Nonunique Stationary States and Broken Universality in Birth Death Diffusion Processes

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Systems with absorbing configurations usually lead to a unique stationary probability measure called quasi stationary state (QSS) defined with respect to the survived samples. We show that the birth death diffusion (BBD) processes exhibit universal phases and phase transitions when the birth and death rates depend on the instantaneous particle density and their time scales are exponentially separated from the diffusion rates. In absence of birth, these models exhibit non-unique QSSs and lead to an absorbing phase transition (APT) at some critical nonzero death rate; the usual notion of universality is broken as the critical exponents of APT here depend on the initial density distribution.

Markovian dynamics is used extensively in Physics, Chemistry, Biology, Economics, and many other branches of science to model real-world processes, both in equilibrium and nonequilibrium conditions [1, 2]. Lattice models in thermal equilibrium [3-5] are usually described by Markov processes (MPs) known as Monte Carlo dynamics, a dynamics that obeys detailed balance condition w.r.t. Gibbs measure, and they have found vast application in polymer physics [6], active and granular media [7], traffic flows [8], protein folding [9] etc.. In the absence of Gibbs measure, nonequilibrium systems are primarily described by Markov jump processes [10] and they exhibit novel nonequilibrium phases, and phase transitions [11, 12]. Ergodicity is crucial for MPs to attain a unique steady state, which is trivially broken if the system has absorbing configurations -the configurations where the system is permanently trapped once reached. The stationary measure of models with absorbing configurations are trivial: the steady-state probability for the system to be any configuration except the absorbing one is zero. An important question is to find a probability measure for the quasi-stationary state (QSS) defined w.r.t. the survived (no absorbed) samples. Simple models like contact process [13], directed percolation [14, 15], pair contact process [16], and fixed energy sandpiles [17-19] are known to have unique QSSs and they exhibit absorbing phase transitions (APT); some of them are realized experimentally [20, 21].

In this letter, we study absorbing-state phase transition in a simple Birth-Death-Diffusion (BDD) process where birth and death rates depend on the instantaneous particle density. We find that irrespective of the nature of diffusing(conserving) dynamics, these models exhibit APT when death rates crosses a critical threshold. In absence of birth, the QSS becomes non-unique in the sense that the probability measure there depends on the initial conditions. The nonuniqueness persists at the critical point resulting in initial condition dependence of the standard critical exponents. We provide A sufficient criteria for the system to have such nonunique QSSs.

The model: Let us associate particle deposition (birth) and evaporation (death) dynamics

\begin{equation}
\frac{D(\rho)}{B(\rho)} \longrightarrow 0,
\end{equation}

to an one dimensional system of size \( L \) which otherwise follows a particle conserving dynamics. The rates \( B(\cdot) \) and \( D(\cdot) \) are analytic functions of instantaneous particle density \( \rho = \frac{N}{L} \), where \( N \) is the number of particles. In particular, we are interested in rate functions of the form \( D(\rho) = e^{-L\theta(\rho)} \), and \( B(\rho) = \rho e^{-L\phi(\rho)} \); \( B(0) = 0 \) ensures that an empty lattice is the only absorbing configuration of the system. Along with these birth-death dynamics (1), we assume the system to have a natural conserving dynamics; the simplest one being the symmetric exclusion process (SEP) on a one-dimensional periodic lattice labeled \( i = 1, 2, ..., L \), with \( s_i = 0, 1 \) representing occupancy of the site \( i \).

\begin{equation}
10 \overset{1}{\rightleftharpoons} 01.
\end{equation}

In absence of birth and death, the conserving dynamics leads the system having \( N \) particles to a stationary probability measure \( g_N(C) \); for SEP in Eq. (2), \( g_N(C) = \left( \frac{L}{N} \right)^{-1} \) is a constant.

The birth-death rates in BBD model, being extremely slow (\( O(e^{-L}) \)) compared to the conserved dynamics (\( O(1) \)), effectuates a separation of timescales between the particle conserving and particle non-conserving dynamics. Between any two density-altering events \( N \rightarrow N \pm 1 \), the system gets enough time to equilibriate to the stationary distribution \( g_N(C) \). This separation of time scale decouples the conserved and nonconserved part of the dynamics, making the nonconserved dynamics effectively a biased walk (RW) on \( N = 0, 1, 2, \ldots, L \), with an absorbing boundary at \( N = 0 \).

\begin{equation}
N - 1 \overset{D_N}{\rightleftharpoons} N, B_N \rightarrow N + 1,
\end{equation}

where \( B_N = (L - N)B(\rho) \) and \( D_N = ND(\rho) \) where \( \rho \equiv \frac{N}{L} \). The additional multiplicative factors in birth and death rates are natural to expect as in any configuration having \( N \) particles, death (of a particle) can occur at any of the \( N \) sites whereas birth can occur at
Figure 1. (Color online) Decay of ρ(t), (a) for different L = 8, 16, 32, with initial density ρi = 0.7 (N = [0.7L]) and (b) different initial conditions ρi = \frac{11}{16} \times \frac{1}{2} \times \frac{1}{4} for L = 16. Three different overlapping curves correspond to Monte Carlo simulations of the BBD model for b = 1.4, d = 0.6 (solid red line), simulation of an effective RW dynamics (dashed green line), and numerical integration of the Master equation for the RW (dotted black line). The horizontal lines are the value of ρs obtained from a QSS simulation proposed in Ref. [22]. Inset of (b) shows that indeed ρs → d = 0.6 following Eq. (5) when L → ∞; asymptotically, ρs − ρs(L) ∼ L^{-0.428} (dashed line).

