Analysis and preconditioning of parameter-robust finite element methods for Biot’s consolidation model

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Abstract
In this paper we consider a three-field formulation of the Biot model which has the displacement, the total pressure, and the pore pressure as unknowns. For parameter-robust stability analysis, we first show a priori estimates of the continuous problem with parameter-dependent norms. Then we study finite element discretizations which provide parameter-robust error estimates and preconditioners. For finite element discretizations we consider standard mixed finite element as well as stabilized methods for the Stokes equations, and the complete error analysis of semidiscrete solutions is given. Abstract forms of parameter-robust preconditioners are investigated by the operator preconditioning approach. The theoretical results are illustrated with numerical experiments.

Keywords Poroelasticity · Error analysis · Preconditioning

Mathematics Subject Classification 65N15 · 65N30

1 Introduction
In poroelastic media saturated by fluids, the behaviors of porous medium and the saturating fluid flow are described by Biot’s consolidation model [5]. Poroelasticity models are widely used in geophysics and petroleum engineering applications, so development of finite element methods for the poroelastic models began more than four decades ago [32, 35] and is still an active research area (see, e.g., [3, 4, 7, 8, 19, 22, 24–26, 28, 29, 33]).
Poroelasticity models for practical applications have various different ranges of parameters. For example, geophysical materials are compressible solids whereas most soft biological tissues are modelled as incompressible or nearly incompressible materials. It turns out that the different parameter ranges are intimately related to accuracy of numerical methods and construction of efficient iterative solvers. Therefore, one of main interests of numerical methods for the Biot model is robustness for model parameter ranges, and there are various recent studies for parameter-robust numerical methods [11, 13, 16, 20, 21, 27] and efficient solvers [2, 14, 17, 22]. Recently, a new three-field formulation for the Biot model was independently introduced in [22] and [27] with different foci of interests. In [22], the main interest is construction of preconditioners robust for various parameters (large bulk and shear moduli, small hydraulic conductivity, and small time step sizes). In [27], the main interest is optimal error estimates robust for large bulk modulus, and they proved stability of the static system using compactness of a linear operator and then proved error estimates. However, complete error analysis for time dependent problems was not given.

The two main purposes of this work are the following: First, we provide comprehensive a priori error analysis of time dependent solutions of the three-field formulation with the total pressure. Compared to the analysis in [30], the analysis in this paper has two new features. First, we included stabilized finite element methods for Stokes equations in [6, 12, 18] for discretization of the elasticity subproblem with error analysis. Second, our error analysis includes an error estimate of fluid flows with the $L^\infty$-in-time and $L^2$-in-space norm (see (23)), which was not given in [30]. We remark that the amount of fluid flows are the quantities of more interest for practical applications. In all of these estimate, we do not use a Grönwall inequality, so there are no exponentially growing factors in the estimates. Second, we provide an abstract form of parameter-robust preconditioners for the stabilized discretization methods. We will use the norm-equivalence preconditioning approach for constructing abstract forms of preconditioners, and present numerical test results.

The paper is organized as follows. In Sect. 2, we introduce preliminary materials including notations, definitions, and the variational formulation of the Biot model. In Sect. 3, we discuss stability of the system and prove energy-type estimates of solutions. In Sect. 4, we discuss finite element discretizations and the a priori error estimates of semidiscrete solutions. In Sect. 5, we prove stability of static system with respect to parameter-dependent norms and propose abstract forms of parameter-robust preconditioners. Finally, we present numerical results illustrating convergence of errors and parameter-robust performances of preconditioners in Sect. 6.

2 Preliminaries

2.1 Notations

Let $\Omega$ be a bounded polygonal domain with Lipschitz continuous boundary in $\mathbb{R}^n$ with $n = 2$ or 3.

For a nonnegative integer $m$, $H^m(\Omega)$, $H^m(\Omega; \mathbb{R}^n)$ denote the standard $\mathbb{R}$ and $\mathbb{R}^n$-valued Sobolev spaces based on $L^2$ norm. For a Banach space $\mathcal{X}$ and $(a, b) \subset \mathbb{R}$,
$C^0(a, b; \mathcal{X})$ denotes the set of functions $f : (a, b) \to \mathcal{X}$ which are continuous in $t \in (a, b)$. For an integer $m \geq 1$ we define

$$C^m(a, b; \mathcal{X}) = \{ f | \partial^i f/\partial t^i \in C^0(a, b; \mathcal{X}), \ 0 \leq i \leq m \},$$

where $\partial f/\partial t$ is the time derivative in the sense of the Fréchet derivative in $\mathcal{X}$ (see e.g., [34]). We also define the space-time norm

$$\| f \|_{L^p(a, b; \mathcal{X})} = \left\{ \begin{array}{ll} \left( \int_a^b \| f \|_\mathcal{X}^p ds \right)^{1/p}, & 1 \leq p < \infty, \\
\operatorname{esssup}_{t \in (a, b)} \| f \|_\mathcal{X}, & p = \infty. \end{array} \right.$$ 

If a time interval $J$ is clear in context, then we use $L^p(J; \mathcal{X})$ to denote $L^p(J; \mathcal{X})$ for simplicity. We define the space-time Sobolev spaces $W^{k,p}(J; \mathcal{X})$ for nonnegative integer $k$ and $1 \leq p \leq \infty$ as the closure of $C^k(J; \mathcal{X})$ with the norm $\| f \|_{W^{k,p}(J; \mathcal{X})} = \sum_{i=0}^k \| \partial^i f/\partial t^i \|_{L^p(J; \mathcal{X})}$. For simplicity we adopt the convention $\| f, g \|_{\mathcal{X}} = \| f \|_{\mathcal{X}} + \| g \|_{\mathcal{X}}$, and $\dot{u}$ is used to denote the time derivative of $f$.

For a triangulation of $\Omega$, $T_h$ is used to denote a shape-regular triangulation for which $h$ is the maximum diameter of triangles (or tetrahedra) and $\mathcal{E}_h$ is the corresponding set of edges (faces), respectively. For $E \in \mathcal{E}_h$ and functions $f, g : \mathcal{E}_h \to \mathbb{R}^n$ we define

$$\langle f, g \rangle_E = \int_E f \cdot g \ ds, \quad \langle f, g \rangle = \sum_{E \in \mathcal{E}_h} \langle f, g \rangle_E.$$ 

For an integer $k \geq 0$ and for each $T \in T_h$, $P_k(T)$ is the space of polynomials of degree $\leq k$ on $T$, and $P_k(T_h)$ denotes the space

$$P_k(T_h) = \begin{cases} \{ q \in H^1(\Omega) : q|_T \in P_k(T), \ T \in T_h \} & \text{if } k \geq 1 \\
\{ q \in L^2(\Omega) : q|_T \in P_k(T), \ T \in T_h \} & \text{if } k = 0 \end{cases}$$ 

For a vector space $\mathbb{X}$, we use $P_k(G; \mathbb{X})$ and $P_k(T_h; \mathbb{X})$ to denote the space of $\mathbb{X}$-valued polynomials with same conditions. We will use $A \lesssim B$ to denote an inequality $A \leq CB$ with a generic constant $C > 0$ which is independent of mesh sizes in the rest of this paper. The constant $C$ can be different in each inequality.

### 2.2 The Biot’s consolidation model

Throughout this paper we restrict our interest on quasistatic consolidation problems and the acceleration term is ignored. In our description of the model, $u$ is the displacement of porous media, $p_p$ is the pore pressure, $f$ is the body force, $g$ is the mass change rate of fluid. The governing equations of Biot’s consolidation model with an isotropic elastic porous medium are

$$- \operatorname{div} \left( 2\mu \varepsilon(u) + (\lambda + \alpha) p_p \right) = f,$$ 

where $\varepsilon(u)$ is the strain tensor of $u$.
\[ s_0 \dot{p}_p + \alpha \text{div} \dot{u} - \text{div}(\kappa \nabla p_p) = g, \quad (1b) \]

where \( \mu \) and \( \lambda \) are the Lamé coefficients, \( s_0 \geq 0 \) is the constrained specific storage coefficient, \( \kappa \) is the hydraulic conductivity tensor, \( \alpha > 0 \) is the Biot–Willis constant which is close to 1, and \( \mathbb{I} \) is the identity matrix. We assume that \( \mu \) is uniformly bounded above and below with positive constants. We assume \( \lambda \) has a uniformly positive lower bound but \( \lambda \) may not have a uniform upper bound and \( \lambda = +\infty \) corresponds to the incompressibility of the solid matrix. We assume that there are constants \( c_0, c_1 \) such that

\[ 0 \leq c_0 \leq s_0(x) \leq c_1, \quad x \in \Omega. \]

We remark that \( s_0 \) is related to \( \alpha \), the porosity \( \phi \), and the bulk moduli of the solid and fluid. Under the assumption that \( \phi \) is uniform with \( 0 < \phi < \alpha \), if the solid is not incompressible, then \( s_0 \geq C/\lambda \) holds with a constant \( C \) of scale 1 (cf. [9]). The hydraulic conductivity tensor \( \kappa = \kappa(x) \) is positive definite with uniform lower and upper bounds \( \kappa_0, \kappa_1 > 0 \), i.e.,

\[ \kappa_0 |\xi|^2 \leq \xi^T \kappa(x) \xi \leq \kappa_1 |\xi|^2, \quad \forall \ 0 \neq \xi \in \mathbb{R}^n, \ a.e. \ x \in \Omega. \]

On details of deriving these equations from physical modelling, we refer to standard porous media texts, e.g., [1].

