On the ‘Stationary Implies Axisymmetric’ Theorem for Extremal Black Holes in Higher Dimensions

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Abstract

All known stationary black hole solutions in higher dimensions possess additional rotational symmetries in addition to the stationary Killing field. Also, for all known stationary solutions, the event horizon is a Killing horizon, and the surface gravity is constant. In the case of non-degenerate horizons (non-extremal black holes), a general theorem was previously established \texttt{gr-qc/0605106} proving that these statements are in fact generally true under the assumption that the spacetime is analytic, and that the metric satisfies Einstein’s equation. Here, we extend the analysis to the case of degenerate (extremal) black holes. It is shown that the theorem still holds true if the vector of angular velocities of the horizon satisfies a certain “diophantine condition,” which holds except for a set of measure zero.

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1 Introduction

In a recent paper \texttt{23}, we proved the following two statements about stationary, asymptotically flat, analytic black hole solutions to the vacuum or electrovacuum Einstein equations with a non-degenerate (non-extremal) event horizon for general spacetime dimension $n \geq 4$: (i) The event horizon is in fact a Killing horizon, and (ii) if it is rotating, then the spacetime

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must also be axisymmetric. Property (i) establishes the zero-th law of black hole thermodynamics as the surface gravity must be constant over a Killing horizon. Property (ii) may be viewed as a “symmetry enhancement” theorem, as it shows that such black holes must have at least one more symmetry than originally assumed. Statements (i) and (ii) are often referred to as rigidity theorem, since they imply in particular that the horizon must be rotating rigidly relative to infinity. An alternative proof of these statements was recently also given in [34].

The rigidity theorem was originally proved for \( n = 4 \) dimensions by [21, 22, 4, 12], and it plays a critical role in the proof of the black hole uniqueness theorem [28, 29, 3, 44, 32, 1] for stationary (electro-)vacuum black hole solutions in \( n = 4 \) dimensions. In higher dimensions, the uniqueness theorem no longer holds as it stands. A variety of explicit stationary black hole solutions have been constructed in recent years but their complete classification is still a major open problem. Properties (i) and (ii) therefore place an important restriction on such black hole solutions in \( n > 4 \). The purpose of the present paper is to establish a version of the rigidity theorem also for the case of degenerate (extremal) black holes. This case corresponds to a vanishing Hawking temperature and is of particular physical importance e.g. for the investigation of the quantum properties of black holes in string theory.

In order to explain why the proof [23] does not carry over straightforwardly to the degenerate case, let us first recall the basic strategy of proof employed in [23]. By assumption, there is a stationary Killing vector field, \( t^a \), which is tangent to the horizon, but not null on the horizon if the latter is rotating. The key step in the proof is to construct another Killing field \( K^a \) which is null on the horizon. This is obtained in turn by finding a distinguished foliation of a neighborhood of the horizon by \( (n-2) \)-dimensional cross sections. To determine that special foliation, one needs to integrate a certain ordinary differential equation along the orbits of the projection \( s^a \) of \( t^a \) onto an arbitrary horizon cross-section, \( \Sigma \). If the orbits of \( s^a \) close on \( \Sigma \), then the integration of this differential equation is straightforward. In \( n = 4 \), the cross section \( \Sigma \) is topologically a two-sphere by the topology theorem [22, 6], implying that the orbits of \( s^a \) must necessarily close. But in higher dimensions, the orbits need not be closed and can in fact be dense on \( \Sigma \). Nevertheless, if the horizon is non-degenerate, then a solution to the desired ordinary differential equation can be obtained using basic methods from ergodic theory. Unfortunately, this method of constructing the desired solution does not seem to generalize straightforwardly to the case of degenerate horizons.

In this paper, we therefore use a different argument which is basically as follows. First, we argue that we can decompose \( s^a = \Omega_1 \psi_1^a + \cdots + \Omega_N \psi_N^a \) locally on \( \Sigma \), where \( N \geq 1 \) and where the vector fields \( \psi_i^a \) commute and have closed orbits with period \( 2\pi \). The constants \( \Omega_i \) can be viewed as a local definition of the angular velocities of the horizon. We now make a Fourier decomposition of the quantities involved in our differential equation on the \( N \)-tori \( \mathbb{T}^N \subset \Sigma \) generated by the \( N \) vector fields \( \psi_i^a \). If this is done, then we can construct the desired solution to our differential equation provided the vector \( \underline{\Omega} = (\Omega_1, \ldots, \Omega_N) \) satisfies

\[
|\underline{\Omega} \cdot \underline{m}| > |\underline{\Omega}| \cdot |\underline{m}|^{-q}
\]

\(^1\)An alternative strategy to prove this result bypassing the rigidity theorem was recently proposed in [20]. However, this argument relies on certain restrictive extra assumptions on the geometry besides stationarity and asymptotic flatness.

\(^2\) For a partial classification see [24, 25], and also [19, 20].
for some number $q$ and for all but finitely many $m \in \mathbb{Z}^N$. We refer to this condition on the angular velocities as a “diophantine condition.” It is satisfied for all $\Omega$ except for a set of measure zero. In summary, if the diophantine condition holds, then we can complete the proof of statements (i) and (ii)—i.e. the rigidity theorem—in the degenerate case. We are unsure whether this condition is a genuine restriction or an artefact of our method of proof.

Our paper is organized as follows. In section 2 we prove statement (i) and (ii) in the extremal case for vacuum black holes. In section 3 we extend these results to include matter fields. The matter fields that we consider consist of a multiplet of scalar fields and abelian gauge fields with a fairly general action, including typical actions characteristic for many supergravity theories. As a by-product, we also generalize our previous results in the non-extremal case [23] to such theories. The rigidity theorem for theories with an additional Chern-Simons term in the action is proved for a typical example in Appendix C. In section 4 we briefly discuss further the nature of the diophantine condition. The decomposition of Einstein’s equation used in the main part of the paper is given in Appendices A and B.

Our signature convention for $g_{ab}$ is $(-, +, +, \cdots)$. The Riemann tensor is defined by $R_{abcd}^k d = 2\nabla_{[a} \nabla_{b]} k_{c}$ and the Ricci tensor by $R_{ab} = R_{acbc}$. We also set $8\pi G = 1$.

## 2 Proof of the rigidity theorem in the vacuum case

Let $(M, g_{ab})$ be an $n$-dimensional, smooth, asymptotically flat, stationary solution to the vacuum Einstein equation containing a black hole. Thus, we assume the existence in the spacetime of a Killing field $t^a$ with complete orbits which are timelike near infinity. Let $H$ denote the portion of the event horizon of the black hole that lies to the future of null infinity $\mathscr{I} \equiv \mathbb{R} \times S^{n-2}$. We assume that $H$ has topology $\mathbb{R} \times \Sigma$, where $\Sigma$ is compact and connected. (If $\Sigma$ is not connected, our arguments can be applied to any connected component of $\Sigma$.) We assume that $t^a$ is not everywhere tangent (and hence normal) to the null generators of $H$. The event horizon $H$ is mapped into itself by a one-parameter group of isometries generated by $t^a$. Following our earlier paper [23], and work of Isenberg and Moncrief [33, 27], our aim in this section is to prove that there exists a vector field $K^a$ defined in a neighborhood of $H$ which is normal to $H$ and on $H$ satisfies

$$\mathcal{L}_t \mathcal{L}_t \cdots \mathcal{L}_t (\mathcal{L}_K g_{ab}) = 0, \quad m = 0, 1, 2, \ldots, \quad (1)$$

where $\ell$ is an arbitrary vector field transverse to $H$. As we shall show at the end of this section, if we assume analyticity of $g_{ab}$ and of $H$ it follows that $K^a$ is a Killing field.

We shall proceed by constructing a candidate Killing field, $K^a$, and then proving that eq. (1) holds for $K^a$. This candidate Killing field is expected to satisfy the following properties: (i) $K^a$ should be normal to $H$. (ii) If we define $S^a$ by

$$S^a = t^a - K^a \quad (2)$$

then, on $H$, $S^a$ should be tangent to cross-sections of $H$. (iii) $K^a$ should commute with $t^a$. (iv) $K^a$ should have constant surface gravity on $H$, i.e., on $H$ we should have $K^a \nabla_a K^b =$

3Note that, since $H$ is mapped into itself by the time translation isometries, $t^a$ must be tangent to $H$, so $S^a$ is automatically tangent to $H$. Condition (iii) requires that there exist a foliation of $H$ by cross-sections $\Sigma(u)$ such that each orbit of $S^a$ is contained in a single cross-section.
κK^b with κ constant on H, since, by the zeroth law of black hole mechanics, this property is known to hold on any Killing horizon in any vacuum solution of Einstein’s equation.

We begin by choosing a cross-section Σ, of H. By arguments similar to those given in the proof of proposition 4.1 of [5], we may assume without loss of generality that Σ has been chosen so that each orbit of τ^a on H intersects Σ at precisely one point, so that τ^a is everywhere transverse to Σ. We extend Σ to a foliation, Σ(u), of H by the action of the time translation isometries, i.e., we define Σ(u) = φ_u(Σ), where φ_u denotes the one-parameter group of isometries generated by τ^a. Note that the function u on H that labels the cross-sections in this foliation automatically satisfies

\[ \mathcal{L}_u u = 1. \]  

Next, we define n^a and s^a on H by

\[ \tau^a = n^a + s^a, \]  

where n^a is normal to H and s^a is tangent to Σ(u). It follows from the transversality of τ^a that n^a is everywhere nonvanishing and future-directed. Note also that \( \mathcal{L}_n u = 1 \) on H. Our strategy is to extend this definition of n^a to a neighborhood of H via Gaussian null coordinates. This construction of n^a obviously satisfies conditions (i) and (ii) above, and it also will be shown below that it satisfies condition (iii). However, it will, in general, fail to satisfy (iv). We shall then modify our foliation so as to produce a new foliation \( \tilde{\Sigma}(\tilde{u}) \) so that (iv) holds as well. We will then show that the corresponding \( K^a = \tilde{n}^a \) satisfies eq. (1).

Given our choice of Σ(u) and the corresponding choice of n^a on H, we can uniquely define a past-directed null vector field \( \ell^a \) on H by the requirements that \( n^a\ell_a = 1 \), and that \( \ell^a \) is orthogonal to each Σ(u). Let r denote the affine parameter on the null geodesics determined by \( \ell^a \), with \( r = 0 \) on H. Let \( x^A = (x^1, \ldots, x^{n-2}) \) be local coordinates on an open subset of Σ. We extend these coordinates to an open neighborhood of H by demanding that they be constant along the orbits of n^a and of \( \ell^a \). The coordinates \((u, r, x^A)\) that are constructed in this manner are referred to as Gaussian null coordinates. If we cover Σ with an atlas of charts, then we obtain a corresponding atlas of Gaussian null coordinates covering an open neighborhood of H. The metric takes the form

\[ ds^2 = 2(dr - r\alpha du - r\beta A dx^A)du + \gamma_{AB}dx^Adx^B. \]  

We write

\[ \beta_a = \beta_A(dx^A)_a, \quad \gamma_{ab} = \gamma_{AB}(dx^A)_a(dx^B)_b, \]  

and we note that \( \beta_a, \gamma_{ab} \) are tensor fields that intrinsically defined in a neighborhood of H, independent of the choice of coordinates \( x^A \) on Σ. Both these tensor fields are by definition orthogonal to \( n^a \) and \( \ell^a \), meaning \( \beta_a n^a = \beta_a \ell^a = 0 \) and \( \gamma_{ab} n^a = \gamma_{ab} \ell^a = 0 \). It follows from the definition of \( u \) and \( r \) that

\[ \mathcal{L}_u u = 1, \quad \mathcal{L}_r r = 0, \]  

and that

\[ \mathcal{L}_u n^a = 0, \quad \mathcal{L}_u \ell^a = 0. \]  

It can also be shown that

\[ \mathcal{L}_u \alpha = 0, \quad \mathcal{L}_u \beta_a = 0, \quad \mathcal{L}_u \gamma_{ab} = 0. \]
We also have

\[ n^a = \left( \frac{\partial}{\partial u} \right)^a, \quad \ell^a = \left( \frac{\partial}{\partial r} \right)^a, \]

and \( n^a \) and \( \ell^a \) commute in particular. Thus, we see that in Gaussian null coordinates the spacetime metric, \( g_{ab} \), is characterized by the quantities \( \alpha, \beta_a, \) and \( \gamma_{ab} \). In terms of these quantities, if we were to choose \( K^a = n^a \), then the condition (1) will hold if and only if the conditions

\[
\underbrace{\mathcal{L}_t \mathcal{L}_t \cdots \mathcal{L}_t}_m (\mathcal{L}_n \gamma_{ab}) = 0, \\
\underbrace{\mathcal{L}_t \mathcal{L}_t \cdots \mathcal{L}_t}_m (\mathcal{L}_n \alpha) = 0, \\
\underbrace{\mathcal{L}_t \mathcal{L}_t \cdots \mathcal{L}_t}_m (\mathcal{L}_n \beta_a) = 0,
\]

hold on \( H \). The next step in the analysis is to use the Einstein equation \( R_{ab} n^a n^b = 0 \) on \( H \), in a manner completely in parallel with the 4-dimensional case [22]. This equation is precisely the Raychaudhuri equation for the congruence of null curves defined by \( n^a \) on \( H \). It yields \( \mathcal{L}_n \gamma_{ab} = 0 \). Thus, the first equation in eq. (11) holds with \( m = 0 \). However, \( n^a \) in general fails to satisfy condition (iv) above. Indeed, from the form, eq. (5), of the metric, we see that the surface gravity, \( \kappa \), associated with \( n^a \) is simply \( \alpha \), and there is no reason why \( \alpha \) need be constant on \( H \). Since \( \mathcal{L}_n \gamma_{ab} = 0 \) on \( H \), the Einstein equation (93) of Appendix A on \( H \) yields

\[ D_a \alpha = \frac{1}{2} \mathcal{L}_n \beta_a, \]

where \( D_a \) denotes the derivative operator on \( \Sigma(u) \), i.e., \( D_a \alpha = \gamma_a^b \nabla_b \alpha \). Thus, if \( \alpha \) is not constant on \( H \), then the last equation in eq. (11) fails to hold even when \( m = 0 \).

