Divisible difference families from Galois rings $GR(4, n)$ and Hadamard matrices

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Abstract

We give a new construction of difference families generalizing Szekeres’s difference families [17]. As an immediate consequence, we obtain some new examples of difference families with several blocks in multiplicative subgroups of finite fields. We also prove that there exists an infinite family of divisible difference families with two blocks in a unit subgroup of the Galois ring $GR(4, n)$. Furthermore, we obtain a new construction method of symmetric Hadamard matrices by using divisible difference families and a new array.

Keywords: difference family; divisible difference family; Hadamard matrix

1 Introduction

Let $(G, \cdot)$ be a finite abelian group with identity 1 and let $N$ be a subgroup of $G$. Let $D_i$ be a $k_i$-subset (called a block) of $G$ for $i = 1, 2, \ldots, b$, and set $\mathcal{F} = \{D_i | 1 \leq i \leq b\}$. For a block $D_i$, let

$$\theta_i(d) = |\{(x, y) | x \cdot y^{-1}, x \neq y, x, y \in D_i \}|, i = 1, 2, \ldots, b.$$ 

Furthermore, we put $\theta_{\mathcal{F}}(d) = \sum_{i=1}^{b} \theta_i(d)$.

Definition 1.1. A family $\mathcal{F} = \{D_i | 1 \leq i \leq b\}$ is called a $(G, N, \{k_1, \ldots, k_b\}, \lambda, \mu)$ divisible difference family if

$$\theta_{\mathcal{F}}(d) = \begin{cases} \lambda & \text{if } d \in N \setminus \{1\}, \\ \mu & \text{if } d \in G \setminus N. \end{cases}$$

In what follows, we abbreviate a divisible difference family as DDF. If $a_i$ blocks have same cardinality $k_i$, we write $k_i^{(a_i)}$ simply. If $k = k_1 = k_2 = \cdots = k_b$, then we will write $(G, N, k, \lambda, \mu)$-DDF. Especially if $N = \{1\}$, it is called a $(G, \{k_1, \ldots, k_b\}, \lambda)$ difference family, briefly denoted as DF. If $b = 1$, then it is called a $(G, N, k, \lambda, \mu)$ divisible difference set. A divisible difference set with $N = \{1\}$ is exactly a $(G, k, \lambda)$ difference set.

A divisible difference family yields a “group divisible design” by developing its blocks. Let $V$ be a set of $vg$ points, $\mathcal{B}$ be a collection of $k$-subsets (blocks) of $V$, and $\mathcal{G}$ be a collection of $g$-subsets (called groups) partitioning $V$. The triple $(V, \mathcal{B}, \mathcal{G})$ is called a $(k, \lambda, \mu)$-group divisible design of type $g^n$ if any two elements from different (resp. same) groups appear exactly $\mu$ (resp. $\lambda$) blocks. If $g = 1$, then the pair $(V, \mathcal{B})$ is called a 2-(v, k, \lambda) design. For any block $B$ of a group divisible design $(V, \mathcal{B}, \mathcal{G})$ and a permutation $\sigma$ on $V$, define $B^\sigma = \{b^\sigma | b \in B\}$. If $B^\sigma \in \mathcal{B}$ for all $B \in \mathcal{B}$, then $\sigma$ is called an automorphism of the design. The set of all such permutations forms a group under composition, which is called the full automorphism group. Any of its subgroups is called an automorphism group. It is obvious that a $(G, N, k, \lambda, \mu)$-DDF $\mathcal{F}$ yields a $(k, \lambda, \mu)$ group divisible

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design of type $|N|^{G/|N|}$ by developing its blocks, i.e., $B = \{D \cdot x \mid D \in \mathcal{F}, x \in G\}$ with $G$ as its automorphism group. Divisible difference families (sets) and difference families (sets) have been extensively studied as a major topic in the design theory in relation to group divisible designs and 2-designs. We refer the reader to [1] [4] [3] [5] [10] [12] [13] [14] [20] on divisible difference families and group divisible designs. However, there are not so many papers about divisible difference families with a few blocks.

In [17], Szekeres constructed difference families with two blocks in finite fields as follows.

**Proposition 1.2.** Let $q \equiv 3 \pmod{4}$ be a prime power. Let $\mathbb{F}_q$ be the finite field with $q$ elements and $N$ be the set of nonzero squares in $\mathbb{F}_q$. Then, the set of

$$D_1 := (N - 1) \cap N \text{ and } D_2 := (N + 1) \cap N$$

forms an $(N, (q - 3)/4, (q - 7)/4)$-DF.

We say that the difference family of Proposition 1.2 is the Szekeres difference family. The Szekeres difference family $\{D_1, D_2\}$ satisfies that

$$-a \notin D_i \text{ if } a \in D_i \text{ for either } i = 1 \text{ or } 2. \tag{1.1}$$

If an $n \times n$-matrix $M$ with entries from $\{1, -1\}$ satisfies $MM^T = nI_n$, $M$ is called an Hadamard matrix, where $I_n$ is the identity matrix of order $n$. An Hadamard matrix $M$ is symmetric or skew if $M = M^T$ or $M - I_n = -(M - I_n)^T$, respectively. For a general background on Hadamard matrices, we refer to [8] [9]. There have been known several constructions of Hadamard matrices from difference families [15] [18] [22] [25]. (These authors used the terminology “supplementary difference sets” instead of difference families.) For example, if a $(G, m, m-1)$-DF with $|G| = 2m + 1$ has the property (1.1), we are able to construct a skew Hadamard matrix as follows.

**Proposition 1.3.** (Theorem 4.4 of p. 321 [18]) Let $\mathcal{F} = \{D_1, D_2\}$ be a $(G, m, m-1)$-DF with $|G| = 2m + 1$ satisfying the condition (1.1). We define $(2m + 1) \times (2m + 1)$-matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ by

$$a_{i,j} := 2f_{D_1}(i - j) - 1 \quad \text{for the characteristic function } f_{D_1} \text{ of } D_1, \quad \text{and}$$

$$b_{i,j} := 2f_{D_2}(i + j) - 1 \quad \text{for the characteristic function } f_{D_2} \text{ of } D_2,$$

where rows and columns $i, j$ are labeled by the elements of $G$. Then, the matrix $M$ defined by

$$M = \begin{bmatrix}
1 & 1 & e^T & e^T \\
-1 & 1 & e^T & -e^T \\
-e & -e & A & B \\
-e & e & -B & A
\end{bmatrix} \tag{1.2}$$

forms a skew Hadamard matrix of order $4(m + 1)$, where $e$ is a column vector of length $2m + 1$ with all one entries.

