Glauber–Sudarshan-type quantizations and their path integral representations for compact Lie groups

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Abstract

In this paper, we consider an arbitrary irreducible unitary representation \((\pi_\lambda, V_\lambda)\) of a compact connected, simply connected semisimple Lie group \(G\) with highest weight \(\lambda\), and apply the idea of Daubechies–Klauder (1985) and Yamashita (2011) on rigorous coherent-state path integrals to this representation, where the orbit of the highest weight vector is interpreted as the manifold of coherent states. Our main theorem is two-fold: the first main theorem is in terms of Brownian motions and stochastic integrals, and proven using the Feynman–Kac–Itô formula on a vector bundle of a Riemannian manifold, due to Güneysu (2010). In the second main theorem, we consider a sequence \((\mu_n)\) of finite measures on the space of smooth paths, and a 'path integral' is defined to be a limit of the integrals with respect to \((\mu_n)\). The formulation and the proof of the second main theorem employ rough path theory originated by Lyons (1998).

1 Introduction

There are several approaches to mathematical foundation of path integrals occurring in quantum physics. Feynman’s original idea is to represent the time evolution of a quantum system, as well as the expectation values of some sort of observables in it, by a integral on the space of paths on the configuration space of the system. As is well known, if we consider the “imaginary time” evolution instead of real time evolution, so-called the Wick rotation, a large part of the idea can be made rigorous by the Feynman–Kac theorem and its generalizations, and this “imaginary time + Feynman–Kac” approach is the most successful one. However, note that in the imaginary-time approaches, it is difficult to deal with time-dependent Hamiltonians, as well as non-unitary time evolutions occurring in open systems. This implies that it is hard to apply the imaginary-time methods to e.g. the theories of quantum information/probability, where time-dependent Hamiltonians and non-unitary time evolutions (e.g. decoherences) frequently occur.

On the other hand, the notion on configuration-space path integrals are believed to be derived from more general notion of phase-space path integrals. Although configuration-space path integrals are preferred to phase-space
path integrals especially in relativistic quantum field theories for their ‘manifest Lorentz covariance,’ the latter ones will be more fundamental if we consider a path integral as a procedure of quantization of a classical system; the main stream of the rigorous studies of quantization (e.g. the theories of geometric/deformation quantization) are formulated on phase spaces. Unlike imaginary-time configuration-space path integrals, little is known about the rigorous justification of general phase-space path integrals (in real or imaginary time).

There is another notion of coherent-state path integrals, which resembles to that of phase-space path integrals; Sometimes the former notion is said to be a part of the latter one, but the precise relation between them is not clear since the rigorous definitions of both have not been given. The notion of coherent states are introduced by Glauber [9], and later generalized by many authors. The original ‘usual’ coherent states are called Glauber coherent states (GCS), to distinguish them from others. Although no widespread rigorous definition of generalized coherent states seems to exist, it seems commonly recognized that if a unitary highest weight irreducible representation of a transformation group of a system is given, the orbit of the highest weight vector is a typical example of the manifold of coherent states (see e.g. [15]).

In 1985, Daubechies and Klauder [4] gave a rigorous GCS path integral formula representing real-time evolution for some class of Hamiltonians, in terms of Brownian motions and stochastic integrals. Yamashita [20] studied GCS path integrals in a similar idea but for other class of Hamiltonians, and with an emphasis on geometric meaning of them. Although an imaginary-time configuration-space path integral can be defined as an integral with respect to a single Wiener measure by the Feynman–Kac theorem, it seems believed that a path integral of other kinds cannot be defined to be an integral with respect to a single Borel measure. Instead we consider a sequence \((\mu_n)_{n \in \mathbb{N}}\) of measures, and regard a path integral as a limit of the form

\[
\lim_{n \to \infty} \int F(\psi) d\mu_n(\psi).
\]

In this paper, we consider an arbitrary irreducible unitary representation of a compact connected, simply connected semisimple Lie group \(G\), and apply the idea of [20] to the orbit of the highest weight state \(G \cdot \mathbf{E}_\lambda\), which is a symplectic manifold with the natural symplectic 2-form \(\omega\), called the Kirillov–Kostant– Souriau 2-form, identifying the orbit \(G \cdot \mathbf{E}_\lambda\) with the coadjoint orbit \(G \cdot \lambda\). Thus \((G \cdot \mathbf{E}_\lambda, \omega)\) can be regarded as a phase space of some classical-mechanical system. However, here we shall deal with the integral on the space of paths on \(G\), not on \(G \cdot \mathbf{E}_\lambda\). The main reason for that is as follows. Consider the usual flat phase space \(M = \mathbb{R}^{2n}\) with a symplectic 2-form \(\omega\). Then there exists a 1-form \(\theta\), called the canonical 1-form, such that \(d\theta = \omega\). If a path \(C\) on \(M\) is given, we can consider the line integral \(\int_C \theta\), interpreted as the “action along \(C\).” On the other hand, for general symplectic manifold \((M, \omega)\), the 1-form \(\theta\) satisfying \(d\theta = \omega\) may not exist; Even if such \(\theta\) exists, the reason for choosing a distinguished \(\theta\), which should be called a ‘natural’ or ‘canonical’ one, may not exist. However, a ‘fairly natural’ 1-form \(\theta\) exists on \(G\), not on \(G \cdot \lambda \cong G \cdot \mathbf{E}_\lambda\); that is, \(\theta\) is the left-invariant 1-form (i.e. the Maurer–Cartan form) w.r.t. the highest weight \(\lambda\). Let \(\tilde{\omega}\) be the pullback of \(\omega\) w.r.t. the map \(G \ni g \mapsto g \cdot \lambda \in G \cdot \lambda\), then we find
\( \hat{\omega} = -d\theta \). Thus our path integral can be said to be nearly a coherent-state or phase-space path integral, but not exactly.

The paper is organized as follows. In Section 2, the statement of the main theorem, together with the definitions of notions (including GS quantization) and symbols needed to state them, is presented. Our main theorem is two-fold: the first theorem is formulated as a limit of the integrals on the space of smooth paths. In Section 8, we prove the first main theorem, and the second theorem is formulated as a limit of the integrals on the space of smooth paths. In Section 9 and 10, we present an outline of the proof of the main theorem. The pre-Borel–Weil theorem, which is a complex-geometric representation of the irreducible unitary representations of \( G \), is derived from the pre-Borel–Weil theorem, but we need only the latter theorem in this paper. In Section 11, we prove the second main theorem, together with the definitions of notions (including GS quantization) and symbols needed to state them, is presented. Our main theorem is two-fold: the first theorem is in terms of Brownian motions and stochastic integrals, and the second theorem is in terms of rough path theory in the style of \[8\].

2 Main theorem

First we recall basic definitions on Lie groups and Lie algebras which we will use in this paper.

Let \( G \) be a compact connected, simply connected semisimple Lie group, that is, \( G \) be one of \( SU(n) \) \((n \geq 2)\), \( Spin(n) \) \((n \geq 3)\), \( Sp(n) \) \((n \geq 1)\) and the five exceptional groups of the types \( E_6, E_7, E_8, F_4 \) and \( G_2 \). Let \( g \) be the Lie algebra of \( G \); \( \mathfrak{g}_C \) be the complexifications of \( G \) and \( g \), respectively. Fix a maximal torus \( T \subset G \) (i.e. \( T \) is a maximal commutative connected compact subgroup of \( G \)). In fact \( T \cong U(1)^{\ell} \) for some \( \ell \). The Lie algebra of \( T \) is denoted by \( t \), and its complexification by \( \mathfrak{t}_C \) (the Cartan subalgebra of \( \mathfrak{g}_C \)). Let \( \ell \) be the rank of \( G \), i.e. \( \ell := \dim t \). Let \( \hat{G} \) denote the unitary dual of \( G \), i.e. the set of (the equivalence classes of) the irreducible unitary presentations of \( G \).

Let \( \kappa(\cdot, \cdot) \) denote the Killing form on \( \mathfrak{g}_C \). Define the linear bijection \( \nu : \mathfrak{g}_C \to \mathfrak{g}_C^* \) by \( \nu(X)(Y) := \kappa(X,Y) \). Define the bilinear form \( (\cdot, \cdot) \) on \( \mathfrak{g}_C^* \) by

\[
(\alpha, \beta) := \kappa(\nu^{-1}(\alpha), \nu^{-1}(\beta)), \quad \alpha, \beta \in \mathfrak{g}_C^*.
\]

For \( \alpha \in \mathfrak{t}_C \), let

\[
\mathfrak{g}_C^\alpha := \{X \in \mathfrak{g}_C| [T, X] = \alpha(T)X, \ \forall T \in \mathfrak{t}_C\}
\]

and \( R := \{\alpha \in \mathfrak{t}_C| \mathfrak{g}_C^\alpha \neq \{0\}\} \setminus \{0\} \), the set of roots of \( \mathfrak{g}_C \). Fix a decomposition \( R = R^+ \cup R^- \), \( R^+ \cap R^- = \emptyset \) such that \( \alpha \in R^+ \) iff \( -\alpha \in R^- \), and that

\[
\alpha, \beta \in R^+, \ \alpha + \beta \in R \implies \alpha + \beta \in R^+
\]

Each element of \( R^+ \) is called a positive root. The subset \( R^+_k \subset R^+ \) of simple roots is defined by

\[
R^+_k := \{\alpha_1, \ldots, \alpha_k\} := \{\alpha \in R^+| \alpha \neq \beta + \gamma, \ \forall \beta, \gamma \in R^+\}
\]
Define the **weight lattice** by

\[ P := \{ \lambda \in \mathfrak{t}^* \mid (\alpha^\vee, \lambda) \in \mathbb{Z}, \quad \forall \alpha \in \mathfrak{t} \} \subset \mathfrak{t}^* \]

where \( \alpha^\vee := 2\alpha/(\alpha, \alpha) \) is the **coroot** corresponding to \( \alpha \). Each element of \( P \) is called an **algebraically integral weight**.

Let \( \ker \exp_t := \{ X \in \mathfrak{t} \mid \exp(X) = 1_G \} \) where \( 1_G \) is the unit in \( G \). The **character lattice** for \( T \) is defined by

\[ \mathcal{X}(T) := \{ \lambda \in \mathfrak{t}^* \mid (\lambda, X) \in 2\pi \mathbb{Z}, \quad \forall X \in \ker \exp_t \}. \]

Each element of \( \mathcal{X}(T) \) is called an **analytically integral weight**. Under the assumption that \( G \) is simply connected, the character lattice \( \mathcal{X}(T) \) equals the weight lattice \( P \) (we have \( \mathcal{X}(T) \subset P \) in general). The set of **dominant weights** \( \mathcal{X}_+(T) \subset \mathcal{X}(T) \) is defined by

\[ \mathcal{X}_+(T) := \{ \lambda \in \mathcal{X}(T) \mid (\lambda, \alpha_i^\vee) \in \mathbb{Z}_+, \quad i = 1, ..., \ell \} \]

It is shown that there is a one-to-one correspondence between \( \hat{G} \) and \( \mathcal{X}_+(T) \); For each \( \lambda \in \mathcal{X}_+(T) \), let \( (d\pi_\lambda, V_\lambda) \) be the irreducible highest weight representation of \( g_C \) on the complex vector space \( V_\lambda \) with highest weight \( \lambda \). Let \( \pi_\lambda \) denote the lift of \( d\pi_\lambda \) to an irreducible representation of \( G \), i.e. \( (\pi_\lambda, V_\lambda) \) be the irreducible representation of \( G \) such that \( d\pi_\lambda \) is the differential representation of \( \pi_\lambda \). Each \( V_\lambda \) has the inner product \( \langle \cdot \vert \cdot \rangle \) where \( \pi_\lambda \) is unitary. (We use the notation \( \langle \cdot \vert \cdot \rangle \) only for usual (positive-definite) inner products, linear in the second variable; on the other hand the notation \( \langle \cdot, \cdot \rangle \) denotes more generic forms, possibly not positive-definite.) We often write the representation \( d\pi_\lambda \) of \( g_C \) simply as \( \pi_\lambda \), unless confusion arises.

