Topological Symmetries of $\mathbb{R}^3$

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March 8, 2016

1 Introduction

Let $G$ be a finite group acting effectively on a smooth manifold $M^n$. In the theory of transformation groups one usually considers two types of actions: topological actions and smooth actions. It was realized very early (cf. [2],[3]) that there are topological actions which can not be smoothed out, i.e. actions which are not conjugate (topologically) to smooth actions.

In the case of an Euclidean space $\mathbb{R}^n$ there is a particularly nice class of smooth actions, namely linear actions of $G$ on $\mathbb{R}^n$. Again there is plenty of examples (cf. [14],[15],[26]) of smooth actions which can not be linearized (even topologically).

It turns out however that in dimension $n \leq 2$ every topological finite group action on $\mathbb{R}^n$ is conjugate to a linear action (cf. [4],[19],[11]). For the Euclidean space $\mathbb{R}^3$ the situation is drastically different. First, there exist topological nonsmoothable actions of cyclic groups on $\mathbb{R}^3$. Second, the existence of uncountably many nonequivalent such actions (cf. [2],[3]) dashes any hope for classification of topological actions of finite groups on $\mathbb{R}^3$. On the other hand, smooth actions on $\mathbb{R}^n$, $n = 3$, are conjugate to linear actions (a highly nontrivial result, cf. [20]), and the linearization fails for smooth actions on $\mathbb{R}^n$, $n \geq 4$ (cf. [14] [15],[26]). In particular it follows that every finite group acting orientation preservingly and smoothly on $\mathbb{R}^n$, $n \leq 3$ is isomorphic to a subgroup of $SO(3)$.

In fact even though there exist nonlinear orientation preserving smooth actions of finite groups on $\mathbb{R}^4$, such group must be isomorphic to a subgroup of $SO(4)$ (cf. [16] and Remark 3.1 in this paper).

In a sharp contrast with smooth actions there are topological orientation preserving actions of finite groups $G$ on $\mathbb{R}^4$ with $G \not\subseteq SO(4)$ (cf. [21] and Remark 3.2 in this paper).

Consequently the main problem left open is the question concerning the structure of finite groups acting topologically on $\mathbb{R}^3$. More specifically:

Problem: Let $G$ be a finite group acting topologically and orientation preservingly on $\mathbb{R}^3$. Is $G$ isomorphic with a finite subgroup of $SO(3)$?

This paper provides the positive answer to this problem. Namely:

Theorem 1 Let $G$ be a finite group acting topologically and orientation preservingly on $\mathbb{R}^3$. Then $G$ is isomorphic with a finite subgroup of $SO(3)$.

*The author acknowledges the support of the Simons Foundation Grant No.261810
It turns out that Theorem 1 extends to actions which are not necessarily orientation preserving, namely:

**Theorem:** Let \( G \) be a finite group acting topologically on \( \mathbb{R}^3 \). Then \( G \) is isomorphic with a finite subgroup of \( O(3) \).

The proof of the above theorem uses heavily Theorem 1 yet it is not exactly a direct consequence of Theorem 1. Consequently, it is treated separately in [27].

There are two main difficulties in dealing with Theorem 1.

The first difficulty is the lack of a systematic study of topological actions of finite groups on 3-manifolds due to the existence of “very exotic” actions on \( \mathbb{R}^3(S^3) \). The most striking results in this area being constructions of R.H.Bing ([2], [3]) of cyclic group actions on \( \mathbb{R}^3(S^3) \) with wildly embedded (and knotted) fixed point sets.

Note that such phenomena can not happen in the smooth setting (i.e., the Smith Conjecture).

These constructions indicate serious problems (impossibility?) with a development of equivariant versions of some powerful tools in 3-dimensional topology which could help to analyze the above problem. Note that in the smooth (locally linear) setting such tools were successfully developed (cf. [23], [10], [18]).

The second difficulty in dealing with the above problem is the low dimension of \( \mathbb{R}^n \) i.e. \( n = 3 \). This low dimension prevents the use of higher dimensional techniques (for example the surgery theory) which could be helpful in purely topological setting (cf. [21], [22] for the case of topological actions in dimension four).

As a consequence our proof of Theorem 1 relies heavily on a careful group theoretic analysis of the structure of finite groups which potentially can act on \( \mathbb{R}^3 \). The techniques employed to accomplish this are those of homological algebra and finite group theory.

The considerations leading to the proof of Theorem 1 turn out to be surprisingly intricate and delicate.

To a large extend this is however expected. For, dealing with purely topological actions, even on surfaces, without any assumption about geometrization of these actions, usually requires a fair amount of work (cf. [11], [24]).

## 2 Proofs

This section contains the proof of Theorem 1. Before providing the details we outline the general strategy of the proof.

Throughout this section, \( G \) is assumed to be a finite group acting topologically and orientation preservingly on \( \mathbb{R}^3 \). All group actions in this paper are assumed to be effective.

### 2.1 Outline of Proof

Our considerations are divided into two cases.

- **Case 1:** \( G \) is solvable.
- **Case 2:** \( G \) is not solvable.

Suppose now that \( G \) is solvable. We will investigate the structure of \( G \) via a subnormal series whose factors are cyclic groups of prime order (the existence of such is guaranteed by solvability). In other words, we will treat \( G \) as obtained by a finite sequence of group
extensions by $\mathbb{Z}_p$, $p$ prime. All possible sequences can be expressed in a “tree diagram” with the trivial group at the origin, possible $G$ at the other end and each line segment representing an extension by $\mathbb{Z}_p$. The existence of an action of $G$ on $\mathbb{R}^3$ significantly reduces the possibilities at each step. We begin with extensions of cyclic groups, and we have:

**A.1.** \[
\{0 \to \mathbb{Z}_n \to G \to \mathbb{Z}_p \to 0, n \not\in \mathbb{Z}^+, p \text{ is odd}\} \implies G \text{ is cyclic.}
\]

**A.2.** \[
\{0 \to \mathbb{Z}_n \to G \to \mathbb{Z}_2 \to 0, n \not\in \mathbb{Z}^+\} \implies G \text{ is cyclic or dihedral.}
\]

Thus we obtain part of the tree diagram

\[
\begin{array}{c}
\{0\} \xrightarrow{\text{odd}} \text{Cyclic} \xrightarrow{\mathbb{Z}_2} \text{Cyclic} \xrightarrow{\mathbb{Z}_2} \text{Dihedral} \xrightarrow{\ldots}
\end{array}
\]

Where the “odd” step is a consecutive extension by $\mathbb{Z}_p$, $p$ odd.

Next we look at branches (extensions) stemming from dihedral groups, here we have:

**B.1.** \[0 \to D_{2n} \to G \to \mathbb{Z}_p \to 0, n \geq 2, p \text{ is odd}\] $\implies G$ is dihedral or the alternating group $A_4$.

**B.2.** \[0 \to D_{2n} \to G \to \mathbb{Z}_2 \to 0, n \geq 2\] $\implies G$ is dihedral.

Consequently such branch assumes the following form:

\[
\text{Dihedral} \xrightarrow{\text{odd}} \text{Dihedral} \xrightarrow{\mathbb{Z}_2} \text{Dihedral} \xrightarrow{\ldots}
\]

Now for a branch stemming from $A_4$, we have:

**C.** \[0 \to A_4 \to G \to \mathbb{Z}_p \to 0, p \text{ prime}\] $\implies p = 2$ and $G \cong S_4$.

Regarding $S_4$, we have:

**D.** There is no extension of $S_4$.

Thus $G$, sitting at the end of the sequence of extensions, must be either cyclic, dihedral, $A_4$ or $S_4$. In particular $G \subset SO(3)$. This finishes Case 1.

Now suppose $G$ is not solvable.

From the above discussions we know that the Sylow $p$-subgroup of $G$, $\text{Syl}_p(G)$, is cyclic for $p$-odd. For the 2-subgroups, it turns out $\text{Syl}_2(G)$ is either cyclic or dihedral.

