VORONOI TESSELLATION AND NON-PARAMETRIC HALO CONCENTRATION

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ABSTRACT

We present and test Tessellation-based Recovery of Amorphous halo Concentrations (TesseRACt), a non-parametric technique for recovering the concentration of simulated dark matter halos using Voronoi tessellation. TesseRACt is tested on idealized N-body halos that are axisymmetric, triaxial, or contain substructure and is compared to traditional least-squares fitting as well as to two non-parametric techniques that assume spherical symmetry. TesseRACt recovers halo concentrations within 3% of the true value regardless of whether the halo is spherical, axisymmetric, or triaxial. Traditional fitting and non-parametric techniques that assume spherical symmetry can return concentrations for non-spherical halos that are systematically off by as much as 10% from the true value. TesseRACt also performs significantly better when there is substructure present outside 0.5 R200. Given that cosmological halos are rarely spherical and often contain substructure, we discuss implications for studies of halo concentration in cosmological N-body simulations including how choice of technique for measuring concentration might bias scaling relations.

Key words: galaxies: fundamental parameters – galaxies: halos – methods: numerical

1. INTRODUCTION

There has long been a disconnect between the way we describe a dark matter halo and the reality of that dark matter structure. In theoretical terms, we think of a halo as a smooth, isotropic, virialized, usually spherical, typically uniformly rotating distribution of mass that obeys a distinct radial distribution. However, while halos in cosmological simulations and observations may conform to this in a statistical sense, any one halo is not really any of these things. Halos are triaxial, anisotropic, and contain significant substructure (e.g., Jing & Suto 2002; Gao et al. 2004; Despali et al. 2014; Groener & Goldberg 2014). Despite this well known fact, many analysis techniques try to extract halo properties by imposing one or more of these assumptions. For example, any procedure that fits a radial profile, be it Hernquist (Hernquist 1990), Navarro–Frenk–White (NFW) (Navarro et al. 1996), or Einasto (Einasto & Haud 1989), to a halo’s mass distribution assumes the halo is both spherical and smooth. In an era of simulations quickly surpassing 10^9 particles, there is a need for physically motivated analysis techniques that do not impose constraints on what a halo should look like.

Halo concentration is a particularly useful statistic for characterizing halos. Since halos that gain the majority of their mass at earlier times (when the mean density of the universe was higher) should be more compact, concentration is believed to encode a great deal of information about halo formation and growth. There have been numerous studies on the relationships between halo concentration in cosmological simulations and halo mass (Navarro et al. 1996; Neto et al. 2007), redshift (Bullock et al. 2001; Gao et al. 2008; Klypin et al. 2011; Prada et al. 2012; Bhattacharya et al. 2013; Dutton & Macciò 2014; Ludlow et al. 2014; Diemer & Kravtsov 2015), environment (Bullock et al. 2001; Macciò et al. 2007), assembly history (Wechsler et al. 2002; Zhao et al. 2009; Ludlow et al. 2013), and cosmology (Colin et al. 2000; Eke et al. 2001; Macciò et al. 2008; Dooley et al. 2014). However, claims are often conflicting and the majority of techniques used to measure concentration fall victim to the above assumptions.

We propose a non-parametric method for estimating halo concentration using Voronoi tessellation which we dub Tessellation-based Recovery of Amorphous halo Concentrations (TesseRACt). Section 2 briefly describes Voronoi tessellation and outlines TesseRACt, Section 3 describes several tests, Section 4 summarizes our findings, and Section 5 discusses some implications and studies that can benefit from TesseRACt.

2. THEORY/BACKGROUND

2.1. Measuring Concentration

The concentration parameter is traditionally defined in an NFW halo as
\[ c_{nfw} = \frac{R_{200}}{R_s} \] (1)

where \( R_{200} \) is the radius enclosing a mean density that is 200 times the critical density of the universe and \( R_s \) is the scale radius. Since \( R_{200} \) can be easily found from the particle data, concentration is typically obtained by fitting Equation (2) to the radially enclosed mass profile to find \( R_s \).

\[ M_{enc}(r) = 4\pi \rho_0 R_s^3 \ln \left( \frac{R_s + r}{R_s} \right) - \frac{r}{R_s + r} \] (2)

Although this is simple in theory, fitting even spherical halos without substructure can be difficult. Fits can be highly sensitive to choice of center, resolution at the center, deviation from the expected NFW power laws, and choice of binning (see Prada et al. 2012). These effects can be alleviated in practice by avoiding fitting altogether. Instead, the unknown profile parameters are related to other halo properties that can be robustly measured. For example, if \( R_{half} \) is the radius enclosing half the mass) and \( R_{200} \) of a halo is known, it is possible to
for $R_i$.

