An inverse problem of determining the volatility in financial mathematics

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Abstract. The paper investigates an inverse problem of determining the volatility coefficient from observed market prices. Based on the linearization technique the problem is translated into an inverse source problem in the parabolic heat function. Using the optimal control framework, the necessary condition which must be satisfied by the minimizer is deduced, and the uniqueness and stability of the solution are also established from the necessary condition.

1. Introduction

Black and Scholes in 1973 established the option pricing model under the assumptions that drift and volatility were constant, and the underlying stock price \( S \) followed a Geometric Brownian Motion, and satisfied the following stochastic differential equation [1]:

\[
dS = \mu S dt + \sigma S dW,
\]

where \( \mu \) is the expected rate of return, \( \sigma \) is the volatility function that may depend on the level of the underlying stock price \( S \), and \( W \) is the standard Brownian process. As shown by the Black-Scholes theory (see [2]), if the stock price \( S \) satisfies a Geometric Brownian Motion the option price \( V \) satisfies the following Partial Differential equation (PDE):

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - rV = 0.
\]

In the above equation, the quantity \( \sigma(S) \) plays a crucial role in option price, but cannot be directly observed from real financial trading market. In the original Black-Scholes theory, the volatility is assumed to be constant. However, in practice, volatility is not constant over time, which is represented by two typical curves: smile and skew (see [3]).

In this paper, we are interested in the problem of determining the volatility coefficient from market option price with different strikes as following:

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - rV &= 0, \quad (S, t) \in \mathbb{R}^+ \times (0, T), \\
V(0, t) &= 0, \\
V(S, T) &= (S - K)^+ = \max(0, S - K), \quad S \in \mathbb{R}^+.
\end{align*}
\]
where $K$ is the strike price, $S$ is the underlying stock price, $T$ is the time of expiry, $\mu$ and $r$ are the risk-neutral drift and the risk-free interest rate, respectively, and are assumed to be constants. The parameter $\sigma(S)$ is the volatility coefficient to be determined, which satisfies the following condition [4] \[ 0 < \sigma < \sigma(S) < \bar{\sigma}, \quad \sigma(S) \in C^1(\omega), \] for some interval $\omega \subset R^+$, $C^1(\omega)$ is the space function on $\omega$ which are H"older continuous with $0 < \lambda < 1$, here $\sigma$ and $\bar{\sigma}$ are continuous. Moreover, it should be pointed out that the value of option $V$ is a function of various parameters in the contract, and will also depend on properties of the underlying asset itself. Such as the strike price $K$, the time of expiry $T$, the current time $t$, the asset price $S$, the drift $\mu$, the volatility $\sigma$, as well as the risk-free interest rate $r$ (see [2]). We can write the option value as $V(S,t;\sigma,\mu;K,T,r)$, however we will not use all these parameters expect when it is important. For our inverse problem we are going to just use $V(S,t;K,T)$ to denote the option value.

In general, some additional conditions are indispensable for the inverse problem. In this paper, we will use the following extra condition: \[ V(S^*,t^*;K,T) = V^*(K), \quad K \in \omega^*, \] where $S^*$ is market price of the stock at time $t^*$, $0 \leq t^* < T$ is the fixed observation time, and $V^*(K)$ denotes market price of options with different strikes $K$ at a given expiry $T$. We attempt to determine the functions $\sigma$ and $V$ in interval $\omega^* \subset (0, +\infty)$, respectively.

To obtain our results we shall use the derivation of a dual problem, the problem can be reduced to a parabolic equation with new variables: \[
\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{1}{2} \sigma^2 K^2 \frac{\partial^2 V}{\partial K^2} - \mu K \frac{\partial V}{\partial K} + (r - \mu) V, \quad (S,t) \in R^+ \times (0,T), \\
V(0,t) &= e^{-(r-t)S}, \\
V(S,t) &= (S - K)^+ = \max(0, S - K), \quad S \in R^+.
\end{aligned}
\] The equation (1.6) was found by Dupire and rigorously justified in [3].

