Quantum Real Lines

— Infinitesimal Structure of $\mathbb{R}$ —

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Abstract

We present in this paper quantum real lines as quantum deformations of the real numbers $\mathbb{R}$. Upon deforming the Heisenberg algebra $\mathcal{L}_H$ generated by $(a, a^\dagger)$ in terms of the Moyal $\ast$-product, we first construct $q$-deformed algebras of $q$-differentiable functions in two cases where $q$ is generic (not a root of unity) and $q$ is the $N$-th root of unity. We then investigate these algebras and finally propose two quantum real lines as the base spaces of these algebras. It is turned out that both quantum lines are discrete spaces and have noncommutative structures. We further find, minimal length, fuzzy structure and infinitesimal structure.

1 Introduction

The long-standing problem, quantization of gravity theory, undoubtedly requires some fundamental modifications to our understanding of geometry. Quantum effects must cause essential changes of some concepts in the classical geometry. Therefore, we should reconsider, for example, the following problems: (i) What is a point? Is it just a stable and localized object, i.e., of null size? (ii) What is a line? Is it a one-dimensional object which is isomorphic to $\mathbb{R}$? The aim of this paper is to challenge these essential problems.

As for the first problem, we expect that a point will get some quantum fluctuation. As the result, it may become something fuzzy and have some extra structure. It is natural to expect further that such a quantum point cannot be expressed just by a
real number. As a set of such quantum points, a *quantum* real line can be defined. Quantum line is, therefore, expected to be something fuzzy or discrete, and have some internal structure. The above observations indicate that quantization of geometry cannot be established on a commutative ring. In recent developments of the string theory, nonstandard structures of geometry have appeared, e.g., noncommutativity of coordinates, fuzzy structure, and so on. Thus, it is quite important to make the concepts of geometry on a noncommutative ring clear more. We will refer such a geometry as noncommutative geometry.

So far, a noncommutative geometry has been presented by Connes [1]. The basic strategy of the Connes’ approach follows from the Gelfand-Naimark theorem. The theorem states the correspondence between a manifold $M$ and a *commutative* ring $A(M)$, i.e., a functional algebra on $M$. In precise, once a commutative algebra $A$ is given, a space or a manifold $M$ is always found and geometrical informations of $M$ can be derived from $A$. As an extension of this theorem, Connes has developed a noncommutative geometry according to the idea; if a noncommutative algebra $A(M)$ is given as a deformation of $A(M)$, the space $M$ which can be called *noncommutative* space will be found and its geometrical informations will be deduced from the algebra $A(M)$.

Here, one of the manipulations to derive $A(M)$ from $A(M)$ is to introduce the Moyal $\star$-product instead of the standard local product $\cdot$ into the algebra $A$ as the multiplication. On the other hand, quantum groups are deformations of classical algebras or groups, and have been expected to give some insight into geometry on a noncommutative ring. In Ref.[2], noncommutative geometry has been discussed within the framework of quantum groups.

The author has deformed the classical mechanics and derived $q$-deformed quantum mechanics in Ref.[3]. Let us review the process briefly, since we will follow it latther. The starting point was the Poisson algebra $A^{CM}(\Gamma)$ on the phase space $\Gamma$, i.e., $A^{CM}(\Gamma)$ is the algebra of the classical mechanics. The algebra $A^{CM}(\Gamma)$ has been deformed into the algebra $A^{qQM}(\Gamma_\gamma)$ of the $q$-deformed quantum mechanics where $\Gamma_\gamma$ is a deformed phase space with the deformation parameter $\gamma$. One of the key steps in the program was the definition of $\Gamma_\gamma$. It was introduced by the product $\Gamma_\gamma = \Gamma \times T_2$. Here $T_2$ was introduced as an two-dimensional space and was regarded as an internal space attached every point on the external space $\Gamma$. Another important step was
the definition of the multiplication to be introduced into $\mathcal{A}^{QM}(\Gamma_{\gamma})$. Here we have derived $\mathcal{A}^{QM}(\Gamma_{\gamma})$ from $\mathcal{A}^{CM}(\Gamma)$ by twice the deformations. The first was performed by introducing the Moyal product $*_h$ into $\mathcal{A}^{CM}(\Gamma)$, and the algebra of the quantum mechanics $\mathcal{A}^{QM}(\Gamma;*_h)$ was obtained. In the second step, $\mathcal{A}^{QM}(\Gamma;*_h)$ was brought to the final algebra $\mathcal{A}^{qQM}(\Gamma_{\gamma};\star)$ by the modifications $*_h \rightarrow \star :=*_h \otimes *_{\gamma}$ together with $\Gamma \rightarrow \Gamma_{\gamma}$. The Moyal products $*_h$ and $*_{\gamma}$ act on the spaces of functions on $\Gamma$ and on $\mathcal{T}_2$, respectively. The procedure is depicted as

\[
\begin{array}{ccc}
\mathcal{A}^{CM}(\Gamma) & \xrightarrow{*_h} & \mathcal{A}^{QM}(\Gamma;*_h) \\
\downarrow *_{\gamma} & & \downarrow *_{\gamma} \\
\mathcal{A}^{qCM}(\Gamma_{\gamma};*_{\gamma}) & \xrightarrow{*_h} & \mathcal{A}^{qQM}(\Gamma_{\gamma};\star)
\end{array}
\]

Notice that there exists the other path to $\mathcal{A}^{qQM}(\Gamma_{\gamma};\star)$ via $\mathcal{A}^{qCM}(\Gamma_{\gamma};*_{\gamma})$, the algebra of $q$-deformed classical mechanics. The essential fact is that the $q$-deformation comes only from the internal sector, i.e., the internal space $\mathcal{T}_2$ and the product $*_{\gamma}$.

In this paper, we will follow the above strategy to quantize the real numbers $\mathbb{R}$, and we will propose quantum real lines. We will start, in Section 2, with deformation of the Heisenberg algebra $\mathcal{L}_H$ generated by the operators $a$ and $a^\dagger$. The deformed algebra $\mathcal{L}_H^q$ will be proposed by introducing the internal sector $\mathcal{A}(\mathcal{T}_2;*_{\gamma})$ which is the algebra of functions on $\mathcal{T}_2$ and is endowed with the Moyal product $*_{\gamma}$. We will further modify $\mathcal{L}_H^q$ to the operator algebra $\hat{\mathcal{L}}_q$ by changing $\mathcal{A}(\mathcal{T}_2;*_{\gamma})$ to $\hat{\mathcal{A}}(\mathcal{T}_1)$ where $\mathcal{T}_1$ is a reduction of $\mathcal{T}_2$. In Section 3, we will construct $q$-deformed algebras $\mathcal{A}$ of $q$-differentiable functions from the algebra $\hat{\mathcal{L}}_q$. We will discuss $\mathcal{A}$ in two cases separately; the case where $q$ is generic (not a root of 1) in Section 3.1 and the case where $q$ is a root of unity in Section 3.2. In Section 4, we will investigate the structures of the quantum real lines proposed as the base spaces of the algebras $\mathcal{A}$. The subsection 4.1 looks at the case when $q$ is generic and the quantum real line $\mathbb{R}_D$ is proposed. We will see that $\mathbb{R}_D$ is a discrete space and has minimal length. On the other hand, the quantum real line $\uparrow \mathbb{R}$ for the case with $q$ at the $N$-th root of unity is proposed in Section 4.2. Here, it will be turned out that $\uparrow \mathbb{R}$ consists of $\mathbb{R}$ and fuzzy internal space. The internal space is called the infinitesimal structure since it is embedded into every infinitesimal crack between two real numbers $x$ and $x+\epsilon \in \mathbb{R}$ with an infinitesimal $\epsilon$. Concluding remarks together with future problems are addressed in Section 5.
2 Deformation of the Heisenberg algebra

Let us first devote to a deformation of the Heisenberg algebra $L_H$, which is the operator algebra generated by the two generators $a$ and $a^\dagger$ satisfying the defining relation

$$aa^\dagger - a^\dagger a = 1. \quad (1)$$

According to the strategy developped in Ref.[3], let us introduce the internal space $T_2$ and the algebra $A(T_2)$ of functions on $T_2$. Here $T_2$ is a two dimensional torus parameterized by $\theta$ and $\tau$. The generators of $A(T_2)$ are, therefore, written as

$$U := e^{i\theta}, \quad V := e^{i\tau}. \quad (2)$$

Now, we have to introduce a multiplication into $A(T_2; \cdot)$. The most familiar and simplest choice is made by the standard pointwise product $\cdot$ and we denote such an algebra as $A(T_2; \cdot)$. Obviously, $A(T_2; \cdot)$ is a closed and commutative algebra, i.e., for $f(U, V), g(U, V) \in A(T_2; \cdot)$, $f \cdot g = g \cdot f \in A(T_2; \cdot)$. In this case, one identifies the base space $T_2$ with a classical torus.

