Quantum states of a generalized time-dependent inverted harmonic oscillator

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Abstract
We discuss the extension of the Lewis and Riesenfeld method of solving the time-dependent Schrödinger equation to cases where the invariant has continuous eigenvalues and apply it to the case of a generalized time-dependent inverted harmonic oscillator. As a special case, we consider a generalized inverted oscillator with constant frequency and exponentially increasing mass.

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I. INTRODUCTION

In the last few years, the study of the time-dependent inverted harmonic oscillator has attracted some attention in the literature[1-4]. Despite its mathematical interest, this problem may find interesting applications in different areas of physics once many quantum-mechanical effects can be treated phenomenologically by means of the time-dependent parameters in the Hamiltonian of the system.

The time-dependent inverted harmonic oscillator is exactly solvable just like the standard time-dependent harmonic oscillator. However, the physics of the time-dependent inverted oscillator is very different: it has a wholly continuous energy spectrum varying from minus to plus infinity; its energy eigenstates are no longer square-integrable and they are doubly degenerate with respect to either the incident direction or, alternatively, the parity.

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The time-dependent inverted harmonic oscillator with an exponentially increasing mass and constant frequency, the so-called inverted Caldirola-Kanai oscillator, has been discussed by Baskoutas et al. On the other hand, the most general case where mass and frequency are both time-dependent has been considered by us in Ref. [4].

In this paper, we discuss the Lewis and Riesenfeld (LR) invariant method [5] to cases where the invariant has continuous eigenvalues and apply it to obtain the exact wave functions of a generalized time-dependent inverted harmonic oscillator. As a special case, we also find the wave functions of a generalized inverted oscillator with constant frequency and exponentially increasing mass. In Sec. II, we briefly outline the LR invariant method to cases where the invariant has continuous eigenvalues. In Sec III, we find the Schrödinger wave functions for a generalized time-dependent inverted oscillator. In addition, as a particular case, we obtain the wave functions of a generalized inverted oscillator with constant frequency and exponentially increasing mass. We end with a summary in Sec. IV.

II. THE LR INVARIANT METHOD: CONTINUOUS EIGENVALUES

The work of LR [5] assumes that the eigenvalue spectrum for the invariant is discrete. Here, we wish investigate the LR method in the case of continuous eigenvalues. To do so, we will consider a quantum system characterized by a time-dependent Hamiltonian \( H(t) \) and whose corresponding Schrödinger equation is

\[
i\hbar \frac{\partial \psi(q, t)}{\partial t} = H(t)\psi(q, t).
\]

A Hermitian operator \( I(t) \) is called an invariant for the system if it satisfies

\[
\frac{dI}{dt} = \frac{1}{i\hbar} [I, H] + \frac{\partial I}{\partial t} = 0.
\]

The eigenvalue equation for \( I(t) \) can be written as [5]

\[
I\phi_\lambda(q, t) = \lambda\phi_\lambda(q, t),
\]

where

\[
\langle \phi_\lambda | \phi_{\lambda'} \rangle = \delta(\lambda - \lambda').
\]

With the aid of Eq. [2], it is easy to show that

\[
\frac{\partial \lambda}{\partial t} = 0.
\]

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The solutions \( \psi_\lambda(q, t) \) for the Schrödinger equation (1) can be expressed as

\[
\psi_\lambda(q, t) = e^{i\alpha_\lambda(t)}\phi_\lambda(q, t),
\]

(6)

where the phases functions \( \alpha_n(t) \) are found from the equation

\[
\hbar \frac{d\alpha_\lambda(t)}{dt} = \left\langle \phi_\lambda \left| i\hbar \frac{\partial}{\partial t} - H(t) \right| \phi_\lambda \right\rangle.
\]

(7)

Therefore, we may write the general solution of Eq. (1) as[6]

\[
\psi(q, t) = \int c(\lambda) e^{i\alpha_\lambda(t)}\phi_\lambda(q, t) d\lambda,
\]

(8)

where

\[
c(\lambda) = \left\langle \psi_\lambda(q, 0) \right| \psi(q, 0) \right\rangle.
\]

(9)

Finally, let us recall that the calculations are similar, though not identical to the discrete cases.

