PERFECT MATCHINGS OF LINE GRAPHS WITH SMALL MAXIMUM DEGREE

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Abstract. Let $G$ be a connected graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_\nu\}$, which may have multiple edges but have no loops, and $2 \leq d_G(v_i) \leq 3$ for $i = 1, 2, \cdots, \nu$, where $d_G(v)$ denotes the degree of vertex $v$ of $G$. We show that if $G$ has an even number of edges, then the number of perfect matchings of the line graph of $G$ equals $2^n/2+1$, where $n$ is the number of 3-degree vertices of $G$. As a corollary, we prove that the number of perfect matchings of a connected cubic line graph with $n$ vertices equals $2^{n/6}+1$ if $n > 4$, which implies the conjecture by Lovász and Plummer holds for the connected cubic line graphs. As applications, we enumerate perfect matchings of the Kagomé lattices, 3.12.12 lattices, and Sierpinski gasket with dimension two in the context of statistical physics.

1. Introduction

Throughout this paper, we suppose that $G = (V(G), E(G))$ is a connected graph with the vertex set $V(G) = \{v_1, v_2, \cdots, v_\nu\}$ and the edge set $E(G)$ which may have multiple edges but have no loops, if not specified. The line graph of $G$, denoted by $L(G)$, is defined as the graph whose vertex set $V(L(G)) = E(G)$ and two vertices $e$ and $f$ in $L(G)$ are joined by $i$ ($i = 0, 1, 2$) edges if and only if two edges $e$ and $f$ in $G$ have $i$ end vertices in common. A perfect matching of $G$ is a set of independent edges of $G$ covering all vertices of $G$. Denote the number of perfect matchings of $G$ by $M(G)$. It is well known that computing $M(G)$ of a graph $G$ is an $NP$-hard problem (see [11, 15, 20]).

Let $G$ be a graph with $\nu$ vertices and let $G_0, G_1, \cdots, G_k$ be graphs such that $G_0 = G$ and, for each $i > 0$, $G_i$ can be obtained from $G_{i-1}$ by subdividing an edge once. Then $G_k$ is said to be a subdivision of $G$. For convenience, we can regard $G$ as a subdivision of $G$. Let $S(G)$ denote the graph obtained from $G$ by subdividing every edge once.

2000 Mathematics Subject Classification. Primary 05C15, 05C16.

Key words and phrases. Perfect matching; Cubic graph; Line graph; Kagomé lattice; 3.12.12 lattice; Sierpinski gasket.

The first author was supported in part by NSFC Grant (10771086) and by Program for New Century Excellent Talents in Fujian Province University.

The second author was supported in part by NSFC Grant #10831001.
A classical theorem of Petersen [17] asserts that every cubic graph without a cut-edge has at least a perfect matching. This result can be derived as a corollary of Tutte’s 1-factor theorem [15]. Edmonds, Lovász, and Pulleyblank [5] and Naddef [16] proved that each cubic graph with \( \nu \) vertices has at least \( \frac{\nu}{4} + 2 \) perfect matchings if it has no cut-edge and has at least \( \frac{\nu}{2} + 1 \) perfect matchings if it is cyclically 4-edge-connected. Similar bounds can be achieved using Lovász’ matching lattice theorem [14]. Recently, Kral, Sereni, and Stiebitz [12] improved the lower bound for graphs without a cut-edge to \( \frac{\nu}{2} \). In the 70’s, Lovász and Plummer conjectured (in the mid-1970s) that every cubic graph with no cut-edge has exponentially many perfect matchings. Voorhoeve [21] proved this conjecture for bipartite cubic graphs. He proved that every cubic bipartite graph \( G \) with no cut-edge has at least \( 6 \cdot \left(\frac{4}{3}\right)^{\frac{\nu}{2} - 3} \) perfect matchings, where \( \nu \) is the number of vertices of \( G \). Furthermore, Lovász and Plummer [15] conjectured that for \( k \geq 3 \) there exist constants \( c_1(k) > 1 \) and \( c_2(k) > 0 \) such that every \( k \)-regular elementary graph (i.e., 1-extendable graph) with \( 2\nu \) vertices contains at least \( c_2(k)c_1(k)^{\nu} \) perfect matchings. Schrijver [19] proved this conjecture for the \( k \)-regular bipartite graphs. He proved that for \( k > 2 \) every \( k \)-regular bipartite graph \( G \) with \( 2\nu \) vertices has at least \( \nu! \left(\frac{k}{\nu}\right)^\nu \left(\geq \sqrt{2\pi} \left(\frac{\nu}{e}\right)^\nu\right) \) perfect matchings.

The above conjectures have been proved to be challenging questions, and are still open. As far as we know the latest result of the first conjecture for the case of the non-bipartite graphs was achieved by Chudnovsky and Seymour [2] who proved that every cubic planar graph \( G \) with no cut-edge has at least \( 2^{\nu/6559782652} \) perfect matchings, where \( \nu \) is the number of vertices (with degree three) of \( G \).