If $\text{Syl}_2(G)$ is cyclic, then $G$ is metacyclic, in particular solvable (cf. [17]), thus $G \subset SO(3)$.

If $\text{Syl}_2(G)$ is dihedral, a theorem of Suzuki (cf. [28]) gives a normal subgroup $G_1 \subset G$ with $[G : G_1] \leq 2$ and $G \cong Z \times L$, where $Z$ is solvable and $L \cong PSL(2, p)$, $p$-prime. It will be shown that in this case $G$ is solvable or $G \cong A_5$. In either case $G \subset SO(3)$.

This concludes the outline of the proof of Theorem 1.

Now we present the necessary details of the above process.
2.2 Preliminaries

Since $G$ acts (orientation preservingly) on $\mathbb{R}^3$, there is an induced orientation preserving action on $S^3$, the one point compactification. In what follows, the action of $G$ on $S^3$ will be assumed to be this one.

The following results will be crucial in our proof.

**Proposition 1** If $G$ is cyclic and nontrivial, then $(S^3)^G = S^1 (= \text{stands for homeomorphism})$.

**Proof** See [25].

The above proposition has the following corollary:

**Corollary 1** If $(S^3)^G = S^1$, then $G$ is cyclic.

**Proof** Let $H \subset G$ be a nontrivial subgroup. Take any nontrivial element $a \in H$, then $\langle a \rangle \subset H \subset G$, where $\langle a \rangle$ is the cyclic subgroup generated by $a$. Thus $(S^3)^{\langle a \rangle} \supset (S^3)^H \supset (S^3)^G$. The two ends of this sequence are homeomorphic to $S^1$ by Proposition 1 and the assumption. Since any embedding of an $S^1$ in another is surjective, $(S^3)^H = S^1 = (S^3)^G$. Now $H$ is taken arbitrarily, which implies that the action of $G$ on $S^3 - (S^3)^G = S^3 - S^1$ is free. It is a well known fact that $S^3 - S^1$ is a cohomological 1-sphere. Thus the Tate cohomology of $G$ has period 2, i.e., $G$ is cyclic (cf. [5]).

Consequently for a finite group $G$ acting orientation preservingly on $\mathbb{R}^3$, $(S^3)^G = S^1$ if and only if $G$ is cyclic and nontrivial.

Another result which plays an important role in our considerations is the following:

**Theorem 2** If $G$ is a 2-group, then $G$ is cyclic or dihedral.

**Proof** By the result of Dotzel and Hamrick (cf. [9]), there is an orthogonal action of $G$ on $\mathbb{R}^4$ such that there exist a map $\phi : \Sigma^k S^3 \longrightarrow \Sigma^k S^3, k \geq 1$ equivariant (with respect to the original action for the domain and the action induced by the orthogonal action on $S^3 \subset \mathbb{R}^4$ for the codomain) which induces $\mathbb{Z}_p$-homology isomorphisms on (suspension of) fixed point sets of non-trivial subgroups. These fixed point sets are spheres, thus their dimension are preserved by $\phi$.

Since $(S^3)^G \neq \emptyset$ in the original action, so the same holds for the orthogonal action. Let $v \in S^0$ be a fixed point. Let $W$ be the orthogonal complement of $v$ in $\mathbb{R}^4$. $G$ acts orthogonally on $W$. Checking the fixed point set, one sees that this action is faithful, thus $G \subset O(3)$.

Now suppose $G$ is neither cyclic or dihedral, then as a subgroup of $O(3)$, $G$ contains $-I_3$. Let $A = -I_3$.

Since $\langle A \rangle$ is cyclic, then $(S^3)^{\langle A \rangle} = S^1$ in the original action by Proposition 1, and hence the same is true for the orthogonal action on $S^3 \subset \mathbb{R}^4$. This implies that $(\mathbb{R}^4)^{\langle A \rangle} = \mathbb{R}^2$. This is impossible. Thus $G$ has to be cyclic or dihedral.

**Remark 1** The first two paragraphs do not need the assumption on the action to be orientation preserving, thus we have shown that any 2-group acting on $\mathbb{R}^3$ is a subgroup of $O(3)$.
2.3 Obstruction Kernels

Our proof of Theorem 1 relies essentially on detecting in a given group a typical subgroup that can not act on $\mathbb{R}^3$. These groups will be called obstruction kernels (abbreviated as O.K.) in what follows. In this subsection we list all of them and prove they cannot act faithfully and orientation preservingly on $\mathbb{R}^3$.

We start with the following lemma:

**Lemma 1** If $G$ acts orientation preservingly on $\mathbb{R}^3$, $H,H'$ are cyclic subgroups of $G$, $H \cap H' \neq \{0\}$, $\langle H \cup H' \rangle = G$ where $\langle H \cup H' \rangle$ denotes the subgroup generated by $H \cup H'$. Then $G$ is cyclic.

**Proof** Consider the inclusions $(S^3)^H \subset (S^3)^{H \cap H'} \supset (S^3)^{H'}$. Since these subgroups are all nontrivial and cyclic, Proposition 1 implies that $(S^3)^H = (S^3)^{H'} = S^1$. Now $G = \langle H \cup H' \rangle$ implies $(S^3)^G = S^1$, thus $G$ is cyclic by Corollary 1.

**Obstruction Kernel of Type 0:** $G = (\mathbb{Z}_p \oplus \mathbb{Z}_q) \rtimes_\varphi \mathbb{Z}_2$ where (the generator of) $\mathbb{Z}_2$ acts as identity on $\mathbb{Z}_p$, and as multiplication by $-1$ on $\mathbb{Z}_q$, $p,q$ distinct odd primes.

**Proof** There is a canonical subgroup $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_2 \subset (\mathbb{Z}_p \oplus \mathbb{Z}_q) \rtimes_\varphi \mathbb{Z}_2 = G$. Assume such action of $G$ on $\mathbb{R}^3$ exists. Since the action on $\mathbb{Z}_p$ is trivial, $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_2 \cong \mathbb{Z}_{2p}$. Letting $H = \mathbb{Z}_{2p}$, $H' = \mathbb{Z}_p \oplus \mathbb{Z}_2 = \mathbb{Z}_{pq}$, we get $H \cap H' = \mathbb{Z}_p$, $\langle H \cup H' \rangle = G$. So by Lemma 1 $G$ is cyclic. But this implies that the subgroup $\mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_2$ is cyclic, contradicting to the assumption.

**Obstruction Kernel of Type 1:** $G = \mathbb{Z}_{2m} \times \mathbb{Z}_2$, $m \geq 2$.

**Proof** Assume the action exists. Let $H = \mathbb{Z}_{2m} \subset G$, $H' = \langle (1,1) \rangle \subset G$. Then $H,H'$ satisfy the assumption of Lemma 1, thus $G = \mathbb{Z}_{2m} \times \mathbb{Z}_2$ is cyclic, which is impossible.

**Obstruction Kernel of Type 2:** $G = \mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_{2k+1}$, $k \geq 1$, $q$ odd prime. $\varphi(1)$ is multiplication by $(-1)$.

**Proof** Assume the action exists. Let $H = \mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_{2k} \subset \mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_{2k+1}$, where $\mathbb{Z}_{2k}$ is generated by $2 \in \mathbb{Z}_{2k+1}$. Then $H$ is cyclic since the action $\varphi$ restricts to a trivial action on $\mathbb{Z}_{2k}$. Let $H'$ be the subgroup $\mathbb{Z}_{2k+1}$ of $G$. Then $H,H'$ satisfy the condition of Lemma 1, and hence $G$ is cyclic, which once again is impossible.

**Obstruction Kernel of Type 3:** Generalized quaternion group $Q_{4m}$, $(m \geq 2)$.

**Proof** Assume the action exists. Let $q$ be a prime factor of $m$. Note that there is a canonical subgroup $Q_{4q} \subset Q_{4m}$.

