On the Shannon capacity and queueing stability of random access multicast

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Abstract

We study and compare the Shannon capacity region and the stable throughput region for a random access system in which source nodes multicast their messages to multiple destination nodes. Under an erasure channel model which accounts for interference and allows for multipacket reception, we first characterize the Shannon capacity region. We then consider a queueing-theoretic formulation and characterize the stable throughput region for two different transmission policies: a retransmission policy and random linear coding. Our results indicate that for large blocklengths, the random linear coding policy provides a higher stable throughput than the retransmission policy. Furthermore, our results provide an example of a transmission policy for which the Shannon capacity region strictly outer bounds the stable throughput region, which contradicts an unproven conjecture that the Shannon capacity and stable throughput coincide for random access systems.

Index Terms

wireless multicast, random access, ALOHA, queueing, stability, throughput, capacity, retransmissions, random linear coding

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I. INTRODUCTION

A fundamental question of communication theory is: at what rate can information be transmitted reliably over a noisy channel? There is more than one way to go about answering this question. For instance, consider an erasure channel where the parameter $\epsilon$ denotes the probability with which a transmission on the channel is lost; with probability $1-\epsilon$ the transmission is received without error. The traditional approach for describing the rate of reliable communication for the erasure channel is to cite its Shannon capacity, which is $1-\epsilon$ bits per channel use for a channel with binary inputs and outputs. If feedback is available to notify the sender when a channel input is erased, then the capacity can be achieved by retransmitting lost inputs [1].

Alternatively, the rate of reliable communication can be described by the maximum stable throughput. In this setting, we view the channel input as a packet of fixed length that arrives at a random time to a sender and is stored in an infinite-capacity buffer while awaiting its turn to be sent over the channel. The packets in the buffer form a queue that is emptied in a specified order, traditionally in first-in-first-out (FIFO) order. A transmission protocol, for instance an Automatic Repeat Request (ARQ) protocol, provides a form of redundancy to ensure that the packet is received correctly. The maximum stable throughput is the highest rate (in packets per channel use) at which packets arrive to the sender while ensuring that the queue remains finite. In the case of the erasure channel, if we assume that packets arrive to the sender according to a Bernoulli process, feedback is available to notify the sender of lost packets, and the transmission protocol consists of retransmitting lost packets, then the buffer at the sender forms a discrete-time $M/M/1$ queue with maximum departure rate $1-\epsilon$. The maximum stable throughput is $1-\epsilon$, which is identical to the Shannon capacity.

For multiuser systems, the relation between the Shannon capacity and the maximum stable throughput for communication over an erasure channel has been explored in the context of random multiple access, where user $n$ transmits with probability $p_n$ at each transmission opportunity. This form of random channel access is a variation on Abramson’s ALOHA protocol [2] and is
particularly attractive for use in mobile ad-hoc networks because it is robust to variations in the network topology and can be implemented without coordination among the transmitting nodes.

The finite-user, buffered random access problem was first formulated by Tsybakov and Mikhailov [3], who provided a sufficient condition for stability and thus a lower bound on the maximum stable throughput. The problem they considered was a system in which finitely many source nodes with infinite-capacity queues randomly access a shared channel to send messages to a central station. Feedback was used to notify the source nodes of failed transmissions and a retransmission scheme was used to ensure eventual successful reception at the central station. The users were assumed to access a collision channel, in which transmission by more than one source results in the loss of all transmitted packets with probability 1. This collision channel model is equivalent to an erasure channel, where for user $n$, $1 - \epsilon = p_n \prod_{j \neq n} (1 - p_j)$. Further progress on this problem was made in [4], in which stochastic dominance arguments were explicitly introduced to find the stable throughput region for a system with 2 source nodes, and in [5], wherein a stable throughput region based on the joint queue statistics was found for 3 source nodes. An exact stability result for arbitrarily (but finitely) many users has not been found, but bounds have been obtained in [5] and [6]. Recently, the authors of [7] improved upon the collision channel model used in all previous works and studied a channel with multipacket reception (MPR) capability. They showed that the stable throughput region transfers from a non-convex region under the collision channel model to a convex region bounded by straight lines for a channel with MPR.

The Shannon capacity region of a random access system was considered in [8] and [9], which both obtained the capacity region for finitely many source nodes transmitting to a central station under the collision channel model. That capacity result can be viewed as the capacity of an asynchronous multiple access channel, which was obtained in [10] and [11]. The more recent contribution of [12] shows explicitly how the random access capacity region in [8], [9] is obtained from the results in [10], [11], in addition to analyzing the capacity for a channel in which packets involved in a collision can be recovered.

It was noted by Massey and Mathys [8] and Rao and Ephremides [4] that the stable throughput and Shannon capacity regions coincide for the special cases of two source nodes and infinitely-many source nodes. As with the point-to-point erasure channel, this result is surprising in that it suggests that the bursty nature of arriving packets, which is captured in the stability problem but not in the capacity problem, is immaterial in determining the limits on the rate of reliable
communication. It has been conjectured that the stable throughput region for finitely-many source nodes transmitting to a central station (which is an unsolved problem) coincides with the corresponding capacity region. This conjecture was explored in [13], in which it was shown to hold in a special case involving correlated arrivals of packets at the source nodes. Recently, further progress was made in [14] towards showing that the stable throughput and Shannon capacity regions coincide for transmission over a channel with MPR. However, a complete proof has still not been found.

In this work we explore the relation between the stable throughput and Shannon capacity for a random access system in which source nodes \textit{multicast} their messages to multiple receivers. Specifically, we consider a system with two source nodes and two destination nodes. The source nodes randomly access the channel to transmit to the two destination nodes. We first characterize the Shannon capacity region of the system, which is similar to Ahlswede’s result [15] on the capacity for two senders and two receivers. We then move to the queueing stability problem and characterize the stable throughput region for our random access multicast system with two different transmission policies. The first is a retransmission policy, which is the policy used in previous works on queueing stability for a (single) central station. Next, we study random access multicast with random linear coding, which is inspired by the recent development of network coding [16], [17] and fountain coding [18], [19]. Our results show that for multicast transmission, the maximum stable throughput under the retransmission policy does not reach the Shannon capacity, however, the random linear coding scheme provides a maximum stable throughput that asymptotically approaches the Shannon capacity.

II. System model

The system we consider throughout this work is shown in Fig. 1. Two source nodes, indexed by $n$, each generate messages to be multicast to two destination nodes, indexed by $m$, over a shared channel. The data generated at source $n = 1$ is assumed independent of the data generated at source $n = 2$. Time is slotted; one time slot corresponds to the amount of time needed to transmit a single packet over the shared channel. A packet is a fixed-length vector of bits. In each time slot, if source $n$ has a packet to transmit, then we refer to the source as being \textit{backlogged}; otherwise the source is \textit{empty}. A backlogged source transmits in a slot with probability $p_n$. We refer to $p_n$ as the transmission probability; it encapsulates random access to the channel. We
Fig. 1. The multicast scenario we consider in this work. Two source nodes randomly access the channel to multicast to two destination nodes. When both sources transmit, which happens in a slot with probability \( p_1 p_2 \), the reception probabilities are as shown above.

Assume the value of \( p_n \) to be fixed in time. (I.e., we do not assume retransmission control, in which \( p_n \) is varied over time according to the history of successful transmissions.)

