ON THE INTERSECTION OF THE CURVES THROUGH A SET OF POINTS IN \( \mathbb{P}^2 \)

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Abstract. Given a set of points in \( \mathbb{P}^2 \), we consider the common zeros of the set of curves of a given degree passing through those points. For general sets of points, these zero sets have the expected dimension and are smooth. In fact, given graded Betti numbers, for any arrangement of points whose ideal has those graded Betti numbers, general among such arrangements, the zero sets have the expected dimension and are smooth.

1. Introduction

There has been a great deal of interest in the linear series of curves in \( \mathbb{P}^2 \) containing a given set of points (see, for example, \[7\], \[9\], \[10\], \[14\], or \[15\]). In this paper, we consider the intersection of all the curves of a given degree containing a given set of points in \( \mathbb{P}^2 \).

Let \( Z \subset \mathbb{P}^2 \) be an arrangement of points in \( \mathbb{P}^2 \) and \( I \) the homogeneous ideal of \( Z \). By “arrangement” we mean a finite set of points. Write \( I_d \) for the degree \( d \) piece of \( I \). If \( d \gg 0 \), then of course \( \text{Zeros}(I_d) = Z \). We ask: what can one say about \( \text{Zeros}(I_d) \) for values of \( d \) smaller than the generating degree of \( I \)? For example: What is the dimension of \( \text{Zeros}(I_d) \)? Is it smooth? The answers to these questions depend partly on the resolution type of the ideal \( I \). We give answers for arrangements which are general of a given resolution type.

Recall that a finite set \( Z \) of points in \( \mathbb{P}^2 \) is defined by a Hilbert–Burch matrix, a matrix whose entries are homogeneous forms on \( \mathbb{P}^2 \), and this matrix determines the minimal free resolution of \( I \) (see section 2.2). Recall also that there are integers \( k \), \( 0 < a_1 \leq \cdots \leq a_{k+1} \), \( 0 < b_1 \leq \cdots b_k \) such that the \((i, j)\)th entry of the Hilbert–Burch matrix has degree \( b_j - a_i \).

In fact, the \( a_i \) are exactly the degrees of the generators in a minimal generating set of \( I \).

Suppose we are given a resolution type as follows. Let us be given some \( (a_1, \ldots, a_{k+1}; b_1, \ldots, b_k) \) such that \( b_j > a_i \) for every \( i, j \). Consider the set of arrangements \( Z \) defined by Hilbert–Burch matrices whose entries have degree \( b_j - a_i \). The requirement \( b_j > a_i \) means that for the ideal \( I \) of an arrangement \( Z \) in this set, every relation (syzygy) of \( I \) has higher degree than every generator of \( I \). For general arrangements \( Z \) in this set, we are able to give answers to the questions above. Explicitly, we prove the following:

**Theorem.** Let us fix \( k \), \( \{a_i\} \), \( \{b_j\} \) as above, such that every \( b_j > a_i \). Consider the set of arrangements defined by Hilbert–Burch matrices whose \((i, j)\)th entries have degree \( b_j - a_i \).

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Let $Z$ be a general arrangement in this set, and let $I$ be the ideal of $Z$. Then for $d \geq 0$, $\text{Zeros}(I_d)$ is smooth and has the expected dimension. Here, expected dimension means the following: If $d < a_1$ then $\text{Zeros}(I_d) = \mathbb{P}^2$. If $a_1 \leq d < a_2$ then $\text{Zeros}(I_d)$ is a curve. If $a_2 \leq d < a_3$ then $\text{Zeros}(I_d)$ is a finite set. If $a_3 \leq d$ then $\text{Zeros}(I_d) = Z$.

(See Theorem 2.8.)

In particular, for any $n > 0$, we give explicit information for general arrangements of $n$ points, see Corollary 2.11.

For simplicity, we work over $\mathbb{C}$, but any algebraically closed field of characteristic zero will do. The restriction on characteristics comes from the use of Kleiman’s generic smoothness theorem [12, III.10.7], in the proof of Proposition 3.11.

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2. Plane arrangements of points

We introduce terminology for the objects of study, the intersection of the curves of a given degree through a given set of points. We also consider families of point arrangements and resolution data.

2.1. Degree envelopes.

Definition 2.1. Let $Z \subset \mathbb{P}^n$ be a non-empty closed subscheme with homogeneous ideal $I$. For $d \geq 0$, we define the $d$th degree envelope, or $d$-envelope, of $Z$ to be the closed subscheme $Z_d = \text{Zeros}(I_d) \subset \mathbb{P}^n$ given by the intersection of all the degree $d$ hypersurfaces containing $Z$. The degree envelopes form a decreasing chain which begins with $\mathbb{P}^n$ and stabilizes at $Z$. If $Z_d \neq Z_{d-1}$, we say $d$ is a geometric generating degree of $I$.

Equivalently, $Z_d$ is the base scheme of the linear series of degree $d$ hypersurfaces containing $Z$.

Example 2.2.  
(1) If $Z$ is a complete intersection of type $(d_1, \ldots, d_r)$ with $d_1 < \cdots < d_r$, then the geometric generating degrees of $I$ are exactly the $d_i$. For each $i$, let $H_i$ be a hypersurface of degree $d_i$ such that $Z = H_1 \cap \cdots \cap H_r$. Then $Z_{d_1} = H_1$, $Z_{d_2} = H_1 \cap H_2$, and so on.
(2) Let $Z$ be five general reduced points in $\mathbb{P}^2$. Then $Z_2$ is the unique conic containing $Z$, and $Z_3 = Z$. The geometric generating degrees are 2 and 3.
(3) Let $Z$ be eight general reduced points in $\mathbb{P}^2$. Then there is a pencil of cubics passing through $Z$, so $Z_3$ consists of the nine basepoints of this pencil. That is, $Z_3$ is the union of $Z$ with an extra ninth point (distinct from $Z$ because $Z$ is general). The geometric generating degrees are 3 and 4.
(4) Let \( Z \) be four reduced points in \( \mathbb{P}^2 \) with three collinear, but not all four. Say the points \( P_1, P_2, \) and \( P_3 \) lie on the line \( L \), and the point \( P_4 \) lies off of \( L \). Then \( Z_3 = L \cup P_4 \) and \( Z_3 = Z \). In this case a degree envelope has components of different dimensions. The geometric generating degrees are 2 and 3.

