Flux creep and flux jumping

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We consider the flux jump instability of the Bean’s critical state arising in the flux creep regime in type-II superconductors. We find the flux jump field, $B_j$, that determines the superconducting state stability criterion. We calculate the dependence of $B_j$ on the external magnetic field ramp rate, $\dot{B}_e$. We demonstrate that under the conditions typical for most of the magnetization experiments the slope of the current-voltage curve in the flux creep regime determines the stability of the Bean’s critical state, i.e., the value of $B_j$. We show that a flux jump can be preceded by the magneto-thermal oscillations and find the frequency of these oscillations as a function of $\dot{B}_e$.

I. INTRODUCTION

The Bean’s critical state model successfully describes the irreversible magnetization in type-II superconductors by introducing the critical current density $j_c(T, B)$, where $T$ is the temperature and $B$ is the magnetic field. In the framework of the Bean’s model the value of the slope of the stationary magnetic field profile is less or equal to $\mu_0 j_c(T, B)$. This nonuniform flux distribution does not correspond to an equilibrium state and under certain conditions flux jumps arise in the critical state. The flux jumping process results in a flux redistribution towards the equilibrium state and is accompanied by a strong heating of the superconductor.

Flux jumping has been numerously studied in conventional and high-temperature superconductors (see the review papers, references therein, and the recent experimental studies). In the general case two types of flux jumps can be considered, namely, the global and the local flux jumps. A global flux jump involves vortices into motion in the entire volume of the sample. A local flux jump happens in a small fraction of the sample volume. Depending on the initial perturbation and the driving parameters there are two qualitatively different types of global flux jumps, namely, complete and partial flux jumps. The first turns the superconductor to the normal state. The second self-terminates when the temperature is still less than the critical temperature.

We illustrate a global flux jump origination treating a superconducting slab with the thickness $2d$ subjected to an external magnetic field $\mathbf{B}_e$ parallel to the sample surface ($yz$ plane). In the framework of the Bean’s critical state model the space distribution of the flux is given by the equation

$$\frac{dB}{dx} = \pm \mu_0 j_c,$$

where the $\pm$ stays for $x > 0$ and $x < 0$ correspondingly. We show the dependence $B(x)$ in Fig. for the case,
when the critical current density depends only on the temperature, i.e., $j_c = j_c(T)$.

Let us now suppose that the temperature of the sample $T_0$ is increased by a small perturbation $\delta T_0$ arising due to a certain initial heat release $\delta Q_0$. The critical current density $j_c(T)$ is a decreasing function of temperature. Thus, the density of the superconducting current screening of the external magnetic field at $T = T_0 + \delta T_0$ is less than at $T = T_0$. This reduction of the screening current leads to a rise of the magnetic flux inside the superconductor as it is shown in Fig. 1. The motion of the magnetic flux into the sample, which occurs as a result of the temperature perturbation $\delta T_0$, induces an electric field perturbation $\delta E_0$. The arise of $\delta E_0$ is accompanied by an additional heat release $\delta Q_1$, an additional temperature rise $\delta T_1$, and, therefore, an additional reduction of the superconducting screening current density $j_c$. Under certain conditions it results in an avalanche-type increase of the temperature and the magnetic flux in the superconductor, i.e., in a global flux jump.

The relative effect of the flux and temperature redistribution dynamics on the flux jumping process depends on the ratio, $\tau$, of the flux, $t_m$, and thermal, $t_\kappa$, diffusion time constants. The value of the dimensionless parameter $\tau$ is determined by the corresponding diffusion coefficients and is equal to

$$\tau = \frac{\mu_0 \lambda \sigma}{C},$$

where $\lambda$ is the heat conductivity, $\sigma$ is the conductivity, and $C$ is the heat capacity.

For $\tau \ll 1$ ($t_m \ll t_\kappa$), rapid propagation of the flux is accompanied by an adiabatic heating of the superconductor, i.e., there is not enough time to redistribute and remove the heat released due to the flux motion. For $\tau \gg 1$ ($t_\kappa \ll t_m$), the space distribution of flux remains fixed during the stage of rapid heating. These adiabatic ($\tau \ll 1$) and dynamic ($\tau \gg 1$) approximations are the basis of the approach to the flux jumping problem and a flux jump scenario significantly depends on the relation between the values of the heat conductivity $\kappa$, heat capacity $C$, and conductivity $\sigma$, that is determined by the slope of the $j$-$E$ curve.

