PARTIALLY ORDERING UNKNOTTING OPERATIONS

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Abstract. In this paper, we introduce an equivalence relation on the set of local moves and classify local moves, called the extended ST-moves, up to the equivalence. Moreover, by inducing a binary relation on the set of equivalence classes of local moves, we show that an extended ST-move realizes the crossing change or the $SH(2)$-move. In addition, for any oriented knot and two extended ST-moves, we discuss the magnitude relation between the unknotting numbers of the knot via the moves, and show that there is an extended ST-move except $SH$-moves so that the knot can be transformed into the trivial knot by the single extended ST-move. Finally, we provide some examples of ST-moves with the binary relation.

1. Introduction

An operation that replaces a tangle diagram on a knot or link diagram with another tangle diagram is referred to as “a local move on a knot or link diagram.” For instance, the Reidemeister moves [10] are local moves on a knot or link diagram. The crossing change, called the X-move, the $\Delta$-move [7], the $\Delta_{ij}$-move [9], the $\sharp$-move [6] and the $n$-gon move [2] are also local moves on a knot or link diagram. We will define local moves as pairs of two tangle diagrams (see Definition 2.5).

In [4], J. Hoste, Y. Nakanishi and K. Taniyama defined an $SH(n)$-move (see FIG. 1), and verified that an $SH(2n-1)$-move is an unknotting operation on an oriented knot or link diagram for $2 \leq n \in \mathbb{N}$. An unknotting operation (see, e.g., [5]) is a local move on a knot or link diagram such that any knot or link diagram can be transformed into a trivial knot or link diagram by a finite sequence of the local move and Reidemeister moves. In [8], we defined an $ST(n)$-move, which is an extension of an $SH(n)$-move, and demonstrated that it realizes the crossing change or the $SH(2)$-move for $2 \leq n \in \mathbb{N}$. A local move is called an $ST(n)$-move if the two oriented $n$-tangle diagrams are both trivial and not equal (see [8] and FIG. 2).

In this paper, we introduce an equivalence relation on the set of local moves and classify local moves, called the extended ST-moves (see Definition 2.8), up to the equivalence. In Theorem 1, we prove that there is a one-to-one correspondence between the set of equivalence classes of extended ST-moves and the set of standard ST-moves (see Definition 3.3). Therefore, any standard ST-move can be chosen as a representative of an equivalence class of extended ST-moves. In Theorem 2, we show that an extended ST-move realizes...
the crossing change or the \(SH(2)\)-move. The former extended \(ST\)-move is called an \(X\)-type and the latter one is called an \(O\)-type. In Theorem 3, it is shown that a local move, which realizes an \(X\)-type, is an unknotting operation. Given any knot and two \(X\)-types, we obtain the magnitude relation between the unknotting numbers of the knot via the two moves (Theorem 4). Moreover, we show that for any oriented knot \(K\), there is an extended \(ST\)-move except \(SH\)-moves so that any diagram of \(K\) can be transformed into a trivial knot diagram by the single \(ST\)-move (Theorem 5).

\[
\begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 \\
\text{ST(3)-move} & \text{ST(3)-move} & \text{ST(3)-move}
\end{array}
\]

**Fig. 2.** Examples of \(ST(3)\)-moves.

Section 2 presents some definitions and a proposition necessary for proving Theorem 1. In section 3, we describe some lemmas and prove Theorem 1. In section 4, we introduce a binary relation \(\preceq\) on the set of equivalence classes of local moves and demonstrate that the binary relation is a partial order on the set. In section 5, we state necessary and sufficient conditions (Lemmas 5.2 and 5.3) for the partial order between the equivalence class of an \(ST\)-move and one of an \(SH\)-move to exist by using their representatives. Section 6 discusses the unknotting numbers of \(ST\)-moves. Finally, we provide some examples of \(ST\)-moves with the relation \(\preceq\).

2. Definitions

Throughout this paper, we work in PL category. Tangles were introduced by J. Conway in [3] in order to help in assembling a knot table and develop symbols of knot diagrams. Since then, tangles have been useful for studying knot theory, DNA topology, quantum topology, and applied fields, such as molecular biology.

We shall begin with some definitions on tangles. First, we will define a tangle as follows.

**Definition 2.1.** Let \(B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}\) be a unit \(3\)-ball. Let \(\hat{T} = \cup_{i=1}^{n} t_i\) be a union of \(n\) pairwise disjoint arcs \(t_i\), embedded properly in \(B\) and let \(\partial \hat{T} = \partial(\cup_{i=1}^{n} t_i) = \cup_{i=1}^{n} \partial t_i = \{(\cos \frac{j\pi}{n}, \sin \frac{j\pi}{n}, 0) \in \mathbb{R}^3 \mid j = 1, 2, \ldots, 2n\}\). Then \((B, \hat{T})\) is called an \(n\)-tangle. An \(n\)-tangle \((B, \hat{T})\) is called to be oriented if each arc \(t_i\) is oriented, where \(i = 1, 2, \ldots, n\).

In this paper, we treat the following regular diagrams.

**Definition 2.2.** Let \((B, \hat{T})\) be an oriented \(n\)-tangle. Let \(p\) be a projection of \(B\) onto the unit disk \(D = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}\). Then \((D, T) := (p(B), p(\hat{T}))\) is called a tangle diagram of \((B, \hat{T})\), where \(p(\hat{T})\) is a regular diagram of \(\hat{T}\), i.e., \(p(\hat{T})\) is regular, and we draw one arc close to a double point (or crossing) so that it appears to have been cut to express that the arc passes under the other arc. Each point \((\cos \frac{j\pi}{n}, \sin \frac{j\pi}{n}, 0)\) is marked \(j\) and called an \(e\)-point of \((B, \hat{T})\) or \((D, T)\).

Henceforth, we let \(p\) be a projection of \(B\) onto the unit disk \(D = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}\), and assume that tangle diagrams are oriented. In the next definition, we will describe the equality of two tangle diagrams.

**Definition 2.3.** Let \((D_1, T_1)\) and \((D_2, T_2)\) be \(n\)-tangle diagrams and let \(I(\partial T_1) = I(\partial T_2)\), where \(I(\partial T_i)\) is the set of initial points of \(\partial T_i\). If we can change \(T_1\) into \(T_2\) by performing a finite number of Reidemeister moves in \(D_1\) and \(D_2\), respectively, keeping the \(2n\) marked
e-points fixed, then the tangle diagrams \((D_1, T_1)\) and \((D_2, T_2)\) are also said to be equal and are denoted by \((D_1, T_1) = (D_2, T_2)\) or \(T_1 = T_2\).

**Definition 2.4.** A tangle diagram \((D, T)\) is trivial if we can change \((D, T)\) into a diagram with no crossings by performing a finite number of Reidemeister moves in \(D\), keeping the marked e-points fixed.

**Definition 2.5.** A local move is a pair of tangle diagrams, \((D_1, T_1)\) and \((D_2, T_2)\), with \(\partial D_1 \cap \partial D_2 \neq \emptyset\) and \(\partial D_1 \cap I(\partial T_1) = I(\partial T_2)\). It is denoted by \(L : (D_1, T_1) \leftrightarrow (D_2, T_2)\), \(L : T_1 \leftrightarrow T_2\) or simply denoted by \(L\). The set of local moves is denoted by \(L\).

Let \((D, T)\) be a tangle diagram and \(f\) be a map from \(D\) to \(\tilde{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1/2\}\) such that \(f((x, y, 0)) = \frac{1}{2}(x, y, 0)\). Then we let \((\tilde{D}, \tilde{T}) := (f(D), f(T))\) and call \((\tilde{D}, \tilde{T})\) the shrunk diagram of \((D, T)\).

In order to describe the equivalence of local moves, we shall define a specific operation, called a braiding operation, as follows.

**Definition 2.6.** Let \(i (\leq 2n)\) be a positive integer and \(A := \{(x, y, 0) \in \mathbb{R}^3 \mid 1/2 \leq x^2 + y^2 \leq 1\}\). Let \(E_i = \bigcup_{k=1}^{2n} e_k\) and \(E_i' = \bigcup_{k=1}^{2n} e_k'\) be unions of 2n pairwise disjoint arcs \(e_k\) and \(e_k'\) embedded properly in \(B \setminus \text{Int}(B)\), respectively, satisfying the following:

(I) Case of \(i = 1, 2, \ldots, 2n-1\),

- \(e_j\) and \(\overline{e_j}\) are both the lines connecting the points \(1/2(\cos \frac{i}{n}, \sin \frac{i}{n}, 0)\) and \((\cos \frac{i}{n}, \sin \frac{i}{n}, 0)\) for \(j \neq i, i+1\),
- \(e_i\) and \(\overline{e_i}\) are arcs connecting the points \(1/2(\cos \frac{i}{n}, \sin \frac{i}{n}, 0)\) and \((\cos \frac{i+1}{n}, \sin \frac{i+1}{n}, 0)\) whose images \(p(e_i)\) and \(p(\overline{e_i})\) are on the annulus \(A\),
- \(e_{i+1}\) and \(\overline{e_{i+1}}\) are arcs connecting the points \(1/2(\cos \frac{i}{n}, \sin \frac{i}{n}, 0)\) and \((\cos \frac{i+1}{n}, \sin \frac{i+1}{n}, 0)\) whose images \(p(e_{i+1})\) and \(p(\overline{e_{i+1}})\) are on the annulus \(A\),
- the diagram \((A, E_i := p(E_i))\) has only one crossing which is an overcrossing (or an undercrossing) on \(p(e_i)\) (or \(p(e_{i+1})\)) and
- the diagram \((A, E_i' := p(E_i'))\) has only one crossing which is an undercrossing (or an overcrossing) on \(p(e_i)\) (or \(p(e_{i+1})\)).