L − N sites. If the QSS of this absorbing RW is \{f_N\}, N = 1, 2, . . . , L, then the same for the whole system can be written assuming a good separation of time scale as, P(\mathcal{C}) = f_N g_N(\mathcal{C}), where N ≈ N(\mathcal{C})

Absorbing phase transition (APT): When the time-scales are well separated, the steady state density of the system can be determined completely from the QSS of the RW. Corresponding APT if steady, and its critical behaviour, are the governed solely by the birth-death dynamics, irrespective of the nature of the diffusive (conserving) dynamics. But, will there be any APT? Consider a region of parameter space where B_N > D_N ∀ N ∈ (0, L). Here the system is drifted towards higher density. Since B_N = e^{−L0.(ρ)^{−1}}L(1−ρ) ≃ B(ρ) and similarly D_N ≃ D(ρ), for any density 0 < ρ < 1 one can make B(ρ) ≫ D(ρ) by increasing L. Then, a super fast drift towards higher density will pin the walker at ρ = 1. On the other hand, for D_N > B_N the system drifts towards ρ = 0 and thus most walkers are absorbed; those who survive will contribute to the QSS. Most of the survived walker will be found at some N∗ = ρ∗L where the dynamics is slowest or the waiting time is large, i.e. where B_N + D_N has its minimum. ρ∗ can be tuned continuously by birth and death rates to obtain an absorbing transition in thermodynamically large systems when ρ∗ falls below \frac{1}{L} (i.e., ρ → 0).

Let us consider a specific example. Along with the conserving dynamics SEP in Eq. (2), we consider dynamics (1) with

\begin{align*}
D(\rho) &= e^{−L ρ(2d−ρ)}; \quad B(\rho) = ρe^{−L ρ(2/b−ρ)}, \quad (4)
\end{align*}

which have two real parameters b and d. We take b ≥ 0 as B(ρ) can not be analytically continued to b < 0. First let us check whether the time-scales are well separated. In Fig. 1(a) we plot ρ(t) for b = 1.4, d = 0.6 obtained from Monte Carlo simulations of the model for L = 8, 16, 32 and compared them with ρ(t) obtained from simulations of the biases RW dynamics (3). For small systems, the Master equation for the absorbing RW can be integrated numerically to obtain ρ(t), which is also plotted in Fig. 1.

An excellent match of all three curves justifies the separation of time scale and the decoupling of the non-conserved dynamics from the conserved one.

For rates (4), B_N > D_N corresponds to bd > 1. In this regime we expect the steady state density to be ρs = 1. For bd < 1, \frac{1}{L}(D_N + B_N) = (1 − ρ)B(ρ) + ρD(ρ), which has its minimum at ρ∗ = d. Thus based on these phenomenological arguments,

\begin{align*}
ρ_s &= \begin{cases} 
0 & d < 0 \\
0 & 0 \leq d < 1 \& bd \leq 1 \\
1 & \text{otherwise}
\end{cases} \quad (5)
\end{align*}

This result, of course, can be shown rigorously (see Supplemental Material).

With ρs in hand, we can proceed to obtain the phase diagram in b-d plane. But there is a problem. For b = 1.4, d = 0.6, we have obtained the steady state density \langle ρ \rangle ≡ ρ_s for different L in Fig. 1; ρ_s for the largest L = 32 was found found to be 0.364 which is far way from the analytical result ρs = 0.6, in (5). The non-monotonic L dependence of ρ_s(L) adds more doubt on whether ρ_s(L) → d as L → ∞. Moreover, simulation for larger L takes astronomically large time as the waiting times are O(e^L). Fortunately a recently proposed simulation method for QSS [22] helps us to obtain ρ_s for reasonably large L. The inset of Fig. 1, for b = 1.4, d = 0.6, shows that indeed ρ_s approaches d = 0.6 following an asymptotic power-law, ρ_s = 0.6 − L^{-0.428}.

The phase diagram, based on Eq. (5), is shown in Fig. 2 where 0 < ρ_s < 1 region is separated from two other regions ρ_s = 0 and ρ_s = 1. The BBD model exhibits continuous absorbing-state phase transition at d = 0 line. For a fixed value of b, ρ_s decreases continuously with decrease of d until a critical threshold d_c = 0 is reached.
Surprisingly, an ordinary non-equilibrium phase transition also takes place along PQR line in Fig. 2. The order parameter $\bar{\rho}_n = 1 - \rho_n$ picks up a nonzero value when, for a fixed $b$, $d$ is decreased below

$$d_c = \begin{cases} 1 & 0 < b < 1 \\ b^{-1} & b \geq 1. \end{cases}$$

(6)

This transition is discontinuous when $b \in (1, \infty)$ (along PQ line in Fig. 2) as $\bar{\rho}_n$ jumps from 0 to $1 - \frac{1}{b}$ at $d = d_c$, whereas it is continuous $b \in (0,1]$, leading to the fact that $Q\ (d(b) = (1,1))$ is a tri-critical point.

