Aharonov-Bohm Effect, Dirac Monopole, and Bundle Theory

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Abstract We discuss the Aharonov-Bohm (A − B) effect and the Dirac (D) monopole of magnetic charge $g = \frac{1}{2}$ in the context of bundle theory, which allows to exhibit a deep geometric relation between them. If $\xi_{A-B}$ and $\xi_D$ are the respective $U(1)$-bundles, we show that $\xi_{A-B}$ is isomorphic to the pull-back of $\xi_D$ induced by the inclusion of the corresponding base spaces. The fact that the A − B effect disappears when the magnetic flux in the solenoid equals an integer number of times the quantum of flux associated with the unit of electric charge, reflects here as a consequence of the pull-back of the Dirac connection in $\xi_D$ to $\xi_{A-B}$, and the Dirac quantization condition.

Keywords: Aharonov-Bohm effect, magnetic monopole, fiber bundles

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1 Introduction

As is well known, the Aharonov-Bohm (A − B) effect [1] and the Dirac (D) magnetic monopole [2],[3] proposal have had a profound influence on the development of the gauge theories of fundamental interactions. The first one of these phenomena was immediately verified experimentally [4] and by many others later on [5], while even if Dirac monopoles have not yet being seen in Nature, both grand unified theories [6] and string theories [7] predict their existence.

The description of both the A − B effect and the D monopole are deeply rooted in the concept of gauge potential and therefore in the concept of connection in fiber bundles. The first one provides an explicit evidence of the non-local character of quantum mechanics describing the motion of electrically charged particles in a non-simply connected space [8],[9], while the second one makes unavoidable the use of at least two charts on manifolds to define the gauge potential, leading to the necessity of a description in terms of a non-trivial bundle [10].

The close relationship between both phenomena consists in the facts that when the magnetic flux $\Phi_{A-B}$ is an integer multiple of the quantum of flux $\Phi_0 = \frac{2\pi}{|e|}$ associated with the electric charge $|e|$, the A − B effect vanishes, and when $\Phi_{A-B}$ also equals the magnetic flux of the monopole, $\Phi_D$, the Dirac quantization condition $(D.Q.C.)$ follows. In this note we want to emphasize this relation at a perhaps deeper level, namely through the relationship between the fiber bundles $\xi_{A-B}$ (trivial) and $\xi_D$ (non-trivial) in which both phenomena occur. After some basic material in section 2., in section 3., we exhibit the bundle morphism $\xi_{A-B} \rightarrow \xi_D$ induced by the inclusion $\iota$ between the corresponding base spaces, and in section 4., we use $\iota$ to construct the pull-back bundle $\iota^*(\xi_D)$, which in turn is proved, in section 5., to be isomorphic to $\xi_{A-B}$ i.e.

$$\xi_{A-B} \cong \iota^*(\xi_D). \quad (1)$$

This is the main result of the present paper, since it exhibits a deep geometric relation between the A − B effect and the magnetic monopole. Of course, the pull-back of the first Chern class $c_1$ of $\xi_D$, $\iota^*(c_1)$, vanishes in $\xi_{A-B}$, what is proved in section 6. In section 7., we show that the pull-back of the Dirac connection from $\xi_D$ to $\xi_{A-B}$ leads to the vanishing of the A − B effect when the $D.Q.C.$ holds, thus setting on purely geometric grounds, one of the basic relations between A − B and D. Section 8 is devoted to final comments.

We use the natural system of units $\hbar = c = 1$. 

