Multivariate Concentration Inequalities with Size Biased Couplings

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Abstract

Let $W = (W_1, W_2, ..., W_k)$ be a random vector with nonnegative coordinates having nonzero and finite variances. We prove concentration inequalities for $W$ using size biased couplings that generalize the previous univariate results. Two applications on local dependence and counting patterns are provided.

1 Introduction

The purpose of this paper is to show how size biased couplings can be used to obtain multivariate concentration inequalities in dependent settings. For a given random vector $W = (W_1, W_2, ..., W_k)$ with nonnegative coordinates having finite, nonzero expectations $\mu_i = \mathbb{E}[W_i]$, another vector $W^i = (W_1^i, W_2^i, ..., W_k^i)$ is said to have $W$ size biased distribution in direction $i$ if

$$\mathbb{E}[W_if(W)] = \mu_i\mathbb{E}[f(W^i)] \quad (1.1)$$

for all functions for which these expectations exist. For a univariate nonnegative random variable $W$ with mean $\mu = \mathbb{E}[W] \in (0, \infty)$, this simplifies to $\mathbb{E}[Wf(W)] = \mu\mathbb{E}[f(W^*)]$ and we say that $W^*$ has $W$ size biased distribution. We refer [2] and [5] for two excellent expository papers on several aspects of size biasing.

In a broad sense, concentration inequalities quantify the fact that a function of a large number of random variables, with certain smoothness conditions, tends to concentrate its values in a relatively narrow range. There is a tremendous literature on inequalities for functions of independent random variables due to their importance in several fields. See, for example [7] and [13] for wonderful surveys, [4] and [12] for book length treatments of the subject. Our approach here will be on the use of couplings from Stein’s method which is a technique introduced by Charles Stein in [14] that is used for obtaining error bounds in distributional approximations. The strength of the method comes from the fact that it can be also used for functions of dependent random variables and various coupling constructions are used for such problems. Sourav Chatterjee in [6] used one important coupling from Stein’s method, exchangeable pairs, to show the concentration of several interesting statistics. Later, in [8], Subhankar Ghosh and Larry Goldstein were able to obtain similar bounds with size biased couplings. Our results in Section 2 will provide multivariate analogues of Ghosh and Goldstein’s results and will also yield a partial improvement in their lower tail inequality.
The paper is organized as follows. In Section 2, we state the multivariate concentration bound with size biased couplings, consider its univariate corollary and discuss briefly the construction of size biased couplings. Proofs of the results are given in Section 3 and we provide two applications, one on local dependence and the other one on counting patterns in random permutations, in Section 4.

2 Main result

We start by fixing some notations. Throughout this paper, for two vectors \( x, y \in \mathbb{R}^k \), we will write

\[
\begin{array}{c}
\mathbf{x} / \mathbf{y} = (x_1/y_1, x_2/y_2, ..., x_k/y_k)
\end{array}
\]

for convenience. Also, we define the partial ordering \( \succeq \) on \( \mathbb{R}^k \) by

\[
\mathbf{x} \succeq \mathbf{y} \iff x_i \geq y_i, \quad \text{for } i = 1, 2, ..., k.
\]

Accordingly, the order \( \preceq \) is defined by \( \mathbf{x} \preceq \mathbf{y} \iff \mathbf{y} \succeq \mathbf{x} \), and the definitions for \( \prec \) and \( \succ \) are similar. Finally, for \( \theta \in \mathbb{R}^k \), \( \theta^t \) will stand for the transpose of \( \theta \) and \( \| \theta \|_2 \) is the \( l^2 \) norm of \( \theta \). Now, we are ready to state our main result.

