Number-conserving rate equation for sympathetic cooling of a boson gas

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We derive a particle number-conserving rate equation for the ground state and for the elementary excitations of a bosonic system which is in contact with a gas of a different species (sympathetic cooling). We use the Girardeau-Arnowitt method and the model derived by Lewenstein et al. with an additional assumption: the high-excited levels thermalize much faster with the cooling agent than the other levels. Evaporation of particles, known to be important in the initial stages of the cooling process, is explicitly included.

I. INTRODUCTION

The development of cooling techniques has opened up the possibility of studying ultracold gases. In particular, the quantum degeneracy of bosons and fermions has been investigated. Basically there are two different processes which can be used to cool an atomic or molecular ensemble, evaporative cooling and sympathetic cooling. However, there are gases in which the interaction is too weak for evaporative cooling to work. In addition, this method fails for identical fermions at low temperatures because of the exclusion principle. In this case, sympathetic cooling can be used as an alternative to evaporative cooling: a cold gas is in thermal contact with another gas to be cooled. The first application of the sympathetic cooling method was in the context of the cooling of charged particles and later it was extended to neutral atoms. Only recently has it been applied in the context of ultracold atoms. In particular, quantum degeneracy of bosons and fermions has been achieved by thermalization of atoms of the same species in different internal states, between two isotopes of the same species, and finally between atoms of different species. Moreover, in contrast to evaporative cooling, sympathetic cooling does not lead to a significant loss of either the cooled gas or the cooling agent.

From the theoretical point of view, classical and quantum models have been derived to describe the dynamics of thermalization of one-species and two-species systems. In the classical Boltzmann regime, an analytical formula for the evolution of the temperature in a mixture of non-equivalent mass atoms and equal mass atoms has been derived in Refs. and later it was extended to neutral atoms. Only recently has it been applied in the context of ultracold atoms. In particular, quantum degeneracy of bosons and fermions has been achieved by thermalization of atoms of the same species in different internal states, between two isotopes of the same species, and finally between atoms of different species. Moreover, in contrast to evaporative cooling, sympathetic cooling does not lead to a significant loss of either the cooled gas or the cooling agent.

The purpose of this work is to describe the sympathetic cooling process of a total-number-conserving system in terms of particles in the ground state and elementary excitations. We use the particle-number-conserving Girardeau-Arnowitt formalism and the sympathetic cooling model developed by M. Lewenstein et al. with the additional assumption, that, the highly excited levels of the trap thermalize much faster with the cooling agent than the other levels (or in other words, the highly excited levels of the trap are in thermal equilibrium with the cooling agent). The advantage of the Girardeau-Arnowitt formalism is that it covers all cases ranging from a total absence of population in the ground state, $n_0 = 0$, to a highly populated ground state $n_0 \approx N$. The assumption of the fast thermalization of the highly excited levels of the trap is related to the separation of the levels of the trap into two different bands, in the same way proposed by C.W. Gardiner et al. in Refs. and the condensate ($B_C$) band and the noncondensate band ($B_{NC}$), where the latter is in thermal equilibrium with the cooling agent. The condensate band includes all trap levels which are directly influenced by the presence of the condensate. The noncondensate band contains energy levels that are sufficiently high for the interaction with

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the condensate to be negligible. As we will show, the master equation for sympathetic cooling can be written in terms of two processes: (a) Creation and annihilation of particles in the ground state and (b) Creation and annihilation of quasiparticles. From this master equation we derive a rate equation for this number-conserving system under sympathetic cooling which includes the growth, scattering and loss (sympathetic evaporation) processes. This result is very important, specially now that sympathetic cooling is used as an alternative cooling method. In addition, this rate equation is more general than the one obtained by Gardiner et al. in refs. [4,5], which is only valid in the regime of a highly populated condensate [20].

The paper is organized as follows: In section I the description of the sympathetic cooling model is given. In section II the master equation describing the dynamics of a number-conserving system in the condensate band is derived. In subsections IV.A, IV.B, and IV.C a rate equation describing the population growth, the effects of the scattered particles as well as the effects of the sympathetic evaporation (trap losses) is derived. Section V contains the conclusion.

II. DESCRIPTION OF THE SYSTEM

For the sake of completeness and notation we briefly review the main results of Refs. [2,3]. However, we introduce some modifications to the model presented in Ref. [2]: the cooling agent is a gas trapped in a harmonic potential, instead of a free gas, and the gas to be cooled is confined in an open trap. This situation is closer to the actual experimental conditions.