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2 Basics

In Ref. [8], the $U(1)$-bundle associated with the $A - B$ effect [1] with an infinitesimally thin and infinitely long solenoid was shown to be the product-and therefore trivial- bundle

$$\xi_{A-B} : S^1 \to (T_0^2)^* \xrightarrow{pr_1} (D_0^2)^*$$

(2)

where $S^1 = U(1) = \{ z \in \mathbb{C}, |z| = 1 \}$ is the structure group, $(D_0^2)^*$ is the punctured open disk in two dimensions, $(T_0^2)^* = (D_0^2)^* \times S^1$ is the open solid 2-torus minus a circle, and $pr_1$ is the projection in the first entry. One has the homeomorphisms $(D_0^2)^* \cong (\mathbb{R}^2)^* = \mathbb{R}^2 \setminus \{0\} \cong \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The reason for (2) is that, in the above conditions, by symmetry reasons the space available to the electrically charged particles (“electrons”) moving around the solenoid is $(\mathbb{R}^2)^*$ which is of the same homotopy type as the circle $S^1$. Then the set of isomorphism classes of $U(1)$-bundles over $(\mathbb{R}^2)^*$ consists of only one element [11]: the class of the product (trivial) bundle $(T_0^2)^*$.

On the other hand, the fiber bundles associated with Dirac monopoles [2],[3] of magnetic charge $g = \# k$ with $k$ an integer and $\#$ a number depending on units, are the Hopf bundles [10],[12]

$$\xi_D^{(k)} : S^1 \to P^3_k \xrightarrow{\tilde{\pi}_k} S^2$$

(3)

where $P^3_0 = S^2 \times S^1$ (the trivial bundle), $P^3_k \cong P^3_{-k}$, $S^2$ is the 2-sphere with $S^2 \cong \mathbb{R}^2 \cup \{\infty\} \cong \mathbb{C} \cup \{\infty\}$. In particular, we are interested in the case $k = 1$ for which $P^3_1 \cong S^3$: the 3-sphere given by

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2, |z_1|^2 + |z_2|^2 = 1\},$$

(4)

$\pi_3 \equiv \pi$ is the Hopf map [13]

$$\pi : S^3 \to S^2, (z_1, z_2) \mapsto \pi(z_1, z_2) = \begin{cases} z_1/z_2, & z_2 \neq 0 \\ \infty, & z_2 = 0 \end{cases}.$$  

(5)

We denote this non-trivial bundle $\xi_D$:

$$\xi_D^{(1)} \equiv \xi_D : S^1 \to S^3 \xrightarrow{\pi} S^2.$$  

(6)

The global connection on $\xi_D$ corresponding to $g = \frac{1}{2}$ ($# = \frac{1}{2}$ and $k = 1$) is the 1-form $\omega \in \Omega^1 S^3 \otimes u(1)$, with $u(1) = \text{Lie}(U(1)) = i\mathbb{R}$, given by [14]

$$\omega = \frac{i}{2}(d\chi + \cos \theta d\varphi),$$

(7)

where $\chi$, $\theta$ and $\varphi$ are the Euler angles in $S^2$ or $\mathbb{R}^3$ ($\theta \in [0, \pi]$ and $\chi, \varphi \in [0, 2\pi]$). The differential of $\omega$ is the 2-form

$$d\omega = \frac{i}{2} \sin \theta d\theta \wedge d\varphi = -F \in \Omega^2 S^3 \otimes u(1)$$

(8)

where $F$ is the field strength

$$F = i[B] = i[B](\sin \theta d\theta \wedge d\varphi)$$

(9)

with

$$B = (\frac{1}{2}, \frac{r}{r^3})$$

(10)

the magnetic field of the monopole in $\mathbb{R}^3 \setminus \{0\}$ (see below).

$\omega$ can be read from the squared length element on $S^3$:

$$dl_{S^3}^2(\chi, \theta, \varphi) = \frac{1}{4}(d\theta^2 + \sin^2 \theta d\varphi^2 + (d\chi + \cos \theta d\varphi)^2)$$

