Cutting plane methods can be extended into nonconvex optimization

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Abstract

We show that it is possible to obtain an $O(\epsilon^{-4/3})$ expected runtime — including computational cost — for finding $\epsilon$-stationary points of smooth nonconvex functions using cutting plane methods. This improves on the best known epsilon dependence achieved by cubic regularized Newton of $O(\epsilon^{-3/2})$ as proved by Nesterov and Polyak (2006). Our techniques utilize the convex until proven guilty principle proposed by Carmon, Duchi, Hinder, and Sidford (2017).

1 Introduction

This paper focuses on finding an $\epsilon$-stationary point $x$ of the function $f : \mathbb{R}^d \to \mathbb{R}$ starting from some point $x^{(0)}$, i.e.,

$$\|\nabla f(x)\| \leq \epsilon$$

under the assumptions that $f(x^{(0)}) - \inf_z f(z)$ is bounded below and the function has Lipschitz first and third derivatives. It is well-known that gradient descent achieves an $\epsilon^{-2}$ runtime when the first derivatives are Lipschitz. This was improved to $\epsilon^{-3/2}$ by Nesterov and Polyak [18] using cubic regularized Newton when the second derivatives are Lipschitz. However, each iteration of cubic regularized Newton is more expensive — it requires Hessian evaluations and solving a linear system. This observation has inspired a line of work developing dimension-free gradient based methods that improve on the worst-case runtime of gradient descent [1, 5, 6, 13, 22]. Dimension-free methods have iteration counts that do not depend on the dimension, only on measures of function regularity i.e., Lipschitz constants. As Carmon et al. [7, 8] showed there are fundamental dimension-free lower bounds for this problem. These lower bounds are dependent on the choice of Lipschitz assumptions for the function and the whether the algorithm evaluates the gradient or the Hessian.

This paper, rather than considering the high-dimensional, low accuracy regime where dimension-free gradient methods are preferred, focuses on the regime where the
dimension is low but we want to obtain high accuracy. In this case it might be acceptable to have iteration costs that scale polynomially with the dimension if that enables an algorithm with significantly less iterations. Our main result, given in Theorem 1, is an algorithm that takes
\[
\tilde{O}((T_1 + d^\omega)d + T_2)\epsilon^{-4/3}
\]
time to find an \(\epsilon\)-stationary point, where \(T_p\) which refers to the cost of one evaluating the function and its first \(p\) derivatives and \(O(d^\omega)\) denotes the runtime for a linear system solve. For simplicity this runtime (and all other runtimes in the introduction) exclude Lipschitz constants, log factors and dependence on the gap \(f(x^{(0)}) - \inf_z f(z)\) where \(x^{(0)}\) is the starting point of the algorithm. See Table 1 for a comparison of our results with known results.

| Lipschitz | method                          | runtime                                                                 | dimension-free lower bound [7, 8] |
|-----------|--------------------------------|-------------------------------------------------------------------------|-----------------------------------|
| \(\nabla f\) | gradient descent               | \(T_1\epsilon^{-2}\)                                                   | \(T_p\epsilon^{-(p+1)/p}\)       |
| \(\nabla f, \nabla^2 f\) | Carmon et al. [6]              | \(T_1\epsilon^{-7/4}\)                                                | \(T_1\epsilon^{-7/4}\)          |
| \(\nabla f, \nabla^3 f\) | Carmon et al. [6]              | \(T_1\epsilon^{-5/3}\)                                                | \(T_1\epsilon^{-5/3}\)          |
| \(\nabla^2 f\) | cubic reg. Nesterov and Polyak [18] | \((T_2 + d^\omega)\epsilon^{-3/2}\)                                   | \(T_2\epsilon^{-3/2}\)          |
| \(\nabla^p f\) | \(p\)th reg. Birgin et al. [4] | \((T_p + ?)\epsilon^{-p+1}/p\)                                       | \(T_p\epsilon^{-(p+1)/p}\)       |
| \(\nabla f, \nabla^3 f\) | This paper. Thm 1               | \(((T_1 + d^\omega)d + T_2)\epsilon^{-4/3}\)                          | \(T_3\epsilon^{-4/3}\)          |
| \(\nabla f, \nabla^3 f\) | This paper. Thm 2               | \((T_3 + d^4)\epsilon^{-4/3}\)                                        |                                   |

Table 1: Comparison of the runtime of different algorithms for finding stationary points of nonconvex functions. The question mark is a placeholder for the time to solve a \(p\)th order regularization problem.

To prove our results we utilizes ideas from Carmon et al. [6], specifically the ‘convex until proven guilty principle’. This is the idea that if one runs an algorithm designed for convex optimization on a nonconvex function, either:

- It will succeed in quickly finding a stationary point.
- It will fail to quickly find a stationary point. In this case a certificate of nonconvexity can be obtained. This certificate of nonconvexity can be exploited to make the algorithm run quickly.

This principle means that by relatively simple modification an algorithm for convex optimization can often be adapted to nonconvex optimization. In [6] the convex algorithm was accelerated gradient descent; here we study cutting plane methods.

There is a rich literature on cutting plane methods for convex optimization both theoretical [2, 14, 15, 23, 24] and empirical [3, 12]. To understand when it makes sense to use a cutting plane method, suppose we wish to solve
\[
\min_{x \in \mathbb{R}^d} f(x)
\]
where \( f \) is smooth, convex and the distance to optimality is bounded. To guarantee a fast runtime under these conditions we have two options: (i) we could use accelerated gradient descent or (ii) a cutting plane method. Accelerated gradient descent has an \( O(T_1/\epsilon^{1/2}) \) runtime [17]; the best known cutting plane method has an \( O(T_1 \log(1/\epsilon) + d^3 \log^O(1)(d)) \) runtime [14]. Therefore, if the dimension is relative low and high accuracy is desired a cutting plane method is recommended. On the other hand, if the dimension is high and low accuracy is desired accelerated gradient descent is recommended. Qualitatively, our results have a similar flavor: our cutting plane method is better than its dimension-free gradient based counterparts [1, 5, 6, 13, 22] when the dimension is small and high accuracy is desired.

Outline Section 1.1 describes the notation used in this paper. Section 1.2 explains why our results improve on pth order regularization. Section 2 reviews cutting plane methods and explains why they cannot be directly applied to nonconvex problems. Section 3 explains how to take failures of the cutting plane algorithm and use them to obtain a certificate of nonconvexity. Section 4 explains how to exploit this certificates of nonconvexity to reduce the function value. Section 5 combines the components from Sections 2-4 to obtain our results. Section 6 discusses possible applications for our method.

