Anti – Kählerian Manifolds

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math-ph/9906012

Abstract

An anti-Kählerian manifold is a complex manifold with an anti-Hermitian metric and a parallel almost complex structure. It is shown that a metric on such a manifold must be the real part of a holomorphic metric. It is proved that all odd Chern numbers of an anti-Kählerian manifold vanish and that complex parallelisable manifolds (in particular the factor space $G/D$ of a complex Lie group $G$ over the discrete subgroup $D$) are anti-Kählerian manifolds. A method of generating new solutions of Einstein equations by using the theory of anti-Kählerian manifolds is presented.

1 Introduction

Kählerian manifolds constitute a major class of Riemannian (complex) manifolds and powerful methods of complex differential geometry have been

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developed to investigate their properties (see e.g. [1]). In this paper we shall consider an apparently new class of pseudo-Riemannian manifolds which will be called anti-Kählerian manifolds and which are also deeply related with complex analysis.

Recall that a Kählerian manifold can be defined as a triple \((M, g, J)\) which consists of a smooth manifold \(M\) endowed with an almost complex structure \(J\) and a Riemannian metric \(g\) such that \(\nabla J = 0\), where \(\nabla\) is the Levi-Civita connection of \(g\) and the metric \(g\) is assumed to be Hermitian: \(g(JX, JY) = g(X, Y)\) for all vectorfields \(X\) and \(Y\) on \(M\). By an anti-Kählerian manifold we mean instead a triple \((M, g, J)\) which consists of a smooth manifold \(M\), an almost complex structure \(J\) and a metric \(g\) such that \(\nabla J = 0\), where \(\nabla\) is the Levi-Civita connection of \(g\) and the metric \(g\) is anti-Hermitian: \(g(JX, JY) = -g(X, Y)\) for all vectorfields \(X\) and \(Y\) on \(M\). An almost-complex anti-Hermitian structure defines in particular an \(O(m, \mathbb{C})\)-structure on \(M\). Structures of this kind have been also studied under the names: almost complex structure with Norden (or \(B\)-) metric \([15, 6, 9, 2, 16]\).

In this paper anti-Kählerian manifolds will be investigated. In particular we will show that any such metric \(g\) should be the real part of a certain holomorphic metric on \(M\). Holomorphic Riemannian metrics on "complex space-times" have been discussed in Plebański \([18]\), Penrose \([17]\). Let us mention that the characterization of complex analytic Riemannian metrics in terms of complex connections has been considered in Ivanov \([14]\).

We also consider the Chern numbers of anti-Kählerian manifolds and anti-Kählerian Einstein manifolds. A family of examples of anti-Kählerian manifolds will be given. It will be shown that all odd Chern numbers of anti-Kählerian manifolds vanish and that a compact Kählerian manifold cannot be anti-Kählerian. It will also be shown that complex parallelisable manifolds (in particular the factor space \(G/D\) of a complex Lie group \(G\) over the discrete subgroup \(D\)) are anti-Kählerian manifolds.

Finally, we show that the complexification of a given Einstein metric leads to a method of generating new solutions of Einstein equations from a given one. In particular, it implies that the class of Einstein metrics with neutral signature is the largest one in the following sense: any Einstein metric, (of arbitrary signature) in dimension \(m\), generates (at least locally) an infinite family of Einstein metrics with neutral signatures in dimensions \(2^k m, k = 1, 2, \ldots\) .
Anti-Kählerian manifolds appeared naturally in a previous work of ours in connection with a variational principle in nonlinear theories of gravity [3, 4].

2 Holomorphic metrics

Let \((M, J)\) be a \(2m\)-dimensional real almost-complex manifold and let \(g\) be an anti-Hermitian metric on \(M\):

\[ g(JX, JY) = -g(X, Y) \]

or equivalently:

\[ g(JX, Y) = g(X, JY) \]

