Extended Dynamics Observer for Linear Systems with Disturbance*

Hongyinping Feng\textsuperscript{a†} and Bao-Zhu Guo\textsuperscript{b,c}

\textsuperscript{a}School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi, 030006, China
\textsuperscript{b}Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, China
\textsuperscript{c}Key Laboratory of System and Control, Academy of Mathematics and Systems Science, Academia Sinica, Beijing, China

Nov.5, 2020

Abstract

This is the last part of four series papers, aiming at stabilization for signal-input-signal-output (SISO) linear finite-dimensional systems corrupted by general input disturbances. A new observer, referred to as Extended Dynamics Observer (EDO), is proposed to estimate both the state and disturbance simultaneously. The working mechanism of EDO consists of two parts: The disturbance with known dynamics is canceled completely by its dynamics and the disturbance with unknown dynamics is absorbed by high-gain. It is found that the high-gain is always working as long as the control plant with unknown input disturbance is observable which is the only assumption for the observer design. When the disturbance dynamics are completely unknown except some boundedness, the EDO is reduced to an extension of the well-known extended state observer or high-gain observer. The main advantage of the developed method is that the prior information about both the control plant and the disturbance can be utilized as much as possible. The more the prior information we have, the better performance the observer would be. An EDO based stabilizing output feedback is also developed in the spirit of estimation/cancellation strategy. The stability of the resulting closed-loop system is established and some of the theoretical results are validated by numerical simulations.

Keywords: Active disturbance rejection control, high-gain, internal model principle, input disturbance, observer.

1 Introduction

The dynamic model of physical systems, as the prior information of control plants, has been used in modern control theory as a starting point of the feedback control design. Since the 1960s when the
control theory was seen as a branch of applied mathematics, a fair amount of control strategies such as adaptive control \[1\], optimal control \[17\] as well as nonlinear control \[12\] have been developed based on mathematical models. These control techniques make use of prior information about the control plant as sufficient as possible in the controller design. However, all the model-based feedback laws must be robust to the control plant uncertainty in engineering applications so that the “engineering approximation” can be made \[22\]. In other words, the unknown parts of the control plant, which serve as the “disturbance”, must be taken into account in the model-based control design.

The tolerance of disturbance and uncertainty is one of the major concerns in modern control theory. There are many well developed control design approaches to cope with disturbance in control systems. The adaptive control can be used for the system with unknown parameters \[23\] and the robust control is an approach to achieve robust performance in the presence of bounded modelling errors \[2\]. The sliding mode control \[26\] and high-gain control \[15\] work for systems with a large scale of uncertainties. The active disturbance rejection control (ADRC) has been recognized as an almost model free control technology \[9\]. Since it was proposed in the late 1980s by \[10\], it has been successfully applied to numerous engineering control problems like typically control of synchronous motors \[20\], high-speed railway \[27\], DC-DC power converter \[18\], flight vehicles control \[24\], and gasoline engines \[25\], among many others.

As an error driven control technology, ADRC is almost free of mathematical models and even works well for those control plants that are almost unknown \[19\]. However, every coin has two sides. On the one hand, the model free characteristic leads to the strong robustness to the uncertainty and disturbance, and on the other hand, it may waste more or less some useful prior information that we have already known. The waste of the prior disturbance information also exists to a varying extent in other control techniques such as the robust control, high-gain control and the sliding mode control. In engineering applications, we are not always completely ignorant of the disturbance. Some rough information like smoothness, boundedness, particularly some dynamic information of the disturbance are available sometimes. This prior information might be useful or even valuable for the observer design. A typical example is the harmonic disturbance where the known frequencies are very useful in internal model principle (IMP) yet are completely wasted in ADRC. The IMP is an elegant approach to robust output regulation, both for finite-dimensional systems \[11\] and for infinite-dimensional ones \[14\]. However, the disturbance in IMP is almost known. Precisely, the dynamics of disturbance are required to be known in IMP, which blocks the general disturbance out the door of the IMP. In one word, a great improvement room still exists for both ADRC and IMP but has not been noticed and emphasized at least in literature.

In this paper, we develop a fundamental principle to design observer via online measurement information and prior information about both the control plant and disturbance. The model of control plant, as the prior information of the system, has been considered sufficiently in literature. However, the disturbance prior information in particular for the dynamic modes of disturbance is usually ignored. We believe that a good observer should possess not only the strong robustness
to the disturbance and control plant but also the ability to make sufficient use of all the valuable prior information. The more the prior information is correctly used, the better performance of the observer would be. When the prior information is insufficient, the observer can still do its best. In this spirit, a new observer, referred to as Extend Dynamics Observer (EDO), is designed to estimate both the disturbance and the system state simultaneously. The EDO inherits almost all the advantages from the extended state observer (ESO) like model free characteristic yet can properly utilize the prior information not only about the control plant but also the disturbance. If all the prior dynamic information about the total disturbance is available, the EDO can admit a zero steady-state error.

Consider the following SISO system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B[d(t) + u(t)], \\
y(t) &= Cx(t),
\end{align*}
\]

(1.1)

where \(A \in \mathbb{R}^{n \times n}\) is the system matrix, \(B \in \mathbb{R}^{n}\) is the control matrix, \(C \in \mathbb{R}^{1 \times n}\) is the output matrix, \(u(t)\) is the control, \(y(t)\) is the measurement and \(d \in L_{\text{loc}}^2[0, +\infty)\) is the disturbance. In this paper, all the unknown signals in the control channel are referred to as disturbances which may contain system uncertainties and external disturbances.

If \(d(t)\) is an estimation of \(d(t)\), a stabilizing feedback control can be naturally designed as

\[
u(t) = -\hat{d}(t) - u_s(t),
\]

(1.2)

where the first term on the right side is obviously used to compensate for the disturbance and the second term \(u_s(t)\) is a stabilizer. This is referred to as an estimation/cancellation strategy and obviously, the key point for such a strategy is the estimation of the state and disturbance. Different from the ESO and IMP, in this work, we decompose the disturbance into two parts: the disturbance with known dynamics and the others otherwise. This decomposition is achieved by the mechanism of the system itself automatically. The disturbance with known dynamics is treated by likewise observer based on IMP and the disturbance with unknown dynamics is dealt with by the high-gain which is the core of disturbance estimation in ADRC. In this way, the prior information can be utilized as sufficient as possible which remedies the deficiency of ADRC and IMP.

The rest of the paper is organized as follows. In the next section, Section 2, we consider the disturbance dynamics and the observability of system (1.1). Section 3 gives a sufficient condition on which the high-gain works. Section 4 is devoted to observer design with known disturbance dynamics and Section 5 is on observer design for general disturbance. In Section 6, we focus on systems where the disturbance dynamics is not available at all. A comparison between EDO and ESO is also presented. Section 7 presents estimation for general period disturbance which contains harmonic disturbance as a special case. An observer based output feedback is proposed in Section 8. The stability of the closed-loop is also considered. Numerical simulations are presented in Section 9 to validate the theoretical results, followed up conclusions in Section 10.
Throughout the paper, the $n$ and $m$ denote the positive integers and the $\mathbb{R}^n$ denotes $n$-dimensional Euclidean space. The identity operator on $\mathbb{R}^n$ will be denoted by $I_n$ and the norm of $\mathbb{R}^n$ is denoted by $\| \cdot \|_{\mathbb{R}^n}$. The spectrum of operator or matrix $A$ is denoted by $\sigma(A)$; the largest real part of eigenvalue of $A$ is denoted as $\Lambda_{\text{max}}(A)$; the transpose of matrix $A$ is represented by $A^\top$.

For simplicity, we denote $\mathbb{C}^+ = \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \geq 0 \}$ and $\| \cdot \|_{\infty} = \| \cdot \|_{L^\infty[0, \infty)}$.

### 2 Disturbance dynamics and observability

We first consider the disturbance dynamics which serve as the prior information to the disturbance estimation. Generally speaking, not all continuous disturbances can be estimated effectively online by a deterministic dynamic system. For instance, if the disturbance is a sample path of the Wiener process, it is differentiable for no time $t \geq 0$. In this case, we do not have any dynamic information about the disturbance and the estimation of such a disturbance by virtue of typical dynamic system observer seems impossible. Based on this observation, we first limit ourselves into an estimable signal space of the following:

$$ S = \{ s \in L^\infty[0, \infty) \mid \dot{s} \text{ exists in the weak sense and belongs to } L^\infty[0, \infty) \}, \quad (2.1) $$

whose norm is given by

$$ \|s\|_S = |s(0)| + \| \dot{s} \|_{\infty}, \quad \forall s \in S. \quad (2.2) $$

A simple computation shows that $(S, \| \cdot \|_S)$ is a Banach space. Noting that the piecewise signal such as

$$ s_T(t) = \begin{cases} e^t, & t \in [0, T], \\ e^T, & t \geq T \end{cases} \quad (2.3) $$

belongs to $S$, the signal space $S$ is quite general and can include the harmonic signals, bounded continuously differentiable periodic signals, piecewise polynomial signals, piecewise exponential signals and their linear combinations.

Let $(G, Q)$ be an observable system with the state space $\mathbb{R}^m$ and output space $\mathbb{R}$. Define

$$ \Omega(G) = \left\{ Qv(t) \mid \dot{v}(t) = Gv(t), \ v(0) \in \mathbb{R}^m, \ t \in \mathbb{R} \right\}. \quad (2.4) $$

By ordinary differential equation theory, we obtain

$$ \Omega(G) = \text{span} \left\{ t^{m\lambda - k} e^{t \lambda} \mid \lambda \in \sigma(G), k = 1, 2, \cdots, m_\lambda, m_\lambda \text{ is the algebraic multiplicity of } \lambda, \ t \in \mathbb{R} \right\}, \quad (2.5) $$

which implies that the space $\Omega(G)$ is independent of $Q$. By $(2.4)$, $\Omega(G) \subset S$ as long as $\sigma(G) \subset i\mathbb{R}$ and each eigenvalue of $G$ is algebraically simple. Define the projection operator $P_G : S \to \Omega(G)$ by

$$ P_Gs = \arg \inf_{g \in \Omega(G)} \| s - g \|_S, \quad \forall s(\cdot) \in S. \quad (2.6) $$
Since $(\mathbb{S}, \| \cdot \|_{\mathbb{S}})$ is a Banach space, the optimal approximation $\mathbb{P}_{\mathbb{G}s} \in \Omega(G)$ always exists, which implies that the operator $\mathbb{P}_{\mathbb{G}}$ is well defined. Let $e = (I - \mathbb{P}_{\mathbb{S}})s$ be the approximation error. A simple computation shows that $e(0) = 0$ and thus

$$\|e\|_{\mathbb{S}} = \|\hat{e}\|_{\infty}. \quad (2.7)$$

In fact, if $s_{\ast}(\cdot) = \mathbb{P}_{\mathbb{G}}s$ with $s_{\ast}(0) \neq s(0)$, i.e., $e(0) \neq 0$, then

$$\|s_{\ast} - s_{\ast}(0) + s(0) - s\|_{\mathbb{S}} = \|\hat{s} - \hat{s}_{\ast}\|_{\infty} < \|s_{\ast} - s\|_{\mathbb{S}}. \quad (2.8)$$

Since $s_{\ast}(\cdot) - s_{\ast}(0) + s(0) \neq s^{\ast}(\cdot)$, (2.8) contradicts to the optimality of $s_{\ast}(\cdot) = \mathbb{P}_{\mathbb{S}}s$ for $s(\cdot)$.

**Definition 1.** Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n}$ and $C \in \mathbb{R}^{1 \times n}$. Suppose that $\Theta$ is a set of signals and we have known that $d \in \Theta$. System (1.1) is said to be observable for the signal set $\Theta$, provided both the initial state and the disturbance are distinguishable in the sense that: For any $T > 0$,

$$u(t) = 0 \text{ and } y(t) = 0 \text{ for a.e. } t \in [0, T] \Rightarrow x(0) = 0 \text{ and } d(t) = 0 \text{ for a.e. } t \in [0, T]. \quad (2.9)$$

**Lemma 2.1.** Suppose that system (1.1) takes on the observability canonical form, i.e.,

$$A = \begin{bmatrix}
0 & 0 & \cdots & 0 & a_1 \\
1 & 0 & \cdots & 0 & a_2 \\
0 & 1 & \cdots & 0 & a_3 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & a_n
\end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\
b_2 \\
b_3 \\
\vdots \\
b_n
\end{bmatrix} \neq 0 \text{ and } C = [0 \ 0 \ \cdots \ 0 \ 1] \in \mathbb{R}^{1 \times n}, \quad (2.10)$$

where $a_j, b_j \in \mathbb{R}$, $j = 1, 2, \cdots, n$. Then, system (1.1) is observable for $\mathbb{S}$ if and only if $b_2 = b_3 = \cdots = b_n = 0$ and $b_1 \neq 0$.