1.1 Notation

Let \( d \) be the dimension of the problem, \( \mathbb{R} \) the set of real numbers, \( \| \cdot \| \) denote the euclidean norm, \( B_R(v) := \{ x \in \mathbb{R}^d : \| x - v \| \leq R \} \), \( \lambda_{\min}(\cdot) \) the minimum eigenvalue of a matrix. Unless otherwise specified \( \log(\cdot) \) is base \( e \) where \( e \) is the exponential constant. Let \( \log^+(\theta) := \max\{1, \log(\theta)\} \). The value \( O(d^\omega) \) denotes the runtime for solving a linear system or computing an SVD with \( \omega \in [2, 3] \) being the fast matrix multiplication constant [11]. We say that a function \( f : \mathbb{R}^d \to \mathbb{R} \) has \( L_p \)-Lipschitz derivatives on the set \( Q \subseteq \mathbb{R}^d \) if

\[
\left| q^{(p)}(0) - q^{(p)}(\theta) \right| \leq L_p |\theta|
\]

for any \( \theta \in \mathbb{R} \), \( x \in Q \), \( s \in B_1(0) \), and \( q : \mathbb{R} \to \mathbb{R} \) with \( q(\theta) := f(x + s\theta) \) and \( x + s\theta \in Q \). It is well-known that this implies

\[
\left| q(0) + \theta q^{(1)}(0) + \cdots + \frac{\theta^p}{p!} q^{(p)}(0) - q(\theta) \right| \leq \frac{L_p}{(p + 1)!} |\theta|^{p+1}
\]

and

\[
\left| q^{(1)}(0) + \cdots + \frac{\theta^p}{(p - 1)!} q^{(p)}(0) - q(\theta) \right| \leq \frac{L_p}{p!} |\theta|^p.
\]

Let \( T_p \) refer to the cost of evaluating the function and its first \( p \) derivatives once. This includes the cost of adding two \( p \)th order tensors or multiplying them by a scalar. We assume \( T_p = \Omega(d) \). For simplicity one can think of \( T_p = \Theta(d^p) \). However, if the derivatives are difficult to evaluate then it might be the case that \( T_p \gg d^p \). Alternatively, if the tensors associated with the \( p \)th derivatives are dense then \( T_p \)
The time for a cutting plane center computation is $T_C$ (see assumption [1]). Given a set $S \subseteq \mathbb{R}^d$, $\text{vol}(S) := \int_S dx$ denotes the volume of that set. The term $\text{Geo}(p)$ denotes the geometric distribution with success probability $p \in [0, 1]$. 

1.2 Review of $p$th order regularization

Since our algorithm is closely related to $p$th order regularization [4] with $p = 3$ we feel it is useful to further discuss this method. In particular, our goal is to explain why this method does not include computation cost in its runtime. This is contrast to our method that does include computational cost.

First let us derive $p$th order regularization. Consider a $p$th order taylor series expansion of a differentiable function $f$ at the point $\bar{x}$:

$$f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T \nabla^2 f(\bar{x})(x - \bar{x}) + \ldots$$

Adding a regularization term, we obtain

$$\tilde{f}_p(\bar{x}; x) := f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x}) + \cdots + \frac{2L_p}{(p+1)!} \| \bar{x} - x \|^p,$$

where $L_p$ is the Lipschitz constant of the $p$th order derivatives. The function $\tilde{f}_p$ is an upper bound on $f$, i.e., $\tilde{f}_p(\bar{x}; x) \geq f(x)$. We define $p$th order regularization method as any sequence $x^{(0)}, \ldots, x^{(k)}$ that satisfies

$$\| \nabla \tilde{f}_p(x^{(k)}; x^{(k+1)}) \| \leq \epsilon/2, \quad \tilde{f}_p(x^{(k)}; x^{(k+1)}) \leq f(x^{(k)}).$$

To meet these conditions it is sufficient to set

$$x^{(k+1)} \leftarrow \arg\min_x \tilde{f}_p(x^{(k)}; x).$$

This method requires

$$O \left( T_p \Delta L_p^{1/p} \epsilon^{-\frac{p+1}{p}} \right)$$

iterations to find stationary points [4], with $\Delta = f(x^{(0)}) - \inf_z f(z)$. For $p = 1$ and $p = 2$ this corresponds gradient descent and cubic regularization respectively. Increasing $p$ improves the $\epsilon$ dependence. However, this improvement in the $\epsilon$ dependence is only with respect to the evaluation complexity — the number of times that we compute the $1, \ldots, p$ derivatives. It excludes the cost of finding a solution to (2). Finding a point satisfying (2) is trivial for gradient descent and well-known for cubic regularization Nesterov and Polyak [18, Section 5]. Unfortunately, for $p \geq 3$ the only available methods have $\epsilon$ dependencies, i.e., cubic regularization, gradient descent, etc. Therefore prior to our work, no method actually improved on the $\epsilon$ dependence of cubic regularization — if one includes computation cost not just evaluation complexity.
2 Cutting plane methods

Cutting plane methods encompass a variety of different algorithms which can be all written in the generic framework given by Algorithm 1. They work by maintaining a region $S^{(t-1)}$ that the optimum is contained in. At each iteration the cutting plane picks a 'center' point $x^{(t)}$ of the region $S^{(t-1)}$. At this point a cut is generated which further reduces the volume of the region. The main difference between different cutting plane methods is how they pick the center point. For example, center of gravity [15] picks the point

$$\frac{\int_S x \, dx}{\int_S dx}$$

but this is different from the volumetric [24] or analytic center [2]. The cost of each center computation varies by method. For example, computing the center of gravity is prohibitively expensive. However, some methods require less expensive centre computations. We make assumption 1 to ensure that our method can generically handle any cutting plane method. The term $1 - \tau$ represents the minimum reduction factor in the volume of $S^{(t)}$ at each iteration. For example, for the center of gravity $\tau = e^{-1}$ and for the Ellipsoid Method $\tau = 1 - e^{-d/2}$ [23]. We remark that $\tau \leq 1/2$ for any possible method [16].

**Algorithm 1: CuttingPlaneMethod**

**Data:** $\hat{f}, x^{(0)}, N, R$

**Result:** $S^{(N)}, x^{(0)}, \ldots, x^{(N)}$

$$S^{(0)} \leftarrow B_R(x^{(0)}) \cap \{x \in \mathbb{R}^d : \nabla \hat{f}(x^{(0)})^T(x - x^{(0)}) \leq 0\};$$

for $t = 1, \ldots, N$

$$x^{(t)} \leftarrow \text{Center}(S^{(t-1)});$$

$$S^{(t)} \leftarrow S^{(t-1)} \cap \{x \in \mathbb{R}^d : \nabla \hat{f}(x^{(t)})^T(x - x^{(t)}) \leq 0\}$$

end

return $S^{(N)}, x^{(0)}, \ldots, x^{(N)}$

An astute reader might notice that Algorithm 1 uses $\hat{f}$ instead of $f$. This is because to obtain our results in Section 5 we need to slightly modify the original function $f$ by adding a proximal term.

**Assumption 1.** There exists some $\tau \in (0, 1/2]$ such that for all $R \in (0, \infty)$ and positive integers $N$, Algorithm 1 satisfies

$$\text{vol} \left( S^{(N)} \right) \leq (1 - \tau)^N \times \text{vol} \left( B_R(0) \right).$$

Furthermore, the time for calling the routine Center is $T_C$.