It is well known that the set $N$ of Proposition 1.2 is a difference set in the additive group of $\mathbb{F}_q$, which yields an Hadamard matrix of order $q + 1$. One can see that this Hadamard matrix is equivalent to that obtained by applying Proposition 1.3 to the difference family of Proposition 1.2.

The main objective of this paper is to obtain an analogy in Galois rings of the relation between difference families and Hadamard matrices of Propositions 1.2 and 1.3. This paper is organized as follows: In Section 2 we give a general construction method of difference families by generalizing Szekeres’s construction of Proposition 1.2. Furthermore, we obtain some new examples of difference families with several blocks in multiplicative subgroups of finite fields. In Section 3, we prove that there exists an infinite family of divisible difference families with two blocks in unit groups of Galois rings of characteristic four. Finally, in Section 4 we give a new construction of symmetric Hadamard matrices by using a divisible difference family satisfying certain conditions and an array obtained by modifying the matrix $M$ of 1.2.
2 Generalization of Szekeres’s construction

At first of this section, we give a general construction of divisible difference families in a unit subgroup of a commutative ring \( \mathbb{R} \) under the assumption of the existence of difference families in the additive group of \( \mathbb{R} \).

**Lemma 2.1.** Let \( \mathbb{R} \) be a commutative ring with identity 1. Let \( \mathbb{R}^+ \) be a additive group and \( \mathbb{R}^* \) be a unit group of \( \mathbb{R} \), and \( I := \mathbb{R} \setminus \mathbb{R}^* \). Let \( N \subseteq \mathbb{R}^* \) and let \( S \) be a complete system of representatives of \( \mathbb{R}^*/N \). Let \( F = \{ D_i \mid 1 \leq i \leq b \} \) be an \( (\mathbb{R}^+, \{k_i \mid 1 \leq i \leq b \}, \lambda) \)-DF. Assume that each \( D_i \) is fixed by \( N \), i.e., \( xD_i = D_i \) for all \( x \in N \). We define subsets \( D_{i,y} \) of \( N \) as follows:

\[
D_{i,y} := y^{-1}(D_i - 1) \cap N
\]

for \( 1 \leq i \leq b \) and \( y \in S \). We put \( F' = \{ D_{i,y} \mid 1 \leq i \leq b; y \in S \} \). Then we have

\[
\theta_{F'}(t) = \lambda - \lambda_t,
\]

where

\[
\lambda_t := \sum_{i=1}^b |D_i \cap (D_i - t + 1) \cap (I + 1)|.
\]

If \( \lambda_t \) is constant for all \( t \in N \setminus \{1\} \), say \( \nu \), then \( F' \) is an \( (N, \{k_i, y \mid 1 \leq i \leq b; y \in S \}, \lambda - \nu) \)-DF. If \( \lambda - \lambda_t \) is a constant value \( \mu \) for \( t \in N \setminus L \) and \( \eta \) for \( t \in L \setminus \{1\} \), where \( L \) is a subgroup of \( N \), then \( F' \) is an \( (N, L, \{k_i, y \mid 1 \leq i \leq b; y \in S \}, \eta, \mu) \)-DDF.

**Proof:** The multiplicity \( \theta_{F'}(t) \) for each \( t \in N \) is given by

\[
\sum_{i=1}^{b} \sum_{y \in S} |D_{i,y} \cap tD_{i,y}| = \sum_{i=1}^{b} \sum_{y \in S} |(D_i - 1) \cap t(D_i - 1) \cap yN|
\]

\[
= \sum_{i=1}^{b} |D_i \cap (D_i - t + 1) \cap (\mathbb{R}^* + 1)|
\]

\[
= \sum_{i=1}^{b} |D_i \cap (D_i - t + 1)| - \sum_{i=1}^{b} |D_i \cap (D_i - t + 1) \cap (I + 1)|.
\]

By the assumption that \( F \) is an \( (\mathbb{R}^+, \{k_i \mid 1 \leq i \leq b \}, \lambda) \)-DF, it holds

\[
\sum_{i=1}^{b} |D_i \cap (D_i - t + 1)| = \lambda,
\]

and hence we obtain the assertion. \( \square \)

We denote the multiplicative group and the additive group of the finite field \( \mathbb{F}_q \) by \( \mathbb{F}_q^* \) and \( \mathbb{F}_q^+ \), respectively, and a primitive element of \( \mathbb{F}_q \) by \( g \). Furthermore, we denote the residue ring \( \mathbb{Z}/s\mathbb{Z} \) of rational integers by \( \mathbb{Z}_s \).

Now, we provide several new examples of difference families in a multiplicative subgroup of the finite field \( \mathbb{F}_q \) by applying Lemma 2.1 to known “cyclotomic” difference sets. We say that a difference set \( D \) in \( \mathbb{F}_q^* \) is cyclotomic if \( D \) is a subgroup of \( \mathbb{F}_q^* \) or the union of a subgroup of \( \mathbb{F}_q^* \) and \( \{0\} \).

**Proposition 2.2.** (i) For any prime power \( q \equiv 3 \pmod{4} \), there exists a \((\mathbb{Z}_{(q-1)/2}, (q-3)/4, (q-7)/4)\)-DF.

(ii) For any prime power \( q = 1 + 4t^2 \) with \( t \equiv 1 \pmod{2} \), there exists a \((\mathbb{Z}_{(q-1)/4}, (q-5)/16, (q-21)/16)\)-DF.
Remark 2.4.  It is known that for a multiplicative subgroup $D,$ $\mathbb{Z}_q^\times$ forms a cyclotomic $(q, [D], \lambda)$ difference set in the following cases: (i) $e = 2$ and $q \equiv 3 \pmod{4}$; (ii) $e = 4$ and $q = 1 + 4t^2$ with $t \equiv 1 \pmod{2}$; and (iii) $e = 8$ and $q = 9 + 64a^2 = 1 + 8b^2$ with $a \equiv b \equiv 1 \pmod{2}$.  

(See [16].)  Now, we apply Lemma 2.1 to these three difference sets $D_1 \cap N$ for $0 \leq i \leq e - 1$.

