Spinons in Magnetic Chains of Arbitrary Spins at Finite Temperatures

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Abstract

The thermodynamics of solvable isotropic chains with arbitrary spins is addressed by the recently developed quantum transfer matrix (QTM) approach. The set of nonlinear equations which exactly characterize the free energy is derived by respecting the physical excitations at $T = 0$, spinons and RSOS kinks. We argue the implication of the present formulation to spinon character formula of level $k = 2S$ SU(2) WZN model.

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1 Introduction

The 1D spin systems have been providing problems of both physical and mathematical interests. Among them, there exists a family of solvable models of Heisenberg type with spin-$S$ \[^{[1, 2]}\]. This includes, for instance,

\begin{equation}
\mathcal{H} = J \sum_{i=1}^{L} \{ \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{4} \}
\end{equation}

(1)

\begin{equation}
\mathcal{H} = \frac{J}{4} \sum_{i=1}^{L} \{ \vec{S}_i \cdot \vec{S}_{i+1} - (\vec{S}_i \cdot \vec{S}_{i+1})^2 + 3 \}
\end{equation}

(2)
as $S = 1/2, 1$, respectively.

Ground state properties as well as low lying excitations have been elucidated by the powerful machinery of solvable models, the Bethe Ansatz Equation (BAE). It has been demonstrated in many contexts \[^{[3, 4, 5]}\] that the underlying field theory is level $k = 2S$ SU(2) WZWN model \[^{[6]}\].

Although this 1D quantum model is equivalent to a 2D vertex model, physical excitations carry both natures of vertex models and restricted SOS models. This is firstly demonstrated in \[^{[7]}\] based on the $S$-- matrix argument. The space of states is identified in \[^{[8]}\]. By decomposition of crystals, such double feature is made explicit in terms of "type of domain walls" and "type of domain". There are independent justifications of this: The double feature in the spectral decomposition is shown by path space approach\[^{[9]}\]. See also \[^{[10]}\] for the decomposition of space picture realized in Fermionic forms.

Here we are interested in the finite temperature problem. Standard arguments employ the string hypothesis\[^{[2, 11]}\]. The excitation is described not in terms of "physical excitations" in the above sense, but in the "string basis". There strings of arbitrary lengths are allowed, which results into infinitely many coupled integral equations among infinitely many unknown functions. The description successfully reproduces the expected specific heat anomaly. It may not be, however, best suitable for practical numerics.

We revisit the problem via the recently developed commuting Quantum Transfer Matrix (QTM) approach \[^{[13]-[20]}\]. The formulation does not reply on the string hypothesis. Rather, it only relies on analyticity structures of the object called QTM\[^{[21, 12]}\]. The problem of the combinatorial summation, i.e., evaluation of the partition function, then reduces to investigations on the analyticity of suitably chosen auxiliary functions. Up to now, two kinds of choices are adopted independently.

(A) The eigenvalue of the QTM as given by the Quantum Inverse Scattering Method consists of several terms. The auxiliary functions are chosen from combinations of products of these terms \[^{[13, 14, 15, 17]}\]. A convenient choice leads to a finite number of coupled non-linear integral equations for a finite number of unknown functions

(B) A set of auxiliary functions may be chosen from the fusion hierarchy among "generalized" QTMs \[^{[2, 16, 18, 20]}\]. Generically, one obtains an infinite number of coupled non-linear integral equations for an infinite number of unknown functions. This can be shown to recover the conventional TBAs based on the string hypothesis. Of course, the new approach is entirely free of any assumption about excitations like string hypothesis.

The spin-1 case is analyzed in the related problem, in the context of finite size corrections\[^{[23]}\]. There six functions are introduced in the spirit of (A). The structure of NLIE among them is
much more involved in comparison to the spin-1/2 case, and seems to defy a simple-minded
generalization to higher S cases. In a sense the most subtle point in the QTM approach appears;
one does not know a priori “better” set of auxiliary functions.

Here, we adopt other choice of auxiliary functions, in particular for the spin 1 case the
number of these functions is 3 in contrast to 6 as in [23]. A simple idea of combining the two
formulations (A) and (B) works well so that the generalization to arbitrary S is possible. The
adopted functions agree with the picture in [7]. Roughly speaking, the fusion part (B) of the
auxiliary functions is related to the RSOS piece of the excitation, while auxiliary functions from
(A) correspond to spinons.

We remark that the fusion hierarchy itself is not truncated, by brute force, into finite set.
Instead, the spinon part makes the functional relations among them strictly closed. Thus we
obtain 2S + 1 coupled integral equations for 2S + 1 unknown functions.

Besides the practical advantage, it implies the universality in the description of thermo-
dynamics of solvable quantum 1D chains, i.e., the description only in terms of objects which
reduce to physical excitations in T \rightarrow 0. This has been already demonstrated for several models
in highly correlated 1D electron systems including the supersymmetric t − J model [14], the
supersymmetric extended Hubbard model [13] and the Hubbard model [17]. There the exact
thermodynamics are formulated in terms of ”spinons” and ”holons”, although they lose sense
at sufficiently high temperatures. The present study adds one successful example even in the
fusion models and gives further supports on the above conjecture.

This paper is organized as follows. In the next section, we define the main object in this
approach, the quantum transfer matrix (QTM). A minimal information of the novel approach is
sketched. Section 3 is devoted to a brief description of the fusion hierarchy of generalized QTMs.
After these preparations, we introduce auxiliary functions and examine functional relations
among them in Section 4. The analytic structure studied numerically leads to nonlinear integral
equations as discussed in Section 5. Based on these equations the low temperature asymtotics is
studied analytically in Section 6. The central charge of the level k = 2S SU(2) WZWN model is
successfully recovered. We also present the numerical evaluation of specific heat of S = 1/2, 1, 3/2
models for wider ranges of temperatures. In Section 7, the implication of the present formulation
to the spinon character formula of the WZWN model [24, 8, 42] is discussed. The summary of
the paper is given in Section 8.

2 QTM Formulation

The present QTM formulation originates from two ingredients: the equivalence theorem between
1D quantum and 2D classical systems [21] on one side and the integrability structure on the
other [25]. Especially, the latter provides the way to introduce commuting QTMs which reduce
the problem of combinatorial counting to that of analyticity of suitable auxiliary functions [12].
Such a strategy has been successfully applied to several interesting models [12]-[20]. We also
mention earlier studies on thermodynamics [26]-[32] which essentially utilize only the former
part of ideas.

A classical analogue to solvable spin-S XXX model is already found as a 2S ± 1 state vertex
model [33]. The Boltzmann weights are identified with the matrix elements of \( \mathfrak{sl}_2 \) invariant \( R \)
matrix:
Figure 1: Graphical representation of the R matrix (its element $R_{\alpha \beta}^{\mu \nu}$).

\[ R^\nu(u) = \sum_{j=0}^{k} \rho_{2k-2j}(u)P_{2k-2j} \]

\[ \rho_{2k-2j} = \prod_{\ell=0}^{j-1} \frac{2(k-\ell)-u}{2(k-\ell)} \prod_{\ell=j}^{k-1} \frac{2(k-\ell)+u}{2(k-\ell)} \]  

where $k = 2S$ and $P_j$ is the projector to $V_j$, the $j + 1$ dimensional irreducible module of $sl_2$. We choose \{-\frac{j}{2}, -\frac{j}{2} + 1, \cdots, \frac{j}{2}\} as basis for $V_j$. The spectral parameter $u$ represents the anisotropy of the vertex weights. The Yang-Baxter equation implies the commutation of row-to-row transfer matrices for arbitrary spectral parameters $u, v$: $T(u)T(v) = T(v)T(u)$ with

\[ T^\alpha_\beta(u) = \sum_{\mu} \prod_{i=1}^{L} R_{\alpha_i \beta_i}^{\mu_i \mu_{i+1}}(u), \]  

where $L$ denotes the real system size, $\alpha_i, \beta_i, \mu_i \in V_k$, $R = PR^\nu$ and $P(x \otimes y) = y \otimes x$.

