Groupoids and the tomographic picture of quantum mechanics

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Abstract

The existing relation between the tomographic description of quantum states and the convolution algebra of certain discrete groupoids represented on Hilbert spaces will be discussed. The realizations of groupoid algebras based on qudit, photon-number (Fock) states and symplectic tomography quantizers and dequantizers will be constructed. Conditions for identifying the convolution product of groupoid functions and the star product arising from a quantization–dequantization scheme will be given. A tomographic approach to construct quasi-distributions out of suitable immersions of groupoids into Hilbert spaces will be formulated and, finally, intertwining kernels for such generalized symplectic tomograms will be evaluated explicitly.

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1. Introduction

Symmetries of quantum and classical systems play an important role in studying their properties [1, 2]. Symmetries are usually associated with ‘symmetry groups’ and in the case of continuous variables with Lie groups. Other mathematical structures very close to groups, such as semigroups and groupoids, have been found to be relevant for describing other properties of a physical system. Semigroups are very important for describing open quantum systems. As for groupoids, we may quote, for instance, from the section ‘Algebra of Physical Quantities’ in the book on Noncommutative Geometry [3] by Alain Connes.

‘The set of frequencies emitted by an atom does not form a group, and it is false that the sum of two frequencies of the spectrum is again one. What experiments dictate is the Ritz–Rydberg combination principle, which permits indexing the spectral lines by the set Δ of all pairs (i, j) of elements of a set I of indices. The frequencies \( v_{ij} \) and \( v_{kl} \) only combine when \( j = k \) to yield \( v_{il} = v_{jj} + v_{jkl} \) (…).

Due to the Ritz–Rydberg combination principle, one is not dealing with a group of frequencies but rather with a groupoid \( \Delta = \{(i,j); i,j \in I\} \) having the composition rule \( (i,j)(j,k) = (i,k) \). The convolution algebra still has a meaning when one passes from a group to a groupoid, and the convolution product may be written

\[
(ab)_{i,k} = \sum_{n} a_{i,n} b_{n,k},
\]

which is identical with the product rule of matrices’.

Thus a key role for the introduction of groupoids in the description of quantum systems is played by the combination principle of frequencies which induces by means of a Fourier expansion, say

\[
a = \sum_{m,n} a_{mn} e^{i(mn)}
\]

and

\[
b = \sum_{k,l} b_{kl} e^{i(kl)}
\]

whose product \( ab \) gives the matrix multiplication rule \( (ab)_{il} = \sum_{n} a_{in} b_{n,k} \), which is just convolution in the groupoid algebra. Then, even if Heisenberg’s matrix mechanics, and its foundations, have only a historical interest today [4], it is nevertheless worthy to discuss the role that groupoids play in the foundations of quantum mechanics because it goes far beyond that of groups.
We would like to mention that the use of groupoids in quantum mechanics has been advocated not only by Connes but by Accardi [5] too.

Recently, we have considered quantum tomography as providing an alternative picture of quantum mechanics [6, 7]. Specifically, we have considered a C*-algebraic approach to quantum tomography where natural instances of C*-algebras are provided by group algebras and the star product is nothing but the convolution product on the group itself. As the convolution product makes sense also on groupoids and, as explained before, the Heisenberg picture originates directly from a groupoid as a carrier space along with the Ritz–Rydberg combination principle, it is quite natural to try to compare the tomographic picture with the Heisenberg one by using the common ground of groupoids.

In quantum mechanics, groups are used practically in all branches of research. Also, semigroups are important for considering evolution of open quantum systems undergoing dissipation or decoherence. The groupoid structure, however, is not so well known in the physicists community, even though there are authoritative examples of its use as we have already stressed. Thus, it seems appropriate from a pedagogical point of view to compare these three different structures: groups, semigroups and groupoids. A binary composition is defined for all elements of a group and a semigroup, while for a groupoid such composition is defined only for particular pairs of elements and is associative when defined. Moreover, any element of a group has an inverse element along with the unit element. In a groupoid there are many different unit elements and any element has its own inverse which may be composed from the left and from the right, yielding different units. In contrast, for a semigroup the unit element is unique, but in general no inverse element exists. Precise definitions and properties of groupoids will be discussed in section 2 and in appendix A.

It is the aim of this paper to review the general ingredients for the existence of a star product on a measure space and to show that, once groupoids are realized in simple terms in Hilbert and Fock spaces, the groupoid algebra convolution coincides with the star product whenever ‘quantizer–dequantizer’ maps are properly defined. This property is the counterpart for groupoids of what happens for groups, there the star product coincides with the group-algebra convolution when quantizer and dequantizer are given by unitary representations of the group. Then such construction will be used to construct quantum tomographies based on groupoid algebras. A more elaborated comparison with tomograms for quantum field theories will be dealt with in future work.

The paper is organized in the following manner. In section 2 we begin with a short discussion of a simple example of groupoid which will be relevant in the whole paper. We will define its groupoid algebra and its convolution product. In section 3 we explain the essential ingredients to construct star products in the quantization–dequantization scheme. In section 4 we compare the two kinds of products present on the groupoid algebra functions, i.e., the convolution product and star products. We will state a proposition giving the conditions for their coincidence. In section 5 we discuss some examples of Hilbert space realizations of groupoids of physical interest satisfying the previous conditions, and a counterexample will also be provided. In section 6 contact is made between certain quasi-distributions arising from realizations of groupoids in the tomographic approach, and well-known tomographic schemes as spin, photon number and symplectic ones. Finally, some conclusions and perspectives are drawn in section 7.

In the mathematical appendix more details and example of groupoids are provided, while the proposition of section 4 is slightly generalized, at least for finite groupoids.

2. A friendly introduction to groupoids

We will start discussing the pair-groupoid of a set, a relevant example for the physical situations considered afterwards and arguably the simplest example of a groupoid, leaving a more rigorous and complete treatment of groupoids to the appendix. In following this example, it is convenient to bear in mind the Ritz–Rydberg combination principle.

Given a set $S$ consider the Cartesian product

$$\Gamma = S \times S = \{(x, y), x, y \in S\}.$$  \hfill (1)

$\Gamma$ may be given the structure of a groupoid as follows. We define first the set $G_0$ of ‘units’ of $\Gamma$ as

$$G_0 = \{(x, x)\}_{x \in S} \subseteq \Gamma$$ \hfill (2)

and two maps, $r$ and $s$, from $\Gamma$ onto $G_0$

$$r, s : \Gamma \to G_0, \quad r(x, y) = (x, x), \quad s(x, y) = (y, y).$$ \hfill (3)

The composition law (product) $\circ$ of two elements $(x, t), (z, y)$ of $\Gamma$ is defined if and only if $t = z$, i.e. $s(x, t) = r(z, y)$, and reads

$$(x, t) \circ (t, y) = (x, y).$$ \hfill (4)

The Ritz–Rydberg frequency indices obey the same composition law. The product $\circ$ is associative, as one can readily check and $r(x, y)$ and $s(x, y)$ are the left and right unity for $(x, y)$, respectively.

Finally, every element $(x, y)$ has a (left and right) inverse given by

$$(x, y)^{-1} = (y, x)$$ \hfill (5)

and one has

$$(x, y) \circ (y, x) = (x, x), \quad (y, x) \circ (x, y) = (y, y).$$ \hfill (6)

Moreover,

$$(y, x) \circ ((x, y) \circ (y, z)) = (y, z),$$

$$(z, x) \circ (y, x)) \circ (y, x) = (z, x).$$ \hfill (7)

Consider the map

$$(r, s) : \Gamma \to G_0 \times G_0, \quad (x, y) \mapsto ((x, x), (y, y)).$$ \hfill (8)

This map is both one-to-one and onto, so the groupoid $\Gamma$ is said to be principal and transitive, respectively.

