THE EFFECTS OF VIEWING ANGLE ON THE MASS DISTRIBUTION OF EXOPLANETS

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ABSTRACT

We present a mathematical method to statistically decouple the effects of unknown inclination angles on the mass distribution of exoplanets that have been discovered using radial-velocity (RV) techniques. The method is based on the distribution of the product of two random variables. Thus, if one assumes a true mass distribution, the method makes it possible to recover the observed distribution. We compare our prediction with available RV data. Assuming that the true mass function is described by a power law, the minimum mass function that we recover proves a good fit to the observed distribution at both mass ends. In particular, it provides an alternative explanation for the observed low-mass decline, usually explained as sample incompleteness. In addition, the peak observed near the low-mass end arises naturally in the predicted distribution as a consequence of imposing a low-mass cutoff in the true distribution. If the low-mass bins below 0.02 $M_T$ are complete, then the mass distribution in this regime is heavily affected by the small fraction of lowly inclined interlopers that are actually more massive companions. Finally, we also present evidence that the exoplanet mass distribution changes form toward low mass, implying that a single power law may not adequately describe the sample population.

Key words: planetary systems – techniques: radial velocities

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1. INTRODUCTION

Samples of exoplanetary systems are increasing rapidly thanks to new ground- and space-based dedicated surveys, thus enabling investigation of their statistical properties. One of these properties is the planetary mass distribution, a key aspect needed to understand the origin of exoplanets and its relation to the initial mass function. Currently, radial-velocity (RV) detections (e.g., Mayor et al. 1983; Butler et al. 1996; Jones et al. 2010) have provided the largest sample of unconstrained systems. However, the RV technique does not provide masses directly because the line-of-sight inclination angles, $i$, cannot be measured unless complementary observations are carried out, for instance transit photometry (Henry et al. 2000) or astrometry (Benedict et al. 2006). Thus, all masses measured with this technique are indeed “minimum” planet masses, $M_{\text{obs}} = M_T \sin i$, where $M_T$, the “true” planet mass, is not known a priori.

Understanding the true mass distribution rather than the minimum mass distribution will allow modelers to compare their mass distributions against a function that is free from one of the largest sources of uncertainty (see Mordasini et al. 2009; Ida & Lin 2005). Also, the sin$i$ degeneracy that plagues RV signals means we can never be fully sure that any individual signature is planetary in nature from the Doppler data alone. This has consequences for a number of aspects of planetary, brown dwarf, and low-mass star studies that deal with inferences drawn from an RV-dominated mass distribution. A prime example of this would be the proper location in mass of the planet–brown-dwarf boundary (see Sahlmann et al. 2011), which allows one to clarify the status of an object as either a planet or a brown dwarf (e.g., Jenkins et al. 2009) and will help us to better understand the formation mechanisms of both classes of objects.

It is thought that the sin$i$ correction is of order unity and would preserve the power-law shape of the observed mass distribution (e.g., Jorissen et al. 2001; Tabachnik & Tremaine 2002; Hubbard et al. 2007; Morton & Johnson 2011). However, no proof has been provided to substantiate this. Methods to recover the $M_T$ distribution from the observed minimum-mass data have been proposed based on: (1) numerically solving an Abel-type integral equation that relates observed and true mass distributions (Jorissen et al. 2001), (2) analytically finding the distribution that maximizes the likelihood of having a given set of minimum masses (Zucker & Mazeh 2001), (3) comparing cumulative distributions of projected and de-projected data with non-parametric statistical tools (Brown 2011), and (4) using the physics of multi-planet systems to resolve the sin$i$ correction (Batygin & Laughlin 2011). With the exception of (4), which is a theoretical prediction, these methods are non-parametric, i.e., they do not assume an a priori model of the data. However, they also suffer from some drawbacks like the need to smooth the data (1 and 3), or the complexity of introducing observational limits (2).

In this paper, we present an alternative method to statistically decouple the sin$i$ dependence in observed exoplanet mass functions. The method is based on the expected distribution of the product of two continuous and independent random variables. Its parametric nature requires an assumption on the shape of the underlying ($M_T$) distribution; but, on the other hand, it offers the possibility of introducing observational constraints in a straightforward fashion. The mathematical problem, applied to planet mass distributions, is stated in Section 2, and its solution is presented in Section 3. In Section 4, the method is implemented on two example distributions and in Section 5, we make a comparison with observational data. In Section 6, we look at the significance of the shape of the true mass distribution and how this may affect current models of planet formation and evolution. Finally, in Section 7 we summarize the results.