(11)

which, in turn, can be obtained from the identification of $S^3$ with the group $SU(2)$ with elements

$$\begin{pmatrix} z_1 & z_2 \\ \bar{z}_2 & \bar{z}_1 \end{pmatrix} = \begin{pmatrix} e^{\frac{i}{2}(\varphi + \chi)} \cos \frac{\theta}{2} & e^{\frac{i}{2}(\varphi - \chi)} \sin \frac{\theta}{2} \\ -e^{-\frac{i}{2}(\varphi - \chi)} \sin \frac{\theta}{2} & e^{-\frac{i}{2}(\varphi + \chi)} \cos \frac{\theta}{2} \end{pmatrix}.$$ 

(12)
Covering $S^2$ with the open sets $U_+$ and $U_-$ respectively defined by $\theta \in [0, \pi)$ (the south pole $S$ excluded) and $\theta \in (0, \pi]$ (the north pole $N$ excluded), considering the pull-back of $\omega$ to $S^2 \setminus \{N, S\}$ with the local sections
\[
s_N : U_+ \setminus \{N\} \rightarrow S^3, \quad s_N(\hat{n}) = (\cos \frac{\theta}{2}, \sin \frac{\theta}{2} e^{i\varphi}), \tag{13a}
\]
\[
s_S : U_- \setminus \{S\} \rightarrow S^3, \quad s_S(\hat{n}) = (\cos \frac{\theta}{2} e^{i\varphi}, \sin \frac{\theta}{2}), \tag{13b}
\]
with $\hat{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, using the inclusion
\[
j : S^3 \rightarrow \mathbb{R}^4, \quad j(z_1, z_2) = (x_1, x_2, x_3, x_4)
\]
\[
= (\cos(\frac{\varphi + x}{2}) \cos \frac{\theta}{2}, \sin(\frac{\varphi + x}{2}) \cos \frac{\theta}{2}, \cos(\frac{\varphi - x}{2}) \sin \frac{\theta}{2}, \sin(\frac{\varphi - x}{2}) \sin \frac{\theta}{2}),
\]
and defining the 1-form $\tilde{\omega} \in \Omega^1(\mathbb{R}^4 \otimes u(1))$ through
\[
\tilde{\omega} = i(x^1 dx^2 - x^2 dx^1 - x^3 dx^4 + x^4 dx^3),
\]
one can prove that $j^*(\tilde{\omega}) = \omega$ and that $s_{N,S}^*(\omega)$ are the usual local 1-forms $A_{\pm}$ on $S^2$, namely
\[
A_+(\theta, \varphi) = s_N^*(\omega)(\theta, \varphi) = (j \circ s_N)^*(\tilde{\omega})(\theta, \varphi) = -\frac{i}{2}(1 - \cos \theta) d\varphi, \tag{16a}
\]
\[
A_-(\theta, \varphi) = s_S^*(\omega)(\theta, \varphi) = (j \circ s_S)^*(\tilde{\omega})(\theta, \varphi) = \frac{i}{2}(1 + \cos \theta) d\varphi. \tag{16b}
\]
The corresponding $u(1)$-valued 3-vector potentials are
\[
A_+ = -\frac{i}{2} \frac{1 + \cos \theta}{2 r \sin \theta} \hat{\varphi}, \quad A_- = +\frac{i}{2} \frac{1 - \cos \theta}{2 r \sin \theta} \hat{\varphi}, \tag{17a}
\]
defined also at $\theta = 0$ ($A_+$) and $\theta = \pi$ ($A_-$):
\[
A_+(\theta = 0) = A_-(\theta = \pi) = 0 \tag{17b}
\]
and on a 2-sphere of arbitrary radius $r > 0$. Clearly, the rotor of $A_+$ and $A_-$ gives the magnetic field $B$.

The first Chern class of $\xi_D$ (taking $S^2$ with unit radius) is given by
\[
c_1(\xi_D) = \frac{i}{2\pi} |F| \tag{18}
\]
where $|F|$ is the cohomology class of $F$ in $H^2(S^2)$; cohomology of the 2-sphere in dimension 2. The integral of $\frac{i}{2\pi} F$ over $S^2$ gives the first Chern number of $\xi_D$:
\[
\frac{i}{2\pi} \int_{S^2} F = 1. \tag{19}
\]
This means that the magnetic charge is a measure of the topological non-triviality of the bundle $\xi_D$ i.e. of the space where it “lives”. In other words, the monopole charge is not a property of the gauge field $A_\pm$ itself, but of the $U(1)$-bundle on which the monopole is a connection.