Theorem 2.1. Let \( W = (W_1, W_2, ..., W_k) \) be a random vector where \( W_i \) is nonnegative with mean \( \mu_i > 0 \) and variance \( \sigma_i^2 \in (0, \infty) \) for each \( i = 1, 2, ..., k \), and suppose that the moment generating function of \( W \) exists everywhere. Assuming that we can find couplings \( \{W^i\}_{i=1}^k \) of \( W \), with \( W^i \) having \( W \) size biased distribution in direction \( i \) and satisfying \( \|W^i - W\|_2 \leq K \) for some constant \( K > 0 \), we have

\[
\begin{align*}
\mathbb{P}
\left( \frac{W - \mu}{\sigma} \succeq -t \right) & \leq \exp \left( -\frac{\|t\|_2^2}{2K_1} \right) \quad (2.1)
\end{align*}
\]

and

\[
\begin{align*}
\mathbb{P}
\left( \frac{W - \mu}{\sigma} \preceq t \right) & \leq \exp \left( -\frac{\|t\|_2^2}{2(2K_1 + K_2\|t\|_2)} \right) \quad (2.2)
\end{align*}
\]

for any \( t \succeq 0 \) where \( K_1 = \frac{2K}{\sigma_1(1)} \|\mu\|_2 \), \( K_2 = \frac{K}{2\sigma(1)} \) with \( \sigma(1) = \min_{i=1,2,...,k} \sigma_i \), \( \mu = (\mu_1, \mu_2, ..., \mu_k) \) and \( \sigma = (\sigma_1, \sigma_2, ..., \sigma_k) \).

Proof of Theorem 2.1 will be given in Section 3. Here we note that the assumption on moment generating function (mgf) can be relaxed to \( \mathbb{E}[e^{\theta^t W}] < \infty \) for \( \|\theta\|_2 \leq 2/K \), as can be checked easily from the proof. As a more general remark, Arratia and Baxendale showed recently for the univariate case that the existence of a bounded coupling for \( W \) assures the existence of the mgf everywhere. See [1] for details. Although a similar result can be given in a multivariate setting, we skip this for now as in applications the underlying random variables are almost always finite (so that mgf exists everywhere).

Noting that the \( k = 1 \) case in Theorem 2.1 reduces to standard size biasing and replacing \( t \) by \( t/\sigma \), we arrive at the following univariate corollary.
Corollary 2.2. Let $W$ be a nonnegative random variable with finite and nonzero mean, and assume that the moment generating function of $W$ exists everywhere. If there exists a size biased coupling $W^s$ of $W$ satisfying $|W^s - W| \leq K$ for some $K > 0$, then for any $t \geq 0$, we have

$$
\mathbb{P}(W - \mu \leq -t) \leq \exp\left(-\frac{t^2}{4K\mu}\right) \quad \text{and} \quad \mathbb{P}(W - \mu \geq t) \leq \exp\left(-\frac{t^2}{4K\mu + Kt}\right). \quad (2.3)
$$

Remark 2.3. For the one dimensional case, the lower tail inequality in (2.3) improves Ghosh and Goldstein's corresponding result (namely, inequality (1) in [8]) by removing the monotonicity condition. However, in both tails the constants are slightly worse than the ones in their theorem, but this is not too surprising as our main result is proven for a multivariate version. We note that this monotonicity condition is also discussed in two recent papers, [1] and [3], where they prove that it is indeed possible to remove the monotonicity condition while keeping the bound exactly the same as in [8].

Remark 2.4. For the upper tail in univariate case, there has been a recent improvement in [1] where the authors show that it is indeed possible to obtain a tail behavior of order $\exp(-ct \log t)$ under bounded size biased coupling assumption. In particular, this result reveals the upper tail inequality given in [8] as a corollary. However, we were not able to obtain a similar bound for the multivariate case yet, and this will be one direction to follow in a subsequent work.

In the rest of this section we briefly review the discussion in [9] which gives a procedure to size bias a collection of nonnegative random variables in a given direction. More on construction of size biased couplings can be found in [11]. Now, as mentioned in the introduction, for a random vector $W = (W_1, W_2, \ldots, W_k)$ with nonnegative coordinates, a random variable $W^i$ is said to have $W$ size bias distribution in direction $i$ if $\mathbb{E}[W^i f(W)] = \mu_i \mathbb{E}[f(W^i)]$ for all functions for which these expectations exist. It is well known that the definition just given is equivalent to the following one.

