On the Halo Velocity Bias

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(Dated: August 6, 2014)

It has been recently shown that any halo velocity bias present in the initial conditions does not decay to unity, in agreement with predictions from peak theory. However, this is at odds with the standard formalism based on the coupled-fluids approximation for the coevolution of dark matter and halos. Starting from conservation laws in phase space, we discuss why the fluid momentum conservation equation for the biased tracers needs to be modified in accordance with the change advocated in Baldauf, Desjacques & Seljak (2014). Our findings indicate that a correct description of the halo properties should properly take into account peak constraints when starting from the Vlasov-Boltzmann equation.

I. INTRODUCTION

While the existence of a spatial bias between the halos hosting galaxies and the underlying dark matter (DM) distribution has been firmly established since the pioneering work of [1], the presence of a halo velocity bias is still being debated. Clearly, whereas galaxy velocities can be physically biased (i.e. on an object-by-object basis) owing to differences between the dark matter and baryon velocity fields (see, e.g. Refs. [2, 3]), by construction halo locally flow with the dark matter (in Einstein’s theory of gravity), so that a halo velocity bias can only arise statistically. Namely, it must be the statistical manifestation of a selection effect, which should naturally arise since virialized structures preferentially trace overdense regions of the Universe [4, 5].

A step forward in the understanding of halo bias has been recently made in Ref. [6] where, through N-body simulations, the authors have measured a halo velocity bias \( b_v(k) \),

\[
v_h(\vec{k}, t) = b_v(k) v_{dm}(\vec{k}, t), \quad b_v(k) = (1 - R_v^2 k^2),
\]

(1.1)

where \( R_v \) is the typical scale of the halo velocity bias

\[
R_v^2 = \frac{\sigma_j^2}{\sigma_i^2}, \quad \sigma_j^2 = \int \frac{d^3k}{(2\pi)^3} k^2 P_{dm}(k) W^2(KR),
\]

(1.2)

\( P_{dm}(k) \) is the DM power spectrum and \( W(x) \) is a spherically symmetric smoothing kernel. This statistical effect is consistent with the relative suppression of the halo velocity divergence power spectrum at late time subsequently reported in Ref. [7], although this type of measurement is more prone to systematics arising from sparse sampling (see e.g. Refs. [8–10]).

Whereas the finding of Ref. [6] is in full agreement with the peak model, which indeed predicts the existence of a linear, statistical halo velocity bias which remains constant with time [11–13], it seems at odds with the prediction based on the coupled-fluids approximation for the coevolution of DM and halos [14]. The latter is widely used to compute the time evolution of bias [15–17] and is based on the idea of following the evolution over cosmic time and in Eulerian space of the halo progenitors - the so-called proto-halos - until their virialization. While their shapes and topology change as a function of time (smaller substructures gradually merge to form the final halo), their centre of mass moves along a well-defined trajectory determined by the surrounding mass density field. Therefore, unlike virialized halos that experience merging, by construction proto-halos always preserve their identity. Their total number is therefore conserved over time, and one can write a continuity equation and an Euler equation for their number density and velocity, respectively. Nevertheless, this approach predicts that any Eulerian velocity bias rapidly decays to unity [14]

\[
b_v^E(k, t) = 1 + D^{-3/2}(t)(b_v(k) - 1),
\]

(1.3)

where \( D(t) \) is the linear growth rate normalized to unity at the collapse redshift. To reconcile these two apparently contradictory results, the authors of Ref. [6] argued that the Euler equation for halos should be changed from
\[ \dot{\theta}_h + H\theta_h + \frac{3}{2} H^2 \Omega_{dm} \delta + \cdots = 0, \quad \theta_h = \vec{\nabla} \cdot \vec{v}_h, \]  

which predicts the incorrect behavior (I.3), to

\[ \dot{\theta}_h + H\theta_h + \frac{3}{2} b_v(k) H^2 \Omega_{dm} \delta_{dm} + \cdots = 0, \quad \theta_h = \vec{\nabla} \cdot \vec{v}_h, \]  

where \( H \) is the Hubble rate and \( \Omega_{dm} \) parametrizes the abundance of DM and the dots stand for higher-order terms. Their physical interpretation is that the gravitational force acting on DM halos is statistically biased. Together with the Euler equation for DM

\[ \dot{\theta}_{dm} + H\theta_{dm} + \frac{3}{2} H^2 \Omega_{dm} \delta_{dm} + \cdots = 0, \quad \theta_{dm} = \vec{\nabla} \cdot \vec{v}_{dm}, \]

one indeed recovers the behavior (I.1) and the Eulerian velocity halo bias does not decay in time to unity.

