Fully Dynamic Maximal Matching in Constant Update Time

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Abstract

Baswana, Gupta and Sen [FOCS’11] showed that fully dynamic maximal matching can be maintained in general graphs with logarithmic amortized update time. More specifically, starting from an empty graph on $n$ fixed vertices, they devised a randomized algorithm for maintaining maximal matching over any sequence of $t$ edge insertions and deletions with a total runtime of $O(t \log n)$ in expectation and $O(t \log n + n \log^2 n)$ with high probability. Whether or not this runtime bound can be improved towards $O(t)$ has remained an important open problem. Despite significant research efforts, this question has resisted numerous attempts at resolution even for basic graph families such as forests.

In this paper, we resolve the question in the affirmative, by presenting a randomized algorithm for maintaining maximal matching in general graphs with constant amortized update time. The optimal runtime bound $O(t)$ of our algorithm holds both in expectation and with high probability.

As an immediate corollary, we can maintain 2-approximate vertex cover with constant amortized update time. This result is essentially the best one can hope for (under the unique games conjecture) in the context of dynamic approximate vertex cover, culminating a long line of research.

Our algorithm builds on Baswana et al.’s algorithm, but is inherently different and arguably simpler. As an implication of our simplified approach, the space usage of our algorithm is linear in the (dynamic) graph size, while the space usage of Baswana et al.’s algorithm is always at least $\Omega(n \log n)$.

Finally, we present applications to approximate weighted matchings and to distributed networks.

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1 Introduction

1.1 Dynamic maximal matching. For any graph $G = (V, E)$, with $n = |V|, m = |E|$, an (inclusion-wise) maximal matching $M = \mathcal{M}(G)$ can be computed in time $O(n + m)$ via a naïve greedy algorithm. A fundamental challenge is to efficiently maintain a maximal matching in a fully dynamic setting.

Starting from an empty graph $G_0$ on $n$ fixed vertices, at each time step $i$ a single edge $(u, v)$ is either inserted to the graph $G_{i-1}$ or deleted from it, resulting in graph $G_i$, and the dynamic algorithm should update the maintained matching $M = \mathcal{M}(G_{i-1})$ to preserve maximality w.r.t. the new graph $G_i$. The holy grail is that the total runtime of the algorithm would be linear in the total number $t$ of update steps. Put in other words, if the total runtime is $T$, the amortized update time $T/t$ should be constant.

There is a naïve deterministic algorithm for maintaining a maximal matching with an update time of $O(n)$. In 1993, Ivković and Lloyd devised a deterministic algorithm with update time $O((n+m)^{\sqrt{2}})$, which, despite being quite involved, is inferior to the naïve algorithm in the regime $m = \omega(n^{\sqrt{2}})$. The state-of-the-art deterministic update time, by Neiman and this author, is $O(\sqrt{m})$.

Baswana, Gupta and Sen used randomization to obtain an exponential improvement in the update time, under the oblivious adversarial model. Specifically, they devised a randomized algorithm for maintaining a maximal matching over any sequence of $t$ edge insertions and deletions with a total runtime of $O(t \log n)$ in expectation and $O(t \log n + n \log^2 n)$ with high probability (w.h.p.). In other words, the (amortized) update time of their algorithm is $O(\log n)$ in expectation and $O(\log n + n \log^2 n)$ w.h.p. (For $t = o(n \log n)$, the high probability bound becomes super-logarithmic; e.g., for $t = \Theta(n)$, it is $O(\log^2 n)$.)

Whether or not the update time of [3] can be improved towards constant has remained an important open problem. Some progress towards its resolution was made for uniformly sparse graphs, such as forests, planar graphs and graphs excluding fixed minors. However, the state-of-the-art update time in such graphs is $O(\sqrt{\log n})$, even in forests, which is still far from constant.

Our contribution. We resolve this basic question in the affirmative, by presenting a randomized algorithm for maintaining maximal matching in general graphs with constant update time. The optimal runtime bound $O(t)$ of our algorithm holds both in expectation and w.h.p. (See Table 1 in App. A.)

Our algorithm builds on Baswana et al.’s algorithm [3], but is inherently different. Also, it is arguably much simpler, both conceptually and technically. This simplification is, in our opinion, an important contribution by itself. As a practical implication of our simplified approach, we implement the algorithm using optimal space $O(n + m)$, where $m$ stands for the dynamic number of edges in the graph. This should be contrasted to the space usage $O(n \log n + m)$ of [3], which is suboptimal when $m = o(n \log n)$. In particular, whenever $m = O(n)$ (which is always the case in uniformly sparse graphs, e.g., planar graphs), this leads to a logarithmic space improvement. Although simpler than [3], our algorithm is far from being simple – indeed, it is tricky and sophisticated. (See Sections 1.4 and 2 for details.)

1.2 Dynamic approximate matching and vertex cover. On static graphs, the classic maximum cardinality matching (MCM) algorithms run in $O(m \sqrt{n})$ time. For dynamic MCMs, Sankowski devised a randomized algorithm with update time $O(n^{1.495})$. On the other hand, finding a minimum

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1In fact, the naïve update time bound holds in the worst case. In this paper we focus on amortized time bounds, and so in some of the literature survey that follows we do not stress the distinction between amortized versus worst-case bounds.

2The oblivious adversarial model is a standard model, which has been used for analyzing randomized data-structures such as universal hashing and dynamic connectivity. The model allows the adversary to know all the edges in the graph and their arrival order, as well as the algorithm to be used. However, the adversary is not aware of the random bits used by the algorithm, and so cannot choose updates adaptively in response to the randomly guided choices of the algorithm.

3Abboud and Vassilevska Williams proved conditional lower bounds for dynamic MCMs, showing that the update time must be $\Omega(m^\epsilon)$ under conjectures concerning the complexity of triangle detection, combinatorial boolean matrix multiplication and 3SUM, where $\epsilon$ is a constant depending on the specific conjecture. For example, under the 3SUM conjecture, it was shown that $\epsilon \geq 1/3$; moreover, under this conjecture, a stronger bound of $\epsilon \geq 1/2 - o(1)$ was given in [18].
cardinality vertex cover (MCVC) is NP-hard. Moreover, under the unique games conjecture (UGC), the MCVC cannot be efficiently approximated within any factor better than 2 \cite{16}. Thus, while the ultimate goal in the context of dynamic MCMs is to efficiently maintain an exact MCM (or maybe a (1+\(\epsilon\))-MCM), the analogous goal for dynamic MCVCs is to maintain a 2-MCVC, where \(t\)-MCM (resp., \(t\)-MCVC) is a shortcut for \(t\)-approximate MCM (resp., \(t\)-approximate MCVC), for any \(t \geq 1\). Despite the inherent difference between these problems, they remain closely related as their LP-relaxations are duals of each other. Note also that (1) the MCM and MCVC are of the same size up to a factor of 2; (2) the set of matched vertices in a maximal matching is a 2-MCVC; (3) a maximal matching is a 2-MCM.

In view of the difficulty of dynamically maintaining exact MCMs and MCVCs, there has been a growing interest in dynamic approximate matching and vertex cover since the pioneering work of Onak and Rubinfeld \cite{23}, who presented a randomized algorithm for maintaining \(c\)-MCM and \(c\)-MCVC, where \(c\) is a large unspecified constant. The update time of the algorithm of \cite{23} is w.h.p. \(O(\log^2 n)\).

The dynamic maximal matching algorithms discussed in Section 1.1 provide 2-MCMs. Moreover, they imply dynamic algorithms for maintaining 2-MCVCs with the same (up to constants) update time. In particular, Baswana et al.’s algorithm \cite{3} can be used to maintain 2-MCM and 2-MCVC with update time \(O(\log n)\) in expectation and \(O(\log n + n\log^2 n)\) w.h.p., significantly improving the result of \cite{23}.

There are many other works on dynamic approximate MCMs and MCVCs; we briefly mention the state-of-the-art results: \cite{6} presented a dynamic algorithm for \((2 + \epsilon)\)-MCVCs with update time \(O(\log n \cdot \epsilon^{-2})\); \cite{10} gave an algorithm for \((1 + \epsilon)\)-MCMs with update time \(O(\sqrt{m} \cdot \epsilon^{-2})\); \cite{4, 5} devised dynamic algorithms for \((3/2 + \epsilon)\)-MCMs in general graphs with update time \(O(m^{1/4} \cdot \epsilon^{-2.5})\). For uniformly sparse graphs, \cite{25} presented algorithms for maintaining \((1 + \epsilon)\)-MCMs and \((2 + \epsilon)\)-MCVCs with update times that depend on the density (or arboricity) of the graph and \(\epsilon\). (See Tables 2 and 3 in App. A.)

Our contribution. Our dynamic maximal matching algorithm can be used to maintain 2-MCM and 2-MCVC in general graphs with update time that is bounded both in expectation and w.h.p. by constant. Under the UGC, a \((2 - \epsilon)\)-MCVC cannot be efficiently maintained, for any \(\epsilon > 0\). Hence, our dynamic 2-MCVC algorithm is essentially the best one can hope for, culminating a long line of research.

1.3 Additional applications. Static and dynamic algorithms for exact and approximate maximum weighted matchings have been extensively studied, both in centralized and distributed networks (see, e.g., \cite{19, 20, 26, 2, 10, 8, 9}). In particular, Anand et al. \cite{2} gave a randomized algorithm for maintaining an 8-approximate maximum weight matching (8-MWM) in general \(n\)-vertex weighted graphs with expected update time \(O(\log n \log \Delta)\), where \(\Delta\) is the ratio between the maximum and minimum edge weights in the graph. The algorithm of \cite{2} employs the dynamic maximal matching algorithm of \cite{3} as a black-box. By plugging our improved algorithm, we shave a factor of \(\log n\) from the update time, obtaining an algorithm for maintaining an 8-MWM in general weighted graphs with expected update time \(O(\log \Delta)\).

Another application of our dynamic maximal matching algorithm is to distributed networks. In a static distributed setting all processors wake up simultaneously, and computation proceeds in fault-free synchronous rounds during which every processor exchanges messages of size \(O(\log n)\). In a dynamic distributed setting, however, upon the insertion or deletion of an edge \(e = (u, v)\), only the affected vertices \((u\ and v)\) are woken up. In this setting, the goal is to optimize (1) the number of communication rounds, and (2) the number of messages, needed for repairing the solution per update operation, over a worst-case sequence of update operations. Following an edge update, \(O(1)\) communication rounds trivially suffice for updating a maximal matching (and thus 2-MCM and 2-MCVC). However, the number of messages sent per update may be as high as \(O(n)\). Our dynamic maximal matching algorithm can be easily distributed, so that the average number of messages sent per update is a small constant.

A more detailed discussion on the applications of our main result is deferred to App. D.\footnote{We consider the standard \textsc{CONGEST} model (cf. \cite{24}), which captures the essence of spatial locality and congestion.}
1.4 Our and Previous Techniques. At the core of Baswana et al. (hereafter, BGS) algorithm is a complex bucketing scheme, where each vertex is assigned a unique level among \( \{-1, 0, \ldots, \log n\} \) according to some stringent criteria. (See Sect. 2 for more details.) The BGS algorithm constantly and persistently changes the level of vertices. Changing the level of a single vertex may trigger a sequence of changes to multiple lower level vertices, referred to in [3] as a fading wave. The cost of a fading wave may be linear in the level of the vertex initiated it, thus the update time is \( O(\log n) \). This is not the only logarithmic-time bottleneck incurred by the fading wave mechanism. Indeed, a central ingredient of this mechanism determines the “right” level to which a vertex should move. To this end, an explicit counter \( \phi_v[j] \) is maintained for each vertex \( v \) and any level \( j \in \{-1, 0, \ldots, \log n\} \). Following a single update on vertices \( u \) and \( v \), the BGS algorithm has to update a logarithmic number of counters for \( u \) and \( v \). Moreover, finding the right level to which a vertex should move using these counters cannot be done in sublogarithmic time (via a binary search on any other search method) due to the special nature of these counters. On top of these logarithmic-time bottlenecks, the fading wave mechanism of BGS is highly intricate, in terms of both the implementation and the analysis.

To break the logarithmic-time barrier, one has to take a significantly different approach, which circumvents the fading wave mechanism, and more importantly, the usage of the counters \( \phi_v[j] \). Although our approach builds on the BGS scheme, it borrows mainly from its simpler ideas, successfully avoiding the use of the more complicated ones. Thus, despite the resemblance between the two approaches, our algorithm deviates significantly from the BGS scheme. We too use a bucketing scheme with \( O(\log n) \) levels. However, to remove the dependency on the number of levels from the update time, our level-maintenance scheme cannot follow stringent criteria as in BGS. In particular, we use a “lazy” approach, which changes the level of a vertex only when necessary. When this happens, it essentially means that the vertex will “rise” to a higher level. As opposed to the BGS algorithm that uses counters to determine the “right” level, our algorithm figures out the “right” level “on the fly” via a certain “level-rising mechanism”. (The “right” level computed by our algorithm is different than that computed by the BGS algorithm.)

The “level-rising mechanism” is a central ingredient in our algorithm, and is obtained by combining ideas from BGS with numerous novel fundamental ideas. (See Sections 2 and 3 for details.) We anticipate that our level-rising mechanism will be applicable to additional dynamic graph problems under the oblivious adversarial model, where some subgraph structure has to be maintained. (At the very least, it should give rise to improved algorithms for maintaining dynamic \((2-\epsilon)\)-MCMs. In fact, we have recently achieved some new results in this context, but they lie outside the scope of the current paper.)

2 A High-Level Technical Overview

The Basics. Let \( M \) be a maximal matching for the graph \( G = (V, E) \) at a certain update step. Any edge of \( M \) (resp., \( E \setminus M \)) is called matched (resp., unmatched). A vertex incident on a matched edge is called matched, and the other endpoint is its mate; otherwise it is free. Consider the following naïve algorithm for maintaining a maximal matching. Following an edge update \( e = (u, v) \), if \( e \) is inserted to the graph and its two endpoints are free, the algorithm adds \( e \) to \( M \). If \( e \) is deleted from the graph and \( e \in M \), the algorithm first removes \( e \) from \( M \). Next, let \( z \in \{u, v\} \); if \( z \) is free, the algorithm scans all neighbors of \( z \) looking for a free vertex. If a free neighbor \( w \) of \( z \) is found, the edge \( (z, w) \) is added to \( M \). Clearly, the update time is constant except for the case that a matched edge gets deleted from the graph, and then the update time is \( O(\deg(u) + \deg(v)) \), which may be as high as \( O(n) \), even in forests.

Following an edge deletion \( (u, v) \), the naïve algorithm matches \( z \in \{u, v\} \) with an arbitrary free neighbor \( w \). What if we match \( z \) with a random neighbor \( w \)? Under the oblivious adversarial model, the expected number of edges incident on \( z \) that are deleted from the graph before deleting edge \((z, w)\) is \( \deg(z)/2 \). Hence, the expected amortized cost of the deletion of edge \((z, w)\) should be \( O(\deg(z)/\deg(w)) \). If \( \deg(w) = O(\deg(z)) \), this cost is constant; however, in the general case it may be as high as \( O(n) \).
Instead of choosing \( w \) randomly among all neighbors of \( z \), one may restrict the attention to the low-degree neighbors of \( z \), i.e., with degree \( O(\deg(z)) \). Denoting by \( \deg_{\text{low}}(z) \) the number of such neighbors, the amortized cost becomes \( \frac{O(\deg(z))}{\deg_{\text{low}}(z)/2} \). However, if \( \deg_{\text{low}}(z) \ll \deg(z) \), this cost may again be too high.