Definition 2.5. Let $W = (W_1, W_2, \ldots, W_k)$ be a random vector where $W_i$'s have finite, nonzero expectations $\mu_j = \mathbb{E}[W_j]$ and joint distribution $dF(x)$. For $i \in \{1, 2, \ldots, k\}$, we say that $W^i = (W^i_1, W^i_2, \ldots, W^i_k)$ has the $W$ size bias distribution in direction $i$ if $W^i$ has joint distribution

$$
dF^i(x) = \frac{x_i dF(x)}{\mu_i}. \quad (2.4)
$$

Note that in univariate case, (2.4) reduces to $dF^*(x) = x dF(x)/\mu$ which explains the name, size biased distribution. Also this latter definition gives insight for a way to construct size biased random variables. Following [9], by the factorization of $dF(x)$, we have

$$
dF^i(x) = \frac{x_i dF(x)}{\mu_i} = \mathbb{P}(W \in dx | W_i = x) \frac{x_i \mathbb{P}(W^i_i \in dx)}{\mu_i} = \mathbb{P}(W \in dx | W_i = x) \mathbb{P}(W^i_i \in dx).
$$

where $W^i_i$ has $W_i$ size biased distribution. Hence, to generate $W^i$ with distribution $dF^i$, first generate a variable $W^i_i$ with $W_i$ size bias distribution. Then, when $W^i_i = x$, we generate the remaining variables according to their original conditional distribution given that $i^{th}$ coordinate takes on the value $x$. 
As an example, the construction just described combined with Theorem 2.1 can be used to prove concentration bounds for random vectors with independent coordinates. To see this in the simplest possible case, let \( W = (W_1, ..., W_k) \) be a random vector where \( W_i \)'s are nonnegative, independent and identically distributed random variables with \( W_i \leq K \) a.s. for some \( K > 0 \), and assume that \( 0 < \sigma^2 = \text{Var}(W_1) < \infty \). To obtain \( W_i \), we let \( W_i \) be on the same space with \( W_i \) size biased distribution and also set \( W_j = W_i \) for \( j \neq i \). Since coordinates of \( W \) are independent, \( W^i = (W_1^i, W_2^i, ..., W_k^i) \) has \( W \) size biased distribution in direction \( i \). Also noting that \( W_i \leq K \) as support of \( W_i \) is a subset of the support of \( W_i \), we obtain \( \|W_i - W\|_2 \leq K \) a.s. and using Theorem 2.1, one can conclude that the lower tail inequality

\[
P\left( \frac{W - \mu}{\sigma} \leq -t \right) \leq \exp \left( -\frac{\sigma^2 \|t\|^2}{4K\sqrt{k}\mu} \right)
\]

and the upper tail inequality

\[
P\left( \frac{W - \mu}{\sigma} \geq t \right) \leq \exp \left( -\frac{\|t\|^2}{4K\sqrt{k}\mu/\sigma^2 + K\|t\|_2/\sigma} \right)
\]

hold for all \( t \geq 0 \).

3 Proofs

Before we begin the proofs, we note the following inequality

\[
|e^y - e^x| \leq |y - x| \left( \frac{e^y + e^x}{2} \right)
\]

which follows from the following observation

\[
e^y - e^x = \int_0^1 e^{y+(1-t)x} dt \leq \int_0^1 (te^y + (1-t)e^x) dt = \frac{e^y + e^x}{2} \quad \text{for all } x \neq y.
\]

