Abstract. Let $X$ be the blow-up of the three dimensional complex projective space along $r$ general points of a smooth elliptic quartic curve $B \subset \mathbb{P}^3$ and let $L \in \text{Pic}(X)$ be any line bundle. The aim of this paper is to provide an explicit algorithm for determining the dimension of $H^0(X, L)$.

1. Introduction

Let $X$ be the blow-up of the three dimensional projective space along $r$ general points lying on a smooth elliptic quartic curve $B$. The aim of this paper is to provide an explicit algorithm for determining the dimension of $H^0(X, L)$ for any $L \in \text{Pic}(X)$. This dimension depends of course on the degree and multiplicities of the general divisor of $\mathbb{P}^3$ corresponding to $L$. In this paper we show that in fact this number is completely determined by the values of the intersections $l_i \cdot L$ and $C \cdot L$, where the $l_i$’s are the strict transforms of lines through pair of the $r$ points and $C$ is the strict transform of $B$.

This work is an attempt to generalize the results of [Har85] to the three dimensional case by extending the techniques used in [DVL]. Recently, in [AMC] and [CEG99], an higher dimensional analog of the same problem is studied under the more restrictive hypothesis that all the points lie on a rational normal curve of $\mathbb{P}^n$. It turns out that this assumption implies the finite generation of the Cox ring of the blow-up variety, while in the case analyzed by the present paper this statement is false.

The paper is organized as follows: in section 2 we fix the necessary notation while section 3 focuses on preliminary results regarding the intersection theory of the varieties which are needed throughout the paper. The main algorithm is explained in section 4, here we show how starting from the linear system $\mathcal{L}$, associated to $L \in \text{Pic}(X)$, one can find a fixed-component free linear system $\mathcal{L}'$ of the same dimension of $\mathcal{L}$. Then we proceed to define three different types of systems, listed in conclusion 4.2.3, which cover the range of all the possibilities. The dimension of a linear system in each one of these classes is given explicitly in theorems 5.1, 6.1 and 7.1.
The aim of this section is to provide the necessary notation for linear systems defined on blow-ups of \( \mathbb{P}^2 \) and \( \mathbb{P}^3 \) and a quadric. In what follows the ground field is assumed to be algebraically closed of characteristic 0.

**The definition of \( X \)** We start with a smooth quadric \( Q \subset \mathbb{P}^3 \) and a general \( B \in | - K_Q | \), i.e. \( B \) is an elliptic curve of degree 4. On this curve we choose \( p_1, \ldots, p_r \) points in general position and \( Z = m_1 p_1 + \cdots + m_r p_r \) is a zero-dimensional subscheme with \( m_i \) non-negative integers and associated ideal sheaf \( \mathcal{I}_Z \). With abuse of notation we denote by

\[
\mathcal{L}_3(d; m_1, \ldots, m_r)
\]
both the sheaf \( \mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_Z \) and its associated linear system.

The *expected dimension* of such a linear system \( \mathcal{L} \) is \( \text{edim}(\mathcal{L}) := \max\{-1, v(\mathcal{L})\} \), where

\[
v(\mathcal{L}) = \binom{d+3}{3} - \sum_{i=1}^{r} \binom{m_i + 2}{3} - 1,
\]

is the *virtual dimension*.

Let \( \pi : X \to \mathbb{P}^3 \) be the blow-up map of \( \mathbb{P}^3 \) along \( p_1, \ldots, p_r \), then we will denote by \( H \) the pull-back of a plane and by \( E_i \) the exceptional divisor corresponding to \( p_i \). In this way the
linear system $L_3(d; m_1, \ldots, m_r)$ will correspond to the complete linear system

$$L_X(d; m_1, \ldots, m_r) := |dH - m_1E_1 - \cdots - m_rE_r|$$

defined on $X$. As before we will use the same notation for the linear system and its associated invertible sheaf. We will say that a system, on $\mathbb{P}^3$ or on $X$, is standard if

$$m_1 \geq \ldots \geq m_r \geq 0 \quad \text{and} \quad 2d \geq m_1 + \cdots + m_4,$$

while we we call it almost standard if it becomes standard after reordering its multiplicities.

The strict transform of $B$ will be denoted by $C$.

The blow-up of $Q$ We have the following commutative diagram

$$\begin{array}{ccc}
Q_r & \longrightarrow & X \\
\pi_Q \downarrow & & \downarrow \pi \\
Q & \longrightarrow & \mathbb{P}^3
\end{array}$$

where $\pi_Q$ is the blow-up map of $Q$ at the $p_i$’s and $e_i := E_i \cap Q_r$ is the exceptional divisor on $Q_r$ corresponding to $p_i$. We will use the notation

$$L_Q(a, b; m_1, \ldots, m_r) := |ah_1 + bh_2 - m_1e_1 - \ldots - m_re_r|$$

where $ah_1 + bh_2 = \pi_Q^*O(a, b)$ to denote both the linear system and its corresponding invertible sheaf. We will say that such a linear system is in standard form if

$$m_1 \geq \ldots \geq m_r \geq 0 \quad \text{and} \quad a + b \geq m_1 + m_2 + m_3 + m_4$$

and it is standard if there exists a base of $\text{Pic}(Q_r)$ such that the system is standard when written in that base.

A proof of the following proposition, which will be used several times in this paper, can be easily obtained by readapting the arguments of [Har85] to the blow-up of a quadric along points on a smooth anticanonical divisor.

**Proposition 2.1.** Let $D$ be an effective divisor of $Q_r$, and let $C \in |-K_Q|$ be a smooth curve, then $C \subseteq \text{Bs}\, |D|$ if and only if $D \cdot C \leq 0$.

**The definition of $Y_I$** Consider the strict transform $l_i$ of the line $p_iq_i$ and denote by $l_1$ the strict transform of the line through $p_2$ and $p_3$. Given a subset $I \subset \{1, \ldots, r\}$, define

$$\pi_I : Y_{rI} \rightarrow X$$

to be the blow-up of $X$ along all the $l_i, i \in I$ and denote by $F_i$, the exceptional divisor corresponding to $l_i$.

When there is no ambiguity about $r$, we will use $Y_I$ instead of $Y_{rI}$. 
By abuse of notation, we will use $H$, resp. $E_i$, on $Y_I$, to denote the pull-back of $H$, resp. $E_i$; and we let $C$, resp. $\tilde{Q}$ to denote the corresponding strict transforms. With

$$\mathcal{L}_Y(d; m_1, \ldots, m_r; \{t_i\}_{i \in I}) := |dH - m_1E_1 - \ldots - m_rE_r - \sum_{i \in I} t_iF_i|,$$

where with $d, m_i, t_i \geq 0$, we will denote the complete linear system and its corresponding invertible sheaf.

