Dimensionally Split Higher Order Semi-discrete Central Scheme for Multi-dimensional Conservation Laws

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Abstract

The dimensionally split reconstruction method as described by Kurganov et al.[4] is revisited for better understanding and a simple fourth order scheme is introduced to solve 3D hyperbolic conservation laws following dimension by dimension approach. Fourth order central weighted essentially non-oscillatory (CWENO) reconstruction methods have already been proposed to study multidimensional problems [10, 20]. In this paper, it is demonstrated that a simple 1D fourth order CWENO reconstruction method by Levy et al.[13] provides fourth order accuracy for 3D hyperbolic nonlinear problems when combined with the semi-discrete scheme by Kurganov et al.[4] and fourth order Runge-Kutta method for time integration.

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I. INTRODUCTION

Many physical phenomena that exhibit discontinuous solutions can be described by hyperbolic conservation laws. Various numerical methods have been proposed to approximate the solutions of these hyperbolic conservation laws. Upwind and central schemes are two of them which are widely used. The advantage of upwind schemes over central schemes is reduced numerical dissipation when dealing with discontinuous solutions. The drawback of upwind scheme is that they involve the solution of local Riemann problems which renders such algorithms numerically more complex and more costly. On the other hand, central schemes are comparatively simpler as they do not require Riemann solvers. One, however, has to adopt higher order central schemes in order to reduce the numerical dissipation. In this paper, we will focus on higher order central schemes due to their intrinsic simplicity.

The very first central scheme was proposed by Lax and Friedrichs. This scheme employs spatial averaging of neighboring grid cells as part of the integration step — a procedure that can be regarded as an imprecise and highly diffusive partial approximation of a Riemann problem. Being first order, the Lax-Friedrichs scheme is numerically too dissipative to be of practical use. In order to reduce the numerical viscosity, a second order central scheme based on a non-oscillatory reconstruction of the linear interpolant was developed. This approach was further improved by Kurganov et al. by introducing second and third order semi-discrete central schemes. The heart of these semi-discrete central schemes is central weighted essentially non-oscillatory (CWENO) reconstruction of the local polynomial under consideration. Various third and fourth order CWENO reconstruction methods have been proposed for 1D, 2D and 3D hyperbolic conservation laws. Kurganov et al. have developed a third order semi-discrete scheme using third order CWENO reconstruction for 1D hyperbolic systems and further followed a dimension by dimension approach to solve 2D hyperbolic problems. While moving towards 2D problems, these authors have introduced an additional term in the central polynomial. However, we find that there is no need of any additional term in the dimension by dimension approach. In fact, if keeping this additional term in the 1D polynomial will introduce an error of second
order when using it dimension by dimension. We will discuss this issue in detail in Section III.

In a recent publication, Lahooti et al. [20] present a fourth order CWENO reconstruction method to solve 3D hyperbolic problems. Here the interpolant polynomial involves twenty seven coefficients. In this paper, it is demonstrated that a simple 1D fourth order CWENO reconstruction method [13] which uses only three coefficient can also be used as an alternative approach to solve 3D hyperbolic problems.

In [4], it has been shown that the proposed semi-discrete scheme is independent of the order of reconstruction. We will demonstrate that this semi-discrete scheme along with 1D fourth order CWENO reconstruction method provides fourth order accuracy for 3D hyperbolic conservation laws by using dimension by dimension approach without the need of additional terms in the coefficients of 1D polynomial. The main advantage of this approach is its simplicity over other multi-dimensional methods [10, 20].

The flow of this paper is as follows: in Section II, we give a brief overview of 3D hyperbolic conservation laws and the semi-discrete scheme. Section III has been devoted to the third order CWENO reconstruction method for 1D, 2D and 3D hyperbolic problems. Section IV covers the fourth order CWENO reconstruction for 3D hyperbolic laws. In Section V, we perform various 2D and 3D convergence tests for linear advection and sound wave to demonstrate the fourth order accuracy. We also perform a two dimensional explosion test to demonstrate the non-oscillatory behavior of fourth order scheme. In Section VI, we provide a brief summary of the work presented in this paper.

II. 3D HYPERBOLIC CONSERVATION LAWS AND THE SEMI-DISCRETE SCHEME

The 3D hyperbolic conservation laws in general form can be expressed as follows:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = 0.$$  \hspace{1cm} (1)

