Self-similar solutions of some model degenerate partial differential equations of the second, third and fourth order

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Abstract—When studying boundary value problems for some partial differential equations arising in applied mathematics, we often have to study the solution of a system of partial differential equations satisfied by hypergeometric functions and find explicit linearly independent solutions for the system. In this study, we construct self-similar solutions of some model degenerate partial differential equations of the second, third, and fourth order. These self-similar solutions are expressed in terms of hypergeometric functions.

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1. INTRODUCTION AND PRELIMINARIES

A self-similar solution is a solution to some system or equation in which independent variables do not appear independently, but only in combination. To obtain the desired combination of variables for a self-similar solution one often uses the methods of the theory of dimensions. The methods of the theory of dimensions originate from the works of J. Bertrand J., A. Vassy, subsequently generalised by H. Weyl.

As an example ([1, p. 113]), we consider the problem of the diffusion of vortices in a viscous incompressible fluid under the assumption that the motion of the fluid is plane-parallel and the fluid occupies the entire plane. The motion in question is transient. Suppose that at the initial moment of time the fluid can potentially move everywhere, except for the pole, which is a trace on the plane of motion of an infinite rectilinear concentrated vortex with circulation. The equation of vortex propagation in this case has the form

\[
\frac{\partial \Omega}{\partial t} = \nu \left( \frac{\partial^2 \Omega}{\partial r^2} + \frac{1}{r} \frac{\partial \Omega}{\partial r} \right),
\]

where \( \Omega \) is the angular velocity of the fluid particles in concentrated circles, and \( \nu \) is the coefficient of kinematic viscosity of the fluid. The solution is sought in the form

\[
\Omega (r, t) = \frac{E}{\nu t} \psi (\xi), \quad \xi = \frac{r^2}{\nu t}. 
\]
Substituting (2) into equation (1), we obtain the solution
\[ \Omega (r, t) = \frac{E}{\nu t} A e^{-\frac{r^2}{4\nu t}}, \]
(3)
where \( A \) is determined from the initial condition of the problem. Thus, (3) is a self-similar solution to equation (1).

It is known that to solve applied problems, one needs to set up a mathematical model of the problem under consideration. In many mathematical models, degenerate differential equations appear (especially in gas dynamics, quantum chemistry, in theoretical physics, in the theory of infinitesimal bending of surfaces of revolution, a momentless theory of shells, etc.).

L.D. Landau and E.M. Lifshits in their article [2] explored the features of the shock wave flow using the Euler-Tricomi equation
\[ u_{xx} + u_{yy} + \frac{1}{3y} u_y = 0, \]
and defined particular solutions of the form
\[ u_k = x^{2k} F \left( -k, -k + \frac{1}{2}, -2k + \frac{5}{6}; 1 + \frac{4y^2}{9x^2} \right), \]
where \( k = \pm \frac{n}{2}, \frac{1}{3} \pm \frac{n}{2}, n \in N_0. \)

Note that the energy absorbed by a non-ferromagnetic conducting sphere placed in an external inhomogeneous magnetic field is calculated explicitly using the hypergeometric functions of many variables ([3]).

The hypergeometric functions of Kampe de Feriet also appear in theoretical physics and quantum chemistry (see e.g. [4]). In the monographs [5] - [7], attention was drawn to the fact that many problems of supersonic gas dynamics are solved using hypergeometric functions.

Using the method of self-similar solutions in articles [8] - [20], fundamental solutions were found and in articles [21] - [24] the main boundary-value problems for the generalised axisymmetric Helmholtz equation were solved.

In this paper, using the method of self-similar solutions, we construct some special solutions of degenerate partial differential equations that are expressed by hypergeometric functions.