**The definition of $\tilde{Y}_I$**

Our last object will be the blow-up of $Y_I$ along $C$ defined as

$$\tilde{\pi}_I : \tilde{Y}_I \to Y_I.$$ 

The exceptional divisor over $C$ will be denoted by $F$ and as before, by abuse of notation, we will denote the pull-back of resp. $H$, $E_i$ and $F_j$ with the same letters and the same for the strict transform of $\tilde{Q}$. With

$$\mathcal{L}_{\tilde{Y}}(d; m_1, \ldots, m_r; \{t_i\}_{i \in I}; t) := |dH - m_1E_1 - \ldots - m_rE_r - \sum_{i \in I} t_iF_i - tF|,$$

where $d, m_i, t_i, t \geq 0$, we will denote the complete linear system and its corresponding invertible sheaf.

In all the above notation, if $m_i = m_{i+1} = \ldots = m_{i+k} = m$, then we will write $m^k$ in stead of $m_i, m_{i+1}, \ldots, m_{i+k}$.

### 3. Preliminaries

In this section we deal with the varieties $X, Y_I$ and $\tilde{Y}_I$ just defined and we will work out their Chern classes and the intersection product of cycles on them.

#### 3.1. A Cremona transformation of type (3,3).

To any four non-collinear points of $\mathbb{P}^3$ we can associate a (3, 3) birational map corresponding to the linear system $\mathcal{L}_3(3; 2^4)$. After a linear change of coordinates this map can be described by:

$$\phi(x_0 : x_1 : x_2 : x_3) = (x_0^{-1} : x_1^{-1} : x_2^{-1} : x_3^{-1}). \quad (3.1)$$

This birational map induces a quasi-isomorphism $\overline{\phi} : X \dashrightarrow X'$ of the blowing-up of two $\mathbb{P}^3$'s along four points. As a consequence, the induced map $\phi^* : \text{Pic}(X') \to \text{Pic}(X)$ is an isomorphism which is described by the following:

**Proposition 3.1 ([LU]).** The action of transformation (3.1) on $\mathcal{L} = \mathcal{L}_X(d; m_1, \ldots, m_4)$ is given by:

$$\phi^*(\mathcal{L}) := \mathcal{L}_X(d + k; m_1 + k, \ldots, m_4 + k), \quad (3.2)$$

where $k = 2d - \sum_{i=1}^{4} m_i$. 
Note that the points \( \phi(p_1), \ldots, \phi(p_4) \) are still in general position \( \phi(C) \), therefore the action of \( \phi \) on a linear system \( \mathcal{L}_3(d; m_1, \ldots, m_r) \) results in a linear system \( \mathcal{L}_3(d + k; m_1 + k, \ldots, m_4 + k, m_5, \ldots, m_r) \) and the same is true for the corresponding systems on \( X \).

Observe that even if \( \dim \phi^*(\mathcal{L}) = \dim \mathcal{L} \), in general the virtual dimensions of the two systems may be different (see e.g. [LU]).

### 3.2. Euler characteristic and Chern classes of the fundamental varieties.

Let \( D \) be a divisor on a smooth rational threefold. According to the Riemann-Roch formula, the Euler characteristic of \( \mathcal{O}_X(D) \) is

\[
\chi(\mathcal{O}_X(D)) = \frac{1}{12} [D(D - K_X)(2D - K_X) + c_2(X)D] + 1,
\]

where \( c_2(X) \) is the second Chern class of \( X \) and \( K_X \) is the canonical class of \( X \).

The threefolds we are going to work on will be \( \mathbb{P}^3, X, Y_I \) and \( \tilde{Y}_I \). Hence we need to know \( K_X, c_2(X) \) and for these varieties.

**Proposition 3.2** (see e.g. [GH94]). Let \( \pi : \tilde{X} \to X \) be the blow-up of a smooth algebraic threefold \( X \) along a smooth, connected subscheme \( V \subset X \) of dimension \( a \leq 1 \) and let \( E \) be the exceptional divisor, then:

\[
c_1(\tilde{X}) = \pi^*c_1(X) + (a - 2)E \quad \text{and} \quad c_2(\tilde{X}) = \pi^*(c_2(X) + a\eta_V) - a \pi^*c_1(X)E,
\]

where \( \eta_V \in H^4(X, \mathbb{Z}) \) is the class of the curve \( V \) in \( X \).

Using the fact that \( c_1(\mathbb{P}^3) = 4H \) and \( c_2(\mathbb{P}^3) = 6H^2 \), we can deduce the following:

**Lemma 3.3.** Let \( I \subset \{1, \ldots, r\} \) be a subset of cardinality \( a \) and denote by \( \epsilon \) a number which is equal to 1 if \( 1 \in I \) and 0 otherwise. The first two Chern classes of \( X, Y_I \) and \( \tilde{Y}_I \) are:

\[
\begin{align*}
X, & \quad c_1 = 4H - 2E_1 - \cdots - 2E_r \\
& \quad c_2 = 6H^2.
\end{align*}
\]

\[
\begin{align*}
Y_I, & \quad c_1 = 4H - 2E_1 - \cdots - 2E_r - \sum_{i \in I} F_i \\
& \quad c_2 = (6 + a)H^2 + aE_1^2 + \sum_{i \in I \setminus \{1\}} E_i^2 + \epsilon (E_2^2 + E_3^2 - E_1^2)
\end{align*}
\]

\[
\begin{align*}
\tilde{Y}_I, & \quad c_1 = 4H - 2E_1 - \cdots - 2E_r - \sum_{i \in I} F_i - F \\
& \quad c_2 = \pi^*c_2(Y_I) + 4H^2 + E_1^2 + \cdots + E_r^2 - 4HF + 2E_1F + \cdots + 2E_rF.
\end{align*}
\]

**Proof.** The first Chern class of all these varieties is easily determined by means of Proposition 3.2 and the same is true for \( c_2(X) \).

To obtain \( c_2(Y_I) \), we will assume for simplicity that \( I = \{2, \ldots, s\} \). In this case, we can consider \( \pi_I \) as the composition \( \pi_s \circ \pi_{s-1} \circ \cdots \circ \pi_2 \), where \( \pi_i : Y_{I_i} \to Y_{I_{i-1}} \) the blow-up map of
$l_i$ and $Y_{l_1} = X$. Since the class of $l_2$ in the Chow ring of $X$ is given by

$$\eta_{l_2} = (H - E_1 - E_2)^2 = H^2 + E_1^2 + E_2^2,$$

then, by applying Proposition 3.2, we obtain the following:

$$c_2(Y_{l_2}) = 6H^2 + H^2 + E_1^2 + E_2^2 - (4H - 2E_1 - \cdots - 2E_r) F_2.$$

Now observe that $(4H - 2E_1 - \cdots - 2E_r) l_2 = 0$ in $X$ so that on $Y_I$ we have:

$$(4H - 2E_1 - \cdots - 2E_r) F_2 \equiv 0.$$

Repeating this argument for all $i = 2, \ldots, s$, and using the fact that $F_i F_j \equiv 0$ for all $i \neq j$ (because the curves $l_i$ and $l_j$ do not intersect), we obtain that

$$c_2(Y_{l_2}) = 6H^2 + \sum_{i=2}^{s} (H^2 + E_1^2 + E_i^2)$$

which gives the desired result.

