We revisit the quantum/classical integrable model correspondence in the context of inhomogeneous finite length XXZ spin-1/2 chains with periodic boundary conditions and show that the Bethe scalar product of an arbitrary state and a Bethe eigenstate is a discrete KP τ-function. The continuous Miwa variables of discrete KP are the rapidities of the arbitrary state.

1. Introduction.

Quantum models of the statistical mechanical type (the only quantum models discussed in this note) such as integrable 1-dimensional quantum spin chains, and classical models such as integrable nonlinear partial differential equations, are related in the sense that the methods used to compute in the former, particularly the quantum inverse scattering transform, also known as the algebraic Bethe Ansatz, are quantum versions of those used to compute in the latter, namely the classical inverse scattering transform. It is therefore natural to expect that the quantum integrable models have classical limits in which they reduce to classical counterparts.

What is less than natural to expect, at least to our minds, is that basic objects in quantum integrable models, such as the correlation functions, turn out to have direct interpretations in terms of objects in classical integrable models, such as solutions of integrable nonlinear partial differential and difference equations, without taking a classical limit. But this turns out to be the case, and it points to a direct connection between quantum and classical integrable models that is distinct from, and to our minds at least as fundamental as that obtained by taking a classical limit.
Notes on the literature. The following is far from a comprehensive survey of the relevant literature. To the best of our knowledge, a direct connection between quantum (statistical mechanical) and classical models of the type that we are interested in first appeared in [1], where Ising spin-spin correlation functions in the scaling limit were shown to satisfy Painlevé equation of the third kind, and subsequently in [2, 3], where critical Ising correlation functions on the lattice were shown to satisfy the Toda lattice equation in Hirota’s bilinear form. Further results, along the same lines as in [1], for the XXZ spin chain at the free fermion point, were obtained in [4], as reviewed in [5].

The fact that \( \tau \)-functions (solutions of Hirota’s bilinear equations) appear in the Ising model as well as in KP theory was discussed in works by the Kyoto group and reviewed in [6] where it was argued that the mathematical reason underlying this coincidence is the fact that both quantum and classical models are based on infinite dimensional Lie algebras that are realized in terms of free fermions.

Closest to the spirit of this note is the work of Krichever et al. [7], reviewed in [8]. The starting point of [7] is the observation that the Bethe eigenvalues satisfy Hirota’s difference equation, various limits of which lead to a large number of integrable differential and difference equations [9]. We will comment on the results of [7] and how they differ from the result in this note in section 6. More recently, studies of the ultra-discrete limit of quantum integrable spin chains revealed many classical integrable structures [10].

Finally, while we are only interested in integrable quantum models in statistical mechanics in this note, it is important to mention bosonisation (the operator formulation of Sato’s theory) as a deep and established correspondence between the quantum field theories of free fermions, which are integrable quantum models, and classical integrable hierarchies, as reviewed in [11]. In this correspondence, expectation values of fermion operators have direct interpretations in terms of solutions of integrable nonlinear partial differential equations. Bosonization was further extended to connect KP theory and conformal field theories on Riemann surfaces (which are integrable quantum models) in [12].

The long term aim of our work is to develop a correspondence between integrable statistical mechanical models and classical integrable hierarchies that is as direct and detailed as that obtained by bosonisation between free fermions and classical integrable hierarchies.
Bethe scalar products and continuous KP $\tau$-functions. Consider the inhomogeneous length-$L$ XXZ spin-$\frac{1}{2}$ chain with periodic boundary conditions. Following [13], the Bethe scalar product $\langle \lambda_1, \cdots, \lambda_N | \mu_1, \cdots, \mu_N \rangle_\beta$ of an arbitrary state $\langle \lambda_1, \cdots, \lambda_N \rangle$ where the auxiliary space rapidities $\{\lambda_1, \cdots, \lambda_N\}$ are free, and a Bethe eigenstate $| \mu_1, \cdots, \mu_N \rangle_\beta$ where the auxiliary space rapidities $\{\mu_1, \cdots, \mu_N\}$ obey the Bethe equations, is a polynomial $\tau$-function of the continuous (differential) KP hierarchy. In this identification, the KP time variables $\{t_1, t_2, \cdots\}$ are power sums of the free rapidities $\{\lambda_1, \cdots, \lambda_N\}$. However, these polynomial KP $\tau$-functions involve by construction more time variables than free rapidities. The reason is as follows.

Expanding the scalar product in terms of Schur polynomials $s_\lambda$, associated to Young diagrams $\{\lambda\}$, that are functions of the rapidities $\{\lambda_1, \cdots, \lambda_N\}$, the maximal number of rows in any Young diagram $\lambda$ is $N$. Switching to KP time variables $\{t_1, t_2, \cdots\}$ that are powers sums in the rapidities, we obtain character polynomials $\chi_\lambda$ that depend on effectively as many time variables as the number of cells in (that is, the size of) $\lambda$ which is larger than $N$. Consequently, the KP time variables $\{t_1, t_2, \cdots\}$ were formally considered in [13] to be independent, and the Bethe scalar product was defined as a restricted KP $\tau$-function obtained by setting $\{t_1, t_2, \cdots\}$ to be power sums of a smaller number of independent variables $\{\lambda_1, \cdots, \lambda_N\}$.

Bethe scalar products and discrete KP $\tau$-functions. In this note, we simplify the correspondence of [13] by working solely in terms of the free rapidities $\{\lambda_1, \cdots, \lambda_N\}$ which are now continuous Miwa variables and the $\tau$-functions that we obtain are those of the discrete KP hierarchy [14, 15].

