A survey of Partizan Misère Game Theory

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1 Introduction

Most research in combinatorial game theory assumes normal play, where the first player unable to move loses, as opposed to misère play, where the first player unable to move wins. It is rather remarkable how much changes when we simply switch the goal from getting the last move to avoiding the last move. At first glance one might think misère play is merely the 'opposite' of normal play, but this is not at all the case. There is actually no relationship between normal outcome and misère outcome: for every pair of (not necessarily distinct) outcome classes $C_1$ and $C_2$, there is a game with normal outcome $C_1$ and misère outcome $C_2$ [9]. Likewise, strategies from normal play are in general neither the same nor reversed for misère play. For example, in normal play, Left would always choose a move to $1 = \{0|\cdot\}$ over a move to $0 = \{\cdot|\cdot\}$; in misère play, there are situations in which Left should choose 1 over 0 and others where Left prefers 0 over 1 — which goes against our intuition that Left is trying to run out of moves before Right\footnote{For example, Left wins playing first on the single game 0 and loses playing first on the single game 1, but loses playing first on $0 + *$ and wins playing first on $1 + *$. So 0 and 1 are incomparable.}.

So we really are in a fog in misère play. We look to the elegant algebra of normal-play games and hope for some semblance of structure, but we are dismayed at every turn:

- Zero is trivial. In normal play we have the wonderful property that every previous-win game is equal to zero. In misère, the zero game is next-win, but our hopes that perhaps every next-win game is equal to zero are more than dashed: in fact, only the game $\{\cdot|\cdot\}$ is equal to zero [9]. In particular, for any game $G \neq 0$, the game $G - G$ is not equal to zero (a very troublesome fact indeed). Consequently, there are no non-zero inverses, and there is no longer an easy test for the equality and inequality of games.
• Equality (and inequality) is rare and difficult to prove. Partly due to
the triviality of zero, equivalence classes induced by the equality rela-
tion are much smaller in misère play, and it is not often possible to
simplify games. Inequality is likewise uncommon, resulting in unfortu-
nate situations like the incomparability of 1 and 0.

• Addition is less intuitive. Disjunctive sum is defined in misère as in
normal play, but much of our intuition for the interaction of games
in a sum is lost. For example, the sum of two left-win games may
be right-win! In fact, nothing can be said about the addition table
of outcomes in misère play: for any three (not necessarily distinct)
outcomes $C_1, C_2, C_3$, there are positions known to satisfy $o(G) \in C_1,$
$o(H) \in C_2,$ and $o(G + H) \in C_3$ under misère play [9]. Other problems
arise with sums, due to the lack of simplification under misère play: for
example, the sum of a game with value $n \in \mathbb{Q}_2$ and a game with value
$m \in \mathbb{Q}_2$ may not even be a number-valued game\(^2\), let alone the game
with value $n + m$.

For these reasons and others, the study of misère games was neglected for
most of the 20th century. One chapter of On Numbers and Games presents
an analysis of ‘How to Lose When You Must’, and Winning Ways extends
this work in their chapter ‘Survival in the Lost World’, but both texts con-
sider only impartial misère games. The genus theory developed in the latter
allowed for the analysis of certain impartial misère games, but left most un-
solvable [17]. A theory for partizan misère games seemed, if possible, even
more elusive.

The fog began to lift when Thane Plambeck [16] and Aaron Siegel [18]
introduced a modified equality relation for games under misère play. Instead
of requiring games to be interchangeable in any sum of games, two games will
be considered equivalent modulo $\mathcal{U}$ if they can be interchanged in any sum
of games from the set $\mathcal{U}$. For example, we might take $\mathcal{U}$ to be the set of all
positions that occur in some particular game, such as domineering, and then
two domineering positions are equivalent ‘modulo domineering’ if they are
interchangeable in any sum of domineering positions. This is a natural and
practical equivalence relation, and its introduction has encouraged renewed
interest in the study of misère games.

\(^2\)This, along with the incomparability of number-valued games, demonstrates that the
numerical value system developed for normal play is virtually meaningless in misère.
Although initially designed only for impartial games, this ‘restricted’ misère analysis works equally well for partizan games [19]. The study of restricted partizan misère games began with the doctoral theses of Paul Ottaway [14] and Meghan Allen [2], and has continued with a relative flurry of recent activity from a number of additional researchers. The present survey of partizan misère game theory highlights the most significant results from recent research, including canonical forms of partizan misère games, the invertibility of games under restricted misère play, and applications to specific partizan misère versions of Nim, Hackenbush, and Kayles. We begin with some prerequisite definitions.

2 Prerequisites

We use the notation $G = \{G^L \mid G^R\}$, where $G^L = \{G^{L_1}, G^{L_2}, \ldots\}$ is the set of left options from $G$ and $G^L$ is a particular left option. Any position which can be reached from a game $G$ is called a follower of $G$.

