Unifying Inflation with Early and Late-time Dark Energy in $F(R)$ Gravity

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In this work we shall present models of $F(R)$ gravity which realize in a unified way the inflationary era along with a post-inflationary early dark energy era, with the late-time dark energy era. We shall use two approach methods in order to realize the unified cosmological eras, firstly we specify a Hubble rate which may describe the three distinct acceleration eras, and then by using well known $F(R)$ gravity reconstruction techniques, we shall find the differential equation which may yield the $F(R)$ gravity that realizes the cosmologies. In our second approach, we shall present in a qualitative way, several $F(R)$ gravities which unify the inflationary era with the early and late-time dark energy eras, and we discuss several qualitative issues related to the terms that realize the post-inflationary early dark energy era. We quantify our analysis by numerically solving the Friedmann equation, using the redshift as the main variable, and expressing all the physical quantities as functions of the statefinder $y_1(z)$, which depends on the redshift and the Hubble rate $H(z)$. For the model studied numerically, we present the behavior of some statefinder quantities, like the deceleration parameter, and we calculate the dark energy density parameter and the dark energy equation of state parameter at present time. After demonstrating that the dark energy era is viable, we investigate when the early dark energy term does not affect the late-time era, and this restricts the free parameters of the model.

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I. INTRODUCTION

The $\Lambda$ Cold Dark Matter ($\Lambda$CDM) model serves as the most successful model of cosmology up to date, since the Cosmic Microwave Background (CMB) polarization data coincide to a great extent with the $\Lambda$CDM predictions [1]. The $\Lambda$CDM model heavily relies on two main components that control Universe’s dynamics, the cosmological constant $\Lambda$ and the assumption of a cold dark matter particle. There are still many issues to be resolved in the cosmology of $\Lambda$CDM model, strongly related with the cosmological constant itself and the dark matter particle. With regard to the cosmological constant, there is no evidence that it is indeed constant, that is, whether the Universe is accelerating due to the presence of a cosmological constant, or a dynamical dark energy evolution is what is actually observed. With regard to the dark matter component, we really do not know whether a particle actually constitutes the dark matter component of the Universe. For the moment, both these issues are still scrutinized by both theoretical cosmologists and observational cosmologists. Apart from these issues, there are also other observational data that challenge theoretical cosmology, one of which is the so-called $H_0$-tension [2–5], which is a mystery. The $H_0$-tension problem casts doubt on the $\Lambda$CDM model, and this tension is actually the discrepancy between the observed values of the Hubble rate based on CMB data [6], and the value of the Hubble rate measured by using low-redshift methods, like the Cepheid variables [7]. The CMB value of the Hubble rate is $H_0 = (67.8 \pm 0.9)$ km/s/Mpc while the Cepheid based value is $H_0 = (73.24 \pm 1.74)$ km/s/Mpc, so the difference is not small. Apart from the Cepheid variables measurements, there exist other measurements [8, 9], that also verify the $H_0$-tension. A theoretical proposal that may explain the $H_0$-tension problem, is the introduction of an early dark energy era [10–12], which relies on the prediction of an post-inflationary acceleration era occurring after the recombination era, so at a redshift $z \sim 1100$.

Modified gravity [13–20], can successfully mimic the $\Lambda$CDM model at late-times [21–27] and also at the same time provide a successful mechanism for generating inflation. It has been shown sometimes ago that early-time inflation maybe successfully unified with late dark energy epoch [23]. After that a number of realistic modified gravities were constructed, in which inflation is unified with dark energy within the same model, see Refs. [24–30]. In some $F(R)$ gravity models [31, 32], dark matter is composed by some weakly interacting massive particle [33] (WIMP), however it is possible that modified gravity can also describe dark matter effects in the Universe [21, 34]. Among all modified gravities, $F(R)$ gravity models are considered to be the most important ones, since the modification uses functions of the scalar curvature, the most fundamental geometric-related quantity, and for a recent review on $F(R)$ gravity
cosmological and astrophysical phenomenology, see for example [13].

In view of the importance of $F(R)$ gravity among modified gravities, in this work we shall investigate how a post-inflationary early dark energy era can be realized in the context of $F(R)$ gravity. Our aim is to unify the inflationary era with the early and late dark energy eras. We shall investigate in a quantitative way how an early dark energy era can be realized, however we shall also provide a qualitative approach, by constructing an appropriate $F(R)$ gravity model which may unify all the acceleration eras. Moreover, we shall also study in detail the phenomenology of one of these models, focusing on the late-time behavior, and we shall examine the effects of the early dark energy generating term on the late-time behavior. This will give us insights on the values of the free parameters of the model. We shall use the statefinder quantity $y_H(z)$ which is a function of the redshift and of the Hubble rate $H(z)$, and by examining the era which corresponds to redshifts $z = [0, 10]$, we shall numerically solve the Friedmann equation in order to check at first hand the late-time phenomenological predictions of the model. In addition, we shall also examine the growth factor and the behavior of the effective gravitational constant as functions of the redshift, and investigate the effects of the early dark energy term on these quantities. The results of the numerical analysis will show us how the free parameters of the early dark energy term can be chosen in such a way so that it does not affect the late-time era, but solely the recombination epoch.

