A new insight into BRST anomalies in string theory

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ABSTRACT

Using the generalized hamiltonian method of Batalin, Fradkin and Vilkovisky, we investigate the algebraic structure of anomalies in the Polyakov string theory that appear as the Schwinger terms in super-commutation relations between BRST charge and total hamiltonian. We obtain the most general form of the anomalies in the extended phase space, without any reference to a two dimensional metric. This pregeometrical result, refered to as the genelarized Virasoro anomaly, independent of the gauge and the regularization under a minor assumption, is a non-perturbative result, and valid for any space-time dimension. In a configuration space, in which the two dimensional metric can be identified, we can geometrize the result without assuming the weak gravitational field, showing that the most general anomaly exactly exhibits the Weyl anomaly.

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1 Introduction

It is well known that in quantizing relativistic strings at subcritical dimensions one encounters anomalies that appear in different forms; the central extension of the Virasoro algebra, the Weyl anomaly, the non-vanishing square of BRST charge etc. Most of the results on these anomalies, as those in the classical references such as refs. [2]-[7], however, are obtained only in particular classes of gauges. A general consideration, in which the gauge dependence of these anomalies is satisfactorily explored, has been lacking, and this may be one of the reasons why the relations among the anomalies above have been only partially revealed [3]-[13].

The phase space of Batalin, Fradkin and Vilkovisky (BFV) [14] is much larger than the usual one, and the gauge-dependence in their hamiltonian formalism can be tracked in a very transparent manner. Therefore, one may expect to obtain the most general form of anomalies in the extended phase space and then to gain a unified understanding on the anomalies [15]. Showing this in the Polyakov string theory [3] is exactly the aim of the present paper. We shall perform an algebraic analysis of anomalies [16, 17, 18] in the extended phase space [14] of the theory in an exhaustive fashion, and we will get a new insight into the anomalies in string theory by revealing their pregeometrical origin and hierarchical relationships.

As we will explain in sect. 2, there is a general, simple criterion for the presence of a gauge anomaly; given a gauge invariant classical system, the anomaly exists if the classical gauge algebra of BRST charge $Q$ and total hamiltonian $H_T$ can not be maintained upon quantization, i.e. if $2Q^2 = [Q, Q] \neq 0$ and/or $[Q, H_T] \neq 0$. We impose super-Jacobi identities [11, 12, 13, 19]-[21] on these anomalous commutators and expand them in $\hbar$ [15]. This leads to a set of consistency conditions on the anomalous terms [16] at each order in $\hbar$, which in turn exhibits the descending nature [22] of the anomalous Schwinger terms for $[Q, Q]$ and $[Q, H_T]$ in the hamiltonian formalism [15, 21, 23]. The fact that in the BFV formalism the BRST charge is directly constructed from classical first-class constraints [14] implies that the algebraic structure of $Q^2$ can be investigated in a completely gauge-independent fashion [15]. The gauge dependence of the theory enters only as a BRST commutator in the total hamiltonian in the extended phase space, and can be easily tracked in the BFV formalisms.

In sect. 3, we shall solve all the consistency conditions in the full expended phase space in an exhaustive manner, under the assumption specified there. Since the BRST cohomology in the extended phase space is the most general one, so is the solution, too. This is the anomaly in the BFV formulation of the Polyakov string theory, and is a pregeometrical result because it

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1See ref. [1] for a review and references therein.
is obtained without any reference to a metric.

This pregeometrical anomaly, called there the generalized Virasoro anomaly with its very simple form, can not be directly identified with the reparametrization or Weyl anomaly in the extended phase space. To do this, we must go to the configuration space and geometrize our result, as we shall do in sect. 4. For a configuration space in which three metric variables can be independently defined, we find a very general result that the generalized Virasoro anomaly exhibits nothing but the Weyl anomaly and completely fixes the relation among the anomalies we mentioned at the beginning. We emphasize that the general form of the Weyl anomaly, derived from the generalized Virasoro anomaly, is a non-perturbative result and independent of a choice of gauges and regularizations, and that it is valid for any space-time dimension D.

From the explicit computation of the Weyl anomaly in the conformal gauge \cite{10}, we fix the overall factor for the generalized Virasoro anomaly. This enables us to derive the $Q^2$ anomaly in the orthonormal gauge \cite{5} including its absolute normalization, which is the BRST version of the conventional Virasoro anomaly. This demonstrates one of the examples that show the hierarchical relations among the anomalies.

The appendix is devoted to prove the uniqueness of the solutions to the consistency conditions of sect. 3.

