Solving the Darwin problem in the first post-Newtonian approximation of general relativity

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Abstract

We analytically calculate the equilibrium sequence of the corotating binary stars of incompressible fluid in the first post-Newtonian (PN) approximation of general relativity. By calculating the total energy and total angular momentum of the system as a function of the orbital separation, we investigate the innermost stable circular orbit for corotating binary (we call it ISCCO). It is found that by the first PN effect, the orbital separation of the binary at the ISCCO becomes small with increase of the compactness of each star, and as a result, the orbital angular velocity at the ISCCO increases. These behaviors agree with previous numerical works.

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I. INTRODUCTION

The laser interferometers such as LIGO [1], VIRGO [2], GEO600 [3] and TAMA300 [4] are currently being constructed and the detection of gravitational waves is expected in this decade. One of the most important astrophysical sources of gravitational waves for these detectors is the coalescing binary neutron stars (BNS’s) because the gravitational waves emitted in the inspiral phase (the so-called last three minutes) [5] have frequencies in the sensitive region of these detectors, i.e., from 10Hz to 1000Hz. We will be able to know each mass, spin, and so on of the BNS’s if we could obtain an accurate theoretical template for data analysis [6]. Hence, much theoretical works have been done to complete it [7] [8] [9].

When the orbital separation of the BNS’s becomes a few times of the neutron star (NS) radius as a result of the radiation reaction of gravitational wave emission, the hydrodynamical effect becomes important. In such a phase, the wave form of gravitational waves is expected to be sensitive to the NS structure, especially the relation between the radius and mass of the NS. Therefore, if gravitational waves from such a phase are detected, we may constrain the equation of state (EOS) of the NS [10] [11] [12]. In particular, the important quantity is the location of the innermost stable circular orbit (ISCO), which will have an information on the EOS of the NS.

There have been many analyses for the ISCO using some approximations of general relativity [13] [14] [15] [16], but in order to determine the precise location of the ISCO, we need fully general relativistic (GR) numerical simulation, which is a very difficult method. Recently, Wilson and his collaborators [17] have performed numerical simulation for obtaining the equilibrium sequence of the BNS and its ISCO solving semi-relativistic equations. The results they have obtained are very interesting, but the accuracy of their results is still in question because they do not show any calibration of their numerical code using test problems. Furthermore, they do not seem to perform any important analysis to their numerical results. When we carry out a large numerical simulation, it is required to perform a detailed analysis after the computing in order to explain the numerical results. The analysis
is desired to be done by comparing the analytical or semi-analytical estimates. If the numerical results qualitatively agree with such an analytical calculation, we can firmly believe the numerical results and also can understand details. Hence, with a large scale simulation, it is favorable to prepare some analytical models in order to understand the essence included in the numerical results.

For that purpose, we here analytically solve the Darwin problem in the first PN approximation. The Darwin problem is concerned with the equilibrium and the stability of a homogeneous fluid star rotating around another one taking into account the mutual tidal interactions [18]. We assume that each NS in the binary system is composed of the incompressible and homogeneous fluid. The reward is that all the calculations can be done analytically. This means that we can obtain a strict solution of a BNS including the GR effect without any large supercomputing. Our results will be very helpful for understanding how the finite-size effects as well as the GR one of the NS’s influence to the location and the orbital angular velocity at the ISCO, and for the analysis of large scale simulation.

This paper is organized as follows. In section II, we show the basic equations to solve the PN Darwin problem. By using the first tensor virial (TV) equations, we derive the angular velocity of corotating binary systems in section III. In section IV, we show the equations of the total energy and the total angular momentum for corotating binary systems. In section V, we calculate the equilibrium sequences and determine the location of the energy and angular momentum minimums (we call it the innermost stable corotating circular orbit (ISCCO) to distinguish it from the innermost stable circular orbit (ISCO) [22]). Section VI is devoted to summary.

Throughout this paper, we use the unit of \( G = 1 \), and \( c \) denotes the light velocity. Latin indices \( i, j, k, \cdots \) take 1 to 3, and \( \delta_{ij} \) denotes the Kronecker’s delta. We use \( I_{ij} \) and \( T_{ij} \) as

\[ ^\dagger\text{This point is the secular instability limit, and not the dynamical instability limit (i.e., ISCO).} \]

The ISCO will locate inside the ISCCO [23].
the quadrupole moment and its trace free part,

\[ I_{ij} = I_{ij} - \frac{1}{3} \delta_{ij} \sum_{k=1}^{3} I_{kk}, \]  

(1.1)
of each star of binary.



II. FORMULATION

Non-axisymmetric equilibrium configurations of uniformly rotating incompressible fluid in the first PN approximation are obtained by solving the integrated form of the Euler equation and the Poisson equations for gravitational potentials consistently. The integrated form of the Euler equation was derived by Chandrasekhar [19] and it can be written as [20]

\[ \frac{P}{\rho} - \frac{1}{2c^2} \left( \frac{P}{\rho} \right)^2 = U - \frac{X_0}{c^2} + \left\{ \frac{\bar{\omega}^2}{2} + \frac{1}{c^2} \left( 2\bar{\omega}^2 U - X_0 + \hat{\beta}\phi \right) \right\} \Omega^2 + \frac{\bar{\omega}^4}{4c^2} \Omega^4 + \text{const.}, \]  

(2.1)

where

\[ \bar{\omega}^2 = \left( x_1 + \frac{R}{2} \right)^2 + x_2^2, \]  

(2.2)

and \( U, X_0, X_\Omega, \) and \( \hat{\beta}\phi \) are the gravitational potentials. In this paper, we consider the equilibrium sequences of BNS’s of equal masses \( M_1 = M_2 = M \) whose coordinate separation is \( R \). We assume that the center of mass of a star (star 1) locates at the origin and the other one (star 2) locates at \( (x_1, x_2, x_3) = (-R, 0, 0) \) (see fig.1). Due to the symmetry, we only pay attention to the equilibrium configuration of star 1 in the following.

The main purpose of this paper is to calculate the PN correction of the angular velocity, the energy and the angular momentum for the Darwin problem. In the Newtonian order, the angular velocity \( \Omega_N \) becomes [23] (see below for derivation)

\[ \text{§The coordinate condition in this paper is the standard PN one [22].} \]
\[ \Omega_N^2 = \frac{2M}{R^3} + \frac{18 I_{11}}{R^5}. \] (2.3)

Thus, in the PN approximation, we can expect that the following types of quantities will be the main terms in the PN order:

\[ \sim \frac{M}{R^3} \times \frac{M}{a_0 c^2}, \quad \sim \frac{M}{R^3} \times \frac{M}{R c^2}, \quad \sim \frac{M a_0^2}{R^5} \times \frac{M}{a_0 c^2}, \quad \text{and} \quad \sim \frac{M a_0^2}{R^5} \times \frac{M}{R c^2}, \] (2.4)

where \( a_0 \) is a typical radius of the star, and we use the relation \( I_{11} \sim M a_0^2 \). We will derive four types of terms shown above and the correction of the energy and the angular momentum by them below.

Since we consider the incompressible fluid, the gravitational potentials inside each star are expressed as the polynomial form of the coordinates \( x_i \). For the purpose of obtaining the PN corrections shown above, we need to take into account the coefficients of the terms such as \( x_1^{m_1} x_2^{m_2} x_3^{m_3} \) in \( X_0, \Omega^2 \Omega_{\hat{\beta} \phi}, \) and so on, where \( 0 \leq m_1, m_2, m_3 \leq 5 \) and \( 0 \leq m_1 + m_2 + m_3 (\equiv m_t) \leq 5 \), up to \( O(R^{-5}) \) for the case \( m_t \) is odd and up to \( O(R^{-3}) \) for the case \( m_t \) is even. In the following subsections, we solve the Poisson equations for the gravitational potentials to derive such terms.

**A. Newtonian Quantities**

Each gravitational potential is composed of two parts; one is the contribution from star 1 and the other is from star 2. In the following, we denote the former part as \( \phi^{1 \rightarrow 1} \), and the latter one as \( \phi^{2 \rightarrow 1} \), where \( \phi \) denotes one of the potentials. We also define \( \phi^{1 \rightarrow 2} \) and \( \phi^{2 \rightarrow 2} \) as the contribution from star 1 to 2 and star 2 itself, respectively.