A system $A$ of $N_A$ bosons is subject to sympathetic cooling due to its interaction with system $B$. The cooling agent of $N_B$ atoms is in thermal equilibrium at temperature $T_B$. The single-particle states of system $B$ are then described by harmonic oscillator wave functions with energy $\epsilon_{\ell} = \hbar \omega_B (\ell_x + \ell_y + \ell_z)$, where $\omega_B$ is the trap frequency and $\vec{\ell} = (\ell_x, \ell_y, \ell_z)$ are the quantum numbers related to the bath. The corresponding Hamiltonian can be written

$$H_B = \sum_{\ell} \hbar \omega_B (\ell_x + \ell_y + \ell_z) b^{\dagger}_{\ell} b_{\ell},$$

with the creation and annihilation operators for the bath $b^\dagger_{\ell}$ and $b_{\ell}$, respectively.

System $A$ is assumed to be confined in an open trap. For energies $\epsilon$ much smaller than the trap depth $\epsilon_t$, the trap can be approximated by a harmonic potential with trap frequency $\omega_A$. The single-particle states in the harmonic oscillator potential have the quantum numbers $\vec{n} = (n_x, n_y, n_z)$. The single-particle eigenfunctions are labelled $\psi_{\vec{n}}(\vec{x})$, the eigenvalues $\epsilon_{\vec{n}} = \hbar \omega_A (n_x + n_y + n_z)$.

The associated creation and annihilation operators are denoted by $a^\dagger_{\vec{n}}$ and $a_{\vec{n}}$, respectively. If, on the other hand, the energy is much larger than the trap depth, the system $A$ can be approximately described by free particles. In this case the states are described by plane waves with energy $\epsilon(k) = \hbar^2 k^2/(2m)$, with $k$ denoting the wavevector of the particle.

$$H_A = \sum_{\vec{n}} U_{\vec{n}} a^\dagger_{\vec{n}} a_{\vec{n}}$$

where

$$U_{\vec{n}} = \begin{cases} \epsilon_{\vec{n}} & \text{if } \epsilon \ll \epsilon_t \\ \epsilon(k) & \text{if } \epsilon \gg \epsilon_t \end{cases}$$

when $\epsilon \approx \epsilon_t$, $U_{\vec{n}}$ cannot be determined in a simple way.

The two-body interaction between the bath and the particles can be written in the form

$$V_{A-B} = \sum_{\vec{n},\vec{n}',\vec{\ell},\vec{\ell}'} \gamma_{\vec{n},\vec{n}',\vec{\ell},\vec{\ell}'} a^\dagger_{\vec{n}} a^\dagger_{\vec{n}'} b_{\vec{\ell}} b_{\vec{\ell}'}.$$
the correlation time for the interaction between systems
A and B is much shorter than the cooling time, and (ii)
rotating-wave approximation: terms rotating at multi-
ples of the trap frequency are neglected.

It was shown in Ref. [13] that if the number $N_B$ of
particles in the bath is very large, decoherence acts very
quickly compared to the equilibration time and reduces
the density matrix to diagonal form. If only scattering
processes involving tightly bound atoms of system A are
involved, the Liouvillian in Eq. (4) becomes

\[ \mathcal{L} \rho_A = \sum_{\vec{m} \neq \vec{n}} \Gamma^{\vec{m}, \vec{n}} \left( 2a_{\vec{m}}^\dagger a_{\vec{n}}^\dagger \rho_A(t) a_{\vec{n}} a_{\vec{m}} - a_{\vec{m}}^\dagger a_{\vec{n}}^\dagger a_{\vec{m}} a_{\vec{n}} \rho_A(t) - \rho_A(t) a_{\vec{n}}^\dagger a_{\vec{m}} a_{\vec{m}} a_{\vec{n}}^\dagger \right), \]

with the rate coefficients

\[ \Gamma^{\vec{m}, \vec{n}} = \frac{1}{2\hbar} \int_{-\infty}^{\infty} \frac{d\tau}{\sum_{\vec{\ell}, \vec{\rho}} \gamma_{\vec{m}, \vec{n}, \vec{\ell}, \vec{\rho}} \gamma_{\vec{m}, \vec{n}, \vec{\rho}, \vec{\ell}}} \times n_T [n_T + 1] \exp \left[ i \left( \Delta \epsilon_{AC} + \Delta \epsilon_{BC} \right) \tau / \hbar \right] \]

where $n_T$ is the average occupation number of the heat
bath oscillator. The energy difference for the system and
the bath is given by $\Delta \epsilon_A = \epsilon_{\vec{m}} - \epsilon_{\vec{n}}$ and $\Delta \epsilon_B = \epsilon_{\vec{\rho}} - \epsilon_{\vec{\ell}}$,
respectively. If particles are scattered into the contin-
um, the rate coefficients have similar form but with the
matrix element defined in Eq. (3). More details about
the evaluation of the rate coefficients are given in Refs.
[12,13].