2 Commutator anomalies and consistency conditions

Here we would like to briefly outline the basic idea in our previous work on commutator anomalies \cite{15}. The BFV method \cite{14} uses two fundamental objects; a BRST charge $Q$ \cite{17} and a total hamiltonian $H_T$. They obey at the classical level the fundamental gauge algebras

$$\{Q, Q\} = 0 , \quad (1)$$

and

$$\frac{d}{dt} Q = \dot{Q} = \{Q, H_T\} = 0 , \quad (2)$$

which entirely express gauge invariance of a given theory: The "nilpotency of $Q$" expressed in eq. (1) means that the underlying constraints in the theory are first-class while eq. (2) implies the consistency of the constraints with the dynamics of the system. At the quantum level, these quantities must be suitably regularized to become well-defined operators. An anomaly arises if

\footnote{See ref. \cite{25} for a review.}
these gauge algebras can not be maintained upon quantization. The anomalous terms may be expanded in $\hbar$ as

$$[Q, Q] \equiv i\hbar^2 \Omega + O(\hbar^3)$$

(3)

$$[Q, H_T] \equiv \frac{i}{2} \hbar^2 \Gamma + O(\hbar^3),$$

(4)

where $[,]$ denotes super-commutator. We must distinguish a super-commutator from a naive one $[,]_0$ which is defined via super-Poisson bracket $\{,\}$:

$$[A, B]_0 \equiv i\hbar \{A, B\}.$$  

(5)

Our basic assumption is that the super-commutation relations between $Q$ and $H_T$ obey the commutation law,

$$[A, B] = (-1)^{|A||B|} [B, A],$$

(6)

the distribution law

$$[A, B + C] = [A, B] + [A, C],$$

(7)

and the super-Jacobi identity

$$[A, [B, C]] + (-1)^{|C|(|A|+|B|)} [C, [A, B]] + (-1)^{|A|(|B|+|C|)} [B, [C, A]] = 0,$$

(8)

where $|A|$ is the grassmann parity of the operator $A$ and can be either even ($= 0 \mod 2$) or odd ($= 1 \mod 2$). The crucial observation is that the outer commutators in the super-Jacobi identities for $Q$ and $H_T$, i.e.

$$[Q, [Q, Q]] = 0,$$

(9)

and

$$2 [Q, [Q, H_T]] + [H_T, [Q, Q]] = 0,$$

(10)

define a set of consistency conditions at each order in $\hbar$ in terms of the naive commutators only. Therefore, the introduction of the naive commutators is essential for the order-by-order investigation. In the lowest order, one finds that \[3\]

$$\delta \Omega = 0$$

(11)

\[3\]The consistency condition (11) has been considered in the context of bosonic string in ref. \[11\]. However, (12) which exhibits the descending nature of anomalies has not been discussed there (see below).
\[ \delta \Gamma = \{ H_T , \Omega \} = -\dot{\Omega} , \]  

(12)

where \( \delta \) is the BRST transformation given by a naive commutator, \( \delta A \equiv -\{ Q , A \} = i[Q , A]_0/\hbar \). The true anomalies \( \Omega \) and \( \Gamma \) should be cohomologically non-trivial; if \( \Omega \) and \( \Gamma \) are solutions, then \( \Omega + \delta X \) and \( \Gamma + \{ H_T , X \} + \delta Y \) also solve (11) and (12), respectively, for any \( X \) and \( Y \), which however can be removed to order \( \hbar^2 \) by redefining \( Q \) and \( H_T \) as \( Q \to Q - (\hbar X/2) \) and \( H_T \to H_T - (\hbar Y/2) \).

As one can see from the definitions (3) and (4), the anomaly in \( Q^2 \), which is basically the commutator anomaly in the algebras of constraints, descends into the anomaly in current divergence. In the chiral Yang-Mills theory, these are the Schwinger terms in the algebra of the Gauß law constraints \[19\] and the axial current divergence, respectively. The descending nature of the anomalies, described here in the hamiltonian formalism, originates from the super-Jacobi identities (9) and (10), which can not easily be recognized in the conventional, pure mathematical formulation of the descent equations \[22\]. To our knowledge, ref. \[15\] is the first one which delivers such a theoretical meaning of the descent equations. With these general discussions in mind we next would like to consider bosonic string theory.

3 Solution in the extended phase space: The generalized Virasoro anomaly

The Polyakov string theory can be described by the lagrangian \[3\]

\[ \mathcal{L} = -\frac{1}{2} \sqrt{-g} g^{\alpha \beta} \partial_{\alpha} X^\mu \partial_{\beta} X_\mu , \quad (\alpha, \beta = 0, 1; \quad \mu = 0, \cdots, D - 1) . \]  

(13)

We choose the parametrization for the metric variables \( g_{\alpha \beta} \) as

\[ \lambda^\pm = \frac{\sqrt{-g} \pm g_{01}}{g_{11}} , \quad \xi = \ln g_{11} . \]  

(14)

Here, the \( \lambda^\pm \) are Weyl invariant, and we shall see below why these variables are taken. A Weyl transformation is described by a translation in \( \xi \)-variable. The conjugate momenta of these variables, which we denote by \( \pi_\lambda^\pm \) and \( \pi_\xi \), vanish identically. These are primary constraints;

\[ \pi_\lambda^\pm \sim 0 , \quad \pi_\xi \sim 0 . \]  

(15)

\footnote{We regard \( \Gamma \) as the descendant of \( \Omega \) because \( \Gamma \) is basically determined by \( \Omega \) via eq. (12). Note that this order of what whose descendant is, is reverse to the usual one.}
The Dirac algorithm further leads to the secondary constraints, the Virasoro constraints,
\[ \varphi_{\pm} \equiv \frac{1}{4}(P \pm X')^2 \sim 0, \] (16)
where \( P_\mu \) denotes the conjugate momentum of the string coordinate \( X_\mu \), and \( X' = \partial X \equiv \partial_1 X \).

