Inequivalent quantization of the rational Calogero model with a Coulomb type interaction

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We consider the inequivalent quantizations of a $N$-body rational Calogero model with a Coulomb type interaction. It is shown that for certain range of the coupling constants, this system admits a one-parameter family of self-adjoint extensions. We analyze both the bound and scattering state sectors and find novel solutions of this model. We also find the ladder operators for this system, with which the previously known solutions can be constructed.

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1. INTRODUCTION

Exactly solvable quantum many body systems like Calogero model and its variants \textsuperscript{1, 2, 3} have found diverse applications in many branches of contemporary physics, including generalized exclusion statistics \textsuperscript{4}, quantum hall effect \textsuperscript{5}, Tomonaga-Luttinger liquid \textsuperscript{6}, quantum chaos \textsuperscript{7}, quantum electric transport in mesoscopic system \textsuperscript{8}, spin-chain models \textsuperscript{9, 10}, Seiberg-Witten theory \textsuperscript{11} and black holes \textsuperscript{12}. The rational Calogero model is described by $N$ identical particles interacting with each other through a long-range inverse-square and harmonic interaction on the line \textsuperscript{1}. The exact spectrum of this rational Calogero model with harmonic confinement has been found through a variety of different techniques \textsuperscript{1, 15}, all of which impose the boundary condition that the wavefunction and the current vanish when any two or more particles coincide. With this boundary condition the Hamiltonian is self-adjoint, which ensures the reality of eigenvalues as well as the completeness of the states. However it was found later that, within a certain region of the parameter space, there exist more general boundary conditions for which the rational Calogero Hamiltonian (with and without harmonic confinement) admits self-adjoint extensions and yields a rich variety of spectra \textsuperscript{17, 18}. As is well known, the possible boundary conditions for an operator are encoded in the choice of its domains, which are classified by the self-adjoint extensions \textsuperscript{19} of the operator. Such self-adjoint extensions play important roles in a variety of physical contexts including Aharonov-Bohm effect \textsuperscript{20}, two and three dimensional delta function potentials \textsuperscript{21}, anyons \textsuperscript{22}, anomalies \textsuperscript{23}, $\zeta$-function renormalization \textsuperscript{24}, particle statistics in one dimension \textsuperscript{25} and black holes \textsuperscript{26}. So it should be interesting to find out more examples of exactly solvable models which can be quantized by using the method of self-adjoint extension.

In this context it may be noted that an exactly solvable variant of the rational Calogero model has been constructed by Khare \textsuperscript{27}, where the confining simple harmonic potential is replaced by a coulomb-like interaction. The bound states of this model can be related to those of the rational Calogero model with harmonic confinement by using the

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underlying $SU(1,1)$ algebra. However, apart from having an infinite number of bound states, this model with coulomb-like interaction also supports continuous scattering states. Similar to the case of original Calogero model, these bound as well as scattering states have been constructed by using the boundary condition that wavefunction and the current vanish when any two or more particles coincide. Due to a factorization property of the eigenfunctions, the eigenvalue problem of this many body system can be reduced to that of the corresponding radial Hamiltonian. In this article, our aim is to find out more general boundary conditions which admit self-adjoint extensions for the radial part of the rational Calogero model with coulomb-like interaction and study the related spectra.

The arrangement of this article is as follows. In Sec.2 we briefly recapitulate how to separate the radial part of the rational Calogero Hamiltonian with coulomb-like interaction. Then we discuss about the most general form of eigenstates associated with this radial Hamiltonian $H_r$. These eigenstates could be singular or nonsingular at $r = 0$ value of the radial coordinate $r$. The bound and scattering states found by Khare are all nonsingular at $r = 0$ [27]. In Sec.3 we show that such nonsingular bound states can also be constructed by using creation annihilation operators associated with the underlying $SU(1,1)$ algebra. In Sec.4 we show that the radial Hamiltonian $H_r$ admits self-adjoint extensions within a certain region of the parameter space. Inequivalent quantizations of $H_r$ by using this method lead to bound and scattering eigenstates which are in general singular at $r = 0$. We explicitly construct such bound states and scattering states in Sec.5 and Sec.6 respectively. In Sec.6, we also derive the scattering matrix for the scattering states and show that the eigenvalues of the bound states can be reproduced from the poles of this scattering matrix. Sec.7 is the concluding section.

2. GENERAL FORM OF EIGENFUNCTIONS OF THE RADIAL HAMILTONIAN

The Hamiltonian for the rational Calogero model with the Coulomb type term is given by

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g \sum_{i<j} \frac{1}{(x_i - x_j)^2} - \frac{\alpha}{\sqrt[4]{\sum_{i<j}(x_i - x_j)^2}}.$$ (1)

It describes the $N$-body problem with equal mass in 1-dimension and in units such that $2mh^{-2} = 1$. It is also understood that the coupling constant $g$ is constrained to satisfy condition $g > -\frac{1}{2}$. The parameter $\alpha$ will be allowed to have any value, positive or negative, which will provide us with the possibility to treat the attractive as well as the repulsive Coulomb potential, respectively. This point is somewhat different in comparison to the treatment made in [27] where only the case for $\alpha > 0$ is considered. Having the Hamiltonian (1), we intend to solve the eigenvalue problem

$$H \Psi = E \Psi.$$ (2)

Following [1], we consider the above eigenvalue equation in a sector of configuration space corresponding to a definite ordering of particles given by $x_1 \geq x_2 \geq \cdots \geq x_N$. The translationally invariant eigenfunctions of the Hamiltonian $H$ can be factorized as

$$\Psi = \prod_{i<j} (x_i - x_j)^{\alpha + \frac{k}{2}} \phi(r) P_k(x),$$ (3)

where $x$ is an abbreviation for $\{x_1, x_2, \ldots, x_N\}$, $a = \pm \frac{1}{2} \sqrt{1 + 2g}$ and $r$ is the collective radial variable defined as

$$r^2 = \frac{1}{N} \sum_{i<j} (x_i - x_j)^2.$$ (4)

Functions $P_k(x)$ are translationally invariant, homogeneous polynomials of degree $k$, $k \geq 0$. They satisfy the equation

$$\left[\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2(a + \frac{1}{2}) \sum_{i<j} \frac{1}{(x_i - x_j)} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\right] P_k(x) = 0.$$ (5)
This equation is analyzed in detail by Calogero [1]. After inserting the factorized form of $\Psi$ from Eq. (3) into Eq. (2) and using Eqs. (4) and (5), we get an equation for $\phi(r)$ as

$$H_r \phi = E\phi,$$

where

$$H_r = -\frac{d^2}{dr^2} - (2k + 2b + 1)\frac{1}{r} \frac{d}{dr} - \frac{\alpha}{\sqrt{N}r}.$$  (7)

The parameter $b$ which enters Eq. (7) is defined as

$$b = \frac{N(N-1)}{2} \left(a + \frac{1}{2}\right) + \frac{N}{2} - \frac{3}{2}.$$  (8)

