Multiplicity of solutions for fractional Schrödinger systems in $\mathbb{R}^N$

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ABSTRACT

In this paper, we deal with the following nonlocal systems of fractional Schrödinger equations

$$
\begin{align*}
\varepsilon^{2s}(-\Delta)^s u + V(x)u &= Q_u(u, v) + \gamma H_u(u, v) \quad \text{in } \mathbb{R}^N \\
\varepsilon^{2s}(-\Delta)^s v + W(x)v &= Q_v(u, v) + \gamma H_v(u, v) \quad \text{in } \mathbb{R}^N \\
u, v > 0 \quad \text{in } \mathbb{R}^N
\end{align*}
$$

where $\varepsilon > 0$, $s \in (0, 1)$, $N > 2s$, $(-\Delta)^s$ is the fractional Laplacian, $V : \mathbb{R}^N \to \mathbb{R}$ and $W : \mathbb{R}^N \to \mathbb{R}$ are continuous potentials, $Q$ is a homogeneous $C^2$-function with subcritical growth, $\gamma \in \{0, 1\}$ and $H(u, v) = (2/(\alpha + \beta))|u|^\alpha |v|^\beta$ with $\alpha, \beta \geq 1$ such that $\alpha + \beta = 2^*_s$. We investigate the subcritical case ($\gamma = 0$) and the critical case ($\gamma = 1$), and using Ljusternik–Schnirelmann theory, we relate the number of solutions with the topology of the set where the potentials $V$ and $W$ attain their minimum values.

1. Introduction

In the last decade, the study of nonlinear partial differential equations involving fractional and nonlocal operators has received a tremendous popularity, due to the fact that such operators have great applications in many areas of the research such as crystal dislocation, finance, phase transitions, material sciences, chemical reactions, minimal surfaces; see for instance [1, 2] for more details.

Motivated by the interest shared by the mathematical community in this topic, the aim of this paper is to investigate the existence and the multiplicity of positive solutions for the following nonlinear fractional Schrödinger system

$$
\begin{align*}
\varepsilon^{2s}(-\Delta)^s u + V(x)u &= Q_u(u, v) + \gamma H_u(u, v) \quad \text{in } \mathbb{R}^N \\
\varepsilon^{2s}(-\Delta)^s v + W(x)v &= Q_v(u, v) + \gamma H_v(u, v) \quad \text{in } \mathbb{R}^N \\
u, v > 0 \quad \text{in } \mathbb{R}^N
\end{align*}
$$

(1)
where \( \varepsilon > 0 \) is a parameter, \( s \in (0, 1) \), \( N > 2s \), \( V: \mathbb{R}^N \to \mathbb{R} \) and \( W: \mathbb{R}^N \to \mathbb{R} \) are continuous potentials, \( Q \) is a homogeneous \( C^2 \)-function with subcritical growth, \( \gamma \in (0, 1) \), and 
\[
H(u, v) = (2/(\alpha + \beta))|u|^{\alpha}|v|^{\beta}
\]
where \( \alpha, \beta \geq 1 \) are such that \( \alpha + \beta = 2^*_s = 2N/(N - 2s) \).

The nonlocal operator \((-\Delta)^s\) is the so-called fractional Laplacian operator which can be defined for any \( u: \mathbb{R}^N \to \mathbb{R} \) smooth enough, by setting
\[
(-\Delta)^s u(x) = -\frac{C(N,s)}{2} \int_{\mathbb{R}^N} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{N+2s}} \, dy \quad (x \in \mathbb{R}^N),
\]
where \( C(N,s) \) is a dimensional constant depending only on \( N \) and \( s \); see for instance [1].

In the scalar case, the problem (1) becomes the well-known fractional Schrödinger equation
\[
\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(x,u) \quad \text{in} \ \mathbb{R}^N.
\]  

(2)

We recall that one of the main reasons of studying (2) is related to the seek of standing wave solutions \( \Phi(t,x) = u(x)e^{-\varepsilon^2\varepsilon t}/\hbar \) for the time-dependent fractional Schrödinger equation
\[
\frac{ih\partial \Phi}{\partial t} = \frac{\hbar^2}{2m}(-\Delta)^s \Phi + V(x)\Phi - g(|\Phi|)|\Phi| \quad \text{for} \ (t,x) \in \mathbb{R} \times \mathbb{R}^N.
\]  

(3)

Equation (3) has been proposed by Laskin [3], and it is a fundamental equation of fractional Quantum Mechanics in the study of particles on stochastic fields modelled by Lévy processes.

When \( s = 1 \), Equation (2) reduces to the classical Schrödinger equation
\[
-\varepsilon^2 \Delta u + V(x)u = f(u) \quad \text{in} \ \mathbb{R}^N,
\]  

(4)

which has been extensively studied in the last 30 years by many authors; see for instance [4–10] and the references therein.

Recently, the study of fractional Schrödinger equations has attracted the attention of many mathematicians. Felmer et al. [11] investigated existence, regularity and qualitative properties of positive solution to (2) when \( V \) is constant, and \( f \) is a smooth function with subcritical growth satisfying the Ambrosetti–Rabinowitz condition. Secchi [12] proved an existence result for a nonlinear fractional Schrödinger equation involving a subcritical nonlinearity and under weak assumptions on the behaviour of the potential \( V \) at infinity. Frank et al. [13] studied uniqueness and nondegeneracy of ground state solutions to (2) with \( f(u) = |u|^\alpha u \), for all \( H^s \)-admissible powers \( \alpha \in (0, \alpha^*) \). The author [14] showed the existence of infinitely many solutions to (2) with \( V(x) = 1 \), and \( f \) is autonomous and satisfies Berestycki–Lions type assumptions. Shang et al. [15] used variational methods to deal with the multiplicity of solutions of a fractional Schrödinger equation with critical growth, and with a continuous and positive potential \( V \). Figueiredo and Siciliano [16] obtained a multiplicity result by means of the Ljusternik–Schnirelmann and Morse theories for (2) involving a superlinear nonlinearity with subcritical growth. Alves and Miyagaki in [17] dealt with the existence and the concentration of positive solutions to (2) via penalization technique and the extension method [18]. We also mention the papers [16, 19–24] where the existence and the multiplicity of solutions to (2) have been investigated under various assumptions on the potential \( V \) and the nonlinearity \( f \), by using suitable variational and topological methods.
Particularly motivated by the papers [15, 16], in this work we aim to extend the multiplicity results for both subcritical and critical cases obtained for the scalar equation (2) to the case of the systems. More precisely, we generalize in the nonlocal setting some existence and multiplicity results appeared in [25–28] in which the authors studied elliptic systems of the type

\[
\begin{align*}
-\varepsilon^2 \Delta u + V(x)u &= Q_u(u, v) + \gamma H_u(u, v) \quad \text{in } \mathbb{R}^N, \\
-\varepsilon^2 \Delta v + W(x)v &= Q_v(u, v) + \gamma H_v(u, v) \quad \text{in } \mathbb{R}^N, \\
u, v > 0 \quad &\text{in } \mathbb{R}^N.
\end{align*}
\]

To the best of our knowledge, there are few results on the nonlinear systems involving the fractional Laplacian in the literature [29–32] and the results presented here seems to be new in the nonlocal framework.

In order to state the main theorems obtained in this work, we come back to our problem (1), and we introduce the assumptions on the potentials \(V, W\) and the function \(Q\).

Firstly, we define the constants

\[
V_0 = \inf_{x \in \mathbb{R}^N} V(x) \quad \text{and} \quad W_0 = \inf_{x \in \mathbb{R}^N} W(x)
\]

and

\[
V_\infty = \liminf_{|x| \to \infty} V(x) \quad \text{and} \quad W_\infty = \liminf_{|x| \to \infty} W(x).
\]

Along the paper, we will assume the conditions on \(V\) and \(W\):

\begin{enumerate}
\item[(H1)] \(V_0 = W_0 > 0\), and \(M = \{x \in \mathbb{R}^N : V(x) = W(x) = V_0\}\) is nonempty;
\item[(H2)] \(V_0 < \max\{V_\infty, W_\infty\}\).
\end{enumerate}

Regarding the function \(Q\), we suppose that \(Q \in C^2(\mathbb{R}_+^2, \mathbb{R})\) and satisfies the conditions:

\begin{enumerate}
\item[(Q1)] there exists \(q \in (2, 2^*_s)\) such that \(Q(tu, tv) = t^q Q(u, v)\) for all \(t > 0, (u, v) \in \mathbb{R}_+^2\);
\item[(Q2)] there exists \(C > 0\) such that \(|Q_u(u, v)| + |Q_v(u, v)| \leq C(u^{q-1} + v^{q-1})\) for all \((u, v) \in \mathbb{R}_+^2\);
\item[(Q3)] \(Q_u(0, 1) = 0 = Q_v(1, 0)\);
\item[(Q4)] \(Q_u(1, 0) = 0 = Q_v(0, 1)\);
\item[(Q5)] \(Q_{uv}(u, v) > 0\) for all \((u, v) \in \mathbb{R}_+^2\).
\end{enumerate}

Since we look for positive solutions of (1), we extend the function \(Q\) to the whole \(\mathbb{R}^2\) by setting \(Q(u, v) = 0\) if \(u \leq 0\) or \(v \leq 0\). We note that the \(q\)-homogeneity of \(Q\) implies that the identity

\[
qQ(u, v) = uQ_u(u, v) + vQ_v(u, v) \quad \text{for any } (u, v) \in \mathbb{R}^2
\]

holds. Moreover, using (Q2), we can see that there exists \(C > 0\) such that

\[
|Q(u, v)| \leq C(|u|^q + |v|^q) \quad \text{for any } (u, v) \in \mathbb{R}^2.
\]
A typical example (see [33]) of function $Q$ which satisfies the above assumptions is the following one. Let $p \geq 1$ and

$$P_p(u, v) = \sum_{\alpha_i + \beta_i = p} a_i u^{\alpha_i} v^{\beta_i},$$

where $i \in \{1, \ldots, k\}$, $\alpha_i, \beta_i \geq 1$ and $a_i \in \mathbb{R}$. The functions

$$Q_1(u, v) = P_q(u, v), \quad Q_2(u, v) = \sqrt{P_l(u, v)}, \quad Q_3(u, v) = \frac{P_{l_1}(u, v)}{P_{l_2}(u, v)},$$

and their possible combinations, with appropriate choice of the coefficients $a_i$, satisfy assumptions $(Q1)$–$(Q5)$ on $Q$ with $r = lq$ and $l_1 - l_2 = q$.

Now, we pass to state our main multiplicity results related to (1). When we take $\gamma = 0$ in (1), we have to deal with a system with subcritical growth, namely

\[
\begin{align*}
\epsilon^{2s}(-\Delta)^s u + V(x)u &= Q_u(u, v) \quad \text{in } \mathbb{R}^N \\
\epsilon^{2s}(-\Delta)^s v + W(x)v &= Q_v(u, v) \quad \text{in } \mathbb{R}^N \\
u, v > 0 \quad &\text{in } \mathbb{R}^N.
\end{align*}
\]

(7)

Since we aim to relate the number of solutions of (7) with the topology of the set $M$ of minima of the potential, it is worth recalling that if $Y$ is a given closed set of a topological space $X$, we denote by $\text{cat}_X(Y)$ the Lusternik–Schnirelmann category of $Y$ in $X$, that is the least number of closed and contractible sets in $X$ which cover $Y$; see [34] for more details.

With the above notations, the first main multiplicity result can be stated.

**Theorem 1.1:** Assume that $(H1)$–$(H2)$ and $(Q1)$–$(Q5)$ hold. Then, for any $\delta > 0$, there exists $\epsilon_\delta > 0$ such that for any $\epsilon \in (0, \epsilon_\delta)$, system (7) admits at least $\text{cat}_{M_\delta}(M)$ solutions, where $M_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta\}$.

It is worth noting that, a common approach to deal with fractional nonlocal problems, is to make use of the Caffarelli–Silvestre method [18], which consists in transforming via a Dirichlet–Neumann map, a given nonlocal problem into a local degenerate elliptic problem set in the half-space $\mathbb{R}^{N+1}_+$ and with a nonlinear Neumann boundary condition. In this work, we prefer to analyse the problem directly in $H^s(\mathbb{R}^N)$ in order to borrow some ideas developed in the case $s = 1$ taking care of the fact that in our situation a more careful analysis is needed due to the nonlocal character of $(-\Delta)^s$.