Another possible product to be introduced in $A(T_2)$ is the, so-called, Moyal product $\ast_\gamma$ with some deformation parameter $\gamma$. Let $A(T_2; \ast_\gamma)$ be the algebra endowed with the product $\ast_\gamma$. Then, let us observe $A(T_2; \ast_\gamma)$ by giving an explicit definition of $\ast_\gamma$. We follow Ref.[3] where $\ast_\gamma$ is defined as

$$\ast_\gamma = \sum_{n=0}^{\infty} \frac{(-i\gamma)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (-)^k \frac{\partial^{n-k} \partial^k}{\partial \theta \partial \tau} \cdot \frac{\partial^{n-k} \partial^k}{\partial \theta \partial \tau}. \quad (3)$$

It is easy to see from (3) that the algebra $A(T_2; \ast_\gamma)$ is no longer commutative but a noncommutative algebra. Indeed, one obtains the basic commutation relations as

$$U \ast_\gamma V = e^{i\gamma} V \ast_\gamma U \quad , \quad (4)$$

$$\theta \ast_\gamma \tau - \tau \ast_\gamma \theta = -i\gamma \quad . \quad (5)$$

Notice that the noncommutativity originates in the nonlocality of the product $\ast_\gamma$. The commutation relations (4,5) suggests that the base space $T_2$ is the so-called noncommutative torus.

Now, it is the time to construct $q$-deformed Heisenberg algebra $L^q_H$. We introduce $L^q_H$ by the product of two algebras $L_H$ and $A(T_2)$,

$$L^q_H = L_H \times A(T_2). \quad (6)$$
In precise, let us choose the generators of $\mathcal{L}_q^H$ as

$$\hat{a}_\theta := a U, \quad \hat{a}_\tau^\dagger := a^\dagger V,$$

i.e., $\mathcal{L}_q^H = \{ \hat{a} | \hat{a} = \hat{a}(\hat{a}_\theta, \hat{a}_\tau^\dagger) \}$ is an operator space spanned by functions of operators $\hat{a}_\theta, \hat{a}_\tau^\dagger$. We then have to define the multiplication to be introduced in $\mathcal{L}_q^H$. Upon assuming here that the sector $\mathcal{L}_H$ possesses the standard operator product, we have only to determine the multiplication associated with the internal sector $\mathcal{A}(\mathcal{T}_2)$. In the above, we have studied two multiplications $\cdot$ and $\ast_\gamma$. The product $\cdot$ makes the algebra $\mathcal{A}(\mathcal{T}_2; \cdot)$ commutative and $\mathcal{L}_q^H$ trivial in the sense that $\mathcal{A}(\mathcal{T}_2; \cdot)$ is factorized out from the sector $\mathcal{L}_H$. Indeed, the commutation relation between $\hat{a}_\theta, \hat{a}_\tau^\dagger \in \mathcal{L}_q^H$ is the same as the relation (1) in $\mathcal{L}_H$. Thus, $\mathcal{A}(\mathcal{T}_2; \cdot)$ does not affect the algebra $\mathcal{L}_q^H$, and $\mathcal{L}_q^H$ is essentially equivalent to $\mathcal{L}_H$. We are not interested in this case.

On the other hand, we have seen that $\mathcal{A}(\mathcal{T}_2; \ast_\gamma)$ is a noncommutative algebra. Owing to the noncommutativity, one expects that some nontrivial corrections will be made in the operator algebra $\mathcal{L}_{q, \ast_\gamma} = \mathcal{L}_H \times \mathcal{A}(\mathcal{T}_2; \ast_\gamma)$. Let us start the investigations of $\mathcal{L}_{q, \ast_\gamma}$ with the commutation relation between the generators $\hat{a}_\theta, \hat{a}_\tau^\dagger \in \mathcal{L}_{q, \ast_\gamma}$. By making use of the commutation relations given in eqs.(1) and (4), one easily obtains the deformed commutation relation

$$\hat{a}_\theta \ast_\gamma \hat{a}_\tau^\dagger - q \hat{a}_\tau^\dagger \ast_\gamma \hat{a}_\theta = U \ast_\gamma V,$$  

where $q$ is also the deformation parameter given by

$$q = e^{i\gamma}.$$  

Note here that, in the limit $\gamma \to 0$, the algebra $\mathcal{A}(\mathcal{T}_2; \ast_\gamma)$ reduces to $\mathcal{A}(\mathcal{T}_2; \cdot)$ and, therefore, $\mathcal{L}_{q, \ast_\gamma} \to \mathcal{L}_H, \cong \mathcal{L}_H$.

Having obtained the operator algebra $\mathcal{L}_{q, \ast_\gamma}$, we can derive an algebra $\mathcal{A}$ on a $q$-deformed real numbers, i.e., $q$-deformed real line. Our interests are in geometrical structure of the line. According to the strategy of the noncommutative geometry $\mathcal{A}$, the structure will be extracted from $\mathcal{A}$. These investigations are the tasks in the following sections. However, for the latter discussions, it is convenient to modify $\mathcal{L}_{q, \ast_\gamma}$ to the new algebra $\hat{\mathcal{L}}_q$ in which the internal sector is not the functionable algebra $\mathcal{A}(\mathcal{T}_2; \ast_\gamma)$ but an operator algebra $\hat{\mathcal{A}}(\mathcal{T}_1)$. The base space of $\hat{\mathcal{A}}(\mathcal{T}_1)$ is a one-dimensional
space $\mathcal{T}_1$. We will define $\hat{\mathcal{A}}(\mathcal{T}_1)$ so that we have a similar commutation relation to (4). Furthermore, upon defining the algebra $\hat{\mathcal{L}}_q$ by

$$\hat{\mathcal{L}}_q := \mathcal{L}_H \times \hat{\mathcal{A}}(\mathcal{T}_1),$$

we require that the defining relation is similar to (8). Before going to the explicit definition of $\hat{\mathcal{A}}(\mathcal{T}_1)$, we should give a comment on the essence of the modification from $\mathcal{L}_q^\gamma H$, $*$, to $\hat{\mathcal{L}}_q$. We have seen that, in the algebra $\mathcal{L}_q^\gamma H$, the $q$-deformation originates in the nonlocal structure of the product $*$. On the contrary, we require that the same $q$-deformed effects come from the noncommutativities originating in the operator nature of $\hat{\mathcal{A}}(\mathcal{T}_1)$. Here, we assume that the standard operator product is introduced in $\hat{\mathcal{A}}(\mathcal{T}_1)$. In precise, upon denoting the generators of $\hat{\mathcal{A}}(\mathcal{T}_1)$ as $\hat{U}$ and $\hat{V}$, the basic commutation relation

$$\hat{U}\hat{V} = q\hat{V}\hat{U}$$

instead of eq. (11) yields $q$-deformed effects in $\hat{\mathcal{L}}_q$.

Let us go ahead to the explicit representations of $\hat{\mathcal{A}}(\mathcal{T}_1)$ and $\hat{\mathcal{L}}_q$. By writing the generators of $\hat{\mathcal{A}}(\mathcal{T}_2)$ as $\hat{U} = e^{i\hat{\theta}}$, $\hat{V} = e^{i\hat{\tau}}$, the required relation (11) is satisfied if

$$\hat{\theta} = \theta, \quad \hat{\tau} = i\gamma \frac{d}{d\theta} := i\gamma \partial_\theta.$$ 

Therefore, the base space $\mathcal{T}_1$ of the algebra $\hat{\mathcal{A}}(\mathcal{T}_1)$ is selected as a one-dimensional torus parameterized by $\theta$. Notice that the representations given in (12) are consistent with the commutation relation corresponding to (3), i.e., $\hat{\theta}\hat{\tau} - \hat{\tau}\hat{\theta} = -i\gamma$. Next, let $\hat{\mathcal{L}}_q$ be generated by the operators $a_q, a_q^\dagger$ which are defined by the substitutions

$$a_\theta \longmapsto a_q, \quad e^{i\theta} a_\tau^\dagger \longmapsto a_q^\dagger.$$ 

The commutation relation between these generators is finally obtained as

$$a_q a_q^\dagger - q a_q^\dagger a_q = q^{i\partial_\theta}.$$ 

We have reached the final $q$-deformed Heisenberg algebra $\hat{\mathcal{L}}_q$ with the deformation parameter $q$ given in (9). It is important to notice that the variable $\theta$ becomes the coordinate of the internal space of the Hilbert space on which $\hat{\mathcal{L}}_q$ acts.