Let \( v_\lambda \in V_\lambda \) (\( ||v_\lambda|| = 1 \)) be a highest weight vector, i.e.

\[ \pi_\lambda(X)v_\lambda = \lambda(X)v_\lambda, \quad \forall X \in \mathfrak{g}. \]

For \( v \in V_\lambda \), define \( v^* \in V_\lambda^* \) by \( v^*(u) := \langle v \vert u \rangle \), \( u \in V_\lambda \). Let \( E_\lambda = v_\lambda v_\lambda^* \) be the orthogonal projection from \( V_\lambda \) onto \( \mathbb{C} v_\lambda \). Let

\[ g \cdot E_\lambda := \pi_\lambda(g)E_\lambda \pi_\lambda(g^{-1}), \quad g \in G, \]

and \( G \cdot E_\lambda := \{ g \cdot E_\lambda \mid g \in G \} \), called the **orbit** through \( E_\lambda \), or the manifold of **coherent states** in the physical context (see e.g. [15]).

For a smooth function \( h : G \cdot E_\lambda \rightarrow \mathbb{C} \), define \( Q(h) \in \text{End}(V_\lambda) \) by

\[ Q(h) := d_\lambda \int_G h(g \cdot E_\lambda)dg, \quad d_\lambda := \dim V_\lambda \]

where \( dg \) denotes the Haar measure on \( G \), normalized so that \( \int_G dg = 1 \). We call the map \( Q : C^\infty(G \cdot E_\lambda) \rightarrow \text{End}(V_\lambda) \) the **Glauber–Sudarshan-type quantization** (or simply, the **GS quantization**). If \( h \) is a real-valued, then the GS quantization \( Q(h) \) is self-adjoint, and so \( \{ e^{itQ(h)} \mid t \in \mathbb{R} \} \) is a one-parameter unitary group. Note that every self-adjoint operator on \( V_\lambda \), possibly not in \( \pi_\lambda(ig) \), is represented as \( Q(h) \) for some \( h \in C^\infty(G \cdot E_\lambda, \mathbb{R}) \). This naming is by an analogue of the **Glauber–Sudarshan representation** (also called the **P-representation**) for the Glauber coherent states, frequently used in quantum optics. A mathematical reason for calling \( Q(h) \) a “quantization of \( h \)” is seen in e.g. [15] [14].
For each \( v \in V_\lambda \), define \( \hat{v} \in L^2(G) \) by

\[
\hat{v}(g) := d_{\lambda}^{1/2} \langle \pi_\lambda(g)v_\lambda | v \rangle, \quad g \in G.
\]

Then the map \( v \mapsto \hat{v} \) turns out to be an isometry.

Since the Killing form \( \kappa \) of \( g \) is negative-definite, \( \langle X | Y \rangle_g := -\kappa(X, Y) \) defines an inner product on \( g \). This induces a Riemannian metric on \( G \). Now consider the Brownian motion \( B \) on the Riemannian manifold \( G \) in the time interval \([0, \infty)\), where the distribution of the starting point is uniform on \( G \), i.e., equals the Haar measure \( d\mu \) on \( G \). Let \( \mu^1 \) be a probability measure on \( C([0, \infty), G) \) which represents such Brownian motion (i.e., a Wiener measure uniform on \( G \)).

For \( r > 0 \), define the probability measure \( \mu^r \) on \( C([0, \infty), G) \) by

\[
d\mu^r(B) := d\mu(B(r^{-1} \cdot)),
\]

i.e., \( \mu^r \) is the time rescaling of \( \mu^1 \), so that the \( \mu^r \)-Brownian motion diffuses \( r \) times faster than the \( \mu^1 \)-Brownian motion.

For \( \alpha \in \mathfrak{g}_C^* \) define the \( C \)-valued 1-form \( \alpha^R \) on \( G \) as the unique right-invariant \( C \)-valued 1-form such that \( \alpha^R_{|_\alpha} = \alpha |_\alpha \). If \( G \) is embedded in the matrix Lie group \( \text{GL}(n, C) \), we have

\[
\alpha^R_g(X) = \alpha\left(X_{g}g^{-1}\right), \quad g \in G, \ X \in \mathfrak{X}(G)
\]

where \( \mathfrak{X}(G) \) is the space of vector fields on \( G \). We naturally view \( t^* \) as a subspace of \( g^* \), and \( g^* \) as a real linear subspace of \( \mathfrak{g}_C^* \) by

\[
g^* \ni \alpha \mapsto \tilde{\alpha} \in \mathfrak{g}_C^*, \quad \tilde{\alpha}(X + iY) := \alpha(X) + i\alpha(Y), \quad X, Y \in g.
\]

Hence we have it \( \leftrightarrow \mathfrak{g}_C^* \), and so \( \alpha^R \) is defined for any \( \alpha \in t^* \), which is a \( \mathbb{C} \)-valued 1-form.

Let \( \rho \in t^* \) be the half sum of positive roots of \( \mathfrak{g}_C^* \): \( \rho := \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} \alpha \). For \( h \in C^\infty(G \cdot E_\lambda, \mathbb{R}) \) and \( t \geq 0 \), let

\[
\mathcal{I}_t(h; B) := \int_0^t \alpha^R(dB_s) - i \int_0^t \dot{h}(B_s) ds, \quad \alpha := -(\lambda + \rho).
\]

Note that \( \mathcal{I}_t(h) \in \mathfrak{i}_\mathbb{R} \), and so \( e^{\mathcal{I}_t(h)} \in \text{U}(1) \). Fix an arbitrary \( v_1 \in V_\lambda \) with \( \|v_1\| = 1 \), and set

\[
Z_{\lambda,t,r} := \int_{C([0, \infty), G)} \left[ e^{\mathcal{I}_t(h; B) \frac{\overline{\tilde{v}_1}}{\mathcal{Z}_0 \tilde{v}_1} (B)} \right] d\mu^r(B).
\]

It is shown that \( Z_{\lambda,t,r} > 0 \), and that \( Z_{\lambda,t,r} \) does not depend on \( v_1 \).

**Theorem 2.1** (Main: Brownian form). Let \( h \in C^\infty(G \cdot E_\lambda, \mathbb{R}) \) be a ‘classical Hamiltonian.’ Then for any \( u, v \in V_\lambda \) and \( t > 0 \), we have

\[
\langle u | e^{it\Phi(h)} v \rangle = \lim_{r \to \infty} \int_{C([0, \infty), G)} \left[ e^{\mathcal{I}_t(h; B) \frac{\overline{u}}{\mathcal{Z}_0 (B)}} \frac{d\mu^r(B)}{Z_{\lambda,t,r}} \right] \]

Consider the problem of generalizing this result to the cases where
(i) $G$ is a finite-dimensional non-compact Lie group;
(ii) $G$ is an infinite-dimensional non-compact Lie group (e.g. infinite-dimensional Heisenberg group, spin group, gauge transformation group, etc.)

In both case, the representation space $V_{\lambda}$ is infinite-dimensional.

In case (i), if $G$ has an invariant Riemannian metric $g$, the ‘standard’ Brownian motion on $(G, g)$ exists, and so it is conjectured that some equation similar to (2.3) holds for an irreducible unitary representation of $G$. (Some positive results concerning this conjecture are given in [1, 20] when $G$ is a finite-dimensional Heisenberg group.)

However, in the other cases of (i), and in all cases of (ii), the standard Brownian motion on $G$ does not exist. Hence any straightforward generalization of (2.3) seems impossible in these cases. To make matters worse, $G$ have no invariant measure in case (ii), and hence the left/right regular representations of $G$ on $L^2(G)$ cannot be defined. Thus it is worth reformulating Theorem 2.1 to a statement which refers to neither Brownian motions nor $L^2(G)$:

**Theorem 2.2 (Main: smooth form).** In the setting of Theorem 2.1, let $(\mu_k)_{k \in \mathbb{N}}$ be a sequence of finite measures on the smooth path space $C^\infty([0, \infty); G)$. If $(\mu_k)_{k \in \mathbb{N}}$ satisfies some conditions given in Sec. 11, then for each $u, v \in V_{\lambda}$ and $t > 0$,

$$\langle u | e^{itQ(h)}v \rangle = \lim_{k \to \infty} \hat{C}^\infty([0, \infty); G) \left[ e^{i\mathcal{I}(h; \varphi)} \langle u | \pi_{\lambda}(\varphi(0))v_{\lambda} \rangle \langle \pi_{\lambda}(\varphi(t))v_{\lambda} | v \rangle \right] d\mu_k(\varphi),$$

(2.4)

where

$$\mathcal{I}_t(h; \varphi) := \int_0^t \alpha^R(d\varphi(s)) - i \int_0^t \dot{h}(\varphi(s))ds, \quad \alpha := -(\lambda + \rho).$$

Here we raise the problem to give a necessary and sufficient condition for $(\mu_k)_{k \in \mathbb{N}}$ to satisfy Eq. (2.4). Although it seems quite difficult to give a perfect answer to this problem, a fairly good sufficient condition is given in terms of rough path theory, originated by Lyons [10].

3 **Pre-Borel–Weil theorem**

Let $G$ be a compact connected, simply connected semisimple Lie group, and $g$ be the Lie algebra of $G$; $G_C$ and $g_C$ be the complexifications of $G$ and $g$, respectively. Fix a maximal torus $T \subset G$, and its Lie algebra $t$.

Define the **adjoint operation** on $g_C$ to be the antilinear map $\ast : g_C \to g_C$ such that $X^\ast = -X$ for all $X \in g$. We see the relation $[X, Y]^\ast = [Y^\ast, X^\ast]$, $X, Y \in g_C$. If $g_C$ is embedded in the matrix Lie algebra $\text{Mat}(n, \mathbb{C}) \cong \mathfrak{gl}(n, \mathbb{C})$, $X^\ast$ is nothing but the adjoint matrix of $X \in g_C$.

Let

$$n^- := \bigoplus_{\alpha \in R^+} g_C^{-\alpha}, \quad b^- := t \oplus n^-.$$

Let $T_1, \ldots, T_\ell \in t$ be a basis of $t_C$ such that

$$\kappa(T_i, T_j) = \delta_{ij}.$$
For each $\alpha \in R$, we can take an element $E_\alpha \in g^2$ such that $E_{-\alpha} = -E_\alpha^*$ and $\kappa(E_\alpha, E_{-\alpha}) = -\kappa(E_\alpha, E_{-\alpha}) = 1$ for all $\alpha \in R$ (Weyl's canonical basis). Then $\{E_\alpha, T_i, E_\alpha^* | \alpha \in R^+\}$ is a basis of $g_C$ with dual basis $\{E_\alpha^*, T_i, E_\alpha | \alpha \in R^+\}$ w.r.t. $\kappa$.

The left and right regular representation $\mathcal{F}_L$ and $\mathcal{F}_R$ of $G$ on $L^2(G)$ are defined by

$$(\mathcal{F}_L(g)f)(x) := f(g^{-1}x), \quad (\mathcal{F}_R(g)f)(x) := f(xg), \quad g \in G, \ f \in L^2(G)$$

For $X \in g$, let $X^L := d\mathcal{F}_L(X)$ and $X^R := d\mathcal{F}_R(X)$. That is, $X^R$ and $X^L$ are the differential operators on $C^\infty(G)$ defined by

$$(X^L f)(g) := \frac{d}{dt} f(e^{-tX} g)|_{t=0}, \quad (X^R f)(g) := \frac{d}{dt} f(ge^{tX})|_{t=0},$$

for $g \in G, \ f \in C^\infty(G)$. For $Z = X + iY \in g_C$ with $X, Y \in g$, let

$$Z^L := X^L + iY^L, \quad Z^R := X^R + iY^R.$$ 

Let $\mathcal{U}(g_C)$ be the universal enveloping algebra of $g_C$. The maps $Z \mapsto Z^L$ and $Z \mapsto Z^R$ are representations of $g_C$ on $C^\infty(G)$, and hence the definitions of $Z^L$ and $Z^R$ are naturally extended for all $Z \in \mathcal{U}(g_C)$.

