Star Products and Geometric Algebra

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Abstract

The formalism of geometric algebra can be described as deformed super analysis. The deformation is done with a fermionic star product, that arises from deformation quantization of pseudoclassical mechanics. If one then extends the deformation to the bosonic coefficient part of superanalysis one obtains quantum mechanics for systems with spin. This approach clarifies on the one hand the relation between Grassmann and Clifford structures in geometric algebra and on the other hand the relation between classical mechanics and quantum mechanics. Moreover it gives a formalism that allows to handle classical and quantum mechanics in a consistent manner.

1 Introduction

Geometric algebra goes back to early ideas of Hamilton, Grassmann and Clifford. But it was first developed into a full formalism by Hestenes in [1] and [2]. The formalism of geometric algebra is based on the definition of the geometric or Clifford product. This product is for vectors defined as the sum of the scalar and the wedge product and equips the vector space with the algebraic structure of a Clifford algebra. This structure then proved to be a very powerful tool, that allows to describe and generalize the structures of vector analysis, of complex analysis and of the theory of spin in a unified and clear formalism. The formalism can then be used to describe classical mechanics in the realm of geometric algebra instead of linear algebra, which is advantageous in many respects [3, 4]. It is also possible to generalize the formalism from the algebra of space to the algebra of spacetime in order to describe electrodynamics and special relativity [1, 4].

In quantum mechanics the Clifford structures of the $\sigma$- and the $\gamma$-matrices correspond to the structures of geometric algebra. So by formulating classical physics and quantum physics with geometric algebra one achieves a formal unification of both areas on a geometric level. Nevertheless this formulation is conceptually not totally unified, because classical mechanics is still formulated on the phase space while quantum mechanics is formulated in Hilbert space. In order to achieve a totally unified formulation we will here combine geometric algebra with the star product formalism. The star product formalism [5] appears in the context

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of deformation quantization where one describes the non-commutativity that enters physics in quantum mechanics not by using non-commuting objects like operators, but by introducing a non-commutative product on the phase space that replaces the conventional product of functions. This star product is so constructed that the quantized star product of two phase space functions corresponds to the operator product of the quantized factors, which then allows to do quantum mechanics on phase space. To include spin in this formalism we used in [6] and [7] fermionic star products that result directly from deformation quantization of pseudoclassical mechanics [8]. Fermionic star products were already discussed in [5], where it was noticed that they lead to a cliffordization of the underlying Grassmann algebra. So it is possible to describe a Clifford algebra as a deformed Grassmann algebra, where this deformation is nothing else than Chevalley cliffordization [7].

In this paper we will use the fact that geometric algebra can be formulated in terms of a Grassmann algebra [9]. We will show that in this context the geometric product can be made explicit as a fermionic star product. It is then straightforward to translate classical mechanics described with geometric algebra into a version where it is described in terms of fermionic deformed super analysis. The fermionic part of the formalism represents hereby the basis vector structure of the space on which the theory is formulated, i.e. the three dimensional space, the phase space or the spacetime. In all cases we consider only the case of flat spaces. In a second step one can then go over to quantum mechanics, where we use here deformation quantization, while in [9] canonical and path integral quantization was used. Combining in this way geometric algebra formulated with a fermionic star product with the bosonic star product of deformation quantization one arrives at a supersymmetric star product formalism that allows to describe quantum mechanics with spin in a unified manner. Moreover by using star products one can immediately give the classical $\hbar \to 0$ limit and see how the spin as a physical observable vanishes. Furthermore one can see that classical mechanics can be described as a half deformed theory, while quantum mechanics is a totally deformed theory, i.e. in classical mechanics the star product acts only on the fermionic basis vector part of the formalism, while for $\hbar > 0$ there exist also a bosonic star product that acts on the coefficients of the basis vectors.

In the second section we will very shortly review the bosonic and fermionic star product formalism and show how quantum mechanics with spin can be described in this context. Then we will show how geometric algebra can be formulated with the fermionic star product. We will therefore formulate well known results of geometric algebra in the formalism of fermionic deformed superanalysis. Afterwards in section 5 and 6 we will extend the formalism to the case of nonrelativistic quantum mechanics and Dirac theory by using the bosonic Moyal product.

2 The Star Product Formalism

We first want to introduce the star product formalism in bosonic and fermionic physics with the example of the harmonic oscillator [5]. The bosonic oscillator with the Hamilton function $H(q, p) = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2$, can be quantized by using the Moyal product

$$f \star_M g = f \exp \left[ \frac{i\hbar}{2} \left( \tilde{\partial}_q \tilde{\partial}_p - \tilde{\partial}_p \tilde{\partial}_q \right) \right] g. \quad (2.1)$$
The star product replaces the conventional product between functions on the phase space and it is so constructed that the star anticommutator, i.e. the antisymmetric part of first order, is the Poisson bracket:

$$\lim_{\hbar \to 0} \frac{1}{\hbar} [f(q, p), g(q, p)]_{\star M} = \lim_{\hbar \to 0} \frac{1}{\hbar} (f(q, p) \star_M g(q, p) - g(q, p) \star_M f(q, p)) = \{f(q, p), g(q, p)\}_{PB}. \quad (2.2)$$

This relation is the principle of correspondence. The states of the quantized harmonic oscillator are described by the Wigner functions $\pi_n^{(M)}(q, p)$. The Wigner functions and the energy levels $E_n$ of the harmonic oscillator can then be calculated with the help of the star exponential

$$\text{Exp}_M(Ht) = e^{-\frac{iHt}{\hbar}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-it}{\hbar}\right)^n H^{n\star_M} = \sum_{n=0}^{\infty} \pi_n^{(M)} e^{-iE_n t/\hbar}, \quad (2.3)$$

where $H^{n\star_M} = H \star_M \cdots \star_M H$ is the $n$-fold star product of $H$. The star exponential fulfills the analogue of the time dependent Schrödinger equation

$$i\hbar \frac{d}{dt} \text{Exp}_M(Ht) = H \star_M \text{Exp}_M(Ht). \quad (2.4)$$

The energy levels and the Wigner functions fulfill the $\star$-genvalue equation

$$H \star_M \pi_n^{(M)} = E_n \pi_n^{(M)} \quad (2.5)$$

and for the harmonic oscillator one obtains $E_n = \hbar \omega (n + \frac{1}{2})$ and

$$\pi_n^{(M)} = 2(-1)^n e^{-2H/\hbar \omega} L_n \left(\frac{4H}{\hbar \omega}\right), \quad (2.6)$$

where the $L_n$ are the Laguerre polynomials. The Wigner functions $\pi_n^{(M)}$ are normalized according to $\frac{1}{2\pi \hbar} \int \pi_n^{(M)} dq dp = 1$ and the expectation value of a phase space function $f$ can be calculated as

$$\langle f \rangle = \frac{1}{2\pi \hbar} \int f \star_M \pi_n^{(M)} dq dp. \quad (2.7)$$

The same procedure can now be used for the grassmannian case [6]. The simplest system in grassmannian mechanics [5] is a two dimensional system with Lagrange function

$$L = \frac{i}{2} \left(\theta_1 \dot{\theta}_1 + \theta_2 \dot{\theta}_2\right) + i\omega \theta_1 \theta_2. \quad (2.8)$$

With the canonical momentum

$$\rho_\alpha = -\frac{i}{2} \theta_\alpha \quad (2.9)$$

the Hamilton function is given by

$$H = \dot{\rho}_\alpha \rho_\alpha - L = -i\omega \theta_1 \theta_2. \quad (2.10)$$

Together with equation (2.4) this Hamiltonian suggests that the fermionic oscillator describes rotation. Indeed, calculating the fermionic angular momentum, which corresponds to the spin, leads to

$$S_3 = \theta_1 \rho_2 - \theta_2 \rho_1 = -i\theta_1 \theta_2, \quad (2.11)$$
so that the Hamiltonian in (2.10) can also be written as $H = \omega S_3$. As a vector the angular momentum points out of the $\theta_1$-$\theta_2$-plane. Therefore we consider the two dimensional fermionic oscillator as embedded into a three dimensional fermionic space with coordinates $\theta_1$, $\theta_2$ and $\theta_3$. Note that we choose both for the fermionic space and momentum coordinates the units $\sqrt{\hbar}$.

Quantizing the fermionic oscillator involves a star product that is given by

$$F \ast_C G = F \exp \left[ \frac{\hbar}{2} \sum_{n=1}^{d} \tilde{\partial}_{\theta_n} \tilde{\partial}_{\theta_n} \right] G.$$ (2.12)

We will call this star product the Clifford star product because it leads to a cliffordization of the Grassmann algebra of the $\theta_i$. This can be seen by considering the star-anticommutator that is given by

$$\{ \theta_i, \theta_j \} \ast_C = \theta_i \ast_C \theta_j + \theta_j \ast_C \theta_i = \hbar \delta_{ij}. $$ (2.13)

Since the Grassmann variables $\sigma^i = \frac{1}{i\hbar} \varepsilon^{ijk} \theta_j \theta_k$ with $i \in \{1, 2, 3\}$, fulfill the relations

$$[\sigma^i, \sigma^j] \ast_C = 2i \varepsilon^{ijk} \sigma^k \quad \text{and} \quad \{ \sigma^i, \sigma^j \} \ast_C = 2 \delta^{ij},$$ (2.15)

with $[\sigma^i, \sigma^j] \ast_C = \sigma^i \ast_C \sigma^j - \sigma^j \ast_C \sigma^i$, they correspond to the Pauli matrices. From equations (2.11) and (2.14) it follows that $S_3 = \frac{\hbar}{2} \sigma^3$ and $H = \omega S_3 = \frac{\hbar \omega}{2} \sigma^3$. Note, that $\{1, \sigma^1, \sigma^2, \sigma^3\}$ is a basis of the even subalgebra of the Grassmann algebra and that this space is also closed under $\ast_C$ multiplication.

