Equalities, inequalities between density flow, spatial and temporal entropies of cellular automata

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Abstract

Defining the density flow of perturbations moving at a given speed for cellular automata, we establish equalities and inequalities between the measurable entropy of a cellular automaton and the measurable entropy of its associated shift. We illustrate our results by different examples and we study some relations between the density flow (with respect to a linear or a sublinear speed) and some properties of cellular automata (positive expansiveness, \(\mu\)-expansiveness and \(\mu\)-equicontinuity). The probability measure we consider must be shift ergodic and invariant for the automaton.

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1 Introduction

A one-dimensional cellular automaton (CA) is a discrete mathematical idealization of a space-time physical system. The space, called configuration space, is the set of doubly infinite sequences of elements of a finite set $A$. The discrete time is represented by the action of a cellular automaton $F$ on this space. Since cellular automata are continuous, shift commuting dynamical systems and depend on a local rule, relations between the spacial (shift) and temporal (automaton) entropies seems natural. In [6] Shereshevsky establishes a first relation between the entropy of the shift and the entropy of the cellular automaton using some discrete analog of Lyapunov exponents for a CA ergodic measure. The product of the shift entropy by the lyapunov exponents give an upper bound of the CA. In [7] Tisseur define average Lyapunov exponents using a CA invariant and shift ergodic measure. Given an infinite configuration, the two Lyapunov exponents (left and right) represent the speed of the faster perturbations moving from the left to the right or from the right to the left coordinates. Remark that it is possible to define these left and right exponents for each points or to take an average value of these speeds.

The Average Lyapunov exponents give the rate of propagation of the faster perturbations and do not take in account the amount of information the automaton carry. This is the reason for which in many cases the inequality is not an equality. Nevertheless the average Lyapunov exponents (when they are equal to zero) are able to prove that the measurable entropy of the cellular automaton is null for CA with equicontinuous points (see [7]) and for some particular sensitive (without equicontinuous point) CA (see [2]).

Here, we propose to introduce the density flow $M_\mu(v)$ of the perturbations with respect to a velocity $v = (v^+, v^-)$ where $v^+$ is the right to left speed and $v^-$ the right to left speed. This average flow depends on a shift-ergodic and CA invariant probability measure $\mu$. In section 3.2 (Theorem 2), we establish that $h_\mu(F) = h_\mu(\sigma) \times M_\mu(v) \times (v^+ + v^-)$ where $h_\mu(F)$ and $h_\mu(\sigma)$ are respectively the entropy of the automaton $F$ and the entropy of the shift $\sigma$ and the speeds $v^+$ and $v^-$ are greater than the radius $r$ of the (CA). More generally, Theorem 2 states that $h_\mu(F) \geq h_\mu(\sigma) \times M_\mu(v) \times (v^+ + v^-)$. These relations can be compared with the inequality that appears in [7] : $h_\mu(F) \leq h_\mu(\sigma) \times (I_\mu^+ + I_\mu^-)$ where $I_\mu^+$ and $I_\mu^-$ are the left and right Average Lyapunov exponents. Remark that the condition $v^+, v^- \geq r$ can be substituted by a weaker one (see Theorem 2) in relation with "maximum
Lyapunov exponents”.

One of our motivations to define the density flow is to understand the dy-

namic of cellular automata with null measurable entropy. From [7], \( I^+ + I^- = 0 \) and \( h_\mu(F) = 0 \), if there exist equicontinuous points in the topological support of the shift ergodic and \( F \)-invariant measure \( \mu \). Having equicontinuous points is equivalent to say that exist some patterns of positive measure, called blocking words that stop the perturbations. In [2], Bressaud and Tisseur show that there exists a sensitive cellular automaton (without equicontinuous points) such that \( I^+ + I^- = 0 \) which implies that \( h_\mu(F) = 0 \). In section 3.3 we give an example of a cellular automaton \( F \), such that the faster pertur-
bations move at positive speed (:math:`I^+ + I^- = 1`) but the density flow at speed \( v^+, v^- := 1 \) is equal to zero. In that case the density flow is equal to zero because the weight of the perturbation moving at a positive speed is equal to zero and Theorem 2 implies that \( h_\mu(F) = 0 \).

Finally we show some relations between some subclasses of CA and the density of flow. We state that there always exists a positive velocity \( v \) such that \( M_\mu(v) = 1 \) when the cellular automaton \( F \) has the positive expansiveness property (\( \mu \) is a shift ergodic and \( F \)-invariant measure). In this case, all the perturbations move with a speed at least equal to \( v \). Furthermore, we prove that \( M_\mu(v) = 0 \) for all linear or ”sublinear speed” \( v \) if the cellular automaton \( F \) has \( \mu \)-equicontinuous points, a measurable equivalent to the existence of equicontinuous points introduced by Gilman in [3]. This class is the complementary class of the \( \mu \)-expansive class which is a measurable equivalent to the positive expansiveness one. We call sublinear speed \( v \) any couple \((v^+, v^-)\) where \( v^+ \) and \( v^- \) represent positive integer sequences \((v^+_n)\) and \((v^-_n)\) that verify \( \lim_{n \to \infty} (v^+_n + v^-_n) = +\infty \) and \( \lim_{n \to \infty} (\frac{v^+_n}{n} + \frac{v^-_n}{n}) = 0 \).

2 Preliminary

2.1 Symbolics systems and cellular automata

Let \( A \) be a finite set or alphabet. Denote by \( A^* \) the set of all concatenations of letters in \( A \). These concatenations are called words. The length of a word \( u \in A^* \) is denoted by \( |u| \). The set of bi-infinite sequences \( x = (x_i)_{i \in \mathbb{Z}} \) is denoted by \( A^\mathbb{Z} \). A point \( x \in A^\mathbb{Z} \) is called a configuration. For \( i \leq j \) in \( \mathbb{Z} \) we denote by \( x(i, j) \) the word \( x_i \ldots x_j \) and by \( x(p, \infty) \) the infinite sequence \((v_i)_{i \in \mathbb{N}}\) such that for all \( i \in \mathbb{N} \) one has \( v_i = x_{p+i-1} \). We endow \( A^\mathbb{Z} \) with the
product topology. The shift \( \sigma: \mathbb{Z}^\mathbb{Z} \to \mathbb{Z}^\mathbb{Z} \) is defined by: \( \sigma(x) = (x_{i+1})_{i \in \mathbb{Z}} \).

