Nonlocal conformal theories have state-dependent central charges

Bora Basa$^1$, Gabriele La Nave$^2$ and Philip Phillips$^1$

Using the recently developed fractional Virasoro algebra [27], we construct a class of nonlocal CFTs with OPE’s of the form

\[
T_k(z)\Phi(w) \sim \frac{h_k\Phi}{(z-w)^{\gamma+1}} + \frac{\partial^\gamma \Phi}{\gamma (z-w)^{\gamma}}, \quad \text{and} \quad T_k(z)T_k(w) \sim \frac{c_kZ}{(z-w)^{3\gamma+1}} + \frac{(1+\gamma)T_k(w)}{(z-w)^{1+\gamma}} + \frac{\partial^\gamma T_k}{\gamma (z-w)^{1+\gamma}},
\]

which naturally results in a central charge, \(c_k\), that is state-dependent, with \(k\) indexing a particular grading. Our work indicates that only those theories which are nonlocal have state-dependent central charges, regardless of the pseudo-differential operator content of their action. All others, including certain fractional Laplacian theories, can be mapped onto an equivalent local one using a suitable covering/field redefinition. In addition, we discuss various perturbative implications of deformations of fractional CFTs that realize a fractional Virasoro algebra through the lens of a degree/state-dependent refinement of the 2 dimensional C-theorem.

1. Introduction

While the locality of the action is a key tenet of field theory, there are numerous settings in which nonlocal operators appear explicitly. A mathematically precise example comes from a physical interpretation of the Caffarelli-Silvestre (CS) extension theorem [7]: Local bulk dynamics can have corresponding boundary dynamics governed by a nonlocal operator, the Laplacian raised to a non-integer power. We will focus exclusively on the fractional Laplacian, but more generally, nonlocal operators of the form \(f(-\Delta)\) appear naturally in many effective theories, for instance, in the context of gravity and cosmology [2, 11].

Quite generally, the fractional Laplacian \((-\Delta)^\gamma\) (or its conformal extension, the Panietz operator [9, 16, 17]) on a function \(f\) in \(\mathbb{R}^n\) provides a Dirichlet-to-Neumann map for a function \(\phi\) in \(\mathbb{R}^{n+1}\) that satisfies a local second-order elliptic differential equation. Paulos, et al. [35], simultaneous with ours [26], noted that the extension theorem allows one to equate the action of a free massive theory in \(d+1\) dimensions in a spacetime such as

\(\text{atmp-fractional-cft-final} — 2021/9/17 — 0:44 — page 1 — #1\)
AdS with a $d$-dimensional nonlocal one given by the first term in Eq. (1.2). That is, the CS extension theorem provides another way of understanding the precise way in which conformal invariance arises in the AdS/CFT conjecture [20, 40].

In the realm of statistical physics, which will furnish a concrete example for the framework of nonlocal CFT we will develop, the long-range Ising (LRI) model,

$$H = -J \sum_{i,j} s_i s_j / r_{ij}^{d+\gamma},$$

where the sum is pairwise over all site, nonlocal kinetic energy operators appear in the corresponding field theory

$$S = \int d^d x \left( \frac{1}{2} \phi(x)(-\Delta)^{\gamma} \phi(x) + \frac{q}{4!} \phi(x)^4 \right).$$

This form of the action is immediate, given

$$(-\Delta)^{\gamma} f(x) = C_{d,\gamma} \int d^d x' \frac{f(x) - f(x')}{|x - x'|^{d+2\gamma}}.$$

The IR physics associated to this continuum model has received a great deal of attention in past decades. As we will review in Sec. 5.2, the fractional/nonlocal \phi^4 theory flows to either the trivial fractional Gaussian fixed point or to a fixed point in either the long-range or short-range Ising (SRI) universality class [12, 37, 38]. Particularly relevant to this work is the conformal invariance of these fixed points. While the fractional Gaussian [36] and SRI fixed points are well known to be conformally invariant, the LRI fixed point has recently been demonstrated to also possess conformal symmetry [35]. Beyond specific interest in LRI physics, conformal invariance has also been studied in nonlocal gauge theories [15, 22].

Regardless of conformal invariance, nonlocal field theories have peculiar properties. For instance, not every case of nonlocal operators appearing in the action of a field theory results in a truly nonlocal theory. We have shown previously that, in some cases [3], the ‘nonlocality’ of the free fractional Gaussian theory can be removed by a suitable field redefinition. In this paper we realize such theories, in the 2 dimensional setting, as ones governed by the tensor product of the ordinary Virasoro algebra and a particular commutative Lie algebra $\mathcal{H}$ of holomorphic functions on $\mathbb{C}$. More generally, we show how they fit in to a broader framework of nonlocal conformal field theory where the stress tensors are governed by an associated nonlocal or, as we
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called it, a fractional Virasoro algebra \(^{27}\) with a \(\mathcal{H}\) valued, or, equivalently, a state dependent central charge (a notion to be made more precise shortly).

By constructing the OPEs of the nonlocal stress tensor with itself and conformal primaries, we find that the OPE’s of a general nonlocal CFT are compatible with the underlying \(\mathcal{H}\) structure of the fractional Virasoro algebra. In particular the \(TT\) OPE realizes the state dependent central charge of the fractional Virasoro algebra \(^{27}\) and the defining OPE of a conformal primary has no state dependence. We then prove that a fractional CFT whose central charge \(c\) is constant (or not state-dependent), is equivalent, in an appropriate covering, to a local CFT which provides a practical probe of locality. We then extend this idea to deformations of such CFTs to formulate an analog of the C-theorem highlighting the importance of interactions in the emergence of nonlocality.

We then explore the relationship between those fractional CFTs that are equivalent to their local, non-fractional counterparts up to a field redefinition. We conjecture, without further discussion in this work, that such fractional CFTs can be realized as brane/defect theories related to the bulk through a CS mechanism. A similar idea has been noted \(^{31}\) where the notion of nonlocal/fractional is relegated to commutativity of fields. In this language, the brane degrees of freedom are commutative but are in correspondence with noncommutative bulk degrees of freedom. Curiously, the local bulk degrees of freedom \(^{31}\) are much akin to our local fields up to field redefinition. Finally, we address a selection of issues that have arisen over the years in the construction of a nonlocal conformal perturbation theory, especially those motivated by interacting nonlocal statistical models to which Refs. \(^{5, 35}\) serve as a detailed, relevant and recent entry point.

2. Fractional Virasoro algebra

We briefly review the construction of the fractional Virasoro algebra of Ref. \(^{27}\). In order to understand a nonlocal CFT, as a first step, one has to understand a stress tensor generated by nonlocal pseudo-differential operators. At the level of OPEs, the fractionalization of the symbol of the differential operators in the stress tensor gives rise to nontrivial monodromy that must somehow be encoded in the central charge. In \(d = 2\), we will opt to work with the fractional holomorphic pseudo-derivative,

\[
\partial_z^\gamma f(z) = \frac{\Gamma(\gamma + 1)}{2\pi i} \int_{\Xi} \frac{f(\xi)}{(\xi - z)^{1+\gamma}} d\xi,
\]
where the contour $\Xi$, around $z$ should be understood as being lifted to the universal covering due to the branch cut associated with the non-integer power. It will soon be made clear that this is the natural object that acts as a derivative on the ring of fractional Laurent polynomials, $\mathbb{C}[[z^{\pm \gamma}]]$.

