Geometry Of The Expected Value Set And
The Set-Valued Sample Mean Process

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Abstract

The law of large numbers extends to random sets by employing Minkowski addition. Above that, a central limit theorem is available for set-valued random variables. The existing results use abstract isometries to describe convergence of the sample mean process towards the limit, the expected value set. These statements do not reveal the local geometry and the relations of the sample mean and the expected value set, so these descriptions are not entirely satisfactory in understanding the limiting behavior of the sample mean process. This paper addresses and describes the fluctuations of the sample average mean on the boundary of the expectation set.

Keywords: Random sets, set-valued integration, stochastic optimization, set-valued risk measures

Classification: 90C15, 26E25, 49J53, 28B20

1 Introduction

Artstein and Vitale [4] obtain an initial law of large numbers for random sets. Given this result and the similarities of Minkowski addition of sets with addition and multiplication for scalars it is natural to ask for a central limit theorem for random sets. After some pioneering work by Cressie [11], Weil [28] succeeds in establishing a reasonable result describing the distribution of the Pompeiu–Hausdorff distance between the sample average and the expected value set. The result is based on an isometry between compact sets and their support functions, which are continuous on some appropriate and adapted sphere (cf. also Norkin and Wets [20] and Li et al. [17]; cf. Kuelbs [16] for general difficulties). However, the Pompeiu–Hausdorff distance of random sets is just an $\mathbb{R}$-valued random variable and its distribution is on the real line. But how do these sample averages, as sets in $\mathbb{R}^d$, converge locally? We address this question for selected points at the boundary of the expected value set.

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This paper elaborates local features of set-valued convergence of sample means and the distribution of particular selections is in focus of our interest. To develop the intuitive understanding we specify and restrict ourselves occasionally to a discrete setting chosen in Cressie [11]; this situation is natural for set-valued risk functionals in mathematical finance as well.

Outline of the paper. We introduce the expectation and the Pompeiu–Hausdorff distance in Section 2. Of particular interest are the boundary points of the expected value. We classify the boundary points in Section 3 and discuss relations between boundary points of the expected value set and corresponding points of the sample means. Section 4 addresses the Law of Large Numbers and Section 5 the Central Limit Theorem. These sections contain our main results, which describe convergence of sample means relative to particular points on the boundary. Section 6 concludes and summarizes the results.

2 Mathematical setting

We work in \( \mathbb{R}^d \) with norm \( \| \cdot \| \). We denote this space by \( X := (\mathbb{R}^d, \| \cdot \|) \), its dual by \( X^* := (\mathbb{R}^d, \| \cdot \|_*) \) and the unit sphere in the dual by \( S^{d-1} := \{ x : \| x \|_* = 1 \} \). The Minkowski sum (also known as dilation) of two subsets \( A \) and \( B \) of \( \mathbb{R}^d \) is \( A + B := \{ a + b : a \in A, b \in B \} \) and the product with a scalar \( p \) is \( p \cdot A := \{ p \cdot a : a \in A \} \). We denote the convex hull of a set \( A \) by \( \text{conv} \, A \) and its topological closure by \( \overline{\text{conv}} \, A \).

Pompeiu–Hausdorff Distance. The appropriate distance on \( \mathcal{C}_d \), the set of compact subsets of \( \mathbb{R}^d \), is the Pompeiu–Hausdorff distance. For this define the point-to-set distance as \( d(a, B) := \inf_{b \in B} \| b - a \| \). The deviation of the set \( A \) from the set \( B \) is \( \mathbb{D}(A, B) := \sup_{a \in A} d(a, B) \). \(^1\) (Some references call \( \mathbb{D}(A, B) \) the excess of \( A \) over \( B \), cf. Hess [14].) The Pompeiu–Hausdorff distance is \( \mathbb{H}(A, B) := \max \{ \mathbb{D}(A, B), \mathbb{D}(B, A) \} \), cf. also Rockafellar and Wets [24].

Note that \( \mathbb{D}(A, B) = 0 \) iff \( A \) is contained in the topological closure of \( B \), \( A \subseteq \overline{B} \), and \( \mathbb{H}(A, B) = 0 \) iff \( \overline{A} = \overline{B} \); moreover \( \mathbb{H}(A, B) = \mathbb{H}(\overline{A}, B) \).

If \( \overline{A} \) and \( \overline{B} \) are compact and convex then it is enough to consider their boundaries \( \partial A \) and \( \partial B \), as we have in addition that \( \mathbb{H}(A, B) = \mathbb{H}(\partial A, \partial B) \) (cf. Wills [29]). In this case we have

\[
\mathbb{H}(A, B) = \| b - a \| \tag{1}
\]

for some \( a \in \partial A \) and \( b \in \partial B \).

\(^1\) An equivalent definition is \( \mathbb{D}(A, B) := \inf \{ \varepsilon > 0 : A \subset B + \text{Ball}_\varepsilon(0) \} \); here, \( B_\varepsilon := B + \text{Ball}_\varepsilon(0) \) is often called \( \varepsilon \)-fattening, or \( \varepsilon \)-enlargement of \( B \).
Lemma 2.1 (Castaing and Valadier [10]). The deviation \( D \) and the Pompeiu–Hausdorff distance \( H \) satisfy the triangle inequality, \( D(A, C) \leq D(A, B) + D(B, C) \) and \( H(A, C) \leq H(A, B) + H(B, C) \). For a Polish space \((X, d)\) the space \((\mathcal{C}, H)\), where \( \mathcal{C} \) is the set of all nonempty, compact and convex subsets of \( X \), is a Polish space again (i.e., a complete, separable and metric space).

By the preceding lemma \((\mathcal{C}, d, H)\), the nonempty compact subsets of \( \mathbb{R}^d \) endowed with the Pompeiu–Hausdorff distance \( H \), is a measurable space. In what follows we equip \( \mathcal{C} \) with the sigma algebra of its Borel subsets generated by the family of closed subsets of \( \mathcal{C} \).

2.1 Expectation

We consider a set-valued random variable \( Y : \Omega \Rightarrow \mathbb{R}^d \) (commonly random sets) on some complete probability space \((\Omega, \mathcal{F}, P)\). Throughout the paper we assume that the set-valued random variable \( Y : \Omega \Rightarrow \mathbb{R}^d \) is compact-valued and measurable, i.e., the associated map \( Y : \Omega \rightarrow (\mathcal{C}, d) \) is measurable.

Definition 2.2 (Expectation, cf. Molchanov [18, Definition 1.12]). The expectation \( \mathbb{E} Y \) of a set-valued random variable \( Y : \Omega \Rightarrow \mathbb{R}^d \) is the collection \( \mathbb{E} Y := \left\{ \int_{\Omega} y dP : y(\cdot) \text{ an integrable selection of } Y \right\} \subseteq \mathbb{R}^d \); (2)

a function \( y : \Omega \rightarrow \mathbb{R}^d \) is an integrable selection of \( Y \) if \( y(\omega) \in Y(\omega) \) for \( P \)-almost every \( \omega \in \Omega \) and \( y(\cdot) \) is \( P \)-integrable, i.e., \( \int \|y(\omega)\| P(d\omega) < \infty \). The expectation (2) (also Aumann expectation) is often denoted \( \mathbb{E} Y = \int Y dP \) as well.

Atomic versus non-atomic probability spaces

Consider a set-valued random variable \( Y \) defined on an atomic space \((\Omega, \mathcal{F}, P)\) such that

\[
P(Y = K_1) = p_1, \ P(Y = K_2) = p_2, \ldots \text{ and } P(Y = K_J) = p_J
\]

(3)

for finitely many sets \((K_j)_{j=1}^J\) with weights \( p_j > 0, \sum_{j=1}^J p_j = 1 \). From the definition of the expected value (2) it is evident that

\[
\mathbb{E} Y = \int Y dP = \sum_{j=1}^J p_j K_j
\]

(cf. Cressie [11] and Figure 1 for illustration). \( \mathbb{E} Y \) is moreover compact, provided that all \( K_j \) are compact.