Let us now estimate the electric field value typical for the magnetization experiments. In this case the external magnetic field ramp rate $\dot{B}_e$ is usually from the interval $\dot{B}_e < 1 \text{ Ts}^{-1}$. The background electric field, $E_b$, induced by the magnetic field variation is of the order of $E_b \sim \dot{B}_e (d - l)$, where $d - l$ is the width of the area occupied by the critical state (see, for example, Fig. 1). We estimate $E_b$ as $E_b < 10^{-6} \text{ Vcm}^{-1}$ using the value $d - l < 10^{-4} \text{ m}$ which is typical for the stability domain of the Bean’s critical state. This electric field interval corresponds to the flux creep regime. Therefore, for the magnetization experiments the background electric field $E_b$ is from the flux creep regime, where the relation between the current density, $j$, and the electric field, $E$, is strongly nonlinear. As a result, the value of $\sigma$, i.e., the slope of the $j$-$E$
curve, is strongly electric field dependent and the flux jumping takes place on a background of a resistive state with a conductivity that strongly depends on the external magnetic field ramp rate $\dot{B}_e$.

In order to calculate the conductivity in the flux creep regime we use the dependence of $j$ on $E$ in the form

$$j = j_c + j_1 \ln \left( \frac{E}{E_0} \right), \quad (3)$$

where $E_0$ is the voltage criterion at which the critical current density $j_c$ is defined, $j_1$ determines the slope of the $j$-$E$ curve and $j_1 \ll j_c$. Note, that the actual choice of $E_0$ is not very essential. Indeed, by taking for the voltage criterion a certain value $\tilde{E}_0$ instead of $E_0$ we change the critical current density from $j_c$ to $\tilde{j}_c = j_c - j_1 \ln(\tilde{E}_0/E_0)$. The difference between $\tilde{j}_c$ and $j_c$ is small as $\ln(\tilde{E}_0/E_0) \sim 1$ and $j_1 \ll j_c$. It is common to define the critical current value as the current density at $E_0 = 10^{-6}$ Vcm$^{-1}$.

Let us also note, that a power law

$$j = j_c \left( \frac{E}{E_0} \right)^{1/n} \quad (4)$$

with $n \gg 1$ is often used to describe the $j$-$E$ curve in the flux creep regime. Expanding the dependence given by Eq. (4) in series in $1/n \ll 1$ and keeping the first two terms we find that if we take $n = j_c/j_1$, then Eqs. (3) and (4) coincide with the accuracy of $1/n^2 \ll 1$.

The relation given by Eq. (3) was first derived in the framework of the Anderson-Kim model considering the thermally activated uncorrelated hopping of bundles of vortices. The vortex-glass and collective creep models result in more sophisticated dependencies of $j$ on $E$. However, these $j$-$E$ curves coincide with the one given by Eq. (3) if $j - j_c \ll j_c$. The recently developed self-organized criticality approach to the critical state also results in Eq. (3) if $j - j_c \ll j_c$. The logarithmic dependence of the current density $j$ on the electric field $E$ in the interval $j - j_c \ll j_c$ is in a good agreement with numerous experimental data. In this paper we use the $j$-$E$ curve given by Eq. (3) to calculate the conductivity $\sigma$ assuming that $j_1/j_c \ll 1$.

It follows from Eq. (3) that for the flux creep regime the conductivity $\sigma$ is given by the formula

$$\sigma = \sigma(E) = \frac{dj}{dE} = \frac{j_1}{E}. \quad (5)$$

We estimate the value of $\sigma$ as $\sigma > 10^{10}$ $\Omega^{-1}$cm$^{-1}$ using the typical data $j_1 > 10^3$ Acm$^{-2}$ and $E < 10^{-7}$ Vcm$^{-1}$. It follows from this estimation that the conductivity $\sigma$ determining the flux jumps dynamics for the magnetization experiments is very high. As a consequence the dimensionless ratio $\tau$ is also very high. Thus, the scenario of a flux jump for the magnetization experiments corresponds to the limiting case when $\tau \gg 1$ and the rapid heating stage takes place on the background of a “frozen-in” magnetic flux.
The electric field dependent conductivity $\sigma(E)$ significantly affects the flux jumping process. In particular, it results in the flux jump field $B_j$ dependence on the magnetic field ramp rate $\dot{B}_e$. This dependence is known from experiments \cite{2,3,5} and was never considered theoretically as a consequence of the logarithmic $j-E$ curve characterizing the flux creep regime in superconductors with high values of the critical current density $j_c$.

