A SUFFICIENT CONDITION FOR THE INSERTION OF A CONTRA-CONTINUOUS (BAIRE-ONE) FUNCTION

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Abstract. A sufficient condition for the insertion of a contra-continuous (resp. Baire-one) function between two comparable real-valued functions is given on the topological spaces that Λ-sets are open (resp. $G_{δ}$-sets).

1. Introduction

Results of Katětov [4], [5] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [1], are used in order to give a sufficient condition for the insertion of a contra-continuous (resp. Baire-one) function between two comparable real-valued functions on the topological spaces that Λ-sets [7] are open (resp. $G_{δ}$-sets).

A generalized class of closed sets was considered by Maki in 1986 [7]. He investigated the sets that can be represented as union of closed sets and called them $V$-sets. Complements of $V$-sets, i.e., sets that are intersection of open sets are called Λ-sets [7].

A real-valued function $f$ defined on a topological space $X$ is called contra-continuous [2] (resp. Baire-one) if the preimage of every open subset of $\mathbb{R}$ is closed (resp. $F_{σ}$-set) in $X$.

If $g$ and $f$ are real-valued functions defined on a space $X$, we write $g \leq f$ in case $g(x) \leq f(x)$ for all $x$ in $X$.

2. The main result

Before giving a sufficient condition for insertability of a contra-continuous (Baire-one) function, the necessary definitions and terminology are stated.

Definition. Let $A$ be a subset of a topological space $(X, \tau)$. We define the subsets $A^{Λ}$ and $A^{V}$ as follows: $A^{Λ} = \bigcap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^{V} = \bigcup\{F : F \subseteq A, F^{c} \in (X, \tau)\}$. In [3], [6], $A^{Λ}$ is called the kernel of $A$.

The following first two definitions are due to, or are modifications of, conditions considered in [4], [5].

2000 Mathematics Subject Classification. Primary 54C08, 54C10; Secondary 26A15, 54C30.

Key words and phrases. Contra-continuous function, Baire-one function, Λ-sets, Lower cut set.
Definition. If $\rho$ is a binary relation in a set $S$ then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any $u$ and $v$ in $S$.

Definition. A binary relation $\rho$ in the power set $\mathcal{P}(X)$ of a topological space $X$ is called a strong binary relation in $\mathcal{P}(X)$ in case $\rho$ satisfies each of the following conditions:

1. If $A_1 \rho B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set $C$ in $\mathcal{P}(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.
2. If $A \subseteq B$, then $A \bar{\rho} B$
3. If $A \rho B$, then $A^\Lambda \subseteq B$ and $A \subseteq B^\Lambda$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [1] as follows:

Definition. If $f$ is a real-valued function defined on a space $X$ and if $\{x \in X : f(x) < l\} \subseteq A(f, l) \subseteq \{x \in X : f(x) \leq l\}$ for a real number $l$, then $A(f, l)$ is a lower indefinite cut set in the domain of $f$ at the level $l$.

We now give the following main result:

**Theorem 1.** Let $g$ and $f$ be real-valued functions on the topological space $X$, that $\Lambda$-sets in $X$ are open (resp. $G_\delta$-sets), with $g \leq f$. If there exists a strong binary relation $\rho$ on the power set of $X$ and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of $f$ and $g$ at the level $t$ for each rational number $t$ such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$, then there exists a contra-continuous (resp. Baire-one) function $h$ defined on $X$ such that $g \leq h \leq f$.

**Proof.** Let $g$ and $f$ be real-valued functions defined on the $X$ such that $g \leq f$. By hypothesis there exists a strong binary relation $\rho$ on the power set of $X$ and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of $f$ and $g$ at the level $t$ for each rational number $t$ such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$.

Define functions $F$ and $G$ mapping the rational numbers $\mathbb{Q}$ into the power set of $X$ by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If $t_1$ and $t_2$ are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1) \rho F(t_2)$, $G(t_1) \rho G(t_2)$, and $F(t_1) \rho G(t_2)$. By Lemmas 1 and 2 of [5] it follows that there exists a function $H$ mapping $\mathbb{Q}$ into the power set of $X$ such that if $t_1$ and $t_2$ are any rational numbers with $t_1 < t_2$, then $F(t_1) \rho H(t_2)$, $H(t_1) \rho H(t_2)$ and $H(t_1) \rho G(t_2)$.

