Introduction. Among the most original and far-reaching of the contributions made by Henri Poincaré to mathematics was his introduction of the use of topological or "qualitative" methods in the study of nonlinear problems in analysis. His starting point was the study of the differential equations of celestial mechanics, and in particular of their periodic solutions. His work on this topic began with his thesis in 1879, and was developed in detail in his great three-volume work, *Méthodes nouvelles de la mécanique céleste*, which appeared in the early 1890s and summarized his many memoirs of the intervening period. It continued until his memoir shortly before his death in 1912 in which he put forward the unproved fixed point result usually referred to as "Poincaré's last geometric theorem".

The ideas introduced by Poincaré include the use of fixed point theorems, the continuation method, and the general concept of global analysis. The writer's acquaintance with Poincaré's influence came through contact with Solomon Lefschetz and Marston Morse, both of whom were very explicit as to the role of Poincaré as an initiator in this direction of mathematical development. In 1934, in the Foreword to his Colloquium volume on *The calculus of variations in the large*, Morse had put this forward very forcefully in the first paragraph:

"For several years the research of the writer has been oriented by a conception of what might be termed macro-analysis. It seems probable to the author that many of the objectively important problems in mathematical physics, geometry, and analysis cannot be solved without radical additions to the methods of what is now strictly regarded as pure analysis."
Any problem which is nonlinear in character, which involves more than one coordinate system or more than one variable, or whose structure is initially defined in the large, is likely to require considerations of topology and group theory in order to arrive at its meaning and its solution. In the solution of such problems classical analysis will frequently appear as an instrument in the small, integrated over the whole problem with the aid of group theory or topology. Such conceptions are not due to the author. It will be sufficient to say that Henri Poincaré was among the first to have a conscious theory of macro-analysis, and of all mathematicians was doubtless the one who most effectively put such a theory into practice.”

I know off-hand of no such written testimony by Lefschetz, though his advice in conversation included the recommendation that everyone strongly interested in nonlinear problems in analysis should read through the Collected Works of Poincaré. How seriously this advice was meant is relatively difficult to determine after all this time. It always seemed to me that this was advice meant to frighten, not to encourage effort in this area, somewhat like the recommendation of André Weil that every aspiring algebraic number theorist should read through Hilbert’s *Zahl-Bericht* once a year. However, we can associate Lefschetz with the analysis offered by Liapounoff in the classical memoir on stability theory of 1892 which Lefschetz caused to be republished in the Annals of Mathematics Studies in 1947. In this work, which was translated and published in French in 1907 under the title *Problème général de la stabilité du mouvement*, Liapounoff wrote

> “L’essai unique, autant que je sache, de solution rigoureuse de la question appartient à M. Poincaré, qui dans le Mémoire remarquable sous bien des rapports *Sur les courbes définies par les équations différentielles* et, en particulier, dans les deux dernières Parties, considère des questions de stabilité relatives au cas d’équations différentielles du second ordre et s’arrête aussi à quelques questions voisines, se rapportant à des systèmes du troisième ordre.

Bien que M. Poincaré se borne à des cas très particuliers, les méthodes dont il se sert permettent des applications beaucoup plus générales et peuvent encore conduire à beaucoup de nouveaux résultats. C’est ce qu’on verra par ce qui va suivre, car, dans une grande partie des mes recherches, je me suis guidé par les idées développés dans le Mémoire cité.”

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Let me conclude this kind of textual testimony by one more quotation, this time from Jurgen Moser's Annals of Mathematics Study, *Stable and random motions in dynamical systems*, published in 1973. In the Introduction (page 4), he writes

"However, the mathematical difficulties connected with this field, inspired more and more the study of basic theoretical problems leading to the development of new mathematical tools. The main influence in this direction came from Poincaré who started a number of new lines of thought, like the qualitative theory of differential equations, the quest for the topology of the energy manifold, he formulated and proved fixed point theorems to establish the existence of periodic solutions. In this connection one may recall that the fixed point theorem by P. Bohl and Poincaré also had its origin in this field, to which Bohl devoted his entire mathematical life."

The topic to which the present paper is devoted, degree theory for nonlinear mappings, is one to which Poincaré made an early and important contribution. His first results appear in a note in the Comptes Rendus [32] and a more detailed development [33] published in the Bulletin Astronomique, both in 1883 only four years after his Thesis. In this pair of papers, Poincaré announces the following result (in my translation): "Let $\xi_1, \xi_2, \ldots, \xi_n$ be $n$ continuous functions of $n$ variables $x_1, x_2, \ldots, x_n$; the variable $x_i$ is subjected to vary between the limits $+a_i$ and $-a_i$. Let us suppose that for $x_i = a_i$, $\xi_i$ is constantly positive, and that for $x_i = -a_i$ constantly negative; I say that there will exist a system of values of $x$ for which all the $\xi$’s vanish."

For the proof, he refers to the celebrated paper by Kronecker in 1869 [25] which began the theory of the topological degree of a mapping. (For a detailed analysis of this paper and of the subsequent literature on the Kronecker index, see the recent historical paper by Siegberg [35]. The reference to the 1883 papers by Poincaré, I owe to Mahwin [27]. Most accounts of Poincaré's role begin with his paper [34] in 1886.) No account of the proof is given in the 1883 papers, but in 1886, Poincaré [34] published the argument on the continuation invariance of the index which is the basis for the proof. This is the memoir cited by Liapounoff, and like the 1883 announcements, applies what amounts to a fixed point argument to prove the existence of periodic solutions of a system of ordinary differential equations.

We can describe the basic idea in the following general terms: Suppose we are given a system of differential equations

$$\frac{du}{dt}(t) = T(u(t))$$
where the right-hand side is independent of time $t$. Suppose this system has the property that for each initial value $u_0$, the system has one and only one solution defined over an interval $[0, T]$. Then the problem of finding a periodic solution $u(t)$ of period $T$ is equivalent to finding an initial value $u_0$ such that the solution $u(t)$ with that given initial value has $u(T) = u_0$. If we set

$$S(t)u_0 = u(t),$$

this means finding a fixed point of the mapping $S(T)$.

The continuation method, a favorite technique of Poincaré, consists of imbedding the problem in a one-parameter family of problems depending upon an auxiliary parameter $s$ and considering the solvability of the problem as $s$ varies. The link between these two devices is the index or topological degree, which assures under appropriate hypotheses that the solvability of a family of problems is invariant under continuous perturbations. This continuity or homotopy invariance is the decisive property of the topological degree of mapping. Thus, in the 1880s, Poincaré was effectively using the basic properties of the topological degree, usually associated with L. E. J. Brouwer [7] in 1912. The result stated by Poincaré has come to be known as the theorem of Miranda because it was proved by C. Miranda [28] in 1940, who showed that it was equivalent to the Brouwer fixed point theorem. (It had earlier been applied by Michal Golomb [21] in 1935, who observed that its proof was implicit in another paper [6] by Brouwer in 1911.)

In the present paper, we give a simple exposition of the classical theory of the degree of a mapping as given by Kronecker and Brouwer, and of its extension by Leray and Schauder [26] in 1934 to mappings in infinite-dimensional Banach spaces of the form $I - g$, with $g$ compact. We discuss in detail the existence and uniqueness of these degrees as defined by the additional properties of additivity, homotopy invariance, and normalization. We also present a self-contained exposition of the recent extension given by the writer [11–14] of these existence and uniqueness results for the degree functions for nonlinear mappings of monotone type from a reflexive Banach space $X$ to its conjugate space $X^*$. We show how such mappings arise from the combination of the ideas of fixed point theory and the somewhat different circle of ideas associated with the direct method of the calculus of variations. The concept of degree of mapping in all these forms is one of the most effective tools for studying the properties of existence and multiplicity of solutions of nonlinear equations.

1. Degree theory in the finite-dimensional case. In the present section, we wish to develop as intuitively as possible, the fundamental properties of the classical topological degree as they were explicitly formulated by Brouwer in 1912 (and implicitly used by Poincaré, Bohl, and others (e.g. Picard [31]) on the basis of Kronecker's earlier results about the index)

Let us begin with some conventions on notation. We shall consider mappings with domain in a topological space $X$ and with values lying in a topological space
If $G$ is an open subset of $X$, we denote by $\text{cl}(G)$ its closure in $X$, and by $\partial G$, its boundary in $X$. Thus $f: \text{cl}(G) \to Y$ will be the prototype of the maps for which a degree function is to be defined. If $f: \text{cl}(G_0) \to Y$ is such a map and $G$ is an open subset of $G_0$, then by $f_G$ we mean the restriction of the given map $f$ to $\text{cl}(G)$.

**Theorem 1.** Let $X = \mathbb{R}^n = Y$ for a given positive integer $n$. For bounded open subsets $G$ of $X$, consider continuous mappings $f: \text{cl}(G) \to Y$, and points $y_0$ in $Y$ such that $y_0$ does not lie in $f(\partial G)$. Then to each such triple $(f, G, y_0)$, there corresponds an integer $d(f, G, y_0)$ having the following properties:

(a) If $d(f, G, y_0) \neq 0$, then $y_0 \in f(G)$. If $f_0$ is the identity map of $X$ onto $Y$, then for every bounded open set $G$ and $y_0 \in G$, we have 
$$d(f_0, G, y_0) = +1.$$ 

(b) (Additivity). If $f: \text{cl}(G) \to Y$ is a continuous map with $G$ a bounded open set in $X$, and $G_1$ and $G_2$ are a pair of disjoint open subsets of $G$ such that $y_0 \notin f(\text{cl}(G) \setminus (G_1 \cup G_2))$ then 
$$d(f, G, y_0) = d(f_1, G_1, y_0) + d(f_2, G_2, y_0).$$ 

(c) (Invariance under homotopy). Let $G$ be a bounded open set in $X$, and consider a continuous homotopy $\{f_t: 0 \leq t \leq 1\}$ of maps of $\text{cl}(G)$ into $Y$. Let $\{y_t: 0 \leq t \leq 1\}$ be a continuous curve in $Y$ such that $y_t \notin f_t(\partial G)$ for any $t$ in $[0, 1]$. Then $d(f_t, G, y_t)$ is constant in $t$ on $[0, 1]$.

**Theorem 2.** The degree function $d(f, G, y_0)$ is uniquely determined by the three conditions of Theorem 1.

Theorem 1 is an appropriately formalized version of the properties of the classical Brouwer degree. Theorem 2 contains an observation made independently in 1972 and 1973 by Fuhrer [20] and Amann and Weiss [2], respectively.

What is this degree function which we describe in Theorems 1 and 2? Intuitively, it is intended to be an algebraic count of the number of solutions $x$ in $G$ for the equation $f(x) = y_0$. We speak of algebraic count because (as we specify more concretely in a moment) some solutions are counted positively, others negatively. The paradigmatic case is that of a mapping $f$ of class $C^1$ with only regular points $x$ as solutions of the equation $f(x) = y_0$, i.e. at each solution $f'(x)$ is a nonsingular linear transformation of $\mathbb{R}^n$. The number of such solutions is then finite, and we describe such solutions as positive if $f'(x)$ preserves orientation and negative if it reverses orientation. Then $d(f, G, y_0)$ equals the number of positive solutions minus the number of negative solutions. Thus defined, the degree function for such mappings is certainly at least an integer. It is also clearly additive and satisfies the normalization condition. The main problem is to show that is extendable to all mappings and is homotopy invariant.

One way of establishing these facts is to identify $d(f, G, y_0)$ for such maps with an integral of the type studied by Kronecker. To do this, let us assume first that $G$
has a \( C^1 \) boundary \( S = \partial G \) which we orient in an appropriate way so as to satisfy the conditions for Stokes theorem:

\[
\int_G d\alpha = \int_S \alpha
\]

for any \( C^1 \) \((n-1)\)-form \( \alpha \) on \( \text{cl}(G) \). We shall describe an \((n-1)\)-form \( \omega \) on \( R^n \setminus \{0\} \) in terms of the fundamental solution \( e_n(r) \) of radial type for the Laplacian on \( R^n \). Thus for \( n > 2 \), \( e_n(r) = c_n r^{2-n} \), while for \( n = 2 \), \( e_2(r) = c_2 \ln(1/r) \). Thus, setting \( \Delta = \sum_{j=1}^n \left( \partial / \partial x_j \right)^2 \), we have

\[
\Delta(e_n) = \delta,
\]

where \( \delta \) is the Dirac delta function at 0. We set

\[
\omega = \sum_{j=1}^n (-1)^j \frac{\partial}{\partial x_j}(e_n(r)) \, dx_j
\]

with \( r = (\sum_{j=1}^n x_j^2)^{1/2} \) and \( dx_j = dx_1 \cdots dx'_{j-1} \cdot dx_{j+1} \cdots dx_n \). Then

\[
\int_{\partial G} \omega = \begin{cases} 
+1 & \text{if } 0 \in G, \\
0 & \text{if } 0 \notin \text{cl}(G).
\end{cases}
\]

Using the form \( f_\ast(\omega) \) on \( S \) induced by the map \( f \) of \( S \) into \( R^n \), we assert that for maps \( f \) having only regular points in \( f^{-1}(0) \)

\[
d(f, G, 0) = \int_{\partial G} f_\ast(\omega).
\]

We need only observe that on any domain \( G_0 \) for which \( f \) has no zeroes,

\[
df_\ast(\omega) = f_\ast(d\omega) = 0.
\]

Hence

\[
\int_{\partial G_0} f_\ast(\omega) = 0
\]

for such \( G_0 \). We can take \( G_0 \) to be the complement in \( G \) of the union of a family of small balls around the various points of the finite set \( \{x_1, \ldots, x_r\} = f^{-1}(0) \). As the balls are taken smaller and smaller (if \( B_j \) is the ball around the \( j \)th point \( x_j \))

\[
\int_G f_\ast(\omega) = \sum_{j=1}^r \int_{\partial B_j} f_\ast(\omega)
\]

while

\[
\int_{\partial B_j} f_\ast(\omega) - \int_{\partial B_j} L_j f_\ast(\omega) \to 0
\]

where \( L_j \) is the linear map \( f'(x_j) \). Finally,

\[
\int_{\partial B_j} L_j f_\ast(\omega) = \text{sgn}(\det(L_j)).
\]

If we combine these facts, we identify \( d(f, G, 0) \) with the corresponding integral.

A similar representation holds for \( d(f, G, y_0) \) if we note that \( d(f, g, y_0) = d(f^{(y_0)}, G, 0) \), where \( f^{(y_0)}(x) = f(x) - y_0 \) for all \( x \) in \( \text{cl}(G) \).
The point of identifying \( d(f, G, y_0) \) with an integral is that the integral makes sense for any \( C^1 \) mapping \( f \), not just for the ones with regular points in \( f^{-1}(y_0) \), provided only that \( f(S) \) does not contain \( y_0 \). Moreover, it is continuous under \( C^1 \) deformations of the mapping \( f \), under the sole restriction that during the deformation \( y_0 \) never appears in \( f(\partial G) \).

We now apply the theorem of Sard-Morse in the case of maps of \( R^n \) into \( R^n \) to approximate \( f \) in the \( C^1 \) topology by a map \( g \) having only regular points in its zero set with \( \| f - g \|_{C^1(\text{cl}(G))} < \varepsilon \). It follows that for any \( f \) and the corresponding \( g \), the integrals for \( f \) and \( g \) are very close. Since \( g \) is one of the paradigmatic maps we were treating earlier, \( d(g, G, 0) \) is an integer. Hence, the integral for \( f \) defining \( d(f, G, 0) \) being arbitrarily close to an integer, must itself be an integer. However, an integer-valued function of a parameter which is continuous in that parameter must be constant. Hence \( d(f, G, y_0) \) is constant under any \( C^1 \) deformation during which it remains defined.

