THE CAUCHY PROBLEM FOR THE NONLINEAR DAMPED WAVE EQUATION WITH SLOWLY DECAYING DATA

MASAHIRO IKEDA, TAKAHISA INUI, AND YUTA WAKASUGI

ABSTRACT. We study the Cauchy problem for the nonlinear damped wave equation and establish the large data local well-posedness and small data global well-posedness with slowly decaying initial data. We also prove that the asymptotic profile of the global solution is given by a solution of the corresponding parabolic problem, which shows that the solution of the damped wave equation has the diffusion phenomena. Moreover, we show blow-up of solution and give the estimate of the lifespan for a subcritical nonlinearity. In particular, we determine the critical exponent for any space dimension.

CONTENTS

1. Introduction 1
   1.1. Main results 3
   1.2. Notations 6
2. Local and global existence of solutions
   2.1. Preliminary estimates 8
   2.2. Proof of Theorems 1.1 and 1.2 in higher dimensional cases 12
   2.3. Proof of Theorems 1.1 and 1.2 in the one-dimensional case 21
3. Asymptotic behavior of the global solution
   3.1. Approximation by an inhomogeneous heat equation 23
   3.2. Preliminary estimates 24
   3.3. Proof of Proposition 3.1 25
   3.4. Proof of Theorem 1.3 28
4. Blow-up and estimates of the lifespan
   4.1. Estimates of the lifespan from below 34
   4.2. Estimates of the lifespan from above 35
Appendix A. Some auxiliary estimates 38
Acknowledgments 40
References 41

1. INTRODUCTION

In this paper, we study the Cauchy problem for the nonlinear damped wave equation

\[
\begin{aligned}
&\partial^2_t u - \Delta u + \partial_t u = \mathcal{N}(u), \\
&u(0, x) = \varepsilon u_0(x), \quad \partial_t u(0, x) = \varepsilon u_1(x), \\
&\quad (t, x) \in [0, \infty) \times \mathbb{R}^n, \\
&\quad x \in \mathbb{R}^n,
\end{aligned}
\]

(1.1)

where \( n \in \mathbb{N} \) and \( u \) is a real-valued unknown function, \( \mathcal{N}(u) \) is a power type nonlinearity, \((u_0, u_1)\) are given data, and \( \varepsilon > 0 \) is a positive parameter, which describes the amplitude of the initial data.

Our purpose is to establish the large data local well-posedness and the small data global well-posedness for the Cauchy problem (1.1) with slowly decaying initial data, that is, we treat the
initial data not belonging to $L^1(\mathbb{R}^n)$ in general. Moreover, we investigate the asymptotic behavior of the global solution and the estimate of the lifespan from both above and below for subcritical nonlinearities.

The equation (1.1) is firstly derived by Oliver Heaviside as the telegrapher’s equation, which describes the current and voltage in an electrical circuit with resistance and inductance. Cattaneo [11] also introduced the equation (1.1) as a modified heat conduction equation which equips the finite propagation speed property. The equation (1.1) also has several background related to biology and stochastic models such as genetics, population dynamics [3, 7] and correlated random walk [6, 29].

The local and global well-posedness, asymptotic behavior of global-in-time solutions and blow-up of local-in-time solutions have been widely studied for a long time. Since a pioneer work by Matsumura [38], it has been well known that solutions of the damped wave equation behaves like that of the heat equation as time tends to infinity. Namely, he established $L^p$-$L^q$ estimates of the linear damped wave equation (Eq. (1.1) with $\mathcal{N}(u) = 0$), whose decay rates are the same as those of the linear heat equation $v_t - \Delta v = 0$ (see also Racke [57] for more general setting). After that, the so-called diffusion phenomena was found by Hsiao and Liu [17] for hyperbolic conservation laws with damping. Namely, they showed that the asymptotic profile of the solution is given by the heat kernel (see also [44, 45, 35, 67]). Later on, Nishihara [46], Marcati and Nishihara [37], Hosono and Ogawa [16], and Narazaki [40] derived more precise $L^p$-$L^q$ estimates for the linear damped wave equation and applied them to semilinear equations to obtain global solutions. Also, the diffusion phenomena for abstract damped wave equations were studied by [21, 23, 2, 58, 59, 52].

For the nonlinear damped wave equation with the absorbing nonlinearity $\mathcal{N}(u) = -|u|^{p-1}u$, Kawashima, Nakao and Ono [32] refined Matsumura’s $L^p$-$L^q$ estimates and applied them to the global well-posedness for the equation (1.1) with arbitrarily initial data $(u_0, u_1) \in H^1 \times L^2$. Based on this result, Karch [30] showed the diffusion phenomena when $p > 1 + \frac{1}{n}$ and $n \leq 3$. After that, Hayashi, Kaikina and Naumkin [12], Ikehata, Nishihara and Zhao [24] and Nishihara [48] treated the case $p > 1 + \frac{2}{n}$ and $n \leq 4$ if the initial data belongs to $(H^1 \cap L^1) \times (L^2 \cap L^1)$. Also, Hayashi, Kaikina and Naumkin [10, 11, 12, 13], Hayashi and Naumkin [14] and Hamza [8] studied the asymptotic profile of solutions for critical and subcritical nonlinearities $1 < p \leq 1 + \frac{2}{n}$.

The nonlinear damped wave equation with the source term $\mathcal{N}(u) = |u|^p$ or $|u|^{p-1}u$ has been widely studied. In this case, Levine [34] showed that the solution in general blows up in finite time for large initial data. Therefore, to obtain the global existence of solutions, we need some smallness condition for the initial data. Nakao and Ono [39] studied the case $\mathcal{N}(u) = |u|^{p-1}u$ with $p \geq 1 + \frac{4}{n}$ and proved the global existence of solutions by the method of modified potential well. Li and Zhou [36] found that when $n \leq 2$, the critical exponent of (1.1) is given by $p = 1 + \frac{2}{n}$, that is, the local-in-time solution can be extended time-globally if $p > 1 + \frac{2}{n}$ and the initial data is sufficiently small, while the finite time blow-up occurs if $p \leq 1 + \frac{2}{n}$ and the initial data has positive integral value. The number $1 + \frac{2}{n}$ is well known as Fujita’s critical exponent named after his seminal work [4], which is the threshold between the global existence and the blow-up of solutions to the semilinear heat equation. Also, in [30], the optimal upper estimate of the lifespan for blow-up solutions was also given (see also [47] for the case $n = 3$, [20] for $n \geq 4$ and $p < 1 + \frac{2}{n}$ and the first author and Ogawa [19] for $n \geq 4$, $p = 1 + \frac{2}{n}$). Later on, Todorova and Yordanov [65] and Zhang [68] determined the critical exponent as $p = 1 + \frac{2}{n}$ for all space dimensions. Moreover, Ono [55, 56] derived $L^m$-decay of solutions for $1 \leq m \leq 2n/(n-2)_+$. The results of [36] and [65] require that the initial data belongs to $H^1 \times L^2$ and has the compact support. Ikehata, Miyakoa and Nakatake [22], Ikehata and Tanizawa [26] and Hayashi, Kaikina and Naumkin [21] removed the compactness assumption and proved the global existence of solutions for the initial data belonging to $L^1$. Moreover, Nakao and Ono [39], Ikehata and Ohta [25] and Narazaki and Nishihara [43] studied the global well-posedness for slowly
decaying initial data not belonging to $L^1$. In particular, in \cite{25}, small data global existence is proved when the nonlinearity is $\mathcal{N}(u) = |u|^{p-1}u$ with $p > 1 + \frac{2r}{n}$ for $n \leq 6$ and $(H^1 \cap L^r) \times (L^2 \cap L^r)$-data, where $r$ satisfies $r \in [1, 2]$ if $n = 1, 2$ and $r \in \left[\frac{n^2 + 16n - n}{4}, \min\{2, \frac{n}{n-2}\}\right]$ if $3 \leq n \leq 6$. Finite time blow-up of local solutions was also obtained for any $n \geq 1$ and $1 < p < \frac{2r}{n}$. However, the above global well-posedness results are restricted to $n \leq 6$ and there are no results for higher dimensional cases. Also, Narazaki \cite{42} considered the slowly decaying data belonging to modulation spaces and proved the global existence when the nonlinearity has integer power.

Concerning the asymptotic profile of global solutions, Gallay and Raugel \cite{5} determined the asymptotic expansion up to the second order when $n = 1$ and the initial data belongs to the weighted Sobolev space $H^{1,1} \times H^{0,1}$ (see Section 1.2 for the definition). Using the expansion of solutions to the heat equation, Kawashima and Ueda \cite{31} extended it to the case of $H^s \times H^1$ and $n \geq 3$. Hayashi, Kaikina and Naumkin \cite{9} obtained the first order asymptotics for all $n \geq 1$ and the initial data belonging to $(H^{s,0} \cap H^{0,0}) \times (H^{s-1,0} \cap H^{0,0})$ with $s > \frac{n}{2}$ (particularly, belonging to $L^1$). Recently, Takeda \cite{63, 64} determined the higher order asymptotic expansion of global solutions. Narazaki and Nishihara \cite{43} studied the case of slowly decaying data and proved that if $n \leq 3$ and the data behaves like $(1 + |x|)^{-kn}$ with $0 < k \leq 1$, then, the asymptotic profile of the global solution is given by $G(t, x) \ast (1 + |x|)^{-kn}$, where $G$ is the Gaussian and $\ast$ denotes the convolution with respect to spatial variables.

Related to the equation (1.1), systems of nonlinear damped wave equation were studied and the critical exponent and the asymptotic behavior of solutions were investigated (see \cite{61, 41, 62, 53, 42, 54, 49, 50, 15, 61}).

In the present paper, we establish the large data local well-posedness and the small data global well-posedness for the nonlinear damped wave equation (1.1) with slowly decaying initial data. Our global well-posedness results extend those of \cite{25, 43} to all space dimensions, and generalize that of \cite{9} to slowly decaying initial data. Moreover, we study the asymptotic profile of the global solution. This also extends those of \cite{43} to all space dimensions. Considering the asymptotic behavior of solutions in weighted norms, we further extended the result of \cite{9} to the asymptotics in $L^m$-norm with $m \leq 2$. Finally, we give an almost optimal lifespan estimate from both above and below. This is also an extension of \cite{36, 47, 20}, in which $L^1$-data were treated.

1.1. Main results. We say that $u \in L^\infty(0, T; L^2(\mathbb{R}^n))$ is a mild solution of (1.1) if $u$ satisfies the integral equation

$$u(t) = (\partial_t + 1) \mathcal{D}(t) \varepsilon u_0 + \mathcal{D}(t) \varepsilon u_1 + \int_0^t \mathcal{D}(t - \tau) \mathcal{N}(u(\tau)) \, d\tau,$$

in $L^\infty(0, T; L^2(\mathbb{R}^n))$, where $\mathcal{D}(t)$ is the solution operator of the damped wave equation defined in (1.8) below.

We assume that there exists $p > 1$ such that the nonlinear term $\mathcal{N}(u)$ satisfies $\mathcal{N} \in C^{p_0}(\mathbb{R})$ with some integer $p_0 \in [0, p]$ and

\[
\begin{align*}
\mathcal{N}^{(l)}(0) &= 0, \\
\left|\mathcal{N}^{(l)}(u) - \mathcal{N}^{(l)}(v)\right| &\lesssim |u - v|(|u| + |v|)^{p-l-1} \quad (l = 0, \ldots, p_0).
\end{align*}
\]
Theorem 1.1 (Local well-posedness for large data). Let \( n \in \mathbb{N} \) and let \( s \geq 0 \) be \( 0 \leq |s| \leq p_0 \). When \( n = 1 \), we also assume that \( 0 \leq s < 1 \). Let \( r \in [1, 2] \) and \( s, p \) satisfy
\[
\min \left\{ \frac{r}{2}, \frac{1}{n} \right\} \leq p < \infty \quad \text{if} \quad 1 \leq n \leq 2s,
\]
\[
1 + \frac{r}{n} \leq p \leq \min \left\{ 1 + \frac{2}{n - 2s}, \frac{2n}{r(n - 2s)} \right\} \quad \text{if} \quad 2s < n, n \geq 2,
\]
\[
1 + \frac{r}{2} \leq p \leq \frac{1}{1 - 2s} \quad \text{if} \quad 2s < n, n = 1.
\]
We take an initial data from
\[
u_0 \in H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n), \quad u_1 \in H^{s-1,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n),
\]
where \( \alpha > n \left( \frac{1}{2} - \frac{1}{n} \right) \). Then, for any \( \varepsilon > 0 \), there exists \( T = T(\varepsilon) \in (0, \infty) \) such that the Cauchy problem (1.1) admits a unique local mild solution \( u \in C([0,T); H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n)) \). Moreover, if \( T < \infty \), \( u \) satisfies
\[
\liminf_{t \to T} \| u(t) \|_{H^{s,0} \cap H^{0,\alpha}} = \infty.
\]

Theorem 1.2 (Global well-posedness for small data). In addition to the assumption in Theorem 1.1, we assume that \( r \in [1, 2] \) and \( s, p \) satisfy
\[
1 + \frac{2r}{n} < p < \infty \quad \text{if} \quad 1 \leq n \leq 2s,
\]
\[
1 + \frac{2r}{n} < p \leq \min \left\{ 1 + \frac{2}{n - 2s}, \frac{2n}{r(n - 2s)} \right\} \quad \text{if} \quad 2s < n, n \geq 2,
\]
\[
1 + 2r < p \leq \frac{1}{1 - 2s} \quad \text{if} \quad 2s < n, n = 1
\]
(when \( r \in (1, 2] \), we may take \( p = 1 + \frac{2r}{n} \)). Then, there exists a constant \( \varepsilon_0 = \varepsilon_0(n, p, r, s, \alpha, \| u_0 \|_{H^{s,0} \cap H^{0,\alpha}}, \| u_1 \|_{H^{s-1,0} \cap H^{0,\alpha}}) > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0] \), the Cauchy problem (1.1) admits a unique global mild solution \( u \in C([0, \infty); H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n)) \). Moreover, the solution \( u \) satisfies the decay estimates
\[
\| u(t) \|_{L^2} \lesssim \varepsilon(t)^{-\frac{q}{2} \left( \frac{1}{2} - \frac{1}{n} \right)}, \quad \| \nabla^s u(t) \|_{L^2} \lesssim \varepsilon(t)^{-\frac{q}{2} \left( \frac{1}{2} - \frac{1}{n} \right) - \frac{s}{2}}, \quad \| |^a u(t) \|_{L^2} \lesssim \varepsilon(t)^{-\frac{q}{2} \left( \frac{1}{2} - \frac{1}{n} \right) + \frac{s}{2}}.
\]

Next, we study the asymptotic behavior of the global solutions. To state our result, we denote \( G(t, x) = (4\pi t)^{-n/2} \exp \left( -\frac{|x|^2}{4t} \right) \) and \( G(t) \phi = G(t) * \phi \).

Theorem 1.3 (Asymptotic behavior of global solutions). Under the assumption of Theorem 1.2, we assume that \( p > 1 + \frac{2r}{n} \) if \( r \in (1, 2] \). Let \( m \) be
\[
r \leq m \leq \frac{2n}{n - 2s} \left( s < \frac{n}{2} \right), \quad r \leq m < \infty \left( s = \frac{n}{2} \right), \quad r \leq m \leq \infty \left( s > \frac{n}{2} \right)
\]
and let \( \zeta \in (0, 1) \) is an arbitrarily small number. Then, the global solution \( u \) of (1.1) constructed in Theorem 1.2 satisfies the following asymptotic behavior: When \( r > 1 \), we have
\[
\| u(t) - \varepsilon G(t)(u_0 + u_1) \|_{L^m} \lesssim \langle t \rangle^{-\frac{n}{2} \left( \frac{1}{2} - \frac{1}{m} \right)} - \min \left( \frac{n}{2} (1 - \frac{1}{r}), \frac{n}{2}, \frac{n}{2} \right) \langle t \rangle^{(p-1) - \frac{1}{m}} + \zeta
\]
for \( t \geq 1 \). When \( r = 1 \), we have
\[
\| u(t) - \theta G(t) \|_{L^m} \lesssim \langle t \rangle^{-\frac{n}{2} \left( 1 - \frac{1}{m} \right)} - \min \left( \frac{n}{2} (1 - \frac{1}{r}), \frac{n}{2}, \frac{n}{2} \right) \langle t \rangle^{(p-1) - \frac{1}{m}} + \zeta
\]
for \( t \geq 1 \), where
\[
\theta = \varepsilon \int_{\mathbb{R}^n} (u_0 + u_1)(x)dx + \int_0^\infty \int_{\mathbb{R}^n} N(u)dxdt.
\]
Remark 1.1. (i) The case $r = 1, m \geq 2$ was studied by Hayashi, Kaikina and Naumkin \cite{43}, though in this paper, we refine their argument (see Section 3.3 and Section 3.4).

(ii) When $r > 1$, as we will see in the proof, the nonlinear term $N(u)$ has better spatial integrability than the linear part. Hence, the nonlinear term decays faster as time tends to infinity and does not affect the asymptotic profile.

(iii) When $n \leq 3$ and the initial data behaves like $(x)^{-k}$ as $|x| \to \infty$, a similar asymptotic behavior was obtained by Narazaki and Nishihara \cite{43}. The above theorem generalizes the result of \cite{43} to all $n \geq 1$ and more general initial data, while the class of the solution is slightly different.

In the critical or subcritical case, we also have the estimate of the lifespan from below. We define the lifespan of the solution of (1.1) by

$$T(\varepsilon) := \sup\{T \in (0, \infty); \text{there exists a mild solution } u \in C([0, T); H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n))\}.$$ 

Theorem 1.4 (Lower bound of the lifespan). In addition to the assumption in Theorem 1.1, we assume that

$$r \in [1, 2], \quad \min\left\{1 + \frac{r}{2}, 1 + \frac{r}{n}\right\} \leq p < 1 + \frac{2r}{n} \quad \text{(Case 1)}$$

or

$$r = 1, \quad p = 1 + \frac{2r}{n}. \quad \text{(Case 2)}$$

Then, there exists $\varepsilon_1 = \varepsilon_1(n, p, r, s, \alpha, \|u_0\|_{H^{s,0} \cap H^{0,\alpha}}, \|u_1\|_{H^{s-1,0} \cap H^{0,\alpha}} > 0$ such that for any $\varepsilon \in (0, \varepsilon_1]$, the lifespan $T = T(\varepsilon)$ of the solution is estimated as

$$T(\varepsilon) \geq \begin{cases} C\varepsilon^{-1/\omega} & \text{in Case 1,} \\ \exp\left(\frac{C\varepsilon^{-1}}{\omega}\right) & \text{in Case 2,} \end{cases}$$

where $\omega = \frac{1}{p-1} - \frac{n}{2r}$ and $C > 0$ is a positive constant independent of $\varepsilon \in (0, \varepsilon_1]$.

Remark 1.2. The estimate in Case 2 was proved by the first author and Ogawa \cite{19}.

Finally, we prove a blow-up result in the subcritical case with the nonlinearity $N(u) = \pm|u|^p$.

Theorem 1.5 (Upper bound of the lifespan). In addition to the assumptions in Theorem 1.1, we assume that

$$N(u) = \pm|u|^p \quad \text{and} \quad p < 1 + \frac{2r}{n} \quad \text{(1.7)}$$

and $\alpha$ satisfies

$$n\left(\frac{1}{r} - \frac{1}{2}\right) < \alpha < \frac{2}{p-1} - \frac{n}{2}.$$ 

Moreover, we take the initial data

$$u_0 \in H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n), \quad u_1 \in H^{s-1,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n)$$

fulfilling

$$\pm(u_0(x) + u_1(x)) \geq \begin{cases} |x|^{-\lambda} & \text{if } |x| > 1, \\
0 & \text{if } |x| \leq 1 \end{cases}$$

(double-sign corresponds to (1.7)) with some $\lambda$ satisfying $\frac{2}{p-1} + \alpha < \lambda < \frac{2}{p-1}$. Then, there exists $\varepsilon_2 = \varepsilon_2(n, p, r, s, \alpha, \lambda) > 0$ such that for any $\varepsilon \in (0, \varepsilon_2]$, the lifespan $T = T(\varepsilon)$ of the solution is estimated as

$$T(\varepsilon) \leq C\varepsilon^{-1/\kappa},$$

where $\kappa = \frac{1}{p-1} - \frac{\lambda}{2}$ and $C$ is a positive constant independent of $\varepsilon \in (0, \varepsilon_2]$. 
Remark 1.3. (i) The case \( r = 1 \) and \( p \leq 1 + \frac{2}{n} \) was studied by Todorova and Yordanov [65], Zhang [68], Li and Zhou [36], Nishihara [47], the first author and Ogawa [19], and the first author and the third author [20]. In particular, Theorem 1.4 shows that when \( 1 + \frac{1}{n} \leq p < 1 + \frac{2}{n} \), the lifespan \( T(\varepsilon) \) is estimates as \( T(\varepsilon) \sim \varepsilon^{-1/\omega} \) with \( \omega = \frac{1}{p - 1} - \frac{n}{2} \).

(ii) When \( r > 1 \), \( p = 1 + \frac{2}{n} \) and the initial data belong to \( L^r(\mathbb{R}^n) \) but not \( H^{0,\alpha}(\mathbb{R}^n) \), it is still an open problem whether the local solution blows up or not.

(iii) When \( N(u) = |u|^{p-1}u \), the blow-up of the solution was proved by Ikehata and Ohta [25], while estimates of lifespan were not obtained. Theorems 1.4 and 1.5 give an almost optimal estimate of lifespan for \( N(u) = \pm |u|^p \).

Our results are summarized in Table 1 below.

| \( r \backslash p \) | \( \min \left\{ 1 + \frac{r}{2}, 1 + \frac{r}{n} \right\} \) | \( 1 + \frac{2r}{n} \) | \( \cdot \cdot \cdot \) | \( 1 + \frac{2}{n - 2s} \) |
|-----------------|------------------|------------------|------------------|------------------|
| 1               | SDBU [65]        | SDBU [68]        | SDGE [20]        | SDGE [68]        |
| \( \varepsilon^{-1/\omega} \leq T(\varepsilon) \leq \varepsilon^{-1/\kappa} \) | \( e^{C\varepsilon^{-(p-1)}} \leq T(\varepsilon) \leq e^{Ce^{-p}} \) | \( T(\varepsilon) = \infty \) | \( T(\varepsilon) = \infty \) |
| \( r > 1 \)     | SDBU [25]        | SDGE             | SDGE             | SDGE             |
| \( \varepsilon^{-1/\omega} \leq T(\varepsilon) \leq \varepsilon^{-1/\kappa} \) | \( T(\varepsilon) = \infty \) | \( T(\varepsilon) = \infty \) | \( T(\varepsilon) = \infty \) |

Table 1. SDBU: Small data blow-up, SDGE: Small data global existence

Our strategy for proving Theorems 1.1 and 1.2 are based on that of Hayashi, Kaikina and Naumkin [9]. However, we have to refine their estimates to fit the slowly decaying data and solutions. The main ingredient is the estimate of the fundamental solution \( D(t) \) (see (1.8) for the definition) of the linear problem, which are given in Lemma 2.1. To prove these estimates, we use a gain of one derivative coming from the high frequency part of the kernel \( L(t,\xi) \) of \( D(t) \) (see (1.10)). Combining these estimates and nonlinear estimates with the contraction mapping principle, we prove the existence of solutions.

To prove Theorem 1.3 we first show that the solution \( u \) of the damped wave equation (1.1) is approximated by the solution of the linear heat equation with the homogeneous term \( N(u) \) (see Proposition 3.1). After that, we investigate the precise asymptotic behavior of solutions to the inhomogeneous linear heat equation (see Proposition 3.4).