It should be noted that, due to Eq. (3), the measure of the quadratically integrable $\phi(r)$ is given by $d\sigma = r^\beta dr$, where $\beta = 2k + 2b + 1$. Thus, the eigenvalue equation (2) of the Calogero model with Coulomb-like interaction is reduced to the eigenvalue equation (6) of the corresponding radial Hamiltonian $H_r$, which may be explicitly written as

$$-\frac{d^2}{dr^2}\phi(r) - (2k + 2b + 1)\frac{1}{r} \frac{d}{dr}\phi(r) - \frac{\alpha}{\sqrt{N}r}\phi(r) = E\phi(r).$$  (9)

However, for the purpose of finding out all possible boundary conditions on the wavefunctions for which $H_r$ admits self-adjoint extensions, in due course we shall also need to study the deficiency subspaces for this radial Hamiltonian. Thus for the sake of convenience we consider a somewhat more general form of Eq. (9), namely,

$$-\frac{d^2}{dr^2}\phi(r) - (2k + 2b + 1)\frac{1}{r} \frac{d}{dr}\phi(r) - \frac{\alpha}{\sqrt{N}r}\phi(r) = \tilde{E}\phi(r),$$  (10)

with $\tilde{E} = E, +i, -i$, depending on whether we are interested in the eigenfunctions or the deficiency subspaces for the radial Hamiltonian (7).

By performing the following transformations

$$\phi(r) = r^{-\frac{\beta}{2}}\psi(r), \quad \beta = 2k + 2b + 1, \quad y = cr,$$

with the parameter $c$ as yet unspecified constant, Eq. (10) can be reduced to Whittaker’s equation, which in turn is related to confluent hypergeometric equation of the form

$$\left[y \frac{d^2}{dy^2} + (2\mu + 1 - y) \frac{d}{dy} - \left(\frac{1}{2} + \mu - \kappa\right)\right]\chi(y) = 0.$$  (12)

The last step, where Whittaker’s equation, satisfied by the function $\psi$, reduces to Eq. (12), is performed by means of the factorization $\psi(r) = e^{-\frac{y}{2}}y^{\frac{\beta}{2}}\chi(y)$. The parameters introduced in Eq. (12) are given as

$$\mu = \frac{\beta}{2} - \frac{1}{2},$$

$$c = 2\sqrt{-\tilde{E}},$$

$$\kappa = \frac{\alpha}{cvN} = \frac{\alpha}{\sqrt{-4NE}}.$$  (13)

The general solution to Eq. (12) is known to be the linear combination of the confluent hypergeometric functions of the first and the second kind [29]

$$\chi(r) = AM \left(\frac{1}{2} + \mu - \kappa, \ 2\mu + 1, \ cr\right) + BU \left(\frac{1}{2} + \mu - \kappa, \ 2\mu + 1, \ cr\right),$$  (14)
where $A$ and $B$ are arbitrary constants. Due to the relations $\psi(r) = e^{-\frac{\beta}{2}(cr)}M(\beta - \kappa, \beta, cr)$ and $y = cr$, Eq.\(14\) yields the general form of $\psi(r)$ as

$$
\psi(r) = e^{-\frac{\beta}{2}(cr)} M\left(\frac{\beta}{2} - \kappa, \beta, cr\right) + B U\left(\frac{\beta}{2} - \kappa, \beta, cr\right).
$$

(15)

Since the measure on the space of quadratically integrable $\phi$ functions is $d\sigma = r^\beta dr$, Eq.\(11\) implies that we have

$$
\int \phi^*\phi d\sigma = \int \psi^*\psi dr,
$$

showing that on the space of quadratically integrable $\psi$ functions the measure is simply $dr$.

It may be noted that, in terms of some generic parameters $a, b$ and the variable $z$, the confluent hypergeometric functions $M$ and $U$ are determined by the expressions \(22\)

$$
M(a, b, z) = 1 + \frac{az}{b} + \frac{a(a + 1)}{b(b + 1)2!} + \cdots + \frac{(a)_n z^n}{(b)_n n!} + \cdots,
$$

(16)

$$
U(a, b, z) = \frac{\pi}{\sin \pi b} \left[ \frac{M(a, b, z)}{\Gamma(1 + a - b)\Gamma(b)} - (z)^{1-b} M(1 + a - b, 2 - b, z) \right],
$$

(17)

where the symbol $(a)_n$ means

$$(a)_n = a(a + 1)(a + 2)\ldots(a + n - 1), \quad (a)_0 = 1.
$$

(18)

These expressions show that while $M(a, b, z)$ is nonsingular at $z = 0$, $U(a, b, z)$ could be singular at $z = 0$ with leading power $z^{1-b}$. Thus, for $B \neq 0$, the solution of $\psi(r)$ given in Eq.\(15\) is singular at $r = 0$. In this article, our main aim is to employ such singular solutions of $\psi(r)$ to construct bound and scattering states through the method of self-adjoint extension. On the other hand it may be observed that, the solution of $\psi(r)$ given in Eq.\(15\) would be nonsingular at $r = 0$ for the case $B = 0$ and $\beta \geq 0$. Bound and scattering state solutions found in Ref. \(27\) all correspond to such nonsingular solutions of $\psi(r)$.

3. CONSTRUCTION OF NONSINGULAR BOUND STATES THROUGH LADDER OPERATORS

Let us now concentrate on the set of bound states corresponding to the spectrum found by Khare. These states are solutions to Eq.\(10\) when $\tilde{E} = E$, $E < 0$. In this case, parameters $c$ and $\kappa$, introduced in \(13\), become real parameters $c = 2\sqrt{|E|}$ and $\kappa = \frac{a}{\sqrt{4N|E|}}$. Since $U$ is singular at $r = 0$, after utilizing equation \(14\), we are left with the wavefunctions of the form

$$
\phi(r) = r^{-\frac{\beta}{2}} e^{-\frac{\beta}{2}cr} M\left(\frac{\beta}{2} - \kappa, \beta, cr\right).
$$

(19)

Due to the fact that $M$ comprises an infinite diverging series, it has to be truncated and this can be achieved by setting $\frac{\beta}{2} - \kappa = -n$, $n = 0, 1, 2, 3, \ldots$. With this truncation condition $M$ reduces to the associated Laguerre polynomials, $L_n^{(\alpha)}(x) = \frac{(\alpha + 1)_n}{n!} M(-n, \alpha + 1, x)$, and wavefunctions \(19\) belong to the following discrete set of normalized eigenfunctions, labeled by the quantum number $n$,

$$
\phi_n(r) = \sqrt{\frac{n!}{(2n + \beta)\Gamma(n + \beta)}} c_n e^{-\frac{\beta}{2}cr} L_n^{(\beta-1)}(cr) = C_n e^{-\frac{\beta}{2}cr} L_n^{(\beta-1)}(cr), \quad n = 0, 1, 2, 3, \ldots
$$

(20)

They are normalized to unity with respect to the measure $d\sigma = r^\beta dr$, i.e. they satisfy $\int \phi_n^*\phi_n d\sigma = 1$. The corresponding bound state energies follow from the aforementioned truncation condition and are given as

$$
E_n = -\frac{1}{4N} \frac{\alpha^2}{(k + b + n + \frac{\beta}{2})^2}, \quad n = 0, 1, 2, 3, \ldots
$$