2. A PARABOLIC EQUATION WITH ONE LINE OF DEGENERATION

Consider in the domain \( \Omega = \{(x,t): x > 0, t > 0\} \), the degenerate parabolic equation
\[ Lu \equiv u_t - u_{xx} - \frac{2\alpha}{x} u_x = 0, \quad \alpha = \text{const} > 0. \]
(4)
We seek self-similar solutions of equation (4) in the form
\[ u = P \omega (\sigma), \]
(5)
where \( \omega = \omega (\sigma) \) is an unknown function, and where \( \sigma = -\frac{x^2}{4t}, \quad P = t^{-\frac{1}{2}}. \) Substituting (5) into equation (4), we have
\[ P \omega \sigma^2 x^2 + \left[ 2P \sigma_x + P \left( \sigma_{xx} + \frac{2\alpha x}{\sigma_x - \sigma_t} \right) \right] \omega + \left( P_{xx} + \frac{2\alpha x}{P_x - P_t} \right) \omega = 0. \]
(6)
After elementary calculations, we find
\[ \sigma^2 x = \frac{1}{t} \sigma, \quad 2P \sigma_x + P \left( \sigma_{xx} + \frac{2\alpha x}{\sigma_x - \sigma_t} \right) = -P \frac{1}{t} \left( \frac{1 + 2\alpha}{2} - \sigma \right), \]
\[ P_{xx} + \frac{2\alpha x}{P_x - P_t} = \frac{1}{2} P \frac{1}{t}. \]
Therefore, in view of the indicated equalities, the ordinary differential equation (6) has the form
\[
\sigma \omega_{\sigma\sigma} + \left(\frac{1 + 2\alpha}{2} - \sigma\right) \omega_{\sigma} - \frac{1}{2} \omega = 0. \tag{7}
\]
It is known ([25]) that the equation
\[
x w_{xx} + (c - x) w_x - aw = 0, \tag{8}
\]
has two linearly independent solutions
\[
w_1 = c_1 1F_1 (a; c; x) = c_1 e^x 1F_1 (c - a; c; -x),
\]
\[
w_2 = c_2 x^{1-c} 1F_1 (a - c + 1; 2 - c; x) = c_2 e^x x^{1-c} 1F_1 (1 - a; 2 - c; -x),
\]
where the hypergeometric function \(1F_1 (a; c; x)\) has the form
\[
1F_1 (a; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m m!} x^m,
\]
and
\[
(a)_m = \Gamma (a + m) / \Gamma (a) = a (a + 1) (a + 2) \cdots (a + m - 1)
\]
is the Pochhammer symbol ([25]). Therefore, taking into account (9) and (7), we define
\[
\omega_1 = c_1 1F_1 \left(\frac{1}{2}; \frac{1 + 2\alpha}{2}; \sigma\right), \quad \omega_2 = c_2 \sigma 1F_1 \left(1 - \frac{3 - 2\alpha}{2}; \sigma\right). \tag{10}
\]
Substituting (10) into (5), we finally obtain
\[
u_1 (x, t) = c_1 \frac{1}{\sqrt{t}} 1F_1 \left(\frac{1}{2}; \frac{1 + 2\alpha}{2}; -\frac{x^2}{4t}\right), \tag{11}
\]
\[
u_2 (x, t) = c_2 \frac{1}{\sqrt{t}} x^2 1F_1 \left(1 - \frac{3 - 2\alpha}{2}; -\frac{x^2}{4t}\right), \tag{12}
\]
two self-similar solutions of equation (4), where \(c_1, c_2\) are constants. Note that in [26], the boundary value problems for the equation \(L^m u = 0\) were considered.

3. A PARABOLIC EQUATION WITH TWO LINES OF DEGENERACY

In the domain \(\Omega = \{(x, y, t) : x > 0, y > 0, t > 0\}\), we consider the equation
\[
Lu \equiv u_t - u_{xx} - u_{yy} - \frac{2\alpha}{x} u_x - \frac{2\beta}{y} u_y = 0, \quad \alpha, \beta = \text{const.} \tag{13}
\]
The solution to equation (13) is sought in the form
\[
u = P \omega (\xi, \eta), \tag{14}
\]
where \(\xi = -\frac{x^2}{8t}, \eta = -\frac{y^2}{8t}, P = t^{-\frac{1}{4}},\) and \(\omega (\xi, \eta)\) is an unknown function. Substituting (14) into (13), we have
\[
A_1 \omega_{\xi}\xi + A_2 \omega_{\xi}\eta + A_3 \omega_{\eta}\eta + A_4 \omega_{\xi} + A_5 \omega_{\eta} + A_6 \omega = 0, \tag{15}
\]
where
\begin{align*}
A_1 &= P \left( \xi_x^2 + \xi_y^2 \right), \quad A_2 = 2P \left( \xi_x \eta_x + \xi_y \eta_y \right), \quad A_3 = P \left( \eta_x^2 + \eta_y^2 \right), \\
A_4 &= \frac{2\alpha}{x} P \xi_x + \frac{2\beta}{y} P \xi_y - P \xi_t + 2P_x \xi_x + P \xi_{xx} + 2P_y \xi_y + P \xi_{yy}, \\
A_5 &= \frac{2\alpha}{x} P \eta_x + \frac{2\beta}{y} P \eta_y - P \eta_t + 2P_x \eta_x + P \eta_{xx} + 2P_y \eta_y + P \eta_{yy}, \\
A_6 &= -P_t + P_{xx} + P_{yy} + \frac{2\alpha}{x} P_x + \frac{2\beta}{y} P_y.
\end{align*}

Calculating the values of the coefficients of equation (15), we obtain a system of partial differential equations
\begin{align}
\begin{cases}
\xi \omega_{\xi\xi} + \left( \frac{1 + 2\alpha}{2} - \xi \right) \omega_{\xi} - \eta \omega_{\eta} - \frac{1}{2} \omega = 0 \\
\eta \omega_{\eta\eta} + \left( \frac{1 + 2\beta}{2} - \eta \right) \omega_{\eta} - \xi \omega_{\xi} - \frac{1}{2} \omega = 0.
\end{cases}
\end{align}

(16)

In the monograph [25], [27] the following system of hypergeometric equations was considered
\begin{align}
\begin{cases}
x w_{xx} + (c_1 - x) w_x - y w_y - a w = 0 \\
y w_{yy} + (c_2 - y) w_y - x w_x - a w = 0
\end{cases}
\end{align}

(17)

and 4 linearly independent solutions were found, expressed in terms of the confluent Kummer functions,
\begin{align}
w_1 &= \lambda_1 \Psi_2 \left( a; c_1, c_2; x, y \right), \\
w_2 &= \lambda_2 x^{1 - c_1} \Psi_2 \left( a + 1 - c_1; 2 - c_1, c_2; x, y \right), \\
w_3 &= \lambda_3 y^{1 - c_2} \Psi_2 \left( a + 1 - c_2; c_1, 2 - c_2; x, y \right), \\
w_4 &= \lambda_4 x^{1 - c_1} y^{1 - c_2} \Psi_2 \left( a + 2 - c_1 - c_2; 2 - c_1, 2 - c_2; x, y \right),
\end{align}