Moreover, even under the optimistic assumption that \( \deg(z) = O(\deg_{\text{low}}(z)) \), there is still a fundamental problem: A random neighbor of \( z \) (of low degree or not) may be matched. To match \( z \) with a neighbor \( w \) that is matched to say \( w' \), we must first delete edge \( (w, w') \) from \( M \). However, the cost of such induced deletions cannot be bounded in the amortized sense, as the adversarial argument does not hold.

**Baswana et al.’s scheme.** The BGS scheme [3] can be described as follows. Each edge is assigned an orientation, yielding a directed graph, and the vertices are given preferences according to their out-degrees. The edge’s orientation also induces responsibility for one of its endpoints: If edge \( (u, v) \) is oriented as \( u \to v \), then it is within \( u \)’s responsibility to notify \( v \) about any change concerning \( u \). Therefore, at any point in time, each vertex knows the updated information of its incoming neighbors. In order to obtain a complete information of all its neighbors, the vertex just needs to scan its outgoing neighbors. Thus, the point in time, each vertex knows the updated information of its incoming neighbors. In order to obtain a logarithmic number of optional levels. Besides these logarithmic-time bottlenecks, recall that the adversarial argument does not apply to the induced deletions (by the algorithm) of matched edges. Whenever a vertex \( z \) is matched to a random neighbor \( w \) that is already matched, say to vertex \( w' \), the
cost of deleting edge \((w,w')\) from the matching can be charged to the new matched edge \((z,w)\). For this charging argument to work, it is crucial to have \(\ell_w < \ell_z\), i.e., a mate should not be chosen randomly among all outgoing neighbors, but rather only among those with \textit{strictly lower} level. The charging argument of [3] shows that the cost of induced deletions, when applied recursively, is logarithmic. Indeed, as the “fading wave” may start at level \(\Omega(\log n)\), it is only natural to incur a logarithmic time bound.

Our approach. Our algorithm builds on the BGS scheme [3], but is inherently different. We too use a bucketing scheme with \(O(\log n)\) levels, and orient each edge towards the lower level endpoint. However, our level-maintenance scheme is, in some sense, opposite to that of BGS. The invariant of [3] translates into \(\log(n-1)-\ell_v\) inequalities for each vertex \(v\), which are maintained by the algorithm at all times. By maintaining these inequalities, the level \(\ell_v\) of each vertex \(v\) is kept close to \(\log(d_{\text{out}}(v))\), with a persistent attempt to rise to a higher level. As opposed to [3], we use a “lazy” approach, and in particular, we do not try to satisfy any of these inequalities. The level of a vertex \(v\) will be changed by our algorithm only when absolutely necessary, namely, following deletions or insertions of matched edges incident on \(v\).

Consider a deletion of a matched edge \((z,w)\) at update step \(l\), suppose that \(w\) was chosen uniformly at random among \(z\)’s outgoing neighbors as a mate for \(z\) at some update step \(l'\), for \(l' < l\), and denote \(z\)’s out-degree at step \(l'\) by \(d'_{\text{out}}(z)\). Our algorithm sets \(z\)’s level \(\ell_z\) at step \(l'\) to roughly \(\log(d'_{\text{out}}(z))\), and leaves it unchanged throughout update steps \(l' + 1, l' + 2, \ldots , l\). The expected number of outgoing edges of \(z\) that are deleted from the graph during this time interval is \(d'_{\text{out}}(z)/2\), thus in the amortized sense, we should be able to cover an expected cost of \(O(d'_{\text{out}}(z))\), for handling the deletion of edge \((z,w)\).

One consequence of our “lazy” approach is that we cannot bound the out-degree \(d_{\text{out}}(z)\) of \(z\) at step \(l\) in terms of \(\ell_z\) as in [3], and in particular, it may be that \(d_{\text{out}}(z) \gg 2^{\ell_z} \approx d'_{\text{out}}(z)\). If it turns out that \(d_{\text{out}}(z) = O(d'_{\text{out}}(z))\), then we can afford to scan all \(z\)’s outgoing neighbors. Hence, we can handle \(z\) deterministically, i.e., match it with an arbitrary free outgoing neighbor, or leave it free if none exists. (We will guarantee that no incoming neighbor of \(z\) is free, hence we do not ignore free neighbors of \(z\).)

The interesting case is when \(d'_{\text{out}}(z) \gg d_{\text{out}}(z)\). In this case we let \(z\) rise to a higher level. However, we cannot let \(z\) rise to the \textit{highest possible} level \(\ell\) for which \(\phi_z(\ell) \geq 2^\ell\) as in [3], since we do not maintain the values of \(\phi_z(\cdot)\). Instead, we let \(z\) rise \textit{gradually}, as long as \(\phi_z(\ell) \geq 2^\ell\), stopping at the lowest level \(\ell^*\) after \(\ell_z\) for which \(\phi_z(\ell^* + 1) < 2^{\ell^*+1}\). If careless, the runtime of this rising process may be prohibitively large. However, here we crucially exploit the fact that we have complete information of \(z\)’s incoming neighbors. Indeed, using this information, we can restrict the attention to \(z\)’s neighbors of level at most \(\ell^*\), which enables us to implement the entire “rising process” in \(\phi_z(\ell^* + 1) = O(2^{\ell^*})\) time. While this runtime may still be too large, we are guaranteed that the number of new outgoing neighbors of \(z\) of level strictly lower than \(\ell^*\) is \(\phi_z(\ell^*) \geq 2^{\ell^*}\). This means that we can cover the cost of the rising process by matching \(z\) to a random such neighbor \(w\). Indeed, since the matched edge \((z,w)\) is chosen with probability at most \(1/2^{\ell^*}\), we can cover an expected cost of \(O(2^{\ell^*})\) in the amortized sense. Summarizing:

\textbf{Observation 2.1} For every neighbor of \(z\) scanned during this rising process, we spend \(O(1)\) time. To compensate for this, \(z\)’s mate is chosen uniformly at random among a constant fraction of these neighbors.

After matching \(z\) with \(w\), we must let \(w\) rise to the same level \(\ell^*\) as \(z\), which requires flipping all incoming edges of \(w\) from vertices at levels between \(\ell_w\) and \(\ell^* - 1\). Alas, as another consequence of our lazy approach, \(w\)’s new out-degree may be much larger than \(2^{\ell^*}\). We can cover the cost of these flips by applying our level-rising mechanism on \(w\), i.e., we let \(w\) rise to yet a higher level, stop at the lowest level \(\ell^*\) after \(\ell\) for which \(\phi_w(\ell^* + 1) < 2^{\ell^*+1}\), and then match \(w\) to a random outgoing neighbor of level lower than \(\ell^*\). To this end, however, we must first delete edge \((z,w)\) from the matching, which requires handling \(z\) (as a free vertex) from scratch. Consequently, in contrast to the rather orderly manner in which vertices’ levels are changed by the BGS algorithm (particularly by the fading wave mechanism), the manner in which vertices’ levels are changed by our algorithm (particularly by the level-rising mechanism) is rather chaotic. Nevertheless, we show that our mechanism can be implemented via an elegant recursive
algorithm, whose analysis boils down to a sophisticated application of Observation 2.1. In this way we bypass the use of the $\phi_v(\cdot)$ values, ultimately achieving the optimal runtime and space bounds.

As mentioned in Sect. 1.4, we believe that our level-rising mechanism will be applicable (after proper adjustments) to additional dynamic graph problems, and it would be interesting to find such applications.

3 The Update Algorithm

The update algorithm is applied following edge insertions and deletions to and from the graph.

3.1 Invariants and data structures. Our algorithm maintains for each vertex $v$ a level $\ell_v$, with $-1 \leq \ell_v \leq \log_3(n - 1)$. (We use logarithms in base 3, whereas [3] use logarithms in base 2; this change simplifies the analysis, but is not crucial.) The algorithm will maintain the following invariants.

Invariant 1 An edge $(u, v)$ with $\ell_u > \ell_v$ is oriented by the algorithm as $u \to v$. (In the case that $\ell_u = \ell_v$, the orientation of $(u, v)$ will be determined suitably by the algorithm.)

Invariant 2 (a) Any free vertex has level -1 and out-degree 0. (Thus the maintained matching is maximal.) (b) Any matched vertex has level at least 0. (c) The two endpoints of any matched edge are of the same level, and this level remains unchanged until the edge is deleted from the matching. (We may henceforth define the level of a matched edge, which is at least 0 by item (b), as the level of its endpoints.)

For each vertex $v$, we maintain linked lists $N_v$ and $O_v$ of its neighbors and outgoing neighbors, respectively. The information about $v$’s incoming neighbors will be maintained via a more detailed data structure $I_v$: A hash table, where each element corresponds to a distinct level $\ell \in \{-1, 0, \ldots, \log_3(n - 1)\}$. Specifically, an element $I_v[\ell]$ of $I_v$ corresponding to level $\ell$ holds a pointer to the head of a non-empty linked list that contains all incoming neighbors of $v$ with level $\ell$. If that list is empty, then the corresponding pointer is not stored in the hash table. While the number of pointers stored in the dynamic hash table $I_v$ is bounded by $\log_3(n - 1) + 2$, it may be much smaller than that. (Indeed, there is no pointer in $I_v$ corresponding to level smaller than $\ell_v$, thus the number of pointers stored in $I_v$ is at most $\log_3(n - 1) + 1 - \ell_v$. Also, by Invariant 2 no incoming neighbor of $v$ is of level -1, thus there is no pointer in $I_v$ corresponding to level -1.) In particular, the total space over all hash tables is linear in the dynamic number of edges in the graph. We can use a static array of size $\log_3(n - 1) + 2$ instead of a dynamic hash table, but the total space usage over all these arrays will be $\Omega(n \log n)$. In the case that the dynamic graph is usually dense (i.e., having $\Omega(n \log n)$ edges), it is advantageous to use arrays, as it is easier to implement all the basic operations (delete, insert, search), and the time bounds become deterministic.

Note that no information whatsoever on the levels of $v$’s outgoing neighbors is provided by the data structure $O_v$. In particular, to determine if $v$ has an outgoing neighbor at a certain level (most importantly at level -1, i.e., a free neighbor), we need to scan the entire list $O_v$. On the other hand, $v$ has an incoming neighbor at a certain level $\ell$ iff the corresponding list $I_v[\ell]$ is non-empty. It will not be in $v$’s responsibility to maintain the data structure $I_v$, but rather within the responsibility of $v$’s incoming neighbors.

We keep mutual pointers between the elements in the various data structures: For any vertex $v$ and any outgoing neighbor $u$ of $v$, we have mutual pointers between all elements $u \in O_v, v \in I_u[\ell_u], v \in N_u, u \in N_v$. For example, when an edge $(u, v)$ oriented as $v \to u$ is deleted from the graph, we get a pointer to either $v \in N_u$ or $u \in N_v$, and through this pointer we delete all elements $u \in O_v, v \in I_u[\ell_u], v \in N_u, u \in N_v$. As another example, when the orientation of edge $(u, v)$ is flipped from $u \to v$ to $v \to u$, then assuming we have a pointer to $v \in O_u$ (which is the case in our algorithm), we can reach element $u \in I_v[\ell_u]$ through the pointer, delete them both from the respective lists, and then create elements $u \in O_v$ and $v \in I_u[\ell_v]$ with mutual pointers. We also keep mutual pointers between a matched edge and its endpoints. (We do not provide a complete description of the trivial maintenance of these pointers for the sake of brevity.)

Following [3], we define $\phi_v(\ell)$ to be the number of neighbors of $v$ with level strictly lower than $\ell$. 
3.2 Procedure set-level($v, \ell$). Whenever the update algorithm examines a vertex $v$, it may need to re-evaluate its level. After the new level $\ell$ is determined, the algorithm calls Procedure set-level($v, \ell$). (See Figure 3.) The procedure starts by updating the outgoing neighbors of $v$ about $v$’s new level. Specifically, we scan the entire list $O_v$, and for each vertex $w \in O_v$, we move $v$ from $I_w[\ell_v]$ to $I_w[\ell]$. Suppose first that $\ell < \ell_v$. In this case the level of $v$ is decreased by at least one. As a result, we need to flip the outgoing edges of $v$ towards vertices of level between $\ell + 1$ and $\ell_v$ to be incoming to $v$. Specifically, we scan the entire list $O_v$, and for each vertex $w \in O_v$ such that $\ell + 1 \leq \ell_w \leq \ell_v$, we perform the following operations: Delete $w$ from $O_v$, add $w$ to $I_v[\ell_w]$, delete $v$ from $I_w[\ell]$, and add $v$ to $O_w$. If $\ell > \ell_v$, the level of $v$ is increased by at least one. As a result, we flip $v$’s incoming edges from vertices of level between $\ell_u$ and $\ell - 1$ to be outgoing of $v$. Specifically, for each non-empty list $I_v[i]$, with $\ell_v \leq i \leq \ell - 1$, and for each vertex $w \in I_v[i]$, we perform the following operations: Delete $w$ from $I_v[i]$, add $w$ to $O_v$, delete $v$ from $O_w$, and add $v$ to $I_w[\ell]$. Note, however, that we do not know for which levels $i$ the corresponding list is non-empty; the time overhead needed to verify this information is $O(\ell)$. Only after the data structures have been updated, we set $\ell_v = \ell$.

The next observation is immediate from the description of the procedure, assuming Invariants 1 and 2 hold. In particular, it shows that the runtime of this procedure is at most $O(d_{out}^{old}(v) + d_{out}^{new}(v) + \ell)$, where $d_{out}^{new}(v)$ and $d_{out}^{old}(v)$ denote $v$’s out-degree before and after the execution of this procedure, respectively.

**Observation 3.1** Let $\ell_v$ denote the out-degree of $v$ before the execution of this procedure.

1. If $\ell = \ell_v$, the procedure does nothing, and the runtime is constant.
2. If $\ell < \ell_v$, then $d_{out}^{new}(v) \leq d_{out}^{old}(v)$ and the procedure’s runtime is $O(d_{out}^{old}(v))$.
3. If $\ell > \ell_v$, then $d_{out}^{new}(v) \geq d_{out}^{old}(v)$ and the procedure’s runtime is $O(d_{out}^{new}(v) + \ell)$. Moreover, after the execution of the procedure, all outgoing neighbors of $v$ are of level at most $\ell - 1$. In the particular case of $\ell_v = -1$ and $\ell = 0$, which occurs when a free vertex $v$ becomes matched at level 0, we have $d_{out}^{new}(v) = 0$; hence the procedure’s runtime in this case is constant.

