A COMMON FIXED POINT THEOREM FOR
A COMMUTING FAMILY OF WEAK∗ CONTINUOUS
NONEXPANSIVE MAPPINGS

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Abstract. It is shown that if \( S \) is a commuting family of weak∗ continuous nonexpansive mappings acting on a weak∗ compact convex subset \( C \) of the dual Banach space \( E \), then the set of common fixed points of \( S \) is a nonempty nonexpansive retract of \( C \). This partially solves a long-standing open problem in metric fixed point theory in the case of commutative semigroups.

1. Introduction

A subset \( C \) of a Banach space \( E \) is said to have the fixed point property if every nonexpansive mapping \( T : C \to C \) (i.e., \( \|Tx - Ty\| \leq \|x - y\| \) for \( x, y \in C \)) has a fixed point. A general problem, initiated by the works of F. Browder, D. Göhde and W. A. Kirk and studied by numerous authors for over 40 years, is to classify those \( E \) and \( C \) which have the fixed point property. For a fuller discussion on this topic we refer the reader to [3, 6].

In this paper we concentrate on weak∗ compact convex subsets of the dual Banach space \( E \). In 1976, L. Karlovitz (see [5]) proved that if \( C \) is a weak∗ compact convex subset of \( \ell_1 \) (as the dual to \( c_0 \)) then every nonexpansive mapping \( T : C \to C \) has a fixed point. His result was extended by T.C. Lim [11] to the case of left reversible topological semigroups. On the other hand, C. Lennard showed the example of a weak∗ compact convex subset of \( \ell_1 \) with the weak∗ topology induced by its predual \( c \) and an affine contractive mapping without fixed points (see [12, Example 3.2]). This shows that, apart from nonexpansiveness, some additional assumptions have to be made to obtain the fixed points.

Let \( S \) be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that for each \( t \in S \), the mappings \( s \to t \cdot s \) and \( s \to s \cdot t \) from \( S \) into \( S \) are continuous. Consider the following fixed point property:

\[(F_s): \text{Whenever } S = \{T_s : s \in S\} \text{ is a representation of } S \text{ as norm-nonexpansive mappings on a non-empty weak∗ compact convex set } C \text{ of a dual Banach space } E \text{ and the mapping } (s, x) \to T_s(x) \text{ from } S \times C \text{ to } C \text{ is jointly continuous, where } C \text{ is equipped with the weak∗ topology of } E, \text{ then there is a common fixed point for } S \text{ in } C.\]

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It is not difficult to show (see, e.g., [9, p. 528]) that property $(F_*)$ implies that $S$ is left amenable (in the sense that $LUC(S)$, the space of bounded complex-valued left uniformly continuous functions on $S$, has a left invariant mean). Whether the converse is true is a long-standing open problem, posed by A. T.-M. Lau in [8] (see also [9, Problem 2], [10, Question 1]).

It is well known that all commutative semigroups are left amenable. The aim of this paper is to give a partial answer to the above problem by showing that every commuting family $S$ of weak$^*$ continuous nonexpansive mappings acting on a weak$^*$ compact convex subset $C$ of the dual Banach space $E$ has common fixed points. Moreover, we prove that the set $Fix S$ of fixed points is a nonexpansive retract of $C$.

Note that the structure of $Fix S$ (with $S$ commutative) was examined by R. Bruck (cf. [1, 2]) who proved that if every nonexpansive mapping $T : C \to C$ has a fixed point in every nonempty closed convex subset of $C$ which is invariant under $T$, and $C$ is convex and weakly compact or separable, then $Fix S$ is a nonexpansive retract of $C$. We are able to mix the elements of Bruck’s method with some properties of $w^*$-continuous and nonexpansive mappings to get the desired result.

