“Massless” vector field in de Sitter Universe

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Abstract

In the present work the massless vector field in the de Sitter (dS) space has been quantized. “Massless” is used here by reference to conformal invariance and propagation on the dS light-cone whereas “massive” refers to those dS fields which contract at zero curvature unambiguously to massive fields in Minkowski space. Due to the gauge invariance of the massless vector field, its covariant quantization requires an indecomposable representation of the de Sitter group and an indefinite metric quantization. We will work with a specific gauge fixing which leads to the simplest one among all possible related Gupta-Bleuler structures. The field operator will be defined with the help of coordinate independent de Sitter waves (the modes) which are simple to manipulate and most adapted to group theoretical matters. The physical states characterized by the divergencelessness condition will for instance be easy to identify. The whole construction is based on analyticity requirements in the complexified pseudo-Riemannian manifold for the modes and the two-point function.

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I. INTRODUCTION

In previous work, the so-called “massive” vector field in dS space has been considered \[1\]. Unitary irreducible representation (UIR) in the principal series of the de Sitter group \(SO_0(1,4)\), with Casimir operator eigenvalue \(<Q^{(1)}_\nu>=\nu^2+\frac{1}{4}\), \(\nu \geq 0\) and the corresponding “mass” \(m_p^2=H^2(\nu^2+\frac{1}{4})\), are associated with that vector field. The interpretation in terms of mass of this field is made possible by carrying out the null curvature limit. Indeed, the principal series of UIR’s admits a massive Poincaré group UIR in the limit \(H=0\) \[2, 3\]. However, there is another vector UIR of the de Sitter group with a minkowskian limit. In other words it is the UIR having a natural extension to the conformal group \(SO_0(2,4)\), which is equivalent to the massless spin 1 UIR of the conformal extension of the Poincaré group \[4, 5\]. The corresponding field obeys a conformal invariant field equation and the minkowskian interpretation is that of a massless field. This UIR belongs to the discrete series of UIR of the dS group corresponding to eigenvalue \(<Q^{(1)}_\nu>=0\) of the Casimir operator, and this value characterizes the field we call “massless” vector field.

The covariant quantization of the massless vector field raises various problems, analogous to those encountered in the quantization of the electromagnetic field in Minkowski space. First of all one should note that the field equation admits gauge solutions. Therefore one is free to use a gauge fixing parameter \(c\). Now it is known \[6, 7\] that the quantization of gauge invariant theories usually requires quantization `a la Gupta-Bleuler. It has in fact been proved that the use of an indefinite metric is an unavoidable feature if one insists on the preserving of causality (locality) and covariance in gauge quantum field theories \[5\]. This means that one cannot restrict the state space of the massless vector field to a Hilbert space, the emergence of states with negative or null norm necessitates indefinite metric quantization.

An indecomposable group representation structure is needed (exactly like for the electromagnetic field in Minkowski space) where the physical states belong to a subspace (characterized by the divergencelessness condition of the field operator \[10\]) \(V\) of solutions, but where the field operator must be defined on a larger gauge dependent space \(V_c\) (which contains negative norm states as well), as shown in Fig\[14\]. The physical subspace \(V\) is invariant but not invariantly complemented in \(V_c\). The same feature repeats in \(V\) where one finds the invariant (but again not invariantly complemented) subspace of gauge solutions \(V_g\). The latter reveal to be orthogonal to all the elements of \(V\) including themselves \[10\]. Consequently one must eliminate them from the physical states, by considering the physical state space as the coset \(V/V_g\). We will see that the physical states propagate on the light-cone and correspond to vector massless Poincaré field in the null curvature limit.

In previous studies, the massless vector field was considered in flat coordinate system covering only the one-half of the dS hyperboloid \[11\]. In Ref. \[12\], Allen and Jacobson calculated the massless vector two-point functions in terms of the geodesic distance. These functions are independent of the choice of the coordinate system. The Hilbert space structure, the vector field operator and the corresponding two-point function are studied in the present paper in terms of coordinate-independent de Sitter waves. We will adopt a very convenient value for the gauge fixing parameter \(c\) and it is not the usual Feynman value \(c=0\) \[7\]. This choice will eliminate from our solutions additional logarithmic divergent terms which on the contrary appear in Ref. \[12\].

The gauge invariant de Sitter vector field equation is presented in Section 2 in terms of the Casimir operator. We start from its expression given in intrinsic coordinates and rewrite it by
FIG. 1: Gupta-Bleuler structure lying behind indecomposable representation of Poincaré or dS group

using the ambient space formalism more convenient when group theoretical considerations are involved. The Gupta-Bleuler triplet is discussed in Section 3. The invariant space is defined with an indecomposable representation of the dS group. Physical states correspond to the UIR’s $\Pi^{\pm}_{1,1}$. It is the central part of the indecomposable representation.

Section 4 is devoted to the solutions of the field equation (which we shall call de Sitter waves) in terms of a scalar field $\phi$ and a generalized polarization vector $\mathcal{E}_\alpha$, according to the following expression

$$K_\alpha(x) = \mathcal{E}_\alpha(x, \partial)\phi(x).$$

The dS vector waves are only locally defined since they are singular on lower dimensional subsets in dS space-time (they are also multivalued in general, but not in the present case). For a global definition, they must be viewed as distributions which are boundary values of analytic continuations of the solutions to tubular domains in the complexified de Sitter space.

In Section 5, we give two different methods for getting a two-point function $\mathcal{W}_{\alpha\alpha'}(x, x')$. On one hand we introduce the two-point function in terms of vector dS waves, on the other hand we define it as a maximally symmetric bivector. Of course we indicate under which circumstances both definition coincide. We then show that the two-point function satisfies the minimal conditions of field equation, locality, covariance, normal analyticity. The normal analyticity allows us to define the two-point function $\mathcal{W}_{\alpha\alpha'}(x, x')$.

Finally, in Section 6 we construct the field operator. We compute the commutator and show that the field operator must be well chosen in order to yield a causal field.

II. DE SITTER FIELD EQUATION

A. De Sitter ambient space description

The de Sitter solution to the cosmological Einstein field equation (with positive constant curvature) can be viewed as a one-sheeted hyperboloid embedded in a five dimensionnal Minkowski space $M^5$:

$$X_H = \{ x \in \mathbb{R}^5, x^{2} = \eta_{\alpha\beta}x^\alpha x^\beta = -H^{-2} = -\frac{3}{\Lambda}, \ \alpha, \beta = 0, 1, 2, 3, 4 \} \quad (\text{II.1})$$
where \( \eta_{\alpha \beta} = \text{diag}(1, -1, -1, -1, -1) \) and \( \Lambda \) is the cosmological constant (in units \( c = 1 \)). The de Sitter metric is given by
\[
\left. ds^2 = \eta_{\alpha \beta} dx^\alpha dx^\beta \right|_{x^2 = -H^2} = g_{\mu \nu} dX^\mu dX^\nu , \quad \nu, \mu = 0, 1, 2, 3 ,
\]
where \( X^\mu \) are the four local space time coordinates on the dS hyperboloid. This way of describing the dS space as a (pseudo-)sphere in a higher-dimensional Minkowski space constitutes the ambient space approach. Two crucial advantages favor the ambient space formalism: the expressions throughout the present paper will have the simplest possible minkowskian-like form (for obvious reasons), and the link with group theory is easily readable in this context.

In ambient space notations, a vector field \( K_\alpha(x) \) can be viewed as a homogeneous function in the \( \mathbb{R}^5 \)-variables \( x^\alpha \) with some arbitrarily chosen degree \( \sigma \) which therefore satisfies:
\[
x^\alpha \frac{\partial}{\partial x^\alpha} K(x) = x \cdot \partial K(x) = \sigma K(x).
\]

The choice for \( \sigma \) will be dictated by simplicity reasons when one has to deal with field equations. In the following we set \( \sigma = 0 \) so that the d’Alembertian operator \( \square_H \equiv \nabla_\mu \nabla^\mu \) on dS space (\( \nabla_\mu \) being the covariant derivative) coincides with the d’Alembertian operator \( \square_5 \equiv \partial^2 \) on \( \mathbb{R}^5 \). We will prove this shortly.

Of course, not every homogeneous vector field of \( \mathbb{R}^5 \) represents a physical dS entity! In order to ensure that \( K_\alpha(x) \) lies in the de Sitter tangent space-time it also must satisfy the transversality condition
\[
x \cdot K(x) = 0.
\]

Given the importance of this transversality property for dS fields let us introduce the symmetric, transverse projector \( \theta_{\alpha \beta} = \eta_{\alpha \beta} + H^2 x_\alpha x_\beta \) which satisfies \( \theta_{\alpha \beta} x^\alpha = \theta_{\alpha \beta} x^\beta = 0 \). It is the transverse form of the dS metric in ambient space notation and it is used in the construction of transverse entities like the transverse derivative \( \partial_\alpha = \theta_{\alpha \beta} \partial^\beta = \partial_\alpha + H^2 x_\alpha \partial \).

Since in most of the works devoted to dS field theory the tensor fields are written using local coordinates, it is very important to provide the link between the two approaches. The “intrinsic” vector field \( A_\mu(X) \) is locally determined by the field \( K_\alpha(x) \) through the relation
\[
A_\mu(X) = \frac{\partial x^\alpha}{\partial X^\mu} K_\alpha(x(X)) \quad \quad K_\alpha(x) = \frac{\partial X^\mu}{\partial x^\alpha} A_\mu(X(x)).
\]

In the same way one can show that the transverse projector \( \theta \) is the only symmetric and transverse tensor which is linked to the dS metric \( g_{\mu \nu} \):
\[
\frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x^\beta}{\partial X^\nu} \theta_{\alpha \beta} = g_{\mu \nu}.
\]

The next step is to explain how the covariant derivatives \( \nabla \) are related to the transverse derivative denoted by \( \partial \). In general, covariant derivatives acting on a \( l \)-rank tensor are transformed according to
\[
\nabla_\mu \nabla_\nu \ldots \nabla_\rho h_{\lambda_1 \ldots \lambda_l} = \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x^\beta}{\partial X^\nu} \frac{\partial x^\gamma}{\partial X^\rho} \frac{\partial x^\eta_1}{\partial X^{\lambda_1}} \ldots \frac{\partial x^\eta_l}{\partial X^{\lambda_l}} \text{Trpr} \partial_\alpha \text{Trpr} \partial_\beta \ldots \text{Trpr} \partial_\lambda K_{\eta_1 \ldots \eta_l},
\]
where the transverse projection defined by
\[
(\text{Trpr} K)_{\lambda_1 \ldots \lambda_l} \equiv \theta_{\lambda_1}^{\eta_1} \ldots \theta_{\lambda_l}^{\eta_l} K_{\eta_1 \ldots \eta_l},
\]
guarantees the transversality in each index. Let us indicate how this works for the scalar and vector fields respectively, since these cases only will be considered in the next.
a. The scalar case

\[ \nabla_\mu \phi = \frac{\partial x^\alpha}{\partial X_\mu} \tilde{\partial}_\alpha \phi \quad \text{and} \quad \nabla_\mu \nabla_\nu \phi(X) = \frac{\partial x^\alpha}{\partial X_\mu} \frac{\partial x^\beta}{\partial X_\nu} (\tilde{\partial}_\alpha \tilde{\partial}_\beta \phi - H^2 x_\beta \tilde{\partial}_\alpha \phi) . \]

The d’Alembertian can be calculated

\[ \Box_H \phi = g^{\mu \nu} \nabla_\mu \nabla_\nu \phi = g^{\mu \nu} \frac{\partial x^\alpha}{\partial X_\mu} \frac{\partial x^\beta}{\partial X_\nu} (\tilde{\partial}_\alpha \tilde{\partial}_\beta \phi - H^2 x_\beta \tilde{\partial}_\alpha \phi) = \theta^{\alpha \beta} (\tilde{\partial}_\alpha \tilde{\partial}_\beta \phi - H^2 x_\beta \tilde{\partial}_\alpha \phi) = \tilde{\partial}^2 \phi . \]

Note that for a homogeneous function \( \phi \) of degree \( \sigma \) one gets

\[ \Box_H \phi \equiv \nabla_\mu \nabla^\mu \phi = \tilde{\partial}^2 \phi = \partial^2 \phi + 3H^2 (x \cdot \partial) \phi + H^2 (x \cdot \partial)(x \cdot \partial) \phi = (\Box_5 + H^2 \sigma (\sigma + 3)) \phi , \]

which motivates our choice \( \sigma = 0 \).

b. The vector case

For a transverse vector field one easily obtains

\[ \nabla_\mu A_\nu = \frac{\partial x^\alpha}{\partial X_\mu} \frac{\partial x^\beta}{\partial X_\nu} (\tilde{\partial}_\alpha \tilde{\partial}_\beta \phi - H^2 x_\beta \tilde{\partial}_\alpha \phi) \quad \text{which implies} \quad \nabla \cdot A = \tilde{\partial} \cdot \mathcal{K} . \quad (II.7) \]

Moreover one gets

\[ \nabla_\mu \nabla_\nu A_\rho = \frac{\partial x^\alpha}{\partial X_\mu} \frac{\partial x^\beta}{\partial X_\nu} \frac{\partial x^\gamma}{\partial X_\rho} \text{Trpr} \tilde{\partial}_\alpha \text{Trpr} \tilde{\partial}_\beta \mathcal{K}_\gamma = \frac{\partial x^\alpha}{\partial X_\mu} \frac{\partial x^\beta}{\partial X_\nu} \frac{\partial x^\gamma}{\partial X_\rho} (\tilde{\partial}_\alpha \tilde{\partial}_\beta \mathcal{K}_\gamma - H^2 \theta_{\alpha \beta} \mathcal{K}_\gamma) \]

\[ - H^2 \theta_{\alpha \gamma} \mathcal{K}_\beta - H^2 x_\beta \tilde{\partial}_\alpha \mathcal{K}_\gamma + H^2 x_\gamma S \left[ H^2 x_\alpha \mathcal{K}_\beta - \tilde{\partial}_\alpha \mathcal{K}_\beta \right] , \]

with \( S \) the non-normalized symmetrization operator. The d’Alembertian becomes:

\[ \Box_H A_\mu = \nabla^\lambda \nabla_\lambda A_\mu = \frac{\partial x^\alpha}{\partial X_\mu} \left[ \tilde{\partial}^2 \mathcal{K}_\alpha - H^2 \mathcal{K}_\alpha - 2H^2 x_\alpha \tilde{\partial} \cdot \mathcal{K} \right] . \quad (II.9) \]

In the following we will recall the “massless” vector field equation on dS background and show how the ambient space formalism is so well adapted to the group theoretical content.

