ON AN ANALOG OF SELBERG’S EIGENVALUE CONJECTURE FOR $SL_3(\mathbb{Z})$

Sultan Catto
The Graduate School and University Center, and Baruch College
The City University Of New York
17 Lexington Avenue
New York, NY 10010
(e-mail: catto@gursey.baruch.cuny.edu)
and
Physics Department, The Rockefeller University
1230 York Avenue, New York, NY 10021-6399
(e-mail: cattos@rockvax.rockefeller.edu)

Jonathan Huntley
Department of Mathematics, Baruch College, CUNY
17 Lexington Avenue, New York, NY 10010
(e-mail: huntley@gursey.baruch.cuny.edu)

Jay Jorgenson
School of Mathematics
Institute for Advanced Study
Princeton, NJ 08540

David Tepper
Department of Mathematics, Baruch College, CUNY
17 Lexington Avenue, New York, NY 10010
(e-mail: tepper@gursey.baruch.cuny.edu)

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Abstract

Let $H$ be the homogeneous space associated to the group $PGL_3(\mathbb{R})$. Let $X = \Gamma \backslash H$ where $\Gamma = SL_3(\mathbb{Z})$ and consider the first non-trivial eigenvalue $\lambda_1$ of the Laplacian on $L^2(X)$. Using geometric considerations, we prove the inequality $\lambda_1 > 3\pi^2/10$. Since the continuous spectrum is represented by the band $[1, \infty)$, our bound on $\lambda_1$ can be viewed as an analogue of Selberg’s eigenvalue conjecture for quotients of the hyperbolic half space.

§1. Statement of the main theorem

A fundamental question in the spectral theory of automorphic forms is whether small eigenvalues exist. More specifically, let $G$ be a noncompact reductive group with finite center, $\Gamma$ a nonuniform lattice, $K$ a maximal compact subgroup of $G$, and set $X = \Gamma \backslash G/K$. It is well known from the theory of Eisenstein series that $L^2(X)$ has continuous spectrum for the ring of invariant differential operators, and in particular for the positive Laplacian, $\Delta$. The continuous spectrum will be, in cases of interest such as $PGL_n(\mathbb{R})$, an interval $[a, \infty)$ with $a > 0$. The question we referred to above is: Do non-constant square integrable eigenforms exist with eigenvalue $\lambda < a$? This problem is important for various considerations in number theory. In the case $G = PGL_2(\mathbb{R})$ and $\Gamma$ is a congruence subgroup, Selberg conjectured that no such nontrivial small eigenvalues exist.

In this paper, we consider the case when

$$G = PGL_3(\mathbb{R}) \quad \text{and} \quad \Gamma = SL_3(\mathbb{Z}).$$

Our main result is the following.

Theorem Let $\lambda_1$ be the eigenvalue for the first nontrivial eigenform on $L^2(X)$. Then

$$\lambda_1 > 3\pi^2/10 > 2.96088.$$

§2. Notation.

Let $\mathcal{H} = G/K$ and set $X = \Gamma \backslash \mathcal{H}$. Explicit coordinates for $\tau \in \mathcal{H}$ via the Iwasawa decomposition are given by

$$\tau = \begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
with \( y_1, y_2 > 0 \), from which one can compute that the (positive) Laplacian \( \Delta \) can be written as

\[
-\Delta = y_1^2 \frac{\partial^2}{\partial y_1^2} - y_1 y_2 \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} + y_2^2 \frac{\partial^2}{\partial y_2^2} + y_1^2 \frac{\partial^2}{\partial x_2^2} + y_1 \frac{\partial^2}{\partial x_1^2} + y_2 \frac{\partial^2}{\partial x_2^2} + 2 y_1^2 x_2 \frac{\partial^2}{\partial x_1 \partial x_3}
\]

(see pages 17 and 33 of [1]). The ring of invariant differential operators is spanned by the identity, the Laplacian \( \Delta \), and a third order operator \( \Delta_3 \) (see [1]). The invariant volume element is given by

\[
dV = \frac{dx_1 dx_2 dx_3 dy_1 dy_2}{y_1^3 y_2^3}.
\]

We shall not use the explicit formula for the invariant volume element; however, the above expression for the Laplacian will be necessary in our proof of the main theorem.