(II) Case of \(i = 2n\),

- \(e_j\) and \(\overline{e_j}\) are both the lines connecting the points \(1/2(\cos \frac{i}{n}, \sin \frac{i}{n}, 0)\) and \((\cos \frac{i}{n}, \sin \frac{i}{n}, 0)\) for \(j \neq 2n, 1\),
- \(e_{2n}\) and \(\overline{e_{2n}}\) are arcs connecting the points \(1/2(0, 0, 0)\) and \((0, 0, 0)\) whose images \(p(e_{2n})\) and \(p(\overline{e_{2n}})\) are on \(A\),
- \(e_1\) and \(\overline{e_1}\) are arcs connecting the points \(1/2(\cos \frac{i}{n}, \sin \frac{i}{n}, 0)\) and \((1, 0, 0)\) whose images \(p(e_1)\) and \(p(\overline{e_1})\) are on \(A\),
- the diagram \((A, E_2 := p(E_2))\) has only one crossing which is the overcrossing (or undercrossing) on the arc \(p(e_2)\) (or \(p(e_1)\)) and
- the diagram \((A, E_{2n} := p(E_{2n}))\) has only one crossing which is the undercrossing (or overcrossing) on the arc \(p(e_{2n})\) (or \(p(\overline{e_1})\)).

If \((D, T)\) is an n-tangle diagram and \((\tilde{D}, \tilde{T})\) is the shrunk diagram of \((D, T)\), then we can regard \((D_1, T_1) := (A \cup \tilde{D}, E_i \cup \tilde{T})\) and \((D_2, T_2) := (A \cup \tilde{D}, E_i' \cup \tilde{T})\) as n-tangle diagrams. The operation that transforms \((D, T)\) into \((D_1, T_1)\) is called the braiding operation \(\sigma_i\) on \((D, T)\) and we write \(T_1 = \sigma_i T\). Similarly, the operation that transforms \((D, T)\) into \((D_2, T_2)\) is called the braiding operation \(\overline{\sigma_i}\) on \((D, T)\) and we write \(T_2 = \overline{\sigma_i} T\) (see FIG. 3). Here the orientation of \((D_j, T_j)\), \(j = 1, 2\), is induced from the orientation of \((D, T)\).

From the definition, we see that \(\sigma_2(\sigma_1 T) = (\sigma_2 \sigma_1) T\). Therefore, we denote \(\sigma_2(\sigma_1 T)\) by \(\sigma_2 \sigma_1 T\).

**Definition 2.7.** Two local moves, \(L : T_1 \leftrightarrow T_2\) and \(L' : T_1' \leftrightarrow T_2'\), are equivalent, denoted by \(L \cong L'\), if there exists a finite sequence of braiding operations \(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m}\) such
that $T'_1 = \sigma_{i_m} \cdots \sigma_{i_2} \sigma_{i_1} T_1$ and $T'_2 = \sigma_{i_m} \cdots \sigma_{i_2} \sigma_{i_1} T_2$. Then the operation that transforms $L$ into $L'$ is called a sequence of braiding operations $\sigma_{i_1}, \sigma_{i_2}, \cdots, \sigma_{i_m}$ on $L$ and we say that $L'$ can be obtained from $L$ using a finite sequence of braiding operations $\sigma_{i_1}, \sigma_{i_2}, \cdots, \sigma_{i_m}$.

This relation $\cong$ is clearly an equivalence relation, i.e., it satisfies the following properties (i) – (iii): (i) $L \cong L$, (ii) $L \cong L'$ implies $L' \cong L$ and (iii) $L \cong L'$ and $L' \cong L''$ imply $L \cong L''$. If $T'_1 = \sigma_{i_m} \cdots \sigma_{i_2} \sigma_{i_1} T_1$ and $T'_2 = \sigma_{i_m} \cdots \sigma_{i_2} \sigma_{i_1} T_2$, then $T_1 = \sigma_{i_m} \cdots \sigma_{i_1} T'_1$ and $T_2 = \sigma_{i_m} \cdots \sigma_{i_1} T'_2$.

For example, the crossing change (X-move) is equivalent to the local move, as shown in the center-side or lower-side diagram of FIG. 4. Even if a rotation is performed on two tangle diagrams of a local move, the (old) local move and the new local move are equivalent.

The next proposition demonstrates that a local move and the local move after a rotation are equivalent.

![Fig. 4. Equivalent local moves.](image-url)
Proposition 2.1. Let $\mathcal{L} : T_i \leftrightarrow T_2$ be a local move and let $T'_i$ be the $n$-tangle diagram rotated $T_i$ $\frac{m \pi}{2}$ radians around its origin ($m \in \mathbb{Z}$, $i = 1, 2$). Then $\mathcal{L}' : T'_i \leftrightarrow T'_2$ is a local move such that $\mathcal{L} \cong \mathcal{L}'$.

Proof. In order to prove the proposition, it is helpful to separate our proof into three cases: $m = 0$, $m > 0$ and $m < 0$.

Case $m = 0$: Then $T'_i = T_i$. Thus, $\mathcal{L} = \mathcal{L}'$.

Case $m > 0$: Let $T''_i = \sigma_1 \sigma_2 \cdots \sigma_{2n-1} T_i$. Then the tangle diagram $T''_i$ is one rotated $T_i$, $\frac{m \pi}{2}$ radians around its origin. Thus, $T''_i = (\sigma_1 \sigma_2 \cdots \sigma_{2n-1})^m T_i$. Thus, we have $\mathcal{L} \cong \mathcal{L}'$.

Case $m < 0$: Let $T''_i = \sigma_{2n-1} \sigma_{2n-2} \cdots \sigma_{i+1} T_i$. Then the diagram $T''_i$ is one rotated $T_i$, $-\frac{m \pi}{2}$ radians around its origin. Thus, $T''_i = (\sigma_{2n-1} \sigma_{2n-2} \cdots \sigma_{i+1})^{-m} T_i$. Thus, we have $\mathcal{L} \cong \mathcal{L}'$. \hfill $\square$

We extend the set of $ST(n)$-moves to the set of local moves that are equivalent to them. A local move that is equivalent to an $ST(n)$-move is called an extended $ST(n)$-move as follows:

Definition 2.8. A local move $\mathcal{L} : (D_1, T_1) \leftrightarrow (D_2, T_2)$ is called an extended $ST(n)$-move if there is a local move $\mathcal{T} : (D_3, T_3) \leftrightarrow (D_4, T_4)$ so that $(D_3, T_3)$ and $(D_4, T_4)$ are both trivial, $\mathcal{L} \cong \mathcal{T}$ and $(D_3, T_3) \neq (D_4, T_4)$ (see FIG. 5). When we take no notice of the number of arcs in each tangle diagram, an extended $ST(n)$-move is simply called an extended $ST$-move. The sets of extended $ST(n)$ and extended $ST$-moves are denoted by $\mathbb{T}_n$ and $\mathbb{T}$, respectively.

![Diagram](image)

FIG. 5

Clearly an $ST(n)$-move is an extended $ST(n)$-move.

Definition 2.9. Let $\mathcal{L} : (D_1, T_1) \leftrightarrow (D_2, T_2)$ be a local move. Let $p_S$ be a stereographic projection from $D_1$ onto $S^2 \subset S^2$, e.g., $p_S(x, y, 0) = (2x/(1+x^2+y^2), 2y/(1+x^2+y^2), (1-x^2-y^2)/(1+x^2+y^2))$ and $p_N$ is a stereographic projection from $D_2$ onto $S^2 \subset S^2$, e.g., $p_N(x, y, 0) = (2x/(1+x^2+y^2), 2y/(1+x^2+y^2), (-1+x^2+y^2)/(1+x^2+y^2))$. Then $(S^2, T_1 \cup T_2) := \left(p_S(D_1) \cup p_N(D_2), p_S(T_1) \cup p_N(T_2)\right)$ is called the union of the local move $\mathcal{L}$. Here $p_S(T_1)$ and $p_N(T_2)$ are regular diagrams of $T_1$ and $T_2$ into $S^2$ and $S^2$, respectively. Each $e$-point marked $j$ of $(D_1, T_1)$ is identified with an $e$-point marked $j$ of $(D_2, T_2)$ in the union of $\mathcal{L}$. Each point $(\cos \frac{j \pi}{n}, \sin \frac{j \pi}{n}, 0)$ in (the equator of) the union of $\mathcal{L}$ is also marked $j$ and called an $e$-point in the union of $\mathcal{L}$. The union $(S^2, T_1 \cup T_2)$ of $\mathcal{L}$ can be regarded as a link diagram in $S^2$. We call each component of the link diagram
“a component with e-points (in the union of ℳ).” If an e-point in the union of ℳ is on a component ℳ with e-points in the union of ℳ, then we call it an e-point of ℳ. Here all e-points in the union of ℳ are fixed.

3. A classification of ST-moves

The SH(n)-move, as illustrated in FIG. 1, is denoted by ℳn. It is easy to see that ℳn ∈ ℳn ⊂ ℳ ⊂ ℳ. In this section, we shall state some lemmas and Theorem 1. Before describing them, we define some notations.

