The capacity of a quantum channel for simultaneous transmission of classical and quantum information

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Abstract  
An expression is derived characterizing the set of admissible rate pairs for simultaneous transmission of classical and quantum information over a given quantum channel, generalizing both the classical and quantum capacities of the channel. Although our formula involves regularization, i.e. taking a limit over many copies of the channel, it reduces to a single-letter expression in the case of generalized dephasing channels. Analogous formulas are conjectured for the simultaneous public-private capacity of a quantum channel and for the simultaneously 1-way distillable common randomness and entanglement of a bipartite quantum state.

1 Introduction

In the paper that marked the beginning of information theory [21], C. E. Shannon introduced the notion of a (classical) channel $W$, a stochastic map modeling the effect of noise experienced by a classical message on its way from sender to remote receiver. There he defined and computed the key property of the channel $W$: its capacity $C(W)$ to convey classical information, expressed in bits per channel use. Many decades later, in the context of quantum information theory, the notion of a quantum channel $N$, a cptp (completely positive trace preserving) map, was introduced as the most general bipartite dynamic resource consistent with quantum mechanics. There are now two basic capacities one may define for $N$: classical $C(N)$ and quantum $Q(N)$. Intuitively, these correspond to the maximum number of bits (respectively qubits) per use of $N$ that can be faithfully transmitted over the channel. The classical capacity theorem was independently proved by Holevo [17], and Schumacher and Westmoreland [23]. The quantum capacity theorem was originally stated by Lloyd [19], although it was only recently generally realized that his proof could be made rigorous [18]. It has also been proved by Shor [25] and subsequently, via the private classical capacity, by Devetak [8]. In the present paper we unify the two capacities by investigating the capacity of $N$ for simultaneously transmitting classical and quantum information, given in the form of a trade-off curve.

Let the sender Alice and receiver Bob be connected via a quantum channel $N : H_{A'} \rightarrow H_B$, where $H_{A'}$ denotes the Hilbert space of Alice’s input system $A'$ and $H_B$ that of Bob’s output system $B$. We shall define three distinct information processing scenarios which will turn out to be equivalent.

Scenario Ia (subspace transmission) Alice’s task is to convey to Bob, in some large number $n$ uses of the channel, one of $\mu$ equiprobable classical messages with low error probability and
simultaneously an arbitrarily chosen quantum state from some Hilbert space $\mathcal{H}$ of dimension $\kappa$ with high fidelity. More precisely, we define a (classical, quantum) channel code to consist of:

- An ordered set $(\mathcal{E}_m)_m \in [\mu]$, $[\mu] = \{1, 2, \ldots, \mu\}$, of cptp maps $\mathcal{E}_m : \mathcal{H} \to \mathcal{H}^\otimes n$. Such an ordered set is the most general function with two inputs, classical and quantum, and one quantum output.

- A decoding quantum instrument $\mathbb{D} = (D_m)_{m \in [\mu]}$, an ordered set of cp (completely positive) maps $D_m : \mathcal{H}^\otimes n \to \mathcal{H}$, the sum of which $D = \sum_{m \in [\mu]} D_m$ is trace preserving. The probability of outcome $m$ for input $\rho$ is $\operatorname{Tr} D_m(\rho)$, while the effective quantum map is $D$. The instrument has one quantum input and two outputs, classical and quantum. It is a natural generalization of a POVM (positive operator valued measure), which cares only about the classical output, and quantum cptp map, which only has a quantum output.

Alice’s classical message is represented by a random variable $M$ uniformly distributed on the set $[\mu]$. Conditional on $M$ taking on a particular value $m$, Alice encodes the quantum state of $A''$ with $\mathcal{E}_m$ and sends it through $n$ copies of the channel $\mathcal{N}$. Bob performs the instrument $\mathbb{D}$ on the channel output, resulting in the classical outcome random variable $M'$ and a quantum output system $B'$. Note that $\mathcal{H} = \mathcal{H} = \mathcal{H}$. We call the ordered pair $((\mathcal{E}_m)_m, \mathbb{D})$ an $(n, \epsilon)$ code if

1. $\Pr\{M' \neq m | M = m\} \leq \epsilon$, $\forall m$,
2. $\min_{|\varphi\rangle \in \mathcal{H}} F(\varphi, (D \circ N^\otimes n \circ \mathcal{E}_m)(\varphi)) \geq 1 - \epsilon$, $\forall m$,

where the fidelity is defined by $F(\rho, \sigma) = \|\sqrt{\rho} \sqrt{\sigma}\|_1^2$. Condition 1 above means that each message should be correctly decoded by Bob with high probability. Condition 2 corresponds to the subspace transmission criterion of $\mathbb{D}$: each pure input state $|\varphi\rangle$ supported on $\mathcal{H}$ should be almost perfectly transmitted to Bob. The (classical, quantum) rate pair of the code is $(r, R)$, with $r = \frac{1}{n} \log \mu$ and $R = \frac{1}{n} \log \kappa$. They represent the number of bits and qubits, respectively, per use of the channel that can be faithfully transmitted simultaneously. A rate pair $(r, R)$ is called achievable if for all $\epsilon, \delta > 0$ and all sufficiently large $n$ there exists an $(n, \epsilon)$ code with rate pair $(r - \delta, R - \delta)$. The simultaneous (classical, quantum) scenario Ia capacity region of the channel $S_{\text{II}}(\mathcal{N})$ is the set of all achievable positive rate pairs.

**Scenario Ib (entanglement transmission)** This scenario is very similar to the first one, but instead of transmitting an arbitrary pure state of $A''$, Alice is required to preserve entanglement $\mathbb{I}$ between $A''$ and some reference system $A$ she has no access to. Here condition 2 is replaced by

$2'. F(\Phi, \Omega_m) \geq 1 - \epsilon$, $\forall m$,

where

$$\Omega_m^{AB'} = |1^A \otimes (D \circ N^\otimes n \circ \mathcal{E}_m)(\Phi^{AA''})|,$$

and

$$|\Phi\rangle = \sqrt{\frac{T}{\kappa}} \sum_{k=1}^\kappa |k\rangle \otimes |k\rangle$$

is the standard maximally entangled state on $\mathcal{H} \otimes \mathcal{H}$. We denote the corresponding capacity region by $S_{\text{II}}(\mathcal{N})$.

