ON FINITE TIME BLOWUP FOR THE MASS-CRITICAL HARTREE EQUATIONS

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Abstract. We consider the fractional Schrödinger equations with focusing Hartree type nonlinearities. When the energy is negative, we show that the solution blows up in a finite time. For this purpose, based on Glassey’s argument, we obtain a virial type inequality.

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1. Introduction

In this paper we consider the Cauchy problem of the focusing fractional nonlinear Schrödinger equations:

\[
\begin{align*}
    i\partial_t u &= |\nabla|^{\alpha} u + F(u), \quad \text{in } \mathbb{R}^{1+n} \times \mathbb{R}, \\
    u(x,0) &= \varphi(x) \quad \text{in } \mathbb{R}^n,
\end{align*}
\]

(1.1)
where $|\nabla| = (-\Delta)^{\frac{\alpha}{2}}$, $n \geq 2$, $\alpha \geq 1$, and $F(u)$ is a nonlocal nonlinear term of Hartree type given by

$$F(u)(x) = -\left(\frac{\psi(\cdot)}{|\cdot|^\gamma} * |u|^2\right)(x) u(x) \equiv -V_\gamma(|u|^2)(x) u(x).$$

Here $0 \leq \psi \in L^\infty(\mathbb{R}^n)$ and $0 < \gamma < n$. We say that (1.1) is focusing since $-V_\gamma(|u|^2)$ serves as an attractive self-reinforcing potential. We also use a simplified notation $V_\gamma$ to denote $V_\gamma(|u|^2)$.

When $\psi$ is homogeneous of degree zero (e.g. $\psi \equiv 1$), the equation (1.1) has scaling invariance. In fact, if $u$ is a solution of (1.1), $u_\lambda$, $\lambda > 0$, given by

$$u_\lambda(t, x) = \lambda^{-\frac{2-\alpha}{2} + \frac{n}{2}} u(\lambda^{\alpha} t, \lambda x),$$

is also a solution. We denote the critical Sobolev exponent $s_c = \frac{2-\alpha}{2}$. Under scaling $u \to u_\lambda$, $H^{s_c}$-norm of data is preserved. The solution $u$ of (1.1) formally satisfies the mass and energy conservation laws:

$$m(u) = \|u(t)\|^2_{L^2},$$
$$E(u) = K(u) + V(u),$$

(1.2)

where

$$K(u) = \frac{1}{2} \langle u, |\nabla|^{\alpha} u \rangle, \quad V(u) = -\frac{1}{4} \langle u, V_\gamma(|u|^2) u \rangle.$$ 

Here $\langle \cdot, \cdot \rangle$ is the complex inner product in $L^2$. In view of scaling invariance and the conservation laws - when each conserved quantity is invariant under scaling - we say the equation (1.1) is mass-critical if $\gamma = \alpha$ and energy-critical if $\gamma = 2\alpha$.

The purpose of this paper is to show the finite time blow-up of solutions to the fractional or higher order equations when (1.1) is mass-critical. If the energy is negative (i.e. the magnitude of the potential energy $V(u)$ is larger than that of kinetic part $K(u)$), then self-attracting power dominates overall dynamics and so it may result in a collapse in a finite time. For the usual Schrödinger equations ($\alpha = 2$), Glassey [7] introduced a convexity argument to show existence of finite time blow-up solutions. Indeed, if $\psi \equiv 1$, $2 \leq \gamma < \min(n, 4)$, $n \geq 3$ and $\varphi \in H^\frac{\gamma}{2}(\mathbb{R}^n)$ with $x\varphi \in L^2$, then

$$\|xu(t)\|^2_{L^2} \leq 8t^2 E(\varphi) + 4t \langle \varphi, A\varphi \rangle + \|x\varphi\|^2_{L^2},$$

where $A$ is the dilation operator $\frac{1}{2}(\nabla \cdot x + x \cdot \nabla)$. This implies that if $E(\varphi) < 0$, then the maximal time of existence $T^* < \infty$. For details, see Section 6.5 in [1] and Section 5 below.

In the fractional or high order equations, a variant of the second moment is the quantity

$$\mathcal{M}(u) := \langle u, x \cdot |\nabla|^{2-\alpha} xu \rangle.$$
This was first utilized by Fröhlich and Lenzmann \[6\] in their study of the semirelative nonlinear Schrödinger equations \((\alpha = 1)\). More precisely, they obtained

\[
\mathcal{M}(u(t)) \leq 2t^2 E(\varphi) + 2t(\langle \varphi, A\varphi \rangle + C\|\varphi\|_{L^2}^4) + \mathcal{M}(\varphi)
\]

for \(\psi = e^{-|x|} (\mu \geq 0)\), \(\gamma = 1\), \(\varphi \in H^2_{rad}(\mathbb{R}^3)\) with \(|x|\varphi \in L^2\). Here the function space \(X_{rad}\) denotes the subspace \(X\) of radial functions. The quartic term \(\|\varphi\|_{L^2}^4\) appears due to the commutator \([|x|^2V_\gamma, |\nabla|]\), and in \(\mathbb{R}^3\) it is controlled by Newton’s theorem.

