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To cite this version:
Elie Aïdékon, Yueyun Hu, Zhan Shi. An infinite-dimensional representation of the Ray-Knight theorems. Science China Mathematics, 2022, 10.1007/s11425-022-2068-0. hal-03035292

HAL Id: hal-03035292
https://hal.science/hal-03035292
Submitted on 2 Dec 2020

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An infinite-dimensional representation of the Ray–Knight theorems

by

Elie Aïdékon1, Yueyun Hu2, and Zhan Shi3

Summary. The classical Ray–Knight theorems for Brownian motion determine the law of its local time process either at the first hitting time of a given value $a$ by the local time at the origin, or at the first hitting time of a given position $b$ by Brownian motion. We extend these results by describing the local time process jointly for all $a$ and all $b$, by means of stochastic integral with respect to an appropriate white noise. Our result applies to $\mu$-processes, and has an immediate application: a $\mu$-process is the height process of a Feller continuous-state branching process (CSBP) with immigration (Lambert [10]), whereas a Feller CSBP with immigration satisfies a stochastic differential equation driven by a white noise (Dawson and Li [7]); our result gives an explicit relation between these two descriptions and shows that the stochastic differential equation in question is a reformulation of Tanaka’s formula.

Keywords. Ray–Knight theorem, $\mu$-process, white noise, Tanaka’s formula.

2010 Mathematics Subject Classification. 60J65, 60J55.

1 Introduction

Let $(B_t)_{t \geq 0}$ be standard one-dimensional Brownian motion associated with its completed natural filtration $(\mathcal{F}_t)_{t \geq 0}$. Denote by $(\mathcal{L}_t)_{t \geq 0}$ a continuous version of local times of $(B_t)$ at position 0. Let $\mu \in \mathbb{R}\setminus\{0\}$. The $\mu$-process $X := (X_t)_{t \geq 0}$ is defined as follows:

$$X_t := |B_t| - \mu \mathcal{L}_t, \quad t \geq 0.$$ 

There are two important special cases of $\mu$-processes: Brownian motion ($\mu = 1$, this is seen using Lévy’s identity), and the three-dimensional Bessel process ($\mu = -1$, seen by means of Lévy’s and Pitman’s identities).

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The $\mu$-process, also referred to as perturbed reflecting Brownian motion, has attracted much attention in the nineties: Lévy’s arc sine law, Ray–Knight theorems as well as pathwise uniqueness of doubly perturbed Brownian motion, see for example \[12, 22, 3, 4, 18, 21, 5, 6, 14, 15\].

The local time of the $\mu$-process at suitable stopping times, as a process of the space variable, turns out to be a squared Bessel process. This is referred to as a Ray–Knight theorem. More precisely, let us fix $\mu > 0$ from now on, so the process $(X_t, t \geq 0)$ is recurrent on $\mathbb{R}$. Since $X$ is a continuous semimartingale, we may define

$$L(t, r) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{r \leq X_s \leq r + \varepsilon\}} \, ds,$$

as the local time of $X$ at time $t$ and position $r$. Moreover, we may and will take a bicontinuous version of local times $L(\cdot, \cdot)$, see [17], Theorem VI.1.7. Let

$$(1.1) \quad \tau^r_t := \inf\{s \geq 0 : L(s, r) > t\},$$

be the inverse local time of $X$. Denote by

$$(1.2) \quad T_r := \inf\{t \geq 0 : X_t = r\},$$

the hitting time of $r$. The following Ray–Knight theorems were established by Carmona, Petit and Yor [4] (see also Yor [22], Chapter 9) and by Le Gall and Yor [12] respectively.

**Theorem 1.1.** Fix $\mu > 0$.

(i) ([4], [22]) Let $a > 0$. The process $(L(\tau^0_a, \cdot), h \geq 0)$ is a squared Bessel process of dimension $2 - \frac{2}{\mu}$, starting from $a$ and absorbed at 0.

(ii) ([12]) Let $b < 0$. The process $(L(T_{b}, \cdot + h), 0 \leq h \leq |b|)$ is a squared Bessel process of dimension $\frac{2}{\mu}$, starting from 0 and reflected at 0.

In the special case $\mu = 1$: the process $X$ is Brownian motion by Lévy’s identity, so Theorem 1.1 boils down to the classical Ray–Knight theorem for Brownian motion, originally proved by Ray [16] and Knight [9] independently. Werner [21] gave an alternative proof of Theorem 1.1 using a result of Lamperti [11] on semi-stable Markov processes. Perman [14] gave another proof of (i) by establishing a path–decomposition result of $X$.

The aim of this work is to describe the underlying Brownian motion, jointly for all $a$ and $b$, in the local time processes in Theorem 1.1. We do this by means of Tanaka’s formula and Walsh’s stochastic integral with respect to a white noise $W$; see Theorem 1.2 below. The idea of using Tanaka’s formula to prove Ray–Knight theorems is not new, and can be found for example in Jeulin [8] (for diffusion processes) and in Norris, Rogers and Williams [13] (for Brownian motion with a local time drift); our main contribution is to show how the white noise $W$ explicitly gives the Brownian part jointly for all $a$ and $b$ in Theorem 1.1.
The aforementioned white noise $W$ is defined as follows. For any Borel function $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}_+} d\ell \int_{\mathbb{R}} g^2(\ell, x) dx < \infty$, let

$$W(g) := \int_0^\infty g(L(t, X_t), X_t) \text{sgn}(B_t) dB_t.$$  

(1.3)

It is easily seen that $W$ is a white noise on $\mathbb{R}_+ \times \mathbb{R}$; indeed, by the occupation time formula (Exercise VI.1.15 in [17]),

$$\int_0^\infty g^2(L(t, X_t), X_t) dt = \int_{\mathbb{R}} dx \int_0^\infty g^2(L(t, x), x) dt L(t, x) = \int_{\mathbb{R}} dx \int_0^\infty g^2(\ell, x) d\ell.$$  

The exponential martingale for Brownian motion implies that

$$E [e^{W(g) - \frac{1}{2} \int_0^\infty g^2(L(t, X_t), X_t) dt}] = 1,$$

showing that $W(g)$ is a centered Gaussian random variable with variance $\int_{\mathbb{R}_+} \int_{\mathbb{R}} g^2(\ell, x) dx d\ell$.

The main result of this work is the following theorem:

**Theorem 1.2.** Fix $\mu > 0$. Let $W$ be the white noise defined via (1.3).

(i) Almost surely for all $a > 0$,

$$L(\tau_a^0, -h) = a - 2 \int_{-h}^0 W([0, L(\tau_a^0, x)], dx) + \left( 2 - \frac{2}{\mu} \right) h, \quad h \in [0, |I_{\tau_a^0}|],$$  

(1.4)

where $I_t := \inf_{0 \leq s \leq t} X_s$, $t \geq 0$, denotes the infimum process of $X$.

