A numerical boundary integral method for cross-sectional distribution of gas/water two phase flows

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Abstract. A clearer understanding of two-phase flows requires getting the distribution of gas/water two-phase flows non-invasively. A numerical method to get the cross-sectional distribution of gas/water two-phase flows, via acquiring the common boundary of the two domains, is presented. This approach is based on the boundary integral equations reduced from Green’s formula and their derivatives, considering that the gas phase is isolating and the conductivity of water phase is known. The forward problem is solved by boundary element method, which shows considerable accuracy but much smaller matrix size compared with finite element method. The boundary integral equations with non-smooth boundaries are discreted, and their Frechet derivatives are linearized. An iterative method is adopted, and the initial guess is estimated with the aid of the Linear Back Projection Algorithm for electrical resistance tomography. Numerical results show that the method can locate the gas bubbles and determine their shape effectively. Residual error is defined to investigate the convergence processes. In the anterior steps, the processes converge more quickly for fewer elements; while in the posterior steps more mesh elements are needed to get better results. Element increment contributes to decrease the final residual errors, but it tends to be faint when the number of elements is considerably large.

1. INTRODUCTION
Gas/water two-phase flows are widely encountered in nature and modern industry. A clearer understanding of two-phase flows requires getting the distribution of gas/water two-phase flows non-invasively, since it has an influence up on the characteristics and behaviors of the mixtures of the two fluids. Numerous techniques have been developed for this purpose over the years: radiation radiography techniques (Uchimura, 1998; Takenaka, 1998), X-ray computed tomography (Toye, 1998; Harvel, 1999), ultrasonic tomography (Xu, 1997), electrical tomography techniques (Bennett, 2002; Kim, 2002), optical techniques (Ursenbacher, 2004) and so on.

Electrical resistance tomography (ERT) is a novel method to determine the electrical conductivity distribution of a cross-section by applying currents at the boundary and measuring the resulting voltages at the electrodes. Comparing to computerized tomography (CT), which is a standard tool in medical diagnostics and non-destructive testing of materials, ERT has many advantages in cost, speed and safety, which makes it more suitable in two-phase or multi-phase flow context. Applications of ERT on two-phase flows have been presented by Dong (2001), Fransolet (2005) and Jin (2007).

ERT system operates in the following way: A number of electrodes are distributed around the cross-section at equal intervals. A constant current of low frequency is applied to an adjacent pair of
electrodes, and potentials are measured on the electrodes. It then switches the current to another pair of electrodes. When the distribution of gas differs, the measured voltages vary. Therefore, the measured potentials contain the information of the gas distribution. After the excited pair of electrodes makes a rotation, a set of potential data is collected for further processing.

Fig.1 A sketch map of ERT sensors

The following difficulty makes it hard to apply ERT to online quantitative multi-phase flow measurement. As the adjacent excitation mode is generally adopted, electrical currents mainly travel from one excited electrode directly to the other. As a result, variation in the conductivity near the centre of the cross-section has little effect on the boundary potentials measured. And concerning the inverse problem, small measurement errors easily cause vibration of the conductivity distribution in the interior of the body. Furthermore, total independent measurements are much fewer than the number of conductivity mesh due to capability constraints of the hardware system. Mathematically speaking the electrical impedance tomography is a nonlinear and severely ill-posed problem, which makes it hard to get a precise image of the electrical conductivity distribution. So it is important to incorporate as much a priori knowledge about the object as possible.

As to the gas/water two-phase flow, the discrete gas phase is isolating, and the conductivity of water phase is constant and can be measured using additional equipment. We only have to find out the interior boundary which separates out the gas and water. Thus priori knowledge is fully utilized, and makes the problem much easier to approach and more likely to be eventually solved numerically.

This paper is partly based on recent work of Kress (2005), Ivanyshyn (2006) and Eckel (2007) where a new integral equation method has been developed to recover inclusions with piecewise constant conductivities, inclusions, and perfectly conducting, respectively. The main work of this paper is to generate and linearize the boundary integral equations (BIEs), and also their derivatives, of 2D laplace equation with non-smooth boundaries, concerning that non-smooth points exist in realistic ERT systems. Another reason is that the discretization of boundary into some set of elements is often used for numerical solutions, which inevitably makes the boundary unsmooth. Thus it can be regard as a supplement to their theory.

