In this paper, we study a large deviation principle for the solution of a backward stochastic differential equation driven by $G$-Brownian motion with subdifferential operator.

1. Introduction

The large deviation principle (LDP in short) characterizes the limiting behavior, as $\varepsilon \to 0$, of family of probability measures $\{\mu_\varepsilon\}_{\varepsilon > 0}$ in terms of a rate function. Several authors have considered large deviations and obtained different types of applications mainly to mathematical physics. General references on large deviations are: Varadhan (1984); Deuschel and Stroock (1989); Dembo and Zeitouni (1998).

Let $X_{s,x,\varepsilon}^t$ be the diffusion process that is the unique solution of the following stochastic differential equation (SDE in short)

$$X_{t}^{s,x,\varepsilon} = x + \int_{s}^{t} \beta(X_{r}^{s,x,\varepsilon})dr + \sqrt{\varepsilon} \int_{s}^{t} \sigma(X_{r}^{s,x,\varepsilon})dW_{r}, \quad 0 \leq s \leq t \leq T$$

where $\beta$ is a Lipschitz function defined on $\mathbb{R}^{n}$ with values in $\mathbb{R}^{n}$, $\sigma$ is a Lipschitz function defined on $\mathbb{R}^{n \times d}$, and $W$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The existence and uniqueness of the strong solution $X_{s,x,\varepsilon}^t$ of (1.1) is standard. Thanks to the work of Freidlin and Wentzell (1984), the sequence $(X_{s,x,\varepsilon}^t)_{\varepsilon > 0}$ converges in probability, as $\varepsilon$ goes to 0, to $(\varphi_{s,x}^{t})_{s \leq t \leq T}$ solution of the following deterministic equation

$$\varphi_{s,x}^{t} = x + \int_{s}^{t} \beta(\varphi_{r,x}^{t})dr, \quad 0 \leq s \leq t \leq T$$

and satisfies a LDP.

Rainero (2006) extended this result to the case of backward stochastic differential equations (BSDEs in short) and Essaky (2008) to BSDEs with subdifferential operator.

Gao and Jiang (2010) extended the work of Freidlin and Wentzell (1984) to stochastic differential equations driven by $G$-Brownian motion ($G$-SDEs in short). The authors considered the following $G$-SDE: for every $0 \leq t \leq T$,

$$X_{t}^{s,x,\varepsilon} = x + \int_{0}^{t} b\varepsilon(X_{r}^{s,x,\varepsilon})dr + \varepsilon \int_{0}^{t} h\varepsilon(X_{r}^{s,x,\varepsilon})d\langle B, B \rangle_{r/\varepsilon} + \varepsilon \int_{0}^{t} \sigma\varepsilon(X_{r}^{s,x,\varepsilon})dB_{r/\varepsilon}$$

2010 Mathematics Subject Classification. Primary 60F10; Secondary 60H10, 60H30.

Key words and phrases. Large deviations, contraction principle, backward stochastic differential equation, $G$-Brownian motion, subdifferential operator, variational inequality.
and use discrete time approximation to establish LDP for G-SDEs.

Hu et al. (2014a) proved the existence and uniqueness of the solutions for BSDEs driven by G-Brownian motion. Moreover, Hu et al. (2014b) showed the comparison theorem, Feynman-Kac formula, and Girsanov transformation for G-BSDEs and established the probabilistic interpretation for the viscosity solutions of a class of fully nonlinear partial differential equations (PDEs in short).

Yang et al. (2017) proved the existence and uniqueness of a solution for a class of nonlinear variational inequalities.

Recently, Dakaou and Hima (2021) established a LDP for the corresponding process.

They studied the asymptotic behavior of the solution of the backward equation and established a LDP for the corresponding process.

Motivated by the aforementioned works, we aim to establish LDP for G-BSDEs with subdifferential operator. More precisely, we consider the following forward-backward stochastic differential equation driven by G-Brownian motion: for every $s \leq t \leq T,$

$$
\begin{align*}
X_t^{s,x,\varepsilon} &= x + \int_s^t b(X_r^{s,x,\varepsilon})dr + \varepsilon \int_s^t h(X_r^{s,x,\varepsilon})d\langle B, B \rangle_r + \varepsilon \int_s^t \sigma(X_r^{s,x,\varepsilon})dB_r, \\
Y_t^{s,x,\varepsilon} &= \Phi(X_T^{s,x,\varepsilon}) + \int_t^T f(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon})dr - \int_t^T Z_r^{s,x,\varepsilon}dB_r \\
& \quad + \int_t^T g(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon})d\langle B, B \rangle_r - (K_t^{s,x,\varepsilon} - K_t^{s,x,\varepsilon})
\end{align*}
$$

They studied the asymptotic behavior of the solution of the backward equation and established a LDP for the corresponding process.

The remaining part of the paper is organized as follows. In Section 2, we present some preliminaries that are useful in this paper. Section 3 is devoted to the large deviations for G-SDEs obtained by Gao and Jiang (2010). The large deviations for G-MBSDEs are given in Section 4.

2. Preliminaries

We review some basic notions and results about G-expectation, G-Brownian motion and G-stochastic integrals (see Peng, 2019; for more details).

Let $\Omega$ be a complete separable metric space, and let $\mathcal{H}$ be a linear space of real-valued functions defined on $\Omega$ satisfying: if $X_i \in \mathcal{H},$ $i = 1, \ldots, n,$ then

$$
\varphi(X_1, \ldots, X_n) \in \mathcal{H}, \quad \forall \varphi \in \mathcal{C}_{l, lip}(\mathbb{R}^n),
$$

where $\mathcal{C}_{l, lip}(\mathbb{R}^n)$ is the space of real continuous functions defined on $\mathbb{R}^n$ such that for some $C > 0$ and $k \in \mathbb{N}$ depending on $\varphi,$

$$
|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \quad \forall x, y \in \mathbb{R}^n.
$$
In multi-dimensional case, the function \( \in C_{S} \) where \( \mathcal{S} \) is a bounded and closed subset of \( \mathbb{R}^{d} \), is a viscosity solution of the following functional equality on \( \Omega \):

\[
\mathbb{E} [X^{1} + \cdots + X^{n}] = \mathbb{E} [X^{1}] + \cdots + \mathbb{E} [X^{n}];
\]

for all \( \lambda \geq 0 \).

\((\Omega, \mathcal{H}, \mathbb{E})\) is called a sublinear expectation space.

**Definition 2.2.** (Independence). Fix the sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\). A random variable \( Y \in \mathcal{H} \) is said to be independent of \((X_1, X_2, \ldots, X_n)\), \( X_i \in \mathcal{H} \), if for all \( \varphi \in \mathcal{C}_{l,lip}(\mathbb{R}^{n+1}) \),

\[
\mathbb{E} [\varphi(X_1, X_2, \ldots, X_n, Y)] = \mathbb{E} \left[ \mathbb{E} [\varphi(x_1, x_2, \ldots, x_n, Y)] \mid (x_1, x_2, \ldots, x_n) = (X_1, X_2, \ldots, X_n) \right].
\]

Now we introduce the definition of \( G \)-normal distribution.

**Definition 2.3.** \((G\text{-normal distribution})\). A random variable \( X \in \mathcal{H} \) is called \( G \)-normally distributed, noted by \( X \sim \mathcal{N}(0, [\sigma^2, \overline{\sigma}^2]) \), if for any function \( \varphi \in \mathcal{C}_{l,lip}(\mathbb{R}) \), the function \( u \) defined by \( u(t, x) := \mathbb{E} [\varphi(x + \sqrt{T}X)] \), \((t, x) \in [0, \infty) \times \mathbb{R}) \), is a viscosity solution of the following \( G \)-heat equation:

\[
\partial_t u - G(D_x^2 u) = 0, \quad u(0, x) = \varphi(x),
\]

where

\[
G(a) := \frac{1}{2} (\sigma^2 a^+ - \overline{\sigma}^2 a^-).
\]

In multi-dimensional case, the function \( G(\cdot) : S_d \rightarrow \mathbb{R} \) is defined by

\[
G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}(\gamma \gamma^\top A),
\]

where \( S_d \) denotes the space of \( d \times d \) symmetric matrices and \( \Gamma \) is a given nonempty, bounded and closed subset of \( \mathbb{R}^{d \times d} \) which is the space of all \( d \times d \) matrices.

Throughout this paper, we consider only the non-degenerate case, i.e., \( \sigma^2 > 0 \).