For each integer \( t \) and each word \( u \), we call cylinder the set \([u]_t = \{ x \in \mathbb{Z}^\mathbb{Z} : x_t = u_1 \ldots ; x_{t+|u|} = u_{|u|} \} \). For this topology \( \mathbb{Z}^\mathbb{Z} \) is a compact metric space. A metric compatible with this topology can be defined by the distance \( d(x, y) = 2^{-i} \) where \( i = \min\{|j| \text{ such that } x(j) \neq y(j)\} \). The dynamical system \( (\mathbb{Z}^\mathbb{Z}, \sigma) \) is called the full shift. A subshift \( X \) is a closed shift-invariant subset \( X \) of \( \mathbb{Z}^\mathbb{Z} \) endowed with the shift \( \sigma \). It is possible to identify \( (X, \sigma) \) with the set \( X \). If \( \alpha = \{A_1, \ldots, A_n\} \) and \( \beta = \{B_1, \ldots, B_m\} \) are two partitions denote by \( \alpha \lor \beta \) the partition \( \{A_i \cap B_j : i = 1 \ldots n, j = 1, \ldots, m\} \). Consider a probability measure \( \mu \) on the Borel sigma-algebra \( \mathcal{B} \) of \( \mathbb{Z}^\mathbb{Z} \). If \( \mu \) is \( \sigma \)-invariant then the topological support of \( \mu \) is a subshift denoted by \( S(\mu) \). The metric entropy \( h_\mu(T) \) of a transformation \( T \) is an isomorphism invariant between two \( \mu \)-preserving transformations. Put \( H_\mu(\alpha) = \sum_{A \in \alpha} \mu(A) \log \mu(A) \). The entropy of the partition \( \alpha \) is defined as \( h_\mu(\alpha) = \lim_{n \to \infty} 1/nH_\mu(\bigvee_{i=0}^{n-1} T^{-i} \alpha) \) and the entropy of \( (X, T, \mu) \) as \( \sup_\alpha h_\mu(\alpha) \).

A cellular automaton (CA) is a continuous self-map \( F \) on \( \mathbb{Z}^\mathbb{Z} \) commuting with the shift. The Curtis-Hedlund-Lyndon theorem states that for every cellular automaton \( F \) there exist an integer \( r \) and a block map \( f \) from \( A^{2^r+1} \) to \( A \) such that: \( F(x)_i = f(x_{i-r}, \ldots, x_i, \ldots, x_{i+r}) \). The integer \( r \) is called the radius of the cellular automaton. If the block map of a cellular automaton is such that \( F(x)_i = f(x_i, \ldots, x_{i+r}) \), the cellular automaton is called one-sided and can be extended a map on a two-sided shift \( \mathbb{Z}^\mathbb{Z} \) or a map on a one-sided shift \( \mathbb{N}^\mathbb{N} \). If \( X \) is a subshift of \( \mathbb{Z}^\mathbb{Z} \) and one has \( F(X) \subset X \), the restriction of \( F \) to \( X \) determines a dynamical system \( (X, F) \); it is called a cellular automaton on \( X \).

### 2.2 Other definitions and one inequality

**Topological and measurable properties:**

For all \( x \in \mathbb{A}^\mathbb{Z} \) and \( \epsilon > 0 \) denote by \( D(x, \epsilon) \) be the set of all points \( y \) such that for all \( i \in \mathbb{N} \) one has \( d(F^i(x), F^i(y)) < \epsilon \) and for all positive integer \( n \), write \( B_n(x) = D(x, 2^{-n}) \). For all \( x \in \mathbb{A}^\mathbb{Z} \), the set \( C_n(x) \) represents the set of all points \( y \) such that \( y(-n, n) = x(-n, n) \).

**Definition 1** Equicontinuity

- A point \( x \) is called an equicontinuous point if for all positive integer \( n \), there exists another integer \( m \geq n \) such that \( B_n(x) \supset C_n(x) \).
A point $x$ is called a $\mu$-almost equicontinuous point if $\mu(B_n(x)) > 0$ for all integer $n \geq r$.

From [3], and [8], if $F$ is a CA of radius $r$, $\mu$ is a shift ergodic measure and $x$ a point which verifies $\mu(B_r(x)) > 0$ then there exist a set of measure one of $\mu$-equicontinuous points. In this case we say that $F$ is a $\mu$-equicontinuous CA.

**Definition 2 Expansiveness**

- A Cellular automaton is positively expansive if there exists a positive integer $n$ such that for all $x \in A^\mathbb{Z}$ one has $B_n(x) = \{x\}$.

- Let $\mu$ be a shift ergodic measure. A cellular automaton $F$ is $\mu$-almost expansive if there exists a positive integer $n$ such that for all $x \in A^\mathbb{Z}$, $\mu(B_n(x)) = 0$ (see [3], [8]).

- A cellular automaton is sensitive if for all points $x \in A^\mathbb{Z}$ and each integer $m > 0$ one has $C_m(x) \subset B_r(x)$.

**Lyapunov exponents:**

The Average Lyapunov exponents represent the average speed of the faster perturbations on the one dimensional lattice $A^\mathbb{Z}$.

Let $W_i^+(x)$ and $W_i^-(x)$ are respectively the set of all the perturbations (any element in $W_i^+(x)$ different of $x$) at the left side and at the right side of a coordinate $i$ in a point $x$. More formaly

$W_i^+(x) = \{y \in A^\mathbb{Z} : \forall i \geq s, y_i = x_i\}$, $W_i^-(x) = \{y \in A^\mathbb{Z} : \forall i \leq s, y_i = x_i\}$.

Let’s define two continuous sequences of functions $I_n^-$ and $I_n^+$ from $A^\mathbb{Z} \to \mathbb{N}$ by

$I_n^-(x) = \min\{s \in \mathbb{N} | \forall 1 \leq i \leq n, \ | F^i(W_s^-(x)) \subset W_0^- (F^i(x))\}$,

$I_n^+(x) = \min\{s \in \mathbb{N} | \forall 1 \leq i \leq n, \ | F^i(W_s^+(x)) \subset W_0^+ (F^i(x))\}$.

Remark that $I_n^+$ and $I_n^-$ are bounded by $rn$ where $r$ is the radius of the CA. Set $I_{n,\mu}^+ = \int_X I_n^+(x) d\mu(x)$ and $I_{n,\mu}^- = \int_X I_n^-(x) d\mu(x)$. The Birkhoff’s theorem implies that for almost all $x$ one has $I_{n,\mu}^+ = \lim_{n \to \infty} \sum_{i=-m}^{m} \frac{1}{2m+1} I_n^+(\sigma^i(x))$ and $I_{n,\mu}^- = \lim_{n \to \infty} \sum_{i=-m}^{m} \frac{1}{2m+1} I_n^-(\sigma^i(x))$. 
Definition 3 Call average Lyapunov exponents the limits
\[ I^+_\mu = \liminf_{n \to \infty} \frac{I^+_n}{n} \quad \text{and} \quad I^-_{\mu} = \liminf_{n \to \infty} \frac{I^-_n}{n}. \]

Theorem 1 Let \( F \) be a cellular automaton on \( \mathbb{Z} \) and \( \mu \) is a shift ergodic and \( F \)-invariant measure, then \( h_\mu(F) \leq h_\mu(\sigma)(I^+_\mu + I^-_{\mu}) \) where \( h_\mu(F) \) and \( h_\mu(\sigma) \) are respectively the measurable entropies of the automaton and the shift \( \sigma \).

3 Results

This aim of this work is double:

Firstly, establish an equality between the spatial and the temporal measurable entropy of a cellular automaton \( F \) for a shift ergodic and \( F \)-invariant measure. In order to do that, we introduce the density flow with respect to a velocity \( v \).

