VARIATIONAL FORMULATION OF LINEAR TIME-DEPENDENT INVARIANTS *

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Abstract. It is shown that linear time-dependent invariants for arbitrary multi-dimensional quadratic systems can be obtained from the Lagrangian and Hamiltonian formulation procedures by considering a variation of coordinates and momenta that follows the classical trajectory and defines a noetherian symmetry transformation.

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There are several physical systems which in different approximations can be represented by Hamiltonians which are quadratic forms of coordinates and momenta variables, \textit{i.e.}, the classical small oscillations problem. Thus the applicability of multidimensional quadratic Hamiltonians ranges from solid-state physics to particle physics, as for example in the relativistic oscillator models of elementary particles \cite{1,2} and the particle creation in non-stationary metrics \cite{3}. The exact solutions for stationary multidimensional quadratic systems have been found and discussed in the framework of canonical transformations \cite{4}.

Recently for the classical small oscillations problem by means of non-noetherian transformations, noetherian quadratic time-independent integrals of motion have been found \cite{5}. For the one-mode parametric classical oscillator Ermakov found an invariant, which is quadratic in the momentum and position variables \cite{6}. In the last years on the base of a geometric approach related to the Noether’s theorem the last invariant has been rederived \cite{7}.

In the case of non-stationary quantum systems there are different procedures for solving time-dependent Hamiltonians and calculating evolution operators, but the simplest, at least for quadratic systems, is the method of time-dependent integrals of motion \cite{8}.

The theory of integrals of motion which do not depend on time in Schrödinger representation is well known \cite{9}. On the other hand, for the one-dimensional quantum oscillator a time-dependent integral of motion quadratic in position and momentum operators has been found \cite{10}, which in form coincides with the Ermakov invariant. The linear time-dependent invariants have been found and applied to get the solutions for the quantum parametric oscillator and a charge moving under a time-dependent magnetic field \cite{11}. In the most general multidimensional case, the time-dependent invariants of the non-stationary quadratic quantum systems also have been found and used to get the solutions of the time-dependent Schrödinger equation in the form of gaussian coherent states and Hermite polynomials of several variables \cite{8, 12}. These invariants are linear combinations of coordinates and momenta operators, with time-dependent coefficients.

We think it is necessary to understand the existence of these time-dependent constants of motion within the framework of a variational formulation. Up to now for one-mode parametric oscillator, the linear invariants of Ref. \cite{11}, were obtained in \cite{13}, \cite{14} and \cite{15} through the Noether’s theorem procedure. Also with this formalism
in Refs. [13] and [14] the quadratic time-dependent invariants of this system were derived. Nevertheless until now it was unknown what symmetry corresponds to linear time-dependent integrals of motion in the frame of Noether’s theorem prescription for multidimensional parametric oscillators.

The aim of this paper is to get the linear multidimensional time-dependent invariants by means of the Lagrangian and Hamiltonian formalisms of the Noether’s theorem procedure. We give explicitly such a variation or noetherian symmetry transformation, which has a clear physical interpretation. Preliminary results, of the Lagrangian Noether’s theorem procedure, for an arbitrary multidimensional forced parametric oscillators with a detailed discussion of a two dimensional quantum system was presented in Ref. [15].

Next we present the active viewpoint of the Lagrangian and Hamiltonian formalisms of the Noether’s theorem procedure for a given Hamiltonian system.

**Lagrangian Formulation**

Let us consider an arbitrary time-dependent Hamiltonian, which is given in terms of a general quadratic form, *i.e.*, 

\[ H = \frac{1}{2} Q_a B_{ab}(t) Q_b + C_a(t) Q_a, \]  

(1)

where hereafter latin indices run from 1 to 2n and sum over repeated indices is understood. Besides, we defined the coordinates and momenta like a 2n-dimensional vector

\[ Q = \begin{pmatrix} p_1 \\ \vdots \\ p_n \\ q_1 \\ \vdots \\ q_n \end{pmatrix}. \]  

(2)

The 2n-dimensional matrix **B** and vector **C**, which define the quadratic form of the Hamiltonian, can be rewritten in terms of the n × n submatrices, A, B, C, and D, and the n—column vectors F, and G, respectively

\[ B = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad C = \begin{pmatrix} F \\ G \end{pmatrix}. \]  

(3)