The outcome classes are partially ordered as in normal play: that is, $\mathcal{L} > \mathcal{P} > \mathcal{R}$, $\mathcal{L} > \mathcal{N} > \mathcal{R}$, and $\mathcal{P} \parallel \mathcal{N}$.

To distinguish between the normal and misère outcomes of a game, the superscripts $+$ and $-$ are introduced: $G \in \mathcal{N}^+$ means that $G$ is next-win under normal play, while $H \in \mathcal{L}^-$ means $H$ is left-win under misère play. The outcome functions $o^-(G)$ and $o^+(G)$ are also used to identify the misère or normal outcome, respectively, of a game $G$.

In normal play, the negative of a game is defined recursively as $-G = \{-G^R \mid -G^L\}$, and is so-called because $G + (-G) = 0$ for all games $G$ under normal play. As mentioned in the introduction, this property holds in misère play only if $G$ is the zero game $\{\cdot \mid \cdot\}$ [9]. To avoid confusion and inappropriate cancellation, we generally write $\overline{G}$ instead of $-G$ and refer to this game as the conjugate of $G$.

Most other definitions from normal-play game theory are used without modification for misère games, including disjunctive sum, equality, and inequality. In this paper, when equality and inequality relations are used, misère play is assumed unless otherwise stated. The equivalence relation developed by Plambeck and Siegel is formalized in Definition 2.1 below.

Definition 2.1. For games $G$ and $H$ and a set of games $\mathcal{U}$, the terms equiv-
alence and inequality, modulo $\mathcal{U}$, are defined by

$$G \equiv H \pmod{\mathcal{U}} \text{ if and only if } o^-(G + X) = o^-(H + X) \text{ for all games } X \in \mathcal{U},$$

$$G \geq H \pmod{\mathcal{U}} \text{ if and only if } o^-(G + X) \geq o^-(H + X) \text{ for all games } X \in \mathcal{U}.$$ 

The words equivalent and indistinguishable are used interchangeably, and if $G \not\equiv H \pmod{\mathcal{U}}$ then $G$ and $H$ are said to be distinguishable modulo $\mathcal{U}$. In this case there must be a game $X \in \mathcal{U}$ such that $o^-(G + X) \neq o^-(H + X)$, and we say that $X$ distinguishes $G$ and $H$. The set $\mathcal{U}$ is called the universe. All universes in this survey are closed under followers and disjunctive sum, and most are also closed under conjugation. Although we usually assume $G$ and $H$ are games in $\mathcal{U}$, this stipulation is unnecessary, and it is sometimes useful to compare games modulo a universe $\mathcal{U}$ even when the games do not belong to $\mathcal{U}$.

Notice that $G \equiv H \pmod{\mathcal{U}}$ implies $G \equiv H \pmod{\mathcal{V}}$ for any subset $\mathcal{V} \subseteq \mathcal{U}$, but in general games can be equivalent in the smaller universe and distinguishable in the larger.

A number of additional definitions are required in order to discuss specific game universes in the sections to follow. Firstly, we identify games where one or both players have no move: a left end is a position with no first move for Left (that is, $G$ with $G^L = \emptyset$), a right end is a position with no first move for Right ($G^R = \emptyset$), and an end is a position that is either a left end or a right end or both (the zero game).

A left (right) end is called dead if each of its followers is also a left (right) end. Games in which every end follower is a dead end are called dead-ending. Figures 1 and 2 provide examples to illustrate these definitions. By definition, in dead-ending games, if Left has no move at some point, then Left will never have a move again. This is a natural property held by well-studied games such as hackenbush, domineering, and other so-called placement games (where players move by placing pieces on a board). The set of all dead-ending games is denoted $\mathcal{E}$ and has proven to be rich in interesting results for misère play. The set of all dead ends and sums of dead ends is denoted $\mathcal{E}_e$.

Games which have no left or right ends — more precisely, where Left can move if and only if Right can move — are called all-small in normal play and dicot in misère. The set of all dicot games is denoted $\mathcal{D}$. Note that $\mathcal{D}$ is a proper subset of $\mathcal{E}$.

Finally, a position is called alternating if neither player can make consecutive moves; that is, if $G^{LL}$ and $G^{RR}$ are empty for all $G^L$ and all $G^R$. This
Figure 1: A dead left end and a left end that is not dead.

Figure 2: A dead-ending game and a game that is not dead-ending.
restriction allows for easier analysis under misère play. The set of all sums of alternating games is denoted $\mathcal{A}$, and the set of all alternating ends and their sums is denoted $\mathcal{A}_e$.

3 Canonical forms

Given the relative lack of structure in misère play, it is perhaps surprising that we have canonical forms here just as in normal play, with precisely the same definitions of domination and reversibility (with inequality under misère play instead of normal play). This was shown in the collaborative paper of G.A. Mesdal [9] and subsequent work by Aaron Siegel [19]. The latter also demonstrated that, as in normal play, the simplified game obtained by removing dominated options and bypassing reversible ones is unique.