Let ℳ ∈ ℳ and ℳ be a component with e-points in the union of ℳ. Let E(ℳ) be the set of integers marked to the e-points of ℳ. We assign the positive integers marked to the e-points of ℳ to the following notations: Let A(1) = min E(ℳ). Suppose that the component with an e-point marked A(1) contains 2k e-points in total. Now, starting with this e-point, move along the arc in D1. When we arrive at an e-point, we denote the number marked to the e-point by A(2). Moving on from the e-point marked A(2) along the arc in T2, when we arrive at the next e-point, we denote the positive integer of this e-point by A(3). In this way, we assign the positive integers from 1 to 2k to the e-points of ℳ by using the above sequences of braiding operations, are denoted by ℳ′ and ℳ′, respectively. Let ℳ′(1), ℳ′(2), . . . , ℳ′(2k) be the positive integers marked to the e-points of ℳ′ in the manner described above. Let S(ℳ′) = {i | A′(i) − A′(i − 1) ̸= 1, i = 2, 3, . . . , 2k} and L(ℳ′) = {i | B′(i) − B′(i − 1) ̸= 1, i = 2, 3, . . . , 2k}. If S(ℳ′) = ∅ (or L(ℳ′) = ∅, resp.), then we let s(ℳ′) = min S(ℳ′) (or l(ℳ′) = min L(ℳ′), resp.). And then, we apply the following sequence of braiding operations to both tangle diagrams of ℳ:

\[ \sigma_{A(s−1)+1} \sigma_{A(s−2)+2} \cdots \sigma_{A(s)+s−1} \]
\[ (or \ \sigma_{B(l−1)+1} \sigma_{B(l−2)+2} \cdots \sigma_{B(l)+l−1}, \ \text{resp.}) \]

where s = s(ℳ′) and l = l(ℳ′). Note that 2 ≤ s, l ≤ 2k, 2 ≤ A(s) − A(s − 1) and 2 ≤ B(l) − B(l − 1). The local move and the component with e-points obtained from ℳ and ℳ using the above sequences of braiding operations, are denoted by ℳ′ and ℳ′, respectively. Let ℳ′(1), ℳ′(2), . . . , ℳ′(2k) be the positive integers marked to the e-points of ℳ′ in the manner described above. Let S(ℳ′) = {i | A′(i) − A′(i − 1) ̸= 1, i = 2, 3, . . . , 2k} and L(ℳ′) = {i | B′(i) − B′(i − 1) ̸= 1, i = 2, 3, . . . , 2k}. If S(ℳ′) = ∅ (or L(ℳ′) = ∅, resp.), then we let s(ℳ′) = min S(ℳ′) (or l(ℳ′) = min L(ℳ′), resp.). Then we see that s(ℳ′) < s(ℳ′) (or l(ℳ′) < l(ℳ′), resp.).

Proposition 3.1. Let ℳ be a component with 2k e-points in the union of a local move ℳ such that S(ℳ) = ∅. Let ℳ′ and ℳ′(+) be the notations described above. If the following (a) or (b) holds for ℳ, then the following (a)' or (b)' holds for ℳ':

(a) The tangle diagrams of ℳ are both trivial.

(b) The condition (a) does not hold. Let ℳ′ be the notation described above. If the following (a)' or (b)' holds for ℳ':

(a)' The tangle diagrams of ℳ′ are both trivial.

(b) The condition (a)' does not hold. Each tangle diagram of ℳ′ is equal to a tangle diagram satisfying the following: All overcrossings are on the arc, say α1, whose e-points are marked A(s(ℳ) − 1) and A(s(ℳ)), and all undercrossings of the diagrams are on arcs except α1.

(a) The tangle diagrams of ℳ′ are both trivial.

(b)' The condition (a)' does not hold. Each tangle diagram of ℳ′ is equal to a tangle diagram satisfying the following: All overcrossings are on the arc, say β2, whose e-points are marked A'(s(ℳ)) and A'(s(ℳ) + 1), and all undercrossings are on arcs except β2.

Here, if s(ℳ) = 2k, then we let A'(s(ℳ) + 1) := A'(1).

Proof. Let ℳ : (D1, T1) ↔ (D2, T2) and ℳ′ : (D′1, T′1) ↔ (D′2, T′2). Let (D̄i, T̄i) be the shrunk diagram of (Di, Ti) obtained from (Di, Ti) by the sequence of braiding operations, where i = 1, 2. We can assume w. l. o. g. that α1 is on D1. Let α′1 ∈ D̄1 be the arc whose end points are marked A(s − 1) ∈ ∂D̄1 and A(s) ∈ ∂D̄1. Let α2 ∈ D′1 \ Int(D̄1) be the arc whose end points are marked A(s) ∈ ∂D̄1 and A′(s) ∈ ∂D′1, and let α3 ∈ D′2 \ Int(D̄2) be
the arc whose end points are marked $A'(s) \in \partial D'_2$ and $A(s) \in \partial D'_2$. Let $\beta_1 \in D'_1$ be the arc whose end points are marked $A'(s - 1)$ and $A'(s)$ (see FIG. 6-8). Here $s = s(\mathcal{C})$.

(I) Case $2 \leq s \leq 2k - 1$ (see FIG. 6 and FIG. 7): Let $\alpha_4 \in D_2$ be the arc whose end points are marked $A(s)$ and $A(s + 1)$. Let $\beta_2 \in D'_2$ be the arc whose e-points are marked $A'(s)$ and $A'(s + 1)$.

If the condition (a) holds for $\mathcal{L}$, then $\alpha'_1$ and $\alpha_4$ possess no crossings, $\alpha_2$ and $\alpha_3$ may do overcrossings and no undercrossings. If the condition (b) holds for $\mathcal{L}$, then $\alpha'_1$ has overcrossings, $\alpha_2$ and $\alpha_3$ may possess overcrossings, $\alpha_4$ does no crossings. Therefore, even if crossings exist or do not exist on $\alpha'_1$, all crossings on $\beta_1$ are overcrossings and are removed using a finite sequence of Reidemeister moves because the e-points marked $A'(s - 1)$ and $A'(s)$ are adjacent. Thus, if a crossing exists in a tangle diagram of $\mathcal{T}'$, then it is on the arc $\alpha_1$. Therefore, it is on $\beta_2$. Remark that if $\beta_2$ has a crossing with itself, then only one self-crossing exists and we have $A(s + 1) < A(s)$. Since $\alpha_3$ passes over and $A'(s + 1) > A'(s)$ by construction, the only one self-crossing can be removed using Reidemeister moves (see FIG. 7). Hence, the condition (a)' or (b)' holds for $\mathcal{L}'$.

(II) Case $s = 2k$ (see FIG. 8): Let $\alpha_4 \in D_2$ be the arc whose end points are marked $A(s)$ and $A(1)$. Let $\beta_2 \in D'_2$ be the arc whose e-points are marked $A'(s)$ and $A(1)$. Even if crossings exist on $\alpha'_1$, $\alpha_2$ or $\alpha_3$, they are removed using a finite sequence of Reidemeister moves because $\beta_1$ and $\beta_2$ are overpasses, and the e-points marked $A'(2k - 1)$ and $A'(2k)$ are adjacent to each other. Thus, the tangle diagrams of $\mathcal{L}'$ are both trivial. Hence, the condition (a)' holds for $\mathcal{L}'$. □

![Fig. 6. Union of $\mathcal{L}'$; Case (I) of $A(s) < A(s + 1)$.

Note that if (b) or (b)' holds for $\mathcal{L}$ or $\mathcal{L}'$, then $S(\mathcal{C}) \neq \emptyset$ or $S(\mathcal{C}') \neq \emptyset$, respectively.

**Proposition 3.2.** Let $\mathcal{C}$ be a component with $2k$ e-points in the union of a local move $\mathcal{L}$ such that $L(\mathcal{C}) \neq \emptyset$. Let $\mathcal{L}'$ and $B'(s)$ be the notations described in the beginning of this section. If the following (c) or (d) holds for $\mathcal{L}$, then the following (c)' or (d)' holds for $\mathcal{L}'$:

(c) The tangle diagrams of $\mathcal{L}$ are both trivial.

(d) The condition (c) does not hold. All overcrossings of the tangle diagrams of $\mathcal{L}$ are on the arc, say $\alpha_1$, whose e-points are marked $B(l(\mathcal{C}) - 1)$ and $B(l(\mathcal{C}))$, and all undercrossings of the diagrams are on arcs except $\alpha_1$.

(c)' The tangle diagrams of $\mathcal{L}'$ are both trivial.

(d)' The condition (c)' does not hold. Each tangle diagram of $\mathcal{L}'$ is equal to a tangle diagram satisfying the following: All overcrossings are on the arc, say $\beta_1$, whose e-points are marked $B'(l(\mathcal{C}))$ and $B'(l(\mathcal{C}) + 1)$, and all undercrossing are on arcs except $\beta_2$.

Here, if $l(\mathcal{C}) = 2k$, then we let $B'(l(\mathcal{C}) + 1) := B'(1)$.

**Proof.** This follows from Proposition 3.1. □
Lemma 3.1. Let

Proof. Let

Therefore, suppose that

Thus, for

Proposition 3.1. If

Note that if (d) or (d′) holds for \( T \) or \( T' \), then \( L(C) \neq \emptyset \) or \( L(C') \neq \emptyset \), respectively.

Lemma 3.1. Let \( T \in \mathbb{T}_n \) and \( c(T) = 1 \). Then \( T \cong \mathcal{H}_n \).

Proof. Let \( C \) be the component with e-points in the union of \( T \). If \( T = \mathcal{H}_n \), then \( T \cong \mathcal{H}_n \). Therefore, suppose that \( T \neq \mathcal{H}_n \), i.e., \( T \) is not an \( ST \)-move or \( S(C) \neq \emptyset \). We may assume that \( T \) is an \( ST(n) \)-move and \( S(C) \neq \emptyset \) because \( T \) is an extended \( ST(n) \)-move and \( T \neq \mathcal{H}_n \).

Since \( S(C) \neq \emptyset \) and the condition (a) in Proposition 3.1 holds for \( T \), the condition (a)' or (b)' holds for \( T' \) by Proposition 3.1. If \( S(C') = \emptyset \), then the condition (a)' holds for \( T' \). Thus, \( T' = \mathcal{H}_n \). So we have \( T \cong \mathcal{H}_n \). Otherwise, the assumption of Proposition 3.1 holds for \( T' \) and that \( s(C) < s(C') \).

