{J}|\) of triplets (or moment conditions) and on the estimation precision of \( \hat{V}_{n} \), which is proportional to the sample size, that is, to \( n \) for the most off-diagonal elements of \( V_{n} \) (see Theorem 1). To select relevant moment conditions, this criterion has to be minimized:

\[
\hat{\mathcal{J}} = \arg \min_{\mathcal{J} \subseteq \mathcal{J}_0} RMSC(\mathcal{J}).
\]

Two examples of the penalization term used by Hall et al. (2007) are the Bayesian information criterion (BIC) with \( \kappa(c, n) = (c - K) \cdot \ln(\sqrt{n})/\sqrt{n} \) and the Hannann-Quinn information criterion (HQIC) with \( \kappa(c, n) = (c - K) \cdot \kappa_c \ln(\ln(\sqrt{n}))/\sqrt{n} \), where the number of estimated parameters \( K = 1 \) in model (1) and constant \( \kappa_c > 2 \).

As in Section 2.3, the proposed robust estimator (11) should minimize the MSE error rather than just the variance of the estimates. We therefore suggest to use the relevant robust moment selection criterion (RRMSC),

\[
RRMSC(\mathcal{J}) = \ln(|\hat{W}_{n,\mathcal{J}}|) + \kappa(|\mathcal{J}|, n),
\]

(12)

which is based on the determinant of an estimate \( \hat{W}_{n} \) of the MSE matrix \( W_{n} \) rather than on the variance matrix estimate \( \hat{V}_{n} \) of the moment conditions. The relevant robust moment conditions are then obtained by minimizing

\[
\hat{\mathcal{J}} = \arg \min_{\mathcal{J} \subseteq \mathcal{J}_0} RRMSC(\mathcal{J}).
\]
3 Robustness properties

There are many measures of robustness that are related to the bias of an estimator, or more typically, the worst-case bias of an estimator due to an unknown form of outlier contamination. In this section, various kinds of contamination are introduced and some relevant measures of robustness are defined (Section 3.1). Using these measures, we characterize the robustness of moment conditions (6) in Section 3.2 and the robustness of the GMM estimator (8) in Section 3.3. Next, we use these results to estimate the bias of the moment conditions (6) as discussed in Section 3.4. Finally, the whole estimation procedure is summarized in Section 3.5.

3.1 Measures of robustness

Given that the analyzed data from model (1) are dependent, the effect of outliers can depend on their structure. Therefore, we first describe the considered contamination schemes and then the relevant measures of robustness.

More formally, let \( Z \) be the set of all possible samples \( Z = \{z_{it}\} \) of size \((n, T)\) following model (1) and let \( Z_\epsilon = \{z^\epsilon_{it}\} \) be a contaminating sample of size \((n, T)\) following a fixed data-generating process, where the index \( \epsilon \) of \( Z_\epsilon \) indicates the probability that an observation in \( Z_\epsilon \) is different from zero. The observed contaminated sample is \( Z + Z_\epsilon = \{z_{it} + z^\epsilon_{it}\}_{i=1,t=1}^{n_T} \). Similarly to Dhaene and Zhu (2009), we consider the contamination by independent additive outliers following some distribution \( G_\zeta \) with a parameter \( \zeta \),

\[
Z^{1}_{\epsilon, \zeta} = \{z^\epsilon_{it}\}_{i=1,t=1}^{n_T}, \quad P(z^\epsilon_{it} \neq 0) = \epsilon, \quad P(z^\epsilon_{it} \leq u | z^\epsilon_{it} \neq 0) = G_\zeta(u), \quad (13)
\]

and by patches of \( k \) additive outliers,

\[
Z^{2}_{\epsilon, \zeta} = \{\zeta \cdot I(\nu^\epsilon_{it} = 1 \text{ or } \ldots \text{ or } \nu^\epsilon_{it-k+1} = 1)\}_{i=1,t=1}^{n_T}, \quad (14)
\]

where \( \nu^\epsilon_{it} \) follows the Bernoulli distribution with the parameter \( \tilde{\epsilon} \) such that \((1 - \tilde{\epsilon})^k = \epsilon\). Additionally, a third contamination scheme \( Z^{3}_{\epsilon, \zeta} = \{z^\epsilon_{it}\}_{i=1,t=1}^{n_T} \) is considered, where

\[
z^\epsilon_{it} = \begin{cases} 
a_{it-l}(-1)^l & \text{if the smallest index } l \geq 0 \text{ with } \nu^\epsilon_{it-l} = 1 \text{ satisfies } l \leq k - 1, \\ 0 & \text{otherwise}, \end{cases} \quad (15)
\]

where \( \Pr(a_{it-l} = \zeta) = 1/2 \) and \( \Pr(a_{it-l} = -\zeta) = 1/2 \) and where \( \nu^\epsilon_{it} \) is defined as in \( Z^{2}_{\epsilon, \zeta} \). Note that (14) and (15) are special cases of a more general type of contamination \( Z^{4}_{\epsilon, \zeta} = \{z^\epsilon_{it}\}_{i=1,t=1}^{n_T} \).
where

$$z_{it}^\epsilon = \begin{cases} 
  a_{it-l\rho}^l & \text{if the smallest index } l \geq 0 \text{ with } \nu_{it-l}^\epsilon = 1 \text{ satisfies } l \leq k - 1, \\
  0 & \text{otherwise},
\end{cases}$$

(16)

and $-1 \leq \rho \leq 1$. Note that this general type of contamination closely corresponds to the contamination by innovation outliers for large $k$ and $\rho = \alpha$. As we can conjecture from Dhaene and Zhu (2009)’s results for $s = p = 1$ that the contamination scheme $Z_{\epsilon,\zeta}^4$ biases estimates towards $\rho$ for $\zeta \to +\infty$ and $\rho$ is unknown in practice, we are not analysing this most general case with $\rho \in [-1, 1]$. Instead, we concentrate on the most extreme cases of $\rho = 1$ and $\rho = -1$ as they can arguably bias the estimate most. Hence, the contamination schemes $Z_{\epsilon,\zeta}^1, Z_{\epsilon,\zeta}^2,$ and $Z_{\epsilon,\zeta}^3$ bias the DZ estimates of $\alpha$ towards 0, 1, and $-1$, respectively – see Section 3.2 and Dhaene and Zhu (2009).

Given the contamination schemes, one of the traditional measures of the global robustness of an estimator is the breakdown point. It can be defined as the smallest fraction of the data that can be changed in such a way that the estimator will not reflect any information concerning the remaining (non-contaminated) observations. Following Genton and Lucas (2003), the estimator as a function of random data is considered non-informative if its distribution function becomes degenerate: the breakdown point $\epsilon^*$ of an estimator $T$ is defined as

$$\epsilon^*_n(T) = \inf_{\epsilon \geq 0} \left\{ \epsilon \bigg| \sup_{Z \in Z} T(Z + Z) = \inf_{Z \in \epsilon} T(Z + Z) \right\}. $$

(17)

Aquaro and Čížek (2014) derived the breakdown points of the estimators $\hat{r}_j, j \in J$, for contamination schemes $Z_{\epsilon,\zeta}^1, Z_{\epsilon,\zeta}^2, \text{ and } Z_{\epsilon,\zeta}^3,$ and under some regularity conditions, proved that the breakdown point of the GMM estimator (8) equals the breakdown point of the DZ estimator $\hat{r}_{(1,1,1)}$ if $(1, 1, 1)' \in J$. While such results characterize the global robustness of the PD-DZ estimators, they are not informative about the size of the bias caused by outliers.

We therefore base the estimation of the bias due to contamination on the influence function. It is a traditional measure of local robustness and can be defined as follows. Let $T(Z + Z)$ denote a generic estimator of an unknown parameter $\theta$ based on a contaminated sample $Z + Z = \{z_{it} + z_{it}^\epsilon\}_{i=1}^n,_{t=1}$, where $Z$ and $Z$ have been defined at the beginning of Section 3. As the definition is asymptotic, let $T(\theta, \zeta, \epsilon, T)$ be the probability limit of $T(Z + Z)$ when $T$ is fixed and $n \to \infty$. Note that $T(\theta, \zeta, \epsilon, T)$ depends on the unknown parameter $\theta$ describing the data generating process, on the fraction $\epsilon$ of data contamination, on the non-zero value $\zeta$ characterizing the outliers, and on the number of time periods $T$. Assume $T$ is consistent under non-contaminated data, that is, $T(\theta, \zeta, 0, T) = \theta$. The influence function (IF) of estimator $T$
at data generating process $Z$ due to contamination $Z_\epsilon$ is defined as

$$\text{IF}(T; \theta, \zeta, T) := \lim_{\epsilon \to 0} \frac{T(\theta, \zeta, \epsilon, T) - \theta}{\epsilon} = \left. \frac{\partial \text{bias}(T; \theta, \zeta, \epsilon, T)}{\partial \epsilon} \right|_{\epsilon=0}, \quad (18)$$

where the equality follows by the definition of asymptotic bias of $T$ due to the data contamination $Z_\epsilon$, $\text{bias}(T; \theta, \zeta, \epsilon, T) := T(\theta, \zeta, \epsilon, T) - \theta$. (If IF does not depend on the number $T$ of time periods, $T$ can be omitted from its arguments.)

Clearly, the knowledge of the influence function allows us to approximate the bias of an estimator $T$ at $Z + Z_\epsilon$ by $\epsilon \cdot \text{IF}(T; \theta, \zeta, T)$. Although such an approximation is often valid only for small values of $\epsilon > 0$ (e.g., in the linear regression model, where the bias can get infinite), it is relevant in a much wider range of contamination levels $\epsilon$ in model (1) given that the parameter space $(-1, 1)$ is bounded and so is the bias.

The disadvantage of approximating bias by $\epsilon \cdot \text{IF}(T; \theta, \zeta, T)$ is that it depends on the unknown magnitude $\zeta$ of outliers. We therefore suggest to evaluate the supremum of the influence function, the gross error sensitivity (GES)

$$\text{GES}(T; \theta, T) = \sup_{\zeta} |\text{IF}(T; \theta, \zeta, T)| \quad (19)$$

and approximate the worst-case bias by $\epsilon \cdot \text{GES}(T; \theta, T)$. For the PD-DZ estimator and the corresponding moment conditions, IF and GES are derived in the following Sections 3.2 and 3.3, where $T$ will equal to $\hat{\alpha}$ and $\hat{r}_j$, respectively (without the subscript $n$ since the IF and GES definitions depend only on the probability limits of the estimators).