**Scenario II (entanglement generation)** In this scenario, simultaneously with transmitting classical information, Alice wishes to generate entanglement $\mathbb{I}$ shared with Bob rather than preserving it as in scenario Ib. Alice prepares, without loss of generality, a pure bipartite state
\[ |\gamma_m\rangle^{AA^n} \] in her lab \((\mathcal{H}_{A^n} := H_{A^n})\), depending on the classical information \(m\), and sends it through the channel. Bob decodes as above, yielding the output state

\[
\Omega_m^{AB'} = [1^A \otimes (\mathcal{D} \circ \mathcal{N}^{\otimes n})](|\gamma_m\rangle^{AA^n}),
\]

shared by Alice and Bob. Everything else is defined as in scenario Ib. The corresponding capacity region is denoted by \(S_{II}(\mathcal{N})\).

In the next section we state our main result, a unique expression for the capacity regions defined above, investigate its properties and relate it to previous work. The proof of our main theorem is relegated to section 3. Some remarks on related problems are collected in section 4. We conclude in section 5 with suggestions for future research.

## 2. Main result

Recall the notion of an ensemble of quantum states \(E = \{p_x, |\phi_x\rangle^{AA'}\}\): the quantum system \(AA'\) is in the state \(|\phi_x\rangle^{AA'}\) with probability \(p_x\). The ensemble \(E\) is equivalently represented by a classical-quantum system of \(X^{AA'}\) in the state

\[
\sum_x p_x |x\rangle \langle x| \otimes |\phi_x\rangle \langle \phi_x|^{AA'}.
\]

\(X\) plays the dual role of an auxiliary quantum system in the state \(\sum_x p_x |x\rangle \langle x|\) and of a random variable with distribution \(p\). Sending the \(A'\) system through the channel \(\mathcal{N}\) gives rise to a classical-quantum system \(X^{AB}\) in some state \(\sigma^{X^{AB}}\)

\[
\sigma^{X^{AB}} = \sum_x p_x |x\rangle \langle x| \otimes [(1 \otimes \mathcal{N})(|\phi_x\rangle \langle \phi_x|)]^{AB}.
\]

For such a state we say that it “arises from” the channel \(\mathcal{N}\). For a multi-party state such as \(\sigma^{X^{AB}}\) the reduced density operator \(\sigma^X\) is defined by \(\text{Tr}_{X'B} \sigma^{X^{AB}}\). Conversely, we call \(\sigma^{X^{AB}}\) an extension of \(\sigma^X\). A pure extension is conventionally called a purification. Define the von Neumann entropy of a quantum state \(\rho\) by \(H(\rho) = -\text{Tr}(\rho \log \rho)\). We write \(H(A)_\sigma = H(\sigma^X)\), omitting the subscript when the reference state is clear from the context. The Shannon entropy \(-\sum_x p_x \log p_x\) of the random variable \(X\) is equal to the von Neumann entropy \(H(X)\) of the system \(X\). Define the conditional entropy

\[
H(A|B) = H(B) - H(AB),
\]

(quantum) mutual information

\[
I(A; B) = H(A) + H(B) - H(AB),
\]

and conditional mutual information

\[
I(A; B|X) = I(A; BX) - I(A; X).
\]

The coherent information \(I(A|B)\) is defined as \(-H(A|B)\). Whenever the state \(\rho^{AB}\) comes about by sending some pure state \(|\phi\rangle^{AA'}\) through the channel \(\mathcal{N}\), we may use the alternative notation \(I_c(\phi^{AA'}, \mathcal{N}) := I(A|B)_{\rho}\), since this quantity is independent of the particular purification \(|\phi\rangle^{AA'}\) of \(|\phi\rangle^{AA'}\). In what follows all information theoretical quantities will refer to the state \(\sigma^{X^{AB}}\), unless stated otherwise.

Our main result is the following theorem.
Theorem 1 The simultaneous capacity regions of $\mathcal{N}$ for the various scenarios Ia, Ib and II are all equal and given by

$$S(\mathcal{N}) = \bigcup_{l=1}^{\infty} \frac{1}{l} S^{(1)}(\mathcal{N}^\otimes l)$$

where $S^{(1)}(\mathcal{N})$ is the union, over all $\sigma^{XAB}$ arising from the channel $\mathcal{N}$, of the $(r, R)$ pairs obeying

$$0 \leq r \leq I(X ; B)$$
$$0 \leq R \leq I(A)B X).$$

Furthermore, in computing $S^{(1)}(\mathcal{N})$ one only needs to consider random variables $X$ defined on some set $\mathcal{X}$ of cardinality $|\mathcal{X}| \leq (\dim \mathcal{H}_A)^2 + 2$.

Since the three scenarios are equivalent we shall speak of a single capacity region. The generic shape of the capacity region is shown in figure 1. We shall informally refer to the outer boundary of the capacity region in the $(r > 0, R > 0)$ quadrant as the “trade-off curve”. In scenarios Ib and II, for any $0 < \lambda < 1$, combining a $(\lambda n, \epsilon)$ code of rate pair $(r_1, R_1)$ with a $((1- \lambda)n, \epsilon)$ code of rate pair $(r_2, R_2)$ one obtains an $(n, 2\epsilon)$ code of rate pair $(\lambda r_1 + (1- \lambda)r_2, \lambda R_1 + (1- \lambda)R_2)$. This construction is known as time-sharing and implies the concavity of the capacity region. In fact, the “single-letter” region $S^{(1)}(\mathcal{N})$ is already concave for all channels $\mathcal{N}$ (see appendix A). The points $(C(\mathcal{N}), 0)$ and $(0, Q(\mathcal{N}))$ represent the classical and quantum capacities, respectively. By time-sharing one may achieve the line segment interpolating between the two, giving an inner bound on the capacity region. An outer bound given by the line segment connecting $(C(\mathcal{N}), 0)$ and $(0, C(\mathcal{N}))$ is obtained by observing that, in scenario Ia, the transmitted quantum subspace may always be used to encode classical information at 1 bit/qubit.
Figure 2: The trade-off curve for the dephasing qubit channel with dephasing parameter 0.2 (i.e., the channel obtained by applying the identity operator with probability 0.9 and $\sigma_z$ with probability 0.1). In the left-hand plot, the trade-off curve is plotted with a solid line and the time-sharing bound with a dashed line. The right-hand plot gives the difference between the optimal strategy and time-sharing.

