Segre maps and entanglement for multipartite systems of indistinguishable particles

Janusz Grabowski¹, Marek Kus² and Giuseppe Marmo³

¹ Faculty of Mathematics and Natural Sciences, College of Sciences, Cardinal Stefan Wyszyński University, Wóycickiego 1/3, 01-938 Warszawa, Poland
² Center for Theoretical Physics, Polish Academy of Sciences, Aleja Lotników 32/46, 02-668 Warszawa, Poland
³ Dipartimento di Scienze Fisiche, Università ‘Federico II’ di Napoli and Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Complesso Universitario di Monte Sant Angelo, Via Cintia, I-80126 Napoli, Italy

E-mail: jagrab@impan.pl, marek.kus@cft.edu.pl and marmo@na.infn.it

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Abstract
We elaborate the concept of entanglement for a multipartite system with bosonic and fermionic constituents and its generalization to systems with arbitrary parastatistics. The entanglement is characterized in terms of generalized Segre maps, thus supplementing an algebraic approach to the problem from a more geometric point of view.

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1. Introduction
The possibility of identifying subsystems states in a given total state of a composite quantum system is known as separability. In the case of pure states, such a possibility is guaranteed if the composite state takes the form of the tensor product of subsystems states.

On the other hand, with the advent of quantum field theory, we have identified elementary particles that are either bosons or fermions. As a matter of fact, according to the spin-statistics theorem, all particles are either bosons or fermions. The difference is that a state is unchanged by the interchange of two identical bosons, whereas it changes the sign under the interchange of two identical fermions. The characterization of fermionic states contains already the lack of the factorization of the total state of the composite system. According to usual wisdom, this would always imply the presence of an entanglement. In our opinion, this state of affairs cannot be maintained, so there is a need of a refinement of the notion of entanglement that better describes the situation when we are dealing with bosons and fermions or even with ‘parabosons’ or ‘paraferrions’ arising from potentially meaningful parastatistics [1, 2].
In [3], we analyzed a concept of entanglement for a multipartite system with bosonic and fermionic constituents in a purely algebraic way using the representation theory of the underlying symmetry groups. Correlation properties of indistinguishable particles become relevant when subsystems are no longer separated by macroscopic distances, such as, e.g., in quantum gates based on quantum dots, where they are confined to the same spatial regions [4]. In our approach to bosons and fermions, we adopted the concept of entanglement put forward in [4, 5] for fermionic systems and extended in [6, 7] in a natural way to bosonic ones.

The problem of quantifying and measuring entanglement in systems of many indistinguishable particles remains a topic of vivid investigations from different points of view. Like in the case of distinguishable particles, also nonclassical correlations of indistinguishable particles can be studied with the use of various characteristics, e.g., a suitably adapted concept of the entropy of a state [14, 15] or by mapping the Fock spaces onto spaces of qubits [16, 17], elaborating thus a concept of the so-called mode entanglement proposed earlier by Zanardi [18, 19]. Another area of the current theoretical research on indistinguishable particles concentrates around problems of identifying and measuring the entanglement in realistic experimental circumstances [20, 21]. An exhaustive exposition of these aspects of the theory, out of the scope of this paper, is contained in a recent review article by Tichy et al [22].

Our approach appeared to be sufficiently general to define entanglement also for systems with an arbitrary parastatistics in a consistent and unified way. For pure states, we defined the S-rank, generalizing the notion of the Schmidt rank for distinguishable particles and playing an analogous role in the characterization of the degree of entanglement among particles with arbitrary exchange symmetry (parastatistics).

In the algebraic geometry, a canonical embedding of the product $\mathbb{CP}^{n-1} \times \mathbb{CP}^{m-1}$ of complex projective spaces into $\mathbb{CP}^{nm-1}$ is known as the Segre embedding (or the Segre map). In the quantum mechanical context, the complex projective space $\mathbb{CP}^{n-1}$ represents pure states in the Hilbert space $\mathbb{C}^n$ and $\mathbb{CP}^{nm-1}$ represents pure states in $\mathbb{C}^n \otimes \mathbb{C}^m$ so that the Segre embedding gives us a geometrical description of separable pure states, and as shown in [8, 9], this description can be extended also to mixed states.

In this paper, we give a geometric description of the entanglement for systems with arbitrary symmetry (with respect to exchanging of subsystems) in terms of generalized Segre embeddings associated with particular parastatistics. This description is complementary to the one presented in [3] in terms of the S-rank. For systems with arbitrary exchange symmetries, unlike for the systems of distinguishable particles, the spaces of states are not, in general, projectivizations of the full tensor products of the underlying Hilbert spaces of subsystems, but rather some parts of them. We show in the following how to extend properly the concept of the Segre embedding to achieve a geometric description analogous to that for distinguishable particles. This approach uses a unifying mathematical framework based on the representation theory and strongly suggesting certain concepts of the separability, thus of the entanglement, in the case of indistinguishable particles. For physicists, this approach may be viewed as being too mathematical and abstract, but in our opinion it covers exactly the logic structure of the notion of entanglement for systems of particles with some symmetries.

In section 2, we briefly review the relevant concepts of composite systems of distinguishable particles, their description in terms of the classical Segre maps, and entanglement measures for systems of distinguishable particles. In sections 3 and 4, we give a brief review of the algebraic description of the entanglement for bosons and fermions in terms of the S-rank of tensors presented in [3]. The main results are contained in

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4 For discussion of a slightly different treatment of bosons in [10–13], see the introduction and conclusion sections of [3].
sections 5–7, where we construct the Segre maps, first, for bosons and fermions and, finally, for systems with an arbitrary parastatistics. The constructions are based on theorem 6.1 that relates a simple tensor of a given parastatistics to the corresponding Young diagram.

2. Composite systems, separability and entanglement

Let \( \mathcal{H} \) be a Hilbert space with a Hermitian product \( \langle \cdot | \cdot \rangle \), \( gl(\mathcal{H}) \) be the vector space of complex linear operators on \( \mathcal{H} \), \( \text{GL}(\mathcal{H}) \) be the group of invertible operators from \( gl(\mathcal{H}) \) and \( U(\mathcal{H}) \) be its subgroup of unitary operators on \( \mathcal{H} \). For simplicity, we will assume that \( \mathcal{H} \) is finite dimensional, say \( \dim(\mathcal{H}) = n \), but a major part of our work remains valid also for Hilbert spaces of infinite dimensions. Note only that in the infinite dimensions the corresponding tensor product \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) is the tensor product in the category of Hilbert spaces, i.e. corresponding to the Hilbert–Schmidt norm.

With \( u(\mathcal{H}) \) we will denote the Lie algebra of the Lie group \( U(\mathcal{H}) \) consisting of anti-Hermitian operators, while \( u^*(\mathcal{H}) \) will denote its dual interpreted as the Euclidean space of Hermitian operators with the scalar product

\[
\langle A, B \rangle_{\rho} = \frac{1}{2} \text{Tr}(AB).
\]

The space of non-negatively defined operators from \( gl(\mathcal{H}) \), i.e. of those \( \rho \in gl(\mathcal{H}) \) that can be written in the form \( \rho = T^*T \) for a certain \( T \in gl(\mathcal{H}) \), we denote as \( P(\mathcal{H}) \). It is a convex cone in \( gl(\mathcal{H}) \) and the set of density states \( D(\mathcal{H}) \) is distinguished in \( P(\mathcal{H}) \) by the normalizing condition \( \text{Tr}(\rho) = 1 \). We will regard \( P(\mathcal{H}) \) and \( D(\mathcal{H}) \) as embedded in \( u^*(\mathcal{H}) \) so that the space \( D(\mathcal{H}) \) of density states is a convex set in the affine hyperplane in \( u^*(\mathcal{H}) \) determined by the equation \( \text{Tr}(\rho) = 1 \). As the difference of two Hermitian operators of trace 1 (the vector connecting two points in the affine hyperplane) is a Hermitian operator of trace 0, the model vector space of this affine hyperplane is therefore canonically identified with the space of Hermitian operators with the trace 0.

