We review the theory of higher spin gauge fields in 2+1 and 3+1 dimensional anti-de Sitter space and present some new results on the structure of higher spin currents and explicit solutions of the massless equations. A previously obtained $d=3$ integrating flow is generalized to $d=4$ and is shown to give rise to a perturbative solution of the $d=4$ nonlinear higher spin equations. A particular attention is paid to the relationship between the star-product origin of the higher spin symmetries, AdS geometry and the concept of space-time locality.

1 Introduction

The concept of supersymmetry as an extension of the space–time symmetries originally introduced by Yuri Golfand and Evgeny Likhtman in 1971 plays tremendously important role in the modern high energy physics. Nowadays it is a text-book example of how important is to know a true symmetry when investigating a theory of fundamental interactions. In this contribution we focus on some attempt to find a larger (infinite-dimensional) symmetry that extends ordinary supersymmetry discovered by Golfand and Likhtman and underlies the theory of higher spin gauge fields. The main motivation is that this further extension of supersymmetry may lead to a most symmetric phase of a theory of fundamental interactions. In particular we emphasize that quantum mechanical nonlocality of star-product algebras of auxiliary spinor variables, identified with the higher spin symmetry algebras, results in space-time nonlocality of higher spin interactions.

A theory of fundamental interactions is presently identified with still mysterious M-theory, which should possess a number of properties such as:

(i) M-theory is some relativistic theory in $d = 11$. $d=11$ SUGRA is a low-energy limit of M-theory.

(ii) M-theory gives rise to superstring models in $d \leq 10$ providing a geometric
explanation to dualities.

(iii) Star-product (Moyal bracket) plays important role in a certain phase of M-theory with nonvanishing vacuum expectation value of the antisymmetric field \( B_{mn} \). In the limit \( \alpha' \to 0 \), \( B_{mn} = \text{const} \) string theory reduces to noncommutative Yang-Mills Theory.

(iv) A particularly interesting version of the M-theory is expected to have anti-de Sitter (AdS) geometry explaining duality between AdS SUGRA and conformal models at the boundary of the AdS space.

The most intriguing question is: “what is M-theory?”. It is instructive to analyze the situation from the perspective of spectrum of elementary excitations. Superstrings describe massless modes of lower spins \( s \leq 2 \) like graviton \( (s = 2) \), gravitino \( (s = 3/2) \), vector bosons \( (s = 1) \) and matter fields with spins \( 1 \) and \( 1/2 \), as well as certain antisymmetric tensors. On the top of that there is an infinite tower of massive excitations of all spins. Since the corresponding massive parameter is supposed to be large, massive higher spin excitations are not directly observed at low energies. They are important however for the consistency of the theory. Assuming that M-theory is some relativistic theory admitting a covariant perturbative interpretation we conclude that it should necessarily contain higher spin modes to describe superstring models as its particular vacua. There are two basic alternatives: (i) \( m \neq 0 \): higher spin modes in M-theory are massive or (ii) \( m = 0 \): higher spin modes in M-theory are massless. Since M-theory is supposed to be formulated in eleven (or may be higher) dimensions in the both cases massive higher spin modes in the compactified superstring models may in principle result from compactification of extra dimensions.

Each of these alternatives is not straightforward. In the massive case it is generally believed that no consistent superstring theory exists beyond ten dimensions and therefore there is no good guiding principle towards M-theory from that side. For the massless option the situation is a sort of opposite: there is a very good guiding principle but it looks like it might be too strong. Indeed, massless fields of high spins are gauge fields. Therefore this type of theories should be based on some higher spin gauge symmetry principle with the symmetry generators corresponding to various representations of the Lorentz group. It is very well known however that it is a hard problem to build a nontrivial theory with higher spin gauge symmetries. One argument is due to the Coleman-Mandula theorem and its generalizations which claim that symmetries of S-matrix in a non-trivial (i.e., interacting) field theory in a flat space can only have sufficiently low spins. Direct arguments come from the explicit attempts to construct higher spin gauge interactions in physically interesting situations (e.g. when the gravitational interaction is included).
These arguments convinced most of experts that no consistent nontrivial higher spin gauge theory can exist at all.

However, some positive results were obtained on the existence of consistent interactions of higher spin gauge fields in the flat space with the matter fields and with themselves but not with gravity. Somewhat later it was realized that the situation changes drastically once, instead of the flat space, the problem is analyzed in the AdS space with nonzero curvature $\Lambda$. This generalization led to the solution of the problem of consistent higher spin gravitational interactions in the cubic order at the action level and, later, in all orders in interactions at the level of equations of motion.

The role of AdS background in higher spin gauge theories is very important. First it cancels the Coleman-Mandula argument which is hard to implement in the AdS background. From the technical side the cosmological constant plays a crucial role as well, allowing new types of interactions with higher derivatives which have a structure $\Delta S_{p,n,m,k}^{\text{int}} \sim \Lambda^p \partial^n \phi \partial^m \phi \partial^k \phi$, where $\phi$ denotes any of the fields involved and $p$ can take negative powers to compensate extra dimension carried by higher derivatives of fields in the interactions (an order of derivatives which appear in the cubic interactions increases linearly with spin). An important general conclusion is that $\Lambda$ should necessarily be nonzero in the phase with unbroken higher spin gauge symmetries. In that respect higher spin gauge theories are analogous to gauged supergravities with charged gravitinos which also require $\Lambda \neq 0$.

Higher spin gauge theories contain infinite sets of spins $0 \leq s < \infty$. This implies that higher spin symmetries are infinite-dimensional. Suppose now that higher spin gauge symmetries are spontaneously broken by one or another mechanism. Then, starting from the phase with massless higher spin gauge fields, one will end up with a spontaneously broken phase with all fields massive except for a subset corresponding to an unbroken subalgebra. The same time a value of the cosmological constant will be redefined because fields acquiring a nonvanishing vacuum expectation value may contribute to the vacuum energy. So, there is in principle a possibility to have a spontaneously broken phase with $m \neq 0$ for higher spins and $\Lambda = 0$ (or $\Lambda$ small). A most natural mechanism for spontaneous breakdown of higher spin gauge symmetries is via dimensional compactification. It is important that in the known $d=3$ and $d=4$ examples the maximal finite-dimensional subalgebras of the higher spin superalgebras coincide with the ordinary AdS SUSY superalgebras giving rise to gauged SUGRA models. Provided that the same happens in higher dimensional models, this opens a natural way for obtaining superstring type theories in $d \leq 10$ starting from some maximally symmetric higher spin gauge theory in $d \geq 11$.  

3
In this contribution we would like to draw attention to a deep parallelism of some of the properties of M-theory with the theory of higher spin gauge fields, focusing mainly on the higher spin symmetries and the closed nonlinear higher spin equations of motion. For more detail on the Lagrangian formulation we refer the reader to\textsuperscript{18} and original papers.\textsuperscript{11}

2 Higher Spin Currents

Usual inner symmetries are related via the Noether theorem to the conserved spin 1 current that can be constructed from different matter fields. For example, a current constructed from scalar fields in an appropriate representation of the gauge group

\[ J^{ij} = \bar{\phi}_i \partial^j \phi^i - \partial^i \phi^j \]  \hspace{1cm} (1)

is conserved on the solutions of the scalar field equations

\[ \partial^i J^{ij} = \bar{\phi}_i (\Box + m^2) \phi^j - (\Box + m^2) \phi^i \phi^j . \]  \hspace{1cm} (2)

(Underlined indices are used for differential forms and vector fields in d-dimensional space-time, i.e. 0 = 0, \ldots, d − 1 while i and j are inner indices. Conventions used throughout the paper are summarized in the Appendix).

Translational symmetry is associated with the spin 2 current called stress tensor. For scalar matter it has the form

\[ T^{mn} = \partial^m \phi \partial^n \phi - \frac{1}{2} \chi^{mn} \left( \partial_n \phi \partial^m \phi - m^2 \phi^2 \right) . \]  \hspace{1cm} (3)

Supersymmetry is based on the conserved current called supercurrent. It has fermionic statistics and is constructed from bosons and fermions. For massless scalar \( \phi \) and massless spinor \( \psi_\nu \) it has the form

\[ J^{\underline{\nu}} = \partial^m \phi (\gamma^m \gamma^{\underline{\nu}} \psi) , \]  \hspace{1cm} (4)

where \( \gamma^{\underline{\nu}} \) are Dirac matrices in d dimensions.

The conserved charges, associated with these conserved currents, correspond, respectively, to generators of inner symmetries \( T^{ij} \), space-time translations \( P^n \) and supertransformations \( Q^i_\nu \). The conserved current associated with Lorentz rotations can be constructed from the symmetric stress tensor

\[ S^{\underline{m} \underline{l}} = T^{mn} x^l - T^{ml} x^n , \quad T^{mn} = T^{nm} . \]  \hspace{1cm} (5)

These exhaust the standard lower spin conserved currents usually used in the field theory.
The list of lower spin currents admits a natural extension to higher spin currents containing higher derivatives of the physical fields. The higher spin currents associated with the integer spin \( J_{n_{m_1...m_t, n_1...n_{s-1}}} \)

are vector fields (index \( n \)) taking values in all representations of the Lorentz group described by the traceless two-row Young diagrams

\[
\begin{array}{cccccccc}
0 & & & & & & & 1 \\
& & & & & & &
\end{array}
\]

with \( 0 \leq t \leq s - 1 \). This means that the currents \( J_{m_{m_1...m_t, n_1...n_{s-1}}} \) are symmetric in the indices \( n \) and \( m \), satisfy the relations

\[
(s - 1)(s - 2)J_{m_{1...m_t, r} r_{n_3...n_{s-1}}} = 0,
\]

\[
t(s - 1)J_{m_{m_2...m_t, r} r_{n_2...n_{s-1}}} + (t - 1)J_{m_{m_3...m_t, r} r_{n_1...n_{s-1}}} = 0,
\]

and obey the antisymmetry property

\[
J_{m_{m_2...m_t} n_{s}, n_{1...n_{s-1}}} = 0,
\]

implying that symetrization over any \( s \) indices \( n \) and/or \( m \) gives zero.

Let us now explain notation, which simplifies analysis of complicated tensor structures and is useful in the component analysis. Following we combine the Einstein rule that upper and lower indices denoted by the same letter are to be contracted with the convention that upper (lower) indices denoted by the same letter imply symmetrization which should be carried out prior contractions. With this notation it is enough to put a number of symmetrized indices in brackets writing e.g. \( X_{n(p)} \) instead of \( X_{n_1...n_p} \).

Now, the higher spin currents are \( J_{m(t),n(s-1)} (1 \leq t \leq s - 1) \) while the conditions (8)-(10) take the form

\[
J_{m(t),n(s-2)} n = 0, \quad J_{m(t-1),n(s-1)} n = 0, \quad J_{m(t),n(s-1)} m = 0,
\]

and

\[
J_{m(t-1),n(s-1)} n = 0.
\]

The higher spin supercurrents associated with half-integer spins

\[
J_{m_{m_1...m_t, n_1...n_{s-3/2}} \nu}
\]
are vector fields (index \(n\)) taking values in all representations of the Lorentz group described by the \(\gamma\)-transversal two-row Young diagrams

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\ddots \\
\vdots \\
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
at
\end{array}
\]
\(s-3/2\)

(14)
i.e., the irreducibility conditions for the higher spin supercurrents \(J^\mu_{m(t), n(s-3/2); \nu}\) read

\[t J^\mu_{m(t-1), n, n(s-3/2); \nu} = 0\]  
(15)
and

\[(s - 5/2) \gamma^n_{\mu} \nu J^\mu_{m(t), n(s-3/2); \nu} = 0 .\]  
(16)

From these conditions it follows that

\[(s - 5/2) \gamma^n_{m} \nu J^\mu_{m(t), n(s-3/2); \nu} = 0\]  
(17)
and all tracelessness conditions (8) and (9) are satisfied.

To avoid complications resulting from the projection to the space of irreducible (i.e. traceless or \(\gamma\)--transversal) two–row Young diagrams we study the currents

\[J^n(\xi) = J^n_{m(t), n(s-1); \xi(m(t), n(s-1))}, \quad J^n(\xi) = \xi_{m(t), n(s-3/2); \nu} J^\mu_{m(t), n(s-3/2); \nu},\]  
(18)
where \(\xi_{m(t), n(s-1)}\) and \(\xi_{m(t), n(s-3/2); \nu}\) are some constant parameters which themselves satisfy analogous irreducibility conditions. The conservation law then reads

\[\partial^\nu J^\nu(\xi) = 0 .\]  
(19)

The currents corresponding to one–row Young diagrams (i.e. with \(t = 0\)) generalize the spin 1 current (1), supercurrent (4) and stress tensor (3). An important fact is that they can be chosen in the form

\[J^m_{n(s-1)} \xi_{n(s-1)} = T^m_{n(s-1)} \xi_{n(s-1)}\]  
(20)
\[J^m_{n(s-3/2); \nu} \xi_{n(s-3/2); \nu} = T^m_{n(s-3/2); \nu} \xi_{n(s-3/2); \nu},\]  
(21)
with totally symmetric conserved currents \(T^n(s)\) or supercurrents \(T^n(s-1/2); \nu\),

\[\xi_{n(s-1)} \partial_n T^n(s) = 0 , \quad \xi_{n(s-3/2); \nu} \partial_n T^n(s-1/2); \nu = 0\]  
(22)
\( (\xi^n_{n(s-2)} = 0, (\xi_{n(s-3/2)}^n \gamma^\nu) = 0). \)

Analogously to the formula (5) for the angular momenta current, the symmetric (super)currents \( T \) allow one to construct explicitly \( x \)-dependent higher spin “angular” currents. An observation is that the angular higher spin (super)currents

\[
J^\mu(\xi) = T_{\mu}^{n(s-1)} x^{m(t)} \xi_{m(t),n(s-1)}, \quad J^\mu(\xi) = T_{\mu}^{n(s-3/2)} x^{m(t)} \xi_{m(t),n(s-3/2); \nu}
\]

where we use the shorthand notation

\[
x^{m(s)} = x^{m} \cdots x^{m}.
\]

also conserve as a consequence of (22) because when the derivative in (19) hits a factor of \( x^m \), the result vanishes by symmetrization of too many indices in the parameters \( \xi \) forming the two-row Young diagrams.

Since the parameters \( \xi_{m(t),n(s-1)} \) and \( \xi_{m(t),n(s-3/2); \nu} \) are traceless and \( \gamma \)-transversal, only the double traceless part of \( T^{n(s-1)} \)

\[
T^{n(2)}_{n(s-2)} = 0, \quad s \geq 4
\]

and triple \( \gamma \)-transversal part of \( T^{n(s-3/2)} \):

\[
\gamma^n T^{n}_{n(s-3/2)} = 0, \quad s \geq 7/2
\]

contribute to (23). These are the (super)currents of the formalism of symmetric tensors (tensor-spinors)\(^{20,9}\) The currents with integer spins \( T^{n(s)} \) were considered in\(^{21,22}\) for the particular case of massless matter fields.

Integer spin currents built from scalars of equal masses

\[
(\Box + m^2) \phi^i = 0
\]

have the form

\[
T^{(2k)ij} = (\partial^{(k)} \phi^i \partial^{(k)} \phi^j - \frac{k}{2} \eta^{nn} \partial^{(k-1)} \partial_m \phi^i \partial^{(k-1)} \partial^m \phi^j + \frac{k}{2} m^2 \eta^{nn} \partial^{(k-1)} \phi^i \partial^{(k-1)} \phi^j + i \leftrightarrow j)
\]

for even spins and

\[
T^{(2k+1)ij} = (\partial^{(k+1)} \phi^i \partial^{(k)} \phi^j - \frac{k}{2} \eta^{nn} \partial^{(k)} \partial_m \phi^i \partial^{(k-1)} \partial^m \phi^j + \frac{k}{2} m^2 \eta^{nn} \partial^{(k)} \phi^i \partial^{(k-1)} \phi^j - i \leftrightarrow j)
\]

\(\Box = \frac{k}{2} m^2 \eta^{nn} \partial^{(k)} \phi^i \partial^{(k-1)} \phi^j \) - i \leftrightarrow j
for odd spins, where we ignore terms containing more than one flat metric $\eta^{nn}$, which do not contribute to the charges (23), and use notation analogous to (24)

$$\partial^{n(s)} = \partial^n \cdots \partial^n_{s}.$$  \hfill (30)

Higher spin supercurrents built from scalar and spinor with equal masses

$$\Box + m^2 \phi = 0, \quad (i\partial_{\underline{n}} \gamma^{\underline{n}} + m)\psi_{\nu} = 0$$  \hfill (31)

read

$$T^{n(k+1)}_{\mu} = \partial^{n(k+1)} \phi \psi_{\nu} - \frac{k + 1}{2} \left( (\gamma^{n} \gamma^{m} \psi)_{\nu} \partial^{n(k)} \partial_{m} \phi + im(\gamma^{n} \psi)_{\nu} \partial^{n(k)} \phi \right).$$  \hfill (32)

Inserting these expressions into (23) we obtain the set of conserved “angular” higher spin currents of even, odd and half-integer spins. The usual angular momentum current corresponds to the case $s = 2, t = 1$.

Remarkably, the conserved higher spin currents listed above are in one-to-one correspondence with the higher spin gauge fields (1-forms) $\omega_{\underline{n},m(t),n(s-1)}$ and $\omega_{\underline{n},m(t),n(s-3/2)}$ introduced for the boson and fermion cases in arbitrary $d$. To the best of our knowledge, the fact that any of the higher spin gauge fields has a dual conserved current was never discussed before. Of course, such a correspondence is expected because, like the gauge fields of the supergravitational multiplets, the higher spin gauge fields should take their values in a (infinite-dimensional) higher spin algebra identified with the global symmetry algebra in the corresponding dynamical system (this fact is explicitly demonstrated below for the cases of $d = 3$ and $d = 4$). The higher spin currents can then be derived via the Noether theorem from the global higher spin symmetry and give rise to the conserved charges identified with the Hamiltonian generators of the same symmetries.

A few comments are now in order.

Higher spin currents contain higher derivatives. Therefore, higher spin symmetries imply, via the Noether procedure, the appearance of higher derivatives in interactions. The immediate question is whether higher spin gauge theories are local or not. As we shall see the answer is “yes” at the linearized level and “probably not” at the interaction level.

It is well known that if the stress tensor is traceless this indicates a larger (conformal) symmetry. This property extends to the higher spin currents provided that the higher spin currents are traceless.

Nontrivial (interacting) theories exhibiting higher spin symmetries are formulated in AdS background rather than in the flat space. Therefore an impor-
tant problem is to generalize the constructed currents to the AdS geometry. This problem was solved recently\textsuperscript{26} for the case d=3.

Explicit form of the higher spin algebras is known for \( d \leq 4 \) although a conjecture was made in \textsuperscript{24} on the structure of higher spin symmetries in any \( d \). The knowledge of the structure of the higher spin currents in arbitrary dimension may be very useful for elucidating a structure of the higher spin symmetries in any \( d \).

3 Higher-Spin Symmetries

The key element of the theory of massless higher spin fields is the higher spin gauge symmetry principle. Its role is as fundamental as that of the Poincare superalgebra discovered by Golfand and Likhtman for supersymmetric theories. From the \( d = 4 \)\textsuperscript{27,28,29} and \( d = 3 \)\textsuperscript{30,31,32,33,34} analysis it is known that the relevant higher spin symmetry algebras \( h \) are certain infinite-dimensional Lie superalgebras which give rise to infinite chains of spins and contain \( AdS_d \) algebras \( o(d-1,2) \) and their superextensions as (maximal) finite-dimensional subalgebras. To fix conventions let us note that the generators of the \( AdS_d \) algebra \( o(d-1,2) \) can be identified with the Lorentz generators \( L^{mn} \) of \( o(d-1,1) \) and the generators of \( AdS_d \) translations \( P^m \in o(d-1,2)/o(d-1,1) \) with the commutation relations

\[
\begin{align*}
[L^{mn}, L^{kl}] &= \eta^{nk}L^{ml} - \eta^{mk}L^{nl} + \eta^{ml}L^{nk} - \eta^{nl}L^{mk}, \\
[L^{mn}, P^k] &= \eta^{nk}P^m - \eta^{mk}P^n, \\
[P^m, P^n] &= -\lambda^2 L^{mn}.
\end{align*}
\]  

(33)

Here \( \lambda^{-1} \) is identified with the AdS radius. It serves as the Inönü-Wigner contraction parameter: \( \lambda \rightarrow 0 \) in the flat limit.

A structure of higher spin algebras \( h \) is such that no higher spin \((s > 2)\) field can remain massless unless it belongs to an infinite chain of massless higher spins with infinitely increasing spins. Unbroken higher spin symmetries require AdS background. One can think however of some spontaneous breakdown of the higher spin symmetries followed by a flat contraction via a shift of the vacuum energy in the broken phase. In a physical phase with \( \lambda = 0 \) and \( m \gg m_{\text{exp}} \) for higher spin fields, \( h \) should break down to a finite-dimensional subalgebra \( g = \{M^{mn}, P^m, Q^i_\nu\} \oplus T^i_j \), giving rise to usual lower spin gauge fields. Here the first set of the generators corresponds to a SUSY algebra while the second one describes some inner (Yang-Mills) part. From this perspective the Coleman-Mandula type theorems can be re-interpreted as statements concerning a possible structure of \( g \) rather than the whole higher spin algebra \( h \).
which requires AdS geometry. These arguments are based on the $d \leq 4$ experience but we expect them to large extend to be true for higher dimensions. The origin of higher spin symmetries can be traced back to the higher spin conserved currents with higher derivatives discussed in the Sec. 2. Let us summarize the main results for $d = 3$ and $d = 4$ following to $^{27,28,29,32,33}$. 

3.1 $d=3$ Higher Spin Symmetries and Deformed Oscillators

The $AdS_3$ algebra is semisimple, $o(2,2) \sim sp(2;R) \oplus sp(2;R)$ with the diagonal subalgebra $sp(2;R) \sim o(2,1)$ identified with the Lorentz subalgebra. A particularly useful realization of the $AdS_3$ generators is

$$L_{\alpha\beta} = \frac{1}{4i}\{\hat{y}_\alpha, \hat{y}_\beta\}, \quad P_{\alpha\beta} = \frac{1}{4i}\{\hat{y}_\alpha, \hat{y}_\beta\}\psi$$

with the generating elements $\hat{y}_\alpha$ and $\psi$ obeying the relations $[\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta}$, $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{12} = 1$ and

$$\psi^2 = 1, \quad [\psi, \hat{y}_\alpha] = 0.$$ 

The projectors to the simple components of $o(2,2)$ are identified with $\Pi_{\pm} = \frac{1}{2}(1 \pm \psi)$.

In $^{35,31,36,32,37}$ it was shown that there exists a one-parametric class of infinite-dimensional algebras which we denote $hs(2;\nu)$ ($\nu$ is an arbitrary real parameter), all containing $sp(2)$ as a subalgebra. This allows one to define a class of higher spin algebras $g = hs(2;\nu) \oplus hs(2;\nu)$. The supertrace operation was defined in $^{32}$ where also a useful realization of the supersymmetric extension of $hs(2;\nu)$ was given, based on a certain deformed oscillator algebra. Since this construction will play a key role below let us explain its properties in somewhat more detail.

Consider an associative algebra $Aq(2;\nu)$ with generic element of the form

$$f(\hat{y},k) = \sum_{n=0}^{\infty} \sum_{A=0,1} \frac{1}{n!} f^{A\alpha_1...\alpha_n}(k)^A \hat{y}_{\alpha_1}...\hat{y}_{\alpha_n},$$

under condition that the coefficients $f^{A\alpha_1...\alpha_n}$ are symmetric with respect to the indices $\alpha_j = 1, 2$ and that the generating elements $\hat{y}_\alpha$ satisfy the relations

$$[\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta}(1 + \nu k), \quad k\hat{y}_\alpha = -\hat{y}_\alpha k, \quad k^2 = 1,$$

where $\nu$ is an arbitrary constant (central element). In other words, $Aq(2;\nu)$ is the enveloping algebra for the relations (37) often called deformed oscillator algebra.
An important property of this algebra is that, for all \( \nu \), the bilinears

\[
T_{\alpha\beta} = \frac{1}{4i} \{ \hat{y}_\alpha, \hat{y}_\beta \}
\]

have \( sp(2) \) commutation relations and rotate \( \hat{y}_\alpha \) as a \( sp(2) \) vector

\[
[T_{\alpha\beta}, T_{\gamma\eta}] = \epsilon_{\alpha\gamma} T_{\beta\eta} + \epsilon_{\beta\gamma} T_{\alpha\eta} + \epsilon_{\alpha\eta} T_{\beta\gamma} + \epsilon_{\beta\eta} T_{\alpha\gamma},
\]

\[
[T_{\alpha\beta}, \hat{y}_\gamma] = \epsilon_{\alpha\gamma} \hat{y}_\beta + \epsilon_{\beta\gamma} \hat{y}_\alpha.
\]

The deformed oscillators described above have a long history and were originally discovered by Wigner who addressed a question whether it is possible to modify the commutation relations for the normal oscillators \( a^\pm \) in such a way that the basic commutation relations \([H, a^\pm] = \pm a^\pm, H = \frac{1}{2} \{a^+, a^-\}\) remain valid. By analyzing this problem in the Fock-type space Wigner found a one-parametric deformation of the standard commutation relations which corresponds to a particular realization of the commutation relations (37) with the identification \( a^+ = \hat{y}_1, a^- = \frac{1}{2} i \hat{y}_2, H = T_{12} \) and \( k = (-1)^N \) where \( N \) is the particle number operator. These commutation relations were discussed later by many authors in particular in the context of parastatistics (see, e.g., \( 39 \)).