- If $q$ is odd, $Q_{4q} = \mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_4$, $\varphi(1)$ is multiplication by $(-1)$. This is O.K. of Type 2.
- If $q = 2$, then $Q_{4q} = Q_8$ is a 2-group. This case is excluded by Theorem 2.

**Obstruction Kernel of Type 4:** $G = \mathbb{Z}_{2k} \rtimes_\varphi \mathbb{Z}_2$, $k \geq 3$, $\varphi(1)$ is multiplication by $2^{k-1} \pm 1$.

**Proof** This is a corollary of Theorem 2.
Obstruction Kernel of Type 5: $G = \mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_4$, $p$ prime, $p \equiv 1(\mod 4)$, $\varphi(1) = n \in \mathbb{Z}_p^*$, $\mathbb{Z}_p$ where $n^2 = -1(\mod 4)$.

Proof Assume the action exists. Then $\mathbb{Z}_p$ is a normal subgroup of $G$, and $\mathbb{Z}_4$ acts on the fixed point set $(S^3)^{\mathbb{Z}_p} = S^1$ (Proposition 1). Let $f$ be the restriction of $\varphi(0, 1)$ on $(S^3)^{\mathbb{Z}_p} = S^1$. Note that $f^4$, being the restriction of $\varphi(0, 4)$, is the identity, and a homeomorphism $f$ of $S^1$ satisfying this must also satisfy $f^2 = \text{id}$. That is, $(0, 2) \in \mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_4$ acts on $(S^3)^{\mathbb{Z}_p} = S^1$ by the identity. Thus $(S^3)^{\mathbb{Z}_p} \rtimes_{\varphi} \mathbb{Z}_2 = (S^3)^{\mathbb{Z}_p} = S^1$, where $\mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_2 \subset \mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_4$ is the canonical subgroup. Hence $\mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_2$ is cyclic. But the action $\varphi$ restricted to $\mathbb{Z}_2 \subset \mathbb{Z}_4$ is nontrivial by definition which is a contradiction.

Before presenting the O.K. of Type 6, we need to address the convention regarding the automorphism group of the dihedral group $D_{2n}$. It is a well known fact that $\text{Aut} D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$ where the semi-direct product is with respect to the canonical identification $\text{Aut} \mathbb{Z}_n \cong \mathbb{Z}_n^*$. An element $(t, s) \in \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$ maps $a^i b^j$ to $a^{i+j} b^j$ and $a^i$ to $a^{i+2d}$. Here $a, b$ are standard generators of $D_{2n}$ ($a^n = b^2 = \text{id}, aba = b$).

Obstruction Kernel of Type 6: $G = D_{2n} \rtimes_{\varphi} \mathbb{Z}_2$, $n$ even, $n \geq 4$, $\varphi(1)$ is $(t, -1) \in \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$, $t$ even.

Proof Assume the action exists. Let $i = \frac{n+2}{2}$, then $2i - t = 2$. Now $(a^i b, 1)^2 = (a^{2i-t}, 0) = (a^2, 0)$. Define $H = \langle (a^i b, 1) \rangle$, and note that $|H| = n$. Define $H' = \langle (a, 0) \rangle$ with $|H'| = n$. Then $H \cap H' = \langle (a^2, 0) \rangle \neq \{0\}$. Let $G' = \langle H \cup H' \rangle$, then Lemma 1 can be applied to $G', H, H'$. Thus $G'$ is cyclic, of order $> n$. But examining the elements of $G = D_{2n} \rtimes_{\varphi} \mathbb{Z}_2$, none has order exceeding $n$, a contradiction.

2.4 The Solvable Case

Now we consider the case of solvable $G$. We will determine what happens for each extension step, then an induction will produce the desired result.

2.4.1 Extension of Cyclic Groups

We are going to prove A.1 and A.2 of the Section 2.1. Let us start with A.1.

Proposition 2 Suppose there is a short exact sequence $0 \rightarrow \mathbb{Z}_n \rightarrow G \rightarrow \mathbb{Z}_p \rightarrow 0$, $p$ odd prime. Then $G$ is cyclic.

Proof Proposition 1 implies that $(S^3)^{\mathbb{Z}_n} = S^1$. Now $\mathbb{Z}_p$ acts on this $S^1$, but $p$ is odd, so the action has to be trivial. Thus $(S^3)^G = ((S^3)^{\mathbb{Z}_n})^{\mathbb{Z}_p} = S^1$, and $G$ is cyclic by Corollary 1.

Corollary 2 If $|G|$ is odd, then $G$ is cyclic.

Proof This is because all odd order groups are solvable. (cf. [12]).

The assertion A.2 is more delicate, and we have to discuss several cases, divided by the number of 2-factors in the order of the cyclic group.
Proposition 3 Suppose there is a short exact sequence $0 \to \mathbb{Z}_n \to G \to \mathbb{Z}_2 \to 0$, $n$ odd. Then $G$ is cyclic or dihedral.

**Proof** Let $p_1^{n_1} \cdots p_k^{n_k}$ be the prime decomposition of $n$. Since $(n, 2) = 1$, the sequence splits (cf. [5] p.93). Thus

$$G \cong \mathbb{Z}_n \rtimes \varphi \mathbb{Z}_2, \varphi : \mathbb{Z}_2 \to \text{Aut} \prod_i \mathbb{Z}_{p_i^{n_i}} = \prod_i \text{Aut} \mathbb{Z}_{p_i^{n_i}}$$

The component of $\varphi(1)$ on each $\text{Aut} \mathbb{Z}_{p_i^{n_i}}$ is a multiplication by $\pm 1$ (because $\varphi(1)$ has order at most 2, and $\text{Aut} \mathbb{Z}_{p_i^{n_i}}$ is cyclic thus a unique element of order 2).

If all components are $+1$, then $G \cong \mathbb{Z}_n \times \mathbb{Z}_2 \cong \mathbb{Z}_{2n}$ is a cyclic group.

If all components are $-1$, $\varphi(1)$ is multiplication of $-1$ on $\mathbb{Z}_n$, then $G \cong D_{2n}$ is the dihedral group.

If both $\pm 1$ exist, take $p, q$ prime factors of $n$ where $\varphi(1)$ is multiplication of $+1, -1$ respectively on the corresponding components. Now there is a canonical subgroup $G_0 \subset G$, i.e. $G_0 := (\mathbb{Z}_p \oplus \mathbb{Z}_q) \rtimes \varphi \mathbb{Z}_2 \subset G$. But $G_0$ is O.K. of Type 0; a contradiction.

In conclusion, the only possibility is to be a cyclic or dihedral group. \(\square\)

Proposition 4 Suppose there is a short exact sequence $0 \to \mathbb{Z}_{2n} \to G \to \mathbb{Z}_2 \to 0$, $n$ odd. Then $G$ is cyclic or dihedral.

**Proof** Note that in the above extension, there is an induced action $\varphi$ of $\mathbb{Z}_2$ on $\mathbb{Z}_{2n}$. As in the previous proof, $\varphi(1)$ is a multiplication of $\pm 1$ on each factor of the prime decomposition $\mathbb{Z}_{2n}$. For convenience, we name those prime with $\varphi(1)$ being $+1$ as $p_i, 1 \leq i \leq k$, and those corresponding to $-1$ as $q_j, 1 \leq j \leq l$ (the action on the $\mathbb{Z}_2$ component is always trivial).

Let $P = \prod_i p_i^{n_i}, Q = \prod_j p_j^{m_j}, n = 2PQ$. An easy computation shows that $H^2(\mathbb{Z}_2, \mathbb{Z}_{2n})$, the cohomology of $\mathbb{Z}_2$ with coefficient $\mathbb{Z}_{2n}$ (being a $\mathbb{Z}_2$-module via $\varphi$), is $\mathbb{Z}_2$ for any $\varphi$. Thus up to equivalence there are two extensions for each fixed $\varphi$.