For determining $c_2(Y_{\tilde{l}_1})$ we need to know the class of $C$ in $Y_I$, which is given by:

$$\eta_C = (2H - E_1 - \cdots - E_r)^2 = 4H^2 + E_1^2 + \cdots + E_r^2.$$

From Proposition 3.2 and $F_i F \equiv 0$ (because $l_i$ and $C$ do not intersect) we deduce:

$$c_2(Y_{\tilde{l}_1}) = \pi^* c_2(Y_I) + 4H^2 + E_1^2 + \cdots + E_r^2 - \pi^* c_1(Y_I) F$$

which gives the desired result.

\[\square\]

### 3.3. Intersection products.

The aim of this section is to provide explicit formulas for the intersection of divisors on $X$, $Y_{l_1}$ and $\tilde{Y}_{l_1}$. In particular, given any three divisors on one of these varieties, we are interested in evaluating their intersection number. Since this number and that of the intersection of their pull-backs are the same, we need only to work out the calculations for divisors in $\tilde{Y}$.

The following proposition will be needed in what follows.

**Proposition 3.4** (see e.g. [GH94]). Let $X$ be a smooth irreducible threefold and $S \subset X$ a smooth irreducible surface, then $c_1(X)|_S = c_1(S) + S|_S$.

**Lemma 3.5.** The non-vanishing intersection products of any three divisors (or their pull-backs) on $X, Y_{l_1}$ and $\tilde{Y}_{l_1}$ are given below:

- $X$: $H^3 = E_1^3 = 1$
- $Y_{l_1}$: $(H \mid E_2 \mid E_3) F_1^2 = (H \mid E_1 \mid E_i) F_i^2 = -1$ (for $i = 2, \ldots, r$)
- $F_i^3 = 2$ (for $i = 1, \ldots, r$)
\[ \tilde{Y}_I \quad HF^2 = -4 \]

\[ E_i F^2 = -1 \quad (i = 1, \ldots, r) \]

\[ F^3 = 2(r - 8) \]

**Proof.** We begin by observing that the following intersections \((i \neq j):\)

\[ H E_i, E_i E_j, F_i F_j, F_i F, E_1 F_1 \]

are numerically equivalent to 0 and the same is true for \(E_i F_1\) if \(i \neq 2, 3\) and \(E_i F_j\) if \(i, j \neq 1\) and \(i \neq j\). So, any intersection product of three divisors containing one of the above monomials must vanish.

Since \(H^2\) is the pull-back of the class of a line of \(\mathbb{P}^3\), it has non-zero intersection only with \(H\) and this intersection is clearly equal to 1. A general \(W \in |H|\) is the blow-up of a plane along \(#I + 4\) points (because a general \(H\) of \(X\) intersects every \(l_i\) in one point and \(C\) in 4 points). On \(W\), let \(h\) be the class of a line, \(e_i\) the exceptional curve coming from the intersection point with \(l_i\), and \(c_1, c_2, c_3, c_4\) those coming from the intersection points with \(C\). Then \(HF_i^2 = e_i^2 = -1\) and \(HF_1^2 = (c_1 + c_2 + c_3 + c_4)^2 = -4\).

Observe that, on \(X\), one has that \(-E_i^2\) is the class of a line of \(E_i\), hence the same is true when we consider the pull-back of this class to \(\tilde{Y}_I\). This implies that \(E_i^2\) has non-zero intersection only with \(E_i\) and this intersection is 1.

If \(E_i\) and \(F_j\) intersect, it is easy to see that this intersection is a \((-1)\)-curve on \(F_j\), hence we obtain that \(E_i F_j^2 = -1\) and in the same way we have that \(E_i F_i^2 = -1\).

On \(F_i\), which is a ruled rational surface, let \(h\) denote the class of a section and \(f\) the class of a fiber. Then \(c_1(F_i) \equiv 2h + 2f\), and it follows from Proposition 3.4 that

\[ F_i |_{F_i} \equiv (4H - 2E_1 - \cdots - 2E_r - \sum_{j \in I} F_j - F) |_{F_i} - 2h - 2f \]

\[ \equiv 4f - 2f - 2f - F_i |_{F_i} - 2h - 2f \]

\[ \equiv -F_i |_{F_i} - 2h - 2f. \]

So, \(F_i |_{F_i} \equiv -h - f\), which implies that \(F_i^3 = (-h - f)^2 = 2\).

Similarly, on \(F\), which is an elliptic ruled surface, let \(h\) denote the class of a section and \(f\) the (numerical equivalence) class of a fiber. Then \(c_1(F) \equiv 2h\), and from Proposition 3.4 we obtain that

\[ F |_{F} \equiv (4H - 2E_1 - \cdots - 2E_r - \sum_{j \in I} F_j - F) |_{F} - 2h \]

\[ \equiv 16f - \sum_{i=1}^{r} (2f) - F |_{F} - 2h \]

\[ \equiv -F |_{F} - 2h - 2(r - 8)f. \]
So, \( F|_F \equiv -h - (r - 8)f \), which implies that \( F^3 = (-h - (r - 8)f)^2 = 2(r - 8) \). \( \square \)

### 3.4. Cohomology of an invertible sheaf and its pull-back.

The aim of this section is to recall a criterion for comparing the cohomology of line bundles on a smooth projective variety \( M \) together with the cohomology of line bundles on its blow-up \( \tilde{M} \) along a smooth subscheme \( V \subset M \). To this purpose consider the blow-up map \( \pi : \tilde{M} \to M \) of \( V \subset M \), then we have the following:

**Lemma 3.6.** For any positive integer \( i \) we have that \( R^i \pi_* O_{\tilde{M}} = 0 \), which in turn implies:

\[
H^i(\tilde{M}, \pi^* \mathcal{L}) \cong H^i(M, \mathcal{L})
\]

for any \( \mathcal{L} \in \text{Pic}(M) \).

**Proof.** The vanishing of the higher direct images of \( \pi_* O_{\tilde{M}} \) depends on the fact that, for \( p \in M \), the fiber \( F_p := \pi^{-1}(p) \) is either a point or \( \mathbb{P}^r \), where \( r \) is the codimension of \( V \) in \( M \). From the projection formula \([\text{Har}77, \text{III, Exercise 8.3}]\) we obtain:

\[
R^i \pi_*(O_{\tilde{M}} \otimes \pi^* \mathcal{L}) \cong R^i \pi_* O_{\tilde{M}} \otimes \mathcal{L}
\]

which implies that \( R^i \pi_* \pi^* \mathcal{L} = 0 \) for any \( i > 0 \). The vanishing of the higher direct image of \( \pi^* \mathcal{L} \) and the isomorphism \( \pi_* \pi^* \mathcal{L} \cong \mathcal{L} \) imply that \( H^i(\tilde{M}, \pi^* \mathcal{L}) \cong H^i(M, \mathcal{L}) \). \( \square \)

**Remark 3.7.** In the course of this paper, lemma [3.6] will be frequently used without referring to it. For example, the proof of the vanishing of \( \mathcal{L}_X(d; m_1, \ldots, m_r) \) on \( X \) will immediately give the vanishing of \( \mathcal{L}_Y(d; m_1, \ldots, m_r; \{0\}_{i \in I}) \) on \( Y_I \).