Hence, by Lemma 2.1, \{ $D_{1,y}$, $0 \leq i \leq e - 1$ \} forms a $(D, \{ k_{1,y} \mid y \in S \}, \lambda - 1)$-DF.  Since $N \simeq \mathbb{Z}_{q-1}$, this DF is isomorphic to that with the parameters in the statement of the proposition.  Finally, we compute each block size $k_{1,y} = |D_{1,y}|$.  By noting that $\lceil (yN + 1) \cap N \rceil = \lceil (N + y) \cap N \rceil$ (cf. [10] Lemma 3 (b)), we have

$$|D_{1,y}| = |y^{-1}(D - 1) \cap D| = |(yN + 1) \cap N| = \lceil (N + y) \cap N \rceil = \lambda$$

since $D = N$ is a difference set in $\mathbb{F}_q$.  This completes the proof. \hfill $\square$

Proposition 2.3.  (i) For any prime power $q \equiv 3 \pmod{4}$, there exists a $(\mathbb{Z}_{q-1}/2, K, (q - 3)/4)$-DF, where $K = \{(q - 3)/4, (q + 1)/4\}$.

(ii) For any prime power $q = 9 + 4t^2$ with $t \equiv 1 \pmod{2}$, there exists a $(\mathbb{Z}_{q-1}/4, K, (q - 3)/16)$-DF, where $K = \{(q - 13)/16, (q - 3)/16^{(3)}\}$.

(iii) For any prime power $q = 441 + 64a^2 = 49 + 8b^2$ with $a \equiv b \equiv 1 \pmod{2}$, there exists a $(\mathbb{Z}_{q-1}/8, K, (q - 56)/64)$-DF, where $K = \{(q - 56)/64, (q + 8)/64^{(7)}\}$.

Proof: It is known that for a multiplicative subgroup $N$ of index $e$ of $\mathbb{F}_q^\times$, $D = N \cup \{0\}$ forms a cyclotomic $(q, [D], \lambda)$ difference set in the following cases: (i) $e = 2$ and $q \equiv 3 \pmod{4}$; (ii) $e = 4$ and $q = 9 + 4t^2$ with $t \equiv 1 \pmod{2}$; and (iii) $e = 8$ and $q = 441 + 64a^2 = 49 + 8b^2$ with $a \equiv b \equiv 1 \pmod{2}$.  (See [16].)  Now, we apply Lemma 2.1 to these three difference sets $D$.  It is clear that $D$ is fixed by $N$.  Let $S = \{ g^i \mid 0 \leq i \leq e - 1 \}$ and $D_{1,y} = g^{-1}(D - 1) \cap N$ for $0 \leq i \leq e - 1$.  For $t \in D \setminus \{1\}$, we have

$$\lambda_t = |D \cap (D - t + 1) \cap \{0\}| = 1.$$ 

Hence, by Lemma 2.1, \{ $D_{1,y}$, $0 \leq i \leq e - 1$ \} forms a $(D, \{ k_{1,y} \mid y \in S \}, \lambda - 1)$-DF.  Since $N \simeq \mathbb{Z}_{q-1}$, this DF is isomorphic to that with the parameters in the statement of the proposition.  Finally, we compute each block size $k_{1,y} = |D_{1,y}|$.  By noting that $\lceil (yN + 1) \cap N \rceil = \lceil (N + y) \cap N \rceil$ (cf. [10] Lemma 3 (b)), we have

$$\lambda_t = |D \cap (D - t + 1) \cap \{0\}| = 1.$$ 

Remark 2.4.  (1) Let $q \equiv 3 \pmod{4}$ and $N$ be the multiplicative subgroup of index 2 of $\mathbb{F}_q^\times$.  Proposition 2.2 (i) says that the set of $(N - 1) \cap N$ and $\lceil (N + 1) \cap N \rceil$ forms an $(N, (q - 3)/4, (q + 7)/4)$-DF.  On the other hand, Proposition 2.4 (i) says that the set of $(N - 1) \cap N$ and $\lceil (N + 1) \cap N \rceil$ forms an $(N, (q - 3)/4, (q - 7)/4)$-DF.  By noting that

$$x \in ((N + 1) \cap N)^{-1} \iff x \in N^{-1} \text{ and } x \in (N - 1)^{-1} \iff x \in N \text{ and } x^{-1} \in N \iff x \in (N - 1) \cap N,$$
3 Divisible difference families from Hadamard difference sets in Galois rings $GR(4, n)$

Let $q = p^r$ be a power of a prime $p$. We denote the absolute trace from $F_q$ to $F_p$ by $Tr_{F_q}$. An additive character of $F_q$ is written as

$$\chi_b(c) = \zeta_p^{Tr_q(bc)} \text{ for any } b \in F_q,$$

where $\zeta_p = \exp\left(\frac{2\pi i}{p}\right)$. Let $g(x) \in Z_{p^2}[x]$ be a primitive basic irreducible polynomial of degree $n$ and denote the root of $g(x)$ by $\xi$. Then $Z_{p^2}[x]/g(x)$ is called a Galois ring of characteristic $p^2$ and of an extension degree $n$, and denoted by $GR(p^2, n)$. The algebraic extension of $Z_{p^2}$ obtained by adjoining $\xi$ is isomorphic to $Z_{p^2}[x]/g(x)$. For easy reference, we put $R_n = GR(4, n)$ as $p = 2$. $R_n$ has a unique maximal ideal $F_n = pR_n$ and the residue ring $R_n/F_n$ is isomorphic to $F_{p^n}$. We take $T_n = \{0, 1, \xi, \ldots, \xi^{p^n-2}\}$ as a set of representatives of $R_n/F_n$. An arbitrary element $\alpha \in R_n$ is uniquely represented as $\alpha = \alpha_0 + p\alpha_1$, $\alpha_0, \alpha_1 \in T_n$.

We define the map $\tau : R_n \rightarrow T_n := T_n \setminus \{0\}$ as $\tau(\alpha) = \alpha p^n$. The kernel of $\tau$ is the group $U_n$ of principal units, which are elements of the form $1 + 2\beta, \beta \in T_n$. For the element $1 + 2\beta \in U_n$, we may regard $\beta$ as an element of $F_{p^n}$, then $U_n$ is isomorphic to the additive group of $F_{p^n}$. If we denote the set of all units in $R_n$ by $R_n^*$, then $R_n^*$ is the direct product of $U_n$ and the cyclic group $T_n^*$ of order $p^n - 1$. In other words, every element of $R_n^*$ is uniquely represented as $\alpha = \alpha_0(1 + p\alpha_1)$, $\alpha_0, \alpha_1 \in T_n, \alpha_0 \neq 0$.

