VANISHING PROPERTIES OF SIGN CHANGING SOLUTIONS TO $p$-LAPLACE TYPE EQUATIONS IN THE PLANE

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Abstract. We study the nonlinear eigenvalue problem for the $p$-Laplacian, and more general problem constituting the Fučík spectrum. We are interested in some vanishing properties of sign changing solutions to these problems. Our method is applicable in the plane.

1. Introduction

We consider the nonlinear eigenvalue problem

$$- \nabla \cdot (|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u,$$

where $1 < p < \infty$, $\lambda \in \mathbb{R}$ is a spectral parameter, and $u = 0$ on the boundary of a bounded domain $G \subset \mathbb{R}^2$ with smooth boundary $\partial G$. A good introduction to the subject is [20] and the references given there, but see also [8]. In the present paper, we study some vanishing properties of the second eigenfunction of the $p$-Laplacian, i.e. we consider a solution to (1.1) corresponding to the second eigenvalue $\lambda_2$. Our main result is stated in Theorem 4.1. The method presented in the paper is based on some topological properties of the nodal domains and the nodal line of the second eigenfunction; our main tool is to couple the Harnack inequality and Hopf’s lemma with some topological arguments.

Due to lack of the unique continuation property, there is no analogue of the Courant nodal domain theorem [12] for the nonlinear eigenvalue problem (1.1). We refer to Theorem 2.3 and the discussion in Section 2. However, it was proved in [11], without unique continuation, that the second eigenfunction of the $p$-Laplacian has exactly two nodal domains, \{x \in G : u(x) > 0\} and \{x \in G : u(x) < 0\}. It is a profound difficulty that very little seems to be known about the topology and the geometry of these nodal domains, even the case of $p = 2$ in the plane is not completely known.

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For \( p = 2 \) one recovers the linear eigenvalue problem for the Laplacian, and unique continuation is a well-known feature and the structure of the spectrum is fully understood. As was mentioned above, in general the structure of nodal domains is not completely known. A conjecture of L. Payne [26, Conjecture 5] states that any second eigenfunction \( u_2 \) of the Laplacian on a bounded planar domain \( \Omega \) does not have a closed nodal line, i.e.

\[
\{ x \in \Omega : u_2(x) = 0 \} \cap \partial \Omega \neq \emptyset.
\]

To the best of our knowledge, it is not even known whether the conjecture is true for bounded simply-connected planar domains. There are, however, significant contributions. We refer to a discussion after Proposition 3.6 for references.

We remark that the vanishing properties, the unique continuation property and the geometry of the nodal line in particular, are still an open problem for the solutions to non-linear equations, e.g. for the solutions to the \( p \)-Laplace equation

\[
\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0,
\]

although there are some results. We refer to [1], [4], [5], [9], [21], [22].

Our results are applicable also for solutions to a more general non-linear eigenvalue problem which constitutes the Fučík spectrum. Discussions and results on this problem are postponed until Section 5.

Lastly, we mention that the method in this paper is applicable, after some modifications, to the planar solutions of the Dirichlet problem for certain quasilinear elliptic equations

\[
\nabla \cdot A(x, \nabla u) = B(x, \nabla u)
\]

For these results we refer to [16].

**Notation.** Throughout the paper \( G \) is a bounded simply-connected domain, a domain is an open connected set, of \( \mathbb{R}^2 \), and in (1.1) we have \( 1 < p < \infty \). We use the notation \( B_r = B_r(x) = B(x, r) \) for concentric open balls of radii \( r > 0 \) centered at some \( x \in G \). We denote the closure, interior, exterior, and boundary of \( E \) by \( \overline{E} \), \( \text{int}(E) \), \( \text{ext}(E) \), and \( \partial E \), respectively.

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2. **Eigenfunctions and Nodal Domains**

We interpret equation (1.1) in the weak sense. A function \( u \in W_0^{1,p}(G) \), \( u \) nontrivial, is an eigenfunction if there exists \( \lambda \in \mathbb{R} \) such that

\[
\int_G |\nabla u|^{p-2} \nabla u \cdot \nabla \eta \, dx = \lambda \int_G |u|^{p-2} u \eta \, dx, \tag{2.1}
\]
where $\eta$ is a test-function in $W^{1,p}_0(G)$. The corresponding real number $\lambda$ is called an eigenvalue. The elliptic regularity theory implies that $u \in C^{1,\alpha}_{\text{loc}}(G)$ for some $\alpha > 0$, cf. [13, 29]. We refer to Lindqvist [20] and the references therein for an overview on nonlinear eigenfunctions and their properties.