A representative from each equivalence class can be given as:

- Split Case: $0 \to \mathbb{Z}_{2n} \to G \rightarrow \mathbb{Z}_2 \to 0$, the semi-direct product.

- Non-split Case: $0 \to \mathbb{Z}_{2n} \xrightarrow{\alpha} \mathbb{Z}_2 \rtimes \varphi \mathbb{Z}_{2P} \to \mathbb{Z}_2 \to 0$, here $\alpha(1)$ is multiplication by $-1$ on $\mathbb{Z}_Q$, $\alpha$ is induced by $\mathbb{Z}_{2n} \rightarrow \mathbb{Z}_Q \times \varphi \mathbb{Z}_{2P} \mathbb{Z}_2 = \mathbb{Z}_Q \times \varphi \mathbb{Z}_{2P} \subset \mathbb{Z}_Q \rtimes \varphi \mathbb{Z}_{2P}$, $\alpha(1) = (1, 2) \in \mathbb{Z}_Q \times \varphi \mathbb{Z}_{2P}$, $\beta$ is the projection on the quotient.

Now we investigate each case.

Split Case:

1) If $Q = 1$, then $\varphi$ is trivial and $G = \mathbb{Z}_{2n} \times \mathbb{Z}_2$. For $n \geq 2$ this is O.K. of Type 1, which is impossible. If $n = 1$ then $G$ is dihedral.

2) If $P = 1$, then the action of $\varphi$ on $\mathbb{Z}_{2n}$ is multiplication by $-1$, so $G$ is the dihedral group $D_{4n}$.

3) If neither $P$ or $Q$ is 1, then taking primes $p, q$ from their respective family we obtain again $(\mathbb{Z}_p \oplus \mathbb{Z}_q) \rtimes \varphi \mathbb{Z}_2 \subset G$ which is O.K. of Type 0, and hence this case cannot occur.

Non-split Case:

1) If $Q = 1$, $G \cong \mathbb{Z}_{4P}$, cyclic.

2) If $Q > 1$, then take one prime factor $q$ from its family. There is a canonical subgroup $\mathbb{Z}_q \rtimes \varphi \mathbb{Z}_{4} \subset \mathbb{Z}_Q \rtimes \varphi \mathbb{Z}_{4P} = G$. Since $P$ is odd, $\phi(1)$ acts on $\mathbb{Z}_q$ by $-1$. This is O.K. of Type 2, so cannot occur.

In conclusion, $G$ is cyclic or dihedral. \(\square\)
Proposition 5 Suppose there is a short exact sequence $0 \to \mathbb{Z}_{2^k} \to G \to \mathbb{Z}_2 \to 0$, $n$ odd, $k \geq 2$. Then $G$ is cyclic or dihedral.

Proof Again there is an induced action of $\mathbb{Z}_2$ on $\mathbb{Z}_{2^k}$, 

$$\varphi : \mathbb{Z}_2 \longrightarrow \text{Aut } \mathbb{Z}_{2^k} = \prod_i \text{Aut } \mathbb{Z}_{p_i^{n_i}} \times \text{Aut } \mathbb{Z}_{2^k}$$

where $n = 2^k p_1^{n_1} \ldots p_l^{n_l}$ is the prime decomposition of $n$. As before we rename the $p_i$’s as $p_i$ and $q_j$ according to the sign of $\varphi(1)$ on the corresponding components, and define $P = \prod p_i, Q = \prod q_j$.

Now for fixed $n, P, Q, k$, the possible $\varphi$’s are classified by their actions on $\mathbb{Z}_{2^k}$. Since $\text{Aut } \mathbb{Z}_{2^k} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{k-2}}$, there is no more than four elements of order $\leq 2$ (when $k = 2$ there are two). It is not hard to see the possibilities for $\varphi(1)$ are the following multiplications on $\mathbb{Z}_{2^k}$:

1) $+1$, in which case $H^2(\mathbb{Z}_2, \mathbb{Z}_{2^k}) = \mathbb{Z}_2$.
2) $-1$, in which case $H^2(\mathbb{Z}_2, \mathbb{Z}_{2^k}) = 0$.
3) $2^{k-1} + 1$, $(k > 2)$, in which case $H^2(\mathbb{Z}_2, \mathbb{Z}_{2^k}) = 0$.
4) $2^{k-1} - 1$, $(k > 2)$, in which case $H^2(\mathbb{Z}_2, \mathbb{Z}_{2^k}) = 0$.

Therefore the equivalence classes of extensions in each case are:

1) There are two equivalence classes:

Split Case: $0 \to \mathbb{Z}_{2^k} \to \mathbb{Z}_{2^k} \rtimes \varphi \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$, the semidirect product. Note that $G$ contains $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_{2^k} \times \mathbb{Z}_2$ since $\varphi$ is trivial on $\mathbb{Z}_{2^k}$. But this is O.K. of Type 1; impossible.

Non-split Case: $0 \to \mathbb{Z}_{2^k} \to (\mathbb{Z}_P \oplus \mathbb{Z}_Q) \rtimes \varphi \mathbb{Z}_{2^{k+1}} \to \mathbb{Z}_2 \to 0, \phi(1)$ is multiplication by +1, -1 on $\mathbb{Z}_P, \mathbb{Z}_Q$ respectively, and $\mathbb{Z}_{2^{k}} \cong \mathbb{Z}_P \times \mathbb{Z}_Q \times \mathbb{Z}_{2^k} \gets \mathbb{Z}_P \times \mathbb{Z}_Q \rtimes \varphi \mathbb{Z}_{2^{k+1}}$ canonically. This extension induces $\varphi$ and does not split. The group $G$ contains an O.K. of Type 2 $\mathbb{Z}_q \times \mathbb{Z}_{2^{k+1}}$ unless $Q = 1$. So the only possible $G$ here is $\mathbb{Z}_P \rtimes \varphi \mathbb{Z}_{2^{k+1}} \cong \mathbb{Z}_P \times \mathbb{Z}_{2^{k+1}}$, which is cyclic.

2) Again there are two equivalence classes:

Split Case: $0 \to \mathbb{Z}_{2^k} \to \mathbb{Z}_{2^k} \rtimes \varphi \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$. The group $G = (\mathbb{Z}_P \oplus \mathbb{Z}_Q \oplus \mathbb{Z}_{2^k}) \rtimes \varphi \mathbb{Z}_2$ contains $\mathbb{Z}_P \rtimes \mathbb{Z}_Q \times \varphi \mathbb{Z}_2 (\mathbb{Z}_P \oplus \mathbb{Z}_Q = \mathbb{Z}_P \oplus \mathbb{Z}_2 = \text{the fixed point set of the action by } \varphi)$ unless $P = 1$. But $(\mathbb{Z}_P \oplus \mathbb{Z}_2) \rtimes \varphi \mathbb{Z}_2 \cong \mathbb{Z}_{2^P} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_{2P} \times \mathbb{Z}_2$, i.e. O.K. of Type 1. So $P = 1$, and $G = (\mathbb{Z}_Q \oplus \mathbb{Z}_{2^k}) \rtimes \varphi \mathbb{Z}_2, \varphi(1)$ acts as multiplication by $-1$, so G is dihedral.

