Application of the bi-characteristic method for reconstructing the effective frequency of electron collisions in the ionosphere

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Abstract. A method has been developed to reconstruct the effective frequency of electron collisions in the Earth's ionosphere, based on data on the attenuation of short-wave radiation during vertical sounding of the ionosphere by a frequency-modulated radio signal. Numerical modeling was carried out based on the Hamilton-Lukin bi-characteristic method.

1. Introduction. Formulation of the problem

This study is devoted to the development of a method for recovering the effective frequency of collisions of electrons in the Earth's ionosphere from the delay and amplitude characteristics of linear frequency modulated (LFM) signals from a vertical sounding ionosonde [1-3]. The relevance of the work is determined by the need for diagnostics and monitoring of the ionosphere due to the significant influence of the state of the ionospheric layers on the operation of radio systems for various purposes: radio communication, navigation (positioning), radar, as well as for constant monitoring of extreme phenomena in the Earth's atmosphere [4].

In this work, mathematical modeling of the propagation of radio waves in the ionospheric plasma is carried out, focused on the operational data of vertical sounding, since vertical sounding (VS) ionosondes are one of the most tested and effective means of diagnostics of the ionosphere [5].

2. Methods and algorithms

It is assumed in this work that the dependence of the delay of the signal reflected from the ionosphere, as well as the amplitude $A$ of the received signal on the frequency $f$, is known from the vertical sounding data. Attenuation of the amplitude of the probing signal depends on the effective collision frequency in accordance with the formula [6]:

$$A = E_0 D \exp[-\psi] ,$$

in which $E_0$ – is the field amplitude at a distance $r_0$ from the source outside the plasma layers. We will assume that the isotropic radiation source is located at the origin. Then:

$$E_0 = \frac{\sqrt{30W}}{r_0} \text{ (V/m)},$$

here, $W$ is the power of the emitter (in this work, $W = 1 \text{ kW}$). The coefficient $D$ included in formula (1) is the divergence of the radial flux. Since the quantities $A$ and $E_0$ are determined from experiment, then, knowing the divergence, by the formula

$$D = \frac{\ln \left( \frac{r_2}{r_1} \right)}{\ln \left( \frac{r_3}{r_2} \right)} ,$$

where $r_1$, $r_2$, and $r_3$ are the distances from the observer to the source and from the source to the Earth's surface.
\[ \psi = -\ln \frac{A}{DE_0}, \]  

(3)

one can find the absorption \( \psi \) containing the effective collision frequency \( \nu_c \).

The divergence \( D \) included in formulas (1) and (3) can be found using the Lukin's extended system of bi-characteristics [6-10]:

\[
\frac{dr}{dt} = \frac{2k_c^2 - \omega^2 \partial \epsilon / \partial k}{\partial (\omega \omega') / \partial \omega}, \quad \frac{dk}{dt} = \frac{\omega^2 \partial \epsilon / \partial r}{\partial (\omega \omega') / \partial \omega},
\]

(4)

\[
\frac{d\zeta}{dt} = \frac{\partial}{\partial \zeta} \left( \frac{2k_c^2 - \omega^2 \partial \epsilon / \partial k}{\partial (\omega \omega') / \partial \omega} \right), \quad \frac{d\zeta}{dt} = \frac{\partial}{\partial \zeta} \left( \frac{\omega^2 \partial \epsilon / \partial \tilde{r}}{\partial (\omega \omega') / \partial \omega} \right),
\]

(5)

\[
\frac{d\tilde{r}}{dt} = \frac{\partial}{\partial \eta} \left( \frac{2k_c^2 - \omega^2 \partial \epsilon / \partial k}{\partial (\omega \omega') / \partial \omega} \right), \quad \frac{d\tilde{r}}{dt} = \frac{\partial}{\partial \eta} \left( \frac{\omega^2 \partial \epsilon / \partial \tilde{r}}{\partial (\omega \omega') / \partial \omega} \right),
\]

(6)

since the divergence is calculated in an isotropic medium as the root of the ratio of the Jacobian \( J \) to \( J_0 \) by the formulas [9, 11]:

\[
D = \sqrt{\frac{J_0}{J}}, \quad J = \begin{vmatrix} k_x & k_y & k_z \\ x'_x & y'_x & z'_x \\ x'_y & y'_y & z'_y \end{vmatrix}, \quad J_0 = J|_{\zeta_0}.
\]

(7)

System (4) supplemented by the initial conditions [8,10]:

\[
k_x(0) = \frac{\omega}{c} \sqrt{\epsilon_0} \sin \zeta, \quad k_y(0) = \frac{\omega}{c} \sqrt{\epsilon_0} \cos \zeta \cos \eta, \quad k_z(0) = \frac{\omega}{c} \sqrt{\epsilon_0} \cos \zeta \sin \eta, \quad \tilde{r}(0) = 0,
\]

