Inverse problems for Schrödinger equations with Yang-Mills potentials in domains with obstacles and the Aharonov-Bohm effect

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Abstract. We study the inverse boundary value problems for the Schrödinger equations with Yang-Mills potentials in a bounded domain $\Omega_0 \subset \mathbb{R}^n$ containing finite number of smooth obstacles $\Omega_j, 1 \leq j \leq r$. We prove that the Dirichlet-to-Neumann operator on $\partial \Omega_0$ determines the gauge equivalence class of the Yang-Mills potentials. We also prove that the metric tensor can be recovered up to a diffeomorphism that is identity on $\partial \Omega_0$.

1. Introduction
Let $\Omega_0$ be a smooth bounded domain in $\mathbb{R}^n$, diffeomorphic to a ball, $n \geq 2$, containing $r$ smooth nonintersecting obstacles $\Omega_j, 1 \leq j \leq r$. Consider the Schrödinger equation in $\Omega = \Omega_0 \setminus \bigcup_{j=1}^r \Omega_j$ with Yang-Mills potentials

$$\sum_{j=1}^n \left( -i \frac{\partial}{\partial x_j} I_m + A_j(x) \right)^2 u + V(x) u - k^2 u = 0 \quad (1.1)$$

with the boundary conditions

$$u \big|_{\partial \Omega_j} = 0, \quad 1 \leq j \leq r,$$

$$u \big|_{\partial \Omega_0} = f(x'), \quad (1.3)$$

where $A_j(x), V(x), u(x)$ are $m \times m$ matrices, $I_m$ is the identity matrix in $\mathbb{C}^m$. Let $G(\Omega)$ be the gauge group of all smooth nonsingular matrices in $\Omega$. Potentials $A(x) = (A_1, ..., A_n), V$ and $A'(x) = (A'_1, ..., A'_n), V'(x)$ are called gauge equivalent if there exists $g(x) \in G(\Omega)$ such that

$$A'(x) = g^{-1} A g - ig^{-1}(x) \frac{\partial g}{\partial x}, \quad V' = g^{-1} V g. \quad (1.4)$$

Let $A$ be the Dirichlet-to-Neumann (D-to-N) operator on $\partial \Omega_0$, i.e.

$$A f = \left( \frac{\partial u}{\partial \nu} + i (A \cdot \nu) u \right) \big|_{\partial \Omega_0},$$
where \( \nu = (\nu_1, \ldots, \nu_n) \) is the unit outward normal to \( \partial \Omega_0 \) and \( u(x) \) is the solution of (1.1), (1.2), (1.3). We assume that the Dirichlet problem (1.1), (1.2), (1.3) has a unique solution. We shall say that the D-to-N operators \( \Lambda \) and \( \Lambda' \) are gauge equivalent if there exists \( g_0 \in G(\Omega) \) such that

\[
\Lambda' = g_0,\partial \Omega_0 \Lambda g_0^{-1},
\]

where \( g_0,\partial \Omega_0 \) is the restriction of \( g_0 \) to \( \partial \Omega_0 \). We shall prove the following theorem:

**Theorem 1.1.** Suppose that D-to-N operators \( \Lambda' \) and \( \Lambda \) corresponding to potentials \( (A', V') \) and \( (A, V) \) respectively are gauge equivalent for all \( k \in (k_0 - \delta_0, k_0 + \delta_0) \), where \( k_0 > 0 \), \( \delta_0 > 0 \). Then potentials \( (A', V') \) and \( (A, V) \) are gauge equivalent too.

If we replace \( A', V' \) by \( A^{(1)} = g_0^{-1}A'g_0 - ig_0^{-1}g_0Vg_0 \), \( V^{(1)} = g_0^{-1}Vg_0 \) then \( \Lambda = \Lambda_1 \) where \( \Lambda_1 \) is the D-to-N operator corresponding to \( (A^{(1)}, V^{(1)}) \). The proof of Theorem 1.1 gives that if \( \Lambda = \Lambda_1 \) then \( (A, V) \) and \( (A^{(1)}, V^{(1)}) \) are gauge equivalent with a gauge \( g \in G(\Omega) \) such that \( g|_{\partial \Omega_0} = I_m \). We shall denote the subgroup of \( G(\Omega) \) consisting of \( g \) such that \( g(x)|_{\partial \Omega_0} = I_m \) by \( G_0(\Omega) \). In the case when \( \Omega_0 \) contains no obstacles Theorem 1.1 was proven in [E] for \( n \geq 3 \) and in [E3] for \( n = 2 \). Note that the result of [E] is stronger since it requires that \( \Lambda = \Lambda_1 \) for one value of \( k \) only. In the case \( n = 2 \) the proof of Theorem 1.1 is simpler than that in [E3] since it does not rely on the uniqueness of the inversion of the non-abelian Radon transform.

We shall prove Theorem 1.1 in two steps. In §2 we shall prove that \( (A, V) \) and \( (A^{(1)}, V^{(2)}) \) are locally gauge equivalent using the reduction to the inverse problem for the hyperbolic equations as in [B], [B1], [KKL], [KL], [E1], and in §3 we shall prove the global gauge equivalence using the results of §2 and of [E2]. Following Yang and Wu (see [WY]) one can describe the gauge equivalence class of \( A = (A_1, \ldots, A_n) \). Fix a point \( x^{(0)} \in \partial \Omega_0 \) and consider all closed paths \( \gamma \) in \( \Omega \) starting and ending at \( x^{(0)} \). Let \( x = \gamma(\tau) \), \( 0 \leq \tau \leq \tau_0 \), be a parametric equation of \( \gamma \), \( \gamma(0) = \gamma(\tau_0) = x^{(0)} \). Consider the Cauchy problem for the system

\[
\frac{\partial}{\partial \tau} c(\tau, \gamma) = \frac{d\gamma(\tau)}{d\tau} \cdot A(\gamma(\tau))c(\tau, \gamma), \quad c(0, \gamma) = I_m.
\]

By the definition the gauge phase factor \( c(\gamma, A) \) is \( c(\tau_0, \gamma) \). Therefore \( A \) defines a map of the group of paths to \( GL(m, \mathbb{C}) \). The image of this map is a subgroup of \( GL(m, \mathbb{C}) \) which is called the holonomy group of \( A \) (see [V]). It is easy to show (c.f. §3) that \( c(\gamma, A^{(1)}) = c(\gamma, A^{(2)}) \) for all closed paths \( \gamma \) iff \( A^{(1)} \) and \( A^{(2)} \) are gauge equivalent in \( \Omega \). As it was shown by Aharanov and Bohm [AB] the presence of distinct gauge equivalent classes of potentials can be detected in an experiment and this phenomenon is called the Aharanov-Bohm effect. In §4 we consider the recovery of the Riemannian metrics from the D-to-N operator in domains with obstacles.