(21)
By using recursive relations for the associated Laguerre functions

\[(n + \alpha)L_{n-1}^{(\alpha)}(x) = (2n + 1 + \alpha - x)L_n^{(\alpha)}(x) - (n + 1)L_{n+1}^{(\alpha)}(x),\]

\[x \frac{d}{dx}L_n^{(\alpha)}(x) = nL_n^{(\alpha)}(x) - (n + \alpha)L_{n-1}^{(\alpha)}(x),\]

we can find recursions

\[\left(n - \frac{1}{2}y - y \frac{d}{dy}\right) \phi_n(y) = \sqrt{\frac{n(2n + \beta - 2)(n + \beta - 1)}{2n + \beta}} \phi_{n-1}(y),\]

\[\left(n + \frac{1}{2}y + y \frac{d}{dy}\right) \phi_n(y) = \sqrt{\frac{(2n + \beta + 2)(n + \beta)(n + 1)}{2n + \beta}} \phi_{n+1}(y),\]

satisfied by the radial functions (20). If we introduce the number operator defined as

\[\hat{N}\phi_n = n\phi_n,\]

we can easily find ladder operators [15], [30], [31] from the recursive relations (23) and (24). They are

\[b = \left[\hat{N} - \frac{1}{2}y - y \frac{d}{dy}\right] \sqrt{\frac{(2\hat{N} + \beta)}{(\hat{N} + \beta - 1)(2\hat{N} + \beta - 2)}},\]

\[b^\dagger = \left[\hat{N} + \beta - \frac{1}{2}y + y \frac{d}{dy}\right] \sqrt{\frac{(2\hat{N} + \beta)}{(\hat{N} + \beta)(2\hat{N} + \beta + 2)}},\]

Straightforward calculation shows that the ladder operators \(b\) and \(b^\dagger\) are bosonic,

\[[b, b^\dagger] = 1,\]

resulting in the simple relation including the number operator \(\hat{N}\).

Due to relations (10), (21) and (25), the radial Hamiltonian in Eq. (7) can be expressed in terms of the number operator \(\hat{N}\) in the way

\[H_r = -\frac{1}{4\hat{N}} \frac{\alpha^2}{(\hat{N} + k + b + \frac{1}{2})^2}.\]

We would like to factorize the radial Hamiltonian (31), so that it can be written in the form

\[H_r = \frac{1}{\gamma^2} A^\dagger A + h,\]
with $\gamma$ and $h$ as yet undetermined parameters. In order to make such factorization, we have to make a transition from the pair of bosonic oscillators $\{b, b^\dagger\}$ to new pair of operators $\{A, A^\dagger\}$. Although the form of the Hamiltonian \[(32)\] looks much more simple, the price we have payed is that the new deformed oscillators are no more bosonic, but rather they are deformed, obeying a more complicated relation in terms of the number operator, namely,

$$A^\dagger A = \Phi(\hat{N}).$$ \[(33)\]

In the above relation $\Phi(\hat{N})$ is an analytic function which is required to satisfy following three conditions:

\begin{enumerate}
  \item $\Phi(\hat{N}) > 0$, \[(34)\]
  \item $\Phi(0) = 0$, \[(35)\]
  \item $\Phi(1) = 1$. \[(36)\]
\end{enumerate}

The function $\Phi(\hat{N})$ which is consistent with relations \[(32)\] and \[(33)\] and which obeys conditions (i),(ii) and (iii) is given as

$$\Phi(\hat{N}) = \frac{\gamma^2 \alpha^2}{4N} \left( \frac{1}{(\frac{\beta}{2})^2} - \frac{1}{{(\hat{N} + \frac{\beta}{2})}^2} \right).$$ \[(37)\]

The parameters $h$ and $\gamma$ are introduced so as to accommodate for the condition (ii) and the normalization condition (iii), respectively, and are equal to

$$h = -\frac{\alpha^2}{N\beta^2}, \quad \gamma^2 = \frac{\beta^2(\frac{\beta}{2} + 1)^2 N}{\alpha^2(\beta + 1)}.$$ \[(38)\]

The deformed oscillators $A, A^\dagger$ can be related \[(32)\] to the bosonic oscillators \[(26)\] and \[(27)\] in the following way

$$A = b \sqrt{\Phi(\hat{N})} \hat{N}, \quad A^\dagger = \sqrt{\Phi(\hat{N})} b^\dagger.$$ \[(39)\]

If we further introduce the operators $J_+, J_-, J_0$ defined as

$$J_- = A \frac{\hat{N}}{\sqrt{\Phi(\hat{N})}}, \quad J_+ = \frac{\hat{N}}{\sqrt{\Phi(\hat{N})}} A^\dagger, \quad J_0 = \frac{\hat{N} + 1}{2},$$ \[(40)\]

one can show that they are, in fact, generators of $SU(1,1)$ algebra,

$$[J_-, J_+] = 2J_0, \quad [J_0, J_\pm] = \pm J_\pm.$$ \[(41)\]

In papers \[28, 33\] the underlying conformal symmetry of the rational Calogero model with a Coulomb-like term is revealed by constructing an explicit realizations of the corresponding $SU(1,1)$ generators. These realizations happen to be different from those found in \[14, 15\] where the realizations of $SU(1,1)$ generators for the rational Calogero model with the harmonic confining term are considered. Since it is known \[14, 15\] that all models with underlying conformal symmetry can be mapped to the set of decoupled oscillators, one could do the same for the Hamiltonian \[11\] by using the construction of $SU(1,1)$ generators made in \[28, 33\]. After finding an appropriate similarity transformation, one could apply it to $SU(1,1)$ generators to find ladder operators for the Hamiltonian \[11\]. It is possible to carry out such transformation since the all systems with underlying conformal symmetry have radial excitations described by the associated Laguerre polynomials with two of the generators playing the role of creation and annihilation operators in the equivalent problem including decoupled set of oscillators. This approach would lead to ladder operators which would not coincide with the ladder operators \[11\], but would rather be related to them by means of some similarity transformation.
4. DEFICIENCY INDICES OF THE RADIAL HAMILTONIAN

The spectrum of this model discussed above is valid for the usual boundary conditions where the wave function vanishes at \( r = 0 \) and it is square integrable. We shall now find the most general set of boundary conditions for which the radial Hamiltonian \( H_r \) is self-adjoint. For this we follow the method of von Neumann. We start by recalling the essential features of this method [19].

Let \( T \) be an unbounded differential operator acting on a Hilbert space \( \mathcal{H} \) and let \( D(T) \) be the domain of \( T \). The inner product of two elements \( \alpha, \beta \in \mathcal{H} \) is denoted by \( (\alpha, \beta) \). Let \( D(T^*) \) be the set of \( \phi \in \mathcal{H} \) for which there is a unique \( \eta \in \mathcal{H} \) with \( (T\xi,\phi) = (\xi,\eta) \forall \xi \in D(T) \). For each such \( \phi \in D(T^*) \), we define \( T^*\phi = \eta \). Then \( T^* \) then defines the adjoint of the operator \( T \) and \( D(T^*) \) spans the corresponding domain of the adjoint. The operator \( T \) is called symmetric or Hermitian iff \( (T\phi,\eta) = (\phi,T\eta) \forall \phi,\eta \in D(T) \). The operator \( T \) is called self-adjoint iff \( T = T^* \) and \( D(T) = D(T^*) \).

