The Dark side of the torsion: Dark Energy from kinetic torsion

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An extension to the Einstein-Cartan (EC) action is discussed in terms of cosmological solutions. The torsion incorporated in the EC Lagrangian is assumed to be totally anti-symmetric, and written by of a vector $S^\mu$. Then this torsion model, compliant with the Cosmological Principle, is made dynamical by introducing its quadratic, totally anti-symmetric derivative. The EC Lagrangian then splits up into the Einstein-Hilbert portion and a (mass) term $\frac{1}{m^2} S^\mu S^\mu$, plays the role of dark energy. The quadratic torsion term, on the other hand, gives rise to a stiff fluid that leads to a bouncing solution. A bound on the bouncing solution is calculated.

The action integral $S = \int \sqrt{-g} (\mathcal{L}_G + \mathcal{L}_m) d^4 x$ is based in this ansatz on the gravity Lagrangian

$$\mathcal{L}_G = \frac{1}{2} R(\Gamma) - \frac{4!}{2m^2} \partial_{[\mu} \bar{K}_{\nu\alpha\beta]} \partial^{[\mu} \bar{K}^{\nu\alpha\beta]},$$

that extends the Einstein-Hilbert term with a dynamical torsion. $\mathcal{L}_G$ is the gravitational Lagrangian and $\mathcal{L}_m$ is the matter fields Lagrangian. Here $c = \hbar = 8\pi G = 1$, $m_\mu$ is a constant with the dimension of mass, $g$ is the determinant of the metric, and $R(\Gamma)$ is the Ricci scalar. The metric signature is -2. The contortion tensor,$$

K_{\mu\nu\sigma} = \frac{1}{2} (T_{\mu\nu\sigma} + T_{\nu\sigma\mu} - T_{\sigma\mu\nu}),$$

becomes identical with the torsion tensor if the latter is totally anti-symmetric which we assume in the following.

The Ricci scalar then splits up into the Levi-Civita dependent part $\bar{R}$ and the torsional part:

$$R(\Gamma) = \bar{R} + g^{\mu\nu} (K_{\sigma\lambda}^\lambda K^\sigma_{\mu\nu} - K^\sigma_{\beta\nu} K_{\mu\sigma} + \nabla_\sigma K^\sigma_{\mu\nu} - \nabla_\nu K^\lambda_{\mu\lambda})$$

$$= \bar{R} + T^\sigma_{\beta\nu} T^{\beta\sigma}$$

Almost all torsion dependent terms on the r.h.s. vanish since the torsion is taken to be totally anti-symmetric. With this ansatz for torsion applied in the action (1), the torsion acquires a mass term from the Ricci scalar, and retains the quadratic anti-symmetric derivative kinetic term:

$$\mathcal{L}_G = \frac{1}{2} (\bar{R} + T^\sigma_{\beta\nu} T^{\beta\sigma}) - \frac{4!}{2m^2} \partial_{[\mu} T_{\nu\alpha\beta]} \partial^{[\mu} T^{\nu\alpha\beta]}.$$
The metric-affine formulation requires independent variations of the action (1) with respect to the connection and the metric. The symmetric part of the variation of the action (1), w.r.t. the connection gives the metric compatibility:

$$\Gamma^\rho_{(\mu \nu)} = \left\{ \begin{array}{ll} \rho & \mu \nu \\ \frac{1}{2} g^{\rho \lambda} (g_{\lambda \mu, \nu} + g_{\lambda \nu, \mu} - g_{\mu \nu, \lambda}). \end{array} \right.$$ (4)

The symmetric part of the connection is the Levi-Civita symbol. The anti-symmetric part of the variation w.r.t. the metric gives:

$$\Theta_{\mu \nu} = G_{\mu \nu} - T^\rho_{\mu \lambda} T_{\rho \nu}^\lambda + \frac{1}{2} g_{\mu \nu} \left( \frac{4!}{m^2} \frac{1}{\sqrt{-g}} \partial_{\alpha} T^{\alpha \gamma \delta} \right) + g^{\alpha \beta} T_{\rho \lambda} T_{\rho \alpha} = \frac{1}{\sqrt{-g}} \partial_{\alpha} T^{\alpha \beta} + g^{\alpha \beta} T_{\rho \lambda} T_{\rho \alpha}.$$ (5)

