Total Colorings of Some Classes of Four Regular Circulant Graphs

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Abstract

The total chromatic number, $\chi''(G)$ is the minimum number of colors which need to be assigned to obtain a total coloring of the graph $G$. The Total Coloring Conjecture (TCC) made independently by Behzad and Vizing that for any graph, $\chi''(G) \leq \Delta(G) + 2$, where $\Delta(G)$ represents the maximum degree of $G$. In this paper we obtained the total chromatic number for some classes of four regular circulant graphs.

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1 Introduction

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The total coloring of a graph $G$ is an assignment of colors to vertices and edges such that no two adjacent vertices or edges or edges incident to a vertex receives a same color. The total chromatic number of a graph $G$, denoted by $\chi''(G)$, is the minimum number of colors required for its total coloring. It is clear that $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of $G$.

Behzad [1] and Vizing [9] have independently proposed the Total Coloring Conjecture (TCC) which states that any simple graph $G$, $\chi''(G) \leq \Delta(G) + 2$. The graphs that can be totally colored by at-least $\Delta(G) + 1$ colors are said to be Type I graphs whereas the graphs which can be colored by $\Delta(G) + 2$ colors are said to be Type II graphs. The decidability algorithm for total coloring is NP-complete even for cubic bipartite graph [8]. Good survey of techniques and other results on total coloring can be found in Yap [10], Borodin [2] and Geetha et al. [4]. In this paper, we obtain the total chromatic number of some four regular circulant graphs are Type I.
2 Four Regular Circulant Graphs

For a sequence of positive integers $1 \leq d_1 < d_2 < \ldots < d_l \leq \lfloor \frac{n}{2} \rfloor$, the circulant graph $G = C_n(d_1, d_2, \ldots, d_l)$ has vertex set $V(G) = \{0, 1, 2, \ldots, n - 1\}$ and two vertices $x$ and $y$ are adjacent if $x \equiv (y + d_i) \mod n$ for some $i$ where $1 \leq i \leq l$. A power of cycles graph $C_n^k$ is a graph with vertex set $V(G) = \{0, 1, 2, \ldots, n - 1\}$ and two vertices $x$ and $y$ are adjacent if and only if $|x - y| \leq k$. It is easy to see that the four regular circulant graph $C_n(1, 2) \cong C_n^2$. Campos and de Mello [3] proved that $C_n^2, n \neq 7$, are Type I and $C_5^2$ is Type II. We know that $K_{4,4}$ is a four regular circulant graph and it is Type II [10]. A Unitary Cayley graph is a circulant graph with vertex set $V(G) = \{0, 1, 2, \ldots, n - 1\}$ and two vertices $x$ and $y$ are adjacent if and only if $gcd((x - y), n) = 1$. Prajnanaswaroop et al. [7], proved that all most all Unitary Cayley graphs of even order are Type I and odd order satisfies TCC. In this paper, we considered the four regular circulant graphs of the form $C_n(a, b), 1 \leq a < b \leq \lfloor \frac{n-1}{2} \rfloor$.

Mauro and Diana [5] proved that the graphs $C_n(2k, 3)$ are Type I for $n = (8\mu + 6\lambda)k$, with $k \geq 1$ and non-negative integers $\mu$ and $\lambda$. Riadh Khennoufa and Olivier Togni [6] studied total colorings of circulant graphs and proved that every 4-regular circulant graphs $C_{6p}(1, k), p \geq 3$ and $k < 3p$ with $k \equiv 1 \mod 3$ or $k \equiv 2 \mod 3$, are Type I. Other cases are still open.

Also they proved that the total chromatic number of $C_{5p}(1, k)$, with $k < \frac{5p}{2}, k \equiv 2 \mod 5, k \equiv 3 \mod 5$ is 5. In the following theorem, we prove that the graphs $C_{5p}(1, k)$ are Type I for the remaining cases $k \equiv 1 \mod 5$ and $k \equiv 4 \mod 5$.

**Theorem 2.1.** The circulant graphs $C_{5p}(a, b)$, where $p \geq 1, a, b \not\equiv 0 \mod 5$, are Type I.

**Proof.** Let $q_1 = gcd(5p, a)$ and $q_2 = gcd(5p, b)$. The circulant graphs $G = C_{5p}(a, b)$ are four regular graphs with $q_1$ cycles of order $\frac{5p}{q_1}$ and $q_2$ cycles of order $\frac{5p}{q_2}$.

Let $\phi : V(G) \cup E(G) \to C = \{0, 1, 2, 3, 4\}$ be a mapping of $G$ defined as follows. The vertices $v_i$ of $C_{5p}(a, b)$ can be colored by $\phi(v_i) = i \mod 5, 0 \leq i \leq 5p - 1$.

