ASYMPTOTIC DIOPHANTINE APPROXIMATION: THE MULTIPLICATIVE CASE

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1. Introduction

Let α and β be real irrational numbers, and let || · || be the distance to the nearest integer. Littlewood’s conjecture asserts that

$$\lim_{q \rightarrow \infty} q || q\alpha || || q\beta || = 0.$$ 

Or put differently, for all ε > 0 the number of 0 < q < Q with || q\alpha || || q\beta || < ε/q tends to infinity as Q does. Dirichlet’s theorem implies lim inf q →∞ q || q\alpha || ≤ 1. A real number α is called badly approximable if lim inf q →∞ q || q\alpha || > 0. Thus, Littlewood’s conjecture is nontrivial only if α and β are distinct and both badly approximable. Instead of looking at very good approximations as in Littlewood’s conjecture we replace ε/q by a larger value ψ(Q)/Q enabling us to give very precise estimates for the number of q as Q becomes large.

Throughout this article, let ψ : [1, ∞) → [1, ∞) be a function, T > 0, and assume ψ(Q)/(QT^2) ≤ 1/e^2 and ψ(Q)/Q ≤ 1 for all Q ≥ Q_0 for some Q_0. We consider the set

$$M_{α, β}(ψ, T, Q) = \{(p_1, p_2, q) \in \mathbb{Z}^3; || p_1 + q\alpha || || p_2 + q\beta || < \frac{ψ(Q)}{Q}, \max\{|p_1 + q\alpha|, |p_2 + q\beta|\} ≤ T, 0 < q < Q\}.$$ 

Theorem 1.1. Suppose α and β are distinct badly approximable numbers. Then, there exists C_0 = C_0(α, β) such that

$$|| M_{α, β}(ψ, T, Q) || - 4ψ(Q)\log Q ≤ C_0(ψ(Q)^{2/3}\log(QT) + ψ(Q)^{1/3}T)$$ 

for all Q ≥ Q_0.

Choosing T = 1/2 we immediately deduce the following corollary.

Corollary 1.1. Suppose α and β are distinct badly approximable numbers. Then, we have

$$|\{q \in \mathbb{Z}; || q\alpha || || q\beta || < \frac{ψ(Q)}{Q}, 0 < q < Q\}| = 4ψ(Q)\log Q + O(ψ(Q)^{2/3}\log Q).$$

For badly approximable α and β we have max{|p_1 + q\alpha|, |p_2 + q\beta|} > C/q, and hence, 0 < q < Q and || p_1 + q\alpha || || p_2 + q\beta || < ψ(Q)/Q implies max{|p_1 + q\alpha|, |p_2 + q\beta|} ≤ ψ(Q)/C. Thus, for T ≥ ψ(Q)/C the second inequality in the definition of M_{α, β}(ψ, T, Q) is void and we obtain a second corollary.
Corollary 1.2. Suppose \( \alpha \) and \( \beta \) are distinct badly approximable numbers. Then, we have
\[
\left| \{ (p_1, p_2, q) \in \mathbb{Z}^3 ; |p_1 + q\alpha||p_2 + q\beta| < \frac{\psi(Q)}{Q} \} \right| < 4\psi(Q) \log Q + O(\psi(Q)^{2/3} \log Q + \psi(Q)^{4/3}).
\]

2. Counting lattice points

For a vector \( x \) in \( \mathbb{R}^D \) we write \( |x| \) for the Euclidean length of \( x \). The closed Euclidean ball centered at \( x \) with radius \( r \) will be denoted by \( B(x,r) \). Let \( \Lambda \) be a lattice of rank \( D \) in \( \mathbb{R}^D \) then we define the successive minima \( \lambda_1(\Lambda), \ldots, \lambda_D(\Lambda) \) of \( \Lambda \) as the successive minima in the sense of Minkowski with respect to the unit ball. That is
\[
\lambda_i = \inf \{ \lambda ; B(0,\lambda) \cap \Lambda \text{ contains } i \text{ linearly independent vectors} \}.
\]

Definition 1. Let \( M \) and \( D \) be positive integers and let \( L \) be a non-negative real. We say that a set \( S \) is in \( \text{Lip}(D,M,L) \) if \( S \) is a subset of \( \mathbb{R}^D \), and if there are \( M \) maps \( \phi_1, \ldots, \phi_M : [0,1)^D \rightarrow \mathbb{R}^D \) satisfying a Lipschitz condition
\[
|\phi_i(x) - \phi_i(y)| \leq L|x - y| \text{ for } x, y \in [0,1)^D, i = 1, \ldots, M
\]
such that \( S \) is covered by the images of the maps \( \phi_i \). For \( D = 1 \) this is to be interpreted as the finiteness of the set \( S \), and the maps \( \phi_i \) are considered points in \( \mathbb{R}^D \) such that \( S \subset \{ \phi_i ; 1 \leq i \leq M \} \).