Outline of contents. In section 2, we recall basic facts related to symmetric functions, Casoratian matrices and Casoratian determinants. In 3, we recall basic facts related to the XXZ spin-$\frac{1}{2}$ chain, the algebraic Bethe Ansatz, the Bethe scalar product, recall Slavnov’s determinant expression of the Bethe scalar product and show that it is a Casoratian determinant. In 4, we recall basic facts related to the continuous and discrete KP hierarchies and define the Miwa variables that relate the two. In 5, we show that Bethe scalar products in the XXZ spin-$\frac{1}{2}$ chain with periodic boundary conditions are discrete KP $\tau$-functions. In 6, we collect a number of remarks. Space limitations allow us to give no more than the minimal definitions necessary to fix the notation and terminology supplemented by references to relevant sources.
2. Symmetric functions and Casoratians.

The canonical reference to symmetric functions is [16]. Casoratian matrices and determinants are carefully discussed in [15]. The definitions in [15] are more general than those used in this note.

Frequently used notation. We use \{x\} for the set of finitely many variables \{x_1, x_2, \ldots, x_N\}, or infinitely many variables \{x_1, x_2, \ldots\}. The cardinality of the set should be clear from the context. We use \{\hat{x}_m\} for \{x\} but with the element \(x_m\) missing. In the case of sets with a repeated variable \(x_i\), we use the superscript \((m_i)\) to indicate the multiplicity of \(x_i\), as in \(x_i^{(m_i)}\). For example, \{\(x^{(3)}_1, x, x^2_3, x_4, \ldots\}\) is the same as \{\(x_1, x_1, x_1, x_2, x_3, x_4, \ldots\\} and \(f\{\ldots, x_i^{(m_i)}, \ldots\}\) is equivalent to saying that \(f\) depends on \(m_i\) distinct variables all of which have the same value \(x_i\). For simplicity, we use \(x_i\) to indicate \(x_i^{(1)}\). In calculations, it is safer to think of any \(x_i\) with multiplicity \(m_i > 1\) initially as distinct, that is \(\{x_i, 1, x_i, 2, \ldots, x_i, m_i\}\), then set these \(m_i\) variables equal to the same value \(x_i\) at the end.

We use the bracket notation \([x] = e^{x} - e^{-x}\), and
\[
\Delta \{x\} = \prod_{1 \leq i < j \leq N} (x_i - x_j), \quad \Delta_{\text{trig}} \{\lambda\} = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)
\]
for the Vandermonde determinant and its trigonometric analogue.

The elementary symmetric function \(e_i\{x\}\) in \(N\) variables \(\{x\}\) is the coefficient of \(k^i\) in the expansion
\[
\prod_{i=1}^{N} (1 + x_i k) = \sum_{i=0}^{\infty} e_i \{x\} k^i
\]
For example, \(e_0 \{x\} = 1, e_1 \{x_1, x_2, x_3\} = x_1 + x_2 + x_3, e_2 \{x_1, x_2\} = x_1 x_2\).

The complete symmetric function \(h_i\{x\}\) in \(N\) variables \(\{x\}\) is the coefficient of \(k^i\) in the expansion
\[
\prod_{i=1}^{N} \frac{1}{1 - x_i k} = \sum_{i=0}^{\infty} h_i \{x\} k^i
\]
For example, \(h_0 \{x\} = 1, h_1 \{x_1, x_2, x_3\} = x_1 + x_2 + x_3, h_2 \{x_1, x_2\} = x_1^2 + x_1 x_2 + x_2^2,\) and \(h_i \{x\} = 0\) for \(i < 0\).
Useful identities for $h_i\{x\}$. From Equation (3), it is straightforward to show that

$$h_i\{x\} = h_i\{\hat{x}_m\} + x_m h_{i-1}\{x\} \quad (4)$$

and from that, one obtains

$$h_i\{x_1, x_2, \cdots, x_N\} = h_i\{x_1^{(2)}, x_2, \cdots, x_N\} - x_1 h_{i-1}\{x_1^{(2)}, x_2, \cdots, x_N\} \quad (5)$$

$$(x_2 - x_1)h_i\{x_1^{(2)}, x_2^{(2)}, x_3, \cdots, x_N\} = x_2 h_i\{x_1^{(2)}, x_2^{(2)}, \cdots, x_N\} - x_1 h_i\{x_1^{(2)}, x_2, \cdots, x_N\} \quad (6)$$

The discrete derivative $\Delta_m h_i\{x\}$ of $h_i\{x\}$ with respect to any one variable $x_m \in \{x\}$ is defined using Equation (4) as

$$\Delta_m h_i\{x\} = \frac{h_i\{x\} - h_i\{\hat{x}_m\}}{x_m} = h_{i-1}\{x\} \quad (7)$$

Note that the effect of applying $\Delta_m$ to $h_i\{x\}$ is a complete symmetric function $h_{i-1}\{x\}$ of degree $i - 1$ in the same set of variables $\{x\}$. The difference operator in Equation (7) is not the most general definition of a discrete derivative, but it is sufficient for the purposes of this note. For a more general definition, see [15].

The Schur polynomial $s_\lambda\{x\}$ indexed by a Young diagram $\lambda = [\lambda_1, \ldots, \lambda_r]$ with $\lambda_i \neq 0$, for $1 \leq i \leq r$, and $\lambda_i = 0$, for $r + 1 \leq i \leq N$, is

$$s_\lambda\{x\} = \frac{\det \left( x_i^{\lambda_j - j + N} \right)_{1 \leq i, j \leq N}}{\Delta\{x\}} = \det \left( h_{\lambda_i - i + j}\{x\} \right)_{1 \leq i, j \leq N} \quad (8)$$

For example, $s_\emptyset\{x\} = 1$, $s_{[1]}(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $s_{[1,1]}(x_1, x_2) = x_1 x_2$. The first equality in Equation (8) is the definition of $s_\lambda\{x\}$. The second is the Jacobi-Trudi identity for $s_\lambda\{x\}$. $s_\lambda\{x\}$ is symmetric in the elements of $\{x\}$ and requires no more than $r$ (the number of non-zero rows in $\lambda$) variables to be non-vanishing.
The one-row character polynomial $\chi_i\{t\}$ indexed by a one-row Young diagram of length $i$, is the $i$-th coefficient in the generating series

$$\sum_{i=0}^{\infty} \chi_i\{t\} k^i = \exp \left( \sum_{i=1}^{\infty} t_i k^i \right)$$

For example, $\chi_0\{t\} = 1$, $\chi_1\{t\} = t_1$, $\chi_2\{t\} = \frac{t_1^2}{2} + t_2$, $\chi_3\{t\} = \frac{t_1^3}{3} + t_1 t_2 + t_3$, and $\chi_i\{t\} = 0$ for $i < 0$. Since $t_i$ has degree $i$, $\chi_i$ is not symmetric in $\{t\}$ and generally depends on as many $t$-variables as the row-length $i$.