Suppose that \( S(C') \neq \emptyset \) and the condition (a) or (b) in Proposition 3.1 holds for \( T' \) for \( i = 1, 2, \ldots, r - 1 \). Here \( T'^{0} := T \), \( T'^{i} := (T'^{i-1})' \), \( C'^{i} := C \) and \( C'^{i} = (C'^{i-1})' \). We let \( T'^{i} := (T'^{i-1})' \) and \( C'^{i} := (C'^{i-1})' \). Then the condition (a)' or (b)' holds for \( T' \) by Proposition 3.1. If \( S(C') = \emptyset \), then the condition (a)' holds for \( T' \). Thus, \( T' = \mathcal{H}_n \). So we have \( T \cong \mathcal{H}_n \). Otherwise, the assumption of Proposition 3.1 holds for \( T' \) and \( s(C'^{i-1}) < s(C') \).

Hence, by induction there is a positive integer \( m \) such that \( S(C'^{m}) = \emptyset \) since the number of e-points on \( C \) is finite. Then two tangle diagrams of \( T'^{m} \) are trivial. Hence, \( T'^{m} \cong \mathcal{H}_n \) because of \( T \cong T' \cong \cdots \cong T'^{m} \cong \mathcal{H}_n \). This completes the proof.

\[ \square \]

Definition 3.1. Let \( T \in \mathbb{T} \). Let \( C \) be a component with e-points in the union of \( T \). Let \( B(1), B(2), \ldots, B(2k) \) be the integers marked to the e-points of \( C \) satisfying \( B(1) < \]
Lemma 3.2. Let \( \mathcal{T} \in \mathbb{T} \). Then there exists an appropriate \( \text{ST}\)-move that is equivalent to \( \mathcal{T} \).

Proof. We may suppose that \( \mathcal{T} \) is an \( \text{ST}\)-move and is not appropriate. The latter property means that a component \( \mathcal{C} \) with e-points in the union of \( \mathcal{T} \) exists such that \( \mathcal{C} \) is not appropriate, i.e., \( L(\mathcal{C}) \neq \emptyset \).

Let \( B(1), B(2), \ldots, B(2k) \) be the positive integers marked to the e-points of \( \mathcal{C} \) satisfying \( B(1) < B(2) < \cdots < B(2k) \). Now \( L(\mathcal{C}) \neq \emptyset \) and the condition \( (c) \) in Proposition 3.2 holds for \( \mathcal{T} \). Therefore, \( (c)' \) or \( (d)' \) in Proposition 3.2 holds for \( \mathcal{T}' \) by Proposition 3.2. If \( L(\mathcal{C}') = \emptyset \), then the condition \( (c)' \) holds for \( \mathcal{T}' \). That is, \( \mathcal{C}' \) is appropriate and \( \mathcal{T}' \) is an \( \text{ST}\)-move.

If \( L(\mathcal{C}') \neq \emptyset \), then the condition \( (c)' \) or \( (d)' \) in Proposition 3.2 holds for \( \mathcal{T}' \). This means that the assumption of Proposition 3.2 holds for \( \mathcal{T}' \). Note that any arc of \( \mathcal{C}' \) do not have a crossing with an arc of \( \mathcal{C}' \). If a diagram of \( \mathcal{T}' \) has a crossing, then its overcrossing is on an arc of \( \mathcal{C}' \) and its undercrossing is on an arc of a component except \( \mathcal{C}' \).

Suppose that \( L(\mathcal{C}') \neq \emptyset \) and the condition \( (c) \) or \( (d) \) in Proposition 3.2 holds for \( \mathcal{T}' \) for \( i = 1, 2, \ldots, r - 1 \). Here \( \mathcal{T} := \mathcal{T}' := (\mathcal{T}'^{-1})', \mathcal{C}' := \mathcal{C} \) and \( \mathcal{C}'' := (\mathcal{C}''^{-1})' \). We let \( \mathcal{T} := (\mathcal{T}'^{-1})' \) and \( \mathcal{C} := (\mathcal{C}'')' \). Then the condition \( (c)' \) or \( (d)' \) holds for \( \mathcal{T} \) by Proposition 3.2. If \( L(\mathcal{C}') = \emptyset \), then the condition \( (c)' \) holds for \( \mathcal{T} \). Thus, \( \mathcal{C} \) is appropriate. Otherwise, the assumption of Proposition 3.2 holds for \( \mathcal{T}' \). Note that any arc of \( \mathcal{C}'' \) do not have a crossing with an arc of \( \mathcal{C}'' \). If a diagram of \( \mathcal{T} \) has a crossing, then its overcrossing is on an arc of \( \mathcal{C}'' \) and the undercrossing is on an arc of a component except \( \mathcal{C}'' \).

Hence, by induction there is a positive integer \( m \) such that \( L(\mathcal{C}''^m) = \emptyset \) since the number of e-points on \( \mathcal{C} \) is finite. Then two tangle diagrams of \( \mathcal{T}^m \) are trivial.

If \( \mathcal{T}^m \) is not appropriate, then we can continue such an operation until \( \mathcal{T}^n \) becomes an appropriate \( \text{ST}\)-move. Additionally, we should remark that every appropriate component with e-points remains appropriate even after the application of these moves.

Hence, there exists an appropriate \( \text{ST}\)-move \( \mathcal{T}^m \) that is equivalent to \( \mathcal{T} \). This completes the proof.

\[ \square \]

Let \( \mathcal{L} \in \mathbb{L} \). The number of components with e-points in the union of \( \mathcal{L} \) is denoted by \( c(\mathcal{L}) \). Henceforth, suppose that \( j \) is a natural integer such that \( 1 \leq j \leq c(\mathcal{T}) \).

Lemma 3.3. Let \( \mathcal{T} \) be an appropriate \( \text{ST}(n)\)-move. Let \( \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{c(\mathcal{T})} \) be the components with e-points in the union of \( \mathcal{T} \). Let \( k_j \) be the number of e-points of \( \mathcal{C}_j \). Then there exists a finite sequence of braiding operations that transforms \( \mathcal{T} \) into an appropriate \( \text{ST}(n)\)-move and \( \mathcal{C}_j \) into a component with e-points marked \( B'_1(1), B'_1(2), \ldots, B'_1(k_j) \) satisfying the following: \( 1 = B_1(1) < \cdots < B_1'(1) < B_2'(1) < \cdots < B_3'(1) < \cdots < B_{c(\mathcal{T})}'(1) < \cdots < B_{c(\mathcal{T})}'(k_j) = 2n \).

Proof. Proposition 2.1 ensures the existence of a finite sequence of braiding operations, i.e., a rotation, that transforms the marks of the e-points of \( \mathcal{C}_1 \) into 1, 2, \ldots and \( k_1 \). The local move and the component with e-points obtained from \( \mathcal{T} \) and \( \mathcal{C}_j \) using the finite sequence of braiding operations, are once again denoted by \( \mathcal{T} \) and \( \mathcal{C}_j \), respectively.

Let \( B_1(1), B_1(2), \ldots, B_1(k_j) \) be the integers marked to the e-points of \( \mathcal{C}_j \) satisfying \( B_1(1) < B_1(2) < \cdots < B_1(k_j) \). Then \( \mathcal{T} \) is an appropriate \( \text{ST}(n)\)-move, and the following property holds: \( 1 = B_1(1) < 2 = B_1(2) < \cdots < k_1 = B_1(k_1) \). Let \( B(1) = B_1(1), B(2) = B_1(2), \ldots, B(k_1 + 1) = B_2(k_1), \ldots, B(k_1 + k_2) = B_2(k_2) \) and

\[ L(\mathcal{C}_1 \cup \mathcal{C}_2) = \{ i \mid B(i) - B(i - 1) \neq 1, i = 2, 3, \ldots, k_1 + k_2 \}. \]
Now we see that \( L(\mathcal{C}_i) = \emptyset \) for each \( j \).

By the proof of Lemma 3.2, there exists a finite sequence of braiding operations that transforms \( \mathcal{C}_j \) into \( \mathcal{C}'_j \) such that \( L(\mathcal{C}'_1 \cup \mathcal{C}'_2) = \emptyset \). Then the integers \( B'_1(1) < B'_1(2) < \cdots < B'_1(k_i) \) marked to the e-points of \( \mathcal{C}'_j \) satisfy the following property: \( 1 = B'_1(1) < \cdots < B'_1(2) < \cdots < B'_2(k_2) = k_1 + k_2 \). Then we see that the local move \( \mathcal{T}' \) obtained from \( \mathcal{T} \) is an \( ST \)-move.

Suppose that \( \mathcal{T} \) is an appropriate \( ST(n) \)-move and \( 1 = B_1(1) < B_2(1) < \cdots < B_{M-1}(1) < 2n \), where \( 2 \leq M \leq c(\mathcal{T}) \). Let \( B(1) = B_1(1), \ B(2) = B_2(1), \ldots, \ B(k_1) = B_3(k_1), \ B(k_1 + 1) = B_4(k_1), \ldots, \ B(k_1 + k_2) = B_2(k_2), \ldots, \ B(k_1 + k_2 + \cdots + k_{M-1} + 1) = B_M(1), \ldots, \ B(k_1 + k_2 + \cdots + k_M) = B_M(k_M) \) and

\[
L(\mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_M) = \{ i \mid B(i) - B(i-1) \neq 1, i = 2, 3, \ldots, k_1 + k_2 + \cdots + k_M \}.
\]

By the proof of Lemma 3.2, there exists a finite sequence of braiding operations that transforms \( \mathcal{C}_j \) into \( \mathcal{C}'_j \) such that \( L(\mathcal{C}'_1 \cup \mathcal{C}'_2 \cup \cdots \cup \mathcal{C}'_M) = \emptyset \). Then \( \mathcal{T}' \) is an appropriate \( ST \)-move, and the integers \( B'_1(1) < B'_1(2) < \cdots < B'_1(k_i) \) marked to the e-points of \( \mathcal{C}'_j \) satisfy the following property: \( 1 = B'_1(1) < \cdots < B'_1(2) < \cdots < B'_2(k_2) < \cdots < B'_M(k_M) = k_1 + k_2 + \cdots + k_M \). Hence, by induction the proof is completed.

\[\text{Definition 3.2. Let } \mathcal{T} \text{ be an } ST \text{-move. Let } \mathcal{C} \text{ be a component with e-points in the union of } \mathcal{T}. \text{ Let } A(1), A(2), \ldots, A(2k) \text{ be the integers marked to the e-points of } \mathcal{C} \text{ as described in the beginning of this section. If } A(1) \text{ is an initial e-point of } \mathcal{C} \text{ and } S(\mathcal{C}) = \emptyset, \text{ then we say that this component } \mathcal{C} \text{ is an } SH(+) \text{-type. If the components with e-points in the union of } \mathcal{T} \text{ are all } SH(+) \text{-types, then } \mathcal{T} \text{ is said to be an } SH(+) \text{-type.}\]

In Definition 3.2, if \( \mathcal{C} \) is an \( SH(+) \)-type, then its form is uniquely decided and we have \( A(i) = B(i) \) for \( i = 1, 2, \ldots, 2k \). Here the notation \( B(i) \) is the integers marked to the e-points of \( \mathcal{C} \) as in Definition 3.1.