Define $\phi_\lambda : G \rightarrow C$ by

$$\phi_\lambda(g) := \langle v_\lambda | \pi_\lambda(g)v_\lambda \rangle, \quad g \in G.$$ 

We see $\pi_\lambda(e^X)v_\lambda = e^{\lambda X}v_\lambda$ for $X \in t$, and hence $\phi_\lambda(e^X) = e^{\lambda X}$.

Define the subspace $\mathcal{H}_\lambda(G) \subset C^\infty(G)$ to be the set of $f \in C^\infty(G)$ such that

$$X^R f = 0, \forall X \in n^- \text{ and } f(gt) = \phi_\lambda(t)^{-1} f(g), \forall t \in T, \forall g \in G. \quad (3.1)$$

Note that $X^R f = 0$ for all $X \in n^-$ if and only if $(E_\alpha^*)_R f = 0$ for all $\alpha \in R^+$.

The **Borel–Weil theorem** is proven in two ways: analytically or algebraically. (For a concise exposition of the Borel–Weil theorem, see e.g. [1].)

The analytic proof begins with the Cartan–Weyl highest weight theory and the Peter–Weyl theorem, and it is completed via the following **pre-Borel–Weil theorem**:

**Theorem 3.1.** (i) $\mathcal{H}_\lambda(G)$ is invariant under $\mathcal{F}_L(G)$;

(ii) $g \mapsto \mathcal{F}_L(g)|_{\mathcal{H}_\lambda(G)}$ is an irreducible unitary representation of $G$ on $\mathcal{H}_\lambda(G) \subset L^2(G)$ with highest weight $\lambda$.

## 4 Casimir and Laplacian

Define $c_\pm, c_0 \in \mathcal{U}(g_C)$ by

$$c_- := \sum_{\alpha \in R^+} E_\alpha^* E_\alpha, \quad c_+ := \sum_{\alpha \in R^+} E_\alpha E_\alpha^*, \quad c_0 := \sum_{i=1}^l T_i^2.$$ 

Recall $\rho$ is the half sum of positive roots. Then we see

$$c_+ - c_- = 2\nu^{-1}(\rho),$$
The Casimir element $c \in \mathcal{H}(g)$ is defined by

$$c := c_0 + c_+ + c_- = c_0 + 2\mu^{-1}(\rho) + 2c_-$$

$$= c_0 + 2\mu^{-1}(\rho) + 2(c_+ - 2\mu^{-1}(\rho)) = c_0 - 2\mu^{-1}(\rho) + 2c_+.$$ \hspace{1cm} (4.1)

Define the Laplacian $\Delta$ on $G$ by

$$\Delta := c^R = c^L.$$ 

Let $\{X_i\}$ be an orthonormal basis of $g$, i.e. $\langle X_i|X_j \rangle_g = -\kappa(X_i, X_j) = \delta_{ij}$. Then by the basic properties of the Casimir elements, we have

$$\Delta = -\sum_k (X_k^R)^2.$$ 

By the pre-Borel–Weil theorem, we also have

$$\Delta|_{\mathcal{H}_k(G)} = ((\lambda + \rho, \lambda + \rho) - (\rho, \rho)) \text{Id.}$$ 

5 Magnetic Laplacian

Let $M$ be a Riemannian manifold, and $\theta$ be a $i\mathbb{R}$-valued 1-form on $M$. Define the magnetic exterior differentiation $d^\theta : C^\infty(M, \mathbb{C}) \to \Lambda^1(M, \mathbb{C})$ by $d^\theta := d + \theta$, i.e. $d^\theta f := df + f\theta$ for $f \in C^\infty(M, \mathbb{C})$, and the magnetic Laplacian $\Delta^\theta : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C})$ by

$$\Delta^\theta := (d^\theta)^*d^\theta.$$ 

where $(d^\theta)^*$ is the formal adjoint of $d^\theta$ with respect to the $L^2$-inner product of functions and 1-forms. Note that $i\mathbb{R} = u(1)$ (the Lie algebra of $U(1)$), and hence $d^\theta$ can be viewed as a covariant derivative on the trivial line bundle $M \times \mathbb{C}$, associated with the trivial $U(1)$-principal bundle $M \times U(1)$. Thus $\Delta^\theta$ is nothing but the Bochner Laplacian corresponding to this covariant derivative.

For further information on magnetic Laplacians on manifolds, see e.g. [19, 6].

The Lie group $G$ has a Riemannian metric given by the inner product on $g$: $\langle X|Y \rangle_g := -\kappa(X, Y)$. For $\alpha \in i\mathbb{R}^* \subset g^*$, let $\alpha^R$ be the $i\mathbb{R}$-valued 1-form on $G$ defined by $\langle e^\alpha|X \rangle_g := \alpha^R(X)$, and define $d^\alpha : C^\infty(G, \mathbb{C}) \to \Lambda^1(G, \mathbb{C})$ and $\Delta^\alpha : C^\infty(G, \mathbb{C}) \to C^\infty(G, \mathbb{C})$ by

$$d^\alpha \equiv d^\alpha_R := d^\alpha^R, \quad \Delta^\alpha \equiv \Delta^\alpha_R := (d^\alpha)^*d^\alpha = \Delta^\alpha^R.$$ 

Let $\{X_k\}$ be an orthonormal basis of $g$, and $\xi_k := \nu(X_k)$. Let $\alpha = i\sum_k a_k \xi_k \in i\mathbb{R}^*$ ($a_k \in \mathbb{R}$). Then we have

$$d^\alpha f = \sum_k \left( X_k^R f + i a_k f \right) \xi_k^R \quad f \in C^\infty(G) \hspace{1cm} (5.1)$$

For a 1-form $A = \sum_k A_k \xi_k^R \in \Lambda^1(G, \mathbb{R})$ ($A_k \in C^\infty(G, \mathbb{R})$), the adjoint operator $(d^\alpha)^* : \Lambda^1(G, \mathbb{R}) \to C^\infty(G)$ is explicitly expressed by

$$(d^\alpha)^* A = -\sum_k \left( X_k^R + i a_k \right) A_k.$$ 

(5.2)
and hence $\Delta^\alpha$ is written as
\[
\Delta^\alpha = -\sum_k \big(X_k^R + i a_k\big)^2 = -\sum_k \big(X_k^R\big)^2 - 2i \sum_k a_k X_k^R + \sum_k a_k^2. \tag{5.3}
\]

The inner product $\langle \bullet | \bullet \rangle_g$ on $g$ is naturally extended to the Hermitian inner product $\langle \bullet | \bullet \rangle_{g_C}$ on $g_C$. This induces the natural Hermitian inner product $\langle \bullet | \bullet \rangle = \langle \bullet | \bullet \rangle_{g_C}$ on $g_C$. (Although $\langle \alpha | \beta \rangle = \langle \alpha, \beta \rangle$ holds for $\alpha, \beta \in ig^*$, we prefer the notation $\langle \bullet | \bullet \rangle$ to $\langle \bullet, \bullet \rangle$ so as to be more consistent with the Hilbert space structure of $L^2(G)$.) Then $\Delta^\alpha$ is written as the following coordinate-free form:

**Lemma 5.1.** Let $\alpha \in ig^*$. Then
\[
\Delta^\alpha = \Delta - 2\nu^{-1}(\alpha)^R + \langle \alpha | \alpha \rangle_{g_C^\ast}. \tag{5.4}
\]

For $\alpha = i \sum_{i=1}^\ell a_i \nu(i T_i) \in i^* \ (a_i \in \mathbb{R})$, the $t$-partial magnetic Laplacian $\Delta^\alpha_t$ is defined by restricting Eq. (5.3) to $t$, i.e.
\[
\Delta^\alpha_t := -\sum_{i=1}^\ell \big((i T_i)^R + i a_i\big)^2 = -\sum_{i=1}^\ell \big((i T_i)^R - \alpha(i T_i)\big)^2
\]
\[
= \sum_{i=1}^\ell \big(t_i^R + a_i\big)^2 = \sum_{i=1}^\ell \big((T_i)^R - \alpha(T_i)\big)^2
\]

Then we have an analogue of (5.4):
\[
\Delta^\alpha_t = \Delta_t - 2\nu^{-1}(\alpha)^R + \langle \alpha | \alpha \rangle_{g_C^\ast}, \quad \Delta_t := c_0^R. \tag{5.5}
\]

**Lemma 5.2.** For $\lambda \in i^*$ and the half sum of positive roots $\rho$, we have
\[
\Delta^{-(\lambda + \rho)} = \Delta^{\lambda} + 2c_+ + (2\lambda + \rho) |\rho\rangle_{g_C^\ast}. \tag{5.6}
\]

**Proof.** We have
\[
\Delta^{-(\lambda + \rho)} = \Delta - 2\nu^{-1}(\lambda)^R + \langle \lambda | \lambda \rangle_{g_C^\ast}
\]
\[
\begin{aligned}
&= \Delta^R + 2 \big[\nu^{-1}(\lambda)^R + \nu^{-1}(\rho)^R\big] + \langle \lambda + \rho | \lambda + \rho \rangle_{g_C^\ast} \\
&= \Delta^R + \big[c_0 - 2\nu^{-1}(\rho) + 2c_\omega\big] + 2\nu^{-1}(\lambda)^R + 2\nu^{-1}(\rho)^R + \langle \lambda + \rho | \lambda + \rho \rangle_{g_C^\ast} \\
&= \big[c_0 - 2\nu^{-1}(\rho) + 2c_\omega\big] + \langle \lambda | \lambda \rangle_{g_C^\ast} + 2c_+ + 2 \langle \lambda | \rho \rangle_{g_C^\ast} + \langle \rho | \rho \rangle_{g_C^\ast} \\
&= \big[c_0 - 2\nu^{-1}(\rho) + 2c_\omega\big] + \langle \lambda | \lambda \rangle_{g_C^\ast} + 2c_+ + 2 \langle \lambda | \rho \rangle_{g_C^\ast} + \langle \rho | \rho \rangle_{g_C^\ast} \\
&= \Delta^{\lambda} + 2c_+ + 2 \langle \lambda | \rho \rangle_{g_C^\ast} + \langle \rho | \rho \rangle_{g_C^\ast}.
\end{aligned}
\]

**Lemma 5.3.** Let $f \in C^\infty(G)$. Then
\[
f(tg) = \phi(t)^{-1}f(g) \text{ for all } t \in T, \ g \in G \text{ if and only if } \Delta^{\lambda} f = 0. \tag{5.7}
\]
In Sec. 2, we defined the GS quantization $Q_{\lambda}(g)$ on $G$ if and only if

$$\forall X \in \mathfrak{t}, \forall g \in G, \quad \frac{d}{dt} f(g e^{tX}) \bigg|_{t=0} = \frac{d}{dt} \phi_{\lambda}(e^{tX})^{-1} f(g) \bigg|_{t=0},$$

iff $\forall X \in \mathfrak{t}$, $(X_+ + \lambda(X)) f = 0$,

iff $\forall i$, $(i T_i)^R + \lambda(i T_i) f = 0$,

iff $-\sum (i T_i)^R + \lambda(i T_i)^2 f \equiv \Delta_{-\lambda} f = 0$.