The proof of Theorem 1.1 is variational and it is based on the method of the Nehari manifold. After proving some compactness results for the functional associated with (7), and observing that the level of compactness are deeply related to the behaviour of the potentials $V$ and $W$ at infinity, we use some arguments developed in [6, 35], to compare the category of some sub-levels of the functional and the category of the set $M$. 
In the second part of our paper, we consider the critical case $\gamma = 1$, that is

$$
\begin{align*}
\varepsilon^2 s(-\Delta)^{s} u + V(x) u &= Q_u(u, v) + \frac{2\alpha}{\alpha + \beta} |u|^{|\alpha - 2} u|\beta| \quad \text{in } \mathbb{R}^N \\
\varepsilon^2 s(-\Delta)^{s} v + W(x) v &= Q_v(u, v) + \frac{2\beta}{\alpha + \beta} |v|^{|\beta - 2} v \quad \text{in } \mathbb{R}^N
\end{align*}
$$

where $\alpha, \beta \geq 1$ are such that $\alpha + \beta = 2^*_s$.

In this context, we assume that $Q$ fulfills the technical assumption:

(Q6) $Q(u, v) \geq \lambda u^{\tilde{\alpha}} v^{\tilde{\beta}}$ for any $(u, v) \in \mathbb{R}^2_+$ with $1 < \tilde{\alpha}, \tilde{\beta} < 2^*_s$, $\tilde{\alpha} + \tilde{\beta} = q_1 \in (2, 2^*_s)$, and $\lambda$ satisfying

- $\lambda > 0$ if either $N \geq 4s$, or $2s < N < 4s$ and $2^*_s - 2 < q_1 < 2^*_s$;
- $\lambda$ is sufficiently large if $2s < N < 4s$ and $2 < q_1 \leq 2^*_s - 2$.

To obtain the multiplicity of positive solutions to (8), we proceed as in the subcritical case. Clearly, the lack of the compactness due to the presence of the critical Sobolev exponent, creates a further difficulty, and more accurate estimates are needed to localize the energy levels where the Palais–Smale condition fails. To circumvent this hitch, we combine the estimates obtained in [36] with some adaptations of the calculations done in [25], which allow us to prove that the number

$$
\tilde{S}_n(\alpha, \beta) = \inf_{u, v \in H^s(\mathbb{R}^N) \setminus \{0, 0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + |(-\Delta)^{s/2} v|^2 \, dx}{\left(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx\right)^{2/2^*_s}}
$$

is strongly related to the best constant $S_n$ of the Sobolev embedding $H^s(\mathbb{R}^N)$ into $L^{2^*_s}(\mathbb{R}^N)$, and plays a fundamental role when we have to study critical systems like (8).

Our second main result can be stated as follows.

**Theorem 1.2:** Let us assume that (H1)-(H2) and (Q1)-(Q6) hold. If $\alpha, \beta \in [1, 2^*_s)$ are such that $\alpha + \beta = 2^*_s$, then for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon_\delta)$, system (8) possesses at least $\text{cat}_{M_\delta}(M)$ solutions.

We conclude this introduction observing that our results complement the ones obtained in [15, 16], in the sense that now we are considering the multiplicity results in the case of systems.

The structure of the paper is the following. In Section 2 we give some preliminary facts about the fractional Sobolev spaces and we set up the variational framework. In Section 3 we deal with the autonomous problem related to (7). In Section 4 we prove some compactness results for the functional associated with (7). In Section 5 we present the proof of Theorem 1.1. In the last section, we discuss the existence and the multiplicity of solutions for the system (1) in the critical case $\gamma = 1$.

## 2. Preliminaries and variational setting

In this section we collect some preliminary results about the fractional Sobolev spaces, and we introduce the functional setting.
For any \( s \in (0, 1) \) we define \( \mathcal{D}^{s,2}(\mathbb{R}^N) \) as the completion of \( C_0^\infty(\mathbb{R}^N) \) with respect to
\[
\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy,
\]
where the above equality holds up to a positive constant, or equivalently
\[
\mathcal{D}^{s,2}(\mathbb{R}^N) = \left\{ u \in L^2_{s^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx < \infty \right\}.
\]

Let us introduce the fractional Sobolev space
\[
H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx < \infty \right\}
\]
edowed with the natural norm
\[
\|u\|_{H^s(\mathbb{R}^N)} = \sqrt{\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx + \int_{\mathbb{R}^N} |u|^2 \, dx}.
\]

For the convenience of the reader, we recall the following embeddings:

**Theorem 2.1 ([1]):** Let \( s \in (0, 1) \) and \( N > 2s \). Then there exists a sharp constant \( S_* = S(N, s) > 0 \) such that for any \( u \in H^s(\mathbb{R}^N) \)
\[
\left( \int_{\mathbb{R}^N} |u|^{2s^*} \, dx \right)^{2/2s^*} \leq S_* \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx. \tag{9}
\]

Moreover, \( H^s(\mathbb{R}^N) \) is continuously embedded in \( L^q(\mathbb{R}^N) \) for any \( q \in [2, 2s^*] \) and compactly in \( L^q_{loc}(\mathbb{R}^N) \) for any \( q \in [1, 2s^*] \).

We also have a Lions-compactness type lemma.

**Lemma 2.1 ([11]):** Let \( N > 2s \). If \( (u_n) \) is a bounded sequence in \( H^s(\mathbb{R}^N) \) and if
\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 \, dx = 0
\]
for some \( R > 0 \), then \( u_n \to 0 \) in \( L^t(\mathbb{R}^N) \) for all \( t \in (2, 2s^*) \).
Now, we give the variational framework of problem (7). Using the change of variable \( x \mapsto \varepsilon x \), we are led to consider the problem

\[
\begin{aligned}
(-\Delta)^s u + V(\varepsilon x)u &= Q_u(u, v) & \text{in } \mathbb{R}^N \\
(-\Delta)^s v + W(\varepsilon x)v &= Q_v(u, v) & \text{in } \mathbb{R}^N \\
u, v > 0 & & \text{in } \mathbb{R}^N.
\end{aligned}
\] (10)

For any \( \varepsilon > 0 \), we introduce the fractional space

\[
\mathbb{H}_\varepsilon = \left\{ (u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} (V(\varepsilon x)|u|^2 + W(\varepsilon x)|v|^2) \, dx < \infty \right\},
\]
endowed with the norm

\[
\|(u, v)\|_\varepsilon^2 = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + |(-\Delta)^{s/2}v|^2 \, dx + \int_{\mathbb{R}^N} (V(\varepsilon x)|u|^2 + W(\varepsilon x)|v|^2) \, dx.
\]

Let us introduce

\[
\mathcal{J}_\varepsilon(u) = \frac{1}{2} \|(u, v)\|_\varepsilon^2 - \int_{\mathbb{R}^N} Q(u, v) \, dx
\]
for any \((u, v) \in \mathbb{H}_\varepsilon\). We define the minimax level

\[
c_\varepsilon = \inf_{(u, v) \in \mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u, v),
\]

where

\[
\mathcal{N}_\varepsilon = \{(u, v) \in \mathbb{H}_\varepsilon \setminus \{(0, 0)\} : \mathcal{J}_\varepsilon'(u, v), (u, v) = 0\}.
\]

It is standard to check that \( \mathcal{J}_\varepsilon \) possesses a Mountain Pass geometry. Indeed, \( \mathcal{J}_\varepsilon \in C^1(\mathbb{H}_\varepsilon, \mathbb{R}) \) and \( \mathcal{J}_\varepsilon(0, 0) = 0 \). Using (6) and Theorem 2.1, we get for any \((u, v) \in \mathbb{H}_\varepsilon\)

\[
\mathcal{J}_\varepsilon(u, v) \geq \frac{1}{2} \|(u, v)\|_\varepsilon^2 - C\|(u, v)\|_\varepsilon^q,
\]
so there exist \( \mu, \rho > 0 \) such that \( \mathcal{J}_\varepsilon(u, v) \geq \rho \) for \( \|(u, v)\|_\varepsilon = \mu \). From (Q1), we can see that for any \((u, v) \in \mathbb{H}_\varepsilon \setminus \{(0, 0)\}\)

\[
\mathcal{J}_\varepsilon(tu, tv) = \frac{t^2}{2} \|(u, v)\|_\varepsilon^2 - t^q \int_{\mathbb{R}^N} Q(u, v) \, dx \to -\infty \quad \text{as} \quad t \to \infty.
\]

Finally, in view of (6), we can note that there exists \( r > 0 \) such that for any \( \varepsilon > 0 \)

\[
\|(u, v)\|_\varepsilon \geq r \quad \text{for any } (u, v) \in \mathcal{N}_\varepsilon.
\] (11)

Since \( \mathcal{J}_\varepsilon \) satisfies Mountain Pass geometry, we can use the homogeneity of \( Q \) to prove that \( c_\varepsilon \) can be alternatively characterized by

\[
c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0, 1]} \mathcal{J}_\varepsilon(\gamma(t)) = \inf_{(u, v) \in \mathbb{H}_\varepsilon \setminus \{(0, 0)\} \atop t \geq 0} \max_{t \geq 0} \mathcal{J}_\varepsilon(tu, tv) > 0,
\]

where \( \Gamma_\varepsilon = \{ \gamma \in C([0, 1], \mathbb{H}_\varepsilon) : \gamma'(0) = 0, \mathcal{J}_\varepsilon(\gamma(1)) < 0 \} \). Moreover, for any \((u, v) \neq (0, 0)\), there exists a unique \( t > 0 \) such that \((tu, tv) \in \mathcal{N}_\varepsilon\). The maximum of the function \( t \to \mathcal{J}_\varepsilon(tu, tv) \) for \( t \geq 0 \) is achieved at \( t = \tilde{t} \); for more details see [34].
3. The autonomous problem when \( \gamma = 0 \)

In this section we establish an existence result for the autonomous problem associated with (7). Let us consider the subcritical autonomous system

\[
\begin{aligned}
(-\Delta)^s u + V_0 u &= Q_u(u, v) \quad \text{in } \mathbb{R}^N \\
(-\Delta)^s v + W_0 v &= Q_v(u, v) \quad \text{in } \mathbb{R}^N \\
u, v &> 0 \quad \text{in } \mathbb{R}^N.
\end{aligned}
\]

We set \( \mathbb{H}_0 = H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \) endowed with the norm

\[
\|(u, v)\|_0^2 = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + |(-\Delta)^{s/2}v|^2 \, dx + \int_{\mathbb{R}^N} (V_0 u^2 + W_0 v^2) \, dx.
\]

Let us introduce the functional \( J_0 : \mathbb{H}_0 \rightarrow \mathbb{R} \) defined as

\[
J_0(u, v) = \frac{1}{2} \|(u, v)\|_0^2 - \int_{\mathbb{R}^N} Q(u, v) \, dx.
\]

Let

\[
c_0 = \inf_{(u, v) \in \mathcal{N}_0} J_0(u, v) = \inf_{(u, v) \in \mathcal{X}_0} \max_{t \geq 0} J_0(tu, tv),
\]

where

\[
\mathcal{N}_0 = \{(u, v) \in \mathbb{H}_0 \setminus \{(0, 0)\} : (\mathcal{J}'_0(u, v), (u, v)) = 0\}.
\]

We begin by proving a useful lemma.

