A Note on the Gauge Equivalence between the Manin-Radul and Laberge-Mathieu Super KdV Hierarchies

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Abstract

The gauge equivalence between the Manin-Radul and Laberge-Mathieu super KdV hierarchies is revisited. Apart from the Inami-Kanno transformation, we show that there is another gauge transformation which also possess the canonical property. We explore the relationship of these two gauge transformations from the Kupershmidt-Wilson theorem viewpoint and, as a by-product, obtain the Darboux-Backlund transformation for the Manin-Radul super KdV hierarchy. The geometrical interpretation of these transformations is also briefly discussed.
I. INTRODUCTION

Recently, Morosi and Pizzocchero \cite{1,2} discussed the gauge equivalence of the Manin-Radul (MR) \cite{4} and Laberge-Mathieu (LM) \cite{5} super Korteweg-de Vries (sKdV) hierarchies from a bi-Hamiltonian and Lie superalgebraic viewpoint. This approach can be viewed as a superextension of the Drinfeld-Sokolov method \cite{6} for building a KdV-type hierarchy for a simple Lie algebra. They showed \cite{1} that the gauge transformation proposed by Inami and Kanno (IK) \cite{7} not only preserves the Lax formulations but also the bi-Hamiltonian structures corresponding to the MR and LM hierarchies. In particular, they provided a geometrical meaning of the IK transformation which rests on the natural fibered structure appearing in the bi-Hamiltonian reduction of loop superalgebras.

In this paper, in addition to the IK transformation, we find that there is another gauge transformation between the MR and LM sKdV hierarchies preserving the Lax formulations. We investigate the canonical property of this gauge transformation and discuss the connection to the IK transformation from the Kupershmidt-Wilson theorem \cite{8} viewpoint. As a by-product, the Darboux-Bäcklund transformation (DBT) for the MR sKdV hierarchy can be constructed from these two gauge transformations. The geometrical interpretation of these two transformations is also briefly discussed.

Our paper is organized as follows: In Sec. II, the bi-Hamiltonian structures of the MR and LM sKdV hierarchies are briefly reviewed. In Sec. III, we introduce a gauge transformation between these two hierarchies and investigate its canonical property. Then in Sec. IV, we discuss the relationship between this transformation and the IK transformation from the KW theorem viewpoint. Concluding remarks are presented in Sec. V.

II. BI-HAMILTONIAN STRUCTURES OF THE MR AND LM SKDV HIERARCHIES

The MR sKdV hierarchy was defined originally from the reduction of the MR super Kadomtsev-Petviashvili hierarchy \cite{4}. It has Lax equation as follows,

\[
\partial_n L^{MR} = [B^{MR}_n, L^{MR}],
\]

with

\[
L^{MR} = \partial^2 - \phi D + a
\]

\[
B^{MR}_n = (-1)^n 4^n (L^{MR})^{n+1/2}_+;
\]

where the superderivative \(D \equiv \partial_\theta + \theta \partial\) satisfies \(D^2 = \partial\), \(\theta\) is the Grassmann variable \((\theta^2 = 0)\) which, together with the even variable \(x \equiv t_1\), define the \((1|1)\) superspace \cite{10} with coordinate \((x, \theta)\). The formal inverse of \(D\) is introduced by \(D^{-1} = \theta + \partial_\theta \partial^{-1}\), which satisfies \(DD^{-1} = D^{-1}D = 1\). The coefficients \(\phi = \phi(x, \theta)\) and \(a = a(x, \theta)\) are an odd and an even superfield on \((1|1)\) superspace, respectively. We denote the action of the superderivative \(D\) on the superfield \(f\) by \((Df)\).