A function on $\Gamma$ is a (real or complex) function $f(x, y)$. When $S$ is a finite or a countable set, say $\{s_i\}$ with $i, k = 1, \ldots, N \leq \infty$, a function $f(x_i, x_k)$ may be thought of as a finite or an infinite matrix. By thinking of $f(x_i, x_k)$ in these terms makes available the row-by-column product. It is now
easy to see that by using an irreducible groupoid realization, the matrices we associate with any function constitute a realization of the groupoid algebra. Let us expand this statement.

We use an irreducible groupoid realization: the element \((x_i, x_k)\) is associated with the \(N \times N\)-matrix \(E_{ik}\), whose entries are zero, but the entry \(ik\) which is one:

\[
(E_{ik})_{jl} = \delta_{ij}\delta_{kl},
\]

(9)

Moreover,

\[
E_{ik}E_{jl} = \delta_{kj}E_{il},
\]

(10)

so that if \((x_i, x_k)\) and \((x_j, x_l)\) are not composable the product of their representative matrices is zero. This generalizes a group realization to the groupoid case. The transpose \(E^T_{ik}\) of \(E_{ik}\) is \(E_{kj}\), associated with \((x_k, x_j)\), the inverse of \((x_i, x_k)\). The matrices \(E_i\)'s are an orthonormal basis of the corresponding Hilbert–Schmidt space, as

\[
\text{Tr} [E_{ik}E_{ij}^T] = \delta_{ij}\delta_{kl}.
\]

(11)

In the present case, they are just the ‘units’ defined by Weyl [8] long ago, while introducing the concept of group algebra and its realizations. They provide an irreducible realization of the groupoid algebra functions in terms of \(N \times N\)-matrices

\[
f \mapsto A_f, \quad A_f := \sum_{i,k=1}^N f(x_i, x_k)E_{ik}.
\]

(12)

We also notice, in passing, that these aspects contain the basic ingredients of the Schwinger’s measurements algebra [9]. The groupoid algebra can be defined, analogously to Weyl’s definition of a group algebra, as the algebra of \(N\)-dim vectors, associated with the functions on a groupoid. So, the vector \(F\), associated with the function \(f\), has component \(f(x_i, x_k)\) in the ‘direction’ \((x_i, x_k)\). Such a vector can be expressed as a formal linear combination

\[
F = \sum_{i,k=1}^N [f(x_i, x_k)](x_i, x_k),
\]

(13)

so that the groupoid realization in terms of units \((x_i, x_k) \mapsto E_{ik}\) yields at once the irreducible realization of the groupoid algebra of equation (12). The groupoid algebra product can be defined as

\[
F_1F_2 = \sum_{i,j,k,l=1 \text{ allowed }}^N [f_1(x_i, x_j)f_2(x_k, x_l)](x_i, x_j) \circ (x_k, x_l).
\]

(14)

As a consequence, the related algebra of groupoid functions is equipped with a convolution product, which reads

\[
(f_1 \ast f_2)(x_n, x_m) = \sum_{i,j,k,l=1}^N f_1(x_i, x_j)f_2(x_k, x_l).
\]

(15)

Taking into account the fact that \((x_i, x_j) \circ (x_k, x_l)\) is defined only if \(j = k\) and \((x_i, x_k) \circ (x_k, x_l) = (x_n, x_m)\) gives \(i, l = m\), these constraints may be implemented in terms of Kronecker delta symbols, and the convolution formula becomes

\[
(f_1 \ast f_2)(x_n, x_m) = \sum_{i,j,k,l=1}^N f_i(x_j, x_l)f_k(x_j, x_l)\delta_{in}\delta_{kl}\delta_{jm} = \sum_{k=1}^N f_1(x_n, x_k)f_2(x_k, x_m).
\]

(16)

In terms of the representative matrices \(A\), the above convolution formula is realized as the usual row-by-column matrix product. In view of equation (10), we obtain

\[
A_{f_1}A_{f_2} = \sum_{i,j,k,l=1}^N f_i(x_j, x_k)f_j(x_l, x_l)E_{ik}E_{jl} = \sum_{i,j,k,l=1}^N f_i(x_j, x_k)f_j(x_l, x_l)E_{il} = \sum_{i,j,k,l=1}^N (f_1 \ast f_2)(x_i, x_j)E_{il} = A_{f_1 \ast f_2}.
\]

(17)

In conclusion, as for groups, the groupoid structure yields the possibility of introducing an algebra structure in the set \(F(\Gamma)\) of the groupoid functions. Besides, for this kind of principal groupoids, functions are associated with numerical matrices, the groupoid algebra can be realized as the algebra of \(N \times N\) matrices and convolution is the usual matrix product.

Finally, when the groupoid is not countable, the convolution formula (16) entails an integration over the groupoid with respect to a suitable measure. For instance, in the continuous case \(S = \mathbb{R}\), it is natural to generalize matrix multiplication to a ‘continuous matrix multiplication’

\[
(f_1 \ast f_2)(x, y) = \int_{\mathbb{R}} f_1(x, t)f_2(t, y)dt,
\]

(18)

where \(dt\) is the translation-invariant Lebesgue measure and \(f_1, f_2\) are continuous functions of proper support [10], we postpone further details to appendix C.

3. A general framework for a ★ product

Given a vector space \(V\) (for the moment finite-dimensional) and its dual \(V^*\), we can consider immersions of a measure space \((X, dx)\) into \(V\) and \(V^*\), say \(x \mapsto v(x)\) and \(x \mapsto \alpha(x)\), satisfying the relation (which is written using different notation)

\[
\mathbb{I}_V = \int_X v(x)\, dx \alpha(x) \equiv \int_X v_x \, dx \alpha_x \equiv \int_X v_x \otimes \alpha_x \, dx,
\]

(19)

i.e. we can construct a resolution of the identity \(\mathbb{I}_V \in \text{End}(V)\) with the property

\[
\alpha_x = \int_X \alpha_x (v_x) \, dx.
\]
Once the previous immersions are given, it is possible to associate with any vector \( w \in V \) a function \( f_w \) on \( X \) as
\[
 f_w (x) := \alpha_x (w) .
\] (20)

With any function \( f \) in the image of this map we may associate a vector \( w_f \) in \( V \) by setting
\[
 w_f := \int_X f (x) \, v_x \, dx .
\] (21)

Assuming, for simplicity, that any function on \( X \) is in the image of the map \( w \mapsto f_w \), it is possible to endow the space of functions on \( X \) with all additional structures we may have on \( V \). For instance, if \( V \) is a Lie algebra with Lie bracket \( [\cdot, \cdot] \) we may define
\[
 [f_{v_1}, f_{v_2}] := f_{[v_1, v_2]} .
\] (22)

Similarly, if \( V \) is an associative algebra with product \( B : V \times V \rightarrow V \), \( v \cdot u = B (v, u) \in V \) and \( v_i \) is a linear basis, then
\[
 v_j \cdot v_k = \sum_i c_{jk}^i v_i ,
\] (23)

and we may define
\[
 f_{v_j} \cdot f_{v_k} := f_{v_j \cdot v_k} = f_{B (v_j, v_k)} .
\] (24)

