Quantum state and circuit distinguishability with single-qubit measurements

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Abstract

We show that the Quantum State Distinguishability (QSD), which is a QSZK-complete problem, and the Quantum Circuit Distinguishability (QCD), which is a QIP-complete problem, can be solved by the verifier who can perform only single-qubit measurements. To show these results, we use measurement-based quantum computing: the honest prover sends a graph state to the verifier, and the verifier can perform universal quantum computing on it with only single-qubit measurements. If the prover is malicious, he does not necessarily generate the correct graph state, but the verifier can verify the correctness of the graph state by measuring the stabilizer operators.

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I. INTRODUCTION

Measurement-based quantum computing [1] is a new model of quantum computing where universal quantum computing can be realized with only adaptive single-qubit measurements on a certain entangled state such as the graph state. Several applications of measurement-based quantum computing in quantum computational complexity theory have been obtained recently. For example, Ref. [2] used measurement-based quantum computing to construct a multiprover interactive proof system for BQP with a classical verifier. Furthermore, Refs. [3, 4] used measurement-based quantum computing to show that the verifier needs only single-qubit measurements in QMA and QAM. The basic idea in these results is the verification of the graph state: prover(s) generate the graph state, and the verifier performs measurement-based quantum computing on it. By checking the stabilizer operators, the verifier can verify the correctness of the graph state. The idea of testing stabilizer operators was also used in Refs. [5, 6] to construct multiprover interactive proof systems for local Hamiltonian problems.

In this paper, we consider two promise problems, Quantum State Distinguishability (QSD) [7], which is QSZK-complete, and Quantum Circuit Distinguishability (QCD) [8], which is QIP-complete. By using the idea of testing stabilizer operators, we show that these problems can be solved by the verifier who can do only single-qubit measurements. Proofs are similar to those of Refs. [3, 4] for QMA and QAM, but several new considerations are required since in protocols to solve QSD and QCD some parts of graph states are kept by the prover.

A. QSD

Definition: Quantum State Distinguishability (QSD$_{\alpha,\beta}$) [7].

- Input: Quantum circuits $Q_0$ and $Q_1$ each acting on $m$ qubits and having $k$ specified output qubits.

- Promise: Let $\rho_a$ ($a \in \{0, 1\}$) be the mixed state obtained by tracing out the non-output qubits of $Q_a|0^m\rangle$. We have either $\frac{1}{2}\|\rho_0 - \rho_1\|_1 \geq \beta$ or $\frac{1}{2}\|\rho_0 - \rho_1\|_1 \leq \alpha$.

- Output: Accept if $\frac{1}{2}\|\rho_0 - \rho_1\|_1 \geq \beta$, reject if $\frac{1}{2}\|\rho_0 - \rho_1\|_1 \leq \alpha$. 
Here, $\|X\|_1 = \text{Tr}\sqrt{X^\dagger X}$ is the trace norm. It was shown in Ref. [7] that if $0 \leq \alpha < \beta^2 \leq 1$, the gap between $\alpha$ and $\beta$ can be amplified to $\alpha = 2^{-r}$ and $\beta = 1 - 2^{-r}$ for any polynomial $r$. Therefore, in this paper, without loss of generality, we take $\alpha = 2^{-r+1}$ and $\beta = 1 - 2^{-r+1}$ for any polynomial $r$.

The problem is a quantum version of the SZK-complete problem, Statistical Difference [9]. The problem $\text{QSD}_{\alpha,\beta}$ and its complement are QSZK-complete for any constants $\alpha$ and $\beta$ satisfying $0 < \alpha < \beta^2 < 1$ [7]. In fact, as is shown in Ref. [7], the prover can prove that two states $\rho_0$ and $\rho_1$ are far apart in the following way.

1. The verifier uniformly randomly chooses $a \in \{0, 1\}$, and sends $\rho_a$ to the prover.

2. The prover performs any measurement to distinguish $\rho_0$ and $\rho_1$, and sends the result $a' \in \{0, 1\}$ to the verifier.