For given $\sigma=\sigma(K)$, we can directly calculate the option premium $V(\cdot,\cdot;K,T)$, the problem is called the direct problem of option pricing which is well-posed in the sense of Hadamard (see [5]). In this paper, we are interested in the inverse problem, i.e., we shall recover the unknown volatility coefficient using the current market prices. The problem is ill-posed and has been intensively studied in the literature. In [4, 6], the authors introduce the inverse volatility problem with the final observation and show the uniqueness and stability results by using Carleman estimates. They obtain a nonlinear Fredholm integral equation and some numerical examples are also given. Several useful numerical algorithms which recover volatility have been proposed (see [7, 8]), these algorithms are mostly based on regularized least squares fitting. It seems that they require heavy computations, but the convergence is not so fast, which is typical for inverse parabolic problems. Lu and Yi in [9] obtain a Fredholm integral equation from Dupire equation, the local uniqueness and stability of implied volatility are proved. In [10,11], the authors consider the inverse problem of determining the implied volatility $\sigma=\sigma(S)$ using the optimal control framework, the existence and uniqueness of the minimum for the control function are addressed and a new well-posed algorithm is presented. Similar results are obtained in [12], where a new extra condition on the average option premium is assumed to be known. In [13], the authors apply the linearisation technique to convert the problem into an inverse source problem, they obtain the stable numerical solution of the inverse problem using the integral equation method and the Landweber iterations. The authors in [14] consider a problem of calibrating the volatility function using regularization technique and the gradient projection method from given option price data, some numerical experiments are given to show their method is effective. Motivated by their method and results, in this paper, we will investigate the inverse problem of reconstructing...
volatility coefficient. After linearization technique the problem can be translated into a heat conduction function. Based on the optimal control framework, we give some rigorous mathematical analysis about it.

The paper is organized as follows: In Section 2 we make the logarithmic substitution for the problem (1.6) and linearize it around constant volatility, after some change of variables the problem is reduced to a heat conduction function with the source \( f \). The necessary condition which must be satisfied by the minimum is discuss in Section 3. Section 4 proves the local uniqueness and stability of the minimum under some assumptions.

2. Mathematical formulation

We make the logarithmic substitution for the variable:

\[
x = \ln \frac{K}{S^x}, \quad \tau = T - t, \quad U(x, \tau) = V(S^x e^\tau, \tau + t),
\]

and \( a(x) = \sigma(S^x e^\tau) \). Then the function \( U \) satisfies the following inverse parabolic problem with terminal observation:

\[
\begin{cases}
U_\tau - \frac{1}{2} a^2(x)(U_{xx} - U_x) + \mu U_x + (r - \mu)U = 0, \\
U(x, 0) = S^x(1 - e^\mu)^+, \\
U(x, \tau^*) = U^*(x), \quad x \in \omega,
\end{cases}
\]

(2.2)

Here \( \omega \) is the interval \( \omega^* \) in the above logarithmic substitution, and \( \tau^* = T - \tau^* \).

It is difficulty to recovery the volatility of the nonlinear inverse problem (2.2), in order to simplify the problem and linearize it around constant volatility \( \sigma_0 \), we assume

\[
\frac{1}{2} a^2(x) = \frac{1}{2} \sigma_0^2 + f(x),
\]

(2.3)

where \( f(x) \) is a small perturbation of constant \( \sigma_0^2 \) and \( f = 0 \) outside \( \omega \). To obtain the linearized problem, considering the standard estimates of parabolic equation (see [4, 6]), we let \( U = U_0 + V + v \), where \( U_0 \) is the solution of equation (2.2) with \( a(x) = \sigma_0 \), \( V \) is the principal linear term of the perturbed solution \( U \), and satisfies the following equation:

\[
\begin{cases}
\frac{\partial V}{\partial \tau} - \frac{1}{2} \sigma_0^2 \frac{\partial^2 V}{\partial x^2} + \frac{\sigma_0^2}{2} + r \frac{\partial V}{\partial x} + (r - \mu)V = \alpha_0 f, \quad (x, \tau) \in \omega \times [0, \tau^*], \\
V(x, 0) = 0, \\
V(x, \tau^*) = V^*(x), \quad x \in \omega,
\end{cases}
\]

(2.4)

where

\[
\alpha_0(x, \tau) = \frac{\partial^2 U_0}{\partial x^2} - \frac{\partial U_0}{\partial x} = S^x \frac{1}{\sigma_0 \sqrt{2\pi \tau}} e^{-\frac{x^2}{2\sigma_0^2}} + \frac{c x}{\sigma_0^2} + \frac{d x^2}{\sigma_0^4} + \mu - r,
\]

and \( V^* \) is the principal linear part of \( U^* \).