According to (38) and (40) the \( sp(2) \) symmetry generated by \( T_{\alpha\beta} \) extends to \( osp(1, 2) \) by identifying the supergenerators with \( \hat{y}_\alpha \). In fact, as shown in \( 40 \), one can start from the \( osp(1, 2) \) algebra to derive the deformed oscillator commutation relations. Since this construction is instructive in many respects we reproduce it here.

One starts with the (super)generators \( T_{\alpha\beta} \) and \( \hat{y}_\alpha \), which, by definition of \( osp(1, 2) \), satisfy the commutation relations (38)-(40). Since \( \alpha \) and \( \beta \) take only two values one can write

\[
[\hat{y}_\alpha, \hat{y}_\beta] = 2i \epsilon_{\alpha\beta}(1 + Q),
\]

where \( Q \) is some new “operator” while the unit term is singled out for convenience. Inserting this back into (40) with the substitution of (38) and completing the commutations one observes that (40) is true if and only if \( Q \) anti-commutes with \( \hat{y}_\alpha \),

\[
Q \hat{y}_\alpha = -\hat{y}_\alpha Q.
\]

The relation (39) does not add anything new since it is a consequence of (38) and (40). As a result we arrive at the following important fact: the enveloping algebra of \( osp(1, 2) \), \( U(osp(1, 2)) \), is isomorphic to the enveloping algebra of the deformed oscillator relations (41) and (42). In other words, the associative
algebra with the generating elements $\hat{y}_\alpha$ and $Q$ subject to the relations (41) and (42) is the same as the associative algebra with the generating elements $\hat{y}_\alpha$ and $T_{\alpha\beta}$ subject to the $osp(1,2)$ commutation relations (38)-(40).

Computing the quadratic Casimir operator of $osp(1,2)$

$$C_2 = -\frac{1}{2}T_{\alpha\beta}T^{\alpha\beta} - \frac{i}{4}\hat{y}_\alpha\hat{y}^\alpha,$$

one derives using (41) that

$$C_2 = -\frac{1}{4}(1 - Q^2).$$

Let us now consider the factor algebra of $U(osp(1,2))$ with respect to its ideal $I(C_2 + \frac{1}{4}(1 - \nu^2))$ generated by the central element $(C_2 + \frac{1}{4}(1 - \nu^2))$ where $\nu$ is an arbitrary number. In other words, every element of $U(osp(1,2))$ of the form $(C_2 + \frac{1}{4}(1 - \nu^2))a$, $\forall a \in U(osp(1,2))$ is supposed to be equivalent to zero. This factorization can be achieved in terms of the deformed oscillators (41), (42) by setting

$$Q = \nu k, \quad k^2 = 1, \quad k\hat{y}_\alpha = -\hat{y}_\alpha k.$$ 

Thus, it is shown that the algebra $Aq(2; \nu)$ introduced in isomorphic to $U(osp(1,2))/I(C_2 + \frac{1}{4}(1 - \nu^2))$. This fact has a number of simple but important consequences. For example, any representation of $osp(1,2)$ with $C_2 = -\frac{1}{4}(1 - \nu^2)$ forms a representation of $Aq(2; \nu)$ ($\nu \neq 0$) and vice versa (for any $\nu$). In particular this is the case for finite-dimensional representations corresponding to the values $\nu = 2l + 1$, $l \in \mathbb{Z}$ with $C_2 = l(l + 1)$.

Let us note that the even subalgebra of $Aq(2; \nu)$ spanned by the elements of the form (36) with $f(\hat{y}, k) = f(-\hat{y}, k)$ decomposes into a direct sum of two subalgebras $Aq_E^\pm(2; \nu)$ spanned by the elements $P^\pm f(\hat{y}, k)$ with $f(-\hat{y}, k) = f(\hat{y}, k)$, $P^\pm = \frac{1}{2}(1 \pm k)$. These algebras can be shown to be isomorphic to the factor algebras $U(sp(2))/I_{(C_2 + \frac{1}{4}(1 - \nu^2))}$, where $C_2 = -\frac{1}{2}T_{\alpha\beta}T^{\alpha\beta}$ is the quadratic Casimir operator of $sp(2)$ and can be interpreted as (infinite-dimensional) algebras interpolating between the ordinary finite-dimensional matrix algebras. Such interpretation of $U(sp(2))/I_{(C_2 - c)}$ was given by Feigin in and Fradkin and Linetsky in.

An important property of $Aq(2; \nu)$ is that it admits a uniquely defined supertrace operation

$$str(f) = f^0 - \nu f^1,$$

The point $\nu = 0$ is special since it may happen that $Q^2 = 0$, $Q \neq 0$. 

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such that $\text{str}(fg) = (-1)^{r_pr_q} \text{str}(gf)$, $\forall f, g$ having a definite parity, $f(-\hat{y}, k) = (-1)^{\nu} f(\hat{y}, k)$ (i.e. $\text{str}(1) = 1$, $\text{str}(k) = -\nu$ while all higher monomials of $\hat{y}_\alpha$ in (36) do not contribute under the supertrace). This supertrace reduces to the ordinary supertrace of finite-dimensional algebras for the special values of the parameter $\nu = 2l + 1$ corresponding to the values of the Casimir operator related to the finite-dimensional representations of $osp(1, 2)$ ($sp(2)$ in the bosonic case). This property allows one to handle the algebras $A_q(2; \nu)$ very much the same way as ordinary finite-dimensional (super)matrix algebras.

What happens for special values of $\nu = 2l + 1$ is that $A_q(2; \nu)$ acquires ideals $I_l$ such that $A_q(2; \nu)/I_l$ amounts to appropriate (super)matrix algebras. These ideals were described in $\text{32}$ as null vectors of the invariant bilinear form $\text{str}(ab)$, $a, b \in A_q(2; \nu)$. Note that the limit $\nu \to \infty$ corresponds to the algebra of area preserving diffeomorphisms in accordance with the original matrix analysis $\text{41}$.

By construction, $A_q(2; \nu)$ possesses $N = 1$ supersymmetry as inner $osp(1, 2)$ automorphisms. A more interesting property $\text{40}$ is that it admits $N = 2$ supersymmetry $osp(2, 2)$ with the generators

$$T_{\alpha\beta} = \frac{1}{4i} \{\hat{y}_\alpha, \hat{y}_\beta\}, \quad Q_\alpha = \hat{y}_\alpha, \quad S_\alpha = \hat{y}_\alpha k, \quad J = k + \nu. \quad (47)$$

The deformed oscillator algebras are important for the description of the higher spin dynamics not only in $d=3$ but also in $d=4$. The reason is that, as shown in section 8.4, they allow one to formulate a non-linear dynamics with explicit local Lorentz symmetry as a consequence of (39). In its turn, the analysis of the higher spin dynamics in the section 8.3 is interesting in the context of the deformed oscillator algebra itself because algebraically it reduces to the construction of some its embedding into a direct product of two Weyl (i.e. oscillator) algebras equipped with certain twist operators.

Thus, the $AdS_3$ algebra is the algebra of bilinears in the oscillators $\hat{y}$. Extension to higher spin algebras consists of allowing arbitrary powers in the oscillators. Namely the higher spin gauge fields $w(\hat{y}, \psi, k|x) = dx^\nu w_\nu(\hat{y}, \psi, k|x)$ have a form

$$w(\hat{y}, \psi, k|x) = \sum_{B=0,1 \, n=0}^\infty \frac{1}{n!} w^A_{B\alpha_1...\alpha_n}(x) k^A \psi^B \hat{y}_{\alpha_1}...\hat{y}_{\alpha_n}, \quad (48)$$

The components $w^A_{B\alpha_1...\alpha_n}(x)$ are identified with the higher spin gauge fields in the $d=3$ space-time with the coordinates $x^\mu$. The higher spin field strengths have a standard form

$$R(\hat{y}, \psi, k|x) = dw(\hat{y}, \psi, k|x) - w(\hat{y}, \psi, k|x) \wedge w(\hat{y}, \psi, k|x), \quad (49)$$

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where \( d = dx \frac{\partial}{\partial x} \). This construction for the ordinary oscillators was suggested in \(^{30}\).

The structure of the higher spin gauge fields (48) suggests that the d3 higher spin currents should form a similar set

\[
J_{\mu}^{A} B_{\alpha_1 \ldots \alpha_n} (x) .
\]  

Here, the two indices \( A = 0,1 \) and \( B = 0,1 \) play different roles. The label \( A \) describes the doubling of all fields as a consequence of \( N = 2 \) supersymmetry in the theory. This doubling can in principle be avoided in an appropriately truncated theory\(^{34}\). The label \( B \) distinguishes between the Lorentz–type fields \((B = 0)\) and frame–type fields \((B = 1)\) and therefore leads to two different types of currents of any spin \( s = \frac{1}{2} n \).

To see that such a structure of currents fits the systematics of the section 2 one observes that in three dimensions all traceless two-row Young diagrams (7) with \( t > 1 \) vanish. Indeed, consider the quantity

\[
Y_{bc}^{cc} (t,p) = \epsilon^{abc} \epsilon^{abc} X_{b(t+2),a(p+2)}
\]

with some traceless \( X_{b(t+2),a(p+2)} \) satisfying the antisymmetry condition

\[
X_{b(t+1),a(p+2)} = 0 .
\]

On the one hand, one gets

\[
X_{b(t+2),a(p+2)} = \frac{p + 1}{p + 3} \epsilon^{abc} Y_{bc}^{cc} (t,p) .
\]

But on the other hand, from (51) it follows that \( Y_{bc}^{cc} (t,p) = 0 \) since at least one pair of indices of \( X_{b(t+2),a(p+2)} \) will be contracted after the two epsilon symbols are replaced by the combinations of the metric tensors.

As a result, only the currents corresponding to the Young diagrams (7) with \( t = 0 \) and \( t = 1 \) survive in the d3 case. The one-row currents \((t = 0)\) describe stress tensor-type higher spin currents and correspond to the currents (50) with \( B = 1 \) while those with a single box in the second row describe higher spin angular momentum–type currents and correspond to the currents (50) with \( B = 0 \).

As shown in \(^{42}\), the ambiguity in \( \nu \) identifies with the ambiguity in the parameter of mass in the d=3 matter systems. Thus, a structure of global higher spin algebra depends on a particular type of matter systems. This property manifests the fact\(^{34}\) that the parameter of mass in the matter field sector of the d3 higher spin system arises from a vacuum expectation value of a certain scalar field in the model, while the global symmetry algebra identifies with the centralizer of the vacuum solution in the larger gauge algebra having a universal structure (see section 8.3).
A special property of the d3 higher spin systems is that higher spin gauge fields do not propagate in d=3 in analogy with the usual Chern-Simons gravitational and Yang-Mills fields although the higher spin gauge symmetries remain nontrivial, like the gravitational (spin 2) and inner (spin 1) symmetries. The matter fields do propagate. In other words, higher spin gauge fields and matter fields belong to different higher spin multiplets and the parameter of mass of matter fields remains arbitrary. In higher dimensions the situation changes significantly because the higher spin gauge fields are propagating massless fields and matter fields may belong to the same multiplet of higher spin symmetry with the higher spin gauge fields thus necessarily being massless just as in ordinary SUSY supermultiplets.

3.2 d=4 Higher Spin Symmetries

Algebraically, the situation in d=4 is analogous. The isomorphism $o(3,2) \sim sp(4|R)$ allows one to realize the generators of Lorentz transformations $L_{\alpha\beta}$, $\bar{L}_{\dot{\alpha}\dot{\beta}}$ and AdS translations $P_{\alpha\dot{\beta}}$ as bilinears

$$L_{\alpha\beta} = \frac{1}{4i} \{\hat{y}_\alpha, \hat{y}_\beta\}, \quad \bar{L}_{\dot{\alpha}\dot{\beta}} = \frac{1}{4i} \{\hat{\bar{y}}_{\dot{\alpha}}, \hat{\bar{y}}_{\dot{\beta}}\}, \quad P_{\alpha\dot{\beta}} = \frac{1}{2i} \hat{y}_\alpha \hat{\bar{y}}_{\dot{\beta}}$$

(53)

in the oscillators obeying the relations

$$[\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta}, \quad [\hat{\bar{y}}_{\dot{\alpha}}, \hat{\bar{y}}_{\dot{\beta}}] = 2i\epsilon_{\dot{\alpha}\dot{\beta}}, \quad [\hat{y}_\alpha, \hat{\bar{y}}_{\dot{\beta}}] = 0.$$  

(54)

The simplest version of the d=4 higher spin algebra is identified with the algebra of all polynomials of the oscillators and Klein operators $k$ and $\bar{k}$ having the properties

$$k^2 = \bar{k}^2 = 1, \quad k\hat{y}_\alpha = -\hat{y}_\alpha k, \quad \bar{k}\hat{\bar{y}}_{\dot{\alpha}} = -\hat{\bar{y}}_{\dot{\alpha}} \bar{k}$$

(55)

(all barred operators commute to all unbarred ones), i.e. the corresponding gauge fields have a form

$$W = \sum_{A,B=0,1} \sum_{n,m=0}^{\infty} \frac{1}{2im!n!} d^A d^B w_{AB\alpha_1...\alpha_n \dot{\alpha}_1...\dot{\alpha}_m}(x) k^A \bar{k}^B \hat{y}_{\alpha_1}...\hat{y}_{\alpha_n} \hat{\bar{y}}_{\dot{\alpha}_1}...\hat{\bar{y}}_{\dot{\alpha}_m}.$$  

(56)

In accordance with (53), the gauge fields bilinear in the oscillators are identified with the gravitational fields.

The operators $k$ and $\bar{k}$ again play a role in the construction and lead to N=2 supersymmetry $osp(2,4)$ with the supergenerators

$$Q_1^\alpha = \hat{y}_\alpha, \quad Q_2^\alpha = ik\bar{k}\hat{y}_\alpha, \quad \bar{Q}_1^\dot{\alpha} = \hat{\bar{y}}_{\dot{\alpha}}, \quad \bar{Q}_2^\dot{\alpha} = ik\bar{k}\hat{\bar{y}}_{\dot{\alpha}},$$

(57)
and the $o(2)$ generator

$$J = i k \tilde{k}.$$  \hfill (58)

Factoring out the trivial $u(1)$ subalgebra associated with the unit element (as explained in the section 4, the resulting subalgebra is spanned by traceless elements) this $osp(2, 4)$ becomes a maximal finite-dimensional subalgebra of the infinite-dimensional $d=4$ higher spin algebra. According to\textsuperscript{19,45,44} the fields

$$W(y, \hat{y}, k, \bar{k}; x) = W(y, \hat{y}, -k, -\bar{k}; x)$$  \hfill (59)

describe the higher spin fields while the fields

$$W(y, \hat{y}, k, \bar{k}; x) = -W(y, \hat{y}, -k, -\bar{k}; x)$$  \hfill (60)

are auxiliary, i.e. do not describe nontrivial degrees of freedom. (For that reason this version of the higher spin algebra was called in\textsuperscript{44} algebra of higher spins and auxiliary fields $shsa(1)$.) Therefore we have two sets of higher spin potentials

$$u^A A_{\alpha_1 \ldots \alpha_n \dot{\alpha}_1 \ldots \dot{\alpha}_m}(x), \quad A=0 \text{ or } 1.$$  \hfill (61)

The subsets associated with spin $s$ are fixed by the condition\textsuperscript{19} $s = 1 + \frac{1}{2}(n + m)$. The complex conjugation transforms dotted indices into undotted and vice versa, thus mapping $n$ to $m$.

For a fixed value of $A$ we therefore expect a set of currents

$$J^{\mu}_{\alpha_1 \ldots \alpha_n, \dot{\alpha}_1 \ldots \dot{\alpha}_m}(x).$$  \hfill (62)

In accordance with the results of the section 2, it indeed describes in terms of two-component spinors the set of all two-row Young diagrams (7) both in the integer spin case ($n + m$ is even) and in the half-integer spin case ($n + m$ is odd), with the identification

$$s = \frac{1}{2}(n + m) + 1, \quad t = \left[ \frac{1}{2} |n - m| \right],$$  \hfill (63)

where $[a]$ denotes integer part of $a$. The fact that $d3$ and $d4$ higher spin algebras give rise to the sets of gauge fields which exactly match the sets of conserved higher spin currents of the section 2 is very significant.

### 3.3 Extended Higher Spin Algebras

Higher spin dynamical systems admit a natural extension to the case with non-Abelian internal (Yang-Mills) symmetries, as was discovered for the $d4$
case in 46,29 and then confirmed for the $d=3$ case in 33,34. The key observation is that higher spin dynamics remains consistent if components of all fields take their values in an arbitrary associative algebra $M$ with unity $I_M$. For example $W(y, \bar{y}, k, \bar{k}|x) \rightarrow W^{ij}(y, \bar{y}, k, \bar{k}|x)$, $i,j = 1 \ldots n$ for $M = \text{Mat}_n$. The gravitational sector is associated with the fields proportional to $I_M$. Therefore, $M$ describes internal symmetries in the model. For the case of semisimple finite-dimensional inner symmetries, $M$ has to be identified with some matrix algebra. This observation leads to a class of higher spin systems with different semisimple Yang-Mills algebras classified for $d=4$ in 29 and for $d=3$ in 34. Note that, the Yang-Mills coupling constant $g^2$ is the only dimensionless constant in the theory. It identifies with the dimensionless combination of the cosmological constant and the gravitation constant $g^2 \sim \Lambda \kappa^2$ analogously to the case of gauged supergravity 16,17.

In this section, we list following to 29 all global extended $d=4$ higher spin symmetries with finite-dimensional internal symmetries. The case $d=3$ can be considered analogously 34.

To solve the problem it is useful to investigate most general truncations of the extended higher spin theories with arbitrary matrix algebras $M$, which lead to consistent higher spin dynamics. To this end one finds such automorphisms $\tau$ of the higher spin algebras which leave invariant the $\text{AdS}_4$ subalgebra associated with the gravitational sector and the nonlinear equations discussed in the section 8.2. The latter condition reduces to the simple additional requirement that, in presence of fermions, the allowed automorphisms should leave invariant the operator

$$K = k \bar{k},$$

which enters explicitly the nonlinear higher spin equations as explained in the section 8.2. (In the purely bosonic case, $K$ becomes a central element and can be truncated away.) Truncating away all degrees of freedom except for singlets with respect to the discrete symmetries associated with $\tau$ one is left with some consistent truncation of the full system being itself a consistent dynamical system. In particular, in the sector of gauge potentials the truncation conditions read $\tau(w) = w$. The requirement that the $\text{AdS}$ algebra associated with the gravitational sector is $\tau$–invariant is imposed to guarantee that the vacuum gravitational fields necessary for any relativistic perturbative interpretation of the model survive in the truncated system.

For example the automorphism $\tau$ defined according to

$$\tau(k) = -k,$$

$$\tau(\bar{k}) = -\bar{k}$$

(65)
can be used to truncate away the auxiliary fields (60). In the rest of this section
we will be only concerned with the sector of dynamical fields. As a result, the generating elements \( k \) and \( \bar{k} \) only appear in this sector in the combination \( K \).

Using a combination of an appropriate involutive inner automorphism \( t \) of the matrix algebra \( M \) with the boson-fermion parity automorphism \( f \) of the algebra of oscillators one arrives at the class of higher spin algebras realized as \((n + m) \times (n + m)\) matrices with the elements depending on the operators \( \hat{y}_\alpha \) and \( \hat{\bar{y}}_\dot{\alpha} \).

\[
P^I_j(\hat{y}, \hat{\bar{y}}) = \begin{pmatrix}
  P^{E}_{j}E^I(\hat{y}, \hat{\bar{y}}) & P^{O}_{j}O^I(\hat{y}, \hat{\bar{y}}) \\
  P^{O}_{j}O^I(\hat{y}, \hat{\bar{y}}) & P^{E}_{j}E^I(\hat{y}, \hat{\bar{y}})
\end{pmatrix}
\]  

for the conditions that

\[
P^{E}_{j}E^I(\hat{y}, \hat{\bar{y}}) = P^{E}_{j}E^I(-\hat{y}, -\hat{\bar{y}}), \quad P^{O}_{j}O^I(\hat{y}, \hat{\bar{y}}) = -P^{O}_{j}O^I(-\hat{y}, -\hat{\bar{y}}),
\]  

i.e. the diagonal blocks \( P^E \) are bosonic (the power series coefficients carry even numbers of spinor indices) while the off-diagonal blocks \( P^O \) are fermionic (the power series coefficients carry odd numbers of spinor indices). The operator \( K \) that anticommutes to fermions is realized as

\[
K = \begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix}.
\]  

By definition (see e.g. \(^{48}\)), Weyl algebra \( A_n \) is the associative algebra with unity \( I \) and the generating elements \( y_\nu (\nu, \mu = 1 \ldots 2n) \) satisfying \([y_\nu, y_\mu] = C^\nu_\mu I\) for some nondegenerate skewsymmetric matrix \( C^\nu_\mu = -C^\mu_\nu \), i.e. \( A_n \) is the algebra of \( n \) oscillators, \( A_n = (\otimes A_1)^n \). We denote the associative algebra spanned by the elements of the form \([66], [67] A_{l}^{n,m} \). Obviously, \( A_{l}^{n,m} \sim A_{l}^{m,n} \). The complex higher spin Lie superalgebra \( hgl(n, m|2l \mid \mathbb{C}) \) is defined via supercommutators in \( A_{l}^{n,m} \) with respect to the grading \([67] \) (of course, any other field instead of \( \mathbb{C} \) can be formally used). The algebra \( hgl(n; m|2l) \) is not simple, containing a trivial central element associated with the unit element of \( A \otimes M \), i.e.

\[
hgl(n; m|2l) = u(1) \oplus hsl(n; m|2l),
\]  

where \( hsl(n; m|2l) \) is the traceless part of \( hgl(n; m|2l) \) according to the definition of the trace \(^{28}\) explained in the section 4.
The complex algebra $hgl(n,m|2l)$ admits a number of real forms. Of the most interest are those which lead to unitary dynamics and, in particular, to compact inner symmetries. The corresponding real higher spin algebras $hu(n;m|4)$ are spanned by elements obeying the reality conditions

\[ [P^j_i(\hat{y},\hat{\bar{y}})]^\dagger = -(i)^{\pi_i(P^j)} P^j_i(\hat{y},\hat{\bar{y}}) \]  

(70)

with

\[ (\hat{y}_\alpha)^\dagger = \hat{\bar{y}}_{\dot{\alpha}}, \quad \pi(P^E) = 0, \quad \pi(P^O) = 1 \]  

(71)

(the algebras $hu(n,m|2l)$ are defined analogously for $l = 2k$).

Although being non-simple due to (69) the algebra $hsu(n;m|4)$ plays a fundamental role in the higher spin theory because all its gauge fields appear in the spectrum including the spin 1 field associated with the $u(1)$ central subalgebra. This is a manifestation of the general fact that the higher spin gauge theories are based on the associative algebra structure rather than on the Lie (super)algebra structure as lower spin theories.

It turns out that the algebras $hu(n;m|2l)$ exhaust all elementary higher spin superalgebras which can be obtained with the aid of automorphisms of the underlying associative algebra $A_{2n}^{\alpha}$. Further truncations of $hu(n;m|4)$ are based on the antiautomorphisms of $A_{2n}^{\alpha}$ which induce automorphisms of $hu(n;m|4)$ (for more detail on the relationship between antiautomorphisms of associative algebras and automorphisms of the related Lie superalgebras see for example 28). These give rise to the higher spin algebras of orthogonal and symplectic types denoted $ho(n;m|4)$ and $husp(n;m|4)$, respectively, which also lead to consistent equations of motion for infinite sets of massless fields of different spins via appropriate truncations of the higher spin equations corresponding to $hu(n;m|4)^{46,29}$. These subalgebras are extracted from $hu(n;m|4)$ by the conditions

\[ P_k^i(\hat{y},\hat{\bar{y}}) = -(i)^{\pi(P_k^i)} \eta^{\dagger i} \eta^{\nu} P_u^\nu(i\hat{y},i\hat{\bar{y}}) \eta^{-1} \]  

(72)

with some nondegenerate bilinear form $\eta_{kl}$. If $\eta_{kl}$ is symmetric, $\eta_{kl} = \eta_{lk}$, this leads to the orthogonal algebras $ho(n;m|4)$. The skewsymmetric form, $\eta_{kl} = -\eta_{lk}$, gives rise to the symplectic algebras $husp(n;m|4)$ ($n$ and $m$ should be even in the latter case). The isomorphisms $hu(n;m|4) \sim hu(m;n|4)$, $ho(n;m|4) \sim ho(m;n|4)$ and $husp(n;m|4) \sim husp(m;n|4)$ take place as a consequence of the isomorphism $A_{2l}^{n,m} \sim A_{2l}^{m,n}$.

Spin–1 Yang-Mills subalgebras of the higher spin algebras defined this way are spanned by the matrices independent of the operators $\hat{y}_\alpha$ and $\hat{\bar{y}}_{\dot{\alpha}}$. As a result, the Yang-Mills subalgebras coincide with $u(n) \oplus u(m), o(n) \oplus o(m)$
and $usp(n) \oplus usp(m)$ for $hu(n; m|4)$, $ho(n; m|4)$ and $husp(n; m|4)$, respectively. Thus, all types of compact Lie algebras which belong to the classical series $a_n$, $b_n$, $c_n$ and $d_n$ can be realized as spin-1 Yang-Mills symmetries in appropriate higher spin theories.

