Post-Newtonian effects in \( N \)-body dynamics: conserved quantities in hierarchical triple systems

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Abstract
Conventional approaches to incorporating general relativistic effects into the dynamics of \( N \)-body systems containing central black holes, or of hierarchical triple systems with a relativistic inner binary, may not be adequate when the goal is to study the evolution of the system over a timescale related to relativistic secular effects, such as the precession of the pericenter. For such problems, it may be necessary to include post-Newtonian (PN) ‘cross terms’ in the equations of motion in order to capture relativistic effects consistently over the long timescales. Cross terms are PN terms that explicitly couple the two-body relativistic perturbations with the Newtonian perturbations due to other bodies in the system. In this paper, we show that the total energy of a hierarchical triple system is manifestly conserved to Newtonian order over the relativistic pericenter precession timescale of the inner binary if and only if PN cross-term effects in the equations of motion are taken carefully into account.

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1. Introduction and summary
The study of general relativistic effects in orbital dynamics has evolved in recent years well beyond the simple two-body problem that was of such historic importance for the theory. Galactic star clusters with massive central black holes [1, 2], triple systems with a relativistic compact inner binary [3–8], binary black hole coalescence in the presence of a third body [9–11] and even the stability of the solar system [12] have been studied using combinations of \( N \)-body techniques and relativistic dynamics.
In a recent paper [13] (hereafter referred to as paper I), we argued that conventional approaches to incorporating relativistic effects in such analyses may not be adequate when the goal is to study the evolution of the system over a timescale related to relativistic secular effects, notably the precession of the pericenter. In the conventional approach, one typically augments the $N$-body Newtonian equations of motion by two-body post-Newtonian (PN) relativistic corrections, where the two bodies in question might be a chosen star and the central black hole in the galactic core problem, or the tight binary system in the hierarchical three-body, or Kozai–Lidov problem. We argued that, for such problems, it may necessary to include PN ‘cross terms’ in the equations of motion in order to capture the relativistic effects consistently over the long timescales.

In the context of a hierarchical triple system, the idea of cross terms is as follows: a relativistic effect in the inner binary, such as the pericenter advance, is proportional to $Gm/ac^2$, where $m$ and $a$ are the mass and semimajor axis of the binary, and $G$ and $c$ are the gravitational constant and speed of light, respectively. A Newtonian effect due to the third body at a distance $R \gg a$ is proportional to $m_3mR^n/a$, where $n$ is a power depending on the degree to which the field of the third body is expressed in a multipole expansion ($n = 3$ corresponds to the leading quadrupole order). A PN effect due to ‘cross terms’ would be proportional to $(Gm/ac^2) \times (m_3mR^n/a)$. On the face of it, this is a smaller effect than either the pure PN effect or the third-body effect, when $aR \ll 1$, as long as the masses are more or less comparable to one another. However, if it is a secular effect, and if one is interested in how this effect grows over a relativistic timescale induced in the binary, $T_B \sim T_B(ac^2/Gm)$, where $T_B$ is the binary period, then the effect could be ‘boosted’ from a PN-level effect to a Newtonian-level effect. This could have hitherto unforeseen consequences in long-term evolutions of such systems. In the context of stellar clusters with a central black hole, $m$ becomes the mass of the central black hole, $a$ becomes the semimajor axis of a chosen star $b$, $m_3$ becomes $m$, and $R$ becomes $R_{bc}$, summed over the other stars in the cluster.

The origin of these ideas was the simple two-body problem of a test particle moving in the gravitational field of a body with mass $M$ and quadrupole moment $Q_2$, including the standard PN corrections from the Schwarzschild part of the metric, whose main consequence is the advance of the pericenter. The Newtonian conserved energy per unit mass, evaluated at pericenter of the orbit is given by

$$E = -\frac{GM}{2a} + \frac{GQ_2}{2} \left(1 + \frac{e}{p}\right) \left(1 - 3\sin^2 i \sin^2 \omega\right),$$