Thus, we have a degree function \( d(f, G, y_0) \) defined for \( f \) of class \( C^1 \) and \( G \) having a \( C^1 \) boundary. We now proceed to remove these restrictions on the smoothness of \( f \) and \( G \).

Let \( G \) be any bounded open subset of \( R^n \) and consider a continuous map \( f: \text{cl}(G) \to R^n \) which is of class \( C^2 \) in \( G \) and such that \( 0 \notin f(\partial G) \). Then we can find \( \gamma > 0 \) such that on the \( \gamma \)-neighborhood of \( \partial G \),

\[
|f(x)| > \gamma.
\]

We consider a \( C^1 \) function \( \varphi(r) \) which is 1 for \( r > \gamma \), 0 for \( r < \gamma/2 \). Set

\[
\tilde{\omega} = \varphi(r)\omega.
\]

For any open subset \( G_x \) of \( G \) which has a \( C^1 \) boundary and contains the complement in \( G \) of the \( \gamma \)-neighborhood of \( \partial G \), the boundary \( G_x \) is contained in the \( \gamma \)-neighborhood of \( \partial G \). Hence

\[
d(f_{G_x}, G_x, 0) = \int_{\partial G_x} f_*(\omega) = \int_{\partial G_x} f_*(\tilde{\omega}).
\]

However, the form \( f_*(\tilde{\omega}) \) is of class \( C^1 \) in \( \text{cl}(G_x) \). Hence

\[
\int_{\partial G_x} f_*(\tilde{\omega}) = \int_{G_x} df_*(\tilde{\omega}) = \int_{G_x} f_*(d\tilde{\omega}) = \int_{G_x} f_*(d\varphi \wedge \omega).
\]

Since

\[
f_*(d\varphi \wedge \omega) = 0
\]

on the \( \gamma \)-neighborhood of \( \partial G \), we see that the resulting integral is independent of the choice of \( G_x \) and can be expressed as an integral over \( G \setminus N_\gamma(\partial G) \), with \( N_\gamma(\partial G) \) the \( \gamma \)-neighborhood.

Finally, suppose \( f: \text{cl}(G) \to R^n \) is merely continuous. Then we can approximate it arbitrarily closely in the \( C^0 \)-norm by \( C^2 \) maps \( g \). If \( h \) is another such approximation, then so is

\[
g_t = (1 - t)g + th
\]
for any \( t \) in \([0,1]\), and for any \( t \), \( g_t(x) \neq 0 \) for \( x \) in \( \partial G \). Hence

\[
\text{d}(g_t, G, 0)
\]

is independent of \( t \) in \([0,1]\) since \( \{g_t\} \) is a \( C^1 \) homotopy of \( C^1 \) maps. We define

\[
\text{d}(f, G, 0) = d(g, G, 0),
\]

and more generally, \( d(f, G, y_0) = d(g, G, y_0) \) for such approximations \( g \).

The degree function thus defined is immediately seen to satisfy the three conditions of Theorem 1.

For the normalization conditions, this follows immediately for the maps with only regular solutions of \( f(x) = y_0 \) by the definition of the degree function for such maps. (Moreover, the identity map is immediately seen to have the appropriate value for its degree function.) For more general maps \( f \), our process of definition yields a map \( g \) with only regular solutions such that \( g \) is close to \( f \) in the \( C^0 \)-norm and

\[
\text{d}(f, G, y_0) = d(g, G, y_0).
\]

Hence, if \( \|f - g\|_{C^0} < \varepsilon \), we see that \( g \) must have \( y_0 \)-points in \( G \), so that there exist points in \( \text{cl}(G) \) such that \( \|f(x) - y_0\| < \varepsilon \) for any \( \varepsilon > 0 \). Since \( f(\text{cl}(G)) \) is closed, it follows that \( y_0 \in f(\text{cl}(G)) \). Since \( y_0 \) does not lie in \( f(\partial G) \) by assumption, \( y_0 \) must belong to \( f(G) \). Additivity on domain follows immediately by the corresponding additivity for maps with regular \( y_0 \)-points. Finally, given a continuous homotopy \( \{f_t\} \) in the \( C^0 \) topology and a continuous curve \( \{y_t\} \) in \( \mathbb{R}^n \), we can consider \( \{f_t - y_t\} \), and show that \( d(f_t, G, 0) \) is constant after the modification of the \( f_t \). We can find \( \gamma > 0 \) such that \( |f_t(x)| > \gamma \) for \( x \) on \( \partial G \). We now choose a sequence \( 0 = t_0 < t_1 < \cdots < t_r = 1 \) such that \( \|f_{t_j} - f_{t_{j+1}}\|_{C^0} < \gamma/2 \). For each \( j \), choose a \( C^1 \)-map \( g_j : \text{cl}(G) \to \mathbb{R}^n \) such that

\[
\|f_{t_j} - g_j\|_{C^0(\text{cl}(G))} < \gamma/2.
\]

Then for each \( j \), we have

\[
\text{d}(f_{t_j}, G, 0) = d(g_j, G, 0),
\]

and it suffices to show that for each \( j \),

\[
\text{d}(g_j, G, 0) = d(g_{j+1}, G, 0).
\]

We consider the linear homotopy

\[
h_t = (1 - t)g_j + tg_{j+1}
\]

between \( g_j \) and \( g_{j+1} \). This homotopy is of class \( C^1 \), so that to obtain the constancy of the degree under the homotopy, it suffices to show that for all \( x \) in \( \partial G \) and \( t \) in \([0,1]\), \( h_t(x) \neq 0 \). Both points \( g_j(x) \) and \( g_{j+1}(x) \) are contained in the open ball of radius \( \gamma \) about the point \( f_t(x) \), and hence so is their convex linear combination \( h_t(x) \). Since this ball does not contain \( 0 \), \( h_t(x) \neq 0 \).

Remark. Analytical proofs for the existence and properties of the degree function have been given a number of times in the literature in the last forty years, for the first time explicitly by Nagumo in 1941. A recent version with bibliographical references is that of Fenske [19].
We now observe that the argument given above, together with one further observation, yields the proof of Theorem 2. Suppose that we are given a degree function $d_x(f, G, y_0)$ which satisfies the three conditions of Theorem 1. We wish to identify it with the degree function $d_x$ already constructed. We may approximate as closely as we please in the $C^0$-norm by a map $g$ of class $C^1$ which has only finitely many points $\{x_1, \ldots, x_r\}$ in $G$ at which $f(x) = y_0$, and all of these points are regular points. Since $f$ and $g$ are close, the linear homotopy

$$h_t = (1 - t)f + tg$$

is permissible in that it produces no points $x$ on $\partial G$ such that $h_t(x) = y_0$ for any $t$ in $[0, 1]$. By homotopy invariance of the degrees,

$$d_i(f, G, y_0) = d_i(g, G, y_0), \quad d(f, G, y_0) = d(g, G, y_0).$$

Hence, we need only show the equality of the two degrees on such mappings $g$. Moreover, using the additivity on domain, we can replace $G$ by the union of small balls $B_j$ around the various points $x_j$. It thus suffices to take $G = B$ where $B$ is the ball of radius $\epsilon$ about the point $x_0$ which is the unique solution of $g(x) = y_0$ in $B$. If $\epsilon > 0$ is sufficiently small and if $L$ is the derivative of $g$ at $x_0$, the homotopy between $g$ and $g_0$ with $g_0(x) = y_0 + L(x - x_0)$ given by

$$k_t = (1 - t)g + tg_0$$

has the property that it produces no $y_0$ points on $\partial B$. Hence, we need only verify that

$$d(g_0, B, y_0) = d_1(g_0, B, y_0).$$

Furthermore, if we deform $L$ through the nonsingular linear mappings of $R^n$, both degrees remain constant. Since the nonsingular linear mappings of $L$ have two path components, the maps with positive determinant and the maps with negative determinant, it suffices to consider a single map in each class. For the mappings with positive determinant (the orientation-preserving maps), we choose the identity map $I$, and the equality of the two degrees follows from the normalization condition. In the other case, we choose the linear mapping $L$ given by

$$Lx_1 = -x_1, \quad Lx_j = x_j \quad \text{for } j \geq 2.$$

To handle this last case, let us observe that it really amounts to considering the same problem in the one-dimensional case, $n = 1$. Indeed, let $B_{n-1}$ be the unit ball in $R^{n-1}$ and for any mapping $f_1: \text{cl}(G_1) \rightarrow R^1$ with $G_1$ a bounded open subset in $R^1$, let us define

$$G_2 = x_0 + G_1 \times B_{n-1}, \quad f_2(x) = y_0 + h(x), \quad h(x) = (f_1(x_1), x_2, \ldots, x_n).$$

We define two degrees $d$ and $d_2$ on the one-dimensional maps by setting

$$d(f_1, G_1, u) = d(f_2, G_2, y_0 + u), \quad d_2(f_1, G_1, u) = d_2(f_2, G_2, y_0 + u)$$

where $u$ is considered as a vector $(u, 0, \ldots, 0)$. The degree function $d$ is the conventional one on $R^1$; we must show it identical to the degree function $d_2$. 

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defined in terms of $d_1$. For $n = 1$, the problem is best resolved by the following picture:

```
-3  -1  +1
```

The function $h$ whose graph is pictured on the interval $[-3, +1]$ has two zeros at 0 and at $(-2)$. Its total degree over $G_0 = (-3, +1)$ with respect to 0 must be 0, since it is linearly homotopic to the constant function $h_0(x) \equiv 1$ without zeroes on the boundary. Since

$$d_2(h, G_0, 0) = d_2(h, G_1, 0) + d_2(h, G_2, 0)$$

where $G_1 = (-3, -1)$ and $G_2 = (-1, +1)$ and since $d_2(h, G_2, 0) = +1$, $d_2(h, G, 0) = 0$, we have

$$d_2(h, G_1, 0) = -1$$

which verifies our assertion for $L$.

This completes the proof of Theorem 2.

2. General degree theories and their elementary properties. Using the result of §1 as a model, we may formulate the problem of constructing more general degree theories in the following terms:

**Definition 1.** We are given a domain space $X$ and a range space $Y$, both topological spaces.

We are given a class $O$ of open subsets $G$ of $X$.

For each $G$ in $O$, we consider a family of maps $f$: $\text{cl}(G) \to Y$; the collection of all such maps for the various $G$ of $O$, we call $F$, the family of maps over which the degree theory is to be defined.

For each $G$ in $O$, we consider a family of homotopies $\{f(t): 0 < t < 1\}$ of maps in $F$, all having the common domain $\text{cl}(G)$; the total collection of all such homotopies for the various $G$ in $O$, we call $H$, the class of permissible homotopies for the degree theory.

Then by a degree theory over the class $F$ which is invariant with respect to the homotopies of $H$ and which is normalized by a given map $f_0$ of $X$ into $Y$, we mean: For each $y_0$ in $Y$ and for each $f$ in $F$, $f$: $\text{cl}(G) \to Y$ for which $y_0 \not\in f(\partial G)$, an integer $d(f, G, y_0)$ is to be prescribed and the prescription is to satisfy the following three conditions:

(a) (Normalization) If $d(f, G, y_0) \neq 0$, then $y_0 \in f(G)$. For each $G$ in $O$, $f_{0,G}$ lies in $F$, and if $y_0 \in f_0(G)$, then $d(f_{0,G}, G, y_0) = +1$. 

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(b) (Additivity on domain) Suppose that \( f \in F; f: \text{cl}(G) \to Y \), and that \( G_1 \) and \( G_2 \) are a pair of disjoint subsets of \( O \) contained in \( G \). Let \( y_0 \notin f(\text{cl}(G) \setminus (G_1 \cup G_2)) \). Then
\[
d(f, G, y_0) = d(f_{G_1}, G_1, y_0) + d(f_{G_2}, G_2, y_0).
\]

(c) (Invariance under homotopy) If \( \{f_t; 0 \leq t \leq 1\} \) is a homotopy in \( H \) with domain \( \text{cl}(G) \) for \( G \) in \( O \), and if \( \{y_t; 0 \leq t \leq 1\} \) is a continuous curve in \( Y \), with \( y_t \notin f_t(\partial G) \) for all \( t \) in \( [0, 1] \), then \( d(f_t, G, y_t) \) is constant in \( t \) on \([0, 1] \).

The existence of a degree theory for a given class of mappings is not a trivial assertion. Let us verify this fact through several elementary observations.

**PROPOSITION 1.** Let \( X \) and \( Y \) be equal to the same Hilbert space \( H \), let \( O \) be a class of open subsets of \( H \) which includes open balls, and consider a class of mappings \( F \) and a class of homotopies \( H \) which includes the affine homotopies in \( F \) (i.e. homotopies of the form \( f_t = (1-t)f + tf_t \) for \( f \) and \( f_t \) in \( F \)). Suppose a degree theory exists for \( F \) invariant with respect to \( H \) and normalized by \( f_0 = I \). Then

For any \( f \in F \) with domain a closed ball \( B \) about the origin in \( H \) such that \((f(u), u) > 0 \) for all \( u \) in \( B \), \( f \) must have a zero in \( B \).

**PROOF OF PROPOSITION 1.** Let \( G \) be the interior of \( B \). \( G \) lies in \( O \) and by hypothesis, \( f \) lies in the class \( F \). If \( f \) has a zero on \( \partial B \), the assertion follows. Otherwise, \( d(f, G, 0) \) is well defined. Let \( f_0 \) be the restriction of the identity map \( I \) to \( B \). Then \( f_0 \) lies in \( F \) and \( d(f_0, G, 0) = +1 \). Consider the affine homotopy,
\[
f_t = (1-t)f_0 + tf \quad (0 \leq t \leq 1).
\]
By assumption, this homotopy lies in the class \( H \) of permissible homotopies. Hence, if we can show that for all \( t \) in \( (0, 1) \), \( f_t(x) \neq 0 \) on \( \partial B \), then \( d(f_t, G, 0) \) is constant in \( t \) and hence equals \(+1\) for all \( t \) in \([0, 1] \). Suppose \( f_t(x) = 0 \) for \( 0 < t < 1 \), with \( x \) in \( \partial B \). Then
\[
0 = (f_t(x), x) = (1-t)(f_0(x), x) + t(f(x), x).
\]
Since \((f(x), x) > 0\) for all \( x \) in \( \partial G \), it follows that
\[
0 \geq (1-t)(f_0(x), x) = (1-t)\|x\|^2 > 0,
\]
which is impossible. Hence \( d(f, G, 0) = +1 \), and by the normalization property of the degree function, \( 0 \in f(G) \). Q.E.D.

A corollary of Proposition 1 is the following:

**PROPOSITION 2.** Suppose that a degree theory satisfying the hypotheses of Proposition 1 exists over a class of mappings \( F \), and that for a ball \( B \) about the origin, \( g: B \to B \) is a mapping such that \( I - g \) lies in \( F \). Then \( g \) has a fixed point in \( B \).

**PROOF OF PROPOSITION 2.** Since \( g \) maps \( B \) into \( B \), it follows that for \( x \) on \( \partial B \),
\[
\|g(x)\| \leq \|x\|.
\]
Hence for \( f = I - g \),
\[
(f(x), x) = \|x\|^2 - (g(x), x) \geq \|x\|^2 - \|g(x)\| \cdot \|x\| > 0.
\]
Applying Proposition 1, f must have a zero in B, i.e. g must have a fixed point in B. Q.E.D.

We draw two conclusions from Proposition 2. First, because we have a degree theory for all continuous mappings on the finite-dimensional space $H = \mathbb{R}^n$, all continuous self-mappings of balls in that space have a fixed point, i.e. the Brouwer fixed point theorem. The second conclusion is that for an infinite-dimensional Hilbert space $H$, we cannot have a degree theory in the sense of Proposition 1 over all continuous mappings since the only case in which all continuous self-maps of the unit ball of a Banach space have a fixed point is that in which the space is finite dimensional.