Under certain conditions a flux jump is preceded by a series of magneto-thermal oscillations \cite{2}. These oscillations have been observed for the low-temperature \cite{17,18} as well as for the high-temperature superconductors \cite{5}. Theoretically magneto-thermal oscillations were considered for a flux jump developing in the flux flow regime \cite{19}. In this case the $j-E$ curve is linear and the value of the conductivity $\sigma$ is electric field independent. The high and electric field dependent conductivity $\sigma(E)$ significantly affects the flux dynamics and therefore the magneto-thermal oscillations. In particular, it results in the dependence of the frequency of the magneto-thermal oscillations on the magnetic field ramp rate $\dot{B}_e$. The effect of the logarithmic $j-E$ curve on the magneto-thermal oscillations was never treated theoretically.

In this paper we consider the flux jump instability of the Bean’s critical state arising in the flux creep regime on the background of a nonuniform electric field determining the conductivity of the type-II superconductor in the flux creep regime. We find the flux jump field, $B_j$, that determines the critical state stability criterion and the dependence of $B_j$ on the external magnetic field, $B_e$, and the external magnetic field ramp rate, $\dot{B}_e$. We show that a flux jump can be preceded by the magneto-thermal oscillations and find the frequency of these oscillations as a function of $\dot{B}_e$.

The paper is organized in the following way. In Sec. II, we consider the critical state stability qualitatively and obtain the stability criterion. In Sec. III, we derive the equations determining the development of the small temperature and electric field perturbations and calculate the frequency of the magneto-thermal oscillations. In Sec. IV, we summarize the overall conclusions.

**II. QUALITATIVE CONSIDERATION**

In this section we consider the critical state stability qualitatively assuming that the thermomagnetic instability develops much faster than the magnetic flux diffusion process. In other words, we treat the case when the heating accompanying the thermomagnetic instability takes place on the background of a “frozen-in” magnetic flux. In Sec. III we derive the exact criterion of applicability of the following qualitative reasoning.

Let us consider a superconducting slab with the thickness $2d$ subjected to a magnetic field parallel to the sample surface (see Fig. 1) and suppose that the temperature of the sample $T_0$ is increased by a small perturbation $\delta T$. 

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\cite{2,3,5}
To keep the critical state stable, i.e., to keep the screening current at the same level, an electric field perturbation $\delta E$ arises. The additional electric field $\delta E$ causes an additional heat release $\delta Q \propto \delta E$, which is the “price” for keeping the total screening current density at the same level, i.e., the “price” for the “frozen-in” magnetic flux.

The critical state is stable if the additional heat release $\delta Q$ can be removed to the coolant by the additional heat flux $\delta W \propto \delta T$ resulting from the temperature perturbation $\delta T$. Thus, the critical state stability criterion has the form

$$\delta W > \delta Q.$$  \hspace{1cm} (6)

The additional heat release per unit length, $\delta Q$, is given by the integral of $j \delta E$ over the width of the superconducting slab

$$\delta Q = \int_{-d}^{d} j \delta E \, dx.$$  \hspace{1cm} (7)

The additional heat flux, $\delta W$, is determined by the temperature perturbation $\delta T$ at the sample surface, i.e.,

$$\delta W = h \delta T \bigg|_{p},$$  \hspace{1cm} (8)

where $h$ is the heat transfer coefficient to the coolant with the temperature $T_{0}$ and $P$ stays for the sample surface.

Using Eqs. (6), (7) and (8) we find the critical state stability criterion, namely, the inequality

$$\int_{-d}^{d} j \delta E \, dx < 2h \delta T \bigg|_{p}.$$  \hspace{1cm} (9)

To derive the explicit form of this stability criterion we have to find the relation between $\delta T$ and $\delta E$. To do it, we calculate the decrease of the current density $\delta j_-$ resulting from the temperature perturbation $\delta T$ and the increase of the current density $\delta j_+$ resulting from the electric field perturbation $\delta E$. If the critical state is stable then the total screening current density stays constant. As a result, the relation between $\delta E$ and $\delta T$ is given by the equation

$$\delta j = \delta j_- + \delta j_+ = 0.$$  \hspace{1cm} (10)

In the critical state, $j \approx j_c$, thus, the decrease of $j$ due to the temperature perturbation $\delta T$ is equal to

$$\delta j_- = -\left[ \frac{\partial j_c}{\partial T} \right] \delta T$$  \hspace{1cm} (11)

(note that $\partial j_c/\partial T < 0$).

