Another derivation of generalized Langevin equations

R. Dengler

July 12, 2016

Abstract

The formal derivation of Langevin equations (and, equivalently Fokker-Planck equations) with projection operator techniques of Mori, Zwanzig, Kawasaki and others apparently not has widely found its way into textbooks. It has been reproduced dozens of times on the fly with many references to the literature and without adding much substantially new. Here we follow the tradition, but strive to produce a self-contained text. Furthermore, we address questions that naturally arise in the derivation. Among other things the meaning of the divergence of the Poisson brackets is explained, and the role of nonlinear damping coefficients is clarified. The derivation relies on classical mechanics, and encompasses everything one can construct from point particles and potentials: solids, liquids, liquid crystals, conductors, polymers, systems with spin-like degrees of freedom ... Einstein relations and Onsager reciprocity relations come for free.

1 Notation

Coordinates in phase space are denoted by $\Gamma = \{p_n, q_n\}$, where $p$ are the momenta and $q$ the coordinates. A Hamiltonian $H(\Gamma)$ defines a flow in phase space according to

$$
\frac{d\Gamma}{dt} = [\Gamma, H] = iL\Gamma,
$$

where $[X, Y] = \sum \left( \frac{\partial X}{\partial q} \frac{\partial Y}{\partial p} - \frac{\partial X}{\partial p} \frac{\partial Y}{\partial q} \right)$ is the Poisson bracket and $L = i[H, ...]$ is the Liouville operator. The expression

$$
\Gamma(\Gamma, t) = e^{iLt}\Gamma
$$

is a function in phase space mapping every point $\Gamma$ to the value of the point a time step $t$ in the future. A simple example is $H = (p^2 + q^2)/2$ with $\Gamma(\Gamma, t) = (p(t), q(t)) = (p\cos t - q\sin t, q\cos t + p\sin t)$. More generally $A(\Gamma, t) = e^{iLt}A(\Gamma)$ is the value of an arbitrary function $A(\Gamma)$ at time $t$ when the system is in state $\Gamma$ at $t = 0$. The density of the phase space points of an ensemble is defined as

$$
\rho(\Gamma, t) = \langle \delta(\Gamma - \Gamma(\Gamma_0, t)) \rangle_{\Gamma_0} = \prod \delta(p_n - p_n(t)) \delta(q_n - q_n(t))_{\{p_n(0), q_n(0)\}},
$$

where the angular bracket denotes the ensemble average. The time derivative of $\rho$ at a given point $\Gamma$ is

$$
\frac{\partial}{\partial t} \rho(\Gamma, t) = \frac{\partial \rho}{\partial p} (\dot{p}) + \frac{\partial \rho}{\partial q} (\dot{q}) = \frac{\partial \rho}{\partial p} \frac{\partial H}{\partial q} - \frac{\partial \rho}{\partial q} \frac{\partial H}{\partial p} = [H, \varrho] = -iL\rho.
$$
A formal solution of this equation is \( \varrho (\Gamma, t) = e^{-iL_t} \varrho (\Gamma, 0) \). Of particular interest is the probability

\[
p(a, t) = \int d\Gamma \delta (a - A(\Gamma)) \varrho (\Gamma, t) = \int d\Gamma \delta (a - A(\Gamma, t)) \varrho (\Gamma)
\]

to observe values \( a = \{a_i\} \) of observables \( A = \{A_i\} \) at time \( t \) when the ensemble is in a state with phase space density \( \varrho (\Gamma) \) at time \( t = 0 \). The equilibrium distribution function of observables is written in the form

\[
p_0 (a) \, da = \text{const} \times e^{-\beta H_{eff}(a)} \, da
\]

where \( H_{eff}(a) \) is called effective Hamiltonian.

## 2 Projection Operator

The Zwanzig projection\[1\] operator operates on functions in the 6-\(N\)-dimensional phase space of \( N \) point particles and projects onto the linear subspace of "slow" phase space functions. It was introduced by R. Zwanzig to derive a generic master equation. It is mostly used in this or similar context in a formal way to derive equations of motion for some "slow" collective variables. A special subset of these functions is an enumerable set of "slow variables" \( A(\Gamma) = \{A_n(\Gamma)\} \). Candidates for some of these variables might be the long-wavelength Fourier components of conserved quantities like energy, momentum and mass density. Relaxation of such quantities requires a transport of the conserved quantity over the distance of a wavelength, and thus is much slower than the relaxation of local quantities, which typically occurs within a few collision times. Another example is the order parameter of a second order phase transition, the dynamics of which slows down at the critical point. The Zwanzig projection operator relies on these functions but doesn’t tell how to find the slow variables of a given Hamiltonian \( H(\Gamma) \).