As previously indicated, our strategy is repair this problem by choosing a new cross-section \( \Sigma \) so that the corresponding \( \tilde{n}^a \) arising from the Gaussian normal coordinate construction will have constant surface gravity on \( H \). The determination of this \( \Sigma \) requires some intermediate constructions, to which we now turn. First, since we already know that \( \mathcal{L}_t \gamma_{ab} = 0 \) everywhere and that \( \mathcal{L}_n \gamma_{ab} = 0 \) on \( H \), it follows immediately from the fact that \( t^a = s^a + n^a \) on \( H \) that

\[ \mathcal{L}_s \gamma_{ab} = 0 \]

on \( H \) (for any choice \( \Sigma \)). Thus, \( s^a \) is a Killing vector field for the Riemannian metric \( \gamma_{ab} \) on \( \Sigma \). Therefore the flow, \( \phi_r : \Sigma \rightarrow \Sigma \) of \( s^a \) yields a one-parameter group of isometries of \( \gamma_{ab} \), which coincides with the projection of the flow \( \phi_u \) of the original Killing field \( t^a \) to \( \Sigma \). Furthermore, using that \( \mathcal{L}_t \beta_a = 0 \), it similarly follows that

\[ D_a \alpha = -\frac{1}{2} \mathcal{L}_s \beta_a \]

on \( H \). We next define

\[ \kappa = \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} \alpha \, dV, \]

where \( dV \) is the volume element on \( \Sigma \) defined from \( \gamma_{ab} \). In our previous paper [23], we assumed that \( \kappa \neq 0 \), i.e., that the horizon is non-degenerate. Here, we assume that the horizon is degenerate, \( \kappa = 0 \).
We seek a new Gaussian null coordinate system based on a new choice $\tilde{\Sigma}$ of the initial cross section such that the corresponding fields $\tilde{u}, \tilde{r}, \tilde{x}^4, \tilde{\alpha}, \tilde{\beta}_a, \tilde{\gamma}_{ab}$ satisfy all the above properties together with the additional requirement that $\tilde{\alpha} = 0$, i.e., constancy of the surface gravity. Let us determine the conditions that these new coordinates would have to satisfy. Since clearly $\tilde{n}^a$ must be proportional to $n^a$, we have

$$\tilde{n}^a = fn^a, \quad (16)$$

for some positive function $f$. Since $\mathcal{L}_\tilde{t}\tilde{n}^a = \mathcal{L}_t n^a = 0$, we must have $\mathcal{L}_t f = 0$. Since on $H$ we have $n^a \nabla_a n^b = \alpha n^b$ and $\tilde{\alpha}$ is given by

$$\tilde{n}^a \nabla_a \tilde{n}^b = \tilde{\alpha} \tilde{n}^b. \quad (17)$$

we find that $f$ must satisfy

$$\tilde{\alpha} = \mathcal{L}_n f + \alpha f = -\mathcal{L}_s f + \alpha f = 0. \quad (18)$$

The last equality provides an equation that must be satisfied by $f$ on $\Sigma$. Writing $F = \log f$, this equation may be written alternatively in the form

$$\mathcal{L}_s F = \alpha. \quad (19)$$

The new coordinate $\tilde{u}$ must satisfy

$$\mathcal{L}_\tilde{t} \tilde{u} = 1, \quad (20)$$

as before. However, in view of eq. (16), it also must satisfy

$$\mathcal{L}_n \tilde{u} = n^a \nabla_a \tilde{u} = \frac{1}{f} \tilde{n}^a \nabla_a \tilde{u} = \frac{1}{f}. \quad (21)$$

Since $n^a = t^a - s^a$, we find that on $\Sigma$, $\tilde{u}$ must satisfy

$$\mathcal{L}_s \tilde{u} = 1 - e^{-F}. \quad (22)$$

Thus, if our new Gaussian null coordinates exist, there must exist smooth solutions to eqs. (19) and (22), and conversely, any solution to these equations will give us the desired new set of Gaussian null coordinates.

It is not difficult to show that there is always an analytic solution $F$ to eq. (19). To see this, we take the gradient $D_a$ of that equation, we use that $s^a$ is a Killing field of $\gamma_{ab}$ and we use the Einstein equation (14). This shows that $F$ must satisfy

$$\mathcal{L}_s \mathcal{L}_a D_a F = D_a \alpha = -\frac{1}{2} \mathcal{L}_s \beta_a. \quad (23)$$

Taking now a divergence $D^a$ of this equation, it follows that

$$\mathcal{L}_s \left( D^a D_a F + \frac{1}{2} D^a \beta_a \right) = 0. \quad (24)$$

Thus, if we choose $F$ as a solution to the equation $D^a D_a F = -\frac{1}{2} D^a \beta_a$, then this $F$ will satisfy the desired equation (19), up to a term annihilated by $D^a D_a$, i.e. a constant, $\mathcal{L}_s F = \alpha + \text{const}$. But we have shown in our previous paper that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \alpha \circ \hat{\phi}_\tau(x) \, d\tau = \kappa = 0 \quad (25)$$

for some function $\hat{\phi}_\tau$. However, this is precisely the result that we need, since it implies that $\mathcal{L}_s F = \alpha + \text{const}$. Thus, we have shown that there exist smooth solutions to eqs. (19) and (22), and conversely, any solution to these equations will give us the desired new set of Gaussian null coordinates.
from which it follows that the constant vanishes. Thus, we have constructed a solution $F$ to eq. (19). It follows from standard elliptic regularity results on the Laplace operator on a compact Riemannian manifold $(\Sigma, \gamma_{ab})$ that $F$ is smooth and that it is even analytic if $\gamma_{ab}$ and $\beta_a$ are analytic.

We are free to add to our solution $F$ any function $F^*$ on $H$ with the property that $\mathcal{L}_s F^* = 0$. We take

$$
\exp[-F^*(x)] = \lim_{T \to \infty} \frac{1}{T} \int_0^T \exp[-F \circ \hat{\phi}_\tau(x)] d\tau,
$$

(26)

where the limit exists by the ergodic theorem [48], since $\hat{\phi}_\tau$ are isometries of $\gamma_{ab}$ and hence in particular area-preserving. Again by the ergodic theorem, the right side can also be written as the integral over the closure of the orbit of $\hat{\phi}_\tau$. Using precisely the same arguments as below in the proof of lemma 14, it is possible to show that $F^*$ is analytic. By replacing $F$ with $F - F^*$ if necessary, we can hence achieve that our solution $F$ to eq. (19) satisfies eq. (26) with $e^{-F^*} = 1$. This will turn out to be convenient momentarily, as the orbit average of the source term in eq. (22) then vanishes.

We now turn to eq. (22). We note that this equation actually has exactly the same form as eq. (19). Also, in both cases the orbit average of the source term on the right side vanishes. However, a difference is that, for eq. (22), we do not appear to have a differential relation analogous to (14). Hence, it does not appear to be possible to solve that equation by the same type of technique as eq. (19). For this reason, we now turn to a different technique.

For this, we first consider the abelian Lie-group $G$ of isometries of $(\Sigma, \gamma_{ab})$ that is generated by the flow $\hat{\phi}_\tau$, $\tau \in \mathbb{R}$ of the vector field $s^a$. The isometry group of any compact Riemannian manifold is known to be a compact Lie group, so it follows that the closure $K$ of $G$ must be contained in the isometry group. Being the closure of an abelian Lie-group, $K$, too, must be abelian, and hence it must be contained in a maximal torus of the isometry group of $(\Sigma, \gamma_{ab})$. Hence, it must be isomorphic to an $N$-torus, $K \cong \mathbb{T}^N$, for some $N \geq 1$. Let $\psi_1^a, \ldots, \psi_N^a$, be the Killing fields on $(\Sigma, \gamma_{ab})$ corresponding to the $N$ commuting generators of $\mathbb{T}^N$. We assume them to be normalized so that their orbits close after $2\pi$. Then we have

$$
 s^a = \Omega_1 \psi_1^a + \cdots + \Omega_N \psi_N^a,
$$

(27)

for some numbers $(\Omega_1, \ldots, \Omega_N)$, all of which are non-zero. If $N = 1$, then the orbits of $s^a$ are closed. If $N > 1$, then the orbits of $s^a$ are not closed, and the numbers $\Omega_i$ are linearly independent over $\mathbb{Z}$. Since the choice of commuting generators of $\mathbb{T}^N$ is arbitrary, the vector of numbers $(\Omega_1, \ldots, \Omega_N) \in \mathbb{R}^N$ is unique up to

$$
 \Omega_i \to \sum_{j=1}^N A_{ij} \Omega_j,
$$

(28)

$$
 \begin{pmatrix}
  A_{11} & \cdots & A_{1N} \\
  \vdots & \ddots & \vdots \\
  A_{N1} & \cdots & A_{NN}
 \end{pmatrix} \in SL(N, \mathbb{Z}).
$$

The Riemannian manifold $(\Sigma, \gamma_{ab})$ may be identified with the space of null-generators of the horizon. Since this is an invariant concept, the vector of numbers $(\Omega_1, \ldots, \Omega_N) \in \mathbb{R}^N$, too, is invariantly defined in terms of $(M, g_{ab})$, i.e., it does not depend on our choice of $\Sigma$ up to

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4 The statement follows by establishing bounds on the derivatives of $\exp[-F^*(y)]$. These bounds are obtained precisely as in [43], by considering $m = 0$ and replacing $J(y)$ by $\exp[-F(y)]$ in that equation.
the above ambiguity. If it was already known that the vector fields $\psi_i^a$ were extendible to global Killing fields, then $\Omega_i$ would be the corresponding angular velocities of the horizon.

That the desired solution to eq. (22) exists is a consequence of the following lemma:

**Lemma 1.** Let $J$ be a smooth function on $\Sigma$ with the property that

$$
0 = \lim_{T \to \infty} \frac{1}{T} \int_0^T J \circ \hat{\phi}_\tau(x) \, d\tau.
$$

(29)

Let $\Omega = (\Omega_1, \ldots, \Omega_N) \in \mathbb{R}^N$ [see eq. (27)] satisfy the following “diophantine condition”: There exits a number $q$ such that

$$
|\Omega \cdot m| > |\Omega| \cdot |m|^{-q}
$$

(30)

holds for all but finitely many $m \in \mathbb{Z}^N$. Then the equation

$$
\mathcal{L}_s h = J,
$$

(31)

with $s^a$ as in eq. (27), has a smooth solution $h$ on $\Sigma$. Furthermore, if $J$ is real analytic, then the same statements hold true and $h$ is real analytic.

