ON SUBPROJECTIVITY AND SUPERPROJECTIVITY
OF BANACH SPACES

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ABSTRACT. We obtain some results for and further examples of subprojective and superprojective Banach spaces. We also give several conditions providing examples of non-reflexive superprojective spaces; one of these conditions is stable under $c_0$-sums and projective tensor products.

1. Introduction

The classes of subprojective and superprojective Banach spaces were introduced by Whitley [35] to find conditions for the conjugate of an operator to be strictly singular or strictly cosingular. They are relevant in the study of the perturbation classes problem for semi-Fredholm operators [15], which has a positive answer when one of the spaces is subprojective or superprojective [18]. A reflexive space is subprojective (superprojective) if and only if its dual is superprojective (subprojective). In general, however, $X$ being subprojective does not imply that $X^*$ is superprojective, and $X^*$ being subprojective does not imply that $X$ is superprojective, and it is unknown whether the remaining implications are valid [20, Introduction]. Basic examples of subprojective spaces are $\ell_p$ for $1 \leq p < \infty$ and $L_p(0, 1)$ for $2 \leq p < \infty$ [18] Proposition 2.4]; and $C(K)$ spaces with $K$ a scattered compact are both subprojective and superprojective [18] Propositions 2.4 and 3.4]. Moreover, recent systematic studies of subprojective spaces [28] (see also [13]) and superprojective spaces [20] have widely increased the family of known examples in those classes.

Here we continue the study of subprojective and superprojective Banach spaces. In Section 2 we give some characterisations of these classes of spaces in terms of improjective operators, and apply them to analyse the subprojectivity and superprojectivity of spaces with the Dunford-Pettis property, in particular $L_1$-spaces and $L_\infty$-spaces. We show that hereditarily-$\ell_1$ spaces with an unconditional basis and hereditarily-$c_0$ spaces are subprojective, and that $C([0, \lambda], X)$ is subprojective when $X$ is subprojective and $\lambda$ is an arbitrary ordinal. We also study the subprojectivity and superprojectivity of some $L_\infty$-spaces obtained by

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Bourgain and Delbaen [4], which provide counterexamples to some natural conjectures.

In Section 3 we find new examples of non-reflexive superprojective Banach spaces. We show that, if $X$ has property (V) and $X^*$ is hereditarily $\ell_1$, then $X$ is superprojective. In particular, this is true for the spaces in the class that we denote by $Sp(U^{-1} \circ W)$, which includes $C(K)$ spaces with $K$ scattered, the isometric preduals of $\ell_1(\Gamma)$ and Hagler’s space $JH$ [21]. Note that $JH$ is a separable space that contains no copies of $\ell_1$ and has non-separable dual, hence $JH$ does not admit an unconditional basis. The class $Sp(U^{-1} \circ W)$ is shown to be stable under passing to quotients and under taking projective tensor products and $c_0$-sums. We also show that the predual $d(w, 1)_*$ of the Lorentz space $d(w, 1)$ and the Schreier space $S$ are superprojective, although they do not belong to $Sp(U^{-1} \circ W)$, and that their dual spaces are subprojective, but the tensor products $S \hat{\otimes}_\pi S$ and $S \hat{\otimes}_\pi \ell_p$ are not superprojective.

In the sequel, subspaces of a Banach space are assumed to be closed unless otherwise stated. Given a subspace $M$ of a Banach space, $J_M$ and $Q_M$ denote its natural embedding and quotient map. A Banach space $X$ is hereditarily $Z$ if every infinite-dimensional subspace of $X$ contains a subspace isomorphic to $Z$. Given Banach spaces $X$ and $Y$, $L(X, Y)$ denotes the set of all (continuous, linear) operators from $X$ into $Y$, and $K(X, Y)$ denotes the subset of compact operators.

An injection is an isomorphic embedding with infinite-dimensional range, and a surjection is a surjective operator with infinite-dimensional range. A compact space $K$ is said to be scattered, or dispersed, if every nonempty subset of $K$ has an isolated point.