(18) - (21)

where \( \lambda_i = \text{const} \), \( i = 1, 2, 3, 4 \), and
\[ \Psi_2 \left( a; c_1, c_2; x, y \right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}}{(c_1)_m (c_2)_n} \frac{x^m y^n}{m! n!}. \]

In view of (18) - (21), we define
\begin{align*}
\omega_1 &= \lambda_1 \Psi_2 \left( \frac{1}{2}; \frac{1 + 2\alpha}{2}, \frac{1 + 2\beta}{2}; -\frac{x^2}{8t}, -\frac{y^2}{8t} \right), \\
\omega_2 &= \lambda_2 \left( \frac{x^2}{8t} \right)^{\frac{1-2\alpha}{2}} \Psi_2 \left( 1 - \alpha; \frac{3 - 2\alpha}{2}, \frac{1 + 2\beta}{2}; -\frac{x^2}{8t}, -\frac{y^2}{8t} \right), \\
\omega_3 &= \lambda_3 \left( \frac{y^2}{8t} \right)^{\frac{1-2\beta}{2}} \Psi_2 \left( 1 - \beta; \frac{1 + 2\alpha}{2}, \frac{3 - 2\beta}{2}; -\frac{x^2}{8t}, -\frac{y^2}{8t} \right), \\
\omega_4 &= \lambda_4 \left( \frac{x^2}{8t} \right)^{\frac{1-2\alpha}{2}} \left( \frac{y^2}{8t} \right)^{\frac{1-2\beta}{2}} \Psi_2 \left( \frac{3 - 2\alpha - 2\beta}{2}, \frac{3 - 2\alpha}{2}, \frac{3 - 2\beta}{2}; -\frac{x^2}{8t}, -\frac{y^2}{8t} \right).
\end{align*}
Substituting \( \omega_i, \ i = 1, 2, 3, 4, \) in (14), we obtain the following special solutions of equation (13):

\[
\begin{align*}
  u_1(x, y, t) &= \lambda_1 \frac{1}{\sqrt{t}} \Psi_2 \left( \frac{1}{2}; \frac{1 + 2\alpha}{2}, \frac{1 + 2\beta}{2}; -\frac{x^2}{8t}, -\frac{y^2}{8t} \right), \\
  u_2(x, y, t) &= \lambda_2 \frac{x^{1-2\alpha}}{t^{1-\alpha}} \Psi_2 \left( 1 - \alpha; \frac{3 - 2\alpha}{2}, \frac{1 + 2\beta}{2}; -\frac{x^2}{8t}, -\frac{y^2}{8t} \right), \\
  u_3(x, y, t) &= \lambda_3 \frac{y^{1-2\beta}}{t^{1-\beta}} \Psi_2 \left( 1 - \beta; \frac{1 + 2\alpha}{2}, \frac{3 - 2\beta}{2}; -\frac{x^2}{8t}, -\frac{y^2}{8t} \right), \\
  u_4(x, y, t) &= \lambda_4 \frac{x^{1-2\alpha}y^{1-2\beta}}{t^{2-\alpha-\beta}} \Psi_2 \left( \frac{3 - 2\alpha - 2\beta}{2}, \frac{3 - 2\alpha}{2}, \frac{3 - 2\beta}{2}; -\frac{x^2}{8t}, -\frac{y^2}{8t} \right),
\end{align*}
\]

where \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) are constants.

4. A DIFFERENTIAL EQUATION OF THE THIRD ORDER WITH ONE LINE OF DEGENERATION

Consider the equation

\[
Lu \equiv y^m u_{xxx} - u_{yyy} = 0, \quad m = \text{const} > 0,
\]

in the domain of \( \Omega = \{ (x, y) : -\infty < x < +\infty, y > 0 \} \). Special solutions of equation (26) are sought in the form

\[
u = P \omega \left( \sigma \right), \tag{27}
\]

where

\[
P = x^{-3}, \quad \sigma = \left( -\frac{3}{x(m + 3)} y^{m + 3} \right)^3, \quad \beta = \frac{m}{m + 3}.
\]

Substituting (27) into equation (26), we find

\[
A \omega_{\sigma \sigma \sigma} + B \omega_{\sigma \sigma} + C \omega_\sigma + D \omega = 0,
\]

where

\[
A = P \left( y^m \sigma_x^3 - \sigma_y^3 \right), \\
B = 3 \left[ y^m P_x \sigma_x^2 - P_y \sigma_y^2 + P \left( y^m \sigma_x \sigma_{xx} - \sigma_y \sigma_{yy} \right) \right], \\
C = 3 \left( y^m P_x \sigma_x - P_y \sigma_y \right) + 3 \left( y^m P_x \sigma_{xx} - P_y \sigma_{yy} \right) + P \left( y^m \sigma_{xxx} - \sigma_{yyy} \right), \\
D = y^m P_{xxx} - P_{yyy}.
\]

