Monotonicity properties
for Bernoulli percolation on layered graphs
- a Markov chain approach

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Abstract

A layered graph $G^\times$ is the Cartesian product of a graph $G = (V, E)$ with the linear graph $\mathbb{Z}$, e.g. $\mathbb{Z}^2$ is the 2D square lattice $\mathbb{Z}^2$. For Bernoulli percolation with parameter $p \in [0,1]$ on $G^\times$ one intuitively would expect that $P_p((o,0) \leftrightarrow (v,n)) \geq P_p((o,0) \leftrightarrow (v,n+1))$ for all $o, v \in V$ and $n \geq 0$. This is reminiscent of the better known bunkbed conjecture. Here we introduce an approach to the above monotonicity conjecture that makes use of a Markov chain building the percolation pattern layer by layer. In case of finite $G$ we thus can show that for some $N \geq 0$ the above holds for all $n \geq N$ $o, v \in V$ and $p \in [0,1]$. One might hope that this Markov chain approach could be useful for other problems concerning Bernoulli percolation on layered graphs.

1 Introduction

Bernoulli (bond) percolation is a stochastic process that has been studied extensively since it was introduced in the 1950s. For a given connected (simple) graph $G = (V(G), E(G))$ and a parameter value $p \in (0,1)$ we consider independent random variables $Z_e, e \in E(G)$, such that $P(Z_e = 1) = p$ and $P(Z_e = 0) = 1 - p$. Often the value $p$ is indicated in the notation for probabilities in the form of $P_p$. $e$ is called open if $Z_e = 1$, and $e$ is called closed if $Z_e = 0$. The family of random variables $(Z_e)_{e \in E(G)}$ determines a random subgraph $G_Z$ of $G$ via $V(G_Z) := V$ and $E(G_Z) := \{ e \in E : Z_e = 1 \}$. Usually $G$ is assumed to be regular in some sense (e.g. a lattice). For variants of this process and various results and open questions in Bernoulli percolation see the standard reference [G]. For $u, v \in V(G)$ we are interested in the event $\{ u \leftrightarrow v \}$ that $u, v$ are in the same connected component of $G_Z$. Interpreting Bernoulli percolation as a model for the spread of a disease, where an open bond transmits the disease and a closed bond does not, $\{ o \leftrightarrow v \}$ represents the event that $v$ is infected by a disease originating in some fixed vertex $o \in V(G)$. It is not unreasonable to expect that vertices closer to $o$ (in some sense that has to be specified) are more likely to be infected than vertices further away, i.e. we have some sort of

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spatial monotonicity property of the connectivity function (or two-point function) \( f_p(v) := \mathbb{P}_p(o \leftrightarrow v) \). In the following we present three settings, where the above monotonicity property can be formulated in a precise way.

**Conjecture 1** Monotonicity of the connectivity function in case of \( \mathbb{Z}^d \). Let \( d \geq 1 \) and \( p \in (0, 1) \). On the vertex set of the graph \( \mathbb{Z}^d \) (in which any two vertices that have a Euclidean distance of 1 are connected by an edge) we consider the componentwise (partial) order \( \leq \) (i.e. \( u \leq v \) iff \( u_i \leq v_i \) for all \( i \in \{1, \ldots, d\} \)).

For Bernoulli percolation on \( \mathbb{Z}^d \) with parameter \( p \) we have

\[
\forall o, u, v \in \mathbb{Z}^d : \quad o \leq u \leq v \quad \Rightarrow \quad \mathbb{P}_p(o \leftrightarrow u) \geq \mathbb{P}_p(o \leftrightarrow v). \quad (1.1)
\]

This conjecture is mathematical folklore, and we don’t know its precise origin. Similar natural monotonicity conjectures can be made for other regular graphs, and there are variants of the above conjecture for \( \mathbb{Z}^d \): e.g. one might want to consider edge probabilities \( p_i, i \in \{1, \ldots, d\} \), depending on the direction of the edges. In case of \( d = 1 \) the above conjecture is trivially true, but already the case \( d = 2 \) is open (even if it seems to be very compelling to assume it is true). The only pertaining result we know of is the result of [LPS], where the above monotonicity is established for \( o, u, v \in \mathbb{Z} \times \{0\}^{d-1} \) provided that \( p \) is sufficiently close to 0.

A classical generalization of the above conjecture is the so called bunkbed conjecture that applies to graphs of the following structure:

**Definition 1** Let \( G \) be a graph and \( T \subset V(G) \). The bunkbed graph \( G_T \) is given by \( V(G_T) = V(G) \times \{0, 1\} \) and \( E(G_T) = E^h(G_T) \cup E^v(G_T) \), where \( E^h(G_T) = \{(u,l)(v,l) : uv \in E(G), l \in \{0, 1\}\} \) is the set of horizontal edges and \( E^v(G_T) = \{(u,0)(u,1) : u \in T\} \) is the set of vertical edges.

Loosely speaking a bunkbed graph consists of two copies of \( G \) with vertices in the two copies connected to each other (by a vertical edge) iff they are copies of the same vertex in \( T \). The name ‘bunkbed graph’ comes from thinking of the two copies of \( G \) as the two beds of a bunkbed and the vertical edges as the poles connecting the two beds (where the pole positions are given by \( T \)).

**Conjecture 2** Bunkbed conjecture. Let \( G = (V, E) \) be a finite connected graph, \( T \subset V \), and let \( p \in (0, 1) \). For Bernoulli percolation on the bunkbed graph \( G_T \) with parameter \( p \) we have

\[
\forall o, v \in V : \quad \mathbb{P}_p((o,0) \leftrightarrow (v,0)) \geq \mathbb{P}_p((o,0) \leftrightarrow (v,1)). \quad (1.2)
\]

This is a natural extension of [Conjecture 1] and can be traced back to P.W. Kasteleyn (1985) (as noted in [BK]). The bunkbed conjecture also appears in several variants. Sometimes only the case \( T = V \) is considered. More general, one may want to consider the case of infinite connected graphs \( G \), or the case of nonconstant edge probabilities, say \( p_e \in [0,1], e \in E \) for the horizontal edges and \( p_u \in [0,1], u \in V \) for the vertical edges (and indeed, often only the case of deterministic vertical edges, i.e. \( p_u \in \{0,1\} \), is considered.) Again it is
very compelling to assume the conjecture is true, but while it has received a lot of attention, only few partial results have been obtained, see [Ri2] and the references therein. In this paper we would like to focus on the setting of layered graphs that is somewhat in between the two settings considered above.

**Definition 2** Let $G = (V, E)$ be a graph. The **layered graph** $G^\times$ is the Cartesian product of $G$ and $\mathbb{Z}$, i.e. $V(G^\times) = G \times \mathbb{Z}$ and $E(G^\times) = E^h(G^\times) \cup E^v(G^\times)$, where $E^h(G^\times) = \{(u, l)(v, l) : uv \in E, l \in \mathbb{Z}\}$ is the set of horizontal edges and $E^v(G^\times) = \{(u, l - 1)(u, l) : u \in V, l \in \mathbb{Z}\}$ is the set of vertical edges.

Loosely speaking the layered graph $G^\times$ consists of biinfinitely many copies of $G$ (forming the layers of $G^\times$) with vertices in two distinct layers connected to each other (by a vertical edge) iff they are copies of the same vertex in $V$ and the two layers are adjacent. To illustrate this definition we note that $\mathbb{Z}^d$ can be viewed as the layered graph $(\mathbb{Z}^{d-1})^\times$.

**Conjecture 3** Monotonicity of the connectivity function in case of layered graphs. Let $G = (V, E)$ be a finite connected graph, and $p \in (0, 1)$. For Bernoulli percolation on the layered graph $G^\times$ with parameter $p$ we have

$$\forall o, v \in V \forall 0 \leq n \leq n' : \mathbb{P}_p((o, 0) \leftrightarrow (v, n)) \geq \mathbb{P}_p((o, 0) \leftrightarrow (v, n')).$$ (1.3)

This conjecture seems to be new. Again, more generally one may want to consider the case of infinite connected graphs $G$, or the case of nonconstant edge probabilities, say $p_e \in [0, 1], e \in E$ for the horizontal edges (in each layer) and $p_u \in [0, 1], u \in V$ for the vertical edges. As we will show, the above conjecture is somewhat in between the two preceding conjectures in terms of generality. We will use the layered structure to investigate weaker and stronger versions of the above conjecture in this paper, namely the monotonicity of the expected number of infected vertices per layer and the monotonicity of the probability for an infection pattern within a layer. While our results concerning the former rely on combinatorial arguments and classical tools from percolation theory, our results concerning the latter rely on a combination of classical tools with Markov chain methods: Building the percolation structure layer by layer, we obtain a Markov chain, for which asymptotics for transition probabilities can be used to obtain monotonicity results. As far as we know this is a novel approach to investigating percolation problems in general, and we hope that this new perspective will be of some use in the future.

The paper is organized as follows. In Section 2 we present our results and explain how they are connected to the above conjectures. In Section 3 we prove statements on the comparison of monotonicity properties. In Section 4 we investigate the Markov chain on infection patterns. In Sections 5 we prove our main result on the monotonicity of the probability for an infection pattern within a layer. In Section 6 we prove our result on the monotonicity of the expected number of infected points per layer.
2 Results

As a motivation for our interest in Conjecture \( 3 \) let us note its relation to Conjectures \( 1 \) and \( 2 \). Recall that the Cartesian product of graphs \( G_1, \ldots, G_d \) with \( G_i = (V_i, E_i) \) is the graph \( G = (V, E) \) with \( V = V_1 \times \cdots \times V_d \) and \( E = \{(v_1, \ldots, v_d) (w_1, \ldots, w_d) : \exists i : v_i w_i \in E_i, \forall j \neq i : v_i = w_i \in V_i\} \). For \( d \geq 1 \) let \( \mathbb{Z}^d, L_k \) and \( C_k^d \) denote the Cartesian products of \( d \) copies of \( \mathbb{Z} \), the linear graph \( L_k \) and the cycle graph \( C_k \) on \( k \geq 2 \) vertices respectively. Here we adopt the convention that \( C_2 := L_2 \) in order to avoid multi-edges.

**Proposition 1**  
Conjecture \( 2 \) (on bunkbed graphs) implies Conjecture \( 3 \) (on layered graphs), and Conjecture \( 3 \) implies Conjecture \( 1 \) (on layered graphs). More precisely:

(a) Let \( p \in (0, 1) \). If \((1.2)\) holds for Bernoulli percolation with parameter \( p \) on every bunkbed graph \( G_T \) (i.e. for every finite connected graph \( G \) and every \( T \subset V(G) \)), then \((1.3)\) holds for Bernoulli percolation with parameter \( p \) on every layered graph \( G^x \) (i.e. for every finite connected graph \( G \)).