(5) Let \( C \) be a smooth plane cubic and let \( Z \) be eleven general reduced points on \( C \). Then \( Z_3 = C \). There is a unique point \( P \in C \) such that \( Z \cup P \) is the complete intersection of \( C \) with a quartic curve, and \( Z_4 = Z \cup P \), twelve points (\( P \) is distinct from all the points of \( Z \) by generality). Finally, \( Z_5 = Z \). In this case, \( I \) has three geometric generating degrees, 3, 4, and 5.

(6) Let \( Z \) be a set of 18 points in \( \mathbb{P}^2 \) in general position. Then the ideal \( I(Z) \) is minimally generated by three forms of degree 5 and one form of degree 6, but only 5 is a geometric generating degree of \( I(Z) \). That is, 6 is a degree of a generator of \( I(Z) \), but not a geometric generating degree.

The following lemma will clarify the relationship between the geometric generating degrees of \( I \) and the usual degrees of (algebraic) generators of \( I \).

**Lemma 2.3.** Let \( I \subset S = \mathbb{C}[x_0, \ldots, x_n] \) be a saturated homogeneous ideal. Say \( I = (H_1, \ldots, H_s) \), with \( H_i \) homogeneous, \( \deg H_i = d_i \), and \( d_1 \leq \cdots \leq d_s \). Then:

1. every geometric generating degree of \( I \) is one of the integers \( d_i \).
2. \( d_1 \) is a geometric generating degree of \( I \). \( \square \)

**Remark 2.4.** We regard \( \mathbb{P}^n \) and \( Z \) itself as trivial degree envelopes of \( Z \). (We have \( Z_d = \mathbb{P}^n \) for \( d < d_1 \), in the notation of Lemma 2.3, and \( Z_d = Z \) for \( d \gg 0 \).) So \( Z \) has no non-trivial degree envelopes if and only if \( I \) has only one geometric generating degree.

These degree envelopes arise naturally in the following situation. Let us consider an arrangement of lines through the origin of \( \mathbb{C}^3 \). Let \( A \subset \mathbb{C}^3 \) be the union of these lines and let \( I \) be the homogeneous ideal of \( A \). If we blow up the origin, then the total transform of the ideal \( I \) may have embedded components supported in the exceptional divisor of the blowup. The exceptional divisor is a \( \mathbb{P}^2 \) on which the strict transforms of the lines in \( A \) mark out an arrangement of points. It is shown in the companion paper [16] that the non-trivial degree envelopes (in \( \mathbb{P}^2 \)) of this point arrangement are the supports of embedded components of the total transform of \( I \). The geometric generating degrees of \( I \) determine the structure of these embedded components.

This situation arose in the process of computing the multiplier ideals of such an ideal of an arrangement of lines in \( \mathbb{C}^3 \), as explained in [16]. Corollary 2.11 is used in that paper to discuss general arrangements of lines.

### 2.2. Partition of \( (\mathbb{P}^2)^n \) by graded Betti numbers.

The Hilbert–Burch theorem gives a useful description of the defining ideal of a Cohen–Macaulay subvariety of codimension 2 in a smooth projective variety. (See, for example, [2] or [4, Section 20.4].) A configuration of finitely many points in \( \mathbb{P}^2 \) is the first example of such a subvariety. We state the theorem only in this special case.
Theorem 2.5 (Hilbert–Burch). Let $Z \subset \mathbb{P}^2$ be a finite set (a zero-dimensional reduced closed subscheme) with saturated homogeneous ideal $I \subset S = \mathbb{C}[x, y, z]$. Then there is an integer $k > 0$ and integers $0 < a_1 \leq \cdots \leq a_{k+1}$, $0 < b_1 \leq \cdots \leq b_k$ such that the minimal graded free resolution of $I$ has the form

$$0 \rightarrow \bigoplus_{j=1}^k S(-b_j) \xrightarrow{A} \bigoplus_{i=1}^{k+1} S(-a_i) \rightarrow I \rightarrow 0,$$

where $A$ is a $(k+1) \times k$ matrix of homogeneous forms. The ideal $I$ is generated by the determinants of the $k \times k$ minors of $A$.

Proof. See, for example, [3, Theorem 4.3].

The $a_i$ and $b_j$ are the resolution data of $I$. One can verify $\sum a_i = \sum b_j$. The resolution data is equivalent to the graded Betti numbers of $I$ [3]. To be precise, the graded Betti numbers give in degree $d$ the number of times that $d$ occurs on the lists $\{a_i\}$ and $\{b_j\}$.

Definition 2.6. Resolution data is a pair of lists $(\{a_i\}_{i=1}^{k+1}, \{b_j\}_{j=1}^k)$ with $0 < a_1 \leq \cdots \leq a_{k+1}$, $0 < b_1 \leq \cdots \leq b_k$, and $\sum a_i = \sum b_j$.

We say resolution data $R = (\{a_i\}, \{b_j\})$ is positive if $a_{k+1} < b_1$ (so that $a_i < b_j$ for every $i$ and $j$).

Remark 2.7. Let $Z$ be an arrangement of $n$ points with resolution data $R = (\{a_i\}, \{b_j\})$, not necessarily positive. One can show that $n = (\sum b_j^2 - \sum a_i^2)/2$, for example by computing the dimensions of global sections of large twists of the short exact sequence (1). See also [3, Exercise 3.15].

