THE COMPLEX GENERA, SYMMETRIC FUNCTIONS AND MULTIPLE
ZETA VALUES

PING LI

Abstract. We examine the coefficients in front of Chern numbers for complex genera, and
pay special attention to the $\text{Td}^1$-genus, the $\Gamma$-genus as well as the Todd genus. Some
related geometric applications to hyper-Kähler and Calabi-Yau manifolds are discussed. Along
this line and building on the work of Doubilet in 1970s, various Hoffman-type formulas for
multiple-(star) zeta values and transition matrices among canonical bases of the ring of sym-
metric functions can be uniformly treated in a more general framework.

1. Introduction

The notion of oriented genus was introduced by Hirzebruch ([Hir66]), which is a ring ho-
momorphism from the rational oriented cobordism ring to $\mathbb{Q}$. Building on Thom’s pioneer
work ([Th54]), it turns out that oriented genera correspond one-to-one to monic formal power
series and are rationally linear combination of Pontrjagin numbers. Two typical examples
are Hirzebruch’s $L$-genus and the $\hat{A}$-genus. The Hirzebruch signature theorem implies that
$L(\cdot)$ is the signature of the intersection pairing on $H^{2k}(M; \mathbb{R})$ and hence an integer ([Hir66,
§8]). The integrality of $\hat{A}(\cdot)$ for spin manifolds was observed by Borel and Hirzebruch ([BH59,
§25]). This fact both motivated and was later explained by the Atiyah–Singer index theorem,
which showed that the $\hat{A}$-genus of a spin manifold is the index of its Dirac operator
([AS68, §5]). Since then various geometric and topological facets of the $L$-genus and $\hat{A}$-genus
have been extensively investigated. Most notably, the former is related to the question of
the homotopy invariance of the higher signatures, known as the Novikov conjecture, and the
latter is deeply related to the existence of positive scalar curvature metrics on spin manifolds
([Lic63],[Hit74],[GL80],[Sto92]).

In contrast to these, the arithmetic properties of these two genera had not yet been fully
studied, except the obvious connection with Bernoulli numbers ([MS74, p.281]). In a recent
work [BB18], Berglund and Bergström showed that the coefficients in front of the Pontrjagin
numbers of the $\hat{A}$-genus and $L$-genus can be expressed in terms of multiple-star zeta values and
an alternating version respectively ([BB18, Thms 1 and 4]). In particular, all these coefficients
are nonzero and their signs can be explicitly determined. Their main idea is to first express
these coefficients in terms of values of the usual Riemann zeta function on even integers, and
then apply Hoffman-type formulas ([Ho92]) to arrive at the desired results. Note that in this
process some quite deep results in algebraic combinatorics are needed.

Inspired by the work [BB18] and some ideas therein, the main purpose of this article is to
study the coefficients in front of Chern numbers for complex genera and then focus on three

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cases: the $\text{Td}^+$-genus, the $\Gamma$-genus as well as the Todd genus. In order to state some results in this article, let us introduce more notation.

An integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a finite sequence of positive integers in non-increasing order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 1$. Denote by $l(\lambda) := l$ and $|\lambda| := \sum_{i=1}^{l} \lambda_i$ and they are called respectively the length and weight of the partition $\lambda$. These $\lambda_i$ are called parts of $\lambda$. We may write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$. It is also convenient to use another notation which indicates the number of times each integer appears: $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \ldots)$. This means that $i$ appears with multiplicity $m_i(\lambda)$ among the parts $\lambda_i$. We use the notation $2\lambda$ to denote the partition $(2\lambda_1, 2\lambda_2, \ldots, 2\lambda_{l(\lambda)})$.

Recall that complex genera are defined via the complex cobordism ring, correspond one-to-one to monic formal power series and are rationally linear combination of Chern numbers ([Mi60]). Let $\varphi$ be a complex genus whose associated power series is $Q(x)$. Denote by $e_i$ the $i$-th elementary symmetric function in the countably many (commuting) variables $x_1, x_2, \ldots$ (more details can be found in Section 2.2), and

$$1 + \sum_{i=1}^{\infty} Q_i(e_1, \ldots, e_i) := \prod_{i=1}^{\infty} Q(x_i),$$

where $Q_i(e_1, \ldots, e_i)$ denotes the homogeneous part of degree $i$ in $\prod_{i=1}^{\infty} Q(x_i)$ (deg($x_i$) = 1). Let $M$ be a compact, almost-complex manifold of real dimension $2n$ with Chern classes $c_i$. Then the value $\varphi(M)$ is given by

$$\varphi(M) = \int_M Q_n(c_1, \ldots, c_n) =: \sum_{|\lambda|=n} b_\lambda(\varphi) C_\lambda[M],$$

where $C_\lambda[M]$ is the Chern number associated to the partition $\lambda$ and $b_\lambda(\varphi)$ the coefficient in front of it.

For real numbers $t_1, \ldots, t_r > 1$, define the two series

$$\zeta(t_1, \ldots, t_r) := \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{n_1^{t_1} n_2^{t_2} \cdots n_r^{t_r}},$$

and

$$\zeta^*(t_1, \ldots, t_r) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_r \geq 1} \frac{1}{n_1^{t_1} n_2^{t_2} \cdots n_r^{t_r}},$$

which were introduced by Hoffman ([Ho92]) and Zagier ([Za94]) independently and can be viewed as multiple versions of the classical Riemann zeta function

$$\zeta(t) := \sum_{n=1}^{\infty} \frac{1}{n^t}, \quad t > 1.$$  

When these $t_1, \ldots, t_r \in \mathbb{Z}_{>0}$, (1.3) and (1.4) are called multiple zeta values (MZV for short) and multiple-star zeta values (MSZV for short) respectively. We refer to [Zh16] for a thorough treatment on various algebraic facets of MZV and MSZV.

The symmetrization of $\zeta(t_1, \ldots, t_r)$ and $\zeta^*(t_1, \ldots, t_r)$ are defined respectively by

$$\zeta_S(t_1, \ldots, t_r) := \sum_{\sigma \in S_r} \zeta(t_{\sigma(1)}, \ldots, t_{\sigma(r)})$$

and

$$\zeta^*_S(t_1, \ldots, t_r).$$
and
\[(1.7) \quad \zeta_S^*(t_1, \ldots, t_r) := \sum_{\sigma \in S_r} \zeta^*(t_{\sigma(1)}, \ldots, t_{\sigma(r)}),\]
where $S_r$ is the permutation group on $\{1, \ldots, r\}$.

The $\text{Td}^{\frac{1}{2}}$-genus is the complex genus whose formal power series is
\[Q(x) = \left(\frac{x}{1 - e^{-x}}\right)^{\frac{1}{2}},\]
the square root of the usual Todd genus. Our first main result is the following

**Theorem 1.1** (Corollary 5.3). The coefficients $b_{2\lambda}(\text{Td}^{\frac{1}{2}})$ of the Chern numbers $C_{2\lambda}(-\cdot)$ in the $\text{Td}^{\frac{1}{2}}$-genus are given by
\[b_{2\lambda}(\text{Td}^{\frac{1}{2}}) = \frac{(-1)^{|\lambda|-l(\lambda)}}{(2\pi)^{|\lambda|} \prod_i m_i(\lambda)!} \cdot \zeta_S^*(2\lambda_1, \ldots, 2\lambda_{l(\lambda)}) \quad (0! := 1).\]
In particular, the coefficient $b_{2\lambda}(\text{Td}^{\frac{1}{2}})$ is nonzero for every partition $\lambda$. It is positive if $|\lambda|-l(\lambda)$ is even and negative if $|\lambda|-l(\lambda)$ is odd.

