Optimal control of forward-backward mean-field stochastic delayed systems

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Dedicated to Professor Bernt Øksendal on the occasion of his 70th birthday

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Abstract

We study methods for solving stochastic control problems of systems of forward-backward mean-field equations with delay, in finite or infinite horizon. Necessary and sufficient maximum principles under partial information are given. The results are applied to solve a recursive utility optimal problem.

Keywords: Optimal control; Stochastic delay equation; Mean-field; Stochastic maximum principle; Hamiltonian; Advanced stochastic equation; Partial information.

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1 Introduction

Stochastic differential equations involving a large number of interacting particles can be approximated by mean-field stochastic differential equations (MFSDEs). Solutions of MFSDEs typically occur as a limit in law of an increasing number of identically distributed interacting processes, where the coefficient depends on an average of the corresponding processes. See e.g. [6]. Even more general MFSDEs with delay can be used to model brain activity in the sense of interactions between cortical columns (i.e. large populations of neurons). As an example in [27], they consider a model of the form

\[ dX(t,r) = f(t,x)dt + \int_{\Gamma} E[b(r,r',x,X(t-\tau(r,r'))(r'))]_{x=\lambda(t,r)} \lambda(t,r) dt + \sigma(r)dB(t,r) \]

Such equations are also used in systemic risk theory and other areas as it is mentioned in [17]. In [26], they consider a problem of stochastic optimal control

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of forward mean-field delayed equations, they derive necessary and sufficient maximum principles accordingly and by using continuous dependence theorems, they prove existence and uniqueness of MF-FSDDEs and MF-ABSDEs. We emphasize that our paper has similarities with [18] but in our case we include delay and jumps and also our type of mean-field equation is different from theirs.

2 Finite horizon stochastic mean-field optimal control problem

Consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ on which we define a standard Brownian motion $B(\cdot)$ and an independent compensated Poisson random measure $\tilde{N}$, such that $\tilde{N}(dt, de) := N(dt, de) - \nu(de)dt$, where $N(dt, de)$ is the jump measure, $\nu$ is the Lévy measure and $\nu(de)dt$ is the compensator of $N$. The information available to the controller may be less than the overall information.

Let $\delta > 0$. We want to control a process given by a following pair of FBSDEs with delay

$$dX(t) = b(t, X(t), \pi(t), \omega)dt + \sigma(t, X(t), \pi(t), \omega)dB(t) + \int_{\mathbb{R}_0} \gamma(t, X(t^-), e, \omega)\tilde{N}(dt, de), t \in [0, T],$$

$$dY(t) = -g(t, X(t), Y(t), Z(t), \pi(t), \omega)dt + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t, e, \omega)\tilde{N}(de, dt), t \in [0, T],$$

with initial condition $X(t) = X_0(t), t \in [-\delta, 0]$ and terminal condition $Y(T) = aX(T)$ which $a$ is a given constant in $\mathbb{R}_0$, where

$$X(t) := (1, ..., N) := \left( \int_{-\delta}^{0} X(t + s)\mu_1(ds), ..., \int_{-\delta}^{0} X(t + s)\mu_3(ds) \right)$$

for bounded Borel measures $\mu_i(ds), i = 1, 2, 3$ that are either Dirac measures or absolutely constant.

**Example 2.1.** Suppose $\mu := \mu_1$ and $N = 1$

1. If $\mu$ is the Dirac measure concentrated at 0, then $X(t) := X(t)$, and the state equation is a SDE.

2. If $\mu$ is the Dirac measure concentrated at $-\delta$, then $X(t) := X(t - \delta)$, and the state equation is a SDE with discrete delay.

3. If $\mu(ds) = e^{\lambda s}ds$, then $X(t) := \int_{-\delta}^{0} e^{\lambda s}X(s)ds$, and the state equation is a SDE with distributed delay.

Here

$$b = b(\omega, t, x, \pi) : \Omega \times [0, T] \times \mathbb{R}^3 \times \times U \rightarrow \mathbb{R},$$

$$\sigma = \sigma(\omega, t, x, \pi) : \Omega \times [0, T] \times \mathbb{R}^3 \times U \rightarrow \mathbb{R},$$

$$\gamma = \gamma(\omega, t, x, \pi, e) : \Omega \times [0, T] \times \mathbb{R}^3 \times U \times \mathbb{R}_0 \rightarrow \mathbb{R},$$

$$g = g(\omega, t, x, y, z, k(-), \pi) : \Omega \times [0, T] \times \mathbb{R}^3 \times \mathbb{R} \times L^2(\mathbb{R}_0) \times U \rightarrow \mathbb{R},$$

$$K = K(\omega, t, e) : \Omega \times [0, T] \times \mathbb{R}_0 \rightarrow \mathbb{R}.$$
Let $b, \sigma, g$ and $\gamma$ are given $\mathcal{F}_t$-measurable for all $x,y,z \in \mathbb{R}, u \in U$ where $U$ is a convex subset of $\mathbb{R}$ and $e \in \mathbb{R}_0$. The set $U$ consists of the admissible control values. The information available to the controller is given by a subfiltration $\mathcal{G} = \{\mathcal{G}_t\}_{t \geq 0}$ such that $\mathcal{G}_t \subseteq \mathcal{F}_t$. The set of admissible controls, i.e. the strategies available to the controller is given by a subset $A_G$ of the càdlàg, $U$-valued and $\mathcal{G}_t$-adapted processes in $L^2(\Omega \times [0,T])$.

**Assumption (I)**

i) The functions $b, \sigma, g$ and $\gamma$ are assumed to be $C^1$ (Fréchet) for each fixed $t, \omega$ and $e$.

ii) **Lipschitz condition**: The functions $b, \sigma$ and $g$ are Lipschitz continuous in the variables $x,y,z$, with the Lipschitz constant independent of the variables $t,u,w$. Also, there exits a function $L \in L^2(\nu)$ independent of $t,u,w$, such that
\[
|\gamma(t,x,u,e,\omega) - \gamma(t,x',u,e,\omega)| \leq L(e)|x - x'|. \tag{3}
\]

iii) **Linear growth**: The functions $b, \sigma, g$ and $\gamma$ satisfy the linear growth condition in the variables with the linear growth constant independent of the variables $t,u,w$. Also there exists a non-negative function $L' \in L^2(\nu)$ independent of $t,u,w$, such that
\[
|\gamma(t,x,u,e,\omega)| \leq L'(e) \left(1 + |x|\right). \tag{4}
\]

The optimal control associated to this problem is to optimize the objective function of the form
\[
J(u) = \mathbb{E} \left[ \int_0^T f(t,X(t),\mathbb{E}[\Phi(X(t))],Y(t),Z(t),K(t,\cdot),\pi(t,\omega))dt + h_1(Y(0)) + h_2(X(T),\mathbb{E}[\psi(X(T))]) \right],
\]
over the admissible controls, for functions
\[
f : [0,T] \times \mathbb{R}^6 \times \phi \times U \times \Omega \to \mathbb{R},
\]
\[
\Phi : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R},
\]
\[
h_1 : \mathbb{R} \times \Omega \to \mathbb{R},
\]
\[
h_2 : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R},
\]
\[
\psi : \mathbb{R} \times \Omega \to \mathbb{R}.
\]
That is, to find an optimal control $u^* \in A_G$ such that
\[
J(u^*) = \sup_{u \in A_G} J(u). \tag{6}
\]

For now, the functions $f, \Phi, \psi, h_i, i = 1,2$ are assumed to satisfy the following assumptions.

**Assumption (II)**

i) The functions $f(t,\cdot,\omega), \Phi(t,\cdot,\omega), \psi(\cdot,\omega), h_i(t,\cdot,\omega), i = 1,2$ are $C^1$ for each $t$ and $\omega$. 

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ii) Integrability condition

\[
\begin{align*}
E \left[ \int_0^T \left( |f(t, X(t), E[\Phi(X(t))], Y(t), Z(t), K(t, \cdot, \pi(t))] \right. \right. \\
+ \left. \left. \left| \frac{\partial f}{\partial x_i} \bigg| t, X(t), E[\Phi(X(t))], Y(t), Z(t), K(t, \cdot, \pi(t)] \right|^2 \right) \right] \ dt \right] < \infty.
\end{align*}
\]  

(7)

2.1 The Hamiltonian and adjoint equations

Let \( \phi \) denote the set of (equivalence classes) measurable functions \( r : \mathbb{R} \to \mathbb{R}_0^+ \) such that

\[
\int_{\mathbb{R}_0^+} \left\{ \sup_{(x,u) \in L} |\gamma(t, x, u, \epsilon, \omega) r(\epsilon)| + \sup_{(x,u) \in L} |\nabla \gamma(t, x, u, \epsilon, \omega) r(\epsilon)| \right\} \nu(d\epsilon) < \infty
\]  

(8)

for each \( t \in [0, T] \) and every bounded \( L \subset \mathbb{R} \times U, P \)-a.s. This integrability condition ensures that whenever \( r \in \phi \),

\[
\nabla \int_{\mathbb{R}_0^+} \gamma(t, x, u, \epsilon, \omega) r(\epsilon) \nu(d\epsilon) = \int_{\mathbb{R}_0^+} \nabla \gamma(t, x, u, \epsilon, \omega) r(\epsilon) \nu(d\epsilon),
\]  

(9)

and similarly for \( K(t, \cdot, \cdot) \).

Example 2.2. We notice that if the linear growth condition

\[
|\gamma(t, x, u, \epsilon, \omega) + |\nabla \gamma(t, x, u, \epsilon, \omega)| \leq L(\epsilon) \{1 + |x| + |u|\}
\]

holds for some \( L \in L^2(\nu) \) independent of \( t, u \), then \( L^2(\nu) \subset \phi \), and this will be the case in section necessary maximum principle.