(ii) Almost surely for all $b < 0$,

$$L(T_b, b + h) = 2 \int_b^{b+h} W([0, L(T_b, x)], dx) + \frac{2}{\mu} h, \quad h \in [0, |b|].$$  

(1.5)

The precise meaning of stochastic integrals with respect to $W$ is given in Section 3. Indeed, we will show that almost surely (1.4) holds for any fixed $a > 0$, hence for all $a$ belonging to a countable dense set of $\mathbb{R}_+$. By using the regularity of local times, we may and will choose a version of stochastic integral such that (1.4) holds simultaneously for all $a > 0$. The same remark applies to (1.5) as well as to Theorem 5.1 in Section 5.

It is not surprising, at least in the case (ii), that the local times of a $\mu$-process can be represented as solution of an SDE driven by a white noise. As a matter of fact, by duality (see [21]), the process $(X_{T_{b+t}} - b, t \in [0, T_b])$ has the same law as the process $|B_t| + \mu \xi_t$, stopped when leaving $|b|$ for the last time. On one hand, the process $(|B_t| + \mu \xi_t, t \geq 0)$ is the height process of a Feller CSBP with immigration (see [10], remark p. 57 in Section 4). On the other hand, Bertoin and Le Gall showed in [1] that general CSBPs are related to flows of subordinators, constructed in [2] critical CSBPs without Gaussian coefficient as solutions of SDEs driven by compensated Poisson random measures. Dawson and Li [7] generalized this SDE to include a Gaussian coefficient and possible immigration. Applied
to our setting, it is shown that a Feller CSBP with immigration can be constructed as a solution of (1.5). Theorem 1.2 connects directly the local times of the $\mu$-process to equation (1.5), without making use of the framework of CSBPs, and in Section 4 we are going to see Theorem 1.1 as a consequence of Tanaka’s formula for $X$.

The rest of the paper is as follows. In Section 2, we follow Walsh [19] by introducing the excursion filtration, then make an enlargement of the filtration à la Jeulin [8]. Section 3 is devoted to study of the martingale measure associated with the white noise $W$. In particular, stochastic integration with respect to $W$ is defined. Theorems 1.2 and 1.1 are proved in Section 4. Sections 5 presents analogous results for the $\mu$-process defined on $\mathbb{R}$.

## 2 The excursion filtration

We first introduce some notation which will be used throughout the paper.

**Notation 2.1.** Let $x \in \mathbb{R}$. We define the process $X^{-x}$ obtained by gluing the excursions of $X$ below $x$ as follows. Let, for $t \geq 0$,

$$A_t^{-x} := \int_0^t 1_{\{X_s \leq x\}}ds, \quad \alpha_t^{-x} := \inf\{u > 0, A_u^{-x} > t\},$$

with the usual convention $\inf \emptyset := \infty$. Define

$$X_t^{-x} := X_{\alpha_t^{-x}}, \quad t < A_{\infty}^{-x} := \int_0^\infty 1_{\{X_s \leq x\}}ds.$$

Similarly, we define $A_t^{+x}$, $\alpha_t^{+x}$ and $X^{+x}$ by replacing $X_s \leq x$ by $X_s > x$. When the process is denoted by $X$ with some superscript, the analogous quantities keep the same superscript. For example, $L^{+x}(t, y)$ denotes the local time of $X^{+x}$ at position $y$ and time $t$, and $I_t^{+x} = \inf_{0 \leq s \leq t} X_s^{+x}$.

**Remark 2.2.** Let $x \in \mathbb{R}$. One can reconstruct $X$ from $X^{-x}$ and $X^{+x}$ by gluing the excursions of $X^{-x}$ and of $X^{+x}$, indexed by their local time.

The following proposition is adapted from Section 8.5 of [22].

**Proposition 2.3.** Let $x \leq 0$.

(i) Define the filtration $(\mathcal{F}_u^{+x})_{u \geq 0}$ by $\mathcal{F}_u^{+x} := \sigma(X_s^{+x}, s \in [0, u])$ and the process

$$\beta_u^{+x} := \int_0^{\alpha_u^{+x}} 1_{\{X_s > x\}}\text{sgn}(B_s)dB_s, \quad u \geq 0.$$

Then $\beta^{+x}$ is $(\mathcal{F}_u^{+x})$-Brownian motion and $X^{+x}$ is an $(\mathcal{F}_u^{+x})$-semimartingale with decomposition

$$X_u^{+x} = \beta_u^{+x} - \frac{1 - \mu}{\mu}I_u^{+x} + \frac{1}{2}L^{+x}(u, x), \quad u \geq 0.$$
(ii) Define the filtration \((\mathcal{F}_u^{-x})_{u \geq 0}\) by \(\mathcal{F}_u^{-x} := \sigma(X_s^{-x}, s \in [0, u])\) and the process

\[
\beta_u^{-x} := \int_0^\alpha u sgn(B_s) dB_s, \quad u \geq 0.
\]

Then \(\beta^{-x}\) is \((\mathcal{F}_u^{-x})\)-Brownian motion and \(X^{-x}\) is an \((\mathcal{F}_u^{-x})\)-semimartingale with decomposition

\[
X_u^{-x} = x + \beta_u^{-x} - \frac{1}{\mu} (\mathcal{I}_u^{-x} - x) - \frac{1}{2} L^{-x}(u, x), \quad u \geq 0.
\]

(iii) The Brownian motions \(\beta^{+x}\) and \(\beta^{-x}\) are independent.

**Proof.** By Tanaka’s formula,

\[
(X_t - x)^+ = (X_0 - x)^+ + \int_0^t 1_{\{X_s > x\}} dX_s + \frac{1}{2} L(t, x).
\]

Take \(t = \alpha_u^{+x}\). We get

\[
(2.3) \quad X_u^{+x} = \int_0^{\alpha_u^{+x}} 1_{\{X_s > x\}} dX_s + \frac{1}{2} L(\alpha_u^{+x}, x).
\]

Moreover, \(dX_s = dB_s - \mu d\xi_s = sgn(B_s) dB_s + (1 - \mu) d\xi_s\) by another application of Tanaka’s formula. Also observe that \(I_t = -\mu \xi_t\), hence

\[
(2.4) \quad dX_s = sgn(B_s) dB_s - \frac{1 - \mu}{\mu} dI_s, \quad s \geq 0.
\]

Therefore,

\[
X_u^{+x} = \int_0^{\alpha_u^{+x}} 1_{\{X_s > x\}} sgn(B_s) dB_s - \frac{1 - \mu}{\mu} \int_0^{\alpha_u^{+x}} 1_{\{X_s > x\}} dI_s + \frac{1}{2} L(\alpha_u^{+x}, x).
\]

We notice that \(L^{+x}(u, r) = L(\alpha_u^{+x}, r)\) for any \(r \in [x, \infty)\) and \(u \geq 0\). On the other hand,

\[
\int_0^{\alpha_u^{+x}} 1_{\{X_s > x\}} dI_s = I_{\alpha_u^{+x} \wedge T_x}^{+x},
\]

which is also the infimum of \(X^{+x}\) on the time interval \([0, u]\). This yields (2.1). This equation also implies that \(\beta^{+x}\) is adapted to \(\mathcal{F}^{+x}\). Moreover, from the definition of \(\beta_u^{+x}\) and Proposition V.1.5 of [17], \(\beta_u^{+x}\) is a \((\mathcal{B}_{\alpha_u^{+x}})\)-continuous martingale with \(\langle \beta^{+x}, \beta^{+x} \rangle_u = L^{+x}(u, u)\), hence \((\mathcal{B}_{\alpha_u^{+x}})\)-Brownian motion. Since \(\mathcal{F}^{+x} \subset \mathcal{B}_{\alpha_u^{+x}}\), we deduce that \(\beta_u^{+x}\) is also \((\mathcal{F}_u^{+x})\)-Brownian motion. This proves (i).