Let \( \Omega := \mathcal{C}([0, \infty), \mathbb{R}) \), which equipped with the raw filtration \( \mathcal{F} \) generated by the canonical process \((B_t)_{t \geq 0}\), i.e., \( B_t(\omega) = \omega_t \), for \((t, \omega) \in [0, \infty) \times \Omega \). Let \( \Omega_T := \mathcal{C}([0, T], \mathbb{R}) \) and let us consider the function spaces defined by

\[
\text{Lip}(\Omega_T) := \left\{ \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}) : n \geq 1, \ 0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq T, \varphi \in \mathcal{C}_{l,lip}(\mathbb{R}^{n}) \right\}, \quad \text{for} \ T > 0,
\]

\[
\text{Lip}(\Omega) := \bigcup_{n=1}^{\infty} \text{Lip}(\Omega_n).
\]

**Definition 2.4.** \((G\text{-Brownian motion and G-expectation})\). On the sublinear expectation space \((\Omega, \text{Lip}(\Omega), \mathbb{E})\), the canonical process \((B_t)_{t \geq 0}\) is called a \( G \)-Brownian motion if the following properties are verified:

1. \( B_0 = 0 \)
2. For each \( t, s \geq 0 \), the increment \( B_{t+s} - B_t \sim \mathcal{N}(0, [s\sigma^2, s\overline{\sigma}^2]) \) and is independent from \((B_{t_1}, \ldots, B_{t_n})\), for \( 0 \leq t_1 \leq \ldots \leq t_n \leq t \).
Moreover, the sublinear expectation $\hat{E}$ is called G-expectation.

**Remark 2.5.** For each $\lambda > 0$, $\left(\sqrt{\lambda} B_{t_2}/\lambda\right)_{t_2 \geq 0}$ is also a G-Brownian motion. This is the scaling property of G-Brownian motion, which is the same as that of the classical Brownian motion.

**Definition 2.6.** (Conditional G-expectation). For the random variable $\xi \in \text{Lip}(\Omega_T)$ of the following form:

$$\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}), \quad \varphi \in \mathcal{C}, \text{Lip}(\mathbb{R}^n),$$

the conditional G-expectation $\hat{E}_{t_i}[]$, $i = 1, \ldots, n$, is defined as follows

$$\hat{E}_{t_i}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})] = \hat{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}),$$

where

$$\hat{\varphi}(x_1, \ldots, x_i) = \hat{E}\left[\varphi(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i}, \ldots, B_{t_n} - B_{t_{n-1}})\right].$$

If $t \in (t_i, t_{i+1})$, then the conditional G-expectation $\hat{E}_t[\xi]$ could be defined by reformulating $\xi$ as

$$\xi = \hat{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}, B_{t_{i+1}} - B_{t_i}, \ldots, B_{t_n} - B_{t_{n-1}}), \quad \hat{\varphi} \in \mathcal{C}, \text{Lip}(\mathbb{R}^n).$$

For $\xi \in \text{Lip}(\Omega_T)$ and $p \geq 1$, we consider the norm $\|\xi\|_{L^p_G} := \left(\hat{E}\left[|\xi|^p\right]\right)^{1/p}$. Denote by $L^p_G(\Omega_T)$ the Banach completion of $\text{Lip}(\Omega_T)$ under $\|\cdot\|_{L^p_G}$. It is easy to check that the conditional G-expectation $\hat{E}_{t_i}[] : \text{Lip}(\Omega_T) \rightarrow \text{Lip}(\Omega_T)$ is a continuous mapping and thus can be extended to $\hat{E}_{t_i}[] : L^p_G(\Omega_T) \rightarrow L^p_G(\Omega_T)$.

**Definition 2.7.** (G-martingale). A process $M = (M_t)_{t \in [0, T]}$ with $M_t \in L^1_G(\Omega_t)$, $0 \leq t \leq T$, is called a G-martingale if for all $0 \leq s \leq t \leq T$, we have

$$\hat{E}_{s}[M_t] = M_s.$$

The process $M = (M_t)_{t \in [0, T]}$ is called symmetric G-martingale if $-M$ is also a G-martingale.

**Theorem 2.8.** (Representation theorem of G-expectation, see Hu and Peng, 2009; Denis et al., 2011). There exists a weakly compact set $\mathcal{P} \subset M_1(\Omega_T)$, the set of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that

$$\hat{E}[\xi] := \sup_{P \in \mathcal{P}} E_P[\xi] \quad \text{for all} \quad \xi \in L^1_G(\Omega_T).$$

$\mathcal{P}$ is called a set that represents $\hat{E}$.

Let $\mathcal{P}$ be a weakly compact set that represents $\hat{E}$. For this $\mathcal{P}$, we define the capacity of a measurable set $A$ by

$$\hat{C}(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega_T).$$

A set $A \in \mathcal{B}(\Omega_T)$ is a polar if $\hat{C}(A) = 0$. A property holds quasi-surely (q.s.) if it is true outside a polar set.
An important feature of the $G$-expectation framework is that the quadratic variation $\langle B \rangle_t$ of the $G$-Brownian motion is no longer a deterministic process, which is given by

$$\langle B \rangle_t := \lim_{\delta(\pi_N) \to 0} \sum_{j=0}^{N-1} (B_{t_j} - B_{t_{j+1}})^2,$$

where $\pi_N = \{t_0, t_1, \ldots, t_N\}$, $N = 1, 2, \ldots$, are refining partitions of $[0, t]$. For all $s, t \geq 0$, $\langle B \rangle_{t+s} - \langle B \rangle_t \in [\sigma^2, \sigma^2]$, q.s. (see Peng, 2019).

Let $M^p_G(0, T)$ be the collection of processes in the following form: for a given partition $\pi^N := \{t_0, t_1, \ldots, t_N\}$ of $[0, T]$,

$$(2.1) \quad \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in \text{Lip}(\Omega_t)$, for all $i = 0, 1, \ldots, N - 1$. For $p \geq 1$ and $\eta \in M^p_G(0, T)$, let $\| \eta \|_{H^p_G} := \left( \mathbb{E} \left[ \left( \int_0^T |\eta_t|^2 ds \right)^{p/2} \right] \right)^{1/p}$, $\| \eta \|_{M^p_G} := \left( \mathbb{E} \left[ \int_0^T |\eta_t|^p ds \right] \right)^{1/p}$ and denote by $H^p_G(0, T)$, $M^p_G(0, T)$ the completions of $M^p_G(0, T)$ under the norms $\| \cdot \|_{H^p_G}$, $\| \cdot \|_{M^p_G}$ respectively.

Let $S^p_G(0, T) := \{h(t, B_{t_1,\ldots,t}, B_{t_2,\ldots,t}, \ldots, B_{t_n,\ldots,t} - B_{t_{n-1},\ldots,t}) : 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T, h \in C_b, L_{\text{lip}}(\mathbb{R}^{n+1})\}$, where $C_b, L_{\text{lip}}(\mathbb{R}^{n+1})$ is the collection of all bounded and Lipschitz functions on $\mathbb{R}^{n+1}$. For $p \geq 1$ and $\eta \in S^p_G(0, T)$, we set $\| \eta \|_{S^p_G} := \left( \mathbb{E} \left[ \sup_{t \in [0, T]} |\eta_t|^p \right] \right)^{1/p}$. We denote by $S^p_G(0, T)$ the completion of $S^p_G(0, T)$ under the norm $\| \cdot \|_{S^p_G}$.

**Definition 2.9.** For $\eta \in M^p_G(0, T)$ of the form (2.1), the Itô integral with respect to $G$-Brownian motion is defined by the linear mapping $\mathcal{I} : M^p_G(0, T) \to L^p_G(\Omega_T)$,

$$\mathcal{I}(\eta) := \int_0^T \eta_t dB_t = \sum_{k=0}^{N-1} \xi_k(B_{t_{k+1}} - B_{t_k}),$$

which can be continuously extended to $\mathcal{I} : H^1_G(0, T) \to L^p_G(\Omega_T)$. On the other hand, the stochastic integral with respect to $\langle B \rangle_t$ is defined by the linear mapping $\mathcal{Q} : M^p_G(0, T) \to L^p_G(\Omega_T)$,

$$\mathcal{Q}(\eta) := \int_0^T \eta_t d\langle B \rangle_t = \sum_{k=0}^{N-1} \xi_k(\langle B \rangle_{t_{k+1}} - \langle B \rangle_{t_k}),$$

which can be continuously extended to $\mathcal{Q} : H^1_G(0, T) \to L^1_G(\Omega_T)$.

**Lemma 2.10.** (BDG type inequality, see Gao, 2009; Theorem 2.1). Let $p \geq 2$, $\eta \in H^p_G(0, T)$ and $0 \leq s \leq t \leq T$. Then,

$$c_p \mathbb{E} \left[ \left( \int_0^T |\eta_t|^2 ds \right)^{p/2} \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \int_s^t |\eta_r|^2 dB_r \right)^p \right] \leq C_p \mathbb{E} \left[ \left( \int_0^T |\eta_t|^2 ds \right)^{p/2} \right],$$
where \( 0 < \epsilon < C_p < \infty \) are constants independent of \( \eta, \sigma \) and \( \tau \).