We will apply the following counting result.

Lemma 2.1 (Theorem 5.4 [1]). Let \( \Lambda \) be a lattice in \( \mathbb{R}^D \) with successive minima \( \lambda_1(\Lambda), \ldots, \lambda_D(\Lambda) \). Let \( S \) be a set in \( \mathbb{R}^D \) such that the boundary \( \partial S \) of \( S \) is in \( \text{Lip}(D,M,L) \). Then \( S \) is measurable, and moreover,
\[
\left| |S \cap \Lambda| - \frac{\text{Vol} S}{\text{det} \Lambda} \right| \leq D^{2D} M \left( \max_{0 \leq j < D} \frac{L^j}{\lambda_1 \cdots \lambda_j} \right),
\]
where for \( j = 0 \) the maximum is to be understood as 1.

3. Prerequisites

We will use Vinogradov’s \( \ll \)-notation. The implicit constant is only allowed to depend on \( \alpha \) and \( \beta \). For brevity let us set
\[
F = \psi(Q)/Q.
\]

Next we introduce the sets
\[
R = \{(x,y) \in \mathbb{R}^2 ; xy \leq F, 0 < x < T, 0 < y < T \},
\]
\[
Z = R \times (0, Q),
\]
and the lattice
\[
\Lambda = (1,0,0)\mathbb{Z} + (0,1,0)\mathbb{Z} + (\alpha, \beta, 1)\mathbb{Z}.
\]

The irrationality of \( \alpha \) and \( \beta \) implies that the points \( (p_1, p_2, q) \) with \( p_1 + q\alpha = 0 \) or \( p_2 + q\beta = 0 \) are bounded in number by \( 2T + 1 \). Hence, by symmetry,
\[
||M_{\alpha,\beta}(\psi,T,Q)| - 4|\Lambda \cap Z|| \ll T + 1.
\]

Unfortunately, our set \( Z \) is very distorted, and this not only in one but in various directions.
4. Partitioning the counting domain

First let us decompose $R$ into three disjoint pieces. Set

$$A = \{(x, y); 0 < y < (F/T^2)x, 0 < x < T\},$$
$$B = \{(x, y); 0 < (T^2/F)x \leq y < T\},$$
$$S = \{(x, y); 0 < (F/T^2)x \leq y < (T^2/F)x, xy < F\}.$$

Hence, we have

$$|\Lambda \cap Z| = |\Lambda \cap A \times (0, Q)| + |\Lambda \cap B \times (0, Q)| + |\Lambda \cap S \times (0, Q)|.$$

The sets $A$ and $B$ are long and thin triangles, hence distorted only in one direction. Thus, the first two summands are relatively easy to deal with. The set $S$ is more troublesome and requires a further decomposition into about $\log Q$ pieces. We assume that $Q \geq Q_0$ so that by assumption $F/T^2 \leq 1/e^2$. Let $\nu \in [1/e^2, 1/e] \cap [1/e^2, 1/e]$ be maximal such that $N = \log(F/T^2)/\log \nu$ is an integer. Hence,

$$1 \leq N \leq \log Q + 2 \log T - \log \psi(Q) \leq 2 \log(QT).$$

Decompose $S$ into the $2^N$ pieces $S_{-N}, \ldots, S_N$, where

$$S_i = \{(x, y); 0 < \nu^ix \leq y < \nu^{i-1}x, xy < F\}.$$

Then we have the following partition

$$S = \bigcup_{-N+1 \leq i \leq N} S_i.$$

Note that $S_0$ lies in a zero-centered ball of radius $\sqrt{\nu + 1/\nu} \sqrt{F} \leq 3\sqrt{F}$. A straightforward calculation yields

$$\text{Vol}_2(S_0) = \frac{\sqrt{F} \nu \sqrt{F/\nu}}{2} + \int_{\sqrt{F/\nu}}^{\sqrt{T}} \frac{F}{x} dx - \frac{F}{2} = -\frac{F}{2} \log \nu.$$

Hence,

$$V = \text{Vol}_3(S_0 \times (0, Q)) = -\frac{F}{2} Q \log \nu.$$

Thus

$$1/2 \leq \psi(Q)/2 \leq V \leq \psi(Q).$$

5. Applying flows

In this section we construct certain elements of the diagonal flow on $R^3$ that transform our distorted sets into sets of small diameter.