Following Kurganov et al. [4], above equation is semi-discrete form can be written as

$$\frac{d\bar{u}_{i,j,k}}{dt} + \frac{F_{i+1/2,j,k} - F_{i-1/2,j,k}}{\Delta x} + \frac{G_{i,j+1/2,k} - G_{i,j-1/2,k}}{\Delta y} + \frac{H_{i,j,k+1/2} - H_{i,j,k-1/2}}{\Delta z} = 0.$$  \hspace{1cm} (2)
Here \( \bar{u}_{i,j,k} \) is the cell average which is defined as
\[
\bar{u}_{i,j,k} = \frac{1}{\Delta x \Delta y \Delta z} \int_{z_{i,j,k-1/2}}^{z_{i,j,k+1/2}} \int_{y_{i-1/2,j,k}}^{y_{i+1/2,j,k}} \int_{x_{i-1/2,j,k}}^{x_{i+1/2,j,k}} R(x, y, z) \, dx \, dy \, dz. \tag{3}
\]
The function \( R(x, y, z) \) is the higher order central weighted essentially non-oscillatory polynomial and \( F_x, G_y \) and \( H_z \) are the numerical fluxes along \( x \)-, \( y \)- and \( z \)-directions respectively and are defined as
\[
F_{i+1/2,j,k}^x = \frac{f^x(u_{i+1,j,k}^W) + f^x(u_{i,j,k}^E)}{2} - \frac{a_{i+1/2,j,k}^x}{2}(u_{i+1,j,k}^W - u_{i,j,k}^E), \tag{4}
\]
\[
G_{i,j+1/2,k}^y = \frac{g^y(u_{i,j+1,k}^S) + g^y(u_{i,j,k}^N)}{2} - \frac{a_{i,j+1/2,k}^y}{2}(u_{i,j+1,k}^S - u_{i,j,k}^N), \tag{5}
\]
\[
H_{i,j,k+1/2}^z = \frac{h^z(u_{i,j,k+1}^B) + h^z(u_{i,j,k}^T)}{2} - \frac{a_{i,j,k+1/2}^z}{2}(u_{i,j,k+1}^B - u_{i,j,k}^T). \tag{6}
\]
The quantities \( u^E, u^W \) etc. denote the reconstructed point values and \( (f^x)^E, (f^x)^W \) are the corresponding fluxes of the conserved quantities at the cell interfaces. Here \( a^x, a^y, a^z \) are the local maximum speeds of propagation at the cell interfaces which are estimated as
\[
a_{i+1/2,j,k}^x = \max \left\{ \rho \left( \frac{\partial f^x(u_{i+1,j,k}^W)}{\partial u} \right), \rho \left( \frac{\partial f^x(u_{i,j,k}^E)}{\partial u} \right) \right\}, \tag{7}
\]
\[
a_{i,j+1/2,k}^y = \max \left\{ \rho \left( \frac{\partial g^y(u_{i,j+1,k}^S)}{\partial u} \right), \rho \left( \frac{\partial g^y(u_{i,j,k}^N)}{\partial u} \right) \right\}, \tag{8}
\]
\[
a_{i,j,k+1/2}^z = \max \left\{ \rho \left( \frac{\partial h^z(u_{i,j,k+1}^B)}{\partial u} \right), \rho \left( \frac{\partial h^z(u_{i,j,k}^T)}{\partial u} \right) \right\}. \tag{9}
\]
where \( \rho(A) \) is the maximum of the magnitude of the eigenvalues of the Jacobian matrix \( A \).

As proposed by Kurganov et al.\cite{[4]}, the accuracy of the above mentioned scheme is determined by the order of the reconstructed polynomial interpolant and by the order of the time integrator. Third and fourth order Runge-Kutta methods are used for the time integration of Eq.\cite{[2]} in order to achieve third and fourth order accuracy respectively in the temporal evolution.

In the next section, we revisit the third order CWENO reconstruction for 2D hyperbolic problems \cite{[4]} and demonstrate that there is no need of additional terms in the dimension
by dimension approach. We also provide a third order CWENO reconstruction for 3D hyperbolic problems.

III. THIRD ORDER CWENO RECONSTRUCTION

The third order CWENO reconstruction for 1D hyperbolic problems is well described \cite{4, 12}. Multidimensional hyperbolic problems use 1D reconstruction when direction by direction approach is adopted \cite{4}. We, therefore, provide a brief overview of the 1D third order CWENO reconstruction.

A. 1D CWENO Reconstruction

In each cell $I_j$, one has to reconstruct a quadratic polynomial which is a convex combination of three polynomials $P_L(x)$, $P_R(x)$ and $P_C(x)$ such that,

$$R_j(x) = w_LP_L(x) + w_CP_C(x) + w_RP_R(x), \quad \text{where } \sum_k w_k = 1, \ w_k \geq 0, \ \forall k \in \{L, C, R\}.$$  \hspace{1cm} (10)

This reconstructed polynomial $R_j(x)$ has to satisfy three constraints namely accuracy, conservation and non-oscillatory reconstruction. $w_L$, $w_C$, $w_R$ are the non-linear weights and $P_L(x)$, $P_R(x)$ are the linear polynomials, which can be expressed as follows:

$$P_L(x) = a_0 + a_1(x-x_j), \quad P_R(x) = b_0 + b_1(x-x_j).$$ \hspace{1cm} (11)

The coefficients of $P_L(x)$ and $P_R(x)$ are determined uniquely by requiring them to conserve the one sided cell averages $\bar{u}_j$, $\bar{u}_{j-1}$ and $\bar{u}_j$, $\bar{u}_{j+1}$ respectively. Thus, Eq.(11) becomes,

$$P_L(x) = \bar{u}_j + \frac{\bar{u}_{j-1} - \bar{u}_j}{\Delta x}(x-x_j), \quad P_R(x) = \bar{u}_j + \frac{\bar{u}_{j+1} - \bar{u}_j}{\Delta x}(x-x_j).$$ \hspace{1cm} (12)