For the upper estimates of the lifespan, we employ a test function method developed by Zhang [68], while this is based on a contradiction argument and not directly applicable to obtain the lifespan estimate. To avoid the contradiction argument, we use the ideas by Kuiper [33], Sun [60] and [20] to obtain an almost optimal estimate of the lifespan.

The rest of the paper is organized as follows. In Section 2, we give a proof of Theorems 1.1 and 1.2. Section 3 is devoted to the proof of Theorem 1.3. Theorems 1.4 and 1.5 will be proved in Section 4. Finally, we collect some useful lemmas in Appendix.

1.2. Notations. For the reader’s convenience, we collect the notations used throughout this paper. The letter \( C \) indicates a generic constant, which may change from line to line. Let \( \langle x \rangle = (1 + |x|^2)^{1/2} \). We also use the symbol \( f \lesssim g \), which means that \( f \leq Cg \) holds with some constant \( C > 0 \). The relation \( f \sim g \) stands for \( f \lesssim g \) and \( g \lesssim f \). Sometimes we use \( a \vee b := \max\{a, b\} \).
For functions $f = f(x) : \mathbb{R}^n \to \mathbb{R}$ and $\phi = \phi(\xi) : \mathbb{R}^n \to \mathbb{R}$, the Fourier transform and the inverse Fourier transform are defined by

$$
\mathcal{F}[f](\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx, \quad \mathcal{F}^{-1}[\phi](x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \phi(\xi)e^{ix\cdot\xi}d\xi,
$$

respectively.

Let $L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$ and $H^{s,\alpha}(\mathbb{R}^n)$ $(s,\alpha \geq 0)$ be the usual Lebesgue and the weighted Sobolev spaces, respectively, equipped with the norms defined by

$$
\|f\|_{L^p} = \left( \int_{\mathbb{R}^n} |f(x)|^pdx \right)^{1/p} \quad (1 \leq p < \infty), \quad \|f\|_{L^\infty} = \text{ess sup} |f|.
$$

Also, for a real number $s$, we denote by $\langle \nabla \rangle^s f = \mathcal{F}^{-1}[(\xi)^s \hat{f}]$. For an interval $I$ and a Banach space $X$, we define $C^n(I; X)$ as the space of $r$-times continuously differentiable mapping from $I$ to $X$ with respect to the topology in $X$.

We denote by $\mathcal{D}(t)$ and $\mathcal{G}(t)$ the solution operator of the linear damped wave and linear heat equations, respectively, that is,

$$
\mathcal{D}(t) := e^{-\frac{t}{2} L(t, \xi)\mathcal{F}}, \quad (1.8)
$$

$$
\mathcal{G}(t) := \mathcal{F}^{-1}e^{-t|\xi|^2}, \quad (1.9)
$$

where

$$
L(t, \xi) := \begin{cases} 
\frac{\sinh(t\sqrt{1/4 - |\xi|^2})}{\sqrt{1/4 - |\xi|^2}} & \text{if } |\xi| < 1/2, \\
\frac{\sin(t\sqrt{|\xi|^2 - 1/4})}{\sqrt{|\xi|^2 - 1/4}} & \text{if } |\xi| > 1/2.
\end{cases} \quad (1.10)
$$

Also, we use

$$
\tilde{\mathcal{D}}(t) := (\partial_t + 1) \mathcal{D}(t). \quad (1.11)
$$

Throughout this paper, we always use $s, r, \alpha$ as real numbers satisfying $s \geq 0$, $r \in [1, 2]$, $\alpha > n\left(\frac{1}{2} - \frac{1}{r}\right)$, respectively. Also, for a real number $s$, we denote by $[s]$ the integer part of $s$. For $T \in (0, \infty]$, we define

$$
\|\phi\|_{X(T)} := \sup_{0 < t < T} \left[ \langle t \rangle^{\frac{s}{q} - \frac{1}{2}} \|\phi(t)\|_{L^2} + \langle t \rangle^{\frac{s}{q} - \frac{1}{2}} \|\nabla^s \phi(t)\|_{L^2} \right]
$$

$$
+ \langle t \rangle^{\frac{s}{q} - \frac{1}{2}} \|\cdot |\alpha \phi(t)\|_{L^2} \right], \quad (1.12)
$$

$$
\|\psi\|_{Y(T)} := \sup_{0 < t < T} \left[ \langle t \rangle^{\eta} ||\nabla^{[\gamma]} \psi(t)||_{L^p} + \sup_{\gamma \in [\sigma_1, \sigma_2]} \langle t \rangle^{\frac{s}{q} - \frac{1}{2}} \|\psi(t)||_{L^\gamma} \right]
$$

$$
+ \langle t \rangle^{\eta} ||\cdot |\alpha \psi(t)||_{L^2} \right], \quad (1.13)
$$

where the parameters are defined in Table 2. Here $(n - 2s)_+ \denotes 0 \lor (n - 2s)$. 

2. Local and global existence of solutions

2.1. Preliminary estimates.

**Lemma 2.1.** Let \( \gamma, \nu \in [1, 2] \), \( \beta \geq 0 \), \( s_1 \geq s_2 \geq 0 \). Then, we have

\[
\|x^{s_1}D(t)\psi\|_{L^2} \lesssim \langle t \rangle^{-\frac{n}{2} \left( \frac{1}{\gamma} - \frac{1}{2} \right) - \frac{s_1}{2} + \frac{n}{2} \left( \frac{1}{r} - \frac{1}{2} \right)} \|x^{s_2} \psi\|_{L^\gamma} + e^{-\frac{\beta}{2}} \|x^{s_1} \langle x \rangle^{-1} \psi\|_{L^2}, \tag{2.1}
\]

\[
\|\cdot \langle x \rangle^{\beta}D(t)\psi\|_{L^2} \lesssim \langle t \rangle^{-\frac{n}{2} \left( \frac{1}{\gamma} - \frac{1}{2} \right) + \frac{n}{2} \left( \frac{1}{r} - \frac{1}{2} \right)} \|\cdot \langle x \rangle^{\beta} \psi\|_{L^\gamma} + \langle t \rangle^{-\frac{\beta}{2}} \|\cdot \langle x \rangle^{\beta} \psi\|_{L^\gamma} + e^{-\frac{\beta}{2}} \|\cdot \psi\|_{L^\gamma} + e^{-\frac{\beta}{2}} \|\cdot \psi\|_{L^\gamma}. \tag{2.2}
\]

We also have the continuity of \( D(t) \) with respect to \( H^{s_1,0} \cap H^{0,\beta} \)-norm:

\[
\lim_{t_1 \to t_2} \|D(t_1)\psi - D(t_2)\psi\|_{H^{s_1,0} \cap H^{0,\beta}} = 0. \tag{2.3}
\]

When \( \gamma = 1 \), the estimates (2.1) and (2.2) were proved by [9]. Here we give a generalization of it to \( \gamma \in [1, 2] \) with a slightly simpler proof.

**Proof.** By the definition of \( L(t, \xi) \), it is easy to see that

\[
e^{-\frac{\beta}{2}} \|x^{s} L(t, \xi)\|_{L^\gamma(|\xi| \leq 1)} \lesssim \langle t \rangle^{-\frac{n}{2} \left( \frac{1}{\gamma} - \frac{1}{2} \right)}
\]

and

\[
\|\langle \xi \rangle L(t, \xi)\|_{L^\infty(|\xi| \geq 1)} \lesssim 1.
\]
Therefore, applying the Plancherel theorem and the Hölder inequality, we have

\[

\|\nabla |s_1 D(t)\hat{\psi}\|_{L^2} = \|\xi |s_1 e^{-\frac{i}{2} L(t, \xi) \hat{\psi}}\|_{L^2} \\
\lesssim e^{-\frac{i}{2} \|\xi |s_1-s_2 L(t, \xi)\|_{L^{\frac{2n}{n-2}}_{|\xi| \leq 1}}} \|\xi |s_2 \hat{\psi}\|_{L^{\frac{4n}{n+2}}_{|\xi| \leq 1}} + e^{-\frac{i}{2} \|\xi L(t, \xi)\|_{L^{\infty}_{|\xi| \geq 1}}} \|\xi |s_1 (\xi^{-1} \hat{\psi})\|_{L^{2}_{|\xi| \geq 1}} \\
\lesssim \langle t \rangle^{-\frac{s_1-s_2}{2} - \frac{\beta}{2} \left(\frac{1}{2} - \frac{1}{2}\right)} \|\nabla |s_2 \hat{\psi}\|_{L^{\gamma}} + e^{-\frac{i}{2} \|\nabla |s_1 (\nabla)^{-1} \hat{\psi}\|_{L^2}},
\]

which gives (2.1).

Next, we prove (2.2). Let \(\chi(\xi) \in C_0^\infty(\mathbb{R}^n)\) be a cut-off function satisfying \(\chi(\xi) = 1\) for \(|\xi| \leq 1\) and \(\chi(\xi) = 0\) for \(|\xi| \geq 2\). We put

\[
K(t, x) := e^{-\frac{i}{2} \mathcal{F}^{-1} (L(t, \xi) \chi(\xi))}.
\]

The Hausdorff-Young inequality implies

\[
\| |^\beta \mathcal{F}^{-1} \left(e^{-\frac{i}{2} L(t, \xi) \chi(\xi) \hat{\psi}}\right) \|_{L^2} \\
= \left\| |^\beta \int_{\mathbb{R}^n} K(t, x - y) \psi(y) dy \right\|_{L^2} \\
\lesssim \left\| \int_{\mathbb{R}^n} \left(|x - y|^\beta |K(t, x - y)| + |K(t, x - y)||y|^\beta \right) \psi(y) dy \right\|_{L^2} \\
\lesssim \| |^\beta K(t)\|_{L^{2n/(n-2)}} \|\psi\|_{L^{\gamma}} + \|K(t)\|_{L^{4n/(3n-2)}} \| |^\beta \psi\|_{L^\nu}.
\]

Now we prove

\[
\| |^\beta K(t)\|_{L^k} \lesssim \langle t \rangle^{\frac{\beta}{2} - \frac{\beta}{2} \left(1 - \frac{1}{2}\right)}
\]

for \(t > 0\) and \(k \in [1, \infty)\). First, we divide the proof into the cases \(0 < t < 1\) and \(t \geq 1\). For the case \(0 < t < 1\), we easily prove \(\| |^\beta K(t)\|_{L^k} \lesssim 1\). Indeed, noting \(|x|^\beta k \lesssim \langle x \rangle^{2Nk-(n+1)}\) for sufficiently large integer \(N \in \mathbb{N}\), we have

\[
\| |^\beta K(t)\|_{L^k} \lesssim \int_{\mathbb{R}^n} \langle x \rangle^{2Nk-(n+1)} |K(t, x)|^k dx \\
= \int_{\mathbb{R}^n} \langle x \rangle^{-(n+1)} \left| \int_{|\xi| \leq 1/4} e^{ix\xi} e^{-\frac{i}{2} (\nabla)^{2N} (L(t, \xi) \chi(\xi))} d\xi \right|^k dx \\
\lesssim 1.
\]

For the case \(t \geq 1\), we change the variables as \(\sqrt{t} \xi = \eta\) and \(x = \sqrt{t} y\) to obtain

\[
\| |^\beta K(t)\|^k_{L^k} = \int_{\mathbb{R}^n} |x|^\beta k \left| \int_{\mathbb{R}^n} e^{ix\xi} e^{-\frac{i}{2} L(t, \xi) \chi(\xi)} d\xi \right|^k dx \\
= t^{-\frac{\beta}{2}} \int_{\mathbb{R}^n} |x|^\beta k \left| \int_{\mathbb{R}^n} e^{ix\eta/\sqrt{t}} e^{-\frac{i}{2} L(t, \eta/\sqrt{t}) \chi(\eta/\sqrt{t})} d\eta \right|^k dx \\
= t^{\frac{\beta}{2} - \frac{\beta}{2} (k-1)} \int_{\mathbb{R}^n} |y|^\beta k \left| \int_{\mathbb{R}^n} e^{iy\eta/\sqrt{t}} e^{-\frac{i}{2} L(t, \eta/\sqrt{t}) \chi(\eta/\sqrt{t})} d\eta \right|^k dy.
\]
As before, using $|y|^{\beta k} \leq \langle y \rangle^{2Nk-(n+1)}$ with sufficiently large integer $N \in \mathbb{N}$, we have

$$\| | \cdot |^{\beta} K(t) \|_{L^k}^{k} \lesssim t^{\frac{\beta}{2}-\frac{n}{2(k-1)}} \int_{\mathbb{R}^n} \langle y \rangle^{-(n+1)} \left| \int_{\mathbb{R}^n} e^{iy\eta} (\nabla_{\eta})^{2N} \left( e^{-\frac{x}{2}} L(t, \eta/\sqrt{t}) \chi(\eta/\sqrt{t}) \right) d\eta \right|^{k} dy.$$

By the definition of $L(t, \xi)$ (see (1.10)), we can easily see that for $t \geq 1$

$$\left| (\nabla_{\eta})^{2N} \left( e^{-\frac{x}{2}} L(t, \eta/\sqrt{t}) \chi(\eta/\sqrt{t}) \right) \right| \lesssim e^{-\frac{ny^2}{4}}.$$

Thus, we obtain the desired estimate (2.5).

Applying (2.5) to (2.4), we have

$$\| | \cdot |^{\beta} F^{-1} \left( e^{-\frac{x}{2}} L(t, \xi) \chi(\xi) \hat{\psi} \right) \|_{L^2} \lesssim \langle t \rangle^{\frac{\beta}{2}-\frac{n}{2}} \| \hat{\psi} \|_{L^\infty} + \langle t \rangle^{-\frac{n}{2}} \| \cdot |^{\beta} \psi \|_{L^\infty}.$$

Hence, it suffices to show that

$$\| | \cdot |^{\beta} F^{-1} \left( e^{-\frac{x}{2}} L(t, \xi) (1 - \chi(\xi)) \hat{\psi} \right) \|_{L^2} \lesssim e^{-\frac{t}{4}} \| \langle \nabla \rangle^{-\beta} \hat{\psi} \|_{L^2}. \quad (2.6)$$

To prove this, we calculate the fractional derivative. Let $\omega := \beta - [\beta]$. Making use of Lemma [A.1] we see that

$$|\partial_j|^\omega (\phi \hat{\psi})(x) = C \int_{\mathbb{R}} \left( \phi(x) \hat{\psi}(x) - \phi(x+y) \hat{\psi}(x+y) \right) \frac{dy_j}{|y_j|^{1+\omega}}$$

$$= \phi(x) |\partial_j|^\omega \hat{\psi}(x) + C \int_{\mathbb{R}} (\phi(x) - \phi(x+y)) \hat{\psi}(x+y) \frac{dy_j}{|y_j|^{1+\omega}}$$

holds, where $y := (0, \ldots, y_j, \ldots, 0)$. Then, using the Plancherel theorem and the Leibniz rule, we have

$$\| |\nabla|^\beta \hat{\psi} \|_{L^2} \lesssim \sum_{j=1}^n \| |\partial_j|^\beta \partial_j (\phi \hat{\psi}) \|_{L^2} \lesssim \sum_{j=1}^n \sum_{k=0}^{[\beta]} \| \partial_j |^{\beta-k} (\phi \partial_j^k \hat{\psi}) \|_{L^2},$$

and hence,

$$\| |\nabla|^\beta \hat{\psi} \|_{L^2} \lesssim \sum_{j=1}^n \sum_{k=0}^{[\beta]} \left( \| \partial_j^{n-k} \phi \|_{L^\infty} \right) \left( \| \partial_j^{k+\omega} \hat{\psi} \|_{L^2} \right)$$

$$+ \sum_{j=1}^n \sum_{k=0}^{[\beta]} \int_{\mathbb{R}} (\partial_j^{[\beta]-k} \phi(\cdot) - \partial_j^{[\beta]-k} \phi(\cdot + y)) \partial_j^k \hat{\psi}(\cdot + y) \frac{dy_j}{|y_j|^{1+\omega}} \|_{L^2}$$

$$\lesssim \left( \sum_{j=1}^n \sum_{k=0}^{[\beta]+1} \| \partial_j^k \hat{\psi} \|_{L^\infty} \right) \| \langle \nabla \rangle^\beta \hat{\psi} \|_{L^2}. $$

We apply the above estimate to the left-hand side of (2.6) with $\psi = \langle \xi \rangle^{-1} \hat{\psi}$ and $\phi = \langle \xi \rangle e^{-\frac{x}{2}} L(t, \xi)(1 - \chi(\xi))$. Since

$$\| \partial_j^k e^{-\frac{x}{2}} (\xi) L(t, \xi)(1 - \chi(\xi)) \|_{L^\infty} \lesssim e^{-\frac{t}{4}}$$

for any $k \in \mathbb{N}$, we obtain

$$\| | \cdot |^{\beta} F^{-1} \left( e^{-\frac{x}{2}} L(t, \xi)(1 - \chi(\xi)) \hat{\psi} \right) \|_{L^2} \lesssim e^{-\frac{t}{4}} \| \langle \nabla \rangle^\beta \hat{\psi} \|_{L^2}.$$
Finally, the bounds of \( L(t, \xi) \) proved above and the continuity of \( L(t, \xi) \) with respect to \( t \) show \((2.23)\). This completes the proof.

Similarly, we can prove the following lemma.

**Lemma 2.2.** Let \( \gamma, \nu \in [1, 2] \), \( \beta \geq 0 \), \( s_1 \geq s_2 \geq 0 \). Then, we have

\[
\begin{align*}
\| & \nabla \|^s_2 \bar{D}(t) \psi \|_{L^2} \leq \langle t \rangle^{-\frac{n}{4} + \frac{\beta}{4}} \| \nabla \|^s_\beta \psi \|_{L^\gamma} + e^{-\frac{\beta}{4}} \| \nabla \|^s_1 \psi \|_{L^2}, \\
\| & \beta \bar{D}(t) \psi \|_{L^2} \leq \langle t \rangle^{-\frac{n}{4} + \frac{\beta}{4}} \| \psi \|_{L^\gamma} + \langle t \rangle^{-\frac{n}{2} - \frac{\beta}{2}} \| \beta \psi \|_{L^\gamma} + e^{-\frac{\beta}{4}} \| \beta \psi \|_{L^2}.
\end{align*}
\]

We also have the continuity of \( \bar{D}(t) \) with respect to \( H^{s_0,0} \cap H^{0,\beta} \)-norm:

\[
\lim_{t_1 \to t_2} \| \bar{D}(t_1) \psi - \bar{D}(t_2) \psi \|_{H^{s_0,0} \cap H^{0,\beta}} = 0.
\]

Let \( \| \cdot \|_{X(T)} \) be defined by \((1.12)\). In order to prove Theorems \((1.1)\) and \((1.2)\) we frequently use the following interpolation inequalities.

**Lemma 2.3.** Let \( s \geq 0 \), \( r \in [1, 2] \), \( \alpha > n\left(\frac{1}{r} - \frac{1}{4}\right) \) and let \( \| \cdot \|_{X(T)} \) be defined by \((1.12)\). Then, the following interpolation inequalities hold:

(i) We have

\[
\sup_{0 < t < T} \left( \sup_{0 \leq s' \leq s} \langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right) + \frac{\beta}{4}} \| \nabla \|_s \phi(t) \|_{L^\gamma} + \sup_{0 \leq \beta \leq \alpha} \langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right) - \frac{\beta}{4}} \| \beta \phi(t) \|_{L^2} \right) \lesssim \| \phi \|_{X(T)}.
\]

(ii) We have

\[
\sup_{0 < t < T} \langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right)} \| \phi(t) \|_{L^\gamma} \lesssim \| \phi \|_{X(T)} \quad \text{for} \quad \begin{cases} r \leq \gamma \leq \infty & \text{if} \quad 1 \leq n < 2s, \\
 r \leq \gamma < \infty & \text{if} \quad n = 2s, \\
 r \leq \gamma \leq \frac{2n}{n - 2s} & \text{if} \quad 2s < n.
\end{cases}
\]

**Proof.** (i) The Plancherel theorem and the Hölder inequality imply

\[
\langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right) + \frac{\beta}{4}} \| \nabla \|_s \phi(t) \|_{L^\gamma} \leq \langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right) + \frac{\beta}{4}} \| \nabla \|_s \hat{\phi}(t) \|_{L^2}
\]

\[
\leq \left( \langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right)} \| \hat{\phi}(t) \|_{L^2} \right)^{\frac{1}{1 - \frac{\beta}{\alpha}}} \left( \langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right) - \frac{\beta}{4}} \| \beta \hat{\phi}(t) \|_{L^2} \right)^{\frac{\beta}{\alpha}}
\leq \| \phi \|_{X(T)}.
\]

In the same way, we have

\[
\langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right) - \frac{\beta}{4}} \| \beta \phi(t) \|_{L^2} \leq \left( \langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right) - \frac{\beta}{4}} \| \phi(t) \|_{L^2} \right)^{1 - \frac{\beta}{\alpha}} \left( \langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right) - \frac{\beta}{4}} \| \beta \phi(t) \|_{L^2} \right)^{\frac{\beta}{\alpha}}
\leq \| \phi \|_{X(T)}.
\]

(ii) If \( \gamma \) satisfies \( 2 \leq \gamma < \infty \) and the condition in \((2.10)\), we apply the Sobolev inequality and obtain

\[
\langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right)} \| \phi(t) \|_{L^\gamma} \lesssim \langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right)} \| \nabla \|_s \phi(t) \|_{L^2}
\]

with \( s' = n\left(\frac{1}{2} - \frac{1}{\gamma}\right) \). Clearly, \( s' \in [0, s] \) holds under the condition in \((2.10)\) and hence, (i) implies

\[
\langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right)} \| \nabla \|_s \phi(t) \|_{L^2} = \langle t \rangle^{\frac{n}{2} \left(\frac{1}{r} - \frac{1}{4}\right) + \frac{\beta}{4}} \| \nabla \|_s \phi(t) \|_{L^2} \lesssim \| \phi \|_{X(T)}.
\]
When $2s > n$ and $\gamma = \infty$, instead of (2.11), using (2.12) below with $\varphi = \hat{\phi}$, $\gamma = 1$ and $\beta = s$, we have

$$
\langle t \rangle^{\frac{\beta}{2}} \|\phi(t)\|_{L^\infty} \lesssim \langle t \rangle^{\frac{\beta}{n}} \|\hat{\phi}(t)\|_{L^1}
$$

$$
\lesssim \left( \langle t \rangle^{\frac{\beta}{2}} + \frac{2}{\beta} \|\nabla^\beta \phi(t)\|_{L^2} \right)^{\frac{n}{2}} \left( \langle t \rangle^{\frac{\beta}{2}} \|\phi(t)\|_{L^2} \right)^{\frac{n-2}{2}}
$$

$$
\lesssim \|\phi\|_{X(T)}.
$$

On the other hand, to prove (2.10) in the case $r \leq \gamma \leq 2$, we first claim that

$$
\|\varphi\|_{L^{\gamma}} \lesssim \|\varphi\|_{L^2}^{\frac{\beta}{(2-\gamma)2\gamma}} \|\cdot\|_{L^2}^{\frac{\beta}{\gamma}}
$$

(2.12)

with $\beta$ satisfying $\beta > n\left(\frac{1}{\gamma} - \frac{1}{2}\right)$ and for $\varphi \in H^{0,\beta}$. Indeed, let $\theta$ be determined later and we calculate

$$
\|\varphi\|_{L^{\gamma}}^\gamma = \int_\mathbb{R}^n (\theta^2 + |x|^2)^{-\frac{\beta n}{2\gamma}} (\theta^2 + |x|^2)^{\frac{\beta n}{2\gamma}} |\varphi(x)| \, dx
$$

$$
\leq \left( \int_\mathbb{R}^n (\theta^2 + |x|^2)^{\beta} |\varphi(x)|^2 \, dx \right)^\frac{\gamma}{2} \left( \int_\mathbb{R}^n (\theta^2 + |x|^2)^{-\frac{\beta n}{2\gamma}} \, dx \right)^\frac{2-\gamma}{2}\gamma.
$$

Since $\frac{\beta n}{2-\gamma} > \frac{n}{2}$, we have

$$
\int_\mathbb{R}^n (\theta^2 + |x|^2)^{-\frac{\beta n}{2\gamma}} \, dx \lesssim \theta^{-\frac{2\beta n}{2\gamma} + n}
$$

and hence,

$$
\|\varphi\|_{L^{\gamma}} \lesssim \theta^{-\frac{\beta n(2-\gamma)}{2\gamma}} \left( \int_\mathbb{R}^n (\theta^2 + |x|^2)^{\beta} |\varphi(x)|^2 \, dx \right)^\frac{1}{2}
$$

$$
\lesssim \theta^{-\frac{\beta n(2-\gamma)}{2\gamma}} \|\varphi\|_{L^2}^{\frac{\beta}{\gamma}} \|\cdot\|_{L^2}^{\frac{\beta}{\gamma}}.
$$

Taking $\theta = \|\varphi\|_{L^2}^{-\frac{\beta}{2\gamma}} \|\cdot\|_{L^2}^{\frac{\beta}{\gamma}}$, we have (2.12).