3. The verifier accepts if and only if $a = a'$.

Let $\{\Pi_0, \Pi_1\}$ be the POVM performed by the prover. Then, the probability that the verifier accepts is

\[
\begin{align*}
p_{\text{acc}} &= \frac{1}{2} \text{Tr}(\Pi_0 \rho_0) + \frac{1}{2} \text{Tr}(\Pi_1 \rho_1) \\
&= \frac{1}{2} \text{Tr}(\Pi_0 \rho_0) + \frac{1}{2} \text{Tr}((I - \Pi_0) \rho_1) \\
&= \frac{1}{2} \frac{1}{2} \text{Tr}(\Pi_0 \rho_0) - \frac{1}{2} \text{Tr}(\Pi_0 \rho_1) \\
&= \frac{1}{2} \frac{1}{2} \text{Tr}(\Pi_0 (\rho_0 - \rho_1)) \\
&\leq \frac{1}{2} + \frac{1}{4} \|\rho_0 - \rho_1\|_1.
\end{align*}
\]

Therefore, for the YES case, by taking the optimal POVM,

\[
p_{\text{acc}} = \frac{1}{2} + \frac{1}{4} \|\rho_0 - \rho_1\|_1 \\
\geq \frac{1}{2} + \frac{1}{2} (1 - 2^{-r+1}) \\
= 1 - 2^{-r},
\]

and for the NO case, for any POVM,

\[
p_{\text{acc}} \leq \frac{1}{2} + \frac{1}{2} 2^{-r+1} \\
= \frac{1}{2} + 2^{-r}.
\]
The first result of the present paper is that QSD can be solved with the verifier who can do only single-qubit measurements. The idea is that the honest prover generates the graph state and sends a part of it to the verifier. The verifier can remotely generates $\rho_0$ or $\rho_1$ in the prover’s place by measuring his part. The verifier can also check that his part is the correct graph state by measuring stabilizer operators. A trade-off is that, as is shown in Fig. 1 in the above protocol, one polynomial-size quantum message from the verifier to the prover and one single-bit classical message from the prover to the verifier are enough, whereas in our protocol, one polynomial-size quantum message from the prover to the verifier, one polynomial-size classical message from the verifier to the prover, and a single-bit classical message from the prover to the verifier are necessary.

(a)

Prover

(1) poly qubits

Verifier

(2) single bit

(Universal QC)

(b)

Prover

(1) poly qubits

Verifier

(2) poly bits

(3) single bit

(Single-qubit measurements)

FIG. 1: (a) The protocol of Ref. [7]. The verifier is quantum universal. (b) Our protocol for QSD. The verifier does only single-qubit measurements.
B. QCD

Definition: Quantum Circuit Distinguishability (QCD\(_{a,b}\)) \[8\].

- Input: mixed-state quantum circuits, \(Q_0\) and \(Q_1\), both of \(n\)-qubit input \(m\)-qubit output.
- Yes: \(|\|Q_0 - Q_1\||_\diamond \geq a\).
- No: \(|\|Q_0 - Q_1\||_\diamond \leq b\).

Here, \(|\|Q_0 - Q_1\||_\diamond \equiv \max_{X: \|X\|_1 = 1} \| (Q_0 \otimes I^\otimes n)(X) - (Q_1 \otimes I^\otimes n)(X) \|_1\) is the diamond norm. It was shown in Ref. \[8\] that QCD\(_{2-\delta,\delta}\) is QIP-complete for any \(\delta > 0\).

In fact, the prover can proof that \(Q_0\) and \(Q_1\) are far apart in the diamond norm as follows. As is shown in Ref. \[8\], there is a state \(|\psi\rangle\) such that

\[|\|Q_0 - Q_1\||_\diamond = \left\| (Q_0 \otimes I^\otimes s)(|\psi\rangle\langle\psi|) - (Q_1 \otimes I^\otimes s)(|\psi\rangle\langle\psi|) \right\|_1\]

For the YES case, the prover sends a part of \(|\psi\rangle\) to the verifier. The verifier uniformly randomly chooses \(i \in \{0, 1\}\) and applies \(Q_i\) on the part, and returns the state to the prover. The prover now has \((Q_i \otimes I)(|\psi\rangle\langle\psi|)\), and therefore he can learn \(i\) by doing a measurement on the state with the probability \(\frac{1}{2} + \frac{1}{4}|\|Q_0 - Q_1\||_\diamond \geq \frac{1}{2} + \frac{a}{4}\). For the NO case, whatever state the prover sends to the verifier, the acceptance probability is less than \(\frac{1}{2} + \frac{1}{4}|\|Q_0 - Q_1\||_\diamond \leq \frac{1}{2} + \frac{b}{4}\).