This can be done for any two independent halo properties, typically characteristic radii or velocities (Avila-Reese et al. 1999; Thomas et al. 2001; Alam et al. 2002; Gao et al. 2004; Klypin et al. 2011; Prada et al. 2012). While such techniques are more robust against deviations from NFW and yield more accurate results than fitting for dense halos with under-sampled central regions, even these techniques still assume that halos are spherical, centered appropriately, and do not contain substructure.

In principle, more accurate concentration for non-spherical halos could be obtained by fitting to triaxial or ellipsoidal bins. Such techniques have been found to provide more accurate mass estimates in both simulations and observations, but generally assume that the axis ratio and alignment remains constant throughout the halo (Warren et al. 1992; Jing & Suto 2002; Allgood et al. 2006; Despali et al. 2013, 2014). In reality, simulations show that halo shape is highly dependent on radius, becoming less and less spherical as you look deeper into the halo (Allgood et al. 2006; Vera-Ciro et al. 2011). This makes measurements assuming a constant shape dependent upon the location within the halo at which shape is measured. In addition, despite allowing for more freedom in halo shape, non-spherical fitting is still victim to the same caveats as spherical fitting and relies on the additional measurement of halo shape.

While both non-spheroidal binning and non-parametric spherical techniques have advantages, neither is completely free of assumptions or can handle substructure. However, using Voronoi tessellation, we can construct a technique that does not rely on fitting, does not require the halo to be centered, does not make any assumptions of spherical symmetry, and allows for substructure.

### 2.2. Tessellation-based Concentration

Given a set of seed points $\{p_1, \ldots, p_n\}$ in some space, Voronoi tessellation divides the space between the seeds such that each seed is the closest seed to its Voronoi region. In this way, each seed $p_i$ has a corresponding Voronoi region of volume $V_i$ encompassing all points in space that are closest to that seed. In astrophysics, Voronoi tessellation and its dual, Delaunay triangulation, have been used to reconstruct density fields from discrete points (Schaap & van de Weygaert 2000), non-parametrically identify galaxy clusters in galaxy surveys (Soares-Santos et al. 2011), identify dark matter halos (Neyrinck et al. 2004) and voids (Neyrinck 2008) in cosmological simulations, and improve the treatment of hydrodynamics in simulations (Springel 2010; Mocz et al. 2013; Hopkins 2015). We take this one step further. Once halos are identified in cosmological simulations (either by friends-of-friends, spherical over-density, or tessellation), the additional information provided by the particles’ associated volumes can be used to derive halo properties (like concentration) without imposing any additional functional form.

To determine concentration from the particle volumes, a profile is constructed that describes how mass scales with volume rather than radius. For a particle $p_i$ with mass $m_i$ and a Voronoi volume $V_i$, the volume $V_{\text{enc},i}$ “enclosed” by that particle is taken to be the sum of all particle volumes which are smaller than $V_i$ or

$$V_{\text{enc},i} = \sum_{j=0}^{n} V_j [V_j \leq V_i].$$

Generally speaking, a particle’s volume $V_i$ will increase with its distance from the center of the halo as density drops off. This means the above summation is likely to include particles that are interior to particle $p_i$ and will not include particles at the edges of the halo which can have infinite volumes. However, since particle volumes are formally independent of where the center of the halo is, this summation will be the same regardless of how the halo is centered. Similarly, the mass $M_{\text{enc},i}$ “enclosed” by a particle $p_i$ is

$$M_{\text{enc},i} = \sum_{j=0}^{n} m_j [V_j \leq V_i].$$

Each particle can then be assigned a “pseudo radius” $R_i' = (3V_{\text{enc},i}/4\pi)^{1/3}$ which is defined as the radius the particle would have if $V_{\text{enc},i}$ were spherical. The result is a volume-based mass profile $M_{\text{enc}}(R')$. Naïvely, the volume-based concentration could then be defined as

$$c_{\text{vol}} = \left(\frac{V_{200}}{V_i}\right)^{1/3}$$

where $V_{200}$ is the smallest volume containing an average density that is 200 times the critical density of the universe and $V_i$ is some scale volume. If the pseudo radii $(R_{200}^i$ and $R_i'$) associated with these volumes converged to the corresponding physical radii $(R_{200}$ and $R_i$) in the case of a spherical halo, $c_{\text{vol}}$ would equal $c_{\text{nfw}}$ and this would be sufficient. However, this is not strictly true.

