INTEGRAL INEQUALITIES FOR HOLOMORPHIC MAPS AND APPLICATIONS

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Abstract. We derive some integral inequalities for holomorphic maps between complex manifolds. As applications, some rigidity and degeneracy theorems for holomorphic maps without assuming any pointwise curvature signs for both the domain and target manifolds are proved, in which key roles are played by total integration of the function of the first eigenvalue of second Ricci curvature and an almost nonpositivity notion for holomorphic sectional curvature introduced in our previous work. We also apply these integral inequalities to discuss the infinite-time singularity type of the Kähler-Ricci flow. The equality case is characterized for some special settings.

1. Introduction

1.1. Background. A general principle in complex geometry states that negative curvature restricts behaviors of holomorphic maps between complex manifolds, see e.g. [9, page 15]. The most classic result along this line should be Schwarz-Pick-Ahlfors Lemma: a non-constant holomorphic map from the unit disc (equipped with the Poincaré metric) to a smooth Riemann surface with negative curvature decreases distances (up to multiplying a constant factor depending only on the bounds of curvatures). Important generalizations of Schwarz-Pick-Ahlfors Lemma to higher dimensions were developed, including Chern [2], Lu [10], Yau [24], Royden [12], etc.. Here, we in particular recall Yau’s general Schwarz Lemma [24] that a holomorphic map from a complete Kähler manifold of Ricci curvature bounded from below to a Hermitian manifold of holomorphic bisectional curvature bounded from above by a negative constant decreases distances (up to multiplying a constant factor depending only on the bounds of curvatures). Royden [12] proved that similar result holds if the target space is a Kähler manifold of holomorphic sectional curvature bounded from above by a negative constant. In particular, their results imply a fundamental rigidity theorem that a holomorphic map from a compact Kähler manifold of positive Ricci curvature to a Hermitian (resp. Kähler) manifold of nonpositive holomorphic bisectional (resp. sectional) curvature must be constant. Excellent expositions on differential geometric developments of Schwarz-type Lemma can be found in [9]. More recently, there are significant progresses on this topic, which relaxed either the curvature assumptions or Kählerian condition, see [11, 14, 21, 22, 23] and references therein for more details.

1.2. Motivations. Let’s focus on the rigidity theorems. The general philosophy behind rigidity theorems for holomorphic maps is that a holomorphic map from a positively curved space to a negatively curved space should be constant. Our study here is mainly motivated by the following natural question: can we make this philosophy more effective? To be more precise, let’s look at, for example, the aforementioned fundamental rigidity theorem of Yau and Royden: a holomorphic map from a compact Kähler manifold of positive (resp. nonnegative) Ricci curvature to a compact Kähler manifold of nonpositive (resp. nonnegative) Ricci curvature must be constant. This is a very effective result, which is quite powerful in applications. However, it is still possible to relax the curvature assumptions in some cases. For example, using the integral inequalities derived in our previous work, we can prove a result similar to Yau and Royden’s rigidity theorem when the target manifold is a Hermitian manifold of holomorphic sectional curvature bounded from above by a negative constant. In particular, our results imply that a holomorphic map from a compact Kähler manifold of nonnegative Ricci curvature to a Hermitian manifold of negative sectional curvature must be constant. This is a significant improvement over Yau and Royden’s result, since it allows for a more general class of target manifolds.

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negative) holomorphic sectional curvature must be constant. For convenience, here we have assumed that both the domain and target manifolds are compact Kähler. Consequently, given a non-constant holomorphic map \( f : (X, \omega) \to (Y, \eta) \) between two compact Kähler manifolds,

(I) if the Ricci curvature of \( \omega \) is positive, then the supremum of holomorphic sectional curvature, denoted by \( \sup_Y H^\eta \) (see Notation 2.2), must be positive;

(II) if the holomorphic sectional curvature of \( \eta \) is negative, then the infimum of Ricci curvature of \( \omega \), denoted by \( \inf_X \lambda_\omega \) (see Notation 2.1), must be negative.

Then, regarding the effectiveness, we may naturally ask (in the above setting):

(A) in the above case (I), can we have an effective positive lower bound for \( \sup_Y H^\eta \)?

(B) in the above case (II), can we have an effective negative upper bound for \( \inf_X \lambda_\omega \)?

The effectiveness results, if can be obtained, will play crucial roles in several problems, including

- weakening/sharpening the curvature conditions in the rigidity theorems for holomorphic maps;
- dealing with a family of Kähler/Hermitian metrics on either the domain or target manifolds, which naturally arises in the study of complex geometric flows including the Kähler-Ricci flow and the Chern-Ricci flow.

Firstly, we observe that the case (B) readily follows from Yau’s general Schwarz Lemma. Precisely, in the above case (II) there holds

\[
f^* \eta \leq \frac{\inf_X \lambda_\omega}{\frac{m+1}{2m} \sup_Y H^\eta} \cdot \omega
\]

on \( X \) and hence

\[
\inf_X \lambda_\omega \leq \frac{\frac{m+1}{2m} \sup_Y H^\eta}{\int_X f^* \eta \wedge \omega^{n-1}} \int_X \omega^n,
\]

(1.2)
giving an effective negative upper bound for \( \inf_X \lambda_\omega \). One may interpret (1.2) as an integral version of (1.1).

In the following, let’s focus on the case (A). Under the curvature conditions in case (I), we don’t have the “reverse” Schwarz Lemma, (say, \( f^* \eta \geq \frac{\inf_X \lambda_\omega}{\frac{m+1}{2m} \sup_Y H^\eta} \cdot \omega \)), leaving the (integral) inequality (1.2) unclear. Noting that (1.2) should be much weaker than the (“reverse”) Schwarz Lemma, to solve the above case (A) we are naturally led to prove the inequality (1.2) directly. In this paper, we shall prove a general integral inequality for non-constant holomorphic maps in a general setting without assuming any curvature condition (Theorem 1.1), which in particular implies the desired inequality (1.2) in its setting (up to the constant factor \( \frac{m+1}{2m} \)), and hence solves the above case (A) completely. We should mention that for the special case \( (X, \omega) = (\mathbb{CP}^1, \omega_{FS}) \), the inequality (1.2) was previously proved in Tosatti-Y.G.Zhang [18, Section 4, Remark 4.1] by a different method.

