NONLOCAL STOKES-VLASOV SYSTEM: EXISTENCE AND DETERMINISTIC HOMOGENIZATION RESULTS

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Abstract. Our work deals with the systematic study of the coupling between the nonlocal Stokes system and the Vlasov equation. The coupling is due to a drag force generated by the fluid-particles interaction. We establish the existence of global weak solutions for the nonlocal Stokes-Vlasov system in dimensions two and three without resorting to assumptions on higher-order velocity moments of the initial distribution of particles. We then study by the means of the sigma-convergence method, the asymptotic behavior in the general deterministic framework, of the sequence of solutions to the nonlocal Stokes-Vlasov system. In guise of illustration, we provide several physical applications of the homogenization result including periodic, almost-periodic and weakly almost-periodic settings.

1. Introduction

This paper is concerned with the rigorous asymptotic analysis of a system of integro-differential equations modeling the evolution of a cloud of particles immersed in an incompressible viscous fluid. We neglect particle-particle collisions in such a way that at the microscale level, particles’ distribution, $f_\varepsilon$, satisfies the Vlasov equation

$$\frac{\partial f_\varepsilon}{\partial t} + \varepsilon v \cdot \nabla f_\varepsilon + \text{div}_v ((u_\varepsilon - v)f_\varepsilon) = 0 \text{ in } Q \times \mathbb{R}^N,$$

in which the non-dimensional small parameter $\varepsilon > 0$ represents the scale of inhomogeneities, $u_\varepsilon(t, x)$ is fluid’s velocity at time $t$ and position $x$, $f_\varepsilon(t, x, v)dv$ is roughly the odd of finding a particle with velocity $v$ near $x$ at time $t$, $Q = (0, T) \times \Omega$, $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) is a bounded domain with smooth boundary, $T$ is a given positive real number representing the final time, the operator $\nabla$ (resp. $\text{div}_v$) denotes the gradient operator with respect to $x \in \Omega$ (resp. the divergence operator in $\mathbb{R}^N$ with respect to $v \in \mathbb{R}^N$). We posit that the cloud of particles is highly diluted in such a way that we may assume that the density of the fluid is constant. Thus, the particles evolve in a Newtonian fluid governed by the Stokes system. The viscoelastic constitutive law associated to the momentum balance and the fluid-particles interaction give rise to the following Stokes system:

$$\frac{\partial u_\varepsilon}{\partial t} - \text{div} \left( A_0 \nabla u_\varepsilon + \int_0^t A_1(t - \tau, x) \nabla u_\varepsilon(\tau, x)d\tau \right) + \nabla p_\varepsilon = -\int_{\mathbb{R}^N} (u_\varepsilon - v)f_\varepsilon dv \text{ in } Q,$$

$$\text{div} u_\varepsilon = 0 \text{ in } Q,$$

where $p_\varepsilon$ is pressure and the oscillating viscosities $A_0$ and $A_1$ are defined by $A_i(t, x) = A_i(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}) ((t, x) \in Q \text{ and } i = 0, 1)$, with the $A_i$s constrained as follows:

(A1) $A_i \in C(Q; L^\infty(\mathbb{R}^N + 1)^{N^2})$ are symmetric matrices with $A_0$ satisfying the following condition:

$$A_0 \xi \cdot \xi \geq \alpha |\xi|^2 \text{ for all } \xi \in \mathbb{R}^N \text{ and a.e. in } Q \times \mathbb{R}^{N+1},$$

with $\alpha > 0$ a given constant not depending on $x, t, y, \tau$ and $\xi$.

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The system (1.1)-(1.3) is supplemented with the initial data
\[ u_\varepsilon(0, x) = u^0(x), \quad f_\varepsilon(0, x, v) = f^0(x, v), \quad x \in \Omega, v \in \mathbb{R}^N, \] (1.4)
and the boundary conditions
\[ u_\varepsilon = 0 \text{ on } \partial \Omega \text{ and } f_\varepsilon(t, x, v) = f_\varepsilon(t, x, v^*) \text{ for } x \in \partial \Omega \text{ with } v \cdot \nu(x) < 0, \] (1.5)
where \( v^* = v - 2(v \cdot \nu(x))\nu(x) \) is the specular velocity, \( \nu(x) \) is the outward normal to \( \Omega \) at \( x \in \partial \Omega \) and the functions \( u^0 \) and \( f^0 \) are chosen as follows:

\[ (A2) \quad u^0 \in L^2(\Omega)^N \text{ with } \text{div} u^0 = 0, \quad f^0 \geq 0, \quad f^0 \in L^\infty(\Omega \times \mathbb{R}^N) \cap L^1(\Omega \times \mathbb{R}^N). \]

It is opportune to stress that we have not imposed the constraint \( |v|^5 f^0 \in L^1(\Omega \times \mathbb{R}^N) \) as suggested by Yu 35. Indeed, we are going to see that the Lemma 2.1 of Hamdache 18 renders such assumption superfluous provided appropriate regularization and truncation are performed. In particular, the truncation of the initial distribution of particles in the \( v \)-direction relieves us from the assumption on moments. Contrary to contemporary approaches, ours permeates initial distribution of the form \( \alpha(x)/(1 + |v|^2) \), where \( \alpha \geq 0 \) and \( \alpha \in L^\infty(\Omega) \).

The system (1.1)-(1.5) arises in several applications comprising the modeling of reactive flows of sprays 1 [26], atmospheric pollution modeling 14, and waste water treatment 11. When there is no particle evolving in the fluid (i.e. when \( f_\varepsilon \equiv 0 \)) the asymptotic analysis of Eqs. (1.1)-(1.5) reduces to the study of the asymptotics of (1.2)-(1.5) (with of course \( f_\varepsilon \equiv 0 \) therein), which has been very recently undertaken by Woukeng 35 in the almost periodic framework.

There is a fairly extensive literature concerned with the asymptotic analysis of the Vlasov equations coupled with other equations: the works 6, 9, 10 deal with the periodic homogenization of the Vlasov equations, the main thrust of the papers 10, 15 is the asymptotic analysis of the coupling Vlasov-Poisson system, Mellet et al. 17 is concerned with the homogenization of the coupling Vlasov-Fokker-Planck/Compressible Navier-Stokes system, and Goudon et al. 12, 13 treats the asymptotic behavior of the coupling Vlasov-Navier-Stokes equations.

In this work, our objective is twofold: 1) we state and prove an existence result for the system (1.1)-(1.5) without any assumption on the \( v \)-moments of the initial condition \( f^0 \); 2) we carry out the homogenization of (1.1)-(1.5) under suitable structural assumptions on the coefficients of the operators involved in (1.2).

These assumptions cover a wide set of concrete behaviors such as the classical periodicity assumption, the almost periodicity hypothesis, weakly almost periodicity hypothesis and much more. In order to achieve our goal, we shall use the concept of \textit{sigma-convergence} 23, 29 which is roughly a formulation of the well-known two-scale convergence method 22 in the context of \textit{algebras with mean value} 19, 29, 39. This is the so-called \textit{deterministic homogenization} theory which includes the periodic homogenization theory as a special case. As far as we know, our results are new in the context of general deterministic homogenization since the available results deal with either periodic homogenization 6, 9, 10, 15 or rely on the concept of \textit{relative entropy} 12, 13, 17, 25, 28, 37.

The remainder of this paper is structured as follows. In Section 2, we state and outline the proof of an existence result for our \( \varepsilon \)-problem. We also derive some a priori estimates that will be useful in next sections. Section 3 deals with the concept of \( \Sigma \)-convergence and its relation with convolution. We first recall some useful tools related to algebras with mean value and define convolution over the spectrum of an algebra with mean value. In Section 4, we state and prove the main homogenization result. In Section 5, we give some concrete situations in which the result of Section 4 is valid. Finally, we summarize our findings in Section 6.

In the sequel, unless otherwise specified, the field of scalars acting on vector spaces is the set of real numbers and scalar functions are real-valued. If \( X \) and \( F \) respectively denote a locally compact space and a Banach space, then we respectively write \( C(X; F) \) and \( \text{BUC}(X; F) \) for continuous mappings of \( X \) into \( F \) and bounded uniformly continuous mappings of \( X \) into \( F \). We shall always assume that \( \text{BUC}(X; F) \) is equipped with the supremum norm \( ||u||_\infty = \sup_{x \in X} ||u(x)|| \) in which \( ||\cdot|| \) stands for the norm of \( F \). In the notations for functions space, we shall omit the codomain when it is \( \mathbb{R} \). To wit, \( C(X) \) will stand for \( C(X; \mathbb{R}) \) and \( \text{BUC}(X) \) will be a shorthand notation for \( \text{BUC}(X; \mathbb{R}) \). Likewise, the usual Lebesgue spaces \( L^p(X; \mathbb{R}) \) and \( L^p_{\text{loc}}(X; \mathbb{R}) \) where \( X \) is equipped with a positive Radon measure, are respectively abbreviated \( L^p(X) \) and \( L^p_{\text{loc}}(X; \mathbb{R}) \).
and $L^p_{\text{loc}}(X)$. Finally, it will always be assumed that the Euclidean space $\mathbb{R}^N$ ($N \geq 1$) and its open sets are each endowed with Lebesgue measure $dy = dy_1 \ldots dy_N$.

2. Existence result and basic a priori estimates

In this part, we focus on the existence of solutions to our $\varepsilon$-problem. We shall define a regularized problem, solve it and show that the limit (in a sense to be specified) of the solution of this regularized problem solves our $\varepsilon$-problem. In order to implement our program, we shall establish some a priori estimates in some classical functional spaces we introduce below. These estimates will be used in compactness arguments at various stages of this work.

The main classical spaces involved in the mathematical study of incompressible fluid flows are spaces connected to kinetic energy, entropy, the boundary conditions and the conservation of mass. These spaces will be denoted $V$ and $H$ and they may be respectively constructed as closure of $\mathcal{V} = \{\varphi \in C_0^\infty(\Omega)^N : \text{div}\varphi = 0\}$ in $H^1(\Omega)^N$ ($H^1(\Omega)$ the usual Sobolev space on $\Omega$), and $L^2(\Omega)^N$. Since $\partial\Omega$ is smooth, we have that $V = \{u \in H^1_0(\Omega)^N : \text{div}u = 0\}$ and $H = \{u \in L^2(\Omega)^N : \text{div}u = 0\}$ and $u \cdot \nu = 0$ on $\partial\Omega$, $\nu$ being the outward unit vector normal to $\partial\Omega$. We denote by $\langle \cdot, \cdot \rangle$ the inner product in $H$, and by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^N$. The associated norm in $\mathbb{R}^N$ is denoted by $|\cdot|$. All duality pairing are denoted by $\langle \cdot, \cdot \rangle$ without referring to spaces involved. Such spaces will be understood from the context. We set

$$\Sigma^\pm = \{(x,v) \in \partial\Omega \times \mathbb{R}^N : \pm v \cdot \nu(x) > 0\}.$$ 

With the functional framework fixed, we can now specify the type of solutions we will be seeking.

**Definition 2.1.** A pair $(u_\varepsilon, f_\varepsilon)$ (for fixed $\varepsilon > 0$) is called a weak solution to the system [1.1]-[1.5] if the following conditions are satisfied:

- $u_\varepsilon \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap C([0,T];V')$;
- $f_\varepsilon(t,x,v) \geq 0$ for any $(t,x,v) \in Q \times \mathbb{R}^N$;
- $f_\varepsilon \cdot |v|^2 \in L^\infty(0,T;L^1(\Omega \times \mathbb{R}^N))$;
- for all $\phi \in C^1([0,T] \times \Omega \times \mathbb{R}^N)$ with compact support in $v$, such that $\phi(T,\cdot,\cdot) = 0$ and $\phi(t,x,v) = \phi(t,x,v^*)$ on $(0,T) \times \Sigma^+$, we have

$$\int_{Q \times \mathbb{R}^N} f_\varepsilon \left( \frac{\partial \phi}{\partial t} + \varepsilon v \cdot \nabla \phi + (u_\varepsilon - v) \cdot \nabla \phi \right) dx dv dt + \int_{\Omega \times \mathbb{R}^N} f_\varepsilon \phi(0,x,v) dx dv = 0; \quad (2.1)$$

- for all $\psi \in C^1([0,T] \times \Omega)^N$ with div $\psi = 0$ and $\psi(T,\cdot,\cdot) = 0$,

$$\int_Q \left( -u_\varepsilon \cdot \frac{\partial \psi}{\partial t} + (A_0^\varepsilon \nabla u_\varepsilon + A_1^\varepsilon \ast \nabla u_\varepsilon) \cdot \nabla \psi \right) dx dt = - \int_{Q \times \mathbb{R}^N} f_\varepsilon (u_\varepsilon - v) \cdot \psi dx dv dt + \int_{\Omega} u^0 \cdot \psi(0,x) dx. \quad (2.2)$$

In Eq. (2.2) $A_1^\varepsilon \ast \nabla u_\varepsilon$ stands for the function defined by

$$(A_1^\varepsilon \ast \nabla u_\varepsilon)(t,x) = \int_0^t A_1^\varepsilon(t-\tau,x)\nabla u_\varepsilon(\tau,x) d\tau$$

whenever $(t,x) \in Q$.

The main result of this section is summarized in the following theorem.

**Theorem 2.1.** Under assumption (A1)-(A2) and for any fixed $\varepsilon > 0$, there exists a weak solution $(u_\varepsilon, f_\varepsilon)$ of [1.1]-[1.5] in the sense of Definition 2.1. There also exists a $p_\varepsilon \in L^2(0,T;L^2(\Omega)/\mathbb{R})$ such that [1.2] is satisfied.

The proof of Theorem 2.1 will be done in several steps described in the subsections that follow. The general idea is loosely to regularize our problem, solve the regularized problem and take the limit of its solution to obtain a solution of our problem.
2.1. Regularization and truncation. We start by fixing notations that will be useful in the sequel. Let \((\theta_N)(\lambda)\) be a mollifying sequence in \(\mathbb{R}^N\) for all \(\lambda > 0\). Furthermore, especially in inequalities involving \(\lambda\) and \(\epsilon\), we shall assume throughout that \(0 < \lambda \leq 1\). The latter assumption is used when needed to obtain uniform estimates in \(\lambda\).

Let \(w \in L^2(0, T; V)\). The regularized system associated to our \(\epsilon\)-problem takes the following form:

\[
\frac{\partial f_{\epsilon, \lambda}}{\partial t} + \epsilon \cdot \nabla f_{\epsilon, \lambda} + \text{div} \left((w \ast \theta_\lambda - v)f_{\epsilon, \lambda}\right) = 0 \quad \text{in} \quad Q \times \mathbb{R}^N
\]

\[
\frac{\partial u_{\epsilon, \lambda}}{\partial t} - \text{div} \left(A_0^\gamma \nabla u_{\epsilon, \lambda} + \int_0^t A_1^\epsilon(t - \tau, x)\nabla w(\tau, x) \, d\tau\right) + \nabla p_{\epsilon, \lambda} = -\int_{\mathbb{R}^N} (w \ast \theta_\lambda - v)\gamma(\lambda)(v)f_{\epsilon, \lambda} \, dv \quad \text{in} \quad Q,
\]

\[
\text{div} u_{\epsilon, \lambda} = 0 \quad \text{in} \quad Q.
\]

The system (2.3) - (2.4) is supplemented with the following initial and boundary conditions.

\[
a) \quad u_{\epsilon, \lambda}(0, x) = u^0(x);
\]

\[
b) \quad f_{\epsilon, \lambda}(0, x, v) := f_\lambda^0(x, v) = \gamma(\lambda)(v)(f^0 \ast \Theta_\lambda)(x, v), \quad x \in \Omega, v \in \mathbb{R}^N,
\]

and the boundary conditions

\[
a) \quad u_{\epsilon, \lambda} = 0 \quad \text{on} \quad \partial \Omega;
\]

\[
b) \quad f_{\epsilon, \lambda}(t, x, v) = f_{\epsilon, \lambda}(t, x, v^*) \quad \text{for} \quad x \in \partial \Omega \quad \text{with} \quad v \cdot \nu(x) < 0.
\]

2.2. Existence, regularity and estimates of \(f_{\epsilon, \lambda}\). In this part, we focus on Eq. (2.3) coupled with the initial condition (2.6) b and the boundary condition (2.7) b). In what follows, we use \(C\) as a generic name for positive constants independent of both \(\epsilon\) and \(\lambda\). In all the estimates, we suppose when the need arises that \(\epsilon\) and \(\lambda\) are sufficiently small. We shall assume throughout that \(f^0 \in L^p(\Omega \times \mathbb{R}^N) \cap L^\infty(\Omega \times \mathbb{R}^N), p \geq 1\), unless we mention otherwise.