### 4. The algorithm

In this section we provide an algorithm for reducing any linear system \( \mathcal{L} \) to one of four types of standard systems, for which we can determine its dimension explicitly. To obtain the speciality of \( \mathcal{L} \) it is then sufficient to compare \( \text{edim}(\mathcal{L}) \) and \( \text{dim}(\mathcal{L}) \).

**4.1. Reducing to a standard class.** Given a linear system \( \mathcal{L} \) we describe an algorithm for finding a new linear system \( \mathcal{L}' \) which is in standard form and such that \( \text{dim} \, \mathcal{L} = \text{dim} \, \mathcal{L}' \).

**Input:** \( (d, m_1, \ldots, m_r) \).

**Sort** the vector \( (m_1, \ldots, m_r) \) in decreasing order.

**While** \( 2d < m_1 + m_2 + m_3 + m_4 \) and \( d > m_1 \)

\[
\begin{align*}
&k := 2d - m_1 - m_2 - m_3 - m_4.
\end{align*}
\]
\[ d := d + k \quad m_i := \max(m_i + k, 0) \quad (i = 1 \ldots 4) \]

Sort the vector \((m_1, \ldots, m_r)\) in decreasing order.

Output: \((d, m_1, \ldots, m_r)\).

Observe that if \(d < m_1\) the system is empty. In this case the algorithm exits immediately from the main cycle and returns us a list which corresponds to an empty system.

The condition \(2d < m_1 + m_2 + m_3 + m_4\) has to be satisfied in order to decrease the degree of the system by means of a Cremona transformation. As mentioned in the previous paragraph, applying the Cremona transformation does not change the dimension of the linear system.

Finally, observe that if \(m_i < 0\) then \(-m_i E_i\) is in the base locus of \(\mathcal{L}\) so that

\[ \dim \mathcal{L} = \dim \mathcal{L} + m_i E_i \]

and this justifies our redefinition of \(m_i\).

After applying this algorithm, you either obtain that \(\dim(\mathcal{L}) = -1\) or \(\dim \mathcal{L} = \dim(\mathcal{L}')\) where \(\mathcal{L}'\) is a standard class with \(d \geq m_1 \geq 0\).

4.2. Three types of standard classes.

In what follows the symbol \(\mathcal{L}_X\) will always denote a non-empty standard linear system of the form \(\mathcal{L}_X(d; m_1, \ldots, m_r)\). Associated to this system we will consider also its restriction to \(Q_r\) which will be denoted by \(\mathcal{L}_Q\) and is of the form \(\mathcal{L}_Q(d, d; m_1, \ldots, m_r)\). Observe that also this system is standard. We will use the symbol \(\mathcal{L}\) to denote any one of \(\mathcal{L}_X\) or \(\mathcal{L}_Q\). With this notation we have that:

\[ C \cdot \mathcal{L} = 4d - m_1 - \cdots - m_r, \]

where \(C \in |-K_{Q_r}|\) is the strict transform of the smooth quartic curve through the \(p_i\)'s.

4.2.1. Step 1: Splitting up according to \(t\).

Claim 1. If \(C \cdot \mathcal{L} \geq 1\), then \(C\) is not in the base locus of \(\mathcal{L}_Q\).

Proof. Since \(\mathcal{L}_Q\) is standard and \(\mathcal{L}_Q \cdot K_{Q_r} \geq 1\), the result follows from Proposition 2.1 (because the blown up points are general on \(C\)).

Claim 2. If \(C \cdot \mathcal{L} \leq 0\) and \(\mathcal{L}_X\) is not of the form \(\mathcal{L}_X(2m; m_8, m_9, \ldots, m_r)\), define

\[ b := \max\{i \mid 4d - m_1 - \cdots - m_i + m_i(i - 8) \geq 1 \quad \text{and} \quad 9 \leq i \leq r\}, \]

\[ t := \left\lfloor \frac{m_1 + \cdots + m_b - 4d + 1}{b - 8} \right\rfloor \]

Then \(tC\) is contained in the base locus of \(\mathcal{L}\).
Proof. Note that $r$ can not be smaller than 8 because in this case, since $\mathcal{L}_X$ is standard, we would have that either $c \geq 1$ or the system is of the form $\mathcal{L}_X(2m; m^8)$. This implies that $b$ is well defined.

We proceed by observing that since $\mathcal{L}_Q$ is standard and $\mathcal{L}_Q \cdot K_Q \geq 0$ then, by Proposition 2.1 we have that $C \subseteq \text{Bs}(\mathcal{L}_Q)$. Removing $C$ from the system we obtain:

$$\mathcal{L}_Q = C + \mathcal{L}^1_Q,$$

where $\mathcal{L}^1_Q$ is still standard and $\mathcal{L}_Q \cdot K_Q = \mathcal{L}_Q \cdot K_Q - (r - 8)$. We repeat this procedure until the intersection of $C$ with the residual system $\mathcal{L}^t_Q$ is positive or $t = m_r$.

If $b = r$, then, taking $t$ as in the statement of the claim, $\mathcal{L}^t_Q \cdot K_Q \leq 0$ and $\mathcal{L}^{t-1}_Q \cdot K_Q \geq 0$, which implies that $tC \subseteq \text{Bs}(\mathcal{L}_Q)$.

If $b < r$, then $m_r C$ is contained in the base locus of $\mathcal{L}_Q$. Consider now the decomposition $\mathcal{L}_Q = m_r C + \mathcal{L}^{m_r}_{Q}$ and let $r' := \max\{i \mid m_i - m_r > 0\}$, then the class of $\mathcal{L}^{m_r}_{Q}$ is the pull-back of a class on $Q_{r'}$ so that we can repeat the above arguments.

In case $b = r'$, proceeding as before, we obtain that, for

$$t = m_r + \left\lfloor \frac{(m_1 - m_r) + \cdots + (m_{r'} - m_r) - 4d + 8m_r + 1}{r' - 8} \right\rfloor,$$

the divisor $tC$ is contained in the base locus of $\mathcal{L}_Q$.

Since $b \geq 9$ exists, after applying these arguments a sufficient number of times, our procedure comes to an end and we obtain that $tC$ is contained in the base locus of $\mathcal{L}_Q$ for the claimed value of $t$. \qed

Conclusion. We distinguish the following cases for $\mathcal{L}_X$:

1. $C \cdot \mathcal{L}_X \geq 1$, i.e. $t = 0$.
2. $\mathcal{L}_X(2m; m^8, m_9, \ldots, m_r)$, i.e. $t = m$.
3. $C \cdot \mathcal{L}_X \leq 0$ and $\mathcal{L}_X \neq \mathcal{L}_X(2m; m^8, m_9, \ldots, m_r)$, i.e. $t$ is as in claim 2.