The Hamiltonian is obtained as the logarithmic derivative at $u = 0$,

\[ H = J \frac{d}{du} \ln T(u) \bigg|_{u=0} \]  

It is an easy exercise to verify (5) gives (1) and (2) for $S = 1/2, 1$ respectively. This may be most easily done by representing $P_j$ in (3) by

\[ P_j = \prod_{p=0, p \neq j}^{k} \frac{\tilde{S}^{-1} S - x_p}{x_j - x_p}, \]  

and $x_j = 1/2(\frac{\ell}{2} + 1 - k(\frac{j}{2} + 1))$. The Hamiltonian for general $S$ can be extracted similarly. This is the well-known expression of the equivalence between 1D quantum systems and 2D classical models. To utilize the equivalence in evaluating finite $T$ quantities, especially, free energy, we need to proceed further.

Let us introduce Boltzmann weights $\tilde{R}$ ($R$) of models related to (3) by clockwise (anticlockwise) $90^0$ rotations

\[ \tilde{R}_{\alpha \beta}^{\mu \nu}(v) = R_{\mu \nu}^{\beta \alpha}(v), \quad \tilde{R}_{\alpha \beta}^{\mu \nu}(v) = R_{\nu \mu}^{\alpha \beta}(v). \]
The standard initial condition of the $R^\vee$ matrix and (3) lead to significant relations,
\[ T(u) = T_R e^{uH/J + O(u^2)}, \quad T(u) = T_L e^{uH/J + O(u^2)}, \]
where $T$ is defined in analogy to $R$, replacing $R$ by $\widetilde{R}$. $T_{R,L}$ are the right- and left-shift operators, respectively and they commute with the Hamiltonian.

We are ready to apply Trotter formula; by substitution
\[ u = -\frac{J\beta}{N}, \]
we find
\[ \left( T(u) T(u) \right)^{N/2} = e^{-\beta H + O(1/N)}. \]

where $\beta$ denotes the inverse temperature. $N$ is a large integer “Trotter” number, interpreted as a fictitious system size in a virtual direction. Thus the partition function of the quantum system (size $L$, inverse temperature $\beta$)
\[ Z = \lim_{N \to \infty} \text{Tr} \left( T(u) T(u) \right)^{N/2}, \]
is identical to the partition function of an inhomogeneous $2S+1$ vertex model with alternating rows on a virtual 2D lattice of size $L \times N$. See (Fig 3). Although the above mapping is exact, the expression (3) is not yet efficient. The eigenvalues of $T(u) T(u)$ are almost degenerate. Hence it is still difficult task to evaluate the trace. The intriguing point in [21] is to consider a transfer matrix $\tilde{T}(u)$ propagating in the "horizontal" direction.

This novel operator acting on $N$ sites, has gaps in the eigenvalues provided $T > 0$. Here we adopt further sophisticate approach developed in [12]-[20]. Explicitly, we define QTM by
\[ T_{QTM}(u,x) = \sum_{\mu} \prod_{i=1}^{N/2} R_{\alpha_{2i-1} \beta_{2i-1}}^{\mu_{2i-1} \mu_{2i} \alpha_{2i}} (u + ix) \tilde{R}_{\alpha_{2i-1} \beta_{2i}}^{\mu_{2i+1} \mu_{2i-1} \beta_{2i-1}} (u - ix), \]
which reduces to above-mentioned $\tilde{T}(u)$ by putting $x = 0$. Fig 4 represents QTM graphically.

Though the introduction of the extra parameter $x$ seems to be redundant, there is a remarkable property;
\[ [T_{QTM}(u,x), T_{QTM}(u,x')] = 0 \]
Figure 3: Graphic representation of the partition function on a virtual two dimensional lattice of $N \times L$. The operators $\mathcal{T}, \overline{\mathcal{T}}$ transfer states from bottom to top, while $\mathcal{T}(u)$ and $\mathcal{T}_{QTM}$ do from right to left.

Figure 4: Graphic representation of $\mathcal{T}_{QTM}(u,v)$. 
by fixing $u$. This originates from the fact that $R$ and $\tilde{R}$ operators possess the same intertwiner. Thus for each $T$, one can associate a auxiliary complex plane $x$ to the partition function.

Due to the gap in spectra, the free energy $f$ of 1D quantum spin chains is given only by the largest eigenvalue $\Lambda_{\text{QTM}}(u,x)$,

$$f = -\frac{1}{\beta} \lim_{L \to \infty} \frac{1}{L} \ln Z = -\frac{1}{\beta} \lim_{N \to \infty} \ln \Lambda_{\text{QTM}}(u = -\frac{\beta J}{N}, x = 0).$$

(12)

This is the starting point of our analysis. The difficulty in evaluating (12) lies in the $N$ dependence of the vertex weights. The numerical extrapolation through finite $N$ studies may be plagued by marginal perturbations [19]. The prescription is to utilize the existence of complex plane $x$ for each $T$. The analytic properties of $T_{\text{QTM}}$ and suitably chosen auxiliary functions in the $x-$ plane make the evaluation possible and transparent.

Before closing this section, we shall describe how to modify the above relations in the presence of an external magnetic field $H$, namely by inclusion of the Zeeman term $-2H \sum_i S_i^z$ to Hamiltonian (8). This contribution is described by diagonal operator $D(H)$,

$$
\begin{pmatrix}
  e^{-2S\beta H} \\
  e^{-(2S-1)\beta H} \\
  \vdots \\
  e^{2S\beta H}
\end{pmatrix}
\otimes
\begin{pmatrix}
  e^{-2S\beta H} \\
  e^{-(2S-1)\beta H} \\
  \vdots \\
  e^{2S\beta H}
\end{pmatrix}
\otimes \cdots
\begin{pmatrix}
  e^{-2S\beta H} \\
  e^{-(2S-1)\beta H} \\
  \vdots \\
  e^{2S\beta H}
\end{pmatrix}.
$$

Thus one only has to insert this inside the trace (9)

$$Z = \lim_{N \to \infty} \text{Tr}(T(u) \bar{T}(u))^{N/2} D(H).$$

In the rotated frame, the effect of the insertion of $D(H)$ is translated to the boundary weight, $B(\mu_1) = e^{\beta \mu_1 H}$;

$$T_{\text{QTM}}(u,x) = \sum_{\mu} B(\mu_1) \prod_{i=1}^{N/2} R_{\alpha_{2i-1} \beta_{2i-1}}^{\mu_{2i-1} \mu_{2i}} (u + ix) \bar{R}_{\alpha_{2i} \beta_{2i+1}}^{\mu_{2i+1} \mu_{2i}} (u - ix).$$

3 Fusion Hierarchy

We consider a hierarchy of quantum transfer matrices acting on $V_k^{\otimes N}$. Let $T_j(u,x)$ be a member of the hierarchy with the auxiliary space $V_j$. 
In other words, it is the transfer matrix of the vertex model of which spins $S(= k/2)$ are assigned to vertical edges and spins $j/2$ to horizontal edges (Fig 5). The quantity of our interest, $T_{\text{QTM}}(u, x)$ coincides with $T_k(u, x)$ apart from over-all normalization, which is specified later.