A variant of Proposition 1 gives us the Poincaré-Miranda theorem stated in the Introduction.

**Proposition 3.** Let $X = Y$ be a Banach space, and suppose that for a class of mappings $F$ and a class of homotopies $H$, we have a degree function defined on $F$, invariant under $H$, and normalized by the identity mapping, where $H$ includes all affine homotopies in $F$. Let $G$ be a set in $O$, with $0 \in G$, and suppose that $f$: $\text{cl}(G) \rightarrow X$ lies in $F$. Suppose that for each $x$ in $\partial G$, there exists a linear functional $w_x$ in $X^*$ such that $\langle w, f(x) \rangle \geq 0$, $\langle w, x \rangle > 0$.

Then $f$ has a zero in $\text{cl}(G)$.

**Proof of Proposition 3.** Again, we may assume that $0 \notin f(\partial G)$. We construct the homotopy $f_t = (1 - t)I + tf$: $\text{cl}(G) \rightarrow X$. By our hypotheses, this homotopy lies in the class $H$. For $t = 0$, $d(f_0, G, 0) = +1$ and hence to show that $d(f, G, 0) = +1$, it suffices to show that $d(f_t, G, 0)$ is constant in $t$ on $[0, 1]$. This will follow from the property of invariance under homotopy if we can verify that for each $t$ in $(0, 1)$ and all $x$ in $\partial G$, $f_t(x) \neq 0$.

Suppose, however, that for a given $x$ in $\partial G$ and some $t$ in $(0, 1)$, we have $f_t(x) = 0$. Then

$$0 = \langle w_x, f_t(x) \rangle = (1 - t)\langle w_x, x \rangle + t \langle w_x, f(x) \rangle \geq (1 - t)\langle w_x, x \rangle > 0,$$

which is a contradiction. Q.E.D.

**Corollary to Proposition 3.** Let $G$ be the cube $\{x: -a_j < x_j < +a_j, j = 1, \ldots, n\}$ in $\mathbb{R}^n$, and let $f$ be a mapping of $\text{cl}(G)$ into $\mathbb{R}^n$ such that on the face $C_j^+ = \{x_j = a_j\}$, $f_j(x) > 0$, and on the face $C_j^- = \{x_j = -a_j\}$, $f_j(x) \leq 0$. Then $f$ has a zero in $\text{cl}(G)$.

**Proof of the Corollary.** The boundary of $G$ consists of the union of the faces $C_j^+$ and $C_j^-$ for various $j$. We consider $\mathbb{R}^n$ as a Hilbert space, and on the faces $C_j^+$ and $C_j^-$, we choose $w_x$ to be the element which is 0 except in the $j$th place, and $x_j$ in the $j$th place. Then $\langle w_x, f(x) \rangle = x_jf_j(x) > 0$, while $\langle w_x, x \rangle = x_j^2 = a_j^2 > 0$. (On the intersections we use whichever of the vectors $w_x$ we please). Hence the hypotheses of Proposition 3 are satisfied, and $f$ has a zero. Q.E.D.

The classical example of an extension of the degree theory for the finite-dimensional case is the Leray-Schauder degree theory which is defined for the
case in which $X = Y$ is an arbitrary Banach space, $0$ is the class of bounded open subsets of $X$, $F$ is the class of continuous maps $f: \text{cl}(G) \to X$ with $(I - f)(\text{cl}(G))$ relatively compact in $X$. Here the class of homotopies $\{f_t; 0 \leq t \leq 1\}$ is restricted by the assumption that there exists a fixed compact set $K$ in $X$ such that $(I - f_t)(\text{cl}(G)) \subseteq K$ for all $t$ in $[0, 1]$.

A possible natural extension of the finite-dimensional and the Leray-Schauder degree theories might be a degree theory for proper Fredholm mappings of index zero. Such a theory in the sense of Definition 1 cannot exist.

**Proposition 4.** There can be no degree theory in the sense of Definition 1 for proper Fredholm mappings of index zero of class $C^\infty$ in any infinite-dimensional Hilbert space.

**Proof of Proposition 4.** The problem, of course, is that all the nonsingular linear mappings of an infinite-dimensional Hilbert space form a single path component (in fact are contractible to a point by a theorem of Kuiper). Suppose that we had a degree theory of the type described. Then given any finite-dimensional subspace $R^n$ of the Hilbert space, we could suspend all the differentiable maps on $R^n$ to get Fredholm maps of index zero. The resulting degree theory for maps on $R^n$ obtained by taking the degree of their suspension must be the same as the original degree in $R^n$, since the uniqueness argument actually works for differentiable mappings of any class. If $L_0$ is a nonsingular linear mapping of $R^n$, we would then have

$$d(L, B_n, 0) = d(S(L_0), B, 0) = d(I, B, 0) = +1,$$

since there is a path of nonsingular linear maps leading from any given one to $I$. This contradicts the fact that for suitably chosen $L_0$, $d(L_0, B_n, 0) = -1$. Q.E.D.

The proof indicates clearly why the appropriate degree theory over the class of all proper Fredholm operators of index zero must have values in $\mathbb{Z}_2$. (For details of such theories, we refer to the article [18].)

3. **The Leray-Schauder theory.** The most celebrated form of degree theory for application to nonlinear problems in partial differential equations has been that introduced by Leray and Schauder [26] in 1934 for mappings $f$ in infinite-dimensional Banach spaces such that $I - f$ is compact. We shall give a reasonably complete exposition of this theory based upon the results already derived for the finite-dimensional degree.

The basic general result is the following:

**Theorem 3.** Let $X$ be a Banach space, and consider the family $F$ of continuous mappings $f: \text{cl}(G) \to X$, with $G$ a bounded open subset of $X$, and with $(I - f)(\text{cl}(G))$ relatively compact in $X$. Let $H$ be the family of continuous homotopies of maps $\{f_t; 0 \leq t \leq 1\}$ in $F$ with a common domain $\text{cl}(G)$ such that there exists a compact subset $K$ of $X$ with $(I - f_t)(\text{cl}(G)) \subseteq K$ for all $t$ in $[0, 1]$. Then there exists one and only one degree function $d(f, G, y_0)$ in the sense of Definition 1 with the identity mapping $I$ as normalizing mapping.
The uniqueness holds even if the homotopies are restricted to affine homotopies and the mappings to differentiable maps of the form \( I - f \) with \( f \) having a finite-dimensional range.

The proof of Theorem 3 uses several auxiliary devices which we introduce in the following proposition.

**Proposition 5.** Let \( X \) be a finite-dimensional Banach space, \( X_0 \) a subspace of \( X \), such that \( X \) is the direct sum of \( X_0 \) and another subspace \( X_1 \). Let \( B_1 \) be the unit ball about zero in \( X_1 \), and for each bounded open subset \( G_0 \) of \( X_0 \), let \( G = G_0 \times B_1 \). Let \( f \) be a continuous map of \( \text{cl}(G_0) \) into \( X_0 \). Then we define the suspension \( S(f) \) mapping \( \text{cl}(G) \) into \( X \) by setting

\[
S(f)(x_0 + x_1) = f(x_0) + x_1, \quad (x_0 \in \text{cl}(G_0), x_1 \in B_1).
\]

Let \( y_0 \) be a point of \( X_0 \) such that \( y_0 \notin f(\partial G_0) \). Then

1. \( y_0 \notin S(f)(\partial G) \).

2. \( d_0(f, G_0, y_0) = d(S(f), G, y_0) \), where the first degree is calculated in \( X_0 \) and the second in \( X \).

**Proof of Proposition 5.** We set \( d_0(f, G_0, y_0) \) to be equal to \( d(S(f), G, y_0) \).

To do this, we first verify (1). Suppose that \( x = x_0 + x_1 \) has the property that \( S(f)(x) = y_0 \). This means that \( f(x_0) + x_1 = y_0 \) or that \( x_1 = y_0 - f(x_0) \) lies in \( X_0 \). Since \( x_1 \) also lies in \( X_1 \) and the two subspaces are complementary, \( x_1 = 0 \) and \( f(x_0) = y_0 \). Hence \( x \) lies in \( G_0 \) which does not intersect \( G \). Thus \( d(S(f), G, y_0) \) is well defined.

We now examine this new function \( f_1 \) on the class of mappings \( f \) and note that it satisfies the three conditions for an index function. By Theorem 2, it therefore coincides with the degree function \( d(f, G_0, y_0) \). Q.E.D.

**Proposition 6.** Let \( X \) be a finite-dimensional Banach space, \( X_0 \) a subspace of \( X \), \( G \) a bounded open set in \( X \) such that \( G_0 = G \cap X_0 \) is nonempty. Let \( f \) be a continuous mapping of \( \text{cl}(G) \) into \( X \), such that \( f = I - g \) and \( g(\text{cl}(G)) \subset X_0 \). Let \( y_0 \) be a point of \( X_0 \) such that \( y_0 \notin f(\partial G) \), and let \( f_0: \text{cl}(G_0) \rightarrow X_0 \) be the restriction of \( f \) to \( \text{cl}(G_0) \).

Then \( d(f, G, y_0) = d(f_0, G_0, y_0) \).

**Proof of Proposition 6.** Since \( \partial G_0 \) is a subset of \( \partial G \), both degrees are well defined since \( y_0 \notin f(\partial G) \).

Let \( X_1 \) be a subspace of \( X \) complementary to \( X_0 \), \( B_1 \) the unit ball about 0 in \( X_1 \), \( G_2 = G_0 \times B_1 \). Let \( S(f_0) \) be the suspension of \( f_0 \) in the sense defined in Proposition 5, with \( S(f_0) \) mapping \( \text{cl}(G_2) \) into \( X \) by the prescription

\[
S(f)(x_0 + x_1) = f_0(x_0) + x_1.
\]

We shall now consider explicitly for what points \( x \) in their respective domains \( f(x) = y_0 \) and \( S(f_0) = y_0 \). In the first case, we have

\[
x - g(x) = y_0
\]
so that if $x = x_0 + x_1$, $x_1 = y_0 - g(x) - x_0 \in X_0$, so that $x_1 = 0$, $x$ lies in $G_0$, and $f_0(x) = y_0$. Similarly, if $S(f_0(x) = y_0$ then

$$x_1 = y_0 - f_0(x_0) \in X_0,$$

$x_1 = 0$, and $x$ lies in $G_0$ and is a solution of $f_0(x) = 0$. Since $G_0 \subset G \cap G_2$, it follows that

$$d(S(f_0) , G_2 , y_0) = d(S(f_0) , G \cap G_2 , y_0); \quad d(f, G, y_0) = d(f, G \cap G_2 , y_0).$$

We now set up the affine homotopy

$$h_t = (1 - t)f + tS(f_0)$$
on $\text{cl}(G \cap G_2)$. If $h_t(x) = 0$ for any $t$, then we have

$$(1 - t)(x - g(x)) + t(x_1 + f_0(x_0)) = y_0,$$

which may be rewritten as

$$x_1 = y_0 + (1 - t)(g(x) - x_0) - t f_0(x_0) \in X_0.$$

Hence $x_1 = 0$, $x = x_0$, $y_0 = f_0(x)$. Thus $x_0 \in G_0$, $x \notin \partial(G_1 \cap G_2)$. Using the invariance of the degree function under homotopy, we see that

$$d(f, G \cap G_2 , y_0) = d(S(f_0) , G \cap G_2 , y_0).$$

Finally, we have

$$d(f_0, G_0 , y_0) = d(S(f_0) , G_2 , y_0) = d(S(f_0) , G \cap G_2 , y_0)$$

$$= d(f, G \cap G_2 , y_0) = d(f, G, y_0). \quad \text{Q.E.D.}$$

**Lemma.** Let $K$ be a compact subset of a Banach space. Then given any $\varepsilon > 0$, there exists a finite-dimensional subset $K_\varepsilon$ of $X$ and a continuous mapping $p$ of $K$ into $K_\varepsilon$ such that for every $x$ of $K$, $\|x - p(x)\| < \varepsilon$.

**Proof of the Lemma.** We may form the covering of $K$ by open balls of radius $\varepsilon$. We can find a finite subcovering by compactness and a partition of unity corresponding to this subcovering, i.e., a finite set of points $(x_1, \ldots, x_r)$ in $K$ and a family of continuous functions $\alpha_j : K \to [0, 1]$ such that each $\alpha_j$ has its support in the ball of radius $\varepsilon$ about $x_j$ and on $K$, $\sum_{j=1}^r \alpha_j = 1$. We set $p(x) = \sum_{j=1}^r \alpha_j(x) x_j$. Then $p(x)$ lies in the simplex spanned by the $x_j$, and for each $x$, $p(x)$ is a convex linear combination of those $x_j$ lying in the open ball of radius $\varepsilon$ about $x$. Hence $p(x)$ also lies in that ball. \text{Q.E.D.}

**Proof of Theorem 3.** Each $f$ is a proper map of $\text{cl}(G)$ into $X$. Indeed, suppose $K$ is a compact subset of $X$, and consider $x$ in $f^{-1}(K)$. Then $x \in g(\text{cl}(G)) + K \subset K_1$, where $K_1$ is a compact subset of $X$. Hence $f^{-1}(K)$ is closed in $X$ and a subset of a compact subset of $X$, and therefore is compact itself. Thus $f$ is proper, and maps closed subsets of $\text{cl}(G)$ into closed subsets of $X_0$. In particular, $f(\partial G)$ is closed in $X$ and does not contain $y_0$. Therefore there exists $r > 0$ such that $f(\partial G)$ does not intersect the ball of radius $r$ about $y_0$.

Using the preceding Lemma, for any $\varepsilon < r$, if we take the compact set $K$ which is the closure of $g(\text{cl}(G))$, we may find a continuous mapping $p$ of $K$ into a
finite-dimensional subset of $X$ such that $\|p(u) - u\| < \epsilon$ for every $u$ in $K$. We now define a new mapping

$$\bar{f} = I - \bar{g} : \text{cl}(G) \to X$$

where $\bar{g} = p \circ g$. For every $x$ in $\text{cl}(G)$, $\|f(x) - \bar{f}(x)\| < \epsilon$. In particular, $y_0 \notin \bar{f}(\partial G)$.

Take any finite-dimensional subspace $X_0$ of $X$ which contains $g_0$ and $\bar{g}(\text{cl}(G))$, and let $G_0 = G \cap X_0$. Then by Proposition 6, $d(f_{G_0}, G_0, y_0)$ is independent of the choice of the subspace. We propose to show that it is also independent of the choice of the approximation.

Take two such approximations $f_1 = I - g_1$ and $f_2 = I - g_2$ with $\|g_1(x) - g(x)\| < r$, $\|g_2(x) - g(x)\| < r$ for every $x$ in $\text{cl}(G)$ and with the values of $g_1$ and $g_2$ lying in finite-dimensional subspaces of $X$. If we set $g_t = (1 - t)g_1 + tg_2$, then each $g_t$ is another such approximation with the values of all $g_t$ lying in a fixed finite-dimensional subspace $X_0$. If we let $X_0$ also contain $y_0$, then $I - g_t$ restricted to $\text{cl}(G_0)$ with $G_0 = G \cap X_0$, is a permissible homotopy, and $d(I - g_t, G_0, y_0)$ is independent of $t$ in $[0, 1]$. It follows that $d(f_1, G_0, y_0) = d(f_2, G, y_0)$.