From (2.12), for $r \leq \gamma \leq 2$, letting $\beta$ satisfy $n\left(\frac{1}{\gamma} - \frac{1}{2}\right) < \beta \leq \alpha$, we obtain

$$
\langle t \rangle^{\frac{\beta}{2}} \left(\frac{\beta}{2} + \frac{\beta}{\gamma}\right) \|\phi(t)\|_{L^{\gamma}} \leq \left( \langle t \rangle^{\frac{\beta}{2}} \|\phi(t)\|_{L^2} \right)^{\frac{1}{2}} \left( \langle t \rangle^{\frac{\beta}{2}} \|\cdot\|_{L^2} \right) \left( \langle t \rangle^{\frac{\beta}{2}} \|\cdot\|_{L^2} \right)^{\frac{\beta}{2}}
$$

and hence, (i) gives

$$
\langle t \rangle^{\frac{\beta}{2}} \left(\frac{\beta}{2} + \frac{\beta}{\gamma}\right) \|\phi(t)\|_{L^{\gamma}} \lesssim \|\phi\|_{X(T)}.
$$

Thus, we finish the proof.

\[\Box\]

2.2. Proof of Theorems 1.1 and 1.2 in higher dimensional cases. We start with the estimate of the Duhamel term. Let $\|\cdot\|_{Y(T)}$ be defined by (1.13).

**Lemma 2.4.** Under the assumption in Theorem 1.1, we have

$$
\left\| \int_0^t \mathcal{D}(t - \tau) \psi(\tau) \, d\tau \right\|_{X(T)} \lesssim \int_0^T \|\psi\|_{Y(\tau)} \, d\tau
$$

(2.13)

for $0 < T \leq 1$. Moreover, we have

$$
\left\| \int_0^t \mathcal{D}(t - \tau) \psi(\tau) \, d\tau \right\|_{X(T)} \lesssim \|\psi\|_{Y(T)} \begin{cases} 1 & \text{if } p > 1 + \frac{2\gamma}{n}, \\ \log (2 + T) & \text{if } p = 1 + \frac{2\gamma}{n}, \\ (T)^{1 - \frac{\beta}{\gamma}(p-1)} & \text{if } p < 1 + \frac{2\gamma}{n}. \end{cases}
$$

(2.14)
for $r = 1, 0 < T < \infty$, and
\[
\left\| \int_0^t \mathcal{D}(t - \tau)\psi(\tau) d\tau \right\|_{X(T)} \lesssim \|\psi\|_{Y(T)} \begin{cases} 
1 & \text{if } p \geq 1 + \frac{2r}{n}, \\
(T)^{1 - \frac{2r}{n}(p-1)} & \text{if } p < 1 + \frac{2r}{n},
\end{cases} \tag{2.15}
\]
for $r \in (1, 2]$, $0 < T < \infty$. Here we may take $T = \infty$ if $p > 1 + \frac{2r}{n}$, or $p \geq 1 + \frac{2r}{n}$, $r \in (1, 2]$.

Proof. The estimate (2.13) is easily proved by looking at the proof of (2.14) and (2.15) carefully, and we may omit it.

**Step1:** Estimate of $\|\nabla|^s \int_0^t \mathcal{D}(t - \tau)\psi(\tau) d\tau\|_{L^2}$. We have
\[
\|\nabla|^s \int_0^t \mathcal{D}(t - \tau)\psi(\tau) d\tau\|_{L^2} \leq \int_0^t \|\nabla|^s \mathcal{D}(t - \tau)\psi(\tau)\|_{L^2} d\tau + \int_0^t \|\nabla|^s \mathcal{D}(t - \tau)\psi(\tau)\|_{L^2} d\tau
\]
=: $I + II$.

For $II$, we apply Lemma 2.1 with $s_1 = s, s_2 = [s], \gamma = \rho$ (see Table 2 for notations) and have
\[
II \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s-|\sigma|}{2} - \frac{\sigma}{2} (\frac{s}{2} - \frac{1}{2})} \|\nabla|[s] \psi(\tau)\|_{L^\rho} d\tau + \int_0^t e^{-\frac{4\tau}{\rho}} \|\nabla|^s \langle \nabla \rangle^{-1} \psi(\tau)\|_{L^2} d\tau.
\]
The Sobolev inequality implies
\[
\|\nabla|^s \langle \nabla \rangle^{-1} \psi(\tau)\|_{L^2} \lesssim \|\nabla|^s \psi(\tau)\|_{L^\rho}
\]
and hence,
\[
II \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s-|\sigma|}{2} - \frac{\sigma}{2} (\frac{s}{2} - \frac{1}{2})} \|\nabla|[s] \psi(\tau)\|_{L^\rho} d\tau
\]
\[
\lesssim \|\psi\|_{Y(T)} \int_0^t \langle t - \tau \rangle^{-\frac{s-|\sigma|}{2} - \frac{\sigma}{2} (\frac{s}{2} - \frac{1}{2})} \tau^{\eta} d\tau
\]
\[
\lesssim \langle t \rangle^{-\frac{s-|\sigma|}{2} - \frac{\sigma}{2} (\frac{s}{2} - \frac{1}{2}) + 1 - \frac{4\rho}{\rho} (p-1)} \|\psi\|_{Y(T)}.
\]

Next, we estimate $I$. Applying Lemma 2.1 with $s_1 = s, s_2 = 0$ and $\gamma \in [\sigma_1, \sigma_2]$ (see Table 2 for notations) determined later, we have
\[
I \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s}{2} (\frac{s}{2} - \frac{1}{2}) - \frac{\sigma}{2} - \frac{s-|\sigma|}{2} - \frac{\sigma}{2} (\frac{s}{2} - \frac{1}{2})} \|\psi(\tau)\|_{L^\rho} d\tau
\]
\[
+ \int_0^t e^{-\frac{4\tau}{\rho}} \|\nabla|^s \langle \nabla \rangle^{-1} \psi(\tau)\|_{L^2} d\tau
\]
=: $I_1 + I_2$.

We calculate $I_2$. The Sobolev embedding and the definition of $Y(T)$-norm imply
\[
I_2 \lesssim \int_0^t e^{-\frac{4\tau}{\rho}} \|\nabla|[s] \psi(\tau)\|_{L^\rho} d\tau \lesssim \|\psi\|_{Y(T)} \int_0^t e^{-\frac{4\tau}{\rho}} \tau^{\eta} d\tau \lesssim e^{-\frac{4\tau}{\rho}} \|\psi\|_{Y(T)}
\]
We divide the estimate of $I_1$ into three cases.

**Case 1:** When $p < 1 + \frac{2r}{n}$, taking $\gamma = r$, and noting $-\frac{4}{\rho}(p-1) > -1$ and $r \in [\sigma_1, \sigma_2]$, we see that
\[
I_1 \lesssim \|\psi\|_{Y(T)} \int_0^t \langle t - \tau \rangle^{-\frac{s}{2} (\frac{s}{2} - \frac{1}{2}) - \frac{s}{2} - \frac{s-|\sigma|}{2} - \frac{\sigma}{2} (\frac{s}{2} - \frac{1}{2}) - \frac{4}{\rho} (p-1)} d\tau
\]
\[
\lesssim \|\psi\|_{Y(T)} \|t \rangle^{-\frac{s}{2} (\frac{s}{2} - \frac{1}{2}) - \frac{s}{2} - \frac{s-|\sigma|}{2} - \frac{\sigma}{2} (\frac{s}{2} - \frac{1}{2}) - \frac{4}{\rho} (p-1)}.
\]
Case 2: When \( p > 1 + \frac{2r}{n} \), taking \( \gamma = r \) and noting \( -\frac{n}{2r} (p - 1) < -1 \), we infer that
\[
I_1 \lesssim \| \psi \|_{Y(T)} \left( \int_0^\frac{t}{2} (t - \tau)^{-\frac{n}{2r} (1 - \frac{1}{r}) - \frac{q}{2} \langle \tau \rangle} - \frac{1}{2r} \langle \tau \rangle - \frac{1}{2r} (p - 1) d\tau \right) \\
\lesssim \| \psi \|_{Y(T)} (t)^{-\frac{n}{2r} (1 - \frac{1}{r}) - \frac{q}{2} (1 - \frac{1}{r})} \\
= \| \psi \|_{Y(T)} (t)^{-\frac{1}{2} (\frac{r}{q} - \frac{1}{r}) - \frac{q}{2}}.
\]
Case 3-1: When \( p = 1 + \frac{2r}{n} \) and \( r > 1 \), we take \( \gamma = \sigma_1 \) and have \( -\frac{n}{2} (\frac{p}{r} - 1 - \frac{1}{\sigma_1}) = -\frac{n}{2} (\frac{1}{r} - \frac{1}{\sigma_1}) - 1 > -1 \), since \( r > \sigma_1 \). Thus, we obtain
\[
I_1 \lesssim \| \psi \|_{Y(T)} \left( \int_0^\frac{t}{2} (t - \tau)^{-\frac{n}{2} (\frac{1}{\sigma_1} - \frac{1}{r}) - \frac{q}{2} \langle \tau \rangle} - \frac{1}{2r} \langle \tau \rangle - \frac{1}{2r} (p - 1) d\tau \right) \\
\lesssim \| \psi \|_{Y(T)} (t)^{-\frac{n}{2} (\frac{1}{\sigma_1} - \frac{1}{r}) - \frac{q}{2} (\frac{1}{\sigma_1} - \frac{1}{r})} \\
= \| \psi \|_{Y(T)} (t)^{-\frac{1}{2} (\frac{r}{q} - \frac{1}{r}) - \frac{q}{2}}.
\]
Case 3-2: When \( p = 1 + \frac{2r}{n} \) and \( r = 1 \), taking \( \gamma = r \), we see that
\[
I_1 \lesssim \| \psi \|_{Y(T)} \left( \int_0^\frac{t}{2} (t - \tau)^{-\frac{n}{2} (\frac{1}{\sigma_1} - \frac{1}{r}) - \frac{q}{2} \langle \tau \rangle} - \frac{1}{2r} \langle \tau \rangle - \frac{1}{2r} (p - 1) d\tau \right) \\
\lesssim \| \psi \|_{Y(T)} (t)^{-\frac{n}{2} (\frac{1}{\sigma_1} - \frac{1}{r}) - \frac{q}{2} (\frac{1}{\sigma_1} - \frac{1}{r})} \\
= \| \psi \|_{Y(T)} (t)^{-\frac{1}{2} (\frac{r}{q} - \frac{1}{r}) - \frac{q}{2}} \log(1 + t).
\]
Step 2: Estimate of \( \| \int_0^\frac{t}{2} D(t - \tau) \psi(\tau) d\tau \|_{L^2} \). We have
\[
\left\| \int_0^\frac{t}{2} D(t - \tau) \psi(\tau) d\tau \right\|_{L^2} \lesssim \int_0^\frac{t}{2} \| D(t - \tau) \psi(\tau) \|_{L^2} d\tau + \int_\frac{t}{2}^t \| D(t - \tau) \psi(\tau) \|_{L^2} d\tau
\]
For \( IV \), we apply Lemma 2.1 with \( s_1 = s_2 = 0 \) and \( \gamma = q \), where \( q \) is defined in Section 1.2, and obtain
\[
IV \lesssim \int_\frac{t}{2}^t \left( t - \tau \right)^{-\frac{n}{2} (\frac{1}{q} - \frac{1}{r})} \left\| \psi(\tau) \right\|_{L^q} + e^{-\frac{t}{4r}} \left\langle \langle \nabla \rangle^{-1} \psi(\tau) \right\|_{L^2} \right) d\tau \\
\lesssim \int_\frac{t}{2}^t \left( t - \tau \right)^{-\frac{n}{2} (\frac{1}{q} - \frac{1}{r})} \left\| \psi(\tau) \right\|_{L^q} \right) d\tau.
\]
Here we have used the Sobolev embedding \( \| \langle \nabla \rangle^{-1} \psi(\tau) \|_{L^q} \lesssim \| \psi(\tau) \|_{L^q} \). Since \( q = \frac{2n}{n+2} \in [\sigma_1, \sigma_2] \) and \( -\frac{n}{2} (\frac{1}{q} - \frac{1}{r}) = -\frac{1}{2r} \), we have
\[
IV \lesssim \| \psi \|_{Y(T)} \left( \int_\frac{t}{2}^t (t - \tau)^{-\frac{n}{2} (\frac{1}{q} - \frac{1}{r})} (\tau)^{-\frac{1}{2r} (\tau)^{q - \frac{1}{2}} - \frac{1}{2r} (p - 1) d\tau \right) \\
\lesssim \| \psi \|_{Y(T)} (t)^{-\frac{1}{2} (\frac{r}{q} - \frac{1}{r}) + 1 - \frac{1}{2r} (p - 1)}.
\]
Let us estimate the term \( III \). Applying Lemma 2.1 with \( s_1 = s_2 = 0 \) and \( \gamma \in [1, 2] \) determined later, we demonstrate
\[
III \lesssim \int_0^\frac{t}{2} (t - \tau)^{-\frac{n}{2} (\frac{1}{q} - \frac{1}{r})} \| \psi(\tau) \|_{L^q} d\tau + \int_0^\frac{t}{2} e^{-c \frac{t}{4r}} \| \langle \nabla \rangle^{-1} \psi(\tau) \|_{L^2} d\tau
\]
\[
=: III_1 + III_2.
\]
From \( q \in [\sigma_1, \sigma_2] \), the Sobolev embedding and the definition of \( Y(T) \)-norm imply
\[
III_2 \lesssim \int_0^T e^{-\frac{t}{2T}} \| \psi(t) \|_{L^q} \, dt \lesssim \| \psi \|_{Y(T)} \int_0^T e^{-\frac{t}{2T}} \left( \tau - \frac{\varphi}{2} \right)^{-\frac{\varphi}{2} - 1} d\tau \lesssim e^{-\frac{T}{2}} \| \psi \|_{Y(T)}.
\]
Similarly to the estimate of \( I_1 \), we divide the estimate of \( III_1 \) into three cases.

**Case 1:** When \( p < 1 + \frac{2r}{n} \), we take \( \gamma = r \) to obtain
\[
III_1 \lesssim \| \psi \|_{Y(T)} \int_0^T (t - \tau)^{-\frac{n}{2} (\frac{1}{r} - \frac{1}{2})} \langle \tau \rangle^{-\frac{n}{2r} (p - 1)} d\tau
\lesssim \| \psi \|_{Y(T)} (t)^{-\frac{n}{2} (\frac{1}{r} - \frac{1}{2})},
\]
where we have used \( -\frac{\varphi}{2r} (p - 1) > -1 \).

**Case 2:** When \( p > 1 + \frac{2r}{n} \), taking \( \gamma = r \), we see that
\[
III_1 \lesssim \| \psi \|_{Y(T)} \int_0^T (t - \tau)^{-\frac{n}{2} (\frac{1}{r} - \frac{1}{2})} \langle \tau \rangle^{-\frac{n}{2r} (p - 1)} d\tau
\lesssim \| \psi \|_{Y(T)} (t)^{-\frac{n}{2} (\frac{1}{r} - \frac{1}{2})},
\]
since \( -\frac{\varphi}{2r} (p - 1) < -1 \).

**Case 3-1:** When \( p = 1 + \frac{2r}{n} \) and \( r > 1 \), we let \( \gamma = \sigma_1 \) to obtain
\[
III_1 \lesssim \| \psi \|_{Y(T)} \int_0^T (t - \tau)^{-\frac{n}{2} (\frac{1}{r} - \frac{1}{2})} \langle \tau \rangle^{-\frac{n}{2r} (p - 1)} d\tau
\lesssim \| \psi \|_{Y(T)} (1 + t)^{-\frac{n}{2} (\frac{1}{r} - \frac{1}{2})} \log(1 + t),
\]
here we have used \( -\frac{\varphi}{2r} (\frac{p}{2} - \frac{1}{\sigma_1}) = -\frac{n}{2} (\frac{1}{r} - \frac{1}{\sigma_1}) - 1 > -1 \).

**Case 3-2:** When \( p = 1 + \frac{2r}{n} \) and \( r = 1 \), taking \( \gamma = r \) gives
\[
III_1 \lesssim \| \psi \|_{Y(T)} \int_0^T (t - \tau)^{-\frac{n}{2} (\frac{1}{r} - \frac{1}{2})} \langle \tau \rangle^{-\frac{n}{2r} (p - 1)} d\tau
\lesssim \| \psi \|_{Y(T)} (1 + t)^{-\frac{n}{2} (\frac{1}{r} - \frac{1}{2})} \log(1 + t),
\]
since \( -\frac{\varphi}{2r} (p - 1) = -1 \).

**Step 3:** Estimate of \( \| \cdot \|_0 \int_0^t D(t - \tau) \psi(t) \, d\tau \|_{L^2} \). We apply Lemma 21 with \( \beta = \alpha, \nu = q \) and \( \gamma \in [\sigma_1, \sigma_2] \) determined later, and obtain
\[
\left\| \cdot \right\|_0 \int_0^t D(t - \tau) \psi(t) \, d\tau \right\|_{L^2}
\lesssim \int_0^t (t - \tau)^{-\frac{n}{2} (\frac{1}{r} - \frac{1}{2})} \| \psi(t) \|_{L^q} \, d\tau + \int_0^t (t - \tau)^{-\frac{n}{2} (\frac{1}{r} - \frac{1}{2})} \| \cdot \|_0 \psi(t) \|_{L^q} \, d\tau
\lesssim V_1 + V_2 + V_3.
\]
In order to estimate \( V_3 \), we employ Lemma A.2 and deduce that
\[
V_3 \lesssim \int_0^t e^{-\frac{\varphi t}{2r}} \| \langle \cdot \rangle^{\alpha} \psi(t) \|_{L^q} \, d\tau.
\]
Therefore, the estimates of $V_2, V_3$ reduce to that of $\int_0^t (t - \tau)^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right)} \|\langle \cdot \rangle^\alpha \psi(\tau)\|_{L^n} d\tau$. Noting $-\frac{n}{2} \left(\frac{1}{q} - \frac{1}{2}\right) = -\frac{1}{2}$, we have

$$\int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right)} \|\langle \cdot \rangle^\alpha \psi(\tau)\|_{L^n} d\tau \lesssim \|\psi\|_{Y(T)} \int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{q} - \frac{1}{2}\right)} \tau^{-\zeta} d\tau,$$

where $\zeta$ is defined in Section 1.2. We compute

$$\int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right)} \tau^{-\zeta} d\tau \lesssim \begin{cases} \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2} + 1 - \frac{n}{2p}(p-1)} & (\zeta < 1), \\ \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2} + 1 - \frac{n}{2p}(p-1)} \log(1 + t) & (\zeta = 1), \\ \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2}} & (\zeta > 1) \end{cases},$$

and note that $\zeta < 1$ holds if $p < 1 + \frac{2r}{n}$. Therefore, we may summarize them as

$$\int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right)} \tau^{-\zeta} d\tau \lesssim \begin{cases} \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2} + 1 - \frac{n}{2p}(p-1)} & (p < 1 + \frac{2r}{n}), \\ \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2}} & (p \geq 1 + \frac{2r}{n}) \end{cases},$$

and hence, we have

$$V_2 + V_3 \lesssim \|\psi\|_{Y(T)} \begin{cases} \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2} + 1 - \frac{n}{2p}(p-1)} & (p < 1 + \frac{2r}{n}), \\ \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2}} & (p \geq 1 + \frac{2r}{n}) \end{cases},$$

For $V_1$, as before, we divide the estimate into three cases.

**Case 1:** When $p < 1 + \frac{2r}{n}$, by taking $\gamma = r$, the definition of $Y(T)$-norm leads to

$$V_1 \lesssim \|\psi\|_{Y(T)} \int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2}} \langle \tau \rangle^{-\frac{n}{2p}(p-1)} d\tau$$

$$\lesssim \|\psi\|_{Y(T)} \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2}},$$

since $-\frac{n}{2p}(p-1) > -1$ and $-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2} > -1$.

**Case 2:** When $p > 1 + \frac{2r}{n}$, we choose $\gamma = r$ and have

$$V_1 \lesssim \|\psi\|_{Y(T)} \int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2}} \langle \tau \rangle^{-\frac{n}{2p}(p-1)} d\tau$$

$$\lesssim \|\psi\|_{Y(T)} \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2}},$$

since $-\frac{n}{2p}(p-1) < -1$ and $-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2} > -1$.

**Case 3:** When $p = 1 + \frac{2r}{n}$ and $r > 1$, letting $\gamma = \sigma_1$, we see that

$$V_1 \lesssim \|\psi\|_{Y(T)} \int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2}} \langle \tau \rangle^{-\frac{n}{2p}(p-1)} d\tau$$

$$\lesssim \|\psi\|_{Y(T)} \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2} (\frac{1}{\sigma_1} - \frac{1}{2}) + 1}$$

$$= \|\psi\|_{Y(T)} \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2}}.$$

Here we have used that $-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) > -1$ and $-\frac{n}{2} \left(\frac{1}{\sigma_1} - \frac{1}{2}\right) + \frac{n}{2} > -1$.

