Infinite Horizon Noncooperative Differential Games with Non-Smooth Costs

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1 Introduction

This paper deals with the study of a class of non-cooperative differential games in infinite time horizon. Namely, we consider a game with dynamics

\[ \dot{x} = \sum_{i=1}^{n} \alpha_i, \quad x(0) = y, \]  

where each player acts on his control \( \alpha_i \) to minimize an exponentially discounted cost of the form

\[ J_i(\alpha) = \int_{0}^{\infty} e^{-t} \left[ h_i(x(t)) + \frac{\alpha_i^2(t)}{2} \right] dt, \]

both \( h_i \) being integrable functions, whose smoothness will be addressed later. Very few results are known on the subject, except in two particular cases: two players zero-sum games and LQ games (where LQ stands for linear-quadratic). Indeed, a key step in this kind of problems is the study of the value function \( u \).

In the region where \( u \) is smooth, its components satisfy a system of Hamilton-Jacobi equations (see [7]), and this system is usually difficult to solve. In the case of two players zero-sum games, since what one player gains is exactly what the other player loses, the two components of \( u \) are one the opposite of the other. Hence, the Hamilton-Jacobi system (HJ in the following) reduces to a single equation and one can apply the standard theory of viscosity solutions (see [1] for more details) to obtain existence and uniqueness results.

In the case of LQ games, the HJ system can be connected to a Riccati system of ODE for matrices. This system is in general much easier than the original one, and standard ODE techniques can be applied (see [6] for a detailed treatment).

On the other hand, in the present case both approaches fail and therefore one has no established techniques to rely on. However, few results still can be proved.

In the finite horizon setting, the analysis presented in [3, 4] showed that, for a non-cooperative \( n \)-players differential game with general terminal payoffs, the well-posedness is strongly related with the HJ system being hyperbolic. Namely, for \( x \in \mathbb{R} \), thanks to recent advances in the theory of hyperbolic systems of PDE, games with strictly hyperbolic HJ systems are well-posed. On the other hand,
it is possible to produce examples of games, even in one spatial dimension, whose corresponding HJ system is not even weakly hyperbolic and, hence, it is ill-posed.

A first attempt to study this problem in the infinite horizon setting, for two players, was made in [2]. The same simple game was considered, and it was proved that, depending on the monotonicity of the cost functions, very different situations could arise. Indeed, the HJ system in this case takes the following form

\[
\begin{align*}
    u_1(x) &= h_1(x) - u'_1 u'_2 - (u'_1)^2/2, \\
    u_2(x) &= h_2(x) - u'_1 u'_2 - (u'_2)^2/2.
\end{align*}
\]

But with a system of this form, we can end up with too many solutions. We find not only value functions \( u \) that leads to Nash equilibria in feedback form, but also solutions that does not represent equilibria of the game. It is then necessary to introduce a suitable concept of admissibility. In particular we say that a solution \( u \) is \( admissible \), if \( u \) is a Carathéodory solution of (1.3), which grows at most linearly as \( |x| \to \infty \) and satisfies suitable jump conditions in points where its derivatives are discontinuous. For such a kind of solutions, a verification theorem was proved: given an admissible solution \( u \) and denoted by \( u'_i \) the components of its derivatives, then \( \alpha_i = -u'_i \) provide a Nash equilibrium solution in feedback form.

In [2], it turned out that existence and uniqueness of admissible solution for (1.3) heavily depend on the choice of the costs. First, suppose that both the cost functionals are increasing (resp. decreasing). This means that both players would like to steer the game in the same direction, namely the direction along which their costs decreases. In this case an admissible solution always exists, and it is also unique, provided a small oscillations assumption is satisfied. This existence result was in some sense expected, since this case corresponds, in the finite horizon setting, to the hyperbolic one studied in [3].

Suppose now that the cost functionals have opposite monotonicity. This means that the players have conflicting interests, since they would like the game to go in different directions. In this case it is known, see [4], that the finite horizon problem is in general ill-posed. On the same line, for our game, it is enough to consider two linear functionals with opposite slopes (say \( k, -k \), for any real number \( k \neq 0 \)) to find infinitely many admissible solutions, and hence infinitely many Nash equilibria in feedback form. Nevertheless, quite surprisingly, it’s still possible to recover existence and uniqueness of admissible solutions to (1.3) in the case of costs that are small perturbation of linear ones, but with slopes that are not exactly opposite. This richness of different situations reflects in some sense the results found in [5]. Indeed, the exact same dynamics was studied, in the finite horizon case, with only exit costs. Main differences between [5] and [2, 3, 4] lay in the concept of solution. The authors of [5] look for discontinuous feedback controls that not
only leads to Nash equilibria, but also satisfies a sort of programming principle. This resulted in (uncountable) infinitely many solutions, at price of stronger assumptions on the final costs.