The proof of (ii) is similar. Tanaka’s formula applied to \((X_t + T_x - x)^-\) with \(t = \alpha_u^{-x} - T_x\) implies that

\[
X_u^{-x} = x + \int_{T_x}^{\alpha_u^{-x}} 1_{\{X_s \leq x\}} sgn(B_s) dB_s - \frac{1 - \mu}{\mu} \int_{T_x}^{\alpha_u^{-x}} 1_{\{X_s \leq x\}} dI_s - \frac{1}{2} L(\alpha_u^{-x}, x).
\]

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We observe that \( L(\alpha_{u, x}^{-}, x) = L^{-, x}(u, x) \) and \( \int_{R}^{a_{u, x}^{+}} 1_{\{X_{s} \leq x\}} dI_{s} = I_{a_{u, x}^{+}, x}^{-} \) while \( I_{a_{u, x}^{-}, x} = I_{a_{u, x}^{-}, x}^{-} \) which gives (2.2). We conclude as for (i). The statement (iii) is a consequence of Knight’s theorem on orthogonal martingales. \( \square \)

The following result is well-known. It has been proved in Section 8.5 of [22] when \( \mu \in (0, 2) \), in [21] and in [15]. Here, following [22], we choose to see it as a consequence of Proposition 2.3.

**Corollary 2.4.** Let \( x \leq 0 \). The processes \( X^{+, x} \) and \( X^{-, x} \) are independent.

**Proof.** By Proposition 2.3 (iii), the martingale parts of \( X^{+, x} \) and \( X^{-, x} \), namely \( \beta^{+, x} \) and \( \beta^{-, x} \), are independent. It remains to see that \( X^{+, x} \) is measurable with respect to \( \beta^{+, x} \) and \( X^{-, x} \) with respect to \( \beta^{-, x} \), which was established by Chaumont and Doney [5] and Davis [6]. \( \square \)

The excursion filtration, introduced by Walsh [19], is defined as

\[
\mathcal{E}_{x}^{+} := \mathcal{F}_{\infty}^{+, x} = \sigma(X_{s}^{+, x}, s \geq 0), \quad x \in \mathbb{R}.
\]

Similarly we define \( \mathcal{E}_{x}^{-} := \mathcal{F}_{\infty}^{-, x} = \sigma(X_{s}^{-, x}, s \geq 0) \) for \( x \in \mathbb{R} \). It is routine to check, using the time-changes \( \alpha^{-, x} \) and \( \alpha^{+, x} \), that \( \mathcal{E}_{x}^{-} \) is increasing in \( x \) whereas \( \mathcal{E}_{x}^{+} \) is decreasing. \(^4\)

Define, for \( u \geq 0 \),

\[
\mathcal{G}_{u}^{+, x} := \sigma(\mathcal{F}_{u}^{+, x}, \mathcal{E}_{x}^{-}), \quad \mathcal{G}_{u}^{-, x} := \sigma(\mathcal{F}_{u}^{-, x}, \mathcal{E}_{x}^{+}).
\]

The idea of such an enlargement of filtrations goes back at least to Jeulin [8].

**Corollary 2.5.** Consider a random function \( g(\ell, y) = g(\ell, y, \omega) \) such that the process \( t \mapsto g(L(t, X_{t}), X_{t}) \) is \( (\mathcal{B}_{t}) \)-progressively measurable and \( \mathbb{E}\left[ \int_{R_{t}} \int_{R} g(\ell, y)^{2} dy \right] < \infty \). Fix \( x \leq 0 \).

(i) The process \( u \mapsto g(L^{+, x}(u, X_{u}^{+, x}), X_{u}^{+, x}) \) is \( (\mathcal{G}_{u}^{+, x}) \)-progressive and almost surely,

\[
\int_{0}^{\infty} g(L(t, X_{t}), X_{t}) 1_{\{X_{t} > x\}} \text{sgn}(B_{t}) dB_{t} = \int_{0}^{\infty} g(L^{+, x}(u, X_{u}^{+, x}), X_{u}^{+, x}) dB_{u}^{+, x}.
\]

(ii) The process \( u \mapsto g(L^{-, x}(u, X_{u}^{-, x}), X_{u}^{-, x}) \) is \( (\mathcal{G}_{u}^{-, x}) \)-progressive and almost surely,

\[
\int_{0}^{\infty} g(L(t, X_{t}), X_{t}) 1_{\{X_{t} \leq x\}} \text{sgn}(B_{t}) dB_{t} = \int_{0}^{\infty} g(L^{-, x}(u, X_{u}^{-, x}), X_{u}^{-, x}) dB_{u}^{-, x}.
\]

\(^4\)We will be implicitly working with a right-continuous (and complete) version of the filtrations \( (\mathcal{E}_{x}^{-})_{x \in \mathbb{R}} \) and \( (\mathcal{E}_{x}^{+})_{x \in \mathbb{R}} \) — if necessary, by means of the procedure of usual augmentation, as described in Section I.4 of Revuz and Yor [17]. In our work, we only study continuous martingales, which are also martingales with respect to augmented filtrations. The same remark applies to the \( \mu \)-process defined on \( \mathbb{R} \) in Section 5.
Proof. We prove (i). The process \( u \mapsto g(L_{\alpha}^+(u, X_{\alpha}^+, X_{\alpha}^+)) \) is \((\mathcal{B}_\alpha^+, \mathcal{F})\)-progressive (Proposition V.1.4, [17]). Therefore it is also progressive with respect to \((\mathcal{B}_u^+, \mathcal{F})\) because the latter filtration is larger. We prove now (2.5). By a time-change (Proposition V.1.4, [17]),

\[
\int_0^{\alpha_t^+} g(L(s, X_s), X_s) 1_{\{X_s > x\}} sgn(B_s) dB_s = \int_0^t g(L_{\alpha}^+(u, X_{\alpha}^+, X_{\alpha}^+)) dB_u^+.
\]

Letting \( t \to \infty \) yields (2.5). Statement (ii) is proved similarly. \( \square \)

3 The martingale measure associated with \( W \)

Recall the definition of the white noise \( W \) in (1.3). For any Borel set \( A \) of \( \mathbb{R}_+ \) with finite Lebesgue measure and \( r \geq 0 \), we define

\[(3.1) \quad M_r(A) := W(1_{A \times [-r,0]}).\]

Proposition 3.1. In the setting of Walsh [20], \((M_r, r \geq 0)\) is a continuous martingale measure with respect to the filtration \((\mathcal{E}_{-r}^+, r \geq 0)\).

