Interacting massive and massless arbitrary spin fields in 4d flat space

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Abstract

Massive and massless arbitrary integer spin fields propagating in four-dimensional flat space are studied. The massive and massless fields are treated by using a light-cone gauge helicity basis formalism. Cubic cross-interactions between massive and massless fields and cubic interactions between massive fields are investigated. We introduce a classification of such cubic interactions and using our classification we build all cubic interaction vertices. Realization of generators of the Poincaré algebra on space of interacting fields is found. As a by-product, some illustrative examples of light-cone form for 3-point invariant amplitudes of massive and massless fields are also discussed.

Keywords: Higher-spin fields, light-cone gauge helicity basis formalism, cubic interaction vertices.
1 Introduction

The light-cone gauge formulation of relativistic dynamics originally proposed in Ref.[11] turned out to useful for a study of many aspects of quantum field theory and string theory. For example, we mention the application of this approach for the investigation of ultraviolet finiteness of $N = 4$ supersymmetric Yang-Mills theory in Refs.[2, 3]. The light-cone gauge approach offers also certain simplifications of approaches to building string field theory [4, 5]. Use of the light-cone approach for the study of superfield formulation of IIB 10d supergravity and 11d supergravity may be found in Refs.[6, 7].

The extensive use of the light-cone gauge approach for the study of higher-spin massless fields begun in Ref.[8], where the cubic vertices for the Yang-Mills-like interactions of higher-spin massless fields in $R^{3,1}$ space were built. The full list of cubic vertices for arbitrary integer spin massless fields in $R^{3,1}$ space was obtained later on in Ref.[9]. Our aim in this paper is to extend the results in Ref.[9] to the case of arbitrary integer spin massive and massless fields propagating in $R^{3,1}$ space. Namely, in this paper, we are interested in cubic vertices for the cross-interactions between massive and massless fields and cubic vertices for the interactions between massive fields. We provide a classification for such cubic interactions and, using our classification, we build all cubic vertices for the massive and massless fields.\(^1\) To our knowledge, our classification for the cubic vertices of massive fields is novel and has not previously been reported in the literature. We note that, in Ref.[9], to build cubic vertices, the higher-spin massless fields were considered by using the light-cone gauge helicity basis formalism. As it is well known, the light-cone gauge helicity basis formalism for massless fields admits the straightforward generalization to the case of massive fields. We use such generalized light-cone gauge helicity basis formalism for building cubic vertices that involve massive fields.

This paper is organized as follows.

In Sec.2, we review a light-cone gauge helicity basis formulism for massive and massless fields propagating in $R^{3,1}$. Also we review a field-theoretical realization of the Poincaré algebra on space of the massive and massless fields.

In Sec.3, we discuss $n$-point interaction vertices. We present restrictions on $n$-point interaction vertices which are obtained by using kinematical symmetries of the Poincaré algebra.

In Sec.4, we investigate cubic vertices. We start with the presentation of restrictions on the cubic vertices which are obtained by considering kinematical and dynamical symmetries of the Poincaré algebra. After that, we introduce a light-cone gauge dynamical principle and formulate our complete system of equations which allows us to fix all solutions for cubic vertices uniquely.

In Sec.5, we introduce harmonic representatives of the cubic vertices. We reformulate our complete system of equations for the cubic vertices in terms of the harmonic vertices. After that we introduce meromorphic vertices which are in one-to-one correspondence with the harmonic vertices. We reformulate all our equations for the harmonic vertices in terms of the meromorphic vertices.

In Sec.6, we discuss our classification for the cubic interactions.

In Sec.7, we discuss the interactions between two massless fields and one massive field, while, in Secs.[8,9] we discuss the interactions between one massless field and two massive fields. Secs[10, 11] are devoted to the interactions between three massive fields. In Secs.7,11 we present the

\(^1\)Light-cone gauge parity-even cubic vertices for arbitrary spin massive and massless fields in $R^{d-1,1}$, $d \geq 4$, were studied in Ref.[10, 11]. Lorentz covariant parity-even cubic vertices for arbitrary spin massive and massless on-shell TT fields in $AdS_{d+1}$, $d \geq 4$, were considered in Ref.[12]. BRST-BV parity-even cubic vertices for arbitrary spin massive and massless fields in $R^{d-1,1}$, $d \geq 4$, were obtained in Ref.[13].
explicit expressions for the meromorphic vertices and discuss some illustrative examples of 3-
point invariant amplitudes in the light-cone frame.

Our conclusions are summarized in Sec.[12]

In Appendix A, we describe our notation and various identities we use in our study of cubic
certices. In Appendix B, we discuss various helpful realizations for the spin operators of massive
fields. In Appendix C, we discuss the incorporation of internal symmetries in our approach. In
Appendix D, we describe the hermitian conjugation rules for our vertices. Appendix E is devoted
to technical details of the derivation of equations for the harmonic vertices and the meromorphic
vertices. Appendix F is devoted to review of 3-point invariant amplitudes in the light-cone frame.
In Appendix G, we make comments on our classification of cubic vertices. Appendix H is devoted
to the derivation of the meromorphic vertices.

2 Light-cone gauge helicity basis formalism for free massive and massless fields

Light-cone frame of Poincaré algebra $iso(3, 1)$. We follow a method proposed in Ref.[1]. Ac-
cording to this method, the problem of building a new light-cone gauge formulation of a dynamical
system amounts to the problem of building the respective light-cone gauge solution for commuta-
tors of an underlying symmetry algebra. For the massive and massless fields propagating in $R^{3,1}$
space, the underlying symmetries are associated with the Poincaré algebra $iso(3, 1)$. Therefore, let
us review the well known light-cone gauge realization of the Poincaré algebra on a space of the
massive and massless fields in $R^{3,1}$ space.

In $R^{3,1}$ space, the Poincaré algebra $iso(3, 1)$ is spanned by the four translation generators de-
noted as $P^\mu$ and the six generators of the $so(3, 1)$ Lorentz algebra denoted as $J^{\mu\nu}$. We use the
following non-trivial commutators of the Poincaré algebra:

$$[P^\mu, J^{\nu\rho}] = \eta^{\mu\nu} P^\rho - \eta^{\mu\rho} P^\nu, \quad [J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\mu\rho} J^{\nu\sigma} + 3 \text{ terms}, \quad (2.1)$$

where $\eta^{\mu\nu}$ stands for the mostly positive Minkowski metric. Let $x^\mu, \mu = 0, 1, 2, 3$, be the Lorentz
basis coordinates. The light-cone frame coordinates $x^\pm, x^R, x^L$ are defined then as

$$x^\pm \equiv \frac{1}{\sqrt{2}}(x^3 \pm x^0), \quad x^R \equiv \frac{1}{\sqrt{2}}(x^1 + ix^2), \quad x^L \equiv \frac{1}{\sqrt{2}}(x^1 - ix^2). \quad (2.2)$$

The $x^+$ is considered as a light-cone evolution parameter. In the light-cone frame, the $so(3, 1)$
Lorentz algebra vector $X^\mu$ is represented as $X^+, X^-, X^R, X^L$, while a scalar product of two
vectors $X^\mu$ and $Y^\mu$ is represented as

$$\eta_{\mu\nu} X^\mu Y^\nu = X^+ Y^- + X^- Y^+ + X^R Y^L + X^L Y^R. \quad (2.3)$$

Relation (2.3) implies that the non-vanishing elements of $\eta_{\mu\nu}$ are given by $\eta_{+-} = \eta_{-+} = 1, \eta_{BL} = \eta_{LR} = 1$. This implies the following interrelations for covariant and contravariant components of
the vector $X^\mu$: $X^+ = X_-, X^- = X_+, X^R = X_L, X^L = X_R$.

In the light-cone frame, the generators of the Poincaré algebra can be split into two groups:

$$P^+, \quad P^R, \quad P^L, \quad J^{+R}, \quad J^{+L}, \quad J^-, \quad J^{RL}, \quad \text{kinematical generators;} \quad (2.4)$$

$$P^-, \quad J^{-R}, \quad J^{-L}, \quad \text{dynamical generators.} \quad (2.5)$$
Needless to say that, in the light-cone frame, commutation relations of the Poincaré algebra are obtained from the ones in (2.1) by using $\eta^{\mu \nu}$ which has non-vanishing elements $\eta^{+-} = \eta^{-+} = 1$, $\eta^{RL} = \eta^{LR} = 1$. The Poincaré algebra generators are assumed to satisfy the following hermitian conjugation rules:

$$
\begin{align*}
&\, P^\pm = P^\pm, \\
&\, P^{RL} = P^{LR}, \\
&\, J^{RL+} = J^{RL}, \quad J^{+-} = -J^{+-}, \quad J^{\pm \pm} = -J^{\pm \pm}.
\end{align*}
$$

(2.6)

We now discuss the light-cone gauge helicity basis formalism for arbitrary spin massive and massless fields.

**Massless fields.** In the light-cone frame, a spin-$s$ massless field is described by complex-valued helicity basis fields $\phi_{\lambda}(x)$, $\lambda = \pm s$, where the label $\lambda$ stands for a helicity, while the argument $x$ stands for the space-time coordinates $x^+, x^{RL}$ (2.2). The hermitian conjugation rule is given by

$$
(\phi_{\lambda}(x))^\dagger = \phi_{-\lambda}(x).
$$

(2.7)

**Massive spin-$s$ field.** In the light-cone frame, a mass-$m$ and spin-$s$ field can be described by the following set of complex-valued helicity basis fields:

$$
\phi_{m,s;n}(x), \quad n = 0, \pm 1, \ldots, \pm s,
$$

(2.8)

$$
(\phi_{m,s;n}(x))^\dagger = \phi_{m,s;-n}(x).
$$

(2.9)

To obtain the description of the massive field in an easy-to-use form, we introduce the bosonic spinor-like creation operators $u$, $v$ and the respective annihilation operators $\bar{u}$, $\bar{v}$ defined by the relations

$$
[u, u] = 1, \quad [\bar{u}, v] = 1, \quad v^\dagger = \bar{v} \quad u^\dagger = \bar{u}, \quad \bar{u}|0\rangle = 0, \quad \bar{v}|0\rangle = 0.
$$

(2.10)

Often, we refer to $u$ and $v$ as oscillators. Using the oscillators, we collect fields (2.8) into a ket-vector,

$$
|\phi_{m,s}(x)\rangle = \sum_{n=-s}^{s} \frac{u^{s+n}v^{s-n}}{(s+n)!(s-n)!}\phi_{m,s;n}(x)|0\rangle.
$$

(2.11)

Below, we prefer to use momentum space fields which are obtainable from the ones above discussed by making the Fourier transform with respect to the spatial coordinates $x^-$, $x^R$, and $x^L$,

$$
\phi_{\lambda}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} e^{i(\beta x^- + p^R x^L + p^L x^R)} \phi_{\lambda}(x^+, p),
$$

(2.12)

$$
|\phi_{m,s}(x)\rangle = \int \frac{d^3p}{(2\pi)^{3/2}} e^{i(\beta x^- + p^R x^L + p^L x^R)} |\phi_{m,s;n}(x^+, p)\rangle, \quad d^3p \equiv d\beta dp^R dp^L,
$$

(2.13)

where the argument $p$ stands for the momenta $\beta$, $p^R$, $p^L$. In terms of the fields $\phi_{\lambda}(x^+, p)$, $\phi_{m,s;n}(x^+, p)$, the hermicity conditions (2.7), (2.9) are represented as

$$
(\phi_{\lambda}(x^+, p))^\dagger = \phi_{-\lambda}(x^+, -p), \quad (\phi_{m,s;n}(x^+, p))^\dagger = \phi_{m,s;-n}(x^+, -p).
$$

(2.14)

**Realization of the Poincaré algebra on massless and massive fields.** The realization of the Poincaré algebra (2.1) in terms of differential operators acting on the massless field $\phi_{\lambda}(p)$ and the massive field $|\phi_{m,s}(x^+, p)\rangle$ takes the following well known form:

**Realizations on space of $\phi_{\lambda}(x^+, p)$ and $|\phi_{m,s}(x^+, p)\rangle$:**
\[ P^R = p^R, \quad P^L = p^L, \quad P^+ = \beta, \quad P^- = p^-, \quad p^- \equiv -\frac{2p^R p^L + m^2}{2\beta}, \quad (2.15) \]

\[ J^{+R} = ix^+ P^R + \partial_{p^L} \beta, \quad J^{+L} = ix^+ P^L + \partial_{p^R} \beta, \quad (2.16) \]

\[ J^{+-} = ix^+ P^+ - \partial_{\beta} \beta, \quad J^{RL} = p^R \partial_{p^L} - p^L \partial_{p^R} + M^{RL}, \quad (2.17) \]

\[ J^{-R} = -\partial_{\beta} p^R + \partial_{p^L} p^+ + \frac{M^{RL} p^R}{\beta} + \frac{1}{\beta} M^R, \quad (2.18) \]

\[ J^{-L} = -\partial_{\beta} p^L + \partial_{p^R} p^- - \frac{M^{RL} p^L}{\beta} + \frac{1}{\beta} M^L, \quad (2.19) \]

where, for partial derivatives, we use the notation

\[ \partial_{\beta} \equiv \partial/\partial \beta, \quad \partial_{p^R} \equiv \partial/\partial p^R, \quad \partial_{p^L} \equiv \partial/\partial p^L, \quad (2.20) \]

while quantities \( M^{RL}, M^R, M^L \) appearing in (2.17)-(2.19) are defined as

for massless field: \[ M^{RL} = \lambda, \quad M^R = 0, \quad M^L = 0; \quad (2.21) \]

for massive field: \[ M^{RL} = \frac{1}{2}(N_u - N_v), \quad N_u \equiv u\bar{u}, \quad N_v \equiv v\bar{v}, \]

\[ M^R = \frac{m}{\sqrt{2}} u\bar{v}, \quad M^L = -\frac{m}{\sqrt{2}} v\bar{u}. \quad (2.22) \]

We note the following commutators and hermitian conjugation rules for operators (2.22):

\[ [M^R, M^L] = -m^2 M^{RL}, \quad [M^{RL}, M^R] = M^R, \quad [M^{RL}, M^L] = -M^L, \quad (2.23) \]

\[ M^{RL\dagger} = M^{RL}, \quad M^R\dagger = -M^L. \quad (2.24) \]

In Appendix B, we discuss alternative helpful realizations for the operators \( M^R, M^L, M^{RL} \).

Field-theoretical form of Poincaré algebra. To quadratic order in the light-cone gauge fields, a field-theoretical representation of the Poincaré algebra generators for the spin-\( s \), massless field and the spin-\( s \) massive field takes the following respective forms:

\[ G_{[2]} = 2 \int d^3 p \beta \phi_s^\dagger G_{\text{diff}} \phi_s, \quad \text{for massless field;} \quad (2.25) \]

\[ G_{[2]} = \int d^3 p \beta \langle \phi_{m,s} | G_{\text{diff}} | \phi_{m,s} \rangle, \quad \text{for massive field,} \quad (2.26) \]

where a notation \( G_{[2]} \) is used for the field-theoretical representation of the generators (2.4), (2.5), while \( G_{\text{diff}} \) stands for the differential operators described in (2.15)-(2.24).

The massless field \( \phi_\lambda \) and the massive fields \( \phi_{m,s;n} \) satisfy the following respective Poisson-Dirac equal-time commutators:

\[ [\phi_\lambda(x^+, p), \phi_\lambda'(x^+, p')] = \frac{1}{2\beta} \delta_{\lambda + \lambda', 0} \delta^3(p + p'), \quad (2.27) \]

\[ [\phi_{m,s;n}(x^+, p), \phi_{m',s';n'}(x^+, p')] = \frac{1}{2\beta} \delta_{s,s';\delta_{n+n',0}} \delta^3(p + p'). \quad (2.28) \]
By using formulas in (2.25)-(2.28), we can easily check the standard equal-time commutators between the fields and the generators of the Poincaré algebra,

\[
[\phi_\lambda, G_{[2]}] = G_{\text{diff}} \phi_\lambda, \quad [|\phi_{m,s}\rangle, G_{[2]}] = G_{\text{diff}} |\phi_{m,s}\rangle .
\] (2.29)

To conclude this section we note that the light-cone gauge action takes the form

\[
S = S_{\text{free}} + \int dx^+ P_{\text{int}}^{-},
\] (2.30)

where \(S_{\text{free}}\) is an action for free fields, while \(P_{\text{int}}^{-}\) is a light-cone gauge Hamiltonian describing interactions. The actions for the free spin-\(s\), \(s = |\lambda|\), massless field and the free spin-\(s\) massive field take the following respective forms:

\[
S_{\text{massless}}^{\text{free}} = \int dx^+ d^3p \phi_\lambda^\dagger \square \phi_\lambda, \quad S_{\text{massive}}^{\text{free}} = \frac{1}{2} \int dx^+ d^3p \langle \phi_{m,s}|(\square - m^2)|\phi_{m,s}\rangle ,
\] (2.31)

\[
\square \equiv 2i\beta \frac{\partial}{\partial x^+} - 2p^R p^+ .
\] (2.32)

As in string theory, the incorporation of internal symmetries can be done by using the Chan–Paton method (see Appendix C).

### 3 Poincaré algebra kinematical restrictions on \(n\)-point dynamical generators

In general, for theories of interacting massive and massless fields, the dynamical generators of the Poincaré algebra (2.5) denoted as \(G_{\text{dyn}}\) take the following expansion:

\[
G_{\text{dyn}} = \sum_{n=2}^{\infty} G_{[n]}^{\text{dyn}} ,
\] (3.1)

where \(G_{[n]}^{\text{dyn}}\) is a functional that has \(n\) powers of massive and massless fields. For \(n = 2\), the dynamical and kinematical generators are given in (2.25), (2.26). Restrictions on \(G_{[n]}^{\text{dyn}}\), \(n \geq 3\), imposed by the Poincaré algebra commutators between kinematical generators (2.4) and dynamical generators (2.5) we refer to as kinematical restrictions. Our aim in this Section is to review the kinematical restrictions. We start with kinematical restrictions which are obtained by using \(P^R\), \(P^L\), \(P^+\) symmetries in (2.4). Namely, by using commutators between the kinematical generators \(P^R\), \(P^L\), \(P^+\) and the dynamical generators, we get the following expressions for \(G_{[n]}^{\text{dyn}}\), \(n \geq 3\),

\[
P_{[n]}^{-} = \int d\Gamma_{[n]} \langle \Phi_{[n]} | \cdot | p_{[n]}^{-}\rangle ,
\] (3.2)

\[
J_{[n]}^{-R} = \int d\Gamma_{[n]} \langle \Phi_{[n]} | \cdot | j_{[n]}^{-R}\rangle + \langle X_{[n]}^R \Phi_{[n]} | \cdot | p_{[n]}^{-}\rangle ,
\] (3.3)

\[
J_{[n]}^{-L} = \int d\Gamma_{[n]} \langle \Phi_{[n]} | \cdot | j_{[n]}^{-L}\rangle + \langle X_{[n]}^L \Phi_{[n]} | \cdot | p_{[n]}^{-}\rangle ,
\] (3.4)

\[\text{We recall that, with the exception of } J^{++}, \text{ all kinematical generators (2.4) are quadratic in the fields. The kinematical generator } J^{++} \text{ takes the following form } J^{++} = J_0 + ix^+ P^- , \text{ where } J_0 \text{ is also quadratic in the fields, while the } P^- \text{ is the dynamical generator appearing in (2.5).}\]
where we use the notation

\[ d\Gamma_{[n]} = (2\pi)^3 \delta^{(3)} \left( \sum_{a=1}^{n} p_a \right) \prod_{a=1}^{n} \frac{d^3 p_a}{(2\pi)^3/2}, \quad d^3 p_a = dp_a^R dp_a^L d\beta_a, \]  

(3.5)

\[ X^R_{[n]} = -\frac{1}{n} \sum_{a=1}^{n} \partial_{p_a^R}, \quad X^L_{[n]} = -\frac{1}{n} \sum_{a=1}^{n} \partial_{p_a^L}; \]  

(3.6)

\[ \langle \Phi_{[n]} \rangle \equiv \prod_{b} \phi^\dagger_{\lambda_b}(x^+, p_b) \prod_{c} \langle \phi_{m_c,s_c}(x^+, p_c) \rangle, \quad b \cup c = 1, \ldots, n, \quad b \cap c = \emptyset, \]  

(3.7)

\[ |p_{[n]}^{-}\rangle = p_{[n]}^{-} \prod_{c} |0\rangle_c, \quad |j_{[n]}^{-,R}\rangle = j_{[n]}^{-,R} \prod_{c} |0\rangle_c, \quad |j_{[n]}^{-,L}\rangle = j_{[n]}^{-,L} \prod_{c} |0\rangle_c, \]  