Notice that the holomorphic pseudo-derivative is not a real power of a first-order differential operator. Instead, we think of it relative to the fractional Laplacian,

\begin{equation}
\partial^\gamma \bar{\partial}^\gamma f(z, \bar{z}) = |C_{\gamma, 2}|^2 (-\Delta)^\gamma f(z, \bar{z}).
\end{equation}

2.1. Algebraic construction

The standard Witt algebra is the algebra of derivations on the Laurent polynomial ring, $\mathbb{C}[[z^{\pm 1}]]$. We instead consider derivations on $\mathbb{C}[[z^{\pm \gamma}]]$ with $\gamma \in (0, 1)$. By derivation, we mean specifically the fractional holomorphic pseudo-derivative. To better understand these objects, we imagine a graded complex vector space,

\begin{equation}
V = \bigoplus_{k \in \mathbb{Z}} V^\gamma_k,
\end{equation}

with each subspace, $V^\gamma_k$, spanned by $z^{\gamma k}$. In order to define a derivation, we need a degree decrementing linear map

$$P_k : V^\gamma_k \rightarrow V^{\gamma(k-1)}$$

defined as $P(z^{\gamma k}) = z^{\gamma(k-1)}$. Then, the fractional derivative is written

\begin{equation}
\partial^\gamma_z = \bigoplus_k \lambda^{\gamma k} \frac{\Gamma(\gamma k + 1)}{\Gamma(\gamma(k-1) + 1)} P_k.
\end{equation}

The important generalization here is that the coefficients now are meromorphic functions of $k$ rather than being $k$ itself. The term that depends only on $\gamma$, $\lambda_{\gamma}$, is arbitrary so long as $\lambda_{\gamma \rightarrow 1} = 1$. Similarly, the meromorphic coefficients reduce to $k$ under $\gamma \rightarrow 1$. The definitions

$$\Gamma_{\gamma}^s(p) := \frac{\Gamma(1 + (s + p)\gamma)}{\Gamma(1 + (s - 1 + p)\gamma)} , \quad A_{p,q}(s) := \Gamma_{\gamma}^p(s) - \Gamma_{\gamma}^q(s)$$

will be helpful moving forward.

We may now ask if $L^\gamma_n := -z^{(n+1)\gamma} \partial^\gamma_z$ acting on $\mathbb{C}[[z^{\pm \gamma}]]$ generates some Lie algebra analogous to the Witt algebra. A constructive approach one could
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Take here is to compute the bracket $[\phi \otimes L^\gamma_m, \psi \otimes L^\gamma_n]$ with $\phi, \psi$ belonging to some sub-algebra of the full algebra of meromorphic functions. These functions are introduced to keep track of the degree dependence of the action of the generators on basis elements,

$$\phi \otimes L^\gamma_n(z^{\gamma k}) = \phi(k)L^\gamma_n(z^{\gamma k}) = \phi(k)\Gamma_0(k) z^{\gamma(k+n)}.$$  

Now,

$$[\phi \otimes L^\gamma_m, \psi \otimes L^\gamma_n] = (\psi(m+s)\phi(s)\Gamma^\gamma_m - \psi(s)\phi(n+s)\Gamma^\gamma_n) \otimes L^\gamma_{m+n}$$

prompts the definition of the fractional bracket,

$$[\phi(s), \psi(s)]_{m,n} := (\psi(m+s)\phi(s)\Gamma^\gamma_m - \psi(s)\phi(n+s)\Gamma^\gamma_n)(1 - \delta_{m,n}).$$

We thus seek the smallest subalgebra of meromorphic functions that is closed under this bracket to append to the fractional generators to recover the analogous fractional Witt algebra. Calling this subalgebra $\mathcal{H}$, we may write

$$\mathcal{W}_\gamma := \bigoplus_{n \in \mathbb{Z}} \mathcal{H} L^\gamma_n.$$  

$\mathcal{W}_\gamma$ is not exactly a Lie algebra in the traditional sense, however. Let $\mathcal{A}$ be an algebra equipped with integer parameterized product, $\star_{p,q}$. Then an $\mathcal{A}$-Lie algebra is the graded $\mathcal{A}$ module, $\mathcal{W} = \oplus_{n \in \mathbb{Z}} \mathcal{W}_n$ endowed with bracket $[\cdot, \cdot] : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ such that

$$[aL, bK] = a \star_{p,q} b[L, K], \quad a, b \in \mathcal{A}, \ L \in \mathcal{W}_p, \ K \in \mathcal{W}_q.$$  

Equipping the algebra of meromorphic functions holomorphic around $\text{Re}(z) \in \mathbb{Z}$ with

$$\phi(s) \star_{p,q} \psi(s) := \frac{\psi(p+s)\phi(s)\Gamma^\gamma_p - \psi(s)\phi(q+s)\Gamma^\gamma_q}{\Gamma^\gamma_p(s) - \Gamma^\gamma_q(s)},$$

we define the associated bracket in Eq. (2.7) to characterize the fractional Witt algebra as a $\mathcal{H}$-Lie algebra spanned by $L^\gamma_n$ with

$$[L^\gamma_m, L^\gamma_n] f(z) = \sum_k f_k A_{m,n}(k) L^\gamma_{m+n}(z^{\gamma k}), \quad f(z) = \sum_n f_n z^{\gamma n}.$$  

If we take $\gamma \to 1$, $\mathcal{H} \to \mathbb{C}$. Of course, a $\mathbb{C}$-Witt algebra is the Witt algebra itself.
We are now in a position to consider central extensions,

\[ 0 \to \mathcal{H} \to \mathcal{V}_\gamma \to \mathcal{W}_\gamma \to 0. \]

The extensions of the fractional Witt algebra by \( \mathcal{H} \) to the fractional Virasoro algebra, \( \mathcal{V}_\gamma \), are indeed \( \mathcal{H} \) parameterized (\( H^2(\mathcal{W}_\gamma, \mathcal{H}) \cong \mathcal{H} \)). Thus, to a meromorphic function, \( c(s) \in \mathcal{H} \) we associate a fractional Virasoro algebra. This construction indicates that the price we have to pay to localize a nonlocal conformal symmetry is a non-constant central charge. The machinery internal to the construction leads us to interpret this as a degree or state dependence of a CFT (defined through the set of its correlators) that realizes \( \mathcal{V}_\gamma \).

2.2. Geometric interpretation

The structure of a nonlocal CFT modeled on the nonlocal Virasoro algebra that we have defined so far can be given a geometric interpretation which highlights the nonlocality as a sign of a higher dimensional theory. The context of such a geometric viewpoint is understood by first interpreting the algebra \( \mathcal{H} \) as the function space of a punctured Riemann surface. Our choice of \( \mathcal{H} \) and of the nonlocal derivations dictates the Riemann surface to be the complex plane with infinitely many points removed. For definiteness, we set \( D := \mathbb{C} \setminus \{z = x + iy : x \in \gamma \mathbb{Z}, x < 0, y = 0\} \) (here we assume \( \gamma \in (0, 1) \) irrational, for simplicity) and therefore \( \mathcal{H} = \mathcal{O}(D) \) its function space (in the sense of ringed spaces).

We then consider a holomorphic family \( \pi : \mathcal{X} \to D \) of Riemann surfaces. In the context of nonlocal CFTs, the family will be isotrivial indicating that the fibers \( \pi^{-1}(s) \) with \( s \in D \) are all complex-analytically isomorphic. Nonetheless, one can consider the more general case of a non-isotrivial family. In order to avoid unnecessary complications, we will take all fibers \( \Sigma := \pi^{-1}(s) \) to possess the same conformal structure.

Since we can interpret Eq. (2.4) as an operator defined on the Laurent polynomial ring in \( w = z^\gamma, \mathbb{H} = \mathbb{C}[[w^{\pm 1}]] \)

\[ \rho^* \partial^\gamma_z = \bigoplus_k c_k \frac{\Gamma(\gamma k + 1)}{\Gamma(\gamma k - 1 + 1)} P, \]

where \( P = \frac{\partial}{\partial w} \) (as \( P(w^k) = w^{(k-1)} \)), we will consider our operators as defined on the universal covering of \( \Sigma \). This induces a new family \( \hat{\pi} : \hat{\mathcal{X}} \to D \) consisting of the universal covering of the fibers of \( \pi \). This operation of taking the
Nonlocal conformal theories have state-dependent central charges. The universal covering can in fact be rendered more geometric, as we care only about the universal covering of a neighborhood of a point \( p \in \pi^{-1}(s) \) for any \( s \). While this can be accomplished by using stacks, we omit this construction, for the sake of simplicity, and focus on having fixed a point \( p(s) \in \pi^{-1}(s) \) for every \( s \), varying analytically (i.e., we fix a holomorphic section of the map \( \pi \)) and work on the universal covering of a neighborhood of such point(s).