### 2. Inverse problem for the hyperbolic system

Consider two hyperbolic system:

\[
L^{(p)} u = \frac{\partial^2}{\partial t^2} u^{(p)} + \sum_{j=1}^n (-i \frac{\partial}{\partial x_j} I_m + A_j^{(p)}(x))^2 u^{(p)} + V^{(p)}(x) u^{(p)} = 0, \quad p = 1, 2, \quad (2.1)
\]

in \( \Omega \times (0, T_0) \) with zero initial conditions

\[
u^{(p)}(x, 0) = u^{(p)}(x, 0) = 0 \quad (2.2)
\]

and the Dirichlet boundary conditions

\[
u^{(p)}|_{\partial \Omega_0 \times (0, T_0)} = 0, \quad 1 \leq j \leq r, \quad u^{(p)}|_{\partial \Omega_0 \times (0, T_0)} = f(x', t), \quad p = 1, 2. \quad (2.3)
\]
Here $\Omega = \Omega_0 \setminus (\cup_{j=1}^n \Omega_j)$ is the same as in §1, $A^{(p)}(x)$, $1 \leq j \leq n$, $V^{(p)}(x)$, $u^{(p)}(x,t)$, $p = 1,2$, are smooth $m \times m$ matrices. As in §1 introduce D-to-N operators $\Lambda^{(p)} = \left( \frac{\partial}{\partial x} + i \sum_{j=1}^n A^{(p)}(x) \cdot \partial_{x_j} \right) \big|_{\partial \Omega_0 \times (0,T_0)}$, $p = 1,2$.

Making the Fourier transform in $t$ one can show that the D-to-N operator for (2.1) when $T_0 = \infty$ determines the D-to-N operator for (1.1) for all $k$ except a discrete set, and vice versa.

We shall prove the following theorem:

**Theorem 2.1.** Suppose $\Lambda^{(1)} = \Lambda^{(2)}$ and $T_0 > \max_{x \in \overline{\Omega}} d(x, \partial \Omega_0)$ where $d(x, \partial \Omega_0)$ is the distance in $\overline{\Omega}$ from $x \in \overline{\Omega}$ to $\partial \Omega_0$. Then potentials $A^{(1)}(x)$, $1 \leq j \leq n$, $V^{(1)}(x)$ and $A^{(2)}(x)$, $1 \leq j \leq n$, $V^{(2)}(x)$ are gauge equivalent in $\overline{\Omega}$, i.e. (1.4) holds with $g \in G_0(\overline{\Omega})$.

Note that Theorem 2.1 implies Theorem 1.1. We can consider a more general than (2.1) equation when the Euclideanian metric is replaced by an arbitrary Riemannian metric:

$$
\frac{\partial^2 u^{(p)}}{\partial t^2} + \sum_{j,k=1}^n \frac{1}{\sqrt{g_p(x)}} \left(-i \frac{\partial}{\partial x_j} I_m + A^{(p)}(x) \right) \sqrt{g_p(x)} g^{jk}_p(x) \left(-i \frac{\partial}{\partial x_j} I_m + A^{(p)}(x) \right) u^{(p)}(x) + V^{(p)}(x) u^{(p)}(x,t) = 0,
$$

where $\|g^{jk}_p(x)\|^{-1}$ are metric tensors in $\overline{\Omega}^{(p)}$, $g_p(x) = \det \|g^{jk}_p\|^{-1}$, $A^{(p)}(x)$, $V^{(p)}(x)$ are the same as in (2.1), $\Omega^{(p)} = \Omega_0 \setminus \overline{\Omega}_p$, $\Omega'_p = \cup_{j=1}^n \Omega_{jp}$. Let $\Gamma$ be an open subset of $\partial \Omega_0$ and let $0 < T < T_0$ be small. Denote by $\Delta(0,T)$ the intersection of the domain of influence of $\Gamma$ with $\partial \Omega_0 \times [0,T]$. We assume that the domain of influence of $\Gamma$ does not intersect $\overline{\Omega}_p \times [0,T]$.

**Lemma 2.1.** Suppose $\Lambda^{(1)} = \Lambda^{(2)}$ on $\Delta(0,T)$. There exist neighborhoods $U^{(p)} \subset \Omega^{(p)}$, $p = 1,2$, $\overline{U}^{(p)} \cap \partial \Omega_0 = \Gamma$ and the diffeomorphism $\varphi : U^{(1)} \to U^{(2)}$ such that $\varphi|\Gamma = I$ and $\|g^{jk}_2\| = \varphi \circ \|g^{jk}_1\|$. Moreover $A^{(1)}$, $1 \leq j \leq n$, $V^{(1)}$ and $\varphi \circ A^{(2)}$, $1 \leq j \leq n$, $\varphi \circ V^{(2)}$ are gauge equivalent in $U^{(1)}$, i.e. there exists $g(x) \in G_0(\overline{U}^{(1)})$, $g(x) = I$ on $\Gamma$ such that (1.4) holds in $U^{(1)}$.

The proof of Lemma 2.2 is the same as the proof of Lemma 2.1 in [E1]. One should replace only the inner products of the form $\int f(u(x,t) v(x,T)dxdt$ by $\int Tr(u^*u)dxdt$ where $v^*$ is the adjoint matrix to $v(x,t)$. We do not assume that matrices $A^{(p)}$, $V^{(p)}$ are self-adjoint. In the latter case Lemma 2.1 can be obtained by the BC-method (see[B]). Extend $\varphi^{-1}$ from $U^2$ to $\overline{U}^{(2)}$ in such a way that $\varphi = I$ on $\partial \Omega_0$ and $\varphi$ is a diffeomorphism of $\overline{\Omega}^{(2)}$ and $\overline{\Omega}^{(2)} = \varphi^{-1}(\overline{\Omega}^{(2)})$. Also extend $g(x)$ from $U^2$ to $\overline{U}^{(2)}$ so that $g(x) \in G_0(\overline{\Omega}_2)$, $g = I$ on $\partial \Omega_0$. Then we get that $\hat{L}^{(2)} = g \circ \varphi \circ L^{(2)} = L^{(1)}$ in $U^{(1)}$.

