A Note on Weak Hypercyclicity and Linear Fractional Composition Operator

Arman Shokrollahi
Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA.

Abstract: This paper discusses the existence of a sufficient condition for an operator to be weakly hypercyclic. We establish a weak hypercyclicity criterion, and thereupon we can answer questions 5.3 and 5.8 posed by Chan and Sanders in [4]. Lastly, we show that for specific type of composition operator, weak hypercyclicity and hypercyclicity are equivalent.

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1. Introduction

In this note, we try to briefly discuss a sufficient condition in terms of norm and weakly open sets for an operator on a reflexive Banach space to be weakly hypercyclic. This gives a useful criterion for weak hypercyclicity of operators. We apply this to specific classes of weakly hypercyclic operators including bilateral weighted shifts on $\ell^p(\mathbb{Z})$ with $1 \leq p < \infty$. This, in turn, provides a large class of weakly hypercyclic bilateral shift which are not norm hypercyclic and it answers question 5.3 in [4] in a different way than what Sanders presented in [14]. In the last section, we show hypercyclicity and weak hypercyclicity are equivalent for a composition operator on the space $H(U)$ of all complex-valued functions holomorphic on the open unit disk $U$ or the Hardy space $H^2$ that is the collection of functions $f \in H(U)$ with $\sum_{n=1}^{\infty} |\hat{f}(n)|^2 < \infty$. So, there are other classes of operators for which norm hypercyclicity and weak hypercyclicity are equivalent and it answers question 5.8 in [4]. The authors in [1–4, 6, 8–11, 13] have already studied weak hypercyclicity and composition operators in various ways. Dodson [7] has also prepared a survey in this regard.

The following theorem summarizes all the necessary conditions which have been obtained for weakly hypercyclic operators. For a proof, see [4] and [6].

**Theorem 1.1.**

Let $T$ be a bounded operator on Banach space $X$. If $T$ is a weakly hypercyclic operator, then
(i) for every norm open set $G$ and weakly open set $W$, there is some integer $n \geq 0$ such that $T^n G \cap W \neq \emptyset$,
(ii) the set of weakly hypercyclic vectors for $T$ is norm dense in $X$,
(iii) $T^*$ has no eigenvalue,
(iv) every component of spectrum $T$ intersect the unit circle.

The following theorem proposes a sufficient condition in terms of open and weakly set for an operator to be weakly hypercyclic. $\text{Ball}_N(X)$ and $\text{Ball}_W(X)$ represent the subsets $\{x \in X : \|x\| \leq N\}$ and $\{x \in X : \|x\| < N\}$ of $X$, respectively, for any integer $N$.

**Theorem 1.2.**
Let $T$ be a bounded operator on a reflexive Banach space $X$ such that

(i) for every norm open set $G$ and weakly open set $W$, there is some integer $n \geq 0$ such that $T^n G \cap W \neq \emptyset$,
(ii) there exists some integer $M$ such that if $G$ is a norm open set with $G \cap \text{Ball}(X) \neq \emptyset$ and $W$ is a weakly open set with $W \cap \text{Ball}(X) \neq \emptyset$, then there is some integer $n \geq 0$ such that $T^n (G \cap \text{Ball}(X)) \cap W \cap \text{Ball}_M(X) \neq \emptyset$.

Then, $T$ is weakly hypercyclic.