Inspiring by these discussions, we attempt to find some kind of graphs the number of perfect matchings of which can be determined by the number of vertices of degree three. In the next section, we prove that if \( G = (V(G), E(G)) \) is a connected graph with an even number of edges, which may have multiple edges but have no loops, and satisfies \( 2 \leq d_G(v) \leq 3 \), for \( v \in V(G) \), then the number of perfect matchings of the line graph of \( G \) equals \( 2^{n/2+1} \), where \( n \) is the number of 3-degree vertices of \( G \). As a corollary, we also prove that the number of perfect matchings of a connected cubic line graph with \( n \) vertices equals \( 2^{n/6+1} \) if \( n > 4 \). As applications, in Section 3 we enumerate perfect matchings of the Kagomé lattices, 3.12.12 lattices, and Sierpinski gasket with dimension two in the context of statistical physics. Finally, in Section 4 we give some remarks.
2. Main results

We first introduce some lemmas. Let $G$ be a graph and $u$ a vertex of $G$. Let $X \cup Y$ be a partition of the edges incident $u$. For an edge $e$ incident to $u$, let $\phi(e)$ be the other endpoint of $e$. Construct a new graph $G'$ from $G$ as follows (see Figure 1):

(i) remove $u$ and the incident edges and insert three new vertices $u'$, $u''$ and $x$;

(ii) connect $x$ to $u'$ and $u''$ by an edge, and for $e \in X$, connect $u'$ to $\phi(e)$ by an edge, and for $e \in Y$, connect $u''$ to $\phi(e)$ by an edge.

For convenience, we say that $G'$ is obtained from $G$ by splitting vertex $u$. The following lemma is immediate from Lemma 1.3 in [3].

**Lemma 2.1.** Let $G$ and $G'$ be the graphs defined as above. Then

$$M(G) = M(G').$$

Figure 2.

(a) A graph $G$; (b) the line graph $L(G)$ and $G$ (dotted lines); (c) the graph $G^*$; (d) the graph $G_s$; (e) the line graph $L(G_s)$ and $G_s$ (dotted lines).
Lemma 2.2. Suppose $G$ is a connected graph and $|E(G)|$ is even. Let $e = (u, v)$ be an edge of $G$ and $d_G(u) \geq 2, d_G(v) \geq 2$. For any non negative integer $s \geq 0$, let $G_s$ be the graph obtained from $G$ by subdividing edge $e$ $2s$ times ($G_0 = G$). Then

(a) $M(G) = M(G_s)$;

(b) $M(L(G)) = M(L(G_s))$ for $s = 0, 1, 2, \ldots$.

Proof. From Lemma 2.1, (a) is immediate. Hence it suffices to prove (b).

Suppose $d_G(u) = p \geq 2, d_G(v) = q \geq 2$. Let $e, f_1, f_2, \ldots$, and $f_{p-1}$ be the $p$ edges incident with vertex $u$, and let $e, g_1, g_2, \ldots$, and $g_{q-1}$ be the $q$ edges incident with vertex $v$ (see Figure 2(a), where $p = q = 3$). Hence $(e, f_1), (e, f_2), \ldots, (e, f_{p-1}), (e, g_1), \ldots, (e, g_{q-1})$ are $p + q - 2$ edges in $L(G)$ (see Figure 2(b)). Let $G^*$ be the graph obtained from $L(G)$ by splitting vertex $e$, which is illustrated in Figure 2(c). By Lemma 2.1,

$$M(L(G)) = M(G^*). \quad (1)$$

Since $G_s$ is the graph obtained from $G$ by subdividing edge $e$ $2s$ times, denote these $2s$ subdividing vertices by $v_1, v_2, \ldots, v_{2s}$ in turn (see Figure 2(d)). Let $v_0 = u$ and $v_{2s+1} = v,$ and $e_i = (v_{i-1}, v_i), i = 1, 2, \ldots, 2s + 1$. Obviously, $e_1 - e_2 - \cdots - e_{2s+1}$ is a path with $2s + 1$ vertices in $L(G)$ and $d_{L(G)}(e_i) = 2$ for $i = 2, 3, \cdots, 2s$ (see Figure 2(e)). Clearly,

$$M(L(G_s)) = M(G^*). \quad (2)$$

Then, by (1) and (2), (b) holds. \qed

Similarly, we can prove the following:

Lemma 2.3. Suppose $G$ is a connected graph. Let $e = (u, v)$ be an edge of $G$ and $d_G(u) \geq 2, d_G(v) \geq 2$. For any non negative integer $s \geq 0$, let $G^{(s)}$ be the graph obtained from $G$ by subdividing edge $e$ $2s + 1$ times. Then

$$M(L(G^{(1)})) = M(L(G^{(2s+1)})), s \geq 1.$$

Now we state and prove our main result as follows:

Theorem 2.4. Let $G$ be a connected graph with vertex set $V(G)$, which may have multiple edges but have no loops, and $2 \leq d_G(v) \leq 3$, for $v \in V(G)$. If $G$ has an even number of edges, then the number of perfect matchings of the line graph of $G$ equals $2^{n/2+1}$, where $n$ is the number of 3-degree vertices of $G$. 
Proof. We prove the theorem by induction on the number of 3-degree vertices of $G$. If $G$ has no 3-degree vertex, i.e., $n = 0$, then $G$ is a cycle with an even number of edges. Hence $M(L(G)) = 2^{0/2+1} = 2$.

Note that $G$ has an even number of 3-degree vertices. Now we assume that $n \geq 2$. Suppose $u$ and $u'$ are two 3-degree vertices of $G$. Since $G$ is connected, there exists one path $P(u-u')$: $u = v_0 - v_1 - \cdots - v_k = u'$ in $G$. Let $j + 1 = \min\{i | d_G(v_i) = 3, 1 \leq i \leq k\}$ and $v = v_{j+1}$. Then $1 \leq j + 1 \leq k$. Hence there exists one path $P(u-v)$: $u = v_0 - v_1 - \cdots - v_{j+1} = v$ in $G$ such that $d_G(u) = d_G(v) = 3$, and $d_G(v_1) = d_G(v_2) = \cdots = d_G(v_j) = 2$ if $j > 0$. By Lemmas 2.2 and 2.3, it suffices to consider two cases $j = 0$ and $j = 1$.

Case 1 $j = 0$.