The bi-Hamiltonian structure of the MR hierarchy has been obtained in \cite{10} as
The product is defined by structures of these two hierarchies have been tabulated in Ref. [1].

\[
\begin{align*}
\Theta_1^{MR} : \left( \frac{\delta a}{\delta \phi} \right) & \rightarrow \left( \frac{\dot{a}}{\phi} \right) = \left( -D\partial + \phi - \partial \right) \left( \frac{\delta a}{\delta \phi} \right) \\
\Theta_2^{MR} : \left( \frac{\delta a}{\delta \phi} \right) & \rightarrow \left( \frac{\dot{a}}{\phi} \right) = \left( \begin{array}{cc}
P_{aa} & P_{a\phi} \\ P_{\phi a} & P_{\phi\phi} \end{array} \right) \left( \frac{\delta a}{\delta \phi} \right),
\end{align*}
\]

where the operators \( P_{ij} \) are given by

\[
\begin{align*}
P_{aa} &= D\partial^3 - 3\phi \partial^2 + 4aD\partial + (2(Da) - 3\phi_x)\partial + 2a_x D + 3\phi(D\phi) \\
&\quad + (Da)_x - 4a\phi - \phi_{xx} + \phi D^{-1}(Da) - (Da)D^{-1}\phi \\
&\quad - \phi D^{-1}\phi D^{-1}\phi - \phi D^{-1}\phi_x + \phi_x D^{-1}\phi \\
P_{a\phi} &= \partial^3 - 2\phi D\partial + 4a\partial - \phi_x D + 2a_x + \phi D^{-1}(D\phi) \\
P_{\phi a} &= \partial^3 + 2\phi D\partial + (4a - 2(D\phi))\partial + \phi_x + 2a_x - (D\phi)_x + (D\phi)D^{-1}\phi \\
P_{\phi\phi} &= 4\phi\partial + 2\phi_x.
\end{align*}
\]

Here, following the notations in [1], the phase space for the MR theory is a pair \( m = (a, \phi) \). A tangent vector at a point \( m \) is denoted by \( \dot{m} = (\dot{a}, \dot{\phi}) \) and a cotangent vector as a pair \( \delta m = (\delta a, \delta \phi) \) where \( \dot{a} \) and \( \delta \phi \) are even superfields, whereas \( \phi \) and \( \delta a \) are odd. The inner product is defined by \( \langle \delta m, \dot{m} \rangle \equiv \int dxd\theta(\delta \dot{a}\dot{a} + \delta \dot{\phi}\dot{\phi}) \).

For LM hierarchy, the Lax equation is given by

\[
\partial_n L^{LM} = [B_n^{LM}, L^{LM}],
\]

with

\[
\begin{align*}
L^{LM} &= \partial^2 - 2u\partial - ((Du) + \tau)D \\
B_n^{LM} &= (-1)^n A^n (L^{LM})_{n+1/2} > 0,
\end{align*}
\]

where \( \mu = \mu(x, \theta) \) and \( \tau = \tau(x, \theta) \) are even and odd superfields, respectively. It should be mentioned that the LM theory discussed here is obtained from the \( N = 2, \alpha = -2 \) LM sKdV theory [3]. The bi-Hamiltonian structure of the LM hierarchy is also taken from [10], in component form, as [1]

\[
\begin{align*}
(\Theta_1^{LM})^{-1} : \left( \frac{\dot{u}}{\tau} \right) & \rightarrow \left( \frac{\delta u}{\delta \tau} \right) = \left( \begin{array}{cc}
D - D^{-1}\tau D^{-1} u\partial^{-1} + D^{-1}u D^{-1} \\
\partial^{-1}u + D^{-1}u D^{-1} D^{-1} - \partial^{-1}\tau\partial^{-1} \end{array} \right) \left( \frac{\dot{u}}{\tau} \right) \\
\Theta_2^{LM} : \left( \frac{\delta u}{\delta \tau} \right) & \rightarrow \left( \frac{\dot{u}}{\tau} \right) = \left( \begin{array}{cc}
-D\partial + \tau & 2u\partial - (Du)D + 2u_x \\
2u\partial - (Du)D + u_x & -D\partial^2 + 3\tau\partial + (D\tau)D + 2\tau_x \end{array} \right) \left( \frac{\delta u}{\delta \tau} \right)
\end{align*}
\]

