A Trudinger-Moser inequality for conical metric in the unit ball

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Abstract
In this note, we prove a Trudinger-Moser inequality for conical metric in the unit ball. Precisely, let $B$ be the unit ball in $\mathbb{R}^N$ ($N \geq 2$), $p > 1$, $g = |x|^{\frac{p}{N}}(dx_1^2 + \cdots + dx_N^2)$ be a conical metric on $B$, and

$$\lambda_p(B) = \inf \left\{ \int_B |\nabla u|^N : u \in W_0^{1,N}(B), \int_B |u|^p dx = 1 \right\}.$$ 

We prove that for any $\beta \geq 0$ and $\alpha < (1 + \frac{p}{N})\omega_{N-1}$, there exists a constant $C$ such that for all radially symmetric functions $u \in W_0^{1,N}(B)$ with $\int_B |\nabla u|^N dx - \alpha (\int_B |u|g^{\frac{p}{N}} dx)^{\frac{N}{p}} \leq 1$, there holds

$$\int_B e^{\alpha |x|^{\frac{p}{N}} g^{\frac{p}{N}}} |u|^{\alpha N - 1} |x|^\beta dx \leq C,$$

where $|x| g^{\frac{p}{N}} dx = dv_g$, $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, $\omega_{N-1}$ is the area of the unit sphere in $\mathbb{R}^N$; moreover, extremal functions for such inequalities exist. The case $p = N$, $-1 < \beta < 0$ and $\alpha = 0$ was considered by Adimurthi-Sandeep \[1\], while the case $p = 2$, $\beta \geq 0$ and $\alpha = 0$ was studied by de Figueiredo-do ´O-dos Santos \[8\].

Key words: Trudinger-Moser inequality, blow-up analysis, conical metric

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1. Introduction
Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$), $W_0^{1,N}(\Omega)$ be the completion of $C_c^\infty(\Omega)$ under the Sobolev norm

$$\|u\|_{W_0^{1,N}(\Omega)} = \left( \int_\Omega |\nabla u|^N dx \right)^{1/N},$$

where $\nabla$ denotes the gradient operator. Write $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, where $\omega_{N-1}$ stands for the area of the unit sphere in $\mathbb{R}^N$. Then the classical Trudinger-Moser inequality \[39\,29\,28\,33\,24\] says

$$\sup_{u \in W_0^{1,N}(\Omega), \|u\|_{W_0^{1,N}} \leq 1} \int_\Omega e^{\alpha |x|^{\frac{p}{N}} g^{\frac{p}{N}}} dx < \infty, \quad \forall \alpha \leq \alpha_N. \quad (1)$$

This inequality is sharp in the sense that if $\alpha > \alpha_N$, all integrals in (1) are still finite, but the supremum is infinite. While the existence of extremal functions for it was solved by Carleson-Chang \[3\], Flucher \[22\] and Lin \[21\].
Using a symmetrization argument and a change of variables, Adimurthi-Sandeep \([1]\) generalized \(1\) to the following singular version:

\[
\sup_{u \in W^{1,\infty}_0(\Omega), \|u\|_{W^{1,\infty}_0(\Omega)} \leq 1} \int_{\Omega} e^{\alpha x \beta y} |x|^{-N\beta} \, dx \leq \infty, \quad \forall \, 0 \leq \beta < 1, \ 0 < \gamma \leq 1 - \beta.
\] (2)

Also, it is sharp in the sense that if \(\gamma > 1 - \beta\), integrals are still finite, but the above supremum is infinite. The inequality \(2\) was extended to the whole Euclidean space \(\mathbb{R}^N\) by Adimurthi-Yang \([2]\). The existence of extremal functions for \(2\) in the case \(N = 2\) was due to Casto-Roy \([6]\), Yang-Zhu \([38]\) and Iula-Mancini \([13]\). An interesting question is whether or not \(2\) still holds for \(\beta < 0\). Generally, the answer is negative. To see this, we choose \(x_0 \neq 0\), \(r_0 > 0\) such that \(B_{2r_0}(x_0) \subset \Omega \setminus \{0\}\). For any \(0 < \epsilon < r_0\), we write the Moser function

\[
m_\epsilon(x) = \begin{cases} \frac{1}{\omega_{N-1}} \left( \log \frac{x}{\epsilon} \right)^\frac{N-1}{N}, & \text{when } |x - x_0| \leq \epsilon \\ \frac{1}{\omega_{N-1}} \left( \log \frac{r_0}{\epsilon} \right)^\frac{N-1}{N}, & \text{when } \epsilon < |x - x_0| \leq r_0 \\ 0, & \text{when } |x - x_0| > r_0. \end{cases}
\]

An easy computation shows \(\|m_\epsilon\|_{W^{1,\infty}_0(\Omega)} = 1\). Since \(\beta < 0\), we have

\[
\int_{\Omega} e^{\alpha N (1 - \beta) m_\epsilon \frac{x}{\epsilon}} |x|^{-N\beta} \, dx \geq \int_{B_{r_0}(x_0)} e^{\alpha N (1 - \beta) m_\epsilon \frac{x}{\epsilon}} |x|^{-N\beta} \, dx \to \infty \quad \text{as } \epsilon \to 0.
\]

Even worse, the above estimate still holds if \(\alpha_N (1 - \beta)\) is replaced by any \(\alpha > \alpha_N\). In conclusion, the singular Trudinger-Moser inequality \(2\) does not hold for \(\beta < 0\).