Note that \( \mathbb{E} Y \) is not necessarily convex. The expectation \( \mathbb{E} Y \) is convex, provided that all sets \( K_j \) are convex, as \( \mathbb{E} \text{conv} Y = \sum_{j=1}^J p_j \text{conv} K_j = \text{conv} \sum_{j=1}^J p_j K_j = \text{conv} \mathbb{E} Y \), where \( \text{conv} A \) denotes the convex hull of the set \( A \).
The situation notably changes for non-atomic probability spaces. Aumann’s Theorem (cf. Aumann [5, Theorem 2]) ensures that $\mathbb{E}Y$ is non-empty, compact and convex, provided that $P$ does not have atoms and there is an integrable random variable $h(\cdot)$, called an envelope function, such that

$$
\|Y(\omega)\| := \sup_{y \in Y(\omega)} \|y\| \leq h(\omega).
$$

(4)

Unless stated differently we shall assume the standard, non-atomic probability space in what follows. Further, the random set $Y$ is assumed to be compact, convex valued and integrably bounded, i.e., $\|Y(\cdot)\|$ is measurable and $\int \|Y(\cdot)\| \, dP < \infty$ (cf. Molchanov [18, Definition 1.11]): Section 4.1 below outlines why this setting is not an essential restriction in investigating the law of large numbers and the central limit theorem. As well, the chosen setting insures that the expectation $\mathbb{E}Y$ defined in (2) is closed (cf. Molchanov [18, Theorem 1.24]).

### 2.2 Support function

The support function of a set $A \subseteq X$ is

$$
s_A(x^*) := \sup_{a \in A} x^*(a),
$$

(5)

where $x^* \in X^*$ is from the dual $X^* = \left( \mathbb{R}^d, \|\cdot\| \right)^* = \left( \mathbb{R}^d, \|\cdot\|_* \right)$.

By the Fenchel–Moreau-duality theorem (cf. Rockafellar [22]) we have the relation

$$
\text{conv} A = \{ s_A^* < \infty \},
$$

where $s_A^*(a) := \sup_{x^* \in X^*} x^*(a) - s_A(x^*)$ is the convex conjugate of $s_A$. The correspondence $A \mapsto s_A$ is one-to-one (injective) between convex, compact sets $A \in \mathcal{C}_d$ and finite valued convex positively homogeneous functions on $\mathbb{R}^d$ and satisfies the isometry

$$
\sup_{a \in A} \|a\| = \sup_{a \in A} \sup_{\|x^*\|_* = 1} x^*(a) = \sup_{\|x^*\|_* = 1} s_A(x^*) = \|s_A\|_\infty,
$$

(6)

where the norm on the space $C(S^{d-1})$ of bounded and continuous functions defined on the unit sphere in the dual space

$$
S^{d-1} := \{ x^* \in \mathbb{R}^d : \|x^*\|_* = 1 \} = \partial B_{X^*}.
$$

(7)
is \( \|f\|_\infty := \sup_{x \in S^{d-1}} |f(x)| \).

As the support function is positively homogeneous \((s_A(\lambda x^*) = \lambda s_A(x^*) \text{ for } \lambda > 0)\), one may restrict \(s_A\) to the unit sphere of the dual without losing information (cf. (7)). The mapping \(K \mapsto s_K|_{\partial B_{X^*}}\) (the restriction to the sphere \(S^{d-1}\)) is an isometric isomorphism from \(\mathfrak{C}_d\), the convex, compact subsets of \(\mathbb{R}^d\) onto \(C(S^{d-1})\), the Banach space of continuous functions endowed with the norm \(\|f\|_\infty = \sup_{s \in \partial B_{X^*}} |f(s)|\) on the compact set \(S^{d-1} = \partial B_{X^*}\) by (6).

### 2.3 Tangent planes

The subdifferential of an \(\mathbb{R}\)-valued function \(f : X^* \to \mathbb{R}\) at a point \(x^* \in X^*\) is the set

\[
\partial f (x^*) := \{ u \in X : f (z^*) - f (x^*) \geq z^*(u) - x^*(u) \text{ for all } z^* \in X^* \} \subseteq X.
\]

The subdifferential \(\partial f (x^*)\) is a convex subset of \(X\), so \(\partial f\) is a set-valued mapping,

\[
\partial f : X^* \rightrightarrows X \quad x^* \mapsto \partial f (x^*).
\]

With the subdifferential at hand we have the following characterization of the subdifferential of the support function \(s_A\) of a set \(A\) (the bipolar theorem for indicator functions), which will turn out useful in investigating the expected value set.

**Lemma 2.3.** The support function \(s_A\) has the subdifferential

\[
\partial s_A (x^*) = \text{arg max}_{\text{conv } A} x^*,
\]

where \(x^* \in X^*\) and

\[
\text{arg max}_D f := \text{arg max} \{ f(d) : d \in D \} = \{ x \in D : f(x) \geq f(x') \text{ for all } x' \in D \}.
\]

Moreover, \(\partial s_A (x^*) \subseteq \partial \text{conv } A\) for every \(x^* \in X^*\).

**Proof.** Note first that \(s_A = \text{conv } s_A\).

Indeed, it is evident that \(s_A \leq \text{conv } s_A\) by definition; for the converse choose \(a = \sum_{i=1}^n \lambda_i a_i \in \text{conv } A\) with \(a_i \in A\), \(\lambda_i > 0\), \(i = 1, \ldots, n\) and \(\sum_{i=1}^n \lambda_i = 1\) so that \(\text{conv } s_A (x^*) < x^*(a) + \varepsilon\). By linearity we also have that \(\text{conv } s_A (x^*) < x^*(a_{i^*}) + \varepsilon\), where \(i^*\) is chosen so that \(x^*(a_{i^*}) \geq x^*(a_i)\) for all \(i = 1, \ldots, n\).

We deduce then from Rockafellar [21, Corollary 23.5.3] that \(\text{arg max}_{a \in \text{conv } A} x^*(a) = \partial \text{conv } s_A (x^*)\), so that the assertion follows.  \(\square\)
Remark 2.4 (Hörmander’s theorem, cf. Hörmander [15]). The concepts of Hausdorff distance and support functions introduced above link to a nice ensemble, as the deviation $D$ can also be states as $D(A, C) = \sup_{a \in A} \inf_{c \in C} \sup_{\|x^*\|_* \leq 1} x^*(a - c)$. It follows from the max-min inequality that

$$D(A, C) = \sup_{a \in A} \inf_{c \in C} \sup_{\|x^*\|_* \leq 1} x^*(a - c) \geq \sup_{a \in A} \inf_{c \in C} x^*(a - c)$$

Assuming that $A$ and $C$ are convex it follows from compactness of the dual ball and the minimax theorem (Fan [13, Theorem 2]) that equality holds in (9), hence

$$D(\text{conv } A, \text{conv } C) = \sup_{\|x^*\|_* \leq 1} \{s_A(x^*) - s_C(x^*)\},$$

the Pompei–Hausdorff distance thus is

$$H(\text{conv } A, \text{conv } C) = \sup_{\|x^*\|_* \leq 1} |s_A(x^*) - s_C(x^*)|,$$

expressed in terms of seminorms. These observations and (6) convincingly relate the Pompeiu–Hausdorff distance with Minkowski addition of convex sets.

It follows from the preceding discussion and remarks that for relatively compact sets $A$ and $C$ there are $a \in \partial A$, $c \in \partial C$ and $\|x^*\|_* \leq 1$ such that $D(A, C) = \|c - a\| = x^*(a - c)$. $x^*$ is an outer normal for both sets, conv $A$ and conv $C$.