Denote the set of all operators from \( D(\mathcal{H}) \) of rank \( k \) with \( D^k(\mathcal{H}) \). It is well known that the set of extreme points of \( D(\mathcal{H}) \) coincides with the set \( D^1(\mathcal{H}) \) of pure states, i.e. the set of one-dimensional orthogonal projectors \( \rho_x = |x\rangle\langle x| \), \( \|x\| = 1 \). Hence, every element of \( D(\mathcal{H}) \) is a convex combination of points from \( D^1(\mathcal{H}) \). The space \( D^1(\mathcal{H}) \) of all pure states can be identified with the complex projective space \( \mathbb{P}(\mathcal{H}) \simeq \mathbb{C}P^{n-1} \) via the projection

\[
\mathcal{H} \setminus \{0\} \ni x \mapsto \rho_x = \frac{|x\rangle\langle x|}{\|x\|^2} \in D^1(\mathcal{H})
\]

which identifies the points of the orbits of the action in \( \mathcal{H} \) of the multiplicative group \( \mathbb{C} \setminus \{0\} \) by complex homotheties. It is well known that the complex projective space \( \mathbb{P}\mathcal{H} = D^1(\mathcal{H}) \) is canonically a Kähler manifold. The symplectic structure on \( D^1(\mathcal{H}) \subset u^*(\mathcal{H}) \) is the canonical symplectic structure of a \( U(\mathcal{H}) \)-coadjoint orbit and the metric called the Fubini–Study metric is just the metric induced from the embedding of \( D^1(\mathcal{H}) \) into the Euclidean space \( u^*(\mathcal{H}) \). This is the best known compact Kähler manifold in the algebraic geometry.

Suppose now that our Hilbert space has a fixed decomposition into the tensor product of two Hilbert spaces: \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \). This additional input is crucial in studying composite quantum systems. Observe first that the tensor product map

\[
\otimes: \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2
\]

associates the product of rays with a ray, and so it induces a canonical embedding on the level of complex projective spaces

\[
\text{Seg}: \mathbb{P}\mathcal{H}_1 \times \mathbb{P}\mathcal{H}_2 \rightarrow \mathbb{P}\mathcal{H} = \mathbb{P}(\mathcal{H}_1 \otimes \mathcal{H}_2),
\]

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vector space $V$ and entangled states are those which are not separable, are called
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$$\langle x^1 \otimes x^2, y^1 \otimes y^2 \rangle_{\mathcal{H}} = \langle x^1, y^1 \rangle_{\mathcal{H}^1} \cdot \langle x^2, y^2 \rangle_{\mathcal{H}^2}. \quad (6)$$

The above group embedding gives rise to the corresponding embedding of Lie algebras or, by our identification, of their duals that, with some abuse of notation, we will denote by

$$\text{Seg} : u^*(\mathcal{H}^1) \times u^*(\mathcal{H}^2) \to u^*(\mathcal{H}), \quad (A, B) \mapsto A \otimes B. \quad (7)$$

The original Segre embedding is just the latter map restricted to pure states. In fact, a stronger result holds true [8, 9].

**Proposition 2.1.** The embedding (7) maps $D^k(\mathcal{H}^1) \times D^l(\mathcal{H}^2)$ into $D^{kl}(\mathcal{H})$.

Let us denote the image $\text{Seg}(D^k(\mathcal{H}^1) \times D^l(\mathcal{H}^2))$, i.e. the set of *separable pure states*, with $S^l(\mathcal{H})$, and its convex hull $\text{conv}(S^l(\mathcal{H}))$, i.e. the set of all *mixed separable states* in $u^*(\mathcal{H})$, with $S(\mathcal{H})$. The states from

$$\mathcal{E}(\mathcal{H}) = D(\mathcal{H}) \setminus S(\mathcal{H}),$$

i.e. those which are not separable, are called *entangled states*. It is well known (see, e.g., [8]) that $S^l(\mathcal{H})$ is exactly the set of extremal points of $S(\mathcal{H})$. What we have just presented is a very simple geometric interpretation of separability and entanglement.

The entangled states play an important role in quantum computing and one of main problems is to decide effectively whether a given composite state is entangled or not. An abstract measurement of entanglement can be based on the following observation (see also [23]).

Let $E$ be the set of all extreme points of a compact convex set $K$ in a finite-dimensional real vector space $V$, and let $E_0$ be a compact subset of $E$ with the convex hull $K_0 = \text{conv}(E_0) \subset K$. For every non-negative function $f : E \to \mathbb{R}_+$, define its extension (*convex roof*) $f_K : K \to \mathbb{R}_+$ by

$$f_K(x) = \inf_{x = \sum_i \alpha_i} \sum_i t_i f(\alpha_i), \quad (8)$$

where the *infimum* is taken with respect to all expressions of $x$ in the form of convex combinations of points from $E$. Recall that, according to the Krein–Milman theorem, $K$ is the convex hull of its extreme points.

**Proposition 2.2.** For every non-negative continuous function $f : E \to \mathbb{R}_+$ that vanishes exactly on $E_0$, the function $f_K$ is convex on $K$ and vanishes exactly on $K_0$.

An immediate consequence is the following (cf [8, 9]).
Corollary 2.1. Let $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$ and let $F : D^1(\mathcal{H}) \to \mathbb{R}$ be a continuous function that vanishes exactly on the set $S^1(\mathcal{H})$ of separable pure states. Then,

$$\mu = F_{D(\mathcal{H})} : D(\mathcal{H}) \to \mathbb{R},$$

is a measure of entanglement, i.e. $\mu$ is convex and $\mu(\rho) = 0$ if and only if the (mixed) density state $\rho$ is separable. Moreover, if $f$ is taken $U(\mathcal{H}^1) \times U(\mathcal{H}^2)$-invariant, then $\mu$ is $U(\mathcal{H}^1) \times U(\mathcal{H}^2)$-invariant.

Remark 2.1. In the terminology of [23], the convex roof function $F_{D(\mathcal{H})}$ is entanglement monotone if $F$ is entanglement monotone on pure states.

3. Tensor algebras, fermions and bosons

To describe some properties of systems composed of indistinguishable particles and to fix the notation, let us start with introducing corresponding tensor algebras associated with a Hilbert space $\mathcal{H}$.

In the tensor power $\mathcal{H}^{\otimes k} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$, we distinguish the subspaces: $\mathcal{H}^{\sigma^k} = \mathcal{H} \vee \cdots \vee \mathcal{H}$ of totally symmetric tensors and $\mathcal{H}^{\wedge^k} = \mathcal{H} \wedge \cdots \wedge \mathcal{H}$ of totally antisymmetric ones, together with the symmetrization, $\pi^\vee_k : \mathcal{H}^{\otimes k} \rightarrow \mathcal{H}^{\sigma^k}$, and antisymmetrization, $\pi^\wedge_k : \mathcal{H}^{\otimes k} \rightarrow \mathcal{H}^{\wedge^k}$, projectors:

$$\pi^\vee_k(f_1 \otimes \cdots \otimes f_k) = \frac{1}{k!} \sum_{\sigma \in S_k} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(k)},$$

$$\pi^\wedge_k(f_1 \otimes \cdots \otimes f_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(k)}.$$

Here, $S_k$ is the group of all permutations $\sigma : \{1, \ldots, k\} \rightarrow \{1, \ldots, k\}$, and $(-1)^\sigma$ denotes the sign of the permutation $\sigma$. Note that with every permutation $\sigma \in S_k$, there is a canonically associated unitary operator $U_\sigma$ on $\mathcal{H}^{\otimes k}$ defined by

$$U_\sigma(f_1 \otimes \cdots \otimes f_k) = f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(k)},$$

so that the map $\sigma \mapsto U_\sigma$ is an injective unitary representation of $S_k$ in $\mathcal{H}^{\otimes k}$. We will write simply $\sigma$ instead of $U_\sigma$ if no misunderstanding is possible. Symmetric and skew-symmetric tensors are characterized in terms of this unitary action by $\sigma(v) = v$ and $\sigma(v) = (-1)^\sigma v$, respectively, for all $\sigma \in S_k$.