The variation w.r.t. the metric yields in addition the modified field equation of gravity:

$$\Theta_{\mu \nu} = \frac{1}{\sqrt{\bar{g}}} \partial_{\alpha} \left( \bar{g}^{\alpha \beta} T_{\rho \lambda} T_{\rho \alpha} \right).$$ (6)

$$\Theta_{\mu \nu} = G_{\mu \nu} - T^\rho_{\mu \lambda} T_{\rho \nu}^\lambda + \frac{1}{2} g_{\mu \nu} \left( \frac{4!}{m^2} \frac{1}{\sqrt{-g}} \partial_{\alpha} T^{\alpha \gamma \delta} \right) + g^{\alpha \beta} T_{\rho \lambda} T_{\rho \alpha}.$$ (7)

where $\bar{S}^\alpha = \sqrt{-g} S^\alpha$ is the vector density of weight 1. The torsion mass term in the Lagrangian is then written as:

$$T_{\mu \alpha \beta} = S_{\alpha} S^\sigma.$$ (8)

IV. SYMMETRIES

Notice that the kinetic term of the torsion tensor is invariant under the "gauge transformation"

$$T_{\mu \alpha \beta} \rightarrow T_{\mu \alpha \beta} + \partial_{[\mu} \Lambda_{\alpha \beta]}.$$ (9)

where $\Lambda_{\alpha \beta}$ is an arbitrary antisymmetric tensor. The symmetry holds since

$$\partial_{[\mu} \partial_{\nu] \Lambda_{\alpha \beta]} = \partial_{[\mu} \partial_{\nu] \Lambda_{\alpha \beta]} = 0,$$

but it is broken by the mass term in the Lagrangian. Respectively, the gauge symmetry of the term $(S_{\mu}^\alpha)^2$ is

$$S^\mu \rightarrow S^\mu + \frac{1}{3!} \epsilon^{\mu \nu \alpha \beta} \partial_{\nu} \Lambda_{\alpha \beta}.$$ (10)

Both symmetries are broken by the mass term in the action. The kinetic term of the vector field in this action differs from that of the Proca field, $F_{\mu \nu} F^{\mu \nu}$ [41, 42]. The Proca kinetic term does not contribute any density for a homogeneous cosmological solution, though.

V. HOMOGENEOUS SOLUTION

In this section we will show how a homogeneous and isotropic ansatz for the torsion gives rise to dark energy via the kinetic term $(S_{\mu}^\alpha)^2$, and to a bouncing cosmology arising from the mass term in the action. That ansatz, $S^\mu = (A(t), 0, 0, 0)$, restricting the torsion vector to its time-like direction, is compliant with the Copernican principle underlying the FLRW metric. Variation w.r.t. the metric gives the two Friedman equations (see Appendix B),

$$\rho = \frac{1}{2 m^2} \frac{A^2}{a^6} - \frac{1}{2} \frac{A^2}{a^4} + \rho_m$$ (12a)

$$p = -\frac{1}{2 m^2} \frac{A^2}{a^6} - \frac{1}{2} \frac{A^2}{a^4} + p_m$$ (12b)

with respectively density and pressure. The kinetic part has manifestly the equation of state $w = -1$, whereas the mass term behaves like a stiff fluid with $w = 1$. These density and pressure terms are different from the quintessence model [43–45] where the kinetic term has an equation of state $w = 1$ and the potential term has an equations of state of $w = -1$. The variation w.r.t. the vector field gives:

$$m^2 A + 3 H \dot{A} = \dot{A},$$ (13)

which is equivalent to the conservation of the energy momentum tensor.

The complete evolution of the Universe with the dynamical torsion modification is obtained via a dynamical...
system. For the dimensionless densities of the dark energy and stiff fluids we get respectively:

$$\Omega_\Lambda = \frac{A^2}{2m^2_\mu a^6H_0^2}, \quad \Omega_\phi = -\frac{A^2}{2a^6H_0^2},$$

(14)

with the re-defined Hubble parameter:

$$\left(\frac{H(z)}{H_0}\right)^2 = \Omega_\Lambda(z) + \Omega_\phi(z) + \Omega_m(1+z)^3 + \Omega_r(1+z)^4.$$  

(15)

$\Omega_m$ is the matter energy density and $\Omega_r$ is the radiation energy density. $z$ is the redshift, $H_0$ is the Hubble parameter in the late Universe and $H(z)$ is the evaluated Hubble parameter. The dynamics of the torsion related fluids can, after a lengthy calculation, be re-written as an autonomous system,

$$\dot{\Omega}_\Lambda = 2\mu\sqrt{-\Omega_\Lambda\Omega_\phi}, \quad \dot{\Omega}_\phi = -6H\Omega_\phi + 2m_\mu\sqrt{-\Omega_\Lambda\Omega_\phi},$$

(16)

where dot denotes time derivative. When both derivatives are zero, the solution shows the asymptotic limit of the evolution. The solution gives: $\Omega_\phi = 0$ and $\Omega_\Lambda = \text{Const}$ which is a stable attractor, i.e. a dark energy dominated Universe.