Our second result is that QCD can be solved by the verifier who can perform only single-qubit measurements. As is shown in Fig. 2, our protocol has an advantage that the second quantum message from the verifier to the prover can be replaced with the classical message, as well as the fact that the verifier needs only single-qubit measurements.

Let us define the class QIP\(_{\text{single}}\) that is equivalent to QIP except that the verifier can perform only single-qubit measurements. Since quantum computing with measurements can be simulated by a unitary quantum computing, it is obvious that QIP\(_{\text{single}}\) \(\subseteq\) QIP. On the other hand, our protocol that solves QCD is obviously in QIP\(_{\text{single}}\), and therefore our result means QIP \(\subseteq\) QIP\(_{\text{single}}\). Hence, we have the result that QIP = QIP\(_{\text{single}}\). The result QMA\(_{\text{single}}\) = QMA was shown in Ref. \[3\], and the result QAM\(_{\text{single}}\) = QAM was shown in Ref. \[4\]. It was a remaining open problem whether QIP\(_{\text{single}}\) = QIP. The present paper solves it.
II. MEASUREMENT-BASED QUANTUM COMPUTING

For readers who are not familiar with measurement-based quantum computing [1], we here explain basics of it. Let us consider a graph $G = (V, E)$, where $|V| = N$. The graph state $|G\rangle$ on $G$ is defined by

$$
|G\rangle \equiv \left( \prod_{(i,j) \in E} CZ_{i,j} \right) |+\rangle^\otimes N,
$$

where $|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$ and $CZ_{i,j} \equiv |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z$ is the CZ gate on the vertices $i$ and $j$.

According to the theory of measurement-based quantum computing [1], for any $m$-width $d$-depth quantum circuit $U$, there exists a graph $G = (V, E)$ with $|V| = N = \text{poly}(m, d)$ and the graph state $|G\rangle$ on it such that if we measure each qubit in $V - V_o$, where $V_o$ is a
certain subset of $V$ with $|V_o| = m$, in certain bases adaptively, then the state of $V_o$ after the measurements is

$$B^m_{x,z} U |0^m\rangle$$

with uniformly randomly chosen $x \equiv (x_1, \ldots, x_m) \in \{0, 1\}^m$ and $z \equiv (z_1, \ldots, z_m) \in \{0, 1\}^m$, where

$$B^m_{x,z} = \bigotimes_{j=1}^m X_j^{x_j} Z_j^{z_j}.$$  

This operator is called a byproduct operator, and its effect is corrected, since $x$ and $z$ can be calculated from measurement results. Hence we finally obtain the desired state $U |0^m\rangle$.

The graph state $|G\rangle$ is stabilized by

$$g_j \equiv X_j \bigotimes_{i \in S_j} Z_i,$$  

for all $j \in V$, where $S_j$ is the set of nearest-neighbour vertices of $j$th vertex. In other words,

$$g_j |G\rangle = |G\rangle$$

for all $j \in V$.

For $u \equiv (u_1, \ldots, u_N) \in \{0, 1\}^N$, we define the state $|G_u\rangle$ by

$$g_j |G_u\rangle = (-1)^{u_j} |G_u\rangle$$

for all $j \in V$. (Therefore, $|G\rangle = |G_{0^N}\rangle$.) The set $\{ |G_u\rangle \}_u$ is an orthonormal basis of the $N$-qubit Hilbert space. In fact, if $u \neq u'$, there exists $j$ such that $u_j \neq u'_j$. Then,

$$\langle G_{u'} | G_u \rangle = \langle G_{u'} | g_j g_j | G_u \rangle$$

$$= (-1)^{u_j + u'_j} \langle G_{u'} | G_u \rangle$$

$$= -\langle G_{u'} | G_u \rangle,$$

and therefore $\langle G_{u'} | G_u \rangle = 0$.

