MACDONALD CUMULANTS, $G$-INVERSION POLYNOMIALS AND $G$-PARKING FUNCTIONS

MACIEJ DOŁĘGA

ABSTRACT. We prove a combinatorial formula for Macdonald cumulants which generalizes the celebrated formula of Haglund, Haiman and Loehr for Macdonald polynomials. We provide several applications of our formula. Firstly, it allows us to give a new, constructive proof of a strong factorization property of Macdonald polynomials proven recently by the author of this paper. Moreover, we prove that Macdonald cumulants are $q, t$-positive in the monomial and in the fundamental quasisymmetric bases. Furthermore, we use our formula to prove the recent higher-order Macdonald positivity conjecture for the coefficients of the Schur polynomials indexed by hooks. Our combinatorial formula relates Macdonald cumulants to the generating function of $G$-parking functions, or equivalently to a certain specialization of the Tutte polynomials.

1. INTRODUCTION

1.1. Schur–positivity and Macdonald polynomials. An ubiquitous problem in the theory of symmetric functions is to express a given symmetric function $f$ in a given basis $\{s_\lambda\}_\lambda$ of particular interest. In other words, one can ask: what can we say about the coefficients $a_\lambda$ in the following expansion:

$$f = \sum_\lambda a_\lambda s_\lambda?$$

Most often, such a basis is the basis of Schur symmetric functions, which turns out to be the most natural in many different contexts such as the representation theory of the symmetric groups, algebraic geometry, or random discrete models, among others. Indeed, Schur symmetric functions are characters of irreducible representations of $GL_n$, their asymptotic behaviour describe many random processes and they represent Schubert classes in Grassmannian varieties, see [Ful97, BG16]. In many cases it was observed that a beautiful but notoriously difficult to prove phenomenon occurs: all the coefficients $a_\lambda$ are nonnegative integers, or in a more general setting they are polynomials with nonnegative integer coefficients. For instance, the homogeneous symmetric function $h_\mu$, which has positive integer coefficients in the Schur basis expansion has a natural $q$–deformation given by the modified Hall-Littlewood symmetric function $Q'_\mu(q)$. The coefficients of $Q'_\mu(q)$ expanded in the Schur basis are polynomials in $q$ with nonnegative integer coefficients. This phenomenon, called Schur–positivity, has a deep geometric reason and a beautiful combinatorial interpretation in the case of Hall–Littlewood symmetric functions [LS78, Lus81]. Moreover, in the case of other symmetric functions Schur–positivity builds deep connections between many different areas of mathematics and physics such as the representation theory

2010 Mathematics Subject Classification. Primary 05E05; Secondary 05A30, 05C05, 05C31.

Key words and phrases. Macdonald polynomials; Schur polynomials; Cumulants; Tutte polynomials; Parking functions; $q, t$-Kostka numbers.

MD is supported from Narodowe Centrum Nauki, grant UMO-2015/16/S/ST1/00420.
of groups, Hecke algebras, algebraic geometry, or the theory of quantum groups, see for instance [LLT96, KT99, Hai01, LPP07, SW12]. Therefore, deciding whether a given symmetric function is Schur–positive is one of the major questions in the contemporary algebraic combinatorics of symmetric functions.

One of the most prominent examples of Schur–positive symmetric functions which contain the aforementioned modified Hall–Littlewood symmetric functions as a special case is the Macdonald symmetric function \( \tilde{H}_\mu(x; q, t) \), introduced by Macdonald in 1988 [Mac88, Mac95] (here, we use “the transformed form” of Macdonald polynomials sometimes called “the modified form” introduced by Garsia and Haiman - see [GH93] for its initial definition and a relation with other forms of Macdonald polynomials). Strictly from the definition, this is a symmetric function in variables \( x := x_1, x_2, \ldots \) with the coefficients being rational functions in \( q, t \). However, Macdonald conjectured [Mac88] that expanding it in the Schur basis:

\[
\tilde{H}_\mu(x; q, t) = \sum_\lambda \tilde{K}_{\lambda, \mu}(q, t) s_\lambda(x)
\]

the coefficients \( \tilde{K}_{\lambda, \mu}(q, t) \) (called transformed \( q, t \)--Kostka coefficients) are in fact polynomials in \( q, t \) with nonnegative coefficients. In the following Garsia and Haiman gave a conjectural representation–theoretic interpretation of the transformed \( q, t \)--Kostka coefficients [GH93], and it took almost ten years more to Haiman to prove it [Hai01]. He achieved this goal by connecting a representation theoretic interpretation of the transformed \( q, t \)--Kostka coefficients with the problem from algebraic geometry of the Hilbert scheme of \( n \) points in the plane. This result is considered as a great breakthrough in the symmetric functions theory and initiated very active research in the remarkable algebraic combinatorics of the Macdonald polynomials, see the expository textbook of Haglund [Hag08].

Macdonald positivity ex-conjecture has seen many generalizations in different directions up to these days. One example of such a generalization called higher–order Macdonald positivity conjecture was presented in our recent work [Dot17a] and will be the main subject of this paper.

1.2. Cumulants and higher–order Macdonald positivity conjecture.

1.2.1. Cumulants. A classical problem in the symmetric functions theory, which is related to the positivity problem from Section 1.1, is to understand the so-called structure constants \( a_{\mu, \nu}^\lambda \) of a given linear basis \( \{ s_\mu \}_\mu \):

\[
s_\mu \cdot s_\nu = \sum_\lambda a_{\mu, \nu}^\lambda s_\lambda.
\]

Let us look on the structure constants for Macdonald polynomials. For partitions \( \lambda = (\lambda_1, \lambda_2, \ldots) \) and \( \mu = (\mu_1, \mu_2, \ldots) \) we define a new partition \( \lambda \oplus \mu := (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots) \) by adding coordinates of partitions \( \lambda \) and \( \mu \). Since Macdonald polynomials \( \{ \tilde{H}_\mu \}_\mu \) form a linear basis of the algebra \( \Lambda \) of symmetric functions over \( \mathbb{Q}(q, t) \), we can define a multiplication \( \oplus \) on \( \Lambda \) by setting \( \tilde{H}_\mu \oplus \tilde{H}_\nu := \tilde{H}_{\mu \oplus \nu} \) and extending it by linearity. Macdonald showed [Mac95] that algebras \( (\Lambda, \oplus) \) and \( (\Lambda, \cdot) \) coincide in the specialization \( q = 1 \) (this also follows from Haglund, Haiman and Loehr’s formula (17)). Thus, the much simpler algebraic structure \( (\Lambda, \oplus) \) can be interpreted as an approximation of the algebra \( (\Lambda, \cdot) \) of interest, as \( q \to 1 \), therefore it is desirable to understand the difference between these two product structures. A
natural way of measuring the discrepancy between two algebraic structures is provided by conditional cumulants.

Let $A$ be a commutative ring with two different multiplicative structures $\cdot$ and $\oplus$ which define two (different) algebra structures on $A$. For any $X_1, \ldots, X_r \in A$ one can define a conditional cumulant $\kappa(X_1, \ldots, X_r) \in A$ as the coefficient of $t_1 \cdots t_r$ in the following formal power series in $t_1, \ldots, t_k$:

\[ \kappa(X_1, \ldots, X_r) := [t_1 \cdots t_r] \log \left( \exp_{\oplus} \left( t_1 X_1 + \cdots + t_r X_r \right) \right), \]

where $\log$ and $\exp_{\oplus}$ are defined in a standard way with respect to multiplication given by $\cdot$ and $\oplus$ respectively. Thus

\[ \log (1 + A) = \sum_{n \geq 1} (-1)^{n-1} \frac{A^n}{n}, \]

and

\[ \exp_{\oplus}(A) = \sum_{n \geq 0} \frac{A^{\oplus n}}{n!}. \]

Definition (1) can be transformed into an equivalent but more combinatorial definition:

\[ \kappa(X_1, \ldots, X_r) = \sum_{\pi \in \mathcal{P}(\{r\})} (-1)^{\#\pi-1} \frac{(\#\pi - 1)!}{\prod_{B \in \pi} \prod_{b \in B} X_b}. \]

Here we sum over set-partitions of $[r] := \{1, 2, \ldots, r\}$, that is all possible sets $\pi$ of nonempty subsets of $[r]$ such that every element $i \in [r]$ belongs to precisely one element of $\pi$ (i.e., $[r]$ is a disjoint union of the elements in $\pi$); $\#\pi$ denotes the number of elements of $\pi$. It is worth mentioning that the Möbius inversion formula asserts that (2) has an equivalent form:

\[ \bigoplus_{j \in J} X_j = \sum_{\pi \in \mathcal{P}(J)} \prod_{B \in \pi} \kappa(X_i : i \in B). \]

Note that cumulants are multilinear. Thus, in order to understand the discrepancy between $(\Lambda, \oplus)$ and $(\Lambda, \cdot)$ it is enough to study cumulants $\kappa(\tilde{H}_{\lambda_1}, \ldots, \tilde{H}_{\lambda_r})$ of the basic elements. It is clear that $\kappa(\tilde{H}_{\lambda_1}, \ldots, \tilde{H}_{\lambda_r}) \in \mathbb{Z}[q, t]\{m_\mu\}_\mu$ because the cumulant $\kappa(\tilde{H}_{\lambda_1}, \ldots, \tilde{H}_{\lambda_r})$ is a linear combination of products of Macdonald polynomials. It is also clear that $\kappa(\tilde{H}_{\lambda_1}, \ldots, \tilde{H}_{\lambda_r}) \in \mathbb{Z}[q, t]\{(q-1)m_\mu\}_\mu$ for $r > 1$ since $(\Lambda, \oplus)$ and $(\Lambda, \cdot)$ coincide in the specialization $q = 1$, but one check that $\kappa(\tilde{H}_{\lambda_1}, \tilde{H}_{\lambda_2}, \tilde{H}_{\lambda_3}) \in \mathbb{Z}[q, t]\{(q-1)^2m_\mu\}_\mu$, which is quite nontrivial. We can observe a pattern here – it is reasonable to think that “higher-order” cumulants provide the approximation of higher order when $q \to 1$, that is $\kappa(\tilde{H}_{\lambda_1}, \ldots, \tilde{H}_{\lambda_r}) \in \mathbb{Z}[q, t]\{(q-1)^r - 1m_\mu\}_\mu$. This statement was conjectured in [DF16], and this implies the partial solution of the $b$-conjecture, see Section 1.2.2. The proof was found recently by the author:

**Theorem 1.1.** [Dol17a] For any partitions $\lambda^1, \ldots, \lambda^r$ one has

\[ \kappa(\tilde{H}_{\lambda_1}, \ldots, \tilde{H}_{\lambda_r}) \in \mathbb{Z}[q, t]\{(q-1)^r - 1m_\mu\}_\mu. \]

1.2.2. *Motivations.* The initial motivation for studying cumulants $\kappa(\tilde{H}_{\lambda_1}, \ldots, \tilde{H}_{\lambda_r})$ comes from our attempts [DF16] to prove the $b$-conjecture – one of the major open problems in the theory of Jack symmetric functions posed by Goulden and Jackson [GJ96]. The $b$-conjecture states that the coefficients of a certain multivariate generating function $\psi(x, y, z; \beta)$ involving Jack symmetric functions can be interpreted as weighted generating functions of graphs embedded into surfaces. Except some special cases [BJ07, La 09, KV16, Dol17b] not much is
known and the \( b \)-conjecture is still wide open. However, in our recent paper [DF16] the author and Féray were able to rewrite the function \( \psi(x, y, z; \beta) \) as a linear combination of cumulants of Jack symmetric functions, which are specializations of cumulants \( \kappa(\tilde{H}_{\lambda_1}, \ldots, \tilde{H}_{\lambda_r}) \), and we showed that Theorem 1.1 implies a partial solution of the \( b \)-conjecture. In view of this result, understanding of the structure of cumulants \( \kappa(\tilde{H}_{\lambda_1}, \ldots, \tilde{H}_{\lambda_r}) \) is of great interest as a potential tool for solving the \( b \)-conjecture in general.