Following previous authors [18, 23], the configuration of each star of binary in the Newtonian order is assumed to be an ellipsoidal figure of its axial length \( a_1, a_2 \) and \( a_3 \). In this case, the solution of the Poisson equation for the Newtonian potential

\[ \Delta U = -4\pi \rho, \] (2.5)

is written as \( U = U^{1 \rightarrow 1} + U^{2 \rightarrow 1} \), where
\[ U^{1 \rightarrow 1} = \pi \rho \left( A_0 - \sum_i A_i x_i^2 \right), \quad (2.6) \]

\[ U^{2 \rightarrow 1} = \frac{M}{R} \left\{ 1 - \frac{x_1}{R} + \frac{2x_1^2 - x_2^2 - x_3^2}{2R^2} + \frac{-2x_1^3 + 3x_1(x_2^2 + x_3^2)}{2R^3} \right. \]
\[ + \frac{8x_1^4 + 3x_2^4 + 3x_3^4 - 24x_1^2(x_2^2 + x_3^2) + 6x_2^2x_3^2}{8R^4} \left. \right\} \]
\[ + \frac{3\mathcal{I}_{11}}{2R^3} \left( 1 - \frac{3x_1}{R} + \frac{12x_1^2 - 5x_2^2 - 5x_3^2}{2R^2} \right) + \frac{3}{2R^5} \left( \mathcal{I}_{22}x_2^2 + \mathcal{I}_{33}x_3^2 \right), \quad (2.7) \]

and

\[ I_{ijkl \ldots} = \int d^3 x \rho x_i x_j x_k x_l \ldots. \quad (2.8) \]

\[ \rho a^2 \] are index symbols introduced by Chandrasekhar \cite{18}, and \( A_0 = \sum_l A_l a_l^2 \) is calculated from \cite{18}

\[ A_0 = a_1 a_2 a_3 \int_0^\infty \frac{du}{\sqrt{(a_1^2 + u)(a_2^2 + u)(a_3^2 + u)}} \quad (2.9) \]
\[ = a_1^2 a_2 a_3 \int_0^\infty \frac{dt}{\sqrt{(1 + t)(a_2^2 + t)(a_3^2 + t)}} \equiv a_1^2 \tilde{A}_0, \quad (2.10) \]

where \( \alpha_2 = a_2/a_1 \) and \( \alpha_3 = a_3/a_1 \). Note that \( U^{2 \rightarrow 2} \) and \( U^{1 \rightarrow 2} \) are obtained by changing \( x_1 \) into \(- (x_1 + R)\) in \( U^{1 \rightarrow 1} \) and \( U^{2 \rightarrow 1} \), respectively.

In the Newtonian order, the pressure is written as

\[ P = P_0 \left( 1 - \sum_i \frac{x_i^2}{a_i^2} \right). \quad (2.11) \]

\( P_0 \) is calculated from the scalar virial relation as

\[ P_0 = \frac{\rho}{3} \left[ \pi \rho A_0 - \frac{\Omega_2^2}{2} (a_1^2 + a_2^2) - \frac{M}{2R^3} \left( 2a_1^2 - a_2^2 - a_3^2 \right) \right] + O(\mathcal{R}^{-5}), \quad (2.12) \]

and \( \alpha_2 \) and \( \alpha_3 \) are determined from \cite{23}

\[ - \frac{P_0}{\rho a_1^2} = -\pi \rho A_1 + \frac{\Omega_2^2}{2} - \frac{M}{R^3}, \quad (2.13) \]
\[ - \frac{P_0}{\rho a_2^2} = -\pi \rho A_2 + \frac{\Omega_2^2}{2} + \frac{M}{2R^3}, \quad (2.14) \]
\[ - \frac{P_0}{\rho a_3^2} = -\pi \rho A_3 - \frac{M}{2R^3}. \quad (2.15) \]
B. $X_0$

As in the case of $U$, the PN potentials are divided into two parts as $X_0 = X_0^{1\rightarrow 1} + X_0^{2\rightarrow 1}$, and we consider them separately.

- Contribution from star 1:

$X_0^{1\rightarrow 1}$ is derived from the Poisson equation \[\Delta X_0^{1\rightarrow 1} = 4\pi \rho \left[ 2\pi \rho \left( A_0 - \sum_l A_l x_i^2 \right) + \frac{3P_0}{\rho} \left( 1 - \sum_l \frac{x_i^2}{a_i^2} \right) + 2U^{2\rightarrow 1} \right],\] (2.16)
and the solution becomes

$$X_0^{1\rightarrow 1} = -\alpha_0 U^{1\rightarrow 1} + \alpha_1 D_1 + \sum_l \eta_l D_{il}$$
$$-\frac{M}{R^3} \left( 2D_{11} - D_{22} - D_{33} \right) - \frac{M}{R^4} \left( -2D_{111} + 3D_{122} + 3D_{133} \right),$$
(2.17)

where

$$\alpha_0 = 2\pi \rho A_0 + \frac{3P_0}{\rho} + \frac{2M}{R} + \frac{3I_{11}}{R^3},$$
(2.18)

$$\alpha_1 = \frac{2M}{R^2} + \frac{9I_{11}}{R^4},$$
(2.19)

$$\eta_l = 2\pi \rho A_l + \frac{3P_0}{\rho a_l^2}.$$  
(2.20)

$D_i$, $D_{ii}$, and $D_{1ii}$ are the solutions of equations

$$\Delta D_i = -4\pi \rho x_{i},$$
(2.21)

$$\Delta D_{ii} = -4\pi \rho x_{i}^2,$$
(2.22)

$$\Delta D_{1ii} = -4\pi \rho x_{1} x_{i}^2,$$
(2.23)

and the solutions at star 1 are [18]

$$D_i = \pi \rho a_i^2 \left( A_i - \sum_l A_{il} x_i^2 \right) x_i,$$
(2.24)

$$D_{ii} = \pi \rho \left[ a_i^4 \left( A_{ii} - \sum_l A_{iii} x_i^2 \right) x_i^2 \right]$$
Contribution from star 2:

The solutions of

\[ D_{11} = \pi \rho \left[ a_i^2 \left( A_{11} - \sum_l A_{11l} x_l^2 \right) x_i^2 \right. \]

\[ + \frac{3}{4} a_i^4 \left( B_{11} - 2 \sum_l B_{11l} x_l^2 + \sum_m B_{11lm} x_l^2 x_m^2 \right) x_1 \], \hspace{1cm} (2.26) \]

\[ D_{1ii} = \pi \rho \left[ a_i^2 a_i^4 \left( A_{1ii} - \sum_l A_{1iil} x_l^2 \right) x_i x_i^2 \right. \]

\[ + \frac{1}{4} a_i^2 a_i^4 \left( B_{1i} - 2 \sum_l B_{1ii} x_l^2 + \sum_m B_{1iim} x_l^2 x_m^2 \right) x_1 \], \hspace{1cm} (2.27) \]

where \( B_{ijk...} \) are index symbols defined by Chandrasekhar [18].

- Contribution from star 2:

The equation for \( X_0^{2 \rightarrow 1} \) is

\[ \Delta X_0^{2 \rightarrow 1} = 4 \pi \rho \left[ 2 \pi \rho \left( A_0 - \sum_l A_l y_l^2 \right) + \frac{3 \rho}{\rho} \left( 1 - \sum_l \frac{y_l^2}{a_l^2} \right) + 2U^{1 \rightarrow 2} \right] \], \hspace{1cm} (2.28) \]

where \( y_1 = -(x_1 + R) \), \( y_2 = x_2 \) and \( y_3 = x_3 \). The solution is written as

\[ X_0^{2 \rightarrow 1} = -\alpha_0 U^{2 \rightarrow 1} - \alpha_1 D_1^{2 \rightarrow 1} + \sum_l \eta_l D_1^{2 \rightarrow 1} - \frac{M}{R^3} \left( 2D_{11}^{2 \rightarrow 1} - D_{22}^{2 \rightarrow 1} - D_{33}^{2 \rightarrow 1} \right) \], \hspace{1cm} (2.29) \]

where \( D_{ij...}^{2 \rightarrow 1} \) are calculated from the same equations as the case of \( D_{ij...} \) [18], i.e.,

\[ \Delta D_{ij...}^{2 \rightarrow 1} = -4 \pi \rho y_i y_j ... \]. \hspace{1cm} (2.30) \]

The solutions of \( D_{ij...}^{2 \rightarrow 1} \) are

\[ D_1^{2 \rightarrow 1} = \frac{I_{11}}{R^2} \left( 1 - \frac{2x_1}{R} + \frac{6x_1^2 - 3x_2^2 - 3x_3^2}{2R^2} + O(R^{-3}) \right) \], \hspace{1cm} (2.31) \]

\[ D_2^{2 \rightarrow 1} = \frac{I_{22}}{R^2} \left( \frac{x_2}{R} - \frac{3x_1 x_2}{R^2} + O(R^{-3}) \right) \], \hspace{1cm} (2.32) \]

\[ D_{ii}^{2 \rightarrow 1} = \frac{I_{ii}}{R} \left( 1 - \frac{x_1}{R} + \frac{2x_1^2 - x_2^2 - x_3^2}{2R^2} + \frac{-2x_1^3 + 3x_1(x_2 + x_3)}{2R^3} + O(R^{-4}) \right) \]

\[ + \frac{3I_{ii11}}{2R^3} \left( 1 - \frac{3x_1}{R} + O(R^{-2}) \right), \hspace{1cm} (2.33) \]

where

\[ I_{ii1} = I_{ii11} - \frac{1}{3} \sum_l I_{iill}. \hspace{1cm} (2.34) \]
C. $X_\Omega$

- Contribution from star 1:
  
  The equation for $X_\Omega^{1\rightarrow 1}$ is
  \[ \Delta X_\Omega^{1\rightarrow 1} = 8\pi \rho \left( x_1^2 + x_2^2 + Rx_1 + \frac{R^2}{4} \right). \]  
  (2.35)