Proof. Since \( W \) is a white noise, it suffices to show that \( M_r \) is measurable with respect to \( \mathcal{E}_{-r}^+ \) and that \( M_s - M_r \) is independent of \( \mathcal{E}_{-r}^+ \) for any \( 0 \leq r < s \). The first statement comes from (2.5) applied to \( x = -r \) and \( g(\ell, y) = 1_{A \times [-r,0]}(\ell, y) \) for a Borel set \( A \) with finite Lebesgue measure. The second statement comes from (2.6) applied to \( x = -r \) and \( g(\ell, y) = 1_{A \times (-s,-r)}(\ell, y) \). Since the processes \( X_{-\cdot}^r \) and \( \beta_{-\cdot}^r \) are independent of \( \mathcal{E}_{-r}^+ \), the proposition follows. \( \square \)

We are going to extend (1.3), seen as an equality for deterministic functions \( g \), to random functions. To this end, we first recall the construction by Walsh in [20] of stochastic integral with respect to the martingale measure \( M \). A (random) function \( f \) is said to be elementary if it is of the form \( f(\ell, x) := Z 1_{[a,b]}(x) 1_A(\ell) \), where \( a < b \leq 0 \), \( A \subset \mathbb{R}_+ \) is a Borel set of finite Lebesgue measure, and \( Z \) is a bounded \( \mathcal{E}_b^+ \)-measurable real-valued random variable. Denote by \( f \cdot M \) the stochastic integral with respect to \( M \):

\[
f \cdot M := Z (M_{[a]}(A) - M_{[b]}(A)) = Z W(1_{A \times [a,b]}).
\]

A simple function is a (finite) linear combination of elementary functions. We extend by linearity the definition of \( f \cdot M \) to simple functions \( f \) and furthermore by isometry to any \( f \in \mathcal{L}_2 \), where \( \mathcal{L}_2 \) denotes the space of \((\mathcal{E}_{-r}^+, r \geq 0)\)-predictable and square-integrable functions, defined as the closure of the space of simple functions under the norm:

\[
\|f\| := \left[ E \left( \int_{\mathbb{R}_+} d\ell \int_{\mathbb{R}_-} f^2(\ell, x) dx \right) \right]^{1/2}.
\]

For any \( f \in \mathcal{L}_2 \), \( f \cdot M \) is a centered random variable with \( E[(f \cdot M)^2] = \|f\|^2 \). We write \( f \cdot M \equiv \int_{\mathbb{R}_+ \times \mathbb{R}_-} f(\ell, x) W(d\ell, dx) \) and for any \( r \geq 0 \), \( (f 1_{\mathbb{R}_+ \times [-r,0]} \cdot M \equiv \int_{\mathbb{R}_+ \times \mathbb{R}_-} f(\ell, x) W(d\ell, dx) \).

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\[ \int_{\mathbb{R}_+ \times [-r,0]} f(\ell, x) W(d\ell, dx). \] The latter, if furthermore \( f \) is of form \( f(\ell, x) = 1_{\{0 \leq \ell \leq \sigma_x\}} \eta_x \), will be re-written as \( \int_0^\infty \eta_x W([0, \sigma_x], dx) \). By the construction of stochastic integral and Proposition 3.1, \( \int_{\mathbb{R}_+ \times [-r,0]} f(\ell, x) W(d\ell, dx) \) is a continuous martingale with respect to the filtration \( (\mathcal{E}^+_r, r \geq 0) \), of quadratic variation process \( \int_{\mathbb{R}_+ \times [-r,0]} f^2(\ell, x) d\ell dx \).

**Proposition 3.2.** Take \( g \in \mathcal{L}^2 \) such that \( s \mapsto g(L(s, X_s), X_s) \) admits a version which is progressive with respect to the Brownian filtration \((\mathcal{B}_s)\). \(^5\) Then

\[ g \cdot M = \int_0^\infty g(L(s, X_s), X_s) \text{sgn}(B_s) dB_s \quad \text{a.s.} \]

**Proof.** By definition of \( \mathcal{L}^2 \), there exists a sequence of simple functions \( g_n \) such that \( \|g - g_n\| \to 0 \) as \( n \to \infty \). By isometry, \( g \cdot M - g_n \cdot M \to 0 \) in \( L^2 \). Since \( g_n \) is a simple function, \( g_n \) is of the form

\[ g_n(\ell, x) = \sum_{k,j=1}^{\infty} Z_{k,j}^n \mathbf{1}_{[a_k^n, a_{k-1}^n]}(x) \mathbf{1}_{A_{k,j}^n}(\ell), \]

where for each \( n \), \( 0 = a_0^n > ... > a_k^n > a_{k+1}^n > ... \) is a decreasing sequence such that \( a_k^n \to -\infty \) as \( k \to \infty \), \((A_{k,j}^n)_{j \geq 1}\) is a collection of (nonrandom) pairwise disjoint Borel subsets of \( \mathbb{R}_+ \) with finite Lebesgue measures, and for any \( k, j \geq 1 \), \( Z_{k,j}^n \) is a bounded \( \mathcal{E}_{a_{k-1}^n}^+ \)-measurable random variable. Moreover for all large \( k, j \), \( Z_{k,j}^n = 0 \), which means the above double sum runs in fact over a finite index set of \( k \) and \( j \).

Note that for a.e. \( z \leq 0 \), \( g(\cdot, z) \) is measurable with respect to \( \mathcal{E}_z^+ \) (as \( g_n \) is). We may (and will) take a version of \( g \) such that \( g(\cdot, z) \) is measurable with respect to \( \mathcal{E}_z^+ \) for all \( z \leq 0 \).