**Lemma 3.4.** Let \( \mathcal{T} \) be an appropriate \( ST \)-move. Then there is an \( SH(+) \)-type that is equivalent to \( \mathcal{T} \).

**Proof.** Suppose that \( \mathcal{T} \) is not an \( SH(+) \)-type. That is, there is a component, say \( \mathcal{C} \), with e-points in the union of \( \mathcal{T} \) such that \( \mathcal{C} \) is not an \( SH(+) \)-type. Choose an initial e-point of \( \mathcal{C} \). By Proposition 2.1, there exists a finite sequence of braiding operations that transforms the e-point into the e-point marked 1. By the proof of Lemma 3.1, the component \( \mathcal{C} \) with e-points can be transformed into an \( SH(+) \)-type using a finite sequence of braiding operations without affecting the other components with e-points in the union of \( \mathcal{T} \).

We can continue such an operation until every component with e-points in the union becomes an \( SH(+) \)-type. We complete the proof.

**Definition 3.3.** Let \( \mathcal{T} \) be an \( SH(+) \)-type. Let \( \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{c(\mathcal{T})} \) be the components with e-points in the union of \( \mathcal{T} \). Let \( A_1(1), A_1(2), \ldots, A_1(k_1) \) be the integers marked to the e-points of \( \mathcal{C}_j \) satisfying \( A_1(1) < A_2(1) < \cdots < A_{c(\mathcal{T})}(1) \). If \( k_1 \leq k_2 \leq \cdots \leq k_{c(\mathcal{T})} \), then \( \mathcal{T} \) is said to be standard (see FIG. 9).

![Fig. 9. Standard ST(3)-moves.](image-url)
Definition 3.4. Let $\mathcal{T} \in \mathcal{T}_n$. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{c(\mathcal{T})}$ be the components with e-points in the union of $\mathcal{T}$ and $n_j = \frac{1}{4} | E(\mathcal{C}_j) |$. If $n_1 \leq n_2 \leq \cdots \leq n_{c(\mathcal{T})}$, then the partition $n = n_1 + n_2 + \cdots + n_{c(\mathcal{T})}$ of $n$ is called the arc-decomposition of $\mathcal{T}$, where $| \ast |$ is the number of the set $\ast$.

By the definition, we see that a standard $ST(n)$-move $\mathcal{T}$, whose arc-decomposition is $n = n_1 + n_2 + \cdots + n_{c(\mathcal{T})}$, is uniquely decided.

The next theorem states that there is a one-to-one correspondence between the set of equivalence classes of extended $ST(n)$-moves and the set of standard $ST(n)$-moves for each $n$.

Theorem 1. Given $\mathcal{T} \in \mathcal{T}_n$, there is a uniquely standard $ST(n)$-move that is equivalent to $\mathcal{T}$. Conversely, given a standard $ST(n)$-move, there is an extended $ST(n)$-move that is equivalent to the standard one.

Proof. Let $\mathcal{T} \in \mathcal{T}_n$. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{c(\mathcal{T})}$ be the components with e-points in the union of $\mathcal{T}$, in which the number of e-points of $\mathcal{C}_{i+1}$ is greater than or equal to one of $\mathcal{C}_i$, where $i = 1, 2, \ldots, c(\mathcal{T}) - 1$. Then there is a standard $ST(n)$-move $\mathcal{T}'$ such that $\mathcal{T}' \cong \mathcal{T}$, by Lemmas 3.2, 3.3 and 3.4. Suppose that there is a standard $ST(n)$-move $\mathcal{T}''$ such that $\mathcal{T}'' \neq \mathcal{T}'$ and $\mathcal{T}'' \cong \mathcal{T}$. Then the arc-decomposition of $\mathcal{T}''$ is not equal to the arc-decomposition of $\mathcal{T}'$, which is a contradiction. Because a braiding operation on a local move $\mathcal{L}$ does not change the number $c(\mathcal{L})$ and the number of arcs in each tangle diagram of any component with e-points of $\mathcal{L}$. Thus, $\mathcal{T}'$ is uniquely decided.

Conversely, given a standard $ST(n)$-move, it is also an extended $ST(n)$-move. Thus, Theorem 1 holds.

For example, for $n = 3$, there exist two standard $ST(3)$-moves in all, i.e., the two local moves in the second row of FIG. 10, which are not equivalent. Although there are many extended $ST(3)$-moves, each of them is equivalent to the diagram on the lower left or the diagram on the lower right in FIG. 10. A left-side local move and a right-side local move of FIG. 10 are not equivalent. On the other hand, the two $ST(3)$-moves, on the left or on the right of FIG. 10 are equivalent. Thus, the four $ST(3)$-moves as in FIG. 10 are classified into two equivalence classes whose representatives are the lower-side local moves of FIG. 10.

By Theorem 1, we can choose a standard $ST$-move as a representative of an equivalence class of extended $ST$-moves. The standard $ST(n)$-move whose arc-decomposition is $n = n_1 + n_2 + \cdots + n_r$, denoted by $(n_1, n_2, \ldots, n_r)$, where $n_1 \leq n_2 \leq \cdots \leq n_r$.

Next, we have a necessary and sufficient condition for two extended $ST$-moves to be equivalent.

Corollary 1. Let $\mathcal{T}_n \in \mathcal{T}_n$ and $\mathcal{T}_m \in \mathcal{T}_m$. Let $n = n_1 + n_2 + \cdots + n_r$ and $m = m_1 + m_2 + \cdots + m_s$ be the arc-decompositions of $\mathcal{T}_n$ and $\mathcal{T}_m$, respectively. Then $\mathcal{T}_n \cong \mathcal{T}_m$ if and only if $r = s$ and $n_i = m_i$, where $i = 1, 2, \ldots, r$.

Proof. Suppose that $\mathcal{T}_n \cong \mathcal{T}_m$. Since a braiding operation on a local move $\mathcal{L}$ does not change the number $c(\mathcal{L})$ and the number of arcs in each tangle diagram of any component with e-points of $\mathcal{L}$, the equalities $r = s$ and $n_i = m_i$ hold, where $i = 1, 2, \ldots, r$.

Suppose that $r = s$ and $n_i = m_i$ holds, where $i = 1, 2, \ldots, r$. By Theorem 1, there is a uniquely standard $ST(n)$-move $\mathcal{T}$, whose arc-decomposition is $n = n_1 + n_2 + \cdots + n_r$, such that $\mathcal{T} \cong \mathcal{T}_n$ and $\mathcal{T} \cong \mathcal{T}_m$. Therefore, we have $\mathcal{T}_n \cong \mathcal{T}_m$. Hence, we complete the proof.

The next corollary provides the number of equivalence classes of extended $ST(n)$-moves.

Corollary 2. Let $(2 \leq n)$ be a positive integer and $p(n)$ be the partition number of $n$. Then we have $[\mathcal{T}_n] = p(n) - 1$, where $[\ast]$ is the set of equivalence classes of the set $\ast$. 

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This is an immediate consequence of Theorem 1.

For example, the number of equivalence classes of extended $ST(3)$-moves is 2 because the partition number of 3 is 3.

4. A PARTIAL ORDER ON EQUIVALENCE CLASSES

In this section, we shall define a specific operation, called a connecting operation, that transforms an $n$-tangle diagram into an $(n - 1)$-tangle diagram in order to describe a binary relation on the set of local moves.

**Definition 4.1.** Let $i (\leq 2n)$ be a positive integer and $A := \{(x, y, 0) \in \mathbb{R}^3 \mid 1/2 \leq x^2 + y^2 \leq 1\}$. Let $U_i$ be a union of $2n - 1$ pairwise disjoint arcs $u_k$ embedded properly in $B \setminus \text{Int}(\overline{B})$, satisfying the following:

(I) Case of $i = 1, 2, \ldots, 2n - 1$.

- For $j \in \{1, 2, \ldots, i - 1\}$, $u_j$ is an arc connecting the points $1/2(\cos \frac{1}{n+1}\pi, \sin \frac{1}{n+1}\pi, 0)$ and $(\cos \frac{1}{n+1}\pi, \sin \frac{1}{n+1}\pi, 0)$ whose image $p(u_j)$ is on the annulus $A$.
- For $j \in \{i + 2, i + 3, \ldots, 2n\}$, $u_j$ is an arc connecting the points $1/2(\cos \frac{1}{n+1}\pi, \sin \frac{1}{n+1}\pi, 0)$ and $(\cos \frac{1}{n+1}\pi, \sin \frac{1}{n+1}\pi, 0)$ whose image $p(u_j)$ is on $A$.
- $u_i$ is an arc connecting the points $1/2(\cos \frac{1}{n+1}\pi, \sin \frac{1}{n+1}\pi, 0)$ and $1/2(\cos \frac{1}{n+1}\pi, \sin \frac{1}{n+1}\pi, 0)$ whose image $p(u_i)$ is on $A$.
- The diagram $(A, U_i := p(\hat{U}_i))$ has no crossings.