3 The relative boundary of the expected value

We shall use tangent planes to investigate the convex expected value set. To this end let $f \in X^*$ be a linear functional. By Aumann’s Theorem, the set-valued mapping

$$\omega \mapsto \partial s_Y(\omega)(f) \subseteq \mathbb{R}^d$$

is measurable and $\mathbb{E} \partial s_Y(f) = \int \partial s_Y(\omega)(f) P(d\omega)$ is non-empty, compact and convex (cf. Aumann [5, Theorem 2]). We continue with a characterization of this expected value. For a related result on the interchangeability of the differentiation $\partial$ and expectation $\mathbb{E}$ we refer to Rockafellar and Wets [23].

Proposition 3.1. Suppose that $f \in X^*$. Then

$$\mathbb{E} \partial s_Y(f) = \partial s_{\mathbb{E}Y}(f) \subseteq \partial \mathbb{E} Y.$$
Proof. Let $e \in \mathbb{E} \partial s_Y (f)$ have the representation $e = \int e \, dP$ and recall from Lemma 2.3 that $e(\omega) \in \partial s_{Y(\omega)} (f) = \arg \max_{y \in Y(\omega)} f(y)$. Note as well that $e(\cdot)$ can be chosen measurable by the Kuratowski and Ryll–Nardzewski measurable selection theorem, cf. Bogachev [7, Volume II, page 36] or Aumann [5, Theorem 2]. Hence, for every measurable $y$ with $y(\cdot) \in Y(\cdot)$ we have that $f(e(\omega)) \geq f(y(\omega))$. Define $y := \int y \, dP$, then

$$f(e) = f \left( \int e \, dP \right) = \int f(e(\omega)) P(\, d\omega) \geq \int f(y(\omega)) P(\, d\omega) = f \left( \int y \, dP \right) = f(y)$$

by linearity of $f$ for every measurable selection $y$. Hence, $e \in \mathbb{E} \arg \max_{y \in Y} f(y) = \partial s_{E Y} (f)$ by (8), which is the inclusion $\subseteq$ of set-equality in (12).

For the converse assume that $e \in \partial s_{E Y} (f) \setminus \mathbb{E} \partial s_Y (f)$. As $\mathbb{E} \partial s_Y (f)$ is convex and compact it follows from the separation theorem that there is an $\alpha \in \mathbb{R}$ such that

$$f(e) = \int f(e) dP > \alpha > \int f(y) dP \tag{13}$$

for every measurable $y(\omega) \in \partial s_{Y(\omega)} (f)$. Notice that $y(\omega) \in \partial s_{Y(\omega)} (f) = \arg \max_{y \in Y(\omega)} f(y) \subseteq Y(\omega)$. By the particular choice of $e$ it follows for $y := \int y \, dP \in \mathbb{E} Y$ that $f(e) > f(y)$.

However, by (13), on a set of strictly positive $P$-measure we have that

$$P(\{\omega : f(e(\omega)) > f(y(\omega))\}) > 0.$$ 

On this set $y(\omega) \notin \arg \max_{y \in Y(\omega)} f(y) = \partial s_{Y(\omega)} (f)$, because $f(e(\omega)) > f(y(\omega))$. This is a contradiction, because $y \in \partial s_{Y(\omega)} (f) = \arg \max_{y \in Y(\omega)} (f)$ $P$-almost everywhere.

The remaining inclusion follows from Lemma 2.3.

We deduce from the previous proposition that the set-valued subdifferential $\partial$ and the set-valued expectation $\mathbb{E}$ commute. Moreover, the set-valued subdifferential of the support function basically is its arg max-set, which is an element from the boundary of the respective set. This is another hint that the boundary $\partial \mathbb{E} Y$ plays a central role, which we intend to investigate in more detail in what follows.

### 3.1 Extreme and exposed points

It will be convenient to classify the boundary points of the convex set $\mathbb{E} Y$ based on the following definitions.

**Definition 3.2** (Extreme points, exposed points). Let $K$ be a convex set.

(i) $k \in K$ is an extreme point if $k = \frac{1}{2} (k_1 + k_2)$ for $k_1 \in K$ and $k_2 \in K$ implies that $k_1 = k_2$. 


(ii) \( k \in K \) is an exposed point if there is a linear, continuous functional \( f \) such that \( f(k) > f(x) \) for all \( x \in K \setminus \{k\} \). \( f \) is said to expose \( k \in K \). The collection of all exposed points of the set \( K \) is denoted by \( \exp K \).

(iii) \( K \) is strictly convex, if \( \{(1 - \lambda)k_0 + \lambda k_1 : 0 < \lambda < 1\} \subseteq \overset{\circ}{K} \), the interior of \( K \), whenever \( k_0, k_1 \in K \) and \( k_0 \neq k_1 \).

Remark 3.3. The point \( e \) in Figure 2b on page 16 is extreme, but not exposed.

Remark 3.4 (Boundary points of strictly convex sets are exposed). If \( K \) is strictly convex, then every boundary point \( k \in \partial K \) is exposed. Indeed, let \( f \) be a linear, separating functional such that \( f(k) > f(x) \) for all \( x \in \overset{\circ}{K} \) (\( f \) exists by the Hahn–Banach theorem). Suppose there were another \( \tilde{k} \in K \) such that \( f(\tilde{k}) = f(k) \). As \( \frac{1}{2} (k + \tilde{k}) \in \overset{\circ}{K} \) by assumption it follows that \( f(k) > f(\frac{1}{2} (k + \tilde{k})) = \frac{1}{2} (f(k) + f(\tilde{k})) \), which is a contradiction. Hence \( f \) exposes \( k \) and \( \partial K = \exp K \).

3.2 The boundary of \( \mathbb{E} Y \)

We return to the geometry of \( \mathbb{E} Y \) and discuss exposed points of \( \mathbb{E} Y \) first. The next theorem elaborates that exposed points of \( \mathbb{E} Y \) are comparably seldom, as being exposed in \( \mathbb{E} Y \) means that the exposing functional exposes points of \( Y(\omega) \) for almost every \( \omega \in \Omega \).

Theorem 3.5. Let \( e \) be an exposed point of \( \mathbb{E} Y \), exposed by a linear functional \( f \). Then \( f \) exposes a single point of \( Y(\omega) \) \( P \)-almost everywhere.

Moreover, there is just a single measurable selection \( e \) such that \( e = \int e \omega \) \( dP \), i.e., \( e \) is \( P \)-almost everywhere unique.

Proof. Let the exposed point \( e \in \mathbb{E} Y \) have the representation \( e = \int e_1 \omega \) \( dP \), where \( e_1(\cdot) \in Y(\cdot) \) is a measurable selection according (2). By definition of an exposed point \( \{e\} = \arg \max_{y \in \mathbb{E} Y} f(y) \) and by Theorem 3.1 we have that \( e \in \mathbb{E} \partial s_Y (f) \), which means that \( e_1(\omega) \in \arg \max_{y \in Y(\omega)} f(y) P \)-a.e. If this representation were not unique, then there is another measurable selection \( e_2(\omega) \in \arg \max_{y \in Y(\omega)} f(y) \) with \( e = \int e_2 \omega \) \( dP \) and \( P(e_1 \neq e_2) > 0 \). In this situation there is a linear functional \( \ell \) such that \( P(\ell(e_1) \neq \ell(e_2)) > 0 \). Define the random variable \( \tilde{e}_1 := \begin{cases} e_1 & \text{if } \ell(e_1) \leq \ell(e_2), \\ e_2 & \text{if } \ell(e_1) > \ell(e_2) \end{cases} \) and \( \tilde{e}_2 := \begin{cases} e_2 & \text{if } \ell(e_1) \leq \ell(e_2), \\ e_1 & \text{if } \ell(e_1) > \ell(e_2) \end{cases} \). Notice that \( \ell(\tilde{e}_1) \leq \ell(\tilde{e}_2) \), and \( P(\ell(\tilde{e}_1) < \ell(\tilde{e}_2)) > 0 \). Hence \( e = \frac{1}{2} (\int \tilde{e}_1 dP + \int \tilde{e}_2 dP) \) and \( \int \ell(\tilde{e}_1) dP < \int \ell(\tilde{e}_2) dP \), and by linearity of \( \ell \) thus \( e_1 := \int \tilde{e}_1 dP \neq \int \tilde{e}_2 dP =: e_2 \). This is a contradiction, because \( f \) can only expose one unique point \( e \in \mathbb{E} Y \). This proves the second assertion.