They satisfy the algebra under the Poisson bracket:
\[ \{ \varphi_{\pm}(\sigma), \varphi_{\pm}(\sigma') \} = \mp(\varphi_{\pm}(\sigma)\partial - \varphi_{\pm}(\sigma')\partial) \delta(\sigma - \sigma') \] (17)
\[ \{ \varphi_{\pm}(\sigma), \varphi_{\mp}(\sigma') \} = 0. \]

The extended phase space of BFV is defined as including to the classical phase space the ghost-auxiliary field sector
\[ (C^A, \overline{P}^A), (P^A, C_A), \text{ and } (N^A, B_A), \] (18)
where \( A(= \lambda^\pm, \xi, \pm) \) labels the first-class constraints given in (15) and (16). The \( C^A \) and \( P^A \) are the BFV ghost fields carrying one unite of the ghost number, \( \text{gh}(C^A) = \text{gh}(P^A) = 1 \), while \( \text{gh}(\overline{P}^A) = \text{gh}(\overline{C}^A) = -1 \) for their canonical momenta, \( \overline{P}^A \) and \( \overline{C}^A \). The last canonical pairs in (18) are auxiliary fields and carry no ghost number. We assign 0 to the canonical dimension of \( X^\mu, \lambda^\pm \) and \( \xi \), and correspondingly +1 to \( P_\mu, \pi^\lambda_\pm \) and \( \pi^\xi \). The canonical dimensions of \( C^\pm_\lambda, C^\xi, \overline{P}^\pm, \overline{P}^\lambda_\pm \) and \( \overline{C}^\xi \) are fixed only relative to that of \( C^\pm, c \equiv \dim(C^\pm): \)
\[ \dim(C^\pm_\lambda) = \dim(C^\xi) = 1 + c, \dim(\overline{P}^\pm) = 1 - c, \dim(\overline{P}^\lambda_\pm) = \dim(\overline{C}^\xi) = -c. \] (19)

The canonical dimensions of other fields are not needed for our purpose as we will see later.

Given the constraints with the algebra (17) and the corresponding extended phase space (18), we can construct the BRST charge
\[ Q = \int d\sigma[C^+\pi^\lambda_+ + C^-\pi^-_\lambda + C^\xi\pi^\xi + C^+(\varphi_+ + \overline{P}^+\partial C^+) \]
\[ + C^- (\varphi_- - \overline{P}^-\partial C^-) + B^A P^A], \] (20)
with \( \text{gh}(Q) = 1 \) and \( \dim(Q) = 1 + c \). This \( Q \) generates the BRST transformations
\[ \delta X^\mu = \frac{1}{2}[C^+(P + X')^\mu + C^- (P - X')^\mu] \]
\[ \delta P_\mu = \frac{1}{2}\partial[C^+(P + X')^\mu - C^- (P - X')^\mu] \]
\[ \delta C^\pm = \pm C^\pm\partial C^\pm \]
\[ \delta \overline{P}^\pm = -[\varphi_{\pm} \pm 2\overline{P}^\pm \partial C^\pm \pm \partial \overline{P}^\pm C^\pm] \]
\[ \delta \lambda^\pm = C^\lambda_\pm, \delta \xi = C^\xi \] (21)
\[ \delta \pi_\lambda^\pm = \delta \pi_\xi = 0, \ \delta C_\lambda^\pm = \delta C_\xi = 0 \]
\[ \delta \overline{P}_\pm = -\pi_\lambda^\pm, \ \delta \overline{P}_\xi = -\pi_\xi, \]
\[ \delta N^A = P^A, \ \delta \overline{C}_A = -B_A, \ \delta P^A = \delta B_A = 0 \]

We are now ready to solve the consistency condition (11) and seek the solution in the form
\[ \Omega = \int d\sigma \omega, \] (22)
where we assume that \( \omega \) is a polynomial of local operators with \( \text{gh}(\omega) = 2 \) and \( \text{dim}(\omega) = 3 + 2c. \)

According to the general structure of the BFV formalism, the total phase space can be divided, with respect to the action of \( \delta \), into two sectors;

\[ S_1 \text{ consisting of } (X^\mu, P_\mu), \] and
\[ S_2 \text{ consisting of all the other fields}. \] (23)

It is easy to see that on each sector the \( \delta \) operation closes:
\[ \delta_1^2 = \delta_2^2 = 0, \ \delta_1 \delta_2 + \delta_2 \delta_1 = 0, \] (24)
where \( \delta = \delta_1 + \delta_2 \), and \( \delta_1(\delta_2) \) acts on \( S_1 \) (\( S_2 \)) variables only. The \( S_2 \)-sector is BRST trivial because it is made of pairs \( (U^a, V^a) \) with \( \delta_2 U^a = \pm V^a \). As shown in ref. [13], there exists no non-trivial solution to \( \delta \Omega = \int d\sigma \delta \omega = 0 \) if \( \omega \) contains the \( S_2 \)-variables. Therefore, \( \omega \) is a function of \( C_\pm, \overline{P}_\pm, X^\mu, P_\mu \) and their derivatives only.