Our theorem is, alas, difficult to use in practice due to the $l \to \infty$ limit. Two simple examples in which this limit is not required are the noiseless qubit channel and the erasure channel, for which both $C(\mathcal{N})$ and $Q(\mathcal{N})$ were previously known \cite{3}. In both cases the boring time-sharing strategy turns out to be optimal. This is particularly trivial to see for the noiseless channel: since $Q(\mathcal{N}) = C(\mathcal{N})$, the inner and outer bound coincide.

A more interesting case is that of a dephasing channel, for which the large $l$ limit is also not required (we prove this in appendix B), yet the resulting trade-off curve is strictly concave. The $S(\mathcal{N})$ region for the dephasing qubit channel with dephasing parameter 0.2 is shown in figure 2.

For the depolarizing channel, another popular example, the $l \to \infty$ limit is known to be needed when the depolarizing parameter is close to $p = 0.189$, the value making $Q(\mathcal{N}) = 0$ \cite{10}. One can, however, make an interesting observation about the behavior of the trade-off curve near $R = Q(\mathcal{N})$. Although the channel itself is invariant under unitary transformations, the $\rho$ that maximizes the coherent information $I_c(\rho, \mathcal{N})$ breaks this symmetry; indeed there is a whole family of density operators attaining $Q(\mathcal{N})$. One can thus construct an ensemble with $R = Q(\mathcal{N})$ and $r > 0$, so the trade-off curve is parallel to the $r$ axis in a finite region around $r = 0$. For the depolarizing channel with different $p$, we have calculated the trade-off curve assuming $l = 1$ and found some interesting behavior. For $p$ small ($p < 0.04$ or so) it is possible to do better than the time-sharing strategy, whereas for larger $p$ ($p > 0.05$), the time-sharing strategy is optimal, assuming $l = 1$. For these values of $p$, it is not known whether taking large $l$ is advantageous for $Q(\mathcal{N})$.

There is an intriguing connection between our capacity region and the findings of Shor \cite{26} concerning the classical capacity of a quantum channel with limited entanglement assistance. The latter may be thought of as extending scenario II to the negative $R$ axis, since entanglement is consumed rather than generated \cite{14}. The result for the $R \leq 0$ region parallels that from theorem
1, replacing (6) by

\[
0 \leq r \leq I(X; B) + I(A; B|X) \leq -H(A|X) = I(A)B - I(A; B|X).
\]  

(7)

The two expressions on the right hand side have the same sum as in equation (6). There is a simple bijection between the two regions: If \((r, R)\) is a point in \(R \geq 0\) corresponding to the state \(\sigma^{XAB}\), then \((r + I(A; B|X), R - I(A; B|X))\) is a point in the \(R \leq 0\) region, and vice versa. Imagine that 1 ebit of entanglement were a stronger resource that 1 bit of communication, in the sense that the latter could be produced from the former. Then the \(R \leq 0\) region would be trivially achievable by the achievability of the \(R \geq 0\) region. The opposite would hold were 1 bit stronger than 1 ebit. However, it is well known that bits and ebits are incomparable resources. The correspondence between the two regions may be interpreted as providing a limited sense in which bits and ebits may be thought of as equally strong.

One may play the same game in the context of scenario I (a or b), with a somewhat less interesting outcome. Here a negative rate \(R\) is interpreted as assistance by a noiseless quantum channel. It is known [24] that the classical capacity of a noiseless channel combined with a noisy one is just the sum of the individual capacities. Hence the scenario I continuation of our trade-off curve simply follows the linear outer bound into the \(R < 0\) region (see figure 1).

3 Proof of theorem 1

The following lemma from [1] is needed to relate scenarios Ia and Ib.

Lemma 2

1. If

\[
\min_{|\varphi\rangle \in \mathcal{H}} F(\varphi, (D \circ N^{\otimes n} \circ \mathcal{E})(\varphi)) \geq 1 - \frac{2}{3}\epsilon
\]

then

\[
F(\Phi, [1 \otimes (D \circ N^{\otimes n} \circ \mathcal{E})](\Phi)) \geq 1 - \epsilon.
\]  

(8)

2. Conversely, if (8) holds then

\[
\min_{|\varphi\rangle \in \mathcal{H}'} F(\varphi, (D \circ N^{\otimes n} \circ \mathcal{E})(\varphi)) \geq 1 - 2\epsilon,
\]

where \(\mathcal{H}'\) is a subspace of \(\mathcal{H}\) satisfying

\[
\dim \mathcal{H}' \geq \frac{1}{2} \dim \mathcal{H}.
\]  

(9)

Observe that \(S_{Ia}(\mathcal{N}) = S_{Ib}(\mathcal{N}) \subseteq S_{II}(\mathcal{N})\). The equality follows from both parts of lemma 2. The inclusion is obvious since one can always generate entanglement by transmitting half of the maximally entangled state \(|\Phi\rangle\). Therefore, to prove theorem 1 it suffices to show that the region (5) is contained in \(S_{Ib}(\mathcal{N})\) (the “direct coding theorem”) and contains \(S_{II}(\mathcal{N})\) (the “converse”).

To prove the converse we need the following simple lemma [8].