**Proof of Theorem 2.1.** We first prove the upper tail inequality. Let \( \theta \succeq 0 = (0, 0, ..., 0) \in \mathbb{R}^k \) with \( \|\theta\|_2 < 2/K \). Note that an application of (3.1) and Cauchy-Schwarz inequality gives for any \( i = 1, ..., k \)

\[
\mathbb{E}[e^{\theta^i W_i}] - \mathbb{E}[e^{\theta^i W}] \leq \left| \mathbb{E}[e^{\theta^i W_i}] - \mathbb{E}[e^{\theta^i W}] \right| \leq \mathbb{E} \left[ \frac{\|\theta\|_2 \|W_i - W\|_2 (e^{\theta^i W_i} + e^{\theta^i W})}{2} \right] \leq \mathbb{E} \left[ \frac{K\|\theta\|_2 (e^{\theta^i W_i} + e^{\theta^i W})}{2} \right] \leq \frac{K\|\theta\|_2}{2} \mathbb{E}[e^{\theta^i W_i} + e^{\theta^i W}].
\]

Changing sides, since \( \|\theta\|_2 < 2/K \), we obtain

\[
\mathbb{E}[e^{\theta^i W_i}] \leq \frac{1 + \frac{K\|\theta\|_2}{2}}{1 - \frac{K\|\theta\|_2}{2}} \mathbb{E}[e^{\theta^i W}]. \tag{3.2}
\]
Letting \( m(\theta) = \mathbb{E}[e^{\theta^t W}] \), using (3.2) and the size bias relation in (1.1) we have

\[
\frac{\partial m(\theta)}{\partial \theta_i} = \mathbb{E}[W_i e^{\theta^t W}] = \mu_i \mathbb{E}[e^{\theta^t W}] \leq \mu_i \frac{1 + \frac{K\|\theta\|_2^2}{2} e^\theta}{1 - \frac{K\|\theta\|_2^2}{2} e^\theta} m(\theta) = \frac{2 + K\|\theta\|_2^2}{2 - K\|\theta\|_2^2} m(\theta). \tag{3.3}
\]

Now, letting \( M(\theta) = \mathbb{E} \left[ \exp \left( \theta^t \left( \frac{W - \mu}{\sigma} \right) \right) \right] \), observe that we have \( M(\theta) = m \left( \frac{\theta}{\sigma} \right) \exp \left( -\theta^t \frac{\mu}{\sigma} \right) \). Hence denoting

\[
\frac{\partial}{\partial \theta_i} m(\beta) = \left. \frac{\partial m(\theta)}{\partial \theta_i} \right|_{\theta=\beta},
\]

we obtain for \( \|\theta/\sigma\|_2 < 2/K \),

\[
\frac{\partial M(\theta)}{\partial \theta_i} = \frac{1}{\sigma_i} \frac{\partial}{\partial \theta_i} m \left( \frac{\theta}{\sigma} \right) \exp \left( -\theta^t \frac{\mu}{\sigma} \right) - \frac{\mu_i}{\sigma_i} m \left( \frac{\theta}{\sigma} \right) \exp \left( -\theta^t \frac{\mu}{\sigma} \right) \\
\leq \frac{\mu_i}{\sigma_i} \left( \frac{2 + K\|\theta/\sigma\|_2^2}{2 - K\|\theta/\sigma\|_2^2} \right) m \left( \frac{\theta}{\sigma} \right) \exp \left( -\theta^t \frac{\mu}{\sigma} \right) - \frac{\mu_i}{\sigma_i} m \left( \frac{\theta}{\sigma} \right) \exp \left( -\theta^t \frac{\mu}{\sigma} \right) \\
= \frac{\mu_i}{\sigma_i} M(\theta) \left( \frac{2 + K\|\theta/\sigma\|_2^2}{2 - K\|\theta/\sigma\|_2^2} - 1 \right) \\
= \frac{\mu_i}{\sigma_i} M(\theta) \left( \frac{2K\|\theta/\sigma\|_2^2}{2 - K\|\theta/\sigma\|_2^2} \right).
\]

This in particular gives for \( \|\theta/\sigma\|_2 < 2/K \),

\[
\frac{\partial \log M(\theta)}{\partial \theta_i} \leq \frac{\mu_i}{\sigma_i} \frac{2K\|\theta/\sigma\|_2^2}{2 - K\|\theta/\sigma\|_2^2}.
\]