This abstract general framework has been extensively analyzed in the previous work [11] under the evocative name of quantizer–dequantizer for the algebra of operators on some Hilbert space \( \mathcal{H} \), i.e. to be more precise, we may consider the Hilbert space \( V \) of Hilbert–Schmidt operators on \( \mathcal{H} \) and, after identifying \( V \) and \( V^* \) in a natural way, the immersions \( \nu(x) \) and \( \alpha(x) \) above will be written as \( \hat{D}(x) \) and \( \hat{U}(x) \), respectively. We recall that in such a case, the dequantizer \( \hat{U}(x) \) and quantizer \( \hat{D}(x) \) give rise to formulae
\[
 f_A (x) = \text{Tr}[\hat{A} \hat{U}^{-1}(x)]
\] (25)

instead of equation (20) and
\[
 \hat{A} = \int_X f_A (x) \, \hat{D}(x) \, dx
\] (26)

instead of equation (21), while equation (19) becomes
\[
 \text{Tr}[\hat{D}(x) \hat{U}^{-1}(x')] = \delta_x (x') .
\] (27)

Symbol functions \( f_A \) may be composed by a star product kernel
\[
 (f_A \ast f_B) (x) = \int f_A (x_1) \, f_B (x_2) \, K (x_1, x_2, x) \, dx_1 \, dx_2 
 = f_{AB} (x) ,
\] (28)

where
\[
 K (x_1, x_2, x) := \text{Tr}[\hat{D}(x_1) \hat{D}(x_2) \hat{U}^{-1}(x)] .
\] (29)

Previous abstract general framework has also been exploited in the tomographic picture of quantum mechanics, where with any vector of the Hilbert space we associate a probability distribution and the space \( X \) is usually associated with the point spectrum of a maximal set of commuting operators. An additional family of parameters is often provided by a Lie group \( G \) acting by similarity transformations on the selected maximal Abelian set of operators, in such a manner that the union of the transformed operators for various elements of \( G \) constitutes a tomographic set.

When the space \( X \) is an Abelian vector group and the immersion is associated with a Weyl system, the outlined scheme includes the Wigner–Weyl picture along with the Moyal formalism [12, 13]. Other instances are provided by (generalized) coherent states [14].

This picture, when the vector space \( V \) is an associative algebra, provides an interesting way to construct associative products on the space of functions \( \mathcal{F}(X) \) by means of a kernel function \( K (x_1, x_2, x_3) \), representing the ‘density’ of the associative product \( B \), which satisfies the quadratic condition originated by associativity.

When the measure space \( (X, dx) \) carries additional structures which are ‘compatible’ with the immersion in the algebra space of operators, there are interesting and useful properties inherited by the constructed star product. For instance, we may look for the group algebra of a Lie group \( G \) and for the convolution product on \( \mathcal{F}(G) \). In this sense, it would be possible to generalize the picture emerging in the Weyl–Wigner formalism to generic groups.

As a particular generalization, because the convolution product makes sense also for functions defined on groupoids, we would like to analyze to what extent the Weyl–Wigner picture generalizes also to groupoids. In summary, the problem we are analyzing may be formulated in the following manner. To study the properties of the star product when the carrier space carries the structure of a groupoid and the immersion into the space of operators may be considered to be compatible with the product structure.

To this aim, we first need to study some properties of groupoids coming back to the example of section 2.

4. Groupoids: \ast convolution and \ast product

To get more insight, we restrict first to the case of the principal transitive pair-groupoid \( \Gamma = S \times S \), with \( S \) a countable set of order \( N < \infty \), discussed in section 2. We begin by observing that the groupoid algebra realization equation (12) looks like a quantization formula, where an operator \( A_f \) is associated with a function \( f \) by means of a quantizer \( E \). This is not a mere analogy: in fact, in view of equation (11), the dequantizer associated with \( E_{ik} \) is just \( E_{ik} \) (compare with equation (29)). In other words, we have a self-dual pair of quantizer–dequantizer. So, we obtain the dequantization formula
\[
 \text{Tr} [A_f E_{1j}] = \text{Tr} \left[ \sum_{i,k=1}^{N} f(x_i, x_k) E_{ik} E_{1j}^T \right]
\]
\[
 = \sum_{i,k=1}^{N} f(x_i, x_k) \delta_{ij} \delta_{kj} = f(x_i, x_j) .
\] (30)

Thanks to the quantization–dequantization scheme based on Weyl units, functions on the groupoid can be multiplied by
the completeness relation
\[ \sum_{m=-j}^{j} |m\rangle \langle m| = \mathbb{I}_j, \]
which eventually gives
\[ \hat{λ} = \sum_{m,m'=-j}^{j} τ'(m,m') |m\rangle \langle m'|, \]
where
\[ τ'(m,m') = Tr[\hat{λ}(|m\rangle \langle m'|)^*] = \langle m| \hat{λ}|m'\rangle. \]

As the Weyl basis operators \(|m\rangle \langle m'|\) are associated with the Weyl units \(E_{mn}\), because of proposition 1 the star product for the symbols \(τ\)’s coincides with the convolution product of the groupoid algebra of \(Γ_1\).

By means of a direct sum over the label \(j\) we may construct the infinite-dimensional case starting with the countable groupoid
\[ Γ = \bigsqcup_{j} Γ_j \]
with composition law
\[ (jm, jm') \circ (j'm', j'm'') = (jm, jm'') \text{ iff } j' = j, \]
and immersion into the space of Hilbert–Schmidt operators \(\mathcal{D}_2 \subset B(\mathcal{H}), \mathcal{H} = \bigoplus \mathcal{H}_j\), given by
\[ (jm, jm') \in Γ \mapsto |jm\rangle \langle jm'| \in B(\mathcal{H}). \]

The groupoid \(Γ\), as disjoint union of groupoids, is principal nontransitive and the immersion decomposes trivially into the finite-dimensional cases previously analyzed. A less trivial infinite-dimensional case is the following.

Let us consider the discrete space \(\mathbb{N}_0 = \{0, 1, 2, \ldots\}\) with counting measure \(μ(n) = 1\) for any natural number \(n\). Consider again the pair-groupoid \(Γ = \mathbb{N}_0 \times \mathbb{N}_0\) with the product measure \(μ \times μ\); it is a principal transitive groupoid with composition rule \((n, m) \circ (m, p) = (n, p)\). Given a complex separable Hilbert space \(\mathcal{H}\), we can immerse the measure space \((Γ, μ \times μ)\) into the Hilbert space \(\mathcal{D}_2\) of Hilbert–Schmidt operators on \(\mathcal{H}\) as
\[ (n, m) \in Γ \mapsto |n\rangle \langle m| \in \mathcal{D}_2, \]
where \(|n\rangle\) and \(|m\rangle\) are Fock boson states, i.e. eigenvectors of the number operator \(\hat{N}\) associated with a harmonic oscillator
\[ \hat{N}|n\rangle = n|n\rangle, \quad (n|m) = δ_{nm}. \]

The completeness relation
\[ \sum_{n} |n\rangle \langle n| = \mathbb{I}, \]
allows for writing
\[ \hat{A} = \sum_{n,m} F_A(n, m) \langle n \rangle \langle m \rangle, \quad (42) \]

\[ F_A(n, m) = \text{Tr}[\hat{A}(\langle n \rangle \langle m \rangle)] = \langle n|\hat{A}|m\rangle, \]
so that the Weyl basis \(|n\rangle \langle m|\) provides again a self-dual pair of quantizer–dequantizer
\[ \hat{D}(n, m) = \hat{U}(n, m) = |n\rangle \langle m|. \quad (43) \]
The corresponding kernel of the star product for the symbols \(F\), given by equation (29), is expressed in terms of Kronecker deltas as
\[ K(n_1, m_1; n_2, m_2; n, m) = \text{Tr}[\hat{D}(n_1, m_1)\hat{D}(n_2, m_2)\hat{U}^\dagger(n, m)] \]
\[ = \delta_{n_1n}\delta_{m_2m}\delta_{m_1m_2}, \quad (44) \]
and the star product reads
\[ (F_A \star F_B)(n, m) = \sum_{n_1, m_1, n_2, m_2} F_A(n_1, m_1)F_B(n_2, m_2) \]
\[ \times K(n_1, m_1; n_2, m_2; n, m) = \sum_{m_1} F_A(n, m_1)F_B(m_1, m), \quad (45) \]
which is just the convolution product of the groupoid algebra \(\Gamma = \mathbb{N}_0 \times \mathbb{N}_0\) given by the usual product of matrices in agreement with the result of proposition 1.

