Taking Physical Infinity Seriously

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Abstract

The concept of infinity took centuries to achieve recognized status in the field of mathematics, despite the fact that it was implicitly present in nearly all mathematical endeavors. Here I explore the idea that a similar development might be warranted in physics. Several threads will be speculatively examined, including some involving nonstandard analysis. While there are intriguing possibilities, there also are noteworthy difficulties.

1 Introduction

Infinity plays a central role in mathematics, and arguably always has – despite occasional negative characterizations (even by some of the most esteemed practitioners). Today surely there is little question about its importance in the minds of the vast majority of mathematicians. There is also very wide appreciation of the idea that whither goes mathematics, there also goes physics (and often the other way around). And yet in physics the notion of infinity plays a rather curious “fix-it-up” role, rather like duct tape, that is brought out for use whenever needed but then put firmly back in the tool box again. Thus it is not kept front and center in actual physical models, quite unlike its now central and fundamental role in mathematics.

This is part of a much larger issue: how mathematics relates to physical reality. This involves many aspects that we will not touch on here, other than some brief comments. For instance, Wigner [26] regards it as “unreasonable” that there is such a strong connection between math and physics. And Kreisel [14] has considered whether quantities that are physically observable (according to a given physical theory) can be generated by a Turing machine; such a theory he calls “mechanical”. See also [1], [16], [22], all

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1In [8] Martin Davis includes a discussion of infinity in mathematics in terms of imaginative powers of our minds (my words, not his), and (partly) justifies this by analogy with physics – somewhat the reverse of my point here, but one I am equally sympathetic to.

2One prominent example that will not be discussed at any length here are the divergent Feynman integrals (among others) of quantum field theory (QFT). See for instance the excellent Wikipedia entry for Renormalization [28].
of whom discuss cosmological issues such as whether space is infinite in extent; Rovelli \cite{22} in particular distinguishes – similarly to a distinction we shall draw – between infinite divisibility and infinite extent.

A related question is: what sort of universe is needed in order for there to be a possibility of mathematics at all? That is, not actual mathematical practice, but simply the possibility of “stuff” sufficient to allow, for instance, such things as sequences, records, relations. There would seem to be a requisite minimum level of temporality and spatiality even for natural numbers to have any meaningfulness. And, perhaps deeper: what counts as stuff, and what is it for stuff to “be”? But we will leave these questions aside, and return to our main theme\textsuperscript{3}.

Here I will describe a number of examples in which infinity is used explicitly in physics, and possible developments that these might suggest, including a few detours along the way.\textsuperscript{4} Yet I must add that, as a non-physicist, I also approach the broader topic with some trepidation; and while I have consulted a number of physicists in the writing of this paper, still any misconceptions are completely my own. I trust the reader will pardon any sense that I am throwing in the kitchen sink; this essay represents some possibly far-flung imaginings that perhaps do not fall altogether within customary styles in scientific writing.

The rest of this paper is organized as follows: We describe the examples just referred to above, to distinguish several modes of use of infinities in physics; next I review some ideas due to Jose Benardete on a Zeno-like puzzle about infinity, and some related issues concerning particles, densities, and spin; we then turn to nonstandard analysis as one methodology that appears to shed some light (in connection with Dirac delta functions), but has difficulties of its own.

## 2 Multiple uses of infinity in physics

Quantum mechanics provides us with many intriguing examples of our subject; I give three here. First, Schrödinger’s solution of his wave equation for the energy levels of the hydrogen atom involves an argument in which infinity plays the role of a kind of reductio, or proof by contradiction, leading to the rejection of the infinity. Second, that same solution results in an infinite set of energy levels, which are pointedly not rejected. Third, Dirac introduced the (infinite-valued on an infinitesimal interval) delta function because it provided a highly simplifying and intuitively satisfying notation for his vastly influential treatment of quantum mechanics. I briefly summarize each of these uses of infinity below.