III. STABILIZER TEST

We now explain the stabilizer test. (See also Refs. [3, 4, 10].) Consider the graph $G = (V, E)$ of Fig. 3. (For simplicity, we here consider the square lattice, but the result can
be applied to any reasonable graph.) As is shown in Fig. 3, we define two subsets, $V_1$ and $V_2 \equiv V - V_1$, of $V$, where $|V_1| = N_1$ and $|V_2| = N_2$. We also define a subset $V_{\text{connect}}$ of $V_2$ by

$$V_{\text{connect}} \equiv \{ j \in V_2 | \exists i \in V_1 \text{ s.t. } (i, j) \in E \}.$$ 

In other words, $V_{\text{connect}}$ is the set of vertices in $V_2$ that are connected to vertices in $V_1$. We further define two subsets of $E$:

$$E_1 \equiv \{(i, j) \in E | i \in V_1 \text{ and } j \in V_1\},$$

$$E_{\text{connect}} \equiv \{(i, j) \in E | i \in V_1 \text{ and } j \in V_2\}.$$ 

Finally, we define two subgraphs of $G$:

$$G' \equiv (V_1 \cup V_{\text{connect}}, E_1 \cup E_{\text{connect}}),$$

$$G'' \equiv (V_1, E_1).$$

![FIG. 3: (a) The graph $G$. $V_1$ is the set of vertices in the dotted red square, and $V_2$ is the set of other vertices. (b) The subgraph $G'$. (c) The subgraph $G''$.](image)

The stabilizer test is the following test:

1. Randomly generate an $N_1$-bit string $k \equiv (k_1, ..., k_{N_1}) \in \{0, 1\}^{N_1}$.

2. Measure the operator

$$s_k \equiv \prod_{j \in V_1} (g'_j)^{k_j},$$

where $g'_j$ is the stabilizer operator, Eq. 1, of the graph state $|G'\rangle$. 

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3. If the result is $+1$ ($-1$), the test passes (fails).

Let $|\Psi\rangle$ be a pure state on $V$. If the probability $p_{\text{pass}}$ that $|\Psi\rangle$ passes the stabilizer test satisfies $p_{\text{pass}} \geq 1 - \epsilon$, then

$$\frac{1}{2} \left\| |\Psi\rangle\langle\Psi| - |\Psi'\rangle\langle\Psi'| \right\|_1 \leq \sqrt{4\epsilon - 4\epsilon^2},$$

(2)

where

$$|\Psi'\rangle \equiv W(|\Gamma''\rangle \otimes |\xi\rangle_{V_2}).$$

Here, $|\xi\rangle$ is a certain state on $V_2$ and

$$W \equiv \prod_{(i,j) \in E_{\text{connect}}} CZ_{i,j}.$$

The proof is given as follows. The probability $p_{\text{test}}$ that the state $|\Psi\rangle$ on $V$ passes the stabilizer test is

$$p_{\text{test}} = \frac{1}{2^{N_1}} \sum_{k \in \{0,1\}^{N_1}} \langle \Psi | \frac{I + s_k}{2} | \Psi \rangle.$$

If we use the relation

$$\prod_{j \in V_1} \frac{I + g_j'}{2} = \frac{1}{2^{N_1}} \sum_{k \in \{0,1\}^{N_1}} s_k,$$

the condition $p_{\text{test}} \geq 1 - \epsilon$ means

$$\langle \Psi | \prod_{j \in V_1} \frac{I + g_j'}{2} | \Psi \rangle \geq 1 - 2\epsilon.$$

(3)

Let $\{|\phi_t\rangle\}_t$ be an orthonormal basis of $N_2$-qubit Hilbert space, where $t \in \{0,1\}^{N_2}$. Then, $\{W|\Gamma''\rangle \otimes |\phi_t\rangle\}_{u,t}$ is an orthonormal basis of the $N$-qubit Hilbert space, and therefore, $|\Psi\rangle$ can be written as

$$|\Psi\rangle = \sum_{u,t} C_{u,t} W|\Gamma''\rangle \otimes |\phi_t\rangle,$$

for certain coefficients $\{C_{u,t}\}_{u,t}$. Let us define

$$|\Psi'\rangle \equiv W|\Gamma''\rangle \otimes \left( \frac{1}{\sqrt{R}} \sum_t C_{0^{N_1},t} |\phi_t\rangle \right),$$
where

\[ R \equiv \sum_t |C_{0N_1,t}|^2 \leq 1 \]

is the normalization constant.