2. BOUNDARY INTEGRALS

Suppose $D \in \mathbb{R}^2$ is a simply connected bounded domain which represents the conducting medium, the electric potential $u$ satisfies the Laplace equation and boundary conditions:
\[
\begin{aligned}
\Delta u &= 0 & \text{in } D \\
\int_{\partial_i} \sigma \frac{\partial u}{\partial n} \, ds &= g_i & \text{on electrode } i \ (i = 1, 2, \ldots, 16) \\
u &= \text{const} & \text{on each electrode} \\
\sigma \frac{\partial u}{\partial n} &= 0 & \text{on other points of } \partial D
\end{aligned}
\]

where \( g_i \) represents the imposed current on the \( i \)-th electrode, \( n \) the outward unit normal to the boundary \( \partial D \), and \( \sigma \) the electrical conductivity in \( D \).

Introduce the well-known fundamental solution to the Laplace equation in \( \mathbb{R}^2 \):

\[
u^*(x,y) = -\frac{1}{2\pi} \ln |x-y| \quad x \neq y
\]  

Applying Green’s integral theorem, get the integral equation directly:

\[
a(y) \nu(y) = \int_{\partial D} \left( u^*(x,y) \frac{\partial u}{\partial n} - u \frac{\partial u^*(x,y)}{\partial n} \right) \, ds \quad y \in \partial D
\]

where \( \alpha(y) = \theta(y)/2\pi \), and \( \theta(y) \) is the internal angel at point \( y \).

For analytical convenience, the boundary of the domain \( D \) is divided into two curves: \( \Gamma_0 \) (exterior) and \( \Gamma_1 \) (internal), and denote the unit normal of \( \Gamma_1 \), \( n \), orient to the interior \( D \). And rewrite equation (3) to be:

\[
\begin{aligned}
a(y) \nu(y) + \int_{\Gamma_0} u \frac{\partial u^*(x,y)}{\partial n} \, ds - \int_{\Gamma_0} u^*(x,y) \frac{\partial u}{\partial n} \, ds &= \int_{\Gamma_1} u \frac{\partial u^*(x,y)}{\partial n} \, ds \quad y \in \Gamma_0 \\
-a(y) \nu(y) - \int_{\Gamma_1} u \frac{\partial u^*(x,y)}{\partial n} \, ds + \int_{\Gamma_1} u^*(x,y) \frac{\partial u}{\partial n} \, ds &= -\int_{\Gamma_0} u^*(x,y) \frac{\partial u}{\partial n} \, ds \quad y \in \Gamma_1
\end{aligned}
\]

Now define the following functions:

\[
\begin{aligned}
w_0(f,g)(y) &= a(y) \nu(y) + \int_{\Gamma_0} f \frac{\partial u^*(x,y)}{\partial n} \, ds - \int_{\Gamma_0} u^*(x,y) g \, ds \quad y \in \Gamma_0 \\
w_1(f,g)(y) &= \int_{\Gamma_0} f \frac{\partial u^*(x,y)}{\partial n} \, ds - \int_{\Gamma_0} u^*(x,y) g \, ds \quad y \in \Gamma_1 \\
K_f h(y) &= \int_{\Gamma_1} h \frac{\partial u^*(x,y)}{\partial n} \, ds \quad y \in \Gamma_0 \\
K_h y(y) &= -a(y) \nu(y) + \int_{\Gamma_1} h \frac{\partial u^*(x,y)}{\partial n} \, ds \quad y \in \Gamma_1
\end{aligned}
\]

where \( f \), \( h \) and \( g \) is \( u \big|_{\Gamma_0}, u \big|_{\Gamma_1}, \frac{\partial u}{\partial n} \big|_{\Gamma_0} \) respectively.