The first assertion follows, as \( e(\cdot) \in \arg \max_{y \in Y(\cdot)} f(y) \) is \( P \)-almost everywhere unique by the second, and \( f \) thus exposes \( e(\omega) \in Y(\omega) \). \( \square \)
We note the contrapositive statement of the previous theorem, Theorem 3.5.

**Corollary 3.6.** Suppose that the linear functional $f$ does not expose a point from $Y(\cdot)$ almost everywhere. Then $f$ does not expose a point of $\mathbb{E} Y$.

The statement of the preceding theorem of course holds true for discrete distributions as in (3), although the proof simplifies significantly. We record the next lemma to emphasize that the $\arg\max$-set of the sample means in addition is the sample mean of the respective $\arg\max$-sets—an observation of further importance for the sample mean process discussed later.

For the next lemma see also [30, Theorem 2.8.7] or Boţ et al. [6, Theorem 3.5.8].

**Lemma 3.7.** Let $Y$ be a random map according (3) with compact and convex outcome and $f \in X^*$. Then

$$\arg\max_{k \in \mathbb{E} Y} f(k) = \sum_{j=1}^J p_j \arg\max_{k \in K_j} f(k) = \mathbb{E} \arg\max_{k \in Y} f(k).$$  \hspace{1cm} (14)

Moreover

$$\arg\max_{k \in \frac{1}{N} \sum_{i=1}^N K_i} f(k) = \frac{1}{N} \sum_{i=1}^N \arg\max_{k \in K_i} f(k) \quad (\omega \in \Omega)$$  \hspace{1cm} (15)

for any sequence of compact and convex sets $(K_i(\omega))_{i=1}^N$.

**Proof.** As for (14) fix $k \in \arg\max_{\mathbb{E} Y} f \subseteq \mathbb{E} Y = \sum_{j=1}^J p_j K_j$, which may be written as $k = \sum_j p_j k_j$ with $k_j \in K_j$. For any $y_j \in K_j$, $y := \sum_j p_j y_j \in \mathbb{E} Y$. By linearity and $f$-maximality of $k$,

$$\sum_j p_j f(k_j) = f(k) \geq f(y) = \sum_j p_j f(y_j)$$

for any $y_j \in K_j$, hence $k_j \in \arg\max_{K_j} f$. This proves that $\arg\max_{\mathbb{E} Y} f \subseteq \sum_j p_j \arg\max_{K_j} f$.

Conversely observe first that any $k \in \sum_j p_j \arg\max_{K_j} f$ has a representation $k = \sum_j p_j k_j$ for $k_j \in \arg\max_{K_j} f$. As $k_j \in \arg\max_{K_j} f \subseteq K_j$ it is thus obvious that $k = \sum_j p_j k_j \in \sum_j p_j K_j = \mathbb{E} Y$. Now pick any $y \in \mathbb{E} Y$ with representation $y = \sum_j p_j y_j$ and $y_j \in K_j$. By linearity and maximality of $k_j$,

$$f(k) = \sum_j p_j f(k_j) \geq \sum_j p_j f(y_j) = f(y),$$

hence $k \in \arg\max_{\mathbb{E} Y} f$, that is $\sum_j p_j \arg\max_{K_j} f \subseteq \arg\max_{\mathbb{E} Y} f$. Summarizing the inclusions, $\arg\max_{\mathbb{E} Y} f = \sum_j p_j \arg\max_{K_j} f$. By $\mathbb{E} \arg\max_{Y} f = \sum_j p_j \arg\max_{K_j} f$ the assertion finally follows.

Equation (15) verifies along the same lines as the proof for (14), but $p_j$ replaced by $\frac{1}{N}$. \qed
The following two theorems address the other properties introduced in Definition 3.2, which are strict convexity (Theorem 3.8 below) and extreme points (Theorem 3.9).

**Theorem 3.8.** Let $Y$ be strictly convex almost surely. Then $EY$ is strictly convex as well.

*Proof.* Let $k_1, k_2 \in EY$ be chosen so that $k_1 \neq k_2$ and let $k_1$ and $k_2$ be measurable selections so that $k_1 = \int k_1 dP$ and $k_2 = \int k_2 dP$. Note, that there is a measurable set $A$ with $P(A) > 0$ and $A \subset \{\|k_1 - k_2\| > \varepsilon\}$ for some $\varepsilon > 0$. For $x \in B_\varepsilon(0)$ fixed define

$$k := \frac{1}{2}k_1 + \frac{1}{2}k_2 \text{ and } k_x(\omega) := k(\omega) + \begin{cases} x & \text{if } \omega \in A, \\ 0 & \text{if } \omega \not\in A. \end{cases}$$

By construction, $k$ and $k_x$ are measurable selections. However, we have that $\int k_x dP = \int k dP + x \cdot P(A)$. As $x \in B_\varepsilon(0)$ was chosen arbitrarily it follows that $\int k dP + B_{\varepsilon \cdot P(A)}(0) \in EY$, i.e., $\frac{1}{2}k_1 + \frac{1}{2}k_2$ is in the interior of $EY$, which is the assertion. \[\square\]

**Theorem 3.9.** Let $e$ be an extreme point of $EY$. Then there is a unique measurable selection $e(\cdot)$ with $e = \int e dP$ and further, $e(\omega)$ is an extreme point of $Y(\omega)$ $P$-almost everywhere.

*Proof.* We notice first that $k_1 = k_2$ in Definition 3.2 (i) is equivalent to $f_i(k_1) = f_i(k_2)$, where $f_i, i = 1, \ldots, d$ are linearly independent functionals.

As $e$ is contained in the boundary, $e \in \partial EY$, the Hahn–Banach theorem provides a linear functional $f_d$ so that $f_d(e) \geq f_d(y)$ for all $y \in EY$. Then, by Proposition 3.1, we have that

$$e \in \partial s_{EY}(f_d) = E \partial s_Y(f_d). \tag{16}$$

It follows from Lemma 2.3 that $Y_{d-1}(\omega) := \partial s_Y(\omega)(f_d) = \arg\max_{y \in Y(\omega)} f_d(y)$ is contained in an affine subspace of co-dimension 1 parallel to $\{f_d(\cdot) = 0\}$ for each $\omega \in \Omega$, as $f_d$ is linear and $Y_{d-1} \subset \{f_d(\cdot) = \text{const}\}$ for some constant.

From (16) we deduce that $e \in EY_{d-1}$ and $e$, by linearity, is an extreme point of the set $\arg\max_{y \in EY} f_d(y)$, which is contained in an affine subspace, which is of co-dimension 1 and parallel to $\{f_d(\cdot) = 0\}$ as well.

We argue now by induction on the dimension. To this end set $Y_d := Y$ and assume that $Y_i$ is contained in an affine subspace of co-dimension $d - i$ so that $f_j(y) = f_j(y')$ for all $y, y' \in Y_i$ and $j > i$. Then we may repeat the previous argument and find a linear functional $f_i$ separating $e$ and $\bigcap_{j > 1} \arg\max_{y \in EY} f_j(y)$. The linear functions $f_j$ may be chosen linearly independent from $f_j, j > i$, as $\bigcap_{j > i} \arg\max_{y \in EY} f_j(y)$ is contained in an affine subspace of co-dimension $d - i$.

Define recursively the random sets

$$Y_{i-1}(\omega) := \partial s_{Y_i(\omega)}(f_i) \subset Y_i(\omega),$$

10
which are contained in an affine hyperplane of co-dimension \( d - (i - 1) \) parallel to \( \{ f_j(\cdot) = 0 ; j = i, \ldots, d \} \).