The collection of all arrangements of $n$ distinct points on $\mathbb{P}^2$ corresponds naturally to $(\mathbb{P}^2)^n - \Delta$, the open complement of the diagonals in $(\mathbb{P}^2)^n$, up to choosing an ordering for the $n$ points. This open set is partitioned by resolution data (equivalently, by graded Betti numbers) into pieces that are constructible sets in the Zariski topology [3].

The main goal of this paper is to prove the following.

Theorem 2.8. Let $(\{a_i\}_{i=1}^{k+1}, \{b_j\}_{j=1}^k)$ be positive resolution data. Let $T \subset (\mathbb{P}^2)^n$ be the locus of arrangements with this resolution data, where $n = (\sum b_j^2 - \sum a_i^2)/2$ as in 2.7. Then $T$ is irreducible. Let $Z \in T$ be a general arrangement. Let $I$ be the ideal of $Z$. Then the set of geometric generating degrees of $I$ and the degree closures of $Z$ are as follows.

If $k = 1$, then the geometric generating degrees of $I$ are $\{a_1, a_2\}$. In particular, $Z_d = \mathbb{P}^2$ for $d < a_1$ and $Z_d = Z$ for $d \geq a_2$. For $a_1 \leq d < a_2$, $Z_d = Z_{a_1}$ is smooth with codimension 1.

If $k \geq 2$, then the geometric generating degrees of $I$ are $\{a_1, a_2, a_3\}$. In particular, $Z_d = \mathbb{P}^2$ for $d < a_1$ and $Z_d = Z$ for $d \geq a_3$. For $a_1 \leq d < a_2$, $Z_d = Z_{a_1}$ is smooth with codimension 1 and for $a_2 \leq d < a_3$, $Z_d = Z_{a_2}$ is smooth with codimension 2 (that is, a set of reduced points).

Remark 2.9. The case $k = 1$ in Theorem 2.8 corresponds to complete intersections.
Remark 2.10. In Example 2.2(4), the resolution data is (2, 2, 3; 3, 4), hence not positive. Note that the 2-envelope consists of a line plus a point, so this fails the codimension part of the conclusion of the theorem.

In general it is not known what happens when the points have non-positive resolution data.

As a special case, so to speak, we get explicit information for general sets of \( n \) points in \( \mathbb{P}^2 \), meaning all arrangements corresponding to points in some fixed open subset of \((\mathbb{P}^2)^*\).

Corollary 2.11. Let \( n > 1 \). Let \( Z \) be a set of \( n \) general points in \( \mathbb{P}^2 \). Let \( d \) and \( r \) be specified by \( \left( \frac{d+1}{2} \right) \leq n = \left( \frac{d+2}{2} \right) - r \) with \( r > 0 \), so that \( d \) is the lowest degree of a curve passing through \( Z \), and \( r \) is the number of independent curves of degree \( d \) passing through \( Z \). Let \( I \) be the ideal of \( Z \).

1. If \( r = 1 \), the geometric generating degrees of \( I \) are \( \{d, d+1\} \) and the \( d \)-envelope \( Z_d \) is a smooth curve of degree \( d \).
2. If \( r = 2 \) and \( d > 2 \), the geometric generating degrees of \( I \) are \( \{d, d+1\} \), and \( Z_d \) is a set of \( d^2 \) distinct, reduced points in \( \mathbb{P}^2 \), a complete intersection of type \( (d, d) \), containing \( Z \) together with \( d^2 - n = \left( \frac{d-1}{2} \right) \) extra points.
3. If \( r = 2 \) and \( d = 2 \) (so \( n = 4 \)), then 2 is the only geometric generating degree of \( I \).
4. If \( r \geq 3 \), then \( d \) is the only geometric generating degree of \( I \).

Proof. It suffices to note that the partition of \((\mathbb{P}^2)^n\) by graded Betti numbers includes a dense piece, corresponding to certain resolution data given in \([8]\). We repeat this “generic” resolution data here. Let \( r \) and \( d \) be defined as in the statement of the theorem. Then, with notation as in Theorem 2.5, the “generic” values of \( k \), \( \{a_i\} \), and \( \{b_j\} \) are as follows.

- If \( 2r \geq d+2 \) then \( k = d+1-r \), \( a_1 = \cdots = a_{k+1} = d \), \( b_1 = \cdots = b_{2r-d-2} = d+1 \), and \( b_{2r-d-1} = \cdots = b_k = d+2 \).
- If \( 2r \leq d+2 \) then \( k = d+1-r \), \( a_1 = \cdots = a_r = d \), \( a_{r+1} = \cdots = a_{k+1} = d+1 \), and \( b_1 = \cdots = b_k = d+2 \).

A general arrangement of \( n \) points has this resolution data, and we apply Theorem 2.8. If \( r \geq 3 \), then \( a_1 = a_2 = a_3 = d \), so \( d \) is the only geometric generating degree of the ideal \( I \) of the arrangement. The other cases \( r = 1, 2 \) are similar.

To prove Theorem 2.8 we interpret an arrangement \( Z \) and its Hilbert–Burch matrix in terms of a vector bundle and apply general transversality results.

### 3. Arrangements via vector bundles

In this section we reinterpret point arrangements in \( \mathbb{P}^2 \) and their degree envelopes in terms of sections of a vector bundle.
3.1. The Hilbert–Burch vector bundle. Given resolution data $R = (\{a_i\}, \{b_j\})$ as in Definition 2.6, we define the Hilbert–Burch vector bundle

$$E(R) = \text{Hom} \left( \bigoplus \mathcal{O}_{\mathbb{P}^2}(-b_j), \bigoplus \mathcal{O}_{\mathbb{P}^2}(-a_i) \right).$$

Note, $E(R)$ is ample if and only if $R$ is positive. We define the vector space of Hilbert–Burch matrices of type $R$ to be $\text{HB}(R) = H^0(\mathbb{P}^2, E(R))$, the vector space of $(k + 1) \times k$ matrices whose $(i, j)$th entry is a homogeneous form of degree $b_j - a_i$ for each $i, j$. For $A \in \text{HB}(R)$, let $I(A)$ be the ideal generated by the determinants of the $k \times k$ minors of $A$ and let $Z(A) \subset \mathbb{P}^2$ be the subscheme cut out by $I(A)$.