The $\text{Td}^{\frac{1}{2}}$-genus is of geometric importance in hyper-Kähler geometry. An irreducible hyper-Kähler manifold $M$ is a simply-connected compact Kähler manifold and $H^0(M, \Omega^2_M)$ is spanned by an everywhere non-degenerate holomorphic 2-form $\tau$, whose complex dimension is necessarily even, say $2n$. This definition is equivalent to the fact that, as a Riemannian manifold, its holonomy is equal to $\text{Sp}(n)$. Such a $\tau$ yields an isomorphism between the holomorphic tangent and cotangent bundles of $M$ and thus odd Chern classes are torsion elements. This means that only Chern numbers of the forms $C_{2\lambda}[M] (|\lambda| = n)$ are involved. We refer the reader to [GHJ03] for more details on irreducible hyper-Kähler manifolds. It is commonly believed that Chern numbers of irreducible hyper-Kähler manifolds should satisfy strong arithmetic constraints (see Section 6 for more details). We refer the reader to [OSV22, §4] and the references therein for some related open questions and comments. Hitchin and Sawon showed that $\text{Td}^{\frac{1}{2}}(M) > 0$ ([HS01]) and Jiang recently showed that $\text{Td}^{\frac{1}{2}}(M) < 1$ when $n \geq 2$ ([Ji23, Cor. 5.1]). Applying Theorem 1.1, we can reformulate Hitchin–Sawon and Jiang’s results in the following version related to MSZV.

**Corollary 1.2.** Let $M$ be an irreducible hyper-Kähler manifold of complex dimension $2n \geq 4$. Then we have
\[0 < \sum_{i(\lambda) - n \text{ are even}} \frac{\zeta_S^*(2\lambda_1, \ldots, 2\lambda_{l(\lambda)})}{\prod_i m_i(\lambda)!} C_{2\lambda}[M] - \sum_{i(\lambda) - n \text{ are odd}} \frac{\zeta_S^*(2\lambda_1, \ldots, 2\lambda_{l(\lambda)})}{\prod_i m_i(\lambda)!} C_{2\lambda}[M] < (2\pi)^{2n}.

The $\Gamma$-genus is the complex genus whose formal power series is
\[(1.8) \quad Q(x) = \frac{1}{\Gamma(1 + x)} := \exp(\gamma x) \prod_{i=1}^{\infty} \left(1 + \frac{x}{i}\right) \exp\left(-\frac{x}{i}\right),\]
where
\[(1.9) \quad \gamma := \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \ln n\right)

is usually called the Euler constant. This $\Gamma$-genus was introduced by Libgober in [Lib99] in connection with mirror symmetry. It turns out that ([Lib99, p.142-143]) the Chern class
polynomials $Q_i(c_1(X), \ldots, c_i(X))$ (in the notation of (1.1)) of certain Calabi-Yau manifolds $X$ are related to the coefficients of the generalized hypergeometric series expansion of the period of a mirror of $X$. He also showed in [Lib99, p.143] that $Q_1(c_1) = \gamma c_1$ and the coefficient of $c_i$ in $Q_i(c_1, \ldots, c_i)$ is $\zeta(i)$ when $i \geq 2$.

Building on his work [Ho97], Hoffman presented in [Ho02] a closed formula for the coefficients $b_\lambda(\Gamma)$ by introducing a ring homomorphism from the ring of symmetric functions to $\mathbb{Q}$ and explaining $b_\lambda(\Gamma)$ as images of monomial symmetric functions (more details can be found in Section 5.2). We shall extend Hoffman’s observation to show the following

**Theorem 1.3** (⇐ Theorem 5.6). When $m_1(\lambda) = 0$, i.e., $\lambda_{(\lambda)} \geq 2$, the coefficients $b_\lambda(\Gamma)$ of the Chern numbers $C_\lambda[\cdot]$ in the $\Gamma$-genus are given by

$$b_\lambda(\Gamma) = \frac{\zeta_S(\lambda_1, \ldots, \lambda_{(\lambda)})}{\prod_i m_i(\lambda)!},$$

which is always positive. In particular, these determine all the coefficients for Calabi-Yau manifolds.

**Remark 1.4.** In this article, by a Calabi-Yau manifold we mean in the weak sense that its first Chern class is a torsion element.

Among the complex genera, the Todd genus, whose associated power series is $x/(1 - e^{-x})$, is the most classical as it equals to the arithmetic genus of compact complex manifolds due to the Hirzebruch-Riemann-Roch theorem. But, as we shall see in Section 5.3, the arithmetic expressions of the coefficients in the Todd genus cannot be made as compact as those of the $\text{Td}_{\Sigma}^2$-genus and $\Gamma$-genus, as illustrated by Theorems 1.1 and 1.3. Nevertheless, some partial arithmetic information on them can still be obtained, which will be briefly summarized in Proposition 5.8.

The rest of this article is organized as follows. We prepare some preliminaries in Section 2 on the partially ordered set consisting of the partitions of a finite set and four classical bases of the ring of symmetric functions. In Section 3 Doubilet’s constructions and formulas in [Do72] will be extended and applied to uniformly treat various Hoffman-type formulas. With the tools in Sections 2 and 3 in hand, Section 4 is devoted to the study of the coefficients in front of Chern numbers for the general complex genera, and the three aforementioned complex genera will be discussed in detail in Section 5, from which Theorems 1.1 and 1.3 follow as direct consequences. Some more remarks on the Chern numbers of irreducible hyper-Kähler manifolds will be presented in the last section, Section 6.

2. Preliminaries

2.1. Partitions of a finite set. The materials in this subsection can be found in [St97, §3].

A poset (partially ordered set) $(P, \leq)$ is a set $P$, together with a binary relation “$\leq$” on $P$ which is reflexive ($x \leq x$, $\forall x \in P$), antisymmetric ($x \leq y$ and $y \leq x$ imply $x = y$), and transitive ($x \leq y$ and $y \leq z$ imply $x \leq z$). We use the obvious notation $x < y$ to mean $x \leq y$ and $x \neq y$. The poset we shall deal with in this article is the case below.

**Definition 2.1.** (1) A partition of a finite set $S$ is a collection of disjoint nonempty subsets of $S$ whose union is $S$. Denote by $\Pi(S)$ the set consisting of all partitions of $S$. If $\pi = \{\pi_1, \ldots, \pi_l\} \in \Pi(S)$, each $\pi_i$ is called a block of $\pi$. Let $l(\pi) = l$ and
call it the length of $\pi$. As in [St97] we use the convention that $[n] := \{1, \ldots, n\}$ and $\Pi_n := \Pi([n])$.