We mention that by approximation $u$ itself can act as a test-function in (2.1) and one has that $\lambda > 0$. The least eigenvalue $\lambda_1$, called the first eigenvalue, is attained as the infimum of the nonlinear Rayleigh quotient. The corresponding eigenfunction is called the first eigenfunction. Let us list some well-known features, see e.g. [20]: The spectrum is a closed set, $\lambda_1$ is simple, i.e., associated first eigenfunctions are constant multiples of each other, the first eigenfunctions are the only eigenfunctions not changing signs, we stress that all higher eigenfunctions necessarily change their sign, and $\lambda_1$ is isolated.

The structure of the spectrum for the eigenvalue problem for the Laplacian is fully understood and every eigenvalue has a variational characterization via the Rayleigh quotient. In our case of the nonlinear eigenvalue problem there is a second eigenvalue $\lambda_2$, that is $\lambda_2 = \min_{\lambda_1 < \lambda} \lambda$, which has a variational characterization [6].

There are several methods, we refer to [8] for one, to obtain a sequence of variational eigenvalues, $\{\lambda_i^*\}_{i=1}^\infty$, such that
\[ 0 < \lambda_1^* < \lambda_2^* \leq \cdots \leq \lambda_i^* \to \infty \]
as $i \to \infty$, and that $\lambda_1^* = \lambda_1$ and $\lambda_2^* = \lambda_2$. It is not clear whether this sequence gives the entire spectrum. Indeed, it remains a pertinent question how one can exhaust the whole spectrum which, in passing, has not been proved to be discrete.

Let us next turn to study nodal domains of an eigenfunction. A maximal connected component, i.e. one not contained in any other connected set, of the set $\{x \in G : u(x) \neq 0\}$ is called, in what follows, a nodal domain. We denote these components by
\[ N_i^+ = \{x \in G : u(x) > 0\}, \quad \text{and} \quad N_j^- = \{x \in G : u(x) < 0\}, \]
where $i, j = 1, 2, \ldots$. We remark that the restriction of any eigenfunction to a nodal domain is the first eigenfunction with respect to that nodal domain. It is known, in any dimension $n \geq 2$, that any eigenfunction has only a finite number of nodal domains. We refer to Lindqvist [20] for a proof.

The classical Courant nodal domain theorem, we refer to [12, p. 452], states that any eigenfunction of the Laplacian corresponding to the $N$-th eigenvalue has at most $N$ nodal domains. In an interesting paper by Alessandrini [3], the validity of the Courant nodal domain theorem for eigenfunctions of second order self-adjoint elliptic operators with the Lipschitz continuous coefficients in the principal part was verified. He also proves that in the plane the Courant nodal domain theorem holds

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also when the coefficients are just bounded and measurable; in higher
dimensions, he proves a weakened version of the Courant theorem when
the coefficients in the principal part are Hölder continuous. Namely,
an eigenfunction corresponding to the $N$-th eigenvalue has at most
$2(N - 1)$ nodal domains, Theorem 4.5 in [3].

Fundamentally, proofs of the Courant nodal domain theorem are
based on the following three main tools:

1. the **variational characterization of eigenvalues**, i.e. the eigenval-
   ues are characterized as the minimizers of the Rayleigh quotient
   over suitable sets of functions,
2. the **maximum principle**,
3. the **unique continuation** property.

When one considers the nonlinear eigenvalue problem and the corre-
sponding Courant nodal domain theorem, several complications arise.
Apart from the problem of describing higher eigenvalues, the unique
continuation property is still an open question for the $p$-Laplacian.

We want to recall two results related to the nonlinear analogue of
the Courant nodal domain theorem.

**Theorem 2.2** (Cuesta et al. [11]). Suppose $u$ is an eigenfunction cor-
responding to the second eigenvalue $\lambda_2$. Then $u$ has exactly two nodal
domains.

**Theorem 2.3** (Drábek–Robinson [14]).

1. Suppose solutions to (1.1) satisfy the unique continuation prop-
   erty. If $u$ is an eigenfunction corresponding to the $N$-th eigen-
   value, then $u$ has at most $N$ nodal domains.

2. Suppose $u$ is an eigenfunction corresponding to the $N$-th eigen-
   value, then $u$ has at most $2(N - 1)$ nodal domains.

We close this section by recalling the Harnack inequality and the
following version of Hopf’s lemma. For the proof we refer to, e.g., [27,
Lemma A.3] and [28, Proposition 3.2.1].