While the cost functionals considered in [2] were a small perturbation of affine costs, in the present paper we study a wider class of cost functions. Motivated by the theory of hyperbolic systems [?], we now consider piecewise linear cost functionals, whose derivative has jumps. This setting is a natural first step towards the analysis of existence and uniqueness of Nash equilibrium solutions for non-linear costs.

Again, as in [2], we reach different results depending on the signs chosen for $h'_i$. Indeed, as it will be proved in the following sections, if we are in the cooperative situation for all $x$, we can still recover a unique admissible solution for (1.3). On the other hand, any change in the behavior of the costs will translate in some sort of instability of the game, leading either to infinitely many admissible solution, or to one unique admissible solution, or even to no admissible solution at all, only depending on the particular choices of the slopes $h'_i$.

In conclusion, this great variety of arising situations seems to suggest that the present approach is not the most suitable one to deal with the intrinsic issues of the problem. In particular, we can provide examples of very simple differential games where no Carathéodory solution with sublinear growth at infinity exists. Recalling that, in the case of smooth costs (see [2]), this class of solutions was exactly the right one to find Nash equilibria in feedback form, our study strongly suggest that a different approach is needed: either to look for Pareto optima, as in [3], or to introduce some other relaxed concept of equilibrium.

The structure of the present paper is the following. In Section 2 we will introduce main notations and definitions. Moreover we will recall briefly what was proved in the case of smooth costs and provide a couple of useful Lemmas. In Section 3 we will present and prove the main results of this paper, dealing with cooperative players, in the sense of players whose costs always have the same monotonicity. In this case existence and uniqueness results hold for both piecewise linear and piecewise smooth cost functionals. In Section 4 we will prove that a similar extension is not possible in the case of conflicting interests. Actually, we will provide an example in which the games has infinitely many Nash equilibria, as well as an example in which there cannot be any admissible solution to (1.3). Finally, in Section 5, we will discuss a last case that can arise when either one or both the cost functionals are allowed to change monotonicity. From this game which is partially “cooperative” (in the sense above) and partially “conflicting”, infinitely many Nash equilibria can be found.

2 Preliminaries

In this paper we consider a scalar 2-persons differential game, with dynamics

$$
\dot{x} = \alpha_1 + \alpha_2 ,
$$

(2.1)
The functions \( t \mapsto \alpha_i(t), i = 1, 2, \) represent the controls implemented by the \( i \)-th player, chosen within a compact set of admissible controls \( A_i \subset \mathbb{R} \). The game takes place on \([0, +\infty[\) and each player is subject to a running cost, exponentially discounted, of the following form

\[
J_i(\alpha_i) = \int_0^\infty e^{-t} \left[ h_i(x(t)) + \frac{\alpha_i^2(t)}{2} \right] dt .
\]

Assume here that both \( h_i \) are piecewise smooth functions with bounded derivatives. Later we will weaken these requirements.

A couple of feedback strategies \((\alpha_1^*(x), \alpha_2^*(x))\) represents a Nash equilibrium solution for the game \((2.1)–(2.2)\) if the following holds. For \( i \in \{1, 2\} \), the feedback control \( \alpha_i = \alpha_i^*(x) \) provides a solution to the corresponding optimal control problem for the \( i \)-th player,

\[
\min_{\alpha_i(\cdot)} J_i(\alpha_i) ,
\]

where the dynamics of the system is

\[
\dot{x} = f_i(x, \alpha_i) + f_j(x, \alpha_j^*(x)) , \quad \alpha_i(t) \in A_i, j \neq i .
\]

More precisely, we require that, for every initial data \( y \in \mathbb{R} \), the Cauchy problem

\[
\dot{x} = f_1(x, \alpha_1^*(x)) + f_2(x, \alpha_2^*(x)) , \quad x(0) = y ,
\]

should have at least one Caratheodory solution \( t \mapsto x(t) \), defined for all \( t \in [0, +\infty[\). Moreover, for every such solution and each \( i = 1, \ldots, m \), the cost to the \( i \)-th player should provide the minimum for the optimal control problem \((2.4)–(2.5)\).

We recall that a Caratheodory solution is an absolutely continuous function \( t \mapsto x(t) \) which satisfies the differential equation in \((2.6)\) at almost every \( t > 0 \).