Fixing a Hilbert space \( \mathcal{H} \) on \( \Sigma \) and an algebra of operators \( \mathcal{A} \) on \( \mathcal{H} \), we can think of the space of operators on the whole family \( \mathcal{X} \) as \( \mathcal{H} \otimes \mathcal{A} \) with some prescribed action of \( \mathcal{H} \) on \( \mathcal{H} \). In terms of a point \( p \in \Sigma \), an analytic neighborhood of \( p \) can be thought of as the disk (with \( p \) corresponding to the origin). The space of functions on the puncture disk \( \Delta^* = \Delta \setminus \{0\} \) is the Laurent polynomial ring in \( w \), \( \mathcal{H} = \mathbb{C}[[w^{\pm 1}]] \) with, which we take then as the Hilbert space, and we then define a nonlocal (linear) action of \( \mathcal{H} \) on \( \mathcal{H} \) as \( \phi \star w^k = \phi(k) \) for \( \phi \in \mathcal{H} \).

We can envision a form of nonlocal conformal transformations in this context as fiberwise conformal transformations of the family \( \mathcal{X} \twoheadrightarrow D \). This is done by considering a metric family, \( g(s) \), on the fibers \( \pi^{-1}(s) \), varying smoothly in the parameter \( s \) and then considering fiberwise conformal transformations of this family of metrics, with a nonlocal twist. In fact, we think of the exponents \( k \) of the states \( w^k \) as points in \( D \) and for each such point, we consider the fiber \( \pi^{-1}(k) \) endowed with the metric \( g_k \) and "local" conformal transformations (i.e. holomorphic or antiholomorphic maps) of this fiber, but we truncate the Virasoro generator at order \( k \). Via identifying all the fibers, if the coefficients \( \phi \in \mathcal{H} \) of the operators are to be taken constant, this would recover the classical Virasoro algebra. The nonlocality stems from the fact that we modify the Virasoro generators at each level (identifying the level with a point of the base \( D \)) by nontrivial functions of \( D \).

3. Stress tensor OPEs

Since the central charge is no longer a c-number but rather an operator, the center of the new fractional algebra acts on Verma modules in a degree dependent manner (and not as identity). This feature is crucial to the formulation but also rather difficult to interpret physically. However, by the CS extension \[7\] mechanism, one should keep in mind that these fractional theories can be extended to algebraically better-behaved theories. Explicitly, the state-dependent formulation of the nonlocal theory is a lower dimensional incarnation of a state-independent local theory. This consistency condition also demands that the space of states of the nonlocal CFT be parameterized
by the moduli of the higher dimensional theory. For a standard field theoretic application of this idea to quantization, see [13].

We work only in the holomorphic sector for the duration of the paper. The fractional conformal fields acting on a Hilbert space $\mathbb{H}$ of states are fractional Laurent series,

\begin{equation}
\phi(z) = \sum_{k \in \mathbb{Z}} \phi_k z^{-\gamma k} \in \mathbb{C}[z^{\pm \gamma}],
\end{equation}

with finitely many negative modes. The fractional modes are required to have a prescribed $\mathcal{H}$-linear action on the Hilbert space but keep in mind that fractional conformal fields can have trivial (constant) $\mathcal{H}$ structure. Among these fields, the stress tensor,

\begin{equation}
T(z) = \sum_{k \in \mathbb{Z}} L_k^\gamma z^{-\gamma(k+2)}
\end{equation}

is of special importance as it encodes the localized fractional conformal invariance. The fractional Virasoro modes are

\begin{equation}
L_n^\gamma = - z^{\gamma(n+1)} \partial^{\gamma} = \frac{1}{2\pi i} \oint dzz^{\gamma(n+1)} T(z).
\end{equation}

We remind the reader that the contour integral are symbolic and should be understood as being paired with appropriate lifts.

### 3.1. $T\phi$ OPE

We define a fractional conformal field $\Phi(z, \bar{z})$ to be a primary if its variation under a fractional conformal transformation is determined by the fractional Taylor expansion,

\begin{equation}
\delta_{\epsilon, \bar{\epsilon}} \Phi(z, \bar{z}) = (h, \partial^\gamma \epsilon + \epsilon \partial^\gamma + \text{conj.}) \Phi(z, \bar{z}).
\end{equation}

This is to say, the space of primary operators on $\mathcal{X}$ has trivial $\mathcal{H}$ structure. This ensures compatibility between a well defined Hamiltonian, $H := L_0^\gamma + \bar{L}_0^\gamma$, and primary of dimension $h_\gamma$ as we shall demonstrate. The $T\Phi$ OPE for
Nonlocal conformal theories have state-dependent central charges which this variation is recovered is obtained through

\[
Q_\epsilon := \oint \frac{dz}{2\pi i} \epsilon(z)T(z)
\]

\[
\{Q, \Phi(w)\} = \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \epsilon(z)T(z)\Phi(w, \bar{w})
\]

\[
= \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \epsilon(z) \left( \frac{hC_\gamma}{(z-w)^{1+\gamma}} + \frac{\partial_w^\gamma}{z-w} \right) \Phi(w)
\]

\[
= \delta_\epsilon \Phi(w).
\]

Using this OPE, one can compute also

\[
[L^\gamma_n, \Phi(w)] = \frac{1}{2\pi i} \oint dz z^{\gamma(n+1)} T(z)\Phi(w)
\]

\[
(3.5)
\]

\[
= \frac{1}{2\pi i} \oint dz z^{\gamma(n+1)} \left( \frac{h_{\gamma}}{(z-w)^{1+\gamma}} + \frac{\partial_w^\gamma}{z-w} \right) \Phi(w)
\]

\[
= h_{\gamma} \partial_w^\gamma (z^{\gamma(n+1)})|_w \Phi(w) + w^{\gamma(n+1)} \partial_w^\gamma \Phi(w),
\]

\[
= h_{\gamma} \Gamma_1^\gamma(n) w^{\gamma n} \Phi(w) + w^{\gamma(n+1)} \partial_w^\gamma \Phi(w)
\]

for \(\gamma n \geq -1\). Setting \(n = 0\), we see that \(L^\gamma_0|_w h_{\gamma} = h_{\gamma}|_w\) makes sense for \(|h_{\gamma}\rangle := \phi(0)|0\rangle\). The conformal dimension may also be incremented as usual:

\[
(3.6)
\]

\[
[L^\gamma_0, L^\gamma_{-m}] = -(\Gamma_1^\gamma(s) - \Gamma_{-m}^\gamma(s)) \otimes L^\gamma_{-m}.
\]

### 3.2. \(TT\) OPE

In correspondence with the \(T\Phi\) OPE, the \(TT\) OPE may be obtained by demanding consistency with \(V_\gamma\) commutation relations:

\[
(3.7)
\]

\[
[L^\gamma_n, L^\gamma_m] = \left( \oint \frac{dz [dw]}{(2\pi i)^2} \right) z^{\gamma(n+1)} T(z) w^{\gamma(m+1)} T(w)
\]

\[
= \left( \oint \frac{dz [dw]}{(2\pi i)^2} \right) z^{\gamma(n+1)} w^{\gamma(m+1)} \left( \frac{\hat{c}}{(z-w)^{3\gamma+1}} + \frac{(1+\gamma)T(w)}{(z-w)^{1+\gamma}} + \frac{\partial_w^\gamma T(z-w)}{z-w} \right)
\]

\[
= A_{m,n} \otimes L^\gamma_{m+n} + \omega(L^\gamma_n, L^\gamma_m) \otimes Z^\gamma.
\]

\(^1\)Since we have a well defined Hamitonian, we have a notion of local time evolution. Nonlocal time evolution would require us to specify, in a rather ad hoc manner, in and out states at the boundaries of some interval.
We schematically denote by $\hat{c} \in H$ the function that characterizes the particular extension of $W_\gamma$ we are considering. More precisely, \((\omega(L^\gamma_m, L^\gamma_n) \otimes Z^\gamma)(z^\gamma k)\) = $\delta_{m+n,0}\eta_n(k)c(k)$ where $c(k) := Z^\gamma(z^\gamma k)$ is the degree dependent central charge and $\omega$ is a $p,q$ bilinear two-form with respect to which we define the set of 2-cycles. Thus, a module over $V_\gamma$ is characterized by central charge, $c(s) \in H$, if the center, $Z^\gamma$, acts not as identity but as multiplication by $c(k)$ at degree $k$.

Collecting the two elementary OPEs,

$$
T(z)\Phi(w) \sim \frac{h_\gamma \Phi}{(z-w)^{1+\gamma}} + \frac{\partial_\gamma \Phi}{z-w},
$$

$$
T(z)T(w) \sim \frac{\hat{c} + (1+\gamma)T_k(w)}{(z-w)^{3+\gamma}} + \frac{\partial_w T_k}{z-w},
$$

we find that the first specifies a notion of primary weight independent of $H$-structure while the second suggests a $H$ parameterized family of descendants where elements of the same family are $H$-related.

