NON-PARAMETRIC MEAN CURVATURE FLOW WITH PRESCRIBED CONTACT ANGLE IN RIEMANNIAN PRODUCTS

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ABSTRACT. Assuming that there exists a translating soliton \( u_\infty \) with speed \( C \) in a domain \( \Omega \) and with prescribed contact angle on \( \partial \Omega \), we prove that a graphical solution to the mean curvature flow with the same prescribed contact angle converges to \( u_\infty + Ct \) as \( t \to \infty \). We also generalize the recent existence result of Gao, Ma, Wang and Weng to non-Euclidean settings under suitable bounds on convexity of \( \Omega \) and Ricci curvature in \( \Omega \).

1. INTRODUCTION

We study a non-parametric mean curvature flow in a Riemannian product \( N \times \mathbb{R} \) represented by graphs

\[
M_t := \{ (x, u(x,t)) : x \in \bar{\Omega} \} \tag{1.1}
\]

with prescribed contact angle with the cylinder \( \partial \Omega \times \mathbb{R} \).

We assume that \( N \) is a Riemannian manifold and \( \Omega \subset N \) is a relatively compact domain with smooth boundary \( \partial \Omega \). We denote by \( \gamma \) the inward pointing unit normal vector field to \( \partial \Omega \). The boundary condition is determined by a given smooth function \( \phi \in C^\infty(\partial \Omega) \), with \( |\phi| \leq \phi_0 < 1 \), and the initial condition by a smooth function \( u_0 \in C^\infty(\bar{\Omega}) \).

The function \( u \) above in (1.1) is a solution to the following evolution equation

\[
\begin{cases}
\frac{\partial u}{\partial t} = W \text{div} \frac{\nabla u}{W} \quad &\text{in } \Omega \times [0, \infty), \\
\frac{\partial \gamma u}{\partial W} := \frac{\langle \nabla u, \gamma \rangle}{W} = \phi \quad &\text{on } \partial \Omega \times [0, \infty), \\
u(x,0) = u_0 \quad &\text{in } \bar{\Omega},
\end{cases}
\tag{1.2}
\]

where \( W = \sqrt{1 + |\nabla u|^2} \) and \( \nabla u \) denotes the gradient of \( u \) with respect to the Riemannian metric on \( N \) at \( x \in \bar{\Omega} \). The boundary condition above can be written as

\[
\langle \nu, \gamma \rangle = \phi, \tag{1.3}
\]

where \( \nu \) is the downward pointing unit normal to the graph of \( u \), i.e.

\[
\nu(x) = \frac{\nabla u(x, \cdot) - \partial_i}{\sqrt{1 + |\nabla u(x, \cdot)|^2}}, \quad x \in \bar{\Omega}.
\]
The longtime existence of the solution \( u_t := u(\cdot, t) \) to (1.2) and convergence as \( t \to \infty \) have been studied under various conditions on \( \Omega \) and \( \phi \). Huisken [5] proved the existence of a smooth solution in a \( C^{2,\alpha} \)-smooth bounded domain \( \Omega \subset \mathbb{R}^n \) for \( u_0 \in C^{2,\alpha}(\bar{\Omega}) \) and \( \phi \equiv 0 \). Moreover, he showed that \( u_t \) converges to a constant function as \( t \to \infty \). In [1] Altschuler and Wu complemented Huisken’s results for prescribed contact angle in case \( \Omega \) is a smooth bounded strictly convex domain in \( \mathbb{R}^2 \). Guan [4] proved a priori gradient estimates and established longtime existence of solutions in case \( \Omega \subset \mathbb{R}^n \) is a smooth bounded domain. Recently, Zhou [8] studied mean curvature type flows in a Riemannian product \( \mathcal{M} \times \mathbb{R} \) and proved the longtime existence of the solution for relatively compact smooth domains \( \Omega \subset \mathcal{M} \). Furthermore, he extended the convergence result of Altschuler and Wu to the case \( \mathcal{M} \) is a Riemannian surface with nonnegative curvature and \( \Omega \subset \mathcal{M} \) is a smooth bounded strictly convex domain; see [8, Theorem 1.4].