where, similarly, the phase space of the LM theory can be represented as a set of pairs \( n = (u, \tau) \). Then the tangent and cotangent vectors at a point \( n \) are represented as \( \dot{n} = (\dot{u}, \dot{\tau}) \) and \( \delta n = (\delta u, \delta \tau) \), respectively, where \( \dot{u} \) and \( \delta \tau \) are even, while \( \delta u \) and \( \tau \) are odd. The inner product is defined by \( \langle \delta n, \dot{n} \rangle \equiv \int dxd\theta(\delta \dot{u}\dot{u} + \delta \dot{\tau}\dot{\tau}) \). More features about the bi-Hamiltonian structures of these two hierarchies have been tabulated in Ref. [3].
III. GAUGE TRANSFORMATIONS

In Ref. [7], Inami and Kanno showed that the MR sKdV hierarchy can be related to the LM sKdV hierarchy via the following gauge transformation:

\[ L_1^{MR} = S_1^{-1} L^{LM} S_1, \quad S_1 = e^{\int^x u}, \]  

(3.1)

which leads to

\[ \phi_1 = (Du) + \tau, \quad a_1 = u_x - u^2 - ((Du) + \tau)(D^{-1}u). \]  

(3.2)

They also showed that the Lax equation (2.1) of the LM theory is mapped into the Lax equation (2.10) of the MR theory under such transformation. Hence (3.2) provides a gauge equivalence between these two hierarchies and now is referred to as the Inami-Kanno transformation. It can be shown that \( S_1^{-1} \) is an eigenfunction of the MR sKdV hierarchy, i.e. \( \partial_n S_1^{-1} = (B_n^{MR} S_1^{-1}) \). Furthermore, Morosi and Pizzocchero [1] showed that the IK transformation is a canonical map, in the sense that the bi-Hamiltonian structure of the LM sKdV hierarchy is mapped to the bi-Hamiltonian structure of the MR sKdV hierarchy. That is,

\[ S_1^t (\Theta_1^{MR})^{-1} S_1' = (\Theta_1^{LM})^{-1} \]  

(3.3)

\[ S_1'(\Theta_2^{LM}) S_1^t = (\Theta_2^{MR}), \]  

(3.4)

where \( S_1' \) and \( S_1^t \) are linearized map and its transport map respectively of the IK transformation and satisfy

\[ \langle S_1^t \delta m, \dot{n} \rangle = \langle \delta m, S_1' \dot{n} \rangle. \]  

(3.5)

In fact, we can construct another transformation between MR and LM sKdV hierarchies as follows:

\[ L_2^{MR} = S_2^{-1} L^{LM} S_2, \quad S_2 = D^{-1} S_1. \]  

(3.6)

Then a simple calculation leads to

\[ \phi_2 = (Du) - \tau, \quad a_2 = -u^2 - (D\tau) - ((Du) - \tau)(D^{-1}u). \]  

(3.7)

It can be shown that the Lax formulations are preserved under such transformation. Hence the transformation (3.6) also provides a gauge equivalence of the MR and LM sKdV hierarchies. Similarly, we can show that, in this case, \( \partial_n S_1 = -(B_n^{MR})^* S_1 \). That means \( S_1 \) is an adjoint eigenfunction of the MR sKdV hierarchy.

Next, let us discuss the canonical property of the transformation (3.6). From (3.7), the linearized map \( S_2' \) and its adjoint map \( S_2^t \) can be derived as follows:

\[ S_2' = \begin{pmatrix} -2u + (D^{-1}u)D & -\phi_2 D^{-1} \\ D & -D \end{pmatrix}, \]  

(3.8)

\[ S_2^t = \begin{pmatrix} -2u + D^{-1} \phi_2 - D(D^{-1}) & -D \\ -D + (D^{-1}u) & -1 \end{pmatrix}. \]  