The latter is interpreted as a state dependence. To better illustrate this, we consider the form of inner products of descendants in modules of $V_\gamma$. Let

$$|n_1, \ldots, n_m := L^\gamma_{-n_1}\ldots L^\gamma_{-n_m}|h\rangle,$$

where $n_1, \ldots, n_m > 0$ be a generic descendant raised from a primary of weight $h$. Then, the usual procedure of commuting through the generators to evaluate inner products of descendants motivates the inner product structure

$$\langle k_1, \ldots, k_l |n_1, \ldots, n_m\rangle = \sum_{k,n} c_{k,n}(k \cdot n)_h,$$

where $(k \cdot n)_h$ is the part of the inner product that depends only on $h$ with $k$ and $n$ being basis vectors. The coefficients $c_{k,n}$, and hence the inner product, depend on the states that intermediate between the primary and the descendant through the evaluation of an operator valued central charge against the states.

3.3. Comments on developing a representation theory

In general, the multimodule structure preserved by the central extension of the fractional Witt algebra is hard to work with even in simple illustrative examples. In lieu of a concrete construction, we discuss briefly the formal analogs of minimal models in our framework.
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From $M_\gamma(\lambda), \lambda \in \mathcal{H} \oplus \mathcal{H}$, a highest weight $\mathcal{H}$-module containing a singular vector, $v$, one can construct the quotient

$$\frac{M_\gamma(\lambda)}{U(\text{Vir}_\gamma) \cdot v_\lambda},$$

where $U(\mathfrak{h})$ denotes the universal enveloping algebra of $\mathfrak{h}$. This quotient, while identical to what one might construct for a conventional minimal model, identifies states that differ by a zero norm vector in $M_\gamma(\lambda)$ in a manner that does not act homogeneously on the fractional Hilbert space and hence must carry some $\mathcal{H}$ structure for consistency.

Going further, at least formally, the multimodule $\mathcal{H}$ admits a filtration in terms of (finite dimensional) vector spaces for a given (rational) $\gamma$ [27]. It is reasonable to expect that this induces a filtration of Verma modules parameterized by $\lambda$ starting from a highest weight module which conjecturally admits a corresponding filtration into irreducible Verma modules. Verifying the last claim and ultimately pursuing a goal of detailed classification is beyond our scope here.

It is not immediately clear if one can determine unitarity bounds for these families of fractional Virasoro modules like one can in the $\gamma = 1$ case. While the construction of the Hilbert space of a nonlocal CFT implies that the usual unitarity bounds obtained from the Kac determinant will still determine if the CFT is unitary, we have to require these bounds to hold at the level of parameterized families. For instance, one would need $c(k) \geq 1, h \geq 0 \ \forall k \in \mathbb{Z}$ where $c(k) \in \mathbb{R} \subset \mathcal{H}$ is the state-dependent central charge. This is simply a refinement of the usual unitarity arguments to the case where $c_{k,n}$ in Eq. (3.9) is different from 1. While sufficient for unitarity, this assertion is not necessary and, unfortunately, not very useful, given the richness and complexity of the representation theory of the fractional Virasoro algebra. As pointed out in [22], even with actionable field theories, unitarity appears to be hard to assess if the theory is nonlocal. Along the lines of [22, 30], one possible approach could be showing that unitarity is inherited from the Caffarelli-Silvestre extended theory in one higher dimension.

4. Locality

While the fractional Virasoro algebra can be thought of as a purely abstract generalization, it was conceived in order to make sense of nonlocal currents. Since the algebra and the fractional CFTs that realize it are built around a
classically nonlocal pseudo-differential operator\(^2\), the theory is tightly intertwined with the quantum mechanical notion of locality.

A diagnostic of the locality of a CFT is the commutation rule of its operator algebra. We say that two operators, \(A\) and \(B\), are local with respect to one another if it can be established that their commutator, as a formal power series is such that

\[
(4.1) \quad z^{n\gamma} [A(z), B(0)] = 0.
\]

for some positive integer \(n\). The CFT is local if such a relationship holds for all operator pairs. This is the usual notion of locality of 2D CFTs \(^{14}\) extended up to coverings of Riemann surfaces for compatibility with an algebra built on \(\mathbb{C}[[z^{\pm\gamma}]]\). However, notice that in choosing this definition, we relegate any mutual nonlocality to \(A\) and \(B\) being related by an \(\mathcal{H}\)-function.

In general, for non-trivial (non-constant) \(\mathcal{H}\)-valued fusion numbers, the locality criterion fails for any two primaries \(A\) and \(B\) (descendants are mutually nonlocal by construction).

### 4.1. Localizing field redefinitions and a criterion for locality

Within the context discussed here, one has to be mindful of the potential existence of field redefinitions that map seemingly nonlocal dynamics to its local counterpart. In \(^3\) we argue that certain Gaussian partition functions of theories with fractional Laplacian actions are equivalent to their \(\gamma = 1\) counterparts under a dynamical field redefinition. For instance,

\[
Z_\gamma = \int D\phi e^{\phi (-\Delta)^\gamma \phi - m^2 \phi^2}
\]

is, up to a constant, equivalent to \(Z_1\) under \(\psi = (-\Delta)^{-\frac{1}{2}} \phi\) only if \(m = 0\).

More generally, such a localizing field redefinition should be compatible with an integration by parts rule,

\[
\int P \psi P \psi,
\]

for some pseudo-differential operator \(P\). The \(m^2 \phi^2\) deformation, for instance, is an obstruction to finding a field redefinition compatible with some pseudo-differential integration by parts rule. See \(^3\) for more details. We corroborate

\(^2\)We will say an operator is (classically) nonlocal if \(\text{supp} D\phi\) is larger than \(\text{supp} \phi\)
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the claim that the massless fractional Gaussian theory is seemingly local while its massive counterpart is not by looking at the scaling of the geometric entropy of both QFTs (Only the latter exhibits an area-law violation in the UV.) which is also consistent with [28].

In dimension 2 and for a CFT, we are able to explore this idea more robustly. While the fractional Virasoro algebra and its CFT realization are so far unwieldy objects, they provide insight into what quantum (non)locality really is and how it emerges.

Lemma 4.1. The tensor product $H_\gamma \otimes W$ inherits the structure of a multi-Lie algebra which is isomorphic to $W_\gamma$. Furthermore, if $\text{Vir}_{c,\gamma}$ has a central charge in $\mathbb{C}$, then $\text{Vir}_{c,1}$ is a (central) $\mathbb{C}$-extension of $H_\gamma \otimes W$.

Proof. Let $c \in \mathbb{C}$ be a constant and consider the fractional central extension along with the usual integer extension tensored with $H_\gamma$ as vector spaces:

$$
0 \overset{\gamma}{\longrightarrow} H_\gamma \overset{\text{Vir}_{c,\gamma}}{\longrightarrow} W_\gamma \overset{\text{Vir}_{c,1}}{\longrightarrow} 0
$$

Clearly, $H_\gamma \otimes \mathbb{C} \cong H_\gamma$ up to a constant. The Lie bracket on $H_\gamma \otimes W$, with $W$ the integer Witt algebra, is given by

$$
[\phi \otimes L_n, \psi \otimes L_m] = [\phi, \psi]_{m,n} \otimes L_{m+n}.
$$

Thus, $H_\gamma \otimes W \cong W_\gamma$. Then, we have the middle isomorphism,

$$
\text{Vir}_{c,\gamma} \cong H_\gamma \otimes \text{Vir}_{c,1}.
$$

In more physical terms, the fractional theories that are local by our definition (which allows for coverings) are those with constant central charge because the object $\text{Vir}_{c,1} \otimes H_\gamma$ is generated by a branched stress tensor with fixed central charge (which is local with respect to itself). Also, note that the existence of the Witt algebra isomorphism can be understood as the existence of a field redefinition that translates to the existence of a ‘localizing’ redefinition at the level of OPEs. Then, we are in a position to verify the
conjecture of Ref. [3] in the context of 2D CFT, namely that a theory is nonlocal if there does not exist a localizing field redefinition.