Let \( \{g''_j\}_j \) be the set of stabilizer operators of the graph state \( |G''\rangle \). Then, it is easy to check

\[ g'_j W = W g''_j \]

for all \( j \in V_1 \). Therefore,

\[
\left( \prod_{j \in V_1} \frac{I + g'_j}{2} \right) |\Psi\rangle = W \left( \prod_{j \in V_1} \frac{I + g''_j}{2} \left( \sum_{u,t} C_{u,t} |G''_u\rangle \otimes |\phi_t\rangle \right) \right) = W \left( \sum_t C_{0N_1,t} |G''\rangle \otimes |\phi_t\rangle \right) = \sqrt{R} |\Psi\rangle.
\]

Hence Eq. (3) means

\[ 1 - 2\epsilon \leq \sqrt{R} \langle \Psi | \Psi' \rangle \]

\[ \leq \langle \Psi | \Psi' \rangle. \]

Therefore,

\[
\frac{1}{2} \left\| |\Psi\rangle \langle \Psi| - |\Psi'\rangle \langle \Psi'| \right\|_1 = \sqrt{1 - |\langle \Psi | \Psi' \rangle|^2} \leq \sqrt{1 - (1 - 2\epsilon)^2} = \sqrt{4\epsilon - 4\epsilon^2}.
\]

IV. QSD

In this section, we explain our protocol for QSD. Let us consider the graph \( G = (V, E) \) of Fig. 4. Our protocol runs as follows:

1. The prover generates a state \( |\Psi\rangle \) on \( V \), and sends all black qubits to the verifier. If the prover is honest, \( |\Psi\rangle \equiv |G\rangle \). If the prover is malicious, \( |\Psi\rangle \) can be any state.

2. With probability \( q \), which is specified later, the verifier does the following.
2-a The verifier uniformly randomly chooses \( a \in \{0, 1\} \).

2-b The verifier performs the measurement-based quantum computing on the received qubits so that the state of qubits in the blue dotted box becomes \( B^m_{x,z} Q_a |0^m\rangle \), and the reduced state of the qubits in the red dotted box becomes \( B^k_{x,z} \rho_a B^k_{x,z} \).

2-c The verifier sends the prover \((x_1, \ldots, x_k)\) and \((z_1, \ldots, z_k)\).

2-d The verifier measures qubits in the red dotted box and the black star qubits in the \( X \) basis (in order to teleport the state to the white qubits that are connected to the star qubits), and sends the \( X \)-basis measurement results to the prover.

2-e The verifier receives the answer bit \( a' \in \{0, 1\} \) from the prover.

2-f The verifier accepts if and only if \( a = a' \).

We denote the acceptance probability by \( p_{\text{comp}} \).

3. With probability \( 1 - q \), the verifier does the stabilizer test by considering \( V_1 \) as the set of black circle qubits. The verifier accepts if and only if the stabilizer test passes. We denote the acceptance probability by \( p_{\text{test}} \).

![FIG. 4: The graph \( G \) for our protocol solving QSD.](image)

First, let us consider the YES case, i.e., \( \frac{1}{2} \| \rho_0 - \rho_1 \|_1 \geq 1 - 2^{-r+1} \). In this case, the prover is honest, and therefore \( |\Psi\rangle = |G\rangle \), which means \( p_{\text{test}} = 1 \) if the verifier chooses the stabilizer test. If the verifier chooses the computation, after the all verifier’s measurements, the state of the white qubits that are connected to the star qubits becomes \( B^k_{x', z'} \rho_a B^k_{x', z'} \), where \( x' \) and \( z' \) can be calculated from the all classical information from the verifier. Therefore, the
prover finally has $\rho_a$, and the prover can learn $a$ by doing an appropriate POVM with an error probability less than $2^{-r}$. Hence the acceptance probability $p_{\text{acc}}$ of the protocol is

\[
p_{\text{acc}} = q p_{\text{comp}} + (1 - q) p_{\text{test}} \geq q(1 - 2^{-r}) + (1 - q) \equiv \alpha.
\]