Since \(|f_\lambda^0| \leq |f^0 \ast \Theta_\lambda|\), we have

\[
\|f_\lambda^0\|_{L^p(\Omega \times \mathbb{R}^N)} \leq \|f^0 \ast \Theta_\lambda\|_{L^p(\Omega \times \mathbb{R}^N)} \leq \|f^0\|_{L^p(\Omega \times \mathbb{R}^N)} \|\Theta_\lambda\|_{L^1(\Omega \times \mathbb{R}^N)} \leq \|f^0\|_{L^p(\Omega \times \mathbb{R}^N)}
\]

in such a way that

\[
\|f_\lambda^0\|_{L^p(\Omega \times \mathbb{R}^N)} \leq \|f^0\|_{L^p(\Omega \times \mathbb{R}^N)}.
\]

By Theorem 4 of Mischler [21], we infer that \(f_{\epsilon, \lambda}\) uniquely exists and belongs to \(L^\infty(0, T, L^\infty(\Omega \times \mathbb{R}^N) \cap L^p(\Omega \times \mathbb{R}^N))\). Since both \(f_\lambda^0\) and the coefficients of Eq. (2.3) are \(C^\infty\), we can deduce using the method of characteristics that \(f_{\epsilon, \lambda}\) is nonnegative and belongs to \(C^1((0, T) \times \Omega \times \mathbb{R}^N)\). Following [18, p. 54], we have

\[
\frac{d}{dt} \left(e^{N t} \int_{\Omega \times \mathbb{R}^N} (e^{-N t} f_{\epsilon, \lambda})^p \, dx \, dv\right) = 0.
\]

Then, by integrating both sides of Eq (2.9) from 0 to \(t\), we obtain

\[
\|f_{\epsilon, \lambda}(t)\|_{L^p(\Omega \times \mathbb{R}^N)} = e^{N t (1 - \frac{1}{p})} \|f_\lambda^0\|_{L^p(\Omega \times \mathbb{R}^N)}.
\]

Thus, by using the inequality (2.8), we arrive at the estimate

\[
\|f_{\epsilon, \lambda}\|_{L^\infty(0, T; L^p(\Omega \times \mathbb{R}^N))} \leq C(N, T, p) \|f^0\|_{L^p(\Omega \times \mathbb{R}^N)}.
\]
Now, we turn our attention to the estimates of \( v \)-moments of \( f_{\varepsilon, \lambda} \). Lemma 2.1 of Hamdache [18] will be our workhorse. Let us first observe that \( f_{\varepsilon}^0 \in L^\infty(\Omega \times \mathbb{R}^N) \cap L^1(\Omega \times \mathbb{R}^N) \) since \( f_{\lambda}^0 \) is compactly supported and \( f^0 \in L^\infty(\Omega \times \mathbb{R}^N) \cap L^p(\Omega \times \mathbb{R}^N), p \geq 1 \) be fixed. Furthermore, we assert that for any \( m \geq 1, \)

\[
\int_{\Omega \times \mathbb{R}^N} |v|^m f_{\lambda}^0 \, dx \, dv \leq C(N, p) \| f^0 \|_{L^p(\Omega \times \mathbb{R}^N)}, \tag{2.12}
\]

Indeed, the following inequalities

\[
\int_{\Omega \times \mathbb{R}^N} |v|^m f_{\lambda}^0 \, dx \, dv \leq \int_{\Omega \times \{ |v| \leq 2 \}} |v|^m f_{\lambda}^0 \, dx \, dv + \int_{\Omega \times \{ |v| > 2 \}} |v|^m f_{\lambda}^0 \, dx \, dv = \int_{\Omega \times \{ |v| \leq 2 \}} |v|^m f_{\lambda}^0 \, dx \, dv
\]

\[
\leq \int_{\Omega \times \{ |v| \leq 2 \}} f_{\lambda}^0 \, dx \, dv \leq |\Omega| \times \{ |v| \leq 2 \}^1/2 \| f_{\lambda}^0 \|_{L^p(\Omega \times \mathbb{R}^N)}
\]

\[
\leq C(N, p) \| f^0 \|_{L^p(\Omega \times \mathbb{R}^N)} \quad \text{(see (2.8))}, \tag{2.13}
\]

where \( q \) is the conjugate exponent of \( p \), are true. In view of the continuous embedding \( H^1_0(\Omega) \hookrightarrow L^r(\Omega) \) for any \( 1 \leq r \leq 6 \), and since \( N \leq 3 \), we may choose \( m \geq 1 \) such that \( N + m \leq 6 \) (for example \( 1 \leq m \leq 4 \) if \( N = 2 \), and \( 1 \leq m \leq 3 \) for \( N = 3 \)). Thus, for such an \( m \), the fact that \( w \in L^2(0, T; L^p(\Omega \times \mathbb{R}^N)) \) steams from both the above continuous embedding and \( w \in L^2(0, T; V) \). With this in mind and taking into account Eq. (2.12), we see that we are within the hypotheses of [18] Lemma 2.1. Hence, the following estimate holds:

\[
\int_{\Omega \times \mathbb{R}^N} |v|^m f_{\varepsilon, \lambda} \, dx \, dv \leq C(N, T) \left( \int_{\Omega \times \mathbb{R}^N} |v|^m f_{\lambda}^0 \, dx \, dv \right)^{\frac{1}{m}} + (\| f_{\lambda}^0 \|_{L^\infty(\Omega \times \mathbb{R}^N)} + 1) \| w \ast \theta_{\lambda} \|_{L^2(0, T; L^{N+m}(\Omega \times \mathbb{R}^N))} \right)^{N+m} \tag{2.14}
\]

By employing the inequality (2.12) in conjunction with the estimate

\[
\| w(t, \cdot) \ast \theta_{\lambda} \|_{L^{N+m}(\Omega)^N} \leq \| w(t, \cdot) \|_{L^{N+m}(\Omega)^N} \| \theta_{\lambda} \|_{L^1(\Omega)} \leq \| w(t, \cdot) \|_{L^{N+m}(\Omega)^N}, \tag{2.15}
\]

and noting that \( \| f_{\lambda}^0 \|_{L^\infty(\Omega \times \mathbb{R}^N)} \leq \| f^0 \|_{L^\infty(\Omega \times \mathbb{R}^N)} \), we arrive at the inequality

\[
\int_{\Omega \times \mathbb{R}^N} |v|^m f_{\varepsilon, \lambda} \, dx \, dv \leq C(N, m, p, T) \left( \| f^0 \|_{L^\infty(\Omega \times \mathbb{R}^N)}^{\frac{1}{m}} + (\| f^0 \|_{L^\infty(\Omega \times \mathbb{R}^N)} + 1) \| w \|_{L^2(0, T; L^{N+m}(\Omega \times \mathbb{R}^N))} \right)^{N+m}
\]

for any \( m \geq 1 \) satisfying \( N + m \leq 6 \). By employing the Sobolev embedding \( H^1_0(\Omega) \hookrightarrow L^{N+m}(\Omega) \), the latter inequality leads to

\[
\int_{\Omega \times \mathbb{R}^N} |v|^m f_{\varepsilon, \lambda} \, dx \, dv \leq C \left[ \| f^0 \|_{L^p(\Omega \times \mathbb{R}^N)}^{\frac{1}{m}} + (\| f^0 \|_{L^\infty(\Omega \times \mathbb{R}^N)} + 1) \| w \|_{L^2(0, T; V)} \right]^{N+m} \tag{2.16}
\]

for any \( m \geq 1 \) satisfying \( N + m \leq 6 \), where \( C = C(N, m, p, T, \Omega) \).

Remark 2.1. We shall discover in the sequel that the estimate (2.16) makes assumptions on higher-order \( v \)-moments of the initial distribution redundant. It is opportune to emphasize that besides Lemma 2.1 of Hamdache [18], both regularization and truncation have played a fundamental role in deriving the inequality (2.16).

Next, we provide estimates which show among other things that the force field

\[
F_{\varepsilon, \lambda} = G_{\varepsilon, \lambda} + H_{\varepsilon, \lambda}, \tag{2.17}
\]

where

\[
G_{\varepsilon, \lambda}(t, x) = -\int_{\mathbb{R}^N} (w \ast \theta_{\lambda} - v) \gamma_{\lambda}(v) f_{\varepsilon, \lambda} \, dv \tag{2.18}
\]

and

\[
H_{\varepsilon, \lambda}(t, x) = \text{div} \left( \int_0^t A_{\gamma}(t - \tau, x) \nabla w(\tau, x) \, d\tau \right), \tag{2.19}
\]
belongs to $L^2(0, T; H^{-1}(\Omega)\times \mathbb{R}^N)$. So, let $\Phi \in C^\infty_0(\Omega)$. For almost all $t \in [0, T]$ we have
\[
\|G_{\varepsilon, \lambda}(t, \cdot)\|_{L^2(\Omega)\times \mathbb{R}^N} \leq \int_{\Omega \times \{\varepsilon \leq 2\}} (1 + |w \ast \theta_\lambda|) f_{\varepsilon, \lambda} |\Phi| \, dx \, dv
\leq C(N) \left( 1 + \|w(t, \cdot)\|_{L^2(\Omega)\times \mathbb{R}^N} \right) \|f_{\varepsilon, \lambda}(t, \cdot)\|_{L^\infty(\Omega \times \mathbb{R}^N)} \|\Phi(t, \cdot)\|_{L^2(\Omega)\times \mathbb{R}^N}.
\] (2.20)
Thus, for almost all $t \in [0, T]$ we have
\[
\|G_{\varepsilon, \lambda}(t, \cdot)\|_{L^2(\Omega)\times \mathbb{R}^N} \leq C(N) \left( 1 + \|w(t, \cdot)\|_{L^2(\Omega)\times \mathbb{R}^N} \right) \|f_{\varepsilon, \lambda}(t, \cdot)\|_{L^\infty(\Omega \times \mathbb{R}^N)}.
\] (2.21)
Since $f_{\lambda}^0 \in L^\infty(\Omega \times \mathbb{R}^N)$, by the maximum principle applied to the transport equation, we have
\[
\|f_{\varepsilon, \lambda}\|_{L^\infty(0, \infty; L^\infty(\Omega \times \mathbb{R}^N))} \leq C(N, T) \|f_{\lambda}^0\|_{L^\infty(\Omega \times \mathbb{R}^N)} \leq C(N, T) \|f^0\|_{L^\infty(\Omega \times \mathbb{R}^N)}.
\] (2.22)
Therefore, using Eq. (2.22) in Eq. (2.21), we obtain
\[
\|G_{\varepsilon, \lambda}(t, \cdot)\|_{L^2(\Omega)\times \mathbb{R}^N} \leq C(N, T) \|f^0\|_{L^\infty(\Omega \times \mathbb{R}^N)} \left( 1 + \|w(t, \cdot)\|_{L^2(\Omega)\times \mathbb{R}^N} \right)
\] (2.23)
for almost all $t \in [0, T]$. Thus, since $w \in L^2(0, T; L^2(\Omega)\times \mathbb{R}^N)$, so is $G_{\varepsilon, \lambda}$.

Now, we turn our attention to $H_{\varepsilon, \lambda}$:
\[
\langle H_{\varepsilon, \lambda}(t, \cdot), \Phi \rangle \leq N^3 \langle A_1 \|\nabla \Phi\|_{L^\infty(\Omega \times \mathbb{R}^N) N^2} \rangle \int_0^T \int_\Omega |\nabla w| |\nabla \Phi| \, dx \, dt
\leq N^3 \langle A_1 \|\nabla \Phi\|_{L^\infty(\Omega \times \mathbb{R}^N) N^2} \rangle T \|\nabla w\|_{L^2(0, T; L^2(\Omega)\times \mathbb{R}^N)} \|\nabla \Phi\|_{L^2(\Omega)\times \mathbb{R}^N}.
\] (2.24)
Therefore, $H_{\varepsilon, \lambda} \in L^2(0, T; H^{-1}(\Omega)\times \mathbb{R}^N)$ and
\[
\|H_{\varepsilon, \lambda}\|_{L^2(0, T; H^{-1}(\Omega)\times \mathbb{R}^N)} \leq N^3 \langle A_1 \|\nabla \Phi\|_{L^\infty(\Omega \times \mathbb{R}^N) N^2} \rangle T^{3/2} \|\nabla w\|_{L^2(0, T; L^2(\Omega)\times \mathbb{R}^N)}.
\] (2.25)

2.3. Existence of $(u_{\varepsilon, \lambda}, p_{\varepsilon, \lambda})$ and further estimates. We look for $u_{\varepsilon, \lambda} \in L^2(0, T; V)$ such that $\partial u_{\varepsilon, \lambda} / \partial t \in L^2(0, T; V')$ and, for almost all $t \in [0, T]$ and all $\Phi \in V$,
\[
\left\langle \frac{du_{\varepsilon, \lambda}}{dt}, \Phi \right\rangle + a_{\varepsilon}(t; u_{\varepsilon, \lambda}, \Phi) = (F_{\varepsilon, \lambda}(t), \Phi),
\] (2.26)
\[
u_{\varepsilon, \lambda}(0) = u_0 \in H,
\] (2.27)
where
\[
a_{\varepsilon}(t; u, \Phi) = \int_\Omega A^0(t, x) \nabla u(x) \cdot \nabla \Phi(x) \, dx.
\] (2.28)
and $F_{\varepsilon, \lambda}$ is defined in Eq. (2.17). Note that by standard arguments [20], the problem (2.20)-(2.27) makes sense. By direct computations using assumptions made on the $A_i$s, one arrives at the following properties of $a_{\varepsilon}$.

- The function $t \mapsto a_{\varepsilon}(t; u, \Phi)$ is measurable for all $u, \Phi \in V$.
- For almost every $t \in [0, T]$ and for all $u, \Phi \in V$,
\[
|a_{\varepsilon}(t; u, \Phi)| \leq N^3 \langle A_0 \|\nabla \Phi\|_{L^\infty(\Omega \times \mathbb{R}^N) N^2} \rangle \|u\|_V \|\Phi\|_V := M \|u\|_V \|\Phi\|_V.
\] (2.29)
- For almost every $t \in [0, T]$ and for all $w \in V$,
\[
|a_{\varepsilon}(t; u, v)| \geq \alpha \|u\|_V^2.
\] (2.30)
Thus, by Lions’ theorem [20], there is a unique $u_{\varepsilon, \lambda} \in L^2(0, T; V) \cap C(0, T; H)$ satisfying Eqs (2.20)-(2.27). Since $F_{\varepsilon, \lambda} \in L^2(0, T; H^{-1}(\Omega)\times \mathbb{R}^N)$ and $w \in L^2(0, T; V)$, it is a simple matter to check that $\partial u_{\varepsilon, \lambda} / \partial t - \text{div}(A^0
abla u_{\varepsilon, \lambda}) - F_{\varepsilon, \lambda} \in H^{-1}(\Omega)\times \mathbb{R}^N \subset D'(\Omega)\times \mathbb{R}^N$ (the usual space of distributions on $\Omega$) for almost all $t \in [0, T]$. Thus, thanks to Eq. (2.20) and Propositions 1.1 and 1.2 of [33], there is a unique $p_{\varepsilon, \lambda}(t) \in L^2(\Omega)\times \mathbb{R}$ such that Eq. (2.4) holds in the sense of distributions and
\[
\|p_{\varepsilon, \lambda}(t)\|_{L^2(\Omega)\times \mathbb{R}} \leq C(\Omega) \|\nabla p_{\varepsilon, \lambda}(t)\|_{H^{-1}(\Omega)\times \mathbb{R}}.
\] (2.31)
for almost all $t \in [0, T]$.

2.4. Solvability of (2.3)-(2.7) with $w = u_{\varepsilon, \lambda}$. Here, we prove via Schauder’s fixed point theorem that by letting $w = u_{\varepsilon, \lambda}$, the regularized problem is still solvable. In order to do that, we consider the mapping

$$S : L^2(0, T; V) \rightarrow L^2(0, T; V)$$

with $w \mapsto u_{\varepsilon, \lambda},$

where $u_{\varepsilon, \lambda}$ is the unique solution of Eqs. (2.36)-(2.37). The mapping $S$ is well-defined because of the previous step. We need to show that $S$ has a fixed point as asserted in the next result.

**Proposition 2.1.** There exists a function $u_{\varepsilon, \lambda}$ in $L^2(0, T; V)$ such that $Su_{\varepsilon, \lambda} = u_{\varepsilon, \lambda}$.