In a 1926 paper, Schrödinger solved his famous wave equation for the special case of the hydrogen atom. Along the way he had to set to zero certain series terms, since

\textsuperscript{3}I can’t resist noting that in roughly 1968–9 Martin Davis mentioned to me that in his estimation a huge unclarity underlay foundational issues in mathematics and in particular set theory: what counts as a thing?

\textsuperscript{4}That the topic is appropriate to a volume dedicated to Martin Davis, I justify with the observations that (i) Martin helped instill in me a general love for ideas on topics far and wide; and (ii) at least two of Martin’s writings bear on related themes: nonstandard analysis \cite{7} and quantum physics \cite{6}. I note that Rovelli \cite{21} entertains an idea already present in \cite{6}, namely that of observer-dependent reference frames in quantum mechanics; and (personal note from Rovelli) this also apparently has come up in writings of Kochen and Isham as well, all after Martin’s contribution appeared. See also \cite{24} for more on this theme.
otherwise they would lead to variables with infinite values. (The remaining terms provide solutions for energy levels of the hydrogen atom that are the familiar Bohr ones that closely match experiment\textsuperscript{5} – but not quite close enough; later refinements were needed, including spin and relativistic effects.) So in this case, a variable taking on an infinite value is used as a reason to reject it and instead consider only alternative lines of argument. This of course is not new to Schrödinger but in fact is a common form of argument, applicable whenever the variable in question is something one has reason to think should be finite. I provide this particular example of such a \textit{reductio} use of infinity here (as opposed to any number of others) simply because it is curious that it arises in the same setting in which the next example occurs. We may refer to this first as a \textit{dense} physical infinity: a physical variable (that in principle might be measured by means of instruments within certain physical confines) taking on (but perhaps should not do so) an infinite value. This is employed via a \textit{reductio} to eliminate the infinity (sometimes easily as above, sometimes with enormous effort and controversy as in QFT).

Yet a result of Schrödinger’s argument is that the distinct possible energy levels of the hydrogen atom alluded to above are infinite in number, and in fact a specific formula is derived for the possible energies, $E_n$ where $n = 1, 2, \ldots$. \textit{This} infinitude is not shrugged off as unphysical; each and every $E_n$ is taken as representing an in-principle possible physical energy for the atom.\textsuperscript{6} Indeed, it is the excellent match-up with experiment that makes the Schrödinger result so convincing.\textsuperscript{7} Of course, it is similar in kind to the infinitude of possible heights (or potential energies) of a projectile above ground level, which is also not seen as unusual. These perhaps amount, in the end, to little more than the fact that the infinite (unbounded) set of real numbers, $\mathbb{R}$, is taken as the possible range of values for many physical variables (with some limitations as dictated by a given situation – but the infinitude is not in general ruled out). This is a \textit{range-of-values} physical infinity: a mere listing of possible values, of which there may be infinitely many. Yet it is a possibility that, in some sense, describes (a working picture of) the universe: the universe has in it an unbounded range of allowable values for certain variables.\textsuperscript{8}

One way to make these two standard physical uses of infinity more intuitive may be this: if a variable represents a measurable quantity, something that one might detect in an experiment, then the measured value must be finite: we have no means to measure an actual infinity; whereas any – even an infinite – number of finite values might be measured (given enough time). Or: there may be an infinite amount of space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, space, matter, 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or energy, in the universe; but not right where the measuring instruments are located. Note that we are not taking a stand on such a view; in fact, we are exploring alternative possibilities!

Indeed, one can reason: there may be things physically present that we cannot measure. One such that comes to mind is the wavefunction itself; this is sometimes characterized as the fundamental “reality” of which our measurements ferret out (and even modify) some features but never reveal the full thing in itself. If the wavefunction is really there, yet never fully revealed, why not also infinite energies and other quantities? Or consider space and time (or spacetime) themselves: we never measure all of space or time, by any means. Yet in measuring bits and pieces, we convince ourselves that there is a great deal more, and in the case of some theories even that the universe has an infinitude of such pieces, either extended (range-of-values) or densely packed.