For well-posedness of the problem, the equations (1) need appropriate boundary and initial conditions. We assume that there are partitions of \( \partial \Omega \) which are

\[ \partial \Omega = \Gamma_p \cup \Gamma_f, \quad \partial \Omega = \Gamma_d \cup \Gamma_t, \quad |\Gamma_d|, |\Gamma_p| > 0 \]

where \( |\Gamma| \) is the \((n - 1)\)-dimensional Lebesgue measure of \( \Gamma \). We also assume that boundary conditions are given as

\[ p(t) = 0 \text{ on } \Gamma_p, \quad -\kappa \nabla p(t) \cdot n = 0 \text{ on } \Gamma_f, \quad u(t) = 0 \text{ on } \Gamma_d, \quad \sigma(t)n = 0 \text{ on } \Gamma_t, \quad (2) \]

for all \( t \in (0, T] \) where \( n \) is the outward unit normal vector field on \( \partial \Omega \) and \( \sigma := 2\mu \epsilon(u) + (\lambda \text{div} u - \alpha p) \mathbb{I} \), the Cauchy stress tensor. Here we only consider the homogeneous boundary condition for simplicity but our method can be easily extended to problems with nonhomogeneous boundary conditions. We also assume that given initial data \( p(0), u(0) \) and \( f(0) \) satisfy the compatibility condition (1a). Well-posedness of this system under these assumptions can be found in [31].

2.3 The formulation with the displacement, total and pore pressures

In [22, 27], a formulation of the Biot model with three unknowns was introduced in order to obtain a parameter-robust formulation of the Biot model for finite element discretizations. Introduction of a new unknown \( p_t := \lambda \text{div} u - \alpha p_p \), which will
be called total pressure, gives an additional equation $\text{div } u - \lambda^{-1}(p_t + \alpha p_p) = 0$. Therefore, we consider a system

\begin{align*}
- \text{div} (2\mu \epsilon(u)) - \nabla p_t &= f, \\
\text{div } u - \lambda^{-1}(p_t + \alpha p_p) &= 0, \\
-\alpha \lambda^{-1} \dot{p}_t - \left( s_0 + \alpha^2 \lambda^{-1} \right) \dot{p}_p + \text{div}(\kappa \nabla p_p) &= -g.
\end{align*}

Let us define function spaces

\begin{align*}
V &= \left\{ v \in H^1(\Omega; \mathbb{R}^n) : v|_{\Gamma_d} = 0 \right\}, \quad Q_t = L^2(\Omega), \\
Q_p &= \{ q \in H^1(\Omega) : q|_{\Gamma_p} = 0 \},
\end{align*}

and consider the following variational form of (3):

**(VP)** For initial data $(u(0), p_t(0), p_p(0)) \in V \times Q_t \times Q_p$ satisfying

\begin{align*}
(2\mu \epsilon(u(0)), \epsilon(v)) + (p_t(0), \text{div } v) &= (f(0), v) \quad \forall v \in V, \\
\text{div } u(0) - \lambda^{-1}(p_t(0) + \alpha p_p(0)) &= 0,
\end{align*}

find $(u, p_t, p_p) \in C^0(0, T; V) \times C^1(0, T; Q_t) \times C^1(0, T; Q_p)$ such that

\begin{align*}
(2\mu \epsilon(u), \epsilon(v)) + (p_t, \text{div } v) &= (f, v), \\
(\text{div } u, q_t) - \left( \lambda^{-1} p_t, q_t \right) - \left( \alpha \lambda^{-1} p_p, q_t \right) &= 0, \\
-\left( \alpha \lambda^{-1} \dot{p}_t, q_p \right) - \left( \left( s_0 + \alpha^2 \lambda^{-1} \right) \dot{p}_p, q_p \right) - (\kappa \nabla p_p, \nabla q_p) &= -g, q_p
\end{align*}

for all $(v, q_t, q_p) \in V \times Q_t \times Q_f$.

We remark that the $C^1$ time regularity of $p_t$ and $p_p$ can be further weakened in consideration of the distributions of space-time functions. However, the main focus of this work is the stability and error estimates of numerical methods which require sufficient regularity of solutions, so we do not present the formulation with fully general regularity assumptions here.

### 3 Discretization with finite elements

In this section we discuss finite element discretization of (5) and the a priori error analysis of numerical solutions. We are interested in discretizations which are robust for the parameters including arbitrarily large $\lambda > 0$, and only nonnegative $s_0 \geq 0$. Note that the limit case $\lambda = \infty$ decouples (5) into two separate problems, the Stokes equation and a time-dependent Darcy flow problems. Therefore, it is natural to combine two finite element methods, one for the Stokes equation for $(u, p_t)$ and the other for the Darcy flow problems for $p_p$. 
For discretizations of the Stokes equation, standard mixed methods with conforming finite elements are natural choices but stabilized methods for the Stokes equation are sometimes preferred due to their smaller number of degrees of freedom. Therefore we propose formulations covering some low order stabilized methods for discretization of \((u, p_t)\) with the a priori error analysis. The parameter \(\mu\) is assumed to be 1 in the model problem of Stokes equations. However, \(\mu\) is a function in \(\Omega\) with large parameter value in most practical poroelasticity problems, so we assume that \(\mu \lesssim 1\), \(\mu_{\min} \leq \mu \leq \mu_{\max}\) and \(\mu_{\max}/\mu_{\min}\) is bounded above and below in \(\Omega\).

For discretizations of \(p_p\), the standard method with Lagrange finite elements is the simplest numerical method but it does not give numerical solutions with local mass conservation. In this paper we use the enriched Galerkin method that we can obtain a locally mass conservative flux via local post-processing. However, our error analysis can be extended to any discretization methods of the Poisson equation including continuous and various discontinuous Galerkin methods.

### 3.1 Finite element methods for the Stokes and Poisson equations

In this subsection we introduce the mixed and stabilized methods for the Stokes equation of \((u, p_t)\) and the Lagrange finite elements for the Poisson equation of \(p_p\). In this section we denote \(V_h, Q_{t,h}, Q_{p,h}\) the finite element spaces for the unknowns \(u, p_t, p_p\), and assume that \(V_h \subset V, Q_{t,h} \subset Q_t,\) and \(Q_{p,h} \subset Q_p\). We define \(k_u, k_{p_t}, k_{p_p}\) by

\[ k_u, k_{p_t}, k_{p_p} : \text{the optimal convergence rates of } V_h, Q_{t,h}, Q_{p,h} \text{ in } L^2. \tag{6} \]