We put, by convention, $\mathcal{H}^{\otimes 0} = \mathcal{H}^{\sigma 0} = \mathcal{H}^{\wedge 0} = \mathbb{C}$. It is well known that the obvious structure of a unital graded associative algebra on the graded space $\mathcal{H}^g = \bigoplus_{k=0}^\infty \mathcal{H}^{\otimes k}$ (the tensor algebra) induces canonical unital graded associative algebra structures on the spaces $\mathcal{H}^\vee = \bigoplus_{k=0}^\infty \mathcal{H}^{\sigma^k}$ (called the bosonic Fock space) and $\mathcal{H}^\wedge = \bigoplus_{k=0}^\infty \mathcal{H}^{\wedge^k}$ (called the fermionic Fock space) of symmetric and antisymmetric tensors. This simply means that we have associative multiplications

$$v_1 \vee v_2 = \pi^\vee(v_1 \otimes v_2),$$

$$w_1 \wedge w_2 = \pi^\wedge(w_1 \otimes w_2),$$

where

$$\pi^\vee = \bigoplus_{k=0}^\infty \pi^\vee_k : \mathcal{H}^g \rightarrow \mathcal{H}^\vee$$

and

$$\pi^\wedge = \bigoplus_{k=0}^\infty \pi^\wedge_k : \mathcal{H}^g \rightarrow \mathcal{H}^\wedge.$$
and

$$\pi^\wedge = \bigoplus_{k=0}^{\infty} \pi_k^\wedge : \mathcal{H}^\otimes \to \mathcal{H}^\wedge$$  \hspace{1cm} (14)$$

are the symmetrization and antisymmetrization projections. Moreover, these multiplications respect the grading, i.e. if \( v_1 \vee v_2 \in \mathcal{H}^{\wedge (k+l)} \) if \( v_1 \in \mathcal{H}^{\wedge k} \), \( v_2 \in \mathcal{H}^{\wedge l} \), and \( w_1 \wedge w_2 \in \mathcal{H}^{\wedge (k+l)} \) if \( w_1 \in \mathcal{H}^{\wedge k} \), \( w_2 \in \mathcal{H}^{\wedge l} \). Note also that the multiplication in \( \mathcal{H}^{\wedge} \) is commutative, \( v_1 \vee v_2 = v_2 \vee v_1 \), and the multiplication in \( \mathcal{H}^{\wedge} \) is graded commutative, \( w_1 \wedge w_2 = (-1)^{k_1 k_2} w_2 \wedge w_1 \), for \( w_1 \in \mathcal{H}^{\wedge k_1} \).

It is well known that the symmetric tensor algebra \( \mathcal{H}^{\wedge} \) can be canonically identified with the algebra \( \text{Pol}(\mathcal{H}) \) of polynomial functions on \( \mathcal{H} \). Indeed, any \( f \in \mathcal{H} \) can be identified with the linear function \( x_f \) on \( \mathcal{H} \) by means of the Hermitian product: \( x_f(y) = \langle f | y \rangle \). We must stress, however, that the identification \( f \mapsto x_f \) is anti-linear. This can be extended to an anti-linear isomorphism of commutative algebras in which \( f_1 \vee \cdots \vee f_k \) corresponds to the homogeneous polynomial \( x_{f_1} \cdots x_{f_k} \). Similarly, one identifies \( \mathcal{H}^{\wedge} \) with the Grassmann algebra \( \text{Grass}(\mathcal{H}) \) of polynomial (super)functions on \( \mathcal{H} \). Here, however, with \( f \in \mathcal{H} \), we associate a linear function \( \xi_f \) on \( \mathcal{H} \) regarded as an odd function: \( \xi_f \xi_f = -\xi_f \xi_f \). In the language of supergeometry, one speaks about the purely odd vector space \( \Pi \mathcal{H} \) obtained from \( \mathcal{H} \) by changing the parity.

If we fix a basis \( e_1, \ldots, e_n \) in \( \mathcal{H} \) and associate with its elements the even linear functions \( x_1, \ldots, x_n \) on \( \mathcal{H} \) and the odd linear functions \( \xi_1, \ldots, \xi_n \) on \( \Pi \mathcal{H} \), then \( \mathcal{H}^{\wedge} \) can be identified with the algebra of complex polynomials in \( n \) commuting variables, \( \mathcal{H}^{\wedge} \simeq \mathbb{C}[x_1, \ldots, x_n] \). Similarly, \( \mathcal{H}^{\wedge} \simeq \mathbb{C}[\xi_1, \ldots, \xi_n] \), i.e. \( \mathcal{H}^{\wedge} \) can be identified with the algebra of complex Grassmann polynomials in \( n \) anticommuting variables. The subspaces \( \mathcal{H}^{\wedge k} \) and \( \mathcal{H}^{\wedge k} \) correspond to the homogeneous polynomials of degree \( k \). It is straightforward that the homogeneous polynomials \( x_1^{k_1} \cdots x_n^{k_n} \), with \( k_1 + \cdots + k_n = k \), form a basis of \( \mathcal{H}^{\wedge k} \), while the homogeneous Grassmann polynomials \( \xi_{i_1} \wedge \cdots \wedge \xi_{i_k} \), with \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \), form a basis of \( \mathcal{H}^{\wedge k} \). In consequence, \( \dim \mathcal{H}^{\wedge k} = \binom{n}{k} \) and \( \dim \mathcal{H}^{\wedge k} = \binom{n}{k} \), so the gradation in the fermionic Fock space is finite dimensional (for a finite-dimensional \( \mathcal{H} \)).

Note that any basis \( \{e_1, \ldots, e_n\} \) in \( \mathcal{H} \) induces a basis \( \{e_i \otimes e_i \otimes \cdots \otimes e_i \mid i_1, \ldots, i_k \in [1, \ldots, n] \} \) in \( \mathcal{H}^{\wedge k} \). Therefore, any \( u \in \mathcal{H}^{\wedge k} \) can be uniquely written as a linear combination

$$u = \sum_{i_1, \ldots, i_k=1}^{n} u^{i_1 \cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}. \hspace{1cm} (15)$$

If \( u \in \mathcal{H}^{\wedge k} \), then the tensor coefficients \( u^{i_1 \cdots i_k} \) are totally symmetric and, after applying the symmetrization projection to (15), we obtain

$$u = \sum_{i_1, \ldots, i_k=1}^{n} u^{i_1 \cdots i_k} e_{i_1} \vee \cdots \vee e_{i_k}. \hspace{1cm} (16)$$

Similarly, if \( u \in \mathcal{H}^{\wedge k} \), the tensor coefficients \( u^{i_1 \cdots i_k} \) are totally antisymmetric and

$$u = \sum_{i_1, \ldots, i_k=1}^{n} u^{i_1 \cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}. \hspace{1cm} (17)$$

We will refer to the coefficients \( u^{i_1 \cdots i_k} \) as to the coefficients of \( u \) in the basis \( \{e_1, \ldots, e_n\} \).