Then the metric \(g\) has necessarily a neutral (Kleinian) signature \((m, m)\). We extend \(J, g\) and the Levi-Civita connection of \(g\) in the well known way by \(\mathcal{C}\)-linearity to the complexification of the tangent bundle \(TM^C = TM \otimes \mathcal{C}\). Let us now fix a (real) basis \(\{X_1, ..., X_m, JX_1, ..., JX_m\}\) in each tangent space \(T_xM\); then the set \(\{Z_a, Z_{\bar{a}}\}\), where \(Z_a = X_a - iJX_a, \ Z_{\bar{a}} = X_a + iJX_a\), forms a basis for each complexified tangent space \(T_xM \otimes \mathcal{C}\). Unless otherwise stated, little Latin indices \(a, b, c, \ldots\) run from 1 to \(m\), while Latin capitals \(A, B, C, \ldots\) run through 1, ..., \(m\), \(1, \ldots, m\); for notational convenience we shall also bar capital indices and we shall assume \(\bar{A} = A\). One has \(JZ_a = iZ_a\) and \(JZ_{\bar{a}} = -iZ_{\bar{a}}\). We set \(g_{AB} = g(Z_A, Z_B) = g_{BA}\). Then the following holds:

Proposition 2.1. Let \((M, J)\) be an almost-complex manifold and \(g\) be an anti-Hermitian metric on it. Then the complex extended metric \(g\) (in the complex basis introduced above) satisfies the following conditions

\[ g_{\bar{a}b} = g_{ba} = 0 \quad (1) \]

\[ g_{\bar{A}\bar{B}} = \bar{g}_{AB} \quad (2) \]

Conversely, if the complex extended metric \(g_{AB}\) satisfies (1-2) then the initial metric must be anti-Hermitian.

The proof of this result is straightforward.

It will be customary, in this note, to write a metric satisfying (1-2) as

\[ ds^2 = g_{ab}dz^a dz^b + g_{\bar{a}\bar{b}}dz^{\bar{a}} dz^{\bar{b}} \quad (3) \]
since in adapted almost-complex coordinates (see below)
\[ x^\mu = (x^a, y^a \equiv x^{m+a}), \quad z^a = x^a + iy^a \]
one has
\[ g_{\mu\nu} dx^\mu dx^\nu = 2 \text{ Re } [g_{ab} dz^a dz^b] \]
where \( \mu = 1, \ldots , 2m, \ a = 1, \ldots , m \) and \( \text{Re} \) reads Real Part.

We define now the complex Christoffel symbols \( \Gamma^C_{AB} \) by
\[ \nabla Z_A Z_B = \Gamma^C_{AB} Z_C \] (4)
It is known [11] that if \( \nabla J = 0 \) then the torsion \( T \) and the Nijenhuis tensor \( N \) satisfy the identity
\[ T(JX, JY) = \frac{1}{2} N(X, Y) \] (5)
for all vectorfields \( X \) and \( Y \). Since the complex extended Levi-Civita connection \( \nabla \) has vanishing torsion, the complex Christoffel symbols are symmetric, i.e.: \( \Gamma^C_{AB} = \Gamma^C_{BA} \). In this case the complex structure \( J \) is integrable, so that the real manifold \( M \) inherits the structure of a complex manifold. Let us now recall (see e.g. [11]) that there is a one-to-one correspondence between complex manifolds and real manifolds with an integrable complex structure. This means that there exists an atlas of real, adapted (local) coordinates \( (x^1, \ldots , x^m, y^1, \ldots , y^m) \) such that \( J(\partial/\partial x^a) = \partial/\partial y^a, \ J(\partial/\partial y^a) = -\partial/\partial x^a. \) Setting then \( z^a = x^a + iy^a \) and taking \( X_a = \partial/\partial x^a \) one gets
\[ Z_a = X_a - iJX_a = 2\partial_a, \quad Z_{\bar{a}} = X_a + iJX_a = 2\partial_{\bar{a}} \] (6)
where \( \partial_A = \partial/\partial z^A \) and \( z^{\bar{a}} = \bar{z}^a. \) It appears that \( (z^a) \) form an atlas of complex (analytic) coordinate charts on \( M. \) Now, by using Christoffel formulae, one gets
\[ \Gamma^C_{AB} = \frac{1}{2} g^{CD} (Z_A g_{BD} + Z_B g_{DA} - Z_D g_{AB}) \]
\[ = g^{CD} (\partial_A g_{BD} + \partial_B g_{DA} - \partial_D g_{AB}) \] (7)
Then the following holds (see also [11]):

**Theorem 2.2.** Let \( M \) be a \( m \)-dimensional complex manifold, seen as a real \( 2m \)-dimensional manifold with a complex structure \( J. \) Let us further assume that \( M \) is provided with an anti-Hermitian metric \( g. \) We extend \( J, g \)
and the Levi-Civita connection $\nabla$ by $\mathcal{C}$-linearity to the complexified tangent bundle $TM^C$. Then the following conditions are equivalent:

(i)  
\[ \nabla_X(JY) = J(\nabla_XY) \]  
where $X$ and $Y$ are arbitrary real vectorfields;

(ii) in all (local) complex coordinate systems $(z^1, ..., z^m)$ on $M$ the (complex) Christoffel symbols satisfy  
\[ \Gamma^C_{AB} = 0 \quad \text{except for} \quad \Gamma^c_{ab} \quad \text{and} \quad \Gamma^\bar{c}_{\bar{a}\bar{b}} = \bar{\Gamma}^c_{ab} \]  

(iii) in all (local) complex coordinate systems $(z^1, ..., z^m)$ on $M$ the components of the complex extended metric $g_{ab}$ have the canonical form (3) and moreover they are holomorphic functions, i.e.
\[ \partial_c g_{ab} = 0 \]  

Proof. From (4) we have  
\[ \bar{\Gamma}^C_{AB} = \Gamma^\bar{C}_{\bar{A}\bar{B}} \]  
The connection satisfies the conditions  
\[ \nabla_{Z^B}(JZ^c) = J\nabla_{Z^B}Z^c = i\nabla_{Z^B}(Z^c) \]  
\[ \nabla_{Z^B}(J\bar{Z}^\bar{c}) = J\nabla_{Z^B}\bar{Z}^\bar{c} = -i\nabla_{Z^B}(\bar{Z}^\bar{c}) \]  
if and only if  
\[ \Gamma^a_{B\bar{c}} = \Gamma^\bar{a}_{Bc} = 0 \]  
This proves the equivalence between (i) and (ii). Then for the Christoffel symbols (7), by taking (4) into account one gets  
\[ \Gamma^a_{B\bar{c}} = g^{aD}(\partial_b g_{cD} + \partial_c g_{D\bar{b}} - \partial_D g_{bc}) = g^{ad}\partial_c g_{bd} \]  
and from (11) it follows that  
\[ \partial_c g_{bd} = 0 \]  

Also the other relations (11) are reduced to (13) or its complex conjugate. Therefore the relation (13) is equivalent to (10). This proves the equivalence between (i) and (iii). Our claim is thence proved. (Q.E.D.)
3 Chern Classes of anti-Kählerian manifolds

Here we consider some conditions on a manifold for the existence of an anti-Kählerian metric. Let \((M, J, g)\) be an anti-Kählerian manifold. Then \(M\) is a complex manifold and according to Theorem 2.2 there exists a holomorphic metric on \(M\). Therefore there is a complex isomorphism between the complex tangent bundle \(\tau\) and its dual \(\tau^*\). From the known properties of the Chern classes \(c_j(\tau^*) = (-1)^jc_j(\tau)\) one gets the following:

**Proposition 3.1.** All odd Chern classes of an anti-Kählerian manifold \(M\) vanish:
\[
c_{2j+1}(M) = 0 \quad \forall j
\]

The following proposition shows that if a simply connected manifold is Kählerian then it cannot be anti-Kählerian.

**Proposition 3.2.** If \(M\) is a compact simply connected Kählerian manifold then it does not admit an anti-Kählerian metric.

Proof. If \(c_1(M) \neq 0\) then according to proposition 3.1 the manifold \(M\) cannot be anti-Kählerian. Now if \(c_1(M) = 0\) and \(M\) is a compact simply connected Kählerian manifold then by a theorem due to Kobayashi [12] we have \(\Gamma(S^mTM) = \Gamma(S^mT^*M) = 0\) for \(m > 0\) and therefore \(M\) does not admit a holomorphic metric. So it cannot be anti-Kählerian. (Q.E.D.)

4 Examples of anti-Kählerian manifolds

Let us now consider the question: which manifolds may admit an anti-Kählerian structure?

Let us first discuss compact manifolds. According to Proposition 3.1, in a compact complex manifold \((M, J)\) which admits a holomorphic metric all odd Chern numbers must vanish: \(c_{2j+1}(M) = 0\). In complex dimension 1 one has \(\chi(M) = c_1(M) = 0\), where \(\chi(M)\) is the Euler characteristic of \(M\); therefore \(M\) is a torus. On any torus one has a holomorphic metric and the corresponding real anti-Kählerian metric will be Lorentzian.