In the space of Grassmann variables there exists an analogue of complex conjugation, which is called the involution. As in [8] it can be defined as a mapping $F \mapsto \overline{F}$, satisfying the conditions

$$\overline{\overline{F}} = F, \quad \overline{F_1 F_2} = \overline{F_2} \overline{F_1} \quad \text{and} \quad \overline{cF} = \overline{c} \overline{F},$$ (2.16)

where $c$ is a complex number and $\overline{c}$ its complex conjugate. For the generators $\theta_i$ of the Grassmann algebra we assume $\overline{\theta_i} = \theta_i$, so that for $\sigma^i$ defined in (2.14) the relation $\sigma^i = \sigma^i$ holds true. This corresponds to the fact that the $2 \times 2$ Pauli matrices are hermitian.

We now define the Hodge dual for Grassmann numbers with respect to the metric $\delta_{ij}$. The Hodge dual maps a Grassmann monomial of grade $r$ into a monomial of grade $d-r$, where $d$ is the number of Grassmann basis elements (which is in our case three):

$$\ast (\theta_{i_1} \theta_{i_2} \cdots \theta_{i_r}) = \frac{1}{(d-r)!} \varepsilon^{i_{r+1} \cdots i_d} \theta_{i_{r+1}} \cdots \theta_{i_d}. $$ (2.17)

With the help of the Hodge dual one can define a trace as

$$\text{Tr}(F) = \frac{2}{\hbar^3} \int d\theta_1 d\theta_2 d\theta_3 \ast F.$$ (2.18)

The integration is given by the Berezin integral for which we have $\int d\theta_i \theta_j = \hbar \delta_{ij}$, where the $\hbar$ on the right hand side is due to the fact that the variables $\theta_i$ have units of $\sqrt{\hbar}$. The only monomial with a non-zero trace is 1, so that by the linearity of the integral we obtain the trace rules

$$\text{Tr}(\sigma^i) = 0 \quad \text{and} \quad \text{Tr}(\sigma^i \ast_C \sigma^j) = 2 \delta^{ij}. $$ (2.19)
With the fermionic star product (2.12) one can—as in the bosonic case—calculate the energy levels and the \(*\)-eigenfunctions of the fermionic oscillator. This can be done with the fermionic star exponential

$$\text{Exp}_C(\mathcal{H} t) = e^{ - \mathcal{H} t } = \sum_{n=0}^{\infty} \frac{1}{n!} \left( - \frac{i \mathcal{H}}{\hbar} \right)^n H^{n \ast C} = \pi_{1/2}^{(C)} e^{ - i \omega t / 2 } + \pi_{-1/2}^{(C)} e^{ i \omega t / 2 },$$

(2.20)

where the Wigner functions are given by

$$\pi_{\pm 1/2}^{(C)} = \frac{1}{2} \pm \frac{i}{\hbar} \theta_1 \theta_2 = \frac{1}{2} ( 1 \pm \sigma^3 ).$$

(2.21)

The $\pi_{\pm 1/2}^{(C)}$ fulfill the \(*\)-eigenvalue equation $H \ast_C \pi_{\pm 1/2}^{(C)} = E_{\pm 1/2} \pi_{\pm 1/2}^{(C)}$ for the energy levels $E_{\pm 1/2} = \pm \hbar \omega / 2$. The Wigner functions $\pi_{\pm 1/2}^{(C)}$ are complete, idempotent and normalized with respect to the trace, i.e. they fulfill the equations

$$\pi_{1/2}^{(C)} + \pi_{-1/2}^{(C)} = 1 , \quad \pi_{\alpha}^{(C)} \ast_C \pi_{\beta}^{(C)} = \delta_{\alpha \beta} \pi_{\alpha}^{(C)} \quad \text{and} \quad \text{Tr} ( \pi_{\pm 1/2}^{(C)} ) = 1 ,$$

(2.22)

respectively. Furthermore they correspond to spin up and spin down states since (2.21) corresponds to the spin projectors and the expectation values of the angular momentum are

$$\langle S_1 \rangle = \text{Tr} \left( \pi_{\pm 1/2}^{(M)} \ast_C \frac{\hbar}{2} \sigma^1 \right) = 0 \quad \text{,} \quad \langle S_2 \rangle = \text{Tr} \left( \pi_{\pm 1/2}^{(M)} \ast_C \frac{\hbar}{2} \sigma^2 \right) = 0$$

$$\langle S_3 \rangle = \text{Tr} \left( \pi_{\pm 1/2}^{(M)} \ast_C \frac{\hbar}{2} \sigma^3 \right) = \pm \frac{\hbar}{2} \quad \text{,} \quad \langle \vec{S}^2 \rangle = \text{Tr} \left( \pi_{\pm 1/2}^{(M)} \ast_C \frac{\hbar^2}{4} \vec{\sigma}^2 \right) = \frac{3}{4} \hbar^2 .$$

(2.23)

where the spin $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$ was used with components of $\vec{\sigma}$ as defined in (2.14).

In the fermionic $\theta$-space the spin $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$ is the generator of rotations, which are described by the star exponential

$$\text{Exp}_C ( \vec{\varphi} \cdot \vec{S} ) = e^{ \frac{i}{\hbar} \vec{\varphi} \cdot \vec{\sigma} } = \cos \frac{\varphi}{2} - i ( \vec{\sigma} \cdot \vec{n} ) \sin \frac{\varphi}{2} ,$$

(2.24)

where we used the definition $\vec{\varphi} = \varphi \vec{n}$ with rotation angle $\varphi$ and a rotation axis given by the unit vector $\vec{n}$. The vector $\vec{\theta} = ( \theta_1 , \theta_2 , \theta_3 )^T$ transforms passively according to

$$\text{Exp}_C ( \vec{\varphi} \cdot \vec{S} ) \ast_C \vec{\theta} \ast_C \text{Exp}_C ( \vec{\varphi} \cdot \vec{S} ) = e^{ \frac{i}{\hbar} \vec{\varphi} \cdot \vec{\sigma} } \ast_C \vec{\theta} \ast_C e^{ \frac{i}{\hbar} \vec{\varphi} \cdot \vec{\sigma} } = R ( \vec{\varphi} ) \vec{\theta}$$

(2.25)

with $R ( \vec{\varphi} )$ being the well-known $SO(3)$ rotation matrix. The axial vector $\vec{\sigma}$ transforms in the same way. Note that the passive transformation (2.25) of the $\theta_i$ amounts to an active transformation of the components $x_i$ in the vector $x = \sum_{i=1}^{3} x_i \theta_i$.

3 Geometric Algebra and the Clifford Star Product

Starting point for geometric algebra [1, 3] is an $n$-dimensional vector space over the real numbers with vectors $a, b, c, \ldots$. A multiplication, called geometric product, of vectors can then be denoted by juxtaposition of an indeterminate number of vectors so that one gets monomials $A, B, C, \ldots$. These monomials can be added in
Furthermore there exists a null vector $a\mathbf{0} = \mathbf{0}$ and the multiplication with a scalar $\lambda a = a\lambda$, with $\lambda \in \mathbb{R}$. The connection between scalars and vectors can be given if one assumes that the product $ab$ is a scalar iff $a$ and $b$ are collinear, so that $\sqrt{a^2}$ is the length of the vector $a$. These axioms define now the Clifford algebra $C\ell(V)$ and the elements $A,B,C,\ldots$ of $C\ell(V)$ are called Clifford or c-numbers.

Since the geometric product of two collinear vectors is a scalar, the symmetric part of the geometric product $\frac{1}{2}(ab + ba) = \frac{1}{2}((a + b)^2 - a^2 - b^2)$ is a scalar denoted $a \cdot b = \frac{1}{2}(ab + ba)$. The product $a \cdot b$ is the inner or scalar product. One can then decompose the geometric product into its symmetric and antisymmetric part:

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba) = a \cdot b + a \wedge b,$$

where the antisymmetric part $a \wedge b = \frac{1}{2}(ab - ba)$ is formed with the outer product. For the outer product one has obviously $a \wedge b = -b \wedge a$ and $a \wedge a = 0$, so that $a \wedge b$ can be interpreted geometrically as an oriented area. The geometric product is constructed in such a way that it gives information over the relative directions of $a$ and $b$, i.e. $ab = ba = a \cdot b \Rightarrow a \wedge b = 0$ means that $a$ and $b$ are collinear whereas $ab = -ba = a \wedge b \Rightarrow a \cdot b = 0$ means that $a$ and $b$ are perpendicular.