The increase of the current density due to the electric field perturbation $\delta E$ can be written as

$$\delta j_+ = \frac{dj}{dE} \delta E = \sigma \delta E.$$  \hspace{1cm} (12)
Note, that the conductivity \( \sigma \) is the differential conductivity, \( i.e., \) it is determined by the slope of the \( j-E \) curve.

Combining Eqs. (5) and (12), we find the relation between \( \delta j_+ \) and \( \delta E \) in the form

\[
\delta j_+ = \frac{j_1}{E_b} \delta E = \frac{j_c}{nE_b} \delta E, \tag{13}
\]

where \( n = j_c/j_1 \gg 1 \).

It follows from Eqs. (10), (11) and (13) that

\[
\delta E = \frac{1}{\sigma} \frac{\partial j_c}{\partial T} \delta T = \frac{nE_b}{j_c} \frac{\partial j_c}{\partial T} \delta T. \tag{14}
\]

Equations (5) and (14) allow to understand the effect of the background electric field \( E_b \) on the critical state stability. It follows from Eq. (5) that a low electric field \( E_b \) results in a high differential conductivity \( \sigma \propto 1/E_b \). In its turn a high conductivity \( \sigma \) leads to a low electric field perturbation (indeed, it follows from Eq. (14) that \( \delta E \propto 1/\sigma \propto E_b \)). The smaller is \( \delta E \), the less “costly” it is to remove the additional heat release. As a result the lower is the background electric field \( E_b \) the more stable is the superconducting state.

Substituting Eq. (14) into Eq. (9) we find the critical state stability criterion in the form

\[
\int_{-d}^{d} nE_b \left| \frac{\partial j_c}{\partial T} \right| \delta T \, dx < 2h \delta T, \tag{15}
\]

We have to treat the temperature perturbation \( \delta T \) in more detail to derive the final form of Eq. (15). The variation of the function \( \delta T(x) \) on the interval \(-d \leq x \leq d\) depends on the value of the Biot number

\[
Bi = \frac{dh}{\kappa}, \tag{16}
\]

where \( \kappa \) is the heat conductivity of the superconductor. Let us assume that the value of the heat transfer coefficient \( h \) is relatively low. As a result, \( Bi \ll 1 \) and the temperature perturbation \( \delta T(x) \) is almost uniform over the width of the superconducting slab. It means that \( \delta T \) cancels in both sides of Eq. (15) and the Bean’s critical state stability criterion takes the following final form

\[
\mathcal{J} = \frac{n}{2h} \int_{-d}^{d} E_b \left| \frac{\partial j_c}{\partial T} \right| \, dx < 1. \tag{17}
\]

Let us note, that this criterion was first derived in order to calculate the maximum value of a superconducting current under conditions typical for the critical current measurements, \( i.e., \) for a superconducting wire carrying a current that is increased with a given ramp rate.

In addition, we assume for simplicity that the value of \( n \) is temperature and magnetic field independent if \( T < T_c \) and \( B < B_{c2} \), where \( T_c \) is the critical temperature.
and $B_{c2}$ is the upper critical field. This assumption is in a good agreement with numerous experimental data as well as with the self-organized criticality approach to the Bean’s critical state.

Using Eq. (17) we can rewrite the criterion given by Eq. (17) in the following form which is convenient for the further analysis

$$J = \frac{n}{h} \int_0^d E_b \left| \frac{\partial j_c}{\partial T} \right| dx = \frac{n}{\mu_0 h} \int_{B^*}^{B_e} E_b \left| \frac{\partial j_c}{\partial T} \right| dB < 1, \quad (18)$$

where $B^* = B(0)$ is the magnetic field in the middle plane of the superconducting slab.