**Case 3:** When $p = 1 + \frac{2r}{n}$ and $r = 1$, taking $\gamma = r$ implies

$$V_1 \lesssim \|\psi\|_{Y(T)} \int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2}} \langle \tau \rangle^{-\frac{n}{2p}(p-1)} d\tau$$

$$\lesssim \|\psi\|_{Y(T)} \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{r} - \frac{1}{2}\right) + \frac{n}{2} \log(1 + t)},$$
Lemma 2.5. Under the assumptions in Theorem 1.1, we have
\[ \|N(u)\|_{Y(T)} \lesssim \|u\|_{X(T)}^p, \]
\[ \|N(u) - N(v)\|_{Y(T)} \lesssim \|u - v\|_{X(T)} \left( \|u\|_{X(T)} + \|v\|_{X(T)} \right)^{p-1}. \]

Proof. First, we consider the term \( \|\langle \cdot \rangle^\alpha N(u)\|_{L^p} \). By the assumption \( (1.2) \) and the Hölder inequality with \( \frac{1}{q} = \frac{1}{2} + \frac{1}{n} \), we have
\[ \|\langle \cdot \rangle^\alpha N(u)\|_{L^p} \leq \|\langle \cdot \rangle^\alpha u\|_{L^2} \|u\|_{L^{p(n-1)}}^{p-1} \]
and hence,
\[ \langle t \rangle^{\frac{1}{2}} \|\langle \cdot \rangle^\alpha N(u)\|_{L^p} \leq \langle t \rangle^{\frac{1}{2}} \left( \frac{1}{n(p-1)} \right)^{p-1} \|u\|_{L^{p(n-1)}}. \]

From the assumption of Theorem 1.1, \( r \leq n(p-1) \) is valid and \( n(p-1) \) also holds when \( 2s < n \). Therefore, we apply Lemma 2.3 to obtain
\[ \langle t \rangle^{\frac{1}{2}} \|\langle \cdot \rangle^\alpha N(u)\|_{L^p} \lesssim \|u\|_{X(T)}^p. \]

Next, we estimate \( \|N(u)\|_{L^\gamma} \) for \( \gamma \in [\sigma_1, \sigma_2] \). By the assumption \( (1.2) \), we see that
\[ \langle t \rangle^{\frac{1}{2} \left( \frac{1}{r} - \frac{1}{\gamma} \right)} \|N(u)\|_{L^\gamma} \lesssim \langle t \rangle^{\frac{1}{2} \left( \frac{1}{r} - \frac{1}{\gamma} \right)} \|u\|_{L^p}. \]

Since \( p \geq 1 + \frac{r}{n} \) and \( \gamma \geq \sigma_1 \geq \frac{nr}{n+r} \), we have
\[ r = \left( 1 + \frac{r}{n} \right) \frac{nr}{n+r} \leq p \gamma. \]

Also, when \( 2s < n \), the assumption \( \sigma_2 \leq \frac{2n}{p(n-2s)} \) leads to
\[ p \gamma \leq p \sigma_2 \leq \frac{2n}{n-2s}. \]

Therefore, \( p \gamma \in [r, \frac{2n}{n-2s}] \) is valid and we apply Lemma 2.3 to obtain
\[ \langle t \rangle^{\frac{1}{2} \left( \frac{1}{r} - \frac{1}{\gamma} \right)} \|N(u)\|_{L^\gamma} \lesssim \|u\|_{X(T)}^p. \quad (2.17) \]

Finally, we estimate \( \|\nabla^{[s]} N(u)\|_{L^p} \). The case \( [s] = 0 \) reduces to (2.17), because \( \rho \in [\sigma_1, \sigma_2] \). When \( [s] \geq 1 \), by the Faà di Bruno formula, for \( \nu \in \mathbb{Z}_{\geq 0} \) with \( \nu \geq 1 \) we have
\[ \nabla^\nu N(u) = \sum_{l=1}^{[\nu]} N^{(l)}(u) \sum_{\nu_1 \geq 1, \ldots, \nu_l \geq 1, \nu_1 + \cdots + \nu_l = \nu} C_{\nu_1, \ldots, \nu_l} \partial^{\nu_1} \partial^{\nu_2} \cdots \partial^{\nu_l} u. \]

Using this with \( |N^{(l)}(u)| \lesssim |u|^{p-l} = |u|^{p-[s]} |u|^{[s]-l} \) and the Hölder inequality, we see that
\[ \|\nabla^{[s]} N(u)\|_{L^p} \lesssim \|u|^{p-[s]}\|_{L^{p-\nu}} \sum_k \|\nabla^{[k]} u\|_{L^q} \cdots \|\nabla^{[q_0]} u\|_{L^r}, \quad (2.18) \]

where the sum is taken over \( k = (k_1, \ldots, k_{[s]}) \in \mathbb{Z}_{\geq 0}^{[s]} \) satisfying \( |k| = k_1 + \cdots + k_{[s]} = [s] \), and \( q_1(k), \ldots, q_{[s]}(k) \) satisfy
\[ \frac{1}{\rho} = \frac{1}{q_0} + \frac{1}{q_1(k)} + \cdots + \frac{1}{q_{[s]}(k)} \quad (2.19) \]
and are defined in the following way.
For each fixed $k = (k_1, \ldots, k_{[s]})$, let us choose $s_1, \ldots, s_{[s]}$ so that

$$\max\{0, k_j - \mu\} \leq s_j < k_j + \frac{1}{p - 1}, \quad \sum_{j=1}^{[s]} s_j = s - \mu, \quad (2.20)$$

where $\mu = \frac{n}{2} - \frac{2}{p-1}$. This is always possible. Indeed, first, it is obvious that $\max\{0, k_j - \mu\} < k_j + \frac{1}{p-1}$ and the interval $[\max\{0, k_j - \mu\}, k_j + \frac{1}{p-1}]$ is not empty. Next, we demonstrate that

$$\sum_{j=1}^{[s]} \max\{0, k_j - \mu\} \leq s - \mu < \sum_{j=1}^{[s]} \left( k_j + \frac{1}{p - 1} \right). \quad (2.21)$$

To prove this, with a direct calculation we have

$$\sum_{j=1}^{[s]} \left( k_j + \frac{1}{p - 1} \right) = [s] + \frac{[s]}{p - 1} > s - \mu,$$

since $n \geq 2$. Also, when $[s] = 1$, it is trivial that $\max\{0, [s] - \mu\} \leq s - \mu$, since the assumption that $p \leq 1 + \frac{2}{n-2s}$ if $2s < n$ implies $s \geq \mu$. When $[s] \geq 2$, noting that $\mu < 0$ leads to $[s] < p < 1 + \frac{2}{n} \leq 2$, we may assume $\mu \geq 0$. Therefore, we have

$$\sum_{j=1}^{[s]} \max\{0, k_j - \mu\} = 0 \leq s - \mu$$

if $k_j \leq \mu$ for all $j = 1, \ldots, [s]$ and

$$\sum_{j=1}^{[s]} \max\{0, k_j - \mu\} \leq (k_i - \mu) + \sum_{j \neq i} k_j = [s] - \mu \leq s - \mu$$

if $k_i > \mu$ for some $i \in \{1, \ldots, [s]\}$. Thus, we prove (2.21) and we can actually find $s_j$ ($j = 1, \ldots, [s]$) satisfying (2.20).

From these $s_j$, we define

$$\frac{1}{q_j(k)} = \frac{1}{2} - \frac{\mu + s_j - k_j}{n}, \quad \frac{1}{q_0} = \left( \frac{1}{2} - \frac{\mu}{n} \right) (p - [s]).$$

Then, a straightforward calculation shows (2.19). Moreover, by the property (2.20) and the assumption $p \geq 1 + \frac{2}{n}$, we have $2 \leq q_j(k) < \infty$ for $j = 1, \ldots, [s]$ and $r \leq n(p-1) = q_0(p-[s]) < \infty$, respectively. Also, we remark that $\mu + s_j \leq s$ holds for all $j = 1, \ldots, [s]$ due to $s_j \geq 0$ and $\sum_{j=1}^{[s]} s_j = s - \mu$. Hence, we apply the Sobolev embedding and Lemma 2.3 to (2.18) and obtain

$$\langle t \rangle^{\eta} \| \nabla^{[s]} \mathcal{A}(u) \|_{L^p} \lesssim \langle t \rangle^{\eta} \| u \|_{L^p(\mathbb{R}; \mathcal{L})}^{[s]} \sum_{k} ||| |\nabla|^{\mu+s_j} u \|_{L^2} \cdots ||| |\nabla|^{\mu+s_{[s]}} u \|_{L^2}$$

$$\lesssim \left( \langle t \rangle^{\frac{n}{2}} \left( \frac{1}{p-1} \right) || u ||_{L^p(\mathbb{R}; \mathcal{L})} \right)^{p-[s]}

\times \sum_{k} \prod_{j=1}^{[s]} \langle t \rangle^{\frac{n}{2}} \left( \frac{1}{p-1} \right) + \frac{\mu+s_j}{2} ||| |\nabla|^{\mu+s_j} u \|_{L^2}

\lesssim \| u \|_{X(T)}^p.$$
In the same manner, with the assumption (1.2), we can prove the estimate for \( \|\mathcal{N}(u) - \mathcal{N}(v)\|_{Y(T)} \). Indeed, for example, we demonstrate
\[
\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^p} \lesssim \|u - v\|_{L^p} (\|u\|_{L^p} + \|v\|_{L^p})^{p-1}.
\]
Hence, we find that
\[
(\mathbb{E}^{T})(\mathbb{E}^{T})\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^p} \lesssim (\mathbb{E}^{T})(\mathbb{E}^{T})\|u - v\|_{L^p} (\|u\|_{L^p} + \|v\|_{L^p})^{p-1}.
\]
The other terms can be estimated in a similar way.

Now we are in the position to prove Theorem 1.1 when \( n \geq 2 \).

Proof of Theorem 1.1 when \( n \geq 2 \). We apply the contraction mapping principle in
\[
X_\varepsilon(T) = \{ u \in L^\infty(0, T; H^{s,0} \cap H^{0,\alpha}(\mathbb{R}^n)); \|u\|_{X(T)} \leq C_0\varepsilon \},
\]
where \( C_0 > 0 \) is determined later. Also, we define a metric in \( X_\varepsilon(T) \) by
\[
d(u, v) := \|u - v\|_{L^\infty(0, T; H^{s,0} \cap H^{0,\alpha})}.
\]
Then, clearly, \( X_\varepsilon(T) \) becomes a complete metric space.

We define the mapping \( \mathcal{M} \) by
\[
\mathcal{M}v(t) = \tilde{D}(t)\varepsilon u_0 + D(t)\varepsilon u_1 + \int_0^t D(t - \tau)\mathcal{N}(v(\tau))d\tau.
\]
Let \( 0 < T < 1 \). Then, by Lemmas 2.1, 2.2, 2.4 (2.13), and 2.5 we have
\[
\|\mathcal{M}v\|_{X(T)} \leq C\varepsilon (\|u_0\|_{H^{s,0} \cap H^{0,\alpha}} + \|u_1\|_{H^{s-1,0} \cap H^{0,\alpha}}) + CT\|v\|_{X(T)}^p.
\]
Taking a constant \( C_0 \) so that \( C(\|u_0\|_{H^{s,0} \cap H^{0,\alpha}} + \|u_1\|_{H^{s-1,0} \cap H^{0,\alpha}}) \leq C_0^2 \), we have
\[
\|\mathcal{M}v\|_{X(T)} \leq \frac{C_0}{2}\varepsilon + CTC_0^p\varepsilon^p.
\]
Letting \( T \) be sufficiently small so that \( CTC_0^p\varepsilon^p \leq \frac{C_0}{2} \), we conclude \( \|\mathcal{M}v\|_{X(T)} \leq C_0\varepsilon \). Thus, \( \mathcal{M} \) maps \( X_\varepsilon(T) \) to itself.

In a similar way, we have
\[
\|\mathcal{M}(v) - \mathcal{M}(w)\|_{X(T)} \leq C\varepsilon^{p-1}T\|v - w\|_{X(T)}
\]
for \( v, w \in X_\varepsilon(T) \) and hence, taking \( T \) further small, we have
\[
\|\mathcal{M}(v) - \mathcal{M}(w)\|_{X(T)} \leq \frac{1}{2}\|v - w\|_{X(T)}.
\]
Therefore, \( \mathcal{M} \) is a contraction mapping on \( X_\varepsilon(T) \) and there is a unique fixed point \( u \) in \( X_\varepsilon(T) \). By the definition of \( \mathcal{M} \), \( u \) is a solution of the Cauchy problem (1.1).

Finally, we prove the continuity with respect to \( t \) of the solution \( u \). For \( t_1, t_2 \geq 0 \), we have
\[
u(t_1) - u(t_2) = (D(t_1) - D(t_2))u_1 + (\tilde{D}(t_1) - \tilde{D}(t_2))u_0 + \int_{t_2}^{t_1} D(t_1 - \tau)\mathcal{N}(u(\tau))d\tau + \int_0^{t_2} (D(t_1 - \tau) - D(t_2 - \tau))\mathcal{N}(u(\tau))d\tau.
\]
Applying (2.3), (2.9) and the Lebesgue convergence theorem with the bound (2.13), we can easily prove \( \lim_{t_1 \to t_2} \|u(t_1) - u(t_2)\|_{H^{s,0} \cap H^{0,\alpha}} = 0 \), which finishes the proof.
**Lemma 2.6.** Under the assumptions in Theorem 1.1, the mild solution of (1.1) is unique in the class \(C([0, T); H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n))\).

**Proof.** Let \(T_0 > 0\) and let \(u, v \in C([0, T_0); H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n))\) are mild solutions of (1.1) with the same initial data \((u_0, u_1)\). We take arbitrary \(0 < T_1 < \min\{1, T_0\}\) and fix it. Then, we have \(\|u\|_{X(T_1)} + \|v\|_{X(T_1)} \leq M\) with some \(M > 0\). By the first assertion of Lemma 2.4 and applying Lemma 2.5 we see that for \(T \in [0, T_1]\)

\[
\|u - v\|_{X(T)} \lesssim M^{p-1} \int_0^T \|u - v\|_{X(\tau)} d\tau.
\]

Hence, we apply the Gronwall inequality and obtain \(\|u - v\|_{X(T)} \equiv 0\) for \(T \in [0, T_1]\), namely \(u \equiv v\) in \(t \in [0, T_1]\). Applying the same argument starting at \((u(T_1), u(T_1))\) instead of \((u_0, u_1)\), we have \(u \equiv v\) on \([0, 2T_1]\). Continuing this until reaching \(T_0\), we have the uniqueness in \(C([0, T_0); H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n))\).

Finally, we mention about the continuity of the solution with respect to the initial data.

**Lemma 2.7** (Lipschitz continuity of the solution map). The solution map

\[
(H^{s,0} \cap H^{0,\alpha}) \times (H^{s-1,0} \cap H^{0,\alpha}) \to C([0, T); H^{s,0} \cap H^{0,\alpha}); (u_0, u_1) \mapsto u
\]

is locally Lipschitz continuous, that is, for any \(T_1 < T\), we have

\[
\|u(t) - v(t)\|_{H^{s,0} \cap H^{0,\alpha}} \lesssim \|u_0 - v_0\|_{H^{s,0} \cap H^{0,\alpha}} + \|u_1 - v_1\|_{H^{s-1,0} \cap H^{0,\alpha}}
\]

on \(t \in [0, T_1]\), where \(u\) and \(v\) are solutions of (1.1) in \(C([0, T); H^{s,0} \cap H^{0,\alpha})\) with the initial data \((u_0, u_1)\) and \((v_0, v_1)\), respectively.

**Proof.** Let \(u\) and \(v\) are solutions of (1.1) in \(C([0, T); H^{s,0} \cap H^{0,\alpha})\) and fix \(T_1 < T\). Then, we have \(\|u\|_{X(T_1)} + \|v\|_{X(T_1)} \leq M\) with some constant \(M > 0\). Therefore, by Lemmas 2.1, 2.4 and 2.5 we have

\[
\|u(t) - v(t)\|_{X(T_1)} \lesssim \|u_0 - v_0\|_{H^{s,0} \cap H^{0,\alpha}} + \|u_1 - v_1\|_{H^{s-1,0} \cap H^{0,\alpha}} + M^{p-1} \int_0^{T_1} \|u(\tau) - v(\tau)\|_{X(\tau)} d\tau.
\]

The Gronwall inequality implies

\[
\|u(t) - v(t)\|_{X(T_1)} \lesssim \|u_0 - v_0\|_{H^{s,0} \cap H^{0,\alpha}} + \|u_1 - v_1\|_{H^{s-1,0} \cap H^{0,\alpha}},
\]

which completes the proof. □

**Proof of Theorem 1.2.** We assume that \(p > 1 + \frac{2r}{n}, r \in [1, 2]\) or \(p = 1 + \frac{2r}{n}, r \in (1, 2]\) and consider the mapping \(M\) defined on \((2.24)\) in the complete metric space

\[
X_\varepsilon(\infty) = \{v \in L^\infty(0, \infty; H^{s,0} \cap H^{0,\alpha}(\mathbb{R}^n)); \|v\|_{X(\infty)} \leq C_0 \varepsilon\}
\]

with the metric \((2.23)\). Then, by Lemma 2.4 we have

\[
\left\|\int_0^t \mathcal{D}(t - \tau) \mathcal{N}(u(\tau)) d\tau\right\|_{X(\infty)} \lesssim \|\mathcal{N}(u)\|_{Y(\infty)}.
\]

Hence, Lemma 2.5 implies

\[
\|Mv\|_{X(\infty)} \leq C\varepsilon(\|u_0\|_{H^{s,0} \cap H^{0,\alpha}} + \|u_1\|_{H^{s-1,0} \cap H^{0,\alpha}}) + C\|v\|_{Y(\infty)}^p.
\]

As before, taking a constant \(C_0\) so that \(C(\|u_0\|_{H^{s,0} \cap H^{0,\alpha}} + \|u_1\|_{H^{s-1,0} \cap H^{0,\alpha}}) \leq C_0\), we see that

\[
\|Mv\|_{X(\infty)} \leq \frac{C_0}{2} \varepsilon + C\varepsilon^p.
\]
Finally, taking $\varepsilon$ sufficiently small, we conclude $\|Mv\|_{X(\infty)} \leq C_0\varepsilon$ and hence, $M$ maps $X_\varepsilon(\infty)$ to itself.

In a similar way, we have
\[
\|M(v) - M(w)\|_{X(\infty)} \leq C\varepsilon^{p-1}\|v - w\|_{X(\infty)}
\]
for $v, w \in X_\varepsilon(\infty)$ and hence, taking $\varepsilon$ further small, we conclude that $M$ is a contraction mapping. Therefore, $M$ has a unique fixed point $u$ in $X_\varepsilon(\infty)$ and by the definition of $M$, $u$ is a mild solution of (1.1). In the same way as in the proof of Theorem 1.1, we deduce that $u$ belongs to $C([0, \infty) ; H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n))$. The uniqueness and continuity of the solution have already proved in Lemmas 2.7 and 2.6.

2.3. Proof of Theorems 1.1 and 1.2 in the one-dimensional case.

Lemma 2.8. Under the assumption in Theorem 1.1, we have
\[
\left\| \int_0^t D(t - \tau)\psi(\tau) d\tau \right\|_{X(T)} \lesssim \int_0^T \|\psi\|_{Y(\tau)} d\tau
\]
for $0 < T \leq 1$. Moreover, we have
\[
\left\| \int_0^t D(t - \tau)\psi(\tau) d\tau \right\|_{X(T)} \lesssim \|\psi\|_{Y(T)} \begin{cases} 1 & \text{if } p > 1 + 2r, \\ \log(2 + T) & \text{if } p = 1 + 2r, \\ (T)^{\frac{1}{2} - \frac{1}{p} - (p-1)} & \text{if } p < 1 + 2r \end{cases}
\]
for $r = 1$, $0 < T < \infty$, and
\[
\left\| \int_0^t D(t - \tau)\psi(\tau) d\tau \right\|_{X(T)} \lesssim \|\psi\|_{Y(T)} \begin{cases} 1 & \text{if } p \geq 1 + 2r, \\ (T)^{\frac{1}{2} - \frac{1}{p} - (p-1)} & \text{if } p < 1 + 2r \end{cases}
\]
for $r \in (1, 2]$, $0 < T < \infty$.

Proof. The proof is almost the same as that of Lemma 2.4 and hence we present only the outline. The main difference arises in the Sobolev inequality, that is, we shall use
\[
\|\langle \nabla \rangle^s \langle \nabla \rangle^{-1} \psi\|_{L^2} \lesssim \|\psi\|_{L^2}
\]
instead of (2.16), and we use Lemma A.2 with $q = 1$, that is,
\[
\|\langle \gamma \rangle^a \langle \nabla \rangle^{-1} \psi\|_{L^2} \lesssim \|\langle \gamma \rangle^a \psi\|_{L^1}.
\]
We estimate
\[
\left\| \langle \nabla \rangle^s \int_0^t D(t - \tau)\psi(\tau) d\tau \right\|_{L^2} \lesssim \int_0^t \|\langle \nabla \rangle^s D(t - \tau)\psi(\tau)\|_{L^2} d\tau + \int_0^t \|\langle \nabla \rangle^s D(t - \tau)\psi(\tau)\|_{L^2} d\tau =: I + II
\]
and Lemma 2.1 with $s_1 = s$, $s_2 = 0$, $\gamma = 1$ implies
\[
II \lesssim \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{s}{2}} \|\psi(\tau)\|_{L^1} d\tau + \int_0^t e^{-\frac{t - \tau}{\varepsilon}} \|\psi(\tau)\|_{L^2} d\tau \lesssim \|\psi\|_{Y(T)} \langle t \rangle^{-\frac{1}{2} - \frac{s}{2} - \frac{1}{p} - (p-1) + 1}.
\]
Similarly, we have

\[
I \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}(\frac{1}{4} - \frac{1}{2})} \|\psi(\tau)\|_{L^2} d\tau + \int_0^t e^{-\frac{L}{r} \tau} \|\psi(\tau)\|_{L^2} d\tau
\]

\[
\lesssim \|\psi\|_{Y(T)} \langle t \rangle^{-\frac{1}{2}(\frac{1}{4} - \frac{1}{2}) - \frac{1}{2}} \begin{cases} 1 & \text{if } p > 1 + 2r \text{ or } p = 1 + 2r, r > 1, \\ \log (2 + t) & \text{if } p = 1 + 2r, r = 1, \\ \langle t \rangle^{1 - \frac{1}{p}(p-1)} & \text{if } p < 1 + 2r, \end{cases}
\]

where \(\gamma = 1\) if \(p = 1 + 2r, r > 1\) and \(\gamma = r\) otherwise. In the same way, we have

\[
\left\| \int_0^t \mathcal{D}(t - \tau) \psi(\tau) d\tau \right\|_{L^2}
\]

\[
\lesssim \|\psi\|_{Y(T)} \langle t \rangle^{-\frac{1}{2}(\frac{1}{4} - \frac{1}{2})} \begin{cases} 1 & \text{if } p > 1 + 2r \text{ or } p = 1 + 2r, r > 1, \\ \log (2 + t) & \text{if } p = 1 + 2r, r = 1, \\ \langle t \rangle^{1 - \frac{1}{p}(p-1)} & \text{if } p < 1 + 2r. \end{cases}
\]

Finally, we estimate

\[
\left\| \langle \cdot \rangle^\alpha \int_0^t D(t - \tau) \psi(\tau) d\tau \right\|_{L^2}
\]

\[
\leq \int_0^t \langle t - \tau \rangle^{-\frac{\alpha}{2}(\frac{1}{4} - \frac{1}{2}) + \frac{\alpha}{2} \| \cdot \|_{L^1}} \| \psi(\tau) \|_{L^2} d\tau + \int_0^t \langle t - \tau \rangle^{-\frac{1}{2} \| \cdot \|_{L^1}} \| \psi(\tau) \|_{L^2} d\tau
\]

\[
+ \int_0^t e^{-\frac{L}{r} \tau} \| \langle \cdot \rangle^\alpha \langle \nabla \rangle^{-1} \psi(\tau) \|_{L^2} d\tau
\]

\[=: V_1 + V_2 + V_3.\]

Lemma A.2 leads to

\[V_3 \lesssim \int_0^t e^{-\frac{L}{r} \tau} \| \langle \cdot \rangle^\alpha \psi(\tau) \|_{L^4} d\tau.\]

Therefore, the estimates of \(V_2, V_3\) reduce to that of \(\int_0^t \langle t - \tau \rangle^{-\frac{1}{2} \| \cdot \|_{L^1}} \| \psi(\tau) \|_{L^2} d\tau\). We have