(II) Case of $i = 2n$.

- For $j \in \{2, 3, \ldots, 2n - 1\}$, $u_j$ is an arc connecting the points $1/2(\cos \frac{1}{n+1}\pi, \sin \frac{1}{n+1}\pi, 0)$ and $(\cos \frac{1}{n+1}\pi, \sin \frac{1}{n+1}\pi, 0)$ whose image $p(u_j)$ is on $A$.
- $u_{2n}$ is an arc connecting the points $1/2(\cos \frac{1}{n+1}\pi, \sin \frac{1}{n+1}\pi, 0)$ and $1/2(1, 0, 0)$ whose image $p(u_{2n})$ is on $A$.
- The diagram $(A, U_{2n} := p(\hat{U}_{2n}))$ has no crossings.

If $(D, T)$ is an $n$-tangle diagram, the e-point marked $i$ is an initial e-point (or a terminal e-point), the e-point marked $i + 1$ (1 if $i = 2n$) is a terminal e-point (or an initial e-point), and $(\hat{D}, \hat{T})$ is the shrunk diagram of $(D, T)$, then we can regard $(D', T') := (A \cup \hat{D}, U_i \cup \hat{T})$ as an $(n - 1)$-tangle diagram. The operation that transforms $(D, T)$ into $(D', T')$ is called the connecting operation $\text{Con}(i)$ on $(D, T)$ and we write $T' = \text{Con}(i)T$. (see FIG. 11). Here the orientation of $(D', T')$ is induced from the orientation of $(D, T)$.

**Fig. 10. Equivalent moves**
Next, we define a binary relation $\preceq$ on the set of local moves.

**Definition 4.2.** Let $\mathcal{L} : T_1 \leftrightarrow T_2$ and $\mathcal{L}' : T'_1 \leftrightarrow T'_2$ be local moves. If there is a finite sequence of connecting and/or braiding operations $o_1, o_2, \ldots, o_r$ such that $T'_1 = o_r \cdots o_2 o_1 T_1$ and $T'_2 = o_r \cdots o_2 o_1 T_2$, then the operation that transforms $\mathcal{L}$ into $\mathcal{L}'$ is called a sequence of connecting and/or braiding operations $o_1, o_2, \ldots, o_r$ on $\mathcal{L}$ and we say that $\mathcal{L}'$ can be obtained (or realized) from $\mathcal{L}$ by a finite sequence of connecting and/or braiding operations $o_1, o_2, \ldots, o_r$, and we write $\mathcal{L} \preceq \mathcal{L}'$. In particular, if $o_1, o_2, \ldots, o_r$ are all braiding operations, then we say that $\mathcal{L}'$ can be obtained (or realized) from $\mathcal{L}$ by a finite sequence of braiding operations $o_1, o_2, \ldots, o_r$, and we write $\mathcal{L} \simeq \mathcal{L}'$ or $\mathcal{L}' = o_r \cdots o_2 o_1 \mathcal{L}$.

Let $\mathcal{L} \in \mathcal{L}_n$. If there is a positive integer $k_1$ such that $o_i = \sigma_{k_i}$, $o_r = Con(k_r)$, $i = 1, 2, \ldots, r - 1$, where
\[
k_{i+1} = \begin{cases} 
  k_i + 1 & \text{if } k_i + 1 \leq 2n, \\
  k_i + 1 \pmod{2n} & \text{if } k_i + 1 > 2n,
\end{cases}
\]
then we call the sequence of operations $o_1, o_2, \ldots, o_r$, a sequence of connecting and braiding operations on $\mathcal{L}$ that connects the e-points marked $k_1$ and $k_r + 1$.

Let $\mathcal{X}$ be the $X$-move (crossing change), $\mathcal{H}_2$ be the $SH(2)$-move, $\mathcal{T}$ be the $ST(3)$-move as shown in the upper-side diagram of FIG. 12 and $\mathcal{T}'$ be the $ST(3)$-move as in the upper-side diagram of FIG. 13. Then we see that $\mathcal{T} \preceq \mathcal{X}$ and $\mathcal{T}' \preceq \mathcal{H}_2$, which can be seen in FIG. 12 and FIG. 13, respectively. Because $\mathcal{X} = Con(4) \bar{\sigma}_3 \bar{\sigma}_2 \mathcal{T}$ and $\mathcal{H}_2 = Con(2)\mathcal{T}'$.
The \(SH(n)\)-move, as illustrated in the diagram in FIG. 1 is denoted by \(\mathcal{K}_n\). The sets of extended \(ST(n)\)-moves, extended \(ST\)-moves and local moves are denoted by \(T_n\), \(T\) and \(L\), respectively. The subset of \(L\) whose element is a pair of \(n\)-tangle diagrams is denoted by \(L_n\).

Let \([L] := \{L' \in \mathcal{L} | L \cong L'\}\) denote the equivalence class to which \(L\) belongs. Then the relation \(\preceq\) is well-defined under the equivalence relation \(\cong\) as follows: Suppose that \(L_1 \cong L_1'\) and \(L_2 \cong L_2'\). If \(L_1 \preceq L_2\), then we only need to prove that \(L_1' \preceq L_2'\). Since \(L_1 \cong L_1'\) and \(L_2 \cong L_2'\), there are finite sequences of braiding operations \(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_v}\) and \(\sigma_{j_1}, \sigma_{j_2}, \ldots, \sigma_{j_n}\) such that \(L_1 = \sigma_{i_v} \cdots \sigma_{i_1} L_1'\) and \(L_2 = \sigma_{j_v} \cdots \sigma_{j_1} L_2'\). Therefore we see that \(L_1' \preceq \sigma_{i_v} \cdots \sigma_{i_1} L_1' = L_1 \preceq L_2 \preceq \sigma_{j_v} \cdots \sigma_{j_1} L_2 = L_2'\). Hence, \(L_1' \preceq L_2'\).

Next, we show that the binary relation \(\preceq\) is a partial order on the set \(\mathcal{L} := \{[L] | L \in \mathcal{L}\}\) of equivalence classes of local moves. Namely, the relation \(\preceq\) on the set \(\mathcal{L}\) is reflexive, antisymmetric and transitive. This means it satisfies the following \((i) - (iii)\) for any \(\mathcal{F}\), \(\mathcal{G}\) and \(\mathcal{L}\) in \(\mathcal{L}\): \((i)\) \([\mathcal{F}] \preceq [\mathcal{G}]\), \((ii)\) if \([\mathcal{F}] \preceq [\mathcal{G}]\) and \([\mathcal{G}] \preceq [\mathcal{L}]\), then \([\mathcal{F}] \preceq [\mathcal{L}]\), and \((iii)\) if \([\mathcal{F}] \preceq [\mathcal{G}]\) and \([\mathcal{G}] \preceq [\mathcal{F}]\), then \([\mathcal{F}] = [\mathcal{G}]\). According to Definition 4.1, we see that \((i)\) and \((ii)\) hold. Let \(\mathcal{F} \in \mathcal{L}_n\) and \(\mathcal{G} \in \mathcal{L}_m\). Suppose that \([\mathcal{F}] \preceq [\mathcal{G}]\) and \([\mathcal{G}] \preceq [\mathcal{L}]\). It holds that \(m \leq n\) by the relation \(\preceq\) and it holds that \(n \leq m\) by the relation \(\preceq\). Therefore, we have \(n = m\). Namely, \(\mathcal{G}\) (or \(\mathcal{F}\), resp.) can be obtained from \(\mathcal{F}\) (or \(\mathcal{G}\), resp.) by a finite sequence of braiding operations. Therefore, we have \([\mathcal{F}] \preceq [\mathcal{G}]\). Thus, we have \([\mathcal{F}] = [\mathcal{G}]\) and so \((iii)\) holds. Hence, \((\mathcal{L}, \preceq)\) is a partially ordered set. We will regard each standard \(ST(n)\)-move as a representative of an equivalence class of an extended \(ST(n)\)-move.
5. SOME RESULTS

In this section, we discuss partially orderings between equivalence classes of extended ST-moves. We have already the partially orderings between equivalence classes of SH-moves. The following result can be found in [4].

**Lemma 5.1.** ([4, Lemma 16]) For any \( n \), we have \( \mathcal{K}_{2n+1} \preceq \mathcal{K}_{2n-1} \) and \( \mathcal{K}_{2n} \preceq \mathcal{K}_{2n-1} \).

**Proof.** Using the sequence of two connecting operations shown in FIG. 14, the \( SH(n-2) \)-move can be realized from the \( SH(n) \)-move. □

Therefore, we have the following corollary.

**Corollary 3.** For any \( n \), we have \( \mathcal{K}_{2n+1} \preceq \mathcal{K}_{2n-1} \) and \( \mathcal{K}_{2n} \preceq \mathcal{K}_{2n-1} \).

Note that if \( \mathcal{L} \in L_n \), then for any \( \mathcal{L}' \in [\mathcal{L}] \), \( \mathcal{L}' \in L_n \) and \( c(\mathcal{L}') = c(\mathcal{L}) \).

**Proposition 5.1.** Let \( \mathcal{L}_n \in L_n \) and \( \mathcal{L}_m \in L_m \). If \( [\mathcal{L}_n] \preceq [\mathcal{L}_m] \), then \( m - c(\mathcal{L}_m) \equiv n - c(\mathcal{L}_n) \) (mod 2) and \( m \leq n \).