1.3. Main results: integral inequalities for holomorphic maps. In this paper, we shall present several integral inequalities for non-constant or non-degenerate holomorphic maps between two complex manifolds without assuming any curvature condition, in which we assume the domain manifold is compact. Precisely, we shall prove the followings:
**Theorem 1.1.** Let \((X, \omega)\) and \((Y, \eta)\) be two Hermitian manifolds and \((X, \omega)\) be \(n\)-dimensional and compact. Then there exists a smooth real function \(\phi\) on \(X\) such that for any non-constant holomorphic map \(f : X \to Y\) there holds
\[
\int_X \lambda_\omega e^{(n-1)\phi} \omega^n \leq n \int_X f^* \kappa_\eta \cdot e^{(n-1)\phi} f^* \eta \wedge \omega^{n-1},
\]
where
1. \(\lambda_\omega\) is the function of the first eigenvalue of the second Ricci curvature \(\text{Ric}^{(2)}(\omega)\) of \(\omega\) with respect to \(\omega\); and
2. \(\kappa_\eta\) is a continuous real function on \(Y\) such that for \(y \in Y\), \(\kappa_\eta(y)\) is the maximal value of holomorphic bisectional curvature of \(\eta\) at \(y\) when \(\eta\) is not Kählerian, and is the “modified” maximal value of holomorphic sectional curvature of \(\eta\) at \(y\) when \(\eta\) is Kählerian (see Notation 2.2 for precise definition).

If furthermore \(\omega\) is Gauduchon, the \(\phi\) in (1.3) can be chosen to be any constant function.

**Theorem 1.2.** Let \((X, \omega)\) and \((Y, \eta)\) be two \(n\)-dimensional Hermitian manifolds and \((X, \omega)\) compact. Then there exists a smooth real function \(\psi\) on \(X\) such that for any non-degenerate holomorphic map \(f : X \to Y\) there holds
\[
\int_X R_\omega e^{(n-1)\psi} \omega^n \leq n \int_X e^{(n-1)\psi} f^* (\text{Ric}(\eta)) \wedge \omega^{n-1},
\]
where \(R_\omega\) is the Chern scalar curvature of \(\omega\) and \(\text{Ric}(\eta)\) is the Chern Ricci curvature of \(\eta\).

If furthermore \(\omega\) is Gauduchon, the \(\psi\) in (1.4) can be chosen to be any constant function.

**Remark 1.3.** These integral inequalities may be regarded as effective obstructions for a holomorphic map being constant or totally degenerate, and they make the aforementioned philosophy in Subsection 1.2 more effective. In particular, inequality (1.3) implies a complete answer to the case (A) in Subsection 1.2 i.e. (1.2) holds in its setting (also see Subsection 4.4 for more general results).

### 1.4. Outline of applications.

As described in Subsection 1.2, the effectiveness results Theorems 1.1 and 1.2 may have applications in several problems. Let’s outline some of these applications.

A classical differential geometric approach in proving rigidity theorem for holomorphic map (i.e. proving constancy of a holomorphic map) makes use of Chern-Lu formula, pointwise curvature signs and the maximum principle arguments. Roughly speaking, this approach makes use of pointwise curvature signs to destroy (pointwise) Schwarz-type Lemma and hence gets the constancy of holomorphic maps.

As applications of Theorem 1.1 we shall prove rigidity theorems under weaker curvature conditions; in particular, the curvatures are not necessarily pointwise signed. In fact, given Theorem 1.1, to prove constancy of holomorphic map, it suffices to destroy the integral inequality (1.3), which can be achieved by just assuming, for example, suitable signs for curvature in certain integral sense or “almost” sense. Therefore, we obtain new rigidity theorems for holomorphic maps without assuming any pointwise curvature signs for both the domain and target manifolds.

Similarly, Theorem 1.2 can be applied to prove degeneracy theorems for holomorphic maps without assuming any pointwise curvature signs for both the domain and target manifolds.

Moreover, our Theorem 1.1 implies a criterion for type IIb singularities of the Kähler-Ricci flow, generalizing a result of Tosatti-Y.G. Zhang [18, Proposition 1.4].
Also, the equality case in Theorem 1.1 is characterized in some special settings. The details of these applications will be given in Section 4.

1.5. Organization. In the next section, we will introduce some necessary notations and results in complex geometry. Then we prove our main results Theorems 1.1 and 1.2 in Section 3. Finally, in Section 4, we provide several applications, including several rigidity and degeneracy theorems for holomorphic maps, a criterion for type IIb singularities of the Kähler-Ricci flow and characterization of equality case in some settings.

2. Preliminaries

2.1. Curvatures in complex geometry. Let \((X, \omega)\) be a Hermitian manifold of dimension \(\dim \mathbb{C} X = n\), where \(\omega = \omega_g\) is the Kähler form of a Hermitian metric \(g\). In a local holomorphic chart \((z^1, ..., z^n)\), we write

\[
\omega = \sqrt{-1} g_{ij} dz^i \wedge d\bar{z}^j.
\]

Recall the curvature tensor \(R = \{R^\omega_{ijkl}\}\) of the Chern connection is given by

\[
R^\omega_{ijkl} = -\frac{\partial^2 g_{kl}}{\partial z^i \partial \bar{z}^j} + g^{ip} \frac{\partial g_{kj}}{\partial z^i} g^{ql} \frac{\partial g_{pl}}{\partial \bar{z}^j}.
\]

Then the Chern Ricci curvature \(\text{Ric}(\omega) = \sqrt{-1} R_{ij} dz^i \wedge d\bar{z}^j\) and Chern scalar curvature \(R_\omega\) are given by

\[
R_{ij} = g^{jk} R^\omega_{ijkl}
\]

and

\[
R_\omega = g^{ji} R^\omega_{ijkl}
\]

respectively. Note that \(R_{ij} = -\frac{\partial^2 \log \det(g_{ij})}{\partial z^i \partial \bar{z}^j}\).

Also recall that the second Ricci curvature \(\text{Ric}^{(2)}(\omega)\) is given by

\[
R^{(2)}_{kl} = g^{ji} R^\omega_{ijkl}.
\]

Notation 2.1 (Function of the first eigenvalue of the second Ricci curvature). Let \((X, \omega)\) be a Hermitian manifold of \(\dim \mathbb{C} X = n\). Define \(\lambda_\omega\) to be the function of the first eigenvalue of the second Ricci curvature \(\text{Ric}^{(2)}(\omega)\) of \(\omega\) with respect to \(\omega\), which is a continuous function on \(X\) satisfying

\[
\text{Ric}^{(2)}(\omega) \geq \lambda_\omega \omega.
\]

For convenience, we call the integration \(\int_X \lambda_\omega \omega^n\) the total first eigenvalue of second Ricci curvature of \(\omega\).

