Electromagnetic Oscillations in a Spherical Conducting Cavity with Dielectric Layers. Application to Linear Accelerators

Władysław Źakowicz,1 Andrzej A. Skorupski,2 and Eryk Infeld2
1Institute of Physics, Polish Academy of Sciences, Al. Lotników 32/46, 02–668 Warsaw, Poland
2Department of Theoretical Physics, National Centre for Nuclear Research, Hoża 69, 00–681 Warsaw, Poland

We present an analysis of electromagnetic oscillations in a spherical conducting cavity filled concentrically with either dielectric or vacuum layers. The fields are given analytically, and the resonant frequency is determined numerically. An important special case of a spherical conducting cavity with a smaller dielectric sphere at its center is treated in more detail. By numerically integrating the equations of motion we demonstrate that the transverse electric oscillations in such cavity can be used to accelerate strongly relativistic electrons. The electron’s trajectory is assumed to be nearly tangential to the dielectric sphere. We demonstrate that the interaction of such electrons with the oscillating magnetic field deflects their trajectory from a straight line only slightly. The Q factor of such a resonator only depends on losses in the dielectric. For existing ultra low loss dielectrics, Q can be three orders of magnitude better than obtained in existing cylindrical cavities.

Keywords: Spherical Cavity, Spherical Dielectric Layer, TE Mode, TM Mode, Q Factor, Linear Accelerator

I. INTRODUCTION

It has been shown [1–3] that, if a plane electromagnetic wave is scattered on a finite dielectric object, structural resonances can be excited in the object (e.g., whispering gallery modes). They are associated with very high amplitudes of oscillating EM fields in the dielectric and its vicinity. Their maxima exceed values reached in resonant cavities of typical linear accelerators by several orders of magnitude. Therefore, one can think of applying these fields to accelerate charged particles [1–3]. Many other applications of the whispering gallery modes are described in [4–6].

As for the proposals given in [1–3], both light produced by lasers and microwaves are conceivable. However, it is difficult to achieve the required synchronization of wave particle in the optical frequency range. In the microwave frequency range, this mechanism would require excessive total excitation energy and so may not be practical.

In this paper we demonstrate that the last mentioned problem can be overcome by locating the dielectric object in a resonant cavity. This appeals to traditional accelerating structures used in SLAC, see Figure 1. In the latter case, maximum amplitudes of accelerating fields are restricted by Joule heating losses in conducting walls and electric breakdown. In this connection, in existing accelerators (e.g., in LHC) one avoids sharp edges of the walls and uses superconductive resonant cavities. Unfortunately, since superconductivity of the walls disappears if the magnetic field on the wall exceeds a critical value, the maximal values of accelerating fields in these highly complicated cavities are not much higher than those reached in SLAC.

II. A SPHERICAL CONDUCTIVE CAVITY WITH DIELECTRIC LAYERS

Our approach to describe electromagnetic oscillations in a resonant cavity assumes that the cavity can be di-
vided into regions in which the fields can be determined analytically. The resonant frequency is defined by the fact that the fields must satisfy boundary conditions at the cavity wall along with continuity conditions at the interfaces. This frequency will be determined by numerically solving the consistency condition for these requirements.

In general, we assume that the cavity is bounded by a conducting spherical surface, and filled concentrically with \( N \geq 1 \) either dielectric or vacuum layers. Each dielectric layer is assumed to be homogeneous. We introduce a spherical coordinate system \((r, \theta, \phi)\) with its origin at the cavity center. The layers are bounded by \( r = a_1, a_2, \ldots, a_{N-1}, \) up to \( r = a_N \equiv b \) for the metallic boundary, see Figure 2.

FIG. 2: An example of a spherical cavity with dielectric layers \((N = 4)\).

The harmonically oscillating electromagnetic fields in each concentric layer are described by Maxwell’s equations (Gaussian units, magnetic permeability \( \mu = 1 \), and complex fields proportional to \( \exp(-i\omega t) \)):

\[
\nabla \times \mathbf{E} = ik \mathbf{B}, \quad \nabla \times \mathbf{B} = -i \epsilon \kappa \mathbf{E} \tag{1}
\]

where

\[
k = \omega / c \tag{2}
\]

\( \omega \) is the angular frequency, and \( \epsilon \) denotes complex dielectric permittivity

\[
\epsilon = \epsilon' + i \epsilon'' , \quad | \epsilon'' | \ll \epsilon' \tag{3}
\]

These fields split into transverse electric (TE) or transverse magnetic (TM), which have no radial components of either field. In an ideal resonator with perfectly conducting walls and perfect dielectrics, pure TE or TM modes can be excited. They will also be approximately valid in real resonators if their energy losses are not too high.

Using (9.116), and (9.119) in [9], which describe the vacuum TE field in spherical coordinates, and replacing there \( k \rightarrow \sqrt{\epsilon} k \) we obtain the most general form of the TE field in the uniform dielectric:

\[
\mathbf{E}_{lm}(r, \omega) = \mathbf{E}_l(r, \omega)e^{-i\omega t} + \mathbf{E}_l(r, \omega) \tag{4}
\]

where

\[
\mathbf{E}_l(r) = A_l^{(1)} j_l(\sqrt{\epsilon} kr) + A_l^{(2)} y_l(\sqrt{\epsilon} kr) \tag{5}
\]

\( \mathbf{X}_{lm}(\theta, \phi) \) are vector spherical harmonics as defined by Eq. (9.119) in [9], and \( j_l(\rho) = \sqrt{\frac{2}{\pi}} J_{l+\frac{1}{2}}(\rho) \) and \( y_l(\rho) = \sqrt{\frac{2}{\pi}} Y_{l+\frac{1}{2}}(\rho) \) are spherical Bessel and Neumann functions.