The Hermitian product in \( \mathcal{H} \) has an obvious extension to a Hermitian product in \( \mathcal{H}^{\wedge k} \),

$$\langle f_1 \otimes \cdots \otimes f_k | g_1 \otimes \cdots \otimes g_k \rangle = \prod_{i=1}^{k} \langle f_i | g_i \rangle. \hspace{1cm} (18)$$

and viewing symmetric and antisymmetric tensors as canonically embedded in the tensor algebra, we find the corresponding Hermitian products in \( \mathcal{H}^{\wedge k} \) and \( \mathcal{H}^{\wedge k} \).
For \( f_1, \ldots, f_k \in \mathcal{H} \) and \( g_1, \ldots, g_k \in \mathcal{H} \), we obtain
\[
(f_1 \vee \cdots \vee f_k | g_1 \vee \cdots \vee g_k) = \frac{1}{(k!)^2} \sum_{\sigma, \tau \in S_k} \prod_{i=1}^{k} (f_{\sigma(i)} | g_{\tau(i)}) = \frac{1}{k!} \text{per} ((f_i | g_j)).
\]
(19)

Here, \( \frac{1}{k!} \sum_{\tau \in S_k} \prod_{i=1}^{k} a_{\tau(i)} = \text{per}(a_{ij}) \) is the permanent of the matrix \( A = (a_{ij}) \). Similarly,
\[
(f_1 \wedge \cdots \wedge f_k | g_1 \wedge \cdots \wedge g_k) = \frac{1}{k!} \det((f_i | g_j)).
\]
(20)

These Hermitian products can be generalized to certain ‘pairings’ (contractions or inner products) between \( \mathcal{H}^{\vee k} \) and \( \mathcal{H}^{\wedge l} \) on one hand, and \( \mathcal{H}^{\vee l} \) and \( \mathcal{H}^{\wedge l} \) on the other, \( l \leq k \). For the standard simple tensors \( f = f_1 \otimes \cdots \otimes f_k \in \mathcal{H}^{\vee k} \) and \( g = g_1 \otimes \cdots \otimes g_l \in \mathcal{H}^{\wedge l} \), we just put
\[
t_{f, g} = (f_1 \otimes \cdots \otimes f_k | g_1 \otimes \cdots \otimes g_l)
\]
and extend it by linearity to all tensors. It is easy to see now that if \( v = f_1 \vee \cdots \vee f_k \in \mathcal{H}^{\vee k} \subset \mathcal{H}^{\vee l} \) and \( v = g_1 \vee \cdots \vee g_l \in \mathcal{H}^{\wedge l} \subset \mathcal{H}^{\wedge l} \), then \( t_{v, v} \in \mathcal{H}^{\vee (k-l)} \).

Similarly, \( t_{w, w} \in \mathcal{H}^{\wedge (k-l)} \), if \( w \in \mathcal{H}^{\vee k} \subset \mathcal{H}^{\vee l} \) and \( \omega \in \mathcal{H}^{\wedge l} \subset \mathcal{H}^{\wedge l} \). Explicitly,
\[
t_{v, v} = \frac{(k-l)!}{k!} \sum_{\sigma \in S_{k-l}} \prod_{j=1}^{l} (f_{\tau(j)} | g_{\sigma(j)}) f_{\tau(l+1)} \wedge \cdots \wedge f_{\tau(k)},
\]
(21)

where \( S(l, k-l) \) denotes the group of all \((l, k-l)\) shuffles. Recall that a permutation \( \tau \) in \( S_{p+q} \) is a \((p, q)\) shuffle if \( \tau(1) < \cdots < \tau(p) \) and \( \tau(p+1) < \cdots < \tau(p+q) \).

For skew-symmetric tensors,
\[
t_{v, v} = \frac{(k-l)!}{k!} \sum_{\sigma \in S_{k-l}} (-1)^{\sigma} \prod_{j=1}^{l} (f_{\tau(j)} | g_{\sigma(j)}) f_{\tau(l+1)} \wedge \cdots \wedge f_{\tau(k)}.
\]
(22)

In particular,
\[
t_{v, v} = (f_1 \vee \cdots \vee f_k | g_1 \vee \cdots \vee g_k)
\]
(23)

and
\[
t_{v, v} = (f_1 \wedge \cdots \wedge f_k | g_1 \wedge \cdots \wedge g_k).
\]
(24)

Moreover,
\[
t_{v, v} = \frac{1}{k!} \sum_{j=1}^{k} (f_1 \vee \cdots \vee f_j | g_1 \vee \cdots \vee g_{k-1}) f_j
\]
(25)

and
\[
t_{v, v} = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} (f_1 \wedge \cdots \wedge f_j | g_1 \wedge \cdots \wedge g_{k-1}) f_j,
\]
(26)

where \( \cdots \cdots \) stands for the omission.
4. The S-rank and entanglement for multipartite Bose and Fermi systems

There are many concepts of a rank of a tensor used in describing its complexity. One of the simplest and most natural is the one based on the inner product operators defined in the previous section. This rank, called in [3] the S-rank and used there to define the entanglement for systems of indistinguishable particles, is a natural generalization of the Schmidt rank of 2-tensors.

**Definition 4.1.** Let \( u \in \mathcal{H}^{\otimes k} \). By the S-rank of \( u \), we understand the maximum of dimensions of the linear spaces \( \mathcal{H}_{\mathcal{K}} \) for \( \sigma \in S_k \), which are the images of the contraction maps

\[
\mathcal{H}^{\otimes (k-1)} \ni \nu \mapsto i_{\nu} \sigma(u) \in \mathcal{H}.
\]

(27)

Nonzero tensors of minimal S-rank in \( \mathcal{H}^{\otimes k} \) (resp., \( \mathcal{H}^{\vee k} \), \( \mathcal{H}^{\wedge k} \)) will be called simple (resp., simple symmetric, simple antisymmetric).

Note that the above definition has its natural counterpart for distinguishable particles, and so tensors from \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k \). We just do the contractions with tensors from \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1} \) and the corresponding permutations. If particles are identical, \( \mathcal{H}_i = \mathcal{H} \), and indistinguishable, e.g., the tensors are symmetric or skew-symmetric, we can skip using permutations. In other words, for \( u \in \mathcal{H}^{\otimes k} \) (resp., \( u \in \mathcal{H}^{\vee k} \)), the S-rank of \( u \) equals the dimension of the linear space that is the image of the contraction map

\[
\mathcal{H}^{\vee (k-1)} \ni \nu \mapsto i_{\nu} u \in \mathcal{H},
\]

resp.,

\[
\mathcal{H}^{\wedge (k-1)} \ni \nu \mapsto i_{\nu} u \in \mathcal{H}.
\]

(28)

(29)

**Theorem 4.1** ([3]).

(a) The minimal possible S-rank of a nonzero tensor \( u \in \mathcal{H}^{\otimes k} \) equals 1. A tensor \( u \in \mathcal{H}^{\otimes k} \) is of S-rank 1 if and only if \( u \) is decomposable, i.e. it can be written in the form

\[
u = f_1 \otimes \cdots \otimes f_k,
\]

where \( f_i \in \mathcal{H} \), \( f_i \neq 0 \).

Such tensors span \( \mathcal{H}^{\otimes k} \).

(b) The minimal possible S-rank of a nonzero tensor \( v \in \mathcal{H}^{\vee k} \) equals 1. A tensor \( v \in \mathcal{H}^{\vee k} \) is of S-rank 1 if and only if \( v \) can be written in the form

\[
u = f_1 \vee \cdots \vee f_k,
\]

where \( f_i \in \mathcal{H} \), \( f_i \neq 0 \).

Such tensors span \( \mathcal{H}^{\vee k} \).

(c) The minimal possible S-rank of a nonzero tensor \( w \in \mathcal{H}^{\wedge k} \) equals 1. A tensor \( w \in \mathcal{H}^{\wedge k} \) is of S-rank 1 if and only if \( w \) can be written in the form

\[
u = f_1 \wedge \cdots \wedge f_k,
\]

where \( f_1, \ldots, f_k \in \mathcal{H} \) are linearly independent. Such tensors span \( \mathcal{H}^{\wedge k} \).

In particular, the S-rank is 1 for simple and simple symmetric tensors and it is \( k \) for simple antisymmetric tensors from \( \mathcal{H}^{\wedge k} \). Simple tensors have the form (30), simple symmetric tensors have the form (31) and simple antisymmetric tensors have the form (32).

Using the concept of simple tensors, we can define simple (non-entangled or separable) and entangled pure states for multipartite systems of bosons and fermions.
Definition 4.2.

(a) A pure state \( \rho_k \) on \( \mathcal{H}^nk \) (resp., on \( \mathcal{H}^nk \)), \( \rho_k = \frac{|i\rangle\langle i|}{|i\rangle\langle i|} \), with \( x \in \mathcal{H}^nk \) (resp., \( x \in \mathcal{H}^nk \)), \( k \neq 0 \), is called a bosonic (resp., fermionic) simple (or non-entangled) pure state if \( x \) is a simple symmetric (resp., antisymmetric) tensor. If \( x \) is not simple symmetric (resp., antisymmetric), we call \( \rho_k \) a bosonic (resp., fermionic) entangled state.

