Idempotent Completion of $n$-Angulated Categories

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Abstract
Let $C$ be an $n$-angulated category. We prove that its idempotent completion $\tilde{C}$ admits a unique $n$-angulated structure such that the canonical functor $\iota: C \to \tilde{C}$ is $n$-angulated. Moreover, the functor $\iota$ induces an equivalence $\text{Hom}_{n\text{-ang}}(\tilde{C}, D) \cong \text{Hom}_{n\text{-ang}}(C, D)$ for any idempotent complete $n$-angulated category $D$, where $\text{Hom}_{n\text{-ang}}$ denotes the category of $n$-angulated functors.

Keywords $n$-Angulated category · Idempotent completion · Mapping cone

Mathematics Subject Classification 18E30

1 Introduction
Let $n$ be an integer greater than or equal to three. In 2013, Geiss, Keller and Oppermann introduced $n$-angulated categories to axiomatize certain $(n-2)$-cluster tilting subcategories of triangulated categories. By definition, an $n$-angulated category is an additive category $C$ equipped with an automorphism $\Sigma$ of $C$ and a class $\Theta$ of $n$-$\Sigma$-sequences that satisfy four axioms (see Definition 2.4). When $n = 3$, an $n$-angulated category is nothing but a triangulated category. Theorem 1 in [5] provides a standard construction of $n$-angulated categories. Other examples of $n$-angulated categories can be found in [2,6].

Our goal is to construct more examples of $n$-angulated categories. Balmer and Schlichting proved that when $n = 3$, the idempotent completion of a triangulated category admits a natural triangulated structure [[1], Theorem 1.5]. We want to extend the construction in [[1], Theorem 1.5] from 3 to $n$. We will show that the idempotent completion of an $n$-angulated category

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admits a unique n-angulated structure such that the inclusion is an n-angulated functor, which satisfies a universal property (see Theorem 3.1).

We remark that two properties used in the proof of [1], Theorem 1.5 fail in n-angulated categories when n > 3. First, an endomorphism (p, q, r) of a triangle \( X_\bullet \) satisfying \( p^2 = p \) and \( q^2 = q \) can be lifted to an endomorphism (p, q, s) of \( X_\bullet \) such that \( s^2 = s \). Second, each morphism fits into a triangle uniquely up to isomorphism.

The paper is organized as follows. In Sect. 2 we first recall some facts on idempotent completion of an additive category. We then define \( n \)-angulated categories and prove some properties on \( n \)-\( \Sigma \)-sequences. In Sect. 3 we state and prove our main theorem.

## 2 Definitions and Preliminaries

In this section, we first recall the construction of idempotent completion of an additive category and some related facts from [1,4]. We then define \( n \)-angulated categories and prove several properties on \( n \)-\( \Sigma \)-sequences, which will be used in the proof of Theorem 3.1.

### 2.1 Idempotent Completion of Additive Categories

An additive category \( \mathcal{C} \) is said to be idempotent complete if for each object \( A \) in \( \mathcal{C} \) and for each idempotent \( e \) : \( A \to A \), we have \( A = \text{Im}(e) \oplus \text{Ker}(e) \).

Let \( \mathcal{C} \) be an additive category. The idempotent completion of \( \mathcal{C} \) is a category \( \widetilde{\mathcal{C}} \) defined as follows. Objects of \( \widetilde{\mathcal{C}} \) are pairs (\( A, e \)), where \( A \) is an object in \( \mathcal{C} \) and \( e : A \to A \) is an idempotent. A morphism in \( \widetilde{\mathcal{C}} \) from (\( A, e \)) to (\( B, f \)) is in the form of \( fpe : A \to B \) for some morphism \( p : A \to B \) in \( \mathcal{C} \).

Assume that \( \mathcal{C}, \mathcal{D} \) and \( \mathcal{E} \) are additive categories. An additive functor \( F : \mathcal{C} \to \mathcal{D} \) yields an additive functor \( \widetilde{F} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{D}} \), by setting \( \widetilde{F}(A, e) = (FA, Fe) \) and \( \widetilde{F}(fpe) = F(f)F(p)F(e) \). Suppose \( G : \mathcal{D} \to \mathcal{E} \) is another additive functor, then we have \( G\widetilde{F} = \widetilde{G}F \).