**Lemma 2.2.** Let $L^{(1)}$ and $L^{(2)}$ be the operators of the form (2.4) in $\Omega^{(p)} = \Omega_0 \setminus \overline{\Omega}_p$, $p = 1,2$. Let $B \subset \Omega^{(1)} \cap \Omega^{(2)}$, $\partial B \cap \partial \Omega_0 = \Gamma$ is open, $\Omega^{(p)} \setminus \overline{B}$ is smooth, $\overline{B}$ is homotopic to $\Gamma$. Suppose $\hat{L}^{(2)} = L^{(1)}$ in $B$ and $\Lambda^{(1)} = \Lambda^{(2)}$ on $\partial \Omega_0 \times (0,T_0)$ where $\Lambda^{(p)}$ are the D-to-N operators corresponding to $L^{(p)}$, $p = 1,2$. Then $\hat{\Lambda}^{(1)} = \hat{\Lambda}^{(2)}$ where $\hat{\Lambda}^{(p)}$ are the D-to-N operators corresponding to $L^{(p)}$ in the domains $(\Omega^{(p)} \setminus \overline{B}) \times (\delta, T_0 - \delta)$, $\delta = \max_{x \in \partial \Omega_0 \setminus \overline{B}} d(x, \partial \Omega_0)$, $\Lambda^{(p)}$ are given on $\partial (\Omega_0 \setminus \overline{B}) \times (\delta, T_0 - \delta)$.

Therefore Lemma 2.2 reduces the inverse problem in $\Omega^{(p)} \times (0,T_0)$ to the inverse problem in a smaller domain $(\Omega^{(p)} \setminus \overline{B}) \times (\delta, T_0 - \delta)$. Combining Lemmas 2.1 and 2.2 we can prove that for any $x^{(0)} \in \Omega^{(1)}$ there exist a domain $B_1 \subset \Omega^{(1)}$ homotopic to $\partial \Omega_0$, $x^{(0)} \in B_1$, a diffeomorphism $\varphi$ of $\hat{\Omega}^{(2)}$ onto $\overline{\Omega}^{(2)}$, $\varphi = I$ on $\partial \Omega_0$, and $g \in G_0(\overline{\Omega}^{(2)})$ such that $\hat{L}^{(2)} \overset{\text{def}}{=} g \circ \varphi \circ L^{(2)} = L^{(1)}$ in
To prove the global gauge equivalence and global diffeomorphism in the case when $\Omega^{(1)}$ is not simply-connected we shall use some additional global quantities determined by the D-to-N operator (c.f. [E2]).

3. Global gauge equivalence

In this section we shall prove Theorem 2.1. Fix arbitrary point $x^{(0)} \in \partial \Omega_0$. Let $\gamma$ be a path in $\Omega$ starting at $x^{(0)}$ and ending at $x^{(1)} \in \varOmega$. $\gamma(\tau) = x(\tau)$ is the parametric equation of $\gamma$, $0 \leq \tau \leq \tau_1$, $x^{(0)} = x(0)$, $x^{(1)} = x(\tau_1)$. Denote by $c^{(p)}(\tau, \gamma)$, $p = 1, 2$, the solution of the system of differential equations

$$i \frac{\partial c^{(p)}(\tau, \gamma)}{\partial \tau} = \dot{\gamma}(\tau) \cdot A^{(p)}(x(\tau))c^{(p)}(\tau, \gamma), \quad (3.1)$$

where

$$c^{(p)}(0, \gamma) = I_m, \quad p = 1, 2, \quad 0 \leq \tau \leq \tau_1, \quad (3.2)$$

$$\dot{\gamma}(\tau) = \frac{dx}{d\tau}. \quad (3.3)$$

Denote $c^{(p)}(x^{(1)}, \gamma) = c^{(p)}(\tau_1, \gamma)$, $p = 1, 2$.

**Lemma 3.1.** Suppose $A^{(1)}$ and $A^{(2)}$ are locally gauge equivalent. Then matrix $c^{(2)}(x^{(1)}, \gamma)(c^{(1)}(x^{(1)}, \gamma))^{-1}$ depends only on the homotopy class of the path $\gamma$ connecting $x^{(0)}$ and $x^{(1)}$.

**Proof** Let $\gamma_1$ and $\gamma_2$ be two homotopic paths connecting $x^{(0)}$ and $x^{(1)}$. Consider the path $\gamma_0 = \gamma_1 \gamma_2^{-1}$ that starts and ends at $x^{(0)}$. It follows from (3.1) that $c^{(2)}(\tau, \gamma)(c^{(1)}(\tau, \gamma))^{-1}$ satisfies the following system of differential equations:

$$i \frac{\partial (c^{(2)}(\tau, \gamma)(c^{(1)}(\tau, \gamma))^{-1})}{\partial \tau} = \dot{\gamma}(\tau) \cdot A^{(2)}(x(\tau))(c^{(2)}(\tau, \gamma)(c^{(1)}(\tau, \gamma))^{-1})$$

$$- (c^{(2)}(\tau, \gamma)(c^{(1)}(\tau, \gamma))^{-1})A^{(1)}(x(\tau)) \cdot \dot{\gamma}(\tau). \quad (3.4)$$

Let $b(\tau, \gamma) = c^{(2)}(\tau, \gamma)(c^{(1)}(\tau, \gamma))^{-1}$, $b(x^{(1)}, \gamma_1) = b(\tau_1, \gamma_1)$, $b(x^{(1)}, \gamma_2) = b(\tau_2, \gamma_2)$, where $x = x^{(p)}(\tau)$ are parametric equations of $\gamma^{(p)}$, $0 \leq \tau \leq \tau_p$, $p = 1, 2$. We have that $b(x^{(1)}, \gamma_1) = b(x^{(1)}, \gamma_2)$ if $b(x^{(0)}, \gamma_0) = I_m$, where $x^{(0)}$ is the endpoint of path $\gamma_0 = \gamma_1 \gamma_2^{-1}$, and $b(x^{(0)}, \gamma_2)$ is the value at the endpoint of the solution of (3.4) along $\gamma_0$ with the initial value (3.2). If $\gamma_0$ can be contracted to a point in $\varOmega$ there exists closed paths $\sigma_1, ..., \sigma_N$ such that $\gamma_0 = \sigma_1...\sigma_N$ and each $\sigma_j$ is contained in a neighborhood $U_j \subset \Omega$ where $A^{(1)}$ and $A^{(2)}$ a gauge equivalent (see Lemma 2.1). We shall show that $b_j(\tau, \sigma_j)$ is continuous on $\sigma_j$ where $b_j(\tau, \sigma_j)$ is the solution of (3.4) with $\gamma$ replaced by $\sigma_j$, $\sigma_j(\tau) = x^{(j)}(\tau)$, $0 \leq \tau \leq \tau_j$, is the parametric equation of $\sigma_j$, $\sigma_j(0) = \sigma_j(\tau_j)$. The continuity on $\sigma_j$ means that $b(0, \sigma_j) = b(\tau_j, \sigma_j)$. Since $A^{(1)}$ and $A^{(2)}$ are gauge equivalent in $U_j$ there exists $g_j(x) \in C^\infty(U_j)$ such that (1.4) holds in $U_j$. It follows from (3.4) and (1.4) that

$$i \frac{\partial}{\partial \tau} (b_j(\tau, \sigma_j)g_j^{-1}(x^{(j)}(\tau)) = 0 \quad \text{for} \quad 0 \leq \tau \leq \tau_j. \quad (3.5)$$