*Proof.* The mapping $S$ is not linear. However, we can check that it is Lipschitz continuous. Indeed, let $w_1, w_2 \in L^2(0, T; V)$ and set $u_{\varepsilon, \lambda}^i = Sw_i$ $(i = 1, 2)$, $w = w_1 - w_2$. Let us also denote by $p_{\varepsilon, \lambda}^i$ $(i = 1, 2)$ the associated pressures. Then, $u_{\varepsilon, \lambda} = u_{\varepsilon, \lambda}^1 - u_{\varepsilon, \lambda}^2$ and $p_{\varepsilon, \lambda} = p_{\varepsilon, \lambda}^1 - p_{\varepsilon, \lambda}^2$ solve the following Stokes system

$$\frac{\partial u_{\varepsilon, \lambda}}{\partial t} - \text{div} (A_0^\varepsilon \nabla u_{\varepsilon, \lambda}) + \nabla p_{\varepsilon, \lambda} = \text{div} (A_1^\varepsilon \ast \nabla w) - \int_{\mathbb{R}^N} (w \ast \theta_\lambda) \gamma_\lambda(v) f_{\varepsilon, \lambda} dv \quad \text{in } Q$$

$$\text{div } u_{\varepsilon, \lambda} = 0 \quad \text{in } Q$$

$$u_{\varepsilon, \lambda} = 0 \quad \text{on } (0, T) \times \partial \Omega$$

$$u_{\varepsilon, \lambda}(0, x) = 0, x \in \Omega.$$

Multiplying the leading equation above by $u_{\varepsilon, \lambda}$, we find after integrating over $\Omega$ that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_{\varepsilon, \lambda}|^2 \, dx + \int_{\Omega} (A_0^\varepsilon \nabla u_{\varepsilon, \lambda} + A_1^\varepsilon \ast \nabla w) \cdot \nabla u_{\varepsilon, \lambda} \, dx + \int_{\Omega \times \mathbb{R}^N} \gamma_\lambda(v) f_{\varepsilon, \lambda}(w \ast \theta_\lambda) \cdot u_{\varepsilon, \lambda} \, dv \, dx = 0.$$

Integrating the above equation over $(0, t)$ and using assumption (A1), we arrive at the inequality

$$\int_0^t \int_{\Omega} \left| u_{\varepsilon, \lambda} \right|^2 \, dx + 2 \alpha \int_0^t \int_{\Omega} \left| \nabla u_{\varepsilon, \lambda} \right|^2 \, dx \, dt \leq -2 \int_0^t \int_{\Omega} (A_1^\varepsilon \ast \nabla w) \cdot \nabla u_{\varepsilon, \lambda} \, dx \, dt - 2 \int_0^t \int_{\Omega \times \mathbb{R}^N} \gamma_\lambda(v) f_{\varepsilon, \lambda}(w \ast \theta_\lambda) \cdot u_{\varepsilon, \lambda} \, dv \, dx \, dt.$$

Using Young’s inequality yields

$$2 \left| \int_0^t \int_{\Omega \times \mathbb{R}^N} \gamma_\lambda(v) f_{\varepsilon, \lambda}(w \ast \theta_\lambda) \cdot u_{\varepsilon, \lambda} \, dv \, dx \, dt \right| \leq C (N, T) \left| f^0 \right|_{L^\infty(\Omega \times \mathbb{R}^N)} \|B(0, 2)\| \int_0^t \int_{\Omega} |w \ast \theta_\lambda| \left| u_{\varepsilon, \lambda} \right| \, dx \, dt \leq C \int_0^t \int_{\Omega} \left( |w \ast \theta_\lambda|^2 + \left| u_{\varepsilon, \lambda} \right|^2 \right) \, dx \, dt \leq C \|w \ast \theta_\lambda\|_{L^2(\Omega)}^2 + C \int_0^t \left| u_{\varepsilon, \lambda} \right|_{L^2(\Omega)}^2 \, dt \leq C \|w\|_{L^2(\Omega)}^2 + C \int_0^t \left| u_{\varepsilon, \lambda} \right|_{L^2(\Omega)}^2 \, dt.$$

Also, it can be verified that

$$2 \left| \int_0^t \int_{\Omega \times \mathbb{R}^N} \gamma_\lambda(v) f_{\varepsilon, \lambda}(w \ast \theta_\lambda) \cdot u_{\varepsilon, \lambda} \, dv \, dx \, dt \right| \leq C (N, T) \left| f^0 \right|_{L^\infty(\Omega \times \mathbb{R}^N)} \|B(0, 2)\| \int_0^t \int_{\Omega} |w \ast \theta_\lambda| \left| u_{\varepsilon, \lambda} \right| \, dx \, dt \leq C \int_0^t \int_{\Omega} \left( |w \ast \theta_\lambda|^2 + \left| u_{\varepsilon, \lambda} \right|^2 \right) \, dx \, dt \leq C \|w \ast \theta_\lambda\|_{L^2(\Omega)}^2 + C \int_0^t \left| u_{\varepsilon, \lambda} \right|_{L^2(\Omega)}^2 \, dt \leq C \|w\|_{L^2(\Omega)}^2 + C \int_0^t \left| u_{\varepsilon, \lambda} \right|_{L^2(\Omega)}^2 \, dt.$$
where $|B(0, 2)|$ stands for the Lebesgue measure of $B(0, 2)$. Thus,

$$
\int_{\Omega} |u_{\epsilon, \lambda}|^2 \, dx + \alpha \int_0^t \| \nabla u_{\epsilon, \lambda} \|^2_{L^2(\Omega)} \, d\tau \leq C \| w \|^2_{L^2(Q)} + C \int_0^t \left( \int_0^\tau \| \nabla w(s) \|^2_{L^2(\Omega)} \, ds \right) \, d\tau
$$

$$
+ C \int_0^t \| u_{\epsilon, \lambda} \|^2_{L^2(\Omega)} \, d\tau
$$

and

$$
C \| w \|^2_{L^2(Q)} + C \int_0^t \left( \int_0^\tau \| \nabla w(s) \|^2_{L^2(\Omega)} \, ds \right) \, d\tau \leq C \| w \|^2_{L^2(Q)} + CT \int_0^T \| \nabla w(s) \|^2_{L^2(\Omega)} \, ds
$$

$$
\leq C \| w \|^2_{L^2(0,T;V)}.
$$

By conflating the previous estimates, we obtain the inequality

$$
\int_{\Omega} |u_{\epsilon, \lambda}|^2 \, dx + \alpha \int_0^t \| \nabla u_{\epsilon, \lambda} \|^2_{L^2(\Omega)} \, d\tau \leq C \| w \|^2_{L^2(0,T;V)} + C \int_0^t \| u_{\epsilon, \lambda} \|^2_{L^2(\Omega)} \, d\tau.
$$

Then, Gronwall’s Lemma implies the inequality

$$
\int_0^t \| u_{\epsilon, \lambda} \|^2_{L^2(\Omega)} \, d\tau \leq C \| w \|^2_{L^2(0,T;V)}
$$

from which we infer that

$$
\int_0^T \| \nabla u_{\epsilon, \lambda} \|^2_{L^2(\Omega)} \, d\tau \leq C \| w \|^2_{L^2(0,T;V)}
$$

or equivalently,

$$
\| u_{\epsilon, \lambda} \|_{L^2(0,T;V)} \leq C \| w \|_{L^2(0,T;V)}.
$$

(2.32)

Now, let $\varphi \in C_0^\infty (0,T) \otimes V$. Then,

$$
\left< \frac{\partial u_{\epsilon, \lambda}}{\partial t}, \varphi \right> = - \int_Q A^\lambda_0 \nabla u_{\epsilon, \lambda} \cdot \nabla \varphi \, dx \, dt - \int_Q (A^\lambda_1 \ast \nabla w) \cdot \nabla \varphi \, dx \, dt
$$

$$
- \int_R \left( \int_{\mathbb{R}^N} f_{\epsilon, \lambda} \gamma_{\lambda}(v) \, dv \right) (w \ast \theta_{\lambda}) \cdot \varphi \, dx \, dt
$$

and

$$
\left| \left< \frac{\partial u_{\epsilon, \lambda}}{\partial t}, \varphi \right> \right| \leq C \| \nabla u_{\epsilon, \lambda} \|_{L^2(Q)} \| \nabla \varphi \|_{L^2(Q)} + \| A^\lambda_1 \ast \nabla w \|_{L^2(Q)} \| \nabla \varphi \|_{L^2(Q)} + C \| w \|_{L^2(0,T;V)} \| \varphi \|_{L^2(Q)}
$$

$$
\leq C \| w \|_{L^2(0,T;V)} \| \varphi \|_{L^2(0,T;V)}
$$

because of (2.32),

$\int t$ being a positive constant that does not depend on $\varphi$. Therefore, it follows from the density of $C_0^\infty (0,T) \otimes V$ in $L^2(0,T;V)$ that $\partial u_{\epsilon, \lambda}/\partial t \in L^2(0,T;V)$ with

$$
\left< \frac{\partial u_{\epsilon, \lambda}}{\partial t}, \varphi \right> \leq C \| w \|_{L^2(0,T;V)}.
$$

(2.33)

The inequality (2.33) implies that $S$ sends continuously $L^2(0,T;V)$ into itself. Moreover (2.32) and (2.33) show that $S$ transforms bounded sets in $L^2(0,T;V)$ into bounded sets in $W(0,T) = \{ w \in L^2(0,T;V) : \partial w/\partial t \in L^2(0,T;V') \} (W(0,T)$ being endowed with the norm $\| w \|_{W(0,T)} = \| w \|^2_{L^2(0,T;V')} + \| \partial w/\partial t \|^2_{L^2(0,T;V')})^{1/2}$ which makes it a Hilbert space). Furthermore, the range of $S$ is contained in $W(0,T)$ which is compact in $L^2(0,T;H)$ because of the Aubin-Lions lemma. Thus, the range of $S$ is relatively compact in $L^2(0,T;H)$ and hence in $L^2(0,T;V)$ since the latter space is closed in the former. Hence, by Schauder’s fixed point theorem, $S$ admits a fixed point. \qed
We have just proved the following result.

**Proposition 2.2.** For any fixed \( \varepsilon > 0 \) and \( \lambda > 0 \), the problem \((2.34)-(2.38)\) below

\[
\frac{\partial f_{e,\lambda}}{\partial t} + \varepsilon v \cdot \nabla f_{e,\lambda} + \text{div}_v ((w * \theta_\lambda - v)f_{e,\lambda}) = 0 \text{ in } Q \times \mathbb{R}^N \tag{2.34}
\]

\[
\frac{\partial u_{e,\lambda}}{\partial t} - \text{div} \left( A_0^* \nabla u_{e,\lambda} + \int_0^t A_1^*(t - \tau, x) \nabla u_{e,\lambda}(\tau, x) d\tau \right) + \nabla p_{e,\lambda} = - \int_{\mathbb{R}^N} (u_{e,\lambda} * \theta_\lambda - v) \gamma_\lambda(v)f_{e,\lambda} dv \text{ in } Q, \tag{2.35}
\]

\[
\text{div} u_{e,\lambda} = 0 \text{ in } Q, \tag{2.36}
\]

\[
u_{e,\lambda}(0, x) = u^0(x), \quad f_{e,\lambda}(0, x, v) := f^0(x, v) = \gamma_\lambda(v)(f^0 * \Theta_\lambda)(x, v), \quad x \in \Omega, v \in \mathbb{R}^N, \tag{2.37}
\]

\[
E_{e,\lambda}(t, x) = \int_{\mathbb{R}^N} (v \cdot \nabla f_{e,\lambda}) dv = \int_{\Omega} (v \cdot \nabla f_{e,\lambda}) dv + \varepsilon \int_{\mathbb{R}^N} (v \cdot \nabla f_{e,\lambda}) |v|^2 dv + \frac{1}{2} \int_{\mathbb{R}^N} \text{div}_v ((u_{e,\lambda} * \theta_\lambda - v)f_{e,\lambda}) dv. \tag{2.43}
\]

\[
a_n \int_{\mathbb{R}^N} f_{e,\lambda} dv = 0, \tag{2.39}
\]

admits a unique solution \((u_{e,\lambda}, f_{e,\lambda}, p_{e,\lambda})\) such that \(u_{e,\lambda} \in L^2(0, T; V)\) with \(\partial u_{e,\lambda}/\partial t \in L^2(0, T; V')\), \(f_{e,\lambda} \in C^1(Q \times \mathbb{R}^N)\) and \(p_{e,\lambda} \in L^\infty(0, T; L^2(\Omega)/\mathbb{R})\).

The following uniform estimates hold true.

**Lemma 2.1.** Let \((u_{e,\lambda}, f_{e,\lambda}, p_{e,\lambda})\) be the solution to \((2.34)-(2.38)\). Then,

\[
\int_{\Omega \times \mathbb{R}^N} (1 + |v|^2) f_{e,\lambda} dv + \int_{\Omega} |u_{e,\lambda}|^2 dv + 2 \int_0^t \int_{\Omega \times \mathbb{R}^N} f_{e,\lambda} |u_{e,\lambda} * \theta_\lambda - v|^2 dv d\tau + \int_0^t \|\nabla u_{e,\lambda}(\tau)\|^2_{L^2(\Omega)} d\tau \leq C \tag{2.39}
\]

for any \(0 \leq t \leq T, \varepsilon > 0\) and \(\lambda > 0\), where \(C > 0\) is independent of both \(\lambda\) and \(\varepsilon\). Moreover if \(f^0 \in L^p(\Omega \times \mathbb{R}^N), (1 \leq p \leq \infty)\), then

\[
\|f_{e,\lambda}\|_{L^\infty(0, T; L^p(\Omega \times \mathbb{R}^N))} \leq \exp(NT) \|f^0\|_{L^p(\Omega \times \mathbb{R}^N)} \text{ for any } \lambda, \varepsilon > 0. \tag{2.40}
\]

It also holds that

\[
\left\| \frac{\partial u_{e,\lambda}}{\partial t} \right\|_{L^2(0, T; H^{-1}(\mathbb{R}^N))} \leq C \tag{2.41}
\]

and

\[
\sup_{\lambda, \varepsilon > 0} \|p_{e,\lambda}\|_{L^2(0, T; L^2(\Omega))} \leq C. \tag{2.42}
\]

**Proof.** The inequality \((2.40)\) has already been obtained (see Eq. \((2.11)\)). Let us now check \((2.39)\). We multiply \((1.1)\) by \(\frac{1}{2} |v|^2\) and \((1.2)\) by \(u_{e,\lambda}\), and we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega \times \mathbb{R}^N} |v|^2 f_{e,\lambda} dv + \int_{\Omega} |u_{e,\lambda}|^2 dv + \int_{\Omega} (A_0^* \nabla u_{e,\lambda} + A_1^* \nabla u_{e,\lambda}) \cdot \nabla u_{e,\lambda} dv + \int_{\Omega} E_{e,\lambda}(t, x) dx = 0, \tag{2.43}
\]

where we set

\[
E_{e,\lambda}(t, x) = \int_{\mathbb{R}^N} f_{e,\lambda}(u_{e,\lambda} * \theta_\lambda - v) \cdot u_{e,\lambda} dv + \frac{\varepsilon}{2} \int_{\mathbb{R}^N} (v \cdot \nabla f_{e,\lambda}) |v|^2 dv + \frac{1}{2} \int_{\mathbb{R}^N} |v|^2 \text{div}_v ((u_{e,\lambda} * \theta_\lambda - v)f_{e,\lambda}) dv.
\]

But

\[
\int_{\Omega} \int_{\mathbb{R}^N} (v \cdot \nabla f_{e,\lambda}) |v|^2 dv dx = \int_{\mathbb{R}^N} \int_{\partial \Omega} f_{e,\lambda} |v|^2 (v \cdot \nu) d\sigma dv = \int_{\{v \cdot \nu > 0\}} f_{e,\lambda} |v|^2 (v \cdot \nu) d\sigma dv + \int_{\{v \cdot \nu < 0\}} f_{e,\lambda} |v|^2 (v \cdot \nu) d\sigma dv.
\]

Since \(v^* = v - 2(v \cdot \nu) \nu\), it holds that \(v^* \cdot \nu = -v \cdot \nu, |v^*|^2 = |v|^2\) and \(dv^* = dv\). Thus, because of the reflection condition \((2.38)\) on \(f_{e,\lambda}\), we have

\[
\int_{\{v \cdot \nu < 0\}} f_{e,\lambda} |v|^2 (v \cdot \nu) d\sigma dv = -\int_{\{v^* \cdot \nu > 0\}} f_{e,\lambda}(t, x, v^*) |v^*|^2 (v^* \cdot \nu) d\sigma d\nu^*,
\]
so that
\[ \int_{\mathbb{R}^N} \int_{\partial \Omega} f_{\varepsilon,\lambda} |v|^2 (v \cdot \nu) d\sigma dv = 0. \]

Also, the following identity holds
\[ \int_{\mathbb{R}^N} |v|^2 \text{div}_v((u_{\varepsilon,\lambda} \ast \theta_{\lambda} - v) f_{\varepsilon,\lambda}) dv = -2 \int_{\mathbb{R}^N} f_{\varepsilon,\lambda}(u_{\varepsilon,\lambda} \ast \theta_{\lambda} - v) \cdot vd\nu. \]

It therefore follows that
\[ \int_{\Omega} E_{\varepsilon,\lambda}(t,x) dx = \int_{\partial \Omega} f_{\varepsilon,\lambda} |u_{\varepsilon,\lambda} \ast \theta_{\lambda} - v|^2 d\sigma dv. \]

Integrating Eq. (2.43) over \((0,t)\) and using the assumption \((A1)\), we are lead to
\[ \int_{\Omega \times \mathbb{R}^N} |v|^2 f_{\varepsilon,\lambda} dv + 2 \int_{0}^{t} \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} |u_{\varepsilon,\lambda} \ast \theta_{\lambda} - v|^2 dv d\tau + \int_{\Omega} |u_{\varepsilon,\lambda}|^2 dx + 2\alpha \int_{0}^{t} \int_{\Omega} |\nabla u_{\varepsilon,\lambda}|^2 dx d\tau \]
\[ \leq -2 \int_{0}^{t} \int_{\Omega} (A_1^* + \nabla u_{\varepsilon,\lambda} \ast \nabla v_{\varepsilon,\lambda}) dv + \int_{\Omega \times \mathbb{R}^N} |v|^2 f_0^1 dv + \int_{\Omega} |u_0|^2 dx. \]