(2) Define $\pi \leq \rho$ in $\Pi(S)$ if every block of $\pi$ is contained in a block of $\rho$. Note that $(\Pi(S), \leq)$ is a poset by easily checking the above-mentioned three properties of “$\leq$”. Also note that $(\Pi(S), \leq)$ has a unique minimal (resp. maximal) element, denoted by $\hat{0}$ (resp. $\hat{1}$), whose blocks are precisely one-element subsets in $S$, i.e., $l(\hat{0}) = |S|$ (resp. which has only one block $S$, i.e., $l(\hat{1}) = 1$). Here as usual $|\cdot|$ denotes the cardinality of a set.

(3) For $\pi, \rho \in \Pi(S)$, there exists a unique element in $\Pi(S)$ denoted by $\pi \wedge \rho$, such that it is a lower bound of $\pi$ and $\rho$ (i.e., $\pi \wedge \rho \leq \pi$ and $\pi \wedge \rho \leq \rho$) and every lower bound $\tau$ of $\pi$ and $\rho$ satisfies $\tau \leq \pi \wedge \rho$. We call $\pi \wedge \rho$ the greatest lower bound of $\pi$ and $\rho$ ([St97, p.102]). Note that if $\pi = \{\pi_i\}$ and $\rho = \{\rho_j\}$, then $\pi \wedge \rho = \hat{0}$ if and only if $|\pi_i \cap \rho_j| \leq 1$ for all $i$ and $j$.

(4) Each $\pi = \{\pi_1, \ldots, \pi_l\} \in \Pi(S)$ is naturally associated to an integer partition, denoted by $\lambda(\pi)$, whose parts are $|\pi_1|, \ldots, |\pi_l|$ and which is called the type of $\pi$.

**Remark 2.2.** The notion of a partition of a finite set defined here should not be confused with that of an integer partition introduced in the Introduction.

Every poset $(P, \leq)$ can be associated with a Möbius functions $\mu : P \times P \rightarrow \mathbb{Z}$ defined inductively by

\[
\mu(x, y) := \begin{cases} 
1, & \text{if } x = y, \\
-\sum_{z \leq y} \mu(x, z), & \text{if } x < y, \\
0, & \text{otherwise},
\end{cases}
\]

whose importance lies in the Möbius inversion formula ([St97, p.116]). For a thorough treatment on its basic properties and examples we refer the reader to [St97, §3.6-3.10]. In the following lemma we only encode the Möbius function of $(\Pi(S), \leq)$ in the form we shall use ([St97, Example 3.10.4]).

**Lemma 2.3.** Assume that $\pi = \{\pi_1, \ldots, \pi_{|\pi|}\} \leq \rho = \{\rho_1, \ldots, \rho_{|\rho|}\} \in \Pi(S)$, and

\[
\begin{align*}
(\pi \rightarrow \rho_i) & := \{\pi_j \mid \pi_j \subset \rho_i\}, \\
|\pi \rightarrow \rho_i| & := \text{the cardinality of } (\pi \rightarrow \rho_i).
\end{align*}
\]

Then the Möbius function $\mu(\pi, \rho)$ is given by

\[
\mu(\pi, \rho) = (-1)^{|\pi| - |\rho|} \prod_{i=1}^{l(\rho)} (|\pi \rightarrow \rho_i| - 1)!.
\]

In particular,

\[
\mu(\hat{0}, \rho) = (-1)^{|S| - |\rho|} \prod_{i=1}^{l(\rho)} (|\rho_i| - 1)!,
\]

and moreover,

\[
\sum_{\pi \leq \rho} |\mu(\hat{0}, \pi)| = \lambda(\rho)!, \quad \lambda(\rho)! := \prod_i |\rho_i|!.
\]
2.2. Symmetric functions. The standard references of this subsection are [Mac95, §1] and [St99, §7].

Let \( \mathbb{Q}[[x_1, x_2, \ldots ]] \) be the ring of formal power series over \( \mathbb{Q} \) in a countably infinite set of (commuting) variables \( x_i \). An \( f(x_1, x_2, \ldots ) \in \mathbb{Q}[[x_1, x_2, \ldots ]] \) is called a symmetric function if it satisfies

\[
f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots ) = f(x_1, x_2, \ldots ), \quad \forall \sigma \in S_k, \forall k \in \mathbb{Z}_{>0}.
\]

Here, if \( \sigma \in S_k, \sigma(i) = i \) for \( i > k \) is understood.

Let \( \Lambda^k(x) \) be the vector space of symmetric functions of homogeneous degree \( k \) (\( \deg(x_i) := 1 \)). Then the ring of symmetric functions \( \Lambda(x) := \bigoplus_{n=0}^{\infty} \Lambda^n(x) \) consists of all symmetric functions with bounded degree.

In the rest of this subsection we assume that \( \lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)}) \) is an integer partition of weight \( n \).

The elementary symmetric function \( e_k \in \Lambda^k(x) \) is defined by

\[
e_k = e_k(x_1, x_2, \ldots ) := \sum_{1 \leq i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k},
\]

and

\[
e_\lambda(x) := \prod_{i=1}^{l(\lambda)} e_{\lambda_i} \in \Lambda^n(x).
\]

The power sum symmetric function \( p_k \in \Lambda^k(x) \) is defined by

\[
p_k = p_k(x_1, x_2, \ldots ) := \sum_{i=1}^{\infty} x_i^k,
\]

and

\[
p_\lambda(x) := \prod_{i=1}^{l(\lambda)} p_{\lambda_i} \in \Lambda^n(x).
\]

The monomial symmetric function \( m_\lambda(x) \in \Lambda^n(x) \) is defined by

\[
m_\lambda(x) := \sum_{(\alpha_1, \alpha_2, \ldots)} x_{1}^{\alpha_1} x_{2}^{\alpha_2} \cdots,
\]

where the sum is over all distinct permutations \((\alpha_1, \alpha_2, \ldots)\) of the entries of the vector \( \lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)}, 0, \ldots) \). In other words, \( m_\lambda(x) \) is the smallest symmetric function containing the monomial \( x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_{l(\lambda)}^{\lambda_{l(\lambda)}} \).

The complete symmetric function \( h_k \in \Lambda^k(x) \) is defined by

\[
h_k = h_k(x_1, x_2, \ldots ) := \sum_{\text{integer partitions } \mu, l(\mu) = k} m_\mu(x),
\]

and

\[
h_\lambda(x) := \prod_{i=1}^{l(\lambda)} h_{\lambda_i} \in \Lambda^n(x).
\]
It is well-known that ([Mac95, §1.2]) the four sets

\[ \{ e_\lambda \mid |\lambda| = n \}, \quad \{ p_\lambda \mid |\lambda| = n \}, \quad \{ m_\lambda \mid |\lambda| = n \}, \quad \{ h_\lambda \mid |\lambda| = n \} \]

are all bases of the vector space \( \Lambda^n(x) \).

The following fact will be used in the sequel ([St99, p.292]).

**Lemma 2.4.** Let \( \{ x_1, x_2, \ldots \} \) and \( \{ y_1, y_2, \ldots \} \) be two countable sets of (commuting) variables \( x_i \) and \( y_j \). Then we have

\[
\prod_{i,j=1}^{\infty} (1 + x_i y_j) = 1 + \sum_{|\lambda| \geq 1} m_\lambda(x) e_\lambda(y) = 1 + \sum_{|\lambda| \geq 1} m_\lambda(y) e_\lambda(x),
\]

where the sum is over all positive integer partitions.