(b) Let \( p \in (0, 1) \) and \( d \geq 2 \). If \((1.3)\) holds for Bernoulli percolation with parameter \( p \) on the layered graphs \( (C_k^{d-1})^x \) (i.e. for every \( k \geq 2 \), then \((1.1)\) holds for Bernoulli percolation with parameter \( p \) on \( \mathbb{Z}^d \). (The same is true for \( L_k^{d-1} \) instead of \( C_k^{d-1} \).)

The proof will be given in Section \( 3 \). Since \( \mathbb{Z}^2 \) is the most prominent graph on which Bernoulli percolation is studied, in view of (b) we are mostly interested in the layered graphs \( C_k^2 \). While our results are fairly general, one might want to keep this particular example in mind.

From now on we consider the layered graph \( G^x \) for some fixed connected graph \( G = (V, E) \), and we will suppress dependencies on \( G \) (and \( V, E \)) in our notations. We consider a fixed vertex \( o \in V \) such that \( (o, 0) \) is considered to be the origin of an infection. On \( G^x \) the translation

\[ \tau : V \times \mathbb{Z} \to V \times \mathbb{Z}, \tau(v, n) := (v, n + 1) \]

is a graph automorphism, and we note that Bernoulli-percolation is translation invariant in that \((\mathbb{Z}_e)_{e \in E(G^x)} \sim (\mathbb{Z}_{\tau(e)})_{e \in E(G^x)}\) We also introduce notation to refer to vertices and edges in certain layers of \( G^x \). For \( n \in \mathbb{Z} \) let

\[ V_n := V \times \{n\}, \quad E_h^n := \{(u, n)(v, n) : u v \in E\}, \quad E_v^n := \{(u, n-1)(u, n) : u \in V\} \]

and \( E_n := E_h^n \cup E_v^n \). We think of \( V_n, E_h^n, E_v^n \) as the vertices, the horizontal edges and the vertical edges of the \( n \)-th layer of \( G^x \). We also write

\[ E_{k,n} := \bigcup_{k \leq m \leq n} E_m, \quad E_{n} := \bigcup_{m \leq n} E_m \text{ and } E_{n,n} := \bigcup_{m \geq n} E_m. \]

Our main tool for analyzing connection probabilities for Bernoulli percolation on a layered graph will be a Markov chain \( \mathcal{X}_n \) that makes use of the layered structure. Informally \( \mathcal{X}_n \) describes which vertices in layer \( V_n \) are connected to each other and which are infected (i.e. connected to \((o, 0)\)) by means of open paths that are fully contained in \( E_{n,n} \). We will formalize this in the following, but it may be more instructive to look at the example illustrated in Figure \( 1 \).
Figure 1: The left hand side shows a realization of Bernoulli percolation on $L_4^4$ restricted to $\{0, 1, 2, 3\} \times \{-1, 0, 1, 2, 3\}$. The origin $(0,0)$ is marked with a thick dot, open edges are drawn, closed edges are dotted. The right hand side shows the corresponding realization of $X_0, \ldots, X_3$. Infected vertices are marked with a thick dot, connected vertices are joined by lines. With the notation introduced below we have $X_0 = \{\{*, 0, 1\}, \{2, 3\}\}, X_1 = \{\{*, 0\}, \{1, 2\}, \{3\}\}, X_2 = \{\{*, 0, 1, 2\}, \{3\}\}, X_3 = \{\{0, 1, 2, 3\}\}$. We note that in $X_1$ 0 is not connected to 1, since here only edges in $E_{\leq 1}$ may be used in connecting paths.

**Definition 3** Let $\ast/\ast \in V$ be an abstract symbol. A partition $x$ of $V \cup \{\ast\}$ will be called an (infection) pattern on $V$. Let $\sim_x$ denote the corresponding equivalence relation on $V \cup \{\ast\}$. Let $M$ denote the set of all patterns on $V$, and let $M^* := \{x \in M : \exists u \in V : u \sim_x \ast\}$ and $M^\dagger := M \setminus M^*$.

By definition, a pattern $x$ on $V$ is the set of equivalence classes of the equivalence relation $\sim_x$ on $V \cup \{\ast\}$. If for $u, v \in V$ we have $u \sim_x v$ we think of $u, v$ as connected w.r.t. $x$, and if we have $u \sim_x \ast$ we think of $u$ as infected w.r.t. $x$. $M^*$ and $M^\dagger$ can be interpreted as the sets of patterns on $V$ with and without infected vertices respectively. In the following definition and later on we will consider patterns in certain layers of $G^x$ that are built from given (random or nonrandom) edge sets: Such an edge set defines a graph and a corresponding connectivity relation on this graph, and the connectivity relation restricted to the layer under consideration defines a corresponding pattern in this layer, provided we specify, where the infection originates. We note that a random edge set gives a random connectivity relation and thus a random pattern.

**Definition 4** We consider percolation on the layered graph $G^x$. Let $n \geq 0$.

(a) Let $\leftrightarrow_{\leq n}$ denote the connectivity relation induced by percolation on $E_{\leq n}$. Let $X_n$ denote the pattern in $V_n$ induced by $\leftrightarrow_{\leq n}$ with the infection originating in $o' := (o, 0)$.

(b) Let $W_n := |\{v' \in V_n : v' \leftrightarrow o'\}|$ denote the number of infected vertices of $V_n$.

For clarity we spell out part (a) explicitly: We have $u' \leftrightarrow_{\leq n} v'$ iff there is path from $u'$ to $v'$ consisting of open bonds from $E_{\leq n}$, and we have $X_n = x$ iff

$\forall u, v \in V : u \sim_x v \leftrightarrow (u, n) \leftrightarrow_{\leq n} (v, n)$ and $\forall u \in V : u \sim_x \ast \leftrightarrow (u, n) \leftrightarrow_{\leq n} o'$.

We note that $\{X_n \in M^*\} = \{o' \leftrightarrow_{\leq n} V_n\} = \{o' \leftrightarrow V_n\} = \{W_n > 0\}$. We now relate Conjecture 3 to monotonicity properties of the above objects.
Proposition 2 We consider percolation on $G^\times$ for some finite connected graph $G$. For all $n \geq 0$ and $p \in (0,1)$ we have the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) of the following statements:

(i) For all $x \in M^*$ we have $\mathbb{P}_p(X_n = x) \geq \mathbb{P}_p(X_{n+1} = x)$.

(ii) For all $v \in V$ we have $\mathbb{P}_p((o,0) \leftrightarrow (v,n)) \geq \mathbb{P}_p((o,0) \leftrightarrow (v,n+1))$.

(iii) We have $\mathbb{E}_p(W_n) \geq \mathbb{E}_p(W_{n+1})$.

This will be proved in Section 3. The advantage of working with infection patterns instead of the connection events we are mainly interested in, is the following:

Proposition 3 We consider percolation on $G^\times$ for some finite connected graph $G$ with parameter $p \in (0,1)$. $X_n, n \geq 0$, is a (time-homogeneous) Markov chain w.r.t. $\mathbb{P}_p$.

This will be shown in Section 4. An obvious disadvantage of working with infection patterns is that monotonicity property (i) cannot be expected to hold for all $n \geq 0$. Indeed, for any pattern $x \in M^*$ such that $o \not\sim_x$ we have $\mathbb{P}_p(X_0 = x) = 0$, whereas for many such patterns we have $\mathbb{P}_p(X_n = x) > 0$ for some $n > 0$. However, it turns out that it is possible to prove something slightly weaker, which is our main result:

Theorem 1 For any finite connected graph $G$ there is an $N \geq 0$ such that for percolation on the layered graph $G^\times$ we have

$$\forall n \geq N \forall p \in (0,1) \forall x \in M^* : \mathbb{P}_p(X_n = x) \geq \mathbb{P}_p(X_{n+1} = x). \quad (2.1)$$

In particular this implies that for all $n \geq N$, $p \in [0,1]$ and $v \in V$ we have

$\mathbb{P}_p((o,0) \leftrightarrow (v,n)) \geq \mathbb{P}_p((o,0) \leftrightarrow (v,n+1))$ and $\mathbb{E}_p(W_n) \geq \mathbb{E}_p(W_{n+1})$.

Remark 1 Monotonicity of patterns.

- For small graphs the onset of monotonicity occurs relatively early. Let $N(G)$ denote the minimal value of $N$ for which (2.1) holds. Then CAS-aided calculations that will be described in Subsection 4.2 give

$$\begin{array}{c|c|c|c}
G & C_2 & C_3 & C_4 \\
N(G) & 2 & 2 & 4
\end{array} \quad (2.2)$$

These calculations also show that Conjecture 3 is true for the above graphs.

- It would be interesting to obtain explicit upper bounds on $N(G)$ for given graphs $G$. While we don’t see how to extract this information from our proof of Theorem 1 given in Section 4, we will show how to do that in a companion paper ([KR]) making use of explicit bounds on the rate of convergence to the quasi-stationary distribution of our Markov chain (see [CV]). With some effort we obtain $N(C_k) \leq c2^k$ for some constant $c$. 


Concerning the monotonicity of the expected number of infected vertices per layer, we note that in view of Conjecture 3 and Proposition 2 we expect that
\[ \forall n \geq 0 \forall p \in [0, 1] : \mathbb{E}_p(W_n) \geq \mathbb{E}_p(W_{n+1}). \]
It is not too difficult to see that the above theorem implies that this holds for all \( n \geq 0 \) and sufficiently small \( p \). In fact this can be seen more explicitly (and for graphs that are not necessarily finite) by a recursive approach, which we present as a complementary result:

**Theorem 2** Let \( G = (V, E) \) be a connected graph of bounded degree such that \( \deg(x) \leq \Delta \) for all \( x \in V \). Then for percolation on \( G^\times \) we have
\[ \forall n \geq 0, p \in [0, \frac{1}{\Delta + 1.4}] : \mathbb{E}_p(W_n) \geq \mathbb{E}_p(W_{n+1}). \]  

**Remark 2** Monotonicity of the expected number of infected vertices.

- The bound on \( p \) in (2.3) can be improved in particular cases. In case of \( G = C_k \) and \( G = \mathbb{Z} \) we have \( \Delta = 2 \), so the bound in (2.3) is \( p \leq \frac{1}{2} \approx 0.29 \), but with some effort (and combinatorial arguments) one can show that \( p \leq 0.35 \) suffices, see [KR].

- For further monotonicity results for the expectations we refer to [RiT]: If \( G \) is locally finite we have \( \mathbb{E}_p(W_n) = \infty \) for some \( n \) iff \( \mathbb{E}_p(W_n) = \infty \) for all \( n \) (see Lemma 4), and under suitable symmetry assumptions on \( G \) (that are satisfied in case of \( G = C_k \) or \( G = \mathbb{Z} \) we have \( \mathbb{E}_p(W_0) \geq \mathbb{E}_p(W_n) \) for all \( n \geq 0 \) and \( p \in [0, 1] \) (see Corollary 2(a)).