As a second example, we will consider giving an operator \(\hat{A}\), a new symbol \(f_A(x, y)\) defined by
\[ f_A(x, y) = \sum_{n,m} F_A(n, m) \varphi_n(x) \varphi_m^*(y) \]
\[ = \text{Tr} \left[ \sum_{n,m} F_A(n, m) (|n\rangle \langle m|) (|x\rangle \langle y|) \right]^\dagger \]
\[ = \text{Tr}[\hat{A}(\langle x\rangle |\langle y|)^\dagger]. \quad (46) \]
Here \(\varphi_n(x) := \langle x|n\rangle\) is the \(n\)th Hermite function, i.e., the \(n\)th excited state wave function of the harmonic oscillator in the position representation and \(|x\rangle, |y\rangle\) are (improper) eigenvectors of the position operator \(\hat{q}\):
\[ \hat{q} |x\rangle = x |x\rangle, \quad \langle x|y\rangle = \delta (x - y). \quad (47) \]
Vice versa, the old symbol \(F_A(n, m)\) can be recovered from the new one as
\[ F_A(n, m) = \int \text{d}x \text{d}y f_A(x, y) \varphi_n^*(x)\varphi_m(y). \quad (48) \]
The symbol \(f_A(x, y)\) may also be obtained considering the pair-groupoid \(\Gamma = \mathbb{R} \times \mathbb{R}\) with the product measure \(\text{d}x \text{d}y\). We may think of \(\Gamma = \mathbb{R} \times \mathbb{R}\) as a principal transitive groupoid \(\Gamma\) with composition rule \((x, y) \circ (y, z) = (x, z)\) and choose the immersion
\[ (x, y) \mapsto |x\rangle \langle y| \]
in the space of rank-one operators in the Gelfand triple \(S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R})\). Thanks to the completeness relation
\[ \int \text{d}x |x\rangle \langle x| = \mathbb{I}, \quad (50) \]
we may write, for a given operator \(\hat{A}\), formulae similar to (42),
\[ \hat{A} = \int \text{d}x \text{d}y f_A(x, y) |x\rangle \langle y|, \quad (51) \]
\[ f_A(x, y) = \text{Tr}[\hat{A}(\langle x\rangle |\langle y|)] = |x\rangle \langle y|, \quad (52) \]
In other words, the operator \(|x\rangle \langle y|\) provides a self-dual pair of quantizer–dequantizer
\[ \hat{D}(x, y) = \hat{U}(x, y) = |x\rangle \langle y|, \quad (53) \]
satisfying
\[ \text{Tr}[\hat{D}(x, y)\hat{U}^\dagger(x', y')] = \delta (x - x')\delta (y - y'). \quad (54) \]
The kernel of the corresponding star product for the symbols, equation (29), is expressed in terms of Dirac delta functions
\[ K(x_1, y_1; x_2, y_2; x, y) = \text{Tr}[\hat{D}(x_1, y_1)\hat{D}(x_2, y_2)\hat{U}^\dagger(x, y)] \]
\[ = \delta (x - x_1)\delta (y_1 - y_2)\delta (y_2 - y). \quad (55) \]
Thus the star product reads
\[ (f_A \star f_B)(x, y) = \int \text{d}x_1 \text{d}y_1 \text{d}x_2 \text{d}y_2 f_A(x_1, y_1) f_B(x_2, y_2) \]
\[ \times K(x_1, y_1; x_2, y_2; x, y) = \int \text{d}y_1 f_A(x, y_1) f_B(y_1, y). \quad (56) \]
In conclusion, we have recovered the usual product of matrices with continuous labels given by equation (18), which is the convolution product of the groupoid algebra of \(\Gamma = \mathbb{R} \times \mathbb{R}\).

Finally, for the three cases considered above, we remark that the generalization to the multi-mode case \(\Gamma = \times \times \times \times \times \) is straightforward. In fact, as a fourth example, consider a two-mode harmonic oscillator. We have to immerse \(\times \times \) onto a Weyl basis generated by the oscillator eigenstates \(|n_1, n_2| = |n_1\rangle \otimes |n_2\rangle\)
\[ ((n_1, n_2), (m_1, m_2)) \mapsto |n_1, n_2\rangle \langle m_1, m_2|. \quad (57) \]
This immersion realizes the groupoid combination rule
\[ ((n_1, n_2), (m_1', m_2')) \circ ((m_1', m_2'), (m_1, m_2)) \]
\[ = ((n_1, n_2), (m_1, m_2)). \]
\[ ((n_1, n_2), (m_1, m_2)) \]
yielding 0 for the forbidden groupoid products. The self-dual quantizer–dequantizer pair reads
\[ \hat{D}(n_1, n_2; m_1, m_2) = \hat{U}(n_1, n_2; m_1, m_2) = |n_1, n_2\rangle \langle m_1, m_2|, \quad (58) \]
and we obtain at once
\[ \hat{A} = \sum_{n_1, n_2, m_1, m_2} A(n_1, n_2; m_1, m_2) |n_1, n_2\rangle \langle m_1, m_2|, \quad (59) \]
The symbols defined in the previous section, their diagonal parts, however, say \( \hat{a} \hat{a}^\dagger \) creation and annihilation operators, and \( z \) a complex number. Then

\[
D^\dagger(z) \hat{A} D(z) = \sum_{n,m} F^\dagger(n,m)(n,m) |n\rangle \langle m|
\]

which may be read, introducing \( |n,z\rangle = D(z) |n\rangle \), as

\[
\hat{A} = \sum_{n,m} \Phi_A(n,m;z) D(z) |n\rangle \langle m| D^\dagger(z)
\]

6. Groupoid algebras and tomograms

The symbols defined in the previous section, \( f_A(x,y), F_A(n,m) \) and \( t_A^m(n,m) \) may be considered quasi-distributions. When the operator \( \hat{A} \) is Hermitian, positive with trace one, it can be associated with a state; however, in general its symbols cannot be interpreted as a probability distribution. Their diagonal parts, however, say \( f_A(x,x), F_A(n,n) \) and \( t^m_A(m,m) \), are positive but are not sufficient to reconstruct \( \hat{A} \). In fact, the off-diagonal terms are components on base elements orthogonal to the diagonal ones. However, in the tomographic picture of quantum mechanics, it is possible to `rotate' the diagonal base elements in such a way that they form a family of bases, possibly overcomplete, which allows for a state reconstruction from the diagonal part of their symbols. These families very often are generated by acting with a representation of a group, whose elements are labeled by a set of parameters which appear as independent variables in the tomographic (i.e. diagonal) symbols. Inspired by the tomographic procedures, in this section we will recover known tomographic schemes, like the spin, photon number and symplectic tomographies, from the groupoid immersions discussed in the previous section.

