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A polynomial-time approximation to a minimum dominating set in a graph†

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Abstract: A dominating set of a graph \( G = (V, E) \) is a subset of vertices \( S \subseteq V \) such that every vertex \( v \in V \setminus S \) has at least one neighbor in set \( S \). Finding a dominating set with the minimum cardinality in a graph \( G = (V, E) \) is known to be NP-hard. A polynomial-time approximation algorithm for this problem, described here, works in two stages. At the first stage a dominant set is generated by a greedy algorithm, and at the second stage this dominating set is purified (reduced). The reduction is achieved by the analysis of the flowchart of the algorithm of the first stage and a special kind of clustering of the dominating set generated at the first stage. The clustering of the dominating set naturally leads to a special kind of a spanning forest of graph \( G \), which serves as a basis for the second purification stage. The greedy algorithm of the first stage has essentially the same properties as the earlier known state-of-the-art algorithms for the problem. The second purification stage results in an essential improvement of the quality of the dominant set created at the first stage. We have measured the practical behavior of the algorithm of both stages on randomly generated problem instances. We have used two different random methods to generate our graphs, each of them yielding graphs with different structure.

Keywords: graph; dominating set; approximation ratio; approximation algorithm; time complexity

1. Introduction

Problem description. One of the most studied problems in combinatorial optimization and graph theory are covering and partitioning problems in graphs. A subset of vertices in a graph is a dominating set if every vertex of that graph which is not in that subset has at least one neighbor in that subset. More formally, given a simple connected undirected graph \( G = (V, E) \) with \( |V| = n \) vertices and \( |E| = m \) edges, a set \( S \subseteq V \) is called a dominating set of that graph if for all \( v \in V \) either \( v \in S \) or there exists a vertex \( u \) in \( V \setminus S \) such that edge \((v, u) \in E\). A widely studied such problem is the dominating set problem. The MINIMUM DOMINATING SET problem consists in determining the minimum cardinality of a dominating set of \( G = (V, E) \).

The domination number of graph \( G \), denoted as \( \gamma(G) \), is the minimum cardinality of a dominating set for that graph; we shall refer to a corresponding dominating set of cardinality \( \gamma(G) \) as a \( \gamma(G) \)-set. A dominating set is minimal if by removing any of its elements the resultant reduced set becomes non-dominating. In fact, it is not so difficult to construct a polynomial-time algorithm that generates a minimal dominating set. However, as one can easily see, not necessarily a minimal dominating set approximates well a minimum dominating set. For instance, given a minimal dominating set, there may exist a non-minimal dominating set with a much smaller number of vertices.
The reader is referred to Haynes et al. [1] for further details on the basic graph terminology. The problem of domination in graphs was mathematically formalized by Berge and Ore in [2] and [3], respectively. Currently, this topic has been detailed in the two well-known books by Haynes, Hedetniemi, and Slater in [1] and [4]. The theory of domination in graphs is an area of increasing interest in discrete mathematics and combinatorial computing, in particular, because of a number of important real-life applications. Such applications arise, for instance, during the analysis of social networks (see Kelleher and Cozzens in [5]), in efficient identification of web communities (see Flake et al. in [6]), in bioinformatics (see Haynes et al. in [7]) and food webs (see Kim et al. in [8]), in the study of transmission of information in networks associated with defense systems (see Powel in [9]), also in distributed computing (see Kuhn et al. in [10], Kuhn and Wattenhofer in [11]).

Our contributions. In this paper we propose a polynomial-time approximation algorithm, for the MINIMUM DOMINATING SET problem, which runs in two stages. The dominating set created at the first stage is reduced at the second purification stage. Although our greedy algorithm of stage 1 was developed independently from the earlier known algorithm, it turned out that we have used a similar heuristic approach as the above mentioned algorithms from [12] and [13] (a detailed comparison of our algorithm with these two algorithms and their descriptions are given later in Sections 2 and 4). The reduction of the dominating set of stage 1 is achieved at stage 2 by the analysis of the flowchart of the algorithm of the first stage and a special kind of clustering of the dominating set generated at the first stage. The clustering of the dominating set naturally leads to a special kind of a spanning forest of the original graph \( G \), which serves as a basis for the purification stage.

In the next section we introduce the greedy algorithm of stage 1 and we describe our purification procedure of stage 2 moreover, study some useful properties of the dominating set delivered by that procedure. In Section 3 we report our experimental results.