3. **Doubilet's formulas and applications**

Since the vector space \( \Lambda^n(x) \) has four bases in (2.7), a natural question is what the transition matrices are between these four bases. We denote by, for instance, \( M(e, m) \) the transition matrix \( (M_{\lambda\mu}) \) of coefficients in the equations

\[ e_\lambda(x) = \sum_{\mu} M_{\lambda\mu} \cdot m_\mu(x), \]

and other transition matrices are similarly denoted. Except the three cases \( M(e, m) \), \( M(h, m) \) and \( M(p, m) \), whose entries are quite easy to describe by their very definitions ([St99, §7.4-7.7]), the entries in other transition matrices are not so direct to describe. In [Do72] Doubilet applied the Möbius inversion to give a unified and compact treatment on the entries of all these transition matrices.

In this section, we shall explain that Doubilet’s constructions can be extended to a more general framework into which two useful situations fit well. One will yield the transition matrices of the four bases (2.7), as originally considered by Doubilet in [Do72]. The other will lead to various Hoffman-type formulas ([Ho92], [Ho19], [Zh16]) which relate the symmetrization of multiple-(star) zeta functions (1.6), (1.7) and various variants to the original Riemann zeta function (1.5).

The following definition is an extension to [Do72, §3]. The form we adopt below is also inspired by the arguments in [BB18, §3].

**Definition 3.1.**

1. Fix a finite set \( S = \{ a_1, \ldots, a_{|S|} \} \) and a (possibly infinite) set \( D \).
   Let \( x(a_i, n) \) be (commuting) variables parametrized by \( a_i \in S \) and \( n \in D \), and
   \[ x(T, n) := \prod_{a_i \in T} x(a_i, n), \quad \forall T \subset S, \forall n \in D. \]

2. Let \( a_i, a_j \in S \) and \( \pi \in \Pi(S) \). The notation “\( a_i \sim a_j \)” is used to denote that \( a_i \) and \( a_j \) belong to the same block of \( \pi \).

3. For an integer partition \( \lambda = (\lambda_1, \ldots, \lambda_l) \), \( \lambda! := \prod_{i=1}^{l} \lambda_i! \).
(4) Let $\pi = \{\pi_1, \ldots, \pi_l\} \in \Pi(S)$ and define
\[
(3.1) \quad p(\pi) := \sum_{n_1, \ldots, n_l \in D} x(\pi_1, n_1)x(\pi_2, n_2)\cdots x(\pi_l, n_l),
\]
\[
(3.2) \quad m(\pi) := \sum_{n_1, \ldots, n_l \in D, \ n_i \text{ are distinct}} x(\pi_1, n_1)x(\pi_2, n_2)\cdots x(\pi_l, n_l),
\]
\[
(3.3) \quad e(\pi) := \sum_{n_1, \ldots, |S| \in D, \ n_i \neq n_j \text{ if } a_i \sim a_j} x(a_1, n_1)x(a_2, n_2)\cdots x(a_{|S|}, n_{|S|}),
\]
and
\[
(3.4) \quad h(\pi) := \sum_{\rho \in \Pi(S)} \lambda(\pi \land \rho)! \cdot m(\rho),
\]
where recall from Definition 2.1 that $\pi \land \rho$ is the greatest lower bound of $\pi$ and $\rho$, and $\lambda(\cdot)$ is the type of a partition in $\Pi(S)$ introduced in Definition 2.1.

When taking $D = \mathbb{Z}_{>0}$ and $x(a_i, n) = x_n$ independent of $a_i \in S$, Definition 3.1 specializes to the following example, which is exactly the case considered in [Do72, Thms 1 and 5] and justifies the notation in (3.1)-(3.4).

**Example 3.2 (Doubilet).** Take $D = \mathbb{Z}_{>0}$ and $x(a_i, n) = x_n$ in Definition 3.1. Then $x(\pi_i, n_i) = (x_{n_i})^{\pi_i}$ and (3.1)-(3.4) become
\[
\begin{cases}
p(\pi) = p_{\lambda(\pi)}(x), \\
m(\pi) = \left[ \prod_i m_i(\lambda(\pi))! \right] \cdot m_{\lambda(\pi)}(x), \\
e(\pi) = \lambda(\pi)! \cdot e_{\lambda(\pi)}(x), \\
h(\pi) = \lambda(\pi)! \cdot h_{\lambda(\pi)}(x).
\end{cases}
\]

**Remark 3.3.** Since the constructions and symbols used in [Do72] are different from ours, we briefly explain that Example 3.2 is *exactly* the case treated in [Do72] for the reader’s convenience. Following the notation in [Do72, p.379], let
\[
F := \{ f : S \rightarrow \mathbb{Z}_{>0} \}, \quad \gamma(f) := \prod_{i \in \mathbb{Z}_{>0}} (x_i)^{|f^{-1}(i)|}, \quad \ker(f) := \{ f^{-1}(i) \mid i \in \mathbb{Z}_{>0} \} \in \Pi(S).
\]
Then
\[
p(\pi) = \sum_{n_1, \ldots, n_l \in \mathbb{Z}_{>0}} (x_{n_1})^{\pi_1}\cdots (x_{n_l})^{\pi_l} = \sum_{\{ f \in F \mid \ker(f) \geq \pi \}} \gamma(f)
\]
and
\[
m(\pi) = \sum_{n_1, \ldots, n_l \in \mathbb{Z}_{>0}, \ n_i \text{ are distinct}} (x_{n_1})^{\pi_1}\cdots (x_{n_l})^{\pi_l} = \sum_{\{ f \in F \mid \ker(f) = \pi \}} \gamma(f).
\]
Recall from Definition 2.1 that \( \ker(f) \wedge \pi = \hat{0} \) is equivalent to \( |f^{-1}(i) \cap \pi_j| \leq 1 \) for all \( i \) and \( j \). Thus

\[
e(\pi) = \sum_{n_1, \ldots, n_{|\pi|} \in \mathbb{Z}_{>0}, \ n_i \neq n_j \text{ if } a_i \sim a_j} x_{n_1}x_{n_2} \cdots x_{n_{|\pi|}} = \sum_{\{f \in F | \ker(f) \cap \pi = \hat{0}\}} \gamma(f).
\]

The original definition of \( h(\pi) \) in [Do72, §4] is a little complicated, but it turns out that it can be expressed in terms of \( m(\pi) \) like (3.4) ([Do72, Thm 6]), which we take as its definition.

Another example we are interested in is the following

**Example 3.4.** Let real numbers \( t_1, \ldots, t_r > 1 \) be given. Take \( S = [r] \), \( D = \mathbb{Z}_{>0} \) and \( x(a, n) = \frac{1}{n^a} \) in Definition 3.1. For \( \pi = \{\pi_1, \ldots, \pi_l\} \in \Pi_r \), denote by \( t_{\pi_i} := \sum_{j \in \pi_i} t_j \).

Recall the notation in (1.3)-(1.7). It is not difficult to check that in this case (3.1)-(3.4) become

\[
\begin{align*}
p(\pi) &= \prod_{i=1}^l \zeta(t_{\pi_i}), \\
m(\pi) &= \zeta_S(t_{\pi_1}, \ldots, t_{\pi_l}), \\
e(\pi) &= \prod_{i=1}^l \zeta_S(t_{u_{i,1}}, \ldots, t_{u_{i,|\pi_i|}}), \ (\pi_i := \{u_{i,1}, \ldots, s_{u_{i,|\pi_i|}}\}), \\
h(\pi) &= \prod_{i=1}^l \zeta^*_S(t_{u_{i,1}}, \ldots, t_{u_{i,|\pi_i|}}).
\end{align*}
\]

The following transition formulas were obtained in [Do72, Thms 2 and 7] by applying the Möbius inversion on the poset \( \Pi(S) \) as well as some other tricks.