Equations (6)-(10) are not difficult to calculate, given an explicit description of the interior boundary and boundary conditions. However conditions given in equation (1) are not strictly Neumann nor Dirichlet forms, the integrals can be converted to algebraic equations via linearization.
which have a unique solution. Thus, if find a $\Gamma_1$, and the corresponding $h$ satisfy following equations, then the inverse problem is solved.

$$K_0h = w_0(f, g) \quad (10)$$

$$K_1h = w_1(f, g) \quad (11)$$

Linear boundary elements are adopted to carry out above calculations and the forward problem. The potentials on the boundaries are considered to be continuous, but the derivatives of them are discontinuous on ends of electrodes. Fig. 2 compares the boundary potentials computed by two numerical methods: linear boundary element method (BEM) and finite element method (FEM) using commercial software COMSOL. The interior boundary is determined by $r = 0.5$ in the polar coordinate system, the radius of the exterior circle is 1, and electrode coverage rate is 0.5. Imposed current on the electrodes are 1 and -1 for the first and second electrode respectively, and 0 for others. For FEM, the mesh size is up to 18720, while the boundaries are meshed with only 174 elements for BEM, which shows considerable accuracy, but much smaller matrix size and higher speed.

If equations (10) and (11) don’t hold, the Frechet derivatives of the operators $K_0h$ and $K_1h$, with respect to $\Gamma_1$, is needed to get iterative equations. Suppose the interior boundary is meshed into line segments:

$$\Gamma_1 = \bigcup z_i^e (t), \quad e = (1, 2, 3, ..., n_1), t \in [0, 1] \} \quad (12)$$

where $n_1$ is the amount of segments. The derivative of $K_0h$, $dK_0h$, can be calculated by the following formula:
\[ dK_oh^+ = \frac{1}{2\pi} \int_0^1 \left\{ \frac{2|z_i^+(\tau)|z_i^+(\tau) - y}{|z_i^+(\tau) - y|^4} \cdot \zeta(\tau) - \left( \frac{[z_i^+(\tau)]^2 \cdot \zeta(\tau) + [\zeta(\tau)]^2 \cdot [z_i^+(\tau) - y]}{|z_i^+(\tau) - y|^2} \right) h^+(\tau) d\tau \right\} \quad y \in \Gamma_0 \]

where \( \zeta(\tau) \tau \in [0,1] \) is the small spatial increment on \( z_i^+ \). It is much more complicated to get the derivative of \( K_oh \), which requires consideration of both the variation of internal angel and \( \Gamma_i \) in the second part of equation (9).

\[ dK_oh = dh \alpha + dK_oh \quad (14) \]

where \( dK_oh \) may be calculated by:

\[ dK_oh^+ = \frac{1}{2\pi} \int_0^1 \left\{ \frac{2|z_i^+(\tau)|z_i^+(\tau) - y}{|z_i^+(\tau) - y|^4} \cdot \zeta(\tau) - \left( \frac{[z_i^+(\tau)]^2 \cdot \zeta(\tau) + [\zeta(\tau)]^2 \cdot [z_i^+(\tau) - y]}{|z_i^+(\tau) - y|^2} \right) h^+(\tau) d\tau \right\} \quad y \in \Gamma_i^+, e \neq j \]

Thus get the iterative equations:

\[ K_oh + K_ohdh + dK_oh = w_0(f,g) \quad (16) \]

\[ K_oh + K_ohdh + dK_oh = w_1(f,g) + grad w_1(f,g) \quad (17) \]

### 3. Numerical Experiments and Results

A few numerical examples are presented to illustrate the feasibility of the method as described in the previous sections. In all these examples the exterior boundary was chosen according to real equipment, which consists of a circular Plexiglas pipe and 16 rectangular electrodes. We tested the algorithm with one unknown interior boundary, which is confined to: (a) circular, (b) elliptical, (c) heart-like and (d) dumbbell-like. Differential voltages on adjacent electrodes, which constitute a 13*18 matrix, are simulated by the finite element tool COMSOL. As to the inverse problem, for the sake of rapidity, the amount of discrete line segments is confined to 25, and iteration steps to less than 10. Results are shown in Fig. 3. The initial circular guess is roughly estimated according to the image of conductivity distribution, which is acquired by using the Linear Back Projection (LBP) Algorithm of ERT. It is
shown that the method can locate the gas bubbles and determine their shape effectively. Comparing the figures, the accuracy and the convergence speed are in connection with the shape of the gas phase. Since the common boundaries are discretized for approximation, the geometric coefficients on nodes usually don’t equal to 0.5, and hence affect the iterative process. If the geometric coefficients are uniform and close to 0.5, a comparatively accurate result is easily acquired; Otherwise, the accuracy and convergence speed will be little constrained where they pulse. In point of the electric field, the local electrical field strength where exists bigger reconstructive error, is so weak, and hence variation of this part of the boundary has little effect on the boundary measurements.