To proceed, we note that the BRST charge given in (20) has rigid symmetries; it is a Lorentz scalar, invariant under the translation, \( X^\mu \to X^\mu + a^\mu \) with a constant \( a^\mu \), and has a discrete symmetry defined by \( X^\mu \to X^\mu, P_\mu \to P_\mu, C_\pm \to C_\mp, \overline{P}_\pm \to \overline{P}_\mp, \partial \to -\partial \). We therefore assume, without loss of generality, that \( \omega \) also respects the same symmetries. The translational invariance forbids \( X^\mu \) to appear in \( \omega \) without derivatives. It is convenient to introduce the variables
\[ Y^\mu_\pm \equiv (P \pm X^\mu)^\mu, \] (25)
\[ \delta Y^\mu_\pm = \pm \partial (C_\pm Y^\mu_\pm). \]

In the appendix, we show that there is no candidate for \( \omega \) which satisfies \( \delta \omega = 0 \). But there is a unique solution with \( \delta \omega = \) a total derivative:
\[ \Omega = \int d\sigma \omega = \int d\sigma (\omega_0 + \omega_1), \] (26)
\footnote{The algebraic equation (11) to be solved is completely gauge-independent. It is the gauge fixing which may break the manifest Lorentz covariance.}
where
\[ \omega_0 = k \left[ (C^+ \partial^3 C^+) - (+ \rightarrow -) \right] \tag{27} \]
\[ \omega_1 = k' \left[ (C^+ \partial C^+ + C^- \partial C^+) - (+ \leftrightarrow -) \right] Y_+ \cdot Y_- . \tag{28} \]

The \( k \) and \( k' \) are gauge-independent constants which should be calculated in some gauge. The formal solution (26) contains an unfamiliar term, \( \omega_1 \) given in (28), which depends on string coordinates. This term is algebraically allowed as an independent anomaly, but it would have never appeared in explicit computations. Thus it implies that we may demand
\[ k' = 0 . \tag{29} \]

Now we would like to discuss the second consistency condition (12). Since the total Hamiltonian depends on the gauge chosen, the solution of (12) depends on it, too. The gauge-fixed action in the BFV formalism is defined as
\[ S = \int d^2 \sigma \left( P \cdot \dot{X} + \overline{F}_A C^A + \overline{C}_A \dot{F}^A + B_A N^A \right) - \int d\sigma^0 H_T , \tag{30} \]
with
\[ H_T = H_C + \frac{1}{\hbar} [Q , \Psi] , \tag{31} \]
where \( H_C \) is the canonical Hamiltonian, and \( \Psi \) is the gauge fermion \[25]. For the present case, \( H_C = 0 \). Combining the consistency condition (12) with (31), we find that \( \Gamma \) is given by a double commutator, to which we apply a super-Jacobi identity to find
\[ \Gamma = \{ \Omega , \Psi \} . \tag{32} \]

Clearly, we can not go further without any assumption on \( \Psi \), and so we restrict ourselves to the standard form of the gauge fermion
\[ \Psi = \int d\sigma [ \overline{C}_A \chi^A + \overline{F}_A N^A ] , \tag{33} \]
where \( \chi^A \)’s are gauge-fixing functions. We look for the solution again in the form
\[ \Gamma = \int d\sigma \gamma , \tag{34} \]
with \( gh(\gamma) = 1 \) and \( \text{dim}(\gamma) = 3 + c \). Since \( \Omega \) with \( k' = 0 \) contains only \( C^\pm \) (see (26) and (27)), we can compute the naive commutator in (32) unambiguously if \( \chi \)'s in \( \Psi \) do not depend on \( \overline{F}_\pm \). We therefore assume this, and arrive at the unique solution
\[ \gamma = 2k[ (\partial N^+ \partial^2 C^+) - (+ \rightarrow -) ] . \tag{35} \]
This result is independent of the gauge-fixing functions $\chi$’s, as long as the above assumption, which is about the weakest one imposed on $\Psi$, is satisfied.

Although $\Omega$ with $k' = 0$ has exactly the same form as the $Q^2$ anomaly of ref. [5], the theoretical content in $\Omega$ is much richer. This $\Omega$, along with its descendant $\Gamma$, exhibits namely the most general form of anomaly in the extended phase space – we would like to call it the generalized Virasoro anomaly because it must originate from the anomalous commutators for the generalized Virasoro constraints, $\Phi_{\pm} = \{\overline{\mathcal{P}}_{\pm}, Q\}$ [4]. Moreover, the result is pregeometric because it has been obtained without any reference to a two-dimensional metric. Note that the geometrical meaning of $\lambda^{\pm}$ and $\xi$, which is given in (14), has disappeared in the extended phase space, as one can see from their BRST transformations (21); they are no longer related to some metric variables, since the associated ghosts, $C_{\lambda}^{\pm}$ and $C_{\xi}$, are by no means the reparametrization ghosts and the Weyl ghost. So at the present stage, the generalized Virasoro anomaly can not be identified with the reparametrization or Weyl anomaly; to distinguish these anomalies from each other we certainly need a metric. In the next section, we shall describe how to construct the metric variables out of $\lambda^{\pm}$ and $\xi$, and how to geometrize the generalized Virasoro anomaly.