Non-split Case: $0 \to \mathbb{Z}_{2^k} \xrightarrow{\alpha} (\mathbb{Z}_P \oplus \mathbb{Z}_Q) \rtimes \varphi Q_{4m} \xrightarrow{\beta} \mathbb{Z}_2 \to 0$, where $m = 2^{k-1} \geq 2$. Here $\phi$ is defined as follows. Let $x = e^{\frac{i\pi}{m}}, y = j$ in $Q_{4m}$, define $\varphi(x) = \text{id}, \varphi(y)$ be multiplication by $+1, -1$ on $\mathbb{Z}_P, \mathbb{Z}_Q$ respectively (Checking relations of $Q_{4m}$, we see this is well-defined). The embedding $\alpha$ comes from $\mathbb{Z}_{2^k} \cong \mathbb{Z}_P \oplus \mathbb{Z}_Q \oplus \mathbb{Z}_{2^k} \hookrightarrow \mathbb{Z}_P \oplus \mathbb{Z}_Q \rtimes \varphi (x) \subset (\mathbb{Z}_P \oplus \mathbb{Z}_Q) \rtimes \varphi Q_{4m}$, and $\beta$ is projection on the quotient. It can be readily verified that this extension is non-split and realizes $\varphi$. Now $G$ contains $Q_{4m}$, which is O.K. of Type 3; impossible.

3) Since $H^2(\mathbb{Z}_2, \mathbb{Z}_{2^k}) = 0$, there is up to equivalence only one extension, the semi-direct product $(\mathbb{Z}_P \oplus \mathbb{Z}_Q \oplus \mathbb{Z}_{2^k}) \rtimes \varphi \mathbb{Z}_2$. This group contains O.K. of Type 4.

4) Again there is only the semi-direct product. Now $G = (\mathbb{Z}_P \oplus \mathbb{Z}_Q \oplus \mathbb{Z}_{2^k}) \rtimes \varphi \mathbb{Z}_2$ contains $\mathbb{Z}_{2^k} \rtimes \varphi \mathbb{Z}_2$. Checking the definition of $\varphi$, we see that this subgroup is precisely O.K. of Type 4; a contradiction.

In conclusion: $G$ has to be cyclic or dihedral. □
Theorem 3 Suppose there is an extension $0 \rightarrow \mathbb{Z}_n \rightarrow G \rightarrow \mathbb{Z}_p \rightarrow 0$, $n \geq 1$, $p$ prime. Then $G$ is cyclic or dihedral.

2.4.2 Extension of Dihedral Groups

We will prove B.1 and B.2 of the outline. We start with an extension by odd primes.

Proposition 6 Suppose there is an extension $0 \rightarrow D_4 \rightarrow G \rightarrow \mathbb{Z}_p \rightarrow 0$, $p$ odd prime. Then $p = 3$ and $G \cong A_4$.

Proof Since $(4, p) = 1$, the sequence splits and $G = D_4 \rtimes \varphi \mathbb{Z}_p$ for some $\varphi : \mathbb{Z}_p \rightarrow \text{Aut} D_4 \cong S_3$ (here $S_3$ is the permutation group of the nontrivial elements of $D_4$).

If $p > 3$, then $\varphi$ is trivial and $G = D_4 \times \mathbb{Z}_p \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which is an O.K. of Type 1.

Thus $p = 3$. We observe that $\varphi$ cannot be trivial, for in that case we would have again $G = D_4 \times \mathbb{Z}_3 \cong \mathbb{Z}_6 \times \mathbb{Z}_2$; impossible. Thus $\varphi(1)$ is a 3-cycle in $\text{Aut} D_4 = S_3$. This $G = D_4 \rtimes \varphi \mathbb{Z}_p$ is precisely $A_4$. \qed

Proposition 7 Suppose there is an extension $0 \rightarrow D_{2n} \rightarrow G \rightarrow \mathbb{Z}_p \rightarrow 0$, $n > 2$, $p$ odd prime. Then $G$ is dihedral.

Proof If $p \nmid n$, then the sequence splits and $G = D_{2n} \rtimes \varphi \mathbb{Z}_p$. There is a subgroup $\mathbb{Z}_n \rtimes \varphi \mathbb{Z}_p \subset G$ (because any automorphism of $D_{2n}$ preserves the $\mathbb{Z}_n$ in $D_{2n}$). But $\mathbb{Z}_n \rtimes \varphi \mathbb{Z}_p \cong \mathbb{Z}_{np}$ since it is an extension of $\mathbb{Z}_n$ by $\mathbb{Z}_p$ and we have proven this in Proposition 2. Consequently $G$ is an extension of this $\mathbb{Z}_{np}$ by $\mathbb{Z}_2$ and thus has to be dihedral by Theorem 3.

If $p | n$, let $n = p^k \ell$. Then $G$ is a subgroup $\mathbb{Z}_{p^k} \subset \mathbb{Z}_n \subset D_{2n} \subset G$. Let the standard generators of $D_{2n}$ be $a, b$ as mentioned before. Let $P$ be a $p$-Sylow subgroup containing $\mathbb{Z}_{p^k}$. Take any $x \in P - D_{2n}$. Suppose $x^{-1}ax \notin \mathbb{Z}_n$, then $x^{-1}ax = a^kb$ and $(a^kb)^l = x^{-1}a^lx = a^l$, a contradiction. Thus $x^{-1}ax \in \mathbb{Z}_n$, which implies $N_G(\mathbb{Z}_n) = G$, which implies $\mathbb{Z}_n \lhd G$. Let $Q = \langle \mathbb{Z}_n \cup P \rangle$. Now $P \not\subset D_{2n}$, and $[G : \mathbb{Z}_n] = 2p$, and hence $[G : Q] \leq 2$. But $G = Q$ implies $D_{2n} \subset P$; impossible. Thus $[G : Q] = 2$. Now $\mathbb{Z}_n \lhd Q$, $[Q : \mathbb{Z}_n] = p$, i.e. $Q$ is $\mathbb{Z}_n$ extended by $\mathbb{Z}_p$. By Proposition 2 it has to be cyclic. As a result, $G$ is dihedral by Theorem 3 (it is an extension of $Q$ by $\mathbb{Z}_2$). \qed

Next we turn to extension by $\mathbb{Z}_2$. The discussion will be divided into two cases depending whether $n$ is odd or even.

Proposition 8 Suppose there is an extension $0 \rightarrow D_{2n} \rightarrow G \rightarrow \mathbb{Z}_2 \rightarrow 0$, $n \geq 2$ odd. Then $G$ is dihedral.

Proof Let $n = p_1^{n_1} \ldots p_k^{n_k}$ be its prime decomposition.

Since $\text{Aut} D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$ and the inner automorphism $\text{Inn} D_{2n} \cong \mathbb{Z}_n \times \{\pm 1\}$, as a result $\text{Out} D_{2n} = \text{Aut} D_{2n} / \text{Inn} D_{2n} \cong \mathbb{Z}_n^*/\{\pm 1\}$.

Since $\text{center}$ of $D_{2n}$, denoted by $C$, is trivial, then $H^2(\mathbb{Z}_2, C) = 0$, $H^2(\mathbb{Z}_2, C') = 0$. In other words, each abstract kernel $\psi : \mathbb{Z}_2 \rightarrow \text{Out} D_{2n}$ has a unique extension up to equivalence. We will construct explicitly an extension for each given kernel.
If $(2 \ast G)$ is dihedral, however, it cannot exist, since $G$ contains a cyclic subgroup of index 2. Denote by $[\langle (a_1, \ldots, a_k) \rangle]$ the element corresponding to $\psi(1)$ in the middle group, and by $[\langle b_1, \ldots, b_k \rangle]$ the element corresponding to $\psi(1)$ in the bottom group, where $b_i \in \mathbb{Z}^*_{p_i}$, $a_i \in \mathbb{Z}_{(p_i-1)p_i}^*$. Since $\psi(1)$ has order 2, $(2a_1, \ldots, 2a_k) = 0$ or $(2a_1, \ldots, 2a_k) = (\frac{p_i-1}{2}p_i^{n_1-1}, \ldots, \frac{p_k-1}{2}p_k^{n_k-1})$. Let $\phi : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2^* \times \mathbb{Z}_{p_k}^* = \mathbb{Z}_2^* \subset \mathbb{Z}_n \times \mathbb{Z}_n^* \cong \text{Aut } D_{2n}, \psi(1) = (b_1, \ldots, b_k)$.