Obviously, $e_1 = (u, v)$ is an edge of $G$. If $G$ has three multiple edges connecting vertices $u$ to $v$, then $G$ has exactly three edges. This is a contradiction with $|E(G)|$ is even. Hence we need to consider the following two subcases:

Subcase 1.1 $G$ has two multiple edges $e_1$ and $e_2$ connecting vertices $u$ and $v$ (see Figure 3(a)).

Let $e_1, e_2$, and $e_3 = (u, u_1)$ (resp., $e_1, e_2$, and $e_4 = (u, u_2)$) be the three edges incident with vertex $u$ (resp., vertex $v$) in $G$ (see Figure 3(a)), and $u_1 \neq v, u_2 \neq u$. Construct a new graph $G'$ from $G$ by deleting vertex $v$ and connecting $u$ and $u_2$ by an edge. Hence $G'$ is a connected graph the degree of each of whose vertices is two or three. Particularly, the number of the 3-degree vertices of $G'$ equals $n - 2$, where $n$ is the number of the 3-degree vertices of $G$. By induction,

$$M(L(G')) = 2^{(n-2)/2+1} = 2^{n/2}.$$ (3)
Note that \( d_{L(G)}(e_1) = d_{L(G)}(e_2) = 4 \) (see Figure 3(b)). Let \( G'' \) be the graph obtained from \( L(G) \) by splitting vertex \( e_2 \), which is illustrated in Figure 3(c). By Lemma 2.1,

\[
M(L(G)) = M(G'').
\]  \hspace{1cm} (4)

Note that \( M(G'') = M(G'' - f - e_2) + M(G'' - g - e_2) \) (see Figure 3(c)). Since each perfect matching of \( G'' - f - e_2 \) (resp. \( G'' - g - e_2 \)) contains no edge \( (e_1, e_4) \) (resp. \( (e_1, e_3) \)), by the definition of \( G' \), it is not difficult to show that

\[
M(G'' - f - e_2) = M(G'' - g - e_2) = M(L(G')).
\]

Hence

\[
M(G'') = 2M(L(G')).
\]  \hspace{1cm} (5)

By (3), (4), and (5), \( M(L(G)) = 2^{n/2+1} \).

**Subcase 1.2** There exists only one edge \( e_1 \) connecting \( u \) to \( v \) in \( G \).

Let \( e_1 = (u, v), e_2 = (u, u_1), \) and \( e_3 = (u, u_2) \) (resp., \( e_1 = (u, v), e_4 = (v, u_3), \) and \( e_5 = (v, u_4) \)) be the three edges incident with vertex \( u \) (resp., vertex \( v \)) in \( G \), and \( u \notin \{u_1, u_2\}, v \notin \{u_3, u_4\} \).

**Subcase 1.2.1** \( e_1 \) is a cut edge of \( G \).

If \( e_1 \) is a cut edge of \( G \) then \( G - e_1 \) has two connected components \( G_1 \) and \( G_2 \). Assume that \( G_1 \) contains vertex \( u \) and \( G_2 \) contains vertex \( v \). Note that \(|E(G)| = |E(G_1)| + |E(G_2)| + 1\) is even. Without loss of generality, we suppose that \(|E(G_1)|\) is even and \(|E(G_2)|\) is odd. Let \( G_3 = G[V(G_2) \cup \{u\}] \), i.e., \( G_3 \) is the graph obtained from \( G \) by deleting all vertices in \( V(G_1) \setminus \{u\} \). Note that \( e_1 \) is a cut vertex of \( L(G) \). Hence

\[
M(L(G)) = M(L(G_1))M(L(G_3)).
\]  \hspace{1cm} (6)

Let \( n_1 \) and \( n_2 \) be the numbers of 3-degree vertices in \( G_1 \) and \( G_3 \), respectively. So

\[
n_1 + n_2 = n - 1.
\]  \hspace{1cm} (7)

By induction,

\[
M(L(G_1)) = 2^{n_1/2+1}.
\]  \hspace{1cm} (8)

In order to enumerate perfect matchings of \( G_3 \), we first prove the following:

**Claim 1** Let \( H \) be a connected graph with an even number of edges and \( u \) a vertex of \( G \) satisfying \( d_H(u) = 1 \). Suppose \( e = (u, v) \) is the edge incident with \( u \) in \( H \) and there exists only three edges \( e = (v, u), e_1 = (v, v_1), \) and \( e_2 = (v, v_2) \) incident with \( v \) in \( H \) (i.e., \( d_H(v) = 3 \)) satisfying \(|\{v_1, v_2\}| = 2 \) (i.e., \( e_1 \) and \( e_2 \) are not two multiple edges of \( G \)).
Construct a new graph $H'$ from $H$ by deleting vertices $u$ and $v$ and connecting vertices $v_1$ to $v_2$ by an new edge $e' = (v_1, v_2)$. Then $M(L(H)) = M(L(H'))$.

In fact, by the definition of $H$, $d_{L(H)}(e) = 2$ and $(e_1, e_2)$ is an edge of $L(H)$ which can not be an edge of a perfect matching of $L(G)$. Hence $M(L(H)) = M(L(H) - (e_1, e_2))$.

Let $H^*$ be the graph obtained from $L(H) - (e_1, e_2)$ by deleting vertex $e$ and identifying vertices $e_1$ and $e_2$ (the new vertex is denoted by $e^*$). Obviously, $L(H) - (e_1, e_2)$ is the graph obtained from $H^*$ by splitting vertex $e^*$. By Lemma 2.1,

$$M(H^*) = M(L(H) - (e_1, e_2)).$$

By the definition of $H'$, $L(H') = H^*$. Hence

$$M(L(H)) = M(L(H'))$$

implying the claim holds.