  Then the solution is
  \[ X_\Omega^{1\rightarrow 1} = -2 \left( D_{11} + D_{22} + RD_1 + \frac{R^2}{4} U^{1\rightarrow 1} \right). \]  
  (2.36)

- Contribution from star 2:

  The equation $X_\Omega^{2\rightarrow 1}$ may be written as
  \[ \Delta X_\Omega^{2\rightarrow 1} = 8\pi \rho \left( y_1^2 + y_2^2 - Ry_1 + \frac{R^2}{4} \right). \]  
  (2.37)

  Then, using $D_{ij}^{2\rightarrow 1}$, the solution is easily derived as
  \[ X_\Omega^{2\rightarrow 1} = -2 \left( D_{11}^{2\rightarrow 1} + D_{22}^{2\rightarrow 1} - RD_1^{2\rightarrow 1} + \frac{R^2}{4} U^{2\rightarrow 1} \right), \]  
  (2.38)
  \[ = -\frac{R^2}{2} U^{2\rightarrow 1} - \frac{2I_{11}}{R^2} x_1 - \frac{2I_{22}}{R} \left( 1 - \frac{x_1}{R} \right). \]  
  (2.39)

D. $\hat{\beta}_\phi$

The definition of $\hat{\beta}_\phi$ is

\[ \hat{\beta}_\phi = -\frac{7}{2} \left( x_1 P_1 + x_2 P_2 + \frac{R}{2} P_1 \right), \]  
\[ -\frac{1}{2} \left[ \left( x_1 + \frac{R}{2} \right)^2 P_{2,2} + x_2^2 P_{1,1} - \left( x_1 + \frac{R}{2} \right) x_2 (P_{1,2} + P_{2,1}) \right], \]  
(2.40)

where $P_1$ and $P_2$ satisfy,

\[ \Delta P_1 = -4\pi \rho \left( x_1 + \frac{R}{2} \right), \]  
(2.41)

\[ \Delta P_2 = -4\pi \rho x_2. \]  
(2.42)
\( P_i \) is also written as \( P_i^{1\rightarrow 1} + P_i^{2\rightarrow 1} \), where

\[
P_1^{1\rightarrow 1} = D_1 + \frac{R}{2} U_1^{1\rightarrow 1}, \tag{2.43}
\]

\[
P_2^{1\rightarrow 1} = D_2, \tag{2.44}
\]

and

\[
P_2^{2\rightarrow 1} = D_2^{2\rightarrow 1} - \frac{R}{2} U_2^{2\rightarrow 1}, \tag{2.45}
\]

\[
P_2^{2\rightarrow 1} = D_2^{2\rightarrow 1}. \tag{2.46}
\]

\[E. \text{ The Collection}\]

Substituting expressions for the gravitational potentials derived in subsections \( \text{A} - \text{D} \) into Eq. (2.1), the integrated form of the Euler equation is written as

\[
\frac{P}{\rho} = U + \delta U + \frac{1}{2} \omega^2 \Omega^2
+ \frac{1}{c^2} \left[ \gamma_0 + \sum_l \gamma_l x_l^2 + \sum_{l\geq m} \gamma_{lm} x_l^2 x_m^2 + x_1 \left( \beta_0 + \sum_l \beta_l x_l^2 + \sum_{l\geq m} \beta_{lm} x_l^2 x_m^2 \right) \right]
+ \text{const.}, \tag{2.47}
\]

where \( \delta U \) is the PN correction of \( U \) which we will mention in the next section. In the following, we do not need \( \gamma_0, \gamma_i, \) and \( \gamma_{ij} \)(see below), but need \( \beta_0, \beta_i \) and \( \beta_{ij} \), which are

\[
\beta_0 = -\frac{M \pi \rho}{R^2} \left( \frac{6P_0}{5\pi \rho^2} + \frac{11A_0}{10} + \frac{3}{2} a_1^2 A_1 + a_2^2 A_2 \right) + \frac{9M^2}{4R^3}
+ \frac{1}{R^4} \left[ \frac{9}{2} \pi \rho I_{11} (-a_1^2 A_1 - 2a_2^2 A_2 + A_0) + \frac{9}{2} \sum_l \left( 2\pi \rho A_l + \frac{3P_0}{\rho a_l^2} \right) I_{ll}
+ \frac{3}{4} \pi \rho M a_1^2 (-2a_1^2 B_{11} + a_2^2 B_{12} + a_3^2 B_{13}) - \frac{9}{2} \pi \rho I_{11} \left( 2A_0 + \frac{3P_0}{\pi \rho^2} \right) \right]
+ \frac{M}{8R^6} (118I_{11} - 93I_{22} - 59I_{33}), \tag{2.48}
\]

\[
\beta_1 = \frac{M \pi \rho}{R^2} \left( \frac{3}{2} a_1^2 A_1 + a_2^2 A_2 - \frac{1}{2} A_1 \right)
+ \frac{1}{R^4} \left[ \frac{9}{2} \pi \rho I_{11} (a_1^2 A_{11} + 2a_2^2 A_{12} - A_1) + \sum_l \eta_l I_{ll} - \left( 2\pi \rho A_0 + \frac{3P_0}{\rho} \right) M \right].
\]
\[
\beta_2 = \frac{M \pi \rho}{R^2} \left( \frac{1}{2} a_1^2 A_{12} - a_2^2 A_{12} + 3a_2^2 A_{22} + A_1 - \frac{3}{2} A_2 \right) + \frac{1}{R^4} \left[ \frac{9}{2} \pi \rho I_{11} (-a_1^2 A_{12} - 2a_2^2 A_{12} + 6a_2^2 A_{22} - 3A_2 + 2A_1) \right.
\]
\[
\left. - \frac{3}{2} \sum_l \eta_l I_{ll} + \frac{3}{2} \left( 2\pi \rho A_0 + \frac{3P_0}{\rho} \right) M \right]
\]
\[
- \frac{3}{2} \pi \rho M a_1^2 (-2a_1^2 B_{112} - 2a_1^2 A_{122} + a_2^2 B_{122} + a_3^2 B_{133}) \right] + \frac{29M^2}{8R^5}, \tag{2.50}
\]
\[
\beta_3 = \frac{M \pi \rho}{R^2} \left( \frac{3}{2} a_1^2 A_{13} + a_2^2 A_{23} - \frac{1}{2} A_3 \right) + \frac{1}{R^4} \left[ \frac{9}{2} \pi \rho I_{11} (a_1^2 A_{13} + 2a_2^2 A_{23} - A_3) - \frac{3}{2} \sum_l \eta_l I_{ll} + \frac{3}{2} \left( 2\pi \rho A_0 + \frac{3P_0}{\rho} \right) M \right.
\]
\[
\left. - \frac{3}{2} \pi \rho M a_1^2 (-2a_1^2 B_{113} - 2a_1^2 A_{133} + a_2^2 B_{123} + a_3^2 B_{133}) \right] + \frac{39M^2}{8R^5}, \tag{2.51}
\]
\[
\beta_{11} = \frac{M \pi \rho}{R^4} \left[ a_1^6 A_{1111} + \frac{3}{4} a_1^2 (-2a_1^2 B_{1111} + a_2^2 B_{1112} + a_3^2 B_{1113}) \right], \tag{2.52}
\]
\[
\beta_{22} = \frac{M \pi \rho}{R^4} \left[ -3a_1^2 a_2^4 A_{1222} + \frac{3}{4} a_1^2 (-2a_1^2 B_{1122} + a_2^2 B_{1222} + a_3^2 B_{1223}) \right], \tag{2.53}
\]
\[
\beta_{33} = \frac{M \pi \rho}{R^4} \left[ -3a_1^2 a_3^4 A_{1333} + \frac{3}{4} a_1^2 (-2a_1^2 B_{1133} + a_2^2 B_{1233} + a_3^2 B_{1333}) \right], \tag{2.54}
\]
\[
\beta_{12} = \frac{M \pi \rho}{R^4} \left[ 2a_1^6 A_{1112} - 3a_1^2 a_2^4 A_{1122} + \frac{3}{2} a_2^2 (-2a_1^2 B_{1112} + a_2^2 B_{1122} + a_3^2 B_{1123}) \right], \tag{2.55}
\]
\[
\beta_{13} = \frac{M \pi \rho}{R^4} \left[ 2a_1^6 A_{1113} - 3a_1^2 a_3^4 A_{1133} + \frac{3}{2} a_2^2 (-2a_1^2 B_{1113} + a_2^2 B_{1123} + a_3^2 B_{1133}) \right], \tag{2.56}
\]
\[
\beta_{23} = \frac{M \pi \rho}{R^4} \left[ -3a_1^2 (a_2^4 A_{1223} + a_3^4 A_{1333}) + \frac{3}{2} a_1^2 (-2a_1^2 B_{1123} + a_2^2 B_{1223} + a_3^2 B_{1233}) \right]. \tag{2.57}
\]

### III. THE POST-NEWTONIAN ANGULAR VELOCITY

In this section, the orbital angular velocity in the PN order is derived using the first TV equation. The first TV relation is derived from

\[
\int d^3x \frac{\partial P}{\partial x_1} = 0. \tag{3.1}
\]

Substituting Eq.\( (2.47) \) to Eq.\( (3.1) \), we have Eq.\( (2.3) \) for the Newtonian order. In the PN order, the explicit form becomes
\[ 0 = \frac{R}{2} M \delta \Omega^2 + \int d^3 x \rho \xi_1 \Omega_N^2 + M \beta_0 + 3 \beta_1 I_{11} + \beta_2 I_{22} + \beta_3 I_{33} + 5 \beta_{1111} \]
\[ + 3 \beta_{12} I_{1122} + 3 \beta_{13} I_{1133} + \beta_{22} I_{2222} + \beta_{33} I_{3333} + \beta_{23} I_{2233} + \delta \int d^3 x \rho \frac{\partial U}{\partial x_1}, \]  

(3.2)

where \( \xi_1 \) is the \( x_1 \) component of the Lagrangian displacement \( \xi_i \), and \( \delta \Omega^2 \) denotes the PN correction of the orbital angular velocity.