From Assumption 1 we immediately derive Lemma 1. Lemma 1 is a standard result but we include it for exposition. We use assumption 1 to ensure our results are generic. In Section 5 we substitute explicit values for $T_C$.

**Lemma 1.** Let $\hat{f} : \mathbb{R}^d \rightarrow \mathbb{R}$ be differentiable, $N$ be a positive integer, $x^{(0)} \in \mathbb{R}^d$, and $r, R \in (0, \infty)$. Consider Algorithm 1. Suppose Assumption 1 holds. If $N \geq \frac{d}{\tau} \log(R/r)$ then

$$\text{vol} \left( S^{(N)} \right) \leq \frac{1}{2} \text{vol} \left( B_R(0) \right).$$
Proof. Using Assumption \[ \frac{\text{vol}(B_r(0))}{\text{vol}(B_R(0))} = (r/R)^d, \] and \((1 - \tau)^{1/\tau} \leq e^{-1}\) we obtain

\[
\text{vol}\left(S^{(N)}\right) \leq \frac{1}{2}(1 - \tau)^N \text{vol}\left(B_R(0)\right) = \frac{1}{2}(1 - \tau)^N (R/r)^d \text{vol}\left(B_r(0)\right) \\
\leq \frac{1}{2}e^{-d\log(R/r)}(R/r)^d \text{vol}\left(B_r(0)\right) = \frac{1}{2} \text{vol}\left(B_r(0)\right).
\]

Notice that so far we have not used convexity! So why is it a non-trivial task to adapt a cutting plane method to a nonconvex function? Even though by Lemma 1 we can guarantee that \(\text{vol}\left(S^{(N)}\right)\) is small we cannot guarantee that it contains a stationary point. To understand this failure we use Figure 1. In Figure 1 a cutting plane method is applied to the function \((x_1^2 - 1)^2 + (x_2^2 - 1)^2\) with center points \(x^{(i)}\) picked arbitrarily. After three cuts the method has restricted its search to the set \(S^{(2)}\) which does not contain any stationary point! In convex optimization this could not happen — by convexity the intersection of our cutting planes will always contain the optimum.

![Figure 1](image-url)

Figure 1: Failure of cutting plane methods on the function \((x_1^2 - 1)^2 + (x_2^2 - 1)^2\).

To enable the usage of cutting plane methods in nonconvex optimization, the first step is to efficiently detect these failures. This is the subject of Section 3.

3 Detecting nonconvexity

Suppose that we have run our cutting plane algorithm and we have a small set \(S^{(N)}\) which we believe contains a stationary point. How can we check if it contains a stationary point? Furthermore, if it does not contain a stationary point can we produce
a certificate of nonconvexity? This is the purpose of Algorithm 2. This section is to analogous to Section 2.1 of Carmon et al. [6] in the sense we aim to find a certificate of nonconvexity. Our goal is to obtain a certificate when a cutting plane method fails. In contrast, Carmon et al. [6] find a certificate when accelerated gradient descent stalls.

**Data:** \( \hat{f}, S(N), x^{(0)}, \ldots, x^{(N)}, \hat{L}_1, \hat{\epsilon}, R \)

**Result:** \( u, v, K \)

\[
x_{\text{best}} \leftarrow \arg\min_{x \in \{x^{(0)}, \ldots, x^{(N)}\}} \hat{f}(x);
\]

**if** \( \| \nabla \hat{f}(x_{\text{best}}) \| \leq \hat{\epsilon} \) **then**

**return** \( x_{\text{best}}, \emptyset, 0 \)

**end**

\( y \leftarrow x_{\text{best}} - \frac{1}{\hat{L}_1} \nabla \hat{f}(x_{\text{best}}) \)

\( r \leftarrow \hat{\epsilon} / (8\hat{L}_1) \)

**for** \( k = 1, \ldots, \infty \) **do**

\( u \leftarrow \) uniformly random point from \( B_r(y) \);

**if** \( u \notin S(N) \) **then**

\( K \leftarrow k \)

**break** /* Lemma 2 proves \( \hat{f}(u) \leq \hat{f}(x_{\text{best}}) \) implying nonconvexity */

**end**

**if** \( \| u - x^{(0)} \| \leq R \) **then**

/* Find a certificate of this nonconvexity */

**for** \( t \in \{0, \ldots, T\} \) **do**

**if** \( \hat{f}(u) < \hat{f}(x^{(t)}) + \nabla \hat{f}(x^{(t)})^T (u - x^{(t)}) \) **then**

**return** \( u, x^{(t)}, K \)

**end**

**end**

**else**

**return** \( u, \emptyset, K \)

**end**

**Algorithm 2: NonconvexityCertificate**

Algorithm 2 is combined with Algorithm 1 in the following process:

\[
S^{(N)}, x^{(0)}, \ldots, x^{(N)} \leftarrow \text{CuttingPlaneMethod}(\hat{f}, x^{(0)}, N, R) \quad (3a)
\]

\[
u, v, K \leftarrow \text{NonconvexityCertificate}(\hat{f}, S^{(N)}, x^{(0)}, \ldots, x^{(N)}, \hat{L}_1, \hat{\epsilon}, R) \quad (3b)
\]

Lemma 2 summarizes possible outcomes of (3). Figure 2 gives an example of Algorithm 2 detecting nonconvexity. In this example \( N = 1 \) and the set \( S^{(1)} = \{ x \in B_R(x^{(0)}): \nabla \hat{f}(x^{(0)})^T (x - x^{(0)}) \leq 0, \nabla \hat{f}(x^{(1)})^T (x - x^{(1)}) \} \). The point \( x^{(t)} \) with the smallest function value is \( x^{(0)} = x_{\text{best}} \) and we take a gradient step from there to \( y \). After sampling from \( B_R(y) \) we are at the point \( u \) (this is the only randomization used in this paper). At this point we are no longer in the set \( S^{(1)} \) but \( \| u - x^{(0)} \| \leq R \). Therefore we must have violated some hyperplane that makes up the set \( S^{(1)} \). It turns
out this hyperplane corresponds to $x^{(1)}$. Therefore we set $v = x^{(1)}$ and return $u, v$
from Algorithm[2]

Randomly sampling from the set $B_r(y)$ allow us to find a point in the nonempty
set $B_r(y) \setminus S^{(N)}$. Each time we sample the probability the point $u$ is in the set $Z := B_r(y) \setminus S^{(N)}$ is equal to $\text{vol}(Z)/\text{vol}(B_r(y))$, i.e., it is a biased coin toss.

**Lemma 2.** Suppose assumption \[ holds. Let $x^{(0)} \in \mathbb{R}^d$, $N$ be a positive integer greater
than $\frac{d}{\tau} \log \left( \frac{8L_1 R}{\epsilon} \right)$, and $\hat{L}_1, R, \hat{\epsilon} \in (0, \infty)$. Assume $\hat{f} : \mathbb{R}^d \to \mathbb{R}$ has $\hat{L}_1$-Lipschitz
derivatives on the set $Q \subseteq \mathbb{R}^d$. Let \[ hold.

