ZERO MEAN CURVATURE ENTIRE GRAPHS OF MIXED TYPE
IN LORENTZ-MINKOWSKI 3-SPACE

S. FUJIMORI, Y. KAWAKAMI, M. KOKUBU, W. ROSSMAN, M. UMEHARA,
AND K. YAMADA

Dedicated to Professor Osamu Kobayashi for his sixtieth birthday.

Abstract. It is classically known that the only zero mean curvature entire
graphs in the Euclidean 3-space are planes, by Bernstein’s theorem. A surface
in Lorentz-Minkowski 3-space $R^3_1$ is called of mixed type if it changes causal
type from space-like to time-like. In $R^3_1$, Osamu Kobayashi found two zero
mean curvature entire graphs of mixed type that are not planes. As far as the
authors know, these two examples were the only known examples of entire zero
mean curvature graphs of mixed type without singularities. In this paper, we
construct several families of real analytic zero mean curvature entire graphs of
mixed type in Lorentz-Minkowski 3-space. The entire graphs mentioned above
lie in one of these classes.

INTRODUCTION

The Jorge-Meeks $n$-noid ($n \geq 2$) is a complete minimal surface of genus zero
with $n$ catenoidal ends in the Euclidean 3-space $R^3$, which has $(2\pi/n)$-rotation
symmetry with respect to an axis. In the authors’ previous work [5], it was shown
that the corresponding space-like maximal surface $J_n$ in Lorentz-Minkowski 3-space
$(R^3_1; t, x, y)$ has an analytic extension $\tilde{J}_n$ as a properly embedded zero mean cur-
vature surface which changes its causal type from space-like to time-like. Noting
that a smooth function $\lambda : R^2 \to R$ can be realized as a subset
\[ \{(t, x, y) \in R^3_1 ; t = \lambda(x, y), x, y \in R\}, \]
which we call the graph of the function $\lambda(x, y)$, it should be remarked that $\tilde{J}_n$ for
$n \geq 3$ cannot be expressed as a graph, but $\tilde{J}_2$ gives an entire zero mean curvature
graph (cf. [5 Section 1]) associated to
\begin{equation}
\lambda(x, y) := x \tanh 2y.
\end{equation}
Remarkably, until now, only two entire graphs of mixed type with zero mean
curvature were known. One is (0.1) as above, and the other is the Scherk type
entire zero mean curvature graph associated to
\begin{equation}
\lambda(x, y) := \log(\cosh x / \cosh y).
\end{equation}

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Both surfaces were found by Osamu Kobayashi. Recently, Sergienko and Tkachev produced several interesting entire zero mean curvature graphs which admit cone-like singularities (the surface given by (0.2) is contained amongst their examples).

Here, the condition ‘mixed type’ for zero mean curvature entire graphs is important. In fact, Calabi proved that there are no space-like zero mean curvature entire graphs except for planes. On the other hand, there are many time-like zero mean curvature entire graphs. For example, the graph of \( t = y + \mu(x) \) gives such an example if \( \mu'(x) > 0 \) for all \( x \in \mathbb{R} \). It should also be remarked that there are no non-zero constant mean curvature surfaces of mixed type (see [10]).

The purpose of this paper is to construct further examples of entire zero mean curvature graphs of mixed type. In fact, we give a real \((4n - 7)\)-parameter (resp. \(3\)-parameter) family \( \mathcal{K}_n \) of space-like maximal surfaces of genus zero which admit only fold-type singularities for each integer \( n \geq 3 \) (resp. for \( n = 2 \), up to motions and a homothetic transformations in \( \mathbb{R}^3 \). The Jorge-Meeks type maximal surface \( \mathcal{J}_n \) mentioned above belongs to \( \mathcal{K}_n \). Also, the previously known two entire graphs of mixed type given in Kobayashi are contained in \( \mathcal{K}_2 \), so we call the surfaces in the class \( \mathcal{K}_n \) the ‘Kobayashi surfaces’ of order \( n \). Applying the technique used in [5], we show that some (open) subset of \( \mathcal{K}_n \) consists of space-like maximal surfaces which have analytic extensions across their fold singularities, as proper immersions. Moreover, in this subclass of \( \mathcal{K}_n \), we can find a \((4n - 7)\)-parameter family of zero mean curvature entire graphs of mixed type, up to congruence and homothety in \( \mathbb{R}^3 \). As a consequence, the maximal surfaces \( S_n \) \((n \geq 2)\) which correspond to Scherk-type saddle towers in the Euclidean 3-space constructed by Karcher (see also [13]) can be analytically extended as entire graphs in \( \mathbb{R}^3 \). Kobayashi’s entire graph (0.2) coincides with \( S_2 \).

During the production of this paper, the authors received a preprint of Akamine, where the entire graph as in Example 2.9 is obtained by using a different approach. This surface belongs to \( \mathcal{K}_2 \), as well as the entire graphs (0.1) and (0.2). As a consequence of Akamine’s construction, we can observe that the surface is foliated by parabolas.

1. Preliminaries

We denote by \( \mathbb{R}^3_1 \) the Lorentz-Minkowski 3-space of signature \((-+++)\). We introduce the notion of maxface given in [15].

**Definition 1.1.** Let \( M^2 \) be a Riemann surface. A \( C^\infty \)-map \( f : M^2 \to \mathbb{R}^3_1 \) is called a *generalized maximal surface* (cf. [3]) if there exists an open dense subset \( W \) of \( M^2 \) such that the restriction \( f|_W \) of \( f \) to \( W \) gives a conformal (space-like) immersion of zero mean curvature. A *singular point* of \( f \) is a point (on \( M^2 \)) at which \( f \) is not an immersion. A singular point \( p \) satisfying \( df(p) = 0 \) is called a *branch point* of \( f \). A generalized maximal surface \( f \) is called a *maxface* if \( f \) does not have any branch points.

A maxface may have singular points in general. Generic singularities of maxface are classified in [8]. As noted in [3] (Remark 2.8), a maxface induces a holomorphic immersion \( F = (F_0, F_1, F_2) : M^2 \to \mathbb{C}^3 \) satisfying the nullity condition, that is, it satisfies

\[-(dF_0)^2 + (dF_1)^2 + (dF_2)^2 = 0,\]
where $\tilde{M}^2$ is the universal covering of $M^2$. This immersion $F$ is called the \textit{holomorphic lift} of $f$. We set
\begin{equation}
(1.1) \quad g := -\frac{dF_0}{dF_1 - idF_2}, \quad \omega := \frac{1}{2}(dF_1 - idF_2),
\end{equation}
where $i = \sqrt{-1}$. Then $(g, \omega)$ is a pair of a meromorphic function and a meromorphic 1-form on $M^2$, and is called the \textit{Weierstrass data} of $f$. The meromorphic function $g$ can be identified with the Gauss map of $f$. Using $(g, \omega)$, $f$ has the following expression (cf. \cite{12} and \cite{15})
\begin{equation}
(1.2) \quad f = \text{Re} \int_{z_0}^{z} \left( -2g, 1 + g^2, i(1 - g^2) \right) \omega.
\end{equation}

The first fundamental form of $f$ is given by
\begin{equation}
(1.3) \quad ds^2 = (1 - |g|^2)^2|\omega|^2.
\end{equation}
In particular, the singular set of $f$ consists of the points where $|g| = 1$. In this situation, we set
\begin{equation}
(1.3) \quad f_E := \text{Re}(F_1, F_2, iF_0) = \text{Re} \int_{z_0}^{z} \left( 1 + g^2, i(1 - g^2), -2ig \right) \omega
\end{equation}
\begin{equation}
= \text{Re} \int_{z_0}^{z} \left( 1 - (-ig)^2, i(1 + (-ig)^2), 2(-ig) \right) \omega.
\end{equation}
This map $f_E$ gives a conformal minimal immersion called the \textit{companion} of $f$ (cf. \cite{15}). The Weierstrass data of $f_E$ is $(-ig, \omega)$. Even if $f$ is single-valued on $M^2$, its companion $f_E$ is defined only on $\tilde{M}^2$ in general. The first fundamental form of $f_E$ is given by
\begin{equation}
(1.4) \quad ds^2_E = (1 + |g|^2)^2|\omega|^2.
\end{equation}

A maxface $f$ is called \textit{weakly complete} if $ds^2_E$ in (1.4) is a complete Riemann metric on $M^2$ (cf. \cite{15} Definition 4.4). The definition of completeness of maxface is also given in \cite{15} Definition 4.4. A complete maxface is weakly complete. (The relationships between the two types of completeness are written in \cite{14} Remark 6.)

For example, the maxface given by
\begin{equation}
(1.5) \quad f(u, v) = (u, \cos v \sinh u, \sin v \sinh u) \quad (u \in \mathbb{R}, \ v \in [0, 2\pi]),
\end{equation}
is called the \textit{elliptic catenoid}. The surface is rotationally symmetric with respect to the time axis, and has a cone-like singularity at the origin. If we set $z = re^{iu}$ and $u := \log r$, then the map $f$ is the real part of the holomorphic immersion
\begin{equation}
F(z) = \left( \log z, \frac{z - z^{-1}}{2}, -i\frac{z + z^{-1}}{2} \right).
\end{equation}
The companion of $f$ is the helicoid in the Euclidean 3-space. The Weierstrass data of $f$ is given by $(g, \omega) = (-z, dz/(2z^2))$. Several maxfaces defined on non-zero genus Riemann surfaces are known (cf. \cite{6}). In this paper, we are interested in maximal surfaces having only fold singularities defined as follows:

\textbf{Definition 1.2.} Let $M^2$ be a Riemann surface, and $(g, \omega)$ the Weierstrass data of a weakly complete maxface $f : M^2 \to \mathbb{R}^3$. A singular point $p \in M^2$ is called \textit{non-degenerate} if $dg(p) \neq 0$ holds. By the implicit function theorem, there exists a regular curve $\sigma(t)$ ($|t| < \varepsilon$) on $M^2$ for some $\varepsilon > 0$ parameterizing the singular set of
$f$ such that $\sigma(0) = p$. A non-degenerate singular point $p$ is called a fold singularity if it satisfies
\begin{equation}
\text{Re} \left( \frac{dg(\sigma(t))}{g^2(\sigma(t))\omega(\sigma(t))} \right) = 0 \quad (|t| < \varepsilon).
\end{equation}

The following fact follows from [4, Theorem 2.15 and Lemma 2.17]:

**Fact 1.3.** For each fold singular point $p$ of a maxface $f$, there exists a local coordinate system $(U; u, v)$ centered at $p$ such that $f(u, v) = f(u, -v)$ and the $u$-axis consists of the fold singularities of $f$. Moreover, the image of $f$ has an analytic extension across the image of the singular curve $u \mapsto f(u, 0)$.

In this paper, we are interested in maxfaces having only fold singularities on the Riemann surface whose compactification is bi-holomorphic to the Riemann sphere

\[ S^2 := \mathbb{C} \cup \{\infty\}. \]

**Definition 1.4.** A pair $(g, \omega)$ consisting of a meromorphic function and a meromorphic 1-form defined on $S^2$ is called a Weierstrass data on $S^2$ if the metric
\begin{equation}
ds_E^2 := (1 + |g|^2)^2|\omega|^2
\end{equation}
has no zeros on $S^2$. Each point where $ds_E^2$ diverges is called an end of $(g, \omega)$, that is, at least one of the three 1-forms
\begin{equation}
\omega, \ g\omega, \ g^2\omega
\end{equation}
have poles at the ends.

We fix a Weierstrass data $(g, \omega)$ on $S^2$. Let \( \{p_1, \ldots, p_N\} \) be the set of ends of $(g, \omega)$. Then the map
\begin{equation}
f = \text{Re}(F), \quad F := \int_{z_0}^{z} (-2g, 1 + g^2, i(1 - g^2))\omega\]
is defined on the universal covering of $S^2 \setminus \{p_1, \ldots, p_N\}$. We call $f$ the maxface associated to $(g, \omega)$. If $f$ is single-valued on $S^2 \setminus \{p_1, \ldots, p_N\}$, that is, the residues of the three meromorphic 1-forms $-2g\omega, (1 + g^2)\omega, i(1 - g^2)\omega$ are all real numbers at each $p_j$ ($j = 1, \ldots, N$), then we say that $(g, \omega)$ satisfies the period condition.

**Proposition 1.5.** Let $(g, \omega)$ be a Weierstrass data on $S^2$, and $p_1, \ldots, p_N$ its ends. If the maxface $f$ associated to $(g, \omega)$ is single-valued on $S^2 \setminus \{p_1, \ldots, p_N\}$, then $f : S^2 \setminus \{p_1, \ldots, p_N\} \to \mathbb{R}^3$ is a weakly complete maxface.