For brevity, we shall only give matrix elements of $\mathcal{R}(u)$ defining the most fundamental $T_1(u, x)$.

\[
\mathcal{R}_{\ell, \ell}^{\pm 1/2, \pm 1/2}(u) = u + 1 \pm 2\ell, \quad \mathcal{R}_{\ell', \ell}^{\ell'-\ell, \ell'-\ell'}(u) = \sqrt{(k + 2 + 2\min(\ell, \ell'))(k - 2\min(\ell, \ell'))}
\]

where $\ell, \ell' \in \{-k/2, \cdots, k/2\}$ and $|\ell - \ell'| = 1$. Similar to (3), the corresponding $R^\ell$ matrix has decomposition,

\[
R^\ell(u) = (u + k + 1)P_{k+1} + (u - k - 1)P_{k-1}.
\]

$\mathcal{R}$-- matrix for $T_j(u, x)$ is obtained from the above elementary $\mathcal{R}(u)$ by $j - 1$ times fusion in the auxiliary space. By the construction, arbitrary pairs in this hierarchy are commutative if they share the same $u$,

\[
[T_j(u, x), T_{j'}(u, x')] = 0.
\]

This is a generalization of (11). In the following, we fix $u$ for all QTMs and omit the dependency on $u$. Due to the consequential commutativity, one needs not to distinguish operators $T_j$ from their eigenvalues.

Then the explicit eigenvalue of the most elementary transfer matrix $T_1(x)$ reads

\[
T_1(x) = \phi_+(x - (k - 1)i)\phi_-(x - (k + 1)i)e^{\beta H} \frac{Q(x + 2i)}{Q(x)} + \phi_-(x + (k - 1)i)\phi_+(x + (k + 1)i)e^{-\beta H} \frac{Q(x - 2i)}{Q(x)}
\]

\[
\phi_\pm(x) := (x \pm iu)^{N/2}, \quad Q(x) := \prod_{j=1}^{m}(x - x_j)
\]

where $x_j, (j = 1, \cdots, m)$ denotes the solution to the Bethe ansatz equation:

\[
\frac{\phi_-(x_j + (k - 1)i)\phi_+(x_j + (k + 1)i)}{\phi_+(x_j - (k - 1)i)\phi_-(x_j - (k + 1)i)} = -e^{2\beta H} \frac{Q(x_j + 2i)}{Q(x_j - 2i)}.
\]

The number of BAE roots, $m$, differs for different eigenstates generally, and $m = Nk/2$ for the largest eigenvalue case.

By construction of the fusion hierarchy and from singularities of the intertwining operators (13) by tentatively replacing $k \to j$, the following relation is valid:

\[
T_j(x)T_1(x - i(j + 1)) = T_{j+1}(x - i) + g_j(x)T_{j-1}(x + i)
\]

\[
g_j(x) = \phi_-(x - i(k + j + 2))\phi_-(x + i(k - j))\phi_+(x - i(k + j))\phi_+(x + i(k - j + 2)).
\]
From this, one can prove the important functional relation ($T-$ system) by induction \[36\],

\[
T_p(x+i)T_p(x-i) = f_p(x) + T_{p-1}(x)T_{p+1}(x), \quad (p \geq 1)
\]

\[
f_p(x) := \prod_{j=1}^{p} \prod_{\sigma=\pm} \phi_\sigma(x + i\sigma(p - k - 2j + 1))\phi_\sigma(x + i\sigma(k - p + 2j + 1))
\]

and $T_0 = 1$.

By substituting (14) into (15), we can successively obtain $T_p(x), (p \geq 2)$. Explicitly, $T_p(x)$ consists of a sum of $p + 1$ terms,

\[
T_p(x) := \sum_{\ell=1}^{p+1} \lambda_\ell^{(p)}(x),
\]

\[
\lambda_\ell^{(p)}(x) := e^{\beta H(p+2-2\ell)}\psi_\ell^{(p)}(x)\frac{Q(x + i(p+1))Q(x - i(p+1))}{Q(x + i(2\ell - p - 1))Q(x + i(2\ell - p - 3))},
\]

\[
\psi_\ell^{(p)}(x) := \prod_{j=1}^{p-\ell+1} \phi_-(x + i(p - k - 2j))\phi_+(x + i(p - k + 2 - 2j))
\]

\begin{align*}
&\times \prod_{j=1}^{\ell-1} \phi_-(x - i(p - k + 2 - 2j))\phi_+(x - i(p - k - 2j)).
\end{align*}

(16)

As is previously noted, $T_k(x)$ has a normalization trivially different from $T_{QTM}(u, x)$ in the previous section:

\[
T_{QTM}(u, x) = \frac{T_k(x)}{\prod_{p=1}^{k} \phi_0(2ip)}
\]

\[
\phi_0(x) := x^{N/2}
\]

(17)

In the original problem of the spin-$S$ chain, only $T_k(x)$ is of our interest. The auxiliary $T_j$'s, however, make the evaluation of $T_k(x)$ transparent as is shown in the following.

4 Auxiliary Functions and Functional Relations among them

To explore the analyticity of the transfer matrix $T_k(x)$, we introduce $k + 1$ auxiliary functions. The first $k - 1$, functions, $\{y_j(x)\}$ are already found in several literatures and have sound basis in the $sl_2$ fusion hierarchy. They are defined by \[36, 37\]

\[
y_j(x) := \frac{T_{j-1}(x)T_{j+1}(x)}{f_j(x)}, \quad j \geq 1.
\]

The functional relations among them are sometimes referred to as $Y-$ system:

\[
y_j(x+i)y_j(x-i) = Y_{j-1}(x)Y_{j+1}(x), \quad j \geq 1
\]

\[
Y_j(x) := 1 + y_j(x)
\]

(18)
and \( y_0(x) := 0 \) which is a consequence of eq. (15). Note that the \( Y \) - system is not truncated to a finite set in this case. The \( k - 1 \)-th equation, which characterizes \( y_{k-1}(x) \) inevitably contains \( y_k(x) \) in r.h.s., and so on. Thus another device is needed to construct a finite set of auxiliary functions satisfying a complete and closed set of functional relations. The remaining two functions \( b(x) , \bar{b}(x) \) and their "relatives" \( \mathcal{B}(x) := 1 + b(x) , \bar{B}(x) := 1 + \bar{b}(x) \) play this role. We define them by ratios of \( \lambda 's \) in \( T_k(x) \) as,

\[
b(x) := \frac{\lambda_1^{(k)}(x + i) + \cdots + \lambda_k^{(k)}(x + i)}{\lambda_{k+1}^{(k)}(x + i)} \quad \text{and} \quad \bar{b}(x) := \frac{\lambda_1^{(k)}(x - i) + \cdots + \lambda_{k+1}^{(k)}(x - i)}{\lambda_1^{(k)}(x - i)}.
\]

The following relations are direct consequences of the above definitions:

\[
\mathcal{B}(x) \lambda_{k+1}^{(k)}(x + i) = e^{-k\beta H} \mathcal{B}(x) \prod_{\sigma = \pm} \prod_{j = 1}^{k} \phi_\sigma(x + (2j + \sigma)i) \frac{Q(x - ik)}{Q(x + ik)} T_k(x + i) = T_k(x + i),
\]

\[
\bar{B}(x) \lambda_{1}^{(k)}(x - i) = e^{k\beta H} \bar{B}(x) \prod_{\sigma = \pm} \prod_{j = 1}^{k} \phi_\sigma(x - (2j - \sigma)i) \frac{Q(x + ik)}{Q(x - ik)} T_k(x - i).
\]

We have \( k - 1 \) equations for \( y_j \) (\( j = 1, \cdots, k - 1 \)) in terms of \( Y_j(x) \) (\( j = 1, \cdots, k - 1 \), \( \mathcal{B}(x) \)) and \( \bar{B}(x) \). The first \( k - 2 \) equations are chosen directly from the \( Y \) - system. In the \( k - 1 \)-th equation, we rewrite \( Y_k(x) \) in the rhs of \( Y \) - system (\( j = k - 1 \) in (13)) by \( \mathcal{B}(x) \bar{B}(x) \), thanks to (20), the definitions of \( y_k, Y_k \) and the functional relation (15):

\[
y_{k-1}(x - i)y_{k-1}(x + i) = Y_{k-2}(x) \mathcal{B}(x) \bar{B}(x).
\]

