Introduction

Recently I. Macdonald defined a family of systems of orthogonal symmetric polynomials depending on two parameters $q, k$ which interpolate between Schur’s symmetric functions and certain spherical functions on $SL(n)$ over the real and p-adic fields [M]. These polynomials are labeled by dominant integral weights of $SL(n)$, and (as was shown by I. Macdonald) are uniquely defined by two conditions: 1) they are orthogonal with respect to a certain weight function, and 2) the matrix transforming them to Schur’s symmetric functions is strictly upper triangular with respect to the standard partial ordering on weights (“strictly” means that the diagonal entries of this matrix are equal to 1). Another definition of Macdonald’s polynomials is that they are (properly normalized) common eigenfunctions of a commutative set of $n$ self-adjoint partial difference operators $M_1, ..., M_n$ (Macdonald’s operators) in the space of symmetric polynomials.

In this paper we present a formula for Macdonald’s polynomials which arises from the representation theory of the quantum group $U_q(sl_n)$. This formula expresses Macdonald’s polynomials via (weighted) traces of intertwining operators between certain modules over $U_q(sl_n)$.

The paper is organized as follows. In Section 1, we define Macdonald’s inner product, orthogonal polynomials, and commuting difference operators, and compute the eigenvalues of these operators. In Section 2, we review some facts about representations of quantum groups that will be needed in the following sections. In Section 3 we introduce weighted traces of intertwiners (vector-valued characters) and prove an analogue of the Weyl orthogonality theorem for them. In Section 4 we formulate the main result – the explicit formula for Macdonald’s polynomials for positive integer values of $k$ – and give a complete proof of this formula. In Section 5, we generalize the result of Section 4 to the case of an arbitrary $k$. In Section 6, we construct Macdonald’s operators from the generators of the center of the quantum group $U_q(sl_n)$.
\( U_q(\mathfrak{sl}_n) \), and derive an explicit formula for generic (non-symmetric) eigenfunctions of Macdonald’s operators using this construction.

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\section{Macdonald’s polynomials}

Here we give the definition and main properties of Macdonald’s polynomials for the root system of type \( A_{n-1} \), following [M].

Let us fix \( n \in \mathbb{Z}_+ \). A sequence \( \lambda = (\lambda_1 \ldots \lambda_n) \in (\mathbb{Z}_+)^n \) is called a partition if \( \lambda_i \geq \lambda_{i+1} \). We define a partial order on partitions: \( \lambda > \mu \) if \( \sum \lambda_i = \sum \mu_i \) and \( \lambda_1 = \mu_1, \ldots, \lambda_k = \mu_k, \lambda_{k+1} > \mu_{k+1} \) for some \( k < n \).

Let us consider polynomials of \( n \) variables \( x_1 \ldots x_n \): \( \mathcal{A} = \mathbb{C}[x_1, \ldots, x_n] \). For any \( \lambda \in \mathbb{Z}^n \), let \( x^\lambda = x_1^{\lambda_1} \ldots x_n^{\lambda_n} \). We have an obvious action of the Weyl group \( S_n \) on \( \mathcal{A} \). We can take a basis of \( \mathcal{A}^{S_n} \) formed by the orbitsums

\begin{equation}
(1.1) \quad m_{\lambda} = \sum_{\mu \in S_n \lambda} x^\mu,
\end{equation}

where \( \lambda \) runs through the set of all partitions. These functions are orthogonal with respect to the inner product given by

\[ < f, g >_0 = [\bar{f} \bar{g}]_0, \]

where

\[ \bar{g}(x_1, \ldots, x_n) = g(x_1^{-1}, \ldots, x_n^{-1}), \]

and \( [ \cdot ]_0 : \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to \mathbb{C} \) is the constant term:

\[ \left[ \sum a_\lambda x^\lambda \right]_0 = a_0. \]

The main object of our study are Macdonald’s polynomials, defined in [M]. This is a family of polynomials depending on two independent variables \( q, t \) and defined by the following theorem:

\textbf{Theorem.} (Macdonald) There exists a unique family of polynomials \( P_\lambda(x; q, t) \in \mathbb{C}(q, t)[x] \) \( (x = (x_1, \ldots, x_n)) \), where \( \lambda \) is a partition and \( \mathbb{C}(q, t) \) is the field of rational functions in \( q, t \), satisfying the following properties:

1. \( P_\lambda(x; q, t) \) is symmetric under the action of \( S_n \) on the \( x \)'s.
2. \( P_\lambda(x; q, t) = m_\lambda(x) + \sum_{\mu < \lambda} c_{\lambda \mu} m_\mu(x) \)
3. \( P_\lambda(x; q, t) \) are orthogonal with respect to the inner product given by

\begin{equation}
(1.2) \quad < f, g >_{q,t} = [f \bar{g} \Delta_{q,t}]_0,
\end{equation}
where

\[(1.3) \quad \Delta_{q,t}(x) = \prod_{i \neq j} \prod_{m=0}^{\infty} \frac{1 - q^{2m}x_i x_j^{-1}}{1 - q^{2m}t^2 x_i x_j^{-1}} \]

These polynomials are called Macdonald’s polynomials (our notation differs slightly from that of Macdonald: what we denote by \(P_{\lambda}(x; q, t)\) in the notations of [M] would be \(P_{\lambda}(q^2, t^2)\)).

Also, often it is convenient to consider Macdonald’s polynomials for \(t = q^k, k \in \mathbb{Z}_+\); for example, for \(k = 0\) these polynomials reduce to the orbitsums \(m_{\lambda}\), and for \(k = 1\) to Schur’s symmetric functions. However, most of the properties of Macdonald’s polynomials obtained for \(t = q^k\) can be generalized to the case when \(q, t\) are independent variables.

In the future it will be convenient to use the following form of \(\Delta\):

\[(1.4) \quad \Delta_{q,t}(x) = \prod_{\alpha \in R} \prod_{m=0}^{\infty} \frac{1 - q^{2m}x_\alpha}{1 - q^{2m}t^2 x_\alpha} = \delta_{q,t} \tilde{\delta}_{q,t}, \]

where \(R \subset \mathbb{C}^n\) is the root system of type \(A_{n-1}\): \(R = \{\alpha_{ij}\}_{i \neq j}, \alpha_{ij} = \varepsilon_i - \varepsilon_j, \varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n\) (1 in the \(i\)-th place), and

\[(1.5) \quad \delta_{q,t}(x) = \prod_{\alpha \in R^+} \prod_{m=0}^{\infty} \frac{1 - q^{2m}x_\alpha}{1 - q^{2m}t^2 x_\alpha} \]

\(R^+ = \{\alpha_{ij}\}_{i < j}\) being the set of positive roots.

The proof of the theorem is based on the use of the following operator \(M_1\) in the space \(\mathbb{C}(q,t)[x_1, \ldots, x_n]S_n\):

\[(1.6) \quad M_1 = t^{1-n} \sum_{i=1}^{n} \left( \prod_{i \neq j} \frac{t^2 x_i - x_j}{x_i - x_j} \right) T_{q^2, x_i}, \]

where \((T_{q^2, x_i} f)(x_1, \ldots, x_n) = f(x_1, \ldots, q^2 x_i, \ldots, x_n)\). Then one can show that \(M_1\) is self-adjoint with respect to the inner product \(\langle \cdot, \cdot \rangle_{q,t}\) and that the eigenvalues of \(M_1\) are distinct. The Macdonald’s polynomials defined above are just the eigenfunctions of the operator \(M_1\):

\[(1.7) \quad M_1 P_{\lambda}(x; q, t) = c_{\lambda}^1 P_{\lambda}(x; q, t) \]

\(c_{\lambda}^1 = \sum_{i=1}^{n} q^{2\lambda_i t(n+1-2i)}\)

Macdonald showed that the operator \(M_1\) can be included in a commutative family of difference operators (cf. [Ch]). Namely, let

\[(1.8) \quad M_r = t^{r(r-n)} \sum_{i_1 < i_2 < \ldots < i_r} \left( \prod_{j \notin \{i_1 \ldots i_r\}} \frac{t^2 x_{i_l} - x_j}{x_{i_l} - x_j} \right) T_{q^2, x_{i_1}} \ldots T_{q^2, x_{i_r}} \]
Proposition 1.1. (Macdonald)
1. $[M_i, M_j] = 0$
2. $M_r$ is self-adjoint with respect to the inner product $< \cdot, \cdot >_{q,t}$.
3. $M_rP_\lambda(x; q, t) = c_\lambda^r P_\lambda(x; q, t)$, where $c_\lambda^r = \sum_{i_1 < \ldots < i_r} q^{2 \lambda_{i_1} t(n+1-2i_1)}$.