In the next section, we will realize $\hat{\mathcal{L}}_q$ explicitly by $\hat{\mathcal{D}}$, the algebra generated by a coordinate operator and a differential operator on $q$-deformed real numbers. We then represent $\hat{\mathcal{D}}$ by the algebra $\mathcal{A}$ corresponding to the Hilbert space of $\hat{\mathcal{D}}$, i.e., $\mathcal{A}$ is the algebra of $q$-differentiable functions.
3 Deformations of algebra of functions on $\mathbb{R}$

Recall that a possible representation of the Heisenberg algebra $L_H$ is realized by the replacements $a \rightarrow \partial_x$ and $a^\dagger \rightarrow x \in \mathbb{R}$. We will adopt similar replacements to the $q$-deformed algebra $\hat{L}_q$. Let $\hat{D}$ be the operator algebra generated by $q$-deformed coordinate and $q$-deformed differential operators. The algebra $\hat{D}$ can be derived from $\hat{L}_q$ by the replacements

\[ a_q \rightarrow D_q, \quad a_q^\dagger \rightarrow \hat{\chi} \]  

(15)

where $D_q$ and $\hat{\chi}$ stand, respectively, for the $q$-differential and the $q$-coordinate operators on a quantum real line. Our goal is to reveal geometry of the line. In order to reach there, we have to study the $q$-deformed algebra $A$ on which the action of $\hat{D}$ is defined. In the following subsections, we are going to discuss $A$ for two cases, separately; in section 3.1, the case where $q$ is generic and in section 3.2, the case with $q$ at a root of unity.

3.1 The case where $q$ is generic

Let us suppose first that the parameter $q$ is not a root of unity. Upon the substitutions (15), the commutation relation (14) reads the commutation relation between $D_q$ and $\hat{\chi}$ as

\[ D_q \hat{\chi} - q \hat{\chi} D_q = q^{i\partial_\theta}. \]  

(16)

and the explicit expressions of these operators are presented by

\[ \hat{\chi} = \frac{1 + q^{i\partial_\theta}}{1 + q^{-1}} x_\theta, \quad D_q = \frac{q^{-i\partial_\theta} - 1}{x_\theta(q - 1)}. \]  

(17)

Here, we temporarily give the base space of these operators as

\[ \mathcal{R} = \{ x_\theta \mid x_\theta := x_+ e^{i\theta}, \ 0 < x_+ \in \mathbb{R}, \ 0 \leq \theta < 2\pi \}. \]  

(18)

One then construct a functional algebra $\mathcal{A}^q$ on $\mathcal{R}$ as

\[ \mathcal{A}^q = \{ f(x_\theta) \mid x_\theta \in \mathcal{R}, \}, \]  

(19)

where the multiplication is supposed to be the standard pointwise product $\cdot$. The complete basis $\mathcal{B}$ of $\mathcal{A}^q$ is given by

\[ \mathcal{B} = \{ x_\theta^n \mid n \in \mathbb{Z} \} \]  

(20)
and, therefore, \( f(x_\theta) \in \mathbb{S}^q \) is expanded as
\[
f(x_\theta) = \sum_{n \in \mathbb{Z}} f_n x_\theta^n.
\] (21)

The actions of the operators \( \hat{x} \) and \( D_q \) on \( \mathbb{S}^q \) are calculated explicitly by (17), i.e.,
\[
\hat{x} f(x_\theta) = x_\theta f(x_\theta) + q^{-1} f(x_\theta q^{-1}) \quad \text{and} \quad D_q f(x_\theta) = \frac{f(x_\theta q) - f(x_\theta)}{x_\theta (q - 1)}.
\] (22)

Note that, by taking the limit \( q \rightarrow 1 \) together with \( \theta \rightarrow 0 \), \( \hat{x} \) becomes the standard coordinate operator \( \hat{x} \) such as \( \hat{x} f(x) = x f(x) \), and \( D_q \) reduces to the standard differential operator, i.e., \( D_q f(x_\theta) \xrightarrow{q \rightarrow 1} \frac{d}{dx} f(x) \). Furthermore, one shows that the \( q \)-differential operator \( D_q \) satisfies the following properties,
\[
D_q 1 = 0, \quad \text{for the identity } 1 \in \mathbb{S}^q,
\] (23)
\[
D_q (f \cdot g) = (D_q f(x_\theta)) g(x_\theta) + f(qx_\theta) (D_q g(x_\theta)).
\] (24)

The first equation (23) indicates that \( D_q \) vanishes constants in \( \mathbb{S}^q \), and (24) shows the deformed Leibniz rule.

Now, one gives the \( q \)-deformed algebra \( \mathcal{A}(\mathbb{S}^q; D_q) \) as the algebra of functions on which the action of the operator \( D_q \) is defined, i.e., \( \mathcal{A}(\mathbb{S}^q; D_q) \) is the algebra of \( q \)-differentiable functions on \( \mathbb{R} \). The base space \( \mathbb{R} \) is the object which is called “quantum real line” and its geometrical structure is derived from \( \mathcal{A}(\mathbb{S}^q; D_q) \). By the definition given in (18), \( \mathbb{R} \) looks like the complex plane \( \mathbb{C} \). However, we should notice that the algebra \( \mathcal{A}(\mathbb{S}^q; D_q) \) possesses only the difference operator. Therefore, \( \mathbb{R} \) may not be the standard complex plane but have some discrete structure. It will be turned out, in Section 4, that the final quantum real line, which will be derived from \( \mathbb{R} \) by some modifications, has such a discrete structure.

Let us end this subsection with the study of \( q \)-differential structure in \( \mathcal{A}(\mathbb{S}^q; D_q) \). Notice first that, as far as \( \mathbb{S}^q \) is considered, the internal sector does not play any special role. Indeed, for \( f(x_\theta), g(x_\theta) \in \mathbb{S}^q \), \( f \cdot g = g \cdot f \), i.e., \( \mathbb{S}^q \) is still a commutative algebra. However, once the operator \( D_q \) is taken into account, i.e., the algebra \( \mathcal{A}(\mathbb{S}^q; D_q) \) is considered, some nontrivial structures appear as shown above. It is interesting to show further that the exterior derivative \( d_q \) can be introduced in \( \mathcal{A}(\mathbb{S}^q; D_q) \) by
\[
d_q f = d_q x_\theta D_q f(x_\theta).
\] (25)
By operating $d_q$ on the identity $x_\theta \cdot f(x_\theta) = f(x_\theta) \cdot x_\theta$, another noncommutativity
\[
d_q x_\theta f(x_\theta) = f(q x_\theta) d_q x_\theta
\] (26)
is found. Thus, one regards the algebra $\mathcal{A}(\mathcal{N}; D_q)$ as a noncommutative algebra.

### 3.2 The case where $q$ is a root of unity

Let us next turn to the $q$-deformed algebra $\mathcal{A}$ in the case where $q$ is given by the $N$-th root of unity, i.e.,
\[
q = e^{i \frac{2\pi}{N}}. \tag{27}
\]
Let $\mathcal{A}_N(\mathcal{N}; D_q)$ be the algebra of $q$-differentiable functions to be discussed in this case. Here $\mathcal{N}$ represents the functional algebra on the base space $\mathcal{N} \mathcal{R}$ given below. In the case with (27), it is natural to expect that the base space is different from $\mathcal{R}$ for the case with generic $q$. Now, let us denote the base space as
\[
\mathcal{N} \mathcal{R} = \{ y_\xi \mid y_\xi := y \xi, \; \xi = e^{i \theta} \}, \tag{28}
\]
i.e., the algebra $\mathcal{N}$ on $\mathcal{N} \mathcal{R}$ is given by
\[
\mathcal{N} \mathcal{R} = \{ f(y_\xi) \mid f(y_\xi) = \sum_{n \in \mathbb{Z}} \alpha_n y_\xi^n, \; y_\xi \in \mathcal{N} \mathcal{R} \}, \tag{29}
\]
with the basis
\[
\mathcal{B}_N := \{ y_\xi^n \mid n \in \mathbb{Z} \}. \tag{30}
\]
In order to make $\mathcal{N}$ and $\mathcal{A}_N(\mathcal{N}; D_q)$ explicit, we have to observe first the operator algebra $\widehat{D}_N$ which acts on $\mathcal{A}_N(\mathcal{N}; D_q)$. We expect that $\widehat{D}_N$ is drastically different from $\widehat{D}$ in the case with generic $q$. Indeed, it is well-known that the representations of quantum groups or quantum universal enveloping algebras with $q$ at a root of unity are completely different from those with generic $q$. For example, in a quantum representation $\mathcal{R}(G)$ of the group $G$, a classical (undeformed) sector $R(G)$ appears apart from a $q$-deformed sector $\mathcal{R}'(G)$ such as $\mathcal{R}(G) = R(G) \otimes \mathcal{R}'(G)$. We will see later that such a remarkable structure appears in our algebras $\widehat{D}_N$ and $\mathcal{A}_N(\mathcal{N}; D_q)$.