where $a$, $e$, $i$ and $\omega$ are the osculating semimajor axis, eccentricity, inclination and pericenter angle of the orbit, and $p = a(1 - e^2)$ (for a pedagogical introduction to osculating orbit elements see [14]). To this order, $a$, $e$ and $i$ do not experience secular changes, but $\omega$ grows linearly with time at the rate $6\pi GM/c^2p$ because of relativistic effects due to the mass $M$. When $\omega$ changes by a macroscopic amount, say $\pi/2$, the energy apparently changes by a Newtonian-order amount, say $\pi/2$, the energy apparently changes by a Newtonian-order amount, in violation of the basic conservation law. We showed in paper I that when cross terms of order $(GM/ac^2) \times (Q_2/Ma^2)$ were included in the PN equations of motion and when the equations for the perturbed orbit elements were carefully solved (including internally generated cross-term contributions), the semimajor axis $a$ suffered a secular change per orbit that was of PN order and also depended on $Q_2$, which, when integrated over a pericenter precession timescale, was boosted to a Newtonian-order variation in $a$ that exactly compensated for the $\omega$ dependence in equation (1), leaving an energy expression that was manifestly constant over a pericenter precession timescale.
This unusual result motivated us to suggest that such cross-terms should be taken into account in other contexts, such as stellar clusters with central massive black holes and hierarchical triple systems. Accordingly, in paper I we wrote down the truncated PN equations of motion, including all relevant cross terms, in a ready-to-use form either for numerical N-body simulations of clusters with a central black hole or for studies involving perturbations of orbit elements in hierarchical triple systems. For the simple case of a hierarchical triple with the third body in a circular orbit, we solved the osculating orbit element perturbation equations explicitly, including the cross-term effects. In this paper, we shall apply those results to demonstrate explicitly that the total energy \( E \) of the system is conserved over a pericenter precession timescale of the inner binary if and only if the PN cross term effects are included. We will also show a related conservation statement for the component of angular momentum of the system perpendicular to the orbital plane of the third body, \( L_Z \).

In section 2, we review the basic equations of paper I for hierarchical triple systems, and in section 3 we derive the conserved energy \( E \) and total angular momentum \( L \) for the system, to PN order, and show that the lowest-order expression (Newtonian plus PN) for \( E \) is apparently not conserved over a pericenter precession timescale, presenting the same conundrum as in the quadrupole problem. In section 4 we show that incorporating the full secular evolution of the orbit elements including PN cross terms completely resolves the conundrum. Concluding remarks are presented in section 5.

2. PN equations of motion for hierarchical triple systems

We consider a three-body system in which two bodies of mass \( m_1 \) and \( m_2 \) are in a close orbit with separation \( r \), and a third body of mass \( m_3 \) is in a wide orbit with separation \( R \gg r \). We define the relative separation vector of the two-body system and the vector from the center of mass of the two-body system to the third body by
\[
x \equiv x_1 - x_2 , \quad X \equiv x_3 - x_0 ,
\]
where
\[
x_0 \equiv \frac{m_1 x_1 + m_2 x_2}{m} ,
\]
where \( m \equiv m_1 + m_2 \) is the mass of the two-body system. We work in the center-of-mass frame of the entire system, where
\[
m_1 x_1 + m_2 x_2 + m_3 x_3 = m x_0 + m_3 x_3 = O\left(c^{-2}\right) ,
\]
where \( O(c^{-2}) \) represents a PN correction to the center of mass. As a result of these definitions,
\[
x_1 = \frac{m_2}{m} X - \frac{m_3}{M} X , \quad x_2 = -\frac{m_1}{m} X - \frac{m_2}{M} X , \quad x_3 = \frac{m}{M} X ,
\]
where \( M = m_1 + m_2 + m_3 \) is the total mass. The \( O(c^{-2}) \) correction in equation (4) will not be relevant because only differences between position vectors appear in the equations of motion, and because velocities that are derived from these relations appear in terms that are already of PN order. We define the velocities \( v \equiv dx/dt \), \( V \equiv dX/dt \), accelerations \( a \equiv dv/dt \), \( A \equiv dV/dt \), distances \( r \equiv |x| \), \( R \equiv |X| \), and unit vectors \( n \equiv x/r \) and \( N \equiv X/R \). For future use we define the symmetric reduced mass \( \eta \equiv m_1 m_2 / m^2 \) and the dimensionless mass difference \( \Delta \equiv (m_1 - m_2)/m \).