3.3 Procedures handle-insertion($u, v$) and handle-deletion($u, v$).

Following an edge insertion $(u, v)$, we apply Procedure handle-insertion($u, v$); see Figure 4 in App. A. Besides updating the relevant data structures in the obvious way, this procedure matches between $u$ and $v$ if they are both free, or it leaves them unchanged. Matching $u$ and $v$ involves setting their level to 0 by making the calls set-level($u, 0$) and set-level($v, 0$), whose runtime is $O(1)$ by Observation 3.1(3).

Following an edge deletion $(u, v)$, we apply Procedure handle-deletion($u, v$); see Figure 2. If edge $(u, v)$ does not belong to the matching, we only need to update the relevant data structures. In the case that edge $(u, v)$ belongs to the matching, both $u$ and $v$ become temporarily free, meaning that they are not matched to any vertex yet, but their level remains temporarily as before. (We handle them next, one after another, but until each of them is handled, its level will exceed -1.) We handle vertices $u$ and $v$ via Procedure handle-free, specifically, by calling handle-free($u$) and later handle-free($v$); Procedure handle-free is the main ingredient of the update algorithm, and is described in Section 3.4.

3.4 Procedure handle-free($v$). The execution of this procedure splits into two cases. See Figure 4.

**Case 1:** $d_{out}^{old}(v) < 3^{\ell_v+1}$. In other words, the first case is when the out-degree of $v$ is not much greater than $3^{\ell_v}$, and we run Procedure deterministic-settle($v$) described in Section 3.4.1 (see Figure 5).

**Case 2:** $d_{out}^{old}(v) \geq 3^{\ell_v+1}$. We run Procedure random-settle($v$) described in Section 3.4.2 (see Figure 6).

3.4.1 Procedure deterministic-settle($v$). The procedure starts by scanning the list $O_v$ for a free vertex. By Invariant 2(a), if no free vertex is found in $O_v$, then $v$ does not have any free neighbor. (Note that we ignore temporarily free neighbors of $v$. See Sect. 4.1 for more details.) If no free vertex is found in $O_v$, then $v$ becomes free and we set its level $\ell_v$ to -1 by calling set-level($v, -1$).
Otherwise a free vertex is found in $O_v$. In this case we match $v$ to an arbitrary such vertex $w$, and set the levels of $v$ and $w$ to $0$ by calling to set-level($v, 0$) and set-level($w, 0$).

Next, we show that the runtime of the procedure is bounded by $O(3^\ell v)$. Let $d_{\text{out}}^{\text{old}}(v)$ denote $v$'s out-degree before the execution of the procedure, and note that this procedure is invoked only when $d_{\text{out}}^{\text{old}}(v) < 3^\ell v + 1$. Moreover, since $v$ is a temporarily free vertex, its level at this stage is at least $0$.

The procedure starts by scanning the list $O_v$ for a free vertex, which takes $O(d_{\text{out}}^{\text{old}}(v)) = O(3^\ell v)$ time. Next, the procedure calls to either set-level($v, -1$) or set-level($v, 0$). Since the level of $v$ prior to either one of these calls is at least $0$, $v$'s level may only decrease, hence the runtime is bounded by $O(d_{\text{out}}^{\text{old}}(v)) = O(3^\ell v)$ by Observations 3.1(1) and 3.1(2). Finally, there is a potential call to set-level($w, 0$), which increases the level of $w$ from $-1$ to $0$; the runtime of this call is constant by Observation 3.1(3).

We remark that the out-degree of $v$ after the potential call to set-level($v, 0$) may be large, since it may have many outgoing neighbors of level $0$. However, by Observations 3.1(1) and 3.1(2), this out-degree is bounded by $d_{\text{out}}^{\text{old}}(v) < 3^\ell v + 1$. Even though we can afford to flip all the edges leading to those vertices (this would require at most $O(3^\ell v)$ time, which we spend anyway), there is no need for it.

3.4.2 Procedure random-settle($v$). The procedure employs what we refer to as a level-rising mechanism. Roughly speaking, the procedure matches $v$ at some level $\ell^*$ higher than $\ell_v$, with a random (possibly matched) neighbor of level strictly lower than $\ell^*$. More accurately, it attempts to create such a matched edge $\langle v, w \rangle$ at level $\ell^*$ within time $O(3^\ell v)$; upon failure, it calls itself recursively to match $w$ at yet a higher level, in which case $v$ becomes free and handled via Procedure handle-free($v$).

Next, we describe this procedure, and the underlying level-rising mechanism, in detail. We find it instructive to provide this description in two stages. In the first stage we outline the two main challenges that this procedure has to cope with, and the specific manner in which it copes with these challenges. Only after the appropriate intuition is established, we turn to the formal description of the procedure.

Challenge 1. Recall that $\phi_v(\ell)$ is the number of $v$'s neighbors of level strictly lower than $\ell$. The reason we restrict our attention to neighbors of $v$ of level strictly lower than a certain threshold is fundamental. When choosing a random mate $w$ for $v$, we cannot guarantee that $w$ would be free. Assuming $w$ is matched to $w'$, matching $v$ with $w$ triggers the deletion of edge $\langle w, w' \rangle$ from the matching. This edge deletion is induced by the algorithm itself rather than the adversary, i.e., the edge remains in the graph but deleted from the matching. Alas, the adversarial argument, which bounds the expected number of edges that are deleted from the graph until the matched edge is deleted from the matching, does not hold for induced deletions. Coping with induced deletions, which is the crux of the problem, requires an intricate charging argument. As in [3], when choosing a mate $w$ for $v$, we restrict our attention to $v$'s neighbors of level strictly lower than that of $v$, or more accurately, strictly lower than the new level to which $v$ rises following the random match. This restriction, however, poses a nontrivial challenge.

One cannot simply set the new level of $v$ to be sufficiently large. Indeed, for our probabilistic argument to work, it is crucial that $v$'s new level would depend on the probability with which the new matched edge is chosen: If $v$'s new level is $\ell^*$, the probability that $w$ is chosen as $v$'s mate should be $O(1/3^\ell v)$. To guarantee that this condition holds, though, $v$ must have $\Omega(3^\ell v)$ neighbors of level strictly lower than $\ell^*$. It is easy to show that such a level $\ell^*$ exists (see Lemma 3.2(1)). It is much less clear, however, how to compute it efficiently. The challenge that we face is actually more complex: After setting the level $\ell_v$ of $v$ to $\ell^*$, we need to update the data structures accordingly. For this scheme to work, the entire runtime should be $O(3^\ell v)$. To see where the difficulty lies, suppose we take $\ell^*$ to be $\ell_v$. Recalling that $d_{\text{out}}(v) \geq 3^\ell v + 1$, $v$ has many neighbors of level at most $\ell^*$. However, we need that $v$ would have many neighbors of level strictly lower than $\ell^*$, and it is possible that most (or all) neighbors of $v$ have level $\ell^*$. One may try taking $\ell^*$ to be $\ell_v + 1$. This would indeed guarantee that $v$ has sufficiently many neighbors of level strictly lower than $\ell^* + 1$. However, now there in another, somewhat contradictory, problem. After setting $\ell^*$ to $\ell_v + 1$, we need to update the data structures. In particular, we must flip all the incoming
edges of \( v \) from neighbors of level \( \ell_v \) to be outgoing of \( v \). This may be prohibitively expensive.

In general, the “right” level \( \ell^* \) should balance two contradictory requirements: While \( v \) should have sufficiently many neighbors of level strictly lower than \( \ell^* \), it should not have too many of them. Any level balancing these requirements will do the job, but we also need to be able to compute it efficiently. We next show that, somewhat surprisingly, a sequential scan for the right level works smoothly.

**Computing the “right” level for \( v \):** Set \( \ell = \ell_v \), and gradually increase \( \ell \) as long as \( \phi_v(\ell) \geq 3^\ell \). Let \( \ell^* \) be the level in which we stop the process, i.e., the minimum level such that \( \ell^* \geq \ell_v \) and \( \phi_v(\ell^* + 1) < 3^{\ell^*+1} \). We then set \( v \)'s level to \( \ell^* \) by calling \texttt{set-level}(\( v \), \( \ell^* \)), thus updating the data structures, which involves flipping all incoming edges of \( v \) from neighbors of level between \( \ell_v \) and \( \ell^* - 1 \) to be outgoing of \( v \).

The following lemma is crucial to the correctness of the level-rising mechanism. It shows that the level \( \ell^* \) satisfies several somewhat contradictory requirements, each of which is important for our algorithm. We stress that, to be able to efficiently compute the level \( \ell^* \), we make critical use of the fact that we have complete information of the incoming neighbors of \( v \); see the proof of Lemma 3.2 for more details.

**Lemma 3.2** Let \( d^{old}_{\text{out}}(v) \) and \( d^{new}_{\text{out}}(v) \) denote \( v \)'s out-degree before and after the call to \texttt{set-level}(\( v \), \( \ell^* \)), respectively. Here \( \ell_v \) denotes the level of \( v \) before the call, whereas \( \ell^* \) is its level afterwards.

1. \( \ell_v < \ell^* \leq \log_3(n-1) \). In particular, a level \( \ell^* \) as required exists. Moreover, the call to \texttt{set-level}(\( v \), \( \ell^* \)) increases \( v \)'s level by at least one.

2. After this call, all outgoing neighbors of \( v \) are of level at most \( \ell^* - 1 \), i.e., \( \phi_v(\ell^*) = d^{new}_{\text{out}}(v) \).

3. \( d^{old}_{\text{out}}(v) \) and \( 3^{\ell^*} \leq d^{new}_{\text{out}}(v) = \phi_v(\ell^*) \leq \phi_v(\ell^* + 1) < 3^{\ell^*+1} \). (In particular, \( d^{old}_{\text{out}}(v) < 3^{\ell^*+1} \).)

4. The runtime of computing \( \ell^* \) and calling to \texttt{set-level}(\( v \), \( \ell^* \)) is bounded by \( O(3^{\ell^*}) \).

**Proof:** (1) By invariant 1, all outgoing neighbors of \( v \) before the call have out-degree at most \( \ell_v \), so \( \phi_v(\ell_v + 1) \geq d^{old}_{\text{out}}(v) \). We also have \( d^{old}_{\text{out}}(v) \geq 3^{\ell_v+1} \), yielding \( \phi_v(\ell_v + 1) \geq d^{old}_{\text{out}}(v) \geq 3^{\ell_v+1} \), and so \( \ell^* > \ell_v \). Since \( \phi_v(\log_3(n-1) + 1) \leq \deg(v) \leq n - 1 < 3^{\log_3(n-1)+1} \), it follows that \( \ell^* \leq \log_3(n-1) \).

(2) By the first assertion of this lemma, \( \ell^* > \ell_v \). This assertion thus follows from Observation 3.1(3).

(3) The first assertion of this lemma and Observation 3.1(3) yield \( d^{old}_{\text{out}}(v) \leq d^{new}_{\text{out}}(v) \). The second assertion of this lemma and the definitions of \( \ell^* \) and \( \phi_v(\cdot) \) yield \( 3^{\ell^*} \leq d^{new}_{\text{out}}(v) = \phi_v(\ell^*) \leq \phi_v(\ell^* + 1) < 3^{\ell^*+1} \).

(4) Recall that we have complete information of the incoming neighbors of \( v \) via the data structure \( \mathcal{L}_v \). This enables us to restrict our attention to the neighbors of \( v \) up to a certain level, and ignore the others. Specifically, for each non-empty list \( \mathcal{L}_v[\ell] \), with \( \ell = \ell_v, \ell_v + 1, \ldots \), we naïvely count the number of vertices in \( \mathcal{L}_v[\ell] \) in order to construct the values \( \phi_v(\ell) \), for \( \ell = \ell_v, \ell_v + 1, \ldots \). Stopping this process once reaching \( \ell^* + 1 \), the time spent is at most linear in the number of scanned levels plus the number of vertices in the corresponding lists \( \mathcal{L}_v[\ell] \). The number of scanned levels is bounded by \( \ell^* + 1 \) and the number of traversed vertices is bounded by \( \phi_v(\ell^*) + 1 \), which is smaller than \( 3^{\ell^*+1} \) by the third assertion of this lemma.

By being able to restrict the attention to \( v \)'s neighbors of level at most \( \ell^* + 1 \), we have shown that the runtime of computing the “right” level \( \ell^* \) is \( O(3^{\ell^*}) \). Moreover, by Observation 3.1(3), the runtime of the call to \texttt{set-level}(\( v \), \( \ell^* \)) is bounded by \( O(d^{new}_{\text{out}}(v) + \ell^*) \), which is, in turn, at most \( O(3^{\ell^*}) \) by the third assertion of this lemma. We stress that the validity of Observation 3.1(3) is also based on our ability to restrict the attention to the (incoming) neighbors of \( v \) whose level is bounded by some threshold.

**Challenge 2.** Lemma 3.2 implies that \( O(3^{\ell^*}) \) time suffices for computing the “right” level \( \ell^* \) and letting \( v \) rise to that level by making the call \texttt{set-level}(\( v \), \( \ell^* \)). Moreover, \( v \) has at least \( 3^{\ell^*} \) outgoing neighbors after this call, all having level at most \( \ell^* - 1 \). By picking uniformly at random an outgoing neighbor \( w \) of \( v \) to match with, we are guaranteed that the matched edge \((v, w)\) is chosen with probability at most \( 1/3^{\ell^*} \). In the amortized sense, this matched edge can cover the entire cost \( O(3^{\ell^*}) \) spent thus far.
In order to add edge \((v, w)\) to the matching, however, we need to update the data structures accordingly. In particular, if \(w\) is matched, say to \(w'\), we must delete edge \((w, w')\) from the matching; this is an \textit{induced} edge deletion. As mentioned, to cope with induced deletions, our charging argument makes critical use of the fact that \(w\) is of level strictly lower than \(\ell^*\). However, this requirement by itself would suffice only if we were guaranteed that edge \((w, w')\) was chosen to the matching by \(w\). (If the vertex initiating the match is of level \(\ell\), then the cost of creating the matched edge is \(O(3^\ell)\) by Lemma 3.2.) In general, this edge might have been chosen to the matching by \(w'\) rather than \(w\), and so it is critical that both endpoints \(w\) and \(w'\) would be of levels strictly lower than \(\ell^*\) for the charging argument to work. To this end, as in \([3]\), we maintain the stronger invariant that the two endpoints of any matched edge are of the same level; see Invariant \((2)c\). Consequently, to match \(v\) with \(w\), the invariant requires that we let \(w\) rise to the new level \(\ell^*\) of \(v\). This requirement, however, poses another nontrivial challenge.

We set the level \(\ell_w\) of \(w\) to \(\ell^*\) by calling set-level\((w, \ell^*)\). This call updates the data structures accordingly, which involves flipping all incoming edges of \(w\) from neighbors of level between \(\ell_w\) and \(\ell^* - 1\) to be outgoing of \(w\). As before, this may be prohibitively expensive. Specifically, by Observation \((3.1)3\), the runtime of the call to set-level\((w, \ell^*)\) is \(O(d_{\text{out}}(w) + \ell^*)\), where \(d_{\text{out}}(w)\) is the new out-degree of \(w\).