Our third example is Dirac’s delta function. This is in wide use by physicists (and not only in quantum mechanics). Yet the delta function is routinely viewed as a useful fiction, not something to take seriously except as a convenient shorthand for a much more cumbersome and less intuitive set of tools. This mode we then call the useful fiction infinity: we use it but we don’t believe it corresponds to anything physical. Nonetheless, it seems to fall also into the dense mode of infinity.

Thus we have cases where a dense infinity is outlawed (by reductio), and others where it is accepted as a useful fiction; and there are also cases (range-of-values) where infinity is accepted as quite physically sensible. Much of what we are considering here is whether some of the “fiction” cases should perhaps be considered as less fiction and more real physics. Delta functions are one case in point (we shall return to them below) but not the only one.

3 Benardete’s challenge

Jose Benardete [4] discusses novel variants of a paradox of Zeno. Here is a version that suits our purposes: Imagine that an impenetrable barrier is erected at each point \( x = 1/2^n \) for \( n=1,2,\ldots \); we suppose the barriers to be of zero thickness (or of decreasing thickness as they close in on \( x = 0 \), so that they do not overlap or touch each other, and so that they do not overlap or touch \( x = 0 \)). Moreover, imagine that each barrier is immovable once so placed. Finally, imagine that a projectile is aimed at the barriers from a point to the left, i.e., from some \( x < 0 \).

Let us first of all note that this appears to be a case of dense infinity. There is an infinitude of physical entities in a finite region. To be sure, this particular setup is highly implausible; we are bringing it into the discussion as an easy warmup case, before proceeding to more physically plausible cases.

Now, what will happen as the projectile moves rightward? Since there is nothing apparent to impede the projectile at negative positions (\( x < 0 \), it would seem that it

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9 More so some decades ago; it seems now a minority view.
10 This is reminiscent of the early uses of imaginary numbers: they were clearly useful, but it was far less clear that such a number could be a thing in any sense available back then. Eventually two developments helped: (i) the observation that imaginary numbers can be interpreted as rotations, and (ii) formal/abstract methodology: if something has a consistent mathematical use, that is all that is needed in order for it to be an object of mathematical study.
should continue its rightward motion until it strikes a barrier. But before it can strike a
barrier at \( x = 1/2^n \) it must first strike (and pass through) all those to its left (at \( x = 1/2^m \nolimits \) for all \( m > n \)). This is impossible by the conditions of the problem. So it cannot strike
any barrier at all! Hence it must stop its rightward motion, never passing zero, yet
without touching anything that would be a cause for its rightward motion to cease.

This has been debated in various philosophical papers; see \([19, 50, 13]\). In \([18]\)
standard physics is brought to bear on the puzzle in the forms of classical mechanics,
quantum mechanics, and relativity, showing for instance in the classical case that a field
effect in the form of a repulsive force is mandated by Newton’s Laws, so that the pro-
jectile is bounced back to the left before passing zero. But the lesson for us here is that
even a dense infinity need not be paradoxical when seen from within standard physical
timey. (Of course, one can resurrect a paradox by insisting the barriers produce no
forces outside their own immediate locations; and the lesson then would be that this is
inconsistent with standard physics.)

Another version of the puzzle involves a continuous barrier-wall extending from
some point \( b > 0 \) all the way back to, but not including, \( x = 0 \). That is, this is a wall
of width \( b \) but with its left face missing. While a seeming bit of physical nonsense (at
least in terms of materials made of atoms) it is a familiar enough entity in mathematics,
essentially a half-open half-closed interval. And the same form of argument applies
as in the earlier Benardete example. It would seem that physical entities cannot be
isolated quite as well as our imaginations might like: physical interactions will occur
and cannot be dismissed by mere stipulation.