3 Comparing monotonicity properties

3.1 Comparing the monotonicity conjectures

Here we prove Proposition 1. The main idea for (a) is that a suitable truncation of a layered graph is a bunkbed graph. For a detailed proof let \( p \in (0, 1) \). We assume the validity of the bunkbed conjecture. Let \( G \) be a finite connected graph, \( o, v \in V(G) \) and \( n \geq 0 \). We note that for proving (1.3) it suffices to consider \( n' = n + 1 \). For \( m \geq n \) we let the subgraph of \( G^\times \) induced by the vertex set \( V(G) \times \{n-m, \ldots, n+m+1\} \) be denoted by \( G^{(m)} = (V^{(m)}, E^{(m)}) \), and we write \( \leftrightarrow^{(m)} \) for the connectivity relation induced by percolation on \( E^{(m)} \), i.e. \( u' \leftrightarrow^{(m)} v' \) iff there is a path from \( u' \) to \( v' \) consisting of open bonds from \( E^{(m)} \).

We note that \( G^{(m)} \) may be identified with the bunkbed graph \( G_T^{(m)} \), where \( G_T^{(m)} \) is defined to be the subgraph of \( G^\times \) induced by the vertex set \( G \times \{n-m, \ldots, n\} \) and \( T = G \times \{n\} \). Thus the validity of the bunkbed conjecture for \( G_T^{(m)} \) implies
\[ \mathbb{P}_p((o, 0) \leftrightarrow^{(m)} (v, n)) \geq \mathbb{P}_p((o, 0) \leftrightarrow (v, n+1)). \]
Since \( \{(o, 0) \leftrightarrow^{(m)} (v, n)\} \uparrow \{(o, 0) \leftrightarrow (v, n)\} \) for \( m \to \infty \) the above implies that \( \mathbb{P}_p((o, 0) \leftrightarrow (v, n)) \geq \mathbb{P}_p((o, 0) \leftrightarrow (v, n+1)) \) as desired.
The main idea for (b) is that \( \mathbb{Z}^d \) can be considered to be a layered graph \((\mathbb{Z}^{d-1})^\times \) w.r.t. every direction, and \( \mathbb{Z}^{d-1} \) can be approximated by \( C^d \) with large \( k \). For a detailed proof let \( p \in (0,1) \) and \( d \geq 2 \). We assume the validity of the monotonicity conjecture for the layered graphs \((C^d)^\times \) for every \( k \geq 2 \). We note that for proving (1.1) it suffices to show that for all \( C \) of the monotonicity conjecture for the layered graphs \((C^d)^\times \) we consider the subgraph \( G \) via bonds in \( \mathbb{Z}^{d-1} \) and let \( B \) be the connectivity relation induced by percolation on \( (C^d)^\times \). Let \( \mathcal{V}(k) = \{ v' \in V_k : |\Delta : |v'| = k \} \) be the boundary of \( V(k) \) and let \( \leftrightarrow \) be the connectivity relation induced by percolation on \( E(k) \). We set

\[
B_k := \{ o' \leftrightarrow w' \text{ w.r.t. } (C^d)^\times \} \quad \text{and} \quad A := \{ o' \leftrightarrow w' \text{ w.r.t. } \mathbb{Z}^d \}
\]

with \( w' \in \mathbb{Z}^d \) and \( o' = (0,0) \). (Here we label the vertices of \( C_{2k+1} \) as \( \{-k, \ldots, k\} \). To show this convergence we consider the subgraph \( G_k = (V_k, E_k) \) of \( \mathbb{Z}^d \) induced by the vertex set \( \{-k, \ldots, k\} \). \( G_k \) naturally can also be considered a subgraph of \((C^d)^\times \). Let \( \mathcal{V}(k) = \{ v' \in V_k : |\Delta : |v'| = k \} \) be the boundary of \( V(k) \) and let \( \leftrightarrow \) be the connectivity relation induced by percolation on \( E(k) \). We set

\[
B_k := \{ o' \leftrightarrow w' \text{ w.r.t. } \mathbb{Z}^d \}
\]

and note that \( B_k \cap \{ o' \leftrightarrow o \} = \{ o \leftrightarrow w' \} \). Thus both \( B_k \) and \( B_k \cap \{ o' \leftrightarrow w' \} \) only depend on edges in \( E(k) \), i.e. their probabilities w.r.t. percolation on \( \mathbb{Z}^d \) or percolation on \((C^d)^\times \) are the same. This implies that \( |P_p(A_k) - P_p(A)| \leq P_p(A_k \cap B_k) + |P_p(A_k \cap B_k^c) - P_p(A \cap B_k^c)| + P_p(A \cap B_k) \leq 2P_p(B_k). \)

For the last term we note that (interpreting \( B_k \) in terms of percolation on \( \mathbb{Z}^d \)) \( B_k \downarrow B := \bigcap_k B_k \), and on \( B \) the clusters of \( o' \) and \( w' \) are infinite and do not intersect, thus \( P_p(B) = 0 \) by Burton-Keane (e.g. see [G]). So letting \( k \to \infty \) we obtain \( |P_p(A_k) - P_p(A)| \to 0 \) as desired. 

3.2 Comparing monotonicity properties for layered graphs

Here we prove Proposition 2. Let \( n \geq 0 \) and \( p \in (0,1) \). (i) \( \Rightarrow \) (ii) follows from the way connectivity events can be obtained from infection patterns. For \( n \geq 0 \), \( x \in M \) and \( v \in V \) we write \( * \leftrightarrow x, n \ v \) iff \( (v, n) \) is infected from the pattern \( x \) at layer \( n \) via bonds in \( E_{n+1}, \ldots, i.e. \) iff there are \( v_0, \ldots, v_m \in V \times \mathbb{Z} \) such that \( v'_0 = (u, n) \) for some \( u \sim x \), \( v'_m = (v, n) \) and for every \( i \) either \( v'_i v'_{i+1} \in E_{n+1} \) such that \( \mathcal{V}_{v'_i v'_{i+1}} = 1 \) or \( v'_i = (v, i), v'_i = (v_{i+1}, n) \) for some \( v_i \sim x v_{i+1} \). For all \( v \in V \) we have

\[
P_p((o, 0) \leftrightarrow (v, n)) = \sum_{x \in M^*} P_p(\mathcal{X}_n = x, (o, 0) \leftrightarrow (v, n))
\]

\[
= \sum_{x \in M^*} P_p(\mathcal{X}_n = x, * \leftrightarrow x, n \ v) = \sum_{x \in M^*} P_p(\mathcal{X}_n = x) P_p(* \leftrightarrow x, n \ v).
\]

In the first step we have used that every open path from \( (o, 0) \) to \( (v, n) \) contains an open subpath from \( (o, 0) \) to \( V_n \) via bonds in \( E_{n+1} \). In the second step we have
used that every path from \((o,0)\) to \((v,n)\) can be decomposed into a path from \((o,0)\) to \(V_n\) via bonds in \(E_{n-1}\), paths from \(V_n\) to \(V_n\) via bonds in \(E_{n,n}\), and paths from \(V_n\) to \(V_n\) via bonds in \(E_{n,n}\), and that the connectivity of any pair of vertices in \(V_n\) via bonds in \(E_{n,n}\) is encoded in \(X_n\). In the third step we have used independence of the two events, which depend on disjoint sets of bonds. Repeating the above arguments we get a corresponding decomposition for \(P_p((o,0) \leftrightarrow (v,n+1))\). Since \(P_p(\cdot \leftrightarrow_{x,n} v) = P_p(\cdot \leftrightarrow_{x,n+1} v)\) by the translation invariance of percolation and \(P_p(X_n = x) \geq P_p(X_{n+1} = x)\) for all \(x \in M^*\) by (i) we thus see that \(P_p((o,0) \leftrightarrow (v,n)) \geq P_p((o,0) \leftrightarrow (v,n+1))\).

(ii) \(\Rightarrow\) (iii) is a straightforward consequence of the linearity of expectation: We have \(E_p(W_n) = \sum_{v \in V} P_p((o,0) \leftrightarrow (v,n))\) and similarly for \(E_p(W_{n+1})\).

4 Markov chain for infection patterns

4.1 Markov property for infection patterns

In all of this subsection we consider percolation on \(G^X\) with some parameter \(p \in (0,1)\), where \(G\) is a fixed finite connected graph. It will be useful to describe percolation events between two layers that start from a prescribed pattern:

**Definition 5** Let \(0 \leq k \leq n\) and \(y \in M\). Let \(\leftrightarrow_{y,k,n}\) denote the connectivity relation induced by percolation on \(E_{k+1,n}\) and the pattern \(y\) in layer \(V_k\). Let \(X_{y,n}^k\) be the pattern in \(V_n\) induced by \(\leftrightarrow_{y,k,n}\) with the infection originating in \(y\), and let \(X_n^y := X_n^{y,0}\).

Writing out the above definition we have \(u' \leftrightarrow_{y,k,n} u'\) iff there are \(v_0', \ldots, v_m' \in V \times Z\) such that \(v_0' = u', v_m' = u'\) and for every \(i\) either \(v_i'v_{i+1}' = 1\) or \(v_i' = (v_i, k), v_{i+1}' = (v_{i+1}, k)\) with \(v_i \sim_y v_{i+1}\). Furthermore we have \(X_{n}^y = x\) iff

\[
\forall u, v \in V : u \sim_x v \iff (u, n) \leftrightarrow_{y,k,n} (v, n) \quad \text{and} \\
\forall u \in V : u \sim_x * \iff \exists w \in V : w \sim_y *, (u, n) \leftrightarrow_{y,k,n} (w, k).
\]

In simple words, the event \(\{X_{n}^y = x\}\) describes all configurations of open bonds in \(E_{k+1,n}\) such that a prescribed pattern \(y\) at layer \(k\) produces the pattern \(x\) at layer \(n\). We also note that \(X_{n}^y = y\) and in particular \(X_n^y = y\).