### III. MASTER EQUATION FOR THE
### CONDENSATE BAND

In this section, a master equation describing the dy-
namics of the condensate band is derived. Basically,
we extend the formalism developed by Gardiner et al.
[14] to the case of two distinct gases. The trap-levels
of the system A are grouped in two bands: the conden-
sate band ($B_C$) and the band of noncondensed par-
ticles ($B_{NC}$), where the latter is in thermal equilibri-
num with the cooling agent system B (see Figure 1).

| trap depth | Noncondensate Band | Condensate Band |
|------------|---------------------|-----------------|

FIG. 1. Distribution of the trap levels in the condensate
$B_C$ and noncondensate $B_{NC}$ bands.

The condensate band $B_C$ includes the ground state
and all excited levels in the trap which are directly influ-
enced by the presence of the condensate. The non-
condensate band $B_{NC}$ is composed by the highly excited
levels, which include also the continuum states for $\epsilon > \epsilon_1$.
The following assumptions are used in order to derive a
master equation for the condensate band from Eq. (3):

(i) We neglect the correlations between the particles in
the condensate band and the particles in the non-
condensate band. Therefore, we assume that the
complete density matrix of the system A can be
written as a direct product between the density of the
condensate band and the density of the non-
condensate band, i.e., $\rho_A = \rho_C \otimes \rho_{NC}$. Since the
main interest is in the dynamics of the condensate
band, the noncondensate band is eliminated from
the description by tracing out the noncondensate
variable, i.e., $\rho_C = \text{Tr}_{NC}(\rho_A)$.

(ii) The density matrix of the noncondensate band $\rho_{NC}$
is considered in thermal equilibrium with the cool-
ing agent at temperature $T_B$. The levels of this
band thermalize much faster with the cooling agent
than the levels of the condensate band. Due to this
assumption, we will completely neglect collisions
between particles of the noncondensate band and
particles of the cooling agent.

Applying these assumptions to Eq. (3), a master equa-
tion for the condensate band can be derived,

\[ \frac{d\rho_C(t)}{dt} = \sum_{\vec{m} \in B_{NC}} \sum_{\vec{n} \in B_C} \Gamma^{\vec{m}, \vec{n}} \langle N_{\vec{m}} \rangle_{n_0} \left( 2a_{\vec{m}}^\dagger \rho_C(t) a_{\vec{m}} - a_{\vec{m}}^\dagger a_{\vec{m}} \rho_C(t) a_{\vec{m}}^\dagger + a_{\vec{m}}^\dagger a_{\vec{m}} \rho_C(t) a_{\vec{m}} - \rho_C(t) a_{\vec{m}}^\dagger a_{\vec{m}} \right) \]

\[ + \sum_{\vec{m} \in B_{NC}} \sum_{\vec{n} \in B_C} \Gamma^{\vec{m}, \vec{n}} \langle N_{\vec{m}} + 1 \rangle_{n_0} \left( 2a_{\vec{n}}^\dagger \rho_C(t) a_{\vec{n}} - a_{\vec{n}}^\dagger a_{\vec{n}} \rho_C(t) a_{\vec{n}}^\dagger + a_{\vec{n}}^\dagger a_{\vec{n}} \rho_C(t) a_{\vec{n}} - \rho_C(t) a_{\vec{n}}^\dagger a_{\vec{n}} \right) \]

\[ + \sum_{\vec{m} \neq \vec{n}} \sum_{\vec{n}, \vec{m} \in B_C} \Gamma^{\vec{m}, \vec{n}} \left( 2a_{\vec{m}}^\dagger a_{\vec{n}} \rho_C(t) a_{\vec{m}}^\dagger a_{\vec{n}} - a_{\vec{m}}^\dagger a_{\vec{n}} a_{\vec{n}} a_{\vec{m}}^\dagger \right) \rho_C(t) a_{\vec{n}}^\dagger a_{\vec{m}}^\dagger a_{\vec{n}} a_{\vec{m}} - a_{\vec{n}}^\dagger a_{\vec{m}} a_{\vec{m}} a_{\vec{n}}^\dagger \right) \]