Another reason why Eq. (I.4) cannot describe the momentum evolution of halos is the fact that it does not differ at all from (I.6). Consequently, it is not possible that Eq. (I.4) describes clustered objects like halos (or peaks), which are different from the smooth DM distribution as they are supposed to be located at points where \( \vec{\nabla} \delta_{dm} = 0 \) and where the smoothed density contrast is larger than some value \( \nu \sigma_0 \), being \( \nu \) the peak height. In other words, Eq. (I.4) does not contain any information about the fact we are dealing with peaks.

The goal of this short note is to explain – at a somewhat more fundamental level than done in [6] – why and how one needs to modify the momentum fluid equation for the coupled-fluids of halos and DM to obtain Eqs. (I.5) and (I.6) as advocated in [6]. The paper is organized as follows. In section II we present a derivation of the fluid equation from first principles and deal with the effective force felt by peaks in section III. Finally, section IV contains our conclusions and further comments.

II. FROM THE CHAPMAN-KOLMOGOROV (OR KLIMONTOVICH-DUPREE) EQUATION TO THE FLUID EQUATION FOR HALOS

To understand the difference between halos and DM, let us take a step back and remember where the collisionless Boltzmann equation for DM comes from. The basic conservation law determining the phase space distribution \( f(\vec{x}, \vec{p}, t) \) is the Chapman-Kolmogorov equation

\[ f(\vec{x}, \vec{p}, t + \Delta t) = \int d^3 \Delta x d^3 \Delta p P_{\Delta t} (\Delta \vec{x}, \vec{x} - \Delta \vec{x}; \Delta \vec{p}, \vec{p} - \Delta \vec{p}) f(\vec{x} - \Delta \vec{x}, \vec{p} - \Delta \vec{p}, t), \]  

where

\[ 1 = \int d^3 \Delta x d^3 \Delta p P_{\Delta t} (\Delta \vec{x}, \vec{x}; \Delta \vec{p}, \vec{p}). \]  

Here \( P_{\Delta t} \) is the transition (or conditional) probability that a DM particle is at the point \((\vec{x}, \vec{p})\) at the time \((t + \Delta t)\) given that it was at point \((\vec{x} - \Delta \vec{x}, \vec{p} - \Delta \vec{p})\) at time \(t\). It is obtained by computing all possible microscopic trajectories and averaging the results over the statistical ensemble.

Eq. (II.2) states that all particles end up somewhere (with probability equal to unity), irrespective of the initial phase space position \((\vec{x}, \vec{p})\). Note that the Chapman-Kolmogorov equation does not, in general, give a complete description of the system, since \( P_{\Delta t} \) is unspecified and may depend on auxiliary functions in addition to the distribution function. Thus, the system must be augmented by equations or conditions determining the transition probability.

The Chapman-Kolmogorov equation reduces to the Vlasov equation in a singular limit where the conditional probability is taken to be a delta function along the self-consistent Vlasov trajectory, or characteristic,

\[ P_{\Delta t} \xrightarrow{\text{Vlasov}} \delta_D [\vec{x} - \Delta \vec{x} - \vec{x}(\vec{x}, \vec{p}, t + \Delta t; t)] \delta_D [\vec{p} - \Delta \vec{p} - \vec{p}(\vec{x}, \vec{p}, t + \Delta t; t)]. \]
by the Poisson equation. Using Eqs. (II.3) and (II.1) gives $f(\vec{x}, \vec{p}, t + \Delta t) = f(\vec{x}', \vec{p}', t)$, which is the integral form of the Vlasov equation, expressing the constancy of $f$ along particle orbits. The differential form is recovered by taking $\Delta t$ to zero. This shows that the Chapman-Kolmogorov equation contains the Vlasov equation as a limiting case.