The channel model we consider is similar to the model used in [7]. A transmitted packet is received without error with a certain probability. Otherwise, the packet is lost and cannot be recovered. We assume that the channels between different source-destination pairs are independent. We introduce the following reception probabilities for sources \( n = 1, 2 \) and destinations \( m = 1, 2 \).

\[
q_{m|n}^{(m)} = \Pr\{\text{packet from } n \text{ is received at } m | \text{ only } n \text{ transmits}\} \quad (1)
\]

\[
q_{m|1,2}^{(m)} = \Pr\{\text{packet from } n \text{ is received at } m | \text{ both sources transmit}\} \quad (2)
\]

We assume throughout that interference cannot increase the reception probability on the channel, i.e., \( q_{m|n}^{(m)} > q_{m|1,2}^{(m)} \). The reception probabilities inherently account for interference and also allow for multipacket reception (MPR). Note that these probabilities can capture the effects of fading on the wireless channel by setting them equal to the probability that a fading signal, represented by a random variable, exceeds a certain signal to interference plus noise (SINR) threshold. The collision channel model used in a number of previous works is given by \( q_{m|n}^{(m)} = 1, q_{m|1,2}^{(m)} = 0 \).

### III. Shannon capacity region

We first analyze the system under consideration in an information-theoretic framework to determine the Shannon capacity region. Thus we assume in this section that the sources are always backlogged. Coding is performed on the data at the source nodes, and we view each
transmitted packet as a codeword symbol, where a collection of \( N \) codeword symbols constitutes a codeword. In order to allow for random access of the channel, we assume that while codeword symbols are synchronized (equivalently, time is slotted), the codewords for the two sources do not necessarily begin and end in the same time slots. We analyze this system first for a general discrete memoryless channel (DMC) and then apply our random access channel model to obtain the capacity region.

A. Discrete memoryless channel

The discrete memoryless channel we consider consists of discrete alphabets \( X_1, X_2, Y_1, \) and \( Y_2 \) and transition probability matrix \( W(y_1, y_2|x_1, x_2) \). The channel can be decomposed into two multiple access channels, each corresponding to a destination node and defined as follows.

\[
W_1(y_1|x_1, x_2) = \sum_{y_2 \in Y_2} W(y_1, y_2|x_1, x_2) \\
W_2(y_2|x_1, x_2) = \sum_{y_1 \in Y_1} W(y_1, y_2|x_1, x_2)
\]

We assume that there is no feedback available on the channel, which differs from the assumption made in the model for the stable throughput problem presented in Section IV.

Source node \( n \) generates a sequence of messages \( J^1_n, J^2_n, \ldots \) where the \( t \)th message \( J^t_n \) takes values from the message set \( \{1, 2, \ldots, 2^{N R_n}\} \). The messages are chosen uniformly from \( \{1, 2, \ldots, 2^{N R_n}\} \) and independently over \( n \). The encoding function \( f_n \) at source \( n \) is given by the mapping

\[
f_n : \{1, 2, \ldots, 2^{N R_n}\} \rightarrow \mathcal{X}_n^N, \quad n = 1, 2.
\]

The encoder output consists of a sequence of codewords \( X_n(J^t_n) \), \( t \geq 1 \). The system is asynchronous in the following sense. Each source and each destination maintain a clock. Let \( S_{nm} \) denote the amount of time that the clock at source \( n \) is running ahead of the clock at destination \( m \). The \( S_{nm} \) are assumed to be integers, meaning that time is discrete and transmissions are symbol-synchronous. The time at each clock can be divided into periods of length \( N \) corresponding to the length of a codeword. Let \( D_{nm} \) denote the offset between the start of periods at source \( n \) and destination \( m \), where \( 0 \leq D_{nm} \leq N - 1 \). We assume that \( D_{nm} \) are uniform over \([0, 1, \ldots, N-1]\) for all \( N \). The codeword \( X_n(J^1_n) \) is sent at time 0 on the clock at source \( n \).

A sequence of channel outputs are observed at each destination, where the outputs at destination \( m \) each take values from the alphabet \( \mathcal{Y}_m \). The decoder operates on a sequence of \( N(T+1) \)
channel outputs to form an estimate of a sequence of $T + 1$ messages. A decoder is defined as follows.

$$
\phi_{nm} : \mathbb{Y}^{N(T+1)}_m \rightarrow \{1, 2, \ldots, 2^{NR_n}\}^{T+1}, \quad n, m = 1, 2
$$

(6)

where $\{1, 2, \ldots, 2^{NR_n}\}^{T+1}$ denotes the $(T + 1)$-fold Cartesian product of $\{1, 2, \ldots, 2^{NR_n}\}$. Since the decoder must synchronize on a particular source $n$, the decoding function is defined separately for each source. The output of the decoder is a sequence of message estimates $\hat{J}^t_{nm}, \hat{J}^t_{nm}, \ldots, \hat{J}^t_{nm}$, where $\hat{J}^t_{nm}$ denotes the estimate at destination $m$ of the $t^{th}$ message sent by source $n$. The error criterion we consider is the average probability of error $P_e^t$ defined as

$$
P_e^t = \Pr \left\{ \bigcup_m \bigcup_n \{ \hat{J}^t_n \neq \hat{J}^t_{nm} \} \right\}.
$$

(7)

The rate pair $(R_1, R_2)$ is achievable if there exists encoding functions $(f_1, f_2)$ and decoding functions $(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22})$ such that $P_e^t \rightarrow 0$ for all $t$ as $N \rightarrow \infty$. The capacity region is the set of all achievable rate pairs.

The model we consider here is a compound version of the totally asynchronous multiple access channel treated in [10] and [11]. As shown in those works, the asynchrony in the system results in the lack of a convex hull operation, and this holds as well in our compound version of the problem. The capacity region is presented below and the proof of the theorem is described in Appendix [2].

**Theorem 1:** The capacity region of the asynchronous compound multiple access channel is the closure of all rate points $(R_1, R_2)$ that lie in the region

$$
\bigcap_m \left\{ (R_1, R_2) : \begin{array}{c}
R_1 < I(X_1; Y_m | X_2) \\
R_2 < I(X_2; Y_m | X_1) \\
R_1 + R_2 < I(X_1, X_2; Y_m)
\end{array} \right\}
$$

for some product distribution $P(x_1)P(x_2)W$.

**B. Random access system**

We now turn our attention to a random access system and apply Theorem [1] to determine the capacity region of the system. Each codeword symbol corresponds to a packet transmitted over the channel shown in Figure [1]. We define the common input alphabet as $\mathcal{X} = \{0, 1, 2, \ldots, 2^u\}$, where $X_n \in \mathcal{X}$, for $n = 1, 2$. A channel input $X_n$ can be either a packet of length $u$ bits (an
information-bearing symbol) or an idle symbol. The 0 symbol is the idle symbol and we let \( \Pr\{X_n = 0\} = 1 - p_n \) according to the random access transmission probability. We assume a uniform distribution on the information-bearing codeword symbols, \( \Pr\{X_n = x\} = p_n/2^u, x = 1, 2, \ldots, 2^u \), meaning that a packet is equally likely to be any sequence of \( u \) bits. The channel output at receiver \( m \) is given by \( Y_m = (Y_{1m}, Y_{2m}) \in \mathcal{X}' \times \mathcal{X}' \) where \( Y_{nm} \) denotes the packet from source \( n \) and \( \mathcal{X}' = \mathcal{X} \cup \Delta \). The \( \Delta \) symbol denotes a packet in error.

The introduction of the idle symbol 0 results in additional protocol or timing information being transmitted over the channel. The information content of this idle symbol is \( h_b(p_n), n = 1, 2 \), where \( h_b \) denotes the binary entropy function. The term \( h_b(p_n) \) appears in the proof provided below and represents the protocol information that is studied by Gallager in [20]. Because we would like our capacity result to represent the rate of reliable communication of data packets, we will aim to exclude this timing information. We do so by considering capacity in packets/slot in the limit as \( u \to \infty \), meaning that the data packets grow large and the fraction of timing information transmitted approaches 0. The timing information is excluded in previous work on random access capacity. In [8], prior to the start of transmission, a “protocol sequence” indicating the occurrence of idle slots is generated at the source and communicated to the receiver, effectively eliminating timing information. In [12], the capacity for \( u \to \infty \) is presented. The capacity of the random access multicast system is given in the following Corollary to Theorem

**Corollary 2:** The capacity region of the random access system with two sources and two destinations is the closure of \((R_1, R_2)\) for which

\[
R_1 < \min_{m=1,2} p_1(1 - p_2)q_{1|1}^{(m)} + p_1 p_2 q_{1|1,2}^{(m)} \\
R_2 < \min_{m=1,2} (1 - p_1)p_2 q_{2|2}^{(m)} + p_1 p_2 q_{2|1,2}^{(m)}
\]

for some \((p_1, p_2) \in [0, 1]^2\).