**Proof.** The map $F$ given in (1.9) is an immersion if and only if the metric $ds_E^2$ given by (1.7) is positive definite. Thus $f$ gives a maxface associated to $S^2 \setminus \{p_1, \ldots, p_N\}$. Since $ds_E^2$ diverges at $\{p_1, \ldots, p_N\}$, at least one of the three 1-forms as in (1.8) has a pole at $z = p_j$ ($j = 1, \ldots, N$). We let $m_j (\geq 1)$ be the maximum order of the poles of the above three forms at $p_j$. Using this, one can easily show that $|z - p_j|^{2m_j} ds_E^2$ is positive definite on a sufficiently small neighborhood of $z = p_j$. This implies that $ds_E^2$ is a complete Riemannian metric on $S^2 \setminus \{p_1, \ldots, p_N\}$. \( \square \)

For example, the ends of the Weierstrass data $(g, \omega) = (z, dz/(2z^2))$ of the elliptic catenoid (cf. (1.5)) consist of \( \{0, \infty\} \). We are interested in a special class of weakly complete maxface on $(M^2 =) S^2$ as follows:

**Definition 1.6.** The Weierstrass data $(g, \omega)$ on $S^2$ is called of fold-type if
(i) all of its ends $p_1, ..., p_N$ lie in the unit circle $S^1 := \{z \in S^2 : |z| = 1\}$,
(ii) $|g(z)| = 1$ holds if and only if $z \in S^1$,
(iii) $\text{Re} [dg/(g^2 \omega)]$ vanishes identically on $S^1$.

A $C^\infty$-map $\varphi : M^2 \to \mathbb{R}^3$ has a fold singularity at $p$ if there exists a local coordinate system $(u, v)$ centered at $p$ such that $\varphi(u, v) = \varphi(u, -v)$. As shown in [4, Lemma 2.17], the maxface induced by a Weierstrass data of fold-type actually has fold singularities along $S^1$.

**Example 1.7** (Scherk-type entire graph). A typical example of Weierstrass data of fold-type on $S^2$ is

$$g = z, \quad \omega = \frac{2dz}{z^4 - 1}.$$  

The ends of the Weierstrass data of $(g, \omega)$ consist of the roots $\{\pm 1, \pm i\}$ of the equation $z^4 = 1$. One can easily check that $(g, \omega)$ is of fold-type, and induces a maxface

$$f(z) = \left(\log \left| \frac{1 + z^2}{1 - z^2} \right|, \log \left| \frac{1 - z}{1 + z} \right|, \log \left| \frac{1 - iz}{1 + iz} \right| \right).$$

By setting $z = re^{i\theta}$, $f = (t, x, y)$ satisfies

$$\cosh x = \frac{1 + r^2}{1 - z^2}, \quad \cosh y = \frac{1 + r^2}{1 + z^2}.$$  

Thus we have

$$\frac{\cosh x}{\cosh y} = \frac{1 + z^2}{1 - z^2} = e^t,$$

that is, the image of $f$ satisfies the relation $(0.2)$ in the introduction. We remark that the conjugate $f_*$ of $f$ induces the triply periodic maximal surface given in [7, Remark 2.4] with $\theta = 0$, which is obtained as a limit of the family of Schwarz P-type maximal surfaces. The fundamental piece of $f_*$ is bounded by four light-like line segments.

We next give an example of Weierstrass data which is not of fold-type, but the corresponding maximal surface admit only fold singularities:

**Example 1.8** (A maximal helicoid). If we set $g = z$ and $\omega = idz/z^2$, then $(g, \omega)$ induces a maximal helicoid by setting $z = re^{i\theta}$ ($r > 0$, $\theta \in \mathbb{R}$), and

$$f := (2\theta, -(r + r^{-1}) \sin \theta, (r + r^{-1}) \cos \theta)$$

is defined on the universal covering of $\mathbb{C} \setminus \{0\}$, and admits only fold singularities at the unit circle $r = 1$. However, $(g, \omega)$ is not of fold-type, since $\{0, \infty\}$ are ends of the Weierstrass data $(g, \omega)$. It should be noted that if we set $u = (r + r^{-1})/2(> 1)$, the surface $f$ analytically extends even when $|u| < 1$. Moreover, the entire analytic extension of a maximal helicoid coincides exactly with the helicoid as a minimal surface in $\mathbb{R}^3$, and is embedded.

**Proposition 1.9.** Let $(g, \omega)$ be a Weierstrass data of fold-type defined on $S^2$, then the maxface $f$ induced by $(g, \omega)$ satisfies the period condition in $R^3_1$, and gives a weakly complete maxface having fold singularities in $S^1$.

**Proof.** Let $p_1, ..., p_N \in S^1$ be the ends of $(g, \omega)$. It is sufficient to show that the maxface $f$ induced by $(g, \omega)$ is single-valued on a punctured disk at each $p_j$ ($j = 1, ..., N$). The Gauss map $g$ does not have a branch point at $p_j$. In fact, if $dg(p_j) = 0$
for some \( j = 1, \ldots, N \), then the singular set \( |g| = 1 \) must bifurcate at \( p_j \), but this
contradicts that the singular set of \( ds^2 \) (cf. (1.2)) is exactly equal to \( S^1 \). Thus, we
can take a local complex coordinate \((U, \xi)\) centered at \( p_j \) such that \( g = e^{i\xi} \). In this
coordinate, the singular set of \( ds^2 \) coincides with the real axis in the \( \xi \)-plane. We
set \( \omega = \tilde{\omega}(\xi)d\xi \). Then the condition (3) in Definition 1.6 implies the function
\[
-\frac{i \mathrm{d}g}{g^2 \omega} = \frac{e^{i \xi}}{e^{2i \xi} \omega} = \frac{1}{e^{i \xi} \omega}
\]
takes real values on the real axis in the \( \xi \)-plane. Thus, we can write \( \tilde{\omega}(\xi) = e^{-i \xi}h(\xi) \),
where \( h(\xi) \) is a meromorphic function satisfying \( h(\xi) = h(\xi) \). Then
\[
(-2g, 1 + g^2, i(1 - g^2))\omega = 2(-1, \cos \xi, \sin \xi)h(\xi)d\xi
\]
has real residue, since \( \cos \xi, \sin \xi \), and \( h(\xi) \) are real-valued functions on the real
axis. Thus, \( f \) is single-valued on a neighborhood of \( p_j \). Since \( p_j \) is arbitrarily
chosen, \( f \) is single-valued on \( S^2 \setminus \{p_1, \ldots, p_N\} \), proving the assertion, since the weak
completeness of \( f \) follows from Proposition 1.9. \( \square \)

The following assertion holds:

**Corollary 1.10.** Let \( p \) (resp. \( q \)) be a fold singular point (resp. an end) of the
weakly complete maxface \( f \) associated to a Weierstrass data \((g, \omega)\) of fold-type on
\( S^2 \). Then the Hopf differential \( Q := \omega \mathrm{d}g \) does not vanish at \( p \) (resp. has a pole at
\( q \)). In particular, the umbilic points of \( f \) correspond to the zeros of \( Q \).

**Proof.** Since \( f \) is weakly complete, \( \omega(p) \neq 0 \) holds. On the other hand, since \( p \) is
a non-degenerate singular point, we have \( \mathrm{d}g(p) \neq 0 \). Thus we get the assertion for
\( p \). We next consider the case that \( q \) is an end of \( f \). Then, we have \( |g(q)| = 1 \). If
\( \mathrm{d}g(q) = 0 \) holds, then the set \( |g| = 1 \) bifurcates at the point \( q \), which contradicts the
condition (2) in Definition 1.6. So we get \( \mathrm{d}g(q) \neq 0 \). Since \( Q = \omega \mathrm{d}g \), we have
\[
\mathrm{d}s_E^2 = (1 + |g|^2)^2 \frac{|Q|^2}{|\mathrm{d}g|^2}.
\]
Since \( \mathrm{d}s_E^2 \) is complete at \( z = q \), the facts \( |g(q)| = 1 \) and \( \mathrm{d}g(q) \neq 0 \) yield that \( Q \)
must have a pole at \( z = q \). \( \square \)

2. A CHARACTERIZATION OF KOBAYASHI SURFACES

We shall prove the following assertion:

**Theorem 2.1.** Let \( f : S^2 \setminus \{p_1, \ldots, p_N\} \to \mathbb{R}^3_1 \) be a weakly complete maxface
satisfying the following four conditions:

1. The Gauss map \( g : S^2 \setminus \{p_1, \ldots, p_N\} \to S^2 \) of \( f \) has at most poles at
   \( p_1, \ldots, p_N \).
2. There exists an anti-holomorphic involution \( I : S^2 \to S^2 \) such that \( f \circ I = f \).
3. The ends \( p_1, \ldots, p_N \) of \( f \) lie on the fixed-point set of \( I \).
4. The fixed points set \( \Sigma^1 \) of \( I \) coincides with the fold singularities of \( f \). Moreover,
   \( f \) has no singularities on \( S^2 \setminus \Sigma^1 \).

\footnote{This assumption is actually needed. If we set \( g = e^{i\xi} \) and \( \omega = -e^{-i\xi}dz \), then \( z = \infty \) is
an essential singularity of \( g \), and the induced maxface satisfies (2)–(4) by setting \( I(z) = \bar{z} \). It is
congruent to the helicoid as in Example 1.3 by the coordinate change \( \zeta = e^{i\xi} \).}
Then there exist an integer \( n(≥ 2) \) and \( 2n \) real numbers \((\text{called the angular data of } f)\)

\[
0 = α_0 ≤ α_1 ≤ \cdots ≤ α_{2n-1} < 2π
\]

and complex numbers \( b_1, \ldots, b_{n-1} \) (not necessarily mutually distinct) satisfying

\[
|b_1|, \ldots, |b_{n-1}| < 1
\]

such that \( f \) is homothetic to a maxface associated to the following Weierstrass data of fold-type:

\[
g = \prod_{i=1}^{n-1} \frac{z - b_i}{1 - b_i z}, \quad \omega = \frac{1}{\prod_{j=0}^{2n-1} (z - e^{iα_j/2})} dz.
\]

The number of distinct values in \( \{α_0, α_1, \ldots, α_{2n-1}\} \) coincides with \( N \). More precisely, the set \( \{e^{iα_0}, \ldots, e^{iα_{2n-1}}\} \) coincides with the set \( \{p_1, \ldots, p_N\} \). Conversely, for each angular data satisfying (2.1), we get a maxface satisfying (1) – (4) by choosing the Weierstrass data \((\omega, F)\).

Proof. Without loss of generality, we may set \( I \) as the canonical inversion

\[
I(z) = \frac{1}{\bar{z}}.
\]

In this case the fixed point set \( Σ^1 \) of \( I \) is the unit circle \( S^1 := \{z ∈ C : |z| = 1\} \).

By (2), we have \( f(1/\bar{z}) = f(z) \), and so

\[
\text{Re}(F(1/\bar{z})) = \text{Re}(F(z))
\]

holds, where \( F \) is the holomorphic lift of \( f \). By the identity theorem, we can conclude that

\[
F(1/\bar{z}) = F(z) + iC
\]

holds for some constant vector \( C ∈ \mathbb{R}^3 \). In particular, \( d(F ◦ I) = dF \) holds. Let \((g, ω)\) be the Weierstrass data of \( f \). By (1) \( g \) is a meromorphic function on \( S^2 \). Since \( f \) is weakly complete, the metric \( ds^2 \) given in (1.7) is complete. The well-known completeness lemma (see the introduction of (10)) yields that \( ω \) is a meromorphic 1-form on \( S^2 \), and so \((g, ω)\) is a Weierstrass data on \( S^2 \). Since \( f \) admits only fold singularities which lie in \( S^1 \) by (3) and (4), \((g, ω)\) satisfies (ii) and (iii) of Definition (10). On the other hand, the condition (3) corresponds to (i) of Definition (10). Thus, we can conclude that \((g, ω)\) is of fold-type. Using (1.1), we have

\[
g ◦ I = 1/\bar{g} \quad \frac{d(F ◦ I)}{d(F_1 ◦ I) + id(F_2 ◦ I)} = \frac{dF_0}{dF_1 + idF_2} = \frac{dF_0(dF_1 - idF_2)}{(dF_0)^2} = \frac{1}{g},
\]

where \( F = (F_0, F_1, F_2) \). We let \( b_1, \ldots, b_l \) be all the zeros of \( g \) as a meromorphic function on \( S^2 \). We do not need to assume that these numbers are distinct. For example if \( b_1 = b_2 \), then \( g \) has zeros of order more than one. By (2.4),

\[
1/\bar{b}_1, \ldots, 1/\bar{b}_l
\]

must be the set of all poles of \( g \), where \( 1/0 \) would be regarded as \( ∞ \). By the fact

\[
g ◦ I = 1/\bar{g},
\]

we may set \( |b_1| < 1 \), without loss of generality. So we may set

\[
g = c \prod_{i=1}^{l} \frac{z - b_i}{1 - b_i z} \quad (|c| = 1).
\]

On the other hand, since \( Q \) is the \((2, 0)\)-component of the complexification of the second fundamental form, \( Q \) has no poles at each regular point of \( f \). By this
together with Corollary 1.10 we can conclude that $Q$ has a pole at $z = q$ if and only if $q$ is an end of $f$. Let $\tilde{n}$ be the total sum of order of poles of $Q := \omega dg$, and $m$ the total order of zeros of $Q$ which are contained in the interior of the unit disk. Since $Q$ has no zeros on $S^1$ by Corollary 1.10, the fact that $Q \circ I = Q$ yields that the total sum of the orders of zeros of $Q$ is equal to $2m$. By the Riemann-Roch relation for $Q$ we have

\begin{equation}
2m - \tilde{n} = -2\chi(S^2) = -4.
\end{equation}

In particular, $\tilde{n}$ is an even number, and can be denoted by $\tilde{n} = 2n$. Also, as shown in the proof of Corollary 1.10, $dg(q) \neq 0$ holds at each end $q$. So the set of poles of $Q$ coincides with the set of poles of $\omega$, and the total sum of orders of poles of $\omega$ is equal to $\tilde{n}(= 2n)$. Moreover, since all ends of $f$ lie in $S^1$, there exist real numbers $\alpha_1, \ldots, \alpha_{2n-1}$ satisfying (2.1) such that $e^{i\alpha_j}$ ($j = 0, 1, \ldots, 2n - 1$) are poles of $\omega$ counted with multiplicity. In particular, $\{e^{i\alpha_0}, \ldots, e^{i\alpha_{2n-1}}\}$ coincides with $\{p_1, \ldots, p_N\}$. We now use the parameter change

\begin{equation}
z \mapsto e^{-i\alpha_0}z.
\end{equation}

to conclude that we may assume $\alpha_0 = 0$. We next consider the following change of the Weierstrass data given by

\begin{equation}
(g, \omega) \mapsto (ag, \bar{a}\omega) \quad (a \in S^1),
\end{equation}

which preserves the first fundamental form $ds^2$ as in (1.2) as well as the second fundamental form $Q + \overline{Q}$, where $Q := \omega dg$, and gives the same maximal surface up to a rigid motion of $R^3_1$. So we may assume that $c = 1$ in the expression (2.5).