For \( \xi \in \text{Lip}(\Omega_T) \), let
\[
\mathcal{E}(\xi) := \hat{E} \left( \sup_{t \in [0,T]} \hat{E}_t[\xi] \right).
\]
\( \mathcal{E} \) is called the G-evaluation. For \( p \geq 1 \) and \( \xi \in \text{Lip}(\Omega_T) \), define
\[
\|\xi\|_{p,\mathcal{E}} := (\mathcal{E}[|\xi|^p])^{1/p}
\]
and denote by \( L^p_\mathcal{E}(\Omega_T) \) the completion of \( \text{Lip}(\Omega_T) \) under the norm \( \|\cdot\|_{p,\mathcal{E}} \).

**Theorem 2.11.** (See Song, 2011). For any \( \alpha \geq 1 \) and \( \delta > 0 \), we have \( L^{\alpha+\delta}_G(\Omega_T) \subset L^\alpha_\mathcal{E}(\Omega_T) \). More precisely, for any \( 1 < \gamma < \beta := (\alpha + \delta)/\alpha \), \( \gamma \leq 2 \) and for all \( \xi \in \text{Lip}(\Omega_T) \), we have
\[
\hat{E}\left[ \sup_{t \in [0,T]} \hat{E}_t[|\xi|^\gamma] \right] \leq C \left\{ (\hat{E}[|\xi|^{\alpha+\beta}])^{\alpha/\gamma} + (\hat{E}[|\xi|^{\alpha+\beta}])^{1/\gamma} \right\},
\]
where
\[
C = \frac{\gamma}{\gamma - 1} (1 + 14 \sum_{i=1}^\infty i^{-\beta/\gamma}).
\]

**Remark 2.12.** By \( \frac{\alpha}{\alpha + \delta} < \frac{1}{\alpha} < 1 \), we have
\[
\hat{E}\left[ \sup_{t \in [0,T]} \hat{E}_t[|\xi|^\gamma] \right] \leq 2C \left\{ (\hat{E}[|\xi|^{\alpha+\beta}])^{\alpha/\gamma} + \hat{E}[|\xi|^{\alpha+\beta}] \right\}.
\]

Set
\[
C_1 = 2 \inf \left\{ \frac{\gamma}{\gamma - 1} (1 + 14 \sum_{i=1}^\infty i^{-\beta/\gamma}) : 1 < \beta, \gamma \leq 2 \right\},
\]
then
\[
(2.2) \quad \hat{E}\left[ \sup_{t \in [0,T]} \hat{E}_t[|\xi|^\gamma] \right] \leq C_1 \left\{ (\hat{E}[|\xi|^{\alpha+\beta}])^{\alpha/\gamma} + \hat{E}[|\xi|^{\alpha+\beta}] \right\},
\]
where \( C_1 \) is a constant only depending on \( \alpha \) and \( \delta \).

**Lemma 2.13.** (See Hu et al., 2014a). Let \( X \in S^\alpha_G(0,T) \) for some \( \alpha > 1 \) and \( \alpha^* = \frac{\alpha}{\alpha - 1} \). Assume that \( K_j, \ j = 1,2, \) are two decreasing G-martingales with \( K_0^j = 0 \) and \( K_T^j \in L^\alpha_G(\Omega_T) \). Then the processus defined by
\[
\int_0^t X_+^j dK_1^j + \int_0^t X_-^j dK_2^j
\]
is also a decreasing G-martingale.

3. Large deviations for G-SDEs

In this section, we present the large deviations for G-SDEs obtained by Gao and Jiang (2010). The authors use discrete time approximation to obtain their results.

First, we recall the following notations on large deviations under a sublinear expectation.

Let \((\chi, d)\) be a Polish space. Let \((U^\varepsilon, \varepsilon > 0)\) be a family of measurable maps from \(\Omega\) into \((\chi, d)\) and let \(\delta(\varepsilon), \varepsilon > 0\) be a positive function satisfying \(\delta(\varepsilon) \to 0\) as \(\varepsilon \to 0\).
A nonnegative function $I$ on $\chi$ is called to be (good) rate function if \( \{x : I(x) \leq \alpha\} \) (its level set) is (compact) closed for all \( 0 \leq \alpha < \infty \).

\[ \{\tilde{C}(U^\varepsilon \in \cdot)\}_{\varepsilon > 0} \] is said to satisfy large deviation principle with speed $\delta(\varepsilon)$ and with rate function $I$ if for any measurable closed subset $\mathcal{F} \subset \chi$,

\[
\limsup_{\varepsilon \to 0} \delta(\varepsilon) \log \tilde{C}(U^\varepsilon \in \mathcal{F}) \leq - \inf_{x \in \mathcal{F}} I(x),
\]

and for any measurable open subset $\mathcal{O} \subset \chi$,

\[
\liminf_{\varepsilon \to 0} \delta(\varepsilon) \log \tilde{C}(U^\varepsilon \in \mathcal{O}) \geq - \inf_{x \in \mathcal{O}} I(x).
\]

In Gao and Jiang (2010), for any $\varepsilon > 0$, the authors considered the following random perturbation SDEs driven by $d$-dimensional $G$-Brownian motion $B$

\[
X_t^{\varepsilon,\delta} = x + \int_0^t b^\varepsilon(X_s^{\varepsilon,\delta})ds + \varepsilon \int_0^t h^\varepsilon(X_s^{\varepsilon,\delta})dB_t + \varepsilon \int_0^t \sigma^\varepsilon(X_s^{\varepsilon,\delta})dB_t
\]

where $\langle B, B \rangle$ is treated as a $d \times d$-dimensional vector,

\[
b^\varepsilon = (b_{\varepsilon}^1, \ldots, b_{\varepsilon}^n)^T : \mathbb{R}^n \to \mathbb{R}^n, \quad \sigma^\varepsilon = (\sigma_{ij}^\varepsilon) : \mathbb{R}^n \to \mathbb{R}^{n \times d},
\]

and $h^\varepsilon : \mathbb{R}^n \to \mathbb{R}^{n \times d}$.

Consider the following conditions:

(H1): $b^\varepsilon$, $\sigma^\varepsilon$ and $h^\varepsilon$ are uniformly bounded;

(H2): $b^\varepsilon$, $\sigma^\varepsilon$ and $h^\varepsilon$ are uniformly Lipschitz continuous;

(H3): $b^\varepsilon$, $\sigma^\varepsilon$ and $h^\varepsilon$ converge uniformly to $b := b^0$, $\sigma := \sigma^0$ and $h := h^0$ respectively.

Let $\mathcal{C}([0, T], \mathbb{R}^n)$ be the space of $\mathbb{R}^n$-valued continuous functions $\varphi$ on $[0, T]$ and $\mathcal{C}_0([0, T], \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued continuous functions $\tilde{\varphi}$ on $[0, T]$ with $\tilde{\varphi}_0 = 0$. Define

\[
\mathbb{H}^d := \{\phi \in \mathcal{C}_0([0, T], \mathbb{R}^d) : \phi \text{ is absolutely continuous and}
\]

\[
\|\phi\|_{\mathbb{H}}^2 := \int_0^T |\phi'(r)|^2dr < +\infty\},
\]

\[
\mathbb{A} := \{\eta = \int_0^t \eta'(r)dr; \eta' : [0, T] \to \mathbb{R}^{d \times d} \text{ Borel measurable and}
\]

\[
\eta'(t) \in \Sigma \text{ for all } t \in [0, T]\}.
\]

We recall the following result of a joint large deviation principle for $G$-Brownian motion and its quadratic variation process.