The corresponding magnetic induction can be determined from the first Maxwell equation \( \nabla \times \mathbf{E} = \kappa \mathbf{B} \). Using also (10.60) in [9] we obtain

\[
\mathbf{B} \equiv \mathbf{B}_{lm}(r, t) = \mathbf{B}_{lm}(r, \omega)e^{-i\omega t} \tag{6}
\]

where \( \mathbf{B}_{lm}(r, \omega) \) involves both the transverse radial component:

\[
\mathbf{B}_{lm}(r, \omega) = \mathbf{B}_{lm}(r, \omega) + \mathbf{B}_{lm}(r, \omega) \tag{7}
\]

in which

\[
\mathbf{B}_{lm}(r, \omega) = \mathbf{B}_{l}(r) \mathbf{n} \times \mathbf{X}_{lm}(\theta, \phi) \tag{8}
\]

\[
\mathbf{B}_{lm}(r, \omega) = \mathbf{B}_{l}(r) \mathbf{n} \tag{9}
\]

\[
\mathbf{B}_{l}(r) = \frac{i}{kr} \left[ A_l^{(1)} j_l(\sqrt{\epsilon} kr) + A_l^{(2)} y_l(\sqrt{\epsilon} kr) \right] \tag{10}
\]

\[
\mathbf{B}_{l}(r) = \frac{\sqrt{l(l+1)}}{kr} \mathbf{E}_l(r) \tag{11}
\]

Here \( Y_{lm}(\theta, \phi) \) are spherical harmonics, \( \mathbf{n} = r / r \), \( l \) is a positive integer related to the integer \( m \) by \(-l \leq m \leq l\), \( j_l(\rho) \equiv \sqrt{\frac{2}{\pi}} J_{l+\frac{1}{2}}(\rho) \) and \( y_l(\rho) \equiv \sqrt{\frac{2}{\pi}} Y_{l+\frac{1}{2}}(\rho) \) are derivatives of the Riccati–Bessel and Riccati–Neumann functions.

In a similar way, using (9.118), (9.119) and (10.60) in [9] along with the second Maxwell equation \( \nabla \times \mathbf{H} = -\mu \mathbf{J} \), we obtain for the TM modes in the uniform dielectric:

\[
\mathbf{B}_{lm}(r, t) = \mathbf{B}_{lm}(r, \omega)e^{-i\omega t} \tag{12}
\]

where

\[
\mathbf{B}_{l}(r) = A_l^{(1)} j_l(\sqrt{\epsilon} kr) + A_l^{(2)} y_l(\sqrt{\epsilon} kr) \tag{13}
\]

\[
\mathbf{E} \equiv \mathbf{E}_{lm}(r, t) = \mathbf{E}_{lm}(r, \omega)e^{-i\omega t} \tag{14}
\]
where \( \mathbf{E}_{lm}(\mathbf{r}, \omega) \) involves both the transverse, and radial component:

\[
\mathbf{E}_{lm}(\mathbf{r}, \omega) = \mathbf{E}_{lm}(\mathbf{r}, \omega) + \mathbf{E}_{lm}(\mathbf{r}, \omega)
\]

(15)
in which

\[
\mathbf{E}_{lm}(\mathbf{r}, \omega) = \mathbf{E}_r(r) \times \mathbf{X}_{lm}(\theta, \phi)
\]

(16)

\[
\mathbf{E}_{lm}(\mathbf{r}, \omega) = \mathbf{E}_r(r) Y_{lm}(\theta, \phi) \mathbf{n}
\]

(17)

\[
\mathbf{E}_r(r) = \frac{i}{k r e} \left[ j_l^{(1)}(\sqrt{\epsilon_N kr}) + \bar{A}_l^{(2)} y_l^0(\sqrt{\epsilon_N kr}) \right]
\]

(18)

\[
\mathbf{E}_r(r) = -\frac{\sqrt{(l+1)}}{k r e} \bar{B}_l(r)
\]

(19)

In our analysis we will admit small energy losses in both the wall and the dielectric layers. However, when calculating the resonant frequency of the cavity \( \omega_0 \equiv \omega' \), these losses will be neglected. Thus the wall is assumed to be perfectly conducting. This means that it carries no electric or magnetic field. Then continuity of the tangential components of the electric field and the normal components of the magnetic induction at each interface require vanishing of these components at the boundary of the wall and the dielectric layers. However, when both the wall and the dielectric layers. However, when neither the wall nor the dielectric layers. However, when both the wall and the dielectric layers. However, when both the wall and the dielectric layers. However, when both the wall and the dielectric layers. However, when both the wall and the dielectric layers.