The line $b = 0$ corresponds to no birth of particles. In this case, we can solve the master equation of the system exactly. Let us assume that in absence of birth and death, the diffusive dynamics follows a master equation $\frac{d}{dt} \langle \phi_N \rangle = C_N \langle \phi_N \rangle$ with $N = 0, 1, \ldots, L$ conserved. $\langle \phi_N \rangle$ is a $(L^N)$ dimensional vector with components representing the probability of being in $(L^N)$ configurations of the conserved system. For $N = 0$, $L$ the system has only one configuration and evolution has no meaning; we set the scalars $C_0 = 0 = C_L$ for notational convenience. We also denote the steady states as $\langle \phi_N \rangle$ which satisfy $C_N \langle \phi_N \rangle = 0$. Now the master equation on $b = 0$ line can be written as $\frac{d}{dt} \langle \phi_N \rangle = \mathbf{M} \langle \phi_N \rangle$ with $\mathbf{M} = (|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_L\rangle)^T$ and $\mathbf{M}$ in block-form reads,

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_0 & D_1 \mathbf{T}_1 & 0 & 0 & \ldots & 0 \\ 0 & \mathbf{M}_1 & D_2 \mathbf{T}_2 & 0 & \ldots & 0 \\ 0 & 0 & \mathbf{M}_2 & D_3 \mathbf{T}_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \mathbf{M}_L \end{pmatrix}. \quad (7)$$

Here $\mathbf{M}_n = C_n - nD_n \mathbf{I}_n$ are $(L^N)$ dimensional square matrices with $D_n = D(n/L)$ and $\mathbf{I}_n$ is the identity matrix; obviously $\mathbf{M}_0 = 0$ is a scalar. The $\mathbf{T}_n$s are $(L^N_{n-1}) \times (L^N_n)$ matrices such that each column has exactly $n$ number of $1$s corresponding to a surjective mapping of $n$-particle configurations to $n-1$ particle configurations, $C_n \mapsto C_{n-1}$, by operation of removal of a single particle without disturbing any other particle. Clearly the eigenvalues of this upper triangular block matrix $\mathbf{M}$ are $\Lambda[\mathbf{M}] = \bigcup_n \Lambda[\mathbf{M}_n]$. In other words, the set of $2^L$ eigenvalues of $\mathbf{M}$ are $\Lambda_{n\nu} = \Lambda[C_n]_{\nu} - nD_n$, where $\Lambda[C_n]_{\nu}$ are $\nu$-th eigenvalue of $C_n$ in a decreasing order and $\nu = 0, 1, \ldots, (L^n_n)$ for any given $n$. It is now straightforward to calculate both the left and right eigenvectors of $\Lambda_{n\nu}$, respectively $\langle \Psi_{n\nu} \rangle$ and $\langle \Psi_{n\nu} \rangle$. For our purpose calculating them for $\nu = 0$ are enough (see below).

$$\langle \Psi_{n0} \rangle = (|\psi_0\rangle, |\psi_1\rangle, \ldots); \quad \langle \Psi_{n0} \rangle = (|\psi_0\rangle, |\psi_1\rangle, \ldots)^T$$

and

$$\langle \Psi_{k} \rangle = \begin{cases} 1 & k < n \\ 0 & k \geq n \end{cases} \quad (8)$$

$$\langle \Psi_{k} \rangle = \begin{cases} \prod_{i=k+1}^{n-1} \frac{\lambda_{0}}{\lambda_{0} - \lambda_{n0}} & k \geq n \\ \prod_{i=1}^{n-1} \frac{\lambda_{0}}{\lambda_{0} - \lambda_{n0}} & k = 0 \\ \prod_{i=k+1}^{n} \frac{\lambda_{0}}{\lambda_{0} - \lambda_{n0}}\mathbf{T}_i \langle \phi_n \rangle & 0 < k \leq n \\ 0 & k > n \end{cases}$$

where $0$ are null vectors and $(1_k| \equiv (1,1,\ldots)$ in $(L^N_k)$ dimension.

The probability $Q_{mn}$ that system, starting from a configuration with $m$ number of particles at $t = 0$, is not absorbed until time $t$ and it has $n$ particles is

$$Q_{mn}(t) = \frac{\langle n|e^{\mathbf{M}t}|m \rangle}{1 - \langle 0|e^{\mathbf{M}t}|m \rangle} = \sum_{k \neq 0} e^{-kD_d T} \langle n|\Psi_{k}\rangle\langle \Psi_{k}|m \rangle$$

where, in the spectral decomposition we kept only the dominant eigenvectors (corresponding to $\lambda_{mn}$). We also denote $|m\rangle = (L^N_m)^{-1}(0,0,\ldots,|1_m\rangle,\ldots,0)$ and $|m\rangle = (m)^T$. Clearly $|m\rangle$ represents a state of the system where all configurations that has exactly $m$ particles are equally likely. Note that, $|m\rangle$ is a left eigenvector of any Markov matrix in $(L^N_m)$ dimension, whereas $|m\rangle$ is the steady state for certain specific dynamics, like the SEP defined in (2), where all configurations are equally likely in steady state.