**Theorem 3.5** (Doubilet). Let the notation be as in Definition 3.1. Then we have

\[
m(\pi) = \sum_{\pi \leq \rho} \mu(\pi, \rho)p(\rho), \tag{3.5}
\]

and

\[
h(\pi) = \sum_{\rho \leq \pi} |\mu(\hat{0}, \rho)|p(\rho), \tag{3.6}
\]

where \( \mu(\cdot, \cdot) \) is the Möbius function of the poset \( \Pi(S) \) given in Lemma 2.3.

**Proof.** For completeness as well as for the reader’s convenience, we copy the proof from [Do72] verbatim.

The definitions (3.1) and (3.2) themselves imply that

\[
p(\pi) = \sum_{\pi \leq \rho} m(\rho), \tag{3.7}
\]

from which, together with the Möbius inversion ([St97, p.116]), (3.5) follows directly. This is the first part in [Do72, Thm 2], which has also been shown in [BB18, §3] independently.
For (3.6), we have (see [Do72, p.385])

\[ h(\pi) = \sum_{\rho} \lambda(\pi \land \rho)! \cdot m(\rho) \]

\[ = \sum_{\rho} \left( \sum_{\tau \leq \pi \land \rho} |\mu(0, \tau)| \right) \cdot m(\rho) \quad \text{(by (2.3))} \]

\[ = \sum_{\rho} \left( \sum_{\tau \leq \pi \land \rho} |\mu(0, \tau)| \right) \cdot m(\rho) \]

\[ = \sum_{\tau \leq \pi} |\mu(0, \tau)| \sum_{\rho \geq \tau} m(\rho) \]

\[ = \sum_{\tau \leq \pi} |\mu(0, \tau)| p(\tau), \quad \text{(by (3.7))} \]

which is exactly the second part in [Do72, Thm 7]. □

Remark 3.6. (1) Doubilet indeed obtained all the transition matrices among the four basis (2.7) in terms of the Möbius function \( \mu(\cdot, \cdot) \) in [Do72]. For our later purpose we only need and hence state the above two in Theorem 3.5. Doubilet’s constructions in Example 3.2 and various transformation formulas among these four bases have been generalized in [RS04] to the associated algebra of symmetric functions in noncommuting variables.

(2) As mentioned above, (3.5) has also been observed in [BB18, §3] and subsequently applied to obtain the transformation formula (3.8) and the Hoffman-type formula (3.9) below. Our treatment in Definition 3.1 is indeed partially inspired by some arguments in [BB18, §3].

Applying (3.5) to Example 3.2, together with the concrete expression of the Möbius function in (2.1), leads to

Corollary 3.7 (Doubilet). Let \( \lambda = (\lambda_1, \ldots, \lambda_l(\lambda)) = (1^{m_1(\lambda)}2^{m_2(\lambda)} \cdots) \) be an integer partition and \( \lambda_{\pi_i} := \sum_{j \in \pi_i} \lambda_j \). Then we have

\[ m_\lambda(x) = \frac{1}{\prod_i m_i(\lambda)!} \sum_{\pi \in \Pi_{l(\lambda)}} \left\{ (-1)^{l(\lambda)-l(\pi)} \prod_{i=1}^{l(\pi)} \left[ (|\pi_i| - 1)! \cdot p_{\lambda_{\pi_i}}(x) \right] \right\}. \]  

Now applying (3.5) (resp. (3.6)) to Example 3.4 by taking \( \pi = \hat{0} \) (resp. \( \pi = \hat{1} \)), together with the help of (2.2), yields the following formula (3.9) (resp. (3.10)) due to Hoffman ([Ho92, Thms 2.1 and 2.2]) (recall \( \hat{0} \) and \( \hat{1} \) in Definition 2.1).

Corollary 3.8 (Hoffman). For any real \( t_1, \ldots, t_r > 1 \), we have

\[ \zeta_S(t_1, \ldots, t_r) = \sum_{\pi \in \Pi_r} \left\{ (-1)^{r-l(\pi)} \prod_{i=1}^{l(\pi)} \left[ (|\pi_i| - 1)! \cdot \zeta(t_{\pi_i}) \right] \right\}, \]  

and

\[ \zeta^*_S(t_1, \ldots, t_r) = \sum_{\pi \in \Pi_r} \prod_{i=1}^{l(\pi)} \left[ (|\pi_i| - 1)! \cdot \zeta(t_{\pi_i}) \right], \]  

where \( t_{\pi_i} := \sum_{j \in \pi_i} t_j \).
Remark 3.9. These Hoffman-type formulas later have various variants. For example, there is an odd version ([Ho19]), an alternating version or a finite version ([Zh16, §7-8]) and so on. In all these cases, we can suitably choose the set $D$ in Definition 3.1 and apply Theorem 3.5 to arrive at the expected formulas.

4. The Coefficients of Chern Numbers in Complex Genera

With the materials in Sections 2 and 3 in hand, we can now examine the coefficients in front of Chern numbers for general complex genera in this section, and then focus on three cases (the $\mathrm{Td}^{\frac{1}{2}}$-genus, $\Gamma$-genus as well as Todd-genus) in the next section.

Let $\varphi$ be a complex genus whose associated power series is

$$Q(x) = 1 + \sum_{i=1}^{\infty} a_i x^i,$$

and

$$1 + \sum_{n=1}^{\infty} Q_n(e_1(x), \ldots, e_n(x)) := \prod_{i=1}^{\infty} Q(x_i),$$

where as before $e_n(x)$ is the $n$-th elementary symmetric function of the variables $x_1, x_2, \ldots$, and $Q_n$ the homogeneous part of degree $n$. Denote by

$$Q_n(e_1(x), \ldots, e_n(x)) = b_n(\varphi)e_n(x) + \sum_{|\lambda|=n, t(\lambda)\geq 2} b_\lambda(\varphi)e_\lambda(x).$$

Then according to (1.1) and (1.2), for each real $2n$-dimensional compact almost-complex manifold $M$ and an integer partition $\lambda$ of weight $n$, the coefficient in front of the Chern number $C_\lambda[M]$ in the value $\varphi(M)$ is exactly $b_\lambda(\varphi)$. These $b_\lambda(\varphi)$ can be determined by the power series $Q(x)$ in the following manner.