We define the degree function $d(f, G, y_0)$ as the common value of $d(f, G_0, y_0)$ for any of the approximating mappings and the corresponding finite-dimensional subspaces $X_0$ containing $y_0$ and $g(\text{cl}(G))$. The normalization condition and the additivity on domain follow easily by choosing the approximating mappings sufficiently close to $f$. For a permissible homotopy $f_t = I - g_t$, with $g_t(\text{cl}(G)) \subset K$ for a fixed compact set $K$ and all $t$ in $[0, 1]$, we choose $r > 0$ such that the ball of radius $r$ about $y_t$ does not meet $f_t(\partial G)$ for any $t$ in $[0, 1]$ and then apply the approximation operator $p$ for $K$ with $e < r$ to all the $g_t$. Then $\{g_t : 0 \leq t \leq 1\}$ is a continuous homotopy of mappings with values in a finite-dimensional subspace $X_0$ of $X$. If $G_0 = G \cap X_0$, and we have absorbed $y_t$ into the $g_t$, we see that $d(f_t, G_0, 0)$ is independent of $t$ in $[0, 1]$. Thus all the conditions for the degree function have been satisfied.

Suppose now that we have two degree functions $d$ and $d_1$. By the finite-dimensional argument, $d$ and $d_1$ must give the same values for suspensions of mappings of finite-dimensional subspace of $X$. Hence by the proof of Proposition 6, the two degree functions give the same result for mappings $f = I - g$, with $g(\text{cl}(G))$ lying in a finite-dimensional subspace of $X$. Since every $f$ of the form $I - g$ can be uniformly approximated by mappings of this last type and is affinely homotopic to such nearby mappings with homotopy paths on the boundary that avoid $y_0$, it follows that the two degree functions must coincide. Thus the proof of Theorem 3 is complete. Q.E.D.

4. The direct method of the calculus of variations and mappings of monotone type. Let $X$ be a Banach space, $G$ an open subset of $X$. If $\phi$ is a real-valued differentiable function on $G$, then its derivative at any point $x_0$ of $G$ is given by

$$\lim_{\epsilon \to 0} \frac{\phi(x_0 + \epsilon v) - \phi(x_0)}{\epsilon} = \langle \phi'(x_0), v \rangle$$
which depends linearly and continuously upon the direction \( v \) in \( X \) and hence is an element of the conjugate space \( X^* \) of \( X \).

We use here some of the standard notation of this kind of functional analysis. The duality between the real Banach spaces \( X^* \) and \( X \) is given by \( \langle w, x \rangle \) for \( w \) in \( X^* \) and \( x \) in \( X \). We shall use the symbol \( \rightarrow \) for strong convergence, \( \rightharpoonup \) for weak convergence.

The derivative \( \varphi' \) thus gives us a mapping from \( G \), a subset of \( X \), to \( X^* \), the conjugate space of \( X \). Thus mappings from subsets of \( X \) to \( X^* \) are a rather natural framework for any approach from functional analysis which attempts to encompass the direct method of the calculus of variations. The latter term, which goes back to the Dirichlet principle of Riemann and its rigorization by Hilbert in 1900, refers to the process of trying to get solutions of the equation \( \varphi'(x) = 0 \) by finding local maxima or minima of the functional \( \varphi(x) \). The basic assumption which is necessary for the case when the problem is imposed in an infinite-dimensional context goes back in its origins to Hilbert and involves using a convexity assumption (or some form of modified convexity) upon the functional \( \varphi \). The argument appears almost truistic in its modern form because it has been essentially absorbed into the structure of our basic functional analysis.

**Definition 2.** Let \( X \) be a Banach space, \( G \) a subset of \( X \), \( f \) a mapping of \( G \) into \( X^* \). Then

(a) \( f \) is said to be monotone if for all \( u \) and \( v \) of \( G \),

\[
\langle f(u) - f(v), u - v \rangle \geq 0.
\]

(b) \( f \) is said to be of class \((S)_{+}\) if for any sequence \( \{x_j\} \) in \( G \) which converges weakly to \( x \) in \( X \) and for which \( \lim \langle f(x_j), x_j - x \rangle \leq 0 \), we have \( x_j \rightharpoonup x \).

(c) \( f \) is said to be pseudo-monotone, if for any sequence \( \{x_j\} \) in \( G \) for which \( x_j \rightharpoonup x \) for some \( x \) in \( X \) while \( \lim \langle f(x_j), x_j - x \rangle \leq 0 \), we have \( \lim \langle f(x_j), x_j - x \rangle = 0 \), and if \( x \in G \), then \( f(x_j) \rightharpoonup f(x) \).

**Proposition 7.** (a) If \( G \) is a convex open subset of \( X \) and \( \varphi \) is a \( C^1 \) real-valued function on \( G \), then \( \varphi \) is convex on \( G \) if and only if \( f = \varphi' \) is a monotone mapping of \( G \) into \( X^* \).

(b) Suppose that \( \varphi \) is a \( C^1 \) functional on \( G \), \( f = \varphi' \). If \( f \) is bounded and pseudo-monotone, then \( \varphi \) is weakly sequentially lower semicontinuous on \( G \), i.e. if \( \{x_j\} \) is a sequence in \( G \) which converges weakly to \( x \) in \( G \), then \( \varphi(x) \leq \lim \varphi(x_j) \).

**Proof of Proposition 7.** Proof of (a). Suppose first that \( \varphi' \) is monotone. For \( u \) and \( v \) two points of \( G \) and \( s \) in \([0, 1]\), set

\[
q(s) = \varphi(su + (1 - s)v) - s\varphi(u) - (1 - s)\varphi(v).
\]

To show that \( \varphi \) is convex, we must show that for every \( u, v \) and \( s \), \( q(s) \leq 0 \). We see that \( q(0) = 0 = q(1) \). On the other hand, \( q \) is continuously differentiable and

\[
q'(s) = \langle \varphi'(v + s(u - v)), u - v \rangle + \varphi(v) - \varphi(u).
\]
Hence for $0 < s < t < 1$, we have
\[ q'(t) - q'(s) = \langle \varphi'(v_t) - \varphi'(v_s), u - v \rangle \]
where $v_t = v + t(u - v)$, $v_s = v + s(u - v)$. Since $v_t - v_s = (t - s)(u - v)$, it follows from monotonicity that
\[ q'(t) - q'(s) = (t - s)^{-1}\langle \varphi'(v_t) - \varphi'(v_s), v_t - v_s \rangle \geq 0. \]
Suppose that $\max q(s) > 0$. Then $q$ has an interior maximum at $s_0$ with $q'(s_0) = 0$. Since $q(s_0) > 0$, and $q'(t) > 0$ for all $t > s_0$, it is impossible that $q(1)$ could be zero. Hence $q(s) \leq 0$ for all $s$ in $[0, 1]$ and $q$ is a convex function.

Suppose conversely that $\varphi$ is convex. Then for $u$ and $v$ in $G$ and $0 < s < 1$, we would have
\[ \frac{\varphi(v + s(u - v)) - \varphi(v)}{s} \leq \varphi(u) - \varphi(v). \]
Letting $s \to 0^+$, we obtain
\[ \langle \varphi'(v), u - v \rangle \leq \varphi(u) - \varphi(v). \]
Interchanging $u$ and $v$, we also have
\[ \langle \varphi'(u), v - u \rangle \leq \varphi(v) - \varphi(u). \]
Adding we find that
\[ \langle \varphi'(u) - \varphi(v), u - v \rangle \geq 0, \]
i.e. $\varphi'$ is monotone.

**Proof of (b).** For each $t$ in $[0, 1]$, set $v_{j,t} = x + t(x_j - x)$. Then $v_{j,t} \to x$ as $j \to +\infty$ for each fixed $t$, while $v_{j,t} - x = t(x_j - x)$. For $t > 0$, we have therefore by the pseudo-monotonicity of $\varphi'$,
\[ \lim \langle \varphi'(v_{j,t}), x_j - x \rangle = \lim \langle \varphi'(v_{j,t}), t^{-1}(v_{j,t} - x) \rangle \geq 0. \]
For each $j$, we have
\[ \varphi(x_j) - \varphi(x) = \int_0^1 \langle \varphi'(v_{j,t}), x_j - x \rangle dt \geq \int_0^1 \{ \langle \varphi'(v_{j,t}), x_j - x \rangle \}^- dt, \]
where the integrand in the last integral is uniformly bounded and converges to 0 as $j \to +\infty$ for each $t > 0$. Hence $\lim \varphi(x_j) - \varphi(x) \geq 0$. Q.E.D.

The primary interest of the definitions for the class $(S)_+$ and for pseudo-monotonicity arises from the fact that one can verify these properties under suitable concrete hypotheses for the maps of a Sobolev space $W_0^{m,p}(\Omega)$ into its conjugate space $W^{-m,p'}(\Omega)$ obtained from an elliptic operator in generalized divergence form
\[ A(u) = \sum_{|a| \leq m} (-1)^{|a|} D^a A_a(x, u, \ldots, D^m u). \]
Here $\Omega$ is an open subset of a Euclidean space $R^n$, $n \geq 1$, and $p$ is an exponent in the reflexive range $1 < p < +\infty$. The Sobolev space $W^{m,p'}(\Omega)$ is the space of $u$ in $L^p(\Omega)$ (with respect to Lebesgue $n$-measure) with all the derivatives $D^a u$ also in
Let $L^p(\Omega)$ for all distribution derivatives of order $\leq m$. The norm of this space is obtained by injecting it into the product of $L^p$-spaces, one for each derivative, by the jet-mapping
\[ u \rightarrow \{ D^\alpha u : |\alpha| \leq m \}. \]

The closed subspace $W_0^{m,p}(\Omega)$ is obtained by taking the closure in $W^{m,p}(\Omega)$ of the testing functions with compact support in $\Omega$.

The operator $A$ given by the differential expression written above makes sense as a mapping of $W_0^{m,p}(\Omega)$ into its conjugate space $W^{-m,p}(\Omega)$, with $p' = p/(p-1)$ the conjugate exponent to $p$, provided that if we replace $u$ and its derivatives by the algebraic variable
\[ \xi = \{ \xi_\alpha : |\alpha| \leq m \}, \]
then $A_\alpha(x, \xi)$ is measurable in $x$ for fixed $\xi$ and continuous in $\xi$ for fixed $x$ in $\Omega$, and satisfies an inequality of the form
\[ |A_\alpha(x, \xi)| \leq c(|\xi|^{p-1} + k_0(x)) \]
with $k_0$ a function in $L^p(\Omega)$. Under these conditions, if we substitute for $u$ and its derivatives in the formal expression, functions in $L^p(\Omega)$, $A_\alpha(x, u, \ldots, D^m u)$ yields an element of $L^p(\Omega)$.

Further hypotheses of some sort of ellipticity and of weak semiboundedness must be imposed before this operator $A$ from $W = W_0^{m,p}(\Omega)$ to $X$ is either pseudo-monotone or of class $(S)_+$. Such conditions for example are
\[ \sum_{|\alpha| = m} A_\alpha(x, \eta, \xi) - A_\alpha(x, \eta, \xi^\#)(\xi_\alpha - \xi_\alpha^\#) > 0 \]
for $\xi \neq \xi^\#$, where $\xi$ is broken up into the $m$th order piece $\xi$ and the lower order piece $\eta$.
\[ \sum_{|\alpha| \leq m} A_\alpha(x, \xi)\xi_\alpha \geq c_0 |\xi|^{p-1} - k_1(x) \quad (k_1 \in L^1(\Omega)). \]

Under these hypotheses, $A$ is of class $(S)_+$. Under very much weaker conditions replacing the semiboundedness in the second hypothesis, $A$ remains pseudo-monotone.

If $X = X^*$ is a Hilbert space $H$, the mappings of type $(S)_+$ contain as a special case the Leray-Schauder maps $I - g$, with $g$ compact. This follows from the fact that each strongly monotone mapping $f$ is of type $(S)_+$ and the fact that the class $(S)_+$ is always invariant under compact perturbations. We say that $f$ is strongly monotone if there exists a continuous positive increasing function $p(r)$ for $r > 0$ such that
\[ \langle f(u) - f(v), u - v \rangle \geq p(\|u - v\|). \]

For the detailed development of the discussion of later sections, we need to develop the properties of our canonical monotone mapping, the duality mapping $J$, which under suitable hypotheses is also a map of class $(S)_+$. In our further
discussion, we shall always assume that the Banach space \( X \) with which we are dealing is reflexive. This is certainly true for the Sobolev spaces \( X = W_0^{m,p}(\Omega) \) which in fact are uniformly convex with duals which are also uniformly convex. It is not always true that every reflexive Banach space has an equivalent norm which is uniformly convex, and a fortiori that \( X \) is also uniformly convex. By results due to Lindenstrauss, Asplund, and Trojanski however, it is true that \( X \) can be renormed so that \( X \) and \( X^* \) are both locally uniformly convex. We shall use only one consequence of this renorming: In the resulting norms on both \( X \) and \( X^* \), if a sequence \( \{x_j\} \) converges weakly to \( x \) in \( X \) and \( \|x_j\| \to \|x\| \), then \( x_j \) converges strongly to \( x \), with a similar property for sequences \( \{w_j\} \) in \( X^* \).

**Proposition 8.** Let \( X \) be a reflexive Banach space which is normed so that both \( X \) and \( X^* \) are locally uniformly convex (and in particular are strictly convex). Then there exists a unique bicontinuous mapping \( J \) of \( X \) onto \( X^* \) which is given by the conditions that for each \( x \) of \( X \), \( \|J(x)\| = \|x\| \) and \( \langle J(x), x \rangle = \|x\|^2 \). This mapping \( J \), called the duality mapping corresponding to the given norm on \( X \), is both monotone and a mapping of class \((S)_+\).

**Proof of Proposition 8.** For each \( x \) in \( X \), it follows from the Hahn-Banach theorem that there exists \( w \) in \( X \) such that \( \|w\| = \|x\| \) and \( \langle w, x \rangle = \|w\| \cdot \|x\| \). Such elements \( w \) lie on the sphere \( \|w\| = \|x\| \) and are also characterized by the inequalities

\[
\langle w, x \rangle = \|x\|^2, \quad \|w\| \leq \|x\|.
\]

Hence they form a convex subset of a sphere in \( X^* \). If \( X^* \) is strictly convex, such an element \( w \) is unique, and this unique element is \( J(x) \) for the given \( x \).

For each \( x \) and \( u \) in \( X \), we may compute

\[
\langle J(x) - J(u), x - u \rangle = \|x\|^2 + \|u\|^2 - \langle J(x), u \rangle - \langle J(u), x \rangle
\]

\[
= (\|x\| - \|u\|)^2 + \|u\| \cdot \|J(x)\| - \langle J(x), u \rangle
\]

\[
+ \{\|J(u)\| \cdot \|x\| - \langle J(u), x \rangle \}.
\]

All the parentheses are nonnegative. Hence \( J \) is monotone. Moreover, if we replace \( u \) by \( x_j \), we obtain

\[
\langle J(x_j) - J(x), x_j - x \rangle \geq (\|x_j\| - \|x\|)^2
\]

\[
\geq \left| \langle J(x_j), x \rangle - \|J(x_j)\| \cdot \|x\| \right|
\]

\[
\geq \left| \langle J(x), x_j \rangle - \|J(x)\| \cdot \|x_j\| \right|.
\]

Suppose \( x_j \to x \). Then \( \langle J(x), x_j - x \rangle \to 0 \). Suppose moreover that

\[
\lim \langle J(x_j), x_j - x \rangle = 0.
\]

Then \( \|x_j\| \to x \), and it follows that \( x_j \to x \). Hence \( J \) lies in the class \((S)_+\).