$P_C(x)$ is a quadratic polynomial which is chosen in such a way that it satisfies the following property,

$$P_{OPT}(x) = c_LP_L(x) + c_RP_R(x) + (1-c_L-c_R)P_C(x)$$ \hspace{1cm} (13)
where $c_L$, $c_R$ are positive constants and $P_{OPT}$ is quadratic polynomial which can be expressed as follows:

$$P_{OPT}(x) = c_0 + c_1(x - x_j) + c_2(x - x_j)^2. \quad \text{(14)}$$

The coefficients of $P_{OPT}$ are also obtained uniquely by requiring it to conserve the cell averages $\bar{u}_{j-1}$, $\bar{u}_j$ and $\bar{u}_{j+1}$. Thus, Eq.(14) becomes,

$$P_{OPT}(x) = \bar{u}_j - \frac{1}{24}(\bar{u}_{j+1} - 2\bar{u}_j + \bar{u}_{j-1}) + \frac{\bar{u}_{j+1} - \bar{u}_{j-1}}{2\Delta x}(x - x_j) + \frac{(\bar{u}_{j+1} - 2\bar{u}_j + \bar{u}_{j-1})}{2\Delta x^2}(x - x_j)^2. \quad \text{(15)}$$

It is well known that any symmetric combination of $c_L$ and $c_R$ will give the third order accuracy [4, 12]. The nonlinear weights $w_i$ can be computed as follows:

$$w_i = \frac{\alpha_i}{\alpha_L + \alpha_C + \alpha_R}, \text{ where } \alpha_i = \frac{c_i}{(\epsilon + IS_i)^p}, \forall i \in \{L, C, R\}. \quad \text{(16)}$$

Here $\epsilon$, $p$ are chosen to be $10^{-6}$ and 2 respectively and $c_L = c_R = 1/4$, $c_C = 1/2$ [4, 12]. $IS_i$ are the smoothness indicators which are defined as,

$$IS_i = \sum_{l=1}^{2} \int_{x_j-1/2}^{x_{j+1/2}} (\Delta x)^{2l-1}(P_i^{(l)}(x))^2 \, dx, \forall i \in \{L, C, R\} \quad \text{(17)}$$

Smoothness indicators and hence, nonlinear weights can easily be computed using Eq.(17) and Eq.(16) respectively, once we have reconstructed all the polynomials. One can then compute the point values at the cell interfaces using Eq.(10).

**B. 2D CWENO Reconstruction**

There exist several 2D CWENO reconstruction methods [7, 11, 12]. However, these authors suggest that 1D CWENO can be used as an alternative method for the same following the direction by direction approach [4] by adding an additional term in the $P_{OPT}(x)$ for third order accuracy. Here, we will show that there is no need to put any additional term by hand in the dimension by dimension approach. We will also show through the various 2D tests that this additional term lowers the accuracy to second order. In order to understand this, let us start with a 2D quadratic polynomial [12].
\[ P_{OPT}(x, y) = a_0 + a_1(x - x_j) + a_2(y - y_k) + a_3(x - x_j)^2 + a_4(y - y_k)^2 + a_5(x - x_j)(y - y_k). \] (18)

The coefficients of this polynomials are obtained by requiring it to conserve the five cell averages \( \bar{u}_{j,k}, \bar{u}_{j+1,k}, \bar{u}_{j-1,k}, \bar{u}_{j,k+1}, \bar{u}_{j,k-1}, \bar{u}_{j,k} \) and imposing one constraint \([11]\) which is,

\[ \partial_{xy}^2 P_{OPT} = \frac{1}{4 \Delta x \Delta y} (\bar{u}_{j+1,k+1} - \bar{u}_{j-1,k+1} - \bar{u}_{j+1,k-1} + \bar{u}_{j-1,k-1}). \] (19)

We obtain,

\[
\begin{aligned}
a_0 &= \bar{u}_{j,k} - \frac{1}{24} (\bar{u}_{j+1,k} - 2\bar{u}_{j,k} + \bar{u}_{j-1,k}) - \frac{1}{24} (\bar{u}_{j,k+1} - 2\bar{u}_{j,k} + \bar{u}_{j,k-1}), \\
a_1 &= \frac{\bar{u}_{j+1,k} - \bar{u}_{j-1,k}}{2\Delta x}, \quad a_2 = \frac{\bar{u}_{j,k+1} - \bar{u}_{j,k-1}}{2\Delta y}, \\
a_3 &= \frac{(\bar{u}_{j+1,k} - 2\bar{u}_{j,k} + \bar{u}_{j-1,k})}{2\Delta x^2}, \quad a_4 = \frac{(\bar{u}_{j,k+1} - 2\bar{u}_{j,k} + \bar{u}_{j,k-1})}{2\Delta y^2}, \\
a_5 &= \frac{1}{8 \Delta x \Delta y} (\bar{u}_{j+1,k+1} - \bar{u}_{j-1,k+1} - \bar{u}_{j+1,k-1} + \bar{u}_{j-1,k-1}).
\end{aligned}
\] (20)