(3.8)

and the indices \(a, b, c = 1, \ldots, n\) label massive or massless fields entering \(n\)-point dynamical generators. We refer to \(p_{[n]}^{-}\) as \(n\)-point interaction vertex, while \(p_{[n]}^{-}\) is referred to as cubic vertex. Sometimes, \(p_{[n]}^{-}\) and \(j_{[n]}^{-,R,L} \) will be denoted as \(g_{[n]}\) and referred to as densities. The densities \(g_{[n]}\) depend on the momenta \(p_a^R, p_a^L, \beta_a\), and oscillators \(u_a, v_a a = 1, 2, \ldots, n,\)

\[ g_{[n]} = g_{[n]}(p_a^R, p_a^L, \beta_a, u_a, v_a), \quad g_{[n]} = p_{[n]}^{-}, \quad j_{[n]}^{-,R}, \quad j_{[n]}^{-,L}. \]  

(3.9)

In (3.2)-(3.4), the product \(\langle \Phi_{[n]} \rangle \cdot |g_{[n]}\rangle\) stands for the shortcut defined as follows

**for vertices involving only massive fields:**

\[ \langle \Phi_{[n]} \rangle \cdot |p_{[n]}^{-}\rangle \equiv \langle \Phi_{[n]} \rangle |p_{[n]}^{-}\rangle, \quad \langle \Phi_{[n]} \rangle \cdot |j_{[n]}^{-,R}\rangle \equiv \langle \Phi_{[n]} \rangle |j_{[n]}^{-,R}\rangle, \quad \langle \Phi_{[n]} \rangle \cdot |j_{[n]}^{-,L}\rangle \equiv \langle \Phi_{[n]} \rangle |j_{[n]}^{-,L}\rangle; \]  

(3.10)

**for vertices involving massless and massive fields:**

\[ \langle \Phi_{[n]} \rangle \cdot |p_{[n]}^{-}\rangle \equiv \langle \Phi_{[n]} \rangle |p_{[n]}^{-}\rangle + \langle \Phi_{[n]} \rangle |\mathcal{I} p_{[n]}^{-}\rangle, \]  

(3.11)

\[ \langle \Phi_{[n]} \rangle \cdot |j_{[n]}^{-,R}\rangle \equiv \langle \Phi_{[n]} \rangle |j_{[n]}^{-,R}\rangle - \langle \Phi_{[n]} \rangle |\mathcal{I} j_{[n]}^{-,L}\rangle, \]  

(3.12)

\[ \langle \Phi_{[n]} \rangle \cdot |j_{[n]}^{-,L}\rangle \equiv \langle \Phi_{[n]} \rangle |j_{[n]}^{-,L}\rangle - \langle \Phi_{[n]} \rangle |\mathcal{I} j_{[n]}^{-,R}\rangle, \]  

(3.13)

where, the bra-vector \(\langle \Phi_{[n]} \rangle_{\mathcal{I}}\) and the action of the operator \(\mathcal{I}\) on the densities \(g_{[n]}\) are defined as

\[ \langle \Phi_{[n]} \rangle_{\mathcal{I}} \equiv \prod_{b} \phi^\dagger_{-\lambda_b}(x^+, p_b) \prod_{c} \langle \phi_{m_c,s_c}(x^+, p_c) \rangle, \quad b \cup c = 1, \ldots, n, \quad b \cap c = \emptyset, \]  

(3.14)

\[ \mathcal{I} g_{[n]}(p_a^R, p_a^L, \beta_a, u_a, v_a) = g_{[n]}^*(\mathcal{I} p_a^R, \mathcal{I} p_a^L, \mathcal{I} \beta_a, \mathcal{I} u_a, \mathcal{I} v_a), \]  

(3.15)

\[ \mathcal{I} p_a^R = -p_a^L, \quad \mathcal{I} p_a^L = -p_a^R, \quad \mathcal{I} \beta_a = -\beta_a, \quad \mathcal{I} u_a = v_a, \quad \mathcal{I} v_a = u_a. \]  

(3.16)

In (3.15) and throughout this paper, the asterisk \(\ast\) stands for a complex conjugation. Shortly speaking, \(\langle \Phi_{[n]} \rangle_{\mathcal{I}}\) is obtained by making the replacement \(\lambda_b \rightarrow -\lambda_b\) in the expression for \(\langle \Phi_{[n]} \rangle\) (3.7), while the action of the operator \(\mathcal{I}\) on the densities \(g_{[n]}\) is given by

\[ \mathcal{I} g_{[n]}(p_a^R, p_a^L, \beta_a, u_a, v_a) = g_{[n]}^*(-p_a^R, -p_a^L, -\beta_a, v_a, u_a). \]  

(3.17)
Rules in (3.10)-(3.17) are obtained by using the hermitian conjugation conditions for the generators $P^-, J^{-r,l}$ (2.6) and the fields (2.14) (for more detailed discussion, see Appendix D). In (3.3) and (3.4), the respective operators $X_{[n]}^R$ and $X_{[n]}^L$ act only on the arguments of the fields.

The remaining kinematical restrictions are given by

$$J^{+-} \text{-symmetry restrictions:}$$

$$J^{+-} p_{[n]} = 0, \quad J^{+-} j^{-R,L}_{[n]} = 0, \quad J^{+-} \equiv \sum_{a=1}^{n} \beta_a \partial_{\beta_a};$$  \(3.18\)

$$J^{RL} \text{-symmetry restrictions:}$$

$$J^{RL} p_{[n]} = 0, \quad (J^{RL} - 1) j^{-R}_{[n]} = 0, \quad (J^{RL} + 1) j^{-L}_{[n]} = 0,$$

$$J^{RL} \equiv \sum_{a=1}^{n} (p_a^R \partial_{p_a^R} - p_a^L \partial_{p_a^L} + M_a^{RL});$$  \(3.19\)

$$J^{+R}, J^{+L} \text{-symmetry restrictions:}$$

$$g_{[n]} = g_{[n]} (p^R_{ab}, p^L_{ab}, \beta_a, u_a, v_a), \quad g_{[n]} = p^L_{[n]}, \quad j^{-R}_{[n]}, \quad j^{-L}_{[n]},$$

$$p^R_{ab} \equiv p_a^R \beta_b - p_b^R \beta_a, \quad p^L_{ab} \equiv p_a^L \beta_b - p_b^L \beta_a.$$  \(3.20\)

Let us briefly comment the restrictions presented in (3.2)-(3.22).

i) The Poincaré algebra commutators between the dynamical generators $P^-, J^{-r}, J^{-l}$ and the kinematical generators $P^R, P^L, P^+$ imply the delta-functions in (3.5).

ii) The Poincaré algebra commutators between the dynamical generators $P^-, J^{-r}, J^{-l}$ and the kinematical generators $J^{+-}, J^{RL}$ amount to equations given in (3.18)-(3.20).

iii) From the Poincaré algebra commutators between the dynamical generators $P^-, J^{-r}, J^{-l}$ and the kinematical generators $J^{+r}, J^{+l}$, we learn that the $n$-point densities $p_{[n]}^-, j^{-r}_{[n]}, j^{-l}_{[n]}$ depend on the momenta $p_{ab}^R$ and $p_{ab}^L$ in place of the momenta $p_a^R$ and $p_a^L$ respectively.

iv) Making use of the conservation laws for the momenta $p_{ab}^R, p_{ab}^L, \beta_a$, one can verify that there are only $n - 2$ independent momenta $p_{ab}^R$ and $n - 2$ independent momenta $p_{ab}^L$ (3.22). This implies that, for $n = 3$, there is only one independent momentum $p^R$ and one independent momentum $p^L$.

4 Poincaré symmetry restrictions on cubic vertices and light-cone gauge dynamical principle

Making use of the conservation laws for the momenta $p_{ab}^R, p_{ab}^L, \beta_a$,

$$\sum_{a=1,2,3} p_{a}^{R,L} = 0, \quad \sum_{a=1,2,3} \beta_a = 0,$$  \(4.1\)

one can check that momenta $p_{ab}^R, p_{ab}^L$ (3.22) are expressible in terms of new momenta $p_{12}^{R,L}, p_{23}^{R,L}, p_{31}^{R,L}$,

$$p_{12}^{R,L} = p_{23}^{R,L} = p_{31}^{R,L} = p^{R,L}.$$  \(4.2\)

3Considering two terms on r.h.s. in (3.11), we introduce $P^+_{\nu} \equiv \int (\Phi_{[n]} | p_{[n]}^-)$ and $P^-_{\nu} \equiv \int (\Phi_{[n]}, x | I_{p_{[n]}^+})$. Using the hermiticity conditions (2.14) and the map $I$ (3.17), we find $P^+_{\nu} = P^-_{\nu}$ and $P^-_{\nu} = P^+_{\nu}$. Therefore the hermitian $P^-$ is given by $P^- = P^+ + P^-_{\nu}$. If the readers prefer not to use the hermiticity conditions (2.14) and the map $I$ (3.17), then the hermitian $P^-$ is given by $P^- = P^+ - P^-_{\nu}$.

4In (3.18) and below, in place of equations for the ket-vectors $|g_{[n]}\rangle$, we prefer to use equations for the densities $g_{[n]}$ (3.8). This implies that, in equations for $g_{[n]}$, we should replace the annihilation operators $\bar{u}$ and $\bar{v}$ (2.22) by the respective derivatives $\partial / \partial u$ and $\partial / \partial v$. 

8
\[ \mathbb{P}^R \equiv \frac{1}{3} \sum_{a=1,2,3} \tilde{\beta}_a p_a^R, \quad \mathbb{P}^L \equiv \frac{1}{3} \sum_{a=1,2,3} \tilde{\beta}_a p_a^L, \quad \tilde{\beta}_a \equiv \beta_{a+1} - \beta_{a+2}, \quad \beta_a = \beta_{a+3}. \] (4.3)

This implies that the cubic densities \( g^{[3]} \) are functions of the momenta \( \beta_a, \mathbb{P}^{R,L} \), and oscillators \( u_a, v_a \)
\[ g^{[3]} = g^{[3]}(\mathbb{P}^R, \mathbb{P}^L, \beta_a, u_a, v_a), \quad g^{[3]} = p_{-}^{[3]}, \quad j^{[3]}_{-R}, \quad j^{[3]}_{-L}. \] (4.4)

In other words, the momenta \( p_a^R \) and \( p_a^L \) enter the cubic densities \( g^{[3]} \) through the momenta \( \mathbb{P}^R \) and \( \mathbb{P}^L \) respectively. Thank to this feature, a study of the cubic densities is considerably simplified.

In this Section, our aims are as follows.

i) To represent kinematical \( J^{+-}, J^{RL} \) symmetry equations (3.18), (3.19) in terms of \( \mathbb{P}^{R,L} \).

ii) To find restrictions imposed on the cubic vertex by the Poincaré algebra dynamical symmetries.

iii) To formulate so called light-cone gauge dynamical principle and present equations which are required to fix all possible solutions for the cubic vertices uniquely.

**Kinematical \( J^{+-}, J^{RL} \)-symmetries.** Using (4.3) and (4.4), we find that, for \( n = 3 \), equations (3.18), (3.19) take the following form:

- **\( J^{+-} \)-symmetry restrictions:**
  \[ J^{+-} p_{[3]} = 0, \quad J^{+-} j_{[3]}^{R} = 0, \quad J^{+-} j_{[3]}^{L} = 0, \] (4.5)

- **\( J^{RL} \)-symmetry restrictions:**
  \[ J^{RL} p_{[3]}^{[3]} = 0, \quad (J^{RL} - 1) j^{R}_{[3]} = 0, \quad (J^{RL} + 1) j^{L}_{[3]} = 0, \] (4.8)

\[ J^{RL} \equiv N_{pR} - N_{pL} + M^{RL}, \quad M^{RL} \equiv \sum_{a=1,2,3} M_{a}^{RL}. \] (4.9)

**Dynamical symmetries.** Restrictions on the densities imposed by the Poincaré algebra commutators between the dynamical generators (2.5) we refer to as dynamical restrictions. Dynamical restrictions on the cubic densities are obtainable from the following commutators of the Poincaré algebra:
\[ [p_-, J^{-R}] = 0, \quad [p_-, J^{-L}] = 0, \quad [J^{-R}, J^{-L}] = 0. \] (4.10)

In the cubic approximation, the first two commutators in (4.10) lead to the relations for the cubic densities,
\[ \dot{j}^{R}_{[3]} = -\frac{1}{p_-} J^{-R} p_{[3]}, \quad \dot{j}^{L}_{[3]} = -\frac{1}{p_-} J^{-L} p_{[3]}, \] (4.11)

where the notation \( J^{-R}, J^{-L}, p_- \) is used for the following operators:
\[ J^{-R} \equiv \frac{\mathbb{P}^R}{\beta} (-N_\beta + M^{RL}) + \sum_{a=1,2,3} \frac{\tilde{\beta}_a}{6\beta_a} m_a^2 p^R - \frac{1}{\beta_a} M^R_{a}, \] (4.12)

\[ J^{-L} \equiv \frac{\mathbb{P}^L}{\beta} (-N_\beta - M^{RL}) + \sum_{a=1,2,3} \frac{\tilde{\beta}_a}{6\beta_a} m_a^2 p^L - \frac{1}{\beta_a} M^L_{a}. \] (4.13)
\[
\mathbf{P}^- = \frac{1}{\beta} \left( \mathbb{P}^R \mathbb{P}^L - \frac{1}{2} p^2 \right), \quad \rho^2 = \beta \sum_{a=1,2,3} \frac{m_a^2}{\beta_a}, \quad \beta \equiv \beta_1 \beta_2 \beta_3, 
\]
(4.14)

\[
N_\beta \equiv \frac{1}{3} \sum_{a=1,2,3} \bar{\beta}_a \beta_a \beta_\alpha, \quad \mathbb{M}^{RL} \equiv \frac{1}{3} \sum_{a=1,2,3} \bar{\beta}_a \mathbb{M}_a^{RL}, 
\]
(4.15)

and \(\bar{\beta}_a\) is defined in (4.4).

If kinematical equations for \(p^\beta_{[3]}\) in (4.5), (4.8) are satisfied, then, using (4.11), we verify that kinematical equations for \(j^\beta_{[3]}\) in (4.5), (4.8) and the third commutator in (4.10) considered in the cubic approximation are also satisfied. This implies that, in the cubic approximation, the kinematical equations for \(p^\beta_{[3]}\) in (4.5), (4.8) and dynamical equations (4.11) exhaust all restrictions obtainable from the commutators of the Poincaré algebra.

**Light-cone gauge dynamical principle.** The kinematical equations for \(p^\beta_{[3]}\) in (4.5), (4.8) and the dynamical equations (4.11) do not fix all solutions for the cubic vertex \(p^\beta_{[3]}\) unambiguously. To find all solutions for the cubic vertex \(p^\beta_{[3]}\) unambiguously we impose additional restrictions on the cubic vertex \(p^\beta_{[3]}\).

Throughout this paper, these additional restrictions are referred to as light-cone gauge dynamical principle and we formulate this principle in the following way.

**i)** The cubic vertex \(p^\beta_{[3]}\) and the densities \(j^\beta_{[3]} R^L\) should be polynomial in the momenta \(\mathbb{P}^R, \mathbb{P}^L, \).

**ii)** The cubic vertex \(p^\beta_{[3]}\) should obey the following restriction:

\[
p^\beta_{[3]} \neq \mathbf{P}^- W, \quad W \text{ is polynomial in } \mathbb{P}^R, \mathbb{P}^L, 
\]
(4.16)

where \(\mathbf{P}^-\) is given in (4.14). We briefly comment restriction (4.16). Upon field redefinitions, the vertex \(p^\beta_{[3]}\) is changed as \(p^\beta_{[3]} \to p^\beta_{[3]} + \mathbf{P}^- f\) (see, e.g., Appendix B in Ref. [10]). If \(p^\beta_{[3]} = \mathbf{P}^- W,\) then such vertex can be made trivial by using the field redefinition with \(f = -W\). As we are going to find non-trivial solutions for the cubic vertex \(p^\beta_{[3]}\) we impose restriction (4.16). The collection of restrictions imposed by the Poincaré algebra commutators and restrictions of the light-cone dynamical principle we refer to as complete system of equations.

**Complete system of equations.** The complete system of equations for the cubic vertex

\[
p^\beta_{[3]} = p^\beta_{[3]} (\mathbb{P}^R, \mathbb{P}^L, \beta_\alpha, u_a, v_a) 
\]
(4.17)

takes the following form:

\[
\mathbf{J}^+ p^\beta_{[3]} = 0, \quad \text{kinematical } J^+ - \text{symmetry}; 
\]
(4.18)

\[
\mathbf{J}^{RL} p^\beta_{[3]} = 0, \quad \text{kinematical } J^{RL} - \text{symmetry}; 
\]
(4.19)

\[
\tilde{j}^{RL}_{[3]} = -\frac{1}{\mathbf{P}^-} \mathbf{J}^{RL} p^\beta_{[3]} , \quad \text{dynamical } P^-, J^{-RL} \text{ symmetries}; 
\]
(4.20)

**Light-cone gauge dynamical principle:**

\[
p^\beta_{[3]}, j^{RL}_{[3]} \quad \text{are polynomial in } \mathbb{P}^R, \mathbb{P}^L; 
\]
(4.21)

\[
p^\beta_{[3]} \neq \mathbf{P}^- W, \quad W \text{ is polynomial in } \mathbb{P}^R, \mathbb{P}^L; 
\]
(4.22)

where \(J^+, J^{RL}, J^{-RL}, P^-\) are given in (4.6), (4.9) and (4.12)-(4.15).

Equations (4.17)-(4.22) constitute the complete system of equations allowing us to fix all solutions for the cubic vertex \(p^\beta_{[3]}\) and the densities \(j^{RL}_{[3]}\) uniquely up to the freedom related to field
redefinitions. Namely, if a cubic vertex $p_{[3]}^{-\text{fix}}$ obeys the complete system of equations, then the cubic vertex $p_{[3]}^{-\text{fix}}$ which is obtained from $p_{[3]}^{-\text{fix}}$ by using the field redefinition, $p_{[3]}^{-\text{fix}} = p_{[3]}^{-\text{fix}} + P^f$, also obeys the complete system of equations. To determine the cubic vertex uniquely we have to choose some representative of the cubic vertex. After choosing a representative of the cubic vertex, the complete system of equations provides us the possibility to find all solutions for the cubic vertex $p_{[3]}^{-\text{fix}}$ and the densities $j_{[3]}^{-R,L}$ uniquely. Below we choose the representative of the cubic vertex which we refer to as harmonic cubic vertex.

5 Equations for cubic harmonic and meromorphic vertices

According to (4.21), the cubic vertex $p_{[3]}^{-\text{fix}}$ is a degree-$K$ polynomial in the momenta $P^R, P^L, K < \infty$. By using field redefinitions, we obtain various representatives of $p_{[3]}^{-\text{fix}}$. The representative of the cubic vertex which obeys the harmonic equation

$$\partial_{\bar{p}R}\partial_{\bar{p}L}p_{[3]}^{-} = 0$$

will be referred to as harmonic vertex (for more detailed discussion, see Sec.4.1 in Ref.[10]). For the harmonic vertex $p_{[3]}^{-\text{fix}}$, kinematical equations (4.18), (4.19) take the same form. Note also that the harmonic vertex $p_{[3]}^{-\text{fix}}$ automatically obeys restriction (4.22). We then find that restrictions (4.20), (4.21) amount to the following

**Equations for harmonic vertex $p_{[3]}^{-}$:**

$$\left(J_{Th}^{-R} + \frac{1}{N_{pL} + 1} \sum_{a=1,2,3} \frac{m_a^2}{2\beta_a} \left(-N_{\beta} + M_{R^L}\right)\partial_{\bar{p}L}\right)p_{[3]}^{-} = 0,$$  \hspace{1cm} (5.2)

$$\left(J_{Th}^{-L} + \frac{1}{N_{pR} + 1} \sum_{a=1,2,3} \frac{m_a^2}{2\beta_a} \left(-N_{\beta} - M_{R^R}\right)\partial_{\bar{p}R}\right)p_{[3]}^{-} = 0,$$  \hspace{1cm} (5.3)

$$J_{Th}^{-R} \equiv \frac{P_{Th}^R}{\beta} \left(-N_{\beta} + M_{R^L}\right) + \sum_{a=1,2,3} \frac{\tilde{\beta}_a}{6\beta_a} m_a^2 \partial_{\bar{p}L} - \frac{1}{\beta_a} M_{R}^a,$$  \hspace{1cm} (5.4)

$$J_{Th}^{-L} \equiv \frac{P_{Th}^L}{\beta} \left(-N_{\beta} - M_{R^R}\right) + \sum_{a=1,2,3} \frac{\tilde{\beta}_a}{6\beta_a} m_a^2 \partial_{\bar{p}R} - \frac{1}{\beta_a} M_{L}^a,$$  \hspace{1cm} (5.5)

$$P_{Th}^R \equiv P^R\Pi^R, \hspace{1cm} \Pi^R \equiv \left(1 - \frac{P^L}{N_{pL} + 1}\right)\partial_{\bar{p}L};$$  \hspace{1cm} (5.6)

$$P_{Th}^L \equiv P^L\Pi^L, \hspace{1cm} \Pi^L \equiv \left(1 - \frac{P^R}{N_{pR} + 1}\right)\partial_{\bar{p}R};$$  \hspace{1cm} (5.7)

**Explicit local representation for densities $j_{[3]}^{-R,L}$:**

$$j_{[3]}^{-R} = \frac{1}{N_{pL} + 1} \left(N_{\beta} - M_{R^L}\right)\partial_{\bar{p}L}p_{[3]}^{-},$$  \hspace{1cm} (5.8)

$$j_{[3]}^{-L} = \frac{1}{N_{pR} + 1} \left(N_{\beta} + M_{R^R}\right)\partial_{\bar{p}R}p_{[3]}^{-}.$$  \hspace{1cm} (5.9)

---

5Equations for the harmonic cubic vertices of light-cone gauge fields in $R^{d-1,1}$, $d \geq 4$, were obtained in Ref.[10]. Here, for the case of $d = 4$, we represent equations in Ref.[10] by using the light-cone gauge helicity basis framework.
The definition of the notation we use in (5.2)-(5.10) may be found in (4.3), (4.7), (4.14), and (4.15). For the derivation of (5.2)-(5.10), see Appendix E.

