Reliability of $K \to \pi\pi$ matrix element calculations

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Abstract

I demonstrate that the short distance contribution to $K \to \pi\pi$ decays must be supplemented with large distance effects. A hybrid calculation is outlined based on QCD diagrams supplemented by chiral contributions and $\pi-\pi$ phaseshifts.

\footnote{Invited talk presented at the International Conference on CP Violation: KAON 2001 (June 12–17, 2001; Pisa, Italy)}
At this meeting the KTeV group reported a new and smaller value of $\varepsilon'/\varepsilon$. The experimental values of the two groups are now consistent with each other:

$$\varepsilon'/\varepsilon = (15.3 \pm 2.6) \times 10^{-4} \quad \text{NA48} \ [1]$$
$$\varepsilon'/\varepsilon = (20.7 \pm 2.8) \times 10^{-4} \quad \text{KTeV} \ [2].$$

On the theoretical side there were, for a long time, two types of predictions:

i) those which predicted small, almost unmeasurable, or negative values, and

ii) those which predicted large, positive and observable values.

It is interesting to examine the reasons that led to the diverse predictions, and also the changes required in order to bring them into agreement with the experiment. For my part, since I belong to the second group, I will present our efforts and work that led to large values for the ratio. The starting point of the analysis is the effective $\Delta S = 1$ Hamiltonian derived from QCD. The Wilson coefficients in the Hamiltonian have been calculated to NLO and I shall use them. It has become evident that tree diagram values for the matrix elements of operators are not sufficient and we need to include loop corrections, which are in fact unitarity corrections. The study and calculation of the loops modify the values for the amplitudes and improve the accuracy of the predictions. All groups which included loop corrections to matrix elements predicted large and positive values for $(\varepsilon'/\varepsilon)$. Now that this fact has been recognized we face the next problem of estimating the uncertainties and improving the reliability of the results. This is a difficult problem because it requires defining a low energy limit of QCD, including loops.

The first article to include loop corrections to matrix elements [3] studied the operators $Q_1$ and $Q_2$ and found an enhancement of the $I = 0$ amplitude. Subsequently, it was recognized that loops are also important for the operators $Q_6$ and $Q_8$ [4], because they enhance $\langle Q_6 \rangle_0$ and reduce $\langle Q_8 \rangle_2$ [4], thus eliminating the cancellation between the two terms and predicting a large ratio $(\varepsilon'/\varepsilon)$. Loop corrections have also been calculated in the chiral quark model [5] with similar results. Finally, they were calculated in chiral perturbation theory, paying special attention to the matching of the chiral loops to the
QCD–scale $[3]$ residing in the Wilson coefficients.

The fact that loop corrections are important is well accepted now. The remaining issue is the accuracy of such calculations, especially the dependence of the results on the cut–off of the chiral theory, since chiral theory is not renormalizable. It was shown in article $[3]$ that to $O(p^2)$ the infinities of the factorizable diagrams are absorbed into the renormalized coupling constants

$$L^r_5 = 2.07 \times 10^{-3} \quad \text{and} \quad L^r_8 = 1.09 \times 10^{-3}. $$

This happens for all operators – those that include currents, as well as those with densities of quarks. The non–factorizable terms bring to $O(p^2)$ a quadratic dependence on the cut–off, which makes the predictions sensitive functions of the cut–off. In the meanwhile, the experience gained in the calculations, together with the fact that we must account for five physical quantities $|A_0(m_K)|$, $|A_2(m_K)|$, $\delta^0_0(m_K)$, $\delta^2_0(m_K)$ and $(\varepsilon'/\varepsilon)_K$, allow me to abstract four general properties that must be included in explanations of the quantities. The aim here is to arrive at a low energy realization of QCD which reproduces current algebra at the 1-loop level, since current algebra is already satisfied at the tree level of chiral theory.

I summarize the properties as four benchmarks.

**Benchmark 1.** The imaginary parts of the amplitude can be calculated by unitarity to be

$$\Im A(K^+ \to \pi^+\pi^0) = |A(K^+ \to \pi^+\pi^0)| \left(-\frac{1}{8}\right) \frac{m^2_K - m^2_\pi}{4\pi F^2_\pi} \sqrt{1 - \frac{4m^2_\pi}{m^2_K}}$$

and similarly

$$\Im A_0(K \to \pi\pi) = |A_0(K \to \pi\pi)| \left(-\frac{1}{8}\right) \frac{m^2_K - 0.5m^2_\pi}{4\pi F^2_\pi} \sqrt{1 - \frac{4m^2_\pi}{m^2_K}}$$

where $|A(K^+ \to \pi^+\pi^0)|$ and $|A_0(K \to \pi\pi)|$ are absolute values of the complete decay amplitudes for $K^+ \to \pi^+\pi^0$ and the $I = 0$ decay, respectively. Taking the ratios $\Im A/|A|$, one arrives at