Theorem 3.1. (See Gao and Jiang, 2010; p. 2225). \( \{\tilde{C}(\varepsilon B_t \in \cdot, \varepsilon \langle B_t \rangle \in \cdot) | t \in [0, T] \in \cdot\}_{\varepsilon > 0} \) satisfies large deviation principle with speed $\varepsilon$ and with rate function

\[
J(\phi, \eta) = \begin{cases} \frac{1}{2} \int_0^T \langle \phi'(r), (\eta'(r))^{-1} \phi'(r) \rangle dr, & \text{if } (\phi, \eta) \in \mathbb{H}^d \times \mathbb{A}, \\ +\infty, & \text{otherwise}. \end{cases}
\]
For any \((\phi, \eta) \in \mathbb{H}^d \times A\), let \(\Psi(\phi, \eta) \in C([0, T], \mathbb{R}^n)\) be the unique solution of the following ordinary differential equation (ODE in short)

\[
\Psi(\phi, \eta)(t) = x + \int_0^t b(\Psi(\phi, \eta)(r))dr + \int_0^t \sigma(\Psi(\phi, \eta)(r))\phi'(r)dr + \int_0^t h(\Psi(\phi, \eta)(r))\eta'(r)dr.
\]

For \(0 \leq \alpha < 1\) given and \(n \geq 1\), for each \(\psi \in C_0([0, T], \mathbb{R}^n)\), set

\[
\|\psi\|_\alpha := \sup_{s, t \in [0, T]} \frac{|\psi(s) - \psi(t)|}{|s - t|^\alpha}
\]

and

\[
C_\alpha^0([0, T], \mathbb{R}^n) := \left\{ \psi \in C_0([0, T], \mathbb{R}^n) : \lim_{\delta \to 0} \sup_{|s - t| < \delta} \frac{|\psi(s) - \psi(t)|}{|s - t|^\alpha} = 0, \|\psi\|_\alpha < \infty \right\}.
\]

**Theorem 3.2.** (See Gao and Jiang, 2010; p. 2227). Let \(0 \leq \alpha < 1/2\) and let \((H_1), (H_2) and (H_3)\) hold. Then for any closed subset \(F\) and any open subset \(O\) in \((C_\alpha^0([0, T], \mathbb{R}^n), \| \cdot \|_\alpha)\),

\[
\limsup_{\varepsilon \to 0} \varepsilon \log \widehat{C} \left( (X_t^{\varepsilon, x} - x) |_{t \in [0, T]} \in F \right) \leq - \inf_{\psi \in F} I(\psi),
\]

and

\[
\liminf_{\varepsilon \to 0} \varepsilon \log \widehat{C} \left( (X_t^{\varepsilon, x} - x) |_{t \in [0, T]} \in O \right) \geq - \inf_{\psi \in O} I(\psi),
\]

where

\[
I(\psi) = \inf \left\{ J(\phi, \eta) : \psi = \Psi(\phi, \eta) - x \right\}.
\]

We immediately have the following result.

**Corollary 3.3.** Let \((H_1), (H_2) and (H_3)\) hold. Then for any closed subset \(F\) and any open subset \(O\) in \(C_0([0, T], \mathbb{R}^n)\),

\[
\limsup_{\varepsilon \to 0} \varepsilon \log \widehat{C} \left( (X_t^{\varepsilon, x} - x) |_{t \in [0, T]} \in F \right) \leq - \inf_{\phi \in F} \Lambda(\bar{\varphi}),
\]

and

\[
\liminf_{\varepsilon \to 0} \varepsilon \log \widehat{C} \left( (X_t^{\varepsilon, x} - x) |_{t \in [0, T]} \in O \right) \geq - \inf_{\phi \in O} \Lambda(\bar{\varphi}),
\]

where

\[
\Lambda(\bar{\varphi}) = \inf \left\{ J(\phi, \eta) : x + \bar{\varphi} = \Psi(\phi, \eta) \right\}.
\]

4. Large deviations for G-BSDEs with subdifferential operator

We consider the \(G\)-expectation space \((\Omega_T, L^1_G(\Omega_T), \widehat{\mathbb{E}})\) with \(\Omega_T = C_0([0, T], \mathbb{R})\) and \(\sigma^2 = \widehat{\mathbb{E}}(B_t^2) \geq -\widehat{\mathbb{E}}(-B_t^2) = \sigma^2 > 0\).
4.1. Assumptions and problem formulation. Yang et al. (2017) obtained the existence and uniqueness of the solution of the following backward stochastic differential equation driven by G-Brownian motion with subdifferential operator

\begin{equation}
(4.1) \left\{ \begin{aligned}
- dY_t + \Pi(Y_t) dt &\geq f(t, Y_t, Z_t) dt - Z_t dB_t + g(t, Y_t, Z_t) d(B)_t - dK_t \\
Y_T &= \xi
\end{aligned} \right.
\end{equation}

where

(A1): $\Pi: \mathbb{R} \rightarrow (-\infty, +\infty]$ is a proper lower semicontinuous (l.s.c. in short) convex function such that $\Pi(y) \geq \Pi(0) = 0$, for all $y \in \mathbb{R}$.

Denote

\begin{align*}
\text{Dom}(\Pi) &= \{ y \in \mathbb{R} : \Pi(y) < \infty \}, \\
\partial \Pi(y) &= \{ u \in \mathbb{R} : \langle u, v - y \rangle + \Pi(y) \leq \Pi(v), \forall v \in \mathbb{R} \}, \\
\text{Dom}(\partial \Pi) &= \{ y \in \mathbb{R} : \partial \Pi(y) \neq \emptyset \}, \\
(y, u) \in \text{Gr}(\partial \Pi) \iff y \in \text{Dom}(\partial \Pi), u \in \partial \Pi(y).
\end{align*}

Note that the subdifferential operator $\partial \Pi: \mathbb{R} \rightarrow 2^\mathbb{R}$ is a maximal monotone operator, that is

$$\langle y - y', u - u' \rangle \geq 0, \forall (y, u), (y', u') \in \text{Gr}(\partial \Pi).$$

(A2): For any $y, z, f(\omega, \ldots, y, z), g(\omega, \ldots, y, z) \in M^2_G(0, T)$.

(A3): The functions $f: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $g: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and there exists a constant $L > 0$ such that for all $t \in [0, T]$, $y, y', z, z' \in \mathbb{R}$,

$$|f(t, y, z) - f(t, y', z')| + |g(t, x, y, z) - g(t, x, y', z')| \leq L(|y - y'| + |z - z'|).$$

Definition 4.1. Let $\xi \in L^2_G(\Omega_T)$, the solution of the $G$-MBSDE (4.1) is a quadruple of processes $(Y, Z, K, U)$ such that

1. $Y \in \mathcal{S}^2_G(0, T)$, $Z \in H^2_G(0, T)$, $K$ is a decreasing $G$-martingale with $K_0 = 0$, $K_T \in L^2_G(\Omega_T)$ and $U \in H^2_G(0, T)$;
2. $\mathbb{E} \left( \int_0^T \Pi(Y_r) dr \right) < +\infty$;
3. For every $0 \leq t \leq T$,
   $$Y_t + \int_t^T U_r dr = \xi + \int_t^T f(r, Y_r, Z_r) dr + \int_t^T g(r, Y_r, Z_r) d(B)_r - \int_t^T Z_r dB_r - (K_T - K_t), \text{ q.s.};$$
4. $(Y_t, U_t) \in \text{Gr}(\partial \Pi)$, q.s. on $\Omega_T \times [0, T]$.

To establish large deviation principle, we consider the following forward-backward stochastic differential equation driven by $G$-Brownian motion with subdifferential operator: for every $s \leq t \leq T$, $x \in \mathbb{R}$,

$$X^{\varepsilon, x, \varepsilon}_r = x + \int_s^t b(X^{\varepsilon, x, \varepsilon}_r) dr + \int_s^t \varepsilon h(X^{\varepsilon, x, \varepsilon}_r) d(B)_r + \int_s^t \varepsilon \sigma(X^{\varepsilon, x, \varepsilon}_r) dB_r.$$
where $\Pi$ is a proper l.s.c. convex function such that $\Pi(y) \geq \Pi(0) = 0$, for all $y \in \mathbb{R}$ and $b, h, \sigma : \mathbb{R} \rightarrow \mathbb{R}; \Phi : \mathbb{R} \rightarrow \mathbb{R}; f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are deterministic functions and satisfy the following assumptions:

(B1): $b, \sigma$ and $h$ are bounded, i.e., there exists a constant $L > 0$ such that
$$
\sup_{x \in \mathbb{R}} \max \left\{ |b(x)|, |\sigma(x)|, |h(x)| \right\} \leq L,
$$

(B2): $f$ and $g$ are continuous in $t$;

(B3): There exist a constant $L > 0$ such that
$$
\begin{align*}
|b(x) - b(x')| + |h(x) - h(x')| + |\sigma(x) - \sigma(x')| &\leq L|x - x'|, \\
|\Phi(x) - \Phi(x')| &\leq L|x - x'|, \\
|f(t, x, y, z) - f(t, x', y', z')| &\leq L(|x - x'| + |y - y'| + |z - z'|), \\
g(t, x, y, z) - g(t, x', y', z') &\leq L(|x - x'| + |y - y'| + |z - z'|).
\end{align*}
$$

It follows from Yang et al. (2017) that, under the assumptions (B1) – (B3), the G-MBSDE (4.2) has a unique solution $\{(Y^s_{t,x,\varepsilon}, Z^s_{t,x,\varepsilon}, K^s_{t,x,\varepsilon}, U^s_{t,x,\varepsilon}) : s \leq t \leq T\}$ such that