To describe the mixed and stabilized methods of \(V_h\) and \(Q_{t,h}\), let us consider an auxiliary problem to find \((u, p) \in V \times Q_t\) such that

\[
(2\mu \epsilon(u), \epsilon(v)) + (p_t, \text{div } v) = (f_1, v), \quad (\text{div } u, q_t) = (f_2, q_t) \tag{7}
\]

for all \((v, q_t) \in V \times Q_t\). First, we can use stable mixed finite elements \((V_h, Q_{t,h})\), i.e., the pair \((V_h, Q_{t,h})\) satisfies the inf-sup condition

\[
\inf_{0 \neq q_t \in Q_{t,h}} \sup_{0 \neq v \in V_h} \frac{(\text{div } v, q_t)}{\|\nabla v\|_0 \|q_t\|_0} \geq C > 0 \tag{8}
\]

with a constant \(C\) independent of \(h\). A similar inf-sup condition holds with denominator \(\|\epsilon(v)\|_V \|q_t\|_{Q_t}\) by rescaling of norms, and the inf-sup constant depends on the constant of Korn’s inequality and \(\mu_{\max}/\mu_{\min}\). For stabilized methods for (7), we consider the stabilized methods of the form

\[
B(u_h, p_{t,h}; v, q_t) := (2\mu \epsilon(u_h), \epsilon(v)) + (p_{t,h}, \text{div } v) + (\text{div } u_h, q_t) - s_h(p_{t,h}, q_t),
\]

\[
F(v, q_t) := (f_1, v) + (f_2, q_t) + \tilde{s}_h(f_1, q_t)
\]
with some bilinear and linear forms \( s_h \) and \( \tilde{s}_h \) on \( V_h \times Q_{t,h} \) such that
\[
|s_h(p_t, q_t)| \leq \| p_t \|_{Q_t} \| q_t \|_{Q_t}.
\]
The discretization of (7) is to find \((u_h, p_{t,h}) \in V_h \times Q_{t,h}\) such that
\[
B(u_h, p_{t,h}; v, q_t) = F(v, q_t) \quad (v, q_t) \in V_h \times Q_{t,h}.
\]
We assume that this discretization is consistent (with sufficiently regular exact solutions) and also assume that an inf-sup condition
\[
\inf_{(u, p_t) \in V_h \times Q_{t,h}} \sup_{(v, q_t) \in V_h \times Q_{t,h}} \frac{B(u, p_t; v, q_t)}{\| u \|_V + \| p_t \|_{Q_t} \| v \|_V + \| q_t \|_{Q_t}} \geq C > 0 \tag{10}
\]
holds with \( C \) independent of \( h \) and parameters.

We here remark that there are known stabilized methods satisfying (10), for example,
\[
V_h = P_1(T_h; \mathbb{R}^n), Q_{t,h} = P_0(T_h),
\]
\[
s_h(p_t, q_t) = \frac{\gamma_2}{2\mu} \sum_{e \in \mathcal{E}_h} h_e^{-1} \langle \| p_t \|, \| q_t \| \rangle_e, \tilde{s}_h = 0, \tag{11b}
\]
with \( \| q_t \| \), the jump of \( q_t \) on edges/faces (cf. [18]), and
\[
V_h = P_1(T_h; \mathbb{R}^n), Q_{t,h} = P_1(T_h),
\]
\[
s_h(p_t, q_t) = \frac{\gamma_2}{2\mu} \sum_{T \in T_h} h_T^2 (\nabla p_t, \nabla q_t)_T, \tilde{s}_h(f, q_t) = -\frac{\gamma_2}{2\mu} \sum_{T \in T_h} h_T^2 (f, \nabla q_t), \tag{12b}
\]
where \( \gamma_2 > 0 \) is a parameter depending on the shape regularity of meshes. These stabilization methods were proposed in [6] and [18], respectively. For more on stabilized methods for the Stokes equation, we refer to [12].

For \( Q_{p,h} \) we use the standard Lagrange finite elements.

### 3.2 Semidiscrete error analysis

The semidiscrete formulation of (5) is to find \((u_h, p_{t,h}, p_{p,h}) \in C^1(0, T; V_h) \times C^1(0, T; Q_{t,h}) \times C^1(0, T; Q_{p,h})\) such that
\[
(2\mu \varepsilon(u_h), \varepsilon(v)) + (p_{t,h}, \text{div } v) = (f, v), \tag{13a}
\]
\[
(\text{div } u_h, q_t) - s_h(p_{t,h}, q_t) - (\lambda^{-1} p_{t,h}, q_t) - (\alpha \lambda^{-1} p_{p,h}, q_t) = \tilde{s}_h(f, q_t), \tag{13b}
\]
\[
- (\alpha \lambda^{-1} \hat{p}_{t,h}, q_p) - \left( (s_0 + \alpha^2 \lambda^{-1}) \hat{p}_{p,h}, q_p \right) - (k \nabla p_{p,h}, \nabla q_p) = (g, q_p) \tag{13c}
\]
for any \( v \in V_h, q_t \in Q_{l,h}, q_p \in Q_{p,h} \). It is obvious that \( s_h = \tilde{s}_h = 0 \) if we use mixed methods for \((u_h, p_t)\).

Suppose that \((u, p_t, p_p)\) is an exact solution of (5) and \((u_h, p_{l,h}, p_{p,h})\) is a numerical solution of (13), and define

\[
eu(t) := u(t) - u_h(t), \quad ep(t) := p_t(t) - p_{l,h}(t), \quad ep_p(t) := p_p(t) - p_{p,h}(t).
\]

For some interpolations \((\Pi_h^V u(t), \Pi_h^Q p_t(t), \Pi_h^{Q_p} p_p(t)) \in V_h \times Q_{l,h} \times Q_{p,h}\), which will be defined below, we split the errors into two parts as

\[
eu(t) = e^{u}_u(t) + e^{h}_u(t) := (u(t) - \Pi_h^V u(t)) + (\Pi_h^V u(t) - u_h(t)),
\]

\[
ep(t) = e^{l}_p(t) + e^{h}_p(t) := (p_t(t) - \Pi_h^Q p_t(t)) + (\Pi_h^Q p_t(t) - p_{l,h}(t)),
\]

\[
ep_p(t) = e^{l}_p(t) + e^{h}_p(t) := (p_p(t) - \Pi_h^{Q_p} p_p(t)) + (\Pi_h^{Q_p} p_p(t) - p_{p,h}(t)).
\]

We define \(\Pi_h^V u(t)\) and \(\Pi_h^Q p_t(t)\) as the solution of auxiliary problem: (AP1) Find \((\Pi_h^V u(t), \Pi_h^Q p_t(t)) \in V_h \times Q_{l,h}\) such that

\[
\left(2\mu \epsilon(\Pi_h^V u(t), \epsilon(v)) + \left(\Pi_h^Q p_t(t), \text{div} \ v\right)\right) + \left(\text{div} \ Pi_h^V u(t), q_t\right) - s_h \left(\Pi_h^Q p_t(t), q_t\right) = (f(t), v),
\]

\[
\left(\text{div} \ Pi_h^V u(t), q_t\right) - s_h \left(\Pi_h^Q p_t(t), q_t\right) = (\text{div} \ u(t), q_t) + \tilde{s}_h(f(t), q_t)
\]

for any \((v, q_t) \in V_h \times Q_{l,h}\).

The stability of mixed methods (when \(s_h = \tilde{s}_h = 0\)) or stabilized methods guarantees the well-posedness of this problem, and furthermore, standard error analyses of mixed or stabilized methods for the Stokes equation [12, 15] give

\[
\|u(t) - \Pi_h^V u(t)\|_V + \|p_t(t) - \Pi_h^Q p_t(t)\|_{Q_l} \\
\lesssim h^m (\|u(t)\|_{m+1} + \|p_t(t)\|_m)
\]

(17)

with \(m \leq \max\{k_u - 1, k_{p_t}\}\) which depends on the regularities of \(u(t)\) and \(p_t(t)\).

We define \(\Pi_h^{Q_p} p_p(t)\) as the solution of another auxiliary problem: (AP2) Find \(\Pi_h^{Q_p} p_p(t) \in Q_{p,h}\) such that

\[
(\kappa \nabla \Pi_h^{Q_p} p_p, \nabla q_p) = (\kappa \nabla p_p, \nabla q_p), \quad \forall q_p \in Q_{p,h}.
\]

It is well-known that

\[
\|p_p(t) - \Pi_h^{Q_p} p_p(t)\|_{1,\kappa} \lesssim \kappa_0^{-2} h^m \|p_p(t)\|_{m+1}
\]

(18)

holds with \(m \leq k_{p_p} - 1\) depending on the regularity of \(p_p(t)\). If \(\Omega\) satisfies the full elliptic regularity assumption and \(\kappa\) is a Lipschitz continuous scalar field on \(\Omega\), then
Suppose that Theorem 1 holds for simplicity of presentation. In addition, we also assume that the exact solutions are sufficiently regular and maximum approximation orders can be achieved in the Bramble–Hilbert lemma for the initial data of the continuous problem. Since this is a stabilized saddle point problem with inf-sup condition, it is rather standard to show that the numerical initial data from this problem holds as well.