Consider the particular case of a spherical cavity filled completely with the dielectric (or vacuum), i.e., \( N = 1 \), the boundary conditions (20) lead to

\[
A_{lN}^{(1)} j_l(\sqrt{\epsilon_N kb}) + A_{lN}^{(2)} y_l(\sqrt{\epsilon_N kb}) = 0
\]

(20) for TE modes. In view of (14)–(19), this will be the case if the following boundary conditions at \( r = b \) are fulfilled:

\[
j_l(\sqrt{\epsilon_1 kb}) = 0 \quad \text{or} \quad j_l^0(\sqrt{\epsilon_1 kb}) = 0
\]

(22) where \( \epsilon_1 \geq 1 \), \( k = \omega_0/c \). This defines the resonant frequency \( \omega_0 \) of either TE or TM modes, depending only on \( \epsilon_1 \) (\( = \epsilon_N \)) and \( b \).

In the presence of layers \( (N > 1) \), \( \omega_0 \) must also depend on \( \epsilon_{N-1}, \epsilon_{N-2} \) etc. and therefore condition (22) cannot be fulfilled. In fact, for the same reason, we can assume that also the remaining functions in conditions (21) are non-vanishing. Therefore these conditions can be satisfied by choosing

\[
A_{lN}^{(1)} = \frac{N_l j_l(\sqrt{\epsilon_N kb})}{j_l(\sqrt{\epsilon_N kb})}, \quad A_{lN}^{(2)} = -\frac{N_l y_l(\sqrt{\epsilon_N kb})}{y_l(\sqrt{\epsilon_N kb})}
\]

(23)

\[
\bar{A}_{lN}^{(1)} = \frac{N_l j_l^0(\sqrt{\epsilon_N kb})}{j_l^0(\sqrt{\epsilon_N kb})}, \quad \bar{A}_{lN}^{(2)} = -\frac{N_l y_l^0(\sqrt{\epsilon_N kb})}{y_l^0(\sqrt{\epsilon_N kb})}
\]

(24)

for TE modes (upper line) or TM modes, where \( N_l \) and \( \bar{N}_l \) are normalization factors.

At the interfaces between dielectrics, the following quantities must be continuous: the tangential components of the electric field and normal ones of the magnetic induction and furthermore, the tangential components of the magnetic induction, due to vanishing of the surface currents at the dielectric surface. For the TE modes, this leads to the following conditions at \( r = a_n, \)

\[
\begin{align*}
A_{lN}^{(1)} j_l(\rho_n) + A_{lN}^{(2)} y_l(\rho_n) &= A_{lN}^{(1)} j_{l+1}(\rho_n^+) + A_{lN}^{(2)} y_{l+1}(\rho_n^+) \\
A_{lN}^{(1)} j_l^0(\rho_n) + A_{lN}^{(2)} y_l^0(\rho_n) &= A_{lN}^{(1)} j_{l+1}^0(\rho_n^+) + A_{lN}^{(2)} y_{l+1}^0(\rho_n^+)
\end{align*}
\]

(25)

This can be written in matrix form

\[
\mathbf{M}_n \cdot \mathbf{A}_n = \mathbf{M}_n^+ \cdot \mathbf{A}_{n+1}
\]

(26)

where

\[
\mathbf{A}_n = \begin{bmatrix} N_l^{(1)} \\ N_l^{(2)} \end{bmatrix}, \quad \mathbf{M}_n = \begin{bmatrix} j_l(\rho_n) & y_l(\rho_n) \\ j_l^0(\rho_n) & y_l^0(\rho_n) \end{bmatrix}
\]

(27)

The \( \mathbf{M} \) matrices are non-singular:

\[
\begin{bmatrix} j_l(\rho) & y_l(\rho) \\ j_l^0(\rho) & y_l^0(\rho) \end{bmatrix} = \frac{\pi}{2} \begin{bmatrix} J_{l+\frac{1}{2}}(\rho) & Y_{l+\frac{1}{2}}(\rho) \\ J_{l+\frac{1}{2}}^0(\rho) & Y_{l+\frac{1}{2}}^0(\rho) \end{bmatrix} = 1 \neq 0
\]

(28)

where last equality follows from the fact that \( J_{l+\frac{1}{2}}(\rho) \) and \( Y_{l+\frac{1}{2}}(\rho) \) are solutions of the Bessel equation

\[
u''(\rho) + \frac{1}{\rho} u'(\rho) + \left[ 1 - \frac{(l+\frac{1}{2})^2}{\rho^2} \right] u(\rho) = 0.
\]

Multiplying (20) by

\[
\mathbf{M}^{-1}_n \rho_n \begin{bmatrix} y_l(\rho_n) \\ -y_l(\rho_n) \end{bmatrix} = \mathbf{J}_n \mathbf{i}_n
\]

(29)

we arrive at the recurrence relation

\[
\mathbf{A}_n = \mathbf{M}^{-1}_n \cdot \mathbf{M}^+_n \cdot \mathbf{A}_{n+1} \quad n = 1, \ldots, N - 1.
\]

(30)

For the \( n \)th interface between dielectrics, this relation defines the vector \( \mathbf{A}_n \) at the lower layer in terms of that at the upper one. Using this relation successively for \( n = N - 1, N - 2, \ldots, 1 \), we can express all \( \mathbf{A}_n \) vectors in terms of

\[
\mathbf{A}_N = N_l \begin{bmatrix} 1 \\ j_l(\rho_N) \end{bmatrix} = N_l \mathbf{i}_N \quad \rho_N = \sqrt{\epsilon_N kb}
\]

(31)
i.e.,

\[ A_n = (M_{n}^{-1} \cdot M_{n}^{+}) \cdot (M_{n+1}^{-1} \cdot M_{n+1}^{+}) \cdot \ldots (M_{N-1}^{-1} \cdot M_{N-1}^{+}) \cdot A_N \equiv N_i a_n. \]  