It is the explicit local representation for the densities \( j^{-R,L}_{[3]} \) given in (5.9), (5.10) that we consider as one of the attractive features for the use of the harmonic vertex. Other attractive feature for the use of the harmonic vertex is that all restrictions on \( p^{-}_{[3]} \) and \( j^{-R,L}_{[3]} \) in (4.18)-(4.22) amount to equations solely for the harmonic vertex given in (4.18), (4.19) and (5.1)-(5.3). Obviously, the harmonic equation (5.1) and the kinematical equations (4.18), (4.19) present no difficulties. A real difficulty is to find solution to equations (5.2), (5.3). Our method for solving equations (5.2), (5.3) is realized in the following 3 steps.

**Step 1.** Solution to harmonic equation (5.1), which is polynomial in \( \mathbb{P}^R \) and \( \mathbb{P}^L \), can be presented as

\[
p^{-}_{[3]} = V_N(\mathbb{P}^R, \beta_a, u_a, v_a) + V_0(\beta_a, u_a, v_a) + V_N(\mathbb{P}^L, \beta_a, u_a, v_a), \tag{5.11}
\]

\[
V_N(\mathbb{P}^R, \beta_a, u_a, v_a) \equiv \sum_{n=1}^{N} (\mathbb{P}^R)^n V_{N,n}(\beta_a, u_a, v_a), \tag{5.12}
\]

\[
V_N(\mathbb{P}^L, \beta_a, u_a, v_a) \equiv \sum_{n=1}^{N} (\mathbb{P}^L)^n V_{N,n}(\beta_a, u_a, v_a). \tag{5.13}
\]

Now, for each harmonic vertex \( p^{-}_{[3]} \), we associate a new vertex \( \tilde{V} \) which is meromorphic function of the momentum \( \mathbb{P}^L \) and independent of the momentum \( \mathbb{P}^R \),

\[
\tilde{V} \equiv V^\otimes_N(\mathbb{P}^L, \beta_a, u_a, v_a) + V_0(\beta_a, u_a, v_a) + V_N(\mathbb{P}^L, \beta_a, u_a, v_a), \tag{5.14}
\]

where a new vertex \( V^\otimes_N \) is defined in terms of the vertex \( V_N \) appearing in (5.11), (5.12) as follows

\[
V^\otimes_N(\mathbb{P}^L, \beta_a, u_a, v_a) \equiv V_N(\mathbb{P}^R, \beta_a, u_a, v_a), \tag{5.15}
\]

\[
\mathbb{P}^R \equiv \frac{\rho^2}{2\mathbb{P}^L}, \quad \mathbb{P}^L \equiv \frac{\rho^2}{2\mathbb{P}^R}, \quad \rho^2 \equiv \beta \sum_{a=1,2,3} \frac{m_a^2}{\beta_a}, \quad \beta \equiv \beta_1 \beta_2 \beta_3. \tag{5.16}
\]

From (5.11)-(5.13), we see that the meromorphic vertex \( \tilde{V} \) (5.14) is obtained from the harmonic vertex \( p^{-}_{[3]} \) by using the replacement \( \mathbb{P}^R \rightarrow \mathbb{P}^R \) in the expression for the vertex \( V_N \) (5.11), where \( \mathbb{P}^R \) is defined in (5.16). Note that the vertex \( V^\otimes_N \) involves terms of negative powers of \( \mathbb{P}^L \). This provides us the following rule for building the harmonic vertex \( p^{-}_{[3]} \) by using the meromorphic vertex \( \tilde{V} \): given the meromorphic vertex \( \tilde{V} \), we make the replacement \( \mathbb{P}^L \rightarrow \mathbb{P}^L \) in the terms of negative powers of \( \mathbb{P}^L \),

\[
p^{-}_{[3]} = V^\otimes_N(\mathbb{P}^L, \beta_a, u_a, v_a) \big|_{\mathbb{P}^R \rightarrow \mathbb{P}^L} + V_0(\beta_a, u_a, v_a) + V_N(\mathbb{P}^L, \beta_a, u_a, v_a), \tag{5.17}
\]

where \( \mathbb{P}^L \) is defined in (5.16). In other words, in view of relations (5.11)-(5.17), there is one-to-one correspondence between the harmonic vertex \( p^{-}_{[3]} \), which is polynomial in \( \mathbb{P}^R \) and \( \mathbb{P}^L \), and the vertex \( \tilde{V} \), which is meromorphic function of \( \mathbb{P}^L \) and independent of \( \mathbb{P}^R \).

**Step 2.** In Appendix E, we show that, in terms of the meromorphic vertex \( \tilde{V} \), equations (5.2), (5.3) take the form

\[
\sum_{a=1,2,3} \left( \frac{m_a^2}{2\beta_a} (N_\beta - M^{RL}_a) - \frac{\beta_a}{6\beta_a} m_a^2 N_{pL} + \frac{\mathbb{P}^L}{\beta_a} M^{RL}_a \right) \tilde{V} = 0, \tag{5.18}
\]
while kinematical equations (4.18), (4.19) can be represented as
\[
(M^{RL} + \sum_{a=1,2,3} \beta_a \partial_\beta_a) \bar{V} = 0 ,
\]
(5.20)
\[
(N_{\bar{P}_{L}} - M^{RL}) \bar{V} = 0 .
\]
(5.21)

Note that, for the derivation of equation (5.20), we used equation (5.21).

\textbf{Step 3.} In Appendix E, we find that the general solution of equations (5.18)-(5.21) can be presented as
\[
\bar{V} = E_m E_\beta \bar{V}^{(2)} , \quad E_m \equiv \prod_{a=1,2,3} E_{ma} , \quad E_\beta \equiv \prod_{a=1,2,3} E_{\beta a} ,
\]
(5.22)
\[
E_{ma} \equiv \exp \left( \frac{1}{\bar{P}_{L} f_a M_a^L} \right) , \quad E_{\beta a} \equiv \left( \frac{\bar{P}_{L}}{\beta_a} \right)^{M_a^{RL}} ,
\]
\[
f_a \equiv -\frac{1}{2} \beta_a - \beta_a c_a , \quad c_a = \frac{m_a^2 - m_{a+1}^2 + m_{a+2}^2}{2m_a^2} , \quad \text{for } m_a \neq 0 ,
\]
(5.23)
\[
E_{ma} \equiv 1 , \quad E_{\beta a} \equiv \left( \frac{\bar{P}_{L}}{\beta_a} \right)^{\lambda_a} , \quad \text{for } m_a = 0 ,
\]
(5.24)
where a new vertex \(\bar{V}^{(2)}\) is independent of the momenta \(\beta_1, \beta_2, \beta_3\), and \(\bar{P}_{L}\),
\[
\bar{V}^{(2)} = \bar{V}^{(2)}(u_a, v_a) ,
\]
(5.25)
and satisfies the following equation:
\[
\sum_{a=1,2,3} \left\{ 2M_a^R + \left( 2c_a m_a^2 - m_{a+1}^2 + m_{a+2}^2 \right) M_a^{RL} \right. \\
+ \left. \left( m_a^2 \left( c_a^2 - \frac{1}{4} \right) - \left( c_a - \frac{1}{2} \right) m_{a+1}^2 + \left( c_a + \frac{1}{2} \right) m_{a+2}^2 \right) M_a^L \right\} \bar{V}^{(2)} = 0 .
\]
(5.26)

To summarize, to each harmonic vertex \(p_{[3]}\) we associated the meromorphic vertex \(\bar{V}\). The harmonic vertex and the meromorphic vertex are in one-to-one correspondence and they are related to each other by simple transformations given in (5.11)-(5.17). For the meromorphic vertex \(\bar{V}\), we find the representation given in (5.22)-(5.24), where \(\bar{V}^{(2)}\) satisfies equation (5.26). Relations (5.22)-(5.24) show the dependence of the vertex \(\bar{V}\) on the momenta \(\beta_1, \beta_2, \beta_3\), and \(\bar{P}_{L}\), while the equation for \(\bar{V}^{(2)}\) in (5.26) involves only the oscillators \(u_a, v_a\). In other words, the problem of finding all cubic vertices is reduced to the problem of finding all solutions to equation (5.26). Finding all solutions to equation (5.26) turns out to be a simple problem.

\section{Classification of cubic vertices}

A structure of the solutions to equation (5.26) depends crucially on the masses. Therefore, before discussing solutions for the meromorphic vertex \(\bar{V}\), we explain our classification of cubic vertices. To this end we find it convenient to introduce a notion of critical and non-critical masses.
Critical and non-critical masses. Consider a cubic vertex for massive and massless fields which have masses \(m_1, m_2, m_3\). We introduce quantities \(D, \mathbf{P}_{\text{em}}\) defined in the following way:

\[
D \equiv m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_2^2m_3^2 - 2m_3^2m_1^2,
\]

\[
\mathbf{P}_{\text{em}} \equiv \sum_{a=1,2,3} \epsilon_a m_a,
\]

\[
\epsilon_1^2 = 1, \quad \epsilon_2^2 = 1, \quad \epsilon_3^2 = 1,
\]

\[
D = (m_1 + m_2 + m_3)(m_1 + m_2 - m_3)(m_1 - m_2 + m_3)(m_1 - m_2 - m_3).
\]

Note that relation (6.4) gives the alternative representation for \(D\) (6.1). If, for given masses \(m_1, m_2, m_3\), we find that \(D = 0\), then we refer to such masses as critical masses, while if, for given masses \(m_1, m_2, m_3\), we find that \(D \neq 0\), then such masses are referred to as non-critical masses. From (6.4), we see that the restriction \(\mathbf{P}_{\text{em}} = 0\) implies the restriction \(D = 0\). Also, from (6.4), we see that the restriction \(D = 0\) implies that there exist \(\epsilon_1, \epsilon_2, \epsilon_3\) (6.3) such that the restriction \(\mathbf{P}_{\text{em}} = 0\) holds true. This is to say that we use the definition

\[
D \neq 0, \quad \text{for non-critical masses;}
\]

\[
D = 0, \quad \mathbf{P}_{\text{em}} = 0, \quad \text{for critical masses.}
\]

All cubic vertices for three massless fields were obtained in Ref. [9] (see (6.11) in Appendix F of our paper). In our paper, we study cubic vertices that involve at least one massive field. This is to say that, depending on masses of fields involved in the cubic vertex, we introduce the following Classification of cubic vertices:

- **Two massless and one massive fields,**

\[
m_1 = 0, \quad m_2 = 0, \quad m_3 \neq 0, \quad D > 0;
\]

- **Two massive and one massless fields,**

\[
m_1 \neq 0, \quad m_2 \neq 0, \quad m_1 \neq m_2, \quad m_3 = 0, \quad D > 0;
\]

\[
m_1 = m, \quad m_2 = m, \quad m \neq 0 \quad m_3 = 0, \quad D = 0, \quad \mathbf{P}_{\text{em}} = 0;
\]

- **Three massive fields,**

\[
m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad D > 0;
\]

\[
m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad D < 0;
\]

\[
m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad D = 0, \quad \mathbf{P}_{\text{em}} = 0.
\]

The following remarks are in order.

- **i)** In (6.7)-(6.9), the restrictions on \(D\) follow from the restrictions on the masses shown explicitly in (6.7)-(6.9). Namely, for \(m_1 = 0, m_2 = 0\) (6.7), the restriction \(m_3 \neq 0\) in (6.7) leads not only to

\[\text{For example, if the masses satisfy the relation } m_1 + m_2 - m_3 = 0, \text{ then the corresponding } \epsilon_a \text{ are given by } \epsilon_1 = 1, \epsilon_2 = 1, \epsilon_3 = -1. \text{ In } R^{2,1} \text{ space, some cubic vertices for massive fields with the critical masses can be obtained via dimensional reduction from vertices for massless fields in } R^{3,1} \text{ space (see Refs. [14]-[15]). It remains to be understood about whether or not, and in what ways, the cubic vertices for massive fields with the critical masses in } R^{3,1} \text{ space can be obtained via dimensional reduction from cubic vertices of massless fields in } R^{4,1} \text{ space.} \]
the restriction \( D \neq 0 \) but also to the restriction \( D > 0 \). For \( m_3 = 0 \), the restriction \( m_1 \neq m_2 \) in (6.8) implies not only the restriction \( D \neq 0 \), but also the restriction \( D > 0 \). For \( m_3 = 0 \), the restriction \( D = 0 \) follows from the restriction \( m_1 = m_2 \) in (6.9).

\textbf{ii}) For three massive fields, the restrictions on \( D \) in (6.10)-(6.12) provide additional restrictions on masses. We note that our use of \( D \) for the classification of cubic vertices of massive fields is related to the following two reasons: 1) a structure of the cubic vertices and hermitian conjugation rules for coupling constants depend on the restrictions on \( D \) and 2) depending on value of \( D \), all processes of a decay of massive particle into two particles can be classified as follows,

\[
\begin{align*}
D > 0 & \quad \text{for real processes with non-zero transfer of momentum; (6.13)} \\
D = 0 & \quad \text{for real processes with zero transfer of momentum; (6.14)} \\
D < 0 & \quad \text{for virtual processes. (6.15)}
\end{align*}
\]

We see that the classification of the processes (6.13)-(6.15) match with the classification we are going to use for cubic vertices in (6.10)-(6.12). For other comments, see Appendix G.

\textbf{iii}) Classification in (6.7)-(6.9) is well known and was already used in Refs.[10, 11] upon building the cross-interactions between massive and massless fields in the framework of light-cone gauge approach. To our knowledge, classification (6.10)-(6.12) has not been used in the earlier literature.

In what follows we use the shortcut \((0, \lambda)\) for the massless field \( \phi^\dagger_\lambda \) and the shortcut \((m, \lambda)\) for the massive field \( \langle \phi_{m,s} | \). We are interested in a decay of a massive particle into two particles with non-collinear momenta for incoming and outgoing particles. For such decay, we have the restrictions \( P_R \neq 0, P_L \neq 0 \) and hence we can represent the on-shell condition (6.18) as the following

\[
\begin{align*}
P_R \longleftrightarrow P_L &= \rho \sqrt{2} e^{i \varphi} , \quad \rho > 0 \, . (6.19)
\end{align*}
\]

On-shell cubic vertex and 3-point amplitudes. Detailed analysis of 3-point amplitudes is beyond the scope of our study. However because our results provide quick and easy access to 3-point amplitudes we briefly comment the light-cone frame representation for the 3-point amplitudes. In the light-cone frame, the energy conservation law amounts to the equation \( P^- = 0 \), where \( P^- \) is given in (4.14). We will refer to this equation as on-shell condition. From (4.14), we see that the on-shell condition amounts to the equation for the momenta \( P_R, P_L \) and \( \beta_1, \beta_2, \beta_3 \),

\[
\begin{align*}
P_R P_L - \frac{\rho^2}{2} &= 0 \, , \quad \text{on-shell condition, (6.18)}
\end{align*}
\]

where \( \rho^2 \) is defined in (4.14). Note that relation (6.18) implies the on-shell restriction \( \rho^2 \geq 0 \). We are interested in a decay of a massive particle into two particles with non-collinear momenta for incoming and outgoing particles. For such decay, we have the restrictions \( P_R \neq 0, P_L \neq 0 \) and hence we can represent the on-shell condition (6.18) as the following

\textbf{On-shell conditions for non-collinear momenta:}

\[
\begin{align*}
P_R &= \frac{\rho}{\sqrt{2}} e^{i \varphi} , \quad P_L = \frac{\rho}{\sqrt{2}} e^{-i \varphi} , \quad \rho > 0 \, . (6.19)
\end{align*}
\]

Plugging the on-shell values of \( P_R, P_L (6.19) \) into the vertices \( p_{[3]}^- (5.11) \) and \( \bar{V} (5.14) \), we get the respective on-shell values of \( p_{[3]}^- \) and \( \bar{V} \). From relations (5.11)-(5.17), we then see that the on-shell values of the harmonic vertex \( p_{[3]}^- \) and the meromorphic vertex \( \bar{V} \) coincide

\[
p_{[3]}^- \, \text{on--sh} = \bar{V}_{\text{on--sh}} \, . (6.20)
\]
Up to the sign, the 3-point invariant amplitude is equal to \( p_{[3] \text{on-sh}} \) (for some details, see Appendix F). Relation (6.20) tells us then that the meromorphic vertex \( \bar{V} \) provides us quick and easy access to the 3-point invariant amplitudes. All that is required is to plug the on-shell value of \( \mathbb{P}^L \) (6.19) into the meromorphic vertex \( \bar{V} \).

7 Vertex \( \bar{V} \) for two massless and one massive fields

Using notation given in (6.16), (6.17), we start with the meromorphic vertex for two massless fields and one massive field (6.7),

\[
(0, \lambda_1) - (0, \lambda_2) - (m_3, s_3), \quad m_3 \neq 0,
\]

i.e. two massless fields carry external line indices \( a = 1, 2 \), while one massive field carries external line index \( a = 3 \). For this particular case, the general expression for the meromorphic vertex \( \bar{V} \) (5.22) takes the following form (for the derivation, see Appendix H):

\[
\bar{V} = C_{\lambda_1, \lambda_2} N_{\lambda_1, \lambda_2} \bar{V}_{\lambda_1, \lambda_2}^{\text{bas}},
\]

\[
\bar{V}_{\lambda_1, \lambda_2}^{\text{bas}} \equiv \beta_1^{-\lambda_1} \beta_2^{-\lambda_2} \beta_{s_3} \mathbb{P}^L_{\lambda_1, \lambda_2 - s_3} L_{3+} L_{3-} L_{s_3} = \beta_1^{-\lambda_1} \beta_2^{-\lambda_2} \beta_{s_3},
\]

\[
L_{3\pm} \equiv \frac{\mathbb{P}^L}{\beta_3} u_3 \pm \frac{g_{3\pm}}{\beta_3} v_3, \quad g_{3+} \equiv -\frac{\beta_3 m_3}{\beta_3}, \quad g_{3-} \equiv \frac{\beta_1 m_3}{\beta_3},
\]

\[
N_{\lambda_1, \lambda_2} \equiv 2^{(\lambda_1 + \lambda_2 + s_3)/2} m_3^{-\lambda_1 - \lambda_2},
\]

where coupling constant \( C_{\lambda_1, \lambda_2} \), the helicities \( \lambda_1, \lambda_2 \), and the spin \( s_3 \) satisfy the restrictions

\[
C_{\lambda_1, \lambda_2} = C_{-\lambda_1, -\lambda_2},
\]

\[
s_3 \geq |\lambda_1 - \lambda_2|.
\]

Vertex \( \bar{V}_{\lambda_1, \lambda_2}^{\text{bas}} \) (7.3) is a unique solution to the equations for the vertex \( \bar{V} \). In the expression for the vertex \( \bar{V} \) (7.2), we inserted the normalization factor \( N_{\lambda_1, \lambda_2} \) (7.5) and the coupling constant \( C_{\lambda_1, \lambda_2} \) (7.6). In general, the coupling constant \( C_{\lambda_1, \lambda_2} \) depends not only on \( \lambda_1, \lambda_2 \) but also on the spin \( s_3 \) and the mass \( m_3 \). We show explicitly the labels \( \lambda_1, \lambda_2 \) because only these labels are not inert under the complex conjugation of the coupling constant (7.6). The following remarks are in order.

i) Restriction (7.7) is obtained by requiring that the powers of \( L_1 \) and \( L_2 \) in (7.3) be non-negative integers. Given values of \( \lambda_1, \lambda_2, \) and \( s_3 \) subjected to restriction (7.7) there is only one vertex \( \bar{V} \)\(^8\).