It follows that \( Y_1(\omega) \) is an interval and the random variable \( \omega \mapsto Y_1(\omega) \), by construction, is measurable. Hence \( e \in \mathbb{E} Y_1 = \int e \, dP \) and \( e \in Y_1 \) is unique, as \( e \) is an extreme point in the interval

\[
\bigcap_{i=1}^{d} \arg \max_{y \in \mathbb{E} Y} f_i(y).
\]

Clearly, \( e \in Y_1 \subset Y_2 \) and \( e \) is unique in \( Y_2 \) as well, as otherwise in conflict with maximality with respect to \( f_2 \). This argument can be repeated (in a backwards recursive way) to see that \( e \in Y \) is unique almost everywhere. \( \square \)

4 The law of large numbers and the central limit theorem

To study the law of large numbers we consider a sequence of independent, set-valued random variables \( Y_i \) with identical distribution (i.i.d.). We are interested in which sense the sample means \( \frac{1}{N} \sum_{i=1}^{N} Y_i(\omega) \) converge to the expected value set \( \mathbb{E} Y \).

We start with general observations regarding the sample mean process.

4.1 Convexification

As was discussed in Section 2.1, the expected value \( \mathbb{E} Y \) is convex in many, but not all situations. However, the sample means \( \frac{1}{N} \sum_{i=1}^{N} Y_i \) always converge to a convex set in the sense of the next lemma.

**Lemma 4.1** (Artstein and Hansen [3]). Let \( (K_i)_{i=1}^{\infty} \) be a sequence of compact sets in a Banach space \( X \) such that

\[
\frac{1}{N} (\mathop{\text{conv}} K_1 + \mathop{\text{conv}} K_2 + \cdots + \mathop{\text{conv}} K_N) \xrightarrow{N \to \infty} K_0
\]

in Pompeiu–Hausdorff distance for some convex and compact set \( K_0 \). Then

\[
\frac{1}{N} (K_1 + K_2 + \cdots + K_N) \xrightarrow{N \to \infty} K_0.
\]

**Remark 4.2** (Shapley-Folkman-Starr). The theorem by Shapley-Folkman-Starr (cf. Arrow and Hahn [2] and also Artstein and Hansen [3]) provides an explicit bound for comparing sums of compacts sets in the space \( X = \mathbb{R}^d \) with finite dimension \( d \). The theorem states that

\[
\mathbb{H} \left( \frac{1}{N} (K_1 + K_2 + \cdots + K_N), \frac{1}{N} \mathop{\text{conv}} (K_1 + K_2 + \cdots + K_N) \right) \leq \frac{\sqrt{d}}{N} \max_{i=1,\ldots,N} \|K_i\|,
\]

where \( \|K\| = \max_{k \in K} \|k\| \) (cf. also Molchanov [18, Section 3.1.1] and Figure 1 again for illustration).
It is thus clear that the sample average $\frac{1}{N} \sum_{i=1}^{N} Y_i$ has the same limiting behavior as $\frac{1}{N} \sum_{i=1}^{N} \text{conv} Y_i$ — the sample averages thus converge to a convex set, particularly in the finite dimensional space $\mathbb{R}^d$. For this we shall specify further and assume the outcomes $Y_i(\omega)$ convex and compact in what follows such that no separate discussion of the discrete setting (3) is necessary.

### 4.2 The set-valued law of large numbers

By the Artstein and Vitale Theorem [4, p. 880], the i.i.d. sample means $Y_N := \frac{1}{N} \sum_{i=1}^{N} Y_i$ with $\mathbb{E} \| Y_i \| < \infty$ converge indeed to the expected value $\mathbb{E} Y$, i.e.,

$$\mathbb{H} \left( \frac{1}{N} \sum_{i=1}^{N} Y_i, \mathbb{E} Y \right) \xrightarrow{N \to \infty} 0 \quad \text{with probability 1.} \quad (17)$$

In view of the representation of the Pompeiu–Hausdorff distance derived in (10) this implies in particular that

$$\frac{1}{N} \sum_{i=1}^{N} s_{Y_i}(x^*) \xrightarrow{N \to \infty} s_{\mathbb{E} Y}(x^*) \quad \text{with probability 1}$$

for every $x^* \in X^*$.

Eq. (17) is referred to as the set-valued law of large numbers. Several extensions are known to this fundamental theorem, we refer the reader to Shapiro and Xu [25] for a uniform law of large numbers.

### 5 The set-valued central limit theorem

The CLT theorem is available in the Banach space $C \left( S^{d-1} \right)$ (cf. Araujo and Giné [1], Li et al. [17]), that is, there is a centered Gaussian random variable $G$ on $C \left( S^{d-1} \right)$ such that

$$\frac{1}{\sqrt{N}} \left( \sum_{i=1}^{N} s_{Y_i} - s_{\mathbb{E} Y} \right) \xrightarrow{D} G,$$

where $\xrightarrow{D}$ indicates convergence in distribution, i.e., $\mathbb{E} f(X_n) \to \mathbb{E} f(X)$ for every $\mathbb{R}$-valued function $f$ which is bounded and continuous. In full generality:

**Theorem 5.1** (Weil [28, Theorem 3]). Let $(Y_i)_{i=1}^n$ and $Y$ be i.i.d. random sets with $\mathbb{E} \| Y \|^2 < \infty$. Then

$$\sqrt{N} \cdot \mathbb{H} \left( \frac{1}{N} \sum_{i=1}^{N} Y_i, \mathbb{E} \text{conv} Y \right) \xrightarrow{D} \| G \|_\infty,$$

where $G$ is a centered Gaussian $C \left( S^{d-1} \right)$-valued random variable.
Proof. Cf. Weil [28, Theorem 8] or Casey [9]. The proof is based on computing the metric entropy of \( S^{d-1} \) and the respective bracketing numbers, it reduces the particular situation here to the general situation described in Aranjo and Giné [1]. Elements of the general theory and proofs can be found in van der Vaart and Wellner [27].

The Gaussian measure \( G \) in Theorem 5.1 is provided by the isometry of convex and compact sets with their respective support function. Moreover \( H (\cdot, \cdot) \in \mathbb{R}_{\geq 0} \) always is just a positive number (as is \( \|G\|_\infty \)), the statement just considers the \( \mathbb{R}_{\geq 0} \)-valued random process \( \sqrt{N} \cdot H \left( \frac{1}{N} \sum_{i=1}^{N} Y_i, \mathbb{E} \text{conv} Y \right) \) and does not reveal anything of the local convergence properties of the sample mean to the expected value.

In view of the latter statements, the preceding discussion and (1), the interesting properties are to be expected on the boundary \( \partial \mathbb{E} Y \). In what follows we shall distinguish and consider three particular situations on the boundary of \( \mathbb{E} Y \), which can be considered to be extremal situations. We discuss the CLT for exposed points, for tangent planes and facets of \( \mathbb{E} Y \) in the following subsections separately.

5.1 The CLT for exposed points

The following theorem ensures that for any exposed point \( k \in \exp \mathbb{E} Y \) there is a particular selection \( y_N \in \mathbb{Y}_N := \frac{1}{N} \sum_{i=1}^{N} Y_i \) from the sample means, such that the process \( \sqrt{N} (y_N - k) \) converges to a Gaussian random variable.

**Theorem 5.2.** Suppose that the envelope function \( h \) (cf. (4)) satisfies \( h \in L^2 \).

Let \( k \in \exp \mathbb{E} Y \) be exposed by the functional \( f \in X^* \) and \( y_N \in \mathbb{Y}_N \) be exposed by the same \( f \). Then there is a unique measurable selection \( k(\cdot) \in Y(\cdot) \) such that

\[
\sqrt{N} (y_N - k) \xrightarrow{d} \mathcal{N}_d (0, \Sigma) \quad \text{as} \quad N \to \infty,
\]

where \( k = \mathbb{E} k \) and \( \Sigma \) is the covariance matrix

\[
\Sigma := \mathbb{E} (k - k)(k - k)^T.
\]

**Proof.** There is a measurable selection \( k \) such that \( k = \int k \, dP \). By Theorem 3.5 the selection \( k \), as \( k \) is exposed, is unique and \( k(\omega) \in \partial s_{Y(\omega)} (f) \subseteq Y(\omega) \). \( k \) is a random variable with expectation \( k \), and as \( \|k\| \leq \|Y\| \leq h \in L^2 \) the covariance matrix

\[
\Sigma := \text{var} k = \mathbb{E} (k - k)(k - k)^T
\]

exists.