**Proof:** The result follows by applying the assumptions about the input distribution and channel reception probabilities to the expressions given in Theorem. We first solve for \( I(X_1; Y_m|X_2) \) by conditioning on \( X_2 \) to obtain

\[
I(X_1; Y_m|X_2) = (1 - p_2)I(X_1; Y_m|X_2 = 0) + p_2 I(X_1; Y_m|X_2 \neq 0).
\]
An expression for $I(X_1; Y_m | X_2 = 0)$ can be found from the following sequence of equalities.

\[
I(X_1; Y_m | X_2 = 0) = H(X_1 | X_2 = 0) - H(X_1 | Y_m, X_2 = 0)
\]

\[
\equiv (a) \quad H(X_1) - \Pr(Y_{1m} = \Delta | X_2 = 0) \log_2 2^u
\]

\[
= -(1 - p_1) \log_2 (1 - p_1) - p_1 \log_2 (p_1 / 2^u) - p_1(1 - q_{11}^{(m)}) u
\]

\[
= h_b(p_1) + p_1 \log_2 2^u - p_1 u + p_1 q_{11}^{(m)} u
\]

\[
= h_b(p_1) + p_1 q_{11}^{(m)} u \tag{9}
\]

where \((a)\) holds since $X_2$ is independent of $X_1$ and since $H(X_1 | Y_{1m} \neq \Delta, X_2 = 0) = 0$. For $(X_1; Y_m | X_2 \neq 0)$ we have

\[
I(X_1; Y_m | X_2 \neq 0) = H(X_1 | X_2 \neq 0) - H(X_1 | Y_m, X_2 \neq 0)
\]

\[
= H(X_1) - \Pr(Y_{1m} = \Delta | X_2 \neq 0) \log_2 2^u
\]

\[
= -(1 - p_1) \log_2 (1 - p_1) - p_1 \log_2 (p_1 / 2^u) - p_1(1 - q_{11,2}^{(m)}) u
\]

\[
= h_b(p_1) + p_1 \log_2 2^u - p_1 u + p_1 q_{11,2}^{(m)} u
\]

\[
= h_b(p_1) + p_1 q_{11,2}^{(m)} u. \tag{10}
\]

Combining expressions (9) and (10) results in

\[
I(X_1; Y_m | X_2) = h_b(p_1) + u p_1 (1 - p_2) q_{11}^{(m)} + u p_1 p_2 q_{11,2}^{(m)} \text{ bits/transmission.} \tag{11}
\]

Since one packet corresponds to $u$ bits, we divide by $u$ to obtain a result in units of packets per slot. We then let $u \to \infty$ to obtain

\[
I(X_1; Y_m | X_2) = p_1 (1 - p_2) q_{11}^{(m)} + p_1 p_2 q_{11,2}^{(m)} \text{ packets/slot.} \tag{12}
\]

By following a similar approach, we can show that $I(X_2; Y_m | X_1)$ is given as follows.

\[
I(X_2; Y_m | X_1) = (1 - p_1) p_2 q_{12}^{(m)} + p_1 p_2 q_{12,2}^{(m)} \text{ packets/slot.} \tag{13}
\]

The bound on the sum rate can be found by breaking up $I(X_1, X_2; Y_m)$ into four terms.

\[
I(X_1, X_2; Y_m) = H(X_1, X_2) - H(X_1, X_2 | Y_m)
\]

\[
= H(X_1 | X_2) + H(X_2) - H(X_1 | Y_m, X_2) - H(X_2 | Y_m)
\]

\[
= H(X_1) + H(X_2) - H(X_1 | Y_{1m}, X_2) - H(X_2 | Y_{1m}, Y_{2m}) \tag{14}
\]
For \( n = 1, 2 \), the terms \( H(X_n) \) can be expressed as

\[
H(X_n) = -(1 - p_n) \log_2(1 - p_n) - p_n \log_2(p_n/2^u)
\]

\[
= h_b(p_n) + p_n \log_2 2^u
\]

\[
= h_b(p_n) + p_n u.
\] (15)

The last two terms in (14) can be found in the following manner.

\[
H(X_1|Y_{1m}, X_2) = H(X_1|Y_{1m} = \Delta, X_2) \Pr(Y_{1m} = \Delta|X_2) 
\]

\[
= (1 - p_2)H(X_1|Y_{1m} = \Delta, X_2 = 0) \Pr(Y_{1m} = \Delta|X_2 = 0)
\]

\[
+ p_2 H(X_1|Y_{1m} = \Delta, X_2 \neq 0) \Pr(Y_{1m} = \Delta|X_2 \neq 0)
\]

\[
= (1 - p_2) \log_2 2^u \Pr(Y_{1m} = \Delta|X_2 = 0) + p_2 \log_2 2^u \Pr(Y_{1m} = \Delta|X_2 \neq 0)
\]

\[
= u(1 - p_2)p_1(1 - q_{1|1}^{(m)}) + up_2p_1(1 - q_{1|2}^{(m)})
\]

\[
= up_1 - up_1(1 - p_2)q_{1|1}^{(m)} - up_1p_2q_{1|2}^{(m)}
\] (16)

\[
H(X_2|Y_{1m}, Y_{2m}) = (1 - p_1)H(X_2|Y_{1m} = 0, Y_{2m}) + p_1 H(X_2|Y_{1m} \neq 0, Y_{2m})
\]

\[
= (1 - p_1)H(X_2|Y_{1m} = 0, Y_{2m} = \Delta) \Pr(Y_{2m} = \Delta|Y_{1m} = 0)
\]

\[
+ p_1 H(X_2|Y_{1m} \neq 0, Y_{2m} = \Delta) \Pr(Y_{2m} = \Delta|Y_{1m} \neq 0)
\]

\[
= (1 - p_1) \log_2 2^u p_2(1 - q_{2|2}^{(m)}) + p_1 \log_2 2^u p_2(1 - q_{2|1,2}^{(m)})
\]

\[
= up_2 - u(1 - p_1)p_2q_{2|2}^{(m)} - up_1p_2q_{2|1,2}^{(m)}
\] (17)

By substituting (15), (16), and (17) in (14), dividing by \( u \) and taking \( u \to \infty \), we obtain

\[
I(X_1, X_2; Y_m) = p_1(1 - p_2)q_{1|1}^{(m)} + p_1p_2q_{1|1,2}^{(m)} + (1 - p_1)p_2q_{2|2}^{(m)} + p_1p_2q_{2|1,2}^{(m)} \text{ packets/slot}.
\] (18)

Since \( I(X_1, X_2; Y_m) = I(X_1; Y_m|X_2) + I(X_2; Y_m|X_1) \), the bound on the sum rate is superfluous in terms of describing the capacity region. The result follows.

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IV. STABLE THROUGHPUT REGION

In this section we treat the system shown in Fig. 1 as a network of queues and state the stable throughput region of the system, which is a generalization of previous results on the stable throughput for a system with a single destination node. The model we consider is as follows.
We no longer assume that the sources are always backlogged; instead a random Bernoulli process with rate $\lambda_n$ packets/slot, $n = 1, 2$ models the arrival of packets at each source. Packets that are not immediately transmitted are stored in an infinite-capacity buffer maintained at each source. Transmissions occur according to the random access protocol with source $n$ transmitting with probability $p_n$ when it is backlogged. If a source is empty in a given slot, it does not access the channel. Each source-destination pair is assumed to maintain an orthogonal feedback channel so that instantaneous and error-free acknowledgements can be sent from the destinations to the sources. We find the stable throughput region for two different transmission schemes: retransmissions and random linear coding.