2. Materials and Methods

2.1. Stage 1: Algorithm_Basic

In this subsection we describe our greedy algorithm of stage 1, Algorithm_Basic. It works on a number of iterations. Denote by \( S_h \) the dominant set formed by the algorithm by iteration \( h \); \( \overline{S_h} = V \setminus S_h \) is the complement of the dominant set of iteration \( h \). Initially, \( S_0 := \emptyset \). At each iteration \( h > 0 \) the dominant set \( S_h \) of iteration \( h \) is obtained by adding vertex \( v_h \in \overline{S_{h-1}} \) to the current dominant set \( S_{h-1} \) of the previous iteration.

At iteration \( h > 0 \), the selection of vertex \( v_h \) is based on the so-called active degree of the vertices in set \( \overline{S_{h-1}} \). An active degree of vertex \( v \in \overline{S_{h-1}} \) at iteration \( h \) is a derivation of the degree of that vertex in graph \( G \) that does not take into account the vertices already belonging to set \( S_{h-1} \) and the vertices adjacent to a vertex in set \( S_{h-1} \). At iteration \( h \), among all edges adjacent to vertex \( v \), edge \((v, x)\) is not counted in the modified degree of vertex \( v \) if either \( x \in S_{h-1} \) or \( x \in S_{h-1} \), but \( x \) is a neighbor of a vertex in the set \( \overline{S_{h-1}} \). So the active degree of vertex \( v \) at iteration \( h \) is the number of neighbors of vertex \( v \) in set \( \overline{S_{h-1}} \), not counting those vertices (neighbors of vertex \( v \)) in set \( \overline{S_{h-1}} \) which are adjacent to a vertex in set \( S_{h-1} \).

Initially at iteration 0, it sets \( S_0 := \emptyset \) (\( \overline{S_0} := V \), respectively). The initial settings are updated iteratively at each iteration \( h > 0 \) resulting in the current sets of vertices \( S_h \) and \( \overline{S_h} \), correspondingly (the partially dominant set and the corresponding complement). At every iteration \( h > 0 \) vertex \( v_h \in \overline{S_{h-1}} \) with the maximum active degree is selected and sets \( S_{h-1} \) and \( \overline{S_{h-1}} \) are modified accordingly resulting in the updated sets \( S_h \) and \( \overline{S_h} \) (vertex \( v_h \) is moved from set \( \overline{S_{h-1}} \) to the set \( S_{h-1} \) resulting in the updated sets \( S_h \) and \( \overline{S_h} \)). So at every iteration \( h > 0 \), the updated set \( S_h \) contains one more vertex than set \( S_{h-1} \) of the previous iteration, whereas the updated set \( \overline{S_h} \) contains one less vertex than the set \( \overline{S_{h-1}} \) of previous iteration.
the previous iteration. The algorithm halts at iteration $h_{\text{max}}$ when $S = S_{h_{\text{max}}}$ is already a dominating set. In Algorithm 1, a formal description is shown.

Denote by $G_h$ the subgraph of graph $G$ induced by set $S_h$. We distinguish two kinds of vertices in graph $G_h$ (in set $S_h$). Consider iteration $h$ when the set $S_h$ is formed according to the made selection of vertex $v_h$. We call a vertex $p \in S_{h-1}$ adjacent to vertex $v_h$ in graph $G_{h-1}$ a pending vertex dependent on vertex $v_h$ if there is no vertex in set $S_{h-1}$ adjacent to vertex $p$ (this iteration will later be used at the purification stage).

Let $p$ be a pending vertex dependent on vertex $v_h$ by iteration $h$ ($h' \leq h$). For the convenience, we shall consider an edge $(p, v) \in G_h$ incident to vertex $p$ in graph $G_h$ as a special kind of a two-directional edge. The direction of edge $(v, p)$ incident to vertex $p$ is marked as dummy, whereas the direction of the edge incident from vertex $p$, $(p, v)$, is treated normally. Accordingly, we shall consider the graph $G_h$ (set $S_h$ empty if it contains only pending vertices (ones which are already covered by at least one vertex of set $S_h$).

The following remark gives a sufficient optimality condition for the algorithm Algorithm_Basic, and will also be useful in the estimation of the approximation ratio of our overall algorithm later on.