**Lemma 1** For all \(n \geq 0\) and \(y, x_0, \ldots, x_n \in M\) we have

\[
P_p(\forall 0 \leq i \leq n : X_i = x_i) = P_p(X_0 = x_0) \prod_{1 \leq i \leq n} P_p(X_i = x_{i-1}) \quad \text{and} \\
P_p(\forall 0 \leq i \leq n : X_i = x_i) = \delta_y(x_0) \prod_{1 \leq i \leq n} P_p(X_i = x_{i-1} = x_i).
\]

In particular \((X_n^y)_{n \geq 0}\) and \((X_n^z)_{n \geq 0}\) are Markov chains with the same transition probabilities \(\pi_p(x, x') := P_p(X_1 = x')\) and initial distributions \(\alpha_p := P_p(X_0 = .)\) and \(\delta_y\) respectively.
Proof: (1.1) follows by noting that we have
\[ \forall 0 \leq i \leq n : X_i = x_i = \{X_0 = x_0, \forall 1 \leq i \leq n : X_i = x_i \} \]
by definition, the random variables \(X_0, X_1^{x_0,0}, \ldots, X_n^{x_{i-1},n-1}\) are independent since they depend on disjoint bond sets, and \(X_i^{x_{i-1},i-1} \sim X_i^{x_1,0}\), which is a consequence of the translation invariance of percolation. (4.2) can be seen similarly.

In particular the above proves Proposition 3. We next collect some simple properties of the above Markov chain of patterns for later reference. Let
\[ x_\uparrow := \{\{\ast\}\} \cup \{\{v\} : v \in V\} \quad \text{and} \quad x_* := \{\{\ast\}\} \cup \{\{v\} : v \in V\} \quad (4.3) \]
denote the pattern without infection and connections and the pattern, where everything is connected and infected, respectively.

Lemma 2 For the above Markov chain of patterns we have:
\[ \forall p, p' \in (0, 1), x, y \in M : \pi_p(x, y) > 0 \Leftrightarrow \pi_{p'}(x, y) > 0. \quad (4.4) \]
\[ \forall x, y \in M : \pi_p(x, y) \text{ is a polynomial function in } p. \quad (4.5) \]
\[ \forall p \in (0, 1), x \in M^1, y \in M^* : \pi_p(x, y) = 0. \quad (4.6) \]
\[ \forall p \in (0, 1), x \in M : \pi_p(x, x) > 0. \quad (4.7) \]
\[ \forall p \in (0, 1), x \in M : \pi_p(x, x_\uparrow) > 0. \quad (4.8) \]

Proof: For (4.4) we note that a specific configuration of bonds in a certain layer has positive probability w.r.t. \(P_p\) if it has positive probability w.r.t. \(P_{p'}\). For (4.5) we similarly note that a specific configuration of bonds in a certain layer has a probability w.r.t. \(P_p\) of the form \(p^k(1 - p)^l\). (4.6) is immediate from the definition of the Markov chain. (4.7) can be seen by considering a layer, where all vertical bonds are open and all horizontal bonds are closed. Similarly, (4.8) can be seen by considering a layer, where all bonds are closed.

As usual, for \(x, y \in M\) we write \(x \rightarrow y\) if \(y\) can be reached from \(x\), i.e. for some \(n\) we have \(\pi^n_p(x, y) > 0\). We note that by (4.4) this does not depend on the value of \(p \in (0, 1)\). By (4.6), \(M^1\) is an absorbing set of states. The decomposition of \(M\) into communicating classes can be quite complicated and in particular for most graphs the set \(M^1\) will not be irreducible. However, if it is irreducible, then (4.7) implies aperiodicity. Finally, in preparation for the proof of Theorem 1, we note the following consequences of the Markovian structure.

Lemma 3 For the above Markov chain of patterns we have:
(a) Let \(n \geq 0\). If for all \(y, x \in M^*\) we have \(P_p(X_n^y = x) \geq P_p(X_n^{y+1} = x)\), then for all \(x \in M^*\) \(P_p(X_n = x) \geq P_p(X_{n+1} = x)\).
(b) Let \(N \geq 0\). If for all \(y, x \in M^*\) we have \(P_p(X_N^y = x) \geq P_p(X_{N+1}^y = x)\), then for all \(n \geq N\) and all \(y, x \in M^*\) \(P_p(X_n^y = x) \geq P_p(X_{n+1}^y = x)\).
Proof: For (a) it suffices to note that for $x \in M^*$

$$\mathbb{P}_p(X_n = x) = \sum_{y \in M^*} \mathbb{P}_p(X_0 = y) \mathbb{P}_p(X'_n = x)$$

and similarly for $n + 1$ instead of $n$. Here we have used that $\mathbb{P}_p(X'_n = x) = 0$ for $y \in M^\uparrow$, $x \in M^*$ by (4.6). For (b) it suffices to note that for all $x, y \in M^*$

$$\mathbb{P}_p(X'_n = x) = \sum_{x' \in M^*} \mathbb{P}_p(X'_N = x') \mathbb{P}_p(X'_{n-N} = x)$$

and similarly for $n + 1$ and $N + 1$ instead of $n$ and $N$, where again we have used (4.6). 

4.2 CAS computations for small graphs

In this subsection we will outline how to obtain Table (2.2), i.e. on how to obtain optimal bounds on the onset of monotonicity with CAS-assistance. Similar calculations are possible for any sufficiently small graph $G$. In some cases, e.g. for $G = C_2 = L_2$ (for which $G^\times$ is the bi-infinite ladder graph), these computations can be carried out by hand. We will use this particular example to illustrate the general procedure.

Since not all partitions of $V$ can be connectivity patterns of percolation on $G^\times$, let us first reduce the state space of the Markov chain. We consider a version $X_n$ of the Markov chain that only records the connectivity in the layer $V_n$, but not which vertices are infected, i.e. $X_n \in M^\uparrow$ and $X_n = x$ for some $x \in M^\uparrow$ if and only if $\forall u, v \in V : u \sim_x v \iff (u, n) \leftrightarrow_n (v, n)$. From the proof of Proposition 3 it can be seen, that $X_n$ indeed is a Markov chain with transition probabilities $\pi^-_p(x, y) = \pi_p(x, y)$ for all $x, y \in M^\uparrow$. $X_n$ is stationary, which follows from its definition and the translation invariance of percolation on $G^\times$. Since a.s. for some $k \leq 0$ all bonds of $E_k$ are closed, $X_n$ only takes values in $M^- := \{x \in M^\uparrow : x_1 \to x\}$. Since $x \to x_1$ for all $x \in M^\uparrow$ by (4.8), the Markov chain $X_n$ is irreducible. In case of $G = C_2$ we have $M^- = \{\{1\}, \{2\}, \{\ast\}\}$ and

$$\pi^-_p = \begin{pmatrix} 1 - p & p \\ (1 - p)(1 - p^2) & p(1 + p - p^2) \end{pmatrix}.$$ 

By definition we have $X_n = f_1(X_n)$, where $f_1 : M \to M^\uparrow$ deletes the infection from a given partition, i.e. $f_1(x) := \{\ast\} \cup \{A \setminus \{\ast\} : A \in x\}$. In particular $X_n$ only takes values in $f_1^{-1}(M^-)$. Since we are only interested in states with infections, we can further reduce the state space by combining all uninfected states into a single state $\dagger$. We thus consider $X'_n := f_\ast(X_n)$, where $f_\ast : M \to M^\ast \cup \{\dagger\}$ is the identification map $f_\ast(x) := \dagger$ for $x \in M^\uparrow$, $f_\ast(x) := x$ for $x \in M^\ast$. $X'_n$ is a Markov chain with state space $M' = (f_1^{-1}(M^-) \cap M^\ast) \cup \{\dagger\}$ and transition probabilities $\pi'_p(x, y) = \pi_p(x, y)$ for all $x, y \in M' \setminus \{\dagger\}$. In case of $G = C_2$ we have $M' = \{\dagger, \{\{1\}, \{2\}\}, \{\{2\}, \{1\}\}, \{\{1, 2\}, \{\ast\}\}\}$ and

$$\pi'_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 - p & p(1 - p) & 0 & p^2 \\ 1 - p & 0 & p(1 - p) & p^2 \\ (1 - p)^2 & p(1 - p)^2 & p(1 - p)^2 & p^2(3 - 2p) \end{pmatrix}.$$ 

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In general, the transition probabilities of $X_n^-$ and $X_n^+$ will be polynomials in $p$ of degree $\leq |E(G)| + |V(G)|$. The initial distribution $\alpha_p^0(x) = \mathbb{P}_p(X_0 = x)$ of the chain $X_n^-$ can be obtained from the chain $X_n^+$: Since $X_n^+$ is irreducible and stationary, the distribution of $X_0^-$ is the unique stationary distribution $\alpha_p^-$ of this chain. It can be computed as a left eigenvector of $\pi_p$ corresponding to the eigenvalue 1, e.g. using Gaussian elimination. Since $X_0 = f_1(X_0')$ and $o \sim X_0^*$, we then can set $\alpha_p^0(x) := 0$ if $o \not\sim x$ and $\alpha_p^0(x) := \alpha_p^-(f_1(x))$ if $o \sim x$. In case of $G = C_2$ we obtain

$$\alpha_p^- = \frac{1}{c_p}((1-p)(1-p^2), p) \quad \text{and} \quad \alpha_p' = \frac{1}{c_p}(0, ((1-p)(1-p^2), 0, p),$$

where $c_p$ is a normalizing constant. By Gaussian elimination in general the entries of $\alpha_p^-$ are rational functions of $p$, and by irreducibility all entries are strictly positive for all $p \in (0, 1)$. Thus we can always choose $c_p$ such that $c_p > 0$ for all $p \in (0, 1)$ and the entries of $c_p\alpha_p^-$ (and thus also of $c_p\alpha_p'$) are polynomials in $p$. In view of (b) of [Lemma 3] it suffices to find an $N$ such that all entries of $A(p, N) := c_p\alpha_p' x_{p, N} - c_p\alpha_p' x_{p, N+1}$ are positive for all $p \in (0, 1)$. The positivity of these polynomials on $(0, 1)$ can be decided using Sturm’s method (e.g. see [BPR]). We thus are left to consider $A(p, n)$ for $n \in \{0, 1, 2, \ldots\}$ until for $n = N$ $A(p, n)$ has the desired property.

If $N$ is not too large, we can thus also check that $\mathbb{P}_p((0, 0) \leftrightarrow (v, n)) \geq \mathbb{P}_p((0, 0) \leftrightarrow (v, n+1))$ for all $v \in V$, $p \in [0, 1]$ and $n \geq 0$. For $n \geq N$ this follows directly from Proposition 2 and the cases $n \in \{0, \ldots, N - 1\}$ can be checked separately using a similar idea to the one used in the proof of Proposition 2.