After elementary calculations, we have

\[
A = \frac{3^3 y^m P x^3}{x^3} \sigma^2 (1 - \sigma), \\
B = \frac{3^3 y^m P x^3}{x^3} \left[ \frac{2 + \beta}{3} + \frac{1 + 2\beta}{3} + 1 - \left( 3 + 1 + \frac{4}{3} + \frac{5}{3} \right) \sigma \right] \sigma, \\
C = \frac{3^3 y^m P x^3}{x^3} \left[ \frac{2 + \beta}{3} \right] \left( 1 + \frac{4}{3} + \frac{5}{3} + 1 \cdot \frac{4}{3} + 1 \cdot \frac{5}{3} + \frac{4}{3} \cdot \frac{5}{3} \right) \sigma, \\
D = -\frac{3^3 y^m P x^3}{x^3} \cdot 1 \cdot \frac{4}{3} \cdot \frac{5}{3}.
\]
By the above equalities, from (28) it follows that
\[
\sigma^2 (1 - \sigma) \omega_{\sigma \sigma} + \left[ \frac{2 + \beta}{3} + \frac{1 + 2\beta}{3} + 1 - \left( 3 + 1 + \frac{4}{3} + \frac{5}{3} \right) \sigma \right] \sigma \omega_{\sigma \sigma} \\
+ \left[ \frac{2 + \beta}{3} 1 + 2\beta \right] - \left( 1 + 1 + \frac{4}{3} + \frac{5}{3} + 1 \cdot \frac{4}{3} + 1 \cdot \frac{5}{3} \right) \sigma \right] \omega_\sigma - 1 \cdot \frac{4}{3} \cdot \frac{5}{3} \omega = 0.
\] (29)

Thus, we have obtained the ordinary Clausen differential equation ([27]), which has the form
\[
x^2 (1 - x) w_{xxx} + [c_1 + c_2 + 1 - (3 + a_1 + a_2 + a_3) x] x w_{xx} \\
+ [c_1 c_2 - (1 + a_1 + a_2 + a_3 + a_1 a_2 + a_1 a_3 + a_2 a_3) x] w_x - a_1 a_2 a_3 w = 0.
\] (30)

The Clausen equation (30) has the following three linearly independent solutions [27]:
\[
w_1 = \lambda_{13} F_2 \left[ \begin{array}{c}
a_1, a_2, a_3 \\
1, 0
\end{array} \right],
\] (31)
\[
w_2 = \lambda_2 x^{1-c_1} F_2 \left[ \begin{array}{c}
a_1 + 1 - c_1, a_2 + 1 - c_1, a_3 + 1 - c_1 \\
2 - c_1, c_2 + 1 - c_1
\end{array} \right],
\] (32)
\[
w_3 = \lambda_3 x^{1-c_2} F_2 \left[ \begin{array}{c}
a_1 + 1 - c_2, a_2 + 1 - c_2, a_3 + 1 - c_2 \\
c_1 + 1 - c_2, 2 - c_2
\end{array} \right].
\] (33)

Note that the Clausen function can be represented as
\[
3 F_2 \left[ \begin{array}{c}
a_1, a_2, a_3 \\
c_1, c_2
\end{array} x \right] = \frac{\Gamma(c_1) \Gamma(c_2)}{\Gamma(a_1) \Gamma(a_2) \Gamma(c_1 - a_1) \Gamma(c_2 - a_2)} \\
\times \int_0^1 \int_0^1 \xi^{a_1-1} \eta^{a_2-1} (1 - \xi)^{c_1-a_1-1} (1 - \eta)^{c_2-a_2-1} (1 - x \xi \eta)^{-a_3} d\xi d\eta,
\]
\[\text{Re } c_i > \text{Re } a_i > 0, \ i = 1, 2,\]

\[3 F_2 (a_1, a_2, a_3; c_1, c_2; x) = 3 F_2 \left[ \begin{array}{c}
a_1, a_2, a_3 \\
c_1, c_2
\end{array} x \right] = \sum_{m=0}^{\infty} \frac{(a_1)_m (a_2)_m (a_3)_m}{(c_1)_m (c_2)_m} x^m.
\]

Taking into account (31) - (33), it follows from (29) that
\[
\omega_1 (\sigma) = \lambda_{13} F_2 \left[ \begin{array}{c}
1, 4, 5 \\
2 + \beta, 3, 2 + \beta \end{array} \sigma \right],
\] (34)
\[
\omega_2 (\sigma) = \lambda_2 \sigma^{1-\beta} \frac{3}{3} F_2 \left[ \begin{array}{c}
4 - \beta, 5 - \beta, 6 - \beta \\
3 - \beta, 3, 2 + \beta \end{array} \sigma \right] = \lambda_2 \sigma^{1-\beta} F \left( \frac{5 - \beta}{3}, \frac{6 - \beta}{3}, \frac{2 + \beta}{3}; \sigma \right),
\] (35)
Finally, we find the following special solutions to equation (26):

\[ \omega_3 (\sigma) = \lambda_3 \sigma^{\frac{2 - 2\beta}{3}} F_2 \left[ \begin{array}{c} 5 - 2\beta, 6 - 2\beta, 7 - 2\beta \\ 3 - 6\beta, 3 - 5\beta, 3 - 2\beta \\ \sigma \end{array} \right] = \lambda_3 \sigma^{\frac{2 - 2\beta}{3}} F \left( \begin{array}{c} 6 - 2\beta, 7 - 2\beta, 4 - \beta \\ \frac{3}{3}, \frac{3}{3}, \frac{3}{3} \end{array}; \sigma \right), \]

where \( F(a, b; c; x) \) is the Gauss hypergeometric function ([25], [27]). Substituting (34) - (36), we finally find the following special solutions to equation (26):

\[ u_1 (x, y) = \lambda_1 x^{-3} F_2 \left[ \begin{array}{c} 1, 4, 5 \\ 2 + \beta, 3 + 1/2 \beta \\ \frac{3}{3}, \frac{3}{3}; \left( \frac{3}{x (m + 3)} \right)^{1/4} \end{array}; \left( \frac{3}{x (m + 3)} y^{m+4} \right)^{3/4} \right], \]