Lemma 3 For two bipartite states \(\rho^{AB}\) and \(\sigma^{AB}\) of a quantum system \(AB\) of dimension \(d\) with fidelity \(f = F(\rho^{AB}, \sigma^{AB})\),

\[
|I(A)B_\rho - I(A)B_\sigma| \leq \frac{2}{\epsilon} + 4 \log d \sqrt{1 - f}.
\]
Proof of theorem 1 (converse for scenario II) Define the classical-quantum state $\omega^{MAB^n}$ to be the result of sending the $A^n$ part of

$$\frac{1}{\mu} \sum_m |m\rangle^M \otimes \Upsilon_{m}^{AA^n}$$

through the channel $\mathcal{N}^{\otimes n}$. We shall prove that, for any $\delta, \epsilon > 0$ and all sufficiently large $n$, if an $(n, \epsilon)$ code has a rate pair $(r, R)$ then

$$r - \delta \leq \frac{1}{n} I(M; B^n)_{\omega},$$

$$R - \delta \leq \frac{1}{n} I(A^n M)_{\omega}. \quad (10)$$

Evidently, it suffices to prove this for $\delta \leq 1, \epsilon \leq \left[ \frac{\delta}{16 \log \dim H_{A'}} \right]^2$ and $n \geq 2$. Fano’s inequality \[5\] says

$$H(M|M') \leq 1 + \Pr\{M' \neq M\} n r.$$

Equation (10) is a consequence of the following string of inequalities

$$nr = H(M) = I(M; M') + H(M|M') \leq I(M; M') + 1 + n\epsilon \log \dim H_{A'},$$

the last line by the Holevo bound \[16\]. On the other hand, defining $\omega^{MAB'}$ to be the state $\omega^{MAB}$ after Bob’s decoding $\mathcal{D}$,

$$I(A^n B^n M)_{\omega} \geq I(A^n B'M_{\omega}) \geq I(A^n B')_{\omega} \geq I(A^n B') + \frac{2}{\epsilon} - 8n R \sqrt{\epsilon} \geq n R - \frac{2}{\epsilon} - 8n \log \dim H_{A'} \sqrt{\epsilon},$$

from which the claim (11) follows. The first inequality is the data processing inequality \[2\], the second follows from the fact that conditioning cannot increase quantum relative entropy \[20\] and the third is an application of lemma 3. It should be noted that we only used a weaker “average” version of conditions 1. and 2′., namely

1. $\Pr\{M' \neq M\} \leq \epsilon,$

2′. $F(\Phi, \frac{1}{m} \sum_m \Omega_m) \geq 1 - \epsilon.$

The bound on the cardinality of $\mathcal{X}$ is proven in appendix C. \[\Box\]

We henceforth restrict attention to scenario I(b). In proving the direct coding theorem, we shall combine purely quantum and purely classical codes. A quantum code is a special case of a (classical, quantum) code defined earlier, for which $\mu = 1$ $(r = 0)$. Quantum codes are characterized by a pair of encoding and decoding maps $(\mathcal{E}, \mathcal{D})$. Define the quantum code density operator \[8\] as $\mathcal{E}(\pi)$, where $\pi = \frac{1}{d} 1^{A''}$.

Often in coding theory is it useful to consider random codes. Alice and Bob have access to an auxiliary resource: a common source of randomness described by some probability distribution.
(P_\alpha). A random quantum code is an ordered set of encoding-decoding pairs ((E^\alpha, D^\alpha))_\alpha, indexed by \alpha. With probability P_\alpha, Alice and Bob choose to employ the deterministic code (E^\alpha, D^\alpha). The average code density operator for the random quantum code is given by \sum_\alpha P_\alpha E^\alpha(\pi). Given a density operator \rho \in \mathcal{H}_A, we say that an (n, \epsilon) random quantum code is “\rho-type” if the average code density operator \omega satisfies

\|\omega - \rho \otimes^n\|_1 \leq \epsilon. \quad (12)

For an ensemble of density operators \{E, \rho_x\} defined on \mathcal{H}_A and sequence x^n = x_1 x_2 \ldots x_n denote \rho_x = \bigotimes^n_{i=1} \rho_{x_i}. We say that an (n, \epsilon) random quantum code is “(E, x^n)-type” if the average code density operator \omega_x^n satisfies

\|\omega_x^n - \rho_x^n\|_1 \leq \epsilon.

The following proposition is a refinement of the quantum channel coding theorem, and was proved in Appendix D of \cite{8}. A perhaps more accessible outline of the proof may be found in \cite{12}.

**Proposition 4** For any \epsilon, \delta > 0 and all sufficiently large n, there exists a random \rho-type (n, \epsilon) quantum code for the channel \mathcal{N} of rate \text{R} = I_c(\rho, \mathcal{N}) - \delta.

Recall the notion of \delta-typical sequences \mathcal{T}_{p, \delta}^n,

\mathcal{T}_{p, \delta}^n = \{x^n : \forall x |N(x|x^n) - np_x| \leq \delta n\},

where \text{N}(x|x^n) counts the number of occurrences of x in \text{x}_n. When the distribution \rho is associated with some random variable X the alternative notation \mathcal{T}_{X, \delta}^n may be used. Proposition \text{4} extends to:

**Proposition 5** For any \epsilon, \delta > 0 and all sufficiently large n, for any typical sequence x^n \in \mathcal{T}_{p, \delta}^n there exists a random \text{(E, x^n)}-type (n, \epsilon) quantum code for the channel \mathcal{N} of rate \text{R} = \sum_x p_x I_c(\rho_x, \mathcal{N}) - c\delta, for some constant c.

**Proof** By proposition \text{4} for sufficiently large n, for all x there exists an \text{(n\rho_x - \delta, \epsilon)} code of rate \text{R}_x = I_c(\rho_x, \mathcal{N}) - \delta, with average density operator \omega_x satisfying

\|\omega_x - \rho_x \otimes^n_{[p_x - \delta]}\|_1 \leq \epsilon.

By “pasting” |X| such codes together (one for each x) an \text{((n - |X|\delta, |X|\epsilon)} code is produced with average code density operator \omega = \bigotimes_x \omega_x. Applying the triangle inequality multiple times,

\|\omega - \bigotimes_x \rho_x \otimes^n_{[p_x - \delta]}\|_1 \leq |X|\epsilon. \quad (13)

Given \text{x} \in \mathcal{T}_{X, \delta}^n, abbreviate \text{n}_x = \text{N}(x|x^n) and \text{\Delta n}_x = \text{n}_x - n[p_x - \delta]. Now transform the above code into the “padded” \text{((n, |X|\epsilon)} quantum code obtained by inserting \rho_x \otimes^{\text{\Delta n}_x} after each \omega_x; its average density operator \omega' obeys

\|\omega' - \bigotimes_x \rho_x \otimes^n_{[p_x - \delta]}\|_1 \leq |X|\epsilon. \quad (14)

The new rate \text{R} is bounded by

\text{R} = \sum_x \text{R}_x[p_x - \delta] \geq \sum x p_x I_c(\rho_x, \mathcal{N}) - \delta(1 + |X| \log \dim \mathcal{H}_A).
Finally, as $\bigotimes_x \rho_x^\otimes n_x \text{ and } \rho_x \otimes n \text{ are related by a permutation of the channel input Hilbert spaces and } \mathcal{N} \otimes n \text{ is invariant under such permutations, there exists an } (n, |X|, \epsilon) \text{ code of the same rate } R \text{ and average code density operator } \omega_{x^n} \text{ such that }

\|\omega_{x^n} - \rho_{x^n}\|_1 \leq |X| \epsilon.

