Two loop results from one loop computations
and non perturbative solutions of exact evolution equations

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Abstract:
A nonperturbative method is proposed for the approximative solution of the exact evolution equation which describes the scale dependence of the effective average action. It consists of a combination of exact evolution equations for independent couplings with renormalization group improved one loop expressions of secondary couplings. Our method is illustrated by an example: We compute the $\beta$-function of the quartic coupling $\lambda$ of an $O(N)$ symmetric scalar field theory to order $\lambda^3$ as well as the anomalous dimension to order $\lambda^2$ using only one loop expressions and find agreement with the two loop perturbation theory. We also treat the case of very strong coupling and confirm the existence of a "triviality bound".
1 Introduction

Nonperturbative field theoretical methods formulated in continuous space and allowing practical calculations are relatively rare. Among the best known figure the Schwinger-Dyson equations \[1\] which constitute a system of exact equations for the 1PI vertices. The momentum integrals appearing in these equations contain, however, not only 1PI vertices but also "bare" couplings. Their solution requires an understanding of physics at very different length scales, englobing the "short-distance physics" and the "long-distance physics" whose scale is characterized by the external momenta of the 1PI green functions. The necessity to integrate over a large momentum range (within which the relevant physics may vary) makes it often hard to find approximate methods which go beyond perturbation theory in some small dimensionless coupling.

The block spin approach \[2\] in lattice theories seems to be an ideal solution for this type of problem: Only the understanding of the physics within a small momentum range is needed in order to make the transition from one length scale to a larger one. The effective average action \[\Gamma_k\] formulates the block spin concept in continuous space. The average action is expressed as a functional integral with a constraint which ensures that only quantum fluctuations with momenta \(q^2 > k^2\) are effectively included. Various field theoretical methods can be used for an approximate solution of this functional integral, as in particular, a renormalization group improved saddle point approximation \[3\]-\[5\]. This method is nonperturbative in the sense that no small dimensionless coupling is required. Nonperturbative phenomena in two and three dimensional scalar theories \[4\], as well as in four dimensional scalar theories at nonvanishing temperature \[6\] have been described successfully by this method.

An infinitesimal analogue of the Schwinger-Dyson equations for the effective average action describes the change of \(\Gamma_k\) with varying scale \(k\). This ensures that only a small momentum range \(q^2 \approx k^2\) must be controlled. The change of the \(k\)-dependent 1PI vertices characterizing \(\Gamma_k\) can now be expressed in terms of such vertices only. Indeed, an exact evolution equation for the scale dependence of the effective average action has been proposed recently \[7\]. It reads for scalar fields \(\phi\)

\[
\frac{\partial}{\partial t} \Gamma_k[\phi] = \frac{1}{2} Tr \left\{ \left( \Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \frac{T}{\partial t} R_k \right\}. \tag{1.1}
\]

Here \(t = \ln \frac{k}{\Lambda}\) where \(\Lambda^{-1}\) is a suitable short distance scale and \(\Gamma_k^{(2)}\) is the exact \(k\) dependent inverse propagator as given by the second functional derivative of \(\Gamma_k\) with respect to the fields \(\phi\). The trace indicates a summation over internal indices and a momentum integration \[1\]. The function \(R_k\) contains details of the averaging procedure and causes the integral to be infrared convergent. Ultraviolet finiteness is guaranted by an appropriate behaviour of \(\frac{T}{\partial t} R_k\). We choose

\[
R_k = \frac{Z_k q^2 f_k^2(q)}{1 - f_k^2(q)} \tag{1.2}
\]

with

\[
f_k^2(q) = \exp \frac{-q^2}{k^2}. \tag{1.3}
\]

We note that for \(q^2 \ll k^2\) the infrared cutoff \(R_k \approx Z_k k^2\) acts like an additional mass term whereas its influence on modes with \(q^2 \gg k^2\) is suppressed exponentially. Although eq. (1.1) has the form of a renormalization group improved one loop equation (and has actually first been proposed in this context \[4\]) it involves no approximation. It has a simple graphical representation (fig.1) as
a loop expression in terms of 1PI-vertices only. Equation (1.1) can be related to earlier versions of "exact renormalization group equations" by an appropriate Legendre transform. For $k \to 0$ all quantum fluctuations are "integrated out" and the effective average action becomes the generating functional for the 1PI green functions (the usual "effective action") in this limit. This distinguishes $\Gamma_k$ from the "effective action with variable ultraviolet cutoff" whose dependence on the cutoff is described by the "exact renormalization group equation" and which becomes a relatively complicated object in the limit $k \to 0$. The simple form of $\Gamma_k$, which can be described, for example, in terms of a scalar potential, kinetic term and higher derivative terms, makes it a reasonable starting point for the development of a nonperturbative method.

A nonperturbative exact evolution equation (1.1) is not yet a nonperturbative method. In order to extract physics one needs in addition a systematic way of solving this equation for $k \to 0$, with initial conditions specified by the "bare action" $\Gamma_\Lambda$ at some short distance scale $\Lambda^{-1}$. As it stands, eq. (1.1) is a partial differential equation for a function $\Gamma_k$ depending on infinitely many variables - for example the Fourier modes $\varphi(q)$ of the scalar field and $k$. This is, in general, impossible to solve. Equivalently, we may expand $\Gamma_k$ in terms of invariants with respect to the symmetries of the theory. Eq. (1.1) is then transformed into an infinite system of coupled nonlinear differential equations for infinitely many couplings (for example 1PI Green functions). Without a systematic approximation method which reduces this infinite system to a finite system (which can then be solved by numerical or analytical methods) not much practical progress is made compared to the simple renormalization group improvement discussed in ref. [4]. From a more systematic point of view a nonperturbative method should be able to compute physical quantities and to give an estimate of the error even for situations where perturbation theory fails. Our paper contains a proposal for such a systematic nonperturbative method.

The basic problem we are confronted with can be seen by taking functional derivatives of eq. (1.1) with respect to $\varphi$. We need $\Gamma^{(2)}_k$ on the r.h.s. and may obtain an evolution equation for this quantity by taking the second functional derivative of eq. (1.1). The corresponding flow equation expresses $\frac{\partial \Gamma^{(2)}_k}{\partial t}$ in terms of $\Gamma^{(2)}_k$, $\Gamma^{(3)}_k$ and $\Gamma^{(4)}_k$. This feature proliferates to higher Green functions: The evolution equation for the n-point function $\Gamma^{(n)}_k$ involves $\Gamma^{(n+1)}_k$ and $\Gamma^{(n+2)}_k$ and the system never closes. For $n$ sufficiently large the couplings $\Gamma^{(n)}_k$ are "irrelevant couplings". They are usually small in a perturbative context since they are proportional to appropriate powers of a small coupling. For nonperturbative problems no such argument is available and we have to find a way to estimate $\Gamma_k^{(n+1)}$ and $\Gamma_k^{(n+2)}$ in order to solve the evolution equation for $\Gamma_k^{(n)}$.

The main point for a proposal for a systematic nonperturbative method for scalar field theories consists in the calculation of $\Gamma_k^{(n)}$ for $n$ sufficiently large by a renormalization group improved one loop approximation. In contrast to the exact expression for $\frac{\partial \Gamma^{(n)}_k}{\partial t}$ which can be derived from (1.1) the one loop expression for $\Gamma_k^{(n)}$ will be approximative. It only involves 1PI vertices $\Gamma_k^{(m)}$ with $m \leq n$. For high enough $n$ the momentum integral in the loop is both ultraviolet and infrared finite and dominated by a small momentum range $q^2 \approx k^2$. We therefore expect this approximation to be quite accurate if the involved vertices $\Gamma_k^{(m)}$ do not depend strongly on $k$. Details of our method and a way of estimating errors will be indicated in the next section.

We also want to test our proposal for a new nonperturbative method for some problem where exact analytical results are known. For this reason we compute in the present paper the $\beta$-function for the quartic coupling $\lambda$ of a $O(N)$-symmetric scalar theory in four dimensions. In section 3 we derive in a short way the general equations. In sections 4 to 6 we calculate the beta

\footnote{For smaller $n$ this holds only for $\frac{\partial \Gamma^{(n)}_k}{\partial t}$, not for the one loop expression of $\Gamma_k^{(n)}$ itself.}
function of the quartic coupling step by step increasing the number of included couplings. We keep all terms in order $\lambda^3$ and find agreement with the standard two loop result. We emphasize, however, that we never perform a two loop calculation but only solve the exact evolution equation (1.1). We also do not employ here an iterative solution which would be equivalent to a loop expansion: All momentum integrals appearing in our calculation have the form of one loop integrals. In our approach the two loop result obtains from an improved one loop calculation! Although this result is remarkable by itself it should be seen here as an illustration and check of our nonperturbative method. The real power of this method will appear once applied to truly nonperturbative problems. A precision calculation of critical exponents in the three dimensional theory is in progress and results will be reported elsewhere. As discussed in more detail in the next section the proposed method has its limitations for a very fast running of the couplings. We will deal explicitly with such a situation in sect. 7, where we discuss the evolution equations for very large values of the quartic scalar coupling. The results of sect. 7 confirm the existence of a triviality bound in the context of continuum field theory.

2 A nonperturbative method for scalar field theories

The evolution equation (1.1) has a close resemblance with a one loop expression:

$$\frac{\partial}{\partial t} \Gamma_k = \frac{1}{2} Tr \left\{ \left( \Gamma_k^{(2)} + R_k \right)^{-1} \frac{\partial}{\partial t} R_k \right\}$$

$$= \frac{1}{2} Tr \left\{ \frac{\partial}{\partial t} \ln \left( \Gamma_k^{(2)} + R_k \right) \right\} - \frac{1}{2} Tr \left\{ \left( \Gamma_k^{(2)} + R_k \right)^{-1} \frac{\partial}{\partial t} \Gamma_k^{(2)} \right\}.$$  \quad (2.1)

Let us define the “renormalization group improved one loop contribution” $\Gamma_k^{(L)}$ by

$$\Gamma_k^{(L)} = \frac{1}{2} Tr \ln \frac{\Gamma_k^{(2)} + R_k}{\Gamma^{(2)}_\Lambda + R_\Lambda}$$

$$= \frac{1}{2} \ln Det \left( \Gamma_k^{(2)} + R_k \right) - \frac{1}{2} \ln Det \left( \Gamma^{(2)}_\Lambda + R_\Lambda \right)$$  \quad (2.2)

and identify $\Gamma^{(L)}_\Lambda$ with the “classical action” $S$. Then we can write

$$\Gamma_k = S + \Gamma_k^{(L)} + \Delta \Gamma_k$$  \quad (2.3)

where the “correction term” $\Delta \Gamma_k$ vanishes in the limit where the last term in eq. (2.1) proportional to $\frac{\partial}{\partial t} \Gamma_k^{(2)}$ can be neglected. We emphasize that $\Gamma_k^{(L)}$ is already an ultraviolet regulated one loop expression and vanishes for $k \rightarrow \Lambda$. This new form of an ultraviolet regulator simply subtracts the contribution from quantum fluctuations with momenta $q^2 > \Lambda^2$ in a way very similar to the infrared regularization discussed above. It can be easily generalized beyond the scalar theory discussed in the present paper, for example to chiral fermions [11] or gauge theories [12], and may therefore be useful for standard one loop calculations in a wider context than the present paper. The main advantage of this regulator is the conservation of all symmetries which are conserved by the quadratic form

$$\Delta S_k = \frac{1}{2} \Omega \sum_q \varphi^+(q) R_k(q) \varphi(q)$$  \quad (2.4)

which provides the infrared cutoff in a functional integral representation of $\Gamma_k$ [3].

The disadvantage of the immediate use of $\Gamma_k^{(L)}$ is linked to the fact that for most theories $\Gamma_k^{(L)}$
depends strongly on \( \Lambda \) for \( \frac{A}{k} \to \infty \). This means that the "ultraviolet properties" encoded in \( \Gamma^{(2)}_\Lambda \) have a strong influence on \( \Gamma^{(L)}_k \), in contradiction to our strategy to describe physics at the scale \( k \) only by properties of \( \Gamma_k \). For this reason the renormalization group improved one loop approximation was used in earlier work \([4]\) only for \( \frac{\partial}{\partial k} \Gamma_k \) rather than for \( \Gamma_k \), since \( \frac{\partial}{\partial k} \Gamma_k \) becomes independent of \( \Lambda \) for \( \Lambda \to \infty \). The situation improves greatly, however, if instead of \( \Gamma_k \) we consider derivatives of \( \Gamma_k \) with respect to the fields, i.e. appropriate 1PI Green functions with \( n \) external fields. For \( n \) sufficiently large the corresponding one loop expression \( \Gamma^{(n)(L)}_k \) becomes independent of \( \Lambda \) in the limit \( \frac{A}{k} \to \infty \) and the momentum integral (2.2) is completely dominated by a narrow range \( q^2 \approx k^2 \). For \( k \) small compared to \( \Lambda \) we can then take the limit \( \Lambda \to \infty \) in order to obtain an expression involving only \( \Gamma_k \). From the evolution equation

\[
\frac{\partial}{\partial t} \Gamma^{(n)}_k = \frac{1}{2} Tr \left\{ \delta^{n} \frac{\partial}{\partial \delta \phi_1 \cdots \delta \phi_n} \Gamma^{(2)}_k \right\}
\]

we obtain

\[
\Gamma^{(n)}_k = \Gamma^{(n)(L)}_k + \Delta \Gamma^{(n)}_k
\]

\[
\Gamma^{(n)(L)}_k = \frac{1}{2} Tr \left\{ \Omega^{-1} \frac{\delta^n}{\delta \phi_1 \cdots \delta \phi_n} ln \left( \Gamma^{(2)}_k + R_k \right) \right\}.
\]

We observe that for \( n \) sufficiently high \( \Gamma^{(n)}_\Lambda \) corresponds to an irrelevant operator such that \( \Gamma^{(n)}_\Lambda \) can be neglected compared to \( \Gamma^{(n)}_k \) for \( \frac{A}{k} \to \infty \). The ultraviolet finiteness (for \( \Lambda \to \infty \)) of the momentum integrals in eq. (2.6) serves as a direct test for which \( \Gamma^{(n)}_k \) this procedure of taking \( \Lambda \to \infty \) is self-consistent.