We note then that the numbers of cubic vertices given in (8.10) - (8.12) coincide with the numbers of 3-point amplitudes given in (5.32) in Ref.\(^{16}\).

ii) As seen from (3.11), to get the hermitian \( P_{[3]}^{-} \), we need not only the vertex \( \bar{V} \) (7.2) but also the vertex \( \mathcal{I} \bar{V} \) which is associated with \( \mathcal{I} P_{[3]}^{-} \). Realization of the operator \( \mathcal{I} \) on the vertex \( \bar{V} \) is given in \( \text{(D.7)} \) in Appendix D. Using (7.2) and (D.7), we get the relation

\[
\mathcal{I} \bar{V} = C_{\lambda_1, \lambda_2}^* N_{-\lambda_1, -\lambda_2} \bar{V}_{-\lambda_1, -\lambda_2}^{\text{bas}},
\]

\(^7\)The normalization factor \( N_{\lambda_1, \lambda_2} \) is used for the convenience (see relations (H.7), (H.8) in Appendix H).

\(^8\)By using the helicity-spinor language, 3-point amplitudes for arbitrary spin massless and massive particles were studied in Refs.\(^{16} \text{[17]} \). In these references, it was noted that, for the case under consideration in this Section, there is only one 3-point amplitude.
which tells us that $\mathcal{I} \tilde{V}$ is associated with $\tilde{V}^{\text{bas}}_{-\lambda_1, -\lambda_2}$. This motivates us to represent $\mathcal{I} \tilde{V}$ as

$$\mathcal{I} \tilde{V} = C_{-\lambda_1, -\lambda_2} N_{-\lambda_1, -\lambda_2} \tilde{V}^{\text{bas}}_{-\lambda_1, -\lambda_2}.$$  \hfill (7.9)

Comparison of (7.8) and (7.9) gives then restriction (7.6).

\textbf{iii) Explicit expression for the harmonic vertex $p_{[3]}^{-}$ associated with the meromorphic vertex $\tilde{V}$ (7.2), (7.3) can be obtained by expanding the meromorphic vertex $\tilde{V}$ (7.2) in Laurent series in $\mathbb{P}^L$ and using (5.14)-(5.17). As the explicit expression for $p_{[3]}^{-}$ is not illuminating let us make comment on the general structure of $p_{[3]}^{-}$. To this end we note that the Laurent series expansion of the meromorphic vertex $\tilde{V}$ (7.2), (7.3) in $\mathbb{P}^L$ can be presented as

$$\tilde{V} = \sum_{n=\lambda_1+\lambda_2+s_3}^{\lambda_1+\lambda_2+s_3} (\mathbb{P}^L)^n \bar{\mathcal{V}}_n.$$  \hfill (7.10)

Using (7.10), it is easy to see that, depending on the values $\lambda_1$, $\lambda_2$, $s_3$, a general form of the harmonic vertex $p_{[3]}^{-}$(5.17) can be presented as

$$p_{[3]}^{-} = V_{s_3-\lambda_1-\lambda_2}(\mathbb{P}^R) + V_0 + \tilde{V}_{\lambda_1+\lambda_2+s_3}(\mathbb{P}^L),$$  \hfill (7.11)

for $\lambda_1 + \lambda_2 - s_3 < 0$, $\lambda_1 + \lambda_2 + s_3 > 0$; \hfill (7.11)

$$p_{[3]} = V_{s_3-\lambda_1-\lambda_2}(\mathbb{P}^R) + V_0,$$  \hfill (7.12)

for $\lambda_1 + \lambda_2 + s_3 \leq 0$; \hfill (7.12)

$$p_{[3]} = V_0 + \tilde{V}_{\lambda_1+\lambda_2+s_3}(\mathbb{P}^L),$$  \hfill (7.13)

for $\lambda_1 + \lambda_2 - s_3 \geq 0$;

where, in (7.11)-(7.13), the dependence of the vertices $V_N$, $V_0$, $\tilde{V}_N$ on the $\beta$-momenta and the oscillators is implicit. In (7.12), $V_0 = 0$ for $\lambda_1 + \lambda_2 + s_3 < 0$, while, in (7.13), $V_0 = 0$ for $\lambda_1 + \lambda_2 - s_3 > 0$. Comparing (5.11) and (7.13), we see that, for $p_{[3]}^{-}$ (7.13), we have $V_N = 0$ and hence $V_N^\infty = 0$. Using (5.14), (5.17), we get then the relation

$$p_{[3]}^{-} = \tilde{V},$$  \hfill (7.14)

for $\lambda_1 + \lambda_2 \geq s_3$.

Thus, for $\lambda_1$, $\lambda_2$, $s_3$ that satisfy the restriction in (7.14), the vertices $\tilde{V}$ and $p_{[3]}^{-}$ coincide. We recall that the restrictions in (7.7) and (7.14) are the well known triangle inequalities.

**On-shell cubic vertex.** 3-point invariant amplitudes are expressed in terms of on-shell cubic vertices. For the decay processes with non-collinear momenta, the on-shell cubic vertex is obtained from off-shell vertex (7.2) by using the on-shell conditions (see (6.19), (6.20)). Consider the decay of the massive particle ($a = 3$) into the two massless particles ($a = 1, 2$),

$$3 \rightarrow 1 + 2.$$  \hfill (7.15)

On-shell value of the off-shell vertex (7.2) is found to be

$$\tilde{V}_{\text{on-sh}} = (-)^{s_3} C_{\lambda_1, \lambda_2} e^{-i(\lambda_1+\lambda_2)\phi} L_{3+}^{s_3-\lambda_1-\lambda_2} L_{3-}^{s_3+\lambda_1-\lambda_2},$$  \hfill (7.16)

$$L_{3+} \equiv -\sqrt{-g_{3+} e^{-\frac{i}{2} \phi} u_3} + \sqrt{g_{3+} e^{\frac{i}{2} \phi} v_3},$$  \hfill (7.17)

$$L_{3-} \equiv -\sqrt{g_{3+} e^{\frac{i}{2} \phi} u_3} - \sqrt{-g_{3+} e^{\frac{i}{2} \phi} v_3},$$  \hfill (7.18)

$$g_{3+} = \frac{m_3}{1 + r}, \quad g_{3-} = -\frac{r m_3}{1 + r},$$  \hfill (7.19)
\[ r \equiv \beta_1 / \beta_2, \quad 0 < r < \infty, \quad (7.20) \]

where an angle variable \( \varphi \) appearing in (7.16)-(7.18) is related to \( \mathbb{P}^L \) as
\[ \mathbb{P}^L = \frac{\rho}{\sqrt{2}} e^{-i\varphi}, \quad \rho = m_3 \beta_2 \sqrt{r}. \quad (7.21) \]

Relations (7.21) are obtained from (6.19) and the definitions of \( \rho^2 \) and \( r \) in (4.14), (7.20). The allowed values of \( r \) in (7.20) are fixed by the condition \( \rho^2 > 0 \). We note the helpful inequalities,
\[ \beta_1 > 0, \quad \beta_2 > 0, \quad \beta_3 < 0, \quad (7.22) \]
\[ g_3^+ > 0, \quad g_3^- < 0, \quad (7.23) \]
where in (7.22), we show the allowed values of the \( \beta \)-momenta for the process (7.15), while restrictions (7.23) are obtained from definitions (7.19). For the reader convenience, we note the following relations which are helpful for the derivation of the on-shell vertex (7.16) from the off-shell vertex (7.2):
\[ L_{3\pm} \bigg|_{\text{on-sh}} = \sqrt{\pm \frac{g_{3\pm}}{2}} e^{-\frac{i}{2} \varphi L_{3\pm}}, \quad \frac{\rho}{\beta_3} = -\sqrt{-g_3^+ g_3^-}. \quad (7.24) \]

8 Vertex \( \bar{\mathcal{V}} \) for two massive fields with non-equal masses and one massless field

Using notation given in (6.16), (6.17), we now consider the meromorphic vertex for two massive fields with non-equal masses and one massless field (6.8),
\[ (m_1, s_1) - (m_2, s_2) = (0, \lambda_3), \quad m_1 \neq m_2, \quad (8.1) \]
i.e. two massive fields carry external line indices \( a = 1, 2 \), while one massless field carries external line index \( a = 3 \). For this particular case, the general expression for the meromorphic vertex \( \bar{\mathcal{V}} \) (5.22) takes the following form (for the derivation, see Appendix H):
\[ \bar{\mathcal{V}} = C_{n_1, n_2, \lambda_3}^{n_1, n_2, \lambda_3} \mathcal{V}_{n_1, n_2, \lambda_3}^{\text{bas}}, \quad (8.2) \]
\[ \mathcal{V}_{n_1, n_2, \lambda_3}^{\text{bas}} = \beta_1^{s_1} \beta_2^{s_2} \beta_3^{-\lambda_3} (\mathbb{P}^L)^{-s_1-s_2+\lambda_3} \prod_{a=1,2} L_{a+n}^{s_a+n_a} L_{a-n}^{s_a-n_a}, \quad (8.3) \]
\[ L_{a\pm} = \frac{\mathbb{P}^L}{\beta_a} u_a + \frac{g_{a\pm}}{\sqrt{2}} v_a, \quad g_{a\pm} = g_a \pm \gamma \frac{\Lambda}{m_a}, \quad g_a = \frac{\tilde{\beta}_a}{2} m_a + c_a m_a, \quad (8.4) \]
\[ c_1 = \frac{m_2^2}{2 m_1^2}, \quad c_2 = -\frac{m_1^2}{2 m_2^2}, \quad \gamma = \frac{1}{2} (m_1^2 - m_2^2), \quad (8.5) \]
where coupling constants \( C_{n_1, n_2, \lambda_3} \) and integers \( n_1, n_2 \in \mathbb{Z} \) satisfy the restrictions
\[ C_{n_1, n_2, \lambda_3}^* = C_{-n_1, -n_2, -\lambda_3}, \quad (8.6) \]
\[ n_1 + n_2 = \lambda_3, \quad n_1, n_2 \in \mathbb{Z}, \quad (8.7) \]
The vertices \( \bar{V}_{n_1,n_2,\lambda_3}^{\text{bas}} \) (8.3) constitute a basis of all solutions for the vertex \( \bar{V} \). In the expression for vertex \( \bar{V} \) (8.2), we inserted the normalization factors \( N_{\lambda_1,\lambda_2} \) (8.5) and the coupling constants \( C_{n_1,n_2,\lambda_3} \). In general, the coupling constants depend not only on \( \lambda_3 \) and the integers \( n_1, n_2 \) but also on the spins \( s_1, s_2 \) and the masses \( m_1, m_2 \). Only \( \lambda_3 \) and the integers \( n_1, n_2 \) are not inert under the complex conjugation of the coupling constants (8.6). The following remarks are in order.

**i)** Restriction (8.7) is obtained by solving equation (5.26), while restrictions (8.8) are obtained by requiring that the powers of \( L_1 \) and \( L_2 \) in (8.3) be non-negative integers.

**ii)** Two integers \( n_1, n_2 \) subjected to restrictions (8.7), (8.8) express the freedom of the solution for \( V \). These two integers label all possible vertices \( V \) that can be built for the fields in (8.1). Using restrictions (8.7), (8.8), we can find a number of the cubic vertices. Let us first note the following general restriction obtained from (8.7), (8.8):

\[-s_1 - s_2 \leq \lambda_3 \leq s_1 + s_2. \tag{8.9}\]

Now using restrictions (8.7), (8.8), we find the following numbers of the cubic vertices:

\[
n = s_1 + s_2 - \lambda_3 + 1 \quad \text{for} \quad |s_1 - s_2| \leq \lambda_3 \leq s_1 + s_2; \tag{8.10}
\]

\[
n = s_1 + s_2 - |s_1 - s_2| + 1 \quad \text{for} \quad -|s_1 - s_2| \leq \lambda_3 \leq |s_1 - s_2|; \tag{8.11}
\]

\[
n = s_1 + s_2 + \lambda_3 + 1 \quad \text{for} \quad -s_1 - s_2 \leq \lambda_3 \leq -|s_1 - s_2|. \tag{8.12}
\]

We note then that the numbers of cubic vertices given in (8.10)-(8.12) coincide with the numbers of 3-point amplitudes given in (5.32) in Ref.[16].

**iii)** From (8.11) we learn that, to get the hermitian \( P_{[\lambda]}^- \), we need not only the vertex \( \bar{V} \) (8.2) but also the vertex \( I \bar{V} \) which is associated with \( I P_{[\lambda]}^- \). Realization of the operator \( I \) on the vertex \( \bar{V} \) is given in (D.7) in Appendix D. Using (8.2), (D.7), and (D.9), we find the expression for \( I \bar{V} \),

\[
I \bar{V} = C_{n_1,n_2,\lambda_3}^{\text{bas}} N_{n_1,n_2,\lambda_3} \bar{V}_{-n_1,-n_2,-\lambda_3}^{\text{bas}}, \tag{8.13}
\]

which tells us that \( I \bar{V} \) is associated with \( \bar{V}_{-n_1,-n_2,-\lambda_3}^{\text{bas}} \). This motivates us to represent \( I \bar{V} \) as

\[
I \bar{V} = C_{-n_1,-n_2,-\lambda_3} N_{-n_1,-n_2,-\lambda_3} \bar{V}_{-n_1,-n_2,-\lambda_3}^{\text{bas}}, \tag{8.14}
\]

The comparison of (8.13) and (8.14) gives restriction (8.6).

**iv)** Explicit expression for the harmonic vertex \( p_{[\lambda]}^- \) associated with the meromorphic vertex \( \bar{V} \) (8.2) can be obtained by expanding the meromorphic vertex \( \bar{V} \) (8.2) in Laurent series in \( \mathbb{P}^L \) and using (5.14)-(5.17). As the explicit expression for \( p_{[\lambda]}^- \) is not illuminating let us make comment on the general structure of \( p_{[\lambda]}^- \). To this end we note that the Laurent series expansion of the meromorphic vertex \( \bar{V} \) (8.2), (8.3) in \( \mathbb{P}^L \) can be presented as

\[
\bar{V} = \sum_{n=-(s_1+s_2+\lambda_3)}^{s_1+s_2+\lambda_3} (\mathbb{P}^L)^n \bar{V}_n. \tag{8.15}
\]

Using (8.9), (8.15), it is easy to see that, a general form of the harmonic vertex \( p_{[\lambda]}^- \) (5.17) can be presented as

\[
p_{[\lambda]}^- = V_{s_1+s_2+\lambda_3}(\mathbb{P}^R) + V_0 + \bar{V}_{s_1+s_2+\lambda_3}(\mathbb{P}^L), \tag{8.16}
\]

In Ref.[16], the authors found it convenient to assume \( s_1 \leq s_2 \). We do not use such assumption.
where, in (8.16), the dependence of the vertices $V_N$, $V_0$, $\bar{V}_N$ on the $\beta$-momenta and the oscillators is implicit. Note also that, for $\lambda_3 = \pm (s_1 + s_2)$, the vertex (8.16) takes the form

$$p_{[3]}^- = V_{2s_1+2s_2}(\mathbb{P}^R) + V_0, \quad \text{for } \lambda_3 = -s_1 - s_2; \quad (8.17)$$

$$p_{[3]}^- = V_0 + \bar{V}_{2s_1+2s_2}(\mathbb{P}^L), \quad \text{for } \lambda_3 = s_1 + s_2. \quad (8.18)$$

Comparing (5.11) and (8.18), we see, that, for the vertex $p_{[3]}^-$ (8.16), we have $V_{N} = 0$ and hence $V_{N}^{\infty} = 0$. Using (5.14), (5.17), we get then the relation

$$p_{[3]}^- = \bar{V}, \quad \text{for } \lambda_3 = s_1 + s_2. \quad (8.19)$$

Thus, for $\lambda_3 = s_1 + s_2$, the vertices $\bar{V}$ and $p_{[3]}^-$ coincide.

v) The quantities $g_{a\pm}$ defined in (8.4) can alternatively be represented in a more explicit form as

$$g_{1+} = -\frac{\beta_3}{\beta_1} m_1, \quad g_{1-} = \frac{\beta_2}{m_1} \sum_{a=1,2} \frac{m_a^2}{\beta_a}, \quad g_{2+} = \frac{\beta_3}{\beta_2} m_2, \quad g_{2-} = -\frac{\beta_1}{m_2} \sum_{a=1,2} \frac{m_a^2}{\beta_a}. \quad (8.20)$$

**On-shell cubic vertex.** 3-point amplitudes are expressed in terms of the on-shell cubic vertices. The on-shell cubic vertex is obtained from off-shell vertex (8.2) by using on-shell conditions (6.19). Consider the decay of the massive particle ($a = 1$) into the massive particle ($a = 2$) and the massless particle ($a = 3$)

$$1 \to 2 + 3, \quad m_1 > m_2. \quad (8.21)$$

The restriction on the masses in (8.21) implies that momenta of the particles are non-collinear. For the non-collinear momenta, the on-shell value of the off-shell vertex (8.2) is found to be

$$\bar{V}^{{\text{on-sh}}} = C_{n_1n_2\lambda_3} (-)^{s_1} e^{-i\lambda_3\varphi} \prod_{a=1,2} L_{a+}^{s_a+n_a} L_{a-}^{s_a-n_a}, \quad (8.22)$$

$$L_{a+} = \varepsilon_a \sqrt{-g_a} e^{-\frac{i}{2} \varphi} u_a + \sqrt{g_a} e^{\frac{i}{2} \varphi} v_a, \quad (8.23)$$

$$g_{1+} = \frac{m_1 r}{1 + r}, \quad g_{1-} = -\frac{m_2 (r_+ - r)}{m_1 (1 + r)}, \quad (8.24)$$

$$g_{2+} = m_2 r, \quad g_{2-} = -m_2 (r_+ - r), \quad (8.25)$$

$$r \equiv \beta_3/\beta_2, \quad r_+ \equiv \frac{m_1^2 - m_2^2}{m_2^2}, \quad 0 < r < r_+, \quad (8.26)$$

where an angle variable $\varphi$ appearing in (8.23) is related to $\mathbb{P}^L$ as

$$\mathbb{P}^L = \frac{\rho}{\sqrt{2}} e^{-i\varphi}, \quad \rho = m_2 \beta_2 \sqrt{r(r_+ - r)}. \quad (8.27)$$

Relations (8.27) are obtained from (6.19) and the definitions of $\rho^2$ and $r$ in (4.14), (8.26). The allowed values of $r$ in (8.26) are fixed by requiring $\rho^2 > 0$. We note the following helpful inequalities

$$\beta_1 < 0, \quad \beta_2 > 0, \quad \beta_3 > 0. \quad (8.28)$$
\[ g_{a+} > 0, \quad g_{a-} < 0, \quad a = 1, 2, \quad (8.29) \]

where in (8.28), we show the allowed values of the \( \beta \)-momenta for the process in (8.21), while restrictions (8.29) follow from restriction (8.21) and the definitions in (8.24), (8.25).