Therefore $b_j(\tau, \sigma_j)g_j^{-1}(x^{(j)}(\tau)) = C$ on $\sigma_j$. We have $b_j(0, \sigma_j) = b_j(\tau_j, \sigma_j) = C g_j(x^{(j)}(0))$. Since $b_j(\tau, \sigma_j)$ is continuous on each $\sigma_j$, $1 \leq j \leq N$, we get that $b(\tau, \gamma_0)$ is continuous on $\gamma_0$, in particular, $b(x^{(0)}, 0, \gamma_0) = I_m$. Therefore $b(x^{(1)}, \gamma_1) = b(x^{(1)}, \gamma_2)$.

Now we shall prove that $b(x^{(1)}, \gamma_1) = b(x^{(1)}, \gamma_2)$ for any two paths connecting $x^{(0)}$ and $x^{(1)}$. As in the case of Lemma 3.1 it is enough to prove that $b(\tau, \gamma)$ is continuous on $\gamma_0 = \gamma_1 \gamma_2^{-1}$ where $b(\tau, \gamma_0)$ is the solution of (3.4) for $\gamma_0$. 

\[ \Box \]
We say that $\tilde{\gamma} = \tilde{\gamma}_1, \ldots, \tilde{\gamma}_N$ is a broken ray in $\overline{\Omega} \times [0, T_0]$ with legs $\tilde{\gamma}_j$, $1 \leq j \leq N$, if it starts at some point $(x^{(1)}, t^{(1)}) \in \partial \Omega_0 \times [0, T_0]$, $x = x^{(1)} + t\omega$, $t = t^{(1)} + \tau$ is the parametric equation of $\tilde{\gamma}_1$ for $0 \leq \tau \leq \tau_1$. Then $\tilde{\gamma}$ makes $N-1$ nontangential reflections at $\partial \Omega \times [0, T_0]$, where $\Omega = \bigcup_{j=1}^r \Omega_j$ and ends at $\partial \Omega_0 \times [0, T_0]$. Denote by $\gamma = \gamma_1, \ldots, \gamma_N$ the projection of $\tilde{\gamma}$ onto the $x$-plane. Let $c^{(p)}(\tau, \gamma)$ be the solution of the system

$$i \frac{\partial}{\partial \tau} c^{(p)}(\tau, \gamma) = A^{(p)}(\gamma(\tau)) \cdot \dot{\gamma}(\tau) c^{(p)}(\tau, \gamma),$$

$$c^{(p)}(0, \gamma) = I_m, \quad p = 1, 2,$$

where $\gamma(\tau)$ is the parametric equation of broken ray, $0 \leq \tau \leq \tau_0$, $\dot{\gamma} = \frac{dx}{d\tau}$ is the direction of the broken ray, $c^{(p)}(\tau, \gamma)$ is continuous on $\gamma(\tau)$. Note that $\frac{dx}{d\tau} = \theta_j$ is constant on $\gamma_j$, $\tau_{j-1} \leq \tau \leq \tau_j$.

The following lemma is the generalization of Theorem 2.1 in [E2]:

**Lemma 3.2.** Let $(x^{(N)}, t^{(N)})$ be the endpoint of the broken ray $\tilde{\gamma} : \tilde{\gamma}(\tau_N) = (x^{(N)}, t^{(N)})$. Denote $c^{(p)}(x^{(N)}, \gamma) = c^{(p)}(\tau_N, \gamma)$, $p = 1, 2$. Then $c^{(2)}(x^{(N)}, \gamma) = c^{(1)}(x^{(N)}, \gamma)$ assuming that $\Lambda^{(1)} = \Lambda^{(2)}$ on $\partial \Omega_0 \times [0, T_0]$.

Assuming that Lemma 3.2 is proven we shall complete the proof of Theorem 2.1.

Let $\gamma$ be a broken ray starting at $x^{(1)}$ and ending at $x^{(N)}$, $x^{(1)} \in \partial \Omega_0$, $x^{(N)} \in \partial \Omega_0$.

Let $c^{(p)}(\tau, \gamma)$ be the solution of (3.6), (3.7). Let $\alpha_1$ be a path on $\partial \Omega_0$ connecting $x^{(0)}$ and $x^{(1)}$ and let $\alpha_2$ be a path on $\partial \Omega_0$ connecting $x^{(N)}$ and $x^{(0)}$. Therefore $\alpha = \alpha_1 \gamma \alpha_2$ is a closed path starting and ending at $x^{(0)}$. If $U_j \cap \partial \Omega_0 \neq \emptyset$ then the gauge $g_j = I_m$ on $\partial \Omega_0$. Therefore $\frac{\partial}{\partial \tau} g_j \cdot \dot{\gamma} = 0$ on $U_j \cap \partial \Omega_0$. Then (1.4) implies that $A^{(2)} \cdot \dot{\gamma} = A^{(1)} \cdot \dot{\gamma}$. It follows from (3.1), (3.2) that $c^{(1)}(\tau, \alpha_1) = c^{(2)}(\tau, \alpha_1)$ and $c^{(1)}(\tau, \alpha_2) = c^{(2)}(\tau, \alpha_2)$. Therefore $b(\tau, \alpha)$ is continuous on $\alpha = \alpha_1 \gamma \alpha_2$. We shall call $\alpha = \alpha_1 \gamma \alpha_2$ an extended broken ray.