It follows from Lemma 3.7 and Theorem 3.5 that \( k(\omega) \) and \( k_i(\omega) \), where

\[
k_i(\omega) \in \partial s_{Y_i(\omega)} (f), \quad (18)
\]
have the same distribution for all \( i \). Hence \( y_N \in \partial s_{Y_N} (f) \subseteq \overline{Y}_N \) and \( \overline{k}_N := \frac{1}{N} \sum_{i=1}^{N} k_i \) have the same distribution as well.

By the central limit theorem (cf. van der Vaart [26]) thus

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} (k_i - \overline{k}) \overset{D}{\rightarrow} N_d (0, \Sigma) \quad (N \to \infty),
\]

which is in turn

\[
\sqrt{N} (y_N - \overline{k}) \overset{D}{\rightarrow} N_d (0, \Sigma)
\]

because

\[
\mathbb{E} k_i = k \quad \text{and} \quad \text{var} k_i = \mathbb{E} (k_i - \overline{k}) (k_i - \overline{k})^\top = \Sigma.
\]

In the proof of Theorem 5.2 it is essential to find a measurable selection \( k_i \in Y_i \) having the same distribution as \( k \), which is possible by means of (18). Similar choices are possible in some other situations, for example again in the binomial setting (3) as in Section 2.1:

**Corollary 5.3.** Let \( Y_i \) be as in Section 2.1 with the additional assumption that \( K_j \neq K_{j'} \) \((j \neq j')\). Then, for any selection \( k \in Y \) with \( k = \mathbb{E} k \in \mathbb{E} Y \) there are selections \( k_i \in Y_i \) with the same distribution as \( k \) such that

\[
\sqrt{N} \sum_{i=1}^{N} (k_i - \overline{k}) \overset{D}{\rightarrow} N_d (0, \text{var} k).
\]

**Proof.** Let

\[
k_j := k(\omega) \in K_j \quad \text{if} \quad Y(\omega) = K_j \quad (j = 1, 2, \ldots J)
\]

and define

\[
k_i(\omega) := k_j \quad \text{if} \quad Y_i(\omega) = K_j \quad (i = 1, 2, 3 \ldots),
\]

such that \( k_i \in \{k_1, \ldots, k_J\} = \text{range}(k) \) and

\[
k_i(\omega) = k(\omega) \quad \text{if} \quad Y_i(\omega) = Y(\omega) \quad (i = 1, 2, \ldots).
\]

Then the random variables have the same distribution, as \( p_j = P(Y = K_j) = P(Y_i = K_j) = P(k_i = k_j) = P(k = k_j) \). It follows that \( \mathbb{E} k = \mathbb{E} k_i = \sum_j p_j k_j = k \) and \( \text{var} k_i = \text{var} k = \mathbb{E} (k - \overline{k}) (k - \overline{k})^\top = \sum_j p_j (k_j - \overline{k}) (k_j - \overline{k})^\top \), from which the rest is immediate.

**Remark 5.4.** A point \( k \in \mathbb{E} Y \) may have various selections \( k \) with \( k = \mathbb{E} k \in \mathbb{E} Y \), there are hence various selections \( k_i \in Y_i \) with possibly different convergence behavior (20). However, if \( k \) is an exposed point, then the selection \( k \) is unique by Theorem 13 and the selections (18) and (21) coincide.
5.2 The CLT along tangent planes

Any compact and convex $K$ can be given as $K = \bigcap_{x^* \in X^*} \{ x^*(\cdot) \leq \max_{k \in K} x^*(k) \}$ and $\{ x^*(\cdot) = \max_{k \in K} x^*(k) \}$ is a tangent plane of co-dimension 1. While the previous subsection addresses exposed points for which $K \cap \{ x^* = \max x^*(K) \} = \partial s_K (x^*)$ is a singleton, we continue in this subsection with the situation that $\partial s_K (x^*)$ is not necessarily a singleton.

The law of large numbers does not only hold for the sequence $Y_i$, it applies for subdifferentials as well.

**Proposition 5.5.** Let $Y$ and $Y_i$ be independent and identically distributed, compact and convex valued random sets with $L^2$-envelope (cf. (4)). Then

$$\mathbb{H} \left( \frac{1}{N} \sum_{i=1}^{N} \partial s_{Y_i} (x^*) , \partial_{\mathcal{E} Y} (x^*) \right) \xrightarrow{N \to \infty} 0 \quad \text{with probability 1} \quad (22)$$

for any $x^*$, and moreover

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\max_{y \in Y_i} x^*(y) - \max_{y \in \mathcal{E} Y} x^*(y)) \xrightarrow{D} \mathcal{N} \left( 0, \sigma^2 (x^*) \right) \quad , \quad (23)$$

where $\sigma^2 (x^*) = \text{var} \max_{y \in Y} x^*(y)$.

**Proof.** Notice first that $\partial s_{Y_i} (x^*) \subseteq Y_i$, and the law of large numbers applies to the sequence $\partial s_{Y_i} (x^*) \subseteq Y_i$ as well: that is to say $\mathbb{H} \left( \frac{1}{N} \sum_{i=1}^{N} \partial s_{Y_i} (x^*) , \mathbb{E} \partial_{\mathcal{E} Y} (x^*) \right) \xrightarrow{N \to \infty} 0$ with probability 1, and the identity $\mathbb{E} \partial_{\mathcal{E} Y} (x^*) = \partial_{\mathcal{E} Y} (x^*)$ already was established in Theorem 3.1.

As for (23) note that

$$\max_{y \in Y_i} x^*(y) = x^* \left( \arg \max_{Y_i} x^* \right)$$

is an $\mathbb{R}$-valued random variable and square integrable, because $|\max_{y \in Y_i(\omega)} x^*(y)| \leq \|x^*\|_* \cdot h(\omega)$. Further, $\mathbb{E} \max_{y \in Y} x^*(y) = \mathbb{E} x^* (\partial s_{Y} (x^*)) = \mathbb{E} (\mathbb{E} \partial_{\mathcal{E} Y} (x^*)) = \mathbb{E} (\partial_{\mathcal{E} Y} (x^*)) = \max_{y \in \mathcal{E} Y} x^*(y)$. The asymptotic distribution (23) thus follows from the classical Central Limit Theorem for $\mathbb{R}$-valued random variables.

**Remark 5.6.** Proposition 5.5 reduces the original problem into two distinct, orthogonal problems, as (22) describes the behavior of parallel sets, all of co-dimension one, whereas (23) is their orthogonal component.
5.3 The CLT for facets

A functional \(f \in X^*\) induces the particular selection (18) above by exposing a single point of the boundary of \(EY\). With this selection it was possible to describe convergence of corresponding exposed points of the sample means.

In what follows we take a kind of dual approach and fix a vector \(x \in X\) first. Then there are nearest points to a compact, convex set \(K\), which we denote by

\[
k_x \in \text{arg min} \left\{ \|x - k'\| : k' \in K \right\}.
\]

In order to have \(k_x(\cdot)\) uniquely defined we shall assume that the unit ball of the norm \(B_{\|\cdot\|}\) is strictly convex (cf. Definition 3.2 (iii) and Figure 2b). We consider the random variable \(k(\omega) := k_x(K(\omega)) \in K(\omega)\), which is a particular selection, whose convergence behavior is being elaborated in what follows.