Second, let us consider the NO case, namely, $\frac{1}{2} \|\rho_0 - \rho_1\|_1 \leq 2^{-r} + 1$. If $p_{\text{pass}} < 1 - \epsilon$, where $\epsilon$ is a certain parameter that will be specified later, there is no guarantee that the prover generated the correct graph state. Therefore, $p_{\text{comp}} = 1$ in the worst case:

\[
p_{\text{acc}} = q p_{\text{comp}} + (1 - q) p_{\text{test}} \leq q + (1 - q)(1 - \epsilon) \equiv \beta_1.
\]

If $p_{\text{pass}} \geq 1 - \epsilon$, on the other hand, $|\Psi\rangle$ is close to

\[|\Psi\rangle \equiv W(|G''\rangle \otimes |\xi\rangle)\]

in the sense of Eq. (2), where $|\xi\rangle$ is a state on the star and white qubits, and $W$ is the unitary operator that applies CZ gates on all edges that connect the qubits in the dotted red box and star qubits. For simplicity, let us assume that $|\Psi\rangle = |\Psi'\rangle$ for the moment. Then, after the step 2-c of the protocol, the state of qubits in the red dotted box, star qubits, white qubits, and prover’s classical memory is

\[
W(B^k_{x,z} \rho_a B^k_{x,z} \otimes |\xi\rangle \langle \xi|) W^\dagger \otimes |x, z\rangle \langle x, z|.
\]

However, since

\[
\frac{1}{2} \|W(B^k_{x,z} \rho_0 B^k_{x,z} \otimes |\xi\rangle \langle \xi|) W^\dagger \otimes |x, z\rangle \langle x, z| - W(B^k_{x,z} \rho_1 B^k_{x,z} \otimes |\xi\rangle \langle \xi|) W^\dagger \otimes |x, z\rangle \langle x, z|\|_1
\]

\[
= \frac{1}{2} \|\rho_0 - \rho_1\|_1 \leq 2^{-r+1},
\]

no POVM can distinguish $\rho_0$ and $\rho_1$ with a probability larger than $\frac{1}{2} + 2^{-r}$. Therefore, for any $|\Psi\rangle$ that satisfies $p_{\text{test}} \geq 1 - \epsilon$, the acceptance probability is

\[
p_{\text{acc}} = q p_{\text{comp}} + (1 - q) p_{\text{test}} \leq q \left( \frac{1}{2} + 2^{-r} + \sqrt{4 \epsilon - 4 \epsilon^2} \right) + (1 - q) \equiv \beta_2.
\]
If we define

\[
\Delta_1(q) \equiv \alpha - \beta_1 = -q2^{-r} + \epsilon(1 - q), \\
\Delta_2(q) \equiv \alpha - \beta_2 = \frac{q}{2} - q2^{-r+1} - q\sqrt{4\epsilon - 4\epsilon^2},
\]

then the optimal value \( q^* \) of \( q \), which satisfies \( \Delta_1(q^*) = \Delta_2(q^*) \), is

\[
q^* = \frac{\epsilon}{\epsilon + \frac{1}{2} - 2^{-r} - \sqrt{4\epsilon - 4\epsilon^2}}
\]

and the gap for this \( q^* \) is

\[
\Delta_2(q^*) = \frac{\epsilon(\frac{1}{2} - 2^{-r+1} - \sqrt{4\epsilon - 4\epsilon^2})}{\epsilon + \frac{1}{2} - 2^{-r} - \sqrt{4\epsilon - 4\epsilon^2}} \\
\geq \frac{\epsilon(\frac{1}{2} - \frac{1}{4} - \sqrt{4\epsilon})}{\epsilon + \frac{1}{2}} \\
= \frac{1}{2} - \frac{1}{4} - \frac{1}{5} \\
= \frac{1}{1020}
\]

if \( r \geq 3 \) and \( \epsilon = \frac{1}{100} \).