Proof. Let us first prove that $M_r m_\lambda = c_\lambda^r m_\lambda + \text{lower order terms}$. The proof is quite similar to that for $M_1$ (see [M]). Let us for simplicity assume that $t = q^k$.

Introduce

$$\delta(x) = \prod_{\alpha \in \mathbb{R}^+} \left( x^{\alpha/2} - x^{-\alpha/2} \right)$$

(1.9)

$$= (x_1 \ldots x_n)^{1/2} \prod_{i < j} (x_i - x_j) = \sum_{\sigma \in S_n} \epsilon(\sigma) x^{\sigma \rho},$$

where $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2})$, and $\epsilon(\sigma)$ is the sign of permutation $\sigma$. Also, for brevity we write $I = \{i_1, \ldots, i_r\}$, where $i_1 < \ldots < i_r$, and $T_{q^2, x_I} = T_{q^2, x_{i_1}} \ldots T_{q^2, x_{i_r}}$.

Then one can easily check that

$$M_r = \sum_{|I| = r} \delta^{-1}(T_{q^2, x_I}) \delta T_{q^2, x_I} = \delta^{-1} \sum_{\sigma} \epsilon(\sigma) x^{\sigma \rho} t^{\sum_{i \in I} (\sigma \rho)_i} T_{q^2, x_I}.$$ 

Now, if we denote $\lambda = \{\sigma \in S_n | \sigma(\lambda) = \lambda\}$, then $|S_\lambda| m_\lambda = \sum_{\sigma' \in S_n} x^{\sigma' \lambda}$, and therefore

$$|S_\lambda| M_r m_\lambda = \sum_{|I| = r} \sum_{\sigma} \sum_{\sigma'} \delta^{-1} \epsilon(\sigma) x^{\sigma \rho} t^{\sum_{i \in I} (\sigma \rho)_i} q^{2 \sum_{i \in I} (\sigma' \lambda)_i} x^{\sigma' \lambda}.$$ 

Introducing $\sigma''$ such that $\sigma' = \sigma \sigma''$ we see that

$$|S_\lambda| M_r m_\lambda = \sum_{\sigma''} \sum_{I} q^{2 \sum_{i \in I} (k \rho + \sigma'' \lambda)_i} \chi_{\sigma'' \lambda},$$

where

(1.10)

$$\chi_\mu = \delta^{-1} \sum_{\sigma \in S_n} \epsilon(\sigma) x^{\sigma(\mu + \rho)},$$

if $\mu$ is a partition, $\chi_\mu$ is the Schur symmetric function, i.e. the character of the corresponding finite-dimensional representation of $\mathfrak{gl}_n$. Obviously, $\chi_{\sigma''(\lambda)} = \sum_{\mu \leq \lambda} a_\mu m_\mu$, and non-zero contribution to $m_\lambda$ is given by the terms with $\sigma'' \lambda = \lambda$. Therefore,

$$M_r m_\lambda = \left( \sum_{I} q^{2 \sum_{i \in I} (k \rho + \lambda)_i} \right) m_\lambda + \text{lower order terms}.$$ 

To complete the proof of the theorem it suffices to show that $M_r$ is self-adjoint with respect to $< \cdot, \cdot >_{q,t}$, which is straightforward. \qed
2. The quantum group $U_q\mathfrak{gl}_n$ and its representations.

Let $q$ be a formal variable. By definition ([D1, J]), quantum group $U_q\mathfrak{gl}_n$ is an associative algebra with unit over the ring $\mathbb{C}(q)$ of rational functions in $q$ with generators $e_i, f_i, i = 1 \ldots n - 1, q^{h_i}, i = 1 \ldots n$ and relations

\[
[\mathfrak{h}_i, \mathfrak{h}_j] = 0 \\
[\mathfrak{h}_i, \mathfrak{e}_i] = \mathfrak{e}_i \\
[\mathfrak{h}_i, \mathfrak{f}_i] = -\mathfrak{f}_i \\
[\mathfrak{h}_i, \mathfrak{e}_{i+1}] = -\mathfrak{e}_{i+1} \\
[\mathfrak{h}_i, \mathfrak{f}_{i+1}] = \mathfrak{f}_{i+1} \\
[\mathfrak{h}_i, \mathfrak{e}_j] = [\mathfrak{h}_i, \mathfrak{f}_j] = 0, \quad j \neq i, i + 1 \\
[\mathfrak{e}_i, \mathfrak{f}_j] = \delta_{ij} \frac{q^{h_i-h_{i+1}}-q^{h_{i+1}-h_i}}{q-q^{-1}}
\]

(2.1)

\[
e_i^2 e_j - (q+q^{-1})e_i e_j e_i + e_j e_i^2 = 0, i = j \pm 1, \quad f_i^2 f_j - (q+q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \\
[f_i, f_j] = 0 = [e_i, e_j], \quad |i-j| > 1
\]

(2.2)

It is known that $U_q\mathfrak{gl}_n$ is a Hopf algebra with the following comultiplication $\Delta$ and antipode $S$:

\[
\Delta e_i = e_i \otimes q^{(h_{i+1}-h_i)/2} + q^{(h_i-h_{i+1})/2} \otimes e_i \\
\Delta f_i = f_i \otimes q^{(h_{i+1}-h_i)/2} + q^{(h_i-h_{i+1})/2} \otimes f_i \\
\Delta h_i = h_i \otimes 1 + 1 \otimes h_i
\]

(2.3)

\[
S(e_i) = -e_i q^{-1} \\
S(f_i) = -f_i q \\
S(h_i) = -h_i
\]

(2.4)

In the limit $q \to 1$, $U_q\mathfrak{gl}_n$ becomes the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_n$: one can identify the generators with the matrix units as follows: $e_i = E_{ii+1}, f_i = E_{i+1,i}, h_i = E_{ii}$.

Like its classical analogue, $U_q\mathfrak{gl}_n$ admits the following polarization: $U_q\mathfrak{gl}_n = U^- \cdot U^0 \cdot U^+$, where $U^\pm$ is the subalgebra generated by $e_i$ (respectively, $f_i$), and $U^0$ is the algebra generated by $q^{h_i}$. $U_q\mathfrak{gl}_n$ also admits an algebra automorphism $\omega$ (Cartan involution), which transposes $U^+$ and $U^-$:

(2.5)

\[
\omega e_i = -f_i, \quad \omega f_i = -e_i, \quad \omega h_i = -h_i
\]

Note that $\omega$ is a coalgebra antiautomorphism.

Representation theory of $U_q\mathfrak{gl}_n$ is quite parallel to the classical case. Unless otherwise stated, we consider only finite-dimensional representations. Define Cartan subalgebra $\mathfrak{h}$ to be the linear span of $h_i$; then every $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ can be considered as a weight, i.e. an element of $\mathfrak{h}^*$ by $\lambda(h_i) = \lambda_i$. We have a bilinear form on the weights given by $< \lambda, \mu > = \sum \lambda_i \mu_i$, which allows us to identify
$\mathfrak{h} \simeq \mathfrak{h}^*$. Define the set of integral weights $P = \{\lambda | \lambda_i - \lambda_j \in \mathbb{Z}\}$ and the set of dominant weights $P_+ = \{\lambda | \lambda_i - \lambda_{i+1} \in \mathbb{Z}_+\}$. Note that $\lambda \in P_+$ iff $\lambda + a(1, \ldots, 1)$ is a partition for some $a \in \mathbb{C}$. We have a natural order on $P$ which is defined precisely in the same way as in Section 1. It is also convenient to introduce fundamental weights $\omega_i = (1, \ldots, 1, 0 \ldots, 0)$ ($i$ ones), $i = 1, \ldots, n - 1$. Then $\lambda \in P_+$ iff $\lambda = a(1, \ldots, 1) + \sum n_i \omega_i, n_i \in \mathbb{Z}_+$.

We also introduce the root system $R = \{\varepsilon_i - \varepsilon_j, i \neq j\} \subset \mathfrak{h}^*$, positive roots $R^+ = \{\varepsilon_i - \varepsilon_j, i < j\}$ and simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, n - 1$, where $\varepsilon_i$ is a basis in $\mathfrak{h}^*$: $\varepsilon_i(h_j) = \delta_{ij}$. Also, let $Q$ be the lattice in $\mathfrak{h}^*$ spanned by all roots, and $Q^+ \subset Q$ be the semigroup spanned by all positive roots.