### 2.1 Slow variables and scalar product

A scalar product between two phase space functions \( f_1(\Gamma) \) and \( f_2(\Gamma) \) is defined by the equilibrium correlation function\[3\]

\[
\langle f_1 f_2 \rangle_0 = \int d\Gamma \rho_0 (\Gamma) \, f_1 (\Gamma) \, f_2 (\Gamma),
\]

where \( \rho_0 (\Gamma) = Z^{-1} e^{-\beta H(\Gamma)} \) is the canonical equilibrium distribution. "Fast" variables, by definition, are orthogonal to all functions \( G(A(\Gamma)) \) of \( A(\Gamma) \) under this scalar product. This definition states that fluctuations of fast and slow variables are uncorrelated, and according to the ergodic hypothesis this also is true for time averages. If a generic function \( f(\Gamma) \) is correlated with some slow variables, then one can subtract functions of slow variables until there remains the uncorrelated fast part of \( f(\Gamma) \). The definition also means that the average of a fast variable vanishes. The product of a slow and a fast variable is a fast variable.

The concise notation

\[
\langle f \rangle_0 = \int d\Gamma' \rho_0 (\Gamma') \, f (\Gamma')
\]

which completely hides the integration variables is used systematically below.
2.2 The projection operator

Consider the continuous set of functions $\Phi_a (A (\Gamma)) = \delta (A (\Gamma) - a) = \prod_n \delta (A_n (\Gamma) - a_n)$ with $a = \{a_n\}$ constant. Any phase space function $G (A (\Gamma))$ depending on $\Gamma$ only through $A (\Gamma)$ is a function of the $\Phi_a$, namely

$$G (A (\Gamma)) = \int da G (a) \delta (A (\Gamma) - a).$$

A generic phase space function $f (\Gamma)$ decomposes according to

$$f (\Gamma) = F (A (\Gamma)) + R (\Gamma),$$

where $R (\Gamma)$ is the fast part of $f (\Gamma)$. To get an expression for the slow part $F (A (\Gamma))$ of $f$ take the scalar product (2) with the slow function $\delta (A (\Gamma) - a)$,

$$\langle \delta (A - a) f \rangle_0 = \langle \delta (A - a) F (A) \rangle_0 = F (a) \langle \delta (A - a) \rangle_0.$$

This gives an expression for $F (a)$, and thus for the operator $P$ projecting an arbitrary function $f (\Gamma)$ to its "slow" part depending on $\Gamma$ only through $A (\Gamma)$,

$$P \cdot f (\Gamma) = F (A (\Gamma)) = \frac{\langle \delta (A - A (\Gamma)) f \rangle_0}{\langle \delta (A - A (\Gamma)) \rangle_0}. \quad (3)$$

This expression agrees with the expression given by Zwanzig[1], except that Zwanzig uses a microcanonical ensemble and subsumes $H (\Gamma)$ under the slow variables. The Zwanzig projection operator fulfills $P \cdot G (A (\Gamma)) = G (A (\Gamma))$ and $P^2 = P$. The fast part of $f (\Gamma)$ is $(1 - P) f (\Gamma)$. Functions of slow variables and in particular products of slow variables are slow variables. The space of slow variables thus is an algebra. The algebra in general is not closed under the Poisson bracket, including the Poisson bracket with the Hamiltonian. The projection operator (3) can also be written in the form

$$P \cdot f (\Gamma) = \int \frac{da}{p_0 (a)} \delta (a - A (\Gamma)) \langle \delta (a - A) f \rangle_0 \quad (4)$$

where $p_0 (a) = \langle \delta (a - A) \rangle_0$ is the equilibrium distribution (1) of $A$.