By the theory of optimal control, see for example [1], we know that if \( u \) is the value function corresponding to \((2.1)–(2.2)\) with costs

\[
J_i(\alpha_i) = \int_0^\infty e^{-t} \psi_i(x(t), \alpha_i(t)) dt ,
\]

then, where \( u \) is smooth, each component \( u_i \) should provide a solution to the corresponding scalar Hamilton-Jacobi-Bellman equation. The vector function \( u \) thus satisfies the stationary system of equations

\[
u_i(x) = H_i(x, u_1', u_2') ,
\]

where the Hamiltonian functions \( H_i \) are defined as follows. For each \( p \in \mathbb{R} \), assume that there exists an optimal control value \( \alpha_j^*(x, p) \) such that

\[
p \cdot \alpha_j^*(x, p) + \psi_j(x, \alpha_j^*(x, p)) = \min_{a \in A_j} \{ p \cdot a + \psi_j(x, a) \} .
\]
Then
\[ H_i(x, p_1, p_2) \equiv p_i \cdot \alpha_i^*(x, p_j) + \psi_i(x, \alpha_i^*(x, p_i)) \cdot (2.9) \]
for \(i, j \in \{1, 2\} \) and \(i \neq j\). In general, even in cases as easy as \(\psi_i = \alpha_i^2/2\), this system will have infinitely many solutions defined on the whole \(\mathbb{R}\) (see Example 1 in [2]). And not every solution corresponds to a Nash equilibrium for the initial game. To single out a (hopefully unique) admissible solution, and therefore a Nash equilibrium for the differential game, additional requirements must be imposed. Namely a solution \(u\) to (2.7) is said to be an \textit{admissible solution} if the following holds:

(A1) \(u\) is absolutely continuous and its derivative \(u'\) satisfies (2.7) at a.e. point \(x \in \mathbb{R}\).

(A2) \(u\) has sublinear growth at infinity; namely, there exists a constant \(C\) such that, for all \(x \in \mathbb{R}\),
\[ |u(x)| \leq C (1 + |x|) . \quad (2.10) \]

(A3) At every point \(y \in \mathbb{R}\), the derivative \(u'\) admits right and left limits \(u'(y+)\), \(u'(y-)\) and at points where \(u'\) is discontinuous, these limits satisfy at least one of the conditions
\[ u_1'(y+) + u_2'(y+) \leq 0 \quad \text{or} \quad u_1'(y-) + u_2'(y-) \geq 0 . \quad (2.11) \]

Because of the assumption on \(h_i'\), the cost functions \(h_i\) are Lipschitz continuous. It is thus natural to require the value functions \(u_i\) to be absolutely continuous, with sub-linear growth as \(x \to \pm \infty\). The motivation for the assumption (A3) is quite simple. Observing that, in (2.8), the feedback controls are \(\alpha_i^* = -u_i'\), the condition (2.11) provides the existence of a local solution to the Cauchy problem
\[ \dot{x} = -u_1'(x) - u_2'(x) , \quad x(0) = y \]
forward in time. In the opposite case, solutions of the O.D.E. would approach \(y\) from both sides, and be trapped.

Notice that, for 2-players games, the assumptions (A3) is equivalent to
\[ u_1'(y+) + u_2'(y+) \leq 0 , \quad u_i'(y-) = -u_i'(y+) \quad (i = 1, 2) . \quad (2.12) \]

This concept of admissibility turns out to be the right one. Indeed, the following verification theorem can be proved (see again [2]).

**Theorem 1** Consider the differential game (2.1)–(2.2). Let \(u : \mathbb{R} \mapsto \mathbb{R}^m\) be an admissible solution to the systems of H-J equations (2.7), so that the conditions (A1)–(A3) hold. Then the controls \(\alpha_i^* = -u_i'\) provide a Nash equilibrium solution in feedback form.
Anyway, this theorem says nothing about the actual existence of admissible solutions to (2.7). To deal with this problem, some manipulations have to be done on (2.7) itself. Indeed, in the present case of costs as in (1.2), the Hamiltonian functions (2.9) lead to

\[
\begin{aligned}
    u_1(x) &= h_1(x) - u_1' u_2' - (u_1')^2 / 2, \\
    u_2(x) &= h_2(x) - u_1' u_2' - (u_2')^2 / 2.
\end{aligned}
\] (2.13)

Differentiating w.r.t. \(x\) and setting \(p_i = u_i'\) one obtains the system

\[
\begin{aligned}
    h_1' - p_1 &= (p_1 + p_2)p_1' + p_1 p_2', \\
    h_2' - p_2 &= p_2 p_1' + (p_1 + p_2)p_2'.
\end{aligned}
\] (2.14)

Set

\[
\Lambda(p) = \begin{pmatrix} p_1 + p_2 & p_1 \\ p_2 & p_1 + p_2 \end{pmatrix}, \quad \Delta(p) = \det \Lambda(p),
\]

and notice that

\[
\frac{1}{2} (p_1^2 + p_2^2) \leq \Delta(p) \leq 2(p_1^2 + p_2^2). \] (2.15)