**Lemma 3.1:** Let \( \{(u_n, v_n)\} \subset \mathbb{H}_0 \) be a sequence such that \( \mathcal{J}'_0(u_n, v_n) \rightarrow 0 \) and \( (u_n, v_n) \rightharpoonup (0, 0) \). Then we have either

(i) \( \|(u, v)\|_0 \rightarrow 0 \), or

(ii) there exist a sequence \( (y_n) \subset \mathbb{R}^N \) and \( R, \gamma > 0 \) such that

\[
\liminf_{n \rightarrow \infty} \int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) \, dx \geq \gamma.
\]

**Proof:** Assume that (ii) is not true. Then, for any \( R > 0 \), we get

\[
\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 \, dx = 0 = \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |v_n|^2 \, dx.
\]

By Lemma 2.1, we can deduce that

\[
u_n, v_n \rightarrow 0 \text{ in } L^t(\mathbb{R}^N) \quad \forall \ t \in (2, 2^*_s).
\]

This fact and (6) give

\[
\int_{\mathbb{R}^N} Q(u_n, v_n) \, dx \rightarrow 0.
\]
Hence, using $\langle \mathcal{J}_0'(u_n, v_n), (u_n, v_n) \rangle \to 0$, (5) and (13) we obtain

$$
\| (u_n, v_n) \|_0^2 = \int_{\mathbb{R}^N} (Q_u(u_n, v_n)u_n + Q_v(u_n, v_n)v_n) \, dx + o_n(1)
$$

$$
= q \int_{\mathbb{R}^N} Q(u_n, v_n) \, dx + o_n(1) = o_n(1),
$$

which implies that (i) holds.  

**Theorem 3.1:** The problem (12) admits a weak solution.

**Proof:** It is clear that $\mathcal{J}_0$ has a Mountain Pass geometry, so, in view of Theorem 1.15 in [34], we can find a sequence $\{(u_n, v_n)\} \subset \mathbb{H}_0$ such that

$$
\mathcal{J}_0(u_n, v_n) \to c_0 \quad \text{and} \quad \mathcal{J}_0'(u_n, v_n) \to 0.
$$

By (5), we can see that

$$
c_0 + o_n(1) \| (u_n, v_n) \|_0 = \mathcal{J}_0(u_n, v_n) - \frac{1}{q} \langle \mathcal{J}_0'(u_n, v_n), (u_n, v_n) \rangle
$$

$$
= \left( \frac{1}{2} - \frac{1}{q} \right) \| (u_n, v_n) \|_0^2,
$$

which implies that $\{(u_n, v_n)\}$ is bounded in $\mathbb{H}_0$. Consequently, thanks to Theorem 2.1, we may assume that

$$
(u_n, v_n) \to (u, v) \quad \text{in } \mathbb{H}_0
$$

$$
u_n \to u, \, v_n \to v \text{ in } L^q_{\text{loc}}(\mathbb{R}^N)
$$

$$(u_n, v_n) \to (u, v) \quad \text{a.e. in } \mathbb{R}^N.
$$

This fact and (Q2) allow us to deduce that $\mathcal{J}_0'(u, v) = 0$.

Now, we assume that $u \neq 0$ and $v \neq 0$. Then, using $(u^-, v^-)$ as test function, where $x^- = -\max\{x, 0\}$, and recalling that $(x - y)(x^- - y^-) \geq (x^--y^-)^2$ for any $x, y \in \mathbb{R}$,
we can see that

$$0 = (\mathcal{J}_0(u, v), (u^-, v^-)) = \int_{\mathbb{R}^N} [(-\Delta)^{s/2}u(-\Delta)^{s/2}u^- + (-\Delta)^{s/2}v(-\Delta)^{s/2}v^-] \, dx$$

$$+ \int_{\mathbb{R}^N} (V_0uu^- + W_0vv^-) \, dx$$

$$- \int_{\mathbb{R}^N} (Q_u(u, v)u^- + Q_v(u, v)v^-) \, dx$$

$$= \int_{\mathbb{R}^{2N}} \left[ \frac{(u(x) - u(y))(u^-(x) - u^-(y))}{|x - y|^{N+2s}} \right] \, dx \, dy$$

$$+ \int_{\mathbb{R}^N} (V_0uu^- + W_0vv^-) \, dx - \int_{\mathbb{R}^N} (Q_u(u, v)u^-$$

$$+ Q_v(u, v)v^-) \, dx$$

$$\geq \int_{\mathbb{R}^{2N}} \left[ \frac{|u^-(x) - u^-(y)|^2}{|x - y|^{N+2s}} + \frac{|v^-(x) - v^-(y)|^2}{|x - y|^{N+2s}} \right] \, dx \, dy$$

$$+ \int_{\mathbb{R}^N} (V_0(u^-)^2 + W_0(v^-)^2) \, dx = \| (u^-, v^-) \|_0^2,$$

where we used the fact that $Q_u = 0$ on $(-\infty, 0) \times \mathbb{R}$ and $Q_v = 0$ on $\mathbb{R} \times (-\infty, 0)$.

Accordingly, $u, v \geq 0$ in $\mathbb{R}^N$. Now, we know that $\nabla Q$ is $(q - 1)$-homogeneous, so using conditions (Q4) and (Q5), and applying the Mean Value Theorem, we can deduce that $Q_u, Q_v \geq 0$. In view of (Q2), we can see that $z = u + v$ is a solution to $(-\Delta)^s z + V_0z \leq Cz^{q-1}$ in $\mathbb{R}^N$, for some constant $C > 0$. Hence, using a Moser iteration argument (see for instance Proposition 5.1.1 in [23] or Theorem 1.2 in [20]) we can prove that $z \in L^\infty(\mathbb{R}^N)$, which implies that $u, v \in L^\infty(\mathbb{R}^N)$. Then $Q_u(u, v)$ and $Q_v(u, v)$ are bounded, and by applying Proposition 2.9 in [37] we have $u, v \in C^{0, \alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. From the Harnack inequality [38], we get $u, v > 0$ in $\mathbb{R}^N$.

At this point, we can show that $\mathcal{J}_0(u, v) = c_0$. Indeed, taking into account $(u, v) \in \mathcal{N}_0$, (5) and using Fatou’s Lemma, we get

$$c_0 \leq \mathcal{J}_0(u, v) = \frac{q - 2}{2} \int_{\mathbb{R}^N} Q(u, v) \, dx$$

$$\leq \liminf_{n \to \infty} \frac{q - 2}{2} \int_{\mathbb{R}^N} Q(u_n, v_n) \, dx$$

$$= \liminf_{n \to \infty} \left[ \mathcal{J}_0(u_n, v_n) - \frac{1}{2} (\mathcal{J}'_0(u_n, v_n), (u_n, v_n)) \right]$$

$$= c_0,$$

which yields $\mathcal{J}_0(u, v) = c_0$. 
Secondly, we consider the case \( u \equiv 0 \) or \( v \equiv 0 \). If \( u \equiv 0 \), we can use \( \langle \mathcal{J}_0'(u, v), (u, v) \rangle = 0 \) and (5) to see that
\[
\| (0, v) \|_0^2 = \int_{\mathbb{R}^N} Q_u(0, v) \, dx = q \int_{\mathbb{R}^N} Q(0, v) \, dx = 0,
\]
that is \( v \equiv 0 \). Analogously, we can prove that \( v \equiv 0 \) implies \( u \equiv 0 \). Therefore, if \( u \equiv 0 \) or \( v \equiv 0 \), we have \( (u, v) = (0, 0) \).

Since \( c_0 > 0 \) and \( \mathcal{J}_0 \) is continuous, we can deduce that \( \| (u_n, v_n) \|_0 \to 0 \). Then, in view of Lemma 3.1, we can find a sequence \( \{y_n\} \subset \mathbb{R}^N \) and constants \( R, \gamma > 0 \) such that
\[
\liminf_{n \to \infty} \int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) \, dx \geq \gamma > 0. \tag{14}
\]
Let us define \( (\tilde{u}_n(x), \tilde{v}_n(x)) := (u_n(x + y_n), u_n(x + y_n)) \). Then, using the invariance of \( \mathbb{R}^N \) by translation, we can infer that \( \mathcal{J}_0(\tilde{u}_n, \tilde{v}_n) \to c_0 \) and \( \mathcal{J}_0'(\tilde{u}_n, \tilde{v}_n) \to 0 \). Since \( \{ (u_n, v_n) \} \) is bounded in \( \mathbb{H}_0 \), we may assume that \( (\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v}) \) in \( \mathbb{H}_0 \), \( \tilde{u}_n \to \tilde{u} \) and \( \tilde{v}_n \to \tilde{v} \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \), for some \( (\tilde{u}, \tilde{v}) \in \mathbb{H}_0 \) which is a critical point of \( \mathcal{J}_0 \).

Thus, in view of (14), we have
\[
\int_{B_R(0)} (|\tilde{u}|^2 + |\tilde{v}|^2) \, dx = \liminf_{n \to \infty} \int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) \, dx \geq \gamma,
\]
which implies that \( \tilde{u} \neq 0 \) or \( \tilde{v} \neq 0 \). Arguing as before, we can obtain that both \( \tilde{u} \) and \( \tilde{v} \) are not identically zero. This ends the proof of theorem. \( \blacksquare \)

4. Compactness properties

In this section we study the compactness properties of the functionals \( \mathcal{J}_\varepsilon \). Firstly, we introduce some notation which we will use in the sequel.

If \( \max \{ V_\infty, W_\infty \} < \infty \), we define the functional \( \mathcal{J}_\infty : \mathbb{H}_0 \to \mathbb{R} \) by setting
\[
\mathcal{J}_\infty(u, v) = \frac{1}{2} \left( \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + |(-\Delta)^{s/2} v|^2 \, dx + \int_{\mathbb{R}^N} (V_\infty u^2 + W_\infty v^2) \, dx \right) - \int_{\mathbb{R}^N} Q(u, v) \, dx,
\]
and we denote by \( c_\infty \) the ground state level of \( \mathcal{J}_\infty \), that is
\[
c_\infty = \inf_{(u, v) \in \mathcal{N}_\infty} \mathcal{J}_\infty(u, v) = \inf_{(u, v) \in \mathbb{H}_0 \setminus \{(0, 0)\}} \max_{t \geq 0} \mathcal{J}_\infty(tu, tv) > 0,
\]
where \( \mathcal{N}_\infty = \{(u, v) \in \mathbb{H}_0 \setminus \{(0, 0)\} : \langle \mathcal{J}_\infty'(u, v), (u, v) \rangle = 0 \} \). If \( \max \{ V_\infty, W_\infty \} = \infty \), we set \( c_\infty = \infty \).

Now, we prove some useful lemmas which allow us to deduce a fundamental compactness result for \( \mathcal{J}_\varepsilon \).

Lemma 4.1: Suppose that \( \max \{ V_\infty, W_\infty \} < \infty \) and let \( d \in \mathbb{R} \). Let \( \{(u_n, v_n)\} \subset \mathbb{H}_\varepsilon \) be a Palais–Smale sequence for \( \mathcal{J}_\varepsilon \) at the level \( d \) such that \( (u_n, v_n) \rightharpoonup (0, 0) \) in \( \mathbb{H}_\varepsilon \). If \( (u_n, v_n) \to (0, 0) \) in \( \mathbb{H}_\varepsilon \), then \( d \geq c_\infty \).
Proof: Let \( \{t_n\} \subset (0, \infty) \) be a sequence such that \((t_n u_n, t_n v_n) \in \mathcal{N}_\infty\). We begin by proving the following claim:

Claim \( t_0 = \limsup_{n \to \infty} t_n \leq 1 \). Assume by contradiction that there exists \( \lambda > 0 \) such that

\[
t_n \geq 1 + \lambda \quad \text{for any } n \in \mathbb{N}.
\]  

(15)

Since \( \{(u_n, v_n)\} \) is bounded in \( \mathbb{H}_\varepsilon \), we get \((\mathcal{J}_\varepsilon'(u_n, v_n), (u_n, v_n)) \to 0\), which together with (5) yields

\[
\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2} u_n\right|^2 + \left|(-\Delta)^{s/2} v_n\right|^2 \, dx + \int_{\mathbb{R}^N} (V(\varepsilon x)|u_n|^2 + W(\varepsilon x)|v_n|^2) \, dx
\]

\[
= q \int_{\mathbb{R}^N} Q(u_n, v_n) \, dx + o_n(1).
\]  

(16)

Using the fact that \((t_n u_n, t_n v_n) \in \mathcal{N}_\infty\) we have

\[
t_n^2 \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2} u_n\right|^2 + \left|(-\Delta)^{s/2} v_n\right|^2 \, dx + \int_{\mathbb{R}^N} (V_\infty|u_n|^2 + W_\infty|v_n|^2) \, dx\right)
\]

\[
= q t_n^2 \int_{\mathbb{R}^N} Q(u_n, v_n) \, dx.
\]  

