AN EXPLICIT REPRESENTATION AND ENUMERATION FOR NEGACYCLIC CODES OF LENGTH $2^k n$ OVER $\mathbb{Z}_4 + u\mathbb{Z}_4$

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Abstract. In this paper, we give an explicit representation and enumeration for negacyclic codes of length $2^k n$ over the local non-principal ideal ring $R = \mathbb{Z}_4 + u\mathbb{Z}_4$ ($u^2 = 0$), where $k, n$ are arbitrary positive integers and $n$ is odd. In particular, we present all distinct negacyclic codes of length $2^k$ over $R$ precisely. Moreover, we provide an exact mass formula for the number of negacyclic codes of length $2^k n$ over $R$ and correct several mistakes in some literatures.

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1. Introduction

Algebraic coding theory deals with the design of error-correcting and error-detecting codes for the reliable transmission of information across noisy channels. The class of constacyclic codes play a very significant role in the theory of error-correcting codes. Since 1999, special classes of constacyclic codes over certain classes of finite commutative chain rings have been studied by numerous authors (see [1], [2], [5], [6], [8]–[11], [15], [17]–[19], [21]–[24], for example). It is an important way and an interesting topic to construct optimal codes (over finite fields or finite rings) from special linear codes over some appropriate rings.

Let $A$ be a finite commutative ring with identity $1 \neq 0$, and denote by $A^\times$ the multiplicative group of units in $A$. For any $a, b \in A$, we denote by $\langle a, b \rangle$ the ideal of $A$ generated by $a$ (resp. $a$ and $b$), i.e. $\langle a \rangle = aA$ (resp. $\langle a, b \rangle = aA + bA$). For any ideal $I$ of $A$, we will identify the element $a + I$ of the residue class ring $A/I$ with $a$ (mod $I$) in this paper.

For any positive integer $N$, let $A^N = \{(a_0, a_1, \ldots, a_{N-1}) \mid a_i \in A, 0 \leq i \leq N-1\}$, which is an $A$-module with componentwise addition and scalar multiplication by elements of $A$. Then an $A$-submodule $C$ of $A^N$ is called a linear code over $A$ of length $N$. For any vectors $a = (a_0, a_1, \ldots, a_{N-1})$, $b = (b_0, b_1, \ldots, b_{N-1}) \in A^N$. The usual Euclidian inner product of $a$ and $b$ is defined by $\langle a, b \rangle = \sum_{j=0}^{N-1} a_j b_j \in A$. Let $C$ be a linear code over $A$ of length $N$. The Euclidian dual code of $C$ is defined by $C^\perp = \{a \in A^N \mid \langle a, b \rangle = 0, \forall b \in C\}$, and $C$ is said to be self-dual if $C = C^\perp$.

Let $\gamma \in A^\times$. Then a linear code $C$ over $A$ of length $N$ is called a $\gamma$-constacyclic code if $(\gamma a_{N-1}, a_0, a_1, \ldots, a_{N-2}) \in C$ for all $(a_0, a_1, \ldots, a_{N-1}) \in C$. In particular, $C$ is a negacyclic code (resp. cyclic code) if $\gamma = -1$ (resp. $\gamma = 1$).

For any vector $a = (a_0, a_1, \ldots, a_{N-1}) \in A^N$, let $a(x) = a_0 + a_1 x + \ldots + a_{N-1} x^{N-1} \in A[x]/(x^N - \gamma)$. We will identify $a$ with $a(x)$ in this paper. It is well known that $C$ is a $\gamma$-constacyclic code of length $N$ over $A$ if and only if $C$ is an ideal of the residue class ring $A[x]/(x^N - \gamma)$. Moreover, its dual code $C^\perp$ is an ideal of the ring $A[x]/(x^N - \gamma^{-1})$ (cf. [18] Propositions 2.2 and 2.3).

In 1999, Wood in [27] showed that for certain reasons finite Frobenius rings are the most general class of rings that should be used for alphabets of codes. Then Dougherty et al. [20] investigated self-dual codes over commutative Frobenius rings. In [28] and [29], Yildiz et al. studied codes over an extension ring of $\mathbb{Z}_4$ and obtained some good $\mathbb{Z}_4$-codes. Here the ring was described as $\mathbb{Z}_4[u]/(u^2) = \mathbb{Z}_4 + u\mathbb{Z}_4 (u^2 = 0)$ which is a local non-principal ring. Then a complete classification and an explicit representation for cyclic codes of odd length over $\mathbb{Z}_4[u]/(u^k) = \mathbb{Z}_4 + u\mathbb{Z}_4 + \ldots + u^{k-1}\mathbb{Z}_4 (u^k = 0)$ were provided by Cao et al. [7] for any integer $k \geq 2$.

Shi et al. in [25] studied $(1+2u)$-constacyclic codes over the ring $\mathbb{Z}_4[u]/(u^2-1) = \mathbb{Z}_4 + u\mathbb{Z}_4 (u^2 = 1)$ of odd length. Then Cao et al. gave a complete description for negacyclic codes of oddly even length and cyclic codes of odd length over the local ring $\mathbb{Z}_4[v]/(v^2 + 2v)$ by [12] and [13], respectively. In the papers, some new and good $\mathbb{Z}_4$-codes were obtained from $\mathbb{Z}_4$-images of codes over $\mathbb{Z}_4[u]/(u^2-1)$ and $\mathbb{Z}_4[v]/(v^2 + 2v)$. Moreover, a complete classification for simple-root cyclic codes over $\mathbb{F}_p[v]/(v^2 - pv)$ was provided in [14] for any prime number $p$ and integer $s \geq 2$.

In [4], Bandi et al. studied negacyclic codes of length $2^k$ over the ring $R = \mathbb{Z}_4 + u\mathbb{Z}_4 (u^2 = 0)$. Moreover, the structure of ideals in $R[u]/(x^{2^k} + 1)$ were described roughly by Discrete Fourier Transform, where $n$ is an odd integer, and the number
of all negacyclic codes of length $2^k n$ over $R$ was given by $\prod_{i \in J} N_{\xi}$, where $J$ denotes a complete set of representatives of the 2-cyclotomic cosets modulo $n$, and

1. $r_\xi$ is the size of the 2-cyclotomic coset modulo $n$ containing $\xi$;
2. $\text{GR}(R, r_{\xi})$ is the Galois extension ring of $R$ with degree $r_{\xi}$;
3. $N_{\xi}$ is the number of all ideals in the ring $S_{r_{\xi}} = \frac{\text{GR}(R, r_{\xi})[x]}{(x^{2^n} + 1)}$,

for every $\xi \in J$. To the best of our knowledge, the following problems have not been completely solved, where $R = \mathbb{Z}_4 + u\mathbb{Z}_4$ ($u^2 = 0)$:

- Give a clear formula to enumerate the number of all negacyclic codes of length $2^k$ over $R$. Although a mass formula for the number of all negacyclic codes of length $2^k$ over $R$ was given by Theorem 12 in [4], this formula is wrong (see Remark 4.4 in this paper).
- Give a clear formula to enumerate the number of negacyclic codes of length $2^k n$ over $R$, for any odd positive integer $n$. In [4], there was no clear formula given to calculate the number $N_{\xi}$ of ideals in $S_{r_{\xi}}$ for any $\xi \in J$.
- Give an explicit representation for every negacyclic codes of arbitrary even length over $R$. Although negacyclic codes over $R$ of even length were studied in [4], the expression for each code is a little complicated. It is not clear enough for the readers to list negacyclic codes over $R$ easily, for specific even lengths.

To solve these problems above, we will adopt a new idea and use some new methods.

In this paper, let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ in which the arithmetic is done modulo 4, and denote $\mathbb{F}_2 = \{0, 1\}$ in which the arithmetic is done modulo 2. We will regard $\mathbb{F}_2$ as a subset of $\mathbb{Z}_4$ in this paper. But $\mathbb{F}_2$ is not a subring of $\mathbb{Z}_4$. Let $a \in \mathbb{Z}_4$. Then $a$ has a unique 2-adic expansion: $a = 2a_0, a_0, a_1 \in \mathbb{F}_2$. It is well known that $a \in \mathbb{Z}_4^*$ if and only if $a_0 \neq 0$. Denote $\overline{a} = a_0 \in \mathbb{F}_2$. Then $- : a \mapsto \overline{a}$ ($\forall a \in \mathbb{Z}_4$) is a ring homomorphism from $\mathbb{Z}_4$ onto $\mathbb{F}_2$, and this homomorphism can be extended to a ring homomorphism from $\mathbb{Z}_4[y]$ onto $\mathbb{F}_2[y]$ by: $\overline{f}(y) = \sum_{i=0}^{n} \overline{b}_i y^i$, for any $f(y) = \sum_{i=0}^{n} b_i y^i \in \mathbb{Z}_4[y]$ where $b_i \in \mathbb{Z}_4$.

Let $f(y)$ be a monic polynomial in $\mathbb{Z}_4[y]$ of degree $d \geq 1$. Then $f(y)$ is said to be basic irreducible if $\overline{f}(y)$ is an irreducible polynomial in $\mathbb{F}_2[y]$ (cf. § 13.4 in [26]).

From now on, we adopt the following notation.

- Let $\mathbb{Z}_4[y]_{\overline{f}(y)} = \{\sum_{i=0}^{d-1} a_i y^i | a_0, a_1, \ldots, a_{d-1} \in \mathbb{Z}_4\}$ in which the arithmetic is done modulo $\overline{f}(y)$.
- Let $\mathbb{F}_2[y]_{\overline{f}(y)} = \{\sum_{i=0}^{d-1} b_i y^i | b_0, b_1, \ldots, b_{d-1} \in \mathbb{F}_2\}$ in which the arithmetic is done modulo $\overline{f}(y)$.