### 3.2 Influence function

The GMM estimator (8) is based on moment conditions depending on the data only by means of the medians $r_j$. We therefore derive first the influence functions of the estimates $\hat{r}_j$ and then combine them to derive the influence function of the GMM estimator. Building on Dhaene and Zhu (2009, Theorems 2 and 7), the IFs of $\hat{r}_j$ in model (1) under contamination schemes $Z^1_{\epsilon, \zeta}$, $Z^2_{\epsilon, \zeta}$, and $Z^3_{\epsilon, \zeta}$ are derived in the following Theorems 2–4. Only the point-mass distribution $G_\zeta$ with the mass at $\zeta \in R$ is considered. In all theorems, $\Phi$ denotes the cumulative distribution function of the standard normal distribution $N(0, 1)$.

**Theorem 2.** Let Assumptions A.1–A.3 hold and $j \in J_0$. Then it holds in model (1) under
Figure 1: Gross-error sensitivity of $\hat{r}_j$, $j = (s, s, p)' \in J_o$, under contamination $Z_{e,\xi}^1$ by independent additive outliers.
Figure 2: Gross-error sensitivity of $\hat{r}_j$, $j = (s, s, p)' \in J_o$, under contamination $Z^2_{e, \xi}$ by patch additive outliers, length of the path $k = 6$. 
Figure 3: Gross-error sensitivity of $\hat{r}_j$, $j = (s, s, p)' \in \mathcal{J}_0$, under contamination $Z_{e, \xi}^3$ by patch additive outliers, length of the path $k = 6$. 
the independent-additive-outlier contamination $Z_{c, \zeta}^1$ with point-mass distribution at $\zeta \neq 0$ that

$$\text{IF}(\hat{r}_j; \alpha, \zeta) = -\pi \sqrt{\frac{1 - \alpha^s}{1 - \alpha^p} - \frac{1}{4}(1 - \alpha^s)^2} \times \left[ \Phi \left( \frac{\zeta(1 + \alpha^s)/2}{\sqrt{2 \frac{\sigma^2}{1 - \alpha^s} \left( 1 - \alpha^s - \frac{(1 - \alpha^s)^2}{4}(1 - \alpha^p) \right)}} \right) - \Phi \left( \frac{\zeta(1 - \alpha^s)/2}{\sqrt{2 \frac{\sigma^2}{1 - \alpha^s} \left( 1 - \alpha^s - \frac{(1 - \alpha^s)^2}{4}(1 - \alpha^p) \right)}} \right) \right]$$

for $j \in J$. As $\zeta$ is unknown, we characterize the worst-case scenario by means of the gross error sensitivity: recall that $\text{GES}(\hat{r}_j; \alpha) = \sup_\zeta |\text{IF}(\hat{r}_j; \alpha, \zeta)|$ by Equation (19).

**Theorem 3.** Let Assumptions A.1–A.3 hold and $j \in J_0$. Then it holds in model (1) under the patched-additive-outlier contamination $Z_{c, \zeta}^2$ with point-mass distribution at $\zeta \neq 0$ and patch length $k \geq 2$ that

$$\text{IF}(\hat{r}_j; \alpha, \zeta) = \frac{\pi}{\kappa} \sqrt{\frac{1 - \alpha^s}{1 - \alpha^p} - \frac{(1 - \alpha^s)^2}{4}} \times \left[ p_C'(0) \left( C(r_j; \zeta, 0) - \frac{1}{2} \right) + p_D'(0) \left( D(r_j; \zeta, 0) - \frac{1}{2} \right) \right],$$

where $p_C'(0)$, $p_D'(0)$, $C(r_j; \zeta, 0)$, and $D(r_j; \zeta, 0)$ are defined in (54), (55), (58), and (59), respectively.

**Theorem 4.** Let Assumptions A.1–A.3 hold and $j \in J_0$. Then it holds in model (1) under the patched-additive-outlier contamination $Z_{c, \zeta}^2$ with point-mass distribution at $\zeta \neq 0$ and patch length $k \geq 2$ that

$$\text{IF}(\hat{r}_j; \alpha, \zeta) = \frac{\pi}{\kappa} \sqrt{\frac{1 - \alpha^s}{1 - \alpha^p} - \frac{(1 - \alpha^s)^2}{4}} \times \left[ p_L'(0) \left( C \left( \frac{1}{2} \right) + p_D \left( \frac{1}{2} \right) + p_E \left( \frac{1}{2} \right) + p_G \left( \frac{1}{2} \right) + p_I \left( \frac{1}{2} \right) \right) \right]$$

where $p_L', L \in \{C, D, E, G, I\}$, are defined in Equations (75), (76), (77), (79), (81), $L(1/2) = L(r_j; \zeta, 0) - 1/2$ for $L \in \{C, D, E, G, I\}$ and $L \in \{C, D, E, G, I\}$, and $L(r_j; \zeta, 0)$ for $L \in \{C, D, E, G, I\}$ are defined in Equations (84)–(88) in Appendix A.3.

The influence functions reported in Theorems 2–4 are complicated objects both due to their algebraic forms and their dependence on the unknown parameter value $\zeta$. As $\zeta$ is unknown, we characterize the worst-case scenario by means of the gross error sensitivity: recall that $\text{GES}(\hat{r}_j; \alpha) = \sup_\zeta |\text{IF}(\hat{r}_j; \alpha, \zeta)|$ by Equation (19).
Given the results in Theorems 2–4, we have to compute the GES of estimators \( \hat{r}_j \) numerically for each \( j = (s, s, p)' \in J_o \) and \( \alpha \in (-1, 1) \). Although this might be relatively demanding if \( T \) is large and a dense grid for \( \alpha \) is used, note that the GES values are asymptotic and independent of a particular data set. They have to be therefore evaluated just once and then used repeatedly during any application of the proposed PD-DZ estimator. We computed the GES of \( \hat{r}_j \) for \( j \in \{ (s, s, p)' ; s = 1, 3, 5, 7 \} \) and \( p = 1, 3, 5, 7, 9, 11 \} \) with the variance \( \sigma^2_{e} \) set equal to one without loss of generality. The results corresponding to Theorems 2–4 are depicted on Figures 1–3. Irrespective of the contamination scheme, most GES curves display typically higher sensitivity to outliers for \( |\alpha| \) close to one than for values of the autoregressive parameter around zero. One can also see that the DZ estimator corresponding to \( s = 1 \) and \( p = 1 \) is indeed biased towards 0, 1, and \(-1\) for the contamination schemes \( Z^1_{\epsilon, \zeta}, Z^2_{\epsilon, \zeta}, \) and \( Z^3_{\epsilon, \zeta}, \) respectively. Concerning the higher-order differences we propose to add to the (AC-)DZ methods, Figure 1 documents they do exhibit high sensitivity to independent outliers. On the other hand, their sensitivity to the patches of outliers on Figure 2, for instance, decreases with an increasing \( s \) and becomes very low (relative to \( s = 1 \) and \( p \geq 1 \)) if \( s \) is larger than the patch length \( k \), for example, \( s = 7 > k = 6 \).

### 3.3 Robust properties of the GMM estimator \( \hat{\alpha}_n \)

Given the results of the previous sections, we will now analyze the robust properties of the general GMM estimator \( \hat{\alpha} \) defined in (8) and based on moment equations (7) for \( j = (s, s, p)' \in J_o \). For the sake of simplicity, we assume now that the weighting matrix of the PD-DZ estimator (8) is sample independent (this result will not be directly used within the estimation procedure).

**Theorem 5.** Consider a particular additive outlier contamination \( Z_{\epsilon} \) occurring with probability \( \epsilon \), where \( 0 < \epsilon < 1 \). Further, let \( J \subseteq J_o \). Finally, assume that \( A_n = A \) is a positive definite diagonal matrix. Then the influence function of the GMM estimator \( \hat{\alpha} \) using moment conditions indexed by \( J \) is given by

\[
\text{IF}(\hat{\alpha}; \alpha, \zeta) = - (d' A d)^{-1} d' A \psi,
\]

where \( d \) is defined in Theorem 1 and \( \psi \) is the \( |J| \times 1 \) vector of the influence function of each single \( \hat{r}_j \), \( \psi = (\text{IF}(\hat{r}_j; \alpha, \zeta))_{j \in J}. \)

Contrary to the breakdown point of Aquaro and Čížek (2014) mentioned earlier, the bias of the proposed PD-DZ estimators is a linear combination of the biases of the individual moment conditions depending on \( \hat{r}_j \). To minimize the influence of outliers on the estimator,
one could theoretically select the moment condition with the smallest IF value, which could however result in a poor estimation if the moment condition is not very informative of the parameter \( \alpha \). As suggested in Section 2.3, we aim to minimize the MSE of the estimates and thus downweight the individual moment conditions if their biases or variances are large. Obviously, this will also lead to lower effects of biased or imprecise moment conditions on the IF in Theorem 5. To quantify the maximum influence of generally unknown outliers on the estimate, the GES function of the GMM estimator, that is, the supremum of IF in (23) with respect to \( \zeta \) can be used again.

### 3.4 Estimating the bias

The IF and GES derived in Section 3.2 characterize only the derivative of the bias caused by outlier contamination. We will refer to them in the case of contamination schemes \( Z_{\epsilon, \zeta}^1, Z_{\epsilon, \zeta}^2, \) and \( Z_{\epsilon, \zeta}^3 \) by \( \text{IF}_k^1 \) and \( \text{GES}_k^1, c = 1, 2, 3 \), respectively, where \( k \) denotes the number of consecutive outliers (patch length) in schemes \( Z_{\epsilon, \zeta}^2 \) and \( Z_{\epsilon, \zeta}^3 \). Whenever the sequence of consecutive outliers is mentioned in this section, we understand by that a sequence of observations \( y_{it}, t = t_1, \ldots, t_2 \), that can all be considered outliers.

To approximate \( \hat{b}_n = \text{Bias}\{g_n(\alpha)\} \) introduced in Section 2.3, we therefore need to estimate the type and amount of outliers in a given sample. Assuming that the consecutive outliers form sequences of length \( k \) and the fraction of such outliers in data is denoted \( \epsilon_k \), the bias can be approximated using the \( \epsilon_k \)-multiple of \( \text{IF}_1^1 \) or \( \text{GES}_1^1 \) if \( k = 1 \) and of \( \max\{ |\text{IF}_k^2|, |\text{IF}_k^3| \} \) or \( \max\{ \text{GES}_k^2, \text{GES}_k^3 \} \) if \( k > 1 \) since we cannot reliably distinguish contamination \( Z_{\epsilon, \zeta}^2 \) and \( Z_{\epsilon, \zeta}^3 \). Given that the outlier locations cannot be reliably computed either, GES is preferred for estimating the bias due to contamination.