The background electric field $E_b$ is induced by the varying external magnetic field $B_e(t)$ and thus the spatial distribution of $E_b$ is given by the Maxwell equation

$$\frac{dE_b}{dx} = \frac{dB}{dt}. \quad (19)$$

Combining Eqs. (1) and (19) and taking into account that $j \approx j_c$ we find that

$$\frac{dE_b}{dB} = \pm \frac{\dot{B}}{\mu_0 j_c(B)}, \quad (20)$$

where the $\pm$ stays for $x > 0$ and $x < 0$ correspondingly. At the same time Eq. (19) results in the relation

$$\frac{\dot{B}_e}{j_c(B_e)} = \frac{\dot{B}}{j_c(B)}. \quad (21)$$

It follows from Eqs. (20) and (21) that the the background electric field $E_b$ dependence on $B$ is given by the formula

$$E_b = \pm \frac{\dot{B}_e(B - B^*)}{\mu_0 j_c(B_e)}, \quad (22)$$

where the $\pm$ stays for $x > 0$ and $x < 0$ correspondingly.

Let us now apply the criterion given by Eq. (18) to calculate the flux jump field $B_j$ assuming that initially there is no flux inside the superconducting slab, i.e., we calculate now the magnetic field of the first flux jump. Using Eqs. (18) and (22) we find the stability criterion in the form

$$J = \frac{n \dot{B}_e}{\mu_0^2 h j_c(B_e)} \int_{B^*}^{B_e} \frac{B - B^*}{j_c(B)} \left| \frac{\partial j_c}{\partial T} \right| dB < 1. \quad (23)$$

The value of the magnetic field $B^*$ is given by the following system of equations

$$B^* = 0, \quad \text{if} \quad B_e < B_p, \quad (24)$$

$$\int_{B^*}^{B_e} \frac{dB}{j_c(B)} = \mu_0 d, \quad \text{if} \quad B_e > B_p, \quad (25)$$
where the value of the magnetic flux penetration field, $B_p$, is determined by

$$\int_0^{B_p} \frac{dB}{j_c(B)} = \mu_0 d. \quad (26)$$

It follows from Eqs. (23) and (25) that $J$ is an increasing function of the external magnetic field $B_e$ if $B_e < B_p$ and $J$ is a decreasing function of $B_e$ if $B_e > B_p$. In other words, if for a given value of $B_e$ the superconducting state is stable in the region $0 < B_e < B_p$ then it is stable for any magnetic field. Thus, if a flux jump occurs it occurs only if $B_e < B_p$. Therefore, we consider now a superconducting slab that is wide enough meaning that $B_j < B_p$.

We have the criterion $J(B_j) = 1$ to find the flux jump field $B_j$ in the case when $B_e < B_p$. Thus, it follows from Eq. (23) that the dependence $B_j(B_e)$ is given by the equation

$$J(B_j) = \frac{nB_e}{\mu_0 hj_c(B_j)} \int_0^{B_j} \frac{B}{j_c(B)} \left| \frac{\partial j_c}{\partial T} \right| dB = 1. \quad (27)$$

Let us approximate the value of $|\partial j_c/\partial T|$ as

$$\left| \frac{\partial j_c}{\partial T} \right| \approx \frac{j_c(B)}{T_c(B) - T_0}. \quad (28)$$

Using Eq. (28) we rewrite Eq. (27) in the following form

$$\frac{nB_e}{\mu_0 hj_c(B_j)} \int_0^{B_j} \frac{B}{T_c(B) - T_0} dB = 1. \quad (29)$$

We treat now the case when $T_0 \ll T_c(B_j)$ or in other words $B_j \ll B_{c2}(T_0)$. It means that $T_c(B) \approx T_c$, where $T_c$ is the critical temperature at zero magnetic field. It follows finally from Eq. (29) that the Bean’s critical state stability criterion determining the the dependence $B_j(B_e)$ is given by the equation

$$\frac{B_j^2}{j_c(B_j)} = \frac{2\mu_0^2 h(T_c - T_0)}{nB_e}. \quad (30)$$

Let us now consider the particular case of the Bean’s critical state model, namely, let us assume that the critical current density is magnetic field independent, i.e.,

$$j_c = j_c(T). \quad (31)$$

Using Eq. (31) we find the following formula for the first flux jump field $B_j$

$$B_j = \sqrt{\frac{2\mu_0^2 j_c(T_0) h(T_c - T_0)}{nB_e}} \propto \frac{1}{B_c^{1/2}}. \quad (32)$$
It follows from Eq. (32) that the value of $B_j$ is inversely proportional to the square root of the magnetic field ramp rate $\dot{B}_e$ and is, therefore, decreasing with the increase of $\dot{B}_e$. The physics of this effect is related to the decrease of the conductivity $\sigma(E)$ in the flux creep regime with the increase of the background electric field $E_b$, i.e., with the increase of $\dot{B}_e$.