\[ + \sum_{\vec{m} \neq \vec{n}} \sum_{\vec{n}, \vec{m} \in B_C} \Gamma^{\vec{m}, \vec{n}} \left( 2a_{\vec{m}}^\dagger a_{\vec{n}} \rho_C(t) a_{\vec{m}}^\dagger a_{\vec{n}} - a_{\vec{m}}^\dagger a_{\vec{n}} a_{\vec{n}} a_{\vec{m}}^\dagger \right) \rho_C(t) a_{\vec{n}}^\dagger a_{\vec{m}}^\dagger a_{\vec{n}} a_{\vec{m}} - a_{\vec{n}}^\dagger a_{\vec{m}} a_{\vec{m}} a_{\vec{n}}^\dagger \right) \]
single-particle states are described by plane waves. Evap-
to continuum states (non-bound states). In this case the
present model these particles are considered to escape
the trap depth $\tilde{\eta}$, given that there are $n_0$ bosons in the ground state. Since the terms with $\tilde{\eta} = \tilde{n}$ cancel, this case is already excluded from the summation. The first two terms, Eqs. (7-10), account for collisions between particles in the condensate band and in the noncondensate band, and are illustrated in Fig. 2. The loss of particles from the trap, due to their interaction with the cooling agent atoms. The phonons operators can be written in terms of the creation and annihilation quasiparticles operators $\hat{\beta}_n^\dagger$, $\hat{\beta}_n$, $\hat{\beta}_n^\dagger$, and $\hat{\beta}_n^\dagger$ defined as $\hat{\beta}_n = \hat{\beta}_n^\dagger - \hat{\beta}_n^\dagger\hat{\beta}_n$, where $\hat{\beta}_n$ is the number operator of particles in the ground-state, $n_\eta$ are usual annihilation and creation operators for the trap level $\tilde{n}$. These operators obey the Bose commutation relations, $[\hat{\beta}_n^\dagger, \hat{\beta}_n^\dagger] = \delta_{n_\eta, n_\eta}$. The phonons operators can be written in terms of the creation and annihilation quasiparticles operators $\hat{b}_n^\dagger$, $\hat{b}_n$, $\hat{b}_n$, $\hat{b}_n^\dagger$, with $|\hat{b}_n|^2 = 1$. The formalism of Girardeau and Arnovitt is more general than the one developed by Gardiner, since it covers all cases ranging from a total absence of population in the ground state $n_0 = 0$ to a highly populated ground state $n_0 \approx N$. A detailed comparison of both formalisms can be found in Ref. [25].

We replace the operators $\hat{a}_\eta, \hat{a}_\eta^\dagger, \hat{a}_\eta, \hat{a}_\eta^\dagger$, in Eqs. (7-10) by the particle-number-conserving Girardeau-Arnovitt operators $\hat{\beta}_n, \hat{\beta}_n^\dagger, \hat{\beta}_n, \hat{\beta}_n^\dagger$. Then, the action of the operators on the density matrix $\rho_\eta(t)$ is computed. Since the decoherence time is very fast (see discussion in Ref. [13]), only the diagonal terms of the reduced density matrix $\rho_\eta$ contribute to the dynamics. Hence, only diagonal terms will be considered. In this way, we obtain a master equation for the diagonal elements $P(n_0, \tilde{n}) = \{n_0, \tilde{n}\}|\rho_\eta(t)|n_0, \tilde{n}\}$. Eqs. (7-10) give an accurate treatment of the internal dynamics of the condensate band, describing how the condensate forms and how the particles are scattered by collisions. Effects of elementary excitations of the condensate, representing the “thermal cloud”, do not appear explicitly, since they are hidden in the single-particle operators $\hat{a}_\eta$ and $\hat{a}_\eta^\dagger$ for $\tilde{n} \in B_C$. In the limit of a highly condensed ground state, i.e. $n_0 \approx N$, the effects of the condensate excitations can be totally neglected. During the process of condensate formation, however, their effects play a crucial role.