- $Y^s_{t,x,\varepsilon} \in S^2_0(0, T)$, $Z^s_{t,x,\varepsilon} \in H^2_0(0, T)$, $K^s_{t,x,\varepsilon}$ is a decreasing $G$-martingale with $K^s_{t,x,\varepsilon} = 0$, $K^s_{T,x,\varepsilon} \in L^2_0(\Omega_T)$ and $U^s_{t,x,\varepsilon} \in H^2_G(0, T)$;

- $
\mathbb{E}\left(\int_s^T \Pi(Y^s_{t,x,\varepsilon}) \, dr\right) < +\infty;
$

- For every $s \leq t \leq T$,
$$
Y^s_{t,x,\varepsilon} + \int_t^T U^s_{r,x,\varepsilon} \, dr = \Phi(X^s_{t,x,\varepsilon}) + \int_t^T f(r, X^s_{r,x,\varepsilon}, Y^s_{r,x,\varepsilon}, Z^s_{r,x,\varepsilon}) \, dr
$$
$$
+ \int_t^T g(r, X^s_{r,x,\varepsilon}, Y^s_{r,x,\varepsilon}, Z^s_{r,x,\varepsilon}) \, dB_r
$$
$$
- \int_t^T Z^s_{r,x,\varepsilon} \, dB_r - (K^s_{t,x,\varepsilon} - K^s_{s,x,\varepsilon}), \text{ q.s. ;}
$$

- $(Y^s_{t,x,\varepsilon}, U^s_{t,x,\varepsilon}) \in \text{Gr}(\partial \Pi)$, q.s. on $\Omega_T \times [s, T]$.

Our purpose is to study the asymptotic behavior of the family $(Y^s_{t,x,\varepsilon})_{\varepsilon > 0}$ as $\varepsilon$ goes to 0.

4.2. Convergence and large deviation principle for the solution of the backward equation. We consider the following decoupled forward-backward stochastic differential equation driven by $G$-Brownian motion with subdifferential operator: for every $s \leq t \leq T$,

$$
\begin{align*}
\left\{ \begin{array}{l}
dX^s_{t,x,\varepsilon} = b(X^s_{t,x,\varepsilon}) \, dt + \varepsilon h(X^s_{t,x,\varepsilon}) \, dB_t + \varepsilon \sigma(X^s_{t,x,\varepsilon}) \, dB_t \\
X^s_{s,x,\varepsilon} = x
\end{array} \right.
\end{align*}
$$

(4.3)
We also consider the following deterministic system: for every 
\[
\begin{align*}
\text{(Contraction principle). Let } & \text{ LDP from one space to another.} \\
& \text{ measures } f: \Omega \rightarrow \mathbb{R}
\end{align*}
\]
with respect to \(\Upsilon\), a Hausdorff topological space. Assume further that 
\[
\text{the function } f \text{ satisfies the large deviation principle with a good rate function } \varepsilon \rightarrow G(\varepsilon).
\]

Consider the following decreasing family: for every \(s \leq t \leq T\), 
\[
\begin{align*}
\psi_0 \rightarrow b(\varphi_t)dt \\
\varphi^{s,x}_t = C_{1,2}.
\end{align*}
\]

Moreover, 
\[
\begin{align*}
\left\{ \tilde{\Psi} \left( (X_t^{s,x} - x) \mid t \in [s,T] \right) \right\}_{\varepsilon > 0}
\end{align*}
\]
satisfies a large deviation principle with speed \(\varepsilon \) and with rate function 
\[
\Lambda(\varphi) = \inf \left\{ J(\phi, \eta) : x + \varphi = \tilde{\Psi}(\phi, \eta) \right\},
\]
where \(\tilde{\Psi}(\phi, \eta) \in C([s,T], \mathbb{R})\) be the unique solution of the following ODE 
\[
\tilde{\Psi}(\phi, \eta)(t) = x + \int_s^t b(\tilde{\Psi}(\phi, \eta)(r))(r)dr.
\]

We recall a very important result in large deviation theory, used to transfer a 
LDP from one space to another. 

**Lemma 4.2.** Let (B1) and (B3) hold. Then 

1. Let \(p \geq 2\). For any \(\varepsilon \in (0,1]\), there exists a constant \(C_p > 0\), independent of \(\varepsilon\), such that 
\[
\tilde{\mathbb{E}}\left( \sup_{s \leq t \leq T} |X_t^{s,x} - \varphi_t^{s,x}|^p \right) \leq C_p \varepsilon^p.
\]

2. Moreover, 
\[
\left\{ \tilde{\mathbb{C}} \left( (X_t^{s,x} - x) \mid t \in [s,T] \right) \right\}_{\varepsilon > 0}
\]
satisfies a large deviation principle with speed \(\varepsilon \) and with rate function 
\[
\Lambda(\varphi) = \inf \left\{ J(\phi, \eta) : x + \varphi = \tilde{\Psi}(\phi, \eta) \right\},
\]
where \(\tilde{\Psi}(\phi, \eta) \in C([s,T], \mathbb{R})\) be the unique solution of the following ODE 
\[
\tilde{\Psi}(\phi, \eta)(t) = x + \int_s^t b(\tilde{\Psi}(\phi, \eta)(r))(r)dr.
\]

We recall a very important result in large deviation theory, used to transfer a 
LDP from one space to another. 

**Lemma 4.3.** (Contraction principle). Let \(\mu_\varepsilon\) be a family of probability measures that satisfies the large deviation principle with a good rate function \(\Lambda\) on a 
Hausdorff topological space \(\chi\), and for \(\varepsilon \in (0,1]\), let \(f_\varepsilon : \chi \rightarrow \Upsilon\) be continuous functions, with \((\Upsilon, d)\) a metric space. Assume that there exists a measurable map 
\(f : \chi \rightarrow \Upsilon\) such that for any compact set \(K \subset \chi\), 
\[
\limsup_{\varepsilon \rightarrow 0} \sup_{x \in K} d(f_\varepsilon(x), f(x)) = 0.
\]
Suppose further that \(\mu_\varepsilon\) is exponentially tight. Then the family of probability measures \(\mu_\varepsilon \circ f_\varepsilon^{-1}\) satisfies the LDP in \(\Upsilon\) with the good rate function 
\[
\Lambda'(y) = \inf \left\{ \Lambda(x) : x \in \chi, y = f(x) \right\}.
\]
The proofs of Lemmas 4.2 and 4.3 can be found in Dakaou and Hima (2021).
Theorem 4.4. Let (B1) – (B3) hold. For any \( \varepsilon \in (0,1] \), there exists a constant \( C > 0 \), independent of \( \varepsilon \), such that

\[
\mathbb{E} \left( \sup_{s \leq t \leq T} |Y_{t}^{s,x,\varepsilon} - \psi_{t}^{s,x}|^2 \right) \leq C \varepsilon^2.
\]

Proof. We consider the following G-BSDE: for every \( s \leq t \leq T, x \in \mathbb{R} \),

\[
Y_{t} = \Phi(\varphi_{T}^{s,x}) + \int_{t}^{T} f(r, \varphi_{r}^{s,x}, Y_{r}, Z_{r})dr - \int_{t}^{T} U_{r}dr + \int_{t}^{T} g(r, \varphi_{r}^{s,x}, Y_{r}, Z_{r})d(B)_{r} - \int_{t}^{T} Z_{r}dB_{r} - (K_{T} - K_{t}).
\]

(4.8)

Thanks to equation (4.6) and the uniqueness of the solution of the G-MBSDEs, it is easy to check that \( \{(\psi_{t}^{s,x}, 0, M_{t}^{s,x}, U_{t}^{s,x}) : s \leq t \leq T\} \) is the solution of the G-MBSDE (4.8). So, we have

\[
Y_{t}^{s,x,\varepsilon} - \psi_{t}^{s,x} = \Phi(\varphi_{T}^{s,x,\varepsilon}) - \Phi(\varphi_{T}^{s,x}) - \int_{t}^{T} \{U_{r}^{s,x,\varepsilon} - U_{r}^{s,x}\} dr + \int_{t}^{T} \{f(r, X_{r}^{s,x,\varepsilon}, Y_{r}^{s,x,\varepsilon}, Z_{r}^{s,x,\varepsilon}) - f(r, \varphi_{r}^{s,x,\varepsilon}, \psi_{r}^{s,x,\varepsilon}, 0)\} dr
\]

\[
+ \int_{t}^{T} \{g(r, X_{r}^{s,x,\varepsilon}, Y_{r}^{s,x,\varepsilon}, Z_{r}^{s,x,\varepsilon}) - g(r, \varphi_{r}^{s,x,\varepsilon}, \psi_{r}^{s,x,\varepsilon}, 0)\} d(B)_{r}
\]

\[
- \int_{t}^{T} Z_{r}^{s,x,\varepsilon}dB_{r} - (K_{T}^{s,x,\varepsilon} - K_{t}^{s,x,\varepsilon}) + (M_{T}^{s,x} - M_{t}^{s,x}).
\]