For the reader convenience, we note the following relations which are helpful for the derivation of the on-shell vertex (8.22) from the off-shell vertex (8.2):

\[ L_{a\pm} \bigg|_{\text{on-sh}} = \sqrt{\pm \frac{g_{a\pm}}{2}} e^{-\frac{i}{2} \rho_{a\pm}} L_{a\pm}, \quad \rho_{a} = \varepsilon_{a} \sqrt{-g_{a+}g_{a-}}, \quad a = 1, 2. \quad (8.30) \]

9 Vertex \( \bar{\mathcal{V}} \) for two massive fields with equal masses and one massless field

Using notation given in (6.16), (6.17), we now consider the meromorphic vertex for two massive fields with equal masses and one massless field (6.9),

\[ (m_1, s_1) - (m_2, s_2) - (0, \lambda_3), \quad m_1 = m, \quad m_2 = m, \quad m \neq 0, \quad (9.1) \]

i.e. two massive fields carry external line indices \( a = 1, 2 \), while one massless field carries external line index \( a = 3 \). For this particular case, the general expression for the meromorphic vertex \( \bar{\mathcal{V}} \) (5.22) takes the following form (for the derivation, see Appendix H):

\[ \bar{\mathcal{V}} = C_{n, \lambda_3} N_{n, \lambda_3} \bar{\mathcal{V}}_{n, \lambda_3}^{\text{bas}}, \quad (9.2) \]

\[ \bar{\mathcal{V}}_{n, \lambda_3}^{\text{bas}} = \beta_1^{s_1} \beta_2^{s_2} \beta_3^{1-\lambda_3} (\mathbb{P}^L)_{-s_1-s_2+\lambda_3} L_1^{2s_1-n} L_2^{2s_2-n} Q^n, \quad (9.3) \]

\[ L_1 \equiv \frac{\mathbb{P}_L}{\beta_1} u_1 - \frac{m \beta_3}{\sqrt{2\beta_1}} v_1, \quad L_2 \equiv \frac{\mathbb{P}_L}{\beta_2} u_2 + \frac{m \beta_3}{\sqrt{2\beta_2}} v_2, \quad (9.4) \]

\[ Q = v_1 L_2 - v_2 L_1 - \frac{1}{\sqrt{2m}} L_1 L_2, \quad (9.5) \]

\[ N_{n, \lambda_3} \equiv i^n \left( \frac{\sqrt{2}}{m} \right)^{\lambda_3}, \quad (9.6) \]

where coupling constants \( C_{n, \lambda_3} \) and integer \( n \in \mathbb{N}_0 \) satisfy the restrictions

\[ C_{n, \lambda_3}^* = C_{n, -\lambda_3}, \quad (9.7) \]

\[ 0 \leq n \leq 2s_{\text{min}}, \quad s_{\text{min}} \equiv \min_{a=1,2} s_a. \quad (9.8) \]

The vertices \( \bar{\mathcal{V}}_{n, \lambda_3}^{\text{bas}} (9.3) \) constitute a basis of all solutions for the vertex \( \bar{\mathcal{V}} \). In the full expression for the vertex \( \bar{\mathcal{V}} (9.2) \), we inserted the normalization factors \( N_{n, \lambda_3} (9.6) \) and the coupling constants \( C_{n, \lambda_3} \). In general, the coupling constants depend not only on \( n, \lambda_3 \) but also on the spins \( s_1, s_2 \) and the mass \( m \). We note that only \( \lambda_3 \) is not inert under the complex conjugation of the coupling constants (9.7). The following comments are in order:

i) The restrictions (9.8) are obtained by requiring that the powers of \( L_1, L_2 \), and \( Q \) in (9.3) be non–negative integers.

ii) The integer \( n \) subjected to restrictions (9.8) expresses the freedom of the solution for \( \bar{\mathcal{V}} \). This integer labels all possible vertices \( \bar{\mathcal{V}} \) that can be built for the fields in (9.1). Now using restrictions (9.8), we find the following number of the cubic vertices:

\[ n = 2s_{\text{min}} + 1, \quad (9.9) \]
where $s_{\text{min}}$ is given in (9.8). We note then that the number of cubic vertices given in (9.9) coincides with the number of 3-point amplitudes given in Ref. [17].

iii) The Laurent series expansion of the meromorphic vertex $\tilde{V}$ (9.2) in $\mathbb{P}^L$ takes the form

$$\tilde{V} = \sum_{n=-s_1-s_2+\lambda_3}^{s_1+s_2+\lambda_3} (\mathbb{P}^L)^n \tilde{V}_n.$$  

(9.10)

Using (9.10), it is easy to see that, depending on the values $s_1$, $s_2$, $\lambda_3$, a general form of the harmonic vertex $p_{[3]}^{-}$ (5.17) can be presented as

$$p_{[3]}^{-} = V_{s_1+s_2-\lambda_3}(\mathbb{P}^R) + V_0 + \tilde{V}_{s_1+s_2+\lambda_3}(\mathbb{P}^L),$$

for $\lambda_3 - s_1 - s_2 < 0$, $s_1 + s_2 + \lambda_3 > 0$;  

(9.11)

$$p_{[3]}^{-} = V_{s_1+s_2-\lambda_3}(\mathbb{P}^R) + V_0,$$

for $s_1 + s_2 + \lambda_3 \leq 0$;  

(9.12)

$$p_{[3]}^{-} = V_0 + \tilde{V}_{s_1+s_2+\lambda_3}(\mathbb{P}^L),$$

for $\lambda_3 - s_1 - s_2 \geq 0$.  

(9.13)

In (9.12), $V_0 = 0$ for $s_1 + s_2 + \lambda_3 < 0$, while, in (9.13), $V_0 = 0$ for $\lambda_3 - s_1 - s_2 > 0$. Comparing (9.11) and (9.13), we see that, for $p_{[3]}^{-}$ (9.13), we have $V_N = 0$ and hence $V_N^{\circ} = 0$. Using (5.14), (5.17), we get then the relation

$$p_{[3]}^{-} = \tilde{V},$$

for $\lambda_3 - s_1 - s_2 \geq 0$.  

(9.14)

iv) From (5.11), we see that, to get the hermitian $P_{[3]}^{-}$, we need not only the vertex $\tilde{V}$ (9.2) but also the vertex $\mathcal{I} \tilde{V}$ which is associated with $\mathcal{I} p_{[3]}^{-}$. Realization of the operator $\mathcal{I}$ on the vertex $\tilde{V}$ is given in (D.7) in Appendix D. Using (9.2), (D.7), (H.17), we find the expression for $\mathcal{I} \tilde{V}$,

$$\mathcal{I} \tilde{V} = C_{n,\lambda_3}^{\ast} N_{n,-\lambda_3}^{\ast} \tilde{V}_{n,-\lambda_3},$$

(9.15)

which tells us that $\mathcal{I} \tilde{V}$ is associated with the vertex $\tilde{V}_{n,-\lambda_3}^{\ast}$. This motivates us to represent $\mathcal{I} \tilde{V}$ as

$$\mathcal{I} \tilde{V} = C_{n,-\lambda_3} N_{n,-\lambda_3} \tilde{V}_{n,-\lambda_3},$$

(9.16)

Comparison of (9.15) and (9.16) gives restriction (9.7).

v) Decay of the massive particle with the mass $m$ into one massive particle with the same mass $m$ and one massless particle is prohibited by the energy-momentum conservation laws. We skip therefore a discussion of the on-shell reduction of vertices (9.2).

10 Vertex $\tilde{V}$ for three massive fields with non-critical masses, $D \neq 0$

Using notation (6.16), we consider the meromorphic vertex for three massive fields (6.10), (6.11),

$$\left(m_1, s_1\right) - \left(m_2, s_2\right) - \left(m_3, s_3\right), \quad D \neq 0, \quad m_a \neq 0, \quad a = 1, 2, 3,$$

(10.1)

i.e. three massive fields carry external line indices $a = 1, 2, 3$. The general expression for the meromorphic vertex $\tilde{V}$ (5.22) takes the following form (for the derivation, see Appendix H):

$$\tilde{V} = C_{n_1,n_2,n_3} N_{n_1,n_2,n_3} \tilde{V}_{n_1,n_2,n_3}^{bas} + C_{-n_1,-n_2,-n_3} N_{-n_1,-n_2,-n_3} \tilde{V}_{-n_1,-n_2,-n_3}^{bas}, \quad D > 0,$$

(10.2)

\footnote{In Ref. [17], the authors use the relation $2s_{\text{min}} = s_1 + s_2 - |s_1 - s_2|$.}
\[ \tilde{V} = C_{n_1,n_2,n_3} N_{n_1,n_2,n_3} \sum_{n_1,n_2,n_3} \tilde{V}_{n_1,n_2,n_3}^{\text{bas}}, \quad \text{for } D < 0, \quad (10.3) \]

\[ \tilde{V}_{n_1,n_2,n_3}^{\text{bas}} = \prod_{a=1,2,3} L_{a_+}^{s_n+n_a} L_{a_-}^{s_n-n_a} \left( \frac{\mathbb{P}_L}{\beta_a} \right)^{-s_n}, \quad (10.4) \]

\[ L_{a\pm} = \frac{\mathbb{P}_L}{\beta_a} u_a + \frac{g_{a\pm}}{\sqrt{2}} v_a, \quad (10.5) \]

\[ g_{a\pm} = g_a \pm \frac{\gamma}{m_a}, \quad g_a = \frac{\beta_a}{\sqrt{2}} m_a + c_a m_a, \quad c_a = \frac{m_{a+1}^2 - m_{a+2}^2}{2m_a^2}, \quad (10.6) \]

\[ \gamma = \frac{1}{2} \sqrt{D} \quad \text{for } D > 0; \quad \gamma = \frac{i}{2} \sqrt{-D} \quad \text{for } D < 0, \quad (10.7) \]

\[ N_{n_1,n_2,n_3} = \prod_{a=1,2,3} 2^{s_a/2} \kappa_a^{(n_a+1-n_a+2)/6}, \quad (10.8) \]

where coupling constants \( C_{n_1,n_2,n_3} \) and integers \( n_1, n_2, n_3 \in \mathbb{Z} \) satisfy the restrictions

\[ C_{n_1,n_2,n_3}^* = C_{-n_1,-n_2,-n_3}, \quad \text{for } D > 0; \quad (10.9) \]

\[ C_{n_1,n_2,n_3}^* = C_{n_1,n_2,n_3}, \quad \text{for } D < 0; \quad (10.10) \]

\[ n_1 + n_2 + n_3 = 0, \quad n_1, n_2, n_3 \in \mathbb{Z}, \quad (10.11) \]

\[ -s_n \leq n_a \leq s_a, \quad a = 1, 2, 3. \quad (10.12) \]

The vertices \( \tilde{V}_{n_1,n_2,n_3}^{\text{bas}} \) (10.4) constitute a basis of all solutions for the vertex \( \tilde{V} \). In the full expressions for the vertices \( \tilde{V} \) (10.2) and (10.3), we inserted the normalization factors \( N_{n_1,n_2,n_3} \) (10.8) and the coupling constants \( C_{n_1,n_2,n_3} \). In general, the coupling constants depend not only on the integers \( n_1, n_2, n_3 \) but also on the spins \( s_a \) and the masses \( m_a, a = 1, 2, 3 \). For \( D > 0 \), only the integers \( n_1, n_2, n_3 \) are not inert under the complex conjugation of the coupling constants (10.9).

The following remarks are in order.

i) Restriction (10.11) is obtained by solving equation (5.26), while restrictions (10.12) are obtained by requiring that the powers of \( L_1, L_2, \) and \( L_3 \) in (10.4) be non-negative integers.

ii) The integers \( n_1, n_2, n_3 \) subjected to restrictions (10.11), (10.12) express the freedom of the solution for \( \tilde{V} \). These integers label all possible vertices \( \tilde{V} \) that can be built for the fields in (10.1).

Now using the restrictions (10.11), (10.12), we find the following number of cubic vertices:

\[ n = (s_1 + s_2 - |s_1 - s_2| + 1)(2s_3 + 1), \quad \text{for } s_3 \leq |s_1 - s_2|; \quad (10.13) \]

\[ n = (2s_1 + 1)(2s_2 + 1) - (s_1 + s_2 - s_3)(s_1 + s_2 - s_3 + 1), \quad \text{for } |s_1 - s_2| \leq s_3 \leq s_1 + s_2; \quad (10.14) \]

\[ n = (2s_1 + 1)(2s_2 + 1), \quad \text{for } s_1 + s_2 \leq s_3. \quad (10.15) \]

To our knowledge our result in (10.13)-(10.15) has not been discussed in the earlier literature.

iii) Explicit expressions for the harmonic vertices \( p_{\alpha} \) can be obtained by expanding the meromorphic vertices \( \tilde{V} \) (10.2), (10.3) in Laurent series in \( \mathbb{P}_L \) and using (5.14)-(5.17). As the explicit Taylor
series expansions for the harmonic vertices $p_{[3]}^- \in \mathbb{P}^R$ and $\mathbb{P}^L$ are not illuminating we briefly comment on the general structure of $p_{[3]}^-$. To this end we note that the Laurent series expansion of the meromorphic vertices $\tilde{\mathcal{V}}$ (10.2), (10.3) in $\mathbb{P}^L$ can be presented as

$$
\tilde{\mathcal{V}} = \sum_{n=-s_1-s_2-s_3}^{s_1+s_2+s_3} (\mathbb{P}^L)^n \tilde{\mathcal{V}}_n .
$$

(10.16)

Using (10.16), it is easy to see that, for all values $s_1$, $s_2$, $s_3$, a general form of the harmonic vertex $p_{[3]}^- (5.17)$ can be presented as

$$
p_{[3]}^- = V_{s_1+s_2+s_3} (\mathbb{P}^R) + V_0 + \tilde{V}_{s_1+s_2+s_3} (\mathbb{P}^L) ,
$$

(10.17)

where, in (10.17), the dependence of the vertices $V_{s_1+s_2+s_3}$, $V_0$, $	ilde{V}_{s_1+s_2+s_3}$ on the $\beta$-momenta and the oscillators is implicit.

**On-shell values of cubic vertex for $D > 0$.** 3-point amplitudes are expressed in terms of on-shell cubic vertices obtained from off-shell vertices (10.2) by using on-shell conditions (6.19). Consider the decay of the massive particle ($a = 1$) into the two massive particles ($a = 2, 3$),

$$
1 \rightarrow 2 + 3 , \quad m_1 > m_2 + m_3 ,
$$

(10.18)

where in (10.18), for the decay with non-collinear momenta, we recall the restriction on the masses.

On-shell value of the off-shell vertex (10.2) is found to be

$$
\tilde{\mathcal{V}}_{\text{on-sh}} = C_{n_1,n_2,n_3} \mathcal{V}_{n_1,n_2,n_3} ^{\text{bas}} + C_{-n_1,-n_2,-n_3} \mathcal{V}_{-n_1,-n_2,-n_3} ^{\text{bas}} ,
$$

(10.19)

$$
\mathcal{V}_{n_1,n_2,n_3} ^{\text{bas}} = (-)^{s_1} \prod_{a=1,2,3} \mathcal{L}_{a+}^{s_a+n_a} \mathcal{L}_{a-}^{s_a-n_a} ,
$$

(10.20)

$$
\mathcal{L}_{a+} = \varepsilon_a \sqrt{-g_{a+} e^{-\frac{1}{2} \varphi} u_a + \sqrt{g_{a+}} e^{\frac{1}{2} \varphi} v_a} ,
$$

$$
\mathcal{L}_{a-} = \varepsilon_a \sqrt{-g_{a-} e^{-\frac{1}{2} \varphi} u_a - \sqrt{-g_{a-}} e^{\frac{1}{2} \varphi} v_a} , \quad \varepsilon_a = \text{sign} \beta_a , \quad a = 1, 2, 3 ;
$$

(10.21)

$$
g_{1\pm} = \frac{Y_{3\pm}}{2m_1 (1 + r)} (r_\mp - r) , \quad g_{2\pm} = m_2 (r - r_\mp) , \quad g_{3\pm} = \frac{m_3}{r_\mp} (r - r_\mp) ,
$$

(10.22)

$$
r \equiv \beta_3 / \beta_2 , \quad r_\pm \equiv (m_1^2 - m_2^2 - m_3^2 \pm \sqrt{D}) / 2m_2 , \quad r_- < r < r_+ ,
$$

(10.23)

$$
Y_{3\pm} = m_3^2 - m_2^2 - m_1^2 \pm \sqrt{D} ,
$$

(10.24)

where an angle variable $\varphi$ appearing in (10.21) is related to $\mathbb{P}^L$ as

$$
\mathbb{P}^L = \frac{\rho}{\sqrt{2}} e^{-i\varphi} , \quad \rho = m_2 \beta_2 \sqrt{(r_+ - r)(r - r_-)} .
$$

(10.25)

Relations (10.25) are obtained from (6.19) and the definitions of $\rho^2$ and $r$ in (4.14), (10.23). The allowed values of $r$ in (10.23) are fixed by requiring $\rho^2 > 0$. We note the helpful inequalities,

$$
\beta_1 < 0 , \quad \beta_2 > 0 , \quad \beta_3 > 0 ,
$$

(10.26)

$$
g_{a+} > 0 , \quad g_{a-} < 0 , \quad a = 1, 2, 3 ; \quad Y_{3\pm} < 0 , \quad r_- > 0 ,
$$

(10.27)
where in (10.26), we show the allowed values of the $\beta$-momenta for the process in (10.18), while restrictions (10.27) follow from restriction (10.18) and the definitions in (10.22)-(10.24). For the reader convenience, we note also the following relations which are helpful for the derivation of the on-shell vertex (10.19) from the off-shell vertex (10.2),

\[
L_{a\pm}|_{\text{on-sh}} = \sqrt{\pm \frac{g_{a\pm}}{2}} e^{-\frac{1}{2}p_{a\pm}^2} L_{a\pm}, \quad \frac{\rho}{\beta_a} = \varepsilon_a \sqrt{-g_a+g_{a-}}, \quad a = 1, 2, 3.
\] (10.28)

### 11 Vertex $\bar{V}$ for three massive fields with critical masses, $D = 0$

Using notation (6.16), we finish with the meromorphic vertex for three massive fields (6.12),

\[
(m_1, s_1) - (m_2, s_2) - (m_3, s_3), \quad D = 0, \quad \mathbf{P}_{em} = 0, \quad m_a \neq 0, \quad a = 1, 2, 3.
\] (11.1)

i.e. three massive fields carry external line indices $a = 1, 2, 3$. For this particular case, the general expression for the vertex $\bar{V}$ (5.22) takes the following form (for the derivation, see Appendix H):

\[
\bar{V} = C_{n,l} N_{n,l} V_{n,l}^{\text{bas}},
\] (11.2)

\[
V_{n,l}^{\text{bas}} = Q_X Q_Y \prod_{a=1,2,3} L_a^{2s_a-n-l} \left( \frac{p_a^L}{\beta_a} \right)^{-s_a},
\] (11.3)

\[
L_a \equiv \frac{p_a^L}{\beta_a} v_a + \frac{\epsilon_a \epsilon_{em}}{\sqrt{2} \beta_a} v_a, \quad \mathbf{P}_{em} \equiv \frac{1}{3} \sum_{a=1,2,3} \beta_a \epsilon_a m_a, \quad \beta_a \equiv \beta_{a+1} - \beta_{a+2},
\] (11.4)

\[
Q_X \equiv \sum_{a=1,2,3} c_a^X v_a L_{a+1} L_{a+2}, \quad c_a^X = \epsilon_a,
\] (11.5)

\[
Q_Y \equiv \frac{1}{\sqrt{2}} L_1 L_2 L_3 + \sum_{a=1,2,3} c_a^Y v_a L_{a+1} L_{a+2}, \quad c_a^Y = \frac{1}{3} \epsilon_a (\epsilon_{a+1} m_{a+1} - \epsilon_{a+2} m_{a+2}),
\] (11.6)

\[
N_{n,l} \equiv i^{n+l},
\] (11.7)

where coupling constants $C_{n,l}$ and integers $n, l \in \mathbb{N}_0$ satisfy the restrictions

\[
C_{n,l}^* = C_{n,l}, \quad n \geq 0, \quad l \geq 0, \quad n + l \leq 2s_{\text{min}}, \quad s_{\text{min}} \equiv \min_{a=1,2,3} s_a.
\] (11.8)

The vertices $V_{n,l}^{\text{bas}}$ (11.3) constitute a basis of all solutions for the vertex $\bar{V}$. In the full expression for the vertex $\bar{V}$ (11.2), we inserted the normalization factors $N_{n,l}$ (11.7) and the coupling constants $C_{n,l}$. In general, the coupling constants depend not only on the integers $n$ and $l$ but also on the spins $s_a$ and the masses $m_a$, $a = 1, 2, 3$. We note that restrictions (11.9) are obtained by requiring that the powers of $Q_X$, $Q_Y$, $L_1$, $L_2$, and $L_3$ in (11.3) be non–negative integers. Two integers $n, l$ subjected to restrictions (11.9) express the freedom of the solution for $\bar{V}$. These integers label all possible vertices $\bar{V}$ that can be built for the fields in (11.1). Using the restrictions (11.9), we find the following number of cubic vertices:

\[
n = (2s_{\text{min}} + 1)(s_{\text{min}} + 1),
\] (11.10)
where \( s_{\text{min}} \) is defined in (11.9). To our knowledge our result in (11.10) has not been discussed in the earlier literature.

The Laurent series expansion of the meromorphic vertex \( \bar{V} \) (11.2) in \( \mathbb{P}^L \) and the corresponding Taylor series expansion of the harmonic vertex \( p_{[3]} \) in \( \mathbb{P}^R \) and \( \mathbb{P}^L \) can schematically be presented as

\[
\bar{V} = \sum_{n = -s_1 - s_2 - s_3}^{s_1 + s_2 + s_3} (\mathbb{P}^L)^n \bar{V}_n, \\
p_{[3]} = V_{s_1 + s_2 + s_3} (\mathbb{P}^R) + V_0 + \bar{V}_{s_1 + s_2 + s_3} (\mathbb{P}^L),
\]

(11.11)

where, in (11.11), the dependence of the vertices \( V_{s_1 + s_2 + s_3}, V_0 \), \( \bar{V}_{s_1 + s_2 + s_3} \) on the \( \beta \)-momenta and the oscillators is implicit.

For \( D = 0 \), the decay of one massive particle into two massive particles is allowed only for collinear momenta of the particles. The collinear momenta lead to the restrictions \( \mathbb{P}^R = 0, \mathbb{P}^L = 0 \). For such \( \mathbb{P}^R, \mathbb{P}^L \), the on-shell cubic vertices are governed by \( V_0 \) (11.11). We skip a discussion of such on-shell cubic vertices.