Now we define the Hamiltonian associated to this problem, for \( \Omega \times [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times L^2(\nu) \times U \times \mathbb{R} \times \mathbb{R} \times L^2(\nu) \times \mathbb{R} \to \mathbb{R} \):

\[
H(t, x, m, y, z, k(\cdot), u, p, q, r(\cdot), \lambda) = f(t, x, m, y, z, k, u) + b(t, x, u)p + \sigma(t, x, u)q + g(t, x, y, z, k) \lambda + \gamma(t, x, u, \epsilon) r(\epsilon) \nu(d\epsilon)
\]

(10)

where \( m = \mathbb{E}[\psi(x)] \).

The adjoint equations for all \( t \in [0, T] \) are defined as follows

\[
dp(t) = E[Y(t) | \mathcal{F}_t] dt + q(t) dB(t) + \int r(t, e) \tilde{N}(dt, de),
\]  

(11)

\[
d\lambda(t) = \frac{\partial H}{\partial y}(t) dt + \frac{\partial H}{\partial z}(t) dB(t) + \int \nabla H(t, e) \tilde{N}(dt, de),
\]  

(12)

with, terminal condition

\[
p(T) = a\lambda(T) + \frac{\partial h_2}{\partial x}(X(T), E[\psi(X(T))])
\]

\[
+ \frac{\partial h_2}{\partial n}(X(T), E[\psi(X(T))]) \psi'(X(T)), a \in \mathbb{R}_0,
\]
initial condition $\lambda(0) = h_1'(Y(0))$ for all $t \in [0, T]$, and

$$
\Upsilon(t) = -\sum_{i=0}^{2} \int_{-\delta}^{0} \left( \frac{\partial H}{\partial x_i}(t-s, \pi) \right) \mu_i(ds) - \mathbb{E} \left[ \frac{\partial H}{\partial m}(t-s, \pi) \right] \Phi'(X(t)).
$$

we denote by $
\frac{\partial H}{\partial x_i}, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial m}, \frac{\partial h_2}{\partial x}, \frac{\partial h_2}{\partial m}
$
the partial derivatives of $H$ and $h$ w.r.t. $(x, y, m)$ and $x_i, n$ resp. and $\nabla_k H$ is the Fréchet derivative of $H$ w.r.t. $k$.

Throughout this work, it would be useful to introduce the simplified notation

$$
h_2(T) = h_2(X(T), \mathbb{E}[\psi(X(T))]).
$$

Example 2.3. Suppose $\mu := \mu_1$

1. If $\mu$ is the Dirac measure concentrated at 0, then

$$
\Upsilon(t) := -\frac{\partial H}{\partial x}(t, \pi) - \mathbb{E} \left[ \frac{\partial H}{\partial m}(t, \pi) \right].
$$

2. If $\mu$ is the Dirac measure concentrated at $\delta$, then

$$
\Upsilon(t) := -\frac{\partial H}{\partial x}(t+\delta, \pi) - \mathbb{E} \left[ \frac{\partial H}{\partial m}(t+\delta, \pi) \right].
$$

3. If $\mu(ds) = e^{\lambda s} ds$, then

$$
\Upsilon(t) := -\int_{-\delta}^{0} \frac{\partial H}{\partial x}(t-s, \pi)e^{\lambda s}(ds) - \mathbb{E} \left[ \frac{\partial H}{\partial m}(t, \pi) \right] \Phi'(X(t)).
$$

Remark 2.4. The existence and uniqueness of mean-field FBSDEs with delay is beyond the scope of this paper, and is a topic for future research. We refer to [26].

2.2 A sufficient maximum principle

When the Hamiltonian $H$ and the functions $(h_i)_{i=1,2}$ are concave, under certain other limitations, it is also possible to derive a sufficient maximum principle.

Theorem 2.5. Let $\pi \in \mathcal{A}_G$ with corresponding state processes $\hat{X}, \hat{Y}, \hat{Z}, \hat{K}(\cdot)$ and adjoint processes $\hat{p}, \hat{q}, \hat{r}(\cdot)$ and $\hat{\lambda}$. Suppose the following holds:

1. (Concavity) The functions

$$
\mathbb{R} \times \mathbb{R} \ni (x, n) \mapsto h_i(x, n), i = 1, 2
$$

and

$$
\mathbb{R}^{10} \times L^2(\nu) \times U \ni (x, m, y, z, k, u) \mapsto H(t, \cdot, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{\lambda}(t))
$$

(17)
are concave $P$-a.s. for each $t \in [0, T]$.

2. (Maximum principle)

\[
\mathbb{E} \left[ H \left( t, \hat{X}(t), \hat{M}(t), \hat{Y}(t), \hat{Z}(t), K(t, \cdot), \hat{\pi}(t), \hat{\rho}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{\lambda}(t) \right) \mid \mathcal{F}_t \right] \\
= \sup_{v \in \mathcal{U}} \mathbb{E} \left[ H \left( t, \hat{X}(t), \hat{M}(t), \hat{Y}(t), \hat{Z}(t), K(t, \cdot), v, \hat{\rho}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{\lambda}(t) \right) \mid \mathcal{F}_t \right],
\]

\[ (18) \]

$P$-a.s. for each $t \in [0, T]$. Then $\pi$ is an optimal control for the problem $\Box$.

Proof. By considering a suitable increasing family of stopping times converging to $T$ we may assume that all the local martingales appearing in the proof below are martingales. See the proof of Theorem 2.1 in [24] for details.

Let $\pi$ be an arbitrary admissible control. Consider the difference

\[
J(\hat{\pi}) - J(\pi) \\
= \mathbb{E} \left[ \int_0^T f(t, \hat{X}(t), \mathbb{E}[\hat{X}(t)], \hat{Y}(t), \hat{Z}(t), K(t, \cdot), \hat{\pi}(t)) \right] dt \\
- f(t, X(t), \mathbb{E}[\Phi(X(t))], Y(t), Z(t), K(t, \cdot), \pi(t)) \right] dt \\
+ h_1(0) + \hat{h}_2(T) - h_2(T) \right] \\
= \mathbb{E} \left[ \int_0^T \triangle \hat{f}(t) dt + \triangle \hat{h}_1(0) + \triangle \hat{h}_2(T) \right], \tag{19}
\]

use the same simplified notation for $\triangle \hat{H}(t), \triangle \hat{X}(t)$ etc. Since $H$ is concave, we have

\[
\triangle \hat{H}(t) \geq \sum_{i=0}^2 \int_{-\delta}^0 \triangle X(t) \left\{ \frac{\partial H}{\partial x_i} (t - s, \pi) \right\} \mu_i(ds) \\
+ \frac{\partial \hat{H}}{\partial m}(t) \mathbb{E}[\triangle \hat{X}(t)] + \frac{\partial \hat{H}}{\partial y}(t) \triangle \hat{Y}(t) \\
+ \frac{\partial \hat{H}}{\partial z}(t) \triangle \hat{Z}(t) + \int_{\mathbb{R}_0} \nabla_k \hat{H}(t, e) \triangle \hat{K}(t, e) \nu(de) + \frac{\partial \hat{H}}{\partial \pi}(t) \triangle \hat{\pi}(t) \\
\geq \sum_{i=0}^2 \int_{-\delta}^0 \triangle X(t) \left\{ \frac{\partial H}{\partial x_i} (t - s, \pi) \right\} \mu_i(ds) \\
+ \frac{\partial \hat{H}}{\partial m}(t) \mathbb{E}[\hat{X}(t)] + \frac{\partial \hat{H}}{\partial y}(t) \triangle \hat{Y}(t) \right] \tag{20} \\
+ \frac{\partial \hat{H}}{\partial z}(t) \triangle \hat{Z}(t) + \int_{\mathbb{R}_0} \nabla_k \hat{H}(t, e) \triangle \hat{K}(t, e) \nu(de) + \frac{\partial \hat{H}}{\partial \pi}(t) \triangle \hat{\pi}(t) \tag{21}
\]

By the concavity of $h_i(\cdot)_{i=1,2}$, we find

\[
\triangle \hat{h}_i(0) \geq \hat{h}_i'(0) \triangle \hat{Y}(0) = \hat{\lambda}(0) \triangle \hat{Y}(0) \tag{22}
\]
Apply Itô’s formula to \( \dot{\lambda}(0) \triangle \dot{Y}(0) \), we get

\[
\begin{align*}
\mathbb{E}[\dot{\lambda}(0) \triangle \dot{Y}(0)] &= \mathbb{E} \left[ \dot{\lambda}(T) \triangle \dot{Y}(T) \right] \\
&= \int_0^T \left\{ -\dot{\lambda}(t) \triangle \dot{g}(t) + \triangle \dot{Y}(t) \frac{\partial \dot{H}}{\partial y}(t) \right. \\
&\quad + \triangle \ddot{Z}(t) \frac{\partial \dot{H}}{\partial z}(t) + \int_{\mathcal{D}} \nabla_k \dot{H}(t, e) \triangle \dot{K}(t, e) \nu(de) \bigg\} dt \\
&= \mathbb{E} \left[ \triangle \dot{X}(T) \left( \dot{p}(T) - \frac{\partial \dot{h}_2}{\partial x}(T) - \frac{\partial \dot{h}_2}{\partial n}(T) \psi'(\dot{X}(T)) \right) \right] \\
&\quad + \int_0^T \left\{ \dot{\lambda}(t) \triangle \dot{g}(t) - \frac{\partial \dot{H}}{\partial y}(t) \triangle \dot{Y}(t) \\
&\quad - \frac{\partial \dot{H}}{\partial z}(t) \triangle \ddot{Z}(t) - \int_{\mathcal{D}} \nabla_k \dot{H}(t, e) \triangle \dot{K}(t, e) \nu(de) \bigg\} dt \\
&= \mathbb{E} \left[ \int_0^T \left\{ \ddot{H}(t) - \triangle \dot{f}(t) + \triangle \dot{X}(t) \ddot{Y}(t) - \triangle \dot{X}(T) \left( \frac{\partial \dot{h}_2}{\partial x}(T) - \frac{\partial \dot{h}_2}{\partial n}(T) \psi'(\dot{X}(T)) \right) \right. \\
&\quad - \frac{\partial \ddot{H}}{\partial y}(t) \triangle \dot{Y}(t) - \frac{\partial \ddot{H}}{\partial z}(t) \triangle \dddot{Z}(t) - \int_{\mathcal{D}} \nabla_k \ddot{H}(t, e) \triangle \ddot{K}(t, e) \nu(de) \bigg\} dt \right] .
\end{align*}
\]