We now state the criterion to determine if a symmetric operator \( T \) is self-adjoint. For this purpose let us define the deficiency subspaces \( K_{\pm} = \ker(iT^*T - i) \) and the deficiency indices \( n_{\pm}(T) = \dim[K_{\pm}] \). Then \( T \) falls in one of the following categories:
1) \( T \) is (essentially) self-adjoint iff \( (n_+,n_-) = (0,0) \).
2) \( T \) has self-adjoint extensions iff \( n_+ = n_- \). There is a one-to-one correspondence between self-adjoint extensions of \( T \) and unitary maps from \( K_+ \) into \( K_- \).
3) If \( n_+ \neq n_- \), then \( T \) has no self-adjoint extensions.

We now return to the discussion of the effective Hamiltonian \( H_r \). This is an unbounded differential operator defined in \( \mathbb{R}^+ \). \( H_r \) is a symmetric operator on the domain
\[
D(H_r) = \{ \phi(0) = \phi'(0) = 0, \ \phi, \ \phi' \text{ absolutely continuous, } \phi \in L^2(\sigma) \},
\]
where \( d\sigma = r^2dr \). We would next like to determine if \( H_r \) is self-adjoint in the domain \( D(H_r) \). To perform such an analysis it is necessary to obtain the square-integrable solutions of the equation
\[
H_r^*\phi_{\pm}(r) = \pm i\phi_{\pm}(r). \quad (42)
\]
The operator \( H_r^* \) is the adjoint of \( H_r \) and is given by the same differential operator as \( H_r \), although their domains might be different. Below we shall give the analysis for the parameter range where \( \mu \neq 0 \), the case for \( \mu = 0 \) being similar. Thus, Eq. (42) is identical to Eq. (10) when \( E = \pm i \),
\[
-\frac{d^2}{dr^2}\phi_{\pm}(r) - (2k + 2b + 1)\frac{d}{dr}\phi_{\pm}(r) - \frac{\alpha}{\sqrt{4N}}\phi_{\pm}(r) = \pm i\phi_{\pm}(r). \quad (43)
\]
We are interested in finding the square-integrable solutions to Eq. (43). The solutions of Eq. (42) or Eq. (43) which are square-integrable at infinity are given by \( \phi_{\pm}(r) = r^{-\frac{d}{2}}\psi_{\pm}(r) \), where
\[
\psi_{\pm}(r) = e^{-\frac{d}{2} \pm r}(c_{\pm}r)\frac{\pi}{\sin \beta} U\left(\frac{\beta}{2} - \kappa_{\pm}, \beta, c_{\pm}r \right) \quad (44)
\]
with
\[
c_+ = c(E = i) = 2\sqrt{-i}, \quad c_- = c(E = -i) = 2\sqrt{i}, \\
\kappa_+ = \frac{\alpha}{c_+\sqrt{N}} = \frac{\alpha}{\sqrt{4N}}, \quad \kappa_- = \frac{\alpha}{c_-\sqrt{N}} = \frac{\alpha}{\sqrt{4N}}. \quad (45)
\]
Since these solutions are also required to be square-integrable at the origin, it is necessary to investigate their behaviour for \( r \to 0 \), which looks as
\[
\psi_{\pm}(r) \to (c_{\pm}r)^{\frac{d}{2}} \frac{\pi}{\sin \beta} \left[ \Gamma(1 - \frac{d}{2} - \kappa_{\pm})\Gamma(\beta) - \frac{(c_{\pm}r)^{1-\beta}}{\Gamma(2 - \beta)\Gamma(2 - \beta)} \right]. \quad (46)
\]

\footnote{In subsequent considerations we shall work on the space of \( \psi(r) \) functions where the measure is \( dr \).}
In the above expression we have restricted ourselves to the lowest order in \( r \), so that we could take \( M(a, b, z) \to 1 \) as the argument \( z \) tends to 0. The square integrability of the wavefunction \( \psi(r) \) near the origin is determined by

\[
|\psi_{\pm}(r)|^2 dr \longrightarrow \left[ A_1 r^\beta + A_2 r + A_3 r^{2-\beta} \right] dr,
\]

where \( A_1, A_2, A_3 \) are some constants independent of \( r \). From Eq.(47) it is seen that near the origin, the functions \( \psi_{\pm} \) (and consequently functions \( \phi_{\pm} \)) are not square-integrable for the parameter \( \beta \) satisfying \( \beta < -1 \) or \( \beta > 3 \). Consequently, in the parameter range \( \beta < -1 \) or \( \beta > 3 \), the functions \( \psi_{\pm} \) are not the elements of the vector space \( L^2[\mathbb{R}^+, dr] \) of quadratically integrable functions defined on the positive real axis. In that case, \( n_+ = n_- = 0 \) and \( H_r \) is essentially self-adjoint in the domain \( D(H_r) \). However, if \( -1 < \beta < 3 \), the functions \( \psi_{\pm} \) (and consequently functions \( \phi_{\pm} \)) are square-integrable. Thus, if \( \beta \) lies in this range, we have \( n_+ = n_- = 1 \) and the Hamiltonian \( H_r \) is not self-adjoint in the domain \( D(H_r) \), but admits self-adjoint extensions. Note that from (13), the allowed range of \( \beta \) implies that the parameter \( \mu \) must lie in the range \( -1 < \mu < 1 \).

The above allowed range of \( \mu \), together with (8), (11) and (13), implies that the values of \( N, k \) and \( a + \frac{1}{2} \) must satisfy the relation

\[
-\frac{N-1+2k}{N(N-1)} < a + \frac{1}{2} < -\frac{N-5+2k}{N(N-1)}.
\]

(48)

for the self-adjoint extension to exist. For \( N \geq 3 \), we have the following classifications of the boundary conditions depending on the value of the parameter \( a + \frac{1}{2} \).

(i) \( a + \frac{1}{2} \geq \frac{1}{2} \) : This corresponds to the boundary condition considered by Khare in [27]. For this choice, both the wave-function and the current vanish as \( x_i \to x_j \). In this case, \( \mu > 1 \) for all values of \( k \geq 0 \). The corresponding Hamiltonian is essentially self-adjoint in the domain \( D(H_r) \), leading to a unique quantum theory.

(ii) \( 0 < a + \frac{1}{2} < \frac{1}{2} \) : For this choice we see that the wave-function in (33) vanishes in the limit \( x_i \to x_j \), although the current may be divergent. In this case \( \mu > 0 \) and \( k \) must be equal to zero so that \( \mu \) may belong to the range \( 0 < \mu < 1 \). The corresponding constraint on \( a + \frac{1}{2} \) is given by \( 0 < a + \frac{1}{2} < \frac{5-N}{N(N-1)} \), which can only be satisfied for \( N = 3 \) and \( N = 4 \). So new quantum states associated with the self-adjoint extension of \( H_r \) exist only in the \( k = 0 \) sector of \( N = 3 \) and \( N = 4 \).