\[ RE = \sum_{i=1}^{16} \sum_{j=1}^{11} |V_i^j - \overline{V_i^j}| \]  \hspace{1cm} (18)

where \( V_i^j \) is the \( j \)-th voltage simulated of \( i \)-th excitation for current computed interior boundary, and \( \overline{V_i^j} \) denotes the given data. Fig. 4 shows the decrease of residual error while the number pf iterative steps increases for each kind of interior boundary in fig. 3. Three mesh sizes (20, 50, and 80 elements) are adopted to discuss the effect on the reconstructions.
The iterative processes converge more quickly in the anterior steps (less than 10 approximately) for relatively less elements, while in the posterior steps more mesh elements are favourable. The residual errors in the final iterations decrease as the number of elements increases, but this effect tends to be faint when the number of elements is large enough. The final residual errors differ little for the three mesh sizes in fig. 4(a), where the curve has a constant function of curvature. While in other subgraphs where the curvatures are similar in total to fig. 4(a), but partially with big bending, the final residual errors vary. Curvature of the boundaries has relation to the required mesh size. If the curve curls sharply we should refine the mesh and vice versa. Further research on mesh algorithm would be carried out.

4. CONCLUSIONS
Gas/water two-phase flows are widely encountered in nature and modern industry. A clearer understanding of two-phase flows requires getting the distribution of gas/water two-phase flows non-invasively. The ill-posed problem makes it hard to apply ERT to online quantitative multi-phase flow measurement. This approach is based on the BIEs of 2D laplace equation. BIEs, and also their derivatives, of non-smooth boundaries are deduced, and hence an iterative method is adopted. Four kinds of interior boundaries are tested which illustrate the feasibility and effectivity of the method. It shows that the accuracy and the convergence speed are in connection with the shape of the gas phase. The accuracy and convergence speed are little constrained where the geometric coefficients vary sharply along the common boundary. The residual error is defined to investigate the convergence process. In the anterior steps, the processes converge more quickly for fewer elements; while in the posterior steps more mesh elements is required to get better results. Element increment contributes to decrease the final residual errors, but it tends to be faint when the number of elements is large enough.
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NOMENCLATURE

\(D\) ~ simply connected bounded domain
\(R\) ~ real-space
\(u\) ~ electric potential
\(g\) ~ current density
\(n\) ~ unit normal
\(x\) ~ global 2-D coordinates
\(y\) ~ global 2-D coordinates
\(f\) ~ electric potential on \(\Gamma_0\)
\(h\) ~ electric potential on \(\Gamma_1\)
\(n_1\) ~ amount of segments of \(\Gamma_1\)
\(z\) ~ curve function
\(w\) ~ curvilinear Integral on \(\Gamma_0\)
\(K\) ~ curvilinear Integral on \(\Gamma_1\)
\(RE\) ~ residual error of boundary data
\(V\) ~ boundary voltage

Greek Letters
\(\Gamma\) ~ boundary
\(\alpha\) ~ geometric coefficient on interval (0,1)
\(\theta\) ~ the internal angel at point \(y\)
\(\sigma\) ~ electrical conductivity
\(\zeta\) ~ natural coordinates

Subscripts
\(i\) ~ \(i\)-th coordinates
\(0\) ~ exterior boundary
\(1\) ~ internal boundary

Superscript
\(e\) ~ \(e\)-th coordinates
\(2\) ~ 2-D
\(j\) ~ \(j\)-th coordinates

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