Let us consider the unit ball \(B \subset \mathbb{R}^N\), which is centered at the origin. Let \(\mathcal{S}\) be a set of all radially symmetric functions. With a slight abuse of notations, we say that \(u\) is radially symmetric if \(u(x) = u(|x|)\) for almost every \(x \in B\). It was proved by Ni \([27]\) that \(W^{1,N}_0(\mathbb{B}) \cap \mathcal{S}\) can be imbedded in \(L^p(\mathbb{B}, |x|^n)\) with \(\alpha > 0\) and \(p = 2(N + \alpha)/(N - 2)\) greater than \(2^* = 2N/(N - 2)\). Motivated by results of Bonheure-Serra-Tarallo \([3]\), Calanchi-Terraneo \([4]\) and de Figueiredo-dos Santos-Miyagaki \([7]\), de Figueiredo-do Ó-dos Santos \([8]\) observed that in the case \(N = 2\), moreover, extremal function for the above supremum exists. Of course they discussed more general weight \(h(|x|)\) and fast growth \(F(u)\) instead of \(|x|^\gamma\) and \(e^{4\pi (1 + \alpha) u^2}\) respectively.

Our aim is to generalize \(3\) to higher dimensional case and to stronger versions. We first have the following:

**Theorem 1.** Let \(B\) be the unit ball in \(\mathbb{R}^N\) \((N \geq 2)\), \(W^{1,N}_0(\mathbb{B})\) and \(\mathcal{S}\) be as above. Then there holds for any \(\beta \geq 0\),

\[
\sup_{u \in W^{1,N}_0(\mathbb{B}) \cap \mathcal{S}, \|u\|_{W^{1,N}_0(\mathbb{B})} \leq 1} \int_{\mathbb{B}} e^{\gamma a(1 + \beta) u^2} |x|^{-N\beta} \, dx \leq \infty, \quad \forall \, \gamma \leq \alpha_N (1 + \beta).
\] (4)

Here \(\alpha_N (1 + \beta)\) is the best constant in the sense that if \(\gamma > \alpha_N (1 + \beta)\), all integrals are finite but the supremum is infinity. Moreover, for any \(\beta \geq 0\) and any \(\gamma \leq \alpha_N (1 + \beta)\), the supremum in \(4\) can be attained.
By a rearrangement argument, for any $\gamma \leq \alpha_N$, there holds
\[
\sup_{u \in W^{1,p}_0(\mathbb{B}), \|u\|_{W^{1,p}_0(\mathbb{B})} \leq 1} \int_{\mathbb{B}} e^{\gamma |u|^{p/N}} dx = \sup_{u \in W^{1,p}_0(\mathbb{B}) \setminus \{0\}, \|u\|_{W^{1,p}_0(\mathbb{B})} \leq 1} \int_{\mathbb{B}} e^{\gamma |u|^{p/N}} dx. \tag{5}
\]

Therefore, when $\Omega = \mathbb{B}$, Theorem 1 includes the classical Trudinger-Moser inequality [1] as a special case and complements Adimurthi-Sandeep’s inequality [2].

Motivated by [34, 41, 40], we would generalize Theorem 1 to a version involving eigenvalue of the $N$-Laplace. For $p > 1$, define
\[
\lambda_p(\mathbb{B}) = \inf_{u \in W^{1,p}_0(\mathbb{B}), \|u\|_{W^{1,p}_0(\mathbb{B})} \neq 0} \frac{\int_{\mathbb{B}} |\nabla u|^p dx}{\int_{\mathbb{B}} |u|^p dx}. \tag{6}
\]

For $\alpha < \lambda_p(\mathbb{B})$, we write for simplicity
\[
\|u\|_{1,\alpha} = \left( \int_{\mathbb{B}} |\nabla u|^\alpha dx - \alpha^{1/N} \right)^{1/N}. \tag{7}
\]

**Theorem 2.** Given $p > 1$. In addition to the assumptions of Theorem 1, let $\lambda_p(\mathbb{B})$ and $\| \cdot \|_{1,\alpha}$ be defined as in (6) and (7) respectively. Then if $\alpha < \lambda_p(\mathbb{B})$, there holds
\[
\sup_{u \in W^{1,p}_0(\mathbb{B}), \|u\|_{1,\alpha} \leq 1} \int_{\mathbb{B}} e^{\gamma \|u\|_{1,\alpha}^p} dx < \infty, \quad \forall \gamma \leq \alpha_N.
\]

Moreover, the above supremum can be attained.