Proof. We see that $f(g t) = \phi_{\lambda}(t)^{-1} f(g)$, $\forall t \in \mathfrak{t}, \forall g \in G$ if and only if

Theorem 5.4. (1) Let $c_{\lambda} := \inf \text{spec} \Delta_{-\lambda}$. Then

$$c_{\lambda} = \langle 2\lambda + \rho | \rho \rangle \in \mathbb{C}.$$

(2) $\mathcal{H}_\lambda(G)$ is the “ground eigenspace” of $\Delta_{-\lambda}$, i.e.

$$\mathcal{H}_\lambda(G) = \ker [\Delta_{-\lambda} - c_{\lambda}]$$

Proof. Since $\Delta_{-\lambda}$ and $c_+^R$ are positive semidefinite operators, we find by [5.0],

$$\Delta_{-\lambda} - (2\lambda + \rho | \rho \rangle \in \mathbb{C} = \Delta_{-\lambda} + 2c_+^R \geq 0.$$

By [5.7] and

$$\forall X \in \mathfrak{n}^-, \quad X_+ f = 0 \iff \forall \alpha \in \mathbb{R}^+, \quad (E_\alpha^+)^R f = 0 \iff 2c_+^R f = 0$$

we have

$$\mathcal{H}_\lambda(G) = \{ f \in C^\infty(G) : \quad 2c_+^R f = 0 & \Delta_{-\lambda} f = 0 \}$$

$$= \{ f \in C^\infty(G) : \quad [2c_+^R + \Delta_{-\lambda}] f = 0 \}$$

$$= \{ f \in C^\infty(G) : \quad \Delta_{-\lambda} - (2\lambda + \rho | \rho \rangle \in \mathbb{C} = \Delta_{-\lambda} + 2c_+^R = 0 \}$$

$$= \ker [\Delta_{-\lambda} - (2\lambda + \rho | \rho \rangle \in \mathbb{C}].$$

6 GS quantization on $\mathcal{H}_\lambda(G)$

In Sec. 2 we defined the GS quantization $Q$ for an irreducible unitary representation $(\pi_\lambda, V_\lambda)$ with highest weight $\lambda \in \mathcal{X}(\mathfrak{t}) \subset \mathfrak{t}$. In the following, we set $(\pi_\lambda, V_\lambda) = (\mathcal{R}, H_\lambda(G))$, and examine the GS quantization there.

Let $\mathfrak{v}_\lambda \in \mathcal{H}_\lambda(G) \subset L^2(G) (||\mathfrak{v}_\lambda|| = 1)$ be the highest weight vector, i.e. $X^\dagger \mathfrak{v}_\lambda = \lambda(X) \mathfrak{v}_\lambda, \forall X \in \mathfrak{t}$, such that $\mathfrak{v}_\lambda(1_G) > 0$. For $u, v \in L^2(G)$, define $L_{u,v} \in C(G)$ by

$$L_{u,v}(g) := \langle u | \mathcal{R}(g) v \rangle, \quad R_{u,v}(g) := \langle u | \mathcal{R}(g) v \rangle.$$

Recall $\phi_{\lambda}(g) := \langle \mathfrak{v}_\lambda | \mathcal{R}(g) \mathfrak{v}_\lambda \rangle = L_{\mathfrak{v}_\lambda, \mathfrak{v}_\lambda}(g)$. 
Lemma 6.1. For any $X \in \mathfrak{g}_C$ and $u, v \in L^2(G)$,

$$X^R L_{u,v} = L_{u, X^v v}, \quad X^L L_{u,v} = L_{-(X^*)^u, u,v}, \quad (6.1)$$

$$X^R L_{u,v} = L_{u, -(X^*)^v u}, \quad X^L L_{u,v} = L_{X^u u,v}. \quad (6.2)$$

Proof. For $X \in \mathfrak{g}$, we have

$$(X^R L_{u,v})(g) = \frac{d}{dt} L_{u,v}(ge^{tX})|_{t=0} = \frac{d}{dt} \langle u| \mathcal{L}(ge^{tX})v \rangle|_{t=0} = \langle u| \mathcal{L}(g)X^t v \rangle = L_{u, Xv}(g).$$

Hence, for $Z = X + iY \in \mathfrak{g}_C$ with $X, Y \in \mathfrak{g}$, we have

$$Z^R L_{u,v} = X^R L_{u,v} + i Y^R L_{u,v} = L_{u, Xv^0} + i L_{u, Xv^1} + i L_{u, Yv^0} = L_{u, Zv^0}.\tag{6.1}$$

Other relations are shown similarly. □

Lemma 6.2. (1) $(\mathcal{L}, \text{span}\{ \mathcal{L}(G)\phi_\lambda \})$ and $(\mathcal{L}, \text{span}\{ \mathcal{L}(G)\bar{\phi}_\lambda \})$ are irreducible unitary representations of $G$ with the highest weight $\lambda$, where $\phi_\lambda$ and $\bar{\phi}_\lambda$ are highest weight vectors, respectively.

(2) $(\mathcal{R}, \text{span}\{ \mathcal{R}(G)\phi_\lambda \})$ and $(\mathcal{R}, \text{span}\{ \mathcal{R}(G)\bar{\phi}_\lambda \})$ are irreducible unitary representations of $G$ with the lowest weight $-\lambda$, where $\phi_\lambda$ and $\bar{\phi}_\lambda$ are lowest weight vectors, respectively.

Proof. Since $v_\lambda$ is the highest weight vector of $(\mathcal{L}, \mathcal{H}_\lambda(G))$, we have

$$\forall \alpha \in \mathbb{R}^+, \quad E^L_\alpha v_\lambda = 0.$$ 

By (6.1), we have

$$E^R_\alpha \phi_\lambda = E^R_\alpha L_{v_\lambda, v_\lambda} = L_{v_\lambda, E^R_\alpha v_\lambda} = L_{v_\lambda, 0} = 0.$$ 

Hence $\phi_\lambda$ is the highest weight vector of $(\mathcal{R}, \text{span}\{ \mathcal{R}(G)\phi_\lambda \})$. By (6.2),

$$E^L_\alpha \phi_\lambda = E^L_\alpha L_{v_\lambda, v_\lambda} = L_{v_\lambda, E^L_\alpha v_\lambda} = L_{v_\lambda, 0} = 0.$$ 

Hence $\bar{\phi}_\lambda$ is the highest weight vector of $(\mathcal{L}, \text{span}\{ \mathcal{L}(G)\bar{\phi}_\lambda \})$. The proof of (2) is similar. □

Lemma 6.3. We have $v_\lambda = d^{1/2}_\lambda \bar{\phi}_\lambda$.

Proof. By Lemma 6.2 $u := \bar{\phi}_\lambda$ is the highest weight vector of $(\mathcal{R}, \text{span}\{ \mathcal{R}(G)u \})$, and also the lowest weight vector of $(\mathcal{R}, \text{span}\{ \mathcal{R}(G)u \})$. Such $u \in L^2(G)$ is unique up to scalar multiple by the Peter–Weyl theorem. If we set $u := v_\lambda$, we have the same statement, since $\mathcal{H}_\lambda(G) = \text{span}\{ \mathcal{L}(G)v_\lambda \}$. Hence $v_\lambda = z \bar{\phi}_\lambda$ for some $z \in \mathbb{C} \setminus \{0\}$. We see $\|\bar{\phi}_\lambda\| = d^{-1/2}_\lambda$, and hence $z = d^{1/2}_\lambda$. □

The following easily shown lemma will not used later, but it will help to understand the relation to the notion of reproducing kernel. (See e.g. [14] for the relations between coherent states, quantizations and reproducing kernels; see [17] for unitary representation theory in terms of reproducing kernels.)
Lemma 6.4 (reproducing kernel). For \( g \in G \), define the \( \mathcal{H}_\lambda(G) \)-delta function \( \delta_{\lambda,g} \in \mathcal{H}_\lambda(G) \) by

\[
\delta_{\lambda,g} := d_\lambda \mathcal{R}_\lambda(g) \phi_\lambda = d_\lambda^{1/2} \mathcal{R}_\lambda(g) \nu_\lambda = d_\lambda L_{\mathcal{R}_\lambda(g) \nu_\lambda, \nu_\lambda}.
\]

Then

\[
v(g) = \langle \delta_{\lambda,g} | v \rangle, \quad \forall v \in \mathcal{H}_\lambda(G), \enspace \forall g \in G.
\]

\( K(g,h) := \delta_{\lambda, h^{-1} g} \) is called the \( \mathcal{H}_\lambda(G) \)-reproducing kernel.

Let \( E_\lambda := \nu_\lambda \nu_\lambda^* \). We view \( E_\lambda \) not as a projection from \( V_\lambda = \mathcal{H}_\lambda(G) \) onto \( \mathbb{C} \nu_\lambda \), but as a projection from \( L^2(G) \) onto \( \mathbb{C} \nu_\lambda \). This view is consistent by the orthogonality relations in the Peter–Weyl theory. Let

\[
E_\lambda(g) := \mathcal{R}_\lambda(g) E_\lambda \mathcal{R}_\lambda(g^{-1}), \quad g \in G.
\]

For \( f \in C^\infty(G, \mathbb{C}) \), define \( f(E_\lambda) \equiv E_\lambda(f) \in \text{End}(\mathcal{H}_\lambda(G)) \) by

\[
f(E_\lambda) \equiv E_\lambda(f) := \int_G f(g)E_\lambda(g)dg.
\]

Note that the definition of \( E_\lambda(f) \) is naturally extended for any \( f \in C^\infty(G, \mathbb{C}) \), the space of Schwartz distributions, since \( \mathcal{H}_\lambda(G) \subset C^\infty(G) \) and \( \text{dim} \mathcal{H}_\lambda(G) < \infty \).

Lemma 6.5. Let \( f \in C(G, \mathbb{R}) \). Then for any \( v_1, v_2 \in \mathcal{H}_\lambda(G) \),

\[
\langle v_1 | E_\lambda(f) v_2 \rangle = d_\lambda^{-1} \langle v_2 | f v_1 \rangle
\]

where \( f \) is regarded as a multiplication operator on \( L^2(G) \) in the rhs.

Proof. Without loss of generality, we can assume that for some \( g_k \in G, \ k = 1, 2 \),

\[
v_k = \mathcal{R}_\lambda(g_k) \nu_\lambda.
\]

Then we have

\[
\langle v_1 | E_\lambda(f) v_2 \rangle = \left( \mathcal{R}_\lambda(g_1) \nu_\lambda \right) \int_G f(g)E_\lambda(g)dg \mathcal{R}_\lambda(g_2) \nu_\lambda
\]

\[
= \int_G f(g) \left( \mathcal{R}_\lambda(g_1) \nu_\lambda \right) \mathcal{R}_\lambda(g_2) \nu_\lambda^* \mathcal{R}_\lambda(g_2^{-1}) \mathcal{R}_\lambda(g_2) \nu_\lambda dg
\]

\[
= \int_G f(g) \left( \nu_\lambda \mathcal{R}_\lambda(g_2^{-1}) \nu_\lambda \right) \left( \nu_\lambda \mathcal{R}_\lambda(g_2^{-1}) \nu_\lambda \right) dg
\]

\[
= \int_G f(g) \left( \mathcal{R}_\lambda(g_1) \phi_\lambda \right) \left( \mathcal{R}_\lambda(g_2) \phi_\lambda \right) dg
\]

\[
= \langle \mathcal{R}_\lambda(g_1) \phi_\lambda | f \mathcal{R}_\lambda(g_2) \phi_\lambda \rangle = \langle \mathcal{R}_\lambda(g_2) \phi_\lambda | f \mathcal{R}_\lambda(g_1) \phi_\lambda \rangle
\]

\[
= \text{Lemma 6.3} \left( \mathcal{R}_\lambda(g_2) d_\lambda^{-1/2} \nu_\lambda | f \mathcal{R}_\lambda(g_1) d_\lambda^{-1/2} \nu_\lambda \right)
\]

\[
= d_\lambda^{-1} \langle \mathcal{R}_\lambda(g_2) \nu_\lambda | f \mathcal{R}_\lambda(g_1) \nu_\lambda \rangle = d_\lambda^{-1} \langle v_2 | f v_1 \rangle.
\]

The following theorem directly follows from the above lemma.
Theorem 6.6 (GS quantization as projection). Let \( f \in C(G, \mathbb{R}) \). Let \( P_\lambda \) be the orthogonal projection from \( L^2(G) \) onto \( H_\lambda(G) \). Then

\[
E_\lambda(f) = d_\lambda^{-1}P_\lambda f P_\lambda,
\]

where \( f \) is regarded as a multiplication operator on \( L^2(G) \) in the rhs. For an orbit function \( f : G \to \mathbb{C}, \) let

\[
Q(f) = d_\lambda E_\lambda(f) = P_\lambda f P_\lambda, \quad \text{i.e.} \quad \forall v \in H_\lambda(G), \quad Q(f)v = P_\lambda f v.
\]

7 Asymptotic representation

Let

\[
\Delta^{-(\lambda + \rho)} := \Delta^{-(\lambda + \rho)} - \text{inf } \text{spec} \Delta^{-(\lambda + \rho)} = \Delta^{-(\lambda + \rho)} - 2(\lambda + \rho|\rho|)\mathbb{C}.
\]

Let \( V \in C^\infty(G, \mathbb{R}) \). For \( r > 0 \), define the operator \( T_r \) by

\[
T_r := r\Delta^{-(\lambda + \rho)} + iv
\]

Then \( T_r \) is a closed operator satisfying

\[
\Re(vT_r v) \geq 0
\]

for all \( v \in \text{dom}(T_r) = \text{dom}(\Delta^{-(\lambda + \rho)}). \) Hence \( T_r \) generates the strongly continuous contraction semigroup \( \{e^{-tT_r} | t \geq 0\} \) by the Hille–Yosida Theorem [18].