The key ingredient, and at the same time the main obstacle, for proving the uniform convergence of \( u_t \) has been a difficulty to obtain a time-independent gradient estimate. We circumvent this obstacle by modifying the method of Korevaar [6], Guan [4] and Zhou [8] and obtain a uniform gradient estimate in an arbitrary relatively compact smooth domain \( \Omega \subset \mathbb{N} \) provided there exists a translating soliton with speed \( C \) and with the prescribed contact angle condition (1.3).

Towards this end, let \( d \) be a smooth bounded function defined in some neighborhood of \( \bar{\Omega} \) such that \( d(x) = \min_{y \in \partial \Omega} \text{dist}(x, y) \), the distance to the boundary \( \partial \Omega \), for points \( x \in \Omega \) sufficiently close to \( \partial \Omega \). Thus \( \gamma = \nabla d \) on \( \partial \Omega \). We assume that \( 0 \leq d \leq 1 \), \( |\nabla d| \leq 1 \) and \( |\text{Hess} d| \leq C_d \) in \( \bar{\Omega} \). We also assume that the function \( \phi \in C^\infty(\partial \Omega) \) is extended as a smooth function to the whole \( \bar{\Omega} \), satisfying the condition \( |\phi| \leq \phi_0 < 1 \).

Our main theorem is the following:

**Theorem 1.1.** Suppose that there exists a solution \( u_\infty \) to the translating soliton equation

\[
\begin{aligned}
\text{div} \frac{\nabla u_\infty}{\sqrt{1 + |\nabla u_\infty|^2}} &= \frac{C_\infty}{\sqrt{1 + |\nabla u_\infty|^2}} \quad \text{in } \Omega, \\
\frac{\partial_t u_\infty}{\sqrt{1 + |\nabla u_\infty|^2}} &= \phi \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.4)

where \( C_\infty \) is given by

\[
C_\infty = \frac{-\int_{\partial \Omega} \phi \, d\sigma}{\int_{\Omega} (1 + |\nabla u_\infty|^2)^{-1/2} \, dx}.
\]

Then the equation (1.2) has a smooth solution \( u \in C^\infty(\bar{\Omega}, [0, \infty)) \) with \( W \leq C_1 \), where \( C_1 \) is a constant depending on \( \phi, u_0, C_d, \) and the Ricci curvature of \( \bar{\Omega} \). Moreover, \( u(x, t) \) converges uniformly to \( u_\infty(x) + C_\infty t \) as \( t \to \infty \).

Notice that the existence of a solution \( u \in C^\infty(\bar{\Omega} \times [0, \infty)) \) to (1.2) is given by [8, Corollary 4.2].

**Remark 1.2.** Very recently, Gao, Ma, Wang, and Weng [3] proved the existence of such \( u_\infty \) and obtained Theorem 1.1 for smooth, bounded, strictly
convex domains $\Omega \subset \mathbb{R}^n$ for sufficiently small $|\phi|$; see [3] Theorem 1.1, Theorem 3.1. It turns out that their proof can be generalized beyond the Euclidean setting under suitable bounds on the convexity of $\Omega$ and the Ricci curvature in $\Omega$.

More precisely, let $\Omega \Subset N$ be a relatively compact, strictly convex domain with smooth boundary admitting a smooth defining function $h$ such that $h < 0$ in $\Omega$, $h = 0$ on $\partial \Omega$,

$$(h_{ij}) \geq k_1(\delta_{ij})$$

for some constant $k_1 > 0$ and sup $\Omega |\nabla h| \leq 1$, $h_{\nu} = -1$ and $|\nabla h| = 1$ on $\partial \Omega$. Furthermore, by strict convexity of $\Omega$, the second fundamental form of $\partial \Omega$ satisfies

$$(\kappa_{ij})_{1 \leq i, j \leq n-1} \geq \kappa_0(\delta_{ij})_{1 \leq i, j \leq n-1},$$

where $\kappa_0 > 0$ is the minimal principal curvature of $\partial \Omega$. In the Euclidean case, $N = \mathbb{R}^n$, such functions $h$ are constructed in [2]. We give some simple examples at the end of Section 3.