Let $X_n$ denote (as before) the pattern in $V_n$ induced by percolation on $E_n$. Let $Y_{n+1}$ denote the uninfected pattern in $V_{n+1}$ induced by percolation on $E_{n+1}$. Let $Z_{n+1}$ denote the configuration of bonds in $E_{n+1}$. We note that $X_n, Z_n, Y_{n+1}, Z_{n+1}$ are independent, $X_n \sim \alpha_p' x_{p, n}, Z_{n+1}$ is a Bernoulli sequence with parameter $p$, and $Y_{n+1} \sim \alpha_p^-$ by reflection invariance of percolation on $G^\times$. Let $A(v, n)$ denote the set of all triples $(x, y, z)$, such that $x$ is an infected pattern, $y$ is an uninfected pattern, $z$ is a bond configuration of a layer of vertical bonds, and there is a path $v'_0, \ldots, v'_l \in V \cup V_{n+1}$ such that $v'_0 = (v_0, n)$ with $v_0 \sim x$, $v'_0 = (v, n)$ and for each $i$ either $v'_i = (v, n)$, $v'_{i+1} = (v, n+1)$, $v_i \sim_y v_{i+1}$, or $v'_i v'_{i+1} \in E_{n+1}$ with $z_{v_i} v'_{i+1} = 1$ or $v'_i = (v, n+1)$, $v'_{i+1} = (v, n+1)$ with $v_i \sim_y v_{i+1}$. Then we have

$$\mathbb{P}_p((0, 0) \leftrightarrow (v, n)) = \sum_{(x, y, z) \in A(v, n)} \mathbb{P}_p(X_n = x, Y_{n+1} = y, Z_{n+1} = z) = \sum_{(x, y, z) \in A(v, n)} \alpha_p' x_{p, n}(x) \alpha_p^-(y) \mathbb{P}_p(Z_{n+1} = z)$$

and a similar expression for $(v, n+1)$. The probabilities $\alpha_p' x_{p, n}(x)$ and $\alpha_p^-(y)$ already have been computed while determining $N$, $\mathbb{P}_p(Z_{n+1} = z) = p^{|z|}(1 - p)^{|z|}$ can easily be computed, and so $c_p^2 \mathbb{P}_p((0, 0) \leftrightarrow (v, n)) - c_p^2 \mathbb{P}_p((0, 0) \leftrightarrow (v, n+1))$ is a polynomial in $p$ that can readily be computed and its positivity for $p \in (0, 1)$ can be checked using Sturm’s method as above.
Remark 3 CAS computations for small graphs $G$.

- We have implemented the above ideas (in Python) to compute $N$ and to verify (1.3) in case of $G = C_k$ for $k \in \{2, 3, 4\}$.

- While it may be possible to extend these calculations to cover $k = 5$, for larger values of $k$ these CAS computations no longer are feasible since the state spaces for the Markov chains are too big. We note that for $k = 5$ we have $|M'| = 127$ and $|M^-| = 42$ and the entries of $\pi'$ and $\pi^-$ are polynomials of degree 10.

5 Partition patterns per layer

Here we prove Theorem 1. We proceed by considering small, intermediate and large values of $p$ separately.

5.1 Intermediate values of $p$

The main ingredient here is a general monotonicity result for Markov chains which relies on asymptotics for transition probabilities. As reference for the needed asymptotics we use [Li1], for more general results see [Li2] and the references therein. While the result of [Li1] is purely algebraic, we would like to point out that it allows for a probabilistic interpretation in terms of quasi-stationary distributions, see [DP].

Proposition 4 Let $\pi$ be the transition matrix of an arbitrary Markov chain on a finite state space $M$. Suppose the chain is locally aperiodic, i.e. for every $x \in M$ either $\pi^n(x, x) = 0$ for all $n \geq 1$ or there is an $N$ such that $\pi^n(x, x) > 0$ for all $n \geq N$. Let $y, x \in M$.

(a) Either there is an $N$ such that $\forall n \geq N : \pi^n(y, x) = 0$ or there are constants $\lambda > 0$, $\sigma \in \{1, 2, \ldots\}$ and $\gamma > 0$ such that

$$\pi^n(y, x) \sim \gamma n^{\sigma-1} \lambda^n \quad \text{for } n \to \infty. \quad (5.1)$$

(b) Suppose that $x$ is transient. Then there is an $N$ such

$$\forall n \geq N : \pi^n(y, x) = 0 \quad \text{or} \quad \forall n \geq N : \pi^{n+1}(y, x) < \pi^n(y, x). \quad (5.2)$$

Proof: For a proof of (a) we refer to [Li1] (Theorem, page 6). We note that this proof is purely algebraic and gives fairly explicit descriptions of the constants $\lambda, \sigma, \gamma$. For a proof of (b) let $x$ be transient. If $\forall n \geq N : \pi^n(y, x) = 0$ we are done. Thus by (a) we may assume that (5.1) holds, which implies

$$\frac{\pi^{n+1}(y, x)}{\pi^n(y, x)} \to \lambda \quad \text{for } n \to \infty.$$ 

It suffices to show that $\lambda < 1$, which follows readily from the transience of $x$: The expected number of visits in $x$ from a chain started in $y$ is finite, i.e.
\[ \sum_{n} \pi^n(y, x) < \infty, \] which implies \( \pi^n(y, x) \to 0 \) for \( n \to \infty \). By (5.1) this implies that \( \gamma \pi^{n-1} \lambda^n \to 0 \), and thus \( \lambda < 1 \).

Now we return to the setting of Theorem 1 with a fixed graph \( G \).

**Proposition 5** For all \( 0 < p_0 < p_1 \) there is an \( N \geq N \) such that
\[ \forall n \geq N \forall p \in [p_0, p_1] \forall y, x \in M^* : P_p(\mathcal{X}_n^y = x) \geq P_p(\mathcal{X}_{n+1}^y = x). \] (5.3)

**Proof:** We first consider a fixed \( p' \in [p_0, p_1] \). The state space \( M \) of the considered Markov chain is finite. The Markov chain is locally aperiodic since \( \pi_{p'}^n(x, x) > 0 \) for all \( n \geq 0 \) by (4.7). For every \( x \in M^* \) we have \( \pi_{p'}(x, x_1) > 0 \) and \( \pi_{p'}^n(x_1, x) = 0 \) for all \( n \geq 0 \) by (4.8) and (4.9), so \( x \) is transient. Thus, by Proposition 2 for all \( y, x \in M^* \) there is an \( N_{p',y,x} \) such that (5.2) holds for \( \pi = \pi_{p'} \) and \( N = N_{p',y,x} \). Since \( M^* \) is finite, we can define \( N_{p'} := \max\{N_{p',y,x} : y, x \in M^*\} \). By definition of \( N_{p'} \) we have
\[ \forall y, x \in M^* : \pi_{p'}^{N_{p'}+1}(y, x) = \pi_{p'}^{N_{p'}}(y, x) = 0 \quad \text{or} \quad \pi_{p'}^{N_{p'}+1}(y, x) < \pi_{p'}^{N_{p'}}(y, x). \]

By (4.4) \( \pi_{p'}^n(y, x) = 0 \) implies that \( \forall p \in (0, 1) : \pi_{p'}^n(y, x) = 0 \), and by (4.5) \( \pi_{p'}^n(y, x) \) is a continuous function of \( p \). Thus the above implies that there is an open neighborhood \( U_{p'} \) of \( p' \) in \( (0, 1) \) such that
\[ \forall p \in U_{p'} \forall y, x \in M^* : \pi_{p'}^{N_{p'}+1}(y, x) \leq \pi_{p'}^{N_{p'}}(y, x). \]

By Lemma 3 this implies that
\[ \forall p \in U_{p'} \forall y, x \in M^* \forall n \geq N_{p'} : P_p(\mathcal{X}_n^y = x) \leq P_p(\mathcal{X}_{n+1}^y = x). \]

By compactness, \([p_0, p_1] \subset \bigcup_{i \in I} U_{p_i}, \) for some finite set \( I \) and some \( p_i \in [p_0, p_1] \). Setting \( N_{p_0,p_1} := \max\{N_{p_i} : i \in I\} \) we have
\[ \forall i \in I \forall p \in U_{p_i} \forall y, x \in M^* \forall n \geq N_{p_0,p_1} : P_p(\mathcal{X}_n^y = x) \leq P_p(\mathcal{X}_{n+1}^y = x). \]

Thus (5.3) holds for \( N = N_{p_0,p_1} \).

**5.2 Small values of \( p \)**

In this case we use that going from a pattern \( y \) to a pattern \( x \) in a fixed large number of steps, in most layers the number of open bonds will be as small as possible. With high probability there will be two consecutive layers with the same configuration of bonds, where all horizontal bonds are closed. By discarding one of these two layers we will be able to compare \( \pi_{p'}^{n+1}(y, x) \) and \( \pi_{p'}^n(y, x) \). To make this into a rigorous argument we introduce some notation and collect some observations.

**Definition 6** Let \( N_n := \{|e \in E_n : Z_e = 1\}| \), for \( y, x \in M^* \) such that \( y \to x \) let
\[ m_{y,x} := \min\{|N_i(\omega) : \exists n \geq 1 : 1 \leq i \leq n, \omega \in \{\mathcal{X}_n^y = x\}\} \text{ and} \]
\[ l_{y,x} := \min\{n \geq 1 : \exists 1 \leq i \leq n : \{\mathcal{X}_n^y = x, N_i = m_{y,x}\} \neq \emptyset\}, \]
and let \( l := \max\{l_{y,x} : y, x \in M^* \text{ such that } y \to x\} \).
Thus $N_n$ is the number of open bonds in layer $E_n$, $m_{y,x}$ is the minimal number of open bonds in some layer of a bond configuration going from pattern $x$ to pattern $y$ in an arbitrary number of steps, $l_{y,x}$ is the minimal number of steps in which you can go from $y$ to $x$ and observe a layer with the minimal number of open bonds in between. We note that $m_{y,x} \geq 1$, $l_{y,x} < \infty$, $l < \infty$ and for every $n \geq l$ and all $y, x \in M^*$ such that $y \rightarrow x$ it is possible to go from $y$ to $x$ in $n$ steps and observe a layer with the minimal number of open bonds in between (making use of (4.7)). Since one might speculate whether values $m_{y,x} > 1$ are possible at all, let us provide a simple example: For $G = L_3$ and $x = y = \{\{*, 1\}, \{0, 2\}\}$ we have $m_{y,x} = 3$.

**Lemma 4** Layers with the minimal number of open bonds. For all $y, x \in M^*$ such that $y \rightarrow x$ we have

\[
\forall 1 \leq i \leq n : \{X^y_n = x, N_i = m_{y,x}\} \subset \{\forall e \in E^h_i : Z_e = 0\}, \quad (5.4)
\]

\[
\forall 1 \leq i < n : \{X^y_n = x, N_i = N_{i+1} = m_{y,x}\} \subset \{\forall e \in E_i : Z_e = Z_{\tau(e)}\}, \quad (5.5)
\]

i.e. in a layer with the minimal number of open bonds all horizontal bonds are closed, and in two consecutive layers with the minimal number open bonds, the open bonds match up.