When $p = N$, Theorem 2 was proved by Nguyen [23] for a smooth bounded domain. As a consequence of Theorem 2 we improve Theorem 1 as follows:

**Theorem 3.** Given $p > 1$. Under the same assumptions of Theorem 2 for any $\beta \geq 0$ and any $\alpha < (1 + \frac{p}{N}\beta)^{N-1} \lambda_p(\mathbb{B})$, there holds
\[
\sup_{u \in W^{1,p}_0(\mathbb{B}) \setminus \{0\}, \int_{\mathbb{B}} |\nabla u|^\beta dx \leq \alpha_N (1 + \frac{p}{N}\beta)} \int_{\mathbb{B}} e^{\gamma \|u\|_{1,\alpha}^p} dx < \infty, \quad \forall \gamma \leq \alpha_N (1 + \frac{p}{N}\beta), \tag{8}
\]

where $\alpha_N (1 + \frac{p}{N}\beta)$ is the best constant in the same sense as in Theorem 1. Furthermore, the supremum in (8) can be attained.

We now explain the geometric meaning of the term $|x|^{\beta} dx$. Let $g_0$ be the standard Euclidean metric, namely $g_0(x) = dx_1^2 + \cdots + dx_N^2$. Define a metric $g(x) = |x|^{2/\beta} g_0(x)$ for $x \in \mathbb{B}$. Then $(\mathbb{B}, g)$ is a conical manifold with the volume element $dv_g = |x|^{\beta} dx$. Moreover, $|\nabla u|^N dx = |\nabla u|^N dv_g$.

The proof of Theorems 1 and 2 is based on a change of variables. While the proof of Theorem 2 is based on blow-up analysis. In the remaining part of this note, we shall prove Theorems 1 and 3 respectively.
2. Proof of Theorem 1

Let $\beta \geq 0$ and $\gamma \leq \alpha N/(1 + \beta)$. Write for simplicity $u(x) = u(r)$ with $r = |x|$. Following Smets-Willems-Su [31] and Adimurthi-Sandeep [1], we make a change of variables. Define a function

$$v(r) = (1 + \beta)^{1-1/N} u(r^{1/(1+\beta)}).$$

A straightforward calculation shows

$$\int_{B} |\nabla v|^N \, dx = \omega_N - \frac{\omega_{N-1}}{1 + \beta} \int_0^1 |u'(r^{1/(1+\beta)})| r^{N-1-\beta/(1+\beta)} \, dr$$

and

$$\int_{B} e^{\gamma |x|^\frac{N}{N-1}} |x|^N \, dx = \omega_N - \frac{\omega_{N-1}}{1 + \beta} \int_0^1 e^{\gamma |v(r)|^{\frac{N}{N-1}}} r^{N-1+\beta} \, dr$$

Then it follows from (9), (10) and (5) that

$$\sup_{u \in W^{1,N}_{0}(B)} \int_{B} e^{\gamma |x|^\frac{N}{N-1}} |x|^N \, dx = \frac{1}{1 + \beta} \sup_{v \in W^{1,N}_{0}(B)} \int_{B} e^{\gamma |v|^\frac{N}{N-1}} |x|^N \, dx.$$

According to Carleson-Chang [5], the supremum on the right-hand side of (11) can be attained, so does the supremum on the left-hand side. This concludes Theorem 1.

3. Proof of Theorem 2

In this section, we use the standard blow-up analysis to prove Theorem 2. This method was originally introduced by Ding-Jost-Li-Wang [9] and Li [17, 18], and extensively employed by Yang [42, 43, 44, 45], Lu-Yang [23], Li-Ruf [19], Zhu [41], do ´O-de Souza [10, 11], Li-Yang [16], Li [15], Nguyen [23, 26] and others. Comparing with the case $p \leq N$ [41, 26], we need more analysis to deal with the general case $p > 1$.

3.1. The existence of maximizers for subcritical functionals

Let $\alpha < \lambda_p(\mathbb{B})$. Denote

$$\Lambda_{\gamma,\alpha} = \sup_{u \in W^{1,p}_{0}(\mathbb{B}) \mid \|u\|_{p,\alpha} \leq 1} \int_{\mathbb{B}} e^{\gamma |u|^\frac{N}{N-1}} \, dx.$$
Lemma 4. For any positive integer $k$, there exists a decreasing radially symmetric function $u_k \in W^{1, N}_0(\mathbb{B}) \cap C^1(\mathbb{B})$ with $\|u_k\|_{1, \alpha} = 1$ such that $\int_{\mathbb{B}} e^{\gamma_k |u_k|^{p-1}} \, dx = \Lambda_{\gamma, \alpha}$, where $\gamma_k = \alpha N - 1/k$. Moreover, $u_k$ satisfies the Euler-Lagrange equation

$$
\begin{cases}
  -\Delta u_k - \alpha \left( \int_{\mathbb{B}} u_k^p \, dx \right)^{N-1} u_k^{p-1} = \frac{1}{2} u_k^{N-1} e^{\gamma_k u_k^{1/N}} \\
u_k \geq 0 \quad \text{in} \quad \mathbb{B} \\
u_k = 0 \quad \text{on} \quad \partial \mathbb{B} \\
\lambda_k = \int_{\mathbb{B}} u_k^{-\frac{N}{p-1}} e^{\gamma_k u_k^{1/N}} \, dx,
\end{cases}
$$

(12)

where $\Delta u_k = \text{div}(\nabla u_k^{N-2} \nabla u_k)$.