This is a well-defined homomorphism, and $0 \rightarrow D_{2n} \rightarrow D_{2n} \times G \rightarrow \mathbb{Z}_2 \rightarrow 0$ induces the abstract kernel $\psi$. Hence $G = D_{2n} \times G \mathbb{Z}_2$.

Recall that $a, b$ denote standard generators of $D_{2n}$. The action $\varphi$ is trivial on $\langle b \rangle$, whence $\langle b \rangle \times \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Now $\mathbb{Z}_n \triangleleft G$, and $G$ is easily seen to be the inner semi-direct product of this copy of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with $\mathbb{Z}_n$. To be precise (checking induced inner automorphisms), $G \cong \mathbb{Z}_n \times \varphi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ where $\varphi(1,0)$ is multiplication by $(-1)$, $\varphi(0,1)$ is determined by $(b_1, \ldots, b_k) \in \mathbb{Z}_2^* \times \mathbb{Z}_{p_k}^* \cong \mathbb{Z}_n$.

Now $b_i = \pm 1$, and there are three subcases:

i) If both $\pm 1$ exist among them, take the semi-direct product of $\mathbb{Z}_{n_i}$ with the second $\mathbb{Z}_{2}$ component, we see there is an O.K. of Type 0 within as before. This case is thus excluded.

ii) If $b_i = +1$ for all $i$, take again the semi-direct product with the second $\mathbb{Z}_2$, $\mathbb{Z}_{2n} \cong \mathbb{Z}_n \times \mathbb{Z}_2 \subset \mathbb{Z}_n \times \varphi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ because the action restricts to trivial action on $\mathbb{Z}_n$ and $n$ is odd. Since $G$ contains a cyclic subgroup of index 2, $G$ has to be dihedral by Theorem 3.

iii) If $b_i = -1$ for all $i$, then $\langle (1,1) \rangle \subset \mathbb{Z}_n \times \varphi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$. Now that $\mathbb{Z}_n \times \langle (1,1) \rangle \cong \mathbb{Z}_{2n}$ since the action is trivial. And conjugation of either copy of $\mathbb{Z}_2$ on $\mathbb{Z}_{2n}$ is multiplication by $-1$. Thus $G$ is dihedral.

This concludes Case 1.

**Case 2:** If $(2a_1, \ldots, 2a_k) = (\frac{p_i-1}{2}p_i^{n_1-1}, \ldots, \frac{p_k-1}{2}p_k^{n_k-1})$, all $p_i \equiv 1 (\text{mod } 4)$ since $2|\frac{p_i-1}{2}$, $b_i = m_i$ where $m_i^2 = -1$ in $\mathbb{Z}_{p_i}^*$. Let $G = \mathbb{Z}_n \times \varphi \mathbb{Z}_4$, where $\varphi : \mathbb{Z}_4 \rightarrow \mathbb{Z}_4^* = \prod_i \mathbb{Z}_{p_i}^*$ sends 1 to $\Pi m_i$.

Restricting $\varphi$ to $\mathbb{Z}_2 \subset \mathbb{Z}_4$, the resulted semi-direct product is precisely $D_{2n}$. The extension

$$0 \rightarrow D_{2n} = \mathbb{Z}_n \times \varphi \mathbb{Z}_2 \rightarrow \mathbb{Z}_n \times \varphi \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0$$

induces the abstract kernel $\psi$, and up to equivalence it is the only one. This extension however can not exist, since $G$ contains $\mathbb{Z}_{p_i} \times \varphi \mathbb{Z}_4$ where $\varphi(1) = m_i \in \mathbb{Z}_{p_i}^*, m_i^2 = -1$, an O.K. of Type 5.

This concludes Case 2.

In conclusion, $G$ has to be dihedral.

\[\square\]

**Proposition 9** Suppose there is an extension $0 \rightarrow D_{2n} \rightarrow G \rightarrow \mathbb{Z}_2 \rightarrow 0$, $n$ even. Then $G$ is dihedral.
Proof If \( n = 2 \), then \( G \) is a 2-group and hence must be dihedral by Theorem 2.

Let \( n > 2 \) and let \( n = 2^i p_1^{k_1} \ldots p_k^{k_k} \) be the prime decomposition. It turns out that the sequence splits. To see this, take a Sylow-2 subgroup of \( D_{2n} \). It has to be of the form \( D_{2^{i+1}} \). Let \( P \) be a Sylow-2 subgroup of \( G \) containing \( D_{2^{i+1}} \), \( [P : D_{2^{i+1}}] = 2 \). By Theorem 2, \( P \) has to be dihedral, in particular, there exist \( c \in P - D_{2^{i+1}} \) of order 2. Note that \( c \not\in D_{2n} \), since otherwise \( P \subset D_{2n} \) which is impossible. Now the map \( \mathbb{Z}_2 \to G \) sending 1 to \( c \) is a splitting.

As a consequence \( G \cong D_{2n} \rtimes \varphi \mathbb{Z}_2 \) for some \( \varphi : \mathbb{Z}_2 \to \text{Aut} D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_n^* \). Let \( \varphi(1) = (t, s) \).

There is a canonical subgroup \( \mathbb{Z}_n \rtimes \varphi \mathbb{Z}_2 \subset D_{2n} \rtimes \varphi \mathbb{Z}_2 \). Since \( \mathbb{Z}_n \rtimes \varphi \mathbb{Z}_2 \) is an extension of \( \mathbb{Z}_n \) by \( \mathbb{Z}_2 \) then it is cyclic or dihedral by previous results (Theorem 3). We discuss these two cases separately:

i)Suppose \( \mathbb{Z}_n \rtimes \varphi \mathbb{Z}_2 \) is cyclic. Then \( G \) is an extension of a cyclic group by \( \mathbb{Z}_2 \), so \( G \) is dihedral.

ii)Suppose \( \mathbb{Z}_n \rtimes \varphi \mathbb{Z}_2 \) is dihedral. Then \( \varphi(1) \in \text{Aut} \mathbb{Z}_n \) is multiplication by \(-1\), and whence \( s = -1 \) (see remark previous to O.K. of Type 6). Now we divide the discussion by the parity of \( t \):

1)If \( t \) is odd, take \( i = \frac{t+1}{2} \). Then \( 2i - t = 1 \) which implies \((a^i b, 1)^2 = (a^{2i-t}, 0) = (a, 0)\); thus \((a^i b, 1)\) is of order \( 2n \). Thus \( G \) contains a cyclic subgroup of order \( 2n \). It is not hard to see that \( G \) is the semidirect product of this cyclic group with \((b, 0)\). An easy computation shows that \((b, 0)(a^i b, 1)(b, 0) = (a, b)^{-1}\), whence \( G \) is dihedral.

2)If \( t \) is even, then \( G \) is O.K. of Type 6; impossible.

In conclusion, \( G \) is dihedral. \( \square \)

Summing up the results in this subsection, we have:

**Theorem 4** Suppose there is an extension \( 0 \to D_{2n} \to G \to \mathbb{Z}_p \to 0 \), \( n \geq 2, p \) prime. Then \( G \) is dihedral or \( A_4 \).