In particular, \(\Delta(p) > 0\) for all \(p = (p_1, p_2) \neq (0, 0)\). Hence, \(\Lambda(p)\) is invertible outside the origin and, for \(p \neq (0, 0)\), we can restrict the study to the equivalent system

\[
\begin{aligned}
    p_1' &= \Delta(p)^{-1} \left[ -p_1^2 + (h_1' - h_2')p_1 + h_1' p_2 \right], \\
    p_2' &= \Delta(p)^{-1} \left[ -p_2^2 + (h_2' - h_1')p_2 + h_2' p_1 \right].
\end{aligned}
\] (2.16)

Now define a new variable \(s\) such that \(ds/dx = \Delta(p)^{-1}\). Using \(s\) as a new independent variable, we write \(p_i = p_i(s)\) and \(h_i = h_i(x(s))\) and study the equivalent system

\[
\begin{aligned}
    \frac{d}{ds} p_1 &= (h_1' - h_2')p_1 + h_1' p_2 - p_1^2, \\
    \frac{d}{ds} p_2 &= (h_2' - h_1')p_2 + h_2' p_1 - p_2^2.
\end{aligned}
\] (2.17)

We underline that it is possible to choose the rescaling in order to map 0 to 0. This choice will be assumed in the following, so that \(s(0) = 0\).

In this new variable, as it was proved \([2]\), every unbounded trajectory \(p(s)\) of (2.17) actually blows up at finite \(s_0\), and it corresponds to an unbounded trajectory \(p(x)\) that tends to \(\infty\) as \(|x| \to \infty\). Since

\[
\left| \frac{dx}{ds} \right| = \Delta(p(s)) \geq \frac{c_0}{(s_0 - s)^2},
\]

we have

\[
\frac{dx}{ds} \geq \frac{c_0}{(s_0 - s)^2},
\]

where \(c_0 > 0\).
it follows that \( u(x) \) increases more than linearly as \( x \to \infty \). Therefore, \( u \) is not admissible.

It remains to consider trajectories of (2.17) that tend to the origin, i.e. to the point where our change of variables is singular. In [2] it was proven that, by (2.15), these solutions satisfy

\[
\left| \frac{dx}{ds} \right| = \Delta(p(s)) = O(1) \cdot e^{-2c,|s|}.
\]

In the original variable \( x \), to the whole trajectory \( s \mapsto p(s) \) there corresponds only a portion of trajectory \( x \mapsto p(x) \), say either for \( x \in \left[ x_0, \infty \right[ \) or \( x \in \left] -\infty, x_0 \right[ \). Another trajectory \( s \mapsto \hat{p}(s) \) has to be constructed to extend the solution to all \( x \in \mathbb{R} \).

For the system (2.17), in the case of smooth functions \( h_1, h_2 \) such that \( |h_i'(x)| \leq C \), we already know the following results (see [2]):

**Theorem 2** Let the cost functions \( h_1, h_2 \) be smooth, and assume that their derivatives satisfy

\[
\frac{1}{C} \leq h_i'(x) \leq C
\]

for some constant \( C > 1 \) and all \( x \in \mathbb{R} \). Then the system (2.17) has an admissible solution and the corresponding functions \( \alpha^*_i = -u'_i \) provide a Nash equilibrium solution to the non-cooperative game (2.1)-(2.2). Assume moreover that the oscillation of their derivatives satisfies

\[
\sup_{x,y \in \mathbb{R}} |h'_i(x) - h'_i(y)| \leq \delta, \quad i = 1, 2
\]

for some \( \delta > 0 \) sufficiently small (depending only on \( C \)). Then the admissible solution is also unique.

**Theorem 3** Let any two constants \( \kappa_1, \kappa_2 \) be given, with

\[
\kappa_1 < 0 < \kappa_2, \quad \kappa_1 + \kappa_2 \neq 0.
\]

Then there exists \( \delta > 0 \) such that the following holds. If \( h_1, h_2 \) are smooth functions whose derivatives satisfy

\[
|h'_1(x) - \kappa_1| \leq \delta, \quad |h'_2(x) - \kappa_2| \leq \delta,
\]

for all \( x \in \mathbb{R} \), then the system of H-J equations (2.17) has a unique admissible solution.

In this paper, we want to look for admissible solutions when smoothness of functions \( h_i \) is relaxed. Namely we consider functions \( h_i \) that are piecewise linear, with a finite number of discontinuity in their derivatives. In other words we require that there exists a finite subdivision

\[
x_0 = -\infty < x_1 < \ldots < x_N < x_{N+1} = +\infty
\]
of \([-\infty, +\infty]\) and two \((N + 1)\)-tuple of constants \((\kappa_i^1, \ldots, \kappa_i^{N+1})\), \(i = 1, 2\), such that

\[
h'_i(x) = \kappa_j^i \quad \text{if} \quad x \in ]x_j, x_{j+1}[, \quad i = 1, 2, \quad j = 0, \ldots, N. \tag{2.18}
\]

Could be of use to remark that this assumption on \(h'_i\) means that the system (2.17) follows different dynamics in each interval \(I_j = ]x_j, x_{j+1}[:\) indeed, in each \(I_j\), (2.17) will have an equilibrium in \((0, 0)\) and a second one in the point \(K^j = (\kappa_j^1, \kappa_j^2)\).

