COMPONENTWISE DIFFERENT TAIL SOLUTIONS FOR BIVARIATE STOCHASTIC RECURRENCE EQUATIONS – WITH APPLICATION TO GARCH(1, 1) PROCESSES –

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ABSTRACT. We study bivariate stochastic recurrence equations (SREs) motivated by applications to GARCH(1, 1) processes. If coefficient matrices of SREs have strictly positive entries, then the Kesten result applies and it gives solutions with regularly varying tails. Moreover, the tail indices are the same for all coordinates. However, for applications, this framework is too restrictive. We study SREs when coefficients are triangular matrices and prove that the coordinates of the solution may exhibit regularly varying tails with different indices. We also specify each tail index together with its constant. The results are used to characterize regular variations of bivariate stationary GARCH(1, 1) processes.

Key words. Regular variation, bivariate GARCH(1, 1), Kesten’s theorem, stochastic recurrence equation.

1. Introduction

We consider the stochastic recurrence equation (SRE)

\[ W_t = A_t W_{t-1} + B_t, \quad t \in \mathbb{N}, \]

where \((A_t, B_t)\) is an i.i.d. sequence, \(A_t\) are \(d \times d\) matrices, \(B_t\) are vectors and \(W_0\) is an initial distribution independent of the sequence \((A_t, B_t)\). Iterations (1.1) generate a Markov chain \((W_t)_{t \geq 0}\) that is not necessarily stationary. Under mild contractivity hypotheses (see e.g. [8, 10]) the sequence \(W_t\) converges in law to a random variable \(W\) that is the unique solution of the equation

\[ W \overset{d}{=} AW + B, \]

where \(W\) is independent of \((A, B)\) and the equation is meant in law. Here \((A, B)\) is a generic element of the sequence \((A_t, B_t)\). If we put \(W_0 = W\) then the chain \(W_t\) becomes stationary. Moreover, extending the set of indices to \(\mathbb{Z}\) and taking an i.i.d. sequence \((A_t, B_t)_{t \in \mathbb{Z}}\) we can have a strictly stationary causal solution \(W_t\) to the equation

\[ W_t = A_t W_{t-1} + B_t, \quad t \in \mathbb{Z}. \]

It is given by

\[ W_t = \sum_{i=-\infty}^{t} A_t \cdots A_{t-i-1} B_{i} + B_{t} \overset{d}{=} W. \]

There is considerable interest in studying various aspects of the iteration (1.1) and, in particular, the tail behaviour of \(W\). The story started with Kesten [25] who obtained fundamental results about tails of \(W_t\) in the case of matrices \(A_t\) having non-negative entries.

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Given $y = (y_1, \ldots, y_d)$ in the unit sphere $S^{d-1}$, let

$$y'W = \sum_{j=1}^{d} y_j W_j, \quad W = (W_1, \ldots, W_d).$$

Under appropriate assumptions Kesten \cite{25} proved that there is $\alpha > 0$ and a function $e_\alpha$ on $S^{d-1}$ such that

$$\lim_{x \to \infty} x^\alpha \mathbb{P}(y'W > x) = e_\alpha(y), \quad y \in S^{d-1}$$

and $e_\alpha(y) > 0$ for $y \in S^{d-1} \cap [0, \infty)^d$. Later on an analogous result was proved by Alsmeyer and Mentemeier \cite{1} for invertible matrices $A$ with some irreducibility and density conditions.

The density assumption was removed by Guivarc’h and Le Page \cite{23} who developed the most general approach to (1.1) with signed $A$ having possibly a singular law. Moreover, their conclusion was stronger i.e. they obtained existence of a measure $\mu$ on $\mathbb{R}^d$ being the week limit of

$$x^\alpha \mathbb{P}(x^{-1}W \in \cdot) \quad \text{when} \quad x \to \infty.$$ 

The latter means regular variation of $W,$ \cite{1} and it was also proved also for (1.1) with $A$ being similarities \cite{13} i.e. when neither assumptions of \cite{23} nor \cite{1} are satisfied. See \cite{14} for an elementary explanation of Kesten’s result and other results mentioned above.

For all the matrices considered above we have the same tail behavior in all directions, one of the reasons being a certain irreducibility or homogeneity of the action of the group generated by the support of the law of $A$. But it does not always have to be like that. We may imagine $A = \text{diag}(A_{11}, \ldots, A_{dd})$ being diagonal such that $\mathbb{E}A_{ii}^{\alpha_i} = 1$ and $\alpha_1, \ldots, \alpha_d$ are different (see e.g. \cite{12}, \cite{13} and \cite{14} Appendix D)). Then $W_1, \ldots, W_d$ are regularly varying with different exponents $\alpha_1, \ldots, \alpha_d$. In such a case, if we want to say that $W$ is regularly varying we need to modify the notion. For more detailed explanation we refer to \cite{14} Chapter 4 as well as the book by Resnick \cite{30} p. 203, where non-standard regular variation appears in various contexts.

Triangular matrices $A$ do not fit into any of the frameworks mentioned above and therefore considering them is a natural next step. However the existing methods cannot be applied and a new approach is needed. This is what we do here. We study $2 \times 2$ upper triangular matrices $A = [A_{ij}]$ with positive entries (i.e. $A_{21}$ is the only one being zero) such that $\mathbb{E}A_{ii}^{\alpha_i} = 1$ and $\alpha_1 \neq \alpha_2$. We prove that

- if $\alpha_1 > \alpha_2$ then $W = (W_1, W_2)$ is regularly varying with index $\alpha_2$.
- if $\alpha_2 > \alpha_1$ then $W_1$ and $W_2$ are regularly varying with indices $\alpha_1$ and $\alpha_2$ respectively.

This is the content of Theorem 3.2. Then we study regular variation of $W_t$ as a time series. In the first case we describe the spectral process $Y_t$ in the sense of \cite{4} corresponding to $W_t$. It is of the form

$$Y_t = A_t \cdots A_1 Y_0,$$

where $Y_0 = \|Y_0\| \Theta_0$, $\mathbb{P}(\|Y_0\| > u) = u^{-\alpha}$, $u \geq 1$ and the law of $\Theta_0$ is the spectral measure of $W$, see Proposition 3.4. In the second case we consider $W_{1,t}$ and $W_{2,t}$ separately (Lemma 3.3).

Our results are interesting from the point of view of financial analysis and they apply to the squared volatility sequence $W_t = (\sigma^2_{1,t}, \sigma^2_{2,t})$ of the bivariate GARCH(1,1) financial model, see

\footnote{1}If $\alpha \notin \mathbb{N}$ then \cite{14} implies regular variation of $W$. If $\alpha \in \mathbb{N}$, the same holds with some additional conditions (see \cite{14} Appendix C). For more on regular variation, we refer to Bingham et al. \cite{5} and Resnick \cite{29,30} in the univariate and multivariate cases, respectively.

\footnote{2}A is a similarity if for every $x \in \mathbb{R}^d$, $|Ax| = \|A\| |x|$.}
Section 4. Then $W_t$ satisfies (1.1) with matrices $A_t$, having non-negative entries. If all the entries of $A_t$ are strictly positive then the theorem of Kesten applies and both $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$ are regularly varying with the same index, see [26], [27]. But if this is not the case then we have to go beyond Kesten’s result and Theorem 3.2 below enters into the picture. From the point of view of applications it is reasonable to relax the assumptions on $A_t$ because it allows us to capture a larger class of financial models.

When matrices $A_t$ are upper triangular we may apply the above results to obtain regular variation of $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$. Namely if $\alpha_1 < \alpha_2$ then $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$ are regularly varying with indices $\alpha_1$ and $\alpha_2$ respectively. If $\alpha_1 > \alpha_2$ then $(\sigma_{1,t}^2, \sigma_{2,t}^2)$ is regularly varying with the index $\alpha_2$ and we have a nice description of its spectral process. Finally, in Propositions 4.2 and 4.3 we study regular variation of the bivariate GARCH$(1,1)$ itself $X_t = (\sigma_{1,t}Z_{1,t}, \sigma_{2,t}Z_{2,t})$.

It turns out that the appearance of triangular matrices in (1.1) generates a lot of technical complications, it is challenging and it is far from being solved in arbitrary dimension. Even for $2 \times 2$ matrices, the case when $\alpha_1 = \alpha_2$ is, in our opinion, out of reach in full generality at the moment. There is a preprint [15] on that case with an extra assumption that $A_{11} = A_{22}$.

2. Bivariate stochastic recurrence equations

We start with the description of the model as well as conditions for stationarity of the related time series.

2.1. The model. We consider the bivariate SRE;

\begin{equation}
W_t = A_t W_{t-1} + B_t, \quad t \in \mathbb{Z},
\end{equation}

where

\begin{equation}
W_t = \begin{pmatrix} W_{1,t} \\ W_{2,t} \end{pmatrix}, \quad A_t = \begin{pmatrix} A_{1,t} & A_{2,t} \\ 0 & A_{4,t} \end{pmatrix} \quad \text{and} \quad B_t = \begin{pmatrix} B_{1,t} \\ B_{2,t} \end{pmatrix},
\end{equation}

and an i.i.d. matrix sequence $(A_t)$ and an i.i.d. vector sequence $(B_t)$. We assume $A_{i,t} > 0$ a.s. $i = 1, 2, 4$ and $B_{i,t} > 0$ a.s. $i = 1, 2$. For our purpose, it is convenient to write the SRE in a coordinate-wise form;

\begin{align}
W_{1,t} &= A_{1,t} W_{1,t-1} + D_t, \\
W_{2,t} &= A_{4,t} W_{2,t-1} + B_{2,t},
\end{align}

where $D_t := B_{1,t} + A_{2,t} W_{2,t-1}$. We sometimes omit the subscript 0 in $A_{i,0}$, $B_{i,0}$ and $W_{i,0}$, etc. and just write $A_i$, $B_i$ and $W_i$ if they are stationary. For further convenience we denote for $t \in \mathbb{Z}$

$$
\Pi_{t,s} = A_t \cdots A_s, \quad t \geq s, \quad \Pi_{t,s} = I, \quad t < s \quad \text{and} \quad \Pi_t = \Pi_{t,1},$$

$$
\Pi_{t,s}^{(i)} = \prod_{j=s}^{t} A_{i,j}, \quad t \geq s, \quad i = 1, 2, 4 \quad \text{and} \quad \Pi_{t,s}^{(i)} = 1, \quad t < s \quad \text{and} \quad \Pi_t^{(i)} = \Pi_{t,1}^{(i)},
$$

where $I$ is the bivariate identity matrix. For a vector $x \in \mathbb{R}^d$, $|x|$ denotes its Euclidean norm and for a $d \times d$ matrix $A$ we use the matrix norm;

$$
||A|| = \sup_{x \in \mathbb{R}^d, |x| = 1} |Ax|.
$$
2.2. Stationarity. Starting from Kesten [25] there is a series of results [10, 8] for the existence of stationary solution for SRE (see also [14]). The notion of the “so called” top Lyapunov exponent

$$\gamma = \inf_{n \geq 1} n^{-1} \mathbb{E} \log \| \Pi_n \|$$

associated with the sequence \((A_i)\) is always essential. If \(\gamma\) is negative and some logarithmic moment conditions are satisfied, then SRE (2.1) has a unique strictly stationary solution ([8] or [14, Theorem 4.1.4]).