In the long time limit, one of the term $k = \bar{k} = \text{arg min}\{kD_d\} \forall k > 0$ is dominant in the sums that appear in $Q_{mn}$ as long as $\langle \Psi_{k}|m \rangle \neq 0$, leading to,

$$Q_{mn} = \frac{\langle n|\Psi_{k}\rangle}{\langle 0|\Psi_{k}\rangle} = \left( \frac{\prod_{k \leq \bar{k}} (1 - \frac{kD_d}{\bar{k}D_d})}{\prod_{k > \bar{k}} (1 - \frac{kD_d}{\bar{k}D_d})} \right) \quad n \leq \bar{k} \quad \bar{k} < n. \quad (9)$$

It appears that $Q_{mn}$ for $b = 0$ does not depend on the initial density $m/L$, however that is not true. Unlike $b > 0$ case where $\langle \Psi_{k}|m \rangle \neq 0 \forall m$, here it is evident from Eq. (8) that $\langle \Psi_{k}|m \rangle$ may become zero for some choice of $m$ leading to a possibility of a non-unique QSS that depends on the initial condition.

In continuum limit, we denote $(m, n, \bar{k}) = (\rho_i, \rho, \rho^*)_L$ and thus the Matrix $Q_{mn}$ takes functional form

$$Q(\rho_i, \rho) = e^{2Ld(\rho - \rho^*)} \Theta(\rho^* - \rho), \quad (10)$$

which is the probability that a non-absorbed (or survived) system has steady state density $\rho$, when it is
initialized at density $\rho_i$. For an arbitrary initial density distribution $\pi(\rho_i)$ with $\rho_i > 0$ the QSS measure is $Q(\rho) = \int_{0}^{1} d\rho_i Q(\rho_i, \rho) \pi(\rho_i)$ and the steady state density, calculated using Laplace method [23], is

$$Q(\rho) = \int_{0}^{1} d\rho_i Q(\rho_i, \rho) \approx \int_{0}^{1} d\rho_i \pi(\rho_i) \rho^*; \ \rho^* = \begin{cases} \rho_i \ & \rho_i \leq d \\ d \ & \rho_i > d \geq 0 \\ 0 \ & d < 0 \end{cases}.$$

Let us consider $\pi(\rho_i) = (1 + a)\rho_i^a$, $a > -1$. Then the moments of the QSS $\langle \rho^k \rangle = d^k - k\frac{d^{k+1}}{k+1} + a \forall k \geq 0$ depend on the initial condition. This features persists at the critical point $d_c = 0$ where the critical exponents too become non-unique. For $d \gtrless 0$, the order parameter $\langle \rho \rangle \sim d^d$ with $\beta = 1$ whereas its variance $\langle \rho^2 \rangle - \langle \rho \rangle^2 \sim d^{-\gamma}$ with a non-unique exponent $\gamma = -(a + 3)$. Continuous variation of critical exponents [24] are not new, they have been observed theoretically [25–27] and experimentally [28], but the exponents there vary w.r.t. certain marginal parameters of the system under study. To the best of our knowledge, dependence of the critical exponents on the initial conditions are new features of the BBD class of models, which has not been observed earlier. What is so special about BBD models at $b = 0$?

In a Markov chain, two configurations belong to the same communicating class (CC) if each one is accessible starting from the other. A “closed” CC is one that is impossible to leave - CCs which are not ‘closed’ are ‘open’. Irreducible Markov chains model many physical systems and they have only one communicating class which is necessarily closed; corresponding Markov matrices are irreducible and the uniqueness of its largest eigenvalue and the uniqueness of the stationary state are well protected by the Perron–Frobenius (PF) theorem [29]. Models like contact process [13], directed percolation [14, 15], and pair contact process [16] has only one open CC and as many closed CCs as there are absorbing configurations. We have seen that the QSS is the eigenvector corresponding to the largest eigenvalue of the Markov sub-matrix defined on the open communicating class (see Supplemental Material for details). For systems having one open CC this sub-matrix is irreducible and corresponding QSS is strictly positive, following PF theorem. On the other hand when a system has many open CCs, like a system described in Fig. 4(a), the Markov sub-matrix becomes reducible, and might have a non-unique QSS. Then, for any finite absorbing Markov chain the quasi-stationary density $\langle \rho \rangle$ may depend on initial condition, except for the special case when $\langle \rho \rangle = 0$. To have a non-unique $\langle \rho \rangle$ in the $L \rightarrow \infty$ limit, we must choose the non-conserving dynamics such that $\langle \rho \rangle$ does not vanish in the thermodynamic limit; birth and death rates in the BDD model in $e^{-\delta \langle \rho \rangle}$ form serves the purpose. Note that in BDD model for $b > 0$ there is only one closed and one open CC (as shown in Fig 4(c)) whereas setting $b = 0$ creates $L$ open CCs resulting in a nonunique QSS. If we consider another conserved dynamics instead SEP, say, the Kawasaki dynamics of Ising model ($H = -J \sum_i \delta_{s_i s_{i+1}}$),

$$0010 \xrightarrow{\delta_{s_i s_{i+1}}} 0011; \ 0101 \xrightarrow{\delta_{s_i s_{i+1}}} 0011; \ 1000 \xrightarrow{\delta_{s_i s_{i+1}}} 1010; \ 1101 \xrightarrow{\delta_{s_i s_{i+1}}} 1111,$$

with $\alpha = e^{-2J}$, or when we extend SEP and Ising models with birth-death dynamics (1) to higher dimensions, the structure of the communicating classes remains same as shown in Fig. 4(b), (c) respectively for $b = 0$ and $b \neq 0$. Thus these models too exhibit an universal phase phase diagram Fig. 2 and a non-unique APT at $b = 0$ (as in Fig. 3).

