Quasi-particles for quantum Hall edges

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We discuss a quasi-particle formulation of effective edge theories for the fractional quantum Hall effect. Fundamental quasi-particles for the Laughlin state with filling fraction \( \nu = \frac{1}{3} \) are edge electrons of charge \(-e\) and edge quasi-holes of charge \(+\frac{e}{3}\). These quasi-particles satisfy exclusion statistics in the sense of Haldane. We exploit algebraic properties of edge electrons to derive a kinetic equation for charge transport between a \( \nu = \frac{1}{3} \) fractional quantum Hall edge and a normal metal.

1 Introduction

In the first paper\(^1\) on the fractional quantum Hall effect, Tsui, Störmer, and Gossard already suggested the existence of fractionally charged quasi-particles over fractional quantum Hall states. They extended Laughlin’s gauge argument from the integer to the fractional quantum Hall effect (fqHe) and argued that fractionally charged quasi-particles could be expected. Soon after that Laughlin constructed an approximate ground state wave function and argued using Schrieffer’s counting argument that a fractionally charged quasi-particle could be constructed by piercing the ground state with an infinitely thin solenoid and adiabatically inserting a flux quantum through this solenoid\(^2\).

In view of the above, the existence of the fqHe at simple filling fractions is equivalent to the existence of fractional charge. In the last few years more direct measurements of the quasiparticle charge have been performed. Goldman et al.\(^3\) used a quantum antidot as an electrometer to measure the charge of the excitations in the \( \nu = \frac{1}{3} \) fractional quantum Hall state. At this conference L. Saminadayar discussed recent shot-noise experiments performed by Saminadayar et al.\(^4\) and de Picciotto et al.\(^5\), which showed that the tunneling current from one \( \nu = \frac{1}{3} \) quantum Hall edge to an other is carried by quasi-particles of fractional charge \( \frac{e}{3} \). M. Reznikov reported the observation of charge \( \frac{e}{5} \) quasi-particles in a similar shot-noise experiment on a \( \nu = \frac{2}{5} \) fractional quantum Hall system.

In the light of these fascinating observations of fractional charge, it might come as a surprise that in most theoretical work on the quantum Hall effect bosonization schemes are used, in which the low energy edge excitations of the fractional quantum Hall system are described by neutral bosonic excitations. At this conference, we reported on an alternative approach that gives a central role to (fractionally) charged quasi-particles at the edge of a fractional quantum Hall system, and we discussed the fractional exclusion statistics of these quasi-particles. In a most interesting contribution, S.B. Isakov proposed how the fractional statistics of quantum Hall quasi-particles can be used for the analysis of shot-noise experiments.

In the now following sections we briefly discuss the properties of charged quasi-particles at a \( \nu = \frac{1}{3} \) fractional quantum Hall edge, with special emphasis on the statistics properties. We refer to our paper\(^6\) for a more detailed discussion.

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2 Charged quasi-particles at a $\nu = \frac{1}{3}$ edge.

2.1 Hall conductance

Before giving any further details, we present a quick argument that illustrates the necessity of assigning fractional exclusion statistics to charged quasi-particles in fqHe edges. One quickly checks that, in an effective edge description, the zero temperature Hall conductance $\sigma_H$ is expressed as

$$\sigma_H = n_{\text{max}} q^2 \frac{e^2}{h}$$

with $q$ the charge of the quasi-particles that carry the edge current and $n_{\text{max}}$ the maximum value of the thermodynamic distribution function of these same quasi-particles. For the $\nu = \frac{1}{3}$ edge, with $\sigma_H = \frac{1}{3} e^2 / h$, both the charge $q = -e$ and charge $q = e / 3$ quasi-particles are seen to have $n_{\text{max}} \neq 1$, implying that both types of quasi-particles are described by exclusion statistics different from Fermi statistics.

2.2 Edge electron states

Before turning our attention to quasi-particles with fractional charge, we first discuss quasi-particles with the charge of an electron and fermion-like exchange statistics. We call these quasi-particles edge electrons.