Notice that for the limit $\mu^2 \to 0$, the kinetic dark energy part in Eqs. (12) dominates. Eq. (13) gives $A \sim a^3$ and the dark energy solution becomes

$$\rho = \frac{\Omega_\Lambda}{2m^2_\mu}H_0^2$$

$$p = -\rho,$$

corresponding to a dynamical Cosmological Constant.

In order to check the viability of the model with the additional stiff fluid, it is natural to use the Big Bang Nucleosynthesis (BBN) constraint that reads $\Delta H^2/H_0^2\text{CDM} < 0.1$ for the BBN epoch ($z \sim 10^9$). Regarding the problem of likelihood maximization, we use an affine-invariant Markov Chain Monte Carlo sampler [46], as it is implemented within the open-source package Polychord [47] with the GetDist package [48] to present the results. The prior we choose is with a uniform distribution, where $\Omega_m \in [0; 1]$, $\Omega_\Lambda \in [0; 1 - \Omega_m]$, $\Omega_\phi \in [-1, 0]$, $m_\mu/H_0 \in [0; 1]$.

The posterior distribution is presented in Fig. 1 with the table underneath. The energy density of the stiff fluid is constraint to be around $10^{-3}$ in the late Universe. The evolution to the past gives much lower energy density that preserves the BBN constraint.

VI. DISCUSSION

This letter extends the EC gravity by a totally anti-symmetric derivative of the contortion tensor. Assuming the torsion tensor to be totally anti-symmetric, that term generates a kinetic term and treats the torsion as an additional degree of freedom which can be expressed as a massive vector field. Its mass comes from the EC curvature scalar that splits up into the Ricci tensor of GR and a mass term $\sim S^\mu S_\mu$.

Of course, further modifications of the mass term by adding a potential $V(K_{\alpha\beta\gamma}K^{\alpha\beta\gamma})$ to the action (9) are possible. This generates a dynamical dark energy [49–51], similarly to the three-form cosmology model [52–54]. The potential

$$V(K_{\alpha\beta\gamma}K^{\alpha\beta\gamma}) = -\frac{1}{2}K_{\alpha\beta\gamma}K^{\alpha\beta\gamma},$$

for example, cancels the bouncing term and gives a Cosmological Constant that emerges from torsion. The total action in that case reads:

$$\mathcal{L}_G = \frac{1}{2}\mathcal{R}(\Gamma) - \frac{4!}{2m^2_\mu} \partial_\mu K^{\nu\alpha\beta}\partial^{[\mu}K^{\nu\alpha\beta]} - \frac{1}{2}K_{\alpha\beta\gamma}K^{\alpha\beta\gamma}$$

$$= \frac{1}{2}\mathcal{R} - \frac{4!}{2m^2_\mu} \partial_\mu K^{\nu\alpha\beta}\partial^{[\mu}K^{\nu\alpha\beta]}.$$  

(17)

This action is equivalent to the Cosmological Constant action, but it is invariant under the gauge symmetry (10). Possible extensions also address a scalar field $\phi$ that couples to the EC term or the torsion terms.

It is important to mention that the kinetic contortion term can emerge from higher curvature theories. The
Riemann-Cartan tensor can be split into the curvature part and into the torsional part via:

\[ R_{\lambda\sigma\mu\nu} + \nabla_{\mu} K_{\lambda\sigma\nu} - \nabla_{\nu} K_{\lambda\sigma\mu} - K_{\lambda\beta\nu} K_{\sigma\mu}^{\beta} + K_{\lambda\beta\mu} K_{\sigma\nu}^{\beta}, \]

The terms \( \partial K \) are part of the Riemann-Cartan tensor. So it is possible to obtain the term and combinations thereof (as well as additional couplings torsion-curvature) from quadratic gravity theories with \( R_{\mu\nu} R^{\mu\nu} \) and/or \( R^{\mu\nu} \alpha R_{\mu\nu} \). Gauge theories of gravity predict these terms naturally [14, 55–61].

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**Appendix A: Identity proof**

We follow here the conventions for the totally antisymmetric symbol with indexes down \( \epsilon_{\mu\nu\alpha\beta} \), or the totally antisymmetric symbol with indexes up \( \epsilon^{\mu\nu\alpha\beta} \) according to the definitions in the book by Anderson [62]. In this case, both of these objects equal 1 for any even permutation of 0123, –1 for any odd permutation of 0123 and zero in the case any index is repeated. For these objects to have the same value after a generic coordinate transformation \( \epsilon_{\mu\nu\alpha\beta} \) has to be considered a covariant tensor density with weight -1, while \( \epsilon^{\mu\nu\alpha\beta} \) has to be considered a contravariant tensor density with weight 1.