The multiplicities of massless particles of different spins in higher spin theories based on extended superalgebras are

| spin algebra     | odd                | even               | half-integer |
|------------------|--------------------|--------------------|--------------|
| $hu(n; m|4)$      | $n^2 + m^2$        | $n^2 + m^2$        | $2nm$        |
| $ho(n; m|4)$      | $\frac{1}{2}(n(n-1) + m(m-1))$ | $\frac{1}{2}(n(n+1) + m(m+1))$ | $nm$ |
| $husp(n; m|4)$    | $\frac{1}{2}(n(n+1) + m(m+1))$ | $\frac{1}{2}(n(n-1) + m(m-1))$ | $nm$ |

Let us note that the fields of all odd spins belong to the adjoint representations of the corresponding Yang-Mills algebras while even spins always belong to a reducible representation which contains a singlet component. This property is important since this singlet component corresponds to the spin–2 colorless field to be identified with graviton. In other words, the finite-dimensional algebra spanned by the elements $(I \otimes$ bilinears in $\hat{y}, \hat{\bar{y}})$ is a proper subalgebra of all higher spin algebras. This is not surprising of course, because only those truncations have been considered that leave invariant the gravitational subalgebra $sp(4)$.

The original higher spin algebra constructed in the section 3.2 (with truncated auxiliary sector by virtue of (65)) is isomorphic to $hu(1; 1|4)$. To see this one identifies $K$ according to (68) and

$$
\begin{pmatrix}
0 & (\hat{y}_\alpha, \hat{\bar{y}}_\dot{\alpha}) \\
(\hat{y}_\alpha, \hat{\bar{y}}_\dot{\alpha}) & 0
\end{pmatrix}
$$

The case with $n = 0$ or $m = 0$ corresponds to the purely bosonic higher spin theories.

From the structure of the higher spin superalgebras it is clear why consistent interaction for a spin $s \geq 2$ field in $d=3+1$ is only possible in presence of infinite sets of massless fields of infinitely increasing spins. The reason is
that any field of spin \( s \geq 2 \) corresponds to generators which are some \( \text{deg} > 2 \) polynomials of \( \hat{y} \) and \( \hat{\bar{y}} \) so that their commutators lead to higher and higher polynomials. The same happens when one replaces the unit matrix \( I \) by some non-Abelian matrix algebra for bilinear polynomials in \( \hat{y} \) and \( \hat{\bar{y}} \), \textit{i.e.} spin-2 particles possessing a non-Abelian structure require higher spin fields.

Although no direct proof that the list of higher spin algebras is complete is known, we conjecture that the three two-parametric classes of elementary algebras described above and their direct sums exhaust all \( d=4 \) higher spin superalgebras with finite-dimensional Yang-Mills symmetries. This conjecture gets a nontrivial support from the analysis of the unitary representations of the higher spin algebras in \( 47,29 \) where it was shown that the algebras listed above admit unitary representations with the spectra of massless states having exactly the same spin multiplicities as it follows from the field theoretical analysis of the higher spin equations of motion based on these superalgebras. Moreover, the algebras which can be obtained from the original higher spin algebras by truncations incompatible with the structure of the higher spin equations do not possess unitary representations containing enough states for all massless fields associated with the gauge fields of these algebras.

A simplest example of a truncation that is well defined at the algebra level but is incompatible with field equations can be obtained by truncating away the dependence on the Klein operators \( k \) and \( \hat{k} \). The resulting algebra gives rise to the set of gauge fields \( w(\hat{y}, \hat{\bar{y}}) \) corresponding to massless fields of all spins \( s \geq 1 \), every spin appears once. However it neither admits a unitary massless representation with this spin spectrum \( 47 \), nor corresponds to any consistent truncation of the higher spin equations \( 46 \). One can however get rid of the operators \( k \) and \( \hat{k} \) in the purely bosonic case with \( w(\hat{y}, \hat{\bar{y}}) = w(-\hat{y}, -\hat{\bar{y}}) \). The resulting bosonic algebra is \( hu(1; 0|4) \).

The higher spin superalgebras \( hu(n; m|4) \), \( ho(n; m|4) \) and \( husp(n; m|4) \) are supersymmetric in the standard sense only if \( n = m \). Indeed, one observes that (averaged) numbers of bosons and fermions in \( 73 \) coincide only for this case. All higher spin superalgebras with \( n = m \) contain the AdS superalgebra \( osp(1; 4) \) subalgebra. The algebras \( hsu(n; n|4) \) contain the \( N=2 \) superalgebra \( osp(2; 4) \). Direct identification of the generators is according to \( 57 \) and \( 58 \) with the realization \( 68 \) and \( 74 \).

For the special values of \( n = m = 2^{[N/2]} \), the higher spin algebras admit larger finite-dimensional supersymmetry subalgebras. This is a particularly appealing special case with \( M \) identified with the Clifford algebra \( C_N \) (\( C_N \sim Mat_{N/2} \) for even \( N \) and \( C_N \sim Mat_{N-1} \oplus Mat_{N-1} \) for odd \( N \)).

The corresponding higher spin algebras can be described by “functions”
of additional operators $\phi_i$, the generating elements of the Clifford algebra

$$\{\phi_i, \phi_j\} = 2\delta_{ij}, \quad (75)$$

$i, j = 1 \ldots N$. A nice feature of these algebras is that they naturally contain extended AdS supersymmetry $osp(N, 4)$ with the supercharges

$$Q^i_\alpha = \hat{y}_\alpha \phi^i, \quad \bar{Q}^i_\dot{\beta} = \hat{\bar{y}}_{\dot{\beta}} \phi^i \quad (76)$$

and $o(N)$ generators

$$T^{ij} = \frac{1}{4} [\phi^i, \phi^j]. \quad (77)$$

For odd $N$, the Clifford algebras are semisimple, thus leading to semisimple higher spin superalgebras. We therefore will only consider the case of even $N$.

These higher spin algebras were originally introduced in $^{28}$, where we used notation $shs^E(N, 4)$ for the algebras spanned by the even “functions”

$$f(-\phi, -y, -\bar{y}) = f(\phi, y, \bar{y}). \quad (78)$$

Further truncation can be achieved by virtue of the natural antiautomorphism $\rho$ of the Clifford algebra defined by $\rho(\phi^i) = \phi^i$. As argued in $^{29}$, this truncation is compatible with the higher spin dynamics for $N = 4p$ when the operator $K$ identified with $\phi_1 \ldots \phi_N$ is invariant. The resulting algebras denoted $shs^E(N, 4|0)$ in $^{28}$, form minimal higher spin algebras possessing $N$-extended supersymmetry and containing the gauge fields of the $N$-extended SUGRA supermultiplet $\epsilon_{\nu\alpha\beta}$, $\omega_{\nu\alpha}\beta$, $\bar{\omega}_{\bar{\nu}\alpha}\bar{\beta}$ (spin 2), $\psi^i_{\nu\alpha}$, $\bar{\psi}^i_{\bar{\nu}\alpha}$ (spin 3/2) and $A^{ij}_{\nu}$ (spin 1) within the set of higher spin gauge fields $W(y, \bar{y}, k, \bar{k}, \phi|x)$. The spin 1/2 and spin 0 fields from the SUGRA multiplets are contained in the sector of 0-forms $^{46,29}$ discussed in the section 8.2.

Let us stress that, in the framework of the higher spin gauge theories, there is no barrier $N \leq 8$ which was a consequence of the restriction $s \leq 2$ in the framework of supergravity. The models with $N > 8$ can be considered equally well. The restriction $N \leq 8$ should be reinterpreted as a restriction on a number of unbroken supersymmetries in the phase with broken higher spin symmetries.

Although the higher spin algebras based on the Clifford algebra $M = C_N$ are distinguished by a larger finite-dimensional supersymmetry, they are still particular cases of the algebras described in the beginning of this section. The identification is as follows $^{29}$:

$$shs^E(2l, 4) = hu(2l-1; 2l-1|4), \quad (79)$$
The particular case of the higher spin theory with $N=8$ that corresponds to a higher spin extension of $N=8$ SUGRA was considered recently in great detail in $^{49}$. Within the classification above this is the case of \[ \text{shs}^E(8p,4|0) = \text{ho}(2^{4p-1},2^{4p-1}|4), \quad \text{shs}^E(8p+4,4|0) = \text{husp}(2^{4p+1},2^{4p+1}|4). \] (80)

The appearance of 8 on the left and right hand sides of this isomorphism is a manifestation of triality of $so(8)$: in $\text{shs}^E(8,4|0)$ 8 stands for the vector representation of $so(8)$ while in $\text{ho}(8;8|4)$ two its spinor representations are referred to.

Two comments are now in order.

The construction can be generalized to infinite-dimensional algebras $M$. In particular, one can consider the higher spin superalgebras $h\ldots(n;m|4)$ with $n \to \infty$ or/and $m \to \infty$, that will lead to theories with infinite numbers of massless particles of every spin. Such theories may be of interest in the context of spontaneous breakdown of higher spin gauge symmetries and a relationship with string theory.

The analysis of extended symmetries in the higher spin theories elucidates their deep parallelism with the string theory and, in particular, with the Chan-Paton structure of inner symmetries. The lesson is that higher spin gauge theories are based on the associative structure rather than on the Lie structure as lower spin theories like Yang-Mills and (super)gravity. In this respect, the $\text{husu}(n,m|4)$ higher spin theories are similar to the oriented open superstring theories while the $\text{ho}(n,m|4)$ and $\text{husp}(n,m|4)$ higher spin theories singled out with the help of antiautomorphisms are analogous to non-oriented open string theories. Remarkably, all higher spin theories contain even spins and, in particular, gravity. From that perspective higher spin theories necessarily contain sectors to be associated with closed superstring.

### 3.4 Towards $d > 4$

The challenging problem is to find higher spin algebras for higher dimensions $d > 4$. At the moment we neither know a complete structure of the higher spin conserved currents nor a structure of the higher spin symmetry superalgebras in arbitrary $d$.

Algebraically, a straightforward generalization of the definition of $d3$ and $d4$ higher spin algebras first conjectured in $^{24}$ seems to be a most natural can-

\footnote{To avoid misunderstandings let us note that unfortunately the authors of $^{49}$ used the notation $\text{shs}^E(8,4)$ for the algebra called $\text{shs}^E(8,4|0)$ in $^{28,29}$ and in this paper.}
didate for higher spin algebras in higher dimensions. It consists of introducing spinors in \(d\) dimensions, \(\hat{y}_\mu\) and \(\hat{\bar{y}}^\nu\) \((\mu, \nu = 1 \ldots \left\lceil \frac{d}{2} \right\rceil)\) satisfying the commutation relations
\[
[\hat{y}_\mu, \hat{\bar{y}}^\nu] = \delta^\nu_\mu \tag{82}
\]
with possible identification
\[
\hat{y}_\mu = \hat{\bar{y}}^\nu C_{\nu\mu} \tag{83}
\]
if antisymmetric charge conjugation matrix \(C_{\nu\mu}\) exists (otherwise one has to require \([\hat{y}_\mu, \hat{y}_\nu] = 0\) and \([\hat{\bar{y}}^\mu, \hat{\bar{y}}^\nu] = 0\) and/or imposing appropriate chirality conditions. The basis of the infinite-dimensional Weyl algebra is formed by monomials
\[
T_{\mu_1 \ldots \mu_n \nu_1 \ldots \nu_m} = \hat{y}_{\mu_1} \ldots \hat{y}_{\mu_n} \hat{\bar{y}}^{\nu_1} \ldots \hat{\bar{y}}^{\nu_m}. \tag{84}
\]
The generators
\[
T_{\mu\nu} = \{\hat{y}_\mu, \hat{\bar{y}}^{\nu}\} \tag{85}
\]
form a finite-dimensional subalgebra \(u(2^\lceil d/2 \rceil)\) (or \(sp(2^\lceil d/2 \rceil)\)) in the Majorana case) which contains standard AdS algebra \(o(d-1,2)\) as the subalgebra spanned by the elements
\[
P_n = \hat{y}_\mu \gamma_n^{\mu\nu} \hat{\bar{y}}^{\nu}, \quad M_{mn} = \hat{y}_\mu [\gamma_m, \gamma_n]^{\nu\mu} \hat{\bar{y}}^{\nu}, \tag{86}
\]
where \(\gamma_n\) denotes Dirac matrices in \(d\) dimensions.

It is an interesting algebraic problem to analyze the field content of the higher spin gauge fields originating from this construction in higher dimensions. To the best of our knowledge a solution of the related problem of decomposition of arbitrary symmetric tensor products of spinor representation into irreducible representations of the Lorentz group has not been yet elaborated for arbitrary \(d\). The situation in lower dimensions is simplified by the isomorphisms of the AdS algebras with the algebras of bilinears of oscillators, \(sp(4) \sim o(3,2)\) and \(sp(2) \oplus sp(2) \sim o(2,2)\).

### 4 Star-Product

In practice, instead of working with the algebras of operators as discussed in the section 3, it is convenient to use usual functions endowed with the product law
\[
(f \ast g)(y) = \frac{1}{(2\pi)^{2p}} \int d^{2p}u d^{2p}v \exp(iu_{\mu}v^{\mu})f(y + u)g(y + v). \tag{87}
\]
Here \(f(y)\) and \(g(y)\) are functions (polynomials or formal power series) of commuting variables \(y_{\mu}\) where \(\mu = 1 \ldots 2p\). This formula defines the associative
algebra with the defining relation

\[ y_\mu \ast y_\nu - y_\nu \ast y_\mu = 2iC_{\mu\nu}, \]  

(88)

where \( C_{\mu\nu} \) is the symplectic form used to raise and lower indices,

\[ u^\mu = C^{\mu\nu}u_\nu, \quad u_\mu = u_\nu C_{\nu\mu}. \]  

(89)

The star-product defined this way describes the product of Weyl ordered (i.e. totally symmetric) polynomials of oscillators in terms of symbols of operators \(^50,51\). Thus, this construction gives a particular realization of the Weyl algebra \( A_p \).

Usual differential versions \(^52,50,51\) can be derived from (87) by elementary Gaussian integration of the Taylor expansion \( f(y) = \exp y^\mu \frac{\partial}{\partial z^\mu} f(z)|_{z=0} \),

\[ A(y) \ast B(y) = e^{i\frac{\theta}{\hbar} \cdot \frac{\vartheta}{\pi}} A(y + y^1)B(y + y^2)|_{y^1 = y^2 = 0}. \]  

(90)

This Weyl product law (often called Moyal bracket for commutators constructed from (90)) is obviously nonlocal. This is of course the ordinary quantum-mechanical nonlocality. Note that the integral formula (87) is sometimes called triangle formula \(^51\) because, for two-component spinors, the term \( u_\mu u^\mu \) in the exponential is equal to the (oriented) area of the triangle with vertices at \( y \), \( y + u \) and \( y + v \). In most cases the triangle formula (87) is more convenient than (90) for practical computations and has a broader area of applicability beyond the class of polynomial functions.

An important property of the star-product is that it admits a uniquely defined supertrace operation \(^28\)

\[ \text{str}(f(y)) = f(0) \]  

(91)

possessing the standard property

\[ \text{str}(f \ast g) = (-1)^{\pi(f)\pi(g)} \text{str}(g \ast f) \]  

(92)

with the parity definition

\[ f(-y) = (-1)^{\pi(f)} f(y). \]  

(93)

This is in accordance with the normal spin-statistics relation once \( \mu \) and \( \nu \) are interpreted as spinor indices. Note that the additional sign factor originates from antisymmetry of the matrix \( C_{\mu\nu} \) leading to

\[ u_\mu u^\mu = -v_\mu u^\mu. \]  

(94)
To prove (92) one takes into account that \( \text{str}(f \ast g) = 0 \) if \( \pi(f) \neq \pi(g) \) because \( \text{str}(h) = 0 \) if \( \pi(h) = 1 \).

The fact of existence of the supertrace operation is very important for the theory of higher spin gauge fields because it allows one to build invariants of the higher spin transformations

\[
\delta A = A \ast \epsilon - \epsilon \ast A
\]

(95)
as supertraces of products \( \text{str}(A \ast B \ldots) \) provided that the “fields” \( A, B \ldots \) have appropriate Grassmann grading for fermions.

The star-product algebras (87) can be interpreted as algebras of differential operators with polynomial coefficients (often identified with \( W_{1+\infty} \) algebras) by setting

\[
y_1 = 2i \frac{\partial}{\partial z}, \quad y_2 = z
\]

(96)
(for the case \( p = 1 \)). From this perspective it may look surprising that the algebra of differential operators with polynomial coefficients admits the uniquely defined supertrace and no usual trace operation.

The star-product formulae (87) and (90) are very handy for practical computations with oscillators. Unfortunately, no useful analog of these formulae is known for the deformed oscillators (37).

For the description of the nonlinear higher spin dynamics we will also need another associative star-product defined on the space of functions of two symplectic (spinor) variables

\[
(f \ast g)(z; y) = \frac{1}{(2\pi)^{2p}} \int d^{2p} u d^{2p} v \exp \left[ iw^\mu v^\nu C_{\mu\nu} \right] f(z + u; y + u)g(z - v; y + v),
\]

(97)
where \( w^\mu \) and \( v^\mu \) are real integration variables. It is a simple exercise with Gaussian integrals to see that this star-product is associative

\[
(f \ast (g \ast h)) = ((f \ast g) \ast h)
\]

(98)
and is normalized such that 1 is a unit element of the star-product algebra, i.e. \( f \ast 1 = 1 \ast f = f \).

The star-product (97) again yields a particular realization of the Weyl algebra

\[
[y_{\mu}, y_{\nu}]_* = -[z_{\mu}, z_{\nu}]_* = 2i C_{\mu\nu}, \quad [y_{\mu}, z_{\nu}]_* = 0
\]

(99)
([a, b]_* = a \ast b - b \ast a). These commutation relations are particular cases of the following simple formulae

\[
[y_{\mu}, f]_* = 2i \frac{\partial f}{\partial y^\mu},
\]

(100)
\[ [z_\mu, f]_* = -2i \frac{\partial f}{\partial z_\mu}, \]  

which are true for an arbitrary \( f(z, y) \).

The star-product (97) corresponds to the normal ordering of the Weyl algebra with respect to the generating elements

\[ a_\mu^+ = \frac{1}{2}(y_\mu - z_\mu), \quad a_\mu = \frac{1}{2}(y_\mu + z_\mu), \]  

(102)

which satisfy the commutation relations

\[ [a_\mu, a_\nu]_* = [a_\mu^+, a_\nu^+]_* = 0, \quad [a_\mu, a_\nu^+]_* = iC_{\mu\nu} \]  

(103)

and can be interpreted as creation and annihilation operators. This is most evident from the relations

\[ a_\mu^+ * f(a^+, a) = a_\mu^+ f(a^+, a), \quad f(a^+, a) * a_\mu = f(a^+, a) a_\mu. \]  

(104)

Star-product (97) admits the supertrace operation

\[ \text{str}(f(z, y)) = \frac{1}{(2\pi)^2} \int d^2p u d^2p v \exp(-iu_\mu v^\mu) f(u, v) \]  

(105)

which can be obtained from the Weyl supertrace operation (91) (for the doubled number of variables) by changing the ordering prescription. Obviously,

\[ \text{str}(f(z, y)) = \text{str}(f(-z, -y)), \]  

(106)

i.e. fermions have zero supertrace once the boson-fermion grading is defined in accordance with the standard relationship between spin and statistics

\[ f(-z, -y) = (-1)^{\tau(f)} f(z, y) \]  

(107)

(for \( y_\mu \) and \( z_\mu \) interpreted as spinors). Then it is elementary to see that (92) holds.

An important property of the star-product (97) is that it admits the inner Klein operator

\[ \Upsilon = \exp iz_\mu y^\mu, \]  

(108)

which behaves as \((-1)^{N_f}\), where \( N_f \) is the fermion number operator. It is a simple exercise to show that

\[ \Upsilon * \Upsilon = 1, \]  

(109)

\[ \Upsilon * f(z; y) = f(-z; -y) * \Upsilon, \]  

(110)
and
\[(\Upsilon * f)(z; y) = \exp iz_{\alpha} y^{\alpha} f(y; z).\] (111)
From (111) we see that, up to the exponential factor, \(\Upsilon\) interchanges the arguments \(z\) and \(y\). For the \(d=4\) problem we will need the left and right inner Klein operators
\[\upsilon = \exp iz_{\alpha} y^{\alpha}, \quad \bar{\upsilon} = \exp i\bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}},\] (112)
which act analogously on the undotted and dotted spinors, respectively:
\[(\upsilon * f)(z, \bar{z}; y, \bar{y}) = \exp iz_{\alpha} y^{\alpha} f(y, \bar{z}; z, \bar{y}),\] (113)
\[\upsilon * f(z, \bar{z}; y, \bar{y}) = f(-z, \bar{z}; -y, \bar{y}) * \upsilon, \quad \bar{\upsilon} * f(z, \bar{z}; y, \bar{y}) = f(z, \bar{z}; y, \bar{y}) * \bar{\upsilon},\] (114)
\[\upsilon * \upsilon = \bar{\upsilon} * \bar{\upsilon} = 1.\] (115)

The star-product (97) is regular: given two polynomials \(f\) and \(g\), \(f * g\) is also some polynomial. Particular star-products corresponding to one or another ordering prescription are of course equivalent in the class of polynomials but may be inequivalent beyond this class. The reason is that reordering of infinite series may lead to divergent coefficients (e.g., to an infinite “vacuum energy” constant term). Moreover, it is not a priori guaranteed that star-product of non-polynomial functions is well defined. The reader can easily construct examples of functions \(f\) and \(g\) such that \(f * g\) will diverge because the bilinear form in the Gaussian integral will degenerate. The special property of the star-product (97) is that it is defined in a certain class of nonpolynomial functions\(^{13,34}\) containing the nonpolynomial Klein operators \(\Upsilon, \upsilon\) and \(\bar{\upsilon}\) in the sense that the product is still associative and no infinities appear in this class. This is not the case for the Weyl ordered star-product in the doubled spinor space where it is not clear how to define the Klein operators \(\Upsilon, \upsilon\) and \(\bar{\upsilon}\) (at least analogous exponentials are ill-defined in the algebra). Since the Klein operators play important role in the construction, the star-product (97) acquires a distinguished role. It is of course not surprising that the star-product associated with the certain normal ordering (104) has better properties with respect to potential divergencies. In fact, as shown in\(^{13,34}\), the star-product (97) is well defined for a class of regular functions which appears in the process of solution of the nonlinear constraints in the section 8. This guarantees that the non-linear higher spin equations are well-defined.

From (97) it follows that functions \(f(y)\) independent of \(z\) form a proper subalgebra. Due to (101) this subalgebra identifies with the centralizer of the elements \(z_{\nu}\). Note that for \(z\)-independent functions the star-product (97) reduces to the Weyl star-product (87). This is why we use the same symbol * for the both product laws.
5 AdS Vacua

AdS background plays distinguished role in the higher spin theories. It appears naturally in the framework of the higher spin algebras as a particular solution of the zero-curvature equation

\[ dw = w \wedge w. \]  

(116)

(From now on we only consider the cases of \( d = 3 \) and \( d = 4 \).)

Any vacuum solution \( w_0 \) of the equation (116) breaks the local higher spin symmetry to its stability subalgebra with the infinitesimal parameters \( \epsilon_0(y|x) \) satisfying the equation

\[ D_0 \epsilon_0 = d \epsilon_0 - w_0 \ast \epsilon_0 + \epsilon_0 \ast w_0 = 0. \]  

(117)

The consistency of the equation (117) is guaranteed by the vacuum equation (116). As a result, (117) admits a unique solution in some neighborhood of an arbitrary point \( x_0 \) with the initial data

\[ \epsilon_0(y|x_0) = \epsilon_0(y), \]  

(118)

where \( \epsilon_0(y) \) is an arbitrary \( x \)-independent element of the Weyl algebra.

In the higher spin theories no further symmetry breaking is induced by other field equations. Therefore, \( \epsilon_0(y) \) parametrizes the global symmetry of the theory, the higher spin global symmetry. As a result, we conclude that global symmetry higher spin algebras identify with the Lie superalgebras constructed from the (anti)commutators of the elements of the Weyl algebra and their extensions with the Klein operators and matrix indices. Note that fields carrying odd numbers of spinor fields are anticommuting thus inducing a structure of superalgebra into (116).

Since functions bilinear in \( y_\alpha \) form a closed subalgebra with respect to commutators it is a consistent ansatz to look for a solution of the vacuum equation (116) in the form

\[ w_0 = \frac{1}{8i} \left( \omega_0^{\alpha\beta}(x) \{ y_\alpha, y_\beta \} + \lambda h_0^{\alpha\beta}(x) \psi \{ y_\alpha, y_\beta \} \right) \]  

(119)

for \( d = 3 \) and

\[ w_0 = \frac{1}{8i} \left( \omega_0^{\alpha\beta}(x) \{ y_\alpha, y_\beta \} + \omega_0^{\dot{\alpha}\dot{\beta}}(x) \{ \bar{y}_\dot{\alpha}, \bar{y}_\dot{\beta} \} + 2\lambda h_0^{\alpha\beta}(x) \{ y_\alpha, \bar{y}_\dot{\beta} \} \right) \]  

(120)

for \( d=4 \), respectively. Here \( \{ a, b \} = a \ast b + b \ast a \). For the star-product (87) we have

\[ \{ y_\alpha, y_\beta \} = 2y_\alpha y_\beta, \quad \{ \bar{y}_\dot{\alpha}, \bar{y}_\dot{\beta} \} = 2\bar{y}_\dot{\alpha} \bar{y}_\dot{\beta}, \quad \{ y_\alpha, \bar{y}_\dot{\beta} \} = 2y_\alpha \bar{y}_\dot{\beta}. \]  

(121)
Inserting these formulae into (116) one finds that the fields \( \omega_0(\bar{\omega}_0) \) and \( h_0 \) identify with the Lorentz connection and the frame field of AdS\(_3\) or AdS\(_4\), respectively, provided that the 1-form \( h_0 \) is invertible. The parameter \( \lambda = r^{-1} \) is identified with the inverse AdS radius. Thus, the fact that the higher spin algebras are star-product (oscillator) algebras leads to the AdS geometry as a natural vacuum solution.