2. THE PROBLEM AND OUR CONCEPT

The problem can be summarized as follows: given an analytical model of the distribution of true planetary masses, is it possible to obtain the distribution of minimum-masses analytically by assuming a random distribution of inclination angles?
The answer to this question is yes, and relies on computing the probability density function (PDF) of the product of two random variables. The problem of finding the PDF of the product of two random variables was first solved by Rohatgi (1976) but its implementation relied on knowledge of the joint PDF of the two variables. In this paper, we apply the approach followed by Glen et al. (2004), which uses the individual PDFs of the two random variables.

To do so, let \( X, Y, \) and \( Z \) be random variables, such that

\[
X = M_T
\]

\[
Y = \sin i
\]

\[
Z = XY.
\]

The above equations state that \( X \) takes the value of any exoplanetary mass, \( M_T \). \( Y \) takes the value of any correction by inclination (viewing) angle \( i \), \( \sin i \), and \( Z \) takes the value of any possible product \( M_T \sin i \). Let also \( f_X, f_Y, \) and \( f_Z \) be their respective PDFs. We wish to obtain \( f_Z \), a prediction for the observed distribution of “minimum” masses, given \( f_X \) (an assumption for the true mass distribution) and \( f_Y \) (which can be calculated analytically).

3. SOLUTION

3.1. PDF of \( X \)

Let \( f_X \) represent the distribution of planet true masses. For simplicity, we assume that \( f_X \) is well described by a power law (Butler et al. 2006). We define a power law of index \(-(1 + \alpha)\), such that it is normalized over the mass interval \([m_{\text{min}}, m_{\text{max}}]\), defined to be the minimum and maximum true masses. Note that a minimum non-zero mass avoids a diverging integral, which is a condition for \( f_X \) to be normalized. Thus, \( f_X \) becomes

\[
f_X(x)dx = \begin{cases} A_X x^{a-1} dx, & 0 < m_{\text{min}} < x < m_{\text{max}} \\ 0, & \text{otherwise,} \end{cases}
\]

where \( x \) represents true masses, \( A_X = \alpha/(m_{\text{min}}^{1+\alpha} - m_{\text{max}}^{1+\alpha}) \) is a normalization constant, and \( \alpha > -1 \). If \( \alpha = 0 \), then \( A_X = (\ln(m_{\text{max}}/m_{\text{min}}))^{-1} \). It is important to note that \( X \) can be treated as a continuous variable and that the present analysis does not require that masses be normalized (i.e., \( x \) can be in any units). An example of this PDF, representing the true mass distribution, is shown in the top panel of Figure 1.

3.2. PDF of \( Y \)

Consider randomly distributed inclination angles, \( i \). The observed inclination angles can be shown to be distributed like \( \sin i \), over the interval \([0, \pi/2]\). This is straightforward to see in spherical coordinates (e.g., Ho & Turner 2011) and implies that higher inclination angles (edge-on) are more probable than lower ones (pole-on).

To get \( f_Y \) for a \( \sin i \) distribution of angles, let \( y = \sin i \). Then \( i = \arcsin y \), with \( i \in [0, \pi/2] \). The corresponding cumulative distribution (CDF) is given by \( 1 - \cos i \). Thus, the differential of this CDF \( d(1 - \cos [i]) = d(1 - \cos [\arcsin y]) \) gives \( f_Y \):

\[
f_Y(y)dy = \frac{y}{\sqrt{1 - y^2}} dy, \quad 0 < y < 1.
\]

The above equation considers all possible angles. However, it may be useful to define a minimum inclination angle, \( i_{\text{min}} \), to account for possible selection effects when comparing with real data. In this case \( f_Y \) becomes

\[
f_Y(y)dy = A_Y \frac{y}{\sqrt{1 - y^2}} dy, \quad \sin i_{\text{min}} < y < 1,
\]

where \( A_Y = 1/\sqrt{1 - \sin (i_{\text{min}})^2} \) is a normalization constant. An example of this PDF, representing the \( \sin i \) distribution, is shown in the bottom panel of Figure 1.