The starting point for this analysis are operators $\Psi^\pm(z)$ that describe the creation and annihilation of edge electrons in a second quantized field theory. This field theory is a so-called Conformal Field Theory, and in what follows we shall exploit special algebraic properties associated to conformal invariance. We use the mode expansion $\Psi^\pm(z) = \sum_t \Psi^\pm_t z^{-t - \frac{3}{2}}$. The mode index $t$ takes half-integer values and corresponds to the dimensionless energy of the mode, i.e., $\epsilon_t = t^2 \frac{\rho_0}{L} \frac{1}{L}$, with $\rho_0 = (\hbar v_F)^{-1}$ and $L$ the length of the edge. Calling $\Psi^\dagger_t = \Psi_{-t}$ and $\Psi_t = \Psi^+_t$, we identify $\Psi^\dagger_t$ and $\Psi_t$, with $t > 0$, with the creation and annihilation operators of an edge-electron of energy $\epsilon_t$, respectively. These operators satisfy the anti-commutation rules

$$\{ \Psi^\dagger_r, \Psi_s \} = (r^2 - \frac{1}{4}) \delta_{-r+s} + 6 L_{-r+s} + 3(r + s) J_{-r+s}.$$  

(2)

In this relation, $L_m$ is the $m$-th Fourier mode of the energy momentum tensor and similarly $J_m$ is a Fourier mode of the current. The zero modes have a simple meaning: $L_0$ is the hamiltonian and $J_0$ measures the charge. The important point to notice here is that for a $\nu = 1$ integer quantum Hall edge, which is a Fermi-liquid, the anti-commutation relations

$$\{ \Psi^\dagger_{r,\nu=1}, \Psi_{s,\nu=1} \} = \delta_{-r+s}$$

are much simpler and only involve the edge electron operators themselves. In a fractional quantum Hall edge, however, the anti-commutator is a non-trivial operator whose value depends on the state in which the expression is evaluated.

By repeatedly acting with the creation operators $\Psi^\dagger_t$, one produces states with more than one edge electron. We will now use the anti-commutation relations to show how some of these states have zero norm. The norm-squared of the one-particle state created by $\Psi^\dagger_t$ is given by

$$\langle 0 | \Psi_t | \Psi^\dagger_t | 0 \rangle = \langle 0 | (t^2 - 1/4) + 6 \epsilon_t + 6 t J_0 | 0 \rangle.$$  

(4)

The zero modes $L_0$, $J_0$ evaluated on the vacuum give zero, and we see that $t = \frac{1}{2}$ leads to a state with norm zero, and that the lowest energy one edge-electron state is $\Psi^\dagger_1 | 0 \rangle$. Continuing,
one may try to add a second quasi-particle to this state. A calculation using the full algebra satisfied by $\Psi^\dagger$, $\Psi$, $L_m$ and $J_n$ (which is a so-called $N=2$ superconformal algebra) shows that the states

$$
\Psi^\dagger(2^\frac{1}{3}) \Psi^\dagger(0) \Psi^\dagger(0) \Psi^\dagger(0) \Psi^\dagger(0) \Psi^\dagger(0) \Psi^\dagger(0)
$$

have zero norm. The lowest-energy two-particle state with non-zero norm is $\Psi^\dagger(2^\frac{1}{3}) \Psi^\dagger(0)$.

Here one sees the start of a pattern: upon adding a third quasi-particle, the first two energy levels directly above $t = \frac{3}{2}$ will be inaccessible, etc. The lowest-energy state with a total of $M$ edge electrons will employ the 1-particle energies $t = \frac{3}{2}, \frac{5}{2}, \ldots, (2M - 1) \frac{3}{2}$. This means that, of all allowed 1-particle states, at the most one out of three can be filled.

A complete basis of multi-electron states is given by

$$
\Psi^\dagger_{(2M-1)^\frac{1}{3}+mM} \Psi^\dagger_{\frac{1}{3}+m_2} \Psi^\dagger_{\frac{1}{3}+m_1} |0\rangle \quad \text{with} \quad m_M \geq \ldots \geq m_2 \geq m_1 \geq 0 .
$$

To this collection of states one may associate a partition function $Z(\mu_k^e, \beta)$, with $\mu_k^e$ a chemical potential for the $k$-th one-particle level. Using a method based on the analysis of truncated partition sums, one shows that in the thermodynamic limit the partition sum $Z(\mu_k^e, \beta)$ factorizes as a product

$$
Z(\mu_k^e, \beta) = \prod_k \Lambda_k \quad \text{with} \quad (\Lambda_k - 1)\Lambda_k^2 = \exp(-\beta(\epsilon_k - \mu_k^e)) .
$$

