LOG MIRROR SYMMETRY AND LOCAL MIRROR SYMMETRY

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ABSTRACT. We study Mirror Symmetry of log Calabi-Yau surfaces. On one hand, we consider the number of “affine lines” of each degree in $\mathbb{P}^2 \setminus B$, where $B$ is a smooth cubic. On the other hand, we consider coefficients of a certain expansion of a function obtained from the integrals of $dx/x \wedge dy/y$ over 2-chains whose boundaries lie on $B_\phi$, where $\{B_\phi\}$ is a family of smooth cubics. Then, for small degrees, they coincide.

We discuss the relation between this phenomenon and local mirror symmetry for $\mathbb{P}^2$ in a Calabi-Yau 3-fold (CKYZ).

In the classification theory of algebraic varieties, one can study non-compact algebraic manifolds by means of Log Geometry.

For a quasi-projective manifold $U$, there exist a projective variety $X$ and a normal crossing divisor $B$ on $X$ such that $X \setminus B$ is isomorphic to $U$. A rational differential form on $X$ is called a logarithmic form if it can be locally written as a linear combination (with regular functions as coefficients) of products of $d \log f$, where $f$ is a regular function that vanishes at most in $B$. Then the space of logarithmic forms is an invariant of $U$ and it plays the role that the space of regular forms plays in the classification theory of projective manifolds.

In particular, we may think of a pair $(X, B)$ (or the open variety $X \setminus B$) as a log Calabi-Yau manifold if $K_X + B$ is trivial. It is expected that some analogue of Mirror Symmetry holds for log Calabi-Yau manifolds.

As an evidence, we studied Mirror Symmetry for the one-dimensional log Calabi-Yau manifold, i.e. $(\mathbb{P}^1, \{0, \infty\})$ or $\mathbb{C}^\times$, in $\mathbb{P}^3$: for $b \geq 0$ and $k, l > 0$, we consider covers of $\mathbb{P}^3$ that have $k$ and $l$ points over 0 and $\infty$ and are simply branched over $b$ prescribed points. We define $F_{b,k,l}(z_1, \ldots, z_k; w_1, \ldots, w_l)$ to be the generating function of the number of such curves. Then $F_{b,k,l}$ coincides with the sum of certain integrals associated to graphs that have $k$ and $l$ ‘initial’ and ‘final’ vertices, have $b$ internal vertices and are trivalent at internal vertices (here, variables $z_1, \ldots, z_k$ and $w_1, \ldots w_l$ are attached to initial and final vertices).

The aim of this paper is to study the two-dimensional case as a further example of Log Mirror Symmetry.

In Section 1, we describe Mirror Symmetry of $\mathbb{P}^2 \setminus B$ where $B$ is a smooth cubic. On A-model side, we count curves of each degree in $\mathbb{P}^2 \setminus B$ whose normalizations are $\mathbb{A}^1$ and construct a power series from those numbers. On B-model side, we construct a function from the integrals of $dx/x \wedge dy/y$ over 2-chains whose boundaries lie on $B_\phi$, where $B_\phi$ is a member of a family of smooth cubics parametrized by $\phi$. Then we see that the coefficients of their expansions coincide up to order 8.

Then, in Section 2, we discuss the relation between our log mirror symmetry and local mirror symmetry studied in (CKYZ). We explain that the number of “affine lines” is in some sense the dual of local Gromov-Witten invariants of $\mathbb{P}^2$ in a Calabi-Yau 3-fold.
1. LOG MIRROR SYMMETRY

1.1. A-model. On the enumerative side, we counted curves of the lowest ‘log genus’ in \([T1]\).

**Definition 1.1.** A plane curve \(C\) is said to satisfy the condition (AL) if it is irreducible and reduced and the normalization of \(C \setminus B\) is isomorphic to the affine line \(\mathbb{A}^1\).

**Remark 1.2.** If \(C\) satisfies (AL), then \(C \cap B\) consists of one point. This point is a \((3 \deg C)\)-torsion for a group structure on \(B\) whose zero element is an inflection point of \(B\).