Our strategy for a precision estimate of \( \Gamma_k \) is composed of two parts. First we compute a set of exact evolution equations for a set of 1PI Green functions \( \Gamma^{(A)}_k \) by taking the appropriate functional derivatives of eq. (2.3). This set denoted by \( \{A\} \) must contain at least all relevant or marginal couplings of the theory but can, in general, be larger. We will call these couplings the "independent" couplings since they are all treated on equal footing for a solution of the evolution equations. The r.h.s. of these evolution equations will depend not only on \( \Gamma^{(A)}_k \) but also on some additional irrelevant couplings \( \Gamma^{(B)}_k \). As a second step the "secondary" couplings in the set \( \{B\} \) are computed by the renormalization group improved one loop approximation (2.4). Here it is important that \( \Gamma^{(2)}_k \) on the r.h.s. of eq. (2.6) is computed for a truncation of \( \Gamma_k \) corresponding to a subset of the 1PI vertices contained in \( \{A\} \). This subset always contains the relevant and marginal couplings. More generally, possible choices of subsets of \( \{A\} \) are determined by the requirement of ultraviolet finiteness of eq. (2.6). This approximation of \( \Gamma^{(B)}_k \) is then inserted in the evolution equation for \( \Gamma^{(A)}_k \). The system is thereby closed and the evolution equation (1.1) reduced to a finite set of coupled nonlinear differential equations for the independent couplings \( \Gamma^{(A)}_k \) which can be solved by numerical or analytical methods.

3Strictly speaking this holds sometimes only for an approximation of \( \Gamma^{(2)}_k \) as discussed below.
Furthermore, we may consider $\Gamma_k^{(B)(L)}$ as the zeroth order of an iterative procedure for the determination of the secondary couplings $\Gamma_k^{(B)}$. In fact, we may compute

$$\frac{\partial}{\partial t} \Delta \Gamma_k^{(n)} = -\frac{1}{2} \text{Tr} \Omega^{-1} \frac{\delta^n}{\delta \varphi_1 \cdots \delta \varphi_n} \left\{ \left( \Gamma_k^{(2)} + R_k \right)^{-1} \frac{\partial}{\partial t} \Gamma_k^{(2)} \right\}$$  \hspace{1cm} (2.7)

approximately by inserting on the r.h.s. the couplings $\Gamma_k^{(A)}$ and $\Gamma_k^{(B)(L)}$. Comparing this expression with $\frac{\partial}{\partial t} \Gamma_k^{(n)(L)}$ computed from eq. (2.4) gives an estimate of the error for the $\beta$-function for couplings $\Gamma_k^{(B)}$. Such an inherent estimate of the error is particularly important for nonperturbative applications of the exact evolution equations where it is not easy to check the accuracy of the results otherwise. On the other hand the improved $\beta$-function $\frac{\partial}{\partial t} \Gamma_k^{(n)(L)} + \frac{\partial}{\partial t} \Delta \Gamma_k$ can be used for an even more precise estimate of $\Gamma_k$ by solving the evolution equations for the combinet set $\Gamma_k^{(A)}$, $\Gamma_k^{(B)}$. Further checks and improvements can be obtained by systematically enlarging the set of independent couplings $\Gamma_k^{(A)}$ for which exact evolution equations are used.

An obvious limitation of our method arises for situations where the contribution from the term $\frac{\partial}{\partial t} \Gamma_k^{(2)}$ in eq. (2.3) becomes comparable or even larger than the corresponding contribution from $\frac{\partial}{\partial t} R_k$. The use of renormalization group improved one loop results for the secondary couplings gives in this case not the correct evolution equations for these couplings. To get an idea when this happens in scalar field theories we write $\Gamma_k^{(2)} = Z_k q^2 + c \lambda(k) \varphi^2 + \ldots$. The first term gives a contribution proportional to the anomalous dimension. The second contributes substantially only for a fast running of $\lambda(k)$. This typically happens for very strong $\lambda$. We will discuss the $\beta$-function for $\lambda$ for the case of strong quartic coupling in sect. 7 and sketch there alternative estimates for the secondary couplings.

### 3 Exact evolution equation for the average potential

We consider an $O(N)$-symmetric scalar field theory in arbitrary dimension $d$. We expand the effective average action in terms of the average potential, kinetic term and higher derivatives.

$$\Gamma_k[\varphi] = \int d^d x \left\{ U_k(\rho) + \frac{1}{2} \partial_\mu \varphi^a : Z_k(\rho, -\partial^2) : \partial^\mu \varphi_a + \frac{1}{4} \partial^\mu \rho : Y_k(\rho, -\partial^2) : \partial_\mu \rho + \ldots \right\}$$  \hspace{1cm} (3.1)

where $\rho = \frac{1}{2} \varphi^a \varphi_a$ and normal ordering indicates that the derivative only acts to the right. Eq. (3.1) is the most general effective action which is $O(N)$-symmetric and contains no more than two derivatives. The exact evolution equation for the effective average potential can be obtained from (3.1) by expanding around a constant background field $\bar{\rho}$. One finds

$$\frac{\partial}{\partial t} U_k(\rho) = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{\partial}{\partial t} R_k(q) \left( \frac{N-1}{M_0} + \frac{1}{M_1} \right)$$  \hspace{1cm} (3.2)

with

$$M_0(\rho, q^2) = Z_k(\rho, q^2) q^2 + R_k(q) + U_k^t(\rho)$$

$$M_1(\rho, q^2) = \tilde{Z}_k(\rho, q^2) q^2 + R_k(q) + U_k^t(\rho) + 2 \rho U_k''(\rho)$$  \hspace{1cm} (3.3)

\footnote{The neglected higher derivatives do not contribute to $\frac{\partial U_k}{\partial t}$.}
and
\[ \tilde{Z}_k(\rho, q^2) = Z_k(\rho, q^2) + \rho Y_k(\rho, q^2). \] (3.4)

Equation (3.2) is a nonlinear partial differential equation for a function of two variables \( t \) and \( \rho \).

One needs, however, the wave function renormalizations \( Z_k(\rho, q^2) \) and \( \tilde{Z}_k(\rho, q^2) \) which must be estimated, in turn, by a solution of evolution equations or by alternative methods. We are mainly interested in the behaviour near the phase transition, i.e. the massless theory or the theory with scalar mass much smaller than the cutoff scale \( \Lambda \). Then the evolution of \( U_k \) is mainly described by the regime with spontaneous symmetry breaking where the minimum of the potential occurs for \( \rho_0 > 0 \) with a \( k \)-dependent location of the minimum \( \rho_0(k) \) determined by \( U_k''(\rho_0) = 0 \). (Primes denote derivatives with respect to \( \rho \).) It is convenient to expand the potential in powers of \( \rho - \rho_0 \). This transforms eq. (3.2) into an infinite system of ordinary differential equations for the infinitely many couplings \( U_k''(\rho_0) \). In particular, the first and second derivative of eq. (3.2) with respect to \( \rho \) yields

\[
\frac{\partial}{\partial t} U_k''(\rho) = v_d(N - 1)k^{d-2}U_k''(\rho)L^d_{1,0}(w_1) + v_d(N - 1)k^d Z_k'\rho L^d_{1,0}(w_1) + v_d k^{d-2} (3U_k''\rho + 2U_k''(\rho)\rho) L^d_{0,1}(w_2) + v_d k^d Z_k'\rho L^d_{0,1}(w_2),
\] (3.5)

\[
\frac{\partial}{\partial t} U_k''(\rho) = -v_d(N - 1)k^{d-4} (U_k''(\rho))^2 L^d_{2,0}(w_1) - v_d k^{d-4} (3U_k''(\rho) + 2U_k''(\rho)\rho)^2 L^d_{0,2}(w_2) - 2v_d(N - 1)k^{d-2}U_k''(\rho) Z_k'\rho L^d_{2,0}(w_1) - 2v_d k^{d-2} (3U_k''(\rho) + 2U_k''(\rho)\rho) Z_k'\rho L^d_{0,2}(w_2)
- v_d(N - 1)k^d (Z_k''(\rho))^2 L^d_{2,0}(w_1) - v_d k^d (Z_k''(\rho))^2 L^d_{0,2}(w_2)
+ v_d(N - 1)k^{d-2}U_k''(\rho) L^d_{1,0}(w_1) + v_d k^{d-2} (5U_k''(\rho) + 2U_k''(\rho)) L^d_{0,1}(w_2)
+ v_d(N - 1)k^d Z_k''(\rho) L^d_{1,0}(w_2) + v_d k^d Z_k''(\rho) L^d_{0,1}(w_2).
\] (3.6)

Here
\[
v_d^{-1} = 2^{d+1} \pi^{d^2} \Gamma(d/2)
\] (3.7)

and we have used the variables
\[
w_1 = U_k',
\]
\[
w_2 = U_k''(\rho) + 2U_k''(\rho)\rho.
\] (3.8)

In terms of the shorthands
\[
P = Z_k(x, x) + R_k(x),
P = \tilde{Z}_k(x, x) + R_k(x)
\] (3.9)

the dimensionless momentum integrals \( L^d_{n_1, n_2} \) are given by
\[
L^d_{n_1, n_2}(w_1, w_2) = k^{2(n_1 + n_2) - d} \int_0^\infty dx x^{d-2} \frac{\partial}{\partial t} \left\{ (P + w_1)^{-n_1}(\tilde{P} + w_2)^{-n_2} \right\}
\] (3.10)

where \( \partial/\partial t \) acts on the r.h.s. only on \( R_k \) and \( x = q^2 \). The functions \( Z_k'(\rho) \) and \( \langle (Z_k''(\rho))^2 \rangle \) correspond to appropriate moments
\[
(Z_k'(\rho))^{d}_{n_1, n_2} = \frac{\int_0^\infty dx x^{d-2} Z_k'(\rho, x) \frac{\partial}{\partial t} \left\{ (P + w_1)^{-n_1}(\tilde{P} + w_2)^{-n_2} \right\}}{\int_0^\infty dx x^{d-2} \frac{\partial}{\partial t} \left\{ (P + w_1)^{-n_1}(\tilde{P} + w_2)^{-n_2} \right\}},
\] (3.11)

6
\[ \langle (Z'_k(\rho))^2 \rangle_{n_1,n_2} = \int_0^\infty dx \int_0^\infty dx x^{d-1} \left( Z'_k(\rho,x) \right)^2 \frac{\partial}{\partial t} \left\{ (P + w_1)^{-n_1}(\bar{P} + w_2)^{-n_2} \right\} \]  \quad (3.12) 

and similar for \( Z''_k(\rho), \tilde{Z}_k(\rho) \) etc. We have omitted for simplicity of notation the labels \( n_1,n_2 \) and \( d \) as well as the dependence of these moments on \( w_1 \) and \( w_2 \). Those can be inferred easily from accompanying functions \( L^d_{n_1,n_2}(w_1,w_2) \).

The minimum \( \rho_0(k) \) of the potential is defined by the condition

\[ U'_k(\rho_0) = 0. \]  \quad (3.13)

We are interested in the running of the minimum \( \rho_0 \). For notational simplicity we denote \( U_k^{(n)}(\rho_0), Z_k^{(n)}(\rho_0), \tilde{Z}_k^{(n)}(\rho_0) \) by \( U^{(n)}, Z^{(n)}, \tilde{Z}^{(n)} \) respectively and understood that \( Z'' \) means the moment \( (3.12) \) evaluated at \( \rho = \rho_0 \). Taking the total derivative of (3.13) we get with (3.13)

\[ \delta \equiv \frac{d}{dt} \rho_0 = -\frac{\partial U'_k(\rho_0)}{\partial \rho} \tilde{\lambda}^{-1} \]

\[ = -v_d(N - 1)k^{d-2}L_{1,0}^d(0) - v_d(N - 1)k^{d-2}L_{1,0}^{d+2}(0)Z'/\tilde{\lambda} \]

\[ -v_1k^{d-2}(3 + 2U''\rho_0/\tilde{\lambda})L_{0,1}^d(2\tilde{\lambda}\rho_0) - v_dk^{d-2}L_{1,0}^{d+2}(2\tilde{\lambda}\rho_0)\tilde{Z}'/\tilde{\lambda}. \]  \quad (3.14)

We also define the quartic coupling \( \tilde{\lambda} \equiv U''_k(\rho_0) \) and obtain

\[ \frac{d}{dt} \tilde{\lambda} = \frac{\partial}{\partial t}U'_k(\rho_0) + U''_k(\rho_0)\delta \]

\[ = -v_d(N - 1)k^{d-2}L_{1,0}^d(0) - v_dk^{d-4}(3\tilde{\lambda} + 2U''\rho_0)^2L_{0,2}^d(2\tilde{\lambda}\rho_0) \]

\[ -2v_d(N - 1)k^{d-2}\tilde{Z}'L_{2,0}^{d+2}(0) - 2v_dk^{d-2}(3\tilde{\lambda} + 2U''\rho_0)\tilde{Z}'L_{0,2}^{d+2}(2\tilde{\lambda}\rho_0) \]

\[ -v_d(N - 1)k^{d}Z''L_{2,0}^{d+4}(0) - v_dk^{d}Z^2L_{0,2}^{d+4}(2\tilde{\lambda}\rho_0) \]

\[ +v_d(N - 1)k^{d}Z''L_{1,0}^{d+2}(0) + v_dk^{d}Z''L_{0,1}^{d+2}(2\tilde{\lambda}\rho_0) \]

\[ +U''\delta. \]  \quad (3.15)

This procedure can be extended straightforward to higher couplings. Now it is useful to set \( Z_k = Z_k(\rho_0,0) \equiv Z_k \) in eq. (3.13) in order to obtain an effective inverse propagator (for massless fields)

\[ Z_k(\rho_0(k),0)q^2 + R_k(q) = \frac{Z_kq^2}{1 - f_k^2(q)}. \]  \quad (3.16)