**Proof:** Let us assume that $J$ is real analytic. It is instructive to first treat the case $N = 1$ separately. In this case, the diophantine condition is trivially fulfilled. If $T = 2\pi/\Omega_1$, then $\hat{\phi}_\tau(x) = x$ for all $x$ in $\Sigma$. We define

$$
h(x) = \frac{1}{T} \int_0^T J \circ \hat{\phi}_\tau(x) \, d\tau.
$$

(32)

This function is analytic, and we claim that it also solves the desired differential equation. Indeed, we have

$$
\mathcal{L}_s h(x) = \frac{1}{T} \int_0^T \mathcal{L}_s J \circ \hat{\phi}_\tau(x) \, d\tau
$$

$$
= \frac{1}{T} \int_0^T \frac{d}{d\tau} J \circ \hat{\phi}_\tau(x) \, d\tau
$$

$$
= -\frac{1}{T} \int_0^T J \circ \hat{\phi}_\tau(x) \, d\tau + \frac{T}{T} J \circ \hat{\phi}_\tau(x) \bigg|_{\tau=0}
$$

$$
= J(x).
$$

(33)

We next treat the case $N > 1$. In that case, we have $\Omega_i/\Omega_j \notin \mathbb{Q}$ for $i \neq j$, and the diophantine condition is non-trivial. Let $\mathbf{\tau} = (\tau_1, \ldots, \tau_N) \in \mathbb{R}^N/(2\pi\mathbb{Z})^N = \mathbb{T}^N$ and let $\Phi_\mathbf{\tau} \in \text{Isom}(\Sigma)$ be the isometry of $\Sigma$ defined as follows. For each $x \in \Sigma$ we let $\Phi_\mathbf{\tau}(x)$ be the point in $\Sigma$ obtained by letting $x$ flow for parameter time $\tau_1$ along the flow line of the Killing field $\psi_1^a$ of $\Sigma$, then for parameter time $\tau_2$ along the flow line of the Killing field $\psi_2^a$ etc. The order in which these flows are applied does not matter as the Killing fields mutually commute. We next define

$$
J(x, m) = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{im\mathbf{\tau}} J \circ \Phi_\mathbf{\tau}(x) \, d\tau_1 \cdots d\tau_N.
$$

(34)

Note that $\Omega \cdot m \neq 0$ if $m \neq 0$, since the entries of $\Omega$ are linearly independent over $\mathbb{Z}$. 

The term under the integral is analytic in \((\tau_1, \ldots, \tau_n)\) for each fixed \(x\), so it may be analytically continued for sufficiently small \(|\text{Im} \tau_i| < c_i(x)\), where \(c_i(x)\) is positive. Because \(\Sigma\) is compact, it follows that the infimum \(c_i\) of \(c_i(x)\) as \(x\) ranges over \(\Sigma\) and as \(i\) ranges from 1, \ldots, \(N\) is a positive constant. By shifting the contours of integration to \(\text{Im} \tau_i = \text{sign}(m_i) c_i\), it then follows that

\[
J(x, m) = \frac{1}{(2\pi)^N} \int_{\pm ic_1}^{2\pi \pm ic_1} \ldots \int_{\pm ic_N}^{2\pi \pm ic_N} e^{im \cdot \Phi_{\tau}(x)} d\tau_1 \ldots d\tau_N,
\]

(35)

and therefore that (setting \(c = \sqrt{N} \inf\{c_i; i = 1, \ldots, N\}\))

\[
|J(x, m)| \leq e^{-c|m|} \sup\{|J \circ \Phi_{\tau}(x)|; \ x \in \Sigma, \ 0 \leq \text{Re} \tau_i \leq 2\pi, \ |\text{Im} \tau_i| = c_i\} = \text{const.} e^{-c|m|},
\]

(36)

for all \(m \in \mathbb{Z}^N\), uniformly in \(x\). We now set

\[
h(x) = i \sum_{m \in \mathbb{Z}^N \setminus \{0\}} \frac{J(x, m)}{\Omega \cdot m}.
\]

(37)

We claim that this is the desired solution. Let us first check that this is well-defined for all \(x\). In view of eq. (30), we can estimate \(|h(x)|\) by pulling the absolute values inside the series (37), to obtain

\[
|h(x)| \leq \sum_{m \in \mathbb{Z}^N \setminus \{0\}} \frac{\text{const.} e^{-c|m|}}{\Omega \cdot m} \leq \frac{\text{const.}}{\Omega} \sum_{m \in \mathbb{Z}^N \setminus \{0\}} |m|^q e^{-c|m|} \\
\leq \frac{\text{const.} \cdot q!}{c^\alpha \Omega^q}. \quad (38)
\]

This estimate is uniform in \(x \in \Sigma\). Hence, the series (37) for \(h(x)\) converges absolutely, uniformly in \(x\). We would next like to show that \(h(x)\) is real analytic. For this, we recall that if a function \(\psi\) on \(\mathbb{R}^k\) is real analytic near the origin in \(\mathbb{R}^k\), then there is an \(r > 0\) and a \(K > 0\) such that

\[
|\partial_\alpha \psi(y)| \leq K^{|\alpha|} |\alpha|!, \quad (39)
\]

for all \(y\) in an open ball of radius \(r\) around the origin. Here we use the multi-index notation \(\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_0^k\),

\[
\partial^\alpha = \frac{\partial^{|\alpha|}}{(\partial y^1)^{\alpha_1} \ldots (\partial y^k)^{\alpha_k}}, \quad |\alpha| = \sum_i \alpha_i, \quad |\alpha|! = \prod_i \alpha_i!.
\]

(40)

This statement follows from the multi-dimensional generalization of the Cauchy integral representation of an analytic function. Conversely, if eq. (39) holds, then \(\psi\) is analytic near the origin. Now let \(\psi\) be a real analytic function on \(\Sigma\), choose a point \(x_0 \in \Sigma\), and let \(y^1, \ldots, y^{n-2}\) be Riemannian normal coordinates centered at \(x_0\). Then there exist \(K, r > 0\) such that eq. (39) holds for \(\psi(y)\) for all \(y\) in a ball of radius \(r\) around the origin (here we identify a neighborhood of \(x_0\) with an open neighborhood of the origin of the Riemann normal coordinates). Furthermore, since \(\Sigma\) is compact, we may choose \(K, r\) to depend only on \(\psi\), but not on the choice of \(x_0\). If \(c_i > 0\) are as above and \(c = (\text{sign}(m_1)c_1, \ldots, \text{sign}(m_N)c_N) \in \mathbb{R}^N\), we have

\[
\partial^\alpha (J \circ \Phi_{\tau_i + \omega}(y)) = \partial^\alpha (J \circ \Phi_{\omega} \circ \Phi_{\tau_i}(y)) = (\partial^\alpha \psi)(y'),
\]

(41)
where the derivatives in the last expression are taken with respect to the Riemann normal coordinates centered at the image of \( x_0 \) under the isometry \( \Phi_{\underline{x}} \), and where \( y' \) is the image of \( y \), identified with the corresponding Riemann normal coordinates. In the last step, we have used that, because \( \Phi_{\underline{x}} \) is an isometry, it takes Riemann normal coordinates to Riemann normal coordinates. Furthermore, we have defined the real analytic function \( \psi \) on \( \Sigma \) by \( \psi = J \circ \Phi_{\underline{x}} \). We now apply the above estimate (39) to obtain

\[
\left| \partial^\alpha (J \circ \Phi_{\underline{x}+\underline{y}}(y)) \right| \leq K^{[\alpha]} \alpha !, \tag{42}
\]

for all \( y \) in a ball of radius \( r \). As above, we next shift the contour of the \( \tau \) integration in the expression for \( \partial \alpha J(y, m) \) by \( i \alpha \), to arrive at

\[
\left| \partial^\alpha J(y, m) \right| = \left| \frac{e^{-cm \alpha}}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{im \tau \alpha} \partial^\alpha (J \circ \Phi_{\underline{x}+\underline{y}}(y)) \, d\tau_1 \cdots d\tau_N \right| \leq e^{-c|\alpha| \alpha !} K^{[\alpha]} \alpha !. \tag{43}
\]

Substituting this bound into the series for \( \partial_\alpha h(y) \) and bounding each term in this series by its absolute value, we obtain \( |\partial^\alpha h(y)| \leq C^{[\alpha]} \alpha ! \) for some constant \( C > 0 \) and all \( y \) in a ball of radius \( r \). Hence, \( h(y) \) is analytic, as we desired to show.

We finally need to check that \( h(x) \) as defined above satisfies the desired differential equation. For this, we first note that \( J(x, 0) = 0 \). Indeed, since \( \Omega_i / \Omega_j \notin \mathbb{Q} \), we know that the orbit of

\[
\mathbb{R} \to \mathbb{T}^N, \quad t \mapsto (t \Omega_1, \ldots, t \Omega_N) \mod (2\pi \mathbb{Z})^N \tag{44}
\]

is dense in \( \mathbb{T}^N \), so application of the ergodic theorem (see e.g. [48]) gives

\[
J(x, 0) = \lim_{T \to \infty} \frac{1}{T} \int_0^T J(x, \Phi_{\Omega_1, \ldots, \Omega_N}(t)) \, dt. \tag{45}
\]

On the other hand, \( \Phi_{(\Omega_1, \ldots, \Omega_N)}(x) \) is by definition equal to \( \hat{\phi}_t(x) \). Hence, in view of our assumption (29), we have \( J(x, 0) = 0 \). Next, we calculate

\[
\mathcal{L}_{\alpha} J(x, m) = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{im \tau \alpha} \mathcal{L}_{\alpha} J \circ \Phi_{\underline{x}}(x) \, d\tau_1 \cdots d\tau_N
\]

\[
= \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{im \tau \alpha} \left( \Omega_1 \frac{\partial}{\partial \tau_1} + \cdots + \Omega_N \frac{\partial}{\partial \tau_N} \right) J \circ \Phi_{\underline{x}}(x) \, d\tau_1 \cdots d\tau_N
\]

\[
= -i m \cdot \Omega J(x, m). \tag{46}
\]

Using \( J(x, 0) = 0 \), we then have

\[
\mathcal{L} \alpha h(x) = i \sum_{m \in \mathbb{Z}^N \setminus 0} \frac{\mathcal{L}_{\alpha} J(x, m)}{\Omega \cdot m} = \sum_{m \in \mathbb{Z}^N} J(x, m) = J \circ \Phi_{\underline{x} \underline{y}}(x) = J(x). \tag{47}
\]

This concludes our proof in the case when \( J \) is real analytic.
Next, suppose $J$ is only smooth. Then the argument in the case $N = 1$ is unchanged and gives a smooth solution $h$. In the diophantine case $N > 1$, we now have for any $k,l \in \mathbb{N}_0$ an estimate
\[
|\Delta^l J(x,m)| \leq \text{const.} (1 + |m|)^{-k}
\] (48)
for a constant only depending on $k,l$, where $\Delta = D^a D_a$. It follows again from the diophantine condition that the sum (37) for $h(x)$ and the corresponding sums for $\Delta^l h(x)$ converge uniformly for all $l$. Thus, $|\Delta^l h(x)|$ is uniformly bounded and hence $h$ is in any of the Sobolev spaces $W^{p,l}(\Sigma, dV)$, and therefore smooth. That $h(x)$ satisfies the desired differential equation follows as in the analytic case.

The lemma shows that the desired new Gaussian null coordinates $(\tilde{u}, \tilde{r}, \tilde{x}^A)$ and corresponding foliation $\tilde{\Sigma}(\tilde{r}, \tilde{u})$ exist under the assumptions stated there. For the rest of the paper, we assume that these hold. Now let $K^a = \tilde{n}^a$. We have previously shown that $\mathcal{L}_n \tilde{\gamma}_{ab} = 0$ on $H$, since this relation holds for any choice of Gaussian null coordinates. However, since our new coordinates have the property that $\tilde{\alpha} = 0$ on $H$, we clearly have that $\mathcal{L}_n \tilde{\alpha} = 0$ on $H$. Furthermore, for our new coordinates, eq. (12) immediately yields $\mathcal{L}_n \tilde{\beta}_a = 0$ on $H$. Thus, we have proven that all of the relations in eq. (11) hold for $m = 0$.

