QUADRICS VIA SEMIGROUPS

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Abstract. Let $M_2$ be the semigroup of linear endomorphisms of a plane. We show that the space of idempotents in $M_2$ is a hyperboloid of one sheet, the set of semigroup-theoretic inverses of a nonzero singular element in $M_2$ is a hyperbolic paraboloid, and the set of nilpotent elements in $M_2$ is a right circular cone.

This is the story of the rediscovery of classical three-dimensional geometry, especially the geometry of quadric surfaces, while studying the semigroup $M_2(\mathbb{R})$ of linear endomorphisms of a real plane. One of the surfaces that appears prominently in this context is the hyperboloid of one sheet, referred to as spaghetti bundle in [8]. In this story the spaghetti presents itself as the set of idempotents in $M_2(\mathbb{R})$, the cone emerges as the set of nilpotent elements and the hyperbolic paraboloid as the set of semigroup-theoretic inverses of a singular element.

This rediscovery was briefly announced in [5]. Generalizations of some of the ideas presented here to semigroups of linear endomorphisms of higher dimensional vector spaces are discussed in [6]. The little bit of semigroup theory quoted below is based on [3].

1. The Semigroup $M_2(\mathbb{R})$

A set $S$ together with an associative binary operation in $S$ is called a semigroup. An element $e$ in $S$ is called an idempotent if $e^2 = e$. If $X \subseteq S$, the set of idempotents in $X$ is denoted by $E(X)$. In any semigroup $S$ we can define certain equivalence relations, denoted by $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{D}$, and $\mathcal{H}$, and called Green’s relations. Let $S^1$ denote $S$ if $S$ has an identity element. Otherwise, let it denote $S$ with an identity element 1 adjoined. For $a, b \in S$ the first three are defined by

$$a \mathcal{L} b \iff aS^1 = bS^1$$

$$a \mathcal{R} b \iff S^1a = S^1b$$

$$a \mathcal{J} b \iff S^1aS^1 = S^1bS^1$$

and the remaining ones by $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

If $a \in S$, the set of all elements in $S$ which are $\mathcal{L}$-equivalent to $a$ is denoted by $L_a$ and is called the $\mathcal{L}$-class containing $a$. The notations $R_a, J_a, D_a, H_a$ have similar meanings. These are the Green classes in $S$. Two elements $a, a'$ in a semigroup $S$ are called inverse elements if $aa'a = a, a'aa' = a'$. A semigroup in which every element has an inverse is called a regular semigroup.

A well-known example of a regular semigroup is $M_n(\mathbb{K})$ (where $\mathbb{K} = \mathbb{R}$, or $\mathbb{K} = \mathbb{C}$) of linear endomorphisms of an $n$-dimensional vector space $V$ over $\mathbb{K}$ under
composition of mappings. The set $S_n$ of singular elements in $M_n(\mathbb{K})$ is a regular subsemigroup of $M_n(\mathbb{K})$. Treating functions as right operators we see that, for $a, b \in M_n(\mathbb{K})$, $a \mathcal{L} b$ if and only if $a$ and $b$ have the same range, and, $a \mathcal{D} b$ if and only if $a$ and $b$ have the same null space. Further, $a \mathcal{D} b$ if and only if $a$ and $b$ have the same rank, and, $a \mathcal{J} b$ is equivalent to $a \mathcal{D} b$.

As already indicated, the semigroup having special interest to us is $M_2(\mathbb{R})$, denoted by $M_2$ in the sequel. We shall represent elements of $M_2$ as square matrices of order 2 relative to some fixed ordered orthonormal basis for $V$. Listing out the entries in the elements of $M_2$ row-wise we get vectors in $\mathbb{R}^4$. In this way we may identify $M_2$ with $\mathbb{R}^4$. The usual inner product in $\mathbb{R}^4$ can be represented using the trace function, defined by $\text{tr}(x) = x_1 + x_4$. If $x = (x_1, \ldots, x_4)$ and $y = (y_1, \ldots, y_4)$ then

$$\langle x, y \rangle = x_1y_1 + \cdots + x_4y_4 = \text{tr}(x^*y).$$

where $x^*$ is the transpose of $x$.