Proof. Let $k$ be a positive integer. By a rearrangement argument, there exists a sequence of decreasing radially symmetric functions $u_j \in W^{1, N}_0(\mathbb{B})$ with $\|u_j\|_{1, \alpha} \leq 1$ and $\int_{\mathbb{B}} e^{\gamma_j |u_j|^{p-1}} \, dx \to \Lambda_{\gamma, \alpha}$ as $j \to \infty$. Since $\alpha < \Lambda_{\gamma}(\mathbb{B})$, $u_j$ is bounded in $W^{1, N}_0(\mathbb{B})$. Without loss of generality, we assume $u_j$ converges to some function $u_0$ weakly in $W^{1, N}_0(\mathbb{B})$, strongly in $L^s(\mathbb{B})$ for any $s > 1$ and almost everywhere in $\mathbb{B}$. If $u_k \equiv 0$, then $\|u_k\|_{W^{1, N}_0(\mathbb{B})} \leq 1 + o(1)$. Thus $e^{\gamma_k u_k^{1/N-1}}$ is bounded in $L^q(\mathbb{B})$ for some $q > 1$. It follows that $e^{\gamma_k u_k^{1/N-1}}$ converges to 1 in $L^1(\mathbb{B})$. This implies that $\Lambda_{\gamma, \alpha} = |\mathbb{B}|$, the volume of $\mathbb{B}$, which is impossible. Therefore $u_k \not\equiv 0$. Clearly $u_k$ is also decreasing radially symmetric and $\|u_k\|_{1, \alpha} \leq 1$. Define a function sequence

$$
v_j = \frac{u_j}{(1 + \alpha(\int_{\mathbb{B}} u_j^p \, dx)^{1/N})^{1/\alpha}}.
$$

It follows that $\|v_j\|_{W^{1, N}_0(\mathbb{B})} \leq 1$, $v_j$ converges to $v_0 = u_0/(1 + \alpha(\int_{\mathbb{B}} u_0^p \, dx)^{1/N})$ weakly in $W^{1, N}_0(\mathbb{B})$. By a result of Lions (22), Theorem I.6), for any $q < 1/(1 - \|v_0\|_{W^{1, N}_0(\mathbb{B})}^{1/(N-1)})$, there holds

$$
\lim_{j \to \infty} \int_{\mathbb{B}} e^{\gamma_j v_j^{1/N}} \, dx < \infty.
$$

(13)

One can easily check that

$$
\left( 1 + \alpha \left( \int_{\mathbb{B}} u_j^p \, dx \right)^{1/N} \right) \left( 1 - \|v_0\|_{W^{1, N}_0(\mathbb{B})}^{1/(N-1)} \right) = 1 - \|u_k\|_{1, \alpha} < 1.
$$

(14)

It follows from (13) and (14) that $e^{\gamma_j v_j^{1/N-1}}$ is bounded in $L^r(\mathbb{B})$ for some $r > 1$, and thus $e^{\gamma_j v_j^{1/N-1}} \to e^{\gamma_k v_0^{1/N}}$ in $L^1(\mathbb{B})$ as $j \to \infty$. Hence $\int_{\mathbb{B}} e^{\gamma_k v_0^{1/N-1}} \, dx = \Lambda_{\gamma, \alpha}$ and $u_0$ is the desired extremal function. Clearly $\|u_0\|_{1, \alpha} = 1$. Moreover, the Euler-Lagrange equation of $u_k$ is (12). According to the regularity theory for degenerate elliptic equations, see Serrin (30), Tolksdorf (32) and Lieberman (20), we have $u_k \in C^1(\mathbb{B})$.

It is indicated by Lemma 3 that for any $\gamma < \alpha N$ and $\alpha < \Lambda_{\gamma}(\mathbb{B})$, the supremum $\Lambda_{\gamma, \alpha}$ can be attained. In particular, for any $\gamma_k = \alpha N - 1/k$, there exists a maximizer $u_k \geq 0$ satisfies (12). It is not difficult to see that

$$
\lim_{k \to \infty} \int_{\mathbb{B}} e^{\gamma_k u_k^{1/N}} \, dx = \Lambda_{\gamma_k, \alpha} = \sup_{u \in W^{1, N}_0(\mathbb{B}), \|u\|_{1, \alpha} \leq 1} \int_{\mathbb{B}} e^{\alpha |u|^{p-1}} \, dx.
$$

(15)
Since \( \|u_k\|_{L^\alpha} = 1 \), without loss of generality, we can assume that \( u_k \) converges to \( u_0 \) weakly in \( W^{1,N}_0(\mathbb{B}) \), strongly in \( L^s(\mathbb{B}) \) for any \( s > 1 \), and almost everywhere in \( \mathbb{B} \). Let \( c_k = u_k(0) = \max_{\mathbb{B}} u_k \). If \( c_k \) is bounded, then applying the Lebesgue dominated convergence theorem to (15), we know that \( u_0 \) is the desired extremal function for the supremum \( \Lambda_{\alpha,N} \). Hereafter we assume

\[
c_k \to \infty \quad \text{as} \quad k \to \infty. \tag{16}
\]

**Lemma 5.** Let \( u_0 \) be the limit of \( u_k \) as above. Then \( u_0 \equiv 0 \) and \( |\nabla u_k|^N dx \to \delta_0 \) weakly in the sense of measure, where \( \delta_0 \) stands for the Dirac measure centered at the origin.