By setting the level of \(w\) to \(\ell^*\), we have created a matched edge \((v, w)\) at level \(\ell^*\). If and when this matched edge is deleted from the graph, we will be able to cover an expected cost of \(O(3^\ell^*)\) in the amortized sense. If \(d_{\text{out}}(w) < 3^\ell^* + 1\), the runtime of the call to set-level\((w, \ell^*)\), and thus of the entire procedure, is \(O(3^\ell^*)\), which can be covered in the amortized sense by the creation of the new matched edge \((v, w)\). However, the complementary case \(d_{\text{out}}(w) \geq 3^\ell^* + 1\) is where the difficulty lies. Indeed, in this case the runtime of the call to set-level\((w, \ell^*)\) may be significantly higher than \(3^\ell^*\). To cover it, we create a matched edge at level higher than \(\ell^*\). To this end we first delete the new matched edge \((v, w)\) from the matching, and then invoke Procedure random-settle recursively, but on \(w\) this time.

The recursive call random-settle\((w)\) creates a matched edge at level higher than \(\ell^*\), which, in the amortized sense, can cover the cost of the call to set-level\((w, \ell^*)\). Thus, in each recursive call we rise to yet a higher level, attempting to charge the yet-uncharged costs of the procedure to the most recently created matched edge. We stress that the level-rising mechanism is not a single computation of a matched edge at some “right” level, but rather a recursive attempt at doing so: Try, rise to a higher level upon failure, and then try again. Note that the maximum level is \(\log_3(n - 1)\). Since \(d_{\text{out}}(w) \leq \text{deg}(w) \leq n - 1 < 3^{\log_3(n - 1) + 1}\), for any vertex \(w\), this recursive attempt eventually succeeds.

The procedure. Procedure random-settle\((v)\) starts by computing the level \(\ell^*\) as described above (i.e., the first level after \(\ell_v\) such that \(\phi_v(\ell^* + 1) < 3^{\ell^* + 1}\)) and setting \(w\)'s level accordingly by calling set-level\((v, \ell^*)\). We then pick uniformly at random an outgoing neighbor \(w\) of \(v\), to match them. (Lemma \((3.2)3\) implies that the matched edge is chosen with probability at most \(1/3^\ell^*\), whereas \(\ell_w \leq \ell^* - 1\) follows from Lemma \((3.2)2\).) If \(w\) is matched, say to \(w'\), then we delete edge \((w, w')\) from the matching. This renders \(w'\) \textit{temporarily free}, meaning that it is not matched to any vertex, but its level remains temporarily as before; \(w'\) will be handled soon, but in the interim, its level will exceed -1. (See Figure \([6]\))

We set the level of \(w\) to \(\ell^*\) by calling set-level\((w, \ell^*)\), which increases its level by at least one, and add edge \((v, w)\) to the matching, thus creating a matched edge of level \(\ell^*\). (If and when this matched edge is deleted from the graph, we will be able to cover an expected cost of \(O(3^\ell^*)\) in the amortized sense.)

The runtime of the call to set-level\((w, \ell^*)\) is \(O(d_{\text{out}}(w) + \ell^*)\), where \(d_{\text{out}}(w)\) is the new out-degree of \(w\). If \(d_{\text{out}}(w) < 3^\ell^* + 1\), this runtime is \(O(3^\ell^*)\), and it can be covered in the amortized sense.

However, in the complementary case \(d_{\text{out}}(w) \geq 3^\ell^* + 1\), the runtime may be significantly higher than \(3^\ell^*\). To cover this runtime, we create a matched edge at level higher than \(\ell^*\). To this end we first delete the new matched edge \((v, w)\) from the matching, thus rendering \(v\) and \(w\) temporarily free, and then invoke Procedure random-settle recursively, by calling random-settle\((w)\). (This recursive call creates a matched edge at level higher than \(\ell^*\), which, in the amortized sense, can cover the cost of the
call to set-level$(w, \ell^*)$.) It is possible that $v$ will become matched as a result of the recursive call to random-settle$(w)$; if $v$ is not matched to any vertex, we invoke Procedure handle-free$(v)$.

Finally, if $w'$ is not matched to any vertex, we invoke Procedure handle-free$(w')$.

4 Analysis

4.1 Invariants. It is easy to verify that our update algorithm satisfies Invariants 1 and 2. The only (technical) exception to Invariant 2 is with temporarily free vertices, which are unmatched, yet their level exceeds $-1$. A vertex becomes temporarily free after its matched edge is deleted, either by the adversary (see line 5(a) in Figure 2) or via Procedure random-settle of the update algorithm (see lines 9(a) in Figure 5). When a free vertex $v$ is handled via Procedure handle-free$(v)$, it may ignore its temporarily free neighbors, as the corresponding edges may be incoming to $v$. In particular, Procedure deterministic-settle$(v)$ deliberately ignores the free neighbors of $v$, and as a result, $v$ may become free although it may have temporarily free neighbors. (Obviously, this is a matter of choice; we can change the procedure to consider temporarily free neighbors of $v$ that belong to $O_v$, but there is no need.) For this reason, our update algorithm makes sure to handle all vertices that become temporarily free later, via appropriate calls to Procedure handle-free; see lines 5(b) and 5(c) in Figure 2 and lines 9(c) and 10 in Figure 6. Hence, if any temporarily free neighbor $w$ of $v$ is ignored by $v$ and $v$ is left free, the subsequent call to handle-free$(w)$ will match $w$, either with $v$ or with another neighbor of $v$.

4.2 Epochs. Given any sequence of edge updates, an edge $(u, v)$ may become matched or unmatched by the algorithm at different update steps. The entire lifespan of an edge $(u, v)$ consists of a sequence of epochs, which refer to the maximal time intervals in which the edge is matched, separated by the maximal time intervals in which the edge is unmatched. (The notion of an epoch was introduced in [3].) Formally, let $e = (u, v)$ be any edge of $\mathcal{M} = \mathcal{M}_0$ at some time step $l$. The epoch $\mathcal{E}(e, l)$ corresponding to edge $e$ at time $l$ refers to the maximal time interval containing $l$ during which $(u, v) \in \mathcal{M}$.

An epoch $\mathcal{E}(e, l)$ is not just a time interval, but rather an object describing a specific edge within that time interval. In particular, for any two distinct (matched) edges $e$ and $e'$ and any time $l$, the respective epochs $\mathcal{E}(e, l)$ and $\mathcal{E}(e', l)$ refer to different objects. On the other hand, for two distinct times $l$ and $l'$ and some edge $e$, it is possible that the respective epochs $\mathcal{E}(e, l)$ and $\mathcal{E}(e, l')$ refer to the same object.

By Invariant 2(c), the endpoints of a matched edge are of the same level, and this level remains unchanged. We henceforth define the level of an epoch to be the level of the corresponding edge.

Any edge update that does not change the matching is processed by our algorithm in constant time. However, an edge update that changes the matching may trigger the creation of some epochs and the termination of some other epochs. The computation cost of creating or terminating an epoch by the algorithm may be large. Moreover, the number of epochs created and terminated due to a single edge update may be large by itself. Therefore, an amortized analysis is required. Following the amortization scheme of [3], we re-distribute the total computation performed at any step $l$ among the epochs created or terminated at step $l$. Specifically, let $\mathcal{E}_1 = \mathcal{E}(e_1, l), \ldots, \mathcal{E}_j = \mathcal{E}(e_j, l)$ (resp., $\mathcal{E}'_1 = (e'_1, l), \ldots, \mathcal{E}'_k = (e'_k, l)$) be the epochs created (resp., terminated) at update step $l$, and let $C_{create}(\mathcal{E}_1), \ldots, C_{create}(\mathcal{E}_j)$ (resp., $C_{term}(\mathcal{E}'_1), \ldots, C_{term}(\mathcal{E}'_k)$) be the respective computation costs charged to the creation (resp., termination) of these epochs. Then we re-distribute the total computation cost performed at update step $l$, denoted by $C^{(l)}$, to the respective epochs in such a way that $C^{(l)} = \sum_{i=1}^{j} C_{create}(\mathcal{E}_i) + \sum_{i=1}^{k} C_{term}(\mathcal{E}'_i)$.

**Claim 4.1** (1) Any epoch created by Procedures handle-insertion or deterministic-settle is of level 0. (2) Any epoch created by Procedure random-settle is of level at least 1.

**Proof:** (1) Immediate. (2) Consider a vertex $v$ that is handled via Procedure random-settle$(v)$. First note that $d_{out}(v) \geq 3^{\ell_w + 1} \geq 1$. By Invariant 2, $\ell_w \geq 0$. By the description of Procedure random-settle$(v)$ and Lemma 5.21(1), we conclude that any epoch created by this procedure is of level at least $\ell^* > \ell_w$. 

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Lemma 3.2 and Claim 4.1 yield the following corollary.

**Corollary 4.2** For any epoch at level $\ell > 0$, initiated by vertex $v$ and corresponding to edge $(v, w)$:

1. The out-degree of $v$ at the time the epoch is created is at least $3^\ell$ and less than $3^{\ell+1}$, though the out-degree of $w$ at that time may be significantly smaller or larger than $3^\ell$.
2. $w$ is chosen as a mate for $v$ uniformly at random among all outgoing neighbors of $v$ at that time, hence the corresponding edge $(v, w)$ becomes matched with probability at most $1/3^\ell$.

### 4.3 Re-distributing the computation costs to epochs.

By re-distributing the total computation cost of the update algorithm to the various epochs, we can visualize the entire update algorithm as a sequence of creation and termination of these epochs. The computation cost associated with an epoch at level $\ell$ (hereafter, level-$\ell$ epoch) includes both its creation cost and its termination cost.

The following lemma plays plays a central role in our analysis. Although a similar lemma was proved in [3], our proof is inherently different than the corresponding proof of [3].

**Lemma 4.3** The total computation cost of the update algorithm can be re-distributed to various epochs so that the computation cost associated with any level-$\ell$ epoch is bounded by $O(3^\ell)$, for any $\ell \geq 0$.

**Proof:** The update algorithm is triggered following edge insertions and edge deletions.

Following an edge insertion, we apply Procedure $\text{handle-insertion}(u, v)$. This procedure occurs at most once per update step, and its runtime is constant. We may disregard this constant cost (formally, we charge this cost to the corresponding update step). Hence, the lemma holds vacuously for edge insertions.

Following an edge deletion, we apply Procedure $\text{handle-deletion}(u, v)$. This procedure occurs at most once per update step; disregarding the calls to $\text{handle-free}(u)$ and $\text{handle-free}(v)$, the runtime of this procedure is constant as well. We henceforth disregard this constant cost (charging it to the corresponding update step), and demonstrate how to re-distribute the costs of the calls to $\text{handle-free}(u)$ and $\text{handle-free}(v)$ to appropriate epochs in a manner satisfying the condition of the lemma.

Let $z \in \{u, v\}$. The execution of Procedure handle-free($z$) splits into two cases. In the first case $d_{\text{out}}(z) < 3^{\ell_z+1}$, and Procedure handle-free($z$) invokes Procedure deterministic-settle($z$), whose runtime is $O(3^{\ell_z})$. Even though Procedure deterministic-settle($z$) may create a new level-$0$ epoch, there is no need to charge this epoch with any costs. The entire cost $O(3^{\ell_z})$ of Procedure deterministic-settle($z$) is charged to the termination cost of epoch $E((u, v), l)$ (triggered by the deletion of edge $(u, v)$ from the graph), which is of level $\ell_z$, thus satisfying the condition of the lemma.

Otherwise $d_{\text{out}}(z) \geq 3^{\ell_z+1}$, and Procedure handle-free($z$) invokes Procedure random-settle($z$).

The runtime of Procedure random-settle($z$) may be much larger than $3^{\ell_z}$ and even than $d_{\text{out}}(z)$. To cover the costs of this procedure, we create a new epoch at a sufficiently high level, attempting to charge the costs of the procedure to the creation cost of the new epoch. However, this charging attempt may sometimes fail, in which case we call the procedure recursively. Each recursive call will create a new epoch at yet a higher level, attempting to charge the yet-uncharged costs of the procedure to the creation cost of the most recently created epoch. Since the levels of epochs created during this process grow by at least one with each recursion level, this recursive process will terminate. Specifically, the depth of this recursive process is bounded by $\log_3(n - 1)$. (Formally, one should also take into account the calls to Procedure handle-free from within Procedure random-settle, which may, in turn, invoke Procedure random-settle.) Next, we describe the charging argument in detail.
For each recursion level of Procedure random-settle, our charging argument may (slightly) over-charge the current level’s costs, so as to cover some of the yet-uncharged costs incurred by previous recursion levels. Denote by random-settle(z(i)) the i-th recursive call, with z(i) being the examined vertex. The initial call to Procedure random-settle may be viewed as the 0th recursion level. As we show below, it may leave a “debt” to the 1st recursion level, but this debt must be bounded by \(O(d_{out}(z(1)))\) units of cost. In general, for each \(i \geq 1\), the debt at recursion level \(i\) (left by previous recursion levels) must be bounded by \(O(d_{out}(z(i)))\) units of cost; in what follows we may view this debt as part of the costs incurred at level \(i\). (Note that the initial call to the procedure has no debt.)

Observe that the i-th recursive call random-settle(z(i)) may also invoke Procedure handle-free. Our charging argument views the call to handle-free as part of the execution of random-settle(z(i)). More concretely, the costs incurred by the call to handle-free are charged to epochs that are created or terminated throughout the execution of the i-th recursive call random-settle(z(i)). Note, however, that Procedure handle-free may, in turn, invoke Procedure random-settle. Our charging argument does not view the new call to random-settle as part of the execution of random-settle(z(i)), but rather as an independent call. More concretely, the costs of the new call to Procedure random-settle are charged only to epochs that are created or terminated throughout the execution of the new call to random-settle.

The following claim completes the charging argument, thus concluding the proof of Lemma 4.3.

Claim 4.4 For any \(i \geq 0\), we can re-distribute the costs of the i-th recursive call random-settle(z(i)) of Procedure random-settle (including the debt from previous recursion levels, if any) to various epoch, allowing a potential debt to the subsequent recursive call random-settle(z(i+1)) (if the procedure proceeds to recursion level \(i + 1\), so that (1) The cost charged to any level-\(\ell\) epoch is bounded by \(O(3^\ell)\), for any \(\ell\), and (2) the debt left for recursion level \(i + 1\) is bounded by \(O(d_{out}(z(i+1)))\). If the procedure terminates at recursion level \(i\), then no debt is allowed, i.e., the entire cost in this case is re-distributed to the epochs.

Proof: The proof is by induction on the recursion level \(i\). Recall that the initial call random-settle(z(0)) of the procedure, which corresponds to the basis \(i = 0\) of the induction, has no debt. Nevertheless, the basis of the induction and the induction step are handled together as follows.