Note that \( \Delta^{-(\lambda + \rho)} \) is a compact operator on \( L^2(G) \). Hence we have the spectrum decomposition

\[
\Delta^{-(\lambda + \rho)} = \sum_{k=1}^{\infty} c_k E_k, \quad 0 = c_1 < c_2 < \cdots
\]

where each \( E_k \) is an orthogonal projection, and \( \sum_k E_k = I \). Let \( \alpha := c_2 \).

If \( V \in C^\infty(G, \mathbb{R}) \) and \( f \in C^\infty(G, \mathbb{C}) \), we see \( f_{r,t} := e^{-tT_r}f \in C^\infty(G, \mathbb{C}) \) and \( \gamma_{r,t} := P_\lambda e^{-tT_r}f = P_\lambda f_{r,t} \) for all \( t \geq 0 \).

Lemma 7.1. Let \( f \in C^\infty(G) \), \( f_{r,t} := e^{-tT_r}f \) and \( \eta(t) := \|(I - P_\lambda)f_{r,t}\| \) then

\[
\frac{d}{dt} \eta(t)^2 \leq -2\alpha \eta(t)^2 - 2 \|V f_{r,t}\| \eta(t) \quad (7.1)
\]

Proof. Let \( A := \Delta^{-(\lambda + \rho)} \). Then we easily find

\[
\frac{d}{dt} \eta(t)^2 = \frac{d}{dt} \|(1 - P_\lambda)e^{-tT_r}f\|^2 = -2r \langle f_{r,t}|Af_{r,t}\rangle - 2\Re \langle V f_{r,t}|(1 - P_\lambda)f_{r,t}\rangle.
\]

Since \( 0 \leq \alpha(I - P_\lambda) \leq A \), we have

\[
\frac{d}{dt} \eta(t)^2 = \frac{d}{dt} \|(1 - P_\lambda)e^{-tT_r}f\|^2 \\
\leq -2r \langle f_{r,t}|(I - P_\lambda)f_{r,t}\rangle - 2\Re \langle V f_{r,t}|(1 - P_\lambda)f_{r,t}\rangle \\
= -2\alpha \|(I - P_\lambda)f_{r,t}\|^2 - 2\Re \langle V f_{r,t}|(1 - P_\lambda)f_{r,t}\rangle \\
\leq -2\alpha \|(I - P_\lambda)f_{r,t}\|^2 + 2 \|V f_{r,t}\| \|(1 - P_\lambda)f_{r,t}\| \\
= -2\alpha \eta(t)^2 - 2 \|V f_{r,t}\| \eta(t).
\]

\( \square \)
Lemma 7.2. Suppose $f \in \ker \Delta^{-(\Lambda+\rho)} (= \mathcal{H}_\Lambda(G))$. Then
\[
\forall t > 0, \quad \|f_{r,t} - g_{r,t}\| \leq \frac{\|V\|_\infty \|f\|}{\alpha r} \tag{7.2}
\]

Proof. Recall $\ker \Delta^{-(\Lambda+\rho)} = \ker(I-P_\Lambda)$ and $\|(I-P_\Lambda)e^{-iTr}f\| = \|(I-P_\Lambda)f_{r,t}\| = \eta(t)$. Assume $\frac{d}{dt}\eta(t)^2 \geq 0$. Then by (7.1), we have
\[
\eta(t)^2 \leq \frac{\|Vf_{r,t}\| \eta(t)}{\alpha r}.
\]
This implies
\[
\eta(t) \leq \frac{\|Vf_{r,t}\|}{\alpha r} \leq \frac{\|V\|\|f\|}{\alpha r},
\]
Thus we find that
\[
\frac{d}{dt}\eta(t)^2 \geq 0 \implies \eta(t) \leq \frac{\|V\|\|f\|}{\alpha r}, \quad \forall t > 0.
\]
Since $\eta(0) = 0$, it follows that
\[
\|f_{r,t} - g_{r,t}\| = \|(I-P_\Lambda)e^{-iTr}f\| = \eta(t) \leq \frac{\|V\|\|f\|}{\alpha r}, \quad \forall t > 0.
\]

Lemma 7.3. We have
\[
\left\| \frac{d}{dt}g_{r,t} - iP_\Lambda VP_\Lambda g_{r,t} \right\| \leq \|V\|_\infty \|(1-P_\Lambda)f_{r,t}\|. \tag{7.3}
\]

Proof. We see
\[
\frac{d}{dt}g_{r,t} + iP_\Lambda VP_\Lambda g_{r,t} = P_\Lambda \frac{d}{dt}f_{r,t} + iP_\Lambda VP_\Lambda f_{r,t} = P_\Lambda (-T_\Lambda f_{r,t}) + iP_\Lambda VP_\Lambda f_{r,t} = -P_\Lambda \left(r\Delta^{-(\Lambda+\rho)} + iV\right)f_{r,t} + iP_\Lambda VP_\Lambda f_{r,t}.
\]
Hence we have
\[
\left\| \frac{d}{dt}g_{r,t} - iP_\Lambda VP_\Lambda g_{r,t} \right\| = \|P_\Lambda V (1-P_\Lambda) f_{r,t}\| \\
\leq \|V (1-P_\Lambda)f_{r,t}\| \leq \|V\|_\infty \|(1-P_\Lambda)f_{r,t}\|.
\]

Proposition 7.4 (Asymptotic representation). Let $f \in \ker \Delta^{-(\Lambda+\rho)} = \mathcal{H}_\Lambda(G)$ and $V \in C^\infty(G, \mathbb{R})$. Then for all $t > 0$,
\[
\lim_{r \to \infty} e^{-iT_\Lambda(V)}f = e^{itP_\LambdaVP_\Lambda}f, \quad T_\Lambda(V) := r\Delta^{-(\Lambda+\rho)} + iV.
\]
Especially, for any classical Hamiltonian $h \in C^\infty(G \cdot E_\Lambda, \mathbb{R})$ and $t > 0$, we have
\[
e^{itQ(h)}f = \lim_{r \to \infty} e^{-iT_\Lambda(h)}f,
\]
where $Q(h)$ is the GS quantization of $h$. 

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Proof. By (7.3),
\[ \lim_{r \to \infty} \left\| \frac{d}{dr} g_{r,t} - iP_{\lambda} \mathcal{V} P_{\lambda} g_{r,t} \right\| \leq \| \mathcal{V} \|_{\infty} \lim_{r \to \infty} \| (1 - P_{\lambda}) f_{r,t} \| = 0 \]
for each \( t > 0 \). This implies
\[ \lim_{r \to \infty} g_{r,t} = e^{i \lambda (x + \mathcal{V} f_{r,0})} = e^{i \lambda (x + \mathcal{V} f)}, \quad \forall t > 0 \]
Thus by (7.2) we have
\[ \lim_{r \to \infty} f_{r,t} = \lim_{r \to \infty} g_{r,t} = e^{i \lambda \mathcal{V} f}. \]

8 Path integral: Brownian form

In this section we give a Brownian path integral representation of the one-parameter unitary group \( \{ e^{it\mathcal{Q}(h)} : t \in \mathbb{R} \} \) where \( \mathcal{Q}(h) \) is the GS quantization of the ‘classical’ Hamiltonian \( h \in C^\infty(\mathfrak{g}, \mathbb{R}) \). The main tool used here is the Feynman-Kac-Itô formula on a vector bundle on a Riemannian manifold, formulated by Güneysu (2010) [11]. The basics of the theory of Brownian motion on a manifold are summarized in [11]. The simplest construction of a Brownian motion on a Riemannian manifold \( M \) will be the one which is based on the Nash embedding \( M \hookrightarrow \mathbb{R}^l \):

Theorem 8.1. Let \( M \hookrightarrow \mathbb{R}^l \) isometrically for some \( l \in \mathbb{N} \) and let the morphism of smooth vector bundles \( A : M \times \mathbb{R}^l \to TM \) be given as the orthogonal projection \( A(x) : \mathbb{R}^l \to T_xM \) for any \( x \in M \). Let \( W \) be a Brownian motion in \( \mathbb{R}^l \). Then the maximal solution of the stochastic differential equation
\[ dX_t = \sum_{j=1}^{l} A_j(X_t) dW^j_t, \quad X_0 = x \]
is a Brownian motion on \( M \) with starting point \( x \). Here \( d \) denotes the Stratonovich differential.

If \( M \) is a compact semisimple Lie group \( G \) embedded in a matrix Lie group \( \text{GL}(n, \mathbb{C}) \), we have a simpler characterization: Let \( W \) be a Brownian motion on \( g \). Then the solution of the left (resp. right) invariant stochastic differential equation \( dX_t = X_t dW_t \) (resp. \( dX_t = (dW_t) X_t \)) is a Brownian motion on \( G \). However, in this section we does not need a specific definitions of a Brownian motion on \( M \).

Let \( M = (M, g) \) be a geodesically and stochastically complete smooth connected Riemannian manifold. (Any compact Lie group \( M \) satisfies this condition. See [11].) Let \( \alpha \) be a \( \mathbb{R} \)-valued smooth 1-form on \( M \). Let \( V : M \to \mathbb{R} \) be a locally square integrable potential which is bounded from below, and
\[ H(\alpha, V) := \frac{1}{2}(d + \alpha)^* (d + \alpha) + V = \frac{1}{2} \Delta + V. \]
The self-adjoint extension of \( H(\alpha, V) \) in \( L^2(M) \) is denoted again by \( H(\alpha, V) \).
Theorem 8.2. (Feynman–Kac–Itô formula on a manifold, Güneysu [11]) Let \( X \) be a Brownian motion in \( G \). Then

\[
e^{-tH(\alpha,V)} f(x) = \mathbb{E} \left[ e^{\mathcal{I}_{\alpha,V}} f(X_t) \middle| X_0 = x \right] \quad \text{a.e. } x \in M,
\]

where

\[
\mathcal{I}_{\alpha,V} := \int_0^t \alpha(\mathsf{d}X_s) - \int_0^t V(X_s)ds.
\]

Here, \( \int_0^t \alpha(\mathsf{d}X_s) \) stands for the Stratonovich line integral of \( \alpha \) along \( B \).

This theorem concerns only the cases where \( V \) is real-valued. However, if we confine ourselves to the cases where \( |V| \) is bounded, it easy to extend to complex-valued \( V \); Its proof is almost same as that of the real-valued cases in [11].