(32)

We recall that in the first layer we must satisfy \( A_1^{(2)} = 0 \), see (21). In view of this requirement, equations (25) for \( n = 1 \) can be written as

\[ A_1^{(1)} j_i(\rho_1) - N_i \left[ a_2^{(1)} j_i(\rho_1^+) + a_2^{(2)} y_i(\rho_1^+) \right] = 0 \]

(33)

\[ A_1^{(1)} j_0(\rho_1) - N_i \left[ a_2^{(1)} j_0(\rho_1^+) + a_2^{(2)} y_0(\rho_1^+) \right] = 0 \]

where \( \rho_1 = \sqrt{\varepsilon_1 k_1}, \rho_1^+ = \sqrt{\varepsilon_2 k_1}, \) and \( a_2^{(1,2)} \) are components of the vector \( a_2 \equiv A_2/N_i \). This vector is defined by (32) and (31) if \( N > 2 \) \((\rho_N = \sqrt{\varepsilon_N k_N})\):

\[ a_2 = (M_1^{-1} \cdot M_1^{+}) \cdot (M_2^{-1} \cdot M_2^{+}) \cdot \ldots (M_{N-1}^{-1} \cdot M_{N-1}^{+}) \]

(34)

For \( N = 2 \), \( a_2 \) is defined by (31), i.e., is given by the last factor in (34).

\[ j_i(\rho_1) \left[ a_2^{(1)} j_i(\rho_1^+) + a_2^{(2)} y_i(\rho_1^+) \right] - j_0(\rho_1) \left[ a_2^{(1)} j_0(\rho_1^+) + a_2^{(2)} y_0(\rho_1^+) \right] = 0. \]

(35)

If this condition is fulfilled, \( A_1^{(1)} \) is given by either of equations (35), which are equivalent. Like all remaining coefficients \( A_n^{(1)} \) and \( A_n^{(2)} \), \( n = 2, \ldots, N \), also \( A_1^{(1)} \) will be proportional to the normalization factor \( N_i \), see (32) and (33).

If there are only two layers \((N = 2)\), \( a_2^{(1)} \) and \( a_2^{(2)} \) in (35) and (36) are given by (31) and the resonant frequency \( \omega_0 \) defined by (35) can be found from

\[ j_i(\sqrt{\varepsilon_1 k_1}) \left[ j_0(\sqrt{\varepsilon_2 k_2}) \frac{1}{j_i(\sqrt{\varepsilon_2 k_2})} - y_i(\sqrt{\varepsilon_2 k_2}) \frac{1}{y_i(\sqrt{\varepsilon_2 k_2})} \right] = 0. \]

(36)

and

\[ A_1^{(1)} = N_i \frac{1}{j_i(\sqrt{\varepsilon_2 k_2})} \left[ j_i(\sqrt{\varepsilon_1 k_1}) \frac{1}{j_i(\sqrt{\varepsilon_2 k_2})} - y_i(\sqrt{\varepsilon_2 k_2}) \frac{1}{y_i(\sqrt{\varepsilon_2 k_2})} \right]. \]

(37)

By replacing in \((25) - (37)\)

\[ A_{1m}^{(1,2)} \rightarrow A_{1m}^{(1,2)}, \quad j_i^0(\sqrt{\varepsilon_m k_1}) \rightarrow j_i^0(\sqrt{\varepsilon_m k_1})/\varepsilon_m, \]

\[ y_i^0(\sqrt{\varepsilon_m k_1}) \rightarrow y_i^0(\sqrt{\varepsilon_m k_1})/\varepsilon_m \]

(38)

for any \( m \) and \( a \), we obtain the corresponding equations for the TM modes.

Any standard software like Mathematica or Maple can be used to solve the non-linear equation \((35)\) or \((36)\) defining the resonant frequency \( \omega_0 \equiv \omega_l \), along with the pertinent linear algebra for \( N > 2 \). We did it for \( N = 2 \), see the following section, and also for \( N = 3 \), by using Mathematica.

Note that for the TM modes, where \( \vec{B}_l(r) \) is continuous in each dielectric interface, \( \vec{E}_l(r) \) will have jumps, due to discontinuities in \( \epsilon \). However, the radial component of the electric displacement \( \vec{D}_l(r) \equiv \epsilon \vec{E}_l(r) \) will be continuous. This will also be true of the TE modes where the radial displacement is identically zero. These facts imply the vanishing of surface charges at each dielectric interface. And this in turn means that the multi-layer dielectric structure resembles (and can approximate) a smooth dielectric with some permittivity profile \( \epsilon(r) \), in spite of jumps in \( \epsilon \).

It was pointed out to us by Paul Martin of SIAM, that our matrix equation (24), which can be used to relate the EM fields of a given mode for two layers of a stratified sphere, is not new. It was probably first used by A. Moroz [10] when calculating forced oscillations in such a sphere but without a conducting wall, induced by an oscillating electric dipole. In this application, the frequency \( \omega \) is arbitrary.

III. A SPHERICAL CONDUCTIVE CAVITY WITH A DIELECTRIC SPHERE

The general theory given in the previous section will now be illustrated by calculations pertinent to the TE modes in a spherical cavity with a dielectric sphere of radius \( a \) and dielectric permittivity \( \epsilon \), i.e., for \( N = 2 \), \( a_1 \equiv a, \epsilon_1 \equiv \epsilon, \) and \( \epsilon_2 = 1 \).