To analyse the role of elementary excitations, we resort to a formalism of Girardeau and Arnovitt, which conserves the number of particles. The method is based on the annihilation $\hat{\beta}_\eta$ and creation $\hat{\beta}_\eta^\dagger$ operators of one particle in the ground state, defined as $\hat{\beta}_\eta = (\hat{N}_\eta + 1)^{-1/2} \hat{a}_\eta$ and $\hat{\beta}_\eta^\dagger = \hat{\beta}_\eta^\dagger \hat{a}_\eta^\dagger/(\hat{N}_\eta + 1)^{-1/2}$, and in the phonon operators $\hat{\beta}_\eta^\dagger$, $\hat{\beta}_\eta^\dagger$ defined as $\hat{\beta}_\eta^\dagger = \hat{\beta}_\eta^\dagger \hat{a}_\eta^\dagger$ and $\hat{\beta}_\eta^\dagger = \hat{\beta}_\eta^\dagger \hat{a}_\eta^\dagger$, where $\hat{N}_\eta$ is the number operator of particles in the ground-state, $\hat{a}_\eta$ and $\hat{a}_\eta^\dagger$ are usual annihilation and creation operators for the trap level $\tilde{n}$. These operators obey the Bose commutation relations, $[\hat{\beta}_\eta^\dagger, \hat{\beta}_\eta^\dagger] = \delta_{n_\eta, n_\eta}$. The phonons operators can be written in terms of the creation and annihilation quasiparticles operators $\hat{b}_n^\dagger$, $\hat{b}_n$, $\hat{b}_n$, $\hat{b}_n^\dagger$, with $|\hat{b}_n|^2 = 1$. The formalism of Girardeau and Arnovitt is more general than the one developed by Gardiner, since it covers all cases ranging from a total absence of population in the ground state $n_0 = 0$ to a highly populated ground state $n_0 \approx N$. A detailed comparison of both formalisms can be found in Ref. [25].

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\[ \frac{d}{dt} P(n_0, \vec{n}, t) = \frac{d}{dt} P_{\text{growth}}(n_0, \vec{n}, t) + \frac{d}{dt} P_{\text{scatt}}(n_0, \vec{n}, t) + \frac{d}{dt} P_{\text{evap}}(n_0, \vec{n}, t). \] (11)

The rate \( \frac{d}{dt} P_{\text{growth}}(n_0, \vec{n}, t) \) represents the contribution of the condensate itself and the quasiparticles for the growth of the condensate. The number of particles in the condensate band changes. \( \frac{d}{dt} P_{\text{scatt}}(n_0, \vec{n}, t) \) describes scattered particles in the condensate band. During the scattering process the number of particles in the condensate band does not change. Finally, \( \frac{d}{dt} P_{\text{evap}}(n_0, \vec{n}, t) \) includes the evaporation of particles with energies larger than the trap depth.

**IV. RATE EQUATIONS FOR THE CONDENSATE BAND**

**A. Growth**

Consider the diagonal elements of the Eqs. (1-3) for \( \vec{n} = 0 \) and \( \vec{n} \neq 0 \) with the replacements described in the previous section, we arrive at a master equation for the occupation probability of the condensate band

\[ \frac{d}{dt} P_{\text{growth}}(n_0, \vec{n}, t) = R_C + R_Q, \] (12)

where \( R_C \) represents the growth rate of the condensate itself

\[ R_C = 2n_0 \Gamma^+(n_0 - 1)P(n_0 - 1, \vec{n}) - 2(n_0 + 1)\Gamma^+(n_0)P(n_0, \vec{n}) + 2(n_0 + 1)\Gamma^-(n_0 + 1)P(n_0 + 1, \vec{n}) - 2n_0 \Gamma^-(n_0)P(n_0, \vec{n}). \] (13)

The coefficients \( \Gamma^+(n_0) \) and \( \Gamma^-(n_0) \) carry information about particles entering and leaving the condensate

\[ \Gamma^+(n_0) \equiv \sum_{\vec{m} \neq \vec{0}} \Gamma_{\vec{m}}^+(N_{\vec{m}})n_0, \]

\[ \Gamma^-(n_0) \equiv \sum_{\vec{m} \neq \vec{0}} \Gamma_{\vec{m}}^-(N_{\vec{m}} + 1)n_0. \]

The rate \( R_Q \) takes into account the contribution of the elementary excitations

\[ R_Q = 2 \sum_{\vec{n}} n_0 \Gamma^+_\vec{n} (n_0 - 1, \vec{n} - \vec{e}_\vec{n}) P(n_0 - 1, \vec{n} - \vec{e}_\vec{n}) + 2 \sum_{\vec{n}} n_0 (n_0 + 1) \Gamma^-_{\vec{n}} (n_0, \vec{n}) P(n_0, \vec{n}) \]

\[ -2 \sum_{\vec{n}} (n_0 + 1) \Gamma^+_{\vec{n}} (n_0, \vec{n} + \vec{e}_\vec{n}) P(n_0 + 1, \vec{n} + \vec{e}_\vec{n}) - 2 \sum_{\vec{n}} n_0 \Gamma^-_{\vec{n}} P(n_0, \vec{n}) + 2 \sum_{\vec{n}} (n_0 + 1) \Gamma^-_{\vec{n}} (n_0 - 1, \vec{n} + \vec{e}_\vec{n}) P(n_0 - 1, \vec{n} + \vec{e}_\vec{n}) \]

\[ + 2 \sum_{\vec{n}} (n_0 + 1) \Gamma^+_{\vec{n}} (n_0 - 1, \vec{n} + \vec{e}_\vec{n}) P(n_0 - 1, \vec{n} + \vec{e}_\vec{n}). \]