We also introduce the following notation (see Figure 1)

\[
A_i = \left\{ \rho (\cos \theta, \sin \theta) \in \mathbb{R}^2 \left| \rho > 0, \ \theta \in \right] (i-1) \frac{\pi}{4}, i \frac{\pi}{4} \right] \right\}, \tag{2.19}
\]

to label regions in \(\mathbb{R}^2\), where we put our non-zero equilibria \(K^j = (\kappa_j^1, \kappa_j^2)\).

Finally, we state a couple of easy properties we will need in the following. They provide expressions for both eigenvalues and eigenvectors of the system obtained linearizing (2.17) around the origin. These expressions were already found in [2], and they follow from simple linear algebra.

**Proposition 1** The linearized system near \((0, 0)\), corresponding to (2.17), has the following form

\[
\begin{pmatrix} p'_1 \\ p'_2 \end{pmatrix} = H \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad H = \begin{pmatrix} \kappa_1 - \kappa_2 & \kappa_1 \\ \kappa_2 & \kappa_2 - \kappa_1 \end{pmatrix}. \tag{2.20}
\]

Moreover the eigenvalues of the matrix \(H\) are

\[
\lambda_- = -\sqrt{(\kappa_1)^2 + (\kappa_2)^2 - \kappa_1 \kappa_2}, \quad \lambda_+ = \sqrt{(\kappa_1)^2 + (\kappa_2)^2 - \kappa_1 \kappa_2}. \tag{2.21}
\]
with corresponding eigenvectors

\[ v_- = \left( 1, \frac{\kappa_2 - \kappa_1 - \sqrt{(\kappa_1)^2 + (\kappa_2)^2 - \kappa_1 \kappa_2}}{\kappa_1} \right), \]

\[ v_+ = \left( 1, \frac{\kappa_2 - \kappa_1 + \sqrt{(\kappa_1)^2 + (\kappa_2)^2 - \kappa_1 \kappa_2}}{\kappa_1} \right). \]  

(2.22)

One can immediately see that the eigenvectors in (2.22) depend actually by the ratio between \( \kappa_2 \) and \( \kappa_1 \) only. Moreover it turns out that this kind of dependence is indeed monotone increasing, as proved in the following Proposition.

**Proposition 2** Set \( \alpha = \frac{\kappa_2}{\kappa_1} \). Then the directions corresponding to the eigenvectors \( v_- \) and \( v_+ \) are given (respectively) by the maps

\[
\begin{align*}
G_-(\alpha) & : \begin{cases}
[0, \infty] \to [0, \infty] \\
\alpha \mapsto \alpha - 1 - \sqrt{\alpha^2 - \alpha + 1}
\end{cases} \\
g_-(\alpha) & : \begin{cases}
[0, \infty] \to [0, \infty] \\
\alpha \mapsto \alpha - 1 + \sqrt{\alpha^2 - \alpha + 1}
\end{cases} \\
G_+(\alpha) & : \begin{cases}
[0, \infty] \to [0, \infty] \\
\alpha \mapsto \alpha - 1 - \sqrt{\alpha^2 - \alpha + 1}
\end{cases} \\
g_+(\alpha) & : \begin{cases}
[0, \infty] \to [0, \infty] \\
\alpha \mapsto \alpha - 1 + \sqrt{\alpha^2 - \alpha + 1}
\end{cases}
\end{align*}
\]

depending on the sign of \( \alpha \) (and hence of \( \kappa_1 \cdot \kappa_2 \)). These maps satisfy

\[
\frac{d}{d\alpha} G_- > 0, \quad \frac{d}{d\alpha} G_+ > 0, \quad \frac{d}{d\alpha} g_- > 0, \quad \frac{d}{d\alpha} g_+ > 0. \]  

(2.23)