2. Preliminaries

Let $E$ be the dual of a Banach space $E_*$. In this paper we focus on the weak$^*$ topology – the smallest one satisfying the condition: for all $e \in E$, the functional $\hat{e}(x) = x(e)$ is continuous (in the strong topology). This definition opens up the possibility to consider the so-called weak$^*$ properties, for example, $w^*$-compactness (compactness in the $w^*$-topology), $w^*$-completeness, etc. In this topology, $E$ becomes a locally convex Hausdorff space. We say that a dual Banach space $E$ has the $w^*$-FPP if every nonexpansive self-mapping defined on a nonempty $w^*$-compact convex subset of $E$ has a fixed point. It is known that $\ell_1 = c_0^*$ and some other Banach lattices have $w^*$-FPP, however $\ell_1 = c^*$ as well as the duals of $C(\Omega)$, where $\Omega$ is an infinite compact Hausdorff topological space, do not possess this property.

A non-void set $D \subset C$ is said to be a nonexpansive retract of $C$ if there exists a nonexpansive retraction $R : C \to D$ (i.e., a nonexpansive mapping $R : C \to D$ such that $R|_D = I$). Since we deal a lot with $w^*$-continuous nonexpansive mappings, we abbreviate them to $w^*$-CN.

We conclude with recalling the following consequence of the Ishikawa theorem (see [3]): if $C$ is a bounded convex subset of a Banach space $X$, $\gamma \in (0, 1)$, and $T : C \to C$ is nonexpansive, then the mapping $T_\gamma = (1-\gamma)I + \gamma T$ is asymptotically regular, i.e., $\lim_{n \to \infty} \|T_\gamma^{n+1}x - T_\gamma^n x\| = 0$ for every $x \in C$. We use this theorem in Lemma 3.4.

3. Fixed-point theorems

We begin with a structural result concerning a single $w^*$-continuous nonexpansive mapping $T : C \to C$. 


Theorem 3.1. Let $C$ be a nonempty weak* compact convex subset of the dual Banach space. Then for any $w^*$-CN self-mapping $T$ on $C$, the set $\text{Fix}T$ of fixed points of $T$ is a (nonempty) nonexpansive retract of $C$.

The proof will follow by constructing gradually (and establishing properties of) three functions, each one defined in the means of the earlier, and the last one being the retraction from $C$ to $\text{Fix}T$.

Proof. Notice first that $C$ is complete in the strong topology. Now, for $x \in C$ and a positive integer $n$, consider a mapping $T_x : C \to C$ defined by

$$T_x z = \frac{1}{n} x + \left(1 - \frac{1}{n}\right) Tz, \; z \in C.$$  

It is not difficult to see that $T_x$ is a contraction:

$$\|T_x y - T_x z\| \leq \left(1 - \frac{1}{n}\right) \|y - z\|.$$  

Hence and from completeness of $C$, it follows from the Banach Contraction Principle that there exists exactly one point $F_n x \in C$ such that $T_x F_n x = F_n x$. This defines a mapping $F_n : C \to C$ by

$$F_n x = \frac{1}{n} x + \left(1 - \frac{1}{n}\right) TF_n x$$  

for $x \in C$. Thus

$$\|TF_n x - F_n x\| = \frac{1}{n} \|TF_n x - x\| \leq \frac{1}{n} \text{diam } C$$

and consequently,

$$\lim_n \|TF_n x - F_n x\| = 0$$

since $C$ is bounded in norm as a weak* compact subset of a Banach space.

Notice that for $x \in \text{Fix}T$ we have

$$T_x x = x$$

and consequently $F_n x = x$.

Furthermore, $F_n x$ is nonexpansive. Indeed,

$$F_n x - F_n y = T_x F_n x - T_y F_n y = \frac{1}{n} (x - y) + \left(1 - \frac{1}{n}\right) (Tx - Ty)$$

which, by putting it into norm and using the triangle inequality and nonexpansiveness of $T$, gives us a desired statement.