3 Exact generalized Langevin Equations

The starting point for the standard derivation of a Langevin equation is the identity $1 = P + Q$, where $Q$ projects onto the fast subspace. It is important in the following that the Liouville operator normally doesn’t commute with the Zwanzig projection operator - time evolution mixes fast and slow variables, $[L, P] \neq 0$.

Consider discrete small time steps $\tau$ with evolution operator $U \simeq 1 + i \tau L$, where $L$ is the Liouville operator. The goal is to express $U^n$ in terms of $U^k P$ and $Q (UQ)^m$. The motivation is that $U^k P$ is a functional of slow variables and that $Q (UQ)^m$ generates expressions which are fast variables at every time step. The expectation is that fast variables isolated in this way can be represented by some model data, for instance a Gaussian white noise. The

1A heat bath in contact with the boundary of the system is irrelevant for the dynamics in the inner parts of the system. The only difference between canonical and microcanonical ensemble is that the microcanonical ensemble doesn’t have the $k = 0$ degree of freedom of the energy density.
decomposition is achieved by multiplying $1 = P + Q$ from the left with $U$, except for the last term, which is multiplied with $U = PU + QU$. Iteration gives

$$1 = P + Q,$$
$$U = UP + PUQ + QUQ,$$
$$U^2 = U^2P + UPUQ + PUQU + QUQUQ,$$
$$... = ...$$
$$U^n = U^nP + \sum_{m=1}^{n} U^{n-m}P (UQ)^m + Q (UQ)^n.$$  

The last line can also be proved by induction. Now assume $U = 1 + itL/n$ and perform the limit $n \to \infty$. This directly leads to the operator identity of Kawasaki\cite{2}

$$e^{itL} = e^{itL}P + i \int_0^t ds e^{i(t-s)L}PLQe^{isLQ} + Qe^{itLQ}.$$  

A generalized exact Langevin equation is obtained by applying this equation to $dA(\Gamma, t)/dt = e^{itL} (dA(\Gamma, t)/dt)_{t=0}$,

$$\frac{d}{dt} A(\Gamma, t) = V + K + R,$$
$$V(\Gamma, t) = e^{itL}P \dot{A}(\Gamma, 0),$$
$$K(\Gamma, t) = i \int_0^t ds e^{i(t-s)L}PLQe^{isLQ} \dot{A}(\Gamma, 0) = i \int_0^t ds e^{i(t-s)L}PLR(\Gamma, s),$$
$$R(\Gamma, t) = Qe^{itLQ} \dot{A}(\Gamma, 0).$$

The contribution $R$ is the fluctuating force. The $K$ term is the damping, it simplifies with the expression for the fluctuating force. The contribution $V$ is called mode coupling. It corresponds to reversible modes like spin or plasma waves. Mode coupling term $V$ and damping term $K$ are functionals of $A(\Gamma, t)$ and $A(\Gamma, t - s)$ respectively and can be simplified considerably.

### 3.1 Mode Coupling

Inserting the expression (4) for the Zwanzig projection into the $V$-term from eq.(5) gives

$$V_i(\Gamma, t) = e^{itL}P [A, H] = \int \frac{da}{p_0(a)} \delta (a - A(\Gamma, t)) \left\langle [A_i, H] \delta (a - A) \right\rangle_0.$$  

The expectation value $\langle ... \rangle_0$ simplifies with $[A_i, H] \rho_0 = -(1/\beta) [A_i, \rho_0]$ and partial integration to

$$\langle ... \rangle_0 (a) = \frac{-1}{\beta} \int d\Gamma' [A_i, \rho_0] \delta (a - A(\Gamma')) = \frac{-1}{\beta} \left\langle \delta (a - A), A_i \right\rangle_0$$
$$= \frac{-1}{\beta} \left\langle \sum_j \frac{d}{dA_j} \delta (a - A) [A_j, A_i] \right\rangle_0$$
$$= \frac{-1}{\beta} \sum_j \frac{d}{da_j} \left\langle \delta (a - A) [A_i, A_j] \right\rangle_0 = \frac{-1}{\beta} \sum_j \frac{d}{dA_j} P[A_i, A_j] \rho_0 (A) \bigg|_{A=a}.$$  