(17)

Putting together (16) and (17) we obtain

\[
q(t_n^q - 1) \int_{\mathbb{R}^N} Q(u_n, v_n) \, dx = \int_{\mathbb{R}^N} [(V_\infty - V(\varepsilon x))|u_n|^2 + (W_\infty
\]

\[- W(\varepsilon x))|v_n|^2] \, dx + o_n(1).
\]  

(18)

Now, we can see that for any \( \eta > 0 \) there exists \( R > 0 \) such that

\[
V(\varepsilon x) \geq V_\infty - \eta, \quad W(\varepsilon x) \geq W_\infty - \eta \quad \text{for any } |x| \geq R.
\]  

(19)

On the other hand, in view of Theorem 2.1, we know that \( u_n \to u \) and \( v_n \to v \) in \( L^t_{\text{loc}}(\mathbb{R}^N) \) for any \( t \in [1, 2^*_s) \). Taking into account this fact, \( \|(u_n, v_n)\|_{\mathcal{H}} \leq C \), (18) and (19) we have

\[
q((1 + \lambda)^q - 1) \int_{\mathbb{R}^N} Q(u_n, v_n) \, dx \leq q(t_n^q - 1) \int_{\mathbb{R}^N} Q(u_n, v_n) \, dx \leq C' \eta + o_n(1).
\]  

(20)

Since \( \|(u_n, v_n)\|_{\mathcal{H}} \to 0 \), we can proceed as in the proof of Lemma 3.1 to deduce that there exist a sequence \( \{y_n\} \subset \mathbb{R}^N \) and constants \( R, \gamma > 0 \) such that

\[
\int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) \, dx \geq \gamma > 0.
\]  

(21)

Let us define \( (\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x + y_n), u_n(x + y_n)) \). Then, we may assume that \( (\tilde{u}_n, \tilde{v}_n) \to (u, v) \) in \( \mathbb{H}_\varepsilon \), for some nonnegative functions \( u \) and \( v \) such that \( \mathcal{J}_\varepsilon'(u, v) = 0 \). From (21), it is easy to see that \( u \not\equiv 0 \) or \( v \not\equiv 0 \). Moreover, arguing as in the proof of
Theorem 3.1, we deduce that \( u \) and \( v \) are positive in \( \mathbb{R}^N \). Then, using Fatou’s Lemma and (20) we get
\[
0 < q((1 + \lambda)^{q-2} - 1) \int_{\mathbb{R}^N} Q(u, v) \, dx \leq C' \eta
\]
for any \( \eta > 0 \), and this gives a contradiction. Therefore we can infer that \( t_0 \leq 1 \).

Now, it is convenient to distinguish the following cases.

Case 1 \( t_0 < 1 \). Then, we may assume that \( t_n < 1 \) for all \( n \in \mathbb{N} \).

From (5) we can see that
\[
c_\infty \leq \mathcal{J}_\infty(t_n u_n, t_n v_n) = \mathcal{J}_\infty(t_n u_n, t_n v_n) - \frac{1}{2} \langle \mathcal{J}'_\infty(t_n u_n, t_n v_n), (t_n u_n, t_n v_n) \rangle
\]
\[
= t_n^q \left( \frac{q - 2}{2} \right) \int_{\mathbb{R}^N} Q(u_n, v_n) \, dx
\]
\[
\leq \left( \frac{q - 2}{2} \right) \int_{\mathbb{R}^N} Q(u_n, v_n) \, dx
\]
\[
= \mathcal{J}_\epsilon(t_n u_n, t_n v_n) - \frac{1}{2} \langle \mathcal{J}'_\epsilon(u_n, v_n), (u_n, v_n) \rangle
\]
\[
= d + o_n(1)
\]
so we deduce that \( d \geq c_\infty \).

Case 2 \( t_0 = 1 \). Up to a subsequence, we may assume that \( t_n \to 1 \). Furthermore we have
\[
d + o_n(1) \geq c_\infty + \mathcal{J}_\epsilon(u_n, v_n) - \mathcal{J}_\infty(t_n u_n, t_n v_n).
\]

Now fix \( \eta > 0 \). Taking into account (19), \( q \)-homogeneity of \( Q \), the boundedness of \( \{(u_n, v_n)\} \) and \( t_n \to 1 \), we can see that
\[
\mathcal{J}_\epsilon(u_n, v_n) - \mathcal{J}_\infty(t_n u_n, t_n v_n) = \frac{(1 - t_n^2)}{2} \left( \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 + |(-\Delta)^{s/2} v_n|^2 \, dx \right)
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}^N} V(\epsilon x)|u_n|^2 + W(\epsilon x)|v_n|^2 \, dx
\]
\[
- \frac{t_n^2}{2} \int_{\mathbb{R}^N} (V_\infty|u_n|^2 + W_\infty|v_n|^2) \, dx
\]
\[
+ (t_n^q - 1) \int_{\mathbb{R}^N} Q(u_n, v_n) \, dx
\]
\[
\geq o_n(1) - C \eta.
\]

Putting together (22) and (23), and from the arbitrariness of \( \eta \) we conclude that \( d \geq c_\infty \). 

\[\blacklozenge\]

**Lemma 4.2:** Assume that \( \max\{V_\infty, W_\infty\} = \infty \). Let \( \{(u_n, v_n)\} \subset \mathbb{H}_\epsilon \) be a Palais–Smale sequence for \( \mathcal{J}_\epsilon \) at the level \( d \) such that \( (u_n, v_n) \rightharpoonup (0, 0) \) in \( \mathbb{H}_\epsilon \). Then \( (u_n, v_n) \to (0, 0) \) in \( \mathbb{H}_\epsilon \).
For any \((a, b) \in \mathbb{R}^2_+\), we define

\[ c_{(a,b)} = \inf_{(u,v) \in H_0 \setminus \{(0,0)\}} \max_{t \geq 0} J_{(a,b)}(tu, tv), \]

where

\[
J_{(a,b)}(u,v) = \frac{1}{2} \left( \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + |(-\Delta)^{s/2} v|^2 \, dx \right) + \frac{1}{2} \int_{\mathbb{R}^N} (a|u|^2 + b|v|^2) \, dx
\]

\[ - \int_{\mathbb{R}^N} Q(u,v) \, dx. \]

We note that if \(a > a'\) then \(c_{(a,b)} > c_{(a',b)}\) and that \(\lim_{n \to +\infty} c_{(a,b)} = \infty\).

Now, for fixed \((a, b) \in \mathbb{R}^2_+\), we can proceed as in the proof of Theorem 3.1 to see that \(c_{(a,b)}\) is achieved in some couple \((u,v)\) where \(u\) and \(v\) are positive functions in \(\mathbb{R}^N\).

Since \(\max\{V_\infty, W_\infty\} = \infty\) we can take \((a, b) \in \mathbb{R}^2_+\) such that \(c_{(a,b)} > d\) and for any fixed \(\eta > 0\) there exists \(R > 0\) such that

\[ V(\varepsilon x) \geq a - \eta, \quad W(\varepsilon x) \geq b - \eta \quad \text{for any } |x| \geq R. \quad (24) \]

We observe that if \(W_\infty < \infty\) we can choose \(b = W_\infty\) and \(a > 0\) large, and when \(V_\infty = \infty\) we take both \(a\) and \(b\) sufficiently large.

If by contradiction \((u_n, v_n) \rightharpoonup (0,0)\) in \(H_\varepsilon\), we argue as in the proof of Lemma 4.1 and using (24) we deduce that \(d \geq c_{(a,b)}\). But this is impossible because we chose \((a,b)\) such that \(c_{(a,b)} > d\). Therefore we can conclude that \((u_n, v_n) \to (0,0)\) in \(H_\varepsilon\).

Now, we are ready to give the proof of a compactness result.

**Theorem 4.1:** The functional \(J_\varepsilon\) constrained to \(N_\varepsilon\) satisfies the Palais–Smale condition at every level \(d < c_\infty\).

**Proof:** Let \(\{(u_n, v_n)\} \subset N_\varepsilon\) be a sequence such that \(J_\varepsilon(u_n, v_n) \to d\) and \(\|J'_\varepsilon(u_n, v_n)\|_* \to 0\). Then (see [34]) there exists \(\{\lambda_n\} \subset \mathbb{R}\) such that

\[ J'_\varepsilon(u_n, v_n) = \lambda_n J'_\varepsilon(u_n, v_n) + o_n(1), \]

where

\[
I_\varepsilon(u, v) := \| (u,v) \|_{\varepsilon}^2 - q \int_{\mathbb{R}^N} Q(u,v) \, dx.
\]

Hence

\[
0 = \langle J'_\varepsilon(u_n, v_n), (u_n, v_n) \rangle = \lambda_n \langle I'_\varepsilon(u_n, v_n), (u_n, v_n) \rangle + o_n(1)
\]

\[
= \lambda_n (2-q) \| (u_n, v_n) \|^2_\varepsilon + o_n(1),
\]

and using (11) we deduce that \(\lambda_n \to 0\). Then \(J'_\varepsilon(u_n, v_n) \to 0\) in the dual of \(H_\varepsilon\).

Since the Palais–Smale of \(J_\varepsilon\) is bounded, we may assume that \((u_n, v_n) \rightharpoonup (u, v)\) in \(H_\varepsilon\), for some \((u, v)\) which is a critical point of \(J_\varepsilon\).
Now, we set \((w_n, z_n) := (u_n - u, v_n - v)\). From the weak convergence of \(\{(u_n, v_n)\}\) and (6), we can apply the Brezis–Lieb Lemma and the splitting Lemma (see for instance Lemma 4.7 in [39]), to deduce that
\[
\mathcal{J}_e(w_n, z_n) = \mathcal{J}_e(u_n, v_n) - \mathcal{J}_e(u, v) + o_n(1)
\]
\[
= d - \mathcal{J}_e(u, v) + o_n(1) =: \tilde{d} + o_n(1)
\]
and
\[
\mathcal{J}'_e(w_n, z_n) = o_n(1).
\]
Since \(\mathcal{J}_e'(u, v) = 0\), we can see that
\[
\mathcal{J}_e(u, v) = \mathcal{J}_e(u, v) - \frac{1}{2} \langle \mathcal{J}_e'(u, v), (u, v) \rangle = \frac{q - 2}{2} \int_{\mathbb{R}^N} Q(u, v) \, dx \geq 0,
\]
which implies that \(\tilde{d} < c_\infty\).

Now, if we assume that \(\max\{V_\infty, W_\infty\} < \infty\), by Lemma 4.1 it follows that \((w_n, z_n) \to (0, 0)\) in \(\mathbb{H}_e\), that is \((u_n, v_n) \to (u, v)\) in \(\mathbb{H}_e\). In the case \(\max\{V_\infty, W_\infty\} = \infty\), we can apply Lemma 4.2 to deduce that \((u_n, v_n) \to (u, v)\) in \(\mathbb{H}_e\). \(\square\)

Arguing as in the above theorem, it is easy to prove that the following result holds true.

**Corollary 4.1:** The critical points of \(\mathcal{J}_e\) constrained to \(\mathcal{N}_e\) are critical points of \(\mathcal{J}_e\) in \(\mathbb{H}_e\)

## 5. Barycenter map and multiplicity of solutions to (10)

In this section, our main purpose is to apply the Ljusternik–Schnirelmann category theory to prove a multiplicity result for system (10). In order to obtain our main result, we first give some useful lemmas.

**Lemma 5.1:** Let \(\varepsilon_n \to 0\) and \(\{(u_n, v_n)\} \subset \mathcal{N}_{\varepsilon_n}\) be such that \(\mathcal{J}_{\varepsilon_n}(u_n, v_n) \to c_0\). Then there exists \(\{\tilde{y}_n\} \subset \mathbb{R}^N\) such that the translated sequence
\[
(\tilde{u}_n(x), \tilde{v}_n(x)) := (u_n(x + \tilde{y}_n), v_n(x + \tilde{y}_n))
\]
has a subsequence which converges in \(\mathbb{H}_0\). Moreover, up to a subsequence, \(\{y_n\} := \{\varepsilon_n \tilde{y}_n\}\) is such that \(y_n \to y \in M\).