For even a spherical halo, the relationship between a particle’s physical radius $R_i$ and pseudo radius $R_i'$ is not 1:1. Due to the intrinsic scatter in the inter-particle spacing at a given physical radius, particles with slightly larger/smaller volumes will be scattered to larger/smaller pseudo radii. Since the density of particles is generally greater toward smaller physical radii, it is more likely for particles inside $R_i$ to be scattered to larger pseudo radii. As a result, there will then be systematically fewer particles considered “enclosed” by particle $p_i$ and $R_i'$ will be systematically lower.

In order to correct for this and preserve the same numerical values for $c_{\text{vol}}$ and $c_{\text{nfw}}$ in the case of spherical halos, the volume based concentration is defined as

$$c_{\text{vol}} = \beta \left(\frac{V_{200}}{V_i}\right)^{\alpha/3}.$$  

$\beta = 0.8062$ and $\alpha = 1.0417$ were obtained by fitting to measurements of $V_{200}$ and $V_i$ for 10 spherical halos with known concentrations between 5 and 70 containing $1 \times 10^6$ particles. Fits for $\alpha$ and $\beta$ were independent of particle number for $N \geq 1 \times 10^5$, indicating that the quoted values are free from contamination by noise from under-sampling the profile. While this treatment is simplistic, tests performed in Section 3.1 indicate that it should be sufficient for most studies.
Equation (7) can then be simplified in terms of the radii a halo would have if it were spherical.

\[ c_{\text{vol}} = \beta \left( \frac{R_{200}'}{R_s} \right)^\alpha. \]  

(8)

Note that since \( V_{200} \) is defined as the smallest volume containing an average density that is 200 times the critical density of the universe, \( R_{200}' \) can be found directly from the Voronoi volumes. However, \( V_s \) and \( R_s' \) are somewhat arbitrary in the absence of fitting. Instead, we choose to define the scale radius and volume in terms of a quantity we can measure. We adopt \( V_{\text{half}} = 4\pi R_{\text{half}}^2/3 \), the smallest volume enclosing \( M_{200}/2 \) (half the mass within \( V_{200} \)).

Once \( R_{\text{half}}' \) is known, the corresponding \( R_s' \) for a spherical halo can be found by numerically solving Equation (3). This relationship yields unique concentrations for each \( R_{\text{half}} \) when \( 1 < c < 100 \). Above \( c = 100 \), \( R_{\text{half}}/R_{200} \) becomes somewhat degenerate about 0.1.

The TasseRACt procedure for finding \( c_{\text{vol}} \) is then:

1. Run Voronoi tessellation to determine the volumes and densities at each particle.
2. Rank particles in order of decreasing density (increasing volume).
3. Calculate enclosed mass \( (M_{\text{enc}}) \) and volume \( (V_{\text{enc}}) \) at each particle by summing the masses and volumes of “denser” particles.
4. Calculate mean enclosed density at each particle \( \left( \frac{\rho_{\text{enc}}}{\rho_c} = M_{\text{enc}}/V_{\text{enc}} \right) \).
5. Find \( V_{200} \) and \( M_{200} \) (the volume and mass enclosed by the particle at which the mean enclosed density reaches 200 times the critical density of the universe).
6. Find \( V_{\text{half}} \) (the volume enclosed by the particle at which the enclosed mass reaches \( M_{200}/2 \)).
7. Calculate \( R_{200}' \) and \( R_{\text{half}}' \) from \( V_{200} \) and \( V_{\text{half}} \).
8. Numerically solve Equation (3) for \( R_s' \).
9. Calculate \( c_{\text{vol}} \) from Equation (8).

Here \( V_{\text{half}} \) is defined in terms of \( M_{200} \), which is calculated from the particle data using the 200\( \rho_{\text{crit}} \) density threshold. While other definitions of halo mass can also be used after making the appropriate substitutions for \( V_{200} \) and \( R_s' \) above, this is equivalent to redefining concentration and can significantly change the measured value.