For \( \gamma > 0 \), by Itô’s formula applied to \( e^{\gamma t} |Y_{t}^{s,x,\varepsilon} - \psi_{t}^{s,x}|^2 \), we have

\[
e^{\gamma t} |Y_{t}^{s,x,\varepsilon} - \psi_{t}^{s,x}|^2 + 2 \gamma \int_{t}^{T} e^{\gamma r} |Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}|^2 dr + \int_{t}^{T} e^{\gamma r} |Z_{r}^{s,x,\varepsilon}|^2 d(B)_{r}
\]

\[
e^{\gamma T} |\Phi(\varphi_{T}^{s,x,\varepsilon}) - \Phi(\varphi_{T}^{s,x})|^2 - 2 \int_{t}^{T} e^{\gamma r} \langle Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}, U_{r}^{s,x,\varepsilon} - U_{r}^{s,x}\rangle dr
\]

\[
+ 2 \int_{t}^{T} e^{\gamma r} \langle Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}, f(r, X_{r}^{s,x,\varepsilon}, Y_{r}^{s,x,\varepsilon}, Z_{r}^{s,x,\varepsilon}) - f(r, \varphi_{r}^{s,x,\varepsilon}, \psi_{r}^{s,x,\varepsilon}, 0)\rangle dr
\]

\[
+ 2 \int_{t}^{T} e^{\gamma r} \langle Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}, g(r, X_{r}^{s,x,\varepsilon}, Y_{r}^{s,x,\varepsilon}, Z_{r}^{s,x,\varepsilon}) - g(r, \varphi_{r}^{s,x,\varepsilon}, \psi_{r}^{s,x,\varepsilon}, 0)\rangle d(B)_{r}
\]

\[
- \int_{t}^{T} e^{\gamma r} \langle Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}, Z_{r}^{s,x,\varepsilon}\rangle dB_{r} - 2 \int_{t}^{T} e^{\gamma r} \{Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}\} dK_{r}^{s,x,\varepsilon}
\]

\[
+ 2 \int_{t}^{T} e^{\gamma r} \{Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}\} dM_{r}^{s,x}.
\]

Since

\[
\langle Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}, U_{r}^{s,x,\varepsilon} - U_{r}^{s,x}\rangle \geq 0,
\]

and

\[-2 \int_{t}^{T} e^{\gamma r} \{Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}\}^{+} dK_{r}^{s,x,\varepsilon} - 2 \int_{t}^{T} e^{\gamma r} \{Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}\}^{-} dM_{r}^{s,x} \geq 0,
\]
by using Young’s inequality and Lipschitz conditions of \( f \) and \( g \), we get

\[
\begin{align*}
e^{-\tau t} |Y_{s,x}^t - \psi_t^s| + \gamma \int_t^T e^{\gamma r} |Y_{r,x}^r - \psi_r^s|^2 dr + \sigma^2 \int_t^T e^{\gamma r} \left| Z_{r,x}^r \right|^2 dr + J_T - J_t & \\
\leq e^{\gamma T} |\Phi (X_{T,x}^s) - \Phi (\varphi_{T,x}^s)|^2 \\
+ 2 \int_t^T e^{\gamma r} |Y_{r,x}^r - \psi_r^s| \left| f (r, X_{r,x}^s, Y_{r,x}^s, Z_{r,x}^s) - f (r, \varphi_{r,x}^s, \psi_{r,x}^s, 0) \right| dr \\
+ 2 \int_t^T e^{\gamma r} |Y_{r,x}^r - \psi_r^s| \left| g (r, X_{r,x}^s, Y_{r,x}^s, Z_{r,x}^s) - g (r, \varphi_{r,x}^s, \psi_{r,x}^s, 0) \right| d \langle B \rangle_r & \\
\leq e^{\gamma T} |\Phi (X_{T,x}^s) - \Phi (\varphi_{T,x}^s)|^2 \\
+ 2L \left( 1 + \sigma^2 \right) \int_t^T e^{\gamma r} |Y_{r,x}^r - \psi_r^s|^2 dr & \\
+ L \left( 1 + \sigma^2 \right) \left( 2 + \frac{L (1 + \sigma^2)}{\sigma^2} \right) \int_t^T e^{\gamma r} |Y_{r,x}^r - \psi_r^s|^2 dr & \\
+ 2 \int_t^T e^{\gamma r} \left| Z_{r,x}^r \right|^2 \sigma^2 dr. &
\end{align*}
\]

where

\[
J_t = \int_s^t e^{\gamma r} Z_{r,x}^r \{ Y_{r,x}^r - \psi_r^s \} d B_r \\
+ 2 \int_s^t e^{\gamma r} \left| Y_{r,x}^r - \psi_r^s \right|^2 d K_{r,x}^r + 2 \int_s^t e^{\gamma r} \left| Y_{r,x}^r - \psi_r^s \right| d M_{r,x}^s.
\]

We have, by setting \( \gamma = L (1 + \sigma^2) \left( 2 + \frac{L (1 + \sigma^2)}{\sigma^2} \right) \),

\[
|Y_{t,x}^s - \psi_t^s|^2 + J_T - J_t \leq e^{\gamma T} |\Phi (X_{T,x}^s) - \Phi (\varphi_{T,x}^s)|^2 \\
+ L \left( 1 + \sigma^2 \right) \int_t^T e^{\gamma r} |X_{r,x}^s - \varphi_{r,x}^s|^2 dr & \\
\leq e^{\gamma T} \left( L^2 + L \left( 1 + \sigma^2 \right) T \right) \sup_{s \leq r \leq T} |X_{r,x}^s - \varphi_{r,x}^s|^2 & \\
\leq C \sup_{s \leq r \leq T} |X_{r,x}^s - \varphi_{r,x}^s|^2.
\]

Since \( J \) is a \( G \)-martingale, taking conditional \( G \)-expectation, we get

\[
|Y_{t,x}^s - \psi_t^s|^2 \leq C \mathbb{E}_t \left[ \sup_{s \leq r \leq T} |X_{r,x}^s - \varphi_{r,x}^s|^2 \right].
\]

Thus we obtain

\[
\mathbb{E} \left[ \sup_{s \leq t \leq \tau} |Y_{t,x}^s - \psi_t^s|^2 \right] \leq C \mathbb{E} \left[ \sup_{s \leq r \leq T} |X_{r,x}^s - \varphi_{r,x}^s|^2 \right].
\]

So, by virtue of (4.7), the proof is complete.

We have an immediate consequence of Theorem 4.4.
Corollary 4.5. For any $\varepsilon \in (0, 1]$ and all $x$ in a compact subset of $\mathbb{R}$, there exists a constant $C > 0$, independent of $s$, $x$ and $\varepsilon$, such that

$$
\hat{\mathbb{E}} \left( \sup_{s \leq t \leq T} |Y_{t}^{s,x,\varepsilon} - \psi_{t}^{s,x}|^2 \right) \leq C \varepsilon^2.
$$

Lemma 4.6. For any $\varepsilon \in (0, 1]$, there exists a constant $C > 0$, independent of $\varepsilon$, such that

$$(4.9) \quad \hat{\mathbb{E}} \left[ |K_{T}^{s,x,\varepsilon}|^2 \right] \leq C.$$ 

Theorem 4.7. Let $(B1) - (B3)$ hold. For any $\varepsilon \in (0, 1]$, there exists a constant $C > 0$, independent of $\varepsilon$, such that

$$
\hat{\mathbb{E}} \left[ \int_{s}^{T} |Z_{r}^{s,x,\varepsilon}|^2 dr \right] \leq C \varepsilon.
$$

Proof. Similarly as in the proof of Theorem 4.4, for $\gamma > 0$, by Itô’s formula applied to $e^{\gamma t} |Y_{t}^{s,x,\varepsilon} - \psi_{t}^{s,x}|^2$, we have

$$
\gamma \int_{s}^{T} e^{\gamma r} |Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}|^2 dr + \sigma^2 \int_{s}^{T} e^{\gamma r} |Z_{r}^{s,x,\varepsilon}|^2 dr + \int_{s}^{T} e^{\gamma r} \langle Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}, Z_{r}^{s,x,\varepsilon} \rangle dB_r
$$

$$
\leq e^{\gamma T} |\Phi (X_{s}^{s,x,\varepsilon}) - \Phi (\psi_{s}^{s,x})|^2
$$

$$
+ 2 \int_{s}^{T} e^{\gamma r} |Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}| |f (r, X_{r}^{s,x,\varepsilon}, Y_{r}^{s,x,\varepsilon}, Z_{r}^{s,x,\varepsilon}) - f (r, \varphi_{r}^{s,x}, \psi_{r}^{s,x}, 0)| dr
$$

$$
+ 2 \int_{s}^{T} e^{\gamma r} |Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}| |g (r, X_{r}^{s,x,\varepsilon}, Y_{r}^{s,x,\varepsilon}, Z_{r}^{s,x,\varepsilon}) - g (r, \varphi_{r}^{s,x}, \psi_{r}^{s,x}, 0)| dB_r
$$

$$
- 2 \int_{s}^{T} e^{\gamma r} \langle Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}, K_{r}^{s,x,\varepsilon} \rangle dK_{r}^{s,x,\varepsilon} + 2 \int_{s}^{T} e^{\gamma r} \langle Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}, M_{r}^{s,x,\varepsilon} \rangle dB_r.
$$