### 3 Bundle Morphism $\xi_{A-} \rightarrow \xi_D$

Using the homeomorphisms $(D^2)^* \cong \mathbb{C}^* \cong \mathbb{C} \cup \{\infty\}$, it can be easily verified that $(i, \bar{i})$ given by
\[
i : \mathbb{C}^* \rightarrow \mathbb{C} \cup \{\infty\}, \quad i(z) = z \tag{20}
\]
and
\[
\bar{i} : \mathbb{C}^* \times S^1 \rightarrow S^3, \quad \bar{i}(z, e^{i\varphi}) = \frac{(z, 1)}{|(z, 1)|} e^{i\varphi} \tag{21}
\]
with \( \| (z, 1) \| = \sqrt{1 + |z|^2} \), and \((\psi_{A-B}, \psi_D)\) the right actions
\[
\psi_{A-B} : (\mathbb{C}^* \times S^1) \times S^1 \to \mathbb{C}^* \times S^1, \quad \psi_{A-B}((z, e^{i\alpha}), e^{i\beta}) = (z, e^{i(\alpha + \beta)})
\] (22)
and
\[
\psi_D : S^3 \times S^1 \to S^3, \quad \psi_D((z_1, z_2), e^{i\lambda}) = (z_1 e^{i\lambda}, z_2 e^{i\lambda})
\] (23)
is the unique bundle morphism
\[
\xi_{A-B} \to \xi_D
\] (24)
induced by the inclusion \( \iota \) i.e.
\[
\pi \circ \iota = \iota \circ pr_1
\] (25)
and
\[
\psi_D \circ (\iota \times Id_{S^1}) = \iota \circ \psi_{A-B}
\] (26)
namely, with lower and upper parts of Diagram 1 commuting.

\[
\begin{array}{c c c c}
(C^* \times S^1) \times S^1 & \xrightarrow{\iota \times Id_{S^1}} & S^3 \times S^1 \\
\psi_{A-B} \downarrow & & \downarrow \psi_D \\
C^* \times S^1 & \xrightarrow{\iota} & S^3 \\
pr_1 \downarrow & & \downarrow \pi \\
C^* & \xrightarrow{\iota} & \mathbb{C} \cup \{\infty\}
\end{array}
\]

Diagram 1

In fact:
\[
\pi \circ \iota z, e^{i\varphi) = \pi \left( \frac{(z, 1)}{\| (z, 1) \|} e^{i\varphi} \right) = z,
\]
\[
\iota \circ pr_1 (z, e^{i\varphi} ) = \iota (z) = z;
\]
\[
\psi_D (\iota \times Id_{S^1})((z, e^{i\varphi}), e^{i\lambda}) = \psi_D (\iota (z, e^{i\varphi}), e^{i\lambda}) = \frac{(z, 1)}{\| (z, 1) \|} e^{i(\varphi + \lambda)},
\]
\[
\iota \circ \psi_{A-B}((z, e^{i\varphi}), e^{i\lambda}) = \iota (z, e^{i(\varphi + \lambda)}) = \frac{(z, 1)}{\| (z, 1) \|} e^{i(\varphi + \lambda)}.
\]