Set \( M = G \), and consider the Brownian motion \( B \) on the Riemannian manifold \( G \) in the time interval \( [0, \infty) \), where the distribution of the starting point is uniform on \( G \), i.e. equals the Haar measure \( \mathsf{d}g \) on \( G \). Let \( \mu^1 \) be a probability measure on \( C([0, \infty), G) \) which represents such Brownian motion (i.e. a Wiener measure uniform on \( G \)). Then the above theorem is restated as

**Proposition 8.3.** Let \( V \in C(G, \mathbb{C}) \). For any \( f_1, f_2 \in L^2(G) \) and \( t \geq 0 \),

\[
\langle f_2 | e^{-tH(\alpha,V)} f_1 \rangle = \int_{C([0, \infty), G)} e^{\mathcal{I}_{\alpha,V}} \mathcal{I}_{\alpha,V}(B) f_1(B) \mathsf{d}\mu^1(B).
\]

For \( r > 0 \), let \( B_t^r := B_{rt} \), and define the probability measure \( \mu^r \) on \( C([0, \infty), G) \) by

\[
d\mu^r(B_t^r) := d\mu^1(B_t),
\]

Recall \( \alpha := -(\lambda + \rho) \) and \( c_\lambda = \inf \text{spec} \Delta^{-(\lambda+\rho)} = \langle 2\lambda + \rho | \rho \rangle g^* \).

**Theorem 8.4.** Let

\[
S_r(h) := \frac{1}{2}(\Delta - (\lambda + \rho) + ih).
\]

Then for \( h \in C^\infty(G \cdot E_\lambda, \mathbb{R}) \), \( f_1, f_2 \in H_\lambda(G) = \ker \Delta^{-(\lambda+\rho)} \) and \( t \geq 0 \),

\[
\langle f_2 | e^{-tS_r(h)} f_1 \rangle = e^{\frac{1}{2} \mathcal{I}_r(h)} \int_{C([0, \infty), G)} e^{\mathcal{I}_r(h)} \mathcal{I}_r(h) f_2(B_0) f_1(B_t) d\mu^r(B).
\]

where

\[
\mathcal{I}_r(h) := \int_0^t \alpha(\mathsf{d}B_s) - i \int_0^t h(B_s)ds.
\]

**Proof.** Let \( V := -\frac{1}{2} c_\lambda + i \frac{1}{r} h \). Then we see

\[
S_r(h) = r \left( \frac{1}{2} \Delta^{\alpha} + V \right) = rH(\alpha, V).
\]
Let $W := C([0, \infty), G)$. Then by Prop. 5.3 we have

$$\langle f_2 | e^{-i S_{t}(h)} f_1 \rangle = \langle f_2 | e^{-i \mathcal{H}(\alpha,V)} f_1 \rangle$$

$$= \int_{\mathcal{W}} \left[ \exp \left( \int_{0}^{t} \alpha(dB_\lambda) - \int_{0}^{t} V(B_\lambda) ds \right) \overline{f_2(B_0)} f_1(B_t) \right] d\mu^I(B)$$

$$= \int_{\mathcal{W}} \left[ \exp \left( \int_{0}^{t} \alpha(dB_\lambda) + \frac{1}{2} r t c_\lambda - i \frac{1}{r} \int_{0}^{t} h(B_\lambda) ds \right) \overline{f_2(B_0)} f_1(B_t) \right] d\mu^I(B)$$

$$= \int_{\mathcal{W}} \left[ \exp \left( \int_{0}^{t} \alpha(dB_\lambda) + \frac{1}{2} r t c_\lambda - i \int_{0}^{t} h(B_\lambda) ds \right) \overline{f_2(B_0)} f_1(B_t) \right] d\mu^I(B)$$

$$= e^{rt c_\lambda} \int_{\mathcal{W}} \left[ \exp \left( \int_{0}^{t} \alpha(dB_\lambda) - i \int_{0}^{t} h(B_\lambda) ds \right) \overline{f_2(B_0)} f_1(B_t) \right] d\mu^I(B).$$

Fix an arbitrary $f \in \mathcal{H}_\lambda(G)$ with $\|f\| = 1$. If we set $h \equiv 0$ in (8.1), since $e^{-i S_{t}(0)} f = f$, we see that the ‘normalization factor’ $e^{-rt c_\lambda}$ can be included in the integral measure: $Z_{\lambda,t,x} := e^{-\frac{1}{2} rt c_\lambda} = \int_{C([0, \infty), G)} \left[ e^{T_{t}(0)} \overline{f(B_0)} f(B_t) \right] d\mu^I(B)$.

**Corollary 8.5** (Brownian path integral). \textit{For $h \in C^\infty(G \cdot E_\lambda, \mathbb{R})$, $f_1, f_2 \in \mathcal{H}_\lambda(G)$ and $t \geq 0$, we have}

$$\langle f_2 | e^{i Q(t)} f_1 \rangle = \lim_{r \to \infty} \int_{C([0, \infty), G)} \left[ e^{T_{t}(0)} \overline{f_2(B_0)} f_1(B_t) \right] \frac{d\mu^I(B)}{Z_{\lambda,t,x}}.$$

**Proof.** Directly follows from the asymptotic representation theorem and Theorem 5.3. \hfill \square

\section{Rough path theory}

In the study of stochastic processes, the Itô Calculus, based on martingale theory, has been the most effective tool for many years. But a few alternative (or additional) approaches are known; e.g. the Malliavin Calculus, and rough paths theory which we use in this paper. Although a rough path theory itself is not a probabilistic theory, the main application of it is to stochastic analysis. Among other things, rough path theories have made a considerable progress on the problem of the (piecewise) smooth approximations of stochastic processes. This problem is an old but also up-to-date one, since it is related to the problem of renormalization occurring mainly in quantum physics. (Another rigorous approach to renormalization is lattice field theory.) When one considers the problem to approximate a martingale by a sequence of other martingales, conventional martingale theory will suffice. However, since a (piecewise) smooth process is not a martingale, it is difficult to deal with smooth approximations in martingale theory (see the complicated analysis in [13]). One will find in next section that the theory of geometric rough paths is the best approach to such problems.

Rough path theory was originated by Lyons [16], and has been extensively developed into several approaches, including the large theories such as the theory
of Gubinelli–Imkeller–Perkowski [10], and that of Hairer [12]. So we do not seem to be able to give a brief overview of rough path theories. (Different approaches use different definitions of the fundamental notions such as ‘rough integral’ and ‘rough differential equation.’) Instead we refer to a single approach of Friz–Victoir book [8]. However, since this 650-pages book is not easily accessible for everyone, we will summarize their approach here for the convenience of readers. See also Baudoin’s lecture note [3], which is more concise and accessible.

Let $V \cong \mathbb{R}^d$ be a vector space with the usual norm, and $T(V)$ be the tensor algebra over $V$, i.e.,
\[
T(V) := \bigoplus_{k=0}^{\infty} T^k(V), \quad T^k(V) := \mathbb{V}^k.
\]

Let
\[
T^{\leq N}(V) := \bigoplus_{k=0}^{N} T^k(V),
\]
and $pr_k$ and $pr_{\leq N}$ denote the projection from $T(V)$ onto $T^k(V)$ and $T^{\leq N}(V)$, respectively. We make $T^{\leq N}(V)$ into an algebra with the product defined by
\[
xy := pr_{\leq N}(x \otimes y) \in T^{\leq N}(V), \quad x, y \in T^{\leq N}(V)
\]
$T^{\leq N}(V)$ is called the truncated tensor algebra. $T^{\leq N}(V)$ is also a Lie algebra with the Lie bracket $[x, y] = xy - yx$. Define $g_N(V) \subset T^{\leq N}(V)$ as the Lie subalgebra of $T^{\leq N}(V)$ generated by $V = T^1(V) \subset T^{\leq N}(V)$. Define the Lie group $G_N(V) \subset T^{\leq N}(V)$ by
\[
G_N(V) := \exp (g_N(V)) = \left\{ \sum_{n=0}^{N} \frac{x^n}{n!} : x \in g_N(V) \right\}.
\]
$G_N(V)$ is called the free nilpotent group of step $N$. We see
\[
G_2(V) = \left\{ 1 + v + \frac{1}{2} v^2 + A : v \in V, A \in \text{Anti} (\mathbb{V}^2) \right\}
\]
where $\text{Anti}(\mathbb{V}^2)$ is the subspace of $\mathbb{V}^\otimes 2$ spanned by $\{ u \otimes v - v \otimes u : u, v \in V \}$.

Let $C^{1-\text{var}}([0, T], V)$ denote the subspace of $C([0, T], V)$ consisting of the functions of bounded variation. Let $x \in C^{1-\text{var}}([0, T], V)$. For $n = 0, 1, \ldots$ and $0 \leq s < t$, define $x^{[n]}_{s,t} \in \mathbb{V}^n$ by
\[
x^{[0]}_{s,t} := 1, \quad x^{[n]}_{s,t} := \int_{s < u_1 < \cdots < u_n < t} dx_{u_1} \otimes \cdots \otimes dx_{u_n}, \quad n \geq 1.
\]
where the integral is of the sense of Riemann–Stieltjes. We see that $x^{[n]}_{s,t}$, $n = 1, 2$ is explicitly written as
\[
x^{[1]}_{s,t} = x_{s,t} := x_t - x_s, \quad x^{[2]}_{s,t} = \int_{s}^{t} (x_r - x_s) \otimes dx_r.
\]
The step-$N$ signature of $x$ is given by
\[
S_N(x)_{s,t} \equiv x^{[\leq N]}_{s,t} := \bigoplus_{k=0}^{N} x^{[k]}_{s,t} \in T^{\leq N}(V), \quad t \in [0, T].
\]
In fact, \( S_N(x) = x^{\leq N} \) is a path on the free nilpotent group \( \mathbb{G}_N(V) \subseteq T^{\leq N}(V) \); Precisely, it is shown that
\[
\mathbb{G}_N(V) \ni S_N(x)_{s,t} : x \in C^\text{1-var}([0, T], V), \ 0 \leq s < t \leq T \\
= \{ S_N(x)_{0,1} : x \in C^\text{1-var}([0, 1], V) \}.
\]

Then we have the following fundamental algebraic relation:

**Theorem 9.1** (Chen’s relation). Given \( x \in C^\text{1-var}([0, T], V) \) and \( 0 \leq s < t < u \leq T \) we have
\[
S_N(x)_{s,u} = S_N(x)_{s,t} S_N(x)_{t,u},
\]
where the rhs is the product in the free nilpotent group \( \mathbb{G}_N(V) \) (= the product in the truncated tensor algebra \( T^{\leq N}(V) \)).

For any \( x \in C^\text{1-var}([0, T], V) \) and \( 0 \leq t_1 < t_2 \leq T \), any path segment
\[
S_N(x)_{0, \bullet} : [t_1, t_2] \ni t \mapsto S_N(x)_{0, t} \in \mathbb{G}_N(V),
\]
as well as its reparametrizations, is said to be horizontal. It is shown that any two points of \( \mathbb{G}_N(V) \) can be connected by a horizontal path, and hence we can define a “geodesic distance” \( d_{CC} \) of two points \( g, h \in \mathbb{G}_N(V) \) as follows:
\[
d_{CC}(g, h) := \inf \left\{ \text{length}(x|_{[t_1, t_2]}): 0 \leq t_1 < t_2 \leq T, \right. \\
x \in C^\text{1-var}([0, T], V), \ x_{0,t_1}^{(\leq N)} = g, \ x_{t_1,t_2}^{(\leq N)} = h \left\}
\]
(9.2)
\[
= \inf \left\{ \text{length}(x|_{[0, 1]}): x \in C^\text{1-var}([0, 1], V), \ x_{0,1}^{(\leq N)} = g^{-1} h \right\}.
\]
(9.3)

where the length of the path \( x|_{[t_1, t_2]} \) is usually defined by the metric on \( V \).

In fact \( d_{CC} \) turns out to be a metric on \( \mathbb{G}_N(V) \), and is called the Carnot–Carathéodory metric.