Compared to the usual case \((\alpha = 2)\), when \(\alpha \neq 2\), the presence of \(|\nabla|^{2-\alpha}\) gives rise to certain types of singular integrals which necessitate use of commutators. So, the main issue is how to estimate the commutator \([|x|^2V_\gamma, |\nabla|^{2-\alpha}]\) since the Newton’s theorem is generally not available except for \(\alpha = 1\) in \(\mathbb{R}^3\). In order to obtain the desired estimate we use the Stein-Weiss inequality \[(2.12)\] and combine this with a convolution estimate in Lemma \[2.3\]. To close our argument, we further need an estimate for the moments \(||xu||_{L^2}\) and \(|||x\nabla u||_{L^2}\) for \(t\) contained in the existence time interval. (See Proposition \[3.1\].) It is done under some regularity assumption \(^1\) which we need to impose to get estimates for the commutators \([|\nabla|^{\alpha}, |x|^2]\) and \([V_\gamma, \nabla \cdot (x \cdot |\nabla|^{2-\alpha}x)\nabla]\).

The Hartree nonlinearity is essentially a cubic one, though it is convolved with the potential. Thus, by fairly standard an argument one can show the local well-posedness of the Cauchy problem for suitably regular initial data. Indeed, we have local well-posedness for \(s \geq \frac{5}{2}\) so that, within the maximal existence time interval \([0, T^*]\), there is a unique solution \(u \in C([0, T^*); H^s) \cap C^1([0, T^*); H^{s-\alpha})\) and \(\lim_{t \uparrow T^*} \|u(t)\|_{H^s} = 0\) if \(T^* < \infty\). For convenience of readers, we append the local well-posedness for general \(\alpha > 0\) in the last section.

Let us define a Sobolev index \(\alpha^*\) by \(\alpha^* = (2k)^2\) where \(k\) is the least integer satisfying \(k \geq \frac{\alpha}{2}\). We separately state our results for the low order case, \(1 \leq \alpha < 2\), and the high order case, \(2 < \alpha < \frac{9}{2} + 1\).

**Theorem 1.1.** Let \(\gamma = \alpha, 1 \leq \alpha < 2,\) and \(n \geq 4\). Assume that \(\psi\) is a nonnegative, smooth, decreasing and radial function with \(|\psi'(\rho)| \leq C\rho^{-1}\) for some \(C > 0\). Additionally, assume that the initial datum satisfies \(\varphi \in H^2_{rad}\) and \(|x|^l \partial^j \varphi \in L^2(\mathbb{R})\) for \(1 \leq l \leq 2, 0 \leq |j| \leq 4 - 2l\). Then, if \(E(\varphi) < 0\), the solution to \[(1.1)\] blows up in a finite time.

**Theorem 1.2.** Let \(\gamma = \alpha, 2 < \alpha < 1 + \frac{9}{2} \) and \(n \geq 4\). Assume that \(\psi\) is a nonnegative, smooth, decreasing and radial function. Additionally, assume that the

\(^1\) Such an assumption is not necessary for the usual Schrödinger equation.
initial datum satisfies \( \varphi \in H_\text{rad}^\alpha \) and \(|x|^\ell \partial^i \varphi \in L^2(\mathbb{R})\) for \(1 \leq \ell \leq 2k, 0 \leq |i| \leq 2k(2k-\ell)\). Then, if \( E(\varphi) < 0 \), the solution to (1.1) blows up in a finite time.

The restriction \( n \geq 4 \) is due to the use of the Stein-Weiss inequality. The technical condition \( \alpha < 1 + \frac{n}{2} \) is imposed because we make use of the convolution estimate (2.10) (Lemma 2.3) and \( n \geq 4 \). For the proof of theorems we show that the mean dilation is decreasing when \( E(\varphi) < 0 \). Clearly, this follows from

\[
\frac{d}{dt} \langle u, Au \rangle \leq 2\alpha E(\varphi),
\]

which holds whenever \( \gamma \geq \alpha \) and \( \psi' \leq 0 \). If \( \gamma = \alpha \), from the estimate (2.10), we have, for \( t \in [0, T^*) \),

\[
\mathcal{M}(u) \leq 2\alpha^2 t^2 E(\varphi) + 2\alpha t \left( \langle \varphi, A\varphi \rangle + C \| \varphi \|^4_{L^2} \right) + \mathcal{M}(\varphi).
\]

In order to validate (1.3) and (1.4), we need estimates for the moments \( \| xu \|_{L^2} \) and \( \| |x| \nabla u \|_{L^2} \) on the time interval \([0, T^*)\).

We finally remark that the argument of this paper dose not readily work for the power type nonlinearity. Since our argument relies on \( H_\text{rad}^\alpha \) regularity assumption and the Stein-Weiss inequality, a different approach seems to be necessary in order to control the commutators.

The rest of paper is organized as follows. In Section 2 we show the finite time blow-up while assuming Proposition 3.1. In Section 3 we provide the proof of Proposition 3.1. The last section is devoted to the local well-posedness.

**Notations.** We use the notations: \( \partial^j = \prod_{1 \leq i \leq n} \partial_{x_i}^{j_i} \) for multi-index \( j = (j_1, \cdots, j_n) \) and \( |j| = \sum_i j_i \). \(|\nabla| = \sqrt{-\Delta}, H_\text{rad}^s = |\nabla|^{-s}L^r, \tilde{H}_\text{rad}^s = \tilde{H}_\text{rad}^s, H_\text{rad}^s = (1 - \Delta)^{-s/2}L^r, H^s = H_\text{rad}^s \). \( A \lesssim B \) and \( A \gtrsim B \) means that \( A \leq CB \) and \( A \geq C^{-1}B \), respectively, for some \( C > 0 \). As usual \( C \) denotes a positive constant, possibly depending on \( n, \alpha \) and \( \gamma \), which may differ at each occurrence.