\]

On the classical side, we shall need the Holevo-Schumacher-Westmoreland (HSW) theorem [17, 23], or rather its “typical codeword” version [8]. Consider the restriction of $\sigma^X_{AB}$ to $XB$:

$$\sigma^X_{XB} = \sum_{x \in X} p_x |x\rangle \langle x^X | \otimes \mathcal{N}(\phi^{A'}_{x^n}).$$

**Proposition 6 (HSW Theorem)** For any $\epsilon, \delta > 0$, define $r = I(X; B) - c' \delta$, for some constant $c'$, and $\mu = 2^{nr}$. For all sufficiently large $n$, there exists a classical encoding map $f : [\mu] \rightarrow T_{X, \delta}$ and a decoding POVM $\Lambda = (\Lambda_m)_{m \in [\mu]}$, such that

$$\text{Tr} \tau_m \Lambda_m \geq 1 - \epsilon, \forall m \in [\mu],$$

where

$$\tau_m = \mathcal{N} \otimes n(\phi^{A'n}_{\Lambda_m})$$

and $\phi^{A'n}_{x^n} = \bigotimes_{i=1}^n \phi^{A'}_{x_i}$.

Proposition 6 says that Bob may reliably distinguish among $\mu$ states of the form $\mathcal{N} \otimes n(\phi^{A'n}_{\Lambda_m})$, with $x^n \in T_{X, \delta}$. The idea behind the proof of the direct coding theorem is for Alice to use a different quantum code depending on the classical message to be sent. Bob first decodes the classical message (while causing almost no disturbance to the quantum system) by taking advantage of the distinguishability of the channel outputs for the different codes. Furthermore, the same information tells him which quantum decoding to perform! Thus, the classical information has been “piggy-backed” on top of the quantum information.

**Proof of Theorem 1 (coding for scenario Ib)** Recall, in scenario Ib Alice is transmitting half of the maximally entangled state $|\Phi\rangle$ through the channel. Define $\mu$, $f$, $\tau_m$ and $\Lambda$ as in proposition 6. For now we shall assume Alice and Bob have access to a common source of randomness with distribution $(P_\alpha)$. For each $m$ define a $\{ (p_x, \phi^{A'}_{x^n}), f(m) \}$-type $(n, \epsilon)$ random quantum code of rate $R = I(A \mid BX) - c \delta$ by the encoding and decoding operators $((E_\alpha, D^{\alpha}_{\Lambda_m}))_\alpha$. By proposition 6 and monotonicity of trace distance [20] we have, for all $m$ and sufficiently large $n$,

$$\| \sum_{\alpha} P_\alpha \tau'^{\alpha}_m - \tau_m \|_1 \leq \epsilon,$$

where $\tau'^{\alpha}_m = (\mathcal{N} \otimes n \circ E^{\alpha}_{\Lambda_m})(\pi)$. By proposition 6

$$\sum_{\alpha} P_\alpha \text{Tr} \tau'^{\alpha}_m \Lambda_m \geq 1 - 2\epsilon. \quad (15)$$

For a specific value of $\alpha$, the encoding map for our (classical, quantum) code is given by $((E^{\alpha}_{\Lambda_m})_{m \in [\mu]}$. The decoding instrument $D^{\alpha}_{\Lambda_m}$ is given by

$$D^{\alpha}_{\Lambda_m} : \rho \mapsto \sum_{m \in [\mu]} \Lambda_m \rho \sqrt{\Lambda_m}.$$
As usual, $D^\alpha = \sum_m D_m^\alpha$ denotes the induced quantum decoding operation. By (15), for all $m$, 
\[ \sum_\alpha P_\alpha p_m^{err,\alpha} \leq 2\epsilon, \]
(16)
where $p_m^{err,\alpha} = 1 - \Tr D_m^\alpha(\tau_m^\alpha)$. Defining an extension of $\tau_m^\alpha$
\[ \xi_m^\alpha = [1 \otimes (N^{\otimes n} \circ E_m^\alpha)][|\Phi\rangle\langle\Phi|], \]
it follows from (15) that
\[ \sum_\alpha P_\alpha \Tr \xi_m^\alpha(1 \otimes \Lambda_m) \geq 1 - 2\epsilon. \]

Invoking the gentle measurement lemma [29] and the concavity of the square root function,
\[ \sum_\alpha P_\alpha \| (1 \otimes D_m^\alpha)(\xi_m^\alpha) - (1 \otimes D_m^\alpha)(\xi_m^\alpha) \|_1 \leq 4\sqrt{\epsilon}, \]
which by the monotonicity of trace distance [20] gives
\[ \sum_\alpha P_\alpha \| (1 \otimes D_m^\alpha)(\xi_m^\alpha) - (1 \otimes D_m^\alpha)(\xi_m^\alpha) \|_1 \leq 4\sqrt{\epsilon}. \]

On the other hand,
\[ \| (1 \otimes D)(\xi_m^\alpha) - (1 \otimes D_m^\alpha)(\xi_m^\alpha) \|_1 \leq \sum_{m' \neq m} \| D_m^\alpha(\xi_m^\alpha) \|_1 \leq 2\epsilon. \]

Since, for all $m, \alpha$,
\[ F(1 \otimes D_m^\alpha(\xi_m^\alpha), \Phi) \geq 1 - \epsilon, \]
putting everything together gives, for all $m$,
\[ \sum_\alpha P_\alpha P_m^{err,\alpha} \leq 3\epsilon + 4\sqrt{\epsilon} \]
(17)
where $P_m^{err,\alpha} = 1 - F((1 \otimes D)^\alpha(\xi_m^\alpha), \Phi)$.