**Theorem 3.1.** Let $R = (a_1, \ldots, a_{k+1}; b_1, \ldots, b_k)$ be positive resolution data as in Definition 2.6. Let $A \in \text{HB}(R)$ be general.

1. $Z(A)$ is an arrangement of distinct, reduced points, with resolution data $R$. The number of points is $n = (\sum b_j^2 - \sum a_i^2)/2$.
2. For each $d \geq 0$, the $d$-envelope $Z(A)_d$ is smooth with codimension determined as follows. Let $r(d) = \#\{a_i \leq d\}$. Explicitly, $r(d)$ is defined by $a_{r(d)} \leq d < a_{r(d) + 1}$, with $r(d) = 0$ for $d < a_1$, and $r(d) = k + 1$ for $d \geq a_{k+1}$. Then $Z(A)_d$ has codimension $r(d)$ if $r(d) \leq 2$.

Furthermore, if $k = 1$, then $Z(A)_d = Z(A)$ if and only if $r(d) = 2$; if $k \geq 2$, then $Z(A)_d = Z(A)$ if and only if $r(d) > 2$.

**Remark 3.2.** Part 1 is already well-known.

This easily implies Theorem 2.8.

**Proof of Theorem 3.1.** Let $R = (\{a_i\}, \{b_j\})$ and $T$ be as in the statement of Theorem 2.8. By the first part of Theorem 3.1, the map $A \mapsto Z(A)$ is a rational map $\text{HB}(R) \dashrightarrow T$. It is surjective, by the Hilbert–Burch theorem 2.5. Since $\text{HB}(R)$ is irreducible, it follows that $T$ is irreducible.

For general $Z \in T$, there is a (general) $A \in \text{HB}(R)$ such that $Z = Z(A)$. Then the claims of Theorem 2.8 regarding $Z$ follow immediately from Theorem 3.1 applied to $A$.

To prove Theorem 3.1, we interpret the degree envelopes $Z(A)_d$ as loci where $A$, as a section of the Hilbert–Burch bundle, meets certain cones. The rest of this section is devoted to developing these tools, and then at the end we prove the theorem.

3.2. Decomposition of determinantal loci. Let $X$ be a generic $(k + 1) \times k$ matrix of variables whose entries $x_{ij}$ are independent variables. For $1 \leq i \leq k + 1$, let $F_i$ be the determinant of the $k \times k$ minor of $X$ obtained by deleting the $i$th row.

**Definition 3.3.** Let $S = \mathbb{C}[x_{1,1}, \ldots, x_{k+1,k}]$. Let $M = M_{(k+1)\times k}(\mathbb{C})$, the vector space of $(k + 1) \times k$ matrices with constant entries. The entries $x_{ij}$ of $X$ give coordinates on $M$. For $1 \leq r \leq k + 1$, we define certain ideals and determinantal loci in $M$, as follows.

1. Let $I_r \subset S$ be the ideal $(F_1, \ldots, F_r)$. 

(2) Let \( J_r \subset S \) be the ideal generated by the determinants of the maximal minors of the 
\((k+1-r) \times k\) matrix consisting of the last \(k+1-r\) rows of \(X\) (all but the first \(r\) rows). In particular, we set \( J_{k+1} = (1) \).
(3) Let \( L_r \subset M \) be the subscheme cut out by \( I_r \).
(4) Let \( N_r \subset M \) be the subscheme cut out by \( J_r \).

By a theorem of Eagon and Hochster [13], \( I_{k+1} \) is prime, as are all the \( J_r \). So \( L_{k+1} \) is irreducible, and so are all the \( N_r \). We have the following very useful decomposition of the determinantal loci \( L_r \):

**Proposition 3.4.** For \( 1 \leq r \leq k+1 \), \( L_r \) is reduced, and \( L_r = L_{k+1} \cup N_r \) as schemes.

I am grateful to M. Hochster for suggesting to me the proof of this statement. It follows from the statement on ideals that \( I_r = I_{k+1} \cap J_r \), which we prove shortly.

**Example 3.5.** Say \( k = 2 \), so
\[
X = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}.
\]

We have
\[
F_1 = cf - de, \quad F_2 = af - be, \quad F_3 = ad - bc,
\]
so
\[
I_1 = (cf - de), \quad I_2 = (cf - de, af - be), \quad I_3 = (cf - de, af - be, ad - bc),
\]
and
\[
J_1 = (cf - de), \quad J_2 = (e, f), \quad J_3 = (1).
\]

Obviously \( I_1 = I_3 \cap J_1 \) and \( I_3 = I_3 \cap J_3 \), but we also have, less obviously, \( I_2 = I_3 \cap J_2 \).

**Lemma 3.6.** Let \( R \) be a ring, \( I \subset R \) be an ideal, and \( e \notin I \). Assume: \( P = I + (e) \) is radical, \( Q = (I : e) = \{ x \mid xe \in I \} \) is prime, and \( e^2 \notin I \) (equivalently, \( e \notin Q \)). Then \( I = P \cap Q \).

**Proof.** First we show \( I \) is radical. Suppose \( x^n \in I \) for \( n \geq 2 \). Then \( x^n \in P \), so \( x \in P \). Therefore \( x = i + ae \) for some \( i \in I \). Since \( x^n \in I \) and \( i \in I \), we get \( (ae)^n \in I \); in particular, \( a^n e^{n-1} \in Q \). Since \( Q \) is prime and \( e \notin Q \), \( a \in Q \). Thus \( ae \in I \), so \( x \in I \).