## 2. Finite time blow-up

In this section we consider finite time blow-up of solutions to the Cauchy problem (1.1) of the mass-critical potentials. We begin with the dilation operator \( A \). With more general assumption for \( \psi \) and \( \gamma \) we obtain an estimate for the time evolution of average of \( A \).

**Lemma 2.1.** Let \( \psi \) be radially symmetric smooth function such that \( \psi' = \partial_r \psi \leq 0 \). Suppose that \( u \in H^\alpha \) and \( xu(t), |x| \nabla u(t) \in L^2 \) for \( t \in [0, T^*) \), where \( T^* \) is the maximal existence time. Then, for \( \gamma \geq \alpha \),

\[
\frac{d}{dt} \langle u, Au \rangle \leq 2\alpha E(\varphi).
\]
Proof. Since $u \in H^\alpha$ and $|x|u, x \cdot \nabla u \in L^2$, $\langle u, Au \rangle$ is well-defined and so is

$$
\frac{d}{dt} \langle u, Au \rangle = i\langle u, [H, A]u \rangle,
$$

where $H = |\nabla|^\alpha - V_\gamma$. Here $[H, A]$ denotes the commutator $HA - AH$. Using the identity $[|\nabla|^\alpha, x] = -\alpha|\nabla|^\alpha - 2\nabla$, we have

$$
[H, A] = -i\alpha|\nabla|^\alpha.
$$

Similarly,

$$
[-V_\gamma, A] = -i(x \cdot \nabla)V_\gamma.
$$

Substituting (2.3) and (2.4) into (2.2), we get

$$
\frac{d}{dt} \langle u, Au \rangle = \alpha \langle u, |\nabla|^\alpha u \rangle + \langle u, (x \cdot \nabla)V_\gamma u \rangle.
$$

For Hartree type $V_\gamma$, we have

$$
\langle u, (x \cdot \nabla)V_\gamma u \rangle = 4\gamma V(u) + \iint \frac{|x-y|\psi'(|x-y|)}{|x-y|}\frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy - \langle u, (x \cdot \nabla)V_\gamma u \rangle,
$$

which implies

$$
\langle u, (x \cdot \nabla)V_\gamma u \rangle = 2\gamma V(u) + \frac{1}{2} \iint \frac{|x-y|\psi'(|x-y|)}{|x-y|}\frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma} dxdy.
$$

Substituting this into (2.5) gives

$$
\frac{d}{dt} \langle u, Au \rangle \leq 2\alpha E(\phi) + 2(\gamma - \alpha)V(u)
$$

$$
+ \frac{1}{2} \iint \frac{|x-y|\psi'(|x-y|)}{|x-y|}\frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma} dxdy.
$$

Since $\gamma \geq \alpha$ and $\psi'(|x|) \leq 0$, we get (2.1). This completes the proof of Lemma 2.1. \qed

Next we consider the nonnegative quantity $M(u) = \langle u, Mu \rangle$ with the virial operator

$$
M := x \cdot |\nabla|^{2-\alpha}x = \sum_{k=1}^{n} x_k |\nabla|^{2-\alpha}x_k.
$$
Suppose \( u(t) \in H^{\alpha^*} \) and \( |x|^{2k}u(t) \in L^2 \) for \( t \in [0, T^*]. \) Then, since \( \mathcal{M}(u) \lesssim \|x|\nabla u|_{L^2}[(1+|x|)^{2k}u|_{L^2}, \) from (3.5) below the quantity \( \mathcal{M}(u) \) is well-defined and finite for all \( t \in [0, T^*], \) and so is

\[
(2.6) \quad \frac{d}{dt}\mathcal{M}(u) = i\langle u, [H, M]u \rangle = i\langle u, [|\nabla|^{\alpha}, M]u \rangle - i\langle u, [V, M]u \rangle.
\]

**Lemma 2.2.** Suppose that \( u(t) \in H^{\alpha^*} \) and \( |x|^{2k}u(t) \in L^2 \) for \( t \in [0, T^*]. \) Then we have

\[
(2.7) \quad \frac{d}{dt}\mathcal{M}(u) \leq 2\alpha\langle u, Au \rangle + C\|\varphi\|_{L^2}^2
\]

for \( t \in [0, T^*], \) where \( C \) is a positive constant depending only on \( n, \alpha \) but not on \( u, \varphi. \)

Now, Theorem 1.1 and Theorem 1.2 immediately follow from Lemma 2.1 and 2.2 once we have Proposition 3.1.

**Proof.** By the identity \( \langle x|\nabla|^{\alpha}x = x|\nabla|^{\alpha} - \alpha|\nabla|^{\alpha - 2}\nabla, \) we have

\[
[|\nabla|^{\alpha}, M] = \langle |\nabla|^{\alpha}x \cdot |\nabla|^{2-\alpha}x - x \cdot |\nabla|^{2-\alpha}x|\nabla|^{\alpha} = -\alpha(x \cdot \nabla + \nabla \cdot x).
\]

Hence, for a smooth function \( v \) we get

\[
[v, M] = vx \cdot |\nabla|^{2-\alpha}x - x \cdot |\nabla|^{2-\alpha}xv
= v|x|^2|\nabla|^{2-\alpha} - (2 - \alpha)|vx \cdot \nabla|^{\alpha} - |\nabla|^{2-\alpha}|x|^2v - (2 - \alpha)|\nabla|^{\alpha} \cdot xv
= \langle x|^2v, |\nabla|^{2-\alpha} \rangle + (\alpha - 2)(vx \cdot \nabla)|\nabla|^{\alpha} + |\nabla|^{\alpha} \cdot xv
\]