A. The Post-Newtonian Deformation

The density profile of an incompressible, homogeneous sphere in the PN order is the same that that in the Newtonian order. However, the density profile of an ellipsoid in the PN order is different from that in the Newtonian order. The Chandrasekhar’s method to calculate the PN effects on the uniformly rotating isolated bodies is as follows [19]. First, he constructs the ellipsoidal figures in the Newtonian order. Then, he regards the PN effect as a small perturbation to the Newtonian configuration, and calculates the deformation from the Newtonian ellipsoid using the Lagrangian displacement. Solving the equations for the Lagrangian displacement and calculating the correction to \( U \) by the deformation, he has obtained the equilibrium configuration in the PN order. In this paper, we follow his method.

Although the deformation from the ellipsoidal figure will really occur by the PN effect, we will show in this subsection that it is a small effect compared with leading terms shown in Eq.(2.4). Hence, for readers who are not interested in such a verification, we recommend to skip to the next subsection.

In order to express Eq.(2.47) using only the coordinates and the index symbols, we have to calculate \( \delta U \) which is defined as

\[ \delta U = - \sum_i \frac{\partial}{\partial x_i} \int d^3 x' \frac{\rho \xi_i |x - x'|}{|x - x'|}. \]

(3.3)

Here, \( \xi_i \) denotes the Lagrangian displacement of the fluid element induced by the PN correction, and is used to guarantee that the pressure of the deformed Darwin ellipsoid at the boundary becomes zero. This means that it is determined from \( \gamma_0, \gamma_i, \gamma_{ij}, \beta_0, \beta_i \) and \( \beta_{ij} \) in Eq.(2.47) in general.
In choosing $\xi_i$, we require it to be divergent free because $\delta \rho = -\sum_i \partial_i (\rho \xi_i)$. To obtain $\delta \Omega^2$ up to $O(R^{-3}) \times \Omega_N^2$, we only need the following:

$$\xi_i^{(0)} = \left( \frac{1}{2}, 0, 0 \right), \quad (3.4)$$

$$\xi_i^{(21)} = \left( \frac{1}{2} x_1^2, 0, -x_1 x_3 \right), \quad (3.5)$$

$$\xi_i^{(22)} = (0, x_1 x_2, -x_1 x_3). \quad (3.6)$$

Thus, we set

$$\xi_i = S_0 \xi_i^{(0)} + S_{21} \xi_i^{(21)} + S_{22} \xi_i^{(22)}, \quad (3.7)$$

where $S_0$, $S_{21}$ and $S_{22}$ are constants. In this case, we can get $\delta U$ of the contribution from star 1 as

$$\delta U^{1 \rightarrow 1} = \pi \rho A_1 S_0 x_1 - \left( \frac{1}{2} \frac{\partial D_{11}}{\partial x_1} - \frac{\partial D_{13}}{\partial x_3} \right) S_{21} - \left( \frac{\partial D_{12}}{\partial x_2} - \frac{\partial D_{13}}{\partial x_3} \right) S_{22}, \quad (3.8)$$

where

$$D_{1i} = \pi \rho a_1^2 a_i^2 \left( A_{1i} - \sum_l A_{1il} x_l^2 \right) x_1 x_i \quad (i \neq 1). \quad (3.9)$$

The contribution from star 2 is defined as

$$\delta U^{2 \rightarrow 1} = -\sum_i \frac{\partial}{\partial x_i} \int_2 d^3 x' \frac{\rho \xi_i'}{|x - x'|}. \quad (3.10)$$

Here, the integral is performed in star 2, and $\xi_i'$ is defined for star 2. Note that $|\xi_i'|$ is the same as $|\xi_i|$, but $\xi_i'$ has opposite sign to $\xi_i$ if components of $\xi_i$ are the even function of $x_i$. Using this definition, we have

**In ref. [19] where Chandrasekhar calculated the PN configuration of the Jacobi ellipsoid, he introduced other type of displacements, such as $\xi^k (k = 1 - 5)$. Coefficients of these displacements are, however, of $O(R^{-3})$ in this paper because the ellipsoidal displacements of a star are generated by the small spin of each star and by the tidal force of the companion star both of which deform the stars only by $O(R^{-3})$. Then, the displacements will contribute to the orbital angular velocity from a higher order of $R^{-1}$ as $\Omega_N^2 \times O(R^{-5})$. This is the reason why we do not need $\gamma_0$, $\gamma_i$ and $\gamma_{ij}$.**
\[
\delta U^{2\to1} = \frac{1}{2} S_0^2 \sum_i \frac{\partial U^{2\to1}}{\partial x_i} + \left( \frac{1}{2} \frac{\partial D^{2\to1}_{11}}{\partial x_1} - \frac{\partial D^{2\to1}_{13}}{\partial x_3} \right) S_{21} + \left( \frac{\partial D^{2\to1}_{12}}{\partial x_2} - \frac{\partial D^{2\to1}_{13}}{\partial x_3} \right) S_{22},
\]

where

\[
D^{2\to1}_{ii} = \frac{3I_{11ii}}{R^3} \left( \frac{x_i}{R} - \frac{4x_1x_i}{R^3} \right) \quad (i \neq 1).
\]

In order to get the orbital angular velocity, we also have to calculate the second and the last terms of the right-hand side of Eq. (3.12) which are occurred by the displacement of the fluid element by the PN correction. The second term is

\[
\frac{M}{2} \Omega^2 \left( S_0 + S_{21} \frac{a_1}{5} \right),
\]

and the last one can be evaluated as follows: First, the contribution from star 1 is zero because

\[
\delta \int d^3x \rho \frac{\partial U^{1\to1}}{\partial x_1} = -\delta \int d^3x \rho(x) \int d^3x' \rho(x') \frac{x_1 - x_1'}{|x - x'|^3} = -\int d^3x \rho(x) \delta \int d^3x' \rho(x') \frac{x_1 - x_1'}{|x - x'|^3} + \int d^3x \rho(x) \delta \int d^3x' \rho(x') \frac{x_1 - x_1'}{|x - x'|^3} = 0.
\]

The contribution from star 2 becomes

\[
\delta \int d^3x \rho \frac{\partial U^{2\to1}}{\partial x_1} = \int d^3x \rho \sum_{i=1}^3 \xi_i \frac{\partial^2 U^{2\to1}}{\partial x_1 \partial x_i} + \int d^3x \rho \frac{\partial \delta U^{2\to1}}{\partial x_1}.
\]

The first term on the right hand side of Eq. (3.15) denotes the force which the displaced element of star 1 receives from the non-displaced potential of star 2. On the other hand, the second term denotes the force which the non-displaced element of star 1 receives from the displaced potential of star 2. For the Lagrangian displacement given in Eqs. (3.4) - (3.6), the first and the second terms of Eq. (3.15) are equal, and using Eq. (3.11), we can calculate Eq. (3.13) as

\[
\delta \int d^3x \rho \frac{\partial U^{2\to1}}{\partial x_1} = 2 \int d^3x \rho \frac{\partial \delta U^{2\to1}}{\partial x_1},
\]

\[
= S_0^2 \frac{2M}{R^3} \left( M + \frac{18I_{11}}{R^2} \right) + S_{21} \frac{2M}{R^3} \left( I_{11} + \frac{9I_{111}}{R^2} + \frac{9I_{11}I_{11}}{MR^2} + \frac{12I_{1133}}{R^2} \right)
\]

\[
+ S_{22} \frac{24M}{R^6} (I_{1133} - I_{1122}).
\]
At the deformed surface of the Darwin ellipsoid, the pressure must vanish. This condition becomes

\[
\left( \frac{P}{\rho} \right)_S = -2 \frac{P_0}{\rho} \sum \frac{\xi_i x_i}{a_i^2} + \delta U + \frac{1}{2} \omega^2 \delta \Omega^2
\]

\[+ \gamma_0 + \sum \gamma l x_l^2 + \sum \gamma_{lm} x_l^2 x_m^2 + x_1 \left( \beta_0 + \sum \beta_l x_l^2 + \sum \beta_{lm} x_l^2 x_m^2 \right) + \text{const.} \]

\[= 0, \quad (3.18)\]

where we use the equation of the boundary surface at the deformed Darwin ellipsoid

\[S = \sum \frac{x_i^2}{a_i^2} - 1 - 2 \sum \frac{\xi_i x_i}{a_i^2} = 0. \quad (3.19)\]