In our setting, \(\gamma < 0\) if there is \(\varepsilon > 0\) such that

$$\mathbb{E} A_i^\varepsilon < 1, \quad \mathbb{E} A_4^\varepsilon < 1 \quad \text{and} \quad \mathbb{E} A_2^\varepsilon < \infty.$$  

We are going to show this. Without loss of generality we may assume that \(\varepsilon < 1\).

First observe that by the Jensen’s inequality

$$\gamma = \inf_{n \geq 1} (n\varepsilon)^{-1} \mathbb{E} \log \| \Pi_n \|^{\varepsilon} \leq \inf_{n \geq 1} (n\varepsilon)^{-1} \log \mathbb{E} \| \Pi_n \|^{\varepsilon}.$$  

Secondly, we decompose the matrix \(A_t = S_t + N_t\) into the sum of a diagonal and a nilpotent one, where

$$S_t = \begin{pmatrix} A_{1,t} & 0 \\ 0 & A_{4,t} \end{pmatrix} \quad \text{and} \quad N_t = \begin{pmatrix} 0 & A_{2,t} \\ 0 & 0 \end{pmatrix},$$

so that \(S_tS_j\) is diagonal and \(S_tN_j, N_tS_j\) are nilpotent. Then we write

$$\Pi_n = A_n \cdots A_1 = (S_n + N_n) \cdots (S_1 + N_1)$$

as the sum of \(2^n\) products. Moreover, observe that the product of bivariate matrices vanishes if the matrices have only zero entries except in the top right corner, and therefore only terms including at most one \(N_i\) are nonzero. Hence we have

$$\| \Pi_n \|^{\varepsilon} = \| \sum_{i=0}^{n} S_{i} \cdots S_{i+1} N_i S_{i-1} \cdots S_1 \|^{\varepsilon} \leq \sum_{i=0}^{n} \| S_{i} \cdots S_{i+1} N_i S_{i-1} \cdots S_1 \|^{\varepsilon},$$

where \(N_0 = S_0 = S_{-1} = I\). Notice that \(S_{i} \cdots S_{i+1} N_i S_{i-1} \cdots S_1, i \neq 0\) is a nilpotent matrix and its only nonzero entry is \(\Pi_{n,i+1}^{(4)} A_{2,i} \Pi_{i-1}^{(1)}\). Moreover, all terms in the product are independent, so we may write

$$\mathbb{E} \| \Pi_n \|^{\varepsilon} \leq \mathbb{E} \sum_{i=1}^{n} \left( \Pi_{n,i+1}^{(4)} A_{2,i} \Pi_{i-1}^{(1)} \right)^{\varepsilon} = \sum_{i=1}^{n} (\mathbb{E} A_i^{\varepsilon})^{n-i} \mathbb{E} A_2^{\varepsilon} (\mathbb{E} A_1^{\varepsilon})^{i-1} \leq nc \rho^n,$$

where \(\rho = \max(\mathbb{E} A_i^{\varepsilon}, \mathbb{E} A_2^{\varepsilon}) < 1\) and \(c = \rho^{-1} \mathbb{E} A_2^{\varepsilon} < \infty\).

Eventually, we can estimate the top Lyapunov exponent,

$$\gamma \leq \inf_{n \geq 1} (n\varepsilon)^{-1} \log ncp^n = \varepsilon^{-1} \inf_{n \geq 1} \left( \frac{\log nc}{n} + \log \rho \right) = \varepsilon^{-1} \log \rho < 0$$

which is what we needed. If we assume additionally that

$$\mathbb{E} \log^+ |B| < \infty,$$

then we may conclude that there exists an a.s. unique causal strictly stationary ergodic solution to SRE (2.1) given by the infinite series,

$$W_t = \sum_{i=-\infty}^{t} \Pi_{t,i+1} B_i;$$

Moreover (2.3) and (2.4) are in agreement with (2.8).
Due to Theorem 1 of [10] (see also [14] Section 2.1), a strictly stationary positive solution for (2.4) exists if
\[ \mathbb{E} \log A_4 < 0 \quad \text{and} \quad \mathbb{E} \log^+ B_2 < \infty. \]
Notice that from stationarity condition of bivariate case, those for component wise SREs are automatically satisfied, i.e. \( \mathbb{E} \log A_1 \) and \( \mathbb{E} \log A_4 \) are smaller than 0, which ensures strictly stationary solution
\[ W_{2,t} = \sum_{i=1}^{\infty} \Pi_{t,t+2,r} B_{2,t+1-i}. \]
Now consider this stationary version \( (W_{2,t}) \) for \( (D_t) \) of SRE (2.3). Since \( W_{2,t} \) is independent of elements \((A_{2,t}, B_{1,t})\), \( \ell > k \) of i.i.d. sequence \((A_{2,t}, B_{1,t})\), we observe that
\[ D_t = B_{1,t} + A_{2,t} W_{2,t-1} \]
is stationary and ergodic, so is the sequence \((A_{1,t}, D_t)\). Then from Theorem 1 of [10] the series \((W_{1,t})\) has the stationary solution given by
\[ W_{1,t} = \sum_{i=0}^{\infty} \Pi_{t,t+1-r} D_{t-i} = \sum_{i=1}^{\infty} \Pi_{t,t+2-r} D_{t+1-i}. \]
Since \((W_{1,t}, W_{2,t})\) satisfies SRE (2.1), then by the uniqueness of the solution, we have

**Lemma 2.1.** Suppose that (2.5) and (2.7) are satisfied and let \( W_{1,t}, W_{2,t} \) be stationary solutions to (2.3) and (2.4) respectively. Then \((W_{1,t}, W_{2,t})\) is the stationary solution to (2.1).

Therefore, in the sequel if the stationarity condition, i.e. (2.5) and (2.7), is satisfied, we may work on these component-wise solutions.

### 3. Main results

#### 3.1. Component-wise tail behavior

Our aim of this section is to describe the tail behavior of \( W_{1,t} \) and \( W_{2,t} \). This is the content of Theorem 3.2 below. We are going to use a fundamental result for one-dimensional SRE formulated below as Theorem 3.1. The statement appeared first in [25] as a corollary of a more general result, but the “right” proof for the one-dimensional case was given later on by Goldie [22] with the constants in the tails specified for the first time. Consider the SRE;
\[ X_t = Q_t X_{t-1} + R_t, \quad t \in \mathbb{Z}, \]
where \(((Q_t, R_t))_{t \in \mathbb{Z}}\) is an \( \mathbb{R}^2 \)-valued i.i.d. sequence. The generic random variable (r.v.) of the sequence \(((Q_t, R_t))\) is denoted by \((Q, R)\).

**Theorem 3.1.** Assume that the following conditions hold.
1. \( Q > 0 \) a.s. and \( \log Q \) is non-arithmetic.
2. There is \( \alpha > 0 \) such that \( \mathbb{E} Q^\alpha = 1, \mathbb{E} R^\alpha < \infty \) and \( \mathbb{E} Q^\alpha \log^+ Q < \infty. \)
3. \( \mathbb{P}(Rx + Q = x) < 1 \) for every \( x \in \mathbb{R}. \)

Then the equation \( X \overset{d}{=} QX + R \) has a solution \( X \) which is independent of \((Q, R)\) and there exist constants \( c_+, c_- \) such that \( c_+ + c_- > 0 \) and
\[ \mathbb{P}(X > x) \sim c_+ x^{-\alpha} \quad \text{and} \quad \mathbb{P}(X \leq -x) \sim c_- x^{-\alpha}, \quad x \to \infty. \]

The constants \( c_+, c_- \) are given by
\[ c_+ = \frac{1}{\alpha m_\alpha} \mathbb{E}[(QX + R)^\alpha] \quad \text{and} \quad c_- = \frac{1}{\alpha m_\alpha} \mathbb{E}[(QX + R)^\alpha - (QX)^\alpha], \]
where \( m_\alpha = \mathbb{E} Q^\alpha \log Q > 0. \)
Due to stationarity it is enough to consider $W_{1,0}$ and $W_{2,0}$. The above result is directly applicable to $W_{2,0}$ but not to $W_{1,0}$. The estimate of the tail of $W_{1,0}$ is more delicate. Due to non-negativity, we may write the stationary solution (2.10) as

$$W_{1,0} = \sum_{i=1}^{\infty} \Pi_{0,2-i}^{(1)} D_{1-i} = \sum_{i=1}^{\infty} \Pi_{0,2-i}^{(1)} A_{2,1-i} W_{2,-i} + \sum_{i=1}^{\infty} \Pi_{0,2-i}^{(1)} B_{1,1-i},$$

and analyze these infinite sums separately. Consider one more SRE;

$$\bar{W}_{1,t} = B_{1,t} + A_{1,t} \bar{W}_{1,t-1}.$$ 

Its stationary solution is

$$\bar{W}_{1,0} = \sum_{i=1}^{\infty} \Pi_{0,2-i}^{(1)} B_{1,1-i}.$$ 

This corresponds to the second term in (3.1).