**Proof.** Suppose that $b_1 \neq 0$ and $b_2 = b_3 = \cdots = b_n = 0$. Then, for any $T > 0$, $u(t) = 0$ for a.e. $t \in [0, T]$ implies that

$$\begin{cases}
\dot{x}_1(t) = a_1 x_n(t) + b_1 d(t), \\
\dot{x}_2(t) = x_1(t) + a_2 x_n(t), \\
\dot{x}_3(t) = \cdots, \quad \text{a.e. } t \in [0, T], \\
\dot{x}_n(t) = x_{n-1}(t) + a_n x_n(t),
\end{cases} \quad (2.11)$$

where $x(t) = [x_1(t) \ x_2(t) \ \cdots \ x_n(t)]^\top$. If $y(t) = x_n(t) = 0$ for a.e. $t \in [0, T]$ (2.11) yields

$$x_n(t) = x_{n-1}(t) = \cdots = x_1(t) = 0, \quad \text{a.e. } t \in [0, T],$$

which implies that $d(t) = 0$ for a.e. $t \in [0, T]$ due to $b_1 \neq 0$. Hence, system (1.1) is observable for $\mathbb{S}$.
Conversely, suppose that system (1.1) is observable for $S$. We first claim that $b_n = 0$. Otherwise, for any $T > 0$,

\[
\begin{aligned}
\dot{x}_1(t) &= a_1 x_n(t) + b_1 d(t), \\
\dot{x}_2(t) &= x_1(t) + a_2 x_n(t) + b_2 d(t), \\
\vdots & \quad \vdots \\
\dot{x}_{n-1}(t) &= x_{n-2}(t) + a_{n-1} x_n(t) + b_{n-1} d(t), \\
y(t) &= x_n(t) = 0,
\end{aligned}
\tag{2.12}
\]

implies that $x_{n-1}(t) + b_n d(t) = 0$ for a.e. $t \in [0, T]$ and hence system (2.12) turns out to be

\[
\begin{aligned}
\dot{x}_1(t) &= -\frac{b_1}{b_n} x_{n-1}(t), \\
\dot{x}_2(t) &= x_1(t) - \frac{b_2}{b_n} x_{n-1}(t), \\
\vdots & \quad \vdots \\
\dot{x}_{n-1}(t) &= x_{n-2}(t) - \frac{b_{n-1}}{b_n} x_{n-1}(t), \\
\end{aligned}
\quad \text{a.e. } t \in [0, T].
\tag{2.13}
\]

Since system (2.13) with the output $x_{n-1}(\cdot)$ is of the observability canonical form, it is always observable for any $b_j \in \mathbb{R}$, $j = 1, 2, \cdots, n$. As a result, each non-zero solution of system (2.13) satisfies $x_{n-1}(t) = -b_n d(t) \neq 0$ for a.e. $t \in [0, T]$ and hence is the zero dynamics of the original system (2.12). This contradicts to the observability of system (1.1). We hence obtain $b_n = 0$.

Similarly, we can prove that $b_{n-1} = 0$. Indeed, in this case, for any $T > 0$,

\[
\begin{aligned}
\dot{x}_1(t) &= a_1 x_n(t) + b_1 d(t), \\
\dot{x}_2(t) &= x_1(t) + a_2 x_n(t) + b_2 d(t), \\
\vdots & \quad \vdots \\
\dot{x}_{n-1}(t) &= x_{n-2}(t) + b_{n-1} d(t), \\
y(t) &= x_{n-1}(t) = 0,
\end{aligned}
\quad \text{a.e. } t \in [0, T]
\tag{2.14}
\]

implies that $x_{n-2}(t) + b_{n-1} d(t) = 0$ for a.e. $t \in [0, T]$ and hence system (2.14) is reduced to

\[
\begin{aligned}
\dot{x}_1(t) &= -\frac{b_1}{b_{n-1}} x_{n-2}(t), \\
\dot{x}_2(t) &= x_1(t) - \frac{b_2}{b_{n-1}} x_{n-2}(t), \\
\vdots & \quad \vdots \\
\dot{x}_{n-3}(t) &= x_{n-4}(t) - \frac{b_{n-2}}{b_{n-1}} x_{n-2}(t), \\
\dot{x}_{n-2}(t) &= x_{n-3}(t) - \frac{b_{n-2}}{b_{n-1}} x_{n-2}(t), \\
\end{aligned}
\quad \text{a.e. } t \in [0, T].
\tag{2.15}
\]

Since system (2.15) with the output $x_{n-2}(\cdot)$ is always observable for any $b_j \in \mathbb{R}$, $j = 1, 2, \cdots, n-1$, each non-zero solution of system (2.15) is a zero dynamics of the original system (2.12). This
contradicts to the observability of system (1.1). We hence obtain \( b_n = b_{n-1} = 0 \). Moreover, we can obtain \( b_n = b_{n-1} = b_2 = 0 \) by repeating the same process. This completes the proof of the lemma due to \( B \neq 0 \).

**Lemma 2.2.** Let \( A \in \mathbb{R}^{n \times n} \) and \( G \in \mathbb{R}^{m \times m} \). Suppose that

\[
\sigma(A) \cap \sigma(G) = \emptyset. \quad (2.16)
\]

Then, system (1.1) is observable for \( \Omega(G) \) if and only if \((A, C)\) is observable and the following transmission zeros condition holds:

\[
C(\lambda - A)^{-1}B \neq 0, \quad \forall \lambda \in \sigma(G). \quad (2.17)
\]

**Proof.** Since we have known that \( d \in \Omega \), there exists a \( Q \in \mathbb{R}^{1 \times m} \) such that \((G, Q)\) is observable and the disturbance can be written as \( \dot{v}(t) = Gv(t) \) and \( d(t) = Qv(t) \) for some initial state. As a result, system (1.1) takes the form

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B[Qv(t) + u(t)], \\
\dot{v}(t) &= Gv(t), \quad y(t) = Cx(t).
\end{align*}
\]

If we define

\[
A_e = \begin{bmatrix} A & BQ \\ 0 & G \end{bmatrix} \quad \text{and} \quad C_e = [C \quad 0],
\]

then system (1.1) is observable for \( \Omega(G) \) if and only if system \((A_e, C_e)\) is observable.

Suppose that \((A_e, C_e)\) is observable and there exists a \( \tau > 0 \) such that \( C_e A_t x = 0 \) for any \( t \in [0, \tau] \). Then, \( C_e A_t x = 0 \) implies that \( x = 0 \) and hence \((A, C)\) is observable. For any \( \lambda \in \sigma(G) \subset \sigma(A_e) \), suppose that \( A_e(x, v)^T = \lambda(x, v)^T \) with \( (x, v)^T \neq 0 \). By exploiting [21, p. 15, Remark 1.5.2] and the observability of \((A_e, C_e)\), a simple computation shows that

\[
C_e(x, v)^T = Cx = C(\lambda - A)^{-1}BQv \neq 0,
\]

which leads to (2.17) easily.

Conversely, for any \( \lambda \in \sigma(G) \subset \sigma(A_e) \), suppose that \( A_e(x, v)^T = \lambda(x, v)^T \) and \( C_e(x, v)^T = 0 \). Then, \( C(\lambda - A)^{-1}BQv = 0 \) and \( Gv = \lambda v \). By assumption (2.17), \( Qv = 0 \) and hence \( v = 0 \) by the observability of \((G, Q)\). As a result, the equations \( A_e(x, v)^T = \lambda(x, v)^T \) and \( C_e(x, v)^T = 0 \) are reduced to \( Ax = \lambda x \) and \( Cx = 0 \). By the observability of \((A, C)\), we obtain \( x = 0 \). Therefore, \((A_e, C_e)\) is observable, or equivalently, system (1.1) is observable for \( \Omega(G) \).

We point out that the observability of disturbance corrupted system (1.1) depends on the disturbance set \( \Theta \) which serves as the prior disturbance information we have known. Different disturbance set may lead to different observability even for the same system. Here is an example to show this point. Let

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad C = [1 \quad -1]. \quad (2.21)
\]
Then, system (1.1) with \( u = 0 \) can be written as
\[
\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = d(t), \quad y(t) = x_1(t) - x_2(t).
\] (2.22)

Suppose that we know nothing about the disturbance except \( d \in S \). Then, system (2.22) is not observable for \( S \). Indeed, a simple computation shows that \( x_1(t) = x_2(t) = d(t) = s_T(t) \) is a nonzero solution of system (2.22) over \([0, T]\), where \( s_T \) is given by (2.3). However, \( d \in S \) and the output satisfies \( y(t) = x_1(t) - x_2(t) \equiv 0 \) on \([0, T]\). By Definition 1, system (2.22) is not observable for \( S \). If we have known the dynamics of the disturbance, the situation becomes completely different.

Suppose that we have known \( d \in \Omega(G) \) for some matrix \( G \) satisfying \( 0, 1 \notin \sigma(G) \). Then, there exists a vector \( Q \) such that system \((G, Q)\) is observable and hence system (2.22) can be written as
\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \quad \dot{x}_2(t) = Qv(t), \\
\dot{v}(t) &= Gv(t), \\
y(t) &= x_1(t) - x_2(t),
\end{align*}
\] (2.23)

which is a disturbance free system. By Lemma 2.2, it is easy to see that system (2.23) is observable. In other words, system (1.1) is observable for \( \Omega(G) \) which is completely different from the observability for \( S \). This fact implies that, if the prior information about the disturbance is enough, we may still estimate the disturbance \( d(\cdot) \) from system (1.1) in terms of the output \( y(\cdot) \) even if it is unobservable for \( S \).

**Remark 2.1.** Definition 1 is different from the observability of disturbance free system where the observability on some finite interval \([0, T]\) implies the observability on entire \([0, \infty)\). Owing to the uncertainty of disturbance, it is almost impossible to estimate the disturbance on \([T, \infty)\) by the information of output over \([0, T]\).

### 3 High-gain for stabilization

In most of the cases, we have to pay prices in estimating disturbance from measured output and the prices are usually characterized by the high-gain. Since it does not need necessarily the prior information about disturbance except for some rough information like boundedness, the high-gain is an effective and practical way to cope with the disturbance. In [4], it has been used to the observer design for the system that represents a chain of \( n \) integrators. The well-known ESO in ADRC is also by means of the high-gain [7], [10]. In this section, we will consider the basic principle of high-gain and investigate the relationship between the observability and the high-gain.

To show the basic principle of high-gain clearly, we begin with the direct propositional feedback for a scalar system with input disturbance:
\[
\dot{x}(t) = u(t) + d(t), \quad u(t) = -\omega x(t),
\] (3.1)
where $d \in L^\infty[0, \infty)$ is the disturbance and $\omega$ is a positive tuning parameter. We solve the closed-loop straightforwardly to get

$$|x(t)| \leq e^{-\omega t}|x(0)| + \int_0^t e^{-\omega(t-s)}|d(s)|ds \leq e^{-\omega t}|x(0)| + \frac{\|d\|_\infty}{\omega}. \quad (3.2)$$

That is

$$\lim_{t \to \infty} |x(t)| \leq \frac{\|d\|_\infty}{\omega}, \quad (3.3)$$

which implies that we can stabilize $x(\cdot)$ as small as possible by increasing the feedback gain $\omega$. In other words, the negative impact of the disturbance $d(\cdot)$ in system (3.1) can be eliminated by increasing the feedback gain $\omega$. However, this property seems not trivial for general linear systems. Here is a sufficient condition under which the high-gain works.