As in the previous case in Section 3.1, the $q$-differential operator $D_q$ is also introduced by the second equation in (22) with the replacement of the variable $x_\theta \to y_\xi$. However, characteristic features of the case with $q$ at a root of unity appear through...
the following equations;

\[ D_q y_\xi^{kN+r} = [r] y_\xi^{kN+r-1}, \quad \text{for} \quad r \neq 0, \]

\[ D_q y_\xi^{kN} = 0. \] (31)

One sees from (31) that \( y_\xi^N \) is a constant with respect to \( D_q \). Further, one finds easily that the action of \( D_q^{N} \) on \( \forall f(y_\xi) \in \mathfrak{B}_N \) is null. However, upon defining another operator \( \partial \) by

\[ \partial := D_N q / [N]!, \]

the actions of \( \partial \) on \( \mathcal{B}_N \) are calculated as

\[ \partial y_\xi^{kN,r} = k y_\xi^{(k-1)N+r}. \quad \text{for} \quad k \geq 1 \]

\[ \partial y_\xi^r = 0, \quad \text{for} \quad r \leq N - 1. \] (32)

where use has been made of the relation \([kN]/[N] = k\). Notice further that \( y_\xi^r, r \leq N - 1 \) behave as constants with respect to the operator \( \partial \). It should be emphasized that, as can be seen from (31) and (32), these two operators \( D_q \) and \( \partial \) are independent of each other on \( \mathfrak{B}_N \). Therefore, in order to define the operator algebra \( \hat{D}_N \), \( \partial \) is needed as well as the \( q \)-differential operator \( D_q \).

Let us consider the algebra \( \mathfrak{B}_N \) on which \( \hat{D}_N \) acts. The above observations indicate that there exists the mapping \( \pi \) such as

\[ \pi : \mathfrak{B}_N \rightarrow \mathfrak{B} \otimes \mathfrak{B}_{\text{int}} \]

\[ f(y_\xi) \rightarrow \pi(f) = \lambda(x) \psi(\xi). \] (33)

Here the variable \( x \) is introduced by

\[ \pi(y_\xi^{kN+r}) = x^k \xi^r \] (34)

with \( k \in \mathbb{Z} \) and \( r \) being valued in

\[ r \in \mathcal{I} := \begin{cases} \{ -p, -p + 1, \cdots, p - 1, p \}, & \text{for} \quad N = 2p + 1, \\ \{ -p + 1, \cdots, p - 1, p \}, & \text{for} \quad N = 2p \end{cases}. \] (35)

The basis \( \mathcal{B}_N \) is, therefore, factorized into two sets as \( \mathcal{B}_N \rightarrow \mathcal{W} \otimes \varpi \) where \( \mathcal{W} := \{ x^k \mid k \in \mathbb{Z} \} \) and \( \varpi := \{ \xi^r \mid r \in \mathcal{I} \} \). Namely, any functions \( \lambda(x) \in \mathfrak{B} \) and \( \psi(\xi) \in \mathfrak{B}_{\text{int}} \) are expanded as

\[ \lambda(x) = \sum_{k \in \mathbb{Z}} \lambda_k x^k, \quad \psi(\xi) = \sum_{r \in \mathcal{I}} \psi_r \xi^r. \] (36)
Now, we have reached the important fact that the base space of $\mathfrak{I}^q_N$ should be introduced instead of $N \mathbb{R}$ as

$$^q\mathbb{R} \ni (x, \xi), \quad (37)$$

whose geometrical structure will be clarified in the next section.

Let us go back again to $\hat{D}_N$ and observe how the operators in $\hat{D}_N$ acts on the factorized space $\mathfrak{I} \otimes \mathfrak{I}_{\text{int}}$. The mapping $\pi$ induces another mapping $\hat{\pi} : \hat{D}_N \to \hat{D}_N^{\text{cl}} \otimes \hat{D}_N^{\text{int}}$. The equations in (32) show that the operator $\partial$ acts on $\mathfrak{I} \otimes \mathfrak{I}_{\text{int}}$ as $\hat{\pi}(\partial) = \partial_x \otimes 1$, $\partial_x$ being the standard differential operator, while from eq.(31), one sees that $\hat{\pi}(D_q) = 1 \otimes D_q$, i.e.,

$$\partial_x \lambda(x) = \frac{d}{dx} \lambda(x), \quad D_q \psi(\xi) = \frac{\psi(q\xi) - \psi(\xi)}{\xi(q - 1)}. \quad (38)$$

We are now at the stage to discuss the $q$-deformed algebra $A_N(\mathfrak{I}^q_N; D_q)$. The investigations given above indicate that $A_N(\mathfrak{I}^q_N; D_q)$ is essentially factorized as

$$A_N(\mathfrak{I}^q_N; D_q) \cong A_N(\mathfrak{I} : \mathfrak{I}_{\text{int}}) := A(\mathfrak{I} ; \partial_x) \otimes A_{\text{int}}(\mathfrak{I}_{\text{int}} ; D_q). \quad (39)$$

The first sector $A(\mathfrak{I} ; \partial_x)$ can be regarded as the external sector and as the standard (undeformed) algebra of differentiable functions, where the variable $x$ is real number. Namely, the base space of $A(\mathfrak{I} ; \partial_x)$ is just $\mathbb{R}$. On the other hand, $A_{\text{int}}(\mathfrak{I}_{\text{int}} ; D_q)$ is regarded as the internal sector and is an algebra of $q$-differentiable functions. The base space of $A_{\text{int}}(\mathfrak{I}_{\text{int}} ; D_q)$ is expected to have some nonstandard structure on which we will discuss later. Thus, in the case where $q$ is a root of unity, the classical sector appears explicitly apart from the $q$-deformed internal sector.

Finally, we should introduce a suitable multiplication in $A_{\text{int}}$. Notice here that the algebra $A_{\text{int}}(\mathfrak{I}_{\text{int}} ; D_q)$ is not closed with respect to the standard pointwise product $\cdot$. Indeed, for $r, r' \in \mathcal{I}$, the value $s$ such as $\xi^r \cdot \xi^{r'} = \xi^s$ is not always in $\mathcal{I}$. In order to make $A_{\text{int}}(\mathfrak{I}_{\text{int}} ; D_q)$ a closed algebra, let us introduce the mapping $\mu : A_{\text{int}} \to \hat{A}(M)$. Here $\hat{A}(M)$ is the algebra of $N \times N$ matrices $M$ generated by the basis

$$\hat{\xi}^r = \mu(\xi^r) = \begin{pmatrix} I_{N-r} \\ I_r \end{pmatrix}, \quad r \in \mathcal{I}, \quad (40)$$

where $I_n$ is the $n \times n$ unit matrix and the convention $I_{N+n} = I_n$ for $n > 0$ is used. Notice that the algebra $\hat{A}(M)$ is closed under the standard matrix multiplication,
\[ \hat{\omega} = \{ \hat{\xi}^r \mid r \in I \} \] is a complete basis. Further, \( \hat{\omega} \) is orthonormal with respect to the pairing \( \langle A, B \rangle := A^\dagger B \). Here we use \( (\hat{\xi}^r)^\dagger = \hat{\xi}^{-r} \) but \( (\hat{\xi}^p)^\dagger = \hat{\xi}^p \) only for the case \( N = 2p \). Since the mapping \( \mu \) is an isomorphism between \( \mathcal{A}_{\text{int}}(\mathfrak{S}_{\text{int}}) \) and \( \hat{\mathcal{A}}(M) \), the multiplication \( \mathcal{A}_{\text{int}}(\mathfrak{S}_{\text{int}}) \) should possess is introduced as follows; upon denoting the multiplication as \( \odot \), the product of two functions \( \psi, \psi' \in \mathcal{A}_{\text{int}}(\mathfrak{S}_{\text{int}}; D_q) \) is defined by

\[ \psi(\xi) \odot \psi'(\xi) := \mu^{-1}(\mu(\psi)\mu(\psi')) \quad (41) \]

Now we have understood the \( q \)-deformed algebra \( \mathcal{A}_N(\mathfrak{S} : \mathfrak{S}_{\text{int}}) \) on \( ^q\mathbb{R} \). The geometrical structure of the base space \( ^q\mathbb{R} \) will be discussed in the next section.