We shall assume for simplicity that extended broken rays generate the homotopy group of $\Omega$. Otherwise we can, as in the end of §2, construct a domain $\Omega^{(0)}$ such that $\Omega^{(0)} \subset \Omega$, $\partial \Omega^{(0)} \supset \partial \Omega_0$, $\partial \Omega^{(0)} \cap \overline{\Omega} = \emptyset$, $\Omega^{(0)}$ is homotopic to $\partial \Omega_0$ and $\Omega \setminus \Omega^{(0)}$ is "thin", i.e. the volume of $\Omega \setminus \Omega^{(0)}$ is small. Since $\Omega^{(0)}$ is homotopic to $\partial \Omega_0$ we get, using Lemmas 2.1 and 2.2 that potentials $A^{(1)}$ and $A^{(2)}$ are globally gauge equivalent in $\Omega^{(0)}$. Therefore the proof of global gauge equivalence in $\Omega$ can be reduced to the proof of the global gauge equivalence in $\Omega^{(0)}$. It is clear that the extended broken rays in $\Omega \setminus \Omega^{(0)}$ generate the fundamental group $\pi_1(\overline{\Omega} \setminus \Omega^{(0)})$. Then $\gamma_0$ is homotopic to $\alpha^{(1)} \ldots \alpha^{(N)}$ where $\alpha^{(j)}$ are extended broken rays. Since $b(\tau, \alpha^{(j)})$ is continuous on $\alpha^{(j)}$, $j = 1, \ldots, N_1$, we get that $b(\tau, \gamma_0)$ is continuous on $\gamma_0$. It follows from Lemma 3.2 that $b(\tau, \sigma_j) = I_m$ on $\partial \Omega_0$ and hence $b(\tau, \gamma_0) = I_m$ on $\partial \Omega_0$. Therefore we can extend $\gamma$ on $\partial \Omega_0$ by reflections without changing $b(\tau, \gamma_0)$. Denote $g(x^{(1)}) = c^{(2)}(x^{(1)}, \gamma) c^{(1)}(x^{(1)}, \gamma)^{-1}$. Then $g(x^{(1)})$ is a single-valued matrix on $\Omega$. Therefore $g(x)$ is nonsingular since $c^{(p)}(x, \gamma)$ are nonsingular, $p = 1, 2$. We have for arbitrary $x^{(1)}$:

$$A^{(2)}(x(\tau)) \cdot \dot{\gamma}(\tau) = i \frac{\partial}{\partial \tau} (c^{(2)}(\tau, \gamma) \cdot (c^{(2)}(\tau, \gamma))^{-1})$$

$$= (i \frac{\partial}{\partial x} g(x(\tau)) \cdot \dot{\gamma}(\tau) c^{(1)} + i g(x(\tau)) \frac{\partial}{\partial \tau} (c^{(1)}(\tau, \gamma)))^{-1} g^{-1}$$

$$= i \frac{\partial}{\partial x} g \cdot \dot{\gamma}^{-1} + g A^{(1)} \cdot \dot{\gamma}^{-1} = (i \frac{\partial}{\partial x} g g^{-1} + g A^{(1)} g^{-1}) \cdot \dot{\gamma}.$$
Since we can choose $\gamma(\tau)$ such that $\gamma(\tau_1) = x^{(1)}$ and $\dot{\gamma}(\tau)$ is arbitrary at $\tau = \tau_1$, we get that
\begin{equation}
A^{(2)}(x^{(1)}) = i \left( \frac{\partial}{\partial x} g \right) g^{-1}(x^{(1)}) + g(x^{(1)}) A^{(1)}(x^{(1)}) g^{-1}(x^{(1)}), \tag{3.9}
\end{equation}
i.e. $A^{(2)}$ is gauge equivalent to $A^{(1)}$ in $\Omega$. We can change $A^{(1)}$ to $A' = gA^{(1)} g^{-1} + i \left( \frac{\partial}{\partial x} gg^{-1} \right)$, $V' = gV^{(1)} g^{-1}$. Then we will have $A^{(2)} = A^{(1)}$. Therefore applying the proof of Theorem 2.1 we get that $V^{(2)} = V'$ in $\Omega$. Therefore $V^{(2)} = gV^{(1)} g^{-1}$ where $g(x)$ is the same as in (3.9).

It remains to prove Lemma 3.2. In the case when the broken ray $\gamma = \gamma_1 \ldots \gamma_M$ does not contain caustics points the proof of Lemma 3.2 is the same as the proof of Theorem 2.1 in [E2]. We shall consider the case when $\gamma$ has some caustics points and we shall simplify also the proof of Theorem 2.1 in [E2]. However in this paper we shall not use rays having caustics points. Consider, for simplicity, the case $n = 2$ and $x_0 \in \gamma_M$ is the only caustics point on $\gamma$. We also assume that the caustics point is generic (see [V]). Note that if $x_0$ is not generic but the broken ray $\gamma$ can be approximated by a sequence of broken rays having generic caustics points, then Lemma 3.2 holds for such $\gamma$ too. This fact suggests that Lemma 3.2 is likely true for any broken ray.

Let $\chi_0(y') \in C^\infty_0(\mathbb{R}^2)$, $y' = (y_1, y_2)$, $\chi_0(y') \geq 0$, $\chi_0(y') = 0$ for $|y'| > 1$, $\chi_0(y') = 1$ for $|y'| < \frac{1}{2}$. \int_{\mathbb{R}^2} \chi_0^2(y')dy' = 1$. Denote $\chi(y') = \frac{1}{\varepsilon} \chi_0 \left( \frac{y'}{\varepsilon} \right)$.

We shall heavily use the notations of [E2, §2]. The difference with [E2] is that in this paper we consider the broken ray $\gamma$ in $\Omega \times [0, T_0]$ and its projection on $\Omega$ will be the broken ray $\gamma$ considered in [E2].

Let $\Pi$ be a plane in $\mathbb{R}^2 \times \mathbb{R}$, $(x, t) \in \Pi$ if $x = x^{(0)}_0 + y_1 \omega_\perp$, $t = y_2 + t^{(0)}$, where $\omega_\perp \cdot \omega = 0$, $x^{(0)} \notin \Omega$ and the plane $\Pi$ does not intersect $\Omega \times \mathbb{R}$. We denote by $\tilde{\gamma}(y') = \tilde{\gamma}_0(y') \tilde{\gamma}_1(y') \ldots \tilde{\gamma}_M(y')$ the broken ray starting at $(x^{(0)} + y_1 \omega_\perp, t^{(0)} + y_2)$ in the direction $(\omega, 1)$, $y' = (y_1, y_2)$. Then the equation of $\tilde{\gamma}_0(y')$ is $x = x^{(0)} + y_1 \omega_\perp + t_0 \omega$, $t = t^{(0)} + y_2 + t_0$, $0 \leq t_0 \leq t_0(y_1)$, where $(x^{(0)} + y_1 \omega_\perp + t_0(y_1) \omega, t^{(0)} + y_2 + t_0(y_1)) = P_{t_0}$ is the point where $\tilde{\gamma}_0$ hits $\partial \Omega' \times (0, T_0)$. As in [E2] we introduce "ray coordinates" $(s_p, t_p)$ in the neighborhood of $\gamma_p$, $0 \leq p \leq M$. Denote by $D_j(x(s_j), t_j)$ the Jacobian of the change of coordinates $x = x^{(j)}(s_j, t_j)$. Let $\tilde{P}_j$ be the points of reflections of $\gamma(y')$ at $\partial \Omega'$, $1 \leq j \leq M$. Denote by $P_j$ the projection of $\tilde{P}_j$ on the $x$-plane. Note that the time coordinate of $\tilde{P}_j$ is $t^{(j)} = t^{(0)} + y_2 + \sum_{r=0}^{j} t_r(y_1)$ where $t_r(y_1)$ is the distance between $P_r$ and $P_{r-1}$. Note that $t = t^{(j)} + t_j$ on $\gamma_j$, $0 \leq t_j \leq t_j(y_1)$.