The queue at each source is described by its arrival process and departure process. As $\lambda_n$ represents the arrival rate, we let $\mu_n$ denote the departure or service rate. In general, a queue is said to be stable if departures occur more frequently than arrivals, i.e., $\lambda_n < \mu_n$. This statement is made precise by Loynes’ result [21], which states that if the arrival and departure processes are non-negative, finite, and strictly stationary, then $\lambda_n < \mu_n$ is a necessary and sufficient condition for stability. Stability of the queue is equivalent to ergodicity of the Markov chain representing the queue length. The stable throughput region for a given transmission policy is defined as the set of all $(\lambda_1, \lambda_2)$ for which there exists transmission probabilities $(p_1, p_2)$ such that both queues remain stable.

The difficulty in finding the stable throughput region for our system (and for any buffered random access system) arises from the interaction of the queues. In particular, the service rate $\mu_n$ of source $n$ will vary according to whether the other source is empty or backlogged and can create interference on the channel. To overcome this difficulty, the technique provided in [4] of introducing a dominant system can be used to decouple the sources. In a dominant system, one of the sources behaves as if it is always backlogged by transmitting “dummy” packets when it empties. The queue length in a dominant system stochastically dominates (i.e., is never smaller than) the queue length in the system of interest, meaning that stability in a dominant system implies stability in the original system. Since one source always behaves as if it is backlogged, the service rate in the dominant system can be easily found. Let $\mu_{nb}$ denote the service rate at source $n$ when the other source is backlogged and $\mu_{ne}$ the service rate when the other source is empty. Using the dominant systems approach, the stable throughput region for a system with two sources can be found exactly. This region is stated in the following theorem, which is a
generalization of the result in [4]. Note that in the stable throughput region presented below, the service rates $\mu_{nb}$ and $\mu_{ne}$ are functions of $p_n$, $n = 1, 2$ (although not explicitly shown in these expressions).

**Theorem 3:** [4] For a network with two sources and two destinations, the stable throughput region is given by the closure of $L(p_1, p_2)$ where

$$L(p_1, p_2) = \bigcup_{i=1,2} L_i(p_1, p_2)$$

and

$$L_1(p_1, p_2) = \left\{ (\lambda_1, \lambda_2) : \begin{array}{l}
\lambda_1 < \frac{\lambda_2}{\mu_{2b}} \mu_{1b} + \left( 1 - \frac{\lambda_2}{\mu_{2b}} \right) \mu_{1e} \\
\lambda_2 < \mu_{2b} \\
\lambda_1 < \mu_{1b}
\end{array} \right\}$$

$$L_2(p_1, p_2) = \left\{ (\lambda_1, \lambda_2) : \begin{array}{l}
\lambda_2 < \frac{\lambda_1}{\mu_{1b}} \mu_{2b} + \left( 1 - \frac{\lambda_1}{\mu_{1b}} \right) \mu_{2e} \\
\lambda_1 < \mu_{1b}
\end{array} \right\}$$

for some $(p_1, p_2) \in [0,1]^2$.

In addition to the stable throughput region, we will be interested in the throughput region of the random access system shown in Fig. 1. The throughput region is the closure of the service rates $\mu_{nb}$ for all $p_n$, $n = 1, 2$, where both sources are assumed to be backlogged. In finding the throughput region, there is no interaction between the sources, and the problem is simpler than finding the stable throughput problem. Previous work on buffered random access systems has shown that in all cases in which the stable throughput region has been found, it is known to coincide with the throughput region, suggesting that the empty state of the buffer is insignificant in determining stability. The relation between stable throughput and throughput is explored in [14]. Additionally, in [22], it is proved that for a random access system with two sources and two destinations, the stable throughput region coincides with the throughput region. This result is restated below.

**Theorem 4:** [22] The stable throughput region of the random access system with two sources and two destinations is equivalent to the throughput region, which is given by the closure of $(\lambda_1, \lambda_2)$ for which

$$\lambda_1 < \mu_{1b}, \quad \lambda_2 < \mu_{2b}$$

for some $(p_1, p_2) \in [0,1]^2$. 
We derive the backlogged and empty service rates $\mu_{nb}$ and $\mu_{ne}$ for two different transmission schemes. Together with Theorems [3] and [4] this provides us a complete characterization of the stable throughput region.

V. STABLE THROUGHPUT: RETRANSMISSIONS

In this section we describe the stable throughput region under assuming that a retransmission protocol is used to ensure reliable communication. In the retransmission scheme, as long as source $n$ has not received feedback acknowledgements from both destinations $m = 1, 2$, it will continue to transmit the packet over the channel with probability $p_n$ in each slot. As soon as the source has received acknowledgements from both destinations, it will remove the packet from its queue and begin transmitting the next packet waiting in its buffer, if any. Let random variable $T_n$ denote the service time for source $n$, given by the total number of slots that transpire before the packet from source $n$ is successfully received at both destinations. (Note that the service time includes slots during which the source does not transmit, which happens with probability $1 - p_n$). Since each completed service in the retransmission scheme results in 1 packet being removed from the queue, the average service rate is given by $\mu_n = 1/E[T_n]$.

We first find the backlogged service rates $\mu_{nb}$. Let $T_n^{(m)}$ denote the number of slots needed for successful reception of a packet from source $n$ at destination $m, m = 1, 2$. The $T_n^{(m)}$ are geometrically distributed according to the transmission probabilities $p_n$ and reception probabilities $q_n^{(m)}$, $q_n^{(m)}|n$,$q_n^{(m)}|1,2$. We introduce the following notation. Let $\phi_n$ denote the probability of successful reception of the packet from source $n$ at destination 1 given that source $n$ transmits and that both sources are backlogged. Similarly, $\sigma_n$ denotes the probability of successful reception at destination 2 given that both sources are backlogged and that source $n$ transmits. For instance, $\phi_1$ and $\sigma_1$ are given by

$$\phi_1 = p_2q_1^{(1)} + p_2q_1^{(1)}$$

$$\sigma_1 = p_2q_1^{(2)} + p_2q_1^{(2)}$$

where $p_n = 1 - p_n$. When source 2 is backlogged, $T_1^{(1)}$ is geometrically distributed with parameter $p_1\phi_1$ and $T_1^{(2)}$ is geometrically distributed with parameter $p_1\sigma_1$. The total service time for source 1 when source 2 is backlogged will be given by the maximum of the service times to each
destination,

\[ T_1 \sim \max_m T_1^{(m)}. \]  

(22)

Similarly, when source 1 is backlogged, the service time for source 2 is given by

\[ T_2 \sim \max_m T_2^{(m)}, \]  

(23)

where

\[ T_2^{(1)} \sim \text{geom} (p_2 \phi_2), \quad T_2^{(2)} \sim \text{geom} (p_2 \sigma_2). \]  

(24)

The expected maximum value \( E[T_n] \) can be readily found and the backlogged service rates are given by

\[ \mu_{nb} = \frac{p_n \phi_n \sigma_n (\phi_n + \sigma_n - \tau_n)}{(\phi_n + \sigma_n) (\phi_n + \sigma_n - \tau_n) - \phi_n \sigma_n} \]  

(25)

where \( \tau_n \) denotes the probability that a packet sent from \( n \) is received at both destinations given that source \( n \) transmits, e.g., \( \tau_1 = p_2 q_1^{(1)} + q_1^{(2)} (1 - p_2) q_1^{(1)} + q_1^{(2)} (1 - p_2) q_1^{(1)}, 2. \)

The empty service rates \( \mu_{ne} \) can be found directly from \( \mu_{nb} \) as

\[ \mu_{1e} = \mu_{1b}|_{p_2 = 0}; \quad \mu_{2e} = \mu_{2b}|_{p_1 = 0}. \]  

(26)

The backlogged and empty service rates for random access multicast with retransmissions have also been found in [22]. In that work, a different approach is used in which a Markov chain is used to model which destinations have received the packet currently under transmission. The result is identical to the one presented above.