We derive the expression for $B_j$ assuming that the rapid heating stage of a flux jump takes place on the background of a “frozen-in” magnetic flux. This approach is valid if $\tau \gg 1$ which is the same as $\dot{B}_e \ll 1$

\begin{equation}
\dot{B}_e \ll \frac{1}{n} \frac{B_p}{t_\kappa},
\end{equation}

where we introduce the typical thermal diffusion time constant $t_\kappa$ as

\begin{equation}
t_\kappa = \frac{d^2 C}{\kappa}.
\end{equation}

Let us now compare the values of $B_j$ and $B_a$, where $B_a$ determines the flux jump field for the adiabatic stability criterion\cite{21}. This well known criterion is based on the suggestion that the heating accompanying a flux jump is an adiabatic process, i.e., it is assumed that there is no heat redistribution during a flux jump. Therefore, the adiabatic stability criterion corresponds to the limiting case of $\tau \ll 1$ that is not the case typical for the magnetization experiments.

The value of $B_a$ is given by the formula\cite{22}

\begin{equation}
B_a = \frac{\pi}{2} \sqrt{\mu_0 C(T_0)(T_c - T_0)}.
\end{equation}

It follows from the comparison of Eqs. (32) and (35) that $B_a < B_j$ if

\begin{equation}
\dot{B}_e < \frac{8}{\pi^2} \frac{\mu_0 j_1 h}{C} = \frac{8}{\pi^2} \frac{\mu_0}{n} \frac{B_p}{t_\kappa}.
\end{equation}

Note, that the inequality given by Eq. (36) is stronger than the one given by Eq. (33) as we assume that $Bi \ll 1$.

The critical current density is decreasing with the increase of the magnetic field. Let us now consider the effect of this dependence on the critical state stability assuming that the value of $B_j$ is relatively high. To do it we use the Kim-Anderson model\cite{10} to describe the function $j_c(B)$. In the case of a high magnetic field this model results in the relation

\begin{equation}
j_c = \frac{\alpha(T)}{B}.
\end{equation}

Using Eqs. (37) and (30) we find for the flux jump field $B_j$ the formula

\begin{equation}
B_j = \left(\frac{2 \mu_0^2 \alpha(T_0) h(T_c - T_0)}{n B_p} \right)^{1/3} \propto \frac{1}{\dot{B}_e^{1/3}}.
\end{equation}

The comparison of Eqs. (38) and (32) shows that the magnetic field dependence of the critical current density slows down the decrease of $B_j$ with the increase of $\dot{B}_e$. 
III. QUANTITATIVE CONSIDERATION

We treat now the critical state stability in more detail and, in particular, we take into consideration the magneto-thermal oscillations. We consider a superconducting slab with the thickness 2\textit{d} subjected to an external magnetic field parallel to the \textit{z} axis (see Fig. [1]).

We use for calculation the Bean’s critical state model assuming that the critical current density is magnetic field independent, \(j_c = j_c(T)\). We suppose also that \(B_e \leq B_p = \mu_0 j_c d\). The space distribution of the background electric field \(E_b\) is then given by the formulae

\[
E_b(x) = \begin{cases} 
B_e(x-l), & \text{if } l < x < d, \\
0, & \text{if } -l < x < l, \\
B_e(x+l), & \text{if } -d < x < -l,
\end{cases}
\]

(39)

where the magnetic field penetration depth, \(l\), is equal to

\[
l = d - \frac{B_e}{\mu_0 j_c}.
\]

(40)

We consider now the stability of the stationary electric field and temperature distributions corresponding to the Bean’s critical state against small electric field and temperature perturbations. To do it let us present the electric field \(E(x,t)\) and the temperature \(T(x,t)\) in the following form

\[
E(x,t) = E_b(x) + \delta E(x) = E_b(x) + \epsilon(x) \exp(\gamma t),
\]

(41)

\[
T(x,t) = \tilde{T}_0 + \delta T(x) = \tilde{T}_0 + \theta(x) \exp(\gamma t),
\]

(42)

where

\[
\epsilon(x) \ll E_b(x), \quad \theta(x) \ll T_0, \quad (\text{Re } \gamma)^{-1}
\]

is the characteristic time of the magneto-thermal instability increase, and

\[
\text{Im } \gamma
\]

is the frequency of the magneto-thermal oscillations.