By the definition of \( Y \) \([13]\) and Fubini’s theorem, we can show that

\[
\begin{align*}
\int_0^T \int_{-\delta}^0 \triangle X(t) \left\{ \frac{\partial H}{\partial x_i}(t - s, \pi) \right\} \mu_i(ds) dt \\
&= \int_0^T \int_{-\delta}^0 \left\{ \frac{\partial H}{\partial x_i}(t, \pi) \right\} \triangle X(t + \delta) \mu_i(ds) dt 
\end{align*}
\]

(25)
Let’s perform the change of variable \( r = t - s \) in the dt-integral to observe that
\[
\mathbb{E}\left[ \int_{-\delta}^{T} \int_{0}^{s} X(t) \left( \frac{\partial H}{\partial x_i}(t-s,\pi) \right) \mu_i(ds) dt \right] 
\]
\[= \mathbb{E}\left[ \int_{-\delta}^{T} \int_{0}^{s} X(t+s) \left( \frac{\partial H}{\partial x_i}(t-s,\pi) \right) \mu_i(ds) dt \right] 
\]
\[= \mathbb{E}\left[ \int_{-\delta}^{T} \int_{0}^{s} X(t+s) \left( \frac{\partial H}{\partial x_i}(r,\pi) \right) \mu_i(ds) dt \right] 
\]
\[= \mathbb{E}\left[ \int_{-\delta}^{T} \int_{0}^{s} X(t+s) \left( \frac{\partial H}{\partial x_i}(t,\pi) \right) \mu_i(ds) dt \right] 
\]
\[= \mathbb{E}\left[ \int_{-\delta}^{T} \int_{0}^{s} X(t+s) \left( \frac{\partial H}{\partial x_i}(t,\pi) \right) \mu_i(ds) dt \right]
\]
Putting (25) in (24), and combining (21) with (19), we obtain
\[J(\hat{x}) - J(\pi) \geq \mathbb{E}\left[ \int_{0}^{T} \frac{\partial H}{\partial \pi}(t) \Delta \hat{x}(t) dt \right] \geq 0,
\]
by the maximum condition of \( H \) (18).

\[\square\]

3 Infinite horizon optimal control problem

In this section, we extend the results obtained in the previous section to infinite horizon. So it can be seen as a generalization to mean-field problems of Theorems 3.1 and 4.1 in [3] and [25] resp. By following the same steps in the previous section but now with infinite time horizon, we consider that the state equations have the forms
\[dX(t) = b(t, X(t), \mathbb{E}[X(t)], \pi(t), \omega)dt + \sigma(t, X(t), \mathbb{E}[X(t)], \pi(t), \omega)dB(t)
\]
\[+ \int_{\mathbb{R}} \gamma(t, X(t^-), \mathbb{E}[X(t^-)], \pi(t^-), \epsilon, \omega) \tilde{N}(dt, d\epsilon), t \in [0, \infty),
\]
\[X(t) = x_0(t), t \in [-\delta, 0],
\]
and
\[dY(t) = -g(t, X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), K(t, \cdot), \pi(t))dt + Z(t)dB(t)
\]
\[+ \int_{\mathbb{R}} K(t, \epsilon, \omega) \tilde{N}(dt, d\epsilon), t \in [0, \infty)
\]
which can be interpreted as in [21] for all finite \( T, \)
\[Y(t) = Y(T) + \int_{t}^{T} g(s, X(s), \mathbb{E}[X(s)], Y(s), \mathbb{E}[Y(s)], Z(s), K(s, \cdot), \pi(s))ds
\]
\[+ \int_{\mathbb{R}} K(t, \epsilon, \omega) \tilde{N}(dt, d\epsilon), 0 \leq t \leq T,
\]
where
\[X(t) := (X_1(t), X_2(t), \ldots X_N(t)) := \left( \int_{-\delta}^{t} X(t+s) \mu_1(ds), \ldots, \int_{-\delta}^{t} X(t+s) \mu_N(ds) \right)
\]
for bounded Borel measures $\mu_1, \ldots, \mu_N(ds)$. We remark if $X$ is a càdlàg process, then $X$ is also càdlàg. We always assume that coefficient functional $\gamma$ is evaluated for the predictable (i.e., left continuous) versions of the càdlàg processes $X, Y$ and $\pi$, and we will omit the minus from the notation.

$$b = b(\omega, t, x, m, u) : \Omega \times [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \times U \rightarrow \mathbb{R},$$

$$\sigma = \sigma(\omega, t, x, m, u) : \Omega \times [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \times U \rightarrow \mathbb{R},$$

$$\gamma = \gamma(\omega, t, x, m, u, e) : \Omega \times [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \times U \times U_0 \rightarrow \mathbb{R},$$

$$g := g(\omega, t, x, m, y, n, z, k(\cdot), u) : \Omega \times [0, \infty) \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times L^2(\nu) \times U \rightarrow \mathbb{R}. $$

We assume that the coefficient functional satisfy the following assumptions:

**Assumptions (III)**

1. The functions $b, \sigma, \gamma, g$ is $C^1$ (Fréchet) with respect to all variables except $t$ and $\omega$.

2. The functions $b, \sigma, \gamma, g$ are jointly measurable.

Let $U$ be a subset of $\phi$. The set $U$ will be the admissible control values. The information available to the controller is given by a sub-filtration $\mathcal{G} = \{\mathcal{G}_t\}_{t \in [0, T]}$ with $\mathcal{F}_0 \subset \mathcal{G}_t \subset \mathcal{F}$. The set of admissible controls, that is, the set of controls that are available to the controller, is denoted by $\mathcal{A}_G$. It will be a given subset of the càdlàg, $U$-valued and $\mathcal{G}_t$-adapted processes in $L^2(\Omega \times [0, \infty))$, such that there exists unique càdlàg adapted processes $X = X^\pi, Y = Y^\pi$, progressively measurable $Z = Z^\pi$, and predictable $K = K^\pi$ satisfying (27) and (28), and if it also satisfies

$$E \left[ \int_0^\infty |X(s)|^2 \, ds \right] + E \left[ \sup_{t \geq 0} e^{\kappa t} (Y(t))^2 + \int_0^\infty (Z(t))^2 + \int_0^t (K(t,e))^2 \nu(de)) \, dt \right] < \infty$$

for some constant $\kappa > 0$.

### 3.1 The Optimization problem

We want to maximise the performance functional

$$J(\pi) = E \left[ \int_0^\infty f(t, X(t), E[X(t)], Y(t), E[Y(t)], Z(t), K(t, \cdot), \pi(t)) \, dt + h(Y(0)) \right]$$

over the set $\mathcal{A}_G$, for some functions

$$f = f(\omega, t, x, m, y, n, z, k(\cdot), u) : \Omega \times [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \phi \times U \rightarrow \mathbb{R},$$

and

$$h =: \mathbb{R} \rightarrow \mathbb{R}.$$ 

That is, we want to find $\pi^* \in \mathcal{A}_G$ such that

$$\sup_{\pi \in \mathcal{A}_G} J(\pi) = J(\pi^*).$$

We assume that the functions $f$ and $h$ satisfy the following assumptions:

**Assumptions (IV)**

1. The functions $f, h$ is $C^1$ (Fréchet) with respect to all variables except $t$ and $\omega$.

2. The function $f$ is predictable, and $h$ is $\mathcal{F}$-measurable for fixed $x, m, y, n, z, k, u$. 

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3.2 The Hamiltonian and the adjoint equation

Define the Hamiltonian function

\[ H : \Omega \times [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times L^2(\nu) \times U \times \mathbb{R} \times L^2(\nu) \times \mathbb{R} \to \mathbb{R} \]

by

\[
H(t, x, m, y, n, z, k(\cdot), u, p, q, r(\cdot), \lambda) = b(t, x, m, u)p + \sigma(t, x, m, u)q + \int_{\mathbb{R}_0} \gamma(t, x, m, u, e)\nu(de)
\]

\[
+ g(t, x, m, y, n, z, k(\cdot), u)\lambda + f(t, x, m, y, n, z, k(\cdot), u).
\]

Now, to each admissible control \( \pi \), we can define the adjoint process by the following system of forward-backward equations:

**Backward equation**

\[
dp(t) = -\mathbb{E}[\Upsilon(t)\mathcal{F}_t]dt + q(t)dB(t) + \int_{\mathbb{R}_0} r(t, e)\tilde{N}(dt, de),
\]

where

\[
\Upsilon(t) = \sum_{i=0}^{N-1} \int_{t-i}^0 \mathbb{E}\left[ \frac{\partial H}{\partial m_i}(t-s, X(t-s), \mathbb{E}[X(t-s)], Y(t-s), \mathbb{E}[Y(t-s)],
\right.
\]

\[
Z(t-s), K(t-s), \pi(t-s), p(t-s), q(t-s), r(t-s) \bigg| \mu_i(ds)
\]

\[
\left. + \sum_{i=0}^{N-1} \int_{t-i}^0 \mathbb{E}\left[ \frac{\partial H}{\partial m_i}(t-s, X(t-s), \mathbb{E}[X(t-s)], Y(t-s), \mathbb{E}[Y(t-s)],
\right. \right.
\]

\[
Z(t-s), K(t-s), \pi(t-s), p(t-s), q(t-s), r(t-s) \bigg| \mu_i(ds)
\]

