KAZHDAN–LUSZTIG POLYNOMIALS FOR THE HERMITIAN
SYMMETRIC PAIR \((B_N, A_{N-1})\)

KEIICHI SHIGECHI

ABSTRACT. We provide combinatorial rules to compute Kazhdan–Lusztig polynomials for the Hermitian symmetric pair \((B_N, A_{N-1})\) when the Hecke algebra has unequal parameters. They are obtained by filling regions delimited by paths with ballot strips. We also extend the binary tree algorithm introduced by Lascoux and Schützenberger to our case.

1. INTRODUCTION

In [9], Kazhdan and Lusztig introduced Kazhdan–Lusztig polynomials \(P_{x,y}\) indexed by two elements \(x\) and \(y\) of an arbitrary Coxeter group. These polynomials are the coefficients of the change of basis from the standard bases of the Hecke algebra to the Kazhdan–Lusztig bases. They play an important role in various research fields such as algebraic combinatorics [11], topology of Schubert varieties [10] and representation theory of Verma modules [2, 6]. In [8], Deodhar introduced the concept of parabolic Kazhdan–Lusztig polynomials \(P_{\alpha,\beta}^\pm\) for a Coxeter group. They are associated with the induced representation of the Hecke algebra by the one-dimensional representations of parabolic subgroups. One of the important examples is the one with the Weyl group of type \(A\) (the symmetric group \(S_N\)) and the maximal parabolic subgroup \(S_K \times S_{N-K}\). This example has been studied as Kazhdan–Lusztig polynomials for Grassmannian permutations [4, 13]. In [18], we provided a unified treatment of maximal parabolic Kazhdan-Lusztig polynomials in the language of paths. In this paper, we continue to investigate the Kazhdan-Lusztig polynomials for the Hermitian symmetric pair \((B_N, A_{N-1})\).

We have two types of parabolic Kazhdan–Lusztig polynomials \(P_{\alpha,\beta}^\pm\) due to the choice of the projection map from \(\mathbb{C}[S_N]\) to \(\mathbb{C}[S_N^\alpha/S_N]\) (see Section 2). In [3], Boe gave a combinatorial description of the Kazhdan-Lusztig polynomials \(P_{\alpha,\beta}^+\) for Hermitian symmetric pairs. He generalized the binary tree algorithm introduced by Lascoux and Schützenberger keeping its flavour. On the other hand in [5], the analysis for \(P_{\alpha,\beta}^-\) was done by using the concept of shifted Dyck partitions. We can identify these cases with the maximally parabolically induced modules of the Hecke algebra with equal Hecke parameters. One of the main results of this paper is to give a combinatorial description of the Kazhdan-Lusztig basis and polynomials \(P_{\alpha,\beta}^\pm\) for the Hecke algebra of type \(B_N\) with unequal Hecke parameters for the Hermitian symmetric pair \((B_N, A_{N-1})\). We provide a unified treatment of \(P_{\alpha,\beta}^\pm\) for the unequal parameter cases and give a generalization of Boe’s algorithm to compute them.

One way of an analysis of Kazhdan–Lusztig polynomials is to solve the recurrence relations of the \(R\)-polynomials, which have the same information as Kazhdan-Lusztig polynomials [4,
In this paper, we study Kazhdan–Lusztig polynomials by using combinatorial properties of the Hermitian symmetric pair \((B_N, A_{N-1})\), namely those of the coset space \(S_N^C/S_N\) where \(S_N\) and \(S_N^C\) are the Weyl group of type A and C. Our analysis has the flavour of the concept of tangles and link patterns used in statistical mechanics and that of Temperley–Lieb algebra \([1, 15, 17, 19]\). We introduce the ballot strips (similar to shifted Dyck partitions in \([5]\)) and graphical rules to compute two types of generating functions \(Q_{\alpha,\beta}^\pm\) in a similar way as \([18]\). These generating functions are shown to be equal to Kazhdan–Lusztig polynomials, that is, \(Q_{\alpha,\beta}^\pm = P_{\alpha,\beta}^\pm\).

The plan of the paper is as follows. In Section 2, we introduce Kazhdan–Lusztig polynomials and their parabolic analogues. Then, we explain their inversion relations. In Section 3, we introduce a concept of ballot strips and new diagrammatic rules for \(m\) and \(n\) integers are \([0, 1, 15, 17, 19]\). We introduce the ballot strips (similar to shifted Dyck partitions in \([5]\)) and of tangles and link patterns used in statistical mechanics and that of Temperley–Lieb algebra \([1, 15, 17, 19]\). We introduce the ballot strips (similar to shifted Dyck partitions in \([5]\)) and graphical rules to compute two types of generating functions \(Q_{\alpha,\beta}^\pm\) in a similar way as \([18]\). These generating functions are shown to be equal to Kazhdan–Lusztig polynomials, that is, \(Q_{\alpha,\beta}^\pm = P_{\alpha,\beta}^\pm\).

The factorization property of the Kazhdan–Lusztig basis is presented. In Section 5, we generalize the binary tree algorithm introduced in \([3, 13]\). This gives an alternative combinatorial algorithm for the computation of \(P_{\alpha,\beta}^\pm\). When the two Hecke parameters are equal, \(i.e., t_N = t\) (see Section 2 for notations), the algorithm coincides with Boe’s. Further, the generating function \(Q_{\alpha,\beta}^+\) introduced in Section 3 is shown to be equal to the generating function of a generalized binary tree.

Notations. we denote by \(N_+\) the set of positive integers and \(N := N_+ \cup \{0\}\). The \(t\)-deformed integers are \([m] := (t^m - t^{-m})/(t - t^{-1})\) for \(m \in \mathbb{Z}\). Also \(\langle 0 \rangle := 1\) and \(\langle m \rangle := t^m + t^{-m}\) for \(m \in \mathbb{N}_+\).

2. Kazhdan–Lusztig polynomials

2.1. Definitions. Let \(S_N^C\) be the finite Weyl group associated with the Dynkin diagram of type \(C\) and generated by \(s_i, 1 \leq i \leq N\) with defining relations \(s_i^2 = 1\) for \(1 \leq i \leq N\), \((s_is_{i+1})^3 = 1\) for \(1 \leq i \leq N - 2\) and \((s_{N-1}s_N)^4 = 1\). Let \(w = s_{i_1} \ldots s_{i_r} \in S_N^C\) be a reduced word. The length functions \(l, l', l_N : S_N^C \rightarrow \mathbb{N}\) is defined as \(l'(w) := \text{Card}\{ij : 1 \leq ij \leq N - 1\}\), \(l_N(w) := \text{Card}\{ij : i_j = N\}\) and \(l(w) := l'(w) + l_N(w) = r\). The symmetric group \(S_N\) of \(N\) letters is a subgroup of \(S_N^C\). The restriction of \(l\) on \(S_N\) is the standard length function of \(S_N\).

We use a natural partial order in \(S_N^C\), known as the (strong) Bruhat order. For a given reduced word \(w = s_{i_1} \ldots s_{i_q}\), a subexpression of \(w\) is of the form \(s_{j_1} \ldots s_{j_q}\) (or empty) with \(1 \leq j_1 < j_2 < \ldots < i_q \leq r\). Then, \(w' \leq w\) if and only if \(w'\) can be obtained as a subexpression of a reduced expression of \(w\).

The Iwahori–Hecke algebra \(\mathcal{H} := \mathcal{H}(S_N^C)\) of type \(B_N\) is an unital, associative algebra over \(\mathbb{C}[t^{\pm 1}, t_N^{\pm 1}]\) satisfying

\[
(T_i - t)(T_i + t^{-1}) = 0, \quad 1 \leq i \leq N - 1,
\]

\[
(T_N - t_N)(T_N + t_N^{-1}) = 0,
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad 1 \leq i \leq N - 2,
\]

\[
T_i T_j = T_j T_i, \quad |i - j| \geq 2.
\]
\[ T_{N-1}T_NT_{N-1}T_N = T_NT_{N-1}T_NT_{N-1}. \]

The set \( \{T_w\}_{w \in S_N^C} \) is the standard monomial basis of \( \mathcal{H} \). Throughout this paper, we consider the two cases for the Hecke parameters \((t, t_N)\):

**Case A:** \( t \) and \( t_N \) are algebraically independent.

**Case B:** \( t_N = t^m \) with some \( m \in \mathbb{N}_+ \).

We define

\[
t^{l(w)} := \begin{cases} 
  t^{l'(w)}t_N^{l'(w)} & \text{for Case A} \\
  t^{l'(w)+ml_N(w)} & \text{for Case B}
\end{cases}
\]

For \( v, w \in S_N^C \), we denote \( t^{l(v)}/t^{l(w)} \) by \( t^{l(v)-l(w)} \).

We define the bar involution of \( \mathcal{H} \), \( \mathcal{H} \ni a \mapsto \bar{a} \) by \( T_i \mapsto T_i^{-1} \), \( 1 \leq i \leq N \), together with \( t^p \mapsto t^{-p} \) for \( p \in \mathbb{N}_+ \) (for Case A & B) and \( t_N \mapsto t_N^{-1} \) (for Case A only).

We consider the abelian groups \( \Gamma^A = \{t^i t_N^j | i, j \in \mathbb{Z}\} \) and \( \Gamma^B = \{t^i | i \in \mathbb{Z}\} \) for Case A and B respectively. The lexicographic order of \( \Gamma_X \) \((X = A, B)\) is defined by \( \Gamma_X = \Gamma^+_X \cup \{1\} \cup \Gamma^-_X \) \((X = A, B)\) where

\[
\Gamma^+_X := \{t^i t_N^j | i > 0, j \in \mathbb{Z}\} \cup \{t_N^i | i > 0\},
\]
\[
\Gamma^-_X := \{t^i | i > 0\}.
\]

Then we have

**Theorem 1** (Lusztig [14]). There exists a unique basis \( \{C^X_w : w \in S_N^C\} \) of \( \mathcal{H} \) and a polynomial \( P^X_{v,w} \) such that \( C^X_w = C^X_v \) and

\[
(1) \quad C^X_w = \sum_{v \leq w} t^{l(v)-l(w)} P^X_{v,w} T_v
\]

where \( t^{l(v)-l(w)} P^X_{v,w} \in \mathbb{Z}(\Gamma_X) \) \((X = A, B)\).