Since, the all the four graphs $C_{5p}(a, b), C_{5p}(b, a), C_{5p}(n - a, b)$ and $C_{5p}(a, n - b)$ are isomorphic to each other, for the edge colorings, we need to consider the three cases.

**Case 1:** $a \equiv 1 \mod 5$ and $b \equiv 1 \mod 5$.

The edges of cycles can be colored by setting $\phi(v_i v_{(i+a) \mod 5}) = (i + 2) \mod 5$ and $\phi(v_i v_{(i+b) \mod 5}) = (i + 4) \mod 5$. If $\phi(v_i) = c$ where $c \in C$, then $\phi(v_i v_{(i+a) \mod 5}) = (c + 2) \mod 5, \phi(v_i v_{(i-a) \mod 5} v_i) = (c + 1) \mod 5, \phi(v_i v_{(i+b) \mod 5}) = (c + 4) \mod 5$ and $\phi(v_i v_{(i-b) \mod 5} v_i) = (c + 3) \mod 5$.

**Case 2:** $a \equiv 2 \mod 5$ and $b \equiv 2 \mod 5$.

The edges of cycles can be colored by setting $\phi(v_i v_{(i+a) \mod 5}) = (i + 3) \mod 5$ and $\phi(v_i v_{(i+b) \mod 5}) = (i + 4) \mod 5$. If $\phi(v_i) = c$ where $c \in C$, then $\phi(v_i v_{(i+a) \mod 5}) = (c + 3) \mod 5, \phi(v_i v_{(i-a) \mod 5} v_i) = (c + 1) \mod 5, \phi(v_i v_{(i+b) \mod 5}) = (c + 4) \mod 5$ and $\phi(v_i v_{(i-b) \mod 5} v_i) = (c + 2) \mod 5$.

**Case 3:** $a \equiv 1 \mod 5$ and $b \equiv 2 \mod 5$.

The edges of cycles can be colored by setting $\phi(v_i v_{(i+a) \mod 5}) = (i + 3) \mod 5$ and $\phi(v_i v_{(i+b) \mod 5}) = (i + 1) \mod 5$. If $\phi(v_i) = c$ where $c \in C$, then $\phi(v_i v_{(i+a) \mod 5}) = (c + 2) \mod 5$ and $\phi(v_i v_{(i-b) \mod 5} v_i) = (c + 3) \mod 5, \phi(v_i v_{(i-a) \mod 5} v_i) = (c + 1) \mod 5, \phi(v_i v_{(i+b) \mod 5}) = (c + 4) \mod 5$ and $\phi(v_i v_{(i-b) \mod 5} v_i) = (c + 2) \mod 5$.
proved that the total chromatic number of $G$ is $5$. Therefore, five colors are used for totally coloring the graph.

In the following theorem, we prove some classes of four regular circulant graphs $C_n(a,b)$ of order $3p$ are Type I.

**Theorem 2.2.** Let $p$ be an odd integer. Then circulant graphs $C_{3p}(a,b)$ with $gcd(a,b) = 1$ and $\frac{3p}{gcd(3p,b)} = 3s$, $s \in N$ are Type I.

**Proof.** Let $q_1 = gcd(3p,a)$ and $q_2 = gcd(3p,b)$. The circulant graphs $G = C_{3p}(a,b)$ are four regular graphs with $q_1$ cycles of order $\frac{3p}{q_1}$ and $q_2$ cycles of order $\frac{3p}{q_2}$. Let $\varphi : V(G) \cup E(G) \to \{0, 1, 2, 3, 4\}$ be a mapping obtained by the following process.

Let $C_i$ be the cycles of order $\frac{3p}{q_i}$ with the vertices $v_{ia}$, $0 \leq i \leq q_2 - 1$. First we consider the cycle $C_0$. If $q_2 = 1$ then the vertices and edges of $C_0$ are colored with the colors 0, 3 and 1 cyclically, starting with $v_0$ receiving the color 0. Otherwise the vertices and edges of $C_0$ are colored with the colors 1, 0 and 4 cyclically, starting with $v_0$ receiving the color 1. Now, consider the cycle $C_1$. The vertices and edges of $C_1$ are colored with the colors 0, 2 and 1 cyclically, starting with $v_{ia}$ receiving the color 0 and $i$ is odd they are colored with the colors 1, 0 and 2 cyclically, starting with $v_{ia}$ receiving the color.

The edges of cycles of order $\frac{3p}{q_i}$ are colored in the following way: if vertex $v_i \in C_0$ then $\varphi(v_iv_{i+a}) = 2$, if $v_i \in C_i$ where $i$ is odd then $\varphi(v_iv_{i+a}) = 3$ and if $v_i \in C_i$ where $i$ is even then $\varphi(v_iv_{i+a}) = 4$. Therefore, only five colors are used for totally coloring the graph. Hence, $\varphi$ is a Type I coloring of $G$.

