Domination Number of Vertex Amalgamation of Graphs

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Domination Number of Vertex Amalgamation of Graphs

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Abstract. For a graph $G = (V, E)$, a subset $S$ of $V$ is called a dominating set if every vertex $x$ in $V$ is either in $S$ or adjacent to a vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of the dominating set of $G$. The dominating set of $G$ with a minimum cardinality is denoted by $\gamma(G)$-set. Let $G₁, G₂, \ldots, Gₜ$ be subgraphs of the graph $G$. If the union of all these subgraphs is $G$ and their intersection is $\{v\}$, then we say that $G$ is the vertex-amalgamation of $G₁, G₂, \ldots, Gₜ$ at vertex $v$. Based on the membership of the common vertex $v$ in the $\gamma(Gᵢ)$-set, there exist three conditions to be considered. First, if $v$ elements of every $\gamma(Gᵢ)$-set, second if there is no $\gamma(Gᵢ)$-set containing $v$, and third if either $v$ is element of $\gamma(Gᵢ)$-set for $1 \leq i \leq p$ or there is no $\gamma(Gᵢ)$-set containing $v$ for $p < i \leq t$. For these three conditions, the domination number of $G$ as vertex-amalgamation of $G₁, G₂, \ldots, Gₜ$ at vertex $v$ can be determined.

1. Introduction

Several results about domination number $\gamma(G)$ and operation of graphs have been explored by some researchers. Some of these results were obtained by Pavlic and Zerovnik [6], Go and Canoy [2], and Kuziak, Lemanska, and Yero [5]. In this paper, we determined the domination number of vertex amalgamation of graphs.

Suppose $G₁$ and $G₂$ are subgraphs of a graph $G = (V, E)$ and $v ∈ V$. The graph $G$ is called vertex amalgamation of $G₁$ and $G₂$ at vertex $v$, denoted by $G = G₁ \cup \{v\} G₂$, if $G = G₁ \cup G₂$ and $G₁ \cap G₂ = \{v\}$. This vertex amalgamation definition proposed by Yang and Kong [9] and can be generated over more than two subgraphs. Suppose $G₁, G₂, \ldots, Gₜ$ are subgraphs of $G$ and $v ∈ V$. If $G = \bigcup_{i=1}^{t} Gᵢ$ and $\cap_{i=1}^{t} Gᵢ = \{v\}$, then $G$ is a vertex amalgamation of $G₁, G₂, \ldots, Gₜ$ at vertex $v$, denoted by $G = V\{v\} \{G₁, G₂, \ldots, Gₜ\}$. If $G₁, G₂, \ldots, Gₜ$ has $n₁, n₂, \ldots, nₜ$ vertices respectively, then $V\{v\} \{G₁, G₂, \ldots, Gₜ\}$ has $\sum_{i=1}^{t} n_i - t + 1$ vertices.

For a survey of some family of graphs, see [1]. The open neighborhood of $v ∈ V$ is the set $N(v) = \{w ∈ V; vw ∈ E\}$ and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. For a set $S ⊆ V$, the open neighborhood $N(S)$ is defined as $\bigcup_{v ∈ S} N(v)$ and the closed neighborhood of $S$ is $N(S) = N(S) \cup S$. A set $S$ of vertices of a graph $G$ is called a dominating set if each vertex of $V − S$ is adjacent to a member of $S$. It is equivalent to that $N[S] = V$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of the dominating set of $G$ [3].

Some known results on domination number of some graphs are $\gamma(Pₙ) = \left\lfloor \frac{n}{2} \right\rfloor$ for $n > 1$ and $\gamma(Cₙ) = \left\lfloor \frac{n}{3} \right\rfloor$ for $n > 3$ by Klobucar [4], $\gamma(Kₙ) = 1$, $\gamma(Kₘn) = \gamma(K_{n₁, n₂, \ldots, nₖ}) = 2$ for $m, n > 1$ and $n_i > 1$ by Snyder [7], $\gamma(Fₙ,k) = n$ for $k > 1$, and $\gamma(Bₙ,k) = n + 1$, for $n, k > 1$ by Wardani [8], $\gamma(Lₙ) = \left\lceil \frac{n+1}{2} \right\rceil$, and $\gamma(Pₙ,f) = 1$. The bounds of domination number was given by Berge [3], that is, for a graph with order $n$ and the maximum degree $Δ(G)$ holds $\left\lfloor \frac{n}{1+Δ(G)} \right\rfloor ≤ \gamma(G) ≤ n − Δ(G)$. 