Suppose that \( x_j \to x \). Then \( \langle J(x_j) - J(x), x_j - x \rangle \to 0 \). Hence \( \langle J(x_j), x \rangle \to \|x\|^2 \). Choose any weakly convergent subsequence \( J(x_j) \to w \). Then \( \langle w, x \rangle = \|x\|^2 \)
and \( \|w\| \leq x \). Hence \( w = J(x) \). Moreover \( \|w\| = \lim \|J(x_j)\| \) and therefore \( J(x_j) \to w = J(x) \). Thus \( J \) is continuous from \( X \) to \( X^* \). If on the other hand, \( J(x_j) \to J(x) \), it also follows that \( \langle J(x_j) - J(x), x_j - x \rangle \to 0 \), so that \( \|x_j\| \to \|x\| \), while

\[
\langle J(x), x_j \rangle \to \|x\|^2.
\]

It follows by the same argument that \( x_j \to x \). Hence \( J^{-1} \) is continuous from \( X^* \) to \( X \). Q.E.D.

5. The degree of mapping for mappings of class \((S)_{+}\) . In the present section, we devote our efforts to the proof of the following theorem.

**Theorem 4.** Let \( X \) be a reflexive Banach space, and consider the family \( F \) of maps \( f: \text{cl}(G) \to X^* \), where \( G \) is a bounded open subset of \( X \) and \( f \) is a mapping of class \((S)_{+}\) with \( f \) demicontinuous (i.e. continuous from the strong topology of \( X \) to the weak topology of \( X^* \)). Let \( H \) be the class of affine homotopies in \( F \), and let \( J \) be the duality mapping from \( X \) to \( X^* \) corresponding to an equivalent norm on \( X \) in which both \( X \) and \( X^* \) are locally uniformly convex.

Then there exists one and only one degree function on \( F \) which is invariant under \( H \) and normalized by the map \( J \).

The homotopies described in Theorem 4 are too weak for some important applications of this result. We therefore introduce the following broader class.

**Definition 3.** Let \( G \) be a bounded open subset of \( X \), \( \{f_t: 0 \leq t \leq 1\} \) a one-parameter family of maps of \( \text{cl}(G) \) into \( X^* \). Then \( \{f_t\} \) is said to be a homotopy of class \((S)_{+}\) if it satisfies the following condition: For any sequence \( \{x_j\} \) in \( \text{cl}(G) \) converging weakly to some \( x \) in \( X \) and for any sequence \( \{t_j\} \) in \( [0,1] \) converging to \( t \) for which

\[
\lim \langle f_{t_j}(x_j), x_j - x \rangle \leq 0,
\]

we have \( x_j \) converging strongly to \( x \) and \( f_{t_j}(x_j) \to f_t(x) \).

**Theorem 5.** The degree function described in Theorem 4 is invariant under homotopies of class \((S)_{+}\).

The construction of the degree function for maps of class \((S)_{+}\) and the proofs of Theorems 4 and 5 rest upon the general technique of Galerkin approximation.

**Definition 4.** Let \( X \) be a Banach space, \( X_0 \) a closed subspace of \( X \), \( G \) an open subset of \( X \) such that \( G \cap X_0 = G_0 \) is nonempty. Let \( f: \text{cl}(G) \to X \) be a given mapping. Then the Galerkin approximant for the map \( f \) is the mapping \( f_0: \text{cl}(G_0) \to X_0 \), where \( \varphi \) is the injection mapping of \( X_0 \) into \( X \), \( \varphi^* \) the corresponding projection of \( X^* \) onto \( X_0^* \), and \( f_0 \) is given by

\[
f_0(x) = \varphi^*(f(x)) \quad (x \in \text{cl}(G_0)).
\]

In a sort of inverse relationship to the concept of the Galerkin approximant is the following notion of a generalized suspension operator.
DEFINITION 5. Let $X$ be a reflexive Banach space, $J$ a duality mapping of $X$ into $X^*$ of class $(S)_+$. Suppose that $X_0$ is a closed subspace of $X$ having a closed complement $X_1$ in $X$, $P$ the projection of $X$ on $X_0$ which corresponds to this splitting so that $(I - P)$ is the projection on $X_1$. We consider $P$ as an element of the space of bounded linear maps of $X$, with $P_1$ the same projection considered as a bounded linear map of $X$ on $X_0$. Then $P^*$ is a bounded linear map of $X^*$ into itself, $P_1^*$ the injection of $X_0^*$ into $X^*$ whose image is the same as the range of the projection $P^*$. Let $B_1$ be the unit ball about $0$ in $X_1$.

Then

(1) For each bounded open subset $G_0$ of $X_0$, we define its suspension $S(G_0)$ as the bounded open subset of $X$ given by

$$S(G_0) = P^{-1}(G_0) \cap (I - P)^{-1}(B_1).$$

Thus each element $x$ of $S(G_0)$ can be written uniquely in the form

$$x = x_0 + x_1 \quad (x_0 \in G_0, x_1 \in B_1)$$

where $x_0 = Px$, $x_1 = (I - P)(x)$.

(2) If $g : \text{cl}(G_0) \rightarrow X_0^*$ is given, we define its suspension $S_g$ as the map of $\text{cl}(S(G_0)) \rightarrow X^*$ given by

$$S_g(x) = P_1^* gP(x) + (I - P)^* J(I - P)(x).$$

PROPOSITION 9. Under the circumstances described in Definition 5:

(a) If $g$ is demicontinuous and of class $(S)_+$ from $\text{cl}(G_0)$ to $X_0^*$, then $S_g$ is demicontinuous and of class $(S)_+$ from $\text{cl}(S(G_0))$ to $X^*$.

(b) If $\{g_t\}$ is an affine homotopy on $\text{cl}(G_0)$, then $\{S_{g_t}\}$ is an affine homotopy on $\text{cl}(S(G_0))$.

(c) If the point $y_0$ in $X_0^*$ does not lie in $g(\partial G_0)$, then $P_1^*(y_0)$ does not lie in $S_g(\partial S(G_0))$.

PROOF OF PROPOSITION 9. Proof of (a). Let $\{u_j\}$ be a sequence in $\text{cl}(S(G_0))$ such that $u_j \rightarrow u$ while $\lim S_g(u_j), u_j - u \leq 0$. Let $v_j = Pu_j$, $x_j = (I - P)u_j$. Then $v_j \rightarrow v = Pu$, $x_j \rightarrow x = (I - P)u$, with $v$ in $X_0$, $x$ in $X_1$. Using the definition of $S_g$, we see that

$$\langle S_g(u_j), u_j - u \rangle = \langle g(v_j), v_j - v \rangle + \langle J(x_j), x_j - x \rangle.$$ 

Since $\lim J(x_j), x_j - x \geq 0$, it follows that $\lim \langle g(v_j), v_j - v \rangle \leq 0$. Hence $v_j \rightarrow v$ and $g(v_j) \rightarrow g(v)$, implying that $\langle g(v_j), v_j - v \rangle \rightarrow 0$. Hence

$$\lim \langle J(x_j), x_j - x \rangle \leq 0.$$ 

Since $J$ is of class $(S)_+$, $x_j \rightarrow x$. Therefore $u_j = x_j + v_j \rightarrow x + v = u$. Q.E.D.

Proof of (b). This follows immediately from the linearity of $P$ and $P_1$ and the definition of the suspension of a map.

Proof of (c). Suppose for $x$ in $\text{cl}(S(G_0))$ that $S_g(x) = P_1^*(y_0)$. We remark that $P^*(X^*) = P_1^*(X_0^*)$ is a closed complement to $(I - P)^*(X^*)$. Thus if

$$P_1^*(g(Px) - y_0) + (I - P)^* J(I - P)(x) = 0,$$
it follows that
\[ P^* \left( g(Px) - y_0 \right) = 0, \quad (I - P)^*J(I - P)(x) = 0. \]

Hence, from the second equation, we find that
\[ 0 = \langle (I - P)^*J(I - P)(x), x \rangle = \langle J(I - P)x, (I - P)x \rangle = \| (I - P)x \|^2, \]
so that \( x = Px \). From the other equation, we see that since \( P^* \) is injective, \( g(x) - y_0 = g(Px) - y_0 = 0 \). By the assumption that \( y_0 \notin g(\partial G_0) \), and since \( x = Px \) lies in \( \text{cl}(G_0) \), we see that \( x \) lies in \( G_0 \). In particular, \( P^*(y_0) \) does not lie in \( S_g(\partial S(G_0)) \) since \( G_0 \subset S_g(G_0) \). Q.E.D.

We can use Proposition 9 in the following way to obtain an important methodological conclusion about degree functions for maps of class \( (S)_+ \).

**Proposition 10.** Under the circumstances of Definition 5 and Proposition 9, suppose that for both of the spaces \( X \) and \( X_0 \), we have degree functions \( d \) and \( d_0 \) defined on the demicontinuous mappings of class \( (S)_+ \) and invariant with respect to affine homotopies in each case, with the degree functions normalized by \( J \) and \( J_0 \), respectively, where \( J \) and \( J_0 \) are the appropriate duality maps. Suppose moreover that the degree function \( d_0 \) is uniquely characterized on the maps of class \( (S)_+ \) by these properties. Then for each map \( g: \text{cl}(G_0) \rightarrow X_0^* \) demicontinuous and of class \( (S)_+ \), and for each \( y_0 \) in \( X_0^* \) such that \( y_0 \notin g(\partial G_0) \), we have
\[ d_0(g, G_0, y_0) = d(S_g, S(G_0), P^*(y_0)). \]

**Proof of Proposition 10.** For each \( g \) and \( y_0 \) as above, we define a new degree function using Proposition 9 by setting
\[ d_1(g, G_0, y_0) = d(S_g, S(G_0), P^*(y_0)). \]
This gives a degree function which is invariant under affine homotopies on the demicontinuous maps \( g \) of class \( (S)_+ \). We remark that the normalization condition is always equivalent to
\[ d_1(J_0, B_0, 0) = +1 \]
for the unit ball \( B_0 \) with center 0 in \( X_0 \). The map \( f = S_{J_0} \) satisfies the condition \( \langle f(u), u \rangle > 0 \) for \( u \neq 0 \). If \( G = S(B_0), 0 \in G \) and the affine homotopy \( \{ f_t = (1 - t)f + tf \} \) has no zeroes except at \( u = 0 \). Hence
\[ d(S_{J_0}, S(B_0), 0) = d(J, S(B_0), 0) = +1. \]
Since \( d_0 \) is the unique degree function for maps of class \( (S)_+ \) on \( X_0 \), it follows that \( d_0 = d_1 \). Q.E.D.

We apply this result to the case in which \( X_0 \) is a finite-dimensional Banach space. Then \( X_0 \) is equivalent to a Hilbert space and \( X_0^* \) can be identified with \( X_0 \). By the remark we just made, normalization by the identity mapping on a finite-dimensional space is equivalent to normalization by any mapping \( f \) which is bicontinuous and such that \( \langle f(u), u \rangle > 0 \) for \( u \neq 0 \), and in particular by the duality map corresponding to the original norm on \( X_0 \). Hence
COROLLARY TO PROPOSITION 10. Let \( X \) be a reflexive Banach space, \( X_0 \) a finite-dimensional subspace of \( X \). Suppose that there exists a degree function \( d \) on the demicontinuous maps of class \((S)_+\) in the space \( X \). Then for each continuous map \( g: \text{cl}(G_0) \to X^* \) where \( G_0 \) is a bounded open subset of \( X_0 \) and for each \( y_0 \) outside \( g(\partial G_0) \), we have \( S_g \) demicontinuous and of class \((S)_+\) mapping \( \text{cl}(S(G_0)) \) into \( X^* \) and
\[
d_0(g, G_0, y_0) = d(S_g, S(G_0), P_1^*(y_0)).
\]

PROOF OF THE COROLLARY. Since \( X_0 \) is of finite dimension, it has a closed complement \( X_1 \) in \( X \), so that the suspension operator of Definition 5 is well defined for some projection map \( P \) of \( X \) on \( X_0 \). Moreover, for finite-dimensional spaces \( X_0 \), each continuous map \( g \) is demicontinuous and of class \((S)_+\) (indeed the two classes coincide). Moreover, as we already observed, we know the uniqueness of the degree function by the results of §1. Hence the desired conclusion follows from Proposition 10. Q.E.D.

We use the Corollary to Proposition 10 as an essential tool in the proof of our basic auxiliary result, Proposition 11.

PROPOSITION 11. Let \( X \) be a reflexive Banach space, \( X_0 \) a finite-dimensional subspace of \( X \), and suppose that a degree function \( d \) is given on the demicontinuous maps of class \((S)_+\) defined on the closures of bounded open sets \( G \) in \( X \) with values in \( X^* \). Suppose that this degree is invariant under affine homotopies and is normalized by a duality map \( J \) of class \((S)_+\).

Let \( f: \text{cl}(G) \to X^* \) be a map in the given class, with \( 0 \in G \) and let \( f_0 \) be the Galerkin approximant of \( f \) with respect to \( X_0 \), so that \( f_0: \text{cl}(G_0) \to X_0^* \), with \( G_0 = G \cap X_0 \). Suppose either that \( d(f, G, 0) \) is not defined, or \( d_0(f_0, G_0, 0) \) is not defined, or (if both are well defined), then
\[
d(f, G, 0) \neq d_0(f_0, G_0, 0)
\]
where we use \( d_0 \) to denote the finite-dimensional degree in \( X_0 \).

Then: There exists \( u \) in \( \partial G \) such that
\[
( f(u), u ) \leq 0,
\]
and for all \( v \) in \( X_0 \),
\[
( f(u), v ) = 0.
\]

PROOF OF PROPOSITION 11. Suppose first that for some \( u_0 \) in \( \partial G \), \( f(u_0) = 0 \). Then \( u_0 \) satisfies our conclusion. Suppose next that for some \( u_0 \) in \( \partial G_0 \), \( f_0(u_0) = 0 \). Then again \( u_0 \) satisfies the conclusion. Thus we may assume without loss of generality that both \( d(f, G, 0) \) and \( d(f_0, G_0, 0) \) are well defined (i.e. \( 0 \notin f(\partial G) \), \( 0 \notin f_0(\partial G_0) \)), and
\[
d(f, G, 0) \neq d_0(f_0, G_0, 0).
\]

If we apply the Corollary of Proposition 10, we see that
\[
d_0(f_0, G_0, 0) = d(S_{f_0}, S(G_0), 0).
\]
We define an auxiliary mapping \( f_1 : \text{cl}(G) \to X^* \) by setting
\[
f_1(x) = P*f(x) + (I - P)*J(I - P)(x).
\]
We have already noted in the proof of Proposition 9 that since 0 lies in \( X_0 \), all the solutions \( x \) of the equation \( S_0(x) = 0 \) must lie in \( G_0 \) and hence in \( G \cap S(G_0) \).

Using the additivity property of the degree, we see that
\[
d(S_0, S(G_0), 0) = d(S_0, S(G_0) \cap G, 0).
\]

Similarly, suppose for \( x \) in \( \text{cl}(G) \), we have \( f_1(x) = 0 \). Then we have
\[
(I - P)*J(I - P)(x) = 0,
\]
from which it follows that \( (I - P)x = 0 \), \( x \) lies in \( \text{cl}(G_0) \), and \( P*f(x) = 0 \). For each \( v \) in \( X_0 \), it follows that
\[
\langle f(x), v \rangle = \langle P*f(x), v \rangle = 0,
\]
so that \( f_0(x) = 0 \). Thus \( x \) must lie in \( G_0 \), since by our assumption \( 0 \notin f_0(\partial G_0) \).