The vacuum equation (116) has a form of zero-curvature equation and therefore admits a pure gauge solution

\[
\omega_0 = -g^{-1}(y|x) \star dg(y|x) \tag{122}
\]

with some invertible element \( g(y|x) \) of the Weyl algebra \( g \star g^{-1} = g^{-1} \star g = I \).

The equation (117) then solves as

\[
\epsilon_0(y|x) = g^{-1}(y|x) \star \epsilon_0(y) \star g(y|x). \tag{123}
\]

(Clearly, (118) is true for a point \( x_0 \) such that \( g(y|x_0) = I \).) This fact was known long ago but unless recently it was not used for practical computation. In this presentation we use it for the analysis of the higher spin dynamics in AdS\(_4\) following the recent work of Kirill Bolotin and the author \(^{53}\).

A particular solution of the vacuum equation (116) corresponding to the stereographic coordinates has a form

\[
h_\nu^{\alpha\beta} = -z^{-1} \sigma_\nu^{\alpha\beta}, \tag{124}
\]

\[
\omega_\alpha^{\alpha\alpha} = -\lambda^2 z^{-1} \sigma_\alpha^{\alpha\beta} x_\alpha^{\beta}, \tag{125}
\]

\[
\bar{\omega}_\nu^{\beta\beta} = -\lambda^2 z^{-1} \sigma_\nu^{\alpha\beta} x_\alpha^{\beta}, \tag{126}
\]

where we use notation

\[
x^{\alpha\beta} = x^n \sigma_\alpha^{\alpha\beta}, \quad x^2 = \frac{1}{2} x^{\alpha\beta} x_\alpha^{\beta}, \quad z = 1 + \lambda^2 x^2. \tag{127}
\]

Let us note that \( z \to 1 \) in the flat limit and \( z \to 0 \) at the boundary of AdS.

The form of the gauge function \( g \) reproducing these vacuum background fields (with all \( s \neq 2 \) fields vanishing) turns out to be remarkably simple \(^{53}\)

\[
g(y, \bar{y}|x) = 2 \sqrt{z} \frac{1}{1 + \sqrt{z}} \exp\left[\frac{-i\lambda}{1 + \sqrt{z}} x^{\alpha\beta} y_\alpha \bar{y}_\beta\right] \tag{128}
\]

with the inverse

\[
g^{-1}(y, \bar{y}|x) = 2 \sqrt{z} \frac{1}{1 + \sqrt{z}} \exp\left[\frac{i\lambda}{1 + \sqrt{z}} x^{\alpha\beta} y_\alpha \bar{y}_\beta\right]. \tag{129}
\]
As shown in the section 6.1, when solving relativistic field equations, \( g \) plays a role of a sort of evolution operator. From this perspective \( \lambda \) is analogous to the inverse of the Planck constant \( \bar{h} \),

\[
\lambda \sim \bar{h}^{-1}.
\]

This parallelism indicates that the flat limit \( \lambda \to 0 \) may be essentially singular.

Let us draw attention to an important difference between the \( d=3 \) and \( d=4 \) cases. The ansatz (119) in the \( d=3 \) case solves the vacuum equation not only for the usual oscillators \( y_\alpha \) but also for the deformed oscillators (37)

\[
w_0 = \frac{1}{8i} \left( \omega^{\alpha\beta}(x) \{ \hat{y}_\alpha(\nu), \hat{y}_\beta(\nu) \} + \lambda h^{\alpha\beta}(x) \psi \{ \hat{y}_\alpha(\nu), \hat{y}_\beta(\nu) \} \right).
\]

(131)

The zero-curvature vacuum equation still implies that the fields \( \omega^{\alpha\beta}(x) = d\omega_\alpha^{\alpha\beta}(x) \) and \( h^{\alpha\beta}(x) = d\omega_\alpha^{\alpha\beta}(x) \) identify with the Lorentz connection and dreibein of \( \text{AdS}_3 \) (under condition that \( h_\alpha^{\alpha\beta}(x) \) is non-degenerate). The properties of the deformed oscillator algebra guarantee that this is true for any value of the parameter \( \nu \), i.e. the differential equations for \( \omega_0^{\alpha\beta} \) and \( h_0^{\alpha\beta} \) which follow from (116) are \( \nu \)-independent.

If we try to proceed similarly in the \( d=4 \) case with

\[
w_0 = \frac{1}{8i} \left( \omega^{\alpha\beta}(x) \{ \hat{y}_\alpha(\nu), \hat{y}_\beta(\nu) \} + \bar{\omega}^{\dot{\alpha}\dot{\beta}}(x) \{ \hat{\bar{y}}_{\dot{\alpha}}(\bar{\nu}), \hat{\bar{y}}_{\dot{\beta}}(\bar{\nu}) \} + 2\lambda h^{\dot{\alpha}\dot{\beta}}(x) \{ \hat{y}_\alpha(\nu), \hat{y}_\beta(\nu) \} \right),
\]

(132)

where \( \hat{y}_\alpha(\nu) \) and \( \hat{\bar{y}}_{\dot{\beta}}(\bar{\nu}) \) are two mutually commuting sets of deformed oscillators, the result would be that for \( \nu = \bar{\nu} = 0 \) the zero-curvature vacuum equation indeed describes \( \text{AdS}_4 \) while for \( \nu \neq 0 \) and/or \( \bar{\nu} \neq 0 \) (116) becomes inconsistent, i.e. it admits no solution with the ansatz (132). The replacement of the ordinary oscillators (54) by the deformed oscillators (37) breaks \( sp(4) \) down to its Lorentz subalgebra \( sl_2(\mathbb{C}) \) spanned by the generators \( L_{\alpha\beta} \) and \( \bar{L}_{\dot{\alpha}\dot{\beta}} \). Commutators of AdS translations \( P_{\alpha\beta} \) for non-zero parameters \( \nu \) and \( \bar{\nu} \) give rise to higher and higher polynomials in the deformed oscillator algebra. To construct some consistent vacuum solution for that case one has to introduce infinitely many nonvanishing vacuum higher spin field components in the \( d=4 \) analog of the expansion (48). This fact will have important consequences for the analysis of the section 9.

6 Free Equations

The structure of higher spin currents suggests that higher spin symmetries relate higher derivatives of relativistic fields. Once spin \( s > 2 \) symmetries
are present, the algebra becomes infinite-dimensional. Therefore, higher spin symmetries relate derivatives of physical fields of all orders. To have a natural linear realization of the higher spin symmetries, it is useful to introduce infinite multiplets reach enough to contain dynamical fields along with all their higher derivatives. Such multiplets admit a natural realization in terms of the Weyl algebra.

Namely, in $d=3$ and $d=4$, 0-forms

$$C(Y|x) = \sum_{n=0}^{\infty} \frac{1}{n!} C^{\nu_1,...,\nu_n} Y_{\nu_1} \ldots Y_{\nu_n}$$

(133)

taking their values in the Weyl algebra form multiplets of the higher spin symmetries for lower spin matter fields and Weyl-type higher spin curvature tensors (in $d = 4$). The free equations of motion have a form

$$D_0 C \equiv \left( dC - w_0 \ast C + C \ast \tilde{w}_0 \right) = 0,$$

(134)

where

$$\tilde{f}(y, \bar{y}) = f(y, -\bar{y})$$

(135)

for $d=4$ and

$$\tilde{f}(y, \psi) = f(y, -\psi)$$

(136)

for $d=3$.

In the both cases, tilde denotes an involutive automorphism of the higher spin algebra which changes a sign of the AdS translations (34) and (53). As a result, the covariant derivative $D_0$ corresponds to some representation of the higher spin algebra which we call twisted representation. The consistency of the equation (134) is guaranteed by the vacuum equation (116).

In this formulation, called in $d=4$ “unfolded formulation”, dynamical field equations have a form of covariant constancy conditions. The fact that the equation (134) is invariant under the global higher spin symmetries

$$\delta C = \epsilon_0 \ast C - C \ast \tilde{\epsilon}_0$$

(137)

with the parameters satisfying (117) is obvious. Moreover, one can write down a general solution of the free field equations (134) in the pure gauge form

$$C(Y|x) = g^{-1}(Y|x) \ast C_0(Y) \ast \tilde{g}(Y|x)$$

(138)

analogous to the form of the global symmetry parameters (123), where $C_0(Y)$ is an arbitrary $x$–independent element of the twisted representation with $Y = (y, \bar{y})$ for $d=4$ and $Y = (y, \psi)$ for $d=3$. Here $C_0(Y)$ plays a role of initial data. Let us now explain in more detail a physical content of the equations (134) starting with the $d=4$ case.
In d=4, by virtue of the star-product (87) the system (134) reduces to
\[ D_0 C(y, \bar{y}|x) \equiv D^L C(y, \bar{y}|x) + i\lambda h^{\alpha\beta} \left( y_\alpha \bar{y}_\beta - \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \bar{y}^\beta} \right) C(y, \bar{y}|x) = 0, \quad (139) \]
where
\[ D^L_0 C(y, \bar{y}|x) = dC(y, \bar{y}|x) - \left( \omega^{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta} + \bar{\omega}^{\alpha\beta} \bar{y}_\alpha \frac{\partial}{\partial \bar{y}^\beta} \right) C(y, \bar{y}|x). \quad (140) \]
Rewriting (133) as
\[ C(Y|x) = C(y, \bar{y}|x) = \sum_{n,m=0}^{\infty} \frac{1}{m!n!} C^{\alpha_1\ldots\alpha_n, \dot{\alpha}_1\ldots\dot{\alpha}_m}(x)y_{\alpha_1} \ldots y_{\alpha_n} \bar{y}_{\dot{\alpha}_1} \ldots \bar{y}_{\dot{\alpha}_m}, \quad (141) \]
one arrives at the following infinite chain of equations
\[ D^L C_{(m), \dot{\beta}(n)} = i\lambda h^{\gamma\delta} C_{(m), \beta(n)} - i\nu m \lambda h_{\alpha\beta} C_{(m-1), \beta(n)}, \quad (142) \]
where \( D^L \) is the Lorentz-covariant differential
\[ D^L A_{\alpha\beta} = dA_{\alpha\beta} + \omega_\gamma A_{\gamma\beta} + \bar{\omega}_\dot{\delta} \wedge A_{\alpha\dot{\delta}}. \quad (143) \]
Here we skip the subscript 0 referring to the vacuum AdS solution and use again the convention introduced in the section 2 with symmetrized indices denoted by the same letter and a number of symmetrized indices indicated in brackets.

The system (142) decomposes into a set of independent subsystems with \( n - m \) fixed. It turns out that the subsystem with \(|n - m| = 2s\) describes a massless field of spin \( s \) (note that the fields \( C_{(m), \dot{\beta}(n)} \) and \( C_{\beta(n), \alpha(m)} \) are complex conjugated).

It is instructive to consider the example of \( s = 0 \) associated with the fields \( C_{\alpha(n), \dot{\beta}(n)} \). The equation (142) at \( n = m = 0 \) expresses the field \( C_{\alpha, \dot{\beta}} \) via the first derivative of \( C \),
\[ C_{\alpha, \dot{\beta}} = \frac{1}{2\lambda} h^{\alpha\beta} D^L C, \quad (144) \]
where \( h^{\alpha\beta} \) is the inverse frame field
\[ h_{\alpha\beta} h^{\gamma\delta} = 2\delta_\gamma^\delta \delta_\beta^\alpha, \quad g^{nm} = \frac{1}{2} h_{\alpha\beta} h^{mn\alpha\beta} \quad (145) \]
with the normalization chosen in such a way that it is true for $h_{\alpha \beta} = \sigma_{\alpha \beta}$ and $h_{\alpha \beta} = \sigma_{\alpha \beta}$.

The second equation with $n = m = 1$ contains more information. First, one obtains by contracting indices with the frame field and using (145) that

$$h_{\alpha \beta} (D^L_{\alpha \bar{\beta}} + 8i\lambda C) = 0.$$  \hspace{1cm} (146)

With the aid of (144) this reduces to the Klein-Gordon equation in $AdS_4$

$$\Box C - 8\lambda^2 C = 0.$$  \hspace{1cm} (147)

The rest part of the equation (142) with $n = m = 1$ expresses the field $C_{\alpha \beta \gamma \delta}$ via second derivatives of $C$

$$C_{\alpha \beta \gamma \delta} = \frac{1}{(2i\lambda)^2} h_{\alpha \beta} D^L_{\alpha \beta} D^L_{\gamma \delta} C.$$  \hspace{1cm} (148)

All other equations with $n = m > 1$ either reduce to identities by virtue of the spin 0 dynamical equation (147) or express higher components in the chain of fields $C_{\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_n}$ via higher derivatives in the space-time coordinates as

$$C_{\alpha(n) \beta(n)} = \frac{1}{(2i\lambda)^n} h_{\alpha \beta} D^L_{\alpha \beta} \ldots D^L_{\gamma \delta} C.$$  \hspace{1cm} (149)

This completes the proof of the fact that the system (142) with $n = m$ describes a scalar field. The value of the mass parameter in (147) is such that $C$ describes massless scalar in $AdS_4$.

Spin 1/2 is described by the mutually conjugated chains of fields $C_{\alpha(m), \beta(n)}$ with $|n - m| = 1$. In this case the first equation with $n = 0$ and $m = 1$ has a form

$$D^L_{\alpha} C_{\alpha} = i\lambda \gamma^\delta C_{\alpha \gamma, \delta}.$$  \hspace{1cm} (150)

Dirac equation is a simple consequence of this equation,

$$h_{\alpha \beta} D^L_{\alpha \beta} C = 0.$$  \hspace{1cm} (151)

All the rest equations again do not impose any further restriction on the dynamical field $C_{\alpha}$ just expressing higher members of the chain via higher space-time derivatives of $C_{\alpha}$. The fact that, although overdetermined, the system (142) is consistent takes place because in (134) $(D_0)^2 = 0$ as a consequence of (116).

Analogously, the equations (142) with other values of $n$ and $m$ describe free field equations for spin $s = |n - m|$ massless fields. However, for spins
s ≥ 1 it is more useful to treat these equations not as fundamental ones but as consequences of the higher spin equations formulated in terms of gauge fields (potentials). To illustrate this point let us first consider the example of gravity.

As argued in section 3, Lorentz connection 1–forms \( \omega_{\alpha\beta} \), \( \tilde{\omega}_{\dot{\alpha}\dot{\beta}} \) and vierbein 1–form \( h_{\alpha\beta} \) can be identified with the \( sp(4) \)–gauge fields. The corresponding \( sp(4) \)–curvatures read in terms of two-component spinors

\[
R_{\alpha_1\alpha_2} = d\omega_{\alpha_1\alpha_2} + \omega_{\alpha_1} \gamma \wedge \omega_{\alpha_2} \gamma + \lambda^2 h_{\alpha_1} \, \delta \wedge h_{\alpha_2} \delta, \quad (152)
\]

\[
\tilde{R}_{\dot{\alpha}_1\dot{\alpha}_2} = d\tilde{\omega}_{\dot{\alpha}_1\dot{\alpha}_2} + \tilde{\omega}_{\dot{\alpha}_1} \dot{\gamma} \wedge \tilde{\omega}_{\dot{\alpha}_2} \dot{\gamma} + \lambda^2 h_{\dot{\gamma}_1} \dot{\wedge} h_{\gamma_2} \gamma, \quad (153)
\]

\[
r_{\alpha\beta} = dh_{\alpha\beta} + \omega_{\alpha} \gamma \wedge h_{\gamma\beta} + \tilde{\omega}_{\beta} \dot{\delta} \wedge h_{\alpha\delta}. \quad (154)
\]

The zero-torsion condition \( r_{\alpha\dot{\beta}} = 0 \) expresses the Lorentz connection \( \omega \) and \( \tilde{\omega} \) via derivatives of \( h \). After that, the \( \lambda \)–independent part of the curvature 2–forms \( R \) (152) and \( \tilde{R} \) (153) coincides with the Riemann tensor. Einstein equations imply that the Ricci tensor vanishes up to a constant trace part proportional to the cosmological constant. This is equivalent to saying that only those components of the tensors (152) and (153) are allowed to be nonvanishing which belong to the Weyl tensor. As is well-known \(^{55}\), Weyl tensor is described by the fourth-rank mutually conjugated totally symmetric multispinors \( C_{\alpha_1\alpha_2\alpha_3\alpha_4} \) and \( \tilde{C}_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4} \). Therefore, Einstein equations with the cosmological term can be cast into the form

\[
r_{\alpha\dot{\beta}} = 0, \quad (155)
\]

\[
R_{\alpha_1\alpha_2} = h_{\gamma_1} \delta \wedge h_{\gamma_2} \delta C_{\alpha_1\alpha_2\gamma_1\gamma_2}, \quad \tilde{R}_{\dot{\beta}_1\dot{\beta}_2} = h_{\dot{\gamma}_1} \dot{\delta}_2 \tilde{C}_{\dot{\beta}_1\dot{\beta}_2\dot{\gamma}_1\dot{\delta}_2}. \quad (156)
\]

It is useful to treat the 0–forms \( C_{\alpha(4)} \) and \( \tilde{C}_{\dot{\alpha}(4)} \) on the right hand sides of (156) as independent field variables which identify with the Weyl tensor by virtue of the equations (156). From (156) it follows that the 0–forms \( C_{\alpha(4)} \) and \( \tilde{C}_{\dot{\alpha}(4)} \) should obey certain differential restrictions as a consequence of the Bianchi identities for the curvatures \( R \) and \( \tilde{R} \). It is not difficult to make sure that these differential restrictions can be equivalently rewritten in the form

\[
D^L C_{\alpha(4)} = i \lambda h^{\gamma\delta} C_{\alpha(4)\gamma,\delta}, \quad D^L \tilde{C}_{\dot{\beta}(4)} = i \lambda h^{\gamma\delta} \tilde{C}_{\gamma,\dot{\beta}(4)\delta}. \quad (157)
\]

where \( C_{\alpha(5),\dot{\delta}} \) and \( \tilde{C}_{\gamma,\dot{\beta}(5)} \) are new multispinor field variables totally symmetric in the spinor indices of each type. (The factor of \( i \lambda \) is introduced for future convenience.)

Once again, Bianchi identities for the left hand sides of (157) impose certain differential restrictions on \( C_{\alpha(5),\dot{\delta}} \) and \( \tilde{C}_{\gamma,\dot{\beta}(5)} \) which can be cast into the form
expression for the linearized curvatures following from this definition is

\[ D^L C_{\alpha(n+4),\beta(n)} = i\lambda(h^{\gamma\delta}C_{\alpha(n+4)\gamma,\beta(n)\delta} - n(n+4)h_{\alpha\beta}C_{\alpha(n+3),\beta(n-1)}) + O(C^2), \]

\[ D^L \bar{C}_{\alpha(n),\beta(n+4)} = i\lambda(h^{\gamma\delta}\bar{C}_{\alpha(n)\gamma,\beta(n+4)\delta} - n(n+4)h_{\alpha\beta}\bar{C}_{\alpha(n-1),\beta(n+3)}) + O(C^2), \]

where \( O(C^2) \) denotes nonlinear terms to be discarded in the linearized approximation we are interested in. All these relations contain no new dynamical information in addition to that contained in the original Einstein equations in the form (155), (156). Analogously to the spin 0 case, (158) and (159) merely express highest 0-forms \( C_{\alpha(n+4),\beta(n)} \) and \( \bar{C}_{\alpha(n),\beta(n+4)} \) via derivatives of the lowest 0-forms \( C_{\alpha(4)} \) and \( \bar{C}_{\beta(4)} \) containing at the same time all consistency conditions for (156) and the equations (158), (159) themselves. Thus, the system of equations (155), (156), (158) and (159) turns out to be dynamically equivalent to the Einstein equations with the cosmological term.

As shown in \(^{19,46}\) this construction extends to all spins \( s \geq 1 \). The linearized higher spin equations read

\[ R_{1,\alpha(n),\beta(m)} = \delta(m)h^{\gamma\delta} \wedge h^{\gamma\delta}C_{\alpha(n)\gamma(2)} + \delta(n)h^{\gamma\delta} \wedge h_{\gamma\delta} \bar{C}_{\beta(m)\delta(2)}, \]

(\( \delta(n) = \delta^0_n \)) plus the equations (142). Here the curvatures \( R_{1,\alpha(n),\beta(m)} \) are the components of the linearized higher spin curvature tensor

\[ R_1(y, \bar{y} \mid x) \equiv dw(y, \bar{y} \mid x) - w_0(y, \bar{y} \mid x) \ast w(y, \bar{y} \mid x) + w(y, \bar{y} \mid x) \ast w_0(y, \bar{y} \mid x) = \sum_{n,m=0}^{\infty} \frac{1}{2k n!m!} y_{\alpha_1} \ldots y_{\alpha_n} \bar{y}_{\beta_1} \ldots \bar{y}_{\beta_m} R_{1,\alpha_1 \ldots \alpha_n,\beta_1 \ldots \beta_m}(x), \]

where \( w_0 \) denotes the \( AdS_4 \) background fields (120). One obtains

\[ R_1(y, \bar{y} \mid x) = D^L w(y, \bar{y} \mid x) - \lambda h^{\alpha\beta}\left(y_{\alpha} \frac{\partial}{\partial y_{\beta}} + \frac{\partial}{\partial y_{\alpha}} \bar{y}_{\beta}\right)w(y, \bar{y} \mid x), \]

where the Lorentz covariant derivative is defined in (140). The component expression for the linearized curvatures following from this definition is

\[ R_{1,\alpha(n),\beta(m)} = D^L w_{\alpha(n),\beta(m)} + n\lambda h^{\alpha\beta} \wedge w_{\alpha(n-1),\beta(m+1)} + m\lambda h^{\gamma\delta} \wedge w_{\gamma\alpha(n),\beta(m+1)} \]

(163)
For spins \( s \geq 3/2 \) the equations (142), like in the case of gravity, do not contain any independent dynamical information just expressing the highest multispinors \( C_{\alpha(n)}, \dot{\beta}(m) \) via derivatives of the generalized higher spin Weyl tensors defined through (160),

\[
C_{\alpha(n)}, \dot{\beta}(m) = \frac{1}{(2i\lambda)^{\frac{n}{2}(n+m-2s)}} h^{\frac{n}{2}(n+m-2s)} D^L_{\alpha_1} \ldots h^{\frac{n}{2}(n+m-2s)} D^L_{\alpha_n} C_{\alpha(2s)} \quad n \geq m, \tag{164}
\]

or

\[
C_{\alpha(n)}, \dot{\beta}(m) = \frac{1}{(2i\lambda)^{\frac{n}{2}(n+m-2s)}} h^{\frac{n}{2}(n+m-2s)} D^L_{\alpha_1} \ldots h^{\frac{n}{2}(n+m-2s)} D^L_{\alpha_n} C_{\beta(2s)} \quad n \leq m. \tag{165}
\]

For \( s = 1 \) the equation (160) is just the definition of the field strengths \( C_{\alpha(2)} \) and \( C_{\dot{\beta}(2)} \) while the equations (142) contain Maxwell equations in the bottom part of the chain\(^{46} \). For spins 0 and 1/2, the system (142) is not linked to the equations for the gauge potentials (160).

Thus, it is shown that the free equations of motion for all massless fields in \( AdS_4 \) can be cast into the form

\[
R_1(y, \bar{y}|x) = h^{\gamma \dot{\beta}} \wedge h^{\dot{\gamma} \dot{\alpha}} \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \bar{y}^\beta} C(0, \bar{y}|x) + h^{\alpha \dot{\gamma}} \wedge h^{\dot{\gamma} \dot{\beta}} \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \bar{y}^\beta} C(y, 0|x), \tag{166}
\]

\[
\mathcal{D}_0 C(Y|x) = 0. \tag{167}
\]

This statement, which plays a key role from various points of view, will be referred to as Central On-Mass-Shell Theorem.

The infinite set of the 0–forms \( C \) forms a basis in the space of all on-mass-shell nontrivial combinations of covariant derivatives of matter fields and (lower and higher spin) curvatures.

A spin \( s \geq 1 \) dynamical massless field is identified with the 1-form (potential)

\[
w_{\alpha(n), \dot{\beta}(n)} \quad n = (s - 1), \quad s \geq 1 \quad \text{integer}, \tag{168}
\]

\[
w_{\alpha(n), \dot{\beta}(m)} \quad n + m = 2(s - 1), \quad |n - m| = 1 \quad s \geq 3/2 \quad \text{half-integer}. \tag{169}
\]

The matter fields are described by the 0-forms

\[
C_{\alpha(0), \dot{\alpha}(0)} \quad s = 0, \tag{170}
\]

\[
C_{\alpha(1), \dot{\alpha}(0)} \oplus C_{\alpha(0), \dot{\alpha}(1)} \quad s = 1/2. \tag{171}
\]
Eqs. (166) and (167) contain the free dynamical equations for all massless fields and express all auxiliary components via higher derivatives of the dynamical fields by virtue of (164) and (165) for the 0-forms and by analogous formulae having a structure

$$w_{\alpha(\mathbf{n}),\dot{\beta}(\mathbf{m})} \sim \left(\lambda^{-1} \frac{\partial}{\partial x}\right)^{\mathbf{n}-\mathbf{m}} w^{\text{phys}} + w^{\text{gauge}}$$  \hspace{2cm} (172)

for the gauge 1-forms, where $w^{\text{phys}}$ denotes some field from the list (168), (169) while $w^{\text{gauge}}$ is a pure gauge part.