**Remark 1.** If $|S_{h_{\text{max}}}| \leq 2$ then $S_{h_{\text{max}}}$ is a minimum dominating set.

**Proof.** It will suffice to consider the two cases when $\gamma(G) = 1$ or $\gamma(G) = 2$. Indeed, suppose, first, that $\gamma(G) = 1$. Then there exists vertex $v \in V$ adjacent to every other vertex in $V$. Then at the initial iteration 0 of Algorithm_Basic the (active) degree of vertex $v$ would be the maximum and this vertex will be selected as $v_0$, and, $S_{h_{\text{max}}} := \{v\}$ (among all such vertices, ties can clearly be broken arbitrarily). If now $S_{h_{\text{max}}} = 2$, $\gamma(G)$ cannot be less than 2. □

2.2. Stage 2: The purification procedure for Algorithm_Basic

In this subsection we complement Algorithm_Basic with the second purification stage, which eliminates specially determined redundant vertices from the dominating set $S_{h_{\text{max}}}$ (the output of Algorithm_Basic). A vertex $x \in S_{h_{\text{max}}}$ is said to be purified at stage 2 if it is eliminated from set $S = S_{h_{\text{max}}}$. The purification procedure uses not only the output of Algorithm_Basic but also its flowchart. The following definitions are useful for the analysis of the flowchart of Algorithm_Basic.

Suppose at some iteration $h$ of Algorithm_Basic a pending vertex $p$, dependent on vertex $v \in S_{h-1}$ gets included in set $S_h$. Then we call pair $(v, p)$ tied. Notice that at any iteration $h$ there may arise at most one new tied pair. Alternatively, we consider $(v, p)$ as an edge. Suppose at iteration $h$ of Algorithm_Basic, vertex $z$ adjacent to a pending vertex $p$ dependent on vertex $v$, gets included in set $S_h$. Then we call pair $(v, z)$ semi-tied.

**Proposition 1.** If there arise neither tied nor semi-tied pairs of vertices during the execution of Algorithm_Basic, then $S_{h_{\text{max}}}$ is a minimum dominating set for graph $G$.

**Proof.** By the way of contradiction, suppose there exists a dominating set $S'$ such that $|S'| < |S_{h_{\text{max}}}|$. If $S' \subset S_{h_{\text{max}}}$ then there exist $v \in S_{h_{\text{max}}}$ and $u \in S'$ such that $u$ is adjacent to $v$. If $(v, u)$ is not a tied pair, then there is a vertex $z \in S_{h_{\text{max}}}$ such that edge $(v, z)$ is a tied pair (since consider that $u$ is inserted before $v$ either $v$ was dependent on $u$ or it was dependent on $z$; in any case a tied pair is generated), which is a contradiction.

Otherwise, note that for every vertex $v \in S_{h_{\text{max}}} \setminus S'$, if $N(v) \cap (S_{h_{\text{max}}} \setminus S') \neq \emptyset$, then there is a vertex $u \in S_{h_{\text{max}}} \cap S'$ such that $u$ is adjacent to $v$ and a similar analysis as above leads to a contradiction. Now, by pigeonhole principle there exists a vertex $u$ in $S'$ having at least two neighbors, say $v$ and $w$ in set $S_{h_{\text{max}}}$. Also, notice that $u \notin S_{h_{\text{max}}}$. Similar to previous analysis, if $(v, w)$ is not a semi-tied pair, then there is a vertex $z \in S_{h_{\text{max}}}$ such that $u$ is a dependent vertex on $z$. So, $(v, z)$ or $(w, z)$ is a semi-tied pair, which is a contradiction. It follows that $S_{h_{\text{max}}}$ is a dominating set of minimum cardinality and Algorithm_Basic is optimal. □
Algorithm 1 Algorithm Basic

Input: A graph G.
Output: A dominating set $S = S_h$.

$h := 0$;

$v_0 :=$ any vertex with the maximum degree in graph $G$;

$S_1 := \{ v_0 \}$;

[ iterative step ]

while $S_h$ is not a dominating set of graph $G$ do

$h := h + 1$;

$v_h :=$ any vertex with the maximum active degree in set $S_{h-1}$;

$S_h := S_{h-1} \cup \{ v_h \}$;

end while

Clusters and the induced spanning trees. A collection of tied pairs define independent structural components that reflect the flowchart of Algorithm Basic. We define these components now.