4 Quantum real lines

We have obtained the algebras of \( q \)-differentiable functions; for the case with generic \( q \), we have \( \mathcal{A}(\mathfrak{S}^q; D_q) \) on the base space \( \mathbb{R} \), and for the case with \( q \) at the \( N \)-th root of unity, we have \( \mathcal{A}_N(\mathfrak{S} : \mathfrak{S}_{\text{int}}) \) on \( ^q\mathbb{R} \). Let us go ahead to the main task of our program, i.e., the investigations of geometrical structures of these base spaces. This is performed via the algebras \( \mathcal{A}(\mathfrak{S}^q) \) and \( \mathcal{A}_N(\mathfrak{S} : \mathfrak{S}_{\text{int}}) \). We will finally propose quantum real lines \( \mathbb{R}_D \) when \( q \) is generic and \( \sqrt{\mathbb{R}} \) when \( q \) is a root of unity.

4.1 Quantum real line \( \mathbb{R}_D \); \( q \) is not a root of unity

Notice first that, as we have seen in Section 3.1, the algebra \( \mathcal{A}(\mathfrak{S}^q; D_q) \) possesses the difference structure induced by \( D_q \). We can, therefore, regard the base space \( \mathbb{R} \) as a discrete space. To see this in more explicit, let us change the variable from \( x_\theta \) to \( \chi \) by the relation

\[ \chi = -i \log x_\theta. \quad (42) \]

Now, let us denote the base space parameterized by \( \chi \) as \( \mathbb{R}_\chi \). According to the change of variable, the difference operator \( D_q \) with respect to \( x_\theta \) should be changed to the operator \( \Delta_\chi \) with respect to \( \chi \). One assumes the relation \( D_q = (D_q\chi)\Delta_\chi \) and finds

\[ \Delta_\chi = e^{\gamma \chi \partial_\chi} - 1. \quad (43) \]

Regarding \( \mathfrak{S}^q \) as the space of functions on \( \mathbb{R}_\chi \), the action of \( \Delta_\chi \) on a functions \( f(\chi) \in \mathfrak{S}^q \) is explicitly written as

\[ \Delta_\chi f(\chi) = \frac{f(\chi + \gamma) - f(\chi)}{\gamma}. \quad (44) \]
The key commutation relation between the coordinate \( \chi \) and \( \Delta_\gamma \) is calculated as

\[
\chi \Delta_\gamma - \Delta_\gamma \chi = K,
\]

where \( K \) is the shift operator such as \( K f(\chi) = f(\chi + \gamma) \).

On the base space \( \mathbb{R}_\gamma \), we have the algebra \( \mathcal{A}(\mathbb{S}^q; \Delta_\gamma) \) as the space of functions on which \( \Delta_\gamma \)-action is defined. It should be emphasized that the variable \( \chi \) is, at this stage, thought to be continuous. However, upon choosing a value \( \chi_0 \) arbitrarily, one can reduce \( \mathbb{S}^q \) to the subset \( \mathcal{S}_D(\chi_0) = \{ f(\chi_0 + n\gamma) \mid n \in \mathbb{Z} \} \) owing to the difference operator \( \Delta_\gamma \). Namely, we can restrict ourselves to the subalgebra \( \mathcal{A}(\mathcal{S}_D(\chi_0); \Delta_\gamma) \subset \mathcal{A}(\mathbb{S}^q; \Delta_\gamma) \). According to the restriction of the algebra, the base space is also restricted to a discrete space with equal distance \( \gamma \). The base space of \( \mathcal{A}(\mathcal{S}_D(\chi_0); \Delta_\gamma) \) is, therefore, written as \( \mathbb{R}_D(\chi_0) = \{ \chi_0 + n\gamma \mid n \in \mathbb{Z} \} \). If one choose another point \( \chi'_0 \in \mathbb{R}_\gamma \), then the algebra \( \mathcal{A}(\mathcal{S}_D(\chi'_0); \Delta_\gamma) \) is obtained as another subalgebra of \( \mathcal{A}(\mathbb{S}^q; \Delta_\gamma) \) and the base space \( \mathbb{R}_D(\chi'_0) \) is derived. As a matter of course, the algebras \( \mathcal{A}(\mathcal{S}_D(\chi_0); \Delta_\gamma) \) and \( \mathcal{A}(\mathcal{S}_D(\chi'_0); \Delta_\gamma) \) are equivalent. Dividing \( \mathcal{A}(\mathbb{S}^q; \Delta_\gamma) \) by the equivalence, one has finally reached the following proposition:

**Proposition 1** : Quantum Real Line \( \mathbb{R}_D \)

*In the case where the deformation parameter \( q \) is not a root of unity, the quantum real line \( \mathbb{R}_D \) is a discrete space composed of an infinite number of points \( \chi_n \) such as

\[
\mathbb{R}_D := \{ \chi_n \mid \chi_n = n\gamma + \chi_0, \ n \in \mathbb{Z} \}.
\] (46)

The quantum real line \( \mathbb{R}_D \) is depicted in Fig.1.

\[
\cdots \chi_{n-2} \chi_{n-1} \chi_n \chi_{n+1} \chi_{n+2} \cdots \rightarrow \mathbb{R}_D
\]

**Fig.1** The quantum real line \( \mathbb{R}_D \)

Let us look at the situations in the limit \( \gamma \to 0 \ (q \to 1) \). We find first that the intervals between adjacent two points become zero and the quantum line \( \mathbb{R}_D \) becomes a continuous line. Further, the difference operator given in (43) reduces
to the standard differential operator $\partial_\chi$ with respect to the variable $\chi$. From these observations, one can conclude that, in the limit $\gamma \to 0$, the variable $\chi$ corresponds to the standard real number and, therefore, $\mathbb{R}_D \to \mathbb{R}$.

It is important to discuss the relation between $\mathbb{R}_\gamma$ and $\mathbb{R}_D$. As we have seen, the algebra $\mathcal{A}(\mathfrak{S}; \Delta_\gamma)$ on $\mathbb{R}_\gamma$ is the direct sum of an infinite number of subalgebras $\mathcal{A}(\mathfrak{S}_D(\chi_0); \Delta_\gamma)$ with $0 \leq \chi_0 < \gamma$. This fact immediately derives that $\mathbb{R}_\gamma$ is also the union of $\mathbb{R}_D(\chi_0)$, i.e., $\mathbb{R}_\gamma = \cup_{0 \leq \chi_0 < \gamma} \mathbb{R}_D(\chi_0)$. One can say from this relation between $\mathbb{R}_\gamma$ and $\mathbb{R}_D$ as follows: A quantum real line $\mathbb{R}_D$ is nothing but a representative, in this sense, each point $\chi_n \in \mathbb{R}_D$ is a representative, i.e., an infinite number of $\chi_n$ fill the interval, denoted as $I_n = [n.n+1) \subset \mathbb{R}_\gamma$, of width $\gamma$. One cannot distinguish a point $\chi_n \in I_n$ from others. In other words, the location of the point $\chi_n$ is always specified with the ambiguity of width $\gamma$. This suggests that we always have the uncertainty $\Delta \chi \sim \gamma$ in coordinate of the quantum line, i.e., there appears minimal length $\gamma$.