(3.9)
After a straightforward but tedious calculation, the Poisson structures transform as

\begin{align}
S_i'\dagger (\Theta_1^{MR})^{-1}S_i &= -(\Theta_1^{LM})^{-1} \\
S_i'(\Theta_2^{LM})S_i'\dagger &= -(\Theta_2^{MR}),
\end{align}

which, comparing with (3.3) and (3.4), acquire a minus sign. It seems that this result contradicts the property of preserving the Lax formulations. However, it is not the case. Since the parity of the gauge operator $S_2$ is odd, the Hamiltonian of the LM hierarchy $H_{n}^{LM} = \text{str}((L^{LM})^{n+1/2})$ (up to a multiplicative constant) then is transformed to $H_{n}^{MR} = -H_{n}^{LM}$ due to the following property:

\begin{equation}
\text{str}(PQ) = (-1)^{|P||Q|}\text{str}(QP),
\end{equation}

where $P$ and $Q$ are any super-pseudo-differential operators with gradings $|P|$ and $|Q|$, respectively. Therefore, the gauge equivalence is compatible with the canonical property under the transformation triggered by the gauge operator $S_2$.

Based on the above discussions, the canonical property of the gauge transformations between the MR and LM sKdV hierarchies can be summarized as follows,

\begin{align}
S_i'\dagger (\Theta_1^{MR})^{-1}S_i' &= -(\Theta_1^{LM})^{-1} \\
S_i'(\Theta_2^{LM})S_i'\dagger &= -(\Theta_2^{MR}),
\end{align}

which seems to be the supersymmetric generalization of the bosonic case [11].

**IV. BÄCKLUND TRANSFORMATION AND KUPERSHMIDT-WILSON THEOREM**

From the IK transformation, we know that if we have a solution $\{u, \tau\}$ of the LM sKdV hierarchy, then Eq. (3.2) gives a solution $\{\phi_1, a_1\}$ of the MR sKdV hierarchy. Sometimes, such a transformation of one hierarchy to another is called a Miura transformation. On the other hand, Eq. (3.7) also gives another solution $\{\phi_2, a_2\}$ of the MR sKdV hierarchy. Hence a Darboux-Bäcklund transformation (DBT) of the MR sKdV hierarchy to itself can be constructed from these two gauge transformations. In other words, let $\{\phi_1, a_1\}$ be a solution of the MR sKdV hierarchy, then solving $\{u, \tau\}$ from (3.2) and substituting it into (3.7) we get

\begin{align}
\phi_2 &= -\phi_1 - 2(D^3 \ln S_1^{-1}) \\
a_2 &= a_1 - (D\phi_1) + 2(D \ln S_1^{-1})(\phi_1 + (D^3 \ln S_1^{-1})),
\end{align}

which is just the DBT derived in Ref. [12]. The action of the gauge operators $S_1$ and $S_2$ for the MR and LM sKdV hierarchies are shown as follows

\begin{equation}
\begin{array}{c}
\text{LM} \\
S_1 \\
MR^1 \leftarrow \text{DBT} \rightarrow MR^2 \\
S_2
\end{array}
\end{equation}
In the following, we want to discuss the relationship in (4.3) from the KW theorem viewpoint, in which the gauge operator $S_2$ plays an important and unambiguous role. From (2.11), the Lax operator $L^{LM}$ can be factorized as follows, \[ L^{LM} = \partial^2 - 2u\partial - ((Du) + \tau)D = (D - \Phi_1)(D - \Phi_1 - \Phi_2)(D - \Phi_2)D \quad (4.4) \]

where $u$ and $\tau$ can be expressed in terms of the superfields $\Phi_i$ as

\[ u = \frac{1}{2}[(D\Phi_1) + (D\Phi_2) - \Phi_1\Phi_2] \quad (4.5) \]
\[ \tau = \frac{1}{2}[(D\Phi_1)\Phi_2 - \Phi_1\Phi_2] \quad (4.6) \]