(b) A mixed state \( \rho \) on \( \mathcal{H}^nk \) (resp., on \( \mathcal{H}^nk \)) we call bosonic (resp., fermionic) simple (or non-entangled) mixed state if it can be written as a convex combination of bosonic (resp., fermionic) simple pure states. In the other case, \( \rho \) is called bosonic (resp., fermionic) entangled mixed state.

According to theorem 4.1, bosonic simple pure \( k \)-states are of the form

\[ |e \vee \cdots \vee e| e \vee \cdots \vee e \]

for unit vectors \( e \in \mathcal{H} \), and fermionic simple pure \( k \)-states are of the form

\[ k! |e_1 \wedge \cdots \wedge e_k| e_1 \wedge \cdots \wedge e_k \]

for orthonormal systems \( e_1, \ldots, e_k \) in \( \mathcal{H} \).

Fixing a base in \( \mathcal{H} \) results in defining coefficients \( [u^i_\ldots^j_b] \) of \( u \in \mathcal{H}^nk \). Formulas characterizing simple tensors, thus simple pure states, can be written in terms of quadratic equations with respect to these coefficients as follows. The corresponding characterization of entangled pure states are obtained by negation of the latter.

Theorem 4.2 (\[3\]).

(a) The pure state \( \rho_u \) associated with a tensor \( u = [u^{i_1\ldots i_k}_b] \in \mathcal{H}^nk \), is entangled if and only if there exist \( i_1, \ldots, i_k, j_1, \ldots, j_k \), and \( s = 1, \ldots, k \), such that

\[ u^{i_1\ldots i_k}_{i_1\ldots i_k} u^{j_1\ldots j_k}_{j_1\ldots j_k} \neq u^{i_1\ldots i_k}_{j_1\ldots j_k} u^{i_1\ldots i_k}_{j_1\ldots j_k} \].

(33)

(b) The bosonic pure state \( \rho_v \), associated with a symmetric tensor \( v = [v^{i_1\ldots i_k}_b] \in \mathcal{H}^nk \), is bosonic entangled if and only if there exist \( i_1, \ldots, i_k, j_1, \ldots, j_k \), such that

\[ v^{i_1\ldots i_k}_{i_1\ldots i_k} v^{j_1\ldots j_k}_{j_1\ldots j_k} \neq v^{i_1\ldots i_k}_{j_1\ldots j_k} v^{i_1\ldots i_k}_{j_1\ldots j_k} \].

(34)

(c) The fermionic pure state \( \rho_w \), associated with an antisymmetric tensor \( w = [w^{i_1\ldots i_k}_b] \in \mathcal{H}^nk \), is fermionic entangled if and only if there exist \( i_1, \ldots, i_{k+1}, j_1, \ldots, j_{k+1} \), such that

\[ w^{i_1\ldots i_k}_{i_1\ldots i_k} w^{j_1\ldots j_{k+1}}_{j_1\ldots j_{k+1}} \neq 0 \],

(35)

where the left-hand side is the antisymmetrization of \( w^{i_1\ldots i_k}_{i_1\ldots i_k} w^{j_1\ldots j_{k+1}}_{j_1\ldots j_{k+1}} \) with respect to the indices \( i_1, \ldots, i_{k+1} \).

Note that the equations opposite to (35), \( w^{i_1\ldots i_k}_{i_1\ldots i_k} w^{j_1\ldots j_{k+1}}_{j_1\ldots j_{k+1}} = 0 \), are sometimes called the Pl"ucker relations.

Example 4.1. Assume that we are dealing with qubit systems and that \( |0\rangle, |1\rangle \) is an orthonormal basis in \( \mathcal{H} \). The tensor \( u = |0\rangle \otimes |0\rangle \) has the S-rank 1:

\[ t_{u|0\rangle\langle 0|}^{(1)}(u) = t_{u|0\rangle\langle 0|}^{(1)}(u) = (a|0\rangle + b|1\rangle)|0\rangle = a|0\rangle, \]

while the tensor \( u_{\pm} = |0\rangle \otimes |1\rangle \pm |1\rangle \otimes |0\rangle \) has the S-rank 2:

\[ t_{u|0\rangle\langle 0|}^{(1)}(u) = \pm t_{u|0\rangle\langle 0|}^{(1)}(u)_{\pm} = \pm (a|0\rangle + b|1\rangle)|0\rangle \pm (a|0\rangle + b|1\rangle)|1\rangle = \pm a|1\rangle \pm b|0\rangle. \]
Example 4.2. For the GHZ-states $|GHZ_k⟩$ and W-states $|W_k⟩$, the S-rank is 2 independently on $k \geq 2$. Indeed, it is clear that contractions of $|GHZ_k⟩ = \frac{1}{\sqrt{k}} (|0^k⟩ + |1^k⟩)$ with $(k-1)$-tensors give all linear combinations of $|0⟩$ and $|1⟩$. The same is true for $|W_k⟩ = \frac{1}{\sqrt{k}} (|0 \cdots 01⟩ + |0 \cdots 10⟩ + \cdots + |1 \cdots 00⟩)$.

On the other hand, the S-rank cannot exceed 2 for qubit systems.

Remark 4.1. The S-rank does not distinguish between $|GHZ_k⟩$ and $|W_k⟩$. Note however that we can slightly generalize our notion of the S-rank including also contractions $ι_νu$ with shorter tensors, i.e. tensors $ν \in H \otimes (k-l)$ with $0 < l < k$. We have not insisted on this generalization in order to avoid additional technical complications. The simple version of the S-rank is sufficient for distinguishing simple tensors. The extended version could be useful in measuring the entanglement.

Example 4.3. If $|i_1⟩$ and $|i_2⟩$ are orthonormal sets in $H_1$ and $H_2$, then the S-rank of $u = \sum_{i=1}^{r} \lambda_i |i_1⟩ \otimes |i_2⟩$ is $r$, as $ι_{H_1}u$ is spanned by $|i_2⟩$, $i = 1, \ldots, r$, and $ι_{H_2}σ(u)$ is spanned by $|i_1⟩$, $i = 1, \ldots, r$. In other words, the S-rank equals the Schmidt rank in this case, so the S-rank is a natural generalization of the latter.

5. Entanglement and Segre maps for Bose and Fermi statistics

Similarly to the case of distinguishable particles (see \[8, 9\]), the sets of all bosonic simple pure states (resp., fermionic simple pure states) can be described as the images of certain maps defined on the products of projective Hilbert spaces, the generalized Segre maps, as follows.

Consider first the standard Segre embedding $Seg_k$ induced by the tensor product map:

$$\langle H_o⟩^k \ni (x_1, \ldots, x_k) \mapsto x_1 \otimes \cdots \otimes x_k \in \langle H^k⟩_o$$

where $H_o = H \setminus \{0\}$.

It is clear that the analogous map $Seg^k_{\lor}$ for the Bose statistics should be

$$\langle H_o⟩^k \ni (x_1, \ldots, x_k) \mapsto x_1 \otimes \cdots \otimes x_k \in \langle H^k⟩_o$$

where $x^k = x \lor \cdots \lor x = x \otimes \cdots \otimes x$ (k-factors), and for the Fermi statistics,

$$\langle H_o⟩^k \ni (x_1, \ldots, x_k) \mapsto x_1 \land \cdots \land x_k \in \langle H^k⟩_o$$

where $\langle H^k⟩_o$ denotes

$$\langle H^k⟩_o \setminus \{(x_1, \ldots, x_k) : x_1 \land \cdots \land x_k = 0\}$$
and \((\mathbb{P}\mathcal{H})^{\otimes k}\) is

\[(\mathbb{P}\mathcal{H})^{\otimes k} \setminus \{ (\rho_{x_1}, \ldots, \rho_{x_k}) : x_1 \wedge \cdots \wedge x_k = 0 \}.
\]

Note that the condition \(x_1 \wedge \cdots \wedge x_k \neq 0\) does not depend on the choice of the vectors \(x_1, \ldots, x_k\) in their projective classes and means that \(\rho_{x_1}, \ldots, \rho_{x_k}\) do not lie in a common projective hyperspace. The subset \(\mathcal{H}_{\otimes k}^+\) (resp., \((\mathbb{P}\mathcal{H})_{\otimes k}^+\)) is open and dense in \(\mathcal{H}^{\otimes k}\) (resp., \((\mathbb{P}\mathcal{H})_{\otimes k}^+\)). The following is an immediate consequence of theorem 4.1.

