(Causal)-Activation of Complex Entanglement Structures in Quantum Networks

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**ABSTRACT**

Entanglement represents “the” key resource for several applications of quantum information processing, ranging from quantum communications to distributed quantum computing. Despite its fundamental importance, deterministic generation of maximally entangled qubits represents an on-going open problem. Here, we design a novel generation scheme exhibiting two attractive features, namely, i) deterministically generating genuinely multipartite entangled states, ii) without requiring any direct interaction between the qubits. Indeed, the only necessary condition is the possibility of coherently controlling – according to the indefinite causal order framework – the causal order among some unitaries acting on the qubits. Through the paper, we analyze and derive the conditions on the unitaries for deterministic generation, and we provide examples for unitaries practical implementation. We conclude the paper by discussing the scalability of the proposed scheme to higher dimensional GME states and by introducing some possible applications of the proposal for quantum networks.

**Introduction**

One of the most fundamental concepts within the quantum realm is the notion of quantum entanglement. It is well established that entangled states – even in the simplest form of two-qubit entangled states – are essential to enable the marvels of quantum information processing\textsuperscript{1–5} within the Quantum Internet. And, as a matter of fact, both the theory of entanglement and its experimental generation have been a topic of intensive research.

Indeed, several applications of quantum information processing – ranging from quantum communications through distributed quantum key distribution to distributed quantum computing – rely on the generation and the remote distribution of entangled flying qubits\textsuperscript{6–9}, with a wide consensus within the research community on light being the ideal substrate for quantum information carriers. Nevertheless, given the limitations of current schemes for photonic entanglement generation, the research is still ongoing. In fact, some of the available schemes are probabilistic\textsuperscript{10–12}, relying as instance on some form of parametric down conversion. Other schemes require a tight matter-flying interaction\textsuperscript{13, 14}. Clearly, when it comes to multi-partite entanglement, both the approaches hardly scale to large systems. This has driven a recent interest in designing all-photonic deterministic sources of entanglement\textsuperscript{15, 16}.

In this work, we contribute toward this research direction by resorting to a recently proposed framework for quantum information processing, namely, the superposition of causal orders\textsuperscript{17–25, 25–36}. Specifically, we design an entanglement generation scheme where a superposition of causal orders between local unitaries, acting on qubits in pure product states, deterministically generates genuinely multipartite entangled (GME) states. Interestingly, the proposed scheme efficiently scales to higher dimensional GME states, due to the simplicity and the modularity of the protocol architecture. Furthermore, the scheme does not require any direct interaction among the input qubits or between the input qubit and the qubit governing the quantum control of the causal order between the unitaries. Indeed, the only requirement is the possibility of coherently controlling the causal order among the unitaries.

It is worthwhile to note that – by exploiting the super-map formalism – the design of the proposed scheme has been conducted without any specific assumption on the particulars of the underlying qubit technology. However, when it comes to practical implementation, we can recognize that the proposed scheme for deterministic entanglement generation is achievable in near-term quantum networks, as coherent control of causal orders is affordable by current technology level and it has been successfully implemented for flying qubits\textsuperscript{27, 28, 35}. In this light, through the paper, we discuss some possible applications of the proposal for
With a series of recent works, researchers have shown that quantum placement of quantum channels – namely, placing quantum channels \( \rho \) with \( \{ U_i \} \) and \( \{ \tilde{U}_j \} \) in a superposition of alternative configurations – can provide significant advantages for a number of problems, ranging from quantum computation\(^{26,37,38} \) and quantum information processing\(^{39,40} \) through non-local games\(^{41} \) to communication complexity\(^{42-48} \). Instances of this quantum placement range from superposition of alternative quantum channels traversed by the information carrier to superposition of alternative causal orders between the quantum channels. With reference to the superposition of causal orders between quantum channels, the placement is realized through an higher-order map known as quantum switch\(^{26} \).

Mathematically, the quantum switch is described by a supermap \( \mathcal{S} \) taking two channels \( \mathcal{W}(\cdot) \) and \( \mathcal{\tilde{W}}(\cdot) \) as inputs, and giving as output a channel resulting from the combination of \( \mathcal{W}(\cdot) \) and \( \mathcal{\tilde{W}}(\cdot) \) in a superposition of causal orders, controlled by a quantum degree of freedom \( |\varphi_{c}\rangle \). Its action on quantum states is defined by the Kraus operators\(^{17,26} \) \( S_{ij} = U_i \tilde{U}_j \otimes |0\rangle \langle 0| + \tilde{U}_j U_i \otimes |1\rangle \langle 1|, \) where \( \{ U_i \} \) and \( \{ \tilde{U}_j \} \) denote the Kraus operators of the primitive channels \( \mathcal{W}(\cdot) \) and \( \mathcal{\tilde{W}}(\cdot) \), and \( \{ |0\rangle, |1\rangle \} \) denotes the orthogonal states of the control system. Accordingly, the resultant channel implemented by the quantum switch is given by:

\[
\mathcal{S}_{\rho_c}(\mathcal{W}, \mathcal{\tilde{W}})(\rho) = \sum_{ij} S_{ij}(\rho) S^\dagger_{ij},
\]

(1)

with \( \rho \) and \( \rho_c \) denoting the density matrix of the input and the control, respectively.