For our purposes, it is more convenient to work with functions on \( \mathcal{H} \) that are \( SL_3(\mathbb{Z}) \) invariant rather than considering functions on the quotient space \( X \). To this end, we introduce a fundamental domain for \( \Gamma \backslash \mathcal{H} \). Specifically, computations on page 56 of [2] show that a fundamental domain \( D \) is described through the following set of inequalities:

\[
\begin{align*}
v_3^3 &< v_3^3 (1 - x_2 + x_3)^2 + w (1 - x_1)^2 + w^{-1}; \\
v_2^3 &< v_2^3 (x_2 - x_3)^2 + w (1 - x_1)^2 + w^{-1}; \\
v_1^3 &< v_1^3 x_2^2 + w; \\
1 &< w^{-2} + x_1^2; \\
0 &< x_1 < \frac{1}{2}; \\
0 &< x_2 < \frac{1}{2}; \\
-\frac{1}{2} &< x_3 < \frac{1}{2},
\end{align*}
\]

where we have used the notation \( w^{-1} = y_1 \) and \( v_1^{-3/2} = y_2^2 y_1 \). Let \( S \) denote the Siegel set described via the inequalities

\[
0 < x_1 < \frac{1}{2}, \quad 0 < x_2 < \frac{1}{2}, \quad -\frac{1}{2} < x_3 < \frac{1}{2}, \quad y_1 > \frac{\sqrt{3}}{2}, \quad y_2 > \frac{\sqrt{3}}{2}.
\]

The set \( S \) contains the fundamental domain \( D \). Further, results from page 61 [2] show the existence of elements \( \gamma_1, ..., \gamma_{10} \in SL_3(\mathbb{Z}) \) such that \( S \subset \bigcup_{i=1}^{10} D \gamma_i \) (we have used the notation \( D \gamma \) to denote the image of the fundamental domain \( D \) under left multiplication by \( \gamma \)). The main aspects of the above points which we shall use are the assertions that for any \( \tau \in S \) we have \( y_1(\tau) > \sqrt{3}/2 \) and that \( S \) is contained in ten translates of \( D \).

Recall that an automorphic form is a \( C^\infty \) function \( \phi \) on \( \mathcal{H} \) which satisfies the following properties:

(i) \( \phi(\gamma \circ \tau) = \phi(\tau) \) for \( \gamma \in SL_3(\mathbb{Z}) \);

(ii) \( |\phi(\tau)| \ll y_1^{N_1} y_2^{N_2} \) for \( \tau \in D \) and integers \( N_1, N_2 \).
(iii) φ is an eigenform for the ring of invariant differential operators.

An automorphic form is said to be a cusp form if it satisfies the additional property

(iv) \[
\frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi \left( \begin{bmatrix} 1 & 0 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \tau \right) d\xi_1 d\xi_3 = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi \left( \begin{bmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tau \right) d\xi_2 d\xi_3 = 0
\]

Cusp forms are square integrable. Although we shall not need this fact, let us note that, from the theory of Eisenstein series, the only noncuspidal square integrable automorphic forms on \( X \) are constant.

§3. Proof of the main theorem

Our method of proof is a modification of that used by Roelcke to show that the small eigenvalue \( \lambda_1 \) for the quotient space \( SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}) / SO_2(\mathbb{R}) \) satisfies the bound \( \lambda_1 > 3\pi^2/2 \) (see page 511 of [3]). We shall use the Fourier expansion of automorphic forms associated to \( SL_3(\mathbb{Z}) \), as developed in Chapter IV of [1].

Assume that \( \phi \) is a non-constant automorphic form, so then \( \Delta \phi = \lambda \phi \) and \( \phi \Delta \phi = \lambda \phi^2 \). Through integration by parts, using the automorphic boundary conditions, and the fact that the Siegel domain \( S \) is contained in ten translates of the fundamental domain \( D \), we obtain the inequality

\[
\int_{S} |\nabla \phi|^2 dV < 10 \lambda.
\]

As on page 67 of [1], let us expand \( \phi \) in a Fourier expansion with respect to the abelian group

\[
\left\{ \begin{bmatrix} 1 & 0 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} : \xi_1, \xi_3 \in \mathbb{R} \right\}.
\]

Specifically, we have \( \phi(\tau) = \sum_{n_1, n_3} \phi_{n_1}^{n_3} (\tau) \) where

\[
\phi_{n_1}^{n_3} (\tau) = \int_{0}^{1} \int_{0}^{1} \phi \left( \begin{bmatrix} 1 & 0 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \tau \right) e^{-2\pi i (n_1 \xi_1 + n_3 \xi_3)} d\xi_1 d\xi_3.
\]