\footnote{The factor $UQ$ is rather formal. It means projecting away the slow part of the observable after each time step. Hamiltonian and the trajectory of the system point in phase space are unchanged.}
Inserting \( \langle ... \rangle_0 \) into (6) and writing \( p_0(a) = \text{const} \times e^{-\beta H_{\text{eff}}(a)} \) finally gives

\[
V_i(\Gamma, t) = \frac{-k_B T}{p_0(A)} \sum_j \frac{d}{dA_j} P[A_i, A_j] p_0(A) \Big|_{A=A(\Gamma, t)} = \sum_j P[A_i, A_j] \frac{dH_{\text{eff}}}{dA_j} \Big|_{A(\Gamma, t)} + D_i,
\]

\[
D_i(\Gamma, t) = -k_B T \sum_j \frac{d}{dA_j} P[A_i, A_j] \Big|_{A=A(\Gamma, t)}.
\]

(7)

The mode coupling (7) is a functional of \( A(\Gamma, t) \), and the Langevin equation (5) with only \( V \) on the r.h.s. agrees with the classical (reversible) equation of motion of variables \( A \) except for the purely kinematic divergence \( D(A) \). In fact, \( D \) actually vanishes in most standard cases and it is reasonable to suggest to use variables \( A \) for which \( D = 0 \). The meaning of this condition is explained in detail in the appendix.

### 3.2 Damping term

Inserting the expression (4) for the projection operator into the damping term \( K_i \) from eq.(5) gives

\[
K_i = \int_0^t ds \int \frac{da}{p_0(a)} \delta(A(t-s) - a) \left[ \delta(a - A) iLR_i(s) \right]_0.
\]

The Liouville operator in the expectation value \( \langle ... \rangle_0 \) can be moved to the \( \delta \)-function by means of a partial integration, which gives the negative time derivative of the delta function,

\[
-\frac{d}{dt} \delta(a - A) = -\sum_j \frac{d\delta(a - A)}{dA_j} \dot{A}_j = \sum_j \frac{d\delta(a - A)}{da_j} \dot{A}_j
\]

and thus

\[
K_i = \int_0^t ds \int \frac{da}{p_0(a)} \delta(A(t-s) - a) \sum_j \frac{d}{da_j} \left[ \delta(a - A) \dot{A}_j(0) R_i(s) \right]_0.
\]

The product of the slow part of \( \dot{A}_j \) and \( \delta(a - A) \) is a slow variable and doesn’t contribute to \( \langle ... \rangle_0 \). It therefore can be replaced with \( Q \dot{A}_j(0) = R_j(0) \) from eq.(5). There results the exact expression

\[
K_i(\Gamma, t) = \sum_j \int_0^t ds \left( \frac{1}{p_0(a)} \frac{d}{da_j} p_0(a) \Lambda_{j,i}(a, s) \right)_{a=A(t-s)}, \tag{8}
\]

where

\[
\Lambda_{j,i}(a, s) = \frac{1}{p_0(a)} \left[ \delta(a - A) R_j(0) R_i(s) \right]_0 = \left[ R_j(0) R_i(s) \right]_0 \tag{9}
\]

are the correlation functions of the fluctuating forces restricted to a subspace with given values \( a \) for the slow variables. \( K(\Gamma, t) \) in effect is a functional of \( A(t-s) \) and \( \Lambda_{j,i}(A(t-s), s) \).

The exact evolution \( A(t-s) \) of \( A \) over the time interval \([s, t]\) here is uninteresting, the non-trivial part is the evolution \( \dot{R}(s) \) of the fluctuating force \( R \) over the time interval \([0, s]\). If the variables \( A \) change much more slowly than \( R \) then one can think of \( \Lambda_{j,i}(a, s) \) as a time average in a situation where the variables \( A \) have the value \( a \).
3.3 Fluctuating force

The fluctuating force \( R(t) = R(\Gamma, t) \) is a fast variable according to definition (5) at every time \( t \) and therefore \( \langle R(t) \rangle_0 = 0 \). It also is easy to verify that \( R \) is a stationary process: the correlation function
\[
\langle R_i(s) R_j(t+s) \rangle_0 = \langle Qe^{isL^Q} R_i(0) \cdot Qe^{i(t+s)L^Q} R_j(0) \rangle_0 = \langle R_i(0) \cdot Qe^{itL^Q} R_j(0) \rangle_0 = \langle R_i(0) R_j(t) \rangle_0
\]
is invariant under an arbitrary time shift \( s \). The proof only requires the fairly trivial rules \( Q^2 = Q \), \( \langle XQY \rangle_0 = \langle (QX)Y \rangle_0 \) and \( \langle XLY \rangle_0 = - \langle (LX)Y \rangle_0 \) to move operators from a factor \( X \) to a factor \( Y \).