In the following, giving that we aim at generating maximally entangled states, we focus on pure input states and unitary channels. Furthermore, we set the control \( |\varphi_{c}\rangle \) to \( |+\rangle \), i.e., we place the primitive channels in an even superposition of causal orders, accordingly to\(^{19-21,26,35,44} \). Accordingly, the quantum switch supermap in (1) exhibits a single unitary Kraus operator \( S = U \tilde{U} \otimes |0\rangle \langle 0| + \tilde{U} U \otimes |1\rangle \langle 1|, \) with \( U \) and \( \tilde{U} \) denoting the (single) Kraus operators of the primitive channels, leading to the overall operation on the input state \( |\varphi\rangle \) given by\(^{1} \):

\[
S(|\varphi\rangle \otimes |\varphi_c\rangle) = \frac{1}{2} (U \tilde{U} + \tilde{U} U) |\varphi\rangle \otimes |+\rangle_c + \frac{1}{2} (U \tilde{U} - \tilde{U} U) |\varphi\rangle \otimes |\varphi_c\rangle
\]

(2)

\(^{1}\)The notation has been simplified with respect to the one in (1) to highlight the focus on pure input states and unitary channels.
After performing a measurement on the control qubit in the coherent basis, the following outcome states – highlighting the superposition of causal orders between the two alternative states $|\psi_+^{(2)}\rangle$ emerges as output.

\[
|\psi_\pm\rangle = \frac{1}{\sqrt{L_\pm}} (U\bar{U} \pm \bar{U}U) |\varphi\rangle = \frac{1}{\sqrt{L_\pm}} (\bar{U} \pm \bar{U}U) |\varphi\rangle \tag{3}
\]

where $L_\pm$ is a normalization constant, depending on both the unitaries $U, \bar{U}$ and on the postselected state. In (3), we introduced $\bar{U}, \bar{U}$ as a shorthand notation, with $\bar{U} = U\bar{U}$ denoting the order where $U$ is applied before $\bar{U}$ and $\bar{U} = \bar{U}U$ denoting the alternative order. This is schematized in Fig. 1 where we omitted – as extensively done whenever possible through the rest of the paper – the normalization constant for the sake of simplicity. Indeed, from Fig. 1b, it is intuitive to grasp that, once the control qubit is measured, the output is a coherent superposition of two contributes, where in each contribute the unitaries process the input according to one of the two alternative causal orders, namely, either $\bar{U}$ or $\bar{U}$.

**Bell states generation**

Let us now consider two local unitary operators $V^{(2)} = U_0 \otimes U_1$ and $\bar{V}^{(2)} = U_0 \otimes U_1$ and a 2-qubit input system in the separable state $|\varphi_0\rangle \otimes |\varphi_1\rangle$.

Being the input in a product state and given the assumption of local unitaries, the resulting outcome, for any causal order between the local unitaries such as $V^{(2)}\bar{V}^{(2)}$ or $\bar{V}^{(2)}V^{(2)}$, will be a product state as well. Furthermore, no entanglement can be distilled from such a state in the asymptotic limit with two-way LOCC assistance\(^9\).

Conversely, if we process the separable input through an even superposition of causal orders between the two unitaries – as shown in Fig. 2a – by measuring the control in the coherent basis, the following output emerges:

\[
|\psi_\pm^{(2)}\rangle = \frac{1}{\sqrt{L_\pm}} \left( V^{(2)}\bar{V}^{(2)} \pm \bar{V}^{(2)}V^{(2)} \right) |\varphi_0\varphi_1\rangle \tag{4}
\]

From (4) we note that, once the control qubit is measured, the output is a coherent superposition of two contributes, where in each contribute the unitaries process the separable input according to one of the two alternative – i.e., either $V^{(2)}\bar{V}^{(2)}$ or $\bar{V}^{(2)}V^{(2)}$ – causal orders. This is schematized in Figure 2b.

Now, the main question arises: is there any entanglement within the quantum state $|\psi_\pm^{(2)}\rangle$ emerging out of the controlled superposition of causal orders? The answer to this question is definitely yes. Indeed, the output state is maximally entangled if and only if the following condition on the local unitaries $\{U_i, \bar{U}_i\}_{i=0,1}$ holds (see Theorem 1 in Methods):

\[
\langle \varphi_0 | \bar{U}_0 U_0 | \varphi_0 \rangle = 0 \quad \wedge \quad \langle \varphi_1 | \bar{U}_1 U_1 | \varphi_1 \rangle = 0 \tag{5}
\]
We have shown that the scheme can be implemented through a coherent control of straightforward unitaries: the Pauli-z gate $\lambda$ with the former constraint operating only on $U \land |\psi\rangle$. To this aim, let us assume as input state $|\psi\rangle$.