For the circulant graphs $C_n(a,b)$ where $n = 3p$ and $p$ is odd, which we considered in the above theorem, the value of $b$ is restricted to factors and multiple of $p$. In the following theorem, we consider few classes of circulant graphs $C_n(1,k)$ where $n = 9p$ are Type I.

**Theorem 2.3.** For each $k \in \{2, 3, ..., \left[\frac{9p-1}{2}\right]\}$, every circulant graph $C_{9p}(1,k)$ with $\frac{9p}{gcd(9p,k)} = 3s$, $s \in N$ is Type I.

**Proof.** Let $q = gcd(9p,k)$. The circulant graphs $G = C_{9p}(1,k)$ are four regular graphs with $q$ internal cycles of order $\frac{9p}{q}$, which are disjoint, and one outer cycle of order $9p$. Let $\varphi : V(G) \cup E(G) \to \{0, 1, 2, 3, 4\}$ be a mapping obtained by the following process.

Case 1: $p$ is even.

When $p$ is even, $9p$ will be a multiple of 6. Riadh Khennoufa and Olivier Togni [6] proved that the total chromatic number of $G = C_{6p}(1,k)$, with $k < \frac{5p}{2}$, $k \equiv 1 \mod 3$, $k \equiv 2 \mod 3$ is 5. From this, one can easily see that the circulant graph $C_{9p}(1,k)$, where $p \geq 1$ and $k < \frac{9p}{2}$, $k \equiv 1 \mod 3$, $k \equiv 2 \mod 3$ are Type I.
Now, we consider the remaining case, \( k \equiv 0 \mod 3 \).

The vertices \( v_i \) are colored by \( \varphi(v_i) = (i \mod 3 + \bigl(\lfloor \frac{i}{3} \bigr) \mod 3) \mod 3 \) if \( q = k \), else the vertices \( v_i \) are colored by \( \varphi(v_i) = (2i \mod 3 - \bigl(\lfloor \frac{i}{3} \bigr) \mod 3) \mod 3 \). The edges of the internal cycles can be colored by setting \( \varphi(v_i v_{i+k} \mod 9p) = (2\varphi(v_{i+k} \mod 9p) - \varphi(v_i)) \mod 3 \). In this coloring process, the vertices and the internal edges of \( G \) are colored using only three colors 0, 1 and 2. Now, the edges of the outer cycle can be colored with two colors 3 and 4 as it is of even order. Therefore, five colors are used for total coloring the graph, hence the graph \( C_{9p}(1,k) \) is Type I.

Case 2: \( p \) is odd.

The case when \( q \neq 1 \), follows from Theorem 2.2. Now, we consider the case when \( q = 1 \).

Sub case 2.1: \( k \equiv 1 \mod 9 \)

The vertices \( v_i \) where \( 0 \leq i \leq 9p - 1 \) of \( G \) can be colored by \( \varphi(v_i) = i \mod 3 + (\lfloor \frac{i}{3} \rfloor \mod 3) \lfloor \frac{i}{2} \rfloor \mod 3 \). The edges of the internal cycles can be colored by setting if \( i \equiv 1 \mod 3 \) then \( \varphi(v_i v_{i+k} \mod 9p) = \varphi(v_{i+k} \mod 9p) + 1 - 3\lfloor \frac{v_{i+k} \mod 9p v_i \mod 9p} {3} \rfloor \) else by \( \varphi(v_i v_{i+k} \mod 9p) = \varphi(v_{i+k} \mod 9p) \). The colors used for vertex \( v_i \) and edges incident to it is given in the table below as an ordered triplet \( (\varphi(v_i), \varphi(v_{i-k} \mod 9p v_i), \varphi(v_i v_{i+k} \mod 9p)) \).

| \( x \) | \( x = 0 \) | \( x = 1 \) | \( x = 2 \) | \( x = 3 \) | \( x = 4 \) | \( x = 5 \) | \( x = 6 \) | \( x = 7 \) | \( x = 8 \) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( x \equiv i \mod 9 \) | (0, 1, 2) | (1, 2, 3) | (2, 3, 1) | (0, 1, 3) | (1, 3, 4) | (2, 3, 1) | (0, 1, 4) | (1, 4, 2) | (2, 4, 1) |

The common missing color for vertices \( v_i \) and \( v_{i+1} \) can be used for coloring the edge \( v_i v_{i+1} \).

Therefore, five colors are used for totally coloring the graph, hence \( G = C_{9p}(1,k) \) Type I.