A Banach space $X$ is an $L_{p,\lambda}$-space ($1 \leq p \leq \infty$; $1 \leq \lambda < \infty$) if every finite-dimensional subspace $F$ of $X$ is contained in another finite-dimensional subspace $E$ of $X$ whose Banach-Mazur distance to the space $\ell_p^{\dim E}$ is at most $\lambda$. The space $X$ is an $L_p$-space if it is an $L_{p,\lambda}$-space for some $\lambda$.

## 2. Subprojective and superprojective spaces

We begin by recalling the definitions given in [35] of the concepts we investigate.

**Definition.** A Banach space $X$ is called subprojective if every infinite-dimensional subspace of $X$ contains an infinite-dimensional subspace complemented in $X$, and $X$ is called superprojective if every infinite-codimensional subspace of $X$ is contained in an infinite-codimensional subspace complemented in $X$.

The following result [20, Proposition 3.3] is useful to show that some spaces fail subprojectivity or superprojectivity.

Proposition 2.1. If a Banach space $X$ contains a copy of $\ell_1$, then $X$ is not superprojective and $X^*$ is not subprojective.

An operator $T: X \to Y$ is called strictly singular if there is no infinite-dimensional subspace $M$ of $X$ such that the restriction $TJ_M$ is an isomorphism. The following, more general concept was introduced by Tarafdar [34].

Definition. An operator $T: X \to Y$ is called improjective if there is no infinite-dimensional subspace $M$ of $X$ such that the restriction $TJ_M$ is an isomorphism and $T(M)$ is complemented in $Y$.

An operator $T: X \to Y$ is called strictly cosingular if there is no infinite-codimensional subspace $N$ of $Y$ such that $Q_NT$ is surjective.

The following characterisation, obtained in [1, Theorem 2.3], shows that strictly cosingular operators are improjective.

Proposition 2.2. An operator $T: X \to Y$ is improjective if and only if there is no infinite-codimensional subspace $N$ of $Y$ such that $Q_NT$ is surjective and $T^{-1}(N)$ is complemented in $X$.

Next we give some characterisations of subprojectivity and superprojectivity in terms of improjective operators.

Proposition 2.3. For a Banach space $X$ the following are equivalent:

(i) $X$ is subprojective;
(ii) every improjective operator $T: Z \to X$ is strictly singular;
(iii) there exists no improjective injection $J: Z \to X$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $X$ is subprojective and an operator $T: Z \to X$ is not strictly singular. Then there exists an infinite-dimensional subspace $M$ of $Z$ such that $TJ_M$ is an isomorphism. Let $N$ be an infinite-dimensional subspace of $T(M)$ complemented in $X$; then $T$ is an isomorphism on $M_0 := M \cap T^{-1}(N)$ and $T(M_0) = N$, hence $T$ is not improjective.

(ii) $\Rightarrow$ (iii) It is enough to observe that injections are not strictly singular.

(iii) $\Rightarrow$ (i) Given an infinite-dimensional subspace $M$ of $X$, the injection $J_M: M \to X$ is not improjective, so there exists an infinite-dimensional subspace $N$ of $M$ which is complemented in $X$. Thus $X$ is subprojective. $\square$

Proposition 2.4. For a Banach space $X$ the following are equivalent:

(i) $X$ is superprojective;
(ii) every improjective operator $T: X \to Y$ is strictly cosingular;
(iii) there exists no improjective surjection $Q: X \to Y$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $X$ is superprojective and an operator $T: X \to Y$ is not strictly cosingular. Then there exists an infinite-codimensional subspace $N$ of $Y$ such that $Q_NT$ is surjective. Let $M$ be
an infinite-codimensional subspace complemented in $X$ and containing $T^{-1}(N)$; then $T(M)$ is closed and infinite-codimensional, $Q_{T(M)}T$ is surjective and $T^{-1}T(M) = M$ is complemented in $X$, hence $T$ is not improjective by Proposition 2.2.

(ii) $\Rightarrow$ (iii) It is enough to observe that surjections are not strictly cosingular.

(iii) $\Rightarrow$ (i) Given an infinite-codimensional subspace $N$ of $X$, the surjection $Q_N: X \to X/N$ is not improjective, so there exists an infinite-codimensional subspace $M$ containing $N$ which is complemented in $X$. Thus $X$ is superprojective.