II. REALIZATION OF EARLY DARK ENERGY WITH $F(R)$ GRAVITY: A QUANTITATIVE APPROACH

Let us first consider how the early dark energy era can be realized by $F(R)$ gravity by using well known reconstruction techniques for $F(R)$ gravity. We shall use the notation and formalism of Ref. [14]. In order to render the article self-contained, let us briefly review the reconstruction procedure for $F(R)$ gravity. Our aim is to unify the inflationary $F(R)$ gravity and how an arbitrary cosmological evolution can be realized, for more details we refer the reader in Ref. [14]. The gravitational action of the $F(R)$ gravity in the presence of perfect matter fluids is,

$$S = \int d^4x \sqrt{-g} \left( \frac{F(R)}{2\kappa^2} + \mathcal{L}_{\text{matter}} \right),$$

(1)

where $\mathcal{L}_{\text{matter}}$ is the Lagrangian density of the perfect matter fluids that are present. Assuming the Friedman-Robertson-Walker (FRW) space-time with flat spatial part,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2,$$

(2)

upon varying the action of Eq. (1) with respect to the metric tensor, we obtain,

$$0 = -\frac{F(R)}{2} + 3 \left( H^2 + \dot{H} \right) F'(R) - 18 \left( 4H^2\dot{H} + H\ddot{H} \right) F''(R) + \kappa^2 \rho,$$

(3)

where the scalar curvature for the FRW Universe is given by $R = 6\dot{H} + 12H^2$. We may rewrite Eq. (3) by using the $e$-foldings number $N = \ln \frac{a}{a_0}$, and so we obtain,

$$0 = -\frac{F(R)}{2} + 3 \left( H^2 + H'H' \right) F'(R) - 18 \left( 4H^3H' + H^2(H')^2 + H^3H'' \right) F''(R) + \kappa^2 \rho,$$

(4)

where $H' \equiv dH/dN$ and $H'' \equiv d^2H/dN^2$. We may consider the situation that the energy density $\rho$ of the matter perfect fluids present, is given by a sum of the energy densities of the perfect fluids with constant equation of state (EoS) $w_i$, 

$$\rho = \sum_i \rho_i a^{-3(1+w_i)} = \sum_i \rho_i a^{-3(1+w_i)} e^{-3(1+w_i)N}.$$

(5)

In the FRW Universe, the Hubble rate $H$ can be expressed in terms of the $e$-foldings number $N$, $H = g(N)$ and the scalar curvature has the form, $R = 6g'(N)g(N) + 12g(N)^2$, which could be solved (if the equation can be inverted) with respect to $N$ as $N = N(R)$. Then we may rewrite Eq. (4) in terms of the $e$-foldings number $N$, 

$$0 = -18 \left( 4g(N(R))^3 g'(N(R)) + g(N(R))^2 g'(N(R))^2 + g(N(R))^3 g''(N(R)) \right) \frac{d^2F(R)}{dR^2}.$$
the cosmological constant and the third term to the CDM. The (effective) cosmological constant $\Lambda$ in the present universe is

$$H^2 = \frac{3}{\kappa^2} G (N (R)) \frac{d^2 F(R)}{dR^2} - \frac{F(R)}{2} + \sum_i \rho_i a_i^{-3(1+w_i)} e^{-3(1+w_i)N(R)},$$

which constitutes a differential equation for the $F(R)$ gravity, where the variable is the scalar curvature $R$. The expression (6) can be simplified by using $G(N) \equiv g(N)^2 = H^2$, so we get,

$$0 = -9G(N(R)) (4G'(N(R)) + G''(N(R))) \frac{d^2 F(R)}{dR^2} + \left(3G(N(R)) + \frac{3}{2} G'(N(R))\right) \frac{dF(R)}{dR} - \frac{F(R)}{2} + \sum_i \rho_i a_i^{-3(1+w_i)} e^{-3(1+w_i)N(R)},$$

and also $R = 3G'(N) + 12G(N)$. Before going to the case of realizing the early dark energy era, let us exemplify the method by realizing the $\Lambda$CDM model with $F(R)$ gravity. The $\Lambda$CDM model Hubble rate has the following form (ignoring for the moment the radiation),

$$F(0) = \frac{3}{\kappa^2} H^2 = \frac{3}{\kappa^2} H^2 + \rho_0 a^{-3} = \frac{3}{\kappa^2} H^2 + \rho_m(0) a_0^{-3} e^{-3N},$$

where $H_0$ is the Hubble constant at present time, $\rho_m(0)$ is the energy density of cold dark matter (CDM) at present time, and $a_0$ is the scale factor at present time. The second term in the right hand side of Eq. (8) corresponds to the cosmological constant and the third term to the CDM. The (effective) cosmological constant $\Lambda$ in the present Universe is given by $\Lambda = 12H_0^2$. Then one gets,

$$G(N) = H^2 + \frac{\kappa^2}{3} \rho_m(0) a_0^{-3} e^{-3N},$$

and $R = 3G'(N) + 12G(N) = 12H_0^2 + \kappa^2 \rho_m(0) a_0^{-3} e^{-3N}$, which can be solved with respect to $N$ as follows,

$$N = -\frac{1}{3} \ln \left( \frac{R - 12H_0^2}{\kappa^2 \rho_m(0) a_0^{-3}} \right).$$

Then Eq. (7) takes the following form:

$$0 = 3 \left( R - 9H_0^2 \right) \left( R - 12H_0^2 \right) \frac{d^2 F(R)}{dR^2} - \left( \frac{1}{2} R - 9H_0^2 \right) \frac{dF(R)}{dR} - \frac{1}{2} F(R).$$