Secondly, to progress in the understanding of the dynamic of cellular automata with null measurable entropy.

3.1 The equality for the uniform measure

Let’s introduce the density flow of perturbations moving at velocity \( r \) in the particular case of the uniform measure.

3.1.1 A first equality

For each cellular automaton \( F \) and positive integer \( p \), denote by \( \alpha_p \) the partition of the set \( \mathbb{Z} \) by the \( p \) central coordinates and by \( \{n\} \alpha^F_p \) the partition \( \alpha_p \cup F^{-1}\alpha_p \cup \cdots \cup F^{-n+1}\alpha_p \). Let \( \{n\} \alpha^F_p(x) \) be the element of the partition \( \{n\} \alpha^F_p \) that contains the point \( x \). Likewise define \( \{\pm n\} \alpha^\sigma_p(x) \) as the element of the partition \( \alpha_p \cup \sigma^{-1}\alpha_p \cup \cdots \cup \sigma^{-n+1}\alpha_p \cup \sigma\alpha_p \cup \sigma^2\alpha_p \cup \cdots \cup \sigma^{n-1}\alpha_p \) generated by the shift \( \sigma \) that contains \( x \). We suppose that \( F \) is a surjective CA which implies that the uniform measure \( \mu \) on \( \mathbb{Z} \) is \( F \)-invariant (see \[1\]). From the probabilistic version of the Shannon-McMillan-Breiman Theorem we have \( h_\mu(F, \alpha_p) = \int_{\mathbb{Z}} \lim_{n \to \infty} \frac{1}{n} \log \mu(\{n\} \alpha^F_p(x)) d\mu(x), \) where \( h_\mu(F, \alpha_p) \) is the measurable entropy of the cellular automaton \( F \) with respect to the finite partition \( \alpha_p \). Since the cellular automaton \( F \) is defined thanks to a
local rule acting on words of size $2r + 1$, the set $\{n\} \alpha_p^F(x)$ is a finite union of cylinders $[y(-rn - p, rn + p)]-rn-p$ where $y \in \{n\} \alpha_p^F(x)$. Let $T_{n,p}^{rn}(x)$ be the set of such cylinders. Since $\mu$ is the uniform measure, all the cylinders $[x(-rn - p, rn + p)]-rn-p$ have the same measure $|A|^{-2(p+rn)+1}$ and $\mu(\{n\} \alpha_p^F(x)) = \#T_{n,p}^{rn} \times \mu(\{rn\} \alpha_p^x(x)) = \#T_{n,p}^{rn} \times |A|^{-2(p+rn)+1}$. Therefore, it follows that

$$h_\mu(F, \alpha_p) = \int_{A^Z} \lim_{n \to \infty} \frac{-1}{n} \log \mu(\{n\} \alpha_p^F(x)) d\mu(x)$$

$$= \int_{A^Z} \lim_{n \to \infty} -\frac{\log [\mu(\{\pm n\} \alpha_p^x(x)) \times \#T_{n,p}^{rn}(x)]}{2rn + 2p + 1} d\mu(x)$$

$$= \int_{A^Z} \lim_{n \to \infty} \frac{\log (\mu(\{\pm n\} \alpha_p^x(x)))}{2rn + 2p + 1} \times 2r \left[ 1 - \frac{\log (\#T_{n,p}^{rn}(x))}{-\log (\mu(\{rn\} \alpha_p^x(x))))} \right] d\mu(x).$$

Since the uniform measure is shift ergodic, the extended version of the Shannon-McMillan-Breiman Theorem (see [5]) tell us that

$$\int_{A^Z} \lim_{n \to \infty} \frac{\log (\mu(\{\pm n\} \alpha_p^x(x)))}{2rn + 2p + 1} d\mu(x) = h_\mu(\sigma, \alpha_p).$$

This implies that $M^p(r) := \int_{A^Z} \lim_{n \to \infty} \left[ 1 - \frac{\log (\#T_{n,p}^{rn}(x))}{-\log (\mu(\{\pm n\} \alpha_p^x(x)))} \right] d\mu(x)$ exits and we obtain that $h_\mu(F, \alpha_p) = h_\mu(\sigma, \alpha_p) \times 2r \times M^p(r)$.

Arguing that $(\alpha_p)$ is a generating sequence for the shift $\sigma$, we obtain that

$$h_\mu(F) = h_\mu(\sigma) \times 2r \times M^* (r),$$

where $M^* (r) = \sup_{\alpha_p} M^p(r)$ As $\mu$ is the uniform measure, $\mu(\{\pm n\} \alpha_p^x(x))) = A^{-2(rn+p)-1}$. The terms $\frac{\log (\#T_{n,p}^{rn}(x))}{\log (\mu(\{\pm n\} \alpha_p^x(x)))}$ corresponds to the ratio of the logarithm of the number of all patterns $u$ of length $2(rn + p) + 1$ which verify that if $y \in [u]_{-rn-p}$ then $F^i(y)(-p, p) = F^i(x)(-p, p)$ for $0 \leq i \leq n$ by the logarithm of the number of all patterns of size $2(rn + p) + 1$. The term $M^* (r)$ can be seen as a the limit of the logarithmic proportion of the finite configurations which move at speed $r$.

For these reasons, we call $M^* (r)$, the density flow of perturbations moving at speed $r$. 
3.1.2 A first basic example

Let \( \sigma_1 \) be the shift map on \( X_1 = \{0, 1\}^Z \) and \( \sigma_2 \) the \( n \) iterated shift map on \( X_2 = \{0, 1\}^Z \). For all \((x^1, x^2) \in X_1 \times X_2 =: X\), one has \((\sigma_1(x^1))_i = x^1_i+1\) and \((\sigma_2^n(x^2))_i = x^2_{i+n}\). Denote by \( F_e \) the cellular automaton on \( \{0, 1\}^Z \times \{0, 1\}^Z \) defined by : \( F_e = \sigma_1 \times \sigma_2^n \), by \( \mu \) be the uniform measure on \( \{0, 1\}^Z \times \{0, 1\}^Z \) and by \( \sigma \), the shift \( \sigma = \sigma_1 \times \sigma_2 \) on \( X \). Remark that the uniform measure \( \mu \) is shift and \( F \)-ergodic and that \( \#^pT_n(x) \) does not depends on \( x \) and \( p \). Since during the \( n \) first iterations of the automaton \( F \), the coordinates between \( p + r + 1 \) and \( p + rn \) in \( X_1 \) and the coordinates between \(-p\) and \(-p - rn \) in \( X \) does not reach the central coordinates between \(-p\) and \( p \), we can write that \( \#T_{n,p}(x) = 2^{(r-1)n+2rn} \). We have \( \mu(\sigma_p(x)) = 2^{2rn+2p+1} = 2^{-4rn+4p+2} \)

\[
M^*(r) = \lim_{n \to \infty} 1 - \frac{(3r-1)n \log(2)}{4r} = 1 - \frac{3r-1}{4r} = \frac{r+1}{4r}.
\]

Clearly, the measurable entropy \( h_\mu(\sigma) \) on \( X \), is equal to \( h_\mu(\sigma_1) + h_\mu(\sigma_2) = 2 \log(2) \) and \( h_\mu(F) = h_\mu(\sigma_1) + h_\mu(\sigma_2^r) = (r+1) \log(2) \).