At this point our code relies on Alice and Bob having access to the common random index $\alpha$.
To prove the theorem it remains to “derandomize” the code, i.e. show that $p_m^{err,\alpha}$ and $P_m^{err,\alpha}$ are small for a particular value of $\alpha$, and for $m$ in a sufficiently large subset of $[\mu]$. By (10) and (17),
\[ \sum_\alpha P_\alpha \frac{1}{\mu} \sum_m (p_m^{err,\alpha} + P_m^{err,\alpha}) \leq 5\epsilon + 8\sqrt{\epsilon}. \]

There exists a particular $\alpha$ for which
\[ \frac{1}{\mu} \sum_m (p_m^{err,\alpha} + P_m^{err,\alpha}) \leq 5\epsilon + 4\sqrt{\epsilon}. \]

Fixing $\alpha$, expurgate the worst half of the codewords, i.e. those $m$ with the highest value of $p_m^{err,\alpha} + P_m^{err,\alpha}$. Now we have a code with both $p_m^{err,\alpha}$ and $P_m^{err,\alpha}$ bounded from above by $10\epsilon + 8\sqrt{\epsilon}$ for all remaining $m$, while the classical rate has only decreased by $\frac{1}{n}$. This concludes the proof. ■
4 Remarks on related problems

The first remark we make concerns replacing the classical–quantum dichotomy with the cryptographically relevant public–private one. In [5] quantum codes were built based on private information transmission ones. The purpose of the latter is for sending classical information about which the potential eavesdropper (to which the “environment” of the channel is granted) cannot learn anything. This should be contrasted with HSW codes which may be viewed as transmitting public information. One may now consider the problem of finding the simultaneous (public, private) capacity of $\mathcal{N}$. The answer follows in a straightforward manner from the methods of [5] and those used in proving theorem 1. Viewing the channel $\mathcal{N}$ as being embedded in an isometry $\mathcal{U}_N$ with an enlarged target Hilbert space, $U_N : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ ($\mathcal{H}_E$ is now given to the eavesdropper), the simultaneous (public, private) capacity region is given by the following modification of theorem 1:

- replace the state $\sigma^{XAB}$ by $\sigma^{XYB}$, obtained by sending the $A'$ part of
  \[ \sum_{xy} p_{xy} |x⟩⟨x|^X \otimes |y⟩⟨y|^Y \otimes p^{A'}_{xy} \]
  through the channel,

- replace $I(A)BX$ by $I(Y; B|X) − I(Y; E|X)$.

The corresponding theorem for classical “wire-tap” channels was proven in [6].

Secondly, one may conceive of a “static” analogue of the problem considered here, where Alice and Bob share many copies of some (mixed) state $\rho^{AB}$ instead of being connected by a quantum channel. In [10] the problem of generating common randomness (perfectly correlated bits) from such a resource using limited forward (Alice to Bob) classical communication was considered. There the “distillable common randomness” (DCR) was defined to be the maximum common randomness obtainable in excess of the classical communication invested, and was advertised as an (asymmetric) measure of the classical correlations in $\rho^{AB}$. In [11] the problem of one-way entanglement distillation was solved, yielding a similarly asymmetric measure of quantum correlations in $\rho^{AB}$. The next step is to unify the two results in a trade-off between DCR and distillable entanglement, which could now be argued to quantify the total correlations in the state. Based on the results of [10], [11] and the present paper we put forth the following conjecture: The simultaneously distillable (classical, quantum) resources are given precisely by theorem 1, where now the test states $\sigma^{XAB}$ are obtained by applying general instruments $D = (D_x)_{x \in X}$ to the $A$ part of $\rho^{AB}$, rather than arising from a channel. A sketch of the proof is as follows. The coding strategy involves double blocking. First use the protocol of [10] on a block of length $n$ to establish a good approximation to $X^n$ on Bob’s side using $\approx nH(B|X)$ bits of forward communication. This already gives us the desired DCR rate of $I(X; B)$. Now that Bob’s system includes $X^n$ they may use further blocking to distill entanglement at a rate of $I(A)BX$ [11]. The classical communication involved in this distillation has now turned into common randomness, effecting no net change in the DCR. The converse theorem is left as an exercise. A somewhat more ambitious goal would be to include the classical communication cost in the trade-off, giving a 3-dimensional region!

The final remark we make is that the “piggy-backing” idea used in the proof of theorem 1 provides an alternative coding strategy to the one in [29] for the classical capacity of $\mathcal{N}$ with limited entanglement assistance, thus establishing an additional connection between the two problems. The original paper on the entanglement assisted capacity [4] describes how to achieve the pair $(r, R) = (I(A; B)_\rho, -H(A|\rho)_\rho)$, for some $\rho^{AB} = (1^A \otimes N)(\phi^{AA'})$ arising from the channel. Using a mixture of codes corresponding to different channel inputs $|\phi_x⟩^{AA'}$, one trivially achieves $(r, R) = (I(A; B|X), −H(A|X))$ (with respect to $\omega^{XAB}$). As it turns out, Bob may use the distinguishability of the channel outputs of different code mixtures to send extra classical information at a rate of $I(X; B)$. This gives the region [7]. A detailed version of this argument will appear in [9].
5 Discussion

In conclusion, an information theoretical characterization of the simultaneous (classical, quantum) capacity region has been derived. The key idea was to use a different quantum code depending on the classical information to be sent, thus “piggy-backing” the classical information on top of the quantum one. The formula derived requires optimization over potentially arbitrarily many copies of the channel. We have shown that for a generalized dephasing channel a single copy suffices. We have also presented some ideas on cryptographic as well as static analogues of this problem.