We now compare the stable throughput region for the retransmissions scheme to the Shannon capacity region. We consider the backlogged service rate \( \mu_{nb} \), since, by Theorem 4, this is the term that bounds the stable throughput for source \( n \). The expected service time for source 1 when source 2 is backlogged is bounded as

\[ E[T_1] = E[\max_m T_1^{(m)}] \overset{(a)}{\geq} \max_m E[T_1^{(m)}] \]  

(27)

\[ = \max_m \frac{1}{p_1 (1 - p_2) q_1^{(m)} + p_1 p_2 q_1^{(m)}} \]  

(28)

\[ = \frac{1}{\min_m p_1 (1 - p_2) q_1^{(m)} + p_1 p_2 q_1^{(m)}} \]  

(29)
where (a) follows from Jensen’s inequality and (b) follows from the expected value of a geometrically distributed random variable. Then the backlogged service rate $\mu_{1b}$ is bounded as

$$\mu_{1b} = \frac{1}{E[T_1]} \leq \min_m p_1(1 - p_2)q_{l|m} + p_1p_2q_{l|m_1}$$  \hspace{1cm} (30)$$

and $\mu_{2b}$ can be bounded similarly. Note that the right-hand side of (30) is equal to the bound on the Shannon achievable rate $R_1$ given in Corollary 2. Thus we should expect that the Shannon capacity region outer bounds the stable throughput region for the retransmission scheme.

VI. STABLE THROUGHPUT: RANDOM LINEAR CODING

In this section we present two different approaches to analyzing the stable throughput region for a transmission scheme in which groups of $K$ packets at the front of the queue are randomly encoded and transmitted over the channel. By encoded, we mean that a random linear combination of the $K$ packets is formed, and we refer to this random linear combination as a coded packet.

As soon as a destination has received enough coded packets and is able to decode the original $K$ packets, it does so and sends an acknowledgement to the source over its feedback channel. Once the source receives acknowledgements from both destinations, it removes the $K$ packets it has been encoding and transmitting from its queue and begins encoding and transmission of the next $K$ packets waiting in its buffer. The stable throughput region for a similar system with $K = 2$ is found in [23].

For a given source, let $s_1, s_2, \ldots, s_K$ denote the $K$ (binary) packets at the front of the queue. Random linear coding is performed in the following manner. A coded packet is randomly generated as

$$\sum_{i=1}^{K} a_is_i$$  \hspace{1cm} (31)$$

where the coefficients $a_i$ are generated according to $Pr(a_i = 1) = 1 - Pr(a_i = 0) = 1/2$, $i = 1, 2, \ldots, K$, and $\sum$ denotes modulo-2 addition. A coded packet formed in this way is the same length as one of the original packets $s_i$ and can be transmitted over the channel within the same time period of a slot. For each coded packet sent, the coefficients $a_i$ used in generating the packet will be appended to the header of the packet. In each slot, the source generates a coded
packet according to (31) and sends it over the channel. Regardless of whether a previously-generated coded packet has been received, the source generates a new coded packet in each slot and transmits it with probability $p_n$.

A. Expected service time

We first analyze the random linear coding scheme by examining the expected service time in a manner similar to the approach used for the retransmission scheme in Section V. Let $\widetilde{T}_n$ denote the service time for source $n$, which is the number of slots that elapse from the transmission of the first coded packet until the source has received acknowledgements from both destinations. Since each completed service in the random linear coding scheme results in $K$ packets being removed from the queue, the average service rate is given by $\bar{\mu}_n = K / E[\widetilde{T}_n]$.

The service time will be a random variable dependent upon the random access transmission probabilities $p_n$, the reception probabilities $q_{n|m}'$ and $q_{n|1,2}'$, and the number of coded packets needed to decode. Let $N^{(m)}_n$ denote the number of coded packets received from source $n$ at destination $m$ before destination $m$ can decode the original $K$ packets. The $N^{(m)}_n$ will be identically distributed over $n$ and $m$ since all coded packets are generated in the same way according to (31). Additionally, $N^{(m)}_n$ will be independent over $n$ since the two sources generate their coded packets independently. However, $N^{(m)}_n$ will be correlated over $m$, since a given source $n$ will generate and transmit the same coded packets to both destinations $m = 1, 2$.

For a fixed value of $K$, let $F_K(j)$ denote the common cumulative distribution function (cdf) of $N^{(m)}_n$, or the probability that the number of coded packets needed for decoding is less than or equal to $j$. With each coded packet it receives, the destination collects a binary column of length $K$ in a matrix, where the column consists of the coefficients $a_i$, $i = 1, 2, \ldots, K$. Then $F_K(j)$ is the probability that decoding can be performed (by solving a system of linear equations, or equivalently, by Gaussian elimination) if the matrix has at most $j$ columns. Thus

$$F_K(j) = \Pr\{\text{a random } K \times j \text{ binary matrix has rank } K\}. \quad (32)$$

Note that for $j < K$ the matrix cannot possibly have rank $K$ and $F_K(j) = 0$. For $j \geq K$, we can write an expression for $F_K(j)$ following the procedure in [24] by first counting up the number of $K \times j$ non-singular matrices, which is given by

$$(2^j - 1)(2^j - 2)(2^j - 2^2) \ldots (2^j - 2^{K-1}). \quad (33)$$
In the product above, each term accounts for a row of the $K \times j$ binary matrix and reflects that the row is neither zero, nor equal to any of the previous rows, nor equal to any linear combination of previous rows. Since the total number of $K \times j$ binary matrices is $2^{jK}$, we obtain

$$F_K(j) = \begin{cases} \prod_{i=0}^{K-1} (1 - 2^{-j+i}) & j \geq K \\ 0 & j < K \end{cases}$$ (34)

The probability mass function (pmf) of $N_n^{(m)}$, or the probability that decoding can be performed when the destination has received exactly $j$ columns and no fewer, is given by

$$f_K(j) = F_K(j) - F_K(j-1).$$ (35)

From (34) and (35) the expected value of $N_n^{(m)}$ can be computed, where $E[N_n^{(m)}] \geq K$. The ratio

$$\frac{E[N_n^{(m)}]}{K} = \prod_{i=0}^{K-1} (1 - 2^{-K+i}) + \frac{1}{K} \sum_{j=K+1}^{\infty} j \left( \prod_{i=0}^{K-1} (1 - 2^{-j+i}) - \prod_{i=0}^{K-1} (1 - 2^{-j+1+i}) \right)$$ (36)

$$\longrightarrow 1 \quad \text{as} \quad K \rightarrow \infty$$ (37)

for all $n, m = 1, 2$.