To separate the running due to dimensions and wave function renormalization from the running due to interaction we switch to the dimensionless renormalized couplings (\( Z \equiv Z_k \))

\[ \kappa = k^{2-d}Z\rho_0, \quad \lambda = k^{d-4}Z^{-2}\tilde{\lambda}, \]

\[ u_3 = k^{2d-6}Z^{-3}U'', \quad u_4 = k^{3d-8}Z^{-4}U^{(4)}, \]

\[ z_1 = k^{d-2}Z^{-2}Z', \quad z_2 = k^{2d-4}Z^{-3}Z'', \]

\[ \tilde{z}_1 = k^{d-2}Z^{-2}\tilde{Z}', \quad \tilde{z}_2 = k^{2d-4}Z^{-3}\tilde{Z}''. \]  \quad (3.17)

and obtain the \( \beta \)-functions

\[ \beta_\kappa \equiv \frac{\partial \kappa}{\partial t} = (2 - d - \eta)\kappa + 2v_d(N - 1)(t_{11}^d + t_{12}^{d+2}z_1/\lambda) \]

\[ +2v_d(3 + 2u_3\kappa/\lambda)i_{11}^d s_1^{d+2}(2\kappa) + 2v_d t_{12}^{d+2}s_1^{d+2}(2\lambda\kappa)\tilde{z}_1/\lambda \]  \quad (3.18)
and

\[
\beta_\lambda \equiv \frac{\partial \lambda}{\partial t} = (d - 4 + 2\eta)\lambda + 2v_d(N - 1)\lambda^2 l^d_2 + 2v_d(3\lambda + 2u_3\kappa)^2 l^d_4 s^d_2(2\lambda\kappa) \\
-2v_d(N - 1)u_3l^d_1 - 2v_d(5u_3 + 2u_4\kappa)l^d_1s^d_1(2\lambda\kappa) \\
+4v_d(N - 1)\lambda z_1l^d_2 + 4v_d(3\lambda + 2u_3\kappa)\tilde{z}_1l^d_2 + 2s^d_2(2\lambda\kappa) \\
-2v_d(N - 1)z_2l^d_3 - 2v_d\tilde{z}_2l^d_3 s^d_1(2\lambda\kappa) \\
+2v_d(N - 1)z_1^2l^d_4 + 2v_d\tilde{z}_1^2l^d_4 s^d_2(2\lambda\kappa) \\
+u_3((d - 2 + \eta)\kappa + \frac{\partial\kappa}{\partial t}).
\] (3.19)

Here we have used the expressions

\[
L^d_{n,0}(0) = -2Z^{-n}p^d_n, \\
L^d_{0,n}(2\lambda\rho_0)/L^d_{n,0}(0) = s^d_n(\frac{2\lambda\rho_0}{k^2}) = s^d_n(2\lambda\kappa).
\] (3.20)

The anomalous dimension is defined as

\[
\eta = -\frac{d}{dt}\ln Z_k.
\] (3.21)

Equation (3.19) is our starting point for a computation of \(\beta_\lambda\). So far, no approximation has been made and eq. (3.19) is exact for arbitrary values of \(\lambda\) and arbitrary dimension \(d\). In addition to \(\lambda\) it involves, however, several unknown quantities \(^5\), namely, \(\kappa, u_3, u_4, \eta, z_1, z_2, \tilde{z}_1, \tilde{z}_2, z_1^2\) and \(z_2^2\), and the threshold functions \(s^d_n(2\lambda\kappa)\). In order to make contact with the usual \(\beta\)-function we have to compute all these quantities as functions of \(\lambda\). We will concentrate on the massless theory which corresponds to the (approximate) scaling solution of the evolution equation. This scaling solution is characterized by fixpoints for all couplings except for the running of \(\lambda\) for \(d = 4\). (This running is very slow for sufficiently small \(\lambda\). The true fixpoint is the Gaussian for \(\lambda = 0\).) For one of the couplings, which we may associate with \(\kappa\), this fixpoint is infrared unstable. This corresponds to the relevant parameter of the theory, i.e. the scalar mass term. For all other couplings (the irrelevant couplings) the fixpoint is infrared stable and will be approached rapidly. Due to the existence of the marginal parameter \(\lambda\) for \(d = 4\) the fixpoints for the various couplings depend on \(\lambda\). Inserting the \(\lambda\)-dependent fixpoints for \(\kappa, \eta\) etc. into eq. (3.19) will then yield \(\beta_\lambda\) as a function of \(\lambda\). Using the nonperturbative proposal of the last section we compute \(\kappa, u_3\) and \(\eta\) from corresponding exact evolution equations - these couplings plus \(\lambda\) correspond to the set \(\{A\}\). The remaining parameters \(z_1, z_2, \tilde{z}_1, \tilde{z}_2, z_1^2\) and \(z_2^2\) constitute the set \(\{B\}\) and are computed using the renormalization group improved one loop approximation. The latter can be evaluated as functions of \(\lambda\) and \(\kappa\) in the approximation of a quartic potential \(U_k\) with uniform \(Z_k = \tilde{Z}_k\). In the present paper we want to check our method for four dimensions where the scalar coupling \(\lambda\) is small for \(k \ll \Lambda\) due to the trivality of the \(\varphi^4\)-theory. We want to compute the beta function of the quartic coupling \(\lambda\) to order \(O(\lambda^3)\), i.e. we want to know the r.h.s. of eq. (3.19) to order \(O(\lambda^3)\). Therefore we have to know the expansion of the following quantities in powers of \(\lambda\) to the specified order

\[\kappa \sim O(\lambda^0), \quad u_3 \sim O(\lambda^3),\]

\(^5\)Remember that \(z_1^2\) stands for an appropriate moment and is in general not equal to the square of \(z_1\).
\[ z_1 \sim O(\lambda^2), \quad z_2 \sim O(\lambda^3), \]
\[ \tilde{z}_1 \sim O(\lambda^2), \quad \tilde{z}_2 \sim O(\lambda^3), \]
\[ \eta \sim O(\lambda^2). \]
(3.22)

From the connection between exact evolution equations and one loop expressions we recognize that the orders of the various couplings which are specified in eq. (3.22) are also the lowest orders which appear in their perturbative expansion. To see this, remember that e.g. in leading order of \( \lambda \) and at one loop \( U_{\rho_0}^n(\rho_0) \) is nothing but the contribution to the six-point-function at zero external momenta in the symmetric phase and that \( Z_k(\rho_0) \) is a contribution to the \( q^2 \)-dependence of the 1-loop four-point-function. Similar arguments hold for the other quantities. We see that higher derivatives of the potential (like the \( \phi^8 \)-coupling \( u_3 \)) or of the wave function renormalization don’t contribute to the beta function of the quartic coupling to order \( O(\lambda^3) \). Consequently these quantities can be truncated. From the combination of evolution equations with one loop expressions we get insight in the important couplings and an efficient tool for practical calculations. With this knowledge we have to compute the r.h.s. of

\[ \beta_\lambda = 2\eta \lambda + \frac{1}{16\pi^2} (N-1) t_1^4 \lambda^2 + \frac{9}{16\pi^2} \lambda^2 l_2^4 s_2^4 (2\lambda \kappa) \]
\[ - \frac{1}{16\pi^2} (N-1) u_3 t_1^4 - \frac{5}{16\pi^2} u_3 l_1^4 s_1^4 (2\lambda \kappa) \]
\[ + \frac{1}{8\pi^2} (N-1) \lambda z_1 t_1^6 + \frac{3}{8\pi^2} \lambda \tilde{z}_1 t_2^6 s_2^6 (2\lambda \kappa) \]
\[ - \frac{1}{16\pi^2} (N-1) z_2 t_1^6 - \frac{1}{16\pi^2} \tilde{z}_2 t_1^6 s_1^6 (2\lambda \kappa) \]
\[ + 2u_3 \kappa + O(\lambda^4) \]
(3.23)

where the fixed point values have to be inserted for all quantities except \( \lambda \). So we have to calculate the \( \beta \)-function of the coupling \( u_3 \) and the one loop expressions for the wave function renormalization constants \( Z_1, Z_2, \tilde{Z}_1 \) and \( \tilde{Z}_2 \). Our aim is to reproduce the known result of the two loop perturbation theory, namely

\[ \beta_\lambda = \frac{1}{16\pi^2} (N+8) \lambda^2 - \frac{1}{(16\pi^2)^2} (9N + 42) \lambda^3. \]
(3.24)

For small values of \( \lambda \) several simplifications occur. The most important one is that all couplings contributing to \( \beta_\lambda \) (3.23) in order \( \sim \lambda^3 \) need only be computed in lowest order in \( \lambda \) and that hence these corrections are simply additive. This means that for the computation of a given coupling, say \( u_3 \), we can put \( z_1 \) etc. to zero. This additivity is not crucial for a numerical solution of the nonperturbative evolution equations but it helps considerably for the analytical study of the present paper. It allows us to treat the different "corrections" to \( \beta_\lambda \) separately and to compare their relative importance. This will be done in the next sections. A second simplification uses the fact that the difference between \( \tilde{P} \) and \( P \) is a correction of order \( \sim \lambda^2 \) (3.22). In the order we are interested in we can always use (3.10)

\[ P(x) = \tilde{P}(x) = \frac{Z_k x}{1 - e^{-x}} \]
(3.25)
in the integrals (3.10) and omit the \( k \)-dependence of \( Z_k \) in \( \frac{\partial P}{\partial \eta} \). This permits a simple analytical computation of the constants \( l_\alpha \). In particular, one finds

\[ l_1^4 = 1, \quad l_2^4 = 1, \quad l_3^4 = 3 \ln \left( \frac{4}{3} \right), \quad l_1^6 = 2, \quad l_2^6 = \frac{3}{2}. \]
(3.26)
Finally a third simplification concerns the threshold functions \( s_n^d(2\lambda\kappa) \). They can be expanded for small \( \lambda\kappa \) according to
\[
l_n^d s_n^d(2\lambda\kappa) = l_n^d - 2n\lambda\kappa l_{n+1}^d + \ldots
\]  
(3.27)

We observe that we can put \( s_n^d = 1 \) except for the term \( \sim \lambda^2 s_n^4 \) in eq. (3.23).

### 4 The average potential with uniform wave function renormalization

We first consider in this section a uniform wave function renormalization \( Z_k(\rho, q^2) = \tilde{Z}_k(\rho, q^2) = Z_k \) and therefore set
\[
Z' = Z'' = \ldots = 0,
Y = Y' = \ldots = 0.
\]  
(4.1)

In this approximation one obtains from differentiating eq. (3.6) with respect to \( \rho \)
\[
\frac{\partial}{\partial \rho} U''''(\rho_0) = 2v_d(N - 1)k^{d-6}\bar{\lambda}^3 L_{3,0}^d(0) + 54v_d k^{d-6}\bar{\lambda}^3 L_{3,0}^d(2\lambda\rho_0).
\]  
(4.2)

The lowest order contribution for the running of \( \kappa \) and \( u_3 \) therefore read
\[
\frac{\partial \kappa}{\partial t} = (2 - d)\kappa + 2v_d(N + 2)l_1^d,
\frac{\partial}{\partial t} u_3 = (2d - 6)u_3 - 4v_d(N + 26)l_3^d \lambda^3.
\]  
(4.3)

The fixed points are now easily calculated as the roots of eq. (4.3). One finds in leading order
\[
\kappa_* = 2v_d \frac{N + 2}{d - 2} l_1^d
\]  
(4.4)

which is infrared unstable and
\[
(u_3)_* = 2v_d \frac{N + 26}{d - 3} l_3^d \lambda^3.
\]  
(4.5)

Note that the last value coincides with the one loop expression for \( u_3 \)
\[
u_3^{(1)} = 2(N + 26)v_d \lambda^3 \int_0^\infty dx x^{\frac{d}{2}-1} P^{-3}(x)
\]  
(4.6)

which follows from eq. (2.6) at zero field \( \varphi = 0 \) and \( n = 6 \). This coincidence can be understood by looking at the estimate of error (2.7) of the \( \beta \)-function caused by the use of "renormalization group improved" one loop expression (1.6) instead of the evolution equation (1.3). Comparing the estimate of error for \( u_3 \)
\[
\frac{\partial \Delta u_3}{\partial t} = \frac{\partial}{\partial t} \Delta \Gamma_k^{(6)}[\varphi = 0] = -\frac{1}{2} Tr \frac{\delta^6}{\delta \varphi_1(q = 0) \ldots \delta \varphi_6(q = 0)} \left\{ (\Gamma_k^{(2)} + R_k)^{-1} \frac{\partial}{\partial t} \Gamma_k^{(2)} \right\} \bigg|_{\varphi = 0}
\]  
(4.7)

with its exact evolution equation
\[
\frac{\partial u_3}{\partial t} = \frac{\partial}{\partial t} \Gamma_k^{(6)}[\varphi = 0] = \frac{1}{2} Tr \frac{\delta^6}{\delta \varphi_1(q = 0) \ldots \delta \varphi_6(q = 0)} \left\{ (\Gamma_k^{(2)}[\varphi] + R_k)^{-1} \frac{\partial}{\partial t} R_k \right\} \bigg|_{\varphi = 0}
\]  
(4.8)
we recognize that the estimated error is of higher order in $\lambda$ since $\frac{\partial}{\partial t} \Gamma_k^{(2)}[0] \sim \lambda^2$ whereas $\frac{\partial}{\partial R} R_k \sim \lambda^0$. We also observe that the difference between $\Gamma_k^{(6)}[\varphi = 0]$ and $\Gamma_k^{(6)}$ evaluated at the minimum is of higher order in $\lambda$. It is therefore no surprise that the lowest order contributions eqs. (4.5), (4.6) coincide. Inserting the fixed points (4.4), (4.5) in eq. (3.23) we obtain in four dimensions

$$\beta_\lambda = \frac{1}{16\pi^2} (N + 8) \lambda^2 - \frac{1}{(16\pi^2)^2} \frac{l_1^2}{l_2} \frac{l_3^2}{l_4} \frac{l_5^2}{l_6} \frac{l_7^2}{l_8} (20N + 88) \lambda^3$$

(4.9)

which is numerically

$$\beta_\lambda = \frac{1}{16\pi^2} (N + 8) \lambda^2 - \frac{1}{(16\pi^2)^2} (17.26N + 75.95) \lambda^3$$

(4.10)

to be compared with eq. (3.24).