So canonical forms ‘work’ in misère play; but in general the concept is less useful than in normal play, because it is so hard to find instances of domination or reversibility. We do at least have a slightly-modified hand-tying principle, which can be used to demonstrate domination of options. In normal-play, this principle says that if two games $G$ and $H$ differ only by the addition of one or more extra left options to $G$, then Left can do at least as well playing $G$ as playing $H$ ($G \geq H$, in normal play); at worst, Left can ‘tie her hand’ and ignore the extra options, thereby essentially playing the game $H$ instead of $G$. In misère play, the same argument holds, with one stipulation: the set $H^L$ of left options cannot be empty. If it is, adding a left option is not always beneficial to Left, who is sometimes happy to have no move in a position. However, when there already exists at least one left option, Left can simply ignore any additional ones. This principle was used in [10] and [6] to classify day-2 and day-3 dicot games.

3.1 Restricted canonical forms

If we restrict ourselves to a particular universe of games $\mathcal{U}$, then we may be able to demonstrate domination or reversibility that does not hold in general; that is, we may have $G \geq H \pmod{\mathcal{U}}$ even if $G \not\geq H$ amongst all games. Consequently, a game could have different ‘canonical forms’ in different universes. However, the construction of a canonical form — specifically, how we deal with reversible options — is not quite the same when the universe is restricted in this way. The problem is related to the following result of [19],
which is used in the construction of misère canonical forms.

**Lemma 3.1.** [[19], Lemma 3.5] If $H$ is a Left end and $G$ is not, then $G \not\geq H$.

This result holds in the context of all misère games; however, it may be the case that a non-Left-end $G$ can be greater than a Left end $H$ modulo some universe $U$. For example, in the universe of dicot games $D$, we have $\{0, \ast|\ast\} \geq 0 \pmod{D}$ [6].

Why is this a problem? In general, Lemma 3.5 means we never have to worry about reversibility through an end; it cannot happen that $G \geq G^{LR}$ if $G^{LR}$ is a Left end, and so in such a case $G^L$ could not be reversible. This fact is exploited in the proof that reversibility works in misère play: that $G' = G$ when $G'$ is obtained from $G$ by replacing a reversible option $G^L$ with the left options of $G^{LR}$ [19]. Since the same fact does not hold in restricted misère play, the result from [19] no longer applies, and so we cannot necessarily bypass all reversible options. In the example above from [6], even though $G = \{0, \ast|\ast\} \geq 0 \pmod{D}$ and $0 = \ast^R = G^{LR}$, it is not the case that $\{0, \ast|\ast\} \equiv \{0|\ast\} \pmod{D}$. Left’s only good move in $G$ is to $\ast$, so removing $\ast$ and replacing it with no options does not result in a game that is just as good for Left.

In [6], the proof from [19] of uniqueness of misère canonical forms was adapted to construct unique ‘restricted’ canonical forms in the universe of dicot games. The problem of reversibility through ends is dealt with as follows: if $G^L$ is reversible through a left end, then replacing $G^L$ with $\ast$ results in an equivalent game.

## 4 Invertibility

We have seen that no non-zero game has an additive inverse in general misère play. However, in a restricted universe $U$, a game $G$ may satisfy $G + \overline{G} \equiv 0 \pmod{U}$, or perhaps even $G + H \equiv 0 \pmod{U}$ for $H \not\equiv \overline{G} \pmod{U}$; the latter situation is discussed in Section 4.1. In either of these cases, the game $G$ is said to be *invertible*. The first result of this kind was Meghan Allen’s demonstration that $\ast+\ast+\ast \equiv 0$ in any universe of dicot games [3]. Allen’s result is generalized in [8] with the following sufficient condition for invertibility in the universe of dicots.

**Theorem 4.1.** [8] If $G + \overline{G} \in \mathcal{N}^-$ and $H + \overline{H} \in \mathcal{N}^-$ for all followers $H$ of $G$, then $G + \overline{G} \equiv 0(\pmod{D})$. 

Theorem 4.1 was used to show that the ordinal sum of * and a number\(^3\), *:\(x\), is invertible in the universe of dicots. This result and others are presented in Table 4, which lists some of the positions known to be invertible in the universes of alternating games (\(A\)), dicots (\(D\)), or dead-ending games (\(E\)).

Many of these instances of invertibility were demonstrated using the following sufficiency condition for invertibility in restricted misère play. Generally, one proves \(G + G \equiv 0 \pmod{U}\) by showing that the outcome of \(G + G\) is the same as the outcome of \(G + G + X\) for any \(X\) in \(U\). Theorem 4.2 essentially says that you need only check the \(X\) positions that are ends.