**Lemma 2.4** (Hopf’s lemma). Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain
with smooth boundary $\partial \Omega$. Let $u \in C^1(\Omega)$ satisfy
\[-\nabla \cdot (|\nabla u|^{p-2}\nabla u) \geq 0\]
interpreted in the weak sense. Assume further that $u > 0$ in $\Omega$ and
$u(x_0) = 0$ at $x_0 \in \partial \Omega$.

Then $\nabla u(x_0) \neq 0$.

The following theorem can be found in Trudinger [30]. The proof is
based on the Moser iteration method.
Theorem 2.5 (Harnack’s inequality). Suppose $u \geq 0$ is an eigenfunction. Then the following inequality is valid
\[ \sup_{B_r} u \leq C \inf_{B_r} u, \]
where $B_{2r} \subset G$ and the constant $C$ depends on $n$ and $p$.

3. A FEW FACTS ABOUT THE PLANE TOPOLOGY

We recall a few facts about the topology of planar sets; a good reference is [25].

We first recall some basic concepts. Let $\Omega$ be any domain in $\mathbb{R}^2$. A Jordan arc is a point set which is homeomorphic with $[0, 1]$, whereas a Jordan curve is a point set which is homeomorphic with a circle. By Jordan’s curve theorem a Jordan curve in $\mathbb{R}^2$ has two complementary domains, and the curve is the boundary of each component. One of these two domains is bounded and this domain is called the interior of the Jordan curve. A domain whose boundary is a Jordan curve is called a Jordan domain.

As a related note, it is well known that the boundary of a bounded simply-connected domain in the plane is connected. In the plane a simply-connected domain $\Omega$ can be defined by the property that all points in the interior of any Jordan curve, which consists of points of $\Omega$, are also points of $\Omega$ [24].

A Jordan arc with one end-point on $\partial \Omega$ and all its other points in $\Omega$, is called an end-cut. If both end-points are in $\partial \Omega$, and the rest in $\Omega$, a Jordan arc is said to be a cross-cut in $\Omega$. A point $x \in \partial \Omega$ is said to be accessible from $\Omega$ if it is an end-point of an end-cut in $\Omega$. Accessible boundary points of a planar domain are aplenty as the following lemma states.

Lemma 3.1 (p. 162, [25]). Let $\Omega$ be any domain in $\mathbb{R}^2$. The accessible points of $\partial \Omega$ are dense in $\partial \Omega$.

Lemma 3.2 (p. 118, [25]). If both end-points of a cross-cut $\gamma$ in a domain $\Omega$ are on the same component of $\partial \Omega$, then $\Omega \setminus \gamma$ has two components, and $\gamma$ is contained in the boundaries of both.

We shall make use of the preceding lemmas, as well as lemmas below, in the proof of our main theorem, Theorem 4.1.

In what follows, $u$ is the second eigenfunction of the $p$-Laplacian. By Theorem 2.2 $u$ has exactly two nodal domains, which we denote by $N^+$ and $N^-$. 

Remark 3.3. Let us define the set
\[ \partial N^+_A = \{ x \in \partial N^+ \cap G : x \text{ is accessible from } N^+ \}, \]
and correspondingly $\partial N^-_A$. Let $x \in \partial N^+ \cap G$ and consider a spherical neighborhood $B_{\delta}(x) \subset G$. It is possible to select points $x_0 \in B_{\delta/2}(x) \cap$
$N^+$ and $x_\delta \in B_{\delta/2}(x) \cap \partial N^+$ such that $x_\delta$ is the closest point to $x_0$. In fact, the line segment $[x_0, x_\delta]$ is contained in $\overline{N^+}$ and so $x_\delta$ is accessible. In addition, $x_\delta \in \partial B_\rho(x_0)$, where $\rho = |x_0 - x_\delta|$. Then by Hopf’s lemma, Lemma 2.4, $\nabla u(x_0) \neq 0$.

Since the preceding procedure can be carried out at any arbitrary small scale $\delta > 0$, we obtain that the set $\{x \in \partial N^+ : x \in \partial N^+_A, u(x) = 0, \nabla u(x) \neq 0\}$ is dense in the relative topology. The case of $\partial N^-$ is treated similarly.

We also remark that neither $N^+$ nor $N^-$ cannot have isolated boundary points, this can be seen by applying Harnack’s inequality.