In what follows, we identify $R_n/pR_n$ with $F_{p^n}$ and denote each element of $R_n/pR_n$ and $F_{p^n}$ by $\bar{\pi}$ in common. In this section, we construct divisible difference families in Galois rings $R_n = GR(4, n)$ by applying Lemma 2.3 to difference sets obtained in [19]. Let $E := \{\pi \in F_{2n} \mid Tr_{2n}(\pi) = 0\}$ for $\pi \in F_{2n}$ such that $Tr_{2n}(\pi) = 0$. Note that $E$ is a subgroup of order $2^{n-1}$ of $F_{2n}$. In [19], it was proved that $D = \{a(1+2b) \mid a \in T_n, b \in T_n, \text{ such that } \bar{b} \in E\}$ forms an $((R_n^*, 2^{n-1}(2^n-1), 2^{n-1}(2^n-1) - 2))$ difference set. (A difference set with parameters $(G, 2^{n-1}(2^n-1), 2^{n-1}(2^n-1) - 2))$ for a group $G$ of order $2^{2n}$ is called Hadamard.) Notice that $D$ is a subgroup of index $2$ of $R_n^*$ isomorphic to $Z_{2^n-1} \times Z_{2^{n-1}}$. We shall use the fact that the characteristic function $\psi(\pi)$ of $E$ in $F_{2^n}$ is given by

$$\psi(\pi) = \frac{1}{2} \sum_{\pi \in F_{2^n}} \chi(\pi^{-1}) = \frac{1}{2} \sum_{\pi \in F_{2^n}} (-1)^{Tr_{2n}(\pi)}$$

where $\chi$ is the canonical additive character of $F_{2^n}$.

**Theorem 3.1.** Let $D$ and $E$ be as in the above. Let $N$ be a subgroup of $D$ and put $L = N \cap U_n$. Then there exists an $(N, L, K, 2^{n}(2^{n-2} - 1), 2^{n-1}(2^{n-1} - 1), 2^{n-2})$-DDF where $K$ is the set of cardinalities of blocks. In particular, if $N = D$, the DDF is isomorphic to a $(Z_{2^n-1} \times Z_{2^{n-1}}, \{0\} \times Z_{2^n-1} \times Z_{2^{n-1}})$-DDF.

**Proof:** Obviously, $D$ is fixed by $N$ since $N \leq D$. Furthermore, we have $D \cap (F_n + 1) = \{1 + 2a | \pi \in F_{2^n} \}$. 

Let \( t = c(1 + 2d) \in N \). By applying Lemma 2.1 to \( D \), the number \( \lambda_t \) is computed as follows:

\[
\lambda_t = |D \cap (D - t + 1) \cap (\mathbb{P}_n + 1)|
\]

\[
= |\{(1 + 2a | \bar{a} \in E \} \cap \{(c'(1 + 2a') | c' \in \mathbb{T}^n_1, \bar{a}' \in E \} - c(1 + 2d) + 1)|
\]

\[
= |E \cap \{c'(a' - d) | a' \in E \}|
\]

\[
= \sum_{\bar{a} \in \mathbb{P}_n} \left( \frac{1}{5} \sum_{\bar{b} \in \mathbb{P}_n} \chi(\overline{uv}) \right) \cdot \left( \frac{1}{2} \sum_{\bar{c} \in \mathbb{P}_n} \chi(\overline{uw}(\bar{a}' + \bar{d})) \right)
\]

\[
= \frac{1}{4} \sum_{\bar{a} \in \mathbb{P}_n} \left( \sum_{\bar{b} \in \mathbb{P}_n} \sum_{\bar{c} \in \mathbb{P}_n} \chi(\overline{uv}(\bar{a} + \bar{c})) + \overline{udv} \right)
\]

\[
= \begin{cases} 
2^n - 2 + 2^n - 2\chi(\overline{ud}) & \text{if } \overline{c} \overline{d} = 1, \\
2^n - 2 & \text{otherwise,}
\end{cases}
\]

\[
= \begin{cases} 
2^n - 1 & \text{if } c = 1, \text{ that is, } t \in L, \\
2^n - 2 & \text{otherwise.}
\end{cases}
\]

The set \( K \) of cardinalities of blocks is determined by the subgroup \( N \). In particular, if \( N = D \), the size \( k_y \) of each block \( y^{-1}(D - 1) \cap N \) is given by

\[
k_y = |(yD + 1) \cap D| = |(D + y) \cap D| = 2^{n-1}(2^n - 1)
\]

since \(|(yD + 1) \cap D| = |(D + y) \cap D|\) and \( D \) is a difference set in \( \mathbb{P}_n^+ \). Since \( D \leq \mathbb{P}_n^+ \) is isomorphic to \( \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2^n \), this DDF is isomorphic to that with the parameters in the statement of the theorem.

\[
\square
\]

**Remark 3.2.** We notice that \( D \cup \mathbb{P}_n \) forms an \((\mathbb{R}_n^+, 2^{n-1}(2^n + 1), 2^{n-2}(2^n + 1)\)) difference set. Let \( N \) and \( L \) be as in Theorem 3.1. We have an \((N, L, K, 2^{n-1}(2^n + 1), 2^{n-2}(2^n + 1))\)-DDF since

\[
|(D \cup \mathbb{P}_n) \cap ((D \cup \mathbb{P}_n) - t + 1) \cap (\mathbb{P}_n + 1)| = |D \cap (D - t + 1) \cap (\mathbb{P}_n + 1)|
\]

for any \( t \in N \setminus \{1\} \). In particular, if \( N = D \), it is isomorphic to a DDF with parameters \((\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2^n, \{0\} \times \mathbb{Z}_{2^{n-1}}^0, 2^{n-1-1}, 2^{n-1}, 2^{n-2}(2^n + 1))\) since

\[
k_y = |y^{-1}(N \cup \mathbb{P}_n - 1) \cap N| = |y^{-1}(N - 1) \cap N| + |y^{-1}(\mathbb{P}_n - 1) \cap N|
\]

\[
= |(N + y) \cap N| + |\mathbb{P}_n \cap (yN + 1)|
\]

\[
= 2^{n-1}(2^n - 1) + 2^{n-1} = 2^{2n-1}.
\]