For every $\lambda \in P_+$ there exists a unique irreducible finite-dimensional representation $V_\lambda$ generated by a highest weight vector $v_\lambda$ of weight $\lambda$. This exhausts all irreducible finite-dimensional representations of $U_q \mathfrak{g}\mathfrak{l}_n$. The characters (and therefore, the dimensions) of the representations $V_\lambda$ are the same as for the classical case. Also, every representation is a direct sum of irreducibles.

We can define a tensor product of two representations by the formula $\pi_{V \otimes W}(x) = (\pi_V \otimes \pi_W) \Delta(x)$. From complete reducibility we know that $V_\lambda \otimes V_\mu \simeq \bigoplus N_{\lambda \mu}^\nu V_\nu$, and the coefficients $N_{\lambda \mu}^\nu$ are the same as for $\mathfrak{g}\mathfrak{l}_n$.

It is known that for any $V, W$ the representations $V \otimes W$ and $W \otimes V$ are isomorphic, but the isomorphism is non-trivial. More precisely (see [D1]), there exists a universal $R$-matrix $R \in U_q \mathfrak{g}\mathfrak{l}_n \hat{\otimes} U_q \mathfrak{g}\mathfrak{l}_n$ ($\hat{\otimes}$ should be understood as a completed tensor product) such that

$$\tag{2.6} \tilde{R}_{V, W} = P \circ \pi_V \otimes \pi_W(R); V \otimes W \rightarrow W \otimes V$$

is an isomorphism of representations. Here $P$ is the transposition: $Pv \otimes w = w \otimes v$. Also, it is known that $R$ has the following form:

$$\tag{2.7} R = \begin{cases} q^{-\sum h_i \otimes h_i} R^*, & R^* \in U^+ \otimes U^- \\ (\epsilon \otimes 1)(R^*) = (1 \otimes \epsilon)(R^*) = 1 \otimes 1, & \end{cases}$$

where $c: U_q \mathfrak{g}\mathfrak{l}_n \rightarrow \mathbb{C}$ is the counit. We will also use the following formula for $R^2$:

$$\tag{2.8} (\tilde{R}_{V_\lambda, \nu} \tilde{R}_{V_\mu, \lambda})|_{V_\nu \subset V_\lambda \otimes V_\lambda} = q^{c(\lambda) + c(\mu) - c(\nu)} \text{Id}$$

where $c(\lambda) = <\lambda, \lambda + 2\rho>$, $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2})$.

We will also need the notion of dual representation. Namely, if $V$ is a representation of $U_q \mathfrak{g}\mathfrak{l}_n$ then by definition $V^*$ is the representation of $U_q \mathfrak{g}\mathfrak{l}_n$ in the dual space to $V$ given by

$$<x v^*, v> = <v^*, S(x)v>.$$
Finally, if \( V \) is an irreducible representation of \( U_q\mathfrak{gl}_n \) let us consider the action of \( U_q\mathfrak{gl}_n \) in \( V \) given by \( \pi_{V'}(x) = \pi(\omega x) \), where \( \omega \) is the Cartan involution defined above. We denote \( V \) endowed with this action by \( V^\omega \). One can easily check that \( V \simeq (V^\omega)^* \) (which is, of course, equivalent to saying that \( V^\omega \simeq V^* \)); that is, there exists non-degenerate pairing \( V \otimes V^\omega \to \mathbb{C} \) which commutes with the action of \( U_q\mathfrak{gl}_n \). Another way to say it is to say that there exists a non-degenerate bilinear pairing (Shapovalov form) \((\cdot, \cdot)_V: V \otimes V \to \mathbb{C} \) such that

\[
(xv, v')_V = (v, \omega S(x)v')_V.
\]
This form is symmetric (which relies on \( \omega S \omega = S^{-1} \)).

Note also that \((V \otimes W)^\omega = W^\omega \otimes V^\omega \) and that if \( \Phi: V \to W \) is an intertwiner then \( \Phi \) is also an intertwiner considered as a map \( V^\omega \to W^\omega \).

### 3. Traces of intertwiners and the generalized Weyl orthogonality theorem

Let \( V, U \) be finite-dimensional representations of \( U_q\mathfrak{gl}_n \), and \( \Phi: V \to V \otimes U \) be a non-zero intertwining operator for \( U_q\mathfrak{gl}_n \).

**Definition.** A vector-valued character is the following function of \( x = (x_1, \ldots, x_n) \):

\[
\chi_\Phi(x_1, \ldots, x_n) = \text{Tr}|_V(\Phi x_1^{h_1} \cdots x_n^{h_n}).
\]

From the definition it is clear that \( \chi_\Phi \) is a linear combination of monomials \( x^\mu \) where \( \mu \) runs over the set of weights of \( V \). Thus, we can consider \( \chi \) as an element of the group algebra \( A = \mathbb{C}(q)[P] \simeq \{ \sum_{\lambda \in P} a_{\lambda} x^\lambda | a_\lambda \in \mathbb{C}(q), \text{almost all } a_\lambda = 0 \} \).

We will sometimes call elements of \( A \) generalized Laurent polynomials in \( x_i \); also, we will write \( x^h \) instead of \( x_1^{h_1} \cdots x_n^{h_n} \). Note that the elements of \( A \) can also be interpreted as functions on \( \mathfrak{h} \) by letting \( x_i(\sum z_j h_j) = e^{z_i} \). This is the same as considering the function on \( \mathfrak{h} \) given by \( \chi(h) = \text{Tr}|_V(\Phi e^h) \).

In particular, for \( V = V_\lambda \chi_\Phi \in x^\lambda \mathbb{C}(q) \left[ \frac{x_1}{x_1} \cdots, \frac{x_{n-1}}{x_{n-1}} \right] \otimes U \); the highest term of \( \chi \) is \( ux^\lambda \) and the lowest term is \( u'x^{-\lambda'} \) for some \( u, u' \in U \). Note that in the contrast with the classical case, \( \chi_\Phi \) is not \( S_n \) symmetric if \( U \) is not a trivial representation.

Using the notion of dual representation, we can rewrite \( \chi_\Phi \) as follows: we can identify \( \Phi \) with an intertwiner \( \Phi: V^* \otimes V \to U \); then \( \chi_\Phi(x) = \Phi(1 \otimes x^h) \sum v^i \otimes v_i \), where \( v_i, v^i \) are the dual bases in \( V, V^* \). Note that \( \sum v^i \otimes v_i = (1 \otimes q^{-2\rho})1 \), where \( 1 = i(1) : \mathbb{C} \to V^* \otimes V \) being an embedding of \( U_q\mathfrak{gl}_n \)-modules.

In particular, this implies that if we substitute \( x_i = q^{2\rho_i} \) in \( \chi \) then \( \chi(q^{2\rho}) = 0 \) if \( U \) is a non-trivial irreducible representation.

The space of all intertwiners \( \Phi: V_\lambda \to V_\lambda \otimes U \) is isomorphic to the space \((V^*_\lambda \otimes V_\lambda \otimes U)^{U_q\mathfrak{gl}_n} \). Let us choose a basis \( \Phi_{\lambda,i} \) in this subspace, orthonormal with respect to Shapovalov form. This gives us the basis in the space of \( U \)-valued characters.

**Generalized Weyl Orthogonality Theorem.** The characters \( \chi_{\lambda,i} \) are orthogonal with respect to the following inner product: \(<f, g>_1 = [(f, g)_U \Delta], \) where \( \Delta = \prod_{\alpha \in R} (1 - x^\alpha), (\cdot, \cdot)_U \) is the Shapovalov form and all the other notations are as in Section 1.