5 Field and momentum dependence of the wave function renormalization

We next take the field and momentum dependence of the wave function renormalization into account. We want to compute $\tilde{z}_1$, $\tilde{z}_2$ and $\tilde{z}_2$ in a renormalization group improved one loop approximation. For this purpose we write $Z_k(\rho, Q^2) - Z_k(0,0)$ as an appropriate difference of two point functions and similar for $\tilde{Z}_k$. This difference is evaluated according to (2.3) where on the r.h.s. an approximation for $\Gamma_k^{(2)}$ is inferred from the ansatz (3.1) with $Z_k(\rho, -\partial^2) = Z_k$ and $Y_k(\rho, -\partial^2) = 0$. In a first approach we neglect the momentum dependence and approximate $Z_k(\rho)$ by $Z_k(\rho, q^2 = 0)$. The one loop expressions are derived in appendix A and read

$$Z_k(\rho, 0) = Z_k(0,0) + 8\frac{v_d}{d} k^{d-6} (U''_k(\rho))^2 \rho E^d_{2,2}(w_1, w_2),$$
$$\tilde{Z}_k(\rho, 0) = Z_k(0,0) + 4\frac{v_d}{d} k^{d-6} (3U''_k(\rho) + 2U'''_k(\rho)\rho)^2 \rho E^d_{0,4}(w_1, w_2) + 4(N - 1)\frac{v_d}{d} k^{d-6} (U''_k(\rho))^2 \rho E^d_{1,0}(w_1, w_2).$$

(5.1)

Here we have introduced the dimensionless integrals

$$E^d_{n_1, n_2}(w_1, w_2) = k^{2(n_1+n_2-1)-d} \int_0^\infty dx x^2 \frac{d}{dx} \hat{P}^2 (P + w_1)^{-n_1} (P + w_2)^{-n_2}$$

(5.2)

where $w_{1,2}$ are given by (3.8) and we have used $x = q^2$, $\hat{P} = \frac{\partial P}{\partial x}$ with $P$ given by (3.25). The $\rho$-derivatives of eqs. (5.1) at fixed $\rho_0$ yield in lowest order

$$Z' = 8\frac{v_d}{d} k^{d-6} \lambda^2 E^d_4,$$
$$Z'' = -128\frac{v_d}{d} k^{d-8} \lambda^3 E^d_5,$$
$$\tilde{Z}' = 4(N + 8)\frac{v_d}{d} k^{d-6} \lambda^2 E^d_4,$$
$$\tilde{Z}'' = -32(N + 26)\frac{v_d}{d} k^{d-8} \lambda^3 E^d_5$$

(5.3)

with $E^d_{n_1+n_2} \equiv E^d_{n_1, n_2}(0,0)$. We switch to the dimensionless renormalized quantities and find for $d = 4$

$$z_1 = \frac{1}{16\pi^2} \lambda^2 E^d_4,$$
\[ z_2 = \frac{-1}{\pi^2} \lambda^3 \epsilon_3^4, \]
\[ z_1 = \frac{N + 8}{32\pi^2} \lambda^2 \epsilon_4^4, \]
\[ z_2 = \frac{-N + 26}{4\pi^2} \lambda^3 \epsilon_5^4 \]  \hspace{1cm} (5.4)

where
\[ E_n^d = Z^{2-n} \epsilon_n^d \quad \text{and} \quad \epsilon_4^4 = \frac{1}{2}, \quad \epsilon_5^4 = \frac{1}{2} \ln \left( \frac{4}{3} \right). \]  \hspace{1cm} (5.5)

Again these one loop expressions are in lowest order equal to the appropriate fixpoints of the corresponding evolution equations. As before this holds because we include only the leading order contributions to the one loop expressions or the appropriate evolution equations. The formulae (5.4) are only approximations and do not include the full contributions in leading order contributions to the one loop expressions or the appropriate evolution equations. As before this holds because we include only the leading order contributions to the one loop expressions or the appropriate evolution equations. We therefore have to take the momentum dependence of \( Z_k(\rho, q^2) \) into account and employ the renormalization group improved one loop expressions (A.6) given in appendix A. We notice that \( Z_k(\rho_0, q^2) = 1 + O(\lambda^2) \). This is the well known result from standard perturbation theory. In the \( \beta_\lambda \)-function (3.23) the quantities \( Z', \tilde{Z}' \) enter up to order \( O(\lambda^2) \) and \( Z'', \tilde{Z}'' \) enter up to order \( O(\lambda^3) \). Deriving equations (A.6) with respect to \( \rho \) at the minimum \( \rho_0 \) we obtain in leading order
\[
\begin{align*}
Z'_k(\rho_0, Q^2) &= -2 \frac{\lambda^3}{Q^2} (2\pi)^{-d} \int d^d q P^{-1}(q) \left[ P^{-1}(q + Q) - P^{-1}(q) \right], \\
Z''_k(\rho_0, Q^2) &= 16 \frac{Q_3^3}{Q^2} (2\pi)^{-d} \int d^d q P^{-1}(q) \left[ P^{-1}(q + Q) - P^{-1}(q) \right], \\
\tilde{Z}'_k(\rho_0, Q^2) &= -(N + 8) \frac{\lambda^2}{Q^2} (2\pi)^{-d} \int d^d q P^{-1}(q) \left[ P^{-1}(q + Q) - P^{-1}(q) \right], \\
\tilde{Z}''_k(\rho_0, Q^2) &= 4(N + 26) \frac{\lambda^2}{Q^2} (2\pi)^{-d} \int d^d q P^{-2}(q) \left[ P^{-1}(q + Q) - P^{-1}(q) \right]. \quad (5.6)
\end{align*}
\]

We notice that in the required order in \( \lambda \) the mass term \( \sim 2\lambda\kappa k^2 \) in the denominators can be neglected and we have only to solve the one loop integrals for a massless theory. We evaluate these integrals using the Laplace transform
\[
P^{-1}(q) = \frac{1 - e^{-\frac{q^2}{2k^2}}}{q^2} = \int_0^{k^2} da e^{-\alpha q^2} \]  \hspace{1cm} (5.7)

and obtain
\[
I(Q^2) = \frac{1}{2} (2\pi)^{-d} \int d^d q P^{-1}(q) \left[ P^{-1}(q + Q) - P^{-1}(q) \right] = v_d \Gamma \left( \frac{d}{2} \right) \int_0^{\frac{k^2}{2}} da \int_0^{\frac{k^2}{2}} d\beta \left( e^{-Q^2 \alpha^2 \frac{d+1}{\alpha + \beta}} - 1 \right) (\alpha + \beta)^{-\frac{d}{2}} \]  \hspace{1cm} (5.8)

and
\[
J(Q^2) = \frac{1}{2} (2\pi)^{-d} \int d^d q P^{-2}(q) \left[ P^{-1}(q + Q) - P^{-1}(q) \right] = v_d \Gamma \left( \frac{d}{2} \right) \int_0^{\frac{k^2}{2}} da \int_0^{\frac{k^2}{2}} d\beta \int_0^{\frac{k^2}{2}} d\gamma \left( e^{-Q^2 \alpha^2 \frac{d+1}{\alpha + \beta + \gamma}} - 1 \right) (\alpha + \beta + \gamma)^{-\frac{d}{2}}. \]  \hspace{1cm} (5.9)
After some transformations we obtain in four dimensions (See app. B)

\[ I(Q^2) = v_4 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+2)! (n+1)} \left( \frac{1}{2} \right)^{n+1} \left( \frac{Q^2}{k^2} \right)^{n+1} \]  \hspace{1cm} (5.10)

and

\[ J(Q^2) = v_4 k^{-2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2)! (n+1)} \left( \frac{Q^2}{k^2} \right)^{n+1} \left\{ \int_0^{\frac{1}{2}} dz z^{n+1} (1-z)^2 - \int_{\frac{1}{2}}^1 dz z^{n+1} (1-z)^2 \right\}. \] \hspace{1cm} (5.11)

The dimensionless renormalized quantities read in lowest order in \( \lambda \)

\[ z_1(Q^2) = 2v_4 \lambda^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2)! (n+1)} \left( \frac{1}{2} \right)^n \left( \frac{Q^2}{k^2} \right)^n, \]

\[ \tilde{z}_1(Q^2) = (N+8)v_4 \lambda^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2)! (n+1)} \left( \frac{1}{2} \right)^n \left( \frac{Q^2}{k^2} \right)^n, \]

\[ z_2(Q^2) = 32v_4 \lambda^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2)! (n+1)} \left( \frac{Q^2}{k^2} \right)^n \left\{ \int_0^{\frac{1}{2}} dz z^{n+1} (1-z)^2 - \int_{\frac{1}{2}}^1 dz z^{n+1} (1-z)^2 \right\}, \]

\[ \tilde{z}_2(Q^2) = 8(N+26)v_4 \lambda^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2)! (n+1)} \left( \frac{Q^2}{k^2} \right)^n \left\{ \int_0^{\frac{1}{2}} dz z^{n+1} (1-z)^2 - \int_{\frac{1}{2}}^1 dz z^{n+1} (1-z)^2 \right\}. \]

Notice that the \((n=0)\)-terms of the sum correspond to eqs. (5.4). We use these expressions to calculate the appropriate moments \((3.11)\) which occur in the \(\beta_\lambda\)-function \((3.23)\).

\[ z_1 l_2^6 = 2v_4 \lambda^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2)! (n+1)} \left( \frac{1}{2} \right)^n l_2^{6+2n} \]

\[ = (24 \ln 3 + 16 \ln 2 - 20 \ln 5 - 4)v_4 \lambda^2, \]

\[ \tilde{z}_1 l_2^6 = (N+8)v_4 \lambda^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2)! (n+1)} \left( \frac{1}{2} \right)^n l_2^{6+2n} \]

\[ = (N+8)(12 \ln 3 + 8 \ln 2 - 10 \ln 5 - 2)v_4 \lambda^2, \]

\[ z_2 l_1^6 = 32v_4 \lambda^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2)! (n+1)} l_1^{6+2n} \left\{ \int_0^{\frac{1}{2}} dz z^{n+1} (1-z)^2 - \int_{\frac{1}{2}}^1 dz z^{n+1} (1-z)^2 \right\} \]

\[ = 16(12 \ln 3 - 5 \ln 5 - 8 \ln 2)v_4 \lambda^3, \]

\[ \tilde{z}_2 l_1^6 = 8(N+26)v_4 \lambda^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2)! (n+1)} l_1^{6+2n} \left\{ \int_0^{\frac{1}{2}} dz z^{n+1} (1-z)^2 - \int_{\frac{1}{2}}^1 dz z^{n+1} (1-z)^2 \right\} \]

\[ = 4(N+26)(12 \ln 3 - 5 \ln 5 - 8 \ln 2)v_4 \lambda^3. \] \hspace{1cm} (5.12)

(See app. C for the evaluation of the sums.) Comparing with the expressions \((5.4)\) we see that the \(Q^2\)-dependence of the wave function accounts for 21\%, 21\%, 30\% and 30\% of the moments \(z_1, \tilde{z}_1, z_2\) and \(\tilde{z}_2\), respectively. Inserting the results \((5.12)\) into the \(\beta_\lambda\)-function \((3.23)\) reads

\[ \beta_\lambda = \frac{N+8}{16\pi^2} \lambda^2 - \frac{10N + 44}{(16\pi^2)^2} \lambda^3 + 2\eta \lambda. \] \hspace{1cm} (5.13)

In order to compute the complete coefficient \(\sim \lambda^3\) we only need now the anomalous dimension \(\eta\) in order \(\lambda^2\).
6 The anomalous dimension

The anomalous dimension η is defined by identifying $Z_k$ with $Z_k(\rho_0(k), Q^2 = 0)$

$$\eta = -\frac{d}{dt} \ln Z_k(\rho_0)$$

$$= -Z_k^{-1}(\rho_0) \frac{\partial Z_k(\rho_0)}{\partial t} - Z_k^{-1}(\rho_0) Z'_k(\rho_0) \delta t$$

(6.1)

where

$$Z_k(\rho) \equiv Z_k(\rho, Q^2 = 0) = \Omega^{-1} \lim_{Q^2 \to 0} \frac{\partial}{\partial Q^2} \frac{\delta^2}{\delta \varphi^2(Q) \delta \varphi^3(Q)} \Gamma_k \bigg|_\rho$$

$$= \lim_{Q^2 \to 0} \frac{\partial}{\partial Q^2} \Gamma_k^{(2)}(Q, -Q; \rho)$$

(6.2)

is evaluated for a constant field configuration. With the exact evolution equation eq. (1.1) we obtain an exact equation for the $k$-dependence of $Z_k(\rho)$ (for $N \geq 2$)

$$\frac{\partial Z_k(\rho)}{\partial t} = \frac{1}{2} \Omega^{-1} \lim_{Q^2 \to 0} \frac{\partial}{\partial Q^2} \frac{\delta^2}{\delta \varphi^2(Q) \delta \varphi^3(Q)} \text{Tr} \left\{ \left[ \Gamma_k^{(2)}(q, -q) + R_k(q) \right]^{-1} \partial_t R_k(q) \right\} \bigg|_\rho$$

(6.3)

We perform the functional derivatives (omitting the internal field indices)

$$\frac{\partial Z_k(\rho)}{\partial t} = \frac{1}{2} \Omega^{-1} \lim_{Q^2 \to 0} \frac{\partial}{\partial Q^2} \frac{\delta^2}{\delta \varphi^2(Q) \delta \varphi^3(Q)} \text{Tr} \left\{ \left[ \Gamma_k^{(2)}(q, -q) + R_k(q) \right]^{-1} \partial_t R_k(q) \right\} \bigg|_\rho$$

$$= \frac{1}{2} \Omega^{-1} \lim_{Q^2 \to 0} \frac{\partial}{\partial Q^2} \text{Tr} \left\{ - \left[ \Gamma_k^{(2)}(q, -q) + R_k(q) \right]^{-1} \partial_t R_k(q) \right\}$$

(6.4)

$$\cdot \Gamma_k^{(4)}(Q, -Q, q, -q) \left[ \Gamma_k^{(2)}(q, -q) + R_k(q) \right]^{-1} \partial_t R_k(q)$$

$$+ 2 \left[ \Gamma_k^{(2)}(q, -q) + R_k(q) \right]^{-1} \Gamma_k^{(3)}(Q, q, -q - Q) \left[ \Gamma_k^{(2)}(q + Q, -q - Q) + R_k(q + Q) \right]^{-1}$$

$$\cdot \Gamma_k^{(3)}(-Q, q + Q, -q) \left[ \Gamma_k^{(2)}(q, -q) + R_k(q) \right]^{-1} \partial_t R_k(q) \bigg|_\rho$$

with

$$\Gamma_k^{(n)}(q_1, \ldots, q_n) = \Omega^{-1} \frac{\delta^n}{\delta \varphi(q_1) \cdots \delta \varphi(q_n)} \Gamma_k$$

(6.5)

containing a factor $\sim \delta(q_1 + \ldots + q_n)$. Inserting $\rho = \rho_0$ in eq. (6.4) and combining eq. (6.1)

with eq. (6.4) gives an exact nonperturbative expression for the anomalous dimension η.