\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2} \| \cdot \|_{L^1}} \| \psi(\tau) \|_{L^2} d\tau \lesssim \|\psi\|_{Y(T)} \int_0^t \langle t - \tau \rangle^{-\frac{1}{2} \langle \tau \rangle^{-\zeta}} d\tau,
\]

where \(\zeta\) is defined in Table 2 (see Section 1.2). We compute

\[
\int_0^t \langle t - \tau \rangle^{-\frac{\alpha}{2}(\frac{1}{4} - \frac{1}{2})} \langle \tau \rangle^{-\zeta} d\tau \lesssim \begin{cases} \langle t \rangle^{-\frac{1}{2}(\frac{1}{4} - \frac{1}{2}) + \frac{\alpha}{2} + 1 - \frac{\alpha}{2}(p-1)} & (\zeta < 1), \\ \langle t \rangle^{-\frac{1}{2} \log(1 + t)} & (\zeta = 1), \\ \langle t \rangle^{-\frac{1}{2}} & (\zeta > 1) \end{cases}
\]

and note that \(\zeta < 1\) holds if \(p < 1 + 2r\). Therefore, we may summarize them as

\[
\int_0^t \langle t - \tau \rangle^{-\frac{\alpha}{2}(\frac{1}{4} - \frac{1}{2})} \langle \tau \rangle^{-\zeta} d\tau \lesssim \begin{cases} \langle t \rangle^{-\frac{\alpha}{2}(\frac{1}{4} - \frac{1}{2}) + \frac{\alpha}{2} + 1 - \frac{\alpha}{2}(p-1)} & (p < 1 + \frac{2r}{n}), \\ \langle t \rangle^{-\frac{\alpha}{2}(\frac{1}{4} - \frac{1}{2}) + \frac{\alpha}{2}} & (p \geq 1 + \frac{2r}{n}) \end{cases}
\]

and hence, we have

\[V_2 + V_3 \lesssim \|\psi\|_{Y(T)} \begin{cases} \langle t \rangle^{-\frac{\alpha}{2}(\frac{1}{4} - \frac{1}{2}) + \frac{\alpha}{2} + 1 - \frac{\alpha}{2}(p-1)} & (p < 1 + \frac{2r}{n}), \\ \langle t \rangle^{-\frac{\alpha}{2}(\frac{1}{4} - \frac{1}{2}) + \frac{\alpha}{2}} & (p \geq 1 + \frac{2r}{n}) \end{cases}.
\]
Lemma 2.9. Under the assumptions in Theorem 1.1, we have
\[ \|\mathcal{N}(u)\|_{Y(T)} \lesssim \|u\|_{X(T)}^p. \]
\[ \|\mathcal{N}(u) - \mathcal{N}(v)\|_{Y(T)} \lesssim \|u - v\|_{X(T)} (\|u\|_{X(T)} + \|v\|_{X(T)})^{p-1}. \]

Proof. At first, we consider \(\|\langle \cdot \rangle^\alpha \mathcal{N}(u)\|_{L^1}.\) The Hölder inequality yields
\[ \langle t \rangle^{\alpha} \|\langle \cdot \rangle^\alpha \mathcal{N}(u)\|_{L^1} \lesssim \langle t \rangle^{\frac{\alpha}{2} + \frac{n}{2}} \|\langle \cdot \rangle^\alpha u\|_{L^p} \left( \langle t \rangle^{\frac{\alpha}{2} \frac{1}{p} - \frac{1}{q} \frac{1}{p} (\frac{1}{p} - \frac{1}{q})} \right)^{p-1}. \]

Since \(1 + \frac{\alpha}{2} \leq p\), we see that \(2(p-1) \geq r.\) Moreover, when \(n > 2s,\) the assumption \(p \leq \frac{1}{1 - 2s}\) implies \(2(p-1) \leq \frac{2n}{n-2s}.\) Therefore, we apply Lemma 2.3 to obtain
\[ \langle t \rangle^{\frac{\alpha}{2}} \|\langle \cdot \rangle^\alpha \mathcal{N}(u)\|_{L^1} \lesssim \|u\|_{X(T)}^p. \]

Next, we estimate \(\|\mathcal{N}(u)\|_{L^\gamma}\) with \(\gamma \in [1, 2].\) We first obtain
\[ \|\mathcal{N}(u)\|_{L^\gamma} \lesssim \|u\|_{L^{p\gamma}}^{p\gamma}. \]

It follows from \(p \geq 1 + \frac{\alpha}{2}\) that \(p\gamma \geq r\) for any \(\gamma \in [1, 2].\) Also, when \(n > 2s,\) the assumption \(p \leq \frac{1}{1 - 2s}\) ensures \(p\gamma \leq \frac{2n}{n-2s}.\) Hence, we apply Lemma 2.3 to derive
\[ \langle t \rangle^{\frac{\alpha}{2} \frac{1}{p} - \frac{1}{q} \frac{1}{p} (\frac{1}{p} - \frac{1}{q})} \|\mathcal{N}(u)\|_{L^\gamma} \lesssim \left( \langle t \rangle^{\frac{\alpha}{2} \frac{1}{p} - \frac{1}{q} \frac{1}{p} (\frac{1}{p} - \frac{1}{q})} \right)^{p} \lesssim \|u\|_{X(T)}^p. \]

The above estimates shows the conclusion. \(\square\)

Proof of Theorems 1.2 and 1.3 when \(n = 1.\) The proof of Theorems 1.1 and 1.2 is completely the same as the case \(n \geq 2\) and we omit the detail. \(\square\)

3. Asymptotic behavior of the global solution

3.1. Approximation by an inhomogeneous heat equation. In this section, we study the asymptotic behavior of the global solution. Let \(u\) be the global-in-time solution proved in the previous section, that is,

\[ u(t) = \tilde{D}(t)\varepsilon u_0 + D(t)\varepsilon u_1 + \int_0^t D(t - \tau)\mathcal{N}(u(\tau)) \, d\tau. \]

First, we consider the solution of the inhomogeneous linear heat equation

\[ (\partial_t - \Delta) v = \mathcal{N}(u) \quad (3.1) \]

with the initial data \(v(0, x) = \varepsilon(u_0 + u_1),\) that is,

\[ v(t) = G(t)(\varepsilon u_0 + \varepsilon u_1) + \int_0^t G(t - \tau)\mathcal{N}(u(\tau)) \, d\tau, \]

where \(G(t)\) is defined in (1.9). We first prove that the asymptotic profile of \(u\) is given by \(v\) in \(H^{s,0} \cap H^{0,\alpha}\)-sense.
Proposition 3.1. Under the assumptions of Theorem 1.2, we have
\[
\|\nabla^s (u(t) - v(t))\|_{L^2} \lesssim \langle t \rangle^{-\frac{2s}{2s - n}} \langle t \rangle^{-\frac{2s}{2s - n} - \min\{1, \frac{n}{4p} (p - 1) - \frac{1}{2}\}}, \tag{3.2}
\]
\[
\|u(t) - v(t)\|_{L^2} \lesssim \langle t \rangle^{-\frac{2}{2 - \frac{n}{2} - \min\{1, \frac{n}{4p} (p - 1) - \frac{1}{2}\}}}, \tag{3.3}
\]
\[
\|\cdot |^\beta (u(t) - v(t))\|_{L^2} \lesssim \langle t \rangle^{-\frac{2}{2 - \frac{n}{2} - \min\{1, \frac{n}{4p} (p - 1) - \frac{1}{2}\}}}, \tag{3.4}
\]
where \(\beta\) is arbitrary number satisfying \(n(\frac{1}{p} - \frac{1}{2}) < \beta \leq \alpha\).

3.2. Preliminary estimates. In order to prove Proposition 3.1, we prepare the following lemma.

Lemma 3.2. Let \(\gamma, \nu \in [1, 2], \beta \geq 0\) and \(s_1 \geq s_2 \geq 0\). Then, we have
\[
\|\nabla^{s_1} (D(t) \psi - G(t) \psi)\|_{L^2} \lesssim \langle t \rangle^{-\frac{2s_1 - 2s_2}{2 - \frac{n}{2} - \frac{1}{2}}} \|\nabla^{s_2} \psi\|_{L^\gamma} + e^{-\frac{\beta}{4}} \|\nabla^{s_1} \psi\|_{L^2},
\]
\[
\|\cdot |^\beta (D(t) \psi - G(t) \psi)\|_{L^2} \lesssim \langle t \rangle^{-\frac{2s_1 - 2s_2}{2 - \frac{n}{2} - \frac{1}{2}}} \|\psi\|_{L^\gamma} + \langle t \rangle^{-\frac{2}{2 - \frac{n}{2} - \frac{1}{2}}} \|\cdot |^\beta \psi\|_{L^\nu}
\]
for any \(t \geq 1\).

Proof. Since
\[
\|\xi^s (e^{-\frac{\beta}{4}} L(t, \xi) - e^{-t|\xi|^2})\|_{L^\gamma(|\xi| \leq 1)} \lesssim \langle t \rangle^{-\frac{\beta}{4} - \frac{1}{2}} - 1
\]
and
\[
\|\langle \xi \rangle L(t, \xi)\|_{L^\infty(|\xi| > 1)} \lesssim 1, \quad \|\langle \xi \rangle e^{-t|\xi|^2}\|_{L^\infty(|\xi| > 1)} \lesssim e^{-t/4},
\]
we have
\[
\|\nabla^{s_1} (D(t) \psi - G(t) \psi)\|_{L^2} = \|\xi^{s_1} (e^{-\frac{\beta}{4}} L(t, \xi) - e^{-t|\xi|^2}) \hat{\psi}\|_{L^2}
\]
\[
\lesssim \|\xi^{s_1 - s_2} (e^{-\frac{\beta}{4}} L(t, \xi) - e^{-t|\xi|^2})\|_{L^{\frac{2\nu}{\nu - \gamma}}(|\xi| \leq 1)} \|\xi^{s_2} \hat{\psi}\|_{L^{\frac{2\gamma}{\gamma - \nu}}(|\xi| \leq 1)}
\]
\[
+ e^{-\frac{\beta}{4}} \|\langle \xi \rangle L(t, \xi)\|_{L^\infty(|\xi| \leq 1)} \|\xi^{s_1} \hat{\psi}\|_{L^2(|\xi| \leq 1)}
\]
\[
+ e^{-\frac{\beta}{4}} \|\langle \xi \rangle e^{-t|\xi|^2}\|_{L^\infty(|\xi| > 1)} \|\xi^{s_1} \hat{\psi}\|_{L^2(|\xi| > 1)}
\]
\[
\lesssim \langle t \rangle^{-\frac{s_1 - s_2}{2 - \frac{n}{2} - \frac{1}{2}}} \|\nabla^{s_2} \psi\|_{L^\gamma} + e^{-\frac{\beta}{4}} \|\nabla^{s_1} \psi\|_{L^2},
\]
which implies the first assertion.

To prove the second estimate, we take a cut-off function \(\chi \in C_0^\infty(\mathbb{R}^n)\) satisfying \(\chi(\xi) = 1\) for \(|\xi| \leq 1\) and \(\chi(\xi) = 0\) for \(|\xi| \geq 2\). We put
\[
K(t, x) := F^{-1} \chi(\xi) \left( e^{-\frac{\beta}{4}} L(t, \xi) - e^{-t|\xi|^2} \right).
\]
Then, in the same way to (2.21), we see that
\[
\|\cdot |^\beta F^{-1} \chi(\xi) \left( e^{-\frac{\beta}{4}} L(t, \xi) - e^{-t|\xi|^2} \right) \hat{\psi}\|_{L^2}
\]
\[
\lesssim \|\cdot |^\beta K(t)\|_{L^{\frac{2\nu}{\nu - \gamma}}} \|\psi\|_{L^\gamma} + \|K(t)\|_{L^{\frac{2\gamma}{\gamma - \nu}}} \|\cdot |^\beta \psi\|_{L^\nu}.
\]
Moreover, in a similar way to (2.25), with
\[
e^{-\frac{\beta}{4}} L(t, \xi) - e^{-t|\xi|^2} = e^{-t|\xi|^2} O(|\xi|^2)
\]
as \(|\xi| \to 0\), we can prove
\[
\|\cdot |^\beta K(t)\|_{L^k} \lesssim \langle t \rangle^{\frac{\beta}{2} - \frac{n}{2} (1 - \frac{1}{2}) - 1},
\]
which implies
\[
\left\| \cdot |^\beta \mathcal{F}^{-1} \chi(\xi) \left( e^{-\frac{t}{2}} L(t, \xi) - e^{-t|\xi|^2} \right) \psi \right\|_{L^2} \\
\lesssim \langle t \rangle^{-\frac{n}{2}(\frac{1}{2} - \frac{1}{2}) + \frac{\beta}{2} - 1} ||\psi||_{L^\gamma} + \langle t \rangle^{-\frac{n}{2}(\frac{1}{2} - \frac{1}{2}) - 1} ||\cdot |^\beta \psi||_{L^\nu}.
\]
We also easily obtain
\[
\left\| \cdot |^\beta \mathcal{F}^{-1} \left( e^{-t|\xi|^2} (1 - \chi(\xi)) \psi \right) \right\|_{L^2} \lesssim e^{-\frac{t}{4}} ||\cdot |^\beta \langle \nabla \rangle_{2}^{-1} \psi||_{L^2}.
\]
This and (2.6) lead to
\[
\left\| \cdot |^\beta \mathcal{F}^{-1} \left( e^{-\frac{t}{2}} L(t, \xi) - e^{-t|\xi|^2} \right) (1 - \chi(\xi)) \psi \right\|_{L^2} \lesssim e^{-\frac{t}{4}} ||\cdot |^\beta \langle \nabla \rangle_{2}^{-1} \psi||_{L^2}.
\]
From this, we reach the conclusion. □

In the same way, we have the following.

**Lemma 3.3.** Let \( \gamma, \nu \in [1, 2], \beta \geq 0, s_1 \geq s_2 \geq 0. \) Then, we have
\[
\left\| \nabla |^{s_1} (\tilde{D}(t) - \mathcal{G}(t)) \psi \right\|_{L^2} \lesssim \langle t \rangle^{-\frac{n}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{s_1}{2}} \left\| \nabla |^{s_2} \psi \right\|_{L^\gamma} + e^{-t/4} \left\| \nabla |^{s_1} \psi \right\|_{L^2},
\]
\[
\left\| \cdot |^\beta (\tilde{D}(t) - \mathcal{G}(t)) \psi \right\|_{L^2} \lesssim \langle t \rangle^{-\frac{n}{2}(\frac{1}{2} - \frac{1}{2}) + \frac{\beta}{2} - 1} ||\psi||_{L^\gamma} + \langle t \rangle^{-\frac{n}{2}(\frac{1}{2} - \frac{1}{2}) - 1} ||\cdot |^\beta \psi||_{L^\nu} + e^{-\frac{t}{4}} ||\cdot |^\beta \psi||_{L^2}.
\]

### 3.3. Proof of Proposition 3.1
For simplicity, we treat only the case \( n \geq 2. \) The proof of the case \( n = 1 \) is similar. Let us estimate the nonlinear part of \( u - v. \) We divide
\[
\left\| \nabla |^{s} \int_{0}^{t} (\mathcal{D}(t - \tau) \mathcal{N}(u(\tau)) - \mathcal{G}(t - \tau) \mathcal{N}(u(\tau))) \, d\tau \right\|_{L^2} \leq I + II,
\]
where
\[
I = \int_{0}^{t/2} \left\| \nabla |^{s} (\mathcal{D}(t - \tau) - \mathcal{G}(t - \tau)) \mathcal{N}(u(\tau)) \right\|_{L^2} \, d\tau
\]
and
\[
II = \int_{t/2}^{t} \left\| \nabla |^{s} (\mathcal{D}(t - \tau) - \mathcal{G}(t - \tau)) \mathcal{N}(u(\tau)) \right\|_{L^2} \, d\tau.
\]
We claim that each term has the desired decay rate.

First, we estimate \( II. \) Applying Lemma 3.2 with \( s_1 = s, \) \( s_2 = |s| \) and \( \gamma = \rho, \) we have
\[
II \lesssim \int_{t/2}^{t} \left\| \nabla |^{s} (\mathcal{D}(t - \tau) - \mathcal{G}(t - \tau)) \mathcal{N}(u(\tau)) \right\|_{L^2} \, d\tau
\]
\[
\lesssim \int_{t/2}^{t} \langle t - \tau \rangle^{-\frac{s}{2} - \frac{n}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{\beta}{2}} \left\| \nabla |^{s} \mathcal{N}(u(\tau)) \right\|_{L^\rho} \, d\tau
\]
\[
+ \int_{t/2}^{t} e^{-\frac{t - \tau}{4}} \left\| \nabla |^{s} \langle \nabla \rangle_{2}^{-1} \mathcal{N}(u(\tau)) \right\|_{L^2} \, d\tau.
\]
Using the Sobolev inequality (2.16), we see that
\[
I \lesssim \int_{\frac{t}{2}}^{t} \langle t - \tau \rangle^{-\frac{3}{4} \left(\frac{1}{2} - \frac{1}{q}\right)} \frac{\langle \eta \rangle^{-1}}{\langle \eta \rangle^{-1}} \|\nabla [\xi] \mathcal{N}(u(\tau))\|_{L^p} \, d\tau
\]  

(3.7)

Combining the estimates (3.7) and (3.8), we conclude
\[
\begin{aligned}
\text{the first assertion (3.2).}
\end{aligned}
\]

Next, we give an estimate of I. Applying Lemma 3.2 with \(s_1 = s, s_2 = 0\) and \(\gamma \in [1, 2]\) determined later, we have
\[
I \lesssim \int_{0}^{\frac{t}{2}} \langle t - \tau \rangle^{-\frac{3}{4} \left(\frac{1}{2} - \frac{1}{q}\right)} \frac{\langle \eta \rangle^{-1}}{\langle \eta \rangle^{-1}} \|\nabla [\xi] \mathcal{N}(u(\tau))\|_{L^p} \, d\tau + \int_{0}^{\frac{t}{2}} e^{-\frac{t - \tau}{2}} \|\nabla [\xi] \mathcal{N}(u(\tau))\|_{L^p} \, d\tau
\]

\[
=: I_1 + I_2.
\]

In the same way as the proof of Lemma 2.1, it is easy to see that \(I_2 \lesssim e^{-\frac{t}{2}} \|\mathcal{N}(u)\|_{Y(\infty)}\). Also, for \(I_1\), taking \(\gamma = r\) if \(p > 1 + \frac{2}{\gamma}\) and \(\gamma = \sigma_1\) if \(p = 1 + \frac{2}{\gamma}, r > 1\), respectively, we can prove
\[
I_1 \lesssim \|\mathcal{N}(u)\|_{Y(\infty)} \langle t \rangle^{-\frac{3}{4} \left(\frac{1}{2} - \frac{1}{q}\right)} - \frac{3}{4}.
\]  

(3.8)

Combining the estimates (3.7) and (3.8), we conclude
\[
\left\|\nabla [\xi] \int_{0}^{t} (\mathcal{D}(t - \tau) - \mathcal{G}(t - \tau)) \mathcal{N}(u(\tau)) \, d\tau \right\|_{L^2} \lesssim \langle t \rangle^{-\frac{3}{4} \left(\frac{1}{2} - \frac{1}{q}\right)} - \frac{3}{4} - \min \{1, \frac{p}{2} (p - 1) - \frac{1}{2}\}.
\]

The estimate of the linear part of \(u - v\) is obvious from Lemmas 3.2 and 3.3. Thus, we obtain the first assertion (3.2).

Secondly, we prove (3.3). As before, we divide the nonlinear term of \(u - v\) as
\[
\int_{0}^{t} (\mathcal{D}(t - \tau) - \mathcal{G}(t - \tau)) \mathcal{N}(u(\tau)) \, d\tau
\]

\[
=: III + IV.
\]

Making use of Lemma 3.2 with \(s_1 = s_2 = 0\) and \(\gamma = q\), and then using the Sobolev embedding \(\|\nabla [\xi]^{-1} \psi\|_{L^2} \lesssim \|\psi\|_{L^q}\), we see that
\[
IV \lesssim \int_{\frac{t}{2}}^{t} \langle t - \tau \rangle^{-\frac{3}{4} \left(\frac{1}{2} - \frac{1}{q}\right)} - \frac{3}{4} - \frac{1}{4} \|\nabla [\xi] \mathcal{N}(u(\tau))\|_{L^p} \, d\tau
\]

\[
\lesssim \int_{\frac{t}{2}}^{t} \langle t - \tau \rangle^{-\frac{3}{4} \left(\frac{1}{2} - \frac{1}{q}\right)} - \frac{3}{4} \|\nabla [\xi] \mathcal{N}(u(\tau))\|_{L^p} \, d\tau.
\]

Noting \(-\frac{3}{4} \left(\frac{1}{4} - \frac{1}{2}\right) - 1 = -\frac{3}{2} < -1\), we further estimate
\[
IV \lesssim \|\mathcal{N}(u)\|_{Y(\infty)} \int_{\frac{t}{2}}^{t} \langle t - \tau \rangle^{-\frac{3}{4} \left(\frac{1}{2} - \frac{1}{q}\right)} - \frac{3}{4} \left(\frac{p}{4} - \frac{1}{q}\right) \, d\tau
\]

\[
\lesssim \langle t \rangle^{-\frac{3}{4} \left(\frac{1}{2} - \frac{1}{q}\right) + \frac{1}{2} - \frac{p}{2} (p - 1)}.
\]  

(3.9)
Furthermore, taking $u$ and hence, we have

$$\text{Noting we deduce that the solution assertion (3.3).}$$

$$\beta \approx \int\frac{3}{2}(t - \tau)^{-\frac{\alpha}{2} - \frac{1}{2}} \|N(u(\tau))\|_{L^\gamma} d\tau + \int_0^t e^{-\frac{t}{\alpha}} \|\nabla\|^{-1} N(u(\tau))\|_{L^2} d\tau$$

$$\therefore III_1 + III_2.$$ 

In the same manner as the proof of Lemma [2.4] we see that the term $III_2$ is bounded by $e^{-\frac{t}{\alpha}} \|N(u)\|_{Y(\infty)}$. Furthermore, taking $\gamma = r$ if $p > 1 + \frac{2n}{n}$ and $\gamma = \sigma_1$ if $p = 1 + \frac{2n}{n}$, $r > 1$, respectively, we have

$$\text{III}_1 \lesssim \| N(u) \|_{Y(\infty)}(t)^{-\frac{\alpha}{2} + \frac{1}{2}}.$$  

(3.10)

Summing up the estimates (3.9) and (3.10), we conclude

$$\left\| \int_0^t (D(t - \tau) - G(t - \tau)) N(u(\tau)) d\tau \right\|_{L^2} \lesssim \left( t^{-\frac{\alpha}{2} + \frac{1}{2}} \right)^{-\min(1, \frac{\alpha}{p} - (p - 1) - \frac{1}{2})}.$$ 

The linear part of $u - v$ is easily estimated from Lemmas [3.2] and [3.3] and we reach the second assertion (3.3).