4 Geometrization to derive the Weyl anomaly

The primary constraints being proportional to $\pi_{\lambda}^{\pm}$ and $\pi_{\xi}$, respectively (see (14)), are responsible for the fact that the variables, $\lambda^{\pm}$ and $\xi$, have lost their original geometrical meanings. The aim of this section is to geometrize the theory to express the generalized Virasoro anomaly in terms of the GL(2)-covariant variables. This will enable us to interpret that pregeometric anomaly.

4.1 $GL(2)$-covariant ghosts

We begin by writing the gauge-fixed action defined in (30) with the standard form of $\Psi$ in (33) more explicitly:

$$S = \int d^2 \sigma \left\{ P \cdot \dot{X} + \pi_{\lambda}^{+} \dot{\lambda}^{+} + \pi_{\lambda}^{-} \dot{\lambda}^{-} - \pi_{\xi} \dot{\xi} + \overline{\mathcal{P}}_{A} \dot{C}^{A} - \mathcal{H}_{\text{CL}} - \mathcal{H}_{\text{GF}} - \mathcal{H}_{\text{FP}} \right\} , \ (A = \lambda^{\pm}, \xi, \pm)$$

(36)
where

\[ H_{CL} = \frac{1}{4}(P + X')^2 N^+ + \frac{1}{4}(P - X')^2 N^- + \pi^\lambda N^+ + \pi^\lambda N^- + \pi^\xi N^\xi \]

(37)

\[ H_{GF} = B_A \chi^A \]

(38)

\[ H_{FP} = \bar{C}_A \delta \chi^A + \bar{P}_A \bar{P}^A + [2\bar{P}_+ \partial \bar{C}^+ + \partial \bar{P}_- \bar{C}^+] N^+ - [2\bar{P}_- \partial \bar{C}^- + \partial \bar{P}_+ \bar{C}^-] N^- \]

(39)

Terms in the first line in \( H_{CL} \) suggests that \( N^\pm \) can be related to two of the metric variables. Recalling that \( P_\mu \) defined in the original lagrangian (13) is given by

\[ P_\mu = -\sqrt{-g} g^{0\alpha} \partial_\alpha X_\mu, \]

(40)

and rewriting the action as

\[ L = P \cdot \dot{X} - \frac{\sqrt{-g}}{2g_{11}} (P^2 + (X')^2) - \frac{g_{01}}{g_{11}} P \cdot X', \]

(41)

one can easily find such relations:

\[ \sqrt{-g}/g_{11} \sim N^0, \ g_{10}/g_{11} \sim N^1, \]

(42)

where \( N^\pm \equiv N^0 \pm N^1 \). Note that the l.h.s. of the expressions in (42) are exactly those for \( \lambda^0 \) and \( \lambda^1 \) in (14). Therefore, to recover the original meaning of \( \lambda^\pm \), we use two of the gauge degrees of freedom (there are five in the extended phase space) to impose two gauge conditions

\[ \chi_\lambda^+ = \lambda^+ - N^+, \quad \chi_\lambda^- = \lambda^- - N^-, \]

(43)

without changing the relation, \( \xi = \ln g_{11} \). There are still three gauge degrees of freedom, which we would like to regard as corresponding to two reparametrization symmetries and one Weyl symmetry. This is possible only if we can go to a configuration space which involves among others three independent metric variables, \( g_{\alpha\beta} \), along with two reparametrization (anti-) ghosts, \( C^\alpha (\bar{C}_\alpha) \), and one Weyl (anti-) ghost, \( C_W (\bar{C}_W) \), with the covariant BRST transformations [10]

\[ \delta g_{\alpha\beta} = C^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\beta C^\gamma g_{\alpha\gamma} + \partial_\alpha C^\gamma g_{\beta\gamma} + C_W g_{\alpha\beta} \]

\[ \delta C^\alpha = C^\beta \partial_\beta C^\alpha \]

\[ \delta C_W = C^\alpha \partial_\alpha C_W. \]

\[ ^6 \]

We have suppressed here the Legendre terms, \( \bar{C}_A \dot{\bar{N}}^A + B_A \dot{N}^A = -\delta (\bar{C}_A N^A) \), by shifting the gauge fermion, \( \Psi \rightarrow \Psi + (\bar{C}_A \dot{N}^A) \).
We would like to construct these covariant ghost fields from the BFV ghosts fields, by using various equations of motion. To this end, one has to specify the remaining gauge conditions. However, it is sufficient for us to assume that the gauge-fixing functions $\chi^{\pm}, \chi^{\xi}$ in the gauge fermion (33) do not depend on $\bar{\Pi}_{A}, \pi^{\lambda}_{\pm}$, and $\pi^{\xi}$. They are arbitrary otherwise, and in this sense our analysis given below is still gauge independent. The variations of $\bar{C}^{\lambda}_{\pm}$ yield the equations of motion for $C^{\pm}_{\lambda}$:

$$C^{\pm}_{\lambda} = \mathcal{P}^{\pm},$$

(45)