Assume that there exist $\alpha_1$ and $\alpha_2$ such that

$$\mathbb{E} A_1^{\alpha_1} = 1, \quad \mathbb{E} A_1^{\alpha_1} \log^+ A_1 < \infty \quad \text{and} \quad \mathbb{E} B_1^{\alpha_1} < \infty,$$

$$\mathbb{E} A_4^{\alpha_2} = 1, \quad \mathbb{E} A_4^{\alpha_2} \log^+ A_4 < \infty \quad \text{and} \quad \mathbb{E} B_2^{\alpha_2} < \infty,$$

then due to Theorem 3.2 we have

$$\mathbb{P}(\bar{W}_{1,0} > x) \sim c_1 x^{-\alpha_1},$$

$$\mathbb{P}(W_{2,0} > x) \sim c_2 x^{-\alpha_2},$$

where positive constants $c_1$ and $c_2$ are given by

$$c_1 = \frac{1}{\alpha_1 \mathbb{E} A_1^{\alpha_1} \log A_1} \mathbb{E}[(A_1 \bar{W}_{1,0} + B_1)^{\alpha_1} - (A_1 \bar{W}_{1,0})^{\alpha_1}].$$

$$c_2 = \frac{1}{\alpha_2 \mathbb{E} A_4^{\alpha_2} \log A_4} \mathbb{E}[(A_4 W_{2,0} + B_2)^{\alpha_2} - (A_4 W_{2,0})^{\alpha_2}].$$

Now we are ready to describe the tail behavior of $W_{1,0}$. Its tail index is equal to $\min(\alpha_1, \alpha_2)$.

**Theorem 3.2.** Consider the bivariate SRE (2.1) such that $\log A_1$ and $\log A_4$ are non-arithmetic. Assume that (3.3) holds and $\mathbb{E} A_2^{\min(\alpha_1,\alpha_2)} < \infty$. Then the stationary solution $W_t$ satisfies

$$\mathbb{P}(W_{1,0} > x) \sim \begin{cases} c_1 x^{-\alpha_1} & \text{if } \alpha_1 < \alpha_2 \\ c_1 x^{-\alpha_2} & \text{if } \alpha_1 > \alpha_2, \end{cases}$$

where

$$c_1 = \frac{2}{\alpha_1} \mathbb{E}[(D_0 + A_{1,0} W_{1,-1})^{\alpha_1} - (A_{1,0} W_{1,-1})^{\alpha_1}],$$

$$c_1 = c_2 \mathbb{E} \left( \lim_{s \to \infty} \sum_{i=1}^{r} \Pi_{0,2-i}^{(1)} A_{2,1-i} \Pi_{i,1-s}^{(4)} \right)^{\alpha_2}.$$ 

Note that the limit in (3.6) has a somewhat strange form, and it seems difficult to write it as just an infinite sum. However, its convergence is guaranteed in the proof.
Proof. Stationarity condition (2.5) follows from (3.3). Indeed, the functions $g_i(h) = \mathbb{E}A_i^h$, $i = 1, 4$ are convex and so there exist $\alpha < \min(\alpha_1, \alpha_2)$ such that $\mathbb{E}A_{i}^{\alpha} < 1$. Hence we may work on the stationary version.

Suppose $\alpha_1 > \alpha_2$. Observe that in (3.1), each term of the first infinite sum includes $W_2$, $\mathbb{E}A_{1}^{\alpha_2}$, $\mathbb{E}A_{2}^{\alpha_2} < \infty$, and the second sum equals in distribution to $\tilde{W}_{1,0}$. Then since
\[
\frac{\mathbb{P}(\tilde{W}_{1,0} > x)}{\mathbb{P}(W_{2,0} > x)} \sim \frac{c_1 + x^{\alpha_2 - \alpha_1}}{c_2} \rightarrow 0, \quad as \quad x \rightarrow \infty,
\]
it suffices to consider the sum
\[
\tilde{X} = \sum_{i=1}^{\infty} \Pi_{0,2}^{(i)} A_{2,1-i} W_{2,-i} = \sum_{i=1}^{\infty} \Pi_{0,2}^{(i)} A_{2,1-i} W_{2,-i} + \sum_{i=s+1}^{\infty} \Pi_{0,2}^{(i)} A_{2,1-i} W_{2,-i} =: \tilde{X}_s + \tilde{X}_s'.
\]
Indeed, $\tilde{X} \geq A_{2,0} W_{2,-1}$ and so
\[
\mathbb{P}(\tilde{X} > x) \geq c x^{-\alpha_2}
\]
for a strictly positive $c$. Therefore, we can invoke the property of dependent summands of regularly varying r.v.’s (Lemma B.6.1 of [14]) in order to obtain
\[
\mathbb{P}(W_{1,0} > x) = \mathbb{P}(\tilde{X} + W_{1,0} > x) \sim \mathbb{P}(\tilde{X} > x) + \mathbb{P}(W_{1,0} > x) \sim \mathbb{P}(\tilde{X} > x).
\]
We start with $\tilde{X}_s$ and apply the induction to $W_{2,-i}$ in $\tilde{X}_s$. Since
\[
W_{2,-i} = A_{4,t} W_{2,t-1} + B_{2,t} = A_{4,t} A_{4,t-1} W_{2,t-2} + A_{4,t} B_{2,t-1} + B_{2,t} = A_{4,t} A_{4,t-1} \cdots A_{4,t-j+1} W_{2,t-j} + \sum_{k=1}^{j-1} A_{4,t} \cdots A_{4,t-k+1} B_{2,t-k} + B_{2,t} = \Pi_{t,t-j+1}^{(4)} W_{2,t-j} + \sum_{k=0}^{j-1} \Pi_{t,t-k+1}^{(4)} B_{2,t-k},
\]
we change indices $(t, j) \rightarrow (i, s)$ such that $t = -i$ and $s = j - t$ to obtain
\[
W_{2,-i} = \Pi_{-i,1-s}^{(4)} W_{2,-s} + \sum_{k=0}^{s-i-1} \Pi_{-i,1-i-k}^{(4)} B_{2,-i-k}.
\]
Substitution of the above into $\tilde{X}_s$ yields
\[
\tilde{X}_s = \sum_{i=1}^{s} \Pi_{0,2}^{(i)} A_{2,1-i} (\Pi_{-i,1-s}^{(4)} W_{2,-s} + \sum_{k=0}^{s-i-1} \Pi_{-i,1-i-k}^{(4)} B_{2,-i-k})
\]
\[
= \sum_{i=1}^{s} \Pi_{0,2}^{(i)} A_{2,1-i} \Pi_{-i,1-s}^{(4)} W_{2,-s} + \sum_{i=1}^{s} \Pi_{0,2}^{(i)} A_{2,1-i} \sum_{k=0}^{s-i-1} \Pi_{-i,1-i-k}^{(4)} B_{2,-i-k} = \tilde{X}_{s,1} + \tilde{X}_{s,2}
\]
Now we are going to prove that $\mathbb{E}\tilde{X}_{s,2}^{\alpha_2} < \infty$. Recall that $(A_i, B_i)$ are i.i.d. and hence $\Pi_{0,2}^{(i)} A_{2,1-i}$ and $\sum_{k=0}^{s-i-1} \Pi_{-i,1-i-k}^{(4)} B_{2,-i-k}, i = 1, 2, \ldots, s$ are independent. Moreover, $\mathbb{E}A_{i}^{\alpha_2} = 1$, $\mathbb{E}A_{1}^{\alpha_2} < 1$, $\mathbb{E}A_{2}^{\alpha_2} < \infty$
and $\mathbb{E}B_i^{a_2} < \infty$. By the Minkowski inequality

$$
\mathbb{E}X_{s,2}^{a_2} = \mathbb{E}\left( \sum_{i=1}^{s} \prod_{0,2-i}^{(1)} A_{2,1-i} \prod_{k=0}^{s-i-1} \prod_{i-1-i-k}^{(4)} B_{2,-i-k} \right)^{a_2}
\leq \left[ \sum_{i=1}^{s} \left( (\mathbb{E}A_1^{a_2})^{-1} \mathbb{E}B_2^{a_2} (s - i)^{a_2} \right)^{1/a_2} \right]^{a_2} < \infty
$$

for $a_2 > 1$ and for $a_2 \leq 1$ by the triangle inequality

$$
\mathbb{E}X_{s,2}^{a_2} = \sum_{i=1}^{s} \mathbb{E}(\prod_{0,2-i}^{(1)} A_{2,1-i} \mathbb{E}(\sum_{k=0}^{s-i-1} \prod_{i-1-i-k}^{(4)} B_{2,-i-k} ))^{a_2}
\leq \mathbb{E}A_2^{a_2} \mathbb{E}B_2^{a_2} \sum_{i=1}^{s} (\mathbb{E}A_1^{a_2})^{-1} (s - i) < \infty.
$$

Hence we have

$$
\limsup_{x \to \infty} \frac{\mathbb{P}(\tilde{X}_{s,2} > x)}{\mathbb{P}(W_{2,0} > x)} = 0
$$

for fixed $s$. Since all the terms are positive

$$
\mathbb{P}(\tilde{X}_{s,1} > (1 + \varepsilon)x) \leq \mathbb{P}(\tilde{X} > x) \leq \mathbb{P}(\tilde{X}_{s,1} > (1 - \varepsilon)x) + \mathbb{P}(\tilde{X}_{s,2} > \frac{\varepsilon}{2} x) + \mathbb{P}(\tilde{X} > \frac{\varepsilon}{2} x)
$$

for some $\varepsilon \in (0, 1)$, it follows from regular variation of $W_{2,0}$ that for $s \geq 1$,

$$
(1 + \varepsilon)^{-a_2} w_s \leq \liminf_{x \to \infty} \frac{\mathbb{P}(\tilde{X} > x)}{\mathbb{P}(W_{2,0} > x)} \leq \limsup_{x \to \infty} \frac{\mathbb{P}(\tilde{X} > x)}{\mathbb{P}(W_{2,0} > x)} \leq (1 - \varepsilon)^{-a_2} w_s
$$