**V. QCD**

In this section, we explain our protocol for QCD. Let us consider the graph \( G = (V, E) \) of Fig. 5. Our protocol runs as follows:

1. The prover generates a state \(|\Psi\rangle\) on \( V \) and sends all black qubits to the verifier. If the prover is honest,

\[
|\Psi\rangle = W_1(|G_1\rangle \otimes |\psi\rangle),
\]

where \( G_1 \) is the subgraph of \( G \) that is obtained by removing all square vertices and all edges that connect the black square vertices and black circle vertices, \(|\psi\rangle\) is the state of the square qubits (black square qubits are those on which \( Q_i \) should be acted), and \( W_1 \) is the unitary operator applying \( CZ \) gates on all edges that connect the black square qubits and black circle qubits. If the prover is malicious, \(|\Psi\rangle\) can be any state.

2. With probability \( q \), which is specified later, the verifier does the following.
2-a The verifier uniformly randomly chooses \( i \in \{0, 1\} \).

2-b The verifier does the measurement-based quantum computation so that the black circle qubits in the dotted red box and white square qubits becomes

\[
(B_{x,z}^m \otimes I)[(Q_i \otimes I)(|\psi\rangle\langle\psi|)](B_{x,z}^m \otimes I). 
\]

2-c The verifier sends \( x \) and \( z \) to the prover.

2-d The verifier measures the black circle qubits in the red dotted box and black star qubits in the X basis, and sends the measurement results to the prover.

2-e The verifier receives \( j \in \{0, 1\} \) from the prover. The verifier accepts if and only if \( i = j \). We denote the acceptance probability by \( p_{\text{comp}} \).

3. With probability \( 1 - q \), the verifier does the stabilizer test by considering \( V_1 \) as the set of black circle qubits. The verifier accepts if and only if the test passes. We denote the acceptance probability by \( p_{\text{test}} \).

First, let us consider the YES case, i.e., \( \|Q_0 - Q_1\|_\diamond \geq a \). In this case, the prover is honest, and therefore, \( p_{\text{test}} = 1 \) and

\[
p_{\text{comp}} = \frac{1}{2} + \frac{1}{4}\|Q_0 - Q_1\|_\diamond \\
\geq \frac{1}{2} + \frac{a}{4}. 
\]
Therefore,
\[ p_{\text{acc}} = q p_{\text{comp}} + (1 - q) p_{\text{test}} \geq q \left( \frac{1}{2} + \frac{a}{4} \right) + (1 - q) \equiv \alpha. \]

Next let us consider the NO case, i.e., \( \| Q_0 - Q_1 \|_\diamond \leq b \). If \( p_{\text{test}} < 1 - \epsilon \),
\[ p_{\text{acc}} = q p_{\text{comp}} + (1 - q) p_{\text{test}} \leq q + (1 - q)(1 - \epsilon) \equiv \beta_1. \]