4.2. Fractional CFTs of constant central charge

The most approachable incarnations of our framework are those where the central charge is a constant and hence accommodate localizing field redefinitions. Algebraically, such theories share the much simpler fractional Virasoro algebra, $\mathcal{H} \otimes \text{Vir}_c$, up to isomorphism. The impetus for studying this restricted class of fractional CFTs is (1) the expectation that they will deform into nonlocal theories under local polynomial perturbations (as we will discuss in Sec. 5), (2) their similarity to defect theories obtained by the Caffarelli-Silvestre mechanism [13, 31]. Our field redefinition approach to constructing these theories will give rise to fractional CFTs that are inequivalent, though similar, to parafermionic CFTs (which also furnish branched OPEs). We refer the reader to the appendix for a brief comparison of our framework and parastatistical CFTs.

4.2.1. Fractional bosonic CFT. The simplest actionable fractional CFT one can consider is

\begin{equation}
S = g \int d^2 z \partial \gamma \phi \bar{\partial} \gamma \phi.
\end{equation}

The conformal invariance of the fractional bosonic CFT has been well established [32, 36] as we mentioned previously and can be shown to follow from the CS extension theorem [7] which is of course equivalent to the well studied fractional free Gaussian theory. This theory, while built out of a nonlocal kinetic operator, does not furnish a generalized operator product algebra with an operator-valued central charge. While this fact is not surprising, it is instructive to see how it comes about. To this end, we will exhibit a field redefinition that localizes the action in an approach that resembles techniques such as bosonization/fermionization (or the Jordan-Wigner transformation if there is a lattice in place).
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Consider the field redefinition $\phi = \partial^{1-\gamma} \phi'$ under which one has, up to the singular terms of the binomial expansion,

$$
\partial^{1-\gamma} \phi'(z) \\
\sim \int \frac{d\xi d\eta}{(2\pi i)^2} \frac{\phi'(\eta)}{(\eta - \xi)^{2-\gamma}} \frac{1}{(\eta - z)^{1+\gamma}} \\
\sim \sum_{kl} \int \frac{d\xi d\eta}{(2\pi i)^2} \phi'(\eta) \eta^{-3-k-\gamma} (-\xi)^k (-z)^l \\
\sim \sum_{kl} \int \frac{d\xi d\eta}{(2\pi i)^2} \phi'(\eta) \eta^{-3-k-\gamma} (-\xi)^k (-z)^l \\
\sim \int \frac{d\xi}{2\pi i} \frac{\phi'(\xi)}{(z - \xi)^2} \\
\sim \partial \phi'(z),
$$

where $\sim$ denotes an equivalence of analytical structure. Of course, arbitrary field redefinitions cannot be used. We say that a field redefinition is physically meaningful if the induced transformation of the partition function is simply a rescaling. That such a non-trivial field redefinition exists is an unusual property of QFTs in general.

Under the aforementioned field redefinition, one has

$$
\frac{1}{(z - w)^{n+\gamma}} \mapsto \frac{1}{(z - w)^{n+1}}
$$

and hence the OPE for the fractional free Boson can readily be constructed:

$$
\partial^{1-\gamma} \phi(z) \partial^{\gamma} \phi(w) \sim \frac{1}{(z - w)^{1+\gamma}} \\
T(z) \partial^{\gamma} \phi(w) \sim \frac{(2 - \gamma) \partial_w^2 \phi(w)}{(z - w)^{1+\gamma}} + \frac{\partial_w^{2\gamma} \phi(w)}{z - w} \\
T(z) T(w) \sim \frac{c_{\gamma} / 2}{(z - w)^{1+3\gamma}} + \text{Laurent exp.}
$$

Upon removing the branches of the singular terms by taking a cover of the fractional OPE, one obtains the usual free scalar OPE with $c_{\gamma \to 1} = 1$, which by our definition makes this theory local. Note that for all of the simple examples we will go through the coefficient of the most singular $TT$ OPE is a constant left undetermined. One could obtain it by fixing the coefficients of the field redefinition so that $\gamma \rightarrow 1$ reproduces exactly the local theory.
4.2.2. Fractional fermionic CFT. We now define the fractional Dirac operators acting on spinors. We first recall that the Dirac operator on a trivial vector bundle $\mathbb{R}^n \times V$ is given by the choice of a representation of the Clifford algebra $Cl_{(p,q)}$ on $V$ (here $(p,q)$ indicates the signature of the quadratic form $Q$ defining the Clifford algebra. For us $Q = \eta$, the Minkowski metric). Therefore one can choose linear maps $\gamma_k : V \to V$ such that

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2\eta_{\mu\nu} \quad (4.9)$$

and then one defines the Dirac operator (dependent on this representation of the Clifford algebra) as

$$\slashed{\partial} = \sum_{\eta} \gamma_\eta \nabla_\eta. \quad (4.10)$$

Here we use the covariant derivative notation to emphasize that this construction works in the more general setting in which the bundle presents with curvature, and thus with a nontrivial connection $\nabla$. It is a well known fact that

$$\slashed{\partial}^2 = \Box \cdot 1_V, \quad (4.11)$$

where $\Box$ is the box operator $\eta^{\mu\nu} \frac{\partial^2}{\partial x_\mu \partial x_\nu}$ (in general the Dirac Laplacian, that is an operator whose symbol is $\sigma(\Delta)(\xi) = \|\xi\|^2$). Because $\slashed{\partial}^2$ is an elliptic operator (which has the same kernel as $\slashed{\partial}$ in the space of functions that vanish sufficiently fast) we can define

$$\slashed{\partial}^\gamma \phi = \slashed{\partial} \left( \slashed{\partial}^{\frac{\gamma}{2}} \phi \right) \quad (4.12)$$

where, for any $s > 0$,

$$\slashed{\partial}^s \phi = \frac{1}{\Gamma(-s)} \int_0^\infty \frac{dt}{t^{1+s}} \left( e^{-t\slashed{\partial}^2} - 1 \right) \phi \quad (4.13)$$

and for $s < 0$

$$\slashed{\partial}^s \phi = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1-s}} e^{-t\slashed{\partial}^2} \phi, \quad (4.14)$$
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having denoted by \( e^{-t\partial^2} \) the heat kernel of \( \partial^2 \). The action is

\[
S = \frac{1}{2} g \int d^2 x \rho \partial^0 \partial^\gamma \phi \\
= \frac{1}{2} g \int d^2 x \left( \partial^{\gamma} \phi \right) \gamma^0 \partial \left( \partial^{\gamma} \phi \right),
\]

having integrated by parts in the last line. Specializing to dimension 2, we write the Dirac operator as \( \partial = \sqrt{2}(\bar{\partial} + \partial^*) \). Hence, the Dirac Laplacian in this case coincides with the complex box operator \( \Box_{\bar{\partial}} = \partial \bar{\partial} + \bar{\partial} \partial \) and satisfies \( \Box_{\bar{\partial}} = \Delta/4 \) (the same is true for \( \Box_{\partial} \)). Furthermore, it is a standard fact that \( \partial \phi = \alpha \) is equivalent to \( \Box_{\bar{\partial}} u = \alpha \), where \( u = \partial \Box_{\bar{\partial}}^{-1} \alpha \). This allows us to define

\[
\partial^\gamma \phi = \frac{1}{\Gamma(-s)} \int_0^{+\infty} \frac{dt}{t^{s+1}} \left( e^{t\Box_{\bar{\partial}}} - i t \right) u,
\]

with \( u = \partial^* \Box_{\bar{\partial}}^{-1} \partial \phi \) and an analogous definition for \( \partial^* \), with \( \Box_{\bar{\partial}} \) replaced by \( \Box_{\partial} \). After the field redefinition, \( \psi = \partial^{\gamma} \phi \), we can write the action as

\[
S = g \int d^2 x (\bar{\psi} \partial^\gamma \phi + \psi \partial^\gamma \phi)
\]

which, after the nonlocal field redefinition \( \psi = \Box_{\bar{\partial}}^{1-s} \phi \) and \( \bar{\psi} = \Box_{\partial}^{1-s} \phi \) reads as the usual Dirac action in dimension 2

\[
S = g \int d^2 x (\bar{\psi} \partial \bar{\psi} + \bar{\psi} \partial \psi).
\]