(3.11)

$$
+ \limsup_{x \to \infty} \frac{\mathbb{P}(\tilde{X}_{s,2} > \frac{\varepsilon}{2} x)}{\mathbb{P}(W_{2,0} > x)} + \limsup_{x \to \infty} \frac{\mathbb{P}(\tilde{X} > \frac{\varepsilon}{2} x)}{\mathbb{P}(W_{2,0} > x)},
$$

where

$$
w_s := \mathbb{E}\left( \sum_{i=1}^{s} \prod_{0,2-i}^{(1)} A_{2,1-i} \prod_{i-1-i-k}^{(4)} B_{2,-i-k} \right)^{a_2}.
$$

Hence, in order to obtain the result, it suffices to show that

$$
\lim_{s \to \infty} \limsup_{x \to \infty} \frac{\mathbb{P}(\tilde{X} > x)}{\mathbb{P}(W_{2,0} > x)} = 0.
$$

(3.12)

However, by Markov inequality together with conditioning, it follows that

$$
\frac{\mathbb{P}(\tilde{X} > x)}{\mathbb{P}(W_{2,0} > x)} = \frac{\mathbb{P}(\sum_{i=s+1}^{\infty} \prod_{0,2-i}^{(1)} A_{2,1-i} W_{2,-i} > x)}{\mathbb{P}(W_{2,0} > x)} \\
\leq \sum_{i=1}^{\infty} \mathbb{P}(\prod_{0,2-(s+i)}^{(1)} A_{2,1-i} W_{2,-(s+i)} > x \mu \zeta(\mu)) / \mathbb{P}(W_{2,0} > x) \\
= \sum_{i=1}^{\infty} \mathbb{E}\left( \frac{\mathbb{P}(G_i W_{2,-(s+i)} > x \ | \ G_i)}{\mathbb{P}(W_{2,0} > x)} \right),
$$
where \( G_i = \zeta(\mu) \pi^{(1)}_{0,2-\alpha_1} A_{2,1-\alpha_1} \) with \( \mu > 1 \) and \( \zeta(\mu) \) is the zeta function. Then

\[
\limsup_{x \to \infty} \frac{\mathbb{P}(X > x)}{\mathbb{P}(W_{2,0} > x)} = \sum_{i=1}^\infty \mathbb{E} \left[ \limsup_{x \to \infty} \frac{\mathbb{P}(G_i W_{2,0-\alpha_1} > x | G_i)}{\mathbb{P}(W_{2,0} > x)} \right]
\]

\[
\leq c \sum_{i=1}^\infty \mathbb{E} G_i^{\alpha_2}
\]

\[
= c \sum_{i=1}^\infty \mathbb{E}(\pi^{(1)}_{0,2-\alpha_1})^{\alpha_2} \mathbb{E} A_{2,1-\alpha_1}^{\alpha_2} \zeta(\mu) i^{\alpha_2} \mu
\]

\[
\leq c'(\mathbb{E} A_1^{\alpha_2})^s \sum_{i=1}^\infty (\mathbb{E} A_1^{\alpha_2})^i i^{\alpha_2} \mu,
\]

where \( c \) and \( c' \) are some positive constants. Since \( \mathbb{E} A_1^{\alpha_2} < 1 \) so that \( \sum_{i=1}^\infty (\mathbb{E} A_1^{\alpha_2})^i i^{\alpha_2} \mu < \infty \), we take \( s \to \infty \) and obtain (3.12). Now, as before, if \( \alpha_2 \leq 1 \) then

\[
w_s \leq \sum_{i=1}^\infty (\mathbb{E} A_1^{\alpha_2})^{i-1} \mathbb{E} A_2^{\alpha_2} < \infty
\]

and if \( \alpha_2 > 1 \) then

\[
w_s^{1/\alpha_2} \leq \sum_{i=1}^\infty (\mathbb{E} A_1^{\alpha_2})^{(i-1)/\alpha_2} (\mathbb{E} A_2^{\alpha_2})^{1/\alpha_2} < \infty.
\]

Moreover, in view of (3.8) and (3.11) there is \( c'' > 0 \) such that for every \( s \), \( w_s \geq c'' \). Hence taking a converging subsequence \( w_{s_k} \) of \( w_s \), we obtain

(3.13)

\[
\lim_{s \to \infty} \frac{\mathbb{P}(X > x)}{\mathbb{P}(W_{2,0} > x)} = \lim_{k \to \infty} w_{s_k} =: w > 0,
\]

which, in particular, proves that \( \lim_{s \to \infty} w_s = w \) because we have the same limit for any converging subsequence. Finally, we obtain

\[
\mathbb{P}(W_{1,0} > x) \sim \lim_{s \to \infty} \mathbb{E}(\sum_{i=1}^s \pi^{(1)}_{0,2-i} A_{2,1-\alpha_1}^{\alpha_2} \pi^{(1)}_{i,1-i})^{\alpha_2} \mathbb{P}(W_{2,0} > x).
\]

Suppose now that \( \alpha_1 < \alpha_2 \). Observe that

\[
W_{1,0} = D_0 + \sum_{i=1}^\infty A_{1,0} \cdots A_{1,1-i} D_{-i}
\]

\[
= D_0 + A_{1,0} D_{-1} + \sum_{i=2}^\infty A_{1,0} \cdots A_{1,1-i} D_{-i}
\]

\[
= D_0 + A_{1,0} \left( D_{-1} + \sum_{i=1}^\infty A_{1,1-i} \cdots A_{1,1-i} D_{-i-1} \right)
\]

\[
= D_0 + A_{1,0} W_{1,-1},
\]

Then \( W_{1,-1} \) has the same law as \( W_{1,0} \) and is independent of \( A_{1,0} \). We are going to use Theorem 2.3 of Goldie [22]. We will have to prove that

\[
I = \int_0^\infty \left[ \mathbb{P}(W_{1,-1} > x) - \mathbb{P}(A_{1,0} W_{1,-1} > x) \right] x^{\alpha_1 - 1} dx < \infty.
\]
Then Theorem 2.3 of [22] implies that
\[
\lim_{t \to \infty} \mathbb{P}(W_{1,-1} > t) = C_+ 
\]
with
\[
C_+ = \frac{1}{\mathbb{E}A_1^{\alpha_1} \log A_1} \int_0^\infty (\mathbb{P}(W_{1,-1} > t) - \mathbb{P}(A_{1,0}W_{1,-1} > t))t^{\alpha_1-1}dt.
\]
In view of Lemma 9.4 in [22],
\[
I = \int_0^\infty |\mathbb{P}(W_{1,-1} > t) - \mathbb{P}(A_{1,0}W_{1,-1} > t)t^{\alpha_1-1}dt
\]
\[
= \int_0^\infty |\mathbb{P}(D_0 + A_{1,0}W_{1,-1} > t) - \mathbb{P}(A_{1,0}W_{1,-1} > t)t^{\alpha_1-1}dt
\]
\[
= \frac{1}{\alpha_1} \mathbb{E}[(D_0 + A_{1,0}W_{1,-1})^{\alpha_1} - (A_{1,0}W_{1,-1})^{\alpha_1}] = \mathbb{E}A_1^{\alpha_1} \log A_1 \cdot C_+.
\]
We have to prove that 0 < I < \infty. The first inequality is obvious, since all the variables are positive and D_0 + A_{1,0}W_{1,-1} > A_{1,0}W_{1,-1} with positive probability. If \( \alpha_1 \leq 1 \), then
\[
I \leq \frac{1}{\alpha_1} \mathbb{E}D_0^{\alpha_1} < \infty.
\]
If \( \alpha_1 > 1 \), then
\[
I \leq \mathbb{E}[(D_0 + A_{1,0}W_{1,-1})^{\alpha_1-1}D_0],
\]
where we have used the fact that \( b^\epsilon - a^\epsilon = \int_a^b k x^{\epsilon-1} dx \leq \kappa b \epsilon^{\epsilon-1} (b - a) \) for any \( \kappa > 1 \) and \( b \geq a \geq 0 \). Further we have
\[
I \leq \max(2^{\alpha_1-2}, 1) (\mathbb{E}D_0^{\alpha_1} + \mathbb{E}A_{1,0}W_1^{\alpha_1-1}D_0),
\]
where we use the Minkowski’s inequality and subadditivity of concave functions depending on whether \( \alpha_1 \geq 2 \) or \( \alpha_1 \leq 2 \), and we need to prove that
\[
\mathbb{E}(A_{1,0}W_{1,-1})^{\alpha_1-1}D_0
\]
\[
= \mathbb{E}A_{1,0}W_{1,-1}^{\alpha_1-1}B_{1,0} + \mathbb{E}(A_{1,0}W_{1,-1})^{\alpha_1-1}A_{2,0}W_{2,-1} < \infty.
\]
Since \( A_{1,0}, B_{1,0} \) and \( W_{1,-1} \) are independent and since \( A_{1,0}^{-1}A_{2,0} \) and \( (W_{1,-1})^{\alpha_1-1}W_{2,-1} \) are independent, what we need to show is
\[
\mathbb{E}[A_{1,0}^{-1}B_{1,0}]\mathbb{E}W_{1,0}^{\alpha_1-1} + \mathbb{E}[A_{1,0}^{-1}A_{2,0}]\mathbb{E}(W_{1,-1})^{\alpha_1-1}W_{2,-1} < \infty.
\]
By Hölder’s inequality, for X = B_{1,0} or A_{2,0} we have
\[
\mathbb{E}[A_{1,0}^{-1}X] \leq (\mathbb{E}X^{\alpha_1-1})^{1/p}(\mathbb{E}X^q)^{1/q} = (\mathbb{E}A_{1,0}^{\alpha_1})^{1/p}(\mathbb{E}X^q)^{1/q} < \infty
\]
with \( p = \alpha_1/\alpha_1 \) and \( q = \alpha_1 \). This together with \( \mathbb{E}W_{1,0}^{\alpha_1-1} < \infty \) shows \( \mathbb{E}[A_{1,0}^{-1}B_{1,0}]\mathbb{E}W_{1,0}^{\alpha_1-1} < \infty \). Therefore it suffices to see \( \mathbb{E}[(W_{1,-1})^{\alpha_1-1}W_{2,-1}] < \infty \). For any positive \( \epsilon < \alpha_2 - \alpha_1 \) we can deduce that \( \mathbb{E}W_2^{\alpha_2+\epsilon} < \infty \) by the property (3.4). Let \( q = \alpha_1 + \epsilon \) and its Hölder conjugate \( p = q/(q-1) \). Then
\[
\mathbb{E}[(W_{1,-1})^{\alpha_1-1}W_{2,-1}] \leq (\mathbb{E}W_{1,0}^{\alpha_1-1})^{1/p}(\mathbb{E}W_{2,0}^{q})^{1/q},
\]
where \( \mathbb{E}W^q_2 < \infty \). Notice that \( \beta := p(\alpha_1 - 1) < \alpha_1 \) because \( p \) is decreasing in \( q \) and \( p = \alpha_1/(\alpha_1 - 1) \) implies \( q = \alpha_1 \). Here we may choose \( c_0 \) such that \( 1 < \beta \). Then by convexity \( \mathbb{E}A^{\beta}_1 < 1 \) and moreover, since \( 1 < \beta < \alpha_1 < \alpha_2 \), for \( k \in \mathbb{Z} \)

\[
\mathbb{E}D^\beta_k \leq ((\mathbb{E}B^\beta_{1,k})^{1/\beta} + (\mathbb{E}(A_{2,k}W_{2,k-1})^\beta)^{1/\beta})^\beta < \infty.
\]