Notice that we can view $C^C$ as the product space of copies of $C$, where each copy is endowed with the $w^*$-topology. Then, according to Tychonoff’s theorem, $C^C$ is compact in the product topology generated in this way (“$w^*$-product topology”). It follows that a sequence $(F_n)_{n \in \mathbb{N}}$ of elements from $C^C$ has a convergent subnet $(F_{n_\alpha})_{\alpha \in \Lambda}$ and we can define

$$R = \text{w}^*\lim_{\alpha} F_{n_\alpha},$$
where the above limit should be understood as taken in the aforementioned \(w^*-\)product topology. Now we can treat the application of \(R\) to some \(x \in C\) as the projection of the mapping onto the \(x\)-th coordinate and since such projections are continuous in the product topology, we obtain

\[ Rx = \lim_{\alpha} w^* F_{n_{\alpha}} x, \]

where this limit is an ordinary \(w^*\)-limit. With this approach, we are able to construct one subnet which guarantees convergence for all \(x \in C\).

Notice that

\[ TRx = \lim_{\alpha} w^* TF_{n_{\alpha}} x \]

since \(T\) is weak* continuous. Now, it follows from the weak* lower semicontinuity of the norm that for any \(x \in C,\)

\[ \|TRx - Rx\| = \|w^* \lim_{\alpha} (TF_{n_{\alpha}} x - F_{n_{\alpha}} x)\| \leq \liminf_{\alpha} \|TF_{n_{\alpha}} x - F_{n_{\alpha}} x\| = 0 \]

and hence

\[ TRx = Rx \]

which means that \(Rx \in \text{Fix } T\). Furthermore, \(Rx = x\) if \(x \in \text{Fix } T\).

We can now use \((2)\) and the weak* lower semicontinuity of the norm to prove that \(R\) is nonexpansive:

\[ \|Rx - Ry\| = \|w^* \lim_{\alpha} (F_{n_{\alpha}} x - F_{n_{\alpha}} y)\| \]

\[ \leq \liminf_{\alpha} \left\| \frac{1}{n_{\alpha}} (x - y) + (1 - \frac{1}{n_{\alpha}})(Tx - Ty) \right\| \leq \limsup_{\alpha} \frac{1}{n_{\alpha}} \|x - y\| \]

\[ + \limsup_{\alpha} (1 - \frac{1}{n_{\alpha}}) \|Tx - Ty\| = \|Tx - Ty\| \leq \|x - y\|. \]

Thus we conclude that \(\text{Fix } T\) is indeed a nonexpansive retract of \(C\). \(\square\)

**Remark 3.2.** The \(w^*\)-continuity of \(T\) cannot be omitted in the assumptions of Theorem 3.1. Indeed, otherwise we would conclude that any dual Banach space has \(w^*-\)FPP. But it is known (see, e.g., [12, Example 3.2]) that \(\ell_1\) (as the dual to the Banach space \(c\)) fails the \(w^*-\)FPP, a contradiction.

The following example shows that we would not be able to relax the assumption of the nonexpansiveness of \(T\) to continuity, either, even if we only postulated the existence of a (continuous) retraction.

**Example 1.** Let \(\ell_1 = c_0^*\) and define

\[ T(x_1, x_2, x_3, ...) = ((x_1)^2, 0, x_2, x_3, ...)\]

on the unit ball \(B_{\ell_1}\). Notice that \(T : B_{\ell_1} \to B_{\ell_1}\) is \(w^*-\)continuous and \(\text{Fix } T = \{(\pm 1, 0, ...)\}\). But a non-connected set cannot be a retract of the ball.
Our next objective is to generalize Theorem 3.1 to a commuting family of \(w^*\)-continuous nonexpansive mappings. If \(S = \{T_s : s \in S\}\) is a family of mappings, we denote by

\[
\text{Fix } S = \bigcap_{s \in S} \text{Fix } T_s
\]

the set of common fixed points of \(S\).

We first prove a lemma which resembles [1, Lemma 6].