**Forward equation**

\[
dx(t) = \frac{\partial H}{\partial y}(t, X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), K(t), \pi(t), p(t), q(t), r(t), \lambda(t))
\]

\[
+ \mathbb{E}\left[ \frac{\partial H}{\partial m}(t, X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), K(t), \pi(t), p(t), q(t), r(t), \lambda(t)) \right]dt
\]

\[
+ \frac{\partial H}{\partial z}(t, X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), K(t), \pi(t), p(t), q(t), r(t), \lambda(t))dB(t)
\]

\[
+ \int_{\mathbb{R}_0} \nabla_k H \left( t, X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), K(t), \pi(t), p(t), q(t), r(t), \lambda(t) \right)(e)\tilde{N}(dt, de),
\]

\[
\lambda(0) = h'(Y(0)).
\]

Here \( \nabla_k H \) is used to denote the Fréchet derivative of \( H \) with respect to the variable \( k(\cdot) \), and hence

\[
\nabla_k H \left( t, X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), K(t), \pi(t), p(t), q(t), r(t), \lambda(t) \right) \in L^2(\nu)^* = L^2(\nu),
\]

for fixed \( \omega, \pi \) and corresponding \( X, Y, Z, K, p, q, r \) and \( \lambda \). We notice also that the integrand

\[
\nabla_k H \left( t, X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), K(t), \pi(t), p(t), q(t), r(t), \lambda(t) \right)(e)
\]

\[
= \sum_{i=0}^{N-1} \int_{t-i}^0 \mathbb{E}\left[ \frac{\partial H}{\partial m_i}(t-s, X(t-s), \mathbb{E}[X(t-s)], Y(t-s), \mathbb{E}[Y(t-s)],
\right.
\]

\[
Z(t-s), K(t-s), \pi(t-s), p(t-s), q(t-s), r(t-s) \bigg| \mu_i(ds)
\]

\[
+ \sum_{i=0}^{N-1} \int_{t-i}^0 \mathbb{E}\left[ \frac{\partial H}{\partial m_i}(t-s, X(t-s), \mathbb{E}[X(t-s)], Y(t-s), \mathbb{E}[Y(t-s)],
\right. \right.
\]

\[
Z(t-s), K(t-s), \pi(t-s), p(t-s), q(t-s), r(t-s) \bigg| \mu_i(ds)
\]

\[
+ \int_{\mathbb{R}_0} \nabla_k H \left( t, X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), K(t), \pi(t), p(t), q(t), r(t), \lambda(t) \right)(e)\tilde{N}(dt, de),
\]

\[
\lambda(0) = h'(Y(0)).
\]
is predictable.

Notice also, \( \Upsilon \) may not be adapted to \( \mathcal{F}_t \), as \( \Upsilon(t) \) is defined using values of \( H \) at time \( t - s \), where \( s < 0 \).

Given an admissible control \( \pi \), suppose there exists progressively measurable processes \( p = p^\pi, q = q^\pi \) and \( \lambda = \lambda^\pi \) and \( r = r^\pi \), satisfying (38) and (39) and such that

\[
\mathbb{E} \left[ \sup_{t \geq 0} e^{\kappa t} (p(t))^2 + \int_0^\infty \left\{ |\lambda(t)|^2 + e^{\kappa t} ((q(t))^2 + \int_{\mathbb{R}_0} (r(t,e))^2 \nu(de)) \right\} dt \right] < \infty
\]

for some constant \( \kappa > 0 \). Then, we say that \( p, q, r \) and \( \lambda \) are adjoint equations to the Forward-Backward system (27)-(28).

### 3.3 Short hand notation

When Adjoint processes exist, we will frequently use the following short hand notation:

\[
H(t, \pi) := H(t, X^\pi(t), \mathbb{E}[X^\pi(t)], Y^\pi(t), \mathbb{E}[Y^\pi(t)], Z^\pi(t), \pi(t), p^\pi(t), q^\pi(t), r^\pi(t))
\]

Similar notation will be used for the coefficient functions \( b, \sigma, \gamma \), and \( g \), and the functions \( f, h \) from the performance functional, and for derivatives of the mentioned functions. We will write \( \nabla H \) for the Fréchet derivative of \( H \) with respect to the variables \( x, n, y, n, z, k(\cdot) \). Notice that \( \nabla H(t, \pi) \) applied to

\[
(x, n, \bar{g}, \bar{n}, \bar{k}(\cdot), \bar{u}) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times L^2(\nu) \times U
\]

is given by

\[
\nabla H(t, \pi)(x, n, \bar{g}, \bar{n}, \bar{k}(\cdot), \bar{u}) = \nabla_x H(t, \pi) x^T + \nabla_n H(t, \pi) n^T
+ \frac{\partial H}{\partial y}(t, \pi) \bar{y} + \frac{\partial H}{\partial g}(t, \pi) \bar{n} + \frac{\partial H}{\partial z}(t, \pi) \bar{z}
\int_{\mathbb{R}_0} \nabla_k H(t, \pi) (e) \bar{k}(e) \nu(de) + \frac{\partial H}{\partial u}(t, \pi) \bar{u}
\]

where \( \nabla_x \) is the gradient (as a row vector) with respect to the variable \( x \), etc.

Using this notation, the state equations and the adjoint equations can be written more compactly as

\[
\begin{align*}
\quad dX(t) &= b(t, \pi)dt + \sigma(t, \pi)dB(t) + \int_{\mathbb{R}_0} \gamma(t, \pi, e) \tilde{N}(dt, de), \quad t \in [0, \infty), \\
X(t) &= x_0(t), \quad t \in [-\delta, 0], \\

dY(t) &= -g(t, \pi)dt + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t, e, \omega) \tilde{N}(dt, de), \quad t \in [0, \infty),
\end{align*}
\]

(38)
and
\[ dp(t) = -\mathbb{E}[\Upsilon(t)|\bar{\mathcal{F}}_t] + q(t)dB(t) + \int_{\mathbb{R}_0} r(e,t)\tilde{N}(dt,de), \]

where
\[ \Upsilon(t) = \sum_{i=0}^{N-1} \int_{-\delta}^{0} \frac{\partial H}{\partial x_i}(t-s,\pi) + \mathbb{E}\left[ \frac{\partial H}{\partial m}(t-s,\pi) \right] \mu_i(ds) \]

\[ d\lambda(t) = \left\{ \frac{\partial H}{\partial y}(t,\pi) + \mathbb{E}\left[ \frac{\partial H}{\partial n}(t,\pi) \right] \right\} dt + \frac{\partial H}{\partial z}(t,\pi) dB(t) + \int_{\mathbb{R}_0} \nabla_k H(t,\pi)(e) \tilde{N}(dt,de), \quad t \in [0,\infty) \]

\[ \lambda(0) = h'(Y(0)). \]

**Example 3.1.** Suppose \( N = 1, \mu := \mu_1 \)

1. If \( \mu \) is the Dirac measure concentrated at 0, then
   \[ \Upsilon(t) := \frac{\partial H}{\partial x}(t,\pi) + \mathbb{E}\left[ \frac{\partial H}{\partial m}(t,\pi) \right]. \]

2. If \( \mu \) is the Dirac measure concentrated at \(-\delta\), then
   \[ \Upsilon(t) := \frac{\partial H}{\partial x}(t+\delta,\pi) + \mathbb{E}\left[ \frac{\partial H}{\partial m}(t+\delta,\pi) \right]. \]

3. If \( \mu(ds) = e^{\lambda s}ds \), then
   \[ \Upsilon(t) := \int_{-\delta}^{0} \frac{\partial H}{\partial x}(t-s,\pi) + \mathbb{E}\left[ \frac{\partial H}{\partial m}(t-s,\pi) \right] e^{\lambda s}(ds) \]
   \[ = \int_{t-\delta}^{t} \frac{\partial H}{\partial x}(-s,\pi) + \mathbb{E}\left[ \frac{\partial H}{\partial m}(-s,\pi) \right] e^{\lambda(s-t)}(ds). \]

**4 A necessary maximum principle**

Suppose that a control \( \pi \in \mathcal{A}_G \) is optimal and that \( \eta \in \mathcal{A}_G \). If the function \( s \mapsto J(\pi + s\eta) \) is well defined and differentiable on a neighbourhood of 0, then

\[ \frac{d}{ds}J(\pi + s\eta) |_{s=0} = 0. \]

Under a set of suitable assumptions on the functions \( f, b, \sigma, g, h, \gamma \) and \( K \), we will show that for every admissible \( \pi \), and bounded admissible \( \eta \),

\[ \frac{d}{ds}J(\pi + s\eta) |_{s=0} = \mathbb{E}\left[ \int_0^{\infty} \frac{\partial}{\partial \pi} H(t,\pi) \eta(t)dt \right]. \]
Then, provided that the set of admissible controls \( \mathcal{A}_G \) is sufficiently large,

\[
\frac{d}{ds} J(\pi + s\eta) \bigg|_{s=0} = 0
\]  

(46)
is equivalent to

\[
E \left[ \frac{\partial}{\partial \pi} H(t, \pi) \bigg| \Phi_t \right] = 0 \quad P - a.s. \text{ for each } t \in [0, \infty).
\]  

(47)

Consequently,

\[
E \left[ \frac{\partial}{\partial \pi} H(t, \pi) \bigg| \Phi_t \right] = 0 \quad P - a.s. \text{ for each } t \in [0, \infty),
\]  

(48)
is a necessary condition for optimality of \( \pi \).