We will begin with spin tomography. Let us consider an irreducible unitary \((2j+1)\)-dimensional representation \( D^j(g) \) of the group \( SU(2) \) and introduce the states \( D^j(g)|jm\rangle \). Here \( g \) is a suitable parameterization [7] of the elements of \( SU(2) \). Then, using equation (35) we can write

\[
D^j(g) \hat{A} D^j(g) = \sum_{m,m'=j} \tau^j_{D^j(g)A D^j(g)}(m,m') |m\rangle \langle m'|
\]

which gives

\[
\hat{A} = \sum_{m,m'=j} w^j_A(g;m,m') D^j(g)|jm\rangle \langle jm'|D^j(g)\rangle
\]

where

\[
w^j_A(g;m,m') = \langle g; jm| \hat{A} |g; jm\rangle
\]

In other words, we obtain a new self-dual pair of quantizer–dequantizer, \( |g; jm\rangle \langle g; jm'| = D^j(g)|jm\rangle \langle jm'|D^j(g)\rangle \), which provides a new symbol \( w_A^j(g;m,m') \) and a reconstruction formula, equation (67), for \( \hat{A} \). The diagonal part of the new symbol \( w^j_A(g;m,m) \) is just the spin tomogram of \( A \), the \( m \)-th component of a \((2j+1)\)-dimensional vector. When \( A \) is a density state operator, the vector is stochastic. Spin tomograms have their own quantizer–dequantizer pair, extensively discussed in [7, 15]. The question of self-duality of these spin tomographic pairs has been analyzed in [16].

As for the photon number tomography, let us consider the usual displacement operator \( D(z) = \exp(\hat{a}^\dagger z - z^* \hat{a}) \), with \( \hat{a}^\dagger, \hat{a} \) the creation and annihilation operators, and \( z \) a complex number. Then

\[
D^\dagger(z) \hat{A} D(z) = \sum_{n,m} F^\dagger(n,m)(n,m) |n\rangle \langle m|
\]

where
with

\[ F_{\mathcal{D}(\mathcal{A}(\mathcal{D}(\mathcal{D}(z)\hat{A}\mathcal{D}(\mathcal{D}(z))))(m))^{n}} = \text{Tr}[\hat{A}(|n, z\rangle\langle m, z|)] \]

\[ = \Phi_{\hat{A}}(n, m; z). \quad (70) \]

Again, \( \Phi_{\hat{A}}(n, m; z) \) is a symbol corresponding to a new quantizer–dequantizer self-dual pair, given by

\[ \mathcal{D}(z)|n\rangle\langle m| \mathcal{D}(z) = |n, z\rangle\langle m, z|. \quad (71) \]

The diagonal part of the new symbol is nothing but the photon number tomographic symbol \( \mathcal{P}_{\hat{A}}(n, -z) \) (see for instance [17])

\[ \Phi_{\hat{A}}(n, n; z) = \mathcal{P}_{\hat{A}}(n, -z) \quad (72) \]

which admits the tomographic reconstruction formula

\[ \hat{A} = \int \frac{d^2\pi}{\pi} \sum_{n=0}^{\infty} \mathcal{P}_{\hat{A}}(n, -z) \hat{D}_{\psi}(n, z), \quad (73) \]

based on the following tomographic photon number quantizer:

\[ \hat{D}_{\psi}(n, z) = \frac{2}{1-s} \left( \frac{s+1}{s-1} \right)^{n} \hat{T}(z, -s), \quad (74) \]

where the \( s \)-ordered displaced parity operator, \((-1 \leq s \leq 1)\), see [18]

\[ \hat{T}(z, s) = \frac{2}{1-s} \hat{D}(z) \left( \frac{s+1}{s-1} \right)^{s} \hat{D}^{\dagger}(z). \quad (75) \]

As for symplectic tomography, let us consider the unitary operator associated with a transformation \( \hat{S}_{\mu\nu} \) (see [6, 19] for more details)

\[ \hat{S}_{\mu\nu} \hat{q} \hat{S}_{\mu\nu} = \mu \hat{q} + v \hat{p}, \]

\[ \hat{S}_{\mu\nu} \hat{q} \hat{S}_{\mu\nu} \hat{S}_{\mu\nu} \hat{S}_{\mu\nu} = x \hat{S}_{\mu\nu} \hat{S}_{\mu\nu} |x\rangle \langle x| \]

with \( \mu \) and \( v \) real parameters. Then, by equation (51),

\[ \hat{S}_{\mu\nu} \hat{S}_{\mu\nu} = \int dx \, dy \, f_{\hat{S}_{\mu\nu}, \hat{S}_{\mu\nu}}(x, y) |x\rangle \langle y| \]

or, equivalently,

\[ \hat{A} = \int dx \, dy \, f_{\hat{S}_{\mu\nu}, \hat{S}_{\mu\nu}}(x, y) \hat{S}_{\mu\nu} |x\rangle \langle y| \hat{S}_{\mu\nu} \quad (77) \]

that we may write as

\[ \hat{A} = \int dx \, dy \, W_{\hat{A}}(x, y; \mu, v) |x\rangle \langle y| \langle x, \mu, v|, \quad (78) \]

where

\[ f_{\hat{S}_{\mu\nu}, \hat{S}_{\mu\nu}}(x, y) = \text{Tr}[\hat{S}_{\mu\nu} \hat{A} \hat{S}_{\mu\nu} (|x\rangle \langle y|)] \]

\[ = \text{Tr}[\hat{A} (|x, \mu, v\rangle \langle y, \mu, v|)] \]

\[ = W_{\hat{A}}(x, y; \mu, v). \quad (79) \]

The new symbol \( W_{\hat{A}}(x, y; \mu, v) \) corresponds to a new quantizer–dequantizer self-dual pair, i.e.

\[ \hat{S}_{\mu\nu} |x\rangle \langle y| \hat{S}_{\mu\nu} = |x, \mu, v\rangle \langle y, \mu, v|. \quad (80) \]

By construction, the diagonal part of the new symbol is just the symplectic tomogram \( \mathcal{W}_{\hat{A}}(x, \mu, v) \) of \( \hat{A} \)

\[ \mathcal{W}_{\hat{A}}(x, x; \mu, v) = \frac{1}{4\pi} \int \mathcal{W}_{\hat{A}}(x, \mu, v) e^{i(s-x\mu-y\nu)} dx \, \mu \, dv. \quad (82) \]

All in all, we have generalized the tomographic schemes introducing tomographic quasi-distributions, whose diagonal part coincides with the usual tomographies. Such tomographic quasi-distributions are obtained by acting with families of unitary operators on certain self-dual quantizer–dequantizer pairs, so they yield the same star product kernels of the generating ones, which in turn coincide with the convolution of some groupoid algebras. However, the diagonal parts of these tomographic quasi-distributions are in correspondence with their own tomographic quantizer–dequantizer pairs which yield tomographic star products, and in general not coincident with the groupoid algebra convolutions. The two kinds of kernels can be related by means of the so-called intertwining maps and dual symbols [20]. Here, for instance, we limit ourselves to show how an intertwining map for the symplectic case, defined as the symbol of \( \hat{D}_{\Sigma}(x', \mu', v') \)

\[ \mathcal{W}_{\hat{D}_{\Sigma}(x', \mu', v')} (x, y; \mu, v) = \int dx \, \mu \, dv \, dx' \, \mu' \, dv' \mathcal{W}_{\hat{D}_{\Sigma}(x', \mu', v')} (x, y; \mu, v) \]

allows to reconstruct the off-diagonal quasi-distribution by the diagonal tomographic symbol. In fact, we may write, using equation (82)

\[ \mathcal{W}_{\hat{A}}(x, y; \mu, v) = \frac{1}{4\pi} \int \mathcal{W}_{\hat{A}}(x', \mu', v') \mathcal{W}_{\hat{D}_{\Sigma}(x', \mu', v')} (x, y; \mu, v) \]

\[ = \int (x, y; \mu, v) dx' \, \mu' \, dv'. \quad (84) \]

7. Conclusions

To summarize we point out the main results of our paper. We have contributed to clarify the role of groupoid structures in quantum mechanics by stating a proposition connecting the convolution product for functions on a countable, principal and transitive groupoid with the star product scheme corresponding to quantizer–dequantizer operators associated with Weyl units. Such a proposition has been checked for finite and infinite Hilbert spaces of qudits and modes of
harmonic oscillators, respectively. Also, we have discussed a simple continuous case. Having done this, we have considered a Fock space realization in a two-mode system, to pave the way for an extension to quantum field theory in a future work.