The interaction of the two bodies with the third body depends on \( x_{13} \) and \( x_{23} \), which we will express as
\[
x_{13} = - X + \alpha_2 x = - R \left[ N - \alpha_2 (r/R)n \right], \\
x_{23} = - X - \alpha_2 x = - R \left[ N + \alpha_1 (r/R)n \right],
\]
where \( \alpha_i \equiv m_i/m \); we will use this to expand quantities such as \( 1/r_{13} \) and \( 1/r_{23} \) as power series in \( r/R \). The resulting equations of motion for the binary system have the form
\[
a = - \frac{Gmn}{r^2} - \frac{Gm_3 r}{R^3} [n - 3(n \cdot N)N] + \frac{1}{c^2} [a]_{\text{Binary}} \\
+ \frac{1}{c^2} [a]_{\text{Cross}} + O \left( \frac{G^2 m m_3 r^{1/2}}{c^2 R^{7/2}} \right),
\]
where we have expanded the Newtonian term from the third body to quadrupole order, and where the binary and cross terms are given by
\[
[a]_{\text{Binary}} = \frac{Gmn}{r^2} \left[ (4 + 2\eta) \frac{Gm}{r} - (1 + 3\eta) v^2 + \frac{3}{2} \eta^2 \right] \\
+ (4 - 2\eta) \frac{Gmnv}{r^2},
\]
\[
[a]_{\text{Cross}} = \frac{Gm_3 \Delta}{r^2} \left[ 2n (v \cdot V) + v (n \cdot V) \right] \\
+ \frac{Gm_3}{r^2} \left[ \frac{5GM}{R} + \alpha_3 \left( V^2 + \frac{3}{2} R^2 \right) \right] \\
+ \frac{Gm_3}{R^2} \left[ \frac{Gm}{2r} (N - 9w n) + 4v (v \cdot N) - v^2 N \right] \\
- \frac{Gm_3}{R^2} \left[ (4 - 2\alpha_3) N (v \cdot V) - 4V (N \cdot v) - (3 + \alpha_3) R v \right] \\
+ \frac{G^2 mm_3}{R^3} \left[ (4 - \eta) (n - 3wN) - \frac{1}{2} (4 - 3\eta) n \left( 1 - 3w^2 \right) \right] \\
+ \frac{Gm_3}{R^3} \left[ (1 - 3\eta) \left( 4v \left( \hat{r} - 3w (v \cdot N) - \hat{v}^2 (n - 3wN) \right) \right. \right],
\]
where \( \mu_3 \equiv m_3 m/|M|, \alpha_3 \equiv m_3/M, \hat{r} \equiv n \cdot v, \hat{R} \equiv N \cdot V \) and \( w \equiv n \cdot N \). Recalling that \( v^2 \sim Gm/r \) and \( V^2 \sim GM/R \) we see that, to linear order in \( m_3 \), the six cross terms scale as \( (Gmm_3/R^3 c^2) \times (R/r)^{9/2} \), where \( n = 5, 4, 2, 1, 0, 0 \), respectively.

Treating the third body in the analogous way and defining \( A \equiv d^2 X/dr^2 \), we obtain
\[
A = - \frac{GMN}{R^2} + \frac{3 Gm_3 r^2}{R^4} \left[ N \left( 1 - 5w^2 \right) + 2wn \right] + O \left( \frac{1}{c^2} \right).
\]

Explicit expressions for the PN terms will not be needed for this discussion.

### 3. Conservation of energy and angular momentum

It is straightforward to show, either by truncating the full PN expressions for energy and angular momentum of an \( N \)-body system (see, e.g. paper I, equation (3.2a) for the energy), or by constructing conserved quantities directly from the equations of motion (7) and (10), that the conserved total energy and angular momentum are given by
\[ E = \frac{1}{2} \mu v^2 - \frac{G \mu m}{r} + \frac{1}{2} \mu_3 V^2 - \frac{G \mu_3 M}{R} + \frac{1}{2} \frac{G \mu m_3 r^2}{R^3} (1 - 3w^2) \]
\[ + \frac{1}{c^2} [E]_{\text{Binary}} + O\left( \frac{m_3}{c^2} \right), \quad (11) \]

\[ L = \mu x \times v + \mu_3 X \times V + \frac{1}{c^2} [L]_{\text{Binary}} + O\left( \frac{m_3}{c^2} \right), \quad (12) \]

where \( \mu_3 = m_3 M/M \), and where the PN contributions from the binary system are given by

\[ [E]_{\text{Binary}} = \frac{3}{8} \mu (1 - 3\eta) v^4 + \frac{1}{2} \frac{G \mu u}{r} \left[ (3 + \eta) v^2 + \frac{G m}{r} + \eta f^2 \right], \quad (13) \]

\[ [L]_{\text{Binary}} = \mu x \times v \left[ \frac{1}{2} (1 - 3\eta) v^2 + (3 + \eta) \frac{G m}{r} \right]. \quad (14) \]

Explicit forms for the cross-term contributions to \( E \) and \( L \), of order \( m_3/c^2 \), will not be needed.