By applying (1.3) and (3.1), we deduce from the linearity of the integral that

\[ g_n \cdot M = \sum_{k,j=1}^{\infty} Z_{k,j}^n \int_0^\infty \mathbf{1}_{[a_k^n, a_{k-1}^n]}(X_s) \mathbf{1}_{A_{k,j}^n}(L(s, X_s)) \text{sgn}(B_s) dB_s. \]

Note that we can (and we will) take \((a_k^n, a_{k-1}^n)\) instead of \([a_k^n, a_{k-1}^n)\) without changing the value of \( g_n \cdot M \). By (2.5) and (2.6) respectively,

\begin{align*}
\int_0^\infty \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_s) \mathbf{1}_{A_{k,j}^n}(L(s, X_s)) \text{sgn}(B_s) dB_s \\
= \int_0^\infty \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^- \mathbf{1}_{-x}) \mathbf{1}_{A_{k,j}^n}(L(u, X_u^- \mathbf{1}_{-x})) d\beta_u^- \\
= \int_0^\infty \mathbf{1}_{(a_k^n, a_{k-1}^n)}(X_u^+ \mathbf{1}_{+y}) \mathbf{1}_{A_{k,j}^n}(L^+(u, X_u^+ \mathbf{1}_{+y})) d\beta_u^+ 
\end{align*}

\(^5\)By *version* we mean a \((\mathcal{B}_s)\)-progressive process \((h_s)\) such that \( \int_0^\infty \mathbf{1}_{(g(L(s, X_s), X_s) \neq h_s)} ds = 0 \) a.s.
with \( x = a_{k-1}^n \) and \( y = a_k^n \). Similarly,

\[
\int_0^\infty g(L(s, X_s), X_s)1_{(a_k^n, a_{k-1}^n)}(X_s) \text{sgn}(B_s)dB_s
\]

(3.4) \[
= \int_0^\infty g(L^{-x}(u, X_u^{-x}), X_u^{-x})1_{(a_k^n, a_{k-1}^n)}(X_u^{-x})d\beta_u^{-x},
\]

(3.5) \[
= \int_0^\infty g(L^{+y}(u, X_u^{+y}), X_u^{+y})1_{(a_k^n, a_{k-1}^n)}(X_u^{+y})d\beta_u^{+y}
\]

with \( x = a_{k-1}^n \) and \( y = a_k^n \) as in (3.2) and (3.3). Write

\[
I(g) := \int_0^\infty g(L(s, X_s), X_s) \text{sgn}(B_s)dB_s.
\]

Then

\[
g_n \cdot M - I(g) = \sum_{k=1}^\infty \Delta_n(k),
\]

where, from (3.3) and (3.5),

\[
\Delta_n(k) := \sum_{j=1}^\infty Z_{k,j}^n \int_0^\infty 1_{(a_k^n, a_{k-1}^n)}(X_u^{+y})1_{A_j}(L^{+y}(u, X_u^{+y}))d\beta_u^{+y}
\]

\[
- \int_0^\infty g(L^{+y}(u, X_u^{+y}), X_u^{+y})1_{(a_k^n, a_{k-1}^n)}(X_u^{+y})d\beta_u^{+y},
\]

where as before \( y = a_k^n \). Since \( Z_{k,j}^n \) is \( \mathcal{E}_{a_{k-1}^n}^+ \)-measurable, hence \( \mathcal{E}_{a_k^n}^+ \)-measurable, we see that the sum over \( j \) in the definition of \( \Delta_n(k) \) is \( \mathcal{E}_{a_k^n}^+ \)-measurable. For the last integral in \( \Delta_n(k) \), we use the fact that for \( z \leq 0 \), \( g(\cdot, z) \) is measurable with respect to \( \mathcal{E}_z^+ \). Note that \( X_u^{+y} \) and \( L^{+y}(\cdot, \cdot) \) are measurable with respect to \( \mathcal{E}_z^+ \). Since \( X_u^{+y} \geq y \) and \( \mathcal{E}_z^+ \) decreases on \( z \), we deduce that \( g(\cdot, X_u^{+y}) \) is \( \mathcal{E}_y^+ \)-measurable, and so is \( g(L^{+y}(u, X_u^{+y}), X_u^{+y}) \). It follows that \( \Delta_n(k) \) is measurable with respect to \( \mathcal{E}_{a_k^n}^+ \).

Now we prove that \( (\Delta_n(k))_{k \geq 1} \) is a martingale difference sequence with respect to the filtration \( (\mathcal{E}_{a_k^n}^+)_{k \geq 1} \). Indeed, using (3.2) and (3.4) instead of (3.3) and (3.5), one can also write

\[
\Delta_n(k) = \sum_{j=1}^\infty Z_{k,j}^n \int_0^\infty 1_{(a_k^n, a_{k-1}^n)}(X_u^{-x})1_{A_j}(L^{-x}(u, X_u^{-x}))d\beta_u^{-x}
\]

\[
- \int_0^\infty g(L^{-x}(u, X_u^{-x}), X_u^{-x})1_{(a_k^n, a_{k-1}^n)}(X_u^{-x})d\beta_u^{-x},
\]

with \( x = a_{k-1}^n \). Recall from Proposition 2.3 that \( \beta^{-x} \) is \( (\mathcal{F}_{u}^{-x}) \)-Brownian motion, which is independent of \( \mathcal{E}_{a_k^n}^+ \) by Corollary 2.4. Then \( \beta^{-x} \) can be seen as \( (\mathcal{G}_{u}^{-x}) \)-Brownian motion.
By Corollary 2.5 (ii), \( g(L^{-x}(u, X_u^{-x}), X_u^{-x}) \) is progressive with respect to the filtration \((\mathcal{G}_u^{-x})\), while

\[
1_{(a_n^u, a_{n-1}^u)}(X_u^{-x}) g_n(L^{-x}(u, X_u^{-x}), X_u^{-x}) = \sum_{j=1}^{\infty} Z_{j,k}^n 1_{(a_n^u, a_{n-1}^u)}(X_u^{-x}) 1_{A_j^r}(L^{-x}(u, X_u^{-x}))
\]

is \((\mathcal{G}_u^{-x})\)-progressive as well. Therefore, one can write

\[
(3.6) \quad \Delta_n(k) = \int_0^\infty 1_{(a_n^u, a_{n-1}^u)}(X_u^{-x}) (g_n - g)(L^{-x}(u, X_u^{-x}), X_u^{-x}) \, d\beta_u^{-x},
\]

with \( x = a_{k-1}^u \). It follows that (since \( \mathcal{G}_0^{-x} = \mathcal{E}_x^+ \))

\[
\mathbb{E}[\Delta_n(k) \mid \mathcal{E}_{a_{k-1}^u}^+] = 0.
\]

In other words, the process \( j \to \sum_{k=1}^j \Delta_n(k) \) is a martingale and we have

\[
\mathbb{E}[(g_n \cdot M - I(g))^2] = \sum_{k=1}^{\infty} \mathbb{E}[\Delta_n(k)^2].
\]

From (3.6), we get

\[
\mathbb{E}[\Delta_n(k)^2] = \mathbb{E} \left[ \int_0^\infty 1_{(a_n^u, a_{n-1}^u)}(X_u^{-x}) ((g_n - g)(L^{-x}(u, X_u^{-x}), X_u^{-x}))^2 \, du \right]
\]

\[
= \mathbb{E} \left[ \int_{a_k^u}^{a_{k-1}^u} (g_n(\ell, z) - g(\ell, z))^2 \, d\ell \, dz \right],
\]

where in the second equality we have used the occupation time formula. It follows that

\[
\mathbb{E}[(g_n \cdot M - I(g))^2] = \|g_n - g\|^2
\]

which goes to 0 as \( n \to \infty \). Thus we get that \( g \cdot M = I(g) \). \( \square \)