In complex dimension 2, if \(M\) is a regular (i.e., without holomorphic 1-forms) compact connected (complex) surface with vanishing first Chern

\[1\] This proposition was suggested to us by R. Narasimhan
class, then it is a $K3$ surface; moreover it is known that any $K3$ surface is Kählerian \cite{20} and simply connected and therefore, in virtue of Proposition 3.2, it does not admit a holomorphic metric. We have then to consider irregular surfaces to find anti-Kählerian manifolds.

An interesting open question is whether the Hopf manifolds $S^{2p+1} \times S^{2q+1}$ admit anti-Kählerian metrics.

Let us present now a large class of anti-Kählerian manifolds. A $m$-dimensional complex manifold $M$ is called (complex) parallelisable if there exist $m$ holomorphic vector fields $e_1, \ldots, e_m$ which are everywhere linearly independent in $M$. Every complex Lie group $G$ is parallelisable. If $D$ is a discrete subgroup of $G$ then the complex manifold $G/D$ is also parallelisable \cite{11}. Conversely, Wang proved \cite{21, 11} that every compact parallelisable manifold can be presented as the factor space $G/D$ of a complex Lie group $G$ over a discrete subgroup $D$.

**Proposition 4.1.** Every complex parallelisable manifold $M$ is an anti-Kählerian manifold.

**Proof.** Let us take a complex chart $(z^\mu), \mu = 1, \ldots, m$, on $M$ and let $(e_\alpha^\mu)$ be the components of the independent holomorphic vector fields in this chart, $\alpha = 1, \ldots, m$. Since the vector fields are linearly independent the inverse matrix $f_\alpha^\mu$ defines $m$ holomorphic covector fields $e_\alpha^\mu f_\beta^\mu = \delta_\alpha^\beta$. Now let us set

$$g_{\mu\nu} = f_\mu^a f_\nu^b \delta_{ab}$$

Then $g_{\mu\nu}$ is a holomorphic metric on $M$. Therefore the manifold $M$ is anti-Kählerian.

It would be interesting to know if every (compact) anti-Kählerian manifold is complex parallelisable. (Q.E.D.)

**Remark.** It is known that between complex parallelisable manifolds only the tori admit Kählerian metrics \cite{21}.

### 5 Anti Kählerian Einstein manifolds

In this section we show that by taking the real part of a holomorphic Einstein metric on a complex manifold of complex dimension $m$ one gets a real Einstein manifold of real dimension $2m$. Recall that a metric $g$ is said to be an *Einstein metric* if

$$\text{Ric}(g) = \gamma g$$

(14)
where $\gamma$ is a real constant and $\text{Ric}(g)$ denotes the Ricci tensor of the metric $g$, i.e. $R_{\mu\nu} = R^\tau_{\mu\tau\nu}$, where $\text{Riem}(g) = R_{\mu\nu}$ is the Riemann curvature tensor of the metric $g$.

The following theorem relating the real and complex Einstein equations holds true:

**Theorem 5.1.** Suppose that $(M, g, J)$ is an anti-Kählerian manifold, i.e. a complex manifold of complex dimension $m$ with a holomorphic metric $\hat{g} \equiv (g_{ab}(z)), a, b = 1, \ldots, m$ and a real metric $g \equiv (g_{\mu\nu}(x)), \mu, \nu = 1, \ldots, 2m$ defined by (3). Then the holomorphic metric $g_{ab}(z)$ is Einstein if and only if the real metric $g_{\mu\nu}(x)$ is a solution of the Einstein equations (14). In other words we have: $\text{Ric}(g) = \gamma g$ iff $\text{Ric}(\hat{g}) = \gamma \hat{g}$, i.e. in components:

$$R_{\mu\nu}(g) = \gamma g_{\mu\nu} \quad \text{iff} \quad R_{ab}(\hat{g}) = \gamma g_{ab}$$

To prove Theorem 5.1 we need first the following lemma, whose proof is just a simple modification of the technical proof of Proposition 3.6 in [1] for the Kählerian case.