Furthermore, cumulants \( \kappa(\tilde{H}_{\lambda_1}, \ldots, \tilde{H}_{\lambda_r}) \) are of a special interest from the following reason: the structure of Macdonald polynomials \( \tilde{H}_{\mu} \) is directly related to algebraic geometry [Hai02]. It turns out that cumulants appear naturally in algebraic geometry [DNWZ] and it is interesting to investigate what kind of geometric information is encoded in the structure of \( \kappa(\tilde{H}_{\lambda_1}, \ldots, \tilde{H}_{\lambda_r}) \). Secondly, we recall that one of the most typical applications of cumulants in the context of probability is to show that a certain family of random variables is asymptotically Gaussian. Especially, when one deals with discrete structures, whose “observables” form a nice algebraic structure, the main technique is to show that conditional cumulants have a certain small cumulant property exactly of the same form as in Theorem 1.1; see [Śni06, FM12, Fér13, D´S18]. It is therefore natural to ask for a probabilistic interpretation of Theorem 1.1, which leads to some kind of a central limit theorem. The most natural framework to investigate this problem seems to be related to Macdonald processes introduced by Borodin and Corwin [BC14] and it would be interesting to link our work with this probabilistic aspect.

Finally, the biggest motivation for us to study cumulants \( \kappa(\tilde{H}_{\lambda_1}, \ldots, \tilde{H}_{\lambda_r}) \) is their beautiful and mysterious combinatorial structure. In particular, their Schur–positivity is yet to be resolved. For partitions \( \lambda^1, \ldots, \lambda^r \) we define the Macdonald cumulant \( \kappa(\lambda^1, \ldots, \lambda^r)(x; q, t) \) as

\[
\kappa(\lambda^1, \ldots, \lambda^r)(x; q, t) := \frac{\kappa(\tilde{H}_{\lambda_1}(x; q, t), \ldots, \tilde{H}_{\lambda_r}(x; q, t))}{(q - 1)^{r-1}}.
\]

We recall that monomial symmetric functions have integer coefficients in the Schur basis expansion. Thus, one can reformulate Theorem 1.1 as follows: for any partitions \( \lambda^1, \ldots, \lambda^r \) one has the following expansion

\[
\kappa(\lambda^1, \ldots, \lambda^r) \in \mathbb{Z}[q, t]\{s_{\mu}\}_{\mu}.
\]

Remarkably, extensive computer simulations suggest that Macdonald cumulants are, in fact, Schur–positive, which we conjectured in our recent paper [Doł17a]:

**Conjecture 1.2** (Higher–order Macdonald positivity conjecture). Let \( \lambda^1, \ldots, \lambda^r \) be partitions. Then, for any partition \( \mu \), the multivariate \( q, t \)-Kostka number \( \tilde{K}_{\mu, \lambda^1, \ldots, \lambda^r}^{(q,t)} \) defined by the following expansion

\[
\kappa(\mu, \lambda^1, \ldots, \lambda^r) := \sum_{\mu} \tilde{K}_{\mu, \lambda^1, \ldots, \lambda^r}^{(q,t)} s_{\mu}
\]

is a polynomial in \( q, t \) with nonnegative integer coefficients.

Note that the case \( r = 1 \) corresponds to the Macdonald positivity ex-conjecture, so our conjecture generalizes it from the cumulant of order 1 to cumulants of higher order.

**1.3. The main results.** Our main result is an explicit combinatorial formula for Macdonald cumulants \( \kappa(\lambda^1, \ldots, \lambda^r) \). Before we go into details of the formula, let us briefly summarize its consequences.
First of all, our main result strengthens Theorem 1.1 twofold. On the one hand – Theorem 1.1 is an immediate consequence of our explicit formula, while the original proof relied on some complicated induction and was not constructive; in particular it asserted that the coefficients of the monomial expansion of Macdonald cumulants belong to $\mathbb{Z}[q, t]$ by some abstract argument. On the other hand our formula shows that Macdonald cumulants are monomial–positive and the coefficients in this expansion have an explicit combinatorial interpretation in terms of counting trees with some weights.

Secondly, we deduce from our formula an explicit, $q, t$–positive expansion of Macdonald cumulants in fundamental quasisymmetric functions.

Finally, we would like to comment a relation between our main result and Conjecture 1.2. There is a well-known combinatorial formula expanding Schur polynomials as a linear combination of monomial symmetric functions with nonnegative integer coefficients. One can invert this formula to expand monomial symmetric function in Schur basis and the coefficients in this expansion are no longer positive in general. In particular, Conjecture 1.2 stronger than positivity and integrality of Macdonald cumulants in monomial basis. However, it turned out that our combinatorial formula implies that Conjecture 1.2 holds true in the special case of hooks. In other words, for any partitions $\lambda^1, \ldots, \lambda^r$ and for any partition $\mu$ of the hook shape (of the form $\mu = (r + 1, 1^s)$ for some nonnegative integers $r, s$) the multivariate $q, t$–Kostka number $\widetilde{K}_{(r+1,1^r)}; \lambda^1, \ldots, \lambda^r(q, t)$ is a polynomial in $q, t$ with nonnegative integer coefficients. Here, as before, we interpret the polynomial $\widetilde{K}_{(r+1,1^r)}; \lambda^1, \ldots, \lambda^r(q, t)$ as a generating series of some trees. There is a hope that our combinatorial formula can be transformed into a combinatorial proof of the Schur–positivity of Macdonald cumulants in the future (see Section 8 for more details), but so far this is a big open problem even in the case of Macdonald polynomials.

1.3.1. Graphs and our main theorem. Let us introduce the graph theory terminology necessary for presenting our main result. Let $G = (V, E)$ be a connected multigraph, possibly with loops, where $V = [r]$. The vertex with label 1 is called the root. For any vertices $i, j \in V$ let $e_{i,j}(G)$ denote the number of edges linking $i$ with $j$ in $G$. We say that $H \subset G$ is a spanning subgraph of $G$, if for any vertex $v \in V$ there exists an edge in $H$ containing $v$. We say that $T \subset G$ is a spanning tree of $G$ if it is a spanning subgraph of $G$ and it is a tree (it is connected and has no cycles). For a pair of different vertices $i, j \in V$ of $T$ we say that $j$ is a descendant of $i$ if $i$ lies on the shortest path from $j$ to the root, and we call $i$ an ancestor of $j$. If $i$ is an ancestor of $j$ adjacent to it, we call it a parent of $j$. We say that a pair $(i, j)$ which does not contain the root is a $\kappa$–inversion of a spanning tree $T$ of $G$ if it is an inversion $(i, j)$ is an ancestor of $j$ and $i > j$) and $j$ is adjacent to the parent of $i$ in $G$. Let $G$ be a graph obtained from $G$ by replacing all multiple edges by single ones. We define the $G$–inversion polynomial by

$$I_G(q) = q^{\text{number of loops in } G} \sum_{T \subset G} q^{\kappa(T)} \prod_{\{i,j\} \in T} [e_{i,j}(G)]_q,$$

where the sum runs over all spanning trees of $G$.

$$\kappa(T) = \sum_{\{i,j\} \sim \kappa–\text{inversion in } T} e_{\text{parent}(i),j}(G),$$

and we use a standard notation $[n]_q := \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}$. For example, if we want to compute $I_G(q)$ for $G$ from Fig. 1, we first notice that there are 3 spanning trees of $G$
which we denote by $T_1, T_2$ and $T_3$. A pair $(3, 2)$ is not a $\kappa$–inversion of $T_1$ nor $T_2$, but it is a $\kappa$–inversion of $T_3$. Therefore $\kappa(T_1) = \kappa(T_2) = 0$ and $\kappa(T_3) = e_{\text{parent}(3), 2}(G) = e_{1, 2}(G) = 6$. Thus,

$$I_G(q) = \sum_T q^{\text{inv}(T)}.$$ 

Note that the $G$–inversion polynomial for a complete graph $G = K_r$ is given by $I_G(q) = \sum_{\lambda \vdash r} \lambda^T q^{\text{maj}(\lambda)}$, where $\lambda^T := \bigoplus_{b \in B} \lambda^b$ and $x^\sigma := \prod_{\square \in \lambda^{(r)}} x_{\sigma(\square)}$.

We are ready to formulate our main result.

**Theorem 1.3.** Let $\lambda^1, \ldots, \lambda^r$ be partitions. Then, the following formula holds true:

$$\kappa(\lambda^1, \ldots, \lambda^r) = \sum_{\sigma: \lambda^{(r)} \to \N^+_+} I_{G_{\lambda^1, \ldots, \lambda^r}}(q) t^{\text{maj}(\sigma)} x^\sigma,$$

where $\lambda^B := \bigoplus_{b \in B} \lambda^b$ and $x^\sigma := \prod_{\square \in \lambda^{(r)}} x_{\sigma(\square)}$.

The summation index in (7) runs over all the fillings $\sigma$ of a Young diagram $\lambda^{(r)}$ by positive integers, $G_{\lambda^1, \ldots, \lambda^r}$ is a certain multigraph (we refer for its construction to (4.1)), and $\text{maj}(\sigma)$ is a certain statistic of the filling $\sigma$ (see (3.3) for the precise definition). We finish this section by mentioning that our formula (7) specializes to the celebrated formula of Haglund, Haiman and Loehr [HHL05a] when $r = 1$. The work of Haglund, Haiman and Loehr [HHL05a], where many consequences of the formula (17) were presented, was a source of inspiration for our research and most of the consequences of our formula (7) can be derived in a similar manner as in [HHL05a].

**1.4. Organization of the paper.** In Section 2 we discuss various interpretations of the polynomial $I_G(q)$ and we prove some of its properties. Section 3 introduces the necessary background on the combinatorics of Macdonald polynomials. Section 4 is devoted to the proof of our main result and explains the construction of graphs involved in our formula. Section 5
gives an explicit formula for the fundamental quasisymmetric functions expansion of Macdonald cumulants. Section 6 is devoted to the proof of the Schur–positivity of Macdonald cumulants in the case of hooks. In Section 7 we investigate a certain subfamily of Macdonald cumulants which arises from the study of the $b$-conjecture and we show that this family is a basis of the symmetric functions algebra. We finish with Section 8, where we state some open problems and possible directions for the future research.

2. Tutte polynomials, $G$-inversion polynomials and $G$-parking functions

2.1. Tutte polynomials and $G$-inversion polynomials. From now on each graph $G = (V, E)$ can possibly have multiple edges and loops. Moreover, in this section we additionally assume that $G$ is connected. The Tutte polynomial of $G$ denoted by $T_k G(x, y)$ was introduced by Tutte in [Tut47], and its various specializations give rise to many important graph invariants such as the number of its spanning trees or the number of its acyclic orientations, among others. This remarkable interdisciplinary of Tutte $G(x, y)$ made it one of the most important invariants in modern graph theory, see [Bol98].

Tutte polynomials are defined by the following equality:

$$T_k G(x, y) = \sum_{H \subseteq G} (x - 1)^{c(H)} (y - 1)^{\#E(H) - \#V + c(H)},$$

where we sum over all (possibly disconnected) sub-graphs of $G$, $c(H)$ denotes the number of connected components of $H$, $E(H)$ is the set of edges of $H$. Tutte [Tut54] noticed that for a connected graph $G$ the specialization $T_k G(1, 1)$ counts the number of spanning trees of $G$, and this observation allows him to express $T_k G(x, y)$ as the bivariate generating function of spanning trees of $G$:

$$T_k G(x, y) = \sum_{T \subseteq G} x^{ia(T)} y^{ea(T)} \in \mathbb{N}[x, y],$$

where $ia(T)$, and $ea(T)$ are certain statistics of a spanning tree $T$, called internal and external activities.