The character polynomial $\chi_\lambda\{t\}$ indexed by a Young diagram $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r]$ with $r$ non-zero-length rows, $r \leq N$, is

$$\chi_\lambda\{t\} = \det \left( \chi_{\lambda_i-i+j}\{t\} \right)_{1 \leq i,j \leq n}$$

For example $\chi_{[1,1]}\{t\} = \frac{t_1^2}{2} - t_2$, $\chi_{[2,1]}\{t\} = \frac{t_1^3}{3} - t_2$, $\chi_{[2,2]}\{t\} = \frac{t_1^4}{12} - t_1 t_3 + t_2^2$. Notice that $\chi_\lambda\{t\}$ can depend on all $t_i$, for $i \leq |\lambda|$, where $|\lambda|$ is the sum of the lengths of all rows in (or area of) $\lambda$.

From character polynomials to Schur polynomials. Assuming that the $t$-variables are independent and that we have sufficiently many $x$-variables, then setting $t_m \to \frac{1}{m} \sum_{i=1}^{N} x_i^m$ sends $\chi_\lambda\{t\} \to h_\lambda\{x\}$. In this note, as in [13], we study Bethe scalar products that are polynomials in $N$ variables $\{x_1, x_2, \ldots, x_N\}$. We can expand these scalar products in terms of Schur polynomials $s_\lambda\{x\}$ where $\{\lambda\}$ has at most $N$ rows, or in terms of the corresponding character polynomials $\chi_\lambda\{t\}$ that require more $t$-variables (which are power sums in the $x$-variables) than $N$ and therefore cannot be all independent. We choose to work in terms of the $x$-variables and $s_\lambda\{x\}$.

Casoratian matrices and determinants. A Casoratian matrix $M$ of the type that appears in this note is such that the elements $M_{ij}$ satisfy either

$$M_{i,j+1}\{x\} = \Delta_m M_{ij}\{x\}, \quad M_{i+1,j}\{x\} = \Delta_m M_{ij}\{x\}$$

where the discrete derivative $\Delta_m$ is taken with respect to any one variable $x_m \in \{x\}$. If $M$ is a Casoratian matrix, then $\det M$ is a Casoratian determinant. Casoratian determinants are discrete analogues of Wronskian determinants.
3. The XXZ spin-$\frac{1}{2}$ chain and the Algebraic Bethe Ansatz.

The XXZ spin-$\frac{1}{2}$ chain is discussed in detail in [17, 18]. A standard reference to the algebraic Bethe Ansatz, including the Bethe scalar product and Slavnov’s determinant expression, is [5]. We leave the definition of auxiliary and quantum spaces, auxiliary and quantum rapidities, and the precise action of the various operators to [5].

Frequently used variables. In the following, $L$ is the number of sites in a periodic XXZ spin-$\frac{1}{2}$ chain, and $N$ is the number of Bethe operators $B(\mu_i)$ that act on the reference state $|0\rangle$ to create an XXZ state $|\mu_1, \cdots, \mu_N\rangle$. $N$ is also the rank of the matrix whose determinant is Slavnov’s expression for the Bethe scalar product. We use the set $\{\lambda\}$ for the free auxiliary space rapidities, $\{\mu\}$ or more explicitly $\{\mu_\beta\}$ for the auxiliary space rapidities that satisfy the Bethe equations, and $\{\nu\}$ for the quantum space rapidities (the inhomogeneities). A Bethe eigenstate state whose rapidities satisfy the Bethe equations is also denoted by a subscript $\beta$, such as $|\lambda\rangle_\beta$.

$\gamma$ is the crossing parameter. We use the exponentiated variables $\{x_i, y_i, z_i, q\} = \{e^{\lambda_i}, e^{\mu_i}, e^{\nu_i}, e^\gamma\}$, but still refer to the exponentiated variables $\{x, y, z\}$ as rapidities rather than exponentiated rapidities for simplicity.

The $L$-operator of the XXZ spin-$\frac{1}{2}$ chain is

$$L_{ai}(\lambda, \nu) = \begin{pmatrix} [\lambda - \nu + \gamma] & 0 & 0 & 0 \\ 0 & [\lambda - \nu] & [\gamma] & 0 \\ 0 & [\gamma] & [\lambda - \nu] & 0 \\ 0 & 0 & 0 & [\lambda - \nu + \gamma] \end{pmatrix}_{ai}$$

where $a$ is an auxiliary space index and $i$ is a quantum space index.