(iii) \( -\frac{1}{2} < a + \frac{1}{2} < 0 \) : The lower bound on \( a + \frac{1}{2} \) is obtained from the condition that the wavefunction be square-integrable. The parameter \( a + \frac{1}{2} \) in this range leads to a singularity in the wavefunction \( \Psi \) in Eq. (3) resulting from the coincidence of any two or more particles. Using permutation symmetry, such an eigenfunction can be extended to the whole of configuration space, although not in a smooth fashion. The new quantum states in this case exist for arbitrary \( N \) and even for non-zero values of \( k \). In fact, imposing the condition that the upper bound on \( a + \frac{1}{2} \) should be greater than \( -\frac{1}{2} \), we find from (33) that \( k \) is restricted as \( k < \frac{1}{2} (N^2 - 3N + 10) \). It can also be shown that there are only two allowed values of \( N \) when both \( N \) and \( a + \frac{1}{2} \) are kept fixed.

Von Neumann’s method also provides a prescription for obtaining the domain of self-adjointness of a symmetric operator, which admits a self-adjoint extension. The extended domain \( D_z(H_r) \) in which \( H_r \) is self-adjoint contains all the elements of \( D(H_r) \), together with the elements of the form \( e^{iz} \psi_+ + e^{-iz} \psi_- \), where \( z \in \mathbb{R} \) (mod \( 2\pi \)). Thus the self-adjoint extensions of this model exist when \( -1 < \beta < 3 \), and in that case,

\[
D_z(H_r) = D(H_r) \oplus \{ e^{iz} \psi_+ + e^{-iz} \psi_- \}
\]

is the extended domain in which \( H_r \) is self-adjoint.

5. BOUND STATES OF THE RADIAL HAMILTONIAN WITH SELF-ADJOINT EXTENSION

We shall now find solutions of the physical problem for the range of system parameters where the self-adjoint extension is necessary. In finding the solutions to Eq.(10), we shall first consider the bound state sector of the problem.
In this sector, the energy of the system is negative, \( E < 0 \), and the wavefunctions need to be square-integrable. We consider the solution of the form

\[
\psi(r) = B e^{-\frac{E}{2}} (cr)^{\frac{1}{2}} U \left( \frac{\beta - \kappa}{2}, \alpha, cr \right).
\]  

(49)

In order to make an analysis and to find the spectrum, we should know the behaviour of the \( U \) function near the origin. Using Eqs. (16) and (17), we can expand \( U(a,b,z) \) at \( z \to 0 \) limit as

\[
\frac{z^{1-b}}{\Gamma(a)\Gamma(2-b)} \left[ 1 + \frac{a - b}{2 - b} z \right] (1 + a - b) (2 + a - b) z^2 + (2 - b) (3 - b) 2! + O(z^3) \right] \cdot
\]  

(50)

Consequently, at \( r \to 0 \) limit, \( \psi(r) \) in Eq. (49) behaves as

\[
\psi(r) \to B e^{-\frac{\tilde{E}r}{2}} (cr)^{\frac{1}{2}} \left[ \frac{1}{\Gamma(1 - \frac{\beta - \kappa}{2})} \left( 1 + \frac{\beta - \kappa}{\beta} cr + \frac{\alpha - \kappa + 1}{2\beta} (cr)^2 + O(r^3) \right) - \frac{(cr)^{1-\beta}}{\Gamma(\beta - \kappa)} \left( 1 + \frac{\beta - \kappa}{2 - \beta} cr + \frac{\alpha - \kappa + 1}{2(2 - \beta)(3 - \beta)} (cr)^2 + O(r^3) \right) \right].
\]  

(51)

The parameters \( c \) and \( \kappa \) appearing in (51) are given in (18), except that they are evaluated for \( \tilde{E} = E \). Since the energy is negative, these parameters are real,

\[
c = 2\sqrt{-E} = 2\sqrt{E_b} = p,
\]

\[
\kappa = \frac{\alpha}{c\sqrt{N}} = \frac{\alpha}{\sqrt{-4NE}} = \frac{\alpha}{\sqrt{4NE_b}} = \frac{\alpha}{p\sqrt{N}} = \frac{\alpha}{p\sqrt{N}}.
\]  

(52)

Here, for future convenience, we have introduced the absolute value \( E_b \) of the bound state energy, \( E_b = -E, \ E < 0 \), and the real parameter \( p \) which coincides with \( c \) in the bound state sector. Note that \( c \) and \( \kappa \) will no more be real in the scattering sector.

If the wavefunction (49) is expected to describe a physically acceptable bound state solutions to Eq. (10), it has to belong to the domain of self-adjointness \( \mathcal{D}_z(H_r) \). If \( \psi_0(r) \in \mathcal{D}(H_r) \), then an arbitrary element of the domain \( \mathcal{D}_z(H_r) \) can be written as \( \psi_0(r) + \rho e^{i\tilde{\omega}} \psi_+ + e^{-i\tilde{\omega}} \psi_- \), where \( \rho \) is a constant. If the solution of the physical wavefunction (49) belongs to the domain \( \mathcal{D}_z(H_r) \), the functional form of physical wavefunction must match with that of an arbitrary element of the domain \( \mathcal{D}_z(H_r) \), which is given by

\[
\psi(r) = \psi_0(r) + \rho (e^{i\tilde{\omega}} \psi_+ + e^{-i\tilde{\omega}} \psi_-),
\]  

(53)

Inserting Eqs. (51) and (10) into relation (53), and equating the coefficients of the lowest order powers in \( r \) (for which there is no contribution from \( \psi_0(r) \)), yields the following two conditions

\[
\frac{\tilde{B} c^{\beta}}{(1 - \kappa - \frac{\beta}{2})} = \frac{e^{i\tilde{\omega}} c_+^{\frac{\alpha}{2}} + e^{-i\tilde{\omega}} c_-^{\frac{\alpha}{2}}}{\Gamma(1 - \kappa - \frac{\beta}{2})},
\]

\[
\frac{\tilde{B} c^{1-\beta}}{(\beta - \kappa)} \left( e^{i\tilde{\omega}} c_+^{\frac{1-\beta}{2}} + e^{-i\tilde{\omega}} c_-^{\frac{1-\beta}{2}} \right) \left( \frac{\beta}{\beta - \kappa} \right) \left( \frac{\beta}{\beta - \kappa} \right),
\]  

(54)

where \( \tilde{B} = B/\rho \). After dividing both sides of these two expressions, we get the relation

\[
\frac{\Gamma(1 - \kappa - \frac{\beta}{2})}{\Gamma(\frac{\beta}{2} - \kappa)} c^{1-\beta} = \frac{e^{i\tilde{\omega}} c_+^{\frac{1-\beta}{2}} + e^{-i\tilde{\omega}} c_-^{\frac{1-\beta}{2}}}{\Gamma(1 - \kappa - \frac{\beta}{2})},
\]  