3. TESTS

For the tests that follow, we compare our volume-based non-parametric technique TasseRACt to both least-squares NFW fitting and two non-parametric techniques that assume spherical symmetry. The first infers the scale radius from the \( R_{\text{half}} \) and \( R_{200} \) in the same fashion as Equation (3), but using the physical particle radii. The second uses the relationship between the peak circular velocity and circular velocity at \( R_{200} \) from Prada et al. (2012),

\[ \frac{v_{\text{max}}}{v_{200}} = \left[ \frac{0.216c}{\ln(1 + c) - c/(1 + c)} \right]^{1/2}. \]

(9)

For each test, we run all four techniques on isolated N-body halos generated by sampling spherical NFW profiles of known concentration. Unless otherwise stated, each halo has

\[ N_{\text{part}} = 1 \times 10^6 \text{ particles, } M_{200} = 1 \times 10^{12} M_\odot, \text{ and } R_{200} = 200 \text{ kpc}. \]  

The halos only differ in \( \rho_0 \) and \( R_s \).

To ensure that insufficient sampling of the profile does not affect the accuracy of the least-squares fitting, fits are bounded on the lower end at 0.05 \( R_{200} \) and \( R_{200} \) on the upper end.

Errors on concentration measurements were determined by running each technique on 10 different realizations of each test halo generated by using a different random number seed. In the tests below, the mean concentration returned by each technique is plotted with the standard deviation across the different realizations.

3.1. Concentration

We first tested the performance of each method on halos with concentrations of 5, 10, 25, and 50. Results are shown in Figure 1.

As these are idealized halos, all of the tested techniques return accurate measures of concentration as expected. For the idealized halos, TasseRACt is the most accurate for all but the \( c = 5 \) halo for which the measured concentration is still within 2% of the correct value. Of the techniques which assume spherical symmetry, all three perform similarly with the maximum circular velocity and fitting techniques returning marginally more accurate concentrations at the low and high ends respectively. All of the techniques tested were consistent across the 10 halo realizations with standard deviations of \(<0.6\%\).

The accuracy of the techniques that assume spherical symmetry does not appear to be overly concentration dependent. However, as the modified concentration definition used for TasseRACt in Equation (8) was fit to measurements across a range of concentrations, TasseRACt is slightly more accurate for halos with intermediate concentrations. If Equation (8) were fit to a wider range of concentrations that extended to \( c < 5 \), we would expect performance to improve for \( c = 5 \) halos. However, since the returned concentration is still with 2% of the correct value, this effect would contribute a negligible amount of error for real halos.
3.2. Halo Shape

3.2.1. Oblateness

To test how each method performed on oblate halos, the $c = 10$ halo from above was squeezed along the $z$ axis using the volume preserving transformation

$$
\begin{align*}
\begin{bmatrix}
  x' \\
  y' \\
  z'
\end{bmatrix} &= \left( \frac{a}{b} \right)^{1/3} \begin{bmatrix}
  a & 0 & 0 \\
  0 & a & 0 \\
  0 & 0 & b
\end{bmatrix} \begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix},
\end{align*}
\tag{10}
$$

for 7 different values of ellipticity where $e_{\text{oblate}} = b/a$ and $a = 1$. Results are shown in Figure 2.

For all but the least spherical halos, TesseRACt consistently recovers concentrations within 0.5% of the correct value, with a minor dependence on halo shape (overestimated by 0.4% for $e_{\text{oblate}} = 0.3$ versus underestimated by 0.02% for spherical halos). However, the performance of the techniques that assume spherical symmetry is highly dependent on halo shape. Fitting performs the worst overall, with increasingly underestimated values for the least spherical halos (10% for $e_{\text{oblate}} = 0.3$). The performance of the two non-parametric spherical techniques is dependent on halo shape in a more complicated manner.