**Theorem 5.1.** A bosonic (fermionic) pure state \(\rho \in \mathbb{P}(\mathcal{H}^{\otimes k})\) (resp., \(\rho \in \mathbb{P}(\mathcal{H}^{\otimes k})\)) is entangled if and only if it lies outside the range of the Segre map

\[\text{Seg}^+ : \mathbb{P}\mathcal{H} \to \mathbb{P}(\mathcal{H}^{\otimes k}) \quad \text{famly resp.}, \quad \text{Seg}^- : (\mathbb{P}\mathcal{H})_{\otimes k}^+ \to \mathbb{P}(\mathcal{H}^{\otimes k}).\]

A mixed bosonic (fermionic) state is entangled if and only if it lies outside the convex hull of the range of the corresponding Segre map.

### 6. Entanglement for generalized parastatistics

Our approach to the entanglement of composite systems for identical particles is so general and natural that it allows for an immediate implication also for generalized parastatistics. Parastatistics were introduced by Green [1] as a refinement of the spin-statistics connection introduced by Pauli [25]. Green was motivated by a two-page paper by Wigner addressing the connection between the equations of motion and the commutation relations (Wigner’s problem) [26, 27]. The context of Green’s paper is quantum field theory, while most of the applications that have been proposed deal with thermodynamical aspects, in particular with the calculation of the partition function. In this paper, we are concerned with parastatistics only to show that our proposed scheme applies to all the situations where states of composite systems are by construction not factorizable and are associated with representations of the permutation group acting on the tensor product of states of the subsystems. The reader who wants to know more about parastatistics could refer to [2, 28, 29].

Observe first that simple tensors of length 1 in \(\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k\) form an orbit of the group \(U(\mathcal{H}_1) \times \cdots \times U(\mathcal{H}_k)\) acting on \(\mathcal{H}\) in the obvious way. In fact, each such tensor can be written as \(e^1_j \otimes \cdots \otimes e^k_j\) for certain choice of orthonormal bases \(e^1_j, \ldots, e^k_j\) in \(\mathcal{H}_j, j = 1, \ldots, k\). This means that simple tensors are just vectors of highest (or lowest, depending on the convention) weight of the compact Lie group \(U(\mathcal{H}_1) \times \cdots \times U(\mathcal{H}_k)\) relative to some choice of a maximal torus and Borel subgroups. If indistinguishable particles are concerned, the symmetric and antisymmetric tensors form particular irreducible parts of the ‘diagonal’ representation of the compact group \(U(\mathcal{H})\) in the Hilbert space \(\mathcal{H}^{\otimes k}\), defined by

\[U(x_1 \otimes \cdots \otimes x_k) = U(x_1) \otimes \cdots \otimes U(x_k) . \tag{39}\]

Recall that we identify the symmetry group \(S_k\) with the group of certain unitary operators on the Hilbert space \(\mathcal{H}^{\otimes k}\) in the obvious way:

\[\sigma(x_1 \otimes \cdots \otimes x_k) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}.\]

Note that the operators of \(S_k\) intertwine the unitary action of \(U(\mathcal{H})\). In the cases of symmetric and antisymmetric tensors, we discussed the Bose and Fermi statistics, respectively. But, for \(k > 2\), there are other irreducible parts of the representation (39) associated with invariant subspaces of the \(S_k\)-action. We shall call them (generalized) parastatistics. Any of these \(k\)-parastatistics (i.e. any irreducible subspace of the tensor product \(\mathcal{H}^{\otimes k}\)) is associated with a **Young tableau** \(a\) with \(k\)-boxes (chambers) as follows (see, e.g., [30–32]).

Consider partitions of \(k\): \(k = \lambda_1 + \cdots + \lambda_r\), where \(\lambda_1 \geq \cdots \geq \lambda_r \geq 1\). A partition \(\lambda = (\lambda_1, \ldots, \lambda_r)\) is associated with a **Young diagram** (sometimes called a **Young frame** or a
Ferrers diagram) with \( \lambda_i \) boxes in the \( i \)th row, the rows of boxes lined up on the left. Define a tableau on a given Young diagram to be a numbering of the boxes by the integers 1, \ldots, \( k \), and denote with \( Y_i \) the set of all such Young tableaux. Finally, put \( Y(k) \) to be the set of all Young tableaux with \( k \) boxes. Given a tableau \( \alpha \in Y(k) \) defines two subgroups in the symmetry group \( S_k \):

\[
P = P_\alpha = \{ \sigma \in S_k : \sigma \text{ preserves each row of } \alpha \}
\]

and

\[
Q = Q_\alpha = \{ \tau \in S_k : \tau \text{ preserves each column of } \alpha \}.
\]

In the space of linear operators on \( \mathcal{H}^{\otimes k} \), we introduce two operators associated with these subgroups:

\[
a_\alpha = \sum_{\tau \in Q} (-1)^{\tau} \tau, \quad b_\alpha = \sum_{\sigma \in P} \sigma.
\]

Finally, we define the Young symmetrizer

\[
c_\alpha = a_\alpha \circ b_\alpha = \sum_{\sigma \in P, \tau \in Q} (-1)^{\tau} \tau \circ \sigma.
\]

It is well known (see, e.g., [30–32]) that \( \pi^\alpha = \frac{1}{\mu(\alpha)} c_\alpha \), for some nonzero rational number \( \mu(\alpha) \), is an orthogonal projector and that the image \( \mathcal{H}^{\otimes k} \) of \( c_\alpha \) is an irreducible subrepresentation of \( U(\mathcal{H}) \), i.e. the parastatistics associated with \( \alpha \). As a matter of fact, these representations for the Young tableaux on the same Young diagram are equivalent, so that the constant \( \mu(\alpha) \) depends only on the Young diagram \( \lambda \) of \( \alpha \) and does not depend on the enumeration of boxes. Hence, \( \mu(\alpha) = \mu(\lambda) \) and this number is related to the multiplicity \( m(\lambda) \) of the corresponding irreducible representation in \( \mathcal{H}^{\otimes k} \) by \( \mu(\lambda) \cdot m(\lambda) = k! \). For a given Young diagram (partition) \( \lambda \), the map

\[
\epsilon_\lambda = \frac{1}{\mu(\lambda)^2} \sum_{\alpha \in Y_\lambda} c_\alpha
\]

is an orthogonal projection, called the central Young symmetrizer, onto the invariant subspace being the sum of all copies of the irreducible representations equivalent to that with a parastatistics from \( Y_\lambda \).

The symmetrization \( \pi^\vee \) (antisymmetrization \( \pi^\wedge \)) projection corresponds to a Young tableau with just one row (one column) and arbitrary enumeration. It is well known that any irreducible representation \( \mathcal{H}^{\alpha} \) of \( U(\mathcal{H}) \) contains cyclic vectors that are of highest weight relative to some choice of a maximal torus and Borel subgroups in \( U(\mathcal{H}) \). We will call them \( \alpha \)-simple tensors or simple tensors in \( \mathcal{H}^{\alpha} \). Note that such vectors can be viewed as generalized coherent states [33]. They can be also regarded as the ‘most classical’ states with respect to their correlation properties [34]. These are exactly the tensors associated with simple (non-entangled) pure states for composite systems of particles with (generalized) parastatistics. This is because \( \alpha \)-simple tensors represent the minimal amount of quantum correlations for tensors in \( \mathcal{H}^{\alpha} \), namely the quantum correlations forced directly by the particular parastatistics.
Example 6.1.