Consider the i-th recursive call random-settle(z), \(i \geq 0\), writing \(z = z(i)\) to avoid cluttered notation. Observe that the debt from previous recursion levels is bounded by \(O(d_{out}(z))\) units of cost. (This observation follows from the induction hypothesis for \(i-1\), unless \(i = 0\), in which case it holds vacuously.) To cover this debt, the procedure creates a new epoch of sufficiently high level, determined as the minimum level \(\ell^* = \ell^*(\ell)\) after \(\ell_z\) such that \(\phi_z(\ell^* + 1) < 3^{\ell^* + 1}\). By Lemma 3.2(4), computing this level \(\ell^*\) requires \(O(3^{\ell^*})\) time, which, by Lemma 3.2(3), already supersedes the debt of \(O(d_{out}(z))\) units of cost from previous recursion levels; we may henceforth disregard this debt. The procedure then sets the level of \(z\) to \(\ell^*\) by calling to set-level(z, \(\ell^*)\). By Lemma 3.2(4), the runtime of this call is \(O(3^{\ell^*})\). Then a random outgoing neighbor \(w\) of \(z\) is chosen as a mate for \(z\), and its level is set to \(\ell^*\) by calling to set-level(w, \(\ell^*)\).

By Observation 3.1(3), the runtime of this call is \(O(d_{out}(w) + \ell^*)\), where \(d_{out}(w)\) is \(w\)'s new out-degree, and it may be that \(d_{out}(w) \gg 3^{\ell^*}\). In the case that \(d_{out}(w) < 3^{\ell^* + 1}\), disregarding a potential call to handle-free(w') that is addressed below, the entire cost of the i-th recursive call random-settle(z(i)) of the procedure is \(O(3^{\ell^*})\), and we charge it to the creation cost of the new level-\(\ell^*\) epoch \(E((z, w), l)\). In this case the procedure terminates at recursion level \(i\), and no debt whatsoever it left.

If \(d_{out}(w) \geq 3^{\ell^* + 1}\), Procedure random-settle(z) deletes the new matched edge \((z, w)\) from the matching, thus terminating the respective epoch \(E((z, w), l)\). Then it proceeds to recursion level \(i + 1\) handle-free that it triggers may decrease it. Each such call to handle-free may, in turn, invoke Procedure random-settle. Nevertheless, the potential growth due to each such call to Procedure random-settle is at least 1, regardless of whether it is a recursive call invoked by Procedure random-settle itself or a new call invoked by Procedure handle-free. Since the potential value is upper bounded by \(n \cdot 3^{2\log_3(n-1)} \leq n^3\) at all times, it follows that the total number of calls to Procedure random-settle is upper bounded by \(n^3\); in particular, this number is finite, thus the recursive process must terminate.
by making a recursive call to \texttt{random-settle}(w), with \( w = z^{(i+1)} \). By Lemma \ref{3.2}(1), this recursive call is guaranteed to create a new epoch at level \( \ell' \) higher than \( \ell^* \). Moreover, Lemma \ref{3.2}(3) implies that \( d_{\text{out}}(w) < 3^{\ell'+1} \), and so the cost \( O(d_{\text{out}}(w) + \ell^*) = O(d_{\text{out}}(w)) = O(3^{\ell'}) \) of the call to \texttt{set-level}(w, \ell') made in the \( i \)th recursion level can be charged to the creation cost of the new level-\( \ell' \) epoch that is created in recursion level \( i + 1 \). Formally, we “drag” the cost of the call to \texttt{set-level}(w, \ell'), which is the aforementioned debt, to the \((i + 1)\)th recursion level. Observe that this debt does not exceed the out-degree of the examined vertex \( w = z^{(i+1)} \) by more than a constant factor, as required.

There is also a potential call to \texttt{handle-free}(z); recall that \( z \) is a shortcut for \( z^{(i)} \). This call should not be confused with the original call to \texttt{handle-free}(z), in which case \( z \) is used as a shortcut for \( z^{(0)} \). In what follows we abandon these shortcuts, and write either \( z^{(0)} \) or \( z^{(i)} \) explicitly, to avoid ambiguity. Recall that we charged \( O(3^{\ell^*(0)}) \) units of cost out of the costs of the original call to \texttt{handle-free}(z^{(0)}) to the termination cost of epoch \( \mathcal{E}((u, v), l) \) (triggered by the deletion of edge \((u, v)\) from the graph). Similarly, we charge \( O(3^{\ell^*}) \) units of cost out of the costs of the new call to \texttt{handle-free}(z^{(i)}) to the termination cost of the level-\( \ell^* \) epoch \( \mathcal{E}((z^{(i)}, w), l) \) (triggered by the deletion of edge \((z^{(i)}, w)\) from the matching). Observe that the only way for the costs of \texttt{handle-free}(z^{(i)}) to exceed \( O(3^{\ell^*}) \) is due to a call to \texttt{random-settle}(z^{(i)}). (Indeed, if a call to \texttt{deterministic-settle}(z^{(i)}) is made, then its cost is bounded by \( O(d_{\text{out}}(z^{(i)})) = O(3^{\ell^*}) \).) However, the new call to \texttt{random-settle}(z^{(i)}) should not be confused with the original call \texttt{random-settle}(z^{(0)}), and is analyzed independently of it. In particular, the costs of the new call to \texttt{random-settle}(z^{(0)}) are not viewed as part of the costs of the original call, and are charged to epochs that are created or terminated as part of the new call to Procedure \texttt{random-settle}.

Finally, there is a potential call to \texttt{handle-free}(w'). We charge \( O(3^{\ell_{w'}}) \) units of cost out of the costs of this call to the termination cost of the level-\( \ell_{w'} \) epoch \( \mathcal{E}((w, w'), l) \) (triggered by the deletion of edge \((w, w')\) from the matching). Similarly to above, the only way for the costs of \texttt{handle-free}(w') to exceed \( O(3^{\ell_{w'}}) \) is due to a call to \texttt{random-settle}(w'). However, the costs of the call to \texttt{random-settle}(w') are not viewed as part of the costs of the original call to Procedure \texttt{random-settle}, and are charged only to epochs that are created or terminated as part of this new call to Procedure \texttt{random-settle}.

Summarizing, the costs of the call to \texttt{random-settle}(z^{(i)}) are re-distributed subject to the requirements of the claim. First, at most \( O(3^{\ell^*}) \) units of cost are charged to the creation and termination costs of the level-\( \ell^* \) epoch \( \mathcal{E}((z^{(i)}, w), l) \). Second, at most \( O(3^{\ell_{w'}}) \) units of cost are charged to the termination cost of the level-\( \ell_{w'} \) epoch \( \mathcal{E}((w, w'), l) \). Finally, the debt left for recursion level \( i + 1 \) (if the procedure proceeds to that level) is bounded by \( O(d_{\text{out}}(z^{(i+1)})) \). The induction step follows.

By Claim \ref{4.4} whenever Procedure \texttt{random-settle} terminates, no debt whatsoever is left. Consequently, the entire cost of this procedure (over all recursion levels) can be re-distributed to various epochs in a manner satisfying the conditions of Lemma \ref{4.3}. This completes the proof of Lemma \ref{4.3}.

4.4 Natural versus induced epochs. An epoch corresponding to edge \((u, v)\) is terminated either because edge \((u, v)\) is deleted from the graph, and then it is called a \textit{natural epoch}, or because the update algorithm deleted edge \((u, v)\) from the matching, and then it is called an \textit{induced epoch}. Following \ref{3}, we will charge the computation cost of each induced epoch \( \mathcal{E} \) of level \( \ell \geq 0 \) to the cost of the unique epoch \( \mathcal{E}' \) whose creation “triggered” the termination of \( \mathcal{E} \).

An induced epoch is terminated only by Procedure \texttt{random-settle}. Consider the first call to Procedure \texttt{random-settle}(z), which we also view as the 0th recursion level. First, some vertex \( w \) of level \( \ell_{w} \) lower than \( \ell^* \) is chosen as a random mate for vertex \( z \), and a level-\( \ell^* \) epoch \( \mathcal{E}((z, w), l) \) is created. If \( w \) is matched to \( w' \), the edge \((w, w')\) is deleted from the matching, and we view the creation of the level-\( \ell^* \) epoch \( \mathcal{E}((z, w), l) \) as terminating the level-\( \ell_{w'} \) epoch \( \mathcal{E}((w, w'), l) \). Next, suppose that \( d_{\text{out}}(w) \geq 3^{\ell^*+1} \). In this case the newly created epoch \( \mathcal{E}((z, w), l) \) is terminated, and immediately afterwards we make the 1st recursive call to \texttt{random-settle}(w). By Lemma \ref{3.2}(1), the call to \texttt{random-settle}(w) is guaranteed to create a new epoch \( \mathcal{E}((w, x), l) \) at level \( \ell' \) higher than \( \ell^* \), where \( x \) is a random outgoing neighbor of \( w \),
which is of level $\ell_x$ lower than $\ell'$ by Lemma 3.2 (2). We view the creation of the level-$\ell'$ epoch $E((w,x),l)$ as terminating the level-$\ell^*$ epoch $E((z,w),l)$. Furthermore, if $x$ is matched to $x'$, the edge $(x,x')$ is deleted from the matching, and we view the creation of the level-$\ell'$ epoch $E((w,x),l)$ as terminating the level-$\ell_{x'}$ epoch $E((x,x'),l)$ as well. We have shown that the creation of epoch $E((w,x),l)$ at the 1st recursion level terminates (at most) two epochs of lower levels. Exactly the same reasoning applies to an arbitrary recursion level $i$ by induction. To summarize: (1) The creation of a new epoch terminates at most two epochs. (2) The levels of the terminated epochs are strictly lower than that of the created epoch, i.e., if the level of the created epoch is $\ell$ and the levels of the terminated epochs are $\ell_1$ and $\ell_2$, then $\ell_1, \ell_2 < \ell$.

We henceforth define the recursive cost of an epoch as the sum of its actual cost and the recursive costs of the (at most) two induced epochs terminated by it; thus the recursive cost of a level-0 epoch is its actual cost. Denote the highest possible recursive cost of a level-$\ell$ epoch by $\hat{C}_\ell$, for any $\ell \geq 0$. (Obviously $\hat{C}_\ell$ is monotone non-decreasing with $\ell$.) By Lemma 4.3 we obtain the recurrence $\hat{C}_\ell \leq \hat{C}_{\ell_1} + \hat{C}_{\ell_2} + O(3^\ell) \leq 2\hat{C}_{\ell-1} + O(3^\ell)$, with the base condition $\hat{C}_0 = O(1)$. This recurrence resolves to $\hat{C}_\ell = O(3^\ell)$.

**Corollary 4.5** For any $\ell \geq 0$, the recursive cost of any level-$\ell$ epoch is bounded by $O(3^\ell)$.

By definition, the sum of recursive costs over all natural epochs is equal to the sum of actual costs over all epochs (both natural and induced) that have been terminated throughout the update sequence.

### 4.5 Bounding the algorithm’s runtime.

During any sequence of $t$ updates, the total number of epochs created equals the number of epochs terminated and the number of epochs that remain alive at the end of the $t$ updates. To bound the computation cost charged to all epochs that remain alive at the end of the update sequence, one may employ an argument similar to [3]. However, instead of distinguishing between epochs that have been terminated and ones that remain alive, we find it more elegant to get rid of those epochs that remain alive by deleting all edges of the final graph, one after another. That is, we append additional edge deletions at the end of the original update sequence, so as to finish with an empty graph.

The order in which these edges are deleted from the graph may be random, but it may also be deterministic, as long as it is oblivious to the maintained matching; e.g., it can be determined deterministically according to some lexicographic rules that are fixed at the outset of the algorithm.

This tweak guarantees that no epoch remains alive at the end of the update sequence. As a result, the sum of recursive costs over all natural epochs will bound the sum of actual costs over all epochs (both natural and induced) that have been created (and also terminated) throughout the update sequence, or in other words, it will bound the total runtime of the algorithm. Note that the total runtime of the algorithm may only increase as a result of this tweak. Since we increase the length of the update sequence by at most a factor of 2, an amortized runtime bound of the algorithm with respect to the new update sequence will imply the same (up to a factor of 2) amortized bound with respect to the original sequence.

Define $Y$ to be the r.v. for the sum of recursive costs over all natural epochs terminated during the entire sequence. In light of the above, $Y$ stands for the total runtime of the algorithm. The proof of the next lemma follows similar lines as in [3], and is given mainly for completeness. Nevertheless, since our algorithm is inherently different than the BGS algorithm and as other parts in the analysis are different, the bounds provided by this lemma shave logarithmic factors from the corresponding bounds of [3].

**Lemma 4.6 (Proof in App. B)** (1) $\mathbb{E}(Y) = O(t)$. (2) $Y = O(t + n \log n)$ w.h.p.

Shaving the $O(n \log n)$ term from the high probability bound requires additional new ideas; see App. C.

Finally, the space usage of our algorithm is linear in the dynamic number of edges in the graph.

**Theorem 4.7** Starting from an empty graph on $n$ fixed vertices, a maximal matching (and thus 2-MCM and also 2-MCVC) can be maintained over any sequence of $t$ edge insertions and deletions in $O(t)$ time in expectation and w.h.p., and using $O(n + m)$ space, where $m$ denotes the dynamic number of edges.
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Appendix

A  Tables and Pseudocode

| Reference                                | Update time | Space            | Bound            |
|-----------------------------------------|-------------|------------------|------------------|
| The naïve algorithm                     | $O(n)$      | $O(n + m)$       | deterministic    |
| Ivković and Lloyd (WG’93) [14]          | $O((n + m)\frac{\sqrt{2}}{2})$ | $"$             | $"$             |
| Neiman and Solomon (STOC’13) [22]       | $O(\sqrt{m})$ | $"$             | $"$             |
| Neiman and Solomon (STOC’13) [22]       | $O(\sqrt{m})$ | $"$             | $"$             |
| Neiman and Solomon (STOC’13) [22]       | $O(\inf_{\beta>1}(\alpha\beta + \log_2 n))$ | $"$             | $"$             |
| He et al. (ISAAC’14) [12]               | $O(\log n)$ | $O(\sqrt{\log n})$ | $"$             |
| for $\alpha = O(1)$                     | $O(\alpha + \sqrt{\alpha \log n})$ | $"$             | $"$             |
| Baswana et al. (FOCS’11) [3]            | $O(\log n)$ | $O(\log n + (n \log^2 n)/t)$ | $O(n \log n + m)$ | expected w.h.p. |
| This paper                              | $O(1)$      | $O(n + m)$       | expected and w.h.p. |

Table 1:  A comparison of previous and our results for dynamic maximal matchings. The parameter $t$ (used for describing the result of [3]) designates the total number of updates and the parameter $\alpha$ (used for describing the results of [22, 12]) designates the arboricity bound of the dynamic graph, i.e., it is assumed that the dynamic graph has arboricity at most $\alpha$ at all times. (The arboricity $\alpha(G)$ of a graph $G$ is the minimum number of edge-disjoint forests into which it can be partitioned, and it is close to the density of its densest subgraph; for any $m$-edge graph, its arboricity ranges between 1 and $\sqrt{m}$.)

handle-insertion($u, v$):

1. $N_v \leftarrow N_v \cup \{u\}$;
2. $N_u \leftarrow N_u \cup \{v\}$;
3. If $\ell_u \geq \ell_v$: /* orient the edge from $u$ to $v$ */
   (a) $O_u \leftarrow O_u \cup \{v\}$;
   (b) $I_v[\ell_u] \leftarrow I_v[\ell_u] \cup \{u\}$;
4. Else: /* orient the edge from $v$ to $u */
   (a) $O_v \leftarrow O_v \cup \{u\}$;
   (b) $I_u[\ell_v] \leftarrow I_u[\ell_v] \cup \{v\}$;
5. If $\ell_u = \ell_v = -1$: /* if $u$ and $v$ are free, match them */
   (a) $M \leftarrow M \cup \{(u, v)\}$;
   (b) set-level($u, 0$);
   (c) set-level($v, 0$);

Figure 1: Handling edge insertion $(u, v)$.