In this paper we will be entirely focused on the specialization $T_k G(1, q)$, which is of a special interest as it appears naturally in many different contexts. Gessel, and Gessel with Sagan [Ges95, GS96] interpreted $T_k G(1, q)$ as the generating function of spanning trees of $G$ with respect to the so-called $\kappa$-inversion statistic, which is more natural and simpler than the external activity. We label vertices of $G$ in an arbitrary way by consecutive nonnegative integers. We also set for every pair of distinct vertices of $G$ an arbitrary linear order on the set of edges linking this pair of vertices and for any edge $e \in E(G)$ we define $s(e)$ as the number of edges in $G$ strictly greater then $e$. We extend this definition to any subgraph $H \subseteq G$ by setting $s(H) := \sum_{e \in H} s(e)$. We recall that an inversion $i > j$ in a tree $T \subseteq G$ forms a $\kappa$-inversion if the parent of $i$ is adjacent to $j$ in the graph $G$. It was shown by Gessel and Sagan [GS96] that

$$T_k G(1, q) = q^{\text{number of loops in } G} \sum_{T \subseteq G} q^{\kappa(T) + s(T)},$$

where we sum over all spanning trees of $G$, and $\kappa(T)$ is given by (6). It is easy to show that above formula can be rewritten in the form (5), thus

$$I_G(q) = T_k G(1, q).$$
In particular, the polynomial \( I_G(q) \) depends only on the structure of \( G \) and is invariant under permuting the labels of the vertices. When \( G \) is a graph with no multiple edges nor loops, Gessel [Ges95] found that

\[
I_G(q) = \sum_{T \subseteq \tilde{G}} \prod_{w \in V \setminus \{v\}} [\delta_T(w)]_q,
\]

where \( v \) is the root of \( G \), \( \delta_T(w) \) is the number of descendants of \( w \) (including \( w \)) adjacent to the parent of \( w \) in \( G \). In fact, the same argument as used by Gessel allows us to extend his formula to the general case of graphs (with multiple edges and loops), which will be useful for us later. We recall that a graph \( \tilde{G} \) is obtained from \( G \) by replacing all multiple edges by single ones.

**Proposition 2.1.** Let \( G \) be a graph. Then

\[
I_G(q) = \sum_{T \subseteq \tilde{G}} \prod_{w \in V \setminus \{v\}} [\delta_T(w)]_q,
\]

with

\[
\delta_T(w) = \sum_i e_{i, \text{parent}(w)}(G),
\]

where we sum over all descendants of \( w \) (including \( w \)), and \( v \) is the root of \( G \).

**Proof.** Using (8) we obtain the formula for the specialization

\[
c_G(q) := q^{\#V-1} \text{Tutte}_G(1, q + 1) = \sum_{H \subseteq G} q^{\#E(H)},
\]

where we sum over all connected sub-graphs of \( G \). Let \( U \subset V \) be a non-empty subset of vertices of \( G \), and for \( w \in V \) we define \( d_G(U, w) \) as the number of edges in \( G \) connecting \( w \) with some vertex from \( U \). Note that erasing a vertex \( w \) from the connected graph \( H \) splits this graph into a collection of connected sub-graphs \( H_1, \ldots, H_l \) with the corresponding sets of their vertices \( V_1, \ldots, V_l \). Then \( \{V_1, \ldots, V_l\} \in \mathcal{P}(V \setminus \{w\}) \) and for all \( 1 \leq i \leq l \) there is at least one edge linking \( w \) with \( V_i \). This leads to the following recursion

\[
c_G(q) = q^{\#V} \sum_{\pi \in \mathcal{P}(V \setminus \{w\})} \prod_{B \in \pi} ((q + 1)^{d_G(B, w)} - 1) \ c_B(q),
\]

which can be rewritten as

\[
\text{Tutte}_G(1, q) = q^{\#V} \sum_{\pi \in \mathcal{P}(V \setminus \{w\})} \prod_{B \in \pi} [d_G(B, w)]_q \ \text{Tutte}_B(1, q)
\]

and it holds true for any vertex \( w \in V \). It is therefore enough to show that the right hand side of (10) satisfies the same recursion for \( w = v \) being the root (which implies that also for any other vertex).

Let \( T \) be a spanning tree of \( \tilde{G} \) with \( \kappa(T) = 0 \). If we delete its root \( v \), we obtain a collection of trees \( T_1, \ldots, T_l \) with the corresponding sets of their vertices \( V_1, \ldots, V_l \) and their roots \( v_1, \ldots, v_l \). Then, for each \( 1 \leq i \leq l \) the graph \( T_i \) is a spanning tree of \( G|_{V_i} \) and \( \kappa(T_i) = 0 \).
Thus, for any such a tree $T$ one has
\[
q^{\text{number of loops in } G} \prod_{w \in V \setminus \{v\}} [\delta_T(w)]_q
\]
\[
= q^{\text{number of loops in } v} \prod_{1 \leq i \leq l} \left( \sum_{w \in V_i} e_{v,w}(G) \right)_q q^{\text{number of loops in } G_i} \prod_{w \in V_i \setminus \{v_i\}} [\delta_{T_i}(w)]_q.
\]
Since $\sum_{w \in V_i} e_{v,w}(G) = d_G(V_i, v)$, the right hand side of (10) satisfies recursion (12), which finishes the proof of Proposition 2.1. □

We recall that a rooted tree $T$ is increasing if it contains no inversions.

**Corollary 2.2.** Let $a_2, \ldots, a_r$ be positive integers, and let $G_{a_2, \ldots, a_r} := (V, E)$ be a graph without loops such that $e_{i,j}(G) = a_{\max(i,j)}$ for each $i \neq j \in V = [r]$. Then
\[
I_{G_{a_2, \ldots, a_r}}(q) = P_{a_2, \ldots, a_r}(q) := \sum_{T \text{ increasing tree on } [r]} \prod_{2 \leq i \leq r} [\delta_T(i)]_q,
\]
and $\delta_T(i) := \sum_j a_j$, where $j$ ranges over descendants of $i$ (including $i$ itself).

**Proof.** Proposition 2.1 asserts the following formula
\[
I_{G_{a_2, \ldots, a_r}}(q) = \sum_{T \subset G_{a_2, \ldots, a_r}} \prod_{\kappa(T) = 0} [\delta_T(i)]_q,
\]
where $\delta_T(i) = \sum_j a_{\max(\text{parent}(i), j)}$. Note that $G_{a_2, \ldots, a_r}$ is the complete graph $K_r$, therefore its spanning trees $T$ with $\kappa(T) = 0$ are precisely increasing trees. Thus, for any descendant of $i$ one has $\max(\text{parent}(i), j) = j$, which finishes the proof. □

**Remark.** In this section we assumed that a graph $G$ is connected. Typically when $G$ is not connected the corresponding Tutte polynomial is defined as the product of Tutte polynomials of each connected component. However, for our purposes we extend the definition of the Tutte polynomial to non-connected graphs by setting its value to 0, which agrees with the idea that this is a weighted generating function of spanning trees of $G$.

We finish this section by an important lemma which links Tutte polynomials with cumulants. This lemma can be also found in [JV13, Proposition 4.1], but our proof differs from the one of Josuat-Verges.

**Lemma 2.3.** Let $G = (V, E)$ be a graph. Then
\[
\text{Tutte}_G(1, q) = (q - 1)^{1 - \#V} \sum_{\pi \in \mathcal{P}(V)} (-1)^{\#\pi - 1}(\#\pi - 1)! \prod_{B \in \pi} q^{\#E|_B},
\]
where $E|_B$ denotes the subset of $E$ consisting of the edges with both endpoints from the set $B \subset V$.

**Proof.** Let $G$ be a graph (possibly disconnected), and we define two generating functions
\[
n_{CG}(q) = \sum_{H \subset G} q^{\#E(H)},
\]
where we sum over all (possibly disconnected) sub-graphs of \( G \) and
\[
c_G(q) = \sum_{H \subset G} q^{\# E(H)},
\]
where we sum over all connected sub-graphs of \( G \). Then, clearly
\[
nc_G(q) = \sum_{\pi \in \mathcal{P}(V)} \prod_{B \in \pi} c_{G|_B}(q).
\]
Thus, the Möbius inversion formula ((2)–(3)) implies that
\[
c_G(q) = \sum_{\pi \in \mathcal{P}(V)} (-1)^{\# \pi - 1}(\# \pi - 1)! \prod_{B \in \pi} nc_{G|_B}(q).
\]
Plugging \( nc_{G|_B}(q) = (1 + q)^{\# E|_B} \) and (11) into the above equality yields the desired result. \( \square \)

2.2. 2.2. \( G \)-parking functions and the abelian sandpile model. The polynomial \( \mathcal{I}_G(q) \) is also a generating function of two other objects of interest: \( G \)-parking functions, and recurrent configurations in an abelian sandpile model on \( G \).

Let \( G = (V, E) \) be a graph with the set of vertices \( V = [r] \), where \( r \geq 1 \) is a positive integer and we denote the root of \( G \) by \( v \in [r] \). For any \( i \in U \subset [r] \setminus \{v\} \) we define the outdegree \( \text{outdeg}_U(i) \) of a vertex \( i \) as the number of edges in \( G \) linking \( i \) with some vertex \( j \notin U \). We call a function \( f : [r] \setminus \{v\} \to \mathbb{N} \) a \( G \)-parking function if for any nonempty subset \( U \subset [r] \setminus \{v\} \) there exists \( i \in U \) such that \( f(i) < \text{outdeg}_U(i) \). For example, when \( G = K_r \), the complete graph on \( [r] \), then the set of \( G \)-parking functions is precisely the set of parking functions.

Postnikov and Shapiro noticed that \( G \)-parking functions are directly related to recurrent configurations in the abelian sandpile model for \( G \), which is a model where we are trying to distribute chips among vertices of our graph. A function \( u : [r] \setminus \{v\} \to \mathbb{N} \) giving the number of chips placed in vertices of \( G \) different from the root is called a configuration. We say that a vertex \( i \in [r] \setminus \{v\} \) is unstable if \( u(i) \geq \text{deg}(i) \) — if this is the case, this vertex can topple by sending chips to adjacent vertices one along each incident edge. We say that a configuration is stable if all the vertices \( i \in [r] \setminus \{v\} \) except the root are stable. For the root we set \( u(v) = -\sum_{i \in [r] \setminus \{v\}} u(i) \), and the root can always topple. Finally, we say that a configuration \( u \) is recurrent if there exists a nontrivial configuration \( u' \neq 0 \) such that \( u \) can be obtained from \( u + u' \) by a sequence of topplings. Postnikov and Shapiro noticed that a configuration \( u \) is recurrent if and only if \( f : [r] \setminus \{v\} \to \mathbb{N} \) defined by \( f(i) := \text{deg}(i) - u(i) - 1 \) is a \( G \)-parking function. We define a weight of a \( G \)-parking function \( f \):
\[
\text{wt}(f) := \# E - (r - 1) - \sum_{i \in [r] \setminus \{v\}} f(i),
\]
and we define a \( q \)-generating function of \( G \)-parking functions \( P_G(q) := \sum_f q^{\text{wt}(f)} \) with respect to their weights. We can also interpret \( P_G(q) \) as the generating function of recurrent configurations on \( G \) with respect to their level, where
\[
\text{level}(u) := \sum_{i \in [r] \setminus \{v\}} u(i) + \text{deg}(0) - \# E.
\]

Merino López proved [ML97] that \( P_G(q) = \text{Tutte}_G(1, q) = \mathcal{I}_G(q) \), therefore we have two additional interpretations of the \( G \)-inversion polynomial \( \mathcal{I}(q) \).
3. Preliminaries on Symmetric Functions and Young Diagrams

3.1. Partitions and Young diagrams. We call $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_l)$ a composition of $n$ if it is a sequence of nonnegative integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$ and $\lambda_l > 0$. If $\lambda$ is a weakly decreasing sequence we call it a partition of $n$. Then $n$ is called the size of $\lambda$ and $l$ is its length. We use the notation $\lambda \vdash n$, or $|\lambda| = n$ to indicate its size, and $\ell(\lambda) = l$ for its length.