We next prove that the equation $\mathcal{L}_\ell \mathcal{L}_n \tilde{\gamma}_{ab} = 0$ holds on $H$. Using what we already know about $\tilde{\beta}_a, \tilde{\gamma}_{ab}$ and taking the Lie-derivative $\mathcal{L}_n$ of the components of the Einstein equation tangent to $\tilde{\Sigma}(\tilde{r}, \tilde{u})$ (see eq. (96) of Appendix A), we get
\[
0 = \mathcal{L}_n \mathcal{L}_n \mathcal{L}_\ell \tilde{\gamma}_{ab},
\] (49)
on $H$. Since $t^a = \tilde{n}^a + \tilde{s}^a$, with $\tilde{s}^a$ tangent to $\tilde{\Sigma}(\tilde{u})$, and since all quantities appearing in eq. (19) are Lie derived by $t^a$, we may replace in this equation all Lie derivatives $\mathcal{L}_n$ by $-\mathcal{L}_s$. Hence, we obtain
\[
0 = \mathcal{L}_s \mathcal{L}_s \mathcal{L}_\ell \tilde{\gamma}_{ab},
\] (50)
on $\tilde{\Sigma}$. Now, write $L_{ab} = \mathcal{L}_\ell \tilde{\gamma}_{ab}$. We fix $x_0 \in \tilde{\Sigma}$ and view eq. (50) as an equation holding at $x_0$ for the pullback, $\tilde{\phi}_\tau^* L_{ab}$, of $L_{ab}$ to $x_0$, where $\tilde{\phi}_\tau: \tilde{\Sigma} \rightarrow \tilde{\Sigma}$ now denotes the flow of $\tilde{s}^a$. Then eq. (50) can be rewritten as
\[
\frac{d^2}{d\tau^2} \tilde{\phi}_\tau^* L_{ab} = 0.
\] (51)
Integration of this equation yields
\[
\frac{1}{\tau} (\tilde{\phi}_\tau^* L_{ab} - L_{ab}) = C_{ab},
\] (52)
where $C_{ab}$ is a tensor at $x_0$ that is independent of $\tau$. However, since $\tilde{\phi}_\tau$ is an isometry, each orthonormal frame component of $\tilde{\phi}_\tau^* L_{ab}$ at $x_0$ is uniformly bounded in $\tau$ by $\sup\{(L^{ab} L_{ab}(x))^{1/2}; x \in \tilde{\Sigma}\}$. Consequently, the limit of eq. (52) as $\tau \rightarrow \infty$ immediately yields
\[
C_{ab} = 0.
\] (53)
Thus, we have $\mathcal{L}_s \mathcal{L}_n \tilde{\gamma}_{ab} = 0$, and therefore $\mathcal{L}_n \mathcal{L}_\ell \tilde{\gamma}_{ab} = \mathcal{L}_\ell \mathcal{L}_n \tilde{\gamma}_{ab} = 0$ on $H$, as we desired to show.
Thus, we now have shown that the first equation in (11) holds for \( m = 0, 1 \), and that the other equations hold for \( m = 0 \), for the tensor fields associated with the “tilde” Gaussian null coordinate system, and \( K^a = \tilde{\eta}^a \). In order to prove that eq. (11) holds for all \( m \), we proceed inductively. Let \( M \geq 1 \), and assume inductively that the first of equations (11) holds for all \( m \leq M \), and that the remaining equations hold for all \( m \leq M - 1 \). Our task is to prove that these statements then also hold when \( M \) is replaced by \( M + 1 \). To show this, we apply the operator \((L^\ell)^{M-1} L_n\) to the Einstein equation \( R_{ab} \tilde{\eta}^a \tilde{\eta}^b = 0 \) (see eq. (52)) and restrict to \( H \). Using the inductive hypothesis, one sees that \((L^\ell)^M (L_n \tilde{\eta}) = 0\) on \( H \), thus establishes the second equation in (11) for \( m \leq M \). Next, we apply the operator \((L^\ell)^M L_n\) to the components of Einstein’s equation \( R_{ab} \tilde{\eta}^a = 0 \) tangent to \( \tilde{\Sigma}(\tilde{u}, \tilde{t}) \) (see eq. (95)), and restrict to \( H \). Using the inductive hypothesis and the above results \((L^\ell)^M (L_n \tilde{\eta}) = 0\) and \((L^\ell)^M (L_n \tilde{\beta}) = 0\), one sees that the tensor field \( S_{ab} \equiv (L^\ell)^{M+1} \tilde{\beta}_{ab} \) satisfies a differential equation of the form

\[
L_n L_n S_{ab} = 0
\]  

(54)
on H. By the same argument as given above for \( L_{ab} \), it follows that \( L_n S_{ab} = 0 \). This establishes the first equation in (11) for \( m \leq M + 1 \), and closes the induction loop.

Thus, we have shown (1) for our choice of \( K^a \). In the analytic case, since \( g_{ab} \) and \( K^a \) are analytic, so is \( L_K g_{ab} \). It follows immediately from the fact that this quantity and all of its derivatives vanish at any point of \( H \) that \( L_K g_{ab} = 0 \) where defined, i.e., within the region where the Gaussian null coordinates \((\tilde{u}, \tilde{t}, \tilde{x}^A)\) are defined. This proves existence of a Killing field \( K^a \) in a neighborhood of the horizon. We may then extend \( K^a \) by analytic continuation. Now, analytic continuation need not, in general, give rise to a single-valued extension, so we cannot conclude that there exists a Killing field on the entire spacetime. However, by a theorem of Nomizu [37] (see also [4]), if the underlying domain is simply connected, then analytic continuation does give rise to a single-valued extension. By the topological censorship theorem [13,14], the domain of outer communication has this property. Consequently, there exists a unique, single valued extension of \( K^a \) to the domain of outer communication, i.e., the exterior of the black hole (with respect to a given end of infinity). Thus, in the analytic case, we have proven the following theorem:

**Theorem 1:** Let \((M, g_{ab})\) be an analytic, asymptotically flat \( n \)-dimensional solution of the vacuum Einstein equations containing a black hole and possessing a Killing field \( t^a \) with complete orbits which are timelike near infinity. Assume that the event horizon, \( H \), of the black hole is analytic and is topologically \( \mathbb{R} \times \Sigma \), with \( \Sigma \) compact and connected, and that \( \kappa = 0 \) (where \( \kappa \) is defined by eq. (15) above). Let \( \Omega = (\Omega_1, \ldots, \Omega_N) \) be the angular velocities associated with projection of \( \phi_t \) onto \( \Sigma \), see eq. (27). If these satisfy the diophantine condition

\[
|\Omega \cdot m| > |\Omega| \cdot |m|^{-q}
\]

(55)
for some number \( q \) and for all but finitely many \( m \in \mathbb{Z}^N \), then there exists a Killing field \( K^a \) whose orbits are tangent to the null-generators of \( H \).

**Remarks:** (1) Note that the diophantine condition is trivially satisfied when \( N = 1 \), i.e., when the one-parameter group of symmetries \( \phi_t \) associated with \( t^a \) maps the horizon...
generators to themselves after some fixed period $T$. For $N > 1$, the diophantine condition is non-trivial. We will discuss it in some more detail in section 4.

(2) If the diophantine condition is satisfied for $\Omega$, then it is also satisfied for $A\Omega$ when $\pm A \in SL(N, \mathbb{Z})$. Thus, the diophantine condition is invariant under changes of the form $A$, which as we discussed, constitute the only ambiguity in our definition of $\Omega$ for the given spacetime.

If we are in the situation described in Theorem 1, we can apply the same type of reasoning as in our previous paper [23] to extend the rotational Killing fields $\tilde{\psi}_i^a$ in the decomposition $s^a = \Omega_1 \tilde{\psi}_1^a + \cdots + \Omega_N \tilde{\psi}_N^a$ [see eq. (27)] to Killing fields on the entire exterior of the spacetime, i.e., we have the following theorem.

**Theorem 2:** Let $(M, g_{ab})$ be an analytic, asymptotically flat $n$-dimensional solution of the vacuum Einstein equations containing a black hole and possessing a Killing field $t^a$ with complete orbits which are timelike near infinity. Assume that the event horizon, $H$, of the black hole is analytic and is topologically $\mathbb{R} \times \Sigma$, with $\Sigma$ compact and connected, and that $\kappa = 0$. As above, assume that $(\Omega_1, \ldots, \Omega_N)$ [see eq. (27)] satisfy the diophantine condition (55). If $t^a$ is not tangent to the generators of $H$, then there exist mutually commuting Killing fields $\tilde{\psi}_1^a, \ldots, \tilde{\psi}_N^a$ (where $N \geq 1$) with closed orbits with period $2\pi$ which are defined in a region that covers $H$ and the entire domain of outer communication. Each of these Killing fields commutes with $t^a$, and $t^a$ can be written as

$$t^a = K^a + \Omega_1 \tilde{\psi}_1^a + \cdots + \Omega_N \tilde{\psi}_N^a,$$

where $K^a$ is the horizon Killing field whose existence is guaranteed by Theorem 1.

**Remarks:** (1) If the spacetime is asymptotically flat in the standard sense with asymptotic infinity of type $S^{n-2}$, then there can be at most $N = [(n+1)/2]$ (we mean the integer part of a number) mutually commuting Killing fields including the stationary Killing field. For example, Myers-Perry black holes [36] in arbitrary $n > 4$ possess a stationary Killing field plus $[(n-1)/2]$ rotational Killing symmetries with angular velocities $\Omega_i$, $i = 1, \ldots, [(n-1)/2]$. These solutions admit a regular extremal (degenerate horizon) limit for a wide range of the parameters of the solutions, for example when all the angular velocities are equally large. However, note that when a Myers-Perry hole has only a single non-vanishing angular momentum, the horizon becomes singular in the extremal limit for $n = 5$, and for $n \geq 6$, there is no extremal limit; the angular velocity can be arbitrary large in that case. A black ring solution [3] in $n = 5$ which possesses 3 mutually commuting Killing fields also admits a regular extremal limit if it has two non-vanishing angular velocities. For more details on higher dimensional, extremal black holes see e.g. [30, 7, 10, 31], and references therein.

(2) If the black hole is non-rotating, i.e. if $t^a$ is tangent to the null generators of $H$, then the solution is static [47]. The same result also holds for Einstein-Maxwell theory [47], and more generally presumably also for many of the Einstein-Matter theories described in the next section. In the non-extremal case, the uniqueness theorems [17, 18] for static Einstein-Maxwell-Dilaton black hole solutions then apply. In the extremal case uniqueness of higher dimensional, static Einstein-Maxwell black hole solutions was shown in [45].
3 Matter fields

So far we have focused on vacuum solutions to the Einstein equations for the sake of simplicity. In this section we generalize our results to include certain types of matter fields. We consider theories containing scalar fields \( \phi \) taking values in a target space manifold \( X \) with positive definite metric \( f_{ij}(\phi) \) and vector fields \( A_a \) taking values in a vector bundle over \( X \) with positive definite vector bundle metric \( h_{IJ}(\phi) \). We write the components of the scalar and vector fields as \( \phi^i \) and \( A_a^I \) respectively. We take the action to be

\[
S = \int d^n x \sqrt{-g} \left( R - \frac{1}{2} f_{ij}(\phi) g^{ab} \nabla_a \phi^i \nabla_b \phi^j - U(\phi) - \frac{1}{4} h_{IJ}(\phi) g^{ac} g^{bd} F_{ab}^I F_{cd}^J \right) + S_{\text{top}}, \tag{57}
\]

where \( F_{ab}^I = \nabla_a A_b^I - \nabla_b A_a^I \), where \( U \) is a potential, and where \( S_{\text{top}} \) denotes a topological term. A typical example for such a term is a Chern-Simons action. It does not affect the form of stress-energy tensor but it can modify the equation of motion for the gauge field, eq. (60). In this section we will discard the topological term for simplicity. But we will discuss the minimal supergravity in \( n = 5 \) dimensions as an example of a theory with a Chern-Simons term in appendix C.

The above class of theories obviously includes the case of pure gravity with a cosmological constant, which corresponds to solutions with constant \( \phi \). It also includes many interesting supergravity theories in 5-(and 4)-dimensions arising from supergravity theories in 11-dimensions and string theories in 10-dimensions by appropriate dimensional reductions. In the latter case, one must include a topological term.

Varying the action eq. (57) gives the following equations of motion:

\[
R_{ab} = f_{ij}(\phi) \nabla_a \phi^i \nabla_b \phi^j + h_{IJ}(\phi) g^{cd} F_{ac}^I F_{bd}^J + \frac{2}{n-2} g_{ab} \left[ U(\phi) - \frac{1}{4} h_{IJ}(\phi) F_{cd}^I F_{cd}^J \right] \tag{58}
\]

\[
\nabla_a (f_{ij}(\phi) \nabla^a \phi^j) - \frac{1}{2} f_{j[k]} \nabla^a \phi^i \nabla_a \phi^k - U_i - \frac{1}{4} h_{IJ[i]} F_{ab}^I F_{ab}^J = 0, \tag{59}
\]

\[
\nabla_c \left[ h_{IJ}(\phi) F^{Jca} \right] = 0, \tag{60}
\]

and the Bianchi identities,

\[
\nabla_{[a} F_{bc]} = 0, \tag{61}
\]

where here and in the following the vertical stroke denotes the derivative with respect to a scalar field component, \( \phi^i \), e.g., \( f_{j[k]} = \partial f_{jk}(\phi)/\partial \phi^i \).