Let us stress that the equations (166) and (167) are equivalent to the usual free higher spin equations in the AdS space which follow from the standard actions proposed in. In addition they link together derivatives in the space-time coordinates $x^\mu$ and in the auxiliary spinor variables $y_\alpha$ and $\bar{y}_{\dot{\alpha}}$. In accordance with (164) and (165), in the sector of 0-forms the derivatives in the auxiliary spinor variables can be viewed as a square root of the space-time derivatives,

$$\frac{\partial}{\partial x^\mu} C(y, \bar{y}|x) \sim \lambda h_{\alpha\dot{\beta}} \partial \frac{\partial}{\partial y_\alpha} \partial \frac{\partial}{\partial \bar{y}_{\dot{\beta}}} C(y, \bar{y}|x).$$  \hspace{2cm} (173)

(This is most obvious from (139).) Analogous formula in the sector of higher spin gauge potentials reads in accordance with (162) and (166)

$$\frac{\partial}{\partial x^\mu} w(y, \bar{y}|x) \sim \lambda h_{\alpha\dot{\beta}} \left(\partial \frac{\partial}{\partial y_\alpha} \bar{y}_{\dot{\beta}} w(y, \bar{y}|x) + y_\alpha \partial \frac{\partial}{\partial \bar{y}_{\dot{\beta}}} w(y, \bar{y}|x)\right).$$  \hspace{2cm} (174)

As a result, any nonlocality in the auxiliary variables $y$ may imply a space-time nonlocality.

As argued in the section 4 the associative star-product acting on the auxiliary spinor variables is nonlocal, thus indicating a potential nonlocality in the space-time sense. The higher spin equations contain star-products via terms $C(Y|x) * X(Y|x)$ with some operators $X$ constructed from the gauge and matter fields. Once $X(Y|x)$ is at most quadratic in the auxiliary variables $Y^\nu$, the resulting expressions are local, containing at most two derivatives in $Y^\nu$. This is the case for the AdS background gravitational fields and therefore, in agreement with the analysis of this section, the higher spin dynamics is local at the linearized level. But this may easily be not the case beyond the linearized approximation. We will illustrate this issue in terms of the integration flow defined in the section 9.

Another important consequence of the formulae (173) and (174) is that they contain explicitly the inverse AdS radius $\lambda$ and become meaningless in
the flat limit $\lambda \to 0$. This happens because, when resolving these equations for the derivatives in the auxiliary variables $y$ and $\bar{y}$, the space-time derivatives appear in the combination

$$\lambda^{-1} \frac{\partial}{\partial x^2}$$

that leads to the inverse powers of $\lambda$ in front of the terms with higher derivatives in the higher spin gauge interactions. This is the main reason why higher spin interactions require the cosmological constant to be nonzero as was first concluded in $^{11}$.

To summarize, the following facts are strongly correlated:
(i) higher spin algebras are described by the (Moyal) star-product in the auxiliary spinor space
(ii) relevance of the AdS background
(iii) potential space-time nonlocality of the higher spin interactions due to the appearance of higher derivatives at the nonlinear level.

These properties are in many respects analogous to the superstring picture with the deep parallelism between the cosmological constant and the string tension parameter. The fact that unbroken higher spin symmetries require AdS geometry may provide an explanation why the most symmetric higher spin phase is not seen in the usual superstring picture with the flat background space-time. Interestingly, recent insight into the structure of superstring theory in $^5$ proves that star-product plays a key role in certain regimes.

6.2 “Plane Waves” in $\text{AdS}_4$

The fact that $C(Y|x)$ describes all derivatives of the physical fields compatible with the field equations allows us to solve the dynamical equations in the form (138). The arbitrary parameters $C_0(Y)$ in (138) describe all higher derivatives of the field $C(Y|x_0)$ at the point $x_0$ with $g(Y|x_0) = I$. In other words, (138) describes a covariantized Taylor expansion in some neighborhood of $x_0$. For the gauge function (128), $x_0 = 0$. Let us now illustrate how the formula (138) can be used to produce explicit solutions of the higher spin equations in $\text{AdS}_4$.

Let us set

$$C_0(Y) = \exp i(y^\alpha \eta_\alpha + \bar{y}^{\dot{\alpha}} \bar{\eta}_{\dot{\alpha}}),$$

where $\eta_\alpha$ is an arbitrary commuting complex spinor and $\bar{\eta}_{\dot{\alpha}}$ is its complex conjugate. Taking into account that

$$\tilde{g}^{-1}(Y|x) = g(Y|x),$$

39
inserting $g(Y|x)$ into (138) and using the product law (87) one performs elementary Gaussian integrations to obtain\textsuperscript{53}

$$C(Y|x) = z^2 \exp i \left[ -\lambda(y_\alpha \bar{y}_\beta + \eta_\alpha \bar{\eta}_\beta)x^{\alpha\beta} + z(y_\alpha \eta_\alpha + \bar{y}_\alpha \bar{\eta}_\alpha) \right],$$

(178)

where $z = 1 + \lambda^2 \frac{1}{2} x^{\alpha\beta} x_{\alpha\beta}$. Using

$$C_{\alpha_1 \ldots \alpha_n}(x) = \left. \frac{\partial}{\partial y^{\alpha_1}} \cdots \frac{\partial}{\partial y^{\alpha_n}} C(y, \bar{y}|x) \right|_{y=\bar{y}=0},$$

(179)

one obtains for the matter fields and higher spin Weyl tensors

$$C_{\alpha_1 \ldots \alpha_2 s}(x) = z^{2(s+1)} \eta_{\alpha_1} \ldots \eta_{\alpha_2 s} \exp ik_{\gamma\beta} x^{\gamma\beta},$$

(180)

where

$$k_{\alpha\beta} = -\lambda \eta_{\alpha} \bar{\eta}_{\beta}$$

(181)

is a null vector expressed in the standard way in terms of spinors. (Expressions for the conjugated Weyl tensors carrying dotted indices are analogous).

Since $z \to 1$ in the flat limit, the obtained solution indeed describes plane waves in the flat space limit $\lambda \to 0$ provided that the parameters $\eta_\alpha$ and $\bar{\eta}_\alpha$ are rescaled according to

$$\eta_\alpha \to \lambda^{-1/2} \bar{\eta}_\alpha, \quad \bar{\eta}_\alpha \to \lambda^{-1/2} \bar{\eta}_\alpha,$$

(182)

On the other hand, $z \to 0$ at the boundary of $AdS_4$ and therefore the constructed AdS plane waves tend to zero at the boundary.

This approach is very efficient and can be applied to produce explicit solutions of the field equations in many cases as we hope to demonstrate elsewhere. So far we only focused on the equation (167) for 0-forms $C$ which has a form of the covariant constancy condition and therefore admits an explicit solution (138). Interestingly enough, although the equation (163) does not have a form of a zero-curvature equation, it also can be solved in a rather explicit algebraic way\textsuperscript{53} using a more sophisticated technics explained in the section 9 and inspired by the analysis of the nonlinear higher spin dynamics.

6.3 $AdS_3$

The d=3 linearized system is much simpler then the d=4 one because d3 “higher spin” fields are of Chern-Simons type and do not propagate analogously to the case of d3 gravity\textsuperscript{43}. Equivalent statement is that d3 higher spin fields do not admit nonzero Weyl tensors. In fact, the name “higher spin gauge fields”
is misleading for d=3 because these gauge fields do not carry any degrees of freedom and therefore do not describe any spin. Higher spin gauge symmetries are however nontrivial.

Consequently, the d3 Central On-Mass-Shell Theorem has a form

\[ \mathcal{R}_1(\hat{y}, \psi, k|x) = 0, \quad \mathcal{D}_0 C(\hat{y}, \psi, k|x) = 0, \]  

(183)

where \( \mathcal{R}_1 \) is the linearized part of the d3 curvature tensor (49) and \( \mathcal{D}_0 \) is the covariant derivative (134). As shown in 42, in the sector of 0-forms, (183) describes four massive scalars \( C(\hat{y}, \psi, k|x) = C(-\hat{y}, \psi, k|x) \) and four massive spinors \( C(\hat{y}, \psi, k|x) = -C(-\hat{y}, \psi, k|x) \) arranged into N=2 d3 hypermultiplets. The values of mass \( M \) are expressed in terms of \( \lambda \) and \( \nu \) as follows

\[ M^2 = \lambda^2 \frac{\nu(\nu \mp 2)}{2} \]  

(184)

for bosons, and

\[ M^2 = \lambda^2 \frac{\nu^2}{2} \]  

(185)

for fermions. The signs “±” correspond to the projections

\[ C^\pm = P^\pm C, \quad P^\pm = \frac{1 \pm k}{2}. \]  

(186)

One doubling of a number of fields of the same mass is due to the dependence on \( \psi (\psi^2 = 1) \) while another one, with the mass splitting in the bosonic sector, is due to \( k \). As expected, the flat “higher spin” connections do not describe any local degrees of freedom. The property that the values of masses depend on a free parameter \( \nu \) in the deformed oscillator algebra (37) is quite different from what happens in d=4 where only massless matter fields appear because they all belong to the same multiplet with massless higher spin gauge fields.

The component form of the covariant constancy conditions (183) with appropriately rescaled component fields amounts to

\[ D^\alpha C_{\alpha(n)} = \lambda \left( h^{\beta \gamma} C_{\beta \gamma \alpha(n)} + n(n - 1) \left( \frac{1}{4} - \frac{M^2}{2\lambda^2(n^2 - 1)} \right) h_{\alpha \alpha} C_{\alpha(n-2)} \right) \]  

(187)

for a boson \( (n \) is even), and

\[ D^\beta C_{\alpha(n)} = \lambda \left( h^{\gamma \alpha} C_{\beta \gamma \alpha(n-1)} - \frac{\sqrt{2}M}{\lambda(n+2)} h_{\alpha \beta} C_{\beta \alpha(n-1)} ight. \]  

\[ + \left. n(n - 1) \left( \frac{1}{4} - \frac{M^2}{2\lambda^2 n^2} \right) h_{\alpha \alpha} C_{\alpha(n-2)} \right) \]  

(188)
for a fermion (n is odd).

Note that analogously to the d4 formulae (164) and (165) one gets
\[ C_{\alpha(n)} = \frac{1}{(\lambda^2)^{(n-2s)}} \frac{m_{\alpha m}}{m_{\alpha}} D^L_{\alpha m} \cdots \frac{m_{\alpha(n-2s)}}{m_{\alpha(n-2s)}} D^L_{\alpha m(n-2s)} C_{\alpha(2s)}, \quad s = 0 \text{ or } 1/2. \]

(189)

As in d=4, this means that
\[ \frac{\partial}{\partial x^{\alpha}} C(y|x) \sim \lambda h_{\alpha \beta} \frac{\partial}{\partial y^{\alpha}} \frac{\partial}{\partial y^{\beta}} C(y|x). \]

(190)

The formula (138) is true for any value of \( \nu \) in the d3 case. However it is not straightforward to apply it in the massive d3 case because no practically useful formula is known generalizing (87) to the case of arbitrary \( \nu \). The problem of developing an efficient machinery of the symbols of operators for the case of general \( \nu \) is therefore quite interesting.

### 7 Free Differential Algebras and Unfolded Formulation

Let us consider an arbitrary set of differential \( p \)-forms \( W^A(x) \) with \( p \geq 0 \) (0-forms are included). Let the generalized curvatures \( R^A \) be defined by the relations
\[ R^A = dW^A + F^A(W), \]
where \( d = dx^{\alpha} \frac{\partial}{\partial x^{\alpha}} \) and \( F^A \) are some functions of \( W^B \) built with the aid of the exterior product of differential forms. Given function \( F^A(W) \) satisfying the generalized Jacobi identity
\[ F^B \frac{\delta F^A}{\delta W^B} \equiv 0 \]
(192)
(the derivative with respect to \( W \) is left), we say following to\(^56\) that it defines a free differential algebra. This property guarantees the generalized Bianchi identity
\[ dR^A = R^B \frac{\delta F^A}{\delta W^B}, \]
(193)
which tells us that the differential equations on \( W^A \)
\[ R^A = 0 \]
(194)
are consistent. Clearly, the requirement that the equation (194) is consistent for generic fields \( W^A \) is equivalent to (192).

\(^c\)In this section, the identification of indices may be different from the conventions summarized in the Appendix.
The property (192) allows one to define the gauge transformations
\[ \delta W^A = d \epsilon^A - \epsilon^B \frac{\delta F^A}{\delta W^B}, \] (195)
where \( \epsilon^A(x) \) is a \((\deg(W^A) - 1)\)-form (0-forms do not give rise to any gauge parameters). With respect to these gauge transformations the generalized curvatures transform as
\[ \delta R^A = -R^C \frac{\delta}{\delta W^C} \left( \epsilon^B \frac{\delta F^A}{\delta W^B} \right). \] (196)
This implies gauge invariance of the equations (194). Also, since the equations (194) are formulated entirely in terms of differential forms, they are explicitly general coordinate invariant.

For the particular case when the set \( W^A \) consists of only 1-forms \( w^i \), the function \( F^i \) is bilinear
\[ F^i = f_{jk}^i w^j \wedge w^k \] (197)
and the relation (192) amounts to the usual Jacobi identity for a Lie algebra \( g \) with the structure coefficients \( f_{jk}^i \) (or superalgebra if some of \( w^i \) carry an additional Grassmann grading). The equation (194) is then the zero-curvature equation for \( g \).

If the set \( W^A \) also contains some \( p \)-forms \( C^\alpha \) (e.g. 0-forms) and the functions \( F^\alpha \) are linear in \( C \)
\[ F^\alpha = t^\alpha_{\beta} w^\beta \wedge C^\beta, \] (198)
the relation (192) implies that the matrices \( t^\alpha_{\beta} \) form some representation \( t \) of \( g \) while the equations (194) contain zero-curvature equations of \( g \) along with the covariant constancy equation \( DC = 0 \) for the representation \( t \).

We see that the vacuum equations (116) \( R = 0 \) and free equations (134) \( D_0 C = 0 \) are just of this form. Therefore, the fields \( C \) span some representation of the AdS algebra. Moreover, since the equations (116) and (134) are formally consistent independently of a particular solution \( w_0 \), the 0-forms \( C \) form some representation of the whole infinite-dimensional higher spin algebra. This is the twisted representation defined by (137).

This simple observation suggests the following strategy for the analysis of the higher spin theories (in fact, any dynamical system). Starting from a space-time with some symmetry algebra \( s \) and vacuum gravitational gauge fields (1-forms) \( w_0 \) taking values in \( s \) and satisfying the zero curvature equations \( dw_0 = w_0 \wedge w_0 \), one reformulates field equations of a given free dynamical system in the “unfolded form” \( D_0 C = 0 \). This can always be done in principle.
and the only question is how simple is the explicit expression for $D_0 C$. Indeed, one starts writing

$$D_L C_0^i = h_{m n} C_1^{i, n} \tag{199}$$

for a given dynamical field $C_0^i(x)$ with some set of spinor and/or vector indices $i$. Next, one checks whether the original field equations impose any restrictions on the first derivatives of $C_0^i$. If they do, as it happens for fermions, one expresses the corresponding components of $C_1^{i, n}$ in terms of $C_0^i$ treating the unrestricted part $\tilde{C}_1^{i, n}$ of $C_1^{i, n}$ as new independent fields parametrizing on-mass-shell nonvanishing components of first derivatives. This leads to the equation

$$D_L C_0^i = h_{m n} \tilde{C}_1^{i, n} + A_{1}^{i, n}(C_0) \tag{200}$$

Then one writes

$$D_L \tilde{C}_1^{i, n} = h_{m n} C_2^{i, n, m} \tag{201}$$

where $C_2^{i, n, m}$ parametrizes the second derivatives. Once again one checks, taking into account the Bianchi identities for (200), which components $\tilde{C}_2^{i, n, m}$ of the second level fields remain independent provided that the original equations of motion are true, expressing the rest of the components of $C_2^{i, n, m}$ in terms of the lower derivatives $C_0^i$ and $\tilde{C}_1^{i, n}$. This process continues infinitely leading to a chain of equations having a form of some covariant constancy conditions. By construction, the resulting set of fields $C_2^{i, n, m}$ realizes some representation of the space-time symmetry algebra $s$ (e.g. Poincare or AdS). The interactions are then described by nonlinear deformations of the resulting free differential algebra.

It is useful to address the question which infinite-dimensional extension of $s$ can act on thus derived representation $t$ of $s$. A natural candidate is a Lie superalgebra $g$ constructed via (anti)commutators from the associative algebra $H$

$$H = U(s)/I(t) \tag{202}$$

where $U(s)$ is the universal enveloping of $s$ while $I(t)$ is the ideal of $U(s)$ spanned by the elements which trivialize on the representation $t$. Of course, this strategy is too naive in general because a set of fields $C$ mixed by a higher spin algebra and compatible with the nonlinear higher spin dynamics may take values in a larger representation. In any case, $U(s)$ is the reasonable starting point to look for a higher spin algebra. Based on somewhat different arguments, this idea was put forward by Fradkin and Linetsky in 37. As shown in the section 3.1 it works in d=3 at least for the case without inner symmetries. Similar interpretation of the Weyl algebra (endowed with Klein operators)
underlying the d=4 higher spin superalgebra can be given in terms of $U(sp(4))$ in the bosonic case and $U(osp(1, 4))$ in the supersymmetric case.

The language of free differential algebras is perfectly adequate for the study of interactions of higher spin theories. The Central On-Mass-Shell Theorem is just a right starting point to attack this problem which reduces to searching for such a deformation of the equations (166) and (167)

$$R(w) = F_2(w, C), \quad D C = F_1(w, C)$$

(203)

with a 2-form $F_2$ and 1-form $F_1$, that is consistent in the sense of (192) and reproduces the linearized equations in the lowest order of the expansion in powers of $C$ and $w_1$ identified with the fluctuational part in $w = w_0 + w_1$. From (166) it is clear that the nonlinear deformation is inevitable in $d \geq 4$ because

$$D^2(C) \sim R C \sim C^2$$

(204)

and therefore $F_1$ necessarily starts from some terms bilinear in $C$. In its turn, this may induce some further nonlinear correction to $F_2$ as a result of the differentiation of $C$ in (166).

It is not a priori guaranteed that some deformation $F_{1,2}$ exists at all. If not, this would mean that no consistent nonlinear higher spin equations exist. (Note that dynamics of any consistent system can in principle be rewritten in the unfolded form.) Once the complete form of $F_1$ and $F_2$ is found, the problem is solved because the resulting equations are formally consistent, gauge invariant and generally coordinate invariant as a consequence of the general properties of free differential algebras. By construction, it describes the correct dynamics at the free field level.

Let us stress that one can proceed analogously for other dynamical systems containing gravity either at the dynamical level (i.e. with the zero-curvature equation for the gravitational fields deformed by the Weyl tensor) or at the background level (i.e. with the background gravitational field satisfying the vacuum zero-curvature equation). The “unfolded formulation” has much in common with the Penrose “exact sets of fields” formulation. In this approach some infinite systems of equations also appear, containing both dynamical equations and the relations expressing higher components via higher derivatives of the dynamical fields. The important difference is that the procedure we use (“unfolding”) is formulated in terms of differential forms (gauge potentials)

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\[ ^d \text{This is a consequence of the fact that Weyl algebra acts on the Fock space defined with respect to creation and annihilation operators built from } y_\alpha \text{ and } \bar{y}_\beta \text{ and identified with the metaplectic (singleton) representation of } sp(4|R) \text{ (for more detail see } 29 \text{ and references therein).} \]
containing, in particular, the gravitational field. As a result, our formulation brings together such seemingly different issues as general coordinate invariance, gauge invariance and formal compatibility of the dynamical equations. The common feature of the two approaches is that all higher derivatives compatible with the field equations at any point \( x_0 \) of the space-time are identified with values of certain fields at \( x_0 \), the fields \( C(Y|x_0) \) in the higher spin case.

Coming back to the higher spin problem, let us note that there was one more requirement imposed on the form of the higher spin free differential algebra. It was assumed that the gravitational Lorentz connection \( \omega_{\alpha\beta} \) and \( \bar{\omega}_{\dot{\alpha}\dot{\beta}} \) only appear via Lorentz covariant derivatives (and their commutator in the Riemann tensor). Equivalently, the deformations \( F_{1,2}(w, C) \) were required to be independent of the Lorentz connection. From (195) it follows that this requirement means that the transformation law for local Lorentz rotations remains undeformed. This requirement is natural and restrictive as explained in the section 8.4. The only other symmetry that remains undeformed in the extended higher spin systems is the spin-1 Yang-Mills symmetry. All other symmetries deform, acquiring some \( C \)-dependent corrections to their transformation laws. In particular, this happens with the AdS translations. It is not surprising, since it is known from the SUGRA example that local AdS translations acquire some curvature-dependent corrections which deform them to diffeomorphisms.

This general problem setting was studied for d4 higher spin theories in a series of papers (see e.g. 46,57) within the order by order analysis in powers of the 0-forms \( C \). This perturbation expansion is natural because the 0-forms \( C \) describe deviations of matter fields and higher spin Weyl tensors from their vacuum values \( C = 0 \), i.e. this is just a weak coupling expansion. In every order, a contribution of the 1-forms \( w \) was taken into account completely. Of course, the important difference between 0-forms \( C \) and 1-forms \( w \) is that the latter can appear at most quadratically in this approach while the former can appear in arbitrary power so that the functions \( F_{1,2} \) may be nonpolynomial in \( C \). The concrete form of the functions \( F_{1,2} \) is quite cumbersome and can hardly be written down explicitly in all orders. However, in 13,14 (and references therein) it was suggested that these functions can be described as solutions of certain rather simple nonlinear equations in additional spinor variables which can be proved to have a unique solution modulo field redefinitions, thus, among other things, proving the fact of existence of the higher spin free differential algebra in all orders in interactions. Skipping technical details of the derivation of these equations we instead formulate in the section 8 the final result and show how it reproduces the higher spin free differential algebra.

Let us note that although we know the closed equations for higher spins,
for the case of pure gravity or Yang-Mills theory an explicit form of all terms nonlinear in $C$ on the right hand sides of (158) and (159) is still unknown. The form of $C^2$--type terms was obtained for the case of gravity in $^{57}$. For the (anti)selfdual case these $C^2$ terms were shown to be complete.

The conclusion is that higher spin systems are in some sense simpler than the lower spin systems used in the standard low energy theories. In fact, the lower spin systems like Einstein, Yang-Mills, and others are unlikely to be subsystems of the full higher spin gauge theories, which can be obtained by a consistent truncation. Indeed, lower spin fields form sources for higher spin fields via higher spin currents analogous to those considered in the section 2. To single out the lower spin subsystems one has to implement some limiting procedure, a kind of low-energy expansion. A most natural possibility to achieve this is via spontaneous breaking of the higher spin symmetries down to some lower spin symmetries. The symmetry breaking parameter equal to the inverse higher spin mass scale should then be identified with an expansion parameter reminiscent of $\alpha'$ in superstring theory. Of course, some of the nice properties of the original theory with unbroken higher spin gauge symmetries may be lost in the resulting low-energy lower spin system.

8 Nonlinear Higher-Spin Equations

In this section we resolve the problem of reconstruction of the $d=3$ and $d=4$ higher spin free differential algebra in all orders by formulating some closed consistent system in an appropriately extended space. The resulting formulation, based on certain non-commutative Yang-Mills fields, is interesting on its own right and, in particular, in the context of the recent developments in the superstring theory $^{4,5}$. The content of this section is based on the papers $^{13,14,33,34,59}$.

8.1 Doubling of Spinor Variables and Non-Commutative Gauge Theory

The key element of the construction consists of the doubling of auxiliary Majorana spinor variables $Y_{\nu}$ in the higher spin 1-forms and 0-forms

$$w(Y; Q|x) \rightarrow W(Z; Y; Q|x), \quad C(Y; Q|x) \rightarrow B(Z; Y; Q|x)$$

and formulating equations which determine the dependence on the additional variables $Z_{\nu}$ in terms of “initial data”

$$w(Y; Q|x) = W(0; Y; Q|x), \quad C(Y; Q|x) = B(0; Y; Q|x).$$
The variables $Q$ denote some discrete (Clifford) variables which are different for $d = 3$ and $d = 4$ cases and will be specified later.

To this end we introduce a new compensator-type spinor field $S_{\nu}(Z; Y; Q|x)$ which carries only pure gauge degrees of freedom and plays a role of a covariant differential along the additional $Z_{\nu}$ directions. It is convenient to introduce anticommuting $Z-$differentials $dZ^{\nu}dZ^{\mu} = -dZ^{\mu}dZ^{\nu}$ to interpret $S_{\nu}(Z; Y; Q|x)$ as $Z$ 1-forms,

\[ S = dZ^{\nu}S_{\nu} . \]  

(207)

The full system of equations in $d=3$ and $d=4$ has the following form

\[ dW = W * W , \]  

(208)

\[ dB = W * B - B * W , \]  

(209)

\[ dS = W * S - S * W , \]  

(210)

\[ S * S = B * S , \]  

(211)

\[ S * S = dZ^{\nu}dZ^{\mu} \left( -iC_{\nu\mu} + 4R_{\nu\mu}(B; c) \right) , \]  

(212)

where $C_{\nu\mu}$ is the charge conjugation matrix and $R_{\nu\mu}(B; c)$ is certain star-product function of the field $B$ and some central elements $c$ of the algebra. The function $R_{\nu\mu}(B; c)$ encodes all information about the higher spin dynamics and will be specified later.

In the analysis of the higher spin dynamics, a typical vacuum solution for the field $S$ is

\[ S_{0} = dZ^{\nu}Z_{\nu} . \]  

(213)

From (101) it follows then that

\[ [S_{0}, f]_{*} = -2i\partial f , \]  

(214)

where

\[ \partial = dZ^{\nu} \frac{\partial}{\partial Z^{\nu}} . \]  

(215)

Interpreting the deviation of the full field $S$ from the vacuum value $S_{0}$ as a $Z-$component of the gauge field $W$,

\[ S = S_{0} + 2i dZ^{\nu}W_{\nu} , \]  

(216)

one rewrites the equations (208), (210) and (212) as

\[ R = dZ^{\nu}dZ^{\mu} R_{\nu\mu}(B; c) \]  

(217)
and the equations (209) and (211) as

$$DB = 0.$$  

(218)

Here the generalized curvatures and covariant derivative are defined by the relations

$$R = (d + \partial)(d\omega W_n + dZ^\nu W_\nu) - (d\omega W_n + dZ^\nu W_\nu) \wedge (d\omega W_n + dZ^\nu W_\nu),$$  

(219)

$$D(A) = (d + \partial)A - (d\omega W_n + dZ^\nu W_\nu) \ast A + A \ast (d\omega W_n + dZ^\nu W_\nu).$$  

(220)

(It is assumed that $d\omega dZ^\nu = -dZ^\nu d\omega$.) We see that the function $R_{\nu\mu}$ in (212) identifies with the $ZZ$ components of the generalized curvatures, while $xx$ and $xZ$ components of the curvature vanish. The equation (218) means that the curvature $R_{\nu\mu}$ is covariantly constant.