By density argument we may assume that \( v = V_\alpha \) in the above identity. Thus it suffices to show that

\[
(2.8) \quad \langle u, [x]^2V_\alpha, |\nabla|^{2-\alpha}u \rangle + \langle u, [V_\alpha x \cdot \nabla|\nabla|^{\alpha} + |\nabla|^{\alpha} \cdot xV_\alpha u \rangle.
\]

**Case \( \alpha > 2. \)** We first consider the higher order case \( \alpha > 2. \) The first term of LHS of (2.8) is rewritten as

\[
(2.9) \quad 2|\text{Im}\langle u, [x]^2V_\alpha |\nabla|^{2-\alpha}u \rangle|.
\]

To handle this we recall the following weighted convolution estimate (see (1.2):

**Lemma 2.3.** Let \( 0 < \gamma < n - 1 \) and \( n \geq 2. \) Then, for any \( f \in L^1_{\text{rad}} \) and \( x \neq 0, \)

\[
(2.10) \quad \int |x - y|^{-\gamma}|f(y)|\,dy \lesssim |x|^{-\gamma}\|f\|_{L^1}.
\]
Applying (2.12) with the kernel of pseudo-differential operator $|\nabla| f$

To estimate this, we make use of the following inequality due to Stein-Weiss [11]:

For $f \in L^p$ with $1 < p < \infty$, $0 < \lambda < n$, $\beta < \frac{n}{p}$, and $n = \lambda + \beta$

Applying (2.12) with $p = 2$, $\beta = \alpha - 2$ and $\lambda = n - (\alpha - 2)$, (2.11) is bounded by $C\|\varphi\|_{L^2}$.

We write the second term of RHS of (2.8) as

Applying (2.12) with $p = 2$, $\beta = \alpha - 1$ and $\lambda = n - (\alpha - 1)$, and Plancherel’s theorem, we get the desired bound (2.8).

Case $1 \leq \alpha < 2$. Now we consider the fractional case $1 \leq \alpha < 2$. The second term of RHS of (2.8) can be treated in the same way as the high order case, and it is bounded by $C\|\varphi\|_{L^2}$. Hence, it suffices to consider the first term. Let us set $g = |x|^2 V_{\alpha}$. Then, we need only to obtain

which gives $\langle u, [x|\varphi|V_{\alpha},|\nabla|^{2-\alpha}u] \rangle \lesssim \|\varphi\|_{L^2}$, and thus (2.8). The kernel $K(x,y)$ of the commutator $[|\nabla|^{2-\alpha},g]$ can be written as $k(x-y)(g(y) - g(x))$, where $k$ is the kernel of pseudo-differential operator $|\nabla|^{2-\alpha}$. Let $K^*$ be the kernel of the dual operator of $[|\nabla|^{2-\alpha},g]$. Then, obviously $K^*(x,y) = -K(x,y)$.

Suppose that $\|g\|_{L^{2-\alpha}} = \sup_{x \neq y \in \mathbb{R}^n} \frac{|g(x) - g(y)|}{|x-y|^{2-\alpha}} < \infty$. Since $|k(x-y)| \lesssim |x-y|^{-n-1-2(2-\alpha)}$, and $0 < 2 - \alpha \leq 1$, it is easy to see the following:

\[ |K(x,y)| \lesssim |x-y|^{-n}, \]
\[ |K(x,y) - K(x',y)| \lesssim \frac{|x-x'|^{2-\alpha}}{|x-y|^{n+2-\alpha}}, \text{ if } |x-x'| \leq |x-y|/2, \]
\[ |K(x,y) - K(x,y')| \lesssim \frac{|y-y'|^{2-\alpha}}{|x-y|^{n+2-\alpha}}, \text{ if } |y-y'| \leq |x-y|/2, \]
and so does $K^*$ (because $K^*(x,y) = -K(x,y)$). Let $\zeta$ be a normalized bump function supported in the unit ball and set $\zeta^x_0, N(x) = \zeta((x-x_0)/N)$. By Theorem 3 in p. 294 of [10], in order to prove (2.13), it is sufficient to show that

$$
\| [\nabla^{2-\alpha}, g] \zeta^x_0, N \|_{L^2} \leq C \| \varphi \|^2_{L^2 N^{\frac{\alpha}{2}}}
$$

with $C$, independent of $x_0, N, \zeta$.

We now show (2.14). The commutator $[\nabla^{2-\alpha}, g]$ can be written as

$$
\sum_{j=1}^n [T_j, g] \partial_j + \sum_{j=1}^n T_j (\partial_j g),
$$

where $T_j = -[\nabla^{2-\alpha}(-\Delta)^{-1}] \partial_j$. For the first sum of (2.15) we obtain

$$
\| [T_j, g] \partial_j (\zeta^x_0, N) \|_{L^2} \leq C \| \varphi \|^2_{L^2 N^{\frac{\alpha}{2}}},
$$

Indeed, let $k_j$ be the kernel of $T_j$. If $\alpha = 1$, $k_j$ is the kernel of Riesz transform. If $1 < \alpha < 2$, it is easy to see $|k_j(x,y)| \lesssim |x-y|^{-n+\alpha-1}$ (note that $|\hat{k}_j(\xi)| \lesssim |\xi|^{-(\alpha-1)}$). Thus it follows that

$$
|K_j(x,y)| = |k_j(x-y)|\|g(y) - g(x)\| \lesssim \|g\|_{L^{2-\alpha}} |x-y|^{-n+(\alpha-1)}.
$$