In such a sense, we shall refer to a sequence of the tied pairs of the form $(v, p_1), (p_1, p_2), (p_2, p_3), \ldots, (p_{l-1}, p_l)$ as a chain. We may look at a chain as a sequence of the corresponding edges, a path from vertex $v$ to vertex $p_l$. Two different chains of tied pairs have either (i) no vertex in common or (ii) solely one vertex in common or (iii) one or more tied pairs in common (a sub-chain of tied pairs of vertices). We shall refer to a maximum set (one with the maximum cardinality) of two or more different chains such that, for any two different chains from the set either (ii) or (iii) is satisfied, as a cluster of chains, or a cluster, for short.

As it is easily seen, a cluster $C$ possesses a single vertex such that all chains of that cluster have that vertex in common. We call this vertex the root of cluster $C$ and denote it by $r(C)$. We note that the structure of a cluster does not generate cycles among its vertices. The following remark is straightforward.

Description of the purification procedure. We are ready to describe our purification procedure that purifies (reduces) dominant set $S$, the output of Algorithm Basic. First we describe the basic version of the procedure which we extend a bit later. The procedure purifies trees $T(C_1), \ldots, T(C_k)$ in this order. Iteratively, it purifies the vertices of the next tree $T(C_i)$ and outputs the purified tree, that we denote by $T^1(C_i)$; $T^k(C_i)$ is the partially purified tree of iteration $h$, and $T^h$ is the corresponding forest. The output of all the $k$ calls of the procedure is a purified forest $T'$.

At iteration $h$ of the procedure for cluster $C_i$, let $v \in T^h(C_i)$. We call vertex $x \in V(G) \setminus V(T(C_i) \cup \cdots \cup T(C_k))$ a semi-private neighbor of vertex $v$ if vertex $v$ is the only (remained) neighbor of vertex $x$ in the current purified forest $T^1(C_1), \ldots, T^1(C_{i-1}), T^h(C_i), T^1(C_{i+1}), \ldots, T^k(C_k)$. Note that, not necessarily, $x$ is a private neighbor of vertex $v$ in graph $G$ (a private neighbor is also a semi-private one for vertex $v$, but not necessarily vice-versa).

At iteration $h$, we distinguish two types of the vertices in tree $T^h(C_i)$: the currently purified ones and ones which were set as non-purified at some previous iteration, the so-called firm ones. The procedure carries out a four-degree bottom-up look-ahead checking. Given a yet unpurified firm vertex, its parent and grandparent are purified and the grand-grandparent is set to be firm (this rule has an exception which is explicitly stated a bit later). Thus an up-going chain of four vertices $(a, b, c, d)$, that we call a quadruple, is considered at once so that the endpoint $a$ of that quadruple is already firm, the endpoint $d$, is set firm and the intermediate vertices are set purified (in general, a processed chain $(a, \ldots, d)$ may contain from two to four vertices; for the sake of simplicity we shall not distinguish them and shall refer to them as quadruples).

The purification procedure starts by purifying all the leaves with no (private or) semi-private neighbor in tree $T(C_i)$. If leaf $v$ is so purified, then its parent is set to be firm. Otherwise (leaf $v$ has a semi-private neighbor), leaf $v$ is set firm. Once all the leaves are so processed, the corresponding firm vertices form the initial set $P^0(C_i)$ of the firm vertices in tree $T^0(C_i)$, the pending firm ones.
Iteratively, at an iteration \( h > 0 \), the highest level leftmost pending firm vertex with a non-firm parent is looked for. We denote the chosen in this way vertex at iteration \( h \) by \( a_h \). If there exists no vertex \( a_h \), i.e., the parent of any pending firm vertex is already firm or there is no pending firm vertex having a parent (the highest level pending firm vertex is the root), then the purification procedure halts. Once vertex \( a \) is selected, the quadruple \((a, b, c, d)\) with \( a = a_h \) is determined and is processed as follows: The parent \( b \) of vertex \( a \) is purified and its grandparent \( c \) (if it exists) is also purified if vertex \( c \) has not earlier been set firm (from another up-going branch) or \( c \) is the root. In the latter case, the root is set firm. If vertex \( c \) is not the root, its parent \( d \) is set to be firm (unless it was already set firm at an earlier iteration). In this way, at most two vertices \( b \) and \( c \), predecessors of vertex \( a \), are purified from vertex \( a_h \) at iteration \( h \).