Given \(x \in X\) and \(W, V \in T^{1,0}_x X \setminus \{0\}\), the holomorphic bisectional curvature of \(\omega\) at \(x\) determined by \(W, V\) is

\[
BK^\omega_x(W, V) := \frac{R^\omega(W, \overline{W}, V, \overline{V})}{|W|_\omega^2 |V|_\omega^2},
\]

and the holomorphic sectional curvature of \(\omega\) at \(x\) in the direction \(W\) is

\[
H^\omega_x(W) := \frac{R(W, \overline{W}, W, \overline{W})}{|W|_\omega^4}.
\]
Notation 2.2 (Supremum of holomorphic curvature). Let \((X, \omega)\) be a Hermitian manifold of \(\dim \mathbb{C} X = n\) and \(x \in X\). We set
\[
BK^\omega_x := \sup \{ BK^\omega_x(W, V) | W, V \in T_x^{1,0} X \setminus \{0\}\}
\]
and
\[
H^\omega_x := \sup \{ H^\omega_x(W) | W \in T_x^{1,0} X \setminus \{0\}\}.
\]
Then we define a continuous real function \(\kappa_\omega\) for \((X, \omega)\) as follows.

1. when \(\omega\) is not Kählerian, we define \(\kappa_\omega(x) := BK^\omega_x\);
2. when \(\omega\) is Kählerian, we define \(\kappa_\omega(x) := \rho(H^\omega_x) \cdot H^\omega_x\), where \(\rho : \mathbb{R} \to \{\frac{n+1}{2n}, 1\}\) is a function with \(\rho(s) = \frac{n+1}{2n}\) for \(s \leq 0\) and \(\rho(s) = 1\) for \(s > 0\).

Definition 2.3 (Compact Kähler manifold of almost nonpositive holomorphic sectional curvature [28]). Let \((X, \omega)\) be a compact Kähler manifold.

1. Let \(\alpha\) be a Kähler class on \(X\). We define a number \(\mu_\alpha\) for \(\alpha\) in the following way:
\[
\mu_\alpha := \inf \{ \sup_X H^\omega | \omega \text{ is a Kähler metric in } \alpha \},
\]
where \(H^\omega\) is the continuous function on \(X\) with \(H^\omega(x) = H^\omega_x\).
2. We say \(X\) is of almost nonpositive holomorphic sectional curvature if for any number \(\epsilon > 0\), there exists a Kähler class \(\alpha_\epsilon\) on \(X\) such that \(\mu_\alpha, \alpha_\epsilon < \epsilon[\hat{\omega}]\).

One may find more motivations and discussions about the above almost nonpositivity notion for holomorphic sectional curvature in [28]. Here we mention that it is not a pointwise notion, but a notion at the level of \((1,1)\)-classes, and the number \(\mu_\alpha\) up to multiplying a constant factor depending only on dimension of manifold, turns out to be an upper bound for the nef threshold of \(\alpha\) [28, Proposition 1.9].

2.2. Royden's trick. Let \(f : (X, \omega) \to (Y, \eta)\) be a holomorphic map between two Hermitian manifolds and \(\dim \mathbb{C} X = n\) and \(\dim \mathbb{C} Y = m\). Given \(x \in X\) and holomorphic charts \((z^1, ..., z^n)\) on \(X\) centered at \(x\) and \((w^1, ..., w^m)\) on \(Y\) centered at \(f(x)\). Write \(f = (f^1, ..., f^m)\) and \(f^i := \frac{\partial f^i}{\partial z^j}, 1 \leq i \leq n, 1 \leq \alpha \leq m,\) in these local charts. Assuming that \(\eta\) is Kähler, Royden [12, page 552] proved that at \(x\),
\[
R^{\eta}_{\alpha, \beta, \gamma, \delta}(g^{i\bar{j}} f^i \bar{f}^j g^{\bar{k} \bar{l}}) \leq f^* \kappa_\eta \cdot (tr_\omega f^* \eta)^2,
\]
where \(\kappa_\eta\) is the function defined in Notation 2.2 (2).

2.3. Gauduchon metrics. Let \(X\) be a compact complex manifold of \(\dim \mathbb{C} X = n\). A Hermitian metric \(\omega\) on \(X\) is called Gauduchon if
\[
\bar{\partial \partial} (\omega^{n-1}) = 0.
\]
Obviously, for a Gauduchon metric \(\omega\) and a smooth function \(u\) on \(X\) we have
\[
\int_X (\Delta_\omega u) \omega^n = 0,
\]
where \(\Delta_\omega u\) is the complex Laplacian defined by \(\Delta_\omega u = tr_\omega (\sqrt{-1} \bar{\partial \partial} u)\). A classical result of Gauduchon [5] states that, for every Hermitian metric \(\omega\), there is a \(\phi \in C^\infty(X, \mathbb{R})\) (unique up to scaling, when \(n \geq 2\)) such that \(e^\phi \omega\) is Gauduchon.

2.4. Non-degenerate holomorphic maps. Let \(f : X \to Y\) be a holomorphic map between two complex manifolds of the same dimension. If there exists some point \(x \in X\) such that the Jacobian \(J(f)\) of \(f\) satisfies \(|\det J(f)(x)| > 0\), then we say \(f\) is non-degenerate; otherwise, we say \(f\) is totally degenerate.
3. Proofs of Integral Inequalities

In this section we prove Theorems 1.1 and 1.2. Our proofs do not involve any maximum principle arguments, since the curvatures of both the domain and target spaces may not be signed in pointwise sense. Instead, we will make use of a perturbation method involving Lebesgue’s Dominated Convergence Theorem.

Proof of Theorem 1.1. We only consider the case that \( \eta \) is a Kähler metric on \( Y \), as the other case can be proved similarly.