Lemma 4.1. Let $f(x)$ be the formal power series determined by

$$Q(x) = 1 + \sum_{i=1}^{\infty} a_i x^i =: \frac{x}{f(x)}.$$

(1) If the coefficients $a_i$ in $Q(x)$ are viewed as $a_i := e_i(y) = e_i(y_1, y_2, \ldots)$, the $i$-th elementary symmetric functions of the variables $y_1, y_2, \ldots$, then

$$b_\lambda(\varphi) = m_\lambda(y).$$

(2) The coefficients $b_n(\varphi)$ in (4.2) are determined by

$$1 + \sum_{n=1}^{\infty} (-1)^n \cdot b_n(\varphi) \cdot x^n = x \cdot \frac{f'(x)}{f(x)}.$$

Remark 4.2. Our notation $f(x)$ here is compatible with that used in the definition of elliptic genus ([HBJ92, p.17]).
Proof. By (4.1) and (4.2) we have

\[ 1 + \sum_{|\lambda| \geq 1} b_\lambda(\varphi)e_\lambda(x) = \prod_{i=1}^{\infty} Q(x_i) \]

\[ = \prod_{i=1}^{\infty} \left( 1 + \sum_{k=1}^{\infty} a_k x_i^k \right) \]

\[ = \prod_{i=1}^{\infty} \left[ 1 + \sum_{k=1}^{\infty} e_k(y_i)x_i^k \right] \]

\[ = \prod_{i,j=1}^{\infty} (1 + x_i y_j) \]

\[ = \prod_{i=1}^{\infty} \left[ 1 + \sum_{k=1}^{\infty} e_k(y_i)x_i^k \right] \]

\[ (2.8) = 1 + \sum_{|\lambda| \geq 1} m_\lambda(y)e_\lambda(x), \]

from which (4.3) follows. The identity (4.4) is well-known ([Hir66, p.11]). We provide a proof here for the sake of completeness.

\[ x \cdot f'(x) = x \cdot \frac{d}{dx} \log Q(x) \]

\[ = 1 - x \sum_{i=1}^{\infty} \frac{y_i}{1 + y_i x} \]

\[ = 1 + \sum_{n=1}^{\infty} (-1)^n(\sum_{i=1}^{\infty} y_i^n)x^n. \]

Thus

\[ \sum_{i=1}^{\infty} y_i^n = p_n(y) = m(n)(y), \]

which, together with (4.3), leads to (4.4).

Combining (4.3), (3.8) and the fact that \( p_i(x) = m_{(i)}(x) \) yield the following closed formula for \( b_\lambda(\varphi) \) in terms of those \( b_n(\varphi) \) determined by (4.4).

Lemma 4.3. The coefficients \( b_\lambda(\varphi) \) in front of the Chern numbers \( C_\lambda[\cdot] \) for the genus \( \varphi \) is given by

\[ (4.5) b_\lambda(\varphi) = \frac{1}{\prod_{i} m_i(\lambda)!} \sum_{\pi \in \Pi(\lambda)} \left\{ (-1)^{l(\lambda) - l(\pi)} \prod_{i} \left[ (|\pi_i| - 1)! \cdot b_{\lambda_{\pi_i}}(\varphi) \right] \right\}, \]

where \( \lambda_{\pi_i} = \sum_{j \in \pi_i} \lambda_j \) and \( b_i(\varphi) \) are determined by the identity (4.4).

Remark 4.4. This has been observed in [BB18, Thm 5] for the case of oriented genera. However, the essence of both proofs is the same.
5. The three classical complex genera

Before treating the three cases in detail, we fix the following notation. The Bernoulli numbers \( B_{2n} \) \((n \geq 1)\) are defined by

\[
\frac{x}{e^x - 1} =: 1 - \frac{1}{2}x + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n},
\]

and it is well-known that ([HBJ92, p.131])

\[
\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2 \cdot (2n)!} B_{2n}.
\]

5.1. The \( Td^\frac{1}{2} \)-genus.

Example 5.1. If \( Q(x) = \left(\frac{x}{1-e^{-x}}\right)^{\frac{1}{2}} \), then the coefficients \( b_n = b_n(Td^{\frac{1}{2}}) \) are given by

\[
b_1 = \frac{1}{4}, \quad b_{2n+1} = 0, \quad (n \geq 1) \quad b_{2n} = \frac{(-1)^{n-1}}{(2\pi)^{2n}} \zeta(2n). \quad (n \geq 1)
\]

Proof. In this case the associated \( f(x) \) is \( \left[ x(1-e^{-x})\right]^{\frac{1}{2}} \). Direct calculations show that

\[
1 + \sum_{n=1}^{\infty} (-1)^n b_n \cdot x^n \stackrel{(4.4)}{=} x \cdot \frac{f'(x)}{f(x)}
\]

\[
= \frac{1}{2} \left( 1 + \frac{x}{e^x - 1} \right)
\]

\[
\stackrel{(5.1)}{=} 1 - \frac{1}{4}x + \sum_{n=1}^{\infty} \frac{B_{2n}}{2 \cdot (2n)!} x^{2n}
\]

\[
\stackrel{(5.2)}{=} 1 - \frac{1}{4}x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2\pi)^{2n}} \zeta(2n) x^{2n},
\]

which yields (5.3).

Our main observation for the coefficients \( b_\lambda(Td^{\frac{1}{2}}) \) is

**Theorem 5.2.** Assume that \( m_1(\lambda) = 0 \). Then \( b_\lambda(Td^{\frac{1}{2}}) = 0 \) unless \( |\lambda| \) is even. If \( |\lambda| \) is even, we have

\[
b_\lambda(Td^{\frac{1}{2}}) = \frac{(-1)^{|l(\lambda)|-|\lambda|}}{(2\pi)^{|l(\lambda)|} \cdot \prod_{i=1}^{l(\lambda)} m_i(\lambda) \cdot \prod_{\pi \in \Pi_{l(\lambda)}} \prod_{i \in \lambda_i} \frac{l(\pi)}{(|\pi_i| - 1)! \cdot \zeta(\pi_i)}}.
\]

**Proof.** If \( |\lambda| \) is odd, then for each \( \pi = \{\pi_1, \ldots, \pi_{l(\lambda)}\} \in \Pi_{l(\lambda)} \) at least some \( \lambda_{\pi_{i_0}} \) is odd since \( |\lambda| = \sum \lambda_{\pi_i} \). As \( m_1(\lambda) = 0 \), this implies that \( \lambda_{\pi_{i_0}} \geq 3 \) and hence \( b_{\lambda_{\pi_{i_0}}} = 0 \) due to (5.3). This tells us that each summand on the right hand side (RHS for short) of (4.5) is zero and consequently \( b_\lambda(Td^{\frac{1}{2}}) = 0 \).
If now \(|\lambda|\) is even, the above analysis says that only those \(\pi = \{\pi_1, \ldots, \pi_{l(\lambda)}\} \in \Pi_{l(\lambda)}\) such that all \(\lambda_{\pi_i}\) are even can contribute to the RHS of (4.5). In this case the summand on the RHS of (4.5) is

\[
(-1)^{l(\lambda) - l(\pi)} \prod_{i=1}^{l(\pi)} \left[\frac{1}{|\pi_i| - 1} \cdot b_{\lambda_{\pi_i}}(\varphi)\right]
\]

\[= (1)^{l(\lambda) - l(\pi)} \prod_{i=1}^{l(\pi)} \left[\frac{1}{|\pi_i| - 1} \cdot \frac{(-1)^{\lambda_{\pi_i} - 1}}{(2\pi)^{\lambda_{\pi_i}} - \zeta(\lambda_{\pi_i})}\right] \tag{5.3}
\]

and leads to the desired (5.4).

Together with (3.10), a direct consequence of Theorem 5.2 is a compact expression in terms of MSZV for those integer partitions whose parts are all even, which is exactly Theorem 1.1 in the Introduction.