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2. Results

We start this section by the observation concerning on the domination number of vertex amalgamation of some complete graphs.

Observation 1. If \( G = V^1_{[\omega]} \{K_{n_1}, K_{n_2}, ..., K_{n_t}\} \) then \( \gamma(G) = 1 \).

The domination number of vertex amalgamation of a complete \( k \)-partite graphs is presented in the following theorem.

Theorem 1.

a. For a complete bipartite graph \( K_{m,n} \), where \( m, n \geq 2 \) if \( G = V^1_{[\omega]} \{K_{m_1,n_1}, K_{m_2,n_2}, ..., K_{m_t,n_t}\} \) then \( \gamma(G) = t + 1 \).

b. Let a complete multipartite graph \( H_i \) has cardinality for each partite be more than one. If \( G = V^1_{[\omega]} \{H_1, H_2, ..., H_t\} \) then \( \gamma(G) = t + 1 \).

Proof.

a. If \( G = V^1_{[\omega]} \{K_{m_1,n_1}, K_{m_2,n_2}, ..., K_{m_t,n_t}\} \) where \( m_i, n_i \geq 2 \) and \( m_i \leq n_i \) then the order of \( G \) is \( |V(G)| = \sum_{i=1}^{t} (m_i + n_i) - t + 1 \) and the maximum degree is \( \Delta(G) = \sum_{i=1}^{t} n_i \). Using Berge, the upper bound of the domination number of \( G \) is \( (\sum_{i=1}^{t} (m_i + n_i) - t + 1) - \sum_{i=1}^{t} n_i = \sum_{i=1}^{t} m_i - t + 1 \). Because \( m_i \geq 2 \) then \( \sum_{i=1}^{t} m_i - t + 1 \geq t + 1 \) so that the least upper bound is \( \gamma(G) \leq t + 1 \). Suppose there is \( T \subseteq V(G) \) which \( |T| = t \). Let \( V(K_{m_i,n_i}) = V_i \cup V_j \) where \( V_i \) is the vertex partite which consists of \( v \). If \( v \in T \) then there is \( V_j \) such that \( x_j \notin T \) for every \( x_j \in V_j \). So, for every \( y_j \in V_j \) which \( y_j \neq v \) holds \( y_j \in V(G) - T \), \( y_j \sim x_j \) but \( y_j \neq v \). If \( v \notin T \) then there are two conditions. First, there exist a \( z_i \in V(K_{m_i,n_i}) \) for every \( i \) such that \( z_i \notin T \). In this case, every vertex which belongs to the same partite of \( v \) adjacent with no element of \( T \). Second, there exists \( K_{m_i,n_i} \) such that every vertex of \( K_{m_i,n_i} \) belongs to \( T \). In this case, every vertex of \( K_{m_i,n_i} \) except \( v \) is not adjacent to the element of \( T \). It’s means that \( T \) is not dominating set.