On the other hand, if $\omega$ has a zero at a regular point of $f$, it must be a pole of $g$. Since $1/b_i$ ($i = 1, \ldots, l$) are regular points of $f$, the regularity of $ds^2$ at $1/b_i$ implies that $\omega$ has the factor $(z - 1/b_i)^2$ for each $i = 1, \ldots, l$. Thus, we may set

\begin{align*}
\omega &= \frac{B \prod_{i=1}^l (1 - b_i z)^2}{\prod_{j=0}^{2n-1} (e^{-i\alpha_j}/2 z - e^{i\alpha_j}/2)} dz \\
&= B \frac{z^n \prod_{i=1}^l (1 - b_i z)^2}{\prod_{j=0}^{2n-1} (e^{-i\alpha_j}/2 z^{1/2} - e^{i\alpha_j}/2 z^{-1/2})} dz,
\end{align*}

where $B \in C \setminus \{0\}$. For the sake of simplicity, we set

\begin{equation}
c_j := e^{i\alpha_j} / 2 \quad (j = 0, \ldots, 2n - 1).
\end{equation}

Then

\begin{align*}
\frac{dg}{g^2 \omega} &= \frac{1}{g \omega} \frac{dg}{g} = \frac{z^n \prod_{i=1}^l (1 - b_i z) \prod_{j=0}^{2n-1} (\bar{c}_j \sqrt{z} - c_j / \sqrt{z})}{B} \frac{dg}{dz} \\
&= \frac{z^{n-1}}{B} \frac{1}{\prod_{i=1}^l (z - b_i)} \frac{\prod_{j=0}^{2n-1} (\bar{c}_j \sqrt{z} - c_j / \sqrt{z})}{\prod_{i=1}^l (z^{-1} - b_i)} \frac{dg}{dz}.
\end{align*}

If we set $z = e^{i\theta}$ ($\theta \in R$), then

\begin{align*}
\prod_{i=1}^l (z - b_i)(z^{-1} - \bar{b}_i), \quad \prod_{j=0}^{2n-1} (\bar{c}_j \sqrt{z} - c_j / \sqrt{z}).
\end{align*}
are real-valued. So there exists a function \( C_1(z) \) satisfying \( C_1(e^{it}) \in \mathbb{R} \) for \( t \in \mathbb{R} \) such that
\[
\frac{dg}{g^2 \omega} = C_1 z^{n-l} \frac{dg/dz}{B - g}.
\]
Since
\[
\frac{dg/dz}{g} = \frac{d(\log g)}{dz} = \sum_{i=1}^{l} \frac{1 - |b_i|^2}{(z - b_i)(1 - b_i z)} = \frac{1}{z} \sum_{i=1}^{l} \frac{1 - |b_i|^2}{(z - b_i)(z^{-1} - b_i)}.
\]
the function \( C_2(z) := zg_z/g \) has the property that \( C_2(e^{it}) \in \mathbb{R} \) for \( t \in \mathbb{R} \), where \( g_z = dg/dz \). Hence, we can write
\[
\frac{dg}{g^2 \omega} = C_1 C_2 z^{n-l-1} \frac{dg}{B - g}.
\]
On the other hand, \( dg/g^2 \omega \) is \( i\mathbb{R} \)-valued on \( S^1 \) because of (iii) of Definition 1.6. Thus, we can conclude that \( z^{n-l-1}/B \) must also be \( i\mathbb{R} \)-valued for all \( z \in S^1 \). In particular, we have \( n-l-1 = 0 \) and \( B \in i\mathbb{R} \). By a homothety of \( f \), we may assume that \( B = i \). Since 0 is neither a pole nor a zero of \( \omega \), we have \( l = n-1 \), and so we get the expression (2.3). Suppose that \( b_1, \ldots, b_{n-1} \) satisfies (2.2), then the fact
\[
\left| \frac{z - b_i}{1 - b_i z} \right| < 1 \quad (|z| < 1)
\]
yields that \( |g(z)| < 1 \) whenever \( |z| < 1 \). By our definition, \( |b_1| < 1 \) and \( g(z) = 0 \) if \( z = b_1 \). On the other hand, if one of \( b_i \) (\( i = 2, \ldots, n-1 \)) satisfies \( |b_i| > 1 \), then \( g \) has a pole at \( z = 1/b_i \). In particular, \( |g(z)| > 1 \) if \( z \) is sufficiently close to \( 1/b_i \). Since \( b_1 \) and \( 1/b_i \) lie in the unit disk, by the intermediate value theorem, there exists a point \( z_0 \) (\( |z_0| < 1 \)) such that \( |g(z_0)| = 1 \). Hence, all of \( b_1, \ldots, b_{n-1} \) must lie in the unit disk. Conversely, one can easily verify that the Weierstrass data \((g, \omega)\) given in (2.3) is of fold-type, and induces a maxface satisfying (1)–(4) by Proposition 1.9.

**Definition 2.2.** We call a maxface given in this theorem a **Kobayashi surface** of order \( n \). We denote by \( \mathcal{K}_n \) the set of congruent classes of all Kobayashi surfaces of order \( n \).

**Remark 2.3.** As seen in the proof of Theorem 2.1 (cf. (2.7) and (2.8)), the circular rotation of the angular data \((\alpha_0, \ldots, \alpha_{2n-1}) \mapsto (\alpha_1, \ldots, \alpha_{2n-1}, \alpha_0)\) and the translation \((\alpha_0, \ldots, \alpha_{2n-1}) \mapsto (\alpha_0 + \beta, \ldots, \alpha_{2n-1} + \beta) \quad (\beta \in \mathbb{R})\) do not change the resulting Kobayashi surface whenever \( b_1 = \cdots = b_n = 0 \).

**Corollary 2.4.** Let \( f \) be a Kobayashi surface as in Theorem 2.1. If \( f \) has at most two umbilics, then there exist an integer \( n(\geq 2) \) and \( 2n \) real numbers \((\alpha_0, \ldots, \alpha_{2n-1})\) satisfying (2.4) such that \( f \) is homothetic to a maxface associated to the following Weierstrass data of fold-type:
\[
g = z^{n-1}, \quad \omega = \frac{i\lambda z}{\prod_{j=0}^{2n-1}(z - e^{i\alpha_j})},
\]
where
\[
\Lambda := \exp i \left( \frac{\alpha_0 + \cdots + \alpha_{2n-1}}{2} \right).
\]
Proof. Let \( n \) be the order of the Kobayashi surface \( f \). If \( n = 2 \), then \( f \) has no umbilics (cf. Corollary 1.10) and the degree of Gauss map is equal to 1. In this case, by a suitable Möbius transformation in \( S^2 \) preserving the unit circle \( S^1 \), we may assume that \( g = z \). Namely, we may set \( b_1 = 0 \), and get the assertion. So we may assume that \( n \geq 3 \). Since \( Q \) has no zeros on \( S^1 \) (cf. Corollary 1.10), the fact that \( f \circ I = f \) implies there are at least two umbilics on \( f \). So by our assumption, the number of umbilics must be exactly two. By a suitable motion in \( R^4 \), we may assume that \( g(0) = 0 \), that is, \( b_1 = 0 \). By a suitable Möbius transformation in \( S^2 \) preserving the unit circle \( S^1 \), we may also assume that \( z = 0 \) is an umbilic of \( f \). In this case, we may set
\[
(2.12) \quad b_1 = \cdots = b_\mu = 0, \quad b_{\mu+1}, \ldots, b_{n-1} \neq 0,
\]
where \( 2 \leq \mu \leq n - 1 \). If \( \mu = n - 1 \), then \( b_i = 0 \) \( (i = 1, \ldots, n - 1) \) and \( g = z^{n-1} \).

Suppose that \( \mu < n - 1 \), by way of contradiction. Using \((2.3)\) we write \( g = z^{\mu}X/Y \), where
\[
X := \prod_{i=\mu+1}^{n-1} (z - b_i), \quad Y := \prod_{i=\mu+1}^{n-1} (1 - \overline{b}_i z).
\]
Since
\[
(2.13) \quad dg = \frac{\mu z^{\mu-1}XY dz + z^{\mu}(dX)Y - z^{\mu}X(dY)}{Y^2}, \quad \omega = \frac{iY^2dz}{\prod_{j=0}^{n-1} (e^{-i\alpha_j/2}z - e^{i\alpha_j/2})},
\]
we have
\[
Q := \omega dg = \frac{i\mu z^{\mu-1}XY dz + z^{\mu}(dX)Y - z^{\mu}X(dY)}{\prod_{j=0}^{n-1} (e^{-i\alpha_j/2}z - e^{i\alpha_j/2})} dz.
\]
Since the zeros of \( Y \) are not umbilics of \( f \), the umbilics of \( f \) correspond to the zeros of \( dg \) (cf. Corollary 1.10). Since \( f \) has exactly two umbilics, we can write
\[
(2.14) \quad dg = \frac{C(z)z^{n-2}}{Y^2} dz \quad (C(z) \neq 0).
\]
Comparing \((2.13)\) and \((2.14)\), we have that
\[
(2.15) \quad C(z)z^{n-\mu-1} = \mu XY + z(dX)Y - zX(dY).
\]
By \((2.12)\), the right hand side does not vanish at \( z = 0 \), a contradiction. Hence, we have \( \mu = n - 1 \). In particular, \((2.13)\) reduces to \((2.10)\).

A Kobayashi surface (cf. Definition 2.2) associated to the Weierstrass data as in \((2.10)\) is called a Kobayashi surface of principal type with angular data \((\alpha_0, \ldots, \alpha_{2n-1})\). In other words, Kobayashi surfaces of principal type are obtained by setting \( b_1 = \cdots = b_{n-1} = 0 \) in \((2.3)\). We denote by \( \mathcal{K}_n^0 (\subset \mathcal{K}_n) \) the set of Kobayashi surfaces of principal type (cf. Definition 2.2 for \( \mathcal{K}_n \)). A Kobayashi surface which is not of principal type is called of general type.

Proposition 2.5. The set \( \mathcal{K}_2 \) of Kobayashi surfaces of order 2 has 3 degrees of freedom, and the set \( \mathcal{K}_n \) Kobayashi surfaces of order \( n \geq 3 \) has \( 4n - 7 \) degrees of freedom, up to congruence and homothety. Moreover, there are \( 2n - 1 \) degrees of freedom for \( \mathcal{K}_n^0 \), up to congruence and homothety. Furthermore, Kobayashi surfaces of order less than 4 are all of principal type.
Proof: We suppose \( n \geq 3 \). Then \( f \) has umbilics. Let \( z = b \) be such an umbilic of \( f \), then it is a zero of \( Q \). We may assume that the maximum value of the orders of zeros of \( Q \) is attained at \( z = b \). By a suitable Möbius transformation, we may set \( b = 0 \). Then it holds that \( dg(0) = 0 \). Without loss of generality, we may also assume \( g(0) = 0 \), and have the following expression like as in the proof of Corollary 2.4:

\[
(2.16) \quad g = z^n \prod_{i=1}^{n-\mu-1} \frac{z - b_i}{1 - b_i z}, \quad \omega = \frac{i \prod_{i=1}^{n-\mu-1} (1 - b_i z)^2}{\prod_{j=0}^{2n+1} (e^{-i\alpha_j / 2 z} - e^{i\alpha_j / 2})} \frac{dz}{z^{n+1}} \quad (2 \leq \mu \leq n-1).
\]

We may fix \( \mu \) equal to 2 in (2.16) by allowing some of the \( b_i \) to be zero. Thus (2.16) has \( n-3 \) complex parameters \( b_1, ..., b_{n-3} \) in the unit disk and \( 2n-1 \) (real) parameters \( \alpha_1, ..., \alpha_{2n-1} \) on the interval \([0, 2\pi]\). Therefore, \( K_n (n \geq 3) \) has \( 4n-7 \) degrees of freedom, up to congruence and homothety. On the other hand, the surfaces of principal type satisfy \( \mu = n-1 \), and they can be controlled by the \( 2n-1 \) angular parameters \( \alpha_1, ..., \alpha_{2n-1} \), since \( \alpha_0 = 0 \). If \( n = 3 \),

\[
2 \leq \mu \leq n-1 = 2
\]

holds, and so \( \mu = n-1 \), in particular, \( K_3 \) consists only of surfaces of principal type.