**Proof:** For (5.4) let $z = (z_e)_{e \in E_{1..n}}$ be a bond configuration contributing to the event $\{X^y_n = x, N_i = m_{y,x}\}$. Let $\bar{z} = (\bar{z}_e)_{e \in E_{1..n+1}}$ be the bond configuration obtained from $z$ by extending the open vertical bonds in $E_i$, i.e. $\bar{z}_e = z_e$ for $e \in E_{1..i-1} \cup E^v_i$, $\bar{z}_e = 0$ for $e \in E^h_i$, $\bar{z}_e = z_{\tau(e)}$ for $e \in E_{i+1..n+1}$. By construction $\bar{z}$ contributes to $\{X^y_{n+1} = x, N_i = m, N_{i+1} = m_{y,x}\}$ for some $m$. If $z$ has an open bond in $E^h_i$, i.e. $\bar{z}$ has an open bond in $E^h_{i+1}$, then $m < m_{y,x}$ by construction, contradicting the minimality of $m_{y,x}$.

For (5.5) let $(z_e)_{e \in E_{1..n}}$ be a bond configuration contributing to the event $\{X^y_n = x, N_i = N_{i+1} = m_{y,x}\}$. Suppose that the open bonds in the two layers do not match up. Since the number of open bonds in these layers is the same and the horizontal bonds match by (5.4), there must be an open bond in $E^v_{i+1}$ such that for all $y, x, N_i = N_{i+1} = m_{y,x}$, contradicting the minimality of $m_{y,x}$. \hfill \Box

**Proposition 6** There are constants $c_1, c_2, c_3 \geq 1$ depending only on $G$ such that for all $n \geq l$, $p \in (0, \frac{1}{2})$ and $y, x \in M^*$ such that $y \rightarrow x$ we have

\[
P_p(X^y_{n+1} = x, A_{y,x,n+1}) \leq p \cdot P_p(X^y_n = x), \quad (5.6)
\]

\[
P_p(X^y_{n+1} = x, A_{y,x,n+1}) \leq c_1\cdot p^n(2m_{y,x}+1), \quad (5.7)
\]

\[
P_p(X^y_n = x) \geq p^2c_3^{-n}p^{n_{y,x}}, \quad (5.8)
\]

where $A_{y,x,n+1} := \{\exists 1 \leq i \leq n+1 : N_{2i-1} = m_{y,x} = N_{2i}\}$.

**Proof:** Let $b := |E_i|$. For (5.6) we note that by definition of $A_{y,x,n+1}$

\[
P_p(X^y_{n+1} = x, A_{y,x,n+1}) \leq \sum_{1 \leq i \leq \frac{n+1}{2}} P_p(X^y_{n+1} = x, N_{2i-1} = m_{y,x} = N_{2i})
\]
using a union bound, and we note that for every \( i \)
\[
\mathbb{P}_p(X_{y_{n+1}} = x, N_{2i-1} = m_{y,x} = N_{2i}) \leq \mathbb{P}_p(X_{n}^y = x)p.
\]
To see this, we consider a bond configuration \( z = (z_e)_{e \in E_{1,n+1}} \) contributing to \( \{X_{y_{n+1}} = x, N_{2i-1} = m_{y,x} = N_{2i}\} \). We decompose this configuration into \( (z_e)_{e \in E_{2i-1}} \) and \( \bar{z} = (\bar{z}_e)_{e \in E_{1,n}} \), which is obtained from \( z \) by deleting layer \( E_{2i-1} \), i.e. \( \bar{z}_e = z_e \) for \( i \in E_{1,2i-2} \) and \( \bar{z}_e = z_{r(e)} \) for \( e \in E_{2i-1,n} \). By the previous lemma \( \bar{z} \) contributes to \( \{X_{y}^y = x\} \), and since \( m_{y,x} \geq 1 \) we have
\[
\mathbb{P}_p(\forall e \in E_{2i-1} : \bar{z}_e = z_e) = p^{m_{y,x}}(1 - p)^{b - m_{y,x}} \leq p.
\]
This establishes the above estimate and using \( \frac{n+1}{2} \leq n \) we thus get (5.6). For (5.7) we note that
\[
\mathbb{P}_p(X_{y_{n+1}} = x, A_{y,x,n+1}^c) \leq \mathbb{P}_p(\forall 1 \leq i \leq \frac{n + 1}{2} : N_{2i-1} + N_{2i} > 2m_{y,x})
\]
by definition of \( A_{y,x,n+1} \) and \( m_{y,x} \). Furthermore for every \( i > 0 \)
\[
\mathbb{P}_p(N_{2i-1} + N_{2i} > 2m_{y,x}) = \sum_{2m_{y,x} + 1 \leq k \leq 2b} {2b \choose k} p^k (1 - p)^{2b - k}
\leq \sum_{0 \leq k \leq 2b} {2b \choose k} p^{2m_{y,x} + 1} = 2^{2b} p^{2m_{y,x} + 1}.
\]
Combining this with the independence of \( N_{2i-1} + N_{2i} (i > 0) \), setting \( c_1 := 2^{2b} \) and using \( \frac{n+1}{2} \leq n, \frac{n+1}{2} \leq n \) we obtain (5.7). For (5.8) we note that by definition of \( l \), for some \( i \in \{1, ..., l\} \) we have \( \{X_{y}^y = x, N_{i} = m_{y,x}\} \neq \emptyset \). Let \( z = (z_e)_{e \in E_{1,i}} \) be a bond configuration contributing to this event, and let \( \bar{z} = (\bar{z}_e)_{e \in E_{1,n}} \) be the bond configuration obtained from \( z \) by inserting \( n - l \) additional layers above layer \( E_{i} \), such that in these additional layers the open bonds precisely match the positions of the open bonds in \( E_{i} \). By the above lemma \( \bar{z} \) contributes to \( \{X_{y}^y = x\} \) and thus
\[
\mathbb{P}_p(X_{y}^y = x) \geq \mathbb{P}_p(\forall e \in E_{1,n} : \bar{z}_e = \bar{z}_e)
= \mathbb{P}_p(\forall e \in E_{1,l} : \bar{z}_e = \bar{z}_e)(p^{m_{y,x}}(1 - p)^{b - m_{y,x}})^{n-l}.
\]
Noting that \( p \leq \frac{1}{2} \) we estimate \( \mathbb{P}_p(\forall e \in E_{1,l} : \bar{z}_e = \bar{z}_e) \geq p^{bl}, (1 - p)^{b - m_{y,x}} \geq (\frac{1}{2})^{b} \) and \( n - l \leq n \). Thus we obtain (5.8) by setting \( c_2 := bl \) and \( c_3 := 2^{b} \). □

The above ideas enable us to treat the case of small values of \( p \):

**Proposition 7** There is a \( p_0 \in (0, \frac{1}{2}) \) and an \( N \geq 0 \) such that
\[
\forall n \geq N \forall p \in (0, p_0] \forall y, x \in M^* : \mathbb{P}_p(X_{n}^y = x) \geq \mathbb{P}_p(X_{n+1}^y = x). \quad (5.9)
\]
**Proof:** Choosing \( c_1, c_2, c_3 \geq 1 \) according to the above proposition, we fix \( N \geq \max\{1, 4c_2\} \) and consider \( p \leq \frac{1}{2} \) and \( y, x \in M^* \). In case of \( y \to x \) we get

\[
\mathbb{P}_p(X_N^{y+1} = x) = \mathbb{P}_p(X_N^{y} = x, A_{y,x,N+1}) + \mathbb{P}_p(X_N^{y} = x, A_{y,x,N+1}^c)
\]

\[
\leq \mathbb{P}_p(X_N^{y} = x)\left(pN + c_1^y p^N c_2^y p^{-c_2}\right) \leq \mathbb{P}_p(X_N^{y} = x)\left(pN + (c_1 c_3)N p^N\right).
\]

If \( p \) is sufficiently small, we thus have \( \mathbb{P}_p(X_N^{y+1} = x) \leq \mathbb{P}_p(X_N^{y} = x) \). In case of \( y \not\to x \) we have \( \mathbb{P}_p(X_N^{y} = x) = 0 = \mathbb{P}_p(X_N^{y} = x) \). In conclusion there is a value \( p_0 \in (0, \frac{1}{2}) \) (depending on \( G \) only) such that for all \( p \in (0, p_0] \) and \( y, x \in M^* \) we have \( \mathbb{P}_p(X_N^{y} = x) \geq \mathbb{P}_p(X_N^{y} = x) \), and (5.9) follows using Lemma 3. \( \square \)

### 5.3 Large values of \( p \)

In this case we will use that going from a pattern \( y \) to a pattern \( x \) in a fixed large number of steps, in most layers the number of open bonds will be as large as possible. If it is not possible that all bonds in a layer are open, we can mimic the argument of the previous subsection: With high probability there are two consecutive layers with the same configuration of bonds, where all vertical bonds are open, and we proceed by discarding one of these layers.

**Definition 7** Let \( N_n := |\{e \in E_n : \tau_e = 0\}| \), for \( y, x \in M^* \) such that \( y \to x \) let

\[
m_{y,x}^{'} := \min\{N_n(\omega) : \exists n \geq 1 : 1 \leq n, \omega \in \{X_n^{y} = x\}\} \quad \text{and}
\]

\[
l_{y,x}^{'} := \min\{n \geq 1 : \exists 1 \leq i \leq n : \{X_n^{y} = x, N_i = m_{y,x}^{'}\} \neq \emptyset\}.
\]

and let \( l' := \max\{l_{y,x}^{'} : y, x \in M^* \text{ such that } y \to x\} \).

Thus \( N_n \) is the number of closed bonds in layer \( E_n \), \( m_{y,x}^{'} \) is the minimal number of closed bonds in some layer of a bond configuration going from pattern \( y \) to pattern \( x \) in an arbitrary number of steps, \( l_{y,x}^{'} \) is the minimal number of steps in which you can go from \( y \) to \( x \) and observe a layer with the minimal number of closed bonds in between. We note that \( l_{y,x}^{'} < \infty \), \( l' < \infty \) and for every \( n \geq l' \) and all \( y, x \in M^* \) such that \( y \to x \) it is possible to go from \( y \) to \( x \) in \( n \) steps and observe a layer with the minimal number of closed bonds in between (making use of (4.7)). We stress that here \( m_{y,x}^{'} = 0 \) cannot be ruled out.