b. Let \( S = \{v, a_1, a_2, ..., a_t\} \) where \( a_i \notin H_i \) and \( a_i \) belongs to the different partite with \( v \). For every \( x \in (V(G) - S) \) holds: (i) if \( x \) belongs to the same partite with \( v \) then \( x \sim a_i \) for some \( i \); (ii) if \( x \) belongs to the same partite with \( a_i \) then \( x \sim v \); and (iii) if \( x \) belongs to the different partite with neither \( v \) and \( a_i \) then both \( x \sim v \) and \( x \sim a_i \) for some \( i \). So \( S \) is a dominating set. Suppose there is \( T \subseteq V(G) \) which \( |T| = t \). If \( v \in T \) then there is \( H_i \) such that no element of this set belongs to \( T \) except \( v \). It implies that the elements of \( H_i \) which belong to the same partite with \( v \) adjacent just with the vertex in the partite that not consist of \( v \). If \( v \notin T \) then there are two cases. First, there is exactly a vertex \( x \) of \( H_i \) belongs to \( T \). In this case, every vertex belongs to the same partite with \( x \) has not adjacent with \( x \). Second, there exists \( H_i \) such that no vertex of \( H_i \) belongs to \( T \). So, every vertex of this \( H_i \) was not adjacent with any element of \( T \).

The following theorem presents the domination number of vertex amalgamation of some cycles.

Theorem 2. If \( G = V^1_{[\omega]} \{C_{n_1}, C_{n_2}, ..., C_{n_t}\} \) then \( \gamma(G) = \sum_{i=1}^{t} \left\lceil \frac{n_i}{3} \right\rceil - t + 1 \).

Proof. Let \( V(C_{n_i}) = \{v_{ij}; i = 1, ..., t, \text{ and } j = 1, ..., n_i\} \). Without loss of generality let \( v = v_{i1} \) be the common vertex. Let \( S = \{v_{i1,3j}; i = 1, ..., t \text{ and } j = 0, 1, ..., \left\lceil \frac{n_i}{3} \right\rceil - 1\} \). For every \( x \in (V(G) - S) \) then \( x = v_{ik} \) where \( i = 1, ..., t \) and \( k \in \{2, 3, ..., n_i; k \not\equiv 1 \mod 3\} \) such that there is \( y \in S \) with \( x \sim y \). So \( S \) is a dominating set. It is clear that \( |S| = 1 + \sum_{i=1}^{t} \left\lceil \frac{n_i}{3} \right\rceil - 1 = \sum_{i=1}^{t} \left( \frac{n_i}{3} - 1 \right) \). If \( v \in T \) then there is \( C_{n_i} \) such that \( \gamma(C_{n_i}) = \left\lceil \frac{n_i}{3} \right\rceil - 1 \). If \( v \notin T \) then \( \gamma(C_{n_i}) = \left\lceil \frac{n_i}{3} \right\rceil - 1 \) for every \( C_{n_i} \).
A specially case of Theorem 2, if the cycles are isomorphic, that is, if $n_1 = n_2 = \cdots = n_t = n$, then
\[
g(G) = t \left( \frac{n}{3} \right) - t + 1.
\]

The following theorem presents the domination number of vertex amalgamation of some paths.

**Theorem 3.** If $G = V^t_{\{v\}} \{P_{n_1}, P_{n_2}, \ldots, P_{n_t}\}$ then
\[
g(G) = \left\{ \begin{array}{ll}
\frac{\sum_{i=1}^t n_i}{3} - t + 1, & \text{if } v \text{ belongs to every } g(P_{n_i}) \text{- set} \\
\frac{\sum_{i=1}^t n_i}{3}, & \text{if } v \text{ has not belongs to any } g(P_{n_i}) \text{- set}.
\end{array} \right.
\]

**Proof.** Let $U = \{v\} \cup \{w \in V(G) : w \sim v\}$. For induced subgraph $(U)$, it holds $G - (U) = \bigcup_{i=1}^t (P_{n_i} - U)$ such that $g(G - (U)) = g\left( \bigcup_{i=1}^t (P_{n_i} - U) \right) = \sum_{i=1}^t g(P_{n_i} - U)$. If $v$ belongs to every $g(P_{n_i})$-set then $g(P_{n_i} - U) = g(P_{n_i}) - 1 = \frac{n_i}{3} - 1$. It is clear that $g((U)) = 1$. So we have $g(G) = g(G - (U)) + g((U)) = \sum_{i=1}^t \left( \frac{n_i}{3} - 1 \right) + 1 = \sum_{i=1}^t \frac{n_i}{3} - t + 1$. If $v$ has not belongs to any $g(P_{n_i})$-set then $v \sim z_i$ which $z_i$ belongs to $g(P_{n_i})$-set. In this case $v$ is independent to the $g(P_{n_i})$, so for $g(G)$ too. It implies $g(G) = ty(P_{n_i})$.