(14)

with \( \vec{e}_\vec{n} \equiv (..., 0, 0, 1, 0, 0, ...) \). The coefficients \( \Gamma^+_\vec{n} (n_0, \vec{n}), \Gamma^-_{\vec{n}} (n_0, \vec{n}), \Gamma^+_{\vec{n}} (n_0, \vec{n}) \) and \( \Gamma^-_{\vec{n}} (n_0, \vec{n}) \) are defined as

\[ \Gamma^+_\vec{n} (n_0, \vec{n}) \equiv \sum_{\vec{m} \neq \vec{0}} \Gamma_{\vec{m}}^+(N_{\vec{m}})n_0|u_{\vec{m}}|^2, \]

\[ \Gamma^-_{\vec{n}} (n_0, \vec{n}) \equiv \sum_{\vec{m} \neq \vec{0}} \Gamma_{\vec{m}}^-(N_{\vec{m}} + 1)n_0|u_{\vec{m}}|^2, \]

\[ \Gamma^+_{\vec{n}} (n_0, \vec{n}) \equiv \sum_{\vec{m} \neq \vec{0}} \Gamma_{\vec{m}}^+(N_{\vec{m}})n_0|v_{\vec{m}}|^2, \]

\[ \Gamma^-_{\vec{n}} (n_0, \vec{n}) \equiv \sum_{\vec{m} \neq \vec{0}} \Gamma_{\vec{m}}^-(N_{\vec{m}} + 1)n_0|v_{\vec{m}}|^2, \]

where \( \vec{m} \in B_{NC} \) and \( \vec{n} \in B_C \). These coefficients describe processes of creation and annihilation of phonons inside the condensate.

In order to obtain the dynamics of the population in the condensate band, the mean number of particles is evaluated. We will start with the contribution of the condensate mode itself. In Eq. (13) we use the thermodynamic relation between the rate coefficients

\[ \Gamma^+(n_0) = \Gamma^- (n_0) e^{\beta(\epsilon_0 - \mu_0)} \]

in combination with the factorization assumption \( P(n_0, \vec{n}) = P(n_0) \otimes P(\vec{n}) = P(n_0) \ldots \otimes P(n_0) \otimes \ldots \). Here \( \beta = 1/k_B T_B \), \( \epsilon_0 \) is the energy of the ground state and \( \mu_0 \equiv \mu(n_0) \) the chemical potential for \( n_0 \) particles in the ground state. For the rate of change of the condensate number one then finds

\[ \frac{dn_0}{dt} \mid_{\text{growth}} = 2 \left( (n_0 + 1)\Gamma^+(n_0) - n_0 \Gamma^-(n_0) \right) \]

\[ = 2\Gamma^+(n_0) \left( n_0 (1 - e^{-\beta(\mu_0 - \epsilon_0)}) + 1 \right). \] (15)

In the limit of \( n_0 \gg N \), with \( N \gg 1 \), the equation above can be approximated by

\[ \frac{dN}{dt} \approx 2\Gamma^+(N) \left( 1 - e^{\beta(\mu_0 - \epsilon_0)} \right), \] (16)

where the coefficient \( \Gamma^+(N) \) contains information about the sympathetic cooling process between system \( A \) and cooling agent \( B \).

If we apply the formalism described here to a gas of only one species, we find a rate equation for the growth of the condensate which has a form similar to the one
obtained by Gardiner and coworkers based on Eq. (23) in Ref. [13] and Eq. (168) in Ref. [14]. However, due to the properties of the Giradeau-Arnovitt formalism, the rate equation in our case is not limited only to highly populated condensates but also includes condensates of arbitrary particle number.

We now explore the contribution of the elementary excitations for the growth process. We use the thermodynamic relation between the coefficients

\[ \Gamma_{\vec{n}}^{++}(n_0, \vec{n}) = \Gamma_{\vec{n}}^{--}(n_0, \vec{n})e^{2(\epsilon_\alpha-\mu_0)}, \]

\[ \Gamma_{\vec{n}}^{+-}(n_0, \vec{n}) = \Gamma_{\vec{n}}^{-+}(n_0, \vec{n})e^{2(\epsilon_\alpha-\mu_0)}, \]

and factorize all correlations i.e., \( P(n_0, \vec{n}) = P(n_0) \otimes \ldots P(n_n) \ldots \) in Eq. (4). Thus, the rate equation for the elementary excitations is given by

\[
\frac{d\vec{n}}{dt}_{\text{growth}} = 2 \sum \vec{n} \Gamma_{\vec{n}}^{++}(n_0, \vec{n}) \left( 1 - e^{2(\mu_0-\epsilon_\alpha)} \right) n_{\vec{n}} + 1 \\
-2 \sum \vec{n} \Gamma_{\vec{n}}^{+-}(n_0, \vec{n}) \left( 1 - e^{2(\mu_0-\epsilon_\alpha)} \right) n_{\vec{n}} + 1.
\]