(a) For $k = 2$, we have just the obvious splitting of $\mathcal{H}^\otimes 2$ into symmetric and antisymmetric tensors: $\mathcal{H}^\otimes 2 = \mathcal{H}^\wedge 2 \oplus \mathcal{H}^\vee 2$.

(b) For $k = 3$, besides symmetric and antisymmetric tensors associated with the Young tableaux

$\alpha_0 = \begin{array}{ccc} 1 & 2 & 3 \end{array}$ and $\alpha_3 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$,

we have two additional irreducible parts associated with the Young tableaux

$\alpha_1 = \begin{array}{cc} 1 & 2 \\ 3 \end{array}$ and $\alpha_2 = \begin{array}{c} 1 \\ 3 \\ 2 \end{array}$,

namely

$\mathcal{H}^\otimes 3 = \mathcal{H}^\wedge 3 \oplus \mathcal{H}^{\alpha_1} \oplus \mathcal{H}^{\alpha_2} \oplus \mathcal{H}^\vee 3$.

Since $p_{\alpha_1} = \{id, (1, 2)\}$ and $Q_{\alpha_1} = \{id, (1, 3)\}$, we obtain $a_{\alpha_1} = id - (1, 3)$ and $b_{\alpha_1} = id + (1, 2)$, so that

$c_{\alpha_1} = a_{\alpha_1} \circ b_{\alpha_1} = (id - (1, 3)) \circ (id + (1, 2)) = id + (1, 2) - (1, 3) - (123)$.

As the multiplicity of the representation is 2, we have $\mu(\alpha_1) = 3!/2 = 3$ and the projection

$\pi^{\alpha_1}: \mathcal{H}^\otimes 3 \to \mathcal{H}^{\alpha_1}$

takes the form

$\pi^{\alpha_1}(x_1 \otimes x_2 \otimes x_3) = \frac{1}{3}(x_1 \otimes x_2 \otimes x_3 + x_2 \otimes x_1 \otimes x_3 - x_3 \otimes x_2 \otimes x_1 - x_3 \otimes x_1 \otimes x_2)$.

Similarly,

$\pi^{\alpha_2}: \mathcal{H}^\otimes 3 \to \mathcal{H}^{\alpha_2}$,

$\pi^{\alpha_2}(x_1 \otimes x_2 \otimes x_3) = \frac{1}{3}(x_1 \otimes x_2 \otimes x_3 + x_3 \otimes x_2 \otimes x_1 - x_2 \otimes x_1 \otimes x_3 - x_2 \otimes x_3 \otimes x_1)$.

The simple tensors (the highest weight vectors) in $\mathcal{H}^{\alpha_1}$ can be written as

$v^{\alpha_1}_\lambda = \lambda(e_1 \otimes e_1 \otimes e_2 - e_2 \otimes e_1 \otimes e_1)$,

for certain choice of an orthonormal basis $e_i$ in $\mathcal{H}$ and $\lambda \neq 0$. Analogously, the simple tensors in $\mathcal{H}^{\alpha_2}$, in turn, take the form

$v^{\alpha_2}_\lambda = \lambda(e_1 \otimes e_2 \otimes e_1 - e_2 \otimes e_1 \otimes e_1)$.

For $\dim(\mathcal{H}) = 3$, the simple tensors of length 1 form an orbit of the unitary group $U(\mathcal{H})$ of the (real) dimension 7 in $\mathcal{H}^{\alpha_1}$ and $\mathcal{H}^{\alpha_2}$. The simple symmetric tensors of length 1 form an orbit of dimension 5, and the simple antisymmetric ones (of length 1)—an orbit of dimension 1. The dimensions of the irreducible representations are $\dim(\mathcal{H}^\wedge 3) = 1$, $\dim(\mathcal{H}^\vee 3) = 10$ and $\dim(\mathcal{H}^{\alpha_1}) = \dim(\mathcal{H}^{\alpha_2}) = 8$.

A fundamental observation is that $\alpha$-simple tensors can also be characterized in terms of the $S$-rank.
Theorem 6.1. A tensor \( v \in \mathcal{H}^\alpha, \alpha \in Y(k) \), is simple if and only if it has the minimal S-rank among nonzero tensors from \( \mathcal{H}^\alpha \). This minimal S-rank equals the number \( r \) of rows in the corresponding Young diagram and the simple tensor reads
\[
v = \pi^\alpha(e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(k)}),
\]
where \( e_1, \ldots, e_r \) are some linearly independent vectors in \( \mathcal{H} \) and \( \alpha(i) \) is the number of the row in which the box with the number \( i \) appears in the tableaux \( \alpha \). In other words, the tensor
\[
E_a = e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(k)}
\]
is the tensor product of \( k \) vectors from the sequence \( E = (e_1, \ldots, e_r) \) obtained by putting \( e_j \) in the places indicated by the number of boxes in the \( j \)th row.

Proof. Assume that \( v \) is of the form (51). Passing to the complexification \( GL(\mathcal{H}) \) of \( U(\mathcal{H}) \) and using a basis \( e_1, \ldots, e_n \) extending the linear independent family \( e_1, \ldots, e_r \in \mathcal{H} \) to identify \( GL(\mathcal{H}) \) with \( GL(n; \mathbb{C}) \), we easily see that \( E_{a_\alpha} \), thus \( v \), is an eigenvector for any diagonal matrix and is killed by any upper-triangular matrix. Moreover, \( v \neq 0 \), and hence, \( v \) is a vector of highest weight. Indeed, by definition, \( b_\alpha(E_{a_\alpha}) \) is nonzero and proportional to \( E_{a_\alpha} \) so that \( v = t \cdot a_\alpha(E_{a_\alpha}) \) for a nonzero constant \( t \). It is now easy to see that \( \{ \tau(E_a) : \tau \in Q \} \) is a family of linearly independent tensors in \( \mathcal{H}^{\otimes k} \) so that \( v = \sum_{\tau \in Q} (-1)^{\tau} \tau(E_a) \) is nonzero. As tensors from \( \{ \tau(E_a) : \tau \in Q \} \) are linearly independent and have one of \( e_1, \ldots, e_r \) as the last factor, the S-rank of \( v \) is at least \( r \). On the other hand, \( v \) is composed of tensor products of \( e_1, \ldots, e_r \) only, so the S-rank is at most \( r \).

Conversely, let \( v \in \mathcal{H}^\alpha, v \neq 0 \). Without loss of generality, we can assume that the numbers in the first column of the Young tableaux \( \alpha \) are \( k - r + 1, \ldots, k \). Since \( \epsilon_{\lambda}(v) = v \), we have
\[
v = \frac{1}{\mu(\lambda)^2} \sum_{\sigma \in S_{\lambda'}} (-1)^{\varsigma} \left( \sum_{\tau \in Q'} (-1)^{\tau} \tau(b_\alpha(v)) \right),
\]
where
\[
Q' = \{ \tau \in Q : \tau \ is \ identical \ on \ the \ first \ column \ of \ \alpha \}\n\]
and \( S_r \) is the permutation group of \( \{ k - r + 1, \ldots, k \} \). This means that \( v \) is skew-symmetric with respect to the last \( r \) positions, so the contractions \( i_vw \), with \( w \in (\mathcal{H}^{\otimes (k-1)}) \), can be written as contractions of a skew-symmetric \( r \)-tensor; thus, they span a vector space of dimension \( \geq r \) (see theorem 4.1 (c)). If this dimension is exactly \( r \), then \( v \) can be written as a combination of linearly independent tensor products \( e_{i(1)} \otimes \cdots \otimes e_{i(k)} \) of vectors from \( \{ e_1, \ldots, e_r \} \). Since the tensor is symmetric with respect to permutations preserving rows and skew-symmetric with respect to permutations of the first column, each tensor product \( e_{i(1)} \otimes \cdots \otimes e_{i(k)} \) in this combination should satisfy \( e_{i(j)} = e_{e_{x(j)}} \) if \( i \) and \( j \) are in the same row of \( \alpha \). Hence, \( v \) is proportional to \( \pi^\alpha(E_{a_\alpha}) \).