Now, using Young’s inequality, we infer that
\[ 2 \int_{0}^{t} \int_{\Omega} (A_1^* + \nabla u_{\varepsilon,\lambda} \ast \nabla v_{\varepsilon,\lambda}) dv + \int_{0}^{t} \int_{\Omega \times \mathbb{R}^N} |v|^2 f_{\varepsilon,\lambda} dv d\tau \]
\[ \leq -2 \int_{0}^{t} \int_{\Omega} (A_1^* + \nabla u_{\varepsilon,\lambda} \ast \nabla v_{\varepsilon,\lambda}) dv + \int_{0}^{t} \int_{\Omega \times \mathbb{R}^N} |v|^2 f_{\varepsilon,\lambda} dv + \int_{\Omega} |u_0|^2 dx \]
\[ \leq -2 \int_{0}^{t} \int_{\Omega} (A_1^* + \nabla u_{\varepsilon,\lambda} \ast \nabla v_{\varepsilon,\lambda}) dv + \int_{0}^{t} \int_{\Omega \times \mathbb{R}^N} |v|^2 f_{\varepsilon,\lambda} dv + \int_{\Omega} |u_0|^2 dx \]
\[ \leq -2 \int_{0}^{t} \int_{\Omega} (A_1^* + \nabla u_{\varepsilon,\lambda} \ast \nabla v_{\varepsilon,\lambda}) dv + \int_{0}^{t} \int_{\Omega \times \mathbb{R}^N} |v|^2 f_{\varepsilon,\lambda} dv + \int_{\Omega} |u_0|^2 dx \]

where \(c_1 = \sup_{(t,x) \in \Omega} |A_1(t,x,\cdot,\cdot)|^2_{L^\infty(\mathbb{R}^{N+1})} < \infty\). Thus
\[ \int_{\Omega \times \mathbb{R}^N} |v|^2 f_{\varepsilon,\lambda} dv + 2 \int_{0}^{t} \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} |u_{\varepsilon,\lambda} \ast \theta_{\lambda} - v|^2 dv d\tau + \int_{\Omega} |u_{\varepsilon,\lambda}|^2 dx + \alpha \int_{0}^{t} \int_{\Omega} |\nabla u_{\varepsilon,\lambda}|^2 d\tau \]
\[ \leq \int_{\Omega \times \mathbb{R}^N} |v|^2 f_0^1 dv + \int_{\Omega} |u_0|^2 dx + \frac{c_1}{\alpha} \int_{0}^{t} \int_{\Omega} |\nabla u_{\varepsilon,\lambda}|^2 d\tau. \]

(2.44)

We infer from Eq. (2.44) that
\[ \alpha \int_{0}^{t} \int_{\Omega} |\nabla u_{\varepsilon,\lambda}|^2_{L^2(\Omega)} d\tau \leq c_2 + \frac{c_1}{\alpha} \int_{0}^{t} \left( \int_{0}^{t} \int_{\Omega} |\nabla u_{\varepsilon,\lambda}(s)|^2_{L^2(\Omega)} ds \right) d\tau \]

where
\[ c_2 = \int_{\Omega \times \mathbb{R}^N} |v|^2 f_{\varepsilon,\lambda} dv + \int_{\Omega} |u_0|^2 dx \leq C(N,p) \|f_0^0\|_{L_p(\Omega \times \mathbb{R}^N)} + \int_{\Omega} |u_0|^2 dx < \infty. \]

It readily follows from Gronwall’s inequality that
\[ \int_{0}^{t} |\nabla u_{\varepsilon,\lambda}|^2_{L^2(\Omega)} d\tau \leq \exp\left( \int_{0}^{t} \frac{c_1}{\alpha} d\tau \right) \left[ \int_{0}^{t} \frac{c_2}{\alpha} \exp\left( -\int_{0}^{s} \frac{c_1}{\alpha} d\tau \right) ds \right] \]
\[ = \frac{c_2}{\alpha} \exp\left( \int_{0}^{t} \frac{c_1}{\alpha} ds \right) \left[ \frac{c_1}{\alpha} \int_{0}^{t} \exp\left( -\int_{0}^{s} \frac{c_1}{\alpha} ds \right) ds \right] \]
\[ \leq \frac{c_2}{\alpha} \exp\left( \int_{0}^{t} \frac{c_1}{\alpha} ds \right) \int_{0}^{\infty} \exp\left( -\frac{c_1}{\alpha} s^2 \right) ds \]
\[ = \frac{c_2}{\alpha} \sqrt{\frac{2\pi}{c_1}} \exp\left( \frac{c_1 T^2}{2\alpha^2} \right) \text{ for all } 0 \leq t \leq T. \]

Setting
\[ c_3 = \frac{c_2}{2} \sqrt{\frac{2\pi}{c_1}} \exp\left( \frac{c_1 T^2}{2\alpha^2} \right), \]
we get
\[ \int_0^1 \| \nabla u_{\epsilon,\lambda} \|^2_{L^2(\Omega)} \, d\tau \leq C_3, \]
and the above inequality entails
\[ \int_{\Omega \times \mathbb{R}^N} |v|^2 f_{\epsilon,\lambda} \, dx \, dv + \int_{\Omega} |u_{\epsilon,\lambda}|^2 \, dx + 2 \int_0^1 \int_{\Omega \times \mathbb{R}^N} f_{\epsilon,\lambda} |u_{\epsilon,\lambda} \ast \theta_{\lambda} - v|^2 \, dx \, dv \, d\tau \leq c_2 + c_3 T. \]
We deduce Eq. (2.39) by letting \( C = c_2 + c_3 T \).

The uniform estimate (2.41) is obtained as Eq. (2.33), and Eq. (2.42) follows in a trivial manner (see e.g. [7]). This concludes the proof. □

2.5. **Passing to the limit \( \lambda \to 0 \).** We wish to pass to the limit as \( \lambda \to 0 \) in the sequence of solutions \((u_{\epsilon,\lambda}, f_{\epsilon,\lambda})\) in order to prove the existence of the solution to our initial problem (1.1)-(1.5). Owing to Lemma 2.1, we have
\[ \| f_{\epsilon,\lambda} \|_{L^2(0,T;L^p(\Omega \times \mathbb{R}^N))} \leq C \text{ for all } 1 \leq p \leq \infty, \]
\[ \| u_{\epsilon,\lambda} \|_{L^2(0,T;L^2(\Omega)^N)} \leq C, \quad \| \nabla u_{\epsilon,\lambda} \|_{L^2(\Omega)} \leq C \quad \text{and} \quad \left\| \frac{\partial u_{\epsilon,\lambda}}{\partial t} \right\|_{L^2(0,T;H^{-1}(\Omega)^N)} \leq C. \]
Using the above uniform estimates (in \( \lambda \)), we deduce that, given an ordinary sequence \( \lambda = (\lambda_n)_n \) (with \( 0 < \lambda_n \leq 1, \lambda_n \to 0 \) when \( n \to \infty \), which we denote by \( \lambda \to 0 \)), there exist a subsequence of \( \lambda \) (still denoted by \( \lambda \)), functions \( u_\epsilon \in L^2(0,T;V) \cap L^\infty(0,T;H) \), \( f_\epsilon \in L^2(0,T;L^p(\Omega \times \mathbb{R}^N)) \) and \( p_\epsilon \in L^2(0,T;L^2(\Omega)/\mathbb{R}) \) such that, as \( \lambda \to 0 \),
\[ f_{\epsilon,\lambda} \to f_\epsilon \text{ in } L^\infty(0,T;L^p(\Omega \times \mathbb{R}^N)) \text{ weak *}, \]
\[ u_{\epsilon,\lambda} \to u_\epsilon \text{ in } L^2(0,T;V) \text{ weak}, \]
\[ u_{\epsilon,\lambda} \to u_\epsilon \text{ in } L^2(0,T;H) \text{ strong}, \]
\[ \frac{\partial u_{\epsilon,\lambda}}{\partial t} \to \frac{\partial u_\epsilon}{\partial t} \text{ in } L^2(0,T;H^{-1}(\Omega)^N) \text{ weak}, \]
and
\[ p_{\epsilon,\lambda} \to p_\epsilon \text{ in } L^2(0,T;L^2(\Omega)/\mathbb{R}) \text{ weak}. \]
Let \( \phi \in C_0^\infty(\Omega) \) where \( \Omega = Q \times \mathbb{R}^N \). We multiply the Vlasov equation (2.34) by \( \phi \) and integrate by parts to get
\[ -\int_\Omega f_{\epsilon,\lambda} \left[ \frac{\partial \phi}{\partial t} + \epsilon v \cdot \nabla \phi + (u_{\epsilon,\lambda} \ast \theta_{\lambda} - v) \cdot \nabla_v \phi \right] \, dx \, dv \, dt = 0. \]
We consider the terms in (2.50) respectively. It is easy to see that, as \( \lambda \to 0 \),
\[ \int_\Omega f_{\epsilon,\lambda} \frac{\partial \phi}{\partial t} \, dx \, dv \, dt \to \int_\Omega f_\epsilon \frac{\partial \phi}{\partial t} \, dx \, dv \, dt. \]
For the second and fourth terms, since the functions \((t,x,v) \mapsto v \cdot \nabla \phi \) and \((t,x,v) \mapsto v \cdot \nabla_v \phi \) belong to \( C_0^\infty(\Omega) \), we use them as test functions to get, as \( \lambda \to 0 \),
\[ \int_\Omega f_{\epsilon,\lambda} [\epsilon v \cdot \nabla \phi - v \cdot \nabla_v \phi] \, dx \, dv \to \int_\Omega f_\epsilon [\epsilon v \cdot \nabla \phi - v \cdot \nabla_v \phi] \, dx \, dv. \]
Now, as for the term \( \int_\Omega f_{\epsilon,\lambda} (u_{\epsilon,\lambda} \ast \theta_{\lambda}) \cdot \nabla_v \phi dx dv dt \), we claim that
\[ \int_\Omega f_{\epsilon,\lambda} (u_{\epsilon,\lambda} \ast \theta_{\lambda}) \cdot \nabla_v \phi dx dv dt \to \int_\Omega f_\epsilon u_\epsilon \cdot \nabla_v \phi dx dv dt. \]
Indeed it is sufficient to prove that, under the convergence result (2.47) and for any \( \psi \in C_0^\infty(\Omega)^N \),
\[ \int_\Omega f_{\epsilon,\lambda} u_{\epsilon,\lambda} \cdot \psi dx dv dt \to \int_\Omega f_\epsilon u_\epsilon \cdot \psi dx dv dt \]
and to apply it with \( \psi = \nabla_v \phi \). For the proof of (2.52), we refer to the proof of a more involved result, Lemma 4.1 in Section 4.
Returning to (2.51), we have
\[
\int Q f_{\varepsilon, \lambda}(u_{\varepsilon, \lambda} \ast \phi) \cdot \nabla v \phi dxdt dv = \int Q f_{\varepsilon, \lambda} [(u_{\varepsilon, \lambda} - u_{\varepsilon}) \ast \theta_{\lambda}] \cdot \nabla v \phi dxdt dv \\
+ \int Q f_{\varepsilon, \lambda}(u_{\varepsilon} \ast \theta_{\lambda}) \cdot \nabla v \phi dxdt dv = (I) + (II).
\]
For the term (I), we have the estimate
\[
|\langle I \rangle | \leq \| f_{\varepsilon, \lambda} \|_{L^\infty(Q)} \| \nabla_v \phi \|_\infty \| u_{\varepsilon, \lambda} - u_{\varepsilon} \|_{L^2(Q)} \| \theta_{\lambda} \|_{L^1(Q)} \\
\leq C \| u_{\varepsilon, \lambda} - u_{\varepsilon} \|_{L^2(Q)},
\]
in which C is a positive constant independent of \( \varepsilon \) and \( \lambda \). Thus, it follows from (2.47) that \( \langle I \rangle \to 0 \).

Regarding (II), we have that \( u_{\varepsilon} \ast \theta_{\lambda} \to u_{\varepsilon} \) in \( L^2(Q) \)-strong (use once again (2.47)), so that by (2.52) we arrive at (II) \( \to \int Q f_{\varepsilon} u_{\varepsilon} \cdot \nabla_v \phi \ dx dt \ dv \).

Taking into account all the above convergence results and passing to the limit in (2.50) as \( \varepsilon \to 0 \), we obtain
\[
- \int f_{\varepsilon} \left[ \frac{\partial \phi}{\partial t} + \varepsilon \nabla \cdot \phi + (u_{\varepsilon} - v) \cdot \nabla_v \phi \right] dx dt dv = 0,
\]
which amounts to
\[
\frac{\partial f_{\varepsilon}}{\partial t} + \varepsilon \nabla \cdot f_{\varepsilon} + \text{div}_v ((u_{\varepsilon} - v) f_{\varepsilon}) = 0 \text{ in } D'(O).
\]

Proceeding as in [21] Section 4] we recover the reflection boundary condition
\[
f_{\varepsilon}(t, x, v) = f_{\varepsilon}(t, x, v^+) \quad \text{for } x \in \partial \Omega \text{ with } v \cdot \nu(x) < 0.
\]
Next using the inequality \( 1 - \gamma_{\lambda}(v) \leq 1_{\{v \geq 1/2\lambda\}} \), we get
\[
\left| \int_{\Omega \times \mathbb{R}^N} (1 - \gamma_{\lambda}(v))(f_0 \ast \Theta_{\lambda}) dx dv \right| \leq \int_{\Omega \times \mathbb{R}^N} 1_{\{v \geq 1/2\lambda\}}(f_0 \ast \Theta_{\lambda}) dx dv \\
\leq 4\lambda^2 \int_{\Omega \times \mathbb{R}^N} |v|^2 (f_0 \ast \Theta_{\lambda}) dx dv \\
\leq C\lambda^2 \| f_0 \|_{L^1(\Omega \times \mathbb{R}^N)}; \text{ see (2.12)}.
\]

So, the functions \( f_0 \ast \Theta_{\lambda} \) and \( \gamma_{\lambda}(v)(f_0 \ast \Theta_{\lambda}) \) have the same \( L^1 \)-limit \( f_0 \) as \( \lambda \to 0 \). Hence, letting \( \lambda \to 0 \), we arrive at
\[
f_{\varepsilon}(0, x, v) = f_0(x, v) \text{ for } (x, v) \in \Omega \times \mathbb{R}^N.
\]

Let us now deal with the Stokes system (2.35). We choose \( \Phi \in C_0^\infty(\Omega)^N \) and multiply (2.35) by \( \psi \) and integrate over \( Q \);
\[
- \int_Q u_{\varepsilon, \lambda} \cdot \frac{\partial \psi}{\partial t} dx dt + \int_Q A_0^* \nabla u_{\varepsilon, \lambda} \cdot \nabla \Phi dx dt + \int_Q (A_1^* \ast \nabla u_{\varepsilon, \lambda}) \cdot \nabla \Phi dx dt \\
- \int_Q p_{\varepsilon, \lambda} \text{div} \Phi dx dt = - \int_Q \gamma_{\lambda}(v)(f_{\varepsilon, \lambda}(u_{\varepsilon, \lambda} - v) \cdot \Phi dx dt dv.
\]

In Eq. (2.54), only the right-hand side is more involved. However, proceeding as in (2.53), one can check that
\[
\int Q \gamma_{\lambda}(v)f_{\varepsilon, \lambda}(u_{\varepsilon, \lambda} - v) \cdot \Phi dx dt dv
\]
and
\[
\int Q f_{\varepsilon}(u_{\varepsilon} - v) \cdot \Phi dx dt dv
\]
have the same limit, which is, using (2.52) and the convergence results (2.45)-(2.49), nothing else but
\[
\int Q f_{\varepsilon}(u_{\varepsilon} - v) \cdot \Phi dx dt dv.
\]
Thus, passing to the limit in (2.54), we realize that \( \mathbf{u}_\varepsilon \) solves the equation
\[
\frac{\partial \mathbf{u}_\varepsilon}{\partial t} - \text{div} \left( A_0^\varepsilon \nabla \mathbf{u}_\varepsilon + \int_0^t A_1^\varepsilon(t-\tau,x) \nabla \mathbf{u}_\varepsilon(\tau,x) d\tau \right) + \nabla p_\varepsilon = - \int_{\mathbb{R}^N} (\mathbf{u}_\varepsilon - \mathbf{v}) f_\varepsilon dv \text{ in } Q.
\]
We also obtain the initial condition \( \mathbf{u}_\varepsilon(0,x) = \mathbf{u}^0(\varepsilon), \ x \in \Omega \).

We have just shown that \((\mathbf{u}_\varepsilon, f_\varepsilon, p_\varepsilon)\) solves the system (1.1)-(1.5). This concludes the proof of Theorem 2.1.

It remains to check that the above triple verifies the same estimates as in Lemma 2.1. As we are going to see below, this is a mere consequence of the following well known result:

- If \( B \) is a Banach space with norm \( \| \cdot \| \) and \( f_n \to f \) in \( B \)-weak or weak*, then \( \| f \| \leq \lim \inf \| f_n \| \).