B. Field equation

The action for free “massless” vector fields \( A_\mu(X) \) propagating on de Sitter space reads \((h = 1) \)

\[ S(A) = \int_{X_H} \frac{1}{4} F^{\mu \nu} F_{\mu \nu} \, d\sigma , \quad (II.10) \]

where \( F^{\mu \nu} = \nabla^\mu A^\nu - \nabla^\nu A^\mu \) and \( d\sigma \) is the \( O(1,4) \)-invariant measure on \( X_H \). The variational principle applied to \((II.10)\) yields the field equation

\[ \nabla_\mu F^{\mu \nu} = \nabla_\mu (\nabla^\nu A^\mu - \nabla^\mu A^\nu) = 0 . \quad (II.11) \]

Since \( [\nabla_\mu, \nabla_\nu] A_\lambda = -H^2 (g_{\mu \lambda} A_\nu - g_{\nu \lambda} A_\mu) \) one obtains the wave equation

\[ (\Box_H + 3H^2) A_\mu(X) - \nabla_\mu \nabla \cdot A(X) = 0 . \quad (II.12) \]
This field equation is identically satisfied by the gauge vector fields of the form \( A_\mu = \nabla_\mu \phi \) because of the property \([\Box_H \nabla_\mu - \nabla_\mu \Box_H] \phi = -3H^2 \nabla_\mu \phi\). Thus (II.12) is invariant under the gauge transformation
\[
A_\mu \rightarrow A'_\mu = A_\mu + \nabla_\mu \phi,
\]
where \( \phi \) is an arbitrary scalar field. The wave equation with gauge fixing parameter \( c \) reads
\[
(\Box_H + 3H^2)A_\mu(X) - c \nabla_\mu \nabla \cdot A(X) = 0.
\]
Our aim is now to write the field equation (II.14) in terms of the Casimir operator of the dS group SO\(_{0}(1,4)\).

C. Casimir operators in the field equation

The kinematical group of the de Sitter space is the 10-parameter group SO\(_{0}(1,4)\) (connected component of the identity in O\((1,4)\)), which is one of the two possible deformations of the Poincaré group. There are two Casimir operators
\[
Q^{(1)}_1 = -\frac{1}{2}L_{\alpha\beta}L^{\alpha\beta}, \quad Q^{(2)}_1 = -W_\alpha W^\alpha,
\]
where
\[
W_\alpha = -\frac{1}{8}\epsilon_{\alpha\beta\gamma\delta\eta}L^{\beta\gamma}L^{\delta\eta}, \quad \text{with 10 infinitesimal generators} \quad L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}.
\]
The subscript 1 in \( Q^{(1)}_1, Q^{(2)}_1 \) reminds that the carrier space is constituted by vectors. The orbital part \( M_{\alpha\beta} \), and the action of the spinorial part \( S_{\alpha\beta} \) on a vector field \( K \) defined on the ambient space read respectively
\[
M_{\alpha\beta} = -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha), \quad S_{\alpha\beta}K_\gamma = -i(\eta_{\alpha\gamma}K_\beta - \eta_{\beta\gamma}K_\alpha).
\]
The symbol \( \epsilon_{\alpha\beta\gamma\delta\eta} \) holds for the usual antisymmetrical tensor. The action of the Casimir operator \( Q^{(1)}_1 \) on \( K \) can be written in the more explicit form
\[
Q^{(1)}_1 K(x) = \left( Q^{(1)}_0 - 2 \right) K(x) + 2x \bar{\partial} \cdot K(x) - 2 \partial x \cdot K(x),
\]
where, \( Q^{(1)}_0 = -\frac{1}{2}M_{\alpha\beta}M^{\alpha\beta} \) is the scalar Casimir operator. We are now in position to express the wave equation (II.14) by using the Casimir operators. This can be done with the help of equation (II.9) since \( Q^{(1)}_0 = -H^{-2}(\bar{\partial})^2 \). The d’Alembertian operator becomes
\[
\Box_H A_\mu = \nabla^\lambda \nabla_\lambda A_\mu = -\frac{\partial x^\alpha}{\partial X^\mu} \left[ Q^{(1)}_0 H^2 K_\alpha + H^2 K_\alpha + 2H^2 x_\alpha \bar{\partial} \cdot K \right],
\]
and the equation (II.14) with this new notation reads
\[
\left( Q^{(1)}_0 - 2 \right) K(x) + 2x \bar{\partial} \cdot K(x) + cH^{-2} \bar{\partial} \partial x \cdot K(x) = 0.
\]
Finally using (II.18) one obtains the field equation formulated in terms of the Casimir operator $Q^{(1)}_1$:

$$Q^{(1)}_1 \mathcal{K}(x) + cD_1 \partial \cdot \mathcal{K}(x) = 0, \quad \text{where} \quad D_1 = H^{-2}\bar{\partial}.$$ (II.21)

But, as we will see, the “minimal” (or optimal) choice of $c$ is not zero, contrary to the flat space case (Feynman gauge). This is because the choice $c = 0$ yields logarithmic divergent terms in the vector field expression. The “minimal” choice on the contrary is chosen so that it allows to eliminate those terms. Before coming back to this point, let us turn to the group-theoretical content of this equation.

### D. Group theoretical notions

The operator $Q^{(1)}_1$ which commutes with the action of the group generators can be used to classify the UIR’s i.e.,

$$(Q^{(1)}_1 - \langle Q^{(1)}_1 \rangle) \mathcal{K}(x) = 0.$$ (II.22)

Following Dixmier in Reference [16] we get a classification scheme by using a pair $(p, q)$ of parameters involved in the following possible spectral values of the Casimir operators:

$$Q^{(1)} = (-p(p + 1) - (q + 1)(q - 2)) I_d, \quad Q^{(2)} = (-p(p + 1)q(q - 1)) I_d.$$ (II.23)

As comprehensively described in Appendix A, three types of scalar, tensorial or spinorial UIR are distinguished for $SO_0(1,4)$ according to the range of values of the parameters $q$ and $p$ [16, 17], namely the principal, the complementary and the discrete series. In the following, we shall restrict the list to those among all unitary representations which precisely have a minkowskian physical spin-1 interpretation in the limit $H = 0$. The flat limit tells us that for the principal and the complementary series it is the value of $p$ which has a spin meaning, and that, in the case of the discrete series, the only representations which have a physically meaningful minkowskian counterpart are those with $p = q$. The spin-1 tensor representations relevant to the present work are the following:

i) The UIR’s $U^{1,\nu}$ in the principal series where $p = s = 1$ and $q = \frac{1}{2} + i\nu$ corresponds to the Casimir spectral values:

$$\langle Q^{(1)}_1 \rangle = \nu^2 + \frac{1}{4},$$ (II.24)

with the parameter $\nu \in \mathbb{R}$ (note that $U^{1,\nu}$ and $U^{1,-\nu}$ are equivalent). The principal series corresponds to the massive case [11].

ii) The UIR’s $V^{1,q}$ in the complementary series where $p = s = 1$ and $q = \frac{1}{2} + \nu$, corresponds to

$$\langle Q^{(1)}_1 \rangle = \frac{1}{4} - \nu^2, \quad \text{with} \quad 0 < |\nu| < \frac{1}{2}, \quad \text{and} \quad \nu \in \mathbb{R}.$$ (II.25)

iii) The UIR’s $\Pi^{\pm}_{1,1}$ in the discrete series where $q = p = s = 1$ correspond to

$$\langle Q^{(1)}_1 \rangle = 0.$$ (II.26)
By comparing equations (II.21) and (II.22), it is immediately seen that the spin-1 massless field in de Sitter space corresponds to the elements $\Pi_{1,1}^{\pm}$ of the discrete series with the Casimir operator eigenvalue $\langle Q_{1}^{(1)} \rangle = 0$. It is shown in [4] that there are exactly two inequivalent UIR’s of the de Sitter group $SO_{0}(1,4)$ which extend biunivocally to the conformal group $SO_{0}(2,4)$, namely $\Pi_{1,1}^{\pm}$. These two unitary irreducible representations differ in the sign of a helicity-like eigenvalue related to the representation of the subgroup $SO(3)$ which is left unchanged after zero-curvature limit, i.e. the subgroup of space isotropy. It is therefore reasonable to say that both representations are distinguished according to their helicity represented by the symbol $\pm$.

These representations are associated to the subspace of solutions to Eq. (II.21) characterized by $\partial \cdot \mathcal{K} = 0$. Thus, it is natural to use the solution of the equation

$$
(Q_{1}^{(1)} - \langle Q_{1}^{(1)} \rangle)K(x) = 0,
$$

already given in [1] for the massive case. The corresponding vector field solution can be put under the form

$$
\mathcal{K}_\alpha(x) = \mathcal{E}_{1\alpha}(x, \xi)\phi(x) + \frac{1}{\langle Q_{1}^{(1)} \rangle} \mathcal{E}_{2\alpha}(x, \xi)\phi(x),
$$

where $\mathcal{E}_{1\alpha}(x, \xi), \mathcal{E}_{2\alpha}(x, \xi)$ and $\phi(x)$ also contain constant terms involving the parameters $p$ and $q$, but which do not diverge for the specific values $p = q = 1$ corresponding to the massless vector UIR ($\langle Q_{1}^{(1)} \rangle = 0$). Clearly, a singularity appears for the spin 1 massless field due to the term $1/\langle Q_{1}^{(1)} \rangle$. The subspace determined by $\partial \cdot \mathcal{K} = 0$ considered so far is therefore not sufficient for the construction of a quantum massless vector field. One must solve the equation in a larger space which includes the $\partial \cdot \mathcal{K} \neq 0$ types of solutions. As expected, one finds three main types of solutions: the divergencelessness type, the gauge type and the latter solutions which aren’t divergenceless.

### III. THE GUPTA-BLEULER TRIPLET

As stated in [6], “the appearance of [the Gupta-Bleuler] triplet seems to be universal in gauge theories, and crucial for quantization”. The ambient space formalism will allow to exhibit this triplet for the present field in exactly the same manner as it occurs for the electromagnetic field. The Gupta-Bleuler structure of the latter is reminded in Appendix B.

We start with the field equation (II.21). The following dS invariant bilinear form (or inner product) on the space of solutions is defined for two modes (which we also note $\mathcal{K}_{1}, \mathcal{K}_{2}$ whatever their depending on a specific dS coordinate system) in [11] as

$$
(K_{1}, K_{2}) = \frac{i}{H^2} \int_{S^3} [K_{1}^{*} \cdot \partial_{\rho} K_{2} - c((\partial_{\rho} x) \cdot K_{1}^{*})(\partial \cdot K_{2}) - (1^* \equiv 2)] d\Omega,
$$

where we have used the system of bounded global intrinsic coordinates ($X^\mu$, $\mu = 0, 1, 2, 3$) well-suited to describe a compactified version of dS space, namely $S^3 \times S^1$. Let us recall that this coordinate system, known as conformal coordinates, is defined by

$$
\begin{align*}
x^0 &= H^{-1} \tan \rho \\
x^1 &= (H \cos \rho)^{-1} (\sin \alpha \sin \theta \cos \varphi), \\
x^2 &= (H \cos \rho)^{-1} (\sin \alpha \sin \theta \sin \varphi), \\
x^3 &= (H \cos \rho)^{-1} (\sin \alpha \cos \theta), \\
x^4 &= (H \cos \rho)^{-1} (\cos \alpha),
\end{align*}
$$

(III.2)
where $-\pi/2 < \rho < \pi/2$, $0 \leq \alpha \leq \pi$, $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$. For the fields that satisfy the divergencelessness condition, the inner product becomes $c$ independent and KG-like:

$$(K_1, K_2) = \frac{i}{\hbar^2} \int_{S^3} \left[ K_1^* \cdot \partial_\rho K_2 - K_2 \cdot \partial_\rho K_1^* \right] d\Omega.$$ 

Let us now define the Gupta-Bleuler triplet $V_g \subset V \subset V_c$ carrying the undecomposable structure for the unitary representation of the de Sitter group appearing in our problem.

- The space $V_c$ is the space of all square integrable (with respect to $(III.1)$) solutions of the field equation $(II.21)$, including negative norm solutions. It is $c$ dependent so that one can actually adopt an optimal value of $c$ which eliminates logarithmic divergent solutions $[^{10}]$. In the next section, we will show that this value is $c = \frac{2}{3}$, (more generally for a spin $s$ field, $c = (2/(2s + 1))$ $[^7]$).

- It contains a closed subspace $V$ of solutions satisfying the divergencelessness condition. This invariant subspace $V$ is not invariantly complemented in $V_c$. In view of Eq. $(II.21)$, it is obviously $c$ independent.