If \( n = 2 \), we can normalize \( b_1 = 0 \) by a suitable Möbius transformation fixing the unit circle. Then, we may set

\[
(2.17) \quad g = z, \quad \omega = \frac{1}{(z - 1)(z - e^{i\alpha_1})(z - e^{i\alpha_2})(z - e^{i\alpha_3})} \frac{dz}{z^{n+1}},
\]

that is, \( K_2 \) consists only of surfaces of principal type, and can be controlled by the 3 parameters \( \alpha_1, \alpha_2, \alpha_3 \).

Example 2.6 (Scherk-type surfaces). We set

\[
(2.18) \quad \alpha_j := \frac{\pi j}{n} \quad (j = 0, ..., 2n-1).
\]

Regarding the fact that \( \sum_{j=0}^{2n-1} \alpha_j = (2n-1)\pi \), (2.10) reduces to

\[
(2.19) \quad g = z^{n-1}, \quad \omega = \frac{dz}{z^{n+1}} \quad (n = 2, 3, 4, ...).
\]

Its companion \( (-i g, \omega) \) is the Weierstrass data of the minimal surface called the Scherk saddle tower given in [11]. The maximal surface \( S_n \) induced by (2.19) is the most important example of a Kobayashi surface of principal type. (The fact that \( S_2 \) induces an entire graph was shown in Kobayashi [12], see also Example 1.7). We shall show later that \( S_n \) can be real analytically extended as an entire graph for \( n \geq 3 \).

Example 2.7 (Jorge-Meeks type maximal surfaces). We set

\[
(2.20) \quad \alpha_{2j} = \alpha_{2j+1} := \frac{2\pi j}{n} \quad (j = 0, 1, ..., n-1).
\]

Then (2.16) reduces to

\[
(2.21) \quad g = z^{n-1}, \quad \omega = \frac{idz}{(z^{n-1})^2} \quad (n = 2, 3, 4, ...),
\]

and its companion \( (-i g, \omega) \) is the Weierstrass data of the minimal surface called the Jorge-Meeks \( n \)-noid. So, the induced maxface is called the Jorge-Meeks type maximal surface, and we denote it by \( J_n \). As shown in the authors’ previous work
[5], the analytic extension of $J_n$ is properly embedded for all $n \geq 2$. Although $J_n$ ($n \geq 3$) is not, the surface $J_2$ is an entire graph of mixed type, given in (0.1).

We next consider the Kobayashi surface satisfying $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} \subset \{0, \pi\}$. The surface of type $(0, 0, \pi, \pi)$ is $J_2$ as in Example 2.7. The most degenerate case is of type $(0, 0, 0, 0)$ as follows:

**Example 2.8 (The ruled Enneper surface).** In case of $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, 0, 0, 0)$, we have

$$g = z, \quad \omega = \frac{idz}{(1 - z)^4},$$

and it induces the following Kobayashi surface

$$f = \left(\frac{\sin \theta(-6u \cos \theta + \cos(2\theta) + 5)}{12(\cos \theta - u)^3}, \frac{\sin \theta (-3u \cos \theta + \cos(2\theta) + 3u^2 - 1)}{6(\cos \theta - u)^3}, \frac{-u \cos \theta - 1}{2(u - \cos \theta)^2}\right),$$

where $z = re^{i\theta}$ and $u := (r + r^{-1})/2$. Then $f$ can be extended to the region $\{(u, \theta); 1 > u > \cos \theta\}$, which is denoted by $\tilde{f}$. The image of $\tilde{f}$ coincides with the set satisfying (see Figure 2.1, left)

$$\Phi = 0, \quad \Phi := 4t^3 + 4t^2x + 4tx^2 - 2ty + t + \frac{4x^3}{3} - 2xy.$$ 

Since

$$\Phi_y = -2(x + t), \quad \Phi_t|_{(t,x,y)=(t,-t,y)} = 1 - 2y, \quad \Phi_x|_{(t,x,y)=(t,-t,y)} = -2y,$$

the gradient vector $(\Phi_t, \Phi_x, \Phi_y)$ never vanishes on $R^3$. Hence the set $\Phi = 0$ has no self-intersections. Thus the analytic extension of the Kobayashi surface with angular parameter $(0, 0, 0, 0)$ is embedded, that is, the problem is affirmative even for this case. The resulting surface is a ruled surface found in Kobayashi [12], and is called the conjugate of Enneper’s surface of 2nd kind. In fact, $\Phi(t, x, y) = 0$ implies that

$$\Phi \left( t - k, x + k, y - \frac{k}{2(t + x)} \right) = 0$$

for all $k \in R$. Space-like ruled maximal surfaces in $R^3$ are classified in [12].

![Figure 2.1. Example 2.8 (left) and Example 2.9 (right)](image-url)
Example 2.9. We set \( n = 2 \) and \( \alpha_0 = \alpha_1 = \alpha_2 = 0 \) and \( \alpha_3 = \pi \). Then (2.10) for \( b_1 = 0 \) reduces to

\[
(2.22) \quad g = z, \quad \omega = \frac{-dz}{(z - 1)^3(z + 1)}.
\]

By setting \( z = re^{i\theta} \), the induced maximal surface is given by

\[
(2.23) \quad f = \frac{1}{8} \left( A + B, A - B, -8r \sin \theta \right),
\]

where

\[
A = -4\left( \frac{r^3 + r}{D_2^2} + \frac{8r^2}{D_2^2} \right), \quad B = \log \frac{D_+}{D_-}, \quad D_\pm = r^2 \pm 2r \cos \theta + 1.
\]

In particular, the image of \( f \) lies in the set (see Figure 2.1)

\[
\mathcal{G} = \{(t, x, y) \in \mathbb{R}^3; \Phi = 0\}, \quad \Phi := \frac{1}{2} \left( e^{4(t+x)} - 1 \right) + 2(t - x) - 4y^2 = 0.
\]

Since \( \Phi_t = 2 + 2e^{4(t+x)} \geq 2(0) \), [17, Corollary 1] yields that \( \mathcal{G} \) can be realized as the image of an entire graph of \( x, y \). This implies that \( \mathcal{G} \) can be considered as the analytic extension of \( f \). It should be remarked that Akamine [1] independently found this surface. Moreover, he showed that the surface can be foliated by parabolas.

3. Analytic extension of Kobayashi surfaces

A Kobayashi surface \( f \) of order \( n \) is invariant under the symmetry \( r \mapsto 1/r \), where \( z = re^{i\theta} \). The singular set \( S^1 := \{|z| = 1\} \) of \( f \) coincides with the fixed point set under the symmetry. So, like as the case of the Jorge-Meeks type maximal surface (cf. (2.21)) discussed in the authors’ previous work [5], it is natural to expect that we can introduce a new variable \( u \) by

\[
(3.1) \quad u := \frac{r + r^{-1}}{2},
\]

which is invariant under the symmetry \( r \mapsto 1/r \). We set

\[
\tilde{D}_1 : = \{ z \in \mathbb{C}; 0 < |z| \leq 1 \}.
\]

To obtain the analytic extension of the image of \( f \), we define an analytic map

\[
\iota : \tilde{D}_1 \ni z = re^{i\theta} \mapsto \left( \frac{r + r^{-1}}{2}, \theta \right) \in \mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}.
\]

The image of the map \( \iota \) is given by

\[
\tilde{\Omega}_n := \{(u, \theta) \in \mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}; u \geq 1\}.
\]

The map \( \iota \) is bijective, whose inverse is given by

\[
\iota^{-1} : \tilde{\Omega}_n \ni (u, \theta) \mapsto \left( u - \sqrt{u^2 - 1}, \theta \right) \in \tilde{D}_1.
\]

If we regard \( f \) as a function of \( (u, \theta) \), the origin \( z = 0 \) in the original source space of \( f \) does not lie in the \( (u, \theta) \)-plane. To indicate what the origin in the old complex coordinate \( z \) becomes in the new real coordinates \( (u, \theta) \), we attach a new point \( p_\infty \) to \( \tilde{\Omega}_n \) as the ‘point at infinity’, and extend the map \( \iota \) so that

\[
\iota(0) = p_\infty.
\]
Hence we have a one-to-one correspondence between \(|z| \leq 1\) and \(\hat{\Omega}_n \cup \{p_\infty\}\). In particular, \(\hat{\Omega}_n \cup \{p_\infty\}\) can be considered as an analytic 2-manifold. The purpose of this section is to prove the following:

**Theorem 3.1.** Let \(f\) be a Kobayashi surface associated to the Weierstrass data as in (2.3). Then \(df(z) = \text{Re}(-2g, 1 + g^2, i(1 - g^2))\omega\) can be parametrized using new parameters \(u = (r + r^{-1})/2\) and \(\theta\), where \(z = re^{i\theta}\) \((0 > r > 0, \theta \in [0, 2\pi))\). Moreover, \(df \circ \iota\) can be analytically extended on the set

\[
\Omega_{\alpha_0, \ldots, \alpha_{2n-1}} := \left\{(u, \theta) \in \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}; u > \max_{j=1, \ldots, N} \left[\cos(\theta - \alpha_i)\right]\right\} \cup \{p_\infty\},
\]

where

\[
0 = \alpha_{i_1} < \cdots < \alpha_{i_N} < 2\pi \quad (0 = i_1 < i_2 < \cdots < i_N \leq 2n - 1)
\]

are the distinct angular values in (2.1).

For the sake of simplicity, we set

\[
\beta_j := \alpha_{j+1} - \alpha_j \quad (j = 0, \ldots, N - 1), \quad \beta_N := 2\pi
\]

and (cf. Figure 3.1)

\[
\gamma_j := \frac{\beta_j + \beta_{j+1}}{2}, \quad (j = 0, \ldots, N - 1),
\]

\[
I_j := \begin{cases} [\gamma_{j-1}, \gamma_j] & (j = 1, \ldots, N - 1), \\ [0, \gamma_0] \cup [\gamma_{N-1}, 2\pi] & (j = 0). \end{cases}
\]

To prove the theorem, we prepare the following lemma:

**Lemma 3.2.** For each pair of numbers \(i, j \in \{0, \ldots, N - 1\}\) \((i \neq j)\), the following inequality holds

\[
\cos(\theta - \beta_i) \geq \cos(\theta - \beta_j) \quad (\theta \in I_i).
\]

Moreover, equality holds if and only if \((j, \theta) = (i - 1, \gamma_{i-1})\) or \((j, \theta) = (i + 1, \gamma_i)\).
Proof. Instead of \( R \), we consider \( R/2\pi Z \), and then the assertion is invariant under the cyclic permutation of \( \beta_0, \ldots, \beta_{N-1} \). So, it is sufficient to consider the case that \( i \geq 1 \) and \( j \geq 2 \). It holds that

\[
\cos(\theta - \beta_i) - \cos(\theta - \beta_j) = 2 \sin \left( \frac{\beta_i - \beta_j}{2} \right) \sin \left( \frac{\theta - \beta_i + \beta_j}{2} \right).
\]

Since \( \theta \in I_i \), we have

\[
\theta - \frac{\beta_i + \beta_j}{2} \in \left[ \gamma_{i-1} - \frac{\beta_i + \beta_j}{2}, \gamma_i - \frac{\beta_i + \beta_j}{2} \right] = \left[ \frac{\beta_{i-1} - \beta_j}{2}, \frac{\beta_{i+1} - \beta_j}{2} \right].
\]

If \( j \leq i - 1 \), then \( \beta_j \leq \beta_{i-1} < \beta_i \) and \( \beta_{i+1} - \beta_j < 2\pi \) hold, and

\[
\theta - \frac{\beta_i + \beta_j}{2} \in \left[ \frac{\beta_{i-1} - \beta_j}{2}, \frac{\beta_{i+1} - \beta_j}{2} \right] \subseteq [0, \pi), \quad \frac{\beta_i - \beta_j}{2} \in (0, \pi).
\]

Hence, by (3.7) we have (3.6). Equality holds if and only if \( j = i - 1 \) and \( \theta = \gamma_{i-1} \).

On the other hand, if \( j \geq i + 1 \), then \( \beta_j \geq \beta_{i+1} > \beta_i \) and \( \beta_{i-1} - \beta_j > -2\pi \) hold. Thus we have

\[
\theta - \frac{\beta_i + \beta_j}{2} \in \left( \frac{\beta_{i-1} - \beta_j}{2}, \frac{\beta_{i+1} - \beta_j}{2} \right] \subseteq (-\pi, 0], \quad \frac{\beta_i - \beta_j}{2} \in (-\pi, 0).
\]

So (3.7) implies (3.6) and the equality condition.

As a consequence, the following two assertions hold:

**Corollary 3.3.** Suppose that \( \theta \in I_j \). Then \( (u, \theta) \in \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \) if and only if \( u > \cos(\theta - \beta_j) \).