**Theorem 1.3.** Let $\Omega \Subset N$ be a smooth, strictly convex, relatively compact domain associated with constants $k_1 > 0$ and $\kappa_0 > 0$ as in (1.6) and (1.7). Let $\alpha < \min\{\kappa_0, k_1(n-1)/2\}$ and assume that the Ricci curvature in $\Omega$ satisfies $|\text{Ric}| < \alpha(k_1(n-1) - \alpha)/(n+1)$. Then there exists $\varepsilon_0 > 0$ such that if $\phi = \cos \theta \in C^3(\Omega)$ satisfies $|\cos \theta| \leq \varepsilon_0 \leq 1/4$ and $||\nabla \theta||_{C^1(\Omega)} \leq \varepsilon_0$ in $\Omega$, there exist a unique constant $C_\infty$ and a solution $u_\infty$ to (1.4). Furthermore, $u_\infty$ is unique up to an additive constant.

We will sketch the proof of Theorem 1.3 in Section 3.

2. PROOF OF THEOREM 1.3

Let $u$ be a solution to (1.2) in $\bar{\Omega} \times \mathbb{R}$. Given a constant $C_\infty \in \mathbb{R}$ we define, following the ideas of Korevaar [6], Guan [4] and Zhou [8], a function $\eta: \bar{\Omega} \times \mathbb{R} \rightarrow (0, \infty)$ by setting

$$\eta = e^{K(u - C_\infty)} \left( Sd + 1 - \frac{\phi}{W} \langle \nabla u, \nabla d \rangle \right),$$

where $K$ and $S$ are positive constants to be determined later. We start with a gradient estimate.

**Proposition 2.1.** Let $u$ be a solution to (1.2) and define $\eta$ as in (2.1). Then, for a fixed $T > 0$, letting

$$(W\eta)(x_0, t_0) = \max_{x \in \Omega, t \in [0, T]} (W\eta)(x, t),$$

there exists a constant $C_0$ only depending on $C_d, \phi, C_\infty$, and the lower bound for the Ricci curvature in $\Omega$ such that $W(x_0, t_0) \leq C_0$.

**Proof.** Let $g = g_{ij} dx^i dx^j$ be the Riemannian metric of $N$. We denote by $(g^{ij})$ the inverse of $(g_{ij})$, $u_j = \partial u / \partial x^j$, and $u_{ij} = u_{ij} - \Gamma^k_{ij} u_k$. We set

$$a^{ij} = g^{ij} - u^{ij} u / W^2$$

such that $a^{ij} \geq k_1(\delta_{ij})$. If we denote by $(\partial_t + L) u$ the time derivative of $u$ under the evolution (1.2), then

$$(\partial_t + L) u \leq a^{ij} u_{ij} - \phi u + \varepsilon_0,$$

for $\varepsilon_0 > 0$, which will be determined later. If we set $W = e^\eta$, then

$$(W\eta) = W^{-1} \partial_t W + W^{-1} L W,$$

and

$$(W\eta)(x_0, t_0) \leq e^{K(u - C_\infty)} S d(x_0) + 1 - \frac{\phi}{W} \langle \nabla u, \nabla d \rangle.$$
and define an operator $L$ by $Lu = a_{ij}u_{ij} - \partial_t u$. Observe that (1.2) can be rewritten as $Lu = 0$. In all the following, computations will be done at the maximum point $(x_0, t_0)$ of $\eta W$. We first consider the case where $x_0 \in \partial \Omega$. We choose normal coordinates at $x_0$ such that $g_{ij} = \delta^{ij} = \delta_{ij}$ at $x_0$, $\partial_n = \gamma$,

$$u_1 \geq 0, \quad u_i = 0 \quad \text{for} \quad 2 \leq i \leq n - 1.$$ 

This implies that

$$d_i = 0 \quad \text{for} \quad 1 \leq i \leq n - 1, \quad d_n = 1, \quad \text{and} \quad d_{i,n} = 0 \quad \text{for} \quad 1 \leq i \leq n.$$ 

We have

$$0 \geq (W\eta)_n = W_n\eta + W\eta_n$$

$$= e^{K(u-C_\nu)} \left( SW_n d + W_n - \frac{\phi W_n}{W} g^{ij} u_i d_j + SW d_n - \frac{W}{W} \phi_n g^{ij} u_i d_j ight) - W \phi g^{ij} (u_{i,n} d_j + u_n d_{j,n}) + W \frac{W_n}{W^2} \phi g^{ij} u_i d_j + KW u_n (Sd + 1 - \frac{\phi}{W} g^{ij} u_i d_j)$$

$$= e^{K(u-C_\nu)} \left( W_n + SW - \phi_n u_n - \phi u_{n,n} + KW u_n (1 - \phi^2) \right). \quad (2.2)$$