In the following, we still use $\overline{f}$ to denote the homomorphism of rings from $\mathbb{Z}_4[y]_{\overline{f}(y)}$ onto $\mathbb{F}_2[y]_{\overline{f}(y)}$ defined by: $\sum_{i=0}^{d-1} a_i y^i \mapsto \sum_{i=0}^{d-1} \overline{a}_i y^i$, $\forall a_0, a_1, \ldots, a_{d-1} \in \mathbb{Z}_4$.

In the rest of this paper, let $k$ be any positive integer and $n$ be an odd positive integer. We assume

\begin{equation}
\sum_{i=0}^{n} f_1(y) f_2(y) \cdots f_r(y),
\end{equation}

where $f_1(y), f_2(y), \ldots, f_r(y)$ are pairwise coprime monic basic irreducible polynomials in $\mathbb{Z}_4[y]$ and

\[\deg(f_j(y)) = d_j, \quad j = 1, \ldots, r.\]

Then $\overline{f_1}(y), \overline{f_2}(y), \ldots, \overline{f_r}(y)$ are pairwise coprime irreducible polynomials in $\mathbb{F}_2[y]$ and $\deg(\overline{f_j}(y)) = d_j$ for all $j$. We will adopt the following notation, where $1 \leq j \leq r$: 
• Let \( \mathbb{Z}_4 + u\mathbb{Z}_4 = \mathbb{Z}_4[u]/(u^2) = \{a + ub \mid a, b \in \mathbb{Z}_4\} \) (\(u^2 = 0\)) in which the operations are defined by \(\alpha + \beta = (a + b) + u(c + d)\) and \(\alpha\beta = ac + u(ad + bc)\), for any \(\alpha = a + bu, \beta = c + du \in \mathbb{Z}_4 + u\mathbb{Z}_4\) with \(a, b, c, d \in \mathbb{Z}_4\). Then \(\mathbb{Z}_4 + u\mathbb{Z}_4\) is a local non-principal ideal ring (cf. [29]).

• Let \(A = \frac{\mathbb{Z}_4[x]}{(x^k + 1)} = \{\sum_{i=0}^{k-1} a_ix^i \mid a_0, a_1, \ldots, a_{k-1} \in \mathbb{Z}_4\}\) in which the arithmetic is done modulo \(x^k + 1\).

• Let \(R_j = \frac{\mathbb{Z}_4[x]}{(f_j(x^2^j))} = \{\sum_{i=0}^{d_j-1} a_ix^i \mid a_0, a_1, \ldots, a_{d_j-1} \in \mathbb{Z}_4\}\) in which the arithmetic is done modulo \(f_j(-x^{2^j})\), where \(\deg(f_j(-x^{2^j})) = 2^j d_j\).

• Let \(T_j = \{\sum_{i=0}^{d_j-1} t_i x^i \mid t_0, t_1, \ldots, t_{d_j-1} \in \mathbb{F}_2\}\) \(\subset R_j\). Then \(|T_j| = 2^{d_j}\).

This paper is organized as follows. In Section 2, we prove that each \(R_j\) is a finite chain ring, \(1 \leq j \leq r\), and establish an isomorphism of rings from the direct product ring \(R_1 \times \cdots \times R_r\) onto \(A\). In Section 3, we construct a precise isomorphism of rings from the direct product ring \((R_1 + uR_1) \times \cdots \times (R_r + uR_r)\) onto \(\frac{\mathbb{Z}_4 + u\mathbb{Z}_4[x]}{(x^k + 1)}\) first. Then we present all distinct ideals of each ring \(R_j + uR_j\) explicitly. Hence we give an explicit representation and enumeration for all distinct negacyclic codes of length \(2^n\) over \(\mathbb{Z}_4 + u\mathbb{Z}_4\). In Section 4, we give an explicit expression for every negacyclic codes of length \(2^k\) over \(\mathbb{Z}_4 + u\mathbb{Z}_4\) and obtain an exact formula to count the number of all these codes. Then we correct a mistake in the mass formula for the number of negacyclic codes of length \(2^k\) over \(\mathbb{Z}_4 + u\mathbb{Z}_4\) in [4]. In Section 5, we give an explicit representation for all distinct cyclic codes of odd length \(n\) over \(\mathbb{Z}_4 + u\mathbb{Z}_4\) and correct some mistakes in [3] and [23]. Section 6 concludes the paper.

2. Structure of the ring \(A = \frac{\mathbb{Z}_4[x]}{(x^k + 1)}\)

In this section, we consider to decompose the ring \(A\) into a direct sum of finite chain rings first. To do this, we need the following lemmas.

**Lemma 2.1.** ([17] Proposition 2.1) Let \(A\) be a finite associative and commutative ring with identity. Then the following conditions are equivalent:

(i) \(A\) is a local ring and the maximal ideal \(M\) of \(A\) is principal, i.e., \(M = \langle \pi \rangle\) for some \(\pi \in A\);

(ii) \(A\) is a local principal ideal ring;

(iii) \(A\) is a chain ring with ideals \(\langle \pi^i \rangle, 0 \leq i \leq \nu\), where \(\nu\) is the nilpotency index of \(\pi\).

**Lemma 2.2.** ([22] Proposition 2.2) Let \(A\) be a finite commutative chain ring, with maximal ideal \(M = \langle \pi \rangle\), and let \(\nu\) be the nilpotency index of \(\pi\). Then

(i) For some prime \(p\) and positive integer \(m\), \(|A/\langle \pi \rangle| = q\) where \(q = p^m, |A| = q^{\nu}\), and the characteristic of \(A/\langle \pi \rangle\) and \(A\) are powers of \(p\);

(ii) For \(i = 0, 1, \ldots, \nu\), \(|\langle \pi^i \rangle| = q^{\nu-i}\).

**Lemma 2.3.** ([22] Lemma 2.4) Using the notations in Lemma 2.2, let \(V \subseteq A\) be a system of representatives for the equivalence classes of \(A\) under congruence modulo \(\pi\). (Equivalently, we can define \(V\) to be a maximal subset of \(A\) with the property that \(r_1 - r_2 \notin \langle \pi \rangle\) for all \(r_1, r_2 \in V, r_1 \neq r_2\).) Then
(i) Every element \( a \) of \( A \) has a unique \( \pi \)-adic expansion: \( a = \sum_{j=0}^{\nu-1} r_j \pi^j \), \( r_0, r_1, \ldots, r_{\nu-1} \in V \).

(ii) \( |A/(\langle \pi \rangle)| = |V| \) and \( |\langle \pi^i \rangle| = |V|^{\nu-i} \), for all integers \( j \): \( 0 \leq i \leq \nu - 1 \).

Let \( 1 \leq j \leq r \). From now on, we adopt the following notation:

- Let \( \Gamma_j = \frac{Z_4[y]}{\langle f_j(y) \rangle} = \{ \sum_{i=0}^{d_j-1} a_i y^i \mid a_0, a_1, \ldots, a_{d_j-1} \in Z_4 \} \) in which the arithmetic is done modulo \( f_j(y) \).
- Let \( \overline{\Gamma}_j = \frac{\mathbb{F}_2[y]}{\langle f_j(y) \rangle} = \{ \sum_{i=0}^{d_j-1} b_i y^i \mid b_0, b_1, \ldots, b_{d_j-1} \in \mathbb{F}_2 \} \) in which the arithmetic is done modulo \( f_j(y) \).

**Lemma 2.4.** Using the notation above, we have the following conclusions:

(i) (cf. [26] Theorem 14.1) \( \Gamma_j \) is a Galois ring of characteristic 4 and cardinality \( 4^{d_j} \), in symbol as \( \Gamma_j = \text{GR}(4, d_j) \). Moreover, we have \( \Gamma_j = Z_4[\zeta_j] \), where \( \zeta_j = y \in \Gamma_j \) satisfying \( \zeta_j^{2^{d_j} - 1} = 1 \), i.e. \( \zeta_j^{2^{d_j}} = \zeta_j \).

Denote \( \overline{\zeta}_j = y \in \overline{\Gamma}_j \). Then \( \overline{\Gamma}_j = \mathbb{F}_2[\overline{\zeta}_j] \) which is a finite field of cardinality \( 2^{d_j} \), \( \overline{\Gamma}_j(x) = \prod_{i=0}^{d_j-1} (x-\overline{\zeta}_j^i) \in \mathbb{F}_2[x] \) and that \( \overline{\zeta}_j \) can be extended to a ring homomorphism from \( \Gamma_j \) onto \( \overline{\Gamma}_j \) by \( \xi \mapsto \overline{\xi} = \sum_{i=0}^{d_j-1} \overline{\alpha}_i \overline{\zeta}_j^i \), for all \( \xi = \sum_{i=0}^{d_j-1} \alpha_i \zeta_j^i \in \Gamma_j \) where \( \alpha_0, \alpha_1, \ldots, \alpha_{d_j-1} \in \mathbb{Z}_4 \).

(ii) (cf. [6] Lemma 2.3(ii)) \( f_j(x) = \prod_{i=0}^{d_j-1} (x - \zeta_j^i) \) in \( \Gamma_j[x] \).

Now, we determine the algebraic structure of each ring \( \mathcal{R}_j = \frac{\mathbb{Z}_4[x]}{\langle f_j(x) \rangle} \), where \( 1 \leq j \leq r \). The following lemma is the key to this paper.

**Lemma 2.5.** Using the notation in Section 1, let \( 1 \leq j \leq r \). Then

(i) There is an invertible element \( \vartheta_j(x) \) of the ring \( \mathcal{R}_j \) such that

\[
(f_j(x))^{2^k} = 2\vartheta_j(x) \text{ in } \mathcal{R}_j.
\]

Hence \( (2) = (f_j(x)^{2^k}) \) as ideals of \( \mathcal{R}_j \).