We now consider the simplified problem in which the outer star is on a circular orbit on the X–Y plane. The inner binary is described by an osculating Keplerian orbit, defined by the equations

\[ r \equiv a \left( 1 - e^2 \right) / (1 + e \cos f), \]
\[ x \equiv a, \]
\[ n \equiv [\cos \Omega \cos (\omega + f) - \cos \iota \sin \Omega \sin (\omega + f)] e_x \]
\[ + [\sin \Omega \cos (\omega + f) + \cos \iota \cos \Omega \sin (\omega + f)] e_y \]
\[ + \sin \iota \sin (\omega + f) e_z, \]
\[ \dot{\lambda} \equiv \dot{n} / df, \quad \dot{h} = n \times \dot{\lambda}, \]
\[ \dot{h} \equiv x \times v = \sqrt{G ma \left( 1 - e^2 \right)} \dot{h}, \quad (15) \]

where \( f \) is the orbital phase or true anomaly and \( \Omega \) is the angle of the ascending node. From the given definitions, it is evident that \( v = i n + (h/r) \dot{\lambda} \) and \( \dot{r} = (he/p) \sin f \). The orbit elements \( a, e, \omega, \iota \) and \( \Omega \) are functions of \( f \) when the orbit is not purely Keplerian.

The outer binary is described by the equations \( X = RN \) and \( V = \Omega_3 R \Lambda \), where \( \Omega_3 = (GM/R^3)^{1/2} \), and where

\[ N = e_x \cos \Omega_3 t + e_y \sin \Omega_3 t, \]
\[ \Lambda = -e_x \sin \Omega_3 t + e_y \cos \Omega_3 t, \]
\[ H = N \times \Lambda = e_z. \quad (16) \]

We shall ignore perturbations of the third body’s orbit due to the binary; these will not be germane to the present discussion.

Retaining only the Newtonian and PN binary terms in the conserved energy and in the \( Z \) component of the angular momentum, averaging over an orbit of the third body, and expressing the result in terms of the osculating orbit elements of the inner binary, we obtain
\[
E = -\frac{G m_\mu}{2a} - \frac{1}{4} \frac{G m_3 a^2}{R^3} \left(1 - e^2\right)^2 \left(1 - 3 \sin^2 \iota \sin^2 \omega\right) \\
+ \frac{1}{8} \frac{\mu}{c^2} \left(\frac{G m}{a}\right)^2 \left[4 + 4(3 + \eta)(1 + e) + 3(1 - 3\eta)(1 + e)^2\right] \\
+ O\left(\frac{m_3}{c^2}\right),
\]
\[
L_Z = \mu \left[\frac{G m a (1 - e^2)^{3/2}}{c^2} \cos \iota \right] \left[1 + \frac{1}{2} \frac{G m (1 - 3\eta)(1 + e) + 2(3 + \eta)}{1 - e}\right] \\
+ O\left(\frac{m_3}{c^2}\right).
\]

We have ignored the constant contributions to the energy and \(L_Z\) from the circular orbit of the third body alone. Since these quantities are known to be constants, independent of true anomaly \(f\), we have displayed them with all orbit elements evaluated at pericenter, \(f = 0\).

The expression for \(E\) presents us with a conundrum. In the standard Newtonian Kozai problem, the semi-major axis \(a\) suffers no secular variations, and the Newtonian secular variations in \(e\) and \(\iota\) are all of order \((m_3/m)(a/R)^3\), and thus of higher order. When the PN binary effects are included, they do not contribute additional secular variations in \(a\), \(e\) and \(\iota\). However, the pericenter \(\omega\) is not constant, but increases via the standard PN secular effect, which we assume dominates other sources of pericenter precession. But equation (17) shows that the interaction energy between the binary system and the third body varies with pericenter angle \(\omega\). This makes sense physically: when \(\omega = 0\), the eccentric binary orbit lies more or less close to the \(X-Y\) plane, with the pericenter and apocenter lying in the plane, while when \(\omega = \pi/2\), the eccentric orbit extends well above and below the plane of the third body. It makes sense that the interaction between the binary and the third body should be quite different in the two cases. But to the order of approximation shown in equation (17), and over a pericenter precession timescale, the total energy must be constant, while the orbit elements \(a\) and \(e\) are also constant. What has gone wrong?