The stationary temperature \(\tilde{T}_0\) is different from \(T_0\) due to the Joule heating power \(j_c E_b\) produced by the background electric field \(E_b\). Let us note, that the difference between \(\tilde{T}_0\) and \(T_0\) is small, \(\tilde{T}_0 - T_0 \ll T_c - T_0\). Indeed, using Eq. (30) we estimate the value of \(\tilde{T}_0 - T_0\) as

\[
\tilde{T}_0 - T_0 \approx (T_c - T_0)/n \ll T_c - T_0.
\]

The small temperature \(\delta T(x) = \theta(x) \exp(\gamma t)\) and electric field \(\delta E = \epsilon(x) \exp(\gamma t)\) perturbations decay if \(\text{Re } \gamma < 0\). Therefore, the stability margin of the Bean’s critical state is determined by the condition \(\text{Re } \gamma < 0\).

Substituting Eqs. (41) and (42) into the Maxwell equation and the heat diffusion equation

\[
\kappa \frac{\partial^2 T}{\partial x^2} + j_c E = C \frac{\partial T}{\partial t},
\]

(44)

\[
\frac{\partial^2 E}{\partial x^2} = \mu_0 \frac{\partial j}{\partial t}
\]

(45)
we find the system of equations describing \( \epsilon(x) \) and \( \theta(x) \). It takes the form

\[
\theta'' - \frac{\gamma C}{\kappa} \theta = -\frac{j e}{\kappa} \epsilon, \tag{46}
\]

\[
\epsilon'' - \frac{\mu_0 \gamma j c}{n E_b(x)} \epsilon = -\frac{\mu_0 \gamma j c}{T_c - T_0} \theta, \tag{47}
\]

where the prime is to denote the derivative over \( x \). Note, that to derive Eq. (47) we use Eq. (39) and the relation

\[
\frac{\partial j}{\partial t} = \frac{\partial j}{\partial E} \frac{\partial E}{\partial t} - \frac{\partial j}{\partial T} \frac{\partial T}{\partial t} = \frac{\gamma j c}{n E_b} \epsilon - \frac{\gamma j c}{T_c - T_0} \theta. \tag{48}
\]

We assume that the superconducting slab is in a thermal contact with a coolant with the temperature \( T_0 \) and we treat here the case when the external magnetic field ramp rate is given, \( i.e., E'(\pm d) = \dot{B}_e \). In addition, the electric field \( E(x) \) is equal to zero in the inner region of the superconducting slab \( (|x| \leq l) \). As a result, the boundary conditions for Eqs. (46) and (47) are given by

\[
\theta'((\pm d)) = \pm \frac{h}{\kappa} \theta(\pm d), \tag{49}
\]

\[
\epsilon(\pm l) = 0, \quad \epsilon'(\pm d) = 0. \tag{50}
\]

Let us present the solution for \( \epsilon(x) \) in the form

\[
\epsilon = \frac{n \theta}{T_c - T_0} E_b(x) + \epsilon_1(x), \tag{51}
\]

where the first term corresponds to the approximation of a “frozen-in” magnetic flux (see Eq. (14)) and the second term describes the deviation from this approximation. It follows from Eqs. (17), (50), and (51) that with the accuracy of \( Bi \ll 1 \) the equation for \( \epsilon_1(x) \) has the form

\[
\epsilon_1'' - \frac{\mu_0 \gamma j c}{n B_c(|x| - l)} \epsilon_1 = 0 \tag{52}
\]

with the boundary conditions

\[
\epsilon_1(\pm l) = 0, \quad \epsilon_1'(\pm d) = -\frac{n B_c \theta}{T_c - T_0}, \tag{53}
\]