Given two additive functors \( F, H : \mathcal{C} \to \mathcal{D} \), a natural transformation \( \alpha : F \to H \) yields a unique natural transformation \( \widetilde{\alpha} : \widetilde{F} \to \widetilde{H} \) with \( \widetilde{\alpha}(A, e) = H(e)\alpha_A F(e) \).

The assignment \( A \mapsto (A, 1) \) defines a functor \( \iota : \mathcal{C} \to \widetilde{\mathcal{C}} \). We have \( \iota F = \widetilde{F} \iota \), that is, the following diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow{\iota} & & \downarrow{\iota} \\
\widetilde{\mathcal{C}} & \xrightarrow{\widetilde{F}} & \widetilde{\mathcal{D}}
\end{array}
\]

commutes. The following is well-known.

**Proposition 2.1** ([1], Proposition 1.3) The category \( \widetilde{\mathcal{C}} \) is an idempotent complete additive category and the functor \( \iota : \mathcal{C} \to \widetilde{\mathcal{C}} \) is additive and fully faithful. Moreover, the functor \( \iota \) induces an equivalence \( \text{Hom}_{\text{add}}(\widetilde{\mathcal{C}}, \mathcal{D}) \cong \text{Hom}_{\text{add}}(\mathcal{C}, \mathcal{D}) \) for any idempotent complete additive category \( \mathcal{D} \), where \( \text{Hom}_{\text{add}} \) denotes the category of additive functors.

**Remark 2.2** Since the functor \( \iota : \mathcal{C} \to \widetilde{\mathcal{C}} \) is fully faithful, we view \( \mathcal{C} \) as a full subcategory of \( \widetilde{\mathcal{C}} \). Thus for each object \( X \in \mathcal{C} \), there exists an object \( X' \in \widetilde{\mathcal{C}} \) such that \( X \oplus X' \in \mathcal{C} \). In fact, if \( X = (A, e) \), then we can take \( X' = (A, 1 - e) \).
2.2 $n$-Angulated Categories

Assume that $n$ is an integer greater than or equal to three. We recall the definition of $n$-angulated categories from [5]. Let $C$ be an additive category equipped with an automorphism $\Sigma : C \to C$. An $n$-$\Sigma$-sequence in $C$ is a sequence of morphisms

$$X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \ldots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1).$$

Its left rotation is the $n$-$\Sigma$-sequence

$$X_\bullet[1] = (X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \ldots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma f_1} \Sigma X_2).$$

An $n$-$\Sigma$-sequence $X_\bullet$ is exact if the induced sequence

$$\cdots \to \text{Hom}_C(-, X_1) \to \text{Hom}_C(-, X_2) \to \cdots \to \text{Hom}_C(-, X_n) \to \text{Hom}_C(-, \Sigma X_1) \to \cdots$$

is exact. A morphism of $n$-$\Sigma$-sequences is a sequence of morphisms $\varphi_\bullet = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ such that the following diagram

$$
\begin{array}{cccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \ldots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \cdots & & \downarrow \varphi_n & & \downarrow \Sigma \varphi_1 \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 
\end{array}
$$

commutes, where each row is an $n$-$\Sigma$-sequence. In this situation we say $\varphi_\bullet$ is a weak isomorphism if for some $1 \leq i \leq n$ both $\varphi_i$ and $\varphi_{i+1}$ (with $\varphi_{n+1} = \Sigma \varphi_1$) are isomorphisms. In particular, $\varphi_\bullet$ is an isomorphism if $\varphi_1, \varphi_2, \ldots, \varphi_n$ are all isomorphisms.

**Definition 2.3** ([5]) Let $C$ be an additive category, $\Sigma$ an automorphism of $C$ and $\Theta$ a collection of $n$-$\Sigma$-sequences. We call $(C, \Sigma, \Theta)$ a pre-$n$-angulated category and call the elements of $\Theta$ $n$-angles if $\Theta$ satisfies the following three axioms:

(N1) (a) $\Theta$ is closed under isomorphisms, direct sums and direct summands.
(b) For each object $X \in C$, the trivial sequence

$$X_1 \xrightarrow{1} X \xrightarrow{1} 0 \xrightarrow{\cdots} 0 \xrightarrow{1} \Sigma X$$

belongs to $\Theta$.
(c) For each morphism $f_1 : X_1 \to X_2$ in $C$, there exists an $n$-$\Sigma$-sequence in $\Theta$ whose first morphism is $f_1$.