Now, notice that \( \mathbb{E}W^\beta_{1,0} = \mathbb{E}(\sum_{i=0}^\infty \Pi_{0,1-i}^1D_{-i})^\beta \) and so by Minkowski’s inequality, it is enough to prove that

\[
\sum_{i=0}^\infty (\mathbb{E}(\Pi_{0,1-i}^1D_{-i})^\beta)^{1/\beta} = \sum_{i=0}^\infty (\mathbb{E}A^{\beta}_1)^{1/\beta}(\mathbb{E}D^\beta_k)^{1/\beta} < \infty,
\]

which holds, because \( \mathbb{E}A^{\beta}_1 < 1 \).

\[\square\]

3.2. Regular variation. Now we are going to study regular variation of the strictly stationary time series \((W_t) = ((W_{1,t}, W_{2,t}))\). As before, we distinguish two cases: \( \alpha_1 < \alpha_2 \) and \( \alpha_1 > \alpha_2 \). In the first case the tail indices of \((W_{i,t}), i = 1, 2\) are distinct so we consider the components separately.

Let us start with discussing regular variation. A univariate time series is said to be regularly varying if its finite-dimensional distributions are such. The latter is meant in the sense of (1.5). More precisely, let \( X \) be an \( h \)-dimensional r.v. It is called **multivariate regularly varying with index \( \alpha \)** if

\[
\mathbb{P}(|X| > u, X/|X| \in \cdot) \overset{v}{\rightarrow} u^{-\alpha}\mathbb{P}(\Theta \in \cdot), \quad u > 0,
\]

where \( \overset{v}{\rightarrow} \) denotes vague convergence and \( \Theta \) is a random vector on the unit sphere \( \mathbb{S}^{h-1} = \{x \in \mathbb{R}^h \mid |x| = 1\} \). Its distribution is called the spectral measure of the regularly varying vector \( X \). This type of approach to determine the tail behavior of a univariate strictly stationary series was introduced by Davis and Hsing [16] and was used by e.g. Mikosch and Stărică [27]. See also [14], page 273.

To characterize the regular variation of \((W_{i,t}), i = 1, 2\), we use the following notation for \( h \geq 1, i = 1, 2, j_1 = 1 \) and \( j_2 = 4 \):

\[
W_{i,t} = (W_{i,1}, \ldots, W_{i,h}), \quad \Xi_{h}^i = (\Pi_{1}^{(j_1)}, \ldots, \Pi_{h}^{(j_1)}), \quad \text{and} \quad R_{i,t} = (R_{i,1}, \ldots, R_{i,h}),
\]

where \( \Pi_{i}^{(j_1)} = \Pi_{i}^{(j_1)} \) and

\[
R_{1,t} = \sum_{i=0}^{t-1} \Pi_{i,t+1-i}^{(j_1)}D_{-i} \quad \text{and} \quad R_{2,t} = \sum_{i=0}^{t-1} \Pi_{i,t+1-i}^{(j_1)}B_{2,t-i}, \quad t \geq 1.
\]

**Lemma 3.3.** Suppose that \( \alpha_1 < \alpha_2 \) and the conditions of Theorem 3.1 are satisfied. Then strictly stationary series \((W_{1,t})\) and \((W_{2,t})\) are regularly varying with indices \( \alpha_1 \) and \( \alpha_2 \) respectively and the spectral measures for finite dimensional vectors \((W_{1,t}, \ldots, W_{i,h}), h \geq 1, i = 1, 2\) are

\[
\mathbb{P}(\Theta_{i,h} \in \cdot) = \frac{\mathbb{E}[\Xi_{h}^i|\Theta_{i,h}| \mathbf{1}\{\Xi_{h}^i/|\Xi_{h}^i| \in \cdot\}]}{\mathbb{E}[|\Xi_{h}^i|]},
\]

where r.v.’s \( \Theta_{i,h} \) take values on \( \mathbb{S}^{h-1} \).

\[\text{Note that in the univariate case, we say that a positive measurable function } f(x) \text{ is regularly varying with index } \rho \text{ if } \lim_{x \to \infty} f(cx)/f(x) = c^\rho, \ c > 0. \text{ Moreover, r.v. } X \text{ is said to be regularly varying with index } \alpha > 0 \text{ if } f(x) = \mathbb{P}(|X| > x) \text{ is regularly varying with index } -\alpha, \text{ see [14] p.273. A similar definition is used for the multivariate case, see [14] p.279.}\]
Proof. The proofs for both series are very similar, so we give the proof only for \((W_{1,t})\). Since \(W_{1,t} = \Pi^{(1)}_{t} W_{1,0} + R_{1,t}, t = 1, \ldots, h\) by induction, we have a representation \(W_{1,h} = \Xi^{(1)}_{h} W_{1,0} + R_{1,h}\) where both vectors \(\Xi^{(1)}_{h}\) and \(R_{1,h}\) have the moment of order \(\alpha_{1}\). Due to the multivariate Breiman’s lemma [14] Lemma C.3.1 (1)), we have

\[
\lim_{x \to \infty} \frac{\mathbb{P}(x^{-1} W_{1,h} \in \cdot)}{\mathbb{P}(W_{1} > x)} = \lim_{x \to \infty} \frac{\mathbb{P}(x^{-1} \Xi^{(1)}_{h} W_{1} \in \cdot)}{\mathbb{P}(W_{1} > x)},
\]

so that it suffices to study the regular variation of \(\Xi^{(1)}_{h} W_{1}\). Moreover, applying Breiman’s lemma again, we obtain as \(y \to \infty\),

\[
\mathbb{P}(\{|W_{1} \Xi^{(1)}_{h}| > xy, \frac{\Xi^{(1)}_{h} W_{1}}{|\Xi^{(1)}_{h}| W_{1}} \in \cdot\} = \mathbb{P}(W_{1} | \Xi^{(1)}_{h}| 1(\Xi^{(1)}_{h} / |\Xi^{(1)}_{h}| \in \cdot) > xy) \sim \mathbb{E}[\Xi^{(1)}_{h}]^{\alpha_{1}} 1(\Xi^{(1)}_{h} / |\Xi^{(1)}_{h}| \in \cdot)x^{-\alpha_{1}} \mathbb{P}(W_{1} > y)
\]

and

\[
\mathbb{P}(\{|W_{1,h}| > xy\} \sim \mathbb{E}[\Xi^{(1)}_{h}]^{\alpha_{1}} x^{-\alpha_{1}} \mathbb{P}(W_{1} > y).
\]

Hence the conclusion follows. By the same logic the time series \((W_{2,t})\) is shown to be regularly varying with index \(\alpha_{2}\) as a series. \(\square\)

Secondly, we study the case \(\alpha_{2} < \alpha_{1}\) where both component processes have the same tail index \(\alpha_{2}\), so we consider a bivariate time series. However, it is more convenient to modify slightly the definition of regular variation i.e. to adopt the version better to the bivariate case as done by Basrak and Segers [14].

An \(\mathbb{R}^{d}\)-valued strictly stationary time series \((X_{t})\) is regularly varying with index \(\alpha > 0\) if the following limits in distribution exist

\[
\mathbb{P}(x^{-1}(X_{0}, \ldots, X_{h}) \in \cdot | |X_{0}| > x) \xrightarrow{w} \mathbb{P}((Y_{0}, \ldots, Y_{h}) \in \cdot), \quad x \to \infty,
\]

where \(\xrightarrow{w}\) denotes weak convergence. The limit vector \((Y_{0}, \ldots, Y_{h})\) has the same distribution as \(|Y_{0}|(\Theta_{0}, \ldots, \Theta_{h})\), where the distribution of \(|Y_{0}|\) is given by \(\mathbb{P}(|Y_{0}| > y) = y^{-\alpha}, y > 1\), and \(|Y_{0}|\) and \((\Theta_{0}, \ldots, \Theta_{h})\) are independent. The distribution of \(\Theta_{0}\) is the spectral measure of \(X_{0}\) and \((\Theta_{t})_{t \geq 0}\) is the spectral process. Notice that \(\Theta_{t}, t \neq 0\) is not always on \(\mathbb{S}^{d-1}\). The equivalence of definitions (3.14) and (3.16) is proved in [14] but (3.16) is usually easier to handle see e.g. [26].

Proposition 3.4. Assume that \(\alpha_{1} > \alpha_{2} > 0\) and that the conditions of Theorem 3.3 are satisfied. Then the bivariate strictly stationary series \(W_{t} = (W_{1,t}, W_{2,t})\) is regularly varying with index \(\alpha_{2}\) in the sense of (3.16) and

\[
\mathbb{P}(x^{-1}(W_{1}, \ldots, W_{h}) \in \cdot | |W_{0}| > x) \xrightarrow{w} \mathbb{P}(Y(\Pi_{1} \Theta_{0}, \ldots, \Pi_{h} \Theta_{0}) \in \cdot), \quad h \geq 1,
\]

where \(\mathbb{P}(Y > x) = x^{-\alpha_{2}}, x > 1\), \(Y\) is independent of \((\Theta_{0}, \Pi_{1}, \ldots, \Pi_{h})\). The distribution of \(\Theta_{0}\) has the spectral measure of \(W_{0}\). Here \(\Theta_{0}\) and \((\Pi_{1}, \ldots, \Pi_{h})\) are also independent.