**Proof:** Since \(\langle \mathcal{J}'_{\varepsilon_n}(u_n, v_n), (u_n, v_n) \rangle = 0\) and \(\mathcal{J}_{\varepsilon_n}(u_n, v_n) \to c_0\), we can argue as in the proof of Proposition 3.1 to deduce that \(\{(u_n, v_n)\}\) is bounded. Let us observe that \(\|(u_n, v_n)\| \to 0\) since \(c_0 > 0\). Therefore, as in the proof of Lemma 3.1, we can find a sequence \(\{\tilde{y}_n\} \subset \mathbb{R}^N\) and constants \(R, \gamma > 0\) such that
\[
\liminf_{n \to \infty} \int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) \, dx \geq \gamma,
\]
which implies that
\[
(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v})\ 
\text{weakly in}
\mathbb{H}_0,
\]
where \((\tilde{u}_n(x), \tilde{v}_n(x)) := (u_n(x + \tilde{y}_n), v_n(x + \tilde{y}_n))\) and \((\tilde{u}, \tilde{v}) \neq (0, 0)\).
Let \( \{t_n\} \subset (0, +\infty) \) be such that \((\hat{u}_n, \hat{v}_n) := (t_n\hat{u}_n, t_n\hat{v}_n) \in \mathcal{N}_0 \) and set \( y_n := \varepsilon_n\tilde{y}_n \).

Using the change of variables \( z \mapsto x + \tilde{y}_n \) we can see that

\[
\mathcal{J}_0(\hat{u}_n, \hat{v}_n) \leq \frac{t_n^2}{2} \left( \int_{\mathbb{R}^N} |(-\Delta)^{s/2} \hat{u}_n|^2 + |(-\Delta)^{s/2} \hat{v}_n|^2 \, dx \right) - \int_{\mathbb{R}^N} Q(t_n\hat{u}_n, t_n\hat{v}_n) \, dx \\
+ \frac{t_n^2}{2} \int_{\mathbb{R}^N} (V(\varepsilon_n(x + \tilde{y}_n))|\hat{u}_n|^2 + W(\varepsilon_n(x + \tilde{y}_n))|\hat{v}_n|^2) \, dx \\
= \mathcal{J}_{\varepsilon_n}(t_n u_n, t_n v_n) \leq \mathcal{J}_{\varepsilon_n}(u_n, v_n) = c_0 + o_n(1).
\]

Taking into account that \( c_0 \leq \mathcal{J}_0(\hat{u}_n, \hat{v}_n) \), we can infer \( \mathcal{J}_0(\hat{u}_n, \hat{v}_n) \to c_0 \).

Now, the sequence \( \{t_n\} \) is bounded since \( \{(\hat{u}_n, \hat{v}_n)\} \) are bounded and \( (\hat{u}_n, \hat{v}_n) \to 0 \). Therefore, up to a subsequence, \( t_n \to t_0 \geq 0 \). Indeed \( t_0 > 0 \). Otherwise, if \( t_0 = 0 \), from the boundedness of \( \{(\hat{u}_n, \hat{v}_n)\} \), we get \((\hat{u}_n, \hat{v}_n) = t_n(\tilde{u}_n, \tilde{v}_n) \to (0, 0) \), that is \( \mathcal{J}_0(\hat{u}_n, \hat{v}_n) \to 0 \) in contrast with \( c_0 > 0 \). Thus \( t_0 > 0 \) and up to a subsequence we have \((\hat{u}_n, \hat{v}_n) \to t_0(\hat{\hat{u}}, \hat{\hat{v}}) = (\hat{u}, \hat{v}) \) weakly in \( \mathbb{H}_0 \). Hence it holds

\[
\mathcal{J}_0(\hat{u}_n, \hat{v}_n) \to c_0 \quad \text{and} \quad (\hat{u}_n, \hat{v}_n) \to (\hat{u}, \hat{v}) \quad \text{weakly in} \quad \mathbb{H}_0.
\]

From Theorem 3.1 we deduce that \((\hat{u}_n, \hat{v}_n) \to (\hat{u}, \hat{v}) \) in \( \mathbb{H}_0 \), that is \((\tilde{u}_n, \tilde{v}_n) \to (\hat{\hat{u}}, \hat{\hat{v}}) \) in \( \mathbb{H}_0 \).

Now we show that \( \{y_n\} \) has a subsequence such that \( y_n \to y \in M \). Assume by contradiction that \( \{y_n\} \) is not bounded, that is there exists a subsequence, still denoted by \( \{y_n\} \), such that \( |y_n| \to +\infty \).

Firstly, we deal with the case \( \max\{V_\infty, W_\infty\} = \infty \).

Since \((u_n, v_n) \in \mathcal{N}_{\varepsilon_n} \) we can see that

\[
q \int_{\mathbb{R}^N} Q(\tilde{u}_n, \tilde{v}_n) \, dx \geq \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|\tilde{u}_n|^2 \, dx + \int_{\mathbb{R}^N} W(\varepsilon_n x + y_n)|\tilde{v}_n|^2 \, dx.
\]

Applying Fatou’s Lemma, we deduce that

\[
\liminf_{n \to \infty} \int_{\mathbb{R}^N} Q(\tilde{u}_n, \tilde{v}_n) \, dx = \infty,
\]

which is impossible because the boundedness of \( \{(u_n, v_n)\} \) and (6) yield

\[
\left| \int_{\mathbb{R}^N} Q(\tilde{u}_n, \tilde{v}_n) \, dx \right| \leq C \quad \text{for any} \quad n \in \mathbb{N}.
\]

Let us consider the case \( \max\{V_\infty, W_\infty\} < \infty \).

Since \((\tilde{u}_n, \tilde{v}_n) \to (\tilde{u}, \tilde{v}) \) strongly in \( \mathbb{H}_0 \) and \( V_0 < \max\{V_\infty, W_\infty\} \), we have

\[
c_0 = \mathcal{J}_0(\hat{\hat{u}}, \hat{\hat{v}}) < \mathcal{J}_\infty(\hat{\hat{u}}, \hat{\hat{v}})
\]

\[
\leq \liminf_{n \to \infty} \left\{ \frac{1}{2} \left( \int_{\mathbb{R}^N} |(-\Delta)^{s/2} \tilde{u}_n|^2 + |(-\Delta)^{s/2} \tilde{v}_n|^2 \, dx \right) - \int_{\mathbb{R}^N} Q(\tilde{u}_n, \tilde{v}_n) \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^N} (V(\varepsilon_n x + y_n)|\tilde{u}_n|^2 + W(\varepsilon_n x + y_n)|\tilde{v}_n|^2) \, dx \right\}
\]

\[
= \liminf_{n \to \infty} \mathcal{J}_{\varepsilon_n}(t_n u_n, t_n v_n) \leq \liminf_{n \to \infty} \mathcal{J}_{\varepsilon_n}(u_n, v_n) = c_0
\]

which leads to a contradiction.
Thus \( \{y_n\} \) is bounded and, up to a subsequence, we may assume that \( y_n \to y \). If \( y \notin M \) then \( V_0 < \max\{V(y), W(y)\} \) and we have

\[
c_0 = J_0(\hat{u}, \hat{v}) < \frac{1}{2} \left( \int_{\mathbb{R}^N} |(-\Delta)^{s/2} \hat{u}|^2 + |(-\Delta)^{s/2} \hat{v}|^2 \, dx \right) + \frac{1}{2} \int_{\mathbb{R}^N} (V(y)|\hat{u}|^2 + W(y)|\hat{v}|^2) \, dx - \int_{\mathbb{R}^N} Q(\hat{u}, \hat{v}) \, dx.
\]

Repeating the same argument developed in (26), we get a contradiction. Therefore we can conclude that \( y \in M \).

For any \( \delta > 0 \) we set

\[
M_\delta = \{ x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta \}.
\]

Let \( (w_1, w_2) \in \mathbb{H}_0 \) be a solution for (12) (which there exists in view of Theorem 3.1), and, for each \( z \in M \), we define

\[
\Psi_{i, \varepsilon, z}(x) = \eta(|\varepsilon x - z|)w_i \left( \frac{\varepsilon x - z}{\varepsilon} \right) \quad i = 1, 2.
\]

where \( \eta \in C_0^\infty(\mathbb{R}_+, [0, 1]) \) is a non-increasing function satisfying \( \eta(t) = 1 \) if \( 0 \leq t \leq \delta/2 \) and \( \eta(t) = 0 \) if \( t \geq \delta \).

Let \( t_\varepsilon > 0 \) be the unique positive number such that

\[
\max_{t \geq 0} J_\varepsilon(t \Psi_{1, \varepsilon, z}, t \Psi_{1, \varepsilon, z}) = J_\varepsilon(t_\varepsilon \Psi_{2, \varepsilon, z}, t_\varepsilon \Psi_{1, \varepsilon, z}).
\]

Finally, we consider \( \Phi_\varepsilon(z) = (t_\varepsilon \Psi_{1, \varepsilon, z}, t_\varepsilon \Psi_{2, \varepsilon, z}) \). Since \( J_0(w_1, w_2) = c_0 \) and \( M \) is compact, we can prove the following result.

**Lemma 5.2:** The functional \( \Phi_\varepsilon \) satisfies the limit

\[
\lim_{\varepsilon \to 0} J_\varepsilon(\Phi_\varepsilon(y)) = c_0 \quad \text{uniformly in } y \in M. 
\]

**Proof:** Assume by contradiction that there exist \( \delta_0 > 0 \), \( \{y_n\} \subset M \) and \( \varepsilon_n \to 0 \) such that

\[
|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_0| \geq \delta_0.
\]

We first show that \( \lim_{n \to \infty} t_{\varepsilon_n} < \infty \). Let us observe that using the change of variable \( z = (\varepsilon_n x - y_n)/\varepsilon_n \), if \( z \in B_\delta/\varepsilon_n(0) \), it follows that \( \varepsilon_n z \in B_\delta(0) \) and \( \varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \).

Then we have

\[
J_\varepsilon(\Phi_{\varepsilon_n}(y_n)) = \frac{t_{\varepsilon_n}^2}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(\eta(|\varepsilon_n z|)w_1(z))|^2 \, dz
\]

\[
+ \frac{t_{\varepsilon_n}^2}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(\eta(|\varepsilon_n z|)w_2(z))|^2 \, dz
\]

\[
+ \frac{t_{\varepsilon_n}^2}{2} \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z|)w_1(z))^2 \, dz
\]
\[ + \frac{t_{\varepsilon_n}^2}{2} \int_{\mathbb{R}^N} W(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z|)w_2(z))^2 \, dz \]
\[ - \int_{\mathbb{R}^N} Q(t_{\varepsilon_n} \eta(|\varepsilon_n z|)w_1(z), t_{\varepsilon_n} \eta(|\varepsilon_n z|)w_2(z)) \, dz. \quad (29) \]

Now assume that \( t_{\varepsilon_n} \to \infty \). By the definition of \( t_{\varepsilon_n} \), (Q1) and (5) we get

\[ \| (\Psi_{1,\varepsilon_n} y_n, \Psi_{2,\varepsilon_n} y_n) \|_{\varepsilon_n}^2 = q t_{\varepsilon_n}^{-2} \int_{\mathbb{R}^N} Q(\eta(|\varepsilon_n z|)w_1(z), \eta(|\varepsilon_n z|)w_2(z)) \, dz. \quad (30) \]

Since \( \eta = 1 \) in \( B_{\delta/2}(0) \) and \( B_{\delta/2}(0) \subset B_{\delta/2\varepsilon_n}(0) \) for \( n \) big enough, and \( w_1, w_2 \) are continuous and positive in \( \mathbb{R}^N \) (see proof of Theorem 3.1) we obtain

\[ \| (\Psi_{1,\varepsilon_n} y_n, \Psi_{2,\varepsilon_n} y_n) \|_{\varepsilon_n}^2 \geq q t_{\varepsilon_n}^{-2} \int_{B_{\delta/2}(0)} Q(w_1(z), w_2(z)) \, dz \geq C_{\delta,q} t_{\varepsilon_n}^{-2}, \quad (31) \]

where \( C_{\delta,q} = q(\delta/2)^N \omega_N \min_{z \in B_{\delta/2}(0)} Q(w_1(z), w_2(z)) > 0 \). Taking the limit as \( n \to \infty \) in (31) we can deduce that

\[ \lim_{n \to \infty} \| (\Psi_{1,\varepsilon_n} y_n, \Psi_{2,\varepsilon_n} y_n) \|_{\varepsilon_n}^2 = \infty, \]

which is a contradiction because of

\[ \lim_{n \to \infty} \| (\Psi_{1,\varepsilon_n} y_n, \Psi_{2,\varepsilon_n} y_n) \|_{\varepsilon_n}^2 = \| (w_1, w_2) \|_0^2 \in (0, \infty) \]

in view of the Dominated Convergence Theorem and Lemma 5 in [40].