**Example 3.3.** Let \( \xi \) be a root of \( g(x) := x^3 + 3x^2 + 2x + 3 \in \mathbb{Z}_4[x] \), a primitive basic irreducible polynomial of \( GR(4,3) \). Furthermore, let \( xyz \) denote the element \( z\xi^2 + y\xi + z \in GR(4,3) \). Let \( D = \{a(1 + 2b) | a \in \mathbb{T}^3_3, b \in \{0, 1, \xi^2, \xi^3\}\} \) and \( L = D \cup \mathbb{U}_3 \). Note that \( D \) forms an \((\mathbb{R}_n^+, 28, 12)\) difference set. Also, \( D \) is a subgroup of index 2 of \( \mathbb{R}_3^+ \). Then, the family of

\[
D_1 = \{103, 232, 322, 112, 211, 111, 231, 121, 300, 332, 212, 331\}
\]

\[
D_2 = \{233, 322, 332, 113, 213, 121, 010, 333, 103, 300, 112, 030\}
\]

forms a \((D, L, 12, 8, 10)\)-DDF. Since \( \mathbb{R}_3^+ \cong \mathbb{Z}_7 \times \mathbb{Z}_2^2 \) and \( L \cong \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \), the blocks can be expressed as

\[
D_1 = \{(1, 0, 0), (1, 1, 0), (2, 0, 1), (2, 1, 0), (3, 0, 0), (3, 0, 1), \\
(4, 0, 0), (4, 0, 1), (5, 0, 1), (5, 1, 0), (6, 0, 1), (6, 1, 1)\} \subseteq \mathbb{Z}_7 \times \mathbb{Z}_2^2 \text{ and}
\]

\[
D_2 = \{(1, 0, 0), (1, 1, 0), (2, 0, 1), (2, 1, 0), (3, 0, 0), (3, 0, 1), \\
(4, 1, 0), (4, 1, 1), (5, 0, 0), (5, 1, 1), (6, 0, 1), (6, 1, 1)\} \subseteq \mathbb{Z}_7 \times \mathbb{Z}_2^2,
\]

and the family \( \{D_1, D_2\} \) forms a \((\mathbb{Z}_7 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2, 12, 8, 10)\)-DDF.
Next we show some important and interesting properties of divisible difference families obtained in Theorem 3.1.

**Proposition 3.4.** The \((\mathbb{Z}_{2n-1} \times \mathbb{Z}_{2}^{n-1}, \{0\} \times \mathbb{Z}_{2}^{n-1}, 2^{n-1}(2^{n-1} - 1), 2^{n}(2^{n-2} - 1), 2^{n-1}(2^{n-1} - 1) - 2^{n-2})\)-DDF \(F = \{D_1, D_2\}\) of Theorem 3.1 has the following properties:

(i) If \(a \in D_1\), then \(-a \in D_1\).

(ii) If \(a \in D_2\), then \(-a \notin D_2\).

(iii) For each \(i = 1, 2\), it holds that \(|D_1 \cap \{(0) \times \mathbb{Z}_{2}^{n-1}\}| = 0\) and \(|D_1 \cap \{(j) \times \mathbb{Z}_{2}^{n-1}\}| = 2^{n-2}\) for any \(j \in \mathbb{Z}_{2n-1} \setminus \{0\}\).

**Proof:** We consider the \((D, L, K, 2^{n}(2^{n-2} - 1), 2^{n-1}(2^{n-1} - 1) - 2^{n-2})\)-DDF \(F = \{D_1, D_2\}\) in \(GR(4, n)\) obtained in Theorem 3.1, where \(D_1 = (D - 1) \cap D\) and \(D_2 = y^{-1}(D - 1) \cap D\). Here, \(\{1, y\}\) is a set of representatives for \(\mathbb{R}_n/D\). We can assume that \(y \in \mathbb{U}_n\). Then, we have \(y^2 = 1\).

(i) We prove that \(a^{-1} \in (D - 1) \cap D\) if \(a \in (D - 1) \cap D\).

\[
\begin{align*}
\text{\(a \in (D - 1) \cap D\)} & \iff \text{\(a \in D\) and \(a + 1 \in D\)} \\
& \iff \text{\(a^{-1} \in D\) and \(a^{-1} + 1 = a^{-1}(a + 1) \in D\)} \\
& \iff \text{\(a^{-1} \in (D - 1) \cap D\).}
\end{align*}
\]

(ii) We prove that \(a^{-1} \notin y(D - 1) \cap D\) if \(a \in y(D - 1) \cap D\).

\[
\begin{align*}
\text{\(a \in y(D - 1) \cap D\)} & \iff \text{\(a \in D\) and \(ay^{-1} + 1 \in D\)} \\
& \iff \text{\(a^{-1} \in D\) and \(a^{-1}y^{-1} + 1 = a^{-1}y + 1 = a^{-1}y(a^{-1} + 1) \in yD\)} \\
& \iff \text{\(a^{-1} \in y(yD - 1) \cap D\)} \\
& \iff \text{\(a^{-1} \notin y(D - 1) \cap D\).}
\end{align*}
\]

Under the isomorphism \(\phi: \mathbb{R}_n^* \to \mathbb{Z}_{2n-1} \times \mathbb{Z}_{2}^{n-1}\), we obtain the assertions (i) and (ii) in the theorem.

(iii) The image of \(\xi^j\{1 + 2x \mid \overline{x} \in E\}\) by the isomorphism \(\phi\) is \(\{j\} \times \mathbb{Z}_{2}^{n-1}\). Hence it is sufficient to show

\[
(\gamma(j) := \left| y(D - 1) \cap D \cap \xi^j\{1 + 2x \mid \overline{x} \in E\} \right| = \begin{cases} 
0 & \text{if } j = 0, \\
2^{n-2} & \text{if } j \neq 0.
\end{cases}
\]

Assume that \(j = 0\). Since \(\{1 + 2x \mid \overline{x} \in E\} \subseteq D\), we have

\[
\gamma(0) = |\{y(w(1 + 2x) - 1) \mid w \in \mathbb{T}_n^*, \overline{x} \in E\} \cap \{1 + 2x \mid \overline{x} \in E\}|.
\]

Since \(w \in \mathbb{T}_n^*, w - 1 \notin \mathbb{U}_n\), so that \(y(w - 1) \notin \mathbb{U}_n\). For \(w \in \mathbb{T}_n^*, \overline{x} \in E\), we have \(y(w(1 + 2x) - 1) = y(w - 1) + 2ywx \notin \mathbb{U}_n\). It yields \(\gamma(0) = 0\).