(8)

defines \( \tilde{r} = (x, y, z) \) – Cartesian coordinates of rays and \( \tilde{k}(t) \) wave vectors as functions of group time \( t \). The parameters \( \zeta, \eta \) in the initial conditions are the initial angles of the ray exit (\( \zeta = 0, \eta = \pi / 2 \) for a vertical ray), and \( \epsilon_0 \) are the value of the effective permittivity of the medium \( \epsilon(\tilde{r}, \tilde{k}, \omega) \):

\[
\epsilon = 1 - X, \quad X = \left( \frac{\omega_p}{\omega} \right)^2 \frac{4\pi e^2 N(z)}{m_e \omega^2},
\]

(9)

in the radiation source [8]. In formula (9), \( N \) is the electron concentration, \( X \) is the ratio of the square of the plasma circular frequency \( \omega_p \) to the square of the operating circular frequency \( \omega \), \( e \) is the electron charge, \( m_e \) is the electron mass, and \( c \) is the speed of light. For the considered model of the ionosphere \( \epsilon_0 \approx 1 \). The circular frequency \( \omega \) is related to the operating frequency \( f \) of the radiation source by the traditional relationship: \( \omega = 2\pi f \).

The paper considers a plane-layered medium, for which the effective dielectric constant of the medium depends only on the height \( z \) and does not depend on \( \tilde{k} \). Therefore, the components of the wave vector \( k_\nu \) and \( k_\varsigma \) are constant along the trajectory (see (4)), and for a vertical ray they are equal to zero \((k_x = k_y = 0)\). Then the expression for the Jacobian \( J \) (7) can be simplified and represented in the form [12]:

\[
J = k_x \begin{vmatrix} x'_x & y'_x \\ x'_y & y'_y \end{vmatrix},
\]

(10)

After determining the components of the wave vector \( k_\nu \) and \( k_\varsigma \), two equations \((x(t) = y(t) = 0)\) remain in system (4), and two of them with respect to \( z(t) \) and \( k_\varsigma(t) \) form a subsystem:

\[
\frac{dz}{dt} = \frac{2k_c^2}{\partial (\omega \omega') / \partial \omega}, \quad \frac{dk}{dt} = \frac{\omega^2 \partial \epsilon / \partial z}{\partial (\omega \omega') / \partial \omega}
\]

(11)

with initial conditions:
k_{c}(0) = \frac{\omega}{c} \sqrt{\varepsilon_0}, \quad z(0) = 0 \ . \quad (12)

In addition to the components of the wave vector, to determine the Jacobian $J$ according to formula (7), it is necessary to find the derivatives of $x$ and $y$ with respect to the initial angles of the ray exit $\zeta$ and $\eta$. For this, it is necessary to integrate equations (5) – (6) [9], supplemented by the initial conditions:

\begin{align*}
&k'_{\zeta}(0) = \frac{\omega}{c} \sqrt{\varepsilon_0} \cos \zeta, \quad k'_{\eta}(0) = 0, \quad k'_{\zeta}(0) = -\frac{\omega}{c} \sqrt{\varepsilon_0} \sin \zeta \cos \eta, \quad r_{\zeta}'(0) = 0, \quad r_{\eta}'(0) = 0, \\
&k'_{\zeta}(0) = -\frac{\omega}{c} \sqrt{\varepsilon_0} \cos \zeta \sin \eta, \quad k'_{\eta}(0) = -\frac{\omega}{c} \sqrt{\varepsilon_0} \sin \zeta \sin \eta, \quad k'_{\eta}(0) = \frac{\omega}{c} \sqrt{\varepsilon_0} \cos \zeta \cos \eta, \quad (13)
\end{align*}

which in our case after simplification are [12]:

\begin{align*}
&k'_{\zeta}(0) = \frac{\omega}{c} \sqrt{\varepsilon_0}, \quad k'_{\eta}(0) = k'_{\zeta}(0) = k'_{\zeta}(0) = k'_{\eta}(0) = 0, \quad k'_{\eta}(0) = -\frac{\omega}{c} \sqrt{\varepsilon_0} \ . \quad (14)
\end{align*}

From (5) – (6), taking into account (14), it can be established that:

\begin{align*}
&k'_{\zeta} = \frac{\omega}{c} \sqrt{\varepsilon_0}, \quad k'_{\eta} = 0, \quad k'_{\eta} = 0, \quad k'_{\eta} = -\frac{\omega}{c} \sqrt{\varepsilon_0} . \quad (15)
\end{align*}