The equations (217) and (218) are consistent. The Bianchi identities for (217) are satisfied as a consequence of (218). The Bianchi identities for (218) are compatible with the equation (217) reducing to the star-commutators of some functions of $B$ and central elements $c$ which vanish due to the simple fact that $F(B; c) \ast G(B; c) = G(B; c) \ast F(B; c)$ for any $F$ and $G$ since $B$ commutes to itself and central elements.

A related statement is that the equations (217) and (218) are gauge invariant under the transformations

$$\delta(x\omega W_n + dZ^\nu W_\nu) = (d + \partial)\epsilon + \epsilon \ast (d\omega W_n + dZ^\nu W_\nu) - (d\omega W_n + dZ^\nu W_\nu) \ast \epsilon,$$  

(221)

$$\delta B = [\epsilon, B]_*,$$  

(222)

with an arbitrary gauge parameter $\epsilon(Z; Y; Q|x)$. In terms of the original variables the gauge transformations (221) have a form

$$\delta W = d\epsilon + [\epsilon, W]_*,$$  

(223)

$$\delta S = [\epsilon, S]_*.$$  

(224)

Note that the gauge transformations for $Z-$components $W_\nu$ of the gauge field acquire the inhomogeneous term due to the vacuum expectation value $S_0$ of $S$.

The consistency of the system of equations (208)-(212) guarantees that it admits a perturbative solution as a system of differential equations with respect to $Z_{\nu}$. This proves that all fields can be expressed modulo gauge transformations in terms of the “initial data” (206) identified with the physical higher spin fields. Inserting thus obtained expressions into (208) and (209) one finds some (nonlinear) corrections to $Z-$independent parts of the higher
spin curvatures and covariant derivatives. (One has to take into account that \((f \ast g)(0; Y)\) is generically different from zero if \(f(0; Y) = 0\) and/or \(g(0; Y) = 0\) because \(Z\)–dependent terms in \(f\) and/or \(g\) contribute to the \(Z\)–independent part of their star-product). By construction, the resulting system of equations will be consistent as a space-time free differential algebra, thus solving the problem. Concrete details of this procedure for \(d=4\) and \(d=3\) are explained in the sections 8.2 and 8.3 following to the original papers \(^{13,14,33,34}\).

Since all components of curvatures and covariant derivatives along space-time directions vanish, this allows one to solve all those equations, which contain space-time derivatives, in the pure gauge form analogous to (122) and (123)

\[
W = -g^{-1}(Z; Y; Q|x) \ast dg(Z; Y; Q|x),
\]

(225)

\[
B(Z; Y; Q|x) = g^{-1}(Z; Y; Q|x) \ast b(Z; Y; Q) \ast g(Z; Y; Q|x),
\]

(226)

\[
S(Z; Y; Q|x) = g^{-1}(Z; Y; Q|x) \ast s(Z; Y; Q) \ast g(Z; Y; Q|x)
\]

(227)

with some invertible \(g(Z; Y; Q|x)\) and arbitrary \(x\)–independent functions \(b(Z; Y; Q)\) and \(s(Z; Y; Q)\). Due to the gauge invariance of the whole system one is left only with the equations (211) and (212) for \(b(Z; Y; Q)\) and \(s(Z; Y; Q)\). These encode in a coordinate independent way all information about the dynamics of massless fields of all spins. From the perspective of the equations (217) and (218) these equations imply some algebraic (via particular form of \(R_{\nu\mu}\)) and differential (\(dZ_\nu D_\nu B = 0\)) constraints on the \(ZZ\) non-commutative star-product field strength built from the potential \(dZ_\nu W_\nu(Z; Y; Q)\).

The global symmetry of the system is identified with the subalgebra of the local symmetry (222)-(224) that leaves invariant a vacuum solution \(W_0, B_0\) and \(S_0\) of the field equations. From (222)-(224) it follows that the global symmetry parameters satisfy

\[
d\varepsilon^{gl} = [W_0, \varepsilon]_s, \quad [\varepsilon^{gl}, B_0]_s = 0, \quad [\varepsilon^{gl}, S_0]_s = 0.
\]

(228)

The first equation is consistent because \(W_0\) satisfies (208) and can be solved as

\[
\varepsilon^{gl}(Z; Y; Q|x) = g_0^{-1}(Z; Y; Q|x) \ast \varepsilon_0^{gl}(Z; Y; Q) \ast g_0(Z; Y; Q|x).
\]

(229)

The second one will be satisfied trivially since we will only consider vacuum solutions with \(B_0 = const\). The third implies that \(\varepsilon_0^{gl}\) belongs to the centralizer of \(s_0\). We therefore conclude that the global symmetry algebra is isomorphic to the centralizer of \(S_0\). (Note that \(s_0 = S_0(x_0)\) if \(g_0(Z; Y; Q|x_0) = 1\); the centralizers of \(S_0(x)\) for different \(x\) are pairwise isomorphic due to (227).)
In our approach, non-commutative gauge fields appear in the auxiliary spinor space associated with the coordinates $Z^\nu$. The dynamics of the higher spin gauge fields is formulated entirely in terms of the corresponding non-commutative gauge curvatures. For the first sight it is very different from the non-commutative Yang-Mills model considered recently in the context of the description of the new phase of string theory, in which star-product is defined directly in terms of the original space-time coordinates $x^\mu$. However, the difference may be not that significant taking into account the relationships like (173) and (174) between space-time and spinor derivatives, which are themselves consequences of the equations (208) and (209) as will become clear in the sections 8.2 and 8.3. From this perspective, the situation with the higher spin equations is reminiscent of the Fedosov quantization approach developed in to solve the problem of quantization of symplectic structures. In this approach the complicated problem of quantization of some (base) manifold (coordinates $x^\mu$) is reduced to a simpler problem of quantization in the fibre endowed with the Weyl star-product structure (analog of coordinates $Z^\nu$). An important difference between the Fedosov’s approach and the structures underlying the higher spin equations is that the former is based on the vector fiber coordinates $Z^\mu$, while the higher spin dynamics prefers spinor coordinates $Z^\nu$ (see also for a discussion of the parallels between the higher spin gauge theory and Fedosov quantization).

A few comments are now in order.

As explained in the sections 8.2 and 8.3, the discrete variables $Q$ sometimes do not commute with the spinor elements $dZ^\nu$ and $Z^\nu$. One therefore has to be careful with the naive interpretation of $\frac{\partial}{\partial Z^\nu}$ and $dZ^\nu$ taking into account some additional sign factors when necessary.

Sometimes, less trivial vacuum solutions with $S_0 \neq dZ^\nu Z^\nu$ turn out to be relevant. In that case, $[S_0, A]_\star \neq \partial A$, and the interpretation in terms of the usual noncommutative Yang-Mills theory becomes less straightforward. Moreover, as shown in the section 8.3, the same equations may admit different vacuum solutions with essentially different vacuum fields $S_0$. From that perspective it is practically more convenient to treat $S \ast S$ as the fundamental object generalizing the non-commutative Yang-Mills strength.

The fact that the equations of motion (208)-(212) are formulated in terms of fields $W$, $B$ and $S_\nu$ taking values in the associative star-product algebra $A$ allows one to extend the construction to the case with inner symmetries by endowing all fields with the matrix indices, i.e. by the extension $A \rightarrow A \otimes Mat_n$. The analysis of possible reductions of the full system is parallel to that of the section 3.3. The perturbative analysis will work equally well provided that non-zero vacuum fields $W^i_{0j}$, $B^i_{0j}$ and $S_{0i}^{i'}$ take values in the singlet
subalgebra $A \otimes I$ spanned by the $n \times n$ unit matrices.

Consistency of the system (208)-(212) is not sufficient alone to fix a form of the curvature $R_{\nu\mu}$. Some additional arguments taking into account that the system should describe appropriate relativistic dynamics have to be added. We will come back to this point in the section 8.4.

8.2 Nonlinear Equations in $d=4$

In the $d=4$ case higher spin dynamics is described by the field variables $W(Z;Y;\mathcal{K}|x)$, $B(Z;Y;\mathcal{K}|x)$ and $S(Z;Y;\mathcal{K}|x)$ depending on the space-time coordinates $x^\mu (\mu = 0 \div 3)$, spinor variables $Z^\nu = (z_\alpha, \bar{z}^{\dot{\alpha}})$, $Y^\nu = (y_\alpha, \bar{y}^{\dot{\alpha}})$ and two Klein operators $\mathcal{K} = (k, \bar{k})$. $S$ is a 1-form with respect to the auxiliary anticommuting spinorial differentials $dz^\alpha$ and $d\bar{z}^{\dot{\alpha}}$,

$$S = s + \bar{s}, \quad s = dz^\alpha s_\alpha(Z;Y;\mathcal{K}|x), \quad \bar{s} = d\bar{z}^{\dot{\alpha}} \bar{s}_{\dot{\alpha}}(Z;Y;\mathcal{K}|x), \quad (230)$$

$$dz^\alpha dz^{\beta} = -d\bar{z}^{\dot{\beta}} d\bar{z}^{\dot{\alpha}}, \quad d\bar{z}^{\dot{\alpha}} d\bar{z}^{\dot{\beta}} = -dz^\alpha dz^{\beta}. \quad (231)$$

By definition, the Klein operators $k$ and $\bar{k}$ anticommute with all undotted and dotted spinors, respectively,

$$k f(Z;Y;dZ;\mathcal{K}) = f(\bar{Z};\bar{Y};d\bar{Z};\mathcal{K}) k, \quad \bar{k} f(Z;Y;dZ;\mathcal{K}) = f(-\bar{Z};-\bar{Y};-d\bar{Z};\mathcal{K}) \bar{k} \quad (232)$$

with

$$\bar{U}_\nu = (-u_\alpha, \bar{u}^{\dot{\alpha}}) \quad \text{for} \quad U_\nu = (u_\alpha, \bar{u}^{\dot{\alpha}}). \quad (233)$$

In addition, it is required that

$$k^2 = \bar{k}^2 = 1, \quad [k, \bar{k}] = 0, \quad [k, dx^\mu] = [\bar{k}, dx^{\dot{\mu}}] = 0; \quad (234)$$

$$\{dx^\mu, dz^\alpha\} = 0, \quad \{dx^{\dot{\mu}}, d\bar{z}^{\dot{\alpha}}\} = 0. \quad (235)$$

In accordance with (232), we assume in this section that the Klein operators $k$ ($\bar{k}$) anticommute with the differentials $dz^\alpha$ ($d\bar{z}^{\dot{\alpha}}$).

The field variables $W$, $B$ and $S$ obey the (anti-)hermiticity (reality) conditions

$$W^\dagger = -W, \quad B^\dagger = B, \quad S^\dagger = -S \quad (236)$$

with the involution $^\dagger$ (i.e., $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$, $(f^\dagger)^\dagger = f$, $(\lambda f + \mu g)^\dagger = \bar{\lambda} f^\dagger + \bar{\mu} g^\dagger$) defined by the relations

$$(z_\alpha)^\dagger = -\bar{z}^{\dot{\alpha}}, \quad (y_\alpha)^\dagger = \bar{y}^{\dot{\alpha}}, \quad (dz_\alpha)^\dagger = -d\bar{z}^{\dot{\alpha}}, \quad (dx^\nu)^\dagger = dx^{\dot{\nu}}, \quad k^\dagger = \bar{k}. \quad (237)$$

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Following to\textsuperscript{14} we fix the curvature on the right hand side of (212) in the form
\[
dd{Z}^{\nu} dZ^{\mu} R_{\nu\mu}(B) = -\frac{i}{4} \left( d\zeta_\alpha d\zeta^\alpha F(B) \ast \kappa + d\bar{\zeta}_{\dot{\alpha}} d\bar{\zeta}^{\dot{\alpha}} \bar{F}(B) \ast \bar{\kappa} \right).
\] (238)

Here \( F(B) \) is an arbitrary star-product function of the 0-form \( B \)
\[
F(B) = \sum_{n=0}^{\infty} \frac{1}{n!} f_n \, \underbrace{B \ast \cdots \ast B}_{n}.
\] (239)

with some complex coefficients \( f_n \). \( \bar{F}(B) \) is its complex conjugate,
\[
\bar{F}(B) = \sum_{n=0}^{\infty} \frac{1}{n!} \bar{f}_n \, \underbrace{B \ast \cdots \ast B}_{n}.
\] (240)

The operators \( \kappa \) and \( \bar{\kappa} \) have the form
\[
\kappa = k \upsilon, \quad \bar{\kappa} = \bar{k} \bar{\upsilon},
\] (241)

where \( \upsilon \) and \( \bar{\upsilon} \) are the inner Klein operators (112).

From (114) and (232) it follows that
\[
\kappa \ast f(Z;Y;K;dZ) = f(Z;Y;K;d\bar{Z}) \ast \kappa, \quad \bar{\kappa} \ast f(Z;Y;K;dZ) = f(Z;Y;K;-d\bar{Z}) \ast \bar{\kappa},
\] (242)
i.e., \( \kappa(\bar{\kappa}) \) commutes with all quantities except for \( d\zeta^\alpha(d\bar{\zeta}^{\dot{\alpha}}) \). As a result, the equation (212) with the curvature tensor (238) acquires a form of two mutually commuting deformed oscillator relations with \( F(B) \) and \( \bar{F}(B) \) playing a role of \( \upsilon \) and \( \bar{\upsilon} \), respectively (note that \( B \) commutes with \( S \) and is covariantly constant according to (211) and (209)).

In order to prove that the system (208)-(212) is consistent, one has to use the fact that the quantities
\[
c_1 = d\zeta_\alpha d\zeta^\alpha, \quad \bar{c}_1 = d\bar{\zeta}_{\dot{\alpha}} d\bar{\zeta}^{\dot{\alpha}}
\] (243)
and
\[
c_2 = d\zeta_\alpha d\zeta^\alpha \kappa, \quad \bar{c}_1 = d\bar{\zeta}_{\dot{\alpha}} d\bar{\zeta}^{\dot{\alpha}} \bar{\kappa}
\] (244)
behave as central elements commuting with all elements of the algebra. As for \( c_1 \) and \( \bar{c}_1 \), this is trivial. For \( c_2 \) and \( \bar{c}_2 \), this is also the case, but now one has to take into account that spinorial indices take just two values. Indeed, due to the factors of \( \kappa \) (\( \bar{\kappa} \)), \( c_2 \) and \( \bar{c}_2 \) anticommutes with all odd functions of \( d\zeta_\alpha \) (\( d\bar{\zeta}_{\dot{\alpha}} \)). However, when spinorial indices take just two values, all these
potentially dangerous terms vanish since \((dz)^3 \equiv 0\), and therefore \(c_2\) and \(\bar{c}_2\) commute with everything. This property encodes the fact that the equations under investigation make sense for two-component spinors thus restricting our consideration to the four-dimensional dynamics.

The equations (208)-(212) are general coordinate invariant (in the \(x\)-space sense) and invariant under the higher spin gauge transformations (222)-(224).

Now, we are in a position to analyze the equations (208)-(212) in the linearized approximation. Let us assume that \(F(0) = 0\) (i.e. \(f_0 = 0\) in (239)). For this case a vacuum solution can be chosen in the form

\[
B_0 = 0, \quad S_0 = dz^\alpha z_\alpha + d\bar{z}^\dot{\alpha} \bar{z}_{\dot{\alpha}}
\]

and

\[
W_0 = \frac{1}{4i} [\alpha_{0}^{\alpha\beta}(x) y_\alpha y_\beta + \bar{\alpha}_{0}^{\dot{\alpha}\dot{\beta}}(x) \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} + 2\lambda h_0^{\dot{\alpha}\dot{\beta}}(x) y_\alpha \bar{y}_{\dot{\beta}}]
\]

with the \(AdS_4\) gravitational vacuum fields discussed in the section 5. This ansatz solves the equations (208)-(212).

In the lowest order, we obtain from (211) and (101) that \(B\) is \(Z\)-independent

\[
B(Z; Y; K|x) = C(Y; K|x) + \text{higher order terms}.
\]

Inserting this into (212) one arrives at the following differential restrictions on the first-order part \(S_1\) of \(S\) :

\[
\partial S_1 = \frac{1}{2} [dz_\alpha dz^\alpha f_1 C(-z, \bar{y}; K) k \exp itz_\alpha y^{\alpha} + d\bar{z}_\dot{\alpha} d\bar{z}^{\dot{\alpha}} \tilde{f}_1 C(y, -\bar{z}; K) \bar{k} \exp it\bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}],
\]

where we have made use of (113), (241) and (242). Using the simple fact that, for two-component spinors, the general solution of the equation

\[
\frac{\partial}{\partial z^\alpha} f^\alpha(z) = g(z)
\]

is

\[
f_\alpha(z) = \frac{\partial}{\partial z^\alpha} \varepsilon(z) + \int_0^1 dt \ t z_\alpha g(tz),
\]

where \(\varepsilon(z)\) is an arbitrary function, one deduces from (249) that

\[
S_1 = \partial \varepsilon_1 + \int_0^1 dt \ t [dz_\alpha z_\alpha f_1 C(-tz, \bar{y}; K) k \exp itz_\alpha y^{\alpha} + d\bar{z}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}} \tilde{f}_1 C(y, -t\bar{z}; K) \bar{k} \exp it\bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}].
\]
The ambiguity in the function \( \varepsilon_1 = \varepsilon_1(Z; Y; K) \) manifests invariance under the gauge transformations (221) in the \( Z \) sector. It is convenient to fix a gauge by requiring \( \partial \varepsilon_1 = 0 \) in (252). This gauge fixing is not complete as it does not fix the gauge transformations with

\[
\varepsilon_1(Z; Y) = \xi_1(Y) + \text{higher order terms}.
\]

Thus, the field \( S \) is entirely expressed in terms of the 0-form \( B \), disappearing as an independent dynamical variable. In this sense, \( S \) can be thought of as a sort of a pure gauge compensator field. This result is not surprising since it means that the noncommutative gauge connection \( S \) is reconstructed in terms of the noncommutative curvature \( F(B) \) uniquely modulo gauge transformations.

Now, let us analyze the equation (210). In the first order, one gets

\[
2i \partial W_1 = dS_1 - [W_0, S_1],
\]

(254)

Using the fact that generic solution of the equation

\[
\frac{\partial}{\partial z^\alpha} \varphi(z) = \chi_\alpha(z)
\]

(255)

has the form

\[
\varphi(z) = \text{const} + \int_0^1 dt z^\alpha \chi_\alpha(tz)
\]

(256)

provided that \( \frac{\partial}{\partial z^\alpha} \chi_\alpha(z) \equiv 0 \) and \( \alpha = 1, 2 \), one finds from (254)

\[
W_1(Z; Y; K) = w(Y; K) + \frac{i}{2} \int_0^1 dt \left(z^\alpha[W_0, s_{1\alpha}](tz, \bar{z}; Y; K) + \bar{z}^\alpha[W_0, \bar{s}_{1\bar{\alpha}}](z, t\bar{z}; Y; K)\right)
\]

(257)

(the terms with \( z^\alpha dz_1 + \bar{z}^{\bar{\alpha}} d\bar{s}_{1\bar{\alpha}} \) vanish due to the formula (252) with \( \partial \varepsilon_1 = 0 \) because \( z^\alpha z_\alpha \equiv \bar{z}^{\bar{\alpha}} \bar{z}_{\bar{\alpha}} \equiv 0 \)). Let us stress that, as shown in the section 8.1, (254) is consistent as the system of differential equations with respect to \( \frac{\partial}{\partial z} \), \( \frac{\partial}{\partial \bar{z}} \) and \( \frac{\partial}{\partial x} \). As a result, it is enough to analyze the equations (208) and (209) at \( Z^\nu = 0 \). For other values of \( Z^\nu \), (208) and (209) will hold automatically provided that the equations (210)-(212) are solved.

Thus, to derive dynamical higher spin equations, one has to insert (257) into (208) and (209) interpreting \( w(Y; K|x) \) and \( C(Y; K|x) \) as generating functions for the higher spin fields. Note that the gauge transformations (253)
preserving the linearized gauge condition $\partial \varepsilon_1 = 0$ identify with the gauge transformations for $w(Y; \mathcal{K}|x)$, the higher spin gauge transformations.

The straightforward analysis of (208) at $Z = 0$, based on (97), (247) and (252) is elementary. The final result is

$$dw(Y; \mathcal{K}) = w^* \wedge w(Y; \mathcal{K})$$

$$+ \frac{i\lambda^2}{8} \left[ f_1 h_{\alpha \beta} k \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \bar{y}^\beta} (C(0, \bar{y}; k, \bar{k}) - C(0, \bar{y}; -k, -\bar{k}))
\right.

$$+ \left. f_1 h_{\alpha \beta} \bar{y}^\gamma \frac{\partial}{\partial \bar{y}^\delta} (C(y, 0; k, \bar{k}) - C(y, 0; -k, -\bar{k}))
\right]

$$- f_1 h_{\alpha \beta} \bar{y}^\gamma \frac{\partial}{\partial y^\delta} (C(0, \bar{y}; k, \bar{k}) + C(0, \bar{y}; -k, -\bar{k}))
\right]

$$+ \text{higher order terms}. \quad (258)$$

Note that all terms with the Lorentz connection $\omega_{0 \alpha \beta}$ and $\bar{\omega}_{0 \alpha \beta\dot{\gamma}}$ in the $C$-dependent part of this formula cancel. This is a particular manifestation of the general property discussed in the section 7 that the higher spin equations are covariant with respect to the local Lorentz transformations.

The equation (209) reduces to

$$dC = W_0 * C - C * W_0 + \text{higher order terms}. \quad (259)$$

The equations (258) and (259) thus describe all dynamical information contained in the linearized system (208)-(212).

To make contact with the section 6.1 let us expand the fields as

$$w(Y; \mathcal{K}|x) = \sum_{A, B = 0, 1} w^{AB}(Y|x)k^A \bar{k}^B, \quad C(Y; \mathcal{K}|x) = \sum_{A, B = 0, 1} \lambda^{-2} C^{AB}(Y|x)k^A \bar{k}^B. \quad (260)$$

Inserting these expressions back into (258) and (259) one finds that the equations for the fields $w^{AA}$ and $C^{A \nrightarrow A}$ reduce exactly to the Central On-Mass-Shell Theorem (166) and (167).

The fields $w^{A \nrightarrow A}$ and $C^{AA}$ are auxiliary and do not describe nontrivial degrees of freedom as was first shown in [58]. Another way to see this is to observe that the set of free equations for $C^{AA}$ decomposes into an infinite set of finite subsets of equations for homogeneous polynomials in spinor variables. Because $C = C^{AA}(x_0)$ plays a role of initial data in the pure gauge solution of the covariant constancy conditions we conclude that each of the subsystems contained in $C^{AA}$ describes at most a finite number of degrees of freedom.
Moreover, because the $AdS_4$ algebra $o(2, 3)$ is noncompact, it does not admit finite-dimensional unitary representations and, therefore, these degrees of freedom should not appear in a unitary theory (ruled out by appropriate boundary conditions). As a result, $C^{AA}$ describes some topological fields which carry no degrees of freedom in the unitary case. Let us note that the equations for $C^{A1-A}$ decompose into infinite subsystems each identified with the equations for some dynamical fields.

The auxiliary fields can be truncated away from the full system by the discrete symmetry

$$
\tau(W(Z; Y, k, \bar{k}|x)) = W(Z; Y, -k, -\bar{k}|x), \quad \tau(S(Z; Y, k, \bar{k}|x)) = S(Z; Y, -k, -\bar{k}|x),
$$

$$
\tau(B(Z; Y, k, \bar{k}|x)) = -B(Z; Y, -k, -\bar{k}|x)
$$

which takes place at least if $F(B)$ is odd. We therefore do not discuss the auxiliary fields here in more detail (the only comment is that the appropriate sector of the equations (258) and (259) just reproduces the analog of the Central-On-Mass-Shell theorem for the sector of auxiliary fields).

Let us note that the operators $k$ and $\bar{k}$ that appear explicitly in the equations via (238) flip a type of the representation linking gauge fields (1-forms) in the adjoint representation of the higher spin algebra to the $0$-forms $C$ in the twisted representation and vice versa. We see that every massless spin appears twice due to the Klein operator $k\bar{k}$ giving rise to the labels $A$ for the higher spin gauge fields $w^{AA}$. It is impossible to get rid of $k\bar{k}$ by any field redefinition. (In the purely bosonic case however the operator $k\bar{k}$ belongs to the center of the algebra and therefore can be replaced by a constant.) Note that in the recent paper $^6$ it was shown that some particle models formulated in terms of twistor variables exhibit analogous spectra of spins of massless particles. It is an interesting problem to establish some direct connection between these superparticle models and higher spin algebras.

Taking into account higher-order terms one can in principle reconstruct all higher-order corrections to the higher spin equations. Thus the system (208)-(212) indeed solves the problem of reconstruction of the nonlinear higher spin free differential algebra discussed in the section 7. Note that the nonlinear terms in the expansion (239) contribute only to the nonlinear corrections in the resulting free differential algebra and therefore describe some ambiguity in the higher spin interactions.