Note that $G_3$ satisfies $d_{G_3}(u) = 1$ and $d_{G_4}(v) = 3$. The three edges incident with $v$ in $G_3$ are $(v, u), (v, u_3),$ and $(v, u_4)$. We consider the following cases (a) and (b).

**Subcase 1.2.1(a)** $(v, u_3)$ and $(v, u_4)$ are not two multiple edges in $G_3$ (i.e., $u_3 \neq u_4$).

Construct a new graph $G_3'$ from $G_3$ by deleting vertices $u$ and $v$ and connecting vertices $u_3$ to $u_4$ by a new edge $(u_3, u_4)$. By the claim above,

$$M(L(G_3)) = M(L(G_3')).$$

Note that the number of 3-degree vertices of $G_3'$ equals $n_2 - 1$. By induction,

$$M(L(G_3')) = 2^{(n_2-1)/2+1} = M(L(G_3)). \quad (9)$$

Hence, by (6) – (9), we have

$$M(L(G)) = 2^{n/2+1}.$$

**Subcase 1.2.1(b)** $(v, u_3)$ and $(v, u_4)$ are two multiple edges in $G_3$ (i.e., $u_3 = u_4$).

Construct a new graph $G_3^*$ from $G_3$ by replacing edge $(v, u_3)$ in $G_3$ (resp., $(v, u_4)$) with a path $(v - w_1 - w_2 - u_3)$ (resp., $(v - w_3 - w_4 - u_3)$). That is, $G_3^*$ is the graph obtained from $G_3$ by subdividing each of edges $(v, u_3)$ and $(v, u_4)$ twice. By Lemma 2.2,

$$M(L(G_3)) = M(L(G_3^*)).$$

From Subcase 1.2.1(a),

$$M(L(G_3^*)) = 2^{(n_2-1)/2+1}. $$
So we have proved the following:

\[ M(L(G_3)) = M(L(G_3^*)) = 2^{(n_2-1)/2+1}. \]  \hfill (9')

Hence, by (6) – (8) and (9'), we have

\[ M(L(G)) = 2^{n/2+1}. \]

**Figure 4.** (a) A graph \( G \); (b) the graph \( G^* \);

**Subcase 1.2.2** \( e_1 \) is not a cut edge of \( G \).

Construct a new graph \( G^* \) from \( G \) by replacing five edges \( e_1 = (u, v) \), \( e_2 = (u, u_1) \), \( e_3 = (u, u_2) \), \( e_4 = (v, u_3) \), and \( e_5 = (v, u_4) \) by five paths \( (u - w_1 - w_2 - v) \), \( (u - w_3 - w_4 - u_1) \), \( (u - w_5 - w_6 - u_2) \), \( (v - w_7 - w_8 - u_3) \), and \( (v - w_9 - w_{10} - u_4) \), respectively (see Figure 4). That is, \( G^* \) is the graph obtained from \( G \) by subdividing each of five edges \( e_1, e_2, e_3, e_4, \) and \( e_5 \) twice. By Lemma 2.2,

\[ M(L(G)) = M(L(G^*)). \]  \hfill (10)

Let \( G_{(1)} \) (resp., \( G_{(2)} \)) be the graph obtained from \( G^* \) by deleting vertices \( w_1, w_2, \) and \( v \), and connecting vertices \( w_7 \) to \( w_9 \) by a new edge \( (w_7, w_9) \) (resp., by deleting vertices \( w_1, w_2, \) and \( u \), and connecting vertices \( w_3 \) to \( w_5 \) by a new edge \( (w_3, w_5) \)). Let \( f = (u, w_1) \), \( g = (w_1, w_2) \), and \( h = (w_2, v) \) (see Figure 4(b)). Note that \( M(L(G^*)) = M(L(G^*) - f - g) + M(L(G^*) - g - h) \). With a similar method as in Subcase 1.2.1, we may prove that

\[ M(L(G^*) - f - g) = M(L(G_{(1)})), \quad M(L(G^*) - g - h) = M(L(G_{(2)})). \]

Hence

\[ M(L(G^*)) = M(L(G_{(1)})) + M(L(G_{(2)})). \]  \hfill (11)

Note that since \( e_1 \) is not a cut edge of \( G \), both \( G_{(1)} \) and \( G_{(2)} \) are connected graphs with \( n - 2 \) 3-degree vertices. By induction,

\[ M(L(G_{(1)})) = M(L(G_{(2)})) = 2^{(n-2)/2+1}. \]  \hfill (12)
From (10) – (12), it follows that $M(L(G)) = 2^{n/2+1}$.

**Case 2** $j = 1$.

Now $e_1 = (u, v_1)$ and $e_2 = (v_1, v)$ are two edges of $G$, and $d_G(v_1) = 2$, $d_G(u) = d_G(v) = 3$. If $G$ has two multiple edges connecting vertices $u$ and $v$, then $G$ is the graph with three vertices obtained from the graph with two vertices and three multiple edges by subdividing an edge once. It is not difficult to see that $G$ has two 3-degree vertices and $M(L(G)) = 4 = 2^{2/2+1}$. Now we only need to distinguish the following two subcases:

Subcase 2.1 There exists one edge $e_3 = (u, v)$ in $G$ connecting vertices $u$ and $v$ (see Figure 5(a)).