For decreasing values of $e_{\text{oblate}}$ (decreasing spherical symmetry), both non-parametric spherical techniques have moderately increasing positive residuals until around $e_{\text{oblate}} = 0.6-0.7$ where the residuals begin to rapidly decrease resulting in underestimates of 6% and 8% for the half-mass and peak-velocity techniques respectively. These dependencies arise because the deformation of the halo flattens the radial mass profile. As fitting uses the entire profile, the concentration is underestimated for all non-spherical halos. However, non-parametric techniques use only two points in the profile ($R_{200}$ and some inner radius) and their performance will depend on how these parts of the profile are affected by the transformation. $R_{200}$ continually increases as compression of the halo edge decreases the density at the edge of the profile. The inner radius will increase as well, but at a slower rate resulting in an overestimation of the concentration until the edge of the halo is compressed to a size comparable to the inner radius. Once this occurs, the inner radius increases more rapidly than $R_{200}$ resulting in underestimates. The precise value of $e_{\text{oblate}}$ at which this occurs will depend on the radius that a particular non-parametric technique utilizes. Since the velocity peaks at a smaller radius than the half-mass radius for an NFW profile, the technique that uses the peak velocity is less sensitive to this effect.

The test halos here have the same ellipticity at all radii by design. However, real halos are found to be decreasingly spherical at their centers (Allgood et al. 2006; Vera-Ciro et al. 2011). While TesseRACt would not be affected, it is likely that the spherical techniques would produce even less accurate concentrations. This would be particularly pronounced for fitting due to sensitivity to the inner profile slope.

3.2.2. Prolateness

To test how each method performed on prolate halos, the $c = 10$ halo from above was squeezed along its $z$ and $y$ axes using the volume preserving transformation

$$
\begin{align*}
\begin{bmatrix}
  x' \\
  y' \\
  z'
\end{bmatrix} &= \left( \frac{a^2}{b^2} \right)^{1/3} \begin{bmatrix}
  a & 0 & 0 \\
  0 & b & 0 \\
  0 & 0 & b
\end{bmatrix} \begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix},
\end{align*}
\tag{11}
$$

for 7 different values of ellipticity where $e_{\text{prolate}} = b/a = c/a$ and $a = 1$. Results are shown in Figure 3.

The performance across all techniques is similar to the above tests for oblate halos. However, there does appear to be a stronger dependence on ellipticity at intermediate values for prolate halos than for oblate halos for techniques which assume spherical symmetry. This is because the prolate transformation flattens the radial mass profile to a greater degree than the oblate transformation.

3.2.3. Triaxiality

We then tested the performance of each method on triaxial halos (See Figure 4). For 9 different values of triaxiality ($T = \frac{c^2-b^2}{c^2-a^2}$, for $c \leq b \leq a$, Franx et al. 1991), the $c = 10$ halo from above was squeezed along the $z$ and $y$ axes using the
triaxiality, but do not return concentrations that deviate from the fiducial value by more than 4%.

The techniques which assume spherical symmetry display moderate dependence on triaxiality, but do not return concentrations that deviate from the residuals plotted here are with respect to this value. Tesseract recovers the fiducial value for all triaxialities. The spherical techniques return concentrations within 0.2% of the fiducial value, which is itself off by up to 4% from the true concentration, it is reasonable to conclude that only the ratio between the largest and smallest halo axes appears to have a significant impact on the accuracy of the concentration estimate.

3.3. Halos with Substructure

To assess how substructure influences concentration measurement using Tesseract and the other techniques tested here, idealized subhalos of varying mass and concentration were added to the halo at varying radii. In each case the mass of the subhalo was set by downsampling the test halo from the previous section with the desired concentration. The size of the subhalo was then scaled such that the mean density within 0.2 R\text{200} was half that of the parent halo.

It should be noted that, when substructure is present, the traditional definition of concentration loses its physical meaning. While it can be used after removing the substructure, this is problematic in several ways. First, what remains within the parent halo will depend upon the technique used to identify the substructure. Second, from a physical standpoint, the substructure is a part of the parent halo that should contribute to its properties. Instead, concentration can be redefined as a measure of how compact the halo is overall, independent of the number of density peaks the mass is distributed between. While

\begin{equation}
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix} = \left(\frac{a^2}{bc}\right)^{1/3} \begin{bmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix},
\end{equation}

where \(a = 1, \frac{c}{a} = 0.3, \) and \(b = \sqrt{c^2 - T(c^2 - a^2)}.\)

Tesseract outperforms all spherical techniques at <0.2% for all triaxialities. The spherical techniques return concentrations that are only slightly less accurate overall (<4%) than in the case where \(T = 1.\) While Tesseract’s performance does not depend on triaxiality, the accuracy of the techniques which assume spherical symmetry does to a small degree. The two non-parametric techniques become less accurate for lower values of \(T,\) while fitting only exhibits this behavior until \(T \approx 0.3,\) where this trend is reversed. However, since the residuals never exceed 4% of the fiducial value, which is itself off by up to 10% from the true concentration, it is reasonable to conclude that only the ratio between the largest and smallest halo axes appears to have a significant impact on the accuracy of the concentration estimate.