We then recall a few facts about connected sets and $\epsilon$-chains. If $x$ and $y$ are distinct points, then an $\epsilon$-chain of points joining $x$ and $y$ is a finite sequence of points

$$x = a_1, a_2, \ldots, a_k = y$$

such that $|a_i - a_{i+1}| \leq \epsilon$, for $i = 1, \ldots, k - 1$. A set of points is $\epsilon$-connected if every pair of points in it can be joined by an $\epsilon$-chain of points in the set.

Lemma 3.4 (Theorem 5.1, p. 81, [25]). A compact set $F$ in $\mathbb{R}^2$ is connected if and only if it is $\epsilon$-connected for every $\epsilon > 0$.

Lemma 3.5 (Theorem 1.3, p. 73, [25]). If a connected set of points in $\mathbb{R}^2$ intersects both $\Omega$ and $\mathbb{R}^2 \setminus \Omega$ it intersects $\partial \Omega$.

We shall need in the proof of Theorem 4.1 the observation that either $\partial N^+$ or $\partial N^-$ is necessarily a continuum, i.e. a compact connected set with at least two points. To show this we shall require that $G$ is a bounded simply-connected domain.

Proposition 3.6. Suppose that $G$ is a bounded simply-connected domain. Then, at least, either $\partial N^+$ or $\partial N^-$ is a continuum.

Proof. If either $N^+$ or $N^-$ is simply-connected, then the corresponding boundary is a continuum. We consider the nodal domain $N^-$ (the reasoning for $N^+$ is symmetric) and shall conclude that either $\partial N^+$ or $\partial N^-$ is a continuum.

We suppose, therefore, that $N^-$ is not simply-connected. Then there exists a Jordan curve $\gamma \subset N^-$ with its interior $S_\gamma$. Moreover, $S_\gamma \subset G$ since $G$ is simply-connected, and the set $E = \{x \in S_\gamma : u(x) \geq 0\}$ is non-empty. If $\tilde{E} = \{x \in S_\gamma : u(x) > 0\}$ was empty, then $u(x) = 0$ for all $x \in E$, and $u(x) < 0$ for all $x \in S_\gamma \setminus E$. This is impossible by Harnack’s inequality, Theorem 2.5. Hence $N^-$ is simply-connected and $\partial N^-$ a continuum.

We consider the case in which $\tilde{E} = \{x \in S_\gamma : u(x) > 0\} \subset E$ is non-empty. The set $\tilde{E}$ is open and $\tilde{E} \subset N^+$. Since by Theorem 2.2 there exist exactly two nodal domains we must have that $\tilde{E} = N^+$. 
It follows also from Theorem 2.2 that \( N^+ \) must be simply-connected as otherwise, by repeating the preceding reasoning, we would obtain a third nodal domain \( N^- \). Hence it follows that the boundary \( \partial N^+ \) is a continuum.

The topology of the nodal domains is not known in general. We recall here a conjecture due to Payne [26, Conjecture 5] which states that any second eigenfunction \( u_2 \) of the Laplacian on a bounded planar domain \( \Omega \) does not have a closed nodal line, i.e.

\[
\{ x \in \Omega : u_2(x) = 0 \} \cap \partial \Omega \neq \emptyset.
\]

In this case, the nodal line intersects \( \partial \Omega \) in exactly two points. See also Yau [32].

Significant results have been obtained. To name a few, Jerison [18] proved the conjecture for long thin convex sets without any assumption on the smoothness of the sets, Melas [23] for convex domains with \( C^\infty \)-boundary, and Alessandrini [2] for general convex domains. See also the references in these papers.

Hoffmann-Ostenhof et al. [17] constructed a non-convex, not simply-connected planar domain \( \Omega \) for which the nodal line of the second eigenfunction of the Laplacian is closed, i.e.

\[
\{ x \in \Omega : u_2(x) = 0 \} \cap \partial \Omega = \emptyset.
\]

To the best of our knowledge, it is not known, even in this linear case, whether Payne’s conjecture holds for bounded simply-connected planar domains (see Remark 3 in [17]). Fournais [15] constructed a set in \( \mathbb{R}^n \), \( n \geq 2 \), for which the nodal surface of the second eigenfunction of the Laplacian is closed, and thus generalizing the domain in [17] to higher dimensions by an alternative argument.

We also note that a recent paper [19] contains an example of a multiply connected domain in \( \mathbb{R}^2 \) for which the second eigenfunction of the Laplacian with Robin boundary conditions has an interior nodal line.

4. Some vanishing properties of the second eigenfunction

The following is our main theorem.