Finally, we prove (3.4). We take $\beta$ satisfying $n(\frac{1}{2} - \frac{1}{2}) < \beta \leq \alpha$. Then, by the interpolation

$$\| \langle \cdot \rangle^\beta \psi \|_{L^q} \leq \| \langle \cdot \rangle^\alpha \psi \|_{L^q}^{\beta \alpha} \| \langle \cdot \rangle^\beta \psi \|_{L^q}^{1 - \beta \alpha},$$

we deduce that the solution $u$ satisfies

$$\langle \cdot \rangle^\beta \langle \cdot \rangle^\alpha N(u(t))\| L^q \lesssim \left( \langle \cdot \rangle^\alpha \langle \cdot \rangle^\beta N(u(\tau))\| L^q \right)^{\frac{\alpha}{\alpha}} \left( \langle \cdot \rangle^\alpha \| N(u(\tau))\| L^q \right)^{1 - \frac{\alpha}{\alpha}} \lesssim 1$$

with $\zeta_\beta = \frac{\alpha}{2\alpha(p - 1) - \frac{1}{2} + \frac{n}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{\beta}{2}$. From Lemma [3.2] with $\nu = q$ and $\gamma \in [1, 2]$ determined later, we obtain

$$\left\| \int_0^t (D(t - \tau) - G(t - \tau)) N(u(\tau)) d\tau \right\|_{L^2} \lesssim \left( \langle t \rangle^\frac{\alpha}{2} + \frac{1}{2} \right)^{-\frac{\alpha}{2} - 1} \| N(u(\tau))\|_{L^\gamma} d\tau + \int_0^t \langle t - \tau \rangle^{-\frac{\alpha}{2} - \frac{1}{2}} \| \nabla\|^{-1} \| \cdot \| \| N(u(\tau))\|_{L^2} d\tau$$

$$\therefore V_1 + V_2 + V_3.$$ 

Lemma [A.2] implies

$$V_3 \lesssim \int_0^t e^{-\frac{t}{\alpha}} \| \langle \cdot \rangle^\beta N(u(\tau))\|_{L^q} d\tau$$

and hence, we have

$$V_2 + V_3 \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{\alpha}{2} \left( \frac{1}{2} - \frac{1}{2} \right) - 1} \| \langle \cdot \rangle^\beta N(u(\tau))\|_{L^q} d\tau.$$ 

Noting $-\frac{\alpha}{2} \left( \frac{1}{2} - \frac{1}{2} \right) - 1 = -\frac{3}{2}$, we proceed the estimate as

$$V_2 + V_3 \lesssim \| N(u) \|_{Y(\infty)} \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \langle \cdot \rangle^{-\zeta_\beta} d\tau$$

$$\lesssim \langle t \rangle^{-\frac{\alpha}{2} \left( \frac{1}{2} - \frac{1}{2} \right) + \frac{\beta}{\alpha} + \frac{1}{\alpha} - (p - 1) + \frac{1}{2} + \left\{ \begin{array}{ll}
\langle t \rangle^{-\frac{\alpha}{2} \left( \frac{1}{2} - \frac{1}{2} \right) + \frac{\beta}{\alpha} - \frac{1}{\alpha} (p - 1) + \frac{1}{2} + \frac{1}{\alpha} \log(1 + t) & (\zeta_\beta < 1), \\
\langle t \rangle^{-\frac{\alpha}{2} \left( \frac{1}{2} - \frac{1}{2} \right) + \frac{\beta}{\alpha} - \frac{1}{\alpha} (p - 1) + \frac{1}{2} + \frac{1}{\alpha} \log(1 + t) & (\zeta_\beta = 1), \\
\langle t \rangle^{-\frac{3}{2}} & (\zeta_\beta > 1). 
\end{array} \right.$$
For $V_1$, as in the proof of Lemma 2.4 taking $\gamma = r$ if $p > 1 + \frac{2r}{n}$ and $\gamma = \sigma_1$ if $p = 1 + \frac{2r}{n}$, $r > 1$, respectively, we can see that

$$V_1 \lesssim \langle t \rangle^{-\frac{\eta}{2}(1-\frac{1}{2})^+ + \frac{\eta}{2} - 1}.$$ 

Consequently, we obtain

$$\left\| \cdot ^{\beta } \int_0^t (D(t-\tau) - G(t-\tau)) N(u(\tau)) \, d\tau \right\|_{L^2} \lesssim \langle t \rangle^{-\frac{\eta}{2}(1-\frac{1}{2})^+ + \frac{\eta}{2} - \min\{1, \frac{\eta}{2}(p-1) - 1\}},$$

which shows the third assertion (3.4) and finishes the proof of Proposition 3.1.

### 3.4. Proof of Theorem 1.3

By virtue of Proposition 3.1, the proof of Theorem 1.3 reduces to the analysis of the asymptotic behavior of solutions to the inhomogeneous heat equation (3.1) with the initial data $\varepsilon (u_0 + u_1)$. In this subsection, we prove the following:

**Proposition 3.4.** Under the assumption of Theorem 1.3, Let $\beta$ satisfy $\eta(\frac{1}{2} - \frac{1}{2}) < \beta \leq \alpha$. Then, the solution $v$ to the equation (3.1) with the initial data $\varepsilon (u_0 + u_1)$ satisfies the following asymptotic behavior: When $r > 1$, we have

$$\| \nabla \|^s (v(t) - \varepsilon G(t)(u_0 + u_1)) \|_{L^2} \lesssim \langle t \rangle^{-\frac{\eta}{2}(1-\frac{1}{2})^+ - \min\{\frac{\eta}{2}(1-\frac{1}{2}), \frac{1}{2}, \frac{\eta}{2}(p-1) - 1\} + \zeta},$$

$$\| v(t) - \varepsilon G(t)(u_0 + u_1) \|_{L^2} \lesssim \langle t \rangle^{-\frac{\eta}{2}(1-\frac{1}{2})^+ - \min\{\frac{\eta}{2}(1-\frac{1}{2}), \frac{1}{2}, \frac{\eta}{2}(p-1) - 1\} + \zeta},$$

$$\| \cdot ^{\beta } (v(t) - \varepsilon G(t)(u_0 + u_1)) \|_{L^2} \lesssim \langle t \rangle^{-\frac{\eta}{2}(1-\frac{1}{2})^+ + \frac{\eta}{2} - \min\{\frac{\eta}{2}(1-\frac{1}{2}), \frac{1}{2}, \frac{\eta}{2}(p-1) - 1\} + \zeta},$$

for $t \geq 1$. When $r = 1$, we have

$$\| v(t) - \theta G(t) \|_{L^m} \lesssim \langle t \rangle^{-\frac{\eta}{2}(1-\frac{1}{2})^+ - \min\{\frac{\eta}{2}(1-\frac{1}{2}), \frac{1}{2}, \frac{\eta}{2}(p-1) - 1\} + \zeta}$$

for $t \geq 1$.

To prove this proposition, we first prepare the following lemma.

**Lemma 3.5.** Let $s_1 \geq s_2 \geq 0$, $\beta \geq 0$ and $\gamma, \nu \in [1, 2]$. Then, we have

$$\| \nabla \|^s G(t) \psi \|_{L^2} \lesssim \langle t \rangle^{-\frac{\eta}{2}(1-\frac{1}{2})^+ - \min\{\frac{\eta}{2}(1-\frac{1}{2}), \frac{1}{2}, \frac{\eta}{2}(p-1) - 1\} + \zeta} \| \nabla \|_{L^2} \psi \|_{L^\gamma},$$

$$\| \cdot ^{\beta } G(t) \psi \|_{L^2} \lesssim \langle t \rangle^{-\frac{\eta}{2}(1-\frac{1}{2})^+ + \frac{\eta}{2} + \nu \psi \|_{L^\gamma} + \langle t \rangle^{-\frac{\eta}{2}(1-\frac{1}{2})^+ + \frac{\eta}{2} + \nu \psi \|_{L^\nu}$$

for $t \geq 1$.

**Proof.** The first assertion is trivial. Noting $\| \cdot ^{\beta } G(t) \|_{L^k} \lesssim t^{-\frac{\eta}{2}(1-\frac{1}{2})^+ + \frac{\eta}{2}}$, we can easily prove the second one in the same way as Lemma 2.1.

**Proof of Proposition 3.4.** For simplicity, we treat only higher dimensional cases $n \geq 2$. The one-dimensional case can be proved in a similar way. At first we shall consider the case $r > 1$. In this case, it suffices to estimate

$$v(t) - \varepsilon G(u_0 + u_1) = \int_0^t G(t-\tau) N(u(\tau)) \, d\tau.$$
Let us start with

\[ \left\| \nabla^s \int_0^t G(t - \tau) \mathcal{N}(u(\tau)) \, d\tau \right\|_{L^2} \lesssim \int_0^{\frac{1}{4}} \left\| \nabla^s G(t - \tau) \mathcal{N}(u(\tau)) \right\|_{L^2} \, d\tau + \int_{\frac{1}{4}}^t \left\| \nabla^s G(t - \tau) \mathcal{N}(u(\tau)) \right\|_{L^2} \, d\tau =: I + II. \]

Applying Lemma 3.5 with \( s_1 = s, \ s_2 = 0 \) and \( \gamma = \sigma_1 \), we have

\[ I \lesssim \int_0^{\frac{1}{4}} \langle t - \tau \rangle^{-\frac{\eta}{2} \left( \frac{1}{\sigma_1} - \frac{1}{2} \right) - \frac{\eta}{2} \left( \frac{p - 1}{\sigma_1} \right)} \left\| \nabla^s \mathcal{N}(u(\tau)) \right\|_{L^{\sigma_1}} \, d\tau \]

\[ \lesssim \| \mathcal{N}(u) \|_{Y(\infty)} \langle t \rangle^{-\eta} \int_0^{\frac{1}{4}} \langle \tau \rangle^{-\frac{\eta}{2} \left( \frac{1}{\sigma_1} - \frac{1}{2} \right) - \frac{\eta}{2} \left( \frac{p - 1}{\sigma_1} \right)} \, d\tau \]

\[ \lesssim \langle t \rangle^{-\frac{\eta}{2} \left( \frac{1}{\sigma_1} - \frac{1}{2} \right) - \frac{\eta}{2} \cdot \min \left( \frac{1}{\rho_1}, \frac{1}{p} - \frac{1}{2} \right) \left( p - 1 \right) + 1}. \]

For \( II \), we apply Lemma 3.5 with \( s_1 = s - [s], \ s_2 = [s] \) and \( \gamma = \rho \) to obtain

\[ II \lesssim \int_0^{\frac{1}{4}} \langle t - \tau \rangle^{-\frac{\eta}{2} \left( \frac{1}{\rho} - \frac{1}{2} \right) - \frac{\eta}{2} \cdot \min \left( \frac{1}{\rho_1}, \frac{1}{p} - \frac{1}{2} \right) \left( p - 1 \right) + 1} \, d\tau. \]

Summing up the above estimates, we have (3.11).

Next, we prove (3.12). First we have

\[ \left\| \int_0^t G(t - \tau) \mathcal{N}(u(\tau)) \, d\tau \right\|_{L^2} \lesssim \int_0^{\frac{1}{4}} \| G(t - \tau) \mathcal{N}(u(\tau)) \|_{L^2} \, d\tau + \int_{\frac{1}{4}}^t \| G(t - \tau) \mathcal{N}(u(\tau)) \|_{L^2} \, d\tau =: III + IV. \]

Lemma 3.5 with \( s_1 = s_2 = 0 \) and \( \gamma = \sigma_1 \) leads to

\[ III \lesssim \int_0^{\frac{1}{4}} \langle t - \tau \rangle^{-\frac{\eta}{2} \left( \frac{1}{\sigma_1} - \frac{1}{2} \right)} \| \mathcal{N}(u(\tau)) \|_{L^{\sigma_1}} \, d\tau \]

\[ \lesssim \| \mathcal{N}(u) \|_{Y(\infty)} \langle t \rangle^{-\eta} \int_0^{\frac{1}{4}} \langle \tau \rangle^{-\frac{\eta}{2} \left( \frac{1}{\sigma_1} - \frac{1}{2} \right) - \frac{\eta}{2} \left( \frac{p - 1}{\sigma_1} \right)} \, d\tau \]

\[ \lesssim \langle t \rangle^{-\frac{\eta}{2} \left( \frac{1}{\sigma_1} - \frac{1}{2} \right) - \frac{\eta}{2} \cdot \min \left( \frac{1}{\rho_1}, \frac{1}{p} - \frac{1}{2} \right) \left( p - 1 \right) + 1}. \]
Also, taking $s_1 = s_2 = 0$ and $\gamma = \sigma_2$ in Lemma 3.5 we see that

$$IV \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_2} - \frac{1}{2}\right)} \|\mathcal{N}(u(\tau))\|_{L^{\sigma_2}} d\tau$$

$$\lesssim \|\mathcal{N}(u)\|_{Y(\infty)} \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_2} - \frac{1}{2}\right)} \int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_2} - \frac{1}{2}\right)} d\tau$$

$$\lesssim \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_2} - \frac{1}{2}\right)} - \frac{1}{2}(p-1)+1,$$

since $-\frac{n}{2} \left(\frac{1}{\sigma_2} - \frac{1}{2}\right) > -1$. Combining the estimates of $III$ and $IV$, we reach the estimate (3.12). Finally, we give a proof of (3.13).

$$\int_0^t ||| \cdot |^{\beta} G(t - \tau) \mathcal{N}(u(\tau)) |||_{L^2} d\tau \lesssim \int_0^t ||| \cdot |^{\beta} G(t - \tau) \mathcal{N}(u(\tau)) |||_{L^2} d\tau$$

$$+ \int_0^t ||| \cdot |^{\beta} G(t - \tau) \mathcal{N}(u(\tau)) |||_{L^2} d\tau$$

$$= IV + VI.$$

For $V$, applying Lemma 3.5 with $\gamma = \sigma_1$ and $\nu = q$, we have

$$V \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_1} - \frac{1}{2}\right)+\frac{\beta}{2}} ||\mathcal{N}(u)||_{L^{\sigma_1}} d\tau + \int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_1} - \frac{1}{2}\right)} ||| \cdot |^{\beta} \mathcal{N}(u)||_{L^q} d\tau$$

$$\lesssim \|\mathcal{N}(u)||_{Y(\infty)} \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_1} - \frac{1}{2}\right)+\frac{\beta}{2}} \int_0^t \langle \tau \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_1} - \frac{1}{2}\right)} d\tau$$

$$+ \|\mathcal{N}(u)||_{Y(\infty)} \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_1} - \frac{1}{2}\right)} \int_0^t \langle \tau \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_1} - \frac{1}{2}\right)} d\tau$$

$$\lesssim \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_1} - \frac{1}{2}\right)+\frac{\beta}{2} - \min \left\{ \frac{n}{2} \left(\frac{1}{\sigma_1} - \frac{1}{2}\right), \frac{n}{2}, \frac{n}{2} (p-1) - 1, \right\} + \beta.$$

For $VI$, letting $\gamma = \sigma_2$ and $\nu = q$ in Lemma 3.5 we see that

$$VI \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_2} - \frac{1}{2}\right)+\frac{\beta}{2}} ||\mathcal{N}(u(\tau))||_{L^{\sigma_2}} d\tau$$

$$+ \int_0^t \langle t - \tau \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_2} - \frac{1}{2}\right)} ||| \cdot |^{\beta} \mathcal{N}(u(\tau)) |||_{L^q} d\tau$$

$$\lesssim \|\mathcal{N}(u)||_{Y(\infty)} \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_2} - \frac{1}{2}\right)+\frac{\beta}{2}} \int_0^t \langle \tau \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_2} - \frac{1}{2}\right)} d\tau$$

$$+ \|\mathcal{N}(u)||_{Y(\infty)} \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_2} - \frac{1}{2}\right)} \int_0^t \langle \tau \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_2} - \frac{1}{2}\right)} d\tau$$

$$\lesssim \langle t \rangle^{-\frac{n}{2} \left(\frac{1}{\sigma_2} - \frac{1}{2}\right)+\frac{\beta}{2} - \frac{1}{2}(p-1)+1}.$$

Summing up the above estimates, we have (3.13) and complete the proof when $r > 1$.

Next, we give a proof of the case $r = 1$. We put

$$\theta = \theta_1 + \theta_2, \quad \theta_1 = \varepsilon \int_{\mathbb{R}^n} (u_0 + u_1) \, dx, \quad \theta_2 = \int_0^\infty \int_{\mathbb{R}^n} \mathcal{N}(u(\tau)) \, dx \, d\tau.$$
We claim that for \( m \in [1, \infty] \), it follows that
\[
\|\varepsilon G(t)(u_0 + u_1) - \theta_1 G(t)\|_{L^m} = o(t^{-\frac{m}{2}(1 - \frac{1}{m})})
\]  
(3.15)
as \( t \to \infty \) for \( u_0 + u_1 \in L^1 \) and
\[
\|\varepsilon G(t)(u_0 + u_1) - \theta_1 G(t)\|_{L^m} \lesssim \langle t \rangle^{-\frac{m}{2}(1 - \frac{1}{m}) - \min\{\frac{1}{2}, \frac{3}{2}\}}\|\langle \cdot \rangle^\delta(u_0 + u_1)\|_{L^1}
\]  
(3.16)for \( t \geq 1 \) and \( u_0 + u_1 \in H^{0, \alpha} \), where \( \delta \) is an arbitrary number satisfying \( 0 < \delta < \alpha - \frac{n}{2} \). Here we also note that \( \|\langle \cdot \rangle^\delta f\|_{L^1} \lesssim \|f\|_{H^{0, \alpha}} \). To prove (3.15), we first consider the case \( m = 1 \). We write \( \phi = \varepsilon(u_0 + u_1) \) and have
\[
\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (G(t, x - y) - G(t, x))\phi(y) \, dy \right| \, dx 
\]
\[
\lesssim \int_{\mathbb{R}^n} \left| \int_{|y| < gt^{1/2}} (G(t, x - y) - G(t, x))\phi(y) \, dy \right| \, dx 
\]
\[
+ \int_{\mathbb{R}^n} \left| \int_{|y| > gt^{1/2}} (G(t, x - y) - G(t, x))\phi(y) \, dy \right| \, dx =: I + II,
\]
where \( \varrho > 0 \) is an arbitrary small number. For \( I \), we use the mean value theorem
\[
|G(t, x - y) - G(t, x)| \leq 2(4\pi t)^{-\frac{m}{2}} t^{-\frac{1}{2}} |y| \int_0^1 \frac{|x - ay|}{\sqrt{t}} e^{-t|x - ay|^2} \, da.
\]
To prove (3.15), noting that \( t^{-\frac{1}{2}}|y| \leq \varrho \). Thus, we have \( I \leq C\varrho \). For \( II \), we easily estimate
\[
II \lesssim \int_{|y| > gt^{1/2}} \left( \int_{\mathbb{R}^n} |G(t, x - y) - G(t, x)| \, dx \right) |\phi(y)| \, dy \lesssim \int_{|y| > gt^{1/2}} |\phi(y)| \, dy.
\]
Taking \( t \) sufficiently large, we have \( II \leq \varrho \). Consequently, we obtain \( I + II \leq C\varrho \) for sufficiently large \( t \). Since \( \varrho \) is arbitrary, this proves (3.15). Similarly, we can easily prove (3.15) in the case \( m = \infty \), and then, by the interpolation, we can obtain (3.15) for all \( m \in [1, \infty] \).

To prove (3.16), noting \( |y| \leq |y|^{\min\{\delta, 1\}}t^{\frac{1}{2}(1 - \min\{\delta, 1\})} \) on the integral region of \( I \) with \( \varrho = 1 \), and then, by the Fubini theorem, we see that
\[
I \lesssim t^{-\frac{1}{2} + \frac{1}{2}(1 - \min\{\delta, 1\})} \int_{\mathbb{R}^n} |y|^{\min\{\delta, 1\}} |\phi(y)| \, dy \lesssim t^{-\min\{\frac{\delta}{2}, \frac{1}{2}\}} \|\langle \cdot \rangle^\delta \phi\|_{L^1}.
\]
On the other hand, the term \( II \) is easily estimated as
\[
II \lesssim \int_{|y| > gt^{1/2}} \left( \int_{\mathbb{R}^n} |G(t, x - y) - G(t, x)| \, dx \right) |\phi(y)| \, dy 
\]
\[
\lesssim t^{-\frac{\delta}{2}} \int_{\mathbb{R}^n} |y|^{\delta} |\phi(y)| \, dy.
\]
Thus, we obtain (3.16) when \( m = 1 \). In a similar way, we can prove (3.16) when \( m = \infty \) and hence, the interpolation gives (3.16) for all \( m \in [1, \infty] \).

Next, we claim that
\[
\left\| \int_0^t G(t - \tau)\mathcal{N}(u(\tau)) \, d\tau - \theta_2 G(t) \right\|_{L^m} = o(t^{-\frac{m}{2}(1 - \frac{1}{m})})
\]  
(3.17)as \( t \to \infty \) and
\[
\left\| \int_0^t G(t - \tau)\mathcal{N}(u(\tau)) \, d\tau - \theta_2 G(t) \right\|_{L^m} \lesssim \langle t \rangle^{-\frac{m}{2}(1 - \frac{1}{m}) - \min\{\frac{1}{2}, \frac{3}{2}\}(p-1) + \varepsilon}
\]  
(3.18)
for all \( t \geq 1 \) and \( m \) satisfying (1.3). Indeed, we first divide the left-hand side as
\[
\left\| \int_0^t G(t-\tau)\mathcal{N}(u(\tau))\,d\tau - \theta_2 G(t) \right\|_{L^m} \\
\leq \int_0^t \left\| G(t-\tau)\mathcal{N}(u(\tau)) - G(t-\tau) \left( \int_{\mathbb{R}^n} \mathcal{N}(u(\tau))\,dx \right) \right\|_{L^m}\,d\tau \\
+ \int_0^t \left\| (G(t-\tau) - G(t)) \left( \int_{\mathbb{R}^n} \mathcal{N}(u(\tau))\,dx \right) \right\|_{L^m}\,d\tau \\
+ \int_0^t \|G(t-\tau)\mathcal{N}(u(\tau))\|_{L^m}\,d\tau + \int_{\frac{t}{2}}^\infty \|G(t)\left( \int_{\mathbb{R}^n} \mathcal{N}(u(\tau))\,dx \right)\|_{L^m}\,d\tau
=: III_m + IV_m + V_m + VI_m.
\]
Let us start with \( VI_m \). Since \( \|G(t)\|_{L^m} = Ct^{-\frac{\alpha}{2}\left(1 - \frac{1}{m}\right)} \) and \( \|\mathcal{N}(u(\tau))\|_{L^1} \lesssim \langle \tau \rangle^{-\frac{\alpha}{2}(p-1)} \), it is easy to see that
\[
VI_m \lesssim t^{-\frac{\alpha}{2}\left(1 - \frac{1}{m}\right) - \frac{\alpha}{2}(p-1)+1}.
\]
Secondly, we estimate \( V_m \). For simplicity, we only consider the case \( n > 2s \). When \( m = \frac{2n}{n-2s} \), the Sobolev inequality gives
\[
V_{\frac{2n}{n-2s}} \lesssim \int_0^t \|\nabla^s G(t-\tau)\mathcal{N}(u(\tau))\|_{L^2} d\tau \\
\lesssim \int_0^t \langle t-\tau \rangle^{-\frac{\alpha}{2}\left(1 - \frac{1}{2}\right) - \frac{2-s}{2} \|\nabla^s \mathcal{N}(u(\tau))\|_{L^p}} d\tau \\
\lesssim \int_0^t \langle t-\tau \rangle^{-\frac{\alpha}{2}\left(1 - \frac{1}{2}\right) - \frac{2-s}{2} \langle \tau \rangle^{-s}} d\tau \\
\lesssim \langle t \rangle^{-\frac{\alpha}{2}\left(1 - \frac{1}{2}\right) - \frac{\alpha}{2}(p-1)+1}.
\]
Also, by \( \|\mathcal{N}(u(\tau))\|_{L^1} \lesssim \langle \tau \rangle^{-\frac{\alpha}{2}(p-1)} \), we immediately obtain \( V_1 \lesssim \langle t \rangle^{-\frac{\alpha}{2}(p-1)+1} \). Interpolating these two cases, we conclude that
\[
V_m \lesssim \langle t \rangle^{-\frac{\alpha}{2}\left(1 - \frac{1}{2}\right) - \frac{\alpha}{2}(p-1)+1}
\]
for all \( m \in \left[1, \frac{2n}{n-2s}\right] \). The case \( n \leq 2s \) can be also proved by using a similar interpolation as in the proof of Lemma 2.1 (ii).