Substituting \( y = y_k \) in Eq. (18) and using Eq. (20) we get,

\[ P_{OPT}(x, y_k) = \bar{u}_{j,k} - \frac{1}{24} (\bar{u}_{j+1,k} - 2\bar{u}_{j,k} + \bar{u}_{j-1,k}) - \frac{1}{24} (\bar{u}_{j,k+1} - 2\bar{u}_{j,k} + \bar{u}_{j,k-1}) + \frac{\bar{u}_{j+1,k} - \bar{u}_{j-1,k}}{2\Delta x} (x - x_j) + \frac{(\bar{u}_{j+1,k} - 2\bar{u}_{j,k} + \bar{u}_{j-1,k})}{2\Delta x^2} (x - x_j)^2. \] (21)

This is nothing but the polynomial \( P_{OPT} \) suggested for the reconstruction along the \( x \)-direction in the dimension by dimension approach \([4]\). We know that \( P_{OPT} \) has to satisfy the conservation property if we want to use it in the reconstruction. This can easily be checked by integrating Eq. (21) over the interval \( \{(x_{j-1/2}, x_{j+1/2}), (y_{k-1/2}, y_{k+1/2})\} \). We do not recover the cell average \( \bar{u}_{j,k} \), if we do so. Therefore, this polynomial can not be used in the dimension by dimension approach as it breaks the conservation property. This is verified in various (2D linear advection and sound wave) tests presented in Section V.

We propose that multidimensional polynomials can not be reduced to a 1D polynomial to be used in the dimension by dimension approach, if the coefficients of the reduced 1D polynomial are obtained from multidimensional constraints. In the dimension by dimension approach, these coefficients have to be computed by applying 1D constraints in all directions.
separately. For example, if we substitute $y = y_k$ in Eq. (18), the coefficients of the reduced 1D polynomial have to be obtained uniquely by imposing the conservation requirement over cell averages $\bar{u}_{j-1,k}$, $\bar{u}_{j,k}$ and $\bar{u}_{j+1,k}$. Thus, $P_{OPT}$ becomes,

$$P_{OPT}(x, y_k) = \bar{u}_{j,k} - \frac{1}{24}(\bar{u}_{j+1,k} - 2\bar{u}_{j,k} + \bar{u}_{j-1,k}) + \frac{\bar{u}_{j+1,k} - \bar{u}_{j-1,k}}{2\Delta x}(x - x_j) + \frac{(\bar{u}_{j+1,k} - 2\bar{u}_{j,k} + \bar{u}_{j-1,k})}{2\Delta x^2}(x - x_j)^2. \quad (22)$$

The one-sided linear polynomial interpolants $P_L(x, y_k)$ and $P_R(x, y_k)$ are determined uniquely by requiring them to conserve the cell averages $\bar{u}_{j,k}$, $\bar{u}_{j-1,k}$ and $\bar{u}_{j+1,k}$ respectively. The central polynomial $P_C(x, y_k)$ can then be computed as follows:

$$P_{OPT}(x, y_k) = c_L P_L(x, y_k) + c_R P_R(x, y_k) + (1 - c_L - c_R) P_C(x, y_k), \text{ here } c_L = c_R = 1/4. \quad (23)$$

Once we have reconstructed all the polynomials, we can easily compute smoothness using Eq. (17) and nonlinear weights using Eq. (16) which are used to reconstruct the non-oscillatory polynomial interpolant $R_{j,k}(x, y_k)$ as follows:

$$R_{j,k}(x, y_k) = w_L P_L(x, y_k) + w_C P_C(x, y_k) + w_R P_R(x, y_k). \quad (24)$$

This polynomial will provide non-oscillatory point values on the cell interfaces along the $x$-direction. Since it is purely 1D reconstruction, one can easily obtain a similar polynomial along the $y$-direction. The third order accuracy of this approach has been verified for the 2D hyperbolic problems which is shown in Section V.

C. 3D CWENO Reconstruction

Although, it is now straightforward to construct the 3D CWENO reconstruction for the dimension by dimension approach, we provide it for the $x$-direction for completeness.