We therefore suggest to compute the bias vector \( \hat{b}_n \) in the following way, provided that the estimates \( \hat{\epsilon}_k \) of the fractions of outliers forming sequences or patches of length \( k \) are available:

\[
\hat{b}_n = \left\{ \max_{k=1,...,T} \left[ \hat{\epsilon}_k \cdot \max_{c} \text{GES}_k^c(\hat{\alpha}_n^0; \hat{r}_{ij}; \hat{\alpha}_n^0) \right] \right\}_{j \in J},
\]

where \( \hat{\alpha}_n^0 \) is an initial estimate of the parameter \( \alpha \) and the inner maximum is taken over \( c \in \{1\} \) for \( k = 1 \) and \( c \in \{2, 3\} \) for \( k > 1 \). Note that if outliers (or particular types of outliers) are not present, \( \hat{\epsilon}_k = 0 \) and the corresponding bias term is zero.

To estimate \( \hat{\epsilon}_k \), an initial estimate \( \hat{\alpha}_n^0 \) is needed. Once it is obtained by the DZ or AC-DZ estimator, the regression residuals \( \hat{\varepsilon}_{it} \) can be constructed, for example, by \( \hat{u}_{it} = y_{it} - \hat{\alpha}_n^0 y_{it-1} \) and \( \hat{\varepsilon}_{it} = \hat{u}_{it} - \text{med}_{t=2,...,T} \hat{u}_{it} \) for any \( i = 1, \ldots, n \) and \( t = 2, \ldots, T \); the median \( \text{med}_{t=2,...,T} \hat{u}_{it} \) is used here as an estimate of the individual effect \( \eta_i \) similarly to Bramati and Croux (2007).
Having estimated residuals $\hat{\varepsilon}_{it}$, the outliers are detected and the fractions $\hat{\epsilon}_k$ of outliers in data forming the patches or sequences of $k$ consecutive outliers are computed. We consider as outliers all observations with $|\hat{\varepsilon}_{it}| > \gamma \hat{\sigma}_\varepsilon$, where $\hat{\sigma}_\varepsilon$ estimates the standard deviation of $\varepsilon_{it}$, for example, by the median absolute deviation $\hat{\sigma}_\varepsilon = \text{MAD}(\hat{\varepsilon}_{it})/\Phi^{-1}(3/4)$, and $\gamma$ is a cut-off point ($\Phi$ denotes the standard normal distribution function). Although one typically uses a fixed cut-off point such as $\gamma = 2.5$, it can be chosen in a data-adaptive way by determining the fraction of residuals compatible with the normal distribution function of errors, for instance. This approach pioneered by Gervini and Yohai (2002) determines the cut-off point as the quantile of the distribution $F_0^+(t) = \Phi(t) - \Phi(-t)$, $t \geq 0$, of $|\varepsilon_{it}|$, $\varepsilon_{it} \sim N(0,1)$:

$$\hat{\gamma}_n = \min\{t : F_n^+(t) \geq 1 - d_n\} \quad (25)$$

for

$$d_n = \sup_{t \geq 2.5} \max\{0, F_0^+(t) - F_n^+(t)\},$$

where $F_n^+$ denotes the empirical distribution function of $|\hat{\varepsilon}_{it}|$.

### 3.5 Algorithm

The whole procedure of the bias estimation, and subsequently, the proposed GMM estimation with the robust moment selection can be summarized as follows.

1. Obtain an initial estimate $\hat{\alpha}_n^0$ by DZ or AC-DZ estimator.
2. Compute residuals $\hat{u}_{it} = y_{it} - \hat{\alpha}_n^0 y_{it-1}$ and $\hat{\varepsilon}_{it} = \hat{u}_{it} - \text{med}_{i=2,\ldots,T} \hat{u}_{it}$ and their standard deviation $\hat{\sigma}_\varepsilon$.
3. Using the data-adaptive cut-off point (25), determine the fractions $\hat{\epsilon}_k$ of outliers present in the data in the forms of outlier sequences of length $k$.
4. Approximate the bias $\hat{b}_n$ due to outliers by $\hat{b}_n$ using (24) and estimate the variance matrix $V_n$ in Theorem 1 by $V_n$ for all moment conditions (6) defined for indices $j \in J_o$.
5. For all $j = (s,s,p)' \in J_o$,
   
   (a) set $J = \{(k,k,l)' : 1 \leq k \leq s \text{ odd}, 1 \leq l \leq p \text{ is odd}\}$;
   
   (b) compute the GMM estimate $\hat{\alpha}_{n,J}$ defined in (11) using the moment conditions selected by $J$ and the weighting matrix defined as the inverse of the corresponding submatrix of $W_n = \hat{b}_n \hat{b}_n' + V_n$;
   
   (c) evaluate the criterion $\text{RRMSC}(J)$ defined in (12).
6. Select the set of moment conditions by

\[ \hat{J} = \arg \min_{\mathcal{J} \subseteq \mathcal{J}_0} RRMSC(\mathcal{J}). \]

7. The final estimate equals \( \hat{\alpha}_{n, \hat{J}} \).

Let us note that the algorithm in step 5 does not evaluate the GMM estimates for all subsets of indices \( \mathcal{J} \subseteq \mathcal{J}_0 \) and the corresponding moment conditions as that would be very time-consuming. It is therefore suggested to limit the number of \( \mathcal{J}_0 \) subsets and one possible proposal, which always includes the DZ condition in the estimation, is described in point 5 of the algorithm. If an extensive evaluation of many GMM estimators has to be avoided, it is possible to opt for a simple selection between the DZ, AC-DZ, and PD-DZ estimator, where PD-DZ uses all moment conditions defined by \( \mathcal{J}_0 \).

4 Monte Carlo simulation

In this section, we evaluate the finite sample performance of the proposed and existing estimators by Monte Carlo simulations. Let \( \{y_{it}\} \) follow model (1). We generate \( T+100 \) observations for each \( i \) and discard the first 100 observations to reduce the effect of the initial observations and to achieve stationarity. We consider cases with \( \alpha = 0.1, 0.5, 0.9, \) \( n = 25, 50, 100, 200, T = 6, 12, \) \( \eta_i \sim N(0, \sigma_\eta^2) \), and \( \varepsilon_{iit} \sim N(0, 1) \). If data contamination is present, it follows the contamination schemes (13) and (14) for \( \epsilon = 0.05, 0.10, 0.20 \), although we report only \( \epsilon = 0.10 \) due to similarity of other results. More specifically, \( Z_{i,\zeta}^1 \) uses \( G_\zeta = U(10, 90) \) and \( Z_{\epsilon,\zeta}^2 \) employs \( p = 3 \) and \( \zeta \) drawn for each patch randomly from \( U(10, 90) \); \( U(\cdot, \cdot) \) denotes here the uniform distribution. Note that we have also considered mixes of two contamination schemes, for example, mixing equally independent additive outliers and patches of outliers, but the results are not reported as they are just convex combinations of the corresponding results obtained with only the first and only the second contamination schemes.

All estimators are compared by means of the mean bias and the root mean squared error (RMSE) evaluated using 1000 replications. The included estimators are chosen as follows. The non-robust estimators are represented by the Arellano-Bond (AB) two-step GMM estimator\(^1\) (Arellano and Bond, 1991), the system Blundell and Bond (BB) estimator\(^2\) (Blundell and

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\(^1\)The (optimal) inverse weight matrix, which is used here, is \( \sum_i Z_i^{AB} H Z_i^{AB} \), where \( Z_i^{AB} \) is the matrix of instruments per individual and \( H \) is a \((T - 1) \times (T - 1)\) tridiagonal matrix with \(-1\) in the first two sub-diagonals, and zeros elsewhere (see Arellano and Bond, 1991, p. 279).

\(^2\)The inverse weight matrix is \( \sum_i Z_i^{BB} G Z_i^{BB} \), where \( Z_i^{BB} \) is the matrix of instruments per individual and \( G \) is a partitioned matrix, \( G = \text{diag}(H, I) \), where \( H \) is as in Arellano-Bond and \( I \) is the identity matrix (see Kiviet, 2007, Eq. (38)).
Table 1: RMSE for all estimators in model with $\varepsilon_{it} \sim N(0, 1)$ and $\eta_i \sim N(0, 1)$ under different sample sizes.

| $\alpha$ | RMSE | RRMSC |
|----------|------|-------|
|          | $T = 6$ | $T = 12$ |
|          | 25  | 50  | 100 | 200 | 25  | 50  | 100 | 200 |
| 0.1      | XD  | 0.120 | 0.083 | 0.060 | 0.042 | 0.068 | 0.048 | 0.034 | 0.023 |
|          | AB  | 0.160 | 0.117 | 0.082 | 0.057 | 0.098 | 0.065 | 0.045 | 0.030 |
|          | BB  | 0.143 | 0.105 | 0.074 | 0.054 | 0.101 | 0.069 | 0.048 | 0.032 |
|          | DZ  | 0.255 | 0.188 | 0.125 | 0.094 | 0.164 | 0.118 | 0.081 | 0.059 |
|          | AC-DZ | 0.247 | 0.177 | 0.125 | 0.090 | 0.145 | 0.106 | 0.076 | 0.051 |
|          | PD-DZ BIC | 0.258 | 0.183 | 0.125 | 0.090 | 0.155 | 0.108 | 0.071 | 0.050 |
|          | PD-DZ HQIC | 0.251 | 0.179 | 0.124 | 0.089 | 0.152 | 0.100 | 0.069 | 0.050 |
| 0.5      | XD  | 0.135 | 0.093 | 0.066 | 0.047 | 0.065 | 0.047 | 0.034 | 0.023 |
|          | AB  | 0.256 | 0.193 | 0.127 | 0.094 | 0.131 | 0.089 | 0.059 | 0.041 |
|          | BB  | 0.163 | 0.127 | 0.095 | 0.069 | 0.119 | 0.083 | 0.058 | 0.042 |
|          | DZ  | 0.286 | 0.207 | 0.145 | 0.099 | 0.186 | 0.128 | 0.091 | 0.063 |
|          | AC-DZ | 0.226 | 0.167 | 0.115 | 0.080 | 0.120 | 0.086 | 0.061 | 0.044 |
|          | PD-DZ BIC | 0.238 | 0.176 | 0.121 | 0.083 | 0.130 | 0.091 | 0.064 | 0.044 |
|          | PD-DZ HQIC | 0.240 | 0.176 | 0.122 | 0.081 | 0.123 | 0.087 | 0.061 | 0.044 |
| 0.9      | XD  | 0.139 | 0.097 | 0.070 | 0.050 | 0.061 | 0.042 | 0.030 | 0.021 |
|          | AB  | 0.693 | 0.612 | 0.523 | 0.431 | 0.322 | 0.275 | 0.230 | 0.162 |
|          | BB  | 0.096 | 0.086 | 0.083 | 0.068 | 0.058 | 0.056 | 0.051 | 0.044 |
|          | DZ  | 0.292 | 0.219 | 0.153 | 0.106 | 0.197 | 0.139 | 0.095 | 0.067 |
|          | AC-DZ | 0.172 | 0.127 | 0.096 | 0.074 | 0.087 | 0.064 | 0.047 | 0.033 |
|          | PD-DZ BIC | 0.184 | 0.132 | 0.098 | 0.078 | 0.089 | 0.065 | 0.050 | 0.035 |
|          | PD-DZ HQIC | 0.195 | 0.136 | 0.101 | 0.074 | 0.090 | 0.067 | 0.050 | 0.033 |
Table 2: Biases and RMSE for all estimators in data with $\varepsilon_{it} \sim \text{N}(0,1)$, $\eta_i \sim \text{N}(0,1)$, and 10% contamination by independent additive outliers under different sample sizes.