**Lemma 3.1.** Let $A_\omega \in \mathbb{R}^{n \times n}$ be a Hurwitz matrix with $\omega = -\Lambda_{\max}(A_\omega) > 0$. Suppose that $B \in \mathbb{R}^n$ such that

$$\lim_{\omega \to +\infty} \|(s - A_\omega)^{-1}B\|_{\mathbb{R}^n} = 0 \text{ uniformly on } s \in \mathbb{C}_+. \quad (3.4)$$

Then, there exists an $L_B > 0$, independent of $\omega$ and $t$, such that

$$\|e^{A_\omega t}B\|_{\mathbb{R}^n} \leq L_B e^{-\omega t}, \quad t \geq 0. \quad (3.5)$$

As a result, for any $d \in L^\infty[0, \infty)$, the solution of system $\dot{x}(t) = A_\omega x(t) + Bd(t)$ satisfies

$$\lim_{t \to \infty} \|x(t)\|_{\mathbb{R}^n} \leq \frac{L_B\|d\|_\infty}{\omega}. \quad (3.6)$$

**Proof.** Let $\varepsilon_j = [0 \cdots 0 1_{jth} 0 \cdots 0]^T$ denote the $j$-th coordinate vector where $1_{jth}$ means the component in the $j$-th position is 1, $j = 1, 2, \cdots, n$. By the assumption (3.4),

$$\lim_{\omega \to +\infty} \|\varepsilon_j^T (s - A_\omega)^{-1}B\|_{\mathbb{R}^n} = 0 \text{ uniformly on } s \in \mathbb{C}_+. \quad (3.7)$$

for $j = 1, 2, \cdots, n$. Applying the inverse Laplace transform to (3.7), we obtain

$$\lim_{\omega \to +\infty} \|\varepsilon_j^T e^{A_\omega t}B\| = \frac{1}{2\pi i} \lim_{\omega \to +\infty} \lim_{T \to \infty} \int_{\gamma-iT}^{\gamma+iT} e^{st} \varepsilon_j^T (s - A_\omega)^{-1}Bds$$

$$= \frac{1}{2\pi i} \lim_{\omega \to +\infty} \int_{\gamma-iT}^{\gamma+iT} e^{st} \lim_{\omega \to +\infty} \varepsilon_j^T (s - A_\omega)^{-1}Bds$$

$$= \frac{1}{2\pi i} \lim_{\omega \to +\infty} \int_{\gamma-iT}^{\gamma+iT} e^{st}0ds = 0, \quad t \geq 0, \quad (3.8)$$

where $\gamma$ is a real number so that the contour path of the integration is in the region of convergence of $\varepsilon_j^T e^{A_\omega t}B$, $j = 1, 2, \cdots, n$. Since $A_\omega$ is Hurwitz, (3.5) follows from (3.8) easily. Moreover, (3.6) holds due to

$$x(t) = e^{A_\omega t}x(0) + \int_0^t e^{A_\omega s}Bd(t-s)ds.$$
Remark 3.1. We point out that (3.5) does not hold for all controllable systems. For example, if we choose
\[ A_\omega = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\omega \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \omega > 0, \] (3.9)
then \((A_\omega, B)\) is controllable. However, a straightforward computation shows that
\[
e^{A_\omega t}B = e^{-\omega t} \begin{bmatrix} 1 + (\omega + 1)t \\ 1 - (\omega^2 + \omega)t \end{bmatrix}, \quad t \geq 0 \] (3.10)
and in particular,
\[
\|e^{A_\omega \frac{1}{\omega}B}\|_{\mathbb{R}^2} = e^{-1} \left\| \begin{bmatrix} 2 + 1/\omega \\ -\omega \end{bmatrix} \right\|_{\mathbb{R}^2} \to \infty \text{ as } \omega \to +\infty. \] (3.11)

The following Theorem shows that system (1.1) can always be stabilized to zero by high-gain provided it is observable for \(S\).

**Theorem 3.1.** Suppose that system (1.1) is observable for \(S\). Then, system \((A, B)\) is controllable and there exist functions \(f_j \in C[0, \infty), \ j = 1, 2, \ldots, n\) such that the feedback
\[
u(t) = [f_1(\omega) \ f_2(\omega) \ \cdots \ f_n(\omega)]^T x(t), \quad \omega > 0, \] (3.12)
stabilizes system
\[
\dot{x}(t) = Ax(t) + B[d(t) + u(t)], \quad d \in S. \] (3.13)
In other words, the closed-loop system given by (3.13) and (3.12) satisfies:
\[
\lim_{t \to \infty} \|x(t)\|_{\mathbb{R}^n} \leq \frac{M\|d\|_{\infty}}{\omega}, \] (3.14)
where \(M\) is a positive constant that is independent of \(\omega\).

**Proof.** We assume without loss of the generality that \(A, B\) and \(C\) are given by the observability canonical form (2.10). By Lemma 2.1, we conclude that \(b_1 \neq 0\) and \(b_2 = b_3 = \cdots = b_n = 0\). For simplicity, we suppose that \(b_1 = 1\). Since system \((A^T, B^T)\) is a chain of \(n\) integrators, it is observable. So \((A, B)\) is also controllable. As a result, there exists an invertible transformation \(U\) that converts system \((A, B)\) into the controllability canonical form \((A^T, C^T)\). More specifically,
\[
UAU^{-1} = A^T \quad \text{and} \quad UB = C^T. \] (3.15)
It is sufficient to consider the following system:
\[
\dot{z}(t) = A^T z(t) + C^T [d(t) + u(t)]. \] (3.16)
Let
\[
K_\omega = [k_1 \omega^n - a_1 \ k_2 \omega^{n-1} - a_2 \ \cdots \ k_n \omega - a_n], \quad \omega > 0, \] (3.17)
where $K = [k_1-a_1\ k_2-a_2\ \cdots\ k_n-a_n]$ is a vector such that $A^\top + C^\top K$ is Hurwitz. A simple computation shows that

$$A^\top + C^\top K = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ k_1\omega^n & k_2\omega^{n-1} & k_3\omega^{n-2} & \cdots & k_n\omega \end{bmatrix}$$

is Hurwitz as well and

$$\lambda \omega \in \sigma(A^\top + C^\top K)$$

if and only if $\lambda \in \sigma(A^\top + C^\top K)$. Moreover, for any $s \in \mathbb{C}_+$, it follows that

$$\left[s - (A^\top + C^\top K)\omega\right]^{-1} C^\top = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ s^n \end{bmatrix}.$$

Since $A^\top + C^\top K$ is Hurwitz, we have $k_1 \neq 0$ and hence

$$\lim_{\omega \to +\infty} \|s - (A^\top + C^\top K)\omega\|^{-1} C^\top \| \|_{\mathbb{R}^n} = 0 \text{ uniformly on } s \in \mathbb{C}_+.$$ (3.21)

By Lemma 3.1, there exists an $L_{C}$ that is independent of $\omega$ and $t$ such that the solution of system

$$\dot{z}(t) = (A^\top + C^\top K)\omega z(t) + C d(t)$$

satisfies

$$\lim_{t \to \infty} \|z(t)\|_{\mathbb{R}^n} \leq \frac{L_{C} \|d\|_{\infty}}{\omega}. \quad (3.22)$$

By (3.15), we can obtain (3.14) easily and moreover, the feedback (3.12) is given by

$$u(t) = K\omega U^{-1} x(t), \quad \omega > 0. \quad (3.23)$$

\hfill \square

4 Observer design with known disturbance dynamics

In this section, we consider a special case that the disturbance dynamics are known, i.e., $d \in \Omega(G)$ with known $G \in \mathbb{R}^{m \times m}$. Since the prior information about the disturbance is sufficient, this is the simplest case for the observer design yet is the concise situation to demonstrate the new idea of observer design.

Suppose that $(G, Q)$ is observable with output space $\mathbb{R}$. Consider the following Luenberger observer of system (1.1):

$$\begin{cases} \dot{x}(t) = A\dot{x}(t) + BQ\dot{x}(t) - F_1[y(t) - C\dot{x}(t)] + Bu(t), \\ \dot{v}(t) = G\dot{v}(t) + F_2[y(t) - C\dot{x}(t)], \end{cases} \quad (4.1)$$
where \( F_1 \in \mathbb{R}^n \) and \( F_2 \in \mathbb{R}^m \) are the gain vectors to be determined. When system (4.1) is observable, \( F_1 \) and \( F_2 \) can be chosen easily by the pole assignment theorem. However, we will choose \( F_1 \) and \( F_2 \) in another way so that we can cope with the general disturbance by high-gain in Section 5. Let the observer errors be

\[
\tilde{x}(t) = x(t) - \hat{x}(t) \quad \text{and} \quad \tilde{v}(t) = v(t) - \hat{v}(t).
\]

Then, they are governed by

\[
\begin{align*}
\dot{\tilde{x}}(t) &= (A + F_1 C)\tilde{x}(t) + B Q \tilde{x}(t), \\
\dot{\tilde{v}}(t) &= G \tilde{v}(t) - F_2 C \tilde{x}(t).
\end{align*}
\]

(4.3)

If we select \( F_1 \) and \( F_2 \) properly such that system (4.3) is stable, then \((x(t), v(t))\) can be estimated in the sense that

\[
\| (x(t) - \hat{x}(t), v(t) - \hat{v}(t)) \|_{\mathbb{R}^n \times \mathbb{R}^m} \to 0 \quad \text{as} \quad t \to \infty.
\]

(4.4)

Inspired by the first two parts [5] and [6] of this series works, the \( F_1 \) and \( F_2 \) can be chosen easily by decoupling the system (4.3) as a cascade system. The corresponding transformation is

\[
\begin{pmatrix}
I_n & S \\
0 & I_m
\end{pmatrix}
\begin{bmatrix}
A + F_1 C & B Q \\
-F_2 C & G
\end{bmatrix}
\begin{pmatrix}
I_n & S \\
0 & I_m
\end{pmatrix}^{-1}
\]

\[
= \begin{bmatrix}
A + (F_1 - SF_2)C & SG - [A + (F_1 - SF_2)C]S + BQ \\
-F_2 C & G + F_2 CS
\end{bmatrix},
\]

(4.5)

where \( S \in \mathbb{R}^{n \times m} \) is to be determined. If we select \( S \) properly such that

\[
SG - [A + (F_1 - SF_2)C]S + BQ = 0,
\]

(4.6)

then the right side matrix of (4.5) is Hurwitz if and only if the matrices \( A + (F_1 - SF_2)C \) and \( G + F_2 CS \) are Hurwitz.

**Theorem 4.1.** Suppose that system (1.1) is observable for \( \Omega(G) \) and \( G \in \mathbb{R}^{m \times m} \) is known. Then, there exist \( F_1 \in \mathbb{R}^n \), \( F_2 \in \mathbb{R}^m \) and \( Q \in \mathbb{R}^{1 \times m} \) such that \((G, Q)\) is observable and the solution of the observer (4.1) satisfies (4.4). Moreover, \( F_1, F_2 \) and \( Q \) can be selected by the following scheme:

(a) Select \( F_0 \in \mathbb{R}^n \) such that \( A + F_0 C \) is Hurwitz, select \( P \in \mathbb{R}^{1 \times m} \) such that \((G, P)\) is observable and select \( F_2 \) such that \( G + F_2 P \) is Hurwitz; (b) Solve the equations

\[
(A + F_0 C)S - SG = BQ \quad \text{and} \quad CS = P
\]

(4.7)

to get \( S \in \mathbb{R}^{n \times m} \) and \( Q \in \mathbb{R}^{1 \times m} \); (c) Set \( F_1 = F_0 + SF_2 \).

**Proof.** Since system (1.1) is observable for \( \Omega(G) \), it follows from Lemma 2.2 that \((A, C)\) is observable and the transmission zeros condition (2.17) holds. Therefore, there exists an \( F_0 \in \mathbb{R}^n \) such that \( A + F_0 C \) is Hurwitz and

\[
\sigma(A + F_0 C) \cap \sigma(G) = \emptyset.
\]

(4.8)
By [16], (4.8) and (2.17), the equations (4.7) admits a solution \( S \in \mathbb{R}^{n \times m} \) and \( Q \in \mathbb{R}^{1 \times m} \). Since \((G, P)\) is observable, there exists an \( F_2 \) such that \( G + F_2 P \) is Hurwitz. As a result, \( F_1 = F_0 + SF_2 \) is well defined.