Hence, for $|x-x_0| < 2N$, $[T_1, g] \partial_1 (\zeta^x_0, N)(x) \lesssim \|g\|_{L^{2-\alpha}}$. This gives

$$
\| [T_1, g] \partial_1 (\zeta^x_0, N) \|_{L^2(\{|x-x_0|<2N\})} \lesssim \|g\|_{L^{2-\alpha}} N^{\frac{\alpha}{2}}.
$$

If $|x-x_0| \geq 2N$, we have $[T_1, g] \partial_1 (\zeta^x_0, N)(x) \lesssim \|g\|_{L^{2-\alpha}} N^{n-1} |x-x_0|^{-(n-1)}$. Hence,

$$
\| [T_1, g] \partial_1 (\zeta^x_0, N) \|_{L^2(\{|x-x_0|\geq2N\})} \lesssim \|g\|_{L^{2-\alpha}} N^{n-1} \left( \int_{|x|>2N} |x|^{-2(n-1)} \, dx \right) \leq \|g\|_{L^{2-\alpha}} N^{\frac{\alpha}{2}}.
$$

We now show $\|g\|_{L^{2-\alpha}} \leq C \| \varphi \|^2_{L^2}$, which gives (2.16). If $x \neq y$, then

$$
|g(x) - g(y)| \leq |x-y| \int_0^1 |\nabla g(z_s)| \, ds, \quad z_s = x + s(y-x).
$$

Since $|\psi(\rho)| \leq C\rho^{-1}$ for $\rho > 0$, from Lemma 2.8 and mass conservation it follows that

$$
|\nabla g(z_s)| \lesssim |s|^{1-\alpha} \|u\|^2_{L^2} = \|x| - s|x-y|^{1-\alpha}\|\varphi\|^2_{L^2},
$$

provided $\alpha < n-2$. Since $\sup_{a>0} \int_0^1 |a-s|^{-\theta} \, ds \leq C_0$ for $0 < \theta < 1$, we have $|g(x) - g(y)| \lesssim |x-y|^{2-\alpha}|\varphi|^2_{L^2}$. Thus we get (2.16).

Finally we need to handle the second sum of (2.15). If $\alpha = 1$, $T_j$ is a Riesz transform. Thus,

$$
\|T_j (\partial_j g) \zeta^x_0, N \|_{L^2} \leq C \| \partial_j g \|_{L^\infty N^{\frac{\alpha}{2}}}.
$$
By Lemma 2.3 for \( \alpha = 1 \), we get \(|\partial_j g(x)| \leq |x|V_1 + |x|^2|\partial_j V_1| \lesssim \|\varphi\|_L^2\). Hence,

\[
(2.17) \quad \|T_j((\partial_j g)\zeta^{x_0,N})\|_{L^2} \leq C\|\varphi\|_{L^2}^2 N^{\frac{\alpha}{2}}.
\]

For \( 1 < \alpha < 2 \), the kernel \( k_j(x) \) of \( T_j \) is bounded by \( C|x|^{-(n-\alpha+1)} \). So, from the duality and Lemma 2.3 with \( \alpha < n-2 \) we have, for any \( \psi \in L^2 \),

\[
|\langle \psi, T_j((\partial_j g)\zeta^{x_0,N}) \rangle| = |\langle T_j^*\psi, (\partial_j g)\zeta^{x_0,N} \rangle| \\
\leq CN^{\frac{\alpha}{2}}\|\partial_j g(\cdot)\| \int |\cdot-y|^{-(n-\alpha+1)}|\psi(y)| dy\|_{L^2} \\
\leq CN^{\frac{\alpha}{2}}\|\varphi\|_{L^2}^2 \|1^{1-\alpha}\| \int |\cdot-y|^{-(n-\alpha+1)}|\psi(y)| dy\|_{L^2},
\]

where \( T_j^* \) is the dual operator of \( T_j \). Using (2.12) with \( \beta = \alpha - 1 \), \( \lambda = n - \alpha + 1 \) and \( p = 2 \), we get

\[
|\langle \psi, T_j((\partial_j g)\zeta^{x_0,N}) \rangle| \leq C\|\psi\|_{L^2} \|\varphi\|_{L^2}^2 N^{\frac{\alpha}{2}}.
\]

Thus, it follows that

\[
(2.18) \quad \|T_j((\partial_j g)\zeta^{x_0,N})\|_{L^2} \leq C\|\varphi\|_{L^2}^2 N^{\frac{\alpha}{2}}.
\]

Therefore combining the estimates (2.16) (2.17) and (2.18) yields (2.14). This completes the proof of Lemma 2.2. \( \square \)

3. Propagation of the moment

We now discuss estimates for the propagation of moments \( \|x|^{2k}u\|_{L^2} \) when \( |x|^{2k} \varphi \in L^2 \) and the solution \( u \in C([0,T^*);H^\alpha) \). For \( \alpha < 2k \), we use the kernel estimate of Bessel potentials. Let us denote, respectively, the kernels of Bessel potential \( D^{-\beta} \) and \( |\nabla|^{\alpha}D^{-2k} = D^{-\beta} \) by \( G_\beta(x) \) and \( K(x) \), where \( D = \sqrt{1-\Delta} \).