(ii) \( \mathcal{R}_j \) is a finite chain ring with the unique maximal ideal \( \langle f_j(x) \rangle \), where \( \langle f_j(x) \rangle = f_j(x)\mathcal{R}_j \), the nilpotency index of \( f_j(x) \) is equal to \( 2^{k+1} \) and \( \mathcal{R}_j/\langle f_j(x) \rangle \) is a finite field of cardinality \( 4^{d_j} \).

(iii) Each element \( \alpha \in \mathcal{R}_j \) has a unique \( f_j(x) \)-adic expansion:

\[
\alpha = b_0(x) + b_1(x)f_j(x) + \cdots + b_{2^{k+1}-1}(x)f_j(x)^{2^{k+1}-1},
\]

where \( b_i(x) \in \mathcal{T}_j \) for all \( i = 0, 1, \ldots, 2^{k+1} - 1 \).

(iv) All distinct ideals of \( \mathcal{R}_j \) are given by: \( \langle f_j(x)^i \rangle = f_j(x)^i\mathcal{R}_j, \) \( i = 0, 1, 2, \ldots, 2^{k+1} \). Moreover, we have \( |\langle f_j(x)^i \rangle| = 2^{(2^{k+1}-i)d_j} \).

(v) Let \( 1 \leq l \leq 2^{k+1} \). Then \( \mathcal{R}_j/\langle f_j(x)^l \rangle = \{ \sum_{i=0}^{l-1} b_i(x)f_j(x)^{i} \mid b_0(x), \ldots, b_{l-1} \in \mathcal{T}_j \} \)

(in which \( f_j(x)^l = 0 \)) and \( |\mathcal{R}_j/\langle f_j(x)^l \rangle| = 2^{ld_j} \).

(vi) Let \( 0 \leq l \leq t \leq 2^{k+1} - 1 \). Then

\[
f_j(x)^l \cdot \mathcal{R}_j/\langle f_j(x)^l \rangle = \left\{ \sum_{i=0}^{t-1} b_i(x)f_j(x)^i \mid b_0(x), \ldots, b_{t-1} \in \mathcal{T}_j \right\},
\]
where we set $f_j(x)^\gamma \in \mathbb{F}_2[x]$ for convenience. Hence $|f_j(x)^\gamma \in \mathbb{F}_2[x]| = 2^{(t-1)d_j}$.

**Proof.** (i) By Lemma 2.4(ii), we have $f_j(x) = \prod_{i=0}^{d_j-1} (x - \zeta_j^i i)$ in $\Gamma_j[x]$. This implies $f(x)^{2^k} = \prod_{i=0}^{d_j-1} (x - \zeta_j^{2^i i})^{2^k}$, where

$$(x - \zeta_j^{2^i i})^{2^k} = \sum_{t=0}^{2^k} \left( \binom{2^k}{t} \right) x^{2^k-t} (-\zeta_j^{2^i i})^t = -\left( x^{2^k} - (\zeta_j^{2^i i})^{2^k} \right) + 2x^{2^k-1} - (\zeta_j^{2^i i})^{2^k-1},$$

since $\left( \binom{2^k}{t} \right) = 2$ and $\left( \binom{2^k}{t} \right) = 0$ in $\mathbb{Z}_4$ for all $t \notin \{0, 2^{k-1}, 2^k \}$. Hence

$$f(x)^{2^k} = (-1)^{d_j} \prod_{i=0}^{d_j-1} \left( (-x^{2^k}) - (\zeta_j^{2^i i})^{2^k} \right) + 2 \sum_{i=0}^{d_j-1} \left( \prod_{0 \leq t \leq d_j-1, t \neq i} x^{2^k} - (\zeta_j^{2^i i})^{2^k} \right).$$

Denote $g_{j,i}(x) = x^{2^k-1} (-\zeta_j^{2^i i})^{2^k-1} \prod_{0 \leq t \leq d_j-1, t \neq i} \left( x^{2^k} + (\zeta_j^{2^i i})^{2^k} \right) \in \Gamma_j[x]$, where $0 \leq i \leq d_j - 1$, and set $g_j(x) = \sum_{i=0}^{d_j-1} g_{j,i}(x)$. Then we have

$$f(x)^{2^k} = (-1)^{d_j} f_j(-x^{2^k}) + 2g_j(x).$$

This implies $g_j(x) = \frac{f(x)^{2^k} - (-1)^{d_j} f_j(-x^{2^k})}{2} \pmod{4}$ and so $g_j(x) \in \mathbb{Z}_4[x]$. As $4 = 0$, we have $2g_j(x) = 2g_j(x)$.

As stated above, we conclude that $f(x)^{2^k} \equiv 2g_j(x) \pmod{f_j(-x^{2^k})}$. This implies

$$f(x)^{2^k} = 2g_j(x) \in \mathbb{F}_2[x].$$

Here, we regard $\mathbb{F}_2[x]$ as a subset of $\mathbb{Z}_4[x]$, but $\mathbb{F}_2[x]$ is not a subring of $\mathbb{Z}_4[x]$.

Now, let $\overline{g}_j(x) = g_j(x) \in \mathbb{R}_j$. As a polynomial in $\Gamma_j[x]$, we see that

$$\overline{g}_{j,i}(x) = x^{2^k-1} \left( \zeta_j^{2^i} \right)^{2^k-1} \prod_{0 \leq t \leq d_j-1, t \neq i} \left( x + \zeta_j^{2^i} \right)^{2^k}$$

for all $i = 0, 1, \ldots, d_j - 1$, and $\overline{g}_j(x) = \sum_{i=0}^{d_j-1} \overline{g}_{j,i}(x)$. Since $\zeta_j, \zeta_j^2, \ldots, \zeta_j^{2^{d_j-1}}$ are all distinct roots of the polynomial $\overline{f}_j(x)$ in the extension field $\Gamma_j$ of $\mathbb{F}_2$ and $\zeta_j^{2^{d_j-1}} = \zeta_j$, for any integer $\lambda$: $0 \leq \lambda \leq d_j - 1$, we have

$$\overline{g}_{j,i}(\zeta_j^{2^\lambda}) = \left( \zeta_j^{2^\lambda} \right)^{2^k-1} \left( \zeta_j^{2^\lambda} \right)^{2^k-1} \prod_{0 \leq t \leq d_j-1, t \neq i} \left( \zeta_j^{2^\lambda} + \zeta_j^{2^t} \right)^{2^k} = 0, \text{ if } i \neq \lambda;$$

and $\overline{g}_{j,\lambda}(\zeta_j^{2^\lambda}) = \zeta_j^{2^\lambda} \prod_{0 \leq t \leq d_j-1, t \neq \lambda} \left( \zeta_j^{2^\lambda} + \zeta_j^{2^t} \right)^{2^k} \neq 0$. These imply

$$\overline{g}_j(\zeta_j^{2^\lambda}) = \left( \sum_{i=0}^{d_j-1} \overline{g}_{j,i}(\zeta_j^{2^\lambda}) \right) = \overline{g}_{j,\lambda}(\zeta_j^{2^\lambda}) \neq 0.$$
Therefore, we conclude that \( \gcd(\overline{f}_j(x), \overline{g}_j(x)) = 1 \). This implies \( \gcd(f_j(x)^2, g_j(x)) = 1 \) as polynomials in \( F_2[x] \). Then by \( \overline{f}_j(-x^{2^k}) = \overline{f}_j(x)^{2^k} \), we see that \( f_j(-x^{2^k}) \) and \( g_j(x) \) are coprime in \( \mathbb{Z}_4[x] \). This implies that \( a(x)\overline{g}_j(x) + b(x)f_j(-x^{2^k}) = 1 \) for some \( a(x), b(x) \in \mathbb{Z}_4[x] \), i.e., \( a(x)\overline{g}_j(x) \equiv 1 \pmod{f_j(-x^{2^k})} \). Hence \( \overline{g}_j(x) \) is an invertible element in the ring \( R_j \). Moreover, by (i) we have \( \langle \overline{f}_j(x)^{2^k} \rangle = \langle \overline{g}_j(x) \rangle \) as ideals of \( R_j \).

(ii) Let \( M = \langle f_j(x), 2 \rangle \) be the ideal of \( R_j \) generated by \( f_j(x) \) and 2. Then the residue class ring of \( R_j \) modulo \( M \) is given by:

\[
\frac{R_j}{M} = \frac{R_j}{\langle f_j(x), 2 \rangle} \cong \frac{F_2[x]/\langle f_j(x)^{2^k} \rangle}{\langle f_j(x) \rangle} \cong \frac{F_2[x]}{\langle f_j(x) \rangle} = \Gamma_j,
\]

where \( \Gamma_j \) is a finite field of \( 2^{d_j} \) elements. Hence \( M \) is a maximal ideal of \( R_j \).

As \( f_j(x)^{2^{k+1}} = (2\overline{g}_j(x))^2 = 4\overline{g}_j(x)^2 = 0 \) in \( R_j \), we see that every element of \( M \) is nilpotent. Hence each element in \( R_j \setminus M \) must be an invertible element, and so \( M \) is the unique maximal ideal of \( R_j \). Therefore, \( R_j \) is a finite chain ring with the unique maximal ideal \( \langle f_j(x) \rangle \) by Lemma 2.1.

Let \( \nu \) be the nilpotency index of \( f_j(x) \) in \( R_j \). Then \( |R_j| = |\frac{R_j}{\langle f_j(x) \rangle}|^{\nu} \) by Lemma 2.2. From this, by \( |R_j| = 4^{2^{d_j}} \) and \( |\frac{R_j}{\langle f_j(x) \rangle}| = |\frac{R_j}{M}| = |\Gamma_j| = 2^{d_j} \), we deduce that \( \nu = 2^{k+1} \).