Clearly, the quantity $\Lambda_k$ can be viewed as the single-level partition sum associated to the $k$-th one-particle energy level. For non-interacting fermions, the analogous factorization is

$$
Z_F(\mu_k, \beta) = \prod_k |1 + \exp(-\beta(\epsilon_k - \mu_k))|
$$

and we recognize the single-level partition sum $|1 + \exp(-\beta(\epsilon_k - \mu_k))|$ as the one dictated by the Pauli principle. The expected occupation of the $k$-th level is given by the Fermi-Dirac distribution

$$
n_{FD}(k) = \frac{1}{\beta} \frac{\partial_{\mu_k^e} Z_F}{Z_F} = \frac{1}{1 + \exp(\beta(\epsilon_k - \mu_k))} .
$$

In a similar manner, we can read off from the $\Lambda_k$ the distribution of the $\nu = \frac{1}{3}$ edge-electrons

$$
n_{e}(k) = \frac{1}{\beta} \frac{\partial_{\mu_k^e} \Lambda_k}{\Lambda_k} .
$$

Instead of trying to solve the characteristic equations (7), we can derive from them

$$
n_{e}(k) = \frac{1}{3 + w_k} , \quad w_k = (\Lambda_k - 1)^{-1} .
$$

The equations (9), (10) agree with the Isakov-Ouvry-Wu equations describing the thermodynamics associated to fractional exclusion statistics as defined by Haldane (11), with statistics parameter equal to $g = 3$.

2.3 Quasi-hole states

By using a similar reasoning, one can analyze the quasi-hole operator $\phi(z)$ associated with the creation of a fractional charge $q = +\frac{e}{3}$ at the $\nu = \frac{1}{3}$ edge. A basis for the corresponding quasi-hole states is

$$
\phi_{-\frac{2(2N-1)}{3} - n_N} \phi_{-\frac{2}{3} - n_2} \phi_{-\frac{1}{3} - n_1} |0\rangle \quad \text{with} \quad n_N \geq \ldots \geq n_2 \geq n_1 \geq 0 .
$$
The partition sum for these quasi-hole states is again factorizable in the thermodynamic limit,

\[ Z(\mu^\phi, \beta) = \prod_l \lambda_l \quad \text{with} \quad (\lambda_l - 1)^3 \lambda_l^2 = \exp(-3\beta(\epsilon_l - \mu^\phi)) \]  

(13)

and we obtain the distribution function

\[ n_\phi(l) = \frac{1}{3 + w_l}, \quad w_l = (\lambda_l - 1)^{-1}. \]  

(14)

These relations are equivalent to an IOV equation for Haldane statistics, this time with the statistics parameter taking the value \( g = \frac{1}{3} \).

2.4 Duality

Having explained the appearance of the distribution functions for fractional exclusion statistics with \( g = 3 \) and \( g = \frac{1}{3} \), respectively, we recall that there is a particle-hole duality between the two cases \( \nu = 1/3 \), \( \nu = 3 \).

\[ 3n_e(\epsilon) = 1 \frac{1}{3} n_\phi(-\frac{1}{3}\epsilon). \]  

(15)

In our paper \( 6 \), we demonstrated how a complete basis for the \( \nu = 1/3 \) edge theory can be obtained by independently filling the one quasi-particle spectra of the edge-electron and the quasi-hole. The interpretation of the duality relation is now that the positive-energy quasi-hole excitations can be viewed as holes in the ground state distribution of negative energy edge-electrons and vice versa. The relative factor \((-\frac{1}{3})\) between the energy arguments in (15) indicates that the act of taking out a single edge-electron from a filled sea corresponds to creating three quasi-holes.

3 Transport properties

The thermodynamic distribution functions that we described satisfy \( n_e^{\text{max}} = 1/3 \) and \( n_\phi^{\text{max}} = 3 \), in agreement with the expression (1) for the Hall conductance. Other thermodynamic quantities, including the edge contribution to the specific heat, are quickly computed in the quasi-particle formalism. Using Rajagopal's formula for fluctuations \( 12 \), we also reproduced the expressions for the Johnson-Nyquist noise \( 13 \). Here we do not discuss these results, but move on and consider transport properties.