Time inversion symmetry then shows that the correlation function is a symmetric matrix
\[
\chi_{i,j}(t) = \langle R_i(0) R_j(t) \rangle_0 = \langle R_i(0) R_j(-t) \rangle_0 = \langle R_i(t) R_j(0) \rangle_0 = \chi_{j,i}(t).
\]
Because \( R \) is a fast variable \( \chi_{i,j}(t) \) decays relatively fast, in a time of order \( \tau \). But it is not necessarily true that this decay can be described by a process independent of \( A \). The variables \( A \) are quasi constant in a time interval \( \tau \), and the fast variables effectively are restricted to a smaller phase space. A more detailed equation which takes this dependence into account is
\[
\chi_{i,j}(t) = \int d a p_0(a) \Lambda_{i,j}(a,t),
\]
\[
\Lambda_{i,j}(a,t) = \frac{1}{p_0(a)} \langle \delta(a-A) R_i(0) R_j(t) \rangle_0 = \langle R_i(0) R_j(t) ; a \rangle_0,
\]
where \( \Lambda \) is a correlation function in which \( H \) and \( A \) are constant. The interesting aspect is that this correlation is identical with the correlation function in the damping term (8).

4 Markov approximation

Further simplifications are possible if the slow variables really are slow in comparison to the fast variables. In that case it suggests itself to replace the unknown fast variables \( R(\Gamma, t) \) with a Gaussian white noise. The memory effects in the damping (8) disappear and there results the Langevin equation
\[
\frac{dA_i}{dt} = \sum_j P[A_i, A_j] \frac{dH_{eff}}{dA_j} - \sum_j \lambda_{i,j}(A) \frac{dH_{eff}}{dA_j} + k_B T \sum_j \frac{d\lambda_{i,j}}{dA_j} + r_i(t), 
\]
\[
\langle r_i(0) r_j(t) ; A \rangle = 2k_B T \lambda_{i,j}(A) \delta(t) .
\]
This is the final result. It contains Onsager reciprocity \( \lambda_{i,j} = \lambda_{j,i} \) and the Einstein relations (11). A special feature of this Langevin equation is the extra term proportional to \( k_B T \) on the r.h.s. The formal derivation explicitly allows damping coefficients depending on the slow variables \( A \). The extra term is a consequence. This unconventional term is of order \( O(A^0) \)

\[3\text{As usual, if external magnetic fields are important then time inversion symmetry also requires to invert the external currents generating the magnetic fields.}\]
for a linear dependence of $\lambda$ on $A$ and appears to be large, but this is misleading. As can be seen from
\[
\int dA p_0 (A) \left( - \sum_j \lambda_{i,j} (A) \frac{dH_{\text{eff}}}{dA_j} + k_B T \sum_j \frac{d\lambda_{i,j} (A)}{dA_j} \right) = 0
\]
the extra term cannot be seen in isolation and is required to have $\langle dA/dt \rangle = 0$. Furthermore, as can be seen from
\[
\frac{1}{n} \sum \langle A_j \frac{dH_{\text{eff}}}{dA_j} \rangle = -\frac{k_B T}{n} \sum \int d a a_j \frac{d}{d a_j} p_0 (a) = k_B T
\]
the extra term actually is small in comparison to the second term if the $A$-dependence of $\lambda$ is small. In that case it thus is justified to use a conventional Langevin equation with constant damping coefficients and without the extra term.

It also is of interest that the dependence of the damping coefficients on the slow variables in a way is trivial at the level of the Langevin equation. Constant coefficients $\lambda^{(A)} (A)$ transform to non-constant coefficients $\lambda^{(B)} (B)$ in the Langevin equation for variables $B$ which are in (nonlinear) bijection with $A$. However, the transformation law of the extra term is rather complicated, see appendix.