The second Hamiltonian structure of the LM theory can be simplified under the factorization (4.4). From (4.5) and (4.6), it is straightforward to show that

\[ \Theta_2^{LM} = \left[ \frac{\partial(u, \tau)}{\partial(\Phi_1, \Phi_2)} \right] \left( \begin{array}{cc} 0 & 2D \\ 2D & 0 \end{array} \right) \left[ \frac{\partial(u, \tau)}{\partial(\Phi_1, \Phi_2)} \right]^\dagger \quad (4.7) \]

where the Fréchet derivative can be calculated as:

\[ \left[ \frac{\partial(u, \tau)}{\partial(\Phi_1, \Phi_2)} \right] = \left( \begin{array}{cc} -\frac{1}{2}(D + \Phi_2) & -\frac{1}{2}(D - \Phi_1) \\ -\frac{1}{2}(\partial + \Phi_2D + (D\Phi_2)) & \frac{1}{2}(\partial - \Phi_1D - (D\Phi_1)) \end{array} \right) \quad (4.8) \]

and $\left[ \frac{\partial(u, \tau)}{\partial(\Phi_1, \Phi_2)} \right]^\dagger$ is its formal adjoint.

Now applying the IK transformation to (4.4), we obtain the multiplicative form of the Lax operator $L_1^{MR}$ as

\[ L_1^{MR} = (D - \Psi_1)(D - \Psi_2)(D - \Psi_3)(D - \Psi_4), \quad (4.9) \]

where the superfields $\Psi_i$ are given by

\[ \Psi_1 = \frac{1}{2}((D^{-1}\Phi_1\Phi_2) + \Phi_1 - \Phi_2) \quad (4.10) \]
\[ \Psi_2 = \frac{1}{2}((D^{-1}\Phi_1\Phi_2) + \Phi_1 + \Phi_2) \quad (4.11) \]
\[ \Psi_3 = \frac{1}{2}((D^{-1}\Phi_1\Phi_2) + \Phi_2 - \Phi_1) \quad (4.12) \]
\[ \Psi_4 = \frac{1}{2}((D^{-1}\Phi_1\Phi_2) - \Phi_1 - \Phi_2). \quad (4.13) \]

where only two of them are independent variables. The Lax equation for $L_1^{MR}$ then can be expressed in terms of the hierarchy equations of $\Psi_i$.

On the other hand, if we apply the gauge transformation (3.6) to (4.4), the Lax operator $L_2^{MR}$ then is factorized as

\[ L_2^{MR} = (D - \Psi_4)(D - \Psi_1)(D - \Psi_2)(D - \Psi_3), \quad (4.14) \]

which differs from $L_1^{MR}$ only by a cyclic permutation: $\Psi_1 \mapsto \Psi_2, \cdots, \Psi_4 \mapsto \Psi_1$. Such cyclic permutation does not change the hierarchy equations of $\Psi_i$, \[13\] and hence generates the DBT for the MR sKdV hierarchy itself.
V. CONCLUDING REMARKS

We have shown that, in addition to the IK transformation, there is another gauge transformation between the MR and LM sKdV hierarchies. We investigate the canonical property of this new gauge transformation and show that it depends on the grading (or parity) of the gauge operator. Using this new gauge transformation and the IK transformation we rederived the DBT for the MR sKdV hierarchy. We also give an interpretation of this new gauge transformation from the KW theorem viewpoint.

Finally, we would like to remark that the geometrical interpretation of the IK transformation discussed in Ref. [1] can be applied to the new gauge transformation as well. The only thing we have to do is to choose a different cross section $\hat{\Sigma}$, which is matrix in the fiber over $m$ of the form

$$
\hat{\Sigma}(m) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
(Du) - \tau & 0 & 0 & 1 \\
-2u & -1 & 0 & 0 \\
0 & 0 & 0 & 0
end{pmatrix}.
$$

Then the transformation (3.7) comes out naturally from a general equation derived in Ref. [1] which describes the quotient space in the bi-Hamiltonian reduction of a loop superalgebra. Since the IK transformation was also derived from the same equation, thus (3.2) and (3.7) can be treated on an equal footing in the bi-Hamiltonian framework.

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