For Eq. (4.17), the classical equations of motion are

\[
\partial^\gamma \bar{\psi} = 0, \quad \bar{\partial}^\gamma \phi = 0.
\]

Following the standard approach to free field OPEs, albeit fractionalized, we first determine the propagator by expressing the action as

\[
S = g \int d^2 x d^2 y \phi_i(x) A^i_{\gamma j}(x, y) \phi_j(y),
\]

where

\[
A^i_{\gamma j}(x, y) = g \delta(x - y) \left( (\gamma^0 \gamma^\mu)_{ij} \partial^\mu \right)^\gamma.
\]
Note that
\[
\left((\gamma^0 \gamma^\mu)_{ij} \partial_\mu\right)^\gamma = 2^\gamma \begin{pmatrix} \partial_z^\gamma & 0 \\ 0 & \partial_{\bar{z}}^\gamma \end{pmatrix}
\]  
which is the fractional Cauchy-Riemann operator. Therefore the propagators satisfy the system of differential equations
\[
2^\gamma g \begin{pmatrix} \partial_z^\gamma & 0 \\ 0 & \partial_{\bar{z}}^\gamma \end{pmatrix} \left( \begin{array}{c} \langle \phi(z, \bar{z})\phi(w, \bar{w}) \rangle \\ \langle \bar{\phi}(z, \bar{z})\phi(w, \bar{w}) \rangle \end{array} \right) = \frac{1}{\pi} \begin{pmatrix} \partial_z^\gamma & \frac{1}{z-w^\gamma} \\ 0 & \partial_{\bar{z}}^\gamma & \frac{1}{\bar{z}-\bar{w}^\gamma} \end{pmatrix}
\]
where we have expressed the delta function as \( \partial_z^\gamma \frac{1}{z-w^\gamma} \) and as \( \partial_{\bar{z}}^\gamma \frac{1}{\bar{z}-\bar{w}^\gamma} \) which has the solutions
\[
\langle \phi(z, \bar{z})\phi(w, \bar{w}) \rangle = \frac{1}{2^\gamma \pi g} \frac{1}{z-w^\gamma} \\
\langle \bar{\phi}(z, \bar{z})\phi(w, \bar{w}) \rangle = \frac{1}{2^\gamma \pi g} \frac{1}{\bar{z}-\bar{w}^\gamma} \\
\langle \phi(z, \bar{z})\bar{\phi}(w, \bar{w}) \rangle = 0.
\]
Differentiating to obtain \( \langle \partial_z^\gamma \phi(z, \bar{z})\phi(w, \bar{w}) \rangle \) and \( \langle \partial_{\bar{z}}^\gamma \phi(z, \bar{z})\bar{\phi}(w, \bar{w}) \rangle \), we conclude that the OPE is
\[
\phi(z)\phi(w) \sim \frac{1}{2^\gamma \pi g} \frac{1}{z-w^\gamma}
\]
Writing the stress tensor as
\[
T^{z\bar{z}} = 2^\gamma g \partial\phi \partial\bar{\phi} \phi \bar{\phi}, \\
T^{zz} = 2^\gamma g \partial\phi \partial\bar{\phi} \phi \bar{\phi},
\]
one can calculate that
\[
T(z)\phi(w) \sim \frac{1}{2^\gamma (z-w)^{1+\gamma}} \phi(w) + \frac{\partial\phi}{z-w}
\]
and
\[
T(z)T(w) \sim \frac{c_{\gamma=1}/2}{(z-w)^{3+\gamma}} + \text{Laurent exp.}
\]
with \( c_{\gamma \to 1} = 1/2 \).
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4.2.3. Fractional ghost CFT. We now introduce the fractional ghost model. The standard local version of this arises for instance in string theory in BRST quantization. Interpreting the fractional generalization of the ghost CFT this way requires describing nonlocal Weyl transformations (and some form of fractional string theory), which is beyond the scope of our present work. The action is

\[
\frac{1}{2} g \int d^2 x b_{\mu\nu} (\partial^\gamma)^{\mu} c^\nu,
\]

where both \( b \) and \( c \) are fermionic and \( b_{\mu\nu} \) is traceless. Here \( (\partial^\gamma)^{\mu} c^\nu = \partial^{\mu} \Box^{(\gamma-1)} c^\nu \). The classical equations of motion are

\[
\partial^\mu \Box^{\frac{1}{2}} b_{\mu\nu} = 0, \quad \partial^\mu \Box^{\frac{1}{2}} c^\nu + \partial^{\nu} \Box^{\frac{1}{2}} c^{\mu} = 0.
\]

Switching to holomorphic coordinates, we find that the holomorphic component of the stress energy tensor is

\[
T(z) = \pi g (2\partial^\gamma c b + c\partial^\gamma b).
\]

Using the same technique we have used in the previous two examples,

\[
T(z)c(w) \sim -\frac{c(w)}{(z-w)^{1+\gamma}} + \frac{\partial_w c(w)}{(z-w)}
\]

\[
T(z)b(w) \sim \frac{2b(w)}{(z-w)^{1+\gamma}} + \frac{\partial_w b(w)}{(z-w)}
\]

\[
T(z)T(w) \sim \frac{c_{\gamma}/2}{(z-w)^{1+3\gamma}} + \frac{2T(w)}{(z-w)^{1+\gamma}} + \frac{\partial_w^2 T(w)}{(z-w)}
\]

and therefore this theory is also equivalent to a local one, in our sense with \( c_{\gamma\to1} = -26 \).

5. Comments on nonlocal field theories

With the understanding that the free fractional Boson is a fractional CFT with constant central charge, we now motivate the conceptual framework behind a perturbation theory built around such a fixed point to better understand how nonlocality emerges under deformations and, in contrast to the preceding examples, how one can identify fractional CFTs with nonconstant central charge.
5.1. Deformations and Renormalization

Knowledge of the scaling dimensions and the coefficients of the 3-point function are sufficient to generate the (one loop) beta functions for the various relevant couplings of the perturbative QFT. For coupling $g_i$ associated with a deformation of dimension $h_i$, the $\beta$-function reads

$$\beta(g_i) = (d - h_i)g_i - N_i^{ijk} g_j g_k + \cdots.$$ 

This expression encodes the effect of a deformation of the Gaussian fixed point by the relevant local operators at the level of coupling constants. The field redefinition arguments of $[3] \beta(g_i) > 0$ implies the non existence of a localizing field redefinition. In corroboration, we have the following:

**Lemma 5.1.** If a local 2D CFT stress tensor (with $c \in \mathbb{C}$) contains nonlocal operators, at least perturbatively, under deformations of it via local operators, the theory flows to a nonlocal CFT.

**Proof.** (sketch) If $\phi^n$ is a local deformation, $\phi$ is local with respect to itself and its powers with $\gamma = 1$. For $T_0$ the stress tensor of the unperturbed CFT, $\phi$ and its powers are local with respect to $T_0$ with $0 < \gamma < 1$. Given a perturbative polynomial deformation, one has for the stress tensor at a perturbatively accessible conformal fixed point

$$T = T_0^{(\gamma)} + \delta T^{(1)},$$

where the superscript denotes the value of $\gamma$ for which Eq. (4.1) holds for $\phi$ paired either with (powers of) itself or the free stress tensor. The idea here is to realize that if $T_0^{(\gamma)}$ is mutually nonlocal wrt $\delta T^{(1)}$, $T_0^{(\gamma)} \delta T^{(1)}$ differs from $T_0^{(\gamma)} T_0^{(\gamma)}$ or $TT$ by a strictly non-constant $H$ function at leading order. □

The assumption of a perturbative deformation and strictly local deformations, while restrictive, ensure (to some extent) that we are focusing on perturbatively renormalizable theories (a general nonlocal deformation is not expected to be renormalizable without fine tuning). An exotic example that is left outside the scope by these assumptions is a perturbatively nonrenormalizable finite nonlocal theory or a theory that is rendered finite by a nonlocal deformation [32].

In a more concise form, Lemma 5.1 states: $C(g) := z^{3\gamma+1} \langle T(z) T(0) \rangle$, where $g := (g_1, g_2, \ldots) \in \mathcal{G}$ is the set of coupling constants in the space of couplings, is a nontrivial $H$ function even if $C(0) \in \mathbb{C} \subset \mathcal{H}$. The part of the
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Operator product algebra that includes the stress tensor contains important structural information regarding the organization of the states of the CFT. The remainder provides the necessary data to construct dynamical theories around fractional conformal fixed points.