We now consider a stationary black hole solution in the above theory with corresponding Killing field \( t^a \), that is \( \mathcal{L}_t g_{ab} = 0 \). We also assume that the other fields are invariant under \( t^a \), that is \( \mathcal{L}_t \phi^i = 0, \mathcal{L}_t A_a^I = 0 \), and that all fields are real analytic. Which asymptotic conditions on the dynamical fields are reasonable in the above theory will in general depend on the precise choice of the potential \( U(\phi) \) and the metrics \( f_{ij}(\phi), h_{IJ}(\phi) \). In the vacuum case, we assumed asymptotic flatness for the metric with standard infinity \( \mathcal{I}^\pm \cong S^{n-2} \times \mathbb{R} \). This assumption was used implicitly to show that \( t^a \) does not vanish on \( H \), a fact which we needed to obtain the desired foliation \( \Sigma(u, r) \) in our construction of the Gaussian null coordinates. Asymptotic flatness was also implicitly used in the proof of Theorem 2, in combination with the topological censorship theorem [13]. Here, it was needed in order to
establish that the exterior of the black hole is a simply connected manifold, which in turn is essential in order to be able to analytically extend the Killing fields $K^a$ and $\psi^i_a$ to the full exterior of the black hole in a single valued way, cf. [23] for the details of this argument. In the present section, we will simply assume that $t^a$ is nowhere vanishing on $H$, and that the exterior is simply connected. As in the vacuum case, we also assume that the black hole is rotation, i.e. that $t^a$ is not everywhere tangent to the null generators of $H$. For the case when the orbits of $t^a$ are tangent to the generators see Remark 2 following Theorem 2.

As in the vacuum case, we distinguish between extremal and non-extremal black holes. In the non-extremal case we will show that, if the orbits of $t^a$ are not everywhere tangent to the null generators of the horizon $H$, then the analogues of Theorems 1 and 2 hold without any restrictions on the vector of angular velocities $\Omega$. This generalizes previous results in [23] to the above type of theories. In the extremal case we will show the same result under the additional assumption that the vector of angular velocities $\Omega$ verifies the diophantine condition given in the statement of Theorem 1.

Let us now explain how the desired additional Killing field $K^a$ described in Theorems 1 and 2 is constructed in the above types of theories. By analogy to the vacuum case, we must now show that

$$L_t L_t \cdots L_t (L_n g_{ab}) = 0, \quad L_t L_t \cdots L_t (L_n \phi^i) = 0, \quad L_t L_t \cdots L_t (L_n F^I_{ab}) = 0.$$  \hspace{1cm} (62)

Again, we first introduce a Gaussian null coordinate system $(u, r, x^A)$ adapted to the horizon geometry, and we seek to adjust the remaining freedom in choosing this coordinate system in such a way that the desired $K^a$ is given by $n^a = (\partial/\partial u)^a$.

To do this, it is convenient to first decompose the components of $F^I_{ab}$ with respect to the Gaussian null coordinate system. For this, we define

$$F^I_{ab} n_a \ell^b = S^I, \quad F^I_{ac} n_a p^c_{\ell b} = V^I_{\ell b}, \quad F^I_{ac} p^c_a p^d_{\ell b} = W^I_{\ell b}, \quad F^I_{cd} p^c_a p^d_{\ell b} = U^I_{ab},$$  \hspace{1cm} (63)

where $p^a_{\ell b}$ projects on the surfaces $\Sigma(u, r)$ of constant $u, r$, cf. Appendix A for details. The field equations are written in terms of these variables and $\gamma_{\alpha \beta}, \beta_{a}, \alpha$ in Appendix B. It immediately follows from $L_t F^I_{ab} = 0$ that $L_t S^I = 0, L_t V^I_{\ell a} = 0, L_t W^I_{\ell a} = 0, L_t U^I_{ab} = 0$. Our task is now to show eqs. (64) and

$$L_t L_t \cdots L_t (L_n \phi^i) = 0, \quad L_t L_t \cdots L_t (L_n S^I) = 0, \quad L_t L_t \cdots L_t (L_n V^I_{\ell a}) = 0, \quad L_t L_t \cdots L_t (L_n W^I_{\ell a}) = 0, \quad L_t L_t \cdots L_t (L_n U^I_{ab}) = 0.$$  \hspace{1cm} (65)

for a suitable choice of our Gaussian null coordinate system.
First, we consider the Raychaudhuri equation for a congruence of null geodesic generators of the even horizon $H$, i.e. the Einstein equations contracted with $n^a n^b$: 

$$\frac{d}{d\lambda} \theta = -\frac{1}{n-2} \theta^2 - \hat{\sigma}^{ab} \hat{\sigma}_{ab} - f_{ij} (\mathcal{L}_n \phi^i) \mathcal{L}_n \phi^j - h_{IJ} q^{ab} V^I_a V^J_b ,$$

where $\lambda$ is an affine parameter of null geodesic generators of $H$ and where $\theta$ and $\hat{\sigma}_{ab}$ denote, respectively, the expansion and the shear of the null geodesic generators. Because the terms on the right-hand side are negative definite, we may argue as in the proof of the area theorem \[22\] to show that $\theta = 0$. It then also follows that $\hat{\sigma}_{ab} = 0$, and

$$\mathcal{L}_n \phi^j = 0 , \quad V^I_a = 0 , \quad \text{on } H .$$

(67)

The relations $\theta = 0 = \hat{\sigma}_{ab}$ on $H$ are equivalent to

$$\mathcal{L}_n \gamma_{ab} = 0 , \quad \text{on } H ,$$

(68)

which—when substituted into the Einstein equations eqs. (93) and (104) and combined with eqs. (67)—give

$$D_a \alpha = \frac{1}{2} \mathcal{L}_n \beta_a , \quad \text{on } H .$$

(69)

In the non-extremal case, we may now argue as in \[23\] that we can always pass to a modified system of Gaussian null coordinates with associated quantities $\tilde{\alpha}, \tilde{\beta}_a, \tilde{\gamma}_{ab}, \tilde{\phi}^i, \tilde{V}^I_a, \tilde{S}^I$ etc. such that $\tilde{\alpha}$ is constant and non-zero over $H$. In the extremal case, we can use the same arguments as in the previous section to construct a modified system of Gaussian null coordinates such that $\tilde{\alpha} = 0$ on $H$ under the assumption that the vector of angular velocities $\Omega$ verifies the diophantine condition given in the statement of Theorem 1. We assume from now on that our Gaussian null coordinates have been chosen in this way in either case, and we drop the “tilde” from the corresponding quantities again to lighten the notation. Thus it follows that

$$\mathcal{L}_n \beta_a = 0 , \quad \text{on } H .$$

(70)

Next, from the Bianchi identities, eq. (61) [see eq. (114)], and condition, eq. (67), we find that

$$\mathcal{L}_n U^I_{ab} = 0 , \quad \text{on } H .$$

(71)

Using conditions, eqs. (67), we immediately can show that $n_a (\nabla_b \phi^j) F^{Iab} = 0$ on $H$. Then, using the results above and the equation of motion for the gauge field, eq. (60), contracted with $g_{ab} n^b$, we obtain

$$\mathcal{L}_n S^I = 0 , \quad \text{on } H .$$

(72)

At this point, we can show that

$$\mathcal{L}_n \mathcal{L}_\ell \gamma_{ab} = 0 , \quad \text{on } H .$$

(73)

Indeed, if we take a Lie derivative $\mathcal{L}_n$ of eq. (58), and contract with $p^c_a p^d_b$ (see eqs. (96), (98), and (107)), then we obtain

$$\mathcal{L}_n \{ \mathcal{L}_n \mathcal{L}_\ell \gamma_{ab} + \alpha \mathcal{L}_\ell \gamma_{ab} \} = 0 , \quad \text{on } H ,$$

(74)

Here it enters that the target space metrics $h_{IJ}$ and $f_{ij}$ are positive definite.
as in the vacuum case. In the non-extremal case, eq. (73) follows from the argument below eq. (72) of [23]. For the extremal case, i.e. when $\alpha = 0$, the same argument as given around eq. (49) above applies.

We next show that

$$\mathcal{L}_n W^I_a = 0, \quad \text{on } H. \quad (75)$$

First, taking a Lie derivative $\mathcal{L}_n$ of the gauge field equation, eq. (60) and contracting with $p^c_a$ (see eq. (112)), we have

$$\mathcal{L}_n \mathcal{L}_n q^{ab} W^I_b + 2\alpha \mathcal{L}_n q^{ab} W^I_b + \mathcal{L}_n \mathcal{L}_n q^{ab} V^I_b = 0, \quad \text{on } H. \quad (76)$$

Second, taking a Lie derivative $\mathcal{L}_n$ of the Bianchi identities, eq. (61) (see eq. (113)) and using $\mathcal{L}_n S^I = 0$, we have

$$\mathcal{L}_n \mathcal{L}_n V^I_a - \mathcal{L}_n \mathcal{L}_n W^I_a = 0, \quad \text{on } H. \quad (77)$$

Substituting this into the above equation, eq. (76), we find

$$\mathcal{L}_n \{ \mathcal{L}_n W^I_a + \alpha W^I_a \} = 0, \quad \text{on } H. \quad (78)$$

Then eq. (75) follows by the same type of argument as below eq. (72) of [23] for the non-extremal case $\alpha = \kappa \neq 0$, and the type of argument as below eq. (49) for the extremal case $\alpha = \kappa = 0$.

Thus, we have shown that all eqs. (11), (64), and (65), for $m = 0$, and the first of eq. (11) $m = 1$ are satisfied on $H$. The remaining equations for all other values of $m$ are verified by the same type of inductive argument as in [23] for the non-extremal case, and as in the previous section for the extremal case.

In summary, we have verified that Theorems 1 and 2 continue to hold in the presence of matter fields described by the above action (57). In the non-extremal case, the diophantine condition stated in Theorem 1 is not required.

4 Discussion

In this paper, we have considered degenerate (extremal) stationary black hole spacetimes with Killing field $t^a$. We showed that, if the vacuum Einstein equations hold and the spacetime is asymptotically flat, then there exists a Killing field $K^a$ that is tangent and normal to the horizon generators, i.e. the black hole horizon is a Killing horizon. We also proved that if $t^a$ is not everywhere tangent to the null generators (so that $K^a \neq t^a$), then there exist $N$ additional rotational Killing fields, where $N$ is at least one. Our proof relied on two technical assumptions about the nature of the black hole: First we assumed that the spacetime metric is real analytic. Secondly, we had to assume that the corresponding angular velocities $\Omega = (\Omega_1, \ldots, \Omega_N)$ satisfy the “diophantine condition” (55). This condition is automatically satisfied when $N = 1$, in which case the spacetime isometries generated by the timelike Killing field map the horizon generators to themselves after the period $T = 2\pi/\Omega_1$. However, when $N > 1$—which can happen only in $n > 4$ spacetime dimensions—the diophantine condition is non-trivial. In this sense, our theorem is weaker than that obtained
in our previous paper [23] for the non-degenerate case, because no assumption of that type had to be made there. We also considered a class of theories containing scalar and abelian gauge fields and derived similar results in this context.