Now, we claim that \((G, Q)\) is observable. Indeed, if we suppose that \( Gh = \lambda h \) and \( Qh = 0 \) with \( \lambda \in \sigma(G) \). Then, the Sylvester equation in (4.7) turns out to be \((A + F_0 C - \lambda)Sh = BQh = 0\). By (4.8), we conclude that \( \lambda \notin \sigma(A + F_0 C) \) and hence \( A + F_0 C - \lambda \) is invertible. As a result, \( Sh = 0 \) and \( CSh = Ph = 0 \). By [21, p. 15, Remark 1.5.2], we can conclude \( h = 0 \) due to the observability of \((G, P)\). Therefore, we obtain the observable system \((G, Q)\) by which system (1.1) can be written as a cascade system (2.18) for some initial state.

Let \( \tilde{x}(t) = x(t) - \hat{x}(t) \) and \( \tilde{v}(t) = v(t) - \hat{v}(t) \).

Then, the error between (2.18) and the observer (4.1) is governed by
\[
\begin{cases}
\dot{\tilde{x}}(t) = [A + (F_0 + SF_2)C]\tilde{x}(t) + BQ\tilde{v}(t), \\
\dot{\tilde{v}}(t) = G\tilde{v}(t) - F_2 C\tilde{x}(t).
\end{cases}
\]

(4.10)

Thanks to the choice of \( F_1 \) and \( F_2 \), a simple computation shows that the following matrices are similar each other
\[
\begin{bmatrix}
A + (F_0 + SF_2)C & BQ \\
-F_2 C & G
\end{bmatrix}
\] and
\[
\begin{bmatrix}
A + F_0 C & 0 \\
-F_2 C & G + F_2 P
\end{bmatrix}.
\]

(4.11)

Since the matrices \( A + F_0 C \) and \( G + F_2 P \) are Hurwitz, both the matrices in (4.11) are Hurwitz. As a result, (4.4) holds due to (4.9).

\[\square\]

**Remark 4.1.** Equations (4.7) are known as the regulator equations which are instrumental to establishing the linear output regulation theory [11]. The transmission zeros condition (2.17) can be represented as
\[
\text{rank} \begin{bmatrix} A - \lambda & B \\ C & 0 \end{bmatrix} = n + m, \quad \forall \lambda \in \sigma(G).
\]

(4.12)

## 5 Observer design with general disturbance

The disturbance considered in Section 4 is quite ideal. In engineering applications, the disturbance dynamics are usually unknown or at least partially unknown. It is therefore more realistic to consider the disturbance \( d \in \mathbb{S} \). Suppose that we have known that \( G \) is a “rough approximation” of the dynamics of \( d \). In order to make use this prior dynamics, we first represent the disturbance dynamically as an output of a system dominated by \( G \).

**Lemma 5.1.** Let \((G, Q)\) be an observable system with state space \( \mathbb{R}^{m+1} \) and output space \( \mathbb{R} \). Suppose that 0 \( \in \sigma(G) \). Then, for any \( d \in \mathbb{S} \), there exists a \( v_0 \in \mathbb{R}^{m+1} \) such that
\[
\begin{cases}
\dot{v}(t) = Gv(t) + \frac{B_d}{QB_d}\hat{e}(t), \quad v(0) = v_0, \\
d(t) = Qv(t),
\end{cases}
\]

(5.1)
where $B_d \in \mathbb{R}^{m+1}$ is the eigenvector corresponding to the eigenvalue 0 of $G$, $e = (I - \mathbb{P}_G)d$ and $\mathbb{P}_G$ is given by (2.6).

**Proof.** Since $(G, Q)$ is observable and $0 \in \sigma(G)$, it follows from the Hautus test [21, p.15, Remark 1.5.2] that $\text{Ker} G \cap \text{Ker} Q = \{0\}$. This means $QB_d \neq 0$ by the fact $GB_d = 0$ and $B_d \neq 0$. The first equation of (5.1) therefore makes sense.

Since $\mathbb{P}_Gd \in \Omega(G)$, it can be written dynamically as

$$\dot{v}_1(t) = Gv_1(t), \quad (\mathbb{P}_Gd)(t) = Qv_1(t)$$

(5.2)

for some initial state. Since $GB_d = 0$, if we let

$$v_2(t) = \frac{d(t) - (\mathbb{P}_Gd)(t)}{QB_d}B_d = \frac{e(t)}{QB_d}B_d, \quad t \geq 0,$$

(5.3)

then

$$\begin{cases} 
\dot{v}_2(t) = Gv_2(t) + \frac{B_d}{QB_d} \dot{e}(t), & v_2(0) = \frac{e(0)}{QB_d}B_d, \\
\dot{e}(t) = Qv_2(t).
\end{cases}$$

(5.4)

System (5.1) then follows from (5.2) and (5.4) by letting $v(t) = v_1(t) + v_2(t)$. $\square$

Now, we design an observer for system (1.1) under the assumption that $d \in S$. By Lemma 2.1, we can suppose without loss of generality that $A, B, C, G$ and $B_d$ satisfy the following assumptions:

**Assumption 5.1.** Let $n$ and $m$ be positive integers, let the matrices $A, B$ and $C$ be given by (2.10) with $b_1 = 1$ and $b_2 = b_3 = \cdots = b_n = 0$ and let the matrices $G, E$ and $B_d$ be given by

$$G = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & g_1 & g_2 & \cdots & g_m \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\
0 \\
\vdots \\
0 \\
1 \end{bmatrix}, \quad \text{and} \quad B_d = \begin{bmatrix} 1 \\
0 \\
\vdots \\
0 \end{bmatrix} \in \mathbb{R}^{m+1},$$

(5.5)

where $g_j \in \mathbb{R}$, $j = 1, 2, \cdots, m$ such that

$$\sigma(G) \subset \mathbb{C}_+.$$

(5.6)

The assumption (5.5) implies that $0 \in \sigma(G)$ and thus $GB_d = 0$ for any $g_j$, $j = 1, 2, \cdots, m$. Moreover, the dynamics dominated by $G$ of (5.6) contain all signals of harmonic, polynomial signals, exponential signals and their linear combinations. Inspired by Theorem 4.1, the EDO of system (5.1) is designed as

$$\begin{cases} 
\dot{x}(t) = [A + (K_{\omega_0} + SE)C]\dot{x}(t) + BQ\dot{v}(t) - (K_{\omega_0} + SE)y(t) + Bu(t), \\
\dot{v}(t) = G\dot{v}(t) - EC\dot{x}(t) + Ey(t),
\end{cases}$$

(5.7)

where $G, E, K_{\omega_0}, S$ and $Q$ are chosen by the following scheme:
Choose $G$ and $E$ in terms of the Assumption 5.1 and the prior information about the disturbance (This will be considered in Sections 6 and 7);

Choose $K = [k_1 - a_1 \ k_2 - a_2 \ \cdots \ k_n - a_n]^\top$ and $P = [p_0 \ p_1 - g_1 \ \cdots \ p_m - g_m]$ such that the matrices $A + KC$ and $G + EP$ are Hurwitz. Let

$$K_{\omega_o} = [k_1\omega_o^n - a_1 \ k_2\omega_o^{n-1} - a_2 \ \cdots \ k_n\omega_o - a_n]^\top$$

and

$$P_{\omega_o} = [p_0\omega_o^{m+1} \ p_1\omega_o^m - g_1 \ \cdots \ p_m\omega_o - g_m],$$

where $\omega_o$ is a positive tuning parameter;

Solve the equations

$$(A + K_{\omega_o}C)S - SG = BQ, \quad CS = P_{\omega_o},$$

(5.10)
to get $S \in \mathbb{R}^{n \times (m+1)}$ and $Q \in \mathbb{R}^{1 \times (m+1)}$.

Lemma 5.2. Under Assumption 5.1 and the scheme of observer design, the equations (5.10) are always solvable. Moreover, the following assertions are true:

(i) System $(G, Q)$ is observable;

(ii) For any $s \in \mathbb{C}_+$, there exist two positive constants $C_K$ and $C_A$, independent of $\omega_o$ and $s$, such that

$$\|C(s - (A + K_{\omega_o}C))^{-1}B\|_{\mathbb{R}^n} \leq \frac{C_K}{\omega_o},$$

(5.11)

$$\frac{1}{|QB_d|} = \frac{1}{|k_1p_0|\omega_o^{n+m+1}}$$

(5.12)

and

$$\|C(s - (A + K_{\omega_o}C))^{-1}B\|_{\mathbb{R}^m} \leq \frac{C_A}{\omega_o};$$

(5.13)

(iii) For any $s \in \mathbb{C}_+$, there exists a positive constant $C_G$, independent of $\omega_o$ and $s$, such that

$$\|C(s - (G + EP_{\omega_o}))^{-1}E\|_{\mathbb{R}^{m+1}} \leq \frac{C_G}{\omega_o^{m+1}}, \quad \forall \ s \in \mathbb{C}_+;$$

(5.14)

(iv) If $G$ is diagonalizable, then there exist two positive constants $C_S$ and $C_Q$, independent of $\omega_o$ and $s$, such that

$$\|s - (A + K_{\omega_o}C)\|_{\mathbb{R}^n} \leq \frac{C_S}{\omega_o^{m+1}}, \quad \forall \ s \in \mathbb{C}_+;$$

(5.15)

and

$$\|s - (G + EP_{\omega_o})\|_{\mathbb{R}^{m+1}} \leq \frac{C_Q}{\omega_o^{m+n}}, \quad \forall \ s \in \mathbb{C}_+.$$
Moreover, a simple computation shows that

\[
A + K_{\omega_o} C = \begin{bmatrix}
0 & 0 & \cdots & 0 & k_1 \omega_o^n \\
1 & 0 & \cdots & 0 & k_2 \omega_o^{n-1} \\
0 & 1 & \cdots & 0 & k_3 \omega_o^{n-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & k_n \omega_o
\end{bmatrix}
\] (5.18)

is Hurwitz, (5.6) implies that

\[
\sigma(A + K_{\omega_o} C) \cap \sigma(G) = \emptyset.
\] (5.19)

Moreover, a simple computation shows that

\[
[\lambda - (A + K_{\omega_o} C)]^{-1} B = \frac{K_{\lambda}}{\rho_A(\lambda, \omega_o)}, \quad \lambda \in \mathbb{C}_+,
\] (5.20)

where

\[
K_{\lambda} = \begin{bmatrix}
k_n \lambda^{n-2} \omega_o + k_{n-1} \lambda^{n-3} \omega_o^2 + \cdots + k_3 \lambda \omega_o^{n-2} + k_2 \omega_o^{n-1} - \lambda^{n-1} \\
k_n \lambda^{n-3} \omega_o + k_{n-1} \lambda^{n-4} \omega_o^2 + \cdots + k_3 \omega_o^{n-2} - \lambda^{n-2} \\
\vdots \\
k_n \omega_o - \lambda \\
-1
\end{bmatrix}
\] (5.21)

and

\[
\rho_A(\lambda, \omega_o) = k_n \lambda^{n-1} \omega_o + k_{n-1} \lambda^{n-2} \omega_o^2 + \cdots + k_2 \lambda \omega_o^{n-1} + k_1 \omega_o^n - \lambda^n.
\] (5.22)

Hence, the following transmission zeros condition holds:

\[
C[\lambda - (A + K_{\omega_o} C)]^{-1} B = \frac{-1}{\rho_A(\lambda, \omega_o)} \neq 0, \quad \forall \lambda \in \mathbb{C}_+.
\] (5.23)

By [13], (5.19) and (5.23), the equations (5.10) are solvable.