Assume that \(j \neq 0\). By letting \(y = 1 + 2a\), we have

\[
\gamma(j) = |(w - 1) + 2w(x + a) - 2a \mid w \in \mathbb{T}_n^*, \overline{x} \in E\} \cap \xi^j\{1 + 2x \mid \overline{x} \in E\}|.
\]

There is a unique \(w \in \mathbb{T}_n^* \setminus \{1\}\) such that \(w - 1 \in \xi^j \cap \mathbb{U}_n\). We put \(w - 1 = \xi^j(1 + 2c)\) for such \(w\).
Then, we have

\[
\gamma(j) = \left| \{ (\xi^j(1 + 2c) + 1)(1 + 2x + 2a) - 1 - 2a \xi E \} \cap \xi^j\{ 1 + 2x \xi E \} \right|
\]

\[
= \left| \{ 2c\xi^j + \xi^j(1)(2x + 2a) - 2a \xi E \} \cap \{ 2\xi^j x \xi E \} \right|
\]

\[
= \left| ( (1 + \xi^{-j}) E + \xi \overline{w} \right \} \cap E \right|
\]

\[
= \sum_{\xi \in F_{2^n}} \left( \frac{1}{2} \sum_{w \in F_2} \chi(ww\xi) \right) \cdot \left( \frac{1}{2} \sum_{\xi \in F_2} \chi(ww(\xi - \overline{\xi})(1 + \xi^{-j}))^{-1}) \right)
\]

\[
= \frac{1}{4} \sum_{\xi \in F_{2^n}} \left( \sum_{w \in F_2} \sum_{\xi \in F_2} \chi(w\xi w + \overline{\xi}(1 + \xi^{-j}))^{-1} + \overline{\xi}(1 + \xi^{-j}))^{-1} \right)
\]

\[
= 2^{n-2}.
\]

Here, we used that \( 1 + \xi^{-j} \neq 0 \) and \( 1 + (1 + \xi^{-j})^{-1} \neq 0 \) for any \( j \neq 0 \). Under the isomorphism \( \phi : \mathbb{R}_n^* \rightarrow \mathbb{Z}_{2^n-1} \times \mathbb{Z}_2^* \), we obtain the assertion (iii) of the theorem.

The next proposition shows that \( (D + 2) \cap T_n^* \) forms a \( (T_n^*, 2^{n-1} - 1, 2^{n-2} - 1) \) difference set. Then it yields an Hadamard matrix of order \( 2^n \).

**Proposition 3.5.** The set \( (D + 2) \cap T_n^* \) forms a \( (T_n^*, 2^{n-1} - 1, 2^{n-2} - 1) \) difference set.

**Proof:** We show that for any \( \xi^j \in T_n^* \backslash \{ 1 \} \)

\[
| (D + 2) \cap T_n^* \cap \xi^j((D + 2) \cap T_n^*) | = 2^{n-2} - 1.
\]

Since \( D = xD \) for \( x \in T_n^* \), we have

\[
| (D + 2) \cap T_n^* \cap \xi^j((D + 2) \cap T_n^*) | = | (a(1 + 2b) + 2 | a, b \in T_n^*, b \in E \} \cap \{ a(1 + 2b) + 2\xi^j | a, b \in T_n^*, b \in E \} \cap T_n^* |. \quad (3.3)
\]

Here, \( a(1 + 2b) + 2 = a + 2(ab + 1) \) \( E \) \( T_n^* \) if and only if \( a = b^{-1} \) since \( a \in T_n^* \). Similarly, \( a(1 + 2b) + 2\xi^j \in T_n^* \) if and only if \( a = b^{-1}\xi^j \). Thus, (3.3) is reformulated as

\[
= \left| \{ b^{-1}(1 + 2b) + 2 | b \in E \backslash \{ 0 \}, b \in T_n^* \} \cap \{ b^{-1}\xi^j(1 + 2b) + 2\xi^j | b \in E \backslash \{ 0 \}, b \in T_n^* \} \right|
\]

\[
= \left| \{ b^{-1} \overline{b} \in E \backslash \{ 0 \}, b \in T_n^* \} \cap \{ b^{-1}\xi^j \overline{b} \in E \backslash \{ 0 \}, b \in T_n^* \} \right|
\]

\[
= \left| E \backslash \{ 0 \} \cap \xi^{-j}(E \backslash \{ 0 \}) \right|
\]

\[
= \sum_{\xi \in F_{2^n}} \left( \frac{1}{2} \sum_{w \in F_2} \chi(ww\xi) \right) \cdot \left( \frac{1}{2} \sum_{\xi \in F_2} \chi(ww\xi^j) \right)
\]

\[
= \frac{1}{4} \sum_{\xi \in F_{2^n}} \left( \sum_{w \in F_2} \sum_{\xi \in F_2} \chi(ww(\xi + \overline{\xi})) \right)
\]

\[
= 2^{n-2} - 1.
\]

The size of \( (D + 2) \cap T_n^* \) is computed as follows:

\[
| (D + 2) \cap T_n^* | = | \{ a(1 + 2b) + 2 | a, b \in T_n^*, b \in E \} \cap T_n^* | = | \{ b^{-1}(1 + 2b) + 2 \overline{b} \in E \backslash \{ 0 \}, b \in T_n^* \} \cap T_n^* | = | \{ b^{-1} \overline{b} \in E \backslash \{ 0 \}, b \in T_n^* \} \cap T_n^* | = | E \backslash \{ 0 \} | - 1 = 2^{n-1} - 1.
\]

This completes the proof.
4 Construction of symmetric Hadamard matrices

In this section, we give a new construction of symmetric Hadamard matrices.

**Theorem 4.1.** Assume that there exists an Hadamard matrix of order \( n \). Let \( \mathcal{F} = \{D_1, D_2\} \) be a \( (G, N, \frac{n(n-2)}{4}, \frac{n(n-4)}{4}, \frac{n(n-2)}{4}) \)-DDF with \( |G| = \frac{n(n-1)}{2} \) and \( |N| = \frac{n}{2} \).

Further assume that \( \mathcal{F} \) satisfies the following conditions:

(i) If \( a \in D_1 \), then \(-a \in D_1\);

(ii) \( |D_1 \cap N| = 0 \) and \( |D_1 \cap (N + j)| = n/4 \) for any \( j \in G \setminus N \) and for each \( i = 1, 2 \).

Then, there exists a symmetric Hadamard matrix of order \( n^2 \).