**Theorem 4.1.** Suppose \( u \) is an eigenfunction corresponding to the second eigenvalue \( \lambda_2 \) in a bounded simply-connected domain \( G \) in \( \mathbb{R}^2 \). We assume further that for all \( x \in G \) there exists \( r_x > 0 \) such that for all \( r \leq r_x \) the set \( \{ z \in B_r(x) \subset G : u(z) = 0 \} \) is connected. Then if \( u = 0 \) in an open subset of \( G \), then \( u \equiv 0 \) in \( G \).

**Proof.** By Theorem 2.2 \( u \) has exactly two nodal domains \( N^+ \) and \( N^- \). We assume, on the contrary, that

(A) \( u \) vanishes in a maximal open set \( D \subset G \) but is not identically zero in \( G \).
The maximal open set \( D \) is formed as follows: for every \( x \in G \) for which there exists an open neighborhood such that \( u \equiv 0 \) on this neighborhood we denote by \( B(x, r_x) \), \( r_x = \sup \{ t > 0 : u_{|B(x,t)} \equiv 0 \} \), the maximal open neighborhood of \( x \) where \( u \) vanishes identically. Then the maximal open set \( D \) is simply the union of all such neighborhoods. We pick a connected component of \( D \), still denoted by \( D \).

We first show that antithesis (A) implies that any neighborhood of \( x \in \partial D \cap G \) contains points of both nodal domains \( N^+ \) and \( N^- \).

Suppose there existed a point \( x \in \partial D \cap G \) and its spherical neighborhood \( B_\delta(x) \), \( \delta > 0 \), such that \( \overline{B_\delta} \subset G \) and \( B_\delta(x) \cap \text{ext}(D) \) contains only points of either \( N^+ \) or \( N^- \), i.e. points at which either \( u > 0 \) or \( u < 0 \). Assume, without loss of generality, that \( B_\delta(x) \cap \text{ext}(D) \) contains points of \( N^+ \) only. Then \( u \geq 0 \) on \( B_\delta(x) \) and by Harnack’s inequality, Theorem \( 2.5 \), \( u \equiv 0 \) on \( B_{\delta/2}(x) \), which contradicts the maximality of the set \( D \), and hence also the antithesis (A). In this case our claim follows.

We, therefore, assume that for any \( x \in \partial D \cap G \) and for any \( \delta > 0 \) the spherical neighborhood \( B_\delta(x) \) contains points of both nodal domains \( N^+ \) and \( N^- \), and \( B_\delta(x) \subset G \). Hence

\[
(\partial D \cap G) \subset (\partial N^+ \cap \partial N^- \cap G). \tag{4.2}
\]

Let \( \partial D_A = \{ x \in \partial D : x \text{ is accessible from } D \} \). By Lemma \( 3.1 \) accessible boundary points are dense. We select (Lemma \( 3.5 \)) points \( x_1 \) and \( x_2 \) in the set \( \partial D_A \cap G \) and the associated spherical neighborhoods \( B_\delta(x_1) \) and \( B_\delta(x_2) \) such that \( \overline{B_\delta(x_1)} \cap \overline{B_\delta(x_2)} = \emptyset \), and that \( \overline{B_\delta(x_1)}, \overline{B_\delta(x_2)} \subset G \).

By Proposition \( 3.6 \) we may assume, without loss of generality, that \( \partial N^+ \) is a continuum. In addition, we note that it is easy to check that \( \partial N^+ \cap G = \partial N^- \cap G \). Then we select \( x_3 \in B_\delta(x_1) \cap \partial N^+_A \) and \( x_4 \in B_\delta(x_2) \cap \partial N^+_A \). We note that \( \delta \) can be chosen small enough so that the sets \( B_\delta(x_1) \cap \partial N^+_A \) and \( B_\delta(x_2) \cap \partial N^+_A \) are both connected. This is assured by our extra assumption in the formulation of the theorem.

We connect \( x_1 \) to \( x_2 \) by a cross-cut \( \gamma_D \) in \( D \), and \( x_3 \) to \( x_4 \) by a cross-cut \( \gamma_{N^+} \) in \( N^+ \). We remark that \( x_3 \), and analogously \( x_4 \), is accessible in \( N^+ \) with a line segment, see Remark \( 3.3 \). Also \( x_1 \), and analogously \( x_2 \), is accessible in \( D \) with a line segment. We fix such line segments to access the points \( x_1, x_2, x_3, \) and \( x_4 \). In this way the line segments constitute part of the cross-cut \( \gamma_D \) and \( \gamma_{N^+} \), respectively.