We can therefore state the counterpart of Lemma 2.1

**Lemma 2.2.** Let \((\mathbf{u}_\varepsilon, f_\varepsilon, p_\varepsilon)\) be the solution to \((1.1)-(1.5)\) constructed in Subsection 2.5. Then,
\[
\int_{\Omega \times \mathbb{R}^N} \left(1 + |v|^2\right) f_\varepsilon dv + \int_{\Omega} |\mathbf{u}_\varepsilon|^2 dx + 2 \int_0^t \int_{\Omega \times \mathbb{R}^N} f_\varepsilon |\mathbf{u}_\varepsilon - v|^2 dv dt \leq C \tag{2.55}
\]
for any \( 0 \leq t \leq T \) and \( \varepsilon > 0 \), where \( C > 0 \) is independent of \( \varepsilon \). Moreover if \( f^0 \in L^p(\Omega \times \mathbb{R}^N) \), \((1 \leq p \leq \infty)\), then
\[
\| f_\varepsilon \|_{L^\infty(0,T;L^p(\Omega \times \mathbb{R}^N))} \leq \exp(NT) \| f^0 \|_{L^p(\Omega \times \mathbb{R}^N)} \text{ for any } \varepsilon > 0. \tag{2.56}
\]

It also holds that
\[
\left\| \frac{\partial \mathbf{u}_\varepsilon}{\partial t} \right\|_{L^2(0,T;H^{-1}(\Omega)^N)} \leq C \tag{2.57}
\]
and
\[
\sup_{\varepsilon > 0} \| p_\varepsilon \|_{L^2(0,T;L^2(\Omega))} \leq C. \tag{2.58}
\]

**Proof.** We follow arguments similar to those in [4]. First and foremost, we have by (2.39) that
\[
\int_{\Omega \times \mathbb{R}^N} (1 + |v|^2) f_{\varepsilon,\lambda} dv + \int_{\Omega} |\mathbf{u}_{\varepsilon,\lambda}|^2 dx + 2 \int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} |\mathbf{u}_{\varepsilon,\lambda} + \theta_\lambda - v|^2 dv dt + \int_0^t \int_{\Omega \times \mathbb{R}^N} \| \nabla \mathbf{u}_{\varepsilon,\lambda}(\tau) \|^2_{L^2(\Omega)} d\tau \leq C \tag{2.59}
\]
The only term to deal with is actually \( \int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} |\mathbf{u}_{\varepsilon,\lambda} + \theta_\lambda - v|^2 dv dt \) which we write as
\[
\int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} |\mathbf{u}_{\varepsilon,\lambda} + \theta_\lambda - v|^2 dv dt = \int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} |\mathbf{u}_{\varepsilon,\lambda} + \theta_\lambda|^2 dv dt - 2 \int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} (\mathbf{u}_{\varepsilon,\lambda} + \theta_\lambda) : \mathbf{v} dv dt + \int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} |v|^2 dv dt = (I) - 2(II) + (III).
\]
Concerning (III), we know that \( f_{\varepsilon,\lambda} \to f_{\varepsilon} \) in \( L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^N)) \)-weak*. Let \( 0 < \eta < 1 \). Then because of (2.59) we have
\[
\int_{\Omega \times \mathbb{R}^N} |v|^2 f_{\varepsilon,\lambda} \gamma_\eta(v) dv \leq \int_{\Omega \times \mathbb{R}^N} |v|^2 f_{\varepsilon,\lambda} dv \leq C.
\]
Hence there exists a function \( g_\eta \in L^\infty([0,t]) \) such that, up to a subsequence of \( \lambda \to 0 \), setting \( M_2(f_{\varepsilon,\lambda} \gamma_\eta)(\tau) = \int_{\Omega \times \mathbb{R}^N} |v|^2 f_{\varepsilon,\lambda} \gamma_\eta(v) dv \),
\[
M_2(f_{\varepsilon,\lambda} \gamma_\eta) \to g_\eta \text{ in } L^\infty([0,t]) \text{-weak*;}
\]
thus
\[
\| g_\eta \|_{L^\infty([0,t])} \leq \lim \inf_{\lambda \to 0} \int_0^t M_2(f_{\varepsilon,\lambda} \gamma_\eta)(\tau) d\tau \leq \lim \inf_{\lambda \to 0} \int_0^t \int_{\Omega \times \mathbb{R}^N} |v|^2 f_{\varepsilon,\lambda} dv dt.
\]
On the other hand, the weak* convergence $f_{ε,λ} → f_ε$ in $L^∞(0, T; L^∞(Ω × ℝ^N))$ implies

$$M_2(f_{ε,λ} γ_η) → M_2(f_ε,λ γ_η) = \int_{Ω × ℝ^N} |v|^2 f_ε γ_η(v) dv$$

in $L^∞([0, t])$-weak* since the product of a function $χ ∈ L^1([0, t])$ by $|v|^2 γ_η$ lies in $L^1((0, t) × Ω × ℝ^N)$. The uniqueness of the weak* limit yields $M_2(f_{ε,λ} γ_η)(τ) = g_η(τ)$ a.e. $τ ∈ (0, t)$. It therefore follows from the Fatou’s lemma and from the fact that $|v|^2 f_ε γ_η(v) \rightarrow |v|^2 f_ε$ as $η → 0$, that

$$\int_{Ω × ℝ^N} |v|^2 f_ε dv ≤ \lim_{η → 0} \inf M_2(f_{ε,λ} γ_η)(τ) = \lim_{η → 0} \inf M_2(f_ε,λ)(τ) ≤ \lim_{λ → 0} \inf M_2(f_ε,λ) \|L^∞(0, t),$$

i.e.

$$\int_{Ω × ℝ^N} |v|^2 f_ε dv ≤ \lim_{λ → 0} \inf \int_{Ω × ℝ^N} |v|^2 f_ε,λ dv,$$

whence

$$\int_0^t \int_{Ω × ℝ^N} |v|^2 f_ε dv ≤ \lim_{λ → 0} \int_0^t \int_{Ω × ℝ^N} |v|^2 f_ε,λ dv.$$

As for the first term (I), one has

$$\int_0^t \int_{Ω × ℝ^N} f_ε,λ |u_ε,λ * θ_λ|^2 dv = \int_0^t \int_{Ω × ℝ^N} |u_ε,λ * θ_λ|^2 f_ε,λ(1 - γ_η(v)) dv + \int_0^t \int_{Ω × ℝ^N} |u_ε,λ * θ_λ|^2 f_ε,λ γ_η(v) dv \geq \int_0^t \int_{Ω × ℝ^N} |u_ε,λ * θ_λ|^2 f_ε,λ γ_η(v) dv."

For any fixed $η$,

$$\int_0^t \int_{Ω × ℝ^N} |u_ε,λ * θ_λ|^2 f_ε,λ γ_η(v) dv → \int_0^t \int_{Ω × ℝ^N} |u_ε|^2 f_ε γ_η(v) dv + \int_0^t \int_{Ω × ℝ^N} |u_ε|^2 f_ε γ_η(v) dv$$

when $λ → 0$. Indeed, it is easy to see that $|u_ε,λ * θ_λ|^2 γ_η → |u_ε|^2 f_ε γ_η$ in $L^1((0, t) × Ω × ℝ^N)$-strong as $λ → 0$, so that combining this with (2.45) (for $p = ∞$) we get our result. Thus, using once again Fatou’s lemma,

$$\int_0^t \int_{Ω × ℝ^N} |u_ε|^2 f_ε dv + \lim \inf_{η → 0} \int_0^t \int_{Ω × ℝ^N} |u_ε|^2 f_ε γ_η(v) dv + \lim \inf_{λ → 0} \int_0^t \int_{Ω × ℝ^N} |u_ε|^2 f_ε,λ dv.$$

Finally, for (II), we have

$$\int_0^t \int_{Ω × ℝ^N} f_ε,λ(u_ε,λ * θ_λ) · v dv = \int_0^t \int_{Ω × ℝ^N} f_ε,λ(\bar{u}_ε,λ * θ_λ - u_ε) · v dv + \int_0^t \int_{Ω × ℝ^N} f_ε,λ u_ε · v dv = (A) + (B).$$

Dealing with (A), we have, by setting $v_ε,λ = u_ε,λ * θ_λ - u_ε$,

$$(A) = \int_0^t \int_{Ω × ℝ^N} f_ε,λ(1 - γ_η(v)) v_ε,λ · v dv + \int_0^t \int_{Ω × ℝ^N} f_ε,λ γ_η(v) v_ε,λ · v dv,$$
and using the inequality \(1 - \gamma_\eta(v) \leq 1_{\{v \geq 1/2\eta\}}\),
\[
\left| \int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} (1 - \gamma_\eta(v)) \mathbf{v}_{\varepsilon,\lambda} \cdot v dxdvd\tau \right| \leq \int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} 1_{\{|v| \geq 1/2\eta\}} |\mathbf{v}_{\varepsilon,\lambda}| |v| dxdvd\tau \\
\leq \int_0^t \int_{\Omega} \left[ \int_{\mathbb{R}^N} |v| f_{\varepsilon,\lambda} 1_{\{|v| \geq 1/2\eta\}} dv \right] |\mathbf{v}_{\varepsilon,\lambda}| dxd\tau \\
\leq \int_0^t \left\{ \left( \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |v| f_{\varepsilon,\lambda} 1_{\{|v| \geq 1/2\eta\}} dv \right)^{\frac{6}{5}} dx \right)^{\frac{5}{6}} \left( \int_{\Omega} |\mathbf{v}_{\varepsilon,\lambda}|^6 dx \right)^{\frac{1}{6}} \right\} d\tau.
\]

But
\[
\int_\Omega \left( \int_{\mathbb{R}^N} |v| f_{\varepsilon,\lambda} 1_{\{|v| \geq 1/2\eta\}} dv \right)^{\frac{6}{5}} dx \leq C(\Omega) \left( \int_{\mathbb{R}^N} |v| f_{\varepsilon,\lambda} 1_{\{|v| \geq 1/2\eta\}} dv dx \right)^{\frac{6}{5}} \\
\leq C(\Omega) \lambda^{\frac{5}{6}} \left( \int_{\mathbb{R}^N \times \Omega} |v|^2 f_{\varepsilon,\lambda} 1_{\{|v| \geq 1/2\eta\}} dv dx \right)^{\frac{6}{5}} \\
\leq C\lambda^{\frac{5}{6}} \text{ because of (2.39)}.
\]

Recalling that \(\mathbf{v}_{\varepsilon,\lambda} \in L^2(0,T;H^1_0(\Omega)^N) \rightarrow L^1(0,T;L^6(\Omega)^N)\), it follows from (2.39) that
\[
\int_0^t \left( \int_{\Omega} |\mathbf{v}_{\varepsilon,\lambda}|^6 dx \right)^{\frac{1}{6}} d\tau \leq C,
\]
so that
\[
\left| \int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} (1 - \gamma_\eta(v)) \mathbf{v}_{\varepsilon,\lambda} \cdot v dxdvd\tau \right| \leq C\lambda.
\]

It follows that \(\int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} (1 - \gamma_\eta(v)) \mathbf{v}_{\varepsilon,\lambda} \cdot v dxdvd\tau \to 0\) as \(\lambda \to 0\).

We claim that
\[
\int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} \gamma_\eta(v) \mathbf{v}_{\varepsilon,\lambda} \cdot v dxdvd\tau \to 0\) as \(\lambda \to 0\).

Indeed
\[
\left| \int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon,\lambda} \gamma_\eta(v) \mathbf{v}_{\varepsilon,\lambda} \cdot v dxdvd\tau \right| \leq \|f_{\varepsilon,\lambda}\|_{L^\infty(\Omega)} \int_0^t \int_{\Omega} |v| |\mathbf{v}_{\varepsilon,\lambda}| dxdvd\tau \\
\leq C \int_0^t \int_{B(0,2)} |v| |\mathbf{v}_{\varepsilon,\lambda}| dxdvd\tau
\]
and
\[
\int_0^t \int_{\Omega} |v| |\mathbf{v}_{\varepsilon,\lambda}| dxdvd\tau \leq 2 |B(0,2)||\Omega|^{\frac{1}{2}} \|\mathbf{v}_{\varepsilon,\lambda}\|_{L^2(Q)} \to 0\) as \(\lambda \to 0\),
\]
since \(\mathbf{v}_{\varepsilon,\lambda} \to 0\) in \(L^2(Q)\) as \(\lambda \to 0\). It follows that \((A) \to 0\) as \(\lambda \to 0\). We use the same kind of arguments to show that \((B) \to \int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon} \mathbf{u}_\varepsilon \cdot v dxdvd\tau\), that is,
\[
(II) \to \int_0^t \int_{\Omega \times \mathbb{R}^N} f_{\varepsilon} \mathbf{u}_\varepsilon \cdot v dxdvd\tau.
\]

Coming back to (2.59) and taking there the lim inf as \(\lambda \to 0\), we get at once (2.55). The lemma follows thereby.
Remark 2.2. We observe that the sequence \( \left( \int_{Q_N} f_\varepsilon(u_\varepsilon - v) \, dv \right)_{\varepsilon > 0} \) is bounded in \( L^1(Q)^N \). Indeed
\[
\int_Q \left| \int_{Q_N} f_\varepsilon(u_\varepsilon - v) \, dv \right| \, dx \, dt \\
\leq \int_Q \int_{Q_N} f_\varepsilon |u_\varepsilon - v| \, dv \, dx \, dt \\
= \int_Q \sqrt{f_\varepsilon(1 + |v|)} \sqrt{f_\varepsilon \frac{|u_\varepsilon - v|}{(1 + |v|)}} \, dv \, dx \, dt \\
\leq \sqrt{2} \left( \int_Q f_\varepsilon(1 + |v|) \, dv \right)^{1/2} \left( \int_Q \frac{f_\varepsilon |u_\varepsilon - v|^2}{(1 + |v|)^2} \, dv \, dx \, dt \right)^{1/2} \\
\leq C \text{(see estimate (2.55) of Lemma 2.2)}
\]

3. Brief introduction to \( \Sigma \)-convergence

This section is far from being a comprehensive introduction to \( \Sigma \)-convergence. It is rather a pretext for fixing notations and recalling fundamental results pertaining to \( \Sigma \)-convergence. We shall restrict ourselves to concepts relevant to our context.

3.1. Algebras with mean value - An overview. We refer the reader to [5, 23, 36, 39] for an extensive presentation of the concept of algebras with mean value (algebras wmv, in short).

Let \( A \) be an algebra wmv on \( \mathbb{R}^N \), that is, a closed subalgebra of the \( \mathcal{C}^* \)-algebra of bounded uniformly continuous functions on \( \mathbb{R}^N \), BUC(\( \mathbb{R}^N \)), which contains the constants, is translation invariant and is such that any of its elements possesses a mean value in the following sense: for any \( u \in A \), the sequence \( (u^\varepsilon)_{\varepsilon > 0} \) (defined by \( u^\varepsilon(x) = u(x/\varepsilon) \), \( x \in \mathbb{R}^N \)) weakly*-converges in \( L^\infty(\mathbb{R}^N) \) to some constant real function \( M(u) \) (called the mean value of \( u \)) as \( \varepsilon \to 0 \). We denote by \( \Delta(A) \) the spectrum of \( A \) and by \( \mathcal{G} \) the Gelfand transformation on \( A \). Let \( B^p_A(\mathbb{R}^N) \) (\( 1 \leq p < \infty \)) denote the Besicovitch space associated to \( A \), that is, the closure of \( A \) with respect to the Besicovitch seminorm
\[
\|u\|_p = \left( \limsup_{r \to +\infty} \frac{1}{|B_r|} \int_{B_r} |u(y)|^p \, dy \right)^{1/p}
\]
where \( B_r \) is the open ball of \( \mathbb{R}^N \) centered at the origin and of radius \( r > 0 \). We set
\[
B^\infty_A(\mathbb{R}^N) = \{ f \in \cap_{1 \leq p < \infty} B^p_A(\mathbb{R}^N) : \sup_{1 \leq p < \infty} \|f\|_p < \infty \}
\]
and we endow it with the seminorm \( \|f\|_\infty = \sup_{1 \leq p < \infty} \|f\|_p \). So topologized, the spaces \( B^p_A(\mathbb{R}^N) \) (\( 1 \leq p \leq \infty \)) are complete seminormed vector spaces which are not in general Fréchet spaces since they are not separated in general. We denote by \( B^p_{\text{inj}}(\mathbb{R}^N) \) the completion of \( B^p_A(\mathbb{R}^N) \) with respect to \( \|\cdot\|_p \) for \( 1 \leq p < \infty \), and with respect to \( \|\cdot\|_\infty \) for \( p = \infty \). The following hold true [24, 29]:

(1) The Gelfand transformation \( \mathcal{G} : A \to C(\Delta(A)) \) extends by continuity to a unique continuous linear mapping (still denoted by \( \mathcal{G} \)) of \( B^p_A(\mathbb{R}^N) \) into \( L^p(\Delta(A)) \), which in turn induces an isometric isomorphism \( \mathcal{G}_1 \) of \( B^p_A(\mathbb{R}^N)/\mathcal{N} = B^p_{\text{inj}}(\mathbb{R}^N) \) onto \( L^p(\Delta(A)) \) (where \( \mathcal{N} = \{ u \in B^p_A(\mathbb{R}^N) : \mathcal{G}(u) = 0 \} \)). Moreover if \( u \in B^p_A(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) then \( \mathcal{G}(u) \in L^\infty(\Delta(A)) \) and \( \|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty(\mathbb{R}^N)} \).