### 12 Conclusions

In this paper, we used the light-cone gauge helicity basis formalism for the investigation of cubic interactions of arbitrary integer spin massive and massless fields in \( R^{3,1} \). We studied cross-interactions between massive and massless fields and interactions between massive fields. Depending on masses of the fields, we introduced the classification of cubic vertices. As convenient representatives of cubic vertices we used the harmonic vertices. To each harmonic vertex we associated the meromorphic vertex. The harmonic vertex and the meromorphic vertex are in one-to-one correspondence and they are related to each other by simple transformation rule. We found simple explicit expressions for all meromorphic vertices. On-shell expressions for the cubic vertices and the meromorphic vertices coincide. We expect that the methods and the results obtained in this paper might have the following applications and generalizations.

**i)** In this paper, we restricted our study to the bosonic fields. The light-cone gauge helicity basis formalism is well adapted also for a study of fermionic fields. Application of our method for a study of Fermi-Bose couplings would be interesting. The Fermi-Bose couplings of the massive and massless light-cone gauge tensor (tensor-spinor) fields in \( R^{d-1,1} \), \( d > 4 \), were studied in Ref.[11]. The study of Fermi-Bose couplings of massless fields in \( R^{3,1} \) by using the light-cone gauge helicity basis formalism may be found in Ref.[13]. The investigation of electromagnetic and gravitational couplings of fermionic fields by using BRST approach may be found in Ref.[19][11].

**ii)** In Ref.[13], we used the bosonic vector-like oscillators to build all BRST-BV parity-even cubic vertices for interacting massive and massless fields in \( R^{d-1,1} \), \( d \geq 4 \). However our research in this paper convinced us that the bosonic spinor-like oscillators are more convenient for the study of interacting massive and massless fields in \( R^{3,1} \). Use of the bosonic spinor-like oscillators for the investigation of BRST-BV free fields in \( R^{3,1} \) may be found in Ref.[28]. We think that building of BRST-BV counterparts of our light-cone gauge cubic vertices of massive fields by using spinor-like

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11Recent discussion of the interesting formulation of free fermionic fields may be found in Ref.[20].

12BRST-BV cubic vertex for the massive gravity in AdS space was discussed in Ref.[21]. BRST-BV parity-even cubic vertices for arbitrary spin massless fields in flat space were discussed in Refs.[13, 22, 23]. In the earlier literature, BRST-BV cubic vertex for spin-3 massless fields was considered in Ref.[24]. For the interesting recent development, see Ref.[25]. For various metric-like formulations of cubic vertices for arbitrary spin massless fields, see Refs.[26], while, for the metric-like formulation of cubic vertices for spin-2 massive fields, see Ref.[27].
oscillators could be of interest. Also note that the bosonic spinor-like oscillators will provide us the possibility to treat BRST-BV parity-even and parity-odd vertices for fields in $R^{3,1}$ on an equal footing. In Ref.[13], we studied BRST-BV parity-even cubic vertices for arbitrary spin fields. To our knowledge, BRST-BV parity-odd vertices for arbitrary spin fields have not been studied in the literature. For fields of some particular values of spins (so called Curtright fields) in $R^{4,1}$ and $R^{6,1}$, the discussion of BRST-BV parity-odd and parity-even cubic vertices may be found in Ref.[29]. Light-cone gauge parity-odd and parity-even cubic vertices for arbitrary spin massless fields in $R^{4,1}$ and $R^{6,1}$ were considered in Refs.[30, 31]. Lorentz covariant parity-odd (and parity even) cubic vertices for arbitrary spin on-shell massless TT fields in $R^{3,1}$ were built in Ref.[32]. Parity-even and parity-odd cubic vertices of higher-spin massless fields in $R^{2,1}$ were considered in Refs.[33] (see also [34]).

iii) Light-cone gauge approach turns out to be convenient for the study of supersymmetric higher-spin theories. For arbitrary spin massless supermultiplets in $R^{3,1}$ and massive supermultiplets in $R^{2,1}$, the light-cone gauge approach provides us the possibility for the use of the unconstrained light-cone gauge superfields (see, e.g., Ref.[35]-[38]). It would be interesting to extend results and methods in this paper to the case of massive and massless supermultiplets in $R^{3,1}$. The discussion of various methods for building interaction vertices in supersymmetric theories may be found in Refs.[39]-[45].

iv) In this paper, we studied cubic vertices of massive and massless fields. Extension of our study to quartic vertices of massless and massive fields along the lines of the light-cone gauge methods in Refs.[40]-[48] is of great interest. Use of other various methods for the study of quartic vertices of higher-spin fields in flat and AdS spaces may be found in Refs.[49]-[55].

v) In this paper, we considered interacting light-cone gauge massive and massless fields in the flat space. Extension of our results and methods to the case of light-cone gauge fields in AdS space is very interesting problem. In this respect, we note that the light-cone gauge formulation of free arbitrary spin massive and massless fields in AdS space was developed in Refs.[56, 57], while, in Ref.[58], we considered interacting arbitrary spin massless fields in $AdS_4$. We expect that a generalization of the results in Ref.[58] to the case of interacting massive fields should be relatively straightforward. Use of frame-like approach for building interaction vertices for fields in AdS space may be found, e.g., in Refs.[61]-[65].

vi) As noted in Ref.[70], the ordinary-derivative formulation of conformal fields and the gauge invariant formulation of massive fields share certain common features. The light-cone gauge formulations of massive fields and conformal fields in Ref.[71] also share some common features. For this reason we expect that the method for the study of interacting massive fields we developed in this paper can be adopted for a study of interacting conformal fields. We note also the interesting proposal for building action of interacting conformal fields in Ref.[72]. Various interesting recent developments in the topic of conformal fields may be found, e.g., in Refs.[73]-[76].

vii) In the recent time, the interesting investigations for the use of the twistor method in Lagrangian formulation of interacting higher-spin massless fields were carried out in Refs.[77]-[80]. Application of the twistor method in Refs.[77]-[80] for the study of Lagrangian formulation of interacting massless and massive fields in four dimensions seems to be interesting avenue to go.

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13The use of the light-cone approach in AdS for the study of light-front bootstrap of Chern-Simons matter theories may be found in Ref.[59]. For the study of bilocal holography by using the light-cone approach in AdS, see Ref.[60].
14The frame-like approach for free massive fields in AdS was developed in Refs.[65, 67] (see also Ref.[68]). The use of the frame-like cubic vertices in AdS space for deriving the cubic vertices in flat space may be found in Ref.[69].
Appendix A  Notation and useful identities

Throughout this paper we use the following notation:

\[ D \equiv m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_2^2m_3^2 - 2m_3^2m_1^2, \]  
\[ \gamma \equiv \frac{1}{2}\sqrt{D}, \text{ for } D > 0; \quad \gamma \equiv \frac{1}{2}\sqrt{-D}, \text{ for } D < 0; \]  
\[ g_a \equiv \frac{\beta_a}{2\beta_a}m_a + c_am_a, \quad c_a \equiv \frac{m_{a+1}^2 - m_{a+2}^2}{2m_a^2}, \quad \beta_a \equiv \beta_{a+1} - \beta_{a+2}, \]  
\[ g_{a\pm} = g_a \pm \frac{\gamma}{m_a}, \] \[ \rho^2 \equiv \beta \sum_{a=1,2,3} \frac{m_a^2}{\beta_a}, \quad \beta \equiv \beta_1\beta_2\beta_3, \] \[ \mathbb{P}_{em} = \frac{1}{3} \sum_{a=1,2,3} \beta_a \epsilon_a m_a, \quad \mathbb{P}_{em} = \sum_{a=1,2,3} \epsilon_a m_a, \quad \epsilon_1^2 = 1, \quad \epsilon_2^2 = 1, \quad \epsilon_3^2 = 1, \] \[ Y_{a\pm} \equiv m_a^2 - m_{a+1}^2 - m_{a+2}^2 \pm 2\gamma, \quad \kappa_a \equiv \frac{Y_{a+}}{Y_{a-}}, \]  

where \( a = 1, 2, 3 \). Using the definitions in (A.3)-(A.6), we find the relations

\[ c_a = \frac{\epsilon_a}{2m_a} (\epsilon_{a+2}m_{a+2} - \epsilon_{a+1}m_{a+1}), \quad g_a = \frac{\epsilon_a}{\beta_a} \mathbb{P}_{em}, \quad \rho^2 = -\mathbb{P}_{em}^2, \quad \text{for } \mathbb{P}_{em} = 0. \]  

Using the definitions in (A.7), we find the relations,

\[ Y_{a+}Y_{a-} = 4m_{a+1}^2m_{a+2}^2, \quad Y_{a\pm} = \frac{1}{2m_a^2}Y_{a+1\mp}Y_{a+2\mp}, \quad a = 1, 2, 3; \]  
\[ Y_{1+}Y_{2+}Y_{3+} = 8m_1^2m_2^2m_3^2, \quad Y_{1+}Y_{2-}Y_{3-} = 8m_1^2m_2^2m_3^2, \]  
\[ \kappa_1\kappa_2\kappa_3 = 1, \quad \kappa_a = \frac{Y_{a+}Y_{a+}}{4m_{a+1}^2m_{a+2}^2}, \quad a = 1, 2, 3. \]  

For \( D < 0 \), the quantities \( \kappa_a \) and \( Y_{a\pm} \) are complex-valued. The relations for \( \kappa_a \) in (A.11) imply

\[ \kappa_a > 0 \quad \text{for } D > 0, \quad a = 1, 2, 3; \]  

while the definitions in (A.2), (A.7) imply the following relations:

\[ \kappa_a^* = \kappa_a, \quad Y_{a\pm}^* = Y_{a\pm}, \quad \text{for } D > 0; \]  
\[ \kappa_a^* = \kappa_a^{-1}, \quad Y_{a\pm}^* = Y_{a\mp}, \quad \text{for } D < 0. \]  

Using definitions in (A.3)-(A.5), we get the relations

\[ g_{a+}g_{a-} = -\frac{\rho^2}{\beta_a^2}, \quad \frac{g_{a+}g_{a+1} - g_{a-}g_{a+1}}{g_{a+}g_{a+1} - g_{a-}g_{a+1}} = \kappa_{a+1}^{-1}, \quad a = 1, 2, 3, \]  

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where
\( \phi \)
The scalar product for the generating functions
\[ \prod_{a=1,2,3} \frac{g_a^+}{g_a^-} = \prod_{a=1,2,3} \kappa_a^{-(n_{a+1} - n_{a+2})/3}, \quad n_1 + n_2 + n_3 = 0, \quad (A.16) \]
\[ \prod_{a=1,2,3} g_a^{s_a+n_a} g_a^{-s_a-n_a} = \prod_{a=1,2,3} \left( -\frac{\rho}{\mu} \right)^{s_a} \kappa_a^{-(n_{a+1} - n_{a+2})/3}, \quad n_1 + n_2 + n_3 = 0. \quad (A.17) \]

**Appendix B** Various realizations of operators \( M^R, M^L, M^{RL} \) and helpful formulas for operators \( E_m, E_\beta \) (5.23)

**u, v-realization.** Sometimes, in place of \( u, v \)-oscillators (2.10), we find it convenient to use \( c \)-number complex-valued variables \( u, v \). In terms of such variables, the generating function of massive field is defined as
\[ \phi_{m,s}(x^+, p, u, v) = \sum_{n=-s}^{s} \frac{u^{s+n} v^{s-n}}{\sqrt{(s+n)! (s-n)!}} \phi_{m,s;n}(x^+, p). \quad (B.1) \]

On space of generating function (B.1), the spin operators are realized as follows
\[ M^R = \frac{m}{\sqrt{2}} u \partial_v, \quad M^L = -\frac{m}{\sqrt{2}} v \partial_u, \quad M^{RL} = \frac{1}{2} (u \partial_u - v \partial_v), \quad (B.2) \]
where \( \partial_u \equiv \partial / \partial u, \partial_v \equiv \partial / \partial v. \) Using the shortcut \( \phi(u, v) \) for the generating function (B.1), we note that the scalar product for the generating functions \( \phi(u, v) \) and \( \varphi(u, v) \) takes the form
\[ (\phi, \varphi) \equiv \int d^2 u d^2 v e^{-u \bar{u} - v \bar{v}} (\phi(u, v))^\dagger \varphi(u, v), \quad \int d^2 u e^{-u \bar{u}} \equiv 1, \quad (B.3) \]
where \( \phi(u, v) \) and \( \varphi(u, v) \) are degree-2s homogeneous polynomials in \( u, v, \)
\[ (u \partial_u + v \partial_v) \phi(u, v) = 2s \phi(u, v), \quad (u \partial_u + v \partial_v) \varphi(u, v) = 2s \varphi(u, v). \quad (B.4) \]

**Projective realization.** Introducing a projective variable \( \alpha \) and a generating function \( \phi_{prj}(\alpha) \),
\[ \alpha = \frac{v}{u}, \quad \phi(u, v) = u^{2s} \phi_{prj}(\alpha), \quad (B.5) \]
we find that on space of \( \phi_{prj}(z) \) the spin operators (B.2) are realized as
\[ M^R = \frac{m}{\sqrt{2}} \alpha \partial_{\alpha}, \quad M^L = \frac{m}{\sqrt{2}} (\alpha^2 \partial_{\alpha} - 2s \alpha), \quad M^{RL} = s - \alpha \partial_{\alpha}, \quad \partial_{\alpha} \equiv \partial / \partial \alpha. \quad (B.6) \]
The scalar product for the generating functions \( \phi_{prj}(\alpha) \) and \( \varphi_{prj}(\alpha) \) takes the form
\[ (\phi_{prj}, \varphi_{prj}) = \int d\sigma_s(\alpha, \bar{\alpha}) (\phi_{prj}(\alpha))^\dagger \varphi_{prj}(\alpha), \quad d\sigma_s(\alpha, \bar{\alpha}) \equiv \frac{(2s + 1)!}{(1 + \alpha \bar{\alpha})^{2s+2}} d^2 \alpha. \quad (B.7) \]
Scalar products (B.3) and (B.7) are related as \( (\phi, \varphi) = (\phi_{prj}; \varphi_{prj}) \). Relations (B.5) imply that the \( u, v \)-realization and the projective realization of vertices are related as
\[ V(u_1, v_1; u_2, v_2; u_3, v_3) = \prod_{a=1,2,3} u^{2s_a} V_{prj}(z_1; z_2; z_3). \quad (B.8) \]
Action of operators $E_m, E_\beta$. Using $M^L, M^{RL}$ given in (B.6), we find the relations

$$e^{tM^L} \alpha e^{-tM^L} = \frac{\alpha}{1 - \frac{tM^L}{\sqrt{2}}} \alpha, \quad tM^{RL} \alpha^{-M^{RL}} = \frac{1}{t} \alpha,$$  \hspace{1cm} (B.9)

$$e^{tM^L} |0\rangle = (1 - \frac{tM^L}{\sqrt{2}})^{2s} |0\rangle, \quad tM^{RL} |0\rangle = t^s |0\rangle,$$  \hspace{1cm} (B.10)

where $|0\rangle \equiv 1$. Using definitions in (5.23), (A.3) and relations (B.9), (B.10), we get the following relations for massive fields:

$$E_{\beta a} \alpha E^{-1}_{\beta a} = \frac{\beta_a}{\beta a} \alpha; \quad E_{\alpha a} \alpha E^{-1}_{\alpha a} = \frac{\Pi L \alpha a}{\beta a L \alpha a},$$ \hspace{1cm} (B.11)

$$E_{\alpha a} E_{\beta a} \alpha E^{-1}_{\beta a} E^{-1}_{\alpha a} = \frac{\alpha a}{\alpha a} + \frac{g a}{\sqrt{2}} \alpha a,$$ \hspace{1cm} (B.12)

$$E_{\alpha a} |0\rangle = L^{2s} \alpha a \left( \frac{\Pi L}{\beta a} \right)^{-s} |0\rangle, \quad E_{\beta a} |0\rangle = \left( \frac{\Pi L}{\beta a} \right)^{s} a.$$ \hspace{1cm} (B.13)

In turn, for arbitrary function $F(\alpha a)$, relations (B.12), (B.13), (5.24) lead to the relations

$$E_{\alpha a} E_{\beta a} F(\alpha a) |0\rangle = F \left( \frac{\alpha a}{\alpha a} \right) L^{2s} \alpha a \left( \frac{\Pi L}{\beta a} \right)^{-s} |0\rangle,$$ \hspace{1cm} (B.14)

$$E_{\alpha a} E_{\beta a} |0\rangle = \left( \frac{\Pi L}{\beta a} \right)^{s} a |0\rangle,$$ \hspace{1cm} (B.15)

### Appendix C  Incorporation of internal $o(N)$ symmetry

For example, consider an internal $o(N)$ symmetry. Incorporation of the internal $o(N)$ symmetry into our treatment of massive and massless fields can be realized in the following four steps.\(^{15}\)

**Step 1.** Let $a, b, c$ be the matrix indices of the $o(N)$ algebra, $a, b, c = 1, \ldots, N$. In place of the singlet massless field $\phi_\lambda$, we introduce the colored massless fields $\phi_{ab}^\lambda$, while, in place of the singlet massive fields $\phi_{m,s,n}$, we introduce the colored massive fields $\phi_{m,s,n}^a$. By definition, these colored fields obey the relations

$$\phi_{ab}^\lambda = (-)^a \phi_{ba}^\lambda; \quad \phi_{m,s,n}^a = (-)^s \phi_{m,s,n}^a,$$  \hspace{1cm} (C.1)

$$(\phi_{ab}^\lambda(x^+, p))^\dagger = \phi_{-ab}^\lambda(x^+, -p), \quad (\phi_{m,s,n}^a(x^+, p))^\dagger = \phi_{m,s,-n}^a(x^+, -p).$$ \hspace{1cm} (C.2)

**Step 2.** In the scalar products entering actions of free fields (2.31), we make the replacements

$$\phi_s^\dagger \phi_s \rightarrow \phi_s^\dagger \phi_s^b, \quad \langle \phi_{m,s} | \phi_{m,s} \rangle \rightarrow \langle \phi_{m,s}^b | \phi_{m,s}^b \rangle,$$  \hspace{1cm} (C.3)

while, in the cubic vertices, the usual products should be replaced by the traced products,

$$\prod_{a=1,2,3} \phi_{ab}^\lambda \rightarrow \phi_{ab}^\lambda \phi_{ab}^\lambda \phi_{ab}^\lambda; \quad \phi_{m,s,1} | \phi_{m,s,2} | \phi_{m,s,3}^\dagger \rightarrow \phi_{m,s,1} | \phi_{m,s,2}^b | \phi_{m,s,3}^c.$$ \hspace{1cm} (C.4)

\(^{15}\)Discussions of the Chan-Paton gauging for the $U(N)$ and $Usp(N)$ symmetries may be found in Refs. [81].
Step 4. To derive hermitian conjugation rule for vertices, we find it convenient to use the c-number complex-valued variables $u$ and $v$ and the generating function given in (B.1). In terms of generating function (B.1), the hermiticity condition for the massive fields $\phi_{m,s,n}(x^+,p)$ in (2.14) takes the following form:

$$\langle \phi_{m,s,n}(x^+,p,u,v) \rangle = \phi_{m,s,n}(x^+,p,u,v), \quad u^* = \bar{u}, \quad v^* = \bar{v}. \quad (D.1)$$

For the case of three massive fields entering the cubic vertex, the generator $P_{[3]}$ (3.2) can be represented in terms of $\phi_{m,s}$ (B.1) in the following way:

$$P_{[3]}^- = \int d\Gamma_{[3]} |u,v\rangle_{[3]}^* \Phi_{[3]}^\dagger P_{[3]}^- (\mathbb{P}^R, \mathbb{P}^L, \beta_a, u_a, v_a), \quad (D.2)$$

$$\Phi_{[3]}^\dagger \equiv \prod_{a=1,2,3} (\phi_{m_a,s_a}(x^+,p_a,u_a,v_a))^\dagger, \quad d\Gamma_{[3]} = \prod_{a=1,2,3} e^{-u_a\bar{u}_a-v_a\bar{v}_a} d^2 u_a d^2 v_a. \quad (D.3)$$

Using (D.1), (D.2), it is easy to verify that the hermitian conjugation of $P_{[3]}^-$ can be presented as

$$P_{[3]}^- = \int d\Gamma_{[3]} |u,v\rangle_{[3]} \mathcal{I} P_{[3]}^- (\mathbb{I} \mathbb{P}^R, \mathbb{I} \mathbb{P}^L, \beta_a, u_a, v_a), \quad (D.4)$$

where we introduce

**Realization of operator $\mathcal{I}$ on cubic vertex $p_{[3]}^-$**:

$$\mathcal{I} p_{[3]}^- (\mathbb{I} \mathbb{P}^R, \mathbb{I} \mathbb{P}^L, \beta_a, u_a, v_a) = p_{[3]}^{*-\dagger} (\mathcal{I} \mathbb{P}^R, \mathcal{I} \mathbb{P}^L, \mathcal{I} \beta_a, \mathcal{I} u_a, \mathcal{I} v_a),$$

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\[
\mathcal{I} \mathbb{P}^R \equiv \mathbb{P}^L, \quad \mathcal{I} \mathbb{P}^L \equiv \mathbb{P}^R, \quad \mathcal{I} \beta_a \equiv -\beta_a, \quad \mathcal{I} u_a \equiv v_a, \quad \mathcal{I} v_a \equiv u_a. \tag{D.5}
\]