The hermiticity of the Hamiltonian implies that the matrix **B** must be symmetric, and this gives rise to the following symmetry conditions over the four constituents
\( n \times n \) matrices:

\[
A^t = A, \quad B^t = C, \quad D^t = D.
\] (4)

Expanding the Hamiltonian (1) we obtain

\[
H = \frac{1}{2}(A_{\alpha\beta}p_\alpha p_\beta + 2B_{\alpha\beta}p_\alpha q_\beta + D_{\alpha\beta}q_\alpha q_\beta) + F_\alpha p_\alpha + G_{\beta}q_\beta,
\] (5)

where the symmetry conditions (4) were used. Now and hereafter, greek indices run from 1 to \( n \). From Hamiltonian (5), it is straightforward to obtain the Hamilton equations of motion

\[
\dot{p}_\alpha = -C_{\alpha\beta}p_\beta - D_{\alpha\beta}q_\beta - G_\alpha, \quad (6a)
\]

\[
\dot{q}_\alpha = A_{\alpha\beta}p_\beta + B_{\alpha\beta}q_\beta + F_\alpha. \quad (6b)
\]

This coupled set of first order differential equations can be decoupled in the following sets of second order equations

\[
\ddot{p}_\alpha = -\left[ C - \left( \dot{D} - DB \right) D^{-1} \right]_{\alpha\beta} \dot{p}_\beta - \left[ \dot{C} + DA - \left( \dot{D} + DB \right) D^{-1} C \right]_{\alpha\beta} p_\beta \\
+ \left[ \left( \dot{D} + DB \right) D^{-1} \right]_{\alpha\beta} G_\beta + D_{\alpha\beta} F_\beta + \dot{G}_\alpha, \quad (7a)
\]

\[
\ddot{q}_\alpha = \left[ B + (\dot{A} - AC) A^{-1} \right]_{\alpha\beta} \dot{q}_\beta + \left[ \dot{B} - AD - (\dot{A} - AC) A^{-1} B \right]_{\alpha\beta} q_\beta \\
- \left[ (\dot{A} - AC) A^{-1} \right]_{\alpha\beta} F_\beta - A_{\alpha\beta} G_\beta + \dot{F}_\alpha. \quad (7b)
\]

Next, we are going to find the Lagrangian of the system and apply Noether’s procedure to get the time-dependent invariants. The Lagrangian is obtained by making a Legendre transformation to the Hamiltonian (5) and using the equation of motion (6b), resulting

\[
L = \frac{1}{2} \left\{ A_{\alpha\beta} A^{-1} q_\alpha q_\beta - (D - CA^{-1} B)_{\alpha\beta} q_\alpha q_\beta + A_{\alpha\beta}^{-1} F_\alpha F_\beta \right\} \\
- (A^{-1} B)_{\alpha\beta} q_\alpha q_\beta - A_{\alpha\beta}^{-1} F_\alpha q_\beta - G_\alpha q_\beta + (A^{-1} B)_{\alpha\beta} F_\alpha q_\beta. \quad (8)
\]

To get the constants of the motion for this system, let us propose infinitesimal variations of coordinates given by

\[
\delta q_\alpha = h_\alpha(t), \quad (9)
\]
where $h(t)$ is an arbitrary $n$–dimensional vector depending only on time. The corresponding variation induced in the Lagrangian (8) is

$$
\delta L = A^{-1}_{\alpha\beta} h_\alpha (\dot{q}_\beta - B_{\beta\gamma} q_\gamma - F_\beta) - (CA^{-1})_{\alpha\beta} h_\alpha \dot{q}_\beta \\
+ (CA^{-1}B - D)_{\alpha\beta} h_\alpha q_\beta - (CA^{-1})_{\alpha\beta} h_\alpha F_\beta - h_\alpha G_\alpha .
$$

(10)

This expression can be rewritten as

$$
\delta L = \frac{d}{dt} \left\{ A^{-1}_{\alpha\beta} h_\alpha q_\beta - (CA^{-1})_{\alpha\beta} h_\alpha q_\beta - \int^t dt \left( A^{-1}_{\alpha\beta} \dot{h}_\alpha F_\beta + (CA^{-1})_{\alpha\beta} h_\alpha F_\beta + h_\alpha G_\alpha \right) \right\} \\
- \left[ \left( A^{-1}_{\alpha\beta} \dot{h}_\alpha \right) + (A^{-1}B)_{\alpha\beta} \dot{h}_\alpha - ((CA^{-1})_{\alpha\beta} h_\alpha) - (CA^{-1}B - D)_{\alpha\beta} h_\alpha \right] q_\beta .
$$