We treat only ‘primitive’ cases.

**Definition 1.3.** Let \(P\) be a point of order \(3d\) for a group structure on \(B\) whose zero element is a point of inflection.

Then we define \(m_d\) to be the number of curves \(C\) of degree \(d\) which satisfy (AL) and \(C \cap B = \{P\}\) (or, more precisely, the degree of the 0-dimensional scheme parametrizing such curves).

The above definition implicitly assumes that the number \(m_d\) does not depend on the choice of \(P\). Although we haven’t proved it yet, it holds in the cases where we know \(m_d\), i.e. when \(d \leq 8\).

**Theorem 1.4.** ([T1]) We have the following table for \(m_d\):

| \(d\) | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|---|---|---|---|---|---|
| \(m_d\)| 1 | 1 | 3 | 16 | 113 | 948 |

Furthermore, under a technical hypothesis (see [T1]), we have \(m_7 = 8974\) and \(m_8 = 92840\).

Our invariants are apparently related to the relative Gromov-Witten invariants defined in [IP] and [LR] for pairs of a symplectic variety and its subvariety of real codimension 2. In algebraic language:

**Proposition-Definition 1.5.** (1-pointed case of [G]) Let \(\bar{M}_{0,1}(\mathbb{P}^2, d)\) be the moduli stack of 1-pointed genus 0 stable maps of degree \(d\) to \(\mathbb{P}^2\) and \(\bar{M}^B_{1}(\mathbb{P}^2, d)\) the closed subset consisting of points corresponding to \(f : (C, P) \to \mathbb{P}^2\) such that

(i) \(f(P) \in B\) and
(ii) \(f^*B - iP \in A_0(f^{-1}B)\) is effective.

Then the virtual fundamental class \([\bar{M}^B_{1}(\mathbb{P}^2, d)]^{\virt}\) is naturally defined and is of expected dimension \(3d - i\).

Considering that the number of \(3d\)-torsions is \((3d)^2\) and that the relative Gromov-Witten invariants take multiple covers into account, we expect the following to hold.

**Conjecture 1.6.** \([M^B_{3d}(\mathbb{P}^2, d)]^{\virt}\) = \((3d)^2 \sum_k d(-1)^{d-d/k} m_{d/k}/k^4\).

**Remark 1.7.** (1) Although the factor \(k^{-4}\) looks unfamiliar, this is compatible with the factor \(k^{-3}\) for Gromov-Witten invariants, since \(m_d\) is conjecturally equal to \((-1)^{d-1} n_d/(3d)\), where \(n_d\) is the local Gromov-Witten invariant (see Remark 2.2).
(2) A. Gathmann informed the author that this is true for \(d \leq 8\) (i.e. where \(m_d\) is calculated).
1.2. **B-model.** The result of [T2] suggests that the mirror manifold of \( C^x \) is \( C^x \). In A-model, we associate a parameter to each point over 0 or \( \infty \) on a cover of \( C^x \), and in B-model this corresponds to the choice of points on \( C^x \): the former can be considered as “Kähler moduli” and the latter as “complex moduli”. In B-model, we used the points as the boundary of integrals to define coordinate functions.

With this in mind, we look for a function which has an expansion with coefficients \( m_d \). Although the calculation below is essentially the same as in [CKYZ], here we see it from the point of view of Mirror Symmetry of log Calabi-Yau surfaces: we claim that the mirror of \( (P^2 \setminus \{ \text{smooth cubic} \} & \text{Kähler moduli}) \) is \( (C^x \times C^x \& \text{cubic}) \), where the latter cubic has complex moduli and we take it as the boundary of integrals, just as in the case of \( C^x \).

We consider homogeneous coordinates \( X, Y, Z \) and inhomogeneous coordinates \( x, y \) on \( P^2 \). Let \( \Omega \) be the logarithmic 2-form \( dx/x \wedge dy/y \) on \( X_0 := P^2 \{ XYZ = 0 \} \) and write \( B_\phi \) for the plane cubic defined by \( XYZ - \phi(X^3 + Y^3 + Z^3) = 0 \).