Then, since \( T_{[\mu\nu\alpha\beta]} \) is totally anti-symmetric tensor it has only one independent component, say the 0123 component, exactly as \( \epsilon_{\mu\nu\alpha\beta} \), therefore the following relation holds:

\[ T_{[\mu\nu\alpha\beta]} = \epsilon_{\mu\nu\alpha\beta} X \]  

(A1)

where \( X \) is a scalar density.

Contracting both sides of the above equation by \( \epsilon^{\mu\nu\alpha\beta} \) and using that \( \epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu\alpha\beta} = 4! \) we obtain:

\[ \epsilon^{\mu\nu\alpha\beta} T_{[\mu\nu\alpha\beta]} = 4! X, \]  

(A2)

and the solution for \( X \) is then,

\[ X = \frac{1}{4!} \epsilon^{\mu\nu\alpha\beta} T_{[\mu\nu\alpha\beta]}. \]  

(A3)

We can now also relate \( F^{[\mu\nu\alpha\beta]} \) to \( X \),

\[ T^{[\mu\nu\alpha\beta]} = g^{\mu\theta} g^{\nu\kappa} g^{\alpha\gamma} g^{\beta\delta} T_{[\theta\kappa\gamma\delta]} \]

\[ = g^{\mu\theta} g^{\nu\kappa} g^{\alpha\gamma} g^{\beta\delta} \epsilon_{\theta\kappa\gamma\delta} X = \frac{1}{g} \epsilon^{\mu\nu\alpha\beta} X. \]  

(A4)

The kinetic term then reads:

\[ T_{[\mu\nu\alpha\beta]} T^{[\mu\nu\alpha\beta]} = 4! \frac{1}{g} X^2. \]  

(A5)

Inserting Eq. (A3) finally gives:

\[ 4! \cdot T_{[\mu\nu\alpha\beta]} T^{[\mu\nu\alpha\beta]} = \frac{1}{g} (\epsilon^{\mu\nu\alpha\beta} T_{\mu\nu\alpha\beta})^2. \]  

(A6)

The identity is originally used in [63].

**Appendix B: Friedmann equations**

The action in the minisuperspace is reduces to the form:

\[ L_{MSS} = -\frac{3a^2\dot{a}}{n} + \frac{3a^2s^2\dot{s}n}{m^2n^3} + \frac{3a^2\dot{a}n}{n^2} - \frac{3a^2s_0\dot{s}_0}{m^2n^3} - \frac{9a^2s_0^2}{2m^2n^3} - \frac{3a^2}{n} + \frac{a^3s_0^3\dot{s}_0}{m^2n^4} - \frac{a^3s_0^3\dot{n}^2}{m^2n^5} - \frac{a^3s_0^2\dot{s}_0^2}{m^2n^3} - \frac{a^3s_0^2}{2n}, \]

using the Laplace function \( n \) and the scale factor \( a \). The variations w.r.t. \( n \) and \( a \) give:

\[ \rho = -2s_0\left(3s_0\dot{H} + \dot{s}_0\right) + 9H^2s_0^2 + s_0^2 - \frac{s_0^2}{2} + \rho_m, \]  

(B1a)

\[ p = -2s_0\left(3s_0\dot{H} + \dot{s}_0\right) + 12Hs_0s_0 + 9H^2s_0 + s_0^2 - \frac{s_0^2}{2} + p_m, \]  

(B1b)

with the gauge \( n = 1 \). The variation w.r.t. the vector field \( s_0 \) gives:

\[ 3H\dot{s}_0 + \dot{s}_0 = s_0\left(m\mu^2 - 3H\right). \]  

(B2)

With the parameter \( A = a^3s_0 \) the density pressure and the vector field variation yields:

\[ \rho = -\frac{2A\ddot{A} + 6AH\dot{A} + A^2}{2m^2a^6} - \frac{A^2}{2a^6} + \rho_m, \]  

(B3a)

\[ p = -\frac{A^2 + 2A\left(\ddot{A} - 3H\dot{A}\right)}{2m^2a^6} - \frac{A^2}{2a^6} + p_m, \]  

(B3b)

\[ \ddot{A} = 3H\dot{A} + m^2a^6A = 0. \]  

(B3c)

Inserting the vector field variation into the density and the pressure terms gives (12a) and (12b).
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