A NOVEL APPROACH IN SOLVING THE SPINOR-SPINOR BETHE-SALPETER EQUATION

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I. INTRODUCTION

Interpretation of many modern experiments requires a covariant description of the two-body system. This is either due to high precision that calls for an inclusion of all possible corrections to a standard (possibly nonrelativistic) approach or due to the high energies and momenta involved in the processes investigated.

In the spirit of a local quantum field theory the starting point of a relativistic covariant description of bound states of two particles is the Bethe-Salpeter (BS) equation. However, despite the obvious simplicity of two-body systems, the procedure of solving the BS equation encounters difficulties. These are related to singularities and branch points (cuts) of the amplitude along the real axis of the relative energy in Minkowski space. Therefore, up to now the BS equation including realistic interaction kernels has been solved either in Euclidean space within the ladder approximation or utilizing additional approximations of the equation itself.

Unfortunately, our understanding of the mathematical properties of bound states within a relativistic approach is far from being perfect. In mathematical terms the BS equation itself is a quite complicated object, and the technical problem of solving it is still a fundamental issue. Consequently there are very few successful examples of solving the BS equation for fermions including realistic interactions. For example, in Ref. \cite{1} the BS equation for spinor particles was regularly treated by using a two-dimensional Gaussian mesh. That series of studies revealed a high feasibility of the BS approach to describe nucleon-nucleon interactions, in particular, processes involving the deuteron. However, it should be mentioned that the algorithms exploiting the two-dimensional meshes are rather cumbersome and require large computer resources. In addition, the numerical solution is obtained as two-dimensional arrays which are quite awkward in practice when computing matrix elements and in attempts to establish reliable parameterizations and possible analytical continuations of the solution back
to Minkowski space. Therefore, it is necessary to provide a method for solving BS equations that would feature a smaller degree of arbitrariness.

In the present paper we suggest an efficient method to solve the BS equation for fermions involving interaction kernels of one-boson exchange type supplemented by corresponding form factors. It is based on hyperspherical harmonics used to expand partial amplitudes and kernels. We show that this novel technique provides many insights into the BS approach. The current study is partially stimulated by the results reported in Ref. [2]. We explore the structure of $^1S_0$ and $^3S_1^−^3D_1$ bound states for different couplings and study the details of the convergence of solutions and corresponding eigenvalues. In particular, on the basis of the proposed method for solving the BS equation it becomes possible to analyze the specifics of the problem related to the stability of bound states in the BS approach. Besides, the hyperspherical expansion provides an effective parameterization of the amplitude, which is extremely useful in practical calculations of observables and in theoretical investigations of the separability of the BS kernel with one-boson exchange interaction. The detailed description of method may be found in [3].

II. OVERVIEW OF THE METHOD

The BS equation for two spinor particles interacting via one-boson-exchange potentials for vertex $G(p)$ being a $4 \times 4$ matrix in spinor space has the form

$$G(p) = ig^2 \int \frac{d^4k}{(2\pi)^4} V(p, k) \Gamma(1) S(k_1) G(k) \tilde{S}(k_2) \tilde{\Gamma}(2), \quad (1)$$

where the propagator $V(p, k)$ for scalar and pseudoscalar exchange mesons is

$$V(p, k) = \frac{1}{(p - k)^2 - \mu^2 + i\varepsilon}, \quad (2)$$

while for spinor particles the corresponding propagators are

$$S(k_1) = \frac{\hat{k}_1 + m}{k_1^2 - m^2 + i\varepsilon}, \quad \tilde{S}(k_2) \equiv CS(k_2)^T C = \frac{\hat{k}_2 - m}{k_2^2 - m^2 + i\varepsilon},$$

where $C = i\gamma^0\gamma^2$ is the charge conjugation matrix, $k_{1,2}$ and $p_{1,2}$ are the four-momenta of the constituent particles, $k$ and $p$ are the corresponding relative momenta. The meson vertices $\Gamma(1, 2)$ are determined by the effective interaction Lagrangians urged to describe the considered fermion system. For a two nucleon system, within the one-boson exchange approximation, these vertices are $\Gamma(1) = 1; \tilde{\Gamma}(2) = -1$ for scalar and $\Gamma(1) = \gamma_5; \tilde{\Gamma}(2) = -\gamma_5$ for pseudoscalar couplings, respectively. Each interaction vertex $\Gamma$ is augmented with cut-off
monopole form factors $F(q^2) = \Lambda^2/(\Lambda^2 - q^2)$ where $\Lambda$ are free parameters. Note that the coupling constant $g$ in eq. (1) is purely imaginary for the pseudoscalar mesons else purely real.