There exists a canonical involution on the set of partitions which associates with a partition $\lambda$ its conjugate partition $\lambda^t$. By definition, the $j$-th part $\lambda^t_j$ of the conjugate partition is the number of positive integers $i$ such that $\lambda_i \geq j$. We define the partial order called dominance order on the set of partitions of the same size as follows:

$$\lambda \geq \mu \iff \sum_{1 \leq i \leq j} \lambda_i \geq \sum_{1 \leq i \leq j} \mu_i$$

for any positive integer $j$.

A partition $\lambda$ is identified with some geometric object, called Young diagram, defined by:

$$\lambda = \{(i, j) : 1 \leq i \leq \lambda_j, 1 \leq j \leq \ell(\lambda)\}.$$

For any box $\Box := (i, j) \in \lambda$ from a Young diagram we define its arm-length by $a_\lambda(\Box) := \lambda_j - i$, its coarm-length by $a'_\lambda(\Box) := i - 1$, its leg-length by $\ell_\lambda(\Box) := \lambda^t_i - j$ and its coleg-length by $\ell'_\lambda(\Box) := j - 1$ (the same definitions as in [Mac95, Chapter I]), see Fig. 2.

3.2. Macdonald polynomials and plethysm. Let $p_i(x)$ be the power-sum symmetric function, that is

$$p_i(x) := \sum_{j \geq 1} x_j^i.$$

For any formal power series $A$ in the indeterminates $q, t, x$, we define the plethystic substitution $p_i[A]$ as the result of substituting $a^i$ for each indeterminate $a$ appearing in $A$. We extend this definition to any symmetric function $f \in \Lambda$ by expanding it in the power-sum basis, and then applying the plethystic substitution as above, i.e.

$$f[A] := \sum_\lambda c_\lambda p_\lambda[A],$$

where $f(x) = \sum_\lambda c_\lambda p_\lambda(x)$ and $p_\lambda[A] := \prod_i p_{\lambda_i}[A]$. 

Figure 2. Arm-, coarm-, leg- and coleg-lengths of a box in a Young diagram.
Similarly as above, we define a symmetric function \( \omega p_\lambda(x) := (-1)^{|\lambda| + \ell(\lambda)} p_\lambda(x) \) and we extend the action of \( \omega \) on \( \Lambda \) by linearity. We make a convention that a bolded capital letter denotes the sum of countably many indeterminates indexed by positive integers, for example \( X := x_1 + x_2 + \cdots \). Note that if \( f \) is a homogeneous symmetric function of degree \( n \) then
\[
(14) \quad f[-X] = (-1)^n \omega f(x).
\]

There exists a scalar product on \( \Lambda \), called the *Hall scalar product* which is defined on the Schur basis \( \{ s_\lambda \}_\lambda \) by making it the orthonormal basis.

It turned out that there exists a unique family \( \{ \tilde{H}_\mu(x; q, t) \}_\mu \) of the symmetric functions, indexed by partitions, which fulfills the following conditions:

(C1) \( \tilde{H}_\mu[X(q - 1); q, t] \in \mathbb{Q}(q, t) \{ s_\lambda \}_{\lambda \leq \mu^t} \),

(C2) \( \tilde{H}_\mu[X(t - 1); q, t] \in \mathbb{Q}(q, t) \{ s_\lambda \}_{\lambda \leq \mu} \),

(C3) \( \langle \tilde{H}_\mu, s_{(\mu)} \rangle = 1 \).

The elements of the above family are called the *Macdonald polynomials*, and their characterization by conditions (C1)–(C3) is equivalent to the characterization proved by Macdonald [Mac95] (Macdonald used a different normalization; for the proof of this equivalence see [Hai99, Proposition 2.6]).

### 3.3. Fillings of Young diagrams.

For any partition \( \lambda \vdash n \) let \( \sigma : \lambda \rightarrow \mathbb{N}_+ \) be a filling of the boxes of the diagram \( \lambda \) by positive integers. A *descent* of \( \sigma \) is a pair of entries \( \sigma(\square) > \sigma(\square') \) such that \( \square \) lies immediately above \( \square' \), that is \( \square' = (i, j) \) and \( \square = (i, j + 1) \) for some positive integers \( i, j \). We define the *set of descents* as follows:

\[
\text{Des}(\sigma) := \{ \square \in \lambda : \sigma(\square) > \sigma(\square') \text{ is a descent} \}.
\]

The *major index* \( \text{maj}(\sigma) \) of a filling \( \sigma \) is defined as:
\[
(15) \quad \text{maj}(\sigma) := \sum_{\square \in \text{Des}(\sigma)} (\ell_\lambda(\square) + 1).
\]

The second statistic that is of great importance in this paper is a certain generalization of inversions in a permutation. First, we say that two boxes \( \square, \square' \in \lambda \) *attack each other* if either

- they are in the same row: \( \square = (i, j), \square' = (k, j) \), or;
- they are in consecutive rows, with the box in the upper row strictly to the right of the one in the lower row: \( \square = (i, j + 1), \square' = (k, j) \), where \( i > k \).
The reading order is the linear ordering of the entries of \( \lambda \) given by reading them row by row, top to bottom, and left to right within each row. We associate to a filling \( \sigma \) its reading word \( w_\sigma \) by reading its entries in the reading order. An inversion of \( \sigma \) is a pair of entries \( \sigma(\square) > \sigma(\square') \), where \( \square, \square' \) attack each other, and \( \square \) precedes \( \square' \) in the reading order.

We say that the ordered triple of boxes \( \square_1, \square_2, \square_3 \) is counterclockwise increasing, if one of the following conditions holds true:

- \( \sigma(\square_1) \leq \sigma(\square_3) < \sigma(\square_2) \), or
- \( \sigma(\square_3) < \sigma(\square_2) < \sigma(\square_1) \), or
- \( \sigma(\square_2) < \sigma(\square_1) \leq \sigma(\square_3) \).

We define the inversion triple as a pair of boxes \( (\square_1, \square_2) \), where \( \square_1 \) is a box lying in the same row as \( \square_2 \) to its left and such that a triple \( \square_1, \square_2, \square_3 \) is counterclockwise increasing, where \( \square_3 \) is a box lying directly below \( \square_1 \). Here, the convention is that for \( \square_1, \square_2 \) lying in the first row \( \sigma(\square_3) < \min_{\square \in \lambda} \sigma(\square) \). The set of inversion triples of \( \sigma \) is denoted by \( \text{InvT}(\sigma) \).

Fig. 3 presents an example of above defined objects.

**Remark.** Note that an inversion triple \( (\square_1, \square_2) \) is defined a priori as a pair of boxes, not as a triple. However, this pair uniquely determines a counterclockwise triple from the definition, and we decided to pick a name triple to avoid a confusion with an inversion \( \sigma(\square) > \sigma(\square') \), which is also a pair (of entries).

We define

\[
\text{inv}(\sigma) := \# \text{Inv}(\sigma) - \sum_{\square \in \text{Des}(\sigma)} a_\lambda(\square) = \# \text{InvT}(\sigma),
\]

where the second equality was shown in [HHL05a].

It turned out that the statistics \( \text{maj} \), and \( \text{inv} \) can be used to describe the combinatorics of the Macdonald polynomials, by the following explicit combinatorial formula, which from now on we treat as the definition of Macdonald polynomials:

**Theorem 3.1.** [HHL05a]

\[
\tilde{H}_\lambda(x; q, t) = \sum_{\sigma: \lambda \to \mathbb{N}_+} q^{\text{inv}(\sigma)} t^{\text{maj}(\sigma)} x^\sigma.
\]

Finally, we say that a filling \( \sigma : \lambda \to \mathbb{N}_+ \) is standard if the set of entries corresponds to the set \( [\lambda] \). Note that standard fillings of \( \lambda \) are in a natural bijection with permutations from \( \mathfrak{S}_n \) given by the correspondence \( \sigma \leftrightarrow w_\sigma \), where \( w_\sigma \) is the reading word of \( \sigma \).

### 4. Coloring of the Young Diagram \( \lambda^{[\tau]} \) and Graphs
4.1. Coloring of the Young diagram $\lambda^{[r]}$. Let $\lambda^1, \ldots, \lambda^r$ be partitions, and let $\pi \in \mathcal{P}([r])$ be a set partition. For each $B \in \pi$, we are going to color the columns of $\lambda^B$ by numbers $b \in B$ as follows: we observe that the Young diagram $\lambda^B$ can be constructed by sorting the columns of the diagrams $\lambda^{b_1}, \ldots, \lambda^{b_k}$ in decreasing order, where $B = \{b_1, \ldots, b_k\}$ and $b_1 < \cdots < b_k$. When several columns have the same length, we use the total order of $B$, that is we put first the columns of $\lambda^{b_1}$, then those of $\lambda^{b_2}$ and so on. We say that a column of $\lambda^B$ is colored by $b \in B$ if this column is identified with the column of $\lambda^b$ in the above construction. Similarly, we say that a box $\Box \in \lambda^B$ is colored by $b \in B$ if it lies in the column colored by $b$; see Fig. 4 (at the moment, please disregard entries). This gives a way to identify boxes of $\sigma^{\Box}$ from the colored diagram $\lambda^B$. Let $\Box \in \sigma^{\Box}$ be a set partition. For each $\Box \in \pi$, we are ready to construct the graph $G_{\lambda^1, \ldots, \lambda^r}^{[r]} := (V, E)$ as follows: we observe that the Young diagram $\lambda^B$ decomposes as the disjoint sum $\bigoplus_{\lambda \in \pi} \lambda^{B_\lambda}$, that is we put first the columns of $\lambda^B$, then those of $\lambda^{b_2}$ and so on. We say that a column of $\lambda^B$ is colored by $b \in B$ if this column is identified with the column of $\lambda^b$ in the above construction.

This identification leads to a one-to-one correspondence between all the fillings $\sigma^{\Box}$ of $\lambda^{[r]}$ with entries from a given set $A$ and between the sets of fillings $\{\sigma^B : \lambda^B \to A \mid B \in \pi\}$. For a given set of fillings $\{\sigma_b : \lambda^b \to A \mid b \in B\}$ the corresponding filling $\sigma : \lambda^B \to A$ is denoted by $\sigma^{[r]}$, see Fig. 4. In particular, for any filling $\sigma = \sigma^{[r]} : \lambda^{[r]} \to \mathbb{N}_+$ and for any set-partition $\pi \in \mathcal{P}([r])$ we have the following formula:

$$
\text{maj}(\sigma) = \sum_{\Box \in \text{Des}(\sigma)} (\ell_{\lambda^{|\pi|}}(\Box) + 1) = \sum_{B \in \pi} \sum_{\Box \in \text{Des}(\sigma^B)} (\ell_{\lambda^{|B|^\pi}}(\Box) + 1) = \sum_{B \in \pi} \text{maj}(\sigma^B).
$$

Indeed, for any filling $\sigma : \lambda^B \to \mathbb{N}_+$ its descent set $\text{Des}(\sigma)$ decomposes as $\text{Des}(\sigma) = \bigsqcup_{b \in B} \text{Des}(\sigma_b)$, and for any $\Box \in \lambda^B$ colored by $b$ one has $\ell_{\lambda^b}(\Box) = \ell_{\lambda^B}(\Box)$ and $\ell_{\lambda^B}(\Box) = \ell_{\lambda^{|B|^\pi}}(\Box)$.