Now, using the mean value theorem, for all \( 0 \leq \theta \leq \mathbb{R}^k \) with \( \|\theta/\sigma\|_2 < 2/K \),

\[
\log(M(\theta)) = \nabla \log(M(z)) \cdot \theta,
\]

for some \( 0 \leq z \leq \theta \). Noting that \( \|z/\sigma\|_2 \leq \|\theta/\sigma\|_2 < 2/K \) and using Cauchy-Schwarz inequality, we obtain

\[
\log M(\theta) = \nabla \log M(z) \cdot \theta \leq \sum_{i=1}^k \frac{2K\|z/\sigma\|_2 \mu_i}{2 - K\|z/\sigma\|_2^2} \frac{\mu_i}{\sigma_i} \theta_i \leq \frac{2K\|\theta/\sigma\|_2^2 \mu_i}{2 - K\|\theta/\sigma\|_2^2} \|\mu\|_2 \|\theta\|_2^2. \tag{3.4}
\]

Next we observe that

\[
\|\theta\|_2 < \frac{1}{K_2} \implies \left\| \frac{\theta}{\sigma} \right\|_2 < \frac{2}{K}.
\]

Thus if \( \|\theta\|_2 < 1/K_2, \) (3.3) yields

\[
\log M(\theta) \leq \left\| \frac{\mu}{\sigma} \right\|_2 \frac{2K\|\theta/\sigma\|_2^2}{2 - K\|\theta/\sigma\|_2^2} = \frac{K_1\|\theta\|_2^2}{2(1 - K_2\|\theta\|_2^2)}.
\]

Hence if \( t \geq 0 \) and \( \|\theta\|_2 < 1/K_2, \) an application of Markov’s inequality yields

\[
P \left( \frac{W - \mu}{\sigma} \geq t \right) \leq P \left( \theta^t \left( \frac{W - \mu}{\sigma} \right) \geq \theta^t t \right) \leq \exp \left( -\theta^t t \right) M(\theta) \leq \exp \left( -\theta^t t + \frac{K_1\|\theta\|_2^2}{2(1 - K_2\|\theta\|_2^2)} \right).
\]
Using $\theta = \frac{t}{K_1+K_2\|t\|_2} \preceq 0$, and noting that $\|\theta\|_2 < 1/K_2$, we finish the proof of the upper tail inequality.

Next we prove the lower tail bound given in (2.1). Letting $\theta \preceq 0$ and using the size bias relation given in (1.1), we have

$$\frac{\partial m(\theta)}{\partial \theta_i} = E[W_i e^{\theta^i W}] = \mu_i E[e^{\theta^i(W^i-W)}]e^{\theta^i W}.$$  

Using the inequality $e^x \geq 1 + x$, this yields

$$\frac{\partial m}{\partial \theta_i} \geq \mu_i E[(1 + \theta^i(W^i-W))e^{\theta^i W}].$$  

By Cauchy-Schwarz inequality and that $\|W^i-W\|_2 \leq K$, we have

$$|\theta^i(W^i-W)| \leq \|\theta\|_2\|W^i-W\|_2 \leq K\|\theta\|_2$$

which in particular gives $\theta^i(W^i-W) \geq -K\|\theta\|_2$. Combining this observation with (3.5), we arrive at

$$\frac{\partial m}{\partial \theta_i} \geq \mu_i E[(1 - K\|\theta\|_2)e^{\theta^i W}] = \mu_i(1 - K\|\theta\|_2)m(\theta).$$  