In the next section we will demonstrate that nothing has gone wrong. We will show explicitly that \(E\) is in fact conserved over a pericenter advance timescale, if and only if one takes into account the contributions of the PN cross terms in the equations of motion to the secular variation of the orbit elements \(a\), \(e\) and \(\iota\).

4. Conserved quantities on relativistic precession timescales

In paper I, we solved the Lagrange planetary equations for the orbit elements of the inner binary, including the PN cross terms in the equation of motion. As we emphasized in that paper, it is essential to find the periodic perturbations in the orbit elements induced by the Newtonian third-body perturbation and by the PN binary perturbations, and to substitute those effects back into the planetary equations, because they will induce cross-term effects of the same order as those in the equations of motion. In addition, in converting the planetary equations from time derivatives to derivatives with respect to true anomaly \(f\), it is essential to use the proper conversion \(df/d\tau = h/r^2 - \dot{\omega} - \ddot{\Omega} \cos \iota\), which can also introduce cross terms.

We integrated the equations over an orbit of the inner binary and averaged over an orbit of the third body to determine the secular variations in the orbit elements. The results can be divided into PN binary terms, Newtonian terms from the third body, labelled ‘K’ for Kozai, and cross
terms (see paper I, equations (4.13) and (4.14))

\[
\langle \Delta \omega \rangle = \langle \Delta \omega \rangle_{\text{Binary}} + \langle \Delta \omega \rangle_{K} + \langle \Delta \omega \rangle_{\text{Cross}},
\]

\[
\langle \Delta e \rangle = \langle \Delta e \rangle_{K} + \langle \Delta e \rangle_{\text{Cross}},
\]

\[
\langle \Delta i \rangle = \langle \Delta i \rangle_{K} + \langle \Delta i \rangle_{\text{Cross}},
\]

\[
\langle \Delta a \rangle = \langle \Delta a \rangle_{\text{Cross}},
\]  

(19)

where \(\langle \Delta \omega \rangle_{\text{Binary}}\) is the usual PN binary pericenter advance, given by

\[
\langle \Delta \omega \rangle_{\text{Binary}} = \frac{6\pi Gm}{c^3a(1 - e^2)};
\]

(20)

since we are assuming that this is the dominant contribution to pericenter precession, we will not display the smaller Kozai and cross-term contributions. Notice that the semi-major axis suffers no secular changes induced by either the PN binary or the Newtonian third-body perturbations, while the eccentricity and inclination suffer no secular changes from the PN binary terms. The Newtonian Kozai contributions to \(\langle \Delta \omega \rangle, \langle \Delta e \rangle \) and \(\langle \Delta i \rangle\) are given by

\[
\langle \Delta \omega \rangle_{K} = \frac{3\pi m_3}{2m} \left(\frac{a}{R}\right)^3 (1 - e^2)^{-1/2} \times [5 \cos^2 i \sin^2 \omega + (1 - e^2) (5 \cos^2 \omega - 3)],
\]

(21)

\[
\langle \Delta e \rangle_{K} = \frac{15\pi m_3}{2m} \left(\frac{a}{R}\right)^3 e (1 - e^2)^{1/2} \sin^2 i \sin \omega \cos \omega ,
\]

(22)

\[
\langle \Delta i \rangle_{K} = - \frac{e}{(1 - e^2)} \cot i \langle \Delta e \rangle_{K}.
\]

(23)

It is useful to recall that these last relations imply that \(\langle \Delta ((1 - e^2)^{1/2} \cos i) \rangle_{K} = 0\), expressing the conservation of \(L_2\) at Newtonian order. The PN cross-term contributions to \(\langle \Delta a \rangle, \langle \Delta e \rangle \) and \(\langle \Delta i \rangle\) are given by paper I, equation (4.14)

\[
\langle \Delta a \rangle_{\text{Cross}} = - \frac{15\pi Gm_3}{2e} \left(\frac{a}{R}\right)^3 F(e, \eta) \sin^2 i \sin 2\omega ,
\]

(24)

\[
\langle \Delta e \rangle_{\text{Cross}} = - \frac{15\pi Gm_3}{8ae^2} \left(\frac{a}{R}\right)^3 \left[ G(e, \eta) \sin 2\omega \\
- 12\pi \frac{e}{(1 - e^2)^{1/2}} \cos 2\omega \right] \sin^2 i ,
\]