Using our coordinate system, we get

$$0 \geq \frac{W_n}{W} + S - \frac{\phi_n u_n}{W} - \frac{\phi u_{n,n}}{W} + Ku_n (1 - \phi^2)$$

$$= S - \frac{u_1^2 d_{1,1}}{W^2} + u_1 \phi_1 \left( 1 + \frac{2\phi^2}{1 - \phi^2} \right) - \frac{\phi u_1}{W} Ku_1$$

$$- \frac{\phi_n u_n}{W} + Ku_n (1 - \phi^2)$$

$$\geq S - C - \frac{K \phi u_1^2}{W} + Ku_n (1 - \phi^2)$$

$$= S - C - \frac{K \phi}{W} \geq S - C - \frac{K}{W},$$

for some constant $C$ depending only on $C_d$ and $\phi$. So choosing $S \geq C + 1$, we get that

$$W(x_0, t_0) \leq K. \quad (2.3)$$

Next we assume that $x_0 \in \Omega$ and that $S \geq C + 1$, where $C$ is as above. Let us recall from [5], Lemma 3.5] that

$$LW = \frac{2}{W} a^{ij} W_i W_j + \text{Ric}(v_N, v_N) W + |A|^2 W,$$

where $v_N = \nabla u / W$ and $|A|^2 = a^{ik} a_{kj} u_i u_j / W^2$ is the squared norm of the second fundamental form of the graph $M_i$. Since $0 = W_t \eta + W \eta_t$, for every $i = 1, \ldots, n$, we deduce that

$$0 \geq L(W\eta) = WL\eta + \eta \left( LW - 2a^{ij} \frac{W_i W_j}{W} \right)$$

$$= WL\eta + \eta \left( |A|^2 + \text{Ric}(v_N, v_N) \right).$$
This yields to
\[ \frac{1}{\eta} L \eta + |A|^2 + \text{Ric}(v_N, v_N) \leq 0. \] (2.4)

To simplify the notation, we set
\[ h = Sd + 1 - \phi u^k d_k / W = Sd + 1 - \phi v^k d_k. \]

So we have
\[ \frac{1}{\eta} L \eta = K^2 a^{ij} u^i u^j + KL(u - C_\infty t) + \frac{2K}{h} a^{ij} h_i h_j + \frac{1}{h} Lh. \] (2.5)

We can compute \( Lh \) as
\[
Lh = a^{ij}(Sd_{ij} - (\phi d_k)_{ij} v^k - (\phi d_k)_i v^k_j - (\phi d_k)_j v^k_i - \phi d_k L v^k) \\
\geq -C - 2 a^{ij}(\phi d_k)_i v^k_j - \phi d_k L v^k.
\]

Since, by \([8, \text{Lemma 3.5}]\),
\[
L v^k = \text{Ric}(a^{k\ell} \partial_\ell, v_N) - |A|^2 v^k
\]
and, by Young’s inequality for matrices,
\[
a^{ij}(\phi d_k)_i v^k_j = \frac{1}{W}(\phi d_k)_i a^{ij} a^{k\ell} u^{\ell} \leq \frac{|A|^2}{6} + C,
\]
we get the estimate
\[
Lh \geq -C - |A|^2 / 3 + \phi d_k v^k |A|^2
\] (2.6)

by using the assumption that \( \text{Ric} \) is bounded.

Next we turn our attention to the other terms in (2.5). We have
\[
a^{ij} u^i = \frac{u^i}{W^2} \quad \text{and} \quad a^{ij} u^i u^j = 1 - \frac{1}{W^2}. \] (2.7)