Thus the Benardete examples provide a kind of dense infinity, but not apparently
one that “breaks” anything. Perhaps this is because it does not directly involve an
infinite density of standard physical quantities like mass or charge or energy. (A closer
analysis might turn up an infinite sort of potential energy, however.) In any event,
when we turn to something “real” such as an electron, the situation presents itself more
starkly.

4 The electron – getting to the point

An electron presents a somewhat related challenge. An electromagnetic field exists
around any charged particle. If the particle is not in motion, then it is simply an electric
field, \( \vec{E} \), given by Coulomb’s Law. But the same law mandates that the field’s mag-
nitude \( E \) increases at locations closer to the particle, approaching infinity in the limit.
In addition, the charge density is zero outside the immediate location of the electron,
and infinity at that location. Finally, the mass density is also infinite at the location
of the electron, and zero elsewhere. These claims are based on the not uncommon as-
sumption that an electron has no spatial extent and is located at a literal mathematical
point; experimentally, the electron’s radius is less than 10^{-22} meters \([15]\). A similar
situation arises in the case of a black hole, where the mass density becomes infinite at
the mathematical point (singularity) of the hole itself\([12]\).
One way to mathematically represent the situation of an infinite point density is via a Dirac delta function, namely one that is infinite at the point in question, and zero elsewhere. This – usually taken as a convenient fiction as already noted – does the trick really well and surprisingly often, and is now a standard item in the physics toolbox. However, delta functions can quickly turn from convenience to headache, due to the nonlinearity of many applications. That is, the usual way to “precisify” a delta function is as a Schwartz distribution: a linear functional on a space of functions. However – as Wald [25] points out – in many applications (nonlinear ones) delta functions (when viewed as distributions) cannot be sensibly multiplied, and this poses significant difficulties for their use where there are point sources of fields. This is a bit outrageous: why cannot one multiply two functions? The answer is that the Schwartz representation really groups these “fiction-functions” into equivalence classes (ones that provide the same results for certain special integration properties\footnote{Namely: $\int_{-\infty}^{\infty} f_1(x) g(x) = \int_{-\infty}^{\infty} f_2(x) g(x)$ for all “test” functions $g$.}), and, integration does not always respect some of the desired characteristics needed for non-linear applications. Yet once ungrouped from each other and treated as genuine functions, delta functions can indeed be multiplied, as we will see in the next section\footnote{This is not to say that successful application to non-linear differential equations is an automatic benefit; as noted, it is the product per se but rather integration properties of products that is at issue.}

Summarizing a bit, one way that infinity arises in physics is as follows: a vector field (such as gravitational or electrostatic force) depends on the spatial separation between one body and another, in a way that increases without bound as that distance decreases to zero. In particular, in these two instances, the force is proportional to the reciprocal of the square of the distance. When that distance is zero, the expression for the force becomes one divided by zero: $1/0$.

Now, division by zero is extremely problematic; it is not simply that it is not defined, but that it is both overdetermined and underdetermined. $0/0$ can be set equal to any number ($0/0 = x$) with impunity, since $0 = 0x$. And $1/0$ cannot be set equal to any number at all, since $1 \neq 0x$. So there is no non-arbitrary nor even consistent way to define division by zero that respects the basic concept of division: $(a/b)b = a$, that is, as the inverse of multiplication.

It is tempting to say that this is because the real numbers are too restrictive, and that $1/0 = \infty$. But then what is $2/0$? And do we allow $1 = 0 \times \infty$? These notions contain hints of a possible solution. In fact, mathematical physics often employs such intuitions, in the form of infinitesimals and infinities; again think of the standard delta function, that is zero at all non-zero reals, yet when infinitesimally close to zero it rises up to infinity.