\[ u_2 (x, y) = \lambda_2 x^{\beta-4} y F \left( \begin{array}{c} 5 - \beta, 6 - \beta, 2 + \beta \\ 3, 3, \frac{3}{3}; \left( \frac{3}{x (m + 3)} y^{m+4} \right)^{3/4} \end{array}; \left( \frac{3}{x (m + 3)} y^{m+4} \right)^{3/4} \right), \]

\[ u_3 (x, y) = \lambda_3 x^{2\beta-5} y^2 F \left( \begin{array}{c} 6 - 2\beta, 7 - 2\beta, 4 - \beta \\ 3, 3, \frac{3}{3}; \left( \frac{3}{x (m + 3)} y^{m+4} \right)^{3/4} \end{array}; \left( \frac{3}{x (m + 3)} y^{m+4} \right)^{3/4} \right), \]

where \( \lambda_1, \lambda_2, \lambda_3 \) are arbitrary constants.

5. THE THIRD-ORDER DIFFERENTIAL EQUATION OF THREE VARIABLES

In the domain \( \Omega = \{(x, y, t) : x > 0, y > 0, t > 0\} \), we consider the equation

\[ Lu \equiv x^m y^n u_t - t^k y^m u_{xxx} - t^k x^n u_{yyy} = 0, \quad m, n, k = \text{const} > 0. \]

The solution to equation (40) is sought in the form

\[ u = P \omega (\xi, \eta), \]

where

\[ P = \left( \frac{2}{k+1} \right)^{-1} \xi^{k+1}, \quad \xi = -\frac{k+1}{2(n+3)^{3/2} k^{1/2}} x^{n+3}, \quad \eta = -\frac{k+1}{2(m+3)^{3/2} k^{1/2}} y^{m+3}, \]

\[ \alpha = n/(n + 3), \quad \beta = m/(m + 3). \]

Then substituting (41) into (40), we obtain a third-order partial differential equation

\[ A_1 \omega_\xi \xi + A_2 \omega_\eta \eta + A_3 \omega_\xi \xi + A_4 \omega_\xi \eta + A_5 \omega_\xi \xi + A_6 \omega_\xi \eta + A_7 \omega_\xi \eta + A_8 \omega_\xi + A_9 \omega_\eta + A_{10} \omega = 0, \]

where

\[ A_1 = Pt^k \left( y^m \xi_x^3 + x^n \xi_y^3 \right), \quad A_2 = Pt^k \left( y^m \eta_x^3 + x^n \eta_y^3 \right), \quad A_3 = 3t^k P \left( y^m \xi_x^2 \eta_x + x^n \xi_y^2 \eta_y \right), \]

\[ A_4 = 3t^k P \left( y^m \xi_x \eta_x^2 + x^n \xi_y \eta_y^2 \right), \quad A_5 = 3t^k \left[ y^m P_x \xi_x^2 + x^n P_y \xi_y^2 + P (y^m \xi_x \xi_x + x^n \xi_y \xi_y) \right], \]

\[ A_6 = 3t^k \left[ y^m P_x \xi_x \eta_x + x^n P_y \xi_y \eta_y \right] + P (y^m \xi_x \xi_x + x^n \xi_y \xi_y) + P (y^m \xi_x \eta_x + x^n \xi_y \eta_y) \]

\[ A_7 = 3t^k \left[ y^m P_x \eta_x^2 + x^n P_y \eta_y^2 + P (y^m \eta_x \eta_x + x^n \eta_y \eta_y) \right], \]

\[ A_8 = 3t^k \left( y^m P_{xx} \xi_x + x^n P_{yy} \xi_y \right) + 2t^k \left( y^m P_{xx} \xi_x + x^n P_{yy} \xi_y \right) + t^k \left( y^m P_x \xi_x + x^n P_y \xi_y \right) \]

\[ + t^k P (y^m \xi_x + x^n \xi_y) - x^n y^m P \xi_t, \]
\[ A_9 = 3t^k \left(y^m P_{xx} \eta_x + x^n P_{yy} \eta_y \right) + 2t^k \left(y^m P_{x} \eta_{xx} + x^n P_{y} \eta_{yy} \right) + t^k \left(y^m P_{x} \eta_{xx} + x^n P_{y} \eta_{yy} \right) + t^k P \left(y^m P_{xxx} + x^n P_{yy} \eta \right) - x^n y^m P \eta, \]

\[ A_{10} = t^k y^m P_{xxx} + t^k x^n P_{yyy} - x^n y^m P. \]

After some calculations, we have

\[ A_1 = -P t^k x^n y^m \frac{k + 1}{2^{k+1}} \xi^2, \quad A_2 = -P t^k x^n y^m \frac{k + 1}{2^{k+1}} \eta^2, \quad A_3 = 0, \quad A_4 = 0, \]

\[ A_5 = -P t^k x^n y^m (\alpha + 2) \frac{k + 1}{2^{k+1}} \xi, \quad A_6 = 0, \quad A_7 = -P t^k x^n y^m (2 + \beta) \frac{k + 1}{2^{k+1}} \eta, \]

\[ A_8 = -P t^k x^n y^m \frac{k + 1}{2^{k+1}} \left(\frac{2 + \alpha + 2\alpha}{3} - 2\xi\right), \quad A_9 = -P t^k x^n y^m \frac{k + 1}{2^{k+1}} \left(\frac{2 + \beta + 2\beta}{3} - 2\eta\right), \]

\[ A_{10} = P t^k x^n y^m \frac{k + 1}{2^{k+1}}. \]