Note that we haven’t proved that $t = m$ in case (2), but the above arguments will still work to obtain that $m_9 C$ is contained in the base locus of $\mathcal{L}_Q$. Since the residue class is then $\mathcal{L}_Q(2m', 2m'; m^8)$, with $m' = m - m_9$, the fact that $t = m$ follows from [Har85, Proposition 1.2].

4.2.2. Step 2: Reducing to the case $t \leq m_r$.

In case (1) there is nothing to be done since $t = 0$. 


In case (2), since the system is standard, we have that $m \geq m_i$ for any $i$. The fact that $mC$ belongs to the base locus of $\mathcal{L}_X$ implies that

$$\dim \mathcal{L}_X(2m; m^8, m_9, \ldots, m_r) = \dim \mathcal{L}_X(2m; m^r),$$

so it is sufficient to determine the dimension of the last system.

In case (3) we have that $t \leq m_r$ if and only if $b = r$. On the other hand, if $b < r$ then $m_{b+1} < t \leq m_b$ and, since $tC$ is contained in the base locus of $\mathcal{L}_X$ we have:

$$\dim \mathcal{L}_X(d; m_1, \ldots, m_r) = \dim \mathcal{L}_X(d; m_1, \ldots, m_b, t^{r-b}).$$

As before we need just to determine the dimension of the last linear system.

4.2.3. Step 3: Reducing to the case $d \geq m_1 + t$.

The previous part allows us to limit our study to the case:

$$\mathcal{L}_X \neq \mathcal{L}_X(2m; m^8, m_9, \ldots, m_r), \quad C \cdot \mathcal{L} \leq 0 \quad \text{and} \quad t \leq m_r.$$

In what follows we will adopt the notation:

$$t_1 = \max\{0, m_1 + m_2 - d\} \quad \text{and} \quad t_i := \max\{0, m_1 + m_i - d\} \quad \text{for} \quad i = 2, \ldots, r.$$

Claim 4.1. Assume that $d < m_1 + t$, then $r \geq 10$, the divisor $M \in \mathcal{L}_X(3; 3, 1^{r-1})$ is a cone contained in the base locus of $\mathcal{L}_X$ and $t_r > 0$.

**Proof.** Since we reduced to consider the case $t \leq m_r$, the positivity of $t_r$ follows immediately from the hypothesis. Observe that $t$ must be positive and that $\mathcal{L}_X$ can not be of the form $\mathcal{L}_X(2m; m^8, m_9, \ldots, m_r)$. As noticed before this implies that $r \geq 9$. Observe that the equality does not hold, since in this case $t = m_1 + \cdots + m_9 - 4d + 1$ and

$$m_1 + t - d = (m_1 + \cdots + m_4 - 2d) + (m_1 + m_5 \cdots + m_7 - 2d) + (m_8 + m_9 - d) + 1,$$

where the first two terms in parentheses are non-positive and the third is negative because $\mathcal{L}_X$ is in standard form. This implies that $r$ is at least 10.

The linear system of $\mathbb{P}^3$ given by $\mathcal{L}_3(3; 3, 1^9)$ contains a unique divisor which is a cone over a plane cubic. Since

$$C \cdot \mathcal{L}_X(3; 3, 1^9) = 0$$

we have that $C$ is contained in the base locus of this system by Proposition 2.1. This implies that $\dim \mathcal{L}_X(3; 3, 1^{r-1}) = 0$ if $r \geq 10$ and the unique element $M$ is the strict transform of the cone with vertex $p_1$ and base curve $B$.

Let $D \in \mathcal{L}_3(d; m_1, \ldots, m_r)$ be a general element of the system that we are considering. Observe that any line $l$ through $p_1$ and $q \in B$ has an intersection multiplicity with $D$ at least

$$m_1 + t > d.$$

This means that the strict transform of $l$ is contained in the base locus of $\mathcal{L}_X$ and this implies that $M$ is contained in the base locus of $\mathcal{L}_X$. $\Box$
Claim 4.2. Assume that $d < m_1 + t$, then the system $\mathcal{L}'_X := \mathcal{L}_X - \mathcal{L}_X(3; 3, 1^r)$ is almost standard or empty.

Proof. In order to see that $\mathcal{L}'_X$ is almost standard, it is sufficient to check that the multiplicities $m_1 - 3, m_2 - 1, \ldots, m_r - 1$ are non negative and that $2d - 6$ is bigger or equal to the sum of the biggest four multiplicities. Observe that if the inequality

$$m_1 - 3 \geq m_5 - 1$$

holds, then $m_1 - 3$ belongs to the set of biggest four multiplicities and this gives the thesis. Assume that $m_1 - 3 < m_5 - 1$ then we would have $m_1 - 1 \leq m_5 \leq m_1$. The fact that $\mathcal{L}_X$ is standard together with claim [4.1] imply that

$$2d \geq m_1 + \cdots + m_4 \quad \text{and} \quad d < m_1 + m_5.$$ 

By substituting the two possible values for $m_5$ we obtain a contradiction. $\square$

Summarizing the two claims we see that if $d < m_1 + t$ then, either $cl_X$ is empty or it is one of the following:

i. $\mathcal{L}_X(3; 3, 1^r)$ with $r \geq 10$ and $\dim \mathcal{L}_X = 0$;

ii. $M + \mathcal{L}'_X$ where $\mathcal{L}'_X$ is almost standard and $\dim \mathcal{L}_X = \dim \mathcal{L}'_X$.

In case (i) we know the dimension of $\mathcal{L}_X$. For case (ii), after reordering the multiplicities of $\mathcal{L}'_X$ we obtain a new system $\mathcal{L}''_X$ which is standard. To determine the dimension of $\mathcal{L}''_X$ we repeat the procedure starting from Step 1 of section 4.2.1.

Conclusion

Using the above Steps, we see that we are reduced to determing the dimension of $\mathcal{L}_X$ in the following cases:

1. $C \cdot \mathcal{L}_X \geq 1$, i.e. $t = 0$.

2. $\mathcal{L}_X = \mathcal{L}_X(2m; m^r)$ with $r \geq 8$, i.e. $t = m$.

3. $1 + m_r(8 - r) \leq C \cdot \mathcal{L}_X \leq 0$ and $d \geq m_1 + t$ where $t = \left\lfloor \frac{C \cdot \mathcal{L} + 1}{r - 8} \right\rfloor \leq m_r$ and $\mathcal{L}_X \neq \mathcal{L}_X(2m; m^8, m^9, \ldots, m_r)$.

5. Determining the dimension of $\mathcal{L}$ for case (1)

We begin by defining $I$ to be set of indices $i$ such that $t_i > 0$. Observe that if $I$ is non-empty then, since $\mathcal{L}_X$ is standard, it must be one of these two types:

$$\{2, 3, \ldots, s - 1, s\} \quad \text{or} \quad \{1, 2, 3\}.$$ 

The following theorem shows that $h^1(X, \mathcal{L}_X)$ is a function of the numbers

$$t_i = -l_i \cdot \mathcal{L}_X$$
where the $l_i$'s are the strict transforms of lines indexed by $I$. This fact has already been proved in [DVL] for the case $r \leq 8$. We will however make the proofs in this paper self-contained.