A simplified form of Eq. (17) can be derived by using the relation \( \Gamma_{\vec{n}}^{+-}(n_0, \vec{n}) = \Gamma_{\vec{n}}^{-+}(n_0, \vec{n}) = \Gamma_{\vec{n}}^{-+}(n_0, \vec{n}) \).

\[
\frac{d\vec{n}}{dt}_{\text{growth}} = 2 \sum \vec{n} \Gamma_{\vec{n}}^{+}(n_0, \vec{n}) \left( 1 - e^{2(\mu_0-\epsilon_\alpha)} \right) n_{\vec{n}} + 1.
\]

with \( \Gamma_{\vec{n}}^{+}(n_0, \vec{n}) = \sum \vec{m} \Gamma_{\vec{n}, \vec{m}}(N_{\vec{m}})n_{\vec{m}} \).

The total growth equation can be obtained from the contribution of the condensate mode itself Eq. (15) and the elementary excitations Eq. (18).

**B. Scattering**

In this subsection the effects of the scattering processes of atoms inside the condensate band are considered, as described by Eqs. (1)-(3). During the scattering process the number of particles in the condensate band does not change.

We use the same analytical procedure described in the previous section to derive a rate equation related to the scattering process. However, we will omit details of this calculation, since the expressions can be rather large. Thus, the corresponding scattering rate equation for \( n_{\vec{n}} = (n_{\vec{n}}^a, a_{\vec{n}}) \) has the form

\[
\frac{d\vec{n}}{dt}_{\text{scatt}} = 2 \sum \vec{n} \Gamma_{\vec{n}}^{u+}(n_0, \vec{n}) \left[ e^{-2(\epsilon_\alpha-\epsilon_\alpha)} - n_{\vec{n}}(1 - e^{2(\epsilon_\alpha-\epsilon_\alpha)}) \right] \\
+2 \sum \vec{n} \Gamma_{\vec{n}}^{v+}(n_0, \vec{n}) \left[ 1 - e^{2(\epsilon_\alpha-\epsilon_\alpha)} \right] + 1 \\
+2 \sum \vec{n} \Gamma_{\vec{n}}^{u+}(n_0, \vec{n}) \left[ 1 - e^{2(\epsilon_\alpha-\epsilon_\alpha)} \right] + 1 \\
-2 \sum \vec{n} \Gamma_{\vec{n}}^{v+}(n_0, \vec{n}) \left[ 1 - e^{2(\epsilon_\alpha-\epsilon_\alpha)} \right] + 1.
\]

The first two terms of Eq. (15) account for the scattering between the particles in the ground state and the quasiparticles of different levels. The other terms describe the scattering between quasiparticles of different levels.

**C. Evaporation**

We now discuss the evaporation of particles from the trap. Atoms can escape from the condensate band of the trap to unbound states. This loss of particles from the trap is induced by their interaction with the cooling agent. This process is particularly important in the initial stage of the cooling process [13]. To implement evaporation in the present model, particles are assumed to escape to continuum states (unbound states) inside the noncondensate band. Only Eq. (5) contributes to this process, since the rate coefficients of Eq. (1), which includes the overlap of the wave functions involved in the process, is completely negligible. Taking the diagonal elements of Eq. (8), \( P_{\text{evap}} \) is obtained

\[
\frac{d}{dt} P_{\text{evap}}(n_0, \vec{n}, t) = -2n_0 \gamma^-(n_0) P(n_0, \vec{n}) \\
+2(n_0 + 1) \gamma^-(n_0 + 1) P(n_0 + 1, \vec{n}) \\
+2 \sum_{\vec{n} \in B_C} (n_0 + 1) \gamma^-(n_0 + 1, \vec{n} + \vec{e}_n) P(n_0 + 1, \vec{n} + \vec{e}_n) \\
-2 \sum_{\vec{n} \in B_C} n_0 \gamma^-(n_0) P(n_0, \vec{n})
\]
\[ +2 \sum_{\vec{n} \in \mathcal{B}_C} n_{\vec{n}} \gamma^{-\vec{n}}_\vec{n} (n_0 + 1, \vec{n} - \vec{e}_n) P(n_0 + 1, \vec{n} - \vec{e}_n) \]
\[ -2 \sum_{\vec{n} \in \mathcal{B}_C} (n_{\vec{n}} + 1) \gamma^{\vec{n}+}_\vec{n} P(n_0, \vec{n}), \]
(21)