But mathematicians have invented many sorts of numbers, going well beyond the familiar real and complex fields, including some that explicitly contain infinities as first-class objects. Which fits the physical situation best? We shall not attempt to answer this here, nor even to survey the existing options. Instead, we shall discuss just one such option, with particular application to delta functions and – possibly – to point particles.
5 NSA

One well-known approach to making sense of infinite and infinitesimal quantities is nonstandard analysis (NSA), where the real number system $\mathbb{R}$ is extended to $\ast \mathbb{R}$, which includes “numbers” that are larger than every real, and also ones that are smaller than every positive real and yet are themselves larger than 0. The latter (small ones) and their negatives become the infinitesimals in common use in physical reasoning. This was the aim of A. Robinson [20]: to develop $\ast \mathbb{R}$ and to show that in fact the familiar intuitive arguments using infinitesimals then become quite rigorous.

But infinitesimals are not the same thing as zero; they are simply very very close to zero; one might say that they form a kind of fuzzy zero – and more generally, that each real $r$ has about it a band of new numbers ($r$ plus any infinitesimal) that “coat” $r$ so closely that for ordinary purposes $r$ and its coat are indistinguishable.$^{15}$

A key point is that, while being in zero’s coat, an infinitesimal $\varepsilon$ nonetheless has a well-defined reciprocal $1/\varepsilon$, which is infinite (larger than every real). We still do not have a reciprocal for zero itself, but perhaps we can dispense with that, and when a variable “approaches” zero we may try to regard it as being in zero’s coat rather than being zero itself. More generally, the coat of a real $r$ then provides stand-ins for $r$, which are $r$-ish in more or less degree (but all of them are $r$-ish and not $s$-ish for any other real $s$).

As Robinson has shown, $\ast \mathbb{R}$ can be given a very rigorous definition, so that it remains an algebraic field and respects the “usual” mathematical properties of $\mathbb{R}$. These properties are given sharp characterization, roughly as follows: for any sentence $S$ that can be expressed in a particular formal language $L$ (including much of standard math notations, for instance $+$, $\times$, constants, $=,<,\forall$, set-membership, etc – but NOT using a symbol for $\mathbb{R}$ itself), $S$ is true when interpreted as being about elements in $\mathbb{R}$ iff it is true about $\ast \mathbb{R}$.\footnote{Details can get a bit complicated; see [7].}$^{16}$ Now this “transfer principle” between $\mathbb{R}$ and $\ast \mathbb{R}$ is the basis for a great many applications of NSA.$^{17}$ But results of such applications – at least when those results are interpreted as being about $\mathbb{R}$ (or more precisely about the “set-theoretic superstructure for $\mathbb{R}$”) – generally are theorems that can also be proven (though maybe less easily or intuitively) without NSA. One of the suggestions we are raising here is this: perhaps $\ast \mathbb{R}$ (or its superstructure) can be taken seriously as a model of physical reality, to see whether this sheds light on infinities that arise in physics.$^{18}$

One very nice (traditional) application of NSA is the delta function, which now can be defined as an actual (non-fictional) function from $\ast \mathbb{R}$ to $\ast \mathbb{R}$. For instance, given an infinitesimal $\varepsilon$, let $\delta(x) = 0$ for all numbers (in $\ast \mathbb{R}$) that lie outside $[-\varepsilon/2, \varepsilon/2]$, and let $\delta(x) = 1/\varepsilon$ for numbers in that interval. The graph of such a function then is an infinitesimally thin, infinitely high rectangle, and the area under it is exactly $\varepsilon \times 1/\varepsilon = 1$. And then the integral of $\delta(x)$ times any function *f from $\ast \mathbb{R}$ to $\ast \mathbb{R}$ (that is an

\footnote{I apologize for introducing the term coat for this; already in use are: monad, haze, cloud, halo. My excuse is that a coat of paint is thin, hugs close to its target, and is not to be touched by other entities (at least while wet).}

\footnote{There are by now dozens of books and hundreds or articles on the subject of NSA in general and applications of the transfer principle in particular. See for instance [5] and [2].}