**Theorem 5.1.** Let $\mathcal{L}_X$ be a non-empty standard system with $C \cdot \mathcal{L}_X \geq 1$, then

$$\dim \mathcal{L}_X = v(\mathcal{L}_X) + \sum_{i=1}^{r} \left( t_i + 1 \right).$$

**Proof.** Without loss of generality, we may assume that $m_r > 0$. This means our linear system $\mathcal{L}_X$ satisfies the following conditions:

1. $2d \geq m_1 + m_2 + m_3 + m_4$
2. $d \geq m_1 \geq \ldots \geq m_r > 0$
3. $t_1 := \max\{0, m_2 + m_3 - d\}$ and $t_i := \max\{0, m_1 + m_i - d\}$ for $i = 2, \ldots, r$
4. $4d \geq m_1 + \ldots + m_r + 1$

Consider the blow-up $\pi_I : Y_I \to X$ along the $l_i$, with $i \in I$ and let $\mathcal{L}_Y$ be the complete linear system $\mathcal{L}_Y(d; m_1, \ldots, m_r; \{t_i\}_{i \in I})$ (and its corresponding invertible sheaf) on $Y_I$. Since, for all $i \in I$, the curve $t_il_i$ belongs to the base locus of $\mathcal{L}_X$, we have that $\dim \mathcal{L}_X = \dim \mathcal{L}_Y$.

Now, using lemma's 3.3 and 3.5, an easy but tedious calculation shows that

$$\mathcal{X}(Y, \mathcal{L}_Y) = \mathcal{X}(X, \mathcal{L}_X) + \sum_{i=1}^{r} \left( t_i + 1 \right).$$

So, in order to prove theorem 5.1 it is sufficient to show that for $i \geq 1$

$$h^i(Y_I, \mathcal{L}_Y) = 0.$$  \hfill (5.1)

Denote by $\mathcal{S}_r := \mathcal{L}_Y(2; 1^r)$ and consider the exact sequence associated to a smooth $\tilde{Q} \in \mathcal{S}_r$

$$0 \to \mathcal{L}_Y - \mathcal{S}_r \to \mathcal{L}_Y \to \mathcal{L}_Y \otimes \mathcal{O}_{\tilde{Q}} \to 0.$$

Now, since $\tilde{Q}$ does not intersect the exceptional divisor $F_i$ of $l_i$, we see that $\mathcal{L}_Y \otimes \mathcal{O}_{\tilde{Q}} = \mathcal{L}_Q(d, d; m_1, \ldots, m_r)$ is standard. It then follows from condition (4) and [Har85] Theorem 1.1 that for $i \geq 1$

$$h^i(\mathcal{L}_Y \otimes \mathcal{O}_{\tilde{Q}}) = 0.$$

In order to obtain (5.1), we obviously need the vanishing of the higher cohomology groups of $\mathcal{L}_Y - \mathcal{S}_r = \mathcal{L}_Y(d - 2; m_1 - 1, \ldots, m_r - 1; \{t_i\}_{i \in I})$. We begin by distinguishing two cases according to the values of $r$.

**Claim 5.2.** If $r \leq 3$ then $h^i(Y, \mathcal{L}_Y) = 0$ for all $i \geq 1$.

Observe that as long as $r \geq 4$, the system $\mathcal{L}_Y - \mathcal{S}_r$ satisfies conditions (1)-(4) unless one of the following occurs:
(a) $d > m_1$ and $m_r = 1$
(b) $d = m_1$ and $m_r > 1$
(c) $d = m_1$ and $m_r = 1$

Recall that $s$ is defined to be the maximum of $I$ and observe that if $d = m_1$ then $s = r$.

So, if $s < r$, the system $\mathcal{L}_Y - \mathcal{S}_r$ satisfies conditions (1)-(4), unless $m_r = 1$. However, in this case, we can consider it as a linear system on $Y_{r'1}$, where $r' := \max\{i \mid m'_i > 0\}$. If we can prove the vanishing of the cohomology groups of $\mathcal{L}_Y - \mathcal{S}_r$ on $Y_{r'1}$, then, because of lemma 3.6, this implies the vanishing of the cohomology groups on $Y_I$. Note that, $r' \geq s$.

Moreover, if $r = s$, then case (a) cannot occur, because this would imply that $m_1 + m_r - d = m_1 + 1 - d \leq 0$, which contradicts $t_r > 0$.

In any case, if $d' > m'_1$, then we can consider an exact sequence like before (using $\mathcal{S}_{r'}$ in stead of $\mathcal{S}_r$ if $m_r = 1$), and, using arguments as before, we can reduce to proving the vanishing of the higher cohomology groups of $\mathcal{L}_Y - 2\mathcal{S}_r$ (or $\mathcal{L}_Y - \mathcal{S}_r - \mathcal{S}_{r'}$ if $m_r = 1$). Obviously, this procedure can be repeated until we are reduced to proving the vanishing of the higher cohomology groups for a class $\mathcal{L}_Y(d; m_1, \ldots, m_r; \{t_i\}_{i \in I})$ satisfying conditions (1)-(4), and with either $r \leq 3$ or $r = s \geq 4$ and $d = m_1$. Note that $t_i$ and $I$ occurring in $\mathcal{L}_Y$ are in fact the same as the original ones.

We thus end the proof with the following:

**Claim 5.3.** Assume that $\mathcal{L}_Y = \mathcal{L}_Y(m_1; m_1, \ldots, m_r; \{t_i\}_{i \in I})$ satisfies conditions (1)-(4) with $s = r \geq 4$, then $h^i(Y, \mathcal{L}_Y - \mathcal{S}_r) = 0$ for all $i \geq 0$.

\[ \square \]

**Proof of Claim 5.2.** This can be regarded as the “toric case” by putting the three points $p_1, p_2, p_3$ in $(1 : 0 : 0 : 0), (0 : 1 : 0 : 0), (0 : 0 : 1 : 0)$ and observing that $X$ and $Y_I$ are toric varieties. The evaluation of the dimension of $\mathcal{L}_X$ can be worked out by counting the monomials with multiplicities $m_i$ at these points. In the same way the cohomology groups of $\mathcal{L}_X$ and $\mathcal{L}_Y$ can be found by purely combinatorial methods and in particular we have that $h^i(Y, \mathcal{L}_Y) = 0$ for all $i \geq 1$.

\[ \square \]

**Proof of Claim 5.3.** By hypothesis we have that $t_i = m_i$ for $i = 2, \ldots, r$. Now let $a_i := m_i - 1$ for $i = 1, \ldots, r$ so that

$$\mathcal{L}_Y - \mathcal{S}_r = \mathcal{L}_Y(a_1 - 1; a_1, \ldots, a_r; a_2 + 1, \ldots, a_r + 1)$$

with $a_1 \geq \cdots \geq a_r \geq 0$ and denote this class by $\mathcal{A}_r$.