B  Proof of Lemma 4.6

While both endpoints of a matched edge are of the same level, this may not be the case for an unmatched
edge. We say that an edge $e$ is \textit{deleted at level} $\ell$ (from the graph) if \textit{at least one} of its endpoints is at level $\ell$ at the time of the deletion. (Thus each edge is deleted at either one or two levels. While the same edge can be deleted and inserted multiple times in an update sequence, we view different occurrences of the same edge as different objects.) Let $S_\ell$ denote the sequence of edge deletions at level $\ell$, write $|S_\ell| = t_\ell$, and denote by $t_{del}$ the total number of deletions; Note that $t_\ell$ is a random variable (r.v.). Then $\sum_{\ell \geq 0} t_\ell \leq 2 t_{del} \leq t$. Let $X_\ell = X_\ell(S_\ell)$ be the r.v. for the number of natural epochs terminated at level $\ell$ for the update sequence $S_\ell$, and let $Y_\ell$ be the r.v. for the sum of recursive costs over these epochs. By Corollary 4.5 $Y_\ell = O(3^\ell \cdot X_\ell)$. Recall that $Y$ is the r.v. for the sum of recursive costs over all natural epochs terminated during the entire sequence. Thus we have $Y = \sum_{\ell \geq 0} Y_\ell$.

Fix an arbitrary level $\ell, 0 \leq \ell \leq \log_3(n - 1)$. Consider any natural level-$\ell$ epoch initiated by some vertex $u$ at some update step $l$, and let $O^l_u$ denote the set of $u$’s outgoing neighbors at that time. By Corollary 4.2 $|O^l_u| \geq 3^\ell$, and the mate $v$ of $u$ is chosen uniformly at random among all vertices of $O^l_u$. The

| Reference | Approximation | Update time | Bound       |
|-----------|--------------|-------------|-------------|
| Bhattacharya et al. (SODA’15) [6] | $3 + \epsilon$ | $O(min(\sqrt{n}/\epsilon, m^{1/3} \cdot \epsilon^{-2}))$ | deterministic |
| Bernstein and Stein (SODA’16) [5] | $3/2 + \epsilon$ | $O(m^{1/4} \cdot \epsilon^{-2})$ | w.h.p. |
| Neiman and Solomon (STOC’13) [22] | $3/2$ | $O(\sqrt{m})$ | w.h.p. |
| Gupta and Peng (FOCS’13) [10] | $1 + \epsilon$ | $O(\sqrt{m} \cdot \epsilon^{-2})$ | w.h.p. |
| Gupta and Sharma [11] (for trees) | $1$ | $O(\log n)$ | w.h.p. |
| Bernstein and Stein (SODA’16) [5] | $3/2 + \epsilon$ | $O(\alpha + \log n + \epsilon^{-2}) + \epsilon^{-6}$ | w.h.p. |
| Peleg and Solomon (SODA’16) [25] | $1 + \epsilon$ | $O(\alpha + \log n + \epsilon^{-4} \cdot \epsilon^{-6})$ | w.h.p. |
| Peleg and Solomon (SODA’16) [25] | $1 + \epsilon$ | $O(\alpha \cdot \epsilon^{-2})$ | w.h.p. |
| Neiman and Solomon (STOC’13) [22] | $2$ | $O(\inf_{\beta > 1} (\alpha \beta + \log \beta n))$ | w.h.p. |
| He et al. (ISAAC’14) [17] | $c$ | $O(\alpha + \sqrt{\alpha} \log n)$ | w.h.p. |
| Bhattacharya et al. (SODA’15) [6] | $2 + \epsilon$ | $O(\log n)$ | expected |
| Peleg and Solomon (SODA’16) [25] | $2 + \epsilon$ | $O(\alpha / \epsilon)$ | w.h.p. |
| Onak and Rubinfeld [23] | $c$ | $O(\log^2 n)$ | w.h.p. |
| Baswana et al. (FOCS’11) [3] | $2$ | $O(\log n + (n \log^2 n) / t)$ | w.h.p. |
| This paper | $2$ | $O(1)$ | expected and w.h.p. |

Table 2: A comparison of previous and our results for dynamic approximate MCMs. The parameter $t$ designates the total number of updates, $\alpha$ designates the arboricity bound of the dynamic graph and $c$ is a sufficiently large constant. In contrast to all other results, the matching maintained in [1 1] is not represented \textit{explicitly}: In order to determine if an edge belongs to the matching, one should run an $O(\log n)$-time query. This result appears in the table for completeness.

| Reference | Approximation | Update time | Bound       |
|-----------|--------------|-------------|-------------|
| Neiman and Solomon (STOC’13) [22] | $2$ | $O(\sqrt{m})$ | deterministic |
| Neiman and Solomon (STOC’13) [22] | $2$ | $O(\inf_{\beta > 1} (\alpha \beta + \log \beta n))$ | w.h.p. |
| He et al. (ISAAC’14) [17] | $2$ | $O(\alpha + \sqrt{\alpha} \log n)$ | w.h.p. |
| Bhattacharya et al. (SODA’15) [6] | $2 + \epsilon$ | $O(\log n \cdot \epsilon^{-2})$ | w.h.p. |
| Peleg and Solomon (SODA’16) [25] | $2 + \epsilon$ | $O(\alpha / \epsilon)$ | w.h.p. |
| Onak and Rubinfeld [23] | $c$ | $O(\log^2 n)$ | w.h.p. |
| Baswana et al. (FOCS’11) [3] | $2$ | $O(\log n)$ | expected |
| This paper | $2$ | $O(1)$ | expected and w.h.p. |

Table 3: A comparison of previous and our results for dynamic approximate MCMVCs. The parameter $t$ designates the total number of updates, $\alpha$ designates the arboricity bound of the dynamic graph and $c$ is a sufficiently large constant.
handle-deletion($u, v$):
1. $N_v \leftarrow N_v \setminus \{u\}$;
2. $N_u \leftarrow N_u \setminus \{v\}$;
3. If $v \in \mathcal{O}_u$: /* if edge ($u, v$) is oriented from $u$ to $v$ */
   (a) $\mathcal{O}_u \leftarrow \mathcal{O}_u \setminus \{v\}$;
   (b) $\mathcal{I}_v[\ell_u] \leftarrow \mathcal{I}_v[\ell_u] \setminus \{u\}$;
4. Else:
   (a) $\mathcal{O}_v \leftarrow \mathcal{O}_v \setminus \{u\}$;
   (b) $\mathcal{I}_u[\ell_v] \leftarrow \mathcal{I}_u[\ell_v] \setminus \{v\}$;
5. If ($u, v$) $\in M$: /* delete edge ($u, v$) from the matching */
   (a) $M \leftarrow M \setminus \{(u, v)\}$; /* $u$ and $v$ become temporarily free, their levels exceed -1 */
   (b) handle-free($u$);
   (c) handle-free($v$);

Figure 2: Handling edge deletion ($u, v$).

set-level($v$, $\ell$):
1. For all $w \in \mathcal{O}_v$: /* update $\mathcal{I}_w$ regarding $v$’s new level */
   (a) $\mathcal{I}_w[\ell_v] \leftarrow \mathcal{I}_w[\ell_v] \setminus \{v\}$;
   (b) $\mathcal{I}_w[\ell] \leftarrow \mathcal{I}_w[\ell] \cup \{v\}$;
2. If $\ell < \ell_v$: /* in this case the level of $v$ is decreased by at least one */
   (a) For all $w \in \mathcal{O}_v$ such that $\ell + 1 \leq \ell_w \leq \ell_v$: /* flip $v$’s outgoing edge ($v, w$) */
      i. $\mathcal{O}_v \leftarrow \mathcal{O}_v \setminus \{w\}$;
      ii. $\mathcal{I}_w[\ell_w] \leftarrow \mathcal{I}_w[\ell_w] \cup \{w\}$;
      iii. $\mathcal{I}_w[\ell] \leftarrow \mathcal{I}_w[\ell] \setminus \{v\}$;
      iv. $\mathcal{O}_w \leftarrow \mathcal{O}_w \cup \{v\}$;
3. If $\ell > \ell_v$: /* in this case the level of $v$ is increased by at least one */
   (a) For all $i = \ell_v, \ldots, \ell - 1$ and all $w \in \mathcal{I}_v[i]$: /* flip $v$’s incoming edge ($v, w$) */
      i. $\mathcal{I}_v[i] \leftarrow \mathcal{I}_v[i] \setminus \{w\}$;
      ii. $\mathcal{O}_v \leftarrow \mathcal{O}_v \cup \{w\}$;
      iii. $\mathcal{O}_w \leftarrow \mathcal{O}_w \setminus \{v\}$;
      iv. $\mathcal{I}_w[\ell] \leftarrow \mathcal{I}_w[\ell] \cup \{v\}$;
4. $\ell_v \leftarrow \ell$;

Figure 3: Setting the old level $\ell_v$ of $v$ to $\ell$. 

iii
handle-free(v):

1. If $d_{\text{out}}(v) < 3\ell_v + 1$: deterministic-settle(v);
2. Else random-settle(v);

Figure 4: Handling a vertex that becomes temporarily free.

deterministic-settle(v):

1. For all $w \in \mathcal{O}_v$:
   
   (a) If $\ell_w = -1$: /* if $w$ is free, match $v$ with $w$ */
      
      i. $M \leftarrow M \cup \{(v, w)\}$;
   
      ii. set-level($v, 0$);
   
      iii. set-level($w, 0$);
   
      iv. terminate;

2. set-level($v, -1$); /* all outgoing neighbors of $v$ are matched, hence $v$ becomes free */

Figure 5: Matching $v$ with a free neighbor (if exists) deterministically. It is assumed that $d_{\text{out}}(v) < 3\ell_v + 1$.

random-settle(v):

1. $\ell^* \leftarrow \ell_v$;
2. while $\phi_v(\ell^* + 1) \geq 3\ell^* + 1$: $\ell^* \leftarrow \ell^* + 1$; /* $\ell^*$ is the minimum level after $\ell_v$ with $\phi_v(\ell^* + 1) < 3\ell^* + 1$ */
3. set-level($v, \ell^*$); /* after this call $\ell_v = \ell^*$ and $3\ell^* \leq d_{\text{out}}(v) = \phi_v(\ell^*) < 3\ell^* + 1$ */
4. Pick an outgoing neighbor $w$ of $v$ uniformly at random; /* $w$ is chosen with probability at most $1/3\ell^*$ and $\ell_w \leq \ell^* - 1$ */
5. $w' \leftarrow \text{mate}(w)$;
6. If $w' \neq \bot$: $M \leftarrow M \setminus \{(w, w')\}$;
7. set-level($w, \ell^*$); /* in order to match $v$ to $w$, they need to be at the same level */
8. $M \leftarrow M \cup \{(v, w)\}$;
9. If $d_{\text{out}}(w) \geq 3\ell^* + 1$:
   
   (a) $M \leftarrow M \setminus \{(v, w)\}$; /* after this command is executed, $\text{mate}(v) = \text{mate}(w) = \bot$ */

   (b) random-settle($w$); /* before this call, $d_{\text{out}}(w) \geq 3\ell^* + 1 = 3\ell_w + 1$ */

   (c) If $\text{mate}(v) = \bot$: handle-free(v); /* if $v$ is temporarily free, handle it */
10. If $w' \neq \bot$ and $\text{mate}(w') = \bot$: handle-free($w'$); /* if $w'$ is temporarily free, handle it */

Figure 6: Matching $v$ at level $\ell^*$ higher than $\ell_v$, with a random neighbor $w$ of level lower than $\ell^*$. If this requires too much time, the procedure calls itself recursively to match $w$ at yet a higher level, in which case $v$ becomes free and handled via Procedure handle-free($v$). It is assumed that $d_{\text{out}}(v) \geq 3\ell_v + 1$.

epoch is terminated when the matched edge $(u, v)$ gets deleted from the graph; we define the (regular)
duration of the epoch as the number of outgoing edges of \( u \) at time \( l \) that get deleted from the graph between time step \( l \) and the epoch’s termination. (All these edge deletions occur at level \( \ell \) by definition.)

**Observation B.1** If there are \( q \) natural level-\( \ell \) epochs with durations at least \( \delta \), then \( q \leq 2t_\ell/\delta \).

**Proof:** Consider the edge deletions that define the durations of these \( q \) epochs. Any such edge deletion \((u, v)\) is associated with at most two epochs, one initiated by \( u \) and possibly another one initiated by the other endpoint \( v \). Hence the total number of such deletions is bounded by \( 2t_\ell \), and we are done.

**Proof of Lemma 4.6(1):** We say that a level-\( \ell \) epoch is short if its duration is at most \((1/2)3^\ell\); otherwise it is long. By definition, when a short epoch is terminated, at least half of the edges among which the matched edge was randomly chosen are still present in the graph. Let \( X_\ell^{\short} \) and \( X_\ell^{\long} \) be the random variables for the number of short and long epochs terminated at level \( \ell \), respectively. By definition, we have \( X_\ell = X_\ell^{\short} + X_\ell^{\long} \). By Observation B.1, \( X_\ell^{\long} \leq 2t_\ell/(3^\ell/2) = 4t_\ell/3^\ell \).

Let \( e = (u, v) \) be an edge deleted (from the graph) at level \( \ell \) during update step \( l \), and let \( Z_e \) be the indicator random variable that takes value 1 if the deletion of edge \( e \) causes termination of a short epoch at level \( \ell \), and 0 otherwise. Observe that \( X_\ell^{\short} = \sum_{e \in S_\ell} Z_e \).

Suppose that the deletion of edge \( e \) causes termination of a short epoch at level \( \ell \), and assume w.l.o.g. that \( u \) was the initiator of this epoch. Let \( l' \) be the update step at which the corresponding epoch \( E(e, l') \) was created, and let \( O_u^{l'} \) be the set of \( u \)'s outgoing neighbors at that time. By Corollary 4.2, \(|O_u^{l'}| \geq 3^\ell \). Moreover, \( u \) selects \( v \) as its mate at update step \( l' \) uniformly at random among all vertices of \( O_u^{l'} \). By Invariant 2(c), the level of edge \( e \) remains \( \ell \) throughout the epoch’s existence, so if any of \( u \)'s outgoing edges at update step \( l' \) is deleted during this time interval, it is deleted at level \( \ell \) by definition.