| $T$ | $\alpha$ | $n$ | $\text{RRMSC}$ | Bias | RMSE |
|-----|---------|-----|----------------|------|------|
|     |         |     |                |      |      |
| 0.1 | XD      | -0.096 | -0.101 | -0.101 | -0.100 | 0.125 | 0.107 | 0.110 | 0.102 |
|     | AB      | -0.096 | -0.067 | -0.103 | -0.091 | 0.122 | 0.087 | 0.115 | 0.095 |
|     | BB      | -0.094 | -0.086 | -0.129 | -0.104 | 0.127 | 0.096 | 0.139 | 0.107 |
|     | DZ      | -0.007 | -0.005 | -0.001 | -0.003 | 0.226 | 0.116 | 0.147 | 0.073 |
|     | AC-DZ   | 0.005  | -0.006 | -0.002 | -0.003 | 0.220 | 0.113 | 0.136 | 0.069 |
|     | PD-DZ BIC | 0.010 | -0.004 | 0.004 | -0.002 | 0.231 | 0.119 | 0.130 | 0.061 |
|     | PD-DZ HQIC | 0.008 | -0.005 | -0.000 | -0.002 | 0.238 | 0.119 | 0.125 | 0.061 |
| 0.5 | XD      | -0.497 | -0.500 | -0.497 | -0.498 | 0.502 | 0.501 | 0.499 | 0.499 |
|     | AB      | -0.476 | -0.470 | -0.506 | -0.491 | 0.485 | 0.471 | 0.508 | 0.492 |
|     | BB      | -0.493 | -0.485 | -0.527 | -0.502 | 0.500 | 0.487 | 0.530 | 0.503 |
|     | DZ      | -0.020 | -0.020 | -0.015 | -0.022 | 0.242 | 0.127 | 0.154 | 0.082 |
|     | AD-DZ   | -0.023 | -0.014 | -0.013 | -0.014 | 0.198 | 0.106 | 0.114 | 0.059 |
|     | PD-DZ BIC | -0.021 | -0.007 | -0.013 | -0.018 | 0.202 | 0.108 | 0.117 | 0.067 |
|     | PD-DZ HQIC | -0.017 | -0.014 | -0.018 | -0.013 | 0.201 | 0.106 | 0.115 | 0.065 |
| 0.9 | XD      | -0.897 | -0.899 | -0.896 | -0.896 | 0.900 | 0.900 | 0.897 | 0.897 |
|     | AB      | -0.894 | -0.883 | -0.906 | -0.895 | 0.898 | 0.884 | 0.907 | 0.896 |
|     | BB      | -0.896 | -0.891 | -0.926 | -0.905 | 0.900 | 0.892 | 0.927 | 0.905 |
|     | DZ      | -0.096 | -0.066 | -0.076 | -0.053 | 0.210 | 0.124 | 0.150 | 0.083 |
|     | AC-DZ   | -0.086 | -0.055 | -0.051 | -0.034 | 0.164 | 0.098 | 0.087 | 0.050 |
|     | PD-DZ BIC | -0.079 | -0.043 | -0.034 | -0.021 | 0.167 | 0.091 | 0.078 | 0.041 |
|     | PD-DZ HQIC | -0.078 | -0.042 | -0.028 | -0.021 | 0.165 | 0.087 | 0.075 | 0.041 |
Table 3: Biases and RMSE for all estimators in data with $\varepsilon_{it} \sim N(0,1)$, $\eta_i \sim N(0,1)$, and 10% contamination by the patches of 3 additive outliers under different sample sizes.

| $T$ | $\alpha$ | n | RRMSC | Bias | RMSE |
|-----|----------|---|--------|------|------|
|     |          | 50 | 200    | 50   | 200  |
| 0.1 | XD       |    | 0.744  | 0.752| 0.606|
|     | AB       |    | 0.565  | 0.571| 0.556|
|     | BB       |    | 0.587  | 0.584| 0.485|
|     | DZ       |    | 0.201  | 0.195| 0.209|
|     | AC-DZ    |    | 0.257  | 0.263| 0.331|
|     | PD-DZ BIC|    | 0.183  | 0.170| 0.233|
| 0.5 | XD       |    | 0.349  | 0.352| 0.207|
|     | AB       |    | 0.160  | 0.170| 0.146|
|     | BB       |    | 0.182  | 0.189| 0.087|
|     | DZ       |    | 0.194  | 0.209| 0.212|
|     | AC-DZ    |    | 0.208  | 0.235| 0.255|
|     | PD-DZ BIC|    | 0.214  | 0.235| 0.227|
| 0.9 | XD       |    |-0.047 | -0.046|-0.193|
|     | AB       |    |-0.249 | -0.231|-0.258|
|     | BB       |    |-0.190 | -0.182|-0.302|
|     | DZ       |    | 0.055  | 0.072| 0.067|
|     | AC-DZ    |    | 0.038  | 0.068| 0.060|
|     | PD-DZ BIC|    | 0.036  | 0.029| 0.035|
|     | PD-DZ HQIC| | 0.028  | 0.001| -0.005|
Bond, 1998), and the X-differencing (XD) estimator (Han et al., 2014). The globally robust estimators are represented by the original DZ and AC-DZ estimators and by the proposed PD-DZ estimator. For the latter, we consider two different moment selection criteria $RRMSC$: BIC and HQIC introduced in Section 2.4.

Considering the clean data first, most estimators exhibit small RMSEs except of the AB estimator that is usually strongly negatively biased if $\alpha$ is close to 1. The BB estimator performs well under these circumstances as expected, but is outperformed by the XD estimation. Regarding the robust estimators, the results are closer to each other for $T = 6$ than for $T = 12$ since there are only three possible moment conditions (6) if $T = 6$. The DZ estimator based on the first moment condition only is lacking behind AC-DZ and PD-DZ when $\alpha$ is not close to zero and additional higher-order moment conditions thus improve estimation. The results for AC-DZ and PD-DZ are rather similar in most situations, with PD-DZ becoming relatively more precise as $n$ increases due to less noisy moment selection. Overall, the performance of PD-DZ is worse than that of the AB and BB estimators for $\alpha = 0.1$, matches them for $\alpha = 0.5$, and outperforms them for $\alpha = 0.9$.

Next, the two different data contaminations schemes are considered: independent additive outliers and the patches of additive outliers. Considering the independent additive outliers (see Table 2), which generally bias estimates toward zero, AB, BB, and XD are strongly biased in all cases as expected. In the case of robust estimators, the negative biases of DZ, AC-DZ, and PD-DZ are rather small and the proposed PD-DZ estimator generally performs better than DZ both in terms of the bias and RMSE, especially at larger sample sizes. Comparing AC-DZ and PD-DZ, the results are rather similar with PD-DZ being slightly better for $\alpha = 0.1$ and $\alpha = 0.9$ and vice versa. This is a positive result as the inclusion of higher-order differences with $s > 1$ in PD-DZ could lead to large biases due to independent additive outliers especially for $\alpha = 0.9$, see Figure 1.

On the other hand, the higher-order differences with $s > 1$ should provide benefits when the data are contaminated by the patches of additive outliers, see Table 3. This type of contamination leads again to substantially biased non-robust estimates by XD, AB, and BB. Regarding the robust estimates, patches of outliers cause larger biases of all methods, but AC-DZ is the most affected one (unreported experiments indicate that the bias of AC-DZ further increases as $T$ grows, while the bias stays constant or decreases for DZ and PD-DZ for higher $T$). Note that the bias decreases as $\alpha$ increases as the patches of outliers bias the DZ-types of estimators towards 1. The proposed PD-DZ exhibits a bit larger bias than DZ if the sample size is small and the moment selection is thus less reliable or if $\alpha = 0.1$ and the higher-order moment conditions, which are technically resistant to these outliers, have very little identification power. In the other cases, the RMSEs of PD-DZ are smaller, sometimes
substantially, than for the DZ and AC-DZ methods.

5 Concluding remarks

In this paper, we propose an extension of the median-based robust estimator for dynamic panel data model of Dhaene and Zhu (2009) by means of multiple pairwise differences. The newly proposed GMM estimation procedure that uses weights accounting both for the variance and outlier-related bias of the moment conditions is combined with the moment selection method. As a result, the estimator performs well in non-contaminated data as well as in data containing both independent outliers and patches of outliers.

A Appendix

The outlier contamination schemes $Z_{1,\xi,\zeta}$, $Z_{2,\xi,\zeta}$, and $Z_{3,\xi,\zeta}$ are generally described by the contamination fraction $\epsilon$ and the magnitude of outliers $\zeta$ (recall that only the point-mass distribution $G_{\zeta}$ is considered here). Therefore, we will denote the non-contaminated sample observations following model (1) by $y_{it}$ and the contaminated sample observations by $y_{it}^{\zeta,\epsilon}$. By definition of $Z_{1,\xi,\zeta}$, $Z_{2,\xi,\zeta}$, and $Z_{3,\xi,\zeta}$, the difference $w_{it} = y_{it}^{\zeta,\epsilon} - y_{it}$ can only equal $-\zeta$, 0, or $\zeta$.