**Definition 5.7 (Facet).** A (continuous) linear functional \(f \neq 0\) is a **facet at** \(k \in K\) if there is a direction \(d\) (associated with \(f\)) and a neighborhood \(U(k)\) such that \(x - d \cdot f(x - k) \in \text{arg max}_{k' \in K} f(k')\) for all \(x \in U(k)\). Further, we shall say that \(k \in K\) is **contained in a facet** if there exists a facet at \(k \in K\).

We collect the following important features of facets, as they will be of interest in what follows (cf. Figure 2 for a simple, helpful illustration).
Remark 5.8 (Important properties of facets). Let \( f \) be a facet and \( d \) a direction associated with the facet according to Definition 5.7.

(i) The direction \( d \) associated with the facet \( f \) always satisfies \( f(d) = 1 \): to see this note first that necessarily \( k \in \arg\max_K f \), as \( k \in U(k) \). For \( x \in U(k) \) fixed thus, \( f(x) - f(d) \cdot f(x - k) = f(k) \), hence \( f(x - k) = f(d) \cdot f(x - k) \) for all \( x \in U(k) \), which can hold true only if \( f(d) = 1 \).

(ii) Associated with a facet and a direction \( d \) are the projection operators \( P^\perp := \frac{d \otimes f}{f(d)} \) and \( P := 1 - P^\perp \), where \( P^\perp(x) = \frac{d}{f(d)} f(x) \). Indeed, it follows from (i) that \( P^\perp \circ P^\perp = P^\perp \), and thus \( P \circ P = P \). In the context of facets of the expected value set below we consider the shifted projective map \( x \mapsto k + P^\perp(x - k) \).

(iii) A facet—up to a constant—is unique. To accept this let \( f \) be a facet, that is \( x - d \cdot f(x - k) \in \arg\max_{k' \in K} f(k') \subseteq K \). For another facet \( g \) hence \( g(x - d \cdot f(x - k)) \leq g(k) \), that is
\[
g(x - k) \leq g(d) \cdot f(x - k).
\]
(25)

For \( x \in U(k), 2k - x \in U(k) \) as well (at least for \( x \) close enough to \( k \)). Hence
\[
g(2k - x - k) \leq g(d) \cdot f(2k - x - k),
\]
or \( g(x - k) \geq g(d) \cdot f(x - k) \), which, together with (25), implies that \( g(x - k) = g(d) \cdot f(x - k) \), and hence \( g(\cdot) = g(d) \cdot f(\cdot) \).

(iv) We have that \( x - d \cdot f(x - k) \in U_\varepsilon(k) \subseteq \arg\max_K f \subseteq K \) for some small \( \varepsilon > 0 \). Indeed, for \( \varepsilon > 0 \) small enough and \( x \in U_\varepsilon/(1 + \|d\|f_\infty)(k) \subseteq U(k) \) we have that \( \|x - d \cdot f(x - k) - k\| \leq (1 + \|d\|f_\infty) \cdot \|x - k\| < \varepsilon \), and hence \( x - d \cdot f(x - k) \in U_\varepsilon(k) \subseteq \arg\max_K f \subseteq K \).

(v) The direction \( d' \) of the facet can be chosen arbitrarily, as long as \( f(d') = 1 \). Indeed, recall that \( x - d' \cdot f(x - k) \in U_\varepsilon(k) \). As \( f(x - d' \cdot f(x - k)) = f(k) \) we find further that \( x - d' \cdot f(x - k) \in \arg\max_K f \), which is the assertion for the alternative direction \( d' \) whenever \( x \in U_\varepsilon/(1 + \|d'\|f_\infty)(k) \) as above.

We demonstrate next that the expected value set \( \mathbb{E} Y \) inherits all facets from the sample sets \( Y_i \).

**Proposition 5.9.** Let \( K \) be a convex and compact set with facet \( f \) and \( Y \) be a set-valued random variable with \( P(Y = K) > 0 \).

(i) Then \( \mathbb{E} Y \) has a facet as well; more precisely, \( f \) is a facet of \( \mathbb{E} Y \) at each point in the relative interior of \( \mathbb{E} \partial s_Y(f) \);
(ii) Let $Y$ and $(Y_i)$ be i.i.d. random sets as in the discrete setting (3). Then $f$ is a facet of $\overline{Y}_N$ with probability $1 - (1 - p)^N$, where $p := P(Y = K)$.

Proof. Let $K$ have a facet $f$ at some $k \in K$. Then

$$\arg \max_{y \in K} f(y) \supset \{x - d \cdot f(x - k) : x \in B_r(k)\}$$

for some $r > 0$ and $d$ with $f(d) = 1$. Choose again the selection $k(\omega) = \partial s_{Y(\omega)}(f)$. By (3.7) thus

$$\arg \max_{y \in \mathbb{E}Y} f(y) = \mathbb{E} \arg \max_{y \in Y} f(y)$$

$$= P(Y \neq K) \cdot \mathbb{E} \left[ \arg \max_{y \in Y} f(y) \middle| Y \neq K \right] + P(Y = K) \cdot \mathbb{E} \left[ \arg \max_{y \in Y} f(y) \middle| Y = K \right]$$

$$= P(Y \neq K) \cdot \mathbb{E} \left[ \arg \max_{y \in Y} f(y) \middle| Y \neq K \right] + P(Y = K) \cdot \arg \max_{y \in K} f(y)$$

$$\supset P(Y \neq K) \cdot \mathbb{E} \left[ k \middle| Y \neq K \right] + P(Y = K) \cdot \{x - d \cdot f(x - k) : x \in B_r(k)\}$$

$$= P(Y \neq K) \cdot \mathbb{E} k + P(Y = K) \cdot \{k + x - d \cdot f(x) : x \in B_r(0)\}$$

$$= P(Y \neq K) \cdot \mathbb{E} k + P(Y = K) \cdot k + \left\{x - d \cdot f(x) : x \in B_{r, p(Y = K)}(0)\right\},$$

hence $f$ is a facet of $\mathbb{E}Y$ at every $k' \in P(Y \neq K) \cdot \mathbb{E} k + P(Y = K) \cdot k \subset \mathbb{E} k$. (Recall that $P(Y = K) > 0$, the conditional expectation in the previous display thus does not cause difficulties).

As for (ii) note that there is $i^*$ so that $K = K_{i^*}$. By (i), $\overline{Y}_N$ has the facet $f$ as soon as $Y_i = K$, which happens with the probability $P(Y_i = K) \geq 1 - (1 - p)^N$ at the $N$-th draw.

Remark 5.10 (The converse is false). Figure 3 provides an example of two sets $A$ and $B$ without facets, although their average $\frac{1}{2}(A + B)$ has a facet. Hence if $\mathbb{E}Y$ has a facet, then this is not necessarily the case for $Y$, not even for discrete random variables $Y$.

To describe the convergence of set-valued sample means close to a facet of $\mathbb{E}Y$ it will be convenient to have an outer normal available. The facet normal is given by the derivative of the norm (cf. Figure 2b for an illustration with an elliptic unit ball, and Bonetti and Vitale [8] for facet normals).

Definition 5.11 (Derivative of the Norm). We shall denote an element of the derivative of the norm $x \in \mathbb{R}^d$ by $\mathbf{H}B_x$.

$$\mathbf{H}B_x \in \partial s_{B^*}(x) \subseteq \mathbb{R}^d;$$

here, $B^* := \left\{x \in \mathbb{R}^d : \|x\|_* \leq 1\right\}$ is the unit ball in the dual space.

\footnote{The conditional expectation is understood in the naïve sense based on conditional probabilities here: note that the sets $\{Y = K\}$ and $\{Y \neq K\}$ have strictly positive probability.}
Figure 3: \(\frac{1}{2}(A + B)\) has a facet at its top, although \(A\) and \(B\) have no facet (cf. Figure 1) (the depicted solid’s equation is \(x^2z^2 + y^2 \leq z^2\)).