Then $\hat{f}(u) \leq \hat{f}(x_{\text{best}})$ where $x_{\text{best}} = \arg\min_{x \in \{x^{(0)}, \ldots, x^{(N)}\}} \hat{f}(x)$, and $K \sim \text{Geo}(p)$
with probability of success $p \geq 1/2$. Furthermore, one of the following cases applies,

(i) $v = 0, \|\nabla \hat{f}(u)\| \leq \hat{\epsilon}$

(ii) $v = 0, \|u - x^{(0)}\| > R$

(iii) $u$ and $v = x^{(k)}$ certify nonconvexity of $y$, i.e.,

$$\hat{f}(u) < \hat{f}(v) + \nabla \hat{f}(v)^T (u - v). \quad (4)$$

**Proof.** First we show $\hat{f}(u) \leq \hat{f}(x_{\text{best}})$. If $u = x_{\text{best}}$ this occurs trivially by definition of $x_{\text{best}}$. Now,

$$\hat{f}(u) \leq \hat{f}(y) + \nabla \hat{f}(y)^T (u - y) + \frac{\hat{L}_1 \|u - y\|^2}{2}$$

$$\leq \hat{f}(x_{\text{best}}) - \frac{\|\nabla \hat{f}(x_{\text{best}})\|^2}{2 \hat{L}_1} + \frac{\hat{L}_1 \|u - y\|^2}{2}$$

$$\leq \hat{f}(x_{\text{best}}) - \frac{\|\nabla \hat{f}(x_{\text{best}})\|^2}{2 \hat{L}_1} + \frac{\|\nabla \hat{f}(y)\|^2}{8 \hat{L}_1} + \frac{\hat{\epsilon}^2}{128 \hat{L}_1} \leq \hat{f}(x_{\text{best}})$$

where the first two transitions use the inequality $\hat{f}(x') \leq \hat{f}(x) + \nabla \hat{f}(x)^T (x' - x) + \frac{\hat{L}_1 \|x' - x\|^2}{2}$, the third uses $\|u - y\| \leq r \leq \hat{\epsilon}/(8 \hat{L}_1)$, the fourth uses $\|\nabla \hat{f}(y)\| \leq \frac{\|\nabla \hat{f}(x_{\text{best}})\| + \|\nabla \hat{f}(y)\|}{2}$, and the fifth $\|\nabla \hat{f}(x_{\text{best}})\| \geq \hat{\epsilon}$. This proves $\hat{f}(u) \leq \hat{f}(x_{\text{best}})$.

Let us show $K \sim \text{Geo}(p)$. Each event $u \notin S^{(N)}$ occurs independently for each
$k = 1, \ldots, \infty$. Therefore $K \sim \text{Geo}(p)$ with $p \geq 1/2$. We remark that implicitly all
results in this paper hold almost surely.

Let us now show that one of cases (i)-(iii) holds. If $v = 0$ then clearly one of cases (i) or (ii) holds. If $v \neq 0$ then $\|u - x^{(0)}\| \leq R$ and since $u \notin S^{(N)}$ there exists some $t$ for which $\nabla \hat{f}(x^{(t)})^T (u - x^{(t)}) > 0$. Since $\hat{f}(u) \leq \hat{f}(x^{(t)})$ we have $\hat{f}(u) < \hat{f}(x^{(t)}) + \nabla \hat{f}(x^{(t)})^T (u - x^{(t)})$.

Since $\text{vol}(S) < \frac{1}{2} \text{vol}(B_r(0))$ by Lemma[1] with $r = \hat{\epsilon}/(8 \hat{L}_1)$, for each $u$ generated by sampling from $B_r(y)$ the probability that $u \in S^{(N)}$ is at most $1/2$.
Figure 2: Diagram showing an example of Algorithm 2 finding a certificate of nonconvexity. The convexity violation occurs between $u$ and $v$ because (i) $\hat{f}(u) \leq \hat{f}(v)$ and (ii) the point $u$ is not inside the halfspace $\{x \in \mathbb{R}^d : \nabla \hat{f}(v)^T (x - v) \leq 0\}$. Consequently, one can prove \cite{4} holds.

4 Exploiting nonconvexity

Suppose that we run Algorithm 2 and find a certificate of nonconvexity. How do we use this information? This is the purpose of Algorithm 3. In particular, we construct a function $q$ along the direction $s$ of nonconvexity of the function $f$ and then query several points on this function. We draw on the ideas of Carmon, Duchi, Hinder, and Sidford \cite{6} to provide more efficient negative curvature exploitation when the third derivatives are Lipschitz.

\begin{algorithm}
\textbf{Data:} $f, c, s, R$
\textbf{Result:} $x$
$q(\theta) := f(c + \theta s)$;
$\theta_* \leftarrow \arg\min_{\theta \in \{\pm 12R, \pm 9R, \pm 3R\}} q(\theta)$
\textbf{return} $u + \theta_* s$
\end{algorithm}

\textbf{Algorithm 3:} ExploitNC

We need to guarantee if there is sufficient nonconvexity between $u$ and $v$ that Algorithm 3 will reduce the function value. This is the purpose of Lemma 3.

\begin{lemma}
Suppose the function $q : [-12R, 12R] \to \mathbb{R}$ has $L_3$-Lipschitz continuous third derivatives, and for some $\gamma \in [-1, 1]$ we have $q''(R\gamma) \leq -21L_3R^2$ then
\[
\min\{q(12R), q(9R), q(-9R), q(-12R)\} \leq q(0) - 536L_3R^4.
\end{lemma}
The proof of Lemma 3 is given in Section A. We remark that Lemma 3 is similar to Lemma 5 in Carmon et al. [6]. The main difference is that the progress is guaranteed with respect to the function value at the origin rather than the maximum of two function values. This is critical to our result.

Figure 3: Visual depiction of the guarantee of Lemma 3.

Figure 3 illustrates Lemma 3. In particular, given the function $q$ and some nearby point $21R\gamma$ with nonconvexity, if we query four different points (red points) we can guarantee at least $536L_3R^4$ reduction in the function value. Notice that if $f : \mathbb{R}^d \to \mathbb{R}$ has $L_3$-Lipschitz continuous third derivatives and $\|s\| = 1$ then we can immediately use Lemma 3 to analyze Algorithm 3.

5 A cutting plane algorithm for nonconvex optimization

This section combines the Algorithms from Section 2-4 to obtain our improved complexity results. First, we present Algorithm 4 that roughly solves a trust-region problem using cutting planes, i.e.,

$$\arg\min_{x \in B_R(z)} f(x).$$

Inside Algorithm 4 we add a proximal term to the function $f$, i.e. write $\hat{f}(x) := f(x) + \frac{\alpha}{2} \| z - x \|^2$. This a typical strategy in nonconvex optimization theory [1, 5, 13, 22] and ensures that when we detect nonconvexity of $\hat{f}$ this corresponds to a large violation of nonconvexity on the function $f$ which we can use to exploit to reduce the function value using Lemma 3. Following Algorithm 4 we present two different algorithms (6 and 10). These algorithms repeatedly call Algorithm 4 until finding a stationary
point. However, \((6)\) only evaluates first and second derivatives; \((10)\) evaluates the first, second and third derivatives (but makes less frequent evaluations).