In order to prove the theorems concerning the influence function of $\hat{\alpha}$, it is useful to derive first the asymptotic bias of $\hat{r}(r_j, \zeta, \epsilon)$ as an estimator of $r_j$. Similarly to Section 3.1, it is defined as

$$\text{bias}(\hat{r}(r_j, \zeta, \epsilon)) := \lim_{n \to \infty} \hat{r}(r_j, \zeta, \epsilon) - r_j,$$

(26)

where $\text{plim}$ denotes the probability limit operator. Let $b := b(r_j, \zeta, \epsilon)$ be a short-hand notation for (26). Then, $b$ solves the following equation

$$E \left[ \text{sgn} \left( \frac{\Delta^s y_{it}^{\zeta,\epsilon} - r_j \Delta^p y_{it-s}^{\zeta,\epsilon}}{\Delta^p y_{it-s}^{\zeta,\epsilon}} \right) \right] = b,$$

(27)

which can also be written as

$$\Pr \left( \frac{\Delta^s y_{it}^{\zeta,\epsilon} - r_j \Delta^p y_{it-s}^{\zeta,\epsilon}}{\Delta^p y_{it-s}^{\zeta,\epsilon}} \leq b \right) = \frac{1}{2}.$$  

(28)

Since $r_j$ is considered only for $j = (s,s,p) \in J_o$, where both $s$ and $p$ are odd, $r_j = -(1 - \alpha^s)/2$. This mapping of $\alpha$ to $r_j = -(1 - \alpha^s)/2$ has the same important properties for $s = 1$ and any odd $s > 1$: it maps interval $(-1,0)$ to $(-1,-1/2)$ and interval $(0,1)$ to $(-1/2,0)$, it is continuous, and it is strictly increasing on $(-1,1)$. One can thus follow the proofs in
Dhaene and Zhu (2009, Theorems 5 and 8) and apply them not only to the case of \( s = p = 1 \), but any odd \( s \) and \( p \) with only two adjustments: (i) the variables \( \Delta^s y_{it} - r_j \Delta^p y_{it-s} \) and \( \Delta^p y_{it-s} \) have to be standardized (Dhaene and Zhu, 2009, equation (17)) and their variances generally depend on the values of \( s \) and \( p \) and (ii) in the case of patches of outliers, the probability that a patch contaminates the ratio \( \Delta^s y_{it}/\Delta^p y_{it-s} \) needs to be generalized.

As for (i), note that, by Equation (2), the variables \( \Delta^s y_{it} - r_j \Delta^p y_{it-s} \) and \( \Delta^p y_{it-s} \) are uncorrelated, and by Assumption A.3, they are independent and normally distributed around zero. From Aquaro and Čížek (2014, Equation (24)), we also know that

\[
\left( \begin{array}{c}
\Delta^s y_{it} - r_j \Delta^p y_{it-s} \\
\Delta^p y_{it-s}
\end{array} \right) \sim N \left( 0, \frac{2\sigma^2}{1-\alpha^2} \begin{pmatrix}
1 - \alpha^s - r_j^2(1-\alpha^p) & 0 \\
0 & 1 - \alpha^p
\end{pmatrix}\right) \tag{29}
\]

(the diagonal structure of the covariance matrix can be also seen from Equation (2.2) that implies \( \text{cov}(\Delta^s y_{it}, \Delta^p y_{it-s}) = r_j \text{var}(\Delta^p y_{it-s}) \)).

### A.1 Independent additive outlier contamination \( Z_{\epsilon, \zeta} \)

Under independent additive outlier contamination \( Z_{\epsilon, \zeta} \), Equation (28) can be written as

\[
\Pr \left( \frac{\Delta^s y_{it} - r_j \Delta^p y_{it-s}}{\Delta^p y_{it-s}} \leq b \right) = \Pr \left( \frac{u_{itj} + \Delta^s w_{it} - r_j \Delta^p w_{it-s}}{\Delta^p y_{it-s} + \Delta^p w_{it-s}} \leq b \right) = \Pr \left[ f(w_{it}) \leq b \right] = \frac{1}{2}, \tag{30}
\]

where residual \( u_{itj} = \Delta^s y_{it} - r_j \Delta^p y_{it-s} \), \( w_{it} \in \{0, \zeta\} \), \( w_{it} = (w_{it}, w_{it-s}, w_{it-s-p})' \) is a random vector, and \( f(w_{it}) \) is a random scalar. Let \( \Omega_{w_{it}} \) be the set of the eight possible outcomes of \( w_{it} \), that is,

\[
\Omega_{w_{it}} := \left\{ \begin{pmatrix}
0 \\
0 \\
\zeta
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
\zeta
\end{pmatrix}, \cdots, \begin{pmatrix}
\zeta \\
\zeta \\
\zeta
\end{pmatrix} \right\}, \tag{31}
\]
where the number of elements is \( \#\Omega_{it} = 8 \). To simplify the notation, let us refer to (31) as \( \Omega_{itj} \), and denote each of its element as \( \omega_{itj} \), \( j = 1, \ldots, 8 \). Then it holds

\[
\Pr [ f(\omega_{it}) \leq b] = \Pr \left[ (f(\omega_{it}) \leq b) \cap \left( \bigcup_{j=1}^{8} \omega_{itj} \right) \right] = \sum_{j=1}^{8} \Pr \left[ (f(\omega_{it}) \leq b) \cap (\omega_{it} = \omega_{itj}) \right] \]

\[
= \sum_{j=1}^{8} \Pr [ (f(\omega_{it}) \leq b) ] \Pr (\omega_{it} = \omega_{itj}) = \sum_{j=1}^{8} \Pr [ f(\omega_{itj}) \leq b] \Pr (\omega_{it} = \omega_{itj}) . \tag{32}
\]

Note that \( \Pr (\omega_{it} = \omega_{itj}) = \Pr (\omega_{it} = \omega_{itj}') \) for some \( j \) and \( j' \) because the data contamination \( Z_{1,\zeta} \) is characterized by outliers occurring independently from each other. For instance, \( \Pr [(\zeta, 0, 0)'] = \Pr [(0, \zeta, 0)'] = \Pr [(0, 0, \zeta)'] = (1 - \epsilon)^2 \epsilon \). Moreover, \( f[(0, 0, 0)'] = f[(\zeta, \zeta, \zeta)'] \).

Therefore, Equation (30) can be decomposed as

\[
\Pr [ f(\omega_{it}) \leq b] = \left[ (1 - \epsilon)^3 + \epsilon^3 \right] A + (1 - \epsilon)^2 \epsilon B + (1 - \epsilon) \epsilon^2 C = \frac{1}{2}, \tag{33}
\]

where \( A, B, \) and \( C \) are defined for \( r_j, \zeta, \) and \( b \) as follows:

\[
A(r_j, b) := \Pr \left( \frac{u_{itj}}{\Delta^p y_{it-s}} \leq b \right),
\]

\[
B(r_j, \zeta, b) := \Pr \left( \frac{u_{itj} + \zeta}{\Delta^p y_{it-s}} \leq b \right) + \Pr \left( \frac{u_{itj} - \zeta(1 + r_j)}{\Delta^p y_{it-s} + \zeta} \leq b \right) + \Pr \left( \frac{u_{itj} + \zeta r_j}{\Delta^p y_{it-s} - \zeta} \leq b \right),
\]

\[
C(r_j, \zeta, b) := \Pr \left( \frac{u_{itj} - \zeta r_j}{\Delta^p y_{it-s} + \zeta} \leq b \right) + \Pr \left( \frac{u_{itj} + \zeta(1 + r_j)}{\Delta^p y_{it-s} - \zeta} \leq b \right) + \Pr \left( \frac{u_{itj} - \zeta}{\Delta^p y_{it-s}} \leq b \right). \tag{34}
\]

These probabilities are all of the form

\[
L(k, l, b) = \Pr \left( \frac{X + k'}{Y - l'} \leq b' \right) \tag{35}
\]

for given \( k, l, \) and \( b, \) and they can be conveniently standardized by using (29) as follows:

\[
L(k, l, b) = \Pr \left( \frac{X + k'}{Y - l'} \leq b' \right), \tag{36}
\]

26
where \( X \) and \( Y \) are independent \( N(0,1) \) variables and
\[
k' := \frac{k}{\sigma_u}, \quad l' := \frac{l}{\sigma_{\Delta p}}, \quad b' := \sigma^* b, \quad (37)
\]
and
\[
\sigma^* := \frac{\sigma_{\Delta p}}{\sigma_u} = \sqrt{\frac{1 - \alpha^p}{1 - \alpha^s - (1 - \alpha^s)^2(1 - \alpha^p)/4}}, \quad (38)
\]
where \( \sigma_u := \sqrt{\text{var}(u_{itj})} \) and \( \sigma_{\Delta p} := \sqrt{\text{var}(\Delta p_y)} \) can be found in (29). Finally, note that \( L(k, l, b) = L(-k, -l, b) \), hence \( B = C \) and (30) becomes
\[
A + \epsilon(1 - \epsilon)(B - 3A) = \frac{1}{2}. \quad (39)
\]

**Proof of Theorem 2.** As in Dhaene and Zhu (2009, proof of Theorem 2), it follows from the definition of influence function that
\[
\text{IF}^\epsilon(r_j; \hat{r}_j, \zeta) := \left. \frac{\partial \text{bias}(r_j; \hat{r}_j, \zeta)}{\partial \epsilon} \right|_{\epsilon=0} = \frac{3A(r_j, 0) - B(r_j, \zeta, 0)}{A'_b(r_j, 0)}, \quad (40)
\]
where the equality follows from the implicit function theorem applied to (39) and where
\[
A'_b(r_j, 0) := \left. \frac{\partial A(r_j, b)}{\partial b} \right|_{\epsilon=0}. \quad (41)
\]