The domination number of vertex amalgamation of some ladders is presented in the following theorem.

**Theorem 4.** If $G = V^t_{\{v\}} \{L_{n_1}, L_{n_2}, \ldots, L_{n_t}\}$ for ladder graph $L_n$ then
\[
g(G) = \left\{ \begin{array}{ll}
\frac{\sum_{i=1}^t (n_i + 1)}{2} - t + 1, & \text{if } v \text{ belongs to every } g(L_{n_i}) \text{- set} \\
\frac{\sum_{i=1}^t (n_i + 1)}{2}, & \text{if } v \text{ has not belongs to any } g(L_{n_i}) \text{- set}.
\end{array} \right.
\]

Before proving the dominating number of a vertex amalgamation of some graphs, we present a lemma concerning on the dominating number of the graph $G$ obtained by joining all vertices of a graph $H$ to a vertex $K_i$, that is, $G = K_1 + H$.

**Lemma 1.** For every graph $G$, $g(G) = 1$ if and only if $G = K_1 + H$ for some graph $H$.

As the diameter of $G = K_1 + H$ is one, it is very easy to see that any one vertex in $G$ can dominate other vertices in $G$. We now have the following theorem.

**Theorem 5.** Let graph $G_i$ which $g(G_i) = 1$ and $G = V^t_{\{v\}} \{G_1, G_2, \ldots, G_t\}$.

a. If $v$ belongs to every $g(G_i)$-set then $g(G) = 1$.

b. If $v$ has not belongs to any $g(G_i)$-set then $g(G) = t$.

c. If $v$ has not belongs to any $g(G_i)$-set for $1 \leq i \leq p$ and $v$ belongs to every $g(G_i)$-set for $p < i \leq t$ then $g(G) = p + 1$.

**Proof.** From Lemma 1, we have $G_i = K_1 + H_i$. Let $|V(H_i)| = n_i$ so $|V(G_i)| = n_i + 1$ and $|V(G)| = \sum_{i=1}^t n_i + 1$. The degree of every vertex in $g(G_i)$-set is $n_i$ and less than $n_i$ for the others.

a. If $v$ belongs to every $g(G_i)$-set then $v$ has $\sum_{i=1}^t n_i$ degree so $\{v\}$ is dominating set.

b. If $v$ has not belong to every $g(G_i)$-set then $v$ adjacent with $t$ vertices which degrees are $n_1, n_2, \ldots, n_t$ respectively. These $t$ vertices span the dominating set of $G$. The cardinality of this dominating set is minimum, if there exist a set with $t-1$ cardinality then there exists $G_i$ where the $n_i$ vertices which less than $n_i$ degree is not adjacent to these $t-1$ vertices.

c. We combine (b) condition for $p$ first subgraphs $G_i$ and (a) condition for the others then we have $g(G) = p + 1$.

The next lemma describes the dominating set of vertex amalgamation of some graphs.
Lemma 2. Let \( G = V^1_t \{ G_1, G_2, \ldots, G_t \} \). If \( S_i \) is a dominating set of \( G_i \) for every \( i = 1,2,\ldots,t \) then \( \bigcup_{i=1}^t S_i \) is a dominating set of \( G \).

In the theorems below we notice \( S_i \) as a dominating set of \( G_i \), then the set \( P_i \) is as \( \gamma(G_i) \)-set, and the set \( P \) is as \( \gamma(G) \)-set.

Theorem 6. Let \( G = V^1_t \{ G_1, G_2, \ldots, G_t \} \) and \( v \) belongs to every \( \gamma(G_i) \)-set, then \( \gamma(G) = \sum_{i=1}^t \gamma(G_i) - t + 1 \).