Before we prove the a priori error analysis we discuss compatible numerical initial data. Note that (13a), (13b) are algebraic equations, so our problem is a system of differential algebraic equations. When the backward Euler method is used for time discretization, compatible numerical data is not significant because the algebraic equation will be satisfied after one time step. However, numerical initial data satisfying this algebraic equation can be important for stability of numerical methods when other time discretization methods such as the Crank–Nicolson method are used. In order to have compatible numerical initial data, we can use the solution of

\[
\begin{align*}
(2 \mu \epsilon(u_h), \epsilon(v)) + (p_{t,h}, \text{div } v) &= (f(0), v), \\
(\text{div } u_h, q_t) - s_h(p_{t,h}, q_t) - (\lambda^{-1} p_{t,h}, q_t) - (\alpha \lambda^{-1} p_{p,h}, q_t) &= \tilde{s}_h(f(0), q_t), \\
- (\alpha^2 \lambda^{-1} p_{t,h}, q_p) - (\kappa \nabla p_{p,h}, \nabla q_p) &= -(\alpha^2 \lambda^{-1} p_t(0), q_p) \\
- (\kappa \nabla p_p(0), \nabla q_p)
\end{align*}
\]

as numerical initial data. Since this is a stabilized saddle point problem with inf-sup condition, it is rather standard to show that the numerical initial data from this problem is a good approximation of initial data of the continuous problem.

In the theorem below we assume that the exact solutions are sufficiently regular and maximum approximation orders can be achieved in the Bramble–Hilbert lemma for simplicity of presentation. In addition, we also assume that \( \Pi^Q p_p \) is an approximation of \( p_p \) with optimal order in the \( L^2 \) norm, i.e., (19) holds.

**Theorem 1** Suppose that \((u, p_t, p_p)\) is the solution of (5) with initial data \((u(0), p_t(0), p_p(0))\), and \((u_h, p_{t,h}, p_{p,h})\) is the solution of (13) with numerical initial data \((u_h(0), p_{t,h}(0), p_{p,h}(0)) \in V_h \times Q_{t,h} \times Q_{p,h}\) satisfying (13a), (13b), and

\[
\begin{align*}
\|p_t(0) - p_{t,h}(0)\|_{Q_t} &\lesssim h^{k_{p_t}} \|p_t(0)\|_{k_{p_t}}, \\
\|p_p(0) - p_{p,h}(0)\|_0 &\lesssim h^{k_{p_p}} \|p_p(0)\|_{k_{p_p}}.
\end{align*}
\]

Then

\[
\begin{align*}
\|\Pi^V_h u - u_h\|_{L^\infty(0,t:V)} + \|\Pi^Q_h p_t - p_{t,h}\|_{L^\infty(0,t:Q_t)} \\
+ \|\Pi^Q_h p_p - p_{p,h}\|_{L^\infty(0,t:L^2_{0,h})} + \|\Pi^Q_h p_p - p_{p,h}\|_{L^2(0,t;H^k_h)} \\
\lesssim h^k \left( \|p_t(0)\|_{H^k} + \|p_p(0)\|_{H^k} + \|\dot{p}_t\|_{L^1(0,t;H^k)} + \|\dot{p}_p\|_{L^1(0,t;H^k)} \right)
\end{align*}
\]

and

\[
\|\Pi^V_h \dot{u} - \dot{u}_h\|_{L^2(0,t;V)} + \|\Pi^Q_h \dot{p}_t - \dot{p}_{t,h}\|_{L^2(0,t;Q_t)}
\]
\[ + \| \Pi_h^Q p \|_{L^2(0,t;L^2_{\sigma_0})} + \| \Pi_h^Q p - p_{p,h} \|_{L^\infty(0,t;H^k)} \]
\[ \leq \| \Pi_h^Q p(p(0) - p_{p,h}(0)) \|_{1,\kappa} + h^k \| \dot{p}_t, \dot{p}_p \|_{L^2(0,t;H^k)} \] (23)

**Proof** The difference of (5) and (13) gives
\[ (2\mu \epsilon(e_u, \epsilon(v)) + (e_{p_t}, \text{div } v) = 0, \]
\[ (\text{div } e_u, q_t) + s_h(p_{t,h}, q_t) - (\lambda^{-1} e_{p_t}, q_t) - (\alpha \lambda^{-1} e_{p_p}, q_t) = -\tilde{s}_h(f, q_t), \]
\[ - (\alpha \lambda^{-1} \dot{\epsilon}_{p_t}, q_p) - ((s_0 + \alpha^2 \lambda^{-1}) \dot{\epsilon}_{p_p}, q_p) - (\kappa \nabla e_{p_p}, \nabla q_p) = 0. \]

From the decomposition (14)–(16) and the equations of (AP1), (AP2), we have reduced error equations

(24a)
\[ (2\mu \epsilon(e^h_u, \epsilon(v)) + (e^h_{p_t}, \text{div } v) = 0, \]

(24b)
\[ (\text{div } e^h_u, q_t) - s_h(e^h_{p_t}, q_t) - (\lambda^{-1} e^h_e - \alpha e^h_{p_p}, q_t) = (\lambda^{-1} e^h_{p_t}, q_t) + (\alpha \lambda^{-1} e^h_{p_p}, q_t), \]
\[ - (\alpha \lambda^{-1} \dot{\epsilon}^h_{p_t}, q_p) - ((s_0 + \alpha^2 \lambda^{-1}) \dot{\epsilon}^h_{p_p}, q_p) - (\kappa \nabla e^h_{p_p}, \nabla q_p) = (\alpha \lambda^{-1} \dot{\epsilon}^h_{p_t}, q_p) - ((s_0 + \alpha^2 \lambda^{-1}) \dot{\epsilon}^h_{p_p}, q_p) \]

(24c)
for any \( v \in V_h, q_t \in Q_{t,h}, q_p \in Q_{p.h}. \)

**Proof of (22):** We take \( v = \dot{e}^h_{u,h} \) in (24), \( q_t = -e^h_{p_t} \) in the time derivative of (24b), \( q_p = -e^h_{p_p} \) in (24c), and add them altogether. Then we have

\[ \frac{1}{2} \frac{d}{dt} \left( \| e^h_u \|_V^2 + s_h(e^h_{p_t}, e^h_{p_t}) + \| e^h_{p_t} - \alpha e^h_{p_p} \|_{0, \lambda - 1}^2 + \| e^h_{p_p} \|_{0, s_0}^2 \right) + \| e^h_{p_p} \|_{1, \kappa}^2 \]
\[ = - (\lambda^{-1} \dot{\epsilon}^h_{p_t} - \alpha \dot{\epsilon}^h_{p_p})^2, e^h_{p_t} - \alpha e^h_{p_p} \right) + (s_0 \dot{\epsilon}^h_{p_p}, e^h_{p_p}), \]

(25)

Defining
\[ X(s)^2 = \| e^h_u(s) \|_V^2 + s_h(e^h_{p_t}(s), e^h_{p_t}(s)) + \| e^h_{p_t}(s) - \alpha e^h_{p_p}(s) \|_{0, \lambda - 1}^2 + \| e^h_{p_p}(s) \|_{0, s_0}^2, \]
and integrating (25) from 0 to \( t \), we have

\[ \frac{1}{2} (X(t)^2 - X(0)^2) + \int_0^t \| e^h_{p_p}(s) \|_{1, \kappa}^2 ds, \]
\[ = \int_0^t \left[ - (\lambda^{-1} \dot{\epsilon}^h_{p_t}(s) - \alpha \dot{\epsilon}^h_{p_p}(s)), e^h_{p_t}(s) - \alpha e^h_{p_p}(s) \right] ds \]
\[ \quad + (s_0 \dot{\epsilon}^h_{p_p}(s), e^h_{p_p}(s)) \]
Adopting the argument of the estimate of \( \|e_p^h\|_{L^2(0,T;L^2_{\lambda^{-1}})} \) for large \( \lambda \), for large \( \lambda \), we obtain

\[
\frac{1}{2} X(t)^2 \leq \frac{1}{2} X(0)^2 + \max \left\{ \|e_p^I - \alpha e_p^I\|_{L^1(0,T;L^2_{\lambda^{-1}})}, \|e_p^h\|_{L^1(0,T;L^2_{\lambda^{-1}})} \right\} X(t).
\]

By Young’s inequality and the arithmetic–geometric mean inequality, we can obtain

\[
X(t) \leq X(0) + 2 \max \left\{ \|e_p^I - \alpha e_p^I\|_{L^1(0,T;L^2_{\lambda^{-1}})}, \|e_p^h\|_{L^1(0,T;L^2_{\lambda^{-1}})} \right\}.
\]