Remark 5.12. By (5) and (8) it holds that
\[
\|\text{HB}_x(x)\| = \|x\| \quad \text{and} \quad |\text{HB}_x(h)| \leq \|h\|
\]
for all \(h \in \mathbb{R}^d\) (that is to say the norm in the dual is one, \(\|\text{HB}_x\|_* = 1\), where the norm is the Lipschitz constant \(\|\lambda\|_* := \sup_{h \neq 0} \frac{|\lambda(h)|}{\|h\|} = L(\lambda)\).

Theorem 5.13. Given \(x \notin \overline{EY}\), suppose that \(k \in \arg\min \{\|x - y\| : y \in EY\}\), the closest point to \(x\), is contained in a facet \(f\) (cf. (24)). Then the facet satisfies \(f(\cdot) = -\alpha \cdot \text{HB}_{k-x}(\cdot)\) for some \(\alpha > 0\).

Proof. Given \(x\), choose \(k\) the nearest point in \(EY\) such that \(d(x, EY) = \|x - k\|\). Both, \(EY\) and the ball \(B_{\|x-k\|}(x)\) are convex, and \(k\) is a common point. Moreover \(EY\) and the open ball \(\tilde{B}_{\|x-k\|}(x)\) do not intersect. The Hahn–Banach Theorem provides a functional (separating plane) for both sets. As the facet is unique the separating functional \(\text{HB}_{k-x}(\cdot)\) is the facet.

Theorem 5.14. Given \(x \notin \overline{EY}\), suppose that \(k_x(EY)\), the nearest point to \(x\), is contained in a facet of \(\overline{EY}\). Then there is a neighborhood \(V(x)\) such that the Pompeiu–Hausdorff distance is \(\mathbb{H}(\{v\}, \overline{EY}) = \text{HB}_{k-x}(k - v)\) for all \(v \in V\), and moreover \(\mathbb{H}(\{v\}, \overline{EY}) - \mathbb{H}(\{x\}, \overline{EY}) = \text{HB}_{k-x}(x - v)\).

Proof. By the above theorem the facet is \(-\text{HB}_{k-x}\). Let us equip the facet \(-\text{HB}_{k-x}\) with the direction \(d := -\frac{k-x}{\|k-x\|}\), such that \(-\text{HB}_{k-x}(d) = 1\). Being a facet, there is by definition a neighborhood \(U(k)\) such that \(u - d \cdot \text{HB}_{k-x}(u - k) \in \arg\max_{\mathbb{R}^X} (-\text{HB}_{k-x})\) for all \(u \in U(k)\). Define \(V(x) := U(k) - (k - x)\). Then \(v + \frac{\text{HB}_{k-x}(k - v)}{\|k-x\|}(k - x) \in \arg\max_{\mathbb{R}^Y} (-\text{HB}_{k-x})\) for every \(v \in V(x)\). Hence, as \(k \in EY\),
\[
\mathbb{H}(\{v\}, \overline{EY}) = \text{HB}_{k-x}(k - v) .
\]
The latter statement of the theorem follows from linearity, as \( \mathbb{H}(\{v\}, \mathbb{E}Y) - \mathbb{H}(\{x\}, \mathbb{E}Y) = \mathbb{H}_{k-x}(k-v) - \mathbb{H}_{k-x}(k-x) = \mathbb{H}_{k-x}(x-v) \). 

With these preparations we can finally describe the distribution along facets.

**Theorem 5.15.** Given \( x \), suppose that \( k_x(\mathbb{E}Y) \), the nearest point to \( x \), is contained in a facet of \( \mathbb{E}Y \). Then

\[
\sqrt{N} \left( \mathbb{H}(\{x\}, \mathbb{Y}_N) - \mathbb{H}(\{x\}, \mathbb{E}Y) \right) \xrightarrow{d} \mathcal{N} \left( 0, \mathbb{H}_{k-x} \cdot \Sigma \cdot \mathbb{H}_{k-x}^\top \right)
\]

where \( k \) and \( \Sigma \) are as in Theorem 5.2 and \( \mathbb{Y}_N := \frac{1}{N} \sum_{i=1}^N Y_i \).

**Proof.** Note that \( k(\omega) = k_x(Y(\omega)) \) is almost surely uniquely defined as the norm is strictly convex and \( k = \mathbb{E}k \). We define the random quantities \( k_i := k_x(Y_i) \) and

\[
V_i := k_i + x - k
\]

\((\mathbb{K}_N := \frac{1}{N} \sum_{i=1}^n k_i \) and \( \mathbb{V}_N := \frac{1}{N} \sum_{i=1}^N V_i \), resp.), such that

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbb{H}_{k-x}(x - V_i) \xrightarrow{d} \mathcal{N} \left( 0, \mathbb{H}_{k-x} \cdot \Sigma \cdot \mathbb{H}_{k-x}^\top \right), \quad n \to \infty,
\]

where \( \Sigma = \mathbb{E}(k - k)(k - k)^\top \). Note next that \( \mathbb{E} \frac{1}{N} \sum_{i=1}^N V_i = x \). By Theorem 5.14 there is a neighborhood \( V(x) \) such that

\[
\mathbb{H}(\{v\}, \mathbb{E}Y) - \mathbb{H}(\{x\}, \mathbb{E}Y) = \mathbb{H}_{k-x}(x - v) \quad \text{for all } v \in V(x).
\]

Note further that

\[
\mathbb{H}(\{x\}, \mathbb{E}Y) - \mathbb{H}(\{x\}, \mathbb{Y}_N)
\]

\[
= \left( \mathbb{H}(\{x\}, \mathbb{E}Y) - \mathbb{H}(\{\mathbb{V}_N\}, \mathbb{E}Y) \right)
\]

\[
+ \mathbb{H}(\{\mathbb{V}_N\}, \mathbb{E}Y) - \mathbb{H}(\{\mathbb{V}_N\}, \mathbb{Y}_N) + \left( \mathbb{H}(\{\mathbb{V}_N\}, \mathbb{Y}_N) - \mathbb{H}(\{x\}, \mathbb{Y}_N) \right)
\]

\[
= \mathbb{H}_{k-x}(\mathbb{V}_N - x) + \mathbb{H}_{k-x}(k - \mathbb{V}_N) - \|x - k\| + \mathbb{H}_{\mathbb{K}_N - x}(x - \mathbb{V}_N)
\]

where we have used (30), (26), (28) and again (30), provided that \( \mathbb{V}_N \in V(x) \). The assertion of the theorem follows from (29) as \( \mathbb{H}_{\mathbb{K}_N - x} \to \mathbb{H}_{k-x} \) for the strictly convex norm, provided that can ensure that \( \mathbb{V}_N \in V(x) \) almost surely.

As \( x \) is in the interior of \( V(x) \) we apply the large deviation theory (cf. for example Dembo and Zeitouni [12] or Norkin and Wets [19, Theorem 4.1]) to obtain that

\[
\limsup_{N \to \infty} \frac{1}{N} \ln P \left( \frac{1}{N} \sum_{i=1}^N V_i \notin V(x) \right) < 0.
\]
That is, there is $q > 0$ such that $P \left( \frac{1}{N} \sum_{i=1}^{N} V_i \notin V(x) \right) < e^{-qN}$ and thus

$$P \left( \frac{1}{N} \sum_{i=1}^{N} V_i \in V(x) \right) > 1 - e^{-qN} \xrightarrow{N \to \infty} 1.$$ 

The desired distribution (27) follows hence from (29).

\section{Summary}

We discuss convergence properties of random sets. We are particularly interested in fluctuations of the sample means close to the boundary of the limit set, the expected value. It turns out that special properties of points on the boundary of the expected value set can already be seen at the boundary of the sample means, while other properties are inherited from the sample means to the expected value set.

The paper addresses important boundary points of the expected value set separately. Exposed points of the expected value set have a unique measurable selection, and so have the sample means. Convergence thus can be described by a usual process of points in $\mathbb{R}^d$. Tangent planes display a similar behavior, we describe their convergence by identifying the moments to describe their convergence by use of the central limit theorem.

We finally address facets which are inherited by the expected value set, but (perhaps surprisingly) not the other way round.

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