Let \( \epsilon \) be an arbitrary positive number, \( \Lambda = \text{tr}_x f^* \eta = |\partial f|^2 = |df|^2 \). Given an arbitrary \( x \in X \), and holomorphic charts \((z^1, \ldots, z^n)\) on \( X \) centered at \( x \) and \((w^1, \ldots, w^m)\) on \( Y \) centered at \( f(x) \). Write \( \omega = \sqrt{-1} g_{ij} dz^i \wedge d\bar{z}^j \), \( \eta = \sqrt{-1} \eta_{\alpha \beta} dw^\alpha \wedge dw^\beta \), \( \text{Ric}^{(2)}(\omega) = \sqrt{-1} R^{(2)}_{ij} dz^i \wedge d\bar{z}^j \), \( f = (f^1, \ldots, f^m) \) and \( f^\alpha_1 := \frac{\partial f^\alpha}{\partial z_1}, 1 \leq 1 \leq m \), in these local charts. By direct computations and Chern-Lu formula [24, 10, 25] (also see [23] Lemma 4.1) for clearer and simpler discussions on Chern-Lu formula we have

\[
\Delta_\omega \log(\Lambda + \epsilon) = \frac{\Delta_\omega \Lambda}{\Lambda + \epsilon} - \frac{|\partial \Lambda|^2}{(\Lambda + \epsilon)^2} - \frac{R^{(2)}_{\alpha \beta} (g^{\alpha i} f^i_1 f^j_1) (g^{\beta k} f^k_1 f^l_1)}{\Lambda + \epsilon} - \frac{|\nabla df|^2}{(\Lambda + \epsilon)^2} \quad (3.1)
\]

By the definition of \( \lambda_\omega \) in Notation 2.1 and Royden’s trick, we get

\[
\Delta_\omega \log(\Lambda + \epsilon) \geq \lambda_\omega \cdot \frac{\Lambda}{\Lambda + \epsilon} - f^* \kappa_\eta \cdot \Lambda \cdot \frac{\Lambda}{\Lambda + \epsilon} + \frac{|\nabla df|^2}{(\Lambda + \epsilon)^2} - \frac{|\partial \Lambda|^2}{(\Lambda + \epsilon)^2} \quad (3.2)
\]

For the last two terms in (3.2),

\[
\frac{|\nabla df|^2}{\Lambda + \epsilon} - \frac{|\partial \Lambda|^2}{(\Lambda + \epsilon)^2} = \frac{|\nabla df|^2}{\Lambda + \epsilon} - \frac{|\partial |df|^2|^2}{(\Lambda + \epsilon)^2} = \frac{|\nabla df|^2}{\Lambda + \epsilon} - \frac{4|\partial |df|^2|^2}{(\Lambda + \epsilon)^2} \geq 0 \quad (3.3)
\]

Set \( V := \{ x \in X | \text{tr}_x f^* \eta = 0 \text{ at } x \} = \{ x \in X | \partial f = 0 \text{ at } x \} \), which, when \( f \) is non-constant, is a proper subvariety (may be empty) of \( Y \). Now, as in [23] Lemma 4.2, we apply Kato inequality in (3.3) to conclude that, outside \( V \),

\[
\frac{|\nabla df|^2}{\Lambda + \epsilon} - \frac{|\partial \Lambda|^2}{(\Lambda + \epsilon)^2} \geq \epsilon |\nabla df|^2 \quad (3.4)
\]

Putting which into (3.2) gives, on \( X \setminus V \),

\[
\Delta_\omega \log(\Lambda + \epsilon) \geq \lambda_\omega \cdot \frac{\Lambda}{\Lambda + \epsilon} - f^* \kappa_\eta \cdot \Lambda \cdot \frac{\Lambda}{\Lambda + \epsilon} \quad (3.4)
\]

Note that both sides of (3.4) are continuous functions on \( X \) and \( V \) is a proper subvariety, therefore, by continuity (3.4) in fact holds on the whole \( X \).
Next, we fix a \( \phi \in C^\infty(X, \mathbb{R}) \) such that \( e^\phi \omega \) is a Gauduchon metric on \( X \) (see subsection 2.3).

Integrating (3.4) with respect to \( e^{(n-1)\phi} \omega^n \) over \( X \) gives

\[
\int_X \frac{\Lambda}{\Lambda + \epsilon} (\lambda - f^*\kappa \cdot \Lambda) e^{(n-1)\phi} \omega^n \leq \int_X (\Delta_\omega \log(\Lambda + \epsilon)) e^{(n-1)\phi} \omega^n. \tag{3.5}
\]

Since \( e^\phi \omega \) is Gauduchon and \( \log(\Lambda + \epsilon) \) is a smooth real function on \( X \), we have

\[
\int_X (\Delta_\omega \log(\Lambda + \epsilon)) e^{(n-1)\phi} \omega^n = \int_X (\Delta e^\phi \omega \log(\Lambda + \epsilon)) e^{(n-1)\phi} \omega^n = 0,
\]

where \( \Delta e^\phi \omega \) is the complex Laplacian with respect to \( e^\phi \omega \). Then we plug it into (3.5) to see that

\[
\int_X \frac{\Lambda}{\Lambda + \epsilon} (\lambda - f^*\kappa \cdot \Lambda) e^{(n-1)\phi} \omega^n \leq 0.
\]

Moreover, we easily have a positive constant \( L \) such that for any \( \epsilon \in (0,1] \),

\[
\left| \frac{\Lambda}{\Lambda + \epsilon} (\lambda - f^*\kappa \cdot \Lambda) \right| \leq L
\]

on \( X \), and as \( \epsilon \to 0^+ \),

\[
\frac{\Lambda}{\Lambda + \epsilon} (\lambda - f^*\kappa \cdot \Lambda) \to \lambda - f^*\kappa \cdot \Lambda
\]

pointwise on \( X \setminus V \) and so pointwise almost everywhere on \( X \) (note that \( V \) is of zero measure with respect to \( e^{(n-1)\phi} \omega^n \)). Therefore, we can apply Lebesgue’s Dominated Convergence Theorem to conclude that

\[
\int_X (\lambda - f^*\kappa \cdot \Lambda) e^{(n-1)\phi} \omega^n = \lim_{\epsilon \to 0} \int_X \frac{\Lambda}{\Lambda + \epsilon} (\lambda - f^*\kappa \cdot \Lambda) e^{(n-1)\phi} \omega^n \leq 0,
\]

from which Theorem 1.1 follows. \( \square \)

Next we prove Theorem 1.2.