In summary, we introduce and study a class of birth-death diffusion models where the time-scale of birth and death rates are well separated from the diffusion rate. These models exhibit ordinary and absorbing phase transitions with universal critical lines that do not depend on the nature of diffusion (conserved) dynamics. In the absence of birth, the QSS measure of these systems become nonunique, leading to a novel APT with initial condition-dependent critical exponents. Usually, the origin of varying critical exponents is existence of an underlying marginal operator, which does not scale under renormalization. It is impossible to extend these ideas to include the dependence of critical exponents on initial conditions! In this letter we provide sufficient criteria: A nonunique QSS is possible when the system has multiple open communicating classes, and the transition rates between them have a thermodynamically stable minimum at some nonzero value of the order parameter. More theoretical understanding is required to design or access such nonunique QSSs in experiments.

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Supplemental material for “Nonunique Stationary States and Broken Universality in Birth Death Diffusion Processes”

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In this supplemental material, we report on (a) why quasi-stationary states are generally unique, (b) a rigorous derivation of a unique quasi-stationary state of the birth-death diffusion (BDD) processes with non-zero birth rate, and (c) exact analytical solution for BDD processes with zero birth rate.

I. WHY QUASI-STATIONARY STATE ARE GENERALLY UNIQUE

In a Markov process, state \( j \) is accessible from state \( i \) if \( \exists \) a path from \( i \) to \( j \), i.e., \( T_{ij}^n > 0 \) for some \( n \geq 0 \), where \( T_{ij} \) is transition rate from state \( i \) to state \( j \). If two states are each accessible from the other, then they communicate. Consider the Markov dynamics of the state space, \( S = \{0, 1, 2, 3, 4\} \):

The states which communicate are: \( 1 \leftrightarrow 2, 2 \leftrightarrow 3, 1 \leftrightarrow 3, \) and \( i \leftrightarrow i \). The communicating classes (or CCs) are the subsets of the state space such that two states are in the same subset iff (if and only if) they communicate. In the above example, the CCs are: \( \{0\}, \{1, 2, 3\} \) and \( \{4\} \). A ‘closed’ CC is one that is impossible to leave, here the closed CC is \( \{0\} \). The CCs which are not ‘closed’ are ‘open’, here the open CCs are \( \{1, 2, 3\} \) and \( \{4\} \).

The models with single absorbing configurations like directed percolation \([1, 3]\) and contact process \([4]\) have one open CC and one closed CC as in Fig. 1(b). Let \( w_{C_i \rightarrow C_j} \) be the non-negative transition rate from configuration \( C_i \) to configuration \( C_j \) and the configuration \( C_0 \) corresponds to absorbing configuration. The \( n^{th} \) component of the column vector \( |P(t)\rangle \) is the probability that system in configuration \( C_n \) at time \( t \). The Master equation for the dynamics is

\[
\frac{d}{dt}|P(t)\rangle = M|P(t)\rangle, \quad \text{where} \quad M = \begin{pmatrix} 0 & |w\rangle \\ 0 & \mathbf{A} \end{pmatrix}, \tag{1}
\]

and \( \mathbf{A}_{ij} = C_{ij} - \delta_{ij} w_{C_i \rightarrow C_0} \), \( C \) is the irreducible dynamics within the open CC of the process, \( S^+ = S \setminus \{C_0\} \), the sub-state space excluding the absorbing configuration, and \( |w\rangle = (w_{C_1 \rightarrow C_0}, w_{C_2 \rightarrow C_0}, \ldots) \). If the left and right eigenvectors corresponding to largest eigenvalue \( \lambda_i, i = 1, 2, \ldots \) of \( \mathbf{A} \) are \( \phi_i \) and \( \phi_i^* \), s.t. \( 0 > \text{Re}(\lambda_1) \geq \text{Re}(\lambda_2) \ldots \), then eigenvalues of \( M \) are \( \{0\} \cup \{\lambda_i\} \) and corresponding left and right eigenvectors are

\[
|\Psi_0\rangle = (1, 1, \ldots); \quad |\Psi_0\rangle = (1, 0, 0, \ldots)^T, \quad \text{and} \quad |\Psi_{i \neq 0}\rangle = (0, \phi_i); \quad |\Psi_{i \neq 0}\rangle = \left( \frac{\langle \phi_i \rangle \lambda_i}{\lambda_i} \right)^T \phi_i.
\]

The steady state \( |\Psi_0\rangle = (1, 0, 0, \ldots)^T \) of the system is trivial here, i.e., in \( t \to \infty \) limit the system is certainly absorbed. The quasi-stationary probability \( Q_{ij} \) is, starting from a configuration \( C_{i \neq 0} \), probability the system survives in configuration \( C_{j \neq 0} \) is

\[
Q_{ij} = \lim_{t \to \infty} \frac{\sum_n \langle j | e^{\lambda_n t} | \Psi_n \rangle \langle \Psi_n | i \rangle}{\sum_{k \neq 0,n} \langle k | e^{\lambda_n t} | \Psi_n \rangle \langle \Psi_n | i \rangle} \tag{2}
\]

Irreducible dynamics of sub-state space \( S^+ \) ensures positivity of \( \langle \phi_1 \rangle \) and \( \phi_1 \), the left and right eigenvectors corresponding to largest eigenvalue of \( \mathbf{A} \) \([27]\). And positivity of \( \langle \phi_1 \rangle \) ensures \( \langle \Psi_1 | i \neq 0 \rangle > 0 \). And thus, the quasi-stationary measure is the normalized right eigenvector of \( \langle \phi_1 \rangle \), the largest eigenvalue of \( \mathbf{A} \) or the second largest eigenvalue of matrix \( M \), and it does not depend on the initial condition.