(25)

\[
\langle \Delta i \rangle_{\text{Cross}} = - \frac{15\pi Gm_3}{8ae^2} \left(\frac{a}{R}\right)^3 \left[ H(e, \eta) \sin 2\omega \\
+ 12\pi \frac{e^2}{(1 - e^2)^{3/2}} \cos 2\omega \right] \sin i \cos i ,
\]

(26)
where
\[
F(e, \eta) \equiv \frac{e (1 + e)^2 [7 + 3e - \eta (3 + 4e)]}{(1 - e) \left(1 - e^2\right)^{3/2}} + \frac{6}{5} \frac{1 - e}{1 + e},
\]
\[
G(e, \eta) \equiv \frac{(1 + e)^2 [(3 + 7e) - (1 + 6e) \eta - f(e, \eta)]}{(1 - e) \left(1 - e^2\right)^{1/2}}
+ \frac{4}{5} \frac{(1 - e)^2 (2 + 4e - 3e^2)}{e^3},
\]
\[
H(e, \eta) \equiv \frac{e (1 + e)^2 [(3 + 7e) - (1 + 6e) \eta + f(e, \eta)]}{(1 - e) \left(1 - e^2\right)^{3/2}}
- \frac{8}{5} \frac{(1 - e)^3 (1 + 3e)}{e^3 \left(1 - e^2\right)^2},
\]
\[\text{(27)}\]

and
\[
f(e, \eta) \equiv \frac{1}{5e^3 (1 + e)} \left[8 - 16e - 24e^2 + 109e^3 + 114e^4 + 43e^5 + 16e^6
- \eta e^3 \left(15 + 47e + 76e^2 + 37e^3\right)\right].
\]
\[\text{(28)}\]

It is now straightforward to show that the energy \(E\) of equation (17) is constant, notwithstanding the secular variation of \(\omega\) induced by the relativistic pericenter precession. We obtain
\[
\frac{a}{G \mu} \Delta E = \frac{1}{2 \alpha} \langle \Delta a \rangle_{\text{cross}} + \frac{3 m_3}{2 m} \left(\frac{a}{R}\right)^3 (1 - e)^2 \sin^2 \iota \sin \omega \cos \omega \langle \Delta \omega \rangle_{\text{Binary}}
+ \left(\frac{G m}{c^2 a}\right) \frac{7 + 3e - \eta (3 + 4e)}{(1 - e)^3} \langle \Delta e \rangle_{K}.
\]
\[\text{(29)}\]

In the second term of \(E\), we have included only the PN binary variation in \(\omega\). The variations in \(a, e\) and \(\iota\) are already of order \(m_3^2\), which would generate contributions of order \(m_3^2\), so these elements may be treated as constants to this order. Similarly, in the third term of \(E\) we have included only the Kozai variation in \(e\); the variation in \(a\) is of cross-term order and thus would generate a 2PN contribution as would the cross-term variations in \(e\). The final terms in \(E\) of order \(m_3 c^2\) (not displayed explicitly in equation (17)) are already of cross-term order; since they involve orbital elements that are constant to lowest order these terms are automatically constant to the order considered. Now inserting equations (20), (21) and (24), we find a complete cancellation, so that \(\Delta E = 0\). The first term from the function \(F(e, \eta)\) that appears in \(\langle \Delta a \rangle_{\text{Cross}}\) cancels the term arising from the Kozai variation \(\langle \Delta e \rangle_{K}\) in the third term, while the second term in \(F(e, \eta)\) cancels the contribution from \(\langle \Delta \omega \rangle_{\text{Binary}}\). Note that if we had not included the properly computed cross-term variations in \(a\), we would have been unable to prove the constancy of \(E\) in the face of the pericenter precession.
In the same way, we can show that \( L_Z \) of equation (18) is constant. We first write

\[
\frac{\Delta L_Z}{(L_Z)_0} = \left( \frac{\langle \Delta a \rangle \text{Cross}}{2a} - \frac{\epsilon}{1 - \epsilon^2} \langle \Delta e \rangle - \tan \iota \langle \Delta \iota \rangle \right) \\
\times \left[ 1 + \frac{\frac{G m}{2c^2a}}{(1 - \epsilon^2)(1 + \epsilon) + 2(3 + \eta)} \right] \\
+ 2 \frac{G m}{c^2a} \frac{2 - \eta}{(1 - \epsilon^2)^2} \langle \Delta e \rangle ,
\]