Finally, equations for \( b 's \) in terms of \( Y_j(x) \) (\( j = 1, \cdots, k - 1 \)), \( \mathcal{B}(x) \) and \( \bar{B}(x) \) are to be found. By comparing explicit forms, one finds

\[
b(x) = e^{\beta(k+1)H} \prod_{\sigma = \pm} \prod_{j = 1}^{k} \phi_\sigma(x + i\sigma) \frac{Q(x + i(k + 2))}{Q(x - ik)} T_{k-1}(x),
\]

\[
\bar{b}(x) = e^{-\beta(k+1)H} \prod_{\sigma = \pm} \prod_{j = 1}^{k} \phi_\sigma(x - i(2j - \sigma)i) \frac{Q(x - i(k + 2))}{Q(x + ik)} T_{k-1}(x).
\]

Note that \( T_{k-1}(x) \) is presented by \( Y_{k-1}(x) \):

\[
T_{k-1}(x - i)T_{k-1}(x + i) = f_{k-1}(x)Y_{k-1}(x)
\]

which originates directly from definitions of \( y_{k-1}, Y_{k-1} \) and the functional relation (15).
In what follows, we analyze these functional relations via the Fourier transformation. One denotes \( \hat{dl}b[q] \) to mean the Fourier transformation of the logarithmic derivative of \( b(x) \),

\[
\hat{dl}b[q] := \int_{-\infty}^{\infty} \frac{d \log b(x)}{dx} e^{iqx} dx,
\]

and similarly for other functions. Under some assumptions on analytic properties of the auxiliary functions, the above functional relations can be transformed into algebraic equations in the Fourier space. Roughly speaking, one can solve \( dlQ[q] \) functions in terms of \( dl\mathcal{B}[q] \) and \( dl\mathcal{B}[q] \) by deleting \( dlT[k][q] \) from algebraic equations originated from (20). Similarly, \( dlT_{k-1}[q] \) is solved by \( dlY_{k-1}[q] \) from (23). Substituting these results into Fourier transformations of logarithmic derivatives of (22), one finds expressions of \( d\hat{b}[q], \hat{b}[q] \) in terms of \( dl\mathcal{B}[q], dl\mathcal{B}[q] \) and \( dlY_{k-1}[q] \). After the inverse Fourier transforming and the integration over \( x \), we shall find the desired finite set of equations. We will make the above mentioned analytic assumptions explicit and examine them in the next section.

Before going into details, let us discuss the physical interpretation of the above functions. As is argued in [7], the \( S \)-matrix of excitations in the spin-\( S \) model factorizes into two pieces: spin-\( \frac{1}{2} \) SU(2) \( S \)-matrix and RSOS \( S \)-matrix of \( sl_2 \) level \( k = 2S \). This is consistent with the general expectation that the underlying field theory is the level-\( k \) SU(2) WZWN model. The latter is known to “decompose” into Gaussian and \( Z_k \) parafermionic field theories [3]. One finds, see for instance [3], several evidences for the equivalence between the \( sl_2 \) RSOS model in regime II and the \( Z_k \) parafermion field theory in the scaling limit. In the present description, \( b, \bar{b} \) are to be identified with up- and down- spinons. As we will see later, only they couple to the magnetic field directly. For the \( S = 1/2 \) case, there are further direct evidences for this identification [5]. On the other hand, \( \{y_j(x)\} \) are insensitive to the external field. We are led to identify \( y_j(x) \) in our choice as the RSOS piece of the excitations. The RSOS model possesses a subset of the \( Y \)-system [18]. The additional condition \( y_k = 0 \) for the model leads to the truncated set of equations among \( k-1 \) \( y \)'s [38, 3]. In the present problem, at sufficiently low temperatures, one observes, \( |b|, |\bar{b}| \sim 0 \) for \( x << \ln \beta \). Thus the substitution of \( \mathcal{B} = \mathfrak{B} = 1 \) in (21) might be legitimate in the vicinity of the origin. Then the resultant approximated \( Y \)-system coincides with that of the RSOS model. In this sense, (21) represents a gluing relation between spinon and RSOS parts of excitations.

5 Nonlinear Integral Equations

We derive the nonlinear integral equations among auxiliary functions introduced in the previous section. The crucial observation is, that all nontrivial zeros and singularities of these functions are determined by zeros of \( Q(x) \) and \( T_j(x), (j = 1, \cdots, k) \). For the largest eigenvalue sector of \( T_k(x) \), zeros of \( Q(x) \) form so-called \( k \)-strings. Imaginary parts of zeros are approximately located at \( (k+1)-2\ell, \ell = 1, \cdots, k \). For later use, we introduce notations,

\[
\Psi_1(x) := Q(x - ik), \quad \Psi_2(x) := Q(x + ik).
\]

Empirically, similar patterns are found for zeros of \( T_j(x) \): they distribute approximately on lines, \( \Im x = \pm(k+j-2\ell), \ell = 0, \cdots, j-1 \). We assume that these observation from numerics with fixed \( N \) is valid and that the deviations from lines are very small in the limit \( N \to \infty \). Then one
deduces the following ansatz on the strips where auxiliary functions are Analytic, NonZero and have Constant asymptotic behavior (ANZC).

\[ b(x), \Phi(x) \quad -1 < \Im x \leq 0, \]
\[ \tilde{b}(x), \tilde{\Phi}(x) \quad 0 \leq \Im x < 1, \]
\[ y_j(x), Y_j(x) (j = 1, \cdots, k - 1), \quad T_p(x), (p = 1, \cdots, k) \quad -1 \leq \Im x \leq 1 \]
\[ \Psi_1(x), \quad \Im x < 0 \]
\[ \Psi_2(x), \quad \Im x > 0 \]

We find it convenient to shift the definition of the arguments in \( b, \Phi, \tilde{b} \) and \( \tilde{\Phi} \). To avoid confusion, these new functions are denoted as \( a \) etc,

\[ a(x) := b(x - i\gamma), \quad \tilde{a}(x) := \tilde{b}(x + i\gamma), \]

and similarly for capital functions. Here \( 0 < \gamma < 1/2 \) is an arbitrary but fixed parameter. Note that this is equivalent to adopt small shifts in the definition of the integration contours for the Fourier transformation. Due to the ANZC properties of \( b, \tilde{b} \), such modifications are almost trivial in the Fourier space.

Having identified ANZC strips, we revisit eqs(20). Consider the integral,

\[ \int_{C} \frac{d}{dz} \log T_k(z) e^{iz} dz, \]

where \( C \) encircles the edges of ”square” : \( [\infty - i, \infty - i] \cup [\infty - i, \infty + i] \cup [\infty + i, -\infty + i] \cup [-\infty + i, -\infty - i] \) in counterclockwise manner. Due to the ANZC property of \( \frac{d}{dz} \log T_k(z) \) inside \( C \), the following equation is valid from Cauchy’s theorem:

\[ 0 = \int_{-\infty}^{\infty} \frac{d}{dx} \log T_k(x - i) e^{iz} dx - \int_{-\infty}^{\infty} \frac{d}{dx} \log T_k(x + i) e^{iz} dx. \]