(N2) An $n$-$\Sigma$-sequence belongs to $\Theta$ if and only if its left rotation belongs to $\Theta$.

(N3) Each commutative diagram

$$
\begin{array}{cccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \ldots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \cdots & & \downarrow \varphi_n & & \downarrow \Sigma \varphi_1 \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 
\end{array}
$$

with rows in $\Theta$ can be completed to a morphism of $n$-$\Sigma$-sequences.
Furthermore, if $\Theta$ satisfies the following axiom, then $(C, \Sigma, \Theta)$ is called an $n$-angulated category:

(N4) The morphisms $\varphi_3, \varphi_4, \ldots, \varphi_n$ in (N3) can be chosen such that the mapping cone

$$C(\varphi_*) = (X_2 \oplus Y_1 \xrightarrow{(-f_2 0 \varphi_2 g_1)} X_3 \oplus Y_2 \xrightarrow{(-f_3 0 \varphi_3 g_2)} \cdots \xrightarrow{(-f_n 0 \varphi_n g_{n-1})} \Sigma X_1 \oplus Y_n \xrightarrow{(-\Sigma f_1 0 \Sigma \varphi_1 g_n)} \Sigma X_2 \oplus \Sigma Y_1)$$

belongs to $\Theta$.

Let $(C, \Sigma, \Theta)$ and $(C', \Sigma', \Theta')$ be two $n$-angulated categories. An additive functor $F : C \to C'$ is said to be $n$-angulated (see [3]) if there exists a natural isomorphism $\alpha : F \Sigma \to \Sigma' F$ and $F$ preserves $n$-angles, that is, if

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

is an $n$-angle in $\Theta$, then

$$FX_1 \xrightarrow{Ff_1} FX_2 \xrightarrow{Ff_2} \cdots \xrightarrow{Ff_{n-1}} FX_n \xrightarrow{\alpha X_1 Ff_n} \Sigma' FX_1$$

is an $n$-angle in $\Theta'$.

2.3 Some Properties on $n$-$\Sigma$-Sequences

Let $C$ be an additive category, $\Sigma$ an automorphism of $C$ and $\Theta$ a class of $n$-$\Sigma$-sequences.

Lemma 2.4 Assume that $\Theta$ satisfies (N1)(b), (N2), (N3), and the following $n$-$\Sigma$-sequence

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

belongs to $\Theta$. Then the following hold.

1. $f_2 f_1 = 0$.
2. If $gf_1 = 0$ for a morphism $g : X_2 \to Y$, then there exists a morphism $h : X_3 \to Y$ such that $g = hf_2$.

Proof (1) By (N1)(b), (N2) and (N3), the following diagram

$$\begin{array}{c}
X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \cdots \xrightarrow{f_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma f_1} \Sigma X_2 \\
\downarrow f_2 \quad \quad \downarrow f_3 \quad \quad \downarrow f_4 \quad \quad \downarrow f_n \\
X_3 \xrightarrow{1} X_3 \xrightarrow{0} \cdots \xrightarrow{0} \Sigma X_3
\end{array}$$

can be completed to a morphism of $n$-$\Sigma$-sequences. It follows that $f_2 f_1 = 0$.

(2) Since $gf_1 = 0$, the following diagram

$$\begin{array}{c}
X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \\
\downarrow g \quad \quad \downarrow g \quad \quad \downarrow g \\
0 \xrightarrow{1} Y \xrightarrow{1} Y \cdots \xrightarrow{1} 0
\end{array}$$

can be completed to a morphism of $n$-$\Sigma$-sequences by (N1)(b), (N2) and (N3). Thus there exists a morphism $h : X_3 \to Y$ such that $g = hf_2$. \qed
Lemma 2.5 Assume that $C$ is idempotent complete, $\Theta$ satisfies (N1)(b), (N2), (N3), and

$$X_\bullet = (X_1 \xrightarrow{(f_1)} X_2 \oplus Y_2 \xrightarrow{(f_2, g_2)} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1) \in \Theta.$$  