$$\sin \delta^0_0 = \frac{1}{4} \frac{m^2_K - 0.5m^2_\pi}{4\pi F^2_\pi} \sqrt{1 - \frac{4m^2_\pi}{m^2_K}} = 0.46 \quad (5)$$

$$\sin \delta^2_0 = \frac{1}{8} \frac{m^2_K - 2m^2_\pi}{4\pi F^2_\pi} \sqrt{1 - \frac{4m^2_\pi}{m^2_K}} = 0.20 \quad (6)$$
which give the following values for the phaseshifts

\[ \delta_0^0 = 27.4^\circ \] to be compared with \[ 34.2^\circ \] from experiments, and

\[ \delta_0^2 = -11.5^\circ \] to be compared with \[ -7^\circ \text{ to } -12^\circ \] from experiments.

One must conclude that for \( s = m_K^2 \) the absolute values of \( K \to \pi\pi \) amplitudes, combined with the phase space integration (unitarity), gives acceptable phases. It still remains to compute the real parts of the amplitudes at \( s = m_K^2 \), as well as real parts of the matrix elements for various operators.

**Benchmark 2.** Practically all articles use the effective \( \Delta S = 1 \) Hamiltonian derived from QCD. At a high momentum scale, like \( \mu = 2 \text{ GeV} \), the Hamiltonian should give reliable estimates which are then extrapolated to lower momenta. In addition, at the high \( \mu \)-scale, the Wilson coefficients in the various schemes are close to each other. Since the quarks carry large momenta, I will use the lowest order contribution of quark operators between hadrons. The numerical estimates of such calculations are shown in Table 1, where I give numerical values for two momenta – \( \mu = 2.0 \) and \( \mu = 1.0 \text{ GeV} \). The change between the two scales is relatively small, so I have also included in the table the experimental values for comparison. The theoretical estimate for \( A_0(\mu) \) gives only 30% of the experimental value and for \( A_2(\mu) \) it gives 150% of the experimental value. Thus additional (non-perturbative) corrections must be large and they must increase \( A_0(m_K) \) and decrease \( A_2(m_K) \). The corrections found by the Dortmund [4, 6] and Trieste [3] groups, in chiral theory, increase \( \langle Q_6 \rangle_0 \) and decrease \( \langle Q_8 \rangle_2 \) in agreement with the above requirement. Similarly, the \( \pi-\pi \) phase shift for \( I = 0 \) is positive and for \( I = 2 \) negative, bringing corrections with the required sign [8, 4, 11].

| \( \mu \) (GeV) | \( A_0(\mu) \) in GeV | \( A_2(\mu) \) in GeV |
|-----------------|------------------|------------------|
| 2 GeV           | \( 0.62 \times 10^{-7} \) | \( 0.25 \times 10^{-7} \) |
| 1 GeV           | \( 0.83 \times 10^{-7} \) | \( 0.23 \times 10^{-7} \) |
| Exper.          | \( 3.33 \times 10^{-7} \) | \( 0.15 \times 10^{-7} \) |

Table 1: Contribution from QCD alone
Benchmark 3. It is evident from Table 1 that the change of the scale $\mu$ is not large enough to provide the required corrections. In fact, several operators of QCD have a similar development between 2.0 and 1.0 GeV. The running of $C_6(q^2)$ (dashed curve) and $C_6(q^2)/m_s^2(q^2)$ (solid curve) in this momentum region are plotted in figure 1. The variation of the ratio is much smaller than the running of $C_6(q^2)$ alone.

![Figure 1: Evolution of $C_6(q^2)$ (dashed curve) and $C_6(q^2)/m_s^2(q^2)$ (solid curve) as functions of $q$.](image)

This stability was already noticed in [11] and a similar one appears for $C_8(q^2)/m_s^2(q^2)$.

Operators whose anomalous dimensions are zero are called marginal and they remain constant over extended regions of momenta. A similar property appears, numerically, in the sum of coefficients [7]

$$Z_1(\mu) + Z_2(\mu) \sim 0.7 \text{ to } 0.8 \quad \text{for} \quad 0.6 < \mu < 2.0 \text{ GeV}. \quad (7)$$

These results from QCD suggest that marginal quantities of QCD could be extrapolated to low momenta. In this case, large corrections must reside on the values of the matrix elements as non–perturbative effects.
Benchmark 4. We have a lot of data on $\pi - \pi$ scattering, some of them being shown in figures 2a and 2b. The data indicate that the phaseshift for the $I = 0$ channel remains elastic up to 900 MeV and then the resonance $f_0(980)$ appears, which forces the $\delta^0_0$ phaseshift to pass through $180^\circ$, as shown in figures 2a and 2b \[12\]. The behavior of $\delta^0_0$ must help the convergence of the $I = 0$ amplitudes, since $\sin \delta^0_0$ goes through zero at $E = 1$ GeV. The $I = 2$ phaseshift is negative, smaller and has no special structure. The dynamics of the $\pi - \pi$ scattering play a role in the decay amplitudes and must be included in the analysis. Contributions from the phaseshifts will bring corrections with the desired signs: increase $\langle Q_6 \rangle_0$ and decrease $\langle Q_8 \rangle_2$. Suggestions along these lines have already been discussed \[9, 10\].