Since the boundary \( \partial N^+ \) is connected it is also \( \varepsilon \)-connected for every \( \varepsilon > 0 \). Hence for each \( \varepsilon > 0 \) the points \( x_1 \) and \( x_3 \) can be joined by an \( \varepsilon \)-chain \( \{ a_1, \ldots, a_k \} \subset \partial N^+ \cap G, k = k(\varepsilon) \), such that

\[
x_1 = a_1, a_2, \ldots, a_{k-1}, a_k = x_3.
\]
We consider a collection of open balls \( \{B_{\frac{1}{2}\varepsilon}(a_i)\}_{i=1}^{k} \), \( a_i \in \partial N^+ \cap G \), such that \( \overline{B_{\frac{1}{2}\varepsilon}(a_i)} \subset G \), and a domain \( U_\varepsilon^1 \) which is defined to be

\[
U_\varepsilon^1 = \bigcup_{i=1}^{k} B_{\frac{1}{2}\varepsilon}(a_i).
\]

Since \( U_\varepsilon^1 \) is a domain there exists a Jordan arc, \( \gamma_{x_1x_3}^\varepsilon \), connecting \( x_1 \) to \( x_3 \) in \( U_\varepsilon^1 \). Correspondingly, the points \( x_2 \) and \( x_4 \) can be joined by an \( \varepsilon \)-chain in \( \partial N^+ \) and we obtain a domain \( U_\varepsilon^2 \) and a Jordan arc \( \gamma_{x_2x_4}^\varepsilon \) connecting \( x_2 \) to \( x_4 \) in \( U_\varepsilon^2 \).

It is worth noting that we have selected \( \gamma_{x_1x_3}^\varepsilon \) and \( \gamma_{x_2x_4}^\varepsilon \) such that either of them does not intersect \( \gamma_D \) or \( \gamma_{N^+} \), save the points \( x_1 \) and \( x_2 \), and \( x_3 \) and \( x_4 \), respectively. This is possible because of the line segment construction described above.

From the preceding Jordan arcs we obtain a Jordan curve \( \Gamma^\varepsilon \), and by slight abuse of notation we write it as a product

\[
\Gamma^\varepsilon = \gamma_{x_1x_3}^\varepsilon \cdot \gamma_{N^+} \cdot \gamma_{x_2x_4}^\varepsilon \cdot \gamma_D.
\]

The Jordan curve \( \Gamma^\varepsilon \) divides the plane into two disjoint domains, and \( \Gamma^\varepsilon \) constitutes the boundary of both domains. We consider the bounded domain, denoted by \( T_\varepsilon \), enclosed by \( \Gamma^\varepsilon \). See Figure 1.

We next deal with the Jordan domain \( T_\varepsilon \). There exists at least one point \( y \in T_\varepsilon \) such that \( u(y) < 0 \), i.e. \( y \in N^- \). Assume that this is not the case: then \( u(x) \geq 0 \) for every \( x \in T_\varepsilon \). Recall that \( \gamma_D \) is one of the Jordan arcs which constitutes the boundary of \( T_\varepsilon \). It hence follows that \( T_\varepsilon \) contains points of \( D \) (Lemma 3.2). By Harnack’s inequality, Theorem 2.5, \( u \equiv 0 \) in \( T_\varepsilon \). This is, however, impossible since \( \gamma_{N^+} \) constitutes the boundary of \( T_\varepsilon \), thus \( u > 0 \) on a sufficiently small neighborhood of a point in \( \gamma_{N^+} \).

In an analogous way, it is possible to show that there exists a point \( z \in N^- \cap (G \setminus \overline{T_\varepsilon}) \). We then connect \( z \) and \( y \) in \( N^- \) by a Jordan arc \( \gamma_{zy} \).

Observe that \( u(x) < 0 \) for every \( x \in \gamma_{zy} \).

Lemma 3.3 implies that the Jordan arc \( \gamma_{zy} \) as a connected set intersects \( \Gamma^\varepsilon \) at least at one point. We distinguish next four possible cases for the point of intersection.

If the point of intersection is contained in \( \gamma_D \) or in \( \gamma_{N^+} \) we have reached a contradiction as \( u(x) = 0 \) for every \( x \in \gamma_D \) and \( u(x) > 0 \) for every \( x \in \gamma_{N^+} \).