Fields in such a system will be described by equations (4) – (9) both in the sphere and the surrounding vacuum. In view of (21) and (23) their radial profiles will be given by

\[ \mathcal{E}_l(r) = N_i \times \begin{cases} A_l j_1(\sqrt{k_1 r}) & \text{if } 0 \leq r \leq a \\ j_1(k_1 r) - y_1(k_1 r) & \text{if } a \leq r \leq b \end{cases} \]

(39)

\[ \mathcal{B}_l(r) = -\frac{i N_i}{k_1} \times \begin{cases} A_l j_0(\sqrt{k_1 r}) & \text{if } 0 \leq r \leq a \\ j_0(k_1 r) - y_0(k_1 r) & \text{if } a \leq r \leq b. \end{cases} \]

(40)

Replacing \( \epsilon_1 \to \epsilon, \epsilon_2 \to 1 \) and \( A_1^{(1)} \to A_1 \) in (36) and (37), we obtain equations defining the resonant frequency \( \omega_l \) and the amplitude coefficient \( A_l \).

We verified that for \( \omega = \omega_l \), the average energies associated with the electric and the magnetic field in the
cavity are equal:

\[
\int_V \epsilon |\mathbf{E}_{lm}(\mathbf{r}, \omega')|^2 \, d\mathbf{v}
= \int_V (|\mathbf{B}_{lm}(\mathbf{r}, \omega')|^2 + |\mathbf{B}_{lm}(\mathbf{r}, \omega')|^2) \, d\mathbf{v}.
\]  
(41)

(This was a check on the correctness of our formulas and accuracy of calculations.) The normalization constant \(N_l\) was chosen so as to satisfy:

\[
\frac{1}{2} \left( \int_a^b |\mathcal{E}(r)|^2 r^2 \, dr + \int_a^b |\mathcal{E}(r)|^2 r^2 \, dr + \int_0^b (|\mathbf{B}_l(r)|^2 + |\mathbf{B}_l(r)|^2) r^2 \, dr \right) = 1.
\]  
(42)

(The corresponding average energy associated with the electric and the magnetic field over our cavity is \(1/(8\pi)\) erg.)

In Figure 3 \(\nu(l) = \omega_l/(2\pi)\) as a function of \(l\) is presented for three spherical cavities with dielectric spheres (\(a\) and \(b\) in cm). Note that \(\nu\) is \(m\) independent (due to degeneracy).

In Figure 4 (a) field radial functions for a spherical cavity with dielectric sphere and perfect metallic wall (Gaussian units, \(k_0 = \omega/c\), \(a\) and \(b\) in cm). The complete fields are given by (43)–(44). Note that \(\mathcal{E}_l\) and \(\mathbf{B}_l\) vanish at the wall, whereas \(\mathbf{B}_l\) is non-zero but small, see (b).

parametrized by the electron’s closest approach \(r_0 = r(t_0)\) and electron velocity \(c\beta\), \(|\beta| = 1\):

\[
r(t) = r_0 + c\beta(t - t_0).
\]  
(43)

The origin of the Cartesian coordinate system \((x, y, z)\) was chosen at the center of the dielectric sphere, and the electron moving along the \(x\) axis was passing just above the dielectric sphere as shown in Figure 1. We chose

\[
r_0 = 1.01a(0, \sin \theta_0, \cos \theta_0), \quad \beta = (1, 0, 0).
\]  
(44)

The effective accelerating field felt by the electron as it passes through the cavity, \(E_{\text{eff}}\), is equal to the real part of \((c \, dt = dx)\)

\[
E(l, m, \theta) \equiv |\bar{E}| e^{i\varphi} = \frac{e}{d} \int_{t_0 - \frac{\pi}{2w}}^{t_0 + \frac{\pi}{2w}} E_x(r(t), \omega') e^{-i\omega't} \, dt,
\]  
(45)

where \(d = 2\sqrt{b^2 - (1.01a)^2}\) is the electron trajectory segment within the cavity, \(E_x\) is the \(x\) component of the effective field \(E^{lm}(\mathbf{r}, \omega')\) given by (43) and \(r(t)\) is given by (43). Thus

\[
E_{\text{eff}} = |\bar{E}| \cos \varphi, \quad \varphi = f(l, m, \theta) - \omega't_0.
\]  
(46)

Maximal acceleration is obtained \((E_{\text{eff}} = |\bar{E}|)\) if \(t_0\) is chosen so that the accelerating phase \(\varphi = 0\). With this choice, the relativistic electron is never decelerated within the spherical cavity, see Figure 7 where two examples are given. Typical results for \(E_{\text{eff}}|_{\varphi=0}\) obtained with our normalization (42) are shown in Figures 5 and 6.
The electromagnetic field given by the real parts of Eqs. (4), (8) and (9), is strongly non-uniform. Therefore one should check how much the relativistic electron will deflect from the assumed trajectory \( \mathbf{r}(t) \) given by (43), due to interaction with this field. A nice feature of our model is that the field in question is described analytically by Eqs. (4–9) so that the pertinent equations of the transversal motion can easily be integrated numerically.