- The subspace $V_g$ of $V$ consists of the gauge solutions of the form $K_g = D_1 \phi_M$. These are orthogonal to every element in $V$ including themselves. They form an invariant subspace of $V$ but admit no invariant complement in $V$.

The inner product is indefinite in $V_c$, semi-definite in $V$ and is positive definite in the quotient space $V/V_g$. The latter is the physical state space. The de Sitter group acts on the physical (or transverse) space $V/V_g$ through the massless, helicity $\pm 1$ unitary representation $\Pi^+_1 \oplus \Pi^-_{1,1}$. We now characterize the gauge state space $V_g$ and the scalar states belonging to the space $V_c/V$.

### A. The gauge states:

With a solution of the form $K_g = D_1 \phi_M$, equation $(II.21)$ becomes (using $D_1 Q_0 \phi = Q_1 D_1 \phi$)

$$(1 - c)D_1 Q_0 \phi_M = 0. \tag{III.3}$$

At this stage one must distinguish the two cases $c = 1$ and $c \neq 1$.

- If $c = 1$, the scalar field $\phi_M$ is unrestricted, let alone mild differentiability conditions, and the gauge state space is given by vectors of the form $D_1 \phi$ for a differentiable scalar field $\phi$.

- If $c \neq 1$, it is seen that $\phi_M$ corresponds to a massless minimally coupled scalar field characterized by $Q_0 \phi_M = 0$ (possibly up to the addition of a particular solution of the inhomogeneous equation $Q_0 \phi_M = \text{cst}$, and associated with the representation $\Pi_{1,0}$). Moreover, since one has $L_{\alpha\beta} D_1 \phi_M = D_1 M_{\alpha\beta} \phi_M$, this shows that the vector $K_g$ does not carry any spin. Thus it is entirely characterized by its scalar content and can be associated to $\Pi_{1,0}$. Note that the representation structure of the minimally coupled scalar field requires another Gupta-Bleuler type of triplet where the gauge states are the constant fields $[^{14}]$. 


B. The scalar states:

The scalar states belong to the quotient space $V_c/V$. In order to characterize them, let us take the divergence of Eq. (II.21):

$$0 = \bar{\partial} \cdot (Q_1 \mathcal{K}(x) + cD_1 \partial \cdot \mathcal{K}(x)) = Q_0 \bar{\partial} \cdot \mathcal{K}(x) + cH^{-2} \bar{\partial}^2 \mathcal{K}(x),$$  \hspace{1cm} (III.4)

from which one derives

$$(1 - c)Q_0 \bar{\partial} \cdot \mathcal{K}(x) = 0.$$  \hspace{1cm} (III.5)

Again one must distinguish between $c = 1$ and $c \neq 1$.

- If $c = 1$, the vector $\mathcal{K}(x)$ is unrestricted except obvious differentiability conditions. For this special value of $c$ one loses the opportunity of restraining the space $V_c$.

- If $c \neq 1$, the divergence $\bar{\partial} \cdot \mathcal{K}(x)$ again correspond to a massless minimally coupled scalar field associated with the representation $\Pi_{1,0}$.

The representation structure of the full space $V_c$, can be pictured as shown in Figure 2.

The arrows indicate the leaks under the de Sitter group action. The central parts $\Pi_{1,1}^\pm$ are the only spin one unitary irreducible representations of the dS group that admit a minkowskian massless spin 1 interpretation (due to their conformal invariance). A closer look at the scalar and the gauge states solutions reveals a further Gupta-Bleuler triplet described in details in Ref. [14]. Indeed, the scalar and gauge states are associated to the representation $\Pi_{1,0}$ (minimally coupled scalar field) where the space of constant functions assumes the role of gauge space which carries the trivial UIR $\Upsilon_0$ (on which both Casimir operator vanish). The coset spaces $(\text{scalar states})/(\text{constant functions})$ and $(\text{gauge states})/(\text{constant functions})$ are spaces which contain negative norm states and which also carry the UIR $\Pi_{1,0}$.

In the following we present the solutions of Eq. (II.21) and explicitly compute the group actions indicated in the above discussion and illustrated by the figure 2.
IV. DE SITTER VECTOR WAVES, FIELD EQUATION SOLUTION

We now solve the “massless” vector wave equation with gauge fixing term. This equation reads

\[(Q_1 - \langle Q_1 \rangle) K(x) + cD_1 \bar{\partial} \cdot K(x) = 0 \quad \text{with} \quad \langle Q_1 \rangle = 0.\]  

(IV.1)

A general solution can be written in terms of two scalar fields

\[K = \bar{Z} \phi + D_1 \phi_1,\]

(IV.2)

where \(Z\) is a constant vector and \(\bar{Z}_\alpha = \theta_{\alpha\beta} Z^\beta\). The scalar field \(\phi_1\) is defined up to the addition of a scalar field \(\phi_g\). After inserting \(K(x)\) in (IV.1) and with the help of the following relations

\[Q_1 D_1 \phi = D_1 Q_0 \phi, \quad Q_1 \bar{Z}_\alpha \phi = \bar{Z}_\alpha (Q_0 - 2) \phi - 2H^2 D_1 (x \cdot Z) \phi,\]

(IV.3)

one finds from the linear independence of the terms in (IV.2) that

\[(Q_0 - 2) \phi = 0,\]

(IV.4)

\[Q_0 \phi_1 - 2H^2 (x \cdot Z) \phi + c \bar{\partial} \cdot K = 0.\]

(IV.5)

Equation (IV.4) means that the scalar field \(\phi\) obeys

\[(\Box_H + 2H^2) \phi = 0.\]

(IV.6)

It therefore corresponds to the “massless” conformally coupled scalar field [13, 15]. This equation is invariant under conformal transformations, and its solutions are known to be the dS “massless” waves

\[\phi(x) = (Hx \cdot \xi)^\sigma \quad \text{with} \quad \sigma = -1, -2.\]

(IV.7)

These are defined on connected open subsets of \(X_H\) such that \(x \cdot \xi \neq 0\), where \(\xi \in \mathbb{R}^5\) lies on the null cone \(C = \{\xi \in \mathbb{R}^5; \xi^2 = 0\}\). They are homogeneous with degree \(\sigma\) on \(C\) and thus are entirely determined by specifying their values on a well chosen curve (the orbital basis) \(\gamma\) of \(C\). As such, the dS scalar waves are not square integrable. However, physical de Sitter entities like square integrable states can be built as superpositions of such waves (by making \(\xi\) vary in \(C\)). They play in de Sitter space, the role of the plane waves in Minkowski space.

Now since the divergence of (IV.2) reads

\[\bar{\partial} \cdot K = -Q_0 \phi_1 + Z \cdot \bar{\partial} \phi + 4H^2 x \cdot Z \phi,\]

(IV.8)

one obtains from equation (IV.5) that the scalar field \(\phi_1\) satisfies

\[Q_0 \phi_1 = -\frac{c}{1 - c} (H^2 x \cdot Z \phi + Z \cdot \bar{\partial} \phi) + \frac{2 - 3c}{1 - c} Z \cdot \bar{\partial} \phi.\]

(IV.9)

At this stage we could fix the value of \(c\). However it is interesting to consider the general case in order to see in which way a specific value of \(c\) correspond to the simplest case.
A. The general case

Our task is to invert the equation (IV.9) in order to completely determine \( \phi_1 \) in terms of the conformally coupled scalar field \( \phi \). According to Eq. (IV.9), \( \phi_1 \) will be entirely determined by \( \phi \), except for an additional term \( \phi_1 = Q_0^{-1}(0) \). First of all, one can put the equation (IV.9) in the form

\[
\phi_1 = Q_0^{-1}\left( -\frac{c}{1-c}(H^2x \cdot Z\phi + Z \cdot \bar{\partial}\phi) + \frac{2 - 3c}{1-c}H^2x \cdot Z\phi \right) + Q_0^{-1}(0),
\]

\[
= -\frac{c}{2(1-c)}(H^2x \cdot Z\phi + Z \cdot \bar{\partial}\phi) + \frac{2 - 3c}{1-c}H^2Q_0^{-1}x \cdot Z\phi + Q_0^{-1}(0). \tag{IV.10}
\]

This can be verified using the relations

\[
Q_0H^2x \cdot Z\phi = H^2x \cdot Z(Q_0 - 4)\phi - 2Z \cdot \bar{\partial}\phi = -2H^2x \cdot Z\phi - 2Z \cdot \bar{\partial}\phi, \tag{IV.11}
\]

\[
Q_0 Z \cdot \bar{\partial}\phi = Z \cdot \bar{\partial}(Q_0 + 2)\phi + 2H^2x \cdot ZQ_0\phi = 4Z \cdot \bar{\partial}\phi + 4H^2x \cdot Z\phi, \tag{IV.12}
\]

which imply

\[
Q_0\left( H^2x \cdot Z + Z \cdot \bar{\partial}\right) \phi = 2\left( H^2x \cdot Z + Z \cdot \bar{\partial}\right) \phi. \tag{IV.13}
\]

Note also that the term \( Q_0^{-1}(0) \) is a minimally coupled scalar field \( \phi_M \) which satisfies the equation \( Q_0\phi_M = 0 \). Then for the vector mode \( \mathcal{K}(x, \xi, Z) \) the general solution reads

\[
\mathcal{K}(x, \xi, Z) = \bar{Z}\phi - \frac{c}{2(1-c)}D_1\left( H^2x \cdot Z\phi + Z \cdot \bar{\partial}\phi \right) + \frac{2 - 3c}{1-c}H^2D_1Q_0^{-1}x \cdot Z\phi + HD_1\phi_M. \tag{IV.14}
\]

Contrary to the minkowskian QED, the simplest gauge fixing is not the Feynman type of choice \( c = 0 \), which here would yield

\[
\mathcal{K}(x, \xi, Z) = \bar{Z}\phi + 2HD_1Q_0^{-1}Hx \cdot Z\phi + HD_1\phi_M. \tag{IV.15}
\]

Actually, the term \( Q_0^{-1}x \cdot Z\phi \) bears a singularity in the solution \( \phi_1 \), which is responsible for the appearance of a singularity, is removed after choosing \( c = 2/3 \). These solutions will correspond to what we call “minimal case”.

B. The minimal case, \( c = 2/3 \)

For the choice \( c = \frac{2}{3} \), and according to (IV.10), \( \phi_1 \) is determined in terms of \( \phi \) and \( \phi_M \) in the following way

\[
\phi_1 = -Z \cdot \bar{\partial}\phi - H^2x \cdot Z\phi + H\phi_M. \tag{IV.16}
\]

Now starting with formula (IV.14), the “massless” vector wave in this gauge can be written in terms of a generalized polarization vector \( \mathcal{E}_\alpha(x, \xi, Z) \), a “massless” conformally coupled scalar field \( \phi \) and gauge solutions as

\[
\mathcal{K}_\alpha(x, \xi, Z) = [\bar{Z}_\alpha - D_{1\alpha}(Z \cdot \bar{\partial} + H^2x \cdot Z)]\phi(x) + HD_{1\alpha}\phi_M
\]

\[
\equiv \mathcal{E}_\alpha(x, \xi, Z)\phi(x) + HD_{1\alpha}\phi_M, \tag{IV.17}
\]

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with the polarization vector
\[
\mathcal{E}_\alpha(x, \xi, Z) = \left[ -\sigma \tilde{Z}_\alpha - \sigma(\sigma + 1) \frac{Z \cdot x}{x \cdot \xi} \tilde{\xi}_\alpha - \sigma(\sigma - 1) \frac{Z \cdot \xi}{(Hx \cdot \xi)^2} \tilde{\xi}_\alpha \right].
\] (IV.18)

The two possible solutions for \( \mathcal{K}(x, \xi, Z) \) corresponding to \( \sigma = -1, -2 \) are
\[
\mathcal{K}_{1\alpha}(x, \xi, Z) = \mathcal{E}_{1\alpha}(x, Z, \xi) (Hx \cdot \xi)^{-1} + D_1\phi_M,
\]
\[
\mathcal{K}_{2\alpha}(x, \xi, Z) = \mathcal{E}_{2\alpha}(x, Z, \xi) (Hx \cdot \xi)^{-2} + D_1\phi_M,
\]
where
\[
\mathcal{E}_{1\alpha}(x, \xi, Z) = \tilde{Z}_\alpha - 2 \frac{Z \cdot \xi}{(Hx \cdot \xi)^2} \tilde{\xi}_\alpha,
\]
\[
\mathcal{E}_{2\alpha}(x, \xi, Z) = 2 \tilde{Z}_\alpha - 2 \frac{Z \cdot x}{x \cdot \xi} \tilde{\xi}_\alpha - 6 \frac{Z \cdot \xi}{(Hx \cdot \xi)^2} \tilde{\xi}_\alpha.
\]

We are now in position to rewrite the general vector wave solution in the convenient following form
\[
\mathcal{K}^c = \mathcal{K}^{\tilde{4}} + \frac{2}{3} \frac{1 - c}{c} \partial Q_0^{-1} \tilde{\partial} \cdot \mathcal{K}^{\tilde{4}},
\] (IV.19)
where the \( \mathcal{K}^{\tilde{4}} \) is the field solutions for \( c = \frac{2}{3} \). This can be checked using the relations (IV.8), (IV.11) and (IV.12). The term \( Q_0^{-1} \tilde{\partial} \cdot \mathcal{K}^{\tilde{4}} \) is responsible for the singularity. In the next, we shall work essentially with the \( c = 2/3 \) gauge.