Then

\[
K(x) = \sum_{j=0}^{\infty} A_j G_{2j+\beta}(x),
\]

where the coefficients \( A_j \) are given by the expansion \( (1-t)^{\frac{n}{2}} = \sum_{j=0}^{\infty} A_j t^j \) for \( |t| < 1 \) with \( \sum_{j \geq 0} |A_j| < \infty \). One can show that \( (1+|x|)^{\ell} K \in L^1 \) for \( \ell \geq 1 \) and has decreasing radial and integrable majorant. In fact, from the integral representation

\[
G_{2j+\beta}(x) = \frac{1}{(4\pi)^{n/2}} \Gamma(j+\beta/2) \int_0^{\infty} \lambda^{2j+\beta-n+1} e^{-|x|^2/4\lambda} e^{-\lambda} d\lambda,
\]

it follows that, for \( j \) with \( 2j + \beta < n \),

\[
G_{2j+\beta}(x) \leq C(|x|^{-n+2j+\beta} x_{\{|x| \leq 1\}}(x) + e^{-c|x|} x_{\{|x| > 1\}}(x)),
\]

and, for \( j \) with \( 2j + \beta \geq n \),

\[
G_{2j+\beta}(x) \leq C(x_{\{|x| \leq 1\}}(x) + e^{-c|x|} x_{\{|x| > 1\}}(x)).
\]
Here the constants $c$ and $C$ of (3.1) and (3.2) are independent of $j$. So, the function $(1 + |x|)^j G_{2j+\beta}$ has a decreasing radial and integrable majorant, which is chosen uniformly on $j$, and so does $K$. For details see p.132–135 of [3].

**Proposition 3.1.** Let $T^*$ be the maximal time of solution $u \in C([0,T^*];H^{\alpha^*})$ to (1.1). If $|x|^j \partial^j \varphi \in L^2(\mathbb{R})$ for $1 \leq \ell \leq 2k$, $0 \leq |\ell| \leq 2k(2k-\ell)$, then $|x|^j \partial^j u(t) \in L^2(\mathbb{R})$ for all $t \in [0,T^*)$.

Let us set $\psi(x) = e^{-\varepsilon|x|^2}$. For proof of Proposition 3.1 we use the following bootstrapping lemma.

**Lemma 3.2.** Let $\ell, m$ be integers such that $2 \leq \ell \leq 2k$ and $0 \leq m \leq \alpha^* - 2k$. Suppose that $\sup_{0 \leq t' \leq t} (\|u(t')\|_{H^{2k+m}} + |||x|^j \partial^j u(t')|||_{L^2}) < \infty$ for all $t \in [0,T^*)$ and $0 \leq j \leq \ell-1, |\ell| \leq 2k + m$. Then $\sup_{0 \leq t' \leq t} |||x|^j \partial^m u(t')|||_{L^2} < \infty$ for all $t \in [0,T^*)$ and $|m| = m$.

**Proof.** Let $v = \partial^m u$ and

$$m_{\ell}(t) = \langle v(t), |x|^2 \psi^2 \varphi(t) \rangle.$$

From the regularity of the solution $u$ it follows that

$$m'_{\ell}(t) = 2\text{Im} \langle v, |\nabla|^\alpha |x|^2 \psi^2 \varphi(t) \rangle + 2\text{Im} \langle |x|^\ell \psi \varphi, |x|^\ell \psi \partial^m (V^\alpha u) \rangle =: (I + II).$$

We first prove the case, $\alpha < 2k$. We rewrite $I$ as

$$I = \text{Im} \langle |x|^\ell \psi \varphi, |\nabla|^\alpha D^{-2k} |x|^\ell \psi \varphi |D^{2k} u \rangle$$

$$+ \text{Im} \langle |\nabla|^\alpha D^{-2k} (|x|^\ell \psi \varphi), [D^{2k}, |x|^\ell \psi \varphi] \rangle =: I_1 + I_2.$$

By the kernel representation of $|\nabla|^\alpha D^{-2k}$, we have

$$|||D^{2k}, |x|^\ell \psi \varphi ||| D^{2k} u(x) |$$

$$\leq \int K(x-y)||x|^\ell \psi \varphi(x) - ||y|^\ell \psi \varphi(y)|||D^{2k} u(y)| dy$$

$$\leq \int K(x-y)||x-y|||x|^\ell-1 + ||y|^\ell-1|||D^{2k} u(y)| dy$$

$$\leq \int K(x-y)||x-y|||D^{2k} u(y)| dy + \int K(x-y)||x-y|||D^{2k} u(y)| dy.$$

Since $|x|^\ell K$ is integrable, Cauchy-Schwarz inequality gives

$$I_1 \lesssim \sqrt{m} (\|u\|_{L^2} + |||x|^\ell-1 D^{2k} u|||_{L^2}).$$
As for $I_2$ we have

$$I_2 = \sum_{1 \leq j \leq k} c_j \text{Im} \langle |\nabla|^{\alpha} D^{-2k}(|x|^\ell \psi_x v), [\Delta^j, |x|^\ell \psi_x v]\rangle$$

$$= \sum_{1 \leq j \leq k} c_j \text{Im} \langle |\nabla|^{\alpha} D^{-2k}(|x|^\ell \psi_x v), \sum_{|l_1|+|l_2|+|l_3|=2j} c_{l_1, l_2, l_3} \partial^{l_1} (|x|^\ell \partial^{l_2} \psi_x \partial^{l_3} v)\rangle.$$ 