Remark 3.5. (i) The same characterization has been examined in [26].

(ii) As seen above the law of \(\Theta_{0}\) is the most important in our case. Usually analytical expressions for the law of \(\Theta_{0}\) are not available. However, since the law of \(\Theta_{0}\) satisfies a certain invariant relation by index \(\alpha_{2}\) and \(A\), we could simulate \(\Theta_{0}\) (see Proposition 5.1 in [4]). We do not pursue this here and only make the following remark. Although our setting for SRE are different from usual assumption, i.e. \(A\) is triangular, once bivariate regular variation for \(W\) has been proved, we
can use the method in [41]. (We checked the assumptions in Section 5 of [41] and did not find any problems for our case.)

**Proof.** We show that for every $y = (y_1, y_2) \in \mathbb{R}^2$,

$$(3.18) \quad \lim_{x \to \infty} \frac{\mathbb{P}(y'W_0 > x)}{\mathbb{P}(W_2 > x)} = w(y)$$

exists,

$$w(y) > 0 \quad \text{if} \quad y \in [0, \infty)^2 \setminus \{0\}$$

and

$$w(y) = 0 \quad \text{if} \quad y \in (-\infty, 0)^2.$$

Then by Boman and Lindskog [7], see also [14, Appendix C], we may conclude the regular variation of $W_0$ in the sense of (3.14). In view of the proof of Theorem 3.1 ((3.7), (3.9) and (3.10)), we recall that

$$W_{1,0} = \overline{X} + \overline{W}_{1,0} = \overline{X}_{s,1} + \overline{X}_{s,2} + \overline{X}^s + \overline{W}_{1,0} \quad \text{and} \quad W_{2,0} = \Pi_{0,1-s}^{(4)} W_{2,-s} + \sum_{i=0}^{s-1} \Pi_{0,1-i}^{(4)} B_{2,-i}.$$ 

Given $\varepsilon \in (0, 1)$, let

$$M_{x,s} = \{ y_1 \overline{X}_{s,1} + y_2 \Pi_{0,1-s}^{(4)} W_{2,-s} > x \},$$

$$M'_s = \{ y_1 (\overline{W}_{1,0} + \overline{X}_{s,2} + \overline{X}^s) + y_2 \sum_{i=0}^{s-1} \Pi_{0,1-i}^{(4)} B_{2,-i} < -\varepsilon x \},$$

$$M''_s = \{ y_1 (\overline{W}_{1,0} + \overline{X}_{s,2} + \overline{X}^s) + y_2 \sum_{i=0}^{s-1} \Pi_{0,1-i}^{(4)} B_{2,-i} > \varepsilon x \}.$$ 

Then

$$M_{(1+\varepsilon)x,s} \setminus M'_s \subset \{ y_1 W_{1,0} + y_2 W_{2,0} > x \} \subset M_{(1-\varepsilon)x,s} \cup M''_s.$$ 

First we notice that

$$y_1 \overline{X}_{s,1} + y_2 \Pi_{0,1-s}^{(4)} W_{2,-s} = \left( y_1 \sum_{i=1}^{s} \Pi_{0,2-i}^{(4)} A_{2,1-i}^{(4)} \Pi_{-i,1-s}^{(4)} + y_2 \Pi_{0,1-s}^{(4)} \right) W_{2,-s}$$

$$=: J(y; s) W_{2,-s}.$$ 

And so

$$\lim_{x \to \infty} \frac{\mathbb{P}(y_1 \overline{X}_{s,1} + y_2 \Pi_{0,1-s}^{(4)} W_{2,-s} > x)}{\mathbb{P}(W_2 > x)} = \mathbb{E} J(y; s)^{\alpha_2} \mathbb{I}(J(y; s) > 0) =: w_\varepsilon(y).$$ 

Moreover, $w_\varepsilon(y)$ is bounded independently of $s$. Indeed, for $\alpha_2 > 1$, by the Minkowski inequality, we have

$$\mathbb{E} J(y; s)^{\alpha_2} \leq \left\{ |y_1| \sum_{i=1}^{s} \left( \mathbb{E} (\Pi_{0,2-i}^{(4)} A_{2,1-i}^{(4)} \Pi_{-i,1-s}^{(4)})^{\alpha_2} \right) \frac{1}{\alpha_2} + |y_2| \left( \mathbb{E} (\Pi_{0,1-s}^{(4)})^{\alpha_2} \right)^{\alpha_2} \right\}^{\alpha_2}$$

$$\leq \left\{ |y_1| \mathbb{E} A_{2}^{\alpha_2} \sum_{i=1}^{s} \left( \mathbb{E} A_{2}^{\alpha_2} \right)^{i-1} + |y_2| \right\}^{\alpha_2} < \infty.$$ 

For $\alpha_2 \leq 1$

$$\mathbb{E} J(y; s)^{\alpha_2} \leq |y_1|^{\alpha_2} \mathbb{E} A_{2}^{\alpha_2} \sum_{i=1}^{s} \left( \mathbb{E} A_{2}^{\alpha_2} \right)^{i-1} + |y_2|^{\alpha_2} < \infty,$$
which follows from the triangle inequality. Now we have
\[
\limsup_{x \to \infty} \frac{\mathbb{P}(y^t W_0 > x)}{\mathbb{P}(W_2 > x)} \leq \limsup_{x \to \infty} \frac{\mathbb{P}(M_{x(1-\varepsilon),x})}{\mathbb{P}(W_2 > x)} + \limsup_{x \to \infty} \frac{\mathbb{P}(M''_x)}{\mathbb{P}(W_2 > x)} \\
\leq (1-\varepsilon)^{-\alpha_2} w_\varepsilon(y) + \limsup_{x \to \infty} \frac{\mathbb{P}(y_1 \bar{X}^t > \varepsilon x)}{\mathbb{P}(W_2 > x)}
\]
and
\[
\liminf_{x \to \infty} \frac{\mathbb{P}(y^t W_0 > x)}{\mathbb{P}(W_2 > x)} \geq \liminf_{x \to \infty} \frac{\mathbb{P}(M_{x(1+\varepsilon),x})}{\mathbb{P}(W_2 > x)} - \limsup_{x \to \infty} \frac{\mathbb{P}(M'_x)}{\mathbb{P}(W_2 > x)} \\
\geq (1+\varepsilon)^{-\alpha_2} w_\varepsilon(y) - \limsup_{x \to \infty} \frac{\mathbb{P}(y_1 |\bar{X}| \varepsilon > x)}{\mathbb{P}(W_2 > x)}.
\]

It follows from (3.12) that the last term in these inequality vanishes.

Then we take a subsequence $s_\varepsilon$ such that $w_{s_\varepsilon}(y)$ is convergent and we obtain
\[
(1 + \varepsilon)^{-\alpha_2} \lim_{k \to \infty} w_{s_\varepsilon}(y) \leq \liminf_{x \to \infty} \frac{\mathbb{P}(y^t W_0 > x)}{\mathbb{P}(W_2 > x)} \leq \limsup_{x \to \infty} \frac{\mathbb{P}(y^t W_0 > x)}{\mathbb{P}(W_2 > x)} \leq (1 - \varepsilon)^{-\alpha_2} \lim_{k \to \infty} w_{s_\varepsilon}(y)
\]
and so letting $\varepsilon \to 0$ we obtain (3.18). Moreover, if $y = (y_1, y_2) \in [0, \infty)^2 \setminus \{0\}$ then
\[
y^t W \geq y_1 \bar{X} + y_2 W_2 \geq \max\{y_1 \bar{X}, y_2 W_2\}
\]
and since both $\bar{X}, W_2$ are regularly varying with index $\alpha_2$, $\lim_{k \to \infty} w_{s_\varepsilon}(y) > 0$.

Next we see (3.17). By induction
\[
W_t = \Pi_t W_0 + R_t,
\]
where $\Pi_t = A_t \cdots A_1$, $R_t = \sum_{i=1}^{t-1} \Pi_{t-i} B_{t-i}$ for $t \geq 1$, and all vectors are column vectors. With this interpretation we write
\[
(W_1, \ldots, W_h) = (\Pi_1, \ldots, \Pi_h) W_0 + (R_1, \ldots, R_h),
\]
where $(\Pi_1, \ldots, \Pi_h)$, $(R_1, \ldots, R_h)$ have moment of order $\alpha_2$ with respect to the matrix norm and are independent of $W_0$. Indeed, for all $t = 1, \ldots, h$ $\mathbb{E}||\Pi_t||^{\alpha_2} < \infty$ and $\mathbb{E}||R_t||^{\alpha_2} < \infty$. Due to Minkowski’s and triangle inequalities, this implies that each random component in two matrices has $\alpha_2$ th moment. Thus $\alpha_2$ th moment with the matrix norm follows. Hence
\[
\lim_{x \to \infty} x^{\alpha_2} \mathbb{P}(|(W_1, \ldots, W_t) - (\Pi_1, \ldots, \Pi_t) W_0| > x) = 0
\]
holds and an application of Breiman’s lemma ([14] Lemma C.3.1 (1)) concludes that $(W_1, \ldots, W_t)$ and $(\Pi_1, \ldots, \Pi_t) W_0$ have the same tail behavior and are regularly varying with index $\alpha_2$. Finally, (3.17) is concluded from the regular variations of $(\Pi_1, \ldots, \Pi_t) W_0$ and $W_0$ together with the multivariate Breiman’s lemma ([14] Lemma C.3.1 (2))] again.

\[\Box\]

**Remark 3.6.** (i) In Lemma 3.3 although each component process has regular variation, we do not have a device to characterize the joint regular variation with different tail indices i.e. we could not characterize tail dependence between processes with different tail indices. Therefore we only provide that for coordinate-wise series.