**Proof.** Let \( \chi_1 = \text{Tr}|_{V_\lambda}(\Phi_{\lambda} x^h), \chi_2 = \text{Tr}|_{V_\mu}(\Phi_{\mu} x^h) \). As was explained above, we can as well consider \( \Phi_{\mu} \) as an intertwiner \( V_\mu^\omega \to U^\omega \otimes V_\mu^\omega \). Thus, \( (\chi_1(x), \chi_2(x^{-1}))_U = \)
\[
\text{Tr}\left|_{\lambda} (\Psi x^h \otimes x^h) \right| (\text{note the change of sign of } h \text{ in the second factor!}), \text{ where the intertwiner } \Psi : V_\lambda \otimes V_\mu^\omega \to V_\lambda \otimes V_\mu^\omega \text{ is defined as the following composition}
\]

\[
V_\lambda \otimes V_\mu^\omega \xrightarrow{\Phi_\lambda \otimes \Phi_\mu^\omega} V_\lambda \otimes U \otimes U^\omega \otimes V_\mu^\omega \xrightarrow{\Id \otimes (\cdot) U \otimes \Id} V_\lambda \otimes V_\mu^\omega.
\]

Since \( V_\lambda \otimes V_\mu^\omega = \bigoplus N_\nu V_\nu \), we see that \( (\chi_1(x), \chi_2(x^{-1}))_U \) is a linear combination of usual characters \( \chi_\nu(x) \). But since these characters are the same as for \( \mathfrak{gl}_n \), we know that \([\chi_\nu(x) \Delta]_0 = 0\) unless \( \nu = 0 \). On the other hand, it is known that if \( \lambda \neq \nu \) then the decomposition of \( V_\lambda \otimes V_\nu^\omega \) does not contain the trivial representation (i.e. \( N_0 = 0 \)); thus, in this case \( \chi_1 \) and \( \chi_2 \) are orthogonal.

If \( \lambda = \nu \) then \( \chi_\Phi \) are pairwise orthogonal by definition. \( \square \)

4. The main theorem

Through this section, we assume \( k \in \mathbb{N} \) and show how one gets Macdonald’s polynomials \( P_\lambda(x; q, q^k) \) as vector-valued characters. Let \( U \) be the finite-dimensional representation of \( U_q \mathfrak{gl}_n \) with the highest weight \( (k-1)n \omega_1 - (k-1)(1, \ldots, 1) = (k-1)(n-1, -1, \ldots, -1) \); as a \( U_q \mathfrak{sl}_n \)-module, this is a \( q \)-analogue of the representation \( S^{(k-1)n} \mathbb{C}^n \). Note that all the weight subspaces in \( U \) are one-dimensional; this property will be very useful to us.

**Lemma 1.** A \( U_q \mathfrak{gl}_n \)-homomorphism \( \Phi : V_\lambda \to V_\lambda \otimes U \) exists iff \( \lambda - (k-1) \rho \in P_+ \); if it exists, it is unique up to a factor.

As we discussed before, it suffices to prove this lemma for \( \mathfrak{gl}_n \), which is a standard exercise. Therefore, let us consider the (non-zero) intertwiners

\[
\Phi_\lambda : V_{\lambda + (k-1) \rho} \to V_{\lambda + (k-1) \rho} \otimes U, \quad \lambda \in P_+,
\]

and the corresponding traces

\[
\varphi_\lambda(x) = \chi_{\Phi_\lambda}(x) = \text{Tr}|_{V_{\lambda + (k-1) \rho}}(\Phi_\lambda x^h);
\]

As we discussed before, \( \varphi_\lambda(x) \) has the form \( \varphi_\lambda(x) = x^{\lambda + (k-1) \rho} p(x), p(x) \in \mathbb{C}(q)[\frac{x_1}{x_1}, \ldots, \frac{x_n}{x_{n-1}}] \). It takes values in the zero-weight subspace \( U[0] \), which is one-dimensional; therefore, we can consider it as a complex-valued function. We choose the normalization of \( \Phi \) and the identification \( U[0] \simeq \mathbb{C} \) in such a way that

\[
\varphi_\lambda(x) = x^{\lambda + (k-1) \rho} + \ldots
\]

**Proposition 4.1.** \( \varphi_\lambda(x) \) is divisible by \( \varphi_0(x) \), and the ratio is a symmetric generalized Laurent polynomial in \( x_i \) with highest term \( x^\lambda \).

**Proof.** Let \( \lambda = a(1, \ldots, 1) + \sum n_i \omega_i \). We prove the statement by induction in height of \( \lambda \): \( |\lambda| = \sum n_i \). If the height is zero, this is obvious. Suppose we’ve proved the statement for all \( \lambda \) of height \( \leq N \); let us prove it for \( \lambda' \) of height \( N + 1 \). Write \( \lambda' = \lambda + \omega_i \) for some \( \lambda \) of height \( N \). Then

\[
V_{\omega_i} \otimes V_\lambda = V_{\lambda'} + \sum_{|\nu| \leq N} C_\nu V_\nu.
\]
By induction assumption, it suffices to prove that \( \text{Tr}(\Phi x^h) \) is divisible by \( \varphi_0(x) \) and the ratio is symmetric, where \( \Phi: V_{\omega_i} \otimes V_\lambda \to V_{\omega_i} \otimes V_\lambda \otimes U \) is such that its restriction to \( V_\lambda' \) is not zero. Let us take \( \Phi = \text{Id} \otimes \Phi_\lambda \). Then \( \text{Tr}(\Phi x^h) = \chi_{\omega_i}(x) \varphi_\lambda(x) \). Since \( \chi_{\omega_i} \) is a symmetric Laurent polynomial in \( x_i \), and by induction assumption the statement of the proposition holds for \( \varphi_\lambda \), we get the desired result. \( \square \)

Now we can formulate our main theorem:

**Theorem 1.** If \( \lambda \) is a partition then \( \varphi_\lambda(x)/\varphi_0(x) \) is the Macdonald’s polynomial \( P_\lambda(x;q,q^k) \).

**Proof.** We have shown that \( \varphi_\lambda(x)/\varphi_0(x) \) are symmetric Laurent polynomials with highest term \( x^\lambda \). It implies that \( \varphi_\lambda(x)/\varphi_0(x) \in \mathbb{C}[x_1, \ldots, x_n]^{S_n} \). Thus, it suffices to prove that they are orthogonal with respect to the inner product \( \langle \cdot, \cdot \rangle_{q,t} \) given by (1.2). This follows from the generalized Weyl orthogonality theorem and the following lemma:

**Main Lemma.**

\[
\varphi_0(x) = q^{\frac{n(n-1)k(k-1)}{4}} \prod_{i=1}^{k-1} \prod_{\alpha \in R^+} (q^{-i}x^{\alpha/2} - q^i x^{-\alpha/2}) \tag{4.4}
\]

Let us first show how to deduce the statement of the theorem from this lemma. Indeed, we know from generalized Weyl orthogonality theorem that \([\varphi_\lambda \varphi_\mu \Delta]_0 = 0 \) if \( \lambda \neq \mu \). Therefore,

\[
[\varphi_\lambda \varphi_\mu \varphi_0 \Delta]_0 = 0. \tag{4.5}
\]

Since

\[
\varphi_0 \varphi_0 \Delta = \prod_{\alpha \in R} \prod_{i=0}^{k-1} (1 - q^{2i}x^\alpha) = \Delta_{q,t}(x),
\]

(4.5) is just the condition of orthogonality with respect to the inner product \( \langle \cdot, \cdot \rangle_{q,t} \).

Now let us prove the Main Lemma. Unfortunately, straightforward computation encounters serious difficulties; thus, we have to find a bypass. First, we prove the following statement:

**Lemma 1.** \( \varphi_\lambda \varphi_\lambda \) is \( S_n \)-symmetric

This follows from the construction used in the proof of the generalized Weyl orthogonality theorem, where we proved that \( \varphi_\lambda \varphi_\lambda \) is a linear combination of characters of irreducible representations.

Now comes the key statement:

**Lemma 2.** \( \varphi_0(x) \) is divisible by \( (1 - q^{2j}x^{-\alpha_i}) \) for any simple root \( \alpha_i \) and \( 1 \leq j \leq k-1 \).