The first step of deriving a necessary maximum principle is to establish the following equalities.

\[
\frac{d}{ds} J(\pi + s\eta) \bigg|_{s=0} = E \left[ \int_0^\infty \nabla f(t, X^\pi(t), \pi) \cdot (\nabla^\pi(t), \eta(t))^T dt + h'(Y^\pi(0)) \cdot \gamma^\pi(0) \right]
\]

\[
= E \left[ \int_0^\infty \frac{\partial H}{\partial \pi}(t, \pi) \eta(t) dt \right].
\]

We will formalize this through Lemma 4.3 and Lemma 4.6, but first we need to impose a set of assumptions:

**Assumptions (V)**

i) **Assumptions on the coefficient functions**

- The functions \( \nabla b, \nabla \sigma \) and \( \nabla g \) are bounded. The upper bound is denoted by \( D_0 \). Also, there exists a non-negative function \( D \in L^2 \) such that

\[
|\nabla \gamma(t, x, u, e)| + |K(t, e)| \leq D(e)
\]  

(49)

- The functions \( \nabla b, \nabla \sigma \) and \( \nabla g \) are Lipschitz continuous in the variables \( x, m, u \), uniformly in \( t, w \), with the Lipschitz constant \( L_0 > 0 \). Also, there exits a function \( L \in L^2(\nu) \) independent of \( t, w \), such that

\[
|\gamma(t, x, u, e) - \gamma(t, x', u', e)| \leq L(e) \left( |x - x'| + |u - u'| \right).
\]  

(50)

- The function \( L' \) from Assumption (I) is also in \( L^2(\nu) \).

ii) **Assumptions on the performance functional**

- The functions \( \nabla f, \nabla h \) and \( \nabla g \) are bounded. The upper bound is still denoted by \( D_0 \).
• The functions $\nabla f, \nabla h$ and $\nabla g$ are Lipschitz continuous in the variables $(x, y, z, k, u)$, uniformly in $t, w$. The Lipschitz constant is still denoted by $L_0$.

iii) Assumptions on the set of admissible processes

• Whenever $u \in A_0$ and $\eta \in A_2$ is bounded, there exists $\epsilon > 0$ such that

$$u + s\eta \in A_0 \quad \text{for each } s \in (-\epsilon, \epsilon).$$

(51)

• For each $t_0 > 0$ and each bounded $\Theta_{t_0}$-measurable random variables $\alpha$, the process $\eta(t) = \alpha^1_{[t_0, t_0 + h]}(t)$ belongs to $A_2$.

4.1 The derivative processes

Suppose that $\pi, \eta \in A_0$, with $\eta$ bounded. Consider the equations

$$dX(t) = \nabla b(t, \pi) \cdot (X(t), E[X(t)], \eta(t))^T dt$$
$$+ \nabla \sigma(t, \pi) \cdot (X(t), E[X(t)], \eta(t))^T dB(t)$$
$$+ \int_{\mathbb{R}_0} \nabla \gamma(t, \pi, e) \cdot (X(t), E[X(t)], \eta(t))^T \tilde{N}(dt, de), \quad t \in [0, \infty),$$

$$X(t) = 0, \quad \text{for } t \in [-\delta, 0],$$

where

$$X(t) := \left( \int_{-\delta}^0 X(t + s) \mu_1(ds), \ldots, \int_{-\delta}^0 X(t + s) \mu_N(ds) \right)$$

and

$$dY(t) = \left( -\nabla g(t, \pi) \cdot (X(t), E[X(t)], Y(t), E[Y(t)], Z(t), K(t), \eta(t))^T \right) dt$$
$$+ Z(t) dB(t) + \int_{\mathbb{R}_0} K(t, e) \tilde{N}(dt, de), \quad t \in [0, \infty).$$

(53)

We say that a solutions $X = X^{\pi, \eta}, Y = Y^{\pi, \eta}, Z = Z^{\pi, \eta}$ and $K = K^{\pi, \eta}$ associated with the controls $\pi, \eta$ exists if there are processes $X, Y, Z$ and $K$ satisfying (52)-(53), and

$$E \left[ \sup_{t \geq 0} e^{\kappa t} (Y(t))^2 + \int_0^\infty |X(t)|^2 + e^{\kappa t} ((Z(t))^2 + \int_{\mathbb{R}_0} (K(t, e))^2 \nu(de)) dt \right] < \infty$$

for some constant $\kappa > 0$.

(54)
4.1.1 Differentiability of the forward state process

To proofs in this section are similar to e.g. the proofs of Lemmas 3.1, 4.1 and in [3] and [25] resp. However because of our jump term, we need to use Kunita’s inequality instead of Burkholder-Davis-Gundy’s inequality. We also do not require any $L^4$-boundedness and convergence of any of our processes as is done e.g. in [3], to assure the convergence in our Lemma 4.3. Requiring $L^4$ boundedness on the process would have lead to the necessity of additional assumptions on the Lipschitz and boundedness constants, as an example assuming that the function $D$ in assumption (V) is also in $L^4(\nu)$ is a sufficient condition to ensure that $E[\sup_{0 \leq v \leq t} |X(v)|^4] < \infty$ for each $t \in [0, \infty)$. For convenience to the reader, let us recall

**Lemma 4.1** (Kunita’s inequality, corollary 2.12 in [19]). Suppose $\rho \geq 2$ and

$$X(t) = x + \int_0^t b(r)dr + \int_0^t \sigma(r)dB(r) + \int_{\mathbb{R}_0} \mathcal{X}(r,e)\tilde{N}(dr,de). \quad (55)$$

Then there exists a positive constant $C_{\rho,T}$, (depending only on $\rho, T$) such that the following inequality holds

$$E \left[ \sup_{0 \leq s \leq T} |X(s)|^\rho \right] \leq C_{\rho,T} \left[ |x|^\rho + \left\{ \int_0^t E[|b(r)|^\rho] + E[|\sigma(r)|^\rho] \right. \right. + E \left[ \int_{\mathbb{R}_0} |\mathcal{X}(r,e)|^\rho \nu(de) \right] + E \left( \int_0^t |\mathcal{X}(r,e)|^2 \nu(de) \right)^{\frac{\rho}{2}} \right] dr. \quad (56)$$

Now, define the random fields

$$F_{\eta}^\alpha(t) := X^{\pi+\alpha\eta}(t) - X^{\pi}(t) \quad (56)$$

and

$$F_{\alpha}^\eta(t) := X^{\pi+\alpha\eta}(t) - X^{\pi}(t) = \left( \int_{-\delta}^0 F_{\alpha}^\eta(t + r)\mu_1(dr), \ldots, \int_{-\delta}^0 F_{\alpha}^\eta(t + r)\mu_N(dr) \right) \quad (57)$$

**Lemma 4.2.** Let $T \in (0, \infty)$. There exists a constant $C = C_T > 0$, independent of $\pi, \eta$, such that

$$E[\sup_{0 \leq v \leq t}|F_{\alpha}^\eta(v)|^2] \leq C \| \eta \|_{L^2(\mathcal{A} \times [0,T])}^2 \alpha^2. \quad (58)$$

whenever $t \leq T$.

Moreover, there exists a measurable version of the map

$$(t, \alpha, \omega) \mapsto F_{\alpha}(t, \omega) \quad (59)$$

such that for a.e. $\omega$, $F_{\alpha}(t) \to 0$ as $\alpha \to 0$ for every $t \in [0, \infty)$.

**Proof.** The proof follows that of Lemma 4.2 in [10] closely. Define

$$\beta_{\alpha}(t) := E[\sup_{-\delta \leq v \leq t}|F_{\alpha}^\eta(v)|^2]. \quad (60)$$
Observe that using Jensen’s inequality, we find that

\[
\mathbb{E}[ \sup_{0 \leq v \leq t} |F_\alpha^n(v)|^2 ] = \mathbb{E} \left[ \sup_{0 \leq v \leq t} \sum_{i=1}^N \left| \int_{-\delta}^0 F_{\alpha}^n(v + r) \mu_i(dr) \right|^2 \right] \\
\leq \mathbb{E} \left[ \sup_{0 \leq v \leq t} \sum_{i=1}^N |\mu_i| \int_{-\delta}^0 |F_{\alpha}^n(v + r)|^2 \mu_i(dr) \right] \\
\leq \mathbb{E} \left[ \sup_{0 \leq v \leq t} \sum_{i=1}^N |\mu_i|^2 \sup_{-\delta \leq r \leq 0} |F_{\alpha}^n(v + r)|^2 \right] \\
\leq |\mu|^2 \beta_\alpha(t)
\]

where \(|\mu| := \sum_{i=1}^N |\mu_i| := \sum_{i=1}^N \mu_i[-\delta, 0]|. Since \(\nabla b, \nabla \sigma, \nabla \gamma\) are bounded, \(b, \sigma\) and \(\gamma\) are Lipshitz in the variable \(x, m, u\). Now using the integral representation of \(X^{\pi+\alpha}\) and \(X^\pi\), Kunita’s inequality, and finally the Lipshitz conditions on \(b, \sigma\) and \(\gamma\), we find that

\[
\beta_\alpha(t) \leq C_{2,T} \mathbb{E} \left[ \int_0^t \left| b(s, \pi + \alpha \eta) - b(s, \pi) \right|^2 + |\sigma(s, \pi + \alpha \eta) - \sigma(s, \pi)|^2 \\
+ \int_{\mathbb{R}_0^d} |\gamma(s, \pi + \alpha \eta, e) - \gamma(s, \pi, e)|^2 \nu(de) ds \right] \\
\leq C_{2,T} (D_2^2 + \|D\|_\mathcal{L}_2^2(t)) \mathbb{E} \left[ \int_0^t \|F_\alpha(s)\|^2 + \alpha^2 |\eta(s)|^2 ds \right] \\
\leq C_{2,T} (D_2^2 + \|D\|_\mathcal{L}_2^2(t)) \left( \int_0^t \beta_\alpha(s) ds + \alpha^2 \|\eta\|_{\mathcal{L}_2^2([0,T])} \right).
\]

Now \(\text{(55)}\) holds by Gronwall’s inequality. The second part of the lemma follows by the same argument as in \([10]\).

Now, fix \(\pi\) and define

\[
A_{\alpha}^\eta(t) := \frac{X^{\pi+\alpha \eta}(t) - X^\pi(t)}{\alpha} - X^{\pi,\eta}(t) \\
A_{\alpha}^0(t) := \frac{X^\pi(t) - X^{\pi+\alpha \eta}(t)}{\alpha} - X^{\pi,\eta}(t) \\
= \left( \int_{-\delta}^0 A_{\alpha}^0(t + r) \mu_1(dr), \ldots, \int_{-\delta}^0 A_{\alpha}^0(t + r) \mu_N(dr) \right)
\]

for each \(\eta\).