Therefore, using the indicated equalities from (42), we define

\[
\begin{align*}
\xi^2 \omega \xi \xi &+ \left(\frac{2 + \alpha + 1 + 2\alpha}{3} + 1\right) \xi \omega \xi \xi + \left(\frac{2 + \alpha + 2\alpha}{3} - \xi\right) \omega \xi - \eta \omega \eta - \omega = 0 \\
\eta^2 \omega \eta \eta &+ \left(\frac{2 + \beta + 1 + 2\beta}{3} + 1\right) \eta \omega \eta \eta + \left(\frac{2 + \beta + 2\beta}{3} - \eta\right) \omega \eta - \xi \omega \xi - \omega = 0.
\end{align*}
\]

(43)

From the general theory it is easy to determine that the system of hypergeometric equations

\[
\begin{align*}
x^2 w_{xxx} + (c_2 + c_1 + 1) x w_{xx} + (c_1 c_2 - x) w_x - y w_y - a w &= 0 \\
y^2 w_{yyy} + (d_2 + d_1 + 1) y w_{yy} + (d_1 d_2 - y) w_y - x w_x - a w &= 0
\end{align*}
\]

(44)

has 9 linearly independent solutions

\[
w_1 (x, y) = F_{1;0;0}^{1;0;0} \left[ \begin{array}{c}
a; & -; & -; & x, y \\ -; & c_1, c_2; & d_1, d_2; \end{array} \right],
\]

(45)

\[
w_2 (x, y) = y^{1-d_1} F_{0;2;2}^{1;0;0} \left[ \begin{array}{c}
1 - d_1 + a; & -; & -; & x, y \\ -; & c_1, c_2; & 2 - d_1, d_2 - d_1 + 1; \end{array} \right],
\]

(46)

\[
w_3 (x, y) = y^{1-d_2} F_{0;2;2}^{1;0;0} \left[ \begin{array}{c}
1 - d_2 + a; & -; & -; & x, y \\ -; & c_1, c_2; & 2 - d_2, d_1 - d_2 + 1; \end{array} \right],
\]

(47)

\[
w_4 (x, y) = x^{1-c_1} F_{0;2;2}^{1;0;0} \left[ \begin{array}{c}
1 - c_1 + a; & -; & -; & x, y \\ -; & 2 - c_1, c_2 - c_1 + 1; & d_1, d_2; \end{array} \right],
\]

(48)

\[
w_5 (x, y) = x^{1-c_1} y^{1-d_1}
\]

\[
\times F_{0;2;2}^{1;0;0} \left[ \begin{array}{c}
c_1 + d_1 - 2 - a; & -; & -; & x, y \\ -; & 2 - c_1, 1 + c_2 - c_1; & 2 - d_1, 1 + d_2 - d_1; \end{array} \right],
\]

(49)
\[ w_6(x,y) = x^{1-c_1} y^{1-d_2} \times F^{1;0;0}_{0;2;2} \left[ \begin{array}{c} c_1 + d_2 - 2 - a; \\ -; \\ -; \\ -; \end{array} ; \begin{array}{c} \xi, \eta \end{array} \right], \tag{50} \]

\[ w_7(x,y) = x^{1-c_2} F^{1;0;0}_{0;2;2} \left[ \begin{array}{c} 1 - c_2 + a; \\ -; \\ -; \end{array} ; \begin{array}{c} \xi, \eta \end{array} \right], \tag{51} \]

\[ w_8(x,y) = x^{1-c_2} y^{1-d_1} \times F^{1;0;0}_{0;2;2} \left[ \begin{array}{c} c_2 + d_1 - 2 - a; \\ -; \\ -; \end{array} ; \begin{array}{c} \xi, \eta \end{array} \right], \tag{52} \]

\[ w_9(x,y) = x^{1-c_2} y^{1-d_2} \times F^{1;0;0}_{0;2;2} \left[ \begin{array}{c} d_2 + c_2 - 2 - a; \\ -; \\ -; \end{array} ; \begin{array}{c} \xi, \eta \end{array} \right], \tag{53} \]

where

\[ F_{p,q;k}^{\alpha_1, \beta_1, \gamma_1} \left[ \begin{array}{c} (a_p); (b_q); (c_k); \\ (\alpha_1); (\beta_1); (\gamma_1); \end{array} ; x, y \right] = \sum_{r,s=0}^{p} \prod_{j=1}^{p} (a_j)_{r+s} \prod_{j=1}^{q} (b_j)_{r+s} \prod_{j=1}^{k} (c_j)_{q+s} x^r y^s \] \tag{54}

are hypergeometric function of Kampe de Feriet ([27]). Then, in view of (45) - (53), the system of hypergeometric equations (43) has the following special solutions

\[ \omega_1(\xi, \eta) = \frac{1 - \beta}{3} F^{1;0;0}_{0;2;2} \left[ \begin{array}{c} 1; \\ -; \end{array} ; \begin{array}{c} \xi, \eta \end{array} \right], \]

\[ \omega_2(\xi, \eta) = \frac{1 + \alpha}{3} F^{1;0;0}_{0;2;2} \left[ \begin{array}{c} 4 - \beta; \\ -; \end{array} ; \begin{array}{c} \frac{2 + \alpha}{3}, \frac{1 + 2 \alpha}{3}; \end{array} \right], \]

\[ \omega_3(\xi, \eta) = \frac{2 (1 - \beta)}{3} F^{1;0;0}_{0;2;2} \left[ \begin{array}{c} 5 - 2 \beta; \\ -; \end{array} ; \begin{array}{c} \frac{2 + \alpha}{3}, \frac{1 + 2 \alpha}{3}; \end{array} \right], \]