Let us go back to the Chapman-Kolmogorov equation (II.1). If we assume that the phase space distribution $f$ has a characteristic time scale longer than $\Delta t$, we can expand and obtain

$$f(\vec{x}, \vec{p}, t) = \int d^3x d^3p P_{\Delta t}(\Delta \vec{x}, \vec{x}; \Delta \vec{p}, \vec{p}) f(\vec{x}, \vec{p}, t)$$

where the dots stand for higher-order terms in the Taylor expansion. The first term in both sides of Eq. (II.4) cancel by virtue of Eq. (II.2). The integrals over $d^3x d^3p$ can be brought inside the derivatives, so that Eq. (II.4) becomes

$$\frac{\partial f}{\partial t}(\vec{x}, \vec{p}, t) = -\frac{\partial}{\partial \vec{x}} \left[ f(\vec{x}, \vec{p}, t) \int d^3x d^3p P_{\Delta t}(\Delta \vec{x}, \vec{x}; \Delta \vec{p}, \vec{p}) \frac{\Delta \vec{x}}{\Delta t} \right]$$

$$- \frac{\partial}{\partial \vec{p}} \left[ f(\vec{x}, \vec{p}, t) \int d^3x d^3p P_{\Delta t}(\Delta \vec{x}, \vec{x}; \Delta \vec{p}, \vec{p}) \frac{\Delta \vec{p}}{\Delta t} \right] + \cdots$$

where we have defined

$$\langle \frac{\Delta \vec{x}}{\Delta t} \rangle = \int d^3x d^3p P_{\Delta t}(\Delta \vec{x}, \vec{x}; \Delta \vec{p}, \vec{p}) \frac{\Delta \vec{x}}{\Delta t}$$

$$\langle \frac{\Delta \vec{p}}{\Delta t} \rangle = \int d^3x d^3p P_{\Delta t}(\Delta \vec{x}, \vec{x}; \Delta \vec{p}, \vec{p}) \frac{\Delta \vec{p}}{\Delta t}.$$ (II.6)

This derivation highlights the important fact that what enters the Boltzmann equation are the average (in a statistical sense) velocity $\langle \Delta \vec{x}/\Delta t \rangle$ and average force (per unit mass) $\langle \Delta \vec{p}/\Delta t \rangle$. It is precisely through these averages that the difference between the equations for the smooth DM component and the clustered halos arises. Only when the singular probability (II.3) is used to compute the coefficients we do find

$$\langle \frac{\Delta \vec{x}}{\Delta t} \rangle \bigg|_{\Delta t \to 0} = \vec{p},$$

$$\langle \frac{\Delta \vec{p}}{\Delta t} \rangle \bigg|_{\Delta t \to 0} = -\vec{\nabla} \Phi.$$ (II.7)

where $\vec{\nabla} \Phi$ is the deterministic gravitational force. These expressions reflect the fact that, on an object-by-object basis, all the particle species (DM, halos etc.) experience the same force, in agreement with Einstein’s equivalence principle. For generic transition probabilities however, one should compute the average (in a statistical sense) gravitational force (per unit mass) $\langle \vec{\nabla} \Phi \rangle$.

The smooth DM distribution is described by the Vlasov equation, such that the average force $\langle \vec{\nabla} \Phi \rangle$ can be replaced by the Poisson equation

$$\nabla^2 \Phi = \frac{3}{2} H^2 \delta_{\text{dm}}.$$ (II.8)
For peaks however, the average force exerted by the smooth DM component is statistically biased owing to the fact that peaks stream towards (or move apart from) each other more coherently in high (low) density environments. We stress again that individual peaks do not feel any shift in the gravitational force. Only when we correctly interpret the coefficients in the Boltzmann equation as average quantities, e.g. the averaged change of momentum per unit time \( \langle \Delta \vec{p}/\Delta t \rangle \), do we recover the exact result for peaks.

An alternative, and maybe more rigorous, way of getting the same result is the following. Let us start from the single particle phase space density

\[
f_K(\vec{r}, \vec{p}, t) = \sum_i \delta_D [\vec{r} - \vec{r}_i(t)] \delta_D [\vec{p} - \vec{p}_i(t)],
\]

where \( K \) indicate the so-called Klimontovich density [18, 19]. We are following the phase space trajectories of single "particles", in this case DM particles or halos, without any averaging, instead we are considering only one realization of a universe. We have used the cosmic time \( t \) and halos are identified at the points where maxima of the DM density contrast are located and with a smoothed density contrast larger than some \( \nu \sigma_0 \). The Klimontovich density obeys the Klimontovich-Dupree equation

\[
\frac{\partial f_K}{\partial t} + \vec{p} \cdot \frac{\partial f_K}{\partial \vec{r}} - \langle \nabla \Phi \rangle \cdot \frac{\partial f_K}{\partial \vec{p}} + \cdots = 0,
\]

where

\[
\langle \nabla \Phi \rangle = G_N \int d^3r' d^3p' f_K(\vec{r}', \vec{p}', t) \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|},
\]

Exactly because the Klimontovich density follows trajectories of all the single particles, some coarse graining is needed to be able to handle the huge amount of information encoded in this quantity. Macrostates can be identified by averaging in a standard way over a statistical ensemble of microstates with similar phase space density in small volumes containing a sufficient amount of particles.