With the distribution of $N_n^{(m)}$ characterized, we can now describe the service time from source $n$ to destination $m$, which we denote $\tilde{T}_n^{(m)}$. The number of slots needed for the successful reception of each coded packet will be geometrically distributed and in total $N_n^{(m)}$ coded packets must be received. Then $\tilde{T}_n^{(m)}$ can be modeled as the sum of $N_n^{(m)}$ independent geometrically distributed random variables.

$$\tilde{T}_n^{(m)} = g_{n,1}^{(m)} + g_{n,2}^{(m)} + \ldots + g_{n,N_n^{(m)}}^{(m)}$$ (38)

In the above expression, $g_{n,i}^{(m)}$, $i = 1, 2, \ldots, N_n^{(m)}$, will be geometrically distributed with a parameter that depends on the reception probabilities and on the assumption of whether the other source is backlogged. For instance, when source 2 is backlogged

$$g_{1,i}^{(m)} \sim \text{geom} \left( p_1 (1-p_2) q_{1|1}^{(m)} + p_1 p_2 q_{1|1,2}^{(m)} \right), \quad i = 1, 2, \ldots, N_1^{(m)}$$ (39)

and when source 2 is empty,

$$g_{1,i}^{(m)} \sim \text{geom} \left( p_1 q_{1|1}^{(m)} \right), \quad i = 1, 2, \ldots, N_1^{(m)}.$$ (40)
The total service time is given as
\[ \tilde{T}_n = \max_m \tilde{T}_n^{(m)}. \quad (41) \]

As in the case of random access with retransmissions, we argue that the stable throughput region for random access with random linear coding will be outer bounded by the Shannon capacity region. In the case that source 2 is backlogged, the expected service time for source 1 is now bounded as
\[
E[\tilde{T}_1] = E[\max_m \tilde{T}_1^{(m)}] \geq \max_m E[\tilde{T}_1^{(m)}] \quad (42)
\]
\[
= \frac{E[N_1^{(m)}]}{\max_m p_1(1 - p_2)q_1^{(m)} + p_1p_2q_1^{(m)}} \quad (43)
\]
\[
= \frac{E[N_1^{(m)}]}{\min_m p_1(1 - p_2)q_1^{(m)} + p_1p_2q_1^{(m)}} \quad (44)
\]

where (a) again follows from Jensen’s inequality, (b) holds since \( g_1^{(m)} \) are independent, identically distributed, and (c) holds since \( N_1^{(m)} \) is identically distributed over \( m \), meaning that \( E[N_1^{(1)}] = E[N_1^{(2)}] \). The backlogged service rate \( \tilde{\mu}_{1b} \) is bounded as
\[
\tilde{\mu}_{1b} = \frac{K}{E[\tilde{T}_1]} \leq \frac{K}{E[N_1^{(m)}]} \min_m p_1(1 - p_2)q_1^{(m)} + p_1p_2q_1^{(m)} \quad (45)
\]
\[
\leq \min_m p_1(1 - p_2)q_1^{(m)} + p_1p_2q_1^{(m)} \quad (46)
\]

Then the Shannon capacity region outer bounds the stable throughput region for random access with random linear coding.

Unfortunately a difficulty arises in finding the service rates \( \mu_{nb} \) and \( \mu_{ne} \) in closed form from \( E[\max_m \tilde{T}_n^{(m)}] \). This difficulty arises for a number of reasons: \( \tilde{T}_n^{(1)} \) and \( \tilde{T}_n^{(2)} \) are not independent, and \( \tilde{T}_n^{(m)} \) is distributed according to a composite distribution function, for which the pdf is not easily expressed in closed form. In fact, even if these two difficulties were to be removed, \( E[\max_m \tilde{T}_n^{(m)}] \) cannot be easily handled. For instance, let us assume that \( \tilde{T}_n^{(1)} \) and \( \tilde{T}_n^{(2)} \) are independent and that \( N_n^{(m)} = n^{(m)} \) are deterministic (which means that the pdf is no longer composite). In that case, \( \tilde{T}_n^{(m)} \) is the sum of \( n^{(m)} \) iid geometric random variables, meaning that \( \tilde{T}_n^{(m)} \) follows a negative binomial distribution. Let us further make the assumption that \( q_n^{(1)} = q_n^{(2)} \) and \( q_n^{(1)} = q_n^{(2)} \), which means that \( \tilde{T}_n^{(m)} \) are identically distributed over \( m \). In this very simplified case, \( E[\max_m \tilde{T}_n^{(m)}] \) is the expected maximum of two iid negative binomial
random variables. The computation of this expected value is treated in [25], and the result involves a periodic function which is approximated by a Fourier series. Thus, even in this very simplified case, we can at best approximate $E[\max_m \tilde{T}_n^{(m)}]$, and this approximation must be computed numerically.

B. Markov chain approach

As an alternative to the analysis presented above, we now develop a Markov chain model which allows us to find the queueing service rates. For a given source node, we set up a vector Markov chain with state $(i, j, k)$ corresponding to the number of linearly independent coded packets that have been received from the source node. In this model, $i$ represents the number of linearly independent coded packets that have been received at destination 1, and $j$ represents the number of linearly independent packets that have been received at destination 2. Since the coded packets are generated by the same source, some of the coded packets received at destination 1 may also have been received at destination 2, and $k$ represents the number of such packets, where $k \leq \min(i, j)$. The variable $k$ allows us to track the correlation between $N_n^{(1)}$ and $N_n^{(2)}$, which was a difficulty in our previous approach described above. The Markov chain evolves in discrete time over the time slots in our system model.

The state space of the Markov chain is the three-dimensional discrete set of points $[0, K]^3$. There are $K + 1$ absorbing states given by $(K, K, k)$, $0 \leq k \leq K$, which represent the reception of $K$ linearly independent coded packets at both destinations, for which the service of $K$ packets at the source has been completed. Transitions in the Markov chain can only occur “upward”, corresponding to the reception of a new linearly independent packet, and a transition results in an increase of the indices $i, j, k$ by at most 1, meaning that at most 1 new linearly independent packet can be received in a slot. We use the notation $(i_1, j_1, k_1) \rightarrow (i_2, j_2, k_2)$ to denote the transition from state $(i_1, j_1, k_1)$ to state $(i_2, j_2, k_2)$.

The Markov chain is irreducible and aperiodic, and because it has a finite state space, a stationary distribution exists. Let $\pi_{i, j, k}$ denote the steady-state probability of $(i, j, k)$. The steady-state probabilities are found by solving the set of equations

$$\pi_{i_1, j_1, k_1} = \sum_{(i_2, j_2, k_2)} \pi_{i_2, j_2, k_2} \Pr((i_2, j_2, k_2) \rightarrow (i_1, j_1, k_1))$$

(47)
and

$$\sum_{(i,j,k)} \pi_{i,j,k} = 1.$$  \hspace{1cm} (48)

The service rate $\tilde{\mu}_n$ is equal to $K$ times the probability of transitioning into an absorbing state $(K, K, k)$, $0 \leq k \leq K$. There are only a few ways to transition into an absorbing state; let $\mathcal{A}_k$ denote the set of states which have a one-step transition into the absorbing state $(K, K, k)$. For $k \in [0, K - 1]$ we have

$$\mathcal{A}_k = \{(K-1, K, k), (K-1, K, k-1), (K, K-1, k), (K, K-1, k-1), (K-1, K-1, k-1)\},$$ \hspace{1cm} (49)

and for $k = K$,

$$\mathcal{A}_K = \{(K-1, K, K-1), (K-1, K-1, K-1)\}.$$ \hspace{1cm} (50)

Note that we define $\mathcal{A}_K$ in this way since the states $(K-1, K, K)$ and $(K, K-1, K)$ violate $k \leq \min(i, j)$. The service rate for source $n$ is given by

$$\tilde{\mu}_n = K \sum_{k=0}^{K} \sum_{(i,j,k) \in \mathcal{A}_k} \pi_{i,j,k} \Pr((i, j, k) \rightarrow (K, K, k)).$$ \hspace{1cm} (51)

The transition probabilities $(i_1, j_1, k_1) \rightarrow (i_2, j_2, k_2)$ for source $n$ can be written assuming that the other source is either backlogged or empty, leading to the service rates $\tilde{\mu}_{nb}$ and $\tilde{\mu}_{ne}$.