**Remark 3.3.** Fix \( b < 0 \). Similarly to (3.1) we may define a martingale measure \( \widehat{M} \) by

\[
\widehat{M}_r(A) := W(1_{A \times [b, b+r]}),
\]

for any Borel set \( A \subset \mathbb{R}_+ \) of finite Lebesgue measure and \( r \geq 0 \). The analogs of Propositions 3.1 and 3.2 hold for \( \widehat{M} \). Specifically, \((\widehat{M}_r, r \geq 0)\) is a martingale measure with respect to the filtration \((\mathcal{E}_{b+r}^+, r \geq 0)\). Moreover, we may define in a similar way the stochastic integral \( f \cdot \widehat{M} \) for any \( f \in \mathcal{L}_2 \), where \( \mathcal{L}_2 \) denotes the space of \((\mathcal{E}_{b+r}^+, r \geq 0)\)-predictable functions \( f \) such that \( \mathbb{E}\left[ \int_{\mathbb{R}_+} d\ell \int_{[b, \infty)} f^2(\ell, x) \, dx \right] < \infty \). Then for any \( g \in \mathcal{L}_2 \)
such that \( s \mapsto g(L(s, X_s), X_s) \) admits a version which is progressive with respect to the Brownian filtration \((\mathcal{B}_s)\), we have

\[
\tag{3.7}
g \cdot \hat{M} = \int_0^\infty g(L(s, X_s), X_s)dX_s - \frac{1}{\mu} \int_{-\infty}^0 g(0, x)dx, \quad \text{a.s.}
\]

With a slight abuse of notation, we shall write \( g \cdot \hat{M} \equiv \int_{\mathbb{R}^+ \times [b, \infty)} g(\ell, x)W(d\ell, dx) \) and \((g1_{\mathbb{R}^+ \times [b, t]} \cdot \hat{M}) \equiv \int_{\mathbb{R}^+ \times [b, t]} g(\ell, x)W(d\ell, dx)\) for any \( t \geq b \). Then \( \int_{\mathbb{R}^+ \times [b, b+r]} g(\ell, x)W(d\ell, dx) \), is an \((\mathcal{E}_{b+r})\)-continuous martingale with quadratic variation process \( \int_{\mathbb{R}^+ \times [b, b+r]} g^2(\ell, x)d\ell dx \) for \( r \geq 0 \).

4 Proofs of Theorems 1.2 and 1.1

By Tanaka’s formula, for any \( r \geq 0 \) and \( x \in \mathbb{R} \),

\[
\tag{4.1}
(X_r - x)^- = (-x)^- - \int_0^r 1_{\{X_s \leq x\}}dX_s + \frac{1}{2}L(r, x).
\]

Part (i) of Theorems 1.2 and 1.1: Applying (4.1) to \( r = \tau_0^a \) gives that \( L(\tau_0^a, x) = 2\int_0^{\tau_0^a} 1_{\{X_s \leq x\}}dX_s \) for all \( x \in \mathbb{R} \). Let \( h \geq 0 \). Taking \( x = -h \) and \( x = 0 \), and using the fact that \( L(\tau_0^a, 0) = a \), we obtain that

\[
L(\tau_0^a, -h) = a - 2\int_0^{\tau_0^a} 1_{\{-h < X_s \leq 0\}}dX_s
\]

\[
= a - 2\int_0^{\tau_0^a} 1_{\{-h < X_s \leq 0\}}\text{sgn}(B_s)dB_s + \left(2 - \frac{2}{\mu}\right)\min(h, |I_{\tau_0^a}|),
\]

where the second equality follows from (2.4) and the fact that \( \int_0^{\tau_0^a} 1_{\{-h < X_s \leq 0\}}dI_s = I_{\min(T_h, \tau_0^a)} = \min(h, |I_{\tau_0^a}|) \).

To deal with the stochastic integral with respect to \((B_s)\) in (4.2), we shall use Proposition 3.2. First we remark that for \( x \leq 0 \), \( L(\tau_0^a, x) \) is measurable with respect to \( \mathcal{E}_x^+ \). In fact, let \( u := A_{\tau_0^a}^+. \) Then \( \alpha_u^{+, x} = \tau_0^a \) as \( \tau_0^a \) is an increasing time for \( A^{+, x} \). Therefore \( L(\tau_0^a, x) = L^{+, x}(A_{\tau_0^a}^+, x) \). Since \( \{A_{\tau_0^a}^{+, x} \geq t\} = \{L(\alpha_t^{+, x}, 0) \leq a\} = \{L^{+, x}(t, 0) \leq a\}, \) we obtain that \( L(\tau_0^a, x) \) is measurable with respect to \( \mathcal{E}_x^+ \).

Let \( g(\ell, x) := 1_{\{0 \leq \ell \leq L(\tau_0^a, x)\}, \{X_0 \leq x\}} \). Using the continuity of local times \( L(\tau_0^a, x) \) on \( x \) and the fact that \( E[\int_{\mathbb{R}^+ \times \mathbb{R}} g^2(\ell, x)dx] < \infty \), we get that \( g \in \mathcal{L}^2 \).

Observe that a.s., \( g(L(s, X_s), X_s) = 1_{\{0 \leq s \leq \tau_0^a, \{X_0 \leq x\}} \), ds-a.e. This follows from the fact that \( \int_0^\infty 1_{\{L(s, X_s) \leq \tau_0^a, X_s \geq 0\}}ds = \int_0^\infty dx \int_0^\infty 1_{\{L(s, X_s) \leq \tau_0^a, x\}}dxL(s, x) = 0 \), by the occupation time formula. Then \( s \mapsto g(L(s, X_s), X_s) \) admits a version which is \((\mathcal{B}_s)\)-progressive and we are entitled to apply Proposition 3.2 to see that

\[
\int_0^{\tau_0^a} 1_{\{-h < X_s \leq 0\}}\text{sgn}(B_s)dB_s = \int_{\mathbb{R}^+ \times (-h, 0]} 1_{\{0 \leq \ell \leq L(\tau_0^a, x)\}}W(d\ell, dx) = \int_{-h}^0 W([0, L(\tau_0^a, x)], dx).
\]
In view of (4.2), this yields Theorem 1.2 (i) for each fixed $a > 0$ and $h \in [0, |I_{\tau_a^0}|]$. Since
the processes are continuous in $h$ and càdlàg in $a$, they coincide except on a null set.