Then we consider the integral

\[
I := \int_{\Gamma_\phi} \Omega,
\]

where \( \Gamma_\phi \) is a 2-chain in \( X_0 \) whose boundary has support on \( B_\phi \).

**Lemma 1.8.** We have

\[
\phi \frac{dI}{d\phi} = \int_{\partial \Gamma} dx/(x - 3\phi y^2).
\]

**Proof.** If we fix \( x \) and differentiate \( x y - \phi(x^3 + y^3 + 1) = 0 \), we have \( dy/d\phi = xy/\phi(x - 3\phi y^2) \).

So, if we set \( z := \phi^3 \) and \( \theta := z^{4/3} \), we have

\[
\{ \theta^3 - 3z\theta(3\theta + 1)(3\theta + 2) \} I = 0.
\]

The following functions form a basis of the space of solutions:

\[
I_1 = 1, \\
I_2 = \log z + I_2^{(0)}, \\
I_3 = I_2 \log z - (\log z)^2/2 + I_3^{(0)},
\]

where

\[
I_2^{(0)} = 6z + 45z^2 + 560z^3 + \frac{17325}{2}z^4 + \frac{756756}{5}z^5 + 2858856z^6 + \frac{399072960}{7}z^7 + \frac{4732755885}{4}z^8 + \cdots,
\]

\[
I_3^{(0)} = 9z + \frac{423}{4}z^2 + 1486z^3 + \frac{389415}{16}z^4 + \frac{21981393}{50}z^5 + \frac{16973929}{2}z^6 + \frac{8421450228}{49}z^7 + \frac{1616340007953}{448}z^8 + \cdots.
\]

The monodromy of \( I_3 \) for \( z \to e^{2\pi i}z \) is \( 2\pi i(I_2 - \pi i) \), which we denote by \( \tilde{I}_3 \). We denote the monodromy \( (2\pi i)^2 \) of \( \tilde{I}_2 \) by \( \tilde{I}_1 \). Now we write \( I_3 \) in terms of \( q := \)
\[ e^{2\pi i I_2/I_1} = -e^{I_2} = -z e^{I_2^{(0)}}: \]

\[
I_3 = \left( \frac{I_2}{2} \right)^2 + \frac{351}{4} z^2 + 1216 z^3 + \frac{319455}{16} z^4 + \frac{18122643}{50} z^5 + \cdots
+ \frac{3516124}{5} z^6 + \frac{7009518168}{49} z^7 + \frac{1350681750297}{484} z^8 + \cdots
= \left( \frac{\log(-q)}{2} \right)^2 + 0 - 9q + \frac{135}{4} q^2 - 244q^3 + \frac{36999}{16} q^4 - \frac{635634}{25} q^5
+ 307095q^6 - \frac{193919175}{64} q^7 + \frac{3422490759}{29} q^8 + \cdots
\]

\[
= \left( \frac{\log(-q)}{2} \right)^2 - 3^2.1 \sum_{k=1}^{\infty} \frac{q^k}{k^2} + 6^2.1 \sum_{k=1}^{\infty} \frac{q^2k}{k^2} - 9^2.3 \sum_{k=1}^{\infty} \frac{q^{3k}}{k^2} + 12^2.16 \sum_{k=1}^{\infty} \frac{q^{4k}}{k^2}
- 15^2.113 \sum_{k=1}^{\infty} \frac{q^{5k}}{k^2} + 18^2.948 \sum_{k=1}^{\infty} \frac{q^{6k}}{k^2} - 21^2.8974 \sum_{k=1}^{\infty} \frac{q^{7k}}{k^2}
+ 24^2.92840 \sum_{k=1}^{\infty} \frac{q^{8k}}{k^2} + \cdots
\]

Remark 1.9. \( K := (zdI_2/dz)^3 d^2 I_3/dI_2^2 \) satisfies \( dK/dz = (27/(1 - 27z))K \), and therefore we have \( K = 1/(1 - 27z) \).