We will now prove by induction on $b$ that for any $i \geq 0$

$$h^i(\mathcal{A}_b) = 0,$$
for all $b \geq 1$ and $a_1 \geq \cdots \geq a_b \geq 0$. 

An easy calculation, using lemma’s 3.3 and 3.5, shows that $\mathcal{X}(\mathcal{A}_b) = 0$. Moreover, $h^0(\mathcal{A}_b) = 0$ because the degree $a_1 - 1$ is less than the biggest multiplicity $a_1$. In this way it is enough to prove the vanishing of the first and second cohomology group.

If $a_1 = 0$, then $\mathcal{A}_1$ is the system $| - H|$ so that $h^1(\mathcal{A}_1) = h^2(\mathcal{A}_1) = 0$ (see e.g. [Har77, III, Theorem 5.1 (b), p. 225]).

Now assume $b = 1$ and $a_1 > 0$, so that $\mathcal{A}_1 = \mathcal{L}_X(a_1 - 1; a_1)$. Note that, because $b = 1$, we do not blow up lines, so we work on $X_1$. Let $W \in \mathcal{W} = \mathcal{L}_X(1; 1)$ and consider the exact sequence

$$0 \rightarrow \mathcal{A}_1 - W \rightarrow \mathcal{A}_1 \rightarrow \mathcal{A}_1 \otimes \mathcal{O}_W \rightarrow 0.$$  

Since $\mathcal{A}_1 - W = \mathcal{L}_X(a_1 - 2; a_1 - 1)$, our induction hypothesis implies that $h^i(\mathcal{A}_1 - W) = 0$ for $i = 0, 1, 2$. On the other hand, because $\mathcal{A}_1 \otimes \mathcal{O}_W = \mathcal{L}_2(a_1 - 1; a_1)$, one easily checks that $h^i(\mathcal{A}_1 \otimes \mathcal{O}_W) = 0$ for all $i \geq 0$.

Next, assume that $b > 1$ and that the statement is true for $b' \leq b - 1$. On $Y_{bI}$, with $I = \{2, \ldots, b\}$, consider the exact sequence

$$0 \rightarrow \mathcal{A}_b - F_b \rightarrow \mathcal{A}_b - 1 \rightarrow \mathcal{A}_b - 1 \otimes \mathcal{O}_{F_b} \rightarrow 0.$$  

Recall that, since $F_b \cong \mathbb{P}^1 \times \mathbb{P}^1$, the restriction of $\mathcal{A}_{b-1}$ to $F_b$ is given by:

$$\mathcal{A}_{b-1} \otimes \mathcal{O}_{F_b} \equiv \mathcal{O}(-1, 0).$$

A standard argument shows that $h^i(F_b, \mathcal{O}(-1, 0)) = 0$ for all $i \geq 0$ and this, together with the vanishing of $h^i(\mathcal{A}_{b-1}) = 0$ (by induction) implies that $h^i(\mathcal{A}_{b-1} - F_b) = 0$ for $i \geq 0$. But

$$\mathcal{A}_{b-1} - F_b = \mathcal{A}_b \quad \text{with} \quad a_b = 0.$$  

Again we will use induction, now on $a_b$, so we assume that the statement is true for all $a_b' \leq a_b - 1$. Let $\mathcal{A}'_b = \mathcal{L}_Y(a_1 - 1; a_1, \ldots, a_b - 1, a_b - 1; a_2 + 1, \ldots, a_{b-1} + 1, a_b)$ and consider the exact sequences

$$0 \rightarrow \mathcal{A}'_b - E_b \rightarrow \mathcal{A}'_b \rightarrow \mathcal{A}'_b \otimes \mathcal{O}_{E_b} \rightarrow 0$$

and

$$0 \rightarrow \mathcal{A}'_b - E_b - F_b \rightarrow \mathcal{A}'_b - E_b \rightarrow (\mathcal{A}'_b - E_b) \otimes \mathcal{O}_{F_b} \rightarrow 0.$$  

Since $\mathcal{A}'_b \otimes \mathcal{O}_{E_b} = \mathcal{L}_2(a_b - 1; a_b)$, we have that $h^i(\mathcal{A}'_b \otimes \mathcal{O}_{E_b}) = 0$ for $i \geq 0$. So, because of our induction hypothesis, the cohomology of the first exact sequence implies $h^i(\mathcal{A}'_b - E_b) = 0$ for $i = 0, \ldots, 3$. On the other hand $(\mathcal{A}'_b - E_b) \otimes \mathcal{O}_{F_b} \equiv \mathcal{O}(-1, a_b)$, and one easily checks that $h^i((\mathcal{A}'_b - E_b) \otimes \mathcal{O}_{F_b}) = 0$ for $i \geq 0$. So, from the cohomology of the second exact sequence and the fact that $\mathcal{A}'_b - E_b - F_b = \mathcal{A}_b$ we conclude that $h^i(\mathcal{A}_b) = 0$ for $i \geq 0$. □
6. Determining the Dimension of $\mathcal{L}$ for Case (2)

As seen before, the dimension of $\mathcal{L}_X(2m; m^8, m_9, \ldots, m_r)$ is equal to the dimension of $\mathcal{L}_X(2m; m^r)$ and this number is evaluated in the following.

**Theorem 6.1.** Let $r \geq 8$ then $\dim \mathcal{L}_3(2m; m^r) = m$.

**Remark 6.2.** The virtual dimension of the system $\mathcal{L}_3(2m; m^8)$ is equal to $m$ only if $r = 8$. This means that for bigger values of $r$ the system is special.

**Proof.** The statement is trivial for $m = 0$, so assume $m$ to be positive. Because of [DVL06, Theorem 6.2 (2)], we know that the base locus of the linear system $\mathcal{L}_3(2m; m^8)$ on $X_8$, is $mC$. This implies that $\dim \mathcal{L}_3(2m; m^8) = \dim \mathcal{L}_3(2m; m^r)$ for all $r \geq 8$ (because all the $p_i$’s lie on $C$). And according to [DVL, Theorem 5.1], we then obtain the statement. □

**Remark 6.3.** Let us just note that the techniques used throughout this paper can yield an alternative prove of theorem 6.1. More precisely, we can consider the blowing up $Y_r$ of $X$ along $C$, and the linear system $\tilde{\mathcal{L}}_{Y} := \tilde{\mathcal{L}}_{Y}(2; 1^r, 1)$ on $Y_r$. Using induction on $m$ and cohomology of the exact sequence

$$0 \longrightarrow \tilde{\mathcal{L}}_{Y} - \mathcal{S}_r \longrightarrow \tilde{\mathcal{L}}_{Y} \longrightarrow \tilde{\mathcal{L}}_{Y} \otimes \mathcal{O}_{\tilde{Q}} \longrightarrow 0,$$

with $\tilde{Q} \in \mathcal{S}_r := \tilde{\mathcal{L}}_{Y}(2; 1^r, 1)$, one can obtain that $\dim(\tilde{\mathcal{L}}_{Y}) = m$. And, since obviously $\dim \mathcal{L}_3(2m; m^r) = \dim \tilde{\mathcal{L}}_{Y}$, this proves the theorem.