Now, keeping the notations as in the upper tail case and using the estimate in (3.6), we get

$$\frac{\partial M}{\partial \theta_i} = \frac{1}{\sigma_i} \partial m \left( \frac{\theta}{\sigma} \right) \exp \left( -\theta^i \frac{\mu}{\sigma} \right) - \frac{\mu_i}{\sigma_i} m \left( \frac{\theta}{\sigma} \right) \exp \left( -\theta^i \frac{\mu}{\sigma} \right)$$

$$= \frac{1}{\sigma_i} \exp \left( -\theta^i \frac{\mu}{\sigma} \right) \left( \partial m \left( \frac{\theta}{\sigma} \right) - \mu_i m \left( \frac{\theta}{\sigma} \right) \right)$$

$$\geq \frac{1}{\sigma_i} \exp \left( -\theta^i \frac{\mu}{\sigma} \right) \left\{ \mu_i \left( 1 - K \left\| \frac{\theta}{\sigma} \right\|_2^2 \right) m \left( \frac{\theta}{\sigma} \right) - \mu_i m \left( \frac{\theta}{\sigma} \right) \right\}$$

Manipulating the terms in the lower bound, this yields

$$\frac{\partial M}{\partial \theta_i} = \frac{1}{\sigma_i} \exp \left( -\theta^i \frac{\mu}{\sigma} \right) \left( -\mu_i K \left\| \frac{\theta}{\sigma} \right\|_2^2 m \left( \frac{\theta}{\sigma} \right) \right)$$

$$= -\frac{\mu_i}{\sigma_i} K \left\| \frac{\theta}{\sigma} \right\|_2^2 M(\theta)$$

$$\geq -\frac{\mu_i}{\sigma_i} K \left\| \frac{\theta}{\sigma} \right\|_2^2 M(\theta).$$

Now, using the mean value theorem, for $\theta \preceq 0$, one can find $\theta \preceq z \preceq 0$ such that

$$\log M(\theta) = \nabla \log M(z) \cdot \theta.$$

Hence for a given $\theta \preceq 0$, we have

$$\log M(\theta) = \nabla \log M(z) \cdot \theta \leq \sum_{i=1}^{k} \left( \frac{-K\mu_i \left\| \theta \right\|_2}{\sigma(1)\sigma_i} \right) \theta_i$$  

(3.7)
where we used that \( \theta_i \leq 0 \) for each \( i \) for the inequalities. Now, using (3.7) and an application of Cauchy-Schwarz inequality gives

\[
\log M(\theta) \leq \frac{K\|\theta\|_2}{\sigma(1)} \sum_{i=1}^{k} \frac{\mu_i |\theta_i|}{\sigma_i} \leq \frac{K}{\sigma(1)} \left\| \frac{\mu}{\sigma} \right\|_2 \|\theta\|_2^2.
\]

which after exponentiation yields

\[
M(\theta) \leq \exp \left( \frac{K}{\sigma(1)} \left\| \frac{\mu}{\sigma} \right\|_2 \|\theta\|_2^2 \right).
\]

Combining this last observation with Markov’s inequality, we arrive at

\[
P\left( \frac{W - \mu}{\sigma} \preceq -t \right) = P\left( \theta^t \left( \frac{W - \mu}{\sigma} \right) \geq \theta^t t \right) \leq \exp \left( -\theta^t t + \frac{K}{\sigma(1)} \left\| \frac{\mu}{\sigma} \right\|_2 \|\theta\|_2^2 \right).
\]

Substituting \( \theta = \frac{-t}{2\sigma(1)\|\mu\|_2} \leq 0 \), result follows.

\[\square\]

4 Two applications

In this section, we will discuss two applications of Theorem 2.1 which will be on joint distributions of (1) locally dependent random variables and (2) the number of patterns in uniformly random permutations.

4.1 Local dependence

Now we show that our results above can be used to obtain concentration bounds for a random vector \( W = (W_1, W_2, ..., W_k) \) with nonnegative coordinates that are functions of a subset of a collection of independent random variables. First part of the following lemma was used in [9] for univariate concentration results.