Eq. (3.18) must be satisfied at the original surface of the Darwin ellipsoid, \( \sum x_i^2 / a_i^2 = 1 \). Coefficients, \( S_0, S_{21} \) and \( S_{22} \), are determined from the fact that the terms of the odd power of \( x_l \) must vanish. The coefficients of \( x_1, x_1^3, x_1 x_2^2 \) and \( x_1 x_2^2 \) in Eq. (3.18) \( Q_1, Q_{111}, Q_{122}, \) and \( Q_{133} \) are as follows:

\[Q_1 = \left( - \frac{P_0}{\rho a_1^2} + \pi \rho A_1 + \frac{\Omega^2}{2} \right) S_0 + \frac{R}{2} \delta \Omega^2 + \beta_0 \]

\[+ S_21 \pi \rho a_1^2 \left( -a_1^2 A_{11} + \frac{1}{2} B_{11} + a_2^2 A_{13} + O(R^{-3}) \right) - S_22 \pi \rho a_1^2 \left( a_2^2 A_{12} - a_3^2 A_{13} \right), \quad (3.20)\]

\[Q_{111} = \beta_1 + S_{21} \pi \rho a_1^2 \left( - \frac{P_0}{\pi \rho^2 a_1^2} + 2a_1^2 A_{111} - a_2^2 A_{113} - \frac{1}{2} B_{111} \right) \]

\[+ S_{22} \pi \rho a_1^2 \left( a_2^2 A_{112} - a_3^2 A_{113} \right), \quad (3.21)\]

\[Q_{122} = \beta_2 + S_{21} \pi \rho a_1^2 \left( a_1^2 A_{112} - \frac{1}{2} B_{112} - a_3^2 A_{123} \right) \]

\[+ S_{22} \pi \rho a_1^2 \left( - \frac{2P_0}{\pi \rho^2 a_1^2 a_2^2} + 3a_2^2 A_{122} - a_3^2 A_{123} \right), \quad (3.22)\]

\[Q_{133} = \beta_3 + S_{21} \pi \rho a_1^2 \left( \frac{2P_0}{\pi \rho^2 a_1^2 a_3^2} + a_1^2 A_{113} - \frac{1}{2} B_{113} - 3a_3^2 A_{133} \right) \]

\[+ S_{22} \pi \rho a_1^2 \left( \frac{2P_0}{\pi \rho^2 a_1^2 a_3^2} + a_2^2 A_{123} - 3a_3^2 A_{133} \right). \quad (3.23)\]

The conditions to determine \( S_0, S_{21} \) and \( S_{22} \) are

\[Q_1 + Q_{133} a_3^2 = 0, \quad (3.24)\]

\[Q_{111} a_1^2 - Q_{133} a_3^2 = 0, \quad (3.25)\]

\[Q_{122} a_2^2 - Q_{133} a_3^2 = 0. \quad (3.26)\]
It is found that the coefficient of $S_0$ in $Q_1$ vanishes when we substitute Eq.(3.2) to Eq.(3.24) and remove $\delta \Omega^2$. Thus, $S_0$ is indeterminant, and the first equation becomes trivial one. This is also verified by calculating $3Q_{111}a_1^2 + Q_{122}a_2^2 + Q_{133}a_3^2$, which coincides with $-5Q_1$. This fact seems to be very natural because $S_0$ is a simple transformation of the center of mass. As a result, $S_{21}$ and $S_{22}$ are determined by the second and third equations. From the combination $2 \times (3.23) - (3.26)$, i.e., $2Q_{111}a_1^2 - Q_{122}a_2^2 - Q_{133}a_3^2 = 0$, we obtain

$$0 = 2\beta_1 a_1^2 - \beta_2 a_2^2 - \beta_3 a_3^2$$
$$+S_{21} \pi \rho a_1^2 \left[ -\frac{4P_0}{\pi \rho^2 a_1^2} + 4a_1^4 A_{111} - a_1^2 a_2^2 A_{112} - 3a_1^2 a_3^2 A_{113} + a_2^2 a_3^2 A_{123} + 3a_3^4 A_{133} - a_2^2 B_{111} + \frac{1}{2}(a_2^2 B_{112} + a_3^2 B_{113}) \right]$$
$$+S_{22} \pi \rho a_2^2 \left[ 2a_1^2 (a_2^2 A_{112} - a_3^2 A_{113}) - 3a_2^4 A_{122} + 3a_3^4 A_{133} \right], \quad (3.27)$$

where we omit higher order terms of $R^{-1}$. $2\beta_1 a_1^2 - \beta_2 a_2^2 - \beta_3 a_3^2$ is calculated as

$$\frac{M \pi \rho}{R^2} \left[ 2a_1^2 A_1 - \frac{5}{2} a_2^2 A_2 + \frac{1}{2} a_3^2 A_3 + \frac{5}{2} a_4^2 (a_2^2 A_{112} - a_3^2 A_{113}) \right] + O(R^{-4}) = O(R^{-4}), \quad (3.28)$$

where we use the relations

$$a_2^2 A_2 = a_1^2 A_1 + O(R^{-3}), \quad (3.29)$$
$$a_3^2 A_3 = a_1^2 A_1 + O(R^{-3}), \quad (3.30)$$
$$a_1^2 a_2^2 A_{112} = a_3^2 A_3 + O(R^{-3}), \quad (3.31)$$
$$a_1^2 a_3^2 A_{113} = a_2^2 A_2 + O(R^{-3}). \quad (3.32)$$

The coefficient of $S_{22}$ is also of $O(R^{-3})$ because $a_2^2 A_{112} - a_3^2 A_{113} = O(R^{-3})$ and $a_1^4 A_{122} - a_3^4 A_{133} = O(R^{-3})$, but that of $S_{21}$ is $O(1)$.

On the other hand, from Eq.(3.26), we have

$$0 = \beta_2 a_1^2 + \beta_3 a_3^2$$
$$+S_{21} \pi \rho a_1^2 \left[ -\frac{2P_0}{\pi \rho^2 a_1^2} + a_1^2 (a_2^2 A_{112} - a_3^2 A_{113}) - \frac{1}{2}(a_2^2 B_{112} - a_3^2 B_{113}) - a_2^2 a_3^2 A_{123} + 3a_3^4 A_{133} \right]$$
$$+S_{22} \pi \rho a_2^2 \left[ -\frac{4P_0}{\pi \rho^2 a_2^2} + 3a_2^4 A_{122} - 2a_2^2 a_3^2 A_{123} + 3a_3^4 A_{133} \right]. \quad (3.33)$$
The coefficients of \(S_{21}\) and \(S_{22}\) are of \(O(1)\) in this equation, and we can also show \(\beta_1 a_1^2 - \beta_2 a_2^2 = O(R^{-4})\). Thus, we can conclude that both \(S_{21}\) and \(S_{22}\) are of \(O(R^{-4})\). Accordingly, we do not have to consider these Lagrangian displacements in the following.

**B. The Post-Newtonian Orbital Angular Velocity**

Since \(S_0\) is arbitrary, we set \(S_0 = 0\). Hence, Eq. (3.3) is calculated as

\[
- \frac{R}{2} M \delta \Omega^2 = M \beta_0 + 3 \beta_1 I_{11} + \beta_2 I_{22} + \beta_3 I_{33} + 5 \beta_{11} I_{1111} + 3 \beta_{12} I_{1122}
\]

\[
+ 3 \beta_{13} I_{1133} + \beta_{22} I_{2222} + \beta_{33} I_{3333} + \beta_{23} I_{2233}
\]

\[
= - \frac{M^2}{R^2} 2 \pi \rho A_0 + \frac{9 M^3}{4 R^3} - \frac{84}{5 R^3} \pi \rho I_{11} A_0 + \frac{M^3}{10 R^5} (28 a_1^2 - 14 a_2^2 - 9 a_3^2),
\]

(3.34)

where we make use of the relations \(A_0 = \sum_l A_l a_l^2\), Eqs. (2.13) – (2.15), equations for \(I_{iiij}\) and reduction formulas for \(A_{ijk\ldots}\) and \(B_{ijk\ldots}\) shown in the section 21 of Chandrasekhar’s textbook [18]. Then, \(\Omega^2\) is written as

\[
\Omega^2 = \frac{2 M}{R^3} \left[ 1 + \frac{1}{c^2} \left\{ 2 \pi \rho A_0 - \frac{9 M}{4 R} - \frac{M}{10 R^3} (28 a_1^2 - 14 a_2^2 - 9 a_3^2) + O(R^{-4}) \right\} \right]
\]

\[
+ \frac{18 I_{11}}{R^5} \left( 1 + \frac{28}{15 c^2} \pi \rho A_0 + O(R^{-2}) \right).
\]

(3.35)

Before proceeding further, we comment on the mass of the system and definition of the center of mass for each star. This is because there are several definitions of them in the PN approximation, and we should clarify the difference between the similar ones.