The monodromy matrix of the inhomogeneous length-$L$ XXZ spin-$\frac{1}{2}$ chain is

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = \prod_{i=1}^{L} L_{ai}(\lambda, \nu_i)$$

where it is conventional to suppress the dependence on the inhomogeneous quantum space rapidities $\nu_i$ in $T_a$ and its elements, and each of the operators $A, B, C,$ and $D$ acts in the tensor product $V_1 \otimes \cdots \otimes V_L$ where $V_i$ is a vector space isomorphic to $\mathbb{C}^2$. 
The transfer matrix is the trace of the monodromy matrix over the auxiliary space,

\[ \text{Tr}_a T_a(\lambda) = A(\lambda) + D(\lambda) \]  

(14)

An arbitrary state \(|\mu\rangle\) is generated by the action of the \(B(\mu)\) operators on the reference state \(|0\rangle = \otimes^L \begin{pmatrix} 1 \\ 0 \end{pmatrix}\),

\[ |\mu\rangle = B(\mu_1) \cdots B(\mu_N)|0\rangle \]  

(15)

An arbitrary dual state \(\langle \lambda |\) is generated by the action of the \(C(\lambda)\) operators on the dual reference state \(\langle 0 | = \otimes^L \begin{pmatrix} 1 \\ 0 \end{pmatrix}\),

\[ \langle \lambda | = \langle 0 | C(\lambda_1) \cdots C(\lambda_N) \]  

(16)

The scalar product of a state and a dual state is

\[ \langle \lambda | \mu \rangle = \langle 0 | C(\lambda_1) \cdots C(\lambda_N) B(\mu_1) \cdots B(\mu_N)|0\rangle \]  

(17)

A Bethe eigenstate \(|\mu\rangle_\beta\) is an eigenstate of the transfer matrix,

\[ \left( A(\lambda) + D(\lambda) \right) |\mu\rangle_\beta = E(\lambda) |\mu\rangle_\beta \]  

(18)

where \(E(\lambda)\) is the corresponding Bethe eigenvalue. For a state \(|\mu\rangle\) to be a Bethe eigenstate, its auxiliary space rapidities must satisfy a set of Bethe equations.

The Bethe equations that must be satisfied by the \(N\) auxiliary space rapidities of a state \(|\mu\rangle = B(\mu_1) \cdots B(\mu_N)|0\rangle\) in order to be a Bethe eigenstate, in the specific case of the inhomogeneous length-\(L\) spin-\(\frac{1}{2}\) chain, are

\[ \prod_{i=1}^L [\mu - \nu_i + \gamma] \prod_{i=1}^N \prod_{j \neq i} [\mu_i - \mu_j - \gamma] [\mu_i - \mu_j + \gamma] = 1 \]  

(19)

where \(\{\nu_1, \cdots, \nu_L\}\), are the quantum space rapidities, which are taken to be part of the parameters that specify the spin chain, rather than the definition of the Bethe state.
A Bethe scalar product is a scalar product of an arbitrary state $\langle \lambda |$ and a Bethe eigenstate $|\mu\rangle_{\beta}$,

$$\langle \lambda |\mu\rangle_{\beta} = \langle 0 | C(\lambda_1) \ldots C(\lambda_N) B(\mu_{1,\beta}) \ldots B(\mu_{N,\beta}) | 0 \rangle$$  \hspace{1cm} (20)

Bethe scalar products as in Equation (20) play a central role in computing XXZ correlation functions [19], hence their importance.

Slavnov’s determinant expression. In [20], Slavnov obtained an elegant determinant expression for the Bethe scalar product,

$$\langle \lambda |\mu\rangle_{\beta} = [\gamma] \prod_{i,j=1}^{N} \prod_{k=1}^{L} \frac{[\lambda_i - \mu_j + \gamma]}{\Delta\{\lambda\} \Delta\{\mu\}} \prod_{k=1}^{N} \prod_{l=1}^{L} [\lambda_k - \nu_l] [\mu_k - \nu_l] \det \Omega$$  \hspace{1cm} (21)

where the components of the $N \times N$ matrix $\Omega$ are

$$\Omega_{ij} = \frac{1}{[\lambda_i - \mu_j][\lambda_i - \mu_j + \gamma]} - \frac{1}{[\mu_j - \lambda_i][\mu_j - \lambda_i + \gamma]} \prod_{k=1}^{L} \frac{[\lambda_i - \nu_k + \gamma]}{[\nu_k - \lambda_i]} \prod_{l=1}^{N} \frac{[\lambda_l - \mu_i - \gamma]}{[\lambda_l - \mu_i + \gamma]}$$  \hspace{1cm} (22)

Slavnov’s scalar product is the main object of interest in this note. We wish to show that it is a Casoratian determinant and that the latter satisfy the bilinear identities of a discrete KP hierarchy [15].

Re-writing Slavnov’s determinant expression. In [13], it was found useful to rewrite Slavnov’s determinant expression for the Bethe scalar product as follows. First, we change variables and work in terms of exponentials of the original variables as follows

$$\{e^{2\lambda_i}, e^{2\mu_i}, e^{2\nu_i}, e^\gamma\} \to \{x_i, y_i, z_i, q\}$$  \hspace{1cm} (23)

but continue to call the exponentials $\{x, y, z\}$ rapidities as that is simpler and should cause no confusion. Ignoring prefactors that do not depend on $\{x\}$, it was shown in [13] that the relevant part of Slavnov’s determinant expression can be re-written as
\[ \det \Omega' = \frac{\det \Omega}{\Delta \{x\}} , \text{ where } \Omega_{ij} = \sum_{k=1}^{N+L-1} x_i^{k-1} \kappa_{kj} , \quad \kappa_{kj} = - \sum_{l=1}^{k} y_j l^{-1} \rho_{lj} , \]

(24)

and

\[ \rho_{lj} = \left( \prod_{m=1}^{L} (y_j q - z_m q^{-1}) \right) \left( \prod_{n \neq j}^{N} (y_j - y_n q^2) \cdot e_{(L+N-1)} \{ -y_j q^{-2} \} \{ -z \} \right) \]

\[ - \left( \prod_{m=1}^{L} (y_j q - z_m q^{-1}) \right) \left( \prod_{n \neq j}^{N} (y_j - y_n q^{-2}) \cdot e_{(L+N-1)} \{ -y_j q^2 \} \{ -zq^{-2} \} \right) \]

(25)

In Equation (25), \( e_k \{ \tilde{y}_j \} \{ z \} \) is the \( k \)-th elementary symmetric polynomial in the set of variables \( \{ y \} \cup \{ z \} \) with the omission of \( y_j \).