**Proof.** We assume that \( [\mathcal{L}_n] \preceq [\mathcal{L}_m] \). A connecting operation changes \( n \) and \( c(\mathcal{L}_n) \) into \( n-1 \) and \( c(\mathcal{L}_n) +1 \), respectively. Therefore, \( m \leq n \) and it changes \( n-1 \) and \( c(\mathcal{L}_n) +1 \) into \( n-1 \) and \( c(\mathcal{L}_n) +1 \). Thus, \( m - c(\mathcal{L}_m) = n - c(\mathcal{L}_n) - 2l \) for some \( l \in \{0, 1, 2, \ldots \} \), and hence we have \( m - c(\mathcal{L}_m) \equiv n - c(\mathcal{L}_n) \) (mod 2). We complete the proof. □

The next corollary gives a necessary and sufficient conditions for the relation between equivalence classes of \( SH \)-move to exist.

**Corollary 4.** \( \mathcal{K}_m \preceq \mathcal{K}_n \) if and only if \( n \equiv m \) (mod 2) and \( n \leq m \).

**Proof.** This follows from Proposition 5.1 and Lemma 5.1. □

By Corollary 4, \( \bigcup_{n=0}^{\infty} \mathcal{K}_{2n+1} \preceq \bigcup_{n=0}^{\infty} \mathcal{K}_{2n} \) are both totally ordered sets. Also, we can conclude from Proposition 5.1 that there are no binary relations \( \preceq \) between an element of \( \bigcup_{n=0}^{\infty} \mathcal{K}_{2n+1} \) and an element of \( \bigcup_{n=0}^{\infty} \mathcal{K}_{2n} \).

**Corollary 5.** Let \( \mathcal{T}_m \in T_m \), \( \mathcal{T}_n \in T_n \) and \( c(\mathcal{T}_m) = c(\mathcal{T}_n) = 1 \). Then \( [\mathcal{T}_m] \preceq [\mathcal{T}_n] \) if and only if \( n \equiv m \) (mod 2) and \( n \leq m \).

**Proof.** This follows from Corollaries 1 and 4. □

The following two lemmas show necessary and sufficient conditions for the relation between the equivalence class of an \( ST \)-move and one of an \( SH \)-move to exist.

**Lemma 5.2.** Let \( \mathcal{T}_m \in T_m \), \( \mathcal{T}_n \in T_n \) and \( c(\mathcal{T}_n) = 1 \). Then \( [\mathcal{T}_m] \preceq [\mathcal{T}_n] \) if and only if \( n \equiv m - c(\mathcal{T}_m) + 1 \) (mod 2) and \( n \leq m - c(\mathcal{T}_m) + 1 \).

**Proof.** When \( c(\mathcal{T}_m) = 1 \), the proposition holds by Corollary 5. Therefore, we only need to prove the case in which \( c(\mathcal{T}_m) \neq 1 \).

Suppose that \( [\mathcal{T}_m] \preceq [\mathcal{T}_n] \), i.e. \( \mathcal{T}_n \) can be realized from \( \mathcal{T}_m \) by using connecting and/or braiding operations. Since the number of components in the union of \( \mathcal{T}_m \) must be changed into one to construct \( \mathcal{T}_n \) from \( \mathcal{T}_m \) by connecting and/or braiding operations, at least \( c(\mathcal{T}_m) - 1 \) connecting operations must be applied on \( \mathcal{T}_m \). If \( c(\mathcal{T}_m) - 1 \) connecting operations are applied on \( \mathcal{T}_m \), then the number of arcs of two tangle diagrams of new \( \mathcal{T}_m \) must be \( m - (c(\mathcal{T}_m) - 1) \). Therefore, by Corollary 5, it is necessary for \( \mathcal{T}_n \) to be realized from \( \mathcal{T}_m \) that \( n \equiv m - c(\mathcal{T}_m) + 1 \) (mod 2) and \( n \leq m - c(\mathcal{T}_m) + 1 \).

Suppose that \( n \equiv m - c(\mathcal{T}_m) + 1 \) (mod 2) and \( n \leq m - c(\mathcal{T}_m) + 1 \). Let \( \mathcal{K}_n = [\mathcal{T}_n] \) and \( \mathcal{T}_m = \{k_1, k_2, \ldots, k_{c(\mathcal{T}_m)}\} \). Then, we show that
$\mathcal{H}_n \preceq \mathcal{T}_m$. Since $n - 1 \leq m - c(\mathcal{T}_m)$, the inequality $n < m$ holds. Thus, $\mathcal{H}_n$ may be realized from $\mathcal{T}_m$ by a finite sequence of connecting and/or braiding operations.

A sequence of connecting operations must transform $c(\mathcal{T}_m)$ components with e-points of $\mathcal{T}_m$ into a single component with e-points. In order to connect them, we need at least $(c(\mathcal{T}_m) - 1)$ times of connecting operations on $\mathcal{T}_m$. Let $N = m - \{c(\mathcal{T}_m) - 1\}$.

(i) Case $n = N$: Apply the following finite sequence of connecting operations $Con(k_1), Con(k_1 + k_2), \ldots, Con(k_1 + k_2 + \cdots k_{c(\mathcal{T}_m) - 2}), Con(k_1 + k_2 + \cdots k_{c(\mathcal{T}_m) - 1})$ on $\mathcal{T}_m$. Then $\mathcal{T}_m$ can be transformed into $\mathcal{H}_n$. Thus, we have $\mathcal{T}_m \preceq \mathcal{H}_n$.

(ii) Case $n < N$ and $n \equiv N \pmod{2}$: Then there is a positive integer $l$ such that $n = N - 2l$. Therefore, we have $\mathcal{H}_n = \mathcal{H}_{N - 2l}$. Further, the following relation can be
obtained from Case (i) and Lemma 5.1: $\mathcal{T}_m \leq \mathcal{H}_N \leq \mathcal{H}_{N-2} \leq \cdots \leq \mathcal{H}_{N-2k} = \mathcal{H}_n$. Thus, we have $\mathcal{T}_m \leq \mathcal{H}_n$.

Hence, a necessary and sufficient condition is for $\mathcal{T}_m$ to be realized from $\mathcal{T}_n$ that $n \equiv m - c(\mathcal{T}_m) + 1 \pmod{2}$ and $n \leq m - c(\mathcal{T}_m) + 1$. This completes the proof. \hfill $\square$

**Lemma 5.3.** Let $\mathcal{T}_m \in \mathcal{T}_m$, $\mathcal{T}_n \in \mathcal{T}_n$ and $c(\mathcal{T}_n) = 1$. Then $[\mathcal{T}_n] \leq [\mathcal{T}_m]$ if and only if $m \equiv n - c(\mathcal{T}_m) + 1 \pmod{2}$ and $m \leq n - c(\mathcal{T}_m) + 1$.

**Proof.** When $c(\mathcal{T}_m) = 1$, the proposition holds by Corollary 5. Therefore, we assume that $c(\mathcal{T}_n) \neq 1$.

Suppose that $[\mathcal{T}_n] \leq [\mathcal{T}_m]$, i.e. $\mathcal{T}_m$ can be realized from $\mathcal{T}_n$. Then we see that $m - c(\mathcal{T}_m) \equiv n - 1 \pmod{2}$ by Proposition 5.1. Since the number of components in the union of $\mathcal{T}_m$ must be changed into $c(\mathcal{T}_m)$ by the sequence of connecting operations, at least $c(\mathcal{T}_m) - 1$ connecting operations must be applied on $\mathcal{T}_n$. If $c(\mathcal{T}_m) - 1$ connecting operations are applied on $\mathcal{T}_n$, then the number of arcs of two tangle diagrams of new $\mathcal{T}_n$ must be $n - \{c(\mathcal{T}_m) - 1\}$. Therefore, we have $m \equiv n - \{c(\mathcal{T}_m) - 1\}$.

Suppose that $m \equiv n - c(\mathcal{T}_m) + 1 \pmod{2}$ and $m \leq n - c(\mathcal{T}_m) + 1$. Let $\mathcal{H}_n = \langle n \rangle \in [\mathcal{T}_n]$ and $\mathcal{T}_m = \langle k_1, k_2, \ldots, k_{c(\mathcal{T}_m)} \rangle \in [\mathcal{T}_m]$ be the standard ST-moves. Then we show that $\mathcal{H}_n \leq \mathcal{T}_m$. Since $m - 1 \leq n - c(\mathcal{T}_m)$, the inequality $m < n$ holds. Thus, $\mathcal{T}_m$ may be realized from $\mathcal{H}_n$ by using connecting operations.

A sequence of connecting operations can transform a single component with e-points of $\mathcal{H}_n$ into $c(\mathcal{T}_m) = c(\mathcal{T}_m)$ components with e-points. In order to divide the single component of $\mathcal{H}_n$ into $c(\mathcal{T}_m)$ components, at least $(c(\mathcal{T}_m) - 1)$ times of connecting operations must be applied on $\mathcal{H}_n$. Let $N = m + c(\mathcal{T}_m) - 1$.

(i) Case $n = N$: Apply sequences of connecting and braiding operations on $\mathcal{H}_n$ that connect the e-points marked $k_1 + 1$ and $2n, k_1 + 1$ and $k_1 + k_2 + 1$ and $k_1 + k_2 + k_3 + 2, k_1 + k_2 + k_3 + 1$ and $k_1 + k_2 + k_3 + k_4 + 2, \ldots$ and $k_1 + k_2 + \cdots + k_{c(\mathcal{T}_m) - 2} + 1$ and $k_1 + k_2 + \cdots + k_{c(\mathcal{T}_m) - 1} + 2$. Thus, $\mathcal{H}_n$ can be transformed into $\mathcal{T}_m$, i.e. $\mathcal{H}_n \leq \mathcal{T}_m$.