**Corollary 3.4.** If \( (u, \theta) \in \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \) then

\[
u > \min_{j=0,1,\ldots,2n-1} \left\{ \cos \left( \frac{\alpha_{j+1} - \alpha_j}{2} \right) \right\} \quad (\alpha_{2n} := 2\pi).
\]

**Proof.** Without loss of generality, we may assume that \( \theta \in I_1 \). Then \( \cos(\theta - \beta_1) \) takes a minimum at \( \theta = \gamma_0 \) or \( \theta = \gamma_1 \). So we have

\[
u > \min \left\{ \cos \left( \frac{\beta_1 - \beta_0}{2} \right), \cos \left( \frac{\beta_2 - \beta_1}{2} \right) \right\} = \min_{j=0,1,\ldots,2n-1} \left\{ \cos \left( \frac{\alpha_{j+1} - \alpha_j}{2} \right) \right\}.
\]

**Proof of Theorem 3.1.** We set \( 2f_zdz = (\varphi_0, \varphi_1, \varphi_2) \). Then, it holds that

\[
\varphi_0 := -2g\omega, \quad \varphi_1 := (1 + g^2)\omega, \quad \varphi_2 := i(1 - g^2)\omega.
\]

Using (2.8), we have that

\[
\varphi_k = \frac{p_k(z)}{q(z)}dz \quad (k = 0, 1, 2),
\]
where

\[
q(z) := \prod_{j=0}^{2n-1} (e^{-i\alpha_j/2}z - e^{i\alpha_j/2}),
\]

(3.10)

\[
p_0(z) := -2\prod_{i=1}^{n-1} (z - b_i)(1 - \bar{b}_i z),
\]

(3.11)

\[
p_k(z) := i^k \left( \prod_{i=1}^{n-1} (1 - \bar{b}_i z)^2 - (-1)^k \prod_{i=1}^{n-1} (z - b_i)^2 \right) \quad (k = 1, 2).
\]

(3.12)

Moreover, it holds that

\[
|q(z)|^2 = 4^n r^{2n} \prod_{j=0}^{2n-1} \left( u - \cos(\theta - \alpha_j) \right)^2 \quad (u = \frac{r + r^{-1}}{2}),
\]

(3.13)

where \( z = re^{i\theta} \). We now fix an index \( k \in \{0, 1, 2\} \), and set

\[
p(z) := p_k(z), \quad \varphi(z) := \varphi_k(z).
\]

It holds that

\[
p(1/z) = -\frac{p(z)}{z^{2n-2}}, \quad q(1/z) = \frac{q(z)}{z^{2n}}.
\]

(3.14)

Since \( u = (r + r^{-1})/2 \), we have \( du = (r^2 - 1)dr/(2r^2) \) and

\[
\varphi = \left( \frac{zp(z)}{q(z)} \frac{2r}{r^2 - 1} \right) du + \frac{izp(z)}{q(z)} d\theta.
\]

Thus, we can write

\[
\text{Re}(\varphi) = \frac{X(r, \theta) du + iY(r, \theta) d\theta}{2|q(z)|^2},
\]

where

\[
X(r, \theta) := \frac{zp(z)q(z) + \bar{z}q(z)p(z)}{(r - r^{-1})/2}, \quad Y(r, \theta) := zp(z)q(z) - \bar{z}q(z)p(z).
\]

Using (3.14), one can easily verify that

\[
X(1/r, \theta) = \frac{X(r, \theta)}{r^{4n}}, \quad Y(1/r, \theta) = \frac{Y(r, \theta)}{r^{4n}}.
\]

By Proposition A.3 there exist two polynomials \( x(u, \theta) \) and \( y(u, \theta) \) in \( u \) with parameter \( \theta \) such that

\[
X(r, \theta) = r^{2n}x(u, \theta), \quad Y(r, \theta) = r^{2n}y(u, \theta).
\]

Hence we have the expression

\[
\text{Re}(\varphi) = \frac{x(u, \theta) du + y(u, \theta) d\theta}{2 \times 4^n \prod_{j=0}^{2n-1} \left( u - \cos(\theta - \alpha_j) \right)}.
\]

By Lemma 3.2, the denominator of \( \text{Re}(\varphi) \) does not vanish on \( \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \), which proves the assertion.

Thus, the three closed forms \( \text{Re}(\varphi_k) \) \( (k = 0, 1, 2) \) are well-defined on a simply connected 2-manifold \( \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \), and the following assertion immediately follows:
Corollary 3.5. Let $f$ be a Kobayashi surface associated to the Weierstrass data as in (2.3). Then $\tilde{f} := f \circ \iota : \tilde{\Omega}_n \cup \{p_{\infty}\} \to \mathbb{R}^3$ can be analytically extended to the domain $\Omega_{\alpha_0, \ldots, \alpha_{2n-1}}$.

From now on, we consider $\tilde{f}$ as a map defined on $\Omega_{\alpha_0, \ldots, \alpha_{2n-1}}$.

Proposition 3.6. Let $f$ be a Kobayashi surface of principal type associated to the Weierstrass data as in (2.10). Suppose that $\alpha_0, \ldots, \alpha_{2n-1}$ are all distinct. Then $\tilde{f} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2)$ has the following expressions:

\begin{align*}
\tilde{x}_0 &= - \sum_{j=0}^{2n-1} A_j \log \left( u - \cos(\theta - \alpha_j) \right), \\
\tilde{x}_1 &= \sum_{j=0}^{2n-1} A_j \left( \cos(n - 1)\alpha_j \right) \log \left( u - \cos(\theta - \alpha_j) \right), \\
\tilde{x}_2 &= \sum_{j=0}^{2n-1} A_j \left( \sin(n - 1)\alpha_j \right) \log \left( u - \cos(\theta - \alpha_j) \right),
\end{align*}

where

\begin{equation}
A_j := \left( -1 \right)^{n+1} \frac{1}{2^{2n-1}} \left( \prod_{i \in \{0, \ldots, 2n-1\} \setminus \{j\}} \sin \frac{\alpha_j - \alpha_i}{2} \right)^{-1} \in \mathbb{R} \setminus \{0\}, \quad (j = 0, \ldots, 2n-1).
\end{equation}

Proof. We use the expression (3.8). Since $\alpha_0, \ldots, \alpha_{2n-1}$ are all distinct, we have the following partial fractional decomposition:

\begin{equation}
\varphi_k = \sum_{j=0}^{2n-1} B_{k,j} \frac{dz}{z - e^{i\alpha_j}} \quad (k = 0, 1, 2),
\end{equation}

where $B_{k,j} \in \mathbb{R}$ ($j = 0, \ldots, 2n-1$) is the residue of the meromorphic 1-form $\varphi_k$ on $S^2$ at $z = e^{i\alpha_j}$. By a straightforward calculation, we have

\begin{align*}
B_{0,j} &= -2A_j, \quad B_{1,j} = 2A_j \cos(n - 1)\alpha_j, \quad B_{2,j} = 2A_j \sin(n - 1)\alpha_j.
\end{align*}

The residue formula yields that

\begin{equation}
\sum_{j=0}^{2n-1} A_j = \sum_{j=0}^{2n-1} A_j \cos(n - 1)\alpha_j = \sum_{j=0}^{2n-1} A_j \sin(n - 1)\alpha_j = 0.
\end{equation}
By integrating \( (3.17) \), we get

\[
x_0 = \text{Re} \int \phi_0 = -2 \text{Re} \sum_{j=0}^{2n-1} A_j \log(z - e^{i\alpha_j}) = -\sum_{j=0}^{2n-1} A_j \log |z - e^{i\alpha_j}|^2
\]

\[
= -\sum_{j=0}^{2n-1} A_j \log \left( 2r \left( \frac{r^2}{r^2 + 1} - \cos(\theta - \alpha_j) \right) \right)
\]

\[
= -\sum_{j=0}^{2n-1} A_j \log \left( \frac{1}{2} \left( r + \frac{1}{r} \right) - \cos(\theta - \alpha_j) \right) - \log(2r) \sum_{j=0}^{2n-1} A_j
\]

Similarly, we have

\[
x_1 = 2 \text{Re} \sum_{j=0}^{2n-1} A_j \left( \cos(n-1)\alpha_j \right) \log(z - e^{i\alpha_j})
\]

\[
= \sum_{j=0}^{2n-1} A_j \left( \cos(n-1)\alpha_j \right) \log \left( \frac{1}{2} \left( r + \frac{1}{r} \right) - \cos(\theta - \alpha_j) \right)
\]

\[
x_2 = \sum_{j=0}^{2n-1} A_j \left( \sin(n-1)\alpha_j \right) \log \left( \frac{1}{2} \left( r + \frac{1}{r} \right) - \cos(\theta - \alpha_j) \right)
\]

Since \( u = (r + r^{-1})/2 \), we get the assertion. \( \square \)

4. Entire graphs induced by Kobayashi surface

Let \( f \) be a Kobayashi surface associated to the Weierstrass data as in \( (2.3) \). We denote by

\[
\tilde{f} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2) : \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \to \mathbb{R}_+^3
\]

its analytic extension. We firstly consider the principal case:

\textbf{Lemma 4.1.} Let \( \tilde{f} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \) be the analytic extension of the Kobayashi surface \( f \) of principal type associated to the Weierstrass data as in \( (2.10) \). Then the map \( (\tilde{x}_1, \tilde{x}_2) : \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \to \mathbb{R}^2 \) is an immersion if and only if

\[
(4.1) \quad |\alpha_j - \alpha_{j+1}| \leq \frac{\pi}{n-1} \quad (j = 0, \ldots, 2n-1),
\]

where \( \alpha_2n := 2\pi \) and \( \alpha_0, \ldots, \alpha_{2n-1} \) are not necessarily distinct.

\textbf{Proof.} We can use the technique given in \( [5] \), that is, we use the identity

\[
(4.2) \quad \frac{\partial(\tilde{x}_1, \tilde{x}_2)}{\partial(u, \theta)} = \frac{-ir^3}{r^2 - 1} \left| \frac{\phi_j}{\tilde{\phi}_j} \right|,
\]
where \( \varphi_k = \hat{\varphi}_k \, dz \) \((k = 0, 1, 2)\) On the \( z \)-plane, it holds that
\[
\begin{align*}
\frac{\partial (\hat{x}_1, \hat{x}_2)}{\partial (u, \theta)} &= -\frac{\hat{\varphi}_1}{r^2 - 1} \left| \frac{i \Lambda(1 + z^{2n-2}) - i \bar{\Lambda}(1 + \bar{z}^{2n-2})}{\bar{\Lambda}(1 - \bar{z}^{2n-2})} \right| \\
\frac{\partial (\hat{x}_1, \hat{x}_2)}{\partial (u, \theta)} &= -\frac{\varphi_2}{r^2 - 1} \left| \frac{i \Lambda(1 + z^{2n-2}) - i \bar{\Lambda}(1 + \bar{z}^{2n-2})}{\bar{\Lambda}(1 - \bar{z}^{2n-2})} \right| \\
&= \frac{-r^3}{r^2 - 1} \cdot \frac{2 - 2|z|^{4n-4}}{2^{2n} \prod_{j=0}^{2n-1} (u - \cos(\theta - \alpha_j))^2} \geq \frac{U_{2n-3}(u)}{2^{2n} \prod_{j=0}^{2n-1} (u - \cos(\theta - \alpha_j))^2},
\end{align*}
\]
where \( u := (r + r^{-1})/2, \Lambda \) is as in (4.11), and \( U_{2n-3}(u) \) is the Chebyshev polynomial of the second kind of degree \( 2n - 3 \), and we have used the second identity of (4.12). By the real analyticity of \( \hat{x}_1, \hat{x}_2 \), the above identity holds on \( \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \). By (4.11) and Corollary 3.3, we have
\[
\begin{align*}
\frac{\partial (\hat{x}_1, \hat{x}_2)}{\partial (u, \theta)} &= \left| \frac{i \Lambda(1 + z^{2n-2}) - i \bar{\Lambda}(1 + \bar{z}^{2n-2})}{\bar{\Lambda}(1 - \bar{z}^{2n-2})} \right| \\
&= \left| \frac{i \Lambda(1 + z^{2n-2}) - i \bar{\Lambda}(1 + \bar{z}^{2n-2})}{\bar{\Lambda}(1 - \bar{z}^{2n-2})} \right| \\
&= \frac{2 - 2|z|^{4n-4}}{2^{2n} \prod_{j=0}^{2n-1} (u - \cos(\theta - \alpha_j))^2},
\end{align*}
\]
which implies that \( \partial (\hat{x}_1, \hat{x}_2)/\partial (u, \theta) \neq 0 \) on \( \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \), and we can conclude that \((u, \theta) \mapsto (\hat{x}_1, \hat{x}_2)\) is an immersion.

On the other hand, if \((\alpha_0, \ldots, \alpha_{2n-1})\) does not satisfy the condition (4.11), then we may set \( \alpha_0 = 0 \) (cf. Remark 2.3) and \( \alpha_1 > \pi/(n - 1) \). We set
\[
u_0 := \cos \frac{\pi}{2(n - 1)}, \quad \theta_0 := \frac{\alpha_1}{2}.
\]
By Corollary 3.3 we have \((\nu_0, \theta_0) \in \Omega_{\alpha_0, \ldots, \alpha_{2n-1}}\). Since \( U_{2n-3}(\nu_0) = 0 \), we have
\[
\begin{align*}
\frac{\partial (\hat{x}_1, \hat{x}_2)}{\partial (u, \theta)} \Big|_{(u, \theta) = (\nu_0, \theta_0)} &= 0.
\end{align*}
\]
In particular, the induced map \((\hat{x}_1, \hat{x}_2)\) of the Kobayashi surface \( f \) is not an immersion at \((\nu_0, \theta_0)\). So the condition (4.11) is sharp. \( \square \)

**Corollary 4.2.** Let \( f \) be a Kobayashi surface of general type associated to the Weierstrass data as in (2.20). Suppose that the inequality (4.11) holds. If \( b_1, \ldots, b_{n-1} \) are sufficiently close to 0, then the map \((\hat{x}_1, \hat{x}_2) : \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \to \mathbb{R}^2\) is an immersion.