**Corollary 5.3** (=Theorem 1.1). For the integer partitions \(2\lambda = (2\lambda_1, \ldots, 2\lambda_{l(\lambda)})\), we have

\[
b_{2\lambda}(T \mathcal{D}^2) = \frac{(-1)^{|\lambda| - l(\lambda)}}{(2\pi)^{|\lambda|} \cdot \prod_i m_i(\lambda)!} \cdot \zeta_S'(2\lambda_1, \ldots, 2\lambda_{l(\lambda)}).
\]

In particular each \(b_{2\lambda}(T \mathcal{D}^2)\) is nonzero with sign \((-1)^{|\lambda| - l(\lambda)}\).

### 5.2. The \(\Gamma\)-genus.

**Definition 5.4.** Define an algebra homomorphism \(T\) from the algebra of symmetric functions \(\Lambda(x)\) to \(\mathbb{R}\) by requiring the values of the power sum symmetric functions \(p_i(x)\) as follows.

\[
T: \Lambda(x) \to \mathbb{R}, \quad T(p_i(x)) := \begin{cases} 
\zeta(i), & i \geq 2 \\
\gamma, & i = 1 \\
1, & i = 0
\end{cases} \quad (p_0(x) := 1)
\tag{5.5}
\]

where \(\gamma\) is the Euler constant introduced in (1.9). Note that the algebra homomorphism \(T\) is completely determined by (5.5) since \(\{p_\lambda(x)\}\) is a basis of \(\Lambda(x)\).

The following result is due to Hoffman ([Ho02, p.972]) building on [Lib99, §1].

**Theorem 5.5** (Hoffman). The coefficient \(b_\lambda(\Gamma)\) in front of the Chern number \(C_\lambda[\cdot]\) for the \(\Gamma\)-genus defined in (1.8) is given by the image of \(m_\lambda(x)\) under \(T\):

\[
b_\lambda(\Gamma) = T(m_\lambda(x)).
\tag{5.6}
\]

**Proof.** Since the proof in [Ho02] resorts to some materials in [Ho97, §5], for the reader’s convenience we provide a proof here. First we have ([Er53, p.45])

\[
\log(1 - x) = \gamma x + \sum_{i=2}^{\infty} \frac{\zeta(i)}{i} \cdot x^i,
\tag{5.7}
\]

and the symmetric functions \(e_i = e_i(x)\) and \(p_i = p_i(x)\) are related by ([Mac95, p.21-23])

\[
1 + \sum_{i=1}^{\infty} e_i \cdot t^i = \exp \left[- \sum_{i=1}^{\infty} \frac{p_i}{i} \cdot (-t)^i\right].
\tag{5.8}
\]
Thus

\[
1 + \sum_{i=1}^{\infty} T(e_i) \cdot t^i \overset{(5.8)}{=} \exp \left[ - \sum_{i=1}^{\infty} \frac{T(p_i)}{i} \cdot (-t)^i \right]
\]

\[
\overset{(5.5)}{=} \exp \left[ \gamma t - \sum_{i=2}^{\infty} \frac{\zeta(i)}{i} \cdot (-t)^i \right]
\]

\[
\overset{(5.7)}{=} \frac{1}{\Gamma(1 + t)}.
\]

Thus

\[
1 + \sum_{|\lambda| \geq 1} b_{\lambda}(\Gamma) e_{\lambda}(y) = \prod_{i=1}^{\infty} \frac{1}{\Gamma(1 + y_i)}
\]

\[
\overset{(5.9)}{=} \prod_{i=1}^{\infty} \left[ 1 + \sum_{k=1}^{\infty} T(e_k) \cdot y_i^k \right]
\]

\[
= T \left[ \prod_{i,j=1}^{\infty} \left( 1 + y_i x_j \right) \right]
\]

\[
\overset{(2.8)}{=} 1 + \sum_{|\lambda| \geq 1} T(m_{\lambda}) e_{\lambda}(y),
\]

which completes the proof.

With these materials in hand, we can proceed to show the result below, from which Theorem 1.3 in the Introduction follows.

**Theorem 5.6.** The coefficient \(b_{\lambda}(\Gamma)\) in front of the Chern number \(C_{\lambda}[-]\) for the \(\Gamma\)-genus are given by

\[
b_{\lambda}(\Gamma) = \frac{1}{\prod_i m_i(\lambda)!} \sum_{\pi \in \Pi_\ell(\lambda)} \left\{ (-1)^{l(\lambda)} - l(\pi) \prod_{i=1}^{l(\pi)} \left[ (|\pi_i| - 1)! \cdot \zeta(\lambda_{\pi_i}) \right] \right\} \quad (\zeta(1) := \gamma)
\]

\[
\overset{(5.10)}{=} \frac{\zeta_S(\lambda_1, \ldots, \lambda_l(\lambda))}{\prod_i m_i(\lambda)!}.
\]

In the first identity \(\zeta(\lambda_{\pi_i}) = \gamma\) occurs if and only if \(|\pi_i| = 1\), say \(\pi_i = \{j\}\), and \(\lambda_j = 1\). In particular, the second identity holds true for Calabi-Yau manifolds.

**Proof.** Due to (5.6) and (5.5), we apply the ring homomorphism \(T(\cdot)\) on both sides of (3.8) to yield the first identity in (5.10) with the convention that \(\zeta(1) := \gamma\).

If moreover \(m_1(\lambda) = 0\), all these \(\lambda_{\pi_i} \geq 2\) and the second identity follows from the Hoffman-type formula (3.9).

\[\square\]

5.3. **The Todd-genus.** Completely analogous to Example 5.1 and Theorem 5.2 we can deduce that
Example 5.7. If $Q(x) = \frac{x}{1-e^{-x}}$, the coefficients $b_n = b_n(Td)$ are

\begin{align}
\begin{cases}
b_1 &= \frac{1}{2}, \\
b_{2n+1} &= 0, & (n \geq 1) \\
b_{2n} &= \frac{2(-1)^{n+1}}{(2\pi)^{2n}} \zeta(2n), & (n \geq 1)
\end{cases}
\end{align}

and that

Proposition 5.8. Assume that $m_1(\lambda) = 0$. Then $b_\lambda(Td) = 0$ unless $|\lambda|$ is even. If $|\lambda|$ is even, we have

$$b_\lambda(Td) = \frac{(-1)^{l(\lambda)} - \frac{\lambda}{2}}{(2\pi)^{|\lambda|}} \prod_{i \in \mathbb{N}(\lambda)} m_i(\lambda)! \sum_{\lambda_{\pi_i} \text{ are even}} \left\{ 2^{l(\pi)} \prod_{i=1}^{l(\pi)} \left( [\frac{|\pi_i|}{2}! - 1] \cdot \zeta(\lambda_{\pi_i}) \right) \right\}.$$ 

In particular, in the cases of $|\lambda|$ be even and $m_1(\lambda) = 0$, $b_\lambda(Td)$ are nonzero with sign $(-1)^{l(\lambda)} - \frac{|\lambda|}{2}$.

Remark 5.9. (1) Unlike the formula (5.4) in Theorem 5.2, we are not able to deduce a compact formula in terms of MZV for those integer partitions whose parts are all even, as that in Corollary 5.3, due to the appearance of the extra factor $2^{l(\pi)}$ in the RHS of (5.12).