**Lemma 3.3.** Let \(S\) be a family of commuting self-mappings acting on a set \(C\) and suppose that there exists a retraction \(R\) of \(C\) onto \(\text{Fix } S\). If \(\tilde{T}\) commutes with every element of the family \(S\), then

\[
\text{Fix } S \cap \text{Fix } \tilde{T} = \text{Fix}(\tilde{T}R).
\]

**Proof.** The inclusion from left to right follows from the simple observation that if \(x \in \text{Fix } S \cap \text{Fix } \tilde{T}\), then \(Rx = x\) and \(\tilde{T}x = x\).

For the other direction, assume \(x \in \text{Fix}(\tilde{T}R)\) which means \(\tilde{T}Rx = x\).

Then, for every \(T \in S\), it follows from the commutativity and the fact that \(Rx \in \text{Fix } T\) that

\[
T\tilde{T}Rx = \tilde{T}(TRx) = \tilde{T}Rx.
\]

Therefore \(\tilde{T}Rx \in \text{Fix } T\) for every \(T \in S\) and consequently

\[
x = \tilde{T}Rx \in \text{Fix } S.
\]

Since \(R\) is a retraction onto \(\text{Fix } S\), we have \(Rx = x\) and hence \(\tilde{T}x = x\).

It follows that \(x \in \text{Fix } S \cap \text{Fix } \tilde{T}\) which proves the inclusion and the whole lemma. \(\square\)

**Lemma 3.4.** Suppose that \(C\) is as in Theorem 3.1 and \(S_n = \{T_1, ..., T_n\}\) is a finite commuting family of \(w^*\)-CN self-mappings on \(C\). Then \(\text{Fix } S_n\) is a nonexpansive retract of \(C\).

**Proof.** We will show by induction on \(n\) that there exists a nonexpansive retraction \(R_n\) from \(C\) onto \(\text{Fix } S_n\). The base case \(n = 1\) follows directly from Theorem 3.1 since \(\text{Fix } S_1 = \text{Fix } T_1\).

Now assume that that there exists a nonexpansive retraction \(R_n\) of \(C\) onto \(\text{Fix } S_n\). We need to show the existence of a nonexpansive retraction \(R_{n+1}\) of \(C\) onto \(\text{Fix } S_{n+1}\).

Let

\[
\tilde{R}_n x = \frac{1}{2} x + \frac{1}{2} T_{n+1} R_n x, \quad x \in C,
\]

and consider a sequence \((\tilde{R}_n^k)_{k \in \mathbb{N}}\) of successive iterations of \(\tilde{R}_n\). As in the proof of Theorem 3.1 we can view \(C^C\) as the product space, compact with respect to the \(w^*\)-topology on \(C\). Hence the sequence \((\tilde{R}_n^k)_{k \in \mathbb{N}}\) has a convergent subnet \((\tilde{R}_n^{k_\alpha})_{\alpha \in \Lambda}\) and we can define

\[
R_{n+1} x = w^* \lim_{\alpha} \tilde{R}_{n}^{k_\alpha} x
\]

for every \(x \in C\).
Since \( T_{n+1} R_n \) is nonexpansive as a composition of such mappings, it is easy to see that also \( \tilde{R}_n \) is nonexpansive. The nonexpansiveness of \( R_{n+1} \) now follows from the weak* lower semicontinuity of the norm. It is also easy to see that \( \text{Fix} T_{n+1} R_n \subset \text{Fix} R_{n+1} \) and, by using Lemma 3.3, we conclude that
\[
\text{Fix} S_{n+1} \subset \text{Fix} R_{n+1}.
\]
But this still does not prove that \( R_{n+1} \) is a mapping we are looking for, nor that \( \text{Fix} S_{n+1} \) is nonempty. To complete the proof, we must show that \( R_{n+1} \) is a mapping onto \( \text{Fix} S_{n+1} \). The rest of the proof is about showing this fact.