Now, suppose \( y \in P \cap Q \). We may write \( y = i + ae \), with \( i \in I \). Then \( y^2 = iy + aye \), where \( iy \in I \) and \( ye \in I \) because \( y \in Q \). Therefore \( y^2 \in I \). Since \( I \) is radical, \( y \in I \). \( \square \)

**Proof of Proposition 3.4.** We go by downward induction on \( r \), starting from \( r = k+1 \). Since \( J_{k+1} = (1) \), the unit ideal, the initial case is trivial.

For \( r \leq k \), \( I_{r+1} = I_r + (F_{r+1}) \); this ideal is radical by induction. We claim that \( J_r = (I_r : F_{r+1}) \) and \( F_{r+1} \notin J_r \). From these claims and the previous lemma it follows that \( I_r = I_{r+1} \cap J_r \), and in particular that \( I_r \) is radical.

For the second claim, note that
\[
F_{r+1} \notin (x_{r+1,1}, \ldots, x_{r+1,k}) \supset J_r.
\]
For the first claim, if $GF_{r+1} \in I_r \subset J_r$ then, since $J_r$ is prime and $F_{r+1} \notin J_r$, we have $G \in J_r$. This shows $(I_r : F_{r+1}) \subset J_r$.

We have to show $J_rF_{r+1} \subset I_r$. We claim that for any generator $P$ of $J_r$ given as a maximal minor of the last $k+1-r$ rows of $X$, we have $PF_{r+1} \in I_r$. We may reorder the columns of $X$ so that the minor whose determinant gives $P$ is given by the first $k+1-r$ columns of $X$. Take the transpose of these columns and write it in block form as $(UV)$, where $U$ is the first $r$ columns and $V$ is the square matrix of size $k+1-r$ whose determinant is $P$. Let $w = (F_1, -F_2, \ldots, (-1)^iF_i, \ldots)$, and write it also in block form as $w = (w_1, w_2)$ where $w_1$ has size $r$ and $w_2$ has size $k+1-r$. Then by Cramer’s rule,

$$(U \ V) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0,$$

so $Vw_2 = -Uw_1$. Multiplying on the left by the adjoint matrix $V^*$ of $V$ (the transpose of the matrix of cofactors) and applying again Cramer’s rule,

$$Pw_2 = \det(V)w_2 = -V^*Uw_1.$$ 

In particular, $F_{r+1}$ is the first entry of $w_2$, so $PF_{r+1}$ is some combination of the entries of $w_1$, namely $F_1, \ldots, F_r$. This shows $PF_{r+1} \in I_r$. Therefore $J_rF_{r+1} \subset I_r$, and so $(I_r : F_{r+1}) = J_r$.

Applying the previous lemma, we see that $I_r = I_{r+1} \cap J_r$, and by induction,

$$I_r = I_{r+1} \cap J_r = I_{r+2} \cap J_{r+1} \cap J_r = I_{r+3} \cap J_{r+2} \cap J_{r+1} \cap J_r = \ldots$$

Since $J_r \subset J_{r+1} \subset \ldots$, we see that, as claimed, $I_r = I_{k+1} \cap J_r$. \qed

We will take advantage of the following useful facts about $L_{k+1}$ and the $N_r$.

**Proposition 3.7.**

(1) $L_{k+1}$ has codimension 2 in $M$.

(2) The singular locus $\text{Sing } L_{k+1}$ has codimension 6 in $M$.

(3) Each $N_r$ has codimension $r$ in $M$.

(4) Each $\text{Sing } N_r$ has codimension $2(r+1)$ in $M$.

(5) $L_{k+1} \subset N_1 = L_1$, but $L_{k+1} \not\subset N_r$ for any $r > 1$.

(6) $N_{k+1} = 0 \subset L_{k+1}$, but $N_r \not\subset L_{k+1}$ for any $r < k+1$.

(7) For $1 < r < k+1$, $L_{k+1} \cap N_r$ has codimension at least 3 in $M$.

**Proof.** We use the well-known formula that in the space of $m \times n$ matrices, the variety of matrices with rank at most $c$ has codimension equal to $(m-c)(n-c)$ (see, for example, [11, Prop. 12.2]), and singular locus equal to the variety of matrices with rank at most $c-1$ (see, for example, [11, Example 14.16]). We apply this to prove the first four parts as follows.

For (1), $L_{k+1}$ is the variety of matrices with rank at most $k-1$ in the space of $(k+1) \times k$ matrices. For (2), $\text{Sing } L_{k+1}$ is the variety of matrices with rank at most $k-2$, in the same space.

Now, write $M = M_1 \times M_2$, where $M_1$ is the affine space with coordinates given by the entries of the first $r$ rows of $X$, and $M_2$ is the affine space with coordinates given by the last
Abusing notation, we denote this tautological map by $X$. Let $N_r' \subset M_2$ be the locus defined by the vanishing of all the maximal minors of the last $k + 1 - r$ rows of $X$. Then

$$N_r = M_1 \times N_r'.$$

Since $N_r'$ is the variety of matrices of rank at most $k - r$ in the space of $(k + 1 - r) \times k$ matrices, $N_r'$ has codimension $r$ in $M_2$. Therefore $N_r$ has codimension $r$ in $M$, proving (3). Furthermore,

$$\text{Sing } N_r = M_1 \times \text{Sing } N_r',$$

where $\text{Sing } N_r' \subset M_2$ has codimension $2(r + 1)$. This proves (4).