**Lemma 4.3.** For each \(t \in (0, \infty)\), it holds that

\[
\theta_\alpha(t) := \mathbb{E} \left[ \sup_{0 \leq v \leq t} |A_{\alpha}^\eta(v)|^2 \right] \to 0, \\
\mathbb{E} \left[ \sup_{0 \leq v \leq t} |A_{\alpha}^0(v)|^2 \right] \to 0
\]

as \(\alpha \to 0\).

**Proof.** Similarly as in the previous proof, we find that

\[
\mathbb{E} \left[ \sup_{0 \leq v \leq t} |A_{\alpha}^\eta(v)|^2 \right] \leq N |\mu|^2 \theta_s(t).
\]

The rest of the proof follows the exact same steps as the proof of Lemma 4.3 in \([10]\) and is therefore omitted. \(\square\)
4.1.2 Differentiability of the backward state process

We will assume that the following convergence results hold for all \( t \geq 0 \)

\[
E \left[ \sup_{0 \leq r \leq t} \left| \frac{Y^{\pi+\alpha \eta}(r)}{\alpha} - Y^{\pi}(r) \right|^2 \right] \to 0, \tag{61}
\]

\[
E \left[ \int_0^T \left| \frac{Z^{\pi+\alpha \eta}(t)}{\alpha} - Z(t) \right|^2 dt \right] \to 0, \tag{62}
\]

\[
E \left[ \int_0^T \int_{\mathbb{R}_0} \left| \frac{K^{\pi+\alpha \eta}(t,e)}{\alpha} - K^{\pi}(t,e) \right|^2 \nu(de) dt \right] \to 0 \tag{63}
\]

as \( \alpha \to 0 \). In particular, \( Y, Z \) and \( K \) are the solutions of (53). We refer to [26] for more details.

4.2 Differentiability of the performance functional

**Lemma 4.4 (Differentiability of \( J \)).** Suppose \( \pi, \eta \in \mathcal{A}_G \) with \( \eta \) bounded. Suppose there exist an interval \( I \subset \mathbb{R} \) with \( 0 \in I \), such that the perturbation \( \pi + s \eta \) is in \( \mathcal{A}_G \) for each \( s \in I \). Then the function \( s \mapsto \frac{d}{ds} J(\pi + s \eta) \) has a (possibly one-sided) derivative at 0 with

\[
\frac{d}{ds} J(\pi + s \eta) \bigg|_{s=0} = \mathbb{E} \left[ \int_0^\infty \nabla f(t, X^\pi(t), \pi) \cdot (X^\pi(t), \eta(t))^T dt + h'(Y^{\pi}(0)) \cdot Y^{\pi,\eta}(0) \right].
\]

**Proof.** Recall that

\[
J(\pi) = \mathbb{E} \left[ \int_0^\infty f(t, X^\pi(t), \pi) dt + h(Y^{\pi}(0)) \right].
\]

Let

\[
J_n(\pi) = \mathbb{E} \left[ \int_0^\infty f(t, X^\pi(t), \pi) dt + h(Y^{\pi}(0)) \right].
\]

We show that

\[
\frac{d}{ds} J_n(\pi + s \eta) \bigg|_{s=0} = \mathbb{E} \left[ \int_0^\infty \nabla f(t, X^\pi(t), \pi) \cdot (X^\pi(t), \eta(t))^T dt + h'(Y^{\pi}(0)) \cdot Y^{\pi,\eta}(0) \right].
\]

Since \( \nabla f \) and \( \eta \) are bounded and \( X \) is finite, we can use the dominated convergence theorem to get

\[
\frac{d}{ds} J(\pi + s \eta) \bigg|_{s=0} = \lim_{n \to \infty} \frac{d}{ds} J_n(\pi + s \eta) \bigg|_{s=0}.
\]

Now proving

\[
\frac{d}{ds} \mathbb{E} \left[ h(Y^{\pi+\alpha \eta}(0)) \right] \bigg|_{s=0} = \mathbb{E} [h'(Y^{\pi}(0)) \cdot Y^{\pi,\eta}(0)]
\]
\[
\begin{align*}
\mathbb{E} & \left[ \frac{1}{s} h(Y^{\pi+\nu}(0)) - h(Y^{\pi}(0)) - h'(Y^{\pi}(0)) \cdot Y^{\pi,\nu}(0) \right] \\
& = \mathbb{E} \left[ \frac{1}{s} \int_0^1 h'(Y^{\pi}(0)) + \lambda(Y^{\pi+\nu}(0) - Y^{\pi}(0)) \cdot \left( \frac{Y^{\pi+\nu}(s) - Y^{\pi}(s)}{s} \right) \\
& \quad + \{ h'(Y^{\pi}(0)) + \lambda(Y^{\pi+\nu}(0) - Y^{\pi}(0)) - h'(Y^{\pi}(0)) \cdot Y^{\pi,\nu}(0) \} d\lambda \right] \\
& \leq \int_0^1 \mathbb{E} \left[ \left( \frac{Y^{\pi+\nu}(s) - Y^{\pi}(s)}{s} \right) - Y^{\pi,\nu}(0) \right] d\lambda \\
& + \int_0^1 \mathbb{E} \left[ |\lambda(Y^{\pi+\nu}(0) - Y^{\pi}(0))| \cdot |Y^{\pi,\nu}(0)| \right] d\lambda \\
& \to 0 \text{ as } s \to 0
\end{align*}
\]

we obtain the last estimation by, assumption of \( Y \) \ref{61} and apply Cauchy-Schwartz inequality, we obtain
\[
\begin{align*}
\mathbb{E} & \left[ \left| Y^{\pi+\nu}(0) - Y^{\pi}(0) \right| \cdot |Y^{\pi,\nu}(0)| \right] \\
& \leq \mathbb{E} \left[ \left( Y^{\pi+\nu}(0) - Y^{\pi}(0) \right)^2 \right]^\frac{1}{2} \mathbb{E} \left[ \left( Y^{\pi,\nu}(0) \right)^2 \right]^\frac{1}{2} \\
& \to 0 \text{ as } s \to 0.
\end{align*}
\]

\[ \square \]

**Lemma 4.5. [Differentiability of \( J \) in terms of the Hamiltonian]** Let \( \pi, \eta \in A_2 \) with \( \eta \) bounded. Let \( X, Y, Z, K, p, q, r, \lambda \) be the state and corresponding to \( \pi \) adjoint processes, and \( X, Y, Z, K, p, q, r, \lambda \) be derivative processes corresponding to \( \pi, \eta \). Suppose there exists adjoint processes \( p, q, r \) corresponding to \( \pi \) and that
\[
\lim_{T \to \infty} \mathbb{E} [p(T)X(T)] = 0, \quad (64)
\]
\[
\lim_{T \to \infty} \mathbb{E} [\lambda(T)Y(T)] = 0. \quad (65)
\]

Then
\[
\frac{d}{ds} J(\pi + s\eta) \big|_{s=0} = \mathbb{E} \left[ \int_0^\infty \frac{\partial H}{\partial \pi}(t, \pi, \eta(t)) dt \right].
\]

**Proof.** For fixed \( T \geq 0 \), define a sequence of stopping times, as follows
\[
\tau_n(\cdot) := T \wedge \inf \left\{ t \geq 0 : \int_0^t \left( p(t) \nabla \sigma(t, \pi) \cdot \left( X(t), \mathbb{E}[X(t)], \eta(t) \right) + X(t) \nu(t) \right)^2 ds \\
+ \int_0^t \int_{\mathbb{R}^n} \left( \tau(s, e) \nabla \gamma(s, \pi, e) \cdot \left( X(s), \mathbb{E}[X(s)], \eta(s) \right) + \frac{\partial H}{\partial z}(s, \pi) + \lambda(s) \mathbb{E}[X(s)] \right)^2 ds \right\} \nu(de)ds \\
+ \int_0^t \left( Y(s) + \lambda(s) \mathbb{E}[X(s)] + \mathbb{E}[X(s)] \nabla H(s, e) \right)^2 \nu(de)ds \geq n \}, \quad n \in \mathbb{N}
\]

(66)
\[ \nabla H - \lambda \nabla g - \nabla f = p \nabla b + q \nabla \sigma + \int_{\mathbb{R}_0} r \nabla \gamma \, d\nu. \quad (67) \]

(This can be shown using the Chain rule for the Fréchet derivative.) By Itô’s formula, we can compute that

\[
p(\tau_n) X(\tau_n) = \int_0^{\tau_n} \left( p(t) \nabla b(t, \pi) + q(t) \nabla \sigma(t, \pi) + \int_{\mathbb{R}_0} r(t, e) \nabla \gamma(t, \pi, e) \nu(de) \right) \cdot \left( X(t), E[X(t)], \eta(t) \right)^T - \mathcal{X}(t) E[Y(t) | \mathcal{F}_t] \, dt \\
+ \int_0^{\tau_n} p(t) \nabla \sigma(t, \pi) \cdot \left( X(t), E[X(t)], \eta(t) \right)^T + \mathcal{X}(t) q(t) dB(t) \\
+ \int_0^{\tau_n} r(t, e) \nabla \gamma(t, \pi, e) \cdot \left( X(t), E[X(t)], \eta(t) \right)^T + \mathcal{X}(t) r(t, \pi, e) \tilde{N}(dt, de) \quad (68)\]

The stochastic integral parts have zero mean by definition of the stopping time, and we recall that \( \mathcal{X}(0) = 0 \). Observe that since we have required that all solutions of the state and adjoint equations belongs to the spaces \( L^2(\Omega \times [0, \infty)) \) or \( L^2(\Omega \times [0, \infty) \times \mathbb{R}_0) \), and that the gradient of the coefficient functionals are bounded, it holds that

\[
E \left[ \int_0^{\tau_n} \left( p(t) \nabla b(t, \pi) + q(t) \nabla \sigma(t, \pi) + \int_{\mathbb{R}_0} r(t, e) \nabla \gamma(t, \pi, e) \nu(de) \right) \cdot \left( X(t), E[X(t)], \eta(t) \right)^T \right] + |\mathcal{X}(t) Y(t)| \, dt < \infty. \quad (69)\]