Theorem 1.1 (i) follows quite simply from Theorem 1.2 (i): for given $a > 0$, since $h \mapsto \int_{-h}^{0} W([0, L(\tau^0_a, x)], dx)$ is an $(\mathcal{E}^+_h)_{h \geq 0}$-continuous martingale with quadratic variation
process $\int_{-h}^{0} L(\tau^0_a, x)dx$, it follows from the Dambis–Dubins–Schwarz theorem that there
exists $(\mathcal{E}^+_h)$-Brownian motion $\gamma$ such that

$$\int_{-h}^{0} W([0, L(\tau^0_a, x)], dx) = \int_{0}^{h} \sqrt{L(\tau^0_a, s)}d\gamma_s.$$ 

Going back to (4.2), we see that for all $0 \leq h \leq \inf \{ s \geq 0 : L(\tau^0_a, -s) = 0 \} = |I_{\tau_a^0}|$,

$$L(\tau^0_a, -h) = a - 2 \int_{0}^{h} \sqrt{L(\tau^0_a, s)}d\gamma_s + \left(2 - \frac{2}{\mu}\right)h,$$

proving Theorem 1.1 (i).

**Part (ii) of Theorems 1.2 and 1.1**: Let $b < 0$ and $T_b := \inf \{ t \geq 0 : X_t = b \}$. For
$h \in [0, |b|]$, we get from (4.1) that

$$L(T_b, b + h) = 2h + 2 \int_{0}^{T_b} 1_{\{X_s \leq b+h\}}dX_s$$

(4.3)

$$= 2 \int_{0}^{T_b} 1_{\{X_s \leq b+h\}} \text{sgn}(B_s)dB_s + \frac{2}{\mu}h,$$

where the second equality follows from (2.4) again and the fact that $\int_{0}^{T_b} 1_{\{X_s \leq b+h\}}dI_s = \int_{0}^{T_b} 1_{\{I_s \leq b+h\}}dI_s = -h$.

The main difference with Part (i) is the measurability. As a matter of fact, for any $x \geq b$, $L(T_b, x)$ is $\mathcal{E}^-_x$-measurable: observe that $L(T_b, x) = L^{-,-}(A_{T_b}^{-,-}, x)$ and for any $t \geq 0$,

$$\{A_{T_b}^{-,-} > t\} = \{T_b > \alpha(t, x)\} = \{\inf_{0 \leq s \leq \alpha(t, x)} X_s > b\} = \{\inf_{0 \leq s \leq t} X_s^{-,-} > b\}$$

is $\mathcal{E}^-_x$-measurable. Let $g(\ell, x) := 1_{\{0 < \ell \leq L(T_b, x)\} \times \{b \leq x \leq b+h\}}$. We can check as in (i) that we may apply (3.7) to get that

$$\int_{0}^{T_b} 1_{\{X_s \leq b+h\}} \text{sgn}(B_s)dB_s = \int_{[0, L(T_b, x)] \times [b, b+h]} W(d\ell, dx) = \int_{b}^{b+h} W([0, L(T_b, x)], dx),$$

proving, in view of (4.3), Theorem 1.2 (ii). Furthermore by Remark 3.3, the process

$$h \mapsto \int_{b}^{b+h} W([0, L(T_b, x)], dx)$$

is an $(\mathcal{E}^-_{b+h})$-continuous martingale with quadratic variation
process $\int_{b}^{b+h} L(T_b, x)dx$. This easily yields Theorem 1.1 (ii). □

## 5 Extension to the two-sided $\mu$-process

In this Section, we shall explore the strong Markov property at the hitting times of a
$\mu$-process defined on $\mathbb{R}$ and present an analogue of Theorem 1.2. This result, apart from
its own interest, will be useful in a forthcoming work on the duality of Jacobi stochastic
flows.
Let \((B_t)_{t \in \mathbb{R}}\) be a two-sided Brownian motion, which means that for \(t \leq 0\), \(B_t = B'_{-t}\), where \(B'\) is a standard Brownian motion independent of \((B_t)_{t \geq 0}\). Denote by \((\mathcal{L}_t')_{t \geq 0}\) the local time process at position zero of \(B'\).

Recall that \(X_t = |B_t| - \mu t\), for \(t \geq 0\). For \(t \leq 0\), we let \(X_t := |B'_{-t}| + \mu \mathcal{L}_t\). We call \((X_t)_{t \in \mathbb{R}}\) a two-sided \(\mu\)-process. Fix \(\mu > 0\). Notice that \(X_t \to \infty\) as \(t \to -\infty\), and \(T_r < 0\) when \(r > 0\).

We naturally extend the notation \(T_r := \inf\{t \in \mathbb{R} : X_t = r\}\) for \(r \in \mathbb{R}\),

\[
L(t, x) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\infty}^{t} \mathbf{1}_{\{x \leq X_s \leq x + \varepsilon\}} ds, \quad x \in \mathbb{R},
\]

the local time accumulated by \((X_t, t \in \mathbb{R})\) at position \(x\) up to time \(t\), and

\[
\tau_a^x := \inf\{t \in \mathbb{R} : L(t, x) > a\}, \quad a \geq 0,
\]

the inverse local time at position \(x\). We define now for bounded Borel functions \(g\) with compact support,

\[
W(g) := \int_{-\infty}^{\infty} g(L(t, X_t), X_t) \text{sgn}(B_t) dB_t.
\]

The stochastic integral has to be understood as an integral with respect to the Brownian motion \(B^{(r)} := (B_{t+T_r}, t \geq 0)\) where \(r\) is any positive real such that \(g(\ell, x) = 0\) for all \(x \geq r\) (that \(B^{(r)}\) is a standard Brownian motion comes from the fact that \((B'_{T_r+t}, t \in [0, |T_r|])\) is distributed as \((B_t, t \in [0, T_r])\)). We will see in the following theorem that \(W\) defines a white noise.

Similarly to Notation 2.1, for \(x \in \mathbb{R}\), we can consider the process \((X_{u,-}^x, u \geq 0)\) obtained by gluing the excursions of \(X\) below \(x\): that is we set for \(t \in \mathbb{R}\), \(A_t^{-x} := \int_{-\infty}^{t} \mathbf{1}_{\{X_s < x\}} ds\), \(\alpha_{u,-} := \inf\{t \in \mathbb{R} : A_t^{-x} > u\}\), and \(X_{u,-}^x := X_{\alpha_{u,-}}^x\) for \(u \geq 0\). The excursion filtration \(\mathcal{E}^-\) is defined as \(\mathcal{E}_x^- := \sigma(X_{\cdot,-}^x, u \geq 0)\), for all \(x \in \mathbb{R}\).

For \(\mathcal{E}_x^+\), we let \(A_t^{+,x} := t\) if \(t \leq T_x\), and for \(t > T_x\), \(A_t^{+,x} := T_x + \int_{T_x}^{t} \mathbf{1}_{\{X_s > x\}} ds\). For \(s \in \mathbb{R}\), we set \(\alpha_{s,+} := \inf\{t \in \mathbb{R} : A_t^{+,x} > s\}\), and \(X_{s,+}^x := X_{\alpha_{s,+}}^x\). The excursion filtration \(\mathcal{E}^+\) is defined as \(\mathcal{E}_x^+ := \sigma(X_{\cdot,+}^x, s \in \mathbb{R})\), for all \(x \in \mathbb{R}\). Note that \(\mathcal{E}_x^+\) is decreasing in \(x\) whereas \(\mathcal{E}_x^-\) is increasing.