By changing the variable from $R$ to $x$ by $x = \frac{R}{9H_0^2} - 3$, Eq. (11) reduces to the hypergeometric differential equation,

$$0 = x(1-x) \frac{d^2 F}{dx^2} + (\gamma - (\alpha + \beta + 1) x) \frac{dF}{dx} - \alpha \beta F.$$

Here

$$\gamma = -\frac{1}{2}, \alpha + \beta = -\frac{1}{6}, \quad \alpha \beta = -\frac{1}{6}.$$

The analytic solution of Eq. (12) is the Gauss hypergeometric function $F(\alpha, \beta; x)$,

$$F(x) = A F(\alpha, \beta, \gamma; x),$$

where $A$ and $B$ are constant. Thus, we demonstrated that $F(R)$ gravity may realize the $\Lambda$CDM epoch without the need to introduce an effective cosmological constant. The attribute of $F(R)$ gravity is that the dark energy era has not a constant equation of state parameter, but it is dynamical. In [10], a parametrization of the fractional energy density for the early dark energy was introduced,

$$\Omega_{d_\text{e}}(a) = \frac{\Omega^0_{d_\text{e}} - \Omega_c (1 - a^{-3w_0})}{\Omega^0_{d_\text{e}} + \rho_m(0) a^{-3w_0}} + \Omega_c (1 - a^{-3w_0}),$$

where $\Omega^0_{d_\text{e}} + \rho_m(0) \sim 1$ and we choose $\Omega_c = 0.01, \rho_m(0) \sim 0.3$, and $w_0 \sim -1$ (in a later section we shall use more accurate values for the matter densities). Then the Hubble rate is given by

$$\frac{H^2(a)}{H_0^2} = \frac{\Omega_m a^{-3} + \Omega_{d_\text{e}} a^{-4}}{1 - \Omega_{d_\text{e}}(a)},$$

where $\Omega^0_m + \rho_m(0) \sim 1$ and we choose $\Omega_c = 0.01, \rho_m(0) \sim 0.3$, and $w_0 \sim -1$ (in a later section we shall use more accurate values for the matter densities). Then the Hubble rate is given by

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$$\frac{H^2(a)}{H_0^2} = \frac{\Omega_m a^{-3} + \Omega_{d_\text{e}} a^{-4}}{1 - \Omega_{d_\text{e}}(a)},$$
where \( \Omega_{\text{rel}}^0 \) is fractional energy density of relativistic particles at present day, with \( \Omega_{\text{rel}}^0 \sim 5 \times 10^{-5} \). Then expressing everything in terms of the \( e \)-folds, we get,

\[
G(N) = H(N)^2 = H_0^2 \frac{\Omega_m^0 e^{-3N} + \Omega_{\text{rel}}^0 e^{-4N}}{1 - \frac{\Omega_m^0}{\Omega_e^0 + \Omega_{\text{rel}}^0 e^{-3N}} + \Omega_e(1 - e^{3N})}
\]

\[
= H_0^2 \frac{\Omega_m^0 e^{-3N} + \Omega_{\text{rel}}^0 e^{-4N}}{\Omega_m^0 e^{-3N} + \Omega_e(1 - e^{3N}) + \Omega_e(1 - e^{3N})(\Omega_{\text{rel}}^0 + \Omega_m^0 e^{-3N})^2}.
\]

Here we have put \( w_0 = -1 \) and \( a_0 = 1 \), which means that \( N = 0 \) at present time. Then we find,

\[
G'(N) = H_0^2 \left\{ \frac{(-3\Omega_m^0 e^{-3N} - 4\Omega_{\text{rel}}^0 e^{-4N})(\Omega_{\text{de}}^0 + \Omega_m^0 e^{-3N}) - 3(\Omega_m^0 e^{-3N} + \Omega_{\text{rel}}^0 e^{-4N})\Omega_m^0 e^{-3N}}{\Omega_m^0 e^{-3N} + \Omega_e(1 - e^{3N}) + \Omega_e(1 - e^{3N})(\Omega_{\text{rel}}^0 + \Omega_m^0 e^{-3N})^2} \right\}. \tag{17}
\]

Then after solving the algebraic equation \( R = 6g'(N)g(N) + 12g(N)^2 \) with respect to \( N \) as \( N = N(R) \), we may solve the differential equation \( (7) \), which gives the functional form of the \( F(R) \) implicitly. However, it is rather difficult to extract analytic results from the above differential equation. Therefore, in the next section we shall adopt a more qualitative approach in order to provide a model for early dark energy, and simultaneously the same model can describe the late-time era too. We also discuss several models of early and late-time dark energy, and we quantify our qualitative proposals by numerically integrating the Friedman equation.