As a corollary, assuming the exact solution is sufficiently smooth, we obtain

\[
\|e_u^h\|_{L^\infty(0,T;V)} \leq \max_{s \in [0,T]} s_h(e_p^h, e_p^h, e_p^h, e_p^h) \frac{1}{2} + \|e_p^h - \alpha e_p^h\|_{L^\infty(0,T;L^2_{\lambda^{-1}})} + \|e_p^h\|_{L^\infty(0,T;L^2_{\lambda^{-1}})} \lesssim X(0) + h^k \|\hat{p}_t, \hat{p}_p\|_{L^1(0,T;H^k)}\]

where \( k = \min\{k_p, k_p\} \). Note that the implicit constant in this estimate is independent of parameter scales, i.e., for large \( \mu \), arbitrarily large \( \lambda \), small \( \kappa_0 \) and \( \kappa_1 \), and small or degenerate \( s_0 \). For mixed methods, the equation (24) and the inf-sup condition (8) can be used to obtain

\[
\|e_p^h\|_{L^\infty(0,T;Q_t)} \lesssim X(0) + h^k \|\hat{p}_t, \hat{p}_p\|_{L^1(0,T;H^k)}, \quad k = \min(k_p, k_p).
\]

In case of stabilized methods, for any \( t \in (0, T] \), there exists \( (v, q_t) \) such that \( \|v\|_V + \|q_t\|_{Q_t} \leq 1 \) and

\[
\|e_u^h(t)\|_V + \|e_p^h(t)\|_{Q_t} \lesssim (2\mu\epsilon(e_u^h(t), e(v)), e(v)) + (e_p^h(t), \nabla v) + (\nabla e_u^h(t), q_t) - s_h(e_p^h(t), q_t).
\]

Using this \((v, q_t)\) with (24) and (24b), we get

\[
\|e_u^h(t)\|_V + \|e_p^h(t)\|_{Q_t} \\lesssim \left( \lambda^{-1}(e_p^h(t) - \alpha e_p^h(t), q_t) + (\lambda^{-1}e_p^I(t), q_t) + (\alpha \lambda^{-1}e_p^I(t), q_t) \right) \lesssim \|e_p^h(t) - \alpha e_p^h(t)\|_{0,\lambda^{-1}} + \|e_p^I(t) - \alpha e_p^I(t)\|_{0,\lambda^{-1}} \lesssim X(0) + h^k \|\hat{p}_t, \hat{p}_p\|_{L^1(0,T;H^k)}, \quad k = \min(k_p, k_p),
\]

where we used (27) in the last inequality.
To estimate $\|e^h_{pp}\|_{L^2(0,t;H^1)}$, we use (26) and get
\[
\frac{1}{2}X(t)^2 + \int_0^t \|e^h_{pp}(s)\|_{1,\kappa}^2 \, ds \leq \frac{1}{2}X(0)^2 + \|\dot{e}^h_{pt}\|_{L^1(0,t;L^2_{\ell,1-\kappa})} X(t) + \|\ddot{e}^h_{pp}\|_{L^1(0,t;L^2_{\ell,0})} X(t).
\]

By Young’s inequality,
\[
\|e^h_{pp}\|_{L^2(0,t;H^1)} \lesssim X(0) + h^k \|\dot{p}_t, \dot{p}_p\|_{L^1(0,t;H^k)}, \quad k = \min\{k_{pt}, k_{pp}\}. \tag{30}
\]

To complete the proof, we need to estimate $X(0)$. Recall that the numerical initial data $(u_h(0), p_{t,h}(0), p_{p,h}(0))$ satisfies (13a) and (13b) at $t = 0$. Recall also that $(\Pi^V_h u(0), \Pi^Q_h p_t(0))$ satisfies (AP1) at $t = 0$. Noting that $\text{div} \, u(0) = \lambda^{-1} p_t(0) + \lambda^{-1} \alpha p_p(0), (e^h_{u}(0), e^h_{p_t}(0), e^h_{pp}(0))$ satisfies
\[
\begin{align*}
(2 \mu \epsilon(e^h_{u}(0)), \epsilon(v)) + (e^h_{pt}(0), \text{div} \, v) &= 0, \\
(\text{div} \, e^h_{u}(0), q_t) - s_h(e^h_{pt}(0), q_t) &= \left(\lambda^{-1}(e^h_{pt}(0) - \alpha e^h_{pp}(0)), q_t\right)
\end{align*}
\]
for all $v \in V_h, q_t \in Q_{h,t}$, therefore
\[
\|e^h_{u}(0)\|_V + \|e^h_{pt}(0)\|_{Q_t} \lesssim h^k \|p_t(0), p_p(0)\|_{H^k}, \quad k = \min\{k_{pt}, k_{pp}\}.
\]

From the boundedness of $s_h(\cdot, \cdot)$,
\[
X(0) \lesssim \|e^h_{u}(0)\|_V + \|e^h_{pt}(0)\|_{Q_t} + \|e^h_{pp}(0)\|_{L^2_{0,\ell,0}} \lesssim h^k \|p_t(0), p_p(0)\|_{H^k},
\]
with $k = \min\{k_{pt}, k_{pp}\}$.

**Proof of (23):** We now estimate $\|e^h_{pp}(t)\|_{L^\infty(0,t;H^1)}$. For this, we take $v = \dot{e}^h_u$ in the time derivative of (24), $q_t = -\dot{e}^h_{pt}$ in the time derivative of (24b), $q_p = -\dot{e}^h_{pp}$ in (24c), and add the equations altogether. Then
\[
\begin{align*}
\|\dot{e}^h_u(t)\|_V^2 + \|\dot{e}^h_{pt}(t) - \alpha \dot{e}^h_{pp}(t)\|_{L^2_{0,\lambda-1}}^2 + \|\dot{e}^h_{pp}(t)\|_{L^2_{0,\lambda-1}}^2 + \frac{1}{2} \frac{d}{dt} \|e^h_{pp}(t)\|_{1,\kappa}^2 &= - \left(\lambda^{-1}\dot{e}^h_{pt} - \alpha \lambda^{-1}\dot{e}^h_{pp}, \dot{e}^h_{pt} - \alpha \dot{e}^h_{pp}\right) + \left(s_0 \dot{e}^h_{pt}, \dot{e}^h_{pp}\right) \tag{31}
\end{align*}
\]
Integrating it from 0 to $t$ and using Young’s inequality, we get
\[
\begin{align*}
\frac{1}{2} \|e^h_{pp}(t)\|_{1,\kappa}^2 + \int_0^t \left[\|\dot{e}^h_u(s)\|_V^2 + \frac{1}{2} \|\dot{e}^h_{pt}(s) - \alpha \dot{e}^h_{pp}(s)\|_{L^2_{0,\lambda-1}}^2 + \frac{1}{2} \|\dot{e}^h_{pp}(s)\|_{L^2_{0,\lambda-1}}^2 \right] ds \\
&\leq \frac{1}{2} \|e^h_{pp}(0)\|_{1,\kappa}^2 + \frac{1}{2} \int_0^t \left[\|\dot{e}^h_{pt}(s) - \alpha \dot{e}^h_{pp}(s)\|_{L^2_{0,\lambda-1}}^2 + \|\dot{e}^h_{pp}(s)\|_{L^2_{0,\lambda-1}}^2 \right] ds.
\end{align*}
\]
In particular,
\[ \| e^h_{p,P} (t) \|_{1,k} + \| \dot{z}^h_{p} \|_{L^2(0,T;V)} + \| \dot{z}^h_{p_t} - \alpha \dot{e}^h_{p,p} \|_{L^2(0,T;L^2_{\lambda^{-1}})} + \| \dot{e}^h_{p,p} \|_{L^2(0,T;H^{k_1})} \leq \| e^h_{p,p} (0) \|_{1,k} + \| \dot{z}^h_{p_t} - \alpha \dot{e}^h_{p,p} \|_{L^2(0,T;L^2_{\lambda^{-1}})} + \| \dot{e}^h_{p,p} \|_{L^2(0,T;L^2_{\lambda^{-1}})} \]
\[ \leq \| e^h_{p,p} (0) \|_{1,k} + \| \dot{z}^h_{p_t} - \alpha \dot{e}^h_{p,p} \|_{L^2(0,T;L^2_{\lambda^{-1}})} + \| \dot{e}^h_{p,p} \|_{L^2(0,T;L^2_{\lambda^{-1}})} \]
\[ \leq \| e^h_{p,p} (0) \|_{1,k} + h^k \| \dot{p}_t \|_{L^2(0,T;H^k)}, \quad k = \min\{k_{p_t}, k_{p,p}\}. \]

In this estimate, the implicit constants are uniformly bounded for small \( \kappa_0, \kappa_1, \) large \( \mu, \) arbitrarily large \( \lambda, \) and small or degenerate \( s_0. \)

\[ \Box \]

