The Stability of Fake Flat Domain Walls on Kähler Manifold

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Abstract. In this paper, we study the stability of flat fake domain walls solution of fake $N=1$ supergravity in $d+1$ dimensions with Kähler surface as the sigma model. We start with Lagrangian for $N=1$ fake supergravity which is coupling between gravity and complex scalar in $d+1$ dimensions with scalar potential turned on. Then, as in supergravity theory, we demand that the scalar fields span the Kähler manifold. The equations of motion for fields can be reduced into first order equations by defining the superpotential and the resulting equations are called the projection equation and the fake BPS equation. Finally, we discuss about the stability of flat fake domain walls by investigating the critical points of the superpotential and the scalar potential.

1. Introduction

The fake supergravity was introduced by Freedman, Nunez, Schnabl, and Skenderis in [1]. The fake supergravity consist interaction between gravity and scalar field with the scalar potential turned on and defines the spinor energy using fake transformation rules similar to the real supergravity theory. The difference between fake and real theory is the existence of the holomorphic function $W(z)$ which called the holomorphic superpotential on fake theory. Using this fake spinor energy and the relation between the holomorphic superpotential and the scalar potential, we can define the first order equations, namely the projection and the fake BPS equations whose solutions automatically satisfy the second order equations of motion.

In this paper, we discuss about the stability of fake flat domain walls solution of fake $N=1$ supergravity theory in $d+1$ dimensions. We start with the fake supergravity action in $d+1$ dimensions which is a gravity and complex scalar field interaction with potential turned on. Similar with the real theory, we demand that the complex scalar field span a one dimensional Kähler manifold. The domain wall solution is established by taken some specific ansatz metric for the spacetime. Then, we characterize the general properties of the critical point of the potential and discuss about the stability of the solution.

The organization of this paper is as follows. First, we discuss about flat domain walls solution for fake $N=1$ supergravity. Second, we derive the projection and the fake BPS equation. And in the last section, we discuss about the stability of the solution near the critical point.
2. Flat domain walls solution for fake \( N = 1 \) supergravity

Let \( (\mathcal{M}^{d+1}, g_{\mu \nu}) \) be a spacetime manifold in \( d+1 \) dimensions equipped with metric tensor \( g_{\mu \nu} \), where \( \mu, \nu = 0, 1, \ldots, d \) and signature \((-+,+,+,-)\). The fake supergravity action is a scalar-gravity action in \( d+1 \) dimensions,

\[
S = \int d^{d+1}x \sqrt{-g} \left( \frac{1}{2\kappa^2} R - g_{\mu \nu} \partial_\mu z \partial_\nu \bar{z} - V(z, \bar{z}) \right),
\]

where \( g = \det g_{\mu \nu} \), \( R \) denotes a scalar Ricci of \( \mathcal{M}^{d+1} \), The fields \((z, \bar{z})\) is a complex scalar field and \( V(z, \bar{z}) \) denotes a real scalar potential. As in the real \( N=1 \) supergravity theory, we demand that the scalar field span a one dimensional Kähler manifold as the \( \sigma \)-model with the Kähler potential, \( K(z, \bar{z}) \) and the Kähler metric, \( g_{z\bar{z}} = \frac{\partial^2 K}{\partial z \partial \bar{z}} > 0 \).

The field equation of motion consist two equations, namely the Einstein field equation,

\[
\kappa^2 R_{\mu \nu} = g_{z\bar{z}} \left( \partial_\mu z \partial_\nu \bar{z} + \partial_\mu \bar{z} \partial_\nu z \right) + \frac{2}{d-1} g_{\mu \nu} V(z, \bar{z}),
\]

and the scalar field equation,

\[
\frac{g_{z\bar{z}}}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \partial^\mu z \right) + \sqrt{-g} g_{z\bar{z}} \partial_\mu z \partial_\nu \bar{z} - \partial_\nu V(z, \bar{z}) = 0,
\]

together with their complex conjugate.