With the outer product one defines simple $r$-vectors or $r$-blades

$$A_r = a_1 \wedge a_2 \wedge \ldots \wedge a_r,$$

which can be interpreted as $r$-dimensional volume forms. The geometric product can then be generalized to the case of a vector and a $r$-blade:

$$aA_r = a \cdot A_r + a \wedge A_r,$$

which is the sum of a $(r-1)$-blade $a \cdot A_r = \frac{1}{2}(aA_r - (-1)^r A_r a)$ and a $(r+1)$-blade $a \wedge A_r = \frac{1}{2}(aA_r + (-1)^r A_r a)$. Applying this recursively one sees, that each c-number can be written as a polynomial of $r$-blades and using a set of basis vectors $e_1, e_2, \ldots, e_r$ a $c$-number reads:

$$A = a + a^i e_i + \frac{1}{2!} a^{i_1 i_2} e_{i_1} \wedge e_{i_2} + \ldots + \frac{1}{n!} a^{i_1 \ldots i_n} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r}.$$

$A$ is called multivector or $r$-vector if the highest appearing grade is $r$. It decomposes into several blades:

$$A = \langle A \rangle_0 + \langle A \rangle_1 + \ldots + \sum_n \langle A \rangle_n,$$

where $\langle \rangle_n$ projects onto the term of grade $n$. A multivector $A_r$ is called homogeneous if all appearing blades have the same grade, i.e. $A_r = \langle A_r \rangle_r$. The geometric product of two homogeneous multivectors $A_r$ and $B_s$ can be written as

$$A_r B_s = \langle A_r B_s \rangle_{r+s} + \langle A_r B_s \rangle_{r+s-2} + \cdots + \langle A_r B_s \rangle_{|r-s|}.$$

The inner and the outer product stand now for the terms with the lowest and the highest grade:

$$A_r \cdot B_s = \langle A_r B_s \rangle_{|r-s|} \quad \text{and} \quad A_r \wedge B_s = \langle A_r B_s \rangle_{r+s}.$$

One should note that the inner and outer product here in the general case do not correspond anymore to the symmetric and the antisymmetric part of the geometric product. For example in the case of two bivectors
one has $A_2 \wedge B_2 = B_2 \wedge A_2$, so that the outer product is symmetric. Actually one finds for the symmetric and the antisymmetric parts of $A_2 B_2$:

$$\frac{1}{2}(A_2 B_2 + B_2 A_2) = A_2 \cdot B_2 + A_2 \wedge B_2 \quad \text{and} \quad \frac{1}{2}(A_2 B_2 - B_2 A_2) = \langle A_2 B_2 \rangle_2.$$  \hspace{1cm} (3.8)

In general the commutativity of the outer and the inner product is given by:

$$A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r \quad \text{and} \quad A_r \cdot B_s = (-1)^{r(s+1)} B_s \cdot A_r.$$  \hspace{1cm} (3.9)

and both products are always distributive:

$$A \wedge (B + C) = A \wedge B + A \wedge C \quad \text{and} \quad A \cdot (B + C) = A \cdot B + A \cdot C.$$  \hspace{1cm} (3.10)

Only the outer product of $r$-vectors is in general associative, i.e. $A \wedge (B \wedge C) = (A \wedge B) \wedge C$, for the inner product one gets:

$$A_r \cdot (B_s \cdot C_t) = (A_r \cdot B_s) \cdot C_t \quad \text{for} \quad r + t \leq s.$$  \hspace{1cm} (3.11)

If one has to calculate several products of different type, the inner and the outer product always have to be calculated first, i.e.

$$A \wedge BC = (A \wedge B)C \neq A \wedge (BC) \quad \text{and} \quad A \cdot BC = (A \cdot B)C \neq A \cdot (BC).$$  \hspace{1cm} (3.12)

The formalism of geometric algebra briefly sketched so far can now be described with Grassmann variables and the Clifford star product, that turns the Grassmann algebra into a Clifford algebra. In order to make the equivalence even more obvious we go over to the dimensionless Grassmann variables

$$\sigma_n = \sqrt{\frac{\theta_n}{\hbar}}.$$  \hspace{1cm} (3.13)

These variables play here the role of dimensionless basis vectors and will therefore be written in bold face, whereas the $\theta_i$ played in the discussion of the first section the role of dynamical variables with dimension $\sqrt{\hbar}$. In the $\sigma_n$-variables the Clifford star product (2.12) has the form

$$F \ast_C G = F \exp \left[ \sum_{n=1}^{d} \frac{\tilde{\partial}}{\partial \sigma_n} \frac{\tilde{\partial}}{\partial \sigma_n} \right] G.$$  \hspace{1cm} (3.14)

As a star product the Clifford star product is associative and distributive.

In order to show how the geometric algebra described with Grassmann variables and the Clifford star product looks like, we first consider the two dimensional euclidian case. One has then two Grassmann basis elements $\sigma_1$ and $\sigma_2$, so that a general element of the Clifford algebra is a supernumber $A = a_0 + a_1 \sigma_1 + a_2 \sigma_2 + a_{12} \sigma_1 \sigma_2 = \langle A \rangle_0 + \langle A \rangle_1 + \langle A \rangle_2$ and a vector corresponds to a supernumber with Grassmann grade one: $a = a_1 \sigma_1 + a_2 \sigma_2$. The Clifford star product of two of these supernumbers is

$$a \ast_C b = ab + a \left[ \sum_{n=1}^{2} \frac{\tilde{\partial}}{\partial \sigma_n} \frac{\tilde{\partial}}{\partial \sigma_n} \right] b = (a_1 b_2 - a_2 b_1) \sigma_1 \sigma_2 + a_1 b_1 + a_2 b_2 \equiv a \wedge b + a \cdot b.$$  \hspace{1cm} (3.15)
where the symmetric and the antisymmetric part of the Clifford star product is given by:

\[
\frac{1}{2}(a *_C b + b *_C a) = a_1 b_1 + a_2 b_2 \equiv a \cdot b
\]

and

\[
\frac{1}{2}(a *_C b - b *_C a) = (a_1 b_2 - a_2 b_1)\sigma_1 \sigma_2 = ab \equiv a \wedge b,
\]

which are terms with Grassmann grade 0 and 2 respectively. Note that now a juxtaposition like \(ab\) is just as in the notation of superanalysis the product of supernumbers and not the Clifford product, which we want to describe explicitly with the star product (3.14). The \(\sigma_i\) form an orthogonal basis under the scalar product: \(\sigma_i \cdot \sigma_j = \frac{1}{2}(\sigma_i *_C \sigma_j + \sigma_j *_C \sigma_i) = \delta_{ij}\).

The unit 2-blade \(i = \sigma_1 \sigma_2\) can be interpreted as the generator of \(\frac{\pi}{2}\)-rotations because by multiplying from the right one gets

\[
\sigma_1 *_C i = \sigma_1 \cdot i = \sigma_2, \quad \sigma_2 *_C i = \sigma_2 \cdot i = -\sigma_1 \quad \text{and} \quad \sigma_1 *_C i *_C i = -\sigma_1,
\]

so that a vector \(\mathbf{x} = x_1 \sigma_1 + x_2 \sigma_2\) is transformed into \(\mathbf{x}' = x *_C i = x \cdot i = x_1 \sigma_2 - x_2 \sigma_1\). The relation \(i^2 *_C = -1\) describes then a reflection and furthermore one has with (3.10): \(I = \sigma_2 \sigma_1 = -i\), so that \(i\) corresponds to the imaginary unit. The connection between the two dimensional vector space with vectors \(\mathbf{x}\) and the Gauss plane with complex numbers \(z\) is established by star multiplying \(\mathbf{x}\) with \(\sigma_1\):

\[
z = \sigma_1 *_C \mathbf{x} = x_1 + ix_2.
\]

Such a bivector that results from star multiplying two vectors is also called spinor. While the bivector \(i\) generates a rotation of \(\frac{\pi}{2}\) when acting from the right, the spinor \(z\) generates a general combination of a rotation and dilation when acting from the right. One can see this by writing \(z = x + i x_2 = |z| e^{i \phi} \) with \(|z|^2 = z *_C z = x_1^2 + x_2^2\). Acting from the right with \(z\) causes then a dilation by \(|z|\) and a rotation by \(\varphi\), one has for example: \(\sigma_1 *_C z = \mathbf{x}\), which is the inversion of (3.12). Here one can see that the formalism of geometric algebra reproduces complex analysis and gives it a geometric meaning.