**Lemma 4.1.** Let \( \mathcal{V} = \{1, 2, ..., k\} \) and \( \{C_v, v \in \mathcal{V}\} \) be a collection of independent random variables, and for each \( i \in \mathcal{V} \), let \( \mathcal{V}_i \subset \mathcal{V} \) and \( W_i = W_i(C_v, v \in \mathcal{V}_i) \) be a nonnegative random variable with nonzero and finite mean.

i. [9] If \( \{C_v^i, v \in \mathcal{V}_i\} \) has distribution

\[
dF(c_v, v \in \mathcal{V}_i) = \frac{W_i(c_v, v \in \mathcal{V}_i)}{E[W_i(C_v, v \in \mathcal{V}_i)]} \; dF(c_v, v \in \mathcal{V}_i)
\]

and is independent of \( \{C_v, v \in \mathcal{V}\} \), letting

\[
W_j^i = W_j(C_v^i, v \in \mathcal{V}_j \cap \mathcal{V}_i, C_u, u \in \mathcal{V}_j \cap \mathcal{V}_i^c),
\]

the collection \( \mathbf{W}^i = \{W_j^i, j \in \mathcal{V}\} \) has the \( \mathbf{W} \) size biased distribution in direction \( i \).

ii. Further if we assume that \( W_i \leq M \) for each \( i \), then we have

\[
\|\mathbf{W}^i - \mathbf{W}\|_2 \leq \sqrt{b}M
\]

where \( b = \max_i |\{j : \mathcal{V}_j \cap \mathcal{V}_i \neq \emptyset\}|.\)
Proof. Proof of the fact that \( W^i = \{ W_j^i, j \in V \} \) has the \( W \) size biased distribution in direction \( i \) can be found in [9]. For the second part, we note that by the construction in the statement, we have \( W_j = W_j^i \) whenever \( V_j \cap V_i = \emptyset \). Thus,

\[
\|W^i - W\|_2 = \left( \sum_{j=1}^{k} |W_j^i - W_j|^2 \right)^{1/2} \leq (M^2 \max_i |\{ j : V_j \cap V_i \neq \emptyset \}|)^{1/2} = \sqrt{b}M.
\]

In conclusion, we note that in the case of local dependence as described above, we can use Theorem 2.1 to obtain concentration bounds for \( W = (W_1, W_2, ..., W_k) \) with \( K = \sqrt{b}M \). This provides a natural generalization to the argument given in Section 2 for vectors with independent coordinates.

Ghosh and Goldstein [9] also provides two specific applications of this result on sliding window statistics and local extrema on a lattice in a univariate setting. The discussion above immediately yields multivariate concentration bounds for each of these problems, but we do not include the details here as they will be repetitions of the steps done in [9].

4.2 Counting patterns

Let \( \tau_1, \tau_2, ..., \tau_k \in S_m \) be \( k \) distinct permutations from \( S_m \), the permutation group on \( m \geq 3 \) elements. Also let \( \pi \) be a uniformly random permutation in \( S_n \), where \( n \geq m \) and set \( V = \{ 1, 2, ..., n \} \). Denoting \( V_s = \{ s, s + 1, ..., s + m - 1 \} \) for \( s \in V \) where addition of elements of \( V \) is modulo \( n \), we say the pattern \( \tau \) appears at location \( s \in V \) if the values \( \{ \pi(v) \}_{v \in V_s} \) and \( \{ \tau(v) \}_{v \in V_s} \) are in the same relative order. Equivalently, the pattern \( \tau \) appears at \( s \) if and only if \( \pi(\tau^{-1}(v) + s - 1), v \in V_1 \) is an increasing sequence. Our purpose here is to prove concentration bounds using Theorem 2.1 for the multivariate random variable \( W = (W_1, W_2, ..., W_k) \) where \( W_i \) counts the number of times pattern \( \tau_i \) appears in \( \pi \). This problem was previously studied in [9] for the univariate case.