(1) If $g_1 = 0$, then $g_2$ is a section and $X_\bullet \cong X'_\bullet \oplus Y'_\bullet$, where

$$X'_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X'_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1)$$

and

$$Y'_\bullet = (0 \to Y_2 \xrightarrow{1} Y_2 \to 0 \to \cdots \to 0 \to 0).$$

(2) If $g_2 = 0$, then $g_1$ is a retraction and $X_\bullet \cong X''_\bullet \oplus Y''_\bullet$, where

$$X''_\bullet = (X'_1 \xrightarrow{f_{11}} X_2 \xrightarrow{f_5} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X'_1)$$

and

$$Y''_\bullet = (Y_2 \xrightarrow{1} Y_2 \to 0 \to \cdots \to 0 \to \Sigma Y_2).$$

**Proof** We only prove (1), as (2) can be proved similarly. Since (0 1) $(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}) = g_1 = 0$, there exists a morphism $g_2 : X_3 \to Y_2$ such that $(0 1) = g_2(f_2 g_2)$ by Lemma 2.4 (2). Thus $g_2$ is a section. Since $(g_2 g_2)^2 = g_2 g_2$ and $C$ is idempotent complete, we can write $X_3 = X'_3 \oplus X''_3$ and

$$X_\bullet = (X_1 \xrightarrow{(f_1)} X_2 \oplus Y_2 \xrightarrow{(f_2, g_2)} X'_3 \oplus X''_3 \xrightarrow{(f_{31}, f_{32})} X_4 \xrightarrow{f_4} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1)$$

where $g_2$ is an isomorphism. It follows that $f_{32} = 0$ and $f_{22} f_1 = 0$ by Lemma 2.4 (1) and (N2). Since $(f_{22} 0) (\begin{pmatrix} f_1 \\ 0 \end{pmatrix}) = 0$, there exists a morphism $(a b) : X'_3 \oplus X''_3 \to X''_3$ such that $(f_{22} 0) = (a b) (\begin{pmatrix} f_{21} \\ f_{22} g_{22} \end{pmatrix})$ by Lemma 2.4 (2). Thus $b = 0$ and $f_{22} = a f_{21}$. So we have the following commutative diagram

which shows that $X_\bullet \cong X'_\bullet \oplus Y'_\bullet$.  

From now on to the end of this section, we assume that $(C, \Sigma, \Theta)$ is a pre-$n$-angulated category.

Lemma 2.6 Let the following

$$X_\bullet \quad X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

$$Y_\bullet \quad Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \xrightarrow{g_3} \cdots \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} \Sigma Y_1$$

$$Z_\bullet \quad Z_1 \xrightarrow{h_1} Z_2 \xrightarrow{h_2} Z_3 \xrightarrow{h_3} \cdots \xrightarrow{h_{n-1}} Z_n \xrightarrow{h_n} \Sigma Z_1$$

where

$$\begin{align*}
\varphi_1 &= \varphi_1 \\
\varphi_2 &= \varphi_2 \\
\varphi_3 &= \varphi_3 \\
\varphi_n &= \varphi_n \\
\Sigma \varphi_1 &= \Sigma \varphi_1 \\
\psi_1 &= \psi_1 \\
\psi_2 &= \psi_2 \\
\psi_3 &= \psi_3 \\
\psi_n &= \psi_n \\
\Sigma \psi_1 &= \Sigma \psi_1
\end{align*}$$

$\Theta$ Springer
be a commutative diagram with rows in $\Theta$. If $\varphi_\bullet$ is a weak isomorphism, then the mapping cone $C(\psi_\bullet \varphi_\bullet) \in \Theta$ if and only if the mapping cone $C(\psi_\bullet) \in \Theta$. Furthermore, if $\varphi_\bullet$ is an isomorphism, then $C(\psi_\bullet \varphi_\bullet)$ is isomorphic to $C(\psi_\bullet)$.