Finally we can check the equality given in the previous subsection:

\[
h_\mu(F_e) = h_\mu(\sigma) \times 2r \times M^*(r) = 2 \log(2) \times 2r \times \frac{r+1}{4r} = (r+1) \log(2).
\]

Remark that using Theorem 1, we obtain the strict following inequality \( h_\mu(F_e) \leq (I^+ + I^-)h_\mu(\sigma) = r \times 2 \log(2) = 2r \log(2) \) with \( I^+ = 0 \) and \( I^- = r \).

In the following, we are going to extends the equality for a shift ergodic measure.

3.2 The equalities and inequalities for a shift ergodic measure

To extend the results of the previous section, we use the property that each cylinders defined by fixing the same number of coordinates have a similar weight for a shift-ergodic measure. This is due to the Shannon-McMillan-Breiman Theorem used for the shift action.

Denote by \( \alpha_p^x \) the partition of \( A^Z \) into cylinders \([x(-p,p)]_p \) where \( x \) any point in \( A^Z \) and by \( \{\alpha_p^x\}^F \) the partition \( \alpha_p \vee F^{-1} \alpha_p \ldots F^{-n} \alpha_p \) where \( F : A^Z \to A^Z \) is a cellular automaton. Then call \( \{\alpha_p^x\}^F(x) \) the element of the partition \( \{\alpha_p^x\}^F \) that contains the point \( x \). Recall that for a two-sided subshift, the shift \( \sigma \), is a bijective map. For the bijective maps \( \sigma \), we write \( G^\sigma(x) \) the element of the partition \( \vee_{i \in G} \sigma^i \alpha_p \), which contains the point \( x \).
The set $G$ is a finite interval of $\mathbb{Z}$. It follows that each set $G \alpha^\sigma_p(x)$ are cylinders set, while the set $\{n\} \alpha^F_p(x)$ are finite union of cylinders set.

Let $\mu$ be a shift ergodic measure. Given a real $0 < \delta < 1$ and two strictly increasing sequence maps $g^+_n$ and $g^-_n$ (from $\mathbb{A}^D$ to $\mathbb{N}$ with $\lim_{n \to \infty} g^+_n(x) + g^-_n(x) = +\infty$ for $\mu$ almost all $x$), set $G_n(x) = \{-g^-_n(x), \ldots, -1, 0, \ldots g^+_n(x)\}$, $|G_n(x)| = g^-_n(x) + g^+_n(x) + 1$ and define

$$\eta_{n, \delta} = \max \left\{ \epsilon \in \mathbb{R} : \exists S \subset X \mu(S) \geq 1 - \delta \text{ and } S \text{ satisfies} \right. \left. \forall x \in S, i \geq n : \left| -\log \mu(G_n(x) \alpha^\sigma_p(x)) - \log \mu(x) \right| < \epsilon \right\}$$

and for any $n \in \mathbb{N}$

$$X^G_{n, \delta, p} = \left\{ x \in \mathbb{A}^D : \left| -\log \mu(G_n(x) \alpha^\sigma_p(x)) - \log \mu(x) \right| \leq \eta_{n, \delta} \right\}.$$ 

Remark that from the Shannon-Breiman-McMillan Theorem, for all $0 \leq \delta \leq 1$ we obtain that $\lim_{n \to \infty} \eta_{n, \delta} = 0$ and $\lim_{n \to \infty} \mu(X^G_{n, \delta, p}) = 1$ for all sequence of application $G$.

**Definition 4** Let define

$$\langle T^G_{n, p}(x) \rangle = \left\{ w \in A^{G_n(x)} \text{ such that } \exists y \in \{n\} \alpha^\sigma_p(x) \right\}$$

and

$$T^G_{n, p}(x) = \{ y \in \mathbb{A}^D, \text{ such that } y(-p, p) \in \langle T^G_n(x) \rangle \}.$$ 

Let $T^G_{n, \delta, p}(x)$ be the finite union of cylinders $[u]_{|u|} \in \mathbb{A}^D$, with $u \in A^{G_n(x)}$ such that

$$T^G_{n, \delta, p}(x) = T^G_{n, p}(x) \cap X^G_{n, \delta, p}$$

and

$$\langle T^G_{n, \delta, p}(x) \rangle = \left\{ w = y(-g^-_n(x), g^+_n(x)) \text{ with } y \in T^G_{n, \delta, p}(x) \right\}.$$ 

**Remark 1** By taking the intersection with $X^G_{n, \delta}$, we keep the ”good” cylinders which have almost all the same weight. Recall (see section 3.1) that for the uniform measure $\mu_u$ we have

$$\mu_u(\{n\} \alpha^F_p(x)) = \#T^r_{n, p}(x) \times \mu_u(\{n\} \alpha^\sigma_p(x)) = \#T^r_{n, p}(x) \times A^{-2r_{n-1}}.$$ 

In this case we can use $T^G_{n, \delta, p}(x)$ instead of $T^G_{n, \delta, p}(x)$. 

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Let $I^{+,*}_n(x) = \max_{y \in \alpha^+_p(x)} I^+_n(y)$ and $I^{-,*}_n(x) = \max_{y \in \alpha^-_p(x)} I^-_n(y)$.

Recall that $g^+_n$ and $g^-_n$ are two increasing sequences of maps from $X$ to $\mathbb{N}$ whom limit is $+\infty$. These maps are the base of the definition of the sequence of applications $G$.

**Lemma 1** If $g^+_n \geq I^{+,*}_n$ and $g^-_n \geq I^{-,*}_n$ then

$$\mu^{(n)}(x) = \sum_{y \in \alpha^+_p(x)} \mu^{(G_n(x) \alpha^+_p(y))}.$$  

In this case we write that $G_n$ satify the condition (*)

**Proof**

From [7] (Proposition 5.1) one has

$$(I^{+,*}_n(x)+p, I^{-,*}_n(x)+p) \alpha^+_p(x) \subset \{n\} \alpha^+_p(x).$$

Since $g^+_n \geq I^{+,*}_n$ and $g^-_n \geq I^{-,*}_n$, then each cylinder set $G_n(x) \alpha^+_p(y)$ is a subset of $\{n\} \alpha^+_p(x)$ when $y \in \{n\} \alpha^+_p(x)$.

Therefore $\{G_n(x) \alpha^+_p(y)\}$ with $y \in \{n\} \alpha^+_p(x)$ is a partition of $\{n\} \alpha^+_p(x)$.

More generally, remark that $\mu^{(n)}(x) \alpha^+_p(x) \leq \sum_{y \in \alpha^+_p(x)} \mu^{(G_n(x) \alpha^+_p(y))}$ and $I^{+,*}_n(x) \leq n$ for all $x \in A^\mathbb{Z}$.