Let $e_1 = (u, v_1), e_3 = (u, v)$, and $e_4 = (u, u_1)$ (resp., $e_2 = (v, v_1), e_3 = (u, v)$, and $e_5 = (v, u_2)$) be the three edges incident with vertex $u$ (resp., with vertex $v$), and $u_1 \neq v, u_2 \neq u$ (see Figure 5(a)). Construct a new graph $G'_1$ from $G$ by deleting vertex $v_1$. Then $G'_1$ is a connected graph with $n - 2$ 3-degree vertices. By induction,

$$M(L(G'_1)) = 2^{(n-2)/2+1}. \tag{13}$$

Note that $d_L(G)(e_3) = 4$ (see Figure 5(b)). Let $G''_1$ be the graph obtained from $L(G)$ by splitting vertex $e_3$, which is illustrated in Figure 5(c). Thus, by Lemma 2.1,

$$M(G''_1) = M(L(G)). \tag{14}$$

Note that $M(G''_1) = M(G''_1 - f - e_3) + M(G''_1 - g - e_3)$ (see Figure 5(c)). Since each perfect matching of $G''_1 - f - e_3$ (resp. $G''_1 - g - e_3$) contains no edge $(e_2, e_5)$ (resp. $(e_1, e_4)$), by Lemma 2.1, it is not difficult to see that

$$M(G''_1 - f - e_3) = M(G''_1 - g - e_3) = M(L(G'_1)).$$

Hence, from (13) and (14),

$$M(L(G)) = M(G''_1) = 2M(L(G'_1)) = 2^{n/2+1}.$$
Subcase 2.2 There exists no edge in $G$ connecting vertices $u$ and $v$.

Using the same method as in Subcase 1.2 (hence we omit the proof), we can show $M(L(G)) = 2^{n/2+1}$.

So we have finished the proof of the theorem. \hfill \Box

If $G$ is a graph with $n$ vertices ($n \to \infty$), define the entropy of $G$ as [7, 6, 24]

$$E(G) = \lim_{n \to \infty} \frac{2\log(M(G))}{n}.$$  

The following result is immediate from Theorem 2.4.

Corollary 2.5. Suppose $G$ is a connected cubic graph $G$ with an even number of edges. Then the number of perfect matchings of $L(G)$ equals $2^{\nu/2+1}$, and the entropy of $L(G)$ equals $\frac{2\log 2}{3}$, where $\nu$ is the number of vertices of $G$.

Given a connected cubic graph $G$ with $\nu$ vertices, construct a cubic graph $G'$ with $3\nu$ vertices from $G$ by cutting off all “corners” of $G$ such that one third of each edge is cut off at each of both ends (see Figure 6 for an example), which is called the clique-inserted-graph of $G$ in [25]. It is not difficult to see that $G'$ is the line graph $L(S(G))$ of the subdivision $S(G)$ of $G$. That is, $G' = L(S(G))$.

![Figure 6.](image)

Corollary 2.6. Suppose $G$ is a connected cubic graph $G$ with $\nu$ vertices and $S(G)$ denotes the graph obtained from $G$ by subdividing every edge once. Then the number of perfect matchings of the clique-inserted-graph of $G$ (i.e., the line graph of $S(G)$) equals $2^{\nu/2+1}$, and the entropy of $L(S(G))$ equals $\frac{\log 2}{3}$.

Theorem 2.7. Suppose $G$ is a connected cubic line graph $G$ with $\nu$ vertices. Then

(1). if $G = K_4$, then $M(G) = 3$;

(2). if $G \neq K_4$, then there exists a connected cubic graph $G^+$ with $\nu/3$ vertices such that $G = L(S(G^+))$. Moreover, $M(G) = 2^{\nu/6+1}$ and the entropy of $G$ equals $\frac{\log 2}{3}$. 
Proof. Suppose that $G'$ is a connected graph and $G = L(G')$ is a connected cubic graph. Let $e = (u, v)$ be an edge of $G'$. We have $d_{G'}(u) + d_{G'}(v) - 2 = d_G(e) = 3$, where $d_G(u)$ denotes the degree of vertex $u$ of $G'$. That is, $d_{G'}(u) + d_{G'}(v) = 5$. Then $(d_{G'}(u), d_{G'}(v)) = (1, 4), (4, 1), (2, 3)$ or $(3, 2)$.

If $(d_{G'}(u), d_{G'}(v)) = (1, 4)$ or $(4, 1)$, then $G'$ is the star $K_{1,4}$ with five vertices and $G$ is the complete graph $K_4$ with four vertices (otherwise, $G$ is not a cubic graph). Hence $M(G) = 3$.

If $G' \neq K_{1,4}$, then for every edge $e = (u, v) \in E(G')$ we have $(d_{G'}(u), d_{G'}(v)) = (2, 3)$ or $(3, 2)$. This implies that $G'$ is a graph obtained from a connected cubic graph $G^+$ by subdividing every edge once (i.e., $G' = S(G')$). By Theorem 2.4, $M(G) = 2^{\nu/6} + 1$ and hence the entropy of $G$ equals $\log_2 3$.

Remark 2.8. By the theorem above, the conjecture posed by Lovász and Plummer holds for the connected cubic line graphs.

3. Applications

As applications, in this section we enumerate perfect matchings of Sierpinski gasket with dimension two, 3.12.12 lattices, and Kagomé lattices in the context of statistical physics.

![Figure 7](image-url) - The first four stages $n = 0, 1, 2, 3$ of two-dimensional Sierpinski gasket $SG_2(n)$.

3.1. The two-dimensional Sierpinski gasket. Fractals are geometrical structures of non-integer Hausdorff dimension realized by repeated construction of an elementary shape on progressively smaller length scales [8, 10, 9, 4]. A well-known example of fractal is the
The construction of the two-dimensional Sierpinski gasket, denoted by $SG_2(n)$ at stage $n$ is shown in Figure 7. At stage $n = 0$, it is an equilateral triangle; while stage $n + 1$ is obtained by the juxtaposition of three $n$-stage structures. The two-dimensional Sierpinski gasket has fractal dimensionality $\frac{\ln 3}{\ln 2}$ [8]. It is not difficult to see that $SG_2(n)$ has $\frac{3}{2}(3^n + 1)$ vertices and $3^{n+1}$ edges [1]. Now we construct a graph sequence $\{G_n\}_{n \geq 0}$ such that the line graph of $G_n$ is isomorphic to $SG_2(n)$, $n = 0, 1, 2, \cdots$. In fact, at stage $n = 0$, $G_0$ is a star $K_{1,3}$; while stage $n + 1$ is obtained by the juxtaposition of three $n$-stage structures, see Figure 8. It is easily verified that $L(G_n) = SG_2(n)$.