3.3. PDF of \( Z \)

Having defined \( f_X \) and \( f_Y \), the PDF of random variables \( X \) and \( Y \), we can now calculate \( f_Z \), the PDF of the product \( XY \). Following Glen et al.’s (2004) solution for the PDF of the product of two continuous and independent random variables, and also considering the above boundaries, \( f_Z(z) \) can be expressed as

\[
f_Z(z) = \frac{z^{\alpha/2} \sin i_{\text{min}}}{m_{\text{min}}^{1+\alpha} - m_{\text{max}}^{1+\alpha}} f_X(z) f_Y(u) du, \quad m_{\text{min}} < z < m_{\text{min}} \sin i_{\text{min}},
\]

\[
f_Z(z) = \frac{z^{\alpha/2} \sin i_{\text{min}}}{m_{\text{min}}^{1+\alpha} - m_{\text{max}}^{1+\alpha}} f_X(u) f_Y(z) du, \quad m_{\text{min}} < z < m_{\text{max}} \sin i_{\text{min}},
\]

\[
f_Z(z) = \frac{z^{\alpha/2} \sin i_{\text{min}}}{m_{\text{min}}^{1+\alpha} - m_{\text{max}}^{1+\alpha}} f_X(u) f_Y(u) du, \quad m_{\text{min}} \sin i_{\text{min}} < z < m_{\text{max}}.
\]

provided that \( m_{\text{min}} < m_{\text{max}} \sin i_{\text{min}} \). This condition is the equivalent of setting a lower limit on \( i \), such that pole-on orbits, producing very large corrections, are excluded from the observed sample. It also determines three “validity regions.”

Replacing Equation (7) with Equations (4) and (6), after some algebra and getting rid of the integration limits for a moment, Equation (7) reads

\[
f_Z(z) = A_X A_Y z \int \frac{u^{1-\alpha} - 1}{\sqrt{u^2 - z^2}} du.
\]

For any value of \( \alpha \), the improper integral in Equation (8) has a primitive in terms of \( zF_{1} \), the first hypergeometric function.
angles, normal distribution of true masses, (Abramowitz & Stegun 1972). From Equation (7), note also that with
\[
\mu = \frac{\ln \sigma^2}{2},
\]

the predicted minimum-mass distribution becomes
\[
\text{predicted minimum-mass distribution} = f_Z(x) = \frac{1}{x \sqrt{2\pi}} \exp \left( -\frac{\ln x - \mu^2}{2\sigma^2} \right),
\]

(10)
with \( \mu = 0 \) and \( \sigma = 0.5 \).

We conclude by emphasizing that setting a particular value of \( i_{\text{min}} \) simulates observational selection effects, meaning that the model mimics observational samples which have missed objects at angles below that limit. Figure 2 shows that including those lowly inclined systems, despite them being less probable, induces dramatic changes in the shape of the predicted distribution at the low-mass end.

We now proceed to compare our models with RV data.

5. COMPARISON WITH OBSERVATIONAL DATA

Figure 3 displays the observed distribution of minimum masses (Schneider et al. 2011; RV data on 643 planets as of October 17 taken from http://exoplanet.eu). Bin sizes were arbitrarily chosen to have a roughly equal number of systems. Overlaid is the model prediction of minimum masses given by Equation (9), i.e., the prediction that results from assuming a true mass distribution of the form \( \propto M_T^{-1} \). We have chosen the following physical parameters for the true mass distribution: \( m_{\text{min}} = 0.02 M_J \) and \( m_{\text{max}} = 22 M_J \), where the lower limit is chosen from observation and the upper limit marks the planetary–brown-dwarf mass boundary (Sahli et al. 2011). To simulate observational selection effects, \( \sin i_{\text{min}} = 0.1 \) was chosen. We emphasize that these parameters are not the result of a fit, but were adjusted manually to attempt a best match to the data.

It can be seen that the power-law part of the predicted curve (Figure 3) only poorly fits the data. However, interestingly enough, both of its extremes do seem to better reproduce the data.

At the low-mass limit (\( z < 0.02 M_J \)) the model describes mass bins which are not “occupied” in the true mass distribution. As already mentioned, the shape of this tail is affected mostly by the inclusion of large \( \sin i \) corrections, i.e., lowly inclined systems, which in turn produce a decrease at the low-mass end of the observed distribution. Such a decrease mimics observational effects on the data like those ones induced by incompleteness (Udry & Santos 2007), as we discuss further in the next section.

At the high-mass end, there is no apparent reason for a systematic lack of observed systems. Here, the data show a decline which is also in good agreement with our model’s
We have shown that if the true mass distribution is described by a power law with boundaries, there must be a peak in the observed mass distribution of exoplanets, and therefore this peak has implications for planet formation and evolution models. A peak in the planetary mass distribution tells us that most of the mass of the proto-planetary disk that goes into forming planets gets locked up in the formation of the larger gas and ice giants. This agrees with the mass distribution of the planets in the solar system.