## III. REALIZATION OF EARLY DARK ENERGY WITH \( F(R) \) GRAVITY: A QUALITATIVE APPROACH

Since we showed that it is rather difficult to construct the \( F(R) \) gravity model that may realize the early dark energy quantitatively, we now try to construct a model by adopting a qualitative approach. We may consider the following type of \( F(R) \) gravity,

\[
F(R) = R + F_{\text{inf}}(R) + F_{\text{EDE}}(R) + F_{\text{DE}}(R), \tag{19}
\]

where the term \( F_{\text{inf}}(R) \) is chosen in such a way so that it dominates in the high curvature regime, so during the inflationary era. Also the term \( F_{\text{DE}}(R) \) can be chosen in such a way so that it dominates in the late-time era, and can describe the dark energy fluid. For example, we may choose

\[
F_{\text{inf}}(R) = \frac{R^2}{M^2}, \quad F_{\text{DE}}(R) = 2\Lambda \left( e^{-\frac{R}{M}} - 1 \right),
\]

but there are also several other examples which may realize successfully an early and a late-time era. Another interesting model firstly appeared in the end of [31], in which case the functional form of the \( F(R) \) gravity is,

\[
R + F_{\text{inf}}(R) + F_{\text{DE}}(R) = R + \frac{R^2}{M^2} - \gamma \Lambda \left( \frac{R}{3M^2} \right)^{\delta}, \tag{20}
\]

where \( \delta \) is a positive number \( 0 < \delta < 1 \) and the rest of the parameters will be defined later on in this section. In the model of Eq. (21) the \( R^2 \) term drives the inflationary era, while the term \( \sim R^\delta \) drives the late-time era.

Returning our focus on Eq. (19), the term \( F_{\text{EDE}}(R) \) can be chosen in such a way so that it may realize the early dark energy era. In order to construct a model of \( F_{\text{EDE}}(R) \), we should recall the following [18, 26] issues: When \( F(R) \) is written in the neighborhood of \( R \sim R_0 \) as,

\[
\frac{F(R)}{R^2} \sim f_0 + f(R)(R - R_0)^n,
\]

with constants \( f_0 > 0, R_0 > 0 \), and a positive integer \( n \), if a function \( f(R) \) does not vanish at \( R = R_0, f(R_0) \neq 0 \), and \( R = R_0 \) is an exact solution describing the de Sitter space-time. It has been shown that

1. When \( n \) is an even integer and \( f(R_0) < 0 \), the solution describing the de Sitter space-time is stable.
2. When \( n \) is an even integer and \( f(R_0) > 0 \), the solution describing the de Sitter space-time is unstable.

3. When \( n \) is an odd integer, the solution describing the de Sitter space-time is quasi-stable. If \( f(R_0) < 0 \), the curvature \( R \) decreases by crossing \( R = R_0 \).

4. When \( n \) is an odd integer, the solution describing the de Sitter space-time is quasi-stable. If \( f(R_0) > 0 \), the curvature \( R \) increases by crossing \( R = R_0 \).

The third case may describe the early dark energy since the quasi-stable de Sitter solution would realize a post-inflationary early dark energy era. Then for example, we may choose,

\[
f(R) = -\frac{\beta R^{m-n-2}}{R_0^{l+m} + R^{l+m}},
\]

with a constant \( \beta > 0 \) of dimensions \([m]^{-1}\), and we can choose the integers \( m > 0 \) and \( l > 0 \) to be large enough. We also choose \( R_0 \) in such a way so that \( R_0 \) is slightly larger than the curvature of the Universe during the recombination epoch. Then we find,

\[
F_{\text{DE}}(R) = f_0 R^2 - \frac{\beta R^{m-n} (R - R_0)^n}{R_0^{l+m} + R^{l+m}}.
\]

The first term in (24) can be absorbed into the redefinition of \( \frac{1}{M^2} \) in \( F_{\text{inf}}(R) \) if we choose \( F_{\text{inf}}(R) \) as in (20) and therefore we put \( f_0 = 0 \) in the following. We should note that the second term in Eq. (24) decreases in the early Universe where \( R \) is large and in the late Universe where \( R \) is small because \( m \) and \( l \) are large enough. In the early Universe, where the curvature \( R \) could be large and we assume \( R \gg R_0 \), if we choose \( \Lambda \) in (20) small enough and \( l \) in (23) or (23) large enough, so that,

\[
|F_{\text{DE}}(R)| \sim \frac{\beta}{R^2} \ll |F_{\text{inf}}(R)| = \frac{1}{M^2} R^2,
\]

the second term \( F_{\text{inf}}(R) \) in (19) becomes dominant and generates the inflationary era. When the curvature becomes smaller, \( R \sim R_0 \), if we choose \( \frac{1}{M^2} R_0 \) \( \ll \beta R_0^l \) and \( m \) is large enough, the third term \( F_{\text{DE}}(R) \) could dominate except right on the point \( R = R_0 \) and generate the quasi-de Sitter space-time, which may be identified with the early dark energy era. Since the de Sitter space-time solution at the point \( R = R_0 \) is quasi-stable, the curvature becomes smaller very slowly and after that the recombination era occurs. After the recombination era, the curvature \( R \) decreases significantly and eventually, since we chose \( m \) to be large enough, the fourth term \( F_{\text{DE}}(R) \) in (19) becomes much larger than the third term \( F_{\text{DE}}(R) \),

\[
|F_{\text{DE}}(R)| \sim \frac{\beta R^{m-n}}{R_0^{l+m-n}} \ll |F_{\text{DE}}(R)| \sim 2 |\Lambda|.
\]