It involves the $k$-dependent full propagator $\left( \Gamma_k^{(2)} + R_k \right)^{-1}$, the 1PI three point function for arbitrary momenta and the 1PI four point function for two pairs of opposite momenta. One of the momenta (corresponding to $Q^2$) can be taken small and only an expression quadratic in this momentum is needed. (We observe that eq. (6.4) can easily be generalized into an exact evolution equation for the full momentum dependent “wave function renormalization” $Z_k(\rho, Q^2)$ by omitting the operation $\lim_{Q^2 \to 0} \frac{\partial}{\partial Q^2}$, subtracting from the now $Q$ dependent r.h.s. the value for $Q = 0$ and deviding by $Q^2$, i.e. by combining eq. (1.1) and eq. (1.2).)

The ansatz (3.1) only describes $\Gamma_k^{(3)}(0, q, -q)$ and $\Gamma_k^{(4)}(0, 0, q, -q)$ or other combinations with

\footnote{For $N = 1$ one should use functional derivatives with respect to $\varphi \equiv \varphi_1$. This changes the formulae of this section but leads to the same final result for $\eta$.}
only two nonvanishing momenta since higher derivative terms of the structure \( \partial^2 \varphi \partial \varphi \partial \varphi \) and \((\partial \varphi)^4\) are not included. In a first step we neglect the momentum dependence of \( \Gamma^{(3)}_k \) and \( \Gamma^{(4)}_k \) except for the contributions quadratic in the momenta. We can then determine the r.h.s. of eq. (6.4) from the ansatz (3.1) with \( Z_k \) and \( Y_k \) depending only on \( \rho \). In this limit \( \eta \) has been computed in [1] and we quote only the results. The scale dependence of \( Z_k(\rho) \) at fixed \( \rho \) reads in this approximation

\[
\frac{\partial Z_k(\rho)}{\partial t} \bigg|_\rho = \frac{\partial Z_k(\rho)^{(a)}}{\partial t} \bigg|_\rho + \frac{\partial Z_k(\rho)^{(b)}}{\partial t} \bigg|_\rho
\]

(6.6)

with

\[
\frac{\partial Z_k(\rho)^{(a)}}{\partial t} \bigg|_\rho = v_d (N \rho) \left[ (N - 1) Z_k(\rho) + Y_k(\rho) \right] k^{d-2} L_{1,0}^d (w_1) + v_d [Z_k'(\rho) + 2Z_k''(\rho) k^{d-2} L_{0,1}^d (w_2),
\]

(6.7)

\[
\frac{\partial Z_k(\rho)^{(b)}}{\partial t} \bigg|_\rho = 4v_d \left( U_k''(\rho) \right)^2 \frac{\rho^{k-2}}{4} Q_{2,1}^d (w_1, w_2) + 4v_d Y_k(\rho) U_k''(\rho) \rho k^{d-4} Q_{2,1}^d (w_1, w_2) + v_d (Y_k(\rho)) \frac{\rho^{k-2}}{4} Q_{2,1}^d (w_1, w_2) - 8v_d Z_k'(\rho) U_k''(\rho) \rho k^{d-4} L_{1,1}^d (w_1, w_2) - 4v_d Z_k'(\rho) Q_{2,1}^d (w_1, w_2) - 4v_d Z_k'(\rho) Y_k(\rho) \rho k^{d-2} L_{1,1}^d (w_1, w_2) + 16v_d Z_k'(\rho) U_k''(\rho) \rho k^{d-4} N_{2,1}^d (w_1, w_2) + 8v_d Z_k'(\rho) Y_k(\rho) \rho k^{d-2} N_{2,1}^d (w_1, w_2).
\]

(6.8)

Note that eq. (6.7) stems from the part of (6.4) which involves the four point function whereas eq. (6.8) is originated from the part of (6.4) which involves the three point functions. The integrals \( N_{n_1,n_2}^d \) are defined by, with \( \dot{P} = \frac{\partial P}{\partial x} \)

\[
N_{n_1,n_2}^d (w_1, w_2) = k^{2(n_1+n_2-1-d)} \int_0^\infty dx x^\frac{d}{2} \frac{\partial}{\partial t} \left\{ \dot{P} (P + w_1)^{-n_1} (\dot{P} + w_2)^{-n_2} \right\}
\]

(6.9)

and the integrals

\[
Q_{n_1,n_2}^{d,\alpha} (w_1, w_2) = k^{2(n_1+n_2-\alpha-d)} \int_0^\infty dx x^\frac{d}{2} \frac{\partial}{\partial t} \left\{ \dot{P} + \frac{2x}{d} \dot{P} - \frac{4x}{d} \dot{P}^2 (P + w_1)^{-1} (P + w_1)^{-n_1} (\dot{P} + w_2)^{-n_2} \right\}
\]

(6.10)

are related by partial integration to other integrals

\[
M_{n_1,n_2}^d (w_1, w_2) = k^{2(n_1+n_2-1-d)} \int_0^\infty dx x^\frac{d}{2} \frac{\partial}{\partial t} \left\{ \dot{P}^2 (P + w_1)^{-n_1} (\dot{P} + w_2)^{-n_2} \right\}
\]

(6.11)

through

\[
Q_{n_1,n_2}^{d,\alpha} (w_1, w_2) = \left\{ \frac{2n_1 - 4}{d} M_{n_1+1,n_2}^d (w_1, w_2) + \frac{2n_2}{d} M_{n_1,n_2+1}^d (w_1, w_2) + \frac{2n_1 - 4}{d} Y_k(\rho) M_{n_1+n_2+1}^d (w_1, w_2) - \frac{2n_2}{d} N_{n_1,n_2}^{d+2\alpha-2} (w_1, w_2) \right\}
\]

(6.12)
Note, that the $E_{n_1,n_2}^d$-integrals are closely related to the $M_{n_1,n_2}^d$-integrals since \( \frac{d}{dx} g(\frac{x}{k^2}) = -2x \frac{d}{dx} g(\frac{x}{k^2}) \) for an arbitrary function depending only on the ratio \( \frac{x}{k^2} \). In lowest order of \( \lambda \) eq. (6.14) reads at the minimum \( \rho_0 \)

\[
\frac{\partial Z}{\partial t} = v_d(NZ' + Y)k^{d-2}I_{1,0}^d(0) + \frac{8}{d}v_d\lambda^2 \rho_0 k^{d-6} M_{4,0}^d(0). \tag{6.13}
\]

Expressed in the dimensionless renormalized quantities (and \( y = Z^{-2}k^{d-2}Y \)) one obtains for the anomalous dimension (6.1)

\[
\eta = 16 v_d \frac{d}{d} m_n^d \kappa^2 + 2v_d(Nz_1 + y)j_1^d - z_1 \left( \frac{\partial \kappa}{\partial t} + (d - 2) \kappa \right) \tag{6.14}
\]

where

\[
m_n^d = -\frac{1}{2} Z^{n-2} M_{n,0}^d(0). \tag{6.15}
\]

We remember \( y = \hat{z}_1 - z_1 \) and write at the fixpoint of our theory

\[
\eta = 16 v_d \frac{d}{d} m_n^d \kappa^2 + 2v_d j_1^d(\hat{z}_1) \kappa - 6 v_d j_1^d(z_1) \kappa. \tag{6.16}
\]

Since we have neglected the momentum dependence of \( Z_k(\rho, Q^2) \) the appropriate one loop expressions are given by (5.4) and we obtain in the present approximation and for \( d = 4 \)

\[
\eta = \frac{3(N + 2)}{(32\pi^2)^2} \lambda^2. \tag{6.17}
\]

Next we include in eq. (6.4) the momentum dependence of \( \Gamma_k^{(3)} \) and \( \Gamma_k^{(4)} \) beyond the approximation quadratic in momenta. We observe that the momentum dependence of the three point and four point function is always of higher order in \( \lambda \) compared with the momentum independent part. The contribution \( \sim \left( \Gamma_k^{(3)} \right)^2 \) is already \( \sim \lambda^2 \) and we therefore do not need the momentum dependence of the three point function for a computation of \( \eta \) in order \( \lambda^2 \). On the other hand we need the quantity

\[
\Delta_k(Q, -Q, q, -q) = \Gamma_k^{(4)}(Q, -Q, q, -q) - \Gamma_k^{(4)}(0, 0, q, -q) - \Gamma_k^{(4)}(Q, -Q, 0, 0) + \Gamma_k^{(4)}(0, 0, 0, 0) \tag{6.18}
\]

in order \( \lambda^2 \). This summarizes the neglected terms and the insertion of (6.18) into (6.4) gives the complete correction to \( \eta \) in order \( \lambda^2 \). In fact, the relevant \( Q \)-dependence of the first term on the r.h.s. of eq. (6.4) is given by an expansion of \( \Gamma_k^{(4)} \) quadratic in \( Q \). The most general form of the \( Q^2 \) part of \( \Gamma_k^{(4)}(Q, -Q, q, -q) \) can be written as

\[
\Gamma_k^{(4)}(Q, -Q, q, -q) = F + G(q^2 + Q^2) + H_0(q^2) + H_1(q^2)Q^2 + H_2(q^2)(qQ)^2 + O(Q^4) \tag{6.19}
\]

with

\[
H_0(0) = 0, \quad \dot{H}(0) = 0, \quad H_1(0) = 0. \tag{6.20}
\]

The first two parts involving \( F \) and \( G \) have already been included in the approximation used in the first half of this section whereas \( H_0(q^2) \) does not contribute to \( \eta \). It is easy to verify that the remaining term \( H_1(q^2)Q^2 + H_2(q^2)(qQ)^2 \) is just the part quadratic in an expansion of
\[ \Delta_k(Q, -Q, q, -q) \]. As an illustration of this argument we give the expansion of \( \Gamma^{(4)}_k \) in fourth order in the momenta for \( N = 1 \)

\[
\Gamma^{(4)}_k (Q, -Q, q, -q) = 3U''_k(p) + 2\rho U'''_k(p) + 4\rho^2 U^{(4)}_k(p) + (q^2 + Q^2)(Z'_k(p) + 2\rho Z''_k(p)) + (q^2Q^2 + 2(qQ)^2)W_k(p)
\]

(6.21)

and note that the last term \( \sim W_k(p) \) corresponds to the contribution from \( \Delta_k \) neglected so far. We will consider \( \Delta_k \) as one of the "secondary" couplings in the sense of sect. 2 and evaluate it by a "renormalization group improved" one loop expression. In lowest order \( \lambda^2 \) we can neglect the mass terms \( \sim 2\lambda\kappa k^2 \) in the propagators and only need a one loop calculation for the massless case, assuming a purely quartic potential and a constant wave function renormalization \( Z_k \). With these approximations the one loop expression for the four point function reads

\[
\left( \Gamma^{(4)}_k (Q, -Q, q, -q) \right)_{22aa} = -\lambda^2 \text{diag} \left( \frac{(N+4)A + 2B(q, Q), (N+8)A + (N+8)B(q, Q)}{(N+4)A + 2B(q, Q), \ldots, (N+4)A + 2B(q, Q)} \right)
\]

(6.22)

where

\[
A = \frac{1}{2} (2\pi)^{-d} \int d^d p P^{-2}(p),
\]

\[
B(q, Q) = \frac{1}{2} (2\pi)^{-d} \int d^d p P^{-1}(p) \left[ P^{-1}(p - Q - q) + P^{-1}(p - Q + q) \right].
\]

(6.23)

The expressions \( A \) and \( B \) are both ultraviolet divergent in four dimensions but these divergencies cancel for the difference (6.18)

\[
\Delta_k(Q, -Q, q, -q) = -\lambda^2 \text{diag} \left( 2, N + 8, 2, \ldots, 2 \right) \frac{1}{2} (2\pi)^{-d} \int d^d P^{-1}(p) 
\]

\[
\left[ 2P^{-1}(p) + P^{-1}(p - Q - q) + P^{-1}(p - Q + q) - 2P^{-1}(p + Q) - 2P^{-1}(p + q) \right]
\]

(6.24)

which is finite in any dimension less than six. Again we evaluate this expression using the Laplace transform (5.7) and obtain

\[
\Delta_k(Q, -Q, q, -q) = -v_d \Gamma \left( \frac{d}{2} \right) \lambda^2 \text{diag} \left( 2, N + 8, 2, \ldots, 2 \right) \int_0 \int_0 d\alpha d\beta
\]

\[
\cdot \left[ e^{-\frac{\alpha\beta}{\alpha+\beta}(Q+q)^2} - 1 + e^{-\frac{\alpha\beta}{\alpha+\beta}(Q-q)^2} - 1 - 2e^{-\frac{\alpha\beta}{\alpha+\beta}Q^2} + 2 - 2e^{-\frac{\alpha\beta}{\alpha+\beta}Q^2} + 2 \right] (\alpha + \beta)^{-\frac{4}{d}}.
\]

(6.25)

With the \( I(Q^2) \)-integral (5.10) we write in four dimensions

\[
\Delta_k(Q, -Q, q, -q) = -v_d \lambda^2 \text{diag} \left( 2, N + 8, 2, \ldots, 2 \right) \sum_{n=0}^{N-2} \frac{(-1)^{n+1}}{(n+2)! (n+1)} \left( \frac{1}{2} \right)^{n+1}
\]

\[
\cdot \left[ \left( \frac{(Q+q)^2}{k^2} \right)^{n+1} + \left( \frac{(Q-q)^2}{k^2} \right)^{n+1} - 2 \left( \frac{Q^2}{k^2} \right)^{n+1} - 2 \left( \frac{q^2}{k^2} \right)^{n+1} \right].
\]

(6.26)
We now insert this expression into eq. (6.3), perform the \( \lim_{Q^2 \to 0} \frac{\partial}{\partial Q^2} \) operation under the momentum integration corresponding to the trace and obtain with (6.1) the contribution
\[
\Delta \eta = -3(N + 2)v^2 \lambda^2 \int_0^\infty dy \partial_t P(y) P^{-2}(y) \sum_{n=1}^\infty \frac{(-1)^n + 1}{(n+1)!} \left( \frac{y}{2} \right)^{n+1}
\]
where we set \( y = q^2/k^2 \) and used eq. (3.25) to perform the integration. Adding this contribution \( \Delta \eta \) to the result from eq. (6.17) we obtain in order \( \lambda \)
\[
\eta = \frac{N + 2}{2} \frac{\lambda^2}{(16\pi^2)^2}.
\]
This coincides with the perturbative two loop result [14]. Insertion of (6.28) into eq. (5.13) yields for the \( \beta_\lambda \)-function
\[
\beta_\lambda = \frac{N + 8}{16\pi^2} \lambda^2 - \frac{9N + 42}{(16\pi^2)^2} \lambda^3.
\]
Again, the two loop coefficient [14] is exactly reproduced. Our "nested one loop calculation" gives in next to leading order in \( \lambda \) exactly the same results as a two loop calculation!