**Proof.** The reflexivity of $X$ and Alaoglu’s Theorem imply that, for every integer $N$, $\text{Ball}_N(X)$ is weakly compact and metrizable. Let $\Gamma_N$ be a collection of countable bases of $\text{Ball}_N(X)$, then $\Gamma = \bigcup \Gamma_N$ is also countable. Suppose $\Gamma = \{W_1, W_2, \cdots, W_N, \cdots\}$. For each $W_j$, there is a weakly open set $V_j$ and an integer $N \geq 1$ such that $W_j = V_j \cap \text{Ball}_N(X)$. Let $A = \{(m, j) : T^{-m} V_j \cap \text{Ball}_N(X) \neq \emptyset\}$. $A$ is nonempty. In fact, condition (i) ensures that there is some integer $m$ such that $(m, j) \in A$ for every $j$. Note that by condition (ii), for every norm open set $G$ and $(m, j) \in A$, there is some integer $n$ such that the set $T^n (G \cap \text{Ball}(X)) \cap T^{-m} V_j \cap \text{Ball}_M(X)$ is nonempty, and consequently $G \cap \text{Ball}(X) \cap T^{-n} [T^{-m} V_j \cap \text{Ball}_M(X)] \neq \emptyset$. So, the set $\bigcup_{n \in \mathbb{N}} T^{-n} [T^{-m} V_j \cap \text{Ball}_M(X)]$ is norm dense in $\text{Ball}(X)$. The Baire’s Category Theorem implies that the set

$$WHY := \bigcap_{(m, j) \in A} \bigcup_{n \geq 0} T^{-n} [T^{-m} V_j \cap \text{Ball}_M(X)]$$

is also norm dense in $\text{Ball}(X)$. Need to be mentioned that to use the Baire’s Category Theorem, it would not be difficult to show that $T^{-n} [T^{-m} W_j \cap \text{Ball}_M(X)] \cap \text{Ball}(X)$ is open in $\text{Ball}(X)$.

Now, we claim every element of the above set is weakly hypercyclic for $T$. Choosing $x$ in the last set, we show that $x$ is a weakly hypercyclic vector for $T$. For, suppose $W$ is an arbitrary weakly open set in $X$, then by condition (i), $T^m \text{Ball}_0(X) \cap W \neq \emptyset$, for some integer $m \geq 0$. There is some integer $N > M\|T\|^m$ such that $T^m \text{Ball}_0(X) \cap W \cap \text{Ball}_N(X) \neq \emptyset$ and consequently $T^{-m} (W \cap \text{Ball}_N(X)) \cap \text{Ball}_0(X) \neq \emptyset$. Since $\Gamma_N$ is a basis of $\text{Ball}_N(X)$ and $\Gamma_N \subseteq \Gamma$, there exists some $W_j \in \Gamma_N$ with $W_j \cap \text{Ball}_N(X) \subseteq W \cap \text{Ball}_N(X)$. Hence, $T^{-m} (V_j \cap \text{Ball}_N(X)) \cap \text{Ball}_0(X) \neq \emptyset$ when $T^{-m} V_j \cap \text{Ball}_0(X) \neq \emptyset$ for some $j$. It states that $(m, j) \in A$ and since $x \in WHY(T)$, there is some integer $n$ such that $T^{n+m} x \in V_j$ and $T^n x \in \text{Ball}_M(X)$ when $\|T^n x\| \leq M\|T\|^m \leq N$. So, $T^{n+m} x \in V_j \cap \text{Ball}_N(X)$ which is a subset of $W \cap \text{Ball}_N(X)$. Thus, $T^{n+m} x \in W$, and so $\text{Orb}(T, x)$ is
weakly dense in $X$, and $T$ is weakly hypercyclic. (the subject of orbits has been studied in many literature, for example see [12].)

2. Weak Hypercyclicity

Theorem 2.1 (Weak Hypercyclicity Criterion).
Let $X$ be a reflexive, separable Banach space, $T \in B(X)$, and there exist two dense subsets $Y$ and $Z$ in $X$, a sequence $\{n_k\}$ of integers and positive integer $M$ such that

1. $T^{n_k}y \overset{wk}{\to} 0$ for every $y \in Y$,
2. for every $y \in Y$ with $\|y\| \leq 1$, there is some integer $N$ such that
   \[ \sup\{\|T^{n_k}y\| : k \geq N\} \leq M, \]
3. there exists a linear map $S_k : Z \to X$ such that for every $z \in Z$,
   \[ S_k z \to 0, \quad \text{and} \quad T^{n_k}S_{n_k}z \to z. \]

Then, $T$ is weakly hypercyclic.