(iii) Using the notation of Section 1, we know that \( T_j = \{ \sum_{i=0}^{d_j-1} t_i x^i \mid t_0, t_1, \ldots, t_{d_j-1} \in \{0, 1\} \} \subseteq R_j \). As \( f_j(x) \) is a monic basic irreducible polynomial in \( \mathbb{Z}_4[x] \), it follows that \( \gamma_1 - \gamma_2 \notin \langle f_j(x) \rangle \) for all \( \gamma_1, \gamma_2 \in T_j \) satisfying \( \gamma_1 \neq \gamma_2 \). Moreover, we have \( |\frac{R_j}{\langle f_j(x) \rangle}| = 2^{d_j} = |T_j| \). Hence \( T_j \) is a system of representatives for the equivalence classes of \( R_j \) under congruence modulo \( f_j(x) \). Then the conclusion follows from Lemma 2.3 immediately.

(iv)–(vi) The conclusions follow from properties of finite chain rings (cf. [22]). Here, we omit the proofs.

Finally, we decompose the ring \( A = \frac{\mathbb{Z}_4[x]}{(x^{2^k} + 1)} \) into a direct sum of the finite chain rings \( R_j \) under the isomorphism meaning.

Let \( 1 \leq j \leq r \) and denote \( F_j(y) = \frac{y^{2^k}-1}{f_j(y)} \in \mathbb{Z}_4[y] \). As \( \gcd(F_j(y), \overline{f}_j(y)) = 1 \), we see that \( F_j(y) \) and \( f_j(y) \) are coprime in \( \mathbb{Z}_4[y] \) (cf. [26] Lemma 13.5). Hence there are polynomials \( a_j(y), b_j(y) \in \mathbb{Z}_4[y] \) such that

\[
(2) \quad a_j(y)F_j(y) + b_j(y)f_j(y) = 1.
\]

In this paper, we define \( \theta_j(x) \in A \) by:

\[
\bullet \quad \theta_j(x) \equiv a_j(-x^{2^k})F_j(-x^{2^k}) = 1 - b_j(-x^{2^k})f_j(-x^{2^k}) \pmod{x^{2^k} + 1}.
\]

Substituting \(-x^{2^k}\) for \( y \) in Equations (1) of Section 1 and (2) above, we obtain

\[ -(x^{2^k} + 1) = -(x^{2^k})^n - 1 = f_1(-x^{2^k})f_2(-x^{2^k}) \cdots f_r(-x^{2^k}). \]

Hence \( x^{2^k} + 1 = -f_1(-x^{2^k})f_2(-x^{2^k}) \cdots f_r(-x^{2^k}) \) and

\[
a_j(-x^{2^k})F_j(-x^{2^k}) + b_j(-x^{2^k})f_j(-x^{2^k}) = 1.
\]
Then from the definition of $\theta_j(x)$ and the Chinese Remainder Theorem for commutative rings with identity, we deduce the following conclusion.

**Theorem 2.6.** Using the notation above, we have the following conclusions:

(i) $\theta_1(x) + \ldots + \theta_r(x) = 1$, $\theta_j(x)^2 = \theta_j(x)$ and $\theta_i(x)\theta_j(x) = 0$ in $\mathcal{A}$, for all integers $i$ and $j$: $1 \leq i \neq j \leq r$.

(ii) $\mathcal{A} = \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_r$, where $\mathcal{A}_j = \theta_j(x)\mathcal{A}$ and its multiplicative identity is $\theta_j(x)$. Moreover, this decomposition is a direct sum of rings in that $\mathcal{A}_i\mathcal{A}_j = \{0\}$ for all integers $i$ and $j$: $1 \leq i \neq j \leq r$.

(iii) For each $1 \leq j \leq r$ and $a(x) \in \mathcal{R}_j = \frac{\mathbb{Z}_4[x]}{(f_j(-x^2))}$, define a map $\tau_j$ by

$$\tau_j : a(x) \mapsto \theta_j(x)a(x) \pmod{x^{2^k} + 1}.$$

Then $\tau_j$ is a ring isomorphism from $\mathcal{R}_j$ onto $\mathcal{A}_j$. Hence $|\mathcal{A}_j| = 4^{2^kd_j} = 2^{k+1}d_j$.

(iv) Define $\tau : (a_1(x), \ldots, a_r(x)) \mapsto \tau_1(a_1(x)) + \ldots + \tau_r(a_r(x))$, i.e.,

$$\tau(a_1(x), \ldots, a_r(x)) = \sum_{j=1}^r \theta_j(x)a_j(x) \pmod{x^{2^k} + 1},$$

for all $a_j(x) \in \mathcal{R}_j$ and $j = 1, \ldots, r$. Then $\tau$ is a ring isomorphism from the direct product ring $\mathcal{R}_1 \times \ldots \times \mathcal{R}_r$ onto $\mathcal{A}$.

3. **Explicit representation and enumeration for negacyclic codes over the ring $\mathbb{Z}_4 + u\mathbb{Z}_4$ of length $2^kn$**

In this section, we determine all distinct negacyclic codes over the ring $\mathbb{Z}_4 + u\mathbb{Z}_4$ of length $2^kn$, i.e., all distinct ideals of the ring $\frac{(\mathbb{Z}_4 + u\mathbb{Z}_4)[x]}{(x^{2^kn} + 1)}$.

In this paper, for a ring $\mathcal{Y} \in \{\mathcal{A}, \mathcal{A}_j, \mathcal{R}_j\}$ where $1 \leq j \leq r$, we set $\mathcal{Y}[u]/(u^2) = \mathcal{Y} + u\mathcal{Y} (u^2 = 0)$ in which the operations are defined by:

$$(\xi_1 + u\eta_1) + (\xi_2 + u\eta_2) = (\xi_1 + \xi_2) + u(\eta_1 + \eta_2);$$

$$\xi_1 + u\eta_1)(\xi_2 + u\eta_2) = \xi_1\xi_2 + u(\xi_1\eta_2 + \xi_2\eta_1),$$

for any $\xi_1, \eta_1, \xi_2, \eta_2 \in \mathcal{Y}$. Let $\alpha \in \frac{(\mathbb{Z}_4 + u\mathbb{Z}_4)[x]}{(x^{2^kn} + 1)}$. Then $\alpha$ can be uniquely expressed as

$$\alpha = \sum_{i=0}^{2^kn-1} (a_i + b_iu)x^i, \quad a_i, b_i \in \mathbb{Z}_4, \quad i = 0, 1, \ldots, 2^kn - 1.$$

Denote $\xi = \sum_{i=0}^{2^kn-1} a_ix^i$ and $\eta = \sum_{i=0}^{2^kn-1} b_ix^i$. Then we have $\xi, \eta \in \mathcal{A} = \frac{\mathbb{Z}_4[x]}{(x^{2^kn} + 1)}$.

Now, define $\sigma : \alpha \mapsto \xi + u\eta$. It can be verified easily that $\sigma$ is a ring isomorphism from the ring $\mathcal{A} = \frac{\mathbb{Z}_4[x]}{(x^{2^kn} + 1)}$ onto $\mathcal{A} + u\mathcal{A}$.

In the rest of this paper, we will identify $\frac{(\mathbb{Z}_4 + u\mathbb{Z}_4)[x]}{(x^{2^kn} + 1)}$ with $\mathcal{A} + u\mathcal{A}$ under the above isomorphism $\sigma$. Moreover, we have the following conclusions:

**Lemma 3.1.** Let $1 \leq j \leq r$. Using the notations of Theorem 2.6, for any $a(x), b(x) \in \mathcal{R}_j$ we define

$$\tau_j(a(x) + b(x)u) = \tau_j(a(x)) + \tau_j(b(x))u = \theta_j(x)(a(x) + b(x)u) \pmod{x^{2^kn} + 1}.$$

Then $\tau_j$ is a ring isomorphism from $\mathcal{R}_j + u\mathcal{R}_j$ onto $\mathcal{A}_j + u\mathcal{A}_j$. 

Proof. By Theorem 2.6 (iii), the isomorphism $\tau_j : \mathcal{R}_j \to A_j$ induces an isomorphism of polynomial rings from $\mathcal{R}_j[u]$ onto $A_j[u]$ in the natural way that
\[
\sum_i a_i(x)u^i \rightarrow \sum_i \tau_j(a_i(x))u^i \quad (\forall a_i(x) \in \mathcal{R}_j).
\]
Hence $\tau_j$ is a ring isomorphism from $\mathcal{R}_j + u\mathcal{R}_j$ onto $A_j + uA_j$. \hfill $\Box$

Lemma 3.2. The following statements are equivalent:

(i) $C$ is a negacyclic code over $\mathbb{Z}_4 + u\mathbb{Z}_4$ of length $2^k n$.

(ii) $C$ is an ideal of the ring $A + uA$.

(iii) For each integer $1 \leq j \leq r$, there is a unique ideal $C_j$ of the ring $\mathcal{R}_j + u\mathcal{R}_j$ such that $C = \bigoplus_{j=1}^r \theta_j(x)C_j \pmod{x^{2^k n} + 1}$. In this case, we have $|C| = \prod_{j=1}^r |C_j|$.

Proof. (i)$\Leftrightarrow$ (ii) It follows from the identification of $\mathcal{R}_j[u]/(x^{2^k n} + 1)$ with $A + uA$.