In each cell $I_{i,j,k}$, one has to reconstruct a quadratic polynomial which is a convex combination of three polynomials $P_L(x, y_j, z_k)$, $P_R(x, y_j, z_k)$ and $P_C(x, y_j, z_k)$ such that

$$R_{i,j,k}(x, y_j, z_k) = w_L P_L(x, y_j, z_k) + w_C P_C(x, y_j, z_k) + w_R P_R(x, y_j, z_k). \quad (25)$$
Here $w_L$, $w_C$, $w_R$ are the positive non-linear weights which follow the constraint,

\[ w_L + w_C + w_R = 1. \]  

(26)

$P_L(x, y_j, z_k)$ and $P_R(x, y_j, z_k)$ are the linear polynomials whose coefficients are determined uniquely by requiring them to conserve the one-sided cell averages $\bar{u}_{i,j,k}$, $\bar{u}_{i-1,j,k}$ and $\bar{u}_{i,j,k}$, $\bar{u}_{i+1,j,k}$ respectively. Thus $P_L(x, y_j, z_k)$ and $P_R(x, y_j, z_k)$ can be expressed as follows:

\[
P_L(x, y_j, z_k) = \bar{u}_{i,j,k} + \frac{\bar{u}_{i+1,j,k} - \bar{u}_{i,j,k}}{\Delta x} (x - x_i), \quad P_R(x, y_j, z_k) = \bar{u}_{i,j,k} + \frac{\bar{u}_{i,j,k} - \bar{u}_{i-1,j,k}}{\Delta x} (x - x_i).
\]  

(27)

$P_C(x, y_j, z_k)$ is the quadratic polynomial which is chosen so as to satisfy the following property,

\[
P_{OPT}(x, y_j, z_k) = c_L P_L(x, y_j, z_k) + c_R P_R(x, y_j, z_k) + (1 - c_L - c_R) P_C(x, y_j, z_k).
\]  

(28)

Here $c_L$, $c_R$ are positive constants and $P_{OPT}$ is quadratic polynomial whose coefficients are obtained uniquely by requiring it to conserve the cell averages $\bar{u}_{i-1,j,k}$, $\bar{u}_{i,j,k}$ and $\bar{u}_{i+1,j,k}$, i.e.

\[
P_{OPT}(x, y_j, z_k) = \bar{u}_{i,j,k} - \frac{1}{24} \left( \bar{u}_{i+1,j,k} - 2\bar{u}_{i,j,k} + \bar{u}_{i-1,j,k} \right) + \frac{\bar{u}_{i+1,j,k} - \bar{u}_{i-1,j,k}}{2\Delta x} (x - x_i) \right.
\]

\[
+ \left. \left( \bar{u}_{i+1,j,k} - 2\bar{u}_{i,j,k} + \bar{u}_{i-1,j,k} \right) \right) \frac{(x - x_i)^2}{2\Delta x^2}.
\]  

(29)

The nonlinear weights $w_i$ are computed as follows [4, 12]:

\[
w_i = \frac{\alpha_i}{\alpha_L + \alpha_C + \alpha_R}, \quad \text{where} \quad \alpha_i = \frac{c_i}{(\epsilon + IS_i)p}, \quad \forall \ i \in \{L, C, R\}.
\]  

(30)

Here $\epsilon$, $p$ are chosen to be $10^{-6}$ and 2 respectively and $c_L = c_R = 1/4$, $c_C = 1/2$ [4, 12]. $IS_i$ are the smoothness indicators which are defined as,

\[
IS_i = \sum_{l=1}^{2} \int_{x_{i-1/2}}^{x_{i+1/2}} (\Delta x)^{2l-1} \left( P^{(l)}_i(x, y_j, z_k) \right)^2 dx, \quad \forall \ i \in \{L, C, R\}.
\]  

(31)

Once, we have reconstructed all the polynomials $P_L$, $P_C$ and $P_R$, corresponding smoothness indicators can be obtained from Eq. (31) and then nonlinear weights are determined.
from Eq. (30). These weights are finally used to compute the polynomial \( R_{i,j,k}(x, y_j, z_k) \) which ultimately provides the non-oscillatory point values at the cell interfaces along the \( x \)-direction.

Similarly, one can easily reconstruct the the quadratic polynomial \( R_{i,j,k} \) along \( y \)-and \( z \)-directions. Third order accuracy of this approach has been verified in 3D linear advection and sound wave tests which are shown in Section V.

IV. FOURTH ORDER CWENO RECONSTRUCTION

The fourth order CWENO reconstruction has already been suggested for multidimensional hyperbolic conservation laws [10, 20]. These methods employ biquadratic and triple quadratic polynomials for 2D and 3D reconstructions respectively. The biquadratic polynomial needs nine coefficients, however, the 3D reconstruction requires twenty seven coefficients. Thus, for 2D hyperbolic problems, the reconstructed polynomial has to be written as a convex combination of nine biquadratic polynomials in each cell and for 3D problems the reconstructed polynomial would be a convex combination of twenty seven triple quadratic polynomials in each cell.

We propose, an alternative but very simple method, based on direction by direction approach where one can simply use 1D fourth order CWENO reconstruction in order to solve multi-dimensional hyperbolic problems without the need of additional terms in the 1D polynomial.

Fourth order CWENO reconstruction for 1D hyperbolic problems is well described by Levy et al. [13]. We will show how this simple method can be used to solve the 3D hyperbolic conservation laws. In order to do so, we first start with a brief overview of 1D fourth order CWENO reconstruction method.