(ii) Regarding Proposition 3.4 even $\alpha_2 > \alpha_1 > 0$, it is possible to obtain the bivariate Basrak Segers limit representation (3.17), but then the spectral measure lies on one axis i.e. vectors $(W_{1,t}, W_{2,t})$ has the same spectral behavior as $(W_{1,t}, 0)$. This is not desirable since in applications tail asymptotics of the both series are crucial. This is the reason why we adopt Lemma 3.3.
3.3. Upper and lower bounds for constants. Although we have expressions for constants \( \overline{c}_1 \) and \( \overline{c}_1 \) in Theorem 3.2, a direct numerical calculation of the quantities seems difficult. A further research is needed for possible calculation of the constants as it is done in [28]. \( \overline{c}_1 \) of (3.5) is treated there but with i.i.d. sequence \((A_t, D_t)\). It remains to be seen whether the method is applicable in our case or not. Alternatively, we derive the upper and lower bounds for \( \overline{c}_1 \).

**Lemma 3.7.** Assume the conditions of Theorem 3.2 with \( \alpha_1 > \alpha_2 \). If \( \alpha_2 > 1 \) then

\[
\mathbb{E} A_2^{\alpha_2} \leq \overline{c}_1 \leq (1 - \tau^{1/\alpha_2})^{-\alpha_2} \mathbb{E} A_2^{\alpha_2},
\]

where \( \tau = \mathbb{E} A_1^{\alpha_2} \) and if \( \alpha_2 \leq 1 \) then

\[
(1 - \tau^{1/\alpha_2})^{-\alpha_2} \mathbb{E} A_2^{\alpha_2} \leq \overline{c}_1 \leq (1 - \tau)^{-1} \mathbb{E} A_2^{\alpha_2}.
\]

**Proof.** Case \( \alpha_1 > 1 \). Since the limit and expectation are interchangeable, we work on

\[
\lim_{s \to \infty} \mathbb{E} \left( \sum_{i=1}^{s} \Pi_{0,2,-i}^{(1)} A_{2,1,-i}^{(4)} \Pi_{1,-i,-s}^{(4)} \right)^{\alpha_2}.
\]

We take the first term \((i = 1)\) i.e. \( \Pi_{0,2}^{(1)} A_{2,0}^{(4)} \Pi_{1,-1,-s}^{(4)} = A_{2,0} A_{4,-1} \cdots A_{4,1-s} \) in the sum, so that we obtain the lower bound \( \mathbb{E}(A_{2,0} A_{4,-1} \cdots A_{4,1-s})^{\alpha_2} = \mathbb{E} A_2^{\alpha_2} \). As for the upper bound, by the Minkowski’s inequality together with independence of \( \Pi_{0,2,-k}^{(1)} A_{2,1-k} \) and \( \Pi_{1,-k,-s}^{(4)} \), \( 1 \leq k \leq s \) obtain

\[
\mathbb{E} \left( \sum_{i=1}^{s} \Pi_{0,2,-i}^{(1)} A_{2,1,-i}^{(4)} \Pi_{1,-i,-s}^{(4)} \right)^{\alpha_2} \leq \left( \sum_{i=1}^{s} (\mathbb{E}(\Pi_{0,2,-i}^{(1)} A_{2,1,-i}^{(4)} \Pi_{1,-i,-s}^{(4)})^{\alpha_2})^{1/\alpha_2} \right)^{\alpha_2}
\]

\[
= \left( \mathbb{E} A_2^{\alpha_2} \left( \frac{1 - \tau^{s/\alpha_2}}{1 - \tau^{1/\alpha_2}} \right)^{\alpha_2} \right)^{\alpha_2}
\]

where \( \mathbb{E} A_4^{\alpha_1} = 1, \mathbb{E} A_1^{\alpha_2} < 1 \) and \( \mathbb{E} A_2^{\alpha_2} < \infty \). This concludes the first result by taking limit in \( s \).

Case \( \alpha_2 \leq 1 \). We apply the triangle inequality for the upper bound and obtain

\[
\mathbb{E} \left( \sum_{i=1}^{s} \Pi_{0,2,-i}^{(1)} A_{2,1,-i}^{(4)} \Pi_{1,-i,-s}^{(4)} \right)^{\alpha_2} \leq \sum_{i=1}^{s} \mathbb{E}(\Pi_{0,2,-i}^{(1)} A_{2,1,-i}^{(4)} \Pi_{1,-i,-s}^{(4)})^{\alpha_2} \leq \sum_{i=1}^{s} (\tau^{i-1} \mathbb{E} A_2^{\alpha_2}) \leq \frac{1 - \tau}{1 - \tau} \mathbb{E} A_2^{\alpha_2},
\]

which yields the result. The lower bound is implied by the reverse Minkowski’s inequality. \( \Box \)

4. Application to bivariate GARCH(1, 1) processes

There are various extensions of a univariate GARCH model to multivariate ones. We stick here to the constant conditional correlation model of Bollerslev [6] and Jeanthequ [24], which is the most fundamental multivariate GARCH process. A bivariate series \( X_t = (X_{1,t}, X_{2,t})' \), \( t \in \mathbb{Z} \) has the GARCH(1, 1) structure if it satisfies:

\[
(4.1) \quad X_t = \Sigma_t Z_t,
\]

where \( (Z_t) \) constitutes an i.i.d. bivariate noise sequence and

\[
\Sigma_t = \text{diag}(\sigma_{1,t}, \sigma_{2,t}),
\]
with \( \sigma_{i,t} \) being the (non-negative) volatility of \( X_{i,t} \). We also assume that \( Z_t = (Z_{1,t}, Z_{2,t})' \) has mean zero and its covariance matrix (standard correlations) is

\[
P = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},
\]

where \( \rho = \text{Corr}(Z_{1,t}, Z_{2,t}) \). The volatility process \( \sigma_{i,t} \) is defined by the following stochastic equation

\[
\begin{align*}
\left( \frac{\sigma_{1,t}^2}{\sigma_{2,t}^2} \right) &= \begin{pmatrix} \alpha_{01} \\ \alpha_{02} \end{pmatrix} + \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} X_{1,t-1}^2 \\ X_{2,t-1}^2 \end{pmatrix} + \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \begin{pmatrix} \sigma_{1,t-1}^2 \\ \sigma_{2,t-1}^2 \end{pmatrix},
\end{align*}
\]

(4.2)

where the second equality follows from (4.1). Writing \( W_t = (\sigma_{1,t}^2, \sigma_{2,t}^2)' \),

\[
(4.3) \quad B_t = \begin{pmatrix} \alpha_{01} \\ \alpha_{02} \end{pmatrix} \quad \text{and} \quad A_t = \begin{pmatrix} \alpha_{11}Z_{1,t-1}^2 + \beta_{11} & \alpha_{12}Z_{2,t-1}^2 + \beta_{12} \\ \alpha_{21}Z_{1,t-1}^2 + \beta_{21} & \alpha_{22}Z_{2,t-1}^2 + \beta_{22} \end{pmatrix} = \begin{pmatrix} A_{1,t} & A_{2,t} \\ A_{3,t} & A_{4,t} \end{pmatrix},
\]

we see that the process \( (W_t) \) is given by the SRE with vector-valued \( B_t \) and matrix-valued \( A_t \):

\[
(4.4) \quad W_t = A_t W_{t-1} + B_t, \quad t \in \mathbb{Z}.
\]

There is a series of results about regular variation of GARCH processes. They are based on the Kesten theorem [25] and so, the tail indices of the component-wise series are always the same. We have in mind Mikosch and Stărică [27] for the univariate GARCH(1, 1) model and Basrak et al [3] for general univariate GARCH(\( p,q \)) processes. The corresponding results for a vector GARCH(1, 1) were obtained by Stărică [31]. There is also a recent result by Fernández and Muriel [21]. Furthermore, tail dependencies for bivariate GARCH(1, 1) models could be captured by newly defined measure called (cross-) extremogram (Matsui and Mikosch [26]), which is proposed in [17] and further developed in [19, 20].

However, in the financial models, we may be forced to go beyond Kesten’s assumptions when some of the entries of \( A_t \) vanish. Then the results of the previous section become very handy and we are able to treat the GARCH(1, 1) model with component-wise different extremes. Note that in Remark 3.2 of [26] the same assumption of upper triangle matrix was suggested for component-wise different tail modeling, though they did not obtain the exact tail behavior.

We assume that \( \alpha_{21} = \beta_{21} = 0 \) in (4.3), \( \alpha_{ii} > 0, i = 1, 2 \) and \( \alpha_{ij}, \beta_{ij} > 0 \) for \( (i,j) \neq (2,1) \). Then \( A_t \) becomes an upper triangular matrix and component-wise we have the following SREs

\[
(4.5) \quad \sigma_{1,t}^2 = A_{1,t} \sigma_{1,t-1}^2 + D_t,
\]

\[
\sigma_{2,t}^2 = A_{2,t} \sigma_{2,t-1}^2 + \alpha_{02},
\]

where \( D_t := \alpha_{01} + A_{2,t} \sigma_{2,t-1}^2 \).

Then the sufficient condition (2.5) for existence of the strictly stationary solution is: there exists \( \varepsilon > 0 \) such that

\[
\max_i (|\alpha_{ii}^\varepsilon E Z_{i,t}^2| + |\beta_{ii}^\varepsilon|) < 1 \quad \text{if} \quad \varepsilon < 1
\]

\[
\max_i (|\alpha_{ii}^\varepsilon E Z_{i,t}^2|^{1/\varepsilon} + |\beta_{ii}^\varepsilon|) < 1 \quad \text{if} \quad \varepsilon \geq 1
\]

holds. Then by Theorem 3.2 we have...
**Corollary 4.1.** Consider the bivariate SRE (4.4) or equivalently SREs (4.5). Assume that r.v. $Z$ has Lebesgue density in $\mathbb{R}^2$ and there exist $\alpha_1, \alpha_2 > 0$ such that

$$
\mathbb{E}A_1^{\alpha_1} = 1 \quad \text{and} \quad \mathbb{E}A_1^{\alpha_1} \log^+ A_1 < \infty, \\
\mathbb{E}A_4^{\alpha_2} = 1 \quad \text{and} \quad \mathbb{E}A_4^{\alpha_2} \log^+ A_4 < \infty,
$$

then

$$
\mathbb{P}(\sigma_2^2 > x) \sim \left\{ \begin{array}{ll}
\tilde{c}_1 x^{-\alpha_1} & \text{if } \alpha_1 < \alpha_2 \\
\tilde{c}_1 x^{-\alpha_2} & \text{if } \alpha_1 > \alpha_2
\end{array} \right. \quad \text{and} \quad \mathbb{P}(\sigma_2^2 > x) \sim c_2 x^{-\alpha_2},
$$

where the constants are given by (3.4) - (3.6).