**Proof of Theorem 1.2** The proof uses ideas similar to Theorem 1.1. Set \( u := \frac{\sqrt{-1} g^\beta_j dz^j \wedge d\bar{z}^\beta}{\sqrt{-1} g^\beta_j dw^\alpha \wedge d\bar{w}^\alpha} \) (where we have fixed local holomorphic charts \((z^1, ..., z^n)\) on \( X \) and \((w^1, ..., w^n)\) on \( Y \) and write \( \omega = \sqrt{-1} g^i_j dz^i \wedge d\bar{z}^j \) and \( \eta = \sqrt{-1} \eta_{\alpha\beta} dw^\alpha \wedge d\bar{w}^\beta \)). Let \( \epsilon \) be an arbitrary positive constant. At any point \( x \) with \(|\det J(f)|(x) > 0\), i.e. \( u(x) > 0 \), by
computations in Chern [2] and Lu [10] we have

\[ \Delta_\omega \log(u + \epsilon) = \frac{\Delta_\omega u}{u + \epsilon} - \frac{\epsilon |\partial u|^2}{u(u + \epsilon)^2} + \frac{\epsilon |\partial u|^2}{u(u + \epsilon)^2} = \frac{u}{u + \epsilon} (\Delta_\omega \log u) + \frac{\epsilon |\partial u|^2}{u(u + \epsilon)^2} \]

\[ = \frac{u}{u + \epsilon} (-tr_\omega (-f^*Ric(\eta) + Ric(\omega)) + \frac{\epsilon |\partial u|^2}{u(u + \epsilon)^2} = \frac{u}{u + \epsilon} (-tr_\omega f^*Ric(\eta) + R_\omega) + \frac{\epsilon |\partial u|^2}{u(u + \epsilon)^2} \]

\[ \geq \frac{u}{u + \epsilon} (-tr_\omega f^*Ric(\eta) + R_\omega), \quad (3.6) \]

where in the fourth equality we have used \( \partial\bar{\partial} \log |\det J(f)|^2 = 0 \) at \( x \) whenever \( \det J(f)(x) \neq 0 \).

Set \( W := \{ x \in X | \det J(f) = 0 \text{ at } x \} \), which, as \( f \) is non-degenerate, is a proper subvariety (may be empty) of \( X \). Then, the above discussions mean that \( (3.6) \) holds on \( X \setminus W \) and by continuity we know it holds on the whole \( X \).

As before, we fix a \( \psi \in C^\infty(X, \mathbb{R}) \) such that \( e^\psi \omega \) is a Gauduchon metric on \( X \). Therefore,

\[ \int_X \frac{u}{u + \epsilon} (-tr_\omega f^*Ric(\eta) + R_\omega) e^{(n-1)\psi} \omega^n \leq \int_X (\Delta_\omega \log(u + \epsilon)) e^{(n-1)\psi} \omega^n. \]

Now, we can use the same arguments as in Theorem 1.1 to complete the proof.

Theorem 1.2 is proved. \( \square \)

4. Applications

This section contains several applications of our integral inequalities.

4.1. Rigidity theorems for holomorphic maps. We will apply Theorem 1.1 to obtain several rigidity theorems. For example, if we assume \( (X, \omega) \) is a compact Kähler manifold of positive Ricci curvature (implying that \( \int_X \lambda_\omega \omega^n > 0 \)) and \( (Y, \eta) \) is a Kähler manifold of nonpositive holomorphic sectional curvature (implying that \( \kappa_\eta \leq 0 \) on \( Y \)), then we easily recover the aforementioned fundamental rigidity theorems of Yau [24] and Royden [12] from Theorem 1.1.

We may particularly mention that, in general, the function \( \lambda_\omega \) in Theorem 1.1 may has different signs at different points and so the second Ricci curvature may not be nonnegatively signed. Therefore, Theorem 1.1 seems very flexible in applications, and implies new rigidity theorems even in the case that the curvatures of both the domain and target manifolds are not assume to be signed in pointwise sense. Theorem 1.1 and the following applications indicate that the total first eigenvalue of second Ricci curvature should be essential in deriving rigidity theorems for holomorphic maps.

Corollary 4.1. A holomorphic map from a compact Hermitian manifold of quasi-positive second Ricci curvature to a compact Kähler manifold of almost nonpositive holomorphic sectional curvature must be constant.

Here, \( \omega \) has quasi-positive second Ricci curvature if and only if \( \lambda_\omega \geq 0 \) on \( X \) and \( \lambda_\omega > 0 \) at some point in \( X \).
Proof. Assume a contradiction that there is a non-constant holomorphic map $f : X \to Y$ between two compact complex manifolds, where $X$ admits a Hermitian metric $\omega$ of quasi-positive second Ricci curvature and $X$ is a compact Kähler manifold of almost nonpositive holomorphic sectional curvature. Fix $\phi \in C^\infty(X, \mathbb{R})$ such that $e^{\phi} \omega$ is Gauduchon. Since $Ric^{(2)}(\omega)$ is quasi-positive, we know

$$\int_X \lambda \omega e^{(n-1)\phi} \omega^n =: \delta_0 > 0.$$  

Since $Y$ is a compact Kähler manifold of almost nonpositive holomorphic sectional curvature, we fix a sequence of Kähler metrics $\eta_i$, $i = 0, 1, 2, \ldots$, on $Y$ such that $(\sup_X H^n_i) \cdot [\eta_i] < \delta_0$. We may assume $\sup_X H^n_i > 0$ for every $i$. Then $\kappa_{\eta_i} \leq \sup_X H^n_i$ on $Y$, and so

$$n \int_X f^* \kappa_{\eta_i} \cdot e^{(n-1)\phi} f^* \eta_i \wedge \omega^{n-1} \leq n \int_X (\sup_X H^n_i) \cdot e^{(n-1)\phi} f^* \eta_i \wedge \omega^{n-1}.$$  

Recall that $e^{\phi} \omega$ being Gauduchon implies that the integration on the above right hand side depends only on the class of $\eta_i$. Therefore,

$$n \int_X f^* \kappa_{\eta_i} \cdot e^{(n-1)\phi} f^* \eta_i \wedge \omega^{n-1} \leq n \int_X (\sup_X H^n_i) \cdot e^{(n-1)\phi} f^* \eta_i \wedge \omega^{n-1} \leq \frac{\delta_0}{2} \leq \int_X \lambda \omega e^{(n-1)\phi} \omega^n$$

for sufficiently large $i$, which contradicts Theorem 1.1.

Corollary 4.1 is proved. \hfill \Box

Corollary 4.2. A holomorphic map from a compact Gauduchon manifold with positive total first eigenvalue of second Ricci curvature to a compact Kähler manifold of almost nonpositive holomorphic sectional curvature must be constant.

Corollary 4.3. A holomorphic map from a compact Gauduchon manifold with zero total first eigenvalue of second Ricci curvature to a Kähler (resp. Hermitian) manifold of negative holomorphic sectional (resp. bisectional) curvature must be constant.