2.2. The coset space. Let \( W^N \) be the left coset space \( S_N^C/S_N \). The following objects are bijective to each other:

\begin{enumerate}
  \item A binary string \( \alpha \in \{1, 2\}^N \). Let \( \mathcal{P}_N \) be the set of binary strings in \( \{1, 2\}^N \).
  \item A path from \((0, 0)\) to \((N, n)\) with \( |n| \leq N \) and \( N - n \in 2\mathbb{Z} \), where each step is in the direction \((1, \pm 1)\).
  \item A minimal (and maximal) representative of the coset \( W^N \).
  \item A shifted Ferrers diagram specified by a path \( \alpha \).
\end{enumerate}

Before proceeding to show explicit bijections, we introduce some terminologies. For later convenience, we introduce a sign \( \epsilon \in \{+,-\} \). When \( \epsilon = + \) (resp. \( \epsilon = - \)), we consider the maximal (resp. minimal) representatives of the coset \( W^N \).

**Definition 2.** For each binary string \( \alpha \in \mathcal{P}_N \), we denote by \( w^\epsilon(\alpha) \) and \( \lambda^\epsilon(\alpha) \) the representative in \( W^N \) and a (rotated) shifted Ferrers diagram corresponding to \( \alpha \).

A bijection between (i) and (ii). Let \( \alpha \in \mathcal{P}_N \) be a binary string. A path starts from \((0, 0)\) and each move is in the direction \((1, \epsilon)\) if \( \alpha_i = 1 \) or \((1, -\epsilon)\) if \( \alpha_i = 2 \). Reversely, for a given
path, we can read $\alpha_i$ according to the tangent of each step. Hereafter, we identify a path with a binary string.

**Remark 3.** Note that the binary string corresponding to a given path $\alpha$ depends on the choice of the sign $\epsilon$. Suppose that a string $\alpha_+ \epsilon$ for $\epsilon = +$ is associated with the path $\alpha$. Then, the string $\alpha_- \epsilon$ for $\epsilon = -$ is obtained by exchanging 1 and 2 in $\alpha_+$.

A bijection between (ii) and (iv). Let $\alpha \in \mathcal{P}_N$ be a path. For $\epsilon = +$, consider the set of integral points

$$S^+(\alpha) := \{(i,j) : (i,j) \text{ is above the path } \alpha, 0 < i \leq N, |j| \leq i, i + j - 1 \in 2\mathbb{Z}\}.$$  

We put (45 degree rotated) squares of length $\sqrt{2}$ whose center are all points in $S^+(\alpha)$. The set of squares can be regarded as (45 degree rotated) shifted Ferrers diagram $\lambda^+ (\alpha)$. For $\epsilon = -$, we define $S^-(\alpha)$ by replacing “above” by “below” in the set $S^+(\alpha)$ and define $\lambda^- (\alpha)$ similarly.

We call a box $(N,j) \in \lambda/\mu$ for some $j$ an anchor box.

Let $\alpha \in \mathcal{P}_N$ be a path and $\lambda$ be the associated Ferrers diagram $\lambda^+(\alpha)$. We denote by $|\lambda|$ the number of boxes in the skew Ferrers diagram $\lambda$. By abuse of notation, we also denote $|\alpha| := |\lambda(\alpha)|$. Notice that $|\alpha|$ depends on the sign $\epsilon$ and we omit $\epsilon$ when it is obvious.

A bijection between (i) and (iii). A bijection directly follows from [16]. We fix the convention by assigning the binary string $1^N$ to the identity in $W^N$ for $\epsilon = \pm$. A reduced word $w^+(\alpha)$ (resp. $w^-(\alpha)$) is obtained from $\lambda^+(\alpha)$ (resp. $\lambda^- (\alpha)$) as follows. Starting from the top (resp. bottom) box, we read the boxes left downward (resp. upward) according to the column of $\lambda^+(\alpha)$ (resp. $\lambda^- (\alpha)$). If the number of boxes in the column is $k_1$, we assign an ordered product $s_{N-k_1+1} \ldots s_N$ to this column. Then, move to the next column with $k_2$ boxes. Continue until all columns are visited. Therefore, $w^+(\alpha)$ (resp. $w^- (\alpha)$) is of the form $(s_{N-k_1+1} \ldots s_N) \ldots (s_{N-k_2+1} \ldots s_N)(s_{N-k_1+1} \ldots s_N)$ with $1 \leq r \leq N$ and $k_1 > k_2 > \ldots k_r \geq 1$. We denote this ordered product by

$$w^\pm (\alpha) = \prod_{(i,j) \in \lambda^\pm (\alpha)} s_i.$$  

Let us take two paths $\alpha, \beta \in \mathcal{P}_N$ and fix the sign $\epsilon$. We denote by $\alpha \leq \beta$ when corresponding representatives satisfy $w^\epsilon (\alpha) \leq w^\epsilon (\beta)$. Note that when $\alpha$ is above $\beta$, $\alpha < \beta$ (resp. $\alpha > \beta$) for $\epsilon = +$ (resp. $\epsilon = -$).

**Example 4.** Let $\alpha = 221121$ and $\epsilon = +$. The shifted Ferrers diagram $\lambda^+(\alpha)$ is shown below. The path $\alpha$ is the lowest path from $O$ to $B$ and the path 111111 is the up-right one from $O$ to $A$. When $\epsilon = -$, the binary string for the path from $O$ to $B$ is 112212. As a maximal representation in $W^N$, $w^+(\alpha) = s_5s_6s_2s_3s_4s_5s_6s_1s_2s_3s_4s_5s_6$. The boxes with * are anchor boxes.
2.3. Parabolic Kazhdan–Lusztig polynomials. An element \( w \in S_N^C \) is uniquely written as \( w = xw' \) such that \( x \in W^N \) and \( w' \in S_N \). The projection \( \varphi : S_N^C \to W^N \) induces two natural projection maps \( \varphi^\pm : H \cong \mathbb{C}[S_N^C] \to \mathbb{C}[W^N], T_w \mapsto (\pm t^{\pm 1})^l(w')m_{\varphi(w)} \), where \( \{m_w\}_{w \in W^N} \) is the standard basis of \( \mathbb{C}[W^N] \). We require that \( \varphi^\pm \) commute with the action of \( H \).

Let \( \alpha := \alpha_1 \alpha_2 \ldots \alpha_N \in \mathcal{P}^N \) be a binary string and \( \mathcal{M}^\pm := \mathbb{C}[W^N] \) be a vector space spanned by \( \{m_\alpha : \alpha \in \mathcal{P}^N\} \). A simple transposition \( s_i \in S_N^C \) naturally acts on the binary string \( \alpha \), i.e., \( s_i \cdot \alpha = \ldots \alpha_i+1 \alpha_i \ldots, 1 \leq i \leq N - 1 \), and \( s_N \cdot \alpha = \ldots (3 - \alpha_N) \). The action of \( H \) on the module \( \mathcal{M}^\epsilon \) with \( \epsilon \in \{+,-\} \) is given by

\[
T_i m_\alpha = \begin{cases} 
\epsilon t^\epsilon m_\alpha & \alpha_i = \alpha_{i+1}, \\
 m_{s_i \alpha} & \alpha_i < \alpha_{i+1}, \quad \text{for } 1 \leq i \leq N - 1, \\
m_{s_i \alpha} + (t - t^{-1})m_\alpha & \alpha_{i+1} < \alpha_i,
\end{cases}
\]

\[
T_N m_\alpha = \begin{cases} 
 m_{s_N \alpha} & \alpha_N = 1, \\
m_{s_N \alpha} + (t_N - t_N^{-1})m_\alpha & \alpha_N = 2,
\end{cases}
\]

for both Case A and B.

We introduce parabolic analogue of the Kazhdan–Lusztig basis:

**Theorem 5** (Deodhar [8]). There exists a unique basis \( \{C_x^\pm\}_{x \in W^N} \) of \( \mathcal{M}^\pm \) such that \( C_x^\pm = C_x \) and

\[
C_y^\pm = \sum_{x \leq y} t^{l(x) - l(y)} P_{x,y}^\pm m_x
\]

where \( P_{y,y}^\pm = 1 \) and \( t^{l(x) - l(y)} P_{x,y}^\pm \in \mathbb{Z} \Gamma^X \) for Case X (X=A,B).

Hereafter, we denote by \( P_{x,y}^{A,\pm} \) (resp. \( P_{x,y}^{B,\pm} \)) the parabolic Kazhdan–Lusztig polynomials for Case A (resp. Case B).

**Theorem 6.** Let \( X \in \{A, B\} \). We have the inversion formula for \( P_{x,\alpha,\beta}^{X,\pm} \):

\[
\sum_{\alpha \in \mathcal{P}_N} (-1)^{|\alpha|+|\beta|} P_{\alpha,\beta}^{X,-} P_{\alpha,\gamma}^{X,+} = \delta_{\beta,\gamma}.
\]

**Proof.** The relations among \( P_{\alpha,\beta}^{\pm} \) and the original Kazhdan–Lusztig polynomials \( P_{x,y} \) are given by Proposition 3.4 and Remark 3.8 in [8]. Together with the inversion formula given in Theorem 3.1 in [9], we have Eqn.(5). For details, see also Theorem 4 in [18].  \( \square \)
3. Combinatorics

3.1. Ballot strips. A ballot path of length \((l, l') \in \mathbb{N}^2\) is a path from \((x, y) \in \mathbb{Z}^2\) to \((x + 2l + l', y + l')\) and over the horizontal line \(y\).

A ballot strip of length \((l, l') \in \mathbb{N}^2\) is obtained by putting unit boxes (45 degree rotated) whose center are at the vertices of a ballot path of length \((l, l')\) (see some examples on Fig. 1). Note that a single box (corresponding to the length \((l, l') = (0, 0)\)) is also included as a ballot strip.