To address this crucial aspect, we note that the sufficient and necessary condition in (5) consists of two

\[ (a) \, \text{Concurrence } C(\rho_+) \text{ of the state } |\psi_+^{(2)}\rangle \text{ given in (4), obtained as output when the control qubit is measured as } |+\rangle \text{ with } \rho_+ \equiv |\psi_+^{(2)}\rangle \langle \psi_+^{(2)}|. \]

\[ (b) \, \text{Concurrence } C(\rho_-) \text{ of the state } |\psi_-^{(2)}\rangle \text{ given in (4), obtained as output when the control qubit is measured as } |\rangle \text{ with } \rho_- \equiv |\psi_-^{(2)}\rangle \langle \psi_-^{(2)}|. \]

Figure 3. 3D plot for the concurrence of the output states $|\psi^{(2)}\rangle$ given in (4) as function of: i) the y-rotation parameter $\lambda$ controlling the unitaries $\tilde{U}_0$ and $\tilde{U}_1$, and ii) the superposition parameter $\alpha$ controlling the input state $|\phi_0\phi_1\rangle$, assumed real. Unitaries $U_0$ and $U_1$ both set to a Pauli-z gate. Maximally entangled states exhibit unitary concurrence.

with $\land$ denoting the Boolean operator AND, and with $\leftarrow - \rightarrow$ being the introduced shorthand notations for the alternative causal orders among the unitaries, i.e., $\tilde{U}_i \equiv U_i U_i$ and $\tilde{U}_i \equiv U_i U_i$ for $i \in \{0, 1\}$.

Stemming from this result, we derive a lighter condition assuring the separability of the output state (see Proposition 1 in Methods). Specifically, if there exists at least one $i \in \{0, 1\}$ so that $\langle \phi_i | \tilde{U}_i | \tilde{U}_i^\dagger | \phi_i \rangle = 1$, the output state in (4) is separable.

Clearly, one could wonder which are the requirements in terms of unitaries and input state so that condition (5) can be satisfied. Namely, how “easily” entanglement can be obtained out of a superposition of causal orders.

To address this crucial aspect, we note that the sufficient and necessary condition in (5) consists of two separate constraints, with the former constraint operating only on $U_0$, $U_0$ and $|\phi_0\rangle$ whereas the latter one depends only on $U_1$, $\tilde{U}_1$ and $|\phi_1\rangle$. This separability feature allows us to design unitaries $U_0$ and $U_0$ independently from $U_1$, $\tilde{U}_1$. Furthermore, the condition in (5) can be satisfied with practical unitaries, as shown in the following.

To this aim, let us assume as input state $|\phi_0\phi_1\rangle = |\eta\eta\rangle$, with $|\eta\rangle = \sqrt{\alpha} |0\rangle + \sqrt{1-\alpha} |1\rangle$ being an arbitrary superposition of basis states. Furthermore, let us assume both $U_0$ and $U_1$ representing the popular Pauli-z gate, i.e., $U_i = \sigma_z$. Finally, let us assume both $\tilde{U}_0$ and $\tilde{U}_1$ being the y-rotation gate $R_y(\sigma_i) = e^{-i\sigma_{iy}}\lambda$, with $\sigma_y$ denoting the Pauli-y gate. Let’s now consider the two possible events, namely, control qubit measured either as $|+\rangle$ or as $|\rangle$. In the former case, the condition for entangled output state given in (4) translates to $\lambda \neq 0, \pi$. In fact, only when the unitary parameter $\lambda$ is either equal to 0 or $\pi$, the output state generated by the quantum switch shown in Figure 2 is separable. Furthermore, the condition for maximally entangled output given in (5) is translated to $\lambda = \pi$. This is shown in Figure 3a by plotting the concurrence (see Methods) of the output state $|\psi_+^{(2)}\rangle$ as a function of the parameters. As regards to the latter case – namely, whenever the control qubit is measured as $|\rangle$ – the output is either separable or maximally entangled. And the state is separable only if $\lambda$ is either equal to 0 or $\pi$, whereas it is maximally entangled for any $\lambda$ in $(0, \pi)$, as shown in Figure 3b.

We have shown that the scheme can be implemented through a coherent control of straightforward unitaries: the Pauli-z gate and the y-rotation gate $R_y(\pi/2)$.

**GHZ-like states generation**

Similarly to the bipartite case, we consider an even superposition of the two alternative causal orders between two 3-qubit local unitaries $V^{(3)} = U_0 \otimes U_1 \otimes U_2$ and $\bar{V}^{(3)} = \bar{U}_0 \otimes \bar{U}_1 \otimes \bar{U}_2$ acting on an initially pure product tripartite state $|\phi_0\phi_1\phi_2\rangle$, as shown in
As long as all the separable input qubits are all set to the same state |⟩, to generate W-like states, a coherent control of two alternative evolutions – as previously done for both Bell and GHZ-like states – is not sufficient due to distinguishing peculiarities of W-like states.

Indeed, as in the bipartite case, the output is a superposition of two different input processing, with the two processing differing for the causal order between the unitaries. This similarity maps as well into the necessary and sufficient condition for the output in (6) being a GHZ-like state, which is given by (see Theorem 2 in Methods):

\[ \langle \varphi_i | \tilde{U}_i | \varphi_i \rangle = 0 \quad \forall i = 0, 1, 2 \]

(7)

and \( \sim \) being the usual shorthand notations for the alternative causal orders among the unitaries. Hence, the output in (6) is a legitimate GHZ-like state if and only if \( \tilde{U}_i | \varphi_i \rangle \) is orthonormal to \( U_i | \varphi_i \rangle \).