Remark 4.4. (1) In Corollary 4.2 we do not assume any pointwise curvature signs for both the domain and target spaces.

(2) In Corollaries 4.1 and 4.2 the same conclusions hold if the target manifold is assumed to be a (not necessarily compact) Hermitian manifold of nonpositive holomorphic bisectional curvature (or real bisectional curvature, a new curvature notion recently introduced in [23]).

(3) It seems that Corollaries 4.1 and 4.2 are new even if we assume the target space is a Kähler (resp. Hermitian) manifold of nonpositive holomorphic sectional (resp. bisectional) curvature.

(4) Corollary 4.2 is somehow optimal if we further assume the target compact Kähler manifold is of nonpositive holomorphic sectional curvature. More precisely, for any compact Kähler manifold $Y$ of nonpositive holomorphic sectional curvature, there exist a compact Kähler manifold $X$ of zero total first eigenvalue of Ricci curvature and a non-constant
holomorphic map \( f: X \to Y \). In fact, since by [24, 17] (also see [28] for a generalization to almost setting), a compact Kähler manifold of (almost) nonpositive holomorphic sectional curvature has nef canonical line bundle, then by using the nef reduction we have a fibration \( \pi: Y \to Z \) with regular fiber \( Y_z \), up to a finite unramified covering, a flat complex torus (see [7] for details). Then we may choose \( X = Y_z \) and \( f: X \to Y \) the holomorphic embedding.

(5) A very special case of Corollary 4.2 is that a Kähler manifold of almost nonpositive holomorphic sectional curvature contains no rational curves, which has been observed in our previous work [28, Theorem 1.10] by using a result of Tosatti-Y.G. Zhang [18]. Our argument for Corollary 4.2 here provides an alternative proof for it.

4.2. Degeneracy theorems for holomorphic maps. As another natural generalization of Schwarz-Pick-Ahlfors Lemma to higher dimensions, one may also compare the volume forms related by a holomorphic map. The most classical works include Chern [2], Lu [10] and Yau [24] etc.. In particular, Yau proved that a non-degenerate holomorphic map from a compact Kähler manifold of scalar curvature bounded from below by a negative number to a Hermitian manifold of Chern Ricci curvature bounded from above by a negative constant decreases volume forms (up to multiplying a constant factor depending only on the bounds of curvatures), assuming the domain and target manifolds are of the same dimension. Consequently, a holomorphic map from an \( n \)-dimensional compact Kähler manifold of nonnegative scalar curvature to an \( n \)-dimensional Hermitian manifold of Chern Ricci curvature bounded from above by a negative constant must be totally degenerate. More recent developments can be found in [14, 11] etc..

Here we would like to prove some degeneracy theorems using the integral inequality in Theorem 1.2. Given Theorem 1.2 to get degeneracy theorems, we just need to assume curvature conditions that will destroy (1.4). Again, noting that it is an integral inequality, the curvatures may be assumed to be signed only in certain integral or almost sense, not necessarily in pointwise sense.

**Corollary 4.5.** A holomorphic map from an \( n \)-dimensional compact Gauduchon manifold of zero total Chern scalar curvature to an \( n \)-dimensional compact Kähler manifold of semiample canonical line bundle and positive Kodaira dimension must be totally degenerate.

**Proof.** Assume a contradiction that there is a non-degenerate holomorphic map \( f: X \to Y \) between two \( n \)-dimensional compact complex manifolds, where \( X \) admits a Gauduchon metric \( \omega \) of zero total Chern scalar curvature, i.e.

\[
\int_X R_{\omega} \omega^n = 0,
\]

and \( Y \) is a compact Kähler manifold of semiample canonical line bundle and positive Kodaira dimension. Since \( f \) is non-degenerate at some point \( x_0 \in X \), \( f|_U: U \to f(U) \) is biholomorphic for some open neighborhood \( U \) of \( x_0 \). Let \( \pi: Y \to \pi(Y) \subset \mathbb{CP}^N \) be the semiample fibration induced by the pluricanonical linear system of \( Y \), where the Kodaira dimension of \( Y \) equals to \( \dim_{\mathbb{C}}(\pi(Y)) \geq 1 \). Then we fix \( \chi = \pi^* \tilde{\omega} \in 2\pi c_1(K_X) \), where \( \tilde{\omega} \) is a multiple of Fubini-Study metric on \( \mathbb{CP}^N \), and by Yau’s fundamental theorem [25] we fix a Kähler metric \( \eta \) on \( Y \) such that \( \text{Ric}(\eta) = -\chi \). Since \( \chi \) is semipositive on \( Y \) and, at a generic point, \( \chi \) is positive in some directions, up to shrinking \( U \) (and so \( f(U) \)) we may
assume $\chi$ has positive directions for every point in $f(U)$. Therefore,
\[
\begin{align*}
    n \int_X f^*(\text{Ric}(\eta)) \wedge \omega^{n-1} &= -n \int_X f^*\chi \wedge \omega^{n-1} \\
    &\leq -n \int_U f^*\chi \wedge \omega^{n-1} \\
    &< 0 = \int_X R_\omega \omega^n,
\end{align*}
\]
contradicting (1.4).

Corollary 4.5 is proved.

Similar arguments can be used to prove

**Corollary 4.6.** A holomorphic map from an $n$-dimensional compact Gauduchon manifold of positive total Chern scalar curvature to an $n$-dimensional compact Kähler manifold of nef canonical line bundle must be totally degenerate.

**Remark 4.7.** We should point out that Corollary 4.6 is essentially not new. In fact, by a recent work of Yang [20, Theorem 4.1], a compact complex manifold admitting a Gauduchon metric of positive total Chern scalar curvature also admits a Hermitian metric of (pointwise) positive Chern scalar curvature. Therefore, using Yang’s result, Corollary 4.6 follows directly from the classical maximum principle arguments in [2, 10, 25].

Here, using the integral inequality (1.4), we shall provide an alternative proof for Corollary 4.6 without involving Yang’s result.