Figure 1. Examples of ballot strips: The length is \((1, 0)\), \((3, 0)\), \((0, 2)\), \((1, 2)\) and \((2, 2)\) from left.

Remark 7. A ballot path of length \((l, 0)\) is nothing but a Dyck strip in [18].

Hereafter, a box \((x, y)\) means a unit box whose center is \((x, y)\). Let \(b\) a box \((x, y)\). Four boxes \((x \pm 1, y \pm 1)\) are neighbours of \(b\). The box \((x + 1, y + 1)\) is said to be NE (north-east) of \(b\) and similarly the other three boxes are NW, SW and SE of \(b\). The two boxes \((x, y \pm 2)\) are said to be just above or just below \(b\).

Recall the definition of an anchor box in a skew Ferrers diagram. We put a constraint for a ballot strip as follows.

Rule 0 (Case A and B): The rightmost box of a ballot strip of length \((l, l')\) with \(l' \geq 1\) is on an anchor box.

Let \(D, D'\) be ballot strips. We define two rules to pile \(D'\) on top of \(D\) in addition to Rule 0.

Rule I: (a) Case A & B: If there exists a box of \(D\) just below a box of \(D'\), then all boxes just below a box of \(D'\) belong to \(D\).

(b) Case B: Suppose \(l' \geq m\). The number of ballot strips of length \((l, l')\) is even for \(l' - m \in \mathbb{Z}\), and zero for otherwise.

Rule II: (a) Case A& B: If there exists a box of \(D'\) just above, NW, or NE of a box of \(D\), then all boxes just above, NW, and NE of a box of \(D\) belong to \(D\) or \(D'\).

(b) Case B: Suppose \(l' \geq m\). If there exists a ballot strip \(D\) of length \((l, l')\) with \(l' - m \in 2\mathbb{Z}\) (resp. \(l' - m - 1 \in 2\mathbb{Z}\)), then there is a strip of length \((l'', l' + 1), l'' \leq l\) (resp. \((l'', l' - 1), l'' \leq l\)) just above (resp. just below) \(D\).

Roughly, Rule I (resp. Rule II) means that we are allowed to pile ballot strips of smaller or equal (resp. longer) length on top of a ballot strip (see Figure 2).

3.2. Generating function. Let \(\alpha, \beta\) be paths in \(\mathcal{P}_N\) such that \(\alpha < \beta\). These two paths characterize the domains, namely the skew Ferrers diagram \(\lambda(\beta)/\lambda(\alpha)\). We fill these domains with ballot strips. We denote by \(\text{Conf}(\alpha, \beta)\) the set of all such possible configurations of ballot strips, and by \(\text{Conf}^{I/II}(\alpha, \beta)\) the subset of configurations satisfying Rule I/II.
Let $\mathcal{D}$ be a ballot strip of length $(l,l') \in \mathbb{N}^2$. We denote by $\text{wt}^X(\mathcal{D})$, $X \in \{A,B\}$, the weight for a ballot strip $\mathcal{D}$, which is given by

**Case A:**

\[
\text{wt}^A(\mathcal{D}) := \begin{cases} 
  t^{2l+l'}, & \text{if } l' \text{ is even}, \\
  -\sigma t^{2l+l'-1} N, & \text{if } l' \text{ is odd}.
\end{cases}
\]

**Case B:**

\[
\text{wt}^B(\mathcal{D}) := \begin{cases} 
  \sigma l' t^{2l+2l'}, & 0 \leq l' \leq m - 1 \\
  t^{m+l'+2l}, & l' \geq m, l' - m \in 2\mathbb{Z} \\
  t^{m+l'+2l-1}, & l' \geq m, l' - m - 1 \in 2\mathbb{Z},
\end{cases}
\]

where the sign $\sigma = +$ (resp. $-$) in the case of Rule I (resp. Rule II).

**Definition 8.** The generating function of ballot strips for the paths $\alpha < \beta$ is defined by

\[
Q^{X,Y,\epsilon}_{\alpha,\beta} = \sum_{C \in \text{Conf}^X(\alpha,\beta)} \prod_{\mathcal{D} \in C} \text{wt}^X(\mathcal{D}),
\]

where $X \in \{A,B\}$, $Y \in \{I,II\}$ and $\epsilon \in \{+, -\}$.

Note that $\text{Conf}^{II}(\alpha,\beta)$ has at most one configuration. Recall that when two paths satisfy $\alpha < \beta$ with the sign $\epsilon$, the change of the sign $\epsilon \mapsto -\epsilon$ yields $\alpha > \beta$. Therefore, we have

\[
Q^{X,Y,\epsilon}_{\alpha,\beta} = Q^{X,Y,-\epsilon}_{\beta,\alpha}.
\]

**Example 9.** Let $(\alpha, \beta) = (111111, 211212)$ and $\epsilon = +$. The possible configurations of ballot strips for Case A and Case B ($m \geq 2$) are
Since we fix the configuration, the absolute value of the weight $|\beta|$ depends on the configuration $C$. We denote by $\beta P$ for a possible element in $P$. Notice that if $C \in \text{Conf}(\alpha, \gamma)$ then an element of $\beta P$ means that an element of $\beta$ is above $1'$. We show $\sum_{\beta P} (9) = \sum_{\beta P} \text{wt}(\beta) \cdot \text{sign}(\beta, \gamma) \cdot \text{wt}(\beta, \gamma) = 0$. The generating function is

$$Q_{\alpha,\beta}^{B,\beta^+} = 1 + 2t^2 + 2t^4 + t^6.$$  

**Theorem 10** (Inversion Formula). Let $X \in \{A, B\}$. The generating functions $Q_{\alpha,\beta}^{X,Y,z}$ satisfy

$$(8) \sum_{\beta P} Q_{\alpha,\beta}^{X,I,-} Q_{\beta,\gamma}^{X,II,-} (-1)^{|\beta|+|\gamma|} = \delta_{\alpha,\gamma}.$$  

**Proof.** We refer to the proof of Theorem 5 in [18] since we can apply the similar arguments to our case. Below, we give the outline of the proof and the difference from [18].

When $\alpha = \gamma$, the argument holds true. Let us fix two paths $\alpha, \gamma$ ($\alpha < \gamma$ in $\epsilon = -$) and a configuration of ballot strips $C \in \text{Conf}(\alpha, \gamma)$ such that there exists a path $\beta$ dividing $C$ into two configurations $C_I(\beta)$ and $C_{II}(\beta)$ where $C_I(\beta) \in \text{Conf}^I(\alpha, \beta)$ and $C_{II}(\beta) \in \text{Conf}^II(\beta, \gamma)$. Notice that $\beta$ depends on the configuration $C$ and there may be several possible $\beta$'s. We denote by $P(C)$ the set of such paths $\beta$'s for the configuration $C$. The weight of the configuration $C$ is given by $\text{wt}(C) = \text{wt}(C_I) \text{wt}(C_{II})$ (recall the definitions of weights (6) and (7)). Since we fix the configuration, the absolute value of the weight $|\text{wt}(C)|$ is independent of the path $\beta$. We denote by $\text{wt}(C) = |\text{wt}(C)| \text{sign}(C)$, where $\text{sign}(C)$ is a sign associated with the configuration $C$. Therefore, we have

$$\sum_{\beta P} Q_{\alpha,\beta}^{X,I,-} Q_{\beta,\gamma}^{X,II,-} (-1)^{|\beta|+|\gamma|} = \sum_C \sum_{\beta P(C)} \text{wt}(C) (-1)^{|\beta|+|\gamma|} = \sum_C |\text{wt}(C)| \sum_{\beta P(C)} \text{sign}(C) (-1)^{|\beta|+|\gamma|}.$$  

Below, we will show that $\sum_{\beta P(C)} \text{sign}(C) (-1)^{|\beta|+|\gamma|} = 0$.

Define the intersection 

$$\mathcal{I}(C) := \left( \bigcup_{\beta P(C)} C_I(\beta) \right) \cap \left( \bigcup_{\beta P(C)} C_{II}(\beta) \right).$$  

The choice of $\beta$ determines that an element of $\mathcal{I}(C)$ belongs to $C_I$ or $C_{II}$. Here an element $B \in \mathcal{I}(C)$ means a ballot strip or ballot strips which are on top of each other and glued together.

Let $\beta_1, \beta_1'$ be paths such that an element $B \in \mathcal{I}(C)$ belongs to $C_I(\beta_1)$ and $C_{II}(\beta_1')$, which means that $\beta_1$ is above $\beta_1'$. We show $\sum_{\beta = \beta_1, \beta_1'} \text{sign}(B) (-1)^{|\beta_1|-|\beta_1'|} = 0$. There are three cases for a possible element in $\mathcal{I}(C)$. 

---
Case 1. (Case A & B) Let $\mathcal{B}$ be a ballot strip of length $(l, 0)$. We need odd number of boxes to form the strip. Therefore, the difference $|\beta_1| - |\beta'_1|$ is odd and the sign $\text{sign}(\mathcal{B}) = 1$ for both $\epsilon = \pm$. This implies the contributions of $\beta_1$ and $\beta'_1$ cancel each other.

Case 2. (Case A only) Let $\mathcal{B}$ be a ballot strip of length $(l, l')$ with $l' \geq 1$. When $l'$ is even, then the number of boxes to form the strip $\mathcal{B}$ is odd. As in Case 1, the contributions of $\beta_1$ and $\beta'_1$ cancel. When $l'$ is odd, the number of boxes to form $\mathcal{B}$ is even. Therefore, $|\beta_1| - |\beta'_1|$ is even. However, we have

$$\text{sign}(\mathcal{B}) = \begin{cases} +, & \mathcal{B} \in \mathcal{C}_{II}(\beta'_1), \\ -, & \mathcal{B} \in \mathcal{C}_{I}(\beta'_1), \end{cases}$$

which implies the contributions from $\beta_1$ and $\beta'_1$ cancel.