**Corollary 1** Under the same assumptions in Theorem 1 and an additional assumption

\[ \| p_{p}(0) - p_{p,h}(0) \|_{1,k} \lesssim h^{k_{pp}-1} \| p_{p}(0) \|_{H^{k_{pp}}}, \]  

we can show that

\[ \| u - u_h \|_{L^\infty(0,T;V)} + \| p_{p} - p_{p,h} \|_{L^\infty(0,T;Q_t)} + \| p_{p} - p_{p,h} \|_{L^\infty(0,T;L^2_{\lambda^{-1}})} \lesssim h^k \left( \| u \|_{L^\infty(0,T;H^k)} + \| p_{p} \|_{L^1(0,T;H^k)} + \| p_{p} \|_{L^1(0,T;H^k)} + \| \dot{p}_p \|_{L^1(0,T;H^k)} + \| \dot{p}_p \|_{L^1(0,T;H^k)} \right) \]

with \( k = \min\{k_{u} - 1, k_{p_t}, k_{p_{p}}\}, \)

\[ \| p_{p} - p_{p,h} \|_{L^2(0,T;H^k)} \lesssim h^k \left( \| p_{p}(0) \|_{H^k} + \| p_{p}(0) \|_{H^k} + \| \dot{p}_t \|_{L^1(0,T;H^k)} + \| \dot{p}_p \|_{L^1(0,T;H^k)} + \| \dot{p}_p \|_{L^1(0,T;H^k)} \right) \]

with \( k = \min\{k_{p_t}, k_{p_{p}} - 1\}, \) and

\[ \| \dot{u} - \dot{u}_h \|_{L^2(0,T;V)} + \| \dot{p}_t - \dot{p}_{t,h} \|_{L^2(0,T;Q_t)} + \| \dot{p}_p - \dot{p}_{p,h} \|_{L^2(0,T;L^2_{\lambda^{-1}})} \lesssim \| \Pi_Q \| p_{p}(0) - p_{p,h}(0) \|_{1,k} + h^k \| \dot{u} \|_{L^2(0,T;H^k)} + \| \dot{p}_t \|_{L^2(0,T;H^k)} + \| \dot{p}_p \|_{L^2(0,T;H^k)} \]

with \( k = \min\{k_{u} - 1, k_{p_t}, k_{p_{p}}\}, \)

\[ \| p_{p} - p_{p,h} \|_{L^\infty(0,T;H^k)} \lesssim h^k \left( \| p_{p} \|_{L^\infty(0,T;H^{k+1})} + \| \dot{p}_t \|_{L^2(0,T;H^k)} + \| \dot{p}_p \|_{L^2(0,T;H^k)} \right) \]

with \( k = \min\{k_{p_{p}} - 1, k_{p_t}\} \) hold.

\[ \Box \]

**Proof** These assertions can be proved easily from the results in Theorem 1 and the triangle inequality, so we omit details.
4 Parameter-robust preconditioning

In this section we discuss preconditioners of the finite element discretizations robust for certain parameter scales. In most applications, the parameters $\mu$, $\lambda$, $\kappa$ are in the ranges

$$0 < \kappa_0, \kappa_1 \ll 1 \ll \mu \lesssim \lambda \leq +\infty. \quad (33)$$

It turns out that preconditioners efficient for the model problem with unit parameter values do not perform well for problems with realistic parameter values. In fact, construction of preconditioners robust for all variations of parameters in (33) is the motivation of [22], and abstract form of parameter-robust block diagonal preconditioners are studied for discretizations with Taylor–Hood and MINI elements. Therefore we only focus on preconditioners for discretizations with the two stabilized methods in (11) and (12). Following the approach in [22], we first define parameter-dependent discrete norms of $V_h$, $Q_{t,h}$, $Q_{p,h}$, and show that the stability of the system with the parameter-dependent norms. Then we can derive abstract forms of block diagonal preconditioners based on the parameter-dependent norms. The numerical results we will present in the last section show that performances of algebraic multigrid block diagonal preconditioners based on the abstract forms are robust for parameter scales.

Before we define parameter-dependent norms, we consider fully discrete schemes of the system to reduce the preconditioning problem. In fully discretization scheme of (13) with time step size $\Delta t > 0$, we solve a static system

\begin{align}
(2\mu \epsilon(u_h), \epsilon(v)) + (p_{t,h}, \text{div} \, v) &= (\tilde{f}, v), \quad (34a) \\
(\text{div} \, u_h, q_t) - s_h (p_{t,h}, q_t) - (\lambda^{-1} p_{t,h}, q_t) - (\alpha \lambda^{-1} p_{p,h}, q_p) &= (\tilde{f}_t, q_t), \quad (34b) \\
- (\alpha \lambda^{-1} p_{t,h}, q_p) - ((s_0 + \alpha^2 \lambda^{-1}) p_{p,h}, q_p) - (\kappa \nabla p_{p,h}, \nabla q_p) &= (\tilde{g}, q_p) \quad (34c)
\end{align}

for all $(v, q_t, q_p) \in V_h \times Q_{t,h} \times Q_{p,h}$ at each time step but $\kappa$ here is $\kappa \Delta t$ with $\kappa$ in the previous section, and $\tilde{f}$, $\tilde{f}_t$, $\tilde{g}$ are right-hand side terms depending on time discretization schemes.

Let us define norms of $V_h$, $Q_{t,h}$, $Q_{p,h}$ as

$$\|v\|_{V_h}^2 = (2\mu \epsilon(v), \epsilon(v)), \quad \|q_t\|_{Q_{t,h}}^2 = ((2\mu)^{-1} q_t, q_t) + s_h(q_t, q_t),$$

$$\|q_p\|_{Q_{p,h}}^2 = \|q_p\|_{0,s_0}^2 + (\kappa \nabla q_p, \nabla q_p),$$

and let $X_h = V_h \times Q_{t,h} \times Q_{p,h}$ be the Hilbert space with the norm

$$\|(v, q_t, q_p)\|_{X_h}^2 = \|v\|_{V_h}^2 + \|q_t\|_{Q_{t,h}}^2 + \|q_p\|_{Q_{p,h}}^2.$$
We define a linear operator $A$ from $X_h$ to its dual space $X_h^*$ using the left-hand side of (34) as

$$
\langle A(u, p_t, p_p), (v, q_t, q_p) \rangle_{(X_h^*, X_h)} = (2\mu \epsilon(u), \epsilon(v)) + (p_t, \text{div } v) + (\text{div } u, q_t) - s_h (p_t, q_t) - \left( \lambda^{-1} p_t, q_t \right) - \left( \alpha \lambda^{-1} p_t, q_p \right) - \left( (s_0 + \alpha^2 \lambda^{-1}) p_p, q_p \right) - \left( \kappa \nabla p_p, \nabla q_p \right)
$$

for $(u, p_t, p_p), (v, q_t, q_p) \in X_h$, where $\langle \cdot, \cdot \rangle_{(X_h^*, X_h)}$ is the duality pairing of $X_h$ and $X_h^*$. We claim that $A$ is an isomorphism from $X_h$ to $X_h^*$ such that $\|A\|_{L(X_h, X_h^*)}$ and $\|A^{-1}\|_{L(X_h^*, X_h)}$ are independent of mesh sizes and the parameters in the ranges of (33).

**Theorem 2** There exists $\beta > 0$, independent of the scales of $\mu$, $\kappa$, $\lambda$ in (33), and the mesh sizes, such that the following inf-sup condition holds:

$$
\inf_{(u, p_t, p_p) \in X_h} \sup_{(v, q_t, q_p) \in X_h} \frac{\langle A(u, p_t, p_p), (v, q_t, q_p) \rangle_{(X_h^*, X_h)}}{\|A(u, p_t, p_p)\|_{X_h^*} \|v, q_t, q_p\|_{X_h}} \geq \beta.
$$

**Proof** To prove the assertion, for given $(0, 0, 0) \neq (u, p_t, p_p) \in X_h$, we will find $(v, q_t, q_p) \in X_h$ such that

$$
\|v, q_t, q_p\|_{X_h} \leq C \|(u, p_t, p_p)\|_{X_h},
$$

(35)

$$
\langle A(u, p_t, p_p), (v, q_t, q_p) \rangle_{(X_h^*, X_h)} \geq C' \|(u, p_t, p_p)\|_{X_h}^2,
$$

(36)

with $C, C' > 0$ independent of the scales of $\mu, \lambda, \kappa$, and mesh sizes.

Suppose that $(0, 0, 0) \neq (u, p_t, p_p) \in X$ is given.