The first step is the expansion of the vertex function $G(p_0, p)$ into spin-angular harmonics

$$G(p_0, p) = \sum_{\alpha} g_{\alpha}(p_0, |p|) T_{\alpha}(p). \quad (3)$$

For specific bound states with given quantum numbers only some basis matrices contribute to the vertex function $G(p)$. E.g., for the $^1S_0$ state only four matrices are relevant to describe the amplitude, while in the $^3S_1 - ^3D_1$ channel eight basis matrices are needed. In the $^1S_0$ channel the basis is

$$T_1(p) = \frac{1}{\sqrt{16\pi}} \gamma_5; \quad T_2(p) = \frac{1}{\sqrt{16\pi}} \gamma_0 \gamma_5; \quad T_3(p) = \frac{1}{\sqrt{16\pi}} \frac{(p\gamma_5)}{|p|} \gamma_0 \gamma_5; \quad T_4(p) = \frac{1}{\sqrt{16\pi}} \frac{(p\gamma_5)}{|p|} \gamma_5. \quad (4)$$

By employing the Pauli principle and the charge conjugation operation one obtains [3] that in the $^1S_0$ channel the component $g_4$ is of the odd parity while the remaining $g_1, ..., g_3$ are of the even parity. As mentioned, the BS amplitude is a mathematically complicated object and it is more convenient to considered it in Euclidean space, where the analytical properties of the amplitude become simpler and more transparent. For convenience, in Euclidean space we redefine the odd partial components $g_4$ for the $^1S_0$ channel as $g_4 \rightarrow ig_4$. Then the Wick rotated BS equation (1) reads

$$g_n(p_4, |p|) = g^2 \int d\Omega_p \int \frac{d^4k}{(2\pi)^4} S(k_4, |k|) \frac{1}{(p-k)^2 + \mu^2} \sum_m A_{nm}(p, k) g_m(k_4, k), \quad (5)$$

where $m, n = 1 \ldots 4$ for the $^1S_0$ and the scalar part $S(k_4, |k|)$ of the two spinor propagators is defined as

$$S(k_4, |k|) = \frac{1}{(k^2 + m^2 - M^2)^2 + M^2k^2_4}. \quad (6)$$

Next step is the expanding of the interaction kernel into hyperspherical harmonics [4]

$$\frac{1}{(p-k)^2 + \mu^2} = 2\pi^2 \sum_{nlm} \frac{1}{n+1} V_n(\tilde{p}, \tilde{k}) Z_{nlm}(\chi_p, \theta_p, \phi_p) Z_{nlm}^*(\chi_k, \theta_k, \phi_k), \quad (7)$$

with

$$Z_{nlm}(\chi, \theta, \phi) = X_{nl}(\chi) Y_{lm}(\theta, \phi); \quad X_{nl}(\chi) = \frac{2^{l+1} (n+1)(n-l)!}{\pi (n+l+1)!} \sin^l \chi C_{n-l}^{l+1}(\cos \chi),$$

$$V_n(\tilde{p}, \tilde{k}) = \frac{4}{(\Lambda_+ + \Lambda_-)^n} \left(\frac{\Lambda_+ - \Lambda_-}{\Lambda_+ + \Lambda_-}\right)^n; \quad \Lambda_{\pm} = \sqrt{(\tilde{p} \pm \tilde{k})^2 + \mu^2}. \quad (8)$$
where $Y_{lm}(\theta, \phi)$ are the familiar spherical harmonics, and $C_{n-l}^{l+1}$ are the Gegenbauer polynomials. The resulting system of equations reads

$$g_n(p_4, |p|) = g^2 \int \frac{k^3 dk \sin^2 \chi_k d\chi_k}{(4\pi^3)} S(k_4, |k|) W_{l_n}(\vec{p}, \vec{k}, \chi_p, \chi_k) \sum_m a_{nm}(k_4, k) g_m(k_4, k), \quad (8)$$

where

$$W_{l_n}(\vec{p}, \vec{k}, \chi_p, \chi_k) = \sum_l \frac{1}{l+1} V_l(\vec{p}, \vec{k}) X_{l_n}(\chi_p) X_{l_n}(\chi_k).$$