Having calculated the divergence $D$ and knowing the amplitude $A$ and the initial amplitude $E_0$ from the experiment, we proceed to finding the effective collision frequency $\nu_c$. The absorption of the wave along the trajectory is related to the imaginary part of the effective permittivity of the medium by the relation $\varepsilon_2$ [6,13]:

$$
\psi = -\frac{\omega}{2} \int_0^t \varepsilon_2 dt . \quad (16)
$$

The expression for the imaginary part of the effective permittivity of the medium $\varepsilon_2$ approximately has the form [1,6]:

$$
\varepsilon_2 \approx \frac{XZ}{1+Z^2}, \quad Z = \nu_c / \omega . \quad (17)
$$

Assuming that the value $Z^2$ can be neglected in the denominator of (17), then

$$
\varepsilon_2 \approx -XZ . \quad (18)
$$

Let us equate the absorption $\psi$ in formulas (16) and (3). Then we get the Volterra integral equation of the second kind:

$$
\frac{V}{\omega^2} \int_0^t \nu_c N dt = L, \quad V = \frac{4\pi e^2}{m_e} , \quad (19)
$$

The right side of equation (19) $L$

$$
L = -\ln \frac{A}{D E_0} , \quad . \quad (20)
$$

is a function of the operating frequency $f$ and is calculated at the point of signal reception. In (19), $t_m$ denotes the time during which the signal travels the distance from the source of the point of reflection from the ionosphere. This quantity is also a function of frequency. Since the effective collision frequency $\nu_c$ and the electron concentration $N$ are functions of the height $z$, which depends on the group time $t$, the function $z(t)$ also depends on the frequency and is calculated along the ray path.

Let us consider the solution of equation (19) by the iteration method [12]. To shorten the notation, we denote $G = \nu_c N$. Since below a certain height the electron concentration is equal to zero, then up to a certain frequency $f_0$, for which the time $t_m$ is equal to $t_{m0}$

$$
\int_{t_{m0}}^t G dt = 0 . \quad (21)
$$
Divide the frequency interval \((f_0, f_{\text{lev}})\) into \(n\) parts \((f_0, f_1, \ldots, f_s, \ldots, f_n = f_{\text{lev}})\). On each interval, we assume the function \(G\) to be constant \((G = G_j)\) and find the values \(t_{mj}\) for each \(f_j\). As a result, we get:

\[
G_i = L_1 \frac{\omega_1^2}{t_{m1} - t_{n1}}, \quad G_z = L_2 \frac{\omega_2^2}{t_{m2} - t_{n2}} - G_j \frac{t_{m1} - t_{n1}}{t_{m2} - t_{n2}}, \quad \ldots, \quad G_j = L_j \frac{\omega_j^2}{t_{m,j} - t_{m,j+1}} - \sum_{i=1}^{j-1} \frac{t_{m,i} - t_{m,i+1}}{t_{m,j} - t_{m,j+1}}, \quad j \geq 2. \tag{22}
\]

Assuming that the dependence of the electron density \(N(z)\) on the height \(z\) is known, that is, it has already been reconstructed from the dependence of the signal delay on the radiation frequency, and taking into account that the values of \(G_j\) are the products of the effective electron collision frequency and the electron concentration at the reflection point, we find the effective collision frequency as a function of height.

3. Numerical simulation results

Consider an example of the implementation of the above algorithm. It is assumed, using the above formulas, to first calculate the field amplitude at different frequencies at the receiving point, and then, using the algorithm (see formulas (21), (22)), restore the effective collision frequency and compare the model and calculated values. Figure 1 shows the model dependence of the electron concentration on the height. It is shown as both a line and a background. In contrast to [12], this work considers a single-layer model, and in Figure 1a shows the entire ionospheric layer, and Figure 1b that part of the ionosphere that is accessible to the signals of vertical sounding.

In Figure 2a and Figure 2b, the thick yellow line shows the dependence of the effective frequency of electron collisions on the height in the ionospheric plasma in accordance with the experimental data [12].

The thin red line in Figure 2a shows an approximation of the effective collision frequency obtained by the least squares method. The dependence is quite complex and can be described by the formula:

\[
\log v_e = a_0 + a_1 z + a_2 z^2 + \frac{b_1}{z}, \tag{23}
\]

where \(a_0 \approx -1.02341, \ b_1 \approx 498.938, \ a_1 \approx 0.00799524, \ a_2 \approx -8.03841 \times 10^{-6}\). However, the effective collision frequency can be reconstructed from the VS data only in the lower part of the ionosphere, that is, not higher than the maximum of the \(F2\) layer. For this region (Figure 2b) the approximation is simpler and can be described by a hyperbolic dependence:
\[ \lg v_s = a + \frac{b}{z}, \quad a = 0.61677, \quad b = 416.177, \quad (z, \text{ km}). \]  

(24)

This dependence is typical for heights below the maximum of the F2 layer. In formulas (23) and (24), the height \( z \) is expressed in kilometers.