The domain walls solution for the fake supergravity is established by considering a ground states that break partially a Lorentz invariance in our spacetime. Let \( x^\mu = (x^a, u), \ a, b = 0, 1, \ldots, d-1 \) be a local coordinate in \( \mathcal{M}^{d+1} \). Then, we have the following ansatz for the spacetime metric,

\[
d s^2 = A^2(u) \eta_{ab}(x^a) dx^a dx^b + du^2,
\]

where \( \eta_{ab} \) denotes the Minkowski metric in \( d \) dimensions and \( A(u) \) is the warp factor.

The Ricci tensor for our spacetime is given by

\[
R_{\mu \nu} = -\eta_{ab} \delta_\mu^a \delta_\nu^b \left[ (d-1)A^2 + AA^\prime \right] - \delta_\mu^a \delta_\nu^d d \left( \frac{A^\prime}{A} \right),
\]

and the scalar Ricci is given by

\[
R = -d \left[ (d-1) \left( \frac{A^\prime}{A} \right)^2 + 2 \frac{A^n}{A} \right],
\]

where \( A^\prime = \frac{dA}{du} \).

3. The projection and the fake BPS equation

In this section, we derive the projection equation for the warp factor and the fake BPS equation for the complex scalar field.

In this paper, we consider the \( u \)-dependent scalar field for the domain walls solution, \( z = z(u) \).

Using this condition, the equations of motion (2) and (3) are reduced into second order ordinary differential equations,

\[
\frac{A^n}{A} + (d-1) \left( \frac{A^\prime}{A} \right) + \frac{2\kappa^2}{(d-1)} V(z, \bar{z}) = 0,
\]

and

\[
\frac{2\kappa^2}{d(d-1)} V(z, \bar{z}) = 0,
\]
\[ z^* + \frac{d}{A} A'z' + g^\xi \partial_\xi g_{\xi \tau} z'z' - g^\pi \partial_\pi V(z, \bar{z}) = 0. \tag{9} \]

Introducing a holomorphic function, \( W(z) \) called holomorphic superpotential and we consider the set of first order differential equations given by,

\[
\begin{align*}
\frac{A'}{A} &= \mathcal{W}(z, \bar{z}), \\
z' &= -\frac{(d-1)}{\kappa^2} g^\pi \partial_\pi \mathcal{W}, \\
\bar{z}' &= -\frac{(d-1)}{\kappa^2} g^\pi \partial_\pi \mathcal{W},
\end{align*}
\tag{10-12}
\]

where the real function \( \mathcal{W}(z, \bar{z}) \) is defined by

\[
\mathcal{W}(z, \bar{z}) = e^{\frac{-i}{2k(1-1)}} |W(z)|. 
\tag{13}
\]

The equation (10) is known as the projection equation and the equations (11) and (12) are called the fake BPS equations. The projection and the fake BPS equations are the solutions of the equations of motion (7), (8), and (9), if the following relation holds,

\[
k^2 V(z, \bar{z}) = \frac{(d-1)}{\kappa^2} g^\pi \partial_\pi \mathcal{W} \partial_\pi \mathcal{W} - \frac{d(d-1)}{2} \mathcal{W}^2. 
\tag{14}
\]

The projection and the fake BPS equations are well-posed, hence for each initial value, the solution exist and unique. In the next section, we discuss about the stability of the domain walls solution for equations (10), (11), and (12).

4. Stability of fake domain walls

Before we discuss about stability of domain walls solutions of fake supergravity, we discuss about the general properties of critical points of the real function \( \mathcal{W}(z, \bar{z}) \).

Let \( \Phi_c = (z_c, \bar{z}_c) \) be a critical point of \( \mathcal{W}(z, \bar{z}) \). Using definition in equation (13), we can write the eigenvalues and the determinant of Hessian matrix of \( \mathcal{W}(z, \bar{z}) \) evaluated at critical point as

\[
\lambda_{1,2} = \kappa^2 g^\pi(\Phi_c) \mathcal{W}(\Phi_c) \pm 2 |\partial_\pi^2 \mathcal{W}(\Phi_c)|, \\
\det H_W = \kappa^4 g^\pi(\Phi_c)^2 \mathcal{W}^2(\Phi_c) - 4 |\partial_\pi^2 \mathcal{W}(\Phi_c)|^2, 
\tag{15-16}
\]

where

\[
\partial_\pi^2 \mathcal{W}(\Phi_c) = \frac{\kappa^2}{2W(\Phi_c)} \left[ K_{\pi}(\Phi_c) W(\Phi_c) + K_\pi(\Phi_c) \frac{dW}{dz}(z_c) + \frac{1}{\kappa^2} \frac{d^2W}{dz^2}(z_c) \right]. 
\tag{17}
\]