We also choose the parameter \( M \) of \( F_{\text{inf}}(R) \) in (20), in such a way so that the fourth term \( F_{\text{DE}}(R) \) in (19) becomes much larger than the second term \( F_{\text{inf}}(R) \), that is,

\[
|F_{\text{inf}}(R)| = \frac{1}{M^2} R^2 \ll |F_{\text{DE}}(R)| \sim 2 |\Lambda|.
\]

when \( R \ll R_0 \). When \( R_0 \gg R \gg R_I \) as in the Universe at present time, the fourth term \( F_{\text{DE}}(R) \) in (19) becomes dominant and behaves as a constant \( F_{\text{DE}}(R) \sim 2 \Lambda \), which plays the role of the effective cosmological constant in the present Universe. Then by adjusting the parameters in \( F_{\text{DE}}(R) \), we could in principle modify the Hubble constant in the present Universe, in such a way so that the Hubble constant is not conflict with the value given by the observations of the Cepheid variables. Since \( F_{\text{DE}}(R) \) affects strongly in a decreasing way both the the large curvature and the small curvature regimes, the term does not influence the inflationary era, where the curvature is large, and also the accelerating expansion of the present Universe, where the curvature is small. Another model unifying the inflationary and the late-acceleration eras has been proposed in [29],

\[
R + F_{\text{inf}}(R) + F_{\text{DE}}(R) = R - 2 \Lambda_I \left(1 - e^{-\frac{R}{R_I}}\right) - \Lambda_I \left(1 - e^{-\left(\frac{R}{R_I}\right)^n}\right) + \gamma R^\alpha.
\]

The last term with constant \( \gamma \) and \( 1 < \alpha \leq 2 \) are added in order to avoid that the curvature singularity appears in the dense matter regions (see [26], for example). The constants \( R_I \) and \( R_I \) correspond to the curvature in the
inflationary epoch and in the late-time acceleration epoch, respectively, and \( \Lambda_t \) and \( \Lambda_f \) are constants which act as effective cosmological constants during the inflationary and the late-time acceleration epochs, respectively. Then by adding \( F_{\text{EDE}}(R) \) to (28), we may also realize the early dark energy.

Now let us analyze the effects of \( F_{\text{EDE}}(R) \) during the late-time epoch in a more concrete way, and see at first hand the range of values of the free parameters for which the effect of \( F_{\text{EDE}}(R) \) on the late-time era is insignificant. We shall analyze the model (21) with the addition of \( F_{\text{EDE}}(R) \), so that the final \( F(R) \) gravity is,

\[
F(R) = R + \frac{R^2}{M^2} - \gamma \Lambda \left( \frac{R}{3m_s^2} \right)^\delta - \frac{\beta R^{m-n}(R - R_0)^n}{R_0^{n+m} + R^{n+m}}.
\]  

We shall also assume that the Universe is filled with dust and radiation perfect fluids, for the late-time study of the model (29). The Universe’s evolution is affected by the geometry via the \( F(R) \) gravity, so the geometry contributes an energy density and a pressure in the Friedmann and Raychaudhuri equation, which for the FRW space-time is written as follows,

\[
3H^2 = \kappa^2 \rho_{\text{tot}}, \quad -2\dot{H} = \kappa^2 (\rho_{\text{tot}} + P_{\text{tot}}),
\]

with \( \rho_{\text{tot}} = \rho_m + \rho_G + \rho_r \) being the total energy density of the Universe, and accordingly the total pressure is \( P_{\text{tot}} = P_r + P_G \). Also \( \rho_r \) is the energy density of relativistic matter, \( P_r \) is its pressure, and \( \rho_G \) is the energy density of the geometric fluid,

\[
\rho_G = \frac{F_R R - F}{2} + 3H^2(1 - F_R) - 3H \dot{F}_R,
\]

where \( F_R = \frac{\partial F}{\partial R} \). The geometric fluid will drive inflation at early times, and eventually will also produce the early dark energy era, and accordingly will drive the late-time evolution. The corresponding pressure of the geometric fluid is equal to,

\[
P_G = \dot{F}_R - H \dot{F}_R + 2\dot{H}(F_R - 1) - \rho_G.
\]

It is easy to show that all the fluids satisfy the energy conservation conditions,

\[
\dot{\rho}_a + 3H(\rho_a + P_a) = 0, \quad \dot{\rho}_r + 3H(\rho_r + P_r) = 0, \quad \dot{\rho}_G + 3H(\rho_G + P_G) = 0.
\]

Our aim in the rest of this section is to numerically solve the cosmological equations, by using appropriate variables and specific values for the free parameters that may yield a viable late-time phenomenology. We start off by specifying the values of the free parameters of the model (29), so the parameter \( M \) for phenomenological reasons related to the inflationary era must be [35],

\[
M = 1.5 \times 10^{-5} \left( \frac{N}{50} \right)^{-1} M_P,
\]

where \( N \) is the e-foldings number during inflation, and \( M_P \) is the reduced Planck mass \( M_P = 2.435 \times 10^{27} \text{eV} \). Hence for \( N \sim 60, \) \( M \) takes approximately the value \( M \approx 3.04375 \times 10^{22} \text{eV} \). In addition, \( m_s \) in Eq. (29) shall be taken equal to \( m_s^2 = \frac{\kappa^2 \rho_m^{(0)}}{3} \), and in addition we shall assume that \( \Lambda \simeq 11.895 \times 10^{-7} \text{eV}^2 \). The latest Planck data on the cosmological parameters indicate that [6],

\[
H_0 = 67.4 \pm 0.5 \text{km}/(\text{sec} \cdot \text{Mpc}),
\]

hence \( H_0 = 67.4 \text{km}/(\text{sec} \cdot \text{Mpc}) \) or equivalently \( H_0 = 1.37187 \times 10^{-33} \text{eV} \), therefore \( h \simeq 0.67 \). Moreover, the latest Planck data also indicate that \( \Omega_c h^2 \) is,

\[
\Omega_c h^2 = 0.12 \pm 0.001,
\]

in effect, the parameter \( m_s^2 \) is,

\[
m_s^2 = \frac{\kappa^2 \rho_m^{(0)}}{3} = H_0 \Omega_c = 1.37201 \times 10^{-67} \text{eV}^2.
\]  