4 Pull-back of \( \xi_D \) by \( \iota \): \( \iota^* (\xi_D) \)

The total space of the induced or pull-back bundle [14] of \( \xi_D \) by \( \iota \), \( \iota^* (\xi_D) : S^1 \to P_{\iota^* (\xi_D)} \xrightarrow{pr_2} C^* \), is defined by
\[
P_{\iota^* (\xi_D)} = \{(z, (z_1, z_2)) \in C^* \times S^3, \ i(z) = \pi(z_1, z_2)\}
\] (27)
and must be such that both the upper and lower parts of Diagram 2 commute i.e. such that \((\iota, pr_2)\) is a bundle morphism \( \iota^* (\xi_D) \to \xi_D \). In Diagram 2, \( pr_2 \) is the projection in the second entry, and
\[
\psi_{\iota^* (\xi_D)} : P_{\iota^* (\xi_D)} \times S^1 \to P_{\iota^* (\xi_D)}, \quad \psi_{\iota^* (\xi_D)}((z, (z_1, z_2)), e^{i\lambda}) = (z, (z_1, z_2) e^{i\lambda})
\] (28)
is the right action of \( S^1 \) on \( P_{\iota^* (\xi_D)} \).

\[
\begin{array}{c c c c}
P_{\iota^* (\xi_D)} \times S^1 & \xrightarrow{pr_2 \times Id_{S^1}} & S^3 \times S^1 \\
\psi_{\iota^* (\xi_D)} \downarrow & & \downarrow \psi_D \\
P_{\iota^* (\xi_D)} & \xrightarrow{pr_2} & S^3 \\
pr_1 \downarrow & & \downarrow \pi \\
C^* & \xrightarrow{\iota} & \mathbb{C} \cup \{\infty\}
\end{array}
\]

Diagram 2
From

\[ \iota \circ \text{pr}_1 = \pi \circ \text{pr}_2 \]  

one has:

\[ \iota \circ \text{pr}_1((z, (z_1, z_2))) = \iota(z) = z, \]

\[ \pi \circ \text{pr}_2((z, (z_1, z_2))) = \pi(z_1, z_2) = z_1/z_2, \]

so \( z_1 = z_2z \) and \( \|(z_1, z_2)\| = 1 \) implies \( (z_1, z_2) = \frac{(z, 1)}{\|(z, 1)\|} e^{i\varphi}. \) Then,

\[ P^*_{\iota, \xi_D} = \{(z, \frac{(z, 1)}{\|(z, 1)\|} e^{i\varphi}), z \in \mathbb{C}^*, \varphi \in [0, 2\pi)\} \subset \mathbb{C}^* \times S^1. \]  

(30)

On the other hand, it holds

\[ \psi_D \circ (\text{pr}_2 \times Id_{S^1}) = \text{pr}_2 \circ \psi^*_{\iota, \xi_D}. \]  

(31)

In fact:

\[ \psi_D \circ (\text{pr}_2 \times Id_{S^1})(z, (z_1, z_2)), e^{i\lambda} = \psi_D((z_1, z_2)e^{i\lambda}) = (z_1 e^{i\lambda}, z_2 e^{i\lambda}), \]

\[ \psi_{\psi^*_{\iota, \xi_D}}(z, (z_1, z_2)), e^{i\lambda} = \text{pr}_2((z, (z_1, z_2)e^{i\lambda})) = (z_1, z_2)e^{i\lambda} = (z_1 e^{i\lambda}, z_2 e^{i\lambda}). \]

5 Bundle Isomorphism: \( \iota^* (\xi_D) \xrightarrow{\cong} \xi_{A-B} \)

In this section we exhibit a “natural” isomorphism between the \( A - B \) bundle and the pull-back by the inclusion \( \iota : \mathbb{C}^* \to \mathbb{C} \cup \{\infty\} \) (i.e. \( \iota : (\mathbb{D}_0^2)^* \to S^2 \) up to homeomorphisms) of the Dirac bundle \( \xi_D \) corresponding to unit magnetic charge, thus establishing a deep relation between the two systems \((A - B):\) experimentally observed, and \( D: \) only theoretical, up to now).