By Lipschitz conditions of $f$ and $g$, we get

$$
\gamma \int_{s}^{T} e^{\gamma r} |Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}|^2 dr + \sigma^2 \int_{s}^{T} e^{\gamma r} |Z_{r}^{s,x,\varepsilon}|^2 dr + \int_{s}^{T} e^{\gamma r} \langle Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}, Z_{r}^{s,x,\varepsilon} \rangle dB_r
$$

$$
\leq e^{\gamma T} |\Phi (X_{s}^{s,x,\varepsilon}) - \Phi (\psi_{s}^{s,x})|^2
$$

$$
+ 2L (1 + \sigma^2) \int_{s}^{T} e^{\gamma r} |Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}| \{ |X_{r}^{s,x,\varepsilon} - \varphi_{r}^{s,x}| + |Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}| + |Z_{r}^{s,x,\varepsilon}| \} dr
$$

$$
- 2 \int_{s}^{T} e^{\gamma r} \langle Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}, K_{r}^{s,x,\varepsilon} \rangle dK_{r}^{s,x,\varepsilon} + 2 \int_{s}^{T} e^{\gamma r} \langle Y_{r}^{s,x,\varepsilon} - \psi_{r}^{s,x}, M_{r}^{s,x,\varepsilon} \rangle dB_r.$$
Using Young’s inequality, we obtain
\[
\gamma \int_s^T e^{\gamma r} |Y_{r,x}^{s,x} - \psi_{r,x}^{s,x}|^2 dr + \sigma^2 \int_s^T e^{\gamma r} |Z_{r,x}^{s,x}|^2 dr + \int_s^T e^{\gamma r} \langle Y_{r,x}^{s,x}, Z_{r,x}^{s,x} \rangle dB_r 
\leq e^{\gamma T} |\Phi(\gamma_{T}^{s,x}) - \Phi(\varphi_{T}^{s,x})|^2 + L (1 + \sigma^2) \int_s^T e^{\gamma r} |X_{r,x}^{s,x} - \varphi_{r,x}^{s,x}|^2 dr 
+ L (1 + \sigma^2) \left( 3 + \frac{4L(1 + \sigma^2)}{\sigma^2} \right) \int_s^T e^{\gamma r} |Y_{r,x}^{s,x} - \psi_{r,x}^{s,x}|^2 dr + \frac{\sigma^2}{4} \int_s^T e^{\gamma r} |Z_{r,x}^{s,x}|^2 dr 
- 2 \int_s^T e^{\gamma r} \{Y_{r,x}^{s,x} - \psi_{r,x}^{s,x}\}^+ dK_{r,x}^{s,x} + 2 \int_s^T e^{\gamma r} \{Y_{r,x}^{s,x} - \psi_{r,x}^{s,x}\}^- dM_{r,x}^{s,x} 
+ 2 \int_s^T e^{\gamma r} \{Y_{r,x}^{s,x} - \psi_{r,x}^{s,x}\}^- dK_{r,x}^{s,x} - 2 \int_s^T e^{\gamma r} \{Y_{r,x}^{s,x} - \psi_{r,x}^{s,x}\}^+ dM_{r,x}^{s,x} 
\leq e^{\gamma T} |\Phi(\gamma_{T}^{s,x}) - \Phi(\varphi_{T}^{s,x})|^2 + L (1 + \sigma^2) \int_s^T e^{\gamma r} |X_{r,x}^{s,x} - \varphi_{r,x}^{s,x}|^2 dr 
+ L (1 + \sigma^2) \left( 3 + \frac{4L(1 + \sigma^2)}{\sigma^2} \right) \int_s^T e^{\gamma r} |Y_{r,x}^{s,x} - \psi_{r,x}^{s,x}|^2 dr + \frac{\sigma^2}{4} \int_s^T e^{\gamma r} |Z_{r,x}^{s,x}|^2 dr 
+ 2e^{\gamma T} \left[ (-K_{T,x}^{s,x}) + (-M_{T,x}^{s,x}) \right] \sup_{s \leq t \leq T} |Y_{t,x}^{s,x} - \psi_{t,x}^{s,x}|.
\]
Thus
\[
\frac{3\sigma^2}{4} \int_s^T e^{\gamma r} |Z_{r,x}^{s,x}|^2 dr + \int_s^T e^{\gamma r} \langle Y_{r,x}^{s,x}, Z_{r,x}^{s,x} \rangle dB_r 
\leq e^{\gamma T} L^2 |X_{T,x}^{s,x} - \varphi_{T,x}^{s,x}|^2 + LT (1 + \sigma^2) e^{\gamma T} \left( \sup_{s \leq t \leq T} |X_{t,x}^{s,x} - \varphi_{t,x}^{s,x}| \right)^2 
+ 2e^{\gamma T} \left[ (-K_{T,x}^{s,x}) + (-M_{T,x}^{s,x}) \right] \sup_{s \leq t \leq T} |Y_{t,x}^{s,x} - \psi_{t,x}^{s,x}|.
\]
We have, by setting $\gamma = L(1 + \sigma^2) \left( 3 + \frac{4L(1 + \sigma^2)}{\sigma^2} \right)$ and Lipschitz condition of $\Phi$,
\[
\frac{3\sigma^2}{4} \int_s^T e^{\gamma r} |Z_{r,x}^{s,x}|^2 dr + \int_s^T e^{\gamma r} \langle Y_{r,x}^{s,x}, Z_{r,x}^{s,x} \rangle dB_r 
\leq e^{\gamma T} L^2 |X_{T,x}^{s,x} - \varphi_{T,x}^{s,x}|^2 + \frac{L T}{\gamma} \left( \sup_{s \leq t \leq T} |X_{t,x}^{s,x} - \varphi_{t,x}^{s,x}| \right)^2 
+ 2e^{\gamma T} \left[ (-K_{T,x}^{s,x}) + (-M_{T,x}^{s,x}) \right] \sup_{s \leq t \leq T} |Y_{t,x}^{s,x} - \psi_{t,x}^{s,x}|.
\]
Then
\[
\frac{3\sigma^2}{4} \int_s^T e^{\gamma r} |Z_{r,x}^{s,x}|^2 dr + \int_s^T e^{\gamma r} \langle Y_{r,x}^{s,x}, Z_{r,x}^{s,x} \rangle dB_r 
\leq e^{\gamma T} L \left( 1 + \sigma^2 \right) \left( \sup_{s \leq t \leq T} |X_{t,x}^{s,x} - \varphi_{t,x}^{s,x}| \right)^2 
+ 2e^{\gamma T} \left[ (-K_{T,x}^{s,x}) + (-M_{T,x}^{s,x}) \right] \sup_{s \leq t \leq T} |Y_{t,x}^{s,x} - \psi_{t,x}^{s,x}|.
\]
Therefore
\[
\frac{3\sigma^2}{4} \mathbb{E} \left[ \int_s^T e^{\gamma r} |Z^s,x,\varepsilon|^2 \right] dr
\]
\[
\leq e^{\gamma T} L \{ L + T (1 + \sigma^2) \} \mathbb{E} \left[ \sup_{s \leq t \leq T} |X^s,x,\varepsilon - \varphi^s,x|^2 \right]
\]
\[
+ 2 e^{\gamma T} \left( \mathbb{E}(|K^s,x,\varepsilon|^2) \right)^{1/2} + \left( \mathbb{E}(|M^s|^2) \right)^{1/2} \left( \mathbb{E} \left[ \sup_{s \leq t \leq T} |Y^s,x,\varepsilon - \psi^s,x|^2 \right] \right)^{1/2}
\]
\[
\leq e^{\gamma T} L \{ L + T (1 + \sigma^2) \} \mathbb{E} \left[ \sup_{s \leq t \leq T} |X^s,x,\varepsilon - \varphi^s,x|^2 \right]
\]
\[
+ 2Ce^{\gamma T} \left( \mathbb{E}(|K^s,x,\varepsilon|^2) \right)^{1/2} + \left( \mathbb{E}(|M^s|^2) \right)^{1/2} \left( \mathbb{E} \left[ \sup_{s \leq t \leq T} |Y^s,x,\varepsilon - \psi^s,x|^2 \right] \right)^{1/2}.
\]
So, by virtue of (4.7) and (4.9), the proof is complete.