Since \( C \) is convex closed and bounded, and \( \tilde{R}_n \) is the convex combination of a nonexpansive mapping and the identity, it follows from the Ishikawa theorem [4] that \( \tilde{R}_n \) is asymptotically regular, i.e.,
\[
\lim_{k \to \infty} \| \tilde{R}_n^{k+1} x - \tilde{R}_n^k x \| = 0
\]
for every \( x \in C \).

Now, fix \( x \) and notice that \( ( \tilde{R}_n^{k_\alpha} x)_{\alpha \in \Lambda} \) is an approximate fixed point net for the mapping \( T_{n+1} R_n \). To see this, use the equation
\[
\tilde{R}_n^{k_\alpha+1} x = \frac{1}{2} \left( \tilde{R}_n^{k_\alpha} x - T_{n+1} R_n \tilde{R}_n^{k_\alpha} x \right) + T_{n+1} R_n \tilde{R}_n^{k_\alpha} x
\]
and the asymptotical regularity in the following calculations:
\[
\limsup_{\alpha} \left\| T_{n+1} R_n \tilde{R}_n^{k_\alpha} x - \tilde{R}_n^{k_\alpha} x \right\| \leq \limsup_{\alpha} \left\| T_{n+1} R_n \tilde{R}_n^{k_\alpha} x - \tilde{R}_n^{k_\alpha+1} x \right\| + \lim_{\alpha} \left\| \tilde{R}_n^{k_\alpha+1} x - \tilde{R}_n^{k_\alpha} x \right\| = \limsup_{\alpha} \left\| T_{n+1} R_n \tilde{R}_n^{k_\alpha} x - \tilde{R}_n^{k_\alpha+1} x \right\|
\]
\[
= \frac{1}{2} \limsup_{\alpha} \left\| T_{n+1} R_n \tilde{R}_n^{k_\alpha} x - \tilde{R}_n^{k_\alpha} x \right\|.
\]
Thus we conclude that
\[
\lim_{\alpha} \left\| T_{n+1} R_n \tilde{R}_n^{k_\alpha} x - \tilde{R}_n^{k_\alpha} x \right\| = 0,
\]
(3)
as desired.

Now, for brevity, denote \( r_\alpha = \tilde{R}_n^{k_\alpha} x \) and notice that for every \( m \leq n \n\)
\[
T_m T_{n+1} R_n r_\alpha = T_{n+1} T_m R_n r_\alpha = T_{n+1} R_n r_\alpha.
\]
That is, \( T_{n+1} R_n r_\alpha \in \text{Fix} T_m \) which is equivalent to the statement that \( T_{n+1} R_n r_\alpha \) belongs to \( \text{Fix} S_n \). It follows that
\[
T_{n+1} R_n r_\alpha = R_n T_{n+1} R_n r_\alpha.
\]
and using the equation (3), we obtain
\[
\limsup_{\alpha} \left\| R_n r_\alpha - r_\alpha \right\| \leq \limsup_{\alpha} \left\| R_n r_\alpha - T_{n+1} R_n r_\alpha \right\| + \lim_{\alpha} \left\| T_{n+1} R_n r_\alpha - r_\alpha \right\|
\]
\[
= \limsup_{\alpha} \left\| R_n r_\alpha - R_n T_{n+1} R_n r_\alpha \right\| \leq \lim_{\alpha} \left\| r_\alpha - T_{n+1} R_n r_\alpha \right\| = 0.
\]
(4)
In the same manner we can see that for every $m \leq n$,
\[
\limsup_{\alpha} \|T_{m}r_{\alpha} - r_{\alpha}\| \leq \limsup_{\alpha} \|T_{m}r_{\alpha} - T_{m}R_{n}r_{\alpha}\| + \limsup_{\alpha} \|T_{m}R_{n}r_{\alpha} - r_{\alpha}\| \\
\leq \lim_{\alpha} \|r_{\alpha} - R_{n}r_{\alpha}\| + \lim_{\alpha} \|R_{n}r_{\alpha} - r_{\alpha}\| = 0.
\]
Since $T_{m}$ is $w^{*}$-continuous, this easily yields
\[
T_{m}R_{n+1}x = R_{n+1}x
\]
and, consequently,
\[
R_{n+1}x \in \text{Fix}\ S_{n}.
\]
(5)