7 Large quartic scalar coupling

In the preceding sections we have concentrated on small values of \( \lambda \). Our nonperturbative exact evolution equations are by no means restricted to this case. We demonstrate this here by discussing the leading order contribution to \( \beta_\lambda \) for very large values of \( \lambda \). We also combine approximately the two regimes for large and small quartic couplings in order to obtain the \( \beta \)-function for the whole range of \( \lambda \). This is justified by the existence of only a small transition regime which is reached quickly for scales only somewhat below the ultraviolet cutoff. Our starting point are the exact formulae (3.18) and (3.19) for \( \beta_\kappa \) and \( \beta_\lambda \) which we combine to
\[
\beta_\lambda = (d - 4 + 2\eta)\lambda + 2v_d(N - 1)\lambda^2 l_2^{d-2} + 2v_d(N - 1) \left\{ 2z_1 l_2^{d-2} + u_3 z_1 + \frac{1}{d+2} - z_2 l_2^{d+2} + z_1^2 l_2^{d+4} \right\} \\
+ v_d \left\{ 2u_3^2 \kappa + u_3 - 2u_3 - 2u_3 \kappa \right\} \\
+ 2v_d \left\{ 2u_3^2 z_1 + 2u_3^2 - 2u_3^2 \right\} \\
+ 2v_d \left\{ 3\lambda + 2u_3 \kappa \right\} \\
+ v_d \left\{ 3\lambda + 2u_3 \kappa \right\} l_2^{d-2} s_2^{d-2} (2\lambda \kappa) + 2 (3\lambda + 2u_3 \kappa) \tilde{z}_1 l_2^{d+2} s_2^{d+2} (2\lambda \kappa) \\
+ z_1^2 l_2^{d+4} s_2^{d+4} (2\lambda \kappa) \right\}.
\]
Our strategy for an evaluation of \( \beta_\lambda \) will consist of a series of assumptions on the maximal value (in powers of \( \lambda \)) that different quantities on the r.h.s. of eq. (7.1) can take for very large values of \( \lambda \). From there we compute the leading contribution to \( \beta_\lambda \). In a second step we establish the self consistency of our assumptions by discussing explicitly the evolution equations for the quantities of concern.

Eq. (3.18) gives no indication that \( \kappa \) should be suppressed by inverse powers of \( \lambda \) and we assume in the following \( \kappa = O(1) \) in the sense of counting powers of \( \lambda^{-1} \). Then the threshold
functions $s_n^d(2\lambda\kappa)$ give suppression factors $\sim \lambda^{-(n+1)}$. We will restrict the discussion of this section to $N \geq 2$ where the contributions $\sim \lambda^{-(n+1)}$ are small compared to the Goldstone boson contributions $\sim (N-1)$. Neglecting terms $\sim s_n^d(2\lambda\kappa)$ in a first approximation we find

$$\beta_\lambda = 2v_d(N-1)\lambda^2 l_2^d + (d-4+2\eta)\lambda + 2v_d(N-1)\left\{2\lambda z_{1,2}^{d+2} + v_3 z_{1,1}^{d+2} - z_2 t_1^{d+2} + z_1^{2d+4}\right\}. \quad (7.2)$$

We remind that the quantities $l_n^d$ have to be evaluated here with $P = Z_k(\rho, q^2)q^2 + R_k(q^2)$ according to (3.3). The difference between this and the values for small $\lambda$ (where $P \approx Z_k q^2 + R_k(q)$) as well as the last term in (7.2) can be attributed to the wave function renormalization $Z_k(\rho, Q^2)/Z_k$. We therefore need an estimate of this quantity.

Let us start at some ultraviolet cutoff $\Lambda$ with $Z_\lambda(\rho, Q^2)$ depending only weakly on $\rho$ such that $|Z_\lambda'(\rho, Q^2)| \leq O(1)$ and $|Z_\lambda''(\rho, Q^2)| \leq O(\lambda)$. We first assume that this situation remains valid for all scales $k$ for which $\lambda$ remains large. Assuming also that $|v_3|$ is at most $O(\lambda^2)$ we find the simple leading order expression

$$\beta_\lambda = 2v_d(N-1)\lambda^2 l_2^d + (d-4+2\eta)\lambda. \quad (7.3)$$

Here the unspecified $Q^2$-dependence of $Z_k(\rho, Q^2)$ only affects the value of $l_2^d$ but not the general structure of eq. (7.3). Unless the $Q^2$-dependence of $Z_k(\rho_0, Q^2)$ is extremely dramatic $l_2^d$ will always remain of order one. As long as $\frac{\partial}{\partial q^2} \Gamma_k^{(3)}(Q, -Q, q, -q)$ does not exceed a value $\sim O(1)$ and $\frac{\partial}{\partial q^2} \Gamma_k^{(3)}(Q, q, -q - Q) \sim O(1)$, $\frac{\partial}{\partial q^2} \Gamma_k^{(2)}(Q + q) \sim O(1)$ we conclude from the exact relation (3.4) that $\eta$ remains of the order $O(1)$. With all these assumptions one finds the leading order contribution

$$\beta_\lambda = 2v_d(N-1)\lambda^2 l_2^d. \quad (7.4)$$

Neglecting in addition the $Q^2$-dependence of $Z_k(\rho_0, Q^2)$ the coefficients $l_2^d$ are given in appendix C with $l_2^d = 1$. Before a discussion of the validity of our assumptions several comments are in order at this place:

i) For large $N$ we recover the result of the leading contribution in the large $N$ approximation

$$\beta_\lambda = 2v_d N \lambda^2 l_2^d + (d-4)\lambda, \quad \eta = 0. \quad (7.5)$$

We remind that the large $N$ approximation obtains formally by neglecting a collective excitation which roughly corresponds to the radial mode. With our assumptions the contributions from the radial mode are small because of the threshold functions $s_n^d(2\lambda\kappa)$ and similar suppressions from the propagator of the massive radial mode $\sim \lambda^{-1}$ in other expressions. This motivates the qualitative correctness of (7.4) also for moderate values of $N$ (except $N = 1$).

ii) We note that there is no dramatic difference in the behaviour of $\beta_\lambda$ for small and large $\lambda$. We may therefore extrapolate for intermediate values of $\lambda$ by connecting $\beta_\lambda$ for small and large $\lambda$ at the transition value $\lambda_{tr}$ where they take equal values. Roughly speaking $\lambda_{tr}$ marks the transition from the perturbative to the nonperturbative regime. We define

$$\bar{\beta} = (\beta_\lambda - (d-4)\lambda)/(2v_d l_2^d \lambda^2) \quad (7.6)$$
such that, for $d = 4$,

$$\tilde{\beta} = \begin{cases} N + 8 - \frac{9N + 42}{16\pi^2} \lambda & \text{for } \lambda < \lambda_{tr} \\ N + \gamma & \text{for } \lambda > \lambda_{tr} \end{cases}, \quad (7.7)$$

(The following discussion will remain valid with minor quantitative modifications in arbitrary dimension $d$. We also allow for a correction factor $\gamma$ accounting for a possible modification of the leading order result (7.3) which corresponds to $\gamma = -1$.) From (7.7) we obtain the transition point

$$\lambda_{tr} = \frac{16\pi^2(8 - \gamma)}{9N + 42}. \quad (7.8)$$

(For $N = 4$, $\gamma = -1$ the transition point is $\lambda_{tr} = 18.2$.) At $\lambda_{tr}$ the ratio of the two loop to one loop contribution for $\tilde{\beta}$ is given by

$$\frac{\tilde{\beta}(2)}{\tilde{\beta}(1)} = -\frac{8 - \gamma}{N + 8} \quad (7.9)$$

which is of the order one except for very large $N$.

iii) The precise form of $\tilde{\beta}$ for large $\lambda$, e.g. the value of $\gamma$ as long as $\gamma$ is sufficiently large compared to $-N$, is not very important for the qualitative behaviour of our model. Taking into account the leading contribution

$$\frac{\partial \lambda}{\partial t} = 2\nu d l_2^d (N + \gamma) \lambda^2 \quad (7.10)$$

one finds for the scale dependence of $\lambda$

$$\frac{1}{\lambda(k)} = \frac{1}{\lambda(\Lambda)} + 2\nu d l_2^d (N + \gamma) \ln \frac{\Lambda}{k}. \quad (7.11)$$

For $d = 4$ the value $\lambda_{tr}$ for the transition to the perturbative behaviour is reached for $k > k_{tr}$

$$k_{tr} = \Lambda \exp -\frac{9N + 42}{(N + \gamma)(8 - \gamma)}. \quad (7.12)$$

(The scale $k_{tr}$ is obtained for $\lambda^{-1}(\Lambda) = 0$. For positive values of $\lambda^{-1}(\Lambda)$ the transition occurs at higher values of $k$. This is used below to establish an upper bound on the Higgs scalar mass.) For $N = 4$ one finds

$$\frac{k_{tr}}{\Lambda} = \begin{cases} 0.056 & \text{for } \gamma = -1 \\ 0.087 & \text{for } \gamma = 0 \end{cases}. \quad (7.13)$$

Obviously the theory reaches the perturbative regime very fast. In four dimensions there is no $\varphi^4$-theory with large renormalized quartic coupling at scales much smaller than the ultraviolet cutoff. For $k$ below the scale $k_{pt}$ where perturbation theory becomes valid we use the solution of (3.29) with $\lambda_{pt} = \lambda(k_{pt})$

$$\frac{1}{\lambda(k)} + \frac{9N + 42}{16\pi^2(N + 8)} \ln \left( \frac{16\pi^2(N + 8)}{9N + 42} \frac{1}{\lambda(k)} - 1 \right) = \frac{1}{\lambda_{pt}} + \frac{9N + 42}{16\pi^2(N + 8)} \ln \left( \frac{16\pi^2(N + 8)}{9N + 42} \frac{1}{\lambda_{pt}} - 1 \right) + \frac{N + \gamma}{16\pi^2} \ln \frac{k_{pt}}{k}$$

$$= \frac{N + 8}{16\pi^2} \ln \frac{\Lambda}{k} - \frac{9N + 42}{16\pi^2(N + \gamma)} + \frac{9N + 42}{16\pi^2(N + 8)} \ln \frac{N + \gamma}{8 - \gamma}. \quad (7.14)$$
For the last equation we have inserted \( k_{\text{tr}} = k_{\text{tr}}, \) (7.12). This constitutes an upper bound for possible values \( \lambda(k) \) for arbitrary scales \( k < k_{\text{tr}} \). We observe that this upper bound does not depend strongly on \( \gamma \), especially for large ratios \( \Lambda/k \). Let us compare the result (7.14) with the result of high order strong coupling expressions on the lattice [15], namely

\[
\frac{1}{\lambda(k)} - \frac{9N + 42}{16\pi^2(N + 8)} \ln \left( \frac{N + 8}{16\pi^2} \lambda(k) \right) = \frac{N + 8}{16\pi^2} \left( \ln \frac{\Lambda}{k} + C \right),
\]

where \( C = 1.9 \) for \( N = 4 \). We observe agreement in the large \( N \) limit and also in leading order \( \sim \lambda^{-1} \). The constant \( C \) computed from (7.14) obtains as \( C = -2.8 \) and deviates from the lattice results. This may due to our neglect of nonleading terms in the \( \beta \)-function for large \( \lambda \) and in an inaccurate matching between the strong coupling and weak coupling regimes. Also the lattice definition of \( \Lambda \) is not identical with our definition. If we include in the strong coupling \( \beta \)-function (7.4) a nonleading term \( \delta \cdot \lambda \) and determine \( \delta \) such that at \( \lambda_{\text{tr}} \) both \( \beta_\lambda \) and \( \frac{\partial \beta}{\partial \lambda} \) match for the strong and weak coupling regimes, we find (for \( \gamma = -1 \)) \( \delta = 0.26, C = -1.8 \).

iv) The upper bound on \( \lambda(k) \) translates into an upper bound for the scalar mass both in the symmetric phase and in the phase with spontaneous symmetry breaking. This bound depends on the ultraviolet cutoff \( \Lambda \). In the phase with spontaneous symmetry breaking the value of the renormalized field corresponding to the potential minimum goes to a constant \( \rho_0 \) larger than zero for \( k \to 0 \). (The scaling solution with almost constant \( \kappa \) for small enough \( \lambda \) ceases to be valid for \( k \) smaller than some characteristic scale of the order \( k^2 \simeq \lambda(k) \rho_0 \).) The renormalized mass is then approximately given\(^7\) by

\[
M^2 = 2\lambda(\rho_0)\rho_0.
\]

With \( \sqrt{2\rho_0} \simeq 246\text{GeV} \) we obtain an upper bound \( M_{\text{max}} \) as a function of \( \Lambda \) by inserting the upper bound for \( \lambda(\rho_0) \) from eq. (7.14). It is qualitatively similar to the lattice results [13], [16] and deviates only in an overall factor of the constant \( C \).

v.) One should note that \( \beta_\lambda \) can be written as a function of \( \lambda \) on a given trajectory, i.e. for given initial values of \( \Gamma_\lambda \). This is achieved by expressing all other couplings on the r.h.s. of (7.11) in terms of \( \lambda \) by inserting the solution \( \lambda(k) \), i.e. \( u_3(\lambda) = u_3(k(\lambda)) \) etc., provided \( \lambda(k) \) is a monotonic function. We emphasize, however, that this function \( \beta_\lambda(\lambda) \) is not a universal function. In general the details of \( \beta_\lambda(\lambda) \) depend on the particular initial values of the couplings specifying \( \Gamma_\lambda \). Only in the vicinity of a fixpoint where \( \lambda \) varies only slowly, \( \kappa \) is fixed by its (\( \lambda \)-dependent) critical value corresponding to the critical trajectory, and all other couplings take their values at infrared fixpoints as functions of \( \lambda \), the function \( \beta_\lambda(\lambda) \) becomes universal. In four dimensions this happens for small \( \lambda \) (the vicinity of the Gaussian fixpoint). In contrast, for very large values of \( \lambda \) the evolution is extremely rapid and one is far away from any fixpoint behaviour. Details of \( \beta_\lambda(\lambda) \) for large \( \lambda \) are therefore indeed expected to depend on details of the short distance action \( \Gamma_\lambda \). The problem of estimating \( \beta_\lambda(\lambda) \) for the most general form of \( \Gamma_\lambda \) is certainly not solvable. In extreme cases \( \Gamma_\lambda \) may correspond to a theory in a universality class completely different from the \( \varphi^4 \)-theory. For example, the effective action \( \Gamma_\lambda \) for a theory with light fermions and Yukawa couplings to the scalars can be written in a form of a (nonlocal) action involving only scalar fields by integrating out the fermionic degrees of freedom. This would certainly lead to a different function \( \beta_\lambda(\lambda) \) even for small values of \( \lambda \).