System with many absorbing configurations \( S^0 = \{C_{0,1}, C_{0,2}, \ldots\} \), like pair contact process \([5]\) has many
closed CC and one open CC as in Fig. 1(a) whose dynamics is given by Master equation
\[ \frac{d}{dt} |P(t)\rangle = M |P(t)\rangle, \]
where \( M = \begin{pmatrix} 0 & w \\ 0 & A \end{pmatrix} \). \( A \) in upper row of matrix \( M \) is square null matrix of dimension equal to number of absorbing configurations, \( n(S^0) \). A similar calculation as before lead to unique quasi-stationary state in systems with many absorbing configuration. The irreducible dynamics within non-absorbing sub-state space \( S^+ = S/S^0 \), lead to uniqueness of quasi-stationary state.

When there are many open CCs the dynamics within non-absorbing sub-state space \( S^+ \) is no longer irreducible and thus lead to non-unique quasi-stationary state. Let’s take an example, as described in for Fig. 1(c), where absorbing sub-state space \( N(S) \) containing \( n(0) \) and \( n(1) \) absorbing configuration. Then the the matrix \( M \) in the Master equation is
\[ M = \begin{pmatrix} 0 & 0 & 0 & \langle w_3 \rangle \\ 0 & A_{11} & 0 & 0 \\ 0 & A_{21} & A_{22} & 0 \\ 0 & A_{31} & A_{32} & A_{33} \end{pmatrix}, \]
where \( A_{kk'} \) are matrices of dimension \( n_k \times n_{k'} \). If we express \( M \) as a \( 2 \times 2 \) block matrix, as in Eq. (1) then (\( w = (0, 0, \langle w_3 \rangle) \) and
\[ A = \begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{pmatrix}, \]
which is reducible. It is evident that more than one open communicating class certainly make irreducible \( A \) irreducible and and it may break the positivity condition of the left and right eigenvectors of largest eigenvalue \( \lambda \), leading to a possibility of having a non-unique quasi-stationary state.

II. QUASI-STATIONARY STATE OF BDD MODEL WITH \( b > 0 \)

If the birth rate and the death rate in the Birth-death diffusion (BDD) process are extremely slow, then effective dynamics is that of single particle random walk (RW), which is much tractable numerically. In the following section, we calculate the quasi-stationary state of RW.

Consider a biased random walk (RW) on 1D lattice with sites labeled \( N = 0, 1, 2, \ldots, L \),
\[ N \rightarrow N + 1, \]
where \( B_N = (L - N)Ne^{-b(2/L - \rho)} \) and \( D_N = Ne^{-L(2/L - \rho)} \) where \( \rho \equiv \frac{b}{L} \). Clearly, the and site \( N = 0 \) is absorbing boundary as \( B_0 = 0 \). The evolution of RW is given by the master equation
\[ \frac{d}{dt} |P(t)\rangle = M |P(t)\rangle, \]
where \( n^{th} \) component of \( |P(t)\rangle \) corresponds to probability of finding the RW on site \( N = n \) at time \( t \) and the \((L + 1) \times (L + 1)\) dimensional matrix \( M \) is
\[ M = \begin{pmatrix} 0 & D_1 & 0 & \ldots & 0 \\ 0 & -D_1 - B_1 & D_2 & \ldots & 0 \\ 0 & B_1 & -D_2 - B_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -D_L \end{pmatrix}. \]

Let us label the eigenvalues of \( M \) in decreasing order of their real-part, \( 0 = \lambda_0 \geq \Re(\lambda_1) \geq \Re(\lambda_2) \ldots \) and denote the left and right eigenvectors of \( \lambda_n \) as \( \langle \psi_n \rangle \) and \( |\psi_n\rangle \) respectively. The steady state \( |\psi_0\rangle = (1, 0, 0, \ldots)^T \) of the system is trivial here - in \( t \rightarrow \infty \) limit the system is certainly absorbed. Probability that, starting from a site \( i \neq 0 \) the walker survives at time \( t \) is \( S_i(t) = \sum_{k \neq 0} \langle k | e^{M t} | i \rangle \).