(ii)$\Leftrightarrow$ (iii) By Theorem 2.6 (ii), we have $A = \bigoplus_{j=1}^r A_j$. Hence
\[
A + uA = \frac{A[u]}{u^2} = \bigoplus_{j=1}^r \frac{A_j[u]}{u^2} = \bigoplus_{j=1}^r (A_j + uA_j).
\]
This decomposition is a direct sum of rings in that $(A_i + uA_i)(A_j + uA_j) = \{0\}$, for any integers $i, j$: $1 \leq i \neq j \leq r$. Therefore, $C$ is an ideal of $A + uA$ if and only if for each integer $j$: $1 \leq j \leq r$, there is a unique ideal $C_j$ of the ring $A_j + uA_j$ such that $C = \bigoplus_{j=1}^r C_j$. From this and by Lemma 3.1, we deduce that $C_j$ is an ideal of $A_j + uA_j$ if and only if there is a unique ideal $C_j$ of $\mathcal{R}_j + u\mathcal{R}_j$ such that
\[
C_j = \tau_j(C_j) = \theta_j(x)C_j = \{\theta_j(x)c_j(x) \mid c_j(x) \in C_j\} \pmod{x^{2^k n} + 1}.
\]
Hence $C = \bigoplus_{j=1}^r \theta_j(x)C_j$ and $|C| = \prod_{j=1}^r |C_j| = \prod_{j=1}^r |C_j|$. \hfill $\Box$

Therefore, in order to present all negacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ of length $2^k n$, it is sufficient to determine all ideals of the ring $\mathcal{R}_j + u\mathcal{R}_j$ for each $j$.

By Theorem 3.8 in [15], we determined all distinct ideals of the ring $K_j + uK_j$ $(u^2 = 0)$, where $K_j = \frac{F_{p^m}[x]}{(f_j(x))^{p^m}} = \left\{ \sum_{i=0}^{p^m d_{j,i}-1} a_i x^i \mid a_0, a_1, \ldots, a_{p^m d_{j,i}-1} \in F_{p^m} \right\}$ in which the arithmetic is done modulo $f_j(x)^{p^m}$, and

- $p$ is a prime number, $m, s$ are positive integers and $F_{p^m}$ is a finite field of $p^m$ elements.
- $f_j(x)$ is an irreducible polynomial in $F_{p^m}[x]$ with degree $d_j$.
- (15) Lemma 3.1 (i) $K_j$ is a finite chain ring with the unique maximal ideal $(f_j(x))$, and the nilpotency index of $f_j(x)$ in $K_j$ is $p^s$.
- (15) Lemma 3.1 (ii) Let $T_j = \left\{ \sum_{i=0}^{d_{j,i}-1} t_i x^i \mid t_0, t_1, \ldots, t_{d_{j,i}-1} \in F_{p^m} \right\} \subset K_j$. Then each element $x \in K_j$ has a unique $f_j(x)$-expansion:
\[
\xi = \sum_{i=0}^{p^s-1} b_i(x)f_j(x)^i, \ b_0(x), b_1(x), \ldots, b_{p^s-1}(x) \in T_j.
\]

For clarity, using Lemma 2.5 we give a table below:
For any positive integer \(i\), let \(\lfloor \frac{i}{2} \rfloor = \min\{l \in \mathbb{Z}^+ | l \geq \frac{i}{2}\}\) and \(\lceil \frac{i}{2} \rceil = \max\{l \in \mathbb{Z}^+ \cup \{0\} | l \leq \frac{i}{2}\}\). Making the following replacements in Theorem 3.8 of [15]:

\[
\mathcal{K}_j \rightarrow \mathcal{R}_j, \quad p \rightarrow 2, \quad s \rightarrow k + 1, \quad k \rightarrow \lambda, \quad m \rightarrow 1,
\]

we obtain the following conclusion.

**Theorem 3.3.** Using the notations above, all distinct ideals \(C_j\) of the ring \(\mathcal{R}_j + u\mathcal{R}_j\) \((a^2 = 0)\) and the number \(|C_j|\) of elements in \(C_j\) are given by the following five cases:

(I) \(2^{2d_j}\) ideals:

- \(C_j = (f_j(x)b(x) + u)\) with \(|C_j| = 2^{2k+1-d_j}\),

where \(b(x) = \sum_{i=2t-1}^{2k+1-\lambda} b_i(x)f_j(x)^i\) and \(b_{2k+1}(x), \ldots, b_{2k+1-2}(x) \in \mathcal{T}_j\).

(II) \(2^{2k+1-1}\) ideals:

- \(C_j = (uf_j(x)^{2k+1-1})\) with \(|C_j| = 2^{d_j}\);
- \(C_j = (f_j(x)^{2k+1-\lambda}b(x) + uf_j(x)^{2k+1-\lambda})\) with \(|C_j| = 2^{d_j(2k+1-\lambda)}\),

where \(b(x) = \sum_{i=2k+1-\lambda-2}^{2k+1-\lambda-1} b_i(x)f_j(x)^i\), \(b_i(x) \in \mathcal{T}_j\), \(\lfloor \frac{1}{2}(2k+1-\lambda) \rfloor \leq i \leq 2k+1-\lambda - 2\) and \(1 \leq \lambda \leq 2k+1 - 2\).

(III) \(2^{k+1} + 1\) ideals:

- \(C_j = (f_j(x)^\lambda)\) with \(|C_j| = 2^{2d_j(2k+1-\lambda)}\), where \(0 \leq \lambda \leq 2k+1\).

(IV) \(2^{2k+1-1}\) ideals:

- \(C_j = \langle u, f_j(x) \rangle\) with \(|C_j| = 2^{d_j(2k+2-1)}\);
- \(C_j = (f_j(x)b(x) + u, f_j(x)^t)\) with \(|C_j| = 2^{d_j(2k+2-1)}\),

where \(b(x) = \sum_{i=2t-1}^{2k+1-\lambda} b_i(x)f_j(x)^i\), \(b_i(x) \in \mathcal{T}_j\), \(\lfloor \frac{1}{2} \rfloor \leq i \leq t - 2\) and \(2 \leq t \leq 2^{k+1} - 1\).

(V) \(2^{2k+1-2}\) ideals:

- \(C_j = \langle u f_j(x)^\lambda, f_j(x)^\lambda t \rangle\) with \(|C_j| = 2^{d_j(2k+2-2\lambda-1)}\), where \(1 \leq \lambda \leq 2k+1 - 1\);
- \(C_j = (f_j(x)^{\lambda+1}b(x) + uf_j(x)^{\lambda-t})\) with \(|C_j| = 2^{d_j(2k+2-2\lambda-t)}\),

where \(b(x) = \sum_{i=2k+1-\lambda-1}^{2k+1-\lambda} b_i(x)f_j(x)^i\), \(b_i(x) \in \mathcal{T}_j\), \(\lfloor \frac{1}{2} \rfloor \leq i \leq t - 2\), \(2 \leq t \leq 2^{k+1} - 1\) and \(1 \leq \lambda \leq 2k+1 - 3\).

Moreover, let \(N_{(2,d_j,2^{k+1})}\) be the number of ideals in \(\mathcal{R}_j + u\mathcal{R}_j\). Then

\[
N_{(2,d_j,2^{k+1})} = \sum_{i=0}^{2k} (1 + 4i)2^{(2k-i)d_j}.
\]
Proof. A direct proof can be given by an argument paralleling to that of Lemma 3.2, Lemma 3.7 and Theorem 3.8 in [15]. Here, we omitted. □

By the following proposition, we give a simplified expression for the number \( N_{(2,d_j,2k+1)} \) of ideals in \( R_j + uR_j \).

**Proposition 3.4.** The number of ideals in \( R_j + uR_j \) is equal to

\[
N_{(2,d_j,2k+1)} = \frac{(2^{d_j} + 3)2^{2k+1}d_j - 2^{d_j}2^{k+2} + 5 + 2^{k+2} + 1}{(2^{d_j} - 1)^2}.
\]

Especially, we have \( N_{(2,1,2k+1)} = 10 \cdot 2^k - 2^{k+2} - 9 \) when \( d_j = 1 \), and \( N_{(2,d_j,2k+1)} = 4^{d_j} + 5 \cdot 2^{d_j} + 9 \) when \( k = 1 \).

**Proof.** By Theorem 3.3, it follows that

\[
N_{(2,d_j,2k+1)} = \sum_{i=0}^{2^k} (2^{d_j})^{2^k-i} + 2^{2^k-1}d_j + 2 \sum_{i=1}^{2^k} i(\frac{1}{2^{d_j}})^{i-1},
\]

where \( \sum_{i=0}^{2^k} (2^{d_j})^{2^k-i} = \sum_{i=0}^{2^k} (2^{d_j})^i = \frac{2^{(2^k+1)d_j-1}}{2^{d_j} - 1} \). Then by

\[
\sum_{i=1}^{2^k} i(x^{-1})^{i-1} = \frac{d}{dx} \left( \sum_{i=0}^{2^k} x^i \right) = \frac{d}{dx} \left( \frac{x^{2^k+1} - 1}{x - 1} \right) = \frac{(2^k + 1)x^{2^k}(x - 1) - (x^{2^k+1} - 1)}{(x - 1)^2},
\]

we have

\[
\sum_{i=1}^{2^k} i(\frac{1}{2^{d_j}})^{i-1} = (2^k + 1)(\frac{1}{2^{d_j}})^{2^k}(\frac{1}{2^{d_j}} - 1) - (\frac{1}{2^{d_j}})^{2^k+1} - 1 \]

\[
= \frac{2^{-(2^k-1)d_j}}{(2^{d_j} - 1)^2} \left( 2^{(2^k+1)d_j} - 2^{d_j}(2^k + 1) + 2^k \right).
\]