Sub case 2.2: \( k \equiv 4 \mod 9 \)

The vertices \( v_i \) where \( 0 \leq i \leq 9p - 1 \) of \( G \) can be colored by \( \varphi(v_i) = i \mod 3 + (\lfloor \frac{i}{3} \rfloor \mod 3) \lfloor \frac{i}{2} \rfloor \mod 3 \). The edges of the internal cycles can be colored by setting if \( i \equiv 1 \mod 3 \) then \( \varphi(v_i v_{i+k} \mod 9p) = \varphi(v_{i+k} \mod 9p + 1) \) else by \( \varphi(v_i v_{i+k} \mod 9p) = \varphi(v_{i+k} \mod 9p) \). The colors used for vertex \( v_i \) and edges incident to it is given in the table below as an ordered triplet \( (\varphi(v_i), \varphi(v_{i-k} \mod 9p v_i), \varphi(v_i v_{i+k} \mod 9p)) \).

| \( x \) | \( x = 0 \) | \( x = 1 \) | \( x = 2 \) | \( x = 3 \) | \( x = 4 \) | \( x = 5 \) | \( x = 6 \) | \( x = 7 \) | \( x = 8 \) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( x \equiv i \mod 9 \) | (0, 1, 2) | (1, 4, 2) | (3, 4, 1) | (0, 3, 1) | (1, 2, 3) | (2, 4, 1) | (0, 1, 4) | (1, 3, 4) | (2, 3, 1) |

The common missing color for vertices \( v_i \) and \( v_{i+1} \) can be used for coloring the edge \( v_i v_{i+1} \).

Therefore, five colors are used for totally coloring the graph, hence \( G = C_{9p}(1,k) \) Type I.

Sub case 2.3: \( k \equiv 7 \mod 9 \)

The vertices \( v_i \) where \( 0 \leq i \leq 9p - 1 \) of \( G \) can be colored by \( \varphi(v_i) = i \mod 3 + (\lfloor \frac{i}{3} \rfloor \mod 3) \lfloor \frac{i}{2} \rfloor \mod 3 \). The edges of the internal cycles can be colored by setting \( \varphi(v_i v_{i+k} \mod 9p) = \varphi(v_{i+k} \mod 9p) \). The colors used for vertex \( v_i \) and edges incident to it is given in the table below as an ordered triplet \( (\varphi(v_i), \varphi(v_{i-k} \mod 9p v_i), \varphi(v_i v_{i+k} \mod 9p)) \).

| \( x \) | \( x = 0 \) | \( x = 1 \) | \( x = 2 \) | \( x = 3 \) | \( x = 4 \) | \( x = 5 \) | \( x = 6 \) | \( x = 7 \) | \( x = 8 \) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( x \equiv i \mod 9 \) | (0, 1, 4) | (1, 2, 0) | (2, 0, 1) | (0, 1, 2) | (1, 3, 0) | (3, 0, 1) | (0, 1, 3) | (1, 4, 0) | (4, 0, 1) |
The common missing color for vertices $v_i$ and $v_{i+1}$ can be used for coloring the edge $v_iv_{i+1}$. Therefore, five colors are used for the total coloring the graph, hence $G = C_{9p}(1, k)$ is Type I.

In Theorem 2.2, we considered few classes of circulant graph $C_n(a, b)$ of order $n = 3p$, where $p$ is an odd integer. Now, in the following theorem, we consider few classes of four regular circulant graphs $C_n(a, b)$ of order $n = 6p$.

**Theorem 2.4.** Every circulant graph $C_{6p}(a, b)$ where $a, b \not\equiv 0 \mod 3$ is Type I, if $p$ is even. Also, $C_{6p}(a, b)$ where $a, b \not\equiv 0 \mod 3$ is Type I, if $p$ is odd and $\gcd(a, b) = 1$.

**Proof.** Let $q_1 = \gcd(6p, a)$ and $q_2 = \gcd(6p, b)$. The circulant graphs $G = C_{6p}(a, b)$ are four regular graphs with $q_1$ cycles of order $\frac{6p}{q_1}$ and $q_2$ cycles of order $\frac{6p}{q_2}$. The circulant graphs considered in the hypothesis can be colored in a similar manner irrespective of $p$ being odd or even, if $\frac{6p}{q_1}$ is odd we swap the value of $a$ and $b$, as graph $C_n(a, b)$ is isomorphic to $C_n(b, a)$. Let $\varphi : V(G) \cup E(G) \to \{0, 1, 2, 3, 4\}$ be a mapping. The vertices $v_i$ are colored by $\varphi(v_i) = i \mod 3$ and the edges of cycles of order $\frac{6p}{q_2}$ be colored by setting $\varphi(v_iv_{(i+a)} \mod 6p) = (2\varphi(v_{(i+a)} \mod 3p) - \varphi(v_i)) \mod 3$. Now, the edges of cycle $\frac{6p}{q_1}$ with two colors 3 and 4 as it is a cycle with even order. Therefore, five colors are used for the total coloring $\varphi$ of the graph, hence graph $G$ is Type I. □

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