To prove it, recall that \( \varphi_0(x) = \text{Tr}(\Phi_0 x^h) \), where \( \Phi_0: V_{(k-1)\rho} \to V_{(k-1)\rho} \otimes U \). Fix a simple root \( \alpha_i \) and consider \( F_i = f_i q^{(h_{i+1} - h_i)/2} \). Then

\[
\Delta(F_i) = F_i \otimes q^{h_{i+1} - h_i} + 1 \otimes F_i \tag{4.6}
\]
Consider \(\text{Tr}(\Phi_0 F^j_i x^h) \in \mathbb{U}[-j\alpha_i]\). Then, using the fact that \(\Phi\) is an intertwiner and (4.6), we see that
\[
\text{Tr}(\Phi_0 F^j_i x^h) = q^{2j} \text{Tr}((F_i \otimes 1)\Phi_0 F^{j-1}_i x^h) + F_i \text{Tr}(\Phi_0 F^{j-1}_i x^h)
\]
Now we can use the cyclic property of trace and relation \(x^h F_i = F_i x^h \frac{x_i + 1}{x_i}\) to get
\[
\text{Tr}(\Phi_0 F^j_i x^h) = q^{2j} \frac{x_i^{j+1}}{x_i} \text{Tr}(\Phi_0 F^j_i x^h) + F_i \text{Tr}(\Phi_0 F^{j-1}_i x^h)
\]
Repeating this process, we see that
\[
\text{Tr}(\Phi_0 F^j_i x^h) = q^{2j} \prod_{m=1}^{j-1} \frac{1}{1 - q^{2m} x_i^{j+1}} F^j_i \text{Tr}(\Phi_0 x^h)
\]
But we know that \(F^j_i |_{\mathbb{U}[0]} \neq 0\) for \(j \leq k - 1\). Also, it is easy to see that the left-hand side is a Laurent polynomial in \(x_i\). This proves Lemma 2. \(\square\)

Now, let us consider \(\varphi_0 \bar{\varphi}_0\). We know that it is divisible by \(1 - q^{2j} x^{\alpha_i}\) for any simple root \(\alpha_i\). But we also know that it is \(S_n\)-invariant. Since the Weyl group acts transitively on the root system (in the simply-laced case), we see that \(\varphi_0 \bar{\varphi}_0\) is divisible by \(1 - q^{2j} x^{\alpha}\) for any \(\alpha \in R\). Comparing the degrees in each variable, we see that
\[
\varphi_0 \bar{\varphi}_0 = c \prod_{j=1}^{k-1} \prod_{\alpha \in R} (1 - q^{2j} x^{\alpha})
\]
for some constant \(c\) depending on \(q, k\).

This is only possible if
\[
\varphi_0 = \pm q^N \prod_{j=1}^{k-1} \prod_{\alpha \in R'} (q^{-j} x^{\alpha/2} - q^j x^{-\alpha/2})
\]
for some subset \(R' \subset R\) such that \(R = R' \cup -R'\) (normalization of \(\varphi_0\) uniquely determines the factor in front of the product). We want to prove \(R' = R^+\). To do it, let us compare the lowest terms on both sides of (4.7). The lowest term of the right hand side is
\[
\prod_{\alpha \in R'} (-1)^{k-1} q^{\pm k(k-1)} x^{-\lambda_0},
\]
where one takes + sign if \(\alpha \in R^+\) and - otherwise, \(\lambda_0 = (k - 1)\rho\) (if we normalize so that the highest term is \(x^{\lambda_0}\)). Thus, to complete the proof of the theorem it suffices to prove the following lemma:

**Lemma 3.**
\[
\varphi_0(x) = x^{\lambda_0} + \ldots \pm q^{\frac{k(k-1)n(n-1)}{2}} x^{-\lambda_0},
\]
where \(\lambda_0 = (k - 1)\rho\).

This again involves several steps, which we briefly outline. First, note that \(V_{\lambda_0} \simeq V^*_0\). Similar to the discussion at the end of Section 2 and beginning of Section 3, we can rewrite \(\varphi\) as follows:
\[
\varphi_0(x) = \Phi(1 \otimes q^{-2\rho} x^h) 1,
\]
where \( 1 = i(1), i: \mathbb{C} \to V_{\lambda_0} \otimes V_{\lambda_0}, \Phi: V_{\lambda_0} \otimes V_{\lambda_0} \to U \) are non-zero homomorphisms of \( U_q \mathfrak{gl}_n \)-modules (both \( i \) and \( \Phi \) exist and are unique up to a factor).

Let us write \( 1 = v_{\lambda_0} \otimes v_{-\lambda_0} + \ldots + av_{-\lambda_0} \otimes v_{\lambda_0} \), for some \( v_{\lambda_0} \in V_{\lambda_0}[\lambda_0], v_{-\lambda_0} \in V_{\lambda_0}[-\lambda_0] \) and \( a \in \mathbb{C} \).

Now, consider the intertwiner \( \tilde{R} = \tilde{R}_{V_{\lambda_0}, V_{\lambda_0}}: V_{\lambda_0} \otimes V_{\lambda_0} \to V_{\lambda_0} \otimes V_{\lambda_0} \), defined in Section 2. Let us write \( V_{\lambda_0} \otimes V_{\lambda_0} = \bigoplus C_{\mu} V_{\mu} \); in particular, the trivial representation \( \mathbb{C} \) occurs in this decomposition with multiplicity 1. Then formula \((2.8)\) for \( \tilde{R}^2 \) implies

\[
\tilde{R}|1 = \pm q^{c(\lambda_0)}1,
\]

where \( c(\lambda) = <\lambda, \lambda + 2\rho> \). On the other hand, we know (from formula \(2.7\)) that \( \tilde{R}(v_{\lambda_0} \otimes v_{-\lambda_0}) = q^{-\lambda_0, \lambda_0} v_{-\lambda_0} \otimes v_{\lambda_0} \). This implies

\[
1 = v_{\lambda_0} \otimes v_{-\lambda_0} + \ldots + q^{c(\lambda_0) - \lambda_0, \lambda_0} v_{-\lambda_0} \otimes v_{\lambda_0} = v_{\lambda_0} \otimes v_{-\lambda_0} + \ldots + q^{-\lambda_0, 2\rho} v_{-\lambda_0} \otimes v_{\lambda_0}
\]

Denote \( \Phi(v_{\lambda_0} \otimes v_{-\lambda_0}) = u_0 \in U[0] \). Let us find \( \Phi(v_{-\lambda_0} \otimes v_{\lambda_0}) \). Again, since \( U \) occurs in the decomposition of \( V_{\lambda_0} \otimes V_{\lambda_0} \) with multiplicity one, we have

\[
\tilde{R}|_{U \subset V_{\lambda_0} \otimes V_{\lambda_0}} = \pm q^{c(\lambda_0) - \frac{c(\lambda_0) - c(\nu)}{2}} \Phi.
\]

where \( c(\lambda) = <\lambda, \lambda + 2\rho> \) and \( \nu = (k-1)(n-1, -1, \ldots, -1) \) is the highest weight of \( U \). Thus, \( \Phi \tilde{R} = \pm q^{c(\lambda_0) - \frac{c(\nu)}{2}} \Phi \). Considering \( \Phi \tilde{R}(v_{\lambda_0} \otimes v_{-\lambda_0}) \), we find that

\[
\pm q^{c(\lambda_0) - \frac{c(\nu)}{2}} \Phi(v_{\lambda_0} \otimes v_{-\lambda_0}) = \pm q^{-\lambda_0, \lambda_0} \Phi(v_{-\lambda_0} \otimes v_{\lambda_0})
\]

Thus, if choose identification \( U[0] \simeq \mathbb{C} \) so that \( \Phi(v_{-\lambda_0} \otimes v_{\lambda_0}) = 1 \) then

\[
\varphi_0(x) = \Phi(1 \otimes q^{-2\rho} x^h) 1 = \Phi(1 \otimes q^{-2\rho} x^h)(v_{\lambda_0} \otimes v_{-\lambda_0} + \ldots + q^{-\lambda_0, 2\rho} v_{-\lambda_0} \otimes v_{\lambda_0})
\]

\[
= x^{\lambda_0} + \ldots + q^{\frac{c(\nu)}{2}} x^{-\lambda_0}
\]

Since \( c(\nu) = k(k-1)n(n-1) \), we get the statement of Lemma 3. This completes the proof of the theorem. \( \square \)

5. The case of generic \( k \)

In this section we show how to get Macdonald’s polynomials for the case when \( q \) and \( t \) are independent variables. However, it will be convenient to introduce formal variable \( k \) such that \( t = q^k \); thus, \( q \) and \( q^k \) are algebraically independent variables. One can check that all the formulas can be rewritten in such a way that \( k \) appears only in the expression \( q^k \) and thus we could avoid using \( k \), writing everything entirely in terms of \( q, t \); however, this would make our construction less transparent. Also, we must consider the algebra \( U_q \mathfrak{gl}_n \), as well as the representations, over the field \( \mathbb{C}(q, t) \) rather than \( \mathbb{C}(q) \).

Let \( M_\mu \) be a Verma module with highest weight \( \mu \) over \( U_q \mathfrak{gl}_n \), \( v_\mu \) be the corresponding highest weight vector. We choose a homogeneous basis \( a_i \) in \( U^- \); then the basis in \( M_\mu \) is given by \( a_i v_\mu \). In particular, this applies to the module \( M_{\lambda+(k-1)p} \),
which is a natural generalization of the module considered in the previous section. Note that if \( k \) is a formal variable then this module is irreducible.