**Proof.** We have to check the conditions of Theorem 3.2. Since each element of $B_i$ is a positive constant, this together with condition (4.6) implies the condition (3.3). In view of (4.3), $A_2$ has the $\alpha_2$th moment since the random component is the same as that in $A_4$. Then, since $\mathbb{E}A_2^{\alpha_1} < (\mathbb{E}A_2^{\alpha_2})^{\alpha_1/\alpha_2} < \infty$ for $\alpha_1 < \alpha_2$ and $\mathbb{E}A_2^{\alpha_2} < \infty$ for $\alpha_2 < \alpha_1$ we have $\mathbb{E}A_2^{\min(\alpha_1,\alpha_2)} < \infty$. Obviously $A$ has Lebesgue density. Therefore all conditions are satisfied. \(\square\)

Now we are going to characterize regular variation of stationary GARCH$(1,1)$ process. We do not apply Lemma 3.3 and Proposition 3.4 directly because the corresponding SRE is satisfied by the volatility vector not by the GARCH process itself. Therefore, some additional work is needed. First we assume that $\alpha_1 < \alpha_2$ and for $h \geq 0$ and $i = 1, 2$ we define the following lagged vectors.

$$
X_{i,h} = (X_{i,1}, \ldots, X_{i,h}),
$$

$$
X_{i,h}^{(k)} = (|X_{i,1}|^k, \ldots, |X_{i,h}|^k), \quad k = 1, 2,
$$

$$
Y_{1,h} = (|Z_{1,1}|(\Pi_1^{(1)})^{1/2}, \ldots, |Z_{1,h}|(\Pi_1^{(1)})^{1/2}),
$$

$$
Y_{2,h} = (|Z_{2,1}|(\Pi_2^{(4)})^{1/2}, \ldots, |Z_{2,h}|(\Pi_2^{(4)})^{1/2}).
$$

In what follows for a matrix or a vector $A$ and a constant $\alpha > 0$, $A^\alpha$ denotes component-wise $\alpha$th power of $A$.

**Proposition 4.2.** Suppose that the conditions of Corollary 4.1 with $\alpha_1 < \alpha_2$ are satisfied and that additionally $Z$ is symmetric. Then random vector $X_{i,h}$ is regularly varying with index $2\alpha_i$, $i = 1, 2$. The spectral measure is given by the distribution of the vector

$$
(r_{i,1}\theta_{i,1}, \ldots, r_{i,h}\theta_{i,h}),
$$

where r.v. $\Theta_{i,h} = (\theta_{i,1}, \ldots, \theta_{i,h}) \in \mathbb{S}_{h-1}^h$ has distribution

$$
\mathbb{P}(\Theta_{i,h} \in \cdot) = \frac{\mathbb{E}|Y_{i,h}|^{2\alpha_i} \mathbb{1}(Y_{i,h}/|Y_{i,h}| \in \cdot)}{\mathbb{E}|Y_{i,h}|^{2\alpha_i}},
$$

and $(r_{i,t})$ is a sequence of the Bernoulli r.v.’s independent of $\Theta_{i,h}$ such that $\mathbb{P}(r_{i,t} = \pm 1) = 0.5$.

**Proof.** Since proofs for the series $(X_{1,t})$ and $(X_{2,t})$ are almost the same, it suffices to see that for $(X_{1,t})$. In view of (3.15), we may write

$$
X_{1,h}^{(2)} = (Z_{1,1}^2\sigma_{1,1}^2, Z_{1,2}^2\sigma_{1,2}^2, \ldots, Z_{1,h}^2\sigma_{1,h}^2)
$$

$$
= (Z_{1,1}^2\Pi_1^{(1)}, \ldots, Z_{1,h}^2\Pi_1^{(1)}) \sigma_{1,0}^2 + (Z_{1,1}^2R_{1,1}, \ldots, Z_{1,h}^2R_{1,h}).
$$
We recall that $Z_{1,t}^2$ and $\Pi^{(1)}_t$ are independent and have moment of order $\alpha_2$ so that $|Z_{1,t}||\Pi^{(1)}_t|^{1/2}$ has the $2\alpha_1$th moment. Moreover, $D_k = \alpha_{01} + A_{2,k}\sigma_{2,k-1}^2$ in

$$R_{1,t} = \sum_{i=1}^{t-1} A_{1,i} \cdots A_{1,r+1-i} D_{r-i} + D_t$$

has moment of order $\alpha_2 > \alpha_1$ and $Z_{1,t}^2$ and $R_{1,t}$ are independent. Therefore, Minkowski’s or triangle inequality implies that $|Z_{1,t}|(R_{1,t})^{1/2}$ has moment of $2\alpha_1$. Hence

$$\lim_{x \to \infty} x^{2\alpha_1} \mathbb{P}([X^{(1)}_{1,h} - Y_{1,h} \sigma_{1,0}] > x) = 0$$

holds and an applying the multivariate Breiman’s lemma we conclude that $X^{(1)}_{1,h}$ and $Y_{1,h} \sigma_{1,0}$ have the same tail behavior and are regularly varying with index $2\alpha_1$. Now similarly as in the proof of Lemma 3.3 as $y \to \infty$

$$\frac{\mathbb{P}(|X^{(1)}_{1,h}| > xy, X^{(1)}_{1,h}/|X^{(1)}_{1,h}| \in \cdot)}{\mathbb{P}(|X^{(1)}_{1,h}| > y)} \sim \frac{\mathbb{P}(\sigma_{1,0} Y_{1,h} > xy, Y_{1,h}/|Y_{1,h}| \in \cdot)}{\mathbb{P}(\sigma_{1,0} Y_{1,h} > y)} \sim x^{-2\alpha_1} \mathbb{E}[Y_{1,h}]^{2\alpha_1} \mathbb{1}(Y_{1,h}/|Y_{1,h}| \in \cdot) \mathbb{E}[Y_{1,h}]^{2\alpha_1} \sim x^{-2\alpha_1} \mathbb{P}(\Theta_{1,h} \in \cdot).$$

Write

$$X_{1,h} = (\text{sign}(Z_{1,1})|X_{1,1}|, \ldots, \text{sign}(Z_{1,h})|X_{1,h}|),$$

then by symmetry of $Z$, the sequence $(\text{sign}(Z_{1,i}))$ is independent of $(|X_{1,i}|)$. Hence by Proposition 5.13 of [18], $X_{1,h}$ is regularly varying with index $2\alpha_1$ and the spectral measure is given by that of [4.7].

In the case $\alpha_1 > \alpha_2$ we are interested in the spectral process as done in Proposition 3.4. However, we dare to use the spectral process by $W_t = (W_{1,t}, W_{2,t})'$ since this gives explicit expression and since the original definition may yield only closed form representation as in [26].

**Proposition 4.3.** Assume that $\alpha_1 > \alpha_2$ and the conditions of Corollary 4.1 are satisfied. Let $h \geq 0$, then $(X_t)$ is regularly varying with index $2\alpha_2$ in the sense of (3.16). In particular, with $W_t = (\sigma_{1,0}^2, \sigma_{2,0}^2)'$, we have

$$\mathbb{P}(x^{-1/2}(X_1, \ldots, X_h) \in \cdot | |W_0| > x) \overset{w}{\to} \mathbb{P}(V(\text{diag}(\Pi_1 \Theta_0))^{1/2} Z_1, \ldots, (\text{diag}(\Pi_h \Theta_0))^{1/2} Z_h) \in \cdot),$$

where $\mathbb{P}(V > x) = x^{-2\alpha_2}$ for $x > 1$ and $V$ is independent of $(\Theta_0, (\Pi_1, \ldots, \Pi_h))$ and $(Z_1, \ldots, Z_h)$.

**Proof.** Write $\Sigma_t = (\text{diag}(W_t))^{1/2}$ so that $X_t = \Sigma_t Z_t$. First we approximate $X_t$ by $\tilde{\Sigma}_t Z_t$ where $\tilde{\Sigma}_t = (\text{diag}(\Pi_t W_0))^{1/2}$. In view of (3.19) the triangle inequality yields

$$|\Sigma_t - \tilde{\Sigma}_t| Z_t \leq |R_t|^{1/2}|Z_t|.$$ 

Since $|R_t|^{1/2}|Z_t|$ has $2\alpha_2$th moment, we have

$$\lim_{x \to \infty} x^{2\alpha_2} \mathbb{P}((\Sigma_t Z_1, \ldots, \Sigma_t Z_h) - (\tilde{\Sigma}_t Z_1, \ldots, \tilde{\Sigma}_t Z_h) | > x) = 0,$$

which together with the Breiman’s lemma implies that $(X_1, \ldots, X_h)$ and $(\Sigma_t Z_1, \ldots, \Sigma_t Z_h)$ have the same tail behavior and are regularly varying with index $2\alpha_2$. For (4.8) we recall from Proposition
3.4 that $\mathbb{P}(x^{-1}W_0 \in \cdot \mid |W_0| > x) \rightarrow \mathbb{P}(V^2 \Theta_0 \in \cdot)$ with $\mathbb{P}(V^2 > x) = x^{-\alpha_2}$, $x > 1$ and $V^2$ and $\Theta_0$ are independent. Due to the multivariate Breiman’s lemma,

$$\mathbb{P}(x^{-1}(\Pi_1W_0, \ldots, \Pi_nW_0) \in \cdot \mid |W_0| > x) \rightarrow \mathbb{P}(V^2(\Pi_1, \ldots, \Pi_n)\Theta_0 \in \cdot).$$

Finally, applying the continuous mapping theorem and another Breiman’s lemma, we obtain the convergence of

$$\mathbb{P}(x^{-1/2}((\text{diag}(\Pi_1W_0))^{1/2}Z_1, \ldots, (\text{diag}(\Pi_nW_0))^{1/2}Z_n) \in \cdot \mid |W_0| > x)$$

to the right hand side of (4.8).

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