One substitutes eqs(21), rewritten in terms of \( \Psi_{1,2}(x), \Phi(x), \tilde{\Phi}(x) \), into the above equation and derives identities among \( \widetilde{d\!l}\Psi_{1,2}[q], \widetilde{d\!l}\Phi[q], \widetilde{d\!l}\tilde{\Phi}[q], \)

\[ \widetilde{d\!l}\Psi_1[q < 0] = 0 \]
\[ \widetilde{d\!l}\Psi_1[q > 0] = \frac{e^{(1-\gamma)q}}{2 \cosh q} \frac{d\!l\tilde{\Phi}[q]}{d\!l\Phi[q]} - \frac{e^{-(1-\gamma)q}}{2 \cosh q} \frac{d\!l\Phi[q]}{d\!l\tilde{\Phi}[q]} \]
\[ + \pi i N e^{-kq} \sinh q \cosh(1 + u)q \]
\[ \cosh q \sinh q \]
\[ \widetilde{d\!l}\Psi_2[q < 0] = -\frac{e^{(1-\gamma)q}}{2 \cosh q} \frac{d\!l\Phi[q]}{d\!l\tilde{\Phi}[q]} + \frac{e^{-(1-\gamma)q}}{2 \cosh q} \frac{d\!l\tilde{\Phi}[q]}{d\!l\Phi[q]} \]
\[ - \pi i N e^{kq} \sinh q \cosh(1 + u)q \]
\[ \cosh q \sinh q \]
\[ \widetilde{d\!l}\Psi_2[q > 0] = 0, \]
and \( \bar{d} \Psi_1[q = 0] = -\bar{d} \Psi_2[q = 0] = \pi N \bar{r}i \). Similarly, one can derive an identity for \( \bar{d}l y_j[q] \)'s and \( \bar{d}l T_{k-1}[q] \) and \( \bar{d}l Y_{k-1}[q] \) from \([23]\). Substituting these relations into the original definitions of \( a \) and \( \bar{a} \), we obtain \( k + 1 \) algebraic relations in Fourier space. (Remember \( \Psi_{1,2}(x) \) are related to \( Q(x) \) by \([24]\).) After taking the inverse Fourier transformation and integrating over \( x \) once, we arrive at the \( k + 1 \) coupled nonlinear integral equations:

\[
\begin{pmatrix}
\log y_1(x) \\
\vdots \\
\log y_{k-1}(x) \\
\log a(x)
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0 \\
\beta H + d(u, x - i\gamma)
\end{pmatrix} + \mathcal{K} * 
\begin{pmatrix}
\log Y_1(x) \\
\vdots \\
\log Y_{k-1}(x) \\
\log \mathfrak{A}(x)
\end{pmatrix}
\]

where \((\mathcal{K} * g)_i\) denotes the matrix convolution, \( \sum_j K_{i,j}(x - y)(g(y))_j dy \) and the ”driving” function \( d(u, x) \) reads

\[d(u, x) = N \frac{2}{q \cosh q} e^{-i qx} dq.\]

The integration constants \((\pm \beta H)\) are fixed by comparing asymptotic values \(|x| \to \infty\) of both sides.

Explicitly the kernel matrix is given by

\[
\mathcal{K}(x) :=
\begin{pmatrix}
0, K(x), 0, \cdots, 0, 0, 0, 0 \\
K(x), 0, K(x), \cdots, 0, 0, 0 \\
0, K(x), 0, 0, \vdots, \vdots \\
\vdots, \vdots, \vdots, \vdots \\
0, \cdots, 0, K(x), 0, 0, 0 \\
0, \cdots, 0, K(x - i\gamma), K(x), \cdots, 0, 0 \\
0, \cdots, 0, K(x - i\gamma), F(x), -F(x + 2i(1 - \gamma)) \\
0, \cdots, 0, K(x + i\gamma), -F(x - 2i(1 - \gamma)), F(x)
\end{pmatrix}
\]

where

\[
K(x) := \frac{1}{4 \cosh \pi x/2} \\
F(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|q| - i qx} dq.
\]

\( T_k \), in terms of these auxiliary functions, can be derived similarly. Technically, we find it convenient to introduce,

\[
T_k^R(x) := \frac{T_k(x)}{\prod_{p=1}^{k} e^{e(p)(x - 2i(k + 1 - p)) e^{e(p+1)(x + 2i(k + 1 - p))}}}
\]

and \( e(p) = +(-) \) for \( p = \text{even (odd)} \). Then the product of two equations in \([24]\) leads to a simple algebraic relation:

\[
T_k^R(x - i)T_k^R(x + i) = \begin{cases} 
\mathfrak{B}(x)\mathfrak{B}(x), & \text{for } k \text{ even} \\
\mathfrak{B}(x)\mathfrak{B}(x) & \phi_-(x+i)\phi_+(x-i)\phi_-(x+i) & \text{for } k \text{ odd}
\end{cases}
\]
From ANZC property of the both sides of (28) in appropriate strips, the logarithmic derivative of them reduces to a simple algebraic equation in Fourier space. Taking account of the normalization (17) and (27), we have

\[
\log \Lambda_{QM}(u, x) = \log \Lambda_{QM}(0) + \int \frac{\log \mathfrak{A}(y)}{4 \cosh \pi/2 (x - y + i\gamma)} dy + \int \frac{\log \mathfrak{A}(y)}{4 \cosh \pi/2 (x - y - i\gamma)} dy
\]

\[
\log \Lambda_{QM}^{(0)}(u, x) := -\delta_{k=(mod2)} \frac{N}{2} \int \frac{e^{-iqx - |q| \sinh uq}}{q \cosh q} dq + \sum_{p=1}^{k} \log \left\{ \frac{\phi_{e(p)}(x - 2i(k + 1 - p))\phi_{e(p+1)}(x + 2i(k + 1 - p))}{\phi_{0}(2ip)^{2}} \right\}.
\]

Finally put \( u = -\beta J/N \) and send \( N \to \infty \) analytically. This merely amounts to replacements:

\[
d(u, x) \to D(x) = -\frac{\beta J \pi}{2 \cosh \pi/2x}
\]

\[
\log \Lambda_{QM}^{(0)}(u, x = 0) \to -\beta e_{0} = -\frac{\beta J}{2} \sum_{j=1}^{k} \frac{(-1)^{k-j}}{j} + \delta_{k=(mod2)} \beta J \log 2.
\]

Note that \( e_{0} \) coincides with the known ground state energy \( [2] \) after a trivial shift which stems from the difference in the normalization of \( R^{i} \). The \( k + 1 \) coupled nonlinear integral equations and \( \log \Lambda_{QM} \) do not carry the fictitious parameter \( N \) any longer. They efficiently describe the thermodynamics of the solvable spin-S XXX model. For an illustration, the specific heat for \( S = 1/2, 1, 3/2 \) is evaluated for a wide range of temperature and plotted below. Each curve is produced by 10-30 minutes CPU time calculation on a Micro Sparc Work Station. In the next section, we derive the low temperature properties using (25), (29) and (31).
6 Analytic Evaluation of the Low Temperature Asymptotics

We consider $T \to 0$ for the vanishing magnetic field. In sufficiently low temperature regime, $a, \log A$ shows a sharp crossover behavior like a step function: $|a|, |\log A| \ll 1$ for $|x| < \frac{2}{\pi} \log \pi \beta J$ and $|a|, |\log A| = O(1)$ for $|x| > \frac{2}{\pi} \log \pi \beta J$. Thus the following scaling functions \([23]\) control the asymptotic behavior:

$$
\begin{align*}
la^\pm(\xi) & := \log a(\pm \frac{2}{\pi} (\xi + \log \pi \beta J)), & lA^\pm(\xi) & := \log A(\pm \frac{2}{\pi} (\xi + \log \pi \beta J)), \\
l\bar{a}^\pm(\xi) & := \log \bar{a}(\pm \frac{2}{\pi} (\xi + \log \pi \beta J)), & l\bar{A}^\pm(\xi) & := \log \bar{A}(\pm \frac{2}{\pi} (\xi + \log \pi \beta J)), \\
l_y^\pm(\xi) & := \log y(\pm \frac{2}{\pi} (\xi + \log \pi \beta J)), & lY^\pm_p(\xi) & := \log Y(\pm \frac{2}{\pi} (\xi + \log \pi \beta J)).
\end{align*}
$$

(32)

In terms of these scaling functions, NLIE are expressed by,

$$
\begin{pmatrix}
l_y^1(\xi) \\
\vdots \\
l_y^{k-1}(\xi) \\
l_a^\pm(\xi) \\
l\bar{a}^\pm(\xi)
\end{pmatrix} =
\begin{pmatrix}
0 \\
\vdots \\
0 \\
-e^{-\xi \pm \imath \gamma \pi/2} \\
-e^{-\xi \mp \imath \gamma \pi/2}
\end{pmatrix} + \tilde{\mathcal{K}} *
\begin{pmatrix}
lY^\pm_1(\xi) \\
\vdots \\
lY^\pm_{k-1}(\xi) \\
lA^\pm(\xi) \\
l\bar{A}^\pm(\xi)
\end{pmatrix}
$$

$$
\tilde{\mathcal{K}}(\xi) = \frac{2}{\pi} \mathcal{K}\left(\frac{2\xi}{\pi}\right)
$$

(33)

Note that neglect of small corrections $\sim O(T)$ leads to the decoupling equations for $\pm$.