(i). Suppose that \(Gh = \lambda h\) and \(Qh = 0\) with \(\lambda \in \sigma(G)\). Then, the Sylvester equation in (5.10) turns out to be \((A + K_{\omega_o} C - \lambda)Sh = BQh = 0\). By (5.19), we conclude that \(\lambda \notin \sigma(A + K_{\omega_o} C)\) and hence \(A + K_{\omega_o} C - \lambda\) is invertible. As a result, \(Sh = 0\) and \(CSSh = P_{\omega_o} h = 0\). By [21, p. Remark 1.5.2], we can conclude \(h = 0\) provided \((G, P_{\omega_o})\) is observable. Using [21, p. Remark 1.5.2] again, \((G, Q)\) is observable if we can prove \((G, P_{\omega_o})\) is observable. Actually, for any \(Gv = \lambda v\) and \(P_{\omega_o} v = 0\) with \(\lambda \in \sigma(G)\), we have \((G + EP_{\omega_o})v = Gv = \lambda v\). Since \(G + EP\) is Hurwitz, it follows from (5.5) and (5.9) that \(G + EP_{\omega_o}\) is Hurwitz as well and

\[
\lambda \omega_o \in \sigma(G + EP_{\omega_o}) \text{ if and only if } \lambda \in \sigma(G + EP).
\] (5.24)

By (5.6) and the fact \(\lambda \in \sigma(G)\), we obtain \(\lambda \notin \sigma(G + EP_{\omega_o})\). Hence, \((G + EP_{\omega_o})v = \lambda v\) implies that \(v = 0\). By [21, p. Remark 1.5.2], \((G, P_{\omega_o})\) is observable.

(ii). Since \(A + K_{\omega_o} C\) is Hurwitz, it follows from (5.18) that \(k_1 \neq 0\). Hence, (5.11) can be obtained by (5.20), (5.21) and (5.22) easily. Noting the \(GB_d = 0\), it follows from (5.10) that

\[
(A + K_{\omega_o} C)SB_d = BQB_d
\] (5.25)
and hence
\[ P_{\omega_o}B_d = CSB_d = C(A + K_{\omega_o}C)^{-1}BQ_Bd. \] (5.26)

By (5.5) and (5.9),
\[ P_{\omega_o}B_d = p_0\omega_o^{m+1}. \] (5.27)

Since \((G, Q)\) is observable and \(GB_d = 0\), the Hautus test [21, p.15, Remark 1.5.2] implies that \(QB_d \neq 0\). Hence, we combine (5.23), (5.27) and (5.26) to obtain \(p_0\omega_o^{m+1} \neq 0\) and
\[ \frac{1}{QB_d} = \frac{C(A + K_{\omega_o}C)^{-1}B}{P_{\omega_o}B_d} = -\frac{1}{k_1p_0\omega_o^{s+1}}, \] (5.28)

which leads to (5.12) easily. In view of (5.18), a straightforward computation shows that
\[ C[s - (A + K_{\omega_o}C)]^{-1} = \frac{-1}{\rho_A(s, \omega_o)}[1 \ s \ s^2 \ \cdots \ s^{n-1}], \ \forall \ s \in \mathbb{C}_+, \] (5.29)

which, together with (5.22), leads to (5.13) easily.

(iii). By a straightforward computation, it follows that
\[ G + EP_{\omega_o} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ p_0\omega_o^{m+1} & p_1\omega_o^m & p_2\omega_o^{m-1} & \cdots & p_m\omega_o \end{bmatrix}, \] (5.30)

and hence
\[ [s - (G + EP_{\omega_o})]^{-1}E = \frac{-1}{\rho_G(s, \omega_o)} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^m \end{bmatrix}, \ \forall \ s \in \mathbb{C}_+, \] (5.31)

where
\[ \rho_G(s, \omega_o) = p_0\omega_o^{m+1} + p_1\omega_o^m s + \cdots + p_m\omega_o s^m - s^{m+1}. \] (5.32)

Since \(G + EP_{\omega_o}\) is Hurwitz, we have \(p_0 \neq 0\) and hence (5.14) follows from (5.31) and (5.32).

(iv). Since \(G\) is diagonalizable, for any \(v \in \mathbb{R}^{m+1}\), there exists a sequence \(v_0, v_1, v_2, \cdots, v_m\) such that \(v = \sum_{j=0}^m v_j\zeta_j\), where \(G\zeta_j = \lambda_j\zeta_j\) with \(\lambda_j \in \sigma(G), \ j = 0, 1, 2, \cdots, m\). By (5.10), (5.20) and (5.23), we have
\[ S\zeta_j = (A + K_{\omega_o}C - \lambda_j)^{-1}BQ\zeta_j = -\frac{K\lambda_j}{\rho_A(\lambda_j, \omega_o)}Q\zeta_j \] (5.33)
and
\[ P_{\omega_o}\zeta_j = CS\zeta_j = C(A + K_{\omega_o}C - \lambda_j)^{-1}BQ\zeta_j = \frac{Q\zeta_j}{\rho_A(\lambda_j, \omega_o)}. \] (5.34)

Consequently,
\[ Q\zeta_j = P_{\omega_o}\zeta_j\rho_A(\lambda_j, \omega_o), \ j = 0, 1, \cdots, m, \] (5.35)
which, together with (5.33), gives
\[
Sv = \sum_{j=0}^{m} v_j S \varepsilon_j = -\sum_{j=0}^{m} v_j P_{\omega_o} \varepsilon_j K_\lambda j. \tag{5.36}
\]

Combining (5.21), (5.9) and (5.36), we obtain (5.15) easily.

Taking (5.30) and (5.5) into account, a simple computation shows that
\[
[s - (G + EP_{\omega_o})]^{-1} B_d = \frac{1}{\rho_G(s, \omega_o)} \begin{bmatrix}
p_m s^{m-1} \omega_o + p_{m-1}s^{m-2} \omega_o^2 + \cdots + p_1 \omega_o^m - s^m & -p_0 \omega_o^{m+1} \\
-p_0 \omega_o^{m+1} s & -p_0 \omega_o^{m+1} s
\end{bmatrix}
\tag{5.37}
\]
for any \( s \in \mathbb{C}_+ \). By (5.32) and the fact \( p_0 \neq 0 \), there exists a positive constant \( M_1 \), independent of \( \omega_o \) and \( s \), such that
\[
\|G[s - (G + EP_{\omega_o})]^{-1} B_d\|_{\mathbb{R}^{m+1}} \leq M_1 \|G\|, \quad \forall \ s \in \mathbb{C}_+. \tag{5.38}
\]

Consequently, it follows from (5.15), (5.13) and (5.38) that
\[
|C[s - (A + K_{\omega_o} C)]^{-1} SG[s - (G + EP_{\omega_o})]^{-1} B_d| \leq C_S M_1 \|G\| C_A \omega_o^m. \tag{5.39}
\]

Combining (5.9), (5.32) and (5.37), there exists a positive constant \( M_2 \), independent of \( \omega_o \) and \( s \), such that
\[
|P_{\omega_o}[s - (G + EP_{\omega_o})]^{-1} B_d| \leq M_2 \omega_o^m, \quad \forall \ s \in \mathbb{C}_+. \tag{5.40}
\]

For any \( v \in \mathbb{R}^{m+1} \), it follows from (5.10) and (5.23) that
\[
Qv = \frac{P_{\omega_o} v - C(A + K_{\omega_o} C)^{-1} SGv}{C(A + K_{\omega_o} C)^{-1} B} = -\rho_A(\lambda, \omega_o)[P_{\omega_o} v - C(A + K_{\omega_o} C)^{-1} SGv]. \tag{5.41}
\]
As a result, there exists an \( M_3 \), independent of \( \omega_o \), such that
\[
|Qv| = M_3 \omega_o^m \|P_{\omega_o} v\| + \|C(A + K_{\omega_o} C)^{-1} SGv\|, \quad \forall \ v \in \mathbb{R}^{m+1}, \tag{5.42}
\]
which, together with (5.39) and (5.40), leads to (5.16) easily.

**Theorem 5.1.** Under Assumption 5.1, the EDO (5.7) of (1.1) is well-posed: For any \( d \in \mathbb{S} \), \((\hat{x}(0), \hat{v}(0)) \in \mathbb{R}^n \times \mathbb{R}^{m+1} \) and \( u \in L_0^2[0, \infty) \), there exists a positive constant \( M_1 \), independent of \( \omega_o \), such that
\[
\lim_{t \to \infty} \|(x(t) - \hat{x}(t), v(t) - \hat{v}(t))\|_{\mathbb{R}^n \times \mathbb{R}^{m+1}} \leq \frac{M_1 \|e\|_{\mathbb{S}}}{\omega_o}, \tag{5.43}
\]
where \( e = (I - P_G)d \) and \( P_G \) is given by (2.6). In particular, there exists a positive constant \( M_2 \), independent of \( \omega_o \), such that
\[
\lim_{t \to \infty} |d(t) - Q\hat{v}(t)| \leq \frac{M_2 \|e\|_{\mathbb{S}}}{\omega_o}. \tag{5.44}
\]
Proof. By Lemma 5.1, system (1.1) can be written dynamically as

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B[Qv(t) + u(t)], \\
\dot{v}(t) &= Gv(t) + \frac{B_d}{QB_d} \dot{e}(t),
\end{align*}
\]

where \( e = (I - PG)d \) and \( PG \) is given by (2.6). Let

\[
\tilde{x}(t) = x(t) - \hat{x}(t) \quad \text{and} \quad \tilde{v}(t) = v(t) - \hat{v}(t).
\]

Then, the error is governed by

\[
\begin{align*}
\dot{\tilde{x}}(t) &= \left[ A + (K\omega_o + SE)C \right] \tilde{x}(t) + BQ \tilde{v}(t), \\
\dot{\tilde{v}}(t) &= G\tilde{v}(t) - EC\tilde{x}(t) + \frac{B_d}{QB_d} \dot{e}(t).
\end{align*}
\]

System (5.47) can be written as

\[
\frac{d}{dt}(\tilde{x}(t), \tilde{v}(t))^\top = A(\tilde{x}(t), \tilde{v}(t))^\top + B\dot{e}(t),
\]

where

\[
A = \begin{bmatrix} A + (K\omega_o + SE)C & BQ \\ -EC & G \end{bmatrix} \quad \text{and} \quad B = \frac{1}{QB_d} \begin{bmatrix} 0 \\ B_d \end{bmatrix}.
\]

In terms of the solution \( S \) of the Sylvester equation of (5.10), we introduce the transformation

\[
\begin{bmatrix} \tilde{x}(t) \\ \tilde{v}(t) \end{bmatrix} = P \begin{bmatrix} \tilde{x}(t) \\ \tilde{v}(t) \end{bmatrix}, \quad P = \begin{bmatrix} I_m & S \\ 0 & I_{m+1} \end{bmatrix}.
\]

Thanks to the choice of \( K\omega_o, E \) and \( S \), system (5.47) can be converted into the following system:

\[
\begin{align*}
\dot{\tilde{x}}(t) &= (A + K\omega_o C)\tilde{x}(t) + \frac{SB_d}{QB_d} \dot{e}(t), \\
\dot{\tilde{v}}(t) &= (G + EP_{\omega_o})\tilde{v}(t) - EC\tilde{x}(t) + \frac{B_d}{QB_d} \dot{e}(t).
\end{align*}
\]

We denote the system matrix and the input matrix of (5.51) by

\[
A_S = \begin{bmatrix} A + K\omega_o C & 0 \\ -EC & G + EP_{\omega_o} \end{bmatrix} \quad \text{and} \quad B_S = \frac{1}{QB_d} \begin{bmatrix} SB_d \\ B_d \end{bmatrix}.
\]

By a simple computation, it follows that

\[
PAP^{-1} = A_S \quad \text{and} \quad B_S = PB,
\]

where the Sylvester equation in (5.10) has been used. For any \( s \in \mathbb{C}_+ \), a simple computation shows that

\[
P^{-1}(s - A_S)^{-1}B_S = \frac{1}{QB_d} \begin{bmatrix} [s - (A + K\omega_o C)]^{-1}SB_d + SJ(s) \\ -J(s) \end{bmatrix},
\]

where \( P \) is the matrix obtained from the transformation (5.50).
where

\[ J(s) = [s - (G + EP_{\omega_o})]^{-1}EC[s - (A + K_{\omega_o}C)]^{-1}SB_d - [s - (G + EP_{\omega_o})]^{-1}B_d. \]  

(5.55)

Noting the \( GB_d = 0 \), it follows from (5.10) that

\[ SB_d = (A + K_{\omega_o}C)^{-1}BQB_d, \]  

(5.56)

and hence

\[ \frac{[s - (A + K_{\omega_o}C)]^{-1}SB_d}{QB_d} = [s - (A + K_{\omega_o}C)]^{-1}(A + K_{\omega_o}C)^{-1}B. \]  