The homeomorphism between the total spaces of the bundles is given by

\[ \Psi : P^*_{\iota, \xi_D} \to \mathbb{C}^* \times S^1, \Psi(z, \frac{(z, 1)}{\|(z, 1)\|} e^{i\varphi}) = (z, e^{i\varphi}). \]  

(32)

It is clear that \( \Psi \) is continuous, one-to-one and onto, with continuous inverse \( \Psi^{-1}. \) It is easily verified that Diagram 3, corresponding to this isomorphism, commutes in its upper and lower parts i.e.

\[ \text{pr}_1 \circ \Psi = \text{Id}_{\mathbb{C}^*} \circ \text{pr}_1 \]  

(33)

and

\[ \psi_{A-B} \circ (\Psi \times Id_{S^1}) = \Psi \circ \psi^*_{\iota, \xi_D}. \]  

(34)

In fact:

\[ \text{pr}_1 \circ \Psi(z, \frac{(z, 1)}{\|(z, 1)\|} e^{i\varphi}) = \text{pr}_1(z, e^{i\varphi}) = z, \]

\[ \text{Id}_{\mathbb{C}^*} \circ \text{pr}_1(z, \frac{(z, 1)}{\|(z, 1)\|} e^{i\varphi}) = \text{Id}_{\mathbb{C}^*}(z) = z; \]

\[ \psi_{A-B} \circ (\Psi \times Id_{S^1})(z, (z_1, z_2)), e^{i\lambda} = \psi_{A-B}(\Psi((z_1, (z_1, z_2)), e^{i\lambda}) = \Psi(z_1, (z_1, z_2)) e^{i\lambda} = \Psi(z, \frac{(z, 1)}{\|(z, 1)\|} e^{i\varphi}) e^{i\lambda} = (z, e^{i\varphi}) e^{i\lambda} = (z, e^{i(\varphi + \lambda)}), \]

\[ \psi \circ \psi^*_{\iota, \xi_D}(z, (z_1, z_2)), e^{i\lambda} = \Psi(z, (z_1, z_2)), e^{i\lambda} = \Psi(z, \frac{(z, 1)}{\|(z, 1)\|} e^{i(\varphi + \lambda)}) = (z, e^{i(\varphi + \lambda)}). \]
6 Chern Classes

$\xi_{A-B}$ is the pull-back of $\xi_D$ by the inclusion $\iota : (D_0^2)^* \to S^2$; however, since $\xi_{A-B}$ is trivial, then all its Chern classes must vanish. Then, in particular, we must verify the vanishing of the pull-back of $c_1$.

$\xi_{A-B} = \iota^*(\xi_D)$ passes to cohomology [15] in the form

$$\iota^*: H^k(S^2) \to H^k((D_0^2)^*)$$

i.e.

$$\iota^*: H^k(S^2) \to H^k((D_0^2)^*), \ k = 0, 1, 2$$

where

$$H^*(S^2) = (H^0(S^2), H^1(S^2), H^2(S^2)) \cong (\mathbb{R}, 0, \mathbb{R})$$

and

$$H^*((D_0^2)^*) = (H^0((D_0^2)^*), H^1((D_0^2)^*), H^2((D_0^2)^*)) \cong (\mathbb{R}, \mathbb{R}, 0)$$

are the cohomology groups of the 2-sphere and the punctured disk respectively. $H^*((D_0^2)^*) \cong H^*(S^1)$ by homotopy invariance. Since $c_1 \in H^2(S^2)$, then

$$\iota^*(c_1) = 0.$$ 

7 Pull-back of the Dirac Connection and Vanishing of the $A - B$ Effect

In terms of the cartesian coordinates in $\mathbb{R}^3$, $(x, y, z) = r(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ with $\theta \in (0, \pi)$ and $\varphi \in [0, 2\pi)$ which implies $(x, y, z) \neq (0, 0, z)$, the monopole potentials $A_{\pm}$ of equations (16a) and (16b) are given by

$$A_{\pm}(x, y, z) = (A_{\pm})_x dx + (A_{\pm})_y dy$$

with

$$(A_{\pm})_x(x, y, z) = \pm \frac{i}{2} \frac{y}{x^2 + y^2} (1 \mp \frac{z}{\sqrt{x^2 + y^2 + z^2}}), \quad (A_{\pm})_y(x, y, z) = \mp \frac{i}{2} \frac{x}{x^2 + y^2} (1 \mp \frac{z}{\sqrt{x^2 + y^2 + z^2}}).$$