**Proof.** We first prove \( u_0 \equiv 0 \). Suppose not. It follows from Lions’ lemma that \( e^{\gamma |u_0|^{N/(N-1)}} \) is bounded in \( L^q(\mathbb{B}) \) for some \( q > 1 \). Then applying elliptic estimates to (12), we conclude \( u_k \) is uniformly bounded in \( \mathbb{B} \), which contradicts our assumption (16). Therefore \( u_0 \equiv 0 \).

Next we prove \( |\nabla u_k|^N dx \to \delta_0 \). Suppose not. Since \( \|u_k\|_{L^\alpha} = 1 \) and \( u_0 \equiv 0 \), there would hold \( \|u_k\|_{W^{1,N}_0(\mathbb{B})} = 1 + o_k(1) \). Thus there exists some \( 0 < r_0 < 1 \) such that

\[
\limsup_{k \to \infty} \int_{|x| \leq r_0} |\nabla u_k|^N dx < 1.
\]

It follows from the classical Trudinger-Moser inequality (11) that \( e^{\gamma |u_k|^{N/(N-1)}} \) is bounded in \( L^q(B_{r_0}) \) for some \( q > 1 \). Since \( u_k \) is decreasing radially symmetric and \( \|u_k\|_{L^\alpha} = 1 \) with \( \alpha < \lambda_\alpha(\mathbb{B}) \), we have

\[
u_k^N(r_0) \leq \left( \frac{1}{|B_{r_0}|} \int_{|x| \leq r_0} u_k^p \, dx \right)^{N/p} \leq \frac{1}{(\lambda_p(\mathbb{B}) - \alpha)B_{r_0}^{N/p}}.
\]

Hence \( e^{\gamma u_k(r_0)^{N/(N-1)}} \) is also bounded in \( L^{q_1}(B_{r_0}) \) for some \( q_1 > 1 \). Then applying elliptic estimates to (12), we conclude that \( u_k \) is uniformly bounded in \( \mathbb{B} \). This contradicts (16) and ends the proof of the lemma. \( \square \)

### 3.2. Blow-up analysis

Let \( r_k = \lambda_k^{\frac{N}{N-1}} (u_k(r_k x) - c_k) \). Using the same argument as in the proof of (14), Lemma 4.3), one has by Lemma 5 and the classical Trudinger-Moser inequality (11) that

\[
r_k e^{\alpha N} \to 0 \quad \text{as} \quad k \to \infty, \quad \forall \alpha < \alpha_N/N. \tag{17}
\]

For \( x \in B_{r_1} \), we define \( \psi_k(x) = c_k^{-1} u_k(r_k x) \) and \( \varphi_k(x) = c_k^{1/(N-1)} (u_k(r_k x) - c_k) \).

**Lemma 6.** Up to a subsequence, there holds \( \psi_k \to 1 \) in \( C_{loc}^1(\mathbb{R}^N) \) and \( \varphi_k \to \varphi \) in \( C_{loc}^1(\mathbb{R}^N) \) as \( k \to \infty \), where

\[
\varphi(x) = -\frac{N - 1}{\alpha N} \log \left( 1 + \frac{\alpha N}{N^{N/(N-1)} |x|^{N/(N-1)}} \right). \tag{18}
\]

**Proof.** A simple calculation gives

\[
-\Delta_N \psi_k = \alpha \lambda_k^{N-2} \|u_k\|_{L^\alpha}^N \psi_k^{p-1} + c_k^{N-1} e^{\gamma u_k(r_k x) - c_k} \psi_k^{1/(N-1)} \tag{19}
\]
and

\[- \Delta_N \psi_k = \alpha c_k^{p^N} \|u_k\|_p^{N-p} \psi_k^{p-1} + e^{\gamma_k (u_k^{N(N-1)} - c_k^{1/(N-1)})} \psi_k^{1/(N-1)}. \tag{20}\]

Since \(u_k\) is bounded in \(L^p(\mathbb{R})\), one has by (16) and (17) that

\[
\left( \int_{R_{k-1}} (c_k^{p-N} r_k^N \|u_k\|_p^{N-p} \psi_k^{p-1})^{p/(p-1)} \, dx \right)^{p/(p-1)} = c_k^{1-N/p} \|u_k\|_p^{N-1} \to 0 \quad \text{as } k \to \infty. \tag{21}\]