(5.57)

By Lemma 5.2, there exist two positive constants \( C_K \) and \( C_S \) such that

\[ \left\| \frac{[s - (A + K_{\omega_o}C)]^{-1}SB_d}{QB_d} \right\| \leq \frac{C_K}{\omega_o^2}, \]  

(5.58)

and

\[ \frac{\|SJ(s)\|_{\mathbb{R}^n}}{|QB_d|} \leq \frac{C_S\|J(s)\|_{\mathbb{R}^{m+1}\omega_o^{m+n}}}{\omega_o} = \frac{C_S\|J(s)\|_{\mathbb{R}^{m+1}}}{\omega_o}, \quad \forall \ s \in \mathbb{C}_+. \]  

(5.59)

By (5.32), (5.37) and the fact \( p_0 \neq 0 \), there exists a \( C_J > 0 \) such that

\[ \|J(s)\|_{\mathbb{R}^{m+1}} < C_J, \quad \forall \ s \in \mathbb{C}_+. \]  

(5.60)

Combing (5.59), (5.60), (5.55), (5.58), (5.54) and (5.12), we arrive at

\[ \|P^{-1}(s - A_S)^{-1}B_S\|_{\mathbb{R}^n} \leq \frac{C_A}{\omega_o}, \quad \forall \ s \in \mathbb{C}_+. \]  

(5.61)

where \( C_A \) is a positive constant independent of \( \omega_o \) and \( s \). Furthermore, it follows from (5.53) that

\[ \|(s - A)^{-1}B\|_{\mathbb{R}^n} = \|P^{-1}(s - A_S)^{-1}B_S\|_{\mathbb{R}^n} \leq \frac{C_A}{\omega_o}, \quad \forall \ s \in \mathbb{C}_+. \]  

(5.62)

Since both \( A + K_{\omega_o}C \) and \( G + B_dP_{\omega_o} \) are Hurwitz and satisfy (5.24) and (5.17), respectively, the operator \( A \) is also Hurwitz. By virtue of Lemma 3.1, there exists an \( L_B > 0 \), independent of \( \omega_o \), such that

\[ \|e^{At}B\|_{\mathbb{R}^n \times \mathbb{R}^{m+1}} \leq L_B e^{-\omega_o t}, \quad t \geq 0. \]  

(5.63)

We solve (5.48) to obtain

\[ \|(\tilde{x}(t), \tilde{v}(t))\|_{\mathbb{R}^n \times \mathbb{R}^{m+1}} = \left\| e^{At}(\tilde{x}(0), \tilde{v}(0)) \right\| + \int_0^t e^{A(t-s)}B\dot{\bar{e}}(s)ds \]  

\[ \leq L_A e^{-\omega_o t} \|(\tilde{x}(0), \tilde{v}(0))\|_{\mathbb{R}^n \times \mathbb{R}^{m+1}} + L_B \int_0^t e^{-\omega_o(t-s)}\|\dot{\bar{e}}\|_{\mathbb{R}^{m+1}}ds \]  

\[ \leq L_A e^{-\omega_o t} \|(\tilde{x}(0), \tilde{v}(0))\|_{\mathbb{R}^n \times \mathbb{R}^{m+1}} + \frac{\|\bar{e}\|_{\mathbb{R}^{m+1}}L_B}{\omega_o}, \]  

(5.64)

where \( L_A \) is a positive constant. This leads to (5.43) from (5.46).
Now, we prove (5.44). For any $s \in \mathbb{C}_+$, it follows from (5.54) that

$$Q^{-1}(s - A)^{-1}B = -\frac{QJ(s)}{QB}, \quad Q = (0, Q). \tag{5.65}$$

By (5.14), (5.13) and (5.15), there exists an $M_4 > 0$ such that

$$\|s - (G + EP_\gamma)\|^{-1}EC[s - (A + K\gamma C)]^{-1}SBd\|_{\mathbb{R}^{m+1}} \leq \frac{M_4}{\omega_o}, \quad \forall \ s \in \mathbb{C}_+. \tag{5.66}$$

By (5.42), (5.9), (5.13) and (5.15), there exists an $M_5 > 0$ such that

$$|Qv| = M_5\omega_o^{n+m+1}v\|_{\mathbb{R}^{m+1}}, \quad \forall \ v \in \mathbb{R}^{m+1}. \tag{5.67}$$

We combine (5.66) and (5.67) to get

$$|Q(s - (G + EP_\gamma))^{-1}EC[s - (A + K\gamma C)]^{-1}SBd| \leq M_4M_5\omega_o^{n+m}, \quad \forall \ s \in \mathbb{C}_+. \tag{5.68}$$

which, together with (5.16), (5.55), (5.12) and (5.65), leads to

$$|Q^{-1}(s - A)^{-1}B| = \left|\frac{QJ(s)}{QB}\right| \leq \frac{M_6}{\omega_o}, \tag{5.69}$$

where $M_6$ is a positive constant independent of $\omega_o$ and $s$. Owing to (5.53), we arrive at

$$|Q(s - A)^{-1}B| \leq \frac{M_6}{\omega_o}, \quad \forall \ s \in \mathbb{C}_+. \tag{5.70}$$

Similarly to (3.8), we apply the inverse Laplace transform on (5.70) to obtain

$$\lim_{\omega_o \to \infty} |Qe^{At}B| = \frac{1}{2\pi i} \lim_{\omega_o \to \infty} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} Q(s - A)^{-1}B ds$$

$$= \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} \lim_{\omega_o \to \infty} Q(s - A)^{-1}B ds \tag{5.71}$$

$$= \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} ds = 0, \quad t \geq 0,$$

where $\gamma$ is a real number so that the contour path of the integration is in the region of convergence of $Qe^{At}B$. Since $A$ is Hurwitz with $\Lambda_{max}(A) = -\omega_o$, (5.71) implies that

$$|Qe^{At}B| \leq L_Qe^{-\omega_o t}, \quad t \geq 0, \tag{5.72}$$

where $L_Q$ is a positive constant which is independent of $\omega_o$. As a result, the solution of system (5.47) satisfies

$$\lim_{t \to \infty} |Q\tilde{v}(t)| = \lim_{t \to \infty} \left|Qe^{A\tilde{t}(0), \tilde{v}(0)}\right| + \lim_{t \to \infty} \left|Q \int_0^t e^{A\tilde{s}}B\tilde{v}(t - s) ds\right|$$

$$\leq \lim_{t \to \infty} \left|\int_0^t L_Qe^{-\omega_o s}|\tilde{v}(t - s)| ds\right| \leq \frac{L_Q\|\tilde{v}\|_{\infty}}{\omega_o}, \tag{5.73}$$

which, together with (2.7), (5.46) and (5.1), leads to (5.44).\qed
Remark 5.1. Suppose that \( \{ \varepsilon_j \}_{j=0}^m \) is a sequence of eigenvectors corresponding to the eigenvalues \( \lambda_j \) of \( G \), which forms a basis for \( \mathbb{R}^{m+1} \). Then, for any \( v = [v_0 \ v_1 \ \cdots \ v_m] \in \mathbb{R}^{m+1} \),

\[
v = [\varepsilon_0 \ \varepsilon_1 \ \cdots \ \varepsilon_m][\varepsilon_0 \ \varepsilon_1 \ \cdots \ \varepsilon_m]^{-1}\begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{bmatrix}.
\]

By (5.34), we obtain the analytic expression of \( Q \) as

\[
Qv = [Q\varepsilon_0 \ Q\varepsilon_1 \ \cdots \ Q\varepsilon_m][\varepsilon_0 \ \varepsilon_1 \ \cdots \ \varepsilon_m]^{-1}\begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{bmatrix}
\]

with

\[
Q\varepsilon_j = \frac{P_{\omega_0}\varepsilon_j}{C(A + K_{\omega_0}C - \lambda_j)^{-1}B}, \quad j = 0, 1, 2, \cdots, m.
\]

Therefore, we can obtain the parameter of the observer (5.7) explicitly via (5.76) and (5.36).

Remark 5.2. By (5.43), the accuracy of the observer depends both on the optimal approximation of \( d \) on \( \Omega(G) \) and the decay rate \( \omega_0 \). From this perspective, we need to choose \( G \) such that \( \Omega(G) \) is as large as possible so that the approximation error can be as small as possible. The choice of \( G \) depends on the prior information about the disturbance. The more the prior information we have, the higher the steady-state error will be. In particular, if we have known all the dynamics of the disturbance, i.e., we have known \( d \in \Omega(G) \) with known \( G \), the steady-state error of the observer (5.7) becomes zero. Another way to improve the observer accuracy is to increase the gain \( \omega_0 \). However, the large \( \omega_0 \) may lead to peaking phenomenon in transient response and hence it may not be feasible to improve the accuracy by increasing \( \omega_0 \) only. Hence, one of the contributions of the present work is giving a new way to improve the accuracy of the observer without increasing the high-gain \( \omega_0 \).

Remark 5.3. When the error of approximation \( P_G \) is zero, the system matrix of the error system (5.47) is similar to the matrix \( A_S \) in (5.52). Owing to the block-trigonal structure of \( A_S \), the poles of the error system (5.47) can be assigned arbitrarily by adjusting \( K_{\omega_0} \) and \( P_{\omega_0} \). This means that the prior information about the control plant and disturbance can be fully used. Moreover, the control plant considered in this paper is wider than the canonical form of \( [1] \) or \( [4] \). Although this canonical form can be extended technically by using high-gain [8], there still exists a waste of system prior information. In fact, only some boundedness of the elements of system matrix \( A \) was used rather than the matrix itself. As a result of this, the poles of the observer error system without the external disturbance cannot be assigned arbitrarily.
6 Extended dynamic observer with constant dynamics

In this section, we consider the EDO (5.7) with constant dynamics $G = 0$. It adapts to the worst situations where we have nothing prior information about the disturbance dynamics excepted some boundedness. For simplicity, we only consider, without loss of the generality, the following second order control plant, i.e.,

$$A = \begin{bmatrix} 0 & a_1 \\ 1 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad C = [0 \ 1],$$  \hspace{1cm} (6.1)

where $a_j \in \mathbb{R}$, $j = 1, 2$. If we choose $G = 0$, then it follows from (2.4) that

$$\Omega(G) = \left\{ v(t) \mid \dot{v}(t) = 0, \ v(0) \in \mathbb{R}, \ t \in \mathbb{R} \right\} = \left\{ v(t) \equiv v(0) \mid v(0) \in \mathbb{R}, \ t \in \mathbb{R} \right\},$$  \hspace{1cm} (6.2)

which implies that the optimal approximation $P_Gd$ of $d$ on $\Omega(G)$ satisfies $\|(I - P_G)d\|_S = \|\dot{d}(t)\|_\infty$.