Proof. We apply the preceding theorem (Theorem 1.2). Suppose $G$ is a norm open set and $W$ is a weakly open set. Choose $z \in G \cap Z$, $y \in W \cap Y$, and let $y_k = y + S_k z$, then $y_k \to y$ in norm topology while $T^{n_k}y_k \overset{wk}{\to} z$. It gives condition (i) of Theorem 1.2.

Again suppose $G$ is a norm open set and $W$ is a weakly open set with $W \cap \text{Ball}(X) \neq \emptyset$ and $G \cap \text{Ball}(X) \neq \emptyset$. Note that $W \cap \text{Ball}^0(X)$ and $G \cap \text{Ball}^0(X)$ are also nonempty. Let $z \in G \cap Y \cap \text{Ball}^0(X)$ and $y \in W \cap Y \cap \text{Ball}^0(X)$, and $y_k$ be the above sequence in the last paragraph. Then for sufficiently large $k$, $\|y_k\| \leq 1$, $\|T^{n_k}y_k\| \leq M + 1$ and $y_k \to y$, $T^{n_k}y_k \overset{wk}{\to} z$ as $k \to \infty$. It states $T^nG \cap W \cap \text{Ball}_{M+1}(X) \neq \emptyset$. Thus, the necessary conditions of part (ii) of Theorem 1.2 are fulfilled and the proof is completed.

3. The Unilateral and Bilateral Weighted Shift

Let $\{e_j : j \in \mathbb{N}\}$ be the canonical basis of $\ell^p(\mathbb{N})$. Then, the operator $T : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ defined by $T(e_j) = w_j e_{j-1}$ for $j \geq 2$ and $T(e_1) = 0$, for some positive and bounded sequence $\{w_j : j \in \mathbb{N}\}$, is called a unilateral backward shift. Also, if $\{e_j : j \in \mathbb{Z}\}$ is the standard basis of $\ell^p(\mathbb{Z})$, then we define the bilateral backward shift $T$ on $\ell^p(\mathbb{Z})$ by $T(e_j) = w_j e_{j-1}$ for all $j \in \mathbb{Z}$ and for some positive and bounded weights $\{w_j : j \in \mathbb{Z}\}$.

Salas [13] introduced the norm hypercyclic unilateral and bilateral weighted shift in terms of the sequence of their weights.

Theorem 3.1 (Salas Theorem [13]).

1. The unilateral weighted shift $T$ with weight sequence $\{w_j : j \geq 1\}$ is hypercyclic if and only if $\sup\{w_1w_2\cdots w_n : n \geq 1\} = \infty$. 

2. The bilateral weighted shift $T$ with weight sequence $\{w_j : j \in \mathbb{Z}\}$ is hypercyclic if and only if for any given $\epsilon > 0$ and $q \in \mathbb{N}$, there exists an arbitrary large $n$ such that for all $|j| < q$,

$$\prod_{s=1}^{n} w_{j+s} > \frac{1}{\epsilon}, \quad \text{and} \quad \prod_{s=0}^{n-1} w_{j+s} < \epsilon.$$ 

The authors in [4] (Theorem 4.1) showed that $\sup\{w_1 w_2 \cdots w_n : n \geq 1\} = \infty$ is equivalent to weak hypercyclicity, hence, weakly hypercyclic unilateral weighted shift is norm hypercyclic. They also introduced a sufficient condition for a bilateral weighted shift on $\ell^p(\mathbb{Z})$ with $2 \leq p < \infty$ to be weakly hypercyclic. This condition is weaker than Salas' condition and more difficult to state. Someone might claim that the following conjecture provides a sufficient condition for weak hypercyclicity of bilateral weighted shift, based on weak hypercyclicity criterion.