Case 3. (Case B only) Let $\mathcal{B}$ be a ballot strip of length $(l, l')$ with $l' \geq 1$. When $1 \leq l' \leq m - 1$, the contributions cancel as in Case 2. Below, we assume that $l' \geq m$. Suppose that $\mathcal{B} \in \mathcal{C}_I(\beta_1)$. The length of $\mathcal{B}$ satisfies $l' - m \in 2\mathbb{Z}$ due to Rule Ib. Further, there is another $\mathcal{B}' \in \mathcal{C}_I(\beta'_1)$ such that the same length as $\mathcal{B}$, and two strips $\mathcal{B}, \mathcal{B}'$ are on top of each other. Without loss of generality, we assume that $\mathcal{B}'$ is just above $\mathcal{B}$. We want to change the path $\beta_1$ to $\beta'_1$ such that $\mathcal{B}$ belongs to $\mathcal{C}_{II}(\beta'_1)$. Since $\mathcal{B} \in \mathcal{C}_{II}(\beta'_1)$, $l' - m \in 2\mathbb{Z}$ and Rule Ib, we need a strip $\mathcal{B}''$ of length $(l'', l' + 1)$, $l'' \geq l$, just above $\mathcal{B}$. We will show that $\mathcal{B}''$ can be obtained from $\mathcal{B}'$ by gluing two ballot strips. Let $b'$ be the leftmost box of $\mathcal{B}$ and $b$ be the northwest box of $b'$. Since the strip $\mathcal{B}'$ is just above $\mathcal{B}$ and of the same length, the ballot strip $\mathcal{D}$ which contains $b$ satisfies $\mathcal{D} \in \mathcal{C}_I(\beta_1)$. Therefore, we are able to glue the strip $\mathcal{B}'$ and the strip $\mathcal{D}$ to form $\mathcal{B}''$. We regard the region obtained by gluing $\mathcal{B}$ and $\mathcal{B}''$ (equivalently $\mathcal{B}, \mathcal{B}'$ and $\mathcal{D}$) as an element of $\mathcal{I}(C)$. See Figure 3 for an example of this operation. As a final step, we have to compare the weights of these strips. It is clear from (7) that the weight of $\mathcal{B}''$ is equal to the product of the weights $\mathcal{B}'$ and $\mathcal{D}$. Since the total number of boxes in the strip $\mathcal{D}$ is odd, $|\beta_1| - |\beta'_1|$ is odd. Further, the sign of this region is plus. These imply that the contributions of these two paths cancel.

In the above three cases, we change a path $\beta_1$ to $\beta'_1$ locally by involving one element in $\mathcal{I}(C)$. We are also able to show the following two facts (see Proposition 4 and Lemma 5 in [18]): 1) When there exists a path $\beta \in P(C)$, $\mathcal{I}(C)$ is not empty. 2) The distance of two elements in $\mathcal{I}(C)$ is at least two. From these two facts, there are $2^r$ possible paths in $P(C)$ when the cardinality of $\mathcal{I}(C)$ is $r$. Therefore, the contribution $\sum_{\beta \in P(C)} \text{sign}(C)(-1)^{|\beta|+l|} \epsilon$ can

---

**Figure 3.** The region below (resp. above) the path $\beta_1$ (resp. $\beta'_1$) satisfies Rule I (resp. Rule II) in the left (resp. right) picture. The strip $\mathcal{B}'$ and the box $b$ (a strip of length $(0, 0)$) form the strip $\mathcal{B}''$. 
be reduced to the sum of local changes of paths corresponding to an element in $\mathcal{I}(C)$, which is zero. This completes the proof of the theorem. □

4. Kazhdan–Lusztig polynomials $P_{\alpha,\beta}^\pm$

4.1. Module $\mathcal{M}^-$.

4.1.1. Case A. Let $\alpha = \alpha_1 \ldots \alpha_N \in \mathcal{P}^N$ be a binary string of length $N$. We make a pair between adjacent 2 and 1 (in this order) in the string $\alpha$ and remove it from $\alpha$. We repeat this procedure until it becomes a sequence $1 \ldots 12 \ldots 2$. We call these remaining 1’s (resp. 2’s) as unpaired 1’s (resp. 2’s). The $(2i-1)$-th (resp. $2i$-th) unpaired 2 from right is called as o-unpaired (resp. e-unpaired) 2.

For simplicity, we introduce an graphical notation for these pairs, unpaired 1s, e- and o-unpaired 2. Consider a line with $N$ points. If $\alpha_i$ and $\alpha_j$ form a pair, then we connect $i$ and $j$ via an arc. If $\alpha_i$ is an unpaired 1, we put a vertical line with a circled 1. If $\alpha_i$ is an e-unpaired (resp. o-unpaired) 2, we put a vertical line with a mark e (resp. o). We call this graphical notation as link pattern for Case A.

**Example 11.** Let $\alpha = 1221222112$. The link pattern is

![Link Pattern](image)

Recall that the module $\mathcal{M}^-$ is spanned by the set of basis $\{m_\alpha\}_{\alpha \in \mathcal{P}^N}$. The space is isomorphic to $V^\otimes N$ where $V \cong \mathbb{C}^2$ has the standard basis $\{|1\rangle, |2\rangle\}$. When $i$-th component of the tensor product is $x \in \{1, 2\}$, we denote it by $|x\rangle_i$. We simply write $|xx\prime\rangle_{ij}$ for the tensor product $|x\rangle_i \otimes |x\prime\rangle_j$ and sometimes denoted by $|xx\prime\rangle$ if the components are obvious. Hereafter, we identify a base $m_{\alpha}, \alpha \in \{1, 2\}^N$ with $|\alpha_1 \ldots \alpha_N\rangle$.

An arc, vertical line with e,o and a circled 1 are building blocks of a link pattern corresponding to a string $\alpha \in \{1, 2\}^N$. We introduce a map $\varpi^A$ from these building blocks (equivalently a partial binary string of length 1 and 2) to a vector in $V^\otimes 2$ or $V$:

![Building Blocks](image)

Then, we extend the map $\varpi^A$ to a link pattern for a string $\alpha$ since a link pattern for $\alpha$ is regarded as a tensor product of the building blocks.
Example 12.

$$\varpi^A(1212) = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\end{array}
= |1\rangle_1 \otimes (|21\rangle_23 + t^{-1}|12\rangle_23) \otimes (|2\rangle_4 + t_N^{-1}|1\rangle_4)
= m_{1212} + t^{-1}m_{1122} + t_N^{-1}m_{1211} + t^{-1}t_N^{-1}m_{1121}
$$

Remark 13. The coefficients of $m_{\alpha}$ in $\varpi^A(\beta)$ is nothing but the generating function of ballot strips (up to the normalization constant $t^{n(|\beta|-|\alpha|)}$) where the region $\lambda^- (\beta)/\lambda^- (\alpha)$ is filled with ballot strips via Rule IIa. This is because an arc corresponds to a ballot strip of length $(l,0)$ and an e-unpaired (resp. o-unpaired) 2 corresponds to a ballot strip of length $(l,2m)$ (resp. $(l,2m+1)$).

Recall that $w^-(\alpha)$ is a minimal length representative in the coset. Let us fix a reduced word $w^-(\alpha)$ and denote the ordered product by $\prod_{(i,j) \in \lambda^- (\alpha)} s_i$.

Lemma 14. An element $\varpi^A(\alpha)$, $\alpha \in P_N$ is factorized as

$$\varpi^A(\alpha) = \prod_{(i,j) \in \lambda^- (\alpha)} (T_i + t^{-l(s_i)})m_{1...1}.$$

Proof. We prove the statement by induction. From the definition of the map $\varpi^A$, we have $\varpi^A(1...1) = m_{1...1}$ and

$$\varpi^A(1...12) = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\end{array} \cdots \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\end{array}
= (T_N + t_N^{-1})m_{1...1}.$$

Fix $\beta \in P_N$. We assume that the statement holds true for $\varpi^A(\alpha)$ for all $\alpha < \beta$. Then, there exists $\alpha < \beta$ and an integer $i$ such that $\beta = s_i \alpha$ with $1 \leq i \leq N$. Since $\varpi^A(\alpha)$ is a tensor product of the building blocks, it is enough to check the action of $T_i + t^{-l(s_i)}$ on a local part of $\varpi^A(\alpha)$ involving $\alpha_i$ and $\alpha_{i+1}$.

(i) In case of $1 \leq i \leq N-1$, we have $(\alpha_i, \alpha_{i+1}) = (1,2)$. We have four cases:

$$\begin{array}{l}
(T_i + t^{-1}) \begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}, \ x = e,o
\end{array}$$

$$\begin{array}{l}
(T_i + t^{-1}) \begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}, \ x = e,o
\end{array}$$

$$\begin{array}{l}
(T_i + t^{-1}) \begin{array}{c}
\begin{array}{c}
\text{x}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{x}
\end{array}
\end{array}, \ x = e,o,
\end{array}$$

$$\begin{array}{l}
(T_i + t^{-1}) \begin{array}{c}
\begin{array}{c}
\text{x}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\text{x}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{x}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\text{x}
\end{array}
\end{array}, \ x = e,o
\end{array}.$$
Now it is clear that right hand sides of above equation indicate \((\beta_i, \beta_{i+1}) = (2, 1)\) and all coefficients are one.

(ii) In case of \(i = N\), we have \(\alpha_N = 1\). We have two cases:

\[
(T_N + t_N^{-1}) \equiv N \quad = \quad \begin{array}{c} e \\ \hline o \end{array}
\]

\[
(T_N + t_N^{-1}) \equiv 1 \quad = \quad \begin{array}{c} o \\ \hline 1 \end{array}
\]

As expected, \(\beta_N = 2\) and the coefficients are one.