Consider the following substitution \( V = e^c x^d W \), then we can further simplify the problem (2.4) to
\[
\begin{align*}
\frac{\partial W}{\partial \tau} - \frac{1}{2} \sigma_0^2 \frac{\partial^2 W}{\partial x^2} &= \alpha f, \\
W(x, 0) &= 0, \\
W(x, \tau^*) &= W^*(x), \quad x \in \omega,
\end{align*}
\]

(2.5)

where \( \alpha(x, \tau) = S^* \frac{1}{\sigma_0 \sqrt{2\pi \tau}} e^{-\frac{x^2}{2\sigma_0^2}}, \quad W^*(x) = e^{-cx-dx}V^*(x). \)

The new weight function is space independent, it can be regarded as a constant when terminal observation is given by \( \tau = \tau^* \). The above problem is not well-posed either, however, we are able to solve the standardized form in an optimal control framework.

We consider the following optimal control problem \( \mathbf{P} \)

Find \( \tilde{f} \in A \) satisfying

\[
J(\tilde{f}) = \min_{f \in A} J(f),
\]

(2.6)

where

\[
J(f) = \frac{1}{2} \int_{\omega} \left[ W(x, \tau^*; f) - W^*(x) \right]^2 dx + \frac{N}{2} \int_{\omega} |\nabla f|^2 dx,
\]

(2.7)

\[
A = \left\{ f(x) \mid 0 < m \leq f \leq M, \nabla f \in H^1(\omega) \right\}.
\]

(2.8)

Here \( m \) and \( M \) are two given positive constants, \( W(x, \tau; f) \) is the solution of system (2.5) for a given \( f \in A \) and \( N \) is the regularization parameter. From Schauder’s theory for parabolic equations [15], we can know for any \( f \in A \), there exists an unique solution \( W(x, \tau) \), satisfying

\[
W(x, \tau) \in C^{2+\lambda,1+\frac{\lambda}{2}}(\bar{Q}), \quad \| W \|_{C^{2+\lambda,1+\frac{\lambda}{2}}(\bar{Q})} \leq C \| f \|_{C^{2+\lambda,1+\frac{\lambda}{2}}(\bar{O})},
\]

(2.9)

where \( Q = \{(x, \tau) \mid x \in \omega, 0 < \tau < \tau^* \}. \)

3. Necessary condition of optimal control problem

For the solution of the problem (2.6), we derive necessary condition for optimal control.

**Theorem 3.1** Let \( f \in A \) be the solution of the optimal control problem \( \mathbf{P} \), and \( W(x, t) \) is the solution of (2.5) corresponding to this optimal coefficient. Then for any \( h \in A \), the following integral inequality holds:

\[
\int_{\omega} \left[ W(x, \tau^*; f) - W^*(x) \right] \xi(x, \tau^*) dx + N \int_{\omega} \nabla f \cdot \nabla (h - f) dx \geq 0.
\]

(3.1)

Here, \( \xi(x, t) \) satisfies the following equation:

\[
\begin{cases}
\xi_t - \frac{1}{2} \sigma_0^2 \xi_{xx} = \alpha \left( h - f \right), \quad (x, \tau) \in Q, \\
\xi(x, 0) = 0, \quad x \in \omega.
\end{cases}
\]

(3.2)

**Proof.** For any \( h \in A \), \( 0 \leq \delta \leq 1 \), let

\[
f_\delta \equiv (1 - \delta) f + \delta h \in A.
\]

(3.3)

Then we have

\[
J_\delta = J(f_\delta) = \frac{1}{2} \int_{\omega} \left[ W(x, \tau^*; f_\delta) - W^*(x) \right]^2 dx + \frac{N}{2} \int_{\omega} |\nabla f_\delta|^2 dx.
\]

(3.4)
Let $W_\delta$ be the solution of the equation (2.5) with given $f = f_\delta$. Since $f$ is an optimal solution, then we have
\[
\frac{dJ_\delta}{d\delta}\bigg|_{\delta = 0} = \int_\omega \left[ W(x, \tau^*; f) - W^*(x) \right] \frac{\partial W_\delta}{\partial \delta} \bigg|_{\delta = 0} \ dx + N \int_\omega \nabla f \cdot \nabla (h - f) \ dx \geq 0.
\] (3.5)

Let $\tilde{W}_\delta = \frac{\partial W_\delta}{\partial \delta}$, then we have the following equation
\[
\begin{align*}
\frac{\partial}{\partial \tau} (\tilde{W}_\delta) - \frac{1}{2} \sigma_0^2 \frac{\partial^2}{\partial x^2} (\tilde{W}_\delta) &= \alpha (h - f), \\
\tilde{W}_\delta (x, 0) &= 0,
\end{align*}
\] (3.6)

