Exact Harmonic Metric for a Moving Reissner-Nordström Black Hole

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Abstract

The exact harmonic metric for a moving Reissner-Nordström black hole with an arbitrary constant speed is presented. As an application, the post-Newtonian dynamics of a non-relativistic particle in this field is calculated.

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I. INTRODUCTION

The motion of a gravitational source can affect the dynamics of particle passing by it, and this effect has attracted considerable attention over the last two decades [1–12]. There are several methods to calculate this effect. One is to directly solve Liénard-Wiechert gravitational potential from the field equations, which has been used to study light propagation in the gravitational field of an arbitrarily moving N-body system, as well as that with angular momentum [3, 5]. Another method takes advantage of the general covariance of field equations to obtain the metric of the moving source from the known static source’s metric via Lorentz transformation [13]. Recently, this method was employed to derive the time-dependent harmonic metrics of arbitrary-constant moving Schwarzschild and Kerr black holes [14–16].

In this work, we apply a Lorentz transformation to derive the exact harmonic metric for a moving Reissner-Nordström black hole with an arbitrary constant speed. Furthermore, based on the metric, we calculate the dynamics of a photon and a particle in the weak-field limit. In what follows we use geometrized units (\(G = c = 1\)).

II. EXACT HARMONIC METRIC FOR AN ARBITRARILY CONSTANTLY MOVING REISSNER-NORDSTRÖM BLACK HOLE

We start with the harmonic metric of Reissner-Nordström black hole, which can be written as [17]

\[
ds^2 = -\frac{R^2 - m^2 + Q^2}{(R + m)^2} dX_0^2 + \left(1 + \frac{m}{R}\right)^2 \left[\delta_{ij} + \frac{m^2 - Q^2}{R^2 - m^2 + Q^2} \frac{X_iX_j}{R^2}\right] dX_i dX_j ,
\]

where \(m\) and \(Q\) are the rest mass and electric charge of the black hole, respectively. \(i, j = 1, 2, 3\), and \(\delta_{ij}\) denotes Kronecker delta. Notice that here \(X_\mu\) denotes the contravariant vector \(x'^\mu = (t', x', y', z')\) for display convenience, and \(R^2 = X_1^2 + X_2^2 + X_3^2\).

Since Einstein field equations have the property of general covariance, the harmonic metric of a constantly moving R-N black hole can be obtained via applying a Lorentz boost to Eq. (1). We denote the coordinate frame of the background as \((t, x, y, z)\), and assume the velocity of the black hole to be \(\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3\), with \(\mathbf{e}_i\) \((i = 1, 2, 3)\) denoting the unit vector of 3-dimensional Cartesian coordinates. The Lorentz transformation between \((t, x, y, z)\) and the comoving frame \((t', x', y', z')\) of the moving hole can be written as

\[
x'^\alpha = \Lambda_{\beta}^\alpha x^\beta ,
\]
with

\[ \Lambda_0^0 = \gamma, \quad \Lambda_0^i = -v_i \gamma, \\]

\[ \Lambda_j^j = \delta_{ij} + v_i v_j \frac{\gamma - 1}{v^2}, \tag{5} \]

where \( \gamma = (1 - v^2)^{-\frac{1}{2}} \) is the Lorentz factor and \( v^2 = v_1^2 + v_2^2 + v_3^2 \). Therefore, the exact harmonic metric of the moving Reissner-Nordström black hole can be obtained as follows

\[ g_{00} = -\frac{\gamma^2(R^2 - m^2 + Q^2)}{(R + m)^2} + \gamma^2 \left( 1 + \frac{m}{R} \right)^2 \left[ v^2 + \frac{(v \cdot X)^2(m^2 - Q^2)}{R^2(R^2 - m^2 + Q^2)} \right], \tag{6} \]

\[ g_{0i} = v_i \gamma^2 \left[ \frac{R^2 - m^2 + Q^2}{(R + m)^2} - \left( 1 + \frac{m}{R} \right)^2 \right] - \gamma \left( 1 + \frac{m}{R} \right)^2 \frac{m^2 - Q^2}{R^2(R^2 - m^2 + Q^2)} \times \]

\[ \left[ X_i (v \cdot X) + \frac{v_i (\gamma - 1)(v \cdot X)^2}{v^2} \right], \tag{7} \]

\[ g_{ij} = \left( 1 + \frac{m}{R} \right)^2 \left\{ \delta_{ij} + \frac{m^2 - Q^2}{R^2(R^2 - m^2 + Q^2)} \left[ X_i + \frac{v_i (\gamma - 1)(v \cdot X)}{v^2} \right] \times \right. \]

\[ \left. \left[ X_j + \frac{v_j (\gamma - 1)(v \cdot X)}{v^2} \right] \right\} + v_i v_j \gamma^2 \left[ \left( 1 + \frac{m}{R} \right)^2 \frac{R^2 - m^2 + Q^2}{(R + m)^2} \right]. \tag{8} \]