First, we consider the conserved mass which is defined as

\[
M_* = \int d^3x \rho \left[ 1 + \frac{1}{c^2} \left( \frac{v^2}{2} + 3U \right) \right]
\]

\[
= M \left[ 1 + \frac{1}{c^2} \left( \frac{13 M}{4 R} + \frac{12 \pi \rho A_0}{5} \right) + \frac{M}{20 R^3} (34 a_1^2 - 11 a_2^2 - 15 a_3^2) + O(R^{-5}) \right].
\]

(3.37)

Using \(M_*\), \(\Omega^2\) becomes

\[\Omega^2 = 3 M a_1^4 / 35, \quad I_{iiii} = M a_1^2 a_2^2 / 35 \quad (i \neq j)\]
\[\Omega^2 = \frac{2M_*}{R^3} \left[ 1 + \frac{1}{c^2} \left\{ -\frac{2\pi\rho A_0}{5} - \frac{11M_*}{2R} - \frac{M_*}{20R^3}(90a_1^2 - 39a_2^2 - 33a_3^2) \right\} + O(R^{-4}) \right] \\
+ \frac{18(I_{11})_*}{R^5} \left[ 1 + \frac{1}{c^2} \left\{ -\frac{8}{15}\pi\rho A_0 - \frac{13M_*}{4R} + O(R^{-2}) \right\} \right], \quad (3.38)\]

where \((I_{11})_* = M_* I_{11}/M\). Thus, \(\Omega^2\) looks as if it depends on the internal structure of the star even for the limit \(a_i/R \to 0\). Since we believe that in the equation of motion (EOM) for the point particle, the quantities depending on the internal structure does not appear, \(M_*\) is not desirable to describe the EOM for the point particle. Instead, in the case when the EOM is derived, one usually adopts the PPN mass \([24]\) defined as

\[M_{\text{PPN}} = \int d^3x \rho \left[ 1 + \frac{1}{c^2} \left( \frac{v^2}{2} + 3U - \frac{1}{2} U_{\text{self}} + \frac{v_{\text{self}}^2}{2} \right) \right] = M \left[ 1 + \frac{1}{c^2} \left( \frac{13M}{4R} + 2\pi\rho A_0 + \frac{M}{20R^3}(38a_1^2 - 7a_2^2 - 15a_3^2) + O(R^{-5}) \right) \right], \quad (3.39)\]

where \(U_{\text{self}}\) and \(v_{\text{self}}\) are the self gravity part of the Newtonian potential and the spin velocity of each star. When we rewrite Eq. (3.36) using the PPN mass, the orbital angular velocity does not depend on the internal structure of the star and agrees with that for the point particle \([7]\) in the limit \(a_i/R \to 0\) as

\[\Omega^2 = \frac{2M_{\text{PPN}}}{R^3} \left[ 1 + \frac{1}{c^2} \left\{ -\frac{11M_{\text{PPN}}}{2R} - \frac{M_{\text{PPN}}}{20R^3}(94a_1^2 - 35a_2^2 - 33a_3^2) + O(R^{-4}) \right\} \right] \\
+ \frac{18(I_{11})_{\text{PPN}}}{R^5} \left[ 1 + \frac{1}{c^2} \left\{ -\frac{2}{15}\pi\rho A_0 - \frac{13M_{\text{PPN}}}{4R} + O(R^{-2}) \right\} \right], \quad (3.40)\]

where \((I_{11})_{\text{PPN}} = M_{\text{PPN}} I_{11}/M\). Thus, when we compare the present results with the point particle calculations, we should use the PPN mass. In the present case, however, \(M_{\text{PPN}}\) is not a conserved quantity although \(M_*\) is. When we consider a sequence of the equilibrium configuration as an evolutionary sequence, we should fix \(M_*\).

Next, we consider the definition of the center of mass for each star. In the PPN formalism, it is defined as \([24]\)

\[x_{\text{PPN}}^i = \frac{1}{M_{\text{PPN}}} \int d^3x \rho x^i \left[ 1 + \frac{1}{c^2} \left( \frac{v^2}{2} + 3U - \frac{1}{2} U_{\text{self}} + \frac{v_{\text{self}}^2}{2} \right) \right], \quad (3.41)\]

and the \(x_1\) coordinate of the center of mass for star 1 deviates from 0 to
\[ \frac{1}{c^2} \left( -\frac{2Ma_i^2}{5R^2} + O(R^{-4}) \right). \]  (3.42)

Thus, in the PPN formalism, the following orbital separation should be used:

\[ R_{\text{PPN}} = R \left[ 1 + \frac{1}{c^2} \left\{ -\frac{4Ma_i^2}{5R^3} + O(R^{-5}) \right\} \right]. \]  (3.43)

It is worth noting that when we define the center of mass by the conserved mass as

\[ x_*^i = \frac{1}{M_*} \int d^3x \rho_* x^i, \]  (3.44)

the result is the same up to \( O(R^{-4}) \). Thus, in this paper, we do not have to distinguish \( R_* \) from \( R_{\text{PPN}} \). Even in the general cases, the difference between \( R_* \) and \( R_{\text{PPN}} \) is expected to be small.

Using \( R_* \) and/or \( R_{\text{PPN}} \), \( \Omega^2 \) is rewritten as

\[ \Omega^2 = \frac{2M_*}{R_*^3} \left[ 1 + \frac{1}{c^2} \left\{ -\frac{2\pi\rho A_0}{5} - \frac{11M_*}{2R_*} - \frac{M_*}{20R_*^3} (138a_1^2 - 39a_2^2 - 33a_3^2) \right\} + O(R_*^{-4}) \right] \]

\[ + \frac{18(I_{11})_*}{R_*^3} \left[ 1 + \frac{1}{c^2} \left\{ -\frac{8}{15}\pi \rho A_0 - \frac{13M_*}{4R_*} + O(R_*^{-2}) \right\} \right], \]  (3.45)

or

\[ \Omega^2 = \frac{2M_{\text{PPN}}}{R_{\text{PPN}}^3} \left[ 1 + \frac{1}{c^2} \left\{ -\frac{11M_{\text{PPN}}}{2R_{\text{PPN}}} - \frac{M_{\text{PPN}}}{20R_{\text{PPN}}^3} (142a_1^2 - 35a_2^2 - 33a_3^2) + O(R_{\text{PPN}}^{-4}) \right\} \right] \]

\[ + \frac{18(I_{11})_{\text{PPN}}}{R_{\text{PPN}}^3} \left[ 1 + \frac{1}{c^2} \left\{ -\frac{2}{15}\pi \rho A_0 - \frac{13M_{\text{PPN}}}{4R_{\text{PPN}}} + O(R_{\text{PPN}}^{-2}) \right\} \right]. \]  (3.46)

Here, we should note that the effect of the spin-orbit coupling terms will appear in \( \Omega^2 \) from \( O(R_{\text{PPN}}^{-6}) \). According to the PN study, it becomes

\[ \Omega^2 = \frac{2M_{\text{PPN}}}{R_{\text{PPN}}^3} \left[ 1 - \frac{2M_{\text{PPN}}}{c^2 R_{\text{PPN}}^3} (a_1^2 + a_2^2) \right], \]  (3.47)

where we omit other terms which do not concern in this discussion. Eq.(3.47) shows that the terms of \( O(R_{\text{PPN}}^{-6}) \) in Eq.(3.46) cannot be explained only by the spin-orbit coupling term. This means that there is a new effect, say the PN quadrupole one, in Eq.(3.46).
IV. THE ENERGY AND THE ANGULAR MOMENTUM

A. The Total Energy

The PN total energy is calculated from $E = E^\text{def}_N + E^\text{def}_{\text{PN}}/c^2$, where

$$E^\text{def}_N = \int \rho \left( \frac{1}{2} v^2 - \frac{1}{2} U \right) \, d^3 x, \quad (4.1)$$

$$E^\text{def}_{\text{PN}} = \int \rho \left( \frac{5}{8} v^4 + \frac{5}{2} v^2 U + \frac{P}{\rho} v^2 - \frac{5}{2} U^2 + \frac{1}{2} \tilde{\beta}_\rho \Omega^2 \right) \, d^3 x. \quad (4.2)$$