Let $\tilde{\xi} = \tilde{W}_\delta \bigg|_{\delta = 0}$, then $\tilde{\xi}$ satisfies
\[
\begin{align*}
\frac{\partial \tilde{\xi}}{\partial \tau} - \frac{1}{2} \sigma_0^2 \frac{\partial^2 \tilde{\xi}}{\partial x^2} &= \alpha (h - f), \\
\tilde{\xi} (x, 0) &= 0,
\end{align*}
\] (3.7)

From (3.5) and (3.7), we have
\[
\int_\omega \left[ W(x, \tau^*; f) - W^*(x) \right] \tilde{\xi} (x, \tau^*) \ dx + N \int_\omega \nabla f \cdot \nabla (h - f) \ dx \geq 0.
\] (3.8)

By the conjugate theory for parabolic equation we can obtain another form of the necessary condition. Let $L_{\tilde{\xi}} = \tilde{\xi} - \frac{1}{2} \sigma_0^2 \tilde{\xi}^2$, and suppose that $\eta$ is the solution of the following problem:
\[
\begin{cases}
L^* \eta = -\frac{\partial \eta}{\partial \tau} - \frac{1}{2} \sigma_0^2 \frac{\partial^2 \eta}{\partial x^2} = 0, \\
\eta (x, \tau^*) = W(x, \tau^*) - W^*(x),
\end{cases}
\] (3.9)

where $L^*$ is the adjoint operator of the operator $L$. From Green function and (3.8),(3.9) we have
\[
\begin{align*}
&\int_0^{\tau^*} \int_\omega (\eta L_{\tilde{\xi}} - \tilde{\xi} L^* \eta) \ dx \ d\tau \\
= &\int_0^{\tau^*} \int_\omega \left( \eta \tilde{\xi}_\tau - \frac{1}{2} \sigma_0^2 \eta \tilde{\xi}_x + \tilde{\xi} \eta_\tau + \frac{1}{2} \sigma_0^2 \tilde{\xi} \eta_x \right) \ dx \ d\tau \\
= &\int_0^{\tau^*} \int_\omega (\eta \tilde{\xi}_\tau + \tilde{\xi} \eta_\tau) \ dx \ d\tau + \int_0^{\tau^*} \int_\omega \left( \frac{1}{2} \sigma_0^2 \tilde{\xi} \eta_x - \frac{1}{2} \sigma_0^2 \tilde{\xi} \eta_x \right) \ dx \ d\tau \\
= &\int_\omega \tilde{\xi} (x, \tau^*) \int_0^{\tau^*} \left( \tilde{\xi} \eta_x - \eta \tilde{\xi}_x \right) \ dx \ d\tau \\
= &\int_\omega \tilde{\xi} (x, \tau^*) [ W(x, \tau^*) - W^*(x) ] \ dx
\end{align*}
\] (3.10)

Therefore one can easily obtain that
\[
\int_0^{\tau^*} \int_\omega \eta L_{\tilde{\xi}} \ dx \ d\tau = \int_\omega \tilde{\xi} (x, \tau^*) [ W(x, \tau^*) - W^*(x) ] \ dx.
\] (3.11)

Combining (3.8) and (3.11), one can easily obtain that
\[
\int_0^{\tau^*} \int_\omega \eta (h - f) \ dx \ d\tau + N \int_\omega \nabla f \cdot \nabla (h - f) \ dx \geq 0.
\] (3.12)

4. Uniqueness and stability

**Theorem 4.1** Suppose $W_1^*(x), W_2^*(x)$ are two given functions which satisfy (2.5). Let $f_1(x), f_2(x)$ be the minimizers of the optimal control problem $P$ corresponding to $W_1^*(x), W_2^*(x)$, respectively. If there
exists a point \( x_0 \in \omega \) such that \( f_i(x_0) = f_j(x_0) \) then we have the following estimate:

\[
\max_{x \in \omega} |f_i - f_j| \leq \frac{1}{\sqrt{2N}} \left\| W' (x) - W'_j (x) \right\|_{L^2(\omega)}.
\]  \( \text{(4.1)} \)

**Proof** By taking \( h = f_2 \) when \( f = f_1 \) and taking \( h = f_1 \) when \( f = f_2 \) in (3.1), then we have

\[
\int_{\omega} W'(x, \tau^*) - W'_1(x, \tau^*) \xi_i(x, \tau^*) \, dx + N \int_{\omega} \nabla f_i \cdot \nabla (f_j - f_i) \, dx \geq 0,
\]  \( \text{(4.2)} \)

and

\[
\int_{\omega} W'_2(x, \tau^*) - W'_2(x, \tau^*) \xi_j(x, \tau^*) \, dx + N \int_{\omega} \nabla f_2 \cdot \nabla (f_j - f_i) \, dx \geq 0.
\]  \( \text{(4.3)} \)

