Resonant bands, Aomoto complex and real 4-nets

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Introduction
The resonant band is a useful notion for the computation of the nontrivial
monodromy eigenspaces of the Milnor fiber of a real line arrangement. We
develop the resonant band description for the cohomology of the Aomoto complex. As an application, we prove that real 4-nets do not exist.

Let us fix some notation:
- \( k \in \mathbb{Z}, k \geq 3; \)
- \( K \) a field (generally, \( \mathbb{R} \) or \( \mathbb{C} \) and \( \mathbb{KP}^2 \) the projective plane;
- \( \mathcal{A} = \{ R_0, \ldots, R_n \} \) a line arrangement in \( \mathbb{KP}^2; \)
- \( \mathcal{A} = \{ H_1, \ldots, H_n \} \) the affine line arrangement in \( \mathbb{K}^2 = \mathbb{KP}^2 \setminus R_0 \) obtained from \( \mathcal{A}; \)
- \( A_0^p(\mathcal{A}) \) the Orlik-Solomon algebra of \( \mathcal{A} \) over \( \mathbb{F}_2 \) generated by the symbols \( e_1, \ldots, e_n; \)
- For \( S \subset A, \) consider \( e(S) := \sum_{H \in S} e_H \in A_0^\ast(\mathcal{A}) \) and \( \eta_B := e(\mathcal{A}) = \sum_{i=1}^n e_i. \)

Definition (k-nets)
\( \mathcal{A} \) supports a \( k \)-net structure if and only if there exist a partition
\( \mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k \) and a finite set of points \( X \subset \mathbb{K}^2 \) such that
- For all \( i \neq j, \) if \( H \in \mathcal{A}_i \) and \( H' \in \mathcal{A}_j, \) then \( H \cap H' \neq \emptyset; \)
- For all \( p \in X \) and for all \( i = 1, \ldots, k, \) there exists a unique \( H \in \mathcal{A}_i \) such that \( p \in H. \)

Known facts
- If \( k \geq 5 \) there does not exist any \( k \)-net;
- There exist infinitely many \( 3 \)-nets;
- The Hesse arrangement is the only known \( 4 \)-net.

Theorem 2 (Papadima-Suciu)
Consider \( S \subset A. \) Then \( e(S) \wedge \eta_B = 0 \) if and only if \( \forall p \in \mathbb{K}^2 \) one of the following is satisfied:
- If \( |\mathcal{A}_p| \) is odd, then \( |\mathcal{A}_p| = |S_p|; \)
- If \( |\mathcal{A}_p| \) is even then \( \mathcal{A}_p := \{ H \in \mathcal{A} | p \in H \}; \)

Definitions
From now on we consider the case \( \mathbb{K} = \mathbb{R} \) and \( n = \text{odd}; \)
- The connected components of \( \mathbb{K}^2 \setminus \bigcup_{H \in A} H \) are called chambers. The set of all chambers is denoted by \( \text{ch}(\mathcal{A}); \)
- Given \( C_1, C_2 \in \text{ch}(\mathcal{A}), d(C_1, C_2) \) is the number of line that separate the chambers.
- A band is a region bounded by two consecutive parallel lines.
- Each band \( B \) has two unbounded chambers \( U(B) \) and \( U_2(B); \)
- A band \( B \) is called resonant if \( d(U(B), U_2(B)) \) is even. The set of all resonant bands is denoted by \( \text{RB}(\mathcal{A}); \)
- \( \nabla : \mathbb{F}_2[\text{RB}(\mathcal{A})] \rightarrow \mathbb{F}_2[\text{ch}(\mathcal{A})] \) defined by \( [B] \rightarrow \sum_{C \in \text{ch}(\mathcal{A})} C \in \mathcal{B} d(U(B), C) \).

Theorem A (T.-Yoshinaga)
\( \text{Ker}(\nabla) \cong H^1(A_0^\ast(\mathcal{A}), \eta_B). \)

Proposition 1 (T.-Yoshinaga)
When \( |\mathcal{A}_p| = 4, \) then there are four cases:
- \( S_p = \emptyset; \)
- \( S_p = A_p; \)
- \( |S_p| = 2 \) and lines in \( S_p \) are adjacent.
- \( |S_p| = 2 \) and lines in \( S_p \) are separated by lines in \( A_p \setminus S_p. \)
Moreover, if \( e(S) \wedge \eta_B = 0, \) then (4) cannot happen.

Theorem B (T.-Yoshinaga)
There does not exist a real arrangement \( \mathcal{A} \) that supports a \( 4 \)-net structure.

Proof
Suppose \( \mathcal{A} \) supports a \( 4 \)-net structure with partition \( \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4. \) There exists a multiple point \( p \in \mathbb{K}^2 \) of \( \mathcal{A} \) with multiplicity \( 4 \) such that \( p \) is the intersection point of \( 4 \) lines \( H_i \in \mathcal{A}_i. \) The lines are ordered like:
\[
\frac{1}{2} \eta_B \leq 4
\]
We can now define \( S = \mathcal{A}_1 \cup \mathcal{A}_3. \) Then we have \( \eta_B \wedge e(S) = 0. \) By definition, \( S_p = \{ H_1, H_2 \} \) consists of two lines and separated by the other two lines \( H_3, H_4. \) Therefore (4) in the previous Proposition happens. This contradicts the statement of the last Proposition.

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