(Notice that $[(A_{\pm})_x] = [(A_{\pm})_y] = [L]^{-1}$ since $[x] = [y] = [z] = [L]$ while $[A_{\pm}] = [L]^{0}$, $L$: length.)

To pull-back by $\iota$ these 1-forms to $(D_0^2)^*$ we must first restrict $A_{\pm}$ to $z = 0$ and then perform the pull-back operation, which reduces to the identity:

$$\iota^*(A_{\pm}(x, y, 0)) = \pm \frac{i}{2} \frac{y dx - x dy}{x^2 + y^2} := ia_{\pm}(x, y)$$

with

$$a_{\pm}(x, y) = \mp \frac{1}{2} \frac{y dx - x dy}{x^2 + y^2}.$$ 

the real-valued $A - B$ potential 1-forms. Clearly, $a_{\pm}$ are closed ($da_{\pm} = 0$) but not exact since $a_{\pm} = \mp \frac{1}{2} d\varphi$ only for $\varphi \in (0, 2\pi)$. If we surround the thin solenoid in the $A - B$ side with closed curves $\gamma_{\pm}$ with $\gamma_- = -\gamma_+$, then the surrounded magnetic flux is

$$\Phi_{A-B} = \int_{\gamma_+} a_+ + \int_{\gamma_-} a_- = \int_{\gamma_+} a_+ + \int_{\gamma_-} (-a_+) = \int_{\gamma_+} a_+ - \int_{\gamma_+} (-a_+) = 2 \int_{\gamma_+} a_+ = 2 \int_{\gamma_+} (-d\varphi) = -2\pi,$$

which coincides, up to a sign, with the flux of the monopole:

$$\Phi_D = \int_{S^2} B = \frac{1}{2} \int_{S^2} \hat{r} \wedge \hat{r} = \frac{1}{2} 4\pi = 2\pi.$$ 

But this implies that the $A - B$ effect vanishes if and only if the value of the electric charge $|e|$ is an integer: the $D.Q.C.$ for the present case where $g = \frac{1}{2}$. In fact, with $\Phi_0 = \frac{2\pi}{n}$ the quantum of magnetic flux associated with the charge $|e|$, the phase change of the wave function in the $A - B$ experiment due to the presence of magnetic flux is

$$e^{-i|e|\Phi_{A-B}} = e^{-\frac{2\pi i}{n} \frac{\Phi_{A-B}}{\Phi_0}} = e^{2\pi i \frac{\Phi_{A-B}}{\Phi_0}} = e^{i|e|\frac{1}{2} 4\pi} = e^{2\pi i |e|} = 1 \Leftrightarrow |e| = n \in \mathbb{Z}.$$ 

(For arbitrary $g$, the $D.Q.C.$ would be $|e|g = \frac{1}{2}$.)
8 Final Comments

It is well known that the $A - B$ effect and the Dirac monopole are closed related [16]; in particular the disappearance of the Dirac string simultaneously with the vanishing of the $A - B$ effect when appropriate conditions of the magnetic fluxes are fulfilled [17]. In the present paper, the above relation has been described in the context of the fiber bundles associated with both phenomena, respectively $\xi_{A-B}$ (trivial) and $\xi_D$ (non-trivial Hopf bundle). The remarkable fact is that $\xi_{A-B}$ turns out to be the pull-back of $\xi_D$ by the inclusion $\iota$ of the corresponding base spaces, which allows to discuss the above relation in a purely geometric language. It would be interesting to investigate if this bundle theoretic relation exists in non-abelian cases.

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