Consider \( \gamma_{x_1x_3}^\varepsilon \) and \( \gamma_{x_2x_4}^\varepsilon \), and the point of intersection which we denote by \( x_j \) for every \( \varepsilon > 0 \). We can select an appropriate subsequence \( \{x_{\varepsilon_j}\}_{j=1}^{\infty} \), \( \lim_{j \to \infty} \varepsilon_j = 0 \), such that for each \( j \) either \( x_{\varepsilon_j} \in U_\varepsilon^1 \) or \( x_{\varepsilon_j} \in U_\varepsilon^2 \). We assume, without loss of generality, that \( x_{\varepsilon_j} \in U_\varepsilon^1 \).
The sequence \( \{x_{\varepsilon_j}\} \) is clearly bounded, and hence there exists a subsequence, still denoted \( \{x_{\varepsilon_j}\}_{j=1}^{\infty} \), such that
\[
\lim_{j \to \infty} x_{\varepsilon_j} = x_0,
\]
and \( x_0 \in \gamma_{zy} \) since \( \gamma_{zy} \) is a compact set. Observe that each \( x_{\varepsilon_j} \in B_{\frac{1}{2}\varepsilon_j}(a_m) \) for some \( a_m \in \partial N^+ \cap G \) in the \( \varepsilon_j \)-chain. We note that \( u(a_m) = 0 \). Moreover, if there existed \( \delta_0 \) and a subsequence, still denoted \( \{x_{\varepsilon_j}\}_{j=1}^{\infty} \), such that
\[
|u(x_{\varepsilon_j})| \geq \delta_0 > 0
\]
for every \( x_{\varepsilon_j} \), this would contradict with uniform continuity of \( u \) (note that \( u \) is uniformly continuous on compact subsets of \( G \)). We hence have that
\[
u(x_0) = \lim_{j \to \infty} u(x_{\varepsilon_j}) = 0.
\]
In conclusion, we have reached a contradiction since \( u(x_0) = 0 \) but, on the other hand, \( x_0 \in \gamma_{zy} \) and hence \( u(x_0) < 0 \).

All four cases lead to a contradiction. Hence antithesis (A) is false, thus the claim follows.

\[\Box\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Jordan domain \( T_\varepsilon \) and Jordan curve \( \gamma_{zy} \) (dotted line) connecting \( z \) to \( y \) in \( N^- \).}
\end{figure}

Let us discuss our extra assumption in Theorem 4.1.

**Remark 4.3.** We mention that the extra assumption, for all \( x \in G \) there exists \( r_x > 0 \) such that for all \( r \leq r_x \) the set \( \{z \in B_r(x) : u(z) = 0\} \) is connected, could be replaced with the assumption that the set has finitely many components.

**Remark 4.4.** The extra assumption is closely related to the concept of topological monotonicity or quasi-monotonicity introduced by Whyburn in [31]; we also refer to Astala et al. [7, 20.1.1, pp. 530 ff].

Let us try to clarify the role of this assumption in the proof of the preceding theorem. We fix there the point \( x_1 \in \partial D_A \cap G \), its neighborhood \( B_\delta(x_1) \), and the point \( x_3 \in B_\delta(x_1) \cap \partial N^+_A \) (similarly \( x_2 \in \partial D_A \cap G \), \( B_\delta(x_2) \), and \( x_4 \in B_\delta(x_2) \cap \partial N^+_A \)). At \( x_1 \) and \( x_3 \) the function \( u \) is known
to vanish. Using the extra assumption in Theorem 4.1, we may conclude that there indeed exists a continuum $C_\delta$ that connects $x_1$ to $x_3$ in $B_\delta(x_1)$ so that $u(x) = 0$ for every $x \in C_\delta$, or in other words, that the set $B_\delta(x_1) \cap \partial N^+_A$ is connected.

Remark 4.5. The following example of possible spiral-like behavior, kindly provided us by Giovanni Alessandrini, illustrates the role of our extra assumption in Theorem 4.1.

Let $G = B_1(0)$ and consider the function $u$ as follows

$$u(re^{i\theta}) = \begin{cases} (1 - r)(2r - 1)^2 \sin(\theta - \log(2r - 1)), & \frac{1}{2} < r \leq 1, \\ 0, & 0 \leq r \leq \frac{1}{2}. \end{cases}$$

It is important to note that the function $u$ above is not known to be a solution to (1.1) for any $\lambda$. It is differentiable in $G$ and its gradient vanishes on $B_{\frac{1}{2}}(0)$. This function has the following nodal domains

$$N^+ = \left\{ z = re^{i\theta} : 0 < \theta - \log(2r - 1) < \pi, \frac{1}{2} < r < 1 \right\},$$

$$N^- = \left\{ z = re^{i\theta} : \pi < \theta - \log(2r - 1) < 2\pi, \frac{1}{2} < r < 1 \right\}.$$