8.3 Nonlinear Equations in $d=3$

The full nonlinear $d = 2 + 1$ system is formulated in terms of the functions $W(z; y, \psi_{1,2}, k, \rho|x)$, $B(z; y, \psi_{1,2}, k, \rho|x)$, and $S_\alpha(z; y, \psi_{1,2}, k, \rho|x)$ that depend
on the space-time coordinates \( x_n (n = 0, 1, 2) \), auxiliary spinors \( z_\alpha, y_\alpha (\alpha = 1, 2) \), \([z_\alpha, z_\beta] = [z_\alpha, y_\beta] = 0\), a pair of Clifford elements \( \{\psi_i, \psi_j\} = 2\delta_{ij} (i = 1, 2) \) that commute to all other generating elements, and another pair of Clifford-type elements \( k \) and \( \rho \) which have the following properties

\[
k^2 = 1, \rho^2 = 1, k\rho + \rho k = 0, ky_\alpha = -y_\alpha k, kz_\alpha = -z_\alpha k, \rho y_\alpha = y_\alpha \rho, \rho z_\alpha = z_\alpha \rho.
\]

The field variables obey the (anti-)hermiticity (reality) conditions

\[
W^\dagger = -W, \quad B^\dagger = B, \quad S_\alpha^\dagger = -S_\alpha
\]

with the involution \( \dagger \) defined by the relations

\[
(z_\alpha)^\dagger = -z_\alpha, \quad (y_\alpha)^\dagger = y_\alpha, \quad (\psi_i)^\dagger = \psi_i, \quad k^\dagger = k, \quad \rho^\dagger = \rho, \quad (dx_n)^\dagger = dx_n.
\]

To define the \( d = 3 \) equations of motion we have to fix a form of the curvature \( R_{\nu\mu} \) in (212). In \( d=3 \), spinor indices take two values and therefore \( R_{\alpha\beta} \) is proportional to the \( d=3 \) charge conjugation matrix \( \epsilon_{\alpha\beta} \). The appropriate choice is

\[
R_{\alpha\beta} = \frac{i}{4} \epsilon_{\alpha\beta} B \ast \kappa,
\]

where

\[
\kappa = kv \equiv k \exp iz_\alpha y^\alpha.
\]

With the aid of the involutive automorphism \( \rho \rightarrow -\rho, \quad S_\alpha \rightarrow -S_\alpha \) the system (208)-(212) can be truncated to the one with the fields \( W \) and \( B \) independent of \( \rho \) and \( S_\alpha \) linear in \( \rho \),

\[
W(z, y; \psi_{1,2}, k, \rho | x) = W(z, y; \psi_{1,2}, k | x), \quad B(z, y; \psi_{1,2}, k, \rho | x) = B(z, y; \psi_{1,2}, k | x),
\]

\[
S_\alpha(z, y; \psi_{1,2}, k, \rho | x) = \rho s_\alpha(z, y; \psi_{1,2}, k | x).
\]

It is this reduced system that describes higher spin interactions of matter fields in \( d=3 \). For this system one finds taking into account (110), (263), and (267) that

\[
\kappa \ast W = W \ast \kappa, \quad \kappa \ast B = B \ast \kappa,
\]

and

\[
\kappa \ast S_\alpha = -S_\alpha \ast \kappa.
\]

The additional minus sign in (271) is due to the factor of \( \rho \) in (269). We observe that for \( d=3 \) the relations (212) have a form of the deformed oscillator relations (37) with \( B \) playing a role of the central element \( \nu \).
Let us note that in this section we assume that the differentials $dz^\alpha$ commute with all variables except for themselves and the space-time differentials $dx^n$ to which they anticommute. An alternative formulation applied to the $d=4$ analysis in the section 8.2 is to require $dz^\alpha$ to anticommute with the Klein operator $k$. Clearly, this is equivalent to the substitution $dz^\alpha \to \rho dz^\alpha$.

Note that the parameter $\varepsilon = \varepsilon(z, y; \psi_{1,2}, k|x)$ of the higher spin gauge transformations (222)-(224) is independent of $\rho$ and therefore commutes with $\kappa$. Thus, the element $\kappa$, that appears explicitly in the constraints (212), belongs to the center of the gauge algebra.

To elucidate the dynamical content of the system (208)-(212), one first of all has to find an appropriate vacuum solution. We consider vacuum solutions with

$$B_0 = \nu,$$  \hspace{1cm} (272)

where $\nu$ is some constant independent of the space-time coordinates and auxiliary variables. As a result, the equations (209) and (211) hold trivially and the vacuum fields $W_0$ and $S_{0\alpha}$ have to satisfy

$$dW_0 = W_0 \wedge W_0,$$  \hspace{1cm} (273)

$$dS_{0\alpha} = W_0 \wedge S_{0\alpha} - S_{0\alpha} \wedge W_0,$$  \hspace{1cm} (274)

$$[S_{0\alpha}, S_{0\beta}]_* = -2i\epsilon_{\alpha\beta}(1 + \nu \kappa).$$  \hspace{1cm} (275)

For $\nu = 0$, the standard choice is $S_{0\alpha} = \rho z_\alpha$. For general $\nu$, a class of solutions of the equations (275) is found in $34$. Here we reproduce the following three most important solutions

$$S_{0\alpha}^{\pm} = \rho \left( z_\alpha + \nu(z_\alpha \pm y_\alpha) \int_0^1 dt e^{it(z+y)_k} \right),$$  \hspace{1cm} (276)

and

$$S_{0\alpha}^{\text{sym}}(z, y) = \rho z_\alpha - \rho \frac{\nu}{8} \int_{-1}^1 ds (1 - s) \left[ e^{\frac{t}{4}(s+1)(z+y)}(y_\alpha + z_\alpha) \right. * \Phi \left( \frac{1}{2}, 2; -\kappa \ln|s|^{\nu} \right)$$

$$+ e^{\frac{t}{4}(s+1)(z+y)}(y_\alpha - z_\alpha) * \Phi \left( \frac{1}{2}, 2; \kappa \ln|s|^{\nu} \right) \big] * \kappa,$$  \hspace{1cm} (277)

where $\Phi(a, c; x)$ is the degenerate hypergeometric function

$$\Phi(a, c; x) = 1 + \frac{ax}{c1!} + \frac{a(a+1)x^2}{c(c+1)2!} + \ldots.$$  \hspace{1cm} (278)

59
The ambiguity in the solutions of the equation (275) takes its origin in the
gauge transformation (224). All three solutions \( S_{0\alpha}^\pm \) and \( S_{0\alpha}^{sym} \) belong to the
same gauge equivalence class. It is not hard to see that \( S_{0\alpha}^\pm \) solve (275). To
prove that \( S_{0\alpha}^{sym} \) solves (275) is more tricky. \( S_{0\alpha}^\pm \) and \( S_{0\alpha}^{sym} \) have the properties
\[
S_{0\alpha}^\pm (z, y; k, \rho) = -\bar{S}_{0\alpha}^\mp (-z, y; k, \rho), \quad S_{0\alpha}^{sym} (z, y; k, \rho) = iS_{0\alpha}^{sym} (-iz, iy; k, \rho),
\]
(279)
and
\[
S_{0\alpha}^{sym} (z, y; k, \rho) = -\bar{S}_{0\alpha}^{sym} (-z, y; k, \rho), \quad S_{0\alpha}^{sym} (z, y; k, \rho) = iS_{0\alpha}^{sym} (-iz, iy; k, \rho).
\]
(280)
In fact, the solution \( S_{0\alpha}^{sym} \) is fixed by the properties (280). It is particularly
useful for the analysis of truncations of the system being invariant under the
discrete symmetries and reality conditions. When the analysis is independent
of the particular form of a vacuum solution the symbol \( S_{0\alpha} \) will be used for
any one of them.

Now, let us turn to the equation (274). Since \( dS_{0\alpha} = 0 \), we get
\[
[W_0, S_{0\alpha}]_* = 0.
\]
(281)
Thus, \( W_0 \) belongs to the subalgebra \( A_S \subset A \) spanned by elements which
commute to \( S_{0\alpha} \), i.e. \( A_S \) is the centralizer of \( S_{0\alpha} \). For the case of \( \nu = 0 \), \( A_S \)
is the subalgebra of functions independent of \( z \). To find \( A_S \) for general \( \nu \) we
construct generating elements \( \hat{y}_\alpha \) commuting with \( S_{0\alpha}^\pm \) (276) and \( S_{0\alpha}^{sym} \) (277).
The final result is
\[
\hat{y}_\alpha^\pm (z, y) = y_\alpha + \nu(z_\alpha \pm y_\alpha) \int_0^1 dt(t - 1)e^{it(zy)k},
\]
(282)
\[
\hat{y}_\alpha^{sym} (z, y) = y_\alpha + k \frac{\nu}{8} \int_0^1 ds(1 - s) \exp \left\{ \frac{\nu}{2}s + (s + 1)(zy) \right\}
\times \left[ (y_\alpha + z_\alpha) \Phi \left( \frac{1}{2}, 2; -k\ln|s|^\nu \right) - (y_\alpha - z_\alpha) \Phi \left( \frac{1}{2}, 2; k\ln|s|^\nu \right) \right].
\]
(283)
Remarkably, \( \hat{y}_\alpha \) again obey the commutation relations of the form (37) but
now with the Klein operator \( k \) instead of \( \kappa \)
\[
[\hat{y}_\alpha, \hat{y}_\beta]_* = 2i\epsilon_{\alpha\beta}(1 + \nu k), \quad \hat{y}_\alpha k = -k\hat{y}_\alpha.
\]
(284)
It is elementary to check that \([\hat{y}_\alpha^\pm, S_{0\beta}]_* = 0 \). The fact that \([\hat{y}_\alpha^{sym}, S_{0\beta}^{sym}]_* = 0 \)
is less trivial. \( \hat{y}_\alpha \) is a by-product, the deformed oscillator algebra
is realized in terms of the embedding into the tensor product of two Weyl algebras equipped with the generating element \( k \).

Since \( k, \psi_1 \) and \( \psi_2 \) commute with \( S_0 \), the subalgebra \( A_S \) is spanned by the power series of \( \hat{y}_\alpha, \psi_1, \psi_2 \) and \( k \), i.e. its generic element has the form

\[
f(z, y; \psi_1, \psi_2, k) = \sum_{B,C,D=0}^1 \sum_{n=0}^\infty \frac{1}{n!} f_{\alpha_1, \ldots, \alpha_n}^{B,C,D} k^B \psi_1^C \hat{y}_\alpha^{\alpha_1} \ldots * \hat{y}_\alpha^{\alpha_n},
\]

where \( f_{\alpha_1, \ldots, \alpha_n}^{B,C,D} \) are totally symmetric multispinors (i.e. we choose the Weyl ordering). According to (281), \( W_0 \) has a form (285).

Because the commutation relations (284) have a form of the deformed oscillator algebra (37), we recover the \( d3 \) higher spin algebra discussed in the section 3 for an arbitrary value of \( \nu \) as the stability algebra of the chosen vacuum solution characterized by the parameter \( \nu \). One can now choose \( AdS_3 \) solution of the vacuum equations (208) in the form (131) identifying \( \psi \) e.g. with \( \psi_1 \),

\[
w_0 = \frac{1}{8i} \left( \omega^{\alpha\beta}(x) \{ \hat{y}_\alpha(\nu), \hat{y}_\beta(\nu) \} + \lambda h^{\alpha\beta}(x) \psi_1 \{ \hat{y}_\alpha(\nu), \hat{y}_\beta(\nu) \} \right).
\]

This completes the construction of the background solution. Let us emphasize that the form of the constraint (212) leads in a rather non-trivial way to the AdS background geometry via realization of the vacuum centralizer \( A_S \) in terms of the deformed oscillators \( \hat{y}_\alpha \).

Once a vacuum solution is known, one can study the system (208)-(212) perturbatively expanding the fields as

\[
B = B_0 + B_1 + \ldots, \quad S_\alpha = S_{0\alpha} + S_{1\alpha} + \ldots, \quad W = W_0 + W_1 + \ldots.
\]

Substitution of these expansions into (208)-(212) gives in the lowest order

\[
D_0 W_1 = 0,
\]

\[
D_0 C = 0,
\]

\[
D_0 S_{1\alpha} = [W_1, S_{0\alpha}]_*,
\]

\[
[S_{0\alpha}, S_{1\beta}]_* - [S_{0\beta}, S_{1\alpha}]_* = -2i \epsilon_{\alpha\beta} C * \kappa,
\]

\[
[S_{0\alpha}, C]_* = 0,
\]

where we denote \( C = B_1 \) and \( D_0 \) is the background covariant derivative.
The system (288)-(292) is analyzed as follows. From (292), one concludes that \( C \) has a form analogous to (285), i.e. \( C = C(\hat{y}; \psi_1,2, k|x) \). Expanding \( C \) as

\[
C = C^{aux}(\hat{y}; \psi_1, k|x) + C^{dyn}(\hat{y}; \psi_1, k|x) \psi_2 ,
\]

with

\[
C^{dyn}(\hat{y}; \psi_1, k|x) = \sum_{A,B=0,1} \sum_{n=0}^{\infty} \frac{1}{n!} C^{dyn\, A\, B\, A_1...\, A_n}(x)(k)^A(\psi_1)^B \hat{y}_{\alpha_1} \ldots \hat{y}_{\alpha_n},
\]

and

\[
C^{aux}(\hat{y}; \psi_1, k|x) = \sum_{A,B=0,1} \sum_{n=0}^{\infty} \frac{1}{n!} C^{aux\, A\, B\, A_1...\, A_n}(x)(k)^A(\psi_1)^B \hat{y}_{\alpha_1} \ldots \hat{y}_{\alpha_n},
\]

one observes that the covariant derivative \( D_0 \) acts differently on \( C^{aux} \) and \( C^{dyn} \) because the factor \( \psi_1 \) that appears in the vacuum solution (286) anticommutes to \( \psi_2 \). As a result, the equation (289) in the sector of \( C^{dyn} \) turns out to be equivalent to the equation (134). From the analysis of the section 6.3 we conclude that \( C^{dyn} \) describes four spin 0 and four spin 1/2 matter fields with the masses given in (184) and (185). The overall doubling for each mass is due to \( \psi_1 \) while the doubling with the mass splitting in the boson sector is due to \( k \) (cf (186)). Altogether, matter fields form a d3 massive hypermultiplet with respect to the \( N = 2 \) supersymmetry algebra \( osp(2,2) \oplus osp(2,2) \) discussed in the section 3.

In the sector of \( C^{aux} \), the covariant derivative acts as in the adjoint representation of the deformed oscillator algebra,

\[
D_0 C^{aux} = dC^{aux} - w_0 * C^{aux} + C^{aux} * w_0.
\]

An important difference between the adjoint representation and twisted representation is that the set of equations for \( C^{aux} \) decomposes into an infinite set of finite subsets of equations for homogeneous polynomials, i.e. for \( C^{aux\, A\, B\, A_1...\, A_n}(x) \) with any fixed \( n \). This is a simple consequence of (40). (The equations for \( C^{dyn} \) contain a finite set of infinite subsystems, each describing a matter field.) Because \( C = C^{aux}(\hat{y}; \psi_1, k|x_0) \) plays a role of initial data in the pure gauge solution of the covariant constancy conditions we conclude that each of the subsystems contained in \( C^{aux} \) describes at most a finite number of degrees of freedom. Moreover, because the AdS_3 algebra \( o(2,2) \) is noncompact, these degrees of freedom cannot appear in a unitary theory, ruled out by appropriate boundary conditions. As a result, \( C^{aux} \) describes some topological fields which carry no degrees of freedom in the unitary case.
The next step consists of resolution of the constraints (291) to reconstruct the auxiliary field $S_{1\alpha}$ as a linear functional of $C$, $S_{1\alpha} = S_{1\alpha}(C)$, up to a gauge ambiguity. Then, (290) allows one to express a part of degrees of freedom in $W_1$ via $C$, while the rest modes, which belong to the kernel of the mapping $[S_{0\alpha}, \ldots]_*$, remain free. These free modes are again arbitrary functions of $\hat{y}_\alpha$, i.e. $W_1 = \omega(\hat{y}; \psi_{1,2}, k|x) + \Delta W_1(C)$, where $\omega(\hat{y}; \psi_{1,2}, k|x)$ corresponds to the higher spin gauge fields. The dynamical equations on them are imposed by eq. (288) after (290) is solved. Eq. (288) describes the $C$-dependent first-order corrections to the higher spin strengths for $\omega$, which are argued in the section 9 to vanish. As a result one arrives at the d3 Central On-Mass-Shell Theorem.

One proceeds analogously in the highest orders. The equations (288)-(292) are consistent and admit a perturbative solution in powers of $C$ and $\nu$.

8.4 Interaction Ambiguity

Historically, we arrived at the equations for $d=4$ higher spin fields via perturbative analysis of the higher spin free differential algebra. In the end, the system of equations self-organized into the compact form (208)-(212) and (238). In this section we focus on some general features underlying this construction which hopefully will be useful for the analysis of higher spin systems in different dimensions and have already been applied for the $d=3$ and $d=2$ systems.

In principle, there are two ways for possible generalizations of the equations (208)-(212): a generalization of the equations within the same set of fields $W, B$ and $S^\nu$ or modifications by using larger sets of fields. The latter problem, being quite interesting in the context of the off-mass-shell (Lagrangian) formulation of the higher spin theories, is not yet enough clarified and is not discussed here.

Within the set of the variables $W, B$ and $S^\nu$ the most interesting question is why $R_{\nu\mu}$ has the particular form (238). One can generalize (212) to

$$S*S = -i[d\tilde{z}_\alpha dz^\alpha (G(B) + F(B)*\kappa) + d\tilde{z}_\alpha d\tilde{z}^\alpha (\bar{G}(B) + \bar{F}(B)*\bar{\kappa}) + dz^\alpha d\tilde{z}^\beta \tilde{H}_{\alpha\beta}(B)]$$

(297)

with arbitrary complex functions $G(B)$ and $F(B)$ and Hermitian $H_{\alpha\beta}(B)$

$$\tilde{H}_{\alpha\beta} = H_{\beta\alpha}$$

(298)

being some star-product expansions analogous to (239). This form of $R_{\nu\mu}$ is consistent with $x-$ and $Z-$ Bianchi identities provided that the rest equations (208)-(211) are true. The term with $H_{\alpha\beta}$ was not considered in since it looks weird breaking down Lorentz invariance explicitly (because it is impossible to
construct a vector from a scalar field $B$ without introducing some exterior vector. Note that since $\kappa$ and $\bar{\kappa}$ anticommute with $dz^\alpha$ and $d\bar{z}^{\dot{\alpha}}$, they can only appear in the $dzdz$ and $d\bar{z}d\bar{z}$ sectors, respectively (otherwise the system becomes inconsistent).

The ambiguity in $G(B)$ is artificial within the perturbative analysis in powers of $B$ provided that $G(0) \neq 0$. Indeed, one can get rid of the dependence on $G$ by means of the following field redefinition which does not affect the equations (208)-(211)

$$S \rightarrow S' = [G(B)]^{-\frac{1}{2}} dz^\alpha s_\alpha + [\bar{G}(B)]^{-\frac{1}{2}} d\bar{z}^{\dot{\alpha}} \bar{s}_{\dot{\alpha}}.$$  \hspace{1cm} (299)

We therefore set $G = 1$.

One can also use the ambiguity in the invertible field redefinitions $B \rightarrow B' = f(B)$. In particular, if $F(B)$ is some real function, $F(B) = \bar{F}(B)$, one can choose $f(B)$ to coincide with the inverse of $F(B)$, thus reducing the problem to the case with $F(B) = \bar{F}(B) = B$. However, generally, $F(B)$ is some complex function of $B$, while $f(B)$ is real, $f(B) = \bar{f}(B)$, because $B$ itself is real in accordance with the reality conditions (236). Assuming that $F(B)$ starts with the linear term $f_1 B$ one can fix this freedom by requiring that $F(B) \ast \bar{F}(B) = B \ast B$ and, therefore

$$F(B) = B \exp[\varphi(B)], \quad \bar{F}(B) = B \exp[-\varphi(B)],$$  \hspace{1cm} (301)

where $\varphi(B)$ is an arbitrary real function of $B$. Note that one cannot use field redefinitions mixing $s_\alpha$ and $\bar{s}_{\dot{\alpha}}$ because of the properties of the Klein operators $\kappa$ and $\bar{\kappa}$. Therefore, we are left with the formula (297) with $G = 1$, $F(B)$ of the form (301) and arbitrary $H_{\alpha\dot{\beta}}(B)$.

Let us now discuss the question of Lorentz invariance of the higher spin equations. The gauge transformations (224) act on the spinor variables $Z^\nu$ but not on the exterior index $\nu$ of $S^\nu$. Moreover, there is no local symmetry at all rotating this index $\nu$. The generators of the Lorentz symmetry acting on the spinor variables are

$$L^{tot}_{\alpha\beta} = \frac{i}{4} \{\{z_{\alpha}, z_{\beta}\}_s - \{y_{\alpha}, y_{\beta}\}_s\}, \quad L^{\dot{\alpha}\dot{\beta}} = \frac{i}{4} \{\{\bar{z}_{\dot{\alpha}}, \bar{z}_{\dot{\beta}}\}_s - \{\bar{y}_{\dot{\alpha}}, \bar{y}_{\dot{\beta}}\}_s\}. \hspace{1cm} (302)$$

Actually, by (100) and (101), the infinitesimal local Lorentz transformations

$$\delta f = [\eta^{\alpha\dot{\beta}} L^{tot}_{\alpha\beta}, f]_s + [\bar{\eta}^{\dot{\alpha}\dot{\beta}} \bar{L}^{tot}_{\dot{\alpha}\dot{\beta}}, f]_s,$$  \hspace{1cm} (303)

\hspace{1cm} In that respect, there is some difference between non-commutative gauge fields $S_\nu$ and base-space differential forms $dz^\omega W^\omega$, possessing diffeomorphisms rotating the form index $\omega$.\footnote{In that respect, there is some difference between non-commutative gauge fields $S_\nu$ and base-space differential forms $dz^\omega W^\omega$, possessing diffeomorphisms rotating the form index $\omega$.}
with the parameter $\eta^{\alpha\beta}(x)$ rotate properly the spinor generating elements,

$$
\delta z_\alpha = 2\eta_\alpha^\beta z_\beta, \quad \delta y_\alpha = 2\eta_\alpha^\beta y_\beta, \quad \delta \bar{z}_\dot{\alpha} = 2\eta_{\dot{\alpha}\dot{\beta}} \bar{z}_{\dot{\beta}}, \quad \delta \bar{y}_{\dot{\alpha}} = 2\eta_{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\beta}}.
$$

(304)

Although the system (208)-(212) is invariant under the local Lorentz transformations (303) (not acting on the outer spinor index in $S_\nu$), this symmetry is spontaneously broken due to the constraint (212) because the r.h.s. of (212) has a non-vanishing vacuum value and therefore $S_\nu$ itself must have a non-vanishing vacuum expectation value as is manifested by the vacuum solution (246).

As explained in the section 8.1 the global symmetry identifies with the centralizer of the vacuum value $S_0$ (246) isomorphic to the star-product algebra of $Z$-independent functions. It contains the Lorentz subalgebra spanned by the generators bilinear in $y^\alpha$ and $\bar{y}^{\dot{\alpha}}$

$$
l_{\alpha\beta} = -\frac{i}{4}\{y_\alpha, y_\beta\}^*, \quad \bar{l}_{\dot{\alpha}\dot{\beta}} = -\frac{i}{4}\{\bar{y}_{\dot{\alpha}}, \bar{y}_{\dot{\beta}}\}^*.
$$

(305)

By its definition, the full global symmetry acts properly in all orders in interactions. In particular this is true for the AdS subalgebra spanned by the bilinears in $y^\alpha$ and $\bar{y}^{\dot{\alpha}}$ and its Lorentz part (305). So, we conclude that the higher spin field equations with arbitrary $G$, $F$ and $H_{\alpha\dot{\beta}}$ possess global Lorentz symmetry. This happens because this global symmetry leaves invariant $S_0$ and does not act on the indices carried by $H_{\alpha\dot{\beta}}$.

The situation with local symmetries is different because, as is known from the example of supergravity\textsuperscript{62}, their form is deformed by curvature-dependent terms compared to the global symmetry algebra. In particular, AdS translations acquire some curvature-dependent corrections which transform them into diffeomorphisms. The local Lorentz symmetry remains undeformed however, with the Lorentz connection entering only via the usual Lorentz covariant derivatives. The local Lorentz symmetry guarantees the equivalence between the Cartan (frame) and the Riemannian formulations of gravity, providing the meaningful interpretation of spinors and tensors. It is therefore reasonable to require the higher spin equations to have standard (undeformed) local Lorentz symmetry. Surprisingly, this simple requirement is highly restrictive and to large extend fixes a form of the higher spin equations.

The question is whether there exists a local Lorentz symmetry which rotates properly spinor indices of the dynamical fields identified with the “initial data” (206) without a contradiction with the constraints (211) and (212) in all orders of perturbations. The answer is that this is indeed true provided that $H_{\alpha\dot{\beta}} = 0$. 