Indeed, there exists a lighter condition (see Proposition 2 in Methods) assuring the separability of the output state (6), given by:

\[ \exists i \in \{0, 1, 2\} : \tilde{U}_i | \varphi_i \rangle = U_i | \varphi_i \rangle \]

(8)

It is crucial to note that the scheme shown in Figure 4 straightforwardly extends to \( n \)-partite GHZ-like states by considering two \( n \)-qubit local unitaries \( V^{(n)} \) and \( \tilde{V}^{(n)} \) acting on a \( n \)-partite separable state. In such a case, the output is a legitimate GHZ-like state as long as (see Remark following Theorem 2 in Methods):

\[ \langle \varphi_i | \tilde{U}_i | \varphi_i \rangle = 0 \quad \forall i = 0, \ldots, n-1 \]

(9)

Clearly, the higher is the dimension of the GHZ-like state to be generated, the higher is the number of constraints in (9) that must be simultaneously satisfied. However, this is not an issue, given that the set of constraints are separable, namely, the design of the \( i \)-th unitaries \( U_i, \tilde{U}_i \), depends only from the \( i \)-th input \( | \varphi_i \rangle \) and it is completely independent from the other inputs as well as the other unitaries. Indeed, as long as all the separable input qubits are all set to the same state \( | \eta \rangle \) (which is reasonable), the condition for deterministically generating GHZ states\(^2\) reduces to a single constraint regardless of the dimension of the state to be generated. Namely, the unitaries acting on the different qubits can be the same. This key feature makes the protocol highly scalable.

This pivotal separability feature of the necessary and sufficient conditions derived in (9) (as well as in (7)) allows us to easily address the issue of designing units and input state for generating a GHZ-like state. Indeed, similarly to the bipartite case, by assuming \( | \eta \rangle \eta \rangle \rangle \) as input state, with \( | \eta \rangle = \sqrt{\alpha} | 0 \rangle + \sqrt{1-\alpha} | 1 \rangle \), as well as \( U_i = \sigma_i \) and \( \tilde{U}_i = R_y(2\lambda) \) for any \( i \), we have that the condition for GHZ-like output state given in (7) translates to \( \lambda = \frac{\pi}{2} \). This is shown in Figure 5 by plotting the GME concurrence (see Methods) of the output state |\( \psi^{(3)} \rangle \) as a function of the parameters, where we can appreciate that the GME concurrence vanishes for \( \lambda = 0 \), and reaches its maximum when \( \lambda = \frac{\pi}{4} \).

**W-like states generation**

To generate W-like states, a coherent control of two alternative evolutions – as previously done for both Bell and GHZ-like states – is not sufficient due to distinguishing peculiarities of W-like states.

\(^2\)It worthwhile to note that this consideration holds also for Bell and W-like states generation.
Theorem 3 in Methods controlling the unitaries {U} acting on each qubit. And, regardless of the particular expression, the necessary and sufficient condition for the output in (11) being a W-like state is (see Theorem 3 in Methods), the following state emerges:

\[ \rho = \frac{1}{\sqrt{L_{\pm\pm}}} \left( \sum_{\pm} U_0 \otimes U_1 \otimes U_2 \right) |\psi_{\pm} \rangle \langle \psi_{\pm}| \]

with \( L_{\pm\pm} \) being the appropriate normalization constant, with \( \sum_{\pm} \) being the usual shorthand notations for the alternative causal orders among the unitaries, and with \( \pm \) being equal to \(+\) or \(\mp\) depending on the measurement output of the control qubit. Regardless of the particular expression of \( |\psi_{\pm\pm} \rangle \), the state is a superposition of three different input processing, differing for the causal order between the unitaries acting on each qubit. And, regardless of the particular expression, the necessary and sufficient condition for the output in (11) being a W-like state is (see Theorem 3 in Methods):

\[ \langle \phi_i | \tilde{U}_i | \phi_i \rangle = 0 \quad \forall i = 0, 1, 2 \]
Scheme for generating a W-like state through a superposition of the causal order between 1-qubit local unitaries $U_i, \tilde{U}_i$ acting on the $i$-th qubit $|\phi_i\rangle$.

**Figure 6.** W-like state through a superposition of two alternative causal orders between two 1-qubit local unitaries $U_i$ and $\tilde{U}_i$, operating on the $i$-th qubit of a separable input state $|\phi_0\rangle \otimes |\phi_1\rangle \otimes |\phi_2\rangle$.

Furthermore, similarly to the GHZ-like state, there exists a lighter condition for the separability of the output state in (11), given by (see Proposition 3 in Methods)

$$\exists i \in \{0, 1, 2\} : \overrightarrow{U}_i |\phi\rangle = \overleftarrow{U}_i |\phi\rangle$$

(13)

It is worthwhile to note that the scheme in Figure 6 straightforwardly extends to $n$-partite W-like states by simply extending condition (12) to any $i = 0, \ldots, n-1$, by reasoning as highlighted in the Remark following Theorem 2 in Methods. Furthermore, the same considerations in terms of unitaries design made for the GHZ-like states continue to hold. This is confirmed by assuming – as done for the GHZ-like states – as input state $|\eta\eta\eta\rangle$, with $|\eta\rangle = \sqrt{\alpha}|0\rangle + \sqrt{1-\alpha}|1\rangle$, as well as $U_i = \sigma_z$ and $\tilde{U}_i = R_y(2\lambda)$ for any $i$. With this setting, the necessary and sufficient condition for the output being a W-like state translates to $\lambda = \frac{\pi}{4}$. This can be clearly seen from the visualization of the GME concurrence of the states in (11) given in Fig. 7, where the GME concurrence for all states reaches the maximum at the critical value of $\lambda = \frac{\pi}{4}$, whereas it vanishes for $\lambda = 0, \frac{\pi}{2}$ when the output state becomes separable.