The thermal contribution (the second and the third term in (29)) to $\lim_{N \to \infty} \log \Lambda_{\text{QTM}}(u = -\frac{\beta J}{\mathcal{K}}, x)$ reads,

$$
\begin{align*}
&\frac{e^{\pi x/2}}{\pi^2 \beta J} [e^{\imath \gamma \pi/2} \int e^{-\xi lA^+} d\xi + e^{-\imath \gamma \pi/2} \int e^{-\xi l\bar{A}^+} d\xi] \\
&+ \frac{e^{-\pi x/2}}{\pi^2 \beta J} [e^{-\imath \gamma \pi/2} \int e^{-\xi lA^-} d\xi + e^{\imath \gamma \pi/2} \int e^{-\xi l\bar{A}^-} d\xi].
\end{align*}
$$

(34)

The crucial observation in $[23]$ is that one needs not solve (33) to evaluate (34) provided that the kernel matrix function satisfies a symmetry, $\mathcal{K}_{i,j}(x-y) = \mathcal{K}_{j,i}(y-x)$. This property is valid in the present case. See \([26]\).

We define $F_\pm$ by

$$
\begin{align*}
F_\pm & := \int_{-\infty}^{\infty} \sum_{p=1}^{k-1} \left[ \frac{d}{d\xi} l_y^p \right] Y^\pm_p - \left( \frac{d}{d\xi} lY^\pm_p \right) l_y^p + d\xi \\
&+ \int_{-\infty}^{\infty} \left[ \frac{d}{d\xi} l_a^\pm \right] lA^\pm + \left( \frac{d}{d\xi} l\bar{a}^\pm \right) l\bar{A}^\pm - \left( \frac{d}{d\xi} lA^\pm \right) l\bar{a}^\pm - \frac{d}{d\xi} l\bar{A}^\pm l\bar{a}^\pm + \frac{d}{d\xi} l\bar{A}^\pm l\bar{a}^\pm \right] d\xi.
\end{align*}
$$

(35)

Then the trick in \([23]\) is as follows. First, take the derivative of both sides of (33) and multiply them by a row vector

$$
\begin{pmatrix}
lY^\pm_1(\xi), \cdots, lY^\pm_{k-1}(\xi), lA^\pm(\xi), l\bar{A}^\pm(\xi)
\end{pmatrix}.
$$

(36)
We call the resultant equality (A). Second, multiply both sides of (33) by the derivative of the row vector (36), which is referred to as (B). Finally, subtract both sides of (A) and (B), and integrate over ξ. Then the lhs of the resultant equality is nothing but $F_\pm$. Remarkably, the most complicated terms in the rhs, like

$$\int d\xi d\xi' lY^+_i(\xi) \frac{d\bar{K}_{i,j}(\xi - \xi')}{d\xi} lY^+_j(\xi')$$

and

$$- \int d\xi d\xi' \frac{d}{d\xi} lY^+_j(\xi) \bar{K}_{j,i}(\xi - \xi') lY^+_i(\xi')$$

cancel with each other. To be precise, the first integral can be converted step by step,

$$= - \int d\xi d\xi' lY^+_i(\xi) \frac{d\bar{K}_{i,j}(\xi - \xi')}{d\xi'} lY^+_j(\xi')$$

$$= - \int d\xi d\xi' lY^+_i(\xi) \frac{d\bar{K}_{j,i}(\xi' - \xi)}{d\xi} lY^+_j(\xi')$$

$$= \int d\xi d\xi' lY^+_i(\xi) \bar{K}_{j,i}(\xi' - \xi) \frac{d}{d\xi'} lY^+_j(\xi')$$

$$= \int d\xi d\xi' \frac{d}{d\xi} lY^+_j(\xi) \bar{K}_{j,i}(\xi - \xi') lY^+_i(\xi')$$

where the symmetry of the kernel matrix, partial integration and the change of integration variables $\xi \leftrightarrow \xi'$ are used.

Similar cancellation happens for other terms. and the following equality results,

$$F_\pm = 2 \int [e^{-\xi \pm i \gamma \pi / 2} lA^\pm + e^{-\xi / 2} l\bar{A}_\pm] d\xi.$$  

(37)

The first thermal correction (34) is thus given by

$$\frac{e^{\pi x / 2} F_+}{2\pi^2 \beta J} + \frac{e^{-\pi x / 2} F_-}{2\pi^2 \beta J}.  

(38)

To evaluate $F_\pm$ explicitly, one rewrites the integration variable from $\xi$ to $a, \bar{a}, y_p$. For example, the first summation term in $F_\pm$ is transformed to

$$\sum_{p=1}^{k-1} \int_{y_b^\pm(-\infty)}^{y_b^\pm(\infty)} dy \left( \frac{\log(1 + y)}{y} - \frac{\log y}{1 + y} \right) = 2 \sum_{p=1}^{k-1} \{ L_+(y_p^+ (\infty)) - L_+(y_p^+ (-\infty)) \},$$

and similarly for others. $L_+(x)$ is a dilogarithm function and is related to Rogers’ dilogarithm function $L(x)$ by $L_+(x) = L(x/(1 + x))$,

$$L_+(x) := \frac{1}{2} \int_0^x \left( \frac{\log(1 + y)}{y} - \frac{\log y}{1 + y} \right) dy,$$

$$L(x) := - \frac{1}{2} \int_0^x \left( \frac{\log(1 - y)}{y} + \frac{\log y}{1 - y} \right) dy.$$

(39)
The asymptotic values of scaling functions are easily extracted. For \( x \to \infty \), \( a^\pm \) coincides with original \( b \). Thus one derives the limiting value by its definition (19) in terms of \( \lambda_p^{(k)} \)'s. Similar for \( y_p^\pm(\infty) \). For \( x \to -\infty \), one should rather consult (33). We send the argument \( x \to -\infty \) in both side and solve the resultant algebraic equations. The results are summarized as,

\[
\begin{align*}
a^\pm(\infty) &= \bar{a}^\pm(\infty) = 0, \quad a^\pm(\infty) = \bar{a}^\pm(\infty) = k, \\
y_p^\pm(\infty) &= \frac{\sin \frac{\pi p}{k+2} \sin \frac{\pi(p+2)}{k+2}}{\sin^2 \frac{\pi}{k+2}} \quad 1 \leq p \leq k - 1, \\
y_p^\pm(-\infty) &= p(p+2) \quad 1 \leq p \leq k - 1.
\end{align*}
\]

With these pieces of information, \( F^\pm \) is now given by

\[
F_+ = F_- = 2 \sum_{p=1}^{k-1} \left[ L\left( \frac{p(p+2)}{(p+1)^2} \right) - L\left( \frac{\sin \frac{\pi p}{k+2} \sin \frac{\pi(p+2)}{k+2}}{\sin^2 \frac{\pi}{k+2}} \right) \right] + 4L\left( \frac{k}{1+k} \right)
\]