Denoting this averaging using angle brackets \( \langle \cdots \rangle \), the first of the distribution functions is given by

\[
\langle f_K(\vec{r}, \vec{p}, t) \rangle = \langle \sum_i \delta_D [\vec{r} - \vec{r}_i(t)] \delta_D [\vec{p} - \vec{p}_i(t)] \rangle = f(\vec{r}, \vec{p}, t).
\]

We now ensemble-average the Klimontovich equation (II.10) and obtain the equation

\[
\frac{\partial f}{\partial t} + \vec{p} \cdot \frac{\partial f}{\partial \vec{r}} - \langle \nabla \Phi \rangle \cdot \frac{\partial f}{\partial \vec{p}} + \cdots = 0,
\]

where

\[
\langle \nabla \Phi \rangle = G_N \int d^3r' d^3p' f(\vec{r}', \vec{p}', t) \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3},
\]

and the \( \cdots \) stand for terms arising at higher-order in perturbation theory when correlations introduced by gravitational clustering are taken into account. We will comment on this point later on.

### III. THE EFFECTIVE FORCE FOR PEAKS

In order to understand the change in the force felt by the peaks on average, we simply have to calculate the average force (II.14) subject to the peak constraint, that is, \( \vec{\nabla} \delta_{dm} = \vec{0} \) and the corresponding Hessian is negative (since peaks are identified with local maxima of the DM over density field).

We will restrict ourselves at the linear level in perturbation theory in such a way that, at early times, the DM density \( \delta_{dm} \) and \( \vec{\nabla} \Phi \) are Gaussian variables. We can therefore apply the theorem presented, for instance, in Refs. [20, 21]...
stating that the conditional probability of zero-mean Gaussian variables $Y_A$ and $Y_B$ is itself a Gaussian variable with mean

$$\langle Y_B | Y_A \rangle \equiv \frac{\langle Y_B \otimes Y_A \rangle}{\langle Y_A \otimes Y_A \rangle} Y_A \quad \text{(III.1)}$$

and covariance matrix

$$C(Y_B, Y_A) \equiv \langle Y_B \otimes Y_B \rangle - \frac{\langle Y_B \otimes Y_A \rangle}{\langle Y_A \otimes Y_A \rangle} \langle Y_A \otimes Y_B \rangle. \quad \text{(III.2)}$$

Let us identify $Y_A$ with $\vec{\nabla} \delta_{\text{dm}}$ and $Y_B$ with $\vec{\nabla} \Phi$. Therefore, the mean shift in $\langle \vec{\nabla} \Phi \rangle$ in the vicinity of a peak is given by

$$\langle \vec{\nabla} \Phi | \vec{\nabla} \delta_{\text{dm}} \rangle = \frac{\langle \vec{\nabla} \Phi \vec{\nabla} \delta_{\text{dm}} \rangle}{\langle \vec{\nabla} \delta_{\text{dm}} \rangle^2} \vec{\nabla} \delta_{\text{dm}}$$

$$= -\frac{3}{2} H^2 \frac{\langle \delta_{\text{dm}}^2 \rangle}{\langle \vec{\nabla} \delta_{\text{dm}} \rangle^2} \vec{\nabla} \delta_{\text{dm}}$$

$$= -\frac{3}{2} H^2 \frac{\sigma_{\text{dm}}^2}{\sigma_1^2} \vec{\nabla} \delta_{\text{dm}}$$

$$= -\frac{3}{2} H^2 R_v^2 \vec{\nabla} \delta_{\text{dm}}. \quad \text{(III.3)}$$

We reiterate that, in this expression, we have assumed that both $\vec{\nabla} \Phi$ and $\vec{\nabla} \delta_{\text{dm}}$ are Gaussian-distributed and therefore the result is valid only at the linear level. At higher-order in perturbation theory a modification should be expected when going to smaller distances.