If \( p_{\text{test}} \geq 1 - \epsilon \), on the other hand, \( |\Psi\rangle \) is close to
\[ |\Psi'\rangle = W(|G'' \rangle \otimes |\xi\rangle) \]
in the sense of Eq. (2). Here, \( G'' \) is the graph whose vertices are black circle qubits and whose edges are those connecting black circle qubits. The operator \( W \) is the unitary operator applying \( CZ \) gates on all edges that connect black circle qubits and the black star or black square qubits. The state \( |\xi\rangle \) is the state of the black star qubits, black square qubits, and white qubits. For the moment, let us assume that \( |\Psi\rangle = |\Psi'\rangle \). After the step 2-c, the state of white qubits, star qubits, black circle qubits in the red dotted box, and prover's classical memory is
\[ [W_2(B_{x,z}^m \otimes I)(Q_i \otimes I)(|\xi\rangle \langle \xi|)(B_{x,z}^m \otimes I)W_2^\dagger] \otimes |x, z\rangle\langle x, z|, \]
where \( W_2 \) is the unitary operator applying \( CZ \) gates on all edges that connects black circle qubits in the red dotted box and star qubits. However,
\[ \left\| [W_2(B_{x,z}^m \otimes I)(Q_0 \otimes I)(|\xi\rangle \langle \xi|)(B_{x,z}^m \otimes I)W_2^\dagger] \otimes |x, z\rangle\langle x, z| - [W_2(B_{x,z}^m \otimes I)(Q_1 \otimes I)(|\xi\rangle \langle \xi|)(B_{x,z}^m \otimes I)W_2^\dagger] \otimes |x, z\rangle\langle x, z| \right\|_1 \]
\[ = \left\| (Q_0 \otimes I)(|\xi\rangle \langle \xi|) - (Q_1 \otimes I)(|\xi\rangle \langle \xi|) \right\|_1 \]
\[ \leq \| Q_0 - Q_1 \|_\diamond, \]
and therefore, \( p_{\text{comp}} \leq \frac{1}{2} + \frac{b}{4} \). Hence for any \( |\Psi\rangle \) such that \( p_{\text{test}} \geq 1 - \epsilon \), the acceptance probability is
\[ p_{\text{acc}} = q p_{\text{comp}} + (1 - q) p_{\text{test}} \leq q \left( \frac{1}{2} + \frac{b}{4} + \sqrt{4\epsilon - 4\epsilon^2} \right) + (1 - q) \equiv \beta_2. \]
If we define
\[
\Delta_1(q) \equiv \alpha - \beta_1 = -\frac{q}{2} + \frac{qa}{4} + \epsilon(1 - q),
\]
\[
\Delta_2(q) \equiv \alpha - \beta_2 = \frac{q(a - b)}{4} - q\sqrt{4\epsilon - 4\epsilon^2},
\]
the optimal value \( q^* \) of \( q \) is
\[
q^* = \frac{\epsilon}{\frac{1}{2} + \epsilon - \sqrt{4\epsilon - 4\epsilon^2} - \frac{b}{4}},
\]
and the gap is
\[
\Delta_2(q^*) = \frac{\epsilon(\frac{a-b}{4} - \sqrt{4\epsilon - 4\epsilon^2})}{\frac{1}{2} + \epsilon - \sqrt{4\epsilon - 4\epsilon^2} - \frac{b}{4}} \\
\geq \frac{\epsilon(\frac{a-b}{4} - 2\sqrt{\epsilon})}{\frac{1}{2} + \epsilon} \\
= \frac{1}{1020}
\]
if we take \( \epsilon = \frac{1}{100}, \ a = 1.5, \) and \( b = 0.5. \) Note that the error can be reduced by running the protocol in parallel, and using the Markov inequality argument \[11\].

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[1] R. Raussendorf and H. J. Briegel, A one-way quantum computer. Phys. Rev. Lett. 86, 5188 (2001).
[2] M. McKague, Interactive proofs for BQP via self-tested graph states. arXiv:1309.5675
[3] T. Morimae, D. Nagaj, and N. Schuch, Quantum proofs can be verified using only single qubit measurements. Phys. Rev. A 93, 022326 (2016).
[4] T. Morimae, Quantum Arthur-Merlin with single-qubit measurements. Phys. Rev. A 93, 062333 (2016).
[5] J. F. Fitzsimons and T. Vidick, A multiprover interactive proof system for the local Hamiltonian problem. arXiv:1409.0260

[6] Z. Ji, Classical verification of quantum proofs. arXiv:1505.07432

[7] J. Watrous, Limits on the power of quantum statistical zero-knowledge. Proceedings of the 43rd Annual IEEE Symposium on Foundations of Computer Science, pp.459-468 (2002).

[8] B. Rosgen and J. Watrous, On the hardness of distinguishing mixed-state quantum computations. Proceedings of the 20th Annual IEEE Conference on Computational Complexity, pp.344-354 (2005).

[9] A. Sahai and S. Vadhan, A complete promise problem for statistical zero-knowledge. In Proceedings of the 38th Annual IEEE Symposium on the Foundations of Computer Science, pp.448-457 (1997).

[10] M. Hayashi and T. Morimae, Verifiable measurement-only blind quantum computing with stabilizer testing. Phys. Rev. Lett. 115, 220502 (2015).

[11] R. Jain, S. Upadhyay, and J. Watrous, Two-message quantum interactive proofs are in PSPACE. arXiv:0905.1300