Similarly, the variations of $\bar{\Pi}_{\pm}, \bar{\Pi}^{\lambda}_{\pm}$ and those of $\pi^{\lambda}_{\pm}$ and $\pi^{\xi}$ give

$$\mathcal{P}^{\pm} = \dot{\bar{C}}^{\pm} \mp C^{\pm} \hat{\partial} N^{\pm},$$

$$\mathcal{P}^{\lambda}_{\pm} = \dot{\bar{C}}^{\lambda}_{\pm}, \mathcal{P}^{\xi} = \dot{C}^{\xi},$$

(46)

and

$$N^{\pm}_{\lambda} = \dot{\lambda}^{\pm}, N^{\xi} = \dot{\xi}$$

(47)

where $\mathcal{P}^{\pm} = \mathcal{P}^{0} \pm \mathcal{P}^{1}$ and $\hat{\partial} B = A \partial B - \partial AB$.

Note that, under the covariant BRST transformation (44), $\bar{g}_{00} \equiv g_{00}/g_{11}$ and $\bar{g}_{01} \equiv g_{01}/g_{11}$ are Weyl-invariant, i.e. $\delta \bar{g}_{00}$ and $\delta \bar{g}_{01}$ do not contain $C_{W}$, and that these variables have been already expressed in terms of $\lambda^{\pm}$. To find $C^{\alpha}$, therefore, we compare $\delta \bar{g}_{00}$ and $\delta \bar{g}_{01}$ with $\delta \lambda^{\pm} = C^{\pm}_{\lambda}$ (see (21)) with the equations of motion (45) and (46). One easily finds

$$C^{0} = \frac{C^{0}}{\lambda^{0}}, \quad C^{1} = C^{1} - \frac{\lambda^{1}}{\lambda^{0}} C^{0}.$$  

(48)

The relation between $C^{\xi}$ and $C_{W}$ can be found in a similar manner. We compare the BRST transformation obtained from (44)

$$\delta \ln g_{11} = C_{W} + C^{\alpha} \partial_{\alpha} \xi + C^{0} \frac{g_{01}}{g_{11}} + 2 C^{1u}$$

(49)

with the one in terms of the BFV basis

$$\delta \xi = C^{\xi}.$$  

(50)

where we have used (45), (46) and the gauge conditions (43). We find

$$C_{W} = C^{\xi} - V_{C}^{+} + V_{C}^{-},$$

(51)

where $V_{C}^{\pm}$ is defined by

$$V_{C}^{\pm} = \frac{1}{2} G_{\pm} C^{\pm} + (C^{\pm})'$$

(52)
Eqs. (49) and (51) completely fix the relation between the ghosts in the BFV basis and those in the GL(2)-covariant basis.

### 4.2 Derivation of the Weyl anomaly

We come to the central issue of the present work, derivation of the Weyl anomaly from the generalized Virasoro anomaly. We denote the geometrized $Q^2$ anomaly as $\Omega_g = \int d\sigma \omega_g$ and its descendant as $\Gamma_g = \int d\sigma \gamma_g$. Two different expressions for $Q^2$, $\Omega$ given in (26) and $\Omega_g$, should obviously belong to the same cohomology class defined by the BRST transformation (21) in the extended phase space. Therefore, the difference between $\Omega$ and $\Omega_g$ should be a coboundary term:

$$\int d\sigma \omega_g = \int d\sigma \omega_0 - \int d\sigma k \delta \eta .$$  

(54)

It turns out that the desired coboundary term \( \boxed{} \) is given by

$$\eta = U^+ + U^- - \frac{1}{2}(G_+ - G_-)C^\xi,$$

(55)

where

$$U^\pm = \frac{1}{4} G^2 \pm C^\pm + G_\pm (C^\pm)' .$$

(56)

To obtain the geometrized expression for $\omega_g$, one replaces the BFV ghosts in (54) by the covariant ones after performing the BRST transformation (21) on $\eta$. Using the relations

$$V^\pm_C = \frac{\partial U^\pm}{\partial G^\pm},$$

$$\sqrt{-g} R = -(V^+_N + V^-_N)' + \frac{1}{2}(G_+ - G_-),$$

$$V^\pm_N = \frac{1}{2} G \pm N^\pm + (N^\pm)' ,$$

(57)

as well as the equations of motion (46) and (47), one indeed obtains the covariant expression

$$\omega_k = k [ \sqrt{-g} R C^0 + \sqrt{-g} y^{\alpha\alpha} C_W \partial_\alpha C_W ] + (\text{total spatial derivative}).$$

(58)