Thus \( \{ t_{\varepsilon_n} \} \) is bounded, and we can assume that \( t_{\varepsilon_n} \to t_0 \geq 0 \). Clearly, if \( t_0 = 0 \), by limitation of \( \| (\Psi_{1,\varepsilon_n} y_n, \Psi_{2,\varepsilon_n} y_n) \|_{\varepsilon_n}^2 \), the growth assumptions on \( Q \), and (30), we can deduce that \( \| (\Psi_{1,\varepsilon_n} y_n, \Psi_{2,\varepsilon_n} y_n) \|_{\varepsilon_n}^2 \to 0 \) which is impossible. Hence \( t_0 > 0 \).

Now, invoking the Dominated Convergence Theorem, we can see that as \( n \to \infty \)

\[ \int_{\mathbb{R}^N} Q(\Psi_{1,\varepsilon_n} y_n, \Psi_{2,\varepsilon_n} y_n) \, dx \to \int_{\mathbb{R}^N} Q(w_1, w_2) \, dx. \]

Then, taking the limit as \( n \to \infty \) in (30) we obtain

\[ \| (w_1, w_2) \|_0^2 = q t_0^{-2} \int_{\mathbb{R}^N} Q(w_1, w_2) \, dx. \]

Using the fact that \( (w_1, w_2) \in \mathcal{N}_0 \) we deduce that \( t_0 = 1 \). Moreover, from (29) we have

\[ \lim_{n \to \infty} \mathcal{J}_\varepsilon (\Phi_{\varepsilon_n}(y_n)) = \mathcal{J}_0(w_1, w_2) = c_0, \]

which is impossible thanks to (28).
Now we are in the position to define the barycenter map. We take $\rho > 0$ such that $M_\delta \subset B_\rho$, and we consider $\Upsilon: \mathbb{R}^N \rightarrow \mathbb{R}^N$ given by

$$\Upsilon(x) = \begin{cases} x & \text{if } |x| < \rho \\ \rho x/|x| & \text{if } |x| \geq \rho. \end{cases}$$

We define the barycenter map $\beta_\varepsilon: \mathcal{N}_\varepsilon \rightarrow \mathbb{R}^N$ as

$$\beta_\varepsilon(u,v) = \frac{\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)(u^2(x) + v^2(x)) \, dx}{\int_{\mathbb{R}^N} u^2(x) + v^2(x) \, dx}.$$ 

**Lemma 5.3:** The functional $\Phi_\varepsilon$ satisfies the limit

$$\lim_{\varepsilon \to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \text{uniformly in } y \in M. \quad (32)$$

**Proof:** Suppose by contradiction that there exist $\delta_0 > 0, \{y_n\} \subset M$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0. \quad (33)$$

Using the definitions of $\Phi_{\varepsilon_n}(y_n), \beta_{\varepsilon_n}, \eta$ and the change of variable $z = (\varepsilon_n x - y_n)/\varepsilon_n$, we can see that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N} \Upsilon(\varepsilon_n z + y_n) - y_n] \eta(|\varepsilon_n z|)|w_1(z)|^2 + |w_2(z)|^2 \, dx}{\int_{\mathbb{R}^N} \eta(|\varepsilon_n z|)|w_1(z)|^2 + |w_2(z)|^2 \, dx}.$$ 

Taking into account $\{y_n\} \subset M \subset B_\rho$ and the Dominated Convergence Theorem we can infer that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1)$$

which contradicts (33). \qed

At this point, we introduce a subset $\tilde{\mathcal{N}}_\varepsilon$ of $\mathcal{N}_\varepsilon$ by taking a function $h: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$, and setting

$$\tilde{\mathcal{N}}_\varepsilon = \{(u,v) \in \mathcal{N}_\varepsilon : \mathcal{J}_\varepsilon(u) \leq c_0 + h(\varepsilon)\}.$$ 

For fixed $y \in M$, we conclude from Lemma 5.2 that $h(\varepsilon) = |\mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) - c_0| \to 0$ as $\varepsilon \to 0$. Hence $\Phi_\varepsilon(y) \in \tilde{\mathcal{N}}_\varepsilon$, and $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$. Moreover, we have the following lemma.

**Lemma 5.4:**

$$\lim_{\varepsilon \to 0} \sup_{(u,v) \in \tilde{\mathcal{N}}_\varepsilon} \text{dist}(\beta_\varepsilon(u,v), M_\delta) = 0.$$
Proof: Let \( \varepsilon_n \to 0 \) as \( n \to \infty \). For any \( n \in \mathbb{N} \) there exists \( (u_n, v_n) \in \mathring{N}_{\varepsilon_n} \) such that
\[
\sup_{(u, v) \in \mathring{N}_{\varepsilon_n}} \inf_{y \in M_0} |\beta_{\varepsilon_n}(u, v) - y| = \inf_{y \in M_0} |\beta_{\varepsilon_n}(u_n, v_n) - y| + o_n(1).
\]
Therefore it suffices to prove that there exists \( \{y_n\} \subset M_\delta \) such that
\[
\lim_{n \to \infty} |\beta_{\varepsilon_n}(u_n, v_n) - y_n| = 0.
\] (34)

We note that \( \{(u_n, v_n)\} \subset \mathring{N}_{\varepsilon_n} \subset N_{\varepsilon_n} \) from which we deduce that
\[
c_0 \leq c_{\varepsilon_n} \leq J_{\varepsilon_n}(u_n, v_n) \leq c_0 + h(\varepsilon_n).
\]
This yields \( J_{\varepsilon_n}(u_n, v_n) \to c_0 \). By Lemma 5.1 there exists \( \{\tilde{y}_n\} \subset \mathbb{R}^N \) such that \( y_n = \varepsilon_n \tilde{y}_n \in M_\delta \) for \( n \) sufficiently large. By setting \( (\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(\cdot + \tilde{y}_n), v_n(\cdot + \tilde{y}_n)) \) we can see that
\[
\beta_{\varepsilon_n}(u_n, v_n) = y_n + \frac{\int_{\mathbb{R}^N} [\varphi(\varepsilon_n z + y_n) - y_n] (\tilde{u}_n^2 + \tilde{v}_n^2) \, dz}{\int_{\mathbb{R}^N} (\tilde{u}_n^2 + \tilde{v}_n^2) \, dz}.
\]
Since \( (\tilde{u}_n, \tilde{v}_n) \to (u, v) \) in \( H_0^2 \) and \( \varepsilon_n z + y_n \to y \in M \), we deduce that \( \beta_{\varepsilon_n}(u_n, v_n) = y_n + o_n(1) \) that is (34) holds.

Now, we are ready to provide the proof of the first multiplicity result related to (7).

Proof of Theorem 1.1: Given \( \delta > 0 \) we can apply Lemmas 5.2, 5.3 and 5.4 to find some \( \varepsilon_\delta > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_\delta) \), the diagram
\[
M \xrightarrow{\Phi_\varepsilon} \mathring{N}_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta
\]
is well-defined and \( \beta_\varepsilon \circ \Phi_\varepsilon \) is homotopically equivalent to the embedding \( \iota : M \to M_\delta \). By the definition of \( \mathring{N}_\varepsilon \) and taking \( \varepsilon_\delta \) sufficiently small, we may assume that \( J_\varepsilon \) satisfies the Palais–Smale condition in \( \mathring{N}_\varepsilon \). Therefore, standard Ljusternik–Schnirelmann theory [34] provides at least \( \text{cat}_{\mathring{N}_\varepsilon}(\mathring{N}_\varepsilon) \) critical points \( (u_i, v_i) \) of \( J_\varepsilon \) restricted to \( \mathring{N}_\varepsilon \). Using the arguments in [35] we can see that \( \text{cat}_{\mathring{N}_\varepsilon}(\mathring{N}_\varepsilon) \geq \text{cat}_{M_\delta}(M) \). From Corollary 4.1 and the arguments contained in the proof of Theorem 3.1 we can conclude that \( u_i > 0, v_i > 0 \) and \( (u_i, v_i) \) is a solution to (10).

6. Proof of Theorem 1.2

In this last section we deal with the nonlocal system in the critical case. As in Section 3, we consider the autonomous critical system
\[
\begin{align*}
(-\Delta)^s u + V_0 u &= Q_0(u, v) + \frac{2\alpha}{\alpha + \beta} |u|^{\alpha-2} u |v| \beta \quad \text{in} \ \mathbb{R}^N \\
(-\Delta)^s v + W_0 v &= Q_0(u, v) + \frac{2\beta}{\alpha + \beta} |u|^{\alpha} |v|^{\beta-2} v \quad \text{in} \ \mathbb{R}^N \\
u, v &> 0
\end{align*}
\] (35)
and define the energy functional
\[ J_0(u,v) = \frac{1}{2} \| (u,v) \|_0^2 - \int_{\mathbb{R}^N} Q(u,v) \, dx - \frac{2}{\alpha + \beta} \int_{\mathbb{R}^N} (u^+)^\alpha (v^+)^\beta \, dx, \]
and its ground state level
\[ m_0 = \inf_{(u,v) \in N_0} J_0(u,v) = \inf_{(u,v) \in X_0 \setminus \{(0,0)\}} \max_{t \geq 0} J_0(tu, tv) > 0. \]

Now, we denote
\[ \tilde{S}_\ast = \tilde{S}_\ast(\alpha, \beta) = \inf_{u,v \in H^s(\mathbb{R}^N) \setminus \{(0,0)\}} \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} u \right|^2 + \left| (-\Delta)^{s/2} v \right|^2 \, dx \left( \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta} \, dx \right)^{2/2_s}. \]

In the next lemma, we prove an interesting relation between \( S_\ast \) and \( \tilde{S}_\ast \).

**Lemma 6.1:** It holds
\[ \tilde{S}_\ast = S_\ast \left[ \left( \frac{\alpha}{\beta} \right)^{\beta/2_s} + \left( \frac{\beta}{\alpha} \right)^{\alpha/2_s} \right]. \]

Moreover, if \( w \) realizes \( S_\ast \), then \( (Aw, Bw) \) realizes \( \tilde{S}_\ast \) where \( A \) and \( B \) are such that \( A/B = \sqrt{\alpha/\beta} \).