(11)

Asking that the variations of coordinates (9) yield a Noether’s symmetry transformation, one has that (11) must be equal to a total time derivative of a function $\Omega_L(q_\alpha, t)$. This is achieved if the variation $h(t)$ satisfies the differential equation

$$
\left( \dot{h}_\alpha A^{-1}_{\alpha\beta} \right) + \dot{h}_\alpha (A^{-1}B)_{\alpha\beta} \\
- \left( h_\alpha (CA^{-1})_{\alpha\beta} \right) - h_\alpha (CA^{-1}B - D)_{\alpha\beta} = 0,
$$

(12)

and the function $\Omega_L(q_\alpha, t)$ is defined by the curly brackets of expression (11). Developing (12), one sees that it has the same form that the homogeneous part of the differential equation of motion for $q_\alpha$, given in (7b). For these symmetry transformations according to the Noether’s theorem the associated conserved charges are given by $J_L = (\partial L/\partial \dot{q}_\alpha) \delta q_\alpha - \Omega_L$, that is

$$
J_L = p_\alpha h_\alpha - A^{-1}_{\alpha\beta} \dot{h}_\alpha q_\beta + (CA^{-1})_{\alpha\beta} h_\alpha q_\beta \\
+ \int^t dt \left( A^{-1}_{\alpha\beta} \dot{h}_\alpha F_\beta - (CA^{-1})_{\alpha\beta} h_\alpha F_\beta + h_\alpha G_\alpha \right) ,
$$

(13)

with $p_\alpha = \left( A^{-1}_{\alpha\beta} \dot{q}_\beta - (A^{-1}B)_{\alpha\beta} q_\beta - A^{-1}_{\alpha\beta} F_\beta \right)$. The different conserved charges are obtained by the initial conditions established for the system (12).

**Hamiltonian Formulation**

There is also the possibility of using the Noether’s theorem in the Hamiltonian formulation. In this case we consider the action

$$
S = \int_{t_0}^{t_1} dt \left\{ \Sigma_{ab} Q_b \dot{Q}_a - H(Q_a, t) \right\} ,
$$

(14)
where we have introduced the \((2n \times 2n)\)-matrix in block form:

\[
\Sigma_+ = \begin{pmatrix} 0 & \mathbf{I} \\ 0 & 0 \end{pmatrix},
\]

with \(\mathbf{I}\) being the identity matrix in \(n\) dimensions.

Taking an arbitrary variation of (14) with respect to coordinates and momenta defined by (2) we get the expression

\[
\delta S = \int_{t_0}^{t_1} dt \left\{ \delta Q_a \left( \Sigma_{+ab} \dot{Q}_b - \frac{\partial H}{\partial Q_b} \right) + \delta \dot{Q}_a \Sigma_{-ab} Q_b \right\},
\]

where the transpose of the matrix \(\Sigma_+\) was used,

\[
\Sigma_- \equiv \begin{pmatrix} 0 & 0 \\ \mathbf{I} & 0 \end{pmatrix}.
\]

If the transformation \(\delta Q_\alpha\) is a noetherian symmetry, the curly bracket in (15) must be written as a total time derivative of a function \(\Omega_H(Q_a, t)\), i.e.,

\[
\delta Q_a \left( \Sigma_{+ab} \dot{Q}_b - \frac{\partial H}{\partial Q_b} \right) + \delta \dot{Q}_a \Sigma_{-ab} Q_b = \frac{d}{dt} \Omega_H(Q_a, t).
\]

Following now the same procedure as in the Lagrangian case the associated conserved quantity to this variation is given by

\[
J_H = \delta Q_a \Sigma_{-ab} Q_b - \Omega_H.
\]

This can be checked also by taking the total time derivative of (17) and the Hamilton equations of motion together with the condition (16).