**Data:** \(f, z, \epsilon, L_1, L_3, R\)

**Result:** \(z^{(+)} , K\)

\[
\alpha \leftarrow 21L_3R^2, \hat{\epsilon} \leftarrow \epsilon/2, \hat{L}_1 \leftarrow L_1 + \alpha, N \leftarrow \frac{\hat{\epsilon}}{\epsilon} \log(R/r);
\]

\[
f(x) := f(x) + \frac{\hat{\epsilon}}{\epsilon}\|z - x\|^2;
\]

\[
S^{(N)}, x^{(0)}, \ldots, x^{(N)} \leftarrow \text{CuttingPlaneMethod}(f, z, R, N);
\]

\[
u, v, K \leftarrow \text{NonconvexityCertificate}(f, S^{(N)}, x^{(0)}, \ldots, x^{(N)}, \hat{L}_1, \hat{\epsilon}, R);
\]

if \(\|\nabla f(u)\| \leq \epsilon\) then

- \(p \leftarrow \arg\min_{s: \|s\|=1} s^T\nabla^2 f(u)s\) using Singular Value Decomposition.
  - if \(p^T \nabla^2 f(u)p \geq -\alpha\) then
    - return \(u, K\)
  - else if \(v = 0\) then
    - return \(u, K\)
  - else
    - return \(\text{ExploitNC}(f, u, p, R), K\)

**Algorithm 4:** CuttingTrustRegion

Lemma 4 shows that during a call to Algorithm 4 we either find a (second-order) stationary point, as we wanted or we make a significant amount of progress in reducing the function value.

**Lemma 4.** Consider Algorithm 4. Suppose that assumption \([7]\) holds. Let \(z \in \mathbb{R}^d, R, L_1, L_3 \in (0, \infty)\). Assume that \(f : \mathbb{R}^d \to \mathbb{R}\) has \(L_1\)-Lipschitz first derivatives and \(L_3\)-Lipschitz third derivatives on the set \(B_{12R}(z)\).

If \(\|\nabla f(z^{(+)})\| \geq \epsilon\) or \(\lambda_{\min}(\nabla^2 f(z^{(+)}) ) \leq -\alpha\) then

\[
f(z^{(+)}) \leq f(z) - \min \left\{ 10L_3^3R^4, \frac{\epsilon^2}{168R^2L_3^3} \right\}. \tag{5}
\]

Furthermore, the runtime of Algorithm 4 is at most

\[
O \left( \frac{(TC + T_1 + Kd)d}{\tau} \log^+ \left( \frac{RL_1}{\epsilon} \right) + T_2 + d\omega \right).
\]

The proof of Lemma 4 is given in Section 5. It is similar to Lemma 7 of Carmon et al. [6]. Our algorithm simply consists of repeatedly calling Algorithm 4 i.e.,

\[
z^{(t+1)}, K^{(t)} \leftarrow \text{CuttingTrustRegion}(f, z^{(t)}, \epsilon, L_1, L_3, L_3^{-1/3} \epsilon^{1/3}/3). \tag{6}
\]

**Theorem 1.** Suppose that assumption \([7]\) holds. Let \(z^{(0)} \in \mathbb{R}^d, L_1, L_3 \in (0, \infty)\). Assume that \(f : \mathbb{R}^d \to \mathbb{R}\) has \(L_1\)-Lipschitz first derivatives and \(L_3\)-Lipschitz third derivatives. Let \(f(z^{(0)}) - \inf_{x \in \mathbb{R}^d} f(x) \leq \Delta\). Under these conditions, the procedure starting with \(t = 0\) will find a point

\[
\|\nabla f(z^{(m)})\| \leq \epsilon, \quad \lambda_{\min}(\nabla^2 f(z^{(m)})) \geq -L_3^{1/3} \epsilon^{2/3}. \tag{7}
\]
uses computation time bounded above by

\[
O \left( \left( \Delta L_3^{1/3} \epsilon^{-4/3} + 1 \right) \frac{(T_C + T_1 + \bar{K}d)d}{\tau} \log \left( \frac{L_3^3}{\epsilon^2 L_3} \right) + T_2 + d^e \right),
\]

where \(\bar{K}\) is a positive random variable satisfying \(P(\bar{K} \geq y) \leq e^{\frac{1}{20}y}\) for all \(y \in \mathbb{R}\).

**Proof.** Substituting \(R = L_3^{-1/3} \epsilon^{1/3}/3\) (chosen to maximize worst-case progress at each iteration) into Lemma 4 shows we reduce the function by \(L_3^{-1/3} \epsilon^{-4/3}/20\) at each iteration that we do not terminate. Using \(f(z(0)) - \inf_{x \in \mathbb{R}^d} f(x) \leq \Delta\) we deduce the number of iterations is at most \(O(\Delta L_3^{1/3} \epsilon^{-4/3} + 1)\). From the bound on the runtime of CuttingTrustRegion as given in Lemma 4 and using \(\bar{K} = \frac{1}{m} \sum_{t=1}^{m} K(t)\).

\(P(\bar{K} \geq y) \leq e^{\frac{1}{20}y}\) follows from Lemma 6 (Appendix C).

Using the Volumetric center \([24]\) as the \(\text{Center()}\) function makes \(\tau\) a dimension independent constant and \(T_C = \tilde{O}(d^e)\) yielding simplifying the expected runtime bound of Theorem 1 to

\[
\tilde{O} \left( \left( \Delta L_3^{1/3} \epsilon^{-4/3} + 1 \right) ((d^e + T_1)d) + T_2 \right) \tag{8}
\]
as we state in Table 1 where \(\tilde{O}\) omits log factors.

From carefully reading the proof of Lemma 4 one observes that if one replaces the code in Algorithm 4 inside the “if \(\|\nabla f(u)\| \leq \epsilon\)” statement with “\text{return } u” then the runtime improves to \(\tilde{O} \left( \left( \Delta L_3^{1/3} \epsilon^{-4/3} + 1 \right) ((d^e + T_1)d) \right)\). This replacement causes us to lose our second-order guarantees but we no longer evaluate the second derivatives. We also remark that our second-order guarantee given in (7) matches the second-order guarantee given by Cartis et al. \([10]\) for quartic regularization. Our runtime for achieving second-order stationarity is a straightforward consequence of the efficient negative curvature exploitation proposed in Carmon et al. \([6]\).