\footnote{See [12] for a rare exceptional – but alas all too preliminary – treatment of NSA's nonstandard universe itself as having physical significance, in this case to QFT.}
appropriate extension of an integrable function \( f \) on the reals), gives \( f(0) \) – or more precisely, gives the average value of \( *f \) in that interval, which is itself in the coat of \( -f(0) \). But now the product of any two such delta functions from \( *\mathbb{R} \) to \( *\mathbb{R} \) is unproblematically another function from \( *\mathbb{R} \) to \( *\mathbb{R} \). There is a tradeoff, however. For we must choose a particular delta function to use in a given application, rather than opt for the distributional approach that lumps many such together.\[19\]

6 Back to the real world

Now we return to physics, and in particular to the electron. We regard it as being a point, or rather, we take its radius to be in the coat of 0 (or whatever point it is centered on). That is, we will postulate it to be a ball of infinitesimal radius. In particular, let some \( \varepsilon \) be that radius, and assume its mass \( m \) is uniformly distributed.\[20\] Now we will attempt to characterize its spin \((\hbar/2)\) as a physical angular momentum \( L \) of actual rotation, namely with an angular frequency \( \omega \) so that we get the usual classical formula:

\[
L = \frac{\hbar}{2} = I \omega = \frac{(2/5)m\varepsilon^2 \omega}{4mr}
\]

Since \( \varepsilon \) is infinitesimal then \( \omega \) must be infinite, since the LHS is finite.

The idea of treating spin as a possible rotational phenomenon was considered long ago (see below), but taking the radius to be a positive real \( r \); this led to trouble with special relativity (SR). A point on the surface of the electron ball would – in order that the rotation provide the proper angular momentum of spin, have to travel at speeds in excess of the speed of light. But to reach such a speed would require infinite energy, according to SR, and that traditionally is taboo. Here then is a possible advantage of NSA: suppose we allow physical quantities to be infinite.

Let’s calculate the speed \( v \) of a point on the surface of an “electron ball” with (initially real) radius \( r \) that is rotating with angular momentum \( \hbar/2 \). From the above equation, we get

\[
v = \omega(1/2\pi)(2\pi r) = \omega r = 5\hbar/(4mr)
\]

If we insist that \( v < c \) then we find

\[
c > 5\hbar/(4mr)
\]

or

\[
\quad r > 5\hbar/(4mc) = 0.5 \times 10^{-12} \text{ meters}
\]

\[19\] Further investigation (I am unaware of any work on this topic) may reveal advantages to particular “natural” choices for a delta function in particular applications. For now I simply point out one from Robinson’s book (p. 138): \( \frac{1}{\sqrt{\pi \varepsilon}} \exp\left(-\frac{x^2}{\varepsilon^2}\right) \). For real values of \( \varepsilon \) this is just an ordinary Gaussian, which arises quite naturally in many situations, and has very nice mathematical properties. Possibly in the nonstandard realm it will also play a helpful role. Note that this is not claimed to resolve issues about non-linear applications where integration properties of products arise.

\[20\] Note that this means the ball will be a proper subset of the coat, since coats have no boundary; if they did, then for instance \( 2\varepsilon \) would be outside the coat, which makes no sense for it too is infinitesimal.
This is essentially the negative result found by Goudsmit and Uhlenbeck [11] that made them (and others) give up the idea of spin as deriving from an actual physical rotation, since it was known even then that \( r \) is less than \( 3 \times 10^{-15} \) meters.

There is an alternative: allowing \( v \geq c \), and also allowing infinite energies, as well as replacing \( r \) by \( \varepsilon \). But why insist that \( r \) be infinitesimal? This is not strictly necessary. But since as already noted, it is commonly thought that \( r = 0 \) (an electron is an actual point with no extent, no volume) and since we are allowing infinities anyway, it is tempting to go “all the way” (at least all the way to infinitesimals, if not literally to zero).