As an example, consider the transition $(i, j, k) \rightarrow (i+1, j, k)$ in the Markov chain for source 1 when source two is backlogged. Assume first that source two does not transmit, which happens with probability $1 - p_2$. Then there are two ways for the transition $(i, j, k) \rightarrow (i+1, j, k)$ to occur. First, destination 2 could receive no packet, which happens with probability $1 - q_{(1)}^{(1)}$, while destination 1 receives a coded packet which is neither an all-zero packet nor equal to any linear combination of the $i$ packets it has already received, which happens with probability $q_{(1)|1}^{(1)}(1 - 2^{i/2 - K})$. Alternatively, both destinations could receive a coded packet, but that packet is either the all zero packet or some linear combination of the packets that have been received by destination 2 and not by destination 1. This happens with probability $q_{(1)|1}^{(1)} q_{(2)}^{(2)}(2j - 2^{k})2^{-K}$. The same two alternatives are possible in the case that source 2 does transmit, which happens with probability $p_2$, except that the reception probabilities are now given by $q_{(1)|1,2}^{(m)}$. Then the transition
Fig. 2. Closure of the stable throughput and capacity regions for a channel with reception probabilities $q^{(1)} = q^{(2)} = 0.8$, $q^{(2)}_{1|1} = q^{(1)}_{2|2} = 0.7$, and $q^{(m)}_{1|1,2} = q^{(m)}_{2|1,2} = 0.6$, $m = 1, 2$. The stable throughput region for random linear coding is abbreviated RLC.

$(i, j, k) \rightarrow (i + 1, j, k)$ for source 1 when source 2 is backlogged occurs with probability

$$p_1 \left[ p_2 \left\{ q^{(1)}_{1|1}(1 - q^{(2)}_{1|1})(1 - 2^i 2^{-K}) + q^{(1)}_{1|1} q^{(2)}_{2|1}(2^j - 2^k)2^{-K} \right\} 
+ p_2 \left\{ q^{(1)}_{1|1,2}(1 - q^{(2)}_{1|1,2})(1 - 2^i 2^{-K}) + q^{(1)}_{1|1,2} q^{(2)}_{2|1,2}(2^j - 2^k)2^{-K} \right\} \right]. \quad (52)$$

The same transition probability can be used when source 2 is empty by setting $p_2 = 0$. Similar arguments can be used to find all transition probabilities for our Markov chain model; we have stated those probabilities in Appendix II. Ultimately, we would like to find closed-form expressions for the service rates $\hat{\mu}_{nb}$ and $\hat{\mu}_{ne}$, but due to the size of the state-space, this is a difficult task. Instead we have computed some numerical examples based on the Markov chain model presented above, and those are presented next.

VII. NUMERICAL EXAMPLES

We have computed a number of numerical examples of the Shannon capacity region and the stable throughput regions for retransmissions and random linear coding. The results for random
linear coding have been computed with the service rates given by Equation (51). Fig. 2 shows results for a “good” channel with relatively large reception probabilities while Fig. 3 shows results for a “poor” channel with smaller reception probabilities. In both figures, we have plotted the stable throughput for random linear coding (abbreviated “RLC”) with various values of $K$. The results show that the Shannon capacity region is strictly larger than the stable throughput regions for both the retransmissions and random linear coding schemes. Additionally, the stable throughput region for random linear coding grows with $K$ and appears to approach the capacity as $K \rightarrow \infty$.

The random linear coding scheme does not necessarily outperform the retransmission scheme. For small values of $K$, the coding scheme is inefficient in the sense that the ratio $K/E[N_n^{(m)}]$ is small. This inefficiency is largely due to the fact that an all-zero coded packet can be generated and transmitted; this possibility is not precluded in (51) and occurs more often for small values of $K$. The inefficiency of the coding scheme means that the retransmission of $K$ packets requires a service time that is less than $E[N_n^{(m)}]$ for small values of $K$. This effect seems to become more pronounced as the channel improves, since for a “good” channel, packets are more often received correctly and do not need to be retransmitted.

VIII. DISCUSSION

One conclusion to draw from our work is that for sufficiently large blocklengths, given by $K$ in our model, coding over packets in a queue yields a higher stable multicast throughput than simply retransmitting lost packets. In light of the recent developments on network coding and fountain codes, this result is not surprising. However, we have also shown that if the blocklength is too small and the code is not designed appropriately, then a retransmission scheme can provide a higher stable throughput than coding. Furthermore, we have combined random access with random linear coding to yield an efficient multicast scheme that can operate in a completely distributed manner. We have shown that the merging of random access with random linear coding results in good performance in the sense that the stable throughput approaches the Shannon capacity. In the process, we have presented a model that represents random coding of packets in a queue.

Another significant outcome of our work is that we have provided an example of a transmission policy for which the stable throughput region does not coincide with the Shannon capacity region.
of a random access system. This outcome contradicts a widely-held (yet unproved) belief that the stability and capacity regions coincide for random access. This result sheds further light on the relation between the stable throughput and the Shannon capacity as representations for the rate of reliable communication.

**APPENDIX I**

**PROOF OF THEOREM 1**

Let $P^t_m$ denote the error probability for the $t^{th}$ message from the sources at receiver $m$.

$$P^t_m \triangleq \Pr \left\{ \bigcup_n \left\{ J^t_n \neq j^t_{nm} \right\} \right\}. $$

The following lemma provides a condition equivalent to $P^t_e \to 0$ and is used in the proof of Theorem 1.

**Lemma 5:** The average error $P^t_e \to 0$ if and only if $\max_m P^t_m \to 0$. 

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Fig. 3. Closure of the stable throughput and capacity regions for a channel with reception probabilities $q_{1|1}^{(1)} = q_{2|2}^{(2)} = 0.8$, $q_{1|2}^{(2)} = q_{2|1}^{(1)} = 0.7$, and $q_{1|1,2}^{(m)} = q_{2|1,2}^{(m)} = 0.2, m = 1, 2$. The stable throughput region for random linear coding is abbreviated RLC.
Proof:  The average error $P_e^t$ can be upper bounded by the union bound as follows: $P_e^t \leq P_1^t + P_2^t \leq 2 \max_m P_m^t$. A similar lower bound also holds, namely $P_e^t \geq \max_m P_m^t$. Thus $\max_m P_m^t \leq P_e^t \leq 2 \max_m P_m^t$ and the result follows.

Achievability for our system is shown by first establishing achievability for the MAC $W_m$. This is shown in [11] and [10]; the approach presented in [11] is summarized here. Each codeword symbol in the codebook for source $n$ is generated according to the distribution $P(x_n)$, independently over codeword symbols and independently over messages. The following two properties are assumed.

(I) The codewords $x_n(1)$ and $x_n(2)$ are reserved for use as preambles. A preamble is sent after every sequence of $T$ messages and $x_n(1)$ and $x_n(2)$ are used as preambles in an alternating fashion. In [11] it is shown that by using the preamble, the receiver can synchronize on codeword boundaries with arbitrarily small probability of synchronization error.

(II) In a sequence of $T + 1$ messages (including a preamble), no messages are repeated. As a result, any two disjoint subsets of $N(T + 1)$ codeword symbols (corresponding to $T + 1$ messages) are independent. For $T \ll 2^{NR_m}$ the resulting loss in rate is negligible.

By observing the channel outputs, the decoder $\phi_1m$ can detect the preambles $x_1(1)$ and $x_1(2)$ to determine that the output symbols in between correspond to inputs $x_1(j_1^1), x_1(j_2^1), \ldots, x_1(j_T^1)$. Let $x_1^*$ denote the sequence of $N(T + 1)$ symbols corresponding to $x_1(j_1^1), x_1(j_2^2), \ldots, x_1(j_T^T)$ augmented by portions of the preambles $x_1(1) \text{ and } x_2(2)$. At the channel output, $x_1^*$ will overlap with a sequence $x_2^+$ consisting of $N(T + 1)$ symbols input by source 2, including $N$ preamble symbols. Let $y_m^{N(T+1)}$ denote the output sequences corresponding to $x_1^+$ and $x_2^+$ at destination $m$. The decoder $\phi_1m$ outputs the unique sequence of messages $\hat{j}_{1m}^1, \hat{j}_{1m}^2, \ldots, \hat{j}_{1m}^T$ that lies in the set of typical $(x_1^+, x_2^+, y_m^{N(T+1)})$ sequences. With this approach it is shown in [11] that $\Pr\{J_t^1 \neq J_{1m}^1\} \rightarrow 0$ for all $t$. A similar technique can be used by decoder $\phi_2m$ to show that $\Pr\{J_t^2 \neq J_{2m}^2\} \rightarrow 0$. Then by the union bound, $P_m^t \rightarrow 0$ for all $t$. Finally, if the rate pair $(R_1, R_2)$ lies in the intersection of the achievable rates for MACs $W_1$ and $W_2$, then $\max_m P_m^t \rightarrow 0$ and thus $P_e^t \rightarrow 0$ for all $t$ by Lemma 5.