Since \(0 \leq \psi_k \leq 1\), there holds

\[
c_k^{-N} e^{\gamma_k (u_k^{N(N-1)} - c_k^{1/(N-1)})} \psi_k^{1/(N-1)} \to 0 \quad \text{as } k \to \infty. \tag{22}\]

It follows from (21) and (22) that \(\Delta_N \psi_k\) is bounded in \(L^{p/(p-1)}(B_{k-1})\). Applying the regularity theory to (19), one obtains

\[
\psi_k \to \psi \quad \text{in } C^0_\text{loc}(\mathbb{R}^N) \quad \text{as } k \to \infty, \tag{23}\]

where

\[
\psi(0) = 1, \quad 0 \leq \psi(x) \leq 1, \quad \forall x \in \mathbb{R}^N. \tag{24}\]

When \(1 < p \leq N\), one can easily see that

\[
c_k^{p-N} \|u_k\|_p^{N-p} \psi_k^{p-1} \to 0 \tag{25}\]

uniformly in \(x \in B_{k-1}\) as \(k \to \infty\). When \(p > N\), we have for any \(R > 0\) and sufficiently large \(k\)

\[
\|u_k\|_p^{N-p} = \left( \int_{\mathbb{R}^N} u_k^p \, dx \right)^{N/p-1} \leq \left( \int_{B_R} u_k^p \, dx \right)^{N/p-1} = c_k^{N-p} r_k^{N^2/p-N} \left( \int_{B_R} \psi_k^p \, dx \right)^{N/p-1}. \tag{26}\]

In view of (24), we conclude \(\int_{B_R} \psi^p \, dx > 0\), which together with (23) and (26) leads to

\[
\|u_k\|_p^{N-p} \leq 2 \left( \int_{B_R} \psi^p \, dx \right)^{N/p-1} c_k^{N-p} r_k^{N^2/p-N} \tag{27}\]

for sufficiently large \(k\). This together with (17) gives

\[
c_k^{p-N} \|u_k\|_p^{N-p} \psi_k^{p-1} \leq 2 \left( \int_{B_R} \psi^p \, dx \right)^{N/p-1} c_k^{N-p} r_k^{N^2/p-N} \to 0 \quad \text{as } k \to \infty. \tag{27}\]

It then follows from (25) and (22) that \(\Delta_N \psi_k\) is bounded in \(L^{\infty}(B_R)\). Applying again the regularity theory to (19), we conclude that \(\psi_k \to \psi \) in \(C^1(\mathbb{R}_{R/2})\). Since \(R\) is arbitrary, up to a subsequence, there holds

\[
\psi_k \to \psi \quad \text{in } C^1_\text{loc}(\mathbb{R}^N) \quad \text{as } k \to \infty, 
\]

where \(\psi\) is a solution of

\[-\Delta_N \psi = 0 \quad \text{in } \mathbb{R}^N, \quad 0 \leq \psi \leq \psi(0) = 1. \tag{28}\]

The Liouville theorem implies that \(\psi \equiv 1\) in \(\mathbb{R}^N\).
Recalling (25) and (27), $c_k^{p^*N} \|u_k\|_{L^p}^{N-p} \phi_k^{p-1} \to 0$ in $L^1_{\text{loc}}(\mathbb{R}^N)$. Then using the same argument as in ([19], Section 3) or ([16], Lemma 17), we have by applying elliptic estimates to (20),
\[
\phi_k \to \phi \quad \text{in} \quad C^1_{\text{loc}}(\mathbb{R}^N) \quad \text{as} \quad k \to \infty,
\]
where $\phi$ satisfies
\[
\begin{cases}
-\Delta_N \phi = e^{a_N \frac{|x|}{k}} & \text{in} \quad \mathbb{R}^N \\
\sup_{\mathbb{R}^N} \phi = \phi(0) = 0.
\end{cases}
\]
Observing that $\phi$ is radially symmetric, one gets (18) by solving the corresponding ordinary differential equation.

Lemma 6 describes the asymptotic behavior of $u_k$ near the blow-up point $0$. To know $u_k$'s behavior away from $0$, by the same argument as in the proof of ([34], Lemma 4.11), we have that
\[
\begin{cases}
c_1^{1/(N-1)} u_k \to G \quad \text{weakly in} \quad W^{1,q}_0(B), \quad \forall \, 1 < q < N \\
c_1^{1/(N-1)} u_k \to G \quad \text{strongly in} \quad L^q(B), \quad \forall \, 1 < s \leq \frac{Nq}{N-q} \\
c_1^{1/(N-1)} u_k \to G \quad \text{in} \quad C^{1}_{\text{loc}}(B \setminus \{0\}),
\end{cases}
\]
where $G$ is a distributional solution of
\[
-\Delta_N G - \alpha ||G||_{p}^{N-p} G^{p-1} = \delta_0 \quad \text{in} \quad B.
\]
(28)
According to Kichenassamy-Veron [14], $G$ can be represented by
\[
G(x) = -\frac{N}{\alpha_N} \log |x| + A_0 + w(x),
\]
where $A_0$ is a constant, $w \in C^\nu(B)$ for some $0 < \nu < 1$ and $w(0) = 0$. In view of (15), we also have an analog of ([34], Proposition 5.2), namely
\[
\Lambda_{\alpha_N,x} \leq |B| + \frac{\omega N-1}{N} e^{\rho a_N + \sum_{j=1}^{N-1} \frac{1}{j}},
\]
(29)
For its proof, since no new idea comes out, we omit the details but refer the readers to [34] (see also [41, 15, 25]).