**Theorem 5.1.** Fix \(\mu > 0\). Equation (5.1) defines a white noise on \(\mathbb{R}_+ \times \mathbb{R}\).

(i) Almost surely for all \(a > 0\), \(r \in \mathbb{R}\) and \(h \in [0, |I_{\tau_a^x} - r|]\),

\[
L(\tau_a^r, r - h) = a - 2 \int_{r-h}^r W([0, L(\tau_a^r, x)], dx) + \left(2 - \frac{2}{\mu}\right) h,
\]

where \(I_t := \inf_{-\infty < s \leq t} X_s, t \in \mathbb{R}\), denotes the infimum process of \(X\).

(ii) Almost surely for all \(a > 0\), \(r \in \mathbb{R}\) and \(h \geq 0\),

\[
L(\tau_a^r, r + h) = a + 2 \int_{r}^{r+h} W([0, L(\tau_a^r, x)], dx) + \frac{2}{\mu} h.
\]
The stochastic integral \( \int_{r-h}^r W([0, L(\tau^r_x, x)], dx) \) is the stochastic integral with respect to the martingale measure \( M^r_h(\bullet) := W(1_{\bullet \leq r-h},) \), \( h \geq 0 \), associated to the filtration \((\mathcal{E}_{r-h})_{h \geq 0}\). The stochastic integral \( \int_{r}^{r+h} W([0, L(\tau^r_x, x)], dx) \) is the stochastic integral with respect to the martingale measure \( \widehat{M}^r_h(\bullet) := W(1_{\bullet \leq r+h},) \), \( h \geq 0 \), associated to the filtration \((\mathcal{E}_{r+h})_{h \geq 0}\).

**Proof.** Notice that for any \( r \in \mathbb{R} \), \( X^r := (X_{T+r-t} - r, t \geq 0) \) is distributed as \((X_t, t \geq 0)\). Therefore we can apply Theorem 1.2 to \( X^r \). As in (1.3), we define \( W^r(\cdot) \) the white noise associated to \( X^r \): for any Borel function \( g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) such that \( \int_{\mathbb{R}_+} d\ell \int_{\mathbb{R}} g^2(\ell, x) dx < \infty \),

\[
W^r(g) := \int_0^\infty g(L^r(t, X^r_t), X^r_t) \text{sgn}(B^r_t) dB^r_t,
\]

where \( L^r(\cdot, \cdot) \) denote the local times of \( X^r \). Let \( \tau_a^{x,(r)} \) be the associated inverse local times. By Theorem 1.2 (i) applied to \( X^r \), for \( a > 0 \),

\[
L^r(\tau^{x,(r)}_a, -h) = a - 2 \int_{-h}^0 W^r([0, L^r(\tau^{x,(r)}_a, x)], dx) + \left(2 - \frac{2}{\mu}\right) h, \quad h \in [0, |I^{(r)}_a|],
\]

where \( I^r_t := \inf_{0 \leq s \leq t} X^r_s, t \geq 0 \), is the infimum process of \( X^r \). Notice that \( \tau^{x,(r)}_a = \tau^{x}_a - T_r, L^r(t, x) = L(t+T_r, r+x) \) for \( x \leq 0 \), and for \( t \geq 0 \), \( I^{(r)}_t = I_{t+T_r} - r \) where \( I_t := \inf_{-\infty < s \leq t} X_s \). Moreover, for any \( x \leq y \),

\[
W^r(1_{A \times [x,y]}) = \int_0^\infty 1_{A \times [x,y]}(L^r(t, X^r_t), X^r_t) \text{sgn}(B^r_t) dB^r_t
\]

where we recall that \( B^r_t = B_{t+T_r}, t \geq 0 \). We deduce that if \( x \leq y \leq 0 \) and \( A \) is a bounded Borel set of \( \mathbb{R}_+ \),

\[
W^r(1_{A \times [x,y]}) = \int_0^\infty 1_{A \times [x,y]}(L(t+T_r, X_{t+T_r} - r), X_{t+T_r} - r) \text{sgn}(B^r_t) dB^r_t = \int_{T_r}^\infty 1_{A \times [r+x, r+y]}(L(t, X_t), X_t) \text{sgn}(B_t) dB_t = W(1_{A \times [r+x, r+y]}).
\]

We deduce that (5.1) defines a white noise on \( \mathbb{R}_+ \times (-\infty, r) \), hence on \( \mathbb{R}_+ \times \mathbb{R} \) since \( r \) can be made arbitrary large. Equation (5.4) becomes

\[
L(\tau^{r}_a, r-h) = a - 2 \int_{r-h}^r W([0, L(\tau^{r}_a, x)], dx) + \left(2 - \frac{2}{\mu}\right) h, \quad h \in [0, |I^{r}_a| - r].
\]

It is (5.2). We prove now (5.3). Let \( r' > r + h \) arbitrary. Using Tanaka’s formula applied to \((X_{T+r'}, t \geq 0)\), we have for any \( x \in \mathbb{R} \) and \( t \geq T_{r'} \),

\[
(X_t - x^-) = (r' - x^-) - \int_{T_{r'}}^t 1_{x \leq s} dB_s + \frac{1}{2} (L(t, x) - L(T_{r'}, x)).
\]
Taking \( t = \tau_a^r \), and \( x = r \) then \( x = r + h \), we get

\[
L(\tau_a^r, r + h) = 2h + a + 2 \int_{T_r}^{\tau_a^r} 1_{\{r < X_s \leq r + h\}} dX_s = a + 2 \int_{T_r}^{\tau_a^r} 1_{\{r < X_s \leq r + h\}} \text{sgn}(B_s) dB_s + \frac{2h}{\mu}
\]

where the second equality follows from (2.4) and the fact that \( \int_{T_r}^{\tau_a^r} 1_{\{r < X_s \leq r + h\}} dI_s = -h \). By Proposition 3.2 applied to \((X^{(r')}, B^{(r')})\),

\[
\int_{T_r}^{\tau_a^r} 1_{\{r < X_s \leq r + h\}} \text{sgn}(B_s) dB_s = \int_{0}^{\tau_{a-r'}^{r'}} 1_{\{r-r' < X_s^{(r')} \leq r + h - r'\}} \text{sgn}(B_s^{(r')}) dB_s^{(r')}
\]

\[
= \int_{r-r'}^{r+h-r'} W^{(r')}([0, L(\tau_a^r, x)], dx)
\]

\[
= \int_{r}^{r+h} W([0, L(\tau_a^r, x)], dx)
\]

by (5.5) applied to \( r = r' \). This proves (5.3) and completes the proof of the theorem. □

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