where
\[
\gamma^{-}(n_0, \vec{n}) \equiv \sum_{\vec{m} \in \mathcal{B}^{(i)}_{NC}} \Gamma_{\vec{m},0},
\]
\[
\gamma^{-\vec{n}}_\vec{n} (n_0, \vec{n}) \equiv \sum_{\vec{m} \in \mathcal{B}^{(i)}_{NC}} \Gamma_{\vec{m},\vec{n}} |u_{\vec{m}}|^2,
\]
\[
\gamma^{\vec{n}+}_\vec{n} (n_0, \vec{n}) \equiv \sum_{\vec{m} \in \mathcal{B}^{(i)}_{NC}} \Gamma_{\vec{m},\vec{n}} |v_{\vec{m}}|^2
= \gamma^{\vec{n}+}_\vec{n} (n_0, \vec{n}) - \gamma^{-\vec{n}}_\vec{n},
\]
(22)

with \( \mathcal{B}^{(i)}_{NC} \) denoting the noncondensate band with energy \( \epsilon_{\vec{m}} \geq \epsilon_{\vec{n}} \). As mentioned at the end of section [1], these rate coefficients \( \Gamma_{\vec{m},0} \) and \( \Gamma_{\vec{m},\vec{n}} \) have a different form from the ones which appear in the growth and scattering processes, since now \( \vec{m} \) describes unbound states. The first line of Eq. (21) describes the evaporation of an atom out of the ground state while the others describe the evaporation of an atom out of the \( n \)-excited level of the condensate band.

Following the same procedure described in the previous subsections, a rate equation for the evaporation of quasiparticles in the condensate band is obtained:
\[
\frac{dn_{\vec{n}}}{dt}_{\text{evap}} = -2 \sum_{\vec{n}} \gamma^{-\vec{n}}_\vec{n} \, n_{\vec{n}} |_{\text{evap}},
\]
\[
\frac{dn_0}{dt}_{\text{evap}} = -2 \sum_{\vec{n}} \gamma^{-\vec{n}}_\vec{n} \, n_{\vec{n}} |_{\text{evap}},
\]
(23)

with
\[
\gamma^{-\vec{n}}_\vec{n} = \sum_{\vec{m} \in \mathcal{B}^{(i)}_{NC}} \Gamma_{\vec{m},\vec{n}} \gamma^{-\vec{n}}_\vec{m}.
\]
(24)

V. CONCLUSION

The total rate equation describing a number-conserving population of the condensate band can be obtained from Eqs. (15, 17, 19, 23)
\[
\frac{dn_{\vec{n}}}{dt} = \frac{dn_{\vec{n}}}{dt}_{\text{growth}} + \frac{dn_{\vec{n}}}{dt}_{\text{scatt}} + \frac{dn_{\vec{n}}}{dt}_{\text{evap}},
\]
(25)

where \( n_0 \) and \( n_{\vec{n}} \) denote, respectively, the population in the condensate and the population of the elementary excitations, which are commonly referred to as "thermal cloud". The equation above gives a complete description for the thermalization of a system \( A \) which is in thermal contact with a bath \( B \). All the information about the dynamics of the thermalization process is contained in the coefficients \( \Gamma \). Analytical and numerical evaluation of the coefficients for a specific system can be carried out along the lines described in Refs. [12, 13].

The result obtained here can be applied to describe the dynamics of sympathetic cooling of a gas in thermal contact with a cooling agent, in terms of the population in the ground-state and elementary excitations. In particular, it can be applied to the case where the cooling agent thermalize much faster compared to the thermalization of the system. The description remains valid for the quantum degenerate regime \( T \ll T_c \), where \( T_c \) is the critical temperature of the gas.

The use of the Giradeau-Arnovitt method has opened up the possibility to describe sympathetic cooling in terms of particles in the ground state and elementary excitation during the whole cooling process, i.e., from the case of total absence of particles in the condensate \((n_0 = 0)\) to the case of a highly populated ground state \((n_0 = N)\). In addition, we include the effects of evaporation which are especially important during the initial stages of the cooling process. To our knowledge this is the first time that a formal complete number-conserving description of sympathetic cooling in terms of particles in the ground state and elementary excitations is given.

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