From these, we deduce that

\[
N_{(2,d_j,2k+1)} = \frac{2(2^{d_j+1}d_j - 1)}{2^{d_j} - 1} + \frac{4}{(2^{d_j} - 1)^2} \left( 2^{(2^k+1)d_j} - 2^{d_j}(2^k + 1) + 2^k \right)
\]

\[
= \frac{(2^{d_j+3})2^{2^k+1}d_j - 2^{d_j}(2^{k+2} + 5) + 2^{k+2} + 1}{(2^{d_j} - 1)^2}.
\]

Especially, we have \( N_{(2,1,2k+1)} = 10 \cdot 2^k - 2^{k+2} - 9 \) when \( d_j = 1 \). □

Then by Lemma 3.2, Theorem 3.3 and Proposition 3.4, we give an explicit representation and enumeration for all distinct negacyclic codes over \( \mathbb{Z}_4 + u\mathbb{Z}_4 \) of arbitrary even length as follows:

**Theorem 3.5.** Every negacyclic code \( C \) over \( \mathbb{Z}_4 + u\mathbb{Z}_4 \) (\( u^2 = 0 \)) of length \( 2^k \) can be constructed by the following two steps:

(i) For each integer \( j, 1 \leq j \leq r \), choose an ideal \( C_j \) of \( R_j + uR_j \) listed in Theorem 3.3.
(ii) Set $C = \bigoplus_{j=1}^{r} \theta_j(x)C_j = \sum_{j=1}^{r} \theta_j(x)C_j \pmod{x^{2^kn} + 1}$. Moreover, the number of codewords in $C$ is equal to $|C| = \prod_{j=1}^{r} |C_j|$.

Then the number of negacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ of length $2^kn$ is equal to

$$\prod_{j=1}^{r} N_{(2,d_j,2^{k+1})} = \prod_{j=1}^{r} \frac{(2d_j + 3)(2^{k+1})d_j - 2d_j(2^{k+2} + 5) + 2^{k+2} + 1}{(2^{d_j} - 1)^2}.$$  

Especially, the number of negacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ of length $2n$ is equal to \(\prod_{j=1}^{r} (4^{d_j} + 5 \cdot 2^{d_j} + 9)\).

Using the notations of Theorem 3.5, $C = \bigoplus_{j=1}^{r} \theta_j(x)C_j$ is called the canonical form decomposition of the negacyclic code $C$ over $\mathbb{Z}_4 + u\mathbb{Z}_4$.

As an application of Theorem 3.5, we list the number of negacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ of length $2n$ for odd positive integers $3 \leq n \leq 21$ as follows:

| $n$ | The number of negacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ of length $2n$ |
|-----|--------------------------------------------------------------------------|
| 3   | 1035 = $(4^1 + 5 \cdot 2^1 + 9)(4^2 + 5 \cdot 2^2 + 9)$                     |
| 5   | 7935 = $(4^1 + 5 \cdot 2^1 + 9)(4^4 + 5 \cdot 2^4 + 9)$                     |
| 7   | 293687 = $(4^1 + 5 \cdot 2^1 + 9)(4^6 + 5 \cdot 2^6 + 9)^2$                 |
| 9   | 4579875 = $(4^1 + 5 \cdot 2^1 + 9)(4^3 + 5 \cdot 2^3 + 9)(4^6 + 5 \cdot 2^6 + 9)$ |
| 11  | 24235215 = $(4^1 + 5 \cdot 2^1 + 9)(4^{10} + 5 \cdot 2^{10} + 9)$            |
| 13  | 386347215 = $(4^1 + 5 \cdot 2^1 + 9)(4^{12} + 5 \cdot 2^{12} + 9)$           |
| 15  | 42500851875 = $(4^1 + 5 \cdot 2^1 + 9)(4^{14} + 5 \cdot 2^{14} + 9)^4$       |
| 17  | 102708354375 = $(4^1 + 5 \cdot 2^1 + 9)(4^6 + 5 \cdot 2^6 + 9)^2$            |
| 19  | 1580578116695 = $(4^1 + 5 \cdot 2^1 + 9)(4^{18} + 5 \cdot 2^{18} + 9)$       |
| 21  | 258775875646875 = $23 \cdot 45 \cdot (4^3 + 5 \cdot 2^3 + 9)^2(4^6 + 5 \cdot 2^6 + 9)^2$ |

By the following example, we show how to list all distinct negacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ of specific lengths, using Theorem 3.5.

**Example 3.6.** All distinct negacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ of length 14 are given by:

$$C = \theta_1(x)C_1 \oplus \theta_2(x)C_2 \oplus \theta_3(x)C_3 = \sum_{j=1}^{3} \theta_j(x)C_j \pmod{x^{14} + 1}$$

and the number of codewords in $C$ is equal to $|C_1||C_2||C_3|$, where

- $\theta_1(x) = 3 + x^2 + 3x^4 + x^6 + 3x^8 + x^{10} + 3x^{12}$;
- $\theta_2(x) = 1 + x^2 + 3x^4 + 2x^6 + 3x^8 + 2x^{10} + 2x^{12}$;
- $\theta_3(x) = 1 + 2x^2 + 2x^4 + x^6 + 2x^8 + x^{10} + 3x^{12}$;

$\diamond$ $C_1$ is one of the following 23 ideals in $\mathbb{Z}_4 + u\mathbb{Z}_4[\frac{x}{(x^2+1)}] = \mathbb{Z}_4[u][\frac{x}{(x^2+1)}]$.

1. $\text{(1-I)}$ 4 ideals:
- $C_1 = (x-1) \cdot (b_1(x-1) + b_2(x-1)^2) + u$ with $|C_1| = 2^4$, where $b_1, b_2 \in \{0, 1\}$.
2. $\text{(1-II)}$ 5 ideals:
- $C_1 = (x-1)^2 \cdot b_1(x-1) + u(x-1)$ with $|C_1| = 2^3$, where $b_1 \in \{0, 1\}$;
- $C_1 = (x-1)^3 \cdot b_0 + u(x-1)^2$ with $|C_1| = 2^2$, where $b_0 \in \{0, 1\}$;
- $C_1 = (u(x-1))^3$ with $|C_1| = 2$.
3. $\text{(1-III)}$ 5 ideals:
- $C_1 = (x-1)^\lambda$ with $|C_1| = 4^{\lambda-1}$, $0 \leq \lambda \leq 4$.
4. $\text{(1-IV)}$ 5 ideals:
- $C_1 = (u, x-1)$ with $|C_1| = 2^2$. 

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$C_1 = ((x-1)\cdot b_0 + u, (x-1)^2)$ with $|C_1| = 2^6$, where $b_0 \in \{0, 1\}$;  
$C_1 = ((x-1)\cdot b_1(x-1) + u, (x-1)^3)$ with $|C_1| = 2^5$, where $b_1 \in \{0, 1\}$.

(1-V) 4 ideals:  
$C_1 = (u(x-1), (x-1)^2)$ with $|C_1| = 2^5$;  
$C_1 = (u(x-1)^2, (x-1)^3)$ with $|C_1| = 2^3$;  
$C_1 = ((x-1)^2 \cdot b_0 + u(x-1), (x-1)^3)$ with $|C_1| = 2^4$, where $b_0 \in \{0, 1\}$.

$\forall$ Denote $f_2(x) = x^3 + 2x^2 + x + 3$, $f_3(x) = x^3 + 3x^2 + 2x + 3$, and set $T_2 = T_3 = \{t_0 + t_1x + t_2x^2 \mid t_0, t_1, t_2 \in \{0, 1\}\}$.

Let $j = 2, 3$. Then $C_j$ is one of the following 113 ideals in $\frac{\mathbb{Z}_4[x]}{(f_j(x^{-1}))}$:  

(1) 64 codes:  
$C_j = (f_j(x) \cdot (\beta_1 f_j(x) + \beta_2 f_j(x)^2) + u)$ with $|C_j| = 2^{12}$, where $\beta_1, \beta_2 \in T_j$.  

(II) 17 codes:  
$C_j = (f_j(x)^2 \cdot \beta_1 f_j(x) + u f_j(x))$ with $|C_j| = 2^{16}$, where $\beta_1 \in T_j$;  
$C_j = (f_j(x)^3 \cdot \beta_0 + u f_j(x)^2)$ with $|C_j| = 2^{16}$, where $\beta_0 \in T_j$;  
$C_j = (u f_j(x)^3)$ with $|C_j| = 2^{3}$.  

(III) 5 codes:  
$C_j = (f_j(x)^3)$ with $|C_j| = 2^{3}$.  

4. Negacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ of length $2^k$

As an application of Theorem 3.3, we determine all distinct negacyclic codes of length $2^k$ over $\mathbb{Z}_4 + u\mathbb{Z}_4$ ($u^2 = 0$), i.e., all distinct ideals of the ring $\frac{\mathbb{Z}_4 + u\mathbb{Z}_4[x]}{(x^2 + 1)}$.

In this case, we have $n = 1, f_1(x) = x - 1$ with degree $d_1 = 1, r = 1, \theta_1(x) = 1, R_1 = \frac{\mathbb{Z}_4[x]}{(f_1(x^{-1}))} = \frac{\mathbb{Z}_4[x]}{(x^2 + 1)}$ and $T_1 = \{0, 1\} = F_2$ as a subset of $R_1$. Then from these, by Theorem 3.3 and Proposition 3.4, we deduce the following conclusion.

Theorem 4.1. Let $k \geq 1$. Then all distinct negacyclic codes of length $2^k$ over $\mathbb{Z}_4 + u\mathbb{Z}_4$ ($u^2 = 0$) are given by the following five cases:

(I) $2^{2k}$ codes:  
$\bigcirc C = ((x-1)b(x) + u)$ with $|C| = 2^{2k+1}$,  
where $b(x) = \sum_{i=2}^{2k+1} b_i(x-1)^i$ with $b_i \in \{0, 1\}$ for all $i = 2^k - 1, \ldots, 2^{k+1} - 2$.