Let us make a few elementary remarks concerning the diophantine condition (55). First, it is well-known that this condition holds for all \( \Omega = (\Omega_1, \ldots, \Omega_N) \in \mathbb{R}^N \) except for a set of Lebesgue measure zero. This follows immediately from the fact that the set where the condition (55) does \textit{not} hold for any \( q \) is contained in the intersection \( \cap_q \Lambda_q \) of the sets

\[
\Lambda_q = \{ \Omega = (\Omega_1, \ldots, \Omega_N) \in \mathbb{R}^N \mid |\Omega \cdot m| \leq |\Omega| \cdot |m|^{-q} \text{ for some } m \text{ with } |m| > 1 \}. \quad (79)
\]

This intersection has Lebesgue measure zero,

\[
\text{measure} \left( \bigcap_{q=1}^{\infty} \Lambda_q \right) = 0. \quad (80)
\]

For completeness, let us briefly show this: If \( B_r \) denotes the ball of radius \( r \) in \( \mathbb{R}^N \), then we have

\[
\text{measure}(\Lambda_q \cap B_r) \leq \sum_{|m|>1} \text{const.} \cdot r^N |m|^{-q-1} \leq \text{const.} \cdot r^N \int_{|x|>1} |x|^{-q-1} \, d^Nx \leq \frac{\text{const.} \cdot r^N}{q + 1 - N}. \quad (81)
\]

Since this goes to zero for \( q \to \infty \), the claim follows immediately. Thus, it would seem that the assumptions of our theorem are satisfied in almost the entire space of possible parameters \( \Omega \). Unfortunately, our analysis gives no indication exactly what the true parameter space of possible values of \( \Omega \) for \( n \)-dimensional stationary black holes really is. For example, it could still happen that this space is very sparsely populated for certain types of black holes, i.e., it is theoretically possible that extremal black holes could only exist for \( \Omega \) in a set of measure zero. In that case, the statement that the assumptions are satisfied for almost all \( \Omega \in \mathbb{R}^N \) would have little value. Let us look at the example of a 5-dimensional black ring. It admits \( N = 2 \) rotational Killing fields and there is a regular limit in which the horizon becomes degenerate. In this limit, the angular velocities \( \Omega = (\Omega_1, \Omega_2) \) are non-vanishing and satisfy \( \theta = \Omega_1/\Omega_2 = \pm 1 \) for the first branch of solutions, or

\[
\theta = (1 + x)/2\sqrt{x}, \quad 0 \leq x < \infty \quad (82)
\]

for the second branch (see, e.g., [7, 9]). Thus, for the first branch, \( \theta \) is in particular rational, and the orbits of the projection of \( t^a \) onto the space of null-generators of the horizon consequently always close. For the second branch, \( \theta \) varies continuously and may thus be irrational. The vector \( \Omega \) satisfies the diophantine condition for almost all extremal black hole solutions, but there is a measure zero set where it does not, corresponding to certain transcendental values of \( \theta \). However, even in those exceptional cases the black hole

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7 Since we know that the orbits \( t \rightarrow t\Omega \mod \mathbb{Z}^N \) are dense on \( \mathbb{T}^N \), it follows that the entries of \( \Omega \) are linearly independent over \( \mathbb{Z} \). By the Schmidt-subspace theorem [40], if there is an \( i \) such that the ratios \( \Omega_j/\Omega_i \) are algebraic numbers for \( j = 1, \ldots, N \), then \( \Omega \) satisfies the diophantine condition. Of course, the set of \( \Omega \) for which this condition is satisfied is much larger— it has full measure.
continues to have $N = 2$ rotational Killing fields and is a Killing horizon. This suggests that our theorem might be true even dropping the diophantine condition.

Secondly, as we have seen, the diophantine condition is needed in lemma 1 to control the sizes of denominators of the form $|\Omega \cdot m|$ when $m$ becomes large. It appears that this condition cannot easily be lifted for generic analytic functions $J(x)$ in this lemma. Indeed, let us suppose that we have, say $N = 2$, and $\theta = \Omega_1/\Omega_2$ is given by the series

$$\theta = \sum_{i=0}^{\infty} \frac{1}{a_i}$$

where $a_i$ is defined recursively by $a_0 = 1$ and $a_{i+1} = 2^K a_i$, with $K \in \mathbb{N}$. This series is converging rapidly to a transcendental number $1 < \theta < 2$, as $a_{i+1} - a_i \geq 1$ and $a_{i+1}/a_i = 2^K(a_{i+1} - a_i) \leq 2^{-K}$. If $p_k/q_k$ denotes the $k$-th partial sum,

$$p_k = \frac{1 + \sum_{i=1}^{k} \frac{a_i}{a_{i-1}}}{a_k},$$

then h.c.f.$(p_k, q_k) = 1$. Furthermore, we have

$$|\theta - p_k/q_k| = \frac{1}{a_{k+1}} \left(1 + \sum_{i=k+1}^{\infty} \frac{a_{i-1} ... a_{i+k}}{a_i a_{k+2}} \right) \leq \sum_{i \geq 0} 2^K \frac{a_i}{a_{k+1}} \leq 2^{-K} q_k.$$  

This implies that an exponential type sum of the form considered in lemma 1

$$\sum_{p, q \in \mathbb{Z}} e^{-c|p| - c|q|} \frac{1}{|p/q - \theta|}$$

cannot converge for sufficiently large $K$, as there are always terms of size at least

$$\frac{e^{-c|p| - c|q|}}{|p/q - \theta|} \geq \frac{1}{2} a_{k+1} e^{-c|a_k|} \geq 2^k (K - c \log_2 e)^{-1} \geq 2^k (K - c \log_2 e)^{-1} \to \infty \quad (k \to \infty)$$

in this sum. In the proof of lemma 1, $e^{-c|p| - c|q|}$ bounds the Fourier coefficients of $J(x)$. If it is only known that $J(x)$ analytic, then no better bound can be derived, and the solution to the equation $L_s h(x) = J(x)$ consequently cannot be obtained by the method of the lemma. However, in our case $J(x) = 1 - e^{-F(x)}$, where $F$ in turn satisfies $L_s F = \alpha$. It might be possible that further constraints can be derived on the Fourier coefficients of $J(x)$ from such a relation combined with Einstein’s equations. But we have not been able to find such relations.

Let us finally make a remark about the assumption of analyticity. It is known that the Einstein-Maxwell system admits extremal multi-black hole solutions which have non-smooth—hence non-analytic—horizons [16, 49, 2]. Therefore when including Maxwell fields, the analyticity assumption—which is one of the key assumptions in our proof—is not entirely plausible. As shown in [23, 12, 39], the analyticity assumption can be partially removed

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\[8\] By eq. (84), $\theta$ cannot be rational. If $x$ were not transcendental, then by a classic theorem of Liouville, we would have $|\theta - p/q| > \text{const.} |q|^{-d}$, where $d$ is the degree of the algebraic number $\theta$. This condition is not satisfied due to eq. (85).
when the event horizon is non-degenerate. In that case, the horizon can be shown to be isometric to a portion of a bifurcate null hypersurface \([40, 41]\), and one can use the characteristic initial value formulation for Einstein’s equations \([11, 35, 43]\) on the bifurcate null hypersurface in order to extend \(K^a\) defined on the horizon to the black hole interior region. This is, however, not the case for degenerate horizons, since on such a horizon with \(\kappa = 0\), the completeness of the Killing parameter of \(K^a\) on the horizon implies that the horizon generator is affinely complete and hence there is no bifurcate surface. Thus, the key issue when generalizing our results to the Einstein-Maxwell system is whether the diophantine condition holds, and whether the solutions is analytic, including a neighborhood of the horizon.

An interesting generalization of this work would be to consider vacuum spacetimes which are not asymptotically flat in the standard sense (with asymptotic infinity of topology \(S^{n-2}\)), but instead for example asymptotically Kaluza-Klein, with asymptotic infinity of the form \(S^2 \times Y\), with \(Y\) a compact manifold of dimension \(n - 4\). In the non-degenerate case, there would be no change in our analysis of the local horizon geometry, and we could construct a vector field \(K^a\) in a neighborhood of the horizon \(H\) satisfying \([11]\), i.e., the Killing equation \(\mathcal{L}_K g_{ab} = 0\) holds on \(H\) to all orders in a Taylor expansion off of \(H\). The same would also apply in the degenerate case if we assume a diophantine condition \([55]\). However, in both cases it might no longer be possible to construct the desired \(K^a\) globally by analytic continuation: The point is that we are only guaranteed to get a single-valued extension if the exterior part of the spacetime is simply connected \([37]\). Now, the topological censorship theorem \([14, 13]\) guarantees that

\[ \pi_1(M_{\text{exterior}}) \cong \pi_1(S^2 \times Y) \]  

(88)

but unlike in the case of an asymptotically flat spacetime with infinity \(S^{n-2}\), the fundamental group \(\pi_1(S^2 \times Y)\) no longer need to vanish. Nevertheless, if \(\pi_1(Y) = 0\), then it does, and we presumably again get results analogous to Theorems 1 and 2.

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A Ricci tensor in Gaussian null coordinates

In this Appendix, we provide expressions for the Ricci tensor in a Gaussian null coordinate system. As derived in section 2 in Gaussian null coordinates, the metric takes the form

\[ g_{ab} = 2 \left( \nabla_{(a} r - r \alpha \nabla_{(a} u - r \beta_{(a)} \right) \nabla_{b)} u + \gamma_{ab}, \]  

(89)

where the tensor fields \(\beta_a\) and \(\gamma_{ab}\) are orthogonal to \(n^a\) and \(\ell^a\). The horizon, \(H\), corresponds to the surface \(r = 0\). We note that \(\gamma^a_{\ b}\) is the orthogonal projector onto the subspace of
the tangent space orthogonal to $n^a$ and $\ell^a$, and that when $r\beta_a \neq 0$, it differs from the orthogonal projector, $q^a_b$, onto the surfaces $\Sigma(u, r)$. It is worth noting that in terms of the Gaussian null coordinate components of $\gamma_{ab}$, we have $q^{ab} = (\gamma^{-1})^{AB}(\partial/\partial x^A)^a(\partial/\partial x^B)^b$. It also is convenient to introduce the non-orthogonal projector $p^a_b$, uniquely defined by the conditions that $p^a_b n^b = p^a_b \ell^b = 0$ and that $p^a_b$ be the identity map on vectors that are tangent to $\Sigma(u, r)$. The relationship between $p^a_b$ and $\gamma^a_b$ is given by

$$p^a_b = -r\ell^a\beta_b + \gamma^a_b.$$  

(90)

In terms of Gaussian null coordinates, we have $p^a_b = (\partial/\partial x^A)^a(\partial x^A)_b$, from which it is easily seen that $L_a p^a_b = 0 = L_\ell p^a_b$. It also is easily seen that $q^{ac} \gamma_{cb} = p^a_b$ and that $p^a_b q^b_c = q^a_c$.

We define the derivative operator $D_c$ acting on a tensor field $T^{a_1 \ldots a_r b_1 \ldots b_s}$ by the following prescription. First, we project the indices of the tensor field by $q^a_b$, then we apply the covariant derivative $\nabla_c$, and we then again project all indices using $q^a_b$. For tensor fields intrinsic to $\Sigma$, this corresponds to the derivative operator associated with the metric $q_{ab}$. We denote the Riemann and Ricci tensors associated with $q_{ab}$ as $R_{abc}d$ and $R_{ab}$.

The Ricci tensor of $q_{ab}$ can then be written in the following form:

$$n^a n^b R_{ab} = -\frac{1}{2} q^{ab} L_n L_n \gamma_{ab} + \frac{1}{4} q^{ca} q^{db} (L_n \gamma_{ab}) L_n \gamma_{cd} + \frac{1}{2} \alpha q^{ab} L_n \gamma_{ab}$$

$$+ \frac{r}{2} \left[ 4 \alpha L_\ell L_\ell \alpha + 8 \alpha L_\ell \alpha + (L_\ell \alpha) q^{ab} L_n \gamma_{ab}$$

$$+ q^{ab} L_\ell \gamma_{ab} \cdot \left\{ -L_n \alpha - rq^{cd} \beta_c L_n \beta_d$$

$$+ (rq^{cd} \beta_c \beta_d + 2 \alpha) L_\ell (r \alpha) + rq^{cd} \beta_c D_d \alpha \right\}$$

$$+ 2 q^{ab} D_a \left\{ \beta_b L_\ell (r \alpha) + D_b \alpha - L_n \beta_b \right\}$$

$$+ q^{bc} L_\ell (r \beta_c) \cdot \left\{ (rq^{ef} \beta_e \beta_f + 2 \alpha) L_\ell (r \beta_b)$$

$$- 4 D_b \alpha + 2 L_n \beta_b + 4 qr^{ae} \beta_e D_a \beta_b \right\}$$

$$+ 2 (L_\ell \alpha) L_\ell (r^2 q^{ab} \beta_a \beta_b) + 4 r q^{ab} \beta_a \beta_b L_\ell \alpha + 2 r q^{ab} \beta_a \beta_b L_\ell L_\ell \alpha$$

$$+ 2 q^{ab} \beta_a L_\ell (r \beta_b) \cdot \left\{ 2 L_\ell (r \alpha) - \frac{1}{2} qr^{cd} \beta_c L_\ell (r \beta_d) \right\}$$

$$+ 2 r^{-1} L_\ell \left\{ r^2 q^{ab} \beta_a (D_b \alpha - L_n \beta_b) \right\} + 2 r^{-1} \alpha L_\ell (r^2 q^{ab} \beta_a \beta_b) \right\},$$