We choose

$$B_d = E = 1, \quad K_{\omega_0} = [\alpha_1 \omega_0^2 - a_1 \ \alpha_2 \omega_0 - a_2]^T \quad \text{and} \quad P_{\omega_0} = \alpha_3 \omega_0$$  \hspace{1cm} (6.3)

such that $\alpha_3 < 0$ and the following matrix is Hurwitz:

$$\begin{bmatrix} 0 & \alpha_1 \\ 1 & \alpha_2 \end{bmatrix}$$  \hspace{1cm} (6.4)

We solve the equations (5.10) to get

$$S = (A + K_{\omega_0}C)^{-1}B = [-\alpha_2 \alpha_3 \omega_0^2 \ \alpha_3 \omega_0]^T \quad \text{and} \quad Q = \alpha_1 \alpha_3 \omega_0^3.$$  \hspace{1cm} (6.5)

In view of (5.7), the observer of system (1.1) with setting (6.1) is found to be

$$\begin{cases} \dot{x}_1(t) = a_1 \dot{x}_2(t) + \alpha_1 \alpha_3 \omega_0^3 v(t) - [(\alpha_1 - \alpha_2 \alpha_3) \omega_0^2 - a_1][y(t) - \hat{x}_2(t)] + u(t), \\
\dot{x}_2(t) = \dot{x}_1(t) + a_2 \dot{x}_2(t) - [(\alpha_2 + \alpha_3) \omega_0 - a_2][y(t) - \hat{x}_2(t)], \\
\dot{\hat{v}}(t) = [y(t) - \hat{x}_2(t)]. \end{cases} \hspace{1cm} (6.6)$$

In order to make a comparison to ESO [7] and the high-gain observer [4], we consider system

$$\begin{cases} \dot{z}(t) = A^T z(t) + C^T [d(t) + u(t)], \\
y(t) = B^T z(t), \end{cases} \quad z(t) = [z_1(t) \ z_2(t)]^T, \hspace{1cm} (6.7)$$

where $A, B$ and $C$ are still given by (6.1). By virtue of the observer (6.6) and the invertible transformation

$$UAU^{-1} = A^T, \quad UB = C^T, \quad CU^{-1} = B^T, \quad U = \begin{bmatrix} 0 & 1 \\ 1 & a_2 \end{bmatrix}. \hspace{1cm} (6.8)$$

the EDO of system (6.7) becomes

$$\begin{cases} \dot{\hat{z}}_1(t) = \hat{z}_2(t) - [(\alpha_2 + \alpha_3) \omega_0 - a_2][y(t) - \hat{z}_1(t)], \\
\dot{\hat{z}}_2(t) = a_1 \hat{z}_1(t) + a_2 \hat{z}_2(t) + \alpha_1 \alpha_3 \omega_0^3 v(t) \\
- [(\alpha_1 - \alpha_2 \alpha_3) \omega_0^2 + a_2 (\alpha_2 + \alpha_3) \omega_0 - a_2^2 - a_1][y(t) - \hat{z}_1(t)] + u(t), \\
\dot{\hat{v}}(t) = [y(t) - \hat{z}_1(t)] \end{cases} \hspace{1cm} (6.9)$$
where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are constants such that $\alpha_3 < 0$ and the matrix (6.4) is Hurwitz. When $a_1 = a_2 = 0$, observer (6.9) is reduced to

$$\begin{align*}
\dot{\hat{z}}_1(t) &= \hat{z}_2(t) - (\alpha_2 + \alpha_3)\omega_o[y(t) - \hat{z}_1(t)], \\
\dot{\hat{z}}_2(t) &= \alpha_1\alpha_3\omega_o^3\dot{v}(t) - (\alpha_1 - \alpha_2\alpha_3)\omega_o^2[y(t) - \hat{z}_1(t)] + u(t), \\
\dot{\hat{v}}(t) &= [y(t) - \hat{z}_1(t)]
\end{align*}$$

(6.10)

and at the same time, system (6.7) turns to be the canonical form of ESO in [7] or high-gain observer in [4]. In this case, the extended state observer or high-gain observer of system (6.7) is

$$\begin{align*}
\dot{\hat{z}}_1(t) &= \hat{z}_2(t) - \beta_1\omega_o[y(t) - \hat{z}_1(t)], \\
\dot{\hat{z}}_2(t) &= \dot{v}(t) - \beta_2\omega_o^2[y(t) - \hat{z}_1(t)] + u(t), \\
\dot{\hat{v}}(t) &= -\beta_3\omega_o^3[y(t) - \hat{z}_1(t)],
\end{align*}$$

(6.11)

where $\beta_1$, $\beta_2$ and $\beta_3$ are constants such that the following matrix is Hurwitz

$$\begin{bmatrix}
-\beta_1 & 1 & 0 \\
-\beta_2 & 0 & 1 \\
-\beta_3 & 0 & 0
\end{bmatrix}.$$  

(6.12)

By proper choices of $\beta_1$, $\beta_2$ and $\beta_3$, the observers (6.11) and (6.10) are equivalent under an invertible coordinate transformation. From this point, the proposed EDO with constant dynamic $G = 0$ covers the ESO as a special case and improves the ESO to the general observable linear system with input disturbance.

7 Extended dynamic observer with harmonic dynamics

This section devotes to a more general case than the constant dynamics discussed in Section 6. In most of engineering applications, the disturbance is not completely ignorant. Some prior information about the disturbance usually has been known before the observer design. When such a prior information is completely known, i.e., the disturbance dynamics $G$ is known, the observer can be designed by Theorem 4.1. When we only known a roughly prior information about the disturbance, we then need both the high-gain and the known disturbance dynamics to deal with the disturbance.

To make it more easier to use, this section shows how to choose the dynamics of disturbance by proper choice of $G$. By Remark 5.2, the steady-state error of the observer (5.7) is proportional to the error of the optimal approximation $\|(I - P_G)d\|_S$ and is inversely proportional to $\omega_o$. In order to decrease the steady-state error we should choose $G$ such that $\Omega(G)$ is as large as possible. On the other hand, the increment of the order of $G$ may lead to overshoot in the transient response due to the high-gain and the extended order of disturbance dynamics. This, in turn, makes us reduce the order of $G$ as much as possible. Hence, we need to find a trade-off between the observer accuracy and the response performance.
Suppose that we have known that \( d \in \mathbb{S} \) is a continuous periodic signal with roughly known frequencies \( \omega_j, j = 1, 2, \cdots, N \). In other words, the disturbance can be decomposed into \( d(t) = d_1(t) + d_2(t) \), where the non-constant dynamics of \( d_1(\cdot) \) are completely unknown and the dynamics of \( d_2(\cdot) \) are known, i.e.,

\[
d_2(t) = \sum_{j=0}^{N} (a_j \cos \omega_j t + b_j \sin \omega_j t),
\]

where \( a_j, b_j \in \mathbb{R}, j = 1, 2, \cdots, N \) are unknown amplitudes. By virtue of the prior information about the frequencies, we are able to choose \( g_1, g_2, \cdots, g_{2N+1} \) such that the matrix \( G \) given by (5.5) satisfies \( \sigma(G) = \{0, \pm \omega_j i \mid j = 1, 2, \cdots, N\} \). Thanks to the Vieta theorem, the choice of the parameters \( g_0, g_1, g_2, \cdots, g_{2N+1} \) is easy and implementable. Owing to (2.5), we have \( d_2 \in \Omega(G) \). By Theorem 5.1, all the negative effects of \( d_2(\cdot) \) can be eliminated and the steady-state error of observer (5.7) now is proportional to

\[
||(I - P_G) d||_\infty < ||d_1||_\infty.
\]

If some frequencies of \( \omega_j \) are large, then \( ||d||_\infty \) may be large as well. As the result, the ESO or high-gain observer may be invalid since the observer gain can not be arbitrarily large in engineering application. However, the EDO can still work well because the high frequencies disturbance has been removed completely by the extended dynamics.

The main advantage of this approach lies in that we only need rough prior information about the disturbance. All the unknown parts or the wrong prior information can be treated automatically by the high-gain. In this way, we can make use of the prior information as much as possible and at the same time, the strong robustness to the disturbance is possessed by the new proposed EDO.

To make this new methodology more understandable, we give another example to show how to utilize the prior periodic information of the disturbance. Suppose that \( d \in \mathbb{S} \) is a periodic disturbance with known period \( T \). By Fourier expansion,

\[
d(t) = \sum_{j=0}^{N} a_j \cos \frac{j\pi t}{T} + \sum_{j=N+1}^{\infty} a_j \cos \frac{j\pi t}{T} := d_1(t) + d_2(t),
\]

where \( a_j, j = 0, 1, \cdots, \) are the Fourier coefficients. Since \( \dot{d} \in L^\infty[0, \infty) \), we have

\[
\dot{d}(t) = \sum_{j=1}^{\infty} \tilde{a}_j \cos \frac{j\pi t}{T}, \quad \tilde{a}_j = a_j \frac{j\pi}{T}, \quad j = 0, 1, \cdots,
\]

which implies that \( \tilde{a}_j \to 0 \) as \( j \to \infty \). Hence, we can choose \( N \) large enough such that the remainder \( ||d_2||_\infty \) is sufficiently small. By Vieta’s theorem, we can choose \( g_0, g_1, g_2, \cdots, g_{2N} \) such that \( G \) given by (5.5) satisfies \( \sigma(G) = \{\pm \frac{j\pi}{T} i \mid j = 0, 1, 2, \cdots, N\} \). As a result, we have \( d_1 \in \Omega(G) \) and hence the steady-state error of observer (5.7) is proportional to \( ||d_2||_\infty \) that may be much smaller than \( ||\dot{d}||_\infty \). In this way, we have improved the accuracy of the observer without using the high-gain. If we have known the best \( N \)-terms approximation of the Fourier expansion, \( d_1(\cdot) \) in (7.3) can be replaced by its best \( N \)-terms approximation. In this case, we may obtain the higher
accuracy of the observer (5.7) by a smaller order $N$. Due to nonlinear characteristics of the best $N$-terms approximation [3, Section 3.8], the observer is then actually a “nonlinear observer” about the disturbance, although it is still a linear one to the control plant.

**Remark 7.1.** If we choose $g_0 = g_1 = \cdots = g_m = 0$ in (5.5), then
\[
\{a_0 t^m + a_1 t^{m-1} + \cdots + a_m t + a_m \mid a_j \in \mathbb{R}, j = 0, 1, \cdots, m, t \in \mathbb{R}\} = \Omega(G). \quad (7.5)
\]
Therefore, the EDO still works for polynomial signals or polynomial piecewise signal in some sense. Moreover, the exponential signals can still be treated by EDO. For example, if the dynamics $G$ satisfies $\sigma(G) = \{0, \lambda\}$, $\lambda > 0$ and the algebraic multiplicity of the eigenvalue $\lambda$ is $n_{\lambda}$, the signals of the type $t^{n_{\lambda}}e^{\lambda t}$ belong to $\Omega(G)$.

### 8 Feedback linearization

In this section, we discuss output feedback stabilization for system (1.1). Without loss of the generality, we suppose that $A$, $B$ and $C$ are given by (2.10) with $b_1 = 1$ and $b_2 = b_3 = \cdots = b_n = 0$. In this case, system (1.1) is always observable for $S$ due to Lemma 2.1. By Theorem 3.1, for any $d \in S$, system $\dot{x}(t) = Ax(t) + B[d(t) + u(t)]$ admits a feedback
\[
u(t) = F_{\omega_c} x(t), \quad F_{\omega_c} = \left[ f_1(\omega_c) \ f_2(\omega_c) \ \cdots \ f_n(\omega_c) \right], \quad \omega_c > 0,
\]
such that
\[
\lim_{t \to \infty} ||x(t)||_{\mathbb{R}^n} \leq \frac{M ||d||_{\infty}}{\omega_c} \quad (8.2)
\]
where $f_j \in C[0, \infty)$, $j = 1, 2, \cdots, n$ and $M$ is a positive constant which is independent of $\omega_c$.

By Theorem 5.1, $\hat{x}(\cdot)$ and $Q\hat{v}(\cdot)$ are estimations of $x(\cdot)$ and $d(\cdot)$, respectively, where $\hat{x}(\cdot)$ and $\hat{v}(\cdot)$ come from the observer (5.7). Similarly to (1.2), the output feedback stabilizing control can be designed as
\[
u(t) = -Q\hat{v}(t) + F_{\omega_c} \hat{x}(t),
\]
where the first term is used to compensate for the disturbance and the second term is the stabilizer. In view of the observer (5.7), the feedback law (8.3) leads to the closed-loop system of (1.1):
\[
\begin{cases}
\dot{x}(t) = Ax(t) + Bd(t) - BQ\hat{v}(t) + BF_{\omega_c} \hat{x}(t), \\
\dot{\hat{x}}(t) = [A + (K_{\omega_0} + SE)C]\hat{x}(t) - (K_{\omega_0} + SE)Cx(t) + BF_{\omega_c} \hat{x}(t), \\
\dot{\hat{v}}(t) = G\hat{v}(t) - EC\dot{\hat{x}}(t) + ECx(t),
\end{cases}
\]
where $G$, $E$, $K_{\omega_0}$, $S$ and $Q$ are chosen by the scheme of observer (5.7).