A. 1D CWENO Reconstruction

In each cell \( I_j \), one has to reconstruct a quadratic polynomial \( R_j(x) \) which is a convex combination of three quadratic polynomials \( P_{j-1}(x), P_j(x) \) and \( P_{j+1}(x) \) such that,
\[ R_j(x) = \sum_{k=j-1}^{j+1} w_k P_k(x), \quad \text{where} \quad \sum_{k=j-1}^{j+1} w_k = 1, \quad w_k \geq 0, \quad \forall \, k \in (j-1, j, j+1). \] (32)

\( R_j(x) \) is reconstructed so as to satisfy all the three constraints accuracy, conservation and non-oscillatory. The coefficients of the polynomial \( P_k(x) \) are obtained uniquely by requiring it to conserve the cell averages \( \bar{u}_{k-1}, \bar{u}_k \) and \( \bar{u}_{k+1} \), where \( k \in (j-1, j, j+1) \). Thus, each polynomial, \( P_k(x) \), can be written as,

\[
P_k(x) = \bar{u}_k - \frac{1}{24}(\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1}) + \frac{\bar{u}_{k+1} - \bar{u}_{k-1}}{2\Delta x}(x - x_k)
+ \frac{(\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1})}{2\Delta x^2}(x - x_k)^2, \quad k = j-1, j, j+1. \] (33)

The nonlinear weights \( w_k \) are obtained as before, i.e.

\[
w_k = \frac{\alpha_k}{\alpha_{j-1} + \alpha_j + \alpha_{j+1}}, \quad \text{where} \quad \alpha_k = \frac{c_k}{(\epsilon + IS_k)^p}, \quad \forall \, k \in (j-1, j, j+1). \] (34)

Here \( \epsilon, p \) are chosen to be \( 10^{-6} \) and \( 2 \) respectively and \( c_{j-1} = c_{j+1} = 1/6, \, c_j = 2/3 \) [13]. \( IS_k \) are the smoothness indicators which are defined as,

\[
IS_k = \sum_{l=1}^{2} \int_{x_{j-1/2}}^{x_{j+1/2}} (\Delta x)^{2l-1}(P_k^{(l)}(x))^2 \, dx, \quad \forall \, k \in (j-1, j, j+1). \] (35)

Once we have reconstructed all the the polynomials \( (P_{j-1}, P_j, P_{j+1}) \), smoothness indicators can easily be computed using Eq.(35) and hence nonlinear weights using Eq.(34). These weights are finally used to reconstruct non-oscillatory the polynomial \( R_j(x) \) which provides fourth order accurate point values at the cell interfaces in smooth regions.

In the next subsection, we provide the dimension by dimension approach for 3D reconstruction.

### B. 3D CWENO Reconstruction

Let us begin with a triple quadratic polynomial used as a building block for 3D CWENO reconstruction in ref.[20] i.e.
\[ P_{ijk}(x, y, z) = \sum_{l,m,n=0}^{2} a_{lmn} (x - x_i)^l (y - y_j)^m (z - z_k)^n. \]  

(36)

For unsplit 3D reconstruction, the coefficients of this polynomial are obtained uniquely by requiring it to conserve twenty seven cell averages \( \bar{u}_{i+1,j,k} \) where \( l, m, n = -1, 0, 1 \). Expanding the series sum on the right hand side of Eq.

\[ P_{ijk}(x, y_j, z_k) = a_{000} + a_{100}(x - x_i) + a_{200}(x - x_i)^2. \]  

(37)

If we use the same coefficients as obtained from 3D unsplit reconstruction, the polynomial \( P_{ijk}(x, y_j, z_k) \) can not be used in the dimension by dimension approach as it does not satisfy the conservation property there. For the dimension by dimension approach, the coefficients of the polynomial in Eq.

\[ \text{Smoothness indicators can be obtained as follows:} \]

\[ IS_{i+1,j,k} = \sum_{m=1}^{2} \int_{x_{i-1/2}}^{x_{i+1/2}} (\Delta x)^{2m-1} \left( P_{i+1}^{(m)}(x, y_j, z_k) \right)^2 \, dx, \quad \forall \, l \in (-1, 0, 1). \]  

(39)

The nonlinear weights \( w_{i+1,j,k} \) are defined as,

\[ w_{i+1,j,k} = \frac{\alpha_{i+1,j,k}}{\alpha_{i-1,j,k} + \alpha_{i,j,k} + \alpha_{i+1,j,k}}, \quad \text{where} \quad \alpha_{i+1,j,k} = \frac{c_{i+1,j,k}}{\epsilon + IS_{i+1,j,k}}, \quad \forall \, l \in (-1, 0, 1). \]  

(40)

Here, \( c_{i-1,j,k} = c_{i+1,j,k} = 1/6 \) and \( c_{i-1,j,k} = 2/3 \) as chosen in 1D reconstruction. Once we have computed all the polynomials \( P_{i+1,j,k}(x, y_j, z_k) \) and weights \( w_{i+1,j,k}, \forall \, l \in (-1, 0, 1) \), we can reconstruct, in each cell \( I_{ijk} \), a non-oscillatory polynomial along the \( x \)-direction, as follows:

\[ R_{i,j,k}(x, y_j, z_k) = \sum_{l=-1}^{1} w_{i+1,j,k} P_{i+1,j,k}(x, y_j, z_k), \quad \text{where} \quad \sum_{l=-1}^{1} w_{i+1,j,k} = 1, \quad w_{i+1,j,k} \geq 0, \quad \forall \, l \in (-1, 0, 1). \]  

(41)
This polynomial \( R_{i,j,k}(x, y_j, z_k) \) is used to obtain non-oscillatory point values \((u^E, u^W)\) and hence corresponding fluxes \((f^E, f^W)\) at the cell interface along the \(x\)-direction which are used to compute numerical flux \( F^x \).