(55)
Inserting the expressions for $c$ and $\kappa$ we obtain the final condition
\[
\frac{\Gamma(1 - \frac{\beta}{2} - \frac{\alpha}{\sqrt{4N E_b}})}{\Gamma(\frac{\beta}{2} - \frac{\alpha}{\sqrt{4N E_b}})} (2\sqrt{E_b})^{1-\beta} = \frac{e^{i\frac{\pi}{4}} e^{-\frac{2i}{4} \beta}}{\Gamma(\frac{\beta}{2} - \frac{\alpha}{\sqrt{4N E_b}})} + \frac{e^{-i\frac{\pi}{4}} e^{-\frac{2i}{4} \beta}}{\Gamma(\frac{\beta}{2} + \frac{\alpha}{\sqrt{4N E_b}})}
\]
(56)
which determines the spectrum corresponding to bound states of the radial Hamiltonian $H_r$ as well as the initial many-body Hamiltonian $H$. Writing $\frac{e^{i\frac{\pi}{4}} e^{-\frac{2i}{4} \beta}}{\Gamma(\frac{\beta}{2} - \frac{\alpha}{\sqrt{4N E_b}})} = \xi_1 e^{i\theta_1}$ and $\frac{e^{-i\frac{\pi}{4}} e^{-\frac{2i}{4} \beta}}{\Gamma(\frac{\beta}{2} + \frac{\alpha}{\sqrt{4N E_b}})} = \xi_2 e^{i\theta_2}$, (56) can be expressed as
\[
\frac{\Gamma(1 - \frac{\beta}{2} - \frac{\alpha}{\sqrt{4N E_b}})}{\Gamma(\frac{\beta}{2} - \frac{\alpha}{\sqrt{4N E_b}})} (2\sqrt{E_b})^{1-\beta} = \frac{\xi_1 \cos(\theta_1 + \frac{\pi}{4})}{\xi_2 \cos(\theta_2 + \frac{\pi}{4})}
\]
(57)

The above analysis shows that for a given choice of the system parameters, Eq. (57) gives the energy eigenvalue $E = -E_b$ as a function of the self-adjoint extension parameter $z$. For a fixed set of system parameters, different choices of $z$ lead to inequivalent quantization and to the spectrum for this model in the parameter range where the system admits self-adjoint extension. In general, the energy $E = -E_b$ cannot be calculated analytically and has to be obtained numerically by plotting (57). Figures 1 and 2 show l.h.s and r.h.s of Eq. (57) for two different, representative sets of the system parameters as well as for the two different choices of the self-adjoint extension parameter $z$. The curved lines at those figures represent graph of the function $f(E_b)$ which is given by the l.h.s of Eq. (57). On the other hand r.h.s of Eq. (57) is represented by a horizontal straight line. The energy eigenvalues of the system described by the Hamiltonian (1) are obtained by looking at the intersections of these two curves. We see from figures that there is an infinite number of bound states near $E_b \to 0$. For $\alpha > 0$, there are infinite number of bound states for any value of $z$. However, the existence of non-oscillatory part shows that, just like the usual case, the spectrum has a lower bound for all possible values of $z$. The situation when $\alpha > 0$ is shown at figures 1 and 2.

For the choice of the self-adjoint extension parameter $z = z_1$ such that $\theta_1 + \frac{\pi}{2} = \frac{\pi}{2}$, the r.h.s. of (57) is zero. This implies that
\[
\frac{\beta}{2} - \frac{\alpha}{\sqrt{4N E_b}} = -n, \quad n = 0, 1, 2, ..., \quad (58)
\]
which gives the usual energy eigenvalues as expressed in (21). It can be shown that the choice of $z = z_2$ such that $\theta_2 + \frac{\pi}{2} = \frac{\pi}{2}$ gives a similar result. At this point it may be noted that the analytical solution (58) implies that for a certain values of the self-adjoint extension parameter and system parameters, even the repulsive Coulomb potential leads to the formation of only one bound state. It can easily be seen if we write (58) in the form $\alpha = \sqrt{N E_b (2n + \beta)}$. This expression shows that in order to have the repulsive Coulomb potential, that is $\alpha < 0$, one has to restrict $\beta$ within the range $-1 < \beta < 0$ and set $n$ equal to zero, resulting in a single bound state. The same conclusion holds also in the general case where the analytical solution is not possible, and it can be verified by extensive numerical investigation of the general relation (56) (see Figure 3 as an example).

Finally, it is important to emphasize that the system described by (1) has a fundamentally different behaviour depending on the sign of $\alpha$. While for $\alpha > 0$, the l.h.s. of Eq. (57) exhibits oscillatory, as well as non-oscillatory behaviour, leading to infinite number of bound states, for $\alpha \leq 0$, it shows only non-oscillatory behaviour resulting in the existence of at most one bound state. This single bound state, if it exists, shows up only for the certain range of the self-adjoint extension parameter $z$. For $\alpha = 0$, this observation is consistent with the result obtained in [10]. This feature can most easily be seen by looking at the special case (58) where the analytical solution is available. There, in order for $\alpha$ to be less than zero, we must have $n = 0$ together with $\beta$ within the range $-1 < \beta < 0$, resulting in a single bound state $E = -E_b = -\frac{\alpha^2}{N \pi^2}$, as already stated just after Eq.(58).
Figure 1. A plot of Eq. (57) using Mathematica with $N = 1000$, $\alpha = 50$, $\beta = 0.8$, $k = 1$ and $z = 0.1$. The horizontal straight line corresponds to the value of the r.h.s of Eq. (57).

Figure 2. A plot of Eq. (57) using Mathematica with $N = 100$, $\alpha = 1.5$, $\beta = -0.7$, $k = 1$ and $z = -0.73$. The horizontal straight line corresponds to the value of the r.h.s of Eq. (57).

Figure 3. A plot of Eq. (57) using Mathematica with $N = 1000$, $\alpha = -1$, $\beta = 1.5$, $k = 1$ and $z = 0.1$. The horizontal straight line corresponds to the value of the r.h.s of Eq. (57). This graph shows the general feature exhibited for arbitrarily strong repulsive Coulomb potential, i.e. for any $\alpha < 0$. 
6. SCATTERING STATES OF THE RADIAL HAMILTONIAN WITH SELF-ADJOINT EXTENSION

Let us now turn our attention to the scattering sector of the problem described by Eq. (10). Since the scattering states correspond to positive energy solutions of Eq. (10) when \( \hat{E} = E > 0 \), the variable \( y = \kappa r = 2\sqrt{-E}r \) becomes purely imaginary, i.e. \( y = iqr \), where the real parameter \( q \) is defined as \( q = 2\sqrt{E} \). Therefore, for analyzing the \( r \rightarrow \infty \) limit of the scattering states, it is of importance to know the behaviour of the confluent hypergeometric functions \( M(a, b, z) \) and \( U(a, b, z) \) in the asymptotic region \( \text{Re}(z) = 0 \) and \( \text{Im}(z) \rightarrow +\infty \). Following Abramowitz & Stegun, we can expand confluent hypergeometric functions in this asymptotic region as

\[
M(a, b, z) \rightarrow \frac{\Gamma(b)}{\Gamma(a)} e^{z} z^{a-b} \left[ 1 + O(|z|^{-1}) \right] + \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a} \left[ 1 + O(|z|^{-1}) \right],
\]

\[
U(a, b, z) \rightarrow O(|z|^{-a}).
\]