For stabilized methods, there exist $C_1, C_2 > 0$ independent of mesh sizes and parameters such that

$$
\sup_{v \in V_h} \frac{\langle \text{div } v, q_t \rangle}{\|v\|_V} \geq 2C_1 \|q_t\|_{Q_t} - 2C_2 (s_h(q_t, q_t))^{\frac{1}{2}} \forall q_t \in Q_{t,h}.
$$

From this there exists $w \in V_h$ such that

$$
\langle \text{div } w, p_t \rangle \geq \left( C_1 \|p_t\|_{Q_t} - C_2 (s_h(p_t, p_t))^{\frac{1}{2}} \right) \|w\|_V.
$$

(37)

Due to linearity of this inequality in $w$ we may rescale $w$ so that $\|w\|_V = \|p_t\|_{Q_t}$.

To prove (35) and (36), we set $v = u + \delta w, q_t = -p_t, q_p = -p_p$ with a constant $\delta > 0$ which will be determined later. One can check that

$$
\|v, q_t, q_p\|_{X_h} \leq \sqrt{2(1 + \delta^2)} \|(u, p_t, p_p)\|_{X_h},
$$

$$
\langle A(u, p_t, p_p), (v, q_t, q_p) \rangle_{(X_h^*, X_h)} = \langle A(u, p_t, p_p), (u, q_t, q_p) \rangle_{(X_h^*, X_h)} + \langle A(u, p_t, p_p), (\delta w, q_t, q_p) \rangle_{(X_h^*, X_h)}.
$$
and (35) follows if $\delta$ is independent of the parameters and mesh sizes. To establish (36) and determine $\delta$, we use the previously chosen $v, q_t, q_p$, and (37) to have

$$
\langle A(u, p_t, p_p), (v, q_t, q_p) \rangle_{(X^*_h, X_h)}
\leq \|u\|_{V_h}^2 + \delta(2\mu\epsilon(u), \epsilon(w)) + \delta(\text{div} w, p_t) + s_h(p_t, p_t) \\
+ (\lambda^{-1} p_t, p_t) + ((s_0 + \alpha^2 \lambda^{-1}) p_p, p_p) + 2(\alpha \lambda^{-1} p_t, p_p) + (\kappa \nabla p_p, \nabla p_p)
$$

(38)

By Young’s inequality and the fact $\|w\|_V = \|p_t\|_{Q_t}$, we also have

$$
\delta(2\mu\epsilon(u), \epsilon(w)) \leq \frac{\delta \theta}{2} \|u\|_V^2 + \frac{\delta \theta}{2} \|w\|_V^2 \leq \frac{\delta \theta}{2} \|u\|_V^2 + \frac{\delta \theta}{2} \|p_t\|_{Q_t}^2 \quad \forall \theta > 0.
$$

By (37) and Young’s inequality,

$$
\delta(\text{div} u, p_t) \geq \delta \left(C_1 \|p_t\|_{Q_t} - C_2 (s_h(p_t, p_t))^\frac{1}{2} \right) \|w\|_V
\geq \delta C_1 \|p_t\|_{Q_t}^2 - \delta C_2 \left(\frac{\eta}{2} s_h(p_t, p_t) + \frac{1}{2\eta} \|p_t\|_{Q_t}^2 \right)
$$

for any $\eta > 0$. From these we can get

$$
\langle A(u, p_t, p_p), (v, q_t, q_p) \rangle_{(X^*_h, X_h)} = \left(1 - \frac{\delta \theta}{2}\right) \|u\|_V^2 + \delta \left(C_1 - \frac{1}{2\theta} - \frac{C_2}{2\eta} \right) \|p_t\|_{Q_t}^2 + \left(1 - \frac{C_2 \eta}{2} \right) s_h(p_t, p_t) \\
+ \|p_t - \alpha p_p\|_{\lambda^{-1}}^2 + \|p_p\|_{s_0}^2 + \|p_p\|_{1,\kappa}^2.
$$

We now set

$$
\theta = \frac{2}{C_1}, \quad \eta = \frac{2C_2}{C_1}, \quad \delta = \min \left\{ \frac{C_1}{2}, \frac{C_1}{2C_2^2} \right\},
$$

and get

$$
\langle A(u, p_t, p_p), (v, q_t, q_p) \rangle_{(X^*_h, X_h)} \geq \frac{1}{2} \|u\|_V^2 + \frac{1}{2} s_h(p_t, p_t) \\
+ \frac{\delta C_1}{2} \|p_t\|_{Q_t}^2 + \|p_p\|_{s_0}^2 + \|p_p\|_{1,\kappa}^2.
$$

Since $C_1, C_2$ are independent of parameters and mesh sizes, so is $\delta$, and therefore (35) and (36) are proved. ☐
The above stability in the parameter-dependent norm $X_h$ suggests an abstract form of preconditioner

$$P = \begin{pmatrix} P_u & P_{pt} & P_{pp} \end{pmatrix}$$

(39)

with $P_u, P_{pt}, P_{pp}$ which are (approximate) inverses of the maps

$u \mapsto -\text{div}(2\mu\varepsilon(u)), \quad p_t \mapsto (1/\mu)p_t, \quad p_p \mapsto (s_0 + \alpha^2\lambda^{-1})p_p - \text{div}(\kappa\nabla p_p)$.

## 5 Numerical results

In this section we present the results of numerical experiments. All numerical experiments are performed with FEniCS version 2017.2.0. ([23])

In the first numerical experiments, we show convergence of finite element methods. The computational domain $\Omega$ is the unit square $[0, 1] \times [0, 1]$ and is divided into $N \times N$ uniform squares, i.e., $h = 1/N$, and then each square is divided into two triangles to obtain the triangulation $T_h$. To illustrate convergence of errors, we consider a manufactured solution of the problem with

$$u = \left(\begin{array}{c} \sin(\pi x) \sin(1 + t) \\ \sin y \sin t \end{array} \right), \quad p = x^2 y^2 \cos t$$

and parameters $\mu = 10, \lambda = 15, \alpha = 1, s_0 = 1, \kappa = 1$. For boundary conditions we impose Dirichlet boundary conditions of $u$ on $\Gamma_d := \{0\} \times [0, 1] \cup \{1\} \times [0, 1]$ and of $p_p$ on $\Gamma_p := \partial\Omega$.

We use the backward Euler time discretization with time step $\Delta t = h^2$ and the errors are computed at $t = 0.5$. In all experiments, $Q_{p,h}$ is the space of continuous piecewise linear polynomials, and the pair $(V_h, Q_{t,h})$ is chosen as the lowest order Taylor–Hood element, the Brezzi–Pitkäranta stabilized method, and the $P_1 - P_0$ stabilized method. The information for optimal convergence and the degrees of freedom of the three methods in our numerical experiments, are given in Table 1. Numerical results with

### Table 1 $k_u, k_{pt}, k_{pp}$ and DOFs of the three methods in numerical experiments

| $N$  | $k_u = 3$ | $k_{pt} = 2$ | $k_{pp} = 2$ | $k_u = 2$ | $k_{pt} = 2$ | $k_{pp} = 2$ | $k_u = 2$ | $k_{pt} = 1$ | $k_{pp} = 2$ |
|------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| DOFs | 2756   | 1156   | 1379   | 10628  | 4356   | 5315   | 41732  | 16900  | 20867  |
| 16   |        |        |        |        |        |        |        |        |        |
| 32   |        |        |        |        |        |        |        |        |        |
| 64   |        |        |        |        |        |        |        |        |        |
| 128  |        |        |        |        |        |        |        |        |        |
Table 2 Errors and convergence rates with the lowest order Taylor–Hood finite elements

| N  | $\|pt - pt,h\|_{L^2}$ | $\|pp - pp,h\|_{L^2}$ | $\|u - uh\|_{H^1}$ | $\|pp - pp,h\|_{1,k}$ |
|----|----------------------|----------------------|-----------------|------------------------|
|    | Error     | Rate     | Error     | Rate     | Error     | Rate     | Error     | Rate     |
| 8  | 4.342e-02 | —        | 3.527e-03 | —        | 5.725e-02 | —        | 1.127e-01 | —        |
| 16 | 1.071e-02 | 2.02     | 8.826e-04 | 2.00     | 1.424e-02 | 2.01     | 5.642e-02 | 1.00     |
| 32 | 2.669e-03 | 2.00     | 2.207e-04 | 2.00     | 3.559e-03 | 2.00     | 2.822e-02 | 1.00     |
| 64 | 6.668e-04 | 2.00     | 5.519e-05 | 2.00     | 8.897e-04 | 2.00     | 1.411e-02 | 1.00     |
| 128| 1.667e-04 | 2.00     | 1.380e-05 | 2.00     | 2.225e-04 | 2.00     | 7.056e-03 | 1.00     |