We now proceed to compute the covariance matrix for $\vec{\nabla} \Phi$ given the constraint. First, we provide the various elements

$$\langle \vec{\nabla} \Phi \otimes \vec{\nabla} \Phi \rangle = \frac{\alpha^2}{3} \sigma_{-1}^2 1_{3 \times 3},$$

$$\langle \vec{\nabla} \Phi \otimes \vec{\nabla} \delta_{\text{dm}} \rangle = -\frac{\alpha}{3} \sigma_0^2 1_{3 \times 3},$$

$$\langle \vec{\nabla} \delta_{\text{dm}} \otimes \vec{\nabla} \Phi \rangle = -\frac{\alpha}{3} \sigma_0^2 1_{3 \times 3},$$

$$\langle \vec{\nabla} \delta_{\text{dm}} \otimes \vec{\nabla} \delta_{\text{dm}} \rangle = \frac{1}{3} \sigma_{-1}^2 1_{3 \times 3}. \quad \text{(III.4)}$$

where we have defined $\alpha = 3H^2 \Omega_{\text{dm}}/2$. The covariance matrix for $\vec{\nabla} \Phi$ given the constraint is therefore given by

$$C(\vec{\nabla} \Phi, \vec{\nabla} \Phi) = \frac{\alpha^2}{3} \left( \sigma_{-1}^2 - \frac{\sigma_0^2}{\sigma_1^2} \right), \quad \text{(III.5)}$$

a result first derived in Ref. [20]. Even though at the location of the peak the mean shift of the gravitational force vanishes and, therefore, at the peak-by-peak level there is no extra force, its variance is not simply proportional to $\sigma_{-1}^2$, but it receives a correction when the statistical ensemble average is taken. This effect can be simply captured by replacing the force $\vec{\nabla} \Phi$ experienced by the peaks by

$$\vec{\nabla} \Phi_{\text{eff}} = \vec{\nabla} \Phi + \frac{3}{2} H^2 \Omega_{\text{dm}} R_v^2 \vec{\nabla} \delta_{\text{dm}}, \quad \text{(III.6)}$$
which is precisely the velocity bias relation proposed in [11–13] since the force is proportional to the velocity in linear theory. Indeed, the variance of this shifted quantity has zero mean and variance given by

\[
\left\langle \left( \vec{\nabla} \Phi_{\text{eff}} \right)^2 \right\rangle = \frac{\alpha^2}{3} \left( \sigma_4^2 - \sigma_1^4 \right),
\]  

(III.7)

which coincides with Eq. (III.5). Inserting this result into Eq. (II.13) for halos

\[
\frac{\partial f_h}{\partial t} + \vec{p} \cdot \frac{\partial f_h}{\partial \vec{r}} - \vec{\nabla} \Phi_{\text{eff}} \cdot \frac{\partial f_h}{\partial \vec{p}} + \cdots = 0,
\]

(III.8)

multiplying now Eq. (II.13) by \( \vec{p} \), reinstating the expansion of the universe, integrating over the momenta and applying the gradient on both sides, we precisely get Eq. (I.5).

IV. CONCLUSIONS AND FURTHER COMMENTS

In this short note, we have considered conservation laws in phase space to explain why the gravitational force acting on biased tracers of the large-scale structure is itself biased, as was already noted in Refs. [6, 13]. Consequently, the halo momentum conservation equation is modified accordingly, in agreement with the change advocated in Ref. [6]. This modification is responsible for the time constancy of the halo velocity bias (at the linear level).

We conclude by some remarks. Like in Refs. [6, 11–13], our derivation of the shift in the gravitational force felt by halos has made use of the properties of Gaussian random fields, so it applies at early time and on sufficiently large scales, when linear perturbation theory holds. It would be interesting to understand what happens when non-linearities become important. Along these lines, the work done in Ref. [21] might be useful. Furthermore, our derivation makes also clear that, when dealing with halos, the starting Vlasov-Boltzmann equation is highly non-trivial. Indeed, going to higher-order, not only the average force receives extra contribution, but also the equation (III.8) is modified into [19]

\[
\frac{\partial f_h}{\partial t} + \vec{p} \cdot \frac{\partial f_h}{\partial \vec{r}} - \vec{\nabla} \Phi_{\text{eff}} \cdot \frac{\partial f_h}{\partial \vec{p}} = \vec{\nabla} \vec{F},
\]

(IV.1)

where

\[
\vec{F}(\vec{r}, \vec{p}, t) = \text{Cov} \left[ \vec{\nabla} \Phi_K(\vec{r}, t), f_K(\vec{r}, \vec{p}, t) \right] = G_N \int d^3 r' d^3 p' \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|} f_{2c}(\vec{r}, \vec{p}, \vec{r}', \vec{p}', t),
\]