**Proof:** Let \( \{ w_n \} \) be a minimizing sequence for \( S_\ast \). Let \( p \) and \( q \) two positive numbers which will be chosen later. Taking \( u_n = pw_n \) and \( v_n = qw_n \) in the quotient (36), we have
\[ \frac{p^2 + q^2}{(p^\alpha q^\beta)^{2/2_s}} \left( \int_{\mathbb{R}^N} |w_n|^{2_*} \, dx \right)^{2/2_s} \geq \tilde{S}_\ast. \]  

We note that
\[ \frac{p^2 + q^2}{(p^\alpha q^\beta)^{2/2_s}} = \left( \frac{p}{q} \right)^{2\beta/2_*} + \left( \frac{p}{q} \right)^{-2\alpha/2_*}, \]
and consider the function \( g : \mathbb{R}_+ \to \mathbb{R} \) defined as
\[ g(t) = t^{2\beta/2_*} + t^{-2\alpha/2_*}. \]

Then it is easy to verify that \( g \) achieves its minimum at the point \( t = \sqrt{\alpha/\beta} \) and in particular
\[ g \left( \sqrt{\frac{\alpha}{\beta}} \right) = \left( \frac{\alpha}{\beta} \right)^{\beta/2_*} + \left( \frac{\beta}{\alpha} \right)^{\alpha/2_*}. \]

Taking \( p \) and \( q \) in (37) such that \( p/q = \sqrt{\alpha/\beta} \) we get
\[ \left[ \left( \frac{\alpha}{\beta} \right)^{\beta/2_*} + \left( \frac{\beta}{\alpha} \right)^{\alpha/2_*} \right] \left( \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} w_n \right|^2 \, dx \right)^{2/2_s} \geq \tilde{S}_\ast.
which gives

\[
\left[ \left( \frac{\alpha}{\beta} \right)^{\beta/2_s^*} + \left( \frac{\beta}{\alpha} \right)^{\alpha/2_s^*} \right] S_s \geq \tilde{S}_s. \tag{40}
\]

Now, in order to conclude the proof, we consider a minimizing sequence \{(u_n, v_n)\} for \(\tilde{S}_s\). Let us define \(z_n = p_n v_n\), where \(p_n > 0\) is such that

\[
\int_{\mathbb{R}^N} |u_n|^2 \, dx = \int_{\mathbb{R}^N} |z_n|^2 \, dx. \tag{41}
\]

Using Young’s inequality and (41) we can see that

\[
\int_{\mathbb{R}^N} |u_n|^\alpha |z_n|^\beta \, dx \leq \frac{\alpha}{2_s^*} \int_{\mathbb{R}^N} |u_n|^\alpha + \beta \, dx + \frac{\beta}{2_s^*} \int_{\mathbb{R}^N} |z_n|^\alpha + \beta \, dx
\]

\[
= \int_{\mathbb{R}^N} |u_n|^{2_s^*} \, dx = \int_{\mathbb{R}^N} |z_n|^{2_s^*} \, dx. \tag{42}
\]

Therefore, by (39), (42) and \(\alpha + \beta = 2_s^*\) we can deduce that

\[
\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 + |(-\Delta)^{s/2} v_n|^2 \, dx \leq \frac{p_n^{2\beta/2_s^*}}{(\int_{\mathbb{R}^N} |u_n|^\alpha |z_n|^\beta \, dx)^{2/2_s^*}} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 \, dx
\]

\[
= \frac{p_n^{2\beta/2_s^*}}{(\int_{\mathbb{R}^N} |u_n|^{2_s^*} \, dx)^{2/2_s^*}} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 \, dx
\]

\[
+ \frac{p_n^{2\beta/2_s^*}}{p_n^{2\beta/2_s^*}} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} z_n|^2 \, dx
\]

\[
\geq S_s \left( p_n^{2\beta/2_s^*} + \frac{p_n^{2\beta/2_s^*}}{p_n^{2\beta/2_s^*}} \right) = S_s g(p_n)
\]

\[
\geq S_s \left( \frac{\alpha}{\beta} \right)^{\beta/2_s^*} + \left( \frac{\beta}{\alpha} \right)^{\alpha/2_s^*} \right] .
\]

The end of the proof is obtained by passing to the limit in the above inequality.

Next, we prove the ‘critical version’ of Lemma 3.1.

\textbf{Lemma 6.2:} Let \{(u_n, v_n)\} \subset \mathbb{H}_0 be a Palais–Smale sequence for \(J_0\) at the level \(d < \frac{2s}{N} \left( \frac{\tilde{S}_s}{2} \right)^{\frac{N}{2s}}\) and \((u_n, v_n) \rightharpoonup (0, 0)\). Then, one of the following conclusions holds:

(i) \(\| (u_n, v_n) \|_0 \to 0\), or

(ii) there exist a sequence \((y_n) \subset \mathbb{R}^N\) and constants \(R, \gamma > 0\) such that

\[
\liminf_{n \to \infty} \int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) \, dx \geq \gamma.
\]
Proof: Assume that (ii) does not hold. Then, for any $R > 0$, we get

$$
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 \, dx = 0 = \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |v_n|^2 \, dx.
$$

Using Lemma 2.1 it follows that

$$
u_n, v_n \to 0 \text{ in } L^r(\mathbb{R}^N) \quad \forall \ r \in (2, 2^*_s),$$

and in view of (6) we can see that $\int_{\mathbb{R}^N} Q(u_n, v_n) \, dx \to 0$.

Since $\{(u_n, v_n)\}$ is bounded we have $\langle J'_0(u_n, v_n), (u_n, v_n) \rangle \to 0$. Then we obtain

$$
\| (u_n, v_n) \|^2_0 - 2 \int_{\mathbb{R}^N} (u_n^+)^\alpha (v_n^+)^\beta \, dx = o_n(1),
$$

which implies that there exists $L \geq 0$ such that

$$
\| (u_n, v_n) \|^2_0 \to L \quad \text{and} \quad \int_{\mathbb{R}^N} (u_n^+)^\alpha (v_n^+)^\beta \, dx \to \frac{L}{2}. \quad (43)
$$

Since $J_0(u_n, v_n) \to d$ we can use (43) to deduce that $d = \frac{Ls}{N}$. By the definition of $\tilde{S}_s$ we get

$$
\| (u_n, v_n) \|^2_0 \geq \tilde{S}_s \left( \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \, dx \right)^{2/2^*_s} \geq \tilde{S}_s \left( \int_{\mathbb{R}^N} (u_n^+)^\alpha (v_n^+)^\beta \, dx \right)^{2/2^*_s},
$$

which gives $L \geq \tilde{S}_s \left( \frac{L}{2} \right)^{2^*_s}$. Now, if $L > 0$ we obtain $Nd = sL \geq 2s \left( \frac{\tilde{S}_s}{2} \right)^{\frac{N}{2s}}$ which provides a contradiction. Thus $L = 0$ and (i) holds true. 

Now we prove that the critical autonomous system admits a nontrivial solution.

Theorem 6.1: The problem (35) has a weak solution.

Proof: Since $J_0$ has a Mountain Pass geometry, there exists $\{(u_n, v_n)\} \subset \mathbb{H}_0$ such that

$$
J_0(u_n, v_n) \to m_0 \quad \text{and} \quad J'_0(u_n, v_n) \to 0.
$$

We aim to show that

$$
m_0 < \frac{2s}{N} \left( \frac{\tilde{S}_s}{2} \right)^{\frac{N}{2s}}. \quad (44)
$$

Indeed, once proved (44), we can repeat the same arguments developed in the proof of Theorem 3.1 and applying Lemma 6.2 instead of Lemma 3.1, we deduce the existence of a weak solution to (35). By the definition of $m_0$ it is enough to prove that there exists $(u, v) \in \mathbb{H}_0$. 

\[\]
\( \mathbb{H}_0 \) such that
\[
\max_{t \geq 0} J_0(tu, tv) < \frac{2s}{N} \left( \frac{S_*}{2} \right)^{\frac{N}{2s}}.
\]

Let \( A, B > 0 \) such that \( A/B = \sqrt{\alpha/\beta} \). Then, in view of Lemma 6.1 we can deduce that
\[
\tilde{S}_* = S_* \frac{(A^2 + B^2)}{(A^\alpha B^\beta)^{2/2s}}.
\]

Fix \( \eta \in C_0^\infty(\mathbb{R}^N) \) a cut-off function such that \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) on \( B_r \) and \( \eta = 0 \) on \( \mathbb{R}^N \setminus B_{2r} \), where \( B_r \) denotes the ball in \( \mathbb{R}^N \) of center at origin and radius \( r \).

For \( \epsilon > 0 \) let us define \( v_\epsilon(x) = \eta(x)z_\epsilon(x) \), where
\[
z_\epsilon(x) = \frac{\kappa \epsilon^{(N-2s)/2}}{\epsilon^2 + |x|^2} \left( N - 2s \right)^{2/2s}
\]
is a solution to
\[
(-\Delta)^s u = S_* |u|^{2s-2} u \quad \text{in } \mathbb{R}^N,
\]
and \( \kappa \) is a suitable positive constant depending only on \( N \) and \( s \).

Now we set
\[
u_\epsilon = \frac{z_\epsilon}{\left( \int_{\mathbb{R}^N} |z_\epsilon|^{2s} \, dx \right)^{1/2s}}.
\]

By performing similar calculations to those in [36] (see Propositions 21 and 22), we can see that
\[
\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_\epsilon|^2 \, dx \leq S_* + O(\epsilon^{N-2s}),
\]
and
\[
\int_{\mathbb{R}^N} |u_\epsilon|^2 \, dx = \begin{cases}
O(\epsilon^{2s}) & \text{if } N > 4s \\
O(\epsilon^{2s} |\log(\epsilon)|) & \text{if } N = 4s \\
O(\epsilon^{N-2s}) & \text{if } N < 4s,
\end{cases}
\]
and
\[
\int_{\mathbb{R}^N} |u_\epsilon|^q \, dx = \begin{cases}
O(\epsilon^{(2N-(N-2s)q)/2}) & \text{if } q > \frac{N}{N-2s} \\
O(\frac{N}{N-2s}) & \text{if } q = \frac{N}{N-2s} \\
O(\epsilon^{(N-2s)q/2}) & \text{if } q < \frac{N}{N-2s}.
\end{cases}
\]

Thus, by (Q6), we can note that
\[
J_0(tAu_\epsilon, tBu_\epsilon) \leq \left[ \frac{t^2}{2} (A^2 + B^2)D_* - \frac{2t^{2s}}{2s} A^\alpha B^\beta \right] - \lambda \epsilon^q A^{q1} B^{q1} \int_{\mathbb{R}^N} |u_\epsilon|^{q1} \, dx,
\]
where

\[ h_\varepsilon(t) := \frac{t^2}{2} (A^2 + B^2)D_\varepsilon - \frac{2t^{2s}}{2s} A^\alpha B^\beta, \]

and

\[ D_\varepsilon = \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_\varepsilon|^2 \, dx + \int_{\mathbb{R}^N} \max\{V_0, W_0\} u_\varepsilon^2 \, dx. \]

Let us denote by \( t_\varepsilon > 0 \) be the maximum point of \( h_\varepsilon(t) \). Since \( h_\varepsilon'(t_\varepsilon) = 0 \) we have

\[ t_\varepsilon = \left( \frac{D_\varepsilon (A^2 + B^2)}{2(A^\alpha B^\beta)^{2/2s}} \right)^{(N-2s)/4s} \geq t_\varepsilon > 0. \]

Using the fact that \( h_\varepsilon(t) \) is increasing in \((0, \bar{t}_\varepsilon)\), we can see that

\[ \mathcal{J}_\varepsilon(t Au_\varepsilon, t Bu_\varepsilon) \leq \frac{2s}{N} \left( \frac{D_\varepsilon (A^2 + B^2)}{2(A^\alpha B^\beta)^{2/2s}} \right)^{N/2s} - \lambda t^{q_1} A^{q_1} B^{q_1} \int_{\mathbb{R}^N} |u_\varepsilon|^{q_1} \, dx. \]

Now, recalling that \((a + b)^r \leq a^r + r(a + b)^{-1} b\) for any \( a, b > 0 \) and \( r \geq 1 \), we can obtain that

\[ D_\varepsilon^{N/2s} \leq \tilde{S}_\varepsilon^{N/2s} + O(\varepsilon^{N-2s}) + C_1 \int_{\mathbb{R}^N} |u_\varepsilon|^2 \, dx, \]

On the other hand \( h_\varepsilon'(t_\varepsilon) = 0 \) and the Mountain Pass geometry of \( \mathcal{J}_\varepsilon \) imply that there exists \( \sigma > 0 \) such that

\[ t_\varepsilon \geq \sigma \quad \text{for any} \quad \varepsilon > 0, \]

that is \( t_\varepsilon \) can be estimated from below by a constant independent of \( \varepsilon \).

Then we have

\[ \mathcal{J}_\varepsilon(t Au_\varepsilon, t Bu_\varepsilon) \leq \frac{2s}{N} \left( \frac{\tilde{S}_\varepsilon}{2} \right)^{N/2s} + O(\varepsilon^{N-2s}) + C_2 \int_{\mathbb{R}^N} |u_\varepsilon|^2 \, dx - \lambda C_3 \int_{\mathbb{R}^N} |u_\varepsilon|^{q_1} \, dx, \]

where \( C_2, C_3 > 0 \) are independent of \( \varepsilon \) and \( \lambda \).