Point \( \Phi_c \) is a local minimum point if all eigenvalues in (15) are positive, hence

\[
\frac{\kappa^2}{2} g^\pi(\Phi_c) \mathcal{W}(\Phi_c) > |\partial_\pi^2 \mathcal{W}(\Phi_c)|, 
\tag{18}
\]

while a saddle point requires

\[
\frac{\kappa^2}{2} g^\pi(\Phi_c) \mathcal{W}(\Phi_c) < |\partial_\pi^2 \mathcal{W}(\Phi_c)|, 
\tag{19}
\]

Moreover, \( \Phi_c \) is a degenerate critical point if

\[
\frac{\kappa^2}{2} g^\pi(\Phi_c) \mathcal{W}(\Phi_c) = |\partial_\pi^2 \mathcal{W}(\Phi_c)|. 
\tag{20}
\]

In general there is no local maximum point for the model.
In this part, we investigate the critical point of the scalar potential \( V(z, \bar{z}) \) for flat domain walls in equation (14) and its relation to the critical point of \( \mathcal{W}(z, \bar{z}) \). The eigenvalues and the determinant of the Hessian matrix of \( V(z, \bar{z}) \) evaluated at the critical point \( \Phi_c \) can be expressed as follows,

\[
\begin{align*}
\lambda_{1,2} &= \frac{2}{\kappa^2} (d-1)^2 g_\sigma (\Phi_c) |\partial_\sigma V(\Phi_c)|^2 - \frac{1}{2} (d^2 - 1) g_{\sigma\bar{\sigma}} (\Phi_c) W^2 (\Phi_c) \\
\det H &= \frac{4 (g^\sigma (\Phi_c))^2}{\kappa^8} (d-1)^4 |\partial_\sigma V(\Phi_c)|^4 + \frac{1}{4} (d^2 - 1)^2 g^\sigma_{\bar{\sigma}} (\Phi_c) W^4 (\Phi_c) \\
&\quad - \frac{2}{\kappa^2} (d-1)^2 (d^2 + 1) W^2 (\Phi_c) |\partial_\sigma V(\Phi_c)|^2.
\end{align*}
\]

Based on equations (21) and (22), the local minimum of the scalar potential exists if

\[
|\partial_\sigma V(\Phi_c)| > \frac{\kappa^2}{2} \left( \frac{d+1}{d-1} \right) g_{\sigma\bar{\sigma}} (\Phi_c) W(\Phi_c),
\]

whereas the local maximum requires

\[
|\partial_\sigma V(\Phi_c)| < \frac{\kappa^2}{2} g_{\sigma\bar{\sigma}} (\Phi_c) W(\Phi_c).
\]

Furthermore the saddle point occurs if

\[
\frac{\kappa^2}{2} g_{\sigma\bar{\sigma}} (\Phi_c) W(\Phi_c) < |\partial_\sigma V(\Phi_c)| < \frac{\kappa^2}{2} \left( \frac{d+1}{d-1} \right) g_{\sigma\bar{\sigma}} (\Phi_c) W(\Phi_c),
\]

and degenerate critical point achieved if

\[
|\partial_\sigma V(\Phi_c)| = \frac{\kappa^2}{2} g_{\sigma\bar{\sigma}} (\Phi_c) W(\Phi_c),
\]

\[
|\partial_\sigma V(\Phi_c)| = \frac{\kappa^2}{2} g_{\sigma\bar{\sigma}} (\Phi_c) W(\Phi_c).
\]

From equations (18), (19), (24) and (25), any local minima of \( \mathcal{W}(z, \bar{z}) \) are mapped into local maxima of the scalar potential \( V(z, \bar{z}) \) and saddle point of \( \mathcal{W}(z, \bar{z}) \) are mapped into saddles or local minima of the scalar potential \( V(z, \bar{z}) \). Moreover the equation (26) comes naturally from the fact that degenerate critical points of \( \mathcal{W} \) are mapped into degenerate points of \( V(z, \bar{z}) \). Since both Hessian matrix \( H_\mathcal{W} \) and \( H_V \) evaluated at degenerate point are singular, these special points are called intrinsic degenerate critical points of \( V(z, \bar{z}) \).