By the definition of $G_n$, $G_n$ has $3^n + 3$ vertices, where each of $3^n$ vertices has degree three and each of three vertices has degree one. The number of edges of $G_n$ equals the number of vertices of $SG_2(n)$, which is $\frac{3}{2}(3^n + 1)$. Obviously, $G_n$ has an even number of edges if $n$ is odd, and $G_n$ has an odd number of edges otherwise. Since $G_n$ has three vertices of degree one, we can not use directly Theorem 2.4. So we construct a new graph $G'_n$ from $G_n$ ($n > 1$) by deleting the three vertices of degree one and $u, v, w$ illustrated in Figure 8 and replacing each of the three bold edges with two multiple edges. By Claim 1 in the proof of Theorem 2.4,

$$M(L(G_n)) = M(L(G'_n)).$$

Note that $G'_n$ is a cubic graph with $3^n - 3$ vertices which has an even number of edges if $n$ is odd and an even number of edges otherwise. Hence, by Theorem 2.4, if $n$ is odd, then

$$M(L(G'_n)) = 2^{(3^n-3)/2+1} = 2^{(3^n-1)/2}.$$

So we give a new method to prove the following:
Theorem 3.1 (Chang and Chen. \[1\]). Suppose $SG_2(n)$ is the two-dimensional Sierpinski gasket. Then the number of perfect matchings of $SG_2(n)$ equals $2^{(3^n-1)/2}$ if $n$ is odd and zero otherwise. The entropy of $SG_2(n)$ equals $\frac{\log 2}{3}$ if $n$ is odd.

3.2. 3.12.12 lattices. The 3.12.12 lattice $RT(n, m)$ with toroidal boundary condition is shown in Figure 9(a), where $(a_1, a_1^*)$, $(a_2, a_2^*)$, \ldots, $(a_{m+1}, a_{m+1}^*)$, and $(b_1, b_1^*)$, $(b_2, b_2^*)$, \ldots, $(b_{n+1}, b_{n+1}^*)$ are edges in $RT(n, m)$. The 3.12.12 lattice $RT(n, m)$ has been used by Fisher \[7\] in a dimer formulation of the Ising model. If we delete edges $(b_1, b_1^*)$, $(b_2, b_2^*)$, \ldots, $(b_{n+1}, b_{n+1}^*)$ from $RT(n, m)$, the 3.12.12 lattice $RC(n, m)$ with cylindrical boundary condition is obtained (see Figure 9(b)). If we delete edges $(a_1, a_1^*)$, $(a_2, a_2^*)$, \ldots, $(a_{m+1}, a_{m+1}^*)$ from $RC(n, m)$, the 3.12.12 lattice $RF(n, m)$ with free boundary condition is obtained (see Figure 9(c)). By means of Pfaffians, Fisher \[7\] and Wu \[23\] proved that the logarithm of the number of perfect matchings of $RT(n, m)$, divided by $3(m+1)(n+1)$ (the number of edges of each of perfect matchings of $RT(n, m)$), converges $\frac{1}{3} \ln 2$ as $m, n \to \infty$, which is called the entropy of $RT(n, m)$ by statistical physicists. By Theorem 2.4, we derive the exact formulae of the numbers of perfect matchings of $RT(n, m), RC(n, m)$, and $RF(n, m)$ as follows.
**Theorem 3.2.** Let $R^T(n, m)$, $R^C(n, m)$, and $R^F(n, m)$ be the 3.12.12 lattices with toroidal, cylindrical, and free boundary conditions, respectively. Then

$$M(R^T(n, m)) = 2^{mn+m+n+2},$$

$$M(R^C(n, m)) = 2^{mn+m+1},$$

$$M(R^F(n, m)) = 2^{mn}.$$

Hence $R^T(n, m)$, $R^C(n, m)$, and $R^F(n, m)$ have the same entropy $\frac{1}{3}\ln 2$.

**Proof.** In order to prove the theorem, we introduce the hexagonal lattices which have been extensively studied by statistical physicists [6, 7, 23]. The hexagonal lattice $H^T(n, m)$ with toroidal boundary condition is shown in Figure 9(d), where $(d_1, d_1^*)$, $(d_2, d_2^*)$, $\ldots$, $(d_{m+1}, d_{m+1}^*)$ and $(d_1, c_1^*)$, $(c_1, c_2^*)$, $\ldots$, $(c_{n-1}, c_n^*)$, $(c_n, d_{m+1}^*)$ are edges in $H^T(n, m)$. It is not difficult to see that the line graph of $S(H^T(n, m))$ is $R^T(n, m)$, where $S(H^T(n, m))$ is the graph obtained from $H^T(n, m)$ by subdividing each edge of $H^T(n, m)$ once. Note that there exists $2(m+1)(n+1)$ vertices of degree three in $S(H^T(n, m))$. By Theorem 2.4,

$$M(R^T(n, m)) = M(L(S(H^T(n, m)))) = 2^{2(m+1)(n+1)/2+1} = 2^{mn+m+n+2}.