Let $L_p^{(p)}$ be the same as in (2.1), $p = 1, 2$. We construct a solution of $L^{(1)} u = 0$ of the form (c.f. (2.1), (2.9) in [E2], see also the earlier work [I]):
\begin{equation}
u(x, t, \omega) = \sum_{j=0}^{M-1} u_j(x, t, \omega) + u_{M1} + u_{M2} + u_{M3} + u^{(1)}, \tag{3.11}\end{equation}
where the principal part of $u_j$ has a form
\begin{equation}u_{j0} = a_{j0}(x, t, \omega) e^{i k(\psi_j(x, \omega) - t)}, \tag{3.12}\end{equation}
$\psi_j(x, \omega)$ are the same as in (2.2), (2.3), (2.4) in [E2] and
\begin{equation}a_{j0} = \left( a_{j-1,0} D_j \right)^{d_{j}} \left| \frac{1}{D_j} \right| c_j(x, \omega), \tag{3.13}\end{equation}
where \( c_{j1}(x, \omega) \) is the solution of the system

\[
 i \theta_j \cdot \nabla c_{j1} = (A^{(1)} \cdot \theta_j)c_{j1}, \quad t_{j-1}(y_1) \leq t_j \leq t_j(y_1), \quad c_{j1} \big|_{P_j} = I_m, \tag{3.14}
\]

\( 0 \leq j \leq M, \quad \nabla = \frac{\partial}{\partial x} \), \( \theta_j \) is the direction of \( \gamma_j \), \( \theta_0 = \omega \).

We shall assume that

\[
 u_0(x, t, \omega) = \chi(y')\alpha_0 \tag{3.15}
\]
on the plane \( \Pi \), i.e. when \( t_0 = 0 \) and \( x = x^{(0)} + y_1\omega_\perp \), \( t = t^{(0)} + y_2 \). Here \( \alpha_0 \) is an arbitrary constant matrix.

Let \( (x_*, t_*) \in \hat{\gamma}_M \) be such that \( x_* \) is the caustics point in the \( x \)-plane. Note that \( u_{M1} \) has the same form as \( u_{M-1} \) for \( t < t_* - C\varepsilon \), where \( \varepsilon \) is the same as in (3.10), solution \( u_{M2} \) is defined in a \( C\varepsilon \)-neighborhood \( U_\varepsilon \) of \( (x_*, t_*) \). We will not write the explicit form of \( u_{M2} \) (see, for example, [V]) since we will only need an estimate

\[
 |u_{M2}| \leq \frac{Ck^\frac{4}{3}}{1 + k^\frac{4}{3}d^\frac{4}{3}(x)}, \tag{3.16}
\]

where \( d(x) \) is the distance from \( x \in U_{0,\varepsilon} \) to the caustics curve. Such estimate holds in the generic case (see [V]). Moreover,

\[
 |\nabla u_{M2}| \leq \frac{Ck^\frac{2}{3}}{1 + k^\frac{2}{3}d^\frac{2}{3}(x)}.
\]

Finally, \( u_{M3} \) is defined for \( t > t_* + C\varepsilon \) and it has the same form as \( u_{M1} \). The main difference is that the amplitude of \( u_{M3} \) has an extra factor \( e^{i\beta} \) where \( \beta \) is real. The construction and the estimate of \( u^{(1)} \) in (3.11) is similar to [E2, Lemma 2.1] with the simplification that we consider the hyperbolic initial-boundary value problem with the zero initial conditions when \( t = 0 \) and zero boundary conditions on \( \partial\Omega \times (0, T_0) \) instead of (2.9) in [E2]. Since we assumed that \( T_0 \) is large enough we get that the endpoint of \( \hat{\gamma}_M \) belongs to \( \partial\Omega_0 \times (0, T_0) \).

We construct a solution \( v(x, t, \omega) \) of \( L^2 v = 0 \) similar to (3.11) with the same initial data as (3.15) for \( v_0 \) with \( \alpha_0 \) replaced by \( \beta_0 \) where \( \beta_0 \) is an arbitrary constant matrix and with the same phase function \( \psi_j(x, \omega), \ 0 \leq j \leq M \), as in (3.12): We have

\[
 v = \sum_{j=0}^{M-1} v_j(x, t, \omega) + v_{M1} + v_{M2} + v_{M3} + v^{(1)}(x, t, \omega), \tag{3.17}
\]

where the principal term of \( v_j \) has the following form:

\[
 v_{j0} = b_{j0}(x, t, \omega)e^{ik(\psi_j(x, \omega)-t)}, \tag{3.18}
\]

where \( b_{j0} \) are the same as \( a_{j0} \) with \( c_{j1}(x, \omega) \) replaced by \( c_{s,j}(x, \omega) \) where \( c_{s,j} \) is the solution of the system

\[
 i\theta_j \cdot \nabla c_{s,j} = (\theta_j \cdot (A^{(2)})^*)c_{s,j}. \tag{3.19}
\]

Taking the adjoint of (3.19) we get

\[
 -i(\theta_j \cdot \nabla c_{s,j}^* = c_{s,j}^*(\theta_j \cdot A^{(2))). \tag{3.20}
\]

Denote

\[
 c_{j2} = (c_{s,j}^*)^{-1}. \tag{3.21}
\]
Then (3.20) implies that
\[ i\theta_j \cdot \nabla c_{j2} = (\theta_j \cdot A^{(2)})c_{j2}. \]  
(3.22)

We assume that \( v^{(1)} \) satisfies zero initial conditions when \( t = T, \ x \in \Omega \), and zero boundary conditions on \( \partial \Omega \times (0, T_0) \). Substitute (3.11) instead of \( u^{(1)} \) and (3.17) instead of \( v^{(2)} \) in the Green’s formula. Dividing by \( 2k \) and passing to the limit when \( k \to \infty \) we obtain (c.f. [E2]):
\[
0 = \sum_{j=0}^{M-1} \int_0^T \int_\Omega ((A^{(1)} - A^{(2)}) \cdot \nabla (\psi_j - t)a_{j0}b_{j0})dxdt + I_M,  
\]
(3.23)

where \( I_M \) is the integral over a neighborhood of \( \gamma_M \). We make a series of changes of variables as in (2.43) in [E2].