(2) The mean value \( M \) defined on \( A \), extends by continuity to a positive continuous linear form (still denoted by \( M \)) on \( B^p_A(\mathbb{R}^N) \) satisfying \( M(u) = \int_{\Delta(A)} \mathcal{G}(u) \, d\beta \) (\( u \in B^p_A(\mathbb{R}^N) \)). Furthermore, \( M(\tau_a u) = M(u) \) for each \( u \in B^p_A(\mathbb{R}^N) \) and all \( a \in \mathbb{R}^N \), where \( \tau_a u = u(\cdot + a) \). Moreover for \( u \in B^p_A(\mathbb{R}^N) \) we have \( \|u\|_p = \{M(|u|^p)\}^{1/p} \) and for \( u + \mathcal{N} \in B^p_{\text{inj}}(\mathbb{R}^N) \) we may still define its mean value once again denoted by \( M \), as \( M(u + \mathcal{N}) = M(u) \).

For \( u = v + \mathcal{N} \in B^p_{\text{inj}}(\mathbb{R}^N) \) (\( 1 \leq p \leq \infty \)) and \( y \in \mathbb{R}^N \), we define in a natural way the translate \( \tau_y u = v(\cdot + y) + \mathcal{N} \) of \( u \), and as it can be seen in [24, 36], this is well defined and induces a strongly continuous \( N \)-parameter group of isometries \( T(y) : B^p_{\text{inj}}(\mathbb{R}^N) \to B^p_A(\mathbb{R}^N) \) defined by \( T(y)u = \tau_y u \). We denote by \( \partial \mathcal{G}/\partial y_i \) (\( 1 \leq i \leq N \)) the infinitesimal generator of \( T(y) \) along the \( i \)-th coordinate direction. We refer the reader to
Indeed, as proved in [34], we have \(1\) be real numbers satisfying

\[\mu \in \Delta(A) \text{ and } f \in A, \text{ define } T_\mu f \text{ by } T_\mu f(y) = \mu(\tau_y f), \quad y \in \mathbb{R}^N.\]

\(T_\mu\) is well defined as an element of \(\text{BUC}(\mathbb{R}^N)\) since \(A\) is translation invariant. Whence a bounded linear operator \(T_\mu : A \to \text{BUC}(\mathbb{R}^N)\).

**Definition 3.1.** The algebra \(A\) is said to be {	extit{introverted}} if \(T_\mu(A) \subset A\) for any \(\mu \in \Delta(A)\).

Let \(A\) be an introverted algebra \(\text{wmv}\) on \(\mathbb{R}^N\). Then [34, Theorem 3.2] its spectrum \(\Delta(A)\) is a compact topological semigroup. In order to simplify the notations, the semigroup operation in \(\Delta(A)\) is additively written. With this in mind, set

\[K(A) = \cap_{s \in \Delta(A)} (s + \Delta(A)), \text{ the kernel of } \Delta(A).\]

The following result provides us with the structure of \(K(A)\).

**Theorem 3.1 (34, Theorem 3.4).** Let \(A\) be an introverted algebra \(\text{wmv}\) on \(\mathbb{R}^N\). Then

(i) \(K(A)\) is a compact topological group.

(ii) The mean value \(M\) on \(A\) can be identified as the Haar integral over \(K(A)\).

With the help of Theorem 3.1, we can define the convolution over \(\Delta(A)\) in terms of its kernel \(K(A)\). Indeed, as proved in [34], we have \(r + s \in K(A)\) whenever \(r \in \Delta(A)\) and \(s \in K(A)\). Thus, let \(p, q, m \geq 1\) be real numbers satisfying \(\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{m}\). For \(u \in L^p(\Delta(A))\) and \(v \in L^q(\Delta(A))\) we define the convolution product \(u \ast v\) as follows:

\[(u \ast v)(s) = \int_{K(A)} u(r)v(s-r)d\beta(r), \text{ a.e. } s \in \Delta(A),\]

where \(-r\) stands for the inverse of \(r \in K(A)\) (recall that \(K(A)\) is an Abelian group). Then \(\hat{u}\) is well defined since \(K(A)\) is an ideal of \(\Delta(A)\), and we have that \(\int_{K(A)} u(r)v(s-r)d\beta(r) = \int_{\Delta(A)} u(r)v(s-r)d\beta(r)\) since \(\beta\) is supported by \(K(A)\). Indeed for \(s \in \Delta(A)\) and \(r \in K(A)\), \(-r\) exists in \(K(A)\) and \(s - r \in K(\Delta(A))\). It holds that \(\hat{u} \ast v \in L^m(\Delta(A))\) and further:

\[\|u \ast v\|_{L^m(\Delta(A))} \leq \|u\|_{L^p(\Delta(A))}\|v\|_{L^q(\Delta(A))}.\]

Now let \(u \in L^p(\mathbb{R}^N; L^p(\Delta(A)))\) and \(v \in L^q(\mathbb{R}^N; L^q(\Delta(A)))\). We define the double convolution \(u \ast v\) as follows:

\[(u \ast v)(x, s) = \int_{\mathbb{R}^N} [(u(t, \cdot) \hat{\ast} v(x-t, \cdot))(s)] dt\]

\[= \int_{\mathbb{R}^N} \int_{K(A)} u(t, r)v(x-t, s-r)d\beta(r) dt, \text{ a.e. } (x, s) \in \mathbb{R}^N \times \Delta(A).\]

Then \(\ast\ast\) is well defined as an element of \(L^m(\mathbb{R}^N \times \Delta(A))\) and satisfies

\[\|u \ast v\|_{L^m(\mathbb{R}^N \times \Delta(A))} \leq \|u\|_{L^p(\mathbb{R}^N \times \Delta(A))}\|v\|_{L^q(\mathbb{R}^N \times \Delta(A))}.\]

It is to be noted that if \(u \in L^p(\Omega; L^p(\Delta(A))), \) and \(v \in L^q(\mathbb{R}^N; L^q(\Delta(A)))\), we still define \(u \ast v\) by replacing \(u\) by its zero extension over \(\mathbb{R}^N\).

Finally, for \(u \in L^p(\mathbb{R}^N; \mathcal{B}^1_A(\mathbb{R}^N))\) and \(v \in L^q(\mathbb{R}^N; \mathcal{B}^q_A(\mathbb{R}^N))\) we define the double convolution still denoted by \(\ast\ast\) as follows: \(u \ast v\) is that element of \(L^m(\mathbb{R}^N; \mathcal{B}^1_A(\mathbb{R}^N))\) defined by

\[G_1(u \ast v) = \hat{u} \ast \hat{v}.\]
3.2. Σ-convergence method. Throughout this section, \( \Omega \) is an open subset of \( \mathbb{R}^N \), and unless otherwise specified, \( A \) is an algebra with mean value on \( \mathbb{R}^N \).

**Definition 3.2.** (1) A sequence \((u_\varepsilon)_{\varepsilon>0} \subset L^p(\Omega) (1 \leq p < \infty)\) is said to weakly Σ-converge in \( L^p(\Omega) \) to some \( u_0 \in L^p(\Omega; B^p_A(\mathbb{R}^N)) \) if as \( \varepsilon \to 0 \),

\[
\int_\Omega u_\varepsilon(x) f^\varepsilon(x) \, dx \to \int_\Omega \tilde{u}_0(x, s) \tilde{f}(x, s) \, dx \, ds \tag{3.1}
\]

for all \( f \in L^{p'}(\Omega; A) (1/p' = 1 - 1/p) \) where \( f^\varepsilon(x) = f(x, x/\varepsilon) \) and \( \tilde{f}(x, \cdot) = G(f(x, \cdot)) \) a.e. in \( x \in \Omega \). We denote this by \( u_\varepsilon \rightharpoonup u_0 \) in \( L^p(\Omega; B^p_A(\mathbb{R}^N)) \).

(2) A sequence \((u_\varepsilon)_{\varepsilon>0} \subset L^p(\Omega) (1 \leq p < \infty)\) is said to strongly Σ-converge in \( L^p(\Omega) \) to some \( u_0 \in L^p(\Omega; B^p_A(\mathbb{R}^N)) \) if it is weakly Σ-convergent and further satisfies the following condition:

\[
\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega)} \text{ as } \varepsilon \to 0.
\]

We denote this by \( u_\varepsilon \to u_0 \) in \( L^p(\Omega; B^p_A(\mathbb{R}^N)) \).

We recall here that \( \tilde{u}_0 = G_1 \circ u_0 \) and \( \tilde{f} = G \circ f \), \( G_1 \) being the isometric isomorphism sending \( B^p_A(\mathbb{R}^N) \) onto \( L^p(\Delta(A)) \) and \( G \), the Gelfand transformation on \( A \).

In the sequel the letter \( E \) will throughout denote any ordinary sequence \((\varepsilon_n)_n\) (integers \( n \geq 0 \)) with \( 0 < \varepsilon_n \leq 1 \) and \( \varepsilon_n \to 0 \) as \( n \to \infty \). The following two results hold (see e.g. [3, 24, 29] for their justification).

**Theorem 3.2.** (i) Any bounded sequence \((u_\varepsilon)_\varepsilon \in E \) in \( L^p(\Omega) \) (for \( 1 < p < \infty \)) admits a subsequence which is weakly Σ-convergent in \( L^p(\Omega) \).

(ii) Any uniformly integrable sequence \((u_\varepsilon)_\varepsilon \in E \) in \( L^1(\Omega) \) admits a subsequence which is weakly Σ-convergent in \( L^1(\Omega) \).

**Theorem 3.3.** Let \( 1 < p < \infty \). Let \((u_\varepsilon)_\varepsilon \in E \) be a bounded sequence in \( W^{1,p}(\Omega) \). Then there exist a subsequence \( E' \) of \( E \), and a couple \((u_0, u_1) \in W^{1,p}(\Omega; B^p_A(\mathbb{R}^N)) \times L^p(\Omega; B^p_A(\mathbb{R}^N)) \) such that, as \( \varepsilon \to 0 \),

\[
u_\varepsilon \to u_0 \text{ in } L^p(\Omega; B^p_A(\mathbb{R}^N)),
\]

\[
\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \text{ in } L^p(\Omega; B^p_A(\mathbb{R}^N)) \text{ weakly } \Sigma,
\]

\( 1 \leq i \leq N \).

**Remark 3.1.** In the above result, \( B^p_A(\mathbb{R}^N) \) stands for the space of invariant functions in \( B^p_A(\mathbb{R}^N) \) under the group of transformation \( T(y) \) of the preceding subsection: \( u \in B^p_A(\mathbb{R}^N) \) if and only if \( \nabla_y u = 0 \). If we assume the algebra \( A \) to be ergodic, then \( B^p_A(\mathbb{R}^N) \) consists of constant functions, so that the function \( u_0 \) in Theorem 3.3 does not depend on \( y \), that is, \( u_0 \in W^{1,p}(\Omega) \). We thus recover the already known result proved in [29] in the case of ergodic algebras.

The next result deals with the Σ-convergence of a product of sequences.

**Theorem 3.4.** ([30] Theorem 6)). Let \( 1 < p, q < \infty \) and \( r \geq 1 \) be such that \( 1/r = 1/p + 1/q \leq 1 \). Assume \((u_\varepsilon)_\varepsilon \subset L^q(\Omega) \) is weakly Σ-convergent in \( L^q(\Omega) \) to some \( u_0 \in L^q(\Omega; B^p_A(\mathbb{R}^N)) \), and \((u_\varepsilon)_\varepsilon \subset L^p(\Omega) \) is strongly Σ-convergent in \( L^p(\Omega) \) to some \( v_0 \in L^p(\Omega; B^p_A(\mathbb{R}^N)) \). Then the sequence \((u_\varepsilon v_\varepsilon)_\varepsilon \subset E \) is weakly Σ-convergent in \( L^r(\Omega) \) to \( u_0 v_0 \).

As a consequence of the above theorem the following holds.

**Corollary 3.1.** Let \((u_\varepsilon)_\varepsilon \subset L^p(\Omega) \) and \((v_\varepsilon)_\varepsilon \subset L^{p'}(\Omega) \cap L^\infty(\Omega) \) (\( 1 < p < \infty \) and \( p' = p/(p - 1) \)) be two sequences such that: (i) \( u_\varepsilon \to u_0 \) in \( L^p(\Omega) \)-weak Σ; (ii) \( v_\varepsilon \to v_0 \) in \( L^{p'}(\Omega) \)-strong Σ; (iii) \( (v_\varepsilon)_\varepsilon \subset E \) is bounded in \( L^\infty(\Omega) \). Then \( u_\varepsilon v_\varepsilon \to u_0 v_0 \) in \( L^p(\Omega) \)-weak Σ.

Now, assume that the algebra \( A \) is introverted. Then its spectrum is a compact topological semigroup whose kernel is a compact topological group, so that we can define, as in the preceding subsection, the convolution over \( \Delta(A) \). Our aim in the next result is to link the Σ-convergence concept to the convolution over the spectrum \( \Delta(A) \) of \( A \). To see this, let \( p, q, m \geq 1 \) be real numbers such that \( \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{m} \). Let
(u_\varepsilon)_{\varepsilon>0} \subset L^p(\Omega) and (v_\varepsilon)_{\varepsilon>0} \subset L^q(\mathbb{R}^N) be two sequences. One may view u_\varepsilon as defined in the whole \mathbb{R}^N by taking its extension by zero outside \Omega. Define
\[ (u_\varepsilon * v_\varepsilon)(x) = \int_{\mathbb{R}^N} u_\varepsilon(t) v_\varepsilon(x-t) \, dt \quad (x \in \mathbb{R}^N), \]
which lies in \( L^m(\mathbb{R}^N) \). Then

**Theorem 3.5** ([34 Theorem 6.2]). Let \((u_\varepsilon)_{\varepsilon>0}\) and \((v_\varepsilon)_{\varepsilon>0}\) be as above. Assume that, as \( \varepsilon \to 0 \), \( u_\varepsilon \to u_0 \) in \( L^p(\Omega) \)-weak \( \Sigma \) and \( v_\varepsilon \to v_0 \) in \( L^q(\mathbb{R}^N) \)-strong \( \Sigma \), where \( u_0 \) and \( v_0 \) are in \( L^p(\Omega; B^p_{A}(\mathbb{R}^N)) \) and \( L^q(\mathbb{R}^N; B^q_{A}(\mathbb{R}^N)) \) respectively. Assume further that the algebra \( wmv \) \( A \) is introverted. Then, as \( \varepsilon \to 0 \),
\[ u_\varepsilon * v_\varepsilon \to u_0 * v_0 \text{ in } L^m(\mathbb{R}^N) \)-weak \( \Sigma \).

In practice, one deals with the evolutionary version of the concept of \( \Sigma \)-convergence. Such concept requires some further notions such as those related to the product of algebras with mean value. Let \( A_y \) (resp. \( A_r \)) be an algebra with mean value on \( \mathbb{R}^N \) (resp. \( \mathbb{R} \)). We define their product denoted by \( A = A_r \odot A_y \) as the closure in \( BUC(\mathbb{R}^N) \) of the tensor product \( A_r \odot A_y = \{ \sum_{\text{finite}} u_i \otimes v_i : u_i \in A_r, \, v_i \in A_y \} \). It is a well known fact that \( A_r \odot A_y \) is an algebra with mean value on \( \mathbb{R}^{N+1} \) (see e.g. [23, 24]).

With this in mind, let \( A = A_r \odot A_y \) be as above. The same letter \( \mathcal{G} \) will denote the Gelfand transformation on \( A_y, A_r, \) and \( A \), as well. Points in \( \Delta(A_y) \) (resp. \( \Delta(A_r) \)) are denoted by \( s \) (resp. \( s_0 \)). The compact space \( \Delta(A_y) \) (resp. \( \Delta(A_r) \)) is equipped with the \( M \)-measure \( \beta_y \) (resp. \( \beta_r \)), for \( A_y \) (resp. \( A_r \)). We have \( \Delta(A) = \Delta(A_r) \times \Delta(A_y) \) (Cartesian product) and the \( M \)-measure for \( A \), with which \( \Delta(A) \) is equipped, is precisely the product measure \( \beta = \beta_r \otimes \beta_y \) (see [24]). Finally, let \( 0 < T < \infty \). We set \( Q = (0,T) \times \Omega \) as in Section 1 (an open cylinder in \( \mathbb{R}^{N+1} \)) and \( \mathcal{O} = Q \times \mathbb{R}^N \).

This being so, a sequence \((u_\varepsilon)_{\varepsilon>0} \subset L^p(Q) \) (1 \( \leq \) \( p \) \( < \) \( \infty \)) is said to weakly \( \Sigma \)-converge in \( L^p(Q) \) to some \( u_0 \in L^p(Q; B^p_{A}(\mathbb{R}^{N+1})) \) if as \( \varepsilon \to 0 \),
\[ \int_Q u_\varepsilon(t,x) f(t,x,\varepsilon,\frac{x}{\varepsilon},\varepsilon) \, dx \, dt \to \int_{Q \times \Omega} \tilde{u}_0(t,x,s_0,s) \hat{f}(t,x,s_0,s) \, dx \, dt \, d\beta \]
for all \( f \in L^p(Q; A) \). We may also define the weak \( \Sigma \)-convergence in \( L^p(\mathcal{O}) \) as follows: \((u_\varepsilon)_{\varepsilon>0} \subset L^p(\mathcal{O}) \) weakly \( \Sigma \)-converges to \( u_0 \in L^p(\mathcal{O}; B^p_{A}(\mathbb{R}^{N+1})) \) if
\[ \int_\Omega u_\varepsilon(t,x,v) f(t,x,\varepsilon,\frac{x}{\varepsilon},\varepsilon,v) \, dx \, dtdv \to \int_{\mathcal{O} \times \Omega} \tilde{u}_0(t,x,s_0,s,v) \hat{f}(t,x,s_0,s,v) \, dx \, dtdvd\beta \]
for any \( f \in L^p(\mathcal{O}; A) \).