C. Group action and physical subspace

In this part we would like to make more complete the description given in Section III of the Gupta-Bleuler structure. More precisely, we characterize the various solutions appearing in the Gupta-Bleuler triplet. The elements of \( V_g \) and \( V_c \) are already known. These are respectively the gauge solutions \( D_1\phi_M \) and the vectors defined by (IV.14). The subspace \( V \) of solutions is characterized by the divergencelessness condition. The divergence of \( \mathcal{K}(x, \xi, Z) \) defined in equation (IV.14) is given by
\[
\tilde{\partial} \cdot \mathcal{K} = \tilde{\partial} \cdot Z\phi + \frac{c}{2(1 - c)} Q_0 \left( H^2 x \cdot Z\phi + Z \cdot \tilde{\partial} \phi \right) - \frac{2 - 3c}{1 - c} H^2 x \cdot Z\phi - H Q_0 \phi_M \tag{IV.20}
\]
\[
= \frac{1}{1 - c} \left[ (2 + \sigma) H^2 Z \cdot x + \sigma \frac{Z \cdot \xi}{x \cdot \xi} \right] \phi(x),
\] (IV.21)
where we have used the relations \( \tilde{\partial} \cdot \tilde{Z} = 4H^2 x \cdot Z \), and \( \tilde{\partial} (Hx \cdot \xi)^\sigma = \sigma H \tilde{\xi} (Hx \cdot \xi)^{\sigma - 1} \) with Eq.(IV.13). As expected, it is seen that the divergencelessness subspace of solutions is not determined by a specific choice of \( c \) (for \( c \neq 1 \)). We therefore can choose to work with the special value \( c = 2/3 \). More important, one sees that the subspace \( V \) will be characterized by the conditions
\[
\sigma = -2 \quad \text{and} \quad Z \cdot \xi = 0.
\] (IV.22)
The value $\sigma = -2$ selects the family $K_{2\alpha}(x, \xi, Z)$ and $Z \cdot \xi = 0$ is a condition on the polarization five-vector $Z$. Therefore the elements of $V$ will be made of superpositions of the following solutions:

$$K_{2\alpha}(x, \xi, Z) = \mathcal{E}_{2\alpha}(x, Z, \xi)(Hx \cdot \xi)^{-2} + HD_{1}\phi_{M} \quad \text{with} \quad \mathcal{E}_{2\alpha}(x, \xi, Z) = 2 \left( \bar{Z}_{\alpha} - \frac{Z \cdot x}{x \cdot \xi} \bar{\xi}_{\alpha} \right).$$

We have indicated that the Gupta-Bleuler triplet is based on three invariant spaces of solutions. Let us first show that each one of these spaces is effectively invariant. It is trivial that the elements of $V_{c}$ for a given $c$ remain in $V_{c}$ under the group action. Moreover, the gauge solutions also form a closed subspace since $L_{\alpha\beta}D_{1}\phi_{M} = D_{1}M_{\alpha\beta}\phi_{M}$. In order to show that $V$ is invariant under the group action, let us consider the infinitesimal group action. The infinitesimal group action reads

$$L_{\alpha\beta}^{(1)}K_{\gamma} = L_{\alpha\beta}^{(1)}(Z\phi + D_{1}\phi_{1}) = \bar{Z}M_{\alpha\beta}\phi - i(Z_{\beta}\theta_{\alpha\gamma} - Z_{\alpha}\theta_{\beta\gamma})\phi + D_{1}M_{\alpha\beta}\phi_{1}, \quad (IV.23)$$

and, with

$$M_{\alpha\beta}(Z \cdot x)\phi = (Z \cdot x)M_{\alpha\beta}\phi - i(Z_{\beta}x_{\alpha} - Z_{\alpha}x_{\beta})\phi,$$

$$M_{\alpha\beta}Z \cdot \bar{\partial}\phi = Z \cdot \bar{\partial}M_{\alpha\beta}\phi - i(Z_{\beta}\bar{\partial}_{\alpha} - Z_{\alpha}\bar{\partial}_{\beta})\phi, \quad (IV.24)$$

one obtains

$$L_{\alpha\beta}^{(1)}K_{\gamma} = \bar{Z}_{\gamma}M_{\alpha\beta}\phi - D_{1}(Z \cdot \bar{\partial} + H^{2}x \cdot Z)M_{\alpha\beta}\phi + HD_{1}M_{\alpha\beta}\phi_{M}$$

$$- i(x_{\beta}Z_{\alpha} - x_{\alpha}Z_{\beta})\bar{\partial}_{\gamma}\phi - iH^{-2}(Z_{\alpha}\bar{\partial}_{\gamma}\bar{\partial}_{\beta} - Z_{\beta}\bar{\partial}_{\gamma}\bar{\partial}_{\alpha})\phi. \quad (IV.25)$$

It is possible to make explicit the divergence of the latter equation

$$\bar{\partial} L_{\alpha\beta}^{(1)}K_{\gamma} = -3iH^{2}(\sigma + 2)(x_{\beta}Z_{\alpha} - x_{\alpha}Z_{\beta})(Hx \cdot \xi)^{\sigma}$$

$$- 3iH^{2}\sigma(\sigma + 1)(x \cdot Z)(x_{\beta}\xi_{\alpha} - x_{\alpha}\xi_{\beta})(Hx \cdot \xi)^{\sigma-1}$$

$$+ 3iH^{2}\sigma(\sigma + 1)(\xi \cdot Z)(x_{\beta}\xi_{\alpha} - x_{\alpha}\xi_{\beta})(Hx \cdot \xi)^{\sigma-2}, \quad (IV.26)$$

which shows that the subspace of divergencelessness solutions with $\sigma = -2$ and $Z \cdot \xi$ is invariant. Because of divergencelessness and transversality $x \cdot \mathcal{E}_{2\alpha}(x, \xi, Z) = 0$, one finds that the independent components are reduced from the original five to three. The actual physical subspace of solutions, that is the quotient space $V/V_{g}$, corresponds to the two-component polarization vectors $\mathcal{E}_{2\alpha}^{\lambda}(x, \xi, Z), \lambda = 1, 2$. The latter satisfy a kind of transversality condition

$$\bar{\xi} \cdot \mathcal{E}_{2\alpha}^{\lambda}(x, \xi, Z) = \xi \cdot \mathcal{E}_{2\alpha}^{\lambda}(x, \xi, Z) = 0. \quad (IV.27)$$

Note that transversality relations (IV.27) are valid only for the physical states in $V/V_{g}$. They no longer hold for states belonging to $V$: by adding the gauge solutions $D_{1}\phi_{M}$ with for instance $\phi_{M} = (Hx \cdot \xi)^{-3} (\sigma = -3$ corresponds to a minimally coupled scalar field) one obtains

$$\bar{\xi} \cdot D_{1}\phi_{M} = \bar{\xi} \cdot \xi H^{-1}(Hx \cdot \xi)^{-4} = H^{-1}(Hx \cdot \xi)^{-2} \neq 0 \quad \text{since} \quad \bar{\xi} \cdot \xi = (Hx \cdot \xi)^{2}. \quad (IV.28)$$
D. General comments concerning the dS vector waves

An important difference with the minkowskian case is that the polarization vectors \( E_i(x, \xi, Z) \) for \( i = 1, 2 \) are functions of the space-time variable \( x \). Moreover, unlike the minkowskian and the dS “massive” vector cases, these two solutions, in our notations, are not complex conjugate of each other. Note that they satisfy the homogeneity properties

\[
E_i(x, a\xi, Z) = E_i(x, \xi, Z) \quad \text{and} \quad E_i(ax, \xi, Z) = E_i(x, \xi, Z),
\]

and thus the dS waves \( K_\alpha(x, \xi, Z) \) are homogeneous with degree \( \sigma \) as functions of \( \xi \) on the null cone \( \mathcal{C} \) as well as on the dS submanifold \( X_H \). Also note that as functions on \( \mathbb{R}^5 \), these waves are homogeneous with degree zero since in that case \( H(x) = -1/\sqrt{-x \cdot x} \).

The arbitrariness introduced with the constant vector \( Z \) will be removed by comparison with the minkowskian case. Unfortunately, our notations for the “massless” conformally coupled scalar waves are not adapted to the computation of the \( H = 0 \) limit. It is due to the fact that contrary to the “massive case” the values \( \sigma = -1, -2 \) are constant \[19\]. In order to get a hint of the behavior of the field equation solutions in the limit \( H = 0 \) (at least the scalar part), one can use the conformal coordinate system which has been introduced in Eq. (III.2). The square-integrable solutions of the field equation (IV.6) are then given by \[20\]:

\[
\phi(x) = \phi(\rho, \Omega) = \cos \rho \frac{e^{\pm i(L+1)\rho}}{\sqrt{L+1}} y_{L\ell m}(\Omega), \quad \text{(IV.28)}
\]

where \( y_{L\ell m}(\Omega) \) are the hyperspherical harmonics on \( S^3 \). It can be shown that in the \( H = 0 \) limit and with

\[
\rho = Ht, \quad \alpha = Hr; \quad HL = k_0 = |\vec{k}|, \quad \text{with} \quad \theta, \varphi \text{ unchanged}, \quad \text{(IV.29)}
\]

the functions (IV.28) become, when suitably rescaled, the usual massless spherical waves (with \( k^2 = (k_0)^2 - (\vec{k})^2 = 0 \)) \[21\].

With these coordinates, the dS vector square integrable solutions formally read

\[
K_\mu(\rho, \Omega) = \frac{\partial x^\alpha}{\partial X^\mu} \left[ \bar{Z}_\alpha - D_1\alpha \left( Z \cdot \vec{\partial} + H^2 x \cdot Z \right) \right] \cos \rho \frac{e^{\pm i(L+1)\rho}}{\sqrt{L+1}} y_{L\ell m}(\Omega). \quad \text{(IV.30)}
\]

We now discuss the limit \( H = 0 \) in order to fix the constant vector \( Z \). The condition is to recover the minkowskian four polarization vectors \( e_\mu(k) \) with \( (\lambda = 0, 1, 2, 3) \) (we actually drop the usual parentheses for \( \lambda \) in order to manage handy expressions). These minkowskian polarization vectors of course satisfy the usual gauge dependent orthogonality relations \[22\]. In general, the constant 5-vectors \( (Z_\alpha) \) to be selected will be labelled by \( \lambda = 0, 1, 2, 3 \) and written \( Z^\lambda \). This corresponds to the fact that although expressed as vectors with five components, the objects in dS space only have four independent components.

Let us at first consider the solutions in the Feynman gauge \( c = 0 \). Recall that, up to gauge states, the field solutions in that case read as

\[
K(x, \xi, Z) = \bar{Z} \phi + 2 \bar{\partial} Q_0^{-1} x \cdot Z \phi. \quad \text{(IV.31)}
\]
A simple and appropriate choice of $Z$ is then given by

$$Z_\lambda^\alpha = (\epsilon_\mu^\lambda(k), Z_4^\lambda = 0), \quad \text{(IV.32)}$$

where $\epsilon_\mu^\lambda(k)$ are the minkowskian polarizations which satisfy the usual relations [22]:

$$\epsilon^0 = n, \quad \epsilon^3 \cdot n = 0, \quad \epsilon^3 \cdot \epsilon^3 = -1, \quad n \cdot n = 1, \quad n^0 > 0, \quad \text{(IV.33)}$$

$$\epsilon^\lambda \cdot \epsilon^{\lambda'} = -\delta_{\lambda\lambda'}, \quad \epsilon^\lambda \cdot n = \epsilon^\lambda \cdot k = 0, \quad \lambda, \lambda' = 1, 2, \quad \text{(IV.34)}$$

$$\eta_{\lambda\lambda'} \epsilon_\mu^\lambda(k) \epsilon^{*\lambda'}_{\nu}(k) = \eta_{\mu\nu}, \quad \epsilon^\lambda(k) \cdot \epsilon^{*\lambda'}(k) = \eta^{\lambda\lambda'}. \quad \text{(IV.35)}$$

This is because qualitatively one can see using the conformal coordinates and (IV.29), that, in the limit $H = 0$, the leading term in $x \cdot Z\phi$ depends only upon $Z_4$. Moreover, $Q_0$ and $Q_0^{-1}$ do not modify the $H$ dependence of the functions which they act upon. For instance

$$Q_0 x_\alpha = -4 x_\alpha.$$  

Hence by setting $Z_4$ to zero, one gets rid of the logarithmic part of (IV.15) in the flat limit. Thus we are left with the polarization vector $\bar{Z}_\alpha$ which satisfies

$$\lim_{H \to 0} \bar{Z}_\alpha = Z_\mu = \epsilon_\mu^\lambda(k) \quad \text{where} \quad \alpha = 0, 1, 2, 3, 4 \quad \text{and} \quad \mu = 0, 1, 2, 3.$$  