Proof Since $\varphi_\bullet$ is a weak isomorphism, the following commutative diagram

\[
\begin{array}{cccccccc}
X_2 \oplus Z_1 & \xrightarrow{(f_2, 0)} & X_3 \oplus Z_2 & \xrightarrow{(f_3, 0)} & \cdots & X_n \oplus Z_n & \xrightarrow{(f_n, 0)} & \Sigma X_2 \oplus \Sigma Z_1 \\
-\psi_2 h_1 & & -\psi_3 h_2 & & \cdots & -\psi_n h_{n-1} & & -\psi_n h_n \\
Y_2 \oplus Z_1 & \xrightarrow{(g_2, 0)} & Y_3 \oplus Z_2 & \xrightarrow{(g_3, 0)} & \cdots & Y_n \oplus Z_n & \xrightarrow{(g_n, 0)} & \Sigma Y_2 \oplus \Sigma Z_1
\end{array}
\]

implies that $C(\psi_\bullet \varphi_\bullet)$ is weakly isomorphic to $C(\psi_\bullet)$. Both $C(\psi_\bullet \varphi_\bullet)$ and $C(\psi_\bullet)$ are exact since the $n$-angles $X_\bullet$, $Y_\bullet$ and $Z_\bullet$ are exact. By $[5]$, Lemma 2.4, $C(\psi_\bullet \varphi_\bullet)$ is an $n$-angle if and only if so is $C(\psi_\bullet)$. The last assertion follows from the above commutative diagram. □

The following Lemma is a generalization of $[1]$, Lemma 1.16.

Lemma 2.7 Let

\[
\tilde{X}_\bullet = (X_1 \oplus Y_1 \xrightarrow{f_1} X_2 \oplus Y_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n \oplus Y_n \xrightarrow{f_n} \Sigma X_1 \oplus \Sigma Y_1)
\]

be an $n$-angle in $C$. If

\[
X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n \xrightarrow{f_n} \Sigma X_1)
\]

or

\[
Y_\bullet = (Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} \cdots \xrightarrow{g_n} Y_n \xrightarrow{g_n} \Sigma Y_1)
\]

is a contractible $n$-angle, then $\tilde{X}_\bullet$ is isomorphic to the direct sum of $X_\bullet$ with $Y_\bullet$.

Proof Without loss of generality, we assume that $X_\bullet$ is a contractible $n$-angle. By definition there exist morphisms $h_i : X_{i+1} \rightarrow X_i$ for $1 \leq i \leq n - 1$ and $h_n : \Sigma X_1 \rightarrow X_n$ such that $1_{X_1} = h_1 f_1 + \Sigma^{-1} (f_n h_n)$ and $1_{X_j} = h_j f_j + f_{j-1} h_{j-1}$ for $2 \leq j \leq n$. Lemma 2.4 (1) and (N2) implies that $\psi_{i+1} f_i + g_{i+1} \psi_i = 0$ for $1 \leq i \leq n - 1$ and $\Sigma \psi_1 : f_n + g_1 : \psi_n = 0$. We have the following commutative diagram.

\[
\begin{array}{cccccccc}
X_1 \oplus Y_1 & \xrightarrow{(f_1, 0)} & X_2 \oplus Y_2 & \xrightarrow{(f_2, 0)} & \cdots & X_n \oplus Y_n & \xrightarrow{(f_n, 0)} & \Sigma X_1 \oplus \Sigma Y_1 \\
\Sigma^{-1} (\psi_n h_n) & & \Sigma^{-1} (\psi_1 h_1) & & \cdots & \Sigma^{-1} (\psi_{n-1} h_{n-1}) & & \Sigma^{-1} (\psi_n h_n) \\
X_1 \oplus Y_1 & \xrightarrow{(f_1, 0)} & X_2 \oplus Y_2 & \xrightarrow{(f_2, 0)} & \cdots & X_n \oplus Y_n & \xrightarrow{(f_n, 0)} & \Sigma X_1 \oplus \Sigma Y_1
\end{array}
\]

So $\tilde{X}_\bullet \cong X_\bullet \oplus Y_\bullet$. □

Lemma 2.8 Let $\varphi_\bullet = \begin{pmatrix} \alpha_\bullet & \beta_\bullet \\ \gamma_\bullet & \delta_\bullet \end{pmatrix} : X_\bullet \oplus X'_\bullet \rightarrow Y_\bullet \oplus Y'_\bullet$ be a morphism of $n$-angles, where $X'_\bullet$ and $Y'_\bullet$ are contractible, then the following holds:

\[
C(\varphi_\bullet) \cong C(\alpha_\bullet) \oplus X'_\bullet [1] \oplus Y'_\bullet.
\]
Note that we have two canonical inclusions morphism such that

\[ p \] and two canonical projections

\[ \text{such that} \]

\[ \text{Let} \ C \] is n-angulated.