(ii) Case $N < n$ and $N \equiv n \pmod{2}$: Then there is a positive integer $l$ such that $N = n - 2l$. Therefore, we have $\mathcal{H}_n = \mathcal{H}_{N+2l}$. Further, the following relation can be obtained from Lemma 5.1: $\mathcal{H}_n = \mathcal{H}_{N+2l} \leq \cdots \leq \mathcal{H}_N$. Using Case (i), we have $\mathcal{H}_N \leq \mathcal{T}_m$. Thus, $\mathcal{H}_n \leq \mathcal{T}_m$.

Hence, a necessary and sufficient condition is for $\mathcal{T}_m$ to be realized from $\mathcal{T}_n$ that $n - \{c(\mathcal{T}_m) - 1\} \equiv m \pmod{2}$ and $m \leq n - \{c(\mathcal{T}_m) - 1\}$ i.e. $m + c(\mathcal{T}_m) - 1 \leq n$. This completes the proof. \hfill $\square$

From Proposition 5.1, we see that any finite sequence of connecting or braiding operations on a local move $\mathcal{L}$ does not change the number $n - c(\mathcal{L})$ modulo 2. Therefore, we can define as follows.

**Definition 5.1.** Let $\mathcal{T} \in \mathcal{T}_n$. If the integer $n - c(\mathcal{T})$ is even, then we say that $\mathcal{T}$ is an X-type. If $n - c(\mathcal{T})$ is odd, then we say that $\mathcal{T}$ is an O-type.

**Theorem 2.** Let $\mathcal{T} \in \mathcal{T}_n$. If $\mathcal{T}$ is an X-type, then $\mathcal{T}$ can realize the ordinary unknotting operation. Otherwise, $\mathcal{T}$ can realize the $SH(2)$-move.

**Proof.** We note that $1 \leq n - c(\mathcal{T}) \leq n - 1$ because of $1 \leq c(\mathcal{T}) \leq n - 1$. Therefore, if $n - c(\mathcal{T})$ is an even integer i.e. $\mathcal{T}$ is an X-type, then $3 \leq n - c(\mathcal{T}) + 1 \equiv 1 \pmod{2}$. Lemma 5.2 gives us $\mathcal{T} \leq \mathcal{H}_{n-c(\mathcal{T})+1}$. Therefore, from [4], we see that $\mathcal{H}_{n-c(\mathcal{T})+1} \leq \mathcal{T}$ and so we have $\mathcal{T} \leq \mathcal{H}$.

If $n - c(\mathcal{T})$ is an odd integer i.e. $\mathcal{T}$ is an O-type, then $2 \leq n - c(\mathcal{T}) + 1 \equiv 0 \pmod{2}$. Lemma 5.2 tells us that $\mathcal{T} \leq \mathcal{H}_{n-c(\mathcal{T})+1}$. Therefore, from [4], we see that $\mathcal{H}_{n-c(\mathcal{T})+1} \leq \mathcal{H}_2$. Thus, we have $\mathcal{T} \leq \mathcal{H}_2$. This completes the proof. \hfill $\square$

We define $\mathcal{T}_X := \{\mathcal{T} \in \mathcal{T} \mid \mathcal{T}$ is an X-type $\}$ and $\mathcal{T}_O := \{\mathcal{T} \in \mathcal{T} \mid \mathcal{T}$ is an O-type $\}$. 

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Theorem 3. Any local move that realizes an extended ST-move in $\mathbb{T}_X$ is an unknotting operation.

Proof. This theorem follows from Theorem 2. □

6. UNKNOTTING NUMBERS OF ST-MOVES

Let $K$ be an oriented knot in the 3-sphere $S^3$. If an $ST$-move is an $O$-type, then the number of components of $K$ must change when the $ST$-move is applied to a diagram of $K$. So we treat only $X$-type $ST$-moves.

Let $\mathcal{T} \in \mathbb{T}_X$. We denote the minimum number of $\mathcal{T}$ that can transform a diagram of $K$ into a trivial knot diagram by $u_{\mathcal{T}}(K)$, where the minimum is taken over all diagrams of $K$. The proceeding properties follow from section 3.

Remark. Let $\mathcal{T}, \mathcal{T}' \in \mathbb{T}_X$. If $\mathcal{T} \preceq \mathcal{T}'$, then we have $u_{\mathcal{T}}(K) \leq u_{\mathcal{T}'}(K)$ for any oriented knot $K$. Hence, if $\mathcal{T} \cong \mathcal{T}'$, then we have $u_{\mathcal{T}}(K) = u_{\mathcal{T}'}(K)$ for any oriented knot $K$. Because if $\mathcal{T} \sim \mathcal{T}'$, then we have $\mathcal{T} \preceq \mathcal{T}'$ and $\mathcal{T}' \preceq \mathcal{T}$.

Theorem 4. Let $\mathcal{T} \in [\{s_1, s_2, \ldots, s_n\}] \subset \mathbb{T}_X$, $\mathcal{T}' \in [\{s'_1, s'_2, \ldots, s'_m\}] \subset \mathbb{T}_X$. If $\sum_{i=1}^n (s_i - 1) \leq \sum_{i=1}^m (s'_i - 1)$, then we have $u_{\mathcal{T}}(K) \leq u_{\mathcal{T}'}(K)$ for any oriented knot $K$. In particular, if $\sum_{i=1}^n (s_i - 1) = \sum_{i=1}^m (s'_i - 1)$, then we have $u_{\mathcal{T}}(K) = u_{\mathcal{T}'}(K)$ for any oriented knot $K$.

Proof. Let $K$ be an oriented knot, $N = \sum_{i=1}^n (s_i - 1)$ and $M = \sum_{i=1}^m (s'_i - 1)$. From Lemma 5.2, we see that $\mathcal{T} \preceq \mathcal{R}_{N+1}$. Also Remark tells us that $u_{\mathcal{T}}(K) \leq u_{\mathcal{R}_{N+1}}(K)$ and $u_{\mathcal{T}'}(K) \leq u_{\mathcal{R}_{M+1}}(K)$.

On the other hands, if $u_{\mathcal{T}}(K) = l$, then we have $u_{\mathcal{R}_{N+1}}(K) \leq l$. Because we can gather one root of each band near one point of the trivial knot and $l$ times of $SH(N+1)$-moves can produce the trivial knot from $K$, we see that $u_{\mathcal{R}_{N+1}}(K) \leq u_{\mathcal{T}}(K)$. Hence, $u_{\mathcal{T}}(K) = u_{\mathcal{R}_{N+1}}(K)$. Similarly, we have $u_{\mathcal{T}'}(K) = u_{\mathcal{R}_{M+1}}(K)$.

Therefore, if $N \leq M$, then we have $u_{\mathcal{T}'}(K) = u_{\mathcal{R}_{M+1}}(K) \leq u_{\mathcal{R}_{N+1}}(K) = u_{\mathcal{T}}(K)$ by Lemma 5.1 and Remark. Thus, we have $u_{\mathcal{T}}(K) \leq u_{\mathcal{T}'}(K)$. In particular, if $N = M$, then we have $u_{\mathcal{T}}(K) = u_{\mathcal{R}_{N+1}}(K) = u_{\mathcal{T}'}(K)$. Thus, we have $u_{\mathcal{T}}(K) = u_{\mathcal{T}'}(K)$. We complete the proof. □

In the next proposition, we show that for any oriented knot $K$, there is an $ST$-move so that $K$ can be transformed into the trivial knot by the single $ST$-move.

Theorem 5. Let $\mathcal{T} \in [\{s_1, s_2, \ldots, s_n\}] \subset \mathbb{T}_X$, $\mathcal{T}' \in [\{s'_1, s'_2, \ldots, s'_m\}] \subset \mathbb{T}_X$. If $\sum_{i=1}^n (s_i - 1) = u_{\mathcal{T}}(K) \cdot \sum_{i=1}^m (s'_i - 1)$, then we have $u_{\mathcal{T}}(K) = 1$ for any oriented knot $K$.

Proof. Let $K$ be an oriented knot and $N = u_{\mathcal{T}}(K) \cdot \sum_{i=1}^m (s'_i - 1)$. Then $u_{\mathcal{R}_{N+1}}(K) = 1$. Because we can gather one root of each band near one point of the trivial knot as in the proof of Theorem 4, the single $SH(N + 1)$-move can produce the trivial knot from $K$. If $\sum_{i=1}^n (s_i - 1) = N$, then Theorem 3 tells us that $u_{\mathcal{T}}(K) = u_{\mathcal{R}_{N+1}}(K)$. Thus, we have $u_{\mathcal{T}}(K) = 1$. □
7. SOME EXAMPLES

Let \( \mathbb{N} \) be the set of natural numbers and let \( r \in \mathbb{N} \). The local move, shown in FIG. 15, is called an \( SH(m, n) \)-move and denoted by \( \mathcal{H}(n, r) \). In particular, \( \mathcal{H}(n, 1) = \mathcal{H}_n \).

From Corollary 1, we have the following.

**Example 1.** \( \mathcal{H}(n, r) \preceq \mathcal{H}(n, s) \) if and only if the following conditions (i) or (ii) holds:

(i) \( n \) is even, \( s \equiv r \pmod{2} \) and \( s \leq r \), (ii) \( n \) is odd and \( s \leq r \).

Additionally, Lemmas 5.2 and 5.3 give us the following example.

**Example 2.** For any positive integers \( a \) and \( b \) \((2 \leq a)\), we have \( \mathcal{H}_{ab+b-1} \preceq \mathcal{H}(a, b) \preceq \mathcal{H}_{ab-b+1} \).

Lastly, the following example results from Theorem 3.

**Example 3.** Let \( a, c \in \mathbb{N} \) and \( 2 \leq b, d \in \mathbb{N} \). If \( b(a-1) = d(c-1) \), then we have \( u_{\mathcal{H}(a, b)}(K) = u_{\mathcal{H}(c, d)}(K) = u_{\mathcal{H}_{ab-b+1}}(K) \) for any oriented knot \( K \).

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