Recall that the runtime of \(p\)th order regularization is

\[
O \left( (T_p + ?) \Delta L_p^{1/p} \epsilon^{-\frac{p+1}{p}} \right) \tag{9}
\]

where \(\Delta\) denotes the runtime of solving a \(p\)th order regularization problem. Let us compare (8) and (9). Consider the case \(p = 2\), i.e., cubic regularization where \(\Delta\) can be replaced by \(d^e\). Note that if the Lipschitz constants and dimension are fixed and \(\epsilon\) goes to zero then our runtime bound (8) better than (9). Furthermore, consider a problem with \(d^e \leq T_1 \approx dT_2\), i.e., the Hessian is computed via finite differences and the derivatives are expensive to evaluate. In this case, if the Lipschitz constants and the dimension grows, our algorithm has the same dimension dependence as cubic regularization, but an improved \(\epsilon\) dependence. Next, consider (9) with \(p = 3\). As we stated in the introduction unlike \(p\)th order regularization our runtimes incorporate computational cost. However, there are even gains in terms of the evaluation complexity. In particular, suppose the high order derivatives are computed with finite differences of the gradients. In this case \(T_3 = \Theta(d^2T_1)\). Hence quartic regularization requires
O \left( d^3 \Delta L_3 \frac{1}{3} \epsilon^{-4/3} \right) \) gradient evaluations versus \( O \left( d^2 \Delta L_3 \frac{1}{3} \epsilon^{-4/3} \right) \) for our method — a factor of \( d \) improvement.

It is difficult to provide a direct comparison between the runtime from Theorem 1 and \( p \)th order regularization without making assumptions on the values of \( T_1, T_2 \) and \( T_3 \). Therefore to present a direct runtime comparison with Birgin et al. [4], we use (10) which avoids gradient calls by solving quartic regularization models. The ideas is just to run our algorithm on the quartic regularized subproblems as follows

\[
\hat{f}^{(t)}(x) := \tilde{f}_3(z^{(t)}),
\]

\[
z^{(t+1)} \leftarrow \text{CuttingTrustRegion}(\hat{f}^{(t)}, z^{(t)}, \epsilon, 2L_1, 2L_3, L_3^{-1/3} \epsilon^{1/3}/24).
\]

Recall the definition of \( \tilde{f}_p \) from (1) in Section 1. Theorem 2 shows that by invoking (10) we can obtain exactly the same evaluation complexity as quartic regularization while having a computationally runtime with the same \( \epsilon \)-dependence (up to log factors) as the evaluation complexity.

To simplify the analysis and final runtime bounds in Theorem 2 we assume that

\[
\epsilon \leq \frac{L_3}{L_1^{1/2}}.
\]

This ensures that we ignore uninteresting corner cases in our analysis. In particular, if (11) is violated then \( \Delta L_3^{1/3} \epsilon^{-4/3} \geq \Delta L_1 \epsilon^{-2} \). Hence the iteration bound of gradient descent will be better than the runtime bound of quartic regularization — in which case one should run gradient descent.

**Theorem 2.** Suppose that assumption 1 holds. Let \( z^{(0)} \in \mathbb{R}^d, L_1, L_3 \in (0, \infty) \) and \( \epsilon \in (0, L_1^{3/2}/L_3^{1/2}] \). Assume that \( f : \mathbb{R}^d \to \mathbb{R} \) has \( L_1 \)-Lipschitz first derivatives and \( L_3 \)-Lipschitz third derivatives. Let \( f(z^{(0)}) - \inf_{x \in \mathbb{R}^d} f(x) \leq \Delta \). Under these conditions, the procedure (10) starting with \( t = 0 \) finds a point \( z^{(m)} \) such that (7) holds with computational time upper bounded by

\[
O \left( \left( \frac{\Delta L_3^{1/3} \epsilon^{-4/3}}{L_1^{1/2} \epsilon} \right) \left( T_3 + \frac{(T_C + d^3 + \bar{K} d) d}{\tau} \right) \log^+ \left( \frac{L_3^2}{\epsilon^2 L_3} \right) \right)
\]

where \( \bar{K} \) is a positive random variable satisfying \( \mathbb{P}(\bar{K} \geq y) \leq e^{-y^{1/2}} \) for all \( y \in \mathbb{R} \).

The proof of Theorem 2 appears in Appendix D. In Theorem 2 we use the fact that in Lemma 4 we only need the function \( \hat{f} \) to be Lipschitz on \( B_{12R} \left( z^{(t)} \right) \). This allows us to get around the issue that the regularization term \( \frac{2L_p}{(p+1)!} \|z^{(t)} - x\|_p \) has Lipschitz first derivatives on \( B_{12R} \left( z^{(t)} \right) \) but not on \( \mathbb{R}^d \).

Finally, we remark that both Theorem 1 and 2 provide stochastic bounds on the runtime. However, the uncertainty in our runtime bound only occurs in the computational complexity since the random variable \( \bar{K} \) is not multiplied by any \( T_p \) term. Therefore the bound on the number of evaluations of the function and its derivatives is deterministic (given the algorithm terminates which occurs almost surely). Let us contrast the
stochastic nature of our results with literature. The literature contains deterministic results. For example, Birgin et al. [4] has the same $\epsilon$-dependence as our work but only bounds the evaluation complexity, and Carmon et al. [6] has a worse $\epsilon$-dependence but better dependence on the problem dimension. Some literature is stochastic. The work of [1, 5] provides algorithms that, with high probability, find a second-order stationary point. The runtime is deterministic but there is a small probability that they fail to find a second-order stationary point. Our results can also be restated in a similar manner by the following simple modification to our algorithms. Fix some $\delta \in (0, 1)$ and stop our algorithm when the computation time exceeds the upper bound proved in our Theorems by a factor of $1 + 10\log(1/\delta)$. With this modification our algorithms fail with probability at most $\delta$ and the runtime bounds hold almost surely.

6 Discussion

Cutting plane methods for convex optimization have had practical success solving problems poorly conditioned problems of mild dimension. The classic example is the traveling salesperson problem. The linear program solved during branch and bound process can be solved with millions of variables [19]. Another application of cutting plane methods in convex optimization is to two-stage stochastic programs. These problems are decomposed into a smaller but poorly conditioned master problem solved using a cutting plane method [3]. Large-scale nonconvex stochastic programs arise in optimal AC power flow [20]. For reasons similar to why cutting plane methods have been successful in convex optimization, this problem offers an opportunity for the application of cutting plane methods. However, to develop a practical cutting plane method would require overcoming many hurdles not addressed in this theoretical paper. These hurdles include handling constraints and the fact that Lipschitz constants are unknown [9].

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To simplify the argument we first prove Lemma 5, a dimension-free variant of Lemma 3. The main idea of Lemma 5 is that since the function \( h \) has Lipschitz third derivatives we can approximate the function using cubic interpolation. Then using the existence of nonconvexity and the asymmetry of a cubic function we deduce the result.