(3) The functor \( \tilde{\Sigma} \) of \( \tilde{n} \)-sequences in \( \tilde{\Sigma} \) induces an equivalence \( \text{Hom}_n(X_1, X_2) \sim \text{Hom}_n(X_2, X_1) \) for any idempotent complete n-angulated category \( D \), where \( \text{Hom}_n \) denotes the category of n-angulated functors.

Proof (1) We show that \( \tilde{\Theta} \) satisfies the axioms of n-angulated categories. Note that \( \tilde{\Theta} \) satisfies (N1)(a) and (N2) by definition.

(N1)(b) Let \( X \) be an object in \( \tilde{C} \). There exists an object \( X' \in \tilde{C} \) such that \( X \oplus X' \in C \). The trivial n-angle \( (X \oplus X') \rightarrow 0 \rightarrow \cdot \cdot \cdot \rightarrow 0 \rightarrow \Sigma(X \oplus X') \in \Theta \) implies that \( (X \rightarrow X \rightarrow \cdot \cdot \cdot \rightarrow X' \rightarrow 0) \rightarrow \Sigma(X) \in \Theta \).

(N3) Given a commutative diagram

\[
\begin{array}{ccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \widetilde{\Sigma}X_1 \\
\downarrow{\varphi_1} & & \downarrow{\varphi_2} & & & & \downarrow{\varphi_n} & \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \widetilde{\Sigma}Y_1 \\
\end{array}
\] (3.1)

with rows in \( \tilde{\Theta} \), there exist two objects \( X'_1 \) and \( Y'_1 \) such that \( X_1 \oplus X'_1, Y_1 \oplus Y'_1 \in \Theta \). Note that we have two canonical inclusions

\[ i_* : X_1 \rightarrow X_1 \oplus X'_1, j_* : Y_1 \rightarrow Y_1 \oplus Y'_1 \]

and two canonical projections

\[ p_* : X_1 \oplus X'_1 \rightarrow X_1, q_* : Y_1 \oplus Y'_1 \rightarrow Y_1 \]

such that \( p_* i_* = 1 \) and \( q_* j_* = 1 \). By (N3), the pair \((j_1 \varphi_1 p_1, j_2 \varphi_2 p_2)\) can be completed to a morphism \( \tilde{\varphi}_* : X_1 \oplus X'_1 \rightarrow Y_1 \oplus Y'_1 \) of n-angles. So \( q_* i_* : X_1 \rightarrow Y_1 \) is a morphism of \( n \)-\( \Sigma \)-sequences extending \( (\varphi_1, \varphi_2) \).

(N1)(c) For each morphism \( f_1 : X_1 \rightarrow X_2 \) in \( \tilde{C} \), we choose two objects \( X'_1, X'_2 \in \tilde{C} \) such that \( (f_1 0 0) : X_1 \oplus X'_1 \rightarrow X_2 \oplus X'_2 \) is a morphism in \( \tilde{C} \). Assume that

\[
\tilde{X}_2 = (X_1 \oplus X'_1) \xrightarrow{(f_1 0 0)} X_2 \oplus X'_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \rightarrow \Sigma(X_1 \oplus X'_1))
\]
is an $n$-angle in $\Theta$ by (N1)(c). Since $\tilde{C}$ is idempotent complete and $\tilde{X}_\bullet \in \tilde{\Theta}$, it follows from Lemma 2.5 that $\tilde{X}_\bullet \cong \tilde{X}_\bullet \oplus X_\bullet'$, where

$$\tilde{X}_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1)$$

and

$$X_\bullet' = (X_1' \xrightarrow{0} X_2' \xrightarrow{1} X_2' \xrightarrow{0} 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma X_1' \xrightarrow{1} \Sigma X_1').$$

So $\tilde{X}_\bullet$ belongs to $\tilde{\Theta}$ and the first morphism is $f_1$. Therefore, $(\tilde{C}, \tilde{\Sigma}, \tilde{\Theta})$ is a pre-$n$-angulated category.