### III. THE GENERALIZED TIME-DEPENDENT INVERTED HARMONIC OSCILLATOR

#### A. Exact invariants and the solution of the Schrödinger equation

Consider the generalized time-dependent inverted oscillator described by the Hamiltonian

\[
H(t) = \frac{p^2}{2M(t)} - \frac{1}{2}M(t)\omega^2(t)q^2 + \frac{y(t)}{2}(pq + qp),
\]

(10)

where \( q, p \) are canonical coordinates with \([q, p] = i\hbar\), \( M(t) \) and \( \omega(t) \) are time-dependent mass and frequency and \( y(t) \) is an arbitrary function of time. Note that the Hamiltonian (10) can be obtained from the standard generalized time-dependent oscillator by replacement \( \omega(t) \rightarrow i\omega(t) \). The Heisenberg’s equations are

\[
\dot{q} = \frac{1}{i\hbar}[q, H] = \frac{p}{M(t)} + y(t)q,
\]

(11)

\[
\dot{p} = \frac{1}{i\hbar}[p, H] = M(t)\omega^2(t)q - yp.
\]

(12)

From Eqs. (11) and (12) we readily obtain that

\[
\ddot{q} + \gamma(t)\dot{q} - \Omega^2(t)q = 0,
\]

(13)

where

\[
\gamma(t) = \frac{d}{dt}\ln[M(t)]
\]

(14)
and
\[\Omega^2(t) = \omega^2 + y^2 + \gamma y + \dot{y},\] (15)
is the modified frequency. Now, it is easy to verify that an invariant for the Hamiltonian (10) is given by
\[I(t) = \frac{1}{2} \left\{ -\left(\frac{q}{\rho}\right)^2 + [pp - M(\dot{\rho} - y\rho)]^2 \right\},\] (16)
where \(\rho(t)\) is a c-number quantity satisfying the auxiliary equation
\[\ddot{\rho} + \gamma(t)\dot{\rho} - \Omega^2(t)\rho = -\frac{1}{M^2}\rho^3.\] (17)

Next, we want to solve the Schrödinger equation, Eq. (1), with \(H(t)\) given by [see Eq. (10)]
\[H(t) = -\frac{\hbar^2}{2M(t)} \frac{\partial^2}{\partial q^2} - \frac{1}{2} M(t)\omega^2(t) q^2 - i\hbar \frac{y(t)}{2} - i\hbar y(t)q \frac{\partial}{\partial q},\] (18)
where \(p = -i\hbar \frac{\partial}{\partial q}\) has been used. To this end, we consider the unitary transformation
\[\phi'(q, t) = U \phi(q, t),\] (19)
where
\[U = \exp \left[ -\frac{iM(t)}{2\hbar\rho} (\dot{\rho} - y\rho)q^2 \right].\] (20)
Under this unitary transformation, the eigenvalue equation, Eq. (3), with \(I(t)\) given by Eq. (16) becomes
\[I'\phi'(q, t) = \lambda \phi'(q, t),\] (21)
with
\[I' = UIU^\dagger = -\frac{\hbar^2}{2\rho^2} \frac{\partial^2}{\partial q^2} - \frac{1}{2} \left(\frac{q}{\rho}\right)^2.\] (22)
Then, taking \(\sigma = q/\rho\) we can write the eigenvalue equation, Eq. (21), in the form
\[\left[ -\frac{\hbar^2}{2} \frac{\partial^2}{\partial \sigma^2} - \frac{\sigma^2}{2} \right]\varphi_\lambda(\sigma) = \lambda \varphi_\lambda(\sigma),\] (23)
or
\[I'\varphi_\lambda(\sigma) = \lambda \varphi_\lambda(\sigma),\] (24)
where
\[\phi'(q, t) = \frac{1}{\rho^{1/2}} \varphi_\lambda(\sigma) = \frac{1}{\rho^{1/2}} \varphi_\lambda(q/\rho).\] (25)
The factor \(1/\rho^{1/2}\) is introduced into Eq. (25) so that the condition
\[\int \phi^*_\lambda(q, t)\phi'(q, t) dq = \int \varphi^*_\lambda(\sigma)\varphi_\lambda(\sigma) d\sigma\] (26)

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holds. Now, Eq. (23) is similar to the eigenvalue equation of the time-independent inverted harmonic oscillator [7]. Then, by setting

$$\epsilon = \frac{\lambda}{\hbar},$$

(27)

and

$$z = \left(\frac{2}{\hbar}\right)^{1/2} \sigma,$$

(28)

Eq. (23) can be written as

$$\frac{\partial^2 \varphi}{\partial z^2} + \left(\frac{1}{4} z^2 + \epsilon\right) \varphi(z, \epsilon) = 0.$$  

(29)

The solutions of Eq. (29) can be expressed in terms of the parabolic cylinder (or Weber) functions [3, 7, 8]. Thus, using Eqs. (19), (20), (25), (27) and (28) we get

$$\phi_{\lambda}(q, t) = \frac{1}{\rho^{1/2}} \exp \left[\frac{i M(t)}{2 \hbar \rho} (\dot{\rho} - y \rho) q^2 \right] \varphi_{\lambda} \left[\left(\frac{2}{\hbar}\right)^{1/2} \left(\frac{q}{\rho}, \frac{\lambda}{\hbar}\right)\right],$$

(30)

where $\varphi_{\lambda}$ in the Eq. (30) is the Weber’s function.