However, for other choices of gauge ($c \neq 0$), the simplest one to work with is not given by (IV.32). In fact, similarly to the massive spin 2 case [29], we will impose (at least here for the $c = 2/3$ gauge)

$$Z^\lambda \cdot Z^{\lambda'} = \eta^{\lambda\lambda'}, \quad \eta_{\lambda\lambda'} Z_\alpha^\lambda Z^{\lambda}_{\beta} = \eta_{\alpha\beta} \quad \forall \lambda, \lambda' = 0, 1, 2, 3, \quad \text{(IV.36)}$$

so that the vector two-point function assume a maximally symmetric bi-tensor form of in ambient space notation. This choice presents the great advantag e to be covariantly defined for all components $\alpha$. Its form is not directly dictated by the flat limit behavior. Rather, we will see in the following that this choice yields correct expressions on the level of the two-point function. Indeed, it is easy to show that the physical polarization vectors

$$\mathcal{E}_{2\alpha}(x, \xi) \overset{\text{def}}{=} \mathcal{E}_{2\alpha}(x, \xi, Z^\lambda) = 2 \left( \bar{Z}_\alpha^\lambda - \frac{Z^\lambda \cdot x}{x \cdot \xi} \xi_\alpha \right) \quad \text{with} \quad \xi \cdot \mathcal{E}_{2\alpha}(x, \xi) = \xi \cdot Z^\lambda = 0, \quad \text{(IV.37)}$$

satisfy

$$\eta_{\lambda\lambda'} \mathcal{E}_{2\alpha}(x, \xi) \mathcal{E}_{2\beta}^{\lambda'}(x, \xi) = 4 \left( \theta_{\alpha\beta} - \frac{\xi_\alpha \xi_\beta}{(Hx \cdot \xi)^2} \right), \quad \text{(IV.38)}$$

and

$$\mathcal{E}_{2}^{\lambda}(x, \xi) \cdot \mathcal{E}_{2}^{\lambda'}(x, \xi) = 4 Z^\lambda \cdot Z^{\lambda'} = \eta^{\lambda\lambda'}. \quad \text{(IV.39)}$$

Formally, it is possible to compute the $H = 0$ limit of the latter expressions. Unfortunately, this particular form (IV.37) is related to the use of the scalar waves $(Hx \cdot \xi)^\sigma$ with $\sigma = -1, -2$, and we already pointed out the impossibility to get at the $H = 0$ limit nontrivial massless minkowskian entities. Note that this isn’t true in the massive case where $\sigma$ can be made $H$
dependent. In that case, one can adopt for $\xi$ a suitable parametrization when one has in view the link with massive Poincaré UIR’s: it is given by the orbital basis

$$\{\xi \in C^+, \xi^{(d)} = 1\} \cup \{\xi \in C^+, \xi^{(d)} = -1\},$$

where $\xi$ is defined in terms of the four-momentum $(k^0, \vec{k})$ of a minkowskian particle (for details see [2, 29])

$$\xi_{\pm} = \left(\frac{k^0}{mc} = \sqrt{\frac{\vec{k}^2}{m^2c^2} + 1}, \frac{\vec{k}}{mc}, \pm 1\right).$$ (IV.40)

To summarize, one could say that in the ambient space formalism the massless polarization vectors in dS space look very similar to their minkowskian counterparts. Unfortunately, the corresponding $H = 0$ limit yields expressions which aren’t easy to interpret from a minkowskian point of view.

Let us end this part with a remark concerning the gauge states. We have said that any solution to Eq. (IV.1) is defined up to an arbitrary gauge field $D_1\phi_M$. As a matter of fact we already had a gauge solution in the expression $K_2^\alpha(x)$ through $E_2(x, Z, \xi)$ as

$$K_{2\alpha}(x, \xi, Z) = E_1^\alpha(x, \xi, Z)(Hx \cdot \xi)^{-1}, \quad K_{2\alpha}(x, \xi, Z) = E_2^\alpha(x, \xi, Z)(Hx \cdot \xi)^{-2},$$ (IV.41)

where

$$E_{1\alpha}(x, \xi, Z) = \bar{Z}_\alpha - 2\frac{Z \cdot \xi}{(Hx \cdot \xi)^2} \bar{\xi}_\alpha,$$

$$E_{2\alpha}(x, \xi, Z) = 2\bar{Z}_\alpha - 2\frac{Z \cdot x}{x \cdot \xi} \bar{\xi}_\alpha - 6\frac{Z \cdot \xi}{(Hx \cdot \xi)^2} \bar{\xi}_\alpha.$$

E. Analytic vector waves

The vectors $K(x)$ given by formula (IV.14) are not globally defined since the waves $(Hx \cdot \xi)^{\sigma}$ are singular on three dimensional light-like manifolds [13]. For a global definition, they have to be viewed as distributions [22]. More precisely, we will consider the boundary values of analytic continuations of the solutions $K(x)$ to tubular domains in the complexified de Sitter space $X^{(c)}_H$. The complexified dS space is defined by:

$$X^{(c)}_H = \{z = x + iy \in \mathbb{C}^5; \eta_{\alpha\beta}z^\alpha z^\beta = (z^0)^2 - \vec{z} \cdot \vec{z} - (z^4)^2 = -H^{-2}\} = \{(x, y) \in \mathbb{R}^5 \times \mathbb{R}^5; x^2 - y^2 = -H^{-2}, x \cdot y = 0\}.$$

For a generic $\sigma$ and an univalued determination of the expression $(z \cdot \xi)^{\sigma}$ we adopt the principal determination in

$$(z \cdot \xi)^{\sigma} = \exp(\sigma [\log |z \cdot \xi| + i\text{arg}(z \cdot \xi)]) \quad \text{with} \quad \text{arg}(z \cdot \xi) \in ] - \pi, \pi[,$$
and characterize \( z \) so that we fix the sign of the imaginary part of \((z \cdot \xi)\). Let us introduce the forward and backward tubes of \( X^{(c)} \). First of all, let \( T^\pm = \mathbb{R}^5 - iV^\pm \) be the forward and backward tubes in \( \mathbb{C}^5 \). The domain \( V^+ \) (resp. \( V^- \)) stems from the causal structure on \( X^\_\!^\!H \):

\[
V^\pm = \{ x \in \mathbb{R}^5; \; x^0 > \sqrt{\| \vec{x} \|^2 + (x^4)^2} \}. \tag{IV.42}
\]

We then introduce their respective intersections with \( X^{(c)}_H \),

\[
\mathcal{T}^\pm = T^\pm \cap X^{(c)}_H, \tag{IV.43}
\]

which are the tubes of \( X^{(c)}_H \). Finally we define the “tuboid” above \( X^{(c)}_H \times X^{(c)}_H \) by

\[
\mathcal{T}_{12} = \{(z, z'); \; z \in \mathcal{T}^+, z' \in \mathcal{T}^-\}. \tag{IV.44}
\]

Details are given in [13]. When \( z \) varies in \( \mathcal{T}^+ \) (or \( \mathcal{T}^- \)) and \( \xi \) lies in the positive cone \( C^+ \) the wave solutions are globally defined because the imaginary part of \((z \cdot \xi)\) has a fixed sign and \( z \cdot \xi \neq 0 \).

Now, if \( \sigma = -1, -2 \), the wave solutions are univalued, of course, but still singular, and an analytical continuation is still needed in order to view them as well-defined objects. Therefore, we define the de Sitter tensor wave \( K_\alpha(x) \) as the boundary value of the analytic continuation to the future tube of Eq. (IV.14). Hence, for \( z \in \mathcal{T}^+ \) and \( \xi \in C^+ \) one gets the two solutions

\[
K_{1\alpha}(z) = \mathcal{E}_{1\alpha}(z, \xi) (Hz \cdot \xi)^{-1}, \quad \text{and} \quad K^*_{2\alpha}(z^*) = \mathcal{E}^*_{2\alpha}(z^*, \xi) (Hz \cdot \xi)^{-2}. \tag{IV.45}
\]

The corresponding boundary values are

\[
\text{bv } K_{1\alpha}(z) \equiv K_{1\alpha}(x) = \mathcal{E}_{1\alpha}(x, \xi) (H(x + i\epsilon) \cdot \xi)^{-1}, \\
\text{bv } K_{2\alpha}(z) \equiv K_{2\alpha}(x) = \mathcal{E}_{2\alpha}(x, \xi) (H(x + i\epsilon) \cdot \xi)^{-2}, \tag{IV.46}
\]

with \( \epsilon \in V^+ \) arbitrarily small.

V. THE TWO-POINT FUNCTION

In a previous work concerning the massive vector case [1], we have constructed the field theory from the Wightman two-point function. The two-point function had to satisfy the conditions of a) positivity, b) locality, c) covariance, d) normal analyticity, e) transversality and d) divergencelessness in order to properly encode the theory of free fields on dS space. In the “massless” case, the divergencelessness condition cannot be maintained if one wishes to preserve the covariance condition. Consequently the field equation is solved in a larger \( c \)-dependent space endowed with an indefinite inner product. On the level of the two-point function this forces to abandon the positivity. The corresponding two-point function cannot be covariantly separated into a positive (physical) and a negative part.

Given the solutions (IV.41) one defines the analytic two-point function explicitly in terms of the following class of integral representations

\[
W_{\alpha\alpha'}(z, z') = a_0 \gamma \int (H z \cdot \xi)^{-1}(H z' \cdot \xi')^{-2} \eta_{\lambda\lambda'} \mathcal{E}_{1\alpha}(z, \xi) \mathcal{E}^*_{2\alpha'}(z', \xi) d\sigma_\gamma(\xi), \tag{V.1}
\]
where $d\sigma_\gamma(\xi)$ is the natural $C^+$ invariant measure on $\gamma$, induced from the $\mathbb{R}^5$ Lebesgue measure [13] and the normalization constant $a_0$ is fixed by local Hadamard condition. The tensor

$$\eta_{\lambda\mu}\mathcal{E}_{1\alpha}^\lambda(z, \xi)\mathcal{E}_{2\alpha'}^\mu(z', \xi) \equiv T_{\alpha\alpha'}(z, z', \xi) \quad (V.2)$$

is a covariant homogeneous bi-vector, in ambient space notation, of degree 0 in the variables $z, z'$ and $\xi$. Of course, this choice is not unique but it is motivated by several facts. First, in the massive vector case, we have also constructed the two-point function with two vectors based on the product $(Hz' \cdot \xi)^\alpha(Hz \cdot \xi)^{-\sigma -3}$ (see Reference [1]). Moreover, in the conformally scalar case this type of two-point function coincides with the expression found for that field in [2] as we will show below. Finally, given the set of modes $K_1, K_2$, we will actually show that the only simple product yielding a causal two-point function is the one presented in (V.1). The boundary value of (V.1) defines the two-point function in terms of global plane waves on $X_H$.

The analytic two-point function (V.1) can be expressed in terms of an analytic scalar two-point function without resorting to any explicit calculation of the integral. Actually, following Allen and Jacobson in Reference [12], we will write the two-point functions in de Sitter space in terms of bivectors. These are functions of two points $(x, x')$ which behave like vectors under coordinate transformations at each point. The bivectors are called maximally symmetric if they respect the de Sitter invariance. As shown in [29] and also proved in Appendix C, any maximally symmetric bivector can be expressed in ambient space notations as a sum of the two basic bivectors $\theta_\alpha \cdot \theta^\alpha$ and $\bar{\partial}_\alpha \bar{\partial}^\alpha$. Thus we can write

$$W_{\alpha\alpha'}(z, z') = \theta_\alpha \cdot \theta^\alpha W_0(z, z') + H^{-2} \bar{\partial}_\alpha \bar{\partial}^\alpha W_1(z, z'). \quad (V.3)$$

Of course, it will be verified that this expression coincides with the two-point function constructed from the modes (IV.41) in formula (V.1) (in the case $c = 2/3$). By imposing the bivector (V.3) to obey Eq. (IV.1) in variable $z$ or $z'$, one finds from (IV.3) the following relations

$$(Q_0 - 2) W_0(z, z') = 0, \quad Q_0 \bar{\partial} W_1(z, z') = 2H^2 z \cdot \theta' W_0(z, z') - c \bar{\partial} \cdot W(z, z'). \quad (V.4)$$

Let us first examine $W_0(z, z')$. This analytic two-point function corresponds to the massless conformally coupled scalar field associated with the complementary series of unitary representations [13, 15]. The Wightman scalar two-point function $W_0(x, x')$ in that case is given by [13]

$$W_0(x, x') = \text{bv } W_0(z, z') \quad \text{with} \quad W_0(z, z') = c_0 \int_\gamma (Hz \cdot \xi)^{-1} (Hz' \cdot \xi)^{-2} d\sigma_\gamma(\xi). \quad (V.5)$$

The normalization constant $c_0$ is determined by imposing the Hadamard condition on the two-point function. This has been done in Ref. [13] where the scalar two-point function has been rewritten in terms of the generalized Legendre function for well-chosen points $z, z' \in T_{12}$ in the domain defined by $(z - z')^2 < 0$. For instance $z = (-iH^{-1} \cosh \varphi, -iH^{-1} \sinh \varphi, 0, 0, 0)$ and $z' = (iH^{-1}, 0, 0, 0, 0)$ with $z \cdot z' = H^{-2} \cosh \varphi$. It has been established that

$$W_0(z, z') = C_0 P_{-1}^{(5)}(-Z) = \frac{-H^2}{8\pi^2} \frac{1}{1 - Z(z, z')}, \quad (V.6)$$

with $Z = -H^2 z \cdot z'$, $C_0 = -2i\pi^2 c_0$ and

$$c_0 = \frac{iH^2}{2^5\pi^4}. \quad (V.7)$$
In terms of the scalar two-point function, the vector two-point function can be written in the following way

\[ W_0(x, x') = \frac{-H^2}{8\pi^2} \left[ P \frac{1}{1 - Z(x, x')} - i\pi\epsilon(x^0 - x'^0)\delta(1 - Z(x, x')) \right]. \quad (V.8) \]

where \( P \) denotes the principal part and \( \epsilon(x^0 - x'^0) = 1, 0, -1 \) whether one has \( (x^0 - x'^0) >, =, \) or \( < 0 \) respectively. This is exactly the two-point function of the conformally coupled scalar field given in \([24]\).