\[ \omega_4(\xi, \eta) = \frac{1 - \alpha}{3} F^{1;0;0}_{0;2;2} \left[ \begin{array}{c} 4 - \alpha; \\ -; \end{array} ; \begin{array}{c} \frac{2 - \alpha}{3}, \frac{2 + \alpha}{3}; \end{array} \right], \]

\[ \omega_5(\xi, \eta) = \frac{1 - \alpha}{3} \eta F^{1;0;0}_{0;2;2} \left[ \begin{array}{c} \alpha + \beta - 5; \\ -; \end{array} ; \begin{array}{c} \frac{4 - \alpha}{3}, \frac{2 + \alpha}{3}; \end{array} \right] \]
\[ \omega_6 (\xi, \eta) = \xi \frac{1 - \alpha}{3} \eta \frac{2 \beta}{3} F_{0:2;2}^{1:0;0} \left[ \begin{array}{c} \alpha + 2 \beta - 6 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{array} ; \xi, \eta \right], \]

\[ \omega_7 (\xi, \eta) = \xi \frac{1 - \alpha}{3} \eta \frac{2 \beta}{3} F_{0:2;2}^{1:0;0} \left[ \begin{array}{c} 5 - 2 \alpha \\ 3 \\ 4 - \alpha - 2 \alpha \\ 3 \\ 3 \\ 3 \end{array} ; \xi, \eta \right], \]

\[ \omega_8 (\xi, \eta) = \xi \frac{1 - \alpha}{3} \eta \frac{2 \beta}{3} F_{0:2;2}^{1:0;0} \left[ \begin{array}{c} 2 \alpha + \beta - 6 \\ 3 \\ 4 - \alpha - 2 \alpha \\ 3 \\ 3 \\ 3 \end{array} ; \xi, \eta \right], \]

\[ \omega_9 (\xi, \eta) = \xi \frac{1 - \alpha}{3} \eta \frac{2 \beta}{3} F_{0:2;2}^{1:0;0} \left[ \begin{array}{c} 2 \alpha + 2 \beta - 7 \\ 3 \\ 4 - \alpha - 2 \alpha \\ 3 \\ 3 \\ 3 \end{array} ; \xi, \eta \right]. \]

Multiplying each solution by \( P = \left( \frac{2}{k+1} \right)^{(k+1)} \), we finally get special solutions for equation (40):

\[ u_1 (x, y, t) = \lambda_1 PF_{0:2;2}^{1:0;0} \left[ \begin{array}{c} 1 \\ ; 3 \\ 2 + \alpha - 1 + 2 \alpha \\ 3 \\ 3 \\ 3 \end{array} ; \xi, \eta \right], \] (55)

\[ u_2 (x, y, t) = \lambda_2 PF_{0:2;2}^{1:0;0} \left[ \begin{array}{c} 4 - \beta \\ 3 \\ 2 + \alpha - 1 + 2 \alpha \\ 3 \\ 3 \\ 3 \end{array} ; \xi, \eta \right], \] (56)

\[ u_3 (x, y, t) = \lambda_3 PF_{0:2;2}^{1:0;0} \left[ \begin{array}{c} 5 - 2 \beta \\ 3 \\ 2 + \alpha - 1 + 2 \alpha \\ 3 \\ 3 \\ 3 \end{array} ; \xi, \eta \right], \] (57)

\[ u_4 (x, y, t) = \lambda_4 PF_{0:2;2}^{1:0;0} \left[ \begin{array}{c} 4 - \alpha \\ 3 \\ 4 - \alpha - 2 + \alpha \\ 3 \\ 3 \\ 3 \end{array} ; \xi, \eta \right], \] (58)

\[ u_5 (x, y, t) = \lambda_5 PF_{0:2;2}^{1:0;0} \left[ \begin{array}{c} \alpha + \beta - 5 \\ 3 \\ 4 - \alpha - 2 + \alpha \\ 3 \\ 3 \\ 3 \end{array} ; \xi, \eta \right], \] (59)

\[ u_6 (x, y, t) = \lambda_6 PF_{0:2;2}^{1:0;0} \left[ \begin{array}{c} \alpha + 2 \beta - 6 \\ 3 \\ 4 - \alpha - 2 + \alpha \\ 3 \\ 3 \\ 3 \end{array} ; \xi, \eta \right], \] (60)

\[ u_7 (x, y, t) = \lambda_7 PF_{0:2;2}^{1:0;0} \left[ \begin{array}{c} 5 - 2 \alpha \\ 3 \\ 4 - \alpha - 2 + \alpha \\ 3 \\ 3 \\ 3 \end{array} ; \xi, \eta \right]. \] (61)
has the following special solutions

\[ u_8(x, y, t) = \lambda_8 P \xi^3 \eta \right F^{1;0:0}_{0:2:2} \left[ \frac{2\alpha + \beta - 6}{3}; \frac{4 - \alpha}{3}, \frac{5 - 2\alpha}{3}, \frac{4 - \beta}{3}, \frac{2 + \beta}{3}; \xi, \eta \right] , \]

\[ u_9(x, y, t) = \lambda_9 P \xi^3 \eta \right F^{1;0:0}_{0:2:2} \left[ \frac{2\alpha + 2\beta - 7}{3}; \frac{4 - \alpha}{3}, \frac{5 - 2\alpha}{3}, \frac{4 - \beta}{3}, \frac{5 - 2\beta}{3}; \xi, \eta \right] , \]

where \( \lambda_1, \ldots, \lambda_9 \) are constants.