In both cases, the obtained expressions are nothing but \(\varpi^A(\beta)\).

\[\square\]

We describe the action of \(T_i + t^{-l(s_i)}\) on \(\varpi^A(\beta)\). This is reduced to a local action of \(T_i + t^{-l(s_i)}\) on a partial binary strings. Together with the proof of Lemma 14, remaining non-trivial cases are as follows.

\[
(T_i + t^{-1})\varpi^A(21) = [2]\varpi^A(21),
\]

\[
(T_i + t^{-1}) \begin{array}{c} e \\ \hline o \end{array} = (t_N^{-1} + t^{-1}t_N)\varpi^A(21),
\]

\[
(T_i + t^{-1}) \begin{array}{c} o \\ \hline e \end{array} = (t_N^{-1} + t_N)\varpi^A(21),
\]

\[
(T_i + t^{-1}) \begin{array}{c} x \\ \hline i \end{array} = \begin{array}{c} x \\ \hline i \end{array}, \quad x = e, o.
\]

**Theorem 15.** \(\varpi^A(\alpha), \alpha \in \mathcal{P}_N\) is the Kazhdan–Lusztig basis \(C_A^{\alpha,-}\).

*Proof.* From Lemma 14, an element \(\varpi^A(\beta)\) is invariant under the involution, as \(T_i + t^{-l(s_i)}\) is invariant. From the definition of \(\varpi^A(\beta)\), it is clear that the coefficient of \(m_\alpha\) is in \(\mathbb{Z}(\Gamma^A)\) and a monomial. Further, the degree is less than or equal to \(l^{(a)} - l^{(\beta)}\) (as a polynomial in \(t^{-1}\)). This completes the proof of the theorem. \[\square\]

Recall that we associate a link pattern with a path \(\beta\), which is a set of pairs between 2 and 1, o- and e-unpaired 2. Define a set of paths by \(F^A(\beta)\) as

\[
F^A(\beta) := \{\alpha \leq \beta : \text{Some pairs and unpaired 2s are flipped}\}
\]

where by flipped we mean switching 2 and 1 in the pair of the binary string \(\beta\) and changing from 2 to 1 in an unpaired 2 of \(\beta\). For a path \(\alpha \in F^A(\beta)\), define integers

\[
d^A(\alpha, \beta) = \{\text{the number of flipped pairs}\},
\]

\[
d^e(\alpha, \beta) = \{\text{the number of flipped e-unpairs}\},
\]

\[
d^o(\alpha, \beta) = \{\text{the number of flipped o-unpairs}\}.
\]

From Remark 13 and Theorem 15, we have
Corollary 16. The generating functions $Q_{\alpha,\beta}^{A,II,-}$ and $P_{\alpha,\beta}^-$ for Case A is equal:
\[
    t^{[\alpha] - [\beta]} Q_{\alpha,\beta}^{A,II,-} = t^{[\alpha] - [\beta]} P_{\alpha,\beta}^-
    = t^{-d_A(\alpha,\beta)} \frac{d_B(\alpha,\beta)}{t_N} (t/t_N)^{-d_A(\alpha,\beta)}.
\]

4.1.2. Case B. Let $\alpha \in \mathcal{P}_N$. For the graphical notation, we make pairs between 2’s and 1’s. Then, we have remaining unpaired 1’s and 2’s as Case A. If $\alpha_i$ is the $j$-th ($1 \leq j \leq m$) unpaired 2 from right, we put a vertical line with the integer $m + 1 - j$. If $\alpha_i$ and $\alpha'_i$ with $i < i'$ are the $j$-th and $(j + 1)$-th unpaired 2’s with $j \geq m + 1$ and $j - m + 1 \in 2\mathbb{Z}$, we put vertical lines (on the $i$-th and $i'$-th point) whose endpoints are connected by a dotted line. If $\alpha_i$ is an unpaired 1 or a remaining unpaired 2 not classified above, then we put a vertical line with a circled 1 or a circled 2 respectively on the $i$-th point. Note that the number of unpaired 2’s with a circled 2 is at most one. We call this graph as a link pattern for Case B.

Example 17. Let $\alpha = 12221222112$ and $m = 2$. The link pattern is

![Link pattern](image)

We define the map $\varpi^B$ from the building blocks to a vector in $V$ or $V^2$:
\[
    \begin{align*}
    \bigcirc & \quad \mapsto |21\rangle + t^{-1}|12\rangle, \\
    \downarrow & \quad \mapsto |2\rangle + (-1)^{m-p} t^{-p}|1\rangle, \quad 1 \leq p \leq m, \\
    \bigcirc & \quad \mapsto |22\rangle + t^{-1}|11\rangle, \\
    \bigotimes & \quad \mapsto |x\rangle, \quad x \in \{1, 2\}.
    \end{align*}
\]

Together with the map from a binary string to a link pattern, we naturally extend the map $\varpi^B$ from a binary string to a vector in $\mathcal{M}^-$, and denote it by $\varpi^B$.

Remark 18. The coefficients of $m_\alpha$ in $\varpi^B(\beta)$ is nothing but the generating function of ballot strips (up to the normalization constant $t^{[\beta] - [\alpha]}$) where the region $\lambda^-(\beta)/\lambda^-(\alpha)$ is filled with ballot strips via Rule IIa and IIb.

Unlike Case A, there is no factorization property for $\varpi^B(\alpha)$. However, we have

Theorem 19. An element $\varpi^B(\alpha), \alpha \in \mathcal{P}_N$, is the Kazhdan–Lusztig basis $C^\alpha_-$ for Case B.

Proof. From the definition of the map $\varpi^B$, it is clear that the coefficient of $m_\alpha$ in $\varpi^B(\beta)$ is 1 for $\alpha = \beta$ and those of $m_\alpha$ for $\alpha < \beta$ is in $t^{-1}Z[t^{-1}]$. Further, the degree is less than or equal to $t^{[\beta] - [\alpha]}$ (as a polynomial in $t^{-1}$). Therefore, it is enough to prove that $\varpi^B(\beta)$ is invariant under the bar involution. We prove it by induction. The first two elements $\varpi^B(1 \ldots 12)$ and $\varpi^B(1 \ldots 1)$ are invariant since $\overline{m_{1 \ldots 1}} = m_{1 \ldots 1}$, $\varpi^B(1 \ldots 12) = (T_N + t_N^{-1})m_{1 \ldots 1}$ and $T_N + t_N^{-1} = T_N + t_N^{-1}$.

Fix $\beta \in \mathcal{P}_N$. We assume that $\varpi^B(\alpha)$ for all $\alpha < \beta$ are invariant under the bar involution. Then, there exists $\alpha$ ($\alpha < \beta$) and the integer $1 \leq i \leq N$ such that $\beta = s_i \alpha$ where $s_i \in S_N$. 

Since $\varpi^B(\beta)$ is a tensor product of building blocks, it is enough to check the local action of $T_i$ on a partial string and the local invariance under the bar involution.

(i) In case of $1 \leq i \leq N-1$, i.e., $(\alpha_i, \alpha_{i+1}) = (1, 2)$. The local actions of $T_i + t^{-1}$ on a partial binary string are

\[
(T_i + t^{-1}) \begin{array}{c} 1 \end{array} = \begin{array}{c} 1 \end{array}, \quad 1 \leq p \leq m, \\
(T_i + t^{-1}) \begin{array}{c} 1 \end{array} = \begin{array}{c} 1 \end{array}, \quad 1 \leq p \leq m,
\]

where $j < i$ and $k < j$ or $i + 1 < k$ in the fifth equation. In all cases, the right hand sides are $\varpi^B(\beta)$ and invariant under the bar involution.

(ii) In case of $i = N$, i.e., $\alpha_N = 1$ and $\beta_N = 2$. Since the image of $\varpi^B$ is a tensor product of building blocks, it is enough to check the action of $T_N + t^{-m}$ on the binary string of the form $2 \ldots 21$ which has some unpaired 2’s. We have

\[
\varpi^B(2 \ldots 2) = (T_N + t^{-m})\varpi^B(2 \ldots 21) - \sum_{k=k_0}^{m} (-1)^{m-k}(k-1)\varpi^B(2 \ldots 21 \ldots 2)_{l-m+k-1 \ldots m-k+1}
\]

where $k_0 = \max(1, m - l + 1)$. For the proof of Eqn.(10), it is enough to check the following three cases by using induction.

\[
(T_N + t^{-m}) \begin{array}{c} 1 \ldots \end{array} = \begin{array}{c} 1 \ldots \end{array} + \sum_{k=k+1}^{m} (-1)^{m-k}(k-1)\varpi^B(2 \ldots 21 \ldots 2)_{l-m+k-1 \ldots m-k+1}
\]

\[
(T_N + t^{-m}) \begin{array}{c} 1 \ldots \end{array} = \begin{array}{c} 1 \ldots \end{array} + \sum_{k=k+1}^{m} (-1)^{m-k}(k-1)\varpi^B(2 \ldots 21 \ldots 2)_{l-m+k-1 \ldots m-k+1}
\]

\[
(T_N + t^{-m}) \begin{array}{c} 1 \ldots \end{array} = \begin{array}{c} 1 \ldots \end{array} + \sum_{k=k+1}^{m} (-1)^{m-k}(k-1)\varpi^B(2 \ldots 21 \ldots 2)_{l-m+k-1 \ldots m-k+1}
\]

\[
(T_N + t^{-m}) \begin{array}{c} 1 \ldots \end{array} = \begin{array}{c} 1 \ldots \end{array} + \sum_{k=k+1}^{m} (-1)^{m-k}(k-1)\varpi^B(2 \ldots 21 \ldots 2)_{l-m+k-1 \ldots m-k+1}
\]

\[
(T_N + t^{-m}) \begin{array}{c} 1 \ldots \end{array} = \begin{array}{c} 1 \ldots \end{array} + \sum_{k=k+1}^{m} (-1)^{m-k}(k-1)\varpi^B(2 \ldots 21 \ldots 2)_{l-m+k-1 \ldots m-k+1}
\]

\[
(T_N + t^{-m}) \begin{array}{c} 1 \ldots \end{array} = \begin{array}{c} 1 \ldots \end{array} + \sum_{k=k+1}^{m} (-1)^{m-k}(k-1)\varpi^B(2 \ldots 21 \ldots 2)_{l-m+k-1 \ldots m-k+1}
\]
KAZHDAN–LUSZTIG POLYNOMIALS FOR \((B_N, A_{N-1})\)

\[
(T_N + t^{-m}) = \sum_{k=2}^{m} (-1)^{m-k} \langle k - 1 \rangle \begin{array}{cccc}
2 & 3 & \ldots & m \\
\end{array} + (-1)^{m-1} \begin{array}{cccc}
2 & 1 & 2 & \ldots \\
\end{array} \\
+ \sum_{k=2}^{m} (-1)^{m-k} \langle k - 1 \rangle \begin{array}{cccc}
2 & k-1 & k & m \\
\end{array} \\
\]

Note that the binary string \(2 \ldots 2^{l+1}\) is bigger (in the Bruhat order) than any other binary strings in the right hand side of Eqn.(10). From the assumption of the induction together with the fact that \((T_N + t^{-m})\) and \(\langle k - 1 \rangle\) are also invariant, the left hand side of Eqn.(10) is invariant. Therefore, \(\varpi B(2 \ldots 2)\) is invariant.