We note next that \( f_1 \) is a mapping of class \((S)_+\) from \( \text{cl}(G) \) to \( X \). The first summand is compact since \( P \) is of finite dimension, so that it suffices to show that \( (I - P)*J(I - P) \) is of class \((S)_+\). Suppose that \( u_j \to u \), and that
\[
\lim ((I - P)J(I - P)u_j, u_j - u) \leq 0.
\]
If \( x_j = (I - P)u_j \), then \( x_j - x = (I - P)u \) by the fact that \( (I - P) \) is a continuous linear map. Moreover
\[
\langle (I - P)*J(I - P)u_j, u_j - u \rangle = \langle Jx_j, x_j - x \rangle.
\]
Thus
\[
\lim \langle Jx_j, x_j - x \rangle \leq 0.
\]
Since \( J \) is of class \((S)_+\), it follows that \( x_j \to x \). Since \( P \) is compact, we know also that \( Pu_j \to Pu \). Hence \( u_j = x_j + Pu_j \to x + Pu = u \), and we have shown that \( f_1 \) is of class \((S)_+\). Since \( f_1 \) is obviously also continuous, the degree function \( d \) applies to the mapping \( f_1 \). Hence
\[
d(f_1, G, y_0) = d(f_1, G \cap G_2, y_0).
\]
We now consider the affine homotopy \( h_t = (1 - t)f_1 + tS_0 \) on \( G \cap G_2 \). For any \( t \) in \((0, 1)\), suppose that for some \( x \), \( h_t(x) = 0 \). This means that
\[
(1 - t)*P*f(x) + tP*f_0(Px) + (I - P)*J(I - P)x = 0.
\]
Again this means that both summands must equal 0, i.e. in particular that \( (I - P)*J(I - P)x = 0 \). This means once more that \( x = Px \). Thus for every \( v \) in \( X_0 \),
\[
0 = \langle (1 - t)*P*f(x) + tP*f_0(x), v \rangle = \langle f_0(x), v \rangle.
\]
Thus \( x \) lies in \( \text{cl}(G_0) \) and \( f_0(x) = 0 \), so that by our initial assumption \( x \) lies in \( G_0 \). In particular, \( h_t \) has no zeroes for any \( t \) on the boundary of \( G \cap G_2 \). Hence
\[
d(f_1, G \cap G_2, 0) = d(S_0, G \cap G_2, 0).
\]
Combining the various equalities we have proved, we obtain
\[ d(f_0, G_0, 0) = d(S_{f_0}, G_2, 0) = d(S_{f_0}, G \cap G_2, 0) \]
\[ = d(f_1, G \cap G_2, 0) = d(f_1, G, 0). \]
If we combine this with our initial assumption that \( d(f_0, G_0, 0) \neq d(f, G, 0) \), we find that
\[ d(f, G, 0) \neq d(f_1, G, 0). \]

Now form the affine homotopy \( k_t = (1 - t)f + tf_1 \). By the inequality of the degrees on \( G \) for \( f \) and \( f_1 \) over 0, we see that there must exist \( u_0 \) in \( \partial G \) and \( t \) in \( (0, 1) \) such that \( k_t(u_0) = 0 \).

For any \( v \) in \( X_0 \), we see that
\[ 0 = \langle k_t(u_0), v \rangle = (1 - t)\langle f(u_0), v \rangle + t \langle f(u_0), v \rangle = \langle f(u_0), v \rangle. \]
Hence, we obtain the desired equality
\[ \langle f(u_0), v \rangle = 0 \quad (v \in X_0). \]

Now consider the equality
\[ 0 = \langle k_t(u_0), (I - P)u_0 \rangle \]
\[ = (1 - t)\langle f(u_0), (I - P)u_0 \rangle + t \langle J(I - P)u_0, (I - P)u_0 \rangle \]
\[ = (1 - t)\langle f(u_0), u_0 \rangle + t\|I - P\|u_0\|^2. \]
Hence
\[ \langle f(u_0), u_0 \rangle = -t(1 - t)^{-1}\|(I - P)u_0\|^2 \leq 0. \]
Q.E.D.

**Proof of Theorem 4.** We carry through the proof both for the existence and for the uniqueness of the degree function for maps of class \((S)_+\) by applying the Galerkin approximation method over the partially ordered set of finite-dimensional subspaces of \( X \).

Let \( \Lambda \) be this partially ordered set of finite-dimensional subspaces \( X_\lambda \) ordered by inclusion. For each \( \lambda \), let \( \varphi_\lambda \) be the injection map of \( X_\lambda \) into \( X \) and \( \varphi_\lambda^* \) the corresponding projection map of \( X^* \) onto \( X_\lambda^* \). We consider only finite-dimensional spaces such that \( G_\lambda = G \cap X_\lambda \) is nonempty. We assume that \( y_0 = 0 \), which does not affect the generality of the argument since we can replace each map \( f \) by \( f - y_0 \). We now assert

There exists \( \lambda_0 \) in \( \Lambda \) such that for all \( \lambda > \lambda_0 \), \( 0 \notin f_\lambda(\partial G_\lambda) \) and \( d(f_\lambda, G_\lambda, 0) \) is independent of \( \lambda \).

Suppose otherwise. Then for each \( X_\lambda \), there would exist \( X_\mu \supset X_\lambda \) such that
\[ d(f_\mu, G_\mu, 0) \neq d(f_\lambda, G_\lambda, 0) \]
(or one of the two degrees is not defined). In all these cases, we may apply Proposition 11 with \( X \) replaced by \( X_\mu \) and \( X_\lambda \) considered as a subspace of \( X_\mu \). Then \( f_\lambda \) is obviously the Galerkin approximant of \( f_\mu \) with respect to \( X_\lambda \). Since both \( X_\mu \) and \( X_\lambda \) have degree functions defined, and the uniqueness of degree holds for
we apply Proposition 11 and obtain a point $u$ in $\partial G$ such that
\[ \langle f(u), u \rangle = \langle f_\mu(u), u \rangle \leq 0, \]
while for all $v$ in $X$,
\[ \langle f(u), v \rangle = \langle f_\mu(u), v \rangle = 0. \]

Let us now define a subset $V_\lambda$ of $\partial G$ by
\[ V = \{ u | u \in \partial G, \langle f(u), u \rangle \leq 0; \langle f(u), v \rangle = 0 \text{ for all } v \text{ in } X \}. \]
By our preceding paragraph, each $V_\lambda$ is nonempty. The family $(V_\lambda)$ is contained in a fixed bounded set $\partial G$ and it has the finite intersection property. If we denote by $w$-cl($V_\lambda$) the closure of $V$ in the weak topology of $X$, then $w$-cl($V_\lambda$) is weakly compact, and
\[ \bigcap \omega \text{-cl}(V_\lambda) \neq \emptyset. \]
Let $x_0$ be an arbitrary point in this intersection.

Let $v$ be an arbitrary point in $X$, and choose a subspace $X_\lambda$ in the ordered set $\Lambda$ such that $X_\lambda$ contains both $x_0$ and $v$. Since $x_0$ lies in $w$-cl($V_\lambda$) and the space $X$ is reflexive, there exists a sequence $\{u_j\}$ in $V_\lambda$ which converges weakly to $x_0$ in $X$. Such $u_j$ must lie in $\partial G$ by the definition of $V_\lambda$. Moreover, for each $j$,
\[ \langle f(u_j), u_j \rangle \leq 0, \quad \langle f(u_j), x_0 \rangle = 0, \quad \langle f(u_j), v \rangle = 0. \]
In particular,
\[ \langle f(u_j), u_j - x_0 \rangle \leq 0, \]
and a fortiori, $\lim \langle f(u_j), u_j - x_0 \rangle = 0$. Using the assumption that $f$ lies in the class $(S)_+$, we see that $u_j$ converges strongly to $x_0$ and hence $x_0$ lies in $\partial G$. By the demicontinuity of $f$, $f(u_j) \to f(x_0)$. Hence
\[ \langle f(x_0), v \rangle = \lim \langle f(u_j), v \rangle = 0. \]
Since $v$ was an arbitrary element of $X$, it follows that $f(x_0) = 0$. Thus we reach a contradiction with the assumption that $0 \notin f(\partial G)$.

*We define $d(f, G, 0)$ as the common value of $d(f_\lambda, G_\lambda, 0)$ for $X_\lambda$ sufficiently large.*

The normalization properties and the additivity property of the degree function thus defined follow immediately from the definition. The invariance under homotopies which are affine as well as homotopies of class $(S)_+$ asserted in Theorem 5 follows from the following two assertions:

**Proposition 12.** Each affine homotopy between two demicontinuous maps $f$ and $f_1$ of class $(S)_+$ is a homotopy of class $(S)_+$. 

**Proposition 13.** If $\{f_t\}$ is a homotopy of class $(S)_+$ on $\text{cl}(G)$ and suppose that $0 \notin f_t(\partial G)$ for any $t$ in $[0, 1]$, then there exists $\lambda_0$ in $\Lambda$ such that for any $t$ in $[0, 1]$ and all $\lambda$ in $\Lambda$, $\lambda > \lambda_0$, we have $d(f_{t, \lambda}, G_{\lambda}, 0)$ well defined and independent of both $t$ and $\lambda$. 

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PROOF OF PROPOSITION 12. Suppose that \{u_j\} is a sequence in \(\text{cl}(G)\) with \(u_j \to u\), \(\{t_j\}\) a sequence in \([0, 1]\) with \(t_j \to t\), and such that
\[
\lim \left\{ (1 - t_j) \langle f(u_j), u_j - u \rangle + t_j \langle f_1(u_j), u_j - u \rangle \right\} \leq 0.
\]
Since \(f\) and \(f_1\) are demicontinuous maps of class \((S)_+\), we know that
\[
\lim \langle f(u_j), u_j - u \rangle \geq 0, \quad \lim \langle f_1(u_j), u_j - u \rangle \geq 0.
\]
Suppose without loss of generality that \(t > 0\). Since \(H^m(f(u_j), u_j - u) = t_j \langle f(u_j), u_j - u \rangle \leq 0\), we see that
\[
\lim (1 - t_j) \langle f(u_j), u_j - u \rangle \geq 0
\]
we see that
\[
t \lim \langle f_1(u_j), u_j - u \rangle = \lim \left\{ t_j \langle f_1(u_j), u_j - u \rangle \right\} \leq 0.
\]
Since \(f\) is of class \((S)_+\), it follows that \(u_j\) converges strongly to \(u\). Therefore
\[
(1 - t_j)f(u_j) + t_jf_1(u_j) \to (1 - t)f(u) + tf_1(u),
\]
and the proof of Proposition 12 is complete. Q.E.D.

PROOF OF PROPOSITION 13. Suppose that the assertion of Proposition 13 were false. Then by Proposition 11, for each \(X_\lambda\) in \(\Lambda\), there would exist \(X_\mu \supseteq X_\lambda\), a point \(u \in \partial G_u\) and a parameter value \(t\) in \([0, 1]\) such that
\[
\langle f_1(u), u \rangle < 0, \quad \langle f_1(u), v \rangle = 0 \quad (v \in X_\lambda).
\]
For each \(\lambda\) in \(\Lambda\) we form a subset \(W_\lambda\) of the Cartesian product of \(\partial G \times [0, 1]\), where
\[
W = \{ [u, t] \mid u \in \partial G, t \in [0, 1]; \langle f_1(u), u \rangle < 0; \langle f_1(u), v \rangle = 0 \quad (v \in X_\lambda) \}.
\]
By our preceding consideration, each \(W_\lambda\) would be a nonempty subset of the bounded set \(\partial G \times [0, 1]\), and the family \(\{W_\lambda\}\) has the finite intersection property. If we take the closure of \(W_\lambda\) in the product of the weak topology on \(\partial G\) and the ordinary topology on \([0, 1]\) and denote this closure by \(w-cl(W_\lambda)\), then this set is compact in its appropriate topology. Hence
\[
\bigcap_\lambda w-cl(W_\lambda) \neq \emptyset.
\]
Let \([u_0, t_0]\) be a point of this intersection.

For an arbitrary point \(v\) of \(X\), choose \(X_\lambda\) in \(\Lambda\) such that \(X_\lambda\) contains both \(u_0\) and \(v\). Since \(X\) is reflexive, we may choose a sequence \([u_j, t_j]\) in the corresponding \(W_\lambda\) such that \(u_j \to u_0\), \(t_j \to t\). For each index \(j\), we have
\[
\langle f_1(u_j), u_j \rangle \leq 0, \quad \langle f_1(u_j), u_0 \rangle = 0, \quad \langle f_1(u_j), v \rangle = 0.
\]
In particular it follows that
\[
\lim \langle f_1(u_j), u_j - u_0 \rangle \leq 0.
\]
If we apply the assumption that \(\{f_1\}\) is a homotopy of class \((S)_+\), it follows that \(u_j \to u_0\) and \(f_1(u_j) \to f_1(u_0)\). Hence \(u_0 \in \partial G\), and \(\langle f_1(u_j), v \rangle \to \langle f_1(u_0), v \rangle = 0\). Since \(v\) is arbitrary, \(f_1(u_0) = 0\) which contradicts our assumption that \(0 \notin f_1(\partial G)\) for all \(t\). Q.E.D.
We have now completed the proof of the existence of the degree in Theorem 5. It remains to prove the uniqueness of the degree in Theorem 4. We shall prove this fact in the following proposition.

**PROPOSITION 14.** Let $X$ be a reflexive Banach space and let $d_1$ be a degree function on the class $F$ of demicontinuous mappings of class $(S)_+$ of the closures of bounded open sets $G$ in $X$ into $X^*$. Suppose that $d_1$ is invariant under the class of affine homotopies in $F$. Then $d_1$ coincides with the degree $d$ defined above in terms of Galerkin approximants.

**PROOF OF PROPOSITION 14.** Suppose the assertion were false. Then there would exist a bounded open set $G$ in $X$, a demicontinuous map $f$ of $\text{cl}(G)$ into $X^*$ of class $(S)_+$ and a point $y_0$ in $X^*$, $y_0 \not\in f(\partial G)$ such that

$$d(f, G, y_0) \neq d_1(f, G, y_0).$$

We may assume without loss of generality that $y_0 = 0$.

By the definition of $d$ in terms of Galerkin approximants, it follows from this assumption that for each $\lambda$ in $\Lambda$, there would exist $X_\mu \supset X_\lambda$ such that $d_0(\mu, G, 0) \neq d_1(f, G, 0)$. We now apply the conclusion of Proposition 11 to the degree function $d_1$ on $X$ and derive the existence of $u$ in $\partial G$ such that

$$\langle f(u), u \rangle \leq 0; \quad \langle f(u), v \rangle = 0 \quad \text{for all } v \in X_\mu.$$

We define the subset $V_\lambda$ of $\partial G$ by

$$V = \{ u \mid u \in \partial G; \langle f(u), u \rangle \leq 0; \langle f(u), v \rangle = 0 \text{ for all } v \in X_\lambda \}.$$  

By the last paragraph, each $V_\lambda$ is nonempty and the family $\{V_\lambda\}$ has the finite intersection property. By the weak compactness of $w-\text{cl}(V_\lambda)$, it follows that $\cap_{\lambda \in \Lambda} w-\text{cl}(V_\lambda) \neq \emptyset$. Let $u_0$ be a point of this intersection.

Let $X_\lambda$ be an element of $\Lambda$ which contains both $u_0$ and a given arbitrary point $v$ of $X$. By the reflexivity of $X$, there exists a sequence $\{u_j\}$ from the corresponding $V_\lambda$ with $u_j \to u_0$. For each index $j$,

$$\langle f(u_j), u_j \rangle \leq 0; \quad \langle f(u_j), u_0 \rangle = 0; \quad \langle f(u_j), v \rangle = 0$$

and $u_j \in \partial G$. A fortiori $\lim \langle f(u_j), u_j - u_0 \rangle \leq 0$, so that $u_j \to u_0$ and $f(u_j) \to f(u_0)$. Thus $u_0 \in \partial G$, and $\langle f(u_0), v \rangle = \lim \langle f(u_j), v \rangle = 0$. Since $v$ was an arbitrary element of $X$, $f(u_0) = 0$, with $u_0 \in \partial G$. This is a contradiction, and the proof of Proposition 14 is complete.