Once quadruple \((a, b, c, d)\) is processed, the current set of the pending firm vertices \( PF^{h-1}(C_i) \) is updated by deleting from it vertex \( a \) (which becomes a non-pending firm vertex), and by including vertex \( d \) into the updated set (a new pending firm vertex).

**Lemma 1.** If the basic version of the purification procedure purifies vertex \( b \in T^h(C_i) \) at iteration \( h \) then that vertex cannot be set firm at any later iteration.

**Proof.** Let vertex \( b \) be purified from vertex \( a_h \) at iteration \( h \). By way of contradiction, suppose vertex \( b \) is set firm from vertex \( a_g \) at iteration \( g > h \). By the construction of the procedure, there must exist two intermediate vertices between vertices \( a_g \) and \( b \), as otherwise vertex \( b \) would not have been set firm. But then the level of vertex \( a_g \) is greater than that of vertex \( a_h \) and hence the procedure could not select vertex \( a_h \) at iteration \( h \) as the highest level pending firm vertex.

From here on, we will refer to the augmented version of the purification procedure as the purification procedure, and to the overall two-stage algorithm as Algorithm_Extended. It is a known fact that the number of vertices in a minimum dominating set is bounded above by \( \frac{n}{2} \), i.e., \( \gamma(G) \leq \frac{n}{2} \), for every simple graph \( G \) of order \( n \), and this bound is tight. We prove that the same bound is valid for the dominating set delivered by Algorithm_Extended

**Lemma 2.** \( |S^*| \leq \frac{n}{2} \), where \( S^* \) is the dominating set returned by Algorithm_Extended.

**Proof.** Let us consider any (non-purified) vertex \( v \in T^*(C) \). Only the following two cases are possible. Either (1) vertex \( v \) was left non-purified as a son of a purified vertex \( z \in T(C) \) or as the parent of such a vertex, or (2) vertex \( v \) was left non-purified as the semi-private neighbor of some vertex \( x \). In case (1) there is a neighbor of vertex \( v \) in the complement \( S^c \) (the corresponding purified vertex \( z \)), and in case (2) there exist vertex \( x \) from \( S^c \). Thus with any vertex \( v \in S^c \) a unique vertex from \( S^c \) is associated. Hence, \( |S^*| \leq \frac{n}{2} \).

### 3. Results and Discussion

In this final section we describe our computation experiments. We have implemented our algorithms in C++ using Windows 10 operative system for 64 bits on a personal computer with Intel Core i7-9750H (2.6 GHz) and 16 GB of RAM DDR4. We have generated two sets of the problem instances obtained by using two different pseudo-random methods to generate the graphs. For the first class, each new edge was added in between two yet non-adjacent vertices randomly until the corresponding size was attained. For the second class of instances, we have followed a more particular rules while adding each new edge according to the chosen structure (e.g., see the example of a graph in Figure 1).