The statistic $\text{inv}$ is not additive with respect to the operation $\oplus$ but its behaviour is also very simple. Let $\text{InvT}_1(\sigma)$ denote the set of triples $(\Box_1, \Box_2) \in \text{InvT}(\sigma)$ such that $\Box_1, \Box_2$ have the same color, and $\text{InvT}_2(\sigma)$ denotes the set of triples $(\Box_1, \Box_2) \in \text{InvT}(\sigma)$ such that $\Box_1, \Box_2$ have different colors. Then, the set $\text{InvT}(\sigma^{[r]})$ of inversion triples of the colored filling $\sigma^B$ decomposes as the disjoint sum $\text{InvT}(\sigma^B) = \text{InvT}_1(\sigma^B) \sqcup \text{InvT}_2(\sigma^B)$ and

$$
\text{InvT}_1(\sigma^{[r]}) = \bigsqcup_{i \in B} \text{InvT}_1(\sigma^{[r]}), \quad \text{InvT}_2(\sigma^{[r]}) = \bigsqcup_{\{i, j\} \subset B} \text{InvT}_2(\sigma^{[r]}).
$$

Let $\sigma : \lambda^{[r]} \to \mathbb{N}_+$ be a filling. We are ready to construct the graph $G_{\lambda^1, \ldots, \lambda^r}^{[r]} := (V, E)$. For each inversion triple in $\sigma$, we draw an edge linking its boxes, and we color its endpoints by the colors of these boxes from the colored diagram $\lambda^{[r]}$; then we identify all the endpoints of the same color – see Fig. 5 for a construction of $G_{\lambda^1, \ldots, \lambda^r}^{[r]}$ for $r = 3$ and $\sigma, \lambda^1, \lambda^2, \lambda^3$ as in Fig. 4. More formally, $G_{\lambda^1, \ldots, \lambda^r}^{[r]} := (V, E)$ is defined by the following data:

(G1) the set of vertices $V$ is equal to $[r]$;

(G2) $e_{i,j}(G_{\lambda^1, \ldots, \lambda^r}^{[r]}) = \begin{cases} \# \text{InvT}_1(\sigma^{[r]}) & \text{for } i = j, \\ \# \text{InvT}_2(\sigma^{[r]}) & \text{for } i \neq j. \end{cases}$

4.2. Proof of Theorem 1.3. We recall the formula (7) that we need to prove:

$$
\kappa(\lambda^1, \ldots, \lambda^r) = \sum_{\sigma : \lambda^{[r]} \to \mathbb{N}_+} \mathcal{I}_{G_{\lambda^1, \ldots, \lambda^r}^{[r]}(q)} t^{\text{maj}(\sigma)} x^{\sigma}.
$$
Proof of Theorem 1.3. Using definition of Macdonald cumulants ((2) and (4)) and HHL’s formula (17), we rewrite the left hand side of (7) as follows:

\[(q - 1)^{1-r} \sum_{\pi \in P([r])} (-1)^{\#\pi-1}(\#\pi - 1)! \prod_{B \in \pi} \tilde{H}_{\lambda^B}(x; q, t)\]

\[= (q - 1)^{1-r} \sum_{\pi \in P([r])} (-1)^{\#\pi-1}(\#\pi - 1)! \prod_{\sigma: \lambda^r \rightarrow N_+} \sum_{B \in \pi} q^{\text{inv}(\sigma^B)} t^{\text{maj}(\sigma^B)} x^{\sigma^B}\]

\[= \sum_{\sigma: \lambda^r \rightarrow N_+} t^{\text{maj}(\sigma)} \left( (q - 1)^{1-r} \sum_{\pi \in P([r])} (-1)^{\#\pi-1}(\#\pi - 1)! \prod_{B \in \pi} q^{\text{inv}(\sigma^B)} \right) x^{\sigma}.\]

The first equality is a consequence of the one-to-one correspondence between fillings of a given diagram and the sets of fillings of its subdiagrams described in Section 4.1, while the last equality follows from (18). The expression in parentheses is given by the following formula:

\[(q - 1)^{1-\#V} \sum_{\pi \in P(V)} (-1)^{\#\pi-1}(\#\pi - 1)! \prod_{B \in \pi} q^{\#E|B},\]

where \((V, E) = G_{\lambda^1, \ldots, \lambda^r}^x\), which is equal to \(I_{G_{\lambda^1, \ldots, \lambda^r}^x}(q)\) by Lemma 2.3 and (9). Indeed, strictly from the definition ((G1)–(G2)) of \(G_{\lambda^1, \ldots, \lambda^r}^x\), one has \(V = [r]\), and

\[\#E|B = \sum_{i \in B} \# \text{InvT}_1(\sigma^{(i)}) + \sum_{\{i, j\} \subset B} \# \text{InvT}_2(\sigma^{(i, j)}) = \# \text{InvT}_1(\sigma^B) + \# \text{InvT}_2(\sigma^B) = \text{inv}(\sigma^B),\]

where the second equality is given by (19). This concludes the proof. \(\square\)

5. Fundamental quasisymmetric function expansion

In this section we are going to find a formula for Macdonald cumulants in terms of fundamental quasisymmetric functions and their superization by applying the method from [HHL05a, Section 4].
5.1. Fundamental quasisymmetric functions.

Definition 5.1 ([Ges84]). For any nonnegative integer \( n \) and a subset \( D \subset [n-1] \) a fundamental quasisymmetric function \( F_{n,D}(x) \) of degree \( n \) in variables \( x = x_1, x_2, \ldots \) is defined by the formula

\[
F_{n,D}(x) := \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}.
\]

More generally, let \( \mathcal{A} = \mathbb{Z}_+ \cup \mathbb{Z}_- = \{1, \bar{1}, 2, \bar{2}, \ldots \} \) be a “super” alphabet of positive letters \( i \) and negative letters \( \bar{i} \), and let \( (\mathcal{A}, \leq) \) be a total order of \( \mathcal{A} \) which preserves the natural order of positive integers. The “super” quasisymmetric function \( \tilde{F}_{n,D}(x, y) \) in variables \( x = x_1, x_2, \ldots \) and \( y = y_1, y_2, \ldots \) is defined by

\[
\tilde{F}_{n,D}(x, y) := \sum_{i_1 \leq \cdots \leq i_n} z_{i_1} \cdots z_{i_n},
\]

where the indices \( i_1, \ldots, i_n \) run over \( \mathcal{A} \), and we set \( z_i = x_i \) for \( i \in \mathbb{Z}_+ \), and \( z_i = y_i \) for \( \bar{i} \in \mathbb{Z}_- \).

Definition 5.2 ([HHL+05b]). The superization of a symmetric function \( f(x) \) is

\[
\tilde{f}(x, y) := \omega_y f[x + Y]
\]

(the subscript \( Y \) denotes that \( \omega \) acts on \( f[x + Y] = f(x, y) \) considered as a symmetric function of the \( y \) variables only).

Proposition 5.3 ([HHL+05b]). Let \( f(x) \) be a homogeneous symmetric function of degree \( n \), written in terms of fundamental quasisymmetric functions as

\[
f(x) = \sum_D c_D F_{n,D}(x).
\]

Then its ”superization” is given by

\[
\tilde{f}(x, y) = \sum_D c_D \tilde{F}_{n,D}(x, y).
\]

5.2. Fundamental quasisymmetric function expansion and super fillings. For any pair of letters \( x, y \in (\mathcal{A}, \leq) \) and for any sign \( \bullet \in \{+, -\} \) we write \( x \leq_\bullet y \) when \( x < y \), or \( x = y \in \mathbb{Z}_\bullet \). We define \( \geq_\bullet \) similarly.

Given a super alphabet \( \mathcal{A} \), a super filling of \( \mu \) is a function \( \sigma : \mu \to \mathcal{A} \).

We define the set Des(\( \sigma \)) of boxes \( \square \in \mu \) occurring as the upper box in a descent, that is in the pair \( \sigma(\square) \geq_\bullet \sigma(\square') \), where \( \square \) lies directly above \( \square' \) in \( \mu \). The entries of an attacking pair \( (\square, \square') \) such that \( \sigma(\square) \geq_\bullet \sigma(\square') \) and such that \( \square \) precedes \( \square' \) in the reading order form an inversion. The set of positions of all inversions in \( \sigma \) is denoted by \( \text{Inv}(\sigma) \), as before.

We say that the ordered triple of boxes \( \square_1, \square_2, \square_3 \) is counterclockwise increasing, if one of the following conditions holds true:

- \( \sigma(\square_1) \leq_+ \sigma(\square_3) \leq_- \sigma(\square_2) \), or
- \( \sigma(\square_3) \leq_- \sigma(\square_2) \leq_- \sigma(\square_1) \), or
- \( \sigma(\square_2) \leq_- \sigma(\square_1) \leq_+ \sigma(\square_3) \).
We define the \emph{inversion triple} as a pair of boxes \((\square_1, \square_2)\), where \(\square_1\) is a box lying in the same row as \(\square_2\) to its left and such that a triple \(\square_1, \square_2, \square_3\) is \textit{counterclockwise increasing}, where \(\square_3\) is a box lying directly below \(\square_1\).

The statistics \(\text{inv}(\sigma)\) and \(\text{maj}(\sigma)\) are defined in terms of \(\text{Inv}(\sigma)\), \(\text{Des}(\sigma)\) and \(\text{InvT}(\sigma)\) by (15), and (16) as for ordinary fillings. Given a permutation \(\sigma \in S_n\) and an integer \(i < n\), we say that \(i\) is an \textit{inverse descent of} \(\sigma\) if \(i + 1\) lies to the left of \(i\) in \(\sigma\). Let \(\text{iDes}(\sigma)\) denote the set of inverse descents of \(\sigma\).

Let \(\lambda^1, \ldots, \lambda^r\) be partitions and let \(n = |\lambda^1| + \cdots + |\lambda^r|\). Then, using formula (7) and a verbatim argumentation as in [HHL05a, Section 4] we have the expansion

\[
\kappa(\lambda^1, \ldots, \lambda^r)(x) = \sum_{\sigma \in S_n} \mathcal{I}_{G_{\lambda^1, \ldots, \lambda^r}}(q)^{t_{\text{maj}(\sigma)}} F_{n, \text{iDes}(\sigma)}(x),
\]

where we abuse notation by denoting both a permutation by \(\sigma\), and the associated standard filling of \(\lambda^{[r]}\) with the reading word given by \(\sigma\) (see Section 3.3). Thus, by Proposition 5.3

\[
\tilde{\kappa}(\lambda^1, \ldots, \lambda^r)(x, y) = \sum_{\sigma \in S_n} \mathcal{I}_{G_{\lambda^1, \ldots, \lambda^r}}(q)^{t_{\text{maj}(\sigma)}} \tilde{F}_{n, \text{iDes}(\sigma)}(x, y),
\]

or equivalently (which will be more useful in applications)

\[
(21) \quad \tilde{\kappa}(\lambda^1, \ldots, \lambda^r)(x, -y) = \tilde{\kappa}(\lambda^1, \ldots, \lambda^r)[X - Y] = \sum_{\sigma : \lambda^{[r]} \rightarrow A} \mathcal{I}_{G_{\lambda^1, \ldots, \lambda^r}}(q)^{t_{\text{maj}(\sigma)}} z^T,
\]

where we sum over all super fillings of \(\lambda^{[r]}\), and \(z_i = x_i\) for \(i \in \mathbb{Z}_+\), and \(z_i = -y_i\) for \(i \in \mathbb{Z}_-\).

The first equality in (21) follows from (14).