We can also introduce the analogue of the module \( U \). Indeed, let

\[
W_k = \{ (x_1 \ldots x_n)^{k-1} p(x), p(x) \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], \deg p = 0 \}
\]

with the action of \( U_q \mathfrak{gl}_n \) given by

\[
\begin{align*}
h_i &\mapsto x_i \frac{\partial}{\partial x_i} - (k - 1) \\
e_i &\mapsto x_i D_i + 1 \\
f_i &\mapsto x_i D_i + 1
\end{align*}
\]

\[
(D_i f)(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, qx_i, \ldots, x_n) - f(x_1, \ldots, q^{-1}x_i, \ldots, x_n)}{(q - q^{-1})x_i}
\]

\( W_k \) is an irreducible infinite-dimensional module over \( U_q \mathfrak{gl}_n \). The set of weights of \( W_k \) is the root lattice \( Q \), and every weight subspace is one-dimensional:

\[
W_k[\lambda] = \mathbb{C}w_\lambda, \quad w_\lambda = (x_1 \ldots x_n)^{k-1}x^\lambda.
\]

If we replace formal variable \( k \) in the formulas above by a positive integer \( k \) then \( W_k \) has a finite-dimensional submodule \( U_k = W_k \cap \mathbb{C}[x_1, \ldots, x_n] \); it coincides with the module \( U \) defined in Section 4.

**Lemma 5.1.** For every \( \lambda \in P_+ \) there exists a unique up to a constant factor intertwiner

\[
\tilde{\Phi}_\lambda^k : M_{\lambda+(k-1)\rho} \to M_{\lambda+(k-1)\rho} \otimes W_k.
\]

We use the notation \( \tilde{\Phi} \) to distinguish these intertwiners from those for finite-dimensional modules introduced in Section 4; the same convention applies to all other notations.

Proof is based on the general fact: if Verma module \( M_\mu \) is irreducible then the space of intertwiners \( M_\mu \to M_\mu \otimes W \) is in one-to-one correspondence with the zero-weight subspace \( W[0] \).

Let us fix the normalization of \( \tilde{\Phi}_\lambda^k \) by choosing a highest-weight vector \( v_{\lambda+(k-1)\rho} \in V_{\lambda+(k-1)\rho}[\lambda+(k-1)\rho] \) and requiring that \( \tilde{\Phi}_\lambda^k v_{\lambda+(k-1)\rho} = v_{\lambda+(k-1)\rho} \otimes w_0 + \ldots \). Then one can find explicit formulas for matrix elements of \( \tilde{\Phi} \) as follows: write

\[
\tilde{\Phi}_\lambda^k(a_i v_{\lambda+(k-1)\rho}) = \sum \tilde{R}_{ijl}^k a_j v_{\lambda+(k-1)\rho} \otimes w_l
\]

Then the condition for \( \tilde{\Phi} \) to be an intertwiner can be rewritten as a system of linear equations on \( \tilde{R}_{ijl}^k \). Due to Lemma 5.1, this system has a unique solution. From this approach one can easily see that \( \tilde{R}_{ijl}^k \) is a rational function in \( q, q^k \).

Similar to Section 4, define the trace:

\[
\tilde{\varphi}_\lambda^k(x) = \text{Tr}_{M_{\lambda+(k-1)\rho}}(\tilde{\Phi}_\lambda^k x^k)
\]
Again, \( \tilde{\varphi}^k_\lambda(x) \) takes values in \( W_k[0] \), which is one-dimensional; so we consider \( \tilde{\varphi} \) as a scalar-valued function, identifying \( W_k[0] \simeq \mathbb{C} \) so that \( w_0 \mapsto 1 \). Then

\[
\varphi^k_\lambda(x) = x^{\lambda+(k-1)^\rho}(1 + \sum_{\mu \in Q_+} \tilde{R}_{\lambda\mu}(q, q^k)x^{-\mu}),
\]

and \( \tilde{R}_{\lambda\mu} \) are rational functions of \( q, q^k \).

**Theorem 2.** \( \varphi^k_\lambda(x) \) is the Macdonald’s polynomial \( P_\lambda(x; q, q^k) \)

**Proof.** Let us recall the traces considered in Section 4. Let \( k \) be a positive integer, \( \lambda \in P_+ \). Then we have defined

\[
\varphi^k_\lambda = \text{Tr}|_{V_{\lambda+(k-1)^\rho}}(\Phi^k_\lambda x^h),
\]

where

\[
\Phi^k_\lambda: V_{\lambda+(k-1)^\rho} \rightarrow V_{\lambda+(k-1)^\rho} \otimes U
\]

is an intertwiner. First, note that \( U \) is a submodule in the module \( W_k \) defined in the beginning of this section, so we can as well consider \( \Phi \) as an intertwiner \( V_{\lambda+(k-1)^\rho} \rightarrow V_{\lambda+(k-1)^\rho} \otimes W_k \). Next, the irreducible module \( V_{\lambda+(k-1)^\rho} \) is a factor module of Verma module \( M_{\lambda+(k-1)^\rho} \). Moreover, if

\[
\mu = \sum n_i \alpha_i, \quad n_i \in \mathbb{Z}_+, \quad \sum n_i < k
\]

then \( \dim V_{\lambda+(k-1)^\rho}[\lambda + (k-1)^\rho - \mu] = \dim M_{\lambda+(k-1)^\rho}[\lambda + (k-1)^\rho - \mu] \). Thus, if we consider elements \( a_i \) of the basis in \( U^- \) such that \( \mu = -\text{weight} \ a_i \) satisfies condition (5.8) then the vectors \( a_i v_{\lambda+(k-1)^\rho} \) form a basis in the corresponding weight subspaces of \( V_{\lambda+(k-1)^\rho} \). Let us consider the restriction of the operator \( \Phi \) to these subspaces. Then it can be written in the form

\[
\Phi^k_\lambda(a_i v_{\lambda+(k-1)^\rho}) = \sum R^{ijkl; k}_\lambda a_j v_{\lambda+(k-1)^\rho} \otimes w_l
\]

The coefficients \( R^{ijkl; k}_\lambda(q) \) are rational functions of \( q \). They can be found by solving the system of equations expressing the intertwinning property of \( \Phi \). This is the same system which defined the coefficients \( \tilde{R}^{ijkl}_\lambda(q, q^k) \) in the expansion (5.3) of the intertwiners \( \tilde{\Phi} \), but now we consider \( k \) as a positive integer, not a formal variable. Still one can check that if we restrict ourselves to considering only \( R^{ijkl; k}_\lambda(q) \) such that both \( -\text{wt} \ a_i, -\text{wt} \ a_j \) satisfy (5.8) then this system has a unique solution. Thus, we have the following lemma.

**Lemma.** For fixed \( \lambda, i, j, l \) such that \( -\text{weight} \ a_i, -\text{weight} \ a_j \) satisfy (5.8),

\[
R^{ijkl; k}_\lambda(q) = \tilde{R}^{ijkl}_\lambda(q, q^k)
\]

for \( k \in \mathbb{Z}_+, k >> 0 \). Here the right-hand side should be understood as a rational function of \( q \) obtained by substituting \( t = q^k, k \in \mathbb{Z}_+ \) in the rational function of two variables \( \tilde{R}^{ijkl}_\lambda(q, t) \).
Corollary 1. If we write

\[ \varphi^k_\lambda(x) = x^{\lambda+(k-1)\rho} \left( 1 + \sum_{\mu \in Q_+} R^k_{\lambda\mu}(q)x^{-\mu} \right) \]

then for fixed \( \lambda, \mu \)

\[ R^k_{\lambda\mu}(q) = \tilde{R}_{\lambda\mu}(q, q^k) \]

for \( k \in \mathbb{Z}_+, k >> 0 \).

Let us consider the ratio \( \tilde{\varphi}^k_\lambda/\tilde{\varphi}^k_0 \). Clearly, it can be written in the form

\[ \frac{\tilde{\varphi}^k_\lambda(x)}{\tilde{\varphi}^k_0(x)} = x^\lambda \left( 1 + \sum_{\mu \in Q_+} \tilde{Q}_{\lambda\mu}(q, q^k)x^{-\mu} \right) \]

Similarly,

\[ \frac{\varphi^k_\lambda(x)}{\varphi^k_0(x)} = x^\lambda \left( 1 + \sum_{\mu \in Q_+} Q^k_{\lambda\mu}(q)x^{-\mu} \right) \]

(in fact, the latter sum is finite due to Theorem 1).