![Figure 10](image-url)

**Figure 10.** (a) The graph $H^T_1(n, m)$, (b) the graph $H^T_2(n, m)$.

Let $H^T_1(n, m)$ be the graph obtained from $H^T(n, m)$ by replacing each of the $n+1$ edges $(d_1, c_1^*)$, $(c_1, c_2^*)$, $\ldots$, $(c_{n-1}, c_n^*)$, $(c_n, d_{m+1}^*)$ with two independent edges (i.e., replacing edge $(d_1, c_1^*)$ with edges $(d_1, f_0)$ and $(c_1^*, f_0^*)$, replacing edge $(c_1, c_2^*)$ with edges $(c_1, f_1)$ and $(c_2^*, f_1^*)$, $\ldots$, replacing edge $(c_{n-1}, c_n^*)$ with edges $(c_{n-1}, f_{n-1})$ and $(c_n^*, f_{n-1}^*)$, and replacing edge $(c_n, d_{m+1}^*)$ by edges $(c_n, f_n)$ and $(d_{m+1}^*, f_n^*)$, see Figure 10(a)). For the graph $H^T_1(n, m)$, subdivide each edge in $E(H^T_1(n, m)) \setminus A$ once, where $E(H^T_1(n, m))$ is the edge set of $H^T_1(n, m)$ and $A$ is the set of $2n+2$ edges $(d_1, f_0)$, $(c_1^*, f_0^*)$, $(c_1, f_1)$, $(c_2^*, f_1^*)$, $\ldots$, $(c_{n-1}, f_{n-1})$, $f_n$.
Let $H^T_1(n, m)$ be the graph obtained from $H^T_1(n, m)$ by replacing each of the $m+1$ edges $(d_1, d_1^*), (d_2, d_2^*), \ldots, (d_{m+1}, d_{m+1}^*)$ with two independent edges (i.e., replacing edge $(d_1, d_1^*)$ with edges $(d_1, g_1)$ and $(d_1^*, g_1^*)$, replacing edge $(d_2, d_2^*)$ with edges $(d_2, g_2)$ and $(d_2^*, g_2^*)$, \ldots, replacing edge $(d_{m+1}, d_{m+1}^*)$ with edges $(d_{m+1}, g_{m+1})$ and $(d_{m+1}^*, g_{m+1}^*)$, see Figure 10(b)). Let $B = \{(d_i, g_i), (d_i^*, g_i^*)| i = 1, 2, \ldots, m+1\}$. For the graph $H^T_2(n, m)$, subdivide each edge in $E(H^T_2(n, m))\setminus(A \cup B)$ once. The resulting graph is denoted $H^*_2(n, m)$. It is not difficult to see that the line graph of $H^*_2(n, m)$ is just $R^F(n, m)$, i.e., $R^F(n, m) = L(H^*_2(n, m))$.

Let $H^*_3(n, m)$ be the graph obtained from $H^T_3(n, m)$ by deleting six vertices $f_0, d_1, g_1, f_n^*, d_{m+1}^*$ and $g_{m+1}^*$ illustrated in Figure 10(b). Denote by $H^*_3(n, m)$ the graph obtained by subdividing each edge of $H^*_3(n, m)$ once which is not a pendant edge. Hence the number of 3-degree (resp., 1-degree) vertices of $H^*_3(n, m)$ equals $2(m + 1)(n + 1) - 4 = 2mn + 2m + 2n - 2$ (resp., $2m + 2n$). By Theorem 2.4 and Claim 1 in the proof of Theorem 2.4,

$$M(L(H^*_3(n, m))) = 2^{(2mn + 2m + 2n - 2 - 2m - 2n)/2 + 1} = 2^{mn}. \quad (15)$$

Note that every perfect matching of $R^F(n, m)$ contains edges $(a_1, b_1), (a_{m+1}, b_{n+1})$ and the two bold edges $e_1$ and $e_2$ illustrated in Figure 9(c). Let $R^F_1(n, m)$ be the graph obtained from $R^F(n, m)$ by deleting the eight vertices incident with edges $e_1, e_2, (a_1, b_1), (a_{m+1}, b_{n+1})$. Then

$$M(R^F(n, m)) = M(R^F_1(n, m)). \quad (16)$$

Obviously, the line graph of $H^*_3(n, m)$ is $R^F_1(n, m)$. Thus

$$M(R^F_1(n, m)) = M(L(H^*_3(n, m))). \quad (17)$$

By (15), (16), and (17),

$$M(R^F(n, m)) = 2^{mn}.$$
Hence
\[
\lim_{m,n \to \infty} \frac{\ln M(R^T(n,m))}{3(m+1)(n+1)} = \lim_{m,n \to \infty} \frac{\ln M(R^C(n,m))}{3(m+1)(n+1)} = \lim_{m,n \to \infty} \frac{\ln M(R^F(n,m))}{3(m+1)(n+1)} = \frac{1}{3} \ln 2,
\]
implying that Theorem 3.2 holds. □

Remark 3.3. Similarly, we can define the 3.3.12 lattices \(R^K(n,m)\) and \(R^M(n,m)\) with Klein-bottle and Mobius-band boundary conditions, respectively. It is similarly verified that
\[
M(R^K(n,m)) = 2^{mn+m+n^2}, M(R^M(n,m)) = 2^{mn+m+1}.
\]