After having described the geometric algebra of the euclidian 2-space we now turn to the euclidian 3-space with basis vectors \(\sigma_1, \sigma_2\) and \(\sigma_3\) and with the Clifford star product (3.14) for \(d = 3\). The basis vectors are orthogonal: \(\sigma_i \cdot \sigma_j = \delta_{ij}\) and a general c-number written as a supernumber has the form

\[
A = a_0 + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 + a_{12} \sigma_1 \sigma_2 + a_{13} \sigma_1 \sigma_3 + a_{23} \sigma_2 \sigma_3 + a_{123} \sigma_1 \sigma_2 \sigma_3.
\]

This multivector has now four different simple multivector parts. Besides the scalar part \(a_0\) there is the pseudoscalar part corresponding to \(I_3 = \sigma_1 \sigma_2 \sigma_3\), which can be interpreted as a right handed volume form, because a parity operation gives \((-\sigma_1)(-\sigma_2)(-\sigma_3) = -I_3\). Moreover \(I_3\) has also the properties of an imaginary unit: \(I_3 = -I_3\) and \(I_3 *_C I_3 = I_3 \cdot I_3 = -1\). While the pseudoscalar \(I_3\) is an oriented volume element the bivector part with the basic 2-blades

\[
i_1 = \sigma_2 \sigma_3 = I_3 *_C \sigma_1, \quad i_2 = \sigma_3 \sigma_1 = I_3 *_C \sigma_2 \quad \text{and} \quad i_3 = \sigma_1 \sigma_2 = I_3 *_C \sigma_3
\]

describes oriented area elements. Each of the \(i_r\) plays in the plane it defines the same role as the \(i\) of the two dimensional euclidian plane defined above. Star-multiplying with the pseudoscalar \(I_3\) is equivalent to taking the Hodge dual, for example to each bivector \(B = b_1 i_1 + b_2 i_2 + b_3 i_3\) corresponds a vector \(\mathbf{b} = b_1 \sigma_1 + b_2 \sigma_2 + b_3 \sigma_3\), which can be expressed by the equation \(B = I_3 *_C \mathbf{b}\). This duality can for example
be used to write the geometric product of two vectors \( a = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 \) and \( b = b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3 \) as:

\[
a *_c b = a \cdot b + I_3 *_c (a \times b),
\]

(3.22)

where \( a \cdot b = \sum_{k=1}^{3} a_k b_k \) and \( a \times b = \varepsilon^{klm} a_k b_l \sigma_m \). Furthermore one finds:

\[
\sigma_1 \times \sigma_2 = -I_3 *_c \sigma_1 *_c \sigma_2 = -I_3 *_c \sigma_1 \sigma_2 = \sigma_3
\]

(3.23)

and cyclic permutations. Note also that one gets with the nabla operator only the component collinear to \( u \) normal vector. Two successive transformations (3.26) lead to:

\[
\nabla_x *_c f = \nabla_x \cdot f + \nabla_x \wedge f = \text{div} f + I_3 *_c \text{rot} f.
\]

(3.24)

The multivector part of (3.20) with even Grassmann grade have the basis \( 1, i_1, i_2, i_3 \) and form a closed subalgebra under the Clifford star product, namely the quaternion algebra. The multivector part of (3.20) with odd grade does not close under the Clifford star product, but nevertheless one can reinvestigate the definition of the Pauli functions in (2.14). Replacing in (2.14) the scalar \( i \) by the pseudoscalar \( I_3 \) one sees that the basis vectors \( \sigma_i \) fulfill

\[
[\sigma_i, \sigma_j] *_c = 2\varepsilon_{ijk} I_3 *_c \sigma_k \quad \text{and} \quad \{\sigma_i, \sigma_j\} *_c = 2\sigma_{ij},
\]

(3.25)

which justifies denoting them \( \sigma_i \). With the pseudoscalar \( I_3 \) the trace (2.18) can be written as \( \text{Tr}(F) = 2 \int d\sigma_3 d\sigma_2 d\sigma_1 + F = 2 \int d\sigma_3 d\sigma_2 d\sigma_1 I_3 *_c F \). So one has here achieved with the Clifford star product a cliffordization of the three dimensional Grassmann algebra of the \( \sigma_i \).

Just as in the two dimensional case one can also consider in three dimensions the role of spinors and rotations. To this purpose one first considers a vector transformation of the form

\[
x \rightarrow x' = -u *_c x *_c u,
\]

(3.26)

where \( u \) is a three dimensional unit vector: \( u = u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3 \) with \( u = |u| = \sqrt{u_1^2 + u_2^2 + u_3^2} = 1 \). This transformation can be identified as a reflection if one decomposes \( x \) into a part collinear to \( u \) and a part orthogonal to \( u \):

\[
x = x_\| + x_\perp = (x \cdot u) u + (x u) *_c u,
\]

(3.27)

with \( x_\| = (x \cdot u) u \) and \( x_\perp = (x u) *_c u = (x u) \cdot u \). One can easily check that

\[
x_\| *_c u = u *_c x_\| \Rightarrow x_\| *_c u \quad \text{and} \quad x_\perp *_c u = -u *_c x_\perp \Rightarrow x_\perp *_c u.
\]

(3.28)

This decomposition of \( x \) can most easily be obtained if one just star-divides \( x *_c u = x \cdot u + x \wedge u \) by \( u \), which gives with \( u^{-1}_c = u \):

\[
x = (x \cdot u) *_c u^{-1}_c + (x u) *_c u^{-1}_c = (x \cdot u) u + (x u) *_c u = x_\| + x_\perp.
\]

(3.29)

Using (3.25) one sees that the transformation (3.26) turns \( x \) into \( x' = -u *_c x *_c u = -x_\| + x_\perp \), so that only the component collinear to \( u \) is inverted, which amounts to a reflection at the plane where \( u \) is the normal vector. Two successive transformations (3.26) lead to:

\[
x \rightarrow x'' = -v *_c x' *_c v = v *_c u *_c x *_c u *_c v = U *_c x *_c U,
\]

(3.30)
where \( U \) can be written as:

\[
U = \mathbf{v} *_{C} \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \wedge \mathbf{u} = \cos \left( \frac{1}{2} |A| \right) + A_0 \sin \left( \frac{1}{2} |A| \right) = e^\frac{1}{2} A_x ,
\]

where the angle between the unit vectors \( \mathbf{u} \) and \( \mathbf{v} \) is described by an bivector \( A = \mathbf{v} \wedge \mathbf{u} = |\mathbf{v}\mathbf{u}| A_0 \). Hereby the unit bivector \( A_0 = \mathbf{v}\mathbf{u}/|\mathbf{v}\mathbf{u}| \) defines the plane in which the angle lies, while the magnitude \( |\mathbf{v}\mathbf{u}| \) gives the angle in radians, furthermore it fulfills \( A_0 *_{C} A_0 = -1 \). If one chooses for example the basis vectors \( \sigma_k \) for \( \mathbf{u} \) and \( \mathbf{v} \), \( A_0 \) would be given by one of the bivectors in (3.21). The additional factor \( 1/2 \) in (3.31) will become clear if one investigates the action of the transformation (3.30). To this purpose one proceeds analogous to the discussion of the reflection (3.26). One first decomposes the vector \( \mathbf{x} \) into a part \( \mathbf{x}_\parallel \) in the plane defined by \( A \) and a part \( \mathbf{x}_\perp \) perpendicular to that plane. This is done analogous to (3.29) by star-dividing \( \mathbf{x} *_{C} A = \mathbf{x} \cdot A + \mathbf{x} \wedge A \) by \( A \) which leads to

\[
\mathbf{x} = (\mathbf{x} \cdot A) *_{C} A^{-1} *_{C} A + (\mathbf{x} A) *_{C} A^{-1} *_{C} A = \mathbf{x}_\parallel + \mathbf{x}_\perp , \tag{3.32}
\]

with \( \mathbf{x}_\parallel *_{C} A = -A *_{C} \mathbf{x}_\parallel \) and \( \mathbf{x}_\perp *_{C} A = A *_{C} \mathbf{x}_\perp \). We then have for the transformation (3.30):

\[
U *_{C} \mathbf{x} *_{C} \overline{U} = e^{-A/2} *_{C} \mathbf{x} *_{C} e^{A/2} *_{C} = \mathbf{x}_\parallel *_{C} \mathbf{x}_\perp \cdot e^A \cdot e_{s_{c}} ,
\]

(3.33)

So the component perpendicular to the plane defined by \( A \) is not changed while the component inside this plane is rotated in that plane with the help of the spinor \( e^A \cdot e_{s_{c}} \), by an angle of magnitude \( |A| \), just as described in the two dimensional case above. One sees here why the rotation in the two dimensional case could be written just by acting with a spinor from the right. This is due to the fact that when the vector lies in the plane of rotation one has

\[
e^{-A/2} *_{C} \mathbf{x} *_{C} e^{A/2} *_{C} = \mathbf{x}_\parallel *_{C} e^A \cdot e_{s_{c}} .
\]

(3.34)

A rotation can be described with the bivector \( A \), but also with the dual vector \( \mathbf{a} \) defined by \( A = I_d *_{C} \mathbf{a} \), where the direction of \( \mathbf{a} \) defines the axis of rotation, while the magnitude gives the radian \( |\mathbf{a}| = |A| \). So \( U \) can also be written as:

\[
U = e^{-\frac{1}{2} I_d *_{C} \mathbf{a}} ,
\]

(3.35)

which corresponds to the star exponential (3.24).