**Lemma 5.** Suppose the function \( h : [-12, 12] \rightarrow \mathbb{R} \) has 1-Lipschitz continuous third derivatives and \( h'''(\gamma) \leq -21 \) for some \( \gamma \in [-1, 1] \) then

\[
\min\{h(12), h(9), h(-9), h(-12)\} \leq h(0) - 536.
\]

**Proof.** Let \( \tilde{h}(\theta) := h(0) + h'(0)\theta + \frac{h''(0)}{2}\theta^2 + \frac{h'''(0)}{6}\theta^3 \) by Lipschitz continuity we have

\[
\left| h''(\theta) - h''(\theta) \right| \leq \frac{\theta^2}{2} \quad \left| \tilde{h}(\theta) - h(\theta) \right| \leq \frac{\theta^4}{24}.
\]

To ensure there exists \( \gamma \in [-1, 1] \) with \( h'''(\gamma) \leq -21 \) we need \( h'''(0)\theta + h''(0) = \tilde{h}''(\theta) \leq -20 \) for \( \theta = 1 \) or \( \theta = -1 \). Consider the case \( -h'''(0) + h''(0) \leq -20 \), using this inequality and \( \left| \tilde{h}(\theta) - h(\theta) \right| \leq \frac{\theta^4}{24} \) we obtain

\[
h(\theta) - h(0) \leq h'(0)\theta + \frac{h''(0)}{2}\theta^2 + \frac{h'''(0)}{6}\theta^3 + \frac{\theta^4}{24} \leq h'(0)\theta + \frac{h''(0) + \theta h'''(0)/3 + \theta^4/24}{2}.
\]
substituting in $\theta = -12$ and $\theta = 9$ we obtain the following two inequalities
\[
\begin{align*}
    h(-12) - h(0) &\leq -12(h'(0) + 18h'''(0)) - 12^2 \times 10 + 12^4 / 24 \\
    h(9) - h(0) &\leq 9(h'(0) + 18h'''(0)) - 9^2 \times 10 + 9^4 / 24.
\end{align*}
\]
Hence
\[
\min\{h(-12), h(9)\} \leq h(0) - (h'(0) + 18h'''(0)) \min\{-12, 9\} - 536 \leq h(0) - 536.
\]
Finally, the case $h'''(0) + h''(0) \leq -20$ follows by symmetry. In particular, defining $\hat{h}(\theta) := h(-\theta)$ and observing that $-\hat{h}'''(0) = h'''(0)$ and $\hat{h}''(0) = h''(0)$ implies that the argument we just made (replacing $h$ with $\hat{h}$) shows
\[
\min\{h(12), h(-9)\} = \min\{\hat{h}(-12), \hat{h}(9)\} \leq h(0) - 536.
\]

\begin{proof}
\end{proof}

\begin{lemma}
Suppose the function $q : [-12R, 12R] \to \mathbb{R}$ has $L_3$-Lipschitz continuous third derivatives, and for some $\gamma \in [-1, 1]$ we have $q'''(R\gamma) \leq -21L_3R^2$ then
\[
\min\{q(12R), q(9R), q(-9R), q(-12R)\} \leq q(0) - 536L_3R^4.
\]
\end{lemma}

\begin{proof}
Define $h(\theta) := \frac{1}{L_3R^3}q(\theta R)$. Note that the function $h$ has 1-Lipschitz third derivatives. Furthermore, since $q''(\gamma) \leq -21L_3R^2$ it follows that $h'''(\gamma) = \frac{1}{L_3R^3}q'''(R\gamma) \leq -21$. We conclude all the conditions of Lemma 3 are met.
\end{proof}

\section{Proof of Lemma 4}

\begin{lemma}
Consider Algorithm 4. Suppose that assumption 1 holds. Let $z \in \mathbb{R}^d$, $R, L_1, L_3 \in (0, \infty)$. Assume that $f : \mathbb{R}^d \to \mathbb{R}$ has $L_1$-Lipschitz first derivatives and $L_3$-Lipschitz third derivatives on the set $B_{12R}(z)$.

If $\|\nabla f(z^{(+)})\| \geq \epsilon$ or $\lambda_{\min}(\nabla^2 f(z^{(+)})) \leq -\alpha$ then
\[
f(z^{(+)}) \leq f(z) - \min \left\{ 10L_3R^4, \frac{\epsilon^2}{168R^2L_3} \right\}.
\]
Furthermore, the runtime of Algorithm 4 is at most
\[
O \left( \frac{(T_C + T_1 + Kd)d}{\tau} \log^+ \left( \frac{RL_1}{\epsilon} \right) + T_2 + d^w \right).
\]
\end{lemma}

\begin{proof}
Before beginning the proof we recap some useful facts:
\[
\begin{align*}
    \hat{f}(u) &\leq \hat{f}(z) = f(z) \quad (12a) \\
    f(u) - f(z) &= \hat{f}(u) - f(z) - \frac{\alpha}{2}\|u - z\| \leq -\frac{\alpha}{2}\|u - z\|^2, \quad (12b)
\end{align*}
\]
where (12a) is from Lemma 2, (12b) follows from the definition of $\hat{f}$ and (12a).
Consider the three possible outcomes of Algorithm 4 which are

\[ \| \nabla f(u) \| \leq \epsilon \text{ and } p^T \nabla^2 f(u) p \geq -\alpha \]  
(13a)

\[ \| \nabla f(u) \| \leq \epsilon \text{ and } p^T \nabla^2 f(u) p < -\alpha \]  
(13b)

\[ \| \nabla f(u) \| > \epsilon. \]  
(13c)

If (13a) holds then Lemma 4 clearly holds. If (13b) or (13c) holds then we wish to establish (5).

Let us show (5) when (13b) holds. In this case, \( z^{(+) - f(z)} \leq -536L_3R^4 \).

Let us show (5) when (13c) holds. Consider the three cases arising from Lemma 2.

(i) \( v = 0 \) and \( \| \nabla \hat{f}(u) \| \leq \epsilon = \epsilon/2 \). In this case \( u = z^{(+)} \). Therefore

\[ \epsilon \leq \| \nabla f(z^{(+) - z}) \| \leq \| \nabla \hat{f}(z^{(+) - z}) \| + \alpha \| z^{(+) - z} \| \leq \epsilon/2 + \alpha \| z^{(+) - z} \|. \]

Rearranging yields \( \| z^{(+) - z} \| \geq \epsilon/(2\alpha) \). Therefore, using (12b), \( \| z^{(+) - z} \| \geq \epsilon/(2\alpha) \), and \( \alpha = 21L_3R^2 \) we get

\[ f(z^{(+) - f(z)} \leq -\frac{\alpha}{2} \| z^{(+) - z} \|^2 \leq \frac{\epsilon^2}{8\alpha} \leq \frac{\epsilon^2}{168L_3R^2}. \]

(ii) \( v = 0 \) and \( \| \nabla \hat{f}(u) \| > \epsilon = \epsilon/2 \). In this case \( u = z^{(+)} \). By Lemma 2 we have \( \| u - z \| > R \). Therefore using (12b), \( \| u - z \| > R \), and \( \alpha = 21L_3R^2 \) we get

\[ f(z^{(+) - f(z)} \leq -\frac{\alpha}{2} \| z^{(+) - z} \|^2 = -\frac{\alpha R^2}{2} = -\frac{21}{2}L_3R^4. \]

(iii) \( v \neq 0 \). In this case, we have a certificate of nonconvexity:

\[ \hat{f}(u) < \hat{f}(v) + \nabla \hat{f}(v)^T (v - u) \Rightarrow f(u) < f(v) + \nabla f(v)^T (v - u) - \frac{\alpha}{2} \| v - u \|^2. \]

Let \( q(\theta) := f(c + \theta s) \) with \( s = \frac{u - v}{\| u - v \|} \) and \( c = \frac{u + v}{2} \). We deduce there exists some point \( \gamma \in [-1,1] \) with \( q''(\gamma) < -\alpha \). Since \( \alpha = 21L_3R^2 \) in Algorithm 4 we can apply Lemma 3 to show that we reduce the function by at least \( 536L_3R^4 \) during our call to \( \text{ExploitNC}(f,c,s,R) \).