These nodal domains are simply-connected and their boundaries contain each one half of the circle $\partial B_1(0)$. In addition, the nodal “line” is the union of the closed disk $B_{\frac{1}{2}}(0)$ and the two spirals

$$S_1 = \left\{ z = re^{i\theta} : \theta = \log(2r - 1), \frac{1}{2} < r < 1 \right\},$$

$$S_2 = \left\{ z = re^{i\theta} : \theta = \pi + \log(2r - 1), \frac{1}{2} < r < 1 \right\}.$$

It is not known to us if the aforementioned spiral-like scenario can be ruled out in our method. This is why the existence of a continuum $C_\delta$ as discussed in the preceding remark is not guaranteed without an extra assumption, see Figure 2. It is an open research question whether this assumption could be omitted. As the above example shows, even if the number of nodal domains is finite in a domain, in a small spherical neighborhood of a point there can be infinitely many parts of these nodal domains.

5. Fučík spectrum

We may consider the more general equation

$$- \nabla \cdot (|\nabla u|^{p-2} \nabla u) = \alpha |u|^{p-2}u_+ - \beta |u|^{p-2}u_-, \quad (5.1)$$

where $1 < p < \infty$, $u_+ = \max\{u, 0\}$, $u_- = -\min\{u, 0\}$, $\alpha$, $\beta \in \mathbb{R}$ are spectral parameters, and $u = 0$ on the boundary of $G$. If a nontrivial $u \in W_0^{1,p}(G)$ satisfies (5.1) in the weak sense it is called a Fučík eigenfunction. The corresponding pair $(\alpha, \beta)$ is a Fučík eigenvalue. The
A situation in which spiral $S$, on which $u > 0$ or $u < 0$, destroys the existence of a continuum $C_\delta$, on which $u = 0$, for each $\delta > 0$.

The set of all Fučík eigenvalues is the Fučík spectrum $\Sigma_p$ and, clearly, the spectrum of (1.1) is contained in $\Sigma_p$.

The first two curves $C_1$ and $C_2$ of $\Sigma_p$ can be considered as the analogue of the first two eigenvalues $\lambda_1$ and $\lambda_2$ of the spectrum of (1.1). Any pair $(\alpha, \beta) \in \Sigma_p$ satisfies $\alpha \geq \lambda_1$ and $\beta \geq \lambda_1$, moreover, $C_1 = (\{\lambda_1\} \times \mathbb{R}) \cup (\mathbb{R} \times \{\lambda_1\})$ due to the fact that the first eigenfunctions to (1.1) do not change signs. Analogously, any Fučík eigenfunction associated to a Fučík eigenvalue $(\alpha, \beta) \notin (\{\lambda_1\} \times \mathbb{R}) \cup (\mathbb{R} \times \{\lambda_1\})$ changes sign. The second curve $C_2$ was considered and constructed in [10], and roughly it can be defined to be such a continuous decreasing curve in the $(\alpha, \beta)$-plane that passes through $(\lambda_2, \lambda_2)$ and does not intersect $C_1$. Any pair $(\alpha, \beta)$ between $C_1$ and $C_2$ does not belong to $\Sigma_p$.

It is known that any Fučík eigenfunction corresponding to the Fučík eigenvalue $(\alpha, \beta) \in C_2$ has finite number of nodal domains. We refer to [14]. We state the counterpart of Theorem 2.2 in the case of Fučík spectrum as a theorem. We refer to [14] for an alternative proof of the following result.

**Theorem 5.2** (Cuesta et al. [11]). Suppose $u$ is a Fučík eigenfunction corresponding to the eigenvalue $(\alpha, \beta) \in C_2$. Then $u$ has exactly two nodal domains.

We consider (5.1) in a bounded simply-connected domain $G$ in $\mathbb{R}^2$. Having Theorem 5.2 and Harnack’s inequality [30] at our disposal, we may state the following theorem; Theorem 5.2 together with the proof of Theorem 4.1 justifies the claim.
Theorem 5.3. Suppose $u$ is a Fučik eigenfunction corresponding to the eigenvalue $(\alpha, \beta) \in C_2$ in $G$. We assume further that for all $x \in G$ there exists $r_x > 0$ such that for all $r \leq r_x$ the set \( \{ z \in B_r(x) \subset G : u(z) = 0 \} \) is connected. Then if $u = 0$ in an open subset of $G$, then $u \equiv 0$ in $G$.

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