The formalism described so far can easily be generalized to the case of \( d \)-euclidian dimensions. Just as there is a duality inside the space spanned by the \( \sigma_i \), there is also the duality between the spaces spanned by the \( \sigma_i \) and the \( \sigma^j \). This duality is expressed by the relation \( \sigma_i \cdot \sigma^j = \delta_i^j \). The \( \sigma^j \)-vectors can be constructed with the help of the pseudoscalar, which is for the \( d \)-dimensional euclidian case \( I_d = \sigma_1 \sigma_2 \cdots \sigma_d \). The space on which the basis vector \( \sigma_j \) is normal is given for an \( d \)-dimensional euclidian space by the \((-1)^{d-1} \sigma_1 \sigma_2 \cdots \sigma_j \cdots \sigma_d \), where \( \sigma_j \) means that this basis vector is missing. The corresponding dual vector is then given by

\[
\sigma^j = (-1)^{d-1} \sigma_1 \sigma_2 \cdots \sigma_j \cdots \sigma_d *_{C} I_d^{-1} *_{C} ,
\]

(3.36)

where \( I_d^{-1} *_{C} \) is the inverse \( d \)-dimensional pseudoscalar.

Note also that the multiple Clifford star product leads to an expansion of Wick type. For example the Clifford product of four basis vectors is given by

\[
\sigma_{i_1} *_{C} \sigma_{i_2} *_{C} \sigma_{i_3} *_{C} \sigma_{i_4} \quad = \quad \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \sigma_{i_4} + \sigma_{i_1} \sigma_{i_2} \delta_{i_3 i_4} - \sigma_{i_1} \sigma_{i_3} \delta_{i_2 i_4} + \sigma_{i_1} \sigma_{i_4} \delta_{i_2 i_3} + \sigma_{i_2} \sigma_{i_3} \delta_{i_1 i_4} - \sigma_{i_2} \sigma_{i_4} \delta_{i_1 i_3} + \sigma_{i_3} \sigma_{i_4} \delta_{i_1 i_2} + \delta_{i_1 i_2} \delta_{i_3 i_4} - \delta_{i_1 i_3} \delta_{i_2 i_4} + \delta_{i_1 i_4} \delta_{i_2 i_3} ,
\]

(3.37)
where the contraction of $\sigma_i$ and $\sigma_j$ is given by $\delta_{ij}$. This suggests to use the star product formalism also in the realm of quantum field theory [10].

4 Geometric Algebra and Classical Mechanics

It is now straight forward to use the formalism described so far in classical mechanics as it was done in [3]. We will here only give two examples to show where the advantages of geometric algebra lie. Let us first consider the three dimensional harmonic oscillator, which is defined by the differential equation $\ddot{r} = -\frac{k}{m}r$.

We will here only give two examples to show where the advantages of geometric algebra lie. Let us first consider the three dimensional harmonic oscillator, which is defined by the differential equation $\ddot{r} = -\frac{k}{m}r$. The ansatz $q = a * c \ e^{it} \hat{c}$ leads to the equation $\lambda^{2*} + \frac{k}{m} = 0$, which is solved by $\lambda = \pm i \omega_0$ with $\omega_0 = \sqrt{k/m}$. The difference to the conventional formalism is that $\hat{c}$ is here a bivector with $\hat{c}^{2*} = -1$. This gives then the two solutions

$$q_\pm = a_\pm * c \ e^{\pm i \omega_0 t} = a_\pm * c \ (\cos \omega_0 t \pm i \sin \omega_0 t).$$

In the second term appears the expression $a_\pm * c \ 1 = a_\pm \hat{1} + a_\pm \cdot \hat{1}$, which is the sum of a term of Grassmann grade three and a term of Grassmann grade one. But the result $q_\pm$ itself is a quantity of Grassmann grade one, so it follows that $a_\pm \hat{1} = 0$, which is the defining equation of the plane in which the oscillatory movement takes place. This plane is defined by the unit bivector $\hat{1}$ and has to be determined by the initial conditions [3].

As the second example we consider the solution of the Kepler problem by spinors [11]. One uses here the fact that the radial position vector $r = r_1 \sigma_1 + r_2 \sigma_2 + r_3 \sigma_3$ can be written as a rotated and dilated basis vector:

$$r = U * c \ \sigma_1 * c \ \overline{U}.$$ (4.2)

The components $r_i$ of $r$ can then be expressed in terms of the components $u_i$ of $U = u_1 + u_2 \sigma_2 \sigma_3 + u_3 \sigma_3 \sigma_1 + u_4 \sigma_1 \sigma_2$:

$$
\begin{pmatrix}
    r_1 \\
    r_2 \\
    r_3 \\
    0
\end{pmatrix} =
\begin{pmatrix}
    u_1 & u_2 & -u_3 & -u_4 \\
    -u_4 & u_3 & u_2 & -u_1 \\
    u_3 & u_4 & u_1 & u_2 \\
    -u_2 & u_1 & -u_4 & u_3
\end{pmatrix}
\begin{pmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    u_4
\end{pmatrix},
$$ (4.3)

which is the well known Kustaanheimo-Stiefel transformation [12] [13]. Comparing [4,2] and [4,3] leads to the notational correspondence

$$r = U * c \ \sigma_1 * c \ \overline{U} \quad \leftrightarrow \quad \vec{r} = L_{\vec{u}} \, \vec{u},$$ (4.4)

where $\vec{r}$ and $\vec{u}$ are four dimensional space vectors considered as tuples of numbers as in the conventional formalism. One should note here that the KS-transformation increases the degrees of freedom by one, which means that the bivector $U$ in (4.2) is not unique [11]. This gauge freedom can be reduced by imposing an additional constraint on $U$ as will be shown below. Squaring (4.4) leads to the relations

$$U * c \ \overline{U} = |U|^2 = r \quad \leftrightarrow \quad L_{\vec{u}} L_{\vec{u}}^T = \vec{u} \vec{u}^T = r,$$ (4.5)

with $r = |r| = |\vec{r}| = r_1^2 + r_2^2 + r_3^2$. Differentiating (4.4) with respect to $t$ one obtains the KS-transformation for the velocities as

$$\dot{r} = \dot{U} * c \ \sigma_1 * c \ \overline{U} + U * c \ \sigma_1 * c \ \overline{U} \quad \leftrightarrow \quad \dot{\vec{r}} = 2 L_{\vec{u}} \dot{\vec{u}}.$$ (4.6)
One can then choose for the constraint
\[ \dot{U} \ast_C \sigma_1 \ast_C U = U \ast_C \sigma_1 \ast_C U \quad \leftrightarrow \quad r_4 = 0, \quad (4.7) \]
which means that the superfluous fourth component \( r_4 \) stays zero for all times. With this constraint it is possible to invert the geometric algebra relation (4.6) for \( U \). Implementing (4.7) in (4.6) gives
\[ \dot{r} = 2 \dot{U} \ast_C \sigma_1 \ast_C U, \]
which can be solved for \( \dot{U} \), so that the inverse relation to (4.6) is
\[ \dot{U} = \frac{1}{2r} \dot{r} \ast_C U \ast_C \sigma_1 \quad \leftrightarrow \quad \ddot{u} = \frac{1}{2r} L^T \dot{r}. \quad (4.8) \]

By introducing a fictitious time \( s \) which is defined as
\[ \frac{d}{ds} = \frac{d}{dt}, \quad \frac{dt}{ds} = r \quad (4.9) \]
it is then possible to regularize the divergent \( 1/r \)-potential so that (4.8) reads
\[ \frac{d^2U}{ds^2} = \frac{1}{2} \left( \dot{r} \ast_C U \ast_C \sigma_1 + \dot{U} \ast_C \sigma_1 \ast_C U \right) = \frac{1}{2} \frac{U}{U} \left( \dot{r} \ast_C r + \frac{1}{2} \dot{r}^2 \ast_C \right). \quad (4.10) \]
Substituting now the inverse square force
\[ m \ddot{r} = -k \frac{r}{r^3} \quad (4.11) \]
one obtains:
\[ \frac{d^2U}{ds^2} = \frac{1}{2m} U \ast_C \left( \frac{1}{2} m \dot{r}^2 \ast_C - \frac{k}{r} \right) = \frac{E}{2m} U, \quad (4.12) \]
which is the equation of motion for an harmonic oscillator. This equation can be solved in a straightforward fashion and is much easier than the equation for \( r \). The orbit can then be calculated by (4.2).