If \( s \leq t \) we will denote by \( D(s, t) \), the set of subdivisions of the interval \( [s, t] \), that is \( \Pi \subseteq D(s, t) \) can be written
\[
\Pi = \{ s = t_0 < t_1 < \cdots < t_n = t \}.
\]

**Definition 9.2.** Let \( \mathcal{G} \) be a group with the unit \( 1_\mathcal{G} \in \mathcal{G} \), and a left invariant metric \( d \) on \( \mathcal{G} \). For a path \( x : [0, T] \rightarrow \mathcal{G} \), let \( x_{s,t} := x^{-1} x_t \in \mathcal{G} \). For \( x, y : [0, T] \rightarrow \mathcal{G} \) and \( p > 0 \), the \( p \)-variation distance (semi-metric) between \( x \) and \( y \) is defined by
\[
d_{\mathcal{G}, \text{p-var}}([0, T])(x, y) := \left[ \sup_{\{ t_i \} \in D([0, T])} \sum_i d(x_{t_i, t_{i+1}}, y_{t_i, t_{i+1}})^p \right]^{1/p}
\]
(9.4)

A path \( x : [0, T] \rightarrow \mathcal{G} \) is said to be of finite \( p \)-variation if \( d_{\mathcal{G}, \text{p-var}}([0, T])(1_\mathcal{G}, x) < \infty \), where \( 1_\mathcal{G} \) is the constant path with value \( 1_\mathcal{G} \). The space of the paths of finite \( p \)-variation is denoted by \( C^\text{p-var}([0, T], \mathcal{G}) \). The \( p \)-variation metric on \( C^\text{p-var}([0, T], \mathcal{G}) \) is given by
\[
\tilde{d}_{\mathcal{G}, \text{p-var}}([0, T])(x, y) := d(x_0, y_0) + d_{\mathcal{G}, \text{p-var}}([0, T])(x, y),
\]
which determines a topology on \( C^\text{p-var}([0, T], \mathcal{G}) \). Let
\[
C^\text{p-var}_0([0, T], \mathcal{G}) := \{ x \in C^\text{p-var}([0, T], \mathcal{G}) : x_0 = 1_\mathcal{G} \}.
\]
If $G$ is the additive group $V \cong \mathbb{R}^d$, we see
\[ d_{V,p-\text{var}([0,T])}(x,y) = \|y-x\|_{p-\text{var}([0,T])} \]
where $\|\cdot\|_{p-\text{var}([0,T])}$ denotes the $p$-variation seminorm defined by
\[ \|x\|_{p-\text{var}([0,T])} := \left( \sup_{\{t_k \}} \sum_k \|x_{t_{k+1}} - x_{t_k}\|^p \right)^{1/p}. \]

Set $G = G_N(V)$ with the Carnot–Carathéodory metric $d_{CC}$, and consider the path space $C_{p-\text{var}}([0,T], G_N(V))$, called the space of weak geometric $p$-rough paths. For $p \geq 1$, this is a complete, non-separable metric space [S] p.175 Theorem 8.13.

Define $GR_{[0,T],0}^p(V)$ (which is denoted by $C_{p-\text{var}}^0([0,T], G_V)$ in [S], $\Omega G^p_{0}([0,T], V)$ in [S]) to be the set of continuous paths $x : [0,T] \to G_{[p]}(V)$ for which there exists a sequence $x_n \in C^\infty([0,T], V)$ such that
\[ \lim_{n \to \infty} S_n(x_n) = x \text{ in } \left( C_{p-\text{var}}^0([0,T], G_{[p]}(V)), d_{p-\text{var}([0,T])} \right). \]

Let $GR_{[0,T]}^p(V) := \left\{ x : [0,T] \to G_{[p]}(V) : x_0, x_1 \in GR_{[0,T],0}^p(V) \right\}.$

(Recall $x_0, t := x_0^{-1} x_t$.) In other words, $GR_{[0,T]}^p(V)$ is the $d_{p-\text{var}([0,T])}$-closure of $C^\infty([0,T], G_{[p]}(V))$. An element of $GR_{[0,T]}^p(V)$ is called a geometric $p$-rough path. For $p > 1$, a geometric $p$-rough path is characterized as a path $[0,T] \to G_{[p]}(V)$ which is absolutely continuous of order $p$, or $p$-absolutely continuous (in the sense of Wiener–Young–Love) ([S] pp.96,180), see also [S][2].

It is shown that $\left( GR_{[0,T]}^p(V), d_{p-\text{var}([0,T])} \right)$ is a complete separable metric space [S] p.180, Proposition 8.25.

Remark: Since $G_N(V)$ is a subset of the normed linear space $T^{\leq N}(V)$, $G_N(V)$ also has a metric $d_T$ of $T^{\leq N}(V)$, different from $d_{CC}$. However, $d_T$ is not a left invariant metric on $G_N(V)$, and hence we cannot replace $d_{CC}$ with $d_T$.

Let $\varphi : \mathbb{R}^{d_1} \to \mathcal{L}(\mathbb{R}^{d_2}, \mathbb{R}^{d_3})$, where $\mathcal{L}(\mathbb{R}^{d_2}, \mathbb{R}^{d_3})$ is the space of linear maps $\mathbb{R}^{d_2} \to \mathbb{R}^{d_3}$. If $\mathbb{R}^{d_1} \ni x \mapsto \varphi(x) e_k \in \mathbb{R}^{d_3}$ is $\gamma$-Lipschitz for all $k = 1, \ldots, d_2$, where $(e_k)$ is the standard basis of $\mathbb{R}^{d_2}$, we write $\varphi \in \text{Lip}^\gamma(\mathbb{R}^{d_1}, \mathcal{L}(\mathbb{R}^{d_2}, \mathbb{R}^{d_3}))$.

Let $\gamma > p$, $V \in \text{Lip}^\gamma(\mathbb{R}^c, \mathcal{L}(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}))$ and $x \in C^{1-\text{var}}([0,T], \mathbb{R}^d)$. Then there exists a unique solution $y \in C^{1-\text{var}}([0,T], \mathbb{R}^c)$ of the ordinary differential equation (ODE)
\[ dy(t) = V(y(t))dx(t), \quad y(0) = y_0 \in \mathbb{R}^c. \]

Thus we define the map
\[ \mathbb{R}^c \times C^{1-\text{var}}([0,T], \mathbb{R}^d) \to C^{1-\text{var}}([0,T], \mathbb{R}^c), \quad (y_0, x) \mapsto \pi_V(y_0, x) \]
by $\pi_V(y_0, x) := y$.

We assume moreover $V \in C^\infty$ here; In the next section it will suffice to consider only the case where $V$ is smooth.
Theorem 9.3. Let \( p \geq 1 \) and \( N = \lfloor p \rfloor \). Let \( x_n \in C^\infty([0, T], \mathbb{R}^d) \), \( x_n := S_{\lfloor p \rfloor}(x_n) \) for each \( n \in \mathbb{N} \), and \( x \in GR^p_{[0, T]}(\mathbb{R}^d) \), and assume

\[
\lim_{n \to \infty} x_n = x \quad \text{in} \quad \left( GR^p_{[0, T]}(\mathbb{R}^d), \tilde{d}_{p\text{-var};[0, T]} \right).
\]

Then the limit

\[
\pi_{(V)}(y_0, x) := \lim_{n \to \infty} \pi_{(V)}(y_0, x_n)
\]

converges in \( (C^{p\text{-var}}([0, T], \mathbb{R}^e), \tilde{d}_{p\text{-var};[0, T]}) \). Furthermore, for each \( y_0 \in G_{\lfloor p \rfloor}(\mathbb{R}^e) \) with \( \text{pr}_1(y_0) = y_0 \), the limit

\[
\pi_{(V)}(y_0, x) := \lim_{n \to \infty} y_0 S_{\lfloor p \rfloor}(\pi_{(V)}(y_0, x_n))
\]

converges in \( \left( GR^p_{[0, T]}(\mathbb{R}^e), \tilde{d}_{p\text{-var};[0, T]} \right) \), and satisfies

\[
\pi_{(V)}(y_0, x) = \text{pr}_1(\pi_{(V)}(y_0, x)).
\]

These definitions of \( \pi_{(V)}(y_0, x) \) and \( \pi_{(V)}(y_0, x) \) do not depend on the choice of the approximating sequence \( x_n \).

(Make sure to distinguish between \( \pi_{(V)} \) and bold letter \( \pi_{(V)} \).)

We call \( y := \pi_{(V)}(y_0, x) \) (resp. \( y := \pi_{(V)}(y_0, x) \)) the **solution of the rough differential equation (RDE solution)** (resp. the full RDE solution) of

\[
dy(t) = V(y(t))dx(t) \quad \text{resp.} \quad dy(t) = V(y(t))dx(t), \tag{9.5}
\]

with \( y(0) = y_0 \in \mathbb{R}^e \) (resp. \( y(0) = y_0 \)), and call the map \( \pi_{(V)} \) (resp. \( \pi_{(V)} \)) the **Itô–Lyons map** (resp. **full Itô–Lyons map**).

The above definition of (full) RDE solution is slightly modified version of [3] p.70], which is slightly different from that of [8] p.224].

The full Itô–Lyons map is characterized as the extension of the ODE solution map \( \pi_{(V)} \) which satisfies the following continuity:

**Theorem 9.4.** Let \( d_1 := \tilde{d}_{p\text{-var};[0, T]} \) and \( d_2 := d_{\infty;[0, T]} \), where

\[
d_{\infty;[0, T]}(x, y) := \sup_{t \in [0, T]} d_{CC}(x_t, y_t).
\]

Then the full Itô–Lyons map

\[
G_{\lfloor p \rfloor}(\mathbb{R}^e) \times \left( GR^p_{[0, T]}(\mathbb{R}^d), d_k \right) \to \left( GR^p_{[0, T]}(\mathbb{R}^d), d_k \right)
\]

\[
(y_0, x) \mapsto \pi_{(V)}(y_0; x)
\]

is continuous for \( k = 1, 2 \). In fact, these are uniformly continuous on each \( \tilde{d}_{p\text{-var}} \)-bounded sets.

Let \( V_1 = \mathbb{R}^d, V_2 = \mathbb{R}^e \), and \( \varphi \in \text{Lip}^{-1}(V_1, L(V_1, V_2)) \cap C^\infty \). Define \( \Phi : V_1 \oplus V_2 \to L(V_1, V_1 \oplus V_2) \) by

\[
\Phi(x \oplus y)x' := x' \oplus \varphi(x)x', \quad x, x' \in V_1, \ y \in V_2.
\]
Then we easily see
\[
\text{proj}_{V_2} \pi_{\Phi}(0, x) = \int_0^x \varphi(x(s)) \, dx(s), \quad x \in C^{1-\text{var}}([0, T], V_1)
\]
where \( \text{proj}_{V_2} \) is the projection from \( V_1 \oplus V_2 \) onto \( V_2 \). Thus the Riemann–Stieltjes line integral \( \int \varphi(x) \, dx \) can be expressed by the ODE solution map \( \pi_{\Phi} \). Similarly we define the rough line integral \( \Upsilon_{\Phi}(x) \equiv \int_0^x \varphi(x(s)) \, dx(s) \) for \( x \in \text{GR}_p([0, T]; V_1) \) to be a map

\[
\Upsilon_{\Phi}(x) \equiv \text{proj}_{T^\leq N\{V_2\}} \pi_{\Phi}(0, x).
\]

10 Brownian motion as rough path

Let \( B_t \) be a Brownian motion (or more generally a semimartingale) on \( V = \mathbb{R}^d \). Then \( B \notin C^{1-\text{var}}([0, T], V) \) a.s., and hence the step-\( N \) signature \( S_N(B)_{s,t} \equiv B^N_{s,t} \) is not defined by the Riemann–Stieltjes integral of (9.1) when \( N \geq 2 \). However we find that if we set

\[
B_{s,t} := 1 \oplus B_{s,t} \oplus \int_s^t B \otimes dB,
\]

where \( dB \) denotes Stratonovich integration, then \( B_{s,t} \in G_2(V) \). In fact it is shown that \( B \) is a geometric \( p \)-rough path for \( 2 < p < 3 \), i.e., \( B := B_0, \bullet \in \text{GR}_p([0, T]; V) \), almost surely. Thus a Brownian motion \( B \) in \([0, T]\) can be identified with a \( \text{GR}_p([0, T]; V) \)-valued random variable \( B \), called the enhanced Brownian motion. Moreover, the solution of the stochastic differential equation \( dY_t = V(Y_t) \, dB_t, \quad Y(0) = y_0 \) can be identified with the solution of the RDE \( dY_t = V(Y_t) dB_t \); Precisely,

\[ \text{Theorem 10.1. [8, p.510 Theorem 17.3]} \]

Let \( p, \gamma \) be such that \( 2 < p < \gamma \). Let \( V \in \text{Lip}^\gamma(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^e)) \), \( y_0 \in \mathbb{R}^e \) and \( B \) be an \( \mathbb{R}^d \)-valued semimartingale, enhanced to \( B = B(\omega) \in \text{GR}_p([0, T]; \mathbb{R}^d) \) almost surely. Then the (for a.e. \( \omega \) well-defined) RDE solution

\[
Y(\omega) = \pi_{\{V\}}(y_0; B(\omega)),
\]
solves the Stratonovich SDE

\[
dY = V(Y) \, dB, \quad Y(0) = y_0.
\]

Note that the definitions of RDE solution (and rough integral) do not refer to any probability measure, i.e., they are deterministically defined. Hence this viewpoint of SDE differs radically from that of conventional stochastic analysis based on martingales.