Furthermore, the partial vertex functions $g_n$ are expanded over the basis $X_{nl}(\chi_p)$ as

$$g_{1,2}(p_4, |p|) = \sum_{j=1}^{\infty} g_{1,2}(\vec{p}) X_{2j-2,0}(\chi_p); \quad g_3(p_4, |p|) = \sum_{j=1}^{\infty} g_3(\vec{p}) X_{2j-1,1}(\chi_p); \quad (9)$$

$$g_4(p_4, |p|) = \sum_{j=1}^{\infty} g_4(\vec{p}) X_{2j,1}(\chi_p). \quad (10)$$

Placing eqs. (9) and (10) in to eq. (8) and performing integration over hyper-angles analytically, the initial system of four-dimensional integration equations reduces to a (infinite) system of only one-dimensional equations. Eventually, by a proper choice of numerical integration method (the Gaussian formula, in our case) one easily transforms the latter system of integral equations in to an ordinary algebraic system of linear equations. Obviously, further solving procedure is straightforward (see for details Ref. [3]).

### III. RESULTS

By the above procedure we solved the BS equation for a system of two spinors with equal masses in $^1S_0$ state interacting via exchanges of scalar and pseudoscalar mesons. Herebelow, for the sake of brevity, we present results only for scalar exchange mesons; more detailed results can be found in [3]. Note that the accuracy of the proposed method depends up on the dimension $N_G$ of the Gaussian mesh used in calculations and on number of terms $M_{max}$ used in the decomposition (9) and (10). Also the magnitude of the coupling constants governs the existence of the solution itself; at some critical values of the coupling constants the solution of the BS equation may not exist at all.

As an illustration of the stability of the numerical procedure, in Table I we present results for the masses of the bound state $M(g^2)$ depending on the Gaussian mesh $N_G$ and $M_{max}$. Calculations have been performed for the $^1S_0$ state, with a scalar meson exchange of mass $\mu$ for two values $\mu = 0.15 \text{ GeV}/c^2$ and $\mu = 0.5 \text{ GeV}/c^2$; the constituent particles (nucleons) have
been taken with equal masses $m = 1.0 \text{ GeV/c}^2$ for simplicity. Results presented in Table I clearly demonstrate that the approximate solution converges rather rapidly, and already at $M_{\max} \sim 4 - 5$ and $N_G = 64$ the method provides a good solution of the system.

| $g^2 = 15$ | $\mu = 0.15 \text{ GeV/c}^2$ | $\mu = 0.5 \text{ GeV/c}^2$ |
|------------|------------------------------|-------------------------------|
| $M_{\max}$ | $N_G = 32$ $N_G = 64$ $N_G = 96$ | $N_G = 32$ $N_G = 64$ $N_G = 96$ |
| 1          | 1.9399 1.9399 1.9399 | 1.9984 1.9984 1.9984 |
| 2          | 1.9370 1.9370 1.9370 | 1.9982 1.9982 1.9982 |
| 3          | 1.9368 1.9368 1.9368 | 1.9982 1.9982 1.9982 |
| 4          | 1.9368 1.9368 1.9368 | 1.9982 1.9982 1.9982 |

| $g^2 = 30$ | $\mu = 0.15 \text{ GeV/c}^2$ | $\mu = 0.5 \text{ GeV/c}^2$ |
|------------|------------------------------|-------------------------------|
| $M_{\max}$ | $N_G = 32$ $N_G = 64$ $N_G = 96$ | $N_G = 32$ $N_G = 64$ $N_G = 96$ |
| 1          | 1.7932 1.7910 1.7905 | 1.9167 1.9142 1.9137 |
| 2          | 1.7897 1.7875 1.7871 | 1.9152 1.9127 1.9122 |
| 3          | 1.7896 1.7874 1.7870 | 1.9152 1.9127 1.9122 |
| 4          | 1.7896 1.7874 1.7870 | 1.9152 1.9127 1.9122 |

TABLE I: Dependence of the bound state masses on the $M_{\max}$ and on the Gaussian mesh used in actual numerical calculations.