Next, we consider \( IV_m \). It follows from \( \|\mathcal{N}(u(\tau))\|_{L^1} \lesssim \langle \tau \rangle^{-\frac{\alpha}{2}(p-1)} \) and the fundamental theorem of calculus
\[
G(t-\tau, x) - G(t, x) = -\tau \int_0^1 (\partial_t G)(t-a\tau, x)\,da
\]
we deduce that
\[
IV_m \lesssim \int_0^t \langle t-\tau \rangle^{-\frac{\alpha}{2}\left(1 - \frac{1}{2}\right)-1} \langle \tau \rangle^{-\frac{\alpha}{2}(p-1)} d\tau \lesssim \langle t \rangle^{-\frac{\alpha}{2}\left(1 - \frac{1}{2}\right)} \begin{cases} \langle t \rangle^{-1} & (p > 1 + \frac{4}{n}), \\ \langle t \rangle^{-1} \log \langle t \rangle & (p = 1 + \frac{4}{n}), \\ \langle t \rangle^{-\frac{\alpha}{2}(p-1)+1} & (p < 1 + \frac{4}{n}). \end{cases}
\]
Finally, we estimate \( III_m \). To prove (3.17), noting \( \mathcal{N}(u(\tau)) \in L^\infty(0, \infty; L^1(\mathbb{R}^n)) \) we can prove
\( III_m = o(t^{-\frac{\alpha}{2}\left(1 - \frac{1}{2}\right)}) \) as \( t \to \infty \).
To show (3.18), applying (3.16), we have

\[ III_m \lesssim \int_0^t (t - \tau)^{-\frac{\alpha}{\beta} \left(1 - \frac{1}{m}\right) - \min \left(\frac{1}{2}, \frac{1}{2}\right)} \| \psi \|^2 \langle u \rangle \| u \|_{L^2} \, d\tau. \]

Taking \( \delta = \beta \) \((n = 2)\), \( \delta = 1 \) \((n \geq 3)\), we conclude

\[ III_m \lesssim \langle t \rangle^{-\frac{\alpha}{\beta} \left(1 - \frac{1}{m}\right) - \min \left(\frac{1}{2}, \frac{1}{2}\right) (p-1)} + \varsigma. \]

Putting together the estimates above, we reach (3.18).

From Propositions 3.1 and 3.4 we finish up the proof of Theorem 1.3.

**Proof of Theorem 1.3.** We start with the case \( r > 1 \). Let \( m \) be a real number satisfying (1.3). First, we assume that \( m \geq 2 \). In this case, the Sobolev embedding theorem implies \( \| \psi \|_{L^m} \leq \| \nabla [^{s_m} \psi] \|_{L^2} \) with \( s_m = n \left(\frac{1}{2} - \frac{1}{m}\right) \in [0, s] \). This and the interpolation inequality

\[ \| \nabla [^{s_m} \psi] \|_{L^2} \leq \| \nabla [^s \psi] \|_{L^2} \| \psi \|_{L^2}^{1-\frac{s_m}{s}} \]

lead to

\[ \| u(t) - \varepsilon G(t)(u_0 + u_1) \|_{L^m} \leq \| \nabla [^s (u(t) - \varepsilon G(t)(u_0 + u_1))] \|_{L^2} \| u(t) - \varepsilon G(t)(u_0 + u_1) \|_{L^2}^{1-\frac{s_m}{s}}. \]

Using Propositions 3.1 and 3.4 we calculate

\[ \| \nabla [^s (u(t) - \varepsilon G(t)(u_0 + u_1))] \|_{L^2} \leq \| \nabla [^s (u(t) - v(t))] \|_{L^2} + \| \nabla [^s (v(t) - \varepsilon G(t)(u_0 + u_1))] \|_{L^2} \]

\[ \lesssim \langle t \rangle^{-\frac{\alpha}{\beta} \left(1 - \frac{1}{m}\right) - \min \left(\frac{1}{2}, \frac{1}{2}\right) (p-1)} + \langle t \rangle^{-\frac{\alpha}{\beta} \left(1 - \frac{1}{m}\right)} + \min \left(\frac{1}{2}, \frac{1}{2}\right) (p-1)} + \varsigma \]

\[ \lesssim \langle t \rangle^{-\frac{\alpha}{\beta} \left(1 - \frac{1}{m}\right) - \min \left(\frac{1}{2}, \frac{1}{2}\right) (p-1)} + \varsigma. \]

Similarly, we can see that

\[ \| u(t) - \varepsilon G(t)(u_0 + u_1) \|_{L^2} \lesssim \langle t \rangle^{-\frac{\alpha}{\beta} \left(1 - \frac{1}{m}\right) - \min \left(\frac{1}{2}, \frac{1}{2}\right) (p-1)} + \varsigma. \]

These two estimates yield

\[ \| u(t) - \varepsilon G(t)(u_0 + u_1) \|_{L^m} \lesssim \langle t \rangle^{-\frac{\alpha}{\beta} \left(1 - \frac{1}{m}\right) - \min \left(\frac{1}{2}, \frac{1}{2}\right) (p-1)} + \varsigma \]

\[ = \langle t \rangle^{-\frac{\alpha}{\beta} \left(1 - \frac{1}{m}\right) - \min \left(\frac{1}{2}, \frac{1}{2}\right) (p-1)} + \varsigma, \]

which shows the desired estimate.

When \( m \in [r, 2] \), by the interpolation

\[ \| \psi \|_{L^m} \leq \| \psi \|_{L^r}^{\frac{r(2-m)}{m(2-r)}} \| \psi \|_{L^2}^{\frac{2(m-r)}{m(2-r)}} \leq \| \psi \|_{L^r}^{\beta} \| \psi \|_{L^2}^{2(m-r)} \]

with \( \beta \) satisfying \( n \left(\frac{1}{r} - \frac{1}{2}\right) < \beta \leq \alpha \) and \( \beta - n \left(\frac{1}{r} - \frac{1}{2}\right) \frac{r(2-m)}{m(2-r)} < \varsigma \). Thus, it suffices to estimate the right-hand side with \( \psi = u(t) - \varepsilon G(t)(u_0 + u_1) \). In the same way as before, we can deduce that

\[ \| \psi(t) - \varepsilon G(t)(u_0 + u_1) \|_{L^2} \lesssim \langle t \rangle^{-\frac{\alpha}{\beta} \left(1 - \frac{1}{m}\right)} + \beta \min \left(\frac{1}{2}, \frac{1}{2}\right) (p-1)} + \varsigma \]


and hence, we have
\[
\|u(t) - \varepsilon G(t)(u_0 + u_1)\|_{L^m} \lesssim \langle t \rangle^{-\frac{\alpha}{2} \left(\frac{1}{r} - \frac{1}{p} - \frac{1}{2}\right)} + \frac{C}{m(2s-m)} \cdot \min\{\frac{\alpha}{2} (1 - \frac{1}{p}), \frac{\alpha}{2}, \frac{m}{p} (p-1)\} + \varsigma
\]
\[
= \langle t \rangle^{-\frac{\alpha}{2} \left(\frac{1}{r} - \frac{1}{p} - \frac{1}{2}\right)} \cdot \frac{C}{m(2s-m)} \cdot \min\{\frac{\alpha}{2} (1 - \frac{1}{p}), \frac{\alpha}{2}, \frac{m}{p} (p-1)\} + \varsigma
\]
\[
= \langle t \rangle^{-\frac{\alpha}{2} \left(\frac{1}{r} - \frac{1}{p} - \frac{1}{2}\right)} - \min\{\frac{\alpha}{2} (1 - \frac{1}{p}), \frac{\alpha}{2}, \frac{m}{p} (p-1)\} + \frac{3\varsigma}{2}.
\]

Rewriting \(\frac{3\varsigma}{2}\) as \(\varsigma\), we complete the proof.

When \(r = 1\), as before, we first divide
\[
\|u(t) - \theta G(t)\|_{L^m} \lesssim \|u(t) - v(t)\|_{L^m} + \|v(t) - \theta G(t)\|_{L^m}.
\]

Let \(\beta\) satisfy \(\frac{\beta}{2} < \beta \leq \alpha\) and \((\beta - \frac{\beta}{2})(\frac{1}{m} - \frac{1}{2}) < \varsigma\). Then, the first term is estimated as
\[
\|u(t) - v(t)\|_{L^m} \lesssim \langle t \rangle^{-\frac{\beta}{2} (1 - \frac{1}{m})} \begin{cases} \langle t \rangle^{-\min\{1, \frac{\beta}{2} (p-1)\} - \frac{1}{2}} & (m \geq 2), \\ \langle t \rangle^{\beta - \frac{\beta}{2}} (\frac{1}{m} - \frac{1}{2}) - \min\{1, \frac{\beta}{2} (p-1)\} - \frac{1}{2} & (1 \leq m < 2) \end{cases}
\]
\[
\lesssim \langle t \rangle^{-\frac{\beta}{2} (1 - \frac{1}{m})} - \min\{1, \frac{\beta}{2} (p-1)\} - \frac{1}{2} + \varsigma.
\]

For the second term, we apply Proposition 3.4 and comparing the decay rate leads to the conclusion.

\[
□
\]

4. Blow-up and estimates of the lifespan

In this section, we give a proof of Theorems 1.4 and 1.5.

4.1. Estimates of the lifespan from below. At first, we prove Theorem 1.4.

Proof of Theorem 1.4. We first consider Case 1, namely, we assume \(r \in [1, 2]\) and \(\min\{1 + \frac{\beta}{2}, 1 + \frac{r}{2}\} \leq p < 1 + \frac{2\beta}{m}\).

If \(T(\varepsilon) = \infty\), then the assertion of the theorem is obviously true. Hence, in what follows we assume \(T(\varepsilon) < \infty\). Then, Theorem 1.1 shows that
\[
\lim_{t \to T(\varepsilon)} \|u(t)\|_{H^{s,0} \cap H^{0,\alpha}} = \infty.
\]

On the other hand, by Lemmas 2.4 and 2.5 we have
\[
\left\| \int_0^t D(t - \tau) \mathcal{N}(u(\tau)) \, d\tau \right\|_{X(T)} \lesssim \|\mathcal{N}(u)\|_{Y(T)} \langle T \rangle^{1 - \frac{\alpha}{2} (p-1)}
\]
\[
\lesssim \|u\|_{X(T)} \langle T \rangle^{1 - \frac{\alpha}{2} (p-1)}.
\]

This implies
\[
\|u\|_{X(T)} \leq \varepsilon C_0 |I_0| + C_1 \|u\|_{X(T)} \langle T \rangle^{1 - \frac{\alpha}{2} (p-1)}
\]
with some constant \(C_0, C_1 > 0\), where \(I_0 = \|u_0\|_{H^{s,0} \cap H^{0,\alpha}} + \|u_1\|_{H^{s-1,0} \cap H^{0,\alpha}}\). Now, by (4.1) and the continuity of \(\|u\|_{X(T)}\) with respect to \(T\), there exists the smallest time \(T(\varepsilon)\) such that \(\|u\|_{X(T(\varepsilon))} = 2C_0 |I_0| \varepsilon\) holds. Then, letting \(T = T(\varepsilon)\) in the above inequality gives
\[
2C_0 |I_0| \varepsilon \leq C_0 |I_0| \varepsilon + C_1 (2C_0 |I_0| \langle T(\varepsilon) \rangle^{1 - \frac{\alpha}{2} (p-1)}).
\]

We rewrite it as
\[
C_0 |I_0| \varepsilon^{-(p-1)} \leq C_1 (2C_0 |I_0| \langle T(\varepsilon) \rangle^{1 - \frac{\alpha}{2} (p-1)}),
\]
which implies
\[
\langle T(\varepsilon) \rangle \gtrsim \varepsilon^{-1/\omega}
\]
with \( \omega = \frac{1}{p-1} - \frac{n}{2} \). Thus, taking \( \varepsilon \) sufficiently small, we easily see that \( \langle \bar{T}(\varepsilon) \rangle \geq 1 \), which enables us to replace \( \langle \bar{T}(\varepsilon) \rangle \) by \( \bar{T}(\varepsilon) \) in the above estimate. This and \( \bar{T}(\varepsilon) < T(\varepsilon) \) imply the desired estimate.

In Case 2, that is, when \( r = 1 \) and \( p = 1 + \frac{2}{n} \), the first author and Ogawa [19] have already proved the conclusion of Theorem 1.4 by a slightly different argument.

Instead of (4.2), we obtain
\[
\|u\|_{X(T)} \leq \varepsilon C_0 I_0 + C_1 \|u\|^p_{X(T)} \log(T).
\]

In the same manner as above, we can see that
\[
\langle \bar{T}(\varepsilon) \rangle \geq \exp \left( C \varepsilon^{-(p-1)} \right),
\]
which gives the desired conclusion.

4.2. Estimates of the lifespan from above. In this subsection, we give a proof of Theorem 1.5. The proof is based on the test function method introduced by Zhang [68] and refined to estimate the lifespan by Kuiper [33], Sun [60] and [20] in which initial data belonging to \( L^1 \) are treated. Here we further adapt their method to fit initial data not belonging to \( L^1 \) by the argument in [18].

Before proving Theorem 1.5, we introduce the definition of weak solutions of (1.1). Let \( T > 0 \) and \( u_0, u_1 \in L^1_{\text{loc}}(\mathbb{R}^n) \). We say that a function \( u \in L^1_{\text{loc}}([0, T) \times \mathbb{R}^n) \) is a weak solution of (1.1) on the interval \([0, T)\) if the identity
\[
\int_{[0, T) \times \mathbb{R}^n} u \left( \psi_t - \Delta \psi - \psi_t \right) \, dx \, dt = \varepsilon \int_{\mathbb{R}^n} \left( (u_0 + u_1) \psi(0, x) - u_0 \psi(0, x) \right) \, dx + \int_{[0, T) \times \mathbb{R}^n} N(u) \psi \, dx \, dt
\]
is valid for any \( \psi \in C_0^\infty([0, T) \times \mathbb{R}^n) \). We also define the lifespan of the weak solution:
\[
T_w(\varepsilon) := \sup\{T \in (0, \infty); \text{there exists a weak solution } u \text{ on } [0, T)\}.
\]

**Proposition 4.1.** Under the assumptions of Theorem 1.5, we have
\[
T_w(\varepsilon) \lesssim \varepsilon^{-1/\kappa}
\]
for \( \varepsilon \in (0, 1] \), where \( \kappa = \frac{1}{p-1} - \frac{n}{2} \) and \( \lambda \) satisfies \( \frac{n}{2} + \alpha < \lambda < \frac{2}{p-1} \).

**Proof of Proposition 4.1** We assume that \( N(u) = |u|^p \). The case \( N(u) = -|u|^p \) is reduced to the case above by considering \(-u\).

First, we may assume that \( T_w(\varepsilon) \geq 4 \). Because, if \( T_w(\varepsilon) \leq 4 \), then we immediately obtain \( T_w(\varepsilon) \leq 4 \varepsilon^{-1/\kappa} \) for any \( \varepsilon \in (0, 1] \).

Let \( \eta = \eta(t) \in C_0^\infty([0, \infty)) \) be a test function satisfying
\[
0 \leq \eta(t) \leq 1, \quad \eta(t) = \begin{cases} 1 & (0 \leq t \leq 1/2), \\ 0 & (t \geq 1) \end{cases}
\]
and let \( \phi(x) := \eta(|x|) \). Then, we have (see for example, [50])
\[
|\eta^{(j)}(t)| \lesssim \eta(t)^{1/p} \quad (j = 1, 2), \quad |\Delta \phi(x)| \lesssim \phi(x)^{1/p}.
\tag{4.3}
\]

Let \( \tau \in [1, T_w(\varepsilon)] \) and \( R \in [2, \infty) \) be parameters. We define \( \eta_\tau(t) = \eta(\frac{t}{\tau}) \), \( \phi_R(x) = \phi(\frac{x}{R}) \) and
\[
\psi = \psi_{\tau,R}(t, x) = \eta_\tau(t) \phi_R(x).
\]
We also put
\[ I_{\tau,R} := \int_{[0,\tau) \times B_R} |u|^p \psi_{\tau,R} \, dx \, dt, \quad J_R := \varepsilon \int_{B_R} (u_0 + u_1) \phi_R(x) \, dx, \]
where \( B_R = \{ x \in \mathbb{R}^n; |x| < R \} \). Taking \( \psi = \psi_{\tau,R} \) in the definition of the weak solution, we have the identity
\[ I_{\tau,R} + J_R = \int_{[0,\tau) \times B_R} u \frac{\partial^2}{\partial \tau^2} \psi_{\tau,R} \, dx \, dt - \int_{[0,\tau) \times B_R} u \Delta \psi_{\tau,R} \, dx \, dt - \int_{[0,\tau) \times B_R} u \partial_t \psi_{\tau,R} \, dx \, dt \]
\[ =: K_1 + K_2 + K_3. \]
By the Hölder inequality and (4.3), we have
\[ K_1 \lesssim \tau^{-2} \int_{[0,T) \times B_R} |u| |\eta''(t/\tau)||\phi_R(x)| \, dx \, dt \tag{4.4} \]
\[ \lesssim \tau^{-2} \int_{[0,T) \times B_R} |u| |\eta_\tau(t)^{1/p}||\phi_R(x)| \, dx \, dt \]
\[ \lesssim \tau^{-2} \frac{1}{\tau,R} \left( \int_{\frac{T}{2}}^\tau \int_{B_R} \phi_R \, dx \, dt \right)^{1/p'} \]
\[ \lesssim \tau^{-2 + 1/p'} R^{n/p'} \int_{\tau,R}^{1/p}, \]
where \( p' = \frac{p}{p-1} \). In a similar way, we see that
\[ K_2 \lesssim \tau^{1/p'} R^{-2 + n/p'} \int_{\tau,R}^{1/p} \tag{4.5} \]
and
\[ K_3 \lesssim \tau^{-1 + 1/p'} R^{n/p'} \int_{\tau,R}^{1/p}. \tag{4.6} \]
It follows from (4.4)–(4.6) that
\[ I_{\tau,R} + J_R \lesssim \left( \tau^{1/p'} R^{-2 + n/p'} + \tau^{-1 + 1/p'} R^{n/p'} \right) \int_{\tau,R}^{1/p}. \]
By the Young inequality, the right-hand side is bounded by
\[ \frac{1}{2} I_{\tau,R} + C(\tau R^{-2p'+n} + \tau^{-p'+1} R^n). \]
Therefore, we have
\[ J_R \lesssim \tau R^{-2p'+n} + \tau^{-p'+1} R^n. \]
From the assumption on the initial data, we see that
\[ J_R \geq \varepsilon \int_{|x| > 1} (u_0(x) + u_1(x)) \phi \left( \frac{x}{R} \right) \, dx \]
\[ \geq \varepsilon \int_{|x| > 1} |x|^{-\lambda} \phi \left( \frac{x}{R} \right) \, dx \]
\[ \geq \varepsilon R^{-\lambda+n} \int_{|y| > 1/R} |y|^{-\lambda} \phi(y) \, dy \]
\[ \geq \varepsilon R^{-\lambda+n} \int_{|y| > 1/2} |y|^{-\lambda} \phi(y) \, dy, \]
This completes the proof. □

Proposition 4.2. Under the assumptions of Theorem 1.1, the mild solution \( u \) on \([0, T)\) is also a weak solution on \([0, T)\).

Proof. Let \( \psi \in C_0^\infty([0, T) \times \mathbb{R}^n) \) be a test function and we assume \( \text{supp} \psi \subset [0, T_1) \times \mathbb{R}^n \), where \( T_1 < T \). Then, Theorem 1.1 or Lemma 2.5 yields \( \mathcal{N}(u(\tau)) \in L^\infty(0, T_1; L^q(\mathbb{R}^n)) \). Let \( F_j \in C_0^\infty([0, \infty) \times \mathbb{R}^n) \) \((j = 1, 2, \ldots)\) be a sequence such that
\[
\lim_{j \to \infty} \|F_j - \mathcal{N}(u)\|_{L^\infty(0, T_1; L^s(\mathbb{R}^n))} = 0.
\]

We also set
\[
v_j := \int_0^t D(t - \tau)F_j(\tau) \, d\tau, \quad u^{nl} := \int_0^t D(t - \tau)\mathcal{N}(u(\tau)) \, d\tau.
\]

Then, taking \( \psi = F_j - \mathcal{N}(u) \) in the proof of Lemma 2.4, we deduce
\[
\|v_j - u^{nl}\|_{L^\infty(0, T_1; L^2(\mathbb{R}^n))} \lesssim \|F_j - \mathcal{N}(u)\|_{L^\infty(0, T_1; L^q(\mathbb{R}^n))} \to 0 \quad (j \to \infty)
\]

(Indeed, in the proof of Lemma 2.4, take \( \gamma = q \) and use the Sobolev embedding theorem). Therefore, we have
\[
\lim_{j \to \infty} \int_0^{T_1} \int_{\mathbb{R}^n} v_j(\partial_t^2 - \Delta - \partial_x)\psi \, dx \, dt = \int_0^{T_1} \int_{\mathbb{R}^n} \partial_t u^{nl}(\partial_t^2 - \Delta - \partial_x)\psi \, dx \, dt.
\]

On the other hand, since \( F_j \) is smooth and compactly supported with respect to \( x \), so is \( v_j \) and hence, using integration by parts we easily compute
\[
\int_0^{T_1} \int_{\mathbb{R}^n} v_j(\partial_t^2 - \Delta - \partial_x)\psi \, dx \, dt = \int_0^{T_1} \int_{\mathbb{R}^n} F_j\psi \, dx \, dt.
\]

Taking the limit \( j \to \infty \) in the right-hand side and noting \( F_j \to \mathcal{N}(u) \) in \( L^\infty(0, T_1; L^q(\mathbb{R}^n)) \), we have
\[
\lim_{j \to \infty} \int_0^{T_1} \int_{\mathbb{R}^n} v_j(\partial_t^2 - \Delta - \partial_x)\psi \, dx \, dt = \int_0^{T_1} \int_{\mathbb{R}^n} \mathcal{N}(u)\psi \, dx \, dt.
\]

Thus, we conclude
\[
\int_0^{T_1} \int_{\mathbb{R}^n} u^{nl}(\partial_t^2 - \Delta - \partial_x)\psi \, dx \, dt = \int_0^{T_1} \int_{\mathbb{R}^n} \mathcal{N}(u)\psi \, dx \, dt.
\]

In a similar way, approximating \( u_0, u_1 \) by smooth functions and using the integration by parts, we find that
\[
\varepsilon \int_0^{T_1} \int_{\mathbb{R}^n} (\mathcal{D}(t)u_0 + \mathcal{D}(t)u_1) (\partial_t^2 - \Delta - \partial_x) \psi \, dx \, dt = \varepsilon \int_{\mathbb{R}^n} ((u_0 + u_1)\psi(0, x) - u_0\psi_t(0, x)) \, dx.
\]

This completes the proof. □

Now, we are in the position to prove Theorem 1.5.
Proof of Theorem 1.5. By Proposition 4.2, we see that a mild solution is also a weak solution. Therefore, we have $T(\varepsilon) \leq T_w(\varepsilon)$. Hence, Proposition 4.4 implies the desired estimate.

Appendix A: Some auxiliary estimates

Lemma A.1. Let $\omega \in (0, 1)$. Then, there exists a constant $C = C(\omega) > 0$ such that we have

$$|\partial_j^\omega \phi(x)| = C \int_\mathbb{R} (\phi(x - y) - \phi(x)) \frac{dy_j}{|y_j|^{1+\omega}},$$

where $y = (0, \ldots, y_j, \ldots, 0)$. In particular, it follows that

$$|\partial_j^\omega (\phi \psi)(x)| = C \int_\mathbb{R} (\phi(x - y) \psi(x - y) - \phi(x) \psi(x)) \frac{dy_j}{|y_j|^{1+\omega}}$$

$$= \phi(x)|\partial_j^\omega \psi(x) + C \int_\mathbb{R} (\phi(x - y) - \phi(x)) \psi(x - y) \frac{dy_j}{|y_j|^{1+\omega}}.$$  

Proof. Recalling the identity

$$\mathcal{F}^{-1}[\phi](x) - y = \mathcal{F}^{-1}[e^{iy_j \xi_j} \phi](x),$$

we have

$$\int_\mathbb{R} (\phi(x - y) - \phi(x)) \frac{dy_j}{|y_j|^{1+\omega}} = \int_\mathbb{R} \mathcal{F}^{-1} \left[ (e^{iy_j \xi_j} - 1) \phi \right](x) \frac{dy_j}{|y_j|^{1+\omega}}$$

$$= \frac{1}{(2\pi)^{n/2}} \int_\mathbb{R} \int_{\mathbb{R}^n} e^{ix \xi} \left( e^{iy_j \xi_j} - 1 \right) \phi d\xi \frac{dy_j}{|y_j|^{1+\omega}}$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \xi} T(\xi_j) \phi d\xi,$$

where

$$T(\xi_j) = \int_\mathbb{R} \left( e^{iy_j \xi_j} - 1 \right) \frac{dy_j}{|y_j|^{1+\omega}} = 2|\xi_j|^\omega \int_0^\infty (\cos y_j - 1) \frac{dy_j}{y_j^{1+\omega}} = C(\omega)|\xi_j|^\omega.$$  

Therefore, we conclude

$$\int_\mathbb{R} (\phi(x - y) - \phi(x)) \frac{dy_j}{|y_j|^{1+\omega}} = \frac{C}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \xi} |\xi_j|^\omega \phi d\xi = C|\partial_j^\omega \phi(x)|.$$  

Lemma A.2 (Sobolev-type inequality). Let $\beta \geq 0$. Then, we have

$$\|\langle \cdot \rangle^\beta (\nabla)^{-1} \psi\|_{L^2} \lesssim \|\langle \cdot \rangle^\beta \psi\|_{L^q},$$

where $q$ is defined in Section 1.2.