For the PN Darwin problem, they become

$$E^\text{def}_N = M \left[ -\frac{4\pi \rho A_0}{5} + \frac{\Omega^2}{5} (a_1^2 + a_2^2) + \frac{R^2 \Omega^2}{4} - \frac{M}{R} - \frac{3 I_{11}}{R^3} + O(R^{-5}) \right], \quad (4.3)$$

$$E^\text{def}_{\text{PN}} = 2M \left[ -\frac{1}{7} (\pi \rho)^2 (11 A_0^2 + a_1^4 A_1^2 + a_2^4 A_2^2 + a_3^4 A_3^2) - \frac{4M \pi \rho A_0}{R} \right.$$

$$\left. - \frac{M \pi \rho}{7R^3} \left\{ \frac{87}{M} A_0 - (2a_1^4 A_1 - a_2^4 A_2 - a_3^4 A_3) \right\} \right.$$

$$\left. - \frac{5}{2} \left\{ \frac{M^2}{R^2} + \frac{M^2}{5R^4} (5a_1^2 - 2a_2^2 - 2a_3^2) \right\} \right.$$

$$\left. + \Omega^2 \left\{ \frac{17}{16} M R + \frac{M}{80R} (6a_1^2 + 21a_2^2 - 17a_3^2) \right\} \right.$$

$$\left. + \Omega^2 \pi \rho \left\{ \frac{R^2}{20} (3A_0 - a_2^2 A_2) + \frac{3}{7} A_0 (a_1^2 + a_2^2) - \frac{12}{35} (a_1^4 A_1 + a_2^4 A_2) \right\} \right.$$

$$\left. - \frac{1}{35} a_1^2 a_2^2 (A_1 + A_2) \right\}$$

$$+ \frac{P_0 \Omega^2}{\rho} \left\{ \frac{R^2}{10} + \frac{2}{35} (a_1^2 + a_2^2) \right\} + \frac{5}{8} \Omega^4 \left\{ \frac{R^4}{16} + \frac{R^2}{10} (3a_1^2 + a_2^2) \right\} + O(R^{-5}) \right], \quad (4.4)$$

$$= 2M \left[ -\frac{34}{21} (\pi \rho A_0)^2 - \frac{11M \pi \rho A_0}{3R} - \frac{7M^2}{32R^2} + \frac{M \pi \rho A_0}{R^3} \left\{ \frac{68}{105} (a_1^2 + a_2^2) - \frac{61}{7} I_{11} \right\} \right.$$

$$\left. + \frac{M^2}{240R^4} (302a_1^2 + 59a_2^2 - 209a_3^2) + O(R^{-5}) \right], \quad (4.5)$$

where we use the equilibrium equations \((2.12)-(2.15)\) and $\Omega^2$ to reduce Eq.\((4.4)\) to Eq.\((4.5)\).

If we substitute $\Omega^2$ into the above formulas, $E$ may be rewritten as $E_N + E_{\text{PN}}/c^2$ where

$$E_N = M \left[ -\frac{4\pi \rho A_0}{5} - \frac{M}{2R} + \frac{3 I_{11}}{2R^3} + \frac{\Omega^2}{5} (a_1^2 + a_2^2) + O(R^{-5}) \right], \quad (4.6)$$

$$E_{\text{PN}} = 2M \left[ -\frac{34}{21} (\pi \rho A_0)^2 - \frac{19M \pi \rho A_0}{6R} - \frac{25M^2}{32R^2} + \frac{M \pi \rho A_0}{R^3} \left\{ \frac{22}{35} (a_1^2 + a_2^2) - \frac{158}{35} I_{11} \right\} \right.$$

$$\left. + \frac{M^2}{240R^4} (26a_1^2 + 35a_2^2 - 155a_3^2) + O(R^{-5}) \right]. \quad (4.7)$$
B. The Total Angular Momentum

We can calculate the PN total angular momentum from $J = J_N^\text{def} + J_{PN}^\text{def}/c^2$, where

$$J_N^\text{def} = \int \rho v_\varphi d^3x,$$

$$J_{PN}^\text{def} = \int \rho \left[ v_\varphi \left( v^2 + 6U + \frac{P}{\rho} \right) + \hat{\beta}_\varphi \Omega \right] d^3x,$$

and $v_\varphi = \Omega \varpi^2$. For the PN Darwin problem, they become

$$J_N^\text{def} = 2M\Omega \left( \frac{R^2}{4} + \frac{a_1^2 + a_2^2}{5} \right),$$

$$J_{PN}^\text{def} = M\Omega \left[ R^2 \pi \rho A_0 + \frac{P_0}{\rho} \left\{ \frac{R^2}{5} + \frac{4}{35} (a_1^2 + a_2^2) \right\} - \frac{R^2}{5} \pi \rho a_2^2 A_2 \right. $$

$$+ \frac{4\pi \rho}{35} \left\{ 18A_0(a_1^2 + a_2^2) - 13(a_1^4 A_1 + a_2^4 A_2) - a_1^2 a_2^2 (A_1 + A_2) \right\}$$

$$+ \Omega^2 \left( \frac{R^4}{8} + \frac{R^2}{5} (3a_1^2 + a_2^2) \right) + \frac{19MR}{4} + \frac{M}{20R} (10a_1^2 + 27a_2^2 - 19a_3^2),$$

$$= M\Omega_N \left[ R^2 \pi \rho A_0 + 5RM + \frac{164}{105} \pi \rho A_0 (a_1^2 + a_2^2) \right. $$

$$+ \frac{M}{10R} (20a_1^2 + 15a_2^2 - 11a_3^2) + O(R^{-2}) \right],$$

where we use the equilibrium equations (2.12)−(2.15) and $\Omega^2$ to reduce Eq.(4.11) to Eq.(4.12).

If we substitute $\Omega^2$ into the above formulas, $J$ may be rewritten as $J_N + J_{PN}/c^2$ where

$$J_N = 2M\Omega_N \left( \frac{R^2}{4} + \frac{a_1^2 + a_2^2}{5} \right),$$

$$J_{PN} = M\Omega_N \left[ \frac{3}{2} R^2 \pi \rho A_0 + \frac{71}{16} RM + \frac{\pi \rho A_0}{1050} (2018a_1^2 + 2081a_2^2 + 21a_3^2) \right. $$

$$+ \frac{M}{80R} (122a_1^2 + 85a_2^2 - 97a_3^2) + O(R^{-2}) \right].$$

V. EQUILIBRIUM SEQUENCE OF THE POST-NEWTONIAN DARWIN ELLIPSOID

In this section, we construct the equilibrium sequences of the PN Darwin ellipsoids fixing $M_*$ and $\rho$ as follows;
(1) using Eqs. (2.13)–(2.15), we numerically calculate the equilibrium sequences of the Newtonian order. Up to this stage, $\alpha_2$, $\alpha_3$, and $\tilde{R} = R/a_1$ are determined.

(2) $a_1$ is determined from the condition $M_*=\text{constant}$ using Eq. (3.37):

$$a_1^3 = \frac{3M_*}{4\pi \rho \alpha_2 \alpha_3} \left[ 1 - \frac{\pi \rho}{c^2} \left( \frac{3M_*}{4\pi \rho \alpha_2 \alpha_3} \right)^{2/3} \left( \frac{12\tilde{A}_0}{5} + \frac{13\alpha_2 \alpha_3}{3R} + \frac{\alpha_2 \alpha_3}{15R^3} (34 - 11\alpha_2^2 + 15\alpha_3^2) \right) \right]. \quad (5.1)$$

(3) after substituting Eq. (5.1) into Eqs. (3.36), (4.3), (4.5), (4.10), and (4.12), we rewrite the PN expressions for the orbital angular velocity, the energy and the angular momentum as

$$\tilde{\Omega}^2 = \frac{\Omega^2}{\pi \rho} = \tilde{\Omega}_N^2 + \frac{M_*}{c^2 a_*} \tilde{\Omega}_{PN}^2,$$  

$$\tilde{E} = \frac{E}{(M_* a_*)^{1/2}} = \tilde{E}_N + \frac{M_*}{c^2 a_*} \tilde{E}_{PN},$$  

$$\tilde{J} = \frac{J}{(M_* a_*)^{1/2}} = \tilde{J}_N + \frac{M_*}{c^2 a_*} \tilde{J}_{PN},$$

where

$$a_* = \left( \frac{3M_*}{4\pi \rho} \right)^{1/3},$$  

$$\tilde{\Omega}_N^2 = \frac{4}{3} \alpha_2 \alpha_3 \left[ \frac{2}{R^3} + \frac{6}{5R^5} (2 - \alpha_2^2 - \alpha_3^2) \right],$$  

$$\tilde{\Omega}_{PN}^2 = \frac{4}{3} (\alpha_2 \alpha_3)^{4/3} \left[ \frac{3}{R^3 \alpha_2 \alpha_3} - \frac{9}{2R^4} + \frac{42}{25R^5} \frac{\tilde{A}_0}{\alpha_2 \alpha_3} (2 - \alpha_2^2 - \alpha_3^2) - \frac{1}{5R^6} (28 - 14\alpha_2^2 - 9\alpha_3^2) \right],$$  

$$\tilde{E}_N = (\alpha_2 \alpha_3)^{1/3} \left[ -\frac{3}{140} \left( \frac{\tilde{A}_0}{\alpha_2 \alpha_3} \right)^2 + \frac{55}{48R^2} + \frac{1}{700R^3} \frac{\tilde{A}_0}{\alpha_2 \alpha_3} (398 + 401\alpha_2^2 + \alpha_3^2) \right] - \frac{1}{120R^4} (194 + 215\alpha_2^2 + 165\alpha_3^2),$$  

$$\tilde{E}_{PN} = (\alpha_2 \alpha_3)^{2/3} \left[ -\frac{3}{140} \left( \frac{\tilde{A}_0}{\alpha_2 \alpha_3} \right)^2 + \frac{55}{48R^2} + \frac{1}{700R^3} \frac{\tilde{A}_0}{\alpha_2 \alpha_3} (398 + 401\alpha_2^2 + \alpha_3^2) \right] - \frac{1}{120R^4} (194 + 215\alpha_2^2 + 165\alpha_3^2),$$  

$$\tilde{J}_N = 2(\alpha_2 \alpha_3)^{-1/6} \left[ \frac{3\tilde{\Omega}_N^2}{4\alpha_2 \alpha_3} \right]^{1/2} \left( \frac{\tilde{R}^2}{4} + \frac{1 + \alpha_2^2}{5} \right),$$  

$$\tilde{J}_{PN} = (\alpha_2 \alpha_3)^{1/6} \left[ \frac{3\tilde{\Omega}_N^2}{4\alpha_2 \alpha_3} \right]^{1/2} \left[ -\frac{3}{8} \tilde{R}^2 \frac{\tilde{A}_0}{\alpha_2 \alpha_3} + \frac{83}{48} \tilde{R} + \frac{1}{1400 \alpha_2 \alpha_3} (338 + 401\alpha_2^2 + 21\alpha_3^2) - \frac{1}{240\tilde{R}} (494 + 155\alpha_2^2 + 141\alpha_3^2) \right].$$
Then, using the numerical value of $\alpha_2$, $\alpha_3$, and $\tilde{R}$ determined at (1), we calculate the sequence of the angular velocity, the energy and the angular momentum as a function of the orbital separation.