We have already mentioned the open problem of including the classical communication cost in the trade-off for the static analogue. Another interesting extension of our work, which in fact served as our original motivation, is the following joint source-channel coding problem. In [15] the task of quantum compression with classical side information was considered. This is a “visible” source coding problem of a pure-state ensemble $E$. By storing partial information about the identity of the states (classically) at a rate $C$ it is possible to reduce the quantum storage rate to some value $Q(C)$. The joint source-channel coding variant of this problem is: Given $E$ and a channel $\mathcal{N}$, what is the rate at which Alice can send the quantum part of the ensemble over the channel? One approach is to first separate the source into a classical and quantum part using the trade-off of [15] and then send them simultaneously through the channel using the trade-off of theorem 1. This procedure is optimized over the ratio $\lambda$ of the classical and quantum rates which should coincide for the source and channel coding part. There are, however “well matched” source-channel pairs for which such a strategy is known to be suboptimal. The following example is due to Smolin [27]. The source is the equiprobable “trine” ensemble $\left( |0\rangle, |\epsilon^+\rangle, |\epsilon^-\rangle \right)$, where $|\epsilon^\pm\rangle = \frac{1}{2} |0\rangle \pm \sqrt{\frac{3}{2}} |1\rangle$ and the channel $\mathcal{N} : \mathcal{H}_3 \rightarrow \mathcal{H}_2$ has operation elements $\{ |0\rangle\langle 0|, |\epsilon^+\rangle\langle 1|, |\epsilon^-\rangle\langle 2| \}$. The channel has no quantum capacity and a classical capacity of 1. Our strategy of separating the source and channel coding gives a source-channel capacity of $1/\log 3$ transmitted copies of the ensemble per channel use. On the other hand, by simply feeding the identity of the state to the channel one achieves a source-channel capacity of 1. Finding a solution for an arbitrary $(E,\mathcal{N})$ pair remains an open question.

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A Proof of concavity of $S^{(1)}(\mathcal{N})$

Here we provide a proof that the region $S^{(1)}(\mathcal{N})$ defined by (6) is concave. Let $\sigma_0^{XAB}$ and $\sigma_1^{XAB}$ be two different states arising from the channel. For some $\lambda$ between 0 and 1, consider the state

$$\sigma^{UXAB} = \lambda |0\rangle\langle 0| \otimes \sigma_0^{XAB} + (1 - \lambda) |1\rangle\langle 1| \otimes \sigma_1^{XAB},$$

which also arises from the channel. Then

$$\lambda I(X;B)_{\sigma_0} + (1 - \lambda) I(X;B)_{\sigma_1} \leq I(UX;B)_{\sigma},$$

$$\lambda I(A)B_{\sigma_0} + (1 - \lambda) I(A)B_{\sigma_1} = I(A)B_{UX\sigma},$$

from which the claim follows.
B The capacity region for dephasing channels

In this section we define the notion of *degradable channels* and show that for such channels the quantum capacity is given by the single-letter formula

\[ Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) := \max_{\rho_{AB}} I(A)B, \]

where the maximization is over all states \( \rho_{AB} \) arising from the channel \( \mathcal{N} \). For the special case of *dephasing* channels we shall prove that the entire trade-off curve can be single-letterized.

Recall that a channel \( \mathcal{N} : \mathcal{H}_A \to \mathcal{H}_B \) can be defined by an isometric embedding \( U_N : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E \), followed by a partial trace over the “environment” system \( E \), so \( \mathcal{N}(\rho) = \text{Tr}_E U_N(\rho) \). This further induces the complementary channel \( \mathcal{N}^c : \mathcal{H}_A \to \mathcal{H}_E \) defined by \( \mathcal{N}^c(\rho) = \text{Tr}_B U_N(\rho) \).

We call a channel \( \mathcal{N} \) *degradable* when it may be degraded to its complementary channel \( \mathcal{N}^c \), i.e. when there exists a map \( T : \mathcal{H}_B \to \mathcal{H}_E \) so that:

\[ \mathcal{N}^c = T \circ \mathcal{N}. \]

To see that \( Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) \) for degradable channels, note that Bob’s output system \( B \) may be mapped by a fixed isometry onto a composite system \( B'E' \) such that the channels from \( A' \) to \( E' \) and to \( E \) are the same. Thus, for any state arising from the channel,

\[ I(AB) = H(B) - H(E) = H(B'E') - H(E) = H(B'E') - H(E') = H(B'|E'). \]

We can then use the inequality[20]

\[ H(B_1'B_2'|E_1'E_2') \leq H(B_1'|E_1') + H(B_2'|E_2') \]

to prove that single-letter maximization already achieves \( Q(\mathcal{N}) \).

A subclass of degradable channels of particular interest are generalized dephasing channels. The latter are defined on some \( d \)-dimensional Hilbert space with a preferred orthonormal basis \( \{|i\rangle\} \), such that all states belonging to this basis are transmitted without error, but pure superpositions of these basis states may become mixed. This implies that if \( \mathcal{N} \) is a dephasing channel then its isometric embedding \( U_N \) obeys

\[ U_N |i\rangle_{A'} = |i\rangle_{B'}|\phi_i\rangle_{E'}, \]

where the \( |\phi_i\rangle \) are generally not mutually orthogonal. When the \( |\phi_i\rangle \) are mutually orthogonal \( \mathcal{N} \) is the completely dephasing channel \( \Delta_d \):

\[ \Delta_d(\rho) = \sum_{i=1}^{d} |i\rangle\langle i|\rho|i\rangle\langle i|. \]

It is clear from the above that any dephasing channel \( \mathcal{N} \) obeys

\[ \Delta_d \circ \mathcal{N} = \mathcal{N} \circ \Delta_d = \Delta_d \]

\[ \mathcal{N}^c \circ \Delta_d = \mathcal{N}^c \]

Every dephasing channel is degradable, since \( \mathcal{N} \) may be degraded to \( \Delta_d \) which may be further degraded to \( \mathcal{N}^c \). In fact, the map \( T \) can be taken to be \( \mathcal{N}^c \). Therefore, \( Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) \) in what follows, the special properties of dephasing channels will allow us to prove an even stronger statement: that the outer boundary of \( S(\mathcal{N}) \) may be expressed as a single-letter formula.
Consider some state $\sigma^{XABE}$ arising from the channel. Bob may degrade his channel further by replacing his system $B$ by $B^{Y}$, where $Y$ now contains the completely dephased version of $B$ (this is why we label it as a classical system). Set $\lambda \geq 1$ and define

$$f_{\lambda}(\mathcal{N}) = \max_{\sigma^{XYE}} [H(Y) + (\lambda - 1)H(Y|X) - \lambda H(E|X)],$$

where the maximization is over all $\sigma^{XYE}$ arising from the channel ($\sigma^{XYE}$ is implicit in the entropies). We shall make use of the following lemma.