**Discussion**

In the previous section, we have shown that the generation of GME states belonging to different non-equivalent classes of states is deterministically achievable through a proper superposition of causal order between local unitaries. We discuss now the possible applications of the proposed scheme under two complementary perspectives, namely, entanglement generation and entanglement distribution, for the Quantum Internet.

**Local Entanglement Generation**

We first consider the case where the entanglement generation is local rather than distributed. Namely, the local unitaries $U_i, \tilde{U}_i$ as well as the controlling degree of freedom $|\phi_c\rangle$ are all located within the same quantum node, which locally implements the proper supermap – such as (6) illustrated in Figure 4a or (10) illustrated in Figure 6a – for generating GME states.

In this scenario, the entangled states are thus deterministically generated in a (some) network node – acting as entanglement generator – and they are subsequently distributed within the network through proper quantum communication links.

Hence, the network node acting as entanglement generator implements a coherent control strategy on the local unitaries processing some initial product state – with both the unitaries and the input state considered as free resources – for generating
(a) GME concurrence $C_{GME}(\rho_{++})$ of the state $|\psi_{++}^{(3)}\rangle$ given in (11), obtained as output when the control qubit is measured as $|++\rangle$ with $\rho_{++} = |\psi_{++}^{(3)}\rangle\langle \psi_{++}^{(3)}|$.

(b) GME concurrence $C_{GME}$ for the states $|\psi_{+\pm}^{(3)}\rangle$, $|\psi_{-\pm}^{(3)}\rangle$ and $|\psi_{-++}^{(3)}\rangle$ given in (11).

Figure 7. 3D plot for the GME concurrence of the output state $|\psi_{++}^{(3)}\rangle$ given in (11) as function of: i) the y-rotation parameter $\lambda$ controlling the unitaries $\{\tilde{U}_i\}$, and ii) the superposition parameter $\alpha$ controlling the input state $|\phi_0\phi_1\phi_2\rangle$, assumed real. Unitaries $\{U_i\}$ set to Pauli-z gate. W-like states exhibit GME concurrence equal to 0.9.

the GME states. Clearly, as discussed within the paper, the coherent control strategy depends on the desired GME output state. Although it is still technologically unclear whether a node can dynamically change the coherent control strategy for generating entangled states belonging to different non-equivalent classes, the proposed scheme for deterministic entanglement generation is achievable in near-term quantum networks, as the coherent control of orders of operations is affordable by current technology level and it has been successfully implemented for single-qubit channels\cite{18,27,35,36}.

Regardless of the control strategy being dynamic or fixed a-priori, three are the crucial properties of the proposed scheme for deterministic GME state generation. i) First, the individual input qubits don’t interact each others or with the control in any way. Indeed, they only traverse their respective local unitaries $U_i \tilde{U}_i$, in a coherent superposition of two alternative causal orders. ii) Second, the (sufficient and necessary) condition for deterministically generating GHZ- and W-like states consists of separable constraints. Namely, the design of the $i$-th unitaries is completely independent from the other unitaries as well as any input qubit different from $|\phi_i\rangle$. iii) Third, as long as all the separable input qubits are all set to the same state $|\eta\rangle$ (which is reasonable), the condition for deterministically generating GHZ- and W-like states reduces to a single constraint regardless of the dimension of the state to be generated. Namely, the unitaries acting on the different qubits can be the same. These crucial features make the protocol highly scalable.

Distributed Entanglement Generation

When it comes to multipartite entanglement distribution in a quantum network, several issues arises. First, as the number of parties to be entangled increases, the number of required multi-qubit gates increases as well. This not only implies severe error propagation effects, but it also hardly scales – as instance, in W-like states – with the number of parties. Furthermore, regardless of the number of parties, whenever the size of the quantum network grows to moderate- or large-scale\cite{50}, direct entanglement distribution is not feasible anymore due to photon noise and losses. In this context, quantum repeaters\cite{3,51,52} are commonly accepted as the strategy for increasing the entanglement distribution range. Unfortunately, regardless of the repeater particulars that roughly depend on the repeater generation\cite{53}, quantum repeaters require some sort of Bell state measurements for Bell pairs distribution or other projective measurements for multipartite entangled states, which are usually hard to implement and very noisy in practice.

Interestingly, our scheme for entanglement generation could provide an alternative strategy for overcoming such issues in distributing multi-partite entanglement, without the need of multi-qubits gates or any other interaction among the qubits. Let us better clarify this with an example. Specifically, as shown in Figure 8a, multiple quantum switches – referred to as edge entanglers – are geographically distributed through the network, so that each switch is closely located (from the entanglement
Scheme for distributing multipartite entanglement states in a quantum network. The network is composed by nine clients, organized in three sets, each served by an intermediate entangler whose control qubit is controlled by an entanglement coordinator.

(b) A magnification of the intermediate entangler $e_i$, which exploits a superposition of causal orders to distribute multi-partite entanglement to remote nodes.