A fundamental fact on weak convergences in a general setting is as follows:
Theorem 10.2. (Weak approximation of rough SDE [3] p.520 Theorem 17.13) Assume that
(i) $X_k, k = 1, 2, \ldots, \infty$ are $\text{GR}^p_{[0,T]}(\mathbb{R}^d)$-valued random variables, possibly defined on different probability spaces, such that $X_k \to X_\infty$ ($k \to \infty$) in law.
(ii) $V \in \text{Lip}^\gamma(\mathbb{R}^c, L(\mathbb{R}^d, \mathbb{R}^c)), \gamma > p$, and $y_0 \in \mathbb{R}^c$.
Then the $\text{GR}^p_{[0,T]}(\mathbb{R}^d)$-valued random variables $Y_k := \pi(V)(y_0, X_k)$ converge to a $Y_\infty \in \text{GR}^p_{[0,T]}(\mathbb{R}^d)$ as $k \to \infty$ in law.

11 Smooth path integral

Assume the compact connected, simply connected semisimple Lie group $G$ is embedded in the matrix Lie group $\text{GL}(\nu, \mathbb{C}) \subset \text{Mat}(\nu) \cong \mathbb{C}^{\nu^2}$. Then a Brownian motion on $G$ is embedded as a process on the Euclidean space $\mathbb{C}^{\nu^2}$. Let $\{B(t): t \in [0, \infty)\}$ be a standard Brownian motion on $\mathbb{V} = \mathbb{g}$ w.r.t. the inner product $\langle \cdot, \cdot \rangle_\mathbb{g} = -\kappa(\cdot, \cdot)$ with $B(0) = 0$. Then a Brownian motion $X$ on $G$ can be constructed by the left (resp. right) invariant stochastic differential equation (SDE)

$$dX(t) = X(t)dB(t), \quad dX(t) = (dB(t))X(t), \quad X(0) = X_0. \quad (11.1)$$

where $X_0$ is a $G$-valued random variable whose distribution is the Haar measure on $G$. Let $V_L$ (resp. $V_R$) $\in \text{Lip}^\gamma(\text{Mat}(\nu), L(\text{Mat}(\nu), \text{Mat}(\nu))) \cap C^\infty$, and assume

$$V_L(U)M = U M, \quad V_R(U)M = M U, \quad \forall U \in G, \forall M \in \text{Mat}(\nu). \quad (11.2)$$

While $V_L(A)$ (resp. $V_R(A)$) is defined for all $A \in \text{Mat}(\nu) \cong \mathbb{R}^{2\nu^2}$, our concern is about the values on $G \subset \text{Mat}(\nu)$ only. Then we can rewrite (11.1) as a usual SDE on a Euclidean space:

$$dX(t) = V_L(X(t))d\mathbb{B}(t), \quad dX(t) = V_R(X(t))d\mathbb{B}(t), \quad X(0) = X_0 \in G. \quad$$

In the following we consider only the left SDE $dX_t = V_L(X_t)d\mathbb{B}_t$.

For a normed vector space $\mathbb{V}$, let

$$\text{GR}^p_{\text{loc},[0,\infty)}(\mathbb{V}) := \left\{ x: [0, \infty) \to \mathbb{G}^p_{[0,\infty)}(\mathbb{V}) : x|[0,T] \in \text{GR}^p_{[0,T]}(\mathbb{V}), \forall T > 0 \right\},$$

with semi-metrics $d_{\text{var}}([0,T])$, $T > 0$. Note that $\text{GR}^p_{\text{loc},[0,\infty)}(\mathbb{V})$ is a Polish space. The full Itô–Lyons map $\pi_{(\nu, \lambda)}$ is naturally extended to a map

$$\pi_{(\nu, \lambda)}: \mathbb{g} \times \text{GR}^p_{\text{loc},[0,\infty)}(\mathbb{g}) \to \text{GR}^p_{\text{loc},[0,\infty)}(\text{Mat}(\nu)).$$

Let $B$ be the $\text{GR}^p_{\text{loc},[0,\infty)}(\mathbb{g})$-valued random variable which is the enhanced Brownian motion of $B$ on $\mathbb{g}$, that is, $B^T := B|[0,T] \in \text{GR}^p_{[0,T]}(\mathbb{g})$ is the enhanced Brownian motion for all $T > 0$. Let

$$X_0 := 1 \otimes X_0 \oplus \left( \frac{1}{2} X_0 \otimes X_0 \right) \in \mathbb{G}_2(\text{Mat}(\nu)).$$

Then

$$X := \pi_{(\nu, \lambda)}(X_0, B)$$

is a $\text{GR}^p_{\text{loc},[0,\infty)}(\text{Mat}(\nu))$-valued random variable, and $\text{pr}_1(X)$ is identified with the Brownian motion $X$ on $G$. Let $\mu_X$ be the probability measure on $\text{GR}^p_{\text{loc},[0,\infty)}(\text{Mat}(\nu))$ which is the law of $X$, i.e. $\mu_X := X_*\mathbb{P}$. 

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For $\alpha \in g_C$, let $V_\alpha \in \Lip^\infty(\alpha, L(\Mat(\nu), \C)) \cap C^\infty$ be such that

\[ V_\alpha(g)x = \alpha_g^\alpha(x), \quad x \in T_gG, \ g \in G \]

where the tangent space $T_gG$ is naturally embedded in $\Mat(\nu)$, and our concern is the values of $V_\alpha(g)x$ for $x \in T_gG \subseteq \Mat(\nu)$, $g \in G \subseteq \Mat(\nu)$ only. Then we have

\[ \int_0^t \alpha_t^R(dX_s) = \text{pr}_1 \int_0^t V_\alpha(X_s)dX_s \quad \text{a.s.} \quad (11.3) \]

where the rhs is a rough line integral.

Recall that if $\nu_n$, $n = 1, \ldots, \infty$ is a sequence of probability measures on $GR^p_{\infty,0}\loc(\Mat(\nu))$ and if $\lim_{n \to \infty} \int f\nu_n = \int f\nu_\infty$ for all continuous and bounded function $f : GR^p_{\infty,0}\loc(\Mat(\nu)) \to \R$, then we say that $\nu_n$ weakly converges to $\nu_\infty$.

Recall the setting in Sec 2 $\mu^1$ is a probability measure on $C([0, \infty), G)$ which represents a Brownian motion on $G$ (i.e. a Wiener measure uniform on $G$). For $h \in C^\infty(G \cdot \bE_\lambda, \R)$ and a (smooth or Brownian) path $\psi : [0, \infty) \to G$, let

\[ I_t(h; \psi) := \int_0^t \alpha_t^R(d\psi(s)) - i \int_0^t h(\psi(s))ds, \quad \alpha := -(\lambda + \rho), \]

where if $\psi$ is smooth, the integral $\int_0^t \alpha(d\psi(s))$ is of Riemann–Stieltjes, and if $\psi$ is Brownian, it is a Stratonovich line integral (or a rough line integral).

**Proposition 11.1.** Let $\mu_n$, $n \in \N$ be probability measures on $C^\infty([0, \infty), G) \subseteq C^\infty([0, \infty), \Mat(\nu))$. Define the probability measures $\mu_n', n \in \N$ on $GR^p_{\infty,0}\loc(\Mat(\nu))$ by

\[ \mu_n' := (S_{[P]} \mu_n, \ i.e. \ \mu_n'(E) := \mu_n \left( S_{[P]}^{-1}(E) \right). \]

If $\mu_n'$ weakly converges to $\mu_X^\infty$, then we have

\[ \lim_{n \to \infty} \int_{C([0, \infty), G)} \xi(\psi)\mu_n(\psi) = \int_{C([0, \infty), G)} \xi(\psi)\mu^\infty(\psi), \quad (11.5) \]

where $\xi(\psi) := e^{\xi(h; \psi)u(\psi_t)}v(\psi_t)$, for all $u, v \in V_\lambda$ and $t \geq 0$.

**Proof.** Recall that a rough line integral $\int_0^t \alpha(d\psi(s))$ is defined by [113] and [90] with the full Itô–Lyons map $\pi_{[P]}$. Since the $\C$-valued random variable $\xi$ is continuous and bounded for each $u, v \in V_\lambda$, the nth law of $\xi$, i.e. $\xi_n\mu_n$, weakly converges to $\xi_\infty\mu^\infty$ by Theorem [102]. Hence [113] follows. \qed

For $r > 0$, define the probability measure $\mu_r^\infty(\psi)$ on $C^\infty([0, \infty), G)$ to be the time rescaling of $\mu_n$ given by $\psi \mapsto \psi_r := \psi(r^{-1} \bullet)$. Fix an arbitrary $v_1 \in V_\lambda$ with $\|v_1\| = 1$, and set

\[ Z_{\lambda, r, n} := \int_{C^\infty([0, \infty), G)} \left[ e^{\xi(h; \psi)u(\psi_t)}v_1(\psi_t) \right] \mu_r^\infty(\psi). \]

By Corollary [8.5] and Proposition [11.1] we find that for any $u, v \in V_\lambda$ and $t > 0$,

\[ \langle u | e^{it\xi(h; \psi)}v \rangle = \lim_{r \to \infty} \lim_{n \to \infty} \int_{C^\infty([0, \infty), G)} \left[ e^{\xi(h; \psi)u(\psi_t)}v(\psi_t) \right] \mu_r^\infty(\psi) = \int_{C^\infty([0, \infty), G)} \left[ e^{\xi(h; \psi)u(\psi_t)}v(\psi_t) \right] \mu_r^\infty(\psi). \]

Thus we find the following theorem:
Theorem 11.2 (Smooth path integral). Let $\mu_n, n \in \mathbb{N}$ be probability measures on $C^\infty([0, \infty), G) \subset C^\infty([0, \infty), \text{Mat}(\nu))$, and define $\mu'_n$ by (11.4). If $\mu'_n$ weakly converges to $\mu_X$ fast enough, then

$$\langle u | e^{iQ(h)} v \rangle = \lim_{n \to \infty} \int_{C^\infty([0, \infty), G)} \left[ e^{i(h) \tilde{u}(\psi)} \tilde{v}(\psi) \right] d\tilde{\mu}_n. \quad (11.7)$$

where $\tilde{\mu}_n$ is the finite measure on $C^\infty([0, \infty), G)$ given by

$$d\tilde{\mu}_n := \frac{d\mu_n}{Z_{\lambda, t, u, n}}.$$

In the above statement, “$\mu'_n$ weakly converges to $\mu_X$ fast enough” means precisely that if $\nu_n (n \in \mathbb{N})$ are probability measures on $C^\infty([0, \infty), G)$, and if $\nu'_n$ weakly converges to $\mu_X$, then there exists a function $f : \mathbb{N} \to \mathbb{N}$ increasing fast enough such that $\mu_n := \nu_{f(n)}$ satisfy (11.7). Thus this condition is neither quantitative nor constructive; It is an open problem to give a quantitative condition for (11.7).

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