The fractional fusion ring is the $\mathcal{H}$-multimodule with products

$$\phi \otimes \mathcal{O}_i \times \psi \otimes \mathcal{O}_j = \sum_k (\phi_{i,j} \psi) \otimes \mathcal{O}_k.$$  

The fusion coefficients are valued in $\mathcal{H}$ and hence are degree/state dependent.

Suppose $\mathcal{O}_i := \partial \mathcal{L}(g, \Lambda_{UV}, \mu)$ for $\mathcal{L}$ some Lagrangian and define

$$\beta_i := \frac{dg_i}{d\log \mu}.$$  

Renormalizability is equivalent to the condition that

$$T_{\eta \bar{\eta}} = \beta^i(g) \mathcal{O}_i$$

where $\beta^i(g)$ are the components of the vector field that generate the renormalization group (here $T$ is not a local stress energy tensor, in general). The subspaces of the fractional graded vector space of states, $V = \bigoplus_{k \in \mathbb{Z}} V^{\gamma k}$, gives rise to a sequence

$$G_{ij}(k) := \eta^{3\gamma+1} \langle \mathcal{O}_i(\eta)\mathcal{O}_j(0) \rangle \big|_{V^{\gamma k}} \in \mathcal{H}.$$  

which we write in the form $G_{ij}$ when we think of it as state dependent operator. We also define the following quantities, following Zamolodchikov in [41]:

$$C(g) = 2\eta^{3\gamma+1} \langle T(\eta)T(0) \rangle \big|_{V^{\gamma k}}$$  

and

$$H_i(g) = 2\eta^{3\gamma+1} \langle T(\eta)\mathcal{O}_i(0) \rangle \big|_{V^{\gamma k}}$$

One can then formulate a degree-dependent refinement of Zamolodchikov’s C-theorem [41]:

**Theorem 5.2.** If $V^{\gamma k}$ contains no negative norm states, $G_{ij}(k)$ is a sequence of metrics on $\mathcal{G}$. Then the (a priori) state-dependent quantity, for a
given \( k \),

\[
c(g) = C(g) + 4\beta^i H_i - 6\beta^i \beta^j G_{ij}
\]
satisfies the following properties

1) \( c \) is non-increasing along the flow

\[
\dot{c} \leq 0
\]

2) critical points (i.e. points for which \( \beta^i(g^*) = 0 \)) are stationary points for \( c(g) \)

\[
\frac{\partial c(g)}{\partial g^i} \bigg|_{g^*} = 0
\]

3) At a critical \( g^* \) the theory has the symmetry of the fractional Virasoro algebra with central charge equal to the value of \( c(g) \) at \( g = g^* \).

**Proof.** The proof follows the one in [41] mutatis mutandis □

An immediate corollary of this is the following formal version of Lemma 5.1

**Corollary 5.3.** An RG flow fixed point \( g^* \in \mathcal{G} \) is a local CFT if and only if \( c(g^*) \) is not state-dependent.

### 5.2. Renormalizibility and the LRI model

In the case where the Gaussian fixed point has an action,

\[
S_0 = \int_{\mathbb{R}^d} \phi(-\Delta)^\gamma \phi,
\]
a quadratic deformation alone induces nonlocality by Ref. [3] and, further, by Lemma 5.1 in dimension 2, so does any relevant polynomial deformation. While such fixed points and their deformations seem rather esoteric, they have been studied extensively. The IR physics obtained by local deformations of the fractional Gaussian fixed point can be tuned to the long and short range Ising universality classes. We will take up the LRI model specifically upon discussing some QFT generalities of deformations of the fractional Gaussian fixed point.
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In the context of Lemma 5.1, consider a general deformation of the fractional Gaussian fixed point

\[ S = S_0 + \int d^d x P(\phi, \nabla \phi), \]

where \( P \) is a polynomial in classically local fields. One usually demands that \( P \) only have finitely many relevant or marginal terms for renormalizability. This constraint appears to be violated by a perturbation of the form \( P(\phi, \Delta^\gamma \phi), \gamma \in (0, 1) \) because there are infinitely many choices for \( \gamma \) for which the terms of \( P \) remain at least marginal. Thus, under this prescription, only local divergences can be controlled (with the assumption that the kinetic term is not perturbatively renormalized) meaning that we are restricted to local deformations. To our knowledge this remains a rule of thumb and warrants a detailed analysis. Ref. [19] explores issues of renormalizability in greater detail.

Granted this renormalizability argument, our construction implies a slight extension that makes the space of renormalizable nonlocal theories slightly larger. To make contact with what we have done so far, let \( d = 2 \) and write the deformed theory somewhere along its RG trajectory as

\[ S = S_0 + \int d^2 x \sum_i O_i g_i. \]

Suppose we allow these relevant deformations to be nonlocal (functions of the fractional Laplacian)\(^3\). Of course, in agreement with the conventional wisdom, this cannot be allowed in general because there would be infinitely many non-zero \( g_i \)'s. However, suppose every element of the set of relevant deforming operators is mutually local with respect to every other element in the sense of Eq. (4.1) while also being mutually nonlocal with respect to the Gaussian fixed point. This is to say,

\[(z - w)^n [O_i(z)O_j(w)] = 0\]

for some \( n \in \mathbb{N} \) for all \( i, j \in I \), for \( I \) an index set. Alternatively, one can consider deformations that are mutually local with the kinetic term. In either case, the set \( \{O_i\}_{i \in I} \) is clearly finite because they can be labeled by a single value of \( \gamma \) and hence the resulting theory is expected to be renormalizable. Conjecturally, Lemma 5.1 and this observation generalizes to higher dimensions where the mutual locality of two operators is interpreted as the

\(^3 h_\phi > 0 \) in a 2D fractional CFT
existence of a field redefinition that simultaneously localizes both operators. The renormalizability of the nonlocally deformed theory considered in \cite{39} appears to be in alignment with this amendment.

This detail may seem relevant only to esoteric theories, even by the standards of nonlocal QFT. Consider, however, what one might expect to obtain by enforcing the BRST invariance of a nonlocal gauge theory\cite{15, 22}. This procedure generically dictates nonlocal couplings of ghosts to gauge fields that, by construction, are mutually local with the kinetic term. While general nonlocal BRST invariance is largely unexplored, the abelian case is discussed in \cite{22} where the gauge fixing term in the Lagrangian contains the fractional Laplacian. However, in their case, the nonlocal gauge theory is coupled to a local matter theory which puts the model outside the reach of our renormalizability criterion. However, the apparent (up to 2-loop) renormalizability of nonlocal QED seems to indicate that nonlocal gauge invariance imposes further restrictions to allowed perturbations.

Returning to scalar fractional field theories, it is rather fortuitous that there exists a conformal fixed point that is a candidate for being a nontrivial example of a nonlocal CFT that has been studied extensively. The long range Ising model is an extension of the standard Ising model where the spin correlations extend over the entire lattice, $\Lambda$, and decay as a fractional power law

$$H = -J \sum_{i,j \in \Lambda} \sigma_i \sigma_j / r_{ij}^{d+2\gamma}, \quad J > 0.$$  

(5.14)

We fix $d = 2$ to remain consistent with our model. This particular all-to-all interaction of spins, in the continuum limit, is associated with the fractional Laplacian. Just as the usual $\phi^4$ theory belongs to the same universality class as the short-range Ising model, the nonlocal $\phi^4$ theory,

$$S = \int_{\mathbb{R}^2} \phi(-\Delta)^\gamma \phi + g_4 \phi^4,$$

(5.15)

can encode LRI physics in the IR. If the IR fixed point of this theory is conformally invariant, Lemma 5.1 and Corollary 5.3 implies that it is an example of a generalized CFT that is nonlocal by our definition. Ref. \cite{35} demonstrates that LRI fixed point is indeed conformally invariant, albeit without a local stress tensor. Our construction describes the fixed point CFT as one possessing a parameterized family of conformal symmetries captured by a nonlocal stress tensor. There are numerous directions to take a discussion
Nonlocal conformal theories have state-dependent central charges regarding a LRI CFT, especially through the lens of state-dependence arising from nonlocality. We focus on the implication of our theory for the phase diagram.