Therefore if (13b) or (13c) holds then (5) holds.

It remains to derive the runtime of the algorithm per iteration. We can bound the computational cost by

\[ O \left( KNd + N(T_C + T_1) + T_2 + d^\omega \right) \]

where \( K \) is the random variable arising from (3). The term \( O(NKd) \) come from the fact that it requires \( O(Nd) \) to evaluate if \( u \notin S^{(N)} \). The term \( T_C + T_1 \) represents
the cost of each iteration of Algorithm 1. The term $O(T_2 + d^\omega)$ comes from the fact that at each iteration of Algorithm 4 we compute an SVD which takes $O(d^\omega)$ time \cite[Section 2.6]{21} and evaluate the Hessian. Using Lemma 2 with $N = \left\lceil \frac{d^2 \log + \left(\frac{8 L_1 R}{\epsilon}\right)}{\tau}\right\rceil$ we know the computation cost can be bounded by

$$O\left(\frac{d(K d + TC + T_1)}{\tau}\log + \left(\frac{8 L_1 R}{\epsilon}\right) + T_2 + d^\omega\right).$$

\[\square\]

C  Proof of Lemma 6

Lemma 6. Let $\bar{K} = \frac{1}{m} \sum_{t=1}^{m} K^{(t)}$ with independent random variables $K^{(t)} \sim \text{Geo}(p^{(t)})$ and $p^{(t)} \geq 1/2$ then $P(\bar{K} \geq y) \leq e^{-y/\bar{m}}$ for all $y \in \mathbb{R}$.

Proof. Since $K^{(t)} \sim \text{Geo}(p^{(t)})$ with $p^{(t)} \geq 1/2$ we can bound the moment generating function for $\alpha \leq 1/10$: 

$$E[e^{\bar{K}/\alpha}] = \frac{1}{1 - \frac{1}{1 - \alpha p^{(t)}}} \leq \frac{2}{1 + 5\alpha} \leq 1 + 5\alpha.$$

Using a typical Chernoff bound argument,

$$P(\bar{K} \geq y) = P(e^{\bar{K}} \geq e^y) \leq \frac{E[e^{\bar{K}/\alpha}]}{e^y/\alpha} \leq \left(1 + 1/(5m)\right)^m \leq e^{-y/\bar{m}}.$$

\[\square\]

D  Proof of Theorem 2

Theorem 2. Suppose that assumption 1 holds. Let $z^{(0)} \in \mathbb{R}^d$, $L_1, L_3 \in (0, \infty)$ and $\epsilon \in (0, L_3^{3/2} / L_3^{1/2}]$. Assume that $f : \mathbb{R}^d \to \mathbb{R}$ has $L_1$-Lipschitz first derivatives and $L_3$-Lipschitz third derivatives. Let $f(z^{(0)}) - \inf_{x \in \mathbb{R}^d} f(x) \leq \Delta$. Under these conditions, the procedure (10) starting with $t = 0$ finds a point $z^{(m)}$ such that (7) holds with computational time upper bounded by 

$$O\left(\left(\Delta L_3^{1/3} \epsilon^{-4/3} + 1\right)\left(T_3 + \frac{(TC + d^3 + \bar{K} d) d}{\tau}\log + \left(\frac{L_3^3}{\epsilon^2 L_3}\right)\right)\right)$$

where $\bar{K}$ is a positive random variable satisfying $P(\bar{K} \geq y) \leq e^{-y/\bar{m}}$ for all $y \in \mathbb{R}$.

Proof. Let us check the assumptions of Lemma 4 hold. Recall that we defined $\bar{f}$ such that 

$$\bar{f}^{(t)}(x) = f(x^{(t)}) + \nabla f(x^{(t)})^T (x - x^{(t)}) + \cdots + \frac{L_3}{12} \|x - x^{(t)}\|^4.$$

Therefore $\bar{f}^{(t)}(x)$ has $2L_3$-Lipschitz third derivatives.
For $x \in B_{12R}(z(t))$ with $R = L^{-1/3} \epsilon^{1/3}/24$ we have

$$\|\nabla f(x) - \nabla \bar{f}(t)(x)\| \leq \frac{L_1}{3} \|x - z(t)\|^3 \leq \frac{\epsilon}{24}.$$ 

Therefore if $\|\nabla \bar{f}(t)(z(t+1))\| \leq \epsilon/2$ then $\|\nabla f(z(t+1))\| \leq \epsilon$. Similarly, for any $x, x' \in B_{12R}(z(t))$ we have

$$\|\nabla^2 f(x) - \nabla^2 \bar{f}(t)(x)\| \leq L_3 \|x - z(t)\|^2 \leq \frac{L_1^{1/3} \epsilon^{2/3}}{4}.$$ 

it follows that $\lambda_{\text{min}}(\nabla^2 f(z(t+1))) \geq -L_3^{1/3} \epsilon^{2/3}/2$ if $\lambda_{\text{min}}(\nabla^2 \bar{f}(t)(z(t+1))) \geq -L_3^{1/3} \epsilon^{2/3}$. Furthermore, we deduce that $\bar{f}(t)$ has $2L_1$-Lipschitz first derivatives using $\epsilon \leq L_1^{1/2}/L_3^{1/2} \Rightarrow L_3^{1/3} \epsilon^{2/3} \leq L_1$.

With these conditions established we can apply Lemma 4 with $R = L_3^{-1/3} \epsilon^{1/3}/24$ to deduce $\bar{f}(t)(z(t)) - \bar{f}(t)(z(t+1)) = \Omega(L_3^{-1/3} \epsilon^{4/3})$. This translates into a progress bound on $f$ since

$$\bar{f}(t)(z(t)) - \bar{f}(t)(z(t+1)) = f(z(t)) - \bar{f}(t)(z(t+1)) \leq f(z(t)) - f(z(t+1)).$$

Therefore if $m$ is the total number of iterations we have

$$\Delta \geq f(z(0)) - f(z(m)) = \sum_{t=0}^{m-1} (f(z(t)) - f(z(t+1))) = \Omega(mL_3^{-1/3} \epsilon^{4/3}).$$

Rearranging shows $m = O(\Delta^{-1/3} \epsilon^{-4/3} + 1)$. The computational cost per iteration derives from Lemma 4 using $T_1 = O(d^3)$, since we need to evaluate the gradient of a quartic regularized model at each iteration. The bound on $K = \frac{1}{m} \sum_{t=1}^{m} K^{(t)}$ derives from Lemma 6. 

\[\square\]