Finally, by using (3) and (1), we get
\[
\limsup_{\alpha} \|T_{n+1}r_{\alpha} - r_{\alpha}\| \leq \limsup_{\alpha} \|T_{n+1}r_{\alpha} - T_{n+1}R_{n}r_{\alpha}\| + \limsup_{\alpha} \|T_{n+1}R_{n}r_{\alpha} - r_{\alpha}\| \\
+ \lim_{\alpha} \|T_{n+1}R_{n}r_{\alpha} - r_{\alpha}\| \leq \lim_{\alpha} \|r_{\alpha} - R_{n}r_{\alpha}\| = 0.
\]
Then, from the $w^{*}$-continuity of $T_{n+1}$,
\[
T_{n+1}R_{n+1}x = R_{n+1}x
\]
which combined with (5), gives
\[
R_{n+1}x \in \text{Fix}\ S_{n+1}.
\]
That is, $\text{Fix}\ S_{n+1}$ is nonempty and $R_{n+1}$ acts onto it, which completes the proof. \qed

We are now in a position to prove our main theorem.

**Theorem 3.5.** Suppose that $C$ is as in Theorem 3.1 and $S$ is an arbitrary family of commuting $w^{*}$-CN self-mappings on $C$. Then $\text{Fix}\ S$ is a nonexpansive retract of $C$.

**Proof.** If $S$ is finite, we can use lemma 3.4. So assume that $S$ is infinite. First notice that
\[
\text{Fix}\ T = (T - I)^{-1}\{0\}
\]
is closed in the $w^{*}$-topology for every $T \in S$. Let
\[
\Lambda = \{\alpha \subset S : \#\alpha < \infty\}
\]
be a directed set with the inclusion relation $\leq$. Denote by $R_{\alpha}$ the nonexpansive retraction from $C$ to $\text{Fix}\alpha = \bigcap_{T \in \alpha} \text{Fix}\ T$ (a more convenient way of writing $\text{Fix}\alpha$) which existence is guaranteed by Lemma 3.4. Then we have a net $(R_{\alpha})_{\alpha \in \Lambda}$, and we can select a subnet $(R_{\alpha_{\gamma}})_{\gamma \in \Gamma}$, $w^{*}$-convergent for any $x \in C$. Define
\[
Rx = w^{*}\lim_{\gamma} R_{\alpha_{\gamma}}x.
\]
For a fixed $T \in S$, take $\gamma_{0}$ such that $\alpha_{\gamma} \geq \{T\}$ for every $\gamma \geq \gamma_{0}$. It exists, straightforward from the subnet definition. Then
\[
\forall_{\gamma \geq \gamma_{0}} R_{\alpha_{\gamma}}x \in \text{Fix}\alpha_{\gamma} \subset \text{Fix}\alpha_{\gamma_{0}} \subset \text{Fix}\ T
\]
and hence $R_\alpha x$ lies eventually in the $w^*$-closed set $\text{Fix} \ T$. Therefore, $R x \in \text{Fix} \ T$ for every $T \in \mathcal{S}$ which implies $R x \in \text{Fix} \ \mathcal{S}$. It is easy to see that $R$ is nonexpansive. Also, for every $\alpha$,

$$x \in \text{Fix} \ \mathcal{S} \implies x \in \text{Fix}_\alpha \implies R_\alpha x = x,$$

from which follows

$$Rx = x, \ x \in \text{Fix} \ \mathcal{S}. \quad (6)$$

Thus $R$ is a nonexpansive retraction from $C$ onto $\text{Fix} \ \mathcal{S}$.

**Remark 3.6.** In particular, the set $\text{Fix} \ \mathcal{S}$ is non-empty. Thus Theorem 3.5 answers affirmatively [10, Question 1] in the case of commutative semigroups.

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