**Lemma 2** If $\mu$ is a shift-ergodic measure, for $\mu$-almost all points $x$ one has

$$\lim_{n \to \infty} \frac{1}{n} \log \mu^{(n)}(x) \leq \limsup_{n \to \infty} \frac{1}{n} \log \left( \mu^{(G_n(x) \alpha^+_p(x))} \times \#(T^G_{n,\delta,p}(x)) \right).$$

If $G_n$ verify (*) the inequality becomes an equality and we obtain the convergence with respect to the variables $n$ an $\delta$.

**Proof**

For all point $x$, we have $\mu^{(n)}(x) \leq \sum_{y \in \alpha^+_p(x)} \mu^{(G_n(x) \alpha^+_p(y))}$ and from Lemma 1, the inequality becomes an equality if condition (*) hold. Hence, it remains to prove that for almost all points $x$ we have

$$\lim_{\delta \to 0} \lim_{n \to \infty} \left[ -\frac{1}{n} \log \left( \mu^{(G_n(x) \alpha^+_p(x))} \times \#(T^G_{n,\delta,p}(x)) \right) \right] = 0.$$
We will show that for any $0 < \delta' < 1$, there exists a set $S \subset A^\mathbb{Z}$ of measure $1 - \delta$ such that if $x \in S$, the sequence
\[
\left( \frac{1}{n} \log \left( \mu(G_n(x) \alpha_p^\sigma(y)) \times \#(T_n(x)) \right) - \frac{1}{n} \log \left( \sum_{y \in A^\mathbb{Z}} \mu(G_n(x) \alpha_p^\sigma(y)) \right) \right)
\]
converge to $0$ when $n$ goes to infinity. Then we claim that if $x$ belong to a set of measure $1 - \delta'$ ($0 < \delta'' < 1$) we obtain
\[
\lim_{n \to \infty} \frac{1}{n} \left( \log \sum_{y \in (n) \alpha_p^\sigma(x)} \mu(G_n(x) \alpha_p^\sigma(y)) - \log \sum_{y \in T_n(x)} \mu(G_n(x) \alpha_p^\sigma(y)) \right) = 0. \tag{2}
\]
Let’s prove this claim. Fix $k > 1$ and for all integer $n > 0$, denote by $Y_{n,\delta}$ the set of points $x$ such that
\[
\frac{\sum_{y \in (n) \alpha_p^\sigma(x)} \mu(G_n(x) \alpha_p^\sigma(y))}{\sum_{y \in T_n(x)} \mu(G_n(x) \alpha_p^\sigma(y))} \leq k.
\]
It is clear that all points in $Y_\delta = \lim_{n \to \infty} \cap_{i=0}^n \cup_{j=i}^\infty Y_{i,j,\delta}$ verify equality (2). Hence, in order to prove the claim, we need to show that for all $n \in \mathbb{N}$ we have $\mu(Y_{n,\delta}) \leq \delta''$ where $\complement Y_{n,\delta}$ is the complementary set of $Y_{n,\delta}$.

The following complete the proof of the claim: Since
\[
\complement Y_{n,\delta} \subset \left( G_n(x) \alpha_p^\sigma(x) - T_n(x), \ x \in \complement Y_{n,\delta} \right) \subset \left( A^\mathbb{Z} - X_{n,\delta}, x \in \complement Y_{n,\delta} \right),
\]
then
\[
\frac{k}{k+1} \mu(G_n(x) \alpha_p^\sigma(x), x \in \complement Y_{n,\delta}) \leq \mu(A^\mathbb{Z} - X_{n,\delta}),
\]
which implies that
\[
\frac{k}{k+1} \mu(\complement Y_{n,\delta}) \leq (A^\mathbb{Z} - X_{n,\delta}) \text{ and } \mu(\complement Y_{n,\delta}) \leq \frac{k+1}{k} \delta := \delta''.
\]
Remark that if $\mu(Y_{n,\delta}) \leq 1 - \delta''$ for all integer $n$, then $\mu(Y_\delta) \leq 1 - \delta''$.

Next, it remains to prove that for all fixed $0 < \delta < 1$ and point $x$ in $X_{\delta,p} = \lim_{n \to \infty} \cap_{i=0}^n \cup_{j=i}^\infty X_{i,j,\delta}$, verify
\[
\lim_{n \to \infty} \frac{1}{n} \left( \log \left( \sum_{y \in T_n(x)} \mu(G_n(x) \alpha_p^\sigma(y)) \right) - \log \left( \#(T_n(x)) \times \mu(G_n(x) \alpha_p^\sigma(x)) \right) \right) = 0. \tag{3}
\]
Remark that if \( x \in X^G_{n,\delta,p} \) and \( y \in T^G_{n,\delta,p}(x) \) one has
\[
\left| \frac{\mu^{G_n(x)\alpha_p^\sigma(y)}}{\mu^{G_n(x)\alpha_p^\sigma(x)}} \right| \leq e^{n\eta_{n,\delta}}.
\]
It follows that
\[
\frac{1}{n} \left( \log \left( \sum_{y \in T^G_{n,\delta}(x)} \mu^{G_n(x)\alpha_p^\sigma(y)} \right) - \log \left( \#(T^G_{n,\delta,p}(x)) \times \mu^{G_n(x)\alpha_p^\sigma(x)} \right) \right) = \eta_{n,\delta}.
\]
Since for \( 0 < \delta < 1 \) the Shannon-Breiman-McMillan Theorem gives that
\[
\lim_{n \to \infty} \eta_{n,\delta} = 0,
\]
so we have proved the equation (3).

Arguing that equation (1) is true for all \( x \in X^G_{\delta} \cap Y_{\delta} =: S \) and taking into consideration that \( \mu(X^G_{\delta} \cap Y_{\delta}) \geq 1 - \delta - \frac{k+1}{k} \delta =: \delta' \), we can conclude letting \( \delta \to 0 \).

From the Shannon-Breiman-McMillan Theorem, \( \left( \frac{1}{n} \log \mu^{(n)\alpha_p^F(x)} \right)_{n \in \mathbb{N}} \) converge almost everywhere when \( \mu \) is an invariant measure, this implies that when \( G_n \) verify condition (*) we obtain
\[
\lim_{n \to \infty} \frac{1}{n} \log \mu^{(n)\alpha_p^F(x)} = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \left( \mu^{G_n(x)\alpha_p^\sigma(x)} \times \#(T^G_{n,\delta,p}(x)) \right).
\]
\end{proof}

**Definition 5** For each integers \( p > 0 \), \( n > 0 \), real \( 0 < \delta < 1 \) and double sequences of functions \( G = ((g_n^-(x)), (g_n^+(x))) \) from \( A^Z \) to \( \mathbb{N} \) we write
\[
M_{n,p,\delta}(G) = \int_{A^Z} 1 - \frac{\log \#(T^G_{n,\delta,p}(x))}{\log \mu^{G_n(x)\alpha_p^\sigma(x)}} d\mu(x)
\]
and
\[
M_n(G) = \sup_p \lim_{\delta \to 0} \limsup_{n \to \infty} M_{n,p,\delta}(G).
\]
We call \( M_n(G) \) the average flow of information at speed \( G \).