**Proof.** By Lemma 2, \( \bigcup_{i=1}^t S_i \) is a dominating set of \( G \) such that \( \gamma(G) = \gamma(\bigcup_{i=1}^t G_i) \leq |\bigcup_{i=1}^t S_i| \). If \( v \) belongs to every \( \gamma(G_i) \)-set then \( v \in S_i \) for every \( i \) such that \( \bigcap_{i=1}^t S_i = \{ v \} \). It implies \( |\bigcup_{i=1}^t S_i| = 1 + \sum_{i=1}^t (|S_i| - 1) = \sum_{i=1}^t |S_i| - t + 1 \). We have \( \gamma(G) = \gamma(\bigcup_{i=1}^t G_i) \leq \sum_{i=1}^t |S_i| - t + 1 \). The least upper bound reached for \( |S_i| = \gamma(G_i) \), so \( \gamma(G) \leq \sum_{i=1}^t \gamma(G_i) - t + 1 \). Suppose there exists \( P \subset V(G) \) which \( P = \bigcup_{i=1}^t P_i \) for \( P_i \subset V(G_i) \) such that \( |P| = \sum_{i=1}^t \gamma(G_i) - t \). If \( v \in P \) then there exists \( P_i \) such that \( |P_i| = |\gamma(G_i) - 1| \). However, there exists \( x \in V(G_i) - P_i \) such that for every \( y \in P_i \) holds \( x \neq y \). Because \( V(G_i) \subset V(G) \) it means that there is \( x \in V(G) - P \) such that for every \( y \in P \) holds \( x \neq y \). So, \( P \) have not a dominating set. If \( v \notin P \) then \( |P| = |\gamma(G_i) - 1| \) for every \( i \). It means that \( P_i \) is not a dominating set with minimum cardinality. So does \( P \). ♦

Theorem 7. If \( G = V^1_t \{ G_1, G_2, \ldots, G_t \} \) and \( v \) has not belong to every \( \gamma(G_i) \)-set then \( \gamma(G) = \sum_{i=1}^t \gamma(G_i) \).

**Proof.** By Lemma 2, we have \( \gamma(G) = \gamma(\bigcup_{i=1}^t G_i) \leq |\bigcup_{i=1}^t S_i| \). If there is no \( \gamma(G_i) \)-set consist of \( v \) then \( v \notin S_i \) for every \( i \) such that \( \bigcap_{i=1}^t S_i = \emptyset \). It implies \( \gamma(G) = \gamma(\bigcup_{i=1}^t G_i) \leq |\bigcup_{i=1}^t S_i| = \sum_{i=1}^t |S_i| \). The least upper bound reached for \( |S_i| = \gamma(G_i) \), so \( \gamma(G) \leq \sum_{i=1}^t \gamma(G_i) \). Suppose there is vertices subset \( P = \bigcup_{i=1}^t P_i \) which \( |P| = \sum_{i=1}^t \gamma(G_i) - 1 \). There exist \( P_i \) such that \( |P_i| = \gamma(G_i) - 1 \) for every \( i \). It means that \( P_i \) is not dominating set with minimum cardinality. So do \( P \). ♦

Corollary 8. Let \( G = V^1_t \{ G_1, G_2, \ldots, G_t \} \). If \( v \) belongs to every \( \gamma(G_i) \)-set for \( 1 \leq i \leq p \) and \( v \) has not belongs to any \( \gamma(G_i) \)-set for \( p + 1 \leq i \leq t \), then \( \gamma(G) = \sum_{i=1}^t \gamma(G_i) - p + 1 \).

3. Conclusion

We conclude this paper with the domination number of vertex amalgamation of some graphs at a vertex \( v \) is like the order of these graphs especially if \( v \) belongs to every \( \gamma(G_i) \)-set, that is \( \gamma(V^1_t \{ G_1, G_2, \ldots, G_t \}) = \sum_{i=1}^t \gamma(G_i) - t + 1 \). The following open problems for future work.

Open problem. Find the distance domination number of particular classes of graphs and the graphs obtained from graph operations.

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