The Kepler problem can also be treated in the canonical formalism. Therefore one first needs the KS-transformation for the momentum. If \( u = \sum_{n=1}^4 u_n \sigma_n \) is the canonical momentum corresponding to \( u = \sum_{n=1}^4 u_n \sigma_n \) the KS-transformation is given by
\[ p = \frac{1}{4r} \left( W \ast_C \sigma_1 \ast_C U + U \ast_C \sigma_1 \ast_C W \right) \leftrightarrow \quad \ddot{u} = \frac{1}{2r} L^T \dot{u}, \quad (4.13) \]
with \( W = w_1 + w_2 \sigma_2 \sigma_3 + w_3 \sigma_3 \sigma_1 + w_4 \sigma_1 \sigma_2 \). For \( p^{2*} = p_1^2 + p_2^2 + p_3^2 \) one gets with (4.13)
\[ p^{2*} = \frac{1}{4r} |W|^2 - p_4^2, \quad (4.14) \]
where \( |W|^2 = W \ast_C W = w_1^2 + w_2^2 + w_3^2 + w_4^2 \) and
\[ p_4 = \frac{1}{2r} (u_1 w_2 - u_2 w_1 + u_3 w_4 - u_4 w_3). \quad (4.15) \]
Equation (4.13) allows to transform the Hamiltonian into \( u_\perp \) and \( w_\perp \)-coordinates. This is done in several steps \[13\]. Starting from the Hamiltonian \( H = \frac{1}{2m} (p_1^2 + p_2^2 + p_3^2) - \frac{k}{r} \) one first extends the phase space by
a \( q_0 \)- and a \( p_0 \)-coordinate and forms the homogenous Hamiltonian as \( H_1 = H + p_0 \). This leads for the zero component to two additional Hamilton equations

\[
\frac{dq_0}{dt} = \frac{\partial H_1}{\partial p_0} = 1 \quad \text{and} \quad \frac{dp_0}{dt} = -\frac{\partial H_1}{\partial q_0} = -\frac{\partial p_0}{\partial t},
\]

which shows that \( q_0 \) corresponds to the time \( t \) and \( p_0 \) is a constant and corresponds to the negative energy of the system, so that \( H_3 = H + p_0 = 0 \) for a conservative force. Since the time is now a coordinate the development of the system has to be described with a different parameter. This development parameter is the fictitious time \( s \) that is connected to the time by (4.9). The relation (4.9) can be implemented if one chooses \( H_2 = r H_1 \). The Hamilton equations that describe then the development according to \( s \) are differential equations with respect to \( s \):

\[
\frac{dq_i}{ds} = \frac{\partial H_2}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{ds} = -\frac{\partial H_2}{\partial q_i} \quad \text{for} \quad i = 0, 1, 2, 3.
\]

Especially for the zero component one gets

\[
\frac{dq_0}{ds} = \frac{dt}{ds} = \frac{\partial H_2}{\partial p_0} = r \quad \text{which corresponds to (4.9).}
\]

After having so far regularized the Hamiltonian one can then go over to KS-coordinates and obtains with (4.14)

\[
H_3 = \frac{1}{8m} (w_1^2 + w_2^2 + w_3^2 + w_4^2) - \frac{1}{2m} p_4^2 - k - E r.
\]

Imposing now the constraint \( p_4 = 0 \), which for \( w_i = m \dot{u}_i \) in (4.15) is just (4.7), and considering bound states with \( E < 0 \) the Hamiltonian is given by

\[
H_4 = \frac{1}{8m} (w_1^2 + w_2^2 + w_3^2 + w_4^2) + |E| (u_1^2 + u_2^2 + u_3^2 + u_4^2) - k,
\]

which describes a four dimensional harmonic oscillator with fixed energy and frequency \( \omega = (|E|/2m)^{1/2} \).

The formalism of geometric algebra can also be applied to hamiltonian mechanics [14]. The 2\( d \)-dimensional phase space is then spanned by \( d \) basis elements \( \{\eta_i\} \) for the space coordinates and \( d \) basis elements \( \{\rho_i\} \) for the momentum coordinates so that a point in phase space is described by the vector \( x = \sum_{n=1}^{d} (q_n \eta_n + p_n \rho_n) \) and the Clifford star product on the phase space has the from

\[
F \ast_c G = F \exp \left[ \sum_{n=1}^{d} \left( \frac{\partial}{\partial q_n} \frac{\partial}{\partial \eta_n} + \frac{\partial}{\partial p_n} \frac{\partial}{\partial \rho_n} \right) \right] G,
\]

so that \( \eta_m \cdot \eta_n = \rho_m \cdot \rho_n = \delta_{mn} \) and \( \eta_m \cdot \rho_n = 0 \). The two \( d \)-dimensional subspaces are related by a bivector \( j \), which is the generalization of the imaginary structure in two dimensions and is defined as:

\[
j = \sum_{n=1}^{d} \eta_n \rho_n.
\]

This bivector plays the role of the symplectic form that relates the space and momentum part of the phase space according to

\[
\eta_n \cdot j = \rho_n \quad \text{and} \quad j \cdot \rho_n = \eta_n.
\]
In phase space one has then two possibilities to assign a scalar to two phase space vectors \( \mathbf{a} \) and \( \mathbf{b} \), apart from the scalar product \( \mathbf{a} \cdot \mathbf{b} \) one can also form the expression

\[
\mathbf{a} \cdot (\mathbf{j} \cdot \mathbf{b}) = -\mathbf{j} \cdot (\mathbf{a} \mathbf{b}) \equiv \mathbf{a} \cdot \tilde{\mathbf{b}}.
\] (4.23)

With the gradient operator

\[
\nabla_x = \sum_{n=1}^{d} \left( \eta_n \frac{\partial}{\partial q_n} + \rho_n \frac{\partial}{\partial p_n} \right)
\] (4.24)

the Hamilton equation can for example be written as:

\[
\dot{x} = \mathbf{j} \cdot (\nabla_x H) = \tilde{\nabla} x H,
\] (4.25)

or explicitly:

\[
\sum_{n=1}^{d} (\dot{q}_n \eta_n + \dot{p}_n \rho_n) = \mathbf{j} \cdot \sum_{n=1}^{d} \left( \eta_n \frac{\partial H}{\partial q_n} + \rho_n \frac{\partial H}{\partial p_n} \right) = \sum_{n=1}^{d} \left( -\rho_n \frac{\partial H}{\partial q_n} + \eta_n \frac{\partial H}{\partial p_n} \right).
\] (4.26)

With (4.25) one gets for the time derivation of a scalar phase space function \( f(x) \):

\[
\dot{f} = \dot{x} \cdot (\nabla_x f) = \sum_{n=1}^{d} \left( \frac{\partial f}{\partial q_n} \frac{\partial H}{\partial p_n} - \frac{\partial f}{\partial q_n} \frac{\partial H}{\partial p_n} \right) = \{f, H\}_{PB}.
\] (4.27)

The Poisson bracket can be written in a compact way as:

\[
\{f, g\}_{PB} = f \left( -\mathbf{j} \cdot (\tilde{\nabla}_x \tilde{\nabla}_x) \right) g = f \left( \tilde{\nabla}_x \cdot \tilde{\nabla}_x \right) g.
\] (4.28)

## 5 Nonrelativistic Quantum Mechanics

The above discussed transformation of the Kepler problem can now be used to calculate the energy levels of the hydrogen atom as it was described in [15]. To this purpose one introduces holomorphic coordinates

\[
a_n = \frac{1}{\sqrt{2}} \left( \sqrt{4m\omega} w_n + i \frac{1}{\sqrt{4m\omega}} w_n \right)
\] (5.1)

so that the Hamiltonian \( H_4 \) in (4.19) can be written as:

\[
H_4 = \omega \left( \sum_{n=1}^{4} a_n \bar{a}_n \right) - e^2,
\] (5.2)

where \( k = e^2 \). Introducing then holomorphic coordinates for left and right moving quanta

\[
a_{R_{12}} = \frac{1}{\sqrt{2}} (a_1 - ia_2), \quad a_{L_{12}} = \frac{1}{\sqrt{2}} (a_1 + ia_2) \quad \text{and} \quad a_{R_{34}} = \frac{1}{\sqrt{2}} (a_3 - ia_4), \quad a_{L_{34}} = \frac{1}{\sqrt{2}} (a_3 + ia_4)
\] (5.3)
the Hamiltonian (5.2) turns into
\[ H_4 = \omega \left( a_{R12} \bar{a}_{R12} + a_{L12} \bar{a}_{L12} + a_{R34} \bar{a}_{R34} + a_{L34} \bar{a}_{L34} \right) - e^2. \] (5.4)

One can now quantize this system with the Moyal product. The four dimensional Moyal star product transforms under KS-transformation and the above transformations into
\[ \ast_M = \exp \left[ \frac{i \hbar}{2} \left( \tilde{\partial}_{u_n} \tilde{\partial}_{w_n} - \tilde{\partial}_{w_n} \tilde{\partial}_{u_n} \right) \right] = \exp \left[ \frac{\hbar}{2} \sum_{X=R12,L12,R34,L34} \left( \tilde{\partial}_{a_X} \tilde{\partial}_{\bar{a}_X} - \tilde{\partial}_{\bar{a}_X} \tilde{\partial}_{a_X} \right) \right]. \] (5.5)

The energy levels can then be obtained by the \( \ast \)-genvalue equation
\[ H_4 \ast_M \pi^{(M)}_{n1n2n3n4} = 0, \] (5.6)
where \( \pi^{(M)}_{n1n2n3n4} \) is the product of four Wigner functions of the one dimensional harmonic oscillator given in (2.6). Eq. (5.6) gives then
\[ e^2 = \hbar \omega \left( n_{R12} + n_{L12} + n_{R34} + n_{L34} + 2 \right). \] (5.7)

To get the energy levels of the hydrogen atom one has to impose the constraint
\[ p_4 = a_{R12} \bar{a}_{R12} - a_{L12} \bar{a}_{L12} + a_{R34} \bar{a}_{R34} - a_{L34} \bar{a}_{L34} = 0, \] (5.8)
which for the energy levels corresponds to \( n_{R12} - n_{L12} + n_{R34} - n_{L34} = 0 \) or \( n_{R12} + n_{R34} = n_{L12} + n_{L34} \equiv n - 1 \).