Then Corollary 1 above immediately implies the following:

Corollary 2. For fixed \( \lambda, \mu \),

\[ Q^k_{\lambda\mu}(q) = \tilde{Q}_{\lambda\mu}(q, q^k) \]

for \( k \in \mathbb{Z}_+, k >> 0 \).

On the other hand, Theorem 1 in the previous section claims that if one writes Macdonald’s polynomials in the form

\[ P_\lambda(x; q, t) = x^{\lambda}(1 + \sum_{\mu \in Q_+} P_{\lambda\mu}(q, t)x^{-\mu}) \]

then

\[ Q^k_{\lambda\mu}(q) = P_{\lambda\mu}(q, q^k) \quad \text{for all } k \in \mathbb{N} \]

Comparing (5.15) and (5.14), we see that

\[ \tilde{Q}_{\lambda\mu}(q, q^k) = P_{\lambda\mu}(q, q^k) \quad \text{if } k \in \mathbb{Z}_+, k >> 0. \]

But this is possible only if \( P_{\lambda\mu} = \tilde{Q}_{\lambda\mu} \) as functions of two variables \( q, t \). Thus, the ratio (5.12) equals to the Macdonald’s polynomial \( P_\lambda(x; q, t) \). \( \Box \)
6. The center of $U_q\mathfrak{gl}_n$ and Macdonald’s operators

In this section we show how one can get Macdonald’s operators $M_r$ introduced in Section 1 from the quantum group $U_q\mathfrak{gl}_n$. This construction is parallel to the one for $q = 1$ (see [E]).

For simplicity, in this section we assume that $t = q^k, k \in \mathbb{N}$. Consider functions $f$ of $n$ variables $x_1, \ldots, x_n$ and introduce the ring of difference operators, acting on these functions:

\[(6.1) \quad DO = \{ D = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha T_\alpha | \text{almost all } a_\alpha = 0 \},\]

where $(T_\alpha f)(x_1, \ldots, x_n) = f(q^{\alpha_1}x_1, \ldots, q^{\alpha_n}x_n)$, and $a_\alpha$ are rational functions in $x_i, q$ with poles only at the points where $x^\mu q^m = 1$ for some $\mu \in \mathbb{Z}^n, m \in \mathbb{Z}$.

As before, let us consider a non-zero intertwiner $\Phi: V \rightarrow V \otimes W$, where $V$ is a highest-weight module over $U_q\mathfrak{gl}_n$ and $W$ is an arbitrary module with finite-dimensional weight spaces ($V, W$ need not be finite-dimensional), and define the corresponding trace $\varphi(x) = \text{Tr}|_V(\Phi x^h)$. This function takes values in $W$.

**Theorem 3.** For any $u \in U_q\mathfrak{gl}_n$ there exists a difference operator $D_u \in DO \otimes U_q\mathfrak{gl}_n$, independent of the choice of $V, W$ and the intertwiner $\Phi$ such that

\[(6.2) \quad \text{Tr}|_V(\Phi u x^h) = D_u \text{Tr}|_V(\Phi x^h).\]

$D_u$ is defined uniquely modulo the left ideal in $U_q\mathfrak{gl}_n$ generated by $q^{h_i} - 1$; thus, $D_u f$ is well defined for any function $f(x_1, \ldots, x_n)$ with values in $W[0]$.

**Proof.**

Let us assume that $u$ is homogeneous: $u \in U_q\mathfrak{gl}_n[\mu]$. Without loss of generality we can assume that $u$ is a monomial in the generators $e_i, f_i, q^{h_i}$ of the form $u = u^- u^+, u_{\pm} \in U_{\pm}, u^0 \in U^0$. Define $\text{sdeg} u = \text{deg} u^+ - \text{deg} u^-$, where $\text{deg} e_i = -\text{deg} f_i = 1$. We prove the theorem by induction in $\text{sdeg} u$.

If $\text{sdeg} u = 0$ then $u = u^0 = q^{\sum \alpha_i h_i}$ for some $\alpha \in \mathbb{Z}^n$. Then it follows immediately from the definition that $D_u = T_\alpha$, so the theorem holds.

Let us make the induction step. Since $\Phi$ is an intertwiner, $\text{Tr}(\Phi u x^h) = \text{Tr}(\Delta(u)\Phi x^h)$. From the definition of comultiplication one easily sees that

$$\Delta(u) = u \otimes q^{\sum \alpha_i h_i} + \sum u'_j \otimes v_j$$

for some $\alpha \in \mathbb{Z}^n$, and $\text{sdeg} u'_j < \text{sdeg} u$. Thus,

$$\text{Tr}(\Phi u x^h) = q^{\sum \alpha_i h_i} \text{Tr}(\Phi x^h u) + \sum v_j \text{Tr}(\Phi x^h u_j).$$

Since commuting with $x^h$ does not change $\text{sdeg} u_j$, by induction assumption we can write

$$\text{Tr}(\Phi u x^h) = q^{\sum \alpha_i h_i} \text{Tr}(\Phi x^h u) + D' \text{Tr}(\Phi x^h)$$

for some $D' \in DO \otimes U_q\mathfrak{gl}_n$. Since $u \in U_q\mathfrak{gl}_n[\mu]$, $x^h u = x^\mu u x^h$, and $\text{Tr}(\Phi x^h u) \in W[\mu]$, so

$$\text{Tr}(\Phi u x^h) = q^{<\alpha, \mu>} x^\mu \text{Tr}(\Phi u x^h) + D' \text{Tr}(\Phi x^h),$$

where $\langle \alpha, \mu \rangle = \sum \alpha_i \mu_i$.
and thus,
\[ \text{Tr}(\Phi u x^h) = \frac{1}{1 - q^{<\alpha, \mu>}} D'\text{Tr}(\Phi x^h). \]

This proves the existence part of the theorem. Uniqueness follows from the following lemma:

**Lemma.** Let us fix a \( U_q \mathfrak{gl}_n \)-module \( W \) with finite-dimensional weight spaces. If \( D \in DO \otimes \text{Hom}(W[0], W[\mu]) \) is such that
\[ D\varphi = 0 \]
for any \( \varphi(x) = \text{Tr}(\Phi x^h), \Phi : V \rightarrow V \otimes W, V - \text{arbitrary highest-weight module} \) then \( D = 0 \).

**Proof of the lemma.** Let us assume that \( D \neq 0 \). Multiplying \( D \) by a suitable polynomial of \( x \), we can assume that \( D \) has polynomial coefficients: \( D = \sum x^\alpha D(\alpha) \), \( D(\alpha) \) being difference operators with constant matrix-valued coefficients. Let us take the maximal (with respect to the lexicographic ordering) \( \alpha \) such that \( D(\alpha) \neq 0 \). Then if we have a trace \( \varphi \) as above such that \( \varphi(x) = x^\lambda w + \text{lower order terms} \), taking the highest term of \( D\varphi \), we see that \( D(\alpha)(x^\lambda w) = 0 \). On the other hand, if we take \( \lambda \) such that \( \lambda + \rho \in -P_+ \) then Verma module \( M_\lambda \) is irreducible and thus for every \( w \in W[0] \) there exists a non-zero intertwiner \( \Phi: M_\lambda \rightarrow M_\lambda \otimes W \) such that the corresponding trace has the form \( \varphi(x) = x^\lambda w + \text{lower order terms} \). Thus \( D(\alpha)(x^\lambda w) = 0 \) for all \( \lambda \in -P_+ - \rho, w \in W[0] \). Thus, if one writes \( D(\alpha) = \sum \beta a_{\alpha \beta} T_\beta \), then \( \sum \beta a_{\alpha \beta} w q^{<\beta, \lambda>} = 0 \) for all \( w \in W[0], \lambda \in -P_+ - \rho \). This is possible only if all \( a_{\alpha \beta} = 0 \), which contradicts the assumption \( D(\alpha) \neq 0 \). \( \square \)

In general, \( u \mapsto D_u \) is not an algebra homomorphism. However, if \( u \) is central: \( u \in Z(U_q \mathfrak{gl}_n) \) then \( \Phi u \) is also an intertwiner, and thus for every \( v \in U_q \mathfrak{gl}_n \) we have:
\[ D_{uv} \text{Tr}(\Phi x^h) = \text{Tr}(\Phi vx^h) = D_v \text{Tr}(\Phi u x^h) = D_v D_u \text{Tr}(\Phi x^h). \]

This implies the following proposition:

**Proposition 6.1.** \( u \mapsto D_u \) is an algebra homomorphism of \( Z(U_q \mathfrak{gl}_n) \) to \( DO \otimes U_q \mathfrak{gl}_n[0]/I \), where \( I \) is the ideal generated by \( q^{h_\mathfrak{h} - 1} \).