Following the set-up of a number of recent experiments, we consider a situation where electrons (or holes) from a Fermi-liquid reservoir are allowed to tunnel into a \( \nu = 1/3 \) fqHe edge. The DC \( I-V \) characteristics for this set-up, which were first computed by Kane and Fisher \( 14 \) (see also \( 15 \)), show a cross-over from a linear (thermal) regime into a power-law behavior at high voltages and thus present a clear fingerprint of the Luttinger liquid features of the fqHe edge. The experimental results from \( 16 \) are in agreement with these predictions. (We refer to the contributions of A.M. Chang and E. Fradkin to this conference and to \( 17 \) for a further theoretical analysis.) The calculations were based on bosonization and on the Keldysh formalism for non-equilibrium transport. Here we reproduce these results in an approach directly based on the edge quasi-particle formalism.

A careful derivation, based directly on the form of the tunneling hamiltonian

\[ H_{\text{int}} \propto t \int d\epsilon \left[ \Psi_{\nu=1}^\dagger(\epsilon) \Psi_{\nu=\frac{1}{3}}(\epsilon) + \text{h.c.} \right], \]  

(16)

leads to the following kinetic equation (compare with \( 18 \))

\[ I(V, T) \propto e t^2 \int_{-\infty}^{\infty} d\epsilon \left[ f(\epsilon - eV) H(\epsilon) - F(\epsilon - eV) h(\epsilon) \right], \]  

(17)
where $h, H$ are one particle Green’s functions

$$h(\epsilon) = \langle \Psi_{\nu=\frac{1}{3}}(\epsilon)\Psi_{\nu=\frac{1}{3}}(\epsilon) \rangle_{V,T}, \quad H(\epsilon) = \langle \Psi_{\nu=\frac{1}{3}}(\epsilon)\Psi_{\nu=\frac{1}{3}}(\epsilon) \rangle_{V,T}$$

(18)

for edge electrons in the $\nu = \frac{1}{3}$ $fqHe$ edge, taken at $V = 0$. Here $f(\epsilon)$ and $F(\epsilon)$ are the Fermi-Dirac distributions for electrons and holes, respectively. The quantities $H(\epsilon)$ and $h(\epsilon)$ can be determined as follows. The ratio of $H(\epsilon)$ and $h(\epsilon)$ is fixed,

$$H(\epsilon) = e^{\beta(\epsilon-eV)}h(\epsilon),$$

(19)

by detailed balance, which can be phrased as the requirement that at zero voltage there should be no current flowing. The sum $H(\epsilon) + h(\epsilon)$ is fixed by the anti-commutation relation (2), here in the continuum approximation

$$\frac{1}{L}\frac{\pi^{2}}{6\beta^{2}}e^{2}\delta(\epsilon - \epsilon') + 6\frac{E_{\epsilon'-\epsilon}}{\rho_{0}} + 3(\epsilon + \epsilon')\frac{Q_{\epsilon'-\epsilon}}{\epsilon\rho_{0}}.$$  

(20)

In this formula, $E_{0}$ is the operator for the total energy per unit length (proportional to $L_{0}$), and $Q_{0}$ is the operator for the total charge per unit length (proportional to $J_{0}$). The expectation values of energy and charge can now be calculated using the distribution function $n_{e}(\epsilon)$, given in (11). We find

$$\langle E_{0}\rangle_{V,T} = \rho_{0}\left(\frac{\pi^{2}}{6\beta^{2}} + \frac{(eV)^{2}}{6}\right), \quad \langle Q_{0}\rangle_{V,T} = -e\rho_{0}\frac{(eV)}{3}$$

(21)

and obtain the exact expressions

$$H(\epsilon) = \frac{(\epsilon - eV)^{2} + \frac{\pi^{2}}{3\beta^{2}}}{e^{-\beta(\epsilon-eV)} + 1}, \quad h(\epsilon) = \frac{(\epsilon - eV)^{2} + \frac{\pi^{2}}{3\beta^{2}}}{1 + e^{\beta(\epsilon-eV)}}.$$ 

(22)

They lead to $I$-$V$ characteristics

$$I(V,T) \propto e\beta^{3}\frac{\epsilon^{2}}{2\pi^{2}} \left(\frac{\beta eV}{2\pi} + \left(\frac{\beta eV}{2\pi}\right)^{3}\right),$$

(23)

in agreement with the result obtained in different approaches[14,15].

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