We need to bound the probability that the deletion of edge \( e \) at update step \( l \) causes termination of a short epoch at level \( \ell \), given that this epoch has not been terminated yet. Since this epoch is short, it suffices to bound the probability that edge \( e \) was chosen at step \( l' \) among the at least \(|O_u^{l'}|/2 \) edges that are still present in the graph. We know that each of these edges has the same probability of being chosen, hence this probability is bounded by \( 2/|O_u^{l'}| \), and we have \( \mathbb{P}(Z_e = 1) \leq 2/|O_u^{l'}| \leq 2/3^\ell \). Note also that \( \sum_{\ell \geq 0} t_\ell \leq t \), which implies that \( \sum_{\ell \geq 0} \mathbb{E}(t_\ell) = \mathbb{E}(\sum_{\ell \geq 0} t_\ell) \leq \mathbb{E}(t) = t \). It follows that

\[
\mathbb{E}(Y) = \sum_{\ell \geq 0} \mathbb{E}(Y_\ell) = \sum_{\ell \geq 0} \mathbb{O}(3^\ell) \cdot \mathbb{E}(X_\ell) = \sum_{\ell \geq 0} \mathbb{O}(3^\ell) \cdot (\mathbb{E}(X_\ell^{\short}) + \mathbb{E}(X_\ell^{\long}))
\]

\[
\leq \sum_{\ell \geq 0} \mathbb{O}(3^\ell) \cdot \left( \sum_{e \in S_\ell} \mathbb{E}[Z_e] + \mathbb{E}(X_\ell^{\long}) \right) = \sum_{\ell \geq 0} \mathbb{O}(3^\ell) \cdot \left( \sum_{e \in S_\ell} \mathbb{P}[Z_e = 1] + (4/3^\ell)\mathbb{E}(t_\ell) \right)
\]

\[
\leq \sum_{\ell \geq 0} \mathbb{O}(3^\ell) \cdot \left( \sum_{e \in S_\ell} (2/3^\ell) + (4/3^\ell)\mathbb{E}(t_\ell) \right) \leq \mathbb{O}(1) \cdot \sum_{\ell \geq 0} (2t_\ell + 4\mathbb{E}(t_\ell)) = \mathbb{O}(t). \]

**Proof of Lemma 4.6(2):** Fix any level \( \ell, 0 \leq \ell \leq \log_3(n - 1) \), consider a level-\( \ell \) epoch initiated by some vertex \( u \) at some update step \( l \), and let \( O_u^l \) denote the set of \( u \)'s outgoing neighbors at that time. We define the uninterrupted duration of the epoch as the number of outgoing edges of \( u \) at time \( l \) that get deleted from the graph between time step \( l \) and the time that the random matched edge \((u, v)\) is deleted from the graph. (Since we appended edge deletions at the end of the original update sequence to guarantee that the final graph is empty, all the outgoing edges of \( u \) at time \( l \), including edge \((u, v)\), will get deleted throughout the update sequence.) If the epoch is natural, then the uninterrupted duration
of an epoch is equal to its (regular) duration. However, for an induced epoch, its uninterrupted duration may be significantly larger than its duration. (Note also that the outgoing edges of $u$ at time $l$ that get deleted from the graph after the epoch’s termination are not necessarily deleted at level $\ell$.)

We argue that the epoch’s uninterrupted duration is a r.v. uniformly distributed in the range $[1, |O_u^l|]$.

**Claim B.2** For any $1 \leq k \leq |O_u^l|$, the probability that the uninterrupted duration of the epoch is precisely $k$ equals $1/|O_u^l| \leq 1/3^\ell$. This bound remains valid even if it is given that the level of the epoch is $\ell$.

**Proof:** Denote the outgoing edges of $u$ at time $l$ by $e_1, e_2, \ldots, e_\rho$, where $\rho = |O_u^l|$, let $d_1, d_2, \ldots, d_\rho$ denote the times at which these edges are deleted from the graph, respectively, and assume w.l.o.g. that $d_1 < d_2 < \ldots < d_\rho$. Denote the edge associated with the epoch by $e_i$. By time $d_i$, all edges $e_1, \ldots, e_i$ have been deleted from the graph, but all edges $e_{i+1}, \ldots, e_\rho$ remain there. Consequently, for the uninterrupted duration of the epoch to equal $k$, it must hold that $e_i = e_k$. Thus the probability that the epoch’s uninterrupted duration equals $k$ is given by the probability that its associated edge is $e_k$. Since this edge is chosen uniformly at random among $\{e_1, \ldots, e_\rho\}$, the probability of choosing $e_k$ equals $1/\rho = 1/|O_u^l|$. Moreover, this bound remains valid even if it is given that the epoch’s level is $\ell$. \[\square\]

Note that the uninterrupted durations of distinct level-$\ell$ epochs are not necessarily independent:

- First, the number $|O_u^l|$ of outgoing neighbors from which a mate $v$ for $u$ is chosen may well depend on previous coin flips of the algorithm. However, by Corollary 4.2, it must be that $|O_u^l| \geq 3^\ell$.
- Second, although the mate of $u$ is chosen uniformly at random among all vertices in $O_u^l$, some of the optional choices may preclude future events from happening. Hence, future events may depend on this random choice. Moreover, this choice may well effect the (regular) durations of epochs that were created prior to it. However, this choice does not effect the uninterrupted durations of such epochs. Also, as $|O_u^l| \geq 3^\ell$, the probability that a specific vertex $v$ in $O_u^l$ is chosen as $u$’s random mate is bounded by $1/3^\ell$, independently of epochs that were created prior to the current epoch.

It follows that the probability that the uninterrupted duration of a level-$\ell$ epoch equals $k$ is bounded by $1/3^\ell$, for any $k$, even if it is given that the level of the epoch is $\ell$, and independently of the uninterrupted durations of epochs that were created prior to the current epoch. We derive the following corollary.

**Corollary B.3** For any $1 \leq k \leq 3^\ell$, the probability that the epoch’s uninterrupted duration is at most $k$ is bounded by $k/3^\ell$, even if it is given that the level of the epoch is $\ell$, and independently of the uninterrupted durations of epochs that were created prior to the current epoch’s creation.

Let $T_\ell$ be the r.v. for the total number of epochs (both induced and natural) terminated at level $\ell$, and assume that $T_\ell \leq 2X_\ell$; in Section B.1 we demonstrate that this assumption does not lose generality. (Since we made sure that the final graph is empty, any created epoch will get terminated throughout the update sequence. Consequently, $T_\ell$ designates the total number of epochs created at level $\ell$ and $\sum_{\ell \geq 0} T_\ell$ designates the total number of epochs created over all levels throughout the entire update sequence.)

With a slight abuse of notation from the proof of the first assertion of this lemma, we say that a level-$\ell$ epoch is $\mu$-short if its uninterrupted duration is at most $\mu \cdot 3^\ell$, for some parameter $0 \leq \mu \leq 1$.

Write $\eta = 1/16\epsilon$, let $T'_\ell$ be the r.v. for the number of level-$\ell$ epochs that are $\eta$-short, and let $T''_\ell = T_\ell - T'_\ell$ be the r.v. for the number of remaining level-$\ell$ epochs, i.e., those that are not $\eta$-short. Let $A_\ell$ be the event that both $T_\ell > 4\log n$ and $T'_\ell \geq T_\ell/4$ hold, or equivalently, $4T''_\ell \geq T_\ell > 4\log n$.

**Claim B.4** $P(A_\ell) \leq 8/(3n^4)$.
If it is given that the level of the epoch is $\ell$, then either $\eta$-short epochs among these $q$ are $\eta$-short is bounded from above by $\eta^j$. Indeed, by Corollary B.3 the probability of an epoch to be $\eta$-short is at most $\eta$, even if it is given that the level of the epoch is $\ell$, and independently of the uninterrupted durations of epochs that were creator prior to the current epoch’s creation. Thus, if we denote by $B(i)$ the event that the $i$th epoch among these $q$ is $\eta$-short, for $1 \leq i \leq j$, then we have $\mathbb{P}(B(i) | B(1) \cap B(2) \cap \ldots B(i-1)) \leq \eta$. (Moreover, this upper bound of $\eta$ on the probability continues to hold even if it is given that the $\ell$th epoch among these $q$, as well as any previously created epoch, is at level $\ell$.)

Consequently,

$$
\mathbb{P}(B(1) \cap B(2) \cap \ldots \cap B(j)) = \mathbb{P}(B(1)) \cdot \mathbb{P}(B(2) | B(1)) \cdot \ldots \cdot \mathbb{P}(B(j) | B(1) \cap B(2) \cap \ldots B(j-1)) \leq \eta^j.
$$

Next, we argue that $\mathbb{P}(T_q = q \cap T_q' = j) \leq \binom{q}{j} \eta^j$. Instead of (i) going over all possibilities of choosing $q$ level-$\ell$ epochs among all-level epochs, (ii) bounding the probability that each such possibility is chosen and precisely $j$ epochs out of the chosen $q$ are $\eta$-short, and (iii) taking the sum of all these probabilities, we handle all such possibilities together. That is, we restrict our attention to a smaller sample space that consists of just the $q$ level-$\ell$ epochs, without actually choosing or fixing them among all epochs, and bound the probability that precisely $j$ of them are $\eta$-short. Specifically, let $E_1, \ldots, E_q$ denote the $q$ level-$\ell$ epochs, and note that there are $\binom{q}{j}$ possibilities to choose $j$ $\eta$-short epochs among $E_1, \ldots, E_q$. As we have shown, each such possibility occurs with probability at most $\eta^j$, and the assertion follows.

Noting that $\binom{q}{j} \leq \frac{(eq/j)^j}{(4e)^j}$ and recalling that $\eta = 1/16e$, we have $\binom{q}{j} \eta^j \leq (1/4)^j$. Hence

$$
\mathbb{P}(A_{\ell}) = \mathbb{P}(T_q > 4 \log n \cap T_q' \geq T_q/4) = \sum_{q > 4 \log n} \sum_{j=q/4}^q \mathbb{P}(T_q = q \cap T_q' = j) \\
\leq \sum_{q > 4 \log n} \sum_{j=q/4}^q \binom{q}{j} \eta^j \\
\leq \sum_{q > 4 \log n} \sum_{j=q/4}^q (1/4)^j \\
\leq 8/3(1/2)^{4 \log n} \leq 8/(3n^4).
$$

Let $c$ be a sufficiently large constant.

**Claim B.5** If $\neg A_{\ell}$, then $Y_{\ell} < c(t_{\ell} + 3^\ell \cdot \log n)$.

**Proof:** If $\neg A_{\ell}$, then either $T_{\ell} < 4 \log n$ or $T_{\ell}' < T_{\ell}/4$ must hold.

In the former case $X_{\ell} \leq T_{\ell} \leq 4 \log n$, and we have

$$
Y_{\ell} = O(3^\ell) \cdot X_{\ell} \leq O(3^\ell) \cdot 4 \log n < c3^\ell \cdot \log n \leq c(t_{\ell} + 3^\ell \cdot \log n).
$$

Next, suppose that $T_{\ell}' < T_{\ell}/4$. In this case $T_{\ell}'' > 3T_{\ell}/4$, i.e., more than three quarters of the $T_{\ell}$ epochs that are terminated at level $\ell$ are not $\eta$-short. Recall also that we assume that $T_{\ell} \leq 2X_{\ell}$ (see Section B.1), i.e., at least half of the $T_{\ell}$ epochs are natural. It follows that at least a quarter of the $T_{\ell}$ epochs that terminated at level $\ell$ are both natural and not $\eta$-short. Denoting by $X_{\ell}''$ the r.v. for the number of epochs that are both natural and not $\eta$-short, we thus have $X_{\ell}'' \geq T_{\ell}/4 \geq X_{\ell}/4$. Since these $X_{\ell}''$ epochs are natural, the duration of each of them is equal to its uninterrupted duration, and thus it exceeds $\eta 3^\ell$. Therefore, Observation B.1 yields $X_{\ell}'' < 2t_{\ell}/(\eta 3^\ell)$. We conclude that

$$
Y_{\ell} = O(3^\ell) \cdot X_{\ell} \leq O(3^\ell) \cdot 4X_{\ell}'' \leq O(3^\ell) \cdot 8t_{\ell}/(\eta 3^\ell) < ct_{\ell} < c(t_{\ell} + 3^\ell \cdot \log n).
$$

Let $A$ be the event that $Y > c(t + (3/2)n \log n)$. Claim B.5 yields the following corollary.
Corollary B.6 If $A$, then $A_0 \cup A_1 \cup \ldots \cup A_{\log_3(n-1)}$.

Proof: We assume that $\neg A_0 \cap \neg A_1 \cap \ldots \cap \neg A_{\log_3(n-1)}$ holds, and show that $A$ cannot hold. Indeed, by Claim B.5 we have that $Y_{\ell} \leq c(t_{\ell} + 3^{\ell} \cdot \log n)$ for each $\ell \geq 0$. It follows that

$$Y = \sum_{\ell \geq 0} Y_{\ell} < \sum_{\ell \geq 0} c(t_{\ell} + 3^{\ell} \cdot \log n) \leq c(t + (3/2)n \log n).$$

Claim B.4 and Corollary B.6 imply that

$$\mathbb{P}(A) \leq \mathbb{P}(A_0 \cup A_1 \cup \ldots \cup A_{\log_3(n-1)}) \leq \sum_{\ell \geq 0} \mathbb{P}(A_{\ell}) \leq (\log_3(n-1) + 1)(8/(3n^4)) = O(\log n/n^4).$$

It follows that $Y$ is upper bounded by $O(t + n \log n)$ with high probability, as required.

B.1 Justifying the assumption

Consider any level $\ell$ where our assumption does not hold, i.e., $T_\ell > 2X_\ell$, whence the number of induced epochs terminated at level $\ell$ exceeds the number of natural epochs terminated at that level. Next, we show that the computation costs incurred by level $\ell$ can be charged to the computation costs at higher levels, allowing us to disregard any level where the assumption does not hold in the runtime analysis.

As $T_\ell > 2X_\ell$, we may define a one-to-one mapping from the natural to the induced epochs, mapping each natural level-$\ell$ epoch to a unique induced epoch at that level. Then we can (temporarily) charge the costs of any level-$\ell$ natural epoch to the induced epoch to which it is mapped. In Section 4.3, we defined the recursive cost of an epoch as the sum of its actual cost and the recursive costs of the at most two induced epochs (at lower levels) terminated by it. Let us re-define the recursive cost of an epoch as the sum of its actual cost and the recursive costs of the at most two induced epochs terminated by it as well as the at most two natural epochs corresponding to them under the aforementioned mapping.

We henceforth change the constant 3 to 5 throughout the paper; in particular, instead of using $\log_3(n-1)$ levels, we will use $\log_5(n-1)$ levels. Consequently, we allow the computation cost associated with any level-$\ell$ epoch to grow from $O(3^\ell)$ to $O(5^\ell)$ (cf. Lemma 4.3), and obtain the recurrence $\hat{C}_\ell \leq 4\hat{C}_{\ell-1} + O(5^\ell)$, with the base condition $\hat{C}_0 = O(1)$, which resolves to $\hat{C}_\ell = O(5^\ell)$ (cf. Corollary 4.5).