If we set \( Q = 0 \), Eqs. (5) - (8) reduce to the harmonic metric of a moving Schwarzschild black hole with velocity \( v \)

\[ g_{00} = -\frac{\gamma^2(1 + \Phi)}{1 - \Phi} + v^2 \gamma^2(1 - \Phi)^2 + \frac{\gamma^2 \Phi^2(1 - \Phi)(v \cdot X)^2}{R^2}, \tag{9} \]

\[ g_{0i} = v_i \gamma^2 \left[ \frac{1 + \Phi}{1 - \Phi} - (1 - \Phi)^2 \right] - \gamma \Phi^2(1 - \Phi) \left[ \frac{X_i (v \cdot X)}{R^2} + \frac{v_i (\gamma - 1)(v \cdot X)^2}{v^2 R^2} \right], \tag{10} \]

\[ g_{ij} = (1 - \Phi)^2 \delta_{ij} + \Phi^2(1 - \Phi) \left[ \frac{X_i + v_i (\gamma - 1)(v \cdot X)}{v^2} \right] \left[ \frac{X_j + v_j (\gamma - 1)(v \cdot X)}{v^2} \right] \]

\[ + v_i v_j \gamma^2 \left[ (1 - \Phi)^2 \frac{1 + \Phi}{1 - \Phi} \right], \tag{11} \]

which are the extension of the exact metric [14] for a Schwarzschild black hole with \( v = ve_1 \). Here \( R \) is also equal to \( \sqrt{X_1^2 + X_2^2 + X_3^2} \). It is worth pointing out that Eqs. (9) - (11), to the first post-Minkowskian approximation, are in agreement with the gravitational Liénard-Wiechert retarded solution [3].

### III. DYNAMICS OF PARTICLE IN THE WEAK-FIELD LIMIT

As an application, we apply the harmonic metric to derive the post-Newtonian dynamics of a neutral and non-relativistic particle in the far field of the moving Reissner-Nordström black hole.
First, we expand Eqs. (6) - (8) up to an order of $1/R^2$

\begin{align*}
g_{00} &= -1 - 2(1 + 2v^2)\Phi - 2\Phi^2 - \frac{Q^2}{R^2}, \\
g_{0i} &= 4v_i\Phi, \\
g_{ij} &= (1 - 2\Phi)\delta_{ij},
\end{align*}

(12)

(13)

(14)

where the velocity of the black hole has also been assumed to be non-relativistic, i.e., $\gamma \simeq 1$. After tedious but straightforward calculations, up to the order of $v^4/R^2$ ($\overline{v}$ and $\overline{r}$ denote typical values of velocity and separation of a system of particles, respectively), we can obtain the equation of motion of a massive particle as follows

\begin{equation}
\frac{du}{dt} = -\nabla \left( \Phi + 2v^2\Phi + 2\Phi^2 + \frac{Q^2}{2R^2} \right) - \frac{\partial \zeta}{\partial t} + u \times (\nabla \times \zeta) + 3u \frac{\partial \Phi}{\partial t} + 4u (u \cdot \nabla) \Phi - u^2 \nabla \Phi,
\end{equation}

(15)

where $u$ denotes the velocity of the particle, and $\zeta = 4v\Phi$. When the charge of the black hole vanishes, this equation reduces to the post-Newtonian dynamics of a non-relativistic particle in the field of a moving Schwarzschild black hole [14, 18].

IV. CONCLUSION

The metric in harmonic coordinates plays an important role in the post-Newtonian dynamics and gravitational wave radiation. In this work we obtain the exact metric for a moving Reissner-Nordström black hole via applying a Lorentz boost to the Reissner-Nordström metric in the harmonic coordinates. This method can avoid directly solving the Einstein field equations for a moving gravitational source. Based on this metric, we derive the post-Newtonian dynamics of a non-relativistic particle. This metric can also be used to calculate the deflection and time delay of light passing by a non-static Reissner-Nordström black hole, as well as Hawking radiation of the black hole.

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