In both cases, the image of \(\varpi B\) is invariant under the bar involution. This completes the proof that \(\varpi B(\beta)\) is Kazhdan–Lusztig basis \(C_{\alpha,\beta}^\pm\).

Define a set of paths by \(F_B(\beta)\) as

\[F_B(\beta) := \{\alpha \leq \beta : \text{Some pairs and unpaired 2s are flipped}\}\],

where by flipped we mean switching 2 and 1 in the pair of the binary string \(\beta\), changing 2 to 1 in an unpaired 2 of \(\beta\) and changing two 2’s to 1’s (at the same time) in a paired 2’s. For a path \(\alpha \in F_B(\beta)\), define integers

\[d_B(\alpha, \beta) := \{\text{the number of flipped 21 pairs and 22 pairs}\}\],

\[d_p(\alpha, \beta) := \{\text{the number of flipped unpair 2 with integer } p\}\].

Note that \(d_p(\alpha, \beta)\) is either 0 or 1. As a consequence of Theorem 19 with Remark 18, we have

**Corollary 20.** A Kazhdan–Lusztig polynomial \(P_{\alpha,\beta}^{B,-}\) is a monomial of \(t^{-1}\), and equal to the generating function \(Q_{\alpha,\beta}^{B,\pm}\)

\[t^{||\alpha||-||\beta||} Q_{\alpha,\beta}^{B,\pm} = t^{||\alpha||-||\beta||} P_{\alpha,\beta}^{B,-} = (-1)^{\sigma'} t^{-d},\]

where the degrees \(\sigma'\) and \(d\) are given by

\[\sigma' = \sum_{p=1}^{m} (m-p)d_B(\alpha, \beta),\]

\[d = d_B(\alpha, \beta) + \sum_{1 \leq p \leq m} pd_p(\alpha, \beta).\]

**4.2. Module \(M^+\).** We prove that the generating functions \(Q_{\alpha,\beta}^{X,\pm}, X = A, B,\) are equal to the Kazhdan–Lusztig polynomials \(P_{\alpha,\beta}^{-}\). The generating functions \(Q_{\alpha,\beta}^{\pm}\) satisfy the inversion relation (Theorem 10) which is exactly the same as the inversion formula (Theorem 5). Therefore, we have the following:
Theorem 21. The generating functions \( Q_{\alpha,\beta}^{A,+} \) (resp. \( Q_{\alpha,\beta}^{B,+} \)) is equal to the Kazhdan–Lusztig polynomials \( P_{\alpha,\beta} \) for Case A (resp. Case B).

4.2.1. Factorization for Case A. For each binary string \( \alpha \), we define a set of integers \( \{ r_{i,j} : (i,j) \in S^+(\alpha) \} \) (recall the definition in Eqn. (2)). They are defined recursively by

\[
    r_{i,j} := \begin{cases} 
        \max(r_{i-1,j-1}, r_{i+1,j-1}) + 1, & (i,j) \in S^+(\alpha), \\
        0, & \text{otherwise}.
    \end{cases}
\]

We define a factorized element \( \widetilde{C}_\alpha, \alpha \in \mathcal{P}_N \) on \( \mathcal{M}^+ \),

\[
    \widetilde{C}_\alpha := \prod_{(i,j) \in \lambda}(T_i(r_{i,j})m_{1...1},
\]

where

\[
    T_i(p) := \begin{cases} 
        T_i + \frac{t^p}{[p]}, & 1 \leq i \leq N - 1, \\
        T_N - t_N + \frac{[p/2]}{[p]}(t_N^{[p/2]} + t_N^{-1}t^{-[p/2]}), & i = N.
    \end{cases}
\]

Here, \([\cdot]\) and \(\lfloor \cdot \rfloor\) is the ceiling and floor function.

Example 22. Let \( \alpha = 21122 \). The set \( r_{i,j} \) of integers is given by

\[
\begin{array}{ccccccc}
    & & & & 5 \ & \ & \\
    & & & 4 \ & \ & \ & \\
    & & 3 \ & \ & \ & \ & \\
    & 2 \ & \ & \ & \ & \ & \\
    1 \ & \ & \ & \ & \ & \ & \\
\end{array}
\]

The associated factorized expression is

\[
    \widetilde{C}_\alpha = T_5(1)T_4(2)T_5(3)T_1(1)T_2(2)T_3(3)T_4(4)T_5(5)m_{111111}.
\]

Theorem 23. The factorized element \( \widetilde{C}_\alpha \) is the Kazhdan–Lusztig basis \( C_{\alpha}^{A,-} \).

Proof. We omit the details since we can apply the same method in [12] to our case. \( \square \)

Remark 24. The factorized element \( \widetilde{C}_\alpha \) appeared in the study of the quantum Knizhnik–Zamolodchikov equation in [7]. This is a natural generalization of the factorization obtained in [12].

5. Binary tree

5.1. Notations. Following [3, 13], we introduce some terminologies to describe binary trees for both Case A and B.

Let \( \mathcal{Z} \) be a set of binary strings such that \( \emptyset \in \mathcal{Z}, z \in \mathcal{Z} \Rightarrow 1z2 \in \mathcal{Z} \) and if \( z_1, z_2 \in \mathcal{Z} \) then the concatenation \( z_1z_2 \in \mathcal{Z} \). A binary string \( \alpha \in \mathcal{P}_N \) is of the form

\[
    \alpha = 2z_12z_2 \ldots 2z_p1z_{p+1}1z_{p+2} \ldots 1z_q
\]
for some integer \( p, q \geq 0 \) with \( z_i \in \mathbb{Z} \). We call an underlined 1 (resp. 2) as an unpaired 1 (resp. unpaired 2).

We denote by \( ||\alpha|| \) the length of a binary string \( \alpha \) and by \( ||\alpha||_\sigma \) the number of \( \sigma \) in the string \( \alpha \). Let \( \alpha, \beta \in \mathcal{P}_N \) with \( \alpha \leq \beta \) (\( \epsilon = + \)) and \( \alpha = \alpha'v\alpha", \beta = \beta'12\beta" \) with \( ||\alpha'|| = ||\beta'|| \) and \( v, w \in \{1, 2\} \). A capacity of the edge corresponding to the underlined 1 and 2 in \( \beta \) is defined by

\[
\text{cap}(12) := ||\alpha'||_1 - ||\beta'||_1.
\]

Similarly, if \( \alpha = \alpha'v \) and \( \beta = \beta'1 \) with \( v \in \{1, 2\} \), then the capacity of the edge corresponding to the underlined 1 is

\[
\text{cap}(1) := ||\alpha||_1 - ||\beta||_1.
\]

Note that the condition \( \alpha \leq \beta \) implies a capacity is always non-negative.

The capacity of \( \beta \) with respect to \( \alpha \) is the collection of capacities of pairs of adjacent 1 and 2 in \( \alpha \) and that of the rightmost 1 in \( \beta \) if it exists.

We associate a binary tree \( A(\alpha) \) with \( \alpha \in \mathcal{P}_N \) in the following subsections for Case A and B.

5.1.1. Case A. We divide unpaired 1’s into two classes. The \((2i-1)\)-th (resp. \(2i\)-th) unpaired 1 from right is called o-unpaired (resp. e-unpaired) 1.

A binary tree \( A(\alpha) \) satisfies

\[
\text{(\(\lozenge1\))} \quad A(\emptyset) \text{ is the empty tree.}
\]
\[
\text{(\(\lozenge2\))} \quad A(2w) = A(w).
\]
\[
\text{(\(\lozenge3\))} \quad A(zw), z \in \mathbb{Z} \text{ is obtained by attaching the tree for } A(z) \text{ and } A(w) \text{ at their roots.}
\]
\[
\text{(\(\lozenge4\))} \quad A(1z2), z \in \mathbb{Z} \text{ is obtained by attaching an edge just above the tree } A(z).
\]
\[
\text{(\(\lozenge5\))} \quad \text{If unpaired } 1 \text{ in } 1w \text{ is e-unpaired (resp. o-unpaired) 1, } A(1w) \text{ is obtained by attaching an edge just above the tree } A(w) \text{ and mark the edge with “e” (resp. “o”).}
\]

We write the capacities of \( \beta \) with respect to \( \alpha \) as integers on leaves of the binary tree \( A(\beta) \) (See Example 26). Denote by \( A(\beta/\alpha) \) a tree \( A(\beta) \) equipped with capacities with respect to \( \alpha \). A labelling of \( A(\beta/\alpha) \) is a set of non-negative integers on edges of \( A(\beta) \) satisfying

\[
\text{(\(\blacklozenge1\))} \quad \text{An integer on a edge connecting to a leaf is less than or equal to its capacity.}
\]
\[
\text{(\(\blacklozenge2\))} \quad \text{Integers on edges are non-increasing from leaves to the root.}
\]

Let \( \sigma \) be the sum of the labels on edges without marks “e” and “o”, and \( \sigma_e \) (resp. \( \sigma_o \)) be the sum of the labels on edges with a mark “e” (resp. “o”).