3.3. Test function computation
In this subsection, we construct a sequence of functions to show that
\[
\Lambda_{\alpha_N,x} > |B| + \frac{\omega N-1}{N} e^{\rho a_N + \sum_{j=1}^{N-1} \frac{1}{j}}.
\]
(30)
The contradiction between (30) and (29) indicates that $c_k$ is a bounded sequence, and whence the desired extremal function exists. This completes the proof of Theorem 2.

For any positive integer $k$, we set
\[
\phi_k(x) = \begin{cases}
\frac{c + \alpha_N}{\alpha_N} \left( -\frac{N}{\alpha_N} \log(1 + c_N |x|^\omega) + b \right), & |x| < \frac{k}{c N} \\
G \left( \frac{x}{k} \right), & \frac{k}{c N} \leq |x| \leq 1,
\end{cases}
\]
(31)
where $c$ and $b$ are constants, depending only on $k$, to be determined later. To ensure $\phi_k \in W^{1,N}(\mathbb{B})$, we need
\[
c + \frac{1}{c^{1/(N-1)}} \left( -\frac{N-1}{\alpha_N} \log(1 + c_N \log k)^{\frac{N-1}{N}} + b \right) = \frac{G(\log k)}{c^{1/(N-1)}},
\]
which implies that
\[
c^{\frac{N}{N-1}} = G\left(\frac{\log k}{k}\right) + \frac{N-1}{\alpha_N} \log(1 + c_N (\log k)^{\frac{N}{N-1}}) - b. \quad (32)
\]
We now calculate the energy of $\phi_k$. In view of (31), a straightforward calculation gives
\[
\int_{|\nabla \phi_k|^{N}} |\nabla \phi_k|^{N} dx \leq \frac{1}{c^{N/(N-1)}} \left( \frac{N-1}{\alpha_N} \sum_{j=1}^{N-1} \frac{1}{j} \log(1 + c_N (\log k)^{\frac{N}{N-1}}) + O((\log k)^{-\frac{N}{N-1}}) \right).
\]
By (28) and the divergence theorem
\[
\int_{|\nabla \phi_k|^{N}} |\nabla \phi_k|^{N} dx = \int_{|\nabla G|^{N-1}} |\nabla G|^{N-1} ds = \alpha \int_{\mathbb{B}} |\nabla G|^{N-1} ds = G\left(\frac{\log k}{k}\right) + \alpha \left( \int_{\mathbb{B}} G^{\frac{N}{N-1}} ds + O\left(\frac{(\log k)^{N-1}}{k}\right) \right).
\]
As a consequence
\[
\int_{\mathbb{B}} |\nabla \phi_k(x)|^{N} dx = \frac{1}{c^{N/(N-1)}} \left( \frac{N-1}{\alpha_N} \sum_{j=1}^{N-1} \frac{1}{j} \log(1 + c_N (\log k)^{\frac{N}{N-1}}) + O\left(\frac{\log k}{k}\right) \right)
\]
\[
+ O\left(\frac{(\log k)^{N-1}}{k}\right) \right) \right) = \frac{1}{c^{N/(N-1)}} \left( \int_{\mathbb{B}} G^{\frac{N}{N-1}} ds + O\left(\frac{(\log k)^{2N}}{k^N}\right) \right)^{N/p} \right),
\]
we obtain
\[
\|\phi_k\|^{N}_{1,p} = \frac{1}{c^{N/(N-1)}} \left( \frac{N-1}{\alpha_N} \sum_{j=1}^{N-1} \frac{1}{j} \log(1 + c_N (\log k)^{\frac{N}{N-1}}) + O\left(\frac{(\log k)^{N-1}}{k}\right) \right).
\]
Set $\|\phi_k\|_{1,p} = 1$. It then follows that
\[
c^{\frac{N}{N-1}} = \frac{N}{\alpha_N} \log k - \frac{N-1}{\alpha_N} \sum_{j=1}^{N-1} \frac{1}{j} \log c_N + \alpha_0 + O((\log k)^{-\frac{N}{N-1}}). \quad (33)
\]
This together with (32) leads to

\[ b = \frac{N - 1}{\alpha_N} \sum_{j=1}^{\frac{N}{2} - 1} \frac{1}{j} + O((\log k)^{\frac{N}{2}}). \]  

(34)