We denote by \( I^{+,*} \) the value \( \sup_{A^Z} \limsup_{n \to \infty} \frac{I_n^{+,*}(x)}{n} \) and by \( I^{-,*} \) the value \( \sup_{A^Z} \limsup_{n \to \infty} \frac{I_n^{-,*}(x)}{n} \).

**Remark 2** In the following theorem \( M_n(G) = \sup_p \lim_{\delta \to 0} \limsup_{n \to \infty} M_{n,p,\delta}(G) \) when the velocity \( v \) is greater than \( (I^{+,*}, I^{-,*}) \). Remark that \( I^{+,*}, I^{-,*} \leq r \) where \( r \) is the radius of the cellular automaton \( F \) we consider.
Theorem 2 If $\mu$ is a $\sigma$-ergodic and $F$-invariant measure and $v = ([v^{-n}]), ([v^{+n}])$ a double sequence of integers we have the following properties:

(i) $h_\mu(F) = h_\mu(\sigma) \times (v^+ + v^-) \times M_\mu(v)$, if $v^+ \geq I^+, v^- \geq I^-$.  
(ii) $h_\mu(F) \geq h_\mu(\sigma) \times (v^+ + v^-) \times M_\mu(v)$.
(iii) $h_\mu(F) = h_\mu(\sigma) \times M$ where  

$$M = \sup_p \int_{A^\delta} \lim_{n \to \infty} \lim_{\delta \to 0} M_{n,\delta,p}(I_i^+(x), I_i^-(x)) \times \frac{I_n^\pm(x)}{n} \, d\mu(x).$$

and $I_n^\pm(x) = I_n^+(x) + I_n^-(x)$.

Proof

Applying the Shannon-McMillan-Breiman Theorem (probabilistic version) to $F$ we obtain 

$$h_\mu(F, \alpha_p) = \int \lim_{n \to \infty} \frac{1}{n} \log \mu^{(\{n\})} \alpha_p^F(x) \, d\mu(x)$$

In order to simplify the first part of the proof we suppose condition (*) (which is the case in part (i) and (iii)). In the more general case, we have limits superior instead of limits and inequalities instead of equalities.

From Lemma 2, for almost all $x$ we have

$$\lim_{n \to \infty} \frac{1}{n} \log \mu^{(\{n\})} \alpha_p^F(x) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \left( \mu^{(G_n(x))} \alpha_p^\sigma(x) \times \log(\# \langle T^{G_n(x)} \rangle) \right)$$

$$= \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \left( \log \mu^{(G_n(x))} \alpha_p^\sigma(x) + \log(\# \langle T^{G_n(x)} \rangle) \right).$$

We can rewrite the inequality as

$$\lim_{n \to \infty} -\frac{1}{n} \log \mu^{(\{n\})} \alpha_p^F(x) =$$

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{|G_n(x)| - \log \mu^{(G_n(x))} \alpha_p^\sigma(x)}{|G_n(x)| n} - \frac{\log(\# \langle T^{G_n(x)} \rangle(x))}{n}$$

and we obtain that

$$\lim_{n \to \infty} \frac{1}{n} \log \mu^{(\{n\})} \alpha_p^F(x) =$$

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{|G_n(x)|}{n} \times -\frac{\log \mu^{(G_n(x))} \alpha_p^\sigma(x)}{|G_n(x)|} \times \left( 1 - \frac{\log(\# \langle T^{G_n(x)} \rangle(x))}{-\log \mu^{(G_n(x))} \alpha_p^\sigma(x)} \right).$$
Hence from the dominated convergence theorem one has

\[
h_\mu(F, \alpha_p) = \\
\lim_{\delta \to 0} \lim_{n \to \infty} \int \frac{-\log \mu^{(G_n(x))\alpha_p^\sigma(x)}}{|G_n(x)|} \times \frac{|G_n(x)|}{n} \times \left(1 - \frac{\log \left(\langle T^{G_n(x)}_{n,\delta,p}(x) \rangle\right)}{-\log \mu^{G_n(x)\alpha_p^\sigma(x)}}\right) d\mu(x)
\]

Using the extended version of Shannon-Breiman-McMillan Theorem for \(\sigma\) (see [5]), we obtain that \(h_\mu(F, \alpha_p)\)

\[
= h_\mu(\sigma, \alpha_p) \times \lim_{\delta \to 0} \lim_{n \to \infty} \int \frac{|G_n(x)|}{n} \times \left(1 - \frac{\log \left(\langle T^{G_n(x)}_{n,\delta,p}(x) \rangle\right)}{-\log \mu^{G_n(x)\alpha_p^\sigma(x)}}\right) d\mu(x).
\]

Proof of (i)

If \(G(n) = ([v^-n], [v^+n])\) and \(v^+ \geq I^+:*, v^- \geq I^-:*, \) then the equality in Lemma 2 becomes an equality, the limsup in \(n\) is a limit and we can write that

\[
h_\mu(F, \alpha_p) = h_\mu(\sigma, \alpha_p) \times (v^+ + v^-) \times \lim_{\delta \to 0} \lim_{n \to \infty} \int 1 - \frac{\log \left(\langle T^{G_n(x)}_{n,\delta,p}(x) \rangle\right)}{-\log \mu^{G_n(x)\alpha_p^\sigma(x)}} d\mu(x).
\]

and finally

\[
h_\mu(F) = h_\mu(\sigma) \times (v^+ + v^-) \times M_\mu(v)
\]

where

\[
M_\mu(v) = \sup_{\alpha_p} \lim_{\delta \to 0} \lim_{n \to \infty} \int 1 - \frac{\log \#\langle T^{G_n(x)}_{n,\delta,p}(x) \rangle}{-\log \mu^{G_n(x)\alpha_p^\sigma(x)}} d\mu(x).
\]

Remark that the Shannon-Breiman-McMillan theorem implies that the density flow can be defined as a double limit in that case.

Proof of (ii)

Same proof but using Lemma 2 without condition (*). Remark that, in that case the density flow is defined thanks to limits superior instead of simple limits.

Proof of (iii)
From [7] if \( F \) has equicontinuous points then \( \frac{I_n^+(x)}{n} \) and \( \frac{I_n^-(x)}{n} \) are almost everywhere bounded and \( h_\mu(F) = 0 \).

When \( F \) is sensitive (no equicontinuous points) then for all \( x \) we have \( \lim_{n \to \infty} \frac{I_n^+(x) + I_n^-(x)}{n} = +\infty \). Clearly we have the same properties for \( I_n^{+,*}(x) \) and \( I_n^{-,*}(x) \). Using Lemma 2 with condition (*) we establish that

\[
h_\mu(F, \alpha_p) = h_\mu(\sigma, \alpha_p) \times \lim_{\delta \to 0} \lim_{n \to \infty} \int \frac{I_n^{+,*}(x) + I_n^{-,*}(x)}{n} \times \left( 1 - \frac{\log(\#(T^{n,s}_\mu(x)))}{-\log \mu(T^n_{\mu}(x))} \right) d\mu(x)
\]

where \( I_n^*(x) = (I_n^{+,*}(x), I_n^{-,*}(x)) \).