Note that $|\partial^{l_1} (|x|^\ell)| \lesssim |x|^{\ell-|l_1|}$ and $|\partial^{l_2} \psi_x (x)| \lesssim \varepsilon^{\frac{|l_2|}{2}} (1 + \varepsilon |x|^2)^{\frac{|l_2|}{2}} \psi_x (x)$. Hence, it follows that

$$I_2 \lesssim \| |\nabla|^{\alpha} D^{-2k}(|x|^\ell \psi_x v) \|_{L^2} \sum_{1 \leq j \leq k} \left( \sum_{|l_1|+|l_2|+|l_3|=2j} \right) \| |x|^{\ell-|l_1|-|l_2|} \partial^{l_3} v \|_{L^2}$$

$$\lesssim \sqrt{m_x} \sum_{1 \leq j \leq k} \left( \sum_{|l_1|+|l_2|+|l_3|=2j} \right) \| |x|^{(j-\ell)} \partial^{l_3} v \|_{L^2}.$$ 

Here we use the fact that the kernel of $|\nabla|^{\alpha} D^{-2k}$ is integrable. Conventionally, the summand is zero if $j - \ell < 0$. By the Hardy-Sobolev inequality we get, for $0 \leq |l_3| \leq j - \ell$,

$$\| |x|^{|l_3|-(j-\ell)} \partial^{l_3} v \|_{L^2} \lesssim \| \partial^{l_3} v \|_{H^{j-\ell-|l_3|}} \lesssim \| v \|_{H^{j-\ell}} \lesssim \| u \|_{H^{j-\ell+m}}.$$ 

If $j - \ell \leq |l_3| \leq 2j - 1$, then

$$\| |x|^{|l_3|-(j-\ell)} \partial^{l_3} v \|_{L^2} \| |x|^{|l_3|-(j-\ell)} \partial^{l_3+m} u \|_{L^2}.$$ 

Thus we finally obtain

(3.3) \hspace{1cm} I \lesssim \sqrt{m_x} (|u(t)|_{H^{2k+m}} + \sum_{0 \leq ||| \leq 2k+m} \| (1 + |x|^\ell) \partial^{l_3} u(t) \|_{L^2}).$$

In the case $\alpha = 2k$, we do not need the estimate for $I_1$. For the estimate of $I_2 = \text{Im} \langle |x|^\ell \psi_x v, [\Delta^k, |x|^\ell \psi_x v]\rangle$, we estimate similarly to obtain (3.3).

Now we proceed to estimate $II$. For this let us observe that

$$II = \sum_{m_1 + m_2 = m} c_{m_1, m_2} \text{Im} \langle |x|^\ell \psi_x v, |x|^\ell \psi_x \partial^{m_1} V_\alpha \partial^{m_2} u \rangle$$

$$\lesssim \sqrt{m_x} \sum_{m_1 + m_2 = m} \| |x|^\ell \partial^{m_1} V_\alpha \|_{L^\infty} \| |x|^{\ell-1} |\partial^{m_2} u| \|_{L^2}.$$
By Young’s inequality we estimate
\[ |x| |\partial^{m_1} V_\alpha| \lesssim \sum_{m_1 + m_2 = m_3} \int |x - y|^{-\alpha}(|x - y| + |y|)|\partial^{m_1} u(y)||\partial^{m_2} u(y)| \, dy \]
\[ \lesssim \sum_{m_1 + m_2 = m_3} \left( \int |x - y|^{-(\alpha - 1)}(|\partial^{m_1} u(y)|^2 + |\partial^{m_2} u(y)|^2) \, dy \right) + \int |x - y|^{-\alpha}(|y| |\partial^{m_1} u(y)|^2 + |\partial^{m_2} u(y)|^2) \, dy. \]

Using the Hardy-Sobolev inequality, we get
\[ II \lesssim \sqrt{m_3} \sum_{0 \leq |j| \leq k + m} \| (1 + |x|)^{\ell - 1} \partial^j u(t) \|_{L^2}^3. \]

From (3.3) and (3.4) it follows that
\[ m'(t) \leq \sqrt{m(t)} \left( \| u(t) \|_{H^{2k+m}} + \sum_{0 \leq |j| \leq 2k+m} (1 + \| (1 + |x|)^{\ell - 1} \partial^j u(t) \|_{L^2})^3 \right), \]
which implies
\[ \sqrt{m(t)} \lesssim \sqrt{m(0)} + \int_0^t \left( \| u(t') \|_{H^{2k+m}} + \sum_{0 \leq |j| \leq 2k+m} (1 + \| (1 + |x|)^{\ell - 1} \partial^j u(t') \|_{L^2})^3 \right) \, dt'. \]

Letting \( \varepsilon \to 0 \), by Fatou’s lemma we get \( \sup_{0 \leq t' \leq t} \| x | \partial^m u \|_{L^2} < \infty \) for all \( t \in [0, T^*) \).

\section*{Proof of Proposition 3.4}
In view of Lemma 3.2, it suffices to show that
\[ \sup_{0 \leq t' \leq t} \| x | \partial^j u(t') \|_{L^2} < \infty \text{ for all } |j| \leq \alpha^* - 2k \text{ and } t \in [0, T^*), \]
provided \( u \in C([0, T^*]; H^{\alpha^*}) \). In fact, we can obtain the same estimates as (3.3) and (3.4) to \( m_\varepsilon \) for the case \( \ell = 1 \) to get
\[ \sqrt{m_\varepsilon(t)} \lesssim \sqrt{m_\varepsilon(0)} + \int_0^t \left( \| u(t) \|_{H^{\alpha^*}} + \| u(t) \|_{H^{\alpha^*}}^3 \right) \, dt'. \]

A limiting argument implies (3.5). This completes the proof of Proposition 3.1.