5 Summary and perspective

This hopefully sufficiently self-contained article has reproduced the derivation of generalized nonlinear Langevin equations from classical statistical mechanics. Questions addressed are the stationarity of the fluctuating forces and the relevance of unconventional extra terms caused by the divergence of Poisson brackets and nonlinear damping coefficients.

In general the formalism doesn’t leave much to be desired. A possible caveat is the division of the degrees of freedom according to the category fast and slow. There rarely is a clear-cut limit and the division is more or less arbitrary. Repeating the derivation with a shifted limit generates a Langevin equation with modified parameters. This is equivalent to the renormalization group, and the question is whether the formalism generates a simple Langevin equation of definite form in the end. In the case of critical phenomena this definitely is the case, the fixed points of the renormalization group are of the Langevin equation type, albeit with nontrivial coefficients. The formalism, however, in principle works with any projection operator, but interpretation and application of the generalized Langevin equations then may be difficult.

Appendix

Divergence of Poisson brackets

The mode coupling (7) contains a term reproducing the classical mechanics of $A$ with a Hamiltonian $H_{\text{eff}} (A)$ but also an additional purely kinematic term containing the Poisson bracket divergence
\[
D_i (\Gamma, t) = -k_B T \sum \frac{d}{dA_j} P [A_i, A_j].
\]
The meaning of this extra contribution can be elucidated by repeating the derivation of the mode coupling term for slow variables $B$ which are in bijection with $A$. The set of slow variables effectively is the same and the projection operator is unchanged.\(^4\) The new aspect

\(^4\)A special case of this is $P = 1$ where $A$ and $B$ are in bijection with $\Gamma$, without any fast variables.
is that now the probability distribution
\[ p_0^{(B)} (B) = \text{const} \times \exp \left( -\beta H_{\text{eff}}^{(B)} (B) \right) = p_0 (A) \left| dA/dB \right| \]
(12)
of the variables \( B \) can be related to the probability distribution of the variables \( A \). This
means that \( H_{\text{eff}} (A) \) and \( H_{\text{eff}}^{(B)} (B) \) cannot agree if the Jacobian isn’t constant.

Theorem: If \( A \) and \( B \) are in bijection then the following three statements are equivalent:
\[ \forall \sum_i \frac{d}{dA_j} [A_i, A_j] = 0 \iff \forall G(A) \sum_j \frac{d}{dA_j} [G, A_j] = 0 \iff \forall \sum_i \frac{d}{dB_j} [B_i, B_j] \left| \frac{dA}{dB} \right| = 0. \]

The connection between first and second statement is trivial. The geometric meaning is
that every \( G(A) \) generates a divergenceless Hamiltonian flow in \( A \)-space. The first statement
implies that the mode coupling for \( A \) is of standard form (\( D = 0 \)) and that the mode coupling
for \( B \) also can be written in a simple form,
\[ \frac{dA_i}{dt} = \sum_j [A_i, A_j] \frac{dH_{\text{eff}} (A)}{dA_j}, \]
\[ \frac{dB_i}{dt} = \sum_j [B_i, B_j] \frac{dH_{\text{eff}} (A (B))}{dB_j}. \]
(13)
On the other hand, the formal derivation of an equation of motion for \( B \) as above gives the
mode coupling
\[ \frac{dB_i}{dt} = -k_B T \frac{1}{p_0^{(B)} (B)} \sum_j \frac{d}{dB_j} [B_i, B_j] p_0^{(B)} (B) \]
\[ = \sum_j [B_i, B_j] \frac{dH_{\text{eff}} (A (B))}{dB_j} - k_B T \sum_j \frac{d}{dB_j} [B_i, B_j] \left| \frac{dA}{dB} \right|, \]
where (12) was used. Comparison with the mode coupling (13) deduced from the equation
of motion for \( A \) proves the third statement. Equivalence with the first statement follows by
symmetry.

The meaning of all this is that the unconventional extra term \( D \) is associated with the
Jacobian of the variable transformation. If the Jacobian is constant and \( D (A) = 0 \) then
also \( D (B) = 0 \). In other words, the extra term arises when the exponent of the probability
distribution isn’t the energy.