2. Geometry of the Green Classes in $M_2$

$M_2$ has three $\mathcal{D}$-classes, namely, $D_0$, $D_1$ and $D_2$, where $D_k$ is the set of endomorphisms of rank $k$. Obviously we have $D_0 = \{0\}$ which is simply a point. From the well-known fact (p.168 [1]) that the space $M(m, n, k)$ of $m \times n$ matrices of rank $k$ is a manifold of dimension $k(m + n - k)$ we immediately deduce that $D_1$ is a three-dimensional submanifold of $\mathbb{R}^4$. What this means is that sufficiently small neighborhoods of every point in $D_1$ ‘looks like’ a three-dimensional euclidean space. Lastly $D_2$ is the set $\text{GL}(2)$ of all invertible elements in $M_2$. It is well known that $\text{GL}(2)$ is a four-dimensional submanifold of $\mathbb{R}^4$.

To describe the $\mathcal{L}$- and $\mathcal{R}$-classes in $M_2$, we require some geometric terminology. A line in a linear space $U$ is an affine subspace of $U$ generated by two distinct points (that is, vectors) in $U$ and a plane $\mathcal{P}$ in $U$ is an affine subspace generated by three non-collinear points in $U$. If $\mathcal{P}$ passes through the origin in $U$ then the set $\mathcal{P} \setminus \{0\}$ is called a punctured plane in $U$. In a similar way we may define a punctured line in $U$.

If $0 = a \in M_2$, then $L_a = R_a = \{0\}$. Also, if $a \in \text{GL}(2)$ then $L_a = R_a = \text{GL}(2)$. The classes $L_a$ and $R_a$, when $a \neq 0$ and $a \notin \text{GL}(2)$, are the nontrivial $\mathcal{L}$- and $\mathcal{R}$-classes in $M_2$. If $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, a simple argument involving range and null space shows that

$$L_e = \left\{ \begin{bmatrix} \alpha & 0 \\ \beta & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{R}, (\alpha, \beta) \neq (0, 0) \right\},$$

$$R_e = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{R}, (\alpha, \beta) \neq (0, 0) \right\}.$$

These are obviously punctured planes. That this is true for any $0 \neq a \in S_2$ can be easily verified.

**Proposition 1.** If $0 \neq a \in S_2$, then $L_a$ and $R_a$ are punctured planes lying in $S_2$.

Surprisingly, the converse of Proposition [1] is also true, that is, every punctured plane in $S_2$ is a nontrivial $\mathcal{L}$-class or $\mathcal{R}$-class in $S_2$. A nontrivial $\mathcal{H}$-class, being the intersection of a nontrivial $\mathcal{L}$-class and a nontrivial $\mathcal{R}$-class, is a punctured line in $S_2$. 

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3. Intersection of $S_2$ with $x_1 + x_4 = \lambda$

We have seen that $D_1$ is a three-dimensional manifold sitting in $\mathbb{R}^4$. To know more about this manifold we consider intersections of $S_2$ with hyperplanes in $M_2$. A hyperplane in $M_2$ is the set of all $x = [x_1, x_2, x_3, x_4]$ satisfying an equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = \lambda$$

where $a = \left[ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \right] \in M_2$. This equation can be expressed in the form $\text{tr}(ax) = \lambda$.

We denote this hyperplane by $P(I; \lambda)$ and its intersection with $S_2$ by $SP(a; \lambda)$. We first consider the special case $SP(I; \lambda)$ where $I = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]$. 

**Proposition 2.** If $\lambda \neq 0$ then $SP(I; \lambda)$ is a hyperboloid of one sheet. Also, $SP(I; 0)$ is a right circular cone with vertex at the origin in $M_2$.

**Proof.** Consider the following points in $P(I; \lambda)$:

$$O'(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}), \ A(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0), \ B(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \ C(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$$

Using the inner product and the induced norm in $\mathbb{R}^4$, we can see that the vectors $\overrightarrow{O'A}$, $\overrightarrow{O'B}$, $\overrightarrow{O'C}$ form an orthonormal system in $P(I; \lambda)$. We choose $O'$ as the origin and the directed lines joining $O'$ to the points $A, B, C$ as the coordinate axes. We shall refer to the coordinates of a point in $P(I; \lambda)$ relative to these axes as the $\text{Bell-coordinates}$ of the point.

Let the Bell-coordinates of any point $Q(x_1, x_2, x_3, x_4)$ in $P(I; \lambda)$ be $(X, Y, Z)$ so that we have

$$\overrightarrow{O'P} = X\overrightarrow{O'A} + Y\overrightarrow{O'B} + Z\overrightarrow{O'C}.$$ 

Since $\overrightarrow{O'I} = \overrightarrow{O'O}$, etc., we must have

\begin{equation}
\begin{align*}
x_1 &= \frac{1}{\sqrt{2}} + \frac{X}{\sqrt{2}}, \\
x_2 &= \frac{Y}{\sqrt{2}}, \\
x_3 &= \frac{Z}{\sqrt{2}}, \\
x_4 &= \frac{1}{\sqrt{2}} - \frac{Z}{\sqrt{2}}.
\end{align*}
\end{equation}