When \( |x| < \frac{\log k}{N} \), we calculate

\[ a_N \phi_k^\frac{N}{2\pi} (x) = a_N c^{\frac{N}{2\pi}} \left( 1 + \frac{1}{c_N/(N-1)} \left( \frac{N - 1}{\alpha_N} \log(1 + c_N |x|^{\frac{N}{2\pi}}) + b \right) \right)^{\frac{N}{2}} \]

\[ \geq a_N c^{\frac{N}{2\pi}} \left( 1 + \sum_{r=1}^{N-1} \frac{N - 1}{c_N/(N-1)} \left( \frac{N - 1}{\alpha_N} \log(1 + c_N |x|^{\frac{N}{2\pi}}) + b \right) \right) \]

\[ = a_N c^{\frac{N}{2\pi}} + \frac{N a_N b}{N - 1} - N \log(1 + c_N |x|^{\frac{N}{2\pi}}). \]  

(35)

In view of (33) and (34),

\[ a_N c^{\frac{N}{2\pi}} + \frac{N a_N b}{N - 1} = N \log k + \sum_{j=1}^{\frac{N}{2} - 1} \frac{1}{j} + (N - 1) \log c_N + a_N A_0 + O((\log k)^{\frac{N}{2\pi}}). \]  

(36)

Moreover, integration by parts leads to

\[ \int_{|x| < \frac{\log k}{N}} e^{-N \log(1 + c_N |x|^{\frac{N}{2\pi}})} |x|^{\frac{N}{2\pi}} \, dx = k^{-N} \int_{|y| < \log k} \frac{dy}{(1 + c_N |y|^{\frac{N}{2\pi}})^N} \]

\[ = k^{-N} \int_0^{(\log k)^{\frac{N}{2\pi}}} \frac{N - 1}{N} t^{N-2} dt \]

\[ = k^{-N} (1 + O((\log k)^{\frac{N}{2\pi}})). \]  

(37)

Combining (35), (36) and (37), we obtain

\[ \int_{|x| < \frac{\log k}{N}} e^{a_N \phi_k^\frac{N}{2\pi} (x)} \, dx \geq \frac{\omega_{N-1}}{N} e^{a_N (\gamma + \log k)^{\frac{N}{2\pi}} / 2} + O((\log k)^{\frac{N}{2\pi}}). \]  

(38)

Using an inequality \( e^t \geq 1 + t \), we have

\[ \int_{|x| < \frac{\log k}{N}} e^{a_N \phi_k^\frac{N}{2\pi} (x)} \, dx \geq |\mathbb{B}| + \frac{\alpha_N}{c_N/(N-1)^2} \int_{\mathbb{B}} G \phi_k^\frac{N}{2\pi} \, dx + O((\log k)^{\frac{N}{2\pi}}). \]  

(39)

Then (30) follows from (38) and (39) immediately.

4. Proof of Theorem 3

As in the proof of Theorem 1, we set \( v(r) = (1 + \frac{b r}{\beta})^{1-1/N} u(r^{1/(1+\beta)}) \). Note that

\[ \int_{\mathbb{B}} |x|^{p} |x|^{p} \, dx = \omega_{N-1} \int_0^1 |u(r)|^{p} r^{N-1+p/2} \, dr \]

\[ = \frac{\omega_{N-1}}{(1 + \frac{b \beta}{N})^{p-1+p/2}} \int_0^1 |v(r^{1+\beta})|^{p} r^{N-1+p/2} \, dr \]

\[ = \frac{\omega_{N-1}}{(1 + \frac{b \beta}{N})^{p-1+p/2}} \int_0^1 |v(t)|^{p} t r^{N-1} \, dt = \frac{1}{(1 + \frac{b \beta}{N})^{p-1+p/2}} \int_{\mathbb{B}} |x|^p \, dx. \]
Similar calculations as in (9) and (10) tell us that
\[
\sup_{u \in W^{1,q}_0(B) \cap \mathcal{X}} \int_B |\nabla u|^{\beta} |u|^\alpha \, dx = \frac{1}{1 + \frac{\beta}{\alpha}} \sup_{v \in W^{1,q}_0(B) \cap \mathcal{X}} \left( \int_B |\nabla v|^{\beta} |v|^\alpha \, dx \right)^{1+\frac{\beta}{\alpha}} \int_B e^{\frac{\beta}{\alpha} |\nabla v|^{\beta} |v|^\alpha} \, dx. \tag{40}
\]
By a rearrangement argument, we have
\[
\sup_{v \in W^{1,q}_0(B) \cap \mathcal{X}} \left( \int_B |\nabla v|^{\beta} |v|^\alpha \, dx \right)^{1+\frac{\beta}{\alpha}} \int_B e^{\frac{\beta}{\alpha} |\nabla v|^{\beta} |v|^\alpha} \, dx \leq \sup_{v \in W^{1,q}_0(B)} \left( \int_B |\nabla v|^{\beta} |v|^\alpha \, dx \right)^{1+\frac{\beta}{\alpha}} \int_B e^{\frac{\beta}{\alpha} |\nabla v|^{\beta} |v|^\alpha} \, dx. \tag{41}
\]
Since \( \alpha < (1 + \frac{\beta}{\alpha})N-1+N/p_+p(B) \) and \( \gamma \leq \alpha N(1 + \frac{\beta}{\alpha}) \), in view of (40) and (41), Theorem 3 follows from Theorem 2 immediately.

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