We now consider the second relation in \((V.3)\). Since the divergence of \( W(z, z') \) reads as

\[ \bar{\partial} \cdot W(z, z') = 4H^2z \cdot \theta'W_0(z, z') + \theta' \cdot \bar{\partial}W_0(z, z') - Q_0 \bar{\partial}'W_1(z, z'), \]

one easily rewrite the second relation in \((V.3)\) as

\[ Q_0 \bar{\partial}'W_1(z, z') = -\frac{c}{1 - c} \left[ \theta' \cdot \bar{\partial} + H^2z \cdot \theta' \right] W_0(z, z') + \frac{2 - 3c}{1 - c} H^2z \cdot \theta'W_0(z, z'). \quad (V.9) \]

A. The minimal case, \( c = 2/3 \)

Again, we first consider the simple case \( c = \frac{2}{3} \). The above equation simplifies to

\[ Q_0 \bar{\partial}'W_1(z, z') = -2 \left[ \theta' \cdot \bar{\partial} + H^2z \cdot \theta' \right] W_0(z, z'), \quad (V.10) \]

which is satisfied if

\[ \bar{\partial}'W_1(z, z') = - \left[ \theta' \cdot \bar{\partial} + H^2z \cdot \theta' \right] W_0(z, z'). \quad (V.11) \]

Thus, we can write the analytic two-point function in the form

\[ W_{\alpha\alpha'}(z, z') = D_{\alpha\alpha'}W_0(z, z'), \quad (V.12) \]

where

\[ D_{\alpha\alpha'} = \theta_\alpha \cdot \theta'_\alpha' - H^{-2}\bar{\partial}_\alpha \left[ \theta'_{\alpha'} \cdot \bar{\partial} + H^2z \cdot \theta'_{\alpha'} \right]. \quad (V.13) \]

The vector two-point function can be developed to

\[ W_{\alpha\alpha'}(z, z') = -H^{-2}\bar{\partial}_\alpha \theta'_{\alpha'} \cdot \bar{\partial}W_0(z, z') - x \cdot \theta'_{\alpha'}\bar{\partial}_\alpha W_0(z, z'), \quad (V.14) \]

and by simple derivation one finally obtains

\[ W_{\alpha\alpha'}(z, z') = c_0 \int_{\gamma} \left( \theta_{\alpha} \cdot \theta'_{\alpha'} - \frac{2\xi_{\alpha} \xi'_{\alpha'}}{(Hz \cdot \xi)^2} \right) (Hz \cdot \xi)^{-1}(Hz' \cdot \xi)^{-2}d\sigma_\gamma(\xi). \quad (V.15) \]

In terms of the scalar two-point function, the vector two-point function can be written in the following way

\[ W_{\alpha\alpha'}(z, z') = -\theta'_{\alpha'} \cdot \theta_\alpha \frac{d}{dZ} W_0(z, z') + H^2(\theta' \cdot z)(\theta \cdot z') \left( 2\frac{d}{dZ} + 2\frac{d^2}{dZ^2} \right) W_0(z, z'), \]
where we have used

\[ \partial_\alpha W_0(z, z') = -H^2 z' \cdot \theta_\alpha \frac{d}{dZ} W_0(z, z'). \] (V.16)

It is explicitly shown in Appendix D that this two-point function agrees with Eq. (V.1) when the constant vectors \( Z^\lambda \) satisfy \( \eta_{\lambda \lambda'} Z_\alpha^\lambda Z_{\alpha'}^\lambda = \eta_{\alpha \beta} \). Taking a closer look at the two-point function with a \( T_{\alpha \alpha'} \) introduced in Eq. (V.2) given by

\[ T_{\alpha \alpha'}(z, z', \xi) = \theta_\alpha \cdot \theta_{\alpha'} - \frac{2 \bar{\xi}_\alpha \xi_{\alpha'}}{(Hz \cdot \xi)^2}, \] (V.17)

one notices that it is analogous to the minkowskian gauge-dependent polarization sum which in general reads:

\[ T_{\mu \nu}(k) = \eta_{\mu \nu} - c_1 \frac{k_\mu k_\nu}{c^2}, \] (V.18)

with \( c/(1 - c) = 2 \) for \( c = 2/3 \).

Now, taking the boundary value limit, it can be proved, by using the same methods as in [1], that the two-point function \( W_{\alpha \alpha'}(x, x') = W_{\alpha \alpha'}(z, z')_{bv} \) satisfies the following conditions:

a) **Indefinite sesquilinear form**

For any test function \( f_\alpha \in D(X_H) \), we have an indefinite sesquilinear form that is defined by

\[ \int_{X_H \times X_H} f^{* \alpha}(x) W_{\alpha \alpha'}(x, x') f^{\alpha'}(x') d\sigma(x) d\sigma(x'), \] (V.19)

where \( f^* \) is the complex conjugate of \( f \) and \( d\sigma(x) \) denotes the dS-invariant measure on \( X_H \). \( D(X_H) \) is the space of \( C^\infty \) functions with compact support in \( X_H \).

b) **Covariance**

The two-point function satisfies the covariance property

\[ g^{-1} W(gx, gx') g = W(x, x'). \] (V.20)

where \( g \in SO_0(1,4) \).

Indeed, let us first write the group action on the dS modes. From Eq. (V.17), we recall that the latter are given by

\[ K_\alpha(x, \xi, Z) = E_\alpha(x, \xi, Z) \phi(x) + HD_1 \phi_M, \] (V.21)

with

\[ E_\alpha(x, \xi, Z) = \left[ -\sigma \bar{Z}_\alpha - \sigma(\sigma + 1) \frac{Z \cdot x}{x \cdot \xi} \bar{\xi}_\alpha - \sigma(\sigma - 1) \frac{Z \cdot \xi}{(H x \cdot \xi)^2} \bar{\xi}_\alpha \right]. \] (V.22)

Now, one easily shows that (also recall \( \phi_M = (Hx \cdot \xi)^\nu \) with \( \nu = 0, -3 \))

\[ E_\alpha(g^{-1} x, \xi, Z) = (g^{-1})_\alpha^\delta E_\delta(x, g\xi, gZ) \quad \text{and} \quad D_{1\alpha} (H g^{-1} x \cdot \xi)^\nu = (g^{-1})_\alpha^\delta D_{1\delta} (H x \cdot g\xi)^\nu. \] (V.23)

Therefore the group action on the dS modes reads:

\[ (U(g) K)_\alpha(x, \xi, Z) = g_\alpha^\gamma K_\gamma(g^{-1} x, \xi, Z) = K_\alpha(x, g\xi, gZ). \] (V.24)

The simplicity of the group action again shows the efficiency of the ambient space formalism. Finally since the integral (V.1) is independent of a specific choice of \( \xi \) (orbital basis) or \( Z \) this proves the covariance property.
c) **Locality**

For every space-like separated pair \((x, x')\), i.e. \(x \cdot x' > -H^{-2}\),

\[
W_{\alpha \alpha'}(x, x') = W_{\alpha' \alpha}(x', x).
\]  

(V.25)

In order to prove this locality condition, we use the identity

\[
W^*_{\alpha \alpha'}(z^*, z'^*) = W_{\alpha \alpha'}(z', z)
\]

easily checked using (V.1) and the following relation

\[
W_{\alpha \alpha'}(z, z') = W^*_{\alpha' \alpha'}(z'^*, z^*)
\]

(V.26)

The latter is valid for space-like separated points \(z, z'\) since it is based on the fact that, for space-like separated points \(z, z'\), one can rewrite the two-point function as:

\[
W_{\alpha \alpha'}(z, z') = D_{\alpha \alpha'}(z, z') P_{\nu}^\mu(\mathcal{Z}) \quad \text{with} \quad D^*(z'^*, z^*) = D(z', z).
\]

Then we use the relations

\[
P_{-2}^\mu(-\mathcal{Z}) = -4i (\sinh \varphi)^{-1} \Psi^\mu_{-1}(\cosh \varphi) = -4i (\sinh \varphi)^{-1} \Psi^\mu_{0}(\cosh \varphi) = P_{-1}^\mu(-\mathcal{Z}),
\]

which follow from the Legendre function property \(\Psi^\mu_\nu(x) = \Psi^{-\mu}_{-\nu}(x)\). One finally gets

\[
W_{\alpha \alpha'}(z, z') = W^*_{\alpha' \alpha'}(z'^*, z^*) = W_{\alpha' \alpha}(z', z).
\]

Finally, the space-like separated pair \((x, x')\) lies in the same orbit of the complex dS group as the pairs \((z, z')\) and \((z'^*, z^*)\). Therefore the locality condition \(W_{\alpha \alpha'}(x, x') = W_{\alpha' \alpha}(x', x)\) holds for the space-like separated points \(x, x'\).


d) **Normal analyticity**

\(W_{\alpha \alpha'}(x, x')\) is the boundary value (in the sense of distributions) of an analytic function \(W_{\alpha \alpha'}(z, z')\). The analyticity properties of the tensor Wightman two-point function in the tuboid \(\mathcal{T}_{12} = \{(z, z'); \ z \in \mathcal{T}^+, z' \in \mathcal{T}^-\}\) follow from the analyticity properties of the dS tensor waves (IV.45).

e) **Transversality**

\[
x \cdot \mathcal{W}(x, x') = 0 = x' \cdot \mathcal{W}(x, x'),
\]

(V.27)

The transversality with respect to \(x\) and \(x'\) is guaranteed since the dS modes \(\mathcal{K}(x)\) are transverse by construction.

**B. The general case \(c \neq 2/3\)**

Let us briefly consider the case \(c \neq 2/3\). We can write the equation (V.9) in the form

\[
\bar{\theta}' W_1(z, z') = -\frac{c}{2(1-c)} [\theta' \cdot \bar{\theta} + H^2 z \cdot \theta'] W_0(z, z') + \frac{2 - 3c}{1 - c} H^2 Q_{0}^{-1} z \cdot \theta' W_0(z, z'),
\]

(V.28)
and the vector two-point function becomes

$$W_{\alpha \alpha'}(z, z') = \theta_{\alpha} \cdot \theta_{\alpha'} W_0(z, z') - \frac{c}{2(1 - c)} H^{-2} \bar{\partial}_{\alpha} \left[ \theta_{\alpha'} \cdot \bar{\partial} + H^2 z \cdot \theta_{\alpha'} \right] W_0(z, z')$$

$$+ \frac{2 - 3c}{1 - c} \bar{\partial}_{\alpha} Q_0^{-1} z \cdot \theta_{\alpha'} W_0(z, z').$$

(V.29)

In order to distinguish the specific value $c = 2/3$, a convenient form of the above expression is

$$W_{\alpha \alpha'}^{c} = W_{\alpha \alpha'}^\# + \frac{\frac{2}{3} - c}{H^2(1 - c)} \bar{\partial}_{\alpha} Q_0^{-1} \bar{\partial} \cdot W_{\alpha \alpha'}^{\#},$$

(V.30)

where the $W_{\alpha \alpha'}^{\#}$ is the two-point function corresponding to $c = 2/3$. The singularity appears in the term $\bar{\partial} Q_0^{-1} \bar{\partial} \cdot W_{\alpha \alpha'}^{\#}$ of the above equation.

VI. THE QUANTUM FIELD

Let us now write the field corresponding to our two-point function. For any test function $f_\alpha \in \mathcal{D}(X_H)$, we define the vector-valued distributions taking values in the space generated by the modes $K_\alpha(x, \xi) \equiv \text{by} K_\alpha(z, \xi)$ by:

$$x \to p_{1\alpha}(f)(x) = \sum_{\lambda=0}^{3} \zeta_\lambda \int_\gamma d\sigma_\gamma(\xi) K^\lambda_{2\xi}(f) K^\lambda_{1\alpha}(x, \xi),$$

(VI.1)

and

$$x \to p_{2\alpha}(f)(x) = \sum_{\lambda=0}^{3} \zeta_\lambda \int_\gamma d\sigma_\gamma(\xi) K^\lambda_{1\xi}(f) K^\lambda_{2\alpha}(x, \xi),$$

(VI.2)

with $\zeta_0 = +1$ and $\zeta_\lambda = -1$ for $\lambda = 1, 2, 3$ and where $K^\lambda_{n\xi}(f)$ with $n=1, 2$ is the smeared form of the modes:

$$K^\lambda_{n\xi}(f) = \int_{X_H} K^\ast_{n\alpha}(x, \xi) f^\alpha(x) d\sigma(x).$$

(VI.3)

The space generated by the $p(f)$’s is equipped with the indefinite invariant inner product

$$\langle p(f), p(g) \rangle = \int_{X_H \times X_H} f^{\ast \alpha}(x) W_{\alpha \alpha'}(x, x') g^{\alpha'}(x') d\sigma(x') d\sigma(x).$$

(VI.4)

As usual, one could be attempted to define the fields as operator-valued distributions,

$$\mathcal{K}(f) = a(p(f)) + a^\dagger(p(f)) \quad \text{with} \quad p(f) = p_1(f) + p_2(f),$$

(VI.5)

where the operators $a(\mathcal{K}^\lambda(\xi)) \equiv \mathcal{K}^\lambda(\xi)$ and $a^\dagger(\mathcal{K}^\lambda(\xi)) \equiv a^{\dagger \lambda}(\xi)$ are respectively antilinear and linear in their arguments. One would get the hermitian field:

$$\mathcal{K}(f) = \sum_{\lambda=0}^{3} \zeta_\lambda \int_\gamma d\sigma_\gamma(\xi) \left[ K^\ast_{2\xi}(f) \mathcal{K}^\lambda_{\xi}(f) a^\lambda(\xi) + K^\lambda_{\xi}(f) a^{\dagger \lambda}(\xi) \right],$$

(VI.6)

with

$$K^\lambda_{\xi}(f) = \sum_{n=1}^{2} K^\lambda_{n\xi}(f).$$
The unsmeared operator would read

$$\mathcal{K}_\alpha(x) = \sum_{\lambda=0}^{3} \zeta_\lambda \int d\sigma_\gamma(\xi) \left[ \mathcal{K}_\alpha^\lambda(x, \xi) a^\lambda(\xi) + \mathcal{K}_{\alpha}^{*\lambda}(x, \xi) a^{i\lambda}(\xi) \right], \quad (VI.7)$$

where $a^\lambda(\xi)$ satisfies the canonical commutation relations (ccr) and is defined by

$$a^\lambda(\xi)|\Omega> = 0.$$

The field equation (VI.7), however, is problematic, since the integral involved does not have a unique solution due to its degrees of homogeneity. The homogeneity degrees of $a^\lambda(\xi)$ are not fixed, i.e., for $\mathcal{K}_\alpha^\lambda_{1\xi}$ mode the degree is $-2$ and for $\mathcal{K}_\alpha^\lambda_{2\xi}$ mode is $-1$.