**Proof:** We define the matrices \( A = (a_{i,j}), B = (b_{i,j}) \) and \( C = (c_{i,j}) \) of size \(|G| \times |G|\) by

\[
\begin{align*}
&\quad a_{i,j} := f_{D_1}(i - j) \quad \text{for the characteristic function } f_{D_1} \text{ of } D_1, \\
&\quad b_{i,j} := f_{D_2}(i + j) \quad \text{for the characteristic function } f_{D_2} \text{ of } D_2, \\
&\quad c_{i,j} := f_{N}(i + j) \quad \text{for the characteristic function } f_{N} \text{ of } N,
\end{align*}
\]

for \( i, j \in G \). Further we let \( A' := 2A - J_{|G|}\) and \( B' := 2B - J_{|G|}\), where \( J_{|G|}\) is the all one matrix of size \(|G| \times |G|\). Without loss of generality, we may assume that the first column and first row of the assumed Hadamard matrix \( H \) are all one vectors. Let \( H' \) be the \((n - 1) \times n\)-matrix obtained by removing the first row of \( H \). Assume that the rows of the matrix \( H' \) are indexed by the elements of \( S = G/N \) and set \( H' = (h_i)_{i \in S} \). Let \( H_1 = (h_i)_{i \in G} \) be the \((n - 1)/2 \times n\)-matrix defined by \( h_{i+x} := h_i \) for \( i \in S \) and \( x \in N \). Furthermore, let \( H_2 = (-h_{-i})_{i \in G} \), then \( H_2 \) has size \( n(n-1)/2 \times n \). Then the matrix

\[
M = \begin{bmatrix}
-J_n & H_1^T & H_2^T \\
H_1 & B' & A' \\
H_2 & A' & -B' - 2C
\end{bmatrix}
\]

forms the desired symmetric Hadamard matrix of order \( n^2 \). It is clear that

\[
MM^T = \begin{bmatrix}
-J_n & H_1^T & H_2^T \\
H_1 & H_1^T + H_2^T & A' \\
H_2 & A' & -B' - 2C
\end{bmatrix}
\]

To prove \( MM^T = n^2I \), we show the following claims.

**Claim 1.** Write \( H_1 = (h_{i,j}^{(1)}) \) and \( H_2 = (h_{i,j}^{(2)}) \), where \( i \in G, 1 \leq j \leq n \). Then, for each \( \ell = 1, 2 \)

\[
(\text{the } (i, j) \text{ entry of } H_\ell H_\ell^T) = \sum_{k=1}^{n} h_{i,k}^{(\ell)} h_{j,k}^{(\ell)} = \left\{ \begin{array}{ll} n & \text{if } i = j \in N, \\ 0 & \text{otherwise}. \end{array} \right.
\]

**Claim 2.** For each \( \ell = 1, 2 \),

\[
(\text{the } (i, j) \text{ entry of } H_\ell^T H_\ell) = \sum_{k \in G} h_{k,i}^{(\ell)} h_{k,j}^{(\ell)} = \left\{ \begin{array}{ll} n(n-1)/2 & \text{if } i = j, \\ -n/2 & \text{otherwise}. \end{array} \right.
\]
Claim 3. Note that $h_{i,j}^{(2)} = -h_{-i,j}^{(1)}$, by the definition of $H_2$. Then,

$$
\begin{align*}
\text{(the (i, j) entry of } H_1 H_2^T) &= \sum_{k=1}^{n} h_{i,k}^{(1)} h_{j,k}^{(2)} \\
&= -\sum_{k=1}^{n} h_{i,k}^{(1)} h_{-j,k}^{(1)} \\
&= \begin{cases} 
- n & \text{if } i + j \in N, \\
0 & \text{otherwise.} 
\end{cases}
\end{align*}
$$

Claim 4.

(The (i, j) entry of $CC^T$) = \sum_{k \in G} c_{i,k} c_{j,k} \\
= \sum_{x \in S} \sum_{y \in N} c_{i,x+y} c_{j,x+y} \\
= \sum_{x \in S} \sum_{y \in N} c_{i,x} c_{j,x} \\
= \frac{n}{2} \sum_{x \in S} c_{i,x} c_{j,x} \\
= \begin{cases} 
\frac{n}{2} & \text{if } i - j \in N, \\
0 & \text{otherwise.} 
\end{cases}

Claim 5. By the assumption that $A$ and $B$ are incidence matrices of $D_1$ and $D_2$, respectively,

$$
\text{(the (i, j) entry of } AA^T + BB^T) = \begin{cases} 
\frac{n(n-2)}{2} & \text{if } i = j, \\
\frac{n(n-4)}{4} & \text{if } i - j \in N \text{ and } i \neq j, \\
\frac{n(n-3)}{4} & \text{otherwise.} 
\end{cases}
$$

Furthermore, by the definition of $A'$ and $B'$,

$$
\text{(the (i, j) entry of } A'A'^T + B'B'^T) = \begin{cases} 
n(n-1) & \text{if } i = j, \\
n & \text{if } i - j \in N \text{ and } i \neq j, \\
0 & \text{otherwise.} 
\end{cases}
$$

Claim 6. Write $A' = (a'_{i,j})$, where $i, j \in G$. Note that for $x \in S$

$$
\sum_{y \in N} a'_{i,x+y} = \begin{cases} 
-n/2 & \text{if } x - i \in N, \\
0 & \text{if } x - i \notin N, 
\end{cases}
$$

by the assumption (ii).

(The (i, j) entry of $A'C^T$) = \sum_{k \in G} a'_{i,k} c_{j,k} \\
= \sum_{x \in S} c_{j,x} \sum_{y \in N} a'_{i,x+y} \\
= -\frac{nc_{j,i}}{2} = \begin{cases} 
-n/2 & \text{if } i + j \in N, \\
0 & \text{otherwise.} 
\end{cases}
Claim 7. Write $B' = (b'_{i,j})$, where $i, j \in G$. Note that for $x \in S$

$$\sum_{y \in N} b'_{i,x+y} = \begin{cases} -n/2 & \text{if } x + i \in N, \\ 0 & \text{if } x + i \notin N, \end{cases}$$

by the assumption (ii). Then,

\[(\text{the } (i,j) \text{ entry of } B'C^T) = \sum_{k \in G} b'_{i,k}c_{j,k} = \sum_{x \in S} h^{(1)}_{x,j} \sum_{y \in N} b'_{i,x+y} = \sum_{x \in S} h^{(2)}_{x,j} \sum_{y \in N} b'_{i,x+y} = -nc_{j-i}/2 = \begin{cases} -n/2 & \text{if } i - j \in N, \\ 0 & \text{otherwise.} \end{cases}\]