**Figure 8.** Distributed multi-partite generation.

distribution perspective) to a certain group of nodes. Each edge entangler implements the proper supermap – such as (6) illustrated in Figure 4a for GHZ-like states – for generating GME states. But each edge entangler uses, as control degree of freedom, the output of another quantum switch – which acts as entangler coordinator – in order to collectively generate the desired multi-partite entangled state. Clearly, each edge entangler generates the required $k$-partite GHZ-like state according to the number $k$ of nodes that are physically linked to it. As instance, in Figure 8a, $e_1$, $e_2$ and $e_3$ generate a tri-partite entangled state by relying on the coherent control of the causal order between unitaries $V^3$ and $\tilde{V}^3$ as illustrated in Figure 8b, where the coherent control is provided as a 3-GHZ state generated by the entanglement coordinator $e_0$. The overall state distributed through the network is a valid $|GHZ\rangle_9$ state. It is worthwhile to note that the proposed scheme can scale to large networks through a proper hierarchical multi-tier architecture, where additional intermediate entanglers are deployed between the coordinator and the edge entanglers.

It is important to note that, in the above example, we have only discussed the distributed multipartite entanglement generation when the coherent control of remotely located edge entanglers is obtained through a proper multi-partite entangled state, shared between the edge entanglers. However, the scheme in Figure 8a requires only the availability of a coherent control of the unitaries among remotely located edge entanglers, regardless of the specific implementation of such a control. And such a coherent control is considered the genuine quantum feature of a quantum network, where a genuine quantum coherence is an intrinsic property of the communication network.

**Entanglement mapping**

Here we consider a scenario where entanglement – rather than generated – must be mapped between different quantum degrees of freedom. As an example, let us consider deterministic generation of photonic GME states, which may benefits from a matter-photon interface with a quantum degree of freedom – such as superconducting-circuit based qubits – where entanglement can be generated easier than in photonic-circuits. Another example is represented by matter-flying interfaces per-se, which represent a critical component for quantum networks, where matter qubits for information processing/storing – based on heterogeneous technologies ranging from transmons through quantum dots to ion traps – must be interfaced to flying qubits – generally implemented with photons – acting as information carriers.

Regardless of the specific applications for an entanglement mapper, the proposed scheme based on superposition of causal orders provides an interesting approach toward deterministic entanglement mapping, worthwhile of further investigation. As an example, let us consider the scheme shown in Figure 9. The initially-entangled quantum degree of freedom, say $\varphi$ in a GHZ state, is used to implement a coherent control among different quantum switches. Each quantum switch implements the superposition of causal orders between two unitaries given in (3) and shown in Figure 1a, by acting on the individual qubit $\varphi$ of a second quantum degree of freedom, initially in a separable state. As long as the condition for deterministic generation of GHZ-like state in (7) is satisfied, the GHZ state is deterministically mapped from the control degree of freedom to the initially separable second degree of freedom, which becomes maximally entangled. In a nutshell, the scheme harnesses the quantum correlation

\footnote{Clearly, the proposed scheme applies to the deterministic mapping of $W$-like states as well.}
Figure 9. Scheme for entanglement mapping between different quantum degrees of freedom. By implementing a coherent control of three different quantum switches through a proper 3-partite GME state and by measuring each control qubit in the coherent basis, an initially pure product tripartite state $|\psi_0\psi_1\psi_2\rangle$ is deterministically transformed into a 3-partite GME state.

It is worthwhile to note that the proposed mapping scheme does not require any interaction between the input qubits or between the input qubit and the control qubit, which represents a key feature whenever the input qubits weakly interact each others or with the environment, as in the mentioned case of photonic qubits.

Methods

Here we give an overview of the main techniques used to establish our results.

Entanglement measures

Quantum correlations has always been considered as a resource to perform tasks that are unachievable through classical resources, or to enhance other ones. Hence, a characterization and quantification of quantum correlations, in particular entanglement, has been widely studied\(^\text{49, 58}\).

For a two-qubit state with density matrix $\rho$, quantum entanglement is completely characterized by $\text{concurrence} C(\rho) = \max\{0, \mu_1 - \mu_2 - \mu_3 - \mu_4\}$, where $\{\mu_i\}$ denotes the set eigenvalues – in decreasing order – of the operator $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho (\sigma_y \otimes \sigma_y)$. Concurrence $C(\cdot)$ is an entanglement monotone metric, with value equal to 1 for maximally entangled states and value equal to 0 for separable states.

For tripartite systems, entanglement measures are more intricate, and are only analytically found for special classes of states\(^\text{58–60}\). For genuinely multipartite entangled (GME) states – i.e., states that are not separable for any bipartition – a known entanglement measure is the GME concurrence\(^\text{61–63}\) $C_{\text{GME}}(\rho) = \sqrt{2 \min\{1 - \text{Tr}(\rho_i^2), 1 - \text{Tr}(\rho_j^2), 1 - \text{Tr}(\rho_k^2)\}}$, with $\rho_i \triangleq \text{Tr}_{jk}(\rho)$ (with $j, k \neq i$) denoting the reduced density matrix for the $i$-th subsystem.

Bell states

**Theorem 1.** The states in (4) are maximally entangled bipartite states if and only if the conditions in (5) hold.

**Proof.** The proof follows by reasoning as in Theorem 2.

**Proposition 1.** The states in (4) are bi-separable if and only if the conditions: $\exists i \in \{0, 1\}: \tilde{U}_i |\psi_i\rangle = \overrightarrow{U}_i |\psi_i\rangle$ hold.
Proof. The proof follows by reasoning as in Proposition 2.

GHZ-like states
Here we derive in Theorem 2 the necessary and sufficient condition for generating a 3-partite GHZ-like state through superposition of causal orders.