A Bethe scalar product is a Casoratian determinant. We wish to show that Slavnov’s determinant expression is Casoratian in the free rapidities \( \{ x \} \) of the general state. Expanding \( \det \Omega \), using the Cauchy-Binet identity, we obtain

\[ \det \Omega = \det \left( \sum_{k=1}^{N+L-1} x_i^{k-1} \kappa_{kj} \right) \]

\[ = \sum_{1 \leq k_1 < \cdots < k_N \leq N+L-1} \det \left( x_i^{k_j-1} \right) \det \left( \kappa_{k_i,j} \right) \]

\[ = \sum_{0 \leq \lambda_N \leq \cdots \leq \lambda_1 \leq L-1} \det \left( x_i^{\lambda_{N+1-j}+j-1} \right) \det \left( \kappa_{\lambda_i(N-i+1)+i,j} \right) \]

\[ = \sum_{0 \leq \lambda_N \leq \cdots \leq \lambda_1 \leq L-1} \det \left( x_i^{\lambda_{j+N+1-i,j}} \right) \det \left( \kappa_{\lambda_i+N+1-i,j} \right) \]

(26)

From the definition of Schur polynomials that uses the Jacobi-Trudi identity in Equation (8) we obtain
\[
\det \Omega' = \frac{\det \Omega}{\Delta(x)} = \sum_{0 \leq \lambda_N \leq \cdots \leq \lambda_1 \leq L-1} \det \left( h_{\lambda_j - j + i}(x) \right) \det \left( \kappa_{\lambda_i + N+1-i,j} \right) \\
= \sum_{0 \leq \lambda_N \leq \cdots \leq \lambda_1 \leq L-1} \det \left( h_{\lambda_{N+1} - N-1-i+j}(x) \right) \det \left( \kappa_{\lambda_i + N+1-i,j} \right) \\
= \sum_{1 \leq k_i \leq \cdots \leq k_N \leq N+L-1} \det \left( \sum_{k=1}^{N+L-1} h_{k-N-1+i}(x) \kappa_{k,j} \right) \\
= \det \left( \sum_{k=1}^{N+L-1} h_{k-N-1+i}(x) \kappa_{k,j} \right) \\
\text{(27)}
\]

Hence \( \det \Omega' \) is Casoratian in \( \{x\} \). Next, we need to show that a Casoratian determinant is a solution of the bilinear identities of discrete KP, but this requires a number of definitions which we outline in the next section.

4. Continuous KP, Miwa variables and discrete KP.

A standard introduction to the continuous KP hierarchy is [21]. Miwa variables are discussed in detail in [22] where further references to their applications are provided. The discrete KP hierarchy was introduced in [9], and further studied in [23] and [14]. In this note, we follow the treatment in [15].

**Continuous KP** is an infinite hierarchy of integrable partial differential equations generated in Hirota’s bilinear form by expanding the bilinear identity

\[
\oint \int_{k=k_{\infty}} d\xi(t-k) \tau(t - \epsilon(k)) \tau(t + \epsilon(k^{-1})) = 0 \quad \text{(28)}
\]

where \( k \in \mathbb{P}^1 \), the contour integral is around the point at infinity \( k_{\infty} \in \mathbb{P}^1 \), \( \{t\} = \{t_1, t_2, t_3, \cdots \} \), \( \xi(t, k) = \sum_{i=1}^{\infty} t_i k^i \), \( \epsilon(k^{-1}) = \left\{ \frac{1}{k}, \frac{1}{2k}, \frac{1}{3k}, \cdots \right\} \), \( \{t \pm \epsilon(k^{-1})\} = \{t_1 \pm \frac{1}{k}, t_2 \pm \frac{1}{2k}, t_3 \pm \frac{1}{3k}, \cdots \} \). The simplest KP equation in the hierarchy is

\[
\left( D_1^3 + 3D_2^2 - 4D_1 D_3 \right) \tau \cdot \tau = 0 \quad \text{(29)}
\]

where \( D_i \) is the Hirota derivative with respect to \( t_i \). For the precise definition of \( D_i \) and that of the notation \( \tau \cdot \tau \), see [21].
Continuous and discrete Miwa variables. In [23], Miwa introduced two infinite sets of variables, the continuous variables \( \{ x \} = \{ x_1, x_2, \cdots \} \), and the discrete (and integer valued) variables \( \{ m \} = \{ m_1, m_2, \cdots \} \), and showed that setting

\[
t_j = \sum_{i=1}^{\infty} m_i \frac{x_j^i}{j}
\]

transforms \( \tau \)-functions of continuous KP to \( \tau \)-functions of a hierarchy of bilinear difference equations, namely discrete KP, studied in detail in [14]. These variables are now known as continuous and discrete Miwa variables, respectively.

Multiplicities. From Equation (30), one can see that the discrete variables \( \{ m \} \), where \( m_i \in \mathbb{Z} \) are multiplicities of the continuous variables \( \{ x \} \). In other words, \( m_i > 1 \) is equivalent to saying that \( x_i \) occurs \( m_i \) times in \( \{ x \} \), or that there are \( m_i \) continuous variables that have the same value \( x_i \).