As the final discussion, let us observe $\mathbb{R}_\gamma$ and $\mathbb{R}_D$ from another viewpoint. One should notice that the space $\mathbb{R}_\gamma$ parameterized by the continuous variable $\chi$ can be regarded as a cylinder $\mathbb{R} \times S^1$. Each point in $\mathbb{R}$, i.e., each real number has an internal space $S^1$. Conversely, each point in $S^1$ has a real line $\mathbb{R}$. Namely, an infinite number of real lines attach to $S^1$. One can freely pick up just one real line $\mathbb{R}$ among them. The quantum real line $\mathbb{R}_D$ is identified with the real line $\mathbb{R}$ we have picked up and an infinite number of $S^1$ on the $\mathbb{R}$. The interval $I_n$ corresponds to $S^1$, i.e., the size of the internal space $S^1$ is $\sim \gamma$. Taking the minimal length discussed above into account, one may consider that quantum point looks a one-dimensionnal circle.

4.2 Quantum real line $\forall \mathbb{R} ; q$ is a root of unity

Let us turn our attention to the case where the deformation parameter $q$ is the $N$-th root of unity. In Section 3.2, we have obtained the closed algebra $\mathcal{A}_N(\mathfrak{S} ; \mathfrak{S}_{\text{int}})$ and introduced the base space $\forall \mathbb{R} = \{(x, \xi) | x \in \mathbb{R}, \xi = e^{i\theta}, 0 \leq \theta < 2\pi\}$ in (17). We are at the stage to investigate geometrical structure of the space $\forall \mathbb{R}$ and to propose the final quantum real line $\forall \mathbb{R}$. Since $\mathcal{A}_N(\mathfrak{S} ; \mathfrak{S}_{\text{int}})$ is the tensor product of two spaces $\mathcal{A}(\mathfrak{S}; \partial_\chi)$ and $\mathcal{A}_{\text{int}}(\mathfrak{S}_{\text{int}}; D_q)$, the base space $\forall \mathbb{R}$ is expected to be the tensor product of $\mathbb{R}$ and $\mathcal{S}$ which is introduced as the base space of $\mathcal{A}_{\text{int}}$. What we should do first is to study the geometrical structure of the internal space $\mathcal{S}$. 
To show the structure of $S$, one should notice that the basis $\omega$ of $A_{\text{int}}$ is of finite dimension, i.e., $\dim \omega = N$, and recall that $\omega$ is complete. Therefore, any function in $A_{\text{int}}$ can be expanded uniquely in terms of $\xi^r, r \in I$. In particular, there exists the function $\delta(\theta; \theta') \in A_{\text{int}}$ which plays the role of the “delta function” as

$$\delta(\theta; \theta') := \sum_{r \in I} (\xi^*)^r \cdot (\xi')^r = \sum_{r \in I} e^{ir(\theta'-\theta)}. \quad (47)$$

Actually, in the limit of $N \to \infty$, the function $\delta(\theta; \theta')$ returns to the Dirac’s delta function as

$$\lim_{N \to \infty} \delta(\theta; \theta') = \delta(\theta - \theta'). \quad (48)$$

This suggests that, for a point $\theta = \theta'$, the function $\delta(\theta; \theta')$ does not have a sharp peak but spreads around the point. In this sense, the base space $S$ is composed of “fuzzy” points and is regarded as a “fuzzy” circle. Thus, the internal space of $q\mathbb{R}$ is the fuzzy circle and $q\mathbb{R}$ is written as,

$$q\mathbb{R} = \mathbb{R} \times S, \quad (49)$$

i.e., every point $x \in \mathbb{R}$ has the fuzzy internal space $S$.

Having obtained the base space $q\mathbb{R}$ as (19), we should make clear the relationship between the internal space $S$ and the external space $\mathbb{R}$. Recall the following facts:

The difference operator $D_q$ acting on the internal sector $S_{\text{int}}$ generates the finite displacement $\xi \to q\xi$. By the definition $q = e^{2\pi i/N}$, one finds $D_q : (x, \theta) \mapsto (x, \theta + \frac{2\pi}{N})$, where $P \mapsto P'$ stands for the mapping from the point $P$ to the points between $P$ and $P'$ along $S$. Therefore, the $N$-th power of $D_q$ brings $\theta$ to the same position $\theta$ in $S$, i.e., $D_q^N : (x, \theta) \mapsto (x, \theta + 2\pi)$. However, upon taking the external space $\mathbb{R}$ into account, the operator $D_q^N/[N]!$ becomes equivalent to the differential operator $\partial_x$, the generator of the infinitesimal displacement along $\mathbb{R}$. Namely, for an infinitesimal number $\epsilon \in \mathbb{R}$, we find $D_q^N/[N]! = \partial_x : (x, \theta) \mapsto (x + \epsilon, \theta)$. One can, therefore, identify $(x, \theta + 2\pi)$ with $(x + \epsilon, \theta)$. This identification indicates that the internal space $S$ connects two real points $x$ and $x + \epsilon$ which are separated by infinitesimal distance along $\mathbb{R}$. In this sense, the fuzzy space $S$ can be called the infinitesimal structure. In order to make such a structure more concrete and finally obtain the quantum real line $^\wedge\mathbb{R}$, some manipulations are needed.

The procedure for deriving $^\wedge\mathbb{R}$ from $q\mathbb{R}$ follows that given in the preceding subsection where $\mathbb{R}_D$ was obtained from $\mathbb{R}$. We first reduce $q\mathbb{R}$ to a discrete $N$-point
space $q\mathbb{R}_N$ and, then, $q\mathbb{R}_N$ is modified to the final quantum real line $\triangledown \mathbb{R}$. To start with the first step, notice that $S$ is continuous space. Actually, the variable $\xi$ parameterizing $S$ is continuous. However, once a point $\xi_0 \in S$ is chosen, a subalgebra $A_{\text{int}}(S_{\text{int}}(\xi_0))$ is obtained owing to the $q$-differential, i.e., difference structure. Here $S_{\text{int}}(\xi_0) = \{\psi(\xi_r) | \xi_r = \xi_0 e^{2\pi i r/N}, r = 0, 1, \cdots, N - 1\}$. One then finds the discrete space $S_N(\xi_0) = \{\xi_r | r = 0, 1, \cdots, N - 1\}$ as the base space of $A_{\text{int}}(S_{\text{int}}(\xi_0); D_q)$. If another point $\xi'_0$ is chosen, another subalgebra $A_{\text{int}}(S_{\text{int}}(\xi'_0); D_q)$ and its base space $S_N(\xi'_0)$ are obtained. The continuous fuzzy space $S$ is the union of all the discrete spaces as $S = \bigcup_{0 \leq \xi_0 < q} S_N(\xi_0)$. Therefore, the base space $q\mathbb{R}$ is reduced to the discrete space

$$q\mathbb{R}_N = \mathbb{R} \times S_N.$$  

The important fact to be noticed here is that, $S_N$ is also fuzzy space, i.e., each point in $S_N$ is not a stable and localized point. For example, the $r$-th point is not located at the position $\xi_r$ but fluctuating around $\xi_r$.

Let us go ahead to the next step where $q\mathbb{R}_N$ is modified into the final quantum line $\triangledown \mathbb{R}$. This is performed by changing the internal space $S_N$ into another discrete space $\Sigma_\varepsilon$. In the quantum space $q\mathbb{R}_N$, the internal space $S_N$ is attached on every point of $\mathbb{R}$ as the extra dimension. On the other hand, upon considering $\triangledown \mathbb{R}$ to be an extension of $\mathbb{R}$, an infinite number of the internal spaces $\Sigma_\varepsilon$ are embedded into the blanks $\triangledown \mathbb{R} \setminus \mathbb{R}$. The space $\Sigma_\varepsilon$ is defined by the mapping $\bar{\pi} : S_N \rightarrow \Sigma_\varepsilon$ such as

$$S_N \ni \xi_r \mapsto \zeta_r = r\varepsilon \in \Sigma_\varepsilon, \quad r = 0, 1, \cdots, N - 1,$$

with some “number” $\varepsilon$. By writing the algebra of functions on $\Sigma_\varepsilon$ as $\tilde{\mathbb{S}}$ and the difference operator with respect to $\zeta$ as $\Delta_\varepsilon$, we have the algebra $A_{\text{int}}(\tilde{\mathbb{S}}; \Delta_\varepsilon)$. The difference operator $\Delta_\varepsilon$ is given by

$$\Delta_\varepsilon \phi(\zeta_r) = \frac{\phi(\zeta_{r+1}) - \phi(\zeta_r)}{\varepsilon}, \quad \phi(\zeta) \in \tilde{\mathbb{S}},$$

The point we should stress here is that, since $\Sigma_\varepsilon$ is to be embedded into $\triangledown \mathbb{R} \setminus \mathbb{R}$, the number $\varepsilon$ cannot be a real number but $\varepsilon \in \triangledown \mathbb{R} \setminus \mathbb{R}$.