Putting this and \( \omega = \sqrt{|E|/2m} \) into (5.7) one gets the well known energy levels of the hydrogen atom
\[ E_n = -e^4 \frac{m}{2 \hbar} \frac{1}{n^2}. \] (5.9)

Geometric algebra in a fermionic star product formalism is not only useful for calculating the energy levels of the hydrogen atom, it can also be combined straightforwardly with the bosonic star product formalism of quantum mechanics. In classical mechanics described with geometric algebra and the Clifford star product the fermionic part of the underlying superanalysis was deformed and the basis vectors played only a mathematical role by generating the structures of vector analysis. Going over to quantum mechanics means that also the scalar coefficients of superanalysis have to be multiplied by a deformed product, namely the bosonic Moyal star product. This leads then to a deformed version of geometric algebra and describing geometric algebra in terms of star products allows to combine the Clifford star product and the Moyal product into one star product, which should be called Moyal-Clifford product. The Clifford product on the phase space that described the structures of classical Hamilton mechanics was given by (4.20). In quantum mechanics one needs now a product with which general multivector functions on the phase space are multiplied. These multivector functions are the observables of the theory and as such can only be multivectors in the space basis vectors \( \sigma_r \). So one has to go over from the Clifford product (4.20) to the Clifford product (3.14), which can be done by implementing constraints that identify the corresponding basis vectors \( \sigma_r \). The Moyal-Clifford product for a single particle system is then
\[ F \ast_{MC} G = F \exp \left[ \sum_{n=1}^{d} \left( \frac{i \hbar}{2} \left( \tilde{\partial}_{q_n} \tilde{\partial}_{p_n} - \tilde{\partial}_{p_n} \tilde{\partial}_{q_n} \right) + \tilde{\partial}_{\sigma_n} \tilde{\partial}_{\bar{\sigma}_n} \right) \right] G. \] (5.10)
To see the consequences of the additional Moyal deformation in geometric algebra one can for example consider the Moyal-Clifford product of two vectors in \(d = 2\) dimensions. The generalization of (3.15) can be written as

\[
a \ast_{MC} b = (a_1 \ast_M b_2 - a_2 \ast_M b_1)\sigma_1 \sigma_2 + a_1 \ast_M b_1 + a_2 \ast_M b_2.
\]  

(5.11)

Under the Moyal product the coefficients in general do not commute if they are functions of \(q_n\) and \(p_n\). This means that the Moyal-Clifford product of the same vectors \(a \ast_{MC} a\) is in general not a scalar, but has also a bivector part. It is this additional bivector part, which appears only for \(d = 2\) dimensions. There is only one candidate for the imaginary structure, namely the symplectic volume form \(\eta\).

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\[
H = \frac{1}{2m} \left[ (p_1 + eA_1) \sigma_1 + (p_2 + eA_2) \sigma_2 + (p_3 + eA_3) \sigma_3 \right]^{2*MC}_{2*MC}
\]

(5.12)

The first three terms \(H_0 = \frac{1}{2m} \sum_{n=1}^{3} (p_n + eA_n)^{2*MC}\) describe the Landau problem of a charged particle in a magnetic field which can be solved in the star product formalism as described in [16] or [7]. The other three terms that describe the interaction of the spin and the magnetic field appear only because of the Moyal product. If the magnetic field points in \(\sigma_3\)-direction the vector potential is given by \(A = -\frac{B}{2} q_2 \sigma_1 + \frac{B}{2} q_1 \sigma_2\) and only the first Moyal-commutator contributes:

\[
H_S = \frac{1}{2m} \left[ (p_1 + eA_1), (p_2 + eA_2) \right]_{*M} \sigma_1 \sigma_2 = \frac{\hbar \omega}{2} \sigma^3
\]

(5.14)

where \(\omega = \frac{2B}{m}\) and \(\sigma^3 = -i \sigma_1 \sigma_2\) is a real quaternion, which is constructed according to (5.10) and (5.11). The difference between this calculation and the conventional approach is that in the conventional formalism the Clifford structure is introduced by putting in Pauli matrices by hand in (5.12). The Pauli matrices describe the spin and lead analogously to the additional term \(H_S\), this approach is also known as the Feynman trick [17]. In geometric algebra the Clifford structures do not have to be added, they are just the basis vectors that already exist in classical mechanics, but become apparent as physical objects in the quantum case. It is then straightforward to calculate the \(\ast\)-eigenfunctions of \(H_S\) which turn out to be the spin Wigner functions described in the first section [7].

One should note that the Moyal-Clifford product is a product for functions on the phase space, which play the role of observables. As seen above these observables are in general multivectors, where the terms of higher grade are described by the space basis vectors \(\sigma_n\) and not by the phase space basis vectors \(\eta_n\) and \(\rho_n\), because the latter should not be observable quantities. Nevertheless the basis vectors of phase space can be considered to play an indirect role in the expression (2.11) of the Moyal product, because the imaginary structure \(i = \sqrt{-1}\) can be interpreted as a two blade on phase space. If the phase space is only two dimensional there is only one candidate for the imaginary structure, namely the symplectic volume form \(j = \eta p\). That the \(i\) in the Moyal product has to be an area bivector can be seen from the integral representation of the Moyal product [18]:

\[
(f \ast_M g)(q, p) = \frac{1}{\pi \hbar^2} \int dq' dq'' dp' dp'' f(q', p') g(q'', p'') \exp \left( \frac{2i}{\hbar} A_\Delta(\vec{x}, \vec{x}', \vec{x}'') \right)
\]

(5.15)
where \( A_\Delta(\vec{x}, \vec{x}', \vec{x}'') \) is the area of the triangle spanned by the vectors \( \vec{x} = (q, p)^T, \vec{x}' = (q', p')^T \) and \( \vec{x}'' = (q'', p'')^T \). So the \( i \) plays here the role of the unit area bivector in phase space. The two dimensional Moyal product can then be written with the gradient \( \nabla_x = \eta \partial_q + \rho \partial_p \) as:

\[
f \ast_M g = \int e^{\frac{K}{\hbar} (\vec{v}_x \cdot \vec{v}_x)} g = \int e^{\frac{K}{\hbar} (\partial_q e - \partial_p e)} g,
\]

so that the correspondence principle has the form

\[
\lim_{\hbar \to 0} \frac{-1}{\hbar} j_c [f,g]_{PB} = \{f,g\}_{PB}.
\]

One should also note the similarity to the fermionic star product of two vectors \( a = a_1 \eta + a_2 \rho \) and \( b = b_1 \eta + b_2 \rho \):

\[
a \ast_c b = |a||b| e^{2\eta \rho A_\Delta(a,b)},
\]

where \( A_\Delta(a,b) \) is the volume of the triangle spanned by the vectors \( a \) and \( b \).

### 6 Spacetime algebra and Dirac theory

Just as it is possible to describe geometric algebra as a fermionic deformed superanalysis it is also possible to describe spacetime algebra in this context. The basis vectors of space-time are the Grassmann elements \( \gamma_0, \gamma_1, \gamma_2, \gamma_3 \), which fulfill

\[
\gamma_\mu \cdot \gamma_\nu = \frac{1}{2} (\gamma_\mu \ast_c \gamma_\nu + \gamma_\nu \ast_c \gamma_\mu) = g_{\mu\nu},
\]

where we choose here \( g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \). The corresponding Clifford star product in space-time is

\[
F \ast_c G = F \exp \left[ g_{\mu\nu} \frac{\bar{\partial}}{\partial_{\gamma_\mu}} \frac{\partial}{\partial_{\gamma_\nu}} \right] G.
\]

A general supernumber in space-time has the form

\[
A = a_0 + a^\mu \gamma_\mu + a^{\mu\nu} \gamma_\mu \gamma_\nu + a^{\mu\nu\rho} \gamma_\mu \gamma_\nu \gamma_\rho + a_4 I_4,
\]

where \( I_4 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \) and only linear independent terms should appear. With the four dimensional pseudoscalar \( I_4 \) and the Clifford star product \( \ast_c \) it is possible to construct analogous to (3.36) the dual basis \( \gamma^\mu \), which gives \( \gamma^0 = \gamma_0 \) and \( \gamma^i = -\gamma_i \). Furthermore one can define in analogy to the three dimensional case a trace:

\[
\text{Tr}(F) = 4 \int d\gamma_0 d\gamma_2 d\gamma_1 d\gamma_0 \ast F.
\]