Table 3 Errors and convergence rates with the Brezzi–Pitkäranta stabilized method

| N  | $\|pt - pt,h\|_{L^2}$ | $\|pp - pp,h\|_{L^2}$ | $\|u - uh\|_{H^1}$ | $\|pp - pp,h\|_{1,k}$ |
|----|----------------------|----------------------|-----------------|------------------------|
|    | Error     | Rate     | Error     | Rate     | Error     | Rate     | Error     | Rate     |
| 8  | 6.024e+00 | —        | 3.549e-03 | —        | 7.017e+00 | —        | 1.139e-01 | —        |
| 16 | 3.748e+00 | 0.96     | 1.409e-03 | 1.33     | 4.438e+00 | 0.66     | 5.723e-02 | 0.99     |
| 32 | 1.519e+00 | 1.30     | 5.280e-04 | 1.42     | 1.793e+00 | 1.31     | 2.843e-02 | 1.01     |
| 64 | 4.642e-01 | 1.71     | 1.643e-04 | 1.68     | 5.463e-01 | 1.71     | 1.415e-02 | 1.01     |
| 128| 1.259e-01 | 1.88     | 4.586e-05 | 1.84     | 1.568e-01 | 1.80     | 7.061e-03 | 1.00     |

Table 4 Errors and convergence rates with the $P_1$-$P_0$ stabilized method

| N  | $\|pt - pt,h\|_{L^2}$ | $\|pp - pp,h\|_{L^2}$ | $\|u - uh\|_{H^1}$ | $\|pp - pp,h\|_{1,k}$ |
|----|----------------------|----------------------|-----------------|------------------------|
|    | Error     | Rate     | Error     | Rate     | Error     | Rate     | Error     | Rate     |
| 8  | 5.051e+00 | —        | 1.149e-02 | —        | 5.816e+00 | —        | 1.223e-01 | —        |
| 16 | 2.604e+00 | 0.96     | 2.726e-03 | 2.08     | 2.906e+00 | 1.00     | 5.772e-02 | 1.08     |
| 32 | 1.349e+00 | 0.95     | 6.126e-04 | 2.15     | 1.402e+00 | 1.05     | 2.835e-02 | 1.03     |
| 64 | 6.832e-01 | 0.98     | 1.490e-04 | 2.04     | 6.762e-01 | 1.05     | 1.413e-02 | 1.01     |
| 128| 3.296e-01 | 1.05     | 3.787e-05 | 1.98     | 3.082e-01 | 1.13     | 7.058e-03 | 1.00     |

Convergence rates of errors for mesh refinements are given in Tables 2, 3, 4.

We can see that the lowest order Taylor–Hood element show robust optimal convergence rates which correspond to Theorem 1 because $k = \min\{k_{pt}, k_{pp}\} = 2$. Note that $\|pp - pp,h\|_{1,k}$ has first order convergence because the best convergence rate of $\|pp - \Pi_{h}^{Q_p} p_p\|_{1,k}$ is only the first order. In the Brezzi–Pitkäranta stabilized method, $\|pp - pp,h\|_{1,k}$ shows first order convergence rate which is optimal but convergence rates of the other errors are not obvious. However, the convergence rates of available errors are higher than the expected convergence rates in Theorem 1. Therefore, the results are acceptable even if a complete analysis is missing. Finally, the $P_1 - P_0$ stabilized method gives errors of optimal convergence rates for all errors. In fact, the second order convergence of $\|pp - pp,h\|_{L^2}$ is one order higher than the one expected by Theorem 1. In conclusion, we observe that convergence rates or errors are same
or higher than the ones expected in Theorem 1 for any of the three methods. We also want to remark that the fluid flow errors $\| p_p - p_{p,h}\|_{1,\kappa}$ are almost same in any methods. Since the numbers of degrees of freedom of Brezzi–Pitkäranta or the $\mathcal{P}_1 - \mathcal{P}_0$ stabilized method are much smaller (see Table 1), the two stabilized methods can be much more efficient choices if the fluid flow is the main interest of simulations.

In the second part of numerical experiments, we present performances of the abstract preconditioners for each of the three methods. Although parameter-robust preconditioning for mixed methods are already studied in [22], we show the results of mixed method and stabilized methods for comparison. To construct preconditioners based on (39) for mixed methods, we use the algebraic multigrid method for the blocks of $u$ and $p_p$ but use the Jacobi preconditioner for the block of $p_t$ as in [22]:

$$
\begin{pmatrix}
\text{AMG}(A_u) & \text{Jacobi}(A_{p_t}) \\
\text{Jacobi}(A_{p_t}) & \text{AMG}(A_{p_p})
\end{pmatrix}
$$

where $A_u, A_{p_t}, A_{p_p}$ are matrices obtained from the bilinear forms

$$(2\mu \epsilon(u), \epsilon(v)), \quad ((2\mu)^{-1} p_t, q_t), \quad ((s_0 + \alpha^2 \lambda^{-1}) p_p, q_p) + (\kappa \nabla p_p, \nabla q_p).$$

For stabilized methods our preconditioners have the form

$$
\begin{pmatrix}
\text{AMG}(A_u) & \text{AMG}(A_{p_t}) & \text{AMG}(A_{p_p}) \\
\text{AMG}(A_{p_t}) & \text{AMG}(A_{p_p}) & \text{AMG}(A_{p_p})
\end{pmatrix}
$$

where $A_u, A_{p_t}, A_{p_p}$ are matrices obtained by

$$(2\mu \epsilon(u), \epsilon(v)), \quad ((2\mu)^{-1} p_t, q_t) + s_h(p_t, q_t), \quad ((s_0 + \alpha^2 \lambda^{-1}) p_p, q_p) + (\kappa \nabla p_p, \nabla q_p)$$

for each stabilized method, and MinRes algorithm is used for iterative solvers. For algebraic multigrid methods we use the algebraic multigrid package Hypre AMG ([10]).

To test robustness of these preconditioners for mesh refinements, and parameter values, we consider the cases with meshes $N = 16, 32, 64, 128, 256, \mu = 1, 10^3, 10^6, \lambda/\mu = 1, 10^3, 10^6$, and scalar $\kappa = 1, 10^{-3}, 10^{-6}, 10^{-9}$. At each case, we only test the static problem with randomly generated right-hand side vectors, and measured number of iterations with relative tolerance $10^{-6}$. The results are given in Figs. 1, 2, 3. One can see that the numbers of iteration in Figs. 2, 3 are nearly robust for different parameter values and mesh refinements. In addition to the less number of degrees of freedom, stabilized methods need smaller numbers of iterations compared to the ones of Taylor–Hood elements, so they can be advantageous to accelerate simulations. A price to pay is the low accuracy of stabilized methods particularly for the elasticity variables.
Fig. 1 Number of iterations for one solve with the Taylor–Hood element. The correspondences of the number of DOFs and $N$ are $2756$ ($N = 16$), $10628$ ($N = 32$), $41732$ ($N = 64$), $165380$ ($N = 128$), $658436$ ($N = 256$).

Fig. 2 Number of iterations for one solve with the Brezzi–Pitkäranta stabilized method. The correspondences of the number of DOFs and $N$ are $1156$ ($N = 16$), $4356$ ($N = 32$), $16900$ ($N = 64$), $66564$ ($N = 128$), $264196$ ($N = 256$).
Fig. 3 Number of iterations for one solve with the $P_1-P_0$ stabilized method. The correspondences of the number of DOFs and $N$ are 1379 ($N = 16$), 5315 ($N = 32$), 20867 ($N = 64$), 82691 ($N = 128$), 329219 ($N = 256$).

6 Conclusion

In this paper we studied the three-field formulation of the Biot model which has the displacement, the total pressure, and the pore pressure as unknowns. We first carried out a comprehensive investigation of the a priori estimate of the continuous problem. Then we studied finite element discretization with parameter-robust stability, and parameter-robust preconditioning of the discretizations. For finite element discretizations we considered standard mixed finite element as well as stabilized methods for the Stokes equations, and complete error estimates of semidiscrete solutions of the Biot model are proved. For parameter-robust preconditioning, we showed parameter-robust stability of the system and derived an abstract form of robust preconditioners. The theoretical results are illustrated with numerical experiments.

Declarations

Conflict of interest The author has no relevant financial or non-financial interests to disclose.

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