The work of Fisher et. al. and Sak [12, 37, 38] has become the standard theory of the LRI model. Counting powers in Eq. (5.15), one finds that for $\gamma > 1/2$, the theory flows to the LRI fixed point under the relevant $\phi^4$ interaction. Thus, at $\gamma_* := 1/2$ one expects a transition from trivial fractional Gaussian physics to LRI physics. Note, by Lemma 4.1 that this is a local to nonlocal transition. We reiterate here that this claim of locality can be independently tested by computing the UV asymptotics of the entanglement entropy of a free fractional Gaussian fixed point to find that there is no power law dependence on $\gamma$. In other words, the area law holds. Tuning $\gamma$ beyond $\gamma_*$ is believed to eventually drive a continuous localization transition where SRI physics is obtained above $\gamma_{**} \leq 1$. The reasoning, again, is based on dimension counting: There exists a $\gamma_{**}$ such that the dimension of $\phi$ of the long range theory, which is fixed by the kinetic term and not anomalous, decreases to match what it would have been in a short range theory.

From our perspective, the former transition can be understood from simple perturbation theory. The fractional Gaussian theory has the physical content of a local theory with the operator algebra expressed in a peculiar basis. This is not the case at $\gamma = \gamma_{**}$ where there is presumed to be a transition from LRI to SRI physics. We would expect such a crossover to be state-dependent with the operator spectra being related by non-constant $\mathcal{H}$ functions. Assuming the LRI fixed point is characterized by a nonlocal Virasoro algebra, this transition cannot be continuous in the traditional OPE sense. If primary operator spectra are to be continuously related, one expects the descendants and hence Virasoro algebras to map continuously as well. As we have discussed, the simplest fractional Virasoro algebra one can envision is of the form $\mathcal{H} \otimes \text{Vir}_c$ (which is not representative of the LRI CFT). Only in this case can one take a branched covering and relate the two Virasoro algebras continuously. Thus, the continuity of the transition must be weakened to the level of correlators, not operator algebras [4, 6, 23, 29]. While the nature of this transition has long been contested, the objection against the continuity of the transition that raises the contradiction that the LRI spectrum contains two parity odd primaries, $\phi$ and $\phi^3$, while the SRI hosts only a single relevant parity odd primary, $\phi$, from which $\phi^3$ descends appears to be consistent with our heuristic claim. We refer to [4, 5] for further elaboration on this problem and their proposed resolution.
6. Final Remarks

The construction of a CFT arising from the fractional Virasoro algebra points one in the unfamiliar direction of a generally non-constant central charge, one that depends on the state. In particular, the notion of locality is refined to allow for partition functions preserving field redefinitions, placing our work relying on entanglement entropy scaling arguments in [3] on firm mathematical footing. Furthermore, while so far forbiddingly unwieldy to work with explicitly, we establish that perturbative RG techniques, guided by a degree dependent refinement of the C-theorem, applied to nonlocal field theories can yield examples of such nonlocal fractional CFTs. The landscape of theories changes remarkably when conformal fixed points are allowed to be state-dependent. This state-dependence is assumed to be controlled by higher dimensional moduli in a manner consistent with the CS extension theorem.

Our work here motivates a shift in thinking from specific nonlocal theories to the classifying spaces of such theories that, as a scheme of conformal perturbation theory, encodes correctly the parameterized family of conformal symmetries of the UV and IR CFTs implied by a nonlocal Virasoro algebra. There is a rather striking analogy to be made here. Namely, physics of fundamental degrees of freedom is described by the embedding of string world sheets in some geometry UV completes a local QFT by giving it extended, non-singular structure that survives at short distances. Curiously, the corresponding UV dynamics is best understood not through any particular worldsheet but rather the moduli space of complex structures which, geometrically, is rather similar in spirit to Sec. 2.2. It is perhaps meaningful to interpret a nonlocal QFT arising from the fractional Laplacian as an effective UV completion given by extending world lines not to sheets but to $1 + \gamma$ dimensional objects which not only provides justification to invoking rather sophisticated mathematics to describe theories that might initially seem within the grasp of conventional methods of C/QFT but also positions us to make contact with the fractality of quantum chaotic processes and strongly correlated quantum criticality [24]. In the former case, the connection with quantum chaos should build on generalizations of the Cantor sets in the context of von Neumann algebras. It may be possible to explore these avenues by constructing sigma models where the base space is a metric measure space in which the worldline embeds as a $1 + \gamma$ dimensional object.

Beyond this analogy, it would be interesting to interpret our fractional CFTs as those on a (fractional) string worldsheet and perhaps make physical sense of the fractional ghost example and nonlocal BRST quantization. In
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this context, the connection between the nonlocal Virasoro algebra and the spaces that realize their implied symmetry, which we expect to be a stacky geometric structure, may be related to the question of symmetries in bulk quantum gravity [21]. Building on the corresponding geometric interpretation and its implications for the standard string worldsheet and generalizations of these ideas to other vertex algebras will be the focus of future work.

On a similar string-theoretic note, allowing for the state-dependence of maps between bulk and boundary operators has been proposed to address the blackhole information paradox [34]. In this context, the state-dependence is used to refer to a bulk operator that depends on the state of the boundary CFT. It would be interesting to explore possible connections between this notion of state-dependence and the one central to our formulation of a nonlocal CFT.

Appendix A. Statistics and the fractional Dirac operator

The mutual locality criterion we propose (Eq. (4.1)) is, for $0 < \gamma < 1$, the nonlocality criterion used by Zamolodchikov and Fateev [42] in their parafermionic CFT construction. Parafermionic currents are nonlocal in the sense that they are extended objects that give a physical meaning to branch cuts (Wilson lines for instance). Since the fractional CFTs we are proposing are perhaps most similar to parafermionic CFTs, we pursue the connection further in this appendix.

The exchange statistics of conformal fields are encoded in the phase change induced by exchanging operators. For parastatistical operators, $A(z)$ and $B(w)$, one has

$$A(z)B(w) = e^{\pi i \gamma} B(w)A(z)$$

where $\gamma \in \mathbb{Q}$. These OPEs have integer singularities up to a covering [33],

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{\Phi_{n}^{AB}(w)}{(z-w)^{n+\gamma}}$$

$$= (z-w)^{-\gamma} \sum_{n=-\infty}^{N} \Phi_{n}^{AB}(w) (z-w)^{n+\gamma}$$

where $\Phi_{n}^{AB}(w)$ is the composite field. For a CFT that furnishes such an algebraic structure for its current operators, the algebra of conformal generators is still the local Virasoro algebra by construction [42].
Slightly more generally and formally, vertex operator algebras admit a relaxation of their locality constraints to accommodate a generalization of the commutator as in Eq. (A.1) [10]. Such structures make sense as ‘ordinary’ CFTs under the action of the local Virasoro algebra despite violating the \( \gamma = 1 \) mutual locality condition. Our goal is to distinguish between theories with nonlocal current algebras with local Virasoro algebra and theories where the Virasoro algebra is nonlocal. We posit that the latter structure is not just a particular CFT model but rather the setting in which nonlocal field theories should be studied.

A relatively less substantial difference (in dimension 2 where there is no spin-statistics theorem) is that a parafermionic theory is one with rational, non-integer or non-half-integer planar spin, \( h - \bar{h} \). On the other hand, the fractionalization in the sense of \( \partial \rightarrow \partial^\gamma \) results in the fractionalization of scaling dimensions, not spin: \( h_\gamma - \bar{h}_\gamma \in \mathbb{Z}^+ / 2 \). The exchange statistics that one may infer from OPEs like \( \phi(z)\phi(w) \), however, are clearly of the form A.1 (cf. Eq. (4.8)). If one were to extrapolate this to \( d > 2 \), one might expect a contradiction with the spin-statistics theorem. This contradiction has been resolved in the algebraic QFT literature [1, 25]. A striking similarity between the consistency of parastatistical theories and our litmus test of locality (the existence of a localizing field redefinition) is the existence of what are called Klein transformations that map parastatistical degrees of freedom to either Bosons or Fermions without altering the physical content of the theories.

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1Department of Physics and Institute for Condensed Matter Theory, University of Illinois 1110 W. Green Street, Urbana, IL 61801, U.S.A.
2Department of Mathematics, University of Illinois, Urbana, IL. 61801