**Theorem 4.2.** [13] Let \(U\) be a universe of games closed under followers, sum, and conjugation, and let \(S \subseteq U\) be a set of games closed under followers. If \(G + G + X \in \mathcal{L}^\ominus \cup \mathcal{N}^\ominus\) for every game \(G \in S\) and every left end \(X \in U\), then \(G + G \equiv 0 \pmod{U}\) for every \(G \in S\).

### 4.1 Non-conjugal invertibility

A bizarre property of restricted misère play is that a game \(G\) can have an additive inverse modulo some universe \(U\) without that inverse being the conjugate \(\overline{G}\). The only known partizan result of this kind appears in [11], where the games \{0|\} and \{1|0\} sum to zero among the set of all partizan kayles\(^4\) positions, despite neither being equivalent to the conjugate of the other in this universe. This inverse pair is further remarkable for the fact that one position is right-win and the other is previous-win.

In the example from partizan kayles, the actual conjugates of \{0|\} and \{1|0\} do not even belong to the universe. In [10] it was conjectured that being

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\(^3\)By *number (integer)* in misère play, we mean a game that is identical to the normal-play canonical form of a number (integer).

\(^4\)The paper [11] solves a partizan version of the game Kayles, played on rows of pins, where Left can knock down a single pin and Right can knock down two adjacent pins.
closed under conjugation would prevent such occurrences of non-conjugal invertibility; however, a counterexample can be seen in Appendix 6 of [18], for a subset of impartial games.

This leads us to a pressing open question in misère theory: in what universes \( \mathcal{U} \) do we have \( G + H \equiv 0 \pmod{\mathcal{U}} \) only if \( H \equiv -G \pmod{\mathcal{U}} \)? In-progress research [7] suggests that this property holds for the set \( \mathcal{D} \) of dicot games as well as the superset \( \mathcal{E} \) of dead-ending games, which further supports the proposition that \( \mathcal{E} \) is a ‘nice’ universe for misère play.

5 Applications to specific games

A number of specific partizan games have been successfully solved using the theory of restricted misère play. These solutions usually consider equivalence modulo the universe of all positions of the specific game in question, and take advantage of results for broader superset universes (for example, dicots).

Penny Nim is a partizan variant of Nim played with stacks of coins. In each stack, coins are all heads up or all tails up, and the entire stack may be lying sideways. On her turn, Left chooses a stack with tails-up or sideways orientation, removes any number of coins from it, and turn it heads up. Right plays similarly on heads-up or sideways coins stacks, but leaves them tails-up. Notice that any position of this game is alternating, and the potential for sideways stacks means that not all components are initially ends. The game is solved in [10], using the analysis of the alternating universe \( \mathcal{A} \), where most ‘single-stack’ positions are invertible. The solution involves first simplifying single stacks of coins, modulo \( \mathcal{A} \), and then determining outcomes for sums of these simplified positions.

Partizan Kayles is a variant of Kayles, played on a row of pins, where Left can knock down a single pin and Right can knock down exactly two adjacent pins. Notice that any position of this game is dead-ending. The game is solved in [11]. The key is to see that Left should always take an isolated single pin when she can; this allows for removal of dominated options and decomposition of long rows of pins into shorter rows — into isolated single pins and pairs of pins, in fact — and then all that remains is to see who wins on a sum of such positions. This is easily done once it is shown that an isolated single pin and an isolated pair of adjacent pins ‘cancel’ (that is, they are additive inverses).

Hackenbush Sprigs is a particular case of Hackenbush. The game can be
seen as rows of blue, green and red dominoes where each row has exactly one green domino, which is the leftmost domino. A move of Left is to pick a blue or green domino and remove it with all dominoes of the same row to its right. Right plays similarly with red or green dominoes. Notice that any position of this game is dicot. The game is solved in [8]. The authors first show that all games are invertible by finding the canonical forms of all rows, modulo dicot games, and then finding the outcomes of sums of such positions.

6 Open problems

We conclude by summarizing two open problems that were introduced above.

The first is the question of non-conjugal invertibility. In what universes does $G + H \equiv 0$ imply $G \equiv H$? Can we find other examples where this does not hold, besides the one impartial and one partizan example that have been identified? Are there conditions we can impose on a universe — perhaps closure under conjugates and some other properties — that would guarantee all additive inverses are conjugates?

A second open problem concerns ‘restricted’ canonical forms. We have a reduction (using domination and a modified reversibility) for dicots. Does the same reversibility work in elsewhere? Are there other universes in which a yet different kind of reversibility would allow us to obtain unique canonical forms? Are there universes in which the problem of reversibility through ends does not even arise — that is, in which Lemma 3.1 holds?

Answers to these questions would fill important gaps in the theory of partizan misère games and allow the recent surge of interest in misère play to continue.

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