As in Dhaene and Zhu (2009, Equation (18)),
\[
A(r_j, b) = \Pr \left( \frac{X}{Y} \leq \sigma^* b \right) = \frac{1}{2} + \frac{1}{\pi} \arctan \sigma^* b, \quad (42)
\]
where \( \sigma^* \) is defined in (38) and \( X, Y \sim N(0,1) \). Hence, \( A(r_j, 0) = 1/2 \) and
\[
A'_b(r_j, 0) = \frac{1}{\pi \sigma^*} = \frac{1}{\pi} \sqrt{\frac{1 - \alpha^p}{1 - \alpha^s - r_j^2(1 - \alpha^p)}}, \quad (43)
\]
(recall that \( r_j = (1 - \alpha^s)/2 \)). Next, Dhaene and Zhu (2009, Lemma 3) implies that, for \( X,Z \sim N(0,1) \) and constants \( c, c', c'' \), \( P\{(X + c)/Z \leq 0\} = 1/2 \) and \( P\{(X + c')/(Z - c) \leq 0\} + P\{(X + c'')/(Z - c) \leq 0\} = 1 + [\Phi(c') - \Phi(-c'')][\Phi(c) - \Phi(-c)] \). Hence, the definition of \( B(r_j, \zeta, b) \) and the standardization (36) imply
\[
B(r_j, \zeta, 0) = \frac{3}{2} + \left[ \Phi \left( \frac{\zeta (1 + r_j)}{\sigma_u} \right) - \Phi \left( -\frac{\zeta r_j}{\sigma_u} \right) \right] \times \left[ \Phi \left( \frac{\zeta}{\sigma_{\Delta p}} \right) - \Phi \left( -\frac{\zeta}{\sigma_{\Delta p}} \right) \right], \quad (44)
\]
Substituting for $\sigma_u := \sqrt{\text{var}(u_{itj})}$ and $\sigma_{\Delta p} := \sqrt{\text{var}(\Delta^p y_{it-s})}$ from (29) and $r_j = -(1 - \alpha^*)/2$ into (44) and for terms $A(r_j, 0)$, $B(r_j, \zeta, 0)$, and $A'_p(r_j, 0)$ in (40) completes the proof. \hfill \Box

### A.2 Patch additive outlier contamination $Z_{e, \zeta}^2$

As in Section A.1, it is useful to derive first the asymptotic bias of $\hat{r}_j$ under the outlier contamination $Z_{e, \zeta}^2$ as defined in (14). This is given by $b := b(r_j, \zeta, \epsilon, k)$ solving the equation

\[
\Pr \left( \frac{\Delta^s y_{it, \zeta}^\epsilon - r_j \Delta^p y_{it-s, \zeta}^\epsilon}{\Delta^p y_{it-s}} \leq b \right) = \Pr \left( \frac{u_{itj} + \Delta^s w_{it} - r_j \Delta^p w_{it-s}}{\Delta^p y_{it-s} + \Delta^p w_{it-s}} \leq b \right)
\]

(45)

where the notation is defined below. Note that the decomposition in the second equality is in Section A.1, in particular Equation (32). In this case, the only difference is that outliers no longer occur independently but in patches. The number of elements of $\Omega_{it}$ increases to $\#\Omega_{it} = 13$ as now, if we observe multiple outliers, we shall distinguish the event of the outliers belonging to the same patch from the event of these outliers belonging to different patches. For instance, $(0, \zeta, \zeta)'$ may be that result of one patch only, $(0, \zeta_1, \zeta_1)'$, or of two patches, $(0, \zeta_2, \zeta_1)'$, where the subscript of $\zeta$ indicates the patch. Recalling that $(1 - \epsilon)^k = \epsilon$,

\[
p_B := \Pr \left[ \binom{\zeta}{0} \cup \binom{0}{\zeta} \right] = \Pr \left( \binom{\zeta_1}{0} \right) + \Pr \left( \binom{0}{\zeta_1} \right) + \Pr \left( \binom{0}{\zeta_2} \right)
\]

(46)

\[
= (1 - \epsilon)^{k + \min(p, k)} \cdot \epsilon \cdot \min\{s, k\}
\]

\[
+ \sup \left\{ 0, s + k - \max\{s + p, k\} \right\} \cdot (1 - \epsilon)^k
\]

\[
+ \epsilon^2 \cdot \left( p + k - \max\{p, k\} \right) \cdot \min\{0, s + \min\{p, k\} - \max\{s, k\}\} \cdot (1 - \epsilon)^k,
\]

\[
p_C := \Pr \left[ \binom{\zeta}{0} \cup \binom{\zeta}{0} \right] = \Pr \left( \binom{\zeta_1}{0} \right) + \Pr \left( \binom{\zeta_1}{0} \right) + \Pr \left( \binom{\zeta_2}{0} \right)
\]

(47)

\[
= \epsilon \cdot \left( p + k - \max\{p, k\} \right) \cdot (1 - \epsilon)^{k + \min\{s, k\}}
\]

\[
+ (1 - \epsilon)^k \cdot \epsilon \cdot \min\{0, \min\{s + p, k\} - s\}
\]

\[
+ (1 - \epsilon)^k \cdot \epsilon^2 \cdot \min\{0, s + \min\{p, k\} - \max\{s, k\}\} \cdot \min\{s, k\},
\]

28
\[ p_D := \text{Pr} \begin{bmatrix} \zeta & 0 \\ 0 & \zeta \\ \zeta & 0 \end{bmatrix} = \text{Pr} \begin{bmatrix} \zeta_2 \\ 0 \\ \zeta_1 \end{bmatrix} + \text{Pr} \begin{bmatrix} 0 \\ \zeta_1 \\ 0 \end{bmatrix} \]

\[ = \varepsilon^2 \cdot (p + k - \max\{p, k\}) \cdot (1 - \varepsilon)^k \cdot \min\{s, k\} \]

\[ + (1 - \varepsilon)^{2k} \cdot \varepsilon \cdot \max\left\{0, s + \min\{p, k\} - \max\{s, k\}\right\}, \]

and \( p_A = 1 - p_B - p_C - p_D \). Next, the terms \( A, B, C, D \) are defined for \( r_j, \zeta, b \) as follows:

\[ A(r_j, b) := \text{Pr} \left( \frac{u_{itj}}{\Delta p_{yt-it}} \leq b \right), \]

\[ B(r_j, \zeta, b) := \text{Pr} \left( \frac{u_{itj} + \zeta}{\Delta p_{yt-it}} \leq b \right), \]

\[ C(r_j, \zeta, b) := \text{Pr} \left( \frac{u_{itj} + \zeta r_j}{\Delta p_{yt-it} - \zeta} \leq b \right), \]

\[ D(r_j, \zeta, b) := \text{Pr} \left( \frac{u_{itj} + \zeta (1 + r_j)}{\Delta p_{yt-it} - \zeta} \leq b \right), \]

where the symmetry \( L(k, l, b) = L(-k, -l, b) \) has been used, recall Equation (35).

**Proof of Theorem 3.** By the definition of influence function in (18),

\[ \text{IF}(\hat{r}_j; r_j, \zeta) = \frac{\partial b(r_j, \zeta, \epsilon, k)}{\partial \epsilon} \bigg|_{\epsilon=0}, \]

where \( b \) denotes the bias of \( \hat{r}_j \). Given that \( (1 - \varepsilon)^k = 1 - \epsilon \), it holds

\[ \frac{\partial b(r_j, \zeta, \epsilon, k)}{\partial \epsilon} = \frac{\partial b(r_j, \zeta, \epsilon, k)}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial \epsilon} = \frac{\partial b(r_j, \zeta, \epsilon, k)}{\partial \varepsilon} \cdot \frac{1}{k(1 - \varepsilon)^{k-1}}. \]

The derivative in (51) can obtained by applying the implicit function theorem to (45),

\[ \frac{\partial b(r_j, \zeta, \epsilon, k)}{\partial \varepsilon} \bigg|_{\epsilon=0} = -\sum_{j \in \{B, C, D\}} p_j'(0) j(r_j, \zeta, 0) + p_A'(0) A(r_j, 0) A_b'(r_j, 0), \]

where \( A_b'(r_j, 0) \) is the same as in (41) and where \( p_j', j \in \{B, C, D\} \), denote the derivatives of
\[ p_j \text{ in Equations (46)-(48) with respect to } \tilde{\epsilon}, \text{ that is,} \]
\[ p'_B(0) := \left. \frac{\partial p_B(\tilde{\epsilon}; s, p, k)}{\partial \tilde{\epsilon}} \right|_{\tilde{\epsilon}=0} = \min\{s, k\} + \max\left\{0, s + k - \max\{s + p, k\}\right\}, \]
\[ p'_C(0) = p + k - \max\{p, k\} + \max\left\{0, \min\{s + p, k\} - s\right\}, \]  
\[ p'_D(0) = \max\left\{0, s + \min\{p, k\} - \max\{s, k\}\right\}, \]
\[ \text{and} \]
\[ p'_A(0) = -[p'_B(0) + p'_C(0) + p'_D(0)]. \]  
\[ \text{As in Section A.1, } A(r_j; 0) = 1/2. \]

Further, it follows from Dhaene and Zhu (2009, Lemma 3) that, for \( X, Z \sim N(0, 1) \) and constants \( c, c', P\{ (X + c')/(Z - c) \leq 0 \} = \Phi(c')|c| + \Phi(c'|c). \)

Hence, the definition (49) and the standardization (36)-(38) imply
\[ B(r_j; \zeta, 0) = \frac{1}{2}, \]
\[ C(r_j; \zeta, 0) = \Phi\left(\frac{-r_j \zeta}{\sigma_u}\right) \Phi\left(\frac{-\zeta}{\sigma_{\Delta p}}\right) + \Phi\left(\frac{r_j \zeta}{\sigma_u}\right) \Phi\left(\frac{\zeta}{\sigma_{\Delta p}}\right), \]
\[ D(r_j; \zeta, 0) = \Phi\left(\frac{- (1 + r_j) \zeta}{\sigma_u}\right) \Phi\left(\frac{-\zeta}{\sigma_{\Delta p}}\right) + \Phi\left(\frac{(1 + r_j) \zeta}{\sigma_u}\right) \Phi\left(\frac{\zeta}{\sigma_{\Delta p}}\right), \]

where \( \sigma_u := \sqrt{\text{var}(u_{it+j})} \) and \( \sigma_{\Delta p} := \sqrt{\text{var}(\Delta p_{y_{it-s}})} \) are given in (29) and \( r_j = -(1 - \alpha^s)/2. \)

Substituting (51)-(59) in (50) completes the proof.

A.3 Patch additive outlier contamination \( Z^3_{\epsilon, \zeta} \)

This case is a generalization of the \( Z^2_{\epsilon, \zeta} \) contamination. The proof structure is very similar to the one in Sections A.1 and A.2, although the algebra is a bit more lengthy. As before, it is useful to derive first the bias of \( \hat{r}_j \) under the outlier contamination \( Z^3_{\epsilon, \zeta} \) as defined in (15).