Discrete KP is an infinite hierarchy of integrable partial difference equations in an infinite set of continuous Miwa variables \( \{ x \} \), where time evolution is obtained by changing the multiplicities \( \{ m \} \) of these variables. In this note, we are interested in situation where the total number of continuous Miwa variables is finite, and the sum of all multiplicities is \( N \). In this case, the discrete KP hierarchy can be written in bilinear form as \( n \times n \) determinant equations

\[
\det \begin{vmatrix}
1 & x_1 & \cdots & x_1^{n-2} & x_2^{n-2} \tau_{+1} \{ x \} & \tau_{-1} \{ x \} \\
1 & x_2 & \cdots & x_2^{n-2} & x_2^{n-2} \tau_{+2} \{ x \} & \tau_{-2} \{ x \} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & x_n & \cdots & x_n^{n-2} & x_n^{n-2} \tau_{+n} \{ x \} & \tau_{-n} \{ x \}
\end{vmatrix} = 0
\]

where \( 3 \leq n \leq N \), and

\[
\tau_{+i} \{ x \} = \tau \{ x_1^{(m_1)}, \cdots, x_i^{(m_i+1)}, \cdots, x_N^{(m_N)} \}, \\
\tau_{-i} \{ x \} = \tau \{ x_1^{(m_1)}, \cdots, x_i^{(m_i)}, \cdots, x_N^{(m_N+1)} \}
\]

In other words, if \( \tau \{ x \} \) has \( m_i \) copies of the variable \( x_i \), then \( \tau_{+i} \{ x \} \) has \( m_i + 1 \) copies of \( x_i \) and the multiplicities of all other variables remain the same,
while \( \tau_{-i}\{x\} \) has one more copy of each variable except \( x_i \). Equivalently, one can use the simpler notation

\[
\begin{align*}
\tau_{+i}\{x\} &= \tau\{m_1, \ldots, (m_i + 1), \ldots, m_N\}, \\
\tau_{-i}\{x\} &= \tau\{(m_1 + 1), \ldots, m_i, \ldots, (m_N + 1)\}
\end{align*}
\] (33)

The simplest discrete KP bilinear difference equation, in the notation of Equation (33), is

\[
\begin{align*}
x_i(x_j - x_k) &\tau_{+i}\{m_i + 1, m_j, m_k\} \tau\{m_i, m_j + 1, m_k + 1\} \\
+ x_j(x_k - x_i) &\tau_{+i}\{m_i, m_j + 1, m_k\} \tau\{m_i + 1, m_j, m_k + 1\} \\
+ x_k(x_i - x_j) &\tau_{+i}\{m_i, m_j, m_k + 1\} \tau\{m_i + 1, m_j + 1, m_k\} = 0
\end{align*}
\] (34)

where \( \{x_i, x_j, x_k\} \in \{x\} \) and \( \{m_i, m_j, m_k\} \in \{m\} \) are any two (corresponding) triples in the sets of continuous and discrete (integral valued) Miwa variables. Equation (34) is the discrete analogue of Equation (29).

**Discrete time evolution in discrete KP.** Each continuous Miwa variable \( x_i \) corresponds to a time variable in discrete KP. Time evolution in discrete KP, in direction \( x_i \), is given by the discrete changes in the multiplicities \( m_i \) of \( x_i \). Notice that as a multiplicity \( m_i \) changes by \( \pm 1 \), the rank of the matrix \( M_{i+1} \), where \( \det M_{i+1} = \tau_{i+1} \), remains the same as the rank of \( M \), where \( \det M = \tau \).

5. **Bethe scalar products are discrete KP \( \tau \)-functions.**

In this section, we adapt the general treatment of [15] to the specific case of Slavnov’s determinant expressions. We do this in detail to show explicitly that the general (and slightly abstract) identities and theorems in [15] apply to Slavnov’s expressions.

**Re-arranging the elements of Slavnov’s determinant.** Given the \( N \times N \) matrix \( \Omega' \) with elements

\[
\omega'_{ij} = \sum_{k=1}^{N+L-1} h_{k-N-1+i}\{x\} \kappa_{kj}
\] (35)

let us consider the matrix \( \Omega'' \) with elements
\[ \omega_{ij} = \sum_{k=1}^{N+L-1} c_{ik} h_{k-j} \{ x \} \] (36)

which is obtained from \( \Omega' \) by reordering the rows of the latter from bottom to top, changing the rows and the columns and setting \( c_{ik} = \kappa_{ki} \). Notice that we use \( \omega \) rather than \( \omega'' \) for the elements of \( \Omega'' \) to simplify the notation.

Since \( \det \Omega' = (-1)^{N(N-1)/2} \det \Omega \), it is sufficient to show that \( \Omega \) satisfies the difference bilinear identities of discrete KP.

**Identities for the elements \( \omega_{ij} \).** It follows from Equations (4–5) that the elements \( \omega_{ij} \) of \( \Omega'' \) satisfy analogous identities

\[
\omega_{ij} \{ x_1, \ldots, x_{(2)m}, \ldots, x_{N} \} = \omega_{ij} \{ x_1, \ldots, x_N \} + x_m \omega_{i,j+1} \{ x_1, \ldots, x_{(2)m}, \ldots, x_N \} \] (37)

\[
(x_r - x_s) \omega_{ij} \{ x_1, \ldots, x_r^{(2)}, \ldots, x_{N} \} = x_r \omega_{ij} \{ x_1, \ldots, x_r^{(2)}, \ldots, x_N \} - x_s \omega_{ij} \{ x_1, \ldots, x_s^{(2)}, \ldots, x_N \} \] (38)

From Equation (7), we see that

\[
\Delta_m \omega_{ij} \{ x_1, \ldots, x_{N} \} = \omega_{i,j+1} \{ x_1, \ldots, x_{N} \} \] (39)

which is equivalent to the statement that \( \det \Omega \) is Casoratian.

**Notation for column vectors with elements \( \omega_{ij} \).** We need the column vector

\[
\vec{\omega}_j = \left\{ \omega_{1j} \{ x_{1}^{(m_1)}, \ldots, x_{N}^{(m_N)} \}, \omega_{2j} \{ x_{1}^{(m_1)}, \ldots, x_{N}^{(m_N)} \}, \ldots, \omega_{Nj} \{ x_{1}^{(m_1)}, \ldots, x_{N}^{(m_N)} \} \right\} \] (40)

and write
for the corresponding column vector where the multiplicities of the variables $x_{k_1}, \ldots, x_{k_n}$ are increased by 1.