\( \boxed{} \) We shall see below how to find it.
We now derive $\Gamma_g$. The local form of (32) with the phase-space expression of $\omega_g$ becomes

$$
\gamma_g = \{\omega_g, \Psi\} = \gamma + k\{\{Q, \eta\}, \Psi\} = \gamma + k\dot{\eta} - k\delta(\{\eta, \Psi\})
$$

where $\gamma$ is given in (35), and we have used a super-Jacobi identity and the equation of motion $\dot{\eta} = \{\eta, H_T\}$. A little algebraic calculation with the identity $V + N - V - N = N_\xi$ yields

$$
\gamma_g = -k\sqrt{-g}RC_W + \text{(total spatial derivative)}.
$$

It should be remarked that using the relation

$$
\delta(\sqrt{-g}RC_W) = \partial_{\alpha}I^\alpha,
$$

$$
I^\alpha = C^\alpha C_W\sqrt{-g}R + \sqrt{-gg^{\alpha\beta}}C_W\partial_\beta C_W,
$$

one can also confirm that the geometrized $\Omega_g$ and $\Gamma_g$ in (58) and (60) satisfy the descent equation in the configuration space $\delta\Gamma_g = -\dot{\Omega}_g$ with $I^0 \propto \omega_g$. The $\gamma_g$ is just twice of the divergence of the BRST current, $2\partial_{\alpha}J^\alpha$, which was explicitly calculated by Fujikawa in the conformal gauge. Comparing (60) with Fujikawa’s result one can fix

$$
k = -\frac{(D - 26)}{24\pi}.
$$

Given the geometrized form of the generalized Virasoro anomaly expressed by (58) and (60), its theoretical meaning becomes now more transparent. We observe that the geometrized $\Gamma_g$ contains the Weyl ghost only and there is no trivial term which can be added to $\Gamma_g$ (like $\eta$ in (59)) to completely replace $C_W$ by the reparametrization ghosts, $C^\alpha$. We therefore conclude that, unless $D = 26$, the constraint corresponding to the Weyl symmetry is inconsistent with the dynamics of the system and consequently it may not be regarded as a constraint any more. As the result, in the configuration space with three independent metric variables, the generalized Virasoro anomaly should be uniquely identified with the Weyl anomaly. This result on the Weyl anomaly, according to the character of its derivation, is non-perturbative and independent of a choice of gauges and regularizations (under minor assumptions).

It is worth finding the counter term in action needed to shift the generalized Virasoro anomaly $\Omega$ of (21) into its geometrized form $\Omega_g$ of (58). It can be calculated from the coboundary term

---

8 Conversely, this suggests how to choose the coboundary term (55).

9 This should be compared with the case of the chiral anomaly in chiral Yang-Mills theories in which the difference between the axial and vector current anomalies is BRST trivial.
and is given by

\[ L_g = \frac{\hbar k}{4} (V_N^+ G_+ + V_N^- G_- + N^+ G_+ + N^- G_-) \]

\[ = \frac{\hbar k}{4} \left[ \frac{1}{\sqrt{-g g_{11}}} \frac{\dot{g}_{11} - 2g'_{01} + g_{01} g'_{11} g_{11}}{g_{11}} \right] \]  

It may also be instructive to derive a gauge-fixed form of the Weyl anomaly. As an example, we consider the orthonormal gauge which is given by

\[ \chi^\pm = N^\pm - 1, \quad \chi^\xi = \xi. \]  

(64)

Using the naive equations of motion for the ghosts in this gauge

\[ C_W = -\partial_\alpha C^\alpha, \quad \partial_\mp C^\pm = 0, \]  

(65)

we find from (58) and (60) that

\[ [Q, Q] = ik \int d\sigma \left\{ (C^+ \partial^\xi C^+) - \left( \begin{array}{c} + \rightarrow - \\ \end{array} \right) \right\}, \]  

(66)

and

\[ [Q, H_T] = 0. \]  

(67)

This is the classical result of Kato and Ogawa [5], and corresponds to the BRST version of the conventional Virasoro anomaly. [3]

We would like to remark, however, that gauge-fixed forms of an anomaly are a self-contradicting notion [4]; it is not legitimate to fix the gauge and then to use equations of motions in the presence of the anomaly. Note that in contrast to those "on shell" anomalies, the generalized Virasoro anomaly is obtained here completely off shell and the Weyl anomaly is derived from it by using only the "safe" equations of motion which are not affected by the Weyl anomaly.

5 Summary

By applying the generalized hamiltonian method of Batalin, Fradkin and Vilkovisky [14], we have quantized bosonic string theory to perform an exhaustive algebraic analysis on anomalies.

\[ \text{One has to be careful in concluding that (66) and (67) are the gauge-fixed form of the Weyl anomaly. It is because that the same expressions can be obtained as a gauge-fixed form of the reparametrization anomaly in the Nambu-Goto string. See [2] for a more detailed discussion on the similarities and differences between the Polyakov string and the Nambu-Goto string.} \]

\[ \text{We are not allowed to gauge away the Weyl degree of freedom in the presence of the Weyl anomaly, for instance.} \]
in the extended phase space. In doing so, we have obtained, without any reference to a two dimensional metric, the most general form of anomaly, the generalized Virasoro anomaly which is expressed by $\Omega$ and its descendant $\Gamma$ given in (26) and (35), respectively. This pregeometrical anomaly has been uniquely identified with the Weyl anomaly in the configuration space, and at the same time we have derived its most general form, without assuming the weak gravitational field. Our results are non-perturbative, independent of the regularizations and the gauge-fixing functions (under the assumptions specified there), and valid for any $D$.