where \((L_Z)_0 \equiv \mu[Gma(1 - \epsilon^2)]^{1/2} \cos \iota\). In the term within large parentheses in equation (30), \(\langle \Delta e \rangle\) and \(\langle \Delta \iota \rangle\) include both Kozai and cross-term contributions. However, because of equations (22) and (23), the Kozai parts of \(\langle \Delta e \rangle\) and \(\langle \Delta \iota \rangle\) cancel exactly (this is the standard result for the conservation of \(L_Z\) in the Kozai problem). So only cross-term contributions will appear in this set of terms; since these are already of 1PN order, we can drop the PN correction factor. The final term in equation (30) is already of 1PN order, so only the Kozai contribution to \(\langle \Delta e \rangle\) is needed; since \(a\) varies only at cross-term order, it may be treated as constant here. The final terms in \(L_Z\) of order \(m_3/c^2\) are already of cross-term order, so these are automatically constant to the order considered. The variation \(\Delta L_Z\) reduces to

\[
\frac{\Delta L_Z}{(L_Z)_0} = \left( \frac{\langle \Delta a \rangle \text{Cross}}{2a} - \frac{\epsilon}{1 - \epsilon^2} \langle \Delta e \rangle \text{Cross} - \tan \iota \langle \Delta \iota \rangle \text{Cross} \right) \\
+ 2 \frac{G m}{c^2a} \frac{2 - \eta}{(1 - \epsilon^2)^2} \langle \Delta e \rangle_K .
\]

Inserting equations (22), (24), (25) and (26) leads to a complete cancellation, so that \(\Delta L_Z = 0\). Here, the pericenter advance plays no role, since the angular momentum \(L\) is insensitive to the orientation of the binary orbit within its plane. Instead, the cross-term contributions to the orbital elements are essential to maintain constancy of \(L_Z\) in the face of Newtonian Kozai variations in \(e\) acting on the PN correction term.

5. Concluding remarks

We have shown that the total energy of a hierarchical triple system is manifestly conserved to Newtonian order over a relativistic pericenter precession timescale if and only if PN cross-term effects in the equations of motion are taken carefully into account. We found a similar conservation statement for \(L_Z\) that also relies on cross-term effects. Future work will explore the implications of PN cross terms in hierarchical triple systems, along two directions. One is the numerical integration of the orbit evolution equations (19)–(28) to explore the possible long-term effects of PN cross terms. Another is to translate the dynamics of hierarchical triples including cross terms into the language of Hamiltonian dynamics and Delaunay variables. PN terms are embedded differently in the Hamiltonian than they are in the equations of motion, because the relation between velocity and momentum has its own PN corrections. The use of a Hamiltonian and Delaunay variables may yield different insights into the effects of PN cross terms than are obtained from the Lagrange planetary equations and osculating orbit elements. It will also enable comparisons with other work on the Kozai–Lidov problem, which is dominated by the Hamiltonian approach. This will be the subject of future papers.
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References

[1] Hopman C and Alexander T 2006 Astrophys. J. 645 1152–63
[2] Merritt D, Alexander T, Mikkola S and Will C M 2011 Phys. Rev. D 84 044024
[3] Miller M C and Hamilton D P 2002 Astrophys. J. 576 894–8
[4] Blaes O, Lee M H and Socrates A 2002 Astrophys. J. 578 775–86
[5] Wen L 2003 Astrophys. J. 598 419–30
[6] Migaszewski C and Goździewski K 2011 Mon. Not. R. Astron. Soc. 411 565–83
[7] Seto N 2013 Phys. Rev. Lett. 111 061106
[8] Naoz S, Kocsis B, Loeb A and Yunes N 2013 Astrophys. J. 773 187
[9] Galaviz P and Brügmann B 2011 Phys. Rev. D 83 084013
[10] Galaviz P 2011 Phys. Rev. D 84 104038
[11] Antognini J M, Shappee B J, Thompson T A and Amaro-Seoane P 2014 Mon. Not. R. Astron. Soc. 439 1079–91
[12] Benitez F and Gallardo T 2008 Celest. Mech. Dyn. Astron. 101 289–307
[13] Will C M 2014 Phys. Rev. D 89 044043
[14] Poisson E and Will C M 2014 Gravity: Newtonian, Post-Newtonian, Relativistic (Cambridge: Cambridge University Press)