6. Multivariate \(q, t\)-Kostka coefficients for hooks

Let \(\lambda^1, \ldots, \lambda^r\) be partitions, and let \(1 \leq s \leq |\lambda^1| + \cdots + |\lambda^r|\) be a positive integer. For any subset \(\{\square_1, \ldots, \square_s\} \subset \lambda^{[r]}\) of boxes we construct a graph \(G_{\lambda^1, \ldots, \lambda^r}^{\square_1, \ldots, \square_s} : = (V, E)\) as follows: we draw an edge between each \(\square_i\) and each box to its left lying in the same row, and we color its endpoints by the colors of the corresponding boxes in \(\lambda^{[r]}\); then we identify all the endpoints of the same color – see Fig. 6 for a construction of \(G_{\lambda^1, \ldots, \lambda^r}^{\square_1, \ldots, \square_3}\) for \(r = 3\) and \(\lambda^1, \lambda^2, \lambda^3\) as in Fig. 4. In other words

- the set of vertices \(V\) is equal to \([r]\),

Figure 6. The graph \(G_{\lambda^1, \lambda^2, \lambda^3}^{\square_1, \square_2, \square_3}\) for \(\lambda^1, \lambda^2, \lambda^3\) from Fig. 4.
• the number of edges linking vertices \(i, j \in V\) is equal to the number of pairs \((\square_k, \square')\) such that \(\square'\) is in the same row as \(\square_k\) to its left, and the pair \((\square_k, \square')\) is colored by \(\{i, j\}\), where \(1 \leq k \leq s\).

We are ready to prove Conjecture 1.2 in the case of hooks.

**Theorem 6.1.** Let \(\lambda^1, \ldots, \lambda^r\) be partitions with \(|\lambda^{[i]}| = n\). Then, for any nonnegative integer \(s\), the coefficient of \((-u)^s\) in \(\kappa(\lambda^1, \ldots, \lambda^r)[1 - u]\) is equal to

\[
(22) \quad \kappa(\lambda^1, \ldots, \lambda^r)[1 - u]|_{(-u)^s} = \sum_{\{\square_1, \ldots, \square_s\} \subset \lambda^{[r]}} \mathcal{I}_{G_{\lambda^{[1]}, \ldots, \lambda^{[r]}}}(q) t^{\sum_{1 \leq i \leq s} \ell'_{\lambda^{[i]}}(\square_i)}.
\]

Equivalently, the multivariate \(q, t\)-Kostka number \(\widetilde{K}_{(n-s, 1^s)}^{\lambda_1, \ldots, \lambda^r}(q, t)\) is a polynomial in \(q, t\) with nonnegative integer coefficients given by the following formula:

\[
(23) \quad \widetilde{K}_{(n-s, 1^s)}^{\lambda_1, \ldots, \lambda^r}(q, t) = \sum_{\{\square_1, \ldots, \square_s\} \subset \lambda^{[r]} \setminus \{(1, 1)\}} \mathcal{I}_{G_{\lambda^{[1]}, \ldots, \lambda^{[r]}}}(q) t^{\sum_{1 \leq i \leq s} \ell'_{\lambda^{[i]}}(\square_i)}.
\]

**Proof.** Firstly, we sketch an argument of Macdonald showing the equivalence of formulas (22) and (23). Macdonald proved [Mac95, Section VI.8, Example 2] that

\[
s_{\lambda}[1 - u] = \begin{cases} 0 & \text{if } \lambda \text{ is not a hook}, \\ (1 - u)(-u)^s & \text{if } \lambda = (n - s, 1^s).\end{cases}
\]

Thus, for any homogeneous symmetric function

\[
f := \sum_{\lambda \vdash n} c_{\lambda} s_{\lambda},
\]

we obtain the following relation

\[
(24) \quad f[1 - u]|_{(-u)^s} = c_{(n-s, 1^s)} + c_{(n-s+1, 1^{s-1})}.
\]

Note that for any subset \(\{\square_1, \ldots, \square_s\} \subset \lambda^{[r]} \setminus \{(1, 1)\}\) graphs \(G_{\lambda^{[1]}, \ldots, \lambda^{[r]}}^\square\) and \(G_{\lambda^{[1]}, \ldots, \lambda^{[r]}(1, 1)}^\square\) coincide, which is clear from our construction (there are no boxes to the left of \((1, 1)\)). Moreover, \(\ell'_{\lambda^{[i]}}((1, 1)) = 0\), therefore

\[
\sum_{\{\square_1, \ldots, \square_s\} \subset \lambda^{[r]}} \mathcal{I}_{G_{\lambda^{[1]}, \ldots, \lambda^{[r]}}^\square}(q) t^{\sum_{1 \leq i \leq s} \ell'_{\lambda^{[i]}}(\square_i)} = \sum_{\{\square_1, \ldots, \square_s\} \subset \lambda^{[r]} \setminus \{(1, 1)\}} \mathcal{I}_{G_{\lambda^{[1]}, \ldots, \lambda^{[r]}}^\square}(q) t^{\sum_{1 \leq i \leq s} \ell'_{\lambda^{[i]}}(\square_i)} + \sum_{\{\square_1, \ldots, \square_s\} \subset \lambda^{[r]} \setminus \{(1, 1)\}} \mathcal{I}_{G_{\lambda^{[1]}, \ldots, \lambda^{[r]}(1, 1)}^\square}(q) t^{\sum_{1 \leq i \leq s-1} \ell'_{\lambda^{[i]}}(\square_i)},
\]

and we proved that the relation (24) is satisfied by (22) and (23). This proves that formulas (22) and (23) are equivalent.

In order to compute the coefficient of \((-u)^s\) in \(\kappa(\lambda^1, \ldots, \lambda^r)[1 - u]\) we use formula (21), which says that this coefficient is equal to the sum of \(\mathcal{I}_{G_{\lambda^{[1]}, \ldots, \lambda^{[r]}}}(q) t^{\text{maj}_{\lambda^{[i]}}(\sigma)}\) over super fillings \(\sigma\) with \(n - s\) entries equal to 1 and \(s\) entries equal to \(1\). If we use an ordering of \(\mathcal{A}\) in which \(\mathcal{A} = 1 < 1\) then the set of inversion triples consists of the pairs \((\square_1, \square_2)\) such that \(\sigma(\square_2) = 1\) and \(\sigma(\square_1)\) is arbitrary (and, as a part of the definition, \(\square_1\) is in the same row as \(\square_2\) to its left). Moreover, \(\square \in \text{Des}(\sigma)\) if and only if \(\sigma(\square') = 1\) for \(\square'\) strictly below \(\square\). Let \(\sigma\) be
a super filling of $\lambda^{[r]}$ with $s$ boxes $\Box_1, \ldots, \Box_s$ filled by $\bar{1}$ and other boxes filled by $1$. It is obvious from the construction that $G_{\lambda^{[r]}}^{\sigma} = G_{\lambda^{[r]}}^{e_1, \ldots, e_s}$. Indeed, we recall that $G_{\lambda^{[r]}}^\sigma$ is constructed by replacing inversion triples by edges and identifying vertices of the same color. Inversion triples in $\sigma$ are given by $(\Box_i, \Box_j)$, where $1 \leq i \leq s$, and $\Box$ is an arbitrary box in the same row as $\Box_i$ to its left. Thus, the construction of both $G_{\lambda^{[r]}}^\sigma$ and $G_{\lambda^{[r]}}^{e_1, \ldots, e_s}$ coincides. Let $\sigma$ be a super filling of $\lambda^{[r]}$ with $s$ boxes $\Box_1, \ldots, \Box_s$ filled by $\bar{1}$ and other boxes filled by $1$. We associate with it another super filling $\tilde{\sigma}$ of $\lambda^{[r]}$ with $s$ boxes $\Box_1, \ldots, \Box_s$ filled by $\bar{1}$ and other boxes filled by $1$, by setting $(\tilde{x}, \tilde{y}) := (x, (\lambda^{[r]})_x^t + 1 - y) \in \lambda^{[r]}$. In other words $(\tilde{x}, \tilde{y})$ is a box lying in the same column as $(x, y)$, but in the $y$-th row counting from the top of this column. The operation $\tilde{\cdot}$ is an involution on the set of super fillings of $\lambda^{[r]}$ with $s$ entries equal to $\bar{1}$ and $n - s$ entries equal to $1$. Note that $\ell_{\lambda^{[r]}}^{\sigma}(\Box) = \ell_{\lambda^{[r]}}^{\tilde{\sigma}}(\Box)$, and $\maj_{\lambda^{[r]}}^{\sigma}(\sigma) = \sum_{1 \leq i \leq s} \ell_{\lambda^{[r]}}^{\tilde{\sigma}}(\Box_i)$, so

$$\maj_{\lambda^{[r]}}^{\tilde{\sigma}}(\sigma) = \sum_{1 \leq i \leq s} \ell_{\lambda^{[r]}}^{\tilde{\sigma}}(\Box_i) = \sum_{1 \leq i \leq s} \ell_{\lambda^{[r]}}^{\tilde{\sigma}}(\Box_i).$$

Finally, it is straightforward from the construction of $G_{\lambda^{[r]}}^{e_1, \ldots, e_s}$ that for a fixed integer $1 \leq i \leq s$ one can replace the box $\Box_i$ by any other box $\Box'$ from the same column of $\lambda^{[r]}$ and the resulting graph $G_{\lambda^{[r]}}^{e_1, \ldots, e_s}$ is the same as the initial one $G_{\lambda^{[r]}}^{e_1, \ldots, e_s}$. Indeed, for any color $j \in [r]$ the number of boxes colored by $j$ and lying in the same row as $\Box_i$ to its left is the same as the number of boxes colored by $j$ and lying in the same row as $\Box'$ to its left. In particular

$$G_{\lambda^{[r]}}^{e_1, \ldots, e_s} = G_{\lambda^{[r]}}^{e_1, \ldots, e_s}.$$

Concluding, we can compute the coefficient in question as follows

$$\sum_{\sigma: \lambda^{[r]} \to \{1^{n-s}, 1^s\}} \mathcal{I}_{G_{\lambda^{[r]}}^{\sigma}}^\lambda(q) t^{\maj_{\lambda^{[r]}}^{\tilde{\sigma}}(\sigma)} = \sum_{\sigma: \lambda^{[r]} \to \{1^{n-s}, 1^s\}} \mathcal{I}_{G_{\lambda^{[r]}}^{\sigma}}^\lambda(q) t^{\maj_{\lambda^{[r]}}^{\tilde{\sigma}}(\sigma)} = \sum_{\{\Box_1, \ldots, \Box_s\} \subseteq \lambda^{[r]}} \mathcal{I}_{G_{\lambda^{[r]}}^{\sigma}}^\lambda(q) t^{\sum_{1 \leq i \leq s} \ell_{\lambda^{[r]}}^{\tilde{\sigma}}(\Box_i)},$$

which proves (22) and finishes the proof of Theorem 6.1. \(\square\)

7. Fully colored Macdonald polynomials

In this section we focus on the special case of cumulants $\kappa(\lambda^1, \ldots, \lambda^r)$ where all the partitions $\lambda^1, \ldots, \lambda^r$ are columns. These cumulants are directly related to the cumulants we used in our previous work [DF16], where we proved polynomiality part of the $b$-conjecture, and we believe that studying their structure might be an important step toward resolving the $b$-conjecture. Moreover, they seem to carry many remarkable properties and therefore they might be of an independent interest.

**Definition 7.1.** For any partition $\mu$, we define fully colored Macdonald polynomial $\bar{H}_\mu(x; q, t)$ as follows:

$$\bar{H}_\mu(x; q, t) := \kappa(1^{\mu^1}, 1^{\mu^2}, \ldots, 1^{\mu^r}).$$

**Theorem 7.2.** The family of fully colored Macdonald polynomials $\bar{H}_\mu(x; q, t)$ is a linear basis of the algebra $\Lambda$ of symmetric functions.
In order to prove above theorem we find an explicit combinatorial formula for some plethystic substitution in the cumulants of Macdonald polynomials.