On the other hand, substituting Eqs. (18) and (19) into Eq. (7) and following the same steps as those of Refs. [9, 10] we find that the phase functions $\alpha_{\lambda}(t)$ are given by

$$\alpha_{\lambda}(t) = -\frac{\lambda}{\hbar} \int_0^t \frac{1}{M(t') \rho(t')^2} dt'.$$

(31)

Therefore, using Eqs. (6) and (30) we obtain that the exact solutions of the Schrödinger equation for the generalized time-dependent inverted oscillator are

$$\psi_{\lambda}(q, t) = \frac{1}{\rho^{1/2}} e^{i \alpha_{\lambda}(t)} \exp \left[i M(t) (\dot{\rho} - y \rho) q^2 \right] \varphi_{\lambda} \left[\left(\frac{2}{\hbar}\right)^{1/2} \left(\frac{q}{\rho}, \frac{\lambda}{\hbar}\right)\right],$$

(32)

where the phase functions are given by Eq. (31). Now, the general solution of the Schrödinger equation is found by directly substituting Eq. (32) into Eq. (8). For $y(t) = 0$, the wave functions (32) are exactly reduced to those of the inverted oscillator with time-dependent mass and frequency obtained in Ref. [4].

B. Generalized inverted oscillator with constant frequency and exponentially increasing mass

In what follows, we consider the case where the frequency $\omega(t)$ and the function $y(t)$ are constants, i.e., $\omega(t) = \omega_0$ and $y(t) = y_0$ and the mass is given by

$$M(t) = me^{\gamma t}.$$  

(33)
For this case, the Hamiltonian (10) becomes
\[ H(t) = e^{-\gamma t} \frac{p^2}{2m} - \frac{1}{2} m\omega_0 e^{\gamma t} q^2 + \frac{y_0}{2} (pq + qp), \] (34)
with \( \gamma = \text{const} \) [see Eq. (14)]. Note that in this case the modified frequency \( \Omega(t) \), Eq. (15), is constant and given by
\[ \Omega^2(t) = \omega_0^2 + y_0^2 + \gamma y_0 \equiv \Omega_0^2. \] (35)
Here we observe that for \( y_0 = 0 \) the Hamiltonian (34) reduces to that of the inverted Caldirola-Kanai oscillator [3].

Let us now consider a particular solution of Eq. (17) given by
\[ \rho(t) = \frac{1}{(m\Omega_1)^{1/2}} e^{-\gamma t/2}, \] (36)
where
\[ \Omega_1^2 = \Omega_0^2 + \gamma^2/4. \] (37)
On the other hand, inserting Eqs. (33) and (36) into Eq. (31) we find that
\[ \alpha_\lambda(t) = -\frac{\lambda \Omega_1}{\hbar} t. \] (38)
Thus, by substituting Eqs. (33), (36) and (38) into Eq. (32) we get after minor algebra that
\[ \psi_\lambda(q, t) = (m\Omega_1)^{1/4} \exp \left[ \frac{\gamma t}{4} - i \frac{\lambda \Omega_1}{\hbar} t - i m e^{-\gamma t/2} \left( \frac{\gamma}{2} + y_0 \right) q^2 \right] \]
\[ \times \phi_\lambda \left[ \left( \frac{2m\Omega_1}{\hbar} \right)^{1/2} e^{-\gamma t/2} q, \frac{\lambda}{\hbar} \right], \] (39)
which are the exact wave functions of the generalized inverted oscillator with constant frequency and exponentially increasing mass. For \( y_0 = 0 \), the wave functions (39) are reduced to those of the inverted Caldirola-Kanai oscillator [3, 4].

IV. SUMMARY

In this paper, we have discussed the LR invariant method for the cases where the invariant has continuous eigenvalues and have employed it to find the exact wave functions of a generalized time-dependent inverted oscillator. We also have obtained, as a special case, the wave functions of a generalized inverted oscillator with constant frequency and exponentially increasing mass. Furthermore, we have seen that for \( y(t) = 0 \) and \( y_0 = 0 \) our results agree with those obtained in Refs. [3, 4]. Finally, we would like to remark...
that as far as we know the Hamiltonians (10) and (34) and the wave functions (32) and (34) have not yet been exhibited in the literature.

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