Comparing the coefficients with the table in Theorem 1.4, we propose:

Conjecture 1.10.

\[ I_3 = \left( \frac{\log(-q)}{2} \right)^2 + \sum_{d=1}^{\infty} (-1)^d (3d)^2 m_d \sum_{k=1}^{\infty} \frac{q^{dk}}{k^2}. \]

Remark 1.11. If we assume Conjecture 1.7, the previous conjecture is equivalent to

\[ I_3 = \left( \frac{\log(-q)}{2} \right)^2 + \sum_{d=1}^{\infty} (-1)^d [M_3d(P^2, d)]^{\text{virt}} q^d, \]

and in fact this may be a more natural equality.

According to A. Gathmann, his algorithm in [3] can be used to prove this.

2. Log Mirror and Local Mirror

In [CKYZ], the generating function of the “numbers of rational curves in a local Calabi-Yau 3-fold” was given.

Let \( \mathbb{P}^2 \) be embedded in a Calabi-Yau 3-fold and denote by \( n_d \) the contribution of rational curves of degree \( d \) in \( \mathbb{P}^2 \) to the number of rational curves in the Calabi-Yau 3-fold. Let \( \bar{M}_{0,0} \) be the moduli of stable maps of genus 0 curves to \( \mathbb{P}^2 \) with degree \( d \) images and \( U \) the vector bundle over \( \bar{M}_{0,0} \) whose fiber at the point \( [f : C \to \mathbb{P}^2] \) is \( H^1(C, f^*K_{\mathbb{P}^2}) \). Then, the Chern number \( K_d := c_{3d-1}(U) \) is equal to \( \sum_{k|d} n_{d/k} k^3 \).

Theorem 2.1. (CKYZ)

\[
I_3 = \left( \frac{\log(-q)}{2} \right)^2 - \sum_{d=1}^{\infty} 3dK_d q^d
= \left( \frac{\log(-q)}{2} \right)^2 - \sum_{d=1}^{\infty} 3dn_d \sum_{k=1}^{\infty} \frac{q^{dk}}{k^2}.
\]
Remark 2.2. Thus Conjecture 1.10 is equivalent to \( n_d = (-1)^{d-1}3dm_d \), and this equality holds for \( d \leq 8 \) by Theorem 1.3 and the \( q \)-expansion of \( I_3 \) in the previous section.

We give a heuristic argument as to why this should hold, although it contains serious gaps as explained later.

By Serre duality, the dual of \( U \) is isomorphic to the vector bundle \( V \) over \( \bar{M}_{0,0} \) whose fiber at \( [f : C \to \mathbb{P}^2] \) is \( H^0(C, K_C \otimes f^*O_{\mathbb{P}^2}(B)) \). Since the rank of the bundles is \( 3d - 1 \), we have \( c_{3d-1}(V) = (-1)^{d-1}K_d \).

Let \( \bar{M}_{0,1} \) be the moduli of stable maps of 1-pointed genus 0 curves to \( \mathbb{P}^2 \) with degree \( d \) images, \( M_{0,1} \) the open subset of \( \bar{M}_{0,1} \) representing \( f : (\mathbb{P}^1, P) \to \mathbb{P}^2 \) and \( \pi : \bar{M}_{0,1} \to \bar{M}_{0,0} \) the projection. Further, let \( E_1 \) be the line bundle over \( \bar{M}_{0,1} \) whose fiber at \( f : (C, P) \to \mathbb{P}^2 \) is \( H^0(C, f^*O_{\mathbb{P}^2}(B) \otimes O_P) \) and \( L \) the line bundle whose fiber is \( H^0(C, O_P(-P)) \). Then, we have \( c_{3d}(\pi^*V \oplus E_1) = 3dc_{3d-1}(V) \), for the zero set of a section of \( E_1 \) induced by a defining equation of \( B \) is the set of the points corresponding to \( f : (C, P) \to \mathbb{P}^2 \) such that \( f(P) \in B \), and there are \( 3d \) such points for any \( f : C \to \mathbb{P}^2 \).