In a real accelerator, where we are dealing with an electron beam of finite cross section, the electromagnetic fields \( \mathbf{E}(r(t), t) \) and \( \mathbf{B}(r(t), t) \) acting on each electron will be superpositions of the external fields and the fields due to the electron charge and current. However, in the lowest approximation (and particularly for not too large beam densities) the latter fields can be neglected. Furthermore, if as in our case, the transversal deflections are small, the deflecting fields can be calculated on the unperturbed trajectory given by (43). It will also be assumed that the electron mass \( m_e = m_0 \gamma, \gamma \gg 1 \), is time independent within the spherical cavity. With these approximations, and within the Cartesian coordinate system \((\bar{x}, \bar{y}, \bar{z})\) with its center at \( \mathbf{r}_0 \), the \( \bar{x} \) axis along the unperturbed trajectory, and the \( \bar{y} \) axis along \( \mathbf{r}_0 \), the electron’s transversal motion will be described by

\[
\begin{align*}
 \dot{v}_\bar{y} &= \frac{-e}{m_e} [E_\bar{y} - B_\bar{z}], \\
 \dot{v}_\bar{z} &= \frac{-e}{m_e} [E_\bar{z} + B_\bar{y}],
\end{align*}
\]

(47)

where \( \bar{t} = t - t_0 \), the field components \( E_\bar{y}(\bar{x}, \bar{t}), B_\bar{z}(\bar{x}, \bar{t}) \), etc. are given by the real parts of Eqs. (4), (8) and (9), taken at \( \bar{x} = \bar{c} t \), and \( \bar{y} = \bar{z} = 0 \). Integrating these equations with zero initial conditions we end up with

\[
\begin{align*}
 v_\bar{y}(\bar{x}) &= \frac{-e}{m_e} \int_{\bar{c} t_0}^{\bar{x}} F_\bar{y}(\bar{x}') \, d\bar{x}' \\
 v_\bar{z}(\bar{x}) &= \frac{-e}{m_e} \int_{\bar{c} t_0}^{\bar{x}} F_\bar{z}(\bar{x}') \, d\bar{x}'
\end{align*}
\]

(49)

(50)

\[
\bar{y}(\bar{x}) = \frac{1}{c} \int_{-\bar{c} t_0}^{\bar{x}} v_\bar{y}(\bar{x}') \, d\bar{x}', \quad \bar{z}(\bar{x}) = \frac{1}{c} \int_{-\bar{c} t_0}^{\bar{x}} v_\bar{z}(\bar{x}') \, d\bar{x}'
\]

(51)
functions soon reaching their limiting values leading to much larger deflections at \( \bar{t} \) for which \( E \) is given by (52), see Figure 10(a) for the accelerating field will be our reference value

\[
E_{\text{eff}} \bigg|_{\varphi = 0} = E_{\text{eff,ref}} = 1.2366 \text{ kV/m.} \tag{52}
\]

It can be seen that if the accelerating phase \( \varphi = 0 \), both \( F_{\bar{y}}(\bar{x}) \) and \( F_{\bar{z}}(\bar{x}) \) are odd functions. Therefore in this case of maximal acceleration, there will be no transversal velocity increments over the cavity, \( v_{\bar{y}}(d/2) = v_{\bar{z}}(d/2) = 0 \). At the same time the velocity components \( v_{\bar{y}}(\bar{x}) \) and \( v_{\bar{z}}(\bar{x}) \) in (51) will be even functions of \( \bar{x} \) tending to zero as \( \bar{x} \to d/2 \). Hence the transversal deflections \( \bar{y}(\bar{x}) \) and \( \bar{z}(\bar{x}) \) will be increasing functions tending to constants as \( \bar{x} \to d/2 \), see Figure 10(a).

For the worst case of accelerating phase \( (\varphi = \pi/2) \) for which \( E_{\text{eff}} = 0 \), \( v_{\bar{y}}(\bar{x}) \) and \( v_{\bar{z}}(\bar{x}) \) will be increasing functions soon reaching their limiting values \( v_{\bar{y}}(d/2) \) and \( v_{\bar{z}}(d/2) \) for \( \bar{x} > 0 \). The corresponding transversal motions \( \bar{y}(\bar{x}) \) and \( \bar{z}(\bar{x}) \) will soon become uniform for \( \bar{x} > 0 \), leading to much larger deflections at \( \bar{x} = d/2 \), see Figure 10(b).

The actual transversal deflections per cavity in an accelerator involving our spherical cavities can be obtained by multiplying the normalized values given in Figure 10 by the factor

\[
\alpha = \frac{e}{m_e E_{\text{eff}}} \frac{E_{\text{eff}}}{E_{\text{eff,ref}}} \frac{\gamma}{m_c} \text{ cm} \tag{53}
\]

where \( E_{\text{eff,ref}} \) is given by (52), \( E_{\text{eff}} \) is the assumed value of the effective accelerating field, and \( \gamma = m_e/m_0 \). For large values of \( E_{\text{eff}} \), small \( \alpha \) requires \( \gamma \) to be sufficiently large. Assuming that \( \bar{z}_{\max} (= \bar{z}(d/2) > \bar{y}(d/2)) \), see Figure 10(a) is not larger than \( \rho \% \) of the spacing between the dielectric sphere and the electron trajectory, 0.01\( a \), \( a = 0.708 \text{ cm} \), the required minimal electron energy is given by

\[
m_c c^2 \text{[GeV]} = \frac{20.69 \times 100}{P} \times \frac{E_{\text{eff}} \text{[MV/m]}}{100} \tag{54}
\]

Thus, if we assume that \( E_{\text{eff}} = 100 \text{ MV/m} \), the transversal displacements will be smaller than 1% of the spacing in question, if the electron energy \( m_e c^2 \geq 21 \text{ GeV} \), i.e., for typical output energies from SLAC. Whether the real dielectric can withstand this value of \( E_{\text{eff}} \) is another question beyond the scope of this paper. More comments will be given later on.