**Proof.** Applying the same technique as in the proof of Lemma 3.1 we have that
\[
\begin{align*}
\frac{\partial (\hat{x}_1, \hat{x}_2)}{\partial (u, \theta)} &= -4^{-n} \frac{X(r, \theta)/r^{2n-2} \prod_{j=0}^{2n-1} (u - \cos(\theta - \alpha_j))^{-2}}{(r - r^{-1})/2},
\end{align*}
\]
where we used the equations (4.12), (3.5), (3.12) and (3.13), and
\[
X(r, \theta) = \prod_{i=1}^{n-1} |b_i z - 1|^4 - \prod_{i=1}^{n-1} |z - b_i|^4
\]
19
can be considered as a polynomial in $r$ with parameter $\theta$ of degree $4(n-1)$, since $b_1, ..., b_{n-1}$ lie in the unit disk. Moreover, it holds that

$$X(1/r, \theta) = -\frac{1}{r^{4n-4}}X(r, \theta).$$

Hence, $X$ is an anti-self-reciprocal polynomial in $r$ of degree $4n-4$. By Proposition A.3 in the appendix, there exists a polynomial $Y(u, \theta)$ in $u$ of degree $2n-3$ such that

$$X(r, \theta) = r^{2n-2} \left( \frac{r - r^{-1}}{2} \right) Y(u, \theta).$$

Thus, we get the identity

$$\frac{\partial(\tilde{x}_1, \tilde{x}_2)}{\partial(u, \theta)} = 4^{-n} Y(u, \theta) \prod_{j=0}^{2n-1} (u - \cos(\theta - \alpha_j))^{-2}.$$

Since the degree of $Y(u, \theta)$ is the same as that of $U_{2n-3}(u)$. By the continuity of the roots of the equation $Y(u, \theta) = 0$ with respect to the parameters $b_1, ..., b_{n-1}$, we can conclude that $(\tilde{x}_1, \tilde{x}_2)$ is an immersion on $\Omega_{\alpha_0, ..., \alpha_{2n-1}}$.

We next discuss the properness of the map $(\tilde{x}_1, \tilde{x}_2)$:

**Proposition 4.3.** Let $f$ be a Kobayashi surface of principal type associated to the Weierstrass data as in (2.10). Suppose that the inequality (4.1) holds and also that $\alpha_0, ..., \alpha_{2n-1}$ are distinct. Then $(\tilde{x}_1, \tilde{x}_2) : \Omega_{\alpha_0, ..., \alpha_{2n-1}} \to \mathbb{R}^2$ is a proper immersion.

**Proof.** Let $\{(u_m, \theta_m)\}_{m=1,2,3,...}$ be a sequence on $\Omega_{\alpha_0, ..., \alpha_{2n-1}}$ converging to a point $(u_{\infty}, \theta_{\infty})$ on the boundary of $\Omega_{\alpha_0, ..., \alpha_{2n-1}}$ in the $(u, \theta)$-plane. Then it is sufficient to show that $(\tilde{x}_1(u_m, \theta_m), \tilde{x}_2(u_m, \theta_m))$ diverges. We first consider the case that $\theta_{\infty} \neq \gamma_j$ ($j = 0, 1, ..., 2n-1$), where $\gamma_j$ is defined in (3.3). Without loss of generality, we may assume that $\theta_{\infty} \in I_1 \setminus \{\gamma_0, \gamma_1\}$, where $I_1$ is the closed interval as in (3.2). By Lemma 3.2 we have

$$u_{\infty} - \cos(\theta_{\infty} - \alpha_1) = 0, \quad u_{\infty} - \cos(\theta_{\infty} - \alpha_j) > 0 \quad (j \neq 1).$$

By (3.10), we have

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = A_1 \begin{pmatrix} \cos(n-1)\alpha_1 \\ \sin(n-1)\alpha_1 \end{pmatrix} X_1 \text{ (a bounded term)},$$

where

$$X_j = \log \left( u - \cos(\theta - \alpha_j) \right) \quad (j = 0, ..., 2n-1).$$

Thus, $(\tilde{x}_1(u_m, \theta_m), \tilde{x}_2(u_m, \theta_m))$ is unbounded when $m \to \infty$, since $A_1 \neq 0$. We next consider the case that $\theta_{\infty} = \gamma_j$ for some $j$. Without loss of generality, we may assume that $j = 1$ and $\theta_{\infty} = \gamma_1$. In this case $X_1$ and $X_2$ both tend to $-\infty$. We now assume that

$$\alpha_2 - \alpha_1 < \frac{\pi}{n-1}.$$

Then we have

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 \cos(n-1)\alpha_1 & A_2 \cos(n-1)\alpha_2 \\ A_1 \sin(n-1)\alpha_1 & A_2 \sin(n-1)\alpha_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \text{ (a bounded term)},$$

where
which implies that \((\tilde{x}_1(u_m, \theta_m), \tilde{x}_2(u_m, \theta_m))\) is unbounded when \(m \to \infty\). Here, we used the fact that
\[
(4.4) \quad \begin{vmatrix} A_1 \cos(n-1)\alpha_1 & A_2 \cos(n-1)\alpha_2 \\ A_1 \sin(n-1)\alpha_1 & A_2 \sin(n-1)\alpha_2 \end{vmatrix} = A_1A_2 \sin \left( (n-1)(\alpha_2 - \alpha_1) \right) \neq 0.
\]
Finally, we consider the case \(\alpha_2 - \alpha_1 = \pi/(n-1)\). In this case, we have
\[
\cos(n-1)\alpha_2 = - \cos(n-1)\alpha_1,
\]
and
\[
\tilde{x}_1 = A_1X_1 \cos(n-1)\alpha_1 + A_2X_2 \cos(n-1)\alpha_2 + \text{(a bounded term)}
\]
\[
= (A_1X_1 - A_2X_2) \cos(n-1)\alpha_1 + \text{(a bounded term)}.
\]
Similarly, it holds that
\[
\tilde{x}_2 = (A_1X_1 - A_2X_2) \sin(n-1)\alpha_1 + \text{(a bounded term)}.
\]
Regarding the fact that the sign of
\[
\prod_{j \in \{0, \ldots, 2n-1\} \setminus \{i\}} \sin \frac{\alpha_j - \alpha_i}{2}
\]
is equal to \((-1)^{i+1}\), we can conclude that \(A_1A_2 < 0\). Hence
\[
\lim_{m \to \infty} |A_1X_1 - A_2X_2| = \infty.
\]
Since \(\cos(n-1)\alpha_1\) and \(\sin(n-1)\alpha_1\) do not vanish at the same time, either \(\tilde{x}_1(u_m, \theta_m)\) or \(\tilde{x}_2(u_m, \theta_m)\) diverges as \(m \to \infty\), which proves the assertion. \(\square\)

**Corollary 4.4.** Let \(f\) be a Kobayashi surface of general type associated to the Weierstrass data as in (2.23). Suppose that the inequality
\[
(4.5) \quad |\alpha_j - \alpha_{j+1}| < \frac{\pi}{n-1} \quad (j = 0, \ldots, 2n-1)
\]
holds, and also that \(\alpha_0, \ldots, \alpha_{2n-1}\) are distinct. If \(b_1, \ldots, b_{n-1}\) are sufficiently close to 0, then the map \((\tilde{x}_1, \tilde{x}_2): \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \to \mathbb{R}^2\) is a proper immersion.

**Proof.** Regarding the expressions (3.9) and using the assumption that \(\alpha_0, \ldots, \alpha_{2n-1}\) are distinct, we can write
\[
\varphi_k = \sum_{j=0}^{2n-1} \frac{B_{k,j}}{z - e^{i\alpha_j}} \, dz \quad (k = 0, 1, 2),
\]
where \(B_{k,j}\) are real numbers depending on the coefficients of the polynomials \(p_k(z)\) given in (3.11), (3.12). In particular, \(B_{k,j}\) depend continuously on the \(n-1\) parameters \(b_1, \ldots, b_{n-1}\). Since
\[
\tilde{x}_k = \frac{1}{2} \sum_{j=0}^{2n-1} B_{k,j} \log \left( u - \cos(\theta - \alpha_j) \right) \quad (k = 0, 1, 2),
\]
one can prove the assertion using the same argument as in the proof of Proposition 1.3 using the identity
\[
\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} \begin{pmatrix} \log(u - \cos(\theta - \alpha_1)) \\ \log(u - \cos(\theta - \alpha_2)) \end{pmatrix} + \text{(a bounded term)},
\]
and the limit formula

$$\lim_{(b_1, \ldots, b_{n-1}) \to 0} \frac{1}{2} \begin{pmatrix} B_{1,1}, B_{1,2} \\ B_{2,1}, B_{2,2} \end{pmatrix} = \begin{pmatrix} A_1 \cos(n-1)\alpha_1 & A_2 \cos(n-1)\alpha_2 \\ A_1 \sin(n-1)\alpha_1 & A_2 \sin(n-1)\alpha_2 \end{pmatrix}.$$ 

□

We now get the following result:

**Theorem 4.5.** Let $f$ be a Kobayashi surface of principal type associated to the Weierstrass data as in (2.10). Suppose that $(\alpha_{2n} := 2\pi)$

$$|\alpha_j - \alpha_{j+1}| \leq \frac{\pi}{n-1} \quad (j = 0, \ldots, 2n-1),$$

and that $\alpha_0, \ldots, \alpha_{2n-1}$ are distinct. We let $\tilde{f} : \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \to \mathbb{R}^3_1$ denote its analytic extension. Then the map $(\tilde{x}_1, \tilde{x}_2) : \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \to \mathbb{R}^2$ is a diffeomorphism. In particular, the image of $\tilde{f}$ gives a zero-mean curvature entire graph of mixed type.

**Proof.** This theorem follows immediately from Proposition 4.3, applying the fact that proper immersions between the same dimensional manifolds are diffeomorphisms under the assumption that the target space is simply-connected (cf. [9, Corollary]). □

Similarly, the following assertion immediately follows from Corollary 4.4.

**Theorem 4.6.** Let $f$ be a Kobayashi surface of general type associated to the Weierstrass data as in (2.3). Suppose that the inequality

$$|\alpha_j - \alpha_{j+1}| < \frac{\pi}{n-1} \quad (j = 0, \ldots, 2n-1),$$

and also that $\alpha_0, \ldots, \alpha_{2n-1}$ are distinct. We let $\tilde{f} : \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \to \mathbb{R}^3_1$ denote its analytic extension. If $b_1, \ldots, b_{n-1}$ are sufficiently close to 0 then the map

$$(\tilde{x}_1, \tilde{x}_2) : \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \to \mathbb{R}^2$$

is a diffeomorphism. In particular, the image of $\tilde{f}$ gives a zero-mean curvature entire graph of mixed type. As a consequence, we get a $(4n-7)$-parameter family of zero-mean curvature entire graphs up to congruence and homothety.

**Remark 4.7.** The assumption in the theorem that $b_1, \ldots, b_{n-1}$ are sufficiently close to 0 is crucial, and in fact, for larger $b_1, \ldots, b_{n-1}$, the corresponding Kobayashi surface might not give an entire graph. In fact, let $f_b$ be the Kobayashi surface of order 4 with angular data $\alpha_j = \pi j/4$ ($j = 0, 1, \ldots, 7$) whose Weierstrass data is

$$g = \frac{z^2 - \overline{b}}{1 - \overline{b}z}, \quad \omega = \frac{(1 - \overline{b}z)^2}{z^8 - 1} dz.$$

We know that the analytic extension of the image of $f_b$ gives an entire graph if $|b|$ is sufficiently small. However, for example, if we set $b = -0.75$, $f_b$ appears to have self-intersections. (See Figure 4.1).
5. The condition for \( \tilde{f} \) of principal type to be immersed

In this section, we consider Kobayashi surfaces of principal type with an angular data \((\alpha_0, ..., \alpha_{2n-1})\) satisfying the weaker condition (cf. (4.1))

\[
|\alpha_j - \alpha_{j+1}| \leq \frac{2\pi}{n-1} \quad (j = 0, ..., 2n-1),
\]

where \(\alpha_{2n} := 2\pi\). Under this condition, we prove the following:

**Lemma 5.1.** Let \( f \) be a Kobayashi surface of principal type associated to the Weierstrass data as in (2.10). Then \( \tilde{f} : \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \to \mathbb{R}^3 \) is an immersion if and only if \((\alpha_0, ..., \alpha_{2n-1})\) satisfies (5.1). (Here \(\alpha_0, ..., \alpha_{2n-1}\) are not necessarily distinct.)