### 2.4.3 Extension of \( A_4 \)

There is an isomorphism \( \text{Aut} A_4 \cong S_4 \), induced by the conjugation of elements of \( S_4 \) on the normal subgroup \( A_4 \). Let \( D_4 \) denote the unique order 4 subgroup of \( A_4 \). \( D_4 \triangleleft A_4 \), and any automorphism of \( A_4 \) preserves \( D_4 \). Now we are ready the state and prove:

**Theorem 5** Suppose there is an extension \( 0 \to A_4 \to G \to \mathbb{Z}_p \to 0 \), \( p \) prime. Then \( p = 2 \) and \( G \cong S_4 \)

**Proof** The conjugation of any element of \( G \) restricts to an automorphism of \( A_4 \). By the discussion above, such an automorphism has to preserve \( D_4 \), so \( D_4 \triangleleft G \).

i)If \( p > 3 \), \((12, p) = 1\) then the sequence splits, so \( G = A_4 \rtimes \varphi \mathbb{Z}_p \) for some \( \varphi \). In fact \( \varphi \) has to be trivial since \( |\text{Aut} A_4| = |S_4| = 24 \) and \( p > 3 \). Now \( G \) contains \( D_4 \rtimes \varphi \mathbb{Z}_p = D_4 \times \mathbb{Z}_p \cong \mathbb{Z}_{2p} \rtimes \mathbb{Z}_2 \). This is O.K. of Type 1; impossible.

ii)If \( p = 3 \), let \( P \) be the Sylow-3 subgroup. By Corollary 2, \( P \cong \mathbb{Z}_{3^n} \) for some \( n \). Since \( |D_4| = 4, |P| = 9 \), \( D_4 \triangleleft G \), we have \( G \cong D_4 \rtimes \varphi \mathbb{Z}_9 \) for some \( \varphi \). \( \varphi(1)^9 = \varphi(9) = \text{id} \in \text{Aut} D_4 \cong S_3 \). \( |S_3| = 6 \) implies \( \varphi(1)^3 = \text{id} \). Consequently there is a subgroup \( D_4 \times \mathbb{Z}_3 \subset D_4 \rtimes \varphi \mathbb{Z}_9 \). Now \( D_4 \times \mathbb{Z}_3 \cong \mathbb{Z}_6 \times \mathbb{Z}_2 \), an O.K. of Type 1.

iii)If \( p = 2 \), define \( P \) as the Sylow-2 subgroup of \( G \) containing \( D_4 \). The group \( P \) is dihedral by Theorem 2, thus there is \( c \in P - D_4 \) with \( c^3 = \text{id} \). Obviously \( c \not\in A_4 \), so we obtain a
splitting. Therefore \( G \cong A_4 \rtimes \varphi \mathbb{Z}_2 \) for some \( \varphi \), where \( \varphi(1) \in \text{Aut} A_4 \cong S_4 \) is of order 2. We consider two cases:

**Case 1:** If \( \varphi(1) \in A_4 \subset S_4 \), then \( \varphi(1) \in D_4 \) implies \( \varphi(1)|_{D_4} = \text{id} \) (\( \varphi(1) \) is conjugation and \( D_4 \) is abelian). This however implies that the subgroup \( D_4 \rtimes \varphi \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). This is impossible by Theorem 2, so this case is excluded.

**Case 2:** If \( \varphi(1) \notin A_4 \). Consider the canonical \( 0 \to A_4 \to S_4 \to \mathbb{Z}_2 \to 0 \). \( \varphi : \mathbb{Z}_2 \to S_4 \) is a splitting. Thus \( S_4 \cong A_4 \rtimes \varphi \mathbb{Z}_2 \cong G \).

This proves the theorem. \( \square \)

### 2.4.4 Extension of \( S_4 \)

**Theorem 6** There is no extension of \( S_4 \)

**Proof** Suppose there is an extension \( 0 \to S_4 \to G \to \mathbb{Z}_p \to 0 \), \( p \) prime. Let \( C \) be the center of \( S_4 \), \( C = \{0\} \). Now Aut \( S_4 \) = Inn \( S_4 \) \( \cong S_4 \), the second isomorphism being the obvious one. Thus Out \( S_4 \) is trivial and there exist only one abstract kernel: the trivial homomorphism \( \mathbb{Z}_p \to \text{Out} S_4 \). The product extension \( 0 \to S_4 \to S_4 \times \mathbb{Z}_p \to \mathbb{Z}_p \to 0 \), and the fact \( H^2(\mathbb{Z}_p, C) = 0 \) imply that this extension is the only one up to equivalence. Thus \( G = S_4 \times \mathbb{Z}_p \).

If \( p = 2 \), then \( G \) contains \( D_4 \times \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \); impossible by Theorem 2.

If \( p > 2 \), then \( G \) contains \( D_4 \times \mathbb{Z}_p \cong \mathbb{Z}_{2p} \oplus \mathbb{Z}_2 \), which is O.K. of Type 1; impossible. \( \square \)

### 2.4.5 Conclusion

Summing up the previous subsections and using an inductive argument, we obtain:

**Corollary 3** If \( G \) is solvable, then \( G \subset SO(3) \).

### 2.5 Arbitrary Finite Group

Now we can prove Theorem 1:

**Proof** Our first observation is that by previous discussions (Theorem 2 and Corollary 2), the Sylow-\( p \) subgroups of \( G \) are cyclic for odd \( p \), and cyclic or dihedral for \( p = 2 \). If the Sylow-2 subgroups are cyclic, then \( G \) is solvable (cf. \([17]\) p.143), and Corollary 3 takes care of this situation. Thus it suffices to assume that Sylow-2 subgroups of \( G \) are dihedral.

By the result of Suzuki (cf. \([28]\)), there exist a subgroup \( G_1 \) of \( G \) having the following properties: \( G_1 \subset G \), \( [G : G_1] \leq 2 \), \( G_1 \cong Z \times L \) where \( Z \) is a solvable group and \( L = PSL(2, p) \) the projective linear group for prime \( p \).

i)If \( p = 2 \) or 3, then \( L \cong S_3 \) or \( L \cong A_4 \), in either case \( G_1 \) is solvable, thus so is \( G \). Corollary 3 gives the desired result.

ii)If \( p = 5 \), then \( L = A_5 \). \( Z \) has to be trivial. This is because otherwise it contains a copy of \( \mathbb{Z}_q \), \( q \) prime. This \( \mathbb{Z}_q \) together with \( D_4 \subset A_5 \) produce \( \mathbb{Z}_q \times D_4 \subset Z \times L = G_1 \). But \( \mathbb{Z}_q \times D_4 \cong \mathbb{Z}_{2q} \oplus \mathbb{Z}_2 \), an O.K. of Type 1. So \( Z = 1 \), \( G_1 \cong A_5 \).

Suppose \( [G : G_1] = 2 \). We have an extension \( 0 \to A_5 \to G \to \mathbb{Z}_2 \to 0 \). Now Aut \( A_5 = S_5 \), and Inn \( A_5 = A_5 \). Hence there are only two possible abstract kernels \( \mathbb{Z}_2 \to \text{Out} A_5 \cong \mathbb{Z}_2 \). Since \( A_5 \) is centerless (\( C = \{0\} \)), then \( H^2(\mathbb{Z}_2, C) = 0 \) in either case. In
other words, there are only two extensions up to isomorphism. Now \( G \cong S_5 \) and \( G \cong A_5 \times \mathbb{Z}_2 \) (together with obvious short exact sequences) are nonequivalent extensions giving all possibilities.

**Case 1:** \( G \cong S_5 \). In this case \( G \) contains a general affine group \( GA(1, 5) \) (e.g. the subgroup \( \langle (12345), (2354) \rangle \) of \( S_5 \)). Note that \( GA(1, 5) \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_5^* \), where the semidirect product is the canonical one. This subgroup is solvable. By Corollary 3, \( \mathbb{Z}_5 \rtimes \mathbb{Z}_5^* \) is isomorphic to either a cyclic/dihedral group, \( A_4 \) or \( S_4 \). This is however impossible (for the dihedral case it is convenient to use the fact \( \mathbb{Z}_5^* \cong \mathbb{Z}_4 \)).

**Case 2:** \( G \cong A_5 \times \mathbb{Z}_2 \). Then \( G \) contains \( D_4 \times \mathbb{Z}_2 \). As we have seen before, this is impossible.

Thus the assumption \( [G : G_1] = 2 \) lead to contradiction, \( G = G_1 \cong A_5 \subset SO(3) \).