If we substitute the above expressions in the equation defining the space $S_2$, namely, $x_1x_4 - x_2x_3 = 0$, we get

\begin{equation}
X^2 + Y^2 - Z^2 = \frac{\lambda^2}{2}.
\end{equation}

If we consider any set of numbers $(X, Y, Z)$ satisfying Eq. (2), then using the relations in Eq. (1), we can see that $(X, Y, Z)$ are the Bell-coordinates of a point on $SP(I; \lambda)$. Therefore Eq. (2) is the equation of $SP(I; \lambda)$ relative to the Bell-axes. When $\lambda \neq 0$, this equation represents a hyperboloid of one sheet ($\S 64 \[2\]$).

When $\lambda = 0$, Eq. (2) reduces to $X^2 + Y^2 - Z^2 = 0$, which represents a right circular cone with vertex at the origin and semi-vertical angle $\frac{\pi}{4}$. Note that this cone lies in the hyperplane $P(I; 0)$.

By direct computation or otherwise one can easily see that $SP(I; 0)$ is the set of nilpotent elements in $M_2$.

We now explore the relations between the geometrical properties of $SP(I; \lambda)$ and the algebraic properties of $M_2$. Obviously the center of $SP(I; \lambda)$ is $O'$ which is a point on the line joining $O = [0 \ 0 \ 0 \ 0]$ and $I = [1 \ 0 \ 0 \ 0]$. Since this line is independent of the choice of $\lambda$, the centers of the hyperboloids $SP(I; \lambda)$, for various values of $\lambda$, all lie on a fixed line. The axis of rotation of the hyperboloid is given by $X = 0$, $Y = 0$. These equations produce a matrix of the form

\[
\begin{bmatrix}
\frac{1}{2} & x_2 \\
-x_2 & \frac{1}{2}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{bmatrix} + \begin{bmatrix}
0 & x_2 \\
-x_2 & 0
\end{bmatrix}.
\]
The set of all such elements form a line through the center of the hyperboloid parallel to the line formed by the set of skew-symmetric elements in $M_2$. The asymptotic cone of $SP(I; \lambda)$ from its center ($\S 78 [2]$) is $X^2 + Y^2 - Z^2 = 0$, which is independent of $\lambda$.

From the above discussion we can construct an image of $S_2$. If we imagine $\lambda$ as ‘time’, $S_2$ can be thought of as a space generated by the ‘moving’ hyperboloid $SP(I; \lambda)$. Initially, that is when $\lambda = 0$, we have a degenerate hyperboloid, namely, the right circular cone $SP(I; 0)$ with vertex at $O$ and axis the line $\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \alpha \in \mathbb{R} \}$. As time advances numerically, the center of the hyperboloid moves along the line $\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \alpha \in \mathbb{R} \}$. The axis of rotation of the moving hyperboloid is a line through the center parallel to the axis of the ‘initial’ hyperboloid. Further, at time $\lambda$, the radius of the principal circular section of the hyperboloid is $\sqrt{\lambda}$. Clearly, this increases as $\lambda$ increases numerically. Thus, the hyperboloid $SP(I; \lambda)$ expands simultaneously as it moves away from the origin.

4. The Space of Idempotents

Let $E(k)$ be the set of idempotents of rank $k$. Obviously, $E(0) = \{0\}$ and $E(2) = \{I\}$. Also, it is easy to see that $E(1) = SP(I; 1)$. The next result follows from this.

Proposition 3. The space $E(1)$ of idempotents of rank 1 in $M_2$ is a hyperboloid of one sheet.

The generators of $E(1)$ through $e$ turn out to be the sets $E(L_e)$ and $E(R_e)$. To see this it is enough to show that these sets are lines through $e$, for if a line lies wholly on a conicoid it must belong to a system of generating lines of the conicoid ($\S 97 [2]$). That $E(L_e)$ and $E(R_e)$ are lines follows from

$$E(L_e) = e + (1 - e)S_2e \quad E(R_e) = e + eS_2(1 - e),$$

which can be verified by direct computation.

Having found the generating lines on $E(1)$, we next determine how these are organized into two systems of generators as in [2]. Let $L_1$ be the family of all lines of the form $E(L_e)$ and $L_2$, the family of all lines of the form $E(R_e)$. Since $V$ is 2-dimensional, no two distinct members of $L_1$ (or, of $L_2$) intersect. Also, every member of $L_1$ intersects every member (except one) of $L_2$, and vice versa. Thus $L_1$ and $L_2$ are the two families of generating lines of $E(1)$. Though it is not an intrinsic property of $E(1)$, it is interesting to see that members of $L_1$ are members of the $\lambda$-system and those of $L_2$ are members of the $\mu$-system of generators of $E(1)$ (in the terminology of [2]).