Known tomographic probability pictures were recognized as diagonal part of quasi-distributions generated by means of groupoids in the tomographic approach. We have also considered a groupoid associated with symplectic tomography, so exhibiting the connection with phase-space and Weyl systems.

An appendix, containing further mathematical details and generalizations of the statements relating convolution and star products has been provided.

We will end this summary by recalling that category theory provides a convenient setting to discuss groupoids, actually a groupoid is just a (small) category whose all morphisms are invertible. It actually happens that 2-groupoids, a deep generalization of groupoids from the point of view of higher categories, arises naturally when discussing the fundamental structure of quantum mechanics and quantum information theory; as a matter of fact they capture the fundamental structure of Schwinger's measurements algebra [9]. These issues will be considered in a forthcoming paper.

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Appendix A. Groupoids

Bearing in mind the example of the pair-groupoid Gamma of section 2, we will discuss here the definition and some properties of an abstract groupoid G. To define a groupoid G we need [3, 21]

1. a set G with a subset Go ⊆ G;
2. two maps r and s from G onto Go;
3. a binary operation o (called multiplication) which is defined for pairs γ1 and γ2 of elements in G whenever s(γ1) = r(γ2) (we will say then that γ1 and γ2 are composable).

Moreover [r, s, o] must satisfy the following axioms:

(a) r(γ1 o γ2) = r(γ1), s(γ1 o γ2) = s(γ2) for all composable γ1 and γ2.
(b) r(γ0) = s(γ0) = γ0, for all γ0 ∈ Go.
(c) r(γ), s(γ) are left and right units for γ respectively, i.e. r(γ) o γ = γ, γ o s(γ) = γ, for all γ ∈ G.
(d) The multiplication o is associative: if (γ1 o γ2) o γ3 is defined, then γ1 o (γ2 o γ3) exists and (γ1 o γ2) o γ3 = γ1 o (γ2 o γ3).

(e) Any γ has a two-sided inverse γ−1, with γ o γ−1 = r(γ) and γ−1 o γ = s(γ). Moreover, (γ · γ−1 o γ′) = γ′ and γ−1 o (γ · γ′) = γ′ for any γ, γ′ composable with γ. The map inv : γ → γ−1 is an involution, that is (γ−1)−1 = γ.

After condition (e) G0 is called the set of unities of G. Some simple consequences can be easily derived from the above definition:

(i) It is possible to define in the set of unities Go the following equivalence relation: γ0 ∼ γ0′ iff there exists γ ∈ G such that r(γ) = γ0 and s(γ) = γ0′. Any equivalence class is called an orbit of G, and G0 is the union of all these orbits.
(ii) The set Hγ0 = {γ ∈ G : s(γ) = γ0 = r(γ)} is a group, called the isotropy group of γ0 ∈ G0.
(iii) The isotropy groups of the unities of the same orbit in G0 are isomorphic.

Two extreme cases regarding G0 are

(a) G0 is the whole groupoid G: so that s(γ) = r(γ) = γ ∀γ ∈ G and any γ can be composed only with γ yielding γ o γ = γ. The orbits of G are the single elements γ; the isotropy group of any γ is trivial, containing only γ.
(b) The set of unities contains only one element, G0 = {e}, so that G is a group, there is only one orbit [e], and the isotropy group of e is G.

Notice that it is also possible to define a groupoid G starting with a set G0 not included in G, so that r, s map G onto G0 (which is now called the base space of the groupoid). For any x ∈ G0 consider the subset Hx = s−1(x) ∩ r−1(x) ⊂ G. As Hx is a group containing the identity eγ = γ−1 o γ, ∀γ ∈ Hx, we define the injection i(x) = eγ of G0 into G, so that Hx is just the isotropy group of eγ. Both ways of considering G0 either as a subset of unities (inner) or as a base space (outer), are equivalent. Each way can be more conveniently used according to the circumstances.

It is also noticeable that there is an equivalent categorical definition of a groupoid. Such approach will become particularly interesting in further research on the structure of quantum systems that will be pursued elsewhere.

We illustrate all these points by considering the groupoids arising from the action of a group on a set G0. To be concrete consider SO(3) acting on the two-dimensional sphere S2 and define G = S2 × SO(3) with r(x, g) = (x, e) and s(x, g) = (x, e), where x ∈ S2 and g is a rotation. Therefore, (x, e) o (y, g) exists iff y = xg and (x, e) o (y, g) := (x, gg). G0 can be considered either as a base space S2 or as a subset (S2, e) ⊂ G. As S2 is a homogeneous space under the action of SO(3), there is only one orbit, the sphere S2 (or (S2, e) ⊂ G). The isotropy group of a point x ∈ S2 is the subgroup of SO(3) of rotations around the axis through x, which is isomorphic to the isotropy groups of all other points of the sphere.

Finally, a groupoid G is called principal if the map

\[(r, s) : G \rightarrow G0 \times G, \gamma \mapsto (r(\gamma), s(\gamma))\]  (A.1)

is one-to-one; it is called transitive if the map (r, s) is onto. For instance, the product-groupoid Gamma discussed in section 2 is both principal and transitive.
Proposition 2. Two units \( \gamma_0, \gamma'_0 \) of the same orbit are connected by a set of elements constituting a ‘left coset’ \( \gamma H_{\gamma_0} \) of the isotropy group of \( \gamma_0 \) in \( G \):

\[
\gamma \circ \gamma_0 \circ \gamma^{-1} = \gamma'_0, \quad \tilde{\gamma} = \gamma \circ \tilde{g}, \quad \tilde{g} \in H_{\gamma_0}.
\]  

(A.2)

Proof. If the groupoid is principal, the proposition is trivially true, as \( \tilde{\gamma} = \gamma \). In general, \( \gamma \) is chosen such that \((r (\gamma'), s (\gamma')) = (\gamma'_0, \gamma_0)\). Observe that any \( \tilde{\gamma} \) with \((r (\gamma'), s (\gamma')) = (\gamma'_0, \gamma_0)\) can be written as \( g' \circ \gamma \circ g \), where \( g \in H_{\gamma_0} \) and \( g' \in H_{\gamma_0} \) are elements of the isotropy groups of the unities. These groups are isomorphic, so define \( \gamma^{-1} \circ g' \circ \gamma =: g_1 \in H_{\gamma_0} \). Then \( g' = \gamma \circ g_1 \circ \gamma^{-1} \) and we obtain

\[
\tilde{\gamma} = g' \circ \gamma \circ g = \gamma \circ \tilde{g}, \quad \tilde{g} := g_1 \circ g \in H_{\gamma_0}.
\]  

(A.3)

Finally, we observe that an equivalent proposition holds when using the ‘right coset’ \( H_{\gamma_0} \gamma \) of the isotropy group of \( \gamma_0 \) in place of the ‘left coset’ \( \gamma H_{\gamma_0} \).