The results for the first class of instances are shown in Table 1. Over all the tested problem instances, we have obtained in average 0.89% of the reduction of the size of the dominating sets at stage 2. We may also observe that the reduction of the size of the dominating sets at the purification stage becomes more notable as the order of the graphs increases.
| No. | Order | Size  | Clusters | Algorithm_Basic | Algorithm_Extended | Purification |
|-----|-------|-------|----------|-----------------|-------------------|--------------|
| 1   | 841   | 1647  | 32       | 155             | 154               | 1            |
| 2   | 908   | 978   | 76       | 235             | 231               | 4            |
| 3   | 859   | 1106  | 46       | 178             | 178               | 0            |
| 4   | 992   | 1365  | 46       | 226             | 223               | 3            |
| 5   | 984   | 1465  | 58       | 208             | 208               | 0            |
| 6   | 1011  | 1811  | 36       | 205             | 205               | 0            |
| 7   | 1196  | 2189  | 42       | 228             | 227               | 1            |
| 8   | 1333  | 1897  | 93       | 363             | 357               | 6            |
| 9   | 1338  | 1722  | 80       | 377             | 373               | 4            |
| 10  | 1673  | 3116  | 67       | 322             | 319               | 3            |
| 11  | 2041  | 2139  | 141      | 519             | 509               | 10           |
| 12  | 2393  | 2812  | 142      | 578             | 575               | 3            |
| 13  | 2031  | 3157  | 114      | 428             | 424               | 4            |
| 14  | 2562  | 3475  | 148      | 578             | 575               | 3            |
| 15  | 3048  | 3262  | 222      | 791             | 779               | 12           |
| 16  | 3286  | 6146  | 138      | 629             | 624               | 5            |
| 17  | 3089  | 4966  | 128      | 633             | 629               | 4            |
| 18  | 3127  | 4376  | 163      | 818             | 817               | 1            |
| 19  | 3904  | 7687  | 151      | 725             | 724               | 1            |
| 20  | 4041  | 6344  | 170      | 837             | 829               | 8            |
| 21  | 4466  | 6895  | 200      | 930             | 921               | 9            |
| 22  | 4378  | 7988  | 185      | 901             | 894               | 7            |
| 23  | 4389  | 8482  | 163      | 818             | 817               | 1            |
| 24  | 4493  | 8911  | 171      | 818             | 816               | 2            |
| 25  | 5071  | 8385  | 231      | 1003            | 997               | 6            |
| 26  | 5342  | 8713  | 265      | 1326            | 1308              | 18           |
| 27  | 5347  | 7470  | 304      | 1112            | 1105              | 7            |
| 28  | 5427  | 7470  | 304      | 1112            | 1105              | 7            |
| 29  | 5346  | 8036  | 282      | 1133            | 1129              | 4            |
| 30  | 6041  | 6385  | 345      | 1536            | 1502              | 34           |
| 31  | 6786  | 7552  | 378      | 1692            | 1649              | 43           |
| 32  | 6052  | 7097  | 295      | 1452            | 1430              | 22           |
| 33  | 6956  | 11942 | 252      | 1378            | 1372              | 6            |
| 34  | 6820  | 12432 | 282      | 1305            | 1299              | 6            |
| 35  | 7042  | 11426 | 324      | 1419            | 1411              | 8            |
| 36  | 7061  | 11426 | 313      | 1439            | 1422              | 17           |
| 37  | 7922  | 15072 | 269      | 1465            | 1464              | 1            |
| 38  | 7952  | 10747 | 343      | 1480            | 1469              | 11           |
| 39  | 8191  | 8791  | 442      | 1527            | 1513              | 18           |
| 40  | 8031  | 10426 | 449      | 1831            | 1813              | 18           |
| 41  | 8179  | 13192 | 388      | 1664            | 1654              | 10           |
| 42  | 8684  | 17266 | 303      | 1592            | 1588              | 4            |
| 43  | 8547  | 12357 | 461      | 1848            | 1837              | 11           |
| 44  | 8195  | 11525 | 465      | 1794            | 1783              | 11           |
| 45  | 9880  | 10266 | 496      | 2401            | 2399              | 42           |
| 46  | 9041  | 15010 | 415      | 1818            | 1805              | 13           |
| 47  | 9817  | 15763 | 408      | 2011            | 1997              | 14           |
| 48  | 9293  | 14002 | 439      | 1952            | 1937              | 15           |
| 49  | 9590  | 13364 | 519      | 2086            | 2067              | 19           |
| 50  | 5346  | 8036  | 282      | 1133            | 1129              | 4            |

Table 1: The results for the randomly generated graphs

The results for the second class of instances are shown in Table 2. These instances were created with the intention to verify the efficiency of the purification stage for the graphs for which the first stage would deliver a poor solution. Taking as an example the graph depicted in Figure 1, we observe that, if we run Algorithm_Basic for that graph, it will first include the central (bold) vertex into the formed dominant set, then it will add all the (gray)
vertices adjacent to the former vertex to the formed dominant set, and then it will also include all the (bold) vertices adjacent to the latter vertices in that set. We may observe such a behavior of the algorithms of the first and the second stages with a significant reduction of the number of vertices in the purified dominating sets for the problem instances of the second class in Table 2.