For the multidimensional quadratic parametric Hamiltonian (1), the linear time-dependent constants of motion can be obtained by means of the variations

\[
\delta Q_a = (\delta p_\alpha, \delta q_\alpha) = (g_\alpha(t), h_\alpha(t)) \quad .
\]

Substituting these variations into the left hand side of (16), we find a symmetry transformation if the following relations are satisfied:

\[
\dot{g}_\alpha = -C_{\alpha\beta} g_\beta - D_{\alpha\beta} h_\beta, \quad \text{(19a)}
\]

\[
\dot{h}_\alpha = A_{\alpha\beta} g_\beta + B_{\alpha\beta} h_\beta, \quad \text{(19b)}
\]

\[
\Omega_H = g_\alpha q_\alpha - \int dt \left( F_\alpha g_\alpha + G_\alpha h_\alpha \right). \quad \text{(19c)}
\]
It is important to remark that the first two equations have the same form that the homogeneous part of the Hamilton equations of motion given in (6). Therefore it is straightforward to demonstrate that the variations \( h_\alpha \) satisfy the same differential equations (12) as the corresponding variations of the Lagrangian formulation. For this reason, the constant of motion (17) coincides with its Lagrangian counterpart (13), i.e., \( J_L = J_H \).

The expression (13) gives rise to \( 2n \) invariants corresponding to the \( 2n \) independent solutions for the system of equations (12). These \( 2n \) constants of the motion can be rewritten in matrix form

\[
\begin{pmatrix}
    p_0(t) \\
    q_0(t)
\end{pmatrix} = \Lambda(t) \begin{pmatrix}
    p \\
    q
\end{pmatrix} + \Delta(t),
\]

where the transformation matrix \( \Lambda \) takes the form

\[
\Lambda(t) = \begin{pmatrix}
    h^{(1)}(1) & \left( h^{(1)}(1)C - \dot{h}^{(1)}(1) \right) A^{-1} \\
    h^{(2)}(2) & \left( h^{(2)}(2)C - \dot{h}^{(2)}(2) \right) A^{-1} \\
    \vdots & \vdots \\
    h^{(2n)}(2n) & \left( h^{(2n)}(2n)C - \dot{h}^{(2n)}(2n) \right) A^{-1}
\end{pmatrix}
\]

The constants of motion defined in (20) satisfy at \( t = 0 \) the conditions \( \vec{p}_0(0) = \vec{p} \) and \( \vec{q}_0(0) = \vec{q} \). These requirements imply different initial conditions for the system of differential equations (12), and guarantee the existence of \( 2n \) independent solutions, which are denoted by the superscript on the function \( h(t) \). Besides these vector solutions are written horizontally. The time-dependent column vector \( \Delta \) is given by

\[
\Delta_k(t) = \int_0^t dt \left( A^{-1}_{\alpha\beta} \dot{h}_\alpha^{(k)} F_\beta + (CA^{-1})_{\alpha\beta} h_\alpha^{(k)} F_\beta + h_\alpha^{(k)} G_\alpha \right).
\]

It is important to remark that the expressions in Eq. (20) are the linear invariants proposed in Ref. [8,12], where it was shown that \( \Lambda \) is a \( 2n \) dimensional symplectic matrix, as it must be because the constants of motion \( p_0(t) \) and \( q_0(t) \) satisfy the canonical commutation relations. However now the elements of the matrix \( \Lambda \) are given in terms of the \( 2n \) independent solutions of the homogeneous classical equations of motion (12), and their derivatives.

To have that the invariants coincide with the position and momentum operators at \( t = 0 \), the appropriate initial conditions for the solutions of Eq. (12) are

\[
h^{(i)}_j(0) = \begin{cases}
    \delta^i_j, & 1 \leq i, j \leq n \\
    0, & n + 1 \leq i, j \leq 2n
\end{cases}
\]

(23a)
and for their derivatives

\[ h_j^{(i)}(0) = \begin{cases} C_{ij}(0), & 1 \leq i, j \leq n \\ -A_{ij}(0), & n + 1 \leq i, j \leq 2n \end{cases} \quad (23b) \]

So we have proved that for multidimensional stationary or non-stationary classical or quantum systems the linear time-dependent invariants follow from the Lagrangian or Hamiltonian formulations of the Noether’s theorem procedure by considering a variation of coordinates or momenta that follows the classical trajectory.

An additional remark is that the quadratic invariants for the stationary classical small oscillation system found from non-noetherian transformations [5] can be obtained from the linear invariants presented here.

The suggested method can be extended to classical or quantum field theory by considering infinite degrees of freedom and as consequence finding an infinite number of time-dependent invariants linear in fields and their corresponding conjugated momenta.

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