**Proof.** Using (4.2), we have (see (2.11) for the definition of \(\Lambda\))

\[
\frac{\partial(\tilde{x}_0, \tilde{x}_1)}{\partial(u, \theta)} = \frac{-ir^3}{r^2 - 1} \begin{vmatrix}
\hat{\varphi}_0 & \hat{\varphi}_1 \\
\varphi_0 & \varphi_1 \\
\end{vmatrix} = \frac{-ir^3}{r^2 - 1} \begin{vmatrix}
-i\Lambda z^{n-1} & 2i\Lambda z^{n-1} \\
-i\Lambda(1 + z^{2n-2}) & -i\Lambda(1 + z^{2n-2}) \\
\end{vmatrix} \frac{1}{\left| \prod_{j=0}^{2n-1} (z - e^{i\alpha_j})^2 \right|}
\]

\[
= \frac{2ir^3}{r^2 - 1} \frac{z^{n-1} + z^{-1}z^{2n-2} - z^{n-1} - z^{-1}z^{2n-2}}{2^{2n+2n-1} \prod_{j=0}^{2n-1} \left( u - \cos(\theta - \alpha_j) \right)}
\]

\[
= 4(r^{n-1} - r^{-n+1}) \cdot \frac{\sin(n-1)\theta}{\left( \prod_{j=0}^{2n-1} \left( u - \cos(\theta - \alpha_j) \right) \right)^2}
\]

\[
= \frac{U_{n-2}(u) \sin(n-1)\theta}{2^{2n-2} \prod_{j=0}^{2n-1} \left( u - \cos(\theta - \alpha_j) \right)^2}
\]
and
\[
\frac{\partial (\tilde{x}_0, \tilde{x}_2)}{\partial (u, \theta)} = \frac{-i\tau^3}{r^2 - 1} \begin{vmatrix}
\hat{\varphi}_0 & \hat{\varphi}_2 \\
\hat{\varphi}_2 & \hat{\varphi}_0
\end{vmatrix} = \frac{-i\tau^3}{r^2 - 1} \begin{vmatrix}
-2i\Lambda z^{n-1} & 2i\bar{\Lambda} z^{n-1} \\
-\Lambda(1 - z^{2n-2}) & -\bar{\Lambda}(1 - z^{2n-2})
\end{vmatrix} \\
= \frac{-U_{n-2}(u) \cos(n-1)\theta}{2^{2n-2} \prod_{j=0}^{2n-1} (u - \cos(\theta - \alpha_j))^2},
\]
where \(U_{n-2}\) is the Chebyshev polynomial of the second kind of degree \(n - 2\). Since \(|\alpha_j - \alpha_{j+1}| \leq 2\pi/(n - 1)\), Corollary 3.3 yields that
\[
u > \cos \frac{\pi}{n - 1} > \cos \frac{\pi}{n - 2}.
\]
By Proposition A.3, \(U_{n-2}(u)\) is monotone increasing on \([\cos \frac{\pi}{n - 2}, \infty)\) and
\[
U_{n-2}(u) > U_{n-2} \left( \cos \frac{\pi}{n - 1} \right) = \frac{\sin \pi}{\sin(\pi/(n - 1))} = 0
\]
holds. In particular, \(U_{n-2}(u) > 0\) on \(\Omega_{\alpha_0, \ldots, \alpha_{2n-1}}\). Since
\[
\frac{\partial (\tilde{x}_0, \tilde{x}_1)}{\partial (u, \theta)}, \quad \frac{\partial (\tilde{x}_0, \tilde{x}_2)}{\partial (u, \theta)}
\]
do not vanish simultaneously, we can conclude that \(\tilde{f}\) is an immersion.

On the other hand, if \((\alpha_0, \ldots, \alpha_{2n-1})\) does not satisfy the condition (5.1), then \(\tilde{f}\) is not an immersion. In fact, in this case, we may set \(\alpha_0 = 0\) and \(\alpha_1 > 2\pi/(n - 1)\). We set
\[
u_0 := \cos \left( \frac{\pi}{n - 1} \right), \quad \theta_0 := \frac{\alpha_1}{2}.
\]
By Corollary 3.3 \((\nu_0, \theta_0) \in \Omega_{\alpha_0, \ldots, \alpha_{2n-1}}\). Since \(U_{n-1}(\nu_0) = 0\), we have
\[
\frac{\partial (\tilde{x}_0, \tilde{x}_1)}{\partial (u, \theta)} \bigg|_{(u, \theta) = (\nu_0, \theta_0)} = \frac{\partial (\tilde{x}_0, \tilde{x}_2)}{\partial (u, \theta)} \bigg|_{(u, \theta) = (\nu_0, \theta_0)} = 0.
\]
On the other hand, the identity \(\sin(2n - 2)\theta = 2 \sin(n - 1)\theta \cos(n - 1)\theta\) induces the relation \(U_{2n-3} = 2T_{n-1}U_{n-2}\) for the Chebyshev polynomials. In particular, \(U_{2n-3}(\nu_0) = 0\) and (see the proof of Lemma 4.1)
\[
\frac{\partial (\tilde{x}_1, \tilde{x}_2)}{\partial (u, \theta)} \bigg|_{(u, \theta) = (\nu_0, \theta_0)} = 0.
\]
Hence \(\tilde{f}\) is not an immersion at \((\nu_0, \theta_0)\).

We next discuss the properness of the map \(\tilde{f}\):

Proposition 5.2. Let \(f\) be a Kobayashi surface of principal type associated to the Weierstrass data as in (2.10). Suppose that the inequalities
\[
|\alpha_j - \alpha_{j+1}| < \frac{2\pi}{n - 1} \quad (j = 0, \ldots, 2n - 1)
\]
hold, and also that \(\alpha_0, \ldots, \alpha_{2n-1}\) are distinct. Then \(\tilde{f} : \Omega_{\alpha_0, \ldots, \alpha_{2n-1}} \to \mathbb{R}^3\) is a proper immersion.
Proof. By the previous lemma, we know that \( \tilde{f} \) is an immersion. So, we will show that \( \tilde{f} \) is proper. Let \( \{(u_m, \theta_m)\}_{m=1,2,3,\ldots} \) be a sequence on \( \Omega_{\alpha_0,\ldots,\alpha_{2n-1}} \) converging to a point \( (u, \theta) \) on the boundary of \( \Omega_{\alpha_0,\ldots,\alpha_{2n-1}} \). Then it is sufficient to show that \( \tilde{f}(u_m, \theta_m) \) diverges. In the case that \( \theta_\infty \neq \gamma_j \) (\( j = 0, 1, \ldots, 2n - 1 \)), we can conclude that \( \tilde{f}(u_m, \theta_m) \) diverges using the same argument as in the proof of Proposition 4.3 where \( \gamma_j \) is defined in (3.5). (The proof of Proposition 4.3 is given under the assumption that \( \alpha_0, \ldots, \alpha_{2n-1} \) are distinct.) So it is sufficient to consider the case that \( \theta_\infty = \gamma_j \) for some \( j \). Without loss of generality, we may assume that \( j = 1 \) and \( \theta_\infty = \gamma_1 \). In this case, we have

\[
\begin{pmatrix}
\tilde{x}_0 \\
\tilde{x}_1 \\
\tilde{x}_2
\end{pmatrix} = M \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \text{(a bounded term),}
\]

where \( X_1 \) and \( X_2 \) are given in (4.3) and

\[
M := \begin{pmatrix}
A_1 & A_2 \\
A_1 \cos(n-1)\alpha_1 & A_2 \cos(n-1)\alpha_2 \\
A_1 \sin(n-1)\alpha_1 & A_2 \sin(n-1)\alpha_2
\end{pmatrix}.
\]

This implies that \( \tilde{f}(u_m, \theta_m) \) is unbounded as \( m \to \infty \) if the rank of the matrix \( M \) is 2. In fact, since \( A_1, A_2 \neq 0 \) we have

\[
\text{rank}(M) = 1 + \text{rank} \begin{pmatrix}
\cos(n-1)\alpha_2 - \cos(n-1)\alpha_1 \\
\sin(n-1)\alpha_2 - \sin(n-1)\alpha_1
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
\cos(n-1)\alpha_2 - \cos(n-1)\alpha_1 \\
\sin(n-1)\alpha_2 - \sin(n-1)\alpha_1
\end{pmatrix} = 2\sin \left( (n-1)\frac{\alpha_2 - \alpha_1}{2} \right) \frac{-\sin(n-1)\gamma_1}{\cos(n-1)\gamma_1}
\]

never vanishes because \( 0 < |\alpha_2 - \alpha_1| < 2\pi/(n-1) \). So we get the assertion. \( \square \)

In the authors’ numerical experiments, the following question has naturally arisen:

Problem. Let \( f \) be a Kobayashi surface of principal type associated to the Weierstrass data as in (2.10). Suppose that \( (\alpha_0, \ldots, \alpha_{2n-1}) \) satisfies (5.1) but not (4.7). (Here \( \alpha_0, \ldots, \alpha_{2n-1} \) are not necessarily distinct.) Then, can one find a suitable condition for the analytic extension \( \tilde{f} : \Omega_{\alpha_0,\ldots,\alpha_{2n-1}} \to \mathbb{R}^3 \) to be properly embedded?

As a special case, the authors proved the proper embeddability of the analytic extension of the Jorge-Meeks type maximal surface \( J_n \) as a Kobayashi surface of principal type of order \( n \), in the previous work [5]. However, if \( n = 3 \), the Kobayashi surface of principal type with angular data \( (0, 3\pi/4, 3\pi/2, 5\pi/3, 7\pi/4, 11\pi/6) \) has an immersed analytic extension having self-intersections (cf. Figure 5.1). On the other hand, for \( n = 2 \), the condition (4.1) gives no restriction for the angular data \( (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \). We can prove the following, which tells us that the problem is affirmative when \( n = 2 \):

Theorem 5.3. Let \( f \) be a Kobayashi surface of order 2. Then its analytic extension \( \tilde{f} : \Omega_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} \to \mathbb{R}^3 \) (\( \alpha_0 = 0 \)) is properly embedded.

Proof. By Proposition 2.5, \( f \) is a Kobayashi surface of principal type. We may assume that its Weierstrass data is written as in (2.17).
We first consider the case that the angular data $\alpha_0, ..., \alpha_3$ are distinct. By Proposition 5.2, $\tilde{f}$ is a proper immersion. So it is sufficient to show that $\tilde{f}$ is an injection.

Using (3.15) and (3.18), we can write

\[
\begin{pmatrix}
\tilde{x}_0 \\
\tilde{x}_1 \\
\tilde{x}_2
\end{pmatrix} =
\begin{pmatrix}
A_1(X_1 - X_0) + A_2(X_2 - X_0) + A_3(X_3 - X_0) \\
-A_1(X_1 - X_0)\cos\alpha_1 - A_2(X_2 - X_0)\cos\alpha_2 - A_3(X_3 - X_0)\cos\alpha_3 \\
-A_1(X_1 - X_0)\sin\alpha_1 - A_2(X_2 - X_0)\sin\alpha_2 - A_3(X_3 - X_0)\sin\alpha_3
\end{pmatrix}
\]

\[
= R \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}
\]

where

\[
X_j := \log \left( u - \cos(\theta - \alpha_j) \right) \quad (j = 0, 1, 2, 3)
\]

and

\[
R := \begin{pmatrix}
A_1 & A_2 & A_3 \\
-A_1 \cos\alpha_1 & -A_2 \cos\alpha_2 & -A_3 \cos\alpha_3 \\
-A_1 \sin\alpha_1 & -A_2 \sin\alpha_2 & -A_3 \sin\alpha_3
\end{pmatrix}.
\]

Since the determinant of $R$ is equal to

\[
4A_1A_2A_3\sin\frac{\alpha_2 - \alpha_1}{2}\sin\frac{\alpha_3 - \alpha_2}{2}\sin\frac{\alpha_3 - \alpha_1}{2} \neq 0,
\]

the matrix $R$ is regular. Since

\[
Y_j = X_j - X_0 = \log \frac{u - \cos(\theta - \alpha_j)}{u - \cos\theta} \quad (j = 1, 2, 3),
\]

it is sufficient to show that the map $\Phi : \Omega_{0,\alpha_1,\alpha_2,\alpha_3} \rightarrow \mathbb{R}^3$ given by

\[
\Phi(u, \theta) := \frac{1}{u - \cos\theta} \left( u - \cos(\theta - \alpha_1), u - \cos(\theta - \alpha_2), u - \cos(\theta - \alpha_3) \right)
\]

is an injection.
is injective. It holds that
\[
\Phi(u, \theta) = 2 \begin{pmatrix}
\sin \frac{\alpha_1}{2} & 0 & 0 \\
0 & \sin \frac{\alpha_2}{2} & 0 \\
0 & 0 & \sin \frac{\alpha_3}{2}
\end{pmatrix}
\begin{pmatrix}
\sin \frac{\alpha_1}{2} & -\cos \frac{\alpha_1}{2} \\
\sin \frac{\alpha_2}{2} & -\cos \frac{\alpha_2}{2} \\
\sin \frac{\alpha_3}{2} & -\cos \frac{\alpha_3}{2}
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta \\
1
\end{pmatrix}
\]

where
\[
(5.3) \quad \xi := \frac{\cos \theta}{u - \cos \theta}, \quad \eta := \frac{\sin \theta}{u - \cos \theta}
\]
Since
\[
|\sin \frac{\alpha_1}{2} - \cos \frac{\alpha_1}{2}| - |\sin \frac{\alpha_2}{2} - \cos \frac{\alpha_2}{2}| = \sin \frac{\alpha_2 - \alpha_1}{2} \neq 0,
\]
the injectivity of \(\Phi\) is equivalent to the injectivity of the map
\[
(5.4) \quad \Psi : \Omega_{0, \alpha_1, \alpha_2, \alpha_3} \ni (u, \theta) \mapsto \frac{1}{u - \cos \theta}(\cos \theta, \sin \theta) \in \mathbb{R}^2.
\]
This follows from the injectivity of the map
\[
\tilde{\Psi} : \Omega_{0, \alpha_1, \alpha_2, \alpha_3} \ni (u, \theta) \mapsto [u - \cos \theta : \cos \theta : \sin \theta] \in P^2,
\]
where \(P^2\) is the real projective plane. Suppose that \(\tilde{\Psi}(u_1, \theta_1) = \tilde{\Psi}(u_2, \theta_2)\). Since \(u_j - \cos \theta_j > 0\) for \((j = 1, 2)\), we have
\[
[\cos \theta_1 : \sin \theta_1] = [\cos \theta_2 : \sin \theta_2]
\]
and so \(\theta_2 - \theta_1 \equiv 0 \mod \pi\), that is, \(\cos \theta_2 = \pm \cos \theta_1\). However \(\cos \theta_2 \neq -\cos \theta_1\) because of the fact that \(\Omega_{0, \alpha_1, \alpha_2, \alpha_3}\) is contained in the upper half of the \((u, \theta)\)-plane. This implies the injectivity of \(\tilde{\Psi}\).