Due to the fact that we are dealing with the problem where \( \text{Re}(z) = 0 \), both leading terms in the asymptotic expansion (59) of \( M \) approximately have the contribution of the same order, so that both of them have to be taken into account. In order to find the scattering matrix we could equally well take the following linear combination

\[
\chi(y) = AM \left( \frac{1}{2} + \mu - \kappa, 1 + 2\mu, y \right) + By^{-2\mu} M \left( \frac{1}{2} - \mu - \kappa, 1 - 2\mu, y \right),
\]

as a general solution to Eq. (12), instead of the one given in (14). In this case the solution for the function \( \psi \), appearing in (11) would look like

\[
\psi(r) = e^{-\frac{i}{2} cr \kappa} \left( A(q) M \left( \frac{\beta}{2} - \kappa, \beta, cr \right) + B(q) (-1)^{1-\beta} M \left( 1 - \frac{\beta}{2} - \kappa, 2 - \beta, cr \right) \right),
\]

where we have assumed that the coefficients \( A(q) \) and \( B(q) \) depend on the real parameter \( q \).

By using (59), we have the following \( r \rightarrow \infty \) limits

\[
M(\frac{\beta}{2} - \kappa, \beta, cr) \rightarrow \frac{\Gamma(\beta)}{\Gamma(\frac{\beta}{2} - \kappa)} e^{cr \kappa} - \frac{\Gamma(\beta)}{\Gamma(\frac{\beta}{2} + \kappa)} (-cr)^{\frac{\beta}{2} - \kappa},
\]

\[
M(1 - \frac{\beta}{2} - \kappa, 2 - \beta, cr) \rightarrow \frac{\Gamma(2 - \beta)}{\Gamma(1 - \frac{\beta}{2} - \kappa)} e^{cr \kappa} + \frac{\Gamma(2 - \beta)}{\Gamma(1 - \frac{\beta}{2} + \kappa)} (-cr)^{\frac{\beta}{2} + \kappa - 1},
\]

so that the wave function (62), describing the scattering state, in the above limit behaves as

\[
\psi(r) \equiv \psi(\hat{E} = E) \rightarrow A(q) \frac{\Gamma(\beta)}{\Gamma(\frac{\beta}{2} - \kappa)} e^{\frac{i}{2} cr \kappa} - \kappa + A(q) \frac{\Gamma(\beta)}{\Gamma(\frac{\beta}{2} + \kappa)} e^{-\frac{i}{2} cr \kappa} - \kappa + B(q) \frac{\Gamma(2 - \beta)}{\Gamma(1 - \frac{\beta}{2} - \kappa)} e^{\frac{i}{2} cr \kappa} - \kappa + B(q) \frac{\Gamma(2 - \beta)}{\Gamma(1 - \frac{\beta}{2} + \kappa)} e^{-\frac{i}{2} cr \kappa} - \kappa.
\]

Note that the parameter \( \kappa \) is also purely imaginary in the scattering sector. For the coupling constant \( \alpha \) greater than zero, \( \kappa \) can be expressed as \( \kappa = -i \frac{\alpha}{q \sqrt{N}} = -i \frac{\kappa}{q \sqrt{N}} = -i|\kappa| \). By using the relations \( y = cr = iqr \) and \( \kappa = -i|\kappa| \), we can express \( \psi(r) \) in Eq. (65) in terms of oscillatory incoming wave and outgoing wave as

\[
\psi(r) \equiv \psi(\hat{E} = E) \rightarrow e^{-i\frac{i}{2} \kappa q} - \kappa \left( A(q) \frac{\Gamma(\beta)}{\Gamma(\frac{\beta}{2} - \kappa)} + B(q) \frac{\Gamma(2 - \beta)}{\Gamma(1 - \frac{\beta}{2} - \kappa)} \right) e^{i(\frac{1}{2} qr + |\kappa| \ln r)} +
\]

\[
e^{i\frac{i}{2} \kappa q} \left( A(q) \frac{\Gamma(\beta)}{\Gamma(\frac{\beta}{2} + \kappa)} e^{i\pi(\kappa - \frac{\beta}{2})} + B(q) \frac{\Gamma(2 - \beta)}{\Gamma(1 - \frac{\beta}{2} + \kappa)} e^{i\pi(\kappa + \frac{\beta}{2} - 1)} \right) e^{-i(\frac{1}{2} qr + |\kappa| \ln r)}.\]
The scattering matrix and the corresponding phase shift can be obtained from the above limiting form of the wave function as a ratio of its outgoing and incoming amplitudes,

$$S(q) = e^{2i\varphi(q)} = \frac{\left(A(q)\frac{\Gamma(\beta)}{\Gamma(-\frac{\beta}{2} - \kappa)} + B(q)\frac{\Gamma(2 - \beta)}{\Gamma(1 - \frac{\beta}{2} - \kappa)}\right) q^{-2\kappa} e^{-i\kappa} \Gamma(\frac{\beta}{2} + \kappa)}\left(A(q)\frac{\Gamma(\beta)}{\Gamma(-\frac{\beta}{2} + \kappa)} + B(q)\frac{\Gamma(2 - \beta)}{\Gamma(1 - \frac{\beta}{2} + \kappa)} e^{-i\pi\kappa} q^{-2 \kappa - 1}\right).$$

(67)

Next, to find a relationship between so far unspecified constants $A(q)$ and $B(q)$, we use the expansion (68) to obtain the $r \to 0$ limit of the wave function (62), in the lowest order in $r$,

$$\psi(\tilde{E} = E) \rightarrow A(q)(cr)^{\frac{\beta}{2}} + B(q)(cr)^{1 - \frac{\beta}{2}}.$$  

(68)

We recall that the Hamiltonian $H_r$ admits a self-adjoint extension in the parameter range $3 > \beta > -1$. Since the wave function (62) has to belong to the domain of self-adjointness $D_+(H_r) = \mathcal{D}(H_r) \oplus \{e^{i\frac{\beta}{2}} \psi_+ + e^{-i\frac{\beta}{2}} \psi_-\}$ we can write

$$\rho \psi(\tilde{E} = E) = e^{i\frac{\beta}{2}} \psi_+ + e^{-i\frac{\beta}{2}} \psi_-,$$

(69)

where $\rho$ is some constant and, as before, $\psi_{\pm}$ are square integrable solutions of Eq. (11) when $\tilde{E} = \pm i$, respectively. In the limit $r \to 0$, the behaviour of $\psi_{\pm}$ is given by the relation (68). Since according to Eq. (69), the coefficients of appropriate powers of $r$ in (68) and (69) must match, the following two conditions emerge

$$\rho A(q) c^\frac{\beta}{2} = e^{i\frac{\beta}{2}} \frac{\pi}{\sin \frac{\pi \beta}{2}} \frac{c_+ c^{-\frac{\beta}{2}}}{\Gamma(1 - \frac{\beta}{2} - \kappa_+)} + e^{-i\frac{\beta}{2}} \frac{\pi}{\sin \frac{\pi \beta}{2}} \frac{c_- c^{-\frac{\beta}{2}}}{\Gamma(1 - \frac{\beta}{2} - \kappa_-)}$$