Since it is simply 1D reconstruction, one can similarly obtain the polynomials \( R_{i,j,k}(x_i, y, z_k), R_{i,j,k}(x_i, y_j, z) \) and hence numerical fluxes \( G^y, H^z \) along \(y\)-and \(z\)-directions respectively. Once, all the numerical fluxes \((F^x, G^y, H^z)\) are computed, fourth order Runge-Kutta method can be used for the time integration of Eq. (2). Fourth order accuracy and non-oscillatory behavior of this scheme has been verified in various multi-dimensional tests which are presented in the next section.

V. RESULTS FROM THE SIMULATIONS

In this section, we present convergence of errors for 2D and 3D linear advection and sound wave tests to validate accuracy of the proposed third and fourth order schemes. The norm of the error for all tests is defined as,

\[
\delta E = \frac{1}{N} \sum_{i=1}^{N} |E_{i}^f - E_{i}^0|.
\]

\( E_{i}^0, E_{i}^f \) are the initial and final solutions as a function of grid resolution and \(N\) is the number of grid points. Moreover, we also perform 2D explosion test for fourth order scheme to show non-oscillatory behavior.

A. 2D and 3D linear advection tests

3D linear advection equation can be obtained by substituting \(U = F = G = H = \rho\) in Eq.(1) which may be reduced to 2D linear advection equation just by dropping the \(z\) dependency. The initial conditions for 2D and 3D case are selected as follows:

\[
\rho(x, y)|_{t=0} = 1 + 10^{-4} \sin[2\pi(x/l_x + y/l_y)], \ [0, l_x] \times [0, l_y] = [0, 2/\sqrt{3}] \times [0, 2].
\]

\(\rho(x, y, z)|_{t=0} = 1 + 10^{-4} \sin[2\pi(x/l_x + y/l_y + z/l_z)], \ [0, l_x] \times [0, l_y] \times [0, l_z] = [0, 1] \times [0, 2/\sqrt{3}] \times [0, 2]\)
Boundary conditions are periodic and final solutions are obtained after one period. Note here that, these tests are performed on more general numerical grids where grid sizes are different in different directions. In Fig.1, we show the convergence of errors in 2D and 3D linear advection tests for third order schemes (with and with out additional terms in the reconstruction). We can clearly see third order convergence in the absence of additional terms in the reconstructed polynomial, however, the errors scale to second order in the presence of additional terms as they are proportional to second derivatives in the transverse directions.

Fig.2 contains the convergence of errors in 3D linear advection test for fourth order scheme which shows that actual order of convergence is higher than fourth order.

B. 2D and 3D adiabatic sound wave tests

Equations governing the dynamics of 3D adiabatic systems can be described by Eq.(1) where,

\[
U = \begin{pmatrix}
\rho \\
\rho v_x \\
\rho v_y \\
\rho v_z \\
E
\end{pmatrix}, \quad F = \begin{pmatrix}
\rho v_x \\
\rho v_x^2 + p \\
\rho v_x v_y \\
\rho v_x v_z \\
v_x(E + p)
\end{pmatrix}, \quad G = \begin{pmatrix}
\rho v_y \\
\rho v_x v_y \\
\rho v_y^2 + p \\
\rho v_y v_z \\
v_y(E + p)
\end{pmatrix}, \quad H = \begin{pmatrix}
\rho v_z \\
\rho v_x v_z \\
\rho v_y v_z \\
\rho v_z^2 + p \\
v_z(E + p)
\end{pmatrix}. \quad (45)
\]

Here \(\rho\) is the density, \(v_x, v_y, v_z\) are \(x, y\) and \(z\) component of velocity, \(E\) is the total energy and \(p\) is the pressure which is related to \(E\) through \(p = (\gamma - 1)(E - 0.5\rho(v_x^2 + v_y^2 + v_z^2))\).