We next consider the case that
\[
\alpha_0 = \alpha_1 = 0, \quad \alpha_2 = \alpha, \quad \alpha_3 = \beta \quad (0 < \alpha < \beta < 2\pi).
\]
We can write (cf. (3.3))
\[
\varphi_0 = \left[ \frac{iB'}{(z - 1)^2} + A_1' \left( \frac{1}{z - e^{i\alpha}} - \frac{1}{z - 1} \right) + A_2' \left( \frac{1}{z - e^{i\beta}} - \frac{1}{z - 1} \right) \right] dz.
\]
In particular, \(A_1'\) and \(A_2'\) are residues of \(\varphi_0\) at \(z = e^{i\alpha}\) and \(z = e^{i\beta}\), respectively. Then
\[
B' := \frac{1}{2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}, \quad A_1' := \frac{1}{4 \sin^2 \frac{\alpha}{2} \sin \frac{\alpha - \beta}{2}}, \quad A_2' := \frac{1}{4 \sin^2 \frac{\beta}{2} \sin \frac{\beta - \alpha}{2}}
\]
hold. Moreover, we have that
\[
\varphi_1 = \left[ \frac{-iB'}{(z - 1)^2} - A_1' \cos \alpha \left( \frac{1}{z - e^{i\alpha}} - \frac{1}{z - 1} \right) - A_2' \cos \beta \left( \frac{1}{z - e^{i\beta}} - \frac{1}{z - 1} \right) \right] dz,
\]
\[
\varphi_2 = \left[ -A_1' \sin \alpha \left( \frac{1}{z - e^{i\alpha}} - \frac{1}{z - 1} \right) - A_2' \sin \beta \left( \frac{1}{z - e^{i\beta}} - \frac{1}{z - 1} \right) \right] dz.
\]
Integrating them, taking the real parts, and replacing parameter \(r\) by \(u\), we have that
\[
\begin{pmatrix}
\tilde{x}_0 \\
\tilde{x}_1 \\
\tilde{x}_2
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
-B' & A_1' & A_2' \\
B' & -A_1' \cos \alpha & -A_2' \cos \beta \\
0 & -A_1' \sin \alpha & -A_2' \sin \beta
\end{pmatrix}
\begin{pmatrix}
X_0' \\
X_1' \\
X_2'
\end{pmatrix},
\]
where
\[ X_0 := \frac{\sin \theta}{u - \cos \theta}, \quad X_1 := \log \left( \frac{u - \cos(\theta - \alpha)}{u - \cos \theta} \right), \quad X_2 := \log \left( \frac{u - \cos(\theta - \beta)}{u - \cos \theta} \right). \]

Since
\[
\begin{vmatrix}
-B' & A_1' & A_2' \\
B' & -A_1 \cos \alpha & -A_2 \cos \beta \\
0 & -A_1' \sin \alpha & -A_2' \sin \beta
\end{vmatrix} = B' A_1' A_2' \sin \alpha \frac{\beta}{2} \sin \frac{\alpha - \beta}{2},
\]

it is sufficient to show that the map \((u, \theta) \mapsto (X_0', X_1', X_2')\) is a proper embedding. The properness follows from the fact that at least one of the three functions \(X_0', X_1', X_2'\) diverges along the sequence \(\{(u_m, \theta_m)\}\) converging to the boundary of \(\Omega_{0,0,\alpha,\beta}\). We see that
\[
\begin{pmatrix}
X_0' \\
X_1' \\
X_2'
\end{pmatrix} = \begin{pmatrix}
0 & 1 & \xi \\
1 - \cos \alpha & -\sin \alpha & \eta
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix},
\]

where \(\xi, \eta\) are the functions given in (5.3). Thus, the embeddedness reduces to the injectivity of the map \(\Psi\) (cf. (5.4)).

Similarly, we consider the case that
\[ \alpha_0 = \alpha_1 = 0, \quad \alpha_2 = \alpha_3 = \alpha \quad (0 < \alpha < 2\pi). \]

Then
\[
\begin{pmatrix}
\dot{x}_0 \\
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
-A'' & -A'' & -A'' B'' \\
A'' & A'' \cos \alpha & A'' B'' \\
0 & B'' / 2 & A''
\end{pmatrix}
\begin{pmatrix}
X_0'' \\
X_1'' \\
X_2''
\end{pmatrix},
\]

where
\[ A'' := \frac{1}{4 \sin^2 \alpha / 2}, \quad B'' := \cot(\alpha / 2), \]

and
\[ (5.5) \quad X_0'' := \frac{\sin \theta}{u - \cos \theta}, \quad X_1'' := \frac{\sin(\theta - \alpha)}{u - \cos(\theta - \alpha)}, \quad X_2'' := \log \left( \frac{u - \cos(\theta - \alpha)}{u - \cos \theta} \right). \]

Since
\[
\begin{vmatrix}
-A'' & -A'' & -A'' B'' \\
A'' & A'' \cos \alpha & A'' B'' \\
0 & B'' / 2 & A''
\end{vmatrix} = -(A'')^3 (1 - \cos \alpha) \neq 0,
\]

and
\[
\begin{pmatrix}
X_0'' \\
e^{X_2''} - 1
\end{pmatrix} = \begin{pmatrix}
0 & 1 & \xi \\
1 - \cos \alpha & -\sin \alpha & \eta
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix},
\]

the embeddedness reduces to the injectivity of the map \(\Psi\) (cf. (5.3)). The properness follows from the fact that \(\dot{x}_2\) diverges. In fact, either \(X_1''\) or \(X_2''\) diverges along the sequence \(\{(u_m, \theta_m)\}\) in \(\Omega_{0,0,\alpha,\alpha}\) converging to the boundary of \(\Omega_{0,0,\alpha,\alpha}\).

Finally, we consider the case that \((\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, 0, 0, \alpha)\). Then
\[
\dot{x}_0 + \dot{x}_1 = \frac{1}{4 \sin(\alpha / 2)} X''_2, \quad \dot{x}_2 = -\frac{\sin \alpha}{8 \sin^3(\alpha / 2)} X''_2 - \frac{1}{2 \sin(\alpha / 2)} X''_0,
\]

hold, where \(X''_0\) and \(X''_2\) are given in (5.5). So, the assertion reduces to the injectivity of the map \((u, \theta) \mapsto (X''_0, X''_2)\) proved as above. The properness follows from the fact that either \(X''_0\) or \(X''_2\) diverges along the sequence \(\{(u_m, \theta_m)\}\) in \(\Omega_{0,0,0,\alpha}\) converging to the boundary of \(\Omega_{0,0,0,\alpha}\). In this case, we can prove that \(\dot{x}_2\) diverges.
The remaining case that \((\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, 0, 0, 0)\) has been discussed in Example 2.8, and one can easily check that the map \(f\) given in (2.23) is proper. So we get the assertion. \(\square\)

**Appendix A. Generalized (anti-)self-reciprocal polynomials**

Let

\[
(A.1) \quad p(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n
\]

be a polynomial in \(r\) with complex coefficients. Then, the polynomial whose coefficients are written in reverse order is given by

\[
\tilde{p}(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0,
\]

which is called the *reciprocal polynomial* of \(p(r)\). If \(p(r)\) satisfies

\[
\tilde{p}(r) = p(r), \quad \text{(resp. } \tilde{p}(r) = -p(r)),
\]

then it is called a *self-reciprocal polynomial* (resp. an *anti-self-reciprocal polynomial*). The following assertion can be proved easily.

**Lemma A.1.** A given polynomial \(p(r)\) of degree \(n\) is self-reciprocal (resp. anti-self-reciprocal) if and only if \(p(1/r) = p(r)/r^n\) (resp. \(p(1/r) = -p(r)/r^n\)).

Regarding this fact, we give the following definition:

**Definition A.2.** A given polynomial \(p(r)\) is called a *generalized self-reciprocal polynomial* (resp. *generalized anti-self-reciprocal polynomial*) if there exists a positive integer \(m\) such that \(p(1/r) = p(r)/r^m\) (resp. \(p(1/r) = -p(r)/r^m\)). The number \(m\) is called the *reciprocal order* of the polynomial.

One can easily observe that for each generalized self-reciprocal polynomial (resp. generalized anti-self-reciprocal polynomial) \(Q(r)\), there exist a self-reciprocal polynomial (resp. anti-self-reciprocal polynomial) \(p(r)\) and a non-negative integer \(l\) such that \(Q(r) = r^{l}p(r)\).

A *Chebyshev polynomial* \(T_n(r)\) (resp. \(U_{n-1}(r)\)) of the *first kind* (resp. the *second kind*) is a polynomial of degree \(n\) (resp. \(n - 1\)) satisfying the following identity

\[
(A.2) \quad \frac{r^n + r^{-n}}{2} = T_n \left( \frac{r + r^{-1}}{2} \right) \quad \text{(resp. } \frac{r^n - r^{-n}}{2} = \left( \frac{r - r^{-1}}{2} \right) U_{n-1} \left( \frac{r + r^{-1}}{2} \right)) \).
\]

The polynomial

\[
\tilde{p}_+(r) = r^n \cdot \frac{r^n + r^{-n}}{2} \quad \text{(resp. } \tilde{p}_-(r) = r^n \cdot \frac{r^n - r^{-n}}{2})
\]

is a typical example of a self-reciprocal polynomial (resp. anti-self-reciprocal polynomial), and the identity \((A.2)\) is the special case of the following assertion:

**Proposition A.3.** Let \(p(r)\) be a generalized self-reciprocal polynomial (resp. generalized anti-self-reciprocal polynomial) of reciprocal order \(2m\). Then there exists a polynomial \(q(u)\) such that

\[
p(r) = r^m q \left( \frac{r + r^{-1}}{2} \right), \quad \text{(resp. } p(r) = r^m \left( \frac{r - r^{-1}}{2} \right) q \left( \frac{r + r^{-1}}{2} \right)).
\]
Proof. Let \( p(r) \) be a polynomial as in \([A.1]\). In particular, \( p(r) \) is a polynomial in degree \( n \). Using the property of the Chebyshev polynomial \( T_n(r) \) of the first kind, we have that

\[
r^{-m} p(r) = \frac{r^{-m} p(r) + r^m p(1/r)}{2} = \sum_{j=0}^{n} a_j \left(\frac{r^{n-j-m} + r^{m+j-n}}{2}\right) = \sum_{j=0}^{n} a_j T_{|n-j-m|} \left(\frac{r + r^{-1}}{2}\right).
\]

So the polynomial \( q(u) := \sum_{j=0}^{n} a_j T_{|n-j-m|}(u) \) attains the desired property. The case that \( p(r) \) is a generalized anti-self-reciprocal polynomial of reciprocal order \( 2m \) can also proved in the same manner using the properties of the Chebyshev polynomial of the second kind.

\[\square\]

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(Shoichi Fujimori) Department of Mathematics, Okayama University, Tsushima-naka, Okayama 700-8530, Japan
E-mail address: fujimori@math.okayama-u.ac.jp

(Yu Kawakami) Graduate School of Natural Science and Technology, Kanazawa University, Kanazawa, 920-1192, Japan,
E-mail address: y-kwkami@se.kanazawa-u.ac.jp

(Masatoshi Kokubu) Department of Mathematics, School of Engineering, Tokyo Denki University, Tokyo 120-8551, Japan
E-mail address: kokubu@cck.dendai.ac.jp

(Wayne Rossman) Department of Mathematics, Faculty of Science, Kobe University, Rokko, Kobe 657-8501, Japan
E-mail address: wayne@math.kobe-u.ac.jp

(Umehara) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology 2-12-1-W8-34, O-okayama, Meguro-ku, Tokyo 152-8552, Japan.
E-mail address: umehara@is.titech.ac.jp

Department of Mathematics, Tokyo Institute of Technology, O-okayama, Meguro, Tokyo 152-8551, Japan
E-mail address: kotaro@math.titech.ac.jp