(IV.2)

and

\[
f_{2c}(\vec{r}, \vec{p}, \vec{r'}, \vec{p'}, t) = f_2(\vec{r}, \vec{p}, \vec{r'}, \vec{p'}, t) - f(\vec{r}, \vec{p}, t) f(\vec{r'}, \vec{p'}, t)
\]

(IV.3)

is the irreducible DM two-particle correlation function. The latter cannot be neglected because it may be much larger than the product of the single-particle terms due to strong gravitational clustering. The vector \( \vec{F} \) represents a correlated force density and arises from fluctuations in the ensemble averaged gravitational potential due to clustering of the matter distribution. This force density must be computed in the presence of the peak constraint, and generally gives rise to drift forces and diffusion in velocity-space. Added up to the fact that an initial constraint on the halo shifts the mean density and causes the density and velocity to be correlated hence, this changes the halo phase space density. Any theory of bias should therefore account of these effects. Work along these lines is in progress [22].

Acknowledgments

The research of A.K. was implemented under the Aristeia Action of the Operational Programme Education and Lifelong Learning and is co-funded by the European Social Fund (ESF) and National Resources. A.K. is also partially supported by European Union’s Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n.
A.R. is supported by the Swiss National Science Foundation (SNSF), project “The non-Gaussian Universe” (project number: 200021140236). M.B. and V.D. acknowledge support by the Swiss National Science Foundation.

[1] N. Kaiser, Astrophys. J. Lett. 284, L9 (1984).
[2] D. Tseliakhovich, C.M. Hirata, Phys. Rev. D 82, 083520 (2010), arXiv:1005.2416 [astro-ph.CO].
[3] J. Yoo, U. N. Dalal, U. Seljak, JCAP 1107, 018 (2011) [arXiv:1105.3732 [astro-ph.CO]].
[4] J.E. Gunn, J.R. Gott III, Astrophys. J. 176, 1 (1972).
[5] W.H. Press, P. Schechter, Astrophys. J. 187, 425 (1974).
[6] T. Baldauf, V. Desjacques and U. Seljak, arXiv:1405.5885 [astro-ph.CO].
[7] E. Jennings, C.M. Baugh, D. Hatt, arXiv:1407.7296 [astro-ph.CO].
[8] W.J. Percival, M. White, Mon. Not. R. Astron. Soc. 393, 297 (2009) arXiv:0808.0003 [astro-ph.CO].
[9] A. Elia, A.D. Ludlow, C. Porciani, Mon. Not. R. Astron. Soc. 421, 3472 (2012) arXiv:1111.4211 [astro-ph.CO].
[10] P. Zhang, Y. Zheng, Y. Jing, arXiv:1405.7125.
[11] V. Desjacques, Phys. Rev. D 78, 103503 (2008) [arXiv:0806.0007 [astro-ph.CO]].
[12] V. Desjacques and R. K. Sheth, Phys. Rev. D 81, 023526 (2010), arXiv:0909.4544 [astro-ph.CO].
[13] V. Desjacques, M. Crocce, R. Scoccimarro, and R. K. Sheth, Phys. Rev. D 82, 103529 (2010), arXiv:1009.3449 [astro-ph.CO].
[14] K. C. Chan, R. Scoccimarro and R. K. Sheth, Phys. Rev. D 85, 083509 (2012) [arXiv:1201.3614 [astro-ph.CO]].
[15] J. N. Fry, Astrophys. J. 461, L65 (1996).
[16] M. Tegmark and P. J. E. Peebles, Astrophys. J. Lett. 500, L79 (1998), astro-ph/9804067.
[17] A. Elia, S. Kulkarni, C. Porciani, M. Pietroni, and S. Matarrese, Mon. Not. R. Astron. Soc. 416, 1703 (2011), arXiv:1012.4833 [astro-ph.CO].
[18] Yu. L. Klimontovich, The Statistical Theory of Non-Equilibrium Processes in a Plasma (MIT Press, Cambridge, 1967).
[19] E. Bertschinger, in Cosmology and Large Scale Structure, proceedings of the 1996 Les Houches Summer School, Session LX, ed. R. Schaeffer, J. Silk, M. Spiro, and J. Zinn-Justin (Amsterdam: Elsevier Science), 273.
[20] J. M. Bardeen, J. R. Bond, N. Kaiser and A. S. Szalay, Astrophys. J. 304 (1986) 15.
[21] C. -P. Ma and E. Bertschinger, Astrophys. J. 612, 28 (2004) [astro-ph/0311049].
[22] M. Biagetti, V. Desjacques, A. Kehagias and A. Riotto, to appear.