(2) An easy consequence of (5.12) is that if $b_\lambda(Td) \neq 0$ for odd $|\lambda|$, then $m_1(\lambda) \geq 1$, i.e., the first Chern class $c_1$ is involved in the Chern number $C_\lambda[\cdot]$. This fact has been observed by Hirzebruch ([Hir66, p.14]).

(3) Recently Schur multiple zeta values (SMZV for short) were introduced and investigated in [NPY18] by utilizing the semi-standard Young tableaux. They generalize MZV and MSZV in a natural way and also build a bridge to the theory of partitions. Related works can be found in [Ba18], [BY18], [BC20], and [BKSYY23]. One of the referees of our paper pointed out that it would be interesting to see if the RHS of (5.12) can be expressible in terms of SMZV. The author hopes to consider it in the near future.

6. SOME COMMENTS ON CHERN NUMBERS OF HYPER-KÄHLER MANIFOLDS

The known examples of irreducible hyper-Kähler manifolds are quite scarce. Up to deformations they are Hilbert schemes of points on K3 surfaces, generalized Kummer varieties ([Be83]) and two examples in dimensions 6 and 10 respectively ([OG99],[OG03]). Many positivity properties on these examples indicate that Chern numbers of irreducible hyper-Kähler manifolds should satisfy many constraints. The following conjecture was proposed in [Ni02, Appendix B] and raised again recently in [OSV22, Question 4.8].

Conjecture 6.1. All (monomial) Chern numbers of irreducible hyper-Kähler manifolds are positive.

Another positivity conjecture on Chern character numbers was also raised in [OSV22]. Before stating it, let us introduce one more notation. Let $x_1, \ldots, x_n$ be Chern roots of a compact almost-complex manifold $M$ of real dimension $2n$, i.e., the Chern classes $c_i(M)$ are viewed as $e_i(x_1, \ldots, x_n)$, the $i$-th elementary symmetric polynomials of $x_1, \ldots, x_n$. Let

$$\text{ch}_i(M) := \frac{x_1^i + \cdots + x_n^i}{i!} \in H^{2i}(M; \mathbb{Q}).$$
Then \( \sum_{i=0}^{n} \text{ch}_i(M) \) is the usual Chern character of \( M \). Given an integer partition \( \lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)}) \) of weight \( n \), the Chern character number \( \text{Ch}_\lambda[M] \) is defined by

\[
\text{Ch}_\lambda[M] := \int_M \prod_{i=1}^{l(\lambda)} \text{ch}_i(M) \in \mathbb{Q}.
\]

By its definition any Chern character number is a rationally linear combination of Chern numbers and vice versa. It was shown in [OSV22, Prop.3.7] that all signed Chern character numbers \((-1)^n \text{Ch}_{2\lambda}(|\lambda| = n)\) of a \( 2n \)-dimensional generalized Kummer variety are positive. Based on this, the following conjecture was raised ([OSV22, Question 4.7]).

**Conjecture 6.2.** All signed Chern character numbers of irreducible hyper-Kähler manifolds are positive.

The only known positivity result on Chern numbers which holds true for all irreducible hyper-Kähler manifolds seems to be the aforementioned \( \text{Td}^\frac{1}{2} \cdot [\cdot] > 0 \) due to Hitchin–Sawon ([HS01]), to the best knowledge of the author.

One of the primary motivations of this article is to see the relations among Conjectures 6.1, 6.2 and \( \text{Td}^\frac{1}{2} \cdot [\cdot] > 0 \). It is not difficult from the identity

\[
\log \frac{\sinh(x/2)}{x/2} = \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)! \cdot 2k} x^{2k}
\]

to deduce that

\[
(6.1) \quad \text{Td}^\frac{1}{2} \at{c=0} = \exp \left( -\sum_{k=1}^{\infty} \frac{B_{2k}}{4k} \cdot \text{ch}_{2k} \right).
\]

Note that the sign of \( B_{2k} \) is \((-1)^{k-1}\) (see (5.2)) and hence

\[
\text{Conjecture 6.2} \at{\text{(6.1)}} \Rightarrow \text{Td}^\frac{1}{2} \cdot [\cdot] > 0.
\]

Nevertheless, our Theorem 1.1 implies that the signs of coefficients in front of (monomial) Chern numbers for \( \text{Td}^\frac{1}{2} \cdot [\cdot] \) can be both positive and negative. So in general Conjecture 6.1 is not able to deduce that \( \text{Td}^\frac{1}{2} \cdot [\cdot] > 0 \).

The next question is what the relations are between Conjecture 6.1 and Conjecture 6.2. Note that

\[
(6.2) \quad C_\lambda[M] = \int_M e_\lambda(x_1, \ldots, x_n), \quad \text{Ch}_\lambda[M] = \frac{1}{\Pi_{i=1}^{l(\lambda)} \lambda_i!} \int_M p_\lambda(x_1, \ldots, x_n)
\]

in the notation of (2.4) and (2.5), and for irreducible hyper-Kähler manifolds only those partitions whose parts are all even are involved. Simple example in the case of complex 4-dimension reads

\[
C_{(4)} = \frac{1}{2} \text{Ch}_{(2,2)} - 6 \text{Ch}_{(4)}, \quad \text{Ch}_{(4)} = \frac{1}{12} C_{(2,2)} - \frac{1}{6} C_{(4)}
\]

and hence indicate that Conjectures 6.1 and 6.2 are in general independent. Here we shall explain that Doubilet’s results in [Do72] give closed transformation formulas between Chern
numbers and Chern character numbers of irreducible hyper-Kähler manifolds. Recall in [Do72, Thm 3] that

\begin{equation}
\begin{cases}
    e(\pi) = \sum_{\rho \leq \pi} \mu(\hat{0}, \rho) \cdot p(\rho) \\
    p(\pi) = \frac{1}{\mu(0, \pi)} \sum_{\rho \leq \pi} \mu(\rho, \pi) \cdot e(\rho),
\end{cases}
\end{equation}

where the related notation can be found in Definition 3.1 and Lemma 2.3.

Putting Lemma 2.3, Example 3.2, (6.2) and (6.3) together, we have

**Lemma 6.3.** Let $M$ be an irreducible hyper-Kähler manifold of complex dimension $2n$ and $2\nu$ an integer partition of weight $2n$. Fix a partition $\pi \in \Pi_{2n}$ such that $\lambda(\pi) = 2\nu$. Then the Chern numbers and Chern character numbers of $M$ are related by

$$C_{2\nu}[M] = \frac{1}{(2\nu)!} \sum_{\rho \leq \pi \atop |\rho_i| \text{ are even}} \mu(\hat{0}, \rho) \cdot \lambda(\rho)! \cdot Ch_{\lambda(\rho)}[M]$$

and

$$Ch_{2\nu}[M] = \frac{1}{(2\nu)! \cdot \mu(0, \pi)} \sum_{\rho \leq \pi \atop |\rho_i| \text{ are even}} \mu(\rho, \pi) \cdot C_{\lambda(\rho)}[M],$$

where the related notation can be found in Definition 2.1 and Lemma 2.3.

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School of Mathematical Sciences, Fudan University, Shanghai 200433, China

Email address: pinglimath@fudan.edu.cn, pinglimath@gmail.com