We repeat this procedure changing the mean radius of each star $a_*$. Once a sequence is obtained, we search the minimum point of the energy. If we find it, we call it the ISCCO.

In figs. 2(a) and (b), we show $\tilde{E}$ and $\tilde{J}$ as functions of the normalized separation $R_*/a_*$, where $\tilde{E} = E/(M_*^2/a_*)$ and $\tilde{J} = J/(M_*^3 a_*)^{1/2}$. The figures show the important fact that the orbital separation at the ISCCO decreases approximately in proportion to $M_*/c^2 a_*$, i.e., the characteristic value of the compactness of each star.

In fig. 3, we show the normalized orbital angular velocity $\tilde{\Omega} = \Omega/\sqrt{\pi \rho}$ at the ISCCO as a function of compactness. We show $\tilde{\Omega}$ at the energy minimum as well as that at the angular momentum minimum. The figure indicates that two minimums are almost coincident, but slightly different. The deviation between the location of the minimums of the energy and the angular momentum comes from the fact that we assume $a_*/R_*$ is a small parameter, and expand the energy to $O(R^{-4})$ and the angular momentum to $O(R^{-1})$. In any case, we may expect that the ISCCO locates near two minimums. Fig. 3 clearly shows that the orbital angular velocity at the ISCCO increases almost linearly with increase of the compactness. Since the PN approximation is valid only for small compactness, we can not do solid estimate of $\tilde{\Omega}$ at the ISCCO for a realistic NS of $M_*/c^2 a_* \sim 0.2$. Here, we dare to extrapolate this results to the realistic NS's. Then, we can find that the angular velocity at the ISCCO is about 10% larger than that in the Newtonian case.

Before closing this section, we summarize the numerical results in Table I. In Table I, † denotes the point of the ISCCO defined by the energy minimum. We note that in the Newtonian order, the axial ratios $\alpha_2$ and $\alpha_3$ at the ISCCO are 0.8573 and 0.7995, respectively. Thus, the PN effect increases $\alpha_2$ and $\alpha_3$ at the ISCCO; i.e., tidal deformation is weak compared with the Newtonian case. The reason for such a behavior seems to be that each star of binary is forced to be compact due to the PN gravity (see Eq. (5.1)), and as a result,
the tidal force is less effective than in the Newtonian case.

VI. SUMMARY

In this paper, we have calculated the equilibrium sequences of the co-rotating BNS’s of the incompressible fluid using the first PN approximation of general relativity. The conclusions are as follows.

1. due to the PN effect, the orbital separation at the ISCO (secular instability limit) decreases in proportion to the compactness of the star $M_*/c^2 a_*$.
2. the orbital angular velocity at the ISCO increases in proportion to the compactness of the star $M_*/c^2 a_*$.
3. the reason for features (1) and (2) is that each star is forced to be compact due to the PN gravity and as a result, the tidal effect becomes less effective compared with the Newtonian binary.

These results agree with the recent numerical study by Shibata [20].

Since the analysis in this paper is done almost analytically, the equilibrium sequence obtained may be regarded as the exact solution of the Einstein equation for the limit of small compactness $M_*/c^2 a_* \ll 1$ and $a_* \ll R$. Recently, two groups have been performed numerical computations for obtaining the equilibrium state of BNS’s using the semi-relativistic approximation [17] [25]. The present solution is useful to check their numerical solutions as well as to understand the essence in their results.

Finally, we comment on the possibility of extending this work to the compressible fluid case. In this paper, it is found that the deformation of the ellipsoidal figure due to the PN gravity is small effect to the angular velocity, so that the ellipsoidal model for the density configuration may be a good approximation. If we determine the density configuration as the ellipsoidal approximation as was done by Lai, Rasio and Shapiro [23], we only need to carry out the integrals which appear (a) in calculating the angular velocity by the first TV equation, and (b) in calculating the energy and the angular momentum, which can be done
easily. We will publish results of such a calculation in a subsequent paper [26].

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TABLE CAPTIONS

Table.I. Equilibrium sequences of the PN Darwin ellipsoids with $M_*/c^2a_* = 0.01, 0.03 \text{ and } 0.05$. † denotes the point of the ISCCO.
| $R_*/a_*$ | $R_*/(a_1)_*$ | $a_2/a_1$ | $a_3/a_1$ | $\tilde{\Omega}$ | $\tilde{j}$ | $\tilde{E}$ |
|---------|-------------|--------|--------|--------|--------|--------|
| $M_*/c^2a_*=0.01$ |
| 2.80   | 2.190       | 0.7283 | 0.6571 | 0.3645 | 1.574  | -1.303 |
| 3.00   | 2.542       | 0.8135 | 0.7483 | 0.3208 | 1.527  | -1.317 |
| 3.20\dagger | 2.829       | 0.8593 | 0.8020 | 0.2877 | 1.518  | -1.320 |
| 3.25   | 2.891       | 0.8673 | 0.8119 | 0.2809 | 1.518  | -1.320 |
| 3.50   | 3.205       | 0.8999 | 0.8532 | 0.2497 | 1.528  | -1.317 |
| 4.00   | 3.785       | 0.9372 | 0.9040 | 0.2033 | 1.571  | -1.309 |
| $M_*/c^2a_*=0.03$ |
| 2.80   | 2.296       | 0.7776 | 0.7087 | 0.3488 | 1.534  | -1.311 |
| 3.00   | 2.602       | 0.8388 | 0.7775 | 0.3101 | 1.510  | -1.318 |
| 3.12\dagger | 2.768       | 0.8633 | 0.8069 | 0.2906 | 1.507  | -1.319 |
| 3.25   | 2.931       | 0.8828 | 0.8312 | 0.2729 | 1.509  | -1.318 |
| 3.50   | 3.235       | 0.9106 | 0.8673 | 0.2433 | 1.521  | -1.315 |
| 4.00   | 3.805       | 0.9433 | 0.9126 | 0.1987 | 1.565  | -1.307 |
| $M_*/c^2a_*=0.05$ |
| 2.80   | 2.370       | 0.8121 | 0.7467 | 0.3353 | 1.508  | -1.314 |
| 3.00   | 2.651       | 0.8596 | 0.8024 | 0.2999 | 1.496  | -1.318 |
| 3.04\dagger | 2.708       | 0.8674 | 0.8119 | 0.2932 | 1.495  | -1.318 |
| 3.25   | 2.966       | 0.8962 | 0.8484 | 0.2649 | 1.500  | -1.316 |
| 3.50   | 3.262       | 0.9201 | 0.8801 | 0.2368 | 1.514  | -1.313 |
| 4.00   | 3.824       | 0.9488 | 0.9206 | 0.1939 | 1.558  | -1.305 |

TABLE I.
FIGURE CAPTIONS

Fig.1. Sketch of the Darwin ellipsoid. The origin of the coordinate we choose locates at the center of mass of star 1.

Fig.2(a). The total energy of the equilibrium sequence as a function of $R_s/a_s$. Solid, dotted, dashed and long-dashed lines denote $M_\ast/c^2a_\ast = 0$ (the Newtonian case), 0.01, 0.03 and 0.05, respectively.

Fig.2(b). The total angular momentum of the equilibrium sequence as a function of $R_s/a_s$. Solid, dotted, dashed and long-dashed lines denote $M_\ast/c^2a_\ast = 0$ (the Newtonian case), 0.01, 0.03 and 0.05, respectively.

Fig.3. The orbital angular velocity at the ISCCO as a function of the compactness parameter $M_\ast/c^2a_\ast$. Filled circles and open triangles denote $\Omega/\sqrt{\pi\rho}$ for the minimum points of the total energy and total angular momentum, respectively.
Fig. 1

Star 2

Star 1

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Fig. 1
Fig. 2(a)

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Fig. 2(a)
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Fig. 2(b)
Fig. 3

- Energy minimum
- Angular momentum minimum