Finally we use three relations (39):

\[
\begin{align*}
L(1) &= L(x) + L(1-x) = \frac{\pi^2}{6}, \quad x \in [0, 1] \\
2L(1) &= 2L\left( \frac{1}{n+1} \right) + \sum_{j=0}^{n-1} L\left( \frac{1}{(1+j)^2} \right), \quad n \in \mathbb{Z}_{\geq 0}, \\
L(1) \frac{3n}{n+2} &= \sum_{j=0}^{n-1} L\left( \frac{\sin^2 \frac{\pi}{n+2} \sin^2 \frac{\pi(j+1)}{n+2}}{\sin^2 \frac{\pi(j+1)}{n+2}} \right), \quad n \in \mathbb{Z}_{\geq 0}
\end{align*}
\]

which yield the neat result

\[
F = \frac{\pi^2 k}{k + 2}.
\]

Thus we conclude the low temperature asymptotics of free energy

\[
f \sim e_0 - \frac{\pi}{6v_s^2}c(k), \quad c(k) = 3k/(k + 2),
\]

where \( v_s = J\pi/2 \) coincides with the known spin velocity and \( c(k) \) is nothing but the central charge of level-\( k \) SU(2) WZWN model. This is the desired result from the WZWN description of massless quantum spin chains. We remark that the final part of the calculation is quite parallel to that in [11] utilizing the same dilogarithm function identities. There the spin-S XXZ model is discussed via the standard string approach at "root of unity" where the number of strings is truncated finitely from the beginning.

### 7 Spinon Characters

The character formulae obviously depend on the base of the space. Recently, the quasi-particle representation has been attracted much attention in the context of long-range interacting model
spectral decomposition of path space in lattice models \cite{24, 11, 42}, or in the statistical interacting picture of Bethe Ansatz solvable models \cite{13}-\cite{17}. See also \cite{15, 19} in the different view points. For the spin-1/2 case, it has been shown recently that the novel thermodynamics formulation yields a natural spinon character and such character formula is generalized to $\hat{sl}(n)_{k=1}$ \cite{51}. Thus it is tempting to find analogues for $\hat{sl}(n)_{k=2}$. The results given in the previous sections provide the first step for the simplest $n=2$ case as discussed below. Remark that we consider "chiral-half", such that the only + part contributions (say $F_+$) in the previous section are taken into account.

The character needs the description of all excited states. In the present contexts, this information might be encoded in the additional zeros of $a, \bar{a}, y$ and their capitals in their "physical strips". Indeed, some low excitations are identified in such a way \cite{16, 18}, and corresponding excited state TBA are derived. Such an analysis is of considerable interest, however it requires extensive numerical efforts. We leave it as a future problem and make a short-cut detour here employing the strategy in \cite{50}.

The central charge is described by the dilogarithm function of which the integration contour is simple. On the other hand, one can define an analytically continued dilogarithm function $L_C(z)$. This is established by adopting general contour $C$ for the integration contour of the dilogarithm function. We then generalize a successful observation from specific examples; all excitation spectra, or effective central charge $c_{\text{eff}}$, in the conformal limit shall be described by $L_C(z)$. Namely, the replacement of a simple contour in the integral representation of the dilogarithm function by complex one leads to an excited state. Regarding $c_{\text{eff}}$ as a function of $L_C(z)$, the summation of $q^{-c_{\text{eff}}/24}$ over a certain set of $C$ is thus expected to reproduce affine characters. (Readers should not confuse this formal variable $q$, the standard notation in this field, with the Fourier variable used in previous sections.)

Let us be more precise. By $C$ we denote a contour starting from $f_-$ and terminating at $f_+$, such that it crosses firstly $[1, \infty)$ $\eta_1(\neq 0)$ times then crosses $(-\infty, 0]$ $\xi_1$ times then again $\eta_2$ times w.r.t. $[1, \infty)$ and so on. The intersections are counted as $+1(-1)$ if the contour goes across the cut $[1, \infty)$ in the counterclockwise (clockwise) manner and $(-\infty, 0]$ in the clockwise (counterclockwise) manner. (Note the definition is slightly different from \cite{50, 51}.) We denote this by $C[f_-, f_+|\{\xi_1, \xi_2, \cdots\}|\{\eta_1, \eta_2, \cdots\}]$. The set of contours are parameterized by $\mathcal{S} = \{C[f_-(p), f_+(p)|\xi_1(p), \cdots, \eta_1(p), \cdots]\}$, where $f_\pm(p) = y_p^\pm(\pm\infty)/Y_p^\pm(\pm\infty), (1 \leq p \leq k-1)$, $f_+(k) = a^+(\infty)/\mathfrak{a}^+(\infty)$, $f_+(k+1) = \bar{a}^+(\infty)/\bar{\mathfrak{a}}^+(\infty)$ and $f_-(k) = f_-(k+1) = 0$.
In the absence of additional zeros of auxiliary functions in the "physical strip" we have
\[ 0 = \log f_+^{(p)} - \sum_{p'} g_{p,p'} \log (1 - f_+^{(p')}) \tag{42} \]
where the “statistical interaction” matrix \( g \) is related to the zero mode of the Fourier transformation of the kernel matrix \( \mathbf{K} \) by
\[ g = I - \mathbf{K}[q = 0]. \tag{43} \]
This is the situation we have treated in previous sections, and (42) follows from (43). One replaces \( \log \) in (42) by an analytically continued one, \( \log \text{Log} \) so as to match its interpretation as chemical potential \[ \eta \]
in (44), in principle. Though such microscopic derivation is yet to be done, we assume the coincidence in the following.

Now, the excitation spectrum is solely implemented in the effective central charge \( c_{\text{eff}}(S) \)
\[ c_{\text{eff}}(S) = \frac{6}{\pi^2} \sum_p \left( L_{C(p)}(f_+^{(p)}, f_-^{(p)}) - \frac{\pi i}{2} D^{(p)} \log C^{(p)}(1 - f_+^{(p)}) \right). \tag{45} \]
Here \( D^{(p)} \) term is included by hand, so as to match its interpretation as chemical potential \[ \xi \]
\( L_C \) is given by,
\[ L_C(f_-, f_+) = -\frac{1}{2} \int_C \left( \frac{\text{Log} C(1 - z)}{z} + \frac{\text{Log} C(z)}{1 - z} \right) dz. \tag{46} \]
for a path \( C \) from \( f_- \) to \( f_+ \).

After straightforward manipulations, one finds \[ \eta \]
\[ L_{C(p)}(f_-^{(p)}, f_+^{(p)}) = L(f_+^{(p)}) - L(f_-^{(p)}) - \pi i \sum_\ell \xi_\ell^{(p)} \log (1 - f_+^{(p)}) - \pi i \sum_\ell \eta_\ell^{(p)} \log (f_+^{(p)}) \]
\[ + 2\pi^2 \sum_\ell \xi_\ell^{(p)} (\sum_\ell \eta_\ell^{(p)}) - 4\pi^2 \sum_\ell \xi_\ell^{(a)} (\eta_1^{(a)} + \cdots + \eta_\ell^{(a)}). \tag{47} \]

Remember \( L(z) \) is defined in (43). The substitution of (47) in (45), using (44) and the definitions of \( L_{C(p)} \) leads to a remarkable result: \( c_{\text{eff}} \) can only be written in terms of \( c(k) \) (see (44)), \( \{\xi_\ell^{(p)}\} \) and \( \{\eta_\ell^{(p)}\} \),
\[ c_{\text{eff}}(S) = c(k) - 24 \mathcal{T}(S), \quad \mathcal{T}(S) = \frac{1}{2} \mathbf{m} \mathbf{n} + \sum_{a=1}^{k+1} \sum_{\ell \geq 1} \xi_\ell^{(a)} (\eta_1^{(a)} + \cdots + \eta_\ell^{(a)}), \tag{48} \]
\[ \mathbf{m} = (n^{(1)}, \ldots, n^{(k+1)}) \]
Figure 8: Young diagrams $YD^{(p)}$ of $n^{(p)} = 3$ corresponding to a set of winding numbers: $\{\xi\} = \{1, 1, 1\}, \{\eta\} = \{1, 1, 2\}$ (left), $\{\xi\} = \{1, 1, 1\}, \{\eta\} = \{0, 1, 1\}$ (middle) and $\{\xi\} = \{1, 1, 1\}, \{\eta\} = \{1, 1, 0\}$ (right). Arrows are put for a guide to eyes. The most right diagram is termed "with tail" in the text.

where $n^{(p)} = \sum_{l} \eta_{l}^{(p)}$. Note that the explicit forms of $D^{(p)}$ are not needed in the above transformation.