\( \square \)

**Questions 1**

Obviously for all \( x \), one has \( \lim_{n \to \infty} \frac{I_n^{+,*}(x)}{n} \) is greater than \( \lim_{n \to \infty} \frac{I_n^+(x)}{n} \). What are the other relations between these two functions.

**Remark 3**

- Since \( \lim_{n \to \infty} \frac{I_n^{+,*}(x)}{n} \) could be equal to zero almost everywhere, the part (iii) of Theorem 2 gives sens to the study of the density flow for sublinear speed.

- If the constants \( I^+_\mu \) and \( I^-_\mu \) represent the average speed of the faster perturbations, in the point of view of the density flow, the ratio \( \frac{h_\mu(F)}{h_\mu(\sigma)} = (v^+ + v^-)M_\mu(v) \) gives the sum of the average over all the points of the averages perturbations’ speeds when \( v^+ \geq r \).

### 3.3 Some Examples

**Example 1**

We use the same automaton \( F_e \) than in the subsection 3.1.2 (based on the shifts \( \sigma_1 \) on \( X_1 = \{0, 1\}^\mathbb{Z} \) and \( \sigma_2 \) on \( X_2 = \{0, 1\}^\mathbb{Z} \) but we consider the non uniform Bernoulli measure \( \mu_B = \mu_B(1) \times \mu_B(2) = B(\frac{1}{3}, \frac{2}{3}) \times B(\frac{1}{3}, \frac{2}{3}) \). The entropy on the two shifts \( h_{\mu_B}(\sigma_1) = h_{\mu_B}(\sigma_2) = \frac{1}{3} \log(3) + \frac{2}{3} \log(\frac{2}{3}) \) and \( h_\mu(\sigma) = h_\mu(\sigma_1) + h_\mu(\sigma_2) = 2(\frac{1}{3} \log(3) + \frac{2}{3} \log(\frac{3}{2})) \). Considering a set of words \( \langle T^{rn}_{n,p}(x) \rangle \) instead of set of cylinders \( T^{rn}_{n,p}(x) \), and using the same arguments than in section 3.1.2, we obtain that \( \#(T^{rn}_{n,p}(x)) = 4^{rn+2p+1} \times 2^{(r-1)n} = 2^{3(r-1)n} \).
For $\mu_B$ almost all the point $x$, the number $T_{n,\delta,p}^n(x)$ of “good cylinders” (in $X_{n,\delta,p}$) among the $2(3r-1)n = \#(T_{n,p}^n(x))$ possible words is approximatively (when $n$ large enough): $\#(T_{n,\delta,p}^n) \approx e^{h_\mu(\sigma)(3r-1)n}$ (a consequence of Shannon-Breiman-McMillan Theorem, see [9] page 95) and for almost all $x$, $-\log \mu(\tau\sigma_p^\alpha(x)) \approx e^{4rn+4p+2}h_\mu(\sigma_1)$.

Therefore we get,

$$M_\mu(r) = \lim_{\delta \to 0} \lim_{n \to \infty} 1 - \frac{\log \#T_{n,\delta,p}^n(x)}{-\log \mu(\tau\sigma_p^\alpha(x))},$$

$$M_\mu(r) = 1 - \frac{3r - 1}{4r} = \frac{r + 1}{4r}.$$

Finally we obtain

$$h_{\mu_B}(F) = M_\mu(r) \times h_{\mu_B}(\sigma) \times 2r = \frac{r + 1}{4r} \times 2r \times 2\left(\frac{1}{3}\log(3) + \frac{2}{3}\log\left(\frac{3}{2}\right)\right),$$

$$= (r + 1)\left(\frac{1}{3}\log(3) + \frac{2}{3}\log\left(\frac{3}{2}\right)\right) = h_{\mu_B}(\sigma_1) + h_{\mu_B}(\sigma_2^r).$$

**Examples 2**

Let $X_2$ be a Sturmian minimal subset of $\{0,1\}^\mathbb{Z}$. Let $\mu_2^s$ be the unique ergodic invariant measure on $(X_2,\sigma_2)$ where $\sigma_2$ is the shift on $X_2$. We call $\mu_1$, the uniform measure on $X_1 = \{0,1\}^\mathbb{Z}$ and we denote by $x = x^1 \times x^2$ any point in $X = X_1 \times X_2$. Consider the cellular automaton $F = Id_1 \times \sigma_2^r$ where $Id_1$ is the identity map on $X_1$, $r$ any positive integer and denote by $\mu$ the measure $\mu_1 \times \mu_2^s$ defined on $X$. Using well known results, we have $h_{\mu_2^s}(\sigma_2) = 0$, $h_{\mu_1}(Id_1) = 0$ and consequently $h_\mu(F) = 0$. From theorem 2, we have $M_\mu(r) = 0$. Clearly the Lyapunov exponents $I^+_\mu + I^-_\mu = I^+_\mu = r$, so in this case there exist perturbations moving at a positive speed but the average flow is equal to 0.

If we modify slightly the last example and we replace the identity map on $X_1$ by $r$ iterations of the shift $\sigma_1$ on $X_1$ and the shift $\sigma_2^r$ by the identity, we obtain a couple (cellular automaton $F_2 = \sigma_1^r \times Id_2$, measure $\mu$) such that the inequality of Theorem 1 becomes an equality: $h_\mu(F_2) = h_\mu(\sigma) \times (I^+_\mu + I^-_\mu) = r \log(2)$ and in this case $M_\mu(r,0) = 1$ (the maximum value of $M_\mu(v)$).
3.4 The density flow and some subclasses of cellular automata

Clearly, in some cases like the shifts or more generally bipermutative cellular automata (see \[1\]), all the perturbations move at speed $r$ which implies that $M_\mu(r) = 1$ for all shift ergodic and $F$-invariant measure $\mu$. Sometimes, like in the last example $M_\mu(r) = 1$ and almost all perturbation move at speed $r$.

The next Proposition shows that there exists a kind of “minimum speed” for all the perturbations in the case of the positively expansive CA.

**Proposition 1** If $F$ is a positively expansive CA and $\mu$ a shift ergodic and $F$-invariant measure, then there exist a real $s > 0$ such that $M_\mu(s) = 1$

**Proof**

From \[2\](Proposition 2), all positively expansive CA has positive pointwise Lyapunov exponents for all point in $A^Z$. More precisely there exist a real $\eta > 0$ such that $\liminf_{n \to \infty} \frac{I^+(x)}{n} > \eta$ and $\liminf_{n \to \infty} \frac{I^-(x)}{n} > \eta$ for all $x \in A^Z$.