The parameters \( \delta \) and \( \gamma \) will be assumed to take the values \( \delta = 1/100 \) and \( \gamma = 2 \). The phenomenology of the model (29) without the early dark energy term is quite interesting and is studied in detail in [30], however we shall briefly
show that it indeed yields a viable phenomenology at late-times. Then we shall include the early dark energy term in order to investigate when the early dark energy term affects the late-time era, and then constrain the parameter $\beta$. For our late-time era analysis, we shall express all the quantities involved in the cosmological equations as functions of the redshift $z$,

$$1 + z = \frac{1}{a}, \quad (38)$$

by also taking the scale factor of the current Universe at $z = 0$ equal to unity. Moreover, we introduce the statefinder quantity $y_H(z)$ [36, 37], which is equal to,

$$y_H(z) = \frac{\rho_G}{\rho_m}, \quad (39)$$

so by using the first Friedmann equation (30), $y_H(z)$ takes the following form,

$$y_H(z) = \frac{3H^2}{\kappa^2 \rho_m^{(0)}} - \frac{\rho_m^{(0)}}{\rho_m^{(0)}} - \frac{\rho_x}{\rho_m^{(0)}}. \quad (40)$$

Due to the fact that $\rho_x = \rho_x^{(0)} a^{-4}$, which means $\frac{\rho_x}{\rho_m} = \chi (1 + z)^4$, with $\chi = \frac{\rho_x^{(0)}}{\rho_m^{(0)}} \simeq 3.1 \times 10^{-4}$, and $\rho_m = \rho_m^{(0)}$, $y_H(z)$ is equal to,

$$y_H(z) = \frac{H^2}{m_s^2} - (1 + z)^3 - \chi (1 + z)^4. \quad (41)$$

We can easily express the Friedmann equations in terms of $y_H(z)$, so it reads [37],

$$\frac{d^2y_H(z)}{dz^2} + J_1 \frac{dy_H(z)}{dz} + J_2 y_H(z) + J_3 = 0, \quad (42)$$
with the functions $J_1$, $J_2$ and $J_3$ being defined in the following way,

$$J_1 = \frac{1}{z+1} \left( -3 - \frac{1 - F_R}{y_H(z) + (z + 1)^3 + \chi(1 + z)^4} 6 \omega'_R F_{RR} \right),$$

$$J_2 = \frac{1}{(z+1)^2} \left( \frac{2 - F_R}{(y_H(z) + (z + 1)^3 + \chi(1 + z)^4) 3 \omega'_R F_{RR}} \right),$$

$$J_3 = -3(z+1) - \frac{(1 - F_R)(z + 1)^3 + 2(1 + z)^4 + \frac{R - F}{3m^2}}{(1+z)^2(y_H(z) + (1+z)^3 + \chi(1+z)^4) 6 \omega'_R F_{RR}},$$

with $F_{RR} = \frac{\omega''_R}{F_R}$. We shall numerically solve the differential equation (42), in the redshift interval $z = [z_i, z_f] = [0, 10]$ with the initial conditions for the function $y_H(z)$ at redshift $z_f = 10$ chosen as follows,

$$y_H(z_f) = \frac{\Lambda}{3m^2} \left( 1 + \frac{1}{1000(1 + z_f)} \right), \quad \frac{dy_H(z)}{dz} \bigg|_{z = z_f} = \frac{1}{1000} \frac{\Lambda}{3m^2}.$$