![Figure 1. A graph from the second class](image)

| No. | Order | Size | Clusters | Algorithm_Basic | Algorithm_Extended | Purification |
|-----|-------|------|----------|-----------------|---------------------|--------------|
| 1   | 3619  | 4588 | 1        | 1975            | 1646                | 329          |
| 2   | 3651  | 4468 | 1        | 1027            | 833                 | 194          |
| 3   | 3773  | 4775 | 1        | 2059            | 1716                | 343          |
| 4   | 3895  | 4824 | 1        | 2071            | 1726                | 345          |
| 5   | 4016  | 4876 | 1        | 1129            | 916                 | 213          |
| 6   | 4180  | 5286 | 1        | 2281            | 1901                | 380          |
| 7   | 4290  | 5491 | 1        | 1717            | 1431                | 286          |
| 8   | 4485  | 5643 | 1        | 1795            | 1496                | 299          |
| 9   | 4620  | 5885 | 1        | 1849            | 1541                | 308          |
| 10  | 4673  | 5714 | 1        | 1314            | 1066                | 248          |
| 11  | 4740  | 6020 | 1        | 1897            | 1581                | 316          |
| 12  | 4965  | 6039 | 1        | 1396            | 1132                | 264          |
| 13  | 5200  | 6597 | 1        | 2801            | 2401                | 400          |
| 14  | 5525  | 7056 | 1        | 2976            | 2551                | 425          |
| 15  | 5720  | 7279 | 1        | 3081            | 2641                | 440          |
| 16  | 5759  | 7309 | 1        | 3102            | 2659                | 443          |
| 17  | 5768  | 7084 | 1        | 1621            | 1315                | 306          |
| 18  | 5785  | 7403 | 1        | 3116            | 2671                | 445          |
| 19  | 5841  | 7096 | 1        | 1642            | 1332                | 310          |
| 20  | 6019  | 7728 | 1        | 3242            | 2779                | 463          |
| 21  | 6058  | 7664 | 1        | 3263            | 2797                | 466          |
| 22  | 6383  | 8126 | 1        | 3438            | 2947                | 491          |
| 23  | 6409  | 8118 | 1        | 3452            | 2959                | 493          |
| 24  | 6571  | 8035 | 1        | 1847            | 1498                | 349          |
| 25  | 6837  | 8580 | 1        | 3144            | 2695                | 449          |
| 26  | 7350  | 9291 | 1        | 3431            | 2941                | 490          |
| 27  | 7447  | 9015 | 1        | 2093            | 1698                | 395          |
| 28  | 7545  | 9464 | 1        | 3522            | 3019                | 503          |
| 29  | 7560  | 9565 | 1        | 3529            | 3025                | 504          |
| 30  | 7593  | 9242 | 1        | 2134            | 1731                | 403          |
| No. | Order | Size  | Clusters | Algorithm_Basic | Algorithm_Extended | Purification |
|-----|-------|-------|----------|-----------------|-------------------|--------------|
| 31  | 7665  | 9741  | 1        | 3578            | 3067              | 511          |
| 32  | 7680  | 9759  | 1        | 3585            | 3073              | 512          |
| 33  | 7812  | 9545  | 1        | 2195            | 1781              | 414          |
| 34  | 8396  | 10190 | 1        | 2359            | 1913              | 446          |
| 35  | 8350  | 10842 | 1        | 4361            | 3991              | 570          |
| 36  | 8688  | 10584 | 1        | 2441            | 1980              | 461          |
| 37  | 8715  | 10996 | 1        | 4649            | 4068              | 581          |
| 38  | 8820  | 11168 | 1        | 4705            | 4117              | 588          |
| 39  | 8835  | 11173 | 1        | 4713            | 4124              | 589          |
| 40  | 8865  | 11241 | 1        | 4729            | 4138              | 591          |
| 41  | 9076  | 11181 | 1        | 4951            | 4126              | 825          |
| 42  | 9142  | 11246 | 1        | 4987            | 4136              | 831          |
| 43  | 9406  | 11563 | 1        | 5131            | 4276              | 835          |
| 44  | 9626  | 11827 | 1        | 5251            | 4376              | 875          |
| 45  | 9648  | 11847 | 1        | 5263            | 4386              | 877          |
| 46  | 10353 | 17077 | 1        | 3946            | 3196              | 750          |
| 47  | 14910 | 18162 | 1        | 4192            | 3394              | 798          |
| 48  | 15787 | 19238 | 1        | 4439            | 3595              | 844          |
| 49  | 16664 | 20304 | 1        | 4685            | 3795              | 890          |
| 50  | 18418 | 22437 | 1        | 5178            | 4194              | 984          |

Table 2: The results of the generated second class graphs.

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