**Notation for determinants with elements $\omega_{ij}$.** We also need the determinant

$$
\tau = \det \begin{pmatrix} \omega_1 & \omega_2 & \cdots & \omega_N \end{pmatrix} = \left| \begin{array}{cccc} \omega_1 & \omega_2 & \cdots & \omega_N \\ \end{array} \right|$

and the notation

$$
\tau^{[k_1, \ldots, k_n]} = \left| \begin{array}{cccc} \omega_1^{[k_1, \ldots, k_n]} & \omega_2^{[k_1, \ldots, k_n]} & \cdots & \omega_N^{[k_1, \ldots, k_n]} \\ \end{array} \right|
$$

for the determinant with shifted multiplicities. Next, and closely following [15], we derive two identities involving Casoratian determinants with elements $\omega_{ij}$.

**Casoratian identity 1.** The first identity that we need is

$$
x_1^{n-2} \tau^{[1]} = \left| \begin{array}{cccc} \omega_1^{[1]} & \omega_2^{[1]} & \cdots & \omega_{N-1}^{[1]} \end{array} \right|
$$

which is derived as follows. From Equation (43), we have

$$
\tau^{[1]} = \left| \begin{array}{cccc} \omega_1^{[1]} & \omega_2^{[1]} & \cdots & \omega_N^{[1]} \\ \end{array} \right|
$$

In view of Equation (37), subtracting $x_1$ times column $j + 1$ from column $j$ in this determinant for $j = 1, 2, \ldots, N - 1$ allows us to write

$$
\tau^{[1]} = \left| \begin{array}{cccc} \omega_1 & \omega_2 & \cdots & \omega_{N-1}^{[1]} \\ \end{array} \right|
$$

Multiplying column $N$ by $x_1$ and adding column $N - 1$ to the result, we obtain

$$
x_1 \tau^{[1]} = \left| \begin{array}{cccc} \omega_1 & \omega_2 & \cdots & \omega_{N-1}^{[1]} \\ \end{array} \right|
$$
Similarly, multiplying column $N$ in Equation (47) by $x_1$ and subtracting column $N - 2$ yields

$$x_1^2 \tau^{[1]} = |\bar{\omega}_1 \bar{\omega}_2 \cdots \bar{\omega}_{N-1} \bar{\omega}_{N-2}^{[1]}|$$ (48)

Iterating this procedure by multiplying column $N$ by $x_1$ and subtracting column $N - j$, we obtain Equation (44).

**Casoratian identity 2.** The second identity that we need is

$$\prod_{1 \leq r < s \leq n} (x_r - x_s) \tau^{[1, \ldots, n]} =
\begin{vmatrix}
\bar{\omega}_1 & \cdots & \bar{\omega}_{N-n} & \omega_{N-n+1}^{[n]} & \cdots & \omega_{N-n+1}^{[n-1]} & \cdots & \omega_{N-n+1}^{[1]} \\
\end{vmatrix}$$ (49)

which is derived as follows. From Equation (47), it follows that

$$x_1 \tau^{[1,2]} = |\bar{\omega}_1^{[2]} \bar{\omega}_2^{[2]} \cdots \bar{\omega}_{N-1}^{[2]} \bar{\omega}_{N-1}^{[1]}|$$ (50)

which we can rewrite by subtracting $x_2$ times column $j + 1$ from column $j$ for $j = 1, 2, \ldots, N - 2$ as

$$x_1 \tau^{[1,2]} = |\bar{\omega}_1 \bar{\omega}_2 \cdots \bar{\omega}_{N-2} \bar{\omega}_{N-1}^{[2]} \bar{\omega}_{N-1}^{[1]}|$$ (51)

Multiplying column $N$ by $(x_1 - x_2)$ and applying Equation (38), we see that

$$(x_1 - x_2)x_1 \tau^{[1,2]} = x_1 |\bar{\omega}_1 \bar{\omega}_2 \cdots \bar{\omega}_{N-2} \bar{\omega}_{N-1}^{[1]}| - x_2 |\bar{\omega}_1 \bar{\omega}_2 \cdots \bar{\omega}_{N-2} \bar{\omega}_{N-1}^{[2]}|$$ (52)

Since the last two columns of the latter determinant are identical, we obtain

$$(x_1 - x_2) \tau^{[1,2]} = |\bar{\omega}_1 \cdots \bar{\omega}_{N-2} \bar{\omega}_{N-1}^{[2]}, \bar{\omega}_{N-1}^{[1]}|$$ (53)

which establishes Equation (49) for $n = 2$. Now suppose inductively that

$$\prod_{1 \leq r < s \leq n} (x_r - x_s) \tau^{[1, \ldots, n]} =
\begin{vmatrix}
\bar{\omega}_1 & \cdots & \bar{\omega}_{N-n} & \omega_{N-n+1}^{[n]} & \cdots & \omega_{N-n+1}^{[n-1]} & \cdots & \omega_{N-n+1}^{[1]} \\
\end{vmatrix}$$ (54)
then analogously to Equation (47), we have

$$\prod_{i=1}^{n} x_i \prod_{1 \leq r < s \leq n} (x_r - x_s) \tau^{[1, \ldots, 2n]} = $$

$$\prod_{i=1}^{n} \bar{\omega}_1 \ldots \bar{\omega}_{N-n} \bar{\omega}^{n-1}_{N-n+1} \bar{\omega}^{n}_{N-n+1} \cdots \bar{\omega}^{0}_{N-n} = $$

$$\bar{\omega}_1 \ldots \bar{\omega}_{N-n} \bar{\omega}^{n-1}_{N-n} \bar{\omega}^{n}_{N-n} \cdots \bar{\omega}^{0}_{N-n} \quad (55)$$