**Proof.** The properties follow from

\[
\begin{align*}
G'_-(\alpha) &= g'_-(\alpha) = 1 - \frac{2\alpha - 1}{2\sqrt{\alpha^2 - \alpha + 1}} = \frac{\sqrt{(2\alpha - 1)^2 + 3} - (2\alpha - 1)}{2\sqrt{\alpha^2 - \alpha + 1}} > 0, \\
G'_+(\alpha) &= g'_+(\alpha) = 1 + \frac{2\alpha - 1}{2\sqrt{\alpha^2 - \alpha + 1}} = \frac{\sqrt{(2\alpha - 1)^2 + 3} + (2\alpha - 1)}{2\sqrt{\alpha^2 - \alpha + 1}} > 0,
\end{align*}
\]

and from

\[
\lim_{\alpha \to 0^+} G_- (\alpha) = \lim_{\alpha \to 0^-} g_-(\alpha) = -1 - 1 = -2, \]

\[
\lim_{\alpha \to 0^+} G_+ (\alpha) = \lim_{\alpha \to 0^-} g_+(\alpha) = 0 + 0 = 0.
\]
We start considering all $K^j = (\kappa^j_1, \kappa^j_2)$ in $A_1 \cup A_2$. Notice that a similar analysis, with straightforward adaptations, can be done if the $K^j$ are in $A_5 \cup A_6$. This choice implies that our system follows the dynamics depicted in Figure 2.

Theorem 4 Let the cost functions $h_1, h_2$ be as in (2.3), and assume that the constants $(\kappa^j_1, \kappa^j_2)$ are all chosen in $A_1 \cup A_2$. Then the system (2.13) has a unique admissible solution and the corresponding functions $\alpha^*_i = -u'_i$ provide a Nash equilibrium solution to the non-cooperative game (2.1)-(2.2).
Proof. **Existence.** The existence of an admissible solution is very easy to prove. Indeed, it is enough to glue together pieces of admissible solutions in each interval $I_j$. We proceed as follows:

- in $I_o$, we set $p^o \equiv K^o = (\kappa_1^o, \kappa_2^o)$:
- for $j \geq 1$, in $I_j$ we set $p^j$ the unique solution of the Cauchy problem for (2.17) with initial datum $p(s(x_j)) = p^{j-1}(s(x_j))$. Since the set

$$\Gamma^j = \left\{(p_1, p_2) \mid p_1, p_2 \in [0, 2C_1], p_1 + p_2 \geq \frac{C_2}{2}\right\} ,$$

where

$$C_1 = \max\{\kappa_1^j, \kappa_2^j, \frac{p_1^{j-1}(s(x_j)) + p_2^{j-1}(s(x_j))}{2}\} ,$$

$$C_2 = \min\{\kappa_1^j, \kappa_2^j, \frac{p_1^{j-1}(s(x_j)) + p_2^{j-1}(s(x_j))}{2}\} ,$$

is positively invariant for (2.17), each $p^j$ will exists up to $s(x_{j+1})$ without reaching $(0, 0)$ and remaining bounded;

Then, it is well defined the continuous function $\bar{p}$ given by $\bar{p}(x) = p^j(x)$ whenever $x \in I_j$. Its admissibility is an immediate consequence of its continuity and the admissibility of each $p^j$.

**Uniqueness.** To prove that the solution built above is the unique admissible solution to (2.17), we start proving uniqueness on $I_o$.

We know from [2] that, for $s$ negative small enough (eventually for $s \to -\infty$), the only solutions that remain bounded are the equilibrium $K^o$ itself and the unstable orbits exiting from the origin. Therefore, these are the unique possible choices, in order to retain admissibility. If we choose an unstable orbit in place of $K^o$, in the original variable $x$ it would correspond to a solution defined only for $x > x_o$ (for a suitable $x_o$). To define the solution also for $x < x_o$, we should need a solution to

$$\begin{cases} p_1' = (\kappa_1^o - \kappa_2^o)p_1 + \kappa_1^op_2 - p_2^2, \\ p_2' = (\kappa_2^o - \kappa_1^o)p_2 + \kappa_2^op_1 - p_2^2, \end{cases}$$

that tends to the origin as $s \to +\infty$ and remains bounded for all negative $s$. But we know from [2] that no solution with both these properties exists. Hence the uniqueness of the solution follows on $I_o$.

For $s > s(x_1)$, the smoothness of the right hand side of (2.17) in each interval $I_j$ ensures that $\bar{p}$ is the unique continuous solution.

It remains to prove that there exists no solution with admissible jumps in $s > s(x_1)$. But this property follows from (2.12) and from the positive invariance of the sets

$$\Gamma^+ = \left\{(p_1, p_2) \in \mathbb{R}^2 \mid p_1 \geq 0, p_2 \geq 0\right\} ,$$

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Indeed, for $s > s(x_1)$ a solution can have only jumps from $\Gamma^+$ to $\Gamma^-$. Hence, recalling [2], after a first jump the solution would be forced to remain in $\Gamma^-$ and to tend towards $\infty$. In the $x$ variable, this would translate into a solution $u(x)$ that grows more than linearly as $|x| \to \infty$, and this would contradict admissibility.

In light of Theorem 4, on the same line of [2], it is natural to ask whether the result still hold for perturbations of (2.3) or it fails. Actually, we can prove the following Theorem.