**Conjecture 1.** The bilateral weighted shift $T(e_j) = w_j e_{j-1}$ on $\ell^p(\mathbb{Z})$ with $1 \leq p < \infty$ is weakly hypercyclic if and only if there exists some sequence $\{n_k\}$ of integers such that

1. $\sup\{w_j w_{j-1} \cdots w_{j-n_k+1} : k \geq 1, j \in \mathbb{Z}\} < \infty$,
2. for all $j \in \mathbb{Z}$, $w_j w_{j+1} \cdots w_{j+n_k} \to \infty$ as $k \to \infty$.

This conjecture is false, because there does not exist a shift that satisfies both conditions (1) and (2) of the conjecture. For a proof, suppose there is an increasing sequence $(n_k)$ satisfying both conditions. Let $\alpha := \sup\{w_j w_{j-1} \cdots w_{j-n_k+1} : k \geq 1, j \in \mathbb{Z}\} < \infty$. If $\alpha = 0$, then $w_j = 0$ for some $j$, and so condition (2) fails to hold. Assume $\alpha > 0$, then by condition (2), there is $n_k$ such that

$$w_0 w_1 \cdots w_{n_k} > \alpha N. \quad (3.1)$$

For $j = n_k$, we have

$$w_{n_k} w_{n_k-1} \cdots w_1 = w_j w_{j-1} \cdots w_{j-n_k+1} \leq \alpha. \quad (3.2)$$

Combining (3.1) and (3.2) yields

$$\alpha N < w_0 w_1 \cdots w_{n_k} = w_0 (w_1 \cdots w_{n_k}) \leq w_0 \alpha,$$

and therefore, $w_0 > N$. Since this inequality holds for any integer $N \geq 1$, we get $w_0 = \infty$ which is a contradiction.

4. **The Linear Fractional Composition Operator**

Every holomorphic self-map $\varphi$ of $U$ induces a linear composition operator $C : H(U) \to H(U)$ by $C(f)(z) = f(\varphi(z))$ for every $f \in H(U)$ and $z \in U$. Shapiro introduced a complete characterization of hypercyclic operators on $H(U)$. In fact, he showed that $C_\varphi$ is hypercyclic if and only if $\varphi$ has no fixed point in $U$. This condition as we see in the theorem below is equivalent to weak hypercyclicity as well.
Theorem 4.1.
Let \( \varphi \) be a holomorphic self-map on \( U \). Then \( C_\varphi \) is hypercyclic on \( H(U) \) if and only if \( C_\varphi \) is weakly hypercyclic.

Proof. Clearly hypercyclicity implies weak hypercyclicity. To prove the other direction, it is enough to show that if \( C_\varphi \) is weakly hypercyclic, then \( \varphi \) has no fixed point in \( U \). Suppose \( \varphi \) has a fixed point \( p \in U \) and \( \text{Orb}(C_\varphi, f) \) is weakly dense for some \( f \in H(U) \). Let \( g \) be an arbitrary vector in \( H(U) \) and \( \epsilon > 0 \). The linear functional \( \Lambda_p : H(U) \to \mathbb{C} \) by \( \Lambda_p(f) = f(p) \) is continuous on \( H(U) \), so there is some positive integer \( n \) such that

\[
|\Lambda_p(f \circ \varphi_n - g)| < \epsilon \quad \text{and} \quad |f(p) - g(p)| < \epsilon.
\]

It states that any function in \( H(U) \) must have the value \( f(p) \) at \( p \). But the weakly closure of this orbit cannot be all of \( H(U) \) and consequently no \( C_\varphi \)-orbit is weakly dense, i.e., \( C_\varphi \) is not weakly hypercyclic.