In order to set aside this problem, a causal field, which is constructed from the modes (IV.45), i.e.

$$\mathcal{K}(f) = a(p_1(f)) + a^{i}(p_2(f)), \quad (VI.8)$$

should replaced (VI.5). It is clear that this field is not hermitian.

The measure satisfies $d\sigma_\gamma(l\xi) = l^3 d\sigma_\gamma(\xi)$ and the field operator homogeneity is $\mathcal{K}_\alpha^\lambda_{n}(x, l\xi) = l^{-n}\mathcal{K}_\alpha^\lambda_{n}(x, \xi)$. This leads to the homogeneity condition

$$a^\lambda(l\xi) \equiv a(\mathcal{K}_\alpha^\lambda_{n}(l\xi)) = a(l^{-n}\mathcal{K}_\alpha^\lambda(\xi)) = l^{-n}a^\lambda(\xi).$$

The integral representation (VI.7) is independent of the orbital basis $\gamma$ as explained in [13]. For the hyperbolic type submanifold $\gamma_4$ given by

$$\gamma_4 = \{ \xi \in C^+, \xi^{(4)} = 1 \} \cup \{ \xi \in C^+, \xi^{(4)} = -1 \},$$

the measure is $d\sigma_{\gamma_4}(\xi) = d^3\bar{\xi}/\xi_0$ and the ccr’s are represented by

$$[a^\lambda(\xi), a^{i\lambda}(\xi')] = \eta^{\lambda\lambda'}\xi^0\delta^3(\bar{\xi} - \bar{\xi}'). \quad (VI.9)$$

The field commutation relations are

$$[\mathcal{K}_\alpha(x), \mathcal{K}_{\alpha'}(x')] = 2i\text{Im} \langle p_{\alpha}(x), p_{\alpha'}(x') \rangle = 2i\text{Im} \mathcal{W}_{\alpha\alpha'}(x, x') = 2iD_{\alpha\alpha'}\text{Im} \mathcal{W}_0(x, x'), \quad (VI.10)$$

where $\mathcal{W}_{\alpha\alpha'}(x, x')$ and $\mathcal{W}_0(x, x')$ are respectively the vector two-point function and the conformally coupled scalar field two-point function [13, 15, 24]. Using formula (V.8) one has

$$\text{Im} \mathcal{W}_0(x, x') = \frac{H^2}{8\pi} \epsilon(x^0 - x'^0) \delta(\mathcal{Z}(x, x') - 1), \quad (VI.11)$$

where we have used $\mathcal{Z}(x, x') = -H^2x \cdot x' + 1 + \frac{H^2}{2}(x - x')^2 \equiv \cosh H\sigma(x, x')$ and

$$\epsilon(x^0 - x'^0) = \begin{cases} 1 & x^0 > x'^0 \\ 0 & x^0 = x'^0 \\ -1 & x^0 < x'^0. \end{cases} \quad (VI.12)$$

Finally one obtains the commutator :

$$iG_{\alpha\alpha'}(x, x') = [\mathcal{K}_\alpha(x), \mathcal{K}_{\alpha'}(x')] = \frac{iH^2}{4\pi}D_{\alpha\alpha'}\epsilon(x^0 - x'^0) \delta(\mathcal{Z}(x, x') - 1), \quad (VI.13)$$

with $D_{\alpha\alpha'}$ given by (VI.13). Light cone propagation of the vector field is apparent in the r.h.s. of (VI.13).
VII. CONCLUSION

We have quantized the massless vector field in de Sitter space-time by adapting to this specific situation the content of previous works: ambient space formalism, construction of modes, de Sitter covariance and Gupta-Bleuler triplets, construction of the Wightman two-point function, and eventually covariant quantization of the field. The next step will be to examine the possibility of construction of a dS covariant QED, based on the present work and on the explicit construction of dS “massive” Dirac fields which has been carried out in [28]. Of course, the main questions will pertain to the physical interpretation of such a formalism, since Physics in de Sitter space-time is far from being a clear and familiar domain of investigation.
APPENDIX A: THE UNITARY IRREDUCIBLE REPRESENTATIONS OF SO(1,4)

The UIR’s may be labelled by a pair of parameters $\Delta = (p, q)$ with $2p \in \mathbb{N}$ and $q \in \mathbb{C}$, in terms of which the eigenvalues of $Q^{(1)}$ and $Q^{(2)}$ are expressed as follows \cite{16,17}:

\begin{align*}
Q^{(1)} &= [-p(p + 1) - (q + 1)(q - 2)]\text{Id}, \\
Q^{(2)} &= [-p(p + 1)q(q - 1)]\text{Id}.
\end{align*}

According to the possible values for $p$ and $q$, three series of inequivalent unitary representations may be distinguished: the principal, complementary and discrete series.

The Principal series of representations:

Also called “massive” representations, they are denoted by $U_{p,\nu}$, and labelled with $\Delta = (p, q) = (p, 1/2 + i\nu)$ where $p = 0, 1, 2, \ldots$ and $\nu \geq 0$ or $p = 1, 2, 3, \ldots$ and $\nu > 0$.

The operators $Q^{(1)}$ and $Q^{(2)}$ are fixed respectively to the following values:

\begin{align*}
Q^{(1)} &= \left[\left(\frac{9}{4} + \nu^2\right) - p(p + 1)\right]\text{Id}, \\
Q^{(2)} &= \left[\left(\frac{1}{4} - \nu^2\right)p(p + 1)\right]\text{Id}.
\end{align*}

The complementary series representations:

The complementary series is denoted by $V_{p,\nu}$ with $\Delta = (p, q) = (p, 1/2 + \nu)$ and $p = 0$ and $\nu \in \mathbb{R}$, $0 < |\nu| < 3/2$ or $p = 1, 2, 3, \ldots$ and $\nu \in \mathbb{R}$, $0 < |\nu| < 1/2$.

The operators $Q^{(1)}$ and $Q^{(2)}$ assume the following values

\begin{align*}
Q^{(1)} &= \left[\left(\frac{9}{4} - \nu^2\right) - p(p + 1)\right]\text{Id}, \\
Q^{(2)} &= \left[\left(\frac{1}{4} - \nu^2\right)p(p + 1)\right]\text{Id}.
\end{align*}

The discrete series of representations:

The elements of the discrete series of representations are denoted by $\Pi_{p,0}$ and $\Pi_{p,q}^{\pm}$, where the signs $\pm$ would stand for helicity in the massless cases. The relevant values for the couple $\Delta = (p, q)$ are $p = 1, 2, 3, \ldots$ and $q = p, p - 1, \ldots, 1, 0$ or $p = 1, 2, 3, \ldots$ and $q = p, p - 1, \ldots, 1/2$.

Let us add a few precisions concerning the UIR’s which extend to the conformal group $SO_0(2,4)$. First recall that, in our view, these UIR’s will correspond to the massless fields in de Sitter space. Masslessness will in fact be synonymous of conformal invariance throughout this paper. In Ref. \cite{4}, the reduction of the $SO_0(2,4)$ unitary irreducible representations to the de Sitter subgroup $SO_0(1,4)$ UIR’s are examined. It is found that the $SO_0(1,4)$ UIR’s which can be extended to UIR’s of the conformal group are the following:
- The scalar representation with \( p = 0, q = 1 \) and \( \langle Q^{(1)} \rangle = 2 \), which, in the above classification, belongs to the \textbf{complementary series} of UIR. In that case, the \( \text{SO}_0(2, 4) \) representation remains irreducible when restricted to the \( \text{SO}_0(1, 4) \) subgroup.

- The UIR’s characterized by \( p = q = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \), which correspond to some terms of the \textbf{discrete series} of UIR. For any values such that \( p = q \), there are two inequivalent unitary irreducible representations of \( \text{SO}_0(2, 4) \) and both remain irreducible when restricted to \( \text{SO}_0(1, 4) \). These two UIR’s denoted \( \Pi_{p,q}^\pm \) differ in the sign of the parameter \( k_0 = \pm p \) connected to a subgroup \( \text{SO}(3) \) and there is no operator in \( \text{SO}_0(2, 4) \) which changes the value of that sign. Therefore these two UIR’s are distinguished by an entity which we are allowed to name the helicity.

We have pictured these representations (up to \( p = 3 \)) in terms of \( p \) and \( q \) in Figure 3. The symbols ◊ and □ stand for the discrete series with half-integer and integer values of \( p \) respectively. Note that except when \( q = 0 \), each symbol ◊ or □ actually represents two UIR’s due to the ± sign in \( \Pi_{p,q}^\pm \). The complementary series is represented in the same diagram by bold lines. The principal series is represented in the \( \text{Re}(q) = 1/2 \) plane by dashed lines. We have superposed the three discrete series of representation with values \( p = 1/2, 3/2, 5/2 \), \( \text{Re}(q) = 1/2 \) and \( \text{Im}(q) = 0 \) to the principal series in order to show how these two diagrams fit together. Note that the substitution \( p \rightarrow p, q \rightarrow (1 - q) \) or \( p \rightarrow (q - 1), q \rightarrow (p + 1) \) do not alter the eigenvalues of the Casimir operators; the representations (unitary or not) with labels \( \Delta = (p, q) \), \( \Delta = (p, 1 - q) \) and \( \Delta = (q - 1, p + 1) \) are said to be “Weyl equivalent.” Weyl equivalent points can be localized in figure 3. For instance start from the points \( q = \frac{1}{2} \) and \( p = 0, 1, 2, \ldots \): the bold lines (complementary series here) on the right hand side of these points are Weyl equivalent to the bold lines on the left hand side, including the limiting points belonging to the discrete series in the case \( p > 0 \).
APPENDIX B: GUPTA-BLEULER TRIPLET OF THE ELECTROMAGNETIC FIELD

Let us briefly present the Gupta-Bleuler triplet for the electromagnetic field. We actually would like to give a hint of how the indecomposable representation structure of the Poincaré group is implemented starting with the usual plane waves. The electromagnetic field $A_\mu(x)$, defined on the 4-dimensional Minkowski space-time $M$ with metric tensor $\eta_{\mu\nu} = (+1, -1, -1, -1)$, satisfies the Maxwell equation

$$\Box A_\mu(x) - \partial_\mu (\partial^\nu A_\nu(x)) = 0 \quad \text{with} \quad \Box = \eta^{\mu\nu} \partial_\mu \partial_\nu \quad \text{and} \quad x = (t, x).$$

Now the four-vector potential $A_\mu(x)$ is defined up to a gauge transformation

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x).$$

This gauge symmetry couples the components of the vector potential and reduces the effective degrees of freedom from 4 to 2. It is left free to quantize only these two independent (physical) components of the field (Coulomb gauge). Since the vector potential is covariantly described by a four-vector, we assure manifest covariance using the Lorentz gauge, characterized by the condition

$$\partial_\mu A^\mu(x) = 0.$$

Independently of any quantization scheme, we are now in position to define the Gupta-Bleuler triplet $V_g \subset V \subset V'$ carrying the indecomposable structure of the related unitary irreducible representation of the Poincaré group.

- The space $V'$ is the space of all solutions of the field equation including negative norm solutions.

- It contains a closed subspace $V$ of solutions satisfying the Lorentz condition. The invariant subspace $V$ is not invariantly complemented in $V'$.

- The subspace $V_g$ of $V$ consists of all positive energy gauge solutions of the form $A_\mu = \partial_\mu \Lambda$. These are orthogonal to every element in $V$ including themselves. They form an invariant subspace of $V$ but admit no invariant complement in $V$.

The usual Klein-Gordon inner product is indefinite in $V'$, semi-definite in $V$ and gets positive-definiteness in the quotient space $V/V_g$. The latter is the physical state space.