**Remark 3.2.** The conclusions of Theorems 3.2-3.5 are still valid mutatis mutandis in the present context (change \( \Omega \) into \( Q \) in Theorem 3.2, \( W^{1,p}(\Omega) \) into \( L^p(0,T; W^{1,p}(\Omega)) \), \( W^{1,p}(\Omega; B^p_{A}(\mathbb{R}^N)) \times L^p(\Omega; B^p_{A}(\mathbb{R}^{N+1})) \) into \( L^p(0,T; W^{1,p}(\Omega; B^p_{A}(\mathbb{R}^N)))) \times L^p(Q; B^p_{A}(\mathbb{R}; B^1_{A}(\mathbb{R}^N)))) \), provided \( A \) is introverted in Theorem 3.5.

4. Homogenization results

Throughout this section, we consider the algebras \( wmv \) \( A_y \) and \( A_r \) to be as in the end of the preceding section. We further assume that \( A_r \) is introverted.

With this in mind, let \((u_\varepsilon, f_\varepsilon)_{\varepsilon>0}\) be the sequence of solutions to (1.1)-(1.3). In view of Lemma 2.2, there is a positive constant \( C \) independent of \( \varepsilon > 0 \) such that
\[ \sup_{\varepsilon>0} \left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^2(0,T; W')} \leq C. \]

This, together with the inequality (2.30) in Lemma 2.2 entail the precompactness of the sequence \((u_\varepsilon)_{\varepsilon>0}\) in \( L^2(0,T; H) \). Thus, given an ordinary sequence \( E' \), there are a subsequence \( E' \) of \( E \) and a function \( u_0 \in L^2(Q)^N \) such that, as \( E' \ni \varepsilon \to 0 \)
\[ u_\varepsilon \to u_0 \text{ in } L^2(Q)^N. \]
In view of (2.55) and by the diagonal process, one can find a subsequence of \((u_\varepsilon)_{\varepsilon \in E'}\) (not relabeled) which weakly converges in \(L^2(0,T;V)\) to the function \(u_0\) (this means that \(u_0 \in L^2(0,T;V)\)). From Theorem 2.3 we infer the existence of a function \(u_1 = (u_1^k)_{1 \leq k \leq N} \in L^2(Q;B^2_{A_1}(\mathbb{R}^N))\) such that the convergence result

\[
\frac{\partial u_\varepsilon}{\partial x_i} \rightarrow \frac{\partial u_0}{\partial x_i} + \overline{\partial u_1}\]  

holds when \(E' \ni \varepsilon \rightarrow 0\). Still from Lemma 2.2 (see (2.56) for \(m = 2\) and (2.68) therein) there exist a subsequence of \(E'\) (still denoted by \(E'\)) and two functions \(f_0 \in L^\infty(0,T;L^2(\Omega \times \mathbb{R}^N;B^2_{A}(\mathbb{R}^{N+1})))\), \(p \in L^2(0,T;B^2_A(\mathbb{R}^{N+1}))\) with \(\int_\Omega p dx = 0\) such that, as \(E' \ni \varepsilon \rightarrow 0\),

\[
f_\varepsilon \rightarrow f_0 \text{ in } L^2(Q \times \mathbb{R}^N)-\text{weak } \Sigma\]  

and

\[
p_\varepsilon \rightarrow p \text{ in } L^2(Q)-\text{weak } \Sigma.
\]

We recall that \(\frac{\partial u_0}{\partial x_i} = (\overline{\partial u}_k)_{1 \leq k \leq N}\) and \(\frac{\partial u_k}{\partial y_i} = \left(\overline{\partial u}_k\right)_{1 \leq k \leq N}\).

Our goal in this section is the study of the asymptotics (as \(\varepsilon \rightarrow 0\)) of \((u_\varepsilon,f_\varepsilon,p_\varepsilon)_{\varepsilon > 0}\) under the following additional assumption

\[\text{(A3) } A_i(t,x,\cdot,\cdot) \in B^2_A(\mathbb{R}^{N+1})\]  

for \(i = 0, 1\) and for all \((t,x) \in \overline{Q}\).

4.1. **Passing to the limit** \(\varepsilon \rightarrow 0\). Let us first find the equation satisfied by \(f_0\). To that end, let \(\phi \in C^\infty_0(\Omega) \otimes A^\infty\) (where we recall that \(\Omega = Q \times \mathbb{R}^N\) with \(Q = (0,T) \times \Omega\) and define \(\phi^\varepsilon \in C^\infty_0(\Omega)\) by \(\phi^\varepsilon(t,x,v) = \phi(t,x,t/\varepsilon,x/\varepsilon,v)\) for \((t,x,v) \in \Omega\). Multiplying the Vlasov equation (1.1) by \(\phi^\varepsilon\) and integrating by parts,

\[-\int_\Omega f_\varepsilon \left(\varepsilon \frac{\partial \phi^\varepsilon}{\partial t} + \varepsilon v \cdot \nabla \phi^\varepsilon + (u_\varepsilon - v) \cdot \nabla_v \phi^\varepsilon\right) dxdtdv = 0.
\]

The above equation is equivalent to the following one

\[-\int_\Omega f_\varepsilon \left(\varepsilon \frac{\partial \phi^\varepsilon}{\partial t} + \frac{1}{\varepsilon} \left(\frac{\partial \phi}{\partial t}\right)^\varepsilon + \varepsilon v \cdot (\nabla \phi)^\varepsilon + v \cdot (\nabla_y \phi)^\varepsilon + (u_\varepsilon - v) \cdot (\nabla_v \phi)^\varepsilon\right) dxdtdv = 0.\]  

Multiplying the above equation by \(\varepsilon\) and letting \(E' \ni \varepsilon \rightarrow 0\) (where \(E'\) is as above), we end up with (using the fact that the sequence \((f_\varepsilon(u_\varepsilon - v))_{\varepsilon > 0}\) is bounded in \(L^1(\Omega)\); see Remark 2.2)

\[
\int_{\Omega \times \Delta(A)} \hat{f}_0 \partial_\varepsilon \hat{\phi} dxdtdv \beta_y = 0
\]

where \(\partial_\varepsilon = \partial_{\phi} \circ \hat{\phi}\). It follows that \(\frac{\partial \phi}{\partial \tau} = 0\), which amounts to say that \(f_0\) does not depends on \(\tau\). Indeed this is equivalent to \(f_0 \in L^\infty(0,T;L^2(\Omega \times \mathbb{R}^N;B^2_{A_y}(\mathbb{R}^N,F^2_{A_y}(\mathbb{R}))))\), and since \(A_y\) is introverted, it is ergodic [34, Remark 3.3], so that \(F^2_{A_y}(\mathbb{R})\) consists of constants. This means that the test functions \(\phi\) may be chosen independent of \(\tau \in \mathbb{R}\), that is, \(\phi \in C^\infty_0(\Omega) \otimes A^\infty\) and so, \(\phi^\varepsilon(t,x,v) = \phi(t,x,x/\varepsilon,v)\) for \((t,x,v) \in \Omega\). Before we can pass to the limit in (4.5), we notice that the function \((t,x,v,y) \mapsto v \cdot \nabla_v \phi\) lies in \(L^\infty(\Omega;A_y)\) since it trivially lies in \(C^\infty_0(\Omega) \otimes A^\infty\). Thus,

\[
\int_\Omega f_\varepsilon v \cdot (\nabla_v \phi)^\varepsilon dxdtdv \rightarrow \int_{\Omega \times \Delta(A_y)} \hat{f}_0 v \cdot \nabla_v \hat{\phi} dxdtdv \beta_y.
\]

In order to pass to the limit in the term \(\int_\Omega f_\varepsilon u_\varepsilon \cdot (\nabla_v \phi)^\varepsilon dxdtdv\), we need the following

**Lemma 4.1.** Let \(u_0\) and \(f_0\) be as in (1.1) and (4.3), respectively. Then for any \(\psi \in (C^\infty_0(\Omega) \otimes A^\infty)\),

\[
\int_\Omega f_\varepsilon u_\varepsilon \cdot \psi^\varepsilon dxdtdv \rightarrow \int_{\Omega \times \Delta(A_y)} \hat{f}_0 u_0 \cdot \hat{\psi} dxdtdv \beta_y
\]  

as \(E' \ni \varepsilon \rightarrow 0\).
Proof. First assume \( \psi = (\psi_i)_{1 \leq i \leq N} \) with \( \psi_i = \varphi_i \otimes \chi_i \otimes w_i \) with \( \varphi_i \in C^\infty_0(Q) \), \( \chi_i \in C^\infty_0(\mathbb{R}^N) \) and \( w_i \in A_y^\infty \). Then

\[
\int_O f_x u_\varepsilon \cdot \psi \ dx dt dv = \sum_{i=1}^N \int_O f_x (t, x, v) u_\varepsilon (t, x) \chi_i (v) \varphi_i (t, x) w_i \left( \frac{x}{\varepsilon} \right) \ dx dt dv.
\]

Set \( U_\varepsilon (t, x, v) = u_\varepsilon (t, x) \chi_i (v) \) for \( (t, x, v) \in O \). Then

\[
U_\varepsilon \to U_0^1 \equiv u_0 \otimes \chi in L^2(O)-strong as E' \ni \varepsilon \to 0.
\]

Indeed, since

\[
\int |U_\varepsilon - U_0|^2 dx dt dv \leq \int_{\mathbb{R}^N} |\chi_i (v)|^2 \ dv \int_Q |u_\varepsilon - u_0|^2 dx dt,
\]

the claim follows from Eq. (4.11). Thus, we infer from Eq. (4.3) and Theorem 3.4 that

\[
\int f_x U_\varepsilon \to f_0 U_0^1 in L^1(O)-weak \Sigma as E' \ni \varepsilon \to 0.
\]

Hence, by choosing the special test function \( \varphi_i \otimes w_i \otimes 1_{\mathbb{R}^N} \in L^\infty(O; A_y^\infty) \), we are led to

\[
\int_O f_x (t, x, v) u_\varepsilon (t, x) \varphi_i (t, x) w_i \left( \frac{x}{\varepsilon} \right) \ dx dt dv \to \int_Q \varphi_i (t, x) w_i \left( \frac{x}{\varepsilon} \right) \ dx dt dv \delta_y,
\]

or,

\[
\int_O f_x u_\varepsilon \cdot \psi \ dx dt dv \to \int_Q f_0 \varphi_i \cdot \psi \ dx dt dv \delta_y.
\]

Now, by some routine computations, the result follows at once from the density of \( C^\infty_0(Q) \otimes C^\infty_0(\mathbb{R}^N) \otimes A_y^\infty \) in \( C^\infty_0(O) \otimes A_y^\infty \).

As a consequence of Lemma 4.1, we have

\[
\int_O f_x u_\varepsilon \cdot (\nabla_v \phi)^\varepsilon \ dx dt dv \to \int_Q f_0 \varphi_i \cdot \nabla_v \phi \ dx dt dv \delta_y.
\]

Returning to (4.9) and taking the limit when \( E' \ni \varepsilon \to 0 \), we arrive at

\[
- \int_{O \times A_y^\infty} \hat{f}_0 \left[ \frac{\partial \hat{\phi}}{\partial t} + v \cdot \nabla_y f_0 + \nu \cdot (u_0 - v) \cdot \nabla_v \phi \right] \ dx dt dv \delta_y = 0,
\]

where \( \hat{\phi} = \mathcal{G} \circ \nabla_v \phi \). This gives rise to the following equation satisfied by \( f_0 \):

\[
\frac{\partial f_0}{\partial t} + v \cdot \nabla_y f_0 + \nu \cdot (u_0 - v) \cdot \nabla_v f_0 = 0 \quad in \ O \times \mathbb{R}^N_y.
\]

Following the lines of [21] Section 4 we prove, by choosing suitable test functions, that the function \( f_0 \) satisfies the following reflection boundary and initial conditions

\[
f_0(t, x, y, v) = f_0(t, x, y, v^*) \ for \ x \in \partial \Omega \ with \ v \cdot v(x) < 0
\]

where \( v^* = v - 2(v \cdot \nu(x)) \nu(x) \).

\[
f_0(0, x, y, v) = f_0(x, v) \ for \ (x, y, v) \in \Omega \times \mathbb{R}^N_y \times \mathbb{R}^N_v
\]

We consider now the Stokes system (12)-(13). Choosing \( \psi_o = (\psi_{o,k})_{1 \leq k \leq N} \in C^\infty_0(Q) \) and \( \psi_1 = (\psi_{1,k})_{1 \leq k \leq N} \in [C^\infty_0(Q) \otimes A^\infty]^N \), we set \( \Phi = (\Phi_0, \Phi_1) \) and define \( \Phi_\varepsilon = \psi_0 + \varepsilon \Phi_1 \) by

\[
\Phi_\varepsilon (t, x) = \psi_0 (t, x, \frac{x}{\varepsilon}) for \ (t, x) \in Q.
\]

It can be checked that \( \Phi_\varepsilon \in C^\infty_0(Q) \). By plugging \( \Phi_\varepsilon \) into the variational formulation of (12), we obtain

\[
- \int_Q u_\varepsilon \cdot \frac{\partial \Phi_\varepsilon}{\partial t} \ dx dt + \int_Q A_y \nabla u_\varepsilon \cdot \nabla \Phi_\varepsilon \ dx dt + \int_Q (A_y \nabla u_\varepsilon \cdot \nabla \Phi_\varepsilon \ dx dt
\]

\[
- \int_Q \rho_\varepsilon \ div \Phi_\varepsilon \ dx dt = - \int_Q f_x (u_\varepsilon - v) \cdot \Phi_\varepsilon \ dx dt.
\]

This completes the proof.
Our immediate goal is to pass to the limit in the above equation. We deal with its constituents in turn. Owing to Eq. (4.11), as $E' \ni \varepsilon \to 0$ in the first term in the left-hand side of Eq. (4.11), we have,

$$
\int_Q u_{\varepsilon} \cdot \frac{\partial \Phi}{\partial x} \, dx dt \to \int_Q u_0 \cdot \frac{\partial \psi}{\partial t} \, dx dt.
$$

For the next term, one easily shows that as $\varepsilon \to 0$,

$$
\frac{\partial \Phi}{\partial x_i} \to \frac{\partial \psi_0}{\partial x_i} + \frac{\partial \psi_1}{\partial y_i} \text{ in } L^2(Q)^N \text{-strong } \Sigma \quad (1 \leq i \leq N).
$$

Combining the above convergence result with Eq. (4.12), we deduce from Corollary 3.1 that, as $E' \ni \varepsilon \to 0$,

$$
\frac{\partial u_{\varepsilon}}{\partial x_j} \frac{\partial \Phi}{\partial x_i} \to \left( \frac{\partial u_0}{\partial x_j} + \frac{\partial \psi_1}{\partial x_i} \right) \cdot \left( \frac{\partial \psi_0}{\partial x_i} + \frac{\partial \psi_1}{\partial y_i} \right) \text{ in } L^2(Q) \text{-weak } \Sigma.
$$

Passing to the limit in the above mentioned term using $A_0$ as a test function (recall that $A_0 \in C(\overline{Q} ; [B^2_1(R^{N+1}) \cap L^\infty(R^{N+1}))])$ by assumption (A3) so that in view of [30, Proposition 8], it is an admissible test function in the sense of [30, Definition 5]), we get

$$
\int_Q A_0 \hat{\nabla} u_{\varepsilon} \cdot \hat{\nabla} \Phi \, dx dt \to \int_Q \hat{\nabla} \Phi \cdot \hat{\nabla} \Phi \, dx dt as \ E' \ni \varepsilon \to 0
$$

where, setting $u = (u_0, u_1)$, we have $\mathbb{D} u = (\mathbb{D}_j u_1)_{1 \leq j \leq N}$ with $\mathbb{D}_j u = (\mathbb{D}_{ij} u^k)_{1 \leq i \leq N}$ and $\mathbb{D}_j u^k = \frac{\partial u_k}{\partial x_j} + \partial_j \hat{u}_1^k$ ($\partial_j \hat{u}_1^k = g_1(\hat{u}_1^k/\partial y_j)$), and the same definition for $\mathbb{D}\Phi$.