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Indeed, for the curvature $R_{\nu\mu}$ (238) with $H_{\alpha\beta} = 0$, the constraints (212), (211) have a form of the deformed oscillator algebra (37) in the dotted and undotted sectors. As a result, the elements

$$M_{\alpha\beta} = \frac{i}{4}\{s_\alpha, s_\beta\}^*$$

(306)

obey the Lorentz commutation relations and rotate properly $s_\alpha$,

$$[M_{\alpha\beta}, s_\gamma]_* = \epsilon_{\alpha\gamma}s_\beta + \epsilon_{\beta\gamma}s_\alpha, \quad [M_{\alpha\beta}, \bar{s}_\alpha]_* = 0.$$  

(307)

Analogously,

$$\bar{M}_{\dot{\alpha}\dot{\beta}} = \frac{i}{4}\{\bar{s}_{\dot{\alpha}}, \bar{s}_{\dot{\beta}}\}^*$$

(308)

obey the Lorentz commutation relations and rotate properly $\bar{s}_{\dot{\alpha}}$,

$$[\bar{M}_{\dot{\alpha}\dot{\beta}}, \bar{s}_{\dot{\gamma}}]_* = \epsilon_{\dot{\alpha}\dot{\gamma}}\bar{s}_{\dot{\beta}} + \epsilon_{\dot{\beta}\dot{\gamma}}\bar{s}_{\dot{\alpha}}, \quad [\bar{M}_{\dot{\alpha}\dot{\beta}}, s_\alpha]_* = 0.$$  

(309)

Now we require that the gauge fixings used to reconstruct $S_\nu$ in terms of $B$ (i.e. of the type we have used to gauge away $\varepsilon_1$ in (252)) do not contain any exterior objects transforming as nontrivial representations of the Lorentz group. In other words $S_\nu(Z; Y)$ is required to be reconstructed in terms of $C(Y) = B(0; Y)$ only by using $Z_\nu$ and $Y_\nu$. From the analysis of the section (8.2) it is clear that this can be achieved in all orders in interactions.

Let us define

$$l_{\alpha\beta} = L_{\alpha\beta}^{tot} - M_{\alpha\beta}, \quad \bar{l}_{\dot{\alpha}\dot{\beta}} = \bar{L}_{\dot{\alpha}\dot{\beta}}^{tot} - \bar{M}_{\dot{\alpha}\dot{\beta}}$$

(310)

with $l_{\alpha\beta}$ and $\bar{l}_{\dot{\alpha}\dot{\beta}}$ to be identified with the generators of the local Lorentz transformations acting on the physical fields. Taking into account (211), we obtain

$$\delta B = [\eta^{\alpha\beta} l_{\alpha\beta}, B]_* = \eta^{\alpha\beta}[L_{\alpha\beta}^{tot}, B]_*,$$

(311)

i.e. $l_{\alpha\beta}$ rotates properly the field $B$.

For the gauge fields $W$ we have

$$\delta W = D(\eta^{\alpha\beta} l_{\alpha\beta}) = (d\eta^{\alpha\beta}) l_{\alpha\beta} + \eta^{\alpha\beta}[L_{\alpha\beta}^{tot}, W]_*.$$  

(312)

Here $D(f) = df - [W, f]_*$ and therefore $D(\eta^{\alpha\beta}) = d(\eta^{\alpha\beta})$, since $\eta^{\alpha\beta}(x)$ is proportional to the unit element of the star-product algebra. Also, $D(l_{\alpha\beta}) = [L_{\alpha\beta}^{tot}, W]_*$ because $dL_{\alpha\beta} = 0$ and $DM_{\alpha\beta} = 0$ (cf. eq.(210)). From (312) one concludes that the gauge field for a true local Lorentz symmetry is

$$W_L = \omega_L^{\alpha\beta} l_{\alpha\beta},$$

(313)
while the other gauge fields are rotated properly under the local Lorentz transformations. (The analysis in the sector of dotted spinors is analogous.)

By assumption, the auxiliary field \( S_\nu \) is expressed via \( B \) by the constraint (212) in a Lorentz covariant way. We therefore have

\[
\eta^{\alpha\beta} L^\text{tot}_{\alpha\beta}, s_\gamma(B) = \eta^{\alpha\beta} (\epsilon_{\alpha\gamma} s_\beta(B) + \epsilon_{\beta\gamma} s_\alpha(B)) + \frac{\delta s_\gamma(B)}{\delta B} \delta B,
\]

(314)

\[
\eta^{\alpha\beta} L^\text{tot}_{\alpha\beta}, \bar{s}_\dot{\gamma}(B) = \frac{\delta \bar{s}_\dot{\gamma}(B)}{\delta B} \delta B,
\]

(315)

where \( \delta B = \eta^{\alpha\beta} [L^\text{tot}_{\alpha\beta}, B]_* \). Making use of (307), we find

\[
\delta S_\nu = \eta^{\alpha\beta} l_{\alpha\beta}, S_\nu|_* = \frac{\delta S_\nu}{\delta B} \delta B.
\]

(316)

As a result, the local Lorentz rotations generated by \( l_{\alpha\beta} \) and \( \bar{l}_{\dot{\alpha}\dot{\beta}} \) do not act on the index \( \nu \) of \( S_\nu \), acting only on the physical fields \( B \). This is just the desired result that the transformation law respects the solution of the constraints for \( S \) in terms of \( B \).

Thus, the fact that the constraints (212), (211) have a form of the deformed oscillator algebra (37) guarantees that the local Lorentz symmetry remains unbroken. This property restricts a form of the curvature \( R_{\nu\mu} \) ruling out the term with \( H_{\alpha\dot{\beta}} \). We conclude that, modulo perturbative field redefinitions, the most general form of the curvature \( R_{\nu\mu} \) compatible with local Lorentz invariance is (238) with \( F(B) \) of the form (301).

Beyond the perturbative analysis, the most general form of the equation (212) compatible with local Lorentz invariance is

\[
S * S = -i[dz_\alpha dz^\alpha (G(B) + F(B) * \kappa) + d\bar{z}_\dot{\alpha} d\bar{z}^{\dot{\alpha}} (\bar{G}(B) + \bar{F}(B) * \bar{\kappa})].
\]

(317)

In principle, the theory may admit phases with different behavior of the functions \( G \) and \( F \). One interesting possibility is \( G(0) = 0 \). The straightforward interpretation in terms of AdS higher spin fields is not obvious for that case but it may correspond to the \( w_{\infty} \) limit of the underlying algebras \(^{32} \) having therefore some relevance to conformal models. Similar statement concerns the strong coupling limit \( F(B) \to \infty \) equivalent to \( G(B) \to 0 \) by a rescaling of \( S_\nu \).

The meaning of the ambiguity in one real function \( \varphi(B) \) which affects higher spin interactions is not yet clear. For general \( \varphi \), some discrete symmetries of the system turn out to be lost. For more details on this issue we refer the reader to the original paper \(^{14} \). Note that even the simplest choice \( \varphi(B) = 0 \) is interesting enough leading to consistent nontrivial higher spin dynamics.
The analysis of the \( d=3 \) case is parallel to \( d=4 \). The constraint (212) associated with \( R_{\alpha\beta} \) (266) again has a form of the deformed oscillator algebra (37) thus guaranteeing local Lorentz invariance. There is no ambiguity in the function \( \varphi(B) \) because the fields are real and, perturbatively, all nonlinearities in \( R_{\alpha\beta} \) can be compensated by field redefinitions analogous to (299) and (300). Beyond the perturbative analysis, the most general form of the equation (212) is

\[
S^* S = -idz_\alpha dz^\alpha (G(B) + F(B) * \kappa).
\]

(318)

The \( d=3 \) equations are very close to the \( d=4 \) self-dual higher spin equations introduced in \(^{14}\). Indeed, it is the Minkowski signature \((+ - - -)\) that forces the left and right sectors to be conjugated to each other. For the signatures \((- - - -)\) and \((+ + - -)\) these sectors are independent. For that reason, for the cases admitting self-duality, the functions \( F(B) \) and \( \bar{F}(B) \) are no longer conjugated to each other but become independent real functions. One can therefore set

\[
\bar{F}(B) = 0, \quad F(B) = B
\]

(319)
or vice versa. (For \( \bar{F}(B) = 0 \) one can achieve \( F(B) = B \) by the field redefinition (300).)

Note that in the standard selfduality equations a half of (higher spin) Weyl tensors (field strengths in the spin 1 case) vanishes. In the self-dual higher spin equations of the form (319) the situation is different because all higher spin Weyl tensors are still contained in the generating function \( B(0; Y; k, \bar{k} | x) \). However a half of the Weyl tensors decouples from the higher spin curvatures (i.e. only another half of them survive in (166)). It is interesting to clarify a relationship of this form of the self-dual higher spin equations with that proposed in \(^{65}\).

9 Integrating Flow

An interesting property of the \( d=3 \) equations (208)-(212) is \(^{34}\) that they admit a flow which expresses solutions of the full system in terms of free fields. Since we use the perturbation expansion in powers of the physical fields identified with the deviation \( C(Y; Q|x) = B(0; Y; Q|x) \) from its vacuum value \( \nu \), let us introduce a formal perturbation expansion parameter \( \eta \) as follows

\[
B(\eta) = \nu + \eta B(\eta).
\]

(320)

Simultaneously, the rest of the fields acquire a formal dependence on \( \eta \), i.e. \( W = W(\eta) \) and \( S_\alpha = S_\alpha(\eta) \). The \( d=3 \) system takes a form

\[
dW = W * \wedge W, \quad dB = W * B - B * W, \quad dS_\alpha = W * S_\alpha - S_\alpha * W.
\]

(321)

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\[ S_\alpha \ast S^\alpha = -2i(1 + \nu \kappa + \eta B \ast \kappa), \quad S_\alpha \ast B = B \ast S_\alpha. \] (322)

The parameter \( \eta \) drops out all the equations except for the non-commutative curvature in (322). As a result, the expansion in powers of \( \eta \) is equivalent to the expansion in powers of the curvature fluctuations.

Now, one observes that for the limiting case \( \eta = 0 \) the system (321), (322) reduces to the free one. Indeed, setting
\[ B|_{\eta=0} = B_1 \equiv C, \quad W|_{\eta=0} = W_0 \equiv w, \quad S_\alpha|_{\eta=0} = S_0^\alpha, \] (323)
we see that at \( \eta = 0 \) the system (321), (322) solves in terms of \( S_\alpha = S_0^\alpha \) with the gauge field \( W = w(\hat{y}(\nu); \psi_{1,2}, k|x) \) satisfying the vacuum equations
\[ dw = w \ast \wedge w, \quad B = C(\hat{y}(\nu); \psi_{1,2}, k|x) \] satisfying the free field equations (289). This is similar to contractions of Lie algebras. For all values of \( \eta \neq 0 \), the systems of equations (321), (322) are pairwise equivalent since the field redefinition (320) is non-degenerate. On the other hand, although the field redefinition (320) degenerates at \( \eta = 0 \), eqs. (321), (322) still make sense for \( \eta = 0 \), describing the free field dynamics.

Remarkably, the two inequivalent systems turn out to be related by the following flow with respect to \( \eta \):
\[ \frac{\partial W}{\partial \eta} = (1 - \mu) B \ast \frac{\partial W}{\partial \nu} + \mu \frac{\partial W}{\partial \nu} \ast B, \] (324)
\[ \frac{\partial B}{\partial \eta} = (1 - \mu) B \ast \frac{\partial B}{\partial \nu} + \mu \frac{\partial B}{\partial \nu} \ast B, \] (325)
\[ \frac{\partial S_\alpha}{\partial \eta} = (1 - \mu) B \ast \frac{\partial S_\alpha}{\partial \nu} + \mu \frac{\partial S_\alpha}{\partial \nu} \ast B, \] (326)

where \( \mu \) is an arbitrary parameter. By applying \( \frac{\partial}{\partial \eta} \) to both sides of eqs.(321), (322) one concludes that for any \( \mu \) the system (324)-(326) is compatible with (321), (322). Therefore, solving (324)-(326) with the initial data (323) satisfying the free equations of motion we can express solutions of the full nonlinear system at \( \eta = 1 \) via solutions of the free system at \( \eta = 0 \). Note that all fields acquire nontrivial dependence on the parameter \( \nu \) via the deformed oscillators (282) or (283).

This approach is very efficient at least perturbatively and allows one to derive the corresponding nonlinear field redefinitions order by order. In particular, using this flow, one easily proves that the d3 higher spin gauge field strengths do not admit nontrivial sources linear in fields. This is of course expected result because, in accordance with the d3 Central On-Mass-Shell
Theorem (183), d3 higher spin fields do not admit Weyl tensors. Nevertheless, even at the linearized level it is a complicated technical problem to find a form of an appropriate field redefinition for arbitrary $\nu$ without using the flow (324)-(326).

The flows (324)-(326) at different $\mu$ develop within the same gauge equivalence class. To see that any variation of $\mu$ is induced by some gauge transformation one has to find such a gauge parameter $\varepsilon$ that

$$\frac{\partial W}{\partial \mu} = D\varepsilon, \quad \frac{\partial B}{\partial \mu} = [\varepsilon, B]_*, \quad \frac{\partial S_\alpha}{\partial \mu} = [\varepsilon, S_\alpha]_*, \quad \varepsilon|_{\eta=0} = 0,$$

(327)

where $D\varepsilon = d\varepsilon - [W, \varepsilon]_*$. The compatibility condition of (327) with (324)-(326) is satisfied if

$$\frac{\partial \varepsilon}{\partial \eta} = \frac{\partial B}{\partial \nu} + (1 - \mu) B * \frac{\partial \varepsilon}{\partial \nu} + \mu \frac{\partial \varepsilon}{\partial \nu} * B,$$

(328)

which condition just fixes the $\eta$-dependence of $\varepsilon$. Thus, one is free to choose any value of $\mu$.

One has to be careful in making statements on the locality of the mapping induced by the flow (324)-(326). Indeed, although it does not contain explicitly space-time derivatives, it contains them implicitly via highest components $C_\alpha(n)$ of the generating function $C(\hat{y})$ which are identified with the highest derivatives of the matter fields according to (189). For example, the equation (325) at $\mu = 0$ in the zero order in $\eta$ reads

$$\frac{\partial}{\partial \eta} B_1(\varepsilon, y) = C(\hat{y}) * \frac{\partial C(\hat{y})}{\partial \nu}.$$

(329)

Because of nonlocality of the star-product, for each fixed rank multispinorial component of the left hand side of this formula there appears, in general, an infinite series involving bilinear combinations of the components $C_\alpha(n)$ with all $n$ on the right hand side of (329). Therefore, in accordance with (190), the right hand side of (329) effectively involves space-time derivatives of all orders, i.e. the transformation laws (324)-(326) can effectively describe some nonlocal transformation. This means that we cannot treat the system (321), (322) as locally equivalent to the free system. Instead we can only claim that there exists a nonlocal mapping between the free and nonlinear system. This mapping is reminiscent of the Nicolai mapping in supersymmetric models

At the linearized level, however, the transformations induced by the integrating flow (324)-(326) are local for the following simple reason. In this case, all field redefinitions are linear in the matter fields $C$ and only the zero-order
(vacuum) part $W_0$ of the higher spin gauge fields $W$ contributes. Since the background gravitational 1-forms (131) are bilinear in the auxiliary variables $\hat{y}_\alpha$, the transformations for physical fields, induced by (324), contain at most two derivatives in $\hat{y}_\alpha$. In accordance with (190), this is equivalent to the statement that the linearized field transformations contain only a finite number of space-time derivatives and therefore are local. Thus, the fact that the equations of motion for higher spin fields do not acquire sources linear in the matter fields is the well-defined local statement. This is not expected to be true for the second-order analysis of bilinear higher spin currents constructed from the matter fields.

In fact, the 2-forms $J(C)$ dual to currents of an arbitrary spin constructed from massless matter fields in $AdS_3$ are shown in\(^\text{26}\) to be exact

\[ J(C) = DU(C) \]  

in the class of infinite expansions in powers of derivatives (i.e. with $U(C)$ depending on all components of $C(Y)$), thus explaining how a nonlocal field redefinition induced by the flow (324)-(326) can compensate higher spin current interactions of matter fields

\[ R_1 \equiv D w = J(C). \]  

Let us stress that this phenomenon has no analog in the flat space.

The existence of the integrating flow (324) takes its origin in the simple fact that $B$ behaves like a constant in the system (208)-(212): it commutes to $S_\alpha$ and satisfies the covariant constancy condition. Knowledge of the vacuum solution with $B = \nu$ can therefore be used to reconstruct the full dependence on $B$. Indeed, the meaning of (324) is that a derivative with respect to $\eta B$ is the same as that with respect to $\nu$. Since the parameter $\eta$ can be interpreted as the coupling constant, the idea to integrate the higher spin equations by integrating a flow with respect to $\eta$ has much in common with the coupling constant evolution method developed in\(^\text{67}\) in application to quantum mechanics.

One can proceed analogously in the d=4 case by the substitution

\[ F(B) = \nu + \eta F(B), \quad \bar{F}(B) = \bar{\nu} + \bar{\eta} \bar{F}(B) \]  

in (238) or, equivalently, reintroducing nonzero $f_0 = \nu$ and $\bar{f}_0 = \bar{\nu}$. This leads to the two flows commuting with the system (208) and to each other,

\[ \frac{\partial X}{\partial \eta} = (1 - \mu) \, F(B) \ast \frac{\partial X}{\partial \nu} \ast \mu \frac{\partial X}{\partial \nu} \ast F(B), \]  

\[ \frac{\partial X}{\partial \bar{\eta}} = (1 - \bar{\mu}) \, \bar{F}(B) \ast \frac{\partial X}{\partial \bar{\nu}} \ast \bar{\mu} \frac{\partial X}{\partial \bar{\nu}} \ast \bar{F}(B) \]  

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for $X = W, S$ or $B$.

For the first sight the existence of the $d=4$ flows is paradoxical because it establishes a connection between the full nonlinear problem and the free system with only vacuum fields in the sector of gauge fields $W$. That was fine for the Chern-Simons $d=3$ higher spin dynamics but sounds surprisingly for the $d=4$ case. Indeed, let us discuss the example of Einstein gravity. Einstein equations (155), (156) can be rewritten as

$$R_{\alpha\beta} = 0, \quad R_{\alpha_1\alpha_2} = \eta h^{\gamma_1\delta} \wedge h^{\gamma_2\delta} C_{\alpha_1\alpha_2\gamma_1\gamma_2}, \quad \bar{R}_{\dot{\beta}_1\dot{\beta}_2} = \bar{\eta} h^{\dot{\gamma}_1\dot{\delta}} \wedge h^{\dot{\gamma}_2\dot{\delta}} \bar{C}_{\dot{\beta}_1\dot{\beta}_2\dot{\gamma}_1\dot{\gamma}_2},$$

where the parameter $\eta$ is introduced in a way it comes from the system (208)-(212). In the limit $\eta = 0$, Einstein equations therefore reduce to the vacuum equations of the AdS space. The dynamical equations of the massless spin 2 field reappear as equations on the Weyl tensor described by the covariant constancy equation (158) and (159), (equivalently by (209)). To understand how this free vacuum system is related to the nonlinear one with $\eta \neq 0$, one has to take into account the difference between the $d=3$ and $d=4$ problems discussed in the end of the section 5.

For $d=3$, the ansatz (131) describes the vacuum solution with the same $AdS_3$ gravitational fields $\omega^{\alpha\beta}(x)$ and $h^{\alpha\beta}(x)$ for arbitrary $\nu$. Therefore, the differentiation with respect to $\nu$ in the $d=3$ case when using the flow to iterate solutions will only act on the deformed oscillators (282) or (283) but not on the space-time dependent coefficients in (131) identified with the background gravitational fields. As a result, implementation of the integration flow in $d=3$ has a form of some (may be effectively nonlocal) field redefinition.

The $d=4$ ansatz (132) only describes $AdS$ geometry for the case of $\nu = \bar{\nu} = 0$. For $\nu \neq 0$ it does not solve the vacuum equations. Therefore, in the $d=4$ case one first of all has to find a true vacuum solution $w_0(\nu, \bar{\nu})$ of the problem with $\nu \neq 0$ and $\nu \neq 0$, such that $w_0(0,0)$ reduces to the $AdS_4$ solution. From the fact that (132) is inconsistent away from $\nu = \bar{\nu} = 0$ it follows that the dependence on the space-time coordinates $x^\mu$ and spinor coordinates $Z_\mu$ and $Y_\mu$ is mixed nontrivially in such a vacuum solution. In particular, $\frac{\partial w_0(x)}{\partial \nu}|_{\nu=0}$ does not express directly via the $AdS_4$ gravitational fields identified with $W_0(x)|_{\nu=0}$ as was the case for $d=3$. As a result, the $d=4$ flow describes a change of variables explicitly containing some functions of space-time coordinates (via $\frac{\partial w_0(x)}{\partial \nu}$), which depend on a particular gauge choice and cannot be directly expressed via the background gravitational fields. Therefore, in $d=4$, the integration flow generates not a field redefinition but some change of variables containing an explicit dependence on the space-time coordinates. The $d=4$ field transform induced by the flow is nonlocal even at the linearized level. The conclusion
therefore is that the flows (333), (334) indeed allow one to solve the nonlinear system in terms of the free system via some expansion of the form

\[ W = W_0 + \eta \alpha_1(x) C + \eta^2 \alpha_2(x) C^2 + \ldots \] (336)

In particular, such an expansion reconstructs the metric tensor in terms of the curvature tensor and in that sense is analogous to the normal coordinate expansion. In d=3 the coefficients \( \alpha_n(x) \) express locally via the metric tensor.

Thus the integration flow in d=4 higher spin system provides a systematic way for the derivation of the coefficients of the expansions like (336). The resulting procedure is pure algebraic at any given order in \( \eta \) (equivalently \( C \)). In particular, it can be used to reconstruct the potentials \( w_1 \) corresponding to solutions of the free field equations described in the section 6.2.

Note that the fact that some system can be integrated order by order with the help of a nonlocal change of variables is rather trivial. What is special about the higher spin systems is that such changes of variables are described in a systematic and constructive way by a simple flow with respect to an additional evolution parameter.

10 Conclusions

Higher spin gauge theories are based on the infinite-dimensional higher spin symmetries. Their role in the higher spin theories is as fundamental the role of the supersymmetry algebra discovered by Golfand and Likhtman\(^1\) for supersymmetric theories. The higher spin symmetries are realized by the algebras of oscillators carrying spinorial representations of the space-time symmetries. These star-product algebras are nonlocal in the auxiliary spinor spaces in the usual quantum-mechanical sense typical for the Moyal product. One point illustrated in this contribution is that the dynamical higher spin field equations transform this nonlocality into nonlocality in the space-time coordinates, i.e. the quantum mechanical nonlocality of the higher spin algebras may imply some space-time nonlocality of the higher spin gauge theories at the interaction level. The same time the higher spin gauge theories remain local at the linearized level.

Another important implication of the star-product origin of the higher spin algebras is that the space-time symmetries are simple (semisimple for d=3) and therefore correspond to AdS geometry rather than to the flat one. The space-time symmetries are realized in terms of bilinears in spinor oscillators according to the well-known isomorphisms \( o(2, 2) \sim sp(2; R) \oplus sp(2; R) \) and \( o(3, 2) \sim sp(4; R) \). This phenomenon has two consequences. On the one hand, it explains why the theory is local at the linearized level. The reason is that bilinears in
the non-commuting auxiliary coordinates can lead to at most two derivatives in the star-products. On the other hand, the fact that higher spin models require AdS geometry is closely related to their potential nonlocality at the interaction level because it allows expansions with arbitrary high space-time derivatives, in which the coefficients carry appropriate (positive or negative) powers of the cosmological constant fixed by counting of dimensions. As a result, higher spin symmetries link together such seemingly distinct concepts as AdS geometry, space-time nonlocality of interactions and quantum mechanical nonlocality of the star-products in auxiliary spinor spaces. Taking into account the recent developments related to the role of the AdS space and star-product in the string theory this triple looks too natural to be just a coincidence. Another consequence of the star-product origin of the higher spin symmetries is that higher spin theories are based on the associative structure rather than on the Lie-algebraic one. As a result, higher spin gauge theories with non-Abelian symmetries classify in a way analogous to the Chan-Paton symmetries in oriented and non-oriented strings.

The algebraic structures underlying dynamical systems associated with infinite sets of higher spin gauge fields turn out to be interesting and deep. The full geometric understanding is still lacking however. The parallelism with the Fedosov quantization is very suggestive in that respect, although higher spin gauge theories involve spinor oscillators instead of the vector ones used in higher spin gauge theories involve spinor oscillators instead of the vector ones used in string theory.

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Appendix. Notation

Underlined Latin indices are used for differential forms and vector fields in d-dimensional space-time with coordinates $x^\underline{\mu}$,

$$m, n, \ldots = 0, \ldots, d-1, \quad \partial_\underline{m} = \frac{\partial}{\partial x^{\underline{m}}}, \quad d = dx^{\underline{m}} \partial_\underline{m}. \quad (337)$$

Indices from the middle of the Latin Alphabet denote fiber vectors,

$$m, n, \ldots = 0, \ldots, d-1, \quad \eta^{mn} = (1, -1, \ldots, -1). \quad (338)$$
In the flat space, base and fiber indices are sometimes identified.

The indices $i$ and $j$ are often used for inner symmetries.

Letters from the middle of the Greek alphabet are reserved for spinors in d-dimensions

$$\mu, \nu, \ldots = 1, \ldots, 2^{[d/2]}$$

(339)

but sometimes are also used for symplectic indices in the star-product. Dirac gamma matrices $\gamma^{\mu,\nu}$ satisfy

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}.$$  

(340)

Two-component spinorial indices are denoted by the Greek indices from the beginning of the Alphabet

$$\alpha, \beta \ldots = 1, 2, \quad \dot{\alpha}, \dot{\beta} \ldots = 1, 2, \quad \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \quad \epsilon_{12} = \epsilon^{12} = 1, \quad \epsilon_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\beta}\dot{\alpha}},$$

(341)

$$A^{\alpha} = \epsilon^{\alpha\beta} A_{\beta}, \quad A_{\alpha} = A^{\beta} \epsilon_{\beta\alpha}, \quad \dot{A}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} A_{\dot{\beta}}, \quad \dot{A}_{\dot{\alpha}} = A^{\dot{\beta}} \epsilon_{\dot{\beta}\dot{\alpha}}.$$  

(342)

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