\(^7\)We define the mass for nonvanishing momentum in order to avoid infrared problems from the Goldstone bosons. Since \( \lambda(k) \) changes slowly for small enough \( \lambda \) we can take \( \lambda(\rho_0) \) instead of the appropriate four point function at nonvanishing momentum with a good accuracy.
contributions from Yukawa couplings would arise.) We will be satisfied to establish that the qualitative behaviour of (7.7) holds for a large class of initial values $\Gamma_\Lambda$ within the range of attraction of the Gaussian fixpoint. The discussion of the validity of the assumptions leading to (7.7) will also give information on this range of attraction. For initial values in the vicinity of the boundary of the range of attraction of the Gaussian fixpoint we expect qualitatively different evolution. No "triviality bound" of the type discussed before can be derived in this case.

Let us now discuss the assumptions leading to (7.7) in more detail. The most important one was that $\kappa$ is not of the order $\lambda^{-1}$ or smaller such that contributions from the radial mode $\sim s_n^d(2\lambda\kappa)$ can be neglected. For $\kappa \sim \lambda^{-1}$ the leading contribution to the evolution of $\kappa$ would be given by (3.18)

$$\frac{\partial \kappa}{\partial t} = \frac{N-1}{16\pi^2} \left( 1 + \frac{z_1}{\lambda} \right).$$

(7.17)

This would drive $\kappa$ very quickly to zero and we conclude that values $\kappa \sim \lambda^{-1}$ or smaller correspond to the symmetric phase. We are interested in the phase with spontaneous symmetry breaking near the phase transition. Here $\kappa$ cannot be smaller than of order one. Our next worry is the value of the dimensionless $\phi^6$-coupling $u_3$. If $u_3$ would be of the order $\lambda^3$ the suppression due to the threshold functions is insufficient and $\beta_3$ would obtain contributions $\sim \lambda^3$. The validity of the approximation (7.2) requires that $u_3$ is at most of the order $\lambda^2$. At first sight this seems to contradict the one loop estimate (4.6). We should remember, however, that large values of $\lambda$ correspond to a situation with rapidly varying couplings, in contrast to the approximative fixpoint behaviour for small and slowly varying $\lambda$. For an estimate of $u_3$ we integrate its leading order evolution equation which may be obtained from eq. (3.6) after derivation with respect to $\rho$ and introduction of the dimensionless renormalized couplings (3.17)

$$\frac{\partial u_3}{\partial t} = -4v_d l_3^d(N-1)\lambda^3 + 6v_d l_2^d(N-1)\lambda u_3.$$  \hspace{2cm} (7.18)

As long as $u_3$ is small compared to $\lambda^2$ we can use the solution (7.11). The first term in (7.18) leads to an increase in $u_3$ until it reaches a value $\sim \lambda^2$. Then the second term in (7.18) becomes important and $u_3/\lambda^2$ is driven to a zero of the r.h.s. of

$$\frac{\partial}{\partial t} \frac{u_3}{\lambda^2} = \lambda^{-2} \frac{\partial u_3}{\partial t} - 2\lambda^{-3} \frac{\partial \lambda}{\partial t} = \frac{N - 1}{16\pi^2} \lambda \left( -2l_3^d + l_2^d \frac{u_3}{\lambda^2} \right).$$  \hspace{2cm} (7.19)

(for $d = 4, \gamma = -1$). We note that the fixpoint

$$\left( \frac{u_3}{\lambda^2} \right)_* = \frac{2l_3^d}{l_2^d}$$

(7.20)
corresponds now to a value of $u_3$ of the order $\lambda^2$. The difference compared to (4.3) arises from the second term in (7.18) which has been neglected in sect. 4. By similar arguments it is easy to establish that $u_4$ is at most of the order $\lambda^3$ in agreement with our approximations.

Let us finally make an estimate for the moments $z_1$ and $z_2$. The one loop estimate (5.12) suggests values $z_1 \sim \lambda^2, z_2 \sim \lambda^3$ in contradiction to the validity of our approximations. If we write the evolution equations for $z_i$ in the form

$$\frac{\partial}{\partial t} z_i = -A_i(\lambda) + B_i(\lambda) z_i + \ldots$$

(7.21)
only the terms \( A_1 \sim \frac{\lambda^2}{16\pi} \), \( A_2 \sim \frac{\lambda^2}{16\pi} \) are directly related to the one loop formulae of sect. 5. (For the discussion of section 5 it is implicitly assumed that \( B_i \) is dominated by the canonical dimension of \( Z', Z'' \). This is only valid for small \( \lambda \).) In leading order the evolution equation for \( Z_k'(p_0, Q^2 = 0) \) and \( Z_k''(p_0, Q^2 = 0) \) \((6.7)\), \((6.8)\) yields

\[
\frac{\partial}{\partial t}(z_1\kappa) = 2v_d\lambda \left( \frac{4}{d} m_3^d + (N - 1) l_2^d z_1\kappa \right), \tag{7.22}
\]

\[
\frac{\partial}{\partial t}(z_2\kappa) = 4v_d(N - 1) l_2^d z_2\kappa \lambda
\]

\[
- 4v_d\lambda^2 \left( \frac{6}{d} m_4^d + (N - 1) l_3^d z_1\kappa \right)
\]

\[
+ 2v_d u_3 \left( \frac{4}{d} m_3^d + (N - 1) l_2^d z_1\kappa \right). \tag{7.23}
\]

Here we have approximated \( L_{1,1}^d(w_1, w_2) \simeq \frac{1}{w_2} L_{1,0}^d(w_1) \) and similar for other threshold functions and we have omitted terms \( \sim Y_k \). (This becomes exact in the large \( N \) limit). We conclude that for large \( \lambda \) there are infrared fixpoints in \( z_1\kappa \) and the ratio \( z_2\kappa/\lambda \)

\[
(z_1\kappa)_* = -\frac{4m_3^d}{(N - 1) dl_2^d} = -\frac{1}{N - 1}, \tag{7.24}
\]

\[
\left( \frac{z_2\kappa}{\lambda} \right)_* = \frac{2}{(N - 1) dl_2^d} \left( 3m_1^d - 2\frac{l_2^d}{l_3^d} m_3^d \right) = -\frac{1}{N - 1} \left( 3\ln 4 - \frac{3}{4} \right) \tag{7.25}
\]

(Here we have inserted the fixpoint value for \( u_3 \) \((7.20)\) and the last part of the equations is evaluated for \( d = 4 \) with constants \( l_3^d, m_3^d \) given in appendix C.) From this we infer that \( |z_1| \)

grows within the regime where \( \lambda \) is large at most to a value \( \sim O(1) \) and \( |z_2| \) becomes at most \( \sim O(\lambda) \). In consequence the last term in \((7.2)\) can indeed be neglected in leading order in the regime of very strong quartic scalar coupling. With the estimations

\[
|\kappa| \leq O(1) \quad , \quad |u_3| \leq O(\lambda^2) \quad , \quad |u_4| \leq O(\lambda^3),
\]

\[
|z_1| \leq O(1) \quad , \quad |z_2| \leq O(\lambda) \tag{7.26}
\]

and \( s_n^d(2\lambda\kappa) \sim O(\lambda^{-(n+1)}) \) we find that the approximation \((7.3)\) is indeed valid provided

\[
|\tilde{z}_1| \leq O(\lambda^2) \quad , \quad |\tilde{z}_2| \leq O(\lambda^3). \tag{7.27}
\]

We therefore quote the evolution equation for \( \tilde{Z}_k \) which has been computed in sect. 5 of \([4]\) and reads in leading order for large \( \lambda \)

\[
\frac{\partial \tilde{Z}_k(\rho)}{\partial t} = v_d k^{d-2} (N - 1) Y_k'(\rho) \rho L_{1,0}^d(U_k'(\rho)) + 4v_d k^{d-6} (N - 1) U_k''(\rho) \rho M_{1,0}^d(\rho) L_{2,0}^d(U_k'(\rho))
\]

\[-2v_d k^{d-4} (N - 1) Y_k'(\rho) U_k''(\rho) \rho L_{2,0}^d(U_k'(\rho)). \tag{7.28}
\]

Expressed in dimensionless renormalized couplings we obtain for the scale dependence of \( \tilde{Z} \) and its derivatives in leading order (\( \tilde{z}_0 \equiv \tilde{Z}/Z \))

\[
\frac{\partial \tilde{z}_0}{\partial t} = -2v_d(N - 1) l_2^d y_1\kappa - \frac{8v_d}{d} (N - 1) m_1^d \lambda^2 \kappa
\]

\[+ 4v_d(N - 1) l_2^d y_0 \lambda \kappa, \tag{7.29}
\]
\[ \frac{\partial \tilde{z}_1(\rho)}{\partial t} = -2v_d(N-1)l_1^d y_2 \kappa - \frac{16v_d}{d} (N-1)m_4^d u_3 \lambda \kappa \\
+4v_d(N-1)l_2^d y_3 \kappa + 6v_d(N-1)l_2^d y_1 \lambda \kappa \\
+32v_d(N-1)m_5^d \lambda^3 \kappa - 8v_d(N-1)l_3^d y_0 \lambda^2 \kappa \] (7.30)

and

\[ \frac{\partial \tilde{z}_2(\rho)}{\partial t} = -2v_d(N-1)l_1^d y_3 \kappa + 8v_d(N-1)l_2^d y_2 \kappa \lambda \\
+10v_d(N-1)l_3^d y_1 u_3 - \frac{16v_d}{d} (N-1)m_4^d \left( u_3^2 + u_4 \lambda \right) \kappa \\
+\frac{160v_d}{d} (N-1)m_5^d u_3 \lambda^2 \kappa + 4v_d(N-1)l_2^d u_4 y_0 \kappa \\
-20v_d(N-1)l_2^d y_1 \lambda \kappa^2 - 24v_d(N-1)l_3^d u_3 y_0 \kappa \lambda \\
-\frac{160v_d}{d} (N-1)m_6^d \lambda^4 \kappa + 24v_d(N-1)l_3^d y_0 \kappa \lambda^3. \] (7.31)

We recognize that \((y_0 = k^{d-2} Z^{-2} Y)\)

\[ \tilde{z}_n = \kappa y_n \] (7.32)

holds in leading order and that the system above exhibits fixpoints of the orders

\[ (y_n)_* \sim \lambda^{n+1}. \] (7.33)

These results agree with our assumptions made above.

The only point missing for a proof that eq. (7.7) is indeed the leading contribution for large \(\lambda\) concerns the neglection of the \(Q^2\)-dependence of \(Z_k(\rho, Q^2)\) and \(\eta\). If this momentum dependence contributes only in nonleading order in \(\lambda\) one obtains \(\gamma = -1\) for \(d = 4\). It is also conceivable, however, that the momentum dependence modifies the value of \(\gamma\) and this is the reason why we have kept \(\gamma\) as a free parameter at the present stage. In principle, these questions can be answered by establishing an appropriate fixpoint behaviour but we have not performed this task in the present paper. It may even be hoped that rigorous bounds on the maximal size of the various quantities appearing on the r.h.s. of (7.3) could be proven with the use of exact evolution equations, in a spirit similar to our discussion for \(u_3, z_1\) and \(z_2\). Even without going so far we believe that our estimate (7.7) for \(\beta_\lambda\) is reliable for very large values of \(\lambda\), with a certain uncertainty in the region of intermediate \(\lambda\). (There is disagreement about the behaviour of \(\beta_\lambda\) for large \(\lambda\) in the literature [18].) These references find the asymptotic behaviour \(\beta_\lambda \sim \lambda, \beta_\lambda \sim \lambda^2\) and \(\beta_\lambda \sim \lambda^3\). Excluding \(\beta\) to be almost zero for intermediate \(\lambda\) this behaviour is sufficient to derive the "triviality bound" (7.16) for the mass of the Higgs scalar. We emphasize that our treatment establishes the existence of a triviality bound without the use of a lattice regularization. For example, we could use a smooth ultraviolet cutoff \(\Lambda\) within our formulation in continuous space. This determines the "initial value" \(\Gamma_\Lambda\) for a given bare theory. Triviality bounds appear for a wide class of bare actions for which \(\Gamma_\Lambda\) obeys (7.26) and (7.27) and similar restrictions for the momentum dependence of the vertices in \(\Gamma_\Lambda\).

8 Conclusion

We have computed the \(\beta\)-function in order \(\sim \lambda^3\) and the anomalous dimension in order \(\sim \lambda^2\) without ever doing a two loop calculation. We used instead an exact evolution equation which
has a one loop form, with classical propagator replaced by the full propagator and solved it using a nonperturbative method which combines exact evolution equations for "independent" couplings with renormalization group improved one loop expressions for "secondary" ones. It is impressive to see how different properties of the propagator in a background field combine to reproduce the perturbative two loop result in next to leading order in the small quartic coupling.

We emphasize that in the course of our calculation we have in addition gained a lot of information on the momentum dependence of various 1PI Green functions. In contrast to the more formal perturbative loop expansion the different contributions to the anomalous dimension and the $\beta$-function for the quartic coupling are here directly connected with physical properties, i.e. the behaviour of Green functions.

Even though the reproduction of perturbative two loop results by a one loop calculation is interesting by its own, this was not the main motivation of the present work. The two loop results are known since a long time and the present method does not offer calculational simplification in this respect. The main interest lies in the fact that our method is by no means restricted to an expansion in a small coupling $\lambda$. The exact evolution equation is valid for arbitrary values of the quartic coupling. We have demonstrated this by computing the $\beta$-function also for very large values of $\lambda$. Our results confirm the lattice results for the triviality bound for the Higgs scalar mass in the context of a Euclidean field theory formulated in continuous space.