3.3. Kagomé lattices. Let \(G(n,m)\) be the plane lattice graph illustrated in Figure 11, each of whose vertices has degree two or four. For \(G(n,m)\), if we identify each pair of vertices \(u_i\) and \(u_i^*\), \(v_j\) and \(v_j^*\), \(i = 1, 2, \cdots, 2m, j = 1, 2, \cdots, n\), the resulting graph, denoted by \(K^T(n,m)\), is called the Kagomé lattice with toroidal boundary condition by statistical physicists (see [6, 18, 23, 22, 24]). For \(G(n,m)\), if we delete vertices \(v_1^*, v_2^*, \cdots, v_n^*\) and identify each pair of vertices \(u_i\) and \(u_i^*\), \(i = 1, 2, \cdots, 2m\), the resulting graph, denoted by \(K^C(n,m)\), is called the Kagomé lattice with cylindrical boundary condition (see [24]). And the graph obtained from \(G(n,m)\) by deleting vertices \(u_i^*, v_j^*, i = 1, 2, \cdots, 2m, j = 1, 2, \cdots, n\), is called the Kagomé lattice with free boundary condition, denoted by \(K^F(n,m)\). By the definitions of \(K^T(n,m), K^C(n,m)\), and \(K^F(n,m)\), we know that all \(K^T(n,m), K^C(n,m)\), and \(K^F(n,m)\) have \(6mn\) vertices.

The study of the molecular freedom for the kagomé lattice has been a subject matter of interest for many years (see, for example, [6, 18]), but most of the studies have been numerical or approximate. By using Paffian orientation, Wu and Wang [24] obtained the interesting formulae of the numbers of perfect matchings of \(K^T(n,m)\) and \(K^C(n,m)\) as follows:
\[
M(K^T(n,m)) = 2^{2mn+1}, \quad M(K^C(n,m)) = 2^{2mn-n+1}.
\]
In fact, they gave a more general formulae (each edge was weighted, see [24]). Now we give a new method to prove (18). Furthermore, we will prove the following:

\[ M(K^F(n, m)) = 2^{2mn-2m-n+1}. \]  (19)

**Figure 12.** (a) The graph \( K^T(n, m) \), where \( u_i \) and \( u_i^* \) are identified as a single vertex, \( i = 1, 2, \cdots, 2m \), and \( v_j \) and \( v_j^* \) are identified as a single vertex, \( j = 1, 2, \cdots, n \). (b) The graph \( K^C(n, m) \), where \( u_i \) and \( u_i^* \) are identified as a single vertex, \( i = 1, 2, \cdots, 2m \). (c) The graph \( K^F(n, m) \).

In the proof of Theorem 3.2, we have defined the hexagonal lattice \( H^T(n, m) \) with toroidal boundary condition (see Figure 9(d)) and two graphs \( H^T_1(n, m) \) and \( H^T_2(n, m) \) (see Figure 10). It is not difficult to see that \( K^T(n, m) \) is the line graph of \( H^T(n-1, 2m-1) \) (see Figure 12(a), where we embed \( H^T(n-1, 2m-1) \) and \( K^T(n, m) \) simultaneously in the plane), \( K^C(n, m) \) is the line graph of the graph , denoted by \( H_1(n-1, 2m-1) \), which is obtained from \( H^T_1(n-1, 2m-1) \) by deleting vertices \( f^*_0, f^*_1, \cdots, f^*_{n-1} \) (see Figure 12(b), where we embed \( K^C(n, m) \) and \( H_1(n-1, 2m-1) \) simultaneously in the plane), and \( K^F(n, m) \) is the line graph of the graph, denoted by \( H_2(n-1, 2m-1) \), which is obtained from \( H^T_2(n-1, 2m-1) \) by deleting vertices \( f^*_0, f^*_1, \cdots, f^*_{n-1}, g^*_1, g^*_2, \cdots, g^*_{2m} \) (see Figure
12(c), where we embed \( K^F(n, m) \) and \( H_2(n - 1, 2m - 1) \) simultaneously in the plane, respectively. With the same method as in the proof of Theorem 3.2, we can prove (18) and (19). Hence we have the following:

**Theorem 3.4.** Let \( K^T(n, m), K^C(n, m), \) and \( K^F(n, m) \) be the Kagomé lattices with toroidal, cylindrical, and free boundary conditions, respectively. Then \( M(K^T(n, m)) = 2^{2mn+1}, M(K^C(n, m)) = 2^{2mn-n+1}, M(K^F(n, m)) = 2^{2mn-2m-n+1} \). Hence \( K^T(n, m), K^C(n, m), \) and \( K^F(n, m) \) have the same entropy \( \frac{2}{3} \ln 2 \).

**Remark 3.5.** Similarly, we can define the Kagomé lattices \( K^K(n, m) \) and \( K^M(n, m) \) with Klein-bottle and Möbius-band boundary conditions, respectively. It is similarly verified that

\[
M(K^K(n, m)) = 2^{2mn+1}, M(K^M(n, m)) = 2^{2mn-n+1}.
\]

### 4. CONCLUDING REMARKS

Kuperberg [13] showed that the number of perfect matchings of the line graph of a graph with vertices of degree at most 3 (and with an even number of edges) is a power of 2. In this paper, we obtain the exact formula of the number of perfect matchings of the line graph of a graph with vertices of degree equal to two or three and with an even number of edges. Moreover, our result implies that the conjecture of Lovász and Plummer on the perfect matchings of regular graphs holds for the connected cubic line graphs and the line graphs of connected cubic graphs with an even number of edges. Finally, as applications of our result, we use a unified method to prove some known formulae of perfect matchings of the Kagomé lattices with toroidal and cylindrical boundary conditions by Wang and Wu [22, 24], the 3.12.12 lattices with toroidal boundary condition by Fisher [7] and Wu [23], and the Sierpinski gasket with dimension two by Chang and Chen [1], respectively. Furthermore, by using this unified approach we solve the problem of enumeration of perfect matchings of the Kagomé lattices with free, Klein-bottle, and Möbius-band boundary conditions, and the 3.12.12 lattices with cylindrical, free, Klein-bottle, and Möbius-band boundary conditions, respectively.

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