Then we note that by the assumptions, we clearly have
\[
KL(u - C_\infty t) = KC_\infty \geq -KC,
\] (2.8)

and we are left to consider
\[
a^{ij} u^i h_j = \frac{u^i h_j}{W^2} = \frac{u^i(Sd_j - (\phi d_k)_{ij} v^k - \phi d_k v^k_j)}{W^2} \\
\geq -C - \frac{\phi d_k u^i v^k_j}{W^2} \\
= -C + \frac{K \phi a^{k\ell} d_k u^\ell}{W} + \frac{\phi}{hW} a^{k\ell} d_k h^\ell \\
= -C + \frac{K \phi a^{k\ell} d_k u^\ell}{W} \\
+ \frac{S \phi a^{k\ell} d_k d_\ell}{hW} - \frac{\phi a^{k\ell} d_k (\phi d_s)_{i} v^s}{hW} - \frac{\phi^2 a^{k\ell} d_s d_s a^m u_{ms} u_{m\ell}}{hW^2} \\
\geq -C + \frac{CK}{W^2} - \frac{|A|^2}{3K}. \] (2.9)
Plugging the estimates (2.6), (2.7), (2.8), and (2.9) into (2.5) and using (2.4) with the Ricci lower bound we obtain
\[ 0 \geq K^2 \left( 1 - \frac{1}{W^2} \right) - CK^2 \left( C + \frac{2K}{h} + \frac{C}{W^2} + \frac{|A|^2}{3K} \right) \]
\[ - \frac{1}{h} \left( C + \frac{|A|^2}{3} \right) + |A|^2 - C \]
\[ = K^2 \left( 1 - \frac{1}{W^2} - \frac{C}{hW^2} \right) - KC \left( 1 + \frac{1}{h} \right) - \frac{|A|^2}{h} \]
\[ + \frac{\phi h}{K} |A|^2 - C. \]

Then collecting the terms including $|A|^2$ and noticing that
\[ 1 - \frac{1}{h} + \frac{\phi h}{K} = \frac{Sd}{K} \geq 0 \]
we have
\[ 0 \geq K^2 \left( 1 - \frac{1}{W^2} - \frac{C}{hW^2} \right) - CK \left( 1 + \frac{1}{h} \right) - C. \]

Now choosing $K$ large enough, we obtain $W(x_0, t_0) \leq C_0$, where $C_0$ depends only on $C_0$, $d$, $\phi$, the lower bound of the Ricci curvature in $\Omega$, and the dimension of $N$. We notice that the constant $C_0$ is independent of $T$. \)

Since
\[ e^K(u(x, t) - C_0 t) (1 - \phi_0) \leq \eta \leq e^K(u(x, t) - C_0 t) (S + 2), \]
we have
\[ W(x, t) \leq \frac{(W\eta)(x_0, t_0)}{\eta(x, t)} \]
\[ \leq \frac{C_0 \eta(x_0, t_0)}{\eta(x, t)}, \]
\[ \leq \\
\[ \leq \frac{C_0(S + 2)}{1 - \phi_0} e^K(u(x_0, t_0) - C_0 t_0 + u(x, t) + C_0 t) \]
for every $(x, t) \in \bar{\Omega} \times [0, T]$.

We observe that the function $u_\infty(x) + Ct$ solves the equation (1.2) with the initial condition $u_0 = u_{\infty}$ if $u_{\infty}$ is a solution to the elliptic equation (1.4) and $C$ is given by (1.5). As in [1] Corollary 2.7, applying a parabolic maximum principle (1.7) we obtain:

**Lemma 2.2.** Suppose that (1.4) admits a solution $u_{\infty}$ with the unique constant $C$ given by (1.5). Let $u$ be a solution to (1.2). Then, we have
\[ |u(x, t) - Ct| \leq c_2, \]
for some constant $c_2$ only depending on $u_0$, $\phi$, and $\Omega$.

**Proof.** Let $V(x, t) = u(x, t) - u_\infty(x)$, where $u_\infty$ is a solution to (1.4). We see that $V$ satisfies
\[
\begin{cases}
\frac{\partial V}{\partial t} = a^{ij} V_{ij} + b^i V_i + C & \text{in } \Omega \times [0, T) \\
\epsilon^{ij} V_i V_j = 0 & \text{on } \partial \Omega \times [0, T),
\end{cases}
\]

where $\tilde{a}^{ij}$, $\tilde{c}^{ij}$ are positive definite matrices and $b^i \in \mathbb{R}$. Then the proof of the lemma follows by applying the maximum principle.

In view of Lemma 2.2 taking $C_\infty = C$, and observing that the constant $C_0$ is independent of $T$, we get from (2.10) a uniform gradient bound.