\section*{4. Appendix}

In this section we provide a proof of the local well-posedness of Hartree equation (1.1). Here we only assume that \( \alpha > 0 \) and \( \psi \in L^\infty \).

\begin{proposition}
Let \( \psi \in L^\infty \). Let \( \alpha > 0 \), \( 0 < \gamma < n \) and \( n \geq 1 \). Suppose \( \varphi \in H^s(\mathbb{R}^n) \) with \( s \geq \frac{n}{2} \). Then there exists a positive time \( T \) such that Hartree equation (1.1) has a unique solution \( u \in C([0, T]; H^{\gamma'}) \cap C^1([0, T]; H^{\gamma''}) \). Moreover, if \( T^* \) is the maximal existence time and is finite, then \( \lim_{t \uparrow T^*} \| u(t) \|_{H^{\gamma'}} = \infty \).
\end{proposition}
Proof. We use the standard contraction mapping argument. So we shall be brief.

Let \((X(T, \rho), d)\) be a complete metric space with metric \(d\) defined by

\[ X(T, \rho) = \{ u \in L_T^\infty (H^s(\mathbb{R}^n)) : \| u \|_{L_T^\infty H^s} \leq \rho \}, \quad d_X(u, v) = \| u - v \|_{L_T^\infty L^2}. \]

We define a mapping \(\mathcal{N} : u \mapsto \mathcal{N}(u)\) on \(X(T, \rho)\) by

\[ \mathcal{N}(u)(t) = U(t)\varphi - i \int_0^t U(t-t')F(u)(t')\,dt', \]

where \(U(t) = e^{-it|\nabla|^s}\). For \(u \in X(T, \rho)\) and \(s \geq \frac{n}{2}\) we estimate

\[ \|\mathcal{N}(u)\|_{L_T^\infty H^s} \leq \|\varphi\|_{H^s} + T\|F(u)\|_{L_T^\infty H^s} \]
\[ \lesssim \|\varphi\|_{H^s} + T \big( \|V_\gamma(|u|^2)\|_{L_T^\infty L^\infty} \|u\|_{L_T^\infty H^s} \]
\[ + \|V_\gamma(|u|^2)\|_{L_T^\infty H^s} \|u\|_{L_T^\infty L^\infty} \big) \]
\[ \lesssim \|\varphi\|_{H^s} + T \big( \|V_\gamma(|u|^2)\|_{L_T^\infty L^\infty} \|u\|_{L_T^\infty H^s} \]
\[ + \|V_\gamma(|u|^2)\|_{L_T^\infty H^s} \|u\|_{L_T^\infty L^\infty} \big) \]
\[ \lesssim \|\varphi\|_{H^s} + T \big( \|\nabla\|_{L_T^\infty L^\infty} \|u\|_{L_T^\infty H^s} + \|u\|_{L_T^\infty H^s} \|u\|_{L_T^\infty H^s} \big) \]
\[ \lesssim \|\varphi\|_{H^s} + T \|u\|_{L_T^\infty H^s}^2 \lesssim \|\varphi\|_{H^s} + T \rho^3. \]

Here we used the generalized Leibniz rule (Lemma A1-A4 in Appendix of \[\text{[3]}\]) for the second and fifth inequalities, the fractional integration for the fourth one, and the trivial inequality

\[ V_\gamma = \int_{\mathbb{R}^n} \frac{\psi(x-y)}{|x-y|^\gamma} |u(y)|^2 \, dy \leq \|\psi\|_{L^\infty} \int_{\mathbb{R}^n} |x-y|^{-\gamma} |u(y)|^2 \, dy, \]

the Hardy-Sobolev inequality

\[ \sup_{x \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{|u(x-y)|^2}{|y|^\gamma} \, dy \right| \lesssim \|u\|_{H^s_{\gamma}}, \]

and the Sobolev embedding \(H^s_{\gamma} \hookrightarrow L^{\frac{2n}{n-\gamma}}\) for the last one. If we choose \(\rho\) and \(T\) such as \(\|\varphi\|_{H^s} \leq \rho/2\) and \(C T \rho^3 \leq \rho/2\), then \(\mathcal{N}\) maps \(X(T, \rho)\) to itself.
Now we show that $N$ is a Lipschitz map with a sufficiently small $T$. Let $u, v \in X(T, \rho)$. Then we have
\[
d_X(N(u), N(v)) \lesssim T \left( \|V_\gamma(|u|^2)(u - v)\|_{L^T_T L^2} + \|V_\gamma(|v|^2 - |u|^2)v\|_{L^T_T L^2} \right)
\lesssim T \left( \|u\|^2_{L^T_T H^\frac{2}{n} \cap L^\infty_T L^\infty} d_X(u, v) + \|V_\gamma(|u|^2 - |v|^2)\|_{L^T_T L^\frac{2}{n} \cap L^\infty_T L^\infty} \right)
\lesssim T \left( \rho^2 d_X(u, v) + \rho \|2u^2 - v^2\|_{L^T_T L^\frac{4}{2n}} \right)
\lesssim T \rho^2 \rho d_X(u, v).
\]
The above estimate implies that the mapping $N$ is a contraction, if $T$ is sufficiently small. The uniqueness and time regularity follow easily from the equation (1.1) and a similar contraction argument.

Finally, let $T^*$ be the maximal existence time. If $T^* < \infty$, then it is obvious from the estimate (4.2) and the standard local well-posedness theory that $\lim_{t \to T^*} \|u(t)\|_{H^\frac{2}{n}} = \infty$. This completes the proof of Proposition 4.1

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