A Banach space $X$ has the Dunford-Pettis property (DPP in short) if every weakly compact operator $T: X \to Y$ takes weakly convergent sequences to convergent sequences; or, equivalently, if every weakly compact operator $T: X \to Y$ takes weakly compact sets to relatively compact sets. We refer the reader to [2, Section 5.4] and [22, Section 10] for further information on the DPP. Examples of spaces with the DPP are the $L_\infty$-spaces and the $L_1$-spaces [22, Section 10]; in particular, the spaces of continuous functions on a compact $C(K)$ and the spaces of integrable functions $L_1(\mu)$.

The next result establishes some necessary conditions for spaces with the DPP to be subprojective or superprojective.

Proposition 2.5. Let $X$ be a Banach space satisfying the DPP.

1. If $X$ is subprojective, then it contains no infinite-dimensional reflexive subspaces.
2. If $X$ is superprojective, then it admits no infinite-dimensional reflexive quotients.

Proof. (1) Let $R$ be a reflexive subspace of $X$. By Proposition 2.3, it is enough to show that the embedding $J_R: R \to X$ is strictly cosingular, hence improjective, as that would make $R$ finite-dimensional.

Let $Q: X \to Z$ be an operator such that $QJ_R$ is surjective. Then $QJ_R$ is weakly compact, so $Z$ is reflexive and $Q$ itself is weakly compact, hence completely continuous by the DPP of $X$. Thus $QJ_R$ is compact, and $Z$ is finite-dimensional.

(2) We could apply Proposition 2.3 to give a proof similar to that of (1), but we choose an alternative one. Take a bounded sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ whose image in the reflexive quotient $X/M$ is weakly convergent but does not have any convergent subsequences. Then $Q_M$ is weakly compact and $X$ has the DPP, so $Q_M$ takes weakly Cauchy sequences to convergent sequences and $(x_n)_{n \in \mathbb{N}}$ cannot have any weakly Cauchy subsequence. Thus $X$ contains a subspace isomorphic to $\ell_1$ and it is not superprojective by Proposition 2.1. \qed

Corollary 2.6. A $\mathcal{L}_1$-space is subprojective if and only if it contains no infinite-dimensional reflexive subspaces.
Proof. The direct implication is a consequence of Proposition 2.5. For the converse, observe that each $\mathcal{L}_1$-space $X$ is isomorphic to a subspace of some space $L_1(\mu)$ [25]. Therefore, every non-reflexive subspace of $X$ contains a copy of $\ell_1$ complemented in $X$ [2, Proposition 5.6.2].

The analogue of Corollary 2.6 for $\mathcal{L}_\infty$-spaces does not hold. We will see later that there exists a $\mathcal{L}_\infty$-space $Y_{bd}$ admitting no infinite-dimensional reflexive quotient which is not superprojective.

The next result was essentially proved by Díaz and Fernández [7].

**Proposition 2.7.** Every hereditarily-$c_0$ Banach space is subprojective.

Proof. It was proved in [7, Theorem 2.2] that if a Banach space $X$ contains no copies of $\ell_1$, then every copy of $c_0$ in $X$ contains another copy of $c_0$ which is complemented in $X$. [28, Proposition 2.2]. We will prove that $C_0([0, \lambda], X)$ is subprojective by induction in $\lambda$. Assume that $C_0([0, \mu], X)$ is indeed subprojective for all $\mu < \lambda$.

Otherwise, if $\lambda$ is a limit ordinal, let $M$ be an infinite-dimensional subspace of $C_0([0, \lambda], X)$ and define the projections

$$P_\mu : C_0([0, \lambda], X) \to C_0([0, \lambda], X)$$

as $P_\mu(f) = f\chi_{[0,\mu]}$ for each $\mu < \lambda$. If there exists $\mu < \lambda$ such that the restriction $P_\mu|_M$ is not strictly singular, then there exists an infinite-dimensional subspace $N \subseteq M$ such that $P_\mu|_N$ is an isomorphism. Since the range of $P_\mu$ is isometric to $C([0, \mu], X)$, which is subprojective by
our induction hypothesis, $N$ contains an infinite-dimensional subspace complemented in $C_0([0, \lambda], \mathcal{X})$ \cite{28} Corollary 2.4.