In Figures 3 and 4 the results of solving the system of bi-characteristic equations (11) – (12) are shown. Figure 3 shows the vertical ray trajectory in the ionosphere in the coordinates \((x, z)\) (Figure 3a) and the ray paths of the LFM signal in the coordinates \((t, z)\) (Figure 3b). In this case, the frequency \( f \) changes from 1 MHz (purple line) to 6.993 MHz (red line) in accordance with the colors of the spectrum. It is obvious that the propagation time of the signal increases with increasing frequency.

In figure 4 the dependences of the vertical component of the wave vector \( k_0 \) on time \( t \) (figure 4a), and on the height \( z \) (figure 4b) are shown.

The meaning of the color coloring of the line of drawings is the same as in figure 3. When constructing the figures, the wave vector was normalized to a value \( k_0 = k_0(0) \) that is a function of frequency (12). Analyzing figure 4, one can trace how the wave vector changes in the ionosphere,
decreasing to 0 at the point of reflection, changes sign and again increases in magnitude. The ray leaves the ionosphere when the ratio \( \frac{k_z}{k_0} \) becomes \(-1\).

Figure 4a. Time dependence of the vertical component of the wave vector

Figure 4b. Height dependence of the vertical component of the wave vector

Figure 5 shows the dependence of the signal attenuation \( A_m \) on time (figure 5a) and height (figure 5b) along the rays corresponding to different frequencies, calculated by the formula:

\[
A_m = -20 \log \left( \sqrt{\frac{J_0}{J}} \right).
\]  

(25)

Figure 5a. Time dependence of signal amplitude

Figure 5b. Height dependence of signal amplitude

If we trace the behavior of the curve along a certain ray, we can see that when moving along the ray from the source, the attenuation increases sharply (up to 100 dB), then drops sharply at the reflection point corresponding to the caustic [13], and then sharply increases again when the ray tends back to point of radiation. On a caustic the amplitude becomes infinite in the ray approximation. Although this is not obvious from figure 5, more accurate calculations show that in the caustic region (reflection point) and in the source \( A_m \rightarrow -\infty \).
Figure 6 shows the frequency dependence of the signal reflection height, calculated from the initial data (figure 6a), and the dependence of the time of arrival of the ray at the reflection point (figure 6b). Both the height of the signal reflection and the time $t_m$ increase with increasing frequency. A particularly sharp increase in the time of arrival of the ray at the point of reflection is observed at frequencies close to the critical $f_c \approx 7$ MHz.

It is easy to obtain a graph of the dependence of the height of the signal reflection on the delay $t_m$, excluding the frequency $f$ (see figure 7). In contrast to the two-layer model [12], the curve is smooth and one-to-one. In figure 8 shows the dependence of absorption $-\psi$ on the frequency $f$, plotted by formulas (16), (17) in accordance with the graph of the logarithm of the collision frequency (figure 2b).

Figures 9 (a, b) shows the dependences of the signal amplitude $A$ at the receiving point (1) and the function $L$ (20) on the frequency $f$. The amplitude of the signal reflected from the ionosphere (figure 9a) first increases with increasing frequency, then sharply decreases. The dependence of the function $L$ (20) on the frequency $f$ is smoother (figure 9b), since it is proportional to the logarithm of the ratio of the amplitude and divergence, multiplied by a constant coefficient. The red line shows the extrapolation of the curve.

Figures 10a and 10b show the results of modeling the dependence of the effective collision frequency $v_e$ on the height $z$. The yellow line shows the model dependence of the effective collision frequency on the height, and the green line shows the calculated values. It can be seen that the model and calculated values coincide with acceptable accuracy, and the accuracy of the coincidence increases with height. In
In figure 7 the dependences of the rates of phase changes (derived phase with respect to group time) on height are shown:

4. Conclusion

Thus, the paper developed a method for reconstructing the effective frequency of electron collisions in the Earth's ionosphere, based on data on the attenuation of short-wave radiation during vertical sounding of the ionosphere with a frequency-modulated radio signal. Numerical modeling was carried out on the basis of the Hamilton-Lukin bi-characteristic method. Constructed ray paths of the LFM signal in time-height coordinates; investigated the dependences of the vertical component of the wave vector on the
height and time; the dependences of the signal amplitude on time and height along the rays are considered. An acceptable agreement between the model values of the effective collision frequency and the calculated values is obtained.

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