**Lemma 5.2.** The Riemann $\text{Riem}(g)$ and the Ricci $\text{Ric}(g)$ tensors of the (real) anti-Hermitian metric $g$ of an anti-Kählerian manifold $(M, g, J)$ satisfy the conditions:

$$\text{Riem}(g)(X, Y) \circ J = J \circ \text{Riem}(g)(X, Y)$$  \hspace{1cm} (15a)

$$\text{Riem}(g)(X, Y) = - \text{Riem}(g)(JX, JY)$$  \hspace{1cm} (15b)

$$\text{Riem}(g)(JX, Y) = J \circ \text{Riem}(g)(X, Y)$$  \hspace{1cm} (15c)

$$\text{Ric}(g)(JX, JY) = - \text{Ric}(g)(X, Y)$$  \hspace{1cm} (16)

for each $X, Y \in \chi(M)$.

Notice that (15c) is a simple combination of (15a-b) and the first Bianchi identity (see also Lemma 1.1 in [2]).

We can now prove Theorem 5.1. We shall not discuss here the Einstein equations for a generic metric of the form (3) but consider only the case when $g_{ab}$ is a holomorphic function. From (10) and (15a-b) we get then for the Riemann tensor:

$$R^D_{ABC} = 0 \quad \text{except for} \quad R^d_{abc} \quad \text{and} \quad R^j_{\bar{a}\bar{b}\bar{c}} = \bar{R}^d_{abc}$$  \hspace{1cm} (17)
Moreover, the Christoffel symbols and Riemann tensor are given (in complex coordinates) by the classical formulae:

\[ \Gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) \]  

(18a)

and

\[ R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{ec} \Gamma^a_{bd} - \Gamma^a_{ed} \Gamma^e_{bc} \]  

(18b)

(see also [13] p. 174, where complex-analytic self-dual Einstein metrics have been studied with some details). Our aim is to establish a link between the complex Ricci tensor \( R_{ab} \equiv R^c_{acb} \) and the real one \( R_{\mu\nu} \equiv R^\rho_{\mu\rho\nu} \).

Define \( \hat{R}(X,Y)V \equiv \text{Riem}(g)(V,X)Y \) then \( \text{Ric}(g)(X,Y) = \text{tr} \hat{R}(X,Y) \), where \( \text{tr} \) means the (real) trace of the \( \mathfrak{R} \)-linear endomorphism \( \hat{R}(X,Y) : T_xM \to T_xM \). Due to (15c) \( J \circ \hat{R}(X,Y) = \hat{R}(X,Y) \circ J \), i.e. \( \hat{R}(X,Y) \) is \( \mathfrak{C} \)-linear on \( (T_xM, J) \). It implies that the trace of the endomorphism \( \hat{R}(X,Y) \) (or its \( \mathfrak{C} \)-linear extension into \( T_x^C M \)) do satisfy

\[ \hat{R}(X,Y)^a_b = \text{Re} [\hat{R}(X,Y)^a_b] = 2 \hat{R}(X,Y)^a_b \]

Now, from (16) the complex Ricci tensor \( R_{ab} \) is related with the real one \( R_{\mu\nu} \) via Proposition 2.1. In particular, analogously to (1) we have

\[ R_{\bar{a}b} = 0 \]

The (complex) Einstein equations

\[ R_{AB}(g) = \gamma g_{AB} \]

are thus equivalent to a pair of equations

\[ R_{ab}(g_{cd}) = \gamma g_{ab} \]  

(19a)

\[ R_{\bar{a}b}(g_{cd}) = \gamma g_{\bar{a}b} \]  

(19b)

To get a real solution of Einstein equations (14) from (19) one uses then real coordinates \( (x^\mu), \mu = 1, ..., 2m \) on \( M \), i.e. \( z^a = x^a + i x^{m+a}, a = 1, ..., m \) and writes the real Ricci tensor \( R_{\mu\nu} \) in the form (3)

\[ R_{\mu\nu} dx^\mu dx^\nu = R_{\bar{a}b} dz^a dz^b + R_{\bar{a}b} dz^a dz^b \]

The result then follows. (Q.E.D.)
**Remark 5.3.** Recall from [4] that beside the original metric $g$ one has to our disposal, on an anti-Kählerian manifold $(M, G, J)$, another real metric of neutral signature so called *twin metric* $h(X,Y) \equiv g(JX,Y)$. One finds

$$h_{\mu\nu}dx^\mu dx^\nu = -2 \text{Im} \left[ g_{ab}dz^adz^b \right] \quad \text{and} \quad h_{ab} = ig_{ab} \quad (20)$$

Since $\nabla gJ = 0$ and $J^\mu = h^{\mu\alpha}g_{\alpha\nu}$ both metrics have the same (real and complex) Christoffel symbols: $\nabla g = \nabla h$, thus the same (real and complex) Riemann and Ricci tensors. In the real case only one of two twin metrics can be Einsteinian. In complex coordinates $R_{ab}(g_{cd}) = \gamma g_{ab}$ implies $R_{ab}(h_{cd}) = -i\gamma h_{ab}$, i.e. both holomorphic metrics are Einstein metrics at the same time. One can say, in this way, that the metric $h$ is an Einstein metric with an imaginary cosmological constant.