We have an exact sequence

\[
0 \to K_C \to K_C (P) \xrightarrow{\text{resid}} O_C \to 0.
\]

If \( C \) is irreducible, i.e. isomorphic to \( \mathbb{P}^1 \), we obtain exact sequences

\[
0 \to H^0(C, K_C ((k+1)P) \otimes f^*O_{\mathbb{P}^2}(B)) \to H^0(C, K_C(-kP) \otimes f^*O_{\mathbb{P}^2}(B)) \to 0
\]

for \( 0 \leq k \leq 3d - 2 \). We also have \( H^0(C, K_C ((-3d-1)P) \otimes f^*O_{\mathbb{P}^2}(B)) = 0 \).

Thus, on \( M_{0,1} \), we have a filtration of \( \pi^*V \oplus E_1 \) such that the associated graded module is isomorphic to \( \bigoplus_{i=0}^{3d-1} E_1 \otimes L^i \).

On the other hand, there are about \( (3d)^2m_d \) plane curves of degree \( d \) satisfying (AL), since the number of 3d-torsions on \( B \) is \( (3d)^2 \). They are in one-to-one correspondence with points \( (f : (\mathbb{P}^1, P) \to \mathbb{P}^2) \in M_{0,1} \) such that \( f \) is birational onto the image and that \( f^*B = 3dP \).

Consider the vector bundle \( E_{3d} \) of rank \( 3d \) over \( \bar{M}_{0,1} \) whose fiber at \( f : (C, P) \to \mathbb{P}^2 \) is \( H^0(C, f^*O_{\mathbb{P}^2}(B) \otimes O_{3dP}) \). On \( M_{0,1} \), the zero set of a section of \( E_{3d} \) induced by a defining equation of \( B \) is the set of the points \( f : (C, P) \to \mathbb{P}^2 \) such that \( f^*B = 3dP \).

Now, from the exact sequences

\[
0 \to O_P(-kP) \to O_{(k+1)P} \to O_kP \to 0,
\]

we see that \( E_{3d} \) has a filtration such that the associated graded module is isomorphic to \( \bigoplus_{i=0}^{3d-1} E_1 \otimes L^i \). Thus we may expect \( (3d)^2m_d \approx (-1)^{3d-1}3dK_d \approx (-1)^{3d-1}3dn_d \).

Rigorously, however, this argument makes little sense. First, the section of \( E_{3d} \) induced by a defining equation of \( B \) has undesirable zeros in \( \bar{M}_{0,1} \). For example, if \( C = C_1 \cup C_2 \cup C_3 \) (a chain in this order), \( P \in C_2 \) and \( C_2 \) maps to a point \( Q \) in \( B \), we may take any rational curves through \( Q \) as the images of \( C_1 \) and \( C_3 \). Thus the number \( (3d)^2m_d \) may be much different from \( c_{3d}(E_{3d}) \). Second, we have a filtration of \( \pi^*V \oplus E_1 \) merely on \( M_{0,1} \).

Let \( \bar{M}_B^B(\mathbb{P}^2, d) \) be as in Definition 1.3. The section of the line bundle \( (E_1 \otimes L^i)|_{\bar{M}_B^B(\mathbb{P}^2, d)} \) induced by a defining equation of \( B \) vanishes when \( f^{-1}B \supseteq (i+1)P \) is satisfied. Then, [3] describes the difference between \( c_1(E_1 \otimes L^i)|_{\bar{M}_B^B(\mathbb{P}^2, d)} \) and
$\bar{\mathcal{M}}_{i+1}^{B}(\mathbb{P}^2, d)^{\text{virt}}$. This should account for the difference between $c(\bigoplus_{i=0}^{3d-1} E_1 \otimes L^i)$ and $c(\pi^{*} V \oplus E_1)$.

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