The equation of the plane of the principal circular section of $SP(I; 1)$ is $Z = 0$, which is equivalent to $x_2 = x_3$, which, in turn, is equivalent to the statement that the element $x \in M_2$ is symmetric. Thus the principal circular section of $E(1)$ consists of the symmetric elements in $E(1)$.

5. Intersections of $S_2$
with Arbitrary Hyperplanes

The first thing we notice is that the coordinates $(X, Y, Z)$ of any point on $SP(a; \lambda)$ relative to some rectangular Cartesian axes in $P(a; \lambda)$ satisfy a second degree equation. Hence $SP(a; \lambda)$ is a quadric surface in $P(a; \lambda)$. The nature of this
surface depends, naturally, on \( a \) and \( \lambda \). If \( \lambda \neq 0 \), then dividing the equation of the plane \( P(a; \lambda)\) by \( \lambda \), we see that the planes \( P(a; \lambda) \) and \( P(\frac{1}{\lambda} a; 1) \) coincide. Hence we need consider only the special surface \( SP(a;1) \).

Let \( a \) be nonsingular. If \( x \in SP(a;1) \) then \( \text{tr}(ax) = 1 \) and so \( ax \in E(1) \). Hence \( x \in a^{-1}E(1) \). The converse is obvious and so \( SP(a;1) = a^{-1}E(1) \). Since \( E(1) \) is a hyperboloid of one sheet and since the map \( x \mapsto a^{-1}x \) is an affine map of \( P(I;1) \) onto \( P(a;1) \), \( SP(a;1) \) must be a hyperboloid of one sheet.

**Proposition 4.** If \( a \) is nonsingular \( SP(a;1) \) is a hyperboloid of one sheet.

Any regular semigroup \( S \) is equipped with a natural partial order \([7]\) defined by

\[
x \leq y \quad \text{if and only if} \quad R_x \leq R_y \text{ and } x = fy \text{ for some } f \in E(R_x).
\]

If \( a \in M_2 \) is nonsingular then, it can be shown that \( SP(a;1) \) is the set of nonzero elements in \( S_2 \) which are less than \( a \) under the natural partial order in \( M_2 \).

Let \( a \in S_2 \). If \( x \in S_2 \) is an inverse of \( a \) then we have \( axa = a \) and so \( ax \in E(1) \) which implies that \( x \in SP(a;1) \). The converse also holds. Thus, if \( 0 \neq a \in S_2 \), then \( SP(a;1) \) is the set of inverses of \( a \). To understand the geometry of \( SP(a;1) \), we consider the special case \( e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). Clearly, \( x \in P(e;1) \) if and only if \( x_1 = 1 \). Let \( O', A, B, C \) denote \( \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \) respectively. Then the vectors \( O'A, O'B, O'C \) form an orthonormal basis in \( P(e;1) \). If \( Q \) denotes \( \begin{bmatrix} 1 & x_2 & x_3 & x_4 \end{bmatrix} \) then

\[
O'Q = x_2O'A + x_3O'B + x_4O'C
\]

so that the coordinates \((X, Y, Z)\) of \( Q \) relative to this system of axes are \( X = x_2, Y = x_3, Z = x_4 \). Now \( x \) is in \( P(e;1) \) if and only if \( x_1 = 1 \) and \( x_4 = x_2x_3 \) and so \( SP(e;1) \) is determined by the cartesian equation \( XY = Z \) which is the equation of a hyperbolic paraboloid. This argument can be modified for arbitrary \( a \in S_2 \) to yield the following result.

**Proposition 5.** Let \( a \) be singular. Then \( SP(a;1) \) is the set of inverses of \( a \) and is a hyperbolic paraboloid.

We next consider intersections of \( S_2 \) with hyperplanes passing through the origin. These are sets of the form \( SP(a;0) \). If \( a \) is nonsingular then \( SP(a;0) = a^{-1}SP(I;0) \). Since \( SP(I;0) \) is a cone with vertex at the origin, \( SP(a;0) \) is also a cone. If \( a \) is singular and nonzero then using some technical arguments, we can show that \( SP(a;0) \setminus \{0\} \) is the union of an \( \mathcal{F} \)-class and an \( \mathcal{R} \)-class in \( S_2 \), that is, a union of two punctured planes.

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