In terms of \( p_{[3]}^- \), the hermicity condition \( P_{[3]}^- = P_{[3]}^{+-} \) is represented as

\[
p_{[3]}^- = \mathcal{I} p_{[3]}^-, \quad \text{or explicitly as} \quad p_{[3]}^- (\mathbb{P}^R, \mathbb{P}^L, \beta_a, u_a, v_a) = p_{[3]}^-(\mathbb{P}^L, \mathbb{P}^R, -\beta_a, v_a, u_a). \tag{D.6}
\]

Using formulas in (5.11)-(5.17), we find

\[
\text{Derivation of (5.2) and (5.9)}
\]

Realization of operator \( \mathcal{I} \) on meromorphic vertex \( \bar{\mathcal{V}} \):

\[
\mathcal{I} \bar{\mathcal{V}}(\mathbb{P}^L, \beta_a, u_a, v_a) = \bar{\mathcal{V}}^*(\mathcal{I} \mathbb{P}^L, \mathcal{I} \beta_a, \mathcal{I} u_a, \mathcal{I} v_a),
\]

\[
\mathcal{I} \mathbb{P}^L \equiv \frac{\rho^2}{2\mathbb{P}^L}, \quad \mathcal{I} \beta_a \equiv -\beta_a, \quad \mathcal{I} u_a \equiv v_a, \quad \mathcal{I} v_a \equiv u_a, \tag{D.7}
\]

where \( \rho^2 \) is given in (A.5). In terms of \( \bar{\mathcal{V}} \), the hermicity condition (D.6) is realized as

\[
\bar{\mathcal{V}} = \mathcal{I} \mathcal{V}, \quad \text{or explicitly as} \quad \bar{\mathcal{V}}(\mathbb{P}^L, \beta_a, u_a, v_a) = \mathcal{V}^*(\frac{\rho^2}{2\mathbb{P}^L}, -\beta_a, v_a, u_a). \tag{D.8}
\]

For the reader convenience, we present the transformations of \( L_{a\pm} \) (10.5) under the action of the operator \( \mathcal{I} \) given in (D.7),

\[
\mathcal{I} L_{a\pm} = \frac{\beta_a}{\sqrt{2\mathbb{P}^L}} g_{a\pm} L_{a\pm}, \quad \text{for } D > 0; \quad \mathcal{I} L_{a\pm} = \frac{\beta_a}{\sqrt{2\mathbb{P}^L}} g_{a\mp} L_{a\pm}, \quad \text{for } D < 0. \tag{D.9}
\]

### Appendix E Derivation of equations for \( p_{[3]}^- \), \( \mathcal{V} \), \( \mathcal{V}^{(2)} \) in (5.2), (5.18), (5.19), (5.26) and representation for \( j_{[3]}^{-R} \) (5.9)

**Derivation of (5.2) and (5.9).** We represent equation (4.20) as

\[
J_{th}^{-R} p_{[3]}^- + \frac{\mathbb{P}^R \mathbb{P}^L}{\beta} \left( - N_\beta + \mathbb{M}^{RL} \right) \frac{1}{N_\mathbb{P} + 1} \partial_{\mathbb{P}^L} p_{[3]}^- = \left( - \frac{\mathbb{P}^R \mathbb{P}^L}{\beta} + \sum_{a=1,2,3} \frac{m_a^2}{\beta_a^2} \right) j_{[3]}^{-R}, \tag{E.1}
\]

where \( J_{th}^{-R} \) is given in (5.4). For the harmonic function \( p_{[3]}^- \), the \( J_{th}^{-R} p_{[3]}^- \) is also the harmonic function. Taking this into account and considering the non-harmonic \( (\mathbb{P}^R \mathbb{P}^L)^n \)-terms in (E.1), we get the representation for \( j_{[3]}^{-R} \) (5.9). Plugging \( j_{[3]}^{-R} \) (5.9) into (E.1), we get the equation for \( p_{[3]}^- \) (5.2). Derivation of the equation for \( p_{[3]}^- \) (5.3) and the representation for \( j_{[3]}^{-L} \) (5.10) is in complete analogy with the derivation of (5.2) and (5.9) above presented.

**Derivation of equations for vertex \( \mathcal{V} \) (5.18), (5.19).** We split our derivation in the three steps.

**Step 1.** Plugging (5.11) into (5.2), (5.3), we find that, in terms of the vertices \( V_N, V_0, \bar{V}_N, \) equations (5.2), (5.3) can be represented as

\[
\left( \frac{\mathbb{P}^R}{\beta} \left( - N_\beta + \mathbb{M}^{RL} \right) - \sum_{a=1,2,3} \frac{1}{\beta_a} M_a^R \right) (V_N + V_0)
\]

\[
+ \sum_{a=1,2,3} \left( \frac{\beta_a}{6\beta_a} m_a^2 \partial_{\mathbb{P}^L} + \frac{m_a^2}{2\beta_a^2 \mathbb{P}^L} \left( - N_\beta + \mathbb{M}^{RL} - \frac{1}{\beta_a} M_a^R \right) \right) \bar{V}_N = 0, \tag{E.2}
\]

\[
\left( \frac{\mathbb{P}^L}{\beta} \left( - N_\beta - \mathbb{M}^{RL} \right) - \sum_{a=1,2,3} \frac{1}{\beta_a} M_a^L \right) (\bar{V}_N + V_0).
\]

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\[
+ \sum_{a=1,2,3} \left( \frac{\beta_a}{6 \beta_a} m_a^2 \partial_{PR} + \frac{m_a^2}{2 \beta_a} \beta a \right) V_N = 0. \tag{E.3}
\]

**Step 2.** Equations (E.2), (E.3) amount to the following equations:

\[
\left( \frac{\partial}{\beta} (N_\beta - M_{RL}) + \sum_{a=1,2,3} \frac{1}{\beta a} M_a \right) V_N + \frac{\partial}{\beta} (N_\beta - M_{RL}) V_0 = 0, \tag{E.4}
\]

\[
\sum_{a=1,2,3} \left( \frac{m_a^2}{2 \beta a} (N_\beta - M_{RL}) - \frac{\beta a}{6 \beta a} m_a N_P L + \frac{\beta a}{\beta a} M_a^R \right) V_N + \sum_{a=1,2,3} \frac{\beta a}{\beta a} M_a V_0 = 0, \tag{E.5}
\]

\[
\frac{\partial}{\beta} (N_\beta + M_{RL}) + \sum_{a=1,2,3} \frac{1}{\beta a} M_a \right) V_N + \frac{\partial}{\beta} (N_\beta + M_{RL}) V_0 = 0, \tag{E.6}
\]

\[
\sum_{a=1,2,3} \left( \frac{m_a^2}{2 \beta a} (N_\beta + M_{RL}) - \frac{\beta a}{6 \beta a} m_a N_P R + \frac{\beta a}{\beta a} M_a^R \right) V_N + \sum_{a=1,2,3} \frac{\beta a}{\beta a} M_a V_0 = 0. \tag{E.7}
\]

Equations (E.4)-(E.7) are obtained from equations (E.2), (E.3) in the following way. Considering terms of the powers \((\partial/\beta)^n, n > 0\), we see that equation (E.2) amounts to equation (E.4), while considering terms of the powers \((\partial a)^n, n \geq 0\), we see that equation (E.2) amounts to equation (E.5). Considering terms of the powers \((\partial a)^n, n > 0\), we see that equation (E.3) amounts to equation (E.6), while considering terms of the powers \((\partial a)^n, n \geq 0\), we see that equation (E.3) amounts to equation (E.7).

**Step 3.** We note that, in terms of \(V_N^\otimes\) defined in (5.15), equation (E.4) can be represented as the equation given below in (E.8), while equation (E.7) can be represented as the equation given below in (E.9).

\[
\sum_{a=1,2,3} \left( \frac{m_a^2}{2 \beta a} (N_\beta - M_{RL}) - \frac{\beta a}{6 \beta a} m_a N_P L + \frac{\beta a}{\beta a} M_a \right) V_N^\otimes + \frac{m_a^2}{2 \beta a} (N_\beta - M_{RL}) V_0 = 0, \tag{E.8}
\]

\[
\frac{\partial}{\beta} \sum_{a=1,2,3} (N_\beta + M_{RL}) V_N^\otimes + \sum_{a=1,2,3} \frac{1}{\beta a} M_a (V_N^\otimes + V_0) = 0. \tag{E.9}
\]

Finally, we note that, by combining equations (E.5) and (E.8), we get equation (5.18), while, by combining equations (E.6) and (E.9), we get equation (5.19).

**Derivation of (5.26).** What is required is to prove that the vertex \(\bar{V}^{(2)}\) is independent of \(\beta_1, \beta_2, \beta_3,\) and \(\partial a,\) and satisfies equation (5.26). We do this in the following two steps.

**Step 1.** Plugging (5.22) into equation (5.19) and using the relation

\[
(N_\beta - \frac{1}{3} \beta a) f_a = - \frac{\beta}{\beta a}; \tag{E.10}
\]

we find that equation (5.19) leads to the equation

\[
N_\beta \bar{V}^{(2)} = 0. \tag{E.11}
\]

Plugging \(\bar{V}\) (5.22) into equations (5.20), (5.21), we find that equations (5.20) and (5.21) lead to the following respective equations for \(\bar{V}^{(2)}:\)

\[
\sum_{a=1,2,3} \beta a \partial a \bar{V}^{(2)} = 0, \quad N_P L \bar{V}^{(2)} = 0. \tag{E.12}
\]
Equation (E.11) and the 1st equation in (E.12) imply that the vertex \( \bar{V}^{(2)} \) does not depend on \( \beta_1, \beta_2, \beta_3 \), while from the 2nd equation (E.12), we learn that the vertex \( \bar{V}^{(2)} \) is independent of \( \mathbb{P}^L \). In other words, as it is stated in (5.25), the vertex \( \bar{V}^{(2)} \) depends only on the oscillators \( u_a, v_a \).

**Step 2.** Plugging (5.22) into equation (5.18) and using the fact that the vertex \( \bar{V}^{(2)} \) is independent of \( \beta_1, \beta_2, \beta_3 \), and \( \mathbb{P}^L \), we find that equation (5.18) leads to equation (5.26).

### Appendix F  Invariant amplitudes in light-cone frame

First, we explain our notation and conventions for the \( S \)-matrix and amplitudes in the light-cone frame. Second, we explain how our cubic vertex is related to 3-point invariant amplitude.

For simplicity of the presentation, we consider massless fields. To discuss amplitudes we use fields in the Dirac (interaction) picture,

\[
\phi_\lambda(x^+, p) = e^{ix^+p^-} \phi_\lambda(p), \quad p^- = -\frac{p^\mu p^\mu}{\beta},
\]

where the \( \phi_\lambda(p) \) is expressed in terms of annihilation \( \bar{a}_\lambda(p) \) and creation \( a_\lambda(p) \) operators as

\[
\phi_\lambda(p) = \frac{\theta(\beta)}{\sqrt{2\beta}} \bar{a}_\lambda(p) + \frac{\theta(-\beta)}{\sqrt{-2\beta}} a_{-\lambda}(-p),
\]

\[
\theta(\beta) = 1 \text{ for } \beta > 0, \quad \theta(\beta) = 0 \text{ for } \beta < 0,
\]

\[
[a_{-\lambda}(p), a_{\lambda}(p')] = \delta_{\lambda,\lambda'}(p - p'),
\]

\[
\bar{a}_\lambda(p) |0\rangle = 0, \quad (\bar{a}_\lambda(p))^\dagger = a_\lambda(p).
\]

The matrix elements of the \( S \)-matrix are defined as

\[
S_{fi} \equiv \langle f | i \rangle, \quad |i\rangle = \prod_{a_i} a_{-\lambda_{a_i}}(-p_{a_i}) |0\rangle, \quad \langle f | = \langle 0 | \prod_{a_f} \bar{a}_{\lambda_{a_f}}(p_{a_f}),
\]

where \( |i\rangle \) and \( \langle f | \) stand for the respective in- and out-states, while the indices \( a_i \) and \( a_f \) label external lines of in-coming and out-going particles respectively. In terms of an invariant amplitude denoted as \( A_{fi} \), the matrix elements \( S_{fi} \) (F.6) can be presented as

\[
S_{fi} = -i(2\pi)^4 N_{fi}^{-1} \delta^{(4)} \left( \sum_{a_i} p_{a_i} + \sum_{a_f} p_{a_f} \right) A_{fi},
\]

\[
N_{fi} \equiv \prod_{a_f} N_{a_f} \prod_{a_i} N_{a_i}, \quad N_{a_i} = (2\pi)^{3/2} \sqrt{-2\beta_{a_i}}, \quad N_{a_f} = (2\pi)^{3/2} \sqrt{2\beta_{a_f}},
\]

where, for in-coming particles, \( \beta_{a_i} < 0 \), while, for out-going particles, \( \beta_{a_f} > 0 \). To cubic approximation, the \( S \)-matrix and the corresponding 3-point invariant amplitude for different particles are given by

\[
S = 1 + i \int dx^+ P^-_{[3]},
\]

\[
A_{fi} = -\left( P^-_{[3]}(\lambda_{a_i}, \lambda_{a_f}, p_{a_i}, p_{a_f}) + \mathcal{I} P^+_{[3]}(-\lambda_{a_i}, -\lambda_{a_f}, p_{a_i}, p_{a_f}) \right)^{\text{on-sh}},
\]

\[
\text{on-sh}
\]

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where the operator $\mathcal{I}$ is defined in (3.17). Formula (F.10) explains how our cubic vertex $p^{-}_{[3]}$ is related to the 3-point invariant amplitude. For the reader convenience, we present $p^{-}_{[3]}$ for the case when all three fields in the vertex are massless,

$$ p^{-}_{[3]} = C_{\lambda_1, \lambda_2, \lambda_3} \beta^{-\lambda_1} \beta^{-\lambda_2} \beta^{-\lambda_3} \prod^{\lambda_1 + \lambda_2 + \lambda_3}, \quad \lambda_1 + \lambda_2 + \lambda_3 > 0, \tag{F.11} $$

where $C_{\lambda_1, \lambda_2, \lambda_3}$ are coupling constants. Cubic vertex $p^{-}_{[3]}$ (F.11) was obtained in Ref.[9].

**Internal o(N) symmetry and massless fields.** Generalization of above given formulas is straightforward. In place of relations (F.1)-(F.5), we use their obvious generalization given by

$$ \phi^{ab}_\lambda(x^+, p) = e^{ix^+p^−} \phi^{ab}_\lambda(p), \quad p^− = -\frac{p^a p^b}{\beta}, \tag{F.12} $$

$$ \phi^{ab}_\lambda(p) = \sum_A \frac{\theta(\beta)}{\sqrt{2\beta}} t^{ab}_A a_{\lambda A}(p) + \frac{\theta(-\beta)}{\sqrt{-2\beta}} t^{ab}_A a_{-\lambda A}(-p), \tag{F.13} $$

$$ [\bar{a}_{\lambda A}(p), a_{\lambda', A'}(p')] = \delta_{\lambda, \lambda'} \delta_{AA'} \delta^{(3)}(p - p'), \tag{F.14} $$

$$ \bar{a}_{\lambda A}(p)|0\rangle = 0, \quad (\bar{a}_{\lambda A}(p))^\dagger = a_{\lambda A}(p), \tag{F.15} $$

$$ A, B = 1, 2, \ldots, N\lambda, \quad N\lambda \equiv \frac{1}{2} N(N + (-)^\lambda), \tag{F.16} $$

where real-valued quantities $t^{ab}_A$ (F.13) satisfy the following relations:

$$ t^{ab}_A = (-)^{\lambda_A} t^{ba}_A, \quad \sum_A t^{ab}_A t^{a'b'}_A = \frac{1}{2} \left( \delta^{a'a'} \delta^{b'b'} + (-)^\lambda \delta^{ab} \delta^{ba'} \right), \quad \sum_{a,b} t^{ab}_A t^{ba}_B = \delta_{AB}. \tag{F.17} $$

For the case of different colored massless particles, the 3-point invariant amplitude and the respective states of in-coming and out-going particles are given by

$$ A_{fi} = -\text{tr}(t_{A_1} t_{A_2} t_{A_3}) \left( p^{-}_{[3]}(\lambda_{ai}, \lambda_{af}, p_{ai}, p_{af}) + \mathcal{I} p^{-}_{[3]}(-\lambda_{ai}, -\lambda_{af}, p_{ai}, p_{af}) \right)|^\text{on-sh}, \tag{F.18} $$

$$ |i\rangle = \prod_{ai} \bar{a}_{-\lambda_{ai} A_{ai}}(-p_{ai})|0\rangle, \quad \langle f| = \langle 0| \prod_{af} \bar{a}_{\lambda_{af} A_{af}}(p_{af}), \tag{F.19} $$

where the operator $\mathcal{I}$ is defined in (3.17).

**Internal o(N) symmetry and massive fields.** Generalization of the above given formulas to the massive fields that respect the internal o(N) symmetry is straightforward. In place of relations (F.1)-(F.5), we use their generalization given by

$$ \phi^{ab}_{m,s;n}(x^+, p) = e^{ix^+p^−} \phi^{ab}_{m,s;n}(p), \quad p^− = -\frac{2p^a p^b + m^2}{2\beta}, \tag{F.20} $$

$$ \phi^{ab}_{m,s;n}(p) = \sum_A \frac{\theta(\beta)}{\sqrt{2\beta}} t^{ab}_A a_{m,s,n A}(p) + \frac{\theta(-\beta)}{\sqrt{-2\beta}} t^{ab}_A a_{m,s,n A}(-p), \tag{F.21} $$

$$ [\bar{a}_{m,s,n A}(p), a_{m', s', n' A'}(p')] = \delta_{s,s'} \delta_{n,n'} \delta_{AA'} \delta^{(3)}(p - p'), \tag{F.22} $$

$$ \bar{a}_{m,s,n A}(p)|0\rangle = 0, \quad (\bar{a}_{m,s,n A}(p))^\dagger = a_{m,s,n A}(p), \tag{F.23} $$

where $A, B$ take values as in (F.16) with the replacement $\lambda \rightarrow s$. The $t^{ab}_A$ appearing in (F.21) satisfy the relations as in (F.17) with the replacement $\lambda \rightarrow s$. 

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Appendix G  Comments on classification (6.13)-(6.15)

Classification (6.13)-(6.15) via instant form of relativistic dynamics. Consider the decay of the massive particle \((a = 1)\) into the two massive particles \((a = 2, 3)\). In the rest frame of the decaying particle, we can use the following expressions for the particles momenta:

\[
p_1^\mu = (m_1, 0), \quad p_2^\mu = (E_2, \mathbf{p}), \quad p_3^\mu = (E_3, -\mathbf{p}), \quad E_a = \sqrt{m_a^2 + \mathbf{p}^2}, \quad a = 1, 2.
\]  

Using relations (G.1) and the energy conservation law \(m_1 = E_2 + E_3\), we get the well known restrictions on the masses and a square of the momentum \(\rho\):

\[
m_1 > m_2 + m_3, \quad \text{for } \mathbf{p} \neq 0; \quad (G.2)
\]

\[
m_1 = m_2 + m_3, \quad \text{for } \mathbf{p} = 0; \quad (G.3)
\]

\[
\rho^2 = \frac{D}{4m_1^2}. \quad (G.4)
\]

It is the relation (G.4) that, among other things, motivates us to use the quantity \(D\) for the classification in (6.13)-(6.15). Namely, the use of relation (G.4) allows us to split all processes into three groups shown in (6.13)-(6.15). We recall also that, for \(m_1 = m_2 + m_3\), i.e., \(D = 0\), the particles momenta turn out to be collinear [17]

\[
p_1^\mu = \frac{m_1}{m_2} p_2^\mu, \quad p_3^\mu = \frac{m_3}{m_2} p_2^\mu, \quad \text{for } m_1 = m_2 + m_3. \quad (G.5)
\]

Classification (6.13)-(6.15) via light-cone form of relativistic dynamics. The \(\rho^2 (4.14)\) can be represented as

\[
\rho^2 = m_2^2 \beta_2^2 (r_+ - r)(r - r_-) , \quad (G.6)
\]

\[
r \equiv \beta_3 / \beta_2 , \quad r_\pm \equiv (m_1^2 - m_2^2 - m_3^2 \pm \sqrt{D})/2m_2^2, \quad (G.7)
\]

\[
D = (m_1^2 - m_2^2 - m_3^2)^2 - 4m_2^2m_3^2, \quad (G.8)
\]

where in (G.8), we represent \(D\) defined in (6.1). It is the expression for \(\rho^2\) in (G.6) that motivates us to use \(D\) for the classification (6.13)-(6.15). Namely, from (6.18) we see that, for the real processes, the \(\rho^2\) should be non-negative. From (G.6), (G.7), we find then the restriction \(D \geq 0\) for the real processes and the restriction \(D < 0\) for virtual processes. For the real processes with non-collinear momenta, the on-shell condition \(\rho > 0\) (6.19) and formula (G.6) lead to the restrictions

\[
r_- < r < r_+, \quad D > 0. \quad (G.9)
\]

For out-going particles, in view of \(\beta_2 > 0\), \(\beta_3 > 0\) and the definition of \(r\) (G.7), we get the restriction \(r > 0\). In view of \(r > 0\) and \(r_+ > r\) (G.9) we get the restriction \(r_+ > 0\). In turn, the restrictions \(r_+ > 0\) and \(D \geq 0\) lead to the restriction

\[
m_1^2 > m_2^2 + m_3^2. \quad (G.10)
\]

Now using (G.8), we see that the restrictions (G.10) and \(D > 0\) lead to the restriction for the masses in (G.2). For \(D = 0\), we get \(r_+ = r_- \equiv r_0\) (G.7). Requirement \(\rho^2 \geq 0\) is then realized only for \(r = r_0\). However, for \(r = r_0\), we get \(\rho^2 = 0\) and this leads to the collinear momenta. Note also that \(D = 0\) and (G.10) imply the restriction on the masses in (G.3) as it should be for the collinear momenta.