The fact that \( D = 0 \) in standard cases can be understood from another point of view.
Often the variables \( A \) directly are related to the generators of a Lie algebra. For example,
for Heisenberg ferromagnets (model ‘\( J \)’ of critical dynamics) \( A = \{m_1, m_2, m_3\} \), where the
\( m_i \) are the components of the magnetization density. The Poisson brackets then generate the
Lie algebra (decorated with a \( \delta \)-function)
\[ [A_i (x), A_j (x')] = \sum_m h_{j,m}^{(i)} A_m (x) \delta (x - x'), \]
where the structure constants \( h \) are a matrix representation of the algebra (the adjoint
representation). According to a theorem of group theory real representations of finite and
compact groups are equivalent to orthogonal representations and thus have vanishing trace.
This in part explains why standard Langevin equations don’t have the unconventional extra
contribution.
Variable transformations in Langevin equations

Under a bijection of variables $A \leftrightarrow B$ one has

$$\Delta B_i = \sum \frac{dB_i}{dA_m} \Delta A_m + \frac{1}{2} \sum \frac{d^2B_i}{dA_m dA_n} \Delta A_m \Delta A_n + \ldots$$

Dividing by a time interval $\Delta t$ and inserting the Langevin equation (10) leads to the Langevin equation for $B$,

$$\frac{dB_i}{dt} = \sum \frac{dB_i}{dA_m} \cdot \left( \frac{dA_m}{dt} \right) + k_B T \sum \frac{d^2B_i}{dA_m dA_n} \lambda^{(A)}_{m,n}.$$  

A word of caution is in order here: this is not standard analysis! $B$ has to be expanded to second order in $A$ because the product of two fluctuating forces from the r.h.s. of the Langevin equation for $A$ is a delta function, see eq.(10).

This should agree with the Langevin equation for $B$ a la eq.(10) derived directly. There follow the identifications

$$\gamma_i^{(B)}(t) = \sum m \frac{dB_i}{dA_m^{(A)}}(t),$$

$$\lambda^{(B)}_{i,j} = \sum_{m,n} \frac{dB_i}{dA_m} \frac{dB_j}{dA_n} \lambda^{(A)}_{m,n}.$$  

The transformation law of the extra term can be found with the divergence rule

$$\sum_j \frac{d}{dB_j} d_j^{(B)}(t) = \left| \frac{dA}{dB} \right| \sum \frac{d}{dA_n} \frac{dB_{(B)}}{dB_j} \left| \frac{dB}{dA} \right|$$

according to

$$\sum_j \frac{d\lambda^{(B)}_{i,j}}{dB_j} = \left| \frac{dA}{dB} \right| \sum \frac{d}{dA_n} \frac{dB_{(B)}}{dB_j} \lambda^{(B)}_{i,j} \left| \frac{dA}{dB} \right| = \left| \frac{dA}{dB} \right| \sum \frac{d}{dA_n} \frac{dB_i}{dA_m} \lambda^{(A)}_{m,n} \left| \frac{dA}{dB} \right|$$

$$= \sum_{m,n} \frac{dB_i}{dA_m} \left( \frac{d\lambda^{(A)}_{m,n}}{dA_n} + \lambda^{(A)}_{m,n} \frac{d}{dA_n} \ln \left| \frac{dB_{(B)}}{dA} \right| \right) + \sum_{m,n} \frac{d^2B_i}{dA_m dA_n} \lambda^{(A)}_{m,n}.$$  

Even if the original damping coefficients $\lambda^{(A)}$ are constant then a nonlinear variable transformation generates non-constant damping coefficients $\lambda^{(B)}$. However, slow variables normally have a certain meaning (conserved quantities, order parameter, ...), and a nonlinear transformation thus is rather formal.

References

[1] R. Zwanzig, Phys. Rev. 124 (1961) 983. "Memory Effects in Irreversible Thermodynamics"

[2] K. Kawasaki 1973 J. Phys. A: Math. Nucl. Gen. 6 1289. “Simple derivations of generalized linear and nonlinear Langevin equations”

[3] H. Mori, Prog. Theor. Phys. 33 (1965) 423. "Transport, Collective Motion, and Brownian Motion"

Current address: ROHDE & SCHWARZ GMBH & CO KG, MÜHLDORFSTR. 15, 81671 MUNICH, P.O.B. 801469.

E-mail: rdengler@cablemail.de