Now,

\[
E[p(T) \mathcal{X}(T)] = \lim_{n \to \infty} E[p(\tau_n) \mathcal{X}(\tau_n)] \\
= E \left[ \int_0^{\tau_n} \left( p(t) \nabla b(t, \pi) + q(t) \nabla \sigma(t, \pi) + \int_{\mathbb{R}_0} r(t, e) \nabla \gamma(t, \pi, e) \nu(de) \right) \cdot \left( X(t), E[X(t)], \eta(t) \right)^T - \mathcal{X}(t) E[Y(t) | \mathcal{F}_t] \, dt \right] \\
= \lim_{n \to \infty} E \left[ \int_0^{\tau_n} \left( \nabla H(t, \pi) - \lambda(t) \nabla g(t, \pi) - \nabla f(t, \pi) \right) \cdot \left( X(t), E[X(t)], \eta(t) \right)^T \right] - \mathcal{X}(t) Y(t) dt \quad (70)\]

In the first and last equality, we have used Lebesgue’s dominated convergence theorem and that the integrand is dominated by the integrable random variable in \( (69) \). In the second equality, we have used the integral representation \( (68) \) of \( p(\tau_n) \mathcal{X}(\tau_n) \) and that the stochastic integrals have zero mean by definition of
the stopping times $\tau_n$, and in the third equality, we have used $[67]$. From the assumption $[65]$, and again using the fact that the integrands are dominated by the integrable random variable in $[69]$, we find that

$$0 = \lim_{T \to \infty} \mathbb{E}[p(T)X(T)] = \mathbb{E} \left[ \int_0^\infty \left( \nabla H(t, \pi) - \lambda(t) \nabla g(t, \pi) - \nabla f(t, \pi) \right) \cdot \left( X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), \mathcal{K}(t), \eta(t) \right)^T - X'(t)Y(0)dt \right].$$

(71)

Similarly, using Itô’s formula, we compute that

$$\lambda(\tau_n)Y(\tau_n) - \lambda(0)Y(0) = \int_0^{\tau_n} \{ \mathcal{Y}(t) \left( \frac{\partial H}{\partial y}(t, \pi) + \mathbb{E} \left[ \frac{\partial H}{\partial n}(t, \pi) \right] \right) \\
+ \lambda(t) \left( - \nabla g(t, \pi) \right) \cdot \left( X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), \mathcal{K}(t), \eta(t) \right)^T \\
+ \mathcal{Z}(t) \frac{\partial H}{\partial z}(t, \pi) + \int_{\mathbb{R}} \mathcal{K}(t, e) \nabla_k H(t, e) \nu(de) \} dt \\
+ \int_0^{\tau_n} \mathcal{Y}(t) \frac{\partial H}{\partial z}(t, \pi) + \lambda(t)\mathcal{Z}(t)dB(t) \\
+ \int_0^{\tau_n} \int_{\mathbb{R}} \mathcal{Y}(t)\nabla_k H(t, e) + \lambda(t)\mathcal{K}(t, e) + \mathcal{K}(t, e)\nabla_k H(t, e)\tilde{N}(dt, de).$$

(72)

We recall that $\lambda(0) = h'(Y(0))$. Then proceeding as above, we find that

$$0 = \lim_{T \to \infty} \mathbb{E} \left[ \lambda(T)Y(T) \right] \\
= \mathbb{E} \left[ h'(Y(0))Y(0) \right] + \mathbb{E} \left[ \int_0^\infty \left( \mathcal{Y}(t) \left( \frac{\partial H}{\partial y}(t, \pi) + \mathbb{E} \left[ \frac{\partial H}{\partial n}(t, \pi) \right] \right) \\
+ \lambda(t) \left( - \nabla g(t, \pi) \right) \cdot \left( X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), \mathcal{K}(t), \eta(t) \right)^T \\
+ \mathcal{Z}(t) \frac{\partial H}{\partial z}(t, \pi) + \int_{\mathbb{R}} \mathcal{K}(t, e) \nabla_k H(t, e) \nu(de) \} dt \right].$$

(73)
Now, combining Lemma 4.4 with the equations (71) and (73) yields

\[
\frac{d}{da} \mathbb{E} \left[ J(\pi + a\eta) \right]_{a=0} = \mathbb{E} \left[ h'(Y(0)) Y(0) \right] \\
+ \int_0^\infty \nabla f(t, \pi) \cdot \left( X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), K(t, \cdot), \eta(t) \right) ^T dt \\
= \mathbb{E} \left[ \int_0^\infty \lambda(t) \left( \nabla g(t, \pi) \cdot \left( X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), K(t), \eta(t) \right) ^T \\
- \left\{ \mathcal{Y}(t) \left( \frac{\partial H}{\partial g}(t, \pi) + \mathbb{E} \left[ \frac{\partial H}{\partial \alpha}(t, \pi) \right] \right) + \mathcal{Z}(t) \frac{\partial H}{\partial \xi}(t, \pi) + \int_{\mathbb{S}_a} \nabla_k H(t, \pi) K(t, e) \nu(de) \right\} dt \right] \\
+ \mathbb{E} \left[ \int_0^\infty \left( \nabla H(t, \pi) - \lambda(t) \nabla g(t, \pi) \right) \right. \\
\left. \cdot \left( X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), K(t, \cdot), \eta(t) \right) ^T - \mathcal{X}(t) \mathcal{Y}(t) dt \right] \\
= \mathbb{E} \left[ \int_0^\infty \nabla H(t, \pi) \cdot \left( X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), K(t, \cdot), \eta(t) \right) ^T \right. \\
\left. - \left\{ \sum_{i=1}^{N-1} \mathcal{X}(t) \left( \frac{\partial H}{\partial x_i}(t - s, \pi) + \mathbb{E} \left[ \frac{\partial H}{\partial \alpha}(t - s, \pi) \right] \right) \mu_i(ds) \right. \\
+ \mathcal{Y}(t) \left( \frac{\partial H}{\partial y}(t, \pi) + \mathbb{E} \left[ \frac{\partial H}{\partial \alpha}(t, \pi) \right] \right) + \mathcal{Z}(t) \frac{\partial H}{\partial \xi}(t, \pi) + \int_{\mathbb{S}_a} K(t, e) \nabla_k H(t, e) \nu(de) \right\} dt \right] \\
+ \mathbb{E} \left[ \sum_{i=1}^{N-1} \frac{\partial H}{\partial x_i}(t, \pi) \cdot \int_{-\delta}^0 \mathcal{X}(t + s) \mu_i(ds) + \frac{\partial H}{\partial \alpha}(t, \pi) \cdot \mathbb{E} \left[ \int_{0}^{\delta} \mathcal{X}(t + s) \mu_i(ds) \right] \right] \\
+ \frac{\partial H}{\partial y}(t, \pi) \mathcal{Y}(t) + \frac{\partial H}{\partial \alpha}(t, \pi) \mathcal{Y}(t) + \frac{\partial H}{\partial \xi}(t, \pi) \mathcal{Z}(t) + \int_{\mathbb{S}_a} \nabla_k H(t, \pi(e)) K(t, e) \nu(de) \right) \\
+ \frac{\partial H}{\partial u}(t, \pi) \eta(t) \\
- \left\{ \sum_{i=1}^{N-1} \int_{-\delta}^0 \mathcal{X}(t) \left( \frac{\partial H}{\partial x_i}(t - s, \pi) + \mathbb{E} \left[ \frac{\partial H}{\partial \alpha}(t - s, \pi) \right] \right) \mu_i(ds) \right. \\
+ \mathcal{Y}(t) \left( \frac{\partial H}{\partial y}(t, \pi) + \mathbb{E} \left[ \frac{\partial H}{\partial \alpha}(t, \pi) \right] \right) + \mathcal{Z}(t) \frac{\partial H}{\partial \xi}(t, \pi) + \int_{\mathbb{S}_a} \nabla_k H(t, e) K(t, e) \nu(de) \right\} dt \right) \\
= \mathbb{E} \left[ \int_0^\infty \frac{\partial H}{\partial \pi}(t, \pi) \eta(t) dt \right].
\]
change of variable $r = t - s$ in the $dt$-integral to observe that

\[
\begin{align*}
\mathbb{E} \left[ \int_{0}^{\infty} \int_{-\delta}^{0} X(t) \left( \frac{\partial H}{\partial x_i}(t-s, \pi) + \mathbb{E} \left[ \frac{\partial H}{\partial m_i}(t-s, \pi) \right] \right) \mu_i(ds) dt \right]
\end{align*}
\]

\[
= \mathbb{E} \left[ \int_{-\delta}^{0} \int_{0}^{\infty} X(t) \left( \frac{\partial H}{\partial x_i}(t-s, \pi) + \mathbb{E} \left[ \frac{\partial H}{\partial m_i}(t-s, \pi) \right] \right) dt \mu_i(ds) \right]
\]

\[
= \mathbb{E} \left[ \int_{-\delta}^{0} \int_{0}^{\infty} X(r) \left( \frac{\partial H}{\partial x_i}(r, \pi) + \mathbb{E} \left[ \frac{\partial H}{\partial m_i}(r, \pi) \right] \right) dt \mu_i(ds) \right]
\]

\[
= \mathbb{E} \left[ \int_{0}^{\infty} \int_{-\delta}^{0} X(t) \left( \frac{\partial H}{\partial x_i}(t, \pi) + \mathbb{E} \left[ \frac{\partial H}{\partial m_i}(t, \pi) \right] \right) dt \mu_i(ds) \right] \tag{79}
\]

**Theorem 4.6** (Necessary maximum principle). Under the assumptions of Lemma 4.5, we can prove the equivalence between:

(i) For each bounded $\eta \in \mathcal{A}_G$,

\[
0 = \frac{d}{ds} J(\pi + s \eta) \bigg|_{s=0} = \mathbb{E} \left[ \int_{0}^{\infty} \frac{\partial H}{\partial \pi}(t, \pi(t)) \eta(t) dt \right]
\]

(ii) For each $t \in [0, \infty)$,

\[
\mathbb{E} \left[ \frac{\partial H}{\partial \pi}(t, \pi) \bigg| \mathcal{G}_t \right]_{\pi = \pi^*(t)} = 0 \text{ a.s.}
\]

**Proof.** Using Lemma 4.4, the proof is similar to the proof of Theorem 4.1 in [2]. \qed

### 4.3 Sufficient maximum principle

**Theorem 4.7** (Sufficient maximum principle). Let $\pi \in \mathcal{A}_G$ with corresponding solutions $X(t), Y(t), Z(t), K(t, \cdot), p(t), q(t), r(t, \cdot), \lambda(t)$. Assume the following conditions hold:

(i)

\[
\mathbb{E} \left[ H(t, \pi) \bigg| \mathcal{G}_t \right] = \sup_{v \in U} \mathbb{E} \left[ H(t, v) \bigg| \mathcal{G}_t \right], \tag{80}
\]

for all $t \in [0, \infty)$ a.s.