Here \( W, \xi, (i = 1, 2) \) are solutions of the system (2.5)/(3.2) with \( f = f_i, (i = 1, 2) \). Setting

\[
W_1 - W_2 = W, \quad \xi_1 + \xi_2 = \Xi,
\]  \( \text{(4.4)} \)

then \( W \) and \( \Xi \) satisfy

\[
\begin{cases}
W_t - \frac{1}{2} \sigma \sigma \xi_{xx} = \alpha (f_1 - f_2), \\
W(x, 0) = 0,
\end{cases}
\]  \( \text{(4.5)} \)

and

\[
\begin{cases}
\Xi_t - \frac{1}{2} \sigma \sigma \xi_{xx} = 0, \\
\Xi(x, 0) = 0,
\end{cases}
\]  \( \text{(4.6)} \)

By the extremum principle we know that the problem (4.6) only has zero solution and thus

\[
\xi_1(x, \tau) = -\xi_2(x, \tau).
\]  \( \text{(4.7)} \)

Moreover, \( \xi_1 \) satisfies the following equation

\[
\begin{cases}
\xi_{tt} - \frac{1}{2} \sigma \sigma \xi_{xx} = \alpha (f_2 - f_1), \\
\xi_1(x, 0) = 0,
\end{cases}
\]  \( \text{(4.8)} \)

By noticing (4.5) and (4.8), we have

\[
W(x, \tau) = -\xi_1(x, \tau).
\]  \( \text{(4.9)} \)

From (4.2), (4.3) and (4.9) then we have

\[
N \int_{\omega} |\nabla (f_1 - f_2) |^2 \, dx \leq \int_{\omega} \left| W_1(x, \tau^*) - W'_1(x, \tau^*) \right| \xi_1(x, \tau^*) \, dx + \int_{\omega} \left| W'_2(x, \tau^*) - W_2(x, \tau^*) \right| \xi_2(x, \tau^*) \, dx \leq \int_{\omega} W'(x, \tau^*) \xi_1(x, \tau^*) \, dx + \int_{\omega} W'_2(x, \tau^*) \xi_2(x, \tau^*) \, dx + \int_{\omega} \nabla f_2 \cdot \nabla (f_2 - f_1) \, dx
\]  \( \text{(4.10)} \)

From the assumption of Theorem 4.1 and the Hölder inequality, for \( x \in \omega \), we have

\[
|f_i(x) - f_j(x)| \leq \left( \int_{\omega} |f_i(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\omega} |f_j(x)|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\omega} |f_i(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\omega} |f_j(x)|^2 \, dx \right)^{\frac{1}{2}},
\]  \( \text{(4.11)} \)
which yield
\[
\max_{x \in \Omega} |f_1 - f_2| \leq \frac{1}{\sqrt{2N}} \left\| W_1^*(x) - W_2^*(x) \right\|_{L^2(\Omega)}. \tag{4.12}
\]

This completes the proof of the Theorem 4.1.

**Remark:** We have gotten the theoretical analysis results of the problem \(P\) based on the optimal control method (OCM). The OCM used in our work is indeed the Tikhonov regularization method with the \(L^2\) -norm of the gradient. It should be mentioned that the regularization parameter plays a major role in the numerical experiment of ill-posed problem. From Theorem 4.1 we can obtain that if there exist a constant \(\delta\) satisfies
\[
\left\| W_1^*(x) - W_2^*(x) \right\|_{L^2(\Omega)} \leq \delta, \quad \frac{\delta^2}{N} \to 0, \tag{4.13}
\]
then \(|f_1 - f_2|_{L^2(\Omega)} \to 0\), this shows the reconstructed limiting optimal control is unique and stable.

5. Concluding remarks
In this paper, we propose an inverse problem of determining the volatility in the Black-Sc holes model, which is still an interesting issue in financial mathematics. The difficulty is due to the lack of conventional stability and non-convexity. Using the partial differential equation theory of linearization technique the problem is translated into an inverse source problem. Based on the optimal control framework, the inverse problem is reduced to an optimization problem, the necessary condition which must be satisfied by the minimizer is deduced, and the uniqueness and stability of the solution are also established from the necessary condition.

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