Now, suppose \( \varphi \) has a linear fractional self-map of \( U \) with no fixed point in \( U \). Then, we say \( \varphi \) is parabolic if \( \varphi \) has only one fixed point which must lie on the unit circle. Parabolic maps are conjugate to translations of the right half-plane into itself. Also, we say \( \varphi \) is hyperbolic if it has two fixed points, one of them lies on the unit circle and the other one is out of \( U \) which in the automorphism case, both the fixed points must lie on \( \partial U \) [3]. Bourdon and Shapiro [2, 3] characterized the hypercyclic composition operator on Hardy space, \( H^2 \). The obtained results state that for a linear fractional self-map \( \varphi \) of \( U \), \( C_\varphi \) is hypercyclic on \( H^2 \) unless \( \varphi \) is a parabolic non-automorphism [3].

Theorem 4.2.
Let \( \varphi \) be a linear fractional self-map of \( U \). Then \( C_\varphi \) is hypercyclic on \( H^2 \) if and only if \( C_\varphi \) is weakly hypercyclic.

Proof. It is clear that if \( \varphi \) is not parabolic non-automorphism, then hypercyclicity and weak hypercyclicity for \( C_\varphi \) are equivalent. Let \( \varphi \) be parabolic non-automorphism, so it has only one fixed point which lies on \( \partial U \). Without loss of generality, we may take this fixed point to be \( +1 \). Set

\[
\sigma(z) = \frac{1 + z}{1 - z} \quad \text{and} \quad \Phi = \sigma \circ \varphi \circ \sigma^{-1}.
\]

Then, \( \sigma \) is a linear fractional mapping of \( U \) onto the open right half-plane \( \mathbb{P} \), and one can easily check that \( \Phi(w) = w + a \) where \( \text{Re}(a) > 0 \) and \( w \in \mathbb{P} \). An easy computation shows that for each \( z \in U \),

\[
1 - |\varphi_n(z)|^2 = \frac{4\text{Re}(\sigma(z) + na)}{|1 + \sigma(z) + na|^2},
\]

and

\[
\varphi_n(z) - \varphi_n(0) = \frac{2(\sigma(z) - \sigma(0))}{(\sigma(z) + na + 1)(\sigma(0) + na + 1)}.
\]

In addition, for each pair of points \( z, w \in U \), and \( f \in H^2 \), the following estimate holds,

\[
|f(z) - f(w)| \leq 2 \|f\| \frac{|z - w|}{\min\{1 - |w|, 1 - |z|\}^{3/2}}.
\]
By substituting \( z \) and \( w \) by \( \varphi_n(z) \) and \( \varphi_n(0) \), respectively, and using the last estimate, we get

\[
|f(\varphi_n(z)) - f(\varphi_n(0))| \leq \frac{M}{\sqrt{n}},
\]

where the constant \( M \) depends on \( f \), \( z \), and \( \varphi \) (for more details see [2, 3]).

Now, suppose \( \text{Orb}(C_\varphi, f) \) is weakly dense for some vector \( f \in H^2 \). The linear functional \( \Lambda_p : H^2 \to \mathbb{C} \) by \( \Lambda_p(g) = g(p) \) is bounded for every \( p \in U \). Let \( g \) be an arbitrary vector in \( H^2 \), \( z \in U \), and \( \epsilon > 0 \). There exists a large enough \( n \) such that

\[
|f(\varphi_n(z)) - f(\varphi_n(0))| < \frac{\epsilon}{4},
\]

and

\[
|\Lambda_z(f \circ \varphi_n - g)| < \frac{\epsilon}{4}, \quad \text{and} \quad |\Lambda_0(f \circ \varphi_n - g)| < \frac{\epsilon}{4}.
\]

Therefore,

\[
|f(\varphi_n(z)) - g(z)| < \frac{\epsilon}{4}, \quad \text{and} \quad |f(\varphi_n(0)) - g(0)| < \frac{\epsilon}{4}.
\]

Thus, by the triangle inequality, we can deduce \( |g(z) - g(0)| < \epsilon \), and consequently \( g \equiv g(0) \).

Hence, only constant functions can be weak cluster points of the \( C_\varphi \)-orbit of an \( H^2 \) function, and thereof \( C_\varphi \) is not weakly hypercyclic. This fulfills the proof.

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