Among those survived, the probability of finding them at a specific site \( j \neq 0 \) is \( Q_{ij}(t) = \frac{1}{S_i(t)} \langle j | e^{M t} | i \rangle \). In the \( t \rightarrow \infty \) limit, \( Q_{ij}(t) \) reaches a stationary value \( Q_j \),
\[ \lim_{t \rightarrow \infty} \frac{\sum_n \langle j | e^{\lambda_n t} | \psi_n \rangle \langle \psi_n | i \rangle}{\sum_{k \neq 0} \langle j | e^{\lambda_n t} | \psi_n \rangle \langle \psi_n | i \rangle} = \frac{\langle j | \psi_1 \rangle}{\langle \psi_1 | \psi_1 \rangle} \equiv Q_j, \]

Indeed, \( Q_j \) is the stationary measure of quasi-stationary state (QSS). Up to a normalization factor, it is given by the right eigenvector of the 2\(^{nd}\) largest eigenvalue of the Markov matrix, i.e., \( |\psi_1\rangle = (f_0, f_1, f_2, \ldots)^T \). Clearly the QSS is unique in the sense that it does not depend on the initial conditions (here, starting density \( \rho = i/L \). The eigenvalue equation \( M |\psi_1\rangle = \lambda_1 |\psi_1\rangle \) results in a recursion relation,
\[ D_{n+1}f_{n+1} = (D_n + B_n + \lambda_1)f_n - B_n - 1f_{n-1} \]
with boundary conditions \( B_0 = 0 = D_0 = B_L \). When \( bd \leq 1 \), for a large \( L \), the eigenspectrum \( \{\lambda_n\} \approx \{-B_i - D_i\} \), the diagonal elements of \( M \). With largest eigenvalue being \( \lambda_0 = 0 = -B_0 - D_0 \), the 2\(^{nd}\) largest eigenvalue \( \lambda_1 \approx -\min(B_{t \leq 0} + D_{t \geq 0}) \). When \( bd > 1 \), \( \lambda_1 \ll -\min(B_{t < 0} + D_{t \neq 0}) \). Now, let’s consider three different cases \( bd = 1 \), \( bd > 1 \) and \( bd < 1 \) separately. When \( d = 1/b \), \( B_n/D_n = 1 - n/L = O(1) \) and thus \( B_n \approx D_n \). Thus, Eq. (10) results in a solution
\[ f_n = n\Theta(Ld - n)/D_n \propto \Theta(d - \rho) e^{L(2d - \rho)} \]
Here, in the last step we have taken large \( L \) limit. In general for any positive polynomials \( h(\rho) \), \( h(\rho)e^{L(\rho)} \approx e^{L(\rho)}/[h(\rho) \approx e^{L(\rho)} \) for large \( L \); as an approximation,
the expression $h(\rho)e^{L_\theta(\rho)} \approx e^{L_\theta(\rho)}$. Further in this article we repeatedly use this approximation without any warning. The Heaviside step function $\Theta(\Lambda d - n)$ appears because the iteration of (10) ends when $B_n + D_n + \lambda_1 = 0$, and this happens when $n = \arg \min (B_n + D_n) = \Lambda d$.

For $bd < 1$, $B_i \ll D_i$, and then Eq. (10) reduces to $D_n f_n = D_{n-1} f_{n-1}$ which gives $f_n = \Theta(\Lambda d - n)/D_n$. Similarly, for $bd > 1$, $B_i \gg D_i$, and the expression $D_i + B_i + \lambda_1$ in Eq. (10) can be replaced by $B_i$, resulting in a solution,

$$f_n \approx \frac{1}{D_n} \prod_{k<n} \frac{B_k}{D_k} e^{L_\rho(2d - \rho) + L^2(d - 1/b)\rho^2} \quad (12)$$

In summary, in the thermodynamic limit, the quasi-stationary measure of the RW dynamics (6) is given by,

$$f(\rho) = \begin{cases} \Theta(d - \rho) e^{L_\rho(2d - \rho)} & bd \leq 1 \\ e^{L^2(2d - \rho)\rho^2} & bd > 1 \end{cases} \quad (13)$$

Presence of a factor $1/b$ in above expression already indicates that a special care must be taken for $b = 0$ case, which will be studied later in this article. The average steady state density of the system is $\rho_s = \int_0^1 df f(x)/\int_0^1 df f(x)$, which can be calculated straightforwardly using the steepest-descent method [6]; the resulting in

$$\rho_s = \begin{cases} 0 & d < 0 \\ d & 0 \leq d < 1 \& bd \leq 1 \\ 1 & \text{otherwise} \end{cases}$$ \quad (14)

as anticipated from a phenomenological argument, in Eq.(6) of the main text.

III. QUASI-STATIONARY STATE OF BBD MODEL WITH $b = 0$

In the following section, we will do the exact calculation of quasi-stationary state of BBD model when birth rate is zero. For matrix $M$ for no-birth BDD process as in Eq. (7) of main text

$$M = \begin{pmatrix} M_0 & D_1 T_1 & 0 & 0 & \ldots & 0 \\ 0 & M_1 & D_2 T_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & M_L \end{pmatrix},$$ \quad (15)

the eigenvalues are $\lambda_{n\nu} = \Lambda(C_n)_{\nu} - nD_n$, where $\Lambda(C_n)_{\nu}$ are $\nu^{th}$ eigenvalue of Markov matrix of the conserved dynamics $C_n$ in a decreasing order and $\nu = 0, 1, \ldots, (L_n)$ for any given $n$. $|\phi_n|$ is the stationary state of conserved dynamics of $n$-particles which satisfies $C_n|\phi_n| = 0$. We are essentially interested in eigenvectors corresponding to eigenvalue $\lambda_n \equiv \lambda_{n,0}$, since these will dominating term in calculation of QSS. Corresponding to eigenvalue $\lambda_n$, the left and right eigenvectors are $\langle \Psi_n | \equiv \langle \psi_n | = \langle (|0\rangle, |\psi_1\rangle, \ldots \rangle$ and $|\Psi_n| \equiv |\Psi_n| = (|\psi_0\rangle, |\psi_1\rangle, \ldots \rangle^T$.