(70)

$$\rho B(q) c^{1 - \frac{\beta}{2}} = -e^{i\frac{\beta}{2}} \frac{\pi}{\sin \frac{\pi \beta}{2}} \frac{c_+ c^{1 - \frac{\beta}{2}}}{\Gamma(\frac{\beta}{2} - \kappa_+)} - e^{-i\frac{\beta}{2}} \frac{\pi}{\sin \frac{\pi \beta}{2}} \frac{c_- c^{1 - \frac{\beta}{2}}}{\Gamma(\frac{\beta}{2} - \kappa_-)}$$

(71)

The last two equations yield

$$\frac{A(q)}{B(q)} = -\frac{\Gamma(2 - \beta)}{\Gamma(\beta)} e^{i\frac{\beta}{2}} \frac{c_+ c^{-\frac{\beta}{2}}}{\Gamma(1 - \frac{\beta}{2} - \kappa_+)} + e^{-i\frac{\beta}{2}} \frac{c_- c^{-\frac{\beta}{2}}}{\Gamma(1 - \frac{\beta}{2} - \kappa_-)} c^{1 - \beta}.$$  

(72)

By using this expression, the scattering matrix (67) becomes

$$S(q) = e^{2i\varphi(q)} = \frac{F_2(\beta, \alpha, z)}{F_1(\beta, \alpha, z)} \frac{e^{i\frac{\beta}{2}} \frac{c_+ c^{-\frac{\beta}{2}}}{\Gamma(\frac{\beta}{2} - \kappa_+)} + e^{-i\frac{\beta}{2}} \frac{c_- c^{-\frac{\beta}{2}}}{\Gamma(\frac{\beta}{2} - \kappa_-)}}{\frac{\Gamma(\frac{\beta}{2} - \kappa_+)}{\Gamma(\frac{\beta}{2} + \kappa)}} c^{-i\pi \kappa} q^{-2\kappa},$$

(73)

where $c = 2\sqrt{-E} = iq$, and $\kappa = \frac{\alpha}{c\sqrt{N}} = \frac{\alpha}{\sqrt{-4\alpha}}$. In writing the expression for the scattering matrix we have introduced the following two functions

$$F_1(\beta, \alpha, z) = e^{i\frac{\beta}{2}} \frac{c_+ c^{-\frac{\beta}{2}}}{\Gamma(\frac{\beta}{2} - \kappa_+)} + e^{-i\frac{\beta}{2}} \frac{c_- c^{-\frac{\beta}{2}}}{\Gamma(\frac{\beta}{2} - \kappa_-)},$$

(74)

$$F_2(\beta, \alpha, z) = e^{i\frac{\beta}{2}} \frac{c_+ c^{-\frac{\beta}{2}}}{\Gamma(1 - \frac{\beta}{2} - \kappa_+)} + e^{-i\frac{\beta}{2}} \frac{c_- c^{-\frac{\beta}{2}}}{\Gamma(1 - \frac{\beta}{2} - \kappa_-)}.$$  

(75)

where $z$ is the self-adjoint extension parameter and $c_\pm$ and $\kappa_\pm$ are defined in (15). As a remark, one can note that the functions $F_1$ and $F_2$ are simply related as

$$F_2(\beta, \alpha, z) = F_1(2 - \beta, \alpha, z).$$


As it is seen from the form of the scattering matrix, for any given value of $\beta$ in the parameter range which admits self-adjoint extension, the scattering matrix has an infinite set of poles on the positive imaginary axis of the complex $q$-plane. The existence of poles for the scattering matrix means that there are bound states in the system under consideration. By taking $q = ip$ as some arbitrary pole for the scattering matrix (73), one can obtain the following equation determining the bound state energies $E_b = -E = \frac{p^2}{4}$:

$$\frac{F_2(\beta, \alpha, z)}{F_1(\beta, \alpha, z)} e^{i\pi(\kappa - \beta + \frac{1}{2})} \frac{q^{1-\beta}}{\Gamma(\frac{n}{2} + \kappa)} = 0. \quad (76)$$

This expression, after utilizing the set of relations $q = 2\sqrt{E} = ip = i2\sqrt{E_b}$ and $\kappa = -\frac{\alpha}{q\sqrt{N}} = -\frac{\alpha}{p\sqrt{N}} = -\frac{\alpha}{2\sqrt{E_b} N}$, finally gives

$$\frac{\Gamma(1 - \frac{\alpha}{2} - \frac{\alpha}{2\sqrt{E_b} N})}{\Gamma(\frac{n}{2} - \frac{\alpha}{2\sqrt{E_b} N})} b^{1-\beta} = \frac{F_1(\beta, \alpha, z)}{F_2(\beta, \alpha, z)}, \quad (77)$$

which reproduces the bound state condition (55).

7. CONCLUSIONS

In this paper we have analyzed the $N$-body rational Calogero model with a Coulomb-like interaction. We have shown that for certain ranges of the system parameters, the system admits a one-parameter family of self-adjoint extensions. The results obtained here for both bound and scattering state sectors are very different from those obtained by Khare in [27]. However, there is no contradiction between these findings as they refer to different ranges of the system parameters. We have also shown that for specific choices of the self-adjoint extension parameter, the usual results of Khare can be recovered.

It has also been shown that a ladder operator construction exists for this system, which also leads to the solution found by Khare. This construction indicates that $su(1,1)$ can be regarded as a spectrum generating algebra for this system, as it happens in conformal quantum mechanics [34] yielding equispaced energy levels. We think that there is a strong correlation between our and the constructions made in papers [28], [33]. We hope to address this issue in more detail in a future.

In the presence of the self-adjoint extension, the $su(1,1)$ can no longer be implemented as the spectrum generating algebra as the dilatation generator in this case does not in general leave the domain of the Hamiltonian invariant [17, 21, 23, 35]. As a result, the spectrum for a generic choice of the self-adjoint extension parameter is no longer expressed in the Coulomb-like form. However, when $z = z_1$ or $z_2$, the Coulomb-like nature of the spectrum is recovered and $su(1,1)$ can again be implemented as a spectrum generating algebra. This effect is analogous to the quantum anomaly also observed in the pure Calogero type systems [17, 30].

We have also seen that the system exhibits qualitatively different behaviour on two sides of the point $\alpha = 0$. We find that for the attractive Coulomb potential ($\alpha > 0$) there exists an infinite number of bound states. In the case of the repulsive Coulomb potential ($\alpha < 0$), there appears to be at most a single bound state, which exists only for certain values of the self-adjoint extension parameter.

In this paper we have restricted our discussion to the case when the coupling constant $g$ of the inverse square interaction is such that there is no collapse to the centre. It would be interesting to analyze this problem where the coupling is more attractive with $g < -\frac{1}{2}$, which would require renormalization group techniques [21, 37].

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