The widely accepted planetary formation theory is by core accretion and subsequent planet migration (Pollack 1984; Lin & Papaloizou 1986) and this model can broadly explain the currently observed population of exoplanets. The peak in the planetary mass distribution needs to be taken into account when comparing the outcomes from core accretion formation models against the observed mass distribution of exoplanets, unless the true mass distribution changes form toward the lowest masses.

One interesting question that arises is, what does the position of the peak in the mass distribution tell us and what does it mean? As we have seen in Figure 3, our true mass distribution model can provide a good fit to the observed mass distribution of the current population of exoplanets, particularly the peak and subsequent decline.

The low-mass peak we find that best describes the observed data is located around $0.02 \, M_J$ or $\sim 6.5 \, M_\oplus$. When we look at systems that have high inclinations, like an observed mass distribution drawn from transiting planets only, we find that for a complete sample, the observed distribution follows the true distribution with no low-mass peak. Therefore, if we assume that the bins are complete below $0.02 \, M_J$, then this is the regime where we begin to observe the effects of systems with low inclination angles, and hence large mass corrections.

Here, we are assuming we have sampled all angles above a certain limit, $i_{\text{min}}$, but our model considers the lower likelihood of observing systems with low inclinations; therefore this tells us something important: that even a small fraction of systems with low inclinations can produce large changes in the observed mass distribution in the low-mass regime. This result is in line with the implications of the analysis by Ho & Turner (2011). These authors demonstrate, using Bayesian analysis, that the posterior distribution of angles is determined by the particular true mass distribution and so the latter cannot be simply obtained from the observed one. Unlike Ho & Turner, we deal here with the prior distribution of angles and study its effects on the assumed shape of the true mass distribution; however, both approaches lead to the conclusion that the low inclination systems modify the low-mass end of the observed distribution.

Also, since most of the observed distribution above the $0.02 \, M_J$ boundary broadly follows the true distribution then small to medium values of $\sin i$ do not affect the overall mass distribution in a large manner. We make it clear that most of the low-mass systems in these bins are genuine rocky planets, but that the small numbers of low-inclination/high-mass interlopers cause a dramatic change to the observed mass distribution, again, assuming that the low-mass bins are complete.

Finally, we do caution that the true mass distribution we are discussing here applies to planets with small semimajor axes ($\leq 4 \, \text{AU}$s or so). The distribution of mass at ever increasing distances from the central star may change the shape of the true mass distribution, but further analysis on this issue is likely to require many more detections like the directly imaged planets around HR8799 (Marois et al. 2008) or planets discovered by microlensing techniques (e.g., Muraki et al. 2011).

7. SUMMARY AND OUTLOOK

We have applied the formal solution for the PDF of the product of two independent random variables to the observational problem of decoupling the $\sin i$ factor from an observed sample of exoplanet masses. Our approach requires that the true mass distribution is modeled by a continuous function that represents the PDF (within given physical or observational limits).

We have shown that if the true mass function is modeled as a power law, comparison with observed data of 643 RV planets shows a good match with our method’s prediction, specifically at both mass ends. In particular, the prediction agrees well with the decline observed toward the low-mass end, thus providing...
an alternative explanation to the turnover being the result of observational biases.

If the low-mass bins below 0.02 \( M_\text{J} \) are assumed to be complete, then we show that the presence of a small number of systems with low inclinations, and hence much larger true masses, heavily affects the distribution in this regime. Such effects are not seen above this mass region where the \( \sin i \) values are more modest and hence corrections are smaller, meaning the true distribution is being matched more closely.

We also again note a mass paucity around 0.1 \( M_\text{J} \) in the observed mass distribution, as the single power-law model does not describe this region very well at all. In fact, it may be prudent to examine the mass distribution using more than one function, like a double power law for instance. This may indicate that the mass distribution changes form below around 0.1 \( M_\text{J} \), a feature we plan to study more in the future.

In summary, we have provided a practical and intuitive method to decouple the \( \sin i \) effect that is inherent to RV samples. The method offers the possibility of introducing observational constraints in a straightforward fashion, in order to compare predictions with current observations.

Current RV surveys are plagued with biases and incompleteness that affect conclusions drawn from analyzing any Doppler data set. For instance, the RV detection method is heavily biased toward the detection of more massive companions on short period orbits, since they induce a larger reflex motion on the host star in comparison with less massive and longer period companions. Therefore, the detection of very low mass planets is only now being fully realized and requires large data sets and novel detection/characterization methods (e.g., Vogt et al. 2010; Pepe et al. 2011; Anglada-Escude et al. 2012; Jenkins et al. 2012). Hence, studying the low-mass end of the mass distribution is one of the cornerstones of exoplanet research at the present time and will continue to be so in the near future.

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