The right eigenvalue equation $M|\Psi_n| = \lambda_n|\Psi_n|$ leads to recursion relation

$$D_i T_i |\psi_{i-1}| + M_i |\psi_i| = \lambda_i |\psi_i| \quad \text{for} \ n > 0. \quad (16)$$

On substituting $|\psi_n| = |\phi_n|$, the recursion relation reduces to form

$$|\psi_k| = \begin{cases} \prod_{i=1}^{n-1} \frac{\lambda_i - \lambda_k}{\lambda_k} & k = 0 \\ \prod_{i=k+1}^{n} \frac{\lambda_i}{\lambda_k - \lambda_n} (M_{i-1} - \lambda_n)^{-1} T_i |\phi_n| & 0 < k \leq n \\ 0 & k > n \end{cases}$$

where $\Phi$ are null vectors and $\langle 1_k \rangle \equiv (1, 1, \ldots)$ in $(L_k)$ dimension. To evaluate $|\psi_{k=0}|$, we have used the fact that $T_1 = \langle 1_L | = (1, 1, \ldots)$ and repeatedly used the identity

$$\langle 1_k | T_k = \langle 1_{k-1} | k. \quad (17)$$

Above identity is derived from definition of $T_k$ i.e., $T_k$ are $(\frac{L_k}{k}) \times (\frac{L_k}{k})$ matrices such that each column has exactly $k$ number of 1s. Similarly, we can write a recursion relation for left eigenvalue equation, and substituting $|\psi_n| = |\phi_n|$ we arrive at

$$\langle \psi_k | = \begin{cases} 0 & k < n \\ \langle 1_k | \prod_{i=n+1}^{k+1} \frac{\lambda_i}{\lambda_n - \lambda_k} & k \geq n \end{cases} \quad (18)$$

Using above expression we get

$$\langle \Psi_k | m \rangle = \prod_{i=k+1}^{L} \frac{\lambda_i}{\lambda_n - \lambda_k} \quad m \geq \tilde{k}$$ \quad (19)

where $\langle m \rangle = (\frac{L}{m})^{-1} \langle 0, 0, \ldots, |1_m\rangle, \ldots, 0 \rangle$ and $|m\rangle = |m\rangle^T$. The key thing to note here is, $\langle \Psi_k | m \rangle = 0$ for $m < k$ and $\langle \Psi_k | m \rangle \neq 0$ for $m \geq k$. The probability $Q_{mn}$ that system, starting from a configuration with $m$ number of particles at $t = 0$, is not absorbed until time $t$ and it has $n$ particles is

$$Q_{mn} = \frac{\langle n | e^{Mt} | m \rangle}{\langle 0 | e^{Mt} | m \rangle} = \sum_{k \neq 0} e^{-KD_k t} \langle n | \Psi_k \rangle \langle \Psi_k | m \rangle$$

The dominating term in above expression is one which have $\lambda_k = \max (\{\lambda_k \neq 0\})$ and $\langle \Psi_k | m \rangle \neq 0$. In the long time limit,

$$Q_{mn} = \frac{\langle n | \Psi_k \rangle}{\langle 0 | \Psi_k \rangle} = \begin{cases} \frac{\lambda_k}{\lambda_n} \prod_{k \leq n} \left(1 - \frac{\lambda_k}{\lambda_n}\right) & n \leq \tilde{k} \\ 0 & \tilde{k} < n \end{cases} \quad (20)$$
On evaluating above expression above for $\lambda_i = -iD_i$, we arrive at Eq. (9) of the main text:

$$Q_{mn} = \begin{cases} \frac{kD_k}{D_n} \prod_{\bar{k} < n} \left(1 - \frac{kD_k}{\bar{k}D_n}\right) & n \leq \bar{k} \\ 0 & \bar{k} < n \end{cases}.$$  \hspace{1cm} (21)

In order to evaluate the continuum form of $Q_{mn}$, we are essentially interested continuum limit of

$$\prod_{k<n} \left(1 - \frac{kD_k}{\bar{k}D_n}\right) \simeq \prod_{k<n} (1 - e^{L(\theta(\rho_k) - \theta(\rho_{\bar{k}})))})$$  \hspace{1cm} (22)

where we have shorthand $\rho_k \equiv k/L$. The expression above is not a Riemann sum. Consider instead

$$\frac{1}{L} \sum_{k<n} \ln(1 - e^{L(\theta(\rho_k) - \theta(\rho_{\bar{k}})))}) \underset{L \to \infty}{\rightarrow} \int_0^{\rho_n} d\rho \ln(1 - e^{L(\theta(\rho) - \theta(\rho_{\bar{k}})))})$$

After using expression evaluated above and using saddle-point method, the Matrix $Q_{mn}$ takes functional form as in Eq. (10) of the main text:

$$Q_{mn} \rightarrow Q(\rho_m, \rho_n) = e^{2Ld(\rho_n - \rho_{\bar{k}})} \Theta(\rho_k - \rho_n).$$  \hspace{1cm} (23)

where in main text, for convenience and understanding we have used notations $(\rho_m, \rho_n, \rho_{\bar{k}}) \equiv (\rho_i, \rho, \rho^*)$.

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