It is suggestive that an explanation of the $K$–decays must include the above properties. There are interpolations suggested by the above benchmarks. I will outline a method in this context, which tries to improve the high energy region of chiral calculations ($p \approx 1$GeV). I will replace the $K$–meson with the divergence of the $\Delta S = 1$ axial current and try to apply current algebra. The divergence of the axial current is realized below 1.0 GeV by its chiral representation and above that energy by the quark representation. The amplitude is an analytic function of the invariant mass of the two–pions–squared, $s$, and satisfies a dispersion relation

$$
Re A_I(\sigma) = \frac{1}{\pi} \int_{4m^2}^{s_0} \frac{\text{Disc.} A_I(s)}{s - \sigma} ds + \frac{1}{\pi} \int_{s_0}^{m^2} \frac{\text{Disc.} A_I(s)}{s - \sigma} ds. \tag{8}
$$

The value of $s_0$ is an intermediate scale, with $\sqrt{s_0} = 1.0$ to 1.4 GeV. The splitting of the
integral is intentional so that for the high energy term I use the QCD amplitude. For instance, consider the amplitude

\[ A_{Q_6}(s, \mu) = C_6(s, \mu) \langle \pi\pi | Q_6 | K \rangle \quad (9) \]

with the discontinuity given by \[\text{Disc. } A_{Q_6}(s, \mu) = \frac{\pi}{\ln^2 |s/\Lambda^2| + \pi^2} C_6(\mu) \left( \ln \frac{\mu^2}{\Lambda^2} \right) \langle \pi\pi | Q_6 | K \rangle. \quad (10)\]

In this energy region the momenta in $Q_6$ are large and the tree contribution for the matrix element will suffice. The integral over the discontinuity is finite and no subtraction is required \[\text{[14].}\]

The discontinuity in the low energy integral is assumed to be \[\text{[3]} \]

\[ \text{Disc. } A_{Q_6}(s) = -4\sqrt{3} L_5 \left( \frac{m_K^2 - m_\pi^2}{F_\pi} \right) \frac{4m_K^4}{m_\pi^2(s) g_0(s)} \quad (11) \]

with the Goldberger–Treiman factor \[\text{[15]} \]

\[ g_0(s) = \frac{\eta_0 \sin 2\delta^0 \cos \delta^0}{1 + \eta_0 \cos 2\delta^0}, \quad (12) \]

$\eta_0$ the inelasticity and $\delta^0$ the phaseshift. This representation is certainly valid below the inelastic threshold. Since the $I = 0$ channel remains elastic up to 1.0 GeV, I will use this form. The enhancement factor from the first integral in eq. (8) is given by

\[ F_6(s_0, m_k) = \frac{1}{\pi} \int_{4m^2}^{s_0} \frac{g_0(s)}{s - m_k^2} \frac{m_s^2(1 \text{ GeV})}{m_\pi^2(s)} (1 \pm h(s)) \, ds. \quad (13) \]

The phenomenological function $h(s) = \frac{1}{2} \left( \frac{s - 0.25}{1.5} \right)$ in the integrand is introduced in order to study the sensitivity of the enhancement factor on off-the-mass shell effects \[\text{[16]} \] (see discussion). Figure 3 shows the enhancement factor as a function of $E = \sqrt{s_0}$. The factor $F_6(s_0, m_k)$ increases up to 900 MeV and then flattens out. To obtain the solid curve I set $h(s) = 0$ and the strange quark mass in the denominator equal to $m_s(1 \text{ GeV})$. The dashed and dashed–dotted curves include the running strange quark mass and $\pm h(s)$, respectively. The enhancement factor lies between 0.40 and 0.55. A similar calculation for $F_8(s_0, m_k)$ gives a depletion factor with the approximate value of $-0.20$ to $-0.30$. 
The loop corrections increase $\langle Q_6 \rangle_0$ by 50% and decrease $\langle Q_8 \rangle_2$ by 20% to 30%. For these values the ratio attains values

$$\left( \varepsilon'/\varepsilon \right) \approx (15 \ldots 20) \times 10^{-4},$$

which do not include the isospin breaking term yet. The range given in eq. (14) includes a rough estimate of the theoretical uncertainties. A detailed calculation with a description of the current algebra and a precise estimate of the theoretical uncertainties will be presented in a publication.

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