It is the time to give the explicit definition of the quantum real line $\triangledown \mathbb{R}$. First, we put $\zeta_0$ on each real number $x \in \mathbb{R}$ and $\zeta_N := \zeta_{N-1} + \varepsilon$ on the real number $x + \varepsilon$ with an infinitesimal number $\varepsilon$. In this way, an internal space $\Sigma_\varepsilon$ is embedded into
the “crack” between $x$ and $x + \epsilon$ for every real number $x \in \mathbb{R}$, and we define the space $\mathcal{M}(x)$ by

$$\mathcal{M}(x) = \{ \tilde{x}_r \mid \tilde{x}_r = x + \zeta_r = x + r\tilde{\epsilon}, \ r = 0, 1, \cdots, N - 1 \}. \quad (53)$$

We have finally reached the stage to propose $\mathcal{V}\mathbb{R}$ through $\mathcal{M}(x)$,

**Proposition 2**: Quantum Real Line $\mathcal{V}\mathbb{R}$

The quantum real line $\mathcal{V}\mathbb{R}$ for the case with the deformation parameter $q$ at the $N$-th root of unity is given by the union of $\mathcal{M}(x)$ as,

$$\mathcal{V}\mathbb{R} = \bigcup_{x \in \mathbb{R}} \mathcal{M}(x). \quad (54)$$

The quantum real line $\mathcal{V}\mathbb{R}$ is shown in Fig.2.

![Fig.2 The quantum real line $\mathcal{V}\mathbb{R}$](image)

where the dots $\bullet$ stand for the “standard” real numbers, while the circles $\circ$ represent numbers in $\mathcal{V}\mathbb{R} \setminus \mathbb{R}$.

Some remarks on $\mathcal{V}\mathbb{R}$ are in order. We have called the internal space $\mathcal{S}$ in $\mathcal{V}\mathbb{R}$ the infinitesimal structure. In $\mathcal{V}\mathbb{R}$, the infinitesimal structure lying between $x$ and $x + \epsilon$, $x, \epsilon \in \mathbb{R}$, corresponds to the spaces $\mathcal{M}(x) \setminus \{\tilde{x}_0\}$. Thus, the infinitesimal structure spreads the interval between two real numbers $x$ and $x + \epsilon$. As we stated above, the number $\tilde{\epsilon}$ is an element of $\mathcal{V}\mathbb{R} \setminus \mathbb{R}$. Since, $\epsilon$ is represented by $\tilde{\epsilon}$ as $\epsilon = N\tilde{\epsilon}$, one should not consider $\epsilon$ as an infinitesimal number in $\mathbb{R}$ but an element of $\mathcal{V}\mathbb{R}$. In other words, from the viewpoints of the extended real numbers $\mathcal{V}\mathbb{R}$, $\epsilon$ is no longer an infinitesimal but a measurable number. One regards $\epsilon$ as an infinitesimal number only when it is observed from the viewpoints of $\mathbb{R}$.

As can be seen from Fig.1 and Fig.2, both quantum real lines $\mathbb{R}_D$ and $\mathcal{V}\mathbb{R}$ are discrete spaces. However, there are essential differences between the two. In the final section, we will summarize these quantum lines by stressing the differences.
5 Concluding Remarks

In this paper, we have proposed possible deformations of the real numbers $\mathbb{R}$ by taking quantum effects into account, i.e., quantum real lines. Our discussions started from deformation of the Heisenberg algebra $\mathcal{L}_H$ by introducing an internal space $\mathcal{T}_2$. In order to derive nontrivial quantum effects from the internal sector, we further introduce the Moyal product $\ast_\gamma$ as the multiplication of the functional algebra on $\mathcal{T}_2$. As the result, we have obtained the $q$-deformed Heisenberg algebra $\hat{\mathcal{L}}_q$ with the defining commutation relation (14). We have next derived the algebras $\mathcal{A}$ of $q$-differentiable functions for the cases (I) where $q$ is generic, i.e., $q = e^{i\gamma}$ with $\gamma$ being irrational or pure imaginary, (II) where $q$ is a root of unity, i.e., $q = e^{2\pi i/N}$ with an integer $N$. From these algebras, the geometrical structures of the base spaces have been deduced. Through these base spaces, we have proposed $q$-deformed real numbers, i.e., quantum real lines.

Let us summarize our derivations of the quantum real lines. In the case (I), the algebra $\mathcal{A}(\mathbb{S}^q; D_q)$ is the space of functions on which the action of the $q$-differential operator $D_q$ is defined. Here, the base space was introduced as $\mathbb{R} = \{x_\theta \mid x_\theta = x_\gamma e^{i\theta}\}$. We have shown that noncommutative natures of the algebra, e.g., eq.(24), arise from the $q$-differential structure. In order to show the geometrical structures of the base space in explicit, the algebra $\mathcal{A}(\mathbb{S}^q; D_q)$ on $\mathbb{R}$ was modified to the algebra $\mathcal{A}(\mathbb{S}_\gamma; \Delta_\gamma)$, where the base space was modified to $\mathbb{R}_\gamma$ as well. Upon parameterizing $\mathbb{R}_\gamma$ by the variable $\chi$ given in (12), $\mathbb{R}_\gamma$ is still a continuous space. However, we found an infinite number of subalgebras in $\mathcal{A}(\mathbb{S}_\gamma; \Delta_\gamma)$. Actually, by fixing a point $\chi_0$ as an origin, the subspace $\mathbb{S}_D(\chi_0) = \{f(\chi_n) \mid \chi_n = \chi_0 + n\gamma, n \in \mathbb{Z}\}$ is extracted from $\mathbb{S}_\gamma$. Since two subalgebras $\mathbb{S}_D(\chi_0)$ and $\mathbb{S}_D(\chi_0')$, $\chi_0' \neq \chi_0 + n\gamma$ are equivalent, we have divided $\mathcal{A}(\mathbb{S}_\gamma; \Delta_\gamma)$ by the equivalence and finally reached the algebra $\mathcal{A}(\mathbb{S}_D : \Delta_\gamma)$. Hence, the base space of $\mathcal{A}(\mathbb{S}_D : \Delta_\gamma)$ is the discrete space $\mathbb{R}_D$ shown in Fig.1. We have proposed $\mathbb{R}_D$ as the $q$-deformed real line in the case with generic $q$. The process from $\mathbb{R}$ to $\mathbb{R}_D$ is

\[
\mathbb{R} \longrightarrow \mathbb{R}_\gamma \searrow \mathbb{R}_D
\]

\[
\psi \uparrow \uparrow \uparrow \psi
\]

\[
x_\theta \quad \chi \quad \chi_n \quad (n \in \mathbb{Z})
\]

where $\searrow$ stands for the reduction.

Let us next summarize the case where $q$ is the $N$-th root of unity. We have
started with the algebra $\mathbb{S}_N^q$ of functions on the base space $\mathbb{N} \mathbb{R} = \{y_\xi \mid y_\xi = y_\xi, \xi = e^{i\theta}\}$. When the differential structure to be introduced in $\mathbb{S}_N^q$ was considered, we found the identity $D_q^N \equiv 0$ on $\mathbb{S}_N^q$. On the contrary, we further found that the operator $D_q^N/[N]!$ acts on $\mathbb{S}_N^q$ as
\[
\frac{D_q^N}{[N]!} y^{kN+r} = k y^{(k-1)N+r}.
\]
From these observations, we have seen that the algebra $\mathcal{A}_N(\mathbb{S}_N^q; D_q)$ on $\mathbb{N} \mathbb{R}$ decomposes as $\mathcal{A}_N(\mathbb{S}_N^q; D_q) \rightarrow \mathcal{A}_N(\mathbb{S}; \mathbb{S}_\text{int}) = A(\mathbb{S}; \partial_x) \otimes \mathcal{A}_\text{int}(\mathbb{S}_\text{int}; D_q)$ where $A(\mathbb{S}; \partial_x)$ is the standard algebra of differentiable functions on $\mathbb{R} = \{x \mid x = y^N\}$. On the other hand, $\mathcal{A}_\text{int}(\mathbb{S}_\text{int}; D_q)$ is the algebra of $q$-differentiable functions on the internal space $\mathcal{S} = \{\xi \mid \xi = e^{i\theta}, 0 \leq \theta < 2\pi\}$. The functional algebra $\mathbb{S}_\text{int}$ on $\mathcal{S}$ has the nonstandard property that the basis $\varpi = \{\xi^n \mid n \in \mathbb{I}\}$ of $\mathbb{S}_\text{int}$ is of finite dimension, i.e., $\dim \varpi = N$. This property affects the structure of $\mathcal{S}$. Indeed, we have seen that $\mathcal{S}$ is a fuzzy space.