(II) $\sum_{\lambda=1}^{2^{k+1} - 1} 2^{2k+1 - \lambda - \lambda(2^{k+1} - \lambda)}$ codes:  
$\bigcirc C = (u(x-1)^{2^{k+1} - 1})$ with $|C| = 2$;  
$\bigcirc C = ((x-1)^{2^{k+1} - 1} b(x) + u(x-1)^{2^{k+1} - 1})$ with $|C| = 2^{2k+1 - \lambda}$,  
where $b(x) = \sum_{i=2}^{2^{k+1} - \lambda - 1} b_i(x-1)^i$,  
$b_i \in \{0, 1\}$ for all $i = \left[ \frac{2^{k+1} - \lambda}{2} \right] - 1, \ldots, 2^{k+1} - \lambda - 2$, and $1 \leq \lambda \leq 2^{k+1} - 2$.

(III) $2^{k+1} + 1$ codes:
\( \mathcal{C} = \langle (x-1)^\lambda \rangle \) with \( |\mathcal{C}| = 2^{2(2k+1-\lambda)}, \ 0 \leq \lambda \leq 2k+1. \)

(IV) \( \sum_{t=1}^{2k+1-1} 2t-\lceil \frac{t}{2} \rceil \) codes:

- \( \mathcal{C} = \langle u_i (x-1) \rangle \) with \( |\mathcal{C}| = 2^{2k+2-i}, \quad 0 \leq i \leq 2k+1. \)
- \( \mathcal{C} = \langle (x-1)^\lambda b(x) + u, (x-1)^i \rangle \) with \( |\mathcal{C}| = 2^{2k+2-i}, \)
  \[ \text{where } b(x) = \sum_{i=\lceil \frac{t}{2} \rceil -1}^{t-2} b_i (x-1)^i, \]
  \[ b_i \in \{0,1\} \text{ for all } i = \left\lceil \frac{t}{2} \right\rceil -1, \ldots, t-2, \text{ and } 2 \leq t \leq 2k+1-1. \]

Moreover, the number of negacyclic codes of length \( 2^k \) over \( \mathbb{Z}_4 + u\mathbb{Z}_4 \) is

\[ N_{(2,1,2k+1)} = 10 \cdot 2^{2k} - 2^{k+2} - 9. \]

As in Dougherty and Ling [21], let \( h_m(x) \) be a monic basic irreducible polynomial in \( \mathbb{Z}_4[x] \) of degree \( m \) that divides \( x^{2^m} - 1 \) and set

\[ \text{GR}(4, m) = \frac{\mathbb{Z}_4[x]}{\langle h_m(x) \rangle} = \left\{ \sum_{i=0}^{m-1} a_i x^i \mid a_0, a_1, \ldots, a_{m-1} \in \mathbb{Z}_4 \right\} \]

in which the arithmetic is done modulo \( h_m(x) \). Then \( \text{GR}(4, m) \) is a Galois ring of characteristic 4 and cardinality \( 4^m \) (cf. [26] Theorem 14.1). Define

\[ R_4(z, m) = \frac{\text{GR}(4, m)[z]}{(z^{2^k} - 1)} = \left\{ \sum_{i=0}^{2^k-1} \alpha_i z^i \mid \alpha_0, \alpha_1, \ldots, \alpha_{2^k-1} \in \text{GR}(4, m) \right\} \]

in which the arithmetic is done modulo \( z^{2^k} - 1 \). By Theorem 2.6 in [21], the number of all distinct ideals in the ring \( R_4(z, m) \) is equal to

\[ 5 + (2m)^2 - 1 + (5 \cdot 2^m - 1)(2m) \frac{(2m)^{2^k-1} - 1}{(2m - 1)^2} - 4 \cdot \frac{2^{k-1} - 1}{2m - 1} \]

\[ = \frac{(2m + 3)2^{(2^k+1)m} - 2m(2^{k+1} + 5) + 2^k + 1}{(2m - 1)^2} \]

\[ = N_{(2, m, 2^k)}. \]

Especially, if \( m = 1 \), the number all distinct ideals in \( \frac{\mathbb{Z}_4[z]}{(z^{2^k} - 1)} \) is

\[ N_{(2,1,2^k)} = 10 \cdot 2^{2k-1} - 2^{k+1} - 9. \]

Using Proposition 3.4 and by setting \( m = d_j \), we have the following conclusion:

**Corollary 4.2.** Let \( 1 \leq j \leq r, f_j(x) \) be a monic basic irreducible polynomial in \( \mathbb{Z}_4[x] \) of degree \( d_j \), and let \( \mathcal{R}_j = \frac{\mathbb{Z}_4[x]}{\langle f_j(x) \rangle} \) be defined as in Section 1. Then the number of ideals in the ring \( \mathcal{R}_j + u\mathcal{R}_j \) \( (u^2 = 0) \) is the same as the number of ideals in the ring \( \frac{\text{GR}(4,d_j)[z]}{(z^{2^k+1})} \), where \( \text{GR}(4,d_j) = \frac{\mathbb{Z}_4[z]}{(f_j(x))} \).
Especially, let $f_j(x) = x - 1$. Then the number of ideals in the ring $\frac{(\mathbb{Z}_4 + u\mathbb{Z}_4)[x]}{(x^2^k + 1)}$ is the same as the number of ideals in the ring $\frac{\mathbb{Z}_4[x]}{(x^2^k - 1)}$.

Therefore, the number of negacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ with length $2^k$ is the same as the number of cyclic codes over $\mathbb{Z}_4$ with length $2^{k+1}$.

By the following example, we show how to list all distinct negacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ of length $2^k$ for specific positive integer $k$, using Theorem 4.1.

**Example 4.3.** All distinct 135 negacyclic codes of length $2^2$ over $\mathbb{Z}_4 + u\mathbb{Z}_4$ are given by the following five cases:

(I) $2^4 = 16$ codes:
- $C = \langle (x - 1) \cdot (\sum_{i=0}^{6} b_i(x - 1)^i) + u \rangle$ with $|C| = 2^8$, where $b_i \in \{0, 1\}$ for all $i = 3, 4, 5, 6$.

(II) $2 \cdot 2^3 + 2 \cdot 2^2 + 2 \cdot 2 + 1 = 29$ codes:
- $C = \langle u(x - 1)^7 \rangle$ with $|C| = 2$;
- $C = \langle (x - 1)^{\lambda + 1} \cdot (\sum_{i=0}^{6} \lambda^i - 1) b_i(x - 1)^i + u(x - 1)^{\lambda} \rangle$ with $|C| = 2^{8 - \lambda}$, where $b_i \in \{0, 1\}$ for all $i = \left[\frac{8 - \lambda}{2}\right], 1, \ldots, 6 - \lambda$, and $1 \leq \lambda \leq 6$.

(III) 9 codes:
- $C = \langle (x - 1)^{\lambda} \rangle$ with $|C| = 2^{2(8 - \lambda)}$, where $0 \leq \lambda \leq 8$.

(IV) $1 + 2 \cdot 2 + 2 \cdot 2^2 + 2 \cdot 2^3 = 29$ codes:
- $C = \langle u, (x - 1) \rangle$ with $|C| = 2^7$;
- $C = \langle (x - 1) \cdot (\sum_{i=0}^{t - 2} \frac{1}{2}) b_i(x - 1)^i) + u, (x - 1)^{\lambda} \rangle$ with $|C| = 2^{16 - t}$, where $b_i \in \{0, 1\}$ for all $i = \left[\frac{1}{2}\right], 1, \ldots, t - 2$, and $2 \leq t \leq 7$.

(V) $\sum_{\lambda=1}^{k} \sum_{t=1}^{\lambda - 1} 2^{t - \left[\frac{t}{2}\right]} = 20 + 12 + 8 + 4 + 2 + 6 = 52$ codes:
- $C = \langle u(x - 1)^{\lambda}, (x - 1)^{\lambda + 1} \rangle$ with $|C| = 2^{16 - 2\lambda - 1}$, where $1 \leq \lambda \leq 6$;
- $C = \langle (x - 1)^{\lambda + 1} \cdot (\sum_{i=0}^{(t - 2)} \frac{1}{2}) b_i(x - 1)^i) + u(x - 1)^{\lambda}, (x - 1)^{\lambda + t} \rangle$ with $|C| = 2^{16 - 2\lambda - t}$, where $b_i \in \{0, 1\}$ for all $i = \left[\frac{1}{2}\right], 1, \ldots, t - 2$, $2 \leq t \leq 8 - \lambda - 1$ and $1 \leq \lambda \leq 5$.

**Remark 4.4.** By Theorem 4.1, there are $N_{(2,1,2^{t+1})} = 23$ negacyclic codes of length 2 and $N_{(2,1,2^{t+1})} = 135$ negacyclic codes of length $2^2$ over $R = \mathbb{Z}_4 + u\mathbb{Z}_4$ ($u^2 = 0$).

However, in [4, Theorem 12], the authors claimed that: *The number $\mathcal{N}(2^k)$ of negacyclic codes of length $2^k$ over $R$ is*

$$\mathcal{N}(2^k) = 11 \cdot 2^{2^k} + 2^{2^k - 1} (5 \cdot 2^k - 12) - ((2^k)^2 + 5 \cdot 2^k + 4).$$

Using this formula, we have $\mathcal{N}(2) = 44 - 2 - 18 = 24$ when $k = 1$; and $\mathcal{N}(2^2) = 176 + 16 - 40 = 152$ if $k = 2$. Hence the formula (3) is incorrect. See [16] for details.