Observe that \( \phi_0^0 = 0 \) since \( \phi \) is not constant and square integrable, hence cuspidal. Let

\[
\Gamma^2_1 = \left\{ \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \right\}.
\]
and set $\Gamma^2_{\infty}$ to be the subgroup of $\Gamma^2_1$ which stablizes infinity. As on page 69 of [1], we then can write $\phi_{n_1}^{n_3}$ as

$$
\phi_{n_1}^{n_3}(\tau) = \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2_1} \sum_{n_1=1}^{\infty} \phi_{n_1}^{0}(\gamma \circ \tau).
$$

By a standard application of elliptic regularity, $\phi$ is necessarily $C^\infty$, hence we can interchange integration and summation and apply Parseval’s theorem to obtain the inequality

$$
\int \frac{\sum_{S \atop n_1=1}^{\infty} \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2_1} \nabla_{\tau} \phi_{n_1}^{0}(\gamma \circ \tau)^2}{\sum_{S \atop n_1=1}^{\infty} \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2_1} \phi_{n_1}^{0}(\gamma \circ \tau)^2} dV < 10\lambda.
$$

Since $\nabla$ is an invariant operator, we may differentiate the expressions in the numerator and then evaluate at $\gamma \circ \tau$, thus yielding

$$
\int \frac{\sum_{S \atop n_1=1}^{\infty} \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2_1} \nabla_{\tau} \phi_{n_1}^{0}(\gamma \circ \tau)|_{\gamma \circ \tau}^2}{\sum_{S \atop n_1=1}^{\infty} \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2_1} \phi_{n_1}^{0}(\gamma \circ \tau)|_{\gamma \circ \tau}^2} dV < 10\lambda.
$$

We now integrate by parts and consider the action of the Laplacian $\Delta$ on functions of the form $\phi_{n_1}^{0}$. Since each function $\phi_{n_1}^{0}$ is independent of $x_3$, these terms in $\Delta$ annihilate $\phi_{n_1}^{0}$. Observe that all terms involving $y_1$, $y_2$ and $x_2$ are positive operators (compare with line (2.31) on page 32 of [1]), so we obtain the bound

$$
\Delta \phi_{n_1}^{0} \geq -y_1^2 \cdot \frac{\partial^2}{\partial x_1^2} \phi_{n_1}^{0} = y_1^2 \cdot 4\pi^2 n_1^2 \phi_{n_1}^{0}.
$$

Since $y_1^2 > \frac{3}{4}$, we have $\Delta \phi_{n_1}^{0} \geq \frac{3}{4} \cdot 4\pi^2 n_1^2 \phi_{n_1}^{0} = 3\pi^2 n_1^2 \phi_{n_1}^{0}$. Combining this inequality with the above calculations and the cuspidality condition $\phi_0^{0} = 0$, we obtain

$$
10\lambda > \int \frac{\sum_{S \atop n_1=1}^{\infty} \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2_1} \Delta_{\tau} \phi_{n_1}^{0}(\gamma \circ \tau) \cdot \phi_{n_1}^{0}(\tau)|_{\gamma \circ \tau} dV}{\sum_{S \atop n_1=1}^{\infty} \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2_1} \phi_{n_1}^{0}(\tau)|_{\gamma \circ \tau}^2} \geq 3\pi^2,
$$

$$
\int \frac{\sum_{S \atop n_1=1}^{\infty} \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2_1} \phi_{n_1}^{0}(\tau)|_{\gamma \circ \tau} dV}{\sum_{S \atop n_1=1}^{\infty} \sum_{\gamma \in \Gamma^2_\infty \backslash \Gamma^2_1} \phi_{n_1}^{0}(\tau)|_{\gamma \circ \tau}^2}.
$$
hence $\lambda \geq 3\pi^2/10$. Since $\phi$ was any cusp form, we obtain the bound as asserted in the theorem.

§4. Concluding remarks

As the continuous spectrum in this situation is $[1, \infty)$, our theorem implies an analogue of Selberg’s eigenvalue conjecture. Note that our bound is stronger than result for $SL_3(\mathbb{Z})$ from [4] who proved $\lambda_1 \geq 1$. In general, our method applies to $G = SL_n(\mathbb{R})$ with $\Gamma = SL_n(\mathbb{Z})$ to give the bound $\lambda_1 > 3\pi^2/M$ where $M$ is the number of fundamental domains which intersect a Siegel set containing the fundamental domain constructed in [2]; however, for $n \geq 4$, this bound is rather weak. Finally, let us remark that our theorem is indeed a consequence of the Ramanujan conjecture, which asserts that all nontrivial automorphic representations come from tempered representations.

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