An important fact is that the internal space $\mathcal{S}$ is attached on every point of the real line $\mathbb{R}$ and connects two real numbers $x$ and $x + \epsilon$ with an infinitesimal $\epsilon$. Thus, we have concluded that the base space $\mathbb{N} \mathbb{R}$ should be transformed to $\mathbb{R} \mathbb{S}$ of the algebra $\mathcal{A}_N(\mathbb{S}; \mathbb{S}_\text{int})$ and $\mathbb{R} \mathbb{S}$ is the product space as $\mathbb{R} \mathbb{S} = \mathbb{R} \times \mathcal{S}$. To reach the finial quantum real line, two steps were needed for the internal sector $\mathcal{A}_\text{int}(\mathbb{S}_\text{int}; D_q)$.

The first was the reduction to $\mathcal{A}_\text{int}(\mathbb{S}(\xi_0); D_q)$ composed of functions on the discrete space $\mathcal{S}_N = \{\xi_r \mid \xi_r = \xi_0 q^r, r = 0, 1, \ldots, N - 1\}$, and we obtained the base space $\mathbb{R}_N = \mathbb{R} \times \mathcal{S}_N$. The next step modified $\mathcal{A}_\text{int}(\mathbb{S}(\xi_0); D_q)$ to the algebra $\mathcal{A}_\text{D}(\mathbb{S}; \Delta_\epsilon)$ where $\mathbb{S}$ was the algebra of functions on the space $\Sigma_\epsilon = \{\xi_r \mid r = 0, 1, \ldots, N - 1\}$. The space $\Sigma_\epsilon$ was introduced instead of $\mathcal{S}_N$ so that we can extend $\mathbb{R}$ to $\mathbb{R} \mathcal{S}$. The base space $\mathbb{R} \mathcal{S}$ is just the quantum real line we have proposed. To define $\mathbb{R} \mathcal{S}$ explicitly, the space $\mathcal{M}(x)$ was introduced at each point $x \in \mathbb{R}$ as $\mathcal{M}(x) = \{\tilde{x}_r \mid \tilde{x}_r = \tilde{x}_0 + r \tilde{\epsilon}, \tilde{x}_0 = x, r = 0, 1, \ldots, N - 1\}$. Finally, the quantum real line in the case with $q$ at the $N$-th root of unity has been defined by $\mathbb{R} \mathcal{S} = \bigcup_{x \in \mathbb{R}} \mathcal{M}(x)$. Thus, we have found the infinitesimal structure represented by $\mathcal{M}(x)$ between two points $x, x + \epsilon \in \mathbb{R}$ with an infinitesimal number $\epsilon$. In summary, the extension $\mathbb{R} \mathbb{S}$ of real numbers $\mathbb{R}$ has been performed by the procedure as
\[
\begin{align*}
\mathbb{N} \mathbb{R} & \rightarrow (\mathbb{R} \times \mathcal{S}) \quad \mathbb{R} \mathcal{S}_N (\mathbb{R} \times \mathcal{S}_N) & \rightarrow \mathbb{R} \mathcal{S} (\bigcup_{x \in \mathbb{R}} \mathcal{M}(x)) \\
y_\xi & (x, \xi) & (x, \xi_r) & \tilde{x}_r
\end{align*}
\]

Let us give some remarks upon natures of the quantum real numbers $\mathbb{R}_\text{D}$ and $\mathbb{R} \mathcal{S}$, and we would like to stress the differences between the two. Although both have sim-
ilar discrete structures, there exist some essential differences. The fact to be stressed first is that the quantum line $\mathbb{R}_D$ is, in general, composed of standard complex numbers, i.e., $\chi_n \in \mathbb{C}$ and points $\chi_n$ are localized objects. The quantum effect appears in $\mathbb{R}_D$ as the blank between two adjacent points. We have discussed in Section 4.1 the physical picture of the interval, i.e., it is just the inimal length in quantum line. In other words, we always have the uncertainty $\Delta \chi \sim \gamma$ in measurement of position.

On the other hand, each point $\tilde{x}_r \in \check{\mathbb{R}}$ has fuzzy structure. Furthermore, $\tilde{x}_r$ cannot be represented in terms of the standard complex numbers, especially the interval $\tilde{\epsilon}$ between $\tilde{x}_r$ and $\tilde{x}_{r+1}$ is the case. We have stressed that $\check{\mathbb{R}}$ is an extended object of the real number $\mathbb{R}$. However, in $\check{\mathbb{R}}$, one can find $\mathbb{R}$ itself as the subset of $\check{\mathbb{R}}$. Actually, the dots $\bullet$ in Fig.2 compose $\mathbb{R}$. It is quite important to notice that the number $\epsilon$, the interval beteween two adjacent $\bullet$, is an infinitesimally small number as long as we observe it within the viewpoints of $\mathbb{R}$. On the contrary, once we observe $\epsilon$ from the standpoint of $\check{\mathbb{R}}$, it is not an infinitesimal but an measurable number in terms of $\tilde{\epsilon}$ as $\epsilon = N \tilde{\epsilon}$.

Another characteristic feature of $\check{\mathbb{R}}$ to be emphasized again is the existence of the internal space $\mathcal{M}(x) \backslash \{\tilde{x}_0\}$ at each pint $\tilde{x}_0 \equiv x \in \mathbb{R}$. In Fig.2, the numbers in the internal spaces are expressed by $\circ$. We have called the internal space the infinitesimal structure in the sense that $\mathcal{M}(x) \backslash \{\tilde{x}_0\}$ lives between the two real numbers $x$ and $x + \epsilon$. Thus, the quantum effects appearing in $\check{\mathbb{R}}$ is different from those in $\mathbb{R}_D$. The discreteness yielding the noncommutativity and the minimal length is considered as the only quantum effect in $\mathbb{R}_D$, while, in $\check{\mathbb{R}}$, the appearance of the infinitesimal structure is the characteristic quantum effect.

Let us find another difference between $\mathbb{R}_D$ and $\check{\mathbb{R}}$ by looking at them at a long-scale. As for the quantum line $\mathbb{R}_D$, we take the long-scale limit through $q \to 1$, i.e., $\gamma \to 0$. Then, the interval between two points becomes infinitesimally small and, therefore, the quantum line $\mathbb{R}_D$ reduces to $\mathbb{R}$. Namely, at the long-scale, the discreteness of $\mathbb{R}_D$ is not visible and $\mathbb{R}_D$ looks continuous. On the other hand, for the line $\check{\mathbb{R}}$, one obtains the case with $q = 1$ by setting $N = 1$. Then, $\mathcal{M}(x)$ becomes the one-point space $\mathcal{M}(x)|_{N=1} = \{\tilde{x}_0 = x \in \mathbb{R}\}$, i.e., the internal space vanishes and $\check{\mathbb{R}} = \mathbb{R}$ with $\tilde{\epsilon} = \epsilon$ being an infinitesimal. The above situation is quite trivial and not beyond our expectation. However, in $\check{\mathbb{R}}$, there is another interesting case where we take the limit $\epsilon \to 0$ with keeping $N \neq 1$. This case corresponds to the
observation of $\mathbb{R}$ at the long-scale such that the interval $\epsilon$ is infinitesimally small and invisible. In this limit, each internal space $\mathcal{M}(x)$ shrinks to a point $x \in \mathbb{R}$ and we find the real line in which every point $x$ has the internal structure.

Let us end this paper by addressing an interesting problem to be studied further. It is the application of the quantum real line $\mathbb{R}$ with $N = 2$ to supersymmetric model. Actually, the author has shown the equivalence between the quantum universal enveloping algebra $U_q\text{sl}(2, \mathbb{C})$ with $q$ at the 2nd root of unity and the supersymmetric algebra $\text{Osp}(2|1)$ [4]. In the case with $N = 2$, the space $\mathcal{M}(x)$ is the two-point space such as $\mathcal{M}(x) = \{\bullet, \circ\}$. It is natural to expect that one of them corresponds to bosonic space and the other to fermionic space. The investigation will appear elsewhere.

Acknowledgments
The author would like to thank Dr. T. Koyama for valuable comments.

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