We remark that, as a set, any groupoid is the disjoint union of groupoids \( G = \bigcup G_i \) corresponding to the partition of \( G_0 = \bigcup \mathcal{O}_i \) into orbits \( \mathcal{O}_i \). A groupoid is transitive iff it has a single orbit. Each \( G_i \) has only one orbit of unities and elements in \( G_i \) cannot be multiplied by elements in \( G_k, k \neq i \). Furthermore, for each \( G_i \), the mapping

\[
\Psi : G_i \to \mathcal{O}_1 \times \mathcal{O}_1 : \gamma \mapsto (r(\gamma), s(\gamma))
\]  

(A.4)

is a morphism of \( G_i \) onto the principal groupoid. All elements in \( G \) belonging to \( \bigcup_{x \in \mathcal{O}_1} \{\gamma\}_x \), where \( \{\gamma\}_x \) is the isotropy group of \( x \in \mathcal{O}_1 \), are mapped onto the diagonal of \( \mathcal{O}_1 \times \mathcal{O}_1 : \{\gamma\}_x \mapsto (x, x) \); they constitute the kernel of this morphism. Elements of different \( G_i \) may be related by introducing topological requirements on \( G \).

Appendix B. Finite groupoids

We restrict now to the case of a finite transitive groupoid of order \( K \):

\[
G = \{\gamma\}_{i=1}^{K}, \quad \text{ord} (G) = K. \tag{B.1}
\]

The groupoid algebra \( \mathcal{F} (G) \) is the algebra of the (complex or real) functions on the groupoid with the convolution product

\[
(f_1 \ast f_2)(\gamma) = \sum_{\gamma' \circ \gamma = \gamma} f_1(\gamma') f_2(\gamma). \tag{B.2}
\]

Hereafter, any summation label ranges from 1 to \( K \). The functions \( \delta_{\gamma_j} \), defined as

\[
\delta_{\gamma_j}(\gamma_k) = \begin{cases} 
1 & \text{if } \gamma_j = \gamma_k \\
0 & \text{if } \gamma_j \neq \gamma_k
\end{cases}
\]

(B.3)

are a basis of the groupoid algebra. They can be associated with the standard basis of \( K \)-dimensional column vectors: \( \delta_{\gamma_j} \mapsto v_j \), with \( (v_j)_k = \delta_{jk} \). For any \( f \in \mathcal{F} (G) \),

\[
f(\cdot) = \sum_{k} f(\gamma_k) \delta_{\gamma_k}(\cdot). \tag{B.4}
\]

Moreover,

\[
(\delta_{\gamma_j} \ast f)(\gamma_k) = \sum_{\gamma' \circ \gamma = \gamma_k} f(\gamma'). \tag{B.5}
\]

In particular,

\[
(\delta_{\gamma_j} \ast \delta_{\gamma_k})(\gamma_l) = \sum_{\gamma' \circ \gamma = \gamma_l} \delta_{\gamma_j}(\gamma_{l'}), \tag{B.6}
\]

In other words, \( \delta_{\gamma_j} \ast \delta_{\gamma_k} \) is 1 in \( \gamma_l = \gamma_j \circ \gamma_k \) (if \( \gamma_j \) and \( \gamma_k \) are composable) and is 0 elsewhere; so

\[
\delta_{\gamma_j} \ast \delta_{\gamma_k} = \delta_{\gamma_j \circ \gamma_k}. \tag{B.7}
\]

The above equation shows that the convolution product is associative because the multiplication \( \circ \) is associative.

The mapping

\[
\gamma \mapsto D(\gamma) = \delta_{\gamma} \ast \tag{B.8}
\]

is a groupoid realization by means of operators in \( L(\mathcal{F} (G)) \) in the following sense:

\[
D(\gamma)D(\gamma_k) = \begin{cases} 
D(\gamma_j \circ \gamma_k) & \text{if } \gamma_j \circ \gamma_k \text{ exists,} \\
0 & \text{if } \gamma_j \circ \gamma_k \text{ does not exist.}
\end{cases} \tag{B.9}
\]

In general, we may define a groupoid realization as a morphism \( \Phi \) of the groupoid \( G \) in the set of operators \( L(V) \) on some linear space \( V \), such that the above equations are satisfied. This means, in abstract, that we add to the groupoid a zero element which is the result of any forbidden multiplication. The realization of the groupoid is then a realization in the usual sense of this new enlarged structure. In section 2, we have introduced an irreducible realization using the Weyl units \( E \).

Given the groupoid realization \( D \), a realization of the groupoid algebra is immediately obtained by the formula

\[
A_f = \sum_{k} f(\gamma_k) D(\gamma_k), \tag{B.10}
\]

which looks like a quantization formula, where an operator \( A_f \) is obtained by a function \( f \) by means of a quantizer \( D(\gamma_k) \). From the above equation we obtain

\[
A_{f_1}A_{f_2} = \sum_{j,k} f_1(\gamma_j) f_2(\gamma_k) D(\gamma_j \circ \gamma_k) = \sum_{\gamma' \circ \gamma = \gamma} f_1(\gamma_j) f_2(\gamma_k) D(\gamma) = \sum_{\gamma' \circ \gamma = \gamma} (f_1 \ast f_2)(\gamma) D(\gamma) = A_{f_1 \ast f_2}. \tag{B.11}
\]

In other terms, the product of operators corresponds to the convolution product of the associated functions. In the standard basis, which we will use from now on, operators are \( K \times K \)-matrices acting on the \( K \)-dimensional vector space of the functions.

Moreover, we can state the following.
Lemma 3. For a transitive groupoid, the dequantizer associated to the quantizer $D(\gamma)\mu$ is $D(\gamma)\mu/\mathcal{N}$, where $\mathcal{N}$ is a suitable normalization constant, with

$$\frac{1}{\mathcal{N}} \text{Tr} \left[ D(\gamma_f)D^T(\gamma_k) \right] = \delta_{\gamma_f}(\gamma_k). \quad (B.12)$$

$$\mathcal{N} = \text{ord}(G_0) + \text{ord} \left( H_{\gamma_f^{-1}} - \gamma_f^{-1} \circ \gamma_k \right). \quad (B.13)$$

Proof. In the standard basis the trace can be written as

$$\sum_{p,q} \left[ D(\gamma_f) \right]_{pq} \left[ D^T(\gamma_k) \right]_{qp} = \sum_{p,q} \delta_{\gamma_f}(\gamma_k) \delta_{\gamma_f^{-1}}(\gamma_k) \delta_{\gamma_f}(\gamma_k) = \delta_{\gamma_f}(\gamma_k) \sum_{p,q} \delta_{\gamma_f^{-1}}(\gamma_k). \quad (B.14)$$

The evaluation of the last summation amounts essentially to count all the $\gamma_k$ that can be composed with $\gamma_f$: they are those contained in the isotropy group of $\gamma_f^{-1} \circ \gamma_f$, plus the elements connecting $\gamma_f^{-1} \circ \gamma_f$ with all the other unities of the unique orbit. The evaluation, from proposition 2, yields eventually the normalization constant $\mathcal{N}$ of equation (B.13). \hfill $\Box$

In view of equation (B.10), we may write

$$f(\gamma) = \frac{1}{\mathcal{N}} \text{Tr} \left[ A_f D^T(\gamma) \right], \quad (B.15)$$

and the symbol functions may be multiplied using a star product kernel $K(\gamma_f, \gamma_k)$. We have:

Proposition 4. The star product corresponding to the kernel

$$K(\gamma_f, \gamma_k, \gamma_l) = \frac{1}{\mathcal{N}} \text{Tr} \left[ D(\gamma_f)D(\gamma_k)D^T(\gamma_l) \right] \quad (B.16)$$

coincides with the convolution product of the groupoid algebra.