**Theorem 5** Let the cost functions $h'_1, h'_2$ in (2.3) be smooth, and assume that:

1. their derivatives satisfy
   \[ \frac{1}{C} \leq h'_i(x) \leq C \]
   for some constant $C > 1$ and all $x \in \mathbb{R}$;
2. on $\mathcal{I}_0$, the following additional assumption is satisfied
   \[ \sup_{\xi, \eta \in \mathcal{I}_0} |h'_i(\xi) - h'_i(\eta)| \leq \delta = \frac{1}{2} \]
   \[ (3.1) \]
   for some $\delta > 0$ sufficiently small (depending only on $C$).

Then the system (2.13) has a unique admissible solution.

**Proof.** We can proceed as in Theorem 4 using Theorem 2 to deal with the perturbations. Indeed, for $s < s(x_1)$ Theorem 2 implies that there exists a unique admissible solution, say $p^o$. Hence, an admissible solution on the whole real line can be built as in the previous case: for $x \in [x_j, x_{j+1}), j \geq 1$, we define $p(x) = p^j(x)$ where $p^j$ is the unique solution to (2.13) with initial datum $p(s(x_j)) = p^{j-1}(s(x_j))$. Exactly as in Theorem 4 this function is well defined and is a continuous admissible solution to (2.17). Since the sets $\Gamma^+$ and $\Gamma^-$ are still positively invariant, also uniqueness can be proved by means of the same arguments used in Theorem 4.

**Remark 1** We underline that the presence of the small oscillations assumption (3.1) is uniquely motivated by the use of Theorem 2 which requires (3.1) to provide a unique admissible solution for $s < s(x_o)$.

4 **Conflicting interests**

In this section we assume that the two players have conflicting interests, i.e. their costs satisfy $h'_1(x) \cdot h'_2(x) < -C < 0$ for all $x \in \mathbb{R}$. For particular choices of smooth costs, this situation can produce infinitely many Nash equilibria to
the game (see [2]). Nevertheless Theorem 3 shows that, for costs which are not exactly opposite and under suitable assumptions of small oscillations, it is possible to recover existence and uniqueness of Nash equilibria. This is not the case for costs as in [2].

4.1 Case 1

Let us consider $j = 1$ in (2.3), i.e. let us consider cost functionals that have a single jump in their derivatives. In particular, assume this jump is located at $x = s(x) = 0$. Moreover, let us choose the constants $K^j = (\kappa_1^j, \kappa_2^j), j = 0, 1$, so that $K^o \in A_4$ and $K^1 \in A_3$.

Under these assumptions, the dynamics followed by the system are depicted in Figure 3 (for $x < 0$) and Figure 4 (for $x > 0$). We now prove that we could find infinitely many solutions to our problem. Indeed, consider an initial datum $p^{in} = (p^{in}_1, p^{in}_2)$ such that $p^{in}_1 + p^{in}_2 = 0$ and $p^{in}_1 < 0 < p^{in}_2$. Recalling Proposition 2 and setting $\alpha^o = \frac{\kappa_2^o}{\kappa_1^o}, \alpha^1 = \frac{\kappa_2^1}{\kappa_1^1}$, we have

$$g_-(\alpha^1) < -2 < -1 = \frac{p^{in}_2}{p^{in}_1} < -\frac{1}{2} < g_+(\alpha^o),$$

i.e. $p^{in}$ belongs to the region between the stable orbit for the negative system (say $\gamma_S^-$) and the unstable one for the positive system (say $\gamma_U^+$), provided it's
been chosen sufficiently near the origin. Therefore to any choice of $p^{in}$ there corresponds an admissible solution tending respectively to either $K^1$ or $K^o$ as $s \to \pm \infty$.

Moreover, if the unstable orbit for the dynamics in Figure 3 (say $\gamma^-_U$) intersects the stable one for the dynamics in Figure 4 (say $\gamma^+_S$), we can obtain an additional solution considering as initial datum that point of intersection. Indeed the function given by the juxtaposition of $\gamma^-_U$ and $\gamma^+_S$ corresponds, in the original variable $x$, to a solution defined on a bounded interval $[x_-, x_+]$, with $x_- < 0 < x_+$ by the choice of the rescaling. This solution can then be extended to an admissible trajectory defined on the whole real line by using $\gamma^+_S$ for $x < x_-$ and $\gamma^-_U$ for $x > x_+$.

**Remark 2** The same construction can be applied when $K^o \in A_8$ and $K^1 \in A_7$.

### 4.2 Case 2

Now we want to show, by means of a second example, how a simple change between the positive and negative behaviors of the costs, can lead to completely different result. Namely, we consider costs with a single jump in their derivatives, located in $x = s(x) = 0$, and $K^o \in A_3$, $K^1 \in A_4$. This choice produce a game with no admissible solutions to (2.17).