**Proposition 6.2.** Let \( c \in Z(U_q \mathfrak{gl}_n), V \) be a highest-weight module over \( U_q \mathfrak{gl}_n \) (not necessarily finite-dimensional), \( c_V = C \text{Id} \) for some \( C \in \mathbb{C}(q) \), and let \( \Phi : V \rightarrow V \otimes W \) be a non-zero intertwiner. Then the trace \( \varphi(x) = \text{Tr}_V(\Phi x^h) \) satisfies the difference equation
\[ (6.3) \quad D_c \varphi(x) = C \varphi(x). \]

This proposition is an obvious corollary of Theorem 3.

This shows that our construction allows us to construct commutative algebras of difference operators and their eigenfunctions. In general, these functions are vector-valued (they take values in the space \( W[0] \)); however, if we choose \( W \) as in Section 4 so that \( W[0] \) is one-dimensional then we can consider the traces as scalar functions; since every central element in \( U_q \mathfrak{gl}_n \) has weight zero, \( D_c \) preserves \( W[0] \)
and thus can be considered as a difference operator with scalar coefficients. We want to show that for appropriate choice of central elements the operators $D_c$ are precisely Macdonald’s operators (up to conjugation).

To find these central elements we will use Drinfeld’s construction of central elements ([D2]), which is based on the universal R-matrix $\mathcal{R} \in U_q \mathfrak{gl}_n \otimes U_q \mathfrak{gl}_n$ discussed in Section 2 (a similar construction was independently proposed by N. Reshetikhin [R]). Define $\mathcal{R}^{21} = P(\mathcal{R})$, $P(x \otimes y) = y \otimes x$.

**Proposition 6.3.** Define $c_r \in U_q \mathfrak{gl}_n$, $r = 1 \ldots , n$ by

$$c_r = (\text{Id} \otimes \text{Tr}(\Lambda_q^r)) (\mathcal{R}^{21} \mathcal{R}(1 \otimes q^{-2\rho})),$$

where $\Lambda_q^r$ is the $q$-deformation of the representation of $\mathfrak{gl}_n$ in the $r$-th exterior power of the fundamental representation $\Lambda^r \mathbb{C}^n$. Then

1. $c_r \in Z(U_q \mathfrak{gl}_n)$
2. If $V$ is a highest-weight module with highest weight $\lambda$, then

$$c_r|_V = \sum_I q^{2\sum_{i \in I}(\lambda + \rho)} \text{Id},$$

where the sum is taken over all sets $I = \{i_1, \ldots , i_r\} \subset \{1, \ldots , n\}$ such that $i_1 < \ldots < i_r$.

**Proof.**

1. This is based on the following statement (see [D2]): if $\theta: U_q \mathfrak{gl}_n \to \mathbb{C}(q)$ is such that $\theta(xy) = \theta(y S^2(x))$ then the element $c_\theta = (\text{Id} \otimes \theta)(\mathcal{R}^{21} \mathcal{R})$ is central. On the other hand, we know that $S^2(x) = q^{-2\rho} x q^{2\rho}$, so $\theta(x) = \text{Tr}_V(x q^{-2\rho})$, where $V$ is any finite-dimensional representation of $U_q \mathfrak{gl}_n$, satisfies $\theta(xy) = \theta(y S^2(x))$. Taking $V = (\Lambda_q^r)^*$, we get statement 1 of the proposition.

2. Let $v_\lambda$ be a highest-weight vector in $V$; let us calculate $c_r v_\lambda$. Let $w \in (\Lambda_q^r)^*[\mu]$. Then (2.7) implies

$$\mathcal{R}^{21} \mathcal{R}(v_\lambda \otimes w) = q^{-2\lambda , \mu} v_\lambda \otimes w + \sum v'_i \otimes w'_i,$$

where $\text{wt } w'_i < \mu$. Thus, $c_r v_\lambda = (\sum_{\mu} (\text{dim } (\Lambda_q^r)^*[\mu]) q^{-2\lambda , \mu} q^{-2\rho , \mu}) v_\lambda$, where the sum is taken over all the weights of $(\Lambda_q^r)^*$. Since the weights of $(\Lambda_q^r)^*$ are $\mu = (\mu_1, \ldots , \mu_n)$ such that $\mu_i = 0$ or $-1$, $\sum \mu_i = -r$, and multiplicity of each weight is 1, we get the desired formula. □

**Remark.** These central elements are closely related to those constructed in [FRT]. Essentially, the central elements constructed in [FRT] are traces of the powers of $L$-matrix, whereas our central elements are coefficients of the characteristic polynomial of $L$.

**Theorem 4.**

$$M_r = \varphi_0^{-1}(x) \circ D_{c_r} \circ \varphi_0(x),$$

where $M_r$ is Macdonald’s operator introduced in Section 1, $c_r$ is the central element constructed in Proposition 6.3, $\varphi_0$ is the operator of multiplication by the function $\varphi_0$ defined by (4.2).

**Proof.** This follows from the fact that $M_r$ and $\varphi_0^{-1}(x) D_{c_r} \varphi_0(x)$ coincide on the Macdonald’s polynomials $P_\lambda(x) = \varphi_\lambda(x)/\varphi_0(x)$; just compare Proposition 1.1,
Theorem 1 and Proposition 6.3. Repeating the uniqueness arguments outlined in the proof of Theorem 3, but considering \( \lambda \in P_+ + (k-1)\rho \) instead of \( \lambda \in -P_+ - \rho \), we see that it is only possible if \( M_r = \varphi_0^{-1} \circ D_{c_r} \circ \varphi_0 \). □

Thus, we can use the traces of the form (4.2) to find eigenfunctions of Macdonald operators \( M_r \). Indeed, let us consider \( \lambda = (\lambda_1, \ldots, \lambda_n) \) as a formal variable; then \( q, q^\lambda \) are algebraically independent. In this case Verma module \( M_\lambda \) is irreducible, and thus there exists an intertwiner \( \Phi: M_\lambda \rightarrow M_\lambda \otimes U \), where the module \( U \) is the same we used in Section 4.

**Theorem 5.**
1. The function

\[
(6.4) \quad f_\lambda(x) = \frac{\text{Tr}_{M_\lambda} (\Phi x^h)}{\varphi_0(x)},
\]

where \( \Phi : M_\lambda \rightarrow M_\lambda \otimes U \) is a non-zero intertwiner and \( \phi_0(x) \) is defined by (4.2), satisfies the following system of difference equations

\[
(6.5) \quad M_rf_\lambda(x) = \sum_{I:|I|=r} q^{2\sum_{i\in I}(\lambda + \rho)_i} f_\lambda(x)
\]

2. The functions \( f_{\sigma(\lambda + \rho) - \rho}, \sigma \in S_n \) form a basis of solutions of the system (6.5) in the space of generalized Laurent series \( \mathcal{F} = \sum_\nu x^\nu \mathbb{C}(q, q^\lambda)[[x_1, \ldots, x_n]] \)

**Proof.**
1. This is an immediate corollary of Proposition 6.2 and Theorem 4.
2. Suppose that \( f \in \mathcal{F} \) is a solution of (6.5) of the form

\[
f(x) = x^\nu + \text{lower order terms}.
\]

Expanding coefficients of Macdonald’s operators in Laurent series, we find the highest term of \( M_rf \):

\[
(M_rf)(x) = \sum_{I:|I|=r} q^{2\sum_{i\in I}\rho_i} T_{q^2,x_i} f = \sum_{I:|I|=r} q^{2\sum_{i\in I}(\nu + \rho)_i} x^\nu + \ldots
\]

Thus \( f(x) \) can be a solution only if for any \( r \),

\[
\sum_{I:|I|=r} q^{2\sum_{i\in I}(\nu + \rho)_i} = \sum_{I:|I|=r} q^{2\sum_{i\in I}(\lambda + \rho)_i},
\]

which is only possible if \( \nu + \rho = \sigma(\lambda + \rho) \) for some \( \sigma \in S_n \). Then the highest term of \( f \) coincides with the highest term of \( f_{\sigma(\lambda + \rho) - \rho} \). Considering \( f - f_{\sigma(\lambda + \rho) - \rho} \) and repeating the same arguments, we finally see that \( f \) is a linear combination of the functions \( f_{\sigma(\lambda + \rho) - \rho} \). □
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