### B. Quality factors

An important parameter of any linear accelerator is the quality \( Q \) of its resonant cavities:

\[
Q = \frac{\omega_0 U}{P} = 2\pi \frac{U}{T_0 P} \tag{55}
\]

where \( \omega_0 \) is the resonant angular frequency of the ideal cavity \( (\sigma = \infty, c'' = 0) \), \( T_0 \) is the corresponding resonant period, \( U \) is the time-averaged energy stored in the cavity \( (\mu = 1) \)

\[
U = \frac{1}{10\pi} \int_V (|\mathbf{E}|^2 + |\mathbf{H}|^2) \, dv \tag{56}
\]

and \( P \) is time-averaged cavity power loss.

The power loss caused by the skin current in the metallic wall bounded by the surface \( S \) is given by

\[
P_{\text{net}} = \alpha \int_S |\mathbf{H}|^2 \, ds \tag{57}
\]

where

\[
\alpha = \frac{c}{8(2\pi)^{3/2}} \sqrt{\frac{\omega_0}{\sigma}} = \frac{c^2}{32\pi^2 \sigma \delta}, \quad \delta = \frac{c}{\sqrt{2\pi \sigma \omega_0}} \tag{58}
\]

\( \delta \) is the skin depth, \( \sigma \) is conductivity of the wall, and the magnetic field intensity \( \mathbf{H} \) refers to the ideal cavity, i.e., its normal component is vanishing \( (\mathbf{H} = \mathbf{H}_\perp) \).
The quality of the cavity related to losses in the metallic wall is thus given by

$$Q_{\text{met}} = \frac{\omega_0 U}{P_{\text{met}}}. \quad (59)$$

Using the fact that at resonance, the averaged energies stored in the electric and magnetic fields are equal, see (41), we end up with ($\mathbf{H} = \mathbf{B}$):

$$Q_{\text{met}} = \frac{2 \int_V |\mathbf{B}|^2 \, dv}{\int_S |\mathbf{B}|^2 \, ds}. \quad (60)$$

This formula is quite general, and in particular can also be used for a traditional cylindrical cavity of radius $R_c$ and height $h$. In that case, the cylindrically symmetric $n = 0$ TM mode used for acceleration is given by

$$E_x(r, \rho) = N J_0(k_0 r) \exp(-i \omega_0 t) \quad (61)$$

$$B_\phi(r, \rho) = iN J'_0(k_0 \rho) \exp(-i \omega_0 t) \quad (62)$$

where $k_0 = \omega_0/c$, $J_0$ is a Bessel function, and $\rho$ and $\phi$ are cylindrical coordinates (cylindrical axis along $z$).

The vanishing of $E_x$ on an ideally conducting cylindrical wall requires that $(\omega_0/c)R_c = 2.405$ (the smallest zero of $J_0$) which defines the angular resonant frequency in terms of $R_c$. Equations (62) and (61) lead to the well known formula for the quality of the cylindrical pill box cavity

$$Q_c = 2.405 \sqrt{\frac{2 \pi \sigma}{\omega_0}} \frac{1}{1 + R_c/h}. \quad (63)$$

For the SLAC pill box cavity shown in Figure 11 ($R_c = h = d = 4 \text{ cm}$, and $\sigma = 5.294 \times 10^{17} \text{ s}^{-1}$ for copper wall in room-temperature), this formula leads to $Q_c = 1.633 \times 10^4$. The corresponding values for spherical cavities with ideal dielectric spheres and the same values of $d$ and $\sigma$ reach much larger values, see Figure 11.

In $k \neq 0$, for $k$ defined by (53). In view of Eq. (2) this implies a complex value of $\omega = \omega' + i \omega''$. For the fields given by (11) – (14), we obtain

$$U(t) = U(t = 0) \times e^{i \omega' t}$$

where $\omega'' < 0$ for the energy $U$ being dissipated rather than generated. Using this result we find for the power losses in the dielectric:

$$P_{\text{diel}} = -\frac{d U}{dt} = -2 \omega'' U.$$}

In view of (55), the corresponding quality will thus be given by

$$Q_{\text{diel}} = -\frac{\omega_0}{2 \text{Im} \omega}. \quad (64)$$

This value is of the order of $(\tan \delta)^{-1} \equiv \epsilon'/\epsilon''$. It is approximately $l$ independent.

In our calculations we took $\epsilon' = 10$ and $\epsilon'' = 10^{-6}$. Dielectrics with such ultra small losses were investigated in [8].

The total power loss in the spherical cavity encasing the dielectric sphere $P_s$ is due to the power loss in the metallic wall and that in the dielectric sphere:

$$P_s = P_{\text{met}} + P_{\text{diel}}. \quad (65)$$

Dividing both sides of this relation by $\omega_0 U$ and using (53) we obtain

$$\frac{1}{Q_s} = \frac{1}{Q_{\text{met}}} + \frac{1}{Q_{\text{diel}}}. \quad (66)$$

where $Q_s$ is the total Q-factor of the spherical cavity. Values of $Q_s$ versus $l$ for three spherical cavities with dielectric spheres and $d = 4 \text{ cm}$ are shown in Figure 12. They are about three orders of magnitude larger than $Q_c = 1.633 \times 10^4$.