For any $x$ in $X$, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If $x$ is in $H(t)$ then $x$ is in $G(t')$ for any $t' > t$; since $x$ in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If $x$ is not in $H(t)$, then $x$ is not in $F(t')$ for any $t' > t$; since $x$ is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f.$
Also, for any rational numbers \( t_1 \) and \( t_2 \) with \( t_1 < t_2 \), we have \( h^{-1}(t_1, t_2) = H(t_2)^V \setminus H(t_1)^A \). Hence \( h^{-1}(t_1, t_2) \) is closed (resp. \( F_\sigma \)-set) in \( X \), i.e., \( h \) is a contra-continuous (resp. Baire-one) function on \( X \). \( \square \)

The above proof used the technique of Theorem 1 of [4].

3. Applications

**Definition.** A real-valued function \( f \) defined on a space \( X \) is called upper semi-contra-continuous (resp. lower semi-contra-continuous) if \( f^{-1}(-\infty, t) \) (resp. \( f^{-1}(t, +\infty) \)) is closed for any real number \( t \).

**Definition.** A real-valued function \( f \) defined on a space \( X \) is called upper semi-Baire-one (resp. lower semi-Baire-one) if \( f^{-1}(-\infty, t) \) (resp. \( f^{-1}(t, +\infty) \)) is \( F_\sigma \)-set for any real number \( t \).

The abbreviations \( usc \), \( lsc \), \( uscc \), \( lsc \), \( usB_1 \) and \( lsB_1 \) are used for upper semicontinuous, lower semicontinuous, upper contra-continuous, lower contra-continuous, upper semi-contra-continuous, upper semi-Baire-one, and lower semi-Baire-one, respectively.

**Corollary 1.** Let \( g \) and \( f \) be real-valued functions defined on a space \( X \), that \( \Lambda \)-sets in \( X \) are open, such that \( f \) is \( lsc \), \( g \) is \( uscc \), and \( g \leq f \). If \( X \) is a extremally disconnected space, then there exists a contra-continuous function \( h \) defined on \( X \) such that \( g \leq h \leq f \).

**Proof.** Let \( g \) be \( uscc \), let \( f \) be \( lsc \), and \( g \leq f \). If a binary relation \( \rho \) is defined by \( A \rho B \) in case \( A^\Lambda \subseteq B^V \), and if \( X \) is a extremally disconnected space then \( \rho \) is a strong binary relation in the power set of \( X \). For each \( t \) in \( \mathbb{Q} \), let \( A(f, t) \) and \( A(g, t) \) be any lower indefinite cut sets for \( f \) and \( g \) respectively. If \( t_1 \) and \( t_2 \) are any elements of \( \mathbb{Q} \) with \( t_1 < t_2 \), then

\[
A(f, t_1) \subseteq \{ x \in X : f(x) \leq t_1 \} \subseteq \{ x \in X : g(x) < t_2 \} \subseteq A(g, t_2);
\]

since \( \{ x \in X : f(x) \leq t_1 \} \) is open and since \( \{ x \in X : g(x) < t_2 \} \) is closed, it follows that \( A(f, t_1)^A \subseteq A(g, t_2)^V \). Hence \( t_1 < t_2 \) implies that \( A(f, t_1) \rho A(g, t_2) \). The proof follows from Theorem 1. \( \square \)

**Corollary 2.** Let \( g \) and \( f \) be real-valued functions defined on a space \( X \), that \( \Lambda \)-sets in \( X \) are open, such that \( f \) is \( lsc \), \( g \) is \( uscc \), and \( f \leq g \). If \( X \) is a normal space, then there exists a contra-continuous \( h \) defined on \( X \) such that \( f \leq h \leq g \).