C Improving the high probability bound

If the length $t$ of the update sequence is $\Omega(n \log n)$, then our current high probability runtime bound $O(t + n \log n)$ reduces to $O(t)$. It is natural to assume that $t \geq n/2$, otherwise some vertices are “idle”, and we can simply ignore them. (Such an assumption needs to be properly justified. Nevertheless, dropping it usually triggers only minor adjustments.) One may further assume that $t = \Omega(n \log n)$, as this is indeed the case in many practical applications. However, we believe that it is important to address the regime $n/2 \leq t = o(n \log n)$, for both theoretical and practical reasons.

We do not make any assumption on $t$. For $t = \Omega(n^\epsilon)$, we prove that the runtime exceeds $O(t)$ with probability (w.p.) polynomially small in $n$. For smaller values of $t$, we prove that the runtime exceeds $O(t)$ w.p. polynomially small in $t$. Note that in the latter regime most of the $n$ vertices are idle, and it does not make much sense that this probability would depend on the number of idle vertices; nevertheless, for completeness, we show that the runtime exceeds $O(t + \log n \cdot \sqrt{t})$ w.p. polynomially small in $n$.

Let us revisit some of the details in the proof of Lemma 4.6(2). Recall that $A_\ell$ is the event that both $T_\ell > 4 \log n$ and $T'_\ell \geq T_\ell/4$ hold. Instead, let us re-define $A_\ell$ to be the event that both $T_\ell > 4 \log t$ and
This modification triggers several changes. First, Claim B.4 will change to \( \mathbb{P}(A_\ell) \leq 8/(3t^4) \). Second, Claim B.5 will be changed, so that if \( \neg A_\ell \), then \( Y_\ell < c(t_\ell + 3^\ell \cdot \log t) \).

In exactly the same way as before, we will be able to argue that \( Y \) exceeds \( O(t + n \log t) \) with probability \( O(\log n/t^4) \). However, this high probability bound is not what we are looking for.

The key insight for improving the high probability bound is given by the following lemma.

**Lemma C.1** The level of all vertices can be bounded by \( \log_3(2\sqrt{t}) \), while increasing the total runtime of our algorithm by at most a constant factor.

Before proving this lemma, we demonstrate its power. That is, we assume that the level of all vertices is bounded by \( \log_3(2\sqrt{t}) \) and that the runtime required for that is negligible, and show that the high probability bound can be improved under this assumption. (Notice that we do not attempt to bound the out-degree of vertices by \( 2\sqrt{t} \).) In this case Equation (2) in the proof of Corollary B.6 will be changed to

\[
Y = \sum_{\ell=0}^{\log_3(2\sqrt{t})} Y_\ell \leq \sum_{\ell=0}^{\log_3(2\sqrt{t})} c(t_\ell + 3^\ell \cdot \log t) \leq c(t + (3/2)2\sqrt{t}\log t).
\]

Consequently, we can re-define \( A \) to be the event that \( Y > c(t + (3/2)2\sqrt{t}\log t) \), without affecting the validity of Corollary B.6. Finally, Equation (3) will be changed to

\[
\mathbb{P}(A) \leq \mathbb{P}(A_0 \cup A_1 \cup \ldots A_{\log_3(2\sqrt{t})}) \leq \sum_{\ell=0}^{\log_3(2\sqrt{t})} \mathbb{P}(A_\ell) \leq (\log_3(2\sqrt{t}) + 1)(8/(3t^4)) = O(\log t/t^4).
\]

Thus, assuming the level of vertices is bounded by \( \log_3(2\sqrt{t}) \), \( Y \) exceeds \( c(t + (3/2)2\sqrt{t}\log t) = O(t) \) w.p. \( O(\log t/t^4) \), which is polynomially small in \( t \) for all \( n \) and \( t \) and polynomially small in \( n \) for all \( t = \Omega(n^t) \).

For the somewhat degenerate regime \( t = o(n^t) \), we simply use the original events \( A_\ell \) and the original claims (Claim B.3 and B.5). Since the number of levels is bounded by \( \log_3(2\sqrt{t}) \), Equation (2) in the proof of Corollary B.6 will be changed to

\[
Y = \sum_{\ell=0}^{\log_3(2\sqrt{t})} Y_\ell \leq \sum_{\ell=0}^{\log_3(2\sqrt{t})} c(t_\ell + 3^\ell \cdot \log n) \leq c(t + (3/2)2\sqrt{t}\log n).
\]

Consequently, we can re-define \( A \) to be the event that \( Y > c(t + (3/2)2\sqrt{t}\log n) \), without affecting the validity of Corollary B.6. Finally, Equation (3) will be changed to

\[
\mathbb{P}(A) \leq \mathbb{P}(A_0 \cup A_1 \cup \ldots A_{\log_3(2\sqrt{t})}) \leq \sum_{\ell=0}^{\log_3(2\sqrt{t})} \mathbb{P}(A_\ell) \leq (\log_3(2\sqrt{t}) + 1)(8/(3n^4)) = O(\log t/n^4).
\]

Thus, assuming the level of vertices is bounded by \( \log_3(2\sqrt{t}) \), \( Y \) exceeds \( c(t + (3/2)2\sqrt{t}\log n) = O(t + \log n \cdot \sqrt{t}) \) w.p. \( O(\log t/n^4) \), which is polynomially small in \( n \) for all \( t = o(n^t) \).

**Proof of Lemma C.1:**

Upon re-evaluating the level of a vertex, our algorithm may increase its level beyond \( \ell_{\max} := \log_3(2\sqrt{t}) \), as part of the rising process that creates a new epoch of sufficiently high level, within Procedure random-settle. We adjust Procedure random-settle \((u) \) by preventing \( \ell^* \) from growing beyond \( \ell_{\max} \). Specifically, we execute the loop in line 2 of the procedure as long as \( \ell^* < \ell_{\max} \), or in other words, we adapt the continuation condition \( \phi_u(\ell^* + 1) \geq 3^{\ell^*+1} \) of the while loop to be \( \phi_u(\ell^* + 1) \geq 3^{\ell^*+1} \) and \( \ell^* < \ell_{\max} \). As a result, we can no longer argue that the resulting level \( \ell^* \) satisfies \( \phi_u(\ell^* + 1) < 3^{\ell^*+1} \). While this upper bound on \( \phi_u(\ell^* + 1) \) may no longer hold, note that the lower bound on \( \phi_u(\ell^*) \), namely \( \phi_u(\ell^*) \geq 3^{\ell^*} \), remains valid (cf. Lemma 4.2(3)).
 Observation C.2 Any vertex initiating a level-$\ell_{\text{max}}$ epoch has out-degree at least $3^{\ell_{\text{max}}} = 2\sqrt{t}$ at that time. In particular, at least this number of edges incident on such a vertex were inserted to the graph.

As before, the mate $w$ of $v$ is chosen uniformly at random among $v$’s outgoing neighbors, and $\ell_w < \ell^*$. More accurately, as detailed below, in some cases we restrict our attention to a subset of $v$’s outgoing neighbors, and choose the mate $w$ of $v$ uniformly at random among these vertices during this time interval, and we can charge the original of this subset. Hence any vertex of level $\ell_{\text{max}}$, all its outgoing edges do not flip. Suppose that the new matched edge $(v, w)$ is of maximum level $\ell_{\text{max}}$, and consider the newly created epoch corresponding to it. Since we guarantee that a random mate has a strictly lower level than the vertex choosing it, the endpoints $v$ and $w$ of this epoch cannot be chosen as random mates of any vertex during the epoch’s lifespan, implying that such an epoch can be terminated only by deleting its associated edge $(v, w)$ from the graph. Thus any epoch of maximum level is a natural epoch, and all edges that are outgoing of its endpoints do not flip. We have shown that, once a vertex rises to level $\ell_{\text{max}}$, the epoch associated with it becomes somewhat “stagnant”.

Lemma 3.2 implies that the runtime of the call to set-level($v, \ell^*$) (respectively, set-level($w, \ell^*$)) is $O(d_{\text{out}}(v) + \ell^*)$ (resp., $O(d_{\text{out}}(w) + \ell^*) = O(d_{\text{out}}(w)))$, where $d_{\text{out}}(v) = \phi_v(\ell^*)$ (resp., $d_{\text{out}}(w) = \phi_w(\ell^*))$ is the new out-degree of $v$ (resp., $w$). However, if $\ell^* = \ell_{\text{max}}$, we may not be able to upper bound neither $d_{\text{out}}(v)$ nor $d_{\text{out}}(w)$ by $O(3^{\ell^*}) = O(\sqrt{t})$. Suppose w.l.o.g. that $d_{\text{out}}(v) \geq d_{\text{out}}(w)$. If $d_{\text{out}}(v) = O(3^{\ell^*})$, then our original analysis carries through. We henceforth assume that $\ell^* = \ell_{\text{max}}$ and $d_{\text{out}}(v) := D \gg \sqrt{t}$, and show how to charge this $O(D)$ cost without creating an epoch at a higher level.

Consider the next time that vertex $v$ becomes temporarily free, and denote by $d^{\text{new}}_{\text{out}}(v)$ the out-degree of $v$ at that time. As mentioned, $v$ may become temporarily free only as a result of its matched edge $(v, w)$ being deleted from the graph, which terminates the corresponding level-$\ell_{\text{max}}$ epoch. Since $v$ has become temporarily free, our update algorithm handles it by invoking Procedure handle-free($v$).

If $d^{\text{new}}_{\text{out}}(v) < 3^{\ell_{\text{max}} + 1} = 6\sqrt{t}$, Procedure handle-free($v$) calls to deterministic-settle($v$). Due to the stagnation properties discussed above, all outgoing edges of $v$ at the time the epoch was created do not flip until its termination. Hence, $D - d^{\text{new}}_{\text{out}}(v) = \Omega(D)$ edges incident on $v$ must have been deleted from the graph during this time interval, and we can charge the original $O(D)$ cost to these edge deletions.

In the complementary case $d^{\text{new}}_{\text{out}}(v) \geq 3^{\ell_{\text{max}} + 1} = 6\sqrt{t}$, this procedure calls to random-settle($v$). We make another adjustment to Procedure random-settle($v$) for this particular case. Specifically, in this case the procedure scans $3\sqrt{t}$ arbitrary outgoing neighbors of $v$, and picks a mate $\tilde{w}$ for $v$ uniformly at random among $v$’s scanned neighbors that are of level strictly lower than $\ell_{\text{max}}$. The level of $v$ remains $\ell_{\text{max}}$, and the level of $\tilde{w}$ is set to $\ell_{\text{max}}$ by calling set-level($\tilde{w}, \ell_{\text{max}}$), thus creating a level-$\ell_{\text{max}}$ epoch.

Claim C.3 For any vertex $v$, at most $2\sqrt{t}$ of its neighbors may have level $\ell_{\text{max}}$ at any point in time.

Proof: Suppose for contradiction that more than $2\sqrt{t}$ neighbors of some vertex $v$ have level $\ell_{\text{max}}$ at some point in time. Each of these neighbors is part of a single level-$\ell_{\text{max}}$ epoch at that time, and the edges corresponding to these epochs are vertex-disjoint. In particular, at least half of these neighbors of $v$ must have initiated a level-$\ell_{\text{max}}$ epoch, or in other words, more than $\sqrt{t}$ vertices must have initiated a level-$\ell_{\text{max}}$-epoch. Observation C.2 implies that the total number of edges ever incident on these vertices exceeds $(\sqrt{t} \cdot 2\sqrt{t})/2 = t$, contradicting the fact that the total number of edge updates is $t$.

Claim C.3 implies that $v$’s mate $\tilde{w}$ is chosen with probability at most $1/\sqrt{t}$. A key property is that the runtime $O(\sqrt{t})$ of Procedure random-settle($v$) in this case does not depend on $v$’s out-degree.
To summarize, the original $O(D)$ cost needed for rising a vertex $v$ to the maximum level $\ell_{\text{max}}$ is linear in the out-degree of $v$ at that time, which may be prohibitively large. However, the runtime of subsequent calls to Procedure $\text{random-settle}(v)$ is $O(\sqrt{t})$, which is inverse-linear in the probability with which the matched edge is chosen, and then our original analysis carries through. Since the level of $v$ remains $\ell_{\text{max}}$ in all such calls to Procedure $\text{random-settle}(v)$, the original $O(D)$ cost can be charged to the insertions of the $D$ outgoing edges of $v$ at that time. On the other hand, if a subsequent call to Procedure $\text{deterministic-settle}(v)$ is made, then $v$’s level will decrease to $-1$ or $0$, which implies that we may have to spend an additional prohibitively large cost to rise $v$ to the maximum level $\ell_{\text{max}}$ in the future. However, at least $\Omega(D)$ edge deletions incident on $v$ must have occurred until that time, to which we can charge the original $O(D)$ cost, and then this charging argument can be reapplied from scratch.

This completes the proof of Lemma C.1.

D Applications

Dynamic approximate MWMs. As mentioned, Anand et al. [2] gave a randomized algorithm for maintaining an 8-MWM in general $n$-vertex weighted graphs with expected update time $O(\log n \log \Delta)$. Their algorithm maintains a partition of the edges in the graph into $O(\log \Delta)$ buckets according to their weight, with each bucket containing edges of the same weight up to a constant factor. The maximal matching algorithm of [3] is employed (as a black-box) for each bucket separately. By carefully maintaining a matching in the graph obtained from the union of these $O(\log \Delta)$ maximal matchings, an 8-MWM is maintained in [2]. By plugging our improved algorithm, we shave a factor of $\log n$ from the update time.

Theorem D.1 Starting from an empty graph on $n$ fixed vertices, an 8-MWM can be maintained over any sequence of edge insertions and deletions in expected amortized update time $O(\log \Delta)$, where $\Delta$ is the ratio between the maximum and minimum edge weights in the graph.

Distributed networks. Consider an arbitrary sequence of edge insertions and deletions in a distributed network. Note that each vertex $v$ can gather complete information about its neighbors in two communication rounds. Consequently, the naïve (centralized) maximal matching algorithm discussed in Section 1.4 can be distributed in the obvious way, requiring $O(1)$ communication rounds following a single edge update. Moreover, messages of size $O(\log n)$ suffice for communicating the relevant information. On the negative side, the number of messages sent per update may be as high as $O(n)$. Note, however, that the total number of messages sent is upper bounded (up to a constant) by the total number of neighbor scans performed by the centralized algorithm. This phenomenon extends far beyond the naïve maximal matching algorithm. In particular, it holds also w.r.t. Baswana et al.’s algorithm [3] and our algorithm. Our analysis of Section 4 shows that the amortized update time is constant, thus the average number of neighbor scans is also a constant. (We did not try to optimize the latter constant, but it is rather small.)

Theorem D.2 Starting from an empty distributed network on $n$ fixed vertices, a maximal matching (and thus 2-MCM and also 2-MCVC) can be maintained distributively (under the $\text{CONGEST}$ communication model) over any sequence of edge insertions and deletions with a constant amortized message complexity.

Remark. Optimizing the constant behind the amortized message complexity is left as an open question.