Thus the proofs of Theorems 4 and 5 are complete.

If we seek to extend the degree function we have constructed to the broader class of pseudo-monotone mappings, we encounter the significant difficulty that the pseudo-monotone maps are not proper in the general case, and for an arbitrary bounded open subset $G$, it is not true for a pseudo-monotone map $f$ of $\text{cl}(G)$ into $X^*$, that $f(\partial G)$ is closed nor that $f(\text{cl}(G))$ is closed. The latter will indeed be the case if $G$ is convex, but is false in general otherwise. This necessitates a serious modification of the definition of a degree function if we are to obtain one for such mappings. We will not restate the formal definition of a
degree function in its complete form to show the salient features of the new
definition, but merely the modifications that are necessary.

**Definition 6.** We shall speak of a degree function in the extended (or weak)
sense if the following modifications are made in the characterizing properties of the
degree in Definition 1.

1. The degree function \( d(f, G, y_0) \) is to be defined only if \( y_0 \not\in \text{cl}(f(\partial G)) \).
2. If \( d(f, G, y_0) \neq 0 \), then \( y_0 \not\in \text{cl}(f(G)) \).
3. The normalizing map \( f_0 \) is assumed to be proper on bounded sets.
4. In the definition of the additivity property on domains, we must assume that
   \[ y_0 \not\in \text{cl}(f(cl(G) \setminus (G_1 \cup G_2))). \]
5. In the property of homotopy invariance, we must assume that if \( Y \) is a metric
   space, then a fixed ball \( B_r(y_t) \) does not intersect \( f_t(\partial G) \) for all \( t \) in \([0, 1]\).

**Theorem 6.** There exists one and only one degree function in the extended sense on
the class \( F \) of maps \( f: \text{cl}(G) \to X^* \), where \( X \) is a reflexive Banach space and the
maps \( f \) are pseudo-monotone, with the degree invariant under affine homotopies and
normalized by the duality mapping \( J \).

**Theorem 7.** The degree function of Theorem 6 is actually invariant under
pseudo-monotone homotopies, where a homotopy \( \{f_t\} \) is said to be pseudo-monotone
if for a sequence \( \{u_j\} \) converging weakly to \( u \) in \( X \) and a sequence \( \{t_j\} \) converging to
t in \([0, 1] \) for which \( \lim \langle f_t(u_j), u_j - u \rangle = 0 \), we have \( \lim \langle f_t(u_j), u_j - u \rangle = 0 \), and
if \( u \) lies in \( \text{cl}(G) \), then \( f_t(u_j) \to f_t(u) \).

We shall not give the detailed proofs of Theorems 6 and 7 here except to
observe that in this case, the proof by Galerkin approximants does not suffice. We
must instead approximate each pseudo-monotone mapping \( f \) by approximating mappings \( f_\varepsilon \), \( \varepsilon > 0 \), in the class \( (S)_+ \) where \( f_\varepsilon = f + \varepsilon J \) and \( J \) is the duality
mapping of class \( (S)_+ \) which we have considered above. For each \( f \) which is
pseudo-monotone and demicontinuous, the approximating map \( f_\varepsilon \) is demicontinu-
os and of class \( (S)_+ \). If \( y_0 \not\in \text{cl}(f(\partial G)) \), then for \( \varepsilon > 0 \) and sufficiently small,
\( y_0 \not\in f_\varepsilon(\partial G) \). On the other hand, on any compact subinterval of \((0, \varepsilon_0) \), \( \{f_\varepsilon\} \) is an
affine homotopy of maps of class \( (S)_+ \). Therefore, by the invariance of the
degree for maps of class \( (S)_+ \) under affine homotopies, \( d(f_\varepsilon, G, y_0) \) will be
independent of \( \varepsilon \) for \( 0 < \varepsilon < \varepsilon_0 \). This common value, we designate as \( d(f, G, y_0) \),
and it will be our extended degree function for the class of pseudo-monotone
mappings. The invariance under pseudo-monotone homotopies follows from the
fact that for each \( \varepsilon > 0 \), \( f_\varepsilon + \varepsilon J \) yields a homotopy of class \( (S)_+ \). It does not
follow in this case that each affine homotopy is pseudo-monotone (unless \( f \) and \( f_1 \)
are bounded), but it is still the case that for each \( \varepsilon > 0 \), \( \{f_\varepsilon + \varepsilon J\} \) is an affine
homotopy of maps of class \( (S)_+ \) and hence a homotopy of class \( (S)_+ \).

Restricted to the class of maps of class \( (S)_+ \), any degree in the extended sense
just becomes a degree in the original sense of Definition 1 since all the maps
and homotopies involved are proper. Thus an extended degree is unique by Theorem 4
on the dense subclass of maps of class \((S)_+\). By continuity of the degree under the homotopy \(f + \varepsilon J\) as \(\varepsilon\) ranges over \([0, e_0)\), we find the uniqueness of the extended degree on the class of pseudo-monotone mappings.

6. The degree for more general mappings of monotone type. There are a number of classes of mappings of monotone type with domain in a reflexive Banach space \(X\) and with values in the conjugate space \(X^*\) for which suitable extensions of the arguments which we developed above can be used to develop an existence and uniqueness theory for a suitably defined degree of mapping. To complete our presentation here, we consider the simplest and most generally useful of such classes, that of maps of the form \(T + f\) with \(T\) maximal monotone and \(f\) bounded and of class \((S)_+\).

Let \(X\) be a reflexive Banach space. We consider maps \(T\) from \(X\) with values subsets of \(X^*\). With each such map, we associate its graph \(G(T)\) in \(X \times X^*\), i.e.

\[
G(T) = \{[u, w] | u \in X, w \in T(x)\}.
\]

Then

The mapping \(T\) is said to be monotone if for any pair of elements \([u, w]\) and \([x, y]\) in \(G(T)\), we have the inequality

\[
\langle w - y, u - x \rangle \geq 0.
\]

\(T\) is said to be maximal monotone if it is monotone and maximal in the sense of graph inclusion among monotone maps (multivalued) from \(X\) to \(X^*\). An equivalent version of the last clause is that for any \([u_0, w_0]\) in \(X \times X^*\) for which

\[
\langle w_0 - y, u_0 - x \rangle \geq 0
\]

for all \([x, y]\) in \(G(T)\), we have \([u_0, w_0]\) in \(G(T)\). If the Banach space \(X\) is reflexive, an equivalent statement is that the mapping \(T + J\) has all of \(X^*\) as its range. (We assume as before that \(X\) has been given an equivalent norm in which the duality mapping \(J\) is bicontinuous and lies in the class \((S)_+\).)

Maximal monotone mappings occur in a number of useful and important contexts. First of all, all demicontinuous monotone maps of \(X\) into \(X^*\) are maximal monotone. Second, for any maximal monotone map \(T\) of \(X\) into the subsets of \(X^*\), its inverse \(T^{-1}\) is maximal monotone from \(X^*\) to the subsets of \(X\) (this makes maximal monotonicity useful in the study of Hammerstein integral equations). Finally if \(\varphi: X \to R \cup \{+\infty\}\) is a proper lower-semi-continuous convex function, its subgradient \(\partial \varphi\) is maximal monotone, where

\[
(\partial \varphi)(u) = \{w | w \in X^*; \text{ For all } v \in X, \varphi(v) - \varphi(u) \geq \langle w, v - u \rangle\}.
\]

This fact makes maximal monotone mappings a useful tool in the study of variational inequalities.

Theorem 8. Let \(X\) be a reflexive Banach space, \(T\) a maximal monotone mapping from \(X\) to \(2^{X^*}\) with \(0 \in T(0)\). Let \(G\) be a bounded open subset of \(X\), and let \(f\) be a bounded mapping of \(\text{cl}(G)\) into \(X^*\) of class \((S)_+\). For each \(\varepsilon > 0\), consider the
generalized Yosida transformation $T_\varepsilon$ corresponding to $T$ \[3\] given by

$$T_\varepsilon = (T^1 + \varepsilon J^{-1})^{-1}.$$  

(A) Suppose that for a given $y_0$ in $X^*$, $y_0$ does not lie in $(T + f)(\partial G)$. Then there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, $y_0$ does not lie in $(T_\varepsilon + f)(\partial G)$. 

(B) For each $\varepsilon > 0$, the mapping $T_\varepsilon + f$ of $\text{cl}(G)$ into $X^*$ is of class $(S)_+$. Hence for $0 < \varepsilon < \varepsilon_0$, the degree function $d(T_\varepsilon + f, G, y_0)$ is defined by the results of the preceding section. Moreover, for $0 < \varepsilon < \varepsilon_1$, the values of all these degrees coincide. 

**Definition 7.** We set the degree function $d(T + f, G, y_0)$ to be the common value of $d(T_\varepsilon + f, G, y_0)$ for $\varepsilon$ sufficiently small. 

**Proposition 15.** Let $X$ be a reflexive Banach space, \{\(T_t; 0 \leq t \leq 1\)\} a family of maximal monotone maps from $X$ to $2^{X^*}$ with $0 \in T_t(0)$ for all $t$. Let $J$ be a duality mapping from $X$ to $X^*$ which corresponds to a norm on $X$ in which both $X$ and $X^*$ are locally uniformly convex. Consider the following four conditions on the family \{\(T_t\)\}:

1. (generalized pseudo-monotonicity) Suppose that for a sequence \(\{t_j\}\) converging to $t$ in \([0, 1]\), we have a sequence \([u_{t_j}, w_{t_j}]\) in $G(T_{t_j})$ with $u_t$ converging weakly to $u$ in $X$, $w_t$ converging weakly to $w$ in $X^*$. Suppose further $\lim\sup\langle w_{t_j}, u_{t_j} \rangle \leq \langle w, u \rangle$. Then $w \in T_t(u)$, and $\langle w_{t_j}, u_{t_j} \rangle \rightarrow \langle w, u \rangle$.

2. Consider the mapping $\varphi$ of $X^* \times [0, 1]$ into $X$ given by 

$$\varphi(w, t) = (T_t + J)^{-1}(w).$$

Then $\varphi$ is continuous from $X^* \times [0, 1]$ to $X$ (with both $X$ and $X^*$ given their strong topologies).

3. For each $w$ in $X^*$, the mapping $\varphi_w$ of $[0, 1]$ into $X$ given by

$$\varphi_w(t) = (T_t + J)^{-1}(w)$$

is continuous from $[0, 1]$ to the strong topology on $X$.

4. (Strong lower-semi-continuity of $G(T_t)$ in $t$) Given $[x, y]$ in $G(T_t)$ and a sequence $\{t_j\}$ converging to $t$ in $[0, 1]$, there exists a sequence $[x_{t_j}, w_{t_j}]$ in $G(T_{t_j})$ such that $x_{t_j}$ converges strongly to $x$ in $X$, $y_{t_j}$ converges strongly to $y$ in $X^*$. Then, conditions (1), (2), (3), and (4) are mutually equivalent.

**Definition 8.** A family \{\(T_t\)\} of maximal monotone mappings which satisfies the mutually equivalent conditions (1)–(4) is called a pseudo-monotone homotopy of maximal monotone maps.

**Proof of Proposition 15.** We shall prove that (4) $\Rightarrow$ (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4). Obviously (2) $\Rightarrow$ (3).

**Proof that (4) Implies (1).** Let $[x, y]$ be an element of $G(T_t)$. For the sequence $\{t_j\}$ given in (1), choose a family $[x_{t_j}, y_{t_j}]$ as described in condition (4) with $[x_{t_j}, y_{t_j}]$ in $G(T_{t_j})$ and $[x_{t_j}, y_{t_j}]$ converging strongly to $[x, y]$. For each $j$, the monotonicity of $T_{t_j}$ implies that

$$\langle w_{t_j} - y_{t_j}, u_j - x_j \rangle \geq 0.$$
Hence
\[ \langle w, u \rangle \geq \lim \langle w_j, u_j \rangle \geq \lim \langle w_j, u_j \rangle \geq \lim \left( \langle w_j, x_j \rangle + \langle y_j, u_j - x_j \rangle \right) \]
\[ = \langle w, x \rangle + \langle y, u - x \rangle. \]
Thus, \( \langle w - y, u - x \rangle \geq 0. \) Since this inequality holds for all \([x, y] \in G(T_t)\), the maximal monotonicity of \( T_t \) implies that \([u, w] \) lies in \( G(T_t)\). On the other hand, setting \([x, y] = [u, w] \) in the inequality above, we see that
\[ \langle w, u \rangle \geq \lim \langle w_j, u_j \rangle \geq \lim \langle w_j, u_j \rangle \geq \langle w, u \rangle. \]
Hence \( \langle w_j, u_j \rangle \rightarrow \langle w, u \rangle \). Q.E.D.

**Proof that (1) implies (2).** Let \( u_j = (T_t + J)^{-1}(w_j) \) with \( w_j \) converging strongly to \( w \) in \( X^* \) and \( t_j \) converging to \( t \) in \([0, 1]\). Then
\[ w_j = y_j + J(u_j) \]
for elements \( y_j \) in \( G(T_t)\). Since \([0, 0] \in G(T_t)\) for each \( j \), we have
\[ \langle w_j, u_j \rangle = \langle y_j, u_j \rangle + \| u_j \|^2 \leq \| u_j \|^2. \]
Hence the sequence \( \{u_j\} \) is bounded, as is the sequence \( J(u_j)\). We wish to show that \( u_j \) converges strongly to \( u = (T_t + J)^{-1}w \). To do so, it suffices to assume that \( u_j \) converges weakly to \( u \), and show that \( u \) must be this given element and that the convergence is strong. We may also assume that \( J(u_j) \) converges weakly to \( z \) in \( X^* \), while \( y_j = w_j - J(u_j) \) converges weakly to \( w - z = y \).

For each \( j \), we have
\[ \langle w_j, u_j - u \rangle = \langle y_j, u_j - u \rangle + \langle J(u_j), u_j - u \rangle. \]
Since \( w_j \) converges strongly to \( w \) while \( u_j - u \) converges weakly to \( 0 \), it follows that \( \langle w_j, u_j - u \rangle \rightarrow 0 \). On the other hand, since \( J \) is pseudo-monotone,
\[ \lim \langle J(u_j), u_j - u \rangle \geq 0. \]
Hence
\[ \lim \langle y_j, u_j - u \rangle \leq 0 \]
i.e.
\[ \lim \langle y_j, u_j \rangle \leq \langle y, u \rangle. \]
If we apply condition (1), we find that \( y \in T_t(u) \), that \( \langle y_j, u_j \rangle \rightarrow \langle y, u \rangle \), i.e. \( \langle y_j, u_j - u \rangle \rightarrow 0 \). Hence
\[ \langle J(u_j), u_j - u \rangle \rightarrow 0. \]
Since \( J \) is a map of class \((S)_+\) by hypothesis, \( u_j \) must converge strongly to \( u \) and \( J(u_j) \) must converge strongly to \( J(u) \). Hence \( y = w - J(u) \), i.e. \( w \in (T_t + J)(u) \), or \( u = (T_t + J)^{-1}(w) \). Q.E.D.

**Proof that (3) implies (4).** Let \([x, y] \in G(T_t)\). Then \( y + J(x) \in (T_t + J)(x) \), i.e. \( x = (T_t + J)^{-1}(y + J(x)) \). If \( \{t_j\} \) is a sequence converging to \( t \) in \([0, 1]\), let
\[ x_j = (T_{t_j} + J)^{-1}(y + J(x)). \]
By condition (3), \( x_j \) converges strongly to \( x \) in \( X \), so that \( J(x_j) \) converges strongly to \( J(x) \) in \( X^* \). On the other hand, there exists \( y_j \) in \( T(x_j) \) such that \( y + J(x) = y_j + J(x_j) \). However, \( y_j = y + J(x) - J(x_j) \) converges strongly to \( y \) in \( X^* \). Q.E.D.