**Definition 25.** The generating function of labellings on the tree \( A(w/v) \):

\[
R^A_{v,w}(t^2, t^2_N) := \sum_{\nu} t^{2\sigma}(-t^2_N)^{\sigma_o}(-t^2/V^2_N)^{\sigma_e},
\]

where the sum runs over all possible labellings \( \nu \) of \( A(w/v) \).

**Example 26.** Let \((\alpha, \beta) = (111111, 221121)\). The binary tree \( A(\beta) \) and a labelling is
The capacities of a pair $12$ and $o$-unpaired $2$ are $2$ and $3$ respectively. The weight of the labelling is $t^4 t_N^3$.

The generating functions defined in Section 3 are related to the generating functions $R^{A}_{v,w}$ as follows.

**Theorem 27.** We have

$$Q^{A,I,+}_{\alpha,\beta} = R^{A}_{\alpha,\beta}. $$

The proof of Theorem will be given in Section 5.2.

### 5.1.2. Case B.

If $\alpha_i$ is the $(m+1-j)$-th $(1 \leq j \leq m)$ unpaired 1 from right, we call this as $j$-terminal 1. If $\alpha_i$ and $\alpha_{i'}$ with $i < i'$ are the $j$-th and $(j+1)$-th unpaired 1’s with $j \geq m + 1$ and $j - m$ odd, we make a pair these 1’s and call it a 11-pair. If $\alpha_i$ is an unpaired 1 and not classified above, we call this as an extra-unpair 1. Note that there is at most one extra-unpair 1.

$A(\beta)$ is defined recursively by the following rules. The rules (♦1)-(♦4) are the same as Case A. We replace (♦5) by the following four conditions:

(♦5') If the underlined 1 in $1w$ is the $j$-terminal with $1 \leq j \leq m$, $A(1w)$ is obtained by putting an edge just above the tree $A(w)$. Then mark this edge with a plus “+” only when $j = 1$.

(♦6) Suppose underlined 1 in $1z1w$ is a 11-pair. The tree $A(1z1w)$ is obtained by attaching an edge above the root of $\overline{A}(zw)$. We mark the edge with a plus “+”.

(♦7) If the underlined 1 in $1w$ is an extra-unpair 1, we have $A(1w) = A(w)$.

Further, we need an additional information on the tree. See [3] for $m = 1$ case. Suppose $w = w'z_{m+2r-1} \ldots z_11z_0$ with $z_i \in \mathbb{Z}$ and $r \geq 0$ ($z_{m+2r}$ is non-empty and maximal). Set $w'' = 1z_{m+2r-1} \ldots z_11z_0$ such that $w = w'z_{m+2r}w''$ and $z_{m+2r} = x sx_{s-1} \ldots x_1$ with $x_i \in \mathbb{Z}$. Here all $x_i$’s can not be decomposed further into a product of non-empty elements in $\mathbb{Z}$. Then the tree $A(x_i)$ contains a unique maximal edge (the edge connecting to the root) corresponding to a pair 12. $A(w'')$ contains a unique maximal edge corresponding to a 11-pair or a $m$-terminal. Observe that $A(x_i) \subseteq A(w)$, $A(w'') \subseteq A(w)$ as binary trees. We say that the maximal edge of $A(x_i)$ (resp. $A(w'')$) immediately precedes the maximal edge of $A(x_{i+1})$ (resp. $A(x_1)$) for $1 \leq i \leq s$.

(♦8) When an edge $e$ immediately precedes an edge $e'$ in the binary tree $A(w)$, we put a dotted arrow from the edge $e$ to the edge $e'$. 
A labelling of $A(w/v)$ is a set of non-negative integers on edges of $A(w)$ satisfying the following rules. In addition to (♣1) and (♣2) (the same as Case A), we require

♠3) An integer attached to any edge with a plus “+” must be even.
♠4) If the label on an edge is less than or equal to the labels on all “preceding” edges, then the former must be even.

**Example 28.** Let $\alpha = 22111211$. The binary trees for $\alpha$ with $m = 1, 2$ and 3 from left to right.

![Binary Trees](image)

Given a labelling $\nu$, let $|\nu|$ be the sum of the labels on all edges in $A(w/v)$.

**Definition 29.** The generating function $R_{v,w}^B$ of labellings on $A(w/v)$ is defined by

$$R_{v,w}^B := \sum_{\nu} t^{2|\nu|}.$$

**Theorem 30.** We have

$$Q_{\alpha,\beta}^{A,+,I} = R_{\alpha,\beta}^B.$$

The proof will be given in Section 5.2.

From Theorems 30 and 21, we have $P_{\alpha,\beta}^{B,+,I} = R_{\alpha,\beta}^B$. From Definition 29, the polynomials $P_{\alpha,\beta}^{+,I}$ have a positivity similar to the equal parameter case.

**Recurrence relation for $P_{v,w}^{X,+,I}$.** Let $v_1 = v21v', w_1 = w12w'$ be binary strings with $||v_1|| = ||w_1||$ and $||v|| = ||w||$. Denote by $c_1$ the capacity of the underlined pair 12 in $w_1$.

**Proposition 31.** The polynomials $P_{v,w}^{X,+,I}$ satisfy

$$(11) \quad P_{v_1,w_1}^{X,+,I} = t^{2c_1} P_{v,v',w,w'}^{X,+,I} + P_{v_{12},w_{12}}^{X,+,I}, \quad X = A, B.$$

**Proof.** Recall that the generating function $R_{v,w}^X$ is the sum of the weight of a labelling on $A(w/v)$. The edge $e$ connected to the leaf has the integer less than or equal to the capacity. If the label on $e$ is equal to the capacity $c_1$, the contribution to $R_{v,w}^X$ is $t^{2c_1} P_{v,v',w,w'}^{X,+,I}$. Note that the binary tree for $P_{v,v',w,w'}^{X,+,I}$ is obtained from the binary tree for $P_{v_1,w_1}^{X,+,I}$ by deleting the edge $e$ and the capacity of the new leaf is again $c_1$. If the label on $e$ is less than $c_1$, a labelling is bijective to a labelling of the same binary tree with the capacity $c_1 - 1$. The binary tree for $P_{v_{12},w_{12}}^{X,+,I}$ is the same as $P_{v_1,w_1}$ but the capacity is $c_1 - 1$. The sum of the contribution of the two cases leads to Eqn.(11).
Similarly, let \(v_2 = v1, w_2 = w1\) be binary strings with \(||v|| = ||w||\). Denote by \(c_2\) the capacity of the underlined 1 in \(w_2\).

**Proposition 32.** The Kazhdan–Lusztig polynomial for Case A satisfies

\[
P_{v_2, w_2}^{A,+}(t, tN) = (−t^2)^{c_2}P_{v, w}^{A,+}(t, t/tN) + P_{v_2, w_2}^{A,+}(t, tN).
\]

**Proof.** The underlined 1 in \(w_2\) is an o-unpaired 1. A label on the edge \(e\) connected to the leaf associated with this o-unpaired 1 is less than or equal to the capacity \(c_1\). If the label is equal to \(c_1\), the contribution to \(P_{v_2, w_2}^{A,+}\) is \((-t^2)^{c_1}P_{v, w}^{A,+}(t, t/tN)\). Note that the binary tree for \(P_{v, w}^{A,+}\) is obtained from the binary tree for \(P_{v_2, w_2}^{A,+}\) by deleting the edge \(e\) and the capacity of the new leaf is \(c_1\). However, the marks “e” and “o” should be exchanged in the deleted binary tree. This exchange of “e” and “o” is realized in \(R_{v, w}^A\) as \(t \rightarrow t/tN\). If the label on \(e\) is less than \(c_1\), the contribution to \(R_{v, w}^A\) is \(P_{v_2, w_2}^{A,+}\) by a similar argument to Proposition 31. Adding the two contributions, we obtain Eqn.(12).

\(\square\)

5.2. **Proofs of Theorem 27 and Theorem 30.** To prove Theorems, we will construct a bijection between a labelling of \(A(w/v)\) and a configuration of ballot strips by Rule I introduced in Section 3. This will be done by introducing a link pattern with labelling. Then, we will show that the generating functions are the same by counting the power of \(t\). See also Section 4 in [18].

Let \(\beta \in \mathcal{P}_N\) be a binary string. We denote by \(\bar{\beta}\) the binary string which is obtained from \(\beta\) by exchanging 1 and 2 in \(\beta\). We also denote by \(\pi(\beta)\) the link pattern for \(\bar{\beta}\) as in Section 4.1. Then, the link pattern \(\pi(\beta)\) is the dual graph of the binary tree \(A(\beta)\) (see Figure 4). In Case A, an edge without a mark (resp. with “o” or “e”) in a binary tree corresponds to an arc (resp. a vertical line with “o” or “e”) in the link pattern. In Case B, an edge without “+” in a binary tree corresponds to an arc (corresponding to a pair 12) or a vertical line with the integer \(p\) with \(2 \leq p \leq m\) in the link pattern. An edge with “+” in a binary tree corresponds to a vertical line with the integer 1 or to an arc for paired 1’s in the link pattern.

Notice that the map from link patterns to trees is not one-to-one without fixing the string \(\beta\): for some cases in Case B, we cannot distinguish an arc from a vertical line in a link pattern by looking at only the binary tree. In Case A, the map is bijective up to the ignorance of unpaired vertices of a link pattern.

---

**Figure 4.** The link pattern and the binary tree for 22111211. (a) Case A. (b) Case B and \(m = 2\).

We attach a labelling (a set of integers) to a link pattern. Fix a labelling of \(A(w/v)\) and an edge \(e\) with a label \(n(e)\). We put the label \(n' = n(e) - n(e')\) on the corresponding pair.
12, paired 1’s or a vertical line, where the edge $e'$ is the parent edge of $e$, unless there is no parent edge (edge connected to the root) in which case we put $n(e)$. See Figure 5 (a) and (b) for example.