If the velocity of the perturbations is $(\eta, \eta)$, there would exist an integer $N > 0$ such that for all $n \geq N$ we have $\frac{I^+(x)}{n} + \frac{I^-(x)}{n} > 2\eta$.

In this case for all integer $n \geq N$, $p > 0$, real $0 < \delta < 1$ and $x \in A^Z$, we obtain $\#(T_{n,\delta,p}^{(\eta,\eta)}(x)) = 1$ which implies that $\log(\#T_{n,\delta,p}^{(\eta,\eta)}(x)) = 0$ and consequently $M_\mu((\eta, \eta)) = 1$.

$\square$

**Questions 2** From Theorem 2, the function $M_\mu(s)$ from $[r, +\infty) \to [0,1]$ is clearly continuous and decreasing but what happens in the interval $[0,1]$?

Since $\mu$-expansiveness property is a measurable equivalent to positive expansiveness, we can wonder what values can take the density flow in that case. We will see that there is no such strong relation with the density flow but for the complementary class of CA with $\mu$-equicontinuous points, the density flow is always equal to zero.

From \[3\] the existence of equicontinuous points is equivalent to the existence of blocking words that stop completely the propagation of the perturbations. From \[8\], there exist an example of couple (cellular automaton, invariant and shift ergodic measure $\mu$) without equicontinuous points such that there exists $\mu$-equicontinuous points for the cellular automaton action
restricted to the topological support of $\mu$. This example shows clearly that the dynamic of CA with $\mu$-equicontinuous points is clearly more general than those with equicontinuous points when $\mu$ is an invariant measure. The next Proposition shows that the density of flows of the perturbations that can move or not move to infinity (strictly $\mu$-equicontinuous or equicontinuous case) is equal to zero taking in consideration a linear or a sublinear speed.

**Proposition 2** If $F$ is a $\mu$-equicontinuous CA, then for all couple of sequence $G_n = (g_n^+, g_n^-)$ such that $\lim_{n \to \infty} g_n^+ + g_n^- = +\infty$ we have $M_\mu(G) = 0$.

**Proof**

Suppose there exist a couple of sequence $G_n = (g_n^+, g_n^-)$ such that $M_\mu(G) > 0$ with $\lim_{n \to \infty} g_n^+ + g_n^- = +\infty$. It follows that there if we fixe $\delta$ and $p$, there exist a set $S$ of positive measure and a real $0 < \alpha < 1$ such that for all point $x \in S$ we have

$$\lim_{n \to \infty} \log \# \langle T G_{i(n)}^{G, \delta, p}(x) \rangle < \alpha \lim_{n \to \infty} \left[ -\log \mu \left( G_{i(n)}^{G, \sigma, p}(x) \right) \right],$$

where $i(n)$ is a subsequence that allows the convergence.

Then, for $n$ large enough, there exist $\alpha'$ such that

$$\log \# \langle T G_{i(n)}^{G, \delta, p}(x) \rangle < \alpha \left[ -\log \mu \left( G_{i(n)}^{G, \sigma, p}(x) \right) \right].$$

Using the same argument as in the proof of Lemma 2, we obtain that for $n$ large enough, there exist a subset of $S$: $S'$ of positive measure such that there exist a positive $k > 1$ such that

$$\frac{\sum_{y \in \{n\}^{G, \sigma, p}(x)} \mu \left( G_{n}^{G, \sigma, p}(y) \right)}{\sum_{y \in T_{G, n, \delta, p}(x)} \mu \left( G_{n}^{G, \sigma, p}(y) \right)} \leq k. $$

Since $\mu^{i(n)}(x) \leq \sum_{y \in \{n\}^{G, \sigma, p}(x)} \mu \left( G_{n}^{G, \sigma, p}(y) \right)$ and when $n$ is large enough $\mu \left( G_{n}^{G, \sigma, p}(x) \right) \approx e^{G_{n}^{G, \sigma, p}(x)}$, then $\mu^{i(n)}(x) \leq (k + 1) \sum_{y \in T_{G, n, \delta, p}(x)} \mu \left( G_{n}^{G, \sigma, p}(y) \right)$. It follows that

$$\mu^{i(n)}(x) \leq (k + 1) \# \langle T_{G, n, \delta, p}(x) \rangle \times \mu \left( G_{n}^{G, \sigma, p}(x) \right) \leq (k + 1) \mu \left( G_{n}^{G, \sigma, p}(x) \right)^{-\alpha} \mu \left( G_{n}^{G, \sigma, p}(x) \right).$$

This implies that for a set of positive measure $\lim_{n \to \infty} \mu^{i(n)}(x) = 0 = \mu(B_{p}(x))$. From $\mathbb{X}$ and $\mathbb{X}$, if there exists a $\mu$-equicontinuous point, then
there exist a set of measure 1 of these kind of points. Hence there is no
µ-equicontinuous point and we can conclude taking the reverse assumption.

Using Proposition 3 (only for linear speed) and Theorem 2, it follows that

**Corollary 1** If µ is a shift ergodic and F-invariant measure, with F a cell-
ular automaton with µ-equicontinuous points, then the measurable entropy
h_µ(F) = 0.

**Remark 4** If F is a µ-expansive CA then clearly for p, δ fixed and all x in
a set of positive measure we have

\[
\lim_{n \to \infty} \#T_{n,p,δ}^*(x) \times \mu(I_n^*(x) \alpha_p(x)) = 0
\]

where \( I^* = (I_{n,+}^*(x), I = (I_{n,-}^*(x)) \). Remark that this condition is weaker to
the property \( M_\mu(I^*) > 0 \).

**Questions 3** Is it possible that a dynamical system (cellular automaton F,
invariant measure µ), verify that \( M_\mu(I^*) > 0 \) and \( I_{\mu,+}^* + I_{\mu,-}^* = 0 \)? The sum
of the exponents \( I_{\mu,+}^* \) and \( I_{\mu,-}^* \) is defined by

\[
I_{\mu,+}^* + I_{\mu,-}^* = \limsup_{n \to \infty} \int \frac{I_{n,+}^*(x) + I_{n,-}^*(x)}{n} d\mu(x) = 0.
\]

From Proposition 2, this CA have to be µ-expansive. In \[ 2 \], there is an
example of µ-expansive CA with \( I_{\mu,+}^* + I_{\mu,-}^* = 0 \) but \( M_\mu(I^*) \) and \( I_{\mu,+}^* \) and \( I_{\mu,-}^* \)
are not known.

We finish with some more general questions.

**Questions 4** For \( h_\mu(F, \alpha_p) \), the bound given by Theorem 1 using the Lyapunov exponents \( I_{\mu,+}^* + I_{\mu,-}^* \) did not depends on the choice of the finite partition
\( \alpha_p \). In what cases the value of \( M_\mu(v) \) does not depends on a supremum value
over all the partition \( \alpha_p \)?

**Questions 5** For each cellular automaton F, it is possible to defined the
function \( M_\mu(v) \) when µ is shift ergodic and non F-invariant. What is the
meaning of \( M_\mu(v) \) in this case?

**Questions 6** What kind of results can be done in the topological case?
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