Note that the Jacobian \( D_M(x^{(M)}(s_M, t_M)) \) vanishes on the caustics set and therefore \( D_M^{-1} \) has a singularity there. However when we make changes of variables this singularity in \( u_{M1}, v_{M1} \) and in \( u_{M3}, v_{M3} \) cancels. Note also that the estimate (3.16) implies that the integral over the neighborhood \( U_\varepsilon \) is \( O(\sqrt{\varepsilon}) \). Therefore taking into account that \( \alpha_0 \) and \( \beta_0 \) are arbitrary matrices we get
\[
\sum_{j=0}^M \int \int_{\Omega} \chi^2(y') c_{j2}^{-1}(A^{(1)} - A^{(2)}) \cdot \theta_j c_{j1} dt_j dy' + O(\sqrt{\varepsilon}) = 0,  
\]
(3.24)

where \( \gamma(y') \) is the broken ray starting at \( (x^{(0)} + y_1 \omega_\perp, t^{(0)} + y_2) \) and we use in (3.24) that \( c^{*}_{j} = c^{-1}_{j2} \) (see (3.21)). Note that
\[
c_{j2}^{-1}(A^{(1)} - A^{(2)}) \cdot \theta_j c_{j1} = i\theta_j \cdot \nabla(c_{j2}^{-1}c_{j1}),  
\]
(3.25)

since \( -c_{j2}^{-1}(A^{(2)} \cdot \theta_j) = i\theta_j \cdot \nabla c_{j2}^{-1} \). After changes of variables \( c_{j1} \) and \( c_{j2} \) in (3.24) satisfy the differential equations (3.14), (3.22) but the initial conditions are different:
\[
c_{ji} |_{P_j = c_{j-1}i} = 1 \leq j \leq M, \quad i = 1, 2.  
\]
(3.26)

We kept the same notation for the simplicity. Taking the limit in (3.24) when \( \varepsilon \to 0 \) we get
\[
\sum_{j=0}^M \int \gamma_j \cdot \nabla(c_{j2}^{-1}c_{j1}) dt_j = \sum_{j=0}^M \left[ (c_{j2}^{-1}c_{j1}) |_{P_j} - (c_{j2}^{-1}c_{j1}) |_{P_{j-1}} \right] = 0.  
\]

Since \( c_{0i} |_{P_k} = I_m \) and (3.26) holds we get that \( c_{M2}^{-1}c_{M1} |_{P_M} = I_m \), i.e. \( c_{M1} |_{P_M} = c_{M2} |_{P_M} \). Lemma 3.2 is proven.

4. Global diffeomorphism

Let \( L^{(1)}u^{(1)} = 0 \) and \( L^{(2)}u^{(2)} = 0 \) be equations of the form (2.4) in domains \( \Omega^{(p)} = \Omega_0 \setminus \Omega^{(p)}_p \), where \( \Omega^{(p)}_p = \cup_{j=1}^{N_p} \Omega_{jp} \), \( p = 1, 2 \). We assume that the initial conditions (2.2) in \( \Omega^{(p)} \), \( p = 1, 2 \) and the boundary conditions (2.3) with \( \Omega_{jp} \) replaced by \( \Omega_{jp} \), \( p = 1, 2 \), are satisfied.

**Theorem 4.1.** Suppose \( \Lambda^{(1)} = \Lambda^{(2)} \) on \( \partial \Omega_0 \), where \( \Lambda^{(p)} \) are the D-to-N operators corresponding to \( L^{(p)} \), \( p = 1, 2 \). Suppose
\[
T_0 > 2 \min_p \max_{x \in \Omega^{(p)}} d_p(x, \partial \Omega_0)  
\]
where \( d_p \) is the distance with respect to the metric tensor \( \|g^{(p)}_{jk}\|^{-1} \). Then there exists a diffeomorphism \( \varphi \) of \( \Omega^{(1)} \) onto \( \Omega^{(2)} \) such that \( \varphi = I \) on \( \partial \Omega_0 \) and \( \|g^{(2)}_{jk}\| = \varphi \circ \|g^{(1)}_{jk}\| \).
We shall sketch the proof of Theorem 4.1 assuming for the simplicity that $m = 1$, $A_j^{(p)} \equiv 0$, $1 \leq j \leq r$, $p = 1, 2$, and $T_0 = \infty$. By using Lemmas 2.1 and 2.2 we can get a domain $\Omega^{(0)} \subset \Omega^{(1)}$ such that $\Omega^{(0)}$ is homotopic to $\partial \Omega_0$, $\Omega^{(1)} \setminus \Omega^{(0)}$ has a small volume. Moreover there exists a diffeomorphism $\tilde{\varphi}$ of $\Omega^{(2)}$ onto $\tilde{\Omega}^{(2)} = \tilde{\varphi}^{-1}(\Omega^{(2)})$, $\tilde{\varphi} = I$ on $\partial \Omega_0$ such that $\tilde{L}^{(2)} \equiv \tilde{\varphi} \circ L_2$ is equal to $L^{(1)}$ in $\Omega^{(0)}$. Note that $\Omega^{(0)} \subset \Omega^{(1)} \cap \tilde{\Omega}^{(2)}$. We also get from Lemma 2.2 that $\Lambda^{(1)} = \tilde{\Lambda}^{(2)}$ on $\partial \Omega^{(0)} \setminus \partial \Omega_0$ where $\tilde{\Lambda}^{(2)}$ is the D-to-N operator corresponding to $\tilde{L}^{(2)}$. Since $\Omega^{(1)} \setminus \Omega^{(0)}$ is thin, there is an open subset $\Gamma_1$ of $\partial \Omega^{(0)}$ such that the endpoints of geodesics corresponding to $L^{(1)}$ in $\Omega^{(1)} \setminus \Omega^{(0)}$, orthogonal to $\Gamma_1$, form an open subset $\Gamma_2 \subset \partial \Omega^{(0)}$. Denote by $D_1 \subset \Omega^{(1)} \setminus \Omega^{(0)}$ the union of these geodesics. It follows from the proof of Lemma 2.1 (see [E1]) that $\Lambda^{(1)}$ on $\Gamma_1$ uniquely determines the metric tensor $\|q_1^{\Lambda}\|^{-1}$ in the semi-geodesic coordinates in $D_1$. Denote by $\psi_1$ the map of $D_1$ on $\tilde{D}_1 = \psi_1(D_1)$ such that $\psi_1(x)$ are the semi-geodesic coordinates in $\tilde{D}_1$. Analogously let $D_2$ be the union of all geodesics of $\tilde{L}^{(2)}$ orthogonal to $\Gamma_1$ and let $\Gamma_2' \subset \partial \Omega^{(0)}$ be the set of its endpoints. Denote by $\psi_2(x)$ the semi-geodesic coordinates for $L^{(2)}$ and let $\tilde{D}_2 = \psi_2(D_2)$. By Lemma 2.1 $\psi_1 \circ L^{(1)} = \psi_2 \circ L^{(2)}$ in $\tilde{D}_1 \cap \tilde{D}_2$. It follows from Lemma 2.1 that $\psi_2 = I$ on $\Gamma_1$. Note that $\Omega^{(2)} \setminus \Omega^{(0)}$ coincide with $\Omega^{(1)} \setminus \Omega^{(0)}$ near $\Gamma_1$.