One may recover 2D adiabatic systems just by dropping the \(z\) components from Eqs.(1) and (45). The initial conditions, for 2D and 3D cases, are chosen as follows:

\[
\rho(x, y)|_{t=0} = 1 + 10^{-6}\sin[2\pi(x/l_x + y/l_y)], \quad v_x|_{t=0} = 1 = v_y|_{t=0}, \quad p = 3/5, \quad [0, l_x] \times [0, l_y] = [0, 1] \times [0, 1]. \quad (46)
\]

\[
\rho(x, y, z)|_{t=0} = 1 + 10^{-6}\sin[2\pi(x/l_x + y/l_y + z/l_z)], \quad v_x|_{t=0} = 1.0 = v_y|_{t=0} = v_z|_{t=0}, \quad p = 3/5, \quad [0, l_x] \times [0, l_y] \times [0, l_z] = [0, 1] \times [0, 1] \times [0, 1]. \quad (47)
\]
γ is chosen to be $5/3$ for all tests. Boundary conditions are periodic and final solutions are obtained after one period. Fig.3. contains the convergence of errors in third order schemes for 2D and 3D adiabatic sound wave tests which also shows that presence of additional terms lowers the accuracy of the scheme. However, as proposed in section III, we get a proper third order convergence for both 2D and 3D tests when additional terms are absent in the reconstruction. In Fig.4. we present the convergence of errors in our fourth order scheme for 3D adiabatic sound wave tests which confirms fourth order accuracy.

C. 2D explosion test problem

We present this test to demonstrate the non-oscillatory behavior of the fourth order scheme. Equations governing the dynamics of 2D explosion test problem is same as described in section B and the initial conditions are chosen as follows:

$$(\rho, v_x, v_y, p) = (1, 0, 0, 1), \text{ if } \sqrt{(x - 0.5)^2 + (y - 0.5)^2} \leq 0.2;$$
$$(\rho, v_x, v_y, p) = (0.125, 0, 0, 0.1), \text{ if } \sqrt{(x - 0.5)^2 + (y - 0.5)^2} \geq 0.2;$$

Here $[0, l_x] \times [0, l_y] = [0, 1] \times [0, 1]$. \hspace{1cm} (48)

γ is chosen to be $5/3$ and $n_x = n_y = 512$. $n_x$, $n_y$ are the number of grid points along $x$-and $y$-directions respectively. Fig.5, Fig.6 show the contour plot of the density at $t = 0$ and $t = 0.1$ respectively. Fig.7 contains the 3D plot of density in the $xy$-plane at $t = 0.1$ and Fig.8 shows the corresponding 1D cut of density along the $x$-direction at $y = 0.5$. Figs.7 and 8 correspond to 3D and 1D plots respectively shown in Fig.4 of ref.[20]. The non-oscillatory property of the fourth order scheme can be clearly seen in Figs.6 and 8.

VI. CONCLUSION

Through this paper, we have introduced a correction to the previous understanding of dimensionally split reconstruction for multi-dimensional hyperbolic problems [4]. We have demonstrated that multi-dimensional reconstruction polynomials can not be reduced to a polynomial to be used in the dimension by dimension approach, if the coefficients of the reduced polynomial are obtained from multi-dimension constraints. These coefficients have
to be obtained uniquely in all directions by applying 1D constraints only. This concept is used successfully to develop a very simple fourth order non-oscillatory central scheme to study multidimensional hyperbolic problems.

This opens the possibility of implementing even higher-order implementations of the central scheme approach although we find that already at fourth-order accuracy these algorithms start to become an interesting alternative to standard Godunov solvers.

Fruitful discussions with J. Stone are gratefully acknowledged and P.S.V. would also like to thank Tapan Chandra Adhyapak for useful suggestions.

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FIG. 1: Convergence of errors in third order scheme, for 2D and 3D Linear Advection (LA) tests, as a function of number of grid points (N). For 2D tests $N = n_x = n_y$, and for 3D test, $N = n_x = n_y = n_z$. Here $n_x$, $n_y$, $n_z$ are the number of grid points along $x$-, $y$- and $z$-directions respectively.

FIG. 2: Convergence of errors in fourth order scheme, for 3D Linear Advection (LA) test, as a function of number of grid points (N). $N = n_x = n_y = n_z$. Here $n_x$, $n_y$, $n_z$ are the number of grid points along $x$-, $y$- and $z$-directions respectively.
FIG. 3: Convergence of errors in third order scheme, for 2D and 3D Sound Wave (SW) tests, as a function of number of grid points (N). For 2D tests N = n_x = n_y, and for 3D test, N = n_x = n_y = n_z. Here n_x, n_y, n_z are the number of grid points along x-, y- and z-directions respectively. In the absence of additional terms, errors in 2D and 3D Sound Wave tests are overlapping.
FIG. 4: Convergence of errors in fourth order scheme, for 3D Sound Wave (SW) test, as a function of number of grid points (N). N = n_x = n_y = n_z. Here n_x, n_y, n_z are the number of grid points along x-, y- and z-directions respectively.

FIG. 5: Contour plot of density at t = 0 for 2D Blast wave
FIG. 6: Contour plot of density at $t = 0.1$ for 2D Blast wave obtained by fourth order scheme

FIG. 7: 3D plot of density at $t = 0.1$ for 2D Blast wave obtained by fourth order scheme
FIG. 8: 1D cut of density, at $y = 0.5$ and at $t = 0.1$ in 2D Blast wave test problem solved by fourth order scheme