The converse for the MAC $W_m$ is shown in [11] and [10] by using Fano’s inequality, the data processing inequality, and the concavity of mutual information. Then $\max_m P_m^t \rightarrow 0$ implies that the rate pair $(R_1, R_2)$ must lie within the intersection of the capacity regions of $W_1$ and $W_2$. 

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DRAFT
APPENDIX II
MARKOV CHAIN ANALYSIS OF RANDOM LINEAR CODING

In the Markov chain analysis of Section VI-B, the state \((i, j, k)\) represents \(i\) linearly independent coded packets received at destination 1, \(j\) linearly independent coded packets received at destination 2, and \(k\) coded packets which have been received at both destinations, \(k \leq \min(i, j)\). When source 2 is backlogged, the non-zero transition probabilities for source 1 are given as follows for \(i, j = 0, 1, \ldots, K\).

\[
(i, j, k) \rightarrow (i, j, k) : p_1 + p_1 \left[ p_2 \left\{ (1-q_{1|1}^{(1)})(1-q_{1|1}^{(2)}) + (1-q_{1|1}^{(1)})q_{1|1}^{(2)}2^{-i}2^{-K} + q_{1|1}^{(1)}(1-q_{1|1}^{(2)}2^{-i}2^{-K} + q_{1|1}^{(1)}q_{1|1}^{(2)}2^{-i}2^{-K}) \right\} + p_2 \left\{ (1-q_{1|1,2}^{(1)})(1-q_{1|1,2}^{(2)}) + (1-q_{1|1,2}^{(1)})q_{1|1,2}^{(2)}2^{-i}2^{-K} + p_2 \left\{ (1-q_{1|1,2}^{(1)})(1-q_{1|1,2}^{(2)}) + (1-q_{1|1,2}^{(1)})q_{1|1,2}^{(2)}2^{-i}2^{-K} \right\} \left\{ (1-q_{1|1,2}^{(1)})(1-q_{1|1,2}^{(2)}) + (1-q_{1|1,2}^{(1)})q_{1|1,2}^{(2)}2^{-i}2^{-K} \right\} \left\{ (1-q_{1|1,2}^{(1)})(1-q_{1|1,2}^{(2)}) + (1-q_{1|1,2}^{(1)})q_{1|1,2}^{(2)}2^{-i}2^{-K} \right\} \right\} \right]\]

\[
(i, j, k) \rightarrow (i + 1, j, k) : p_1 \left[ p_2 \left\{ (1-q_{1|1}^{(1)})(1-q_{1|1}^{(2)}2^{-i}2^{-K}) + (1-q_{1|1}^{(1)})q_{1|1}^{(2)}2^{-i}2^{-K} \right\} + p_2 \left\{ (1-q_{1|1,2}^{(1)})(1-q_{1|1,2}^{(2)})(1-2^i2^{-K}) + (1-q_{1|1,2}^{(1)})q_{1|1,2}^{(2)}2^{-i}2^{-K} \right\} \right] \]

\[
(i, j, k) \rightarrow (i, j + 1, k) : p_1 \left[ p_2 \left\{ (1-q_{1|1}^{(1)})(1-q_{1|1}^{(2)}2^{-i}2^{-K}) + (1-q_{1|1}^{(1)})q_{1|1}^{(2)}2^{-i}2^{-K} \right\} + p_2 \left\{ (1-q_{1|1,2}^{(1)})(1-q_{1|1,2}^{(2)})(1-2^i2^{-K}) + (1-q_{1|1,2}^{(1)})q_{1|1,2}^{(2)}2^{-i}2^{-K} \right\} \right] \]

\[
(i, j, k) \rightarrow (i + 1, j + 1, k + 1) : p_1 \left[ p_2 \left\{ (1-q_{1|1}^{(1)})(1-q_{1|1}^{(2)}2^{-i}2^{-K}) + (1-q_{1|1}^{(1)})q_{1|1}^{(2)}2^{-i}2^{-K} \right\} + p_2 \left\{ (1-q_{1|1,2}^{(1)})(1-q_{1|1,2}^{(2)})(1-2^i2^{-K}) + (1-q_{1|1,2}^{(1)})q_{1|1,2}^{(2)}2^{-i}2^{-K} \right\} \right] \]

\[
(i, K, k) \rightarrow (i, K, k) : p_1 + p_1 \left[ p_2 \left\{ (1-q_{1|1}^{(1)}2^{-i}2^{-K}) + (1-q_{1|1}^{(1)})2^i2^{-K} \right\} + p_2 \left\{ (1-q_{1|1}^{(1)})(1-q_{1|1}^{(2)}2^{-i}2^{-K}) + (1-q_{1|1}^{(1)})q_{1|1}^{(2)}2^{-i}2^{-K} \right\} \right] \]

\[
(i, K, k) \rightarrow (i + 1, K, k) : p_1 \left[ p_2 \left\{ (1-(2^i + 2^i)(2^i2^{-K})) + (1-q_{1|1}^{(1)})(1-2^i2^{-K}) \right\} + p_2 \left\{ (1-q_{1|1,2}^{(1)})(1-2^i2^{-K}) + (1-q_{1|1,2}^{(1)})q_{1|1,2}^{(2)}2^{-i}2^{-K} \right\} \right] \]

\[
(i, K, k) \rightarrow (i + 1, K, k + 1) : p_1 \left[ p_2 \left\{ (1-(2^i + 2^i2^{-K}))(K-k)2^{-K}) + (1-q_{1|1}^{(1)})(K-k)2^{-K}) \right\} + p_2 \left\{ (1-q_{1|1,2}^{(1)})(K-k)2^{-K}) + (1-q_{1|1,2}^{(1)})q_{1|1,2}^{(2)}(K-k)2^{-K}) \right\} \right] \]

\[
(K, j, k) \rightarrow (K, j, k) : p_1 + p_1 \left[ p_2 \left\{ (1-q_{1|1}^{(2)}2^{-i}2^{-K}) + (1-q_{1|1}^{(2)})2^i2^{-K} \right\} + p_2 \left\{ (1-q_{1|1}^{(2)})(1-q_{1|1}^{(2)}2^{-i}2^{-K}) + (1-q_{1|1}^{(2)})q_{1|1}^{(2)}2^{-i}2^{-K} \right\} \right] \]

\[
(K, j, k) \rightarrow (K, j+1, k) : p_1 \left[ p_2 \left\{ (1-(2^i + 2^i)(2^i2^{-K})) + (1-q_{1|1}^{(2)})(1-2^i2^{-K}) \right\} + p_2 \left\{ (1-q_{1|1,2}^{(2)})(1-2^i2^{-K}) + (1-q_{1|1,2}^{(2)})q_{1|1,2}^{(2)}(1-2^i2^{-K}) \right\} \right] \]
\[(K, j, k) \rightarrow (K, j+1, k+1) : p_1 \left[ p_2 \left( q_{1,1}^{(2)} (K - k) 2^{-K} \right) + p_2 \left( q_{1,1,2}^{(2)} (K - k) 2^{-K} \right) \right] \]

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