Theorem 2. The states in (6) are GHZ-like states if and only if the condition given in (7) holds.

Proof. We first observe that any tripartite GHZ-like state $|\Psi^{(3)}\rangle$ is equivalent to the GHZ state $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ through local unitaries\(^{5}\) - i.e., $|\text{GHZ}\rangle = (U_0^{\text{LU}} \otimes U_1^{\text{LU}} \otimes U_2^{\text{LU}}) |\Psi^{(3)}\rangle$.

Case $\iff$ (sufficient condition). By hypothesis, $\tilde{U}_i |\varphi_i\rangle = (\tilde{U}_i |\varphi_i\rangle)^\perp$ for any $i$. Hence, (6) is equivalent to:

$$|\psi_\pm^{(3)}\rangle = \frac{1}{\sqrt{L_{\pm}}} \tilde{U}_0 |\varphi_0\rangle \tilde{U}_1 |\varphi_1\rangle \tilde{U}_2 |\varphi_2\rangle \pm U_0 |\varphi_0\rangle \mid\left(\tilde{U}_1 |\varphi_1\rangle\right)^\perp \mid\left(\tilde{U}_2 |\varphi_2\rangle\right)^\perp$$

(14)

By defining local unitaries such that $U_i^{\text{LU}} \mid\tilde{U}_i\rangle = |\tilde{U}_i\rangle$ (hence, $U_i^{\text{LU}} \left(\tilde{U}_i |\varphi_i\rangle\right)^\perp = |1\rangle$), the thesis follows.

Case $\implies$ (necessary condition). By hypothesis, (6) is a GHZ-like state. We prove the case with a reductio ad absurdum by supposing that there exists at least one $i$, say $i = 0$, so that $\tilde{U}_0 |\varphi_0\rangle \neq \left(\tilde{U}_0 |\varphi_0\rangle\right)^\perp$. From GHZ-like state definition, there exist $U_i^{\text{LU}}, U_j^{\text{LU}}$ such that $|\psi_\pm^{(3)}\rangle$ is LU-equivalent to the following (by neglecting the normalization factor for the sake of simplicity):

$$(I \otimes U_i^{\text{LU}} \otimes U_j^{\text{LU}}) |\psi_\pm^{(3)}\rangle = \tilde{U}_0 |\varphi_0\rangle \otimes |00\rangle \pm \tilde{U}_0 |\varphi_0\rangle \otimes |11\rangle$$

(15)

Then, there must exist another unitary $U_0^{\text{LU}}$ acting on the first qubit such that (15) is equivalent to the $|\text{GHZ}\rangle$ state, i.e.:

$$(U_0^{\text{LU}} \otimes I \otimes I) \left(\tilde{U}_0 |\varphi_0\rangle \otimes |00\rangle \pm \tilde{U}_0 |\varphi_0\rangle \otimes |11\rangle\right) = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

(16)

Since unitary matrices preserve orthogonality, from (16), it follows that $\tilde{U}_0 |\varphi_0\rangle = \left(\tilde{U}_0 |\varphi_0\rangle\right)^\perp$, which constitutes a reductio ad absurdum. Hence, the thesis follow.

Remark. Clearly, the above result can be straightforwardly extended to $n$-partite GHZ-like states $|\psi^{(n)}\rangle$ generated through an even superposition of the two alternative causal orders between two $n$-qubit local unitaries $V^{(n)} = \bigotimes_{i=0}^{n-1} U_i$ and $V^{(n)} = \bigotimes_{i=0}^{n-1} \tilde{U}_i$ acting on an initially pure product $n$-partite state $\bigotimes_{i=0}^{n-1} |\varphi_i\rangle$ by following the same reasoning and, in such a case, the necessary and sufficient condition becomes: $\langle \varphi_i | \tilde{U}_i \tilde{U}_0 \mid \varphi_i\rangle = 0 \forall i = 0, \ldots, n-1$.

Proposition 2. The states in (6) are bi-separable if and only if the condition given in (8) holds.

Proof. In the following, we directly prove the proposition for the arbitrary states $|\psi^{(n)}_{\pm}\rangle$ generated through an even superposition of the two alternative causal orders between two $n$-qubit local unitaries $V^{(n)} = \bigotimes_{i=0}^{n-1} U_i$ and $V^{(n)} = \bigotimes_{i=0}^{n-1} \tilde{U}_i$ acting on an initially pure product $n$-partite state $\bigotimes_{i=0}^{n-1} |\varphi_i\rangle$.

Case $\iff$ (sufficient condition). It is straightforward to recognize that, whenever there exists at least an $i$ such that $\tilde{U}_i |\varphi_i\rangle = \tilde{U}_i |\varphi_i\rangle$, the states $|\psi^{(n)}_{\pm}\rangle$ in Eq. 6 are separable.

Case $\implies$ (necessary condition). By hypothesis $|\psi^{(n)}_{\pm}\rangle$ are separable. Hence, there exist a partition so that $|\psi^{(n)}_{\pm}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ with $|\psi_A\rangle$ being pure state of the first subsystem $A$. Let us assume, without loss of generality, subsystem $A$ consisting of the first two qubits of (6). We prove the case with a reductio ad absurdum by supposing that $\tilde{U}_i |\varphi_i\rangle \neq \tilde{U}_i |\varphi_i\rangle$ for any $i = \{0, 1, \ldots, n-1\}$.