**Lemma 7** For two general dephasing channels $\mathcal{N}_{1}$ and $\mathcal{N}_{2}$

$$f_{\lambda}(\mathcal{N}_{1} \otimes \mathcal{N}_{2}) = f_{\lambda}(\mathcal{N}_{1}) + f_{\lambda}(\mathcal{N}_{2})$$

**Proof** The “$\geq$” direction follows from the fact that the input ensemble for $\mathcal{N}_{1} \otimes \mathcal{N}_{2}$ may be chosen to be a tensor product of the ones maximizing $f_{\lambda}(\mathcal{N}_{1})$ and $f_{\lambda}(\mathcal{N}_{2})$. To show the opposite inequality, in what follows let us refer to the state $\sigma^{XY:12|E}$ that maximizes $f_{\lambda}(\mathcal{N}_{1} \otimes \mathcal{N}_{2})$. Observe that

$$H(Y_{1}Y_{2}) = H(Y_{1}) + H(Y_{2}|Y_{1})$$

and

$$H(Y_{1}Y_{2}|X) = H(Y_{1}|X) + H(Y_{2}|Y_{1}X)$$

and

$$H(E_{1}E_{2}|X) = H(E_{1}|X) + H(E_{2}|E_{1}X) \leq H(E_{1}|X) + H(E_{2}|Y_{1}X),$$

the latter since $E_{1}$ contains a degraded version of $Y_{1}$ for all values of $X$. Hence

$$f_{\lambda}(\mathcal{N}_{1} \otimes \mathcal{N}_{2}) = H(Y_{1}Y_{2}) + (\lambda - 1)H(Y_{1}Y_{2}|X) - \lambda H(E_{1}E_{2}|X) \leq H(Y_{1}) + (\lambda - 1)H(Y_{1}|X) - \lambda H(E_{1}|X) + H(Y_{2}|Y_{1}) + (\lambda - 1)H(Y_{2}|XY_{1}) - \lambda H(E_{2}|XY_{1}) \leq f_{\lambda}(\mathcal{N}_{1}) + f_{\lambda}(\mathcal{N}_{2}).$$

We shall use Lagrange multipliers to calculate $S(\mathcal{N})$. By theorem 1, the quantity to be maximized is

$$I(X;B) + \lambda I(A)BX,$$

over all states $\sigma$ that arise from $\mathcal{N}^{\otimes n}$. Operationally it is clear that we should restrict attention to $\lambda \geq 1$, since $-\lambda$ is the slope of the boundary of $S(\mathcal{N})$ and a qubit channel may always be used to send classical bits at a unit rate. For any such state we have

$$I(X;B) + \lambda I(A)BX = H(B) + (\lambda - 1)H(B|X) - \lambda H(E|X) \leq H(Y) + (\lambda - 1)H(Y|X) - \lambda H(E|X) \leq f_{\lambda}(\mathcal{N}^{\otimes n}) \leq nf_{\lambda}(\mathcal{N}).$$

The first inequality follows from the fact that complete dephasing increases entropy, and is saturated by completely dephasing the input to $\mathcal{N}^{\otimes n}$ (recall that $\mathcal{N}$ commutes with $\Delta_{d}$). The third inequality is by lemma 6. Thus, for dephasing channels, $S(\mathcal{N}) = S^{(1)}(\mathcal{N})$, which makes the optimization problem tractable.

We now turn to the particular case of the qubit $p$-dephasing channel

$$\mathcal{N} = (1 - p)1_{2} + p\Delta_{2}.$$
It is easily checked that the outer boundary of $S(N)$ is achieved by the $\mu$–parametrized family of ensembles, $\mu \in [0, 1/2]$, consisting of diag$(\mu, 1 - \mu)$ and diag$(1 - \mu, \mu)$ chosen with equal probabilities. The trade-off curve is given by

$$(r, R) = \left(1 - h_2(\mu), h_2(\mu) - h_2\left(1/2 + 1/2\sqrt{1 - 16p(1 - p)\mu(1 - \mu)}\right)\right),$$

where $h_2(\mu) = -\mu \log_2 \mu - (1 - \mu) \log_2 (1 - \mu)$ is the binary entropy function. Figure 2 shows this curve for $p = 0.2$.

### C Proof of the cardinality bound

Here we justify the condition on the cardinality of $X$ in the statement of theorem 1. Caratheodory’s theorem states that in a $t$-dimensional Euclidean space, each point of a connected compact set $K$ can be represented as a convex combination of at most $t + 1$ points in $K$. Let $F(H)$ be the family of all density operators on the Hilbert space $H_A'$ of dimension $d$. Let $K$ be the image of $F(H)$ under some continuous mapping $f$ defined by $f(\rho) = (f_1(\rho), \ldots, f_t(\rho))$. As $F(H)$ is connected and compact, so is $K$. Then for any probability measure $\mu$ on the algebra of density operators of $H_A'$, Caratheodory’s theorem implies the existence of some finite ensemble $\{p_x, \rho_x : x \in X\}$, $|X| = t + 1$, such that

$$\int_{F(H)} \mu(d\rho)f_j(\rho) = \sum_{x \in X} p_x f_j(\rho_x), \quad \forall j \in [t].$$

Turning to our problem, the quantities $I(X; B)$ and $I(A\rangle BX)$ depend solely on the ensemble $E = \{p_x, \rho_x\}$, where $\rho_x := \phi_x^A$, and the channel $N$. Moreover, they only depend on the vector $\sum_x p_x f(\rho_x)$, where the vector valued function $f$ is defined so that $f_1, \ldots, f_{d^2 - 1}$ are the $d^2 - 1$ degrees of freedom of $\rho$ (linear in $\rho$), $f_{d^2}(\rho) = H(N(\rho))$ and $f_{d^2 + 1}(\rho) = I_c(\rho, N)$. Suppose that a particular point in $S^{1(1)}(N)$ corresponds to some ensemble $E' = \{\mu(d\rho), \rho\}$. The above implies that the same point is achievable by a finite ensemble with at most $d^2 + 2$ elements.

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