**Theorem 8.1.** Under the Assumption 5.1, for any $d \in S$, there exist $F_{\omega_c}$, $K_{\omega_0}$, $S$ and $Q$ such that the solution of closed-loop system (8.4) satisfies:
\[
\lim_{t \to \infty} ||x(t)||_{\mathbb{R}^n} \leq \frac{M_0 ||(I - \mathbb{P}_C)d||_{\mathbb{R}}}{\omega_c \omega_0}, \quad \forall \ t \geq 0,
\]

\[87\]
where $\mathbb{P}_G$ is defined by (2.6), $\omega_o$, $\omega_c$ are tuning gains and $M_0$ is a positive constant that is independent of $\omega_o$. Moreover, $F_{\omega_c} \in \mathbb{R}^{1 \times n}$ can be chosen by Theorem 3.1 and the observer parameters $G, E, K_{\omega_o}, S$ and $Q$ can be chosen by the scheme of parameters choice of observer (5.7).

**Proof.** Since $d \in \mathbb{S}$, it can be represented dynamically as (5.1). By Theorem 5.1, the observer (5.7) is well-posed. Define the invertible transformation

$$
\begin{bmatrix}
x \\
v \\
x \\
v
\end{bmatrix} = 
\begin{bmatrix}
I_n & 0 & 0 & 0 \\
0 & I_{m+1} & 0 & 0 \\
I_n & 0 & -I_n & 0 \\
0 & I_{m+1} & 0 & -I_{m+1}
\end{bmatrix}
\begin{bmatrix}
x \\
v \\
x \\
v
\end{bmatrix}.
$$

(8.6)

In view of (5.1), the transformation (8.6) converts the closed-loop system (8.4) into

$$
\begin{aligned}
\dot{x}(t) &= (A + BF_{\omega_e})x(t) + BF_{\omega_e}\hat{x}(t) + BQ\hat{v}(t), \\
\dot{v}(t) &= Gv(t) + \frac{B_d}{QB_d}\hat{e}(t), \\
\hat{x}(t) &= [A + (K_{\omega_o} + SE)C]\hat{x}(t) + BQ\hat{v}(t), \\
\hat{v}(t) &= G\hat{v}(t) - EC\hat{x}(t) + \frac{B_d}{QB_d}\hat{e}(t).
\end{aligned}
$$

(8.7)

By Theorem 5.1, there exists a positive constant $M_1$, independent of $\omega_o$ and $\omega_c$, such that

$$
\lim_{t \to \infty} |F_{\omega_e}\hat{x}(t) + Q\hat{v}(t)| \leq \frac{M_1\|e\|_S}{\omega_o}.
$$

(8.8)

By Theorem 3.1, (8.8) and (8.2), there exists an $M_2 > 0$ such that

$$
\|x(t)\|_{\mathbb{R}^n} \leq \frac{M_2M_1\|e\|_S}{\omega_c\omega_o},
$$

(8.9)

which leads to (8.5). \qed

**Remark 8.1.** When the input disturbance is the nonlinear dynamics of the control plant, the EDO based feedback (8.3) actually achieves the feedback linearization of nonlinear system. After canceling the the nonlinear dynamics by its estimation, the transient performance of the nonlinear system behaves like the nominal linear system.

**Remark 8.2.** The disturbance with unknown dynamics is dealt with essentially by high-gain. It is therefore necessary to consider the sensitiveness to the random measurement noise. However, the strict theoretical analysis is not an easy task. Here we only give a simple numerical analysis in Section 9. A rigorous mathematical analysis is left in our next future works. Moreover, the “peaking phenomenon” caused by high-gain and extended dynamics may take place in the transient response. This drawback should be sufficiently taken into consideration in the practice.

**Remark 8.3.** When $G$, $K$ and $P$ is given, the only tuning parameter of the observer (5.7) is $\omega_o$. Similarly, the only tuning parameter of the feedback (8.1) is $\omega_c$ provided $f_j$ is given $j = 1, 2, \cdots, n$. Therefore, the tuning parameters of the closed-loop system (8.4) can boil down to to $\omega_o$ and $\omega_c$ which are referred to as “bandwidth” of the observer and controller, respectively in ADRC [7].
9 Numerical simulations

In order to validate the developed fundamental principle visually, we present some simulations for the closed-loop system (8.4). The finite difference scheme is adopted in discretization. The numerical results are programmed in Matlab. The time step is taken as 0.0001. Suppose that the control plant is known and is given by (6.1) with \( a_1 = 2 \) and \( a_2 = 1 \). Let \((x_1(0), x_2(0)) = (0, 1)\) and let the initial state of the observer be zero. The tuning parameters are chosen as \( \omega_o = 10 \) and \( \omega_c = 10 \) and the disturbance is chosen as \( d(t) = \sin \omega t + 10 \) with \( \omega = 10 \). In contrast with the simulations in [4], the frequency of disturbance here is much larger but the tuning gain \( \omega_o \) is much smaller.

We consider three cases: a) The only prior information about the disturbance is \( d \in S \); b) There is an estimation 9.5 for \( \omega = 10 \); c) The frequency \( \omega = 10 \) is known. We choose the extended dynamics as \( G_1 = 0 \), \( \sigma(G_2) = \{0, \pm 9.5i\} \) and \( \sigma(G_3) = \{0, \pm 10i\} \), respectively. The state estimation, disturbance estimation and the controller with \( G_1 \) are are plotted in Figure 1. The counterparts for \( G_2 \) and \( G_3 \) are plotted in Figures 2 and 3, respectively. In order to look at the sensitiveness of the measurement noise, the state estimation, disturbance estimation and the controller with \( G_2 \) and corrupted measurement \( y(t) = Cx(t) + 0.01\xi(t) \) are plotted in Figure 4, where \( \xi(t) \) is the standard Gaussian noise generated by the Matlab program command “randn”.

![Figure 1](image1.png)

(a) \( x_1 \) and its estimation \( \hat{x}_1 \)

(b) \( x_2 \) and its estimation \( \hat{x}_2 \)

(c) \( d - Q\hat{v} \) and controller

Figure 1: The dynamics of disturbance is completely unknown except \( d \in S \).

![Figure 2](image2.png)

(a) \( x_1 \) and its estimation \( \hat{x}_1 \)

(b) \( x_2 \) and its estimation \( \hat{x}_2 \)

(c) \( d - Q\hat{v} \) and controller

Figure 2: The dynamics of disturbance is roughly known.

Since the observer gain \( \omega_o \) is relatively small, the error of the disturbance estimation is not very
small for the case $G_1 = 0$. However, if the have known the prior information $\sigma(G_2) = \{0, \pm 9.5i\}$, the accuracy of disturbance estimation is improved significantly. When the disturbance dynamics are completely known, the error of the disturbance estimation is convergent to zero. Moreover, Figure 4 shows that the proposed EDO and its feedback are still insensitive to the measurement noise.

Finally, we point out that the peaking phenomenon takes place when we improve more the convergent rate of the observer. This is caused by the high-gain and the order of extended dynamics. In all simulations, the output is technically chosen as $y(t) = (1 - e^{-t})C[x_1(t) x_2(t)]^\top$ to avoid the peaking phenomenon.

**10 Conclusions**

In this paper, a novel dynamics compensation approach is developed to stabilize linear systems with input disturbance. An extended dynamic observer (EDO) is designed, in terms of both the prior information and the online measurement information, to estimate both the disturbance and the system state simultaneously. The EDO takes almost all advantages from ESO and IMP. More specifically, it possesses strong robustness to the system and disturbance, as the ESO in ADRC, and at the same time, it proposes a feasible way to utilize as much the prior information of the disturbance and the control plant as possible. When there is no information about disturbance
dynamics, the EDO is reduced automatically to an extension of ESO in ADRC which has achieved great success in many engineering applications.

We just present a fundamental principle for the observer and controller design. The technical tunings such as shaping the transient response are still required in engineering applications. From the theoretical point of view, this paper gives a systematic way to utilize the prior disturbance information and the high-gain. The future works are the online computations of the disturbance dynamics.

References

[1] K.J. Astrom and B. Wittenmark, *Adaptive Control*, Addison-Wesley, 1989.

[2] G. Calafiore and M.C. Campi, The scenario approach to robust control design, *IEEE Trans. Automat. Control*, 51(2006), 742-753.

[3] O. Christensen and K.L. Christensen, *Approximation Theory From Taylor Polynomials to Wavelets*, Birkhäuser, Basel, 2004.

[4] L.B. Freidovich and H.K. Khalil, Performance recovery of feedback-linearization-based designs, *IEEE Trans. Automat. Control*, 53(2008), 2324-2334.

[5] H. Feng, X.H. Wu and B.Z. Guo, Actuator dynamics compensation in stabilization of abstract linear systems, *arXiv: 2008.11333*, https://arxiv.org/abs/2008.11333 (*as the first part of a series of studies*).

[6] H. Feng, X.H. Wu and B.Z. Guo, Dynamics compensation in observation of abstract linear systems, *arXiv: 2009.01643*, https://arxiv.org/abs/2009.01643 (*as the second part of this series of studies*).

[7] Z. Gao, Scaling and bandwith-parameterization based controller tuning, *American Control Conference*, 2003, 4989-4996.

[8] B.Z. Guo and Z.L. Zhao, On the convergence of extended state observer for nonlinear systems with uncertainty, *Systems Control Lett.*, 60(2011), 420-430.

[9] B.Z. Guo and Z.L. Zhao, *Active Disturbance Rejection Control for Nonlinear Systems: AnIntroduction*, John Wiley & Sons Inc., New York, 2016.

[10] J. Han, From PID to Active Disturbance Rejection Control, *IEEE Trans. Ind. Electron.*, 56(2009), 900-906.

[11] J. Huang, *Nonlinear Output Regulation: Theory and Applications*, SIAM, Philadelphia, 2004.

[12] H.K. Khalil, *Nonlinear Systems*, Macmillan Co., New York, 1992.
[13] V. Natarajan, D.S. Gilliam, and G. Weiss. The state feedback regulator problem for regular linear systems, *IEEE Trans. Automat. Control*, 59(2014), 2708-2723.

[14] L. Paunonen and S. Pohjolainen, The internal model principle for systems with unbounded control and observation, *SIAM J. Control Optim.*, 52(2014), 3967-4000.

[15] I.R. Petersen and C. V. Hollot, High gain observers applied to problems in the stabilization of uncertain linear systems, disturbance attenuation and $N^\infty$ optimization, *Int. J. Adapt. Control Signal Proc.*, 2(1988), 347-369.

[16] M. Rosenblum, On the operator equation $BX -XA = Q$, *Duke Math. J.*, 23(1956), 263-270.

[17] I.M. Ross, *A Primer on Pontryagin’s Principle in Optimal Control*, Ames, IA, USA: Collegiate, 2009.

[18] B. Sun and Z. Gao, A DSP-based active disturbance rejection control design for a 1-kW H-bridge DC-DC power converter, *IEEE Trans. Ind. Electron.*, 52(2005), 1271-1277.

[19] S. Shao and Z. Gao, On the conditions of exponential stability in active disturbance rejection control based on singular perturbation analysis, *Internat. J. Control*, 90(2017), 2085-2097.

[20] H. Sira-Ramírez, J. Linares-Flores, C. García-Rodríguez, and M. A. Contreras-Ordaz, On the control of the permanent magnet synchronous motor: an active disturbance rejection control-approach, *IEEE Trans. Control Syst. Technol.*, 22(2014), 2056-2063.

[21] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser, Basel, 2009.

[22] H.S. Tsien, *Engineering Cybernetics*, McGraw-Hill, New York, 1954.

[23] H.P. Whitaker, J. Yamron, and A. Kezer, Design of model-reference adaptive control systems for aircraft, Report R-164, *Instrumental Laboratory, Massachusetts Institute of Technology*, 1958.

[24] Y.Q. Xia and M.Y. Fu, *Compound Control Methodology for Flight Vehicles*, Springer-Verlag, Berlin, 2013.

[25] W.C. Xue, W.Y. Bai, S. Yang, K. Song, Y. Huang, and H. Xie, ADRC with adaptive extended state observer and its application to air-fuel ratio control in gasoline engines, *IEEE Trans. Ind. Electron.*, 62(2015), 5847-5857.

[26] Y. Xiong and M. Saif, Sliding mode observer for nonlinear uncertain systems, *IEEE Trans. Autom. Control*, 46(2001), 2012-2017.

[27] G. N. Zhang, Z. Liu, S. Yao, Y. Liao and C. Xiang, Suppression of low-frequency oscillation in traction network of high-speed railway based on auto-disturbance rejection control, *IEEE Trans. Transp. Electr.*, 2(2016), 244-255.