Now we distinguish the following cases:

If \( N > 4s \) then \( q_1 > N/(N-2s) \). Hence, by (46) and (47), we can see that

\[ \sup_{t \geq 0} h_\varepsilon(t) \leq \frac{2s}{N} \left( \frac{\tilde{S}_\varepsilon}{2} \right)^{N/2s} + O(\varepsilon^{N-2s}) + O(\varepsilon^{2s}) - \lambda O(\varepsilon^{(2N-(N-2s)q_1)/2}). \]

Taking into account \((2N-(N-2s)q_1)/2 < 2s < N-2s\) we get the thesis for \( \varepsilon \) small enough.

When \( N = 4s \) then \( q_1 \in (2, 4) \) and in particular \( q_1 > N/(N-2s) = 2 \), so from (46) and (47) we deduce that

\[ \sup_{t \geq 0} h_\varepsilon(t) \leq \frac{2s}{N} \left( \frac{\tilde{S}_\varepsilon}{2} \right)^{N/2s} + O(\varepsilon^{2s}) + O(\varepsilon^{2s} |\log(\varepsilon)|) - \lambda O(\varepsilon^{4s-sq_1}) \]

which implies (11) because of \( \lim_{\varepsilon \to 0} (\varepsilon^{4s-sq_1}/\varepsilon^{2s}(1 + |\log(\varepsilon)|)) = \infty \).
If $2s < N < 4s$ and $q_1 \in (4s/(N - 2s), 2^s)$ then $q_1 > N/(N - 2s)$. Therefore we have

$$
\sup_{t \geq 0} h_\varepsilon(t) \leq \frac{2s}{N} \left( \frac{\tilde{S}_*}{2} \right)^{\frac{N}{2s}} + O(\varepsilon^{N - 2s}) + O(\varepsilon^{N - 2s}) - \lambda O(\varepsilon^{(2N - (N - 2s)q_1)/2})
$$

and we obtain the conclusion for $\varepsilon$ sufficiently small in light of $(2N - (N - 2s)q_1)/2 < N - 2s$.

If $2s < N < 4s$ and $q_1 \in (2, 4s/(N - 2s)]$, we argue as before and using (47) we get

$$
\sup_{t \geq 0} h_\varepsilon(t) \leq \begin{cases}
\frac{2s}{N} \left( \frac{\tilde{S}_*}{2} \right)^{\frac{N}{2s}} + O(\varepsilon^{N - 2s}) - \lambda O(\varepsilon^{(2N - (N - 2s)q_1)/2}) & \text{if } q_1 > \frac{N}{N - 2s} \\
\frac{2s}{N} \left( \frac{\tilde{S}_*}{2} \right)^{\frac{N}{2s}} + O(\varepsilon^{N - 2s}) - \lambda O(\varepsilon^{(N - 2s)q_1/2}) & \text{if } q_1 = \frac{N}{N - 2s} \\
\frac{2s}{N} \left( \frac{\tilde{S}_*}{2} \right)^{\frac{N}{2s}} + O(\varepsilon^{N - 2s}) - \lambda O(\varepsilon^{(N - 2s)q_1/2}) & \text{if } q_1 < \frac{N}{N - 2s}.
\end{cases}
$$

Then we can find $\lambda_0 > 0$ large enough such that for any $\lambda \geq \lambda_0$ and $\varepsilon > 0$ small it holds

$$
\sup_{t \geq 0} h_\varepsilon(t) < \frac{2s}{N} \left( \frac{\tilde{S}_*}{2} \right)^{\frac{N}{2s}}.
$$

Putting together the above estimates we can infer that for any $\varepsilon > 0$ sufficiently small

$$
\max_{t \geq 0} \mathcal{J}_0(t Au_\varepsilon, t Bu_\varepsilon) \leq \max_{t \geq 0} h_\varepsilon(t) = h_\varepsilon(t_\varepsilon) < \frac{2s}{N} \left( \frac{\tilde{S}_*}{2} \right)^{\frac{N}{2s}}.
$$

Since we are interested in weak solutions of (8), we consider the re-scaled system

$$
\begin{align*}
(-\Delta)^s u + V(\varepsilon x) u &= Q_u(u, v) + \frac{2\alpha}{\alpha + \beta} |u|^{\alpha - 2} u |v|^\beta & \text{in } \mathbb{R}^N \\
(-\Delta)^s v + W(\varepsilon x) v &= Q_v(u, v) + \frac{2\beta}{\alpha + \beta} |u|^{\alpha} |v|^{\beta - 2} v & \text{in } \mathbb{R}^N \\
u, v > 0
\end{align*}
$$

(48)

Thus, the corresponding functional $\mathcal{J}_\varepsilon : \mathbb{H}_\varepsilon \to \mathbb{R}$ is given by

$$
\mathcal{J}_\varepsilon(u, v) = \frac{1}{2} \|(u, v)\|_E^2 - \int_{\mathbb{R}^N} Q(u, v) \, dx - \frac{2}{\alpha + \beta} \int_{\mathbb{R}^N} (u_+)^\alpha (v_+)^\beta \, dx.
$$

Clearly, the critical points of $\mathcal{J}_\varepsilon$ belong to the Nehari manifold

$$
\mathcal{M}_\varepsilon := \{(u, v) \in \mathbb{H}_\varepsilon \setminus \{(0, 0)\} : \langle \mathcal{J}_\varepsilon'(u, v), (u, v) \rangle = 0\},
$$

and the ground state level is given by

$$
m_\varepsilon := \inf_{(u, v) \in \mathcal{M}_\varepsilon} \mathcal{J}_\varepsilon(u, v) = \inf_{(u, v) \in \mathbb{H}_\varepsilon \setminus \{(0, 0)\}} \max_{t \geq 0} \mathcal{J}_\varepsilon(tu, tv) > 0.
$$

As in the previous sections, the Palais–Smale condition for the functional $\mathcal{J}_\varepsilon$ is related to $V_\infty$ and $W_\infty$. Then, as in Section 4, when $\max\{V_\infty, W_\infty\} < \infty$, we define the limit
functional $\mathcal{J}_\infty : \mathbb{H}_0 \to \mathbb{R}$ by setting

$$
\mathcal{J}_\infty (u, v) := \frac{1}{2} \left( \iint_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + |(-\Delta)^{s/2} v|^2 \, dx + \iint_{\mathbb{R}^N} (V_\infty u^2 + W_\infty v^2) \, dx \right)
- \int_{\mathbb{R}^N} Q(u, v) \, dx - \frac{2}{\alpha + \beta} \int_{\mathbb{R}^N} (u^+)^\alpha (v^+)^\beta \, dx,
$$

and its ground state level

$$
m_\infty := \inf_{(u, v) \in \mathbb{H}_0 \setminus \{(0, 0)\}} \max_{t \geq 0} \mathcal{J}_\infty (tu, tv) > 0.
$$

If $\max\{V_\infty, W_\infty\} = \infty$ we set $m_\infty := \infty$.

Since the map $(u, v) \mapsto \iint_{\mathbb{R}^N} (u^+)^\alpha (v^+)^\beta \, dx$ is positively $2^\ast$-homogeneous, the arguments developed in Section 4 permit to deduce a compactness result for the functional $\mathcal{J}_\varepsilon$. More precisely, following the lines of the proofs of Theorem 4.1 and Corollary 4.1, replacing Lemma 3.1 by Lemma 6.2, we can prove that the next result holds.

**Theorem 6.2:** The functional $\mathcal{J}_\varepsilon$ constrained to $\mathcal{M}_\varepsilon$ satisfies the $(PS)_d$-condition at any level $d < \min\{m_\infty, (s/N)S_\varepsilon^{N/2s}\}$. Moreover, critical points of $\mathcal{J}_\varepsilon$ constrained to $\mathcal{M}_\varepsilon$ are critical points of $\mathcal{J}_\varepsilon$ in $\mathbb{H}_\varepsilon$.

We conclude this section giving our second multiplicity result. Since many calculations made in Section 5 can be easily adapted in this context, we present only a sketch of the proof.

**Proof of Theorem 1.2:** We proceed as in the proof of Theorem 1.1. Fix $\delta > 0$ and choose $\eta \in C_0^\infty (\mathbb{R}, [0, 1])$ such that $\eta (t) = 1$ if $0 \leq t \leq \frac{\delta}{2}$ and $\eta (t) = 0$ if $t \geq \delta$. Let $(\tilde{w}_1, \tilde{w}_2) \in \mathbb{H}_0$ be the solution of (35) given by Theorem 6.1. For any $y \in M$, we define

$$
\tilde{\Psi}_{i,\varepsilon,y}(x) := \eta (|\varepsilon x - y|) \tilde{w}_i \left( \frac{\varepsilon x - y}{\varepsilon} \right), \quad i = 1, 2,
$$

and we introduce the map $\tilde{\Phi}_\varepsilon (y) := (t_1 \tilde{\Psi}_{1,\varepsilon,y}, t_2 \tilde{\Psi}_{2,\varepsilon,y})$, where $t_\varepsilon$ is the unique positive number satisfying

$$
\max_{t \geq 0} \mathcal{J}_\varepsilon (t \tilde{\Psi}_{1,\varepsilon,y}, t \tilde{\Psi}_{2,\varepsilon,y}) = \mathcal{J}_\varepsilon (t_\varepsilon \tilde{\Psi}_{1,\varepsilon,y}, t_\varepsilon \tilde{\Psi}_{2,\varepsilon,y}).
$$

As in Section 5, we can see that

$$
\lim_{\varepsilon \to 0^+} \mathcal{J}_\varepsilon (\tilde{\Phi}_\varepsilon (y)) = m_0 \quad \text{uniformly for } y \in M.
$$

Moreover, denoted by $\mathcal{Y} : \mathbb{R}^N \to \mathbb{R}^N$ the function defined in Section 4 we can define the barycenter map $\tilde{\beta}_\varepsilon : \mathcal{M}_\varepsilon \to \mathbb{R}^N$ given by

$$
\tilde{\beta}_\varepsilon (u, v) := \frac{\int_{\mathbb{R}^N} \mathcal{Y} (\varepsilon x) (|u(x)|^2 + |v(x)|^2) \, dx}{\int_{\mathbb{R}^N} (|u(x)|^2 + |v(x)|^2) \, dx}.
$$
Then it is easy to check that
\[
\lim_{\varepsilon \to 0^+} \tilde{\beta}_\varepsilon (\Phi_\varepsilon (y)) = y \text{ uniformly for } y \in M
\]
and
\[
\lim_{\varepsilon \to 0^+} \sup_{(u, v) \in \tilde{M}_\varepsilon} \text{dist}(\tilde{\beta}_\varepsilon (u, v), M_\delta) = 0,
\]
where
\[
\tilde{M}_\varepsilon := \{(u, v) \in M_\varepsilon : J_\varepsilon (u, v) \leq m_0 + \tilde{h}(\varepsilon)\}
\]
and \(\tilde{h} : [0, \infty) \to [0, \infty)\) satisfies \(\tilde{h}(\varepsilon) \to 0\) as \(\varepsilon \to 0^+\).

Consequently, there exists \(\varepsilon_\delta > 0\) such that for any \(\varepsilon \in (0, \varepsilon_\delta)\) the diagram
\[
M \xrightarrow{\Phi_\varepsilon} \tilde{M}_\varepsilon \xrightarrow{\tilde{\beta}_\varepsilon} M_\delta
\]
is well defined and \(\tilde{\beta}_\varepsilon \circ \Phi_\varepsilon\) is homotopically equivalent to the embedding \(\iota : M \to M_\delta\).

Therefore \(\text{cat}_{\tilde{M}_\varepsilon} (\tilde{M}_\varepsilon) \geq \text{cat}_{M_\delta} (M)\). From Theorem 6.2 and \(m_0 < s^{\frac{N}{2s}} \tilde{S}^N_{\infty}\), we may suppose that \(\varepsilon_\delta\) is so small that \(J_\varepsilon\) satisfies the Palais–Smale condition in \(\tilde{M}_\varepsilon\). Then the proof goes as in the subcritical case by using Ljusternik–Schnirelmann theory.

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