A computational method giving reliable results also for large $\lambda$ is particularly important in dimensions less than four, where it is well known that the critical behaviour is governed by a large value of the infrared fixpoint for $\lambda$. Our method can be directly applied to arbitrary dimensions and arbitrary values of the coupling. The main modification as compared to the present calculation is the size of the dimensionless mass term $2\lambda\kappa$ in the propagator. It cannot be treated as a small quantity anymore and the threshold functions have to be evaluated numerically. In view of the nonperturbative applications it is crucial to have a consistent nonperturbative scheme for an approximative solution of the exact evolution equation. This was presented in the present paper and amounts to a renormalization group improved one loop calculation of the "secondary" couplings. The successful precision test of our systematic truncation scheme offers the exciting prospect of precision calculations for the critical behaviour in two and three dimensions or the high temperature phase transition in four dimensions.
A One loop calculation of the wave function renormalization

In this section we perform a one loop calculation of the wave function renormalization. From eq. (3.1) we get after functional derivation and inserting a configuration with constant scalar field \( \phi_1 = \sqrt{2\rho} \), \( \varphi_a = 0 \) for \( a \neq 1 \)

\[
\left( \Gamma^{(2)}_k (Q, -Q; \rho) \right)_{ab} = U_k' (\rho) \delta_{ab} + 2\rho U''_k (\rho) \delta_a \delta_b  
+ \rho Q^2 \left[ Z_k (\rho, Q^2) \delta_{ab} + \rho Y_k (\rho, Q^2) \delta_{a1} \delta_{b1} \right].
\]  

(A.1)

From this we see

\[
Q^2 Z_k (\rho, Q^2) = \left( \Gamma^{(2)}_k (Q, -Q; \rho) \right)_{22} - \left( \Gamma^{(2)}_k (0, 0; \rho) \right)_{22},
\]

\[
Q^2 \tilde{Z}_k (\rho, Q^2) = \left( \Gamma^{(2)}_k (Q, -Q; \rho) \right)_{11} - \left( \Gamma^{(2)}_k (0, 0; \rho) \right)_{11}
\]  

(A.2)

and

\[
Z_k (\rho, Q^2 = 0) = \lim_{Q^2 \rightarrow 0} \frac{\partial}{\partial Q^2} \left( \Gamma^{(2)}_k (Q, -Q; \rho) \right)_{22},
\]

\[
\tilde{Z}_k (\rho, Q^2 = 0) = \lim_{Q^2 \rightarrow 0} \frac{\partial}{\partial Q^2} \left( \Gamma^{(2)}_k (Q, -Q; \rho) \right)_{11}.
\]  

(A.3)

We consider the difference

\[
Z_k (\rho, Q^2) - Z_k (0, 0) = \frac{1}{Q^2} \left[ \left( \Gamma^{(2)}_k (Q, -Q; \rho) \right)_{22} - \left( \Gamma^{(2)}_k (0, 0; \rho) \right)_{22} \right] - \frac{\partial}{\partial Q^2} \left( \Gamma^{(2)}_k (Q, -Q; \rho) \right)_{22} \bigg|_{Q^2 = 0}
\]  

(A.4)

as one of the "secondary" couplings \( \Gamma^{(n)}_k \) in the sense of section 2 and similar for \( \tilde{Z}_k \). (Note \( \tilde{Z}_k (0, 0) = Z_k (0, 0) \).) In order to evaluate the one loop formula (2.6) we approximate \( \Gamma^{(2)}_k \) on the r.h.s. by the second functional derivative of the expression

\[
\Gamma_k [\varphi] = \int d^d x \left\{ U_k (\rho) + \frac{1}{2} Z_k \partial_{\mu} \varphi^a \partial^\mu \varphi_a \right\}.
\]  

(A.5)

This yields

\[
Z_k (\rho, Q^2) = Z_k (0, 0) - 2 \frac{U''_k (\rho)}{Q^2} (2\pi)^{-d} \int d^d q \left( P(q) + U'_k (\rho) + 2U''_k (\rho) \rho \right)^{-1} 
\cdot \left[ (P(q + Q) + U'_k (\rho))^{-1} - (P(q) + U'_k (\rho))^{-1} \right],
\]

\[
\tilde{Z}_k (\rho, Q^2) = Z_k (0, 0) - \frac{\rho}{Q^2} (2\pi)^{-d} \int d^d q \left\{ \frac{3U''_k (\rho) + 2U'''_k (\rho) \rho}{P(q) + U'_k (\rho) + 2U''_k (\rho) \rho}
\cdot \left[ (P(q + Q) + U'_k (\rho) + 2U''_k (\rho) \rho)^{-1} - (P(q) + U'_k (\rho) + 2U''_k (\rho) \rho)^{-1} \right] 
+ \frac{(N - 1)U''_k (\rho)}{P(q) + U'_k (\rho)} \left[ (P(q + Q) + U'_k (\rho))^{-1} - (P(q) + U'_k (\rho))^{-1} \right] \right\}
\]  

(A.6)

and

\[
Z_k (\rho, 0) = Z_k (0, 0) + 8 \frac{U''_k (\rho)}{d} \int_0^\infty dx \frac{d}{d^2} \left( P + U'_k + 2U''_k \rho \right)^{-2} (P + U'_k + 2U''_k \rho)^{-2},
\]

\[
\tilde{Z}_k (\rho, 0) = Z_k (0, 0) + 4 \frac{U''_k (\rho)}{d} (3U'_k + 2U''_k \rho)^2 \rho \int_0^\infty dx \frac{d}{d^2} \left( P + U'_k + 2U''_k \rho \right)^{-4}
\]

\[+ 4(N - 1) \frac{U''_k (\rho)}{d} \int_0^\infty dx \frac{d}{d^2} \left( P + U'_k \right)^{-4}
\]

(A.7)
where \( x = q^2, \dot{P} = \frac{\partial P}{\partial q} \) and \( P \) is given by (3.25). The one loop expressions eqs. (A.6) and eqs. (A.7) are ultraviolet finite in any dimension less than six. The renormalization group improved one loop approximation should therefore only be used for \( d < 6 \).

## B Evaluation of some integrals

In this section we evaluate the integrals \( I(Q^2) \) and \( J(Q^2) \) appearing in section 5 in four dimensions. Throughout this section we will use the inverse propagator (3.25) for computations, i.e. we will set \( Z = 1 \). We start evaluating the \( I(Q^2) \) which is defined in eq. (5.8) and reads, with \( a = k^2 \alpha, b = k^2 \beta \) in eq. (5.8)

\[
I(Q^2) = \frac{1}{2} (2\pi)^{-4} \int d^4 q P^{-1}(q) \left[ P^{-1}(q + Q) - P^{-1}(q) \right]
= v_4 \int_0^1 \int_0^1 \frac{d\alpha}{b} (a + b)^{-2}.
\]  

(B.1)

We transform \( y = \frac{Q^2}{\alpha + b}, dy = -\frac{Q^2}{\alpha + b} da \) and perform the \( y \)-integration

\[
I(Q^2) = v_4 k^2 \frac{1}{Q^2} \int_0^1 \frac{1}{b} \left( 1 - e^{-\frac{Q^2}{k^2 (\alpha + b)}} - \frac{Q^2}{k^2 b} \right) + v_4 \ln 2.
\]

(B.2)

In terms of \( z = \frac{b}{\alpha + b} \) one finds

\[
I(Q^2) = v_4 k^2 \frac{1}{Q^2} \int_0^1 d\alpha \left( z^{-2} - \frac{Q^2}{k^2 z^2} - z^{-2} e^{-\frac{Q^2}{k^2 z^2}} \right).
\]

(B.3)

Writing the integrand as a Taylor series one finally obtains

\[
I(Q^2) = v_4 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n + 2)! (n + 1)} \left( \frac{1}{2} \right)^{n+1} \left( \frac{Q^2}{k^2} \right)^{n+1}
\]

which is eq. (5.10). Analogously we proceed with the \( J(Q^2) \)-integral. It reads

\[
J(Q^2) = \frac{1}{2} (2\pi)^{-4} \int d^4 q P^{-2}(q) \left[ P^{-1}(q + Q) - P^{-1}(q) \right]
= v_4 k^{-2} \int_0^1 dc \left[ \ln(2 + c) - 2 \ln(1 + c) + \ln c + k^2 \frac{1+c}{1+c} \int_0^1 dzz^{-2} \left( 1 - e^{-\frac{Q^2}{k^2 z^2}} \right) \right].
\]

(B.5)

We consider the double integral \( K \) and integrate by parts

\[
K = k^2 \frac{1}{Q^2} \int_0^1 dc \int_0^{1+c} dzz^{-2} \left( 1 - e^{-\frac{Q^2}{k^2 z^2}} \right)
= k^2 \frac{1}{Q^2} \left\{ -\ln 2 + \int_0^1 dc \left[ e^{-1} \left( 1 - e^{-\frac{Q^2}{k^2 (1+c)}} \right) - e^{-\frac{Q^2}{k^2 (1+c)}} + 2 + c e^{-\frac{Q^2}{k^2 (2+c)}} + \frac{Q^2}{k^2} \int_0^{1+c} dzz^{-2} e^{-\frac{Q^2}{k^2 z^2}} \right] \right\}.
\]

\[27\]
The brackets $[\cdots]$ contain four terms. In the first one we substitute $z = \frac{1}{1+c}$ and expand in a Taylor series

$$
\frac{k^2}{Q^2} \int_0^1 dce^{-\frac{Q^2}{k^2} \frac{c}{1+c}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \left( \frac{Q^2}{k^2} \right)^{n+1} \int_0^1 dz \frac{z^n}{1-z}.
$$

Similarly we proceed with the next two terms

$$
\frac{k^2}{Q^2} \int_0^1 dce^{-\frac{Q^2}{k^2Le}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{Q^2}{k^2} \right)^n \int_0^1 dz \frac{z^n}{(1-z)^2},
$$

$$
\frac{k^2}{Q^2} \int_0^1 dce^{-\frac{Q^2}{k^2Le}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{Q^2}{k^2} \right)^n \int_0^1 dz \frac{z^n}{(1-z)^2}.
$$

In the last term (the double integral) we expand the integrand in a Taylor series and perform the $\int dz$-integration for every term. Again we substitute $z = \frac{c}{1+c}$ and $z = \frac{1+c}{d+c}$

$$
\int_0^1 dc \int_0^{\frac{1+c}{d+c}} \int_0^{\frac{1+c}{d+c}} dz = 6 \ln 2 - 3 \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} \left( \frac{Q^2}{k^2} \right)^{n+1} \left\{ \int_0^1 dz \frac{z^{n+1}}{(1-z)^2} - \int_0^1 dz \frac{z^{n+1}}{(1-z)^2} \right\}.
$$

Summing up these contributions, one finally obtains

$$
J(Q^2) = v_3 k^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \left( \frac{Q^2}{k^2} \right)^{n+1} \left\{ \int_0^1 dz \frac{z^{n+1}}{(1-z)^2} - \int_0^1 dz \frac{z^{n+1}}{(1-z)^2} \right\}
$$

which is eq. (5.11).

C Contributions to $\beta_\lambda$

Finally we add up all the contributions which enter in the $\beta_\lambda$-function (3.23) in order $\lambda^3$. Inserting eqs. (5.12), (6.17) and (6.27) in (3.23) we write

$$
\beta_\lambda = 2v_4 (N+8) \lambda^2 - 4v_4^2 \lambda^3 \left\{ (10N+44) \left[ 2l^4_4 l^4_3 - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)!} \left( \frac{1}{2} \right)^n l^{n+2}_0 \right.ight.

+ 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)!} \left( 3m_{04}^4 l^4_4 - \frac{1}{2} \right) \left. \right] \right.

- (N+2) \left( 3m_{04}^4 l^4_4 - \frac{1}{2} \right) \right\}.
$$

(C.1)

From eqs. (3.11) and (3.20) we find

$$
l^4_n = n \int_0^\infty dy y^{n+1} e^{-y} (1-e^{-y})^{n-1}
$$

(C.2)
where \( y = x/k^2 \) and \( f_k \) from eq. (1.3). These integrals may be integrated by parts and lead to \( \Gamma \)-functions. From this we get

\[
\begin{align*}
l_1^4 &= l_2^4 = 1, \\
l_3^4 &= 3 \ln \frac{4}{3}, \\
l_1^{6+2n} &= (n+2)!, \\
l_2^{6+2n} &= 2(n+1) - \frac{(n+1)!}{2n+1}.
\end{align*}
\]

(C.3)

From eqs. (6.11) and (6.15) we obtain for \( n \geq \frac{d}{2} + 1 \) with \( \partial_t \left( \frac{x_n}{t_m} \dot{P}^2 \right) = -2x \partial_x \left( \frac{x_n}{t_m} \dot{P}^2 \right) \) and integration by parts

\[
m_n^d = \delta_n, \frac{d}{2} + 1 + (n - 1 - \frac{d}{2}) \int_0^\infty dy y^\frac{d}{2} \left( \frac{\partial p}{\partial y} \right)^2 p^{-n}
\]

\[
= \delta_n, \frac{d}{2} + 1 + (n - 1 - \frac{d}{2}) \epsilon_n^d
\]

(C.4)

with

\[
p(y) = \frac{P}{k^2} = \frac{y}{1 - e^{-y}}.
\]

(C.5)

Directly we find

\[m_3^4 = 1\]

(C.6)

and after further integration by parts

\[m_4^4 = \frac{1}{2}.
\]

(C.7)

(Note that the \( \epsilon_n^d \) are closely related to the \( m_n^d \).) Inserting these results in eq. (C.1) it remains performing two sums, namely

\[
\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)(n+2)} x^n = \sum_{n=0}^\infty \frac{(-1)^n}{n+1} x^n - \sum_{n=0}^\infty \frac{(-1)^n}{n+2} x^n
\]

\[
= x^{-1} \ln(1 + x) + x^{-2} [\ln(1 + x) - x]
\]

(C.8)

with \( x = \frac{1}{2}, x = \frac{1}{4} \) and

\[
\sum_{n=0}^\infty \frac{(-1)^n}{n+1} \int_a^b dz \frac{z^{n+1}}{(1-z)^2}
\]

\[
= \int_a^b \frac{dz}{(1-z)^2} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} z^n
\]

\[
= \left[ \frac{\ln(1+z)}{1-z} - \frac{1}{2} \ln(1+z) + \frac{1}{2} \ln(1-z) \right] \bigg|_a^b.
\]

(C.9)

Putting all this together leads to the desired result.
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Figure Caption

Fig. 1: Graphical representation of the exact evolution equation
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