Let us now present the results of the numerical analysis, focusing first briefly on the model (29) without the presence of the early dark energy term. The full analysis of this model is performed elsewhere [30]. In order to have some standard comparison reference, we shall compare the results of the model (29) without the early dark energy term, with the ΛCDM model. Accordingly, we shall compare the model without the early dark energy term, with the model (29), in the presence of the early dark energy term, in order to see which values of the parameter $\beta$ can affect the late-time behavior. Let us start off our analysis with the phenomenology of the model (29) without the early dark energy term, and by numerically solving the differential equation (42), we present the solution $y_H(z)$ in the left plot of Fig. 1. In the right plot of Fig. 1, we present the dark energy EoS parameter $\omega_G(z)$, and in the bottom plot we present the deceleration parameter $q(z)$. In all the plots, the blue curves correspond to the $F(R)$ gravity model (29) without the early dark energy term, and the red curves correspond to the ΛCDM model. Also at $z = 0$, the value of the dark energy EoS parameter is evaluated to be $\omega_G(0) = -0.995827$, and also the dark energy density parameter is $\Omega_G(0) = 0.683968$, which are within the constraints of the latest Planck data on the cosmological parameters. Moreover, the deceleration parameter is $q(0) = -0.520954$, which is very close to the ΛCDM value $q_\Lambda = -0.5$. Now let us turn our focus on the model (29), by taking into account the presence of the early dark energy term. Since the early dark energy era is engineered to occur around the recombination era, so for $z \sim 1100$, the curvature of the Universe $R_0$ at this era is of the order $O(10^{-59}) eV^2$, we shall assume that $R_0 = 10^{-59} eV^2$. Also $m$ must be an odd integer in order for the early dark energy era to have an unstable de Sitter attractor, hence we take $n = 3$. Furthermore, $l$ and $m$ must be larger from $n$, so we choose $l = m = 8$. As it seems, the early dark energy term has no effect on the late-time era, when $\beta \sim R_0$, however, when $\beta \geq 10^{400}R_0$, the early dark energy term starts to affect the late-time era. Indeed this can be seen in Fig. 2, where we plot the statefinder $y_H(z)$ (left upper plot), the dark energy EoS parameter (right upper plot) and the deceleration parameter $q(z)$ (bottom plot). In all the plots of Fig. 2, the blue curves correspond to the model (29), while the red curves to the model (29) without the early dark energy term. As it can be seen, the effects of the early dark energy term on the late-time evolution are significant, but this occurs only when $\beta \geq 10^{400}R_0$. Also the value of the dark energy EoS parameter is $\omega_G(0) = -0.996237$, the dark energy density parameter $\Omega_G(0) = 0.702882$ and the deceleration parameter $q(0) = -0.55031$, so the effect of the early dark energy term affect the late-time dynamics for $\beta \geq 10^{400}R_0$.

Another important issue to discuss in the matter density perturbations issue and the corresponding growth factor. In fact, the matter density perturbations can be considered as a consistent criterion that can distinguish the cosmic evolution corresponding to different $F(R)$ gravities. This is because the cosmological matter density perturbations actually make possible the distinction between the evolution of each $F(R)$ gravity from the background [38]. With regard to the matter density perturbations, a consistent calculation of these requires the subhorizon approximation, so that the theoretical framework is consistent with the Newtonian gravity [38]. The subhorizon approximation is quantified by the condition,

$$\frac{k^2}{a^2} \gg H^2,$$

where $a(t)$ is the scale factor and $k$ is the wavenumber of the comoving mode. Practically, the subhorizon approximation indicates that we should consider comoving wavelengths $\lambda = a/k$ of a comoving mode $k$, which are much shorter than the corresponding Hubble radius $H^{-1}$. The subhorizon approximation holds true during the matter domination era, and we shall focus on the last stage of the matter domination era and the dawn of the dark energy era, so for redshifts $z = [0, 10]$. The matter density perturbations are quantified by the parameter $\delta = \frac{\delta m}{\rho_m}$, which satisfies the following differential equation [37],

$$\ddot{\delta} + 2H \dot{\delta} - 4\pi G_{\text{eff}}(a, k) \rho_m \delta = 0$$

(46)
FIG. 2: The function $yH$ (left upper plot), the dark energy EoS parameter $\omega_G(z)$ (right upper plot) and the deceleration parameter (bottom plot), as functions of the redshift, for the $F(R)$ gravity model with the early dark energy term (blue curves), for $\beta \geq 10^4 R_0$. The red curves correspond to the model without the early dark energy term.

where $G_{\text{eff}}(a,k)$ is the effective gravitational constant of the $F(R)$ gravity theory, with its analytic form being for $F(R)$ gravity [37],

$$G_{\text{eff}}(a,k) = \frac{G}{F'(R)} \left[ 1 + \frac{k^2 F''(R)}{\alpha^2 F'(R)} \right],$$

and $G$ denotes the present value of Newton’s constant of gravity. In the following we shall focus on the growth factor $f_g(z) = \frac{d\ln \delta}{d\ln a}$, so we can express the differential equation governing the matter density perturbations evolution (46) in terms of the growth factor. By using the following rules,

$$\dot{H} = \frac{dH}{dz} (z + 1)H(z), \quad \dot{\delta} = H f_g \delta, \quad \ddot{\delta} = \dot{H} f_g \delta + H \dot{f}_g \delta + H f_g \dot{\delta},$$

we can express the differential equation that governs the evolution of the matter perturbations (46), in terms of the growth factor,

$$\frac{df_g(z)}{dz} + \left( \frac{1 + z}{H(z)} \frac{dH(z)}{dz} - 2 - f_g(z) \right) \frac{f_g(z)}{1 + z} + \frac{4\pi G_{\text{eff}}(a,k)}{G (z + 1)H(z)^2 \rho_m} = 0.$$  

From the above equation it is apparent that the effect of $F(R)$ gravity on the growth factor is quantified in the term $G_{\text{eff}}(a(z),k)$, and it is notable that the latter depends on the comoving wavenumber $k$. By taking the present time Universe $a = 1$, the comoving number can be constrained by the present time value of the Hubble rate, so since $k \gg H_0$, the wavenumber $k$ must be $k > 0.000124011$ Mpc$^{-1}$. In the following we shall numerically solve the differential equation (49), and we shall investigate two main issues, firstly the general behavior of the growth factor as a function of the redshift for various allowed values of the comoving wavenumber $k$, for the model (21). Accordingly, we shall consider the model (29) in which the early dark energy term is added, and we shall investigate the behavior of the model and examine whether the early dark energy term affects the late-time behavior of the growth factor. This study shall validate the assumptions made in the previous late-time study, which indicated that when $\beta \sim R_0$, the early dark energy term has no effect on the late-time evolution. For the numerical study, we shall assume that...
the growth factor at a redshift $z_f = 10$ is equal to, $f_g(z_f, k) = 0.997595 \ [37]$, which is the value of the growth factor for the $\Lambda$CDM model. For comparison reasons, we shall also compare the $F(R)$ gravity model (21) without the early dark energy term with the $\Lambda$CDM, so at this point we quote the $\Lambda$CDM expression for the growth factor, which is

\[ f_g(z) = \left( \frac{H_0^2 \Omega_m^0 (1 + z)^3}{H(z)^2} \right)^{\frac{1}{2}}, \]