**Proof.** Assume a contradiction that there is a non-degenerate holomorphic map $f : X \to Y$ between two $n$-dimensional compact complex manifolds, where $X$ admits a Gauduchon metric $\omega$ of positive total Chern scalar curvature, i.e.
\[
\int_X R_\omega \omega^n =: \delta_0 > 0,
\]
and $Y$ is a compact Kähler manifold of nef canonical line bundle. By the definition of nefness and Yau’s fundamental theorem [25] we may fix a sequence of Kähler metrics $\eta_i, i = 0, 1, 2, \ldots,$ on $Y$ such that
\[
Ric(\eta_i) \leq i^{-1}\eta_0.
\]
Then we easily see that
\[
\begin{align*}
    n \int_X f^*(\text{Ric}(\eta_i)) \wedge \omega^{n-1} &\leq \frac{n}{i} \int_X f^*(\eta_0) \wedge \omega^{n-1} \\
    &\leq \frac{\delta_0}{2} \\
    &< \int_X R_\omega \omega^n
\end{align*}
\]
for sufficiently large $i$, contradicting (1.4).

Corollary 4.6 is proved. \qed
4.3. Infinite-time singularity types of the Kähler-Ricci flow. We would like to discuss how our integral inequality (1.3) relates to the study of infinite-time singularity types of the Kähler-Ricci flow. Let \((Y, \eta_0)\) be an \(m\)-dimensional compact Kähler manifold. Consider the Kähler-Ricci flow \(\eta = \eta(t), t \in [0, T)\) on \(Y\) running from \(\eta_0\):
\[
\begin{align*}
\partial_t \eta(t) &= -\text{Ric}(\eta(t)) \\
\eta(0) &= \eta_0.
\end{align*}
\] (4.1)

We assume the canonical line bundle \(K_Y\) of \(Y\) is nef, which is equivalent to that the Kähler-Ricci flow running from an arbitrary Kähler metric can be smoothly solved for all \(t \in [0, \infty)\) (see [1, 13, 19]). Recall from Hamilton [6] that the infinite-time singularities of the Kähler-Ricci flow are divided into three types. Precisely, a long-time solution \(\eta(t), t \in [0, \infty)\), to the Kähler-Ricci flow (4.1) is of

- type IIb if
  \[
  \limsup_{t \to \infty} \left( \sup_Y |t| \text{Rm}(\eta(t)) |_{\eta(t)} \right) = \infty,
  \]

- type IIIa if
  \[
  \limsup_{t \to \infty} \left( \sup_Y |t| \text{Rm}(\eta(t)) |_{\eta(t)} \right) \in (0, \infty),
  \]

- type IIIb if
  \[
  \limsup_{t \to \infty} \left( \sup_Y |t| \text{Rm}(\eta(t)) |_{\eta(t)} \right) = 0.
  \]

The infinite-time singularity types are about the long-time boundedness of curvature tensor of the Kähler-Ricci flow, which are crucial in understanding the singularity models of the Kähler-Ricci flow. There are many progresses in classifying infinite-time singularity types of the Kähler-Ricci flow in recent years, assuming the Abundance Conjecture (i.e. the canonical line bundle is semiample), see [4, 8, 18, 26, 27, 3]. In the surface case, a complete classification is obtained by Tosatti-Y.G.Zhang [18]. In general, without assuming the Abundance Conjecture, it seems not much progresses in determining the singularity type of the Kähler-Ricci flow. A conjecture raised by Tosatti [15, Conjecture 6.7] predicts that for the Kähler-Ricci flow on any compact Kähler manifold with nef canonical line bundle, the infinite-time singularity type does not depend on the choice of the initial metric, which is confirmed in our previous work [26 Corollary 1.5] in 3-dimensional case.

Here we would like to first recall a useful criterion for type IIb singularities due to Tosatti-Y.G.Zhang [18] without assuming the Abundance Conjecture.

**Proposition 4.8.** [18, Proposition 1.4] Let \(Y\) be a compact Kähler manifold of nef canonical line bundle. Assume there is a non-constant holomorphic map \(f : \mathbb{C}P^1 \to Y\) such that \(\int_{\mathbb{C}P^1} f^* c_1(Y) = 0\), then any solution to the Kähler-Ricci flow (4.1) on \(Y\) must be of type IIb.

In their proof [18, Section 4, Remark 4.1] for Proposition 4.8, an integral inequality for a non-constant holomorphic map \(f : \mathbb{C}P^1 \to Y\) is derived using “\(\epsilon\)-regularity argument”. Our Theorem 1.1 provides an alternative argument for [18, Section 4, Remark 4.1], and generalizes it to a more general setting. Consequently, we can also generalize the criterion in Proposition 4.8 as follows.

**Corollary 4.9.** Let \(Y\) be a compact Kähler manifold of nef canonical line bundle. Assume there is a non-constant holomorphic map \(f : (X, \omega) \to Y\), where \((X, \omega)\) is a compact Gauduchon manifold of positive total second Ricci curvature, then any solution to the
Kähler-Ricci flow \( (1.1) \) on \( Y \) must be of type IIb or type IIIa. If furthermore \( \int_X f^*c_1(Y) \wedge \omega^{n-1} = 0 \) (where \( n = \dim \mathbb{C}X \)), then any solution to the Kähler-Ricci flow \( (1.1) \) on \( Y \) must be of type IIb.

**Proof.** The proof uses the ideas in Corollaries 4.1 and 4.2. Given a long-time solution \( \eta(t) \) to the Kähler-Ricci flow \( (1.1) \) on \( Y \). Similar to Corollaries 4.1 and 4.2, we have

\[
\sup_Y H^{\eta(t)} \geq \frac{\int_X \lambda_\omega \omega^n}{n \int_X f^*\eta(t) \wedge \omega^{n-1}} = \frac{n^{-1} \int_X \lambda_\omega \omega^n}{\int_X f^*\eta_0 \wedge \omega^{n-1} + t \cdot 2\pi \int_X f^*c_1(K_Y) \wedge \omega^{n-1}},
\]

where we have used that \([\eta(t)] = [\eta_0] + t \cdot 2\pi c_1(K_Y)\) and \( \omega \) is Gauduchon. Therefore, as \( t \to \infty \),

- \( \limsup_{t \to \infty} \sup_Y tH^{\eta(t)} \geq \frac{n^{-1} \int_X \lambda_\omega \omega^n}{\int_X f^*c_1(K_Y) \wedge \omega^{n-1}} > 0 \), if \( \int_X f^*c_1(K_Y) \wedge \omega^{n-1} > 0 \);
- \( \limsup_{t \to \infty} \sup_Y tH^{\eta(t)} = +\infty \), if \( \int_X f^*c_1(K_Y) \wedge \omega^{n-1} = 0 \).