**Lemma 5** Layers with the minimal number of closed bonds. For all \( y, x \in M^* \) such that \( y \to x \) we have

\[
\forall 1 \leq i \leq n : \{X_n^{y} = x, N_i = m_{y,x}^{'}\} \subset \{\forall e \in E_i : \tau_e = 1\}, \quad (5.10)
\]

\[
\forall 1 \leq i < n : \{X_n^{y} = x, N_i = N_{i+1} = m_{y,x}^{'}\} \subset \{\forall e \in E_i : \tau_e = \tau(\varepsilon)\}, \quad (5.11)
\]

i.e. in a layer with the minimal number of closed bonds all vertical bonds are open, and in two consecutive layers with the minimal number closed bonds, the closed bonds match up.
Proof: This is very similar to the proof of Lemma 4. For (5.10) let \( z = (z_e)_{e \in E_{1,n}} \) be a bond configuration contributing to the event \( \{X^y_n = x, N^y_i = m'_{y,x} \} \). Let \( \bar{z} = (\bar{z}_e)_{e \in E_{1,n+1}} \) be the bond configuration obtained from \( z \) by inserting an additional layer in between layers \( E_i \) and \( E_{i+1} \) with all vertical bonds open and the horizontal bonds as in \( E_i \), i.e. \( \bar{z}_e = z_e \) for all \( e \in E_{1,i} \), \( \bar{z}_e = 1 \) for all \( e \in E_{i+1} \) and \( \bar{z}_e = z_{e-1}(e) \) for all \( e \in E_{i+2:n+1} \cup E_{h}^{i+1} \). By construction \( \bar{z} \) contributes to \( \{X^y_n = x, N^y_i = m'_{y,x}, N^y_{i+1} = m'_{y,x} \} \) for some \( m' \). If \( z \) and thus \( \bar{z} \) has a closed bond in \( E^y_i \), then \( m' < m'_{y,x} \) by construction, contradicting the minimality of \( m'_{y,x} \).

For (5.11) let \( (\bar{z}_e)_{e \in E_{1,n}} \) be a bond configuration contributing to the event \( \{X^y_n = x, N^y_i = N^y_{i+1} = m'_{y,x} \} \). Suppose that the closed bonds in the two layers do not match up. Since the number of closed bonds in these layers is the same and the vertical bonds match by (5.10), there must be a closed bond in \( E^h_{i+1} \) such that the corresponding bond in \( E^h_i \) is open. By opening this closed bond we obtain a bond configuration \( (\bar{z}_e')_{e \in E_{1,n}} \), which contributes to \( \{X^y_n = x, N^y_i = m'_{y,x} - 1 \} \), and this contradicts the minimality of \( m'_{y,x} \).

Proposition 8 There are constants \( c_1, c_2, c_3 \geq 1 \) depending only on \( G \) such that for all \( n \geq t', p \in (\frac{1}{2}, 1) \) and \( y, x \in M^* \) such that \( y \rightarrow x \) and \( m'_{y,x} \geq 1 \) we have

\[
\mathbb{P}(X^y_{n+1} = x, A^y_{y,x,n+1}) \leq (1-p)n\mathbb{P}(X^y_{n} = x), \quad (5.12)
\]

\[
\mathbb{P}(X^y_{n+1} = x, (A^y_{y,x,n+1})') \leq c_1^2(1-p)^{\frac{n+1}{2}}, \quad (5.13)
\]

\[
\mathbb{P}(X^y_{n} = x) \geq (1-p)c_2^{-n}(1-p)^{|m'_{y,x}}|, \quad (5.14)
\]

where \( A^y_{y,x,n+1} := \{0 \leq i \leq \frac{n+1}{2} : N^y_{2i-1} = m'_{y,x} = N^y_{2i} \} \)

Proof: This is very similar to the proof of Proposition 6. Let \( b := |E_1| \). For (5.12) we note that by definition of \( A^y_{y,x,n+1} \) we have

\[
\mathbb{P}(X^y_{n+1} = x, A^y_{y,x,n+1}) \leq \sum_{0 \leq i \leq \frac{n+1}{2}} \mathbb{P}(X^y_{n+1} = x, N^y_{2i-1} = m'_{y,x} = N^y_{2i})
\]

and for every \( i \)

\[
\mathbb{P}(X^y_{n+1} = x, N^y_{2i-1} = m'_{y,x} = N^y_{2i}) \leq \mathbb{P}(X^y_{n} = x)(1-p).
\]

To see this, we consider a bond configuration \( z = (z_e)_{e \in E_{1,n+1}} \) contributing to \( \{X^y_{n+1} = x, N^y_{2i-1} = m'_{y,x} = N^y_{2i} \} \). We decompose this configuration into \( (z_e)_{e \in E_{2i-1}} \) and \( \bar{z} = (\bar{z}_e)_{e \in E_{1,n}} \), which is obtained from \( z \) by deleting layer \( E_{2i-1} \). By the previous lemma \( \bar{z} \) contributes to \( \{X^y_{n} = x \} \), and since \( m'_{y,x} \geq 1 \) we have

\[
\mathbb{P}(\forall e \in E_{2i-1} : \bar{z}_e = z_e) = (1-p)^{m'_{y,x}} \geq 1 - p.
\]

This establishes the above estimate and using \( \frac{n+1}{2} \leq n \) we thus get (5.12). For (5.13) we note that

\[
\mathbb{P}(X^y_{n+1} = x, (A^y_{y,x,n+1})') \leq \mathbb{P}(\forall 0 \leq i \leq \frac{n+1}{2} : N^y_{2i-1} + N^y_{2i} > 2m'_{y,x})
\]

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by definition of $A'_{y,x,n+1}$ and $m'_{y,x}$. Furthermore for every $i > 0$

$$\mathbb{P}_p(N_{2i-1} + N_{2i} > 2m'_{y,x}) = \sum_{2m'_{y,x}+1 \leq k \leq 2b} \binom{2b}{k} (1-p)^k p^{2b-k}$$

$$\leq \sum_{0 \leq k \leq 2b} \binom{2b}{k} (1-p)^{2m'_{y,x}+1} = 2^{2b} (1-p)^{2m'_{y,x}+1}.$$ 

Combining this with the independence of $N_{2i-1} + N_{2i}$ ($i > 0$), setting $c_1 := 2^{2b}$ and using $\frac{n+1}{2} \leq n, \frac{n+1}{2} \geq \frac{p}{2}$ we obtain (5.13). For (5.14) we note that by definition of $l'$, for some $i \in \{1, ..., l'\}$ $\{X_n^y = x, N_i = m'_{y,x}\} \neq \emptyset$. Let $z = (z_e)_{e \in E_{1,l'}}$ be a configuration of bonds contributing to this event, and let $\bar{z} = (\bar{z}_e)_{e \in E_{1,n}}$ be the bond configuration obtained from $z$ by inserting $n - l'$ additional layers above layer $E_i$, such that in these additional layers the closed bonds precisely match the positions of the closed bonds in $E_i$. By the above lemma $\bar{z}$ contributes to $\{X_n^y = x\}$ and thus

$$\mathbb{P}_p(X_n^y = x) \geq \mathbb{P}_p(\forall e \in E_{1,n} : \bar{z}_e = \bar{z}_e)$$

$$= \mathbb{P}_p(\forall e \in E_{1,l'} : \bar{z}_e = \bar{z}_e)((1-p)^{m'_{y,x}} p^{b-m'_{y,x}})^{n-l'}.$$ 

Noting that $p \geq \frac{1}{2}$ we estimate $\mathbb{P}_p(\forall e \in E_{1,l'} : \bar{z}_e = \bar{z}_e) \geq (1-p)^{bl'} p^{b-m'_{y,x}} \geq \left(\frac{1}{2}\right)^{bl'}$ and $n - l' \leq n$. Thus we obtain (5.14) by setting $c_2 := bl'$ and $c_3 := 2^{b}$.

If it is possible that all bonds in a layer are open, we have to give a separate argument: Here with high probability there will be such a layer, and we proceed by discarding the layer before that.

**Proposition 9** There are constants $c_1, c_2, c_3, c_4 \geq 1$ depending only on $G$ such that for all $n \geq l'$, $p \in (\frac{1}{2}, 1)$ and $y, x \in M^*$ such that $y \to x$ and $m'_{y,x} = 0$ we have

$$\mathbb{P}_p(X_{n+1}^y = x, B_{n+1}) \leq (1 - (1 - p)^{c_1}) \mathbb{P}_p(X_n^y = x),$$

$$\mathbb{P}_p(X_{n+1}^y = x, B_{n+1}) \leq (c_2(1-p))^n,$$

$$\mathbb{P}_p(X_n^y = x) \geq c_3^n (1-p)^{c_4},$$

where $B_{n+1} := \{ \exists 2 \leq i \leq n + 1 \forall e \in E_i : \bar{z}_e = 1\}$.

**Proof:** Let $b := |E_1|$. For (5.15) let $z = (z_e)_{e \in E_{1,n+1}}$ be a bond configuration contributing to $\{X_{n+1}^y = x, B_{n+1}\}$. Let $i \in \{2, ..., n + 1\}$ denote the last layer of this configuration consisting exclusively of open bonds (which exists by definition of $B_{n+1}$). We decompose this configuration into $\bar{z} = (\bar{z}_e)_{e \in E_{1,n}}$ which is obtained from $z$ by deleting layer $E_{i-1}$, and $z' = (z'_e)_{e \in E_i}$ which equals the bond configuration of the deleted layer (i.e. $z'_e := z_{e-i(e)}$). We note that by $z'$ and $\bar{z}$ the original configuration $z$ is uniquely determined: If $j \in \{1, ..., n\}$ is the last layer of $\bar{z}$ consisting of open bonds only, then $z$ can be recovered from $\bar{z}$ by inserting $z'$ into $\bar{z}$ between the layers $E_{j-1}$ and $E_j$. Furthermore $\bar{z}$ contributes to $\{X_n^y = x\}$ (since after a layer with all bonds open the pattern is $x_*$ provided
the preceding pattern is infected) and \( z' \) contributes to \( \{ \exists e \in E^*_1 : Z_e = 1 \} \) (since the infection has to be transmitted by this layer). By independence we thus obtain

\[
P_p(X^y_{n+1} = x, B_{n+1}) \leq P_p(X^y_n = x)P_p(\exists e \in E^*_1 : Z_e = 1),
\]

which gives (5.16) setting \( c_1 := |E^*_1| = |V| \). For (5.17) we note that by definition of \( E^*_1 \), \( \forall 2 \leq i \leq n + 1 : N_i \neq b \),

by definition of \( B_{n+1} \). Furthermore for every \( i \)

\[
P_p(N_i \neq b) = 1 - p^b = (1 - p) \sum_{0 \leq j < b} p^j \leq b(1 - p).
\]