Since $|\psi^{(n)}_{\pm}\rangle$ are separable, there exist two local unitaries acting on the first two qubits\(^{4}\) such that:

$$\left(U_0^{\text{LU}} \otimes U_1^{\text{LU}} \otimes \bigotimes_{i=0}^{n-2} I \right) |\psi^{(n)}_{\pm}\rangle = U_0^{\text{LU}} \tilde{U}_0 |\varphi_0\rangle \otimes U_1^{\text{LU}} \tilde{U}_1 |\varphi_1\rangle \otimes \tilde{U}_3 \otimes \cdots \otimes \tilde{U}_{n-1} |\varphi_3 \cdots \varphi_{n-1}\rangle \pm U_0^{\text{LU}} \tilde{U}_0 |\varphi_0\rangle \otimes U_1^{\text{LU}} \tilde{U}_1 |\varphi_1\rangle \otimes \tilde{U}_3 \otimes \cdots \otimes \tilde{U}_{n-1} |\varphi_3 \cdots \varphi_{n-1}\rangle$$

(17)

\(^{4}\)The same reasoning – as well as the same result – holds by considering local unitaries acting on the remaining $n - 2$ qubits.
with $|↑↑⟩$ denoting a certain state for subsystem $A$. Since unitary matrices preserve inner product, from (17) it results that $\overline{U}_0 \otimes \overline{U}_1 |\varphi_0 \varphi_1⟩ = c \overline{U}_0 \otimes \overline{U}_1 |\varphi_0 \varphi_1⟩$. But this last equality requires that $\overline{U}_i |\psi⟩ = \overline{U}_i |\psi⟩$ for any $i \in \{0, 1\}$, which constitutes a reductio ad absurdum. Hence, the thesis follow.

W-like states

**Theorem 3.** The states in (11) are W-like states if and only if the condition given in (12) holds.

**Proof.** We first note that any tripartite W-like state $|\Psi^{(3)}⟩$ is equivalent to $|W⟩ = \frac{1}{\sqrt{3}}(|100⟩ + |010⟩ + |001⟩)$ by local unitaries -i.e., $|W⟩ = (U_0^{LU} \otimes U_1^{LU} \otimes U_2^{LU}) |\Psi^{(3)}⟩$. Similarly to the proof of Theorem 2, the proof of the sufficiency of the condition in (12) is straightforward. In the meanwhile, the necessity can be proved by reductio ad absurdum as in Theorem 2, by supposing that there exists at least one $i$ – say, without loss of generality, $i = 0$ – so that $\overline{U}_0 |\varphi⟩ \neq (\overline{U}_0 |\varphi_0⟩)^\perp$. From the W-like state equivalence, there exists a local unitary of the form $I \otimes U_1^{LU} \otimes U_2^{LU}$ such that

$$(I \otimes U_1^{LU} \otimes U_2^{LU}) |\psi^{(3)}⟩ = \overline{U}_0 |\varphi₀⟩ \otimes |00⟩ \pm \overline{U}_0 |\varphi₁⟩ \otimes |10⟩ \pm \overline{U}_0 |\varphi₂⟩ \otimes |01⟩$$

(18)

Then, there must exist another unitary $U_0^{LU}$ acting on the first qubit such that (18) is equivalent to W state, i.e.:

$$(U_0^{LU} \otimes I \otimes I) \overline{U}_0 |\varphi₀⟩ \otimes |00⟩ \pm \overline{U}_0 |\varphi₁⟩ \otimes |10⟩ \pm \overline{U}_0 |\varphi₂⟩ \otimes |01⟩ = \frac{1}{\sqrt{3}} (|100⟩ \pm |010⟩ \pm |001⟩)$$

(19)

Since unitary matrices preserve orthogonality, from (19), it follows that $\overline{U}_0 |\varphi₀⟩ = (\overline{U}_0 |\varphi₀⟩)^\perp$, which constitutes a reductio ad absurdum. Hence, the thesis follow. □

**Remark.** Although the deterministic generation of any $n$-partite GHZ-like state requires only a qubit degree of freedom controlling two different evolutions coherently, the deterministic generation of W-like states requires a higher-order control of the causal orders. Specifically, $n$ local unitaries – with $n$ being a power of 2 – must be arranged in a particular way. The rationale for this requirement lays in the necessity of having a maximally coherent basis of states that serves as a measurement setup on the controlling degrees of freedom, allowing the deterministic generation of the superposition required in the W states on all outputs. This requirement can only be met in Hilbert spaces of dimension which is a power of two. In such space, an orthonormal basis of maximally coherent states exists, and it can be used to coherently control the order of the local unitaries. Therefore, generating $n$-partite W-like states deterministically, where $n = 2^d$, encounters no problem as we can always find a maximally coherent orthonormal basis achieving this. Instead, a slight adjustment on the control strategy needs to be handled in order to achieve the deterministic generation of $n$-partite W states when $n$ is not a power of 2. To overcome this issue, we embed the control degrees of freedom in a larger Hilbert space of dimension $2^d$ where $d = \lceil \log_2 n \rceil$.

**Proposition 3.** The state in (11) is bi-separable if and only if the condition given in (13) holds.

**Proof.** The proof follows the similar steps of Proposition 2. □

**Data availability**

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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Competing interests
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