Corollary 4.9 is proved. \( \square \)

**Remark 4.10.** In Corollary 4.9, if \( Y \) is a compact complex manifold and we are given a long-time solution to the Chern-Ricci flow (see [10] Equation (1.1)), say \( \eta(t) \) on \( Y \), then similar curvature estimates hold if we replace \( H^{\eta(t)} \) by \( BK^{\eta(t)} \) and \( c_1(Y) \) by the first Bott-Chern class \( c_1^{BC}(Y) \).

### 4.4. Lower bounds for \( \mu_\alpha \) and nef threshold.

As a final remark, we mention that the integral inequalities (1.3) and (1.4) imply effective estimates for the number \( \mu_\alpha \) defined in Definition 2.3 and the nef threshold \( \nu_\alpha \) of \( \alpha \), where \( \alpha \) is a Kähler class on a compact Kähler manifold \( Y \) and \( \nu_\alpha := \inf \{ s \in \mathbb{R} | 2\pi c_1(K_Y) + s \alpha \text{ is nef} \} \). Namely,

- given any non-constant holomorphic map \( f : (X, \omega) \to Y \) with \( (X, \omega) \) a compact Gauduchon manifold, we have
  \[ \mu_\alpha \geq \frac{\int_X \lambda_\omega \omega^n}{n \int_X f^*\alpha \wedge \omega^{n-1}}. \]
- given any non-degenerate holomorphic map \( f : (X, \omega) \to Y \) with \( (X, \omega) \) a compact Gauduchon manifold and \( \dim \mathbb{C}X = \dim \mathbb{C}Y \), we have
  \[ \nu_\alpha \geq \frac{\int_X R_\omega \omega^n}{n \int_X f^*\alpha \wedge \omega^{n-1}}. \]

Therefore, we have obtained the desired effectiveness results discussed in Subsection 1.2 which motivate our study, in a much more general setting. These may also be regarded as quantitative versions of Corollaries 4.2 and 4.6 respectively.

### 4.5. Characterizing the equality case.

For integral inequality (1.3) in Theorem 1.1, we may wonder: can we conclude any particular properties if the equality is achieved? While that inequality always holds, we expect that the equality case would give restrictions on the involved data. To be more precise, let’s focus on a special setting as follows.

In Theorem 1.1, if we choose \( X = Y \) be an \( n \)-dimensional compact Kähler manifold and \( f = Id_X : X \to X \) the identity map, then for any Kähler metrics \( \omega \) and \( \eta \) on \( X \) we have

\[
\int_X \lambda_\omega \omega^n \leq n \int_X \kappa_\eta \cdot \eta \wedge \omega^{n-1}. \quad (4.2)
\]
Then it may be natural to ask:

**Question 4.11.** Can we conclude any particular properties if the equality in (4.2) is achieved by two Kähler metrics \( \omega \) and \( \eta \) on \( X \)?

It is possible to answer the above question for some special settings.

**Proposition 4.12** (\( \eta \) being a Kähler-Einstein metric). Let \( (X, \eta) \) be an \( n \)-dimensional compact Kähler-Einstein manifold. For any given Kähler metric \( \omega \) on \( X \) we have

\[
\int_X \lambda \omega^n \leq n \int_X \kappa_\eta \cdot \eta \wedge \omega^{n-1},
\]

and the equality holds for some \( \omega \) if and only if both \( \eta \) and \( \omega \) are of zero or negative constant holomorphic sectional curvature.

**Proof.** Firstly, recall a classical result of Berger: for a Kähler metric \( \eta \) on \( X \) and any point \( x \in X \), we have

\[
\int_{W \in \mathbb{CP}^{n-1}} H^\eta_x(W) \omega_{FS}^{n-1}(W) = \frac{2}{n(n+1)} R_\eta(x),
\]

where \( \omega_{FS} \) is the Fubini-Study metric on \( \mathbb{CP}^{n-1} \) with \( \int_{\mathbb{CP}^{n-1}} \omega_{FS}^{n-1} = 1 \) and \( H^\eta_x \) is regarded as a function defined on \( \mathbb{CP}^{n-1} \). Combining the definition of \( \kappa_\eta \) in Notation 2.2(2) implies

\[
n \kappa_\eta \cdot \eta \geq \frac{n+1}{2} H^\eta \cdot \eta \geq \frac{1}{n} R_\eta \eta = \text{Ric}(\eta),
\]

where the first inequality is strict at those points \( x \) with \( H^\eta_x > 0 \) and in the last equality we used \( \eta \) is Kähler-Einstein. Now, if there is a Kähler metric \( \omega \) satisfying the equality in (4.7), we see

\[
\int_X \lambda \omega^n = n \int_X \kappa_\eta \cdot \eta \wedge \omega^{n-1} \geq \int_X \text{Ric}(\eta) \wedge \omega^{n-1} = \int_X \text{Ric}(\omega) \wedge \omega^{n-1},
\]

implying that

\[
\text{Ric}(\omega) = \lambda \omega
\]

and hence

\[
n \kappa_\eta \cdot \eta = \text{Ric}(\eta).
\]

It turns out that the inequalities in (4.5) are all equalities. Therefore, the holomorphic sectional curvature of \( \eta \) is nonpositive and pointwise constant, equaling to \( \frac{2}{n(n+1)} R_\eta \), and hence is a constant, since \( \eta \) is Kähler-Einstein.

On the other hand, the (4.6) implies that \( \lambda_\omega \) is smooth, and then by differentiating (4.6) we see \( d\lambda_\omega \equiv 0 \), i.e. \( \lambda_\omega \) is a constant and \( \omega \) is a Kähler-Einstein metric on \( X \). Therefore, \( \omega \) is also of constant holomorphic sectional curvature, since \( (X, \eta) \) is a compact complex space form of zero or negative curvature. Note that in the negative curvature case, by uniqueness of negative Kähler-Einstein metric, \( \omega \) must be proportional to \( \eta \). \( \square \)

By almost identical arguments, we also have

**Proposition 4.13** (\( \eta \) being a cscK metric). Let \( (X, \eta) \) be an \( n \)-dimensional compact Kähler manifold of constant scalar curvature. For any given Kähler metric \( \omega \) on \( X \) we have

\[
\int_X \lambda \omega^n \leq n \int_X \kappa_\eta \cdot \eta \wedge \omega^{n-1},
\]

and the equality holds for some \( \omega \in [\eta] \) if and only if \( \eta \) is of zero or negative constant holomorphic sectional curvature and \( \omega = \eta \).
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