**Lemma 2.3.** Suppose that (1.4) admits a solution $u_\infty$ with the unique constant $C$ given by (1.5). Let $u$ be a solution to (1.2). Then $W(x,t) \leq C_1$ for all $(x,t) \in \bar{\Omega} \times [0,\infty)$ with a constant $C_1$ depending only on $\phi_0$, $u_0$, and $\Omega$.

Having a uniform gradient bound in our disposal, applying once more the strong maximum principle for linear uniformly parabolic equations, we obtain:

**Theorem 2.4.** Suppose that (1.4) admits a solution $u_\infty$ with the unique constant $C$ given by (1.5). Let $u_1$ and $u_2$ be two solutions of (1.2) with the same prescribed contact angle as $u_\infty$. Let $u = u_1 - u_2$. Then $u$ converges to a constant function as $t \to \infty$. In particular, if $C$ is given by (1.5), then $u_1(x,t) - u_\infty(x) - Ct$ converges uniformly to a constant as $t \to \infty$.

**Proof.** The proof is given in [1, p. 109]. We reproduce it for the reader’s convenience. One can check that $u$ satisfies

$$
\begin{align*}
\frac{\partial u}{\partial t} &= \tilde{a}^{ij}u_{,ij} + b^i u_i \\
\tilde{c}^{ij} u_{,ij} &= 0
\end{align*}
$$

where $\tilde{a}^{ij}$, $\tilde{c}^{ij}$ are positive definite matrices and $b^i \in \mathbb{R}$. By the strong maximum principle, we get that the function $F_u(t) = \max u(\cdot,t) - \min u(\cdot,t) \geq 0$ is either strictly decreasing or $u$ is constant. Assuming on the contrary that $\lim_{t \to \infty} u$ is not a constant function, setting $u_n(\cdot,t) = u(\cdot, t - t_n)$ for some sequence $t_n \to \infty$, we would get a non-constant solution, say $v$, defined on $\Omega \times (-\infty, +\infty)$ for which $F_v$ would be constant. We get a contradiction with the maximum principle.

Theorem 1.1 now follows from Lemma 2.3 and Theorem 2.4.

### 3. Proof of Theorem 1.3

Theorem 1.3 is essentially proven in [3, Theorem 2.1, 3.1]. The only extra ingredient we must take into account in our non-flat case is the following Ricci identity for the Hessian $\varphi_{ij}$ of a smooth function $\varphi$

$$
\varphi_{k,ij} = \varphi_{i,kj} = \varphi_{ik,j} + K_{kij}^\ell \varphi_{\ell}. 
$$

(3.1)

For the convenience of the reader, we mostly use the same notations as in [3]. Thus let $h$ be a smooth defining function of $\Omega$ such that $h \leq 0$ in $\Omega$, $h = 0$ on $\partial \Omega$, $(h_{ij}) \geq k_1(\delta_{ij})$ for some constant $k_1 > 0$ and $\sup_{\Omega} |\nabla h| \leq 1$, $h_\gamma = -1$ and $|\nabla h| = 1$ on $\partial \Omega$. Furthermore, by strict convexity of $\Omega$, the second fundamental form of $\partial \Omega$ satisfies

$$
(k_{ij})_{1 \leq i,j \leq n-1} \geq \kappa_0(\delta_{ij})_{1 \leq i,j \leq n-1},
$$

where $\kappa_0 > 0$ is the minimal principal curvature of $\partial \Omega$. 