(50)

where $H(z)$ in this case is the $\Lambda$CDM model Hubble rate. Our aim at this point is to firstly indicate explicitly that the $F(R)$ gravity model of Eq. (21) behaves similarly to the $\Lambda$CDM model, and secondly, to investigate if the addition of the early dark energy era term affects the late-time behavior, and to which extent, having in mind that when $\beta \sim R_0^l$ the early dark energy term does not affect the late-time era, as we saw in the previous section. Let us first focus on the model (21) and in Fig. 3 we present the behavior of the fraction $\frac{G_{\text{eff}}}{G}$ for $k = 1 \text{Mpc}^{-1}$ (black curve), $k = 0.0125 \text{Mpc}^{-1}$ (green curve), $k = 0.00125 \text{Mpc}^{-1}$ (blue curve) and $k = 0.000125 \text{Mpc}^{-1}$ (red curve). As it can be seen, as the limiting allowed $k$ value is reached (red and blue curves), the effective gravitational constant approaches the present day Newton’s value. Also in Fig. 4 we present the results of the numerical solution to the equation (49) which yields the growth factor $f_g(z)$ as a function of the redshift, for $k = 1 \text{Mpc}^{-1}$ (black curve), $k = 0.0125 \text{Mpc}^{-1}$ (blue curve), $k = 0.00125 \text{Mpc}^{-1}$ (green curve) and $k = 0.000125 \text{Mpc}^{-1}$ (yellow curve). Also the black dashed curve corresponds to the growth factor of the $\Lambda$CDM model. As it can be seen, the green and yellow curves are indistinguishable from the $\Lambda$CDM model, and deviations occur only for higher values of the wavenumber. The results of the numerical analysis hold true, even when the early dark energy term is added, hence even for the model (29) and remarkably, no deviations occur even when $\beta \geq 10^{400} R_0^l$. Therefore in conclusion, the dark energy era of the model (29) is not affected at all from the early dark energy term. Using the same approach as in this section,
with similar a form of early dark energy, one can show that the realistic $F(R)$ gravities unifying inflation with dark energy of Refs. [13, 24–30], may be easily extended by adding an early dark energy term which does not spoil the inflationary and dark energy behavior of the models.

IV. CONCLUSIONS

In this paper we investigated how a post-inflationary early dark energy era can be realized by $F(R)$ gravity, and how to describe the inflationary era, the early and late-time dark energy era with a single $F(R)$ gravity model. For our study we used two approaches, namely a quantitative approach in which we used a well-known reconstruction technique in order to realize a generalized Hubble rate that describes all the above mentioned cosmological eras. In our second approach, we used several qualitative arguments in order to construct an appropriate $F(R)$ gravity that may realize inflation with the early and late-time dark energy eras. In the process of selecting a suitable early dark energy term, we investigated the constraints that the free parameters of the model must satisfy in order for an early dark energy era to be realized. Moreover, for one of the proposed models, we investigated the cosmological behavior of the model at late times, with and without the early dark energy term. We numerically integrated the Friedmann equation, by using the redshift as variable, and we expressed all the physical quantities as functions of the statefinder function $y_H(z)$, which solely depends on the redshift and the Hubble rate $H(z)$. The results of our numerical analysis indicated that the model is very similar to the ΛCDM model at late-times, and also we demonstrated which values of the free parameters make the early dark energy term to have no effect on the late-time dark energy era. Also we investigated the behavior of the growth factor for the same model, and of the effective gravitational constant focusing on redshifts $z \leq 10$. For all the cases we studied, we found that the early dark energy term does not affect at all neither the effective gravitational constant, nor the growth factor. With regard to the effective gravitational constant, we found that for some values of the wavenumber, the gravitational constant is equal to the present time value of Newton’s gravitational constant, however, there are differences for a limited set of values of the wavenumber, mainly large values. This study should be extended to even larger redshifts, and this in principle could have an observable effect on the CMB, via the gravitational interaction and the Thomson scattering when the CMB was generated, exactly on the last scattering surface. In general the pivot scale dependence of cosmological parameters is a quite confusing issue in modern theoretical cosmology, so caution is required on this topic, therefore we did not go deeper in our analysis. In addition, with regard to the growth factor, since the latter characterizes the growth of the matter perturbations, an important quantity is the so-called growth index $\gamma$, which relates the growth factor $f_g(z)$ with the matter density parameter $\Omega_m(z)$ as $f_g(z) = \Omega_m(z)^{\gamma(z)}$. The form of the growth index is currently unknown, and its value cannot be determined directly, however, its value may be inferred by the observational values of $\Omega_m(z)$ and $f_g(z)$. A cosmographic approach [40] may also shed some light on the exact functional form of the growth index, but we defer this study to a future work.

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