3.3.1. Substructure Mass

Figure 5 shows the results from varying the mass of the subhalo across 1%–20% of the parent halo’s \(M_{200}.\) The subhalos had a concentration of 50% and were placed 0.5 \(R_{200}\) from the center of the parent halo. Since subhalos were scaled to have a central density within 0.2 \(R_{200}\) equal to half that of the parent halo, increasing the mass of the subhalo also increases its size.

As expected, all techniques return concentrations that depend on the mass (and size) of the subhalo, but Tesseract is less dependent overall. For the most massive subhalo, 20% of the mass of the parent, Tesseract estimates the concentration to be 8% larger than that of the parent halo alone, while techniques assuming spherical symmetry estimate the concentration to be up to 30% smaller. Tesseract returns larger concentrations because particles in the subhalo have small enough volumes that they are assumed to be within the inner parts of the parent halo and thus result in a smaller half-mass volume. The techniques assuming spherical symmetry return smaller concentrations in the presence of substructure because the subhalo contributes mass to the outer profile. However, if concentration is meant to measure how compact the matter is within the halo, even if its not all centrally located, halos with more substructure should have higher concentrations.

3.3.2. Substructure Radius

Figure 6 shows the results from varying the distance of the subhalo from the center of the parent halo from 0.05 to 0.5 \(R_{200}.\) The subhalos each had a concentration of 50% and 1% of the mass of the parent halo.

At small radii, the subhalo almost coincides with the center of the parent halo, causing all techniques to return higher concentrations than they would had there been no substructure present. As the subhalo is placed at larger radii, the concentrations returned by all techniques become lower. However, while techniques assuming spherical symmetry return concentrations that are lower than that of the parent halo alone by up to 20% as additional mass is added to the

Tesseract does this naturally, techniques assuming spherical symmetry cannot without first identifying substructure.
outer profile, TesseRACt estimates the concentration to be only 0.7% lower at 0.75 $R_{200}$. This occurs because, when the subhalo is placed at larger radii where the density of the parent is lower, the densities assigned to the subhalo particles using tessellation become lower and contribute less to the half-mass volume. The radius at which the spherical techniques change from returning concentrations that are higher versus lower than the parent halo depends on which part of the mass profile is used to calculate concentration.

3.3.3. Substructure Concentration

Figure 7 shows the results from varying the concentration of the subhalo from 5 to 50. The subhalos were placed 0.5 $R_{200}$ from the center of the parent halo and had 1% the mass of the parent.

For subhalos of equal or greater concentration than the parent ($c_{\text{sub}} \geq 10$), TesseRACt returns concentrations closer to that of the parent halo than the spherical techniques (<3% versus <20%). While the spherical techniques show little dependence on concentration in this regime, TesseRACt does. Since TesseRACt places no constraints on where the center of the halo is, the measured concentration is essentially an average of the concentrations of the two halos present, the parent and the subhalo. As a result, the same dependence on halo concentration that was seen in Section 3.1 is also present for subhalo concentration.

However, the case of a subhalo that is less concentrated than the parent halo ($c_{\text{sub}} = 5$) is less physical. In order to maintain a central density half that of the $c = 10$ parent halo with 1% of the mass, the $c = 5$ halo must be scaled to less than half of its original size. This means that the substructure contributes to a narrower region of the radial mass profile, resulting in lower estimates for $R_{200}$ and $M_{200}$ that are closer to those for the parent halo alone. For fitting, which uses the entire profile, the result is a significantly lower concentration (20%). The non-parametric techniques, which are not dependent on the whole profile, return concentrations that are closer to the halo parent in this case. The half-mass technique is slightly closer (2%) than the peak velocity technique (3%) because while $R_{\text{half}}$ changes with $R_{200}$, the peak velocity occurs well within the radius at which the substructure was placed.