We introduce the following automorphisms of $R^2$

$$g_i(x, y) = (\nu^{i/2}x, \nu^{-i/2}y).$$

Then, for $-N + 1 \leq i \leq N$ we have

$$g_iS_i = S_0.$$

We extend $g_i$ to an automorphism of $R^3$

$$G_i(x, y, z) = (\nu^{i/2}x, \nu^{-i/2}y, z).$$

Next we introduce a further automorphism of $R^3$

$$G_0(x, y, z) = (\theta x, \theta y, \theta^{-2}z),$$

where

$$\theta = \frac{V^{1/3}}{3\sqrt{F}}.$$
We note that
\[ \sqrt{\frac{Q}{\psi(Q)^{1/3}}} \ll \theta \ll \sqrt{\frac{Q}{\psi(Q)^{1/3}}}. \]

Let us write
\[ \varphi_i = G_\theta \circ G_i. \]

Lemma 5.1. The boundary of \( \varphi_i(S_i \times (0, Q)) \), \( \varphi_N(A \times (0, Q)) \) and \( \varphi_{-N+1}(B \times (0, Q)) \) lie in \( \text{Lip}(3, M, L) \) for some \( M \ll 1 \) and \( L \ll V^{1/3} \).

Proof. The boundaries can be covered by planes and the set \( \{ (x, \theta^2 F/x, z) ; \theta \sqrt{vF} \leq x \leq \theta \sqrt{F}, 0 \leq z \leq Q/\theta^2 \} \). For the Jacobian \( J \) of the parameterising map \( (t_1, t_2) \to (at_1 + b, \theta^2 F/(at_1 + b), ct_2) \) with \( a = \theta \sqrt{F}(1 - \sqrt{v}), b = \theta \sqrt{vF}, c = Q/\theta^2 \), and domain \( [0, 1]^2 \) we get for its operator norm \( \| J \| \ll V^{1/3} \). Hence, we are left with linear parameterising maps, and thus, it suffices to show that the diameter of \( \varphi_i(S_i \times (0, Q)) \), \( \varphi_N(A \times (0, Q)) \) and \( \varphi_{-N+1}(B \times (0, Q)) \) is \( \ll V^{1/3} \). First we note that \( G_i(S_i \times (0, Q)) = S_0 \times (0, Q) \). Hence, \( \varphi_i(S_i \times (0, Q)) = \theta S_0 \times (0, \theta^{-2}Q) \). As \( S_0 \) lies in the zero-centered ball of radius \( 3\sqrt{F} \) we conclude that the diameter of \( \varphi_i(S_i \times (0, Q)) \) is \( \ll V^{1/3} \). Similarly, we see that the triangles \( g_N A \) and \( g_{-N+1} B \) lie in the zero-centered ball of radius \( 3\sqrt{F} \), and hence also \( \varphi_N(A \times (0, Q)) \) and \( \varphi_{-N+1}(B \times (0, Q)) \) have diameter \( \ll V^{1/3} \).

6. Controlling the orbits

Our transformations of the previous section have brought our distorted sets into nice shapes. Unfortunately, they transform our lattice \( \Lambda \) in a less favourable manner. Indeed, the corresponding orbit of \( \Lambda \) escapes to infinity, i.e., the first successive minimum gets arbitrarily small. However, the rate of escape is still controllable.