This finishes the case \( p = 5 \).

ii)If \( p > 5 \), then \( L \) contains (as the normalizer of a Sylow-\( p \) subgroup) a group \( \mathbb{Z}_p \rtimes_\varphi (\mathbb{Z}_p^*/\{\pm 1\}) \) where \( \varphi([a]) \) is the multiplication by \( a^2 \) for \( a \in \mathbb{Z}_p^* \). Let \( b \) be a generator of the cyclic group \( \mathbb{Z}_p \). Then \( \mathbb{Z}_p \rtimes_\varphi (\mathbb{Z}_p^*/\{\pm 1\}) \cong \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_{p-1} \) where \( \varphi(1) \) is multiplication by \( b^2 \).

This group is solvable and contains \( \mathbb{Z}_p, p > 5 \), so it is cyclic or dihedral (Corollary 3). The element \( (0, 1) \in \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_{p-1} \) has order \( p-\frac{1}{2} > 2 \), thus the conjugation of \( (0, 1) \) in \( \mathbb{Z}_p \) has to be trivial in either case (in the dihedral case \( \mathbb{Z}_p \) is in the unique cyclic subgroup of index 2, so does \( (0, 1) \)). Thus \( b^2 = 1 \in \mathbb{Z}_p \) which implies \( |\mathbb{Z}_p^*| = 1 \) or 2, consequently \( p = 2 \) or 3; a contradiction.

In conclusion \( G \subset SO(3) \). \( \Box \)

### 3 Remarks

This short section contains the following two results (cf. [16], [21]):

**Remark 3.1:** Let \( G \) be a finite group acting (orientation preservingly), locally linearly or smoothly on \( \mathbb{R}^4 \). Then \( G \) is isomorphic to a subgroup of \( O_4 \) (\( SO(4) \)).

**Remark 3.2:** There are finite groups \( G \) which act topologically and orientation preservingly on \( \mathbb{R}^4 \) and \( G \nsubseteq SO(4) \) (in fact \( G \nsubseteq O(7) \)).

These two remarks are included for the completeness reasons. Our proof of Remark 3.1 is quite different from the one given in [16]. It is much shorter and is very much in the spirit of considerations in our proof of Theorem 1.

**Proof of 3.1:** Our first observation is the following:

**FACT:** The only finite simple group which acts orientation preservingly on \( \mathbb{R}^4 \) is \( A_5 \).

The above fact follows from the direct inspection of all finite non-abelian simple groups, in the atlas of Finite Groups (cf. [7]). The point here is that each such group except \( A_5 \) has a solvable subgroup too large to act effectively on \( \mathbb{R}^4 \). (We recall that each solvable group acting on \( \mathbb{R}^4 \) always has a fixed point). For example, in \( A_6 \) one can take the normalizers of Sylow-3 subgroups.

Suppose now that \( G \) is NOT simple. Let \( H \neq \{0\} \) be a maximal normal proper subgroup of \( G \) (i.e. \( G/H \) is simple).

**Case 1:** \( H \) is a non-abelian simple group (hence \( H \cong A_5 \)). Then we have an extension

\[
0 \to H \to G \to G/H \to 0
\]  \hspace{1cm} (1)
In order to classify such extensions (cf. [5] p.105), let $\psi : G/H \to \text{Out}(A_5) \cong \mathbb{Z}_2$ be a homomorphism. Then $\psi : G/H \to \mathbb{Z}_2$ is trivial except when $G/H \cong \mathbb{Z}_2$. (note that $G/H$ is a simple group).

The set of extensions (1) with fixed $\psi$ is classified by $H^2(G/H; \mathbb{Z}(H))$, where $\mathbb{Z}(H)$ is the center of $H$ (cf. [5] p.105). Consequently there is only one extension $G \cong H \times G/H$ for $G/H \neq \mathbb{Z}_2$ and two extensions $G \cong H \times \mathbb{Z}_2$ and $G \cong S_5$ for $G/H \cong \mathbb{Z}_2$.

It follows that both $G \cong A_5 \times \mathbb{Z}_2$ and $S_5$ are subgroups of $SO(4)$ and it is not difficult to see that $G \cong A_5 \times G/H$, $G/H \neq \mathbb{Z}_2$ cannot act on $\mathbb{R}^4$.

Case 2: $H$ is simple abelian. In this case $(\mathbb{R}^4)^G = ((\mathbb{R}^4)^H)^{G/H} = \{\text{pt}\}$. Consequently $G \subset SO(4)$.

Case 3: $H$ is not simple. Repeating the argument from Case 1 and Case 2 with $H$ replacing $G$ one easily concludes $G \subset SO(4)$.

Case 4: Suppose $G$ has an orientation reversing element. Let $K \triangleleft G$ be the normal subgroup of orientation preserving elements, so that $G/K \cong \mathbb{Z}_2$. Then either $(\mathbb{R}^4)^K \neq \emptyset$ and hence $(\mathbb{R}^4)^G \neq \emptyset$ and consequently $G \subset O(4)$ or $K \cong A_5$ and hence we have an extension

$$0 \longrightarrow K \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \quad (2)$$

This however was handled in Case 1 and hence the proof of Remark 3.1 is concluded.

**Proof of 3.2:** Let $Q(8p,q)$ be the generalized quaternionic group (cf. [8]), given by the extension

$$0 \longrightarrow \mathbb{Z}_p \times \mathbb{Z}_q \longrightarrow Q(8p,q) \longrightarrow Q(8) \longrightarrow 0$$

where $Q(8)$ is the standard quaternionic group of order eight.

It turns out that $\pi = Q(8p,q)$ is a 4-periodic group and hence acts freely on a simply connected CW complex $\tilde{X}$ with $\tilde{X} \cong S^3$. (cf. [8]).

There are conditions on $p,q$ (cf. [8]) (for example $(p,q) = (3,313),(3,433),(7,113),(5,461), \ldots$) which imply that $\pi$ acts freely on a closed 3-manifold $\mathcal{M}^3$ which is a homology 3-sphere. Moreover there is a $\mathbb{Z}[\pi]$-homology equivalence

$$k : \mathcal{M}^3 \longrightarrow X = \tilde{X}/\pi$$

Consider the map $h = k \times \text{id} : \mathcal{M}^3 \times I \longrightarrow X \times I$ and let $\lambda(h) \in L^h_0(\pi)$ be the surgery obstruction for changing $h$ to a homotopy equivalence without modifying anything on the boundaries.

Now let $\mathcal{F} : \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]$ be the identity homomorphism and $\Gamma_0(\mathcal{F})$ be the Cappell-Shaneson homological surgery obstruction group as in [6].

The natural homomorphism $j_* : L^h_0(\pi) \to \Gamma_0(\mathcal{F})$ is an isomorphism (see [6] p.288) and clearly $j_*(\lambda(h)) = 0$ so that $\lambda(h) = 0$ in $L^h_0(\pi)$.

Let $h : (\mathcal{W}^4, \mathcal{M}^3, \mathcal{M}^3) \longrightarrow (X \times I; X, X)$ be a homotopy equivalence. Form a two ended open manifold

$$\mathcal{W}^4_0 := \ldots \cup \mathcal{W}^4 \cup \mathcal{W}^4 \cup \mathcal{W}^4 \cup \ldots$$

by stacking together copies of $\mathcal{W}^4$.

Observe that $\pi_1(\mathcal{W}^4) \cong \pi$ and the universal cover $\tilde{\mathcal{W}}^4_0$ of $\mathcal{W}^4_0$ is a manifold properly homotopy equivalent to $S^3 \times \mathbb{R}$ and hence homeomorphic to $S^3 \times \mathbb{R}$ by [13].

One point compactification of one end of $\mathcal{W}^4_0$ yields an action of $\pi$ on $\mathbb{R}^4$ with one fixed point. Since $\pi$ is not isomorphic to a subgroup of $O(7)$ (cf. [1]) the proof of 3.2 is complete.
Acknowledgments.: We would like to thank Prof. Reinhard Schultz for turning our attention to the problem studied in this paper and suggesting a possible line of attack.

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