Combining this with the independence of \( N_i \) \( (i > 0) \) and setting \( c_2 := b \) we obtain (5.16). For (5.17) we note that by definition of \( l' \), for some \( i \in \{ 1, \ldots, l' \} \) \( \{ X^y_n = x, N_i = b \} \neq \emptyset \). Let \( z = (z_e)_{e \in E_{1,l'}} \) be a configuration of bonds contributing to this event, and let \( (z_e)_{e \in E_{1,n}} \) be the bond configuration obtained from \( z \) by inserting \( n - l' \) additional layers preceding layer \( E_1 \), such that in these additional layers all bonds are open. This new configuration contributes to \( \{ X^y_n = x \} \) and thus

\[
P_p(X^y_n = x) \leq P_p(\forall e \in E_{1,n} : Z_e = z_e) = P_p(\forall e \in E_{1,l'} : Z_e = z_e)p^{b(n-l')}
\]

Noting that \( p \geq \frac{1}{2} \) we estimate \( P_p(\forall e \in E_{1,l'} : Z_e = z_e) \geq (1 - p)^{l'} \) and \( p^{b(n-l')} \geq (\frac{1}{2})^b \) and we obtain (5.17) by setting \( c_3 := 2^b \) and \( c_4 = bl' \). \( \square \)

Combining the two preceding propositions gives the desired result in case of large values of \( p \):

**Proposition 10** There is a \( p_1 \in (\frac{1}{2}, 1) \) and an \( N \geq 0 \) such that

\[
\forall n \geq N \forall y, x \in M^* : P_p(X^y_n = x) \geq P_p(X^y_{n+1} = x).
\]

**Proof:** Choosing \( c_1, c_2, c_3, c_4 \geq 1 \) according to the above proposition and \( c_1', c_2', c_3' \geq 1 \) according to the previous one, we fix \( N \geq \max \{ l', 2(c_4 + 1), 4c_3 \} \) and consider \( p > \frac{1}{2} \) and \( y, x \in M^* \). In case of \( y \rightarrow x \) such that \( m'_{y,x} = 0 \) we get

\[
P_p(X^y_N = x) = P_p(X^y_{N+1} = x, B_{N+1}) + P_p(X^y_{N+1} = x, B^c_{N+1})
\]
\[
\leq P_p(X^y_N = x)(1 - (1 - p)^{c_1} + (c_2(1 - p))^N c_3^N (1 - p)^{-c_4})
\]
\[
\leq P_p(X^y_N = x)(1 - (1 - p)^{c_1} (1 - c_2 c_3)^N (1 - p)^{\frac{N}{2}}).
\]

If \( 1 - p \) is sufficiently small, we thus have \( P_p(X^y_{N+1} = x) \leq P_p(X^y_N = x) \). In case of \( y \rightarrow x \) such that \( m'_{y,x} \geq 1 \) we get

\[
P_p(X^y_{N+1} = x) = P_p(X^y_{N+1} = x, A'_{y,x,N+1}) + P_p(X^y_{N+1} = x, A'_{y,x,N+1}^c)
\]
\[
\leq P_p(X^y_N = x)((1 - p)N + (c_1')^N (1 - p)^{\frac{N}{2}} (c_3')^N (1 - p)^{-c_2})
\]
\[
\leq P_p(X^y_N = x)((1 - p)N + (c_1' c_3')^N (1 - p)^{\frac{N}{2}}).
\]
If $1 - p$ is sufficiently small, we thus have $\mathbb{P}_p(X_N^y = x) \leq \mathbb{P}_p(X_N^y_{n+1} = x)$. In case of $y \not\to x$ we have $\mathbb{P}_p(X_N^y_{n+1} = x) = 0 = \mathbb{P}_p(X_N^y = x)$. In conclusion there is a value $p_1 \in (\frac{1}{2}, 1)$ (depending on $G$ only) such that for all $p \in [p_1, 1)$ and $y, x \in M^*$ we have $\mathbb{P}_p(X_N^y = x) \geq \mathbb{P}_p(X_N^y_{n+1} = x)$, and (5.18) follows using Lemma 3.

5.4 Proof of Theorem 1

Theorem 1 can now easily be proved combining the results from the last three subsections: We first choose $p_0 \in (0, \frac{1}{2})$ and $N_1 \geq 0$ according to Proposition 7, and $p_1 \in (\frac{1}{2}, 1)$ and $N_2 \geq 0$ according to Proposition 10. For these values of $p_0, p_1$ we choose $N_3 \geq 0$ according to Proposition 5. Setting $N := \max\{N_1, N_2, N_3\}$, for all $n \geq N, p \in (0, 1)$, $y, x \in M^*$ we thus get $\mathbb{P}_p(X_n^y = x) \geq \mathbb{P}_p(X_{n+1}^y = x)$, as desired. The two additional claims easily follow from the first one using Proposition 2.

6 Number of infected points per layer

Here we prove Theorem 2. Let $G$ be a connected graph of bounded degree such that $\deg(x) \leq \Delta$ for all $x \in V$. We first show that for $p \in [0, 1]$ and $n \geq 0$ we have

$$\mathbb{E}_p(W_n) \cdot \max_{w \in G} \mathbb{E}_p(\tilde{W}_n^w) \geq \mathbb{E}_p(W_{n+1}),$$

(6.1)

where $\tilde{W}_n^w = |\{v \in V : (v, 1) \leftrightarrow_{\geq n} (w, 0)\}|$ and $\leftrightarrow_{\geq n}$ is the connectivity relation induced by percolation on $E_{n'}$. For this we note that

$$\mathbb{E}_p(W_{n+1}) = \sum_{v \in V} \mathbb{P}_p(o' \leftrightarrow (v, n+1)),$$

and considering a self-avoiding path from $o'$ to $(v, n+1)$ and its last vertex in the layer $V_n$, we see that

$$\{o' \leftrightarrow (v, n+1)\} = \bigcup_{w \in V} \{o' \leftrightarrow (w, n)\} \circ \{(w, n) \leftrightarrow_{\geq n+1} (v, n+1)\}.$$

Thus using a union bound and the BK-inequality (see (6)) we obtain

$$\mathbb{E}_p(W_{n+1}) \leq \sum_{w, v \in V} \mathbb{P}_p(o' \leftrightarrow (w, n)) \mathbb{P}_p((w, n) \leftrightarrow_{\geq n+1} (v, n+1)).$$

While the BK-inequality is usually formulated in a setting with finitely many bonds, this is not a problem since the above estimate thus holds for percolation on $E_{n', n+1+n'}$ for any fixed $n' \geq 0$, and letting $n' \to \infty$ all probabilities and expectations converge to give the above estimate as stated. By translation invariance

$$\sum_{v \in V} \mathbb{P}_p((w, n) \leftrightarrow_{\geq n+1} (v, n+1)) = \sum_{v \in V} \mathbb{P}_p((w, 0) \leftrightarrow_{\geq 1} (v, 1)) = \mathbb{E}_p(\tilde{W}_n^w)$$
and we thus obtain (6.1) follows from (6.1) by estimating $\mathbb{E}_p(\hat{W}_i^w)$. Let $w \in V$ be fixed for the remainder of the proof. Let $W_i$ denote the set of self-avoiding paths in $E_i$, of length $l$ from $(w,0)$ to some vertex in $V_1$, and for any $P \in W_i$ let $A_P$ denote the event, that all edges of $P$ are open. We note that

$$
\mathbb{E}_p(\hat{W}_i^w) \leq \mathbb{E}_p\left(\sum_{l \geq 1} \sum_{P \in W_i} 1_{A_P}\right) = \sum_{l \geq 1} \sum_{P \in W_i} \mathbb{P}(A_P) = \sum_{l \geq 1} |W_i|^p^l.
$$

Any $P \in W_i$ starts with a step going up (to $V_1$) and then never returns to $V_0$ (because it is self-avoiding). It is possible, that it stays within $V_1$, in which case we have $\Delta$ possibilities for the first step and $\Delta - 1$ possibilities for all other steps. On the other hand, if the path also visits $V_2$, then (after the first step up) it remains in $V_1$ for $k_1 \geq 0$ steps, then goes up to $V_2$, then performs another $k \geq 1$ steps to end in $V_2$, then goes down to $V_1$, then stays in $V_1$ for an additional $k_2 \geq 0$ steps. We note that for the last step of the $k$ steps from $V_2$ to $V_2$ there are at most $\Delta$ choices. (After $k - 1$ steps we are either in $V_1, V_2$ or $V_3$. If we are in $V_1$ or $V_3$ we have to go up or down respectively. If we are in $V_2$ we stay in $V_2$ by $\Delta$ choices.) This gives

$$
|W_i| \leq \Delta(\Delta - 1)^{l-2} + \sum_{k_1, k_2} \Delta(\Delta - 1)^{k_1 - 1} \cdot (\Delta + 1)^{k_2 - 1} \Delta(\Delta - 1)^{k_2 - 1},
$$

where the sum is over all $k_1, k_2 \geq 0, k_1 \geq 1, k_2 \geq 0$ such that $k_1 + k + k_2 + 3 = l$, and we interpret $\Delta(\Delta - 1)^{-1}$ as 1. We thus obtain

$$
\mathbb{E}_p(W_i^w) \leq \sum_{l \geq 1} \Delta(\Delta - 1)^{l-2} p^l
$$

$$
+ \sum_{k_1 \geq 0} \Delta(\Delta - 1)^{k_1 - 1} p^{k_1 + 1} \sum_{k_2 \geq 1} (\Delta - 1)^{k_2 - 1} \Delta p^{k_2} \sum_{k_1 \geq 1} (\Delta - 1)^{k_1 - 1} \Delta p^{k_1 - 1}
$$

$$
= \frac{p(1 + p)}{1 - (\Delta - 1)p} + \left(\frac{p(1 + p)}{1 - (\Delta - 1)p}\right)^2 \frac{\Delta p^2}{1 - (\Delta + 1)p} =: f_\Delta(p)
$$

using $p \leq p_\Delta := \frac{1}{\Delta + 1} < \frac{1}{\Delta + 2}$. Since $f_\Delta(p)$ is increasing in $p$, it suffices to check that $g(\Delta) := f_\Delta(p_\Delta) \leq 1$. Using $\frac{1}{p_\Delta} - (\Delta - 1) = 2.4$ and $\frac{1}{p_\Delta} - (\Delta + 1) = 0.4$ we have

$$
g(\Delta) = \frac{1 + \frac{1}{\Delta + 1}}{2.4} + \left(1 + \frac{1}{\Delta + 1}\right)^2 \frac{\Delta}{0.4} \leq \frac{1 + \frac{1}{\Delta + 1}}{2.4} \left(1 + \frac{1}{2.4 \cdot 0.4}\right) =: h(\Delta)
$$

using $(1 + \frac{1}{\Delta + 1})^\Delta / \Delta + 1 \leq 1$. $h$ is decreasing in $\Delta$, and it is easy to check that $g(\Delta) \leq 1$ for $\Delta \in \{0, 1, 2, 3, 4\}$ and $h(5) \leq 1$, thus $g(\Delta) \leq 1$ for all $\Delta$ and we are done.

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