We consider the equation
\[
\begin{align*}
\begin{cases}
\partial_i u := \left( g^{ij} - \frac{u'w_i}{1 + |\nabla u|^2} \right) u_{ij} = \varepsilon u & \text{in } \Omega \\
\partial_\gamma u = \phi \sqrt{1 + |\nabla u|^2} & \text{on } \partial \Omega
\end{cases}
\end{align*}
\] (3.2)
for small $\varepsilon > 0$. Writing $\phi = -\cos \theta$, $v = \sqrt{1 + |\nabla u|^2}$ and
\[\Phi(x) = \log w(x) + ah(x),\]
where $w(x) = v - u^j h_j \cos \theta$ and $\alpha > 0$ is a constant to be determined, we assume that the maximum of $\Phi$ is attained in a point $x_0 \in \Omega$. If $x_0 \in \partial \Omega$, we can proceed as in [3] pp. 34-36. Thus choosing $0 < \alpha < \kappa_0$ and $0 < \varepsilon_0 \leq \varepsilon_\alpha < 1$ such that
\[\kappa_0 - \alpha > \frac{\varepsilon_\alpha (M_1 + 3)}{1 - \varepsilon_\alpha},\] (3.3)
where $M_1 = \sup_\Omega |\nabla^2 h|$, yields an upper bound
\[|\nabla' u(x_0)|^2 \leq \frac{\varepsilon_\alpha (M_1 + 3) + \alpha}{\kappa_0 - \alpha - \varepsilon_\alpha (M_1 + 3)} < \frac{\kappa_0}{\kappa_0 - \alpha - \varepsilon_\alpha (M_1 + 3)}\]
for the tangential component of $\nabla u$ on $\partial \Omega$. Combining this with the boundary condition $u_\gamma = -v \cos \theta$ gives an upper bound for $|\nabla u(x_0)|$ and hence for $\Phi(x_0)$.

The only difference to the Euclidean case occurs when $x_0 \in \Omega$, i.e. is an interior point of $\Omega$. At this point we have, using the same notations as in [3] p. 42,
\[0 = \Phi_i(x_0) = \frac{w_j}{w} + ah_i\]
and
\[0 \geq a^{ij} \Phi_{ij}(x_0) = \frac{a^{ij} w_{ij}}{w} - \alpha^2 a^{ij} h_j + \alpha a^{ij} h_{ij} = : I + II + III.\]

We choose normal coordinates at $x_0$ such that $u_1(x_0) = |\nabla u(x_0)|$ and $(u_{ij}(x_0))_{2 \leq i, j \leq n}$ is diagonal. Then at $x_0$, we have
\[[II + III \geq -\alpha^2 (1 + 1/\nu^2) + \alpha k_1 (n - 1 + 1/\nu^2)].\]

We denote $J = a^{ij} w_{ij} = J_1 + J_2 + J_3 + J_4$, where $J_1, J_3$ and $J_4$ are as in [3] (2.19)]. We have, by [3] (2.22),
\[J_3 + J_4 \geq -C(|\cos \theta| + |\nabla \theta| + |\nabla^2 \theta|) u_1 - C(|\cos \theta| + |\nabla \theta|) \sum_{i=2}^{n} |u_i|,\]
where $C$ depends only on $n, M_1$ and $\sup_\Omega |\nabla^3 h|$. Writing $S^\ell = \frac{u}{\nu} - h_\ell \cos \theta$ and using the Ricci identity
\[a^{ij} u_{k;ij} = a^{ij} u_{ij;k} + \text{Ric}(\partial_k, \nabla u)\]
So combining the previous estimates, we find
\[
H \geq \frac{\partial u}{\partial r} + \frac{u^2}{2} \cos^2 \theta
\]
and (3.1), we get
\[
J = J_2 + S^k \text{Ric} (\partial_k, \nabla u)
\]
where \(J_2\) is as in [3] (2.19)]. Since \(|S^1| \leq 2\) and \(|S^k| \leq 1\) for \(k \geq 2\), we obtain
\[
J \geq J_2 - (n + 1) |\text{Ric}_\Omega| |\nabla u|,
\]
where \(|\text{Ric}_\Omega|\) is the bound for the Ricci curvature in \(\Omega\), i.e. \(|\text{Ric}(x)| \leq |\text{Ric}_\Omega|\) for all unit vectors \(x \in T\Omega\). At this point, we can proceed as in [3] to get that
\[
J_1 + J_2 \geq \sum_{i=2}^{n} u_i^2.
\]
So combining the previous estimates, we find
\[
I = \frac{f}{u} \geq -C(\cos^2 \theta + |\nabla \theta| + |\nabla^2 \theta|) - (n + 1) |\text{Ric}_\Omega|.
\]
Hence we obtain
\[
0 \geq I + II + III \geq -C(\cos^2 \theta + |\nabla \theta| + |\nabla^2 \theta|) - (n + 1) |\text{Ric}_\Omega|
\]
\[
- \alpha^2(1 + 1/v^2) + ak_1(n - 1 + 1/v^2)
\]
\[
=: C_1 + C_2 / v^2,
\]
where
\[
C_1 = -C\epsilon_0 - (n + 1) |\text{Ric}_\Omega| + a(k_1(n - 1) - \alpha)
\]
and \(C_2 = a(k_1 - \alpha)\). If \(C_1 > 0\) and \(C_2 > 0\), we get a contradiction, and therefore the maximum of \(\Phi\) is attained on \(\partial\Omega\). If \(C_1 > 0\) and \(C_2 < 0\), then \(v^2 \leq -C_2 / C_1\) and again we have an upper bound for \(\Phi(x_0)\). To have \(C_1 > 0\) we need
\[
|\text{Ric}_\Omega| < (a(k_1(n - 1) - \alpha) - C\epsilon_0) / (n + 1).
\]
(3.5)
Fixing \(\alpha < \min\{k_0, k_1(n - 1) / 2\}\) and assuming that
\[
|\text{Ric}_\Omega| < (a(k_1(n - 1) - \alpha) / (n + 1))
\]
(3.6)
and, finally, choosing \(0 < \epsilon_0 \leq \min\{\epsilon_a, 1/4\}\) small enough so that (3.5) holds, we end up again with a contradiction, and therefore the maximum of \(\Phi\) is attained on \(\partial\Omega\). All in all, we have obtained a uniform gradient bound for a solution \(u\) to (3.2) that is independent of \(\epsilon\). Once the uniform gradient bound is established the rest of the proof goes as in [1] (or [3]). In some special cases we get sharper estimates than those above.