The Poincaré group acts on the physical (or transverse) space $V/V_g$ through the massless, helicity $\pm 1$ unitary representation $\mathcal{P}(0, 1) \oplus \mathcal{P}(0, -1)$ \cite{34}. Now since $V_g$ and $V'/V$ (scalar states) carry a representation equivalent to $\mathcal{P}(0, 0)$ (massless scalar UIR of the Poincaré group) it is possible to write the representation on positive energy states as

$$\mathcal{P}(0, 0) \rightarrow \mathcal{P}(0, 1) \oplus \mathcal{P}(0, -1) \rightarrow \mathcal{P}(0, 0) \quad ,$$

where the arrows indicates the leak under the group action. We now analyse this indecomposable structure in terms of the (vector) plane waves with components:

$$\phi^r_\mu(x) = \epsilon^r_\mu(k) \frac{e^{i(\vec{k} \cdot \vec{x} - \omega_k t)}}{\sqrt{2 \omega_k}} \quad \text{with} \quad k_0 = \omega_k = \sqrt{|\vec{k}|^2} ,$$

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where the polarization vectors are

\[ \epsilon^0(k) = (\epsilon_\mu^0(k)) = \left( \frac{1}{\omega_k} \right), \quad \epsilon^3(k) = \left( \frac{0}{\tilde{k}/\omega_k} \right), \]

\[ \epsilon^i(k) = \left( \frac{0}{\epsilon_i} \right) \quad \text{with} \quad \tilde{k} \cdot \epsilon_i(k) = 0, \quad \epsilon_i(k) \cdot \epsilon_j(k) = \delta_{ij} \quad \text{for} \quad i = 1, 2. \]

These vectors obey the following identity:

\[ \epsilon^r_\mu(k) \epsilon'^r_\nu(k) \eta^{\mu\nu} = \eta^{rr}. \]

As usual, the polarization \( \epsilon^i(k) \) with \( i = 1, 2 \) are the physical transverse polarizations. The plane waves are normalized (in the Bohr integral sense) to

\[ ||\phi^r||^2 = +1 \quad \text{for} \quad r = 1, 2, 3 \quad \text{and} \quad ||\phi^0||^2 = -1. \]

The gauge states \( \phi_g \in V_g \) and the scalar states \( \phi_s \in V'/V \) can be written

\[ \phi_\mu^g = \frac{1}{\sqrt{2}} \left( \phi_\mu^0 + \phi_\mu^3 \right) = \frac{k_\mu e^{ik \cdot x}}{2\omega_k^{3/2}}, \quad \phi_\mu^s = \frac{1}{\sqrt{2}} \left( \phi_\mu^0 - \phi_\mu^3 \right) = \frac{\tilde{k}_\mu e^{ik \cdot x}}{2\omega_k^{3/2}}, \]

and satisfy

\[ \langle \phi^s, \phi^g \rangle = 1, \quad ||\phi^g||^2 = 0, \quad ||\phi^s||^2 = 0. \]

A general solution of the field equation can be written

\[ \phi(x) = a_g \phi^g + a_1 \phi^1 + a_2 \phi^2 + a_s \phi^s. \]

We now consider the group action in order to display the indecomposable structure. In terms of vector components, the Poincaré group acts as

\[ (U(a, \Lambda) \phi)_\mu(x) = \Lambda_\nu^\rho \phi_\nu(\Lambda^{-1}(x-a)). \]

First of all let us show that starting with a physical state, the group action will yield transverse states and gauge states. For this we choose \( (k_\mu) = (1, 1, 0, 0) \) and the transverse state \( (\phi_\mu^1) \) with \( (\epsilon_\mu^1(k)) = (0, 0, 1, 0) \). Let us consider the group action \( U(0, \Lambda) \)

with \( \Lambda = \begin{pmatrix} \cosh \theta & 0 & \sinh \theta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh \theta & 0 & \cosh \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \) which yields \( \Lambda k = \begin{pmatrix} \cosh \theta \\ 1 \\ \sinh \theta \\ 0 \end{pmatrix}, \quad \Lambda \epsilon^1(k) = \begin{pmatrix} \sinh \theta \\ 0 \\ \cosh \theta \end{pmatrix}. \)

One gets

\[ (U(0, \Lambda) \phi^1)(x) = \Lambda \epsilon^1(k) e^{i\Lambda k \cdot x} = \tanh \theta \left[ \begin{pmatrix} \cosh \theta \\ 1 \\ \sinh \theta \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -1/ \sinh \theta \\ 0 \end{pmatrix} \right] e^{i\Lambda k \cdot x}. \]
and therefore
\[(U(0, \Lambda)\phi^1)(x) = \alpha (\Lambda k) e^{i\Lambda k \cdot x} + \beta \ e^1(\Lambda k)e^{i\Lambda k \cdot x},\]
where \(e^1(\Lambda k)\) is transverse with respect to \(\Lambda k\).

Let us now start with a scalar state in order to see that the group action will generate scalar states, transverse states as well as gauge states. With the same vector \((k_\mu) = (1, 1, 0, 0)\) we define the scalar state \((\phi^s_\mu)\) with \((\epsilon^s_\mu(k)) = (1, -1, 0, 0)\). We again consider the group action \(U(0, \Lambda)\)

\[
\Lambda = \begin{pmatrix}
\cosh \theta & 0 & \sinh \theta & 0 \\
0 & 1 & 0 & 0 \\
\sinh \theta & 0 & \cosh \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

which yields \(\Lambda k = \begin{pmatrix} \cosh \theta \\ 1 \\ \sinh \theta \\ 0 \end{pmatrix}, \quad \Lambda \epsilon^s(k) = \begin{pmatrix} \cosh \theta \\ -1 \\ \sinh \theta \\ 0 \end{pmatrix} \)

One gets
\[
(U(0, \Lambda)\phi^s)(x) = \Lambda \epsilon^s(k)e^{i\Lambda k \cdot x} = \left[ \alpha \begin{pmatrix} \cosh \theta \\ -1 \\ \sinh \theta \\ 0 \end{pmatrix} + \beta \begin{pmatrix} \cosh \theta \\ 1 \\ \sinh \theta \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ \tanh \theta \\ -1/ \cosh \theta \\ 0 \end{pmatrix} \right]e^{i\Lambda k \cdot x}
\]

and therefore
\[
(U(0, \Lambda)\phi^s)(x) = \alpha \epsilon^s(\Lambda k)e^{i\Lambda k \cdot x} + \beta (\Lambda k)e^{i\Lambda k \cdot x} + \gamma \ e^i(\Lambda k)e^{i\Lambda k \cdot x},
\]
where \(e^i(\Lambda k)\) with \(i = 1, 2\) is transverse with respect to \(\Lambda k\) and with \(\alpha = 1/(\cosh^2 \theta), \, \beta = \tanh^2 \theta, \, \gamma = -2 \tanh \theta\).

**APPENDIX C: THE TWO-POINT FUNCTION FROM MAXIMALLY SYMMETRIC BIVECTORS IN AMBIENT SPACE**

Following Allen and Jacobson in reference \[12\] we express here the two-point functions in de Sitter space (maximally symmetric) in terms of bivectors.

Maximally symmetric bivectors are functions of two points \((x, x')\) which behave like vectors under coordinate transformations at either point. The bivectors are called maximally symmetric if they respect the de Sitter invariance.

As shown in reference \[12\], any maximally symmetric bivector can be expressed as a sum of products of three basic tensors. The coefficients in this expansion are functions of the geodesic distance \(\mu(x, x')\), that is the distance along the geodesic connecting the points \(x\) and \(x'\) (note that \(\mu(x, x')\) can be defined by unique analytic extension also when no geodesic connects \(x\) and \(x'\)). In this sense, these fundamental tensors form a complete set. They can be obtained by differentiating the geodesic distance:

\[
n_a = \nabla_a \mu(x, x'), \quad n_{a'} = \nabla_{a'} \mu(x, x')
\]
and through the parallel propagator

\[ g_{ab'} = -c^{-1}(Z) \nabla_a n_{b'} + n_a n_{b'} . \]

The geodesic distance is implicitly defined \[13\] for \( Z = -H^2 x \cdot x' \) by

\[ Z = \cosh(\mu H) \quad \text{for } x \text{ and } x' \text{ timelike separated,} \]
\[ Z = \cos(\mu H) \quad \text{for } x \text{ and } x' \text{ spacelike separated such that } |x \cdot x'| < H^{-2}. \]

The two-point function in terms of the basis bi-vectors reads

\[ T_{ab'}(x, x') = \alpha(\mu) g_{ab'} + \beta(\mu) n_a n_{b'}. \]

Since in this paper we work in ambient space, let us accordingly re-express the basic bi-vectors in terms of the corresponding notations.

**Proposition 1** In ambient space notations \((x \in \mathbb{R}^5 \text{ and the constraint } x \cdot x = -H^{-2})\), the basic bi-vectors corresponding to \( n_a, n_{a'} \) and \( g_{ab'} \) can be chosen as

\[ \bar{\partial}_\alpha \mu(x, x'), \quad \bar{\partial}_{a'} \mu(x, x'), \quad \theta_\alpha \cdot \theta_{a'}. \]

**Proof.**

One merely has to consider the restriction to the hyperboloid given by

\[ T_{ab'}(x, x') = \frac{\partial x^\alpha}{\partial X^a} \frac{\partial x'^{a'}}{\partial X^{b'}} T_{\alpha\beta'}. \]

- **When** \( Z = \cos(\mu H) \), one finds

\[ n_a = \frac{\partial x^\alpha}{\partial X^a} \bar{\partial}_\alpha \mu(x, x') = \frac{\partial x^\alpha}{\partial X^a} \frac{H(\theta_\alpha \cdot x')}{\sqrt{1 - Z^2}}, \quad n_{b'} = \frac{\partial x'^{a'}}{\partial X^{a'}} \bar{\partial}_{a'} \mu(x, x') = \frac{\partial x'^{a'}}{\partial X^{a'}} \frac{H(\theta_{a'} \cdot x)}{\sqrt{1 - Z^2}}, \]

and

\[ \nabla_a n_{b'} = \frac{\partial x^\alpha}{\partial X^a} \frac{\partial x'^{a'}}{\partial X^{b'}} \bar{\partial}_\alpha \bar{\partial}_{a'} \mu(x, x') = c(Z) \left[ Z n_a n_{b'} - \frac{\partial x^\alpha}{\partial X^a} \frac{\partial x'^{a'}}{\partial X^{b'}} \theta_\alpha \cdot \theta_{a'} \right], \]

with \( c(Z) = -\frac{H}{\sqrt{1 - Z^2}} \).

- **When** \( Z = \cosh(\mu H) \), \( n_a, n_{b'} \) are multiplied by \( i \) and \( c(Z) \) becomes \( -\frac{iH}{\sqrt{1 - Z^2}} \).

In both cases we have

\[ \frac{\partial x^\alpha}{\partial X^a} \frac{\partial x'^{a'}}{\partial X^{b'}} \theta_\alpha \cdot \theta_{a'} = g_{ab'} + (Z - 1) n_a n_{b'}. \]
APPENDIX D: THE TWO-POINT FUNCTION FROM THE FIELD MODES

We have shown that the vector two-point function can be expressed as

\[ W_{\alpha\alpha'}(z, z') = c_0 \int_{\gamma} \left( \theta_\alpha \cdot \theta_{\alpha'} - \frac{2 \xi_\alpha' \xi_\alpha}{(Hz \cdot \xi)^2} \right) \phi_1(z) \phi_2(z') d\sigma_\gamma(\xi), \]  

(D.1)

where \( \phi_1(z) = (Hz \cdot \xi)^{-1} \) and \( \phi_2(z) = (Hz \cdot \xi)^{-2} \). This has been done starting with the formula

\[ W_{\alpha\alpha'}(z, z') = -H^{-2} \bar{\theta}_\alpha \cdot \bar{\partial} W_0(z, z') - x \cdot \theta'_{\alpha'} \bar{\partial}_\alpha W_0(z, z'), \]

based on the general maximally symmetric bivector form. This two-point function is also equal to

\[ W_{\alpha\alpha'}(z, z') = -H^{-2} \bar{\theta}_\alpha' \theta_\alpha \cdot \bar{\partial} W_0(z, z') - x' \cdot \theta_\alpha \bar{\partial}_\alpha' W_0(z, z'), \]

which yields

\[ W_{\alpha\alpha'}(z, z') = 2 \theta_\alpha \cdot \theta'_{\alpha'} W_0(z, z') + \theta \cdot x' \bar{\partial} W_0(z, z') - c_0 \int_{\gamma} 6 \frac{\xi_\alpha' \xi_\alpha}{(Hz' \cdot \xi)^2} \phi_1(z) \phi_2(z') d\sigma_\gamma(\xi). \]  

(D.2)

Thus it is established that

\[ -c_0 \int_{\gamma} 6 \frac{\xi_\alpha' \xi_\alpha}{(Hz' \cdot \xi)^2} \phi_1(z) \phi_2(z') d\sigma_\gamma(\xi) = -\theta_\alpha \cdot \theta'_{\alpha'} W_0(z, z') - \theta \cdot x' \bar{\partial} W_0(z, z') - c_0 \int_{\gamma} 2 \frac{\xi_\alpha' \xi_\alpha}{(Hz \cdot \xi)^2} \phi_1(z) \phi_2(z') d\sigma_\gamma(\xi). \]  

We are now in position to calculate the two-point function with the help of the field modes.

\[ W_{\alpha\alpha'}(z, z') = c_0 \int_{\gamma} \eta_{\lambda\lambda'} \left( \bar{Z}_{\alpha}^\lambda - 2 \frac{Z_\lambda \cdot \xi}{(H x' \cdot \xi)^2} \right) \left( 2 \bar{Z}_{\alpha'}^\lambda \cdot \xi - 6 \frac{Z_\lambda \cdot \xi}{(H x' \cdot \xi)^2} \xi_{\alpha'} - 2 \frac{Z_\lambda \cdot x'}{(x' \cdot \xi)} \xi_{\alpha'} \right) \phi_1(z) \phi_2(z') d\sigma_\gamma(\xi). \]

For \( \eta_{\lambda\lambda'} Z_{\alpha}^{\lambda} Z_{\beta}^{\lambda'} = \eta_{\alpha\beta} \) and using \( \xi^2 = 0 \) one gets

\[ W_{\alpha\alpha'}(z, z') = 2 \theta_\alpha \cdot \theta'_{\alpha'} W_0(z, z') + \theta \cdot x' \bar{\partial} W_0(z, z') - c_0 \int_{\gamma} 6 \frac{\xi_\alpha' \xi_\alpha}{(Hz' \cdot \xi)^2} \phi_1(z) \phi_2(z') d\sigma_\gamma(\xi). \]

Now with the help of Eq. (D.2) one finally finds that this expression coincides with the expression (D.1).

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