The absolute normalization for the Weyl anomaly and hence for the generalized Virasoro anomaly can be fixed by an explicit computation. We have used the result of Fujikawa [10] in the conformal gauge, and derived the $Q^2$ anomaly in the orthonormal gauge, which has been computed by Kato and Ogawa [5]. This is one of the examples for showing the hierarchical relations among the anomalies in bosonic string theory; In the unconstrained extended phase space, the generalized Virasoro anomaly is sitting on the top of the hierarchy of anomalies. And in its subspace, in which the two dimensional metric variables can be identified, this pregeometrical anomaly obtains its geometrical meaning and appears as the Weyl anomaly.

We have found the local counter term which shifts the generalized Virasoro anomaly into the Weyl anomaly. If one begins with the total lagrangian consisting of (13) and (63), one may be led to a reparametrization invariant but Weyl non-invariant theory. It means that the results obtained in this paper will be the starting point toward construction of subcritical string theory or 2D gravity as an anomalous gauge theory. We shall discuss this issue in the forthcoming communications [28].

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Appendix

In this appendix we prove the uniqueness of the formal solution for $\omega$ given in (27) with
(28) and (29). As discussed in sect. 3, \( \omega \) is a function of \( C^\pm, Y^\mu_\pm \equiv (P \pm X')^\mu, \overline{P}_\pm \) and their derivatives only. We start by writing the most general form of \( \omega \), allowed by its ghost number, its canonical dimension (see eq. (22)), and the rigid symmetries assumed there:

\[
\omega = \omega_0 + \omega_Y + \omega_\overline{P},
\]

where

\[
\omega_0 = k_0 \left[ (C^+ \partial^3 C^+) - (+ \rightarrow -) \right]
\]

\[
\omega_Y = \left[ C^+ \partial C^+ (k_1 Y_+ \cdot Y_- + k_2 Y_+^2 + k_3 Y_-^2) + C^+ \partial C^- (k_4 Y_+ \cdot Y_-
\right. \\
\left. + k_5 Y_+^2 + k_6 Y_-^2) \right] - \left[ + \leftrightarrow - \right]
\]

\[
\omega_\overline{P} = \left[ C^- \partial^2 C^+ \left( k_7 \overline{P}_+ + k_8 \overline{P}_- \right) + \partial C^+ \partial C^- C^+ \left( k_9 \overline{P}_+ + k_{10} \overline{P}_- \right) \right] + \left[ + \leftrightarrow - \right].
\]

In eqs. (A.2)-(A.4), we have regarded total derivative terms as zero because they do not contribute to \( \Omega \). Since the BRST variation of \( \omega_0 \) is already a total derivative, i.e.

\[
\delta \omega_0 = -k_0 \left[ \partial (C^+ \partial C^+ \partial^2 C^+ + (+ \rightarrow -)) \right],
\]

\( \omega_0 \) is clearly an independent part of \( \omega \).

We next consider \( \delta \omega_\overline{P} \), and find that its \( \overline{P} \)-dependent terms can become a total derivative, namely

\[
\delta \omega_\overline{P} = -k_{10} \partial \left[ C^- \partial C^+ (C^- - C^- \overline{P}_+ - \overline{P}_-) \right] + \text{(terms without } \overline{P}_\pm )
\]

if and only if

\[
k_7 = 0, k_8 = k_9 + k_{10}.
\]

If these relations are satisfied, on the other hand, \( \omega_\overline{P} \) can basically be absorbed into \( \omega_Y \). To see this, one adds to \( \omega_\overline{P} \) a BRST trivial term

\[
- \left[ k_9 \delta (C^+ \partial C^- \overline{P}_+ ) + k_{10} \delta (C^+ \partial C^+ \overline{P}_-) \right] + \left[ + \leftrightarrow - \right]
\]

and then shifts \( k_3 \) and \( k_5 \) in \( \omega_Y \) according to \( k_3 \rightarrow k_3 + \frac{1}{4} k_{10} \) and \( k_5 \rightarrow k_5 + \frac{1}{4} k_9 \), respectively. Therefore, we can discard \( \omega_\overline{P} \) as an independent solution. The BRST transformation of the remaining term, \( \omega_Y \), becomes a total derivative if and only if

\[
k_3 = -k_5 = k_6, k_1 = -k_4.
\]

One indeed finds

\[
\delta \omega_Y = k_1 \partial \left[ C^+ C^- \partial (C^+ - C^-) Y_+ \cdot Y_- \right]
\]
\[ +k_3 \partial[ C^+ C^- \partial (C^+ + C^-) (Y_2^2 - Y_+^2) ] , \tag{A.10} \]

if (A.9) is used. Note however that terms proportional to \( k_2 \) and those proportional to \( k_3 = -k_5 = k_6 \) in \( \omega_Y \) are coboundary terms (up to total derivatives), because

\[ \delta (C^\pm Y_\mp^2) = \mp C^\pm \partial C^\pm Y_\mp^2 \tag{A.11} \]

\[ \delta (C^\pm Y_\mp^2) = \pm \partial (C^+ C^- Y_\mp^2) \pm [(C^\pm \partial C^\pm + C^\pm \partial C^\mp + C^\mp \partial C^\pm) Y_\mp^2]. \tag{A.12} \]

This shows that \( \omega_Y \) with \( k' = k_1 \) and eq. (A.9) - up to coboundary and total derivative terms - is exactly \( \omega_1 \) given in eq. (28).
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