We proceed by contradiction. Assume that an admissible solution $\tilde{p} = (\tilde{p}_1, \tilde{p}_2)$ exists, for a Cauchy problem with initial datum $\tilde{p}(0) = \tilde{p}^{in}$. Then, recalling the
results in [2], we have that
\[ \lim_{s \to +\infty} |\tilde{p}(s)| < +\infty \]
actually implies
\[ \lim_{s \to +\infty} |\tilde{p}(s)| = 0 , \]
and hence \( \tilde{p} \) is one of the stable orbits of the positive system. Now we underline that this means \( \tilde{p}^\circ \notin \gamma_U^+ \). Then, we can repeat the proof of Theorem 3 given in [2], and find
\[ \lim_{s \to s_o+} |\tilde{p}(s)| = +\infty , \]
for a suitable \( s_o < 0 \), eventually \( s_o = -\infty \). Therefore the solution cannot be admissible, and we have a contradiction.

Notice that the previous calculations hold even if the unstable orbit for the dynamics in Figure 3 (say \( \gamma_U^\circ \)) intersects the stable one for the dynamics in Figure 4 (say \( \gamma_S^\circ \)). This means there is no solution as the one built in the previous case, using more trajectories in the \( s \) variable: this is obviously due to the fact that we cannot find solutions bounded at \( +\infty \) (resp. \( -\infty \)) to extend a possible \( \tilde{p} \) when \( x > x_+ \) (resp. \( x < x_- \)).

Remark 3 The same result can be obtained when \( K^o \in A_7 \) and \( K^1 \in A_8 \).

Remark 4 Actually, one can still construct particular cases so that there exist admissible solutions. Fixed \( K^o, K^1 \) as above, assume that the trajectories \( \gamma_U^- \) and \( \gamma_S^+ \) intersect in a point. Moreover, set \( x_- \) and \( x_+ \) the values introduced in the previous example, \( \ell = |x_+ - x_-| \) and \( \mathcal{J}_n = [x_- + n\ell, x_+ + n\ell], n \in \mathbb{Z} \). We can define piecewise linear costs on the whole \( \mathbb{R} \) by repeating on each \( \mathcal{J}_n \) the same 2-value piecewise linear cost. In other words, \( \forall n \in \mathbb{Z} \) set
\[ h_i'(x)|_{\mathcal{J}_n} = \begin{cases} \kappa_i^o & \text{if } x \in [x_- + n\ell, x_+ + n\ell] \\ \kappa_i^1 & \text{if } x \in [n\ell, x_- + n\ell] \end{cases} \quad i = 1, 2, \quad (4.1) \]
Then, we find a solution by simply gluing together periodically \( \gamma_U^- \) and \( \gamma_S^+ \). This solution is admissible, being bounded in the \( p_1, p_2 \) plane. Any way no general results as Theorem 3 is possible.

5 Mixed Cases

In this section we end our presentation of ill-posed problems, with a last example presenting costs that can switch from a situation with conflicting interests into a cooperative one. More precisely, we consider costs with a single jump in their derivative, located again in \( x = s(x) = 0 \), and \( K^o \in A_5 \cup A_6, K^1 \in A_1 \cup A_2 \). Moreover, let us assume
\[ \alpha^1 = \frac{\kappa_2^1}{\kappa_1^1} \neq \frac{\kappa_2^o}{\kappa_1^o} = \alpha^o . \quad (5.1) \]
With these assumptions, the system follows the dynamics depicted in Figure 5 (resp. Figure 2) for $x < 0$ (resp. $x > 0$) and $K^o, K^1$ are not on the same line through the origin.

Again, we observe the existence of infinitely many Nash equilibria. Assume it holds $\alpha^o < \alpha^1$ in (5.1) (the opposite inequality leading to a similar analysis). Then, we can consider the non-empty region

$$\Omega = \left\{ (p_1, p_2) \in \mathbb{R}^2 \mid p_1 < 0 < p_2, \ G^- (\alpha^o) < \frac{p_2}{p_1} < G^- (\alpha^1) \right\}.$$

This region is, at least near the origin, say in a neighborhood $\mathcal{O}$, exactly the region between the stable orbit for the positive system and the unstable one for the negative system. Taking as initial datum any point $p^{in}$ both in $\Omega$ and in $\mathcal{O}$, we can construct an admissible solution in the following way. We take for $s < 0$ the unique solution to the negative system, passing through $p^{in}$ at $s = 0$ and tending to $K^o$ as $s \to -\infty$. In an analogous way, we take for $s > 0$ the unique solution to the positive system, passing through $p^{in}$ at $s = 0$ and tending to $K^1$ as $s \to +\infty$. Every such a solution, being continuous and bounded in $s$, corresponds to an admissible solution $u(x)$.

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