3.4. Dependence on Particle Number

Next, we tested the performance of each technique for different resolutions (See Figure 8). Beginning with the $c = 10$ halo containing $10^9$ particles, transformed to be prolate with an $\epsilon_{\text{prolate}} = 0.3$, random subsets of the particles were selected to create halos of the same size and shape, but lower resolution.

Above $2 \times 10^3$ particles, all of the tested techniques are reasonably convergent (<10%) around the fiducial concentration for $N = 1 \times 10^9$. For fewer particles, the residuals for the peak velocity technique climb quickly to >80% of the fiducial value for 100 particles. TesseRACt and the half-mass non-parametric technique perform similarly on average, returning concentrations within 20% of the fiducial value at the lowest resolution. However, TesseRACt was more precise than the half-mass technique. Across the 10 halo realizations, the standard deviation in the concentrations returned by TesseRACt and the half-mass technique were 40% and 50% respectively at $N = 100$. Fitting was surprisingly accurate at low resolutions with residuals <10% and standard deviations similar to TesseRACt.
3.5. Cosmological Halos

Finally, we tested each technique on four different halos from the Ganesh cosmological simulation described in M. Sinha et al. (2015, in preparation). Figure 9 shows the projected mass density for each of the selected halos. These halos cover a range of masses ($6 \times 10^{11}$ to $1 \times 10^{14} M_\odot$), have a range of shapes, and offer varying amounts of substructure.

Different techniques can measure very different concentrations for the same halo. For each halo, TesseRACt returns a concentration that is systematically larger than that returned by the techniques that assume spherical symmetry, particularly for halos that contain significant substructure (the top two panels in Figure 9). This is consistent with the previous tests on idealized halos and highlights the differences between TesseRACt and traditional techniques for measuring concentration. While TesseRACt measures an increase in the concentration of the halo when the halo contains additional density peaks (subhalos), techniques that assume spherical symmetry do not. Rather, the presence of substructure in the outer halo can cause such techniques to measure a lower concentration. If concentration is interpreted as measuring how compact the halo is, regardless of how many density peaks the mass is distributed between, then TesseRACt is a better tool.

In addition, while the traditional techniques require a choice of center, TesseRACt does not. For the halos in Figure 9, the choice of center had a significant influence on the concentrations measured using the techniques that assume spherical symmetry, particularly for halos with substructure. The concentrations measured using fitting for halos (a) and (b) in Figure 9 each changed by 4% when the center was changed by 5% of the virial radius as measured by TesseRACt. Given that there are a wide variety of methods for centering currently in use (e.g., center of mass, density peak, potential minimum), this could make direct comparison of concentrations from different studies difficult. However, TesseRACt measures the same concentration.
concentration regardless of how the halo is centered as long as it contains the same particles.

It is important to note that, regardless of technique, concentrations measured for halos in cosmological simulations will be dependent on the technique used to identify the halos and the selected definition of concentration. For this study, concentrations were defined in terms of \( R_{200} \), the radius (or pseudo radius) where \( \frac{M_{\text{halo}}}{4\pi R_{200}^3} = 200 \rho_{\text{crit}} \). \( R_{200} \) can be measured directly from the particle data as long as the halo extends past this point and only includes halo particles. However, different halo finders use different criteria for including/excluding particles from a halo and the halo may be truncated interior to \( R_{200} \). An alternate radius may be defined in terms of the mass determined by the halo finder or a larger density threshold, but direct comparison between concentrations defined in terms of different radii should be avoided. Additional work is needed to assess how TesserACT and other techniques for measuring concentration depend upon choice of halo finder and will be the subject of a future paper.

4. SUMMARY

For idealized spherical halos, TesserACT is slightly more accurate at recovering intermediate concentrations for N-body halos than techniques that assume spherical symmetry (\( \sim 0.5\% \) versus \( \sim 1\% \)). However, TesserACT truly shines for non-spherical halos. For the most oblate or triaxial galaxies, even the most accurate spherical technique using the peak circular velocity returned concentrations that underestimated the true value by up to \( \sim 10\% \), while TesserACT had residuals of only \( \sim 0.5\% \).

TesserACT also performs more consistently in the presence of substructure. While traditional techniques for measuring concentration return systematically lower concentrations when there is substructure present in the outer halo, TesserACT returns concentrations which are higher. If there is substructure present, the halo is more compact overall. While TesserACT concentrations reflect this regardless of where the substructure is within the parent halo, the concentrations measured by techniques which assume spherical symmetry will only be larger in the presence of substructure if the substructure is near the center of the halo.