For (5) and (6), the inclusions $L_{k+1} \subset N_1$ and $N_{k+1} \subset L_{k+1}$ are clear. To see the non-inclusions, consider the following $(k + 1) \times k$ matrices, given in block form:

$$A_r = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ I_{k+1-r} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} I_k \\ 0 \end{pmatrix}$$

where $I_{k+1-r}$ and $I_k$ are the identity matrices of the indicated sizes. Then for $r > 1$, $A_r \in L_{k+1}$ but $A_r \notin N_r$. For $r < k + 1$, $B \in N_r$ but $B \notin L_{k+1}$.

Finally, for (7), for $1 < r < k + 1$, $L_{k+1} \cap N_r$ is strictly contained in $L_{k+1}$, which is irreducible and of codimension 2. \qed

3.3. **Cones in $E(R)$ and degree envelopes.** Given positive resolution data $R$ and a Hilbert–Burch matrix $A \in \text{HB}(R)$, recall that $I(A)$ is the ideal of determinants of $k \times k$ minors of $A$. Each of these is obtained by omitting a row of $A$. For $1 \leq i \leq k + 1$, let $F_i(A)$ be the determinant of the $k \times k$ minor of $A$ obtained by omitting the $i$th row. Note that $\text{deg } F_i(A) = a_i$. Then $I(A) = (F_1(A), \ldots, F_{k+1}(A))$, and for $d \geq 0$, the $d$-envelope $Z(A)_d$ is defined by the forms $F_i(A)$ such that $\text{deg } F_i(A) = a_i \leq d$.

The matrix $A$ is also a section of the Hilbert–Burch bundle $E(R)$, and we take advantage of this to give an alternative approach for $Z(A)$ and its degree envelopes. The idea is to define cones in the total space of $E(R)$ analogous to the $L_r \subset M$ considered in the previous section, and then recover $Z(A)$ and the $Z(A)_d$ as the loci in $\mathbb{P}^2$ where $A$ meets these cones.

We denote by $E(R)$ the total space of the vector bundle $E(R)$. Let $\pi : E(R) \to \mathbb{P}^2$ be the projection map. There is a tautological map of bundles on $E(R)$,

$$\pi^* \bigoplus_{j=1}^{k} \mathcal{O}_{\mathbb{P}^2}(-b_j) \to \pi^* \bigoplus_{i=1}^{k+1} \mathcal{O}_{\mathbb{P}^2}(-a_i).$$

Abusing notation, we denote this tautological map by $X$, and for each $i, j$, we denote by $x_{ij}$ the induced map

$$x_{ij} : \pi^* \mathcal{O}_{\mathbb{P}^2}(-b_j) \to \pi^* \mathcal{O}_{\mathbb{P}^2}(-a_i).$$

The $x_{ij}$ are global coordinates on $E(R)$. Suppose over an affine open subset $U \subset \mathbb{P}^2$ one trivializes each of the line bundles $\mathcal{O}_{\mathbb{P}^2}(-b_j), \mathcal{O}_{\mathbb{P}^2}(-a_i)$. We get a trivialization of $E(R)|_U$, hence coordinates on $E(R)|_U = \pi^{-1}(U)$. These coordinates are the $x_{ij}$ (together with coordinates on $U$). In particular, the $x_{ij}$ restrict to coordinates on each fiber of $E(R)$.
We define cones in $E(R)$ by vanishing of determinants of minors of $X = (x_{ij})$, just as in the previous section. As before, for $1 \leq r \leq k + 1$, let $F_r$ be the determinant of the minor of $X$ obtained by omitting the $r$th row. The vanishing-locus $\{F_r = 0\} \subset E(R)$ is the rank-dropping locus of the vector bundle map given by removing the $r$th row of $X$:

$$\pi^* \bigoplus_{\substack{\ell = 1 \leq k \leq r \leq k+1 \atop \ell \neq r}} \mathcal{O}_{\mathbb{P}^2}(-b_j) \rightarrow \pi^* \bigoplus_{\substack{\ell = 1 \leq k \leq r \leq k+1 \atop \ell \neq r}} \mathcal{O}_{\mathbb{P}^2}(-a_i).$$

For $1 \leq r \leq k + 1$, let $L_r(R) \subset E(R)$ be defined by $F_1 = \cdots = F_r = 0$, the scheme-theoretic intersection of the rank-dropping loci.

Similarly, let $N_r(R) \subset E(R)$ be defined by the vanishing of the maximal minors of the last $k + 1 - r$ rows of $X$. Equivalently, $N_r$ is the rank-dropping locus of the map of vector bundles,

$$\pi^* \bigoplus_{\substack{\ell = 1 \leq k \leq r \leq k+1 \atop \ell \neq r}} \mathcal{O}_{\mathbb{P}^2}(-b_j) \rightarrow \pi^* \bigoplus_{\substack{\ell = 1 \leq k \leq r \leq k+1 \atop \ell \neq r}} \mathcal{O}_{\mathbb{P}^2}(-a_i).$$

Now, over an affine open $U \subset \mathbb{P}^2$, trivializing each $\mathcal{O}_{\mathbb{P}^2}(-b_j)$, $\mathcal{O}_{\mathbb{P}^2}(-a_i)$, the resulting trivialization of $E(R)|_U$ gives an isomorphism

$$E(R)|_U \longrightarrow M \times U,$$

which takes

$$L_r(R)|_U \longrightarrow L_r \times U,$$

$$N_r(R)|_U \longrightarrow N_r \times U$$

This leads to the following “global” analogue of Propositions 3.4 and 3.7:

**Proposition 3.8.** Let $R$ be positive resolution data. For $1 \leq r \leq k + 1$, $L_r(R)$ is reduced and $L_r(R) = L_{k+1}(R) \cup N_r(R)$. Also, $L_{k+1}(R)$ is irreducible and reduced, and each $N_r(R)$ is irreducible and reduced. We have the following facts:

1. $L_{k+1}(R)$ has codimension 2 in $E(R)$.
2. $\text{Sing } L_{k+1}(R)$ has codimension 6 in $E(R)$.
3. Each $N_r(R)$ has codimension $r$ in $E(R)$.
4. Each $\text{Sing } N_r(R)$ has codimension $2(r + 1)$ in $E(R)$.
5. $L_{k+1}(R) \subset N_1(R) = L_1(R)$, but $L_{k+1}(R) \not\subset N_r(R)$ for any $r > 1$.
6. $N_{k+1}(R) = 0 \subset L_{k+1}(R)$, but $N_r(R) \not\subset L_{k+1}(R)$ for any $r < k + 1$.
7. For $1 < r < k + 1$, $L_{k+1}(R) \cap N_r(R)$ has codimension at least 3 in $E(R)$.