(ii) Transversality conditions

\[
\lim_{T \to \infty} \mathbb{E} \left[ \hat{p}(T) \left( \hat{X}(T) - X(T) \right) \right] \leq 0 \tag{81}
\]

\[
\lim_{T \to \infty} \mathbb{E} \left[ \hat{\lambda}(T) \left( \hat{Y}(T) - Y(T) \right) \right] \geq 0. \tag{82}
\]
Then $\pi$ is an optimal control for the problem \[51\].

Proof. The proof is similar to the proof of Theorem 2.5 but with infinite time horizon and Theorem 3.1 in \[2\]. \hfill $\square$

5 Optimal consumption with respect to recursive utility

Suppose now that the state equation is a cash flow on the form

\[
\begin{aligned}
dX(t) &= \left[ b_0(t, \mathbb{E}[X(t)]) - \pi(t) \right] dt + \sigma(t, X(t), \mathbb{E}[X(t)], \pi(t)) dB(t) \\
&\quad + \int_{\mathbb{R}_0^d} \mathcal{A}(t, X(t), \mathbb{E}[X(t)], \pi(t), \epsilon) N(dt, de), \quad t \in [0, \infty),
\end{aligned}
\]

where the control $\pi(t) \geq 0$ represents a consumption rate. The function $b_0$ is assumed to be deterministic, in addition to the assumptions from the previous sections. We want to consider an optimal recursive utility problem similar to the one in \[2\]. See also \[11\]. For notational convenience assume that $\mu_0$ is the Dirac measure concentrated at 0.

Define the recursive utility process $Y(t) = Y^\pi(t)$, by the BSDE in the unknown processes $(Y, Z, K) = (Y^\pi, Z^\pi, K^\pi)$, by

\[
\begin{aligned}
dY(t) &= -g(t, X(t), \mathbb{E}[X(s)], Y(t), \mathbb{E}[Y(t)], \pi(t), \omega) dt + Z(t) dB(t) \\
&\quad + \int_{\mathbb{R}_0^d} K(t, e, \omega) N(dt, de), \quad t \in [0, \infty).
\end{aligned}
\]

We assume that equation \[25\] satisfies \[30\] and for all finite $T$ this is equivalent to

\[
Y(t) = \mathbb{E}\left[ Y(T) + \int_t^T g(s, X(s), \mathbb{E}[X(s)], Y(s), \mathbb{E}[Y(s)], \pi(s)) ds \mid \mathcal{F}_t \right]; t \leq T \leq \infty.
\]

Notice that the function $b_0$ from the state equation \[24\] depends only on $\mathbb{E}[X(t)]$ and on the control $\pi(t)$, and that the driver $g$ is independent of $Z$. We have put no further restrictions on the coefficient functionals so far. Let $f = 0, b = 0$ and $h_1(y) = y$, in particular, we want to maximize the performance functional

\[
J(\pi) := Y^\pi(0) = \mathbb{E}[Y^\pi(0)].
\]

The admissible controls are assumed to be the càdlàg, $\mathcal{G}_t$-adapted non-negative processes in $L^2(\Omega \times [0, T])$.

The adjoint processes $(p, q, r) = (p^\pi, q^\pi, r^\pi)$ and $\lambda = \lambda^\pi$ corresponding to $\pi$ are defined by

\[
dp(t) = -\mathbb{E}[\mathcal{Y}(t)|\mathcal{F}_t] dt + q(t) dB(t) + \int_{\mathbb{R}_0^d} r(t, e) N(dt, de),
\]

with

\[
\mathcal{Y}(t) = \frac{\partial}{\partial t} b_0(t) + \mathbb{E}\left[ \frac{\partial}{\partial t} b_0(t) \right] \\
+ \sum_{i=0}^2 \int_{-\delta}^0 \left\{ \frac{\partial H}{\partial t}(t-s, \pi) + \mathbb{E}\left[ \frac{\partial H}{\partial t}(t-s, \pi) \right] \right\} \mu_i(ds)
\]
and
\[
\begin{aligned}
\left\{ \begin{array}{l}
d\lambda(t) = \lambda(t) \left( \frac{\partial}{\partial y} g(t, \pi) + \mathbb{E} \left[ \frac{\partial}{\partial \eta} g(t, \pi) \right] \right) dt, t \in [0, \infty) \\
\lambda(0) = \frac{h'(1)}{Y(0)} = 1.
\end{array} \right.
\end{aligned}
\] (87)

We assume that equation (86) satisfies the decay condition (36).

The Hamiltonian for the forward-backward system is
\[
H(t, \pi) = \left( b_0(t, \mathbb{E}[x]) - \pi \right) p + \sigma(t, \pi) q
+ \int_{\mathbb{R}_0} \gamma(t, \pi, e) r(t, e) \nu(de) + g(t, \pi) \lambda
\] (88)

and
\[
\frac{\partial}{\partial \pi} H(t, \pi) = -p(t) + \frac{\partial}{\partial \pi} \left( \sigma(t, \pi(t)) q(t) \\
+ \int_{\mathbb{R}_0} \gamma(t, \pi(t), e) r(t, e) \nu(de) + g(t, \pi(t)) \lambda(t) \right).
\] (89)

Now, applying the necessary maximum principle to the expression above yields the following:

**Corollary 5.1.** Suppose that \( \hat{\pi}(t) \) is an optimal control. Then
\[
\mathbb{E}[\hat{\pi}(t) | \mathcal{G}_t] = \mathbb{E} \left[ \frac{\partial}{\partial \pi} \left( \sigma(t, \hat{\pi}(t)) \hat{q}(t) \\
+ \int_{\mathbb{R}_0} \gamma(t, \hat{\pi}(t), e) \hat{r}(t, e) \nu(de) + g(t, \hat{\pi}(t)) \hat{\lambda}(t) \right) \right] | \mathcal{G}_t.
\] (90)

We see that if we can put additional conditions on the forward-backward system such that \( \hat{q} = 0, \hat{r} = 0, \lambda \) is deterministic with \( \lambda > 0 \) and that \( \mathcal{G}_t = \mathcal{F}_t \), then (90) reduces to
\[
\frac{\hat{p}(t)}{\hat{\lambda}(t)} = \frac{\partial}{\partial \eta} g(t, \hat{\pi}).
\] (91)

**Example 5.2.** Suppose that the following condition holds:
\( g \) is independent of \( x \) and \( m \), for example let us take
\[
g(t, \pi) = -\alpha Y(t) + \beta \mathbb{E}[Y(t)] - \ln \pi.
\]

Then \( (p, 0, 0) \) where \( p \) solves the deterministic equation
\[
\hat{p}(t) = \hat{p}(T) - \int_t^T \frac{\partial}{\partial m} b_0(s, m) p(s) ds
\]
solves (86) for all finite \( T \). Now by (91) we deduce that
\[
\hat{\pi}(t) = \frac{-\hat{\lambda}(t)}{\hat{p}(t)}.
\] (92)
with
\[
\frac{\partial}{\partial \pi} \rho(t, \pi) = \frac{-1}{\pi(t)}. \tag{93}
\]

Consequently
\[
\lambda(t) = e^{-(\alpha - \beta)t} \text{ for all } t \in [0, \infty). \tag{94}
\]

Combining (93) and (91), if \(\pi\) is bounded away from 0 we have
\[
\hat{p}(T) = \frac{e^{-(\alpha - \beta)T}}{\hat{\pi}(T)} \xrightarrow{T \to \infty} 0 \text{ if } \beta < \alpha. \tag{95}
\]

Put \(\pi = 0\) in equation (83), integrating and taking expectation, we obtain
\[
h(t) := \mathbb{E}[X(t)] = x + \int_0^t b_0(s, \mathbb{E}[X(s)])ds.
\]

First we assume that \(b_0\) has at most linear growth, in the sense that there exists a constant \(c\) such that \(b_0(t, x) \leq cx\). Then we get
\[
h(t) \leq x + c \int_0^t h(s)ds
\]
and hence by the Gronwall inequality it follows that
\[
h(t) \leq xe^{ct} \text{ for all } t. \tag{96}
\]

For given consumption rate \(\pi\), let \(X^{\pi}(t)\) be the corresponding solution of (83). Then since \(\pi(t) \geq 0\) for all \(t\), we always have \(X^{\pi}(t) \leq X^0(t)\). Therefore, to prove the transversality condition it suffices to prove that \(\mathbb{E} [\hat{p}(T)X^0(T)]\) goes to 0 as \(T\) goes to infinity.

Let us compare (96) with the decay of \(\hat{p}(T)\) in (95) we get
\[
\mathbb{E} [\hat{p}(T)X^0(T)] = \mathbb{E} [\hat{p}(T)X(T)] = \hat{p}(T)\mathbb{E}[X(T)] = \frac{x e^{-(\alpha - \beta - c)T}}{\pi} \xrightarrow{T \to \infty} 0 \text{ if } c < (\alpha - \beta). \tag{97}
\]
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