(91)

$$n^a \ell^b R_{ab} = -2 L_\ell \alpha + \frac{1}{4} q^{ca} q^{db} (L_n \gamma_{cd}) L_\ell \gamma_{ab} - \frac{1}{2} q^{ab} L_\ell L_n \gamma_{ab} - \frac{1}{2} \alpha q^{ab} L_n \gamma_{ab} - \frac{1}{2} q^{ab} \beta_a b$$

$$+ \frac{r}{2} \left[ -2 L_\ell L_n \alpha - \frac{1}{2} q^{ab} L_\ell \gamma_{ab} \cdot \left\{ 2 L_\ell \alpha + q^{cd} \beta_c L_\ell (r \beta_d) \right\}$$

$$- q^{ab} \beta_a L_\ell \beta_b - L_\ell (q^{ab} \beta_a L_\ell (r \beta_b)) - q^{ab} D_a (L_\ell \beta_b) \right\},$$

(92)
\[ n^b p_a R_{bc} = -p^b_a D_b \alpha + \frac{1}{2} \mathcal{L}_n \beta_a + \frac{1}{4} \beta_a q^{bc} \mathcal{L}_n \gamma_{bc} - p^d [a p^e_b] D_d (q^{bc} \mathcal{L}_n \gamma_{ce}) \]

\[ + \frac{r}{2} \left[ \frac{1}{2} (q^{bc} \mathcal{L}_n \gamma_{bc}) \mathcal{L}_t \beta_a + \mathcal{L}_n \mathcal{L}_t \beta_a + 2 \alpha \mathcal{L}_t \beta_a \right. \]

\[ - 2 p^b_a D_b (\mathcal{L}_t \alpha) + \mathcal{L}_t (q^{bc} \beta_b \mathcal{L}_n \gamma_{ca}) - 2 r^{-1} \mathcal{L}_t \left( r^2 q^{cd} \beta_c p^b_a D_d \beta_d \right) \]

\[ - \frac{1}{2} q^{bc} \mathcal{L}_t \gamma_{bc} : \left\{ - (r q^{ef} \beta_e \beta_f + 2 \alpha) \mathcal{L}_t (r \beta_a) \right. \]

\[ + 2 p^d_a D_d \alpha - q^{bc} \beta_b \mathcal{L}_n \gamma_{ca} + 2 r q^{ef} \beta_e p^d_a D_d \beta_f \right\} \]

\[ - 2 \mathcal{L}_t (\alpha \beta_a) - 2 r (\mathcal{L}_t \alpha) \mathcal{L}_t \beta_a + p^d_a D_b \left\{ q^{bc} \beta_c \mathcal{L}_t (r \beta_d) \right\} \]

\[ - 2 p^b_a q^{cd} D_d D_{[b \beta_c]} - q^{bc} (\mathcal{L}_t \beta_b) \mathcal{L}_n \gamma_{ca} \]

\[ - q^{bc} \mathcal{L}_t (r \beta_b) : \left\{ (r q^{ef} \beta_e \beta_f + 2 \alpha) \mathcal{L}_t \gamma_{ca} + p^d_a D_c \beta_d \right. \]

\[ + \beta_c \mathcal{L}_t (r \beta_a) - r q^{ef} \beta_e \beta_f \mathcal{L}_t \gamma_{ca} \right\} \]

\[ + q^{bc} (\mathcal{L}_t \gamma_{ca}) \left\{ 2 \beta_b \mathcal{L}_t (r \alpha) + 2 D_b \alpha - \mathcal{L}_n \beta_b + 2 r q^{de} \beta_e D_d \beta_d \right\} \right] , \] (93)

\[ \ell^a \ell^b R_{ab} = - \frac{1}{2} q^{ab} \mathcal{L}_t \mathcal{L}_t \gamma_{ab} + \frac{1}{4} q^{ca} q^{db} (\mathcal{L}_t \gamma_{ab}) \mathcal{L}_t \gamma_{cd} , \] (94)

\[ \ell^b p^c_a R_{bc} = - \frac{1}{4} \beta_a q^{bc} \mathcal{L}_t \gamma_{bc} - \mathcal{L}_t \beta_a + \frac{1}{2} q^{bc} \beta_c \mathcal{L}_t \gamma_{ab} - p^d [a p^e_b] D_d \left( q^{bc} \mathcal{L}_t \gamma_{ce} \right) \]

\[ + \frac{r}{2} \left[ - \mathcal{L}_t \mathcal{L}_t \beta_a + \mathcal{L}_t \left( q^{bc} \beta_c \mathcal{L}_t \gamma_{ab} \right) \right. \]

\[ + \frac{1}{2} (q^{cd} \mathcal{L}_t \gamma_{cd}) \left( - \mathcal{L}_t \beta_a + q^{bc} \beta_c \mathcal{L}_t \gamma_{ab} \right) \right] , \] (95)
\[ p^e_a p^d_b R_{cd} = -\mathcal{L}_e \mathcal{L}_n \gamma_{ab} - \alpha \mathcal{L}_e \gamma_{ab} + p^e_a p^d_b R_{cd} - p^e_\alpha p^d_b D_c \beta_d - \frac{1}{2} \beta_a \beta_b + q^{cd} \left( \mathcal{L}_e \gamma_{d(a)} \mathcal{L}_n \gamma_{b)c} - \frac{1}{4} \left\{ (q^{cd} \mathcal{L}_n \gamma_{cd}) \mathcal{L}_n \gamma_{ab} + (q^{cd} \mathcal{L}_c \gamma_{cd}) \mathcal{L}_n \gamma_{ab} \right\} \right) \]

\[ + \frac{r}{2} \left[ -2 \alpha \mathcal{L}_e \mathcal{L}_c \gamma_{ab} - p^e_a p^d_b D_c (q^{cd} \beta_d \mathcal{L}_e \gamma_{ef}) \right. \]

\[ - \frac{1}{2} (q^{ce} \mathcal{L}_e \gamma_{cd}) \left\{ (rq^{ef} \beta_e \beta_f + 2 \alpha) \mathcal{L}_c \gamma_{ab} + 2 p^e_\alpha p^d_b D_c \beta_f \right\} \]

\[ -2 \mathcal{L}_e \gamma_{ab} - r^{-1} \left\{ \mathcal{L}_e \left( r^{eq} \beta_e \beta_f \right) \right\} \mathcal{L}_c \gamma_{ab} \]

\[ -rq^{ef} \beta_e \beta_f \mathcal{L}_e \mathcal{L}_c \gamma_{ab} - 2 \mathcal{L}_e \left\{ p^e_\alpha p^d_b D_c \beta_a \right\} \]

\[ -2 \beta_a \mathcal{L}_e \gamma_{ab} - r \left( \mathcal{L}_e \beta_a \right) \mathcal{L}_e \gamma_{ab} - rq^{ef} \beta_e \beta_a \left( \mathcal{L}_e \gamma_{ab} \right) \mathcal{L}_e \gamma_{bf} \]

\[ + 2q^{cd} \beta_d \left\{ \mathcal{L}_e \left( r \beta_a \right) \right\} \mathcal{L}_c \gamma_{b)} + 2 p^e_\alpha p^d_b q^{cd} \left( D_d \beta_e \right) \mathcal{L}_e \gamma_{fc} \]

\[ + q^{cd} \left( rq^{ef} \beta_e \beta_f + 2 \alpha \right) \left( \mathcal{L}_e \gamma_{ca} \right) \mathcal{L}_e \gamma_{db} \right] . \]  

(96)

## B Gravity coupled to matter fields

We start from the action

\[ S = \int d^n x \sqrt{-g} \left( R - \frac{1}{2} f_{ij}(\phi) g^{ab} \nabla_a \phi^i \nabla_b \phi^j - U(\phi) - \frac{1}{4} h_{IJ}(\phi) g^{ac} g^{bd} F_{ab}^I F_{cd}^J \right) , \]

(97)

where \( F_{ab}^I = \nabla_a A_{b}^I - \nabla_b A_{a}^I \), and where \( f_{ij}(\phi) \) and \( h_{IJ}(\phi) \) are positive definite metrics on the spaces of scalar fields, \( \phi^i \), and gauge fields, \( A_{a}^I \), respectively. Variation of \( S \) gives

\[ R_{ab} = T_{ab} - \frac{T}{n-2} g_{ab} , \]

(98)

\[ \nabla_a \left( f_{ij}(\phi) \nabla^a \phi^j \right) - \frac{1}{2} f_{jkl} \nabla^a \phi^j \nabla^c \phi^k - U_{ij} - \frac{1}{4} h_{IJ} F_{ab}^I F_{cd}^J = 0 , \]

(99)

\[ \nabla_c \left( h_{IJ}(\phi) F_{Ica}^J \right) = 0 , \]

(100)

where the stress-energy tensor, \( T_{ab} \), is given by

\[ T_{ab} = f_{ij}(\phi) \nabla_a \phi^i \nabla_b \phi^j - \frac{1}{2} g_{ab} \left\{ f_{ij}(\phi) \nabla^c \phi^i \nabla_c \phi^j + 2 U(\phi) \right\} \]

\[ + h_{IJ}(\phi) \left\{ g^{cd} F_{ac}^I F_{bd}^J - \frac{1}{4} g_{ab} F_{cd}^I F_{cd}^J \right\} , \]

(101)

and \( T = T_{c}^{c} \), and where here and in the following the vertical bar denotes the derivative with respect to a scalar field, \( \phi^i \), e.g., \( f_{jkli} = \partial f_{jk}(\phi)/\partial \phi^i \). In terms of the tensor fields, \( S^I, V_a^I, W_a^I U_{ab}^I \), and the metric variables \( \alpha, \beta_a, \gamma_{ab} \) introduced in the context of Gaussian
null coordinates, the right-hand side of eq. (98) can be decomposed as follows:

\[ n^a n^b \left( T_{ab} - \frac{T}{n-2} g_{ab} \right) = f_{ij}(\phi) (\mathcal{L}_n \phi^i) \mathcal{L}_n \phi^j + h_{IJ}(\phi) \cdot \left\{ q^{ab} V^I_a V^J_b + 2r \cdot \left( \alpha S^I S^J + \beta^a V^I_a S^J \right) + r^2 \cdot \beta^b S^I S^J \right\} - r \cdot \frac{4\alpha}{n-2} \cdot T, \]

(102)

\[ n^a \ell^b \left( T_{ab} - \frac{T}{n-2} g_{ab} \right) = f_{ij}(\phi) (\mathcal{L}_n \phi^i) \mathcal{L}_\ell \phi^j + h_{IJ}(\phi) \cdot \left\{ q^{ab} W^I_a W^J_b + r \cdot \beta^a W^I_a S^J \right\} + \frac{2}{n-2} \cdot T, \]

(103)

\[ \ell^a p^c \left( T_{bc} - \frac{T}{n-2} g_{bc} \right) = f_{ij}(\phi) (\mathcal{L}_\ell \phi^i) \mathcal{L}_c \phi^j + h_{IJ}(\phi) \cdot \left\{ -S^I V^a_a + q^{bc} V^I_b U^J_{ac} + r \cdot \left[ -2\alpha S^I W^a_a + \beta^b U^I_{ab} S^J - \beta^b W^I_a W^J_a \right] \right\} - r \cdot \frac{2\beta_a}{n-2} \cdot T, \]

(104)

\[ \ell^c \ell^b \left( T_{ab} - \frac{T}{n-2} g_{ab} \right) = f_{ij}(\phi) (\mathcal{L}_\ell \phi^i) \mathcal{L}_\ell \phi^j + h_{IJ}(\phi) \cdot q^{ab} W^I_a W^J_b, \]

(105)

\[ \ell^b p^a \left( T_{bc} - \frac{T}{n-2} g_{bc} \right) = f_{ij}(\phi) p^c_a (\mathcal{L}_\ell \phi^i) D_c \phi^j + h_{IJ}(\phi) \cdot \left\{ S^I W^a_a + q^{bc} W^I_b U^J_{ac} - r \cdot \beta^b W^I_b W^J_a \right\}, \]

(106)

\[ p^c_a p^d_b \left( T_{cd} - \frac{T}{n-2} g_{cd} \right) = f_{ij}(\phi) p^c_a p^d_b (D_c \phi^i) D_d \phi^j + g_{IJ}(\phi) \cdot \left\{ 2V^I_a W^J_b + q^{cd} U^I_{ca} U^J_{db} + r \cdot \left[ 2\alpha W^I_a W^J_b - \beta^c (W^I_a U^J_{bc} + W^I_b U^J_{ac}) \right] + r^2 \cdot \beta^c \beta^d W^I_a W^J_b \right\} + \frac{2}{n-2} \cdot \gamma_{ab} \cdot T, \]

(107)
where

\[
T = U(\phi) - h_{IJ}(\phi) \cdot \left( -\frac{1}{2} S^I S^J + q^{ab} V^I_a W^J_b + \frac{1}{4} q^{ab} q^{cd} U^I_{ca} U^J_{db} \right) \\
- r \cdot h_{IJ}(\phi) \cdot \left( \beta^a W^I_a S^J - 2 \beta^c q^{ab} W^I_a U^J_{bc} + \alpha q^{ab} W^I_a W^J_b \right) \\
- \frac{r^2}{2} \cdot h_{IJ}(\phi) \cdot \left\{ \left( \beta^c \beta^a q^{ab} - \beta^a \beta^b \right) W^I_a W^J_b \right\}.
\]

(108)

The tensors \( q^{ab} \) and \( p^a_b \) are defined in Appendix A.