We have defined the cones we are interested in. Now we want to show how to use them to get point arrangements in $\mathbb{P}^2$ and degree envelopes.

For positive resolution data $R$ and $A \in \text{HB}(R)$, the arrangement $Z(A)$ and its degree envelopes $Z(A)_d$ are defined by the vanishing of the forms $F_i(A)$. The idea is to see these
Let $F_i(A)$ as pullbacks of the equations $F_i$ on $E(R)$, and then we will see that the $Z(A)$ and $Z(A)_d$ are the loci in $\mathbb{P}^2$ where $A$, as a section of $E(R)$, intersects the cones $L_r(R)$.

Let $s_A : \mathbb{P}^2 \to E(R)$ be the section associated to $A$. For $1 \leq i \leq k + 1$ and $1 \leq j \leq k$, one has the following two maps of line bundles:

$$A_{ij} : \mathcal{O}_{\mathbb{P}^2}(-b_j) \to \mathcal{O}_{\mathbb{P}^2}(-a_i),$$

$$x_{ij} : \pi^*\mathcal{O}_{\mathbb{P}^2}(-b_j) \to \pi^*\mathcal{O}_{\mathbb{P}^2}(-a_i).$$

Evidently $A_{ij} = s_A^*x_{ij}$. This implies, for $1 \leq i \leq k + 1$, $F_i(A) = s_A^*F_i$. We obtain the following:

**Proposition 3.9.** Let $R = (\{a_i\}, \{b_j\})$ be positive resolution data and $A \in \text{HB}(R)$. Let $s_A : \mathbb{P}^2 \to E(R)$ be the map corresponding to $A \in H^0(\mathbb{P}^2, E(R))$. Then $Z(A) = s_A^{-1}(L_{k+1}(R))$, the locus in $\mathbb{P}^2$ where $A$ meets $L_{r}(R)$.

For $d \geq 1$, the $d$-envelope $Z(A)_d$ is the scheme-theoretic preimage $s_A^{-1}(L_r(R))$, the locus where $A$ meets $L_r(R)$, where $r = r(d) = \#\{a_i \leq d\}$ (or, $r$ is defined by $a_r \leq d < a_{r+1}$—the same function $r(d)$ as in Theorem 3.11).

3.4. General transversality for sections of a vector bundle. We recall the following well-known statement:

**Lemma 3.10.** Let $E$ be a globally generated vector bundle of rank $e$ on a projective variety $X$. Let the total space of $E$ be denoted $\mathbf{E}$. Let $L \subset \mathbf{E}$ be a closed subset with $\dim L < \text{rk} E$. Then a general section $s \in H^0(X, E)$ does not meet $L$. □

This is proved by a dimension count. It can be generalized to give the following proposition, reminiscent of the proof of Bertini’s theorem in characteristic zero via Kleiman’s generic smoothness theorem as presented in [12, III.10.9]. It belongs to the folklore, but for lack of a reference we give a statement and proof.

**Proposition 3.11.** Let $E$ be an ample and globally generated vector bundle of rank $e$ on a smooth complex projective variety $X$. Let the total space of $E$ be denoted $\mathbf{E}$. Let $L \subset \mathbf{E}$ be an irreducible reduced closed subset with $\dim L \geq e$ and $\dim \text{Sing} L < e$. Then for a general global section $s$ of $E$, the locus $s^{-1}(L) \subset X$ where $s$ meets $L$ is nonempty, reduced, smooth, and with codimension in $X$ equal to the codimension of $L$ in $\mathbf{E}$.

**Proof.** By Theorem 12.1(c) of [3], for every section $s$ of $E$, $s^{-1}(L)$ is a positive cycle on $X$, so in particular nonempty.

Let $U \subset H^0(X, E)$ be the open subset of sections not meeting $\text{Sing} L$. Consider

$$\tilde{L} = \{ (s, x) \in U \times X \mid s(x) \in L - \text{Sing}(L) \}.$$ 

This is a nonempty open subset of the set

$$(2) \{ (s, x) \in H^0(X, E) \times X \mid s(x) \in L - \text{Sing}(L) \}.$$ 

Note that

$$(3) \{ (s, x) \in H^0(X, E) \times X \mid s(x) \in L - \text{Sing}(L) \} \longrightarrow L - \text{Sing}(L)$$
is an affine bundle. Indeed, it is the restriction to $L - \text{Sing} \, L$ of the affine bundle $H^0(X, E) \times X \to E$ given by $(s, x) \mapsto s(x)$. The restricted bundle $(3)$ has smooth base $L - \text{Sing} \, L$; therefore its total space $(2)$ is reduced and smooth. Hence the open subset $\tilde{L}$ is reduced and smooth.

The projection map $\tilde{L} \to U$ is surjective because every section of $E$ meets $L$. By Keiman’s generic smoothness theorem [12, Theorem III.10.7], there is an open dense subset $W \subset U$ over which the fibers of this projection map are nonempty, reduced, smooth, and all of the same codimension, namely the codimension of $L$ in $E$. Finally, the fiber over $s \in W$ is isomorphic to $s^{-1}(L) \subset X$. □

3.5. Proof of Theorem 3.1. We now use the tools we have just developed to prove Theorem 3.1 in turn implying Theorem 2.8 and Corollary 2.11.