Back to our calculations: if \( \varepsilon \) is infinitesimal then as noted above, the angular frequency \( \omega \) is infinite. But what then is the speed of a point in the electron coat, at distance \( \varepsilon \) from the origin of rotation? It will be as above, but replacing \( r \) with \( \varepsilon \), hence infinite:

\[
v = \omega \varepsilon = \frac{5\hbar}{(4m\varepsilon)}
\]

This infinite speed precisely produces the finite angular momentum \( \frac{\hbar}{2} \). That is, the infinite speed of a point within the electron coat (which itself is at infinitesimal distance \( \varepsilon \) from the origin of rotation), works together with that infinitesimal distance to produce the needed finite angular momentum of spin.

However, not everything works out so nicely. The kinetic energy of mass \( m \) with speed \( v \), in SR, is

\[
T = mc^2\left(\gamma - 1\right)
\]

where \( \gamma = \frac{1}{\sqrt{1 - (v/c)^2}} \).

When \( v = c \), \( \gamma \) is infinite, hence \( T \) would seem to be infinite. This is well-known, of course, and is a primary reason that \( c \) is regarded as an unreachable upper limit on all speeds of massive objects. But it is now even worse: for this infinity (of \( \gamma \)) seems to be of the totally impossible kind: \( 1/0 \).

There is however another interpretation: multiplying through by \( \sqrt{1 - (v/c)^2} \), we get

\[
\sqrt{1 - (v/c)^2}T = mc^2\left(1 - \sqrt{1 - (v/c)^2}\right)
\]

and for \( v = c \) this becomes \( 0 \times T = mc^2 \). A reasonable conclusion now is that \( m = 0 \): a particle traveling at light-speed has no mass. And \( T \) is not further constrained here, infinite or otherwise. Presumably it can (for \( v = c \)) be taken as the energy of an appropriate light-speed particle. Whether this is physical nonsense or not, at least we are getting some sort of “results” from such an approach.

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21 But see for instance [10] and [17], for this is still a topic of dispute.
22 For many purposes; but in QFT for instance this is not quite right.
23 It is no good trying to wriggle out of this by supposing \( T \) is an NSA sort of infinity; that would correspond to \( v \) being “almost” the same as \( c \) (in the same coat, so that \( v/c \) is in the coat of 1). For in fact we need – for the Goudsmit/Uhlenbeck model – that \( v \) be even greater than \( c \). And then \( \gamma \) actually has an imaginary value! This leads into the even stranger physics of tachyons.
24 This can actually be given a positive spin (pun intended). The Higgs field endows particles with mass according to whether they are retarded by it – retarded from traveling at light-speed, that is. Particles that are not so retarded are by definition massless!
7 Summary and Discussion

We have isolated three uses of infinities in physics: dense, range-of-values, and useful-fiction. Range-of-values seems generally unproblematic, but illustrative of the idea that our understanding of the universe can involve infinities of some sort. These are not directly measured, but rather are supported by a mix of inductive reasoning and evidence; and they do not seem to present major difficulties.

The infinities in the Benardete example perhaps lie in between range-of-values and dense: many location values are posited, yet they come close to representing an infinite density of something – but it is not clear just what. And there is no outright paradox if we apply ordinary physics and a little commonsense.

But when we replace imagined barriers with actual physical entities such as fields, things can quickly get bizarre, as in infinite values for charge and force and mass densities. While our discussion focused on a point-model of the electron, any point-source field will do. There are standard tools for representing this – for instance the delta function – but these are usually seen as merely useful calculational devices and not as possible models of what the universe is like. I am arguing that the great success of such tools speaks to the strong possibility of an underlying phenomenon well-worth trying to model.

I am not urging that NSA need be the mathematical physics of the future. There are certainly other directions to consider, such as the surreal numbers studied by Conway, Kruskal and others (see Wikipedia entry [29]). In addition, Bell [3] presents an approach to infinitesimals (but not infinities) based on “smooth worlds” where logic (and geometry) gets even stranger than in NSA yet where physics again comes into play. And indeed infinity (of the dense kind) might happen not to be physically sensible at all. But the idea should not be discarded out of hand.

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