7. Determining the Dimension of $\mathcal{L}$ for Case (3)

We recall that $t_i := \max\{0, m_1 + m_i - d\}$, for $i = 2, \ldots r$ is equal to the opposite of the intersection of the strict transform of $l_i$ with $\mathcal{L}_X$ (if this intersection is negative). The same is true for $t_1 := \max\{0, m_2 + m_3 - d\}$, where the line is $l_1$ through $p_2$ and $p_3$. Finally, the number $t := \lceil C \cdot \mathcal{L} + 1 \rceil$ can be described as

$$t := \max\{i \in \mathbb{N} \mid C \cdot (\mathcal{L}_X - iC) \leq 0\}.$$

The next theorem shows that the speciality of $\mathcal{L}$ only comes from lines $l_i$ for which $t_i > 0$ and from the curve $C$.

**Theorem 7.1.** Let $\mathcal{L}(d; m_1, \ldots, m_r)$ be a standard class which is not of type (2). Assume that

$$1 + m_r(8 - r) \leq C \cdot \mathcal{L} \leq 0$$

and $d \geq m_1 + t$, then

$$\dim \mathcal{L} = v(\mathcal{L}) + \sum_{i=1}^{r} \left( \begin{array}{c} t_i + 1 \\ 3 \end{array} \right) + (r - 8) \left( \begin{array}{c} t + 1 \\ 3 \end{array} \right) + n \left( \begin{array}{c} t + 1 \\ 2 \end{array} \right),$$

where $n \leq r - 9$ is a non-negative integer such that $n = (8 - r)(t - 1) + C \cdot \mathcal{L}$. 


Proof. Note that, since $L_X$ is standard and $C \cdot L \leq 0$ we have that $r$ is at least 9. From the assumptions, it follows that $0 < t \leq m_r$. Define $I$ to be the set of $i$ such that $t_i$ is positive and consider the system $\tilde{L}_Y$ defined on $\tilde{Y}_I$ of the form:

$$L_Y(d; m_1, \ldots, m_r; \{t_i\}_{i \in I}, t).$$

Since $\sum_{i \in I} t_i l_i + tC$ belongs to the base locus of $L_X$, we have that $\dim L = \dim \tilde{L}_Y$. The linear system $\tilde{L}_Y$ thus satisfies the following conditions

1. $2d \geq m_1 + m_2 + m_3 + m_4$
2. $d \geq m_1 \geq \ldots \geq m_r > 0$
3. $t_1 := \max\{0, m_2 + m_3 - d\}$ and $t_i := \max\{0, m_1 + m_i - d\}$ for $i = 2, \ldots, r$
4. $t = \left\lceil \frac{C \cdot L + 1}{r - 8} \right\rceil = \frac{C \cdot L + r - 8 - n}{r - 8}, 0 < t \leq m_r$
5. $d \geq m_1 + t$.

On the other hand, using lemma's 3.3 and 3.5 an easy but tedious calculation shows that $X(Y, L_Y) = X(\tilde{Y}, \tilde{L}_Y) + (r - 8)\left(\frac{t + 1}{3}\right) + y\left(\frac{t + 1}{2}\right)$.

So, in order to prove theorem 7.1 it is sufficient to show that for any $i \geq 1$

$$h^i(\tilde{Y}_I, \tilde{L}_Y) = 0 \quad (7.1)$$

Denote by $S_r := L_Y(2; 1^r; \{0\}_{i \in I}, 1)$ and consider a general element $\tilde{Q} \in S_r$ from which we have the exact sequence

$$0 \rightarrow \tilde{L}_Y - S_r \rightarrow \tilde{L}_Y \rightarrow \tilde{L}_Y \otimes O_{\tilde{Q}} \rightarrow 0.$$

By abuse of notation, let $C$ also denote the anticanonical curve on $\tilde{Q} \subset \tilde{Y}_I$, then $F|_{\tilde{Q}} = C$.

So, since $\tilde{Q}$ and $F_i$ are disjoint, we have that the restriction of $\tilde{L}_Y$ to $\tilde{Q}$ is:

$$\tilde{L}_Y \otimes O_{\tilde{Q}} = L_Q(d - 2t, d - 2t; m_1 - t, \ldots, m_r - t).$$

Observe that this system is standard because $d \geq m_1 + t$ and $m_r \geq t$. On the other hand, condition (4) implies that $L_Q \cdot K_{\tilde{Q}} \leq -1$. So it follows from [Har85, Theorem 1.1] that $h^i(\tilde{L}_Y \otimes O_{\tilde{Q}}) = 0$ for any $i \geq 1$.

In order to obtain (7.1), we obviously need the vanishing of the higher cohomology groups of $\tilde{L}_Y - S_r$, which is of the form:

$$\tilde{L}_Y(d - 2; m_1 - 1, \ldots, m_r - 1, \{t_i\}_{i \in I}; t - 1).$$

First, let us check when $L_Y(d'; m_1', \ldots, m_r', \{t_i\}_{i \in I}; t') = \tilde{L}_Y - S_r$ satisfies conditions (1)-(5).

Observe that conditions (1), (3) and (5) are always satisfied.

(2) is satisfied unless $m_r = 1$ and in this case $t = 1$. 


(4) Since
\[ t' = \frac{m_1 + \cdots + m_r - 4d + r - 8 - n}{r - 8} - 1 \]
\[ = \frac{m'_1 + \cdots + m'_r - 4d' + r - 8 - n}{r - 8}, \]
we obtain \( 0 < t' \leq m'_t \) unless \( t = 1 \).

In this way we see that \( t > 1 \), then the system satisfy all the conditions. This means that, if \( t' \geq 1 \), we can consider an exact sequence as above, and, using arguments as before, we can reduce to proving the vanishing of the higher cohomology groups of \( \tilde{\mathcal{L}}_Y - 2\mathcal{S}_r \). Obviously, this procedure can be repeated \( t \) times, or thus until we are left with proving the vanishing of the higher cohomology groups of
\[ \tilde{\mathcal{L}}_Y - t\mathcal{S}_r = \tilde{\mathcal{L}}_Y(d - 2t; m_1 - t, \ldots, m_r - t; \{t_i\}_{i \in I}; 0). \]

Since \( t = 0 \), we only need to prove the vanishing of the higher cohomology groups on \( Y_I \) of the class \( \tilde{\mathcal{L}}_Y(d - 2t; m_1 - t, \ldots, m_r - t; \{t_i\}_{i \in I}) \), which we will denote as \( \mathcal{L}_Y \). Now, since
\[ C \cdot \mathcal{L}_Y - 1 = 4d - 8t - m_1 - \cdots - m_r + rt - 1 = r - 9 - n \geq 0, \]
we have that the class \( \mathcal{L}_Y \) satisfies the conditions of theorem 5.1 and we conclude that \( h^i(\mathcal{L}_Y) = 0 \) for \( i \geq 1 \). As mentioned before, this is enough to prove the theorem. \( \square \)

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