Proof. In view of equation (B.12), the evaluation of the kernel gives

$$K(\gamma_f, \gamma_k, \gamma_l) = \delta_{\gamma_f}(\gamma_k) \delta_{\gamma_f}(\gamma_k) = \delta_{\gamma_f}(\gamma_k) \delta_{\gamma_f}(\gamma_k) \quad (B.17)$$

so that the star product

$$(f_1 * f_2)(\gamma_l) = \sum_{j,k} K(\gamma_j, \gamma_k, \gamma_l) f_1(\gamma_j) f_2(\gamma_k)$$

$$= \sum_{\gamma_f \gamma_f^{-1} = \gamma_l} f_1(\gamma_f) f_2(\gamma_f) = (f_1 * f_2)(\gamma_l) \quad (B.18)$$

is nothing but the convolution product. \hfill $\Box$

The above proposition generalizes proposition 1 (section 4) to the case of a transitive, non-principal groupoid. We observe that when $G$ is also principal with $N$ equivalent unities, its order $K = N^2$ and $D$-realization is reducible and contains $N$ times the irreducible realization of the Weyl units.

Appendix C. Remarks on the convolution in the continuous case

Let us consider now a measure space $(S, \mu)$ and the principal, transitive pair-groupoid $\Gamma = S \times S$, with the usual composition law $(x, y) \circ (y, z) = (x, z)$. It is natural to choose for $S \times S$ the product measure $\mu \times \mu$, even though this is not mandatory. The convolution formula now would read

$$(f_1 * f_2)(x, y) = \int \mu(dy_1) \mu(dy_2) f_1(x, y_1) f_2(x_2, y_2) \mu(dx_1) \mu(dx_2). \quad (C.1)$$

This is an integral on the subset $M$ of $S \times S \times S$ given by the constraints $y_1 = x_2, x_1 = x$ and $y_2 = y$, induced by the groupoid composition law and this integral in general will be zero. For a countable groupoid, with $S = \{x_k, k = 1, \ldots, N \leq \infty\}$, $\mu = k \mu_k$, where $\mu_k(\Delta) = 1$ if $x_k \in \Delta$ and zero otherwise, yielding the usual convolution formula (16). Therefore it is necessary to make sense of the integral above. To discuss a concrete relevant case, assume $S = \mathbb{R}$, and $\mu(dx) = dx$, the Lebesgue measure on $\mathbb{R}$. The groupoid composition constraint becomes the linear constraints $h_1(x_1, x_2) = x_1 - x_2 = 0$, $h_2(y_1, y_2) = y_1 - y_2 = 0$, $h_3(y_2, y) = y_2 - y = 0$, and then, the convolution integral may be read as

$$(f_1 * f_2)(x, y) = \int f_1(x_1, y_1) f_2(x_2, y_2) \delta(x_1 - x) \delta(x_1 - x) \delta(x_1 - x) \delta(y_2 - y) \mu(dx_1) \mu(dx_2) \mu(dx_2) = \int f_1(x_1, y_1) f_2(y_1, y) \mu(dx_1), \quad (C.2)$$

which is just the continuous matrix product of equation (18) where $f_1, f_2$ are continuous functions with proper support $f$ is a continuous function with proper support $f$ for each compact subset $K$ of $\mathbb{R}$, the intersections of $K \times \mathbb{R}$ and $\mathbb{R} \times K$ with the set $\{x; y\}$ have compact closure).

However, this way to write the convolution is arbitrary to some extent. In fact, the form above depends on the choice of linear constraints, leading to $h(x - y)$ instead of $h(h(x - y))$ with an arbitrary function such that $h(0) = 0, h(0) \neq 0$, realizing the same constraints. For instance, upon substituting $h(x_1 - x_2) = \delta(x_1 - x_2) = \delta(a_1(x_1 - x_2))$ $h(y_1 - y_2) = \delta(y_1 - y_2)$ $\delta(y_1 - y_2)$ $\delta(y_1 - y_2)$ $\delta(\alpha_2(y_1 - y_2))$ with $\alpha_1, \alpha_2 > 0$ the integration domain $M = \mathbb{R}$ does not change, but the convolution formula becomes

$$(f_1 * f_2)(x, y) = \frac{1}{\alpha_1(x) \alpha_2(y)} \int_{\mathbb{R}} f_1(x_1, y_1) f_2(y_1, y) dy_1. \quad (C.3)$$

Moreover, a different measure can be chosen. Suppose $\mu(dx) = \rho(x) dx$, then the space of integrable groupoid functions varies accordingly and the convolution formula reads

$$(f_1 * f_2)(x, y) = \rho(x) \rho(y) \int_{\mathbb{R}} f_1(x_1, y_1) f_2(y_1, y) \rho^2(y_1) dy_1. \quad (C.4)$$

These considerations hold for any principal transitive groupoid with a base space $G_0$, which is a measure space $(S, \mu)$, in this case the morphism of equation (A.4) is an
isomorphism and one easily concludes that for this kind of groupoids one can always reduce the convolution to the form

\[(f_1 * f_2)(x, y) = \rho(x)\rho(y) \int_{s \in S} f_1(s, x - s) f_2(s, y + s) d\mu(s).\]  

(C.5)

To illustrate this point consider the groupoid arising from the action of the group \(\mathbb{R}\) as translations on the base space \((\mathbb{R}, dx)\) (analogous to the action of \(SO(3)\) discussed previously).

Using again the delta functions as constraints, the general expression of convolution (C.1) yields

\[(f_1 * f_2)(x, t) = \int_{\mathbb{R}} f_1(x, s - x) f_2(s, t + x - s) ds.\]  

(C.6)

The morphism (A.4) for this example reads \(\Psi(x, t) = (x, t + x)\) and introducing the pullback \(\Psi^* f\) of a function \(f\) we obtain

\[\Psi^* (f_1 * f_2)(x, t) = \Psi^* \int_{\mathbb{R}} f_1(x, s - x) f_2(s, t + x - s) ds = \int_{\mathbb{R}} \Psi^* f_1(x, s) \Psi^* f_2(s, t) ds\]  

(C.7)

and recover the usual form of convolution.

In the general case the groupoid elements will have an isotropy group \(H_0\) and the morphism \(\Psi\) therefore will have a kernel. To integrate functions on such a groupoid one would need a measure on the set of units and a measure on the isotropy group. For the isotropy group it is natural to use the Haar measure, when available, and one should choose some measure on the set of units \(G_0\).

We will just mention here two cases.

1. \(G\) a transitive groupoid, \(G_0 = \{\text{set of } N \text{ elements}\}, \) and \(H_0\) a group with an invariant measure. In this case it can be shown that functions on \(G\) are represented as \(N \times N\)-matrices, whose entries are functions on \(H_0\). Convolution of two functions is now a row-by-column product of the corresponding matrices, where products of matrix elements are replaced by the usual group convolution on \(H_0\).

2. \(G\) is the groupoid arising from the action of the rigid motion on a plane. In this case, \(G_0\) is the plane and \(H_0\) is the rotation around an axis. It is natural to use the Haar measure \(d\theta\) for \(H_0\) and \(d\chi dy\) on the plane which is invariant under rigid motions. As before, the convolution can be written as a continuous matrix multiplication, but with a group convolution in the \(\theta\) variable.

In the general case of a topological groupoid \(G\) the convolution algebra can be constructed by using a family of measures as done for instance in [21] where a left Haar system of measures is used; however, there is more elegant construction using half-densities as discussed for instance in [3] (section 2.5, page 101) but that goes back to the work of Stachura and Zakrzewski [22].

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