---

16For \(|p| \neq 0\), relation (G.4) is also valid for a decay of massive particle into two massless particles and a decay of massive particle into one massive particle and one massless particle.

17Note that the equation \(D = 0\) has two solutions: \(m_1 = m_2 + m_3\) and \(m_1 = |m_2 - m_3|\).
Appendix H  Derivation of meromorphic vertices \( \bar{\mathcal{V}} \)

We start with the following comment on equation (5.26). Let \( \bar{\mathcal{V}}^{(2)}_{\text{sol}} \) be some solution of equation (5.26). Making the transformation

\[
\bar{\mathcal{V}}^{(2)}_{\text{sol}} = \exp\left( \sum_{a=1,2,3} \omega_a M_a^L \right) \bar{\mathcal{V}}^{(2)}_{\text{sol}}(\omega),
\]

we find that a vertex \( \bar{\mathcal{V}}^{(2)}_{\text{sol}}(\omega) \) should satisfy the following equation:

\[
\sum_{a=1,2,3} \left\{ 2M^R_a + \left( 2c_{\omega,a}m_a^2 - m_{a+1}^2 + m_{a+2}^2 \right) M_a^RL 
+ \left( m_a^2(c_{\omega,a}^2 - \frac{1}{4}) - (c_{\omega,a} - \frac{1}{2})m_{a+1}^2 + (c_{\omega,a} + \frac{1}{2})m_{a+2}^2 \right) M_a^L \right\} \bar{\mathcal{V}}^{(2)}_{\text{sol}}(\omega) = 0,
\]

\( c_{\omega,a} \equiv c_a - \omega_a. \)  

We see that equation (H.2) is obtained from equation (5.26) by the replacement \( c_a \rightarrow c_{\omega,a} \) (H.3). We use this freedom in the choice of \( c_a \). Namely, we find it convenient to use \( c_a \) given in (5.23).

**Derivation of (7.3).** For \( m_1 = 0, m_2 = 0 \), using \( c_3 = 0 \) (5.23) and \( M_a^R, M_a^L, M_a^{RL} \), \( a = 1, 2 \), given in (2.21), we see that equation (5.26) takes the form

\[
\left( 2M_3^R + m_3^2(M_1^{RL} - M_2^{RL}) - \frac{m_3^2}{4}M_3^L \right) \bar{\mathcal{V}}^{(2)} = 0, \quad M_1^{RL} \equiv \lambda_1, \quad M_2^{RL} \equiv \lambda_2.
\]

For the treatment of equation (H.4), we find it convenient to use the \( \alpha \)-representation for the operators \( M_3^R, M_3^L \) given in (6.6). Doing so, we represent equation (H.4) as

\[
\left( \sqrt{2} - \frac{m_3^2}{4\sqrt{2}} \alpha_3^2 \right) \partial_{\alpha_3} + \frac{m_3^2s_3}{2\sqrt{2}} \alpha_3 + m_3(\lambda_1 - \lambda_2) \right) \bar{\mathcal{V}}^{(2)} = 0.
\]

General solution to equation (H.5) is found to be

\[
\bar{\mathcal{V}}^{(2)}_{\lambda_1,\lambda_2} = y_{3+}^{s_3-\lambda_1+\lambda_2} y_{3-}^{s_3+\lambda_1-\lambda_2}, \quad y_{3\pm} \equiv 1 \pm \frac{m_3}{2\sqrt{2}} \alpha_3.
\]

Using (5.22), (B.14), (B.15) and making the transformation from the \( \alpha \)-representation to the \( u, v \)-representation (B.5), (B.8), we find that (H.6) leads to (7.3).

Let us comment on the normalization factor \( N_{\lambda_1\lambda_2} \) in (7.2). Under action of the operator \( \mathcal{I} \), we get

\[
\mathcal{I} \bar{\mathcal{V}}^{\text{bas}}_{\lambda_1,\lambda_2} = m_3^{2\lambda_1+2\lambda_2} - \lambda_1 - \lambda_2 \bar{\mathcal{V}}^{\text{bas}}_{-\lambda_1,-\lambda_2}.
\]

Using (H.7), we note the more attractive relation

\[
\mathcal{I} \left( n_{\lambda_1\lambda_2} \bar{\mathcal{V}}^{\text{bas}}_{\lambda_1,\lambda_2} \right) = n_{-\lambda_1,-\lambda_2} \bar{\mathcal{V}}^{\text{bas}}_{-\lambda_1,-\lambda_2}, \quad n_{\lambda_1,\lambda_2} = 2^{(\lambda_1+\lambda_2)/2} m_3^{\lambda_1-\lambda_2}.
\]

This is to say that using the normalization factor \( n_{\lambda_1\lambda_2} \), we get the complex conjugation rule for the coupling constant in (7.6). Note that \( N_{\lambda_1\lambda_2} = 2^{s_3/2} n_{\lambda_1\lambda_2} \). We inserted the extra factor \( 2^{s_3/2} \) to get simple overall factor for the on-shell vertex in (7.16).
Derivation of (8.3). We use $c_\alpha, \alpha = 1, 2$ (5.23) when $m_3 = 0$, and $M_3^{\mu\nu}, M_3^\nu, M_3^\nu$ given in (2.21),

\[
c_1 = \frac{m_2^2}{2m_1^2}, \quad c_2 = -\frac{m_1^2}{2m_2^2}, \quad M_3^\mu = 0, \quad M_3^\nu = 0, \quad M_3^{\mu\nu} \equiv \lambda_3.
\]

(H.9)

Using (H.9) in equation (5.26), we obtain the equation

\[
\left(2M_1^\nu + 2M_2^\nu - (m_1^2 - m_2^2)M_3^\mu - \frac{\gamma^2}{m_1^2}M_1^\nu - \frac{\gamma^2}{m_2^2}M_2^\nu\right)\bar{\nabla}^{(2)} = 0, \quad \gamma \equiv \frac{1}{2}(m_1^2 - m_2^2).
\]

(H.10)

For the treatment of equation (H.10), we use the operators $M_a^\mu, M_a^\nu, a = 1, 2$, given in (B.6). Doing so, we represent equation (H.10) as

\[
\left((m_1^2 - m_2^2)\lambda_3 + \sum_{\alpha = 1, 2} (\sqrt{2m_\alpha} - \frac{\gamma^2}{\sqrt{2m_\alpha}}\alpha)\partial_\alpha + \frac{\sqrt{2} s_\alpha \lambda^2}{m_\alpha} \partial_\alpha\right)\bar{\nabla}^{(2)} = 0.
\]

(H.11)

General solution to equation (H.11) is found to be

\[
\bar{\nabla}^{(2)} \equiv \lambda_3 \prod_{\alpha = 1, 2} y_{a_+}^{s_\alpha + n_\alpha} y_{a_-}^{s_\alpha - n_\alpha}, \quad y_{a_\pm} \equiv 1 \pm \frac{\gamma}{\sqrt{2m_\alpha}}\alpha, \quad n_1 + n_2 = \lambda_3,
\]

where $n_1, n_2 \in \mathbb{Z}$, while $\gamma$ is given in (H.10). Using (5.22), (B.14), (B.15) and making the transformation from the $\alpha$-representation to the $u, v$-representation (B.5), (B.8), we find that (H.12) leads to (8.3).

Derivation of (9.3). This case is the particular case of the one above considered. Namely, plugging $m_1 = m_2$ into equation (H.10), we see that equation (H.10) is considerably simplified as

\[
\sum_{\alpha = 1, 2} M_a^\nu \bar{\nabla}^{(2)} = 0, \quad \text{or equivalently as} \quad \sum_{\alpha = 1, 2} \partial_\alpha \bar{\nabla}^{(2)} = 0,
\]

(H.13)

where we use operator $M_a^\nu$ (B.6). All solutions to equation (H.13), which are polynomial in $\alpha$, can be chosen as

\[
\bar{\nabla}^{(2)} = X_\alpha^n, \quad X_\alpha \equiv \alpha_1 - \alpha_2 + c_0, \quad n \in \mathbb{N}_0,
\]

(H.14)

where $c_0$ is a real-valued constant. The freedom in the choice of $c_0$ corresponds to the freedom in the choice of a basis of the polynomials $X_\alpha^n$. We fix $c_0$ by looking for the simplest transformation of the vertex $\bar{\nabla}^{bas}_{n, \lambda_3}$ (9.3) under the action of the operator $\mathcal{I}$. To this end, using (5.22), (B.14) and making the transformation from the $\alpha$-representation to the $u, v$-representation (B.5), (B.8), we find that (H.14) leads to

\[
\bar{\nabla}^{bas}_{n, \lambda_3} = X^n \left(\frac{\beta_3}{\beta_4}\right)^{\lambda_3} \prod_{\alpha = 1, 2} L_\alpha^{2s_\alpha} \left(\frac{\beta_3}{\beta_4}\right)^{-s_\alpha}, \quad X \equiv \frac{v_1}{L_1} - \frac{v_2}{L_3} + c_0,
\]

(H.15)

where $L_1, L_2$ are given in (9.4). By acting with the operator $\mathcal{I}$ on $X$, we get

\[
\mathcal{I} X = -X + 2c_0 + \frac{\sqrt{2}}{m}.
\]

(H.16)

From (H.16), we see that the choice $c_0 = -1/\sqrt{2m}$ leads to the simplest transformation, $\mathcal{I} X = -X$. Using such $c_0$, we get $\bar{\nabla}^{bas}$ in (9.3). Note that, in (9.3), in place of $X$ (H.15), we prefer to use $Q \equiv L_1 L_2 X$ (9.5). We note also the following helpful relations:

\[
\mathcal{I} L_1 = -\frac{m\beta_3}{\sqrt{2P_L}} L_1, \quad \mathcal{I} L_2 = \frac{m\beta_3}{\sqrt{2P_L}} L_2, \quad \mathcal{I} Q = \frac{m^2 \beta_3}{2P_L^2} Q.
\]

(H.17)
Derivation of (10.4). Plugging $c_a$ (5.23) into equation (5.26), we get the following equation:

$$\sum_{a=1,2,3} (2M_a^R - \frac{D}{4m_a^2} M_a^L) \bar{V}^{(2)} = 0.$$  \hspace{1cm} (H.18)

For the treatment of equation (H.18), we find it convenient to use the $\alpha$-representation for the operators $M^R, M^L$ given in (B.6). Doing so, we cast equation (H.18) into the form

$$\sum_{a=1,2,3} \left( (\sqrt{2}m_a - \frac{D}{4\sqrt{2}m_a} \alpha_a^2) \partial_{\alpha_a} + \frac{D s_a}{2\sqrt{2}m_a} \alpha_a \right) \bar{V}^{(2)} = 0.$$  \hspace{1cm} (H.19)

Making the transformation to a new vertex $\bar{V}^{(3)}$,

$$\bar{V}^{(2)} = \prod_{a=1,2,3} \left( 1 - \frac{D \alpha_a^2}{8m_a^2} \right) \bar{V}^{(3)},$$  \hspace{1cm} (H.20)

we find that equation (H.19) amounts to the following equation for $\bar{V}^{(3)}$:

$$\sum_{a=1,2,3} (\sqrt{2}m_a - \frac{D}{4\sqrt{2}m_a} \alpha_a^2) \partial_{\alpha_a} \bar{V}^{(3)} = 0.$$  \hspace{1cm} (H.21)

General solution to equation (H.21) is found to be

$$\bar{V}^{(3)}_{n_1,n_2,n_3} = \prod_{a=1,2,3} y_{a,2,3}^{\alpha_a} y_{a,-1}^{-\alpha_a}, \quad n_1 + n_2 + n_3 = 0, \quad y_{a,2,3} \equiv 1 \pm \frac{\gamma}{\sqrt{2m_a}} \alpha_a,$$  \hspace{1cm} (H.22)

where $n_1, n_2, n_3 \in \mathbb{Z}$, while $\gamma$ is defined in (A.2). Using (H.20), we find then

$$\bar{V}^{(2)}_{n_1,n_2,n_3} = \prod_{a=1,2,3} y_{a,2,3}^{\alpha_a} y_{a,-1}^{-\alpha_a}.$$  \hspace{1cm} (H.23)

Using (5.22), (B.14) and making the transformation from the $\alpha$-representation to the $u, v$-representation (B.5), (B.8), we find that (H.23) leads to (10.4).

Derivation of (10.9), (10.10). Using relations for the action of the operator $\mathcal{I}$ given in (D.9) and expressions for the normalization factors $N_{n_1,n_2,n_3}$ (10.8), we find

$$\mathcal{I} N_{n_1,n_2,n_3} \bar{V}^{bas}_{n_1,n_2,n_3} = N_{-n_1,-n_2,-n_3} \bar{V}^{bas}_{-n_1,-n_2,-n_3}, \quad \text{for } D > 0;$$  \hspace{1cm} (H.24)

$$\mathcal{I} N_{n_1,n_2,n_3} \bar{V}^{bas}_{n_1,n_2,n_3} = N_{n_1,n_2,n_3} \bar{V}^{bas}_{n_1,n_2,n_3}, \quad \text{for } D < 0.$$  \hspace{1cm} (H.25)

From (H.24), (H.25), we see that in order to get the vertices $\bar{V}$ that satisfy the hermicity condition (D.8), we should use the vertices $\bar{V}$ given in (10.2) and (10.3), where the coupling constants should satisfy the respective hermicity conditions (10.9) and (10.10).

Derivation of (11.3). Plugging $c_a$ (5.23) into equation (5.26), we get equation (H.18). Using $D = 0$ in equation (H.18), we get the equation

$$\sum_{a=1,2,3} M_a^R \bar{V}^{(2)} = 0, \quad \text{or equivalently} \quad \sum_{a=1,2,3} m_a \partial_{\alpha_a} \bar{V}^{(2)} = 0,$$  \hspace{1cm} (H.26)

where we use the representation for $M^R$ given in (B.6). All solutions to equation (H.26), which are polynomial in $\alpha$, can be chosen as

$$\bar{V}^{(2)}_{n,l} = X^X_{\alpha} Y^L_{\alpha}, \quad X_\alpha \equiv c_0^X + \sum_{a=1,2,3} c_a^X \alpha_a, \quad Y_\alpha \equiv c_0^Y + \sum_{a=1,2,3} c_a^Y \alpha_a.$$  \hspace{1cm} (H.27)
\[ \sum_{a=1,2,3} c_a^X m_a = 0, \quad \sum_{a=1,2,3} c_a^Y m_a = 0, \]  

(H.28)

where \( n, l \in \mathbb{N}_0 \), while \( c_0^{XY}, c_a^{XY} \) are real-valued constants. We fix these constants in the following three steps.

**Step 1.** Using (5.22), (B.14) and making the transformation from the \( \alpha \)-representation to the \( u, v \)-representation (B.5), (B.8), we find that (H.27) leads to

\[ \bar{V}_{n,l}^{\text{bas}} = X^E Y^E \prod_{a=1,2,3} L_a^{2s_a} \left( \frac{\mathbb{P}_\epsilon}{\beta_a} \right)^{-s_a}, \]  

(H.29)

\[ X_E \equiv c_0^X + \sum_{a=1,2,3} c_a^X \frac{v_a}{L_a}, \quad Y_E \equiv c_0^Y + \sum_{a=1,2,3} c_a^Y \frac{v_a}{L_a}, \]  

(H.30)

where \( L_a \) are given in (11.4).

**Step 2.** Introducing the notation \( \mathcal{X} = X_E, Y_E \),

\[ \mathcal{X} \equiv c_0^X + \sum_{a=1,2,3} c_a^X \frac{v_a}{L_a}, \]  

(H.31)

and by acting with the operator \( \mathcal{I} \) on \( \mathcal{X} \), we get

\[ \mathcal{I} \mathcal{X} = -\mathcal{X} + 2c_0^X + \sqrt{2} \mathbb{P}_\epsilon \sum_{a=1,2,3} c_a^X \epsilon_a \beta_a. \]  

(H.32)

Requiring \( \mathcal{I} \mathcal{X} = -\mathcal{X} \), we get the following equation

\[ c_0^X + \frac{1}{\sqrt{2} \mathbb{P}_\epsilon} \sum_{a=1,2,3} c_a^X \epsilon_a \beta_a = 0. \]  

(H.33)

By using \( \mathbb{P}_\epsilon = 0 \), we find that equation (H.33) amounts to the two equations given by

\[ c_1^X = \epsilon_1 \epsilon_3 c_3^X + \sqrt{2} \epsilon_1 \epsilon_2 m_2 c_0^X, \quad c_2^X = \epsilon_2 \epsilon_3 c_3^X - \sqrt{2} \epsilon_1 \epsilon_2 m_1 c_0^X. \]  

(H.34)

**Step 3.** Equations (H.34) have two solutions. We denote the 1st solution and the 2nd solution as \( c_0, c_a^X \) and \( c_0^Y, c_a^Y \) respectively. The 1st solution is fixed by the choice \( c_0^X = 0 \). From (H.34), we get

\[ c_1^X = \epsilon_1 \epsilon_3 c_3^X, \quad c_2^X = \epsilon_2 \epsilon_3 c_3^X. \]  

(H.35)

Choosing \( c_3^X = \epsilon_3 \), we get \( c_a^X \) given in (11.5). For the 1st solution, we find \( \sum_{a=1,2,3} \epsilon_a c_a^X = 3 \). We note then that the 2nd solution can be fixed by imposing the additional restriction

\[ \sum_{a=1,2,3} \epsilon_a c_a^Y = 0. \]  

(H.36)

The additional restriction (H.36) can be satisfied by the following replacement in equation (H.33):

\[ c_a^Y \rightarrow c_a^Y - \frac{1}{3} \epsilon_a \sum_{b=1,2,3} c_b^Y \epsilon_b. \]  

(H.37)
Note that equations (H.28), (H.33), (H.34) are invariant under replacement (H.37). Using (H.34) and (H.36), we get
\[ c^Y_a = \frac{\sqrt{2}}{3} c^Y_0 (\epsilon_{a+1} m_{a+1} - \epsilon_{a+2} m_{a+2}). \] (H.38)
Choosing \( c^Y_0 = 1/\sqrt{2} \), we get \( c^Y_a, c^Y_0 \) given in (11.6). Note that \( c^X_a, c^Y (11.5), (11.6) \) satisfy equations (H.28). Also we note that, in (11.3), in place of \( X_E \) and \( Y_E \) (H.30), we prefer to use \( Q \equiv L_1 L_2 L_3 X_E \) (11.5) and \( Q_Y \equiv L_1 L_2 L_3 Y_E \) (11.6) respectively.

**Derivation of (11.8).** Using relations for the action of the operator \( \mathcal{I} \) given in (D.7) and the relations
\[ \mathcal{I} L_a = \frac{e^a_{\mu} p^\mu}{\sqrt{2} p_L} L_a, \quad \mathcal{I} p^\mu = -\frac{p^\mu}{2 p_L}, \quad \mathcal{I} X_E = -X_E, \quad \mathcal{I} Y_E = -Y_E, \] (H.39)
we find
\[ \mathcal{I} N_{n,l} \bar{V}^{\text{bas}}_{n,l} = N_{n,l} \bar{V}^{\text{bas}}_{n,l}. \] (H.40)

From (H.40), we see that in order to get the vertex \( \bar{V} \) that satisfies the hermicity condition (D.8), we should use the vertex \( \bar{V} \) given in (11.2), where the coupling constants should satisfy the hermicity condition given in (11.8). Note that the relation for \( \mathcal{I} p^\mu \) in (H.39) is the particular case of the relation for \( \mathcal{I} p^\mu \) in (D.7), when \( D = 0, P^\mu = 0 \). This can be seen by noticing the relation \( \rho^2 = -P^\mu P_\mu \) when \( D = 0, P^\mu = 0 \).

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