Similarly, we have

\[(\text{the } (i,j) \text{ entry of } CB^{T}) = \begin{cases} -n/2 & \text{if } i - j \in N, \\ 0 & \text{otherwise.} \end{cases}\]

Claim 8. (The $(i,j)$ entry of $B'H_1 + A'H_2$)

\[= \sum_{k \in G} b'_{i,k}h^{(1)}_{k,j} + \sum_{k \in G} a'_{i,k}h^{(2)}_{k,j} = \sum_{x \in S} h^{(1)}_{x,j} \sum_{y \in N} b'_{i,x+y} + \sum_{x \in S} h^{(2)}_{x,j} \sum_{y \in N} a'_{i,x+y} = -nh^{(1)}_{i,j}/2 - nh^{(2)}_{i,j}/2 = 0.\]

Claim 9. Note that for $x \in S$

$$\sum_{y \in N} c_{i,x+y} = \begin{cases} n/2 & \text{if } x + i \in N, \\ 0 & \text{if } x + i \notin N. \end{cases}$$

Then,

\[(\text{the } (i,j) \text{ entry of } A'H_1 - B'H_2 - 2CH_2) = \sum_{k \in G} a'_{i,k}h^{(1)}_{k,j} - \sum_{k \in G} b'_{i,k}h^{(2)}_{k,j} - 2 \sum_{k \in G} c_{i,k}h^{(2)}_{k,j} = \sum_{x \in S} h^{(1)}_{x,j} \sum_{y \in N} a'_{i,x+y} - \sum_{x \in S} h^{(2)}_{x,j} \sum_{y \in N} b'_{i,x+y} - 2 \sum_{x \in S} h^{(2)}_{x,j} \sum_{y \in N} c_{i,x+y} = -nh^{(1)}_{i,j}/2 + nh^{(2)}_{i,j}/2 - nh^{(2)}_{i,j}/2 = 0.\]

Claim 10. It holds that

$$A'B'^T = B'A'^T$$

since

\[(\text{the } (i,j) \text{ entry of } A'B'^T) = \sum_{k \in G} (2f_{D_1}(i-k) - 1)(2f_{D_2}(j+k) - 1) = \sum_{h \in G} (2f_{D_1}(j-h) - 1)(2f_{D_2}(i+h) - 1) = \text{(the } (i,j) \text{ entry of } B'A'^T),\]
where we put \( h = j - i + k \).

Now, we prove \( MM^T = n^2 I \) using the claims above. From Claim 2, we have

\[
\text{(the } (i, j) \text{ entry of } nJ_n + H_1^T H_1 + H_2^T H_2) = n + \begin{cases} n^2 - n & \text{if } i = j, \\ -n & \text{otherwise}, \end{cases} = \begin{cases} n^2 & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases}
\]

From Claims 1 and 5, we have

\[
\text{(the } (i, j) \text{ entry of } H_1^T H_1 + A'A'^T + B'B'^T) = \begin{cases} n & \text{if } i - j \in N, \\ 0 & \text{otherwise}, \end{cases} + \begin{cases} n(n-1) & \text{if } i = j, \\ -n & \text{if } i - j \in N, i \neq j, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
= \begin{cases} n & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases}
\]

From Claims 1, 4, 5 and 7, we have

\[
\text{(the } (i, j) \text{ entry of } H_2^T H_2 + A'A'^T + B'B'^T + 4CC'^T + 2B'C'^T + 2CB'^T) = \begin{cases} n & \text{if } i - j \in N, \\ 0 & \text{otherwise}, \end{cases} + \begin{cases} n(n-1) & \text{if } i = j, \\ -n & \text{if } i - j \in N, i \neq j, \\ 0 & \text{otherwise}, \end{cases} + \begin{cases} 2n & \text{if } i - j \in N, \\ 0 & \text{otherwise}, \end{cases} + \begin{cases} 2n & \text{if } i - j \in N, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
= \begin{cases} n^2 & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases}
\]

We obtain

\[
-J_1 H_1^T + H_1^T B'^T + H_2^T A'^T = 0, -H_2 J_n + A'H_1 - B'H_2 - 2CH_2 = O \text{ and } H_2 H_1^T + A'B'^T - B'A'^T - 2CA'^T = O \text{ from Claims 8, 9, 10, and } H_1 J_n = O. \text{ Finally, } M \text{ is symmetric by the definitions of } A, B, C \text{ and the assumption (i). Thus we obtain the assertion.}
\]

\[\square\]

**Remark 4.2.**

(1) The known construction theorem by Goethals-Seidel (cf. [6] or [18, Theorem 5.15]) provides the same orders of Hadamard matrices obtained from Theorem 4.1 and requires the existence of an Hadamard matrix only. Though Theorem 4.1 requires further more condition, that is the existence of DDFs, we think that Theorem 4.1 may produces Hadamard matrices nonisomorphic to those by Goethals-Seidel’s theorem according to the assumed DDF.

(2) Note that Yamamoto-Yamada’s Hadamard difference set \( D = \{ a(1 + 2b) | a \in \mathbb{T}^*_n, b \in E \} \) immediately yields a symmetric Hadamard matrix of order \( 2^{2n} \). On the other hand, since the divisible difference family obtained from Theorem 4.1 satisfies the conditions of Theorem 4.1, we have a symmetric Hadamard matrix of order \( 2^{2n} \). One can see that these two symmetric Hadamard matrices are equivalent. A similar thing had happened also for the Szekeres difference families: if we apply Proposition 1.3 to the Szekeres difference family, we obtain a skew Hadamard matrix of order \( q + 1 \), which is equivalent to that obtained from the cyclotomic difference set of index 2 in \( \mathbb{F}_q \) as described in Introduction. However, in [13, 21], the authors succeeded to construct difference families satisfying the conditions of Proposition 1.3 different from the Szekeres difference families. Then, they obtained new skew Hadamard matrices by Proposition 1.3. Hence, we expect that we have divisible difference families different from those of Theorem 4.1 to obtain new symmetric Hadamard matrices. Thus, we have the following natural problem.
Problem 4.3. Construct divisible difference families satisfying the conditions of Theorem 4.1 different from divisible difference families given in Theorem 3.1.

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