(N4) Consider the commutative diagram (3.1) with rows in $\tilde{\Theta}$. For $i = 1, 2$, we choose $X_i'$, $Y_i' \in \tilde{C}$ such that $(f_i \ 0 \ 0) : X_1 \oplus X_1' \rightarrow X_2 \oplus X_2'$ and $(g_i \ 0 \ 0) : Y_1 \oplus Y_1' \rightarrow Y_2 \oplus Y_2'$ are morphisms in $\tilde{C}$. By the proof of (N1)(c), we assume that $\tilde{X}_\bullet$ and $\tilde{Y}_\bullet$ are $n$-angles in $\tilde{\Theta}$ such that the first morphisms are $f_1$ and $g_1$ respectively, moreover, $\tilde{X}_\bullet \oplus X_\bullet'$ and $\tilde{Y}_\bullet \oplus Y_\bullet'$ are $n$-angles in $\tilde{\Theta}$, where

$$X_\bullet' = (X_1' \xrightarrow{0} X_2' \xrightarrow{1} X_2' \xrightarrow{0} 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma X_1' \xrightarrow{1} \Sigma X_1')$$

and

$$Y_\bullet' = (Y_1' \xrightarrow{0} Y_2' \xrightarrow{1} Y_2' \xrightarrow{0} 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma Y_1' \xrightarrow{1} \Sigma Y_1').$$

By (N4), the pair $(f_i \ 0 \ 0), (g_i \ 0 \ 0)$ can be extended to a morphism of $n$-angles

$$\tilde{\varphi}_\bullet = (\tilde{\varphi}_\bullet \ \alpha_\bullet \ \beta_\bullet \ \gamma_\bullet) : \tilde{X}_\bullet \oplus X_\bullet' \rightarrow \tilde{Y}_\bullet \oplus Y_\bullet',$n

in $\tilde{\Theta}$ such that the mapping cone $C(\tilde{\varphi}_\bullet) \in \tilde{\Theta}$. By Lemma 2.8, we have $C(\tilde{\varphi}_\bullet) \cong C(\varphi_\bullet) \oplus X_\bullet'[1] \oplus Y_\bullet'\oplus Y_\bullet'$. We obtain $C(\varphi_\bullet) \in \tilde{\Theta}$ by definition.

By (N3), we have two weak isomorphisms $\phi_\bullet : \tilde{X}_\bullet \rightarrow \tilde{X}_\bullet$ and $\psi_\bullet : \tilde{Y}_\bullet \rightarrow Y_\bullet$, where $\phi_i = 1$ and $\psi_i = 1$ for $i = 1, 2$. Note that $\varphi_\bullet = \psi_\bullet \tilde{\varphi}_\bullet \phi_\bullet : \tilde{X}_\bullet \rightarrow Y_\bullet$ is a morphism of $n$-angles extending $(\varphi_1, \varphi_2)$, moreover, the mapping cone $C(\varphi_\bullet) \in \tilde{\Theta}$ by Lemma 2.6 and its dual. Thus (N4) holds. Consequently, $(\tilde{C}, \tilde{\Sigma}, \tilde{\Theta})$ is an $n$-angulated category.

(2) Note that the functor $i : C \rightarrow \tilde{C}$ is $n$-angulated. Suppose that $\tilde{\Theta}'$ is another $n$-angulation of $(\tilde{C}, \tilde{\Sigma})$ such that the functor $i$ is $n$-angulated. Since $\tilde{\Theta}'$ is closed under direct summands, we have $\tilde{\Theta} \subseteq \tilde{\Theta}'$. Now by [[5], Proposition 2.5 (c)], $\tilde{\Theta} = \tilde{\Theta}'$. This proves the uniqueness of the $n$-angulated structure.

(3) For an idempotent complete $n$-angulated category $\mathcal{D}$, the equivalence

$$\text{Hom}_{n\text{-ang}}(\tilde{C}, \mathcal{D}) \cong \text{Hom}_{n\text{-ang}}(C, \mathcal{D})$$

follows from Proposition 2.1 and the fact that an additive functor $F : \tilde{C} \rightarrow \mathcal{D}$ is $n$-angulated if and only if $Fi : C \rightarrow \mathcal{D}$ is $n$-angulated. 

$\square$

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