For \( \tau \in S_m \), let \( I_j(\tau) \) be the indicator that \( \tau(1), ..., \tau(m - j) \) and \( \tau(j + 1), ..., \tau(m) \) are in the same relative order. Following the calculations in [9], for \( i = 1, ..., k \), we have

\[
\mu_i = \mathbb{E}[W_i] = \frac{n}{m!}, \tag{4.1}
\]

and

\[
\sigma_i^2 = \text{Var}(W_i) = n \left( \frac{1}{m!} \left( 1 - \frac{2m - 1}{m!} \right) + 2 \sum_{j=1}^{m-1} \frac{I_j(\tau_i)}{m + j}! \right). \tag{4.2}
\]

Now we are ready to give our main result.

Theorem 4.2. With the setting as above, if \( W = (W_1, W_2, ..., W_k) \), then the conclusions of Theorem 2.1 hold with mean and variance as in (4.1) and (4.2), and

\[
K_1 = \frac{2k(2m - 1)m!}{m! - 2m + 2} \quad \text{and} \quad K_2 = \frac{\sqrt{k}(2m - 1)m!}{2\sqrt{n}(m! - 2m + 1)}.
\]
Proof. Letting $\pi$ be a uniformly random permutation in $S_n$, and $X_{s,\tau}$ the indicator that $\tau$ appears at $s$,

$$X_{s,\tau}(\pi(v), v \in V_s) = \mathbb{1}(\pi^{-1}(1) + s - 1) < ... < \pi^{-1}(m) + s - 1),$$

the sum $W = \sum_{s \in V} X_{s,\tau}$ counts the number of $m$-element-long segments of $\pi$ that have the same relative order as $\tau$.

Now let $\sigma_s$ the permutation in $S_m$ so that

$$\pi(\sigma_s(1) + s - 1) < ... < \pi(\sigma_s(m) + s - 1)$$

and set

$$\pi_1^s(v) = \begin{cases} \pi(\sigma_s(1) + s - 1)), & \text{if } v \in V_s \\ \pi(v), & \text{if } v \notin V_s \end{cases}$$

In other words, $\pi_1^s$ is the permutation $\pi$ with the values $\pi(v), v \in V_s$ reordered so that $\pi_1^s(\gamma)$ for $\gamma \in V_s$ are in the same relative order as $\tau_1$. Similarly we can define $\pi_2^s, ..., \pi_k^s$ corresponding to $\tau_2, ..., \tau_k$, respectively.

To obtain $W^i$, the $W$ size biased variate in direction $i$ for $i = 1, 2, ..., k$, pick an index $\beta$ uniformly from $\{1, ..., n\}$ and set $W^i = \sum_{s \in V} X_{s,\tau}(\pi_\beta^s)$. Then $W^i = (W_1^i, W_2^i, ..., W_k^i)$ for $i = 1, 2, ..., k$. The fact that we indeed obtain the desired size bias variates follows from results in [10].

Since $\pi_1^\beta, \pi_2^\beta, ..., \pi_k^\beta$ agree with $\pi$ on all the indices leaving out $V_\beta$ and $|V_\beta| = m$, we obtain $|W^i_j - W^i_j| \leq 2m - 1$ for $i, j = 1, 2, ..., k$. Hence, $\|W^i - W\|_2 \leq \sqrt{k}(2m - 1)$ for each $i \in \{1, 2, ..., k\}$.

Now recall from [12] that $\sigma_i^2 = n\left(\frac{1}{m!} \left(1 - \frac{2m-1}{m!}\right) + 2 \sum_{j=1}^{m-1} \frac{I_j(\tau_i)}{(m+j)!}\right)$ for $i = 1, 2, ..., k$. Since $0 \leq I_j \leq 1$, one can obtain a variance lower bound by setting $I_k = 0$. In particular, this yields

$$\sigma_{(1)}^2 \geq n\left(\frac{1}{m!} \left(1 - \frac{2m-1}{m!}\right)\right).$$

Since the constants $K_1$ and $K_2$ in Theorem 2.1 can be replaced by larger constants, result follows from simple computations.

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