A labelling of a link pattern $\pi(\beta)$ which is obtained from a labelling of $A(\beta/\alpha)$ satisfies the following three conditions: 1) All labels are non-negative integers. 2) Given a smallest pair 12, the sum of all labels on planar pairings and paired 1’s which surround the pair 12 is less than or equal to the capacity of the pair. 3) Given the rightmost unpaired 2, the sum of all labels on unpaired 2’s and paired 1’s is less than or equal to the capacity of this unpaired 2.

We consider a pair of paths $\alpha, \beta$ with $\alpha < \beta$ in $\epsilon = +$, and the associated link pattern $\pi(\beta)$ with a labelling. We associate with it a collection of ballot strips between paths $\alpha$ and $\beta$ following the three steps based on the map constructed above.

**Step 1.** In the first step, we associate a collection of ballot paths with a labelling of a link pattern. We stack ballot paths on top of each other forming parallel layers above $\beta$ in the following order.

1-1 Case B: We associate with each paired 1’s of $\pi(\beta)$ two ballot paths (of the same length) which connect a half-step to the left of the left point of the pairing and an anchor box. If the pair has the label $n$, then we stack $2n$ ballot paths. We start this process from the leftmost paired 1’s, then move to the next paired 1’s.

1-2 Take a vertical line with “e” or “o” for Case A or a vertical line with an integer $p, 1 \leq p \leq m$ for Case B. Suppose this vertical line is $n_1$-th vertical line from right end with the label $n_2$ and there are $n_3$ planar arcs between the vertical line and the right end. Then, we stack $n_2$ ballot paths of length $(n_3, n_1)$. In both Case A and B, we first stack ballot strips corresponding to the leftmost vertical line, then move right and repeat the procedure.

1-3 Case A and Case B: With each pair 12 of $\pi(\beta)$, we associate ballot paths of length $(l, 0)$ (that is a Dyck path) which start a half-step to the left of the left point of the pairing and a half-step to the right of its right point. If the pair has a label $n$, then we stack $n$ such ballot paths on top of each other, forming parallel layers above $\beta$. We repeat the process for all pairs starting from the largest arcs and ending with the smallest arcs.

When we stack ballot paths above the path $\beta$, we form parallel layers along the shape of $\beta$. Note that some ballot paths may have common starting or end points in the Step 1, in which case they are merged into a larger ballot path.

**Step 2.** We associate the corresponding ballot strip with each ballot path obtained in Step 1 since a ballot strip is characterized by a ballot path. We will show that these strips remain under the path $\alpha$ for the following two cases.

2-1 Let $p$ be a smallest planar pairing, that is, connecting $i$ and $i + 1$. Then the difference of heights of $\alpha$ and $\beta$ at the center of the pairing $p$ (i.e., the depth of the corner in the skew Ferrers diagram) is by direct computation exactly the capacity of the edge $e$ (corresponding to $p$) in the tree $A(\beta/\alpha)$. In terms of $\pi(\beta)$, the number of ballot strips above this corner is the sum of labels of pairings (pair 12 or paired 1’s) surround $p$. 
This number is the label of $e$, which is less than or equal to the capacity. Therefore, the ballot strips remain below $\alpha$ at this local maximum of $\beta$.

2-2 Let $v$ be the rightmost vertical line if exists. From the construction of the bijection, the binary tree $A(\beta/\alpha)$ has the edge $e'$ (corresponding to $v$) with a capacity. Then the difference of heights of $\alpha$ and $\beta$ at this point is nothing but the capacity of the edge $e'$. In terms of $\pi(\beta)$, the number of ballot strips above this point is the sum of labels of unpaired 1’s and paired 1’s. This number is nothing but the label of $e'$, less than or equal to the capacity. Note that the capacity is equal to the number of anchor boxes in the skew Ferrers diagram. The ballot strips remain below $\alpha$ at the right end.

In both cases, strips are below $\alpha$, which implies the claim of step 2.

**Step 3.** The last step is to fill up the remaining regions by ballot strips of length $(0,0)$ (i.e. a single box). From Step 1 and 2, it is clear that there is no ballot strips of length $(l,l') \in \mathbb{N}^2 \setminus \{(0,0)\}$ on top of a single box. See Figure 5 for an example.

![Figure 5](attachment:figure5.png)

**Figure 5.** Bijection among a configuration of ballot strips, a label of the binary tree and a label of the link pattern ($\alpha = 22111211$ and $m = 2$).

It is easy to show that the correspondence above is bijective by reverting the procedure. Therefore, for both Case A and Case B, we have

**Proposition 33.** Let $\alpha < \beta$ be paths in $\mathcal{P}_N$ with $\epsilon = +$. There exists a bijection between labellings of $A(\beta/\alpha)$ and configurations of ballot strips in the skew Ferrers diagram $\lambda(\beta)/\lambda(\alpha)$ satisfying Rule I.

From the proof of Proposition 33, we also know that a label of a link pattern is bijective to a configuration $C$ of ballot strips. We define the weight of a label of a link pattern as the weight of corresponding configuration $C$. To prove theorems, it is enough to show that the weight of a label of a binary tree and that of the corresponding link pattern are equal. We show this statement for Case A and B simultaneously.

Fix a labelling $L$ of a link pattern $\pi(\beta)$, whose weight is $\text{wt}(L)$. We increase by one the label associated with a link $p$, where a link means one of a pair 12, a vertical line, and paired 1’s. We also assume that the obtained labelling is an allowed labelling on $\pi(\beta)$. Recall that the position of a link $p$ and number of unpaired 1’s right to $p$ uniquely determined the length of the ballot strip $D$ corresponding to $p$. The contribution of this increment to the generating function is the term $\text{wt}(D)\text{wt}(L)$ in the generating function $Q_{\alpha,\beta}^{X,I,+}$. This increment on the label is translated in the language of $A(\beta/\alpha)$ as follows. Suppose that an edge $e$ in $A(\beta/\alpha)$ is associated with the link $p$ and let $n(e)$ be the label of the edge $e$. 
From the construction of the bijection in Proposition 33, we increase by one the labels of all descendants of $e$ in the tree. It is clear that the number of all types of edges (with or without “+”, “e” or “o”) descending to $e$ determines the length of the ballot strip $D$. Together with the weight contribution of all descending edges (see Definitions 25 and 29), the contribution of this binary tree to the generating function is exactly $\text{wt}(D)\text{wt}(L)$. This implies that two generating functions $Q_{X,I}^{\alpha,\beta}$ and $R_{X}^{\alpha,\beta}(X=A,B)$ are equal.

References

[1] R. Baxter, S. Kelland, and F. Wu, Equivalence of the Potts model or Whitney polynomial with an ice-type model, J. Phys. A: Math. Gen. 9 (1976), 397–406.
[2] A. Beilinson and J. Bernstein, Localisation de $\mathfrak{g}$-modules, C. R. Avad. Sci. Paris Sér. I Math. 292 (1981), no. 1, 15–18.
[3] B. D. Boe, Kazhdan–Lusztig polynomials for Hermitian symmetric spaces, Trans. Amer. Math. Soc. 309 (1988), 279–294.
[4] F. Brenti, Kazhdan–Lusztig and $R$-polynomials, Young’s lattice, and Dyck partitions, Pacific Journal of Mathematics 207 (2002), 257–286.
[5] , Parabolic Kazhdan–Lusztig polynomials for Hermitian symmetric spaces, Trans. Amer. Math. Soc. 361 (2009), 1703–1729.
[6] J.-L. Brylinski and M. Kashiwara, Kazhdan–Lusztig conjecture and holonomic systems, Invent. Math. 64 (1981), no. 3, 387–410.
[7] J. de Gier and P. Pyatov, Factorised solutions of Temperley–Lieb qKZ equations on a segment, Adv. Theor. Math. Phys. 14 (2010), no. 3, 795–878, [arXiv:0710.5362].
[8] V. Deodhar, On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan–Lusztig polynomials, J. Algebra 111 (1987), no. 2, 483–506.
[9] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), no. 2, 165–184.
[10] , A topological approach to Springer’s representation, Adv. Math. 38 (1980), 222–228.
[11] , Schubert varieties and Poincaré duality, Proc. Sympos. Pure Math. 36 (1980), 185–203.
[12] A. Kirillov, Jr. and A. Lascoux, Factorization of Kazhdan–Lusztig elements for Grassmannians, Combinatorial methods in representation theory (Kyoto, 1998), Adv. Stud. Pure. Math., vol. 28, Kinokuniya, Tokyo, 2000, pp. 143–154, [arXiv:math/9902072].
[13] A. Lascoux and M.-P. Schützenberger, Polynômes de Kazhdan & Lusztig pour les grassmanniennes, Young tableaux and Schur functions in algebra and geometry (Turuń 1980), Astérisque, vol. 87, Soc. Math. France, Paris, 1981, pp. 249–266.
[14] G. Lusztig, Hecke Algebra with Unequal Parameters, CRM monograph series, vol. 18, American Mathematical Society, 2003.
[15] P. Martin, Temperley–Lieb algebras and the long distance properties of statistical mechanical models, J. Phys. A 23 (1990), no. 1, 7–30.
[16] R. A. Proctor, Classical Bruhat Order and Lexicographic Shellability, J. Algebra 77 (1982), 104–126.
[17] H. Saleur, Virasoro and Temperley Lieb algebras, Knots, topology and quantum field theories (Florence, 1989), World Sci. Publ., River Edge, NJ, 1989, pp. 485–496.
[18] K. Shigechi and P. Zinn-Justin, Path representation of maximal parabolic Kazhdan–Lusztig polynomials, J. pure and appl. algebra 216 (2012), 2533–2548, [arXiv:1001.1080].
[19] H. Temperley and E. Lieb, Relations between the “percolation” and “colouring” problem and other graph-theoretical problems with regular lattices: some exact results for the “percolation” problem, Proc. Roy. Soc. London Ser. A 322 (1971), no. 1549, 251–280.

E-mail address: k1.shigechi at gmail.com