The last Theorem can be used to construct a “tower” of solutions of Einstein equations. Let be given a real analytic $n$-dimensional Einstein manifold $M^n$ with the Einstein metric $g_{\alpha\beta}(x) : R_{\alpha\beta}(g) = \gamma g_{\alpha\beta}$. Let $\hat{M}^{2n}$ be a certain complex analytic extension (complexification) of the manifold $M^n$ ($x^\alpha \mapsto z^a = x^\alpha + iy^\alpha$; $\alpha, a = 1, \ldots, n$) with the complex analytic metric $\hat{g}_{ab}(z)$ which is an analytic continuation of the original metric $g_{\alpha\beta}(x)$ (see [13], [14], [7]). The pair $(\hat{M}^{2n}, \hat{g})$ is an anti-Kählerian manifold since $\hat{g}$ is automatically analytic. Moreover, one has the (complex) Einstein equation $R_{ab}(\hat{g}) = \gamma \hat{g}_{ab}$ where $R_{ab}(\hat{g})$ is obtained from $\hat{g}_{ab}(z)$ by using the standard formulae for the Ricci tensor with partial derivatives with respect to the complex coordinates $z^a$ (see also (18) and [13], p.174). This is so because all steps in the algorithm for calculating the Ricci tensor (e.g., taking the inverse metric, computing partial derivatives, multiplications, etc...) do commute with the operation of analytic continuation $g(x) \mapsto \hat{g}(z)$. Now, by taking the real part of $\hat{g}_{ab}dz^adz^b$ one gets a new analytic (real) metric of neutral signature on the $2n$-dimensional real manifold $\hat{M}^{2n}$. On virtue of Theorem 5.1 this new metric is again Einsteinian. We can continue in this way and get a $4n$- dimensional real analytic Einstein metric, then the $8n$-dimensional and so on.

To produce a whole family of examples one can take the complex analytic continuations of all real analytic solutions of Einstein equations. We give here a simple concrete example. Take the standard Einstein metric on the
$m$-dimensional sphere $S^m \subset \mathbb{R}^{m+1}$. After analytic continuation one gets:

$$ds^2 = dz^a dz^a + \frac{(z^a dz^a)^2}{1 - z^a z^a} + \text{complex conj.} = g_{\mu\nu} dx^\mu dx^\nu$$

(21)

This metric $g_{\mu\nu}$ lives on the complex sphere $S^m_C \ (w_1^2 + ... + w_{m+1}^2 = 1)$, which can be interpreted as a quadric $\zeta_1^2 + ... + \zeta_{m+1}^2 - \zeta_{m+2}^2 = 0$ in $\mathbb{C}P^{m+1}$ if one takes $w_i = \zeta_i/\zeta_{m+2}$. It gives a solution of the Einstein equations (14) and provides an example of an anti-Hermitian Einstein manifold $(\mathcal{M}, g, J)$. As a real manifold the complex sphere $S^m_C$ is diffeomorphic to the tangent bundle $TS^m$. In particular for $m = 2$ we get a real solution of Einstein equations on the 4-dimensional, non-compact real manifold $TS^2$ with a metric of Kleinian signature $(++--)$.

Notice also that any Einstein metric on a compact Riemannian manifold $\mathcal{M}^n$ leads to an anti-Kählerian Einstein metric on another real manifold $\mathcal{M}^{2n}$. It follows from known facts [3] that any Einstein metric is analytic in a certain atlas on $\mathcal{M}^n$. Therefore there exists a complex analytic continuation of the metric to a complex manifold of complex dimension $n$ which is a real anti-Kählerian manifold $\mathcal{M}^{2n}$.

Acknowledgments

We are grateful to G. Alekseev, M. O. Katanaev, R. Narasimhan and Z. Olszak for useful discussion. One of us (A.B.) is supported by Polish KBN and Mexican CONACyT (#27670 E). I.V. is supported by INTAS grant 960698.

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