5. **Negacyclic codes of odd length over $\mathbb{Z}_4 + u\mathbb{Z}_4$**

In previous sections, we always suppose $k \geq 1$. In this section, let $k = 0$. We give a brief discussion on negacyclic codes of odd length $n$ over the ring $R = \mathbb{Z}_4 + u\mathbb{Z}_4$.

As in [29], we define $\phi : R^n \to \mathbb{Z}_4^{2^n}$ by:

$$\phi(\xi) = (b_0, b_1, \ldots, b_{n-1}, a_0 + b_0, a_1 + b_1, \ldots, a_{n-1} + b_{n-1})$$

for any $\xi = (a_0 + ub_0, a_1 + ub_1, \ldots, a_{n-1} + ub_{n-1}) \in R^n$ where $a_i, b_i \in \mathbb{Z}_4$ for all $i = 0, 1, \ldots, n - 1$. Then we define the Lee weight $w_L$ on $R$ by letting

$$w_L(a + ub) = w_L(b, a + b),$$
where $a, b \in \mathbb{Z}_4$ and $w_L(b, a+b)$ describes the usual Lee weight on $\mathbb{Z}_4^2$. Furthermore, the Lee distance is defined accordingly. Note that with this definition of the Lee weight and the Gray map we know the following conclusion.

**Proposition 5.1.** ([29] Theorem 2.3) The map $\phi : R^n \to \mathbb{Z}_4^{2n}$ is a distance preserving linear isometry. Thus, if $C$ is a linear code over $R$ of length $n$, then $\phi(C)$ is a linear code over $\mathbb{Z}_4$ of length $2n$ and the two codes have the same Lee weight enumerators.

In the following, let $n$ be odd. In the ring $\mathbb{Z}_4[x] \subset R[x]$, we have $(-x)^n - 1 = -(x^n + 1)$. Hence the map $\varphi : \frac{R[x]}{(x^n - 1)} \to \frac{R[x]}{(x^n + 1)}$ defined by

$$\varphi(\alpha(x)) = \alpha(-x) = \sum_{i=0}^{n-1} (-1)^i \alpha_i x^i \quad (\forall \alpha(x) = \sum_{i=0}^{n-1} \alpha_i x^i \text{ where } \alpha_i \in R \text{ for all } i)$$

is a ring isomorphism. Therefore, $C$ is a negacyclic code over $R$ of length $n$, i.e. $C$ is an ideal of the ring $\frac{R[x]}{(x^n - 1)}$, if and only if there is a unique cyclic code $D$ over $R$ of length $n$, i.e. $D$ is an ideal of the ring $\frac{R[x]}{(x^n + 1)}$, such that

$$C = \varphi(D) = \left\{ \sum_{i=0}^{n-1} (-1)^i \alpha_i x^i \mid \sum_{i=0}^{n-1} \alpha_i x^i \in D \text{ where } \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in R \right\}.$$

From this, one can easily verify the following conclusion.

**Proposition 5.2.** The isomorphism $\varphi$ is a distance preserving linear isometry on $R^n$. Hence the two codes $\varphi(D)$ and $D$ have the same Lee weight (and Hamming weight) enumerators, for every cyclic code $D$ of length $n$ over $R$.

Therefore, it is sufficient to determine all cyclic codes of length $n$ over $R = \mathbb{Z}_4 + u\mathbb{Z}_4$ ($u^2 = 0$), in order to determine all negacyclic codes of length $n$ over $R$ (corresponding to the case of $k = 0$ in previous sections). There were some literatures on this kind of cyclic codes. Please refer to [3], [23] and [29], for examples. In these papers, the following results were given:

† “There are $7^m$ cyclic codes of length $n$ over $R^n$ (Corollary 11 in [3] and Corollary 4.1 in [23]). Here $R = \mathbb{Z}_4 + u\mathbb{Z}_4$ ($u^2 = 0$), $x^n - 1 = g_1g_2\ldots g_k$ and $g_1, \ldots, g_k$ are basic irreducible pairwise coprime polynomials in $\mathbb{Z}_4[x] \subset R[x]$. In fact, the number $7^m$ of cyclic codes over $R$ with length $n$ is wrong (see Remark 5.4 of this paper).

⊥ Let $C$ be a cyclic code of odd length $n$ over $R$. Then

$$C = \langle f_1(x) + 2f_2(x) + uf_3(x), uf_4(x) + 2uf_4(x) \rangle,$$

where $f_2(x) \mid f_1(x) \mid x^n - 1$ and $f_4(x) \mid f_3(x) \mid x^n - 1$ in $R[x]$ (Theorem 4.4 in [23] and Theorem 4 in [29]).

Now, we provided a new way different from the methods used in [3], [23] and [29] to study cyclic codes of odd length $n$ over $R = \mathbb{Z}_4 + u\mathbb{Z}_4$ ($u^2 = 0$). To do this, we adopt the previous notations in Sections 1 and 2, for any integer $1 \leq j \leq r$ by Equations (1) and (2):

- Let $e_j(x) \in \frac{R[x]}{(x^n - 1)} \subset \frac{R[x]}{(x^n - 1)}$ be defined by
  $$e_j(x) = a_j(x)F_j(x) + 1 - b_j(x)f_j(x) \pmod{x^n - 1}.$$
Let $K_j = \{ \frac{Z_n}{f_j(x)} \} = \{ \sum_{i=0}^{d_j-1} a_i x^i \mid a_0, a_1, \ldots, a_{d_j-1} \in Z_4 \}$ in which the arithmetic is done modulo $f_j(x)$. Then $K_j$ is a Galois ring of $4^{d_j}$ elements.

- Let $F_j = \{ \sum_{i=0}^{d_j-1} b_i x^i \mid b_0, b_1, \ldots, b_{d_j-1} \in F_2 \} \subset K_j$.

- Define a map $\Psi : \frac{Z_n}{f(x)} + u \frac{Z_n}{f(x)} \rightarrow R[x]/(x^n - 1)$ by: for any $\xi = \sum_{i=0}^{n-1} a_i x^i + u \sum_{i=0}^{n-1} a_{i+1} x^i$ with $a_i, a_{i+1} \in Z_4$ for all $i$, we set

$$\Psi(\xi) = \sum_{i=0}^{n-1} a_i x^i,$$

where $a_i = a_{i+1} + ua_{i+1} \in R$ for all $i = 0, 1, \ldots, n - 1$.

Then $\Psi$ is a ring isomorphism from $\frac{Z_n}{f(x)} + u \frac{Z_n}{f(x)}$ onto $R[x]/(x^n - 1)$.

In the following, we think of $\frac{Z_n}{f(x)} + u \frac{Z_n}{f(x)} (u^2 = 0)$ and $R[x]/(x^n - 1)$ as the same under the isomorphism $\Psi$. Then as a direct corollary of Theorems 3.4 and 3.7 in [7], we obtain the following conclusion:

**Theorem 5.3.** All distinct cyclic code over $R$ of odd length $n$ are given by

$$C = \bigoplus_{j=1}^{r} e_j(x)C_j \pmod{x^n - 1},$$

where $C_j$ is an ideal of the ring $K_j + uK_j$ listed by the following table for all $j = 1, \ldots, r$:

| case | number of ideals | $C_j$ | $|C_j|$ |
|------|-----------------|-------|--------|
| I.   | 3               | $\langle u^i \rangle$ ($i = 0, 1, 2$) | $2^{d_j(2-i)}$ |
| II.  | 2               | $\langle 2u^s \rangle$ ($s = 0, 1$) | $2^{d_j(2-s)}$ |
| III. | $2^{d_j} - 1$   | $\langle u + 2h(x) \rangle$ ($h(x) \in F_j \setminus \{0\}$) | $2^{2d_j}$ |
| V.   | 1               | $\langle u, 2 \rangle$ | $2^{2d_j}$ |

Moreover, the number of codewords in $C$ is equal to $|C| = \prod_{j=1}^{r} |C_j|$. Hence the number of all cyclic codes over $R$ of odd length $n$ is $\prod_{j=1}^{r} (2^{d_j} + 5)$.

**Remark 5.4.** (†) By Theorem 5.3, the number of all cyclic codes of length 7 over $Z_4 + uZ_4$ is $(2^3 + 1) \cdot (2^3 + 5)^2 = 1183$. But in [3] and [23], the authors claimed that the number of all cyclic codes of length 7 over $Z_4 + uZ_4$ is equal to $7^3 = 343$. Therefore, the formula for cyclic codes given by [3] and [23] is wrong.

Moreover, we obtained 39 formally self-dual quasi-cyclic codes of length 14 and index 2 over $Z_4$ by the distance preserving isometry from $(Z_4 + uZ_4)^7$ to $Z_4^{14}$ (see Section 6 of [7]).

(†) Using the ring isomorphism $\varphi : \frac{R[x]}{(x^5 - 1)} \rightarrow \frac{R[x]}{(x^5 + 1)}$, by Theorem 5.3, one can list all distinct $\prod_{j=1}^{r} (2^{d_j} + 5)$ negacyclic codes of length $n$ over $Z_4 + uZ_4$ precisely and easily, for any specific odd positive integer $n$.

6. Conclusion

In this paper, we give an explicit expression for every negacyclic code over $Z_4 + uZ_4$ $(u^2 = 0)$ of arbitrary even length $2^kn$, and provide an exact mass formula to enumerate the number of all these codes.

A natural problem is to represent all distinct self-dual negacyclic codes over $Z_4 + uZ_4$ of length $2^kn$ precisely and provide a clear formula to enumerate the
number of all these self-dual codes, for any positive integer $k$ and odd positive integer $n$.

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