**Proof.** Let \( f \) be \( lsc \), \( g \) be \( uscc \), and \( f \leq g \). A binary relation \( \rho \) is defined by \( A \rho B \) in case \( A^\Lambda \subseteq F \subseteq F^\Lambda \subseteq B^V \) for some closed set \( F \) in \( X \). If \( X \) is normal, then \( \rho \) is a strong binary relation in the power set of \( X \). If \( t_1 \) and \( t_2 \) are any elements of \( \mathbb{Q} \) with \( t_1 < t_2 \), then

\[
A(g, t_1) = \{ x \in X : g(x) < t_1 \} \subseteq \{ x \in X : f(x) \leq t_2 \} = A(f, t_2);
\]

since \( \{ x \in X : g(x) < t_1 \} \) is closed and since \( \{ x \in X : f(x) \leq t_2 \} \) is open and \( X \) is normal, it follows that \( A(g, t_1) \rho A(f, t_2) \). The proof follows from Theorem 1. \( \square \)
Corollary 3. Let \( g \) and \( f \) be real-valued functions defined on a space \( X \), that \( \Lambda \)-sets in \( X \) are \( G_\delta \)-sets, such that \( f \) is \( h \)-Baire-one function. If, for each pair of disjoint \( G_\delta \)-sets \( F_0, F_1 \), there are two \( F_\sigma \)-sets \( F_0 \) and \( F_1 \) such that \( F_0 \subseteq F_0 \), \( G_1 \subseteq F_1 \) and \( F_0 \cap F_1 = \emptyset \), then there exists a \( \Lambda \)-set such that \( g \leq h \leq f \).

Proof. Let \( f \) be \( h \)-Baire-one function. If a binary relation \( \rho \) is defined by \( A \rho B \) in case \( A \subseteq B \), then by hypothesis \( \rho \) is a strong binary relation in the power set of \( X \). If \( t_1 \) and \( t_2 \) are any elements of \( \mathbb{Q} \) with \( t_1 < t_2 \), then

\[
A(f, t_1) \subseteq \{ x \in X : f(x) \leq t_1 \} \subseteq \{ x \in X : g(x) < t_2 \} \subseteq A(g, t_2);
\]

since \( \{ x \in X : f(x) \leq t_1 \} \) is \( G_\delta \)-set and \( \{ x \in X : g(x) < t_2 \} \) is \( F_\sigma \)-set, it follows that \( A(f, t_1) \subseteq A(g, t_2) \). Hence \( t_1 < t_2 \) implies that \( A(f, t_1) \rho A(g, t_2) \). The proof follows from Theorem 1.

Corollary 4. Let \( g \) and \( f \) be real-valued functions defined on a space \( X \), that \( \Lambda \)-sets in \( X \) are \( G_\delta \)-sets, such that \( f \) is \( h \)-Baire-one function. If, for each pair of disjoint \( F_\sigma \)-sets \( F_0, F_1 \), there are two \( G_\delta \)-sets \( G_0 \) and \( G_1 \) such that \( F_0 \subseteq G_0 \), \( G_1 \subseteq F_1 \) and \( G_0 \cap G_1 = \emptyset \), then there exists a \( \Lambda \)-set such that \( f \leq h \leq g \).

Proof. Let \( f \) be \( h \)-Baire-one function. If a binary relation \( \rho \) is defined by \( A \rho B \) in case \( A \subseteq B \), then by hypothesis \( \rho \) is a strong binary relation in the power set of \( X \). If \( t_1 \) and \( t_2 \) are any elements of \( \mathbb{Q} \) with \( t_1 < t_2 \), then

\[
A(g, t_1) = \{ x \in X : g(x) < t_1 \} \subseteq \{ x \in X : f(x) \leq t_2 \} = A(f, t_2);
\]

since \( \{ x \in X : g(x) < t_1 \} \) is a \( F_\sigma \)-set and \( \{ x \in X : f(x) \leq t_2 \} \) is a \( G_\delta \)-set, by hypothesis it follows that \( A(g, t_1) \rho A(f, t_2) \). The proof follows from Theorem 1.

Remark 1 ([4], [5]). If \( g \) and \( f \) be real-valued functions defined on a normal space \( X \) such that \( f \) is \( h \)-Baire-one function. If a binary relation \( \rho \) is defined by \( A \rho B \) in case \( A \subseteq B \), then by hypothesis \( \rho \) is a strong binary relation, is also a necessary condition for the stated insertion property.

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