At critical point \( \Phi_c \), the Ricci tensor is given by,

\[
R_{\mu\nu} = -D W^2 (\Phi_c) g_{\mu\nu},
\]

and we have the following condition for the holomorphih superpotential,

\[
\nabla_{\bar{z}} W(z_c) \equiv \frac{dW}{dz} + \kappa^2 K W \bigg|_{z_c} = 0.
\]

Therefore, there are two possibilities of Lorentz invariant \( N=1 \) vacuums related to the critical point \( \Phi_c \), namely Minkowskian and AdS vacuums.

The Minkowskian vacuums satisfies the following conditions,

\[
W(z_c) = \frac{dW}{dz_c} (z_c) = 0.
\]
Thus, we can simplify the eigenvalues and the determinant for both $W(z, \bar{z})$ and $V(z, \bar{z})$ in equations (15), (16), (21) and (22) as follow,

$$\lambda_{1,2}^{W} = e^{\frac{2}{\kappa} \kappa_{(\Phi_{c})}} \left| \frac{d^{2}W}{dz^{2}}(z_{c}) \right|,$$

$$\det H_{W} = -e^{\frac{2}{\kappa} \kappa_{(\Phi_{c})}} \left| \frac{d^{2}W}{dz^{2}}(z_{c}) \right|^{2},$$

$$\lambda_{1,2}^{V} = \frac{(d-1)^{2}}{2\kappa^{4}} g^{2}(\Phi_{c}) e^{2\kappa_{(\Phi_{c})}} \left| \frac{d^{2}W}{dz^{2}}(z_{c}) \right|^{2},$$

$$\det H_{V} = \frac{(d-1)^{4}}{4\kappa^{8}} \left( g^{2}(\Phi_{c}) \right)^{2} e^{2\kappa_{(\Phi_{c})}} \left| \frac{d^{2}W}{dz^{2}}(z_{c}) \right|^{4}.$$  

The non-degeneracy requires that

$$\frac{d^{2}W}{dz^{2}}(z_{c}) \neq 0.$$  

Hence, it implies that the superpotential $W(z)$ has to be at least a quadratic polynomial. In other words, the family of Minkowskian ground states satisfying equation (35) is said to be isolated. Moreover, in these vacuums the possible non-degenerate critical points of $W(z, \bar{z})$ are saddles which are mapped into the minima of the scalar potential, $V(z, \bar{z})$. This properties will not change even if we set $\kappa \to 0$. This means that there is an isomorphism between Minkowskian vacuums in local and global supersymmetric theories for $W(z_{c}) = 0$.

The AdS vacuums satisfies the following conditions,

$$W(z_{c}) \neq 0, \quad \nabla W(z_{c}) = 0.$$  

Isolated AdS ground states demand that the function $W(z, \bar{z})$ must not satisfy equation (26) and (27). Unlike Minkowskian vacuums, the first order analysis of AdS vacuums does depend on the $U(1)$-connection i.e., the term $\kappa^{2} K_{x}(\Phi)$ which is non-holomorphic, beside superpotential $W(z)$. To solve such non-holomorphic equation we choose the critical points of the real function $W(z, \bar{z})$ are determined by the critical points of the holomorphic superpotential $W(z)$, i.e., that

$$\frac{dW}{dz}(z_{c}) = 0,$$

which implies that

$$K_{x}(\Phi_{c}) = K_{x}(\Phi_{c}) = 0,$$

since $W(z_{c}) \neq 0$. This also means that at the ground states the $U(1)$-connection does not appear in any order analysis.

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