The Berezin integral acts hereby again like a projector on the scalar part of \( F \). The definition of the trace by projecting on the scalar part was already given in [19] and it was also stated that the use of geometric algebra greatly simplifies all the trace calculations usually done in the matrix formalism. An explicit expression for the trace can now in the formalism of deformed superanalysis be given by the Berezin integral.
The question is now how a spacetime vector \( x = x^\mu \gamma_\mu \) is related to its space vector part \( x = x^i \gamma_i \). In the \( \gamma_0 \)-system this can be seen by a space-time split which amounts to star-multiplying by \( \gamma_0 \):

\[
x *_C \gamma_0 = x \cdot \gamma_0 + x \gamma_0 = t + x.
\]

(6.5)

One should note that \( x = x \gamma_0 = x^1 \gamma_1 \gamma_0 + x^2 \gamma_2 \gamma_0 + x^3 \gamma_3 \gamma_0 \) is a spacetime bivector, but on the other hand it is also a space vector because the two-blades \( \gamma_i \gamma_0 \) behave like \( \gamma_i \):

\[
\sigma_i \cdot \sigma_j = \frac{1}{2} (\gamma_i \gamma_0 *_C \gamma_j \gamma_0 + \gamma_j \gamma_0 *_C \gamma_i \gamma_0) = \delta_{ij},
\]

\[
I_3 = \sigma_1 *_C \sigma_2 *_C \sigma_3 = \gamma_1 \gamma_0 *_C \gamma_2 \gamma_0 *_C \gamma_3 \gamma_0 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = I_4,
\]

\[
\sigma_i \sigma_j = \frac{1}{2} (\gamma_i \gamma_0 *_C \gamma_j \gamma_0 - \gamma_j \gamma_0 *_C \gamma_i \gamma_0)
\]

\[
= \gamma_j \gamma_i = I_4 *_C \gamma_k \gamma_0 = I_3 *_C \sigma_k \quad \text{for cyclic } i, j, k.
\]

(6.6)

Where the four dimensional star product (6.2) and the three dimensional star product (6.1) is used in (6.6) should be clear from the context. The square of the position four vector is \( x^{2*} = t^2 - x^{2*} \).

If a particle is moving in the \( \gamma_0 \)-system along \( x(\tau) \), where \( \tau \) is the proper time, the proper velocity is given by \( u(\tau) = \frac{dx}{d\tau} x(\tau) \), with \( u^{2*} = 1 \). For the space-time split of the proper velocity one obtains:

\[
u *_C \gamma_0 = u \cdot \gamma_0 + u \gamma_0 = \frac{d}{d\tau} (x(\tau) *_C \gamma_0) = \frac{d}{d\tau} (t + x) = \frac{dt}{d\tau} + \frac{dx}{dt} \frac{dt}{d\tau}.
\]

(6.7)

Comparing the scalar and the bivector part leads to

\[
u_0 = u \cdot \gamma_0 = \frac{dt}{d\tau} \quad \text{and} \quad u = \frac{dx}{dt} = \frac{dx}{d\tau} \frac{d\tau}{dt} = \frac{u \gamma_0}{u \cdot \gamma_0}
\]

(6.8)

and with \( 1 = u^{2*} = u_0^2 (1 - u^{2*} \) one gets

\[
u_0 = u \cdot \gamma_0 = \frac{1}{\sqrt{1 - u^{2*}}} = \gamma.
\]

(6.9)

It is now also possible to specify a Lorentz transformation from a coordinate system \( \gamma_\mu \) to an in \( \gamma_1 \)-direction moving coordinate system \( \gamma'_\mu \). For the coefficients this transformation is given by \( t = \gamma(t' + \beta x^1) \), \( x^1 = \gamma(x'^1 + \beta t') \), \( x^2 = x'^2 \), and \( x^3 = x'^3 \). The condition \( x = x^\mu \gamma_\mu = x'^\mu \gamma'_\mu \) leads then to

\[
\gamma' = \gamma(\gamma_0 + \beta \gamma_1) \quad \text{and} \quad \gamma'_1 = \gamma(\gamma_1 + \beta \gamma_0).
\]

(6.10)

Introducing the angle \( \alpha \) so that \( \beta = \tanh(\alpha) \) this can be written as

\[
\gamma'_0 = \cosh(\alpha) \gamma_0 + \sinh(\alpha) \gamma_3 = e^{2 \gamma_3 \gamma_0} *_C \gamma_0 \quad \text{and} \quad \gamma'_1 = \cosh(\alpha) \gamma_1 + \sinh(\alpha) \gamma_0 = e^{\gamma_3 \gamma_0} *_C \gamma_1
\]

(6.11)

or with \( L_1 = e^{\gamma_3 \gamma_0} \) as \( \gamma'_\mu = L_1 *_C \gamma_\mu *_C \gamma_1 \). In general the generators of a passiv Lorentz transformation can be calculated with

\[
\sigma_{\mu \nu} = \frac{I_4}{2} *_C \left[ \gamma_\mu, \gamma_\nu \right] *_C,
\]

(6.12)
so that the generators for the boosts and the rotations are
\[ K_i = \frac{1}{2} \sigma_0 i \quad \text{and} \quad S_i = \frac{1}{2} \sum_{j<k} \varepsilon_{ijk} \sigma_{jk}. \] (6.13)

These generators satisfy
\[ [S_i, S_j] = I_4 \varepsilon_{ijk} S_k, \quad [S_i, K_j] = I_4 \varepsilon_{ijk} K_k, \quad \text{and} \quad [K_i, K_j] = -I_4 \varepsilon_{ijk} S_k \] (6.14)
and a passive Lorentz transformation is given by
\[ \gamma'_{\mu} = \Lambda_{\nu}^\mu \gamma_{\nu} = e_1 4 I_4 * C \sigma_{\mu \nu \omega} \nu \omega * C \gamma_{\mu} * C e_1 I_4 * C \sigma_{\mu \nu \omega} \nu \omega * C, \] (6.15)
which is a generalization of (6.11).

The Dirac equation can then be written down immediately as \[ 7 \]
\[ (p \mp m) \ast_{MC} \pi^{(MC)}_{\pm m} (p) = 0, \] (6.16)
where no slash notation is needed, because one naturally has \( p = p^\mu \gamma_\mu \). The Wigner function \( \pi^{(MC)}_{\pm m} (p) \) for the Dirac equation is the functional analog of the well known energy projector of Dirac theory:
\[ \pi^{(MC)}_{\pm m} (p) = \frac{\pm p + m}{2m}. \] (6.17)

Besides the energy one also has the spin as an observable, which is here given by
\[ S_s = \frac{\hbar}{2} \gamma_5 * C s, \] (6.18)
where \( s = s^\mu \gamma_\mu \) is a vector which fulfills \( s^2 * C = -1 \) and \( s \cdot p = 0 \). \( \gamma_5 \) is here \( \gamma_5 = i I_4 \). With \( S_s * C S_s = \left( \frac{\hbar}{2} \right)^2 \) and \( [S_s, p \mp m] * C = 0 \) one sees that the spin Wigner function is given by the functional analog of the spin projector in Dirac theory
\[ \pi^{(C)}_{\pm s} (s) = \frac{1}{2} \pm \frac{1}{\hbar} S_s \] (6.19)
and fulfills \( S_s * C \pi^{(C)}_{\pm s} (s) = \pm \frac{\hbar}{2} \pi^{(C)}_{\pm s} (s) \). The total Wigner function is then the Clifford star product of the two single Wigner functions.

7 Conclusions

There are two formal and conceptual barriers that separate quantum theory from classical theory. The first barrier is that classical theory is described on the phase space while quantum theory is described on the Hilbert space. This conceptual barrier is overcome by the program of deformation quantization that describes quantum theory on the phase space. The second barrier is that one uses in classical mechanics the Gibbs-Heaviside formalism, which can not take spin into account. In quantum theory where spin is a physical observable it is described in the nonrelativistic case by the Feynman trick, which substitutes \( \vec{p} \) by \( \vec{p} \cdot \vec{\sigma} \) and in the relativistic case it is introduced by writing \( p_\mu \gamma^\mu \). Both notations clearly indicate that the \( \sigma_i \) and
the $\gamma_\mu$ are basis vectors, but this is obscured by representing them by matrices. The work of Hestenes has clarified this point by formulating classical and quantum theory in the same formalism of geometric algebra. The astonishing thing is now that also this second barrier can be overcome in terms of the star product formalism, so that classical and quantum theory can be unified on a formal level. Both can be described by the formalism of deformed superanalysis, where classical mechanics is a half deformed formalism, that means the deformation only takes place in the Grassmann sector of superanalysis, while quantum mechanics leads to a totally deformed formalism, where also the product of the scalar coefficients is deformed. This shows at least on a formal level in which way quantum theory is a more fundamental theory compared to the classical theory.

The star product formalism has also advantages in the context of geometric calculus, because it gives an explicit expression for the geometric product. Geometric algebra in the way Hestenes constructed it, is formulated with respect to the scalar and the wedge product, which represent the lowest and the highest order terms of the geometric product. All other terms of the geometric product are then formulated with the help of these two products. This approach is very practical, especially if one has only terms that are at most bivectors. But in the general case the highest and the lowest terms of an expansion have on a formal level the same status as all other terms. The star product gives now all these terms of different grade as terms of an expansion, that can be calculated in a straightforward fashion.

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