**Lemma 4.1.** The following equalities hold: $\tilde{D}_1 = \tilde{D}_2$ and $\psi = \psi_2^{-1} \psi_1 = I$ on $\Gamma_2$.

Proof: Since we assume that $T_0 = \infty$ we can switch to the inverse problem for the equations of the form (1.1). Choose parameter $k \in \mathbb{C}$ such that the boundary value problem of the form (1.1), (1.2), (1.3) has a unique solution $u_p$ for any $f \in H^1_2(\partial \Omega^{(0)} \setminus \partial \Omega_0)$ where $f$ is the same for $p = 1$ and $p = 2$. Choose $f$ nonsmooth. Denote $\Gamma_2 = \psi_1^{-1}(\Gamma_2)$, $\tilde{\Gamma}_2 = \psi_2^{-1}(\Gamma_2')$. It follows from the unique continuation theorem that $\psi_1 \circ u_1 = \psi_2 \circ u_2$ in $\tilde{D}_2 \cap D_1$ since the Cauchy data of $u_1$ and $u_2$ coincide on $\Gamma_1$. Here $L^{(p)} u_p = 0$, $p = 1, 2$. If $\Gamma_2 \neq \Gamma_2'$ we get a contradiction since $\psi_2 \circ u_1$ is $C^\infty$ outside $\Gamma_2$ and $\psi_1 \circ u_2$ is $C^\infty$ outside $\Gamma_2'$. Therefore $\tilde{D}_1 = \tilde{D}_2$ and $\psi = \psi_2^{-1} \psi_1$ is a diffeomorphism of $D_1$ onto $D_2$. Since $\Gamma_2 = \Gamma_2'$ we have $\Gamma_2' = \psi(\Gamma_2)$. Since $\psi_1 \circ u_1 = \psi_2 \circ u_2$ in $D_1 = \tilde{D}_2$ and $u_1 = u_2 = f(x)$ on $\Gamma_2$ we get $f(x) = f(\psi(x))$ on $\Gamma_2$. Since $f$ is arbitrary this implies that $\psi = I$ on $\Gamma_2$.

Therefore $\psi = I$ on $\partial \tilde{D}_1 \cap \partial \Omega^{(0)}$. Define $\varphi^{(1)} = \tilde{\varphi}$ on $\Omega^{(0)}$, $\varphi^{(1)} = \psi \circ \tilde{\varphi}$ on $D_1$. We get that $\varphi^{(1)} \circ L^{(2)} = L^{(1)}$ in $\Omega^{(0)} \cup \tilde{D}_1$.

Note that the proof of Lemma 4.1 uses the fact that $\Lambda^{(1)} = \tilde{\Lambda}^{(2)}$ on $\partial \Omega^{(0)} \setminus \partial \Omega_0$ whereas the Lemma 2.1 uses only that $\Lambda^{(1)} = \Lambda^{(2)}$ near $\Gamma$. Applying Lemma 2.2 to $\Omega^{(p)} \setminus (\tilde{\Omega}^{(0)} \cup \tilde{\Omega}^{(1)} \setminus \Gamma)$ and using again Lemmas 4.1, 2.1 and 2.2 we prove Theorem 4.1.

Remark 4.1. We shall show now that the obstacles can be recovered up to the diffeomorphism.

Let $\gamma_0$ be an open subset of $\partial \Omega^{(0)}$ close to the obstacle $\Omega_1'$. Denote by $\Delta_1$ the union of all geodesics in $\Omega^{(0)}$ orthogonal to $\gamma_0$ and ending on $\Omega_1'$. Denote by $\gamma_1$ the intersection of $\Sigma \Gamma$ and $\tilde{\Omega}_1'$. Introduce semi-geodesic coordinates for $L^{(1)}$ in $\Delta_1$. Let $\varphi_1$ be the change of variables to the semi-geodesic coordinates and let $\Delta_1 = \varphi_1(\Delta_1)$. Let $\varphi_2$ be the change of variables to the semi-geodesic coordinates for $\tilde{L}^{(2)}$ in $\Delta_2$ where $\Delta_2$ is the union of all geodesics of $\tilde{L}^{(2)}$ orthogonal to $\gamma_0$ and ending on $\Omega_2'$. Let $\gamma_2 = \Sigma \Gamma \cap \partial \Omega_2'$, $\gamma_2 = \varphi_2(\gamma_2)$, $\Delta_2 = \varphi_2(\Delta_2)$. Let $L^{(1)} u_1 = 0$ be a geometric optics solution in $\Omega^{(1)} \setminus \Omega^{(0)}$ similar to constructed in §3 that starts on $\gamma_0$, reflects at $\partial \Omega_1'$ and leaves $\Omega^{(1)} \setminus \Omega^{(0)}$ again on $\gamma_0$. Let $u_2$ be the solution of $L^{(2)} u_2 = 0$ in $\Omega^{(2)} \setminus \Omega^{(0)}$ having the same boundary data as $u_1$. Since $\varphi_1 \circ L^{(1)} = \varphi_2 \circ L^{(2)}$ in $\Delta_1 \cap \Delta_2$ and since $\varphi_1 \circ u_1$ and $\varphi_2 \circ u_2$ have the same Cauchy data on $\gamma_0$ we get by the uniqueness continuation theorem that $\varphi_1 \circ u_1 = \varphi_2 \circ u_2$ in $\Delta_1 \cap \Delta_2$. If $\tilde{\gamma}_1 \neq \tilde{\gamma}_2$ then we can find $u_1$ such that $\varphi_1 \circ u_1$ and $\varphi_2 \circ u_2$ will have different point of reflection and this will contradict that $\varphi_1 \circ u_1 = \varphi_2 \circ u_2$ in $\Delta_1 \cap \Delta_2$. Since $\tilde{\gamma}_1 = \tilde{\gamma}_2$ we get that $\varphi(\gamma_1) = \gamma_2 \subset \partial \Omega_2'$ and $\varphi(\Delta_1) = \Delta_2$ where $\varphi = \varphi_2^{-1} \varphi_1$. \qed
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