This is given by \( b := b(r_j, \zeta, \epsilon, k) \) solving the equation
\[ \text{Pr}\left( \frac{\Delta^s y_{it} \zeta - r_j \Delta^p y_{it} \zeta}{\Delta p y_{it-s}} \leq b \right) = \text{Pr}\left( \frac{u_{itj} + \Delta^p w_{it} - r_j \Delta^p w_{it-s}}{\Delta p y_{it-s} + \Delta^p w_{it-s}} \leq b \right) \]
\[ = p_A A + p_B B + p_C C + p_D D + p_E E + p_F F + p_G G + p_H H + p_I I + p_J J = \frac{1}{2}, \]

where the notation is explained below. Note that the set \( \Omega_{it} \) in (31) is different than it was for previous types of contaminations as now outliers can be either negative or positive multiple
of \( \zeta \). Also recall that \((1 - \varepsilon)^k = \epsilon\).

Table 4: Configurations of patch outliers and their probabilities.

| Configuration | Formula |
|---------------|---------|
| \((|\zeta|, \ldots, 0, \ldots, 0)')\) | \((1 - \varepsilon)^{k+\min\{p, k\}} \cdot \varepsilon \cdot \min\{s, k\}\) |
| \((0, \ldots, 0, \ldots, |\zeta|)')\) | \(\varepsilon \cdot \left(p + k - \max\{p, k\}\right) \cdot (1 - \varepsilon)^{k+\min\{s, k\}}\) |
| \((0, \ldots, |\zeta|, \ldots, 0)')\) | \((1 - \varepsilon)^{2k} \cdot \varepsilon \cdot \max\left\{0, s + \min\{p, k\} - \max\{s, k\}\right\}\) |
| \((|\zeta|, \ldots, |\zeta|, \ldots, 0)')\) | \((1 - \varepsilon)^k \cdot \varepsilon \cdot \max\left\{0, \min\{s + p, k\} - s\right\}\) |
| \((|\zeta|, \ldots, |\zeta|, \ldots, 0)')\) | \((1 - \varepsilon)^k \cdot \varepsilon^2 \cdot \max\left\{0, s + \min\{p, k\} - \max\{s, k\}\right\} \cdot \min\{s, k\}\) |
| \((0, \ldots, |\zeta|, \ldots, |\zeta|)')\) | \(\varepsilon \cdot \max\left\{0, s + k - \max\{s + p, k\}\right\} \cdot (1 - \varepsilon)^k\) |
| \((0, \ldots, |\zeta|, \ldots, |\zeta|)')\) | \(\varepsilon^2 \cdot \left(p + k - \max\{p, k\}\right) \cdot \max\left\{0, s + \min\{p, k\} - \max\{s, k\}\right\} \cdot (1 - \varepsilon)^k\) |
| \((|\zeta|, \ldots, 0, \ldots, |\zeta|)')\) | \(\varepsilon^2 \cdot \left(p + k - \max\{p, k\}\right) \cdot (1 - \varepsilon)^k \cdot \min\{s, k\}\) |
| \((|\zeta|, \ldots, |\zeta|, \ldots, |\zeta|)')\) | \(\varepsilon \cdot \max\{0, k - s - p\}\) |
| \((|\zeta|, \ldots, |\zeta|, \ldots, |\zeta|)')\) | \(\varepsilon^2 \cdot \max\{0, k - p\} \cdot \min\{s, k\}\) |
| \((|\zeta|, \ldots, |\zeta|, \ldots, |\zeta|)')\) | \(\varepsilon^2 \cdot k \cdot \max\left\{0, \min\{s + p, k\} - s\right\}\) |
| \((|\zeta|, \ldots, |\zeta|, \ldots, |\zeta|)')\) | \(\varepsilon^3 \cdot k \cdot \max\left\{0, s + \min\{p, k\} - \max\{s, k\}\right\} \cdot \min\{s, k\}\) |

By using the results in Table 4, we have that

\[
p_B := \Pr \left[ \left( \begin{array}{c} \zeta \\ 0 \\ 0 \end{array} \right) \cup \left( \begin{array}{c} -\zeta \\ 0 \\ 0 \end{array} \right) \cup \left( \begin{array}{c} 0 \\ \zeta \\ 0 \end{array} \right) \cup \left( \begin{array}{c} 0 \\ -\zeta \\ 0 \end{array} \right) \right]
\]

\[
= \frac{1}{2} \Pr \left( \begin{array}{c} \zeta \\ 0 \\ 0 \end{array} \right) + \frac{1}{2} \Pr \left( \begin{array}{c} -\zeta \\ 0 \\ 0 \end{array} \right) + \frac{1}{4} \Pr \left( \begin{array}{c} 0 \\ \zeta \\ 0 \end{array} \right) + \frac{1}{4} \Pr \left( \begin{array}{c} 0 \\ -\zeta \\ 0 \end{array} \right)
\]

(61)

\[
= \Pr \left( \begin{array}{c} |\zeta| \\ 0 \\ 0 \end{array} \right) + \frac{1}{2} \Pr \left( \begin{array}{c} 0 \\ |\zeta| \\ |\zeta| \end{array} \right),
\]

31
\[ p_C := \Pr \left[ \begin{pmatrix} 0 \\ \zeta \\ -\zeta \\ 0 \end{pmatrix} \cup \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cup \begin{pmatrix} \zeta \\ \zeta \\ 0 \\ 0 \end{pmatrix} \cup \begin{pmatrix} -\zeta \\ -\zeta \\ 0 \\ 0 \end{pmatrix} \right] \]

\[ = \Pr \begin{pmatrix} 0 \\ 0 \\ -\zeta \\ 0 \end{pmatrix}_{\zeta_1} + \Pr \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\zeta_1} + \Pr \begin{pmatrix} \zeta_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\zeta_1} + \Pr \begin{pmatrix} -\zeta_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\zeta_1} \] (62)

\[ p_D := \Pr \left[ \begin{pmatrix} \zeta \\ -\zeta \\ 0 \\ 0 \end{pmatrix} \cup \begin{pmatrix} -\zeta \\ -\zeta \\ 0 \\ 0 \end{pmatrix} \cup \begin{pmatrix} 0 \\ 0 \\ \zeta \\ -\zeta \end{pmatrix} \cup \begin{pmatrix} 0 \\ 0 \\ -\zeta \\ -\zeta \end{pmatrix} \right] \]

\[ = \Pr \begin{pmatrix} \zeta_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\zeta_1} + \Pr \begin{pmatrix} -\zeta_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\zeta_1} + \Pr \begin{pmatrix} 0 \\ 0 \\ \zeta_1 \\ -\zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} 0 \\ 0 \\ -\zeta_1 \\ -\zeta_1 \end{pmatrix} \] (63)

\[ p_E := \Pr \left[ \begin{pmatrix} 0 \\ -\zeta \\ \zeta \\ -\zeta \end{pmatrix} \right] \]

\[ = \Pr \begin{pmatrix} 0 \\ -\zeta_1 \\ 0 \\ -\zeta_1 \end{pmatrix}_{\zeta_1} + \Pr \begin{pmatrix} 0 \\ -\zeta_2 \\ 0 \\ -\zeta_2 \end{pmatrix}_{\zeta_1} + \Pr \begin{pmatrix} 0 \\ 0 \\ \zeta_1 \\ -\zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} 0 \\ 0 \\ -\zeta_2 \\ -\zeta_2 \end{pmatrix}_{\zeta_1} \] (64)

\[ p_F := \Pr \left[ \begin{pmatrix} -\zeta \\ 0 \\ \zeta \\ -\zeta \end{pmatrix} \right] = \Pr \begin{pmatrix} -\zeta_2 \\ 0 \\ 0 \\ \zeta_1 \end{pmatrix}_{\zeta_1} + \Pr \begin{pmatrix} \zeta_2 \\ 0 \\ 0 \\ -\zeta_1 \end{pmatrix}_{\zeta_1} \] (65)

\[ = \frac{1}{2} \Pr \begin{pmatrix} 0 \\ 0 \\ \zeta_2 \\ 0 \\ -\zeta_1 \end{pmatrix}_{\zeta_1} \]
\[ p_G := \Pr \left[ \begin{pmatrix} \zeta \\ -\zeta \\ 0 \end{pmatrix} \cup \begin{pmatrix} -\zeta \\ \zeta \\ 0 \end{pmatrix} \right] \]

\[ = \Pr \begin{pmatrix} \zeta_1 \\ -\zeta_1 \\ 0 \end{pmatrix} + \Pr \begin{pmatrix} \zeta_2 \\ -\zeta_1 \\ 0 \end{pmatrix} + \Pr \begin{pmatrix} -\zeta_1 \\ \zeta_1 \\ 0 \end{pmatrix} + \Pr \begin{pmatrix} -\zeta_2 \\ \zeta_1 \\ 0 \end{pmatrix} \]

\[ = \Pr \begin{pmatrix} |\zeta_1| \\ |\zeta_1| \\ 0 \end{pmatrix} + \frac{1}{2} \Pr \begin{pmatrix} |\zeta_2| \\ |\zeta_1| \\ 0 \end{pmatrix}, \tag{66} \]

\[ p_H := \Pr \left[ \begin{pmatrix} -\zeta \\ \zeta \\ \zeta \end{pmatrix} \cup \begin{pmatrix} -\zeta \\ \zeta \\ -\zeta \end{pmatrix} \right] \]

\[ = \Pr \begin{pmatrix} -\zeta_2 \\ -\zeta_1 \\ \zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} -\zeta_3 \\ -\zeta_2 \\ \zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} \zeta_2 \\ \zeta_1 \\ -\zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} \zeta_3 \\ \zeta_2 \\ -\zeta_1 \end{pmatrix} \]

\[ = \frac{1}{2} \Pr \begin{pmatrix} |\zeta_2| \\ |\zeta_1| \\ |\zeta_1| \end{pmatrix} + \frac{1}{4} \Pr \begin{pmatrix} |\zeta_3| \\ |\zeta_2| \\ |\zeta_1| \end{pmatrix}, \tag{67} \]

\[ p_I := \Pr \left[ \begin{pmatrix} \zeta \\ -\zeta \\ \zeta \end{pmatrix} \cup \begin{pmatrix} -\zeta \\ \zeta \\ -\zeta \end{pmatrix} \right] \]