6. A FOURTH-ORDER DIFFERENTIAL EQUATION WITH TWO LINES OF DEGENERACY

In the domain \( \Omega = \{(x, t) : x > 0, t > 0 \} \), we consider the equation

\[ Lu = x^n u_t - t^k u_{xxxx} = 0, \quad n, k = \text{const} > 0. \]  

(64)

Special solutions of equation (64) will be sought in the form

\[ u(x, t) = P(t) \omega(\sigma), \]

(65)

where

\[ P = \left( \frac{1}{k + 1} \right)^{-1}, \quad \sigma = -\frac{k + 1}{(n + 4)t^{k+1}} x^{n+4}. \]

(66)

Substituting (65) into equation (64), we define

\[ t^k P \omega_{\sigma\sigma\sigma}\sigma_x^4 + 6t^k P \sigma_{xx} \sigma_x^2 \omega_{\sigma\sigma\sigma} + t^k \left[ 3P \sigma_{xx}^2 + 4P \sigma_x \sigma_{xxx} \right] \omega_{\sigma\sigma} + \left[ t^k P \sigma_{xxxx} - x^n P t \sigma_t \right] \omega_\sigma - x^n P_t \omega = 0. \]

After some calculations, we have

\[ \sigma^3 \omega_{\sigma\sigma\sigma} + \left( \frac{3 + 3 + \alpha}{4} + \frac{2 + 2\alpha}{4} + \frac{1 + 3\alpha}{4} \right) \sigma^2 \omega_{\sigma\sigma\sigma} + \left( \frac{1 + 3 + \alpha}{4} + \frac{2 + 2\alpha}{4} + \frac{1 + 3\alpha}{4} \right) \sigma \omega_{\sigma\sigma} \]

\[ + \left( \frac{3 + 2 + 2\alpha}{4} + \frac{3 + 1 + 3\alpha}{4} + \frac{2 + 2a + 2\alpha}{4} \right) \sigma \omega_{\sigma} + \left( \frac{3 + 2 + 2\alpha}{4} + \frac{1 + 3\alpha}{4} \right) \omega_{\sigma} - \omega = 0, \]

(67)

where \( \alpha = n/(n + 4) \). From the general theory ([27]) it is known that the equation

\[ x^3 w_{xxxx} + (3 + c_1 + c_2 + c_3) x^2 w_{xxx} + \\
+ (1 + c_1 + c_2 + c_3 + c_1 + c_2 + c_1 c_2 + c_2 c_3) x w_{xx} + (c_1 c_2 c_3 - x) w_x - \omega w = 0, \]

(68)

has the following special solutions

\[ w_1 = \lambda_{11} \right F_3 \left( a; c_1, c_2, c_3; x \right), \]

(69)

\[ w_2 = \lambda_2 x^{1-c_1} \right F_3 \left( 1 - c_1 + a; 2 - c_1, 1 + c_2 - c_1, 1 + c_3 - c_1; x \right), \]

(70)

\[ w_3 = \lambda_3 x^{1-c_1} \right F_3 \left( 1 - c_2 + a; 1 + c_1 - c_2, 2 - c_2, 1 + c_3 - c_2; x \right), \]

(71)

\[ w_4 = \lambda_4 x^{1-c_1} \right F_3 \left( 1 - c_3 + a; 1 + c_1 - c_3, 1 + c_2 - c_3, 2 - c_3; x \right), \]

(72)
where \( \lambda_i \) are constants, \( i = 1, 2, 3, 4 \), and

\[
1F_3(a; c_1, c_2, c_3; x) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c_1)_m(c_2)_m(c_3)_m m!} x^m.
\]  

(73)

Then from equation (67), in view of (69) - (72), taking into account representation (65), it is easy to determine special solutions to equation (64):

\[
u_1(x, t) = \tilde{\lambda}_1 \left( \frac{1}{k+1} t^{k+1} \right)^{-1} 1F_3 \left( \frac{3 + \alpha}{4}, \frac{2 + 2\alpha}{4}, \frac{1 + 3\alpha}{4}; - \frac{k+1}{(n+4)^4 t^{k+1}} x^{n+4} \right),
\]

(74)

\[
u_2(x, t) = \tilde{\lambda}_2 \left( \frac{1}{k+1} t^{k+1} \right)^{-\frac{9+\alpha}{4}} x_0^{2F_2} \left( \frac{3 + \alpha}{4}, \frac{2 + 2\alpha}{4}; - \frac{k+1}{(n+4)^4 t^{k+1}} x^{n+4} \right),
\]

(75)

\[
u_3(x, t) = \tilde{\lambda}_3 \left( \frac{1}{k+1} t^{k+1} \right)^{-\frac{9-\alpha}{4}} x_0^{2F_2} \left( \frac{5 - \alpha}{4}, \frac{3 + \alpha}{4}; - \frac{k+1}{(n+4)^4 t^{k+1}} x^{n+4} \right),
\]

(76)

\[
u_4(x, t) = \tilde{\lambda}_4 \left( \frac{1}{k+1} t^{k+1} \right)^{-\frac{11-3\alpha}{4}} x_0^{2F_2} \left( \frac{6 - 2\alpha}{4}, \frac{5 - \alpha}{4}; - \frac{k+1}{(n+4)^4 t^{k+1}} x^{n+4} \right),
\]

(77)

where \( \tilde{\lambda}_i \) are constants, \( i = 1, 2, 3, 4 \).

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