We consider here the case of \( \tau \gg 1 \), \( i.e., \) the case when in the first approximation in \( \tau^{-1} \ll 1 \) the magnetic flux is “frozen-in” in the bulk of the superconducting slab. It means that during the rapid heating stage the magnetic flux redistribution takes place only in a thin surface layer with the thickness \( \delta_s \ll d - l \). In other words, the function \( \epsilon_1(x) \) decays inside the superconducting slab and differs from zero only if \( d - |x| \) is less or of the order of the skin depth \( \delta_s \). In the region \( d - |x| \ll d - l \) Eq. (52) takes the form

\[
\epsilon_1'' - \frac{\gamma B_p^2}{n B_c B_e d^2} \epsilon_1 = 0. \tag{54}
\]
The solution of Eq. (54) matching the boundary conditions given by Eq. (53) reads

\[ \epsilon_1(x) = -\frac{n\dot{B}_e \delta_s \theta}{T_c - T_0} \exp\left(\frac{|x| - d}{\delta_s}\right), \] (55)

where we introduce the value of the skin depth \(\delta_s\) as

\[ \delta_s = d \sqrt{\frac{nB_e\dot{B}_e}{\gamma B_p^2}}. \] (56)

To find the values of Re \(\gamma\) and Im \(\gamma\) we integrate now Eq. (46) over \(x\) from \(-d\) to \(d\). Using Eqs. (49), (51), and (55) we find the equation determining \(\gamma\) in the form:

\[ h - \frac{n\dot{B}_e B^2_e}{2\mu_0 j_c(T_c - T_0)} = -\gamma C_d - \frac{B_e n^2 \dot{B}_e^2}{\gamma \mu_0 j_c(T_c - T_0)}. \] (57)

We show schematically the dependencies of Re \(\gamma\) and Im \(\gamma\) on \(B_e\) in Fig. 2, where the field \(B_e\) is determined by the equation

\[ \frac{B^2_i}{B^2_j} - 1 = \sqrt{\frac{8C_d n\dot{B}_e B_e}{h} \frac{B_a}{B^2_j}}. \] (58)

The difference between \(B_i\) and \(B_j\) is small in the case when the ramp rate \(\dot{B}_e\) is low, i.e., \(B_i - B_j \ll B_j\) if

\[ \dot{B}_e < \frac{B_i}{2\pi^{2/3} n} \frac{B_p}{t_\kappa} \left(\frac{B_a}{B_p}\right)^{2/3}. \] (59)

It follows from Eq. (57) that Re \(\gamma = 0\) if \(B_e = B_j\), i.e., the Bean’s critical state is stable if \(B_e < B_j\), where the flux jump field \(B_j\) is given by the Eq. (32). We find also that at the stability threshold (for \(B_e = B_j\)) the value of \(\gamma\) is imaginary, i.e., \(\gamma = i\omega\). Thus the magneto-thermal instability is preceded by the magneto-thermal oscillations with the frequency, \(\omega\), given by the formula

\[ \omega = \left(\frac{2n^3 \dot{B}_e^3 h}{\mu_0^2 j_c d^2 C(T_c - T_0)}\right)^{1/4} \propto \dot{B}_e^{3/4}. \] (60)

The “frozen-in” magnetic flux approximation is valid if the surface layer where \(\epsilon_1(x) \neq 0\) is thin, i.e., if \(\delta_s \ll d - l\). Using Eqs. (32), (41), (56), and (60) we find the applicability criterion of the above approach in the form

\[ B_j > B_a \left(\frac{B_p}{\pi^2 B_a}\right)^{1/3}. \] (61)

IV. SUMMARY

To summarize, we consider the flux jump instability of the Bean’s critical state in type-II superconductors. We show that under the conditions typical for most of the magnetization experiments this instability arise in the
flux creep regime. We find the flux jump field $B_j$ that determines the critical state stability criterion. We show that the Bean’s critical state stability is determined by the slope of the current-voltage curve. We calculate the dependence of $B_j$ on the external magnetic field ramp rate $\dot{B}_e$. We find the frequency of the magneto-thermal oscillations preceding a flux jump as a function on the external magnetic field ramp rate $\dot{B}_e$.

Acknowledgments

I am grateful to I. Rosenman and L. Legrand for useful discussions stimulated this work.

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FIG. 1. Magnetic field $B(x)$ distribution at different temperatures: $T = T_0$ (solid line), $T = T_0 + \delta T$ (dashed line).

FIG. 2. The dependencies of Re $\gamma$ (solid line) and Im $\gamma$ (dashed line) on $B_e$. 
Fig. 1. R.G. Mints, Flux creep and flux jumping.
Fig. 2. R.G. Mints, Flux creep and flux jumping.