5. DISCUSSION

The poor performance of traditional techniques for measuring concentration is troubling given that halos in simulations are not often spherical and can contain a great deal of substructure.

Studies of halos in cosmological simulations indicate that halos tend to be prolate overall, halos become increasingly triaxial at higher redshifts, and the most massive halos at all redshifts are the least spherical (Allgood et al. 2006; Bett et al. 2007; Vera-Ciro et al. 2011; Despali et al. 2014). Therefore, concentrations resulting from techniques that assume spherical symmetry would result in median concentrations that are biased overall to be lower than the true value and especially biased for halos that are massive or at high redshift. This bias could be expressed in the scaling relations between concentration and other intrinsic halo properties like mass, formation redshift, and environment.

The concentration of a halo is believed to reflect the state of the universe at its collapse (Navarro et al. 1996). Therefore, on average, more massive halos should have lower concentrations, halos at higher redshift should have lower concentrations than those of the same mass at lower redshift, and halos in denser environments should have higher concentrations. However, the exact relationship between halo concentration, mass, redshift, and environment is still under debate (Bullock et al. 2001; Eke et al. 2001; Neto et al. 2007; Gao et al. 2008; Klypin et al. 2011; Prada et al. 2012; Bhattacharya et al. 2013; Dutton & Maccio 2014; Ludlow et al. 2014; Diemer & Kravtsov 2015).

Since halo shape also varies systematically with mass and redshift, it is possible that imposing the assumption of spherical symmetry could systematically bias such studies. Use of a non-parametric technique like TesserACT that is independent of halo shape could be instrumental in eliminating this bias and pinning down these relations.

In addition to biasing the median measured concentration for halos in a given mass and redshift bin, the error introduced by the choice of centering or assumption of spherical symmetry would also increase scatter in the distribution of concentration for these halos due to scatter in halo shape. Intrinsic scatter in this relationship is expected to result from differences between any two halos’ mass assembly histories that result in slightly different concentrations. However, if scatter is also being introduced by the technique used to measure concentration or center the halo, measurement of this scatter becomes less informative. Since scatter in the concentration-mass relationship is greatest for those halos that are the least virialized, have the poorest NFW fits, and have ill-defined centers (Jing 2000; Neto et al. 2007), removing these halos can help to remove some of this bias. However, even virialized halos can be non-spherical and it is likely that the assumption of spherical symmetry and choice of center would still contribute to the remaining scatter. TesserACT could help to identify what the true intrinsic scatter in the concentration-mass relationship is for both virialized and un-virialized halos.

TesserACT could be improved for future studies in a number of ways. First, it may be possible to improve TesserACT’s performance by more accurately parameterizing the relationship between traditional concentration and that calculated using tessellation-based volumes as in Equation (8). Fitting over a wider range of concentrations and allowing a more complex relationship should improve the slight dependence of TesserACT’s performance on concentration. However, since this dependence only affects concentrations by \( < 3\% \), the parameterization presented here should be sufficient for most studies.

Another possibility is extending the tessellation to other parameters. For example, tessellation in the full six-dimensional phase space could be used to place better constraints on how deep a particle resides within the host galaxy’s potential. This would be particularly useful for identifying substructure. While substructure particles may be confused with parent halo particles on the basis of spatial volumes alone, the two would be more easily differentiated in velocity space. In addition, for simulations which include gas, tessellation over parameters like star formation rate, temperature, and metallicity in addition to position and velocity can help to identify structure and explore formation scenarios. However, tessellating over non-commensurate parameters requires a choice of scaling. In the case of phase space tessellation, results have been shown to be highly dependent on the choice of scaling whether global
(Maciejewski et al. 2009) or local (Sharma & Steinmetz 2006; Ascasibar 2010).

6. CODE

The TesseRACt method for estimating concentrations from Voronoi volumes has been assembled into a publicly available Python package that can be found on the Python Package Index.3 This release includes example halos, routines for running and handling output from an adapted version of VOBOZ (Neyrinck et al. 2004), and routines for estimating concentration using the spherical techniques tested here. The TesseRACt code currently uses the Qhull libraries (Barber et al. 2013) to compute the Voronoi volumes associated with a set of particle positions.

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3 https://pypi.python.org/pypi/tesseract