Let \( \mathcal{H}^\alpha \subset \mathcal{H}^{\otimes k} \) denotes the irreducible component of the tensor representation of the unitary group \( U(\mathcal{H}) \) in \( \mathcal{H}^{\otimes k} \) associated with a Young diagram \( \alpha \in Y(k) \).

Definition 6.1.

(a) We say that a pure state \( \rho_\alpha \) on \( \mathcal{H}^{\otimes k} \) obeys a parastatistics \( \alpha \in Y(k) \) (pure \( \alpha \)-state for short) if \( v \in \mathcal{H}^\alpha \), i.e. \( \rho \) is a pure state on the Hilbert space \( \mathcal{H}^\alpha \).

(b) A pure state \( \rho \) on \( \mathcal{H}^{\otimes k} \) obeying a parastatistics \( \alpha \) is called a simple pure state for the parastatistics \( \alpha \) (simple pure \( \alpha \)-state for short) if \( \rho \) is represented by an \( \alpha \)-simple tensor in \( \mathcal{H}^\alpha \). If \( \rho \) is not a simple \( \alpha \)-state, we call it an entangled pure \( \alpha \)-state.
(c) A mixed state $\rho$ on $\mathcal{H}^\alpha$ we call a simple (mixed) state for the parastatistics $\alpha$ (simple $\alpha$-state for short), if it can be written as a convex combination of simple pure $\alpha$-states. In the other case, $\rho$ is called an entangled mixed $\alpha$-state.

7. Segre maps for generalized parastatistics

In general, for an arbitrary parastatistics (the Young tableau) $\alpha \in Y(k)$ with the partition (the Young diagram) $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, we define the generalized Segre map $\text{Seg}^\alpha$ (in $\alpha$-Segre map) as a map $\text{Seg}^\alpha : (\mathcal{PH})_\alpha^{\otimes r} \rightarrow \mathcal{P}(\mathcal{H}^\alpha)$ described as follows.

Let us first consider the map
$$i_\alpha : \mathcal{H}^{\otimes r} \rightarrow \mathcal{H}_\alpha^{\otimes k}, \quad (x_1, \ldots, x_r) \mapsto x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(k)},$$
where $\alpha(i)$ is the number of the row in which the box with the number $i$ appears in the tableaux $\alpha$. In other words, we make a tensor product of $k$ vectors from $\{x_1, \ldots, x_r\}$ by putting $x_j$ in the places indicated by the number of boxes in the $j$th row. For instance, the Young tableaux from example 6.1 give $i_{\alpha}, (x_1, x_2) = x_1 \otimes x_1 \otimes x_2$ and $i_{\alpha}, (x_1, x_2) = x_1 \otimes x_2 \otimes x_1$. It is clear that $i_{\alpha}(x_1, \ldots, x_r)$ is an eigenvector of $b_{\alpha}$.

The Segre map $\text{Seg}^\alpha$ associates with $(\rho_1, \ldots, \rho_r) \in (\mathcal{PH})_\alpha^{\otimes r}$ the pure state $\rho_{\text{Seg}^\alpha(x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(k)})}$ in $\mathcal{H}^\alpha$, as shown in the following diagram:

$$\begin{align*}
\mathcal{H}_\alpha^{\otimes r} & \ni (x_1, \ldots, x_r) \xrightarrow{\pi^\alpha} \pi^\alpha(x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(k)}) \in \mathcal{H}_\alpha^{\otimes r} \\
(\mathcal{PH})_\alpha^{\otimes r} & \ni (\rho_1, \ldots, \rho_r) \xrightarrow{\text{Seg}^\alpha} \rho_{\pi^\alpha(x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(k)})} \in \mathcal{P}(\mathcal{H}^\alpha)
\end{align*}$$

Note that $\pi^\alpha(x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(k)})$ is proportional to the antisymmetrization of the tensor $x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(k)}$ and that the construction is correct, since $\pi^\alpha(x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(k)})$ is nonzero if and only if $x_1 \wedge \cdots \wedge x_r \neq 0$, and its projective class is uniquely determined by the projective classes of $x_1, \ldots, x_r$. Note also that we can always take $x_1, \ldots, x_r$ orthogonal, say $x_1 = e_1, \ldots, x_r = e_r$, since the antisymmetrization kills the part of $x_i$ that is the orthogonal projection of $x_i$ onto the linear subspace spanned by the rest of the vectors $x_j$. Now, according to theorem 6.1, $\pi^\alpha(x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(k)})$ is $\alpha$-simple so that $\rho_{\pi^\alpha(x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(k)})}$ is a simple pure $\alpha$-state. Moreover, each simple pure $\alpha$-state is of this form, and for symmetric and antisymmetric tensors, this construction agrees with (37) and (38). We therefore obtain the following.

**Theorem 7.1.** A pure $\alpha$-state $\rho$ in $\mathcal{P}(\mathcal{H}^\alpha)$ is an entangled $\alpha$-state if and only if it is not in the range of the Segre map

$$\text{Seg}^\alpha : (\mathcal{PH})_\alpha^{\otimes r} \rightarrow \mathcal{P}(\mathcal{H}^\alpha).$$

The set $\mathcal{S}(\mathcal{H}^\alpha)$ of mixed non-entangled $\alpha$-states is the convex hull of the range of the $\alpha$-Segre map:

$$\mathcal{S}(\mathcal{H}^\alpha) = \text{conv} \{ \text{Seg}^\alpha (\mathcal{P}(\mathcal{H})_\alpha^{\otimes r}) \},$$

and mixed entangled $\alpha$-states are exactly the members of

$$\mathcal{D}(\mathcal{H}^\alpha) \setminus \mathcal{S}(\mathcal{H}^\alpha).$$

Let us observe that, for unit vectors $x_1, \ldots, x_r$,

$$\|\pi^\alpha(x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(k)})\|_2^2 : \rho_{\pi^\alpha(x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(k)})} = \pi^\alpha \circ (\rho_{x_{\alpha(1)}} \otimes \cdots \otimes \rho_{x_{\alpha(k)}}) \circ \pi^\alpha,$$
as operators on $H^\otimes k$. Indeed, since $\pi^\alpha$ is an orthogonal projection, the left-hand side, applied to $y$, equals
\[
|\pi^\alpha(x_\alpha^{(1)} \otimes \cdots \otimes x_\alpha^{(k)})|\langle y | (\pi^\alpha(x_\alpha^{(1)} \otimes \cdots \otimes x_\alpha^{(k)}) |\pi^\alpha(y)|\rangle = (\pi^\alpha \circ (\rho^\alpha_{x_\alpha^{(1)}} \otimes \cdots \otimes \rho^\alpha_{x_\alpha^{(k)}}) \circ \pi^\alpha)(y).
\]
This suggests to look at the map (the big $\alpha$-Segre map)
\[
\tilde{\text{Seg}}^\alpha : (u^*(H))^r \rightarrow u^*(H^\alpha) \subset u^*(H^\otimes k),
\]
where $u^*(H)$ denotes the (real) vector space of self-adjoint operators on $H$, defined by
\[
\tilde{\text{Seg}}^\alpha(u_1, \ldots, u_r) = \pi^\alpha \circ (h_\alpha^{(1)} \otimes \cdots \otimes h_\alpha^{(k)}) \circ \pi^\alpha.
\]
The big $\alpha$-Segre map is a natural generalization of the map (7). We postpone a closer study of the big Segre maps to a separate paper.

8. Conclusions

The presented geometric description, in terms of Segre maps, of entanglement properties for systems with arbitrary statistics parallels our previous algebraic approach to such systems based on the concept of the S-rank of a tensor. It puts on equal and unifying footing systems of distinguishable particles, for which both algebraic and geometric descriptions were known, and systems with indistinguishable particles. What is more, this description provides an explicit form of simple (separable) pure states for an arbitrary parastatistics and effective procedures to check the simplicity in the bosonic and the fermionic cases. Such procedures for arbitrary parastatistics are not known to us and may be the subject of forthcoming papers. Also, the problem of the decomposition of the total algebra of operators, associated with the S-rank, is an interesting and open question.

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