**Remark 4.8.** As a consequence of Theorems 4.4 and 4.7, we get
\[
\mathbb{E} \left[ \sup_{s \leq t \leq T} |Y^s,x,\varepsilon - \psi^s,x|^2 + \int_s^T |Z^s,x,\varepsilon|^2 dr \right] \leq Ce, \]
where $C$ is a positive constant.

We now want to prove that the process $Y^{s,x,\varepsilon}$ satisfies a LDP. For that reason, we recall the link between Variational Inequality (VI in short) and G-MBSDEs. For all $\varepsilon > 0$, we consider the following VI
\begin{equation}
(4.10)
\begin{cases}
\partial_t u^\varepsilon + \mathcal{L}^\varepsilon (D_x^2 u^\varepsilon, D_x u^\varepsilon, u^\varepsilon, x, t) \in \partial \Pi(u^\varepsilon(t, x)), \\
u^\varepsilon(T, x) = \Phi(x), \ x \in \mathbb{R}
\end{cases}
\end{equation}
where
\[
\mathcal{L}^\varepsilon (D_x^2 u^\varepsilon, D_x u^\varepsilon, u^\varepsilon, x, t) = G \left( H \left( D_x^2 u^\varepsilon, D_x u^\varepsilon, u^\varepsilon, x, t \right) + \langle b(x), D_x u^\varepsilon \rangle \right)
\]
\[
+ f \left( t, x, u^\varepsilon, \langle \varepsilon \sigma(x), D_x u^\varepsilon \rangle \right),
\]
and
\[
H \left( D_x^2 u^\varepsilon, D_x u^\varepsilon, u^\varepsilon, x, t \right) = D_x^2 u^\varepsilon \varepsilon^2 \sigma \sigma^T + 2 \langle D_x u^\varepsilon, \varepsilon h(x) \rangle
\]
\[
+ 2g \left( t, x, u^\varepsilon, \langle \varepsilon \sigma(x), D_x u^\varepsilon \rangle \right)
\]
Now consider
\begin{equation}
(4.11)
u^\varepsilon(t, x) = Y^{t,x,\varepsilon}_t, \ (t, x) \in [0, T] \times \mathbb{R}.
\end{equation}
\begin{equation}
(4.12)
u^0(t, x) = \psi^t,x, \ (t, x) \in [0, T] \times \mathbb{R}.
\end{equation}
In Yang et al. (2017) it is shown that $u^\varepsilon$ is a viscosity solution of VI (4.10) and we have
\begin{equation}
(4.13)Y_{t}^{s,x,\varepsilon} = u^\varepsilon(t, X_{t}^{s,x,\varepsilon}), \ \forall t \in [s, T].
\end{equation}
Let $C_{0,s}([s, T], \mathbb{R})$ be the space of $\mathbb{R}$-valued continuous functions $\varphi$ on $[s, T]$ with $\tilde{\varphi}_s = 0$.

Let $s \in [0, T]$ and $\varepsilon \geq 0$. We define the mapping $F^\varepsilon : C_{0,s}([s, T], \mathbb{R}) \rightarrow C([s, T], \mathbb{R})$ by
\begin{equation}
(4.14)F^\varepsilon(\varphi) = [t \mapsto u^\varepsilon(t, x + \varphi_t)], \ s \leq t \leq T, \ \varphi \in C_{0,s}([s, T], \mathbb{R}).
\end{equation}
where \( u^\varepsilon \) is given by (4.11) and \( u^0 \) by (4.12).

By virtue of (4.14) and (4.13), for any \( \varepsilon > 0 \) and all \( x \in \mathbb{R} \), we have \( Y^{s,x,\varepsilon} = F^\varepsilon (X^{s,x,\varepsilon} - x) \).

We have the following result of large deviations

**Theorem 4.9.** Let (B1) – (B3) hold. Then for any closed subset \( F \) and any open subset \( O \) in \( C([s,T],\mathbb{R}) \),

\[
\limsup_{\varepsilon \to 0} \varepsilon \log \hat{C}(Y^{s,x,\varepsilon} \in F) \leq -\inf_{\psi \in F} \Lambda'(\psi),
\]

and

\[
\liminf_{\varepsilon \to 0} \varepsilon \log \hat{C}(Y^{s,x,\varepsilon} \in O) \geq -\inf_{\psi \in O} \Lambda'(\psi),
\]

where

\[
\Lambda'(\psi) = \inf \left\{ \Lambda(\tilde{\psi}) : \psi_t = F^0(\tilde{\psi})(t) = u^0(t, x + \tilde{\psi}_t), t \in [s,T], \tilde{\psi} \in C_{0,x}([s,T],\mathbb{R}) \right\}.
\]

**Proof.** Since the family \( \left\{ \hat{C}((X^{s,x,\varepsilon}_t - x)_{t \in [s,T]} \in \cdot) \right\}_{\varepsilon > 0} \) is exponentially tight (see Lemma 3.4 p. 2235 in Gao and Jiang (2010)), by virtue of Lemma 4.3 (contraction principle) and Lemma 4.2, we just need to prove that \( F^\varepsilon, \varepsilon > 0 \) are continuous and \( \{F^\varepsilon\}_{\varepsilon > 0} \) converges uniformly to \( F^0 \) on every compact subset of \( C_{0,x}([s,T],\mathbb{R}) \), as \( \varepsilon \to 0 \). Since \( u^\varepsilon \) is continuous, it is not hard to prove that \( F^\varepsilon \) is also continuous. The uniform convergence of \( \{F^\varepsilon\}_{\varepsilon > 0} \) is a consequence of Corollary 4.5.

\[\square\]

**References**

I. Dakaou and A. S. Hima. Large Deviations for Backward Stochastic Differential Equations Driven by G-Brownian Motion. *Journal of Theoretical Probability*, 34 (2):499–521, 2021.

A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*. Springer, Berlin, 2nd edition, 1998.

L. Denis, M. Hu, and S. Peng. Function spaces and capacity related to a sublinear expectation: application to G-brownian motion paths. *Potential Anal.*, 34:139–161, 2011.

J.-D. Deuschel and D. W. Stroock. *Large Deviations*. Academic Press Inc., Boston, 1989.

E. H. Essaky. Large deviation principle for a backward stochastic differential equation with subdifferential operator. *C. R. Acad. Sci. Paris*, 346:75–78, 2008.

M. I. Freidlin and A. D. Wentzell. *Random Perturbations of Dynamical Systems*. Springer, Berlin, 1984.

F. Gao. Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-brownian motion. *Stochastic Processes and their Applications*, 119:3356–3382, 2009.

F. Gao and H. Jiang. Large deviations for stochastic differential equations driven by G-brownian motion. *Stochastic Processes and their Applications*, 120:2212–2240, 2010.

M. Hu and S. Peng. On representation theorem of G-expectations and paths of G-Brownian motion. *Acta Mathematicae Applicatae Sinica, English Series*, 25:539–546, 2009.
M. Hu, S. Ji, S. Peng, and Y. Song. Backward stochastic differential equations
driven by $G$-brownian motion. *Stochastic Processes and their Applications*, 124:
759–784, 2014a.

M. Hu, S. Ji, S. Peng, and Y. Song. Comparison theorem, Feynman-Kac formula
and Girsanov transformation for BSDEs driven by $G$-brownian motion. *Stochastic
Processes and their Applications*, 124:1170–1195, 2014b.

S. Peng. *Nonlinear expectations and stochastic calculus under uncertainty with ro-
bust CLT and G-Brownian motion*, volume 95. Probability Theory and Stochastic
Modelling, Springer, 2019.

S. Rainero. Un principe de grandes déviations pour une équation différentielle
stochastique progressive rétrograde. *C. R. Acad. Sci. Paris*, 343:141–144, 2006.

Y. Song. Some properties on $G$-evaluation and its applications to $G$-martingale
decomposition. *SCIENCE CHINA Mathematics*, 54 No. 2:287–300, 2011.

S. R. S. Varadhan. *Large Deviations and Applications*. Society for Industrial and
Applied Mathematics (SIAM), Philadelphia, 1984.

F. Yang, Y. Ren, and L. Hu. Multi-valued backward stochastic differential equations
driven by $G$-brownian motion and its applications. *Mathematical Methods in the
Applied Sciences*, 40:4696–4708, 2017.

Département de Mathématiques, Université Dan Dicko Dankoulodo de Maradi, BP
465, Maradi, Niger

*Email address*, A. S. Hima: abdoulaye.hima@uddm.edu.ne

*Email address*, I. Dakaou: ibrahim.dakaou@uddm.edu.ne