**Theorem 9.** Let \( X \) be a reflexive Banach space, \( G \) a bounded open subset of \( X \), \( \{f_t: 0 \leq t \leq 1\} \) a homotopy of class \((S)_+\) of maps of \( \text{cl}(G) \) into a bounded subset of \( X^* \). Let \( \{T_t: 0 \leq t \leq 1\} \) be a family of maximal monotone mappings of \( X \) into \( 2^{X^*} \) with \( 0 \in T_t(0) \) for all \( t \). Suppose \( \{T_t\} \) is a pseudo-monotone homotopy in the sense of Definition 1. Let \( \{y(t): t \in [0,1]\} \) be a continuous path in \( X^* \) such that \( y(t) \in (T_t + f_t)(\partial G) \) for all \( t \in [0,1] \). Then

(a) For each \( \varepsilon > 0 \), \( T_{t,e} = (T_t^{-1} + \varepsilon J^{-1})^{-1} \) is a bounded pseudo-monotone mapping of \( X \) into \( X^* \), while the family \( \{T_{t,e} + f_t: 0 \leq t \leq 1\} \) is a homotopy of class \((S)_+\) from \( \text{cl}(G) \) to \( X^* \).

(b) There exists \( \varepsilon_0 > 0 \), such that for \( 0 < \varepsilon < \varepsilon_0 \),

\[
y(t) \in (T_{t,e} + f_t)((\partial G),
\]

\[
d(T_{t,e} + f_t, G, y(t)) \text{ is well defined and independent of } t \text{ and } \varepsilon.
\]

**Theorem 10.** The degree function defined by Definition 1 on the class of maps of the form \( T + f \) with \( T \) maximal monotone and \( f \) bounded and of class \((S)_+\) is a classical degree function with respect to the class of homotopies described in Theorem 2 with the canonical map \( f_0 = J \).

**Proof of Theorems 8 and 9.** As stated, Theorem 9 includes Theorem 8 as a special case since the constant homotopy \( T_t = T \) for all \( t \) is indeed pseudo-monotone in the sense of Definition 8. Hence it suffices to prove Theorem 9. The proof rests on the following auxiliary result.

**Proposition 16.** Let \( X \) be a reflexive Banach space, \( \{T_t: t \in [0,1]\} \) a pseudo-monotone homotopy of maximal monotone maps from \( X \) to \( 2^{X^*} \) with \( 0 \in T_t(0) \) for all \( t \). Let \( \{v_j\} \) be a sequence in \( X \) converging weakly to \( v \), and for sequences \( \{t_j\} \) in \( [0,1] \) converging to \( t \), \( 0 < \varepsilon_j, 0 < \delta_j \rightarrow 0, \delta_j \rightarrow 0, \) let

\[
v_j = T_{t_j, \varepsilon_j}(u_j), \quad z_j = T_{t_j, \delta_j}(u_j).
\]

Suppose further that for another sequence \( \{s_j\} \) in \( [0,1] \), and for

\[
w_j = (1 - s_j)v_j + s_jz_j,
\]

we have \( w_j \) converging weakly to \( w \) in \( X^* \), while \( \lim \langle w_j, u_j \rangle \leq \langle w, u \rangle \).

Then \( w \in T_t(u) \), and \( \langle w_j, u_j \rangle \rightarrow \langle w, u \rangle \).

**Proof of Proposition 16.** Since \( v_j = T_{t_j, \varepsilon_j}(u_j) \), it follows that

\[
v_j \in T_{t_j}(u_j - \varepsilon_jJ^{-1}(v_j)).
\]

Similarly, \( z_j \in T_{t_j}(u_j - \delta_jJ^{-1}(z_j)) \). Since \( 0 \in T_{t_j}(0) \), and \( T_{t_j} \) is monotone,

\[
(v_j, u_j - \varepsilon_jJ^{-1}(v_j)) \geq 0, \quad (z_j, u_j - \delta_jJ^{-1}(z_j)) \geq 0.
\]
Hence
\[ e_j \| v_j \|^2 \leq \langle v_j, u_j \rangle, \quad \delta_j \| z_j \|^2 \leq \langle z_j, u_j \rangle. \]
If we multiply each of these inequalities by \((1 - s_j)\) and \(s_j\), respectively and add, we obtain
\[ (1 - s_j) e_j \| v_j \|^2 + s_j \delta_j \| z_j \|^2 \leq \langle w_j, u_j \rangle \leq M. \]
Hence \((1 - s_j) e_j \| v_j \|^2\) is bounded, as is \(s_j \delta_j \| z_j \|^2\), so that
\[ (1 - s_j) e_j \| v_j \| \to 0, \quad s_j \delta_j \| z_j \| \to 0. \]

Let \([x, y]\) be any element of \(G(T_t)\). By the condition (4) for the pseudo-monotone homotopy \(\{T_t\}\), for each \(j\), there exist elements \([x_j, y_j]\) of \(G(T_j)\) with \(x_j\) converging strongly to \(x\), \(y_j\) converging strongly to \(y\). By the monotonicity of \(T_t\),
\[ \langle v_j - y_j, u_j - e_j J^{-1}(v_j) - x_j \rangle \geq 0, \quad \langle z_j - y_j, u_j - \delta_j J^{-1}(z_j) - x_j \rangle \geq 0. \]
Hence,
\[ \langle v_j - y_j, u_j - x_j \rangle \geq e_j \langle v_j - y_j, J^{-1}(v_j) \rangle \geq -\|v_j\| e_j \|v_j\|, \]
and
\[ \langle z_j - y_j, u_j - x_j \rangle \geq \delta_j \langle z_j - y_j, J^{-1}(z_j) \rangle \geq -\|v_j\| \delta_j \|z_j\|. \]
If we multiply these inequalities by \((1 - s_j)\) and \(s_j\), respectively, and add, we get
\[ \langle w_j - y_j, u_j - x_j \rangle \geq -M \{ (1 - s_j) e_j \|v_j\| + s_j \delta_j \|z_j\| \} \to 0. \]
Thus
\[ \langle w, u \rangle \geq \lim \langle w_j, u_j \rangle \geq \lim \langle w_j, u_j \rangle \geq \lim \{ \langle w_j - y_j, +x_j \rangle + \langle y_j, +u_j \rangle \} \]
\[ = \langle w - y, x \rangle + \langle y, u \rangle. \]
Hence
\[ \langle w - y, u - x \rangle \geq 0 \quad ([x, y] \text{ in } G(T_t)), \]
and by the maximal monotonicity of \(T_t\), \(w \in T_t(u)\). Substituting \([u, w]\) for \([x, y]\) in the preceding chain of inequalities, we see that
\[ \langle w, u \rangle \geq \lim \langle w_j, u_j \rangle \geq \lim \langle w_j, u_j \rangle \geq \langle w, u \rangle. \]
Thus, finally, \(\langle w_j, u_j \rangle \to \langle w, u \rangle\). Q.E.D.

**Proof of Theorem 9 completed.** By Proposition 12, it follows that if \(\{T_t\}\) is a pseudo-monotone homotopy, then \(T_t^{-1}\) is a pseudo-monotone homotopy. Similarly, if \(\{T_t^{-1}\}\) is a pseudo-monotone homotopy and if \(\{g_t\}\) is a pseudo-monotone homotopy of bounded monotone maps, then \(\{T_t^{-1} + g_t\}\) is a pseudo-monotone homotopy. In particular, \(\{T_t^{-1} + \varepsilon J^{-1}\}\) is a pseudo-monotone homotopy, and \(T_{t, \varepsilon} = (T_t^{-1} + \varepsilon J^{-1})^{-1}\) is also a pseudo-monotone homotopy. Hence \(T_{t, \varepsilon} + f_t\) is a pseudo-monotone homotopy of class \((S)_{+}\) for each \(\varepsilon > 0\). Thus part (a) holds.

Suppose the conclusion of part (b) were false. Then there would exist a sequence \(\{t_j\}\) in \([0, 1]\) and two sequences of positive numbers \(\varepsilon_j, \delta_j \to 0\) and points
u_j on ∂G, such that for suitable s_j in [0, 1]

\[ y_{t_j} = (1 - s_j)T_{t_j,s_j}(u_j) + s_jT_{t_j,\delta}(u_j) + f(t_j). \]

We may assume that t_j converges to t, and that y_{t_j} converges strongly to y_t. If we set v_j = T_{t_j,s_j}(u_j), z_j = T_{t_j,\delta}(u_j), and w_j = y_{t_j} - f(t_j), we may assume that u_j converges weakly to u and that w_j converges weakly to w. Moreover,

\[ \lim \langle w_j, u_j - u \rangle = \lim \langle y_{t_j}, u_j - u \rangle - \lim \langle f(t_j), u_j - u \rangle \leq 0 \]

since

\[ \lim \langle f(t_j), u_j - u \rangle \geq 0 \]

by the pseudo-monotonicity of the homotopy \{f_t\}. If we apply Proposition 2, it follows that \( w \in T_t(u) \), and that \( \langle w_j, u_j - u \rangle \rightarrow 0 \). Hence

\[ \langle f(t_j), u_j - u \rangle \rightarrow 0, \]

and since the homotopy \{f_t\} is of class \((S)_+\), \( u_j \) converges strongly to \( u \), and \( f(t_j) \) converges weakly to \( f(t) \). Thus \( u \) lies in \( \partial G \), \( y_t - f(t) = w \in T_t(u) \), i.e. \( y_t \in (T_t + f(t))(\partial G) \), which contradicts the hypothesis. Q.E.D.

**Theorem 11.** The degree function described by Theorem 3 can be extended to the class of mappings of the form \( T + f \), with \( f \) pseudo-monotone and bounded by letting

\[ d(T + f, G, y_0) = \lim_{\delta \to 0} d(T + f + \delta J, G, y_0), \]

where the mapping \( f + \delta J \) is of class \((S)_+\). This extension has properties similar to the corresponding extension of the degree function to pseudo-monotone mappings in §5.

We turn now to the uniqueness result for the degree function over the class \( T + f \), with \( T \) maximal monotone, \( f \) of class \((S)_+\) and bounded.

**Theorem 12.** Let \( X \) be a reflexive Banach space. Then there exists exactly one degree function on the class of maps \( T + f \), with \( T \) maximal monotone and \( f \) bounded and of class \((S)_+\), which is normalized by \( J \) and such that the degree is invariant under all affine homotopies of the form

\[ (1 - t_t)(T + f) + tf, \]

with \( T \) maximal monotone, \( f \) and \( f_t \) of class \((S)_+\).

**Proof of Theorem 12.** Let \( d_1 \) be such a degree function. When restricted to the maps of class \((S)_+\), it coincides with the unique degree function on that class. Hence, we must show that this unique identification persists when we pass to the broader class \((T + f)\).

Suppose \( y_0 \notin (T + f)(\partial G) \). Consider the affine homotopy between \( (T + f) \) and the approximating map \( (T_\varepsilon + f) \) for small \( \varepsilon > 0 \) which we employed above in proving the existence of the degree. If \( d_1 \neq d \), then for arbitrary small \( \varepsilon > 0 \), we can find \( u_\varepsilon \) on \( \partial G \) and \( t_\varepsilon \) in \([0, 1]\) such that

\[ (1 - t_\varepsilon)T(u_\varepsilon) + t_\varepsilon T_\varepsilon(u_\varepsilon) + f(u_\varepsilon) \equiv y_0. \]
We may choose a sequence \( u_j \) in \( \partial \mathcal{G} \), \( u_j \to u \), \( t_j \to t \), \( \varepsilon_j \to 0 \), \( f(u_j) \to v_0 \). For each \( j \), we have \( w_j \in T(u_j) \) such that

\[
(1 - t_j)w_j + t_j T_{\varepsilon_j}(u_j) + f(u_j) = y_0.
\]

We now apply a variant of the argument used in the proof of Proposition 16. If \( v_j = T_{\varepsilon_j}(u_j) \), then

\[
v_j \in T(u_j - \varepsilon_j J^{-1}(v_j)).
\]

Similarly, \( w_j \in T(u_j) \). Hence

\[
\varepsilon_j \|v_j\|^2 \leq \langle v_j, u_j \rangle, \quad 0 \leq \langle w_j, u_j \rangle.
\]

If we multiply these inequalities by \((1 - t_j)\) and \(t_j\), respectively, and add, we obtain

\[
t_j \varepsilon_j \|v_j\|^2 \leq \langle y_0 - f(u_j), u_j \rangle \leq M.
\]

Hence \( t_j \varepsilon_j \|v_j\|^2 \) is bounded, so that \( t_j \varepsilon_j \|v_j\| \to 0 \).

Let \([x, y]\) be any element of \( G(T) \). Then

\[
\langle v_j - y, u_j - \varepsilon_j J^{-1}(v_j) - x \rangle \geq 0, \quad \langle w_j - y, u_j - x \rangle \geq 0.
\]

Thus,

\[
\langle v_j - y, u_j - x \rangle \geq \langle v_j - y, \varepsilon_j J^{-1}(v_j) \rangle \geq \langle -y, \varepsilon_j J^{-1}(v_j) \rangle \geq -\varepsilon_j \|v_j\| \|y\|.
\]

If we multiply the inequalities by \( t_j \) and \((1 - t_j)\), respectively, and add, we obtain

\[
\langle y_0 - f(u_j) - y, u_j - x \rangle \geq -\varepsilon_j t_j \|v_j\| \|y\|,
\]

where the term on the right approaches zero as \( j \to +\infty \). Hence, if \( z_j = y_0 - f(u_j) \), then \( z_j \to z = y_0 - v \), and

\[
\lim \langle z_j - y, u_j - x \rangle \geq 0.
\]

On the other hand, since \( z_j + f(u_j) = y_0 \), we have

\[
\langle z_j, u_j - u \rangle + \langle f(u_j), u_j - u \rangle = \langle y_0, u_j - u \rangle,
\]

i.e.

\[
\lim \langle z_j, u_j - u \rangle + \lim \langle f(u_j), u_j - u \rangle \leq 0.
\]

Since \( f \) is of class \((S)_+\), \( \lim \langle f(u_j), u_j - u \rangle \geq 0 \). Hence

\[
\lim \langle z_j, u_j - u \rangle \leq 0, \quad \lim \langle z_j, u_j \rangle \leq \langle z, u \rangle.
\]

Thus

\[
\langle z, u \rangle \geq \lim \langle z_j, u_j \rangle \geq \lim \langle z_j, u_j \rangle \geq \langle y, u - x \rangle + \langle z, x \rangle.
\]

It follows that \( \langle z - y, u - x \rangle \geq 0 \) for all \([x, y]\) in \( G(T) \). By the maximal monotonicity of \( T, z \in T(u) \). Replacing \([x, y]\) by \([u, z]\), we find that \( \langle z_j, u_j \rangle \to \langle z, u \rangle \). Hence

\[
\lim \langle f(u_j), u_j - u \rangle \leq \lim \langle z_j, u - u_j \rangle = 0.
\]
Since \( f \) is of class \((S)_+\), \( u_j \to u \) so that \( u \) lies on \( \partial G \), and \( f(u_j) \to f(u) \). Thus \( f(u) = y_0 - z \), or in other words, \( y_0 \in (T + f)(u) \). This contradicts the assumption that \( y_0 \notin (T + f)(\partial G) \). The contradiction is to the assumption that the two degrees are different for \((T + f)\). Hence the two degrees coincide. Q.E.D.

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