It follows that

$$\prod_{i=1}^{n} x_i \prod_{1 \leq r < s \leq n} (x_r - x_s) \tau^{[1, \ldots, 2n+1]} = $$

$$\prod_{i=1}^{n} \bar{\omega}_1 \ldots \bar{\omega}_{N-n-1} \bar{\omega}^{n-1}_{N-n} \bar{\omega}^{n}_{N-n} \cdots \bar{\omega}^{0}_{N-n} = $$

$$\bar{\omega}_1 \ldots \bar{\omega}_{N-n-1} \bar{\omega}^{n-1}_{N-n} \bar{\omega}^{n}_{N-n} \cdots \bar{\omega}^{0}_{N-n} \quad (56)$$

Using Equation (38) repeatedly gives

$$\prod_{1 \leq i \leq n} (x_i - x_{n+1}) \times $$

$$\bar{\omega}_1 \ldots \bar{\omega}_{N-n-1} \bar{\omega}^{n-1}_{N-n} \bar{\omega}^{n}_{N-n} \cdots \bar{\omega}^{0}_{N-n} = $$

$$\prod_{i=1}^{n} x_i \bar{\omega}_1 \ldots \bar{\omega}_{N-n-1} \bar{\omega}^{n-1}_{N-n} \bar{\omega}^{n}_{N-n} \cdots \bar{\omega}^{0}_{N-n} \quad (57)$$

Combining this with Equation (56) shows that

$$\prod_{1 \leq r < s \leq n+1} (x_r - x_s) \tau^{[1, \ldots, 2n+1]} = $$

$$\bar{\omega}_1 \ldots \bar{\omega}_{N-n-1} \bar{\omega}^{n-1}_{N-n} \bar{\omega}^{n}_{N-n} \cdots \bar{\omega}^{0}_{N-n} \quad (58)$$

thereby completing the proof of Equation (49). We are finally in a position to complete the proof that Slavnov’s determinant expressions are discrete KP $\tau$-functions.

**Bilinear identities from Laplace expansions.** Following [15], we consider the $2N \times 2N$ determinant, which is identically zero,
where we have used subscripts to label the zero elements with the positions of the columns that they are in for notational clarity. Performing a Laplace expansion of the left hand side of Equation (59) in \( N \times N \) minors along the top \( N \times N \) block, we obtain

\[
\sum_{\nu=1}^{n} (-)^{\nu-1} \left| \omega_1 \cdots \omega_{n-1} \omega^{[\nu]}_{n-n+2} \omega_1 \cdots \omega_{n-n+2+n+2} \omega^{[\nu-1]}_{n-n+2+2} \cdots \omega^{[1]}_{n-n+2} \right| = 0 \quad (60)
\]

Using Equations (44–49), Equation (60) becomes

\[
\sum_{\nu=1}^{n} (-)^{\nu-1} x_{\nu}^{n-2} \prod_{1 \leq r < s \leq n, \tau, \nu \neq s, \nu} (x_r - x_s) \tau^{[1]} \cdots \tau^{[n]} = 0 \quad (61)
\]

which we recognise as the cofactor expansion of the determinant in Equation (31) using the last column. Hence we conclude that Slavnov’s determinant expression for the XXZ Bethe scalar product is a \( \tau \)-function of discrete KP.

6. Remarks.

**Shifted \( \tau \)-functions are not Bethe scalar products.** A Bethe scalar product that involves \( m_i \) rapidities \( x_i \), for \( i = 1, 2, \ldots, i_{\text{max}} \), is a Casoratian determinant of a matrix of rank \( r = \sum_{i=1}^{i_{\text{max}}} m_i \). Let us denote the corresponding \( \tau \)-function by \( \tau = \tau \{ x^{(m_1)}_1, \ldots, x^{(m_{i_{\text{max}}})}_i \} \). Now let us consider a time evolution of the latter, for example \( \tau_{i+1} = \tau \{ x^{(m_1)}_1, \ldots, x^{(m_{i+1})}_i, x^{(m_{i+1})}_{i+1}, \ldots, x^{(m_{i_{\text{max}}})}_i \} \). Time evolution has increased the multiplicities by 1, but kept the rank of the corresponding Casoratian determinant the same, thus we cannot interpret \( \tau_{i+1} \) as a Bethe scalar product and it remains unclear to us how to interpret the discrete time evolution of a Bethe scalar product in the language of the XXZ spin chain.
Fermionization remains valid. Continuous KP $\tau$-functions can be written as expectation values of charged free fermion operators [21]. This remains the case for discrete KP $\tau$-functions and was the starting point of the results of [14,23]. In [13], the fermion expectation value version of Slavnov’s determinant expression was obtained based on an earlier result [26]. It is straightforward to show that this result remains the same as the continuous KP time variables are restricted to be power sums of a finite and smaller number of continuous Miwa variables.

Relation to the work of Krichever et al. As mentioned earlier, our result is close in spirit to that of Krichever et al. [7, 8] and works that followed including [24,25]. The starting point of [7] is that the Bethe eigenvalues satisfy a bilinear identity that has the same structure as Hirota’s bilinear difference equation and hence can be identified with $\tau$-functions of a discrete hierarchy. From this, a large number of interesting results follow, including an identification of the fusion rules of the transfer matrices of the quantum spin chain with Hirota’s difference equations, that each step in the nested Bethe Ansatz approach to the spin chain is identified with a classical Bäcklund transformation, and most interestingly that the eigenvalues of Baxter’s $Q$ operator are classical (suitably normalized) Baker-Akhiezer functions. On the other hand, our result is that it is the Bethe scalar product of a Bethe eigenstate rather than the corresponding Bethe eigenvalue that is identified with a discrete KP $\tau$-function, and we are far from obtaining further results that are analogous to those of [7]. We hope that our identification is compatible with and complements that of [7].

Relation to the work of Sato and Sato. Equation (61) also follows from Theorem 3 of Sato and Sato [27]. We didn’t know this when we obtained our proof, and the existence of more than one proof can only shed more light on the result obtained.

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