Proof. By the Plancherel theorem, it suffices to estimate $\|\langle \nabla \rangle^\beta \langle \xi \rangle^{-1} \hat{\psi}\|_{L^2}$. First, we note that

$$\|\langle \nabla \rangle^\beta \langle \xi \rangle^{-1} \hat{\psi}\|_{L^2} \lesssim \|\langle \xi \rangle^{-1} \hat{\psi}\|_{L^2} + \|\nabla |\xi\rangle^{-1} \hat{\psi}\|_{L^2}.$$  

The Sobolev embedding implies $\|\langle \nabla \rangle^{-1} \psi\|_{L^2} \lesssim \|\psi\|_{L^q}$ with $q = \max\{1, \frac{2n}{\beta + 2}\}$, which gives the estimate for the first term of the right-hand side. Let us estimate the second term. When $\beta \in \mathbb{Z}$,
the Leibniz rule and the Sobolev embedding and noting that $|\partial^k_j \langle \xi \rangle^{-1}| \leq \langle \xi \rangle^{-1}$ immediately imply $\|\nabla^\beta \langle \xi \rangle^{-1} \hat{\psi} \|_{L^2} \lesssim \|\langle \xi \rangle^{\beta} \hat{\psi} \|_{L^\infty}$. When $\beta \notin \mathbb{Z}$, letting $\omega = \beta - [\beta] > 0$ and using Lemma [A.1], we have

\[
\|\nabla^\beta \langle \xi \rangle^{-1} \hat{\psi} \|_{L^2} \lesssim \sum_{j=1}^n \sum_{k=0}^{[\beta]} \left\| \left( \partial^\beta_j \langle \xi \rangle^{-1} \right) |\partial_j|^{k+\omega} \hat{\psi} \right\|_{L^2} + \sum_{j=1}^n \sum_{k=0}^{[\beta]} \left\| \int_\mathbb{R} \left( \partial^\beta_j \langle \xi \rangle^{-1} - \partial^\beta_j \langle \xi + \eta \rangle^{-1} \right) \partial^k_j \hat{\psi}(\xi + \eta) \frac{d\eta_j}{|\eta_j|^{1+\omega}} \right\|_{L^2},
\]

(A.1)

where $\eta = (0, \ldots, \eta_j, \ldots, 0)$. Then, a straight forward calculation shows

\[
\left\| \left( \partial^\beta_j \langle \xi \rangle^{-1} \right) |\partial_j|^{k+\omega} \hat{\psi} \right\|_{L^2} \lesssim \|\langle \xi \rangle^{-1} |\partial_j|^{k+\omega} \hat{\psi} \|_{L^2} \lesssim \|x_j |^{k+\omega} \psi \|_{L^q},
\]

(A.2)

where we have used the Sobolev embedding again. Moreover, we claim that

\[
\left\| \int_\mathbb{R} \left( \partial^\beta_j \langle \xi \rangle^{-1} - \partial^\beta_j \langle \xi + \eta \rangle^{-1} \right) \partial^k_j \hat{\psi}(\xi + \eta) \frac{d\eta_j}{|\eta_j|^{1+\omega}} \right\|_{L^2} \lesssim \|x_j |^{k} \hat{\psi} \|_{L^q}.
\]

(A.3)

Indeed, we first note that the left-hand side is bounded by

\[
\left\| \int_{|\eta_j| \leq 1} \left( \partial^\beta_j \langle \xi \rangle^{-1} - \partial^\beta_j \langle \xi + \eta \rangle^{-1} \right) \partial^k_j \hat{\psi}(\xi + \eta) \frac{d\eta_j}{|\eta_j|^{1+\omega}} \right\|_{L^2} + \left\| \int_{|\eta_j| > 1} \left( \partial^\beta_j \langle \xi \rangle^{-1} - \partial^\beta_j \langle \xi + \eta \rangle^{-1} \right) \partial^k_j \hat{\psi}(\xi + \eta) \frac{d\eta_j}{|\eta_j|^{1+\omega}} \right\|_{L^2} =: I + II.
\]

For $I$, the fundamental theorem of calculus implies

\[
I \lesssim \int_{|\eta_j| \leq 1} \left\| \int_0^1 \frac{d}{da} \partial^\beta_j \langle \xi + a\eta \rangle^{-1} \partial^k_j \hat{\psi}(\xi + \eta) \right\|_{L^2} \frac{d\eta_j}{|\eta_j|^{1+\omega}} \lesssim \int_{|\eta_j| \leq 1} \left\| \int_0^1 \partial^\beta_j \langle \xi + a\eta \rangle^{-1} \partial^k_j \hat{\psi}(\xi + \eta) \right\|_{L^2} \frac{d\eta_j}{|\eta_j|^{1+\omega}} = \int_{|\eta_j| \leq 1} \left\| \langle \xi \rangle^{-1} \partial^k_j \hat{\psi}(\xi) \right\|_{L^2} \frac{d\eta_j}{|\eta_j|^{\omega}} \lesssim \left\| \langle \xi \rangle^{-1} \partial^k_j \hat{\psi}(\xi) \right\|_{L^2}.
\]

This and the Sobolev embedding lead to $I \lesssim \|x_j |^{k} \hat{\psi} \|_{L^q}$. For $II$, we first have

\[
II \lesssim \int_{|\eta_j| > 1} \left\| \left( \partial^\beta_j \langle \xi \rangle^{-1} - \partial^\beta_j \langle \xi + \eta \rangle^{-1} \right) \partial^k_j \hat{\psi}(\xi + \eta) \right\|_{L^2} \frac{d\eta_j}{|\eta_j|^{1+\omega}} \lesssim \int_{|\eta_j| > 1} \left\| \partial^\beta_j \langle \xi - \eta \rangle^{-1} \partial^k_j \hat{\psi}(\xi) \right\|_{L^2(|\xi - \eta| \leq |\xi|/2)} \frac{d\eta_j}{|\eta_j|^{1+\omega}} + \int_{|\eta_j| > 1} \left\| \partial^\beta_j \langle \xi - \eta \rangle^{-1} \partial^k_j \hat{\psi}(\xi) \right\|_{L^2(|\xi - \eta| > |\xi|/2)} \frac{d\eta_j}{|\eta_j|^{1+\omega}} + \int_{|\eta_j| > 1} \left\| \partial^\beta_j \langle \xi + \eta \rangle^{-1} \partial^k_j \hat{\psi}(\xi + \eta) \right\|_{L^2(\mathbb{R}^n)} \frac{d\eta_j}{|\eta_j|^{1+\omega}} =: II_1 + II_2 + II_3.
\]
The third term $\Pi_3$ is easily estimated as
\[
\Pi_3 \lesssim \| \langle \xi \rangle^{-1} \partial_j^k \hat{\psi}(\xi) \|_{L^2} \lesssim \| x_j \|^k \psi \|_{L^q}.
\] (A.4)

For $\Pi_2$, we note that
\[
| \partial_j^{[\beta] - k}(\xi - \eta)^{-1} | \lesssim (\xi - \eta)^{-1} \lesssim (\xi - \eta)^{-1}
\]
holds for $| \xi - \eta | > | \xi |/2$ and hence,
\[
\Pi_2 \lesssim \int_{|\eta_j| > 1} \| \langle \xi - \eta \rangle^{-1} \partial_j^k \hat{\psi}(\xi) \|_{L^2(|\xi|/2 \leq |\xi|)} \frac{d\eta_j}{|\eta_j|^{1+\omega}} \lesssim \| \langle \xi \rangle^{-1} \partial_j^k \hat{\psi}(\xi) \|_{L^2(\mathbb{R}^n)} \lesssim \| x_j \|^k \psi \|_{L^q}.
\] (A.5)

Finally, for $\Pi_1$, we first remark that $| \eta_j | \geq | \xi | - | \xi - \eta | \geq \frac{1}{2} | \xi |$ holds. Noting again that $| \partial_j^{[\beta] - k}(\xi - \eta)^{-1} | \lesssim (\xi - \eta)^{-1}$ and applying the Hölder inequality, we see that
\[
\Pi_1 \lesssim \int_{|\eta_j| > 1} \| \langle \xi - \eta \rangle^{-1} \|_{L^2(|\xi|/2 \leq |\xi|)} \frac{d\eta_j}{|\eta_j|^{1+\omega}}
\]
for $n \geq 2$, where $q'$ is the conjugate of $q$, and
\[
\Pi_1 \lesssim \int_{|\eta_j| > 1} \| \langle \xi - \eta \rangle^{-1} \|_{L^2(|\xi|/2 \leq |\xi|)} \frac{d\eta_j}{|\eta_j|^{1+\omega}}
\]
for $n = 1$. In what follows, we only consider the case $n \geq 2$, because the case $n = 1$ is similar. Using $| \eta_j | = | \eta | > \frac{1}{2} | \xi |$, we have
\[
\| \langle \xi - \eta \rangle^{-1} \|_{L^2(|\xi|/2 \leq |\xi|)} \leq \| \langle \xi - \eta \rangle^{-1} \|_{L^2(|\xi| \leq |\eta_j|)} = \| (\Xi)^{-1} \|_{L^2(|\Xi| \leq |\eta_j|)} \lesssim (\log |\eta_j| + 1)^{\frac{1}{q'}}.
\]

Also, we immediately obtain
\[
\| \partial_j^k \hat{\psi}(\xi) \|_{L^{q'}(|\xi| \leq |\xi|/2)} \lesssim \| \partial_j^k \hat{\psi}(\xi) \|_{L^{q'}(\mathbb{R}^n)} \lesssim \| x_j \|^k \psi \|_{L^q(\mathbb{R}^n)}.
\]

Combining them with (A.6), we conclude
\[
\Pi_1 \lesssim \| x_j \|^k \psi \|_{L^q} \int_{|\eta_j| > 1} (\log |\eta_j| + 1)^{\frac{1}{q'}} \frac{d\eta_j}{|\eta_j|^{1+\omega}} \lesssim \| x_j \|^k \psi \|_{L^q}.
\] (A.7)

By (A.4), (A.5) and (A.7), we have $\Pi \lesssim \| x_j \|^k \psi \|_{L^q}$. Consequently, we obtain
\[
\| \langle \xi \rangle^\beta \langle \nabla \rangle^{-1} \psi \|_{L^2} \lesssim \| \psi \|_{L^q} + \sum_{j=1}^{n} \sum_{k=0}^{[\beta]} \left( \| x_j \|^k \psi \|_{L^q} + \| x_j \|^{k+\omega} \psi \|_{L^q} \right) \lesssim \| \langle x \rangle^\beta \psi \|_{L^q},
\]
which shows the assertion.

\begin{flushright}
\Box
\end{flushright}

\section*{Acknowledgments}

The authors are deeply grateful to Professor Mitsuru Sugimoto for useful suggestions and comments. This work was supported in part by Grant-in-Aid for JSPS Fellows 15J01600 (Y. Wakasugi), 15J02570 (T. Inui), 26.1884 (M. Ikeda), and Grant-in-Aid for Young Scientists (B) 15K17571 (M. Ikeda) of Japan Society for the Promotion of Science.
NONLINEAR DAMPED WAVE EQUATION WITH SLOWLY DECAYING DATA

References

[1] C. R. Cattaneo, Sur une forme de l’équation de la chaleur éliminant le paradoxe d’une propagation instantanée, Compte. Rend. 247 (1958), 431–433.

[2] R. Chill, A. Haraux, An optimal estimate for the difference of solutions of two abstract evolution equations, J. Differential Equations 193 (2003), 385–395.

[3] S. R. Dunbar, H. G. Othmer, On a nonlinear hyperbolic equation describing transmission lines, cell movement, and branching random walks, in “Nonlinear Oscillations in Biology and Chemistry,” (H. G. Othmer, Ed.), Lecture Notes in Biomathematics, Vol. 66, Springer-Verlag, New York/Berlin, 1986.

[4] H. Fujita, On the blowing up of solutions of the Cauchy problem for \( u_t = \Delta u + u^{1+\alpha} \), J. Fac. Sci. Univ. Tokyo Sec. I, 13 (1966), 109–124.

[5] Th. Gallay, G. Raugel, Scaling variables and asymptotic expansions in damped wave equations, J. Differential Equations 150 (1998), pp. 42–97.

[6] S. Goldstein, On diffusion by discontinuous movements and the telegraph equation, Quart. J. Mech. Appl. Math. 4 (1951), 129–156.

[7] K. P. Hadeler, Reaction transport systems in biological modelling, in: V. Capasso et al., editor, Mathematics inspired by biology, pp. 95–150. CIME Lectures 1997, Lecture Notes in Mathematics 1714, Springer, 1999.

[8] M. Hamza, Asymptotically self-similar solutions of the damped wave equation, Nonlinear Analysis 73 (2010), 2897–2916.

[9] N. Hayashi, E. I. Kaikina, P. I. Naumkin, Damped wave equation with super critical nonlinearities, Differential Integral Equations 17 (2004), 637–652.

[10] N. Hayashi, E. I. Kaikina, P. I. Naumkin, Damped wave equation in the subcritical case, J. Differential Equations 207 (2004), 161–194.

[11] N. Hayashi, E. I. Kaikina, P. I. Naumkin, Damped wave equation with a critical nonlinearity, Trans. Amer. Math. Soc. 358 (2006), 1165–1185.

[12] N. Hayashi, E. I. Kaikina, P. I. Naumkin, Asymptotics for nonlinear damped wave equations with large initial data, Sib. Èlektroń. Mat. Izv. 4 (2007), 249–277.

[13] N. Hayashi, E. I. Kaikina, P. I. Naumkin, On the critical nonlinear damped wave equation with large initial data, J. Math. Anal. Appl. 334 (2007), 1400–1425.

[14] N. Hayashi, P. I. Naumkin, Damped wave equation with a critical nonlinearity in higher space dimensions, preprint.

[15] N. Hayashi, P. I. Naumkin, M. Tominaga, Remark on a weakly coupled system of nonlinear damped wave equations, J. Math. Anal. Appl. 428 (2015), 490–501.

[16] T. Hosono, T. Ogawa, Large time behavior and \( L^p-L^q \) estimate of solutions of 2-dimensional nonlinear damped wave equations, J. Differential Equations 203 (2004), 82–118.

[17] L. Hsiao, T. -P. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, Comm. Math. Phys. 43 (1992), 599–605.

[18] M. Ikeda, T. Inui, Small data blow-up of \( L^2 \) or \( H^1 \)-solution for the semilinear Schrödinger equation without gauge invariance, J. Evol. Equ. 15 (2015), 571–581.

[19] M. Ikeda, T. Ogawa, Lifespan of solutions to the damped wave equation with a critical nonlinearity, J. Differential Equations 261 (2016), 1880–1903.

[20] M. Ikeda, Y. Wakasugi, A note on the lifespan of solutions to the semilinear damped wave equation, Proc. Amer. Math. Soc. 143 (2015), 163–171.

[21] R. Ikehata, Diffusion phenomenon for linear dissipative wave equations in an exterior domain, J. Differential Equations 186 (2002), 633–651.

[22] R. Ikehata, Y. Miyaoaka, T. Nakatake, Decay estimates of solutions for dissipative wave equations in \( \mathbb{R}^N \) with lower power nonlinearities, J. Math. Soc. Japan 56 (2004), 365–373.

[23] R. Ikehata, K. Nishihara, Diffusion phenomenon for second order linear evolution equations, Studia Math. 158 (2003), 153–161.

[24] R. Ikehata, K. Nishihara, H. Zhao, Global asymptotics of solutions to the Cauchy problem for the damped wave equation with absorption, J. Differential Equations 226 (2006), 1–29.

[25] R. Ikehata, M. Ohta, Critical exponents for semilinear dissipative wave equations in \( \mathbb{R}^N \), J. Math. Anal. Appl. 269 (2002), 87–97.

[26] R. Ikehata, K. Tanizawa, Global existence of solutions for semilinear damped wave equations in \( \mathbb{R}^N \) with noncompactly supported initial data, Nonlinear Anal. 61 (2005), 1189–1208.

[27] R. Ikehata, G. Todorova, B. Yordanov, Optimal decay rate of the energy for wave equations with critical potential, J. Math. Soc. Japan 65 (2013), 183–236.
[28] R. Ikeda, G. Todorova, B. Yordanov, Wave equations with strong damping in Hilbert spaces, J. Differential Equations 254 (2013), 3352–3368.

[29] M. Kac, A stochastic model related to the telegrapher’s equation, Rocky Mountain J. Math. 4 (1974), 497–509.

[30] G. Karch, Selfsimilar profiles in large time asymptotics of solutions to damped wave equations, Studia Math. 143 (2000), 175–197.

[31] T. Kawakami, Y. Ueda, Asymptotic profiles to the solutions for a nonlinear damped wave equation, Differential Integral Equations 26 (2013), 781–814.

[32] S. Kawashima, M. Nakao, K. Ono, On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term, J. Math. Soc. Japan 47 (1995), 617–653.

[33] H. Kuiper, Life span of nonnegative solutions to certain quasilinear parabolic Cauchy problems, Electronic J. Diff. Eqs. 2003 (2003), 1–11.

[34] H. A. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + F(u)$, Trans. Amer. Math. Soc. 192 (1974), 1–21.

[35] T.-T. Li, Nonlinear heat conduction with finite speed of propagation, in: Proceedings of the China-Japan Symposium on Reaction Diffusion Equations and Their Applications to Computational Aspects, World Scientific, Singapore, 1997.

[36] T.-T. Li, Y. Zhou, Breakdown of solutions to $\Delta u + u_t = |u|^{1+\alpha}$, Discrete Contin. Dyn. Syst. 1 (1995), 503–520.

[37] P. Marcati, K. Nishihara, The $L^p-L^q$ estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media, J. Differential Equations 191 (2003), 445–469.

[38] A. Matsumura, On the asymptotic behavior of solutions of semi-linear wave equations, Publ. Res. Inst. Math. Sci. 12 (1976), 169–189.

[39] M. Nakao, K. Ono, Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations, Math. Z. 214 (1993), 325–342.

[40] T. Narazaki, $L^p-L^q$ estimates for damped wave equations and their applications to semi-linear problem, J. Math. Soc. Japan 56 (2004), 585–626.

[41] T. Narazaki, Global solutions to the Cauchy problem for the weakly coupled system of damped wave equations, Discrete Contin. Dyn. Syst. Suppl. (2009), 592–601.

[42] T. Narazaki, Global solutions to the Cauchy problem for a system of damped wave equations, Differential Integral Equations 24 (2011), 569–600.

[43] T. Narazaki, K. Nishihara, Asymptotic behavior of solutions for the damped wave equation with slowly decaying data, J. Math. Anal. Appl. 338 (2008), 803–819.

[44] K. Nishihara, Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping, J. Differential Equations 131 (1996), 171–188.

[45] K. Nishihara, Asymptotic behavior of solutions of quasi-linear hyperbolic equations with linear damping, J. Differential Equations 137 (1997), 384–395.

[46] K. Nishihara, $L^p-L^q$ estimates of solutions to the damped wave equation in 3-dimensional space and their application, Math. Z. 244 (2003), 631–649.

[47] K. Nishihara, $L^p-L^q$ estimates for the 3-D damped wave equation and their application to the semi-linear problem, Seminar Notes of Math. Sci. 6, Ibaraki Univ., (2003), 69–83.

[48] K. Nishihara, Global asymptotics to the damped wave equation with absorption in higher dimensional space, J. Math. Soc. Japan 58 (2006), 805–836.

[49] K. Nishihara, Asymptotic behavior of solutions for a system of semilinear heat equations and the corresponding damped wave system, Osaka J. Math. 49 (2012), 331–348.

[50] K. Nishihara, Y. Wakasugi, Critical exponent for the Cauchy problem to the weakly coupled damped wave system, Nonlinear Anal. 108 (2014), 249–259.

[51] K. Nishihara, Y. Wakasugi, Global existence of solutions for a weakly coupled system of semilinear damped wave equations, J. Differential Equations 259 (2015), 4172–4201.

[52] H. Nishiyama, Remarks on the asymptotic behavior of the solution to an abstract damped wave equation, arXiv:1505.01791v2.

[53] T. Ogawa, H. Takeda, Global existence of solutions for a system of nonlinear damped wave equations, Differential Integral Equations 23 (2010), 635–657.

[54] T. Ogawa, H. Takeda, Large time behavior of solutions for a system of nonlinear damped wave equations, J. Differential Equations 251 (2011), 3090–3113.

[55] K. Ono, Global existence and asymptotic behavior of small solutions for semilinear dissipative wave equations, Discrete Contin. Dyn. Syst. 9 (2003), 651–662.
[56] K. Ono, Global solvability and $L^p$ decay for the semilinear dissipative wave equations in four and five dimensions, Funkcial. Ekvac. 49 (2006), 215–233.

[57] R. Racke, Non-homogeneous non-linear damped wave equations in unbounded domains, Math. Methods Appl. Sci. 13 (1990), 481–491.

[58] P. Radu, G. Todorova, B. Yordanov, Diffusion phenomenon in Hilbert spaces and applications, J. Differential Equations 250 (2011), 4200–4218.

[59] P. Radu, G. Todorova, B. Yordanov, The generalized diffusion phenomenon and applications, SIAM J. Math. Anal. 48 (2016), 174–203.

[60] F. Sun, Life span of blow-up solutions for higher-order semilinear parabolic equations, Electronic J. Diff. Eqs. 2010 (2010), 1-9.

[61] F. Sun, M. Wang, Existence and nonexistence of global solutions for a nonlinear hyperbolic system with damping, Nonlinear Anal. 66 (2007), 2889–2910.

[62] H. Takeda, Global existence and nonexistence of solutions for a system of nonlinear damped wave equations, J. Math. Anal. Appl. 360 (2009) 631–650.

[63] H. Takeda, Higher-order expansion of solutions for a damped wave equation, Asymptotic Analysis 94 (2015), pp. 1–31.

[64] H. Takeda, Large time behavior of solutions for a nonlinear damped wave equation, Commun. Pure Appl. Anal. 15 (2016), 41–55.

[65] G. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Differential Equations 174 (2001), 464–489.

[66] G. Todorova, B. Yordanov, Weighted $L^2$-estimates for dissipative wave equations with variable coefficients, J. Differential Equations 246 (2009), 4497–4518.

[67] H. Yang, A. Milani, On the diffusion phenomenon of quasilinear hyperbolic waves, Bull. Sci. Math. 124 (2000), 415–433.

[68] Qi S. Zhang, A blow-up result for a nonlinear wave equation with damping: the critical case, C. R. Acad. Sci. Paris Sér. I Math. 333 (2001), 109–114.

(M. Ikeda and T. Inui) Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

(Y. Wakasugi) Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya 464-8602, Japan