Let us now tackle the term involving convolution. First we know that $A_1^\tau \to A_1$ in $L^1(R^{N+1})$-strong $\Sigma$ and $\nabla u_{\varepsilon} \to \nabla u_0 + \nabla_y u_1$ in $L^2(Q)^N$-weak $\Sigma$; hence by virtue of Theorem 3.5, we conclude that

$$
A_1^\tau \ast \nabla u_{\varepsilon} \to A_1 \ast (\nabla u_0 + \nabla_y u_1) \text{ in } L^2(Q)^N \text{-weak } \Sigma \text{ as } E' \ni \varepsilon \to 0
$$

where the double convolution is defined with respect to the time variable as follows:

$$
\left( A_1 \ast (\nabla u_0 + \nabla_y u_1) \right) (t, x, s_0, s) = \int_0^t \int_{\overline{A_1}} \nabla u_0(t - \tau, x, s_0, s - \tau) \, d\beta_{s_0}(\tau),
$$

in which the function $\nabla u_0$ is assumed to be defined on the whole of $R^N$ by taking its zero-extension off $\Omega$. Therefore, repeating the same reasoning as for the preceding term, we arrive at (as $E' \ni \varepsilon \to 0$)

$$
\int_Q A_1^\tau \ast \nabla u_{\varepsilon} \cdot \hat{\nabla} \Phi \, dx dt \to \int_{Q \times \Delta(A_\varepsilon) \times K(A_\varepsilon)} (A_1 \ast \mathbb{D} u) \cdot \mathbb{D} \Phi \, dx dt d\beta.
$$

or, as $\beta_\varepsilon$ is supported by $K(A_\varepsilon)$,

$$
\int_Q A_1^\tau \ast \nabla u_{\varepsilon} \cdot \hat{\nabla} \Phi \, dx dt \to \int_{Q \times \Delta(A)} (A_1 \ast \mathbb{D} u) \cdot \mathbb{D} \Phi \, dx dt d\beta.
$$

As for the term with the pressure, we have

$$
\int_Q p_{\varepsilon} \text{ div } \Phi \, dx dt = \int_Q p_{\varepsilon} \text{ div } \psi_0 \, dx dt + \int_Q p_{\varepsilon} (\text{div}_y \psi_1)^{\tau} \, dx dt \quad (4.12)
$$

$$
+ \varepsilon \int_Q p_{\varepsilon} (\text{div}_y \psi_1)^{\varepsilon} \, dx dt.
$$

Set $p_0(x, t) = \int_{\Delta(A)} \hat{p}(t, x, s_0, s) \, d\beta$ for a.e. $(t, x) \in Q$, where $p$ is as in (1.4). Then we know that $p_{\varepsilon} \to p_0$ in $L^2(Q)$-weak, and, passing to the limit in (4.12) as $E' \ni \varepsilon \to 0$ yields

$$
\int_Q p_{\varepsilon} \text{ div } \Phi \, dx dt \to \int_Q p_0 \text{ div } \psi_0 \, dx dt + \int_{Q \times \Delta(A)} \hat{p} \text{ div}_y \psi_1 \, dx dt d\beta.
$$
For Eq. (4.14), reasoning as in the proof of (4.6) and (4.7), we get
\[ \int_{\Omega} f_\varepsilon(u_\varepsilon - v) \cdot \Phi_\varepsilon \, dx dt dv = \int_{\Omega \times \Delta(A)} \tilde{f}_0(u_0 - v) \cdot \psi_\varepsilon \, dx dt dv. \]

Putting together the previous convergence results, we are led to the fact that the quadruple \((u_0, u_1, f_0, p)\) determined by (4.1)–(4.4) solves the system consisting of equation (4.8) and

\[
\begin{align*}
- \int_{\Omega} u_0 \cdot \psi_1' \, dx dt + & \int_{\Omega \times \Delta(A)} \left( \tilde{A}_1 \Delta u + \tilde{A}_1 * * \nabla u \right) \cdot \nabla \psi_1 \, dx dt dv \\
- \int_{\Omega} p_0 \, \nabla \psi_0 \cdot dx dt = & \int_{\Omega \times \Delta(A)} \tilde{p} \nabla \psi_0 \, dx dt dv - \int_{\Omega \times \Delta(A)} \tilde{f}_0 (u_0 - v) \cdot \psi_0 \, dx dt dv,
\end{align*}
\]

for all \(\Phi = (\psi_0, \psi_1) \in C^\infty_0(Q) \times C^\infty_0(Q) \otimes A^\infty)^N.\)

From the equality \(\nabla u_\varepsilon = 0\) we easily deduce that \(\nabla \psi_1 u_\varepsilon = 0\). Next, we need to uncouple Eq. (4.13), which is equivalent to the system (4.14) below:

\[
\begin{align*}
& \int_{\Omega \times \Delta(A)} \left( \tilde{A}_1 \nabla u + \tilde{A}_1 * * \nabla u \right) \cdot \Delta \psi_1 \, dx dt dv = \int_{\Omega \times \Delta(A)} \tilde{p} \nabla \psi_1 \, dx dt dv = 0 \\
& \text{for all } \psi_1 \in [C^\infty_0(Q) \otimes A^\infty]^N, \quad (4.14)
\end{align*}
\]

and

\[
\begin{align*}
& \int_{\Omega} u_0 \cdot \psi_1' \, dx dt + \int_{\Omega \times \Delta(A)} \left( \tilde{A}_0 \Delta u + \tilde{A}_1 * * \nabla u \right) \cdot \nabla \psi_1 \, dx dt dv \\
& \int_{\Omega} p_0 \, \nabla \psi_0 \cdot dx dt = \int_{\Omega \times \Delta(A)} \tilde{p} \nabla \psi_0 \, dx dt dv - \int_{\Omega \times \Delta(A)} \tilde{f}_0 (u_0 - v) \cdot \psi_0 \, dx dt dv, \quad (4.15)
\end{align*}
\]

For Eq. (4.14), we choose \(\psi_1(x, t) = \varphi(x, t) w\) with \(\varphi \in C^\infty_0(Q)\) and \(w \in (A^\infty)^N.\) Then (4.14) becomes

\[
\begin{align*}
& \int_{\Delta(A)} \left( \tilde{A}_0 \Delta u + \tilde{A}_1 * * \nabla u \right) \cdot \Delta \tilde{w} \, dx dt dv = \int_{\Delta(A)} \tilde{p} \nabla \tilde{w} \, dx dt dv = 0, \\
& \text{for all } \tilde{w} \in (A^\infty)^N, \quad (4.16)
\end{align*}
\]

Now, fix \(\xi \in \mathbb{R}^{N+N}\) and consider the following cell problem:

\[
\begin{align*}
& \text{Find } u_\xi \in B^{1,2}_{A_1}(\mathbb{R}_r; B^{1,2}_{\text{div}}(\mathbb{R}_y)^N), p_\xi \in B^{2}_{A_1}(\mathbb{R}_r; B^{2}_{A_2}(\mathbb{R}_y)^N)/\mathbb{R} \text{ such that} \\
& \int_{\Delta(A)} \left( \tilde{A}_0 (\xi + \partial \tilde{u}_\xi) + \tilde{A}_1 * (\xi + \partial \tilde{u}_\xi) \right) \cdot \Delta \tilde{w} \, dx dt dv = \int_{\Delta(A)} \tilde{p} \nabla \tilde{w} \, dx dt dv = 0, \quad (4.17)
\end{align*}
\]

where \(B^{1,2}_{\text{div}}(\mathbb{R}_y)^N = \{v \in B^{1,2}_{A_2}(\mathbb{R}_y)^N : \nabla \text{div } v = 0\}.\) Then Eq. (4.17) is the variational formulation of the problem

\[
\begin{align*}
& \nabla \text{div } y (A_0 \nabla _y u_\xi + A_1 * \nabla _y u_\xi) + \nabla _y p_\xi = \text{div } y (A_0 \xi + A_1 + * \xi) \text{ in } \mathbb{R}^{N+1}_y, \\
& \nabla \text{div } y u_\xi = 0.
\end{align*}
\]

Thanks to the properties of the functions \(A_1 \) (i.e., 0, 1), the above problem is classically solved and possesses a unique solution \((u_\xi, p_\xi) \in B^{2}_{A_2}(\mathbb{R}_r; B^{2}_{\text{div}}(\mathbb{R}_y)^N)/\mathbb{R}.\)

Now, returning to Eq. (4.17) where we set \(\xi = \nabla \text{div } u_\xi \) and \(x, t \) we find out that Eqs. (4.10) and (4.17) are the variational formulation of the same problem (say Eq. (4.18)). Owing to the uniqueness of the solution of Eq. (4.17), we have \(u_1 = u \nabla \text{div } u_\xi \) and \(p = p \nabla \text{div } u_\xi \) where \(u \nabla u_\xi \) (resp. \(p \nabla u_\xi \)) denotes the function \((t, x) \mapsto u \nabla u_\xi (t, x) \) (resp. \((t, x) \mapsto p \nabla u_\xi (t, x) \)) defined from \(Q \) into \(B^{2}_{A_2}(\mathbb{R}_r; B^{2}_{A_2}(\mathbb{R}_y)^N)/\mathbb{R}.\)

4.2. **Homogenization result.** Let us first define the effective coefficients. Let the matrices \(C_k = (c_{ij}^k)_{1 \leq i, j \leq N} \) (i.e., 0, 1) be defined as follows: for any \(\xi = (\xi_{ij})_{1 \leq i, j \leq N}(k = 0, 1)\) be defined as follows:

\[
\begin{align*}
C_0 \xi = & \int_{\Delta(A)} \tilde{A}_0 (\xi + \partial \tilde{u}_\xi) \, dx dt dv, \\
C_1 \xi = & \int_{\Delta(A)} \left( \tilde{A}_1 * (\xi + \partial \tilde{u}_\xi) \right) \, dx dt dv.
\end{align*}
\]

Then, thanks to the uniqueness of \(u_\xi \) (for a given \(\xi \)), the matrices \(C_k \) are well defined and are symmetric. It is obvious that the \(c_{ij}^k \) are obtained by choosing in Eq. (4.19) \(\xi = (\delta_{ij})_{1 \leq i, j \leq N} \) (the identity matrix), \(\delta \) being the Kronecker delta.

The matrix \(C_k \) are the effective homogenized viscosities which depend continuously on \((t, x) \in Q \) as seen in the next result whose classical proof is omitted.
Proposition 4.1. It holds that

(i) $C_i$ ($i = 0, 1$) are symmetric and further $C_i \in \mathcal{C}(Q)^{N^2}$;
(ii) $C_0 \lambda \cdot \lambda \geq \alpha |\lambda|^2$ for all $(x, t) \in Q$ and all $\lambda \in \mathbb{R}^N$, where $\alpha$ is the same as in assumption (A1).

We can now formulate the homogenized problem. To this end, consider Eq. (4.15) in which we take $u_1 = u \psi_0 u_0$. We get

\[
\begin{aligned}
- \int_\Omega u_0 \cdot \psi_0' dx dt + \int_\Omega \left[ \left( \widehat{A}_0(\nabla u_0 + \partial \nu \psi_0 u_0) + \widehat{A}_1 \ast (\nabla u_0 + \partial \nu \psi_0 u_0) \right) \right] \cdot \nabla \psi_0 dx dt \\
- \int_\Omega \rho_0 \text{div} \psi_0 dx dt = - \int_\Omega \left[ \int_{\mathbb{R}^N} \left( \int_{\Delta(A)} \widehat{f}_0 d\beta \right) (u_0 - v) \right] \cdot \psi_0 dx dt \quad \forall \psi_0 \in C_0^\infty(Q),
\end{aligned}
\]

which is just the variational formulation (where accounting of $\text{div} u_0 = 0$) of the following anisotropic nonlocal Stokes system

\[
\begin{aligned}
\frac{\partial u_0}{\partial t} - \text{div}(C_0 \nabla u_0 + f_0 C_1(t - \tau, x) \nabla u_0(x, \tau) d\tau) + \nabla p_0 = - \int_{\mathbb{R}^N} f(u_0 - v) dv & \quad \text{in } Q \\
u_0(0, x, y) = u^0(x) & \quad \text{in } \Omega
\end{aligned}
\]

where $f = \int_{\Delta(A)} \widehat{f}_0 d\beta$. Finally, to be more concise, let us put together Eq. (4.8)-(4.10):

\[
\begin{aligned}
\frac{\partial f_0}{\partial t} + v \cdot \nabla f_0 + \text{div}_v ((u_0 - v) f_0) = 0 & \quad \text{in } \mathcal{O} \times \mathbb{R}^N \\
f_0(t, x, y, v) = f_0(t, x, y, v^*) & \quad \text{for } x \in \partial \Omega \text{ with } v \cdot \nu(x) < 0 \\
f_0(0, x, y, v) = f_0^0(x, v) & \quad \text{for } (x, y, v) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^N
\end{aligned}
\]

where $v^* = v - 2(v \cdot \nu(x)) \nu(x)$.

In view of what has been done above, we see that the system (4.20)-(4.21) possesses at least solution $(u_0, f_0, p_0)$ such that $u_0 \in L^2([0, T); V) \cap \mathcal{C}([0, T]; H)$, $f_0 \in L^\infty(0, T; L^\infty(\Omega \times \mathbb{R}^N) \cap L^2(\Omega \times \mathbb{R}^N))$ and $p_0 \in L^2(0, T; L^2(\Omega)/\mathbb{R})$. We are therefore led to the following homogenization result which is the second main result of this work.

**Theorem 4.1.** Assume that (A1)-(A3) hold. For each $\varepsilon > 0$, let $(u_\varepsilon, f_\varepsilon, p_\varepsilon)$ be a solution to (1.1)-(1.5). Then up to a subsequence, the sequence $(u_\varepsilon)_{\varepsilon > 0}$ strongly converges in $L^2(Q)^N$ to $u_0$, the sequence $(f_\varepsilon)_{\varepsilon > 0}$ weakly converges in $L^2(Q \times \mathbb{R}^N)$ towards $f_0$ and the sequence $(p_\varepsilon)_{\varepsilon > 0}$ weakly converges in $L^2(0, T; L^2(\Omega)/\mathbb{R})$ towards $p_0$, where $(u_0, f_0, p_0)$ is a solution to the system (4.20)-(4.21). Moreover any weak $\Sigma$-limit point $(u_0, f_0, p_0)$ in $L^2(0, T; V) \cap \mathcal{C}([0, T]; H) \times L^\infty(0, T; L^\infty(\Omega \times \mathbb{R}^N) \cap L^2(\Omega \times \mathbb{R}^N)) \times L^2(0, T; L^2(\Omega)/\mathbb{R})$ of $(u_\varepsilon, f_\varepsilon, p_\varepsilon)_{\varepsilon > 0}$ is a solution to Problem (4.20)-(4.21).

5. Some applications

A look at the previous section reveals that the homogenization process has been made possible thanks to Assumption (A3). This assumption is formulated in a general manner encompassing a variety of concrete behaviors of the coefficients of the operator involved in (1.2). We aim at providing in this section some natural situations leading to the homogenization of (1.1)-(1.5). First and foremost, it is an easy task (using (34)) to see that all the algebras wmv involved in the following problems are introverted.

**5.1. Problem 1 (Periodic homogenization).** The homogenization of (1.1)-(1.5) can be achieved under the periodicity assumption

\[
(A3)_1 \quad \text{The functions } A_i(t, x, \cdot, \cdot) \quad (i = 0, 1) \text{ are periodic of period } 1 \text{ in each scalar coordinate.}
\]

This leads to (A3) with $A = C_{\text{per}}(Z \times Y) = C_{\text{per}}(Z) \otimes C_{\text{per}}(Y)$ (the product algebra, with $Y = (0, 1)^N$ and $Z = (0, 1)$), and hence $B^2_A(\mathbb{R}^{N+1}) = L^2_{\text{per}}(Z \times Y)$.

**5.2. Problem 2 (Almost periodic homogenization).** The above functions in (A3)$_1$ are both almost periodic in $(\tau, y)$ in the sense of Besicovitch [2]. This amounts to (A3) with $A = AP(\mathbb{R}^{N+1}) = AP(\mathbb{R}^r) \otimes AP(\mathbb{R}^N)$, the Bohr almost periodic functions on $\mathbb{R}^N$ [3].
5.3. Problem 3 (Weak almost periodic homogenization). The homogenization problem for (1.1)-(1.5) may also be considered under the assumption 

\[(A3)_2 \quad A_i(t, x, \cdot, \cdot) \quad (i = 0, 1) \quad \text{is weakly almost periodic} \quad \mathbb{R}^d.\] 
This leads to (A3) with \( A = WAP(\mathbb{R}_\tau) \odot WAP(\mathbb{R}^N_y) \quad \text{(WAP)} \), the algebra of continuous weakly almost periodic functions on \( \mathbb{R}_y^N \); see e.g., [8].

5.4. Problem 4. Let \( F \) be a Banach subalgebra of BUC(\( \mathbb{R}^m \)). Let \( B_\infty(\mathbb{R}^d; F) \) denote the space of all continuous functions \( \psi \in C(\mathbb{R}^d; F) \) such that \( \psi(\zeta) \) has a limit in \( F \) as \( |\zeta| \to \infty \). In particular, it is known that \( B_\infty(\mathbb{R}^d, \mathbb{R}) = B_\infty(\mathbb{R}^d) \).

With this in mind, our goal here is to study the homogenization for problem (1.1)-(1.5) under the hypothesis 

\[(A3)_3 \quad A_i(t, x, \cdot, \cdot) \in B_\infty(\mathbb{R}^d; L^2_{\text{per}}(Y)) \quad \text{for any} \quad (t, x) \in Q, \quad \text{where} \quad Y = (0, 1)^N.

It is an easy task to see that the appropriate algebra here is the product algebra \( A = B_\infty(\mathbb{R}_\tau) \odot C_{\text{per}}(Y) \).

6. Conclusion

In this work, we have constructed weak solutions of a nonlocal Stokes-Vlasov system without any assumptions on high-order velocity moments of the initial distribution of particles. Our approach consisted in applying Schauder’s fixed point theorem to a carefully regularized version of the Stokes-Vlasov system and passing to the limit by means of compactness arguments. Our investigation culminated with the homogenization of Stokes-Vlasov system under generous structural assumptions on coefficients encompassing various forms of classical behaviors.

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