Let \( \sigma : \mu \to A \) be a super filling of \( \mu \). We say that it is compatible with \( \mu \) if \(|\sigma(x, y)| \geq y \) for all \((x, y) \in \mu\). We also denote by \( m(\sigma) \) and \( p(\sigma) \), respectively, the number of negative and positive, respectively, entries in \( \mu \). We fix the following ordering of \( A \):

\[
1 < 2 < \cdots < 2 < 1,
\]

and we set \( x^{[\sigma]} := \prod_{\square \in \mu} x^{[\sigma(\square)]} \).

**Lemma 7.3.** For any positive integer \( r \) and partitions \( \lambda^1, \ldots, \lambda^r \) we have

\[
\kappa(\lambda^1, \ldots, \lambda^r)[X(t-1); q, t] = \sum_{\sigma; \lambda^\alpha} \mathcal{I}_{G^\alpha_{\lambda^1, \ldots, \lambda^r}}(q) (-1)^{m(\sigma)} \qquad \text{for some } \sigma \text{ compatible with } \lambda^r.
\]

**Proof.** The proof of [HHL05a, Lemma 5.2] which corresponds to the case \( r = 1 \) works without any changes in the general case, so we only recall the main argument. Using (21) we get the formula

\[
\kappa(\lambda^1, \ldots, \lambda^r)[X(t-1); q, t] = \sum_{\sigma; \lambda^\alpha} \mathcal{I}_{G^\alpha_{\lambda^1, \ldots, \lambda^r}}(q) (-1)^{m(\sigma)} \qquad \text{for some } \sigma \text{ compatible with } \lambda^r.
\]

Therefore, it is enough to show that there exists an involution \( \phi \) on the set of super fillings \( \sigma : \lambda^r \to A \) which fixes super fillings compatible with \( \mu \), and for other super fillings preserves \( G^\sigma_{\lambda^1, \ldots, \lambda^r} \) and \( p(\sigma) + \text{maj}(\sigma) \) but increases/decreases \( m(\sigma) \) by one. Let \( \sigma \) be a super filling not compatible with \( \lambda^r \), and let \( a \) be the smallest integer such that \( a = |\sigma(x, y)| < y \) for some \((x, y) \in \lambda^r\). Let \( \square \) be the first box in the reading order with \(|\sigma(\square)| = a \). We define

\[
\phi(\sigma)(\square) = \begin{cases} 
\sigma(\square) & \text{for } \square \neq \square', \\
\sigma(\square) & \text{for } \square = \square'.
\end{cases}
\]

It was shown in [HHL05a, Proof of Lemma 5.2] that

\[
p(\sigma) + \text{maj}(\sigma) = p(\phi(\sigma)) + \text{maj}(\phi(\sigma)),
\]

and

\[
\text{InvT}(\sigma) = \text{InvT}(\phi(\sigma)).
\]

It implies that \( G^\sigma_{\lambda^1, \ldots, \lambda^r} = G^{\phi(\sigma)}_{\lambda^1, \ldots, \lambda^r} \), which finishes the proof.

**Proof of Theorem 7.2.** Observe that for any partitions \( \lambda^1, \ldots, \lambda^r \) one has

\[
\kappa(\lambda^1, \ldots, \lambda^r)[X(t-1)] \in \mathbb{Q}(q, t)\{s_\lambda\}_{\lambda \subseteq \Delta^{|r|}};
\]

which is an immediate corollary from (C1) and (C2). Indeed, it is enough to use definition of Macdonald cumulants ((2) and (4)), and a well-known property of Schur functions: \( s_\mu s_\nu \in \mathbb{Z}\{s_\lambda\}_{\mu \cup \nu \subseteq \lambda \subseteq \mu \cup \nu} \) (alternatively, it also follows from (25)).

In particular, \( \tilde{H}_\mu[X(t-1); q, t] \in \mathbb{Q}(q, t)\{s_\lambda\}_{\lambda \subseteq \mu} \) and it is enough to show that

\[
[s_\mu] \tilde{H}_\mu[X(t-1); q, t] \neq 0.
\]

We claim that

\[
[s_\mu] \tilde{H}_\mu[X(t-1); q, t] = (1)^{|\mu|}P_{\mu_2, \mu_3, \ldots, \mu_{\mu_1}}(q) + O(t),
\]
where $P_{\mu_2, \mu_3, \ldots, \mu_{t_1}}$ is given by (13). Indeed, using (25) we end up with the following expansion
\[
[s_\mu] \tilde{H}_\mu [X(t-1); q, t] = (-1)^{|\mu|} \mathcal{I}_{G^\sigma_{1^{\mu_2}, 1^{\mu_3}, \ldots, 1^{\mu_{t_1}}}} (q) + O(t),
\]
where $\sigma$ is the unique super filling compatible with $\mu$ with entries $\{1^{\mu_1}, \ldots, \ell(\mu)^{\mu(t)}\}$. This filling is given by the explicit formula:
\[
\sigma(i, j) := \overline{j}
\]
for any box $(i, j) \in \mu$. Thus, each pair of boxes lying in the same row belongs to $\text{Inv}_T(\sigma)$, since $\overline{\ell} \leq -\overline{\ell} \leq +\overline{\ell} - 1$. This implies that the number of edges $e_{i, j} \left( G^\sigma_{1^{\mu_2}, 1^{\mu_3}, \ldots, 1^{\mu_{t_1}}} \right)$ linking vertices $i \neq j$ is equal to the number $\mu^t_{\text{max}(i, j)}$, and (26) follows from Corollary 2.2. In particular $[s_\mu] \tilde{H}_\mu [X(t-1); q, t] \neq 0$, which finishes the proof.

Remark. Note that $[s_1]|\mu| \tilde{H}_\mu = t^{n(\mu)} P_{\mu_2, \mu_3, \ldots, \mu_{t_1}} (q)$, where $n(\mu) := \sum_{i \geq 1} (i - 1) \mu_i$. This follows from Theorem 6.1 and from the above analysis of the polynomial $\mathcal{I}_{G^\sigma_{1^{\mu_2}, 1^{\mu_3}, \ldots, 1^{\mu_{t_1}}}} (q)$. A polynomial $P_{\alpha_1, \ldots, \alpha_r} (q)$ already appeared in the literature in the context of Macdonald polynomials, and it would be interesting to find a connection between these results and our work – see Section 8 for more details.

**Proposition 7.4.** For any partition $\mu$ the fully colored Macdonald polynomial $\tilde{H}_\mu (x; -1, t)$ is $t$–positive in the monomial basis, and in the fundamental quasisymmetric basis.

**Proof.** This is straightforward from formulas (7) and (20) and from Proposition 2.1. □

8. **Open problems**

We decided to conclude the paper by mentioning several possible directions for the future research that arise naturally from Theorem 1.3 which actually raises more questions than it answers.

8.1. **Schur positivity, $G$-parking functions and geometry of Hilbert schemes.** The first topic is related to the standard technique of proving Schur–positivity of a given function $f$ by constructing a certain $\mathfrak{S}_n$–module $V$ and interpreting $f$ as the Frobenius characteristic of $V$. We are going to quickly review this technique in the following. Let $V$ be a $\mathfrak{S}_n$-module and we decompose it as its direct sum of its irreducible submodules:
\[
V = \bigoplus_\lambda V^\lambda. \]
Then, we define the Frobenius characteristic of $V$ as
\[
\text{Fr}(V) := \sum_\lambda c_\lambda s_\lambda.
\]
If, additionally, $V$ is a $k$-graded $\mathfrak{S}_n$–module, that is
\[
V = \bigoplus_{i_1, \ldots, i_k \geq 0} V^{i_1, \ldots, i_k},
\]
and each summand in this decomposition is a $G_n$–module, then
\[
\text{Fr}(V) := \sum_{i_1, \ldots, i_k \geq 0} t_1^{i_1} \cdots t_k^{i_k} \text{Fr}(V_{i_1, \ldots, i_k}).
\]
Equivalently, let $\mu \vdash n$ be a partition of $n$, and let $V^{i_1, \ldots, i_k}_{\mu}$ denote the subspace of $V^{i_1, \ldots, i_k}$ consisting of fix-points of the action of the subgroup
\[
G_{\mu} := G_{\{1, \ldots, \mu_1\}} \times G_{\{\mu_1+1, \ldots, \mu_1+\mu_2\}} \times \cdots \times G_{\{\mu_1+\cdots+\mu_{l-1}+1, \ldots, \mu_l\}} < G_n,
\]
where $\ell(\mu) = l$. Then
\[
(27) \quad \text{Fr}(V) = \sum_{i_1, \ldots, i_k \geq 0} t_1^{i_1} \cdots t_k^{i_k} \sum_{\mu \vdash n} \dim(V^{i_1, \ldots, i_k}_{\mu}) \ m_\mu.
\]

The celebrated result of Haiman [Hai01] proves that the Macdonald polynomial $\tilde{H}_\mu(x; q, t)$ can be interpreted as the Frobenius characteristic of a certain $G_n$–module $D_\mu$ (associated with a partition $\mu \vdash n$), which carries a natural structure of a bigraded module. Thanks to our explicit, combinatorial formula (7) it is natural to use (27) and try to prove Schur–positivity of Macdonald cumulant $\kappa(\lambda_1, \ldots, \lambda_r)$ by constructing a bigraded $G_n$–module $D^{\lambda_1, \ldots, \lambda_r}$ such that
\[
\text{Hilb}_{q,t}(D^{\lambda_1, \ldots, \lambda_r}) = \sum_{\sigma: \lambda^{(r)} \vdash \{\mu_1, 2\mu_2, \ldots\}} I_G^{(r)}(q) t^{\text{maj}_{\lambda^{(r)}}(\sigma)},
\]

where
\[
\text{Hilb}_{q,t}(D) = \sum_{i,j} q^i t^j \dim(D^{i,j})
\]
is the Hilbert series of a bigraded vector space $D$ with respect to its gradation. We mention here that Postnikov and Shapiro [PS04] introduced $G$-parking functions in order to construct certain graded vector spaces, whose Hilbert series are given by $I_G(q)$. Is it possible to merge ideas of Haiman, and Postnikov with Shapiro to construct a module $D^{\lambda_1, \ldots, \lambda_r}$ as in question?

In fact, Haiman’s representation-theoretical interpretation of Macdonald polynomials was a corollary of another result of him – Haiman showed that a certain geometric object, called isospectral Hilbert scheme has “nice” geometric properties, that is it is normal, Cohen–Macaulay, and Gorenstein (see [Hai02], which explains all these terms and much more in an available way for non-experts). What kind of geometric properties (if any) of isospectral Hilbert schemes or related geometric objects assure Schur–positivity of Macdonald cumulants? The other way round – does Schur–positivity of Macdonald cumulants imply that some geometric object has nice properties? One can ask a weaker question by using various specializations of Macdonald cumulants carrying geometric interpretations.