**Example 3.1.** As the first example let us consider the hyperbolic space \(H^n\) and a geodesic ball \(\Omega = B(0, R)\). Furthermore, we choose
\[
h(x) = \frac{r(x)^2}{2R} - \frac{R}{2}
\]
as a defining function for $\Omega$. Here $r(\cdot) = d(\cdot, o)$ is the distance to the center $o$. Then $\kappa_0 = \coth R$ and we may choose $k_1 = 1/R$. Since $\text{Ric}(\partial_k, \partial_1) = -(n-1)\delta_{k1}$, (3.4) can be replaced by
\[
\tilde{J}_2 \geq J_2 - 2(n-1)|\nabla u|
\]
and consequently (3.6) can be replaced by
\[
2(n-1) < \alpha \left( \frac{n-1}{R} - \alpha \right),
\]
where $\alpha < \min\{\coth R, \frac{n-1}{2R}\}$. Hence we obtain an upper bound for the radius $R$. For instance, if $n = 2$, then $\alpha < \frac{1}{2R}$ and we need $R < \frac{1}{2\sqrt{2}}$. For all dimensions, $\alpha = 1$ and $R < \frac{n-1}{2n-1}$ will do.

**Example 3.2.** As a second example let $N$ be a Cartan-Hadamard manifold with sectional curvatures bounded from below by $-K^2$, with $K > 0$. Again we choose $\Omega = B(o, R)$ and
\[
h(x) = \frac{r(x)^2}{2R} - \frac{R}{2}.
\]
Now $1/R \leq \kappa_0 \leq K \coth(KR)$ and again we may choose $k_1 = 1/R$. This time $\text{Ric}(\partial_k, \partial_1) \geq -(n-1)K^2$ and $\text{Ric}(\partial_k, \partial_1) \geq -\frac{1}{2}(n-1)K^2$ for $k = 2, \ldots, n$, and therefore instead of (3.4) and (3.6) we have
\[
\tilde{J}_2 \geq J_2 - K^2 \left( \frac{(n+1)^2}{2} - 2 \right)|\nabla u|
\]
and
\[
K^2 \left( \frac{(n+1)^2}{2} - 2 \right) < \alpha \left( \frac{n-1}{R} - \alpha \right),
\]
where $\alpha < \min\{1/R, \frac{n-1}{2K}\}$. Again we obtain upper bounds for the radius $R$. If $n \geq 3$ we need
\[
R < \left( \frac{n-2}{K^2 \left( \frac{(n+1)^2}{2} - 2 \right)} \right)^{1/2}
\]
whereas for $n = 2$ the bound
\[
R < \frac{1}{2\sqrt{2}K}
\]
is enough since now $\text{Ric}(\partial_2, \partial_1) = 0$.

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