The solution of the BS equation, eq. (3), can be obtained from the known numerical values of the partial components $g_j$. In Fig. 1 we present the behavior of the partial coefficients in the $^1S_0$ channel $g_1^j(\tilde{\rho})$, $j = 1 \ldots 4$, eq. (9), as a function of the euclidian relative momentum $\tilde{\rho} = \sqrt{\rho^2 + \mathbf{p}^2}$. Calculations have been performed for a bound system with $M = 1.937 \text{ GeV/c}^2$. The actual parameters used in numerical calculations are $N_G = 96$, $M_{\max} = 4$ and $g^2 = 15$. Closed squares correspond to $g_1^1$, closed circles - $g_1^2$ multiplied by 10, triangles - $g_1^3$ multiplied by 100, open circles - $g_1^4$ multiplied by 1000. It can be seen that at large $\tilde{\rho}$ each function decreases as inverse powers of $\tilde{\rho}$, which allows for a relatively simple parametrization of the result; the solid lines correspond to such a fit. An analysis of the obtained results shows that the numerical solution is rather sensitive to the magnitude of the coupling constants. Moreover, it has been found that there are some critical values of the coupling constants, $g_{\text{cr}}^2$, above which the solution of the BS equation does not exist (in absence of the cut-off form factors). At values of the coupling constants close the their critical values $g^2 \sim g_{\text{cr}}^2$ the numerical solution becomes unstable and
FIG. 1: Left panel: The coefficients $g_j^i, j = 1 \ldots 4$, eq. (3), as functions of the euclidian momentum $\tilde{p} = p_1^2 + p^2$. Right panel: The mass of the bound state $^1S_0$ as a function of the cut-off parameter $\Lambda$ at different values of the coupling constant $g$. At large $g > g_{cr}$ the solution is strongly dependent on the cut-off parameter, i.e. becomes rather unstable.

strongly dependent on the Gaussian mesh, $M_{max}$ and $\Lambda$. Such a situation is illustrated in Fig. 1 right panel, where we present the dependence of the mass $M$ on the values of the cut-off parameter $\Lambda$ at different coupling constants $g^2$. It is clearly seen that at relatively low values of $g^2$ below its critical value ($g_{cr}^2 \sim 40$) the solution is practically independent on $\Lambda$, i.e. it converges rather rapidly. Contrarily, at $g^2 > g_{cr}^2$ the solution $M(g^2)$ becomes manifestly dependent on $\Lambda$, i.e., it can disappear at all. Such a behavior of the solution at $g^2 \sim g_{cr}^2$ exactly reproduces the peculiarities of the well-known collapse phenomenon for potentials like $-\alpha/r^2$ in nonrelativistic quantum mechanics. The magnitude of the critical value $g_{cr}^2$ may be estimated analytically and it turns out to be $g_{cr}^2 \sim 4\pi^2$ (see Ref [3]). As mentioned, the obtained solution has a rather simple dependence on the euclidian momentum $\tilde{p}$ and allows for a simple analytical parametrization of the solution, which can be exploited further to perform a Wick rotation.

FIG. 2: The mass of the bound state $M$ in the $^1S_0$ channel within different models, non relativistic (NR), light front (LF) dynamics and present (BS) approach.
back to Minkowsky space. In this context is is instructive to compare our solution with ones obtained within other known approaches. In Fig. 2 a comparison of our solution (solid line) with the results of non relativistic calculations (dotted line) as well as with the ones within the Light Front dynamics (LF) \cite{2} (dashed line) is presented. As expected, at low values of the coupling constant different approaches provide similar results. As $g$ increases the difference becomes more significant, reflecting the role of relativistic effects. The difference between LF and BS approaches is due to different treatment of the vertex function, namely within the LF the vertex function consists on three components, while within the BS formalism four partial components describe the solution. Obviously, the role of the fourth component increases with $g^2$.

IV. CONCLUSION

We generalize a method based on hyperspherical harmonics to solve the homogeneous spinor-spinor Bethe-Salpeter equation in Euclidean space. To do so, we introduce a new basis of spin-angular harmonics, suitable to expand the Bethe-Salpeter vertex into four-dimensional hyperspherical harmonics. We obtain an explicit form of the corresponding system of one-dimensional integral equations for the partial components and formulate a proper numerical algorithm to solve this system of equations. The BS vertex functions are studied in detail for the $^1S_0$ and $^3S_1 - ^3D_1$ bound states with scalar, pseudoscalar and vector meson exchanges. Our results are in a good agreement with calculations within the non relativistic and Light Front Dynamics approaches. Within the novel method the effectiveness of the numerical procedure is analyzed for the scalar, pseudoscalar and vector meson exchanges and conditions for stability of the solution are established.

An advantage of the method is the possibility to present the numerical solution in a reliable and simple analytical parameterized form, extremely convenient in practical calculations of matrix elements within the BS formalism and for analytical continuation of the solution back to Minkowski space. The method allows for a covariant description of two-body systems, such as the deuteron, positronium and the variety of known mesons, as relativistic bound states and, by solving the inhomogeneous BS equation, to describe the scattering states.

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