Proof of Theorem 3.1. Let $R = (\{a_i\}_{i=1}^{k+1}, \{b_j\}_{j=1}^{r})$ be the positive resolution data given in the hypothesis of the statement of the theorem. Let $E(R)$ be the Hilbert–Burch vector bundle as defined above, with total space $E(R)$, and for $1 \leq r \leq k+1$ let $L_r(R), N_r(R) \subset E(R)$ be the cones defined in section 3.3. The positivity of $R$ means $E(R)$ is a direct sum of ample line bundles on $\mathbb{P}^2$, hence ample.

We saw that for every $A \in \text{HB}(R)$, with corresponding section $s_A : \mathbb{P}^2 \to E(R)$, the subscheme $Z(A) \subset \mathbb{P}^2$ is the locus where $s_A$ meets $L_{k+1}(R)$. Recall that by Proposition 3.8, $L_{k+1}(R)$ has codimension 2 in $E(R)$ and $\text{Sing} \, L_{k+1}(R)$ has codimension 6 in $E(R)$. Therefore, for general sections $A \in \text{HB}(R)$, Proposition 3.11 shows that $Z(A)$ is nonempty, reduced, and smooth, with codimension 2 in $\mathbb{P}^2$. One checks easily that the Hilbert–Burch short exact sequence as in Theorem 2.5 is a resolution of the ideal $I(A)$, so $Z(A)$ has the resolution data $R$, as claimed. The number of points is $n = (\sum b_j^2 - \sum a_i^2)/2$ of $Z(A)$ by an argument as in Remark 2.7. This proves the first part of the theorem, which was nevertheless known previously.

Now let $d \geq 0$. Recall our earlier notation, that for $A \in \text{HB}(R)$ and $1 \leq r \leq k+1$, $F_r(A)$ is the homogeneous form of degree $a_r$ given by the determinant of the $k \times k$ minor of $A$ obtained by omitting the $r$th row. Then the $d$-envelope $Z(A)_d$ is defined by the vanishing of those forms $F_r(A)$ such that $\deg F_r(A) = a_r \leq d$. Since $a_1 \leq \cdots \leq a_{k+1}$, we see that $Z(A)_d$ is defined by $F_1(A) = \cdots = F_r(A) = 0$, where $r = r(d) = \#\{a_i \leq d\}$ is given, as in the statement of the theorem, by $a_r \leq d < a_{r+1}$, with $r = 0$ for $d < a_1$ and $r = k+1$ for $d \geq a_{k+1}$.

First of all, if $r = 0$, then $Z(A)_d = \mathbb{P}^2$ is clear.

Suppose $r \geq 1$. We saw in section 3.3 that $Z(A)_d$ is the locus where the corresponding section $s_A : \mathbb{P}^2 \to E(R)$ meets $L_r(R)$. We now apply Proposition 3.8 as follows.

If $r = 1$ (equivalently, $a_1 \leq d < a_2$), then $L_1(R) = N_1(R)$, which is irreducible and of codimension 1 in $E(R)$, with singularities $\text{Sing} \, (N_1(R))$ of codimension 4 in $E(R)$. Then by Proposition 3.11, for general $A \in \text{HB}(R)$, $Z(A)_d$ is smooth and reduced, with codimension 1. (Note that $Z(A)_d$ is defined by the single equation $F_1(A)$.)
If \( r = 2 \) (equivalently, \( a_2 \leq d < a_3 \)), then \( L_2(R) = L_{k+1}(R) \cup N_2(R) \). In this case \( L_{k+1}(R) \) and \( N_2(R) \) both have codimension 2 in \( E(R) \). Since \( L_{k+1}(R) \cap N_2(R) \) has codimension at least 3, by Proposition 3.10 we see that for general \( A \in \text{HB}(R) \), the section \( s_A \) does not meet \( L_{k+1}(R) \cap N_2(R) \). Therefore for such \( A \), \( Z(A)_d \) is the disjoint union

\[
Z(A)_d = s_A^{-1}(L_{k+1}(R)) \cup s_A^{-1}(N_2(R)) = Z(A) \cup s_A^{-1}(N_2(R)).
\]

Since \( N_2(R) \) has codimension 2 in \( E(R) \) and \( \text{Sing} \ N_2(R) \) has codimension 6, Proposition 3.11 shows \( s_A^{-1}(N_2(R)) \) is nonempty, smooth, reduced, and of codimension 2 in \( \mathbb{P}^2 \). Therefore \( Z(A)_d \) is a reduced set of points, strictly larger than \( Z(A) \).

If \( r \geq 3 \) (equivalently, \( a_3 \leq d \)), then \( L_r(R) = L_{k+1}(R) \cup N_r(R) \). Since \( N_r(R) \) has codimension \( r > \dim \mathbb{P}^2 \) in \( E(R) \), general sections \( A \in \text{HB}(R) \) do not meet \( N_r(R) \). Therefore the \( d \)-envelope \( Z(A)_d \), the locus where \( s_A \) meets \( L_r(R) \), is just the locus where \( s_A \) meets \( L_{k+1}(R) \). This is \( Z(A) \). Therefore for \( d \geq a_3 \), \( Z(A)_d = Z(A) \).

This completes the proof of the theorem. \( \square \)

**Remark 3.12.** It is natural to consider similar questions in higher dimension. One expects similar results for general arrangements of points in \( \mathbb{P}^s \): that the geometric generating degrees and degree envelopes are determined by the number \( n \) of points.

One may also consider more special arrangements of points in \( \mathbb{P}^s \). For example, Gorenstein point arrangements in \( \mathbb{P}^3 \) are defined by the Pfaffians of a skew-symmetric matrix (see [1]), and certain point arrangements in \( \mathbb{P}^s \) are defined by the minors of a \( k \times (k + s - 1) \) matrix.

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