8.2. $G$-inversion polynomials and Macdonald polynomials. We recall Section 7 which points that the coefficient $[s_1^\mu]\tilde{H}_\mu(x; q, t)$ is given by the polynomial $P_{\mu_1^{j_1}, \ldots, \mu_r^{j_r}}(q)$, where
\[
P_{\mu_1^{j_1}, \ldots, \mu_r^{j_r}}(q) := \sum_{T \text{ increasing tree on } [n]} \prod_{2 \leq i \leq n} [\delta_T(i)]_q,
\]
and $\delta_T(i) = \sum_j a_j$ (here $j$ ranges over descendants of $i$ including $i$ itself ). When $a_2 = \cdots = a_n = 1$ this is the inversion polynomial but this also corresponds to the generating function of parking functions with respect to the statistic called area. This function appeared in the context of the Shuffle ex-conjecture [HHL+05b] proved recently by Carlsson and Mellit [CM18].
We define two sets $B_{\mu}(q, t) := \{q^{\alpha(t)}t^{\beta(t)}| \square \in \mu \}$, $T_{\mu}(q, t) := B_{\mu}(q, t) \setminus \{1\}$. Given any symmetric function $f$, we define two operators $\Delta_f$, $\Delta'_f$ acting on the space of symmetric functions by describing their action on the Macdonald basis and extending this by linearity:

$$\Delta_f \tilde{H}_{\mu}(x; q, t) := f(B_{\mu}(q, t)) \cdot \tilde{H}_{\mu}(x; q, t), \quad \Delta'_f \tilde{H}_{\mu}(x; q, t) := f(B_{\mu}(q, t)) \cdot \tilde{H}_{\mu}(x; q, t).$$

The Shuffle ex-conjecture expresses the function $\Delta'_{\nu_{e_n-1}} e_n$ in terms of parking functions with respect to two statistics called area and dinv and the polynomial $P_{t_1, \ldots, t_{n-1}}(q)$ appears as $\langle \Delta'_{\nu_{e_n-1}} e_n, h_1^n \rangle$, where $e_n, h_n$ are elementary and complete symmetric functions, respectively.

The Shuffle ex-conjecture was generalized in two directions. The first generalization is given by the Rational Shuffle ex-conjecture of Bergeron, Garsia, Leven, and Xin [BGSLX16] proved very recently by Mellit [Mel16], and the second one is given by the Delta conjecture of Haglund, Remmel and Wilson [HRW18]. Both conjectures are related to the combinatorics of Tesler matrices and their generalizations, where polynomials $P_{a_2, \ldots, a_n}(q)$ appear naturally, see [AGH+12, Wil17]. Moreover, polynomials $P_{a_2, \ldots, a_n}(q)$ correspond to $(q, t)$–Ehrhart functions of certain flow polytopes [LMM18].

We believe that all these similarities are not coincidental and finding a missing link between Macdonald cumulants and the aforementioned problems would be of great importance.

We leave all these questions wide open for future research.

ACKNOWLEDGMENTS

We would like to thank to the anonymous referees for their valuable comments.

REFERENCES

[AGH+12] D. Armstrong, A. Garsia, J. Haglund, B. Rhoades, and B. Sagan, Combinatorics of Tesler matrices in the theory of parking functions and diagonal harmonics, J. Comb. 3 (2012), no. 3, 451–494. MR 3029443

[BC14] A. Borodin and I. Corwin, Macdonald processes, Probab. Theory Related Fields 158 (2014), no. 1-2, 225–400.

[BG16] A. Bufetov and V. Gorin, Fluctuations of particle systems determined by Schur generating functions, Preprint arXiv:1604.01110 to appear in Adv. Math., 2016.

[BGSLX16] F. Bergeron, A. Garsia, E. Sergel Leven, and G. Xin, Compositional $(km, kn)$-shuffle conjectures, Int. Math. Res. Not. IMRN (2016), no. 14, 4229–4270. MR 3556418

[BJ07] D. R. L. Brown and D. M. Jackson, A rooted map invariant, non-orientability and Jack symmetric functions, J. Combin. Theory Ser. B 97 (2007), no. 3, 430–452. MR 2305897 (2007m:05225)

[Bol98] B. Bollobás, Modern graph theory, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York, 1998. MR 1633290

[CM18] E. Carlsson and A. Mellit, A proof of the shuffle conjecture, J. Amer. Math. Soc. 31 (2018), no. 3, 661–697. MR 3787405

[DNWZ] E. Di Nardo, H. P. Wynn, and P. Zwiernik, Cumulants: Theory, computation and applications, Wiley-Interscience, ISBN: 978-1-119-10313-4, Book in preparation.

[DF16] M. Dolega and V. Féray, Gaussian fluctuations of Young diagrams and structure constants of Jack characters, Duke Math. J. 165 (2016), no. 7, 1193–1282. MR 3498863

[DoH17a] M. Dolega, Strong factorization property of Macdonald polynomials and higher-order Macdonald’s positivity conjecture, J. Algebraic Combin. 46 (2017), no. 1, 135–163. MR 3666415

[DoH17b] M. Dolega, Top degree part in $b$-conjecture for unicellular bipartite maps, Electron. J. Combin 24 (2017), no. 3, Paper 3.24.

[DS18] M. Dolega and P. Śniady, Gaussian fluctuations of Jack-deformed random Young diagrams, Probab. Theory Related Fields (2018). https://doi.org/10.1007/s00440-018-0854-9
[Fér13] V. Féray, Asymptotic behavior of some statistics in Ewens random permutations, Electron. J. Probab. 18 (2013), no. 76, 1–32. 4

[FM12] V. Féray and P.-L. Méliot, Asymptotics of q-plancherel measures, Probab. Theory Related Fields 152 (2012), no. 3-4, 589–624. 4

[Ful97] William Fulton, Young tableaux, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997, With applications to representation theory and geometry. MR 1464693 1

[Ges84] I. M. Gessel, Multivariate P-partitions and inner products of skew Schur functions, Combinatorics and algebra (Boulder, Colo., 1983), Contemp. Math., vol. 34, Amer. Math. Soc., Providence, RI, 1984, pp. 289–317. MR 777705 16

[Ges95] ———, Enumerative applications of a decomposition for graphs and digraphs, Discrete Math. 139 (1995), no. 1-3, 257–271, Formal power series and algebraic combinatorics (Montreal, PQ, 1992). MR 1336842 7, 8

[GH93] A. Garsia and M. Haiman, A graded representation model for Macdonald’s polynomials, Proc. Nat. Acad. Sci. U.S.A. 90 (1993), no. 8, 3607–3610. MR 1214091 2

[GH99] I. P. Goulden and D. M. Jackson, Connection coefficients, matchings, maps and combinatorial conjectures for Jack symmetric functions, Trans. Amer. Math. Soc. 348 (1996), no. 3, 873–892. MR 1325917 (96m:05196) 3

[GS96] I. M. Gessel and B. E. Sagan, The Tutte polynomial of a graph, depth-first search, and simplicial complex partitions, Electron. J. Combin. 3 (1996), no. 2, Research Paper 9, approx. 36 pp. The Foata Festschrift. MR 1392494 7

[Hag08] J. Haglund, The q,t-Catalan numbers and the space of diagonal harmonics, University Lecture Series, vol. 41, American Mathematical Society, Providence, RI, 2008, With an appendix on the combinatorics of Macdonald polynomials. MR 2371044 2

[Hai99] M. Haiman, Macdonald polynomials and geometry, New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97), Math. Sci. Res. Inst. Publ., vol. 38, Cambridge Univ. Press, Cambridge, 1999, pp. 207–254. MR 1731818 12

[Hai01] ———, Hilbert schemes, polygraphs and the Macdonald positivity conjecture, J. Amer. Math. Soc. 14 (2001), no. 4, 941–1006. MR 1839919 2, 22

[Hai02] ———, Notes on Macdonald polynomials and the geometry of Hilbert schemes, Symmetric functions 2001: surveys of developments and perspectives, NATO Sci. Ser. II Math. Phys. Chem., vol. 74, Kluwer Acad. Publ., Dordrecht, 2002, pp. 1–64. MR 2059359 4, 22

[HHL05a] J. Haglund, M. Haiman, and N. Loehr, A combinatorial formula for Macdonald polynomials, J. Amer. Math. Soc. 18 (2005), no. 3, 735–761. MR 2138143 6, 13, 15, 17, 20

[HHL+05b] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov, A combinatorial formula for the character of the diagonal coinvariants, Duke Math. J. 126 (2005), no. 2, 195–232. MR 2115257 16, 22

[HRW18] J. Haglund, J. B. Remmel, and A. T. Wilson, The delta conjecture, Trans. Amer. Math. Soc. 370 (2018), no. 6, 4029–4057. MR 3811519 23

[JV13] M. Josuat-Vergès, Cumulants of the q-semicircular law, Tutte polynomials, and heaps, Canad. J. Math. 65 (2013), no. 4, 865–878. MR 3071084 9

[KT99] A. Knutson and T. Tao, The honeycomb model of $GL_n(\mathbb{C})$ tensor products. I. Proof of the saturation conjecture, J. Amer. Math. Soc. 12 (1999), no. 4, 1055–1090. MR 1671451 2

[KV16] A. L. Kanunnikov and E. A. Vassilieva, On the matchings-Jack conjecture for Jack connection coefficients indexed by two single part partitions, Electron. J. Combin. 23 (2016), no. 1, Paper 1.53, 30. MR 3484758 3

[La 09] M. A. La Croix, The combinatorics of the Jack parameter and the genus series for topological maps, Ph.D. thesis, University of Waterloo, 2009. 3

[LLT96] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, Comm. Math. Phys. 181 (1996), no. 1, 205–263. MR 1410572 2

[LMM18] R. I. Liu, K. Mészáros, and A. H. Morales, Flow polytopes and the space of diagonal harmonics, Canad. J. Math. (2018). https://doi.org/10.4153/CJM-2018-007-3 23

[LPP07] T. Lam, A. Postnikov, and P. Pylyavskyy, Schur positivity and Schur log-concavity, Amer. J. Math. 129 (2007), no. 6, 1611–1622. MR 2369890 2
[LS78] A. Lascoux and M.-P. Schützenberger, *Sur une conjecture de H. O. Foulkes*, C. R. Acad. Sci. Paris Sér. A-B 286 (1978), no. 7, A323–A324. MR 0472993

[Lus81] G. Lusztig, *Green polynomials and singularities of unipotent classes*, Adv. in Math. 42 (1981), no. 2, 169–178. MR 641425

[Mac88] I. G. Macdonald, *A new class of symmetric functions*, Publ. IRMA Strasbourg 372 (1988), 131–171.

[Mac95] , *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications. MR 1354144

[Mel16] A. Mellit, *Toric braids and \((m,n)\)-parking functions*, arXiv:1604.07456, 2016.

[ML97] C. Merino López, *Chip firing and the Tutte polynomial*, Ann. Comb. 1 (1997), no. 3, 253–259. MR 1630779

[PS04] A. Postnikov and B. Shapiro, *Trees, parking functions, syzygies, and deformations of monomial ideals*, Trans. Amer. Math. Soc. 356 (2004), no. 8, 3109–3142. MR 2052943

[Sni06] P. Śniady, *Gaussian fluctuations of characters of symmetric groups and of Young diagrams*, Probab. Theory Related Fields 136 (2006), no. 2, 263–297. MR 2240789

[SW12] J. Shareshian and M. L. Wachs, *Chromatic quasisymmetric functions and Hessenberg varieties*, Configuration spaces, CRM Series, vol. 14, Ed. Norm., Pisa, 2012, pp. 433–460. MR 3203651

[Tut47] W. T. Tutte, *A ring in graph theory*, Proc. Cambridge Philos. Soc. 43 (1947), 26–40. MR 0018406

[Tut54] , *A contribution to the theory of chromatic polynomials*, Canadian J. Math. 6 (1954), 80–91. MR 0061366

[Wil17] A. T. Wilson, *A weighted sum over generalized Tesler matrices*, J. Algebraic Combin. 45 (2017), no. 3, 825–855. MR 3627505

**WYDZIAŁ MATEMATYKI I INFORMATYKI, UNIWERSYTET IM. ADAMA MICKIEWICZA, COLLEGIUM MATHEMATICUM, UJMULTOWSKA 87, 61-614 POZNAN, POLAND.**

**INSTYTUT MATEMATYCZNY, UNIWERSYTET WROCŁAWSKI, PL. GRUNWALDZKI 2/4, 50-384 WROCŁAW, POLAND.**

**E-mail address:** maciej.dolega@amu.edu.pl