Lemma 6.1. For \( -N + 1 \leq i \leq N \) we have
\[ \lambda_1(\varphi_i \Lambda) \gg \min \{ 1, \theta^{|i|/2} \}, \]
\[ \lambda_2(\varphi_i \Lambda) \gg \min \{ 1, \theta^{-|i|/2} \}. \]

Proof. Suppose \( v = (p_1 + qa, p_2 + q\beta, q) \in \Lambda \), \( v \neq 0 \). We distinguish two cases. First we assume \( q \neq 0 \). Then by the arithmetic-geometric mean inequality we have
\[ |v|^2 \geq 3(|p_1 + qa||p_2 + q\beta||q)|^{2/3}. \]

Using that \( \alpha \) and \( \beta \) are distinct badly approximable numbers we conclude \( |\varphi_i v| \gg 1 \). Suppose now that \( q = 0 \). Then \( p_1 \) and \( p_2 \) are not both zero, and hence
\[ |\varphi_i v| \geq \max \{ \theta v^{i/2} |p_1|, \theta^{-i/2} |p_2| \} \gg \theta v^{i/2}. \]

This shows the first inequality. To prove the second inequality it suffices now to note that for \( \Lambda_0 = (\theta v^{i/2}, 0)Z + (0, \theta v^{-i/2})Z \) we have \( \lambda_2(\Lambda_0) = \theta v^{-i/2} \).

7. Proof of Theorem 1.1

We have
\[ |\Lambda \cap Z| = |\Lambda \cap A \times (0, Q)| + |\Lambda \cap B \times (0, Q)| + \sum_{i = -N+1}^{N} |\Lambda \cap S_i \times (0, Q)| \]
\[ = |\varphi_N \Lambda \cap \varphi_N (A \times (0, Q))| \]
\[ + |\varphi_{-N+1} \Lambda \cap \varphi_{-N+1} (B \times (0, Q))| \]
\[ + \sum_{i = -N+1}^{N} |\varphi_i \Lambda \cap \varphi_i (S_i \times (0, Q))|. \]
Applying Lemma 2.1 to each summand, using Lemma 5.1, and collecting the main terms gives the main term
\[
\frac{\text{Vol}_3(Z)}{\det \Lambda} = \psi(Q) \log Q,
\]
and collecting the error terms bounds the error term as
\[
\ll \sum_{i=-N+1}^{N} \left( 1 + \frac{L}{\lambda_1(\varphi_i A)} + \frac{L^2}{\lambda_1(\varphi_i A)\lambda_2(\varphi_i A)} \right).
\]
Then applying Lemma 6.1 and not forgetting that \(\nu < 1\), we can bound this by
\[
\ll N \sum_{i=0}^{N} \left( 1 + \sum_{i=0}^{N} L + \sum_{i=0}^{N} \frac{L}{\theta \nu/2} + \sum_{i=0}^{N} \frac{L^2}{\theta \nu/2} + \sum_{i=0}^{N} L^2 + \sum_{i=0}^{N} \frac{L^2}{\theta^2} \right)
= (N+1) \left( 1 + L + L^2 + \frac{L^2}{\theta^2} \right) + \left( \frac{L + L^3}{\theta} \right) \frac{\nu^{-\frac{N+1}{2}} - 1}{\nu^{-\frac{1}{2}} - 1}.
\]
Next we note that \(L^2/\theta^2 \ll \psi(Q)/Q \leq 1\). Moreover, recalling the definition of \(N\) and \(\nu\) we see that \(\nu^{-\frac{1}{2}} - 1 \geq \sqrt{e} - 1 > 0\) and \(\nu^{-\frac{N+1}{2}} = \frac{\nu^{-\frac{1}{2}} - 1}{\nu^{-\frac{1}{2}} - 1} \leq e^{-\frac{T}{\sqrt{F}}/\sqrt{F}} \). Hence, the error term is bounded by
\[
\ll NV^{2/3} + TV^{1/3}
\]
which, thanks to (4.1) and (4.2), is bounded by
\[
\ll \log(QT)\psi(Q)^{2/3} + T\psi(Q)^{1/3}.
\]
Hence, we have shown that for \(Q \geq Q_0\)
\[
|M_{\alpha,\beta}(\psi, T, Q)| - 4\psi(Q) \log Q| \ll T + 1 + \log(QT)\psi(Q)^{2/3} + T\psi(Q)^{1/3}.
\]
Finally, as \(Q \geq Q_0\) we have \(QT^2/\psi(Q) \geq e^2\), and thus \(QT \geq e\). Hence, \(T + 1 + \log(QT)\psi(Q)^{2/3} + T\psi(Q)^{1/3} \ll \log(QT)\psi(Q)^{2/3} + T\psi(Q)^{1/3}\), and this completes the proof of Theorem 1.1.

ACKNOWLEDGEMENTS

This article was initiated during a visit at the University of York, and motivated by a question of Sanju Velani. I am very grateful to Victor Beresnevich, Alan Haynes and Sanju Velani for many interesting and stimulating discussions and their encouragement.

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