Similarly, the equation for scalar fields, eq. (99), are explicitly written as

\[
0 = U_{ji} \\
+ h_{JIi} \left\{ -\frac{1}{2} S^I S^J + q^{ab} V^I_a W^J_b + \frac{1}{4} q^{ab} q^{cd} U^I_{ca} U^J_{db} \right. \\
+ r \cdot \left( \beta^a W^I_a S^J - 2 \beta^c q^{ab} W^I_a U^J_{bc} + \alpha q^{ab} W^I_a W^J_b \right) \\
+ \frac{r^2}{2} \cdot \left\{ \left( \beta^c \beta^a q^{ab} - \beta^a \beta^b \right) W^I_a W^J_b \right\} \\
- f_{ij} \cdot \left[ 2 \mathcal{L}_\phi \mathcal{L}_\phi^j + \frac{1}{2} (q^{ab} \mathcal{L}_\phi^I \gamma^{ab}) \mathcal{L}_\phi^j + \frac{1}{2} (q^{ab} \mathcal{L}_\phi^I \gamma_{ab}) \mathcal{L}_\phi^j \right. \\
+ \frac{1}{2} (q^{ab} \mathcal{L}_\phi^I \gamma_{ab}) \left\{ q^{ab} \beta_a D_b \phi^j + \left( 2 \alpha + r^2 \beta^c \beta_c \right) \mathcal{L}_\phi^j \right\} \\
+ 2r \left( \beta^a \beta_c \mathcal{L}_\phi^I \mathcal{L}_\phi^j + \left( 2 \alpha + r^2 \beta^c \beta_c \right) \mathcal{L}_\phi^j \mathcal{L}_\phi^j \right) \mathcal{L}_\phi^j \\
+ \frac{r^2}{2} \mathcal{L}_\phi^j \mathcal{L}_\phi^j + q^{ab} D_a D_b \phi^j \\
+ \left( \frac{1}{2} f_{jkl} - f_{kij} \right) \left[ 2 \mathcal{L}_\phi^j \mathcal{L}_\phi^k + \left( 2 \alpha + r^2 \beta^c \beta_c \right) \mathcal{L}_\phi^j \mathcal{L}_\phi^k \mathcal{L}_\phi^k \right. \\
+ 2r q^{ab} \beta_a \mathcal{L}_\phi^I \mathcal{L}_\phi^j \mathcal{L}_\phi^j \mathcal{L}_\phi^k \mathcal{L}_\phi^k \\
\right\}. 
\]

(109)

The equations of motion for the gauge fields, \( F^I_{ab} \), are given by

\[
0 = h_{JIi} \left\{ S^I (\mathcal{L}_\phi^j) - q^{ab} (D_a \phi^j) W^J_b \right\} \\
+ h_{IJ} \left\{ - D_a (q^{ab} W^J_b) + \mathcal{L}_\phi^I S^J - \beta^a W^J_a + \frac{1}{2} S^J (q^{ab} \mathcal{L}_\phi^I \gamma_{ab}) \right\} \\
+ r \cdot \left[ - h_{IJi} (\mathcal{L}_\phi^I \beta^a W^J_a + h_{IJ} \left\{ - \mathcal{L}_\phi^I (\beta^a W^J_a) + \frac{1}{2} \beta^a W^J_a (q^{ab} \mathcal{L}_\phi I \gamma_{ab}) \right\} \right],
\]

(110)
\[
0 = h_{IJ} \frac{1}{2} \left\{ - (\mathcal{L}_n \phi^i) S^J - q^{ab} \left( D_a \phi^i \right) V_b^J \right\} \\
+ h_{IJ} \left\{ - \mathcal{L}_n S^J - \frac{1}{2} S^J (q^{ab} \mathcal{L}_n \gamma_{ab}) - D_a (q^{ab} V_b^J) \right\} \\
+ r \cdot \left[ h_{IJ} \left( \mathcal{L}_n \phi^i \right) \beta^a W_a^J - h_{IJ} (D_a \phi^i) q^{ab} \left( \beta_b S^J - \beta^c U_{bc}^J + 2 \alpha W_b^J \right) \right] \\
+ h_{IJ} \left\{ \mathcal{L}_n (\beta^a W_a^J) + \frac{1}{2} (\beta^c W_c^J) (q^{ab} \mathcal{L}_n \gamma_{ab}) \\
- D_a (q^{ab} \beta_b S^J) + q^{ab} D_a (\beta^c U_{bc}^J) - 2 D_a (\alpha q^{ab} W_b^J) \right\} \\
+ r^2 \cdot \left[ - h_{IJ} (D_a \phi^i) q^{ab} \left( \beta^c \beta_b W_b^J - \beta_b \beta^c W_c^J \right) \right. \\
\left. + h_{IJ} D_a \left( \beta^c \beta_b q^{ab} W_b^J - q^{ab} \beta_b \beta^c W_c^J \right) \right],
\]

(111)

\[
0 = h_{IJ} \left\{ \mathcal{L}_n (q^{ab} W_b^J) + \frac{1}{2} q^{ad} W_d^J (q^{bc} \mathcal{L}_n \gamma_{bc}) + D_c (q^{bc} q^{ad} U_{bd}) \\
+ q^{ab} \beta_b S^J - q^{ab} \beta^c U_{bc}^J + 2 \alpha q^{ab} W_b^J + \mathcal{L}_\ell (q^{ab} V_b^J) + \frac{1}{2} q^{ab} V_b^J (q^{cd} \mathcal{L}_\ell \gamma_{cd}) \right\} \\
+ h_{IJ} \left\{ (\mathcal{L}_n \phi^i) q^{ab} W_b^J + (\mathcal{L}_\ell \phi^i) q^{ab} V_b^J + q^{ad} q^{bc} (D_b \phi^i) U_{cd}^J \right\} \\
+ r \cdot \left[ h_{IJ} \left( 2 D_c (q^{[bc]} q^{ad] \beta_b W_d^J) + \mathcal{L}_\ell (q^{ab} \beta_b S^J) - \mathcal{L}_\ell (q^{ab} \beta^c U_{bc}^J) \right) \\
+ 2 \mathcal{L}_\ell (\alpha q^{ab} W_b^J) + 2 \beta^c \beta_b q^{ab} W_b^J - 2 \beta^a \beta^c W_b^J \right] \\
+ \left\{ \frac{1}{2} h_{IJ} (q^{de} \mathcal{L}_\ell \gamma_{de}) + h_{IJ} (\mathcal{L}_\ell \phi^i) \right\} \cdot \left( \beta^c \beta_b q^{ab} S^J - q^{ab} \beta^c U_{bc}^J + 2 \alpha q^{ab} W_b^J \right) \\
+ 2 h_{IJ} (q^{[a} q^{b]} (D_b \phi^i) \beta_c W_d^J) \right] \\
+ r^2 \cdot \left[ h_{IJ} \mathcal{L}_\ell \left( \beta^c \beta_b q^{ab} W_b^J - q^{ab} \beta_b \beta^c W_c^J \right) \\
+ \left\{ \frac{1}{2} h_{IJ} (q^{de} \mathcal{L}_\ell \gamma_{de}) + h_{IJ} (\mathcal{L}_\ell \phi^i) \right\} \cdot q^{ab} \beta^c \left( \beta_c W_b^J - \beta_b W_c^J \right) \right].
\]

(112)
The Bianchi identities $\nabla_a F_{bc} = 0$, are written as
\begin{align}
\mathcal{L}_n W^I_a - \mathcal{L}_\ell V^I_a + p^c a D_c S^I = 0, \quad (113) \\
\mathcal{L}_n U^I_{ab} - 2p^c [a p^d b] D_c V^I_d = 0, \quad (114) \\
\mathcal{L}_\ell U^I_{ab} - 2p^c [a p^d b] D_c W^I_d = 0, \quad (115) \\
p^d [a p^e b p^f c] D_d U^I_{ef} = 0. \quad (116)
\end{align}

### C Chern-Simons theories in $n = 5$

Here we outline how the rigidity theorem can be proved in the presence of an additional Chern-Simons term in the action. For simplicity and concreteness, we restrict attention to the example of minimal supergravity theory in $n = 5$ dimensions. This theory has a metric and a single gauge field with field strength tensor $F_{ab} = \nabla_a A_b - \nabla_b A_a$; we set the Fermi-fields equal to zero. Its action is
\begin{equation}
S = \int d^5 x \sqrt{-g} \left( R - F_{ab} F^{ab} + \frac{2}{3\sqrt{3}} \epsilon^{abcde} F_{ab} F_{cd} A_e \right). \quad (117)
\end{equation}

The last term in this action is a Chern-Simons term. The resulting Einstein equations (i.e., varying $g_{ab}$) are precisely the same as those given previously in eqs. and with $\phi^i = 0$ and $h_{IJ} = \delta_{IJ}$, as the stress-energy tensor is not modified by the addition of the Chern-Simons term, whereas the equations of motion for the gauge field (i.e., varying $A_a$) are modified to
\begin{equation}
\nabla_b F^{ba} + \frac{1}{2\sqrt{3}} \epsilon^{abcde} F_{bc} F_{de} = 0. \quad (118)
\end{equation}

We decompose this equation by contracting the free index into $n^a$, $\ell^a$, and $p^a b$, respectively. The first term of the left-hand side of the above equation is given by eqs., and, respectively (with $h_{IJ} = \delta_{IJ}$), and we have introduced $\epsilon^{abc} = q^{ad} q^{be} q^{cf} n^p q^q \epsilon_{pqdef}$.

We will now outline how to prove the rigidity theorem in the presence of the Chern-Simons term. For brevity, we only outline the main changes compared to the case without

\[\text{In the } n = 5 \text{ minimal supergravity theory described by (117) (and also in other supergravity theories), it is common to consider solutions possessing a covariantly constant spinor field. By forming a suitable bi-linear combination of this spinor field, one obtains an everywhere non-spacelike Killing vector field, which in particular must be null at the event horizon. Therefore, the event horizon in minimal supergravity theories in 5-dimensions must be a Killing horizon for such solutions [13, 42].}\]
Chern-Simons term described in Sect. 3. Recall that in our proof we need to use the equation of motion for the gauge field only when we show the following equations:

$$\mathcal{L}_n S = 0, \quad \mathcal{L}_n W_a = 0.$$  \hspace{3cm} (122)

First we note that since $V_a = 0$ on $H$ from the Raychaudhuri equation, eq. (120) must vanish on $H$. Thus, when contracted with $(dr)_a$, the Chern-Simons terms in eq. (118) is irrelevant to the equation of motion at least on $H$. We can then show that the first of eq. (122) is satisfied for $m = 0$, so $\mathcal{L}_n S = 0$ on $H$. We can then easily show that the Lie-derivative $\mathcal{L}_n$ of eq. (121) vanishes on $H$ and hence does not contribute to the Lie-derivative $\mathcal{L}_n$ of eq. (118) contracted with $p^c$, on $H$. Then, from these results, we find that eq. (76) also holds in the presence of a Chern-Simons term. By the same argument as after eq. (76), we then conclude that the second of eq. (122) holds for $m = 0$. Next, taking the Lie-derivative $\mathcal{L}_n$ of eq. (118) contracting it with $\ell_a = (du)_a$ and using the results derived so far (in particular $\mathcal{L}_n W_a = 0$ on $H$), we can show that the first of eq. (122) holds for $m = 1$. Furthermore, taking $\mathcal{L}_n \mathcal{L}_l$ of eq. (118) contracted with $p^c$, and using what we already know (in particular $\mathcal{L}_n \mathcal{L}_l S = 0$ on $H$), we can show that the second of eq. (122) holds for $m = 1$. Then, using inductive method as in a similar manner for the vacuum case, we can show that eqs. (122)—as well as eqs. (11), (64), and (65)—hold for all $m = 0, 1, 2, \ldots$. Thus, our rigidity theorems also applies to the theory described by the action (117).

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