FIG. 11: The quality $Q_{\text{met}}$ versus $l$ for three spherical cavities with ideal dielectric spheres (see Figure 6).

FIG. 12: The quality $Q_s$ versus $l$ for three spherical cavities with dielectric spheres (see Figure 6).

In a real accelerator, openings in the metallic wall are necessary for free penetration of the cavity by the electron beam, and to enable coupling between neighboring cavities. This will lower the quality $Q_{\text{met}}$ but should have
little effect on the total quality of the spherical cavity \( Q_s \). The latter is defined by losses in the dielectric, see \((66)\) where \( Q_{\text{diel}} \ll Q_{\text{net}} \). The resonant frequency should not be drastically changed either, as the EM fields at the iris \((r = b)\) are very small fractions of their maxima, see Figure 4.

C. Discussion and summary

When comparing the effective accelerating fields in the traditional pill box cavity with that in our spherical cavities with dielectric spheres, we first assume that \( U \), the time-averaged energy stored in the cavity, see \((50)\), is the same in both situations. Therefore the normalization factor \( N \) in \((61)\) and \((62)\) will first be chosen so that

\[
2 \pi h \frac{1}{2} \int_0^{R_e} \rho \, d \rho \left[ |E_x(\rho, t)|^2 + |B_y(\rho, t)|^2 \right] = 1 \quad (67)
\]

see \((12)\) \((U = 1/(8\pi) \text{ erg})\).

The complex effective accelerating field \( \bar{E} \) for the cylindrical resonator shown in Figure 1 \((R_e = h = d)\) is given by the right hand side of \((45)\) in which \( E_x \) is defined by \((61)\) with \( \rho = 0 \), and \( \omega_0 = \omega_0 \). The result is

\[
\bar{E}_c = N \frac{\sin \alpha}{\alpha} e^{-i \omega_0 t_0}, \quad \alpha = \frac{\omega_0 d}{2 c} \quad (68)
\]

where \( t_0 \) is the time at which the electron passes the center of the cavity.

We now denote by \( E_{\text{eff}} \) and \( E_{\text{eff}} \) the maximal effective accelerating fields \((\text{equal to } |\bar{E}|)\) for the spherical cavity with a dielectric sphere and the traditional cylindrical cavity, for any values of the average energies in the cavities, \( U_c \) and \( U_c \). In view of the fact that \( E_x \) in \((61)\) is proportional to \( \sqrt{U} \) we can write, using the definition \((59)\) of \( Q \),

\[
\frac{E_{\text{eff}}}{E_{\text{eff}}_c} = \sqrt{\frac{P_c}{P_c}} G(l) \quad (69)
\]

where \( G(l) \), the “gain factor”, is given by

\[
G(l) = \left. \frac{E_{\text{eff}}}{E_{\text{eff}}_c} \right|_{U_c = U_c} \sqrt{\frac{Q_s}{Q_c}}\frac{\nu_c}{\nu_s} \quad (70)
\]

Here \( \nu_s \) and \( \nu_c \) are the resonant frequencies of the spherical and the cylindrical cavities \((\nu = \omega_0/(2\pi))\) and \( P_s \) and \( P_c \) are the corresponding power losses. They are equal to the powers that must be supplied from external sources to sustain the oscillations. They should be as large as possible to avoid breakdown in the dielectric or at the metallic wall. Further research is necessary to give an estimate of the ratio \( P_s/P_c \). We can only hope that it is not smaller than unity.

For our typical SLAC pill box cavity shown in Figure \( \#1 \) \((R_e = h = d = 4 \text{ cm})\) we obtain \( \nu_c = 2.87 \text{ GHz} \) and

\[
E_{\text{eff}} \|_{U_c = 1/(8\pi) \text{ erg}} = 3.16 \text{ kV/m}, \quad \text{to be used in } (70). \]

The results of our calculation are shown in Figures \( \#3 \) \( \#13 \) for three reasonable values of \( b/a = 3, 2, 1/2 \). The electron trajectory segment inside the cavity shown in Figure 1 was equal to the typical length of the pill-box cavity of SLAC \((4 \text{ cm})\). Calculations were performed for various values of \( l \) and \( m \). The optimal parameters found were: \( a = 0.708 \text{ cm}, b = 2.124 \text{ cm}, \) and \((l, m) = (6, -2)\).

IV. CONCLUSIONS

An electric field, intensified by structural resonance, can be used to accelerate electrons. This is demonstrated here by placing a dielectric sphere concentrically inside a spherical resonator, in which an appropriate whispering gallery mode is excited. A strong, accelerating field appears next to the surface of the dielectric. At the same time, the tangential component of the magnetic field at the wall of the resonator is minimal. This makes losses at the metallic walls negligible without engaging expensive cryogenic systems ensuring superconductivity of the walls. The \( Q \) factor of the resonator only depends on losses in the dielectric. For existing dielectrics, this gives a \( Q \) factor three orders of magnitude better than obtained in existing cylindrical cavities. Furthermore, for the proposed spherical cavity, all field components at the metallic wall are either zero or very small, see Figure 4.

Therefore, one can expect the proposed spherical cavity to be less prone to electrical breakdowns than the traditional cylindrical cavity.

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