In the following, we shall argue that the summation of $q^{-c_{\text{eff}}}$ over some subset $O$ of all possible contours reproduce the character $\text{ch}_j(z, q)$ of level $k$ WZN model with spin-$j$ ($j = \text{some fixed integer or half-integer}$).

We present necessary conditions for such $O$ below.

- $\eta'$s and $\xi'$s are non-negative.

Such path can be parameterized by

$$C[f^{(p)}, f^{(p)}_+ | \xi^{(p)}_1, \ldots, \xi^{(p)}_{n^{(p)}} | 1, \ldots, 1]$$

and $\xi^{(p)}_{l} \geq 0, (p = 1, \ldots, n^{(p)})$.

- For $p = k, k + 1$, we require $\frac{n^{(k)} + n^{(k-1)}}{2} - j \in \mathbb{Z}_{\geq 0}$ in addition.

- For $p \leq k - 1$, $\xi^{(p)}_{n^{(p)}} \geq 1$.

Graphically, one can associate a Young diagram $YD^{(p)}$ to a set of winding numbers: $\{\xi^{(p)}_{l}\}, \{\eta^{(p)}_{l}\}$ (or a Young diagram with tail for some cases in $p = k, k + 1$). First draw a line of length 1 downwards. Next draw a line of length $\xi^{(p)}_{1}$ to the left. Then again draw a line of length 1 downwards. We continue this procedure $n^{(p)}$ times. Finally draw a horizontal line from the starting point to the left and also draw a vertical line from the end point upwards. See the Figure.

Obviously, the number of boxes in the diagram is equal to the second term in $T(S)$.

We allow for contours which are isomorphic to a set of Young diagrams $YD^{(p)}, (p = 1, \ldots, k - 1)$ such that

- $n^{(k-2p+1)} = \text{odd}$, $n^{(k-2p)} = \text{even}$.

- By the definition, the depth of $YD^{(p)}$ is $n^{(p)}$. The width is restricted by the maximum value $u^{(p)}_{\text{max}}$ which is determined by depths of "adjacent" diagrams:

$$u^{(p)}_{\text{max}} = \frac{1}{2}(n^{(p-1)} + n^{(p+1)} - 2n^{(p)} + \delta_{2j,k-p}), \quad (p = 1, \ldots, k - 2)$$

for the fixed $j$ and $n^{(0)} = 0$. The case $p = k - 1$ is exceptional, $u^{(k-1)}_{\text{max}} = \frac{1}{2}(n^{(k-2)} + n^{(k)} + n^{(k+1)} - 2n^{(k-1)} + \delta_{j,1/2})$. 

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of certainly these constraints on winding numbers have "microscopic" origins in patterns of zeros we obtain the product of Gaussian q-binomial equivalent to the one in $\Psi$.

The latter, however, is derived under several assumptions on the possible homotopy class of $S_{\mathcal{O}}$. This has been demonstrated by the calculation of the low-temperature specific heat as well as the spinon character formula. This has been actually demonstrated for few cases: the superintegrable formulation\cite{2} and is obviously efficient in numerics. The resultant formulation has an interpretation in terms of physical excitations, spinons and RSOS kinks. This has been demonstrated for few cases: the superintegrable

**$S_{\mathcal{O}}$** stands for contributions from $(M,N)$ spinons;

$$S_{M,N}(z;q) = \frac{1}{(q)_{M}(q)_{N}^{2}} z^{(M-N)/2}$$

resulting from summations over "nodes" $z_{\ell}^{(p)} \geq 0$, $\ell = 1, \ldots, n_{(p)}$ and $p = k, k+1$. $A_k$ is the Cartan matrix for $sl_{k+1}$ and $\Psi_{A_k}$ denotes

$$\Psi_{A_k}^{m_1}(u_j;q) = \sum_{m_2,m_3,\ldots,m_k} q^{1/4m_1m_k} \prod_{i=2}^{k} \left[ \frac{1}{2^n((2-A_k).m + u_j)_i} \right]$$

and $(u_j)_i = \delta_{i,2j+1}$. The summations are taken over odd (even) positive integers for $m_{even(odd)}$. Note we redefine $n_{(k-\ell)} = m_{\ell+1}$, $1 \leq \ell \leq k-1$. The appearance of the Gaussian q-binomial usually originates from combinatorics on the truncated Bratteli diagram and is the reminiscence of the RSOS model. Here the origin is also simple. It comes from the restriction on the width of Young diagrams. We denote the number of boxes in a Young diagram $YD^{(p)}$ by $b^{(p)}$. For fixed $\{n^{(a)}\}$, $T(S)$ assumes the same value for diagrams having identical $b^{(p)}$. This multiplicity is given by $p(n^{(p)},w_{\text{max}}^{(p)},b^{(p)})$ = the number of partition of $b^{(p)}$ into at most $w_{\text{max}}^{(p)}$ part, each $\leq n^{(p)}$. Thanks to the generating relation,

$$\sum_{b^{(p)}} p(n^{(p)},w_{\text{max}}^{(p)},b^{(p)}) q^{b^{(p)}} = \left[ \frac{n^{(p)} + w_{\text{max}}^{(p)}}{n^{(p)}} \right] = \left[ \frac{(n^{(p-1)} + n^{(p+1)} + \delta_{p,k-2j})/2}{n^{(p)}} \right]$$

we obtain the product of Gaussian q-binomial equivalent to the one in $\Psi_{A_k}^{m_1}(u;q)$.

Our result implies that the Hilbert space of level $k$ SU(2) WZW model is isomorphic to a part of homotopy space of Rogers’ dilogarithm function. We believe that this provides a novel interesting view point in the prototype of CFT and deserves further examinations.

**8 Summary and Discussion**

In this report, we formulate a novel description of the thermodynamics of solvable spin-$S$ XXX models. The suitable choices of auxiliary functions yield a natural generalization of the strategy in $\Psi$. The nonlinear integral equations close finitely, which clearly differs from the string formulation\cite{4} and is obviously efficient in numerics. The resultant formulation has an interpretation in terms of physical excitations, spinons and RSOS kinks. This has been demonstrated by the calculation of the low-temperature specific heat as well as the spinon character formula. The latter, however, is derived under several assumptions on the possible homotopy class of $L_C$. Certainly these constraints on winding numbers have "microscopic" origins in patterns of zeros of $y$—(or $T$) functions. This has been actually demonstrated for few cases: the superintegrable
3 state chiral Potts model \[ 52 \] and the simplest case of \( sl_2 \) RSOS models with open boundaries \[ 53 \]. We hope to report on extensive numerical investigations on zeros in the present context in the near future.

Finally we mention spinon pictures arisen from different view points \[ 54, 55, 56 \], where explicit "spinon" bases are constructed by vertex operators. Thus the meaning of "spinon" is more transparent in comparison to the present approach. Less obvious is their concrete relation to eigenstates of spin Hamiltonians. The method also involves an uncontrolled approximation, a truncation procedure, in evaluating the partition function as well as "one body" distribution functions. This contrasts to the present approach which involves no approximation. In a sense they are complementary, and their relations are to be explored.

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