\[ = \Pr \begin{pmatrix} \zeta_1 \\ -\zeta_1 \\ \zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} \zeta_2 \\ -\zeta_1 \\ \zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} \zeta_2 \\ -\zeta_2 \\ \zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} \zeta_3 \\ -\zeta_2 \\ \zeta_1 \end{pmatrix} \]

\[ + \Pr \begin{pmatrix} \zeta_1 \\ -\zeta_1 \\ \zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} \zeta_2 \\ -\zeta_1 \\ \zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} \zeta_2 \\ -\zeta_2 \\ \zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} \zeta_3 \\ -\zeta_2 \\ \zeta_1 \end{pmatrix} \]

\[ = \Pr \begin{pmatrix} |\zeta_1| \\ |\zeta_1| \\ |\zeta_1| \end{pmatrix} + \frac{1}{2} \Pr \begin{pmatrix} |\zeta_2| \\ |\zeta_1| \\ |\zeta_1| \end{pmatrix} + \frac{1}{2} \Pr \begin{pmatrix} |\zeta_2| \\ |\zeta_1| \\ |\zeta_1| \end{pmatrix} + \frac{1}{4} \Pr \begin{pmatrix} |\zeta_3| \\ |\zeta_2| \\ |\zeta_1| \end{pmatrix}, \tag{68} \]
\[
    p_J := \Pr \left[ \begin{pmatrix} \zeta \\ -\zeta \\ -\zeta \end{pmatrix} \cup \begin{pmatrix} -\zeta \\ \zeta \\ \zeta \end{pmatrix} \right]
    = \Pr \begin{pmatrix} \zeta_2 \\ -\zeta_2 \\ -\zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} \zeta_3 \\ -\zeta_2 \\ \zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} -\zeta_2 \\ \zeta_2 \\ \zeta_1 \end{pmatrix} + \Pr \begin{pmatrix} -\zeta_3 \\ \zeta_2 \\ -\zeta_1 \end{pmatrix}
    = \frac{1}{2} \Pr \begin{pmatrix} |\zeta|_2 \\ |\zeta|_2 \\ |\zeta|_1 \end{pmatrix} + \frac{1}{4} \Pr \begin{pmatrix} |\zeta|_3 \\ |\zeta|_2 \\ |\zeta|_1 \end{pmatrix},
\]

and
\[
    p_A = 1 - \sum_{j \in I \setminus \{A\}} p_j,
\]

where \( I := \{A, B, C, D, E, F, G, H, I, J\} \). Moreover,

\[
    A(r_j, b) := \Pr \left( \frac{u_{itj}}{\Delta p_{y_{it-s}}} \leq b \right),
    B(r_j, \zeta, b) := \Pr \left( \frac{u_{itj} + \zeta}{\Delta p_{y_{it-s}}} \leq b \right),
    C(r_j, \zeta, b) := \Pr \left( \frac{u_{itj} + \zeta r_j}{\Delta p_{y_{it-s}} - \zeta} \leq b \right),
    D(r_j, \zeta, b) := \Pr \left( \frac{u_{itj} + \zeta (1 + r_j)}{\Delta p_{y_{it-s}} - \zeta} \leq b \right),
    E(r_j, \zeta, b) := \Pr \left( \frac{u_{itj} + \zeta (1 + 2r_j)}{\Delta p_{y_{it-s}} - 2\zeta} \leq b \right),
    F(r_j, \zeta, b) := \Pr \left( \frac{u_{itj} + \zeta (r_j - 1)}{\Delta p_{y_{it-s}} - \zeta} \leq b \right),
    G(r_j, \zeta, b) := \Pr \left( \frac{u_{itj} + \zeta (2 + r_j)}{\Delta p_{y_{it-s}} - \zeta} \leq b \right),
    H(r_j, \zeta, b) := \Pr \left( \frac{u_{itj} + 2\zeta r_j}{\Delta p_{y_{it-s}} - 2\zeta} \leq b \right),
    I(r_j, \zeta, b) := \Pr \left( \frac{u_{itj} + 2\zeta (1 + r_j)}{\Delta p_{y_{it-s}} - 2\zeta} \leq b \right),
    J(r_j, \zeta, b) := \Pr \left( \frac{u_{itj} + 2\zeta}{\Delta p_{y_{it-s}}} \leq b \right),
\]

where the symmetry \( L(k, l, b) = L(-k, -l, b) \) has been used, recall Equation (35).
Proof of Theorem 4. Denote

$$p'_j(0) := \frac{\partial p_j(\hat{\epsilon}; s, p, k)}{\partial \hat{\epsilon}} \bigg|_{\hat{\epsilon} = 0}, \quad j \in \mathcal{I} := \{A, B, C, D, E, F, G, H, I, J\},$$

where $p_j(\cdot)$, $j \in \mathcal{I}$, are defined in (61)–(70). Given that $(1 - \hat{\epsilon})^k = 1 - \epsilon$, it holds

$$\text{IF}(\hat{r}_j; r_j, \zeta) = \frac{\partial \text{bias}(\hat{r}_j; r_j, \zeta)}{\partial \epsilon} = \frac{\partial b(r_j, \zeta, \epsilon, k)}{\partial \epsilon} = \frac{\partial b(r_j, \zeta, \epsilon, k)}{\partial \epsilon} \frac{1}{k(1 - \epsilon)^{k-1}}.$$  

Differentiating (60) with respect to $\epsilon$ and evaluating it at $\epsilon = 0$ yields

$$\frac{\partial b(r_j, \zeta, \epsilon, k)}{\partial \epsilon} \bigg|_{\epsilon = 0} = -\frac{\sum_{j \in \mathcal{I} \setminus \{A\}} p'_j(0) j(r_j, \zeta, 0) - A(r, 0) \sum_{j \in \mathcal{I} \setminus \{A\}} p'_j(0)}{A'_b(r_j, 0)},$$

where $A'_b(r_j, 0)$ is defined in (41) and where (see results in Table 4)

$$p'_B(0) = \min\{s, k\},$$
$$p'_C(0) = p + k - \max\{p, k\},$$
$$p'_D(0) = \max\{0, s + \min\{p, k\} - \max\{s, k\}\},$$
$$p'_E(0) = \max\{0, s + k - \max\{s + p, k\}\},$$
$$p'_F(0) = 0,$$
$$p'_G(0) = \min\{0, \min\{s + p, k\} - s\},$$
$$p'_H(0) = 0,$$
$$p'_I(0) = \max\{0, k - s - p\},$$
$$p'_J(0) = 0.$$

As in Section A.1, $A(r_j; 0) = 1/2$. Further, it follows from Dhaene and Zhu (2009, Lemma 3) that, for $X, Z \sim \mathcal{N}(0, 1)$ and constants $c, c'$, $P\{(X + c')/(Z - c) \leq 0\} = \Phi(c')\Phi(-c) + \Phi(c')\Phi(c).$
Hence, the definition (71) and the standardization (36)–(38) imply

\[ B(r_j; \zeta, 0) = \frac{1}{2}, \quad (83) \]

\[ C(r_j; \zeta, 0) = \Phi \left( -\frac{r_j \zeta}{\sigma_u} \right) \Phi \left( -\frac{\zeta}{\sigma_{\Delta^p}} \right) + \Phi \left( \frac{r_j \zeta}{\sigma_u} \right) \Phi \left( \frac{\zeta}{\sigma_{\Delta^p}} \right), \quad (84) \]

\[ D(r_j; \zeta, 0) = \Phi \left( -\frac{(1 + r_j) \zeta}{\sigma_u} \right) \Phi \left( -\frac{\zeta}{\sigma_{\Delta^p}} \right) + \Phi \left( \frac{(1 + r_j) \zeta}{\sigma_u} \right) \Phi \left( \frac{\zeta}{\sigma_{\Delta^p}} \right), \quad (85) \]

\[ E(r_j; \zeta, 0) = \Phi \left( -\frac{(1 + 2 r_j) \zeta}{\sigma_u} \right) \Phi \left( -\frac{2 \zeta}{\sigma_{\Delta^p}} \right) + \Phi \left( \frac{(1 + 2 r_j) \zeta}{\sigma_u} \right) \Phi \left( \frac{2 \zeta}{\sigma_{\Delta^p}} \right), \quad (86) \]

\[ G(r_j; \zeta, 0) = \Phi \left( -\frac{(2 + r_j) \zeta}{\sigma_u} \right) \Phi \left( -\frac{\zeta}{\sigma_{\Delta^p}} \right) + \Phi \left( \frac{(2 + r_j) \zeta}{\sigma_u} \right) \Phi \left( \frac{\zeta}{\sigma_{\Delta^p}} \right), \quad (87) \]

\[ I(r_j; \zeta, 0) = \Phi \left( -\frac{2(1 + r_j) \zeta}{\sigma_u} \right) \Phi \left( -\frac{2 \zeta}{\sigma_{\Delta^p}} \right) + \Phi \left( \frac{2(1 + r_j) \zeta}{\sigma_u} \right) \Phi \left( \frac{2 \zeta}{\sigma_{\Delta^p}} \right), \quad (88) \]

where \( \sigma_u := \sqrt{\text{var}(u_{itj})} \) and \( \sigma_{\Delta^p} := \sqrt{\text{var}(\Delta^p y_{it-s})} \) are given in (29) and \( r_j = -(1 - \alpha^s)/2 \). Substituting (73)–(88) in (72) completes the proof.

A.4 General results

Proof of Theorem 5. Given a non-stochastic weighting matrix \( A \), the proof follows directly from Equation (10). The estimator \( \hat{\alpha} \) is defined by the solution of the sample analogs of equations (6), which are deterministic functions of \( \hat{r}_j \). Thus the influence function of \( \hat{\alpha} \) is fully determined by the influence functions of each \( \hat{r}_j \) being an element of \( g(\alpha) \):

\[ \text{IF}(\hat{\alpha}; \alpha, \zeta) = -(d' A d)^{-1} d' A \psi, \quad (89) \]

where \( \psi := \{ \text{IF}(\hat{r}_j; r_j, \zeta) \}_{j \in J_o} \) is a \#\( J_o \) \times 1 vector whose elements \( \text{IF}(\hat{r}_j; r_j, \zeta) \), \( j \in J_o \), are derived for each considered data contamination \( Z_{1,\zeta}^1 \), \( Z_{2,\zeta}^2 \), and \( Z_{3,\zeta}^3 \) in Theorem 2, 3, and 4, respectively.

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37
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