Assume now, on the other hand, that $P_{\mu|M}$ is strictly singular for every $\mu < \lambda$. We will construct a strictly increasing sequence of ordinals $\lambda_1 < \lambda_2 < \ldots$ and a sequence $(f_n)_{n \in \mathbb{N}}$ of normalised functions in $M$ such that $\|P_{\lambda_k-1}(f_k)\| < 2^{-k}/8$ and $\|P_{\lambda_k}(f_k) - f_k\| < 2^{-k}/8$ for every $k \in \mathbb{N}$, where we write $\lambda_0 = 0$ for convenience. To this end, let $k \in \mathbb{N}$, and assume that $\lambda_{k-1}$ has already been obtained. By hypothesis, $P_{\lambda_{k-1}|M}$ is not an isomorphism, so there exists $f_k \in M$ such that $\|f_k\| = 1$ and $\|P_{\lambda_{k-1}}(f_k)\| < 2^{-k}/8$, and then there is $\lambda_k \in (\lambda_{k-1}, \lambda)$ such that $\|P_{\lambda_k}(f_k) - f_k\| < 2^{-k}/8$, which finishes the inductive construction process. Let $F = [f_k : k \in \mathbb{N}] \subseteq M$, which is infinite-dimensional, and define the intervals $I_k = (\lambda_{k-1}, \lambda_k]$ and the operators $T_k = P_{\lambda_k} - P_{\lambda_{k-1}}$ for every $k \in \mathbb{N}$; then $T_k(f) = f \chi_{I_k}$, so each $T_k$ is a norm-one projection and $T_kT_j = 0$ if $i \neq j$.

Let now $g_k = T_k(f_k) = P_{\lambda_k}(f_k) - P_{\lambda_{k-1}}(f_k)$ for each $k \in \mathbb{N}$; then

$$\|g_k - f_k\| \leq \|P_{\lambda_k}(f_k) - f_k\| + \|P_{\lambda_{k-1}}(f_k)\| < 2^{-k}/4,$$

so $1/2 < \|g_k\| < 3/2$ for every $k \in \mathbb{N}$. Note that $C_0([0, \lambda])^* = \mathcal{E}_1([0, \lambda])$ \cite{11} Theorem 14.24] and $C_0([0, \lambda], \mathcal{X})^* = (C_0([0, \lambda]) \hat{\otimes}_e \mathcal{X})^* = C_0([0, \lambda])^* \hat{\otimes}_\pi \mathcal{X}^*$ \cite{33} Theorem 5.33], so

$$C_0([0, \lambda], \mathcal{X})^* = \mathcal{E}_1([0, \lambda]) \hat{\otimes}_\pi \mathcal{X}^* = \mathcal{E}_1([0, \lambda], \mathcal{X}^*)$$

and for each $k \in \mathbb{N}$ we can take $x_k \in C_0([0, \lambda], \mathcal{X})$ with norm $\|x_k\| < 2$ such that $x_k(g_k) = 1$ and $x_k$ is concentrated on $I_k$, which makes $(g_n, x_n)_{n \in \mathbb{N}}$ a biorthogonal sequence in $(C_0([0, \lambda], \mathcal{X}), C_0([0, \lambda], \mathcal{X})^*)$. In the spirit of the principle of small perturbations \cite{34}, let $K$ be the operator defined on $C_0([0, \lambda], \mathcal{X})$ as $K(f) = \sum_{n=1}^{\infty} x_n(f)(f_n - g_n)$; then

$$\sum_{n=1}^{\infty} \|x_n\| \|f_n - g_n\| < \sum_{n=1}^{\infty} 2^{-n}/2 = 1/2,$$

so $K$ is well defined and $U = I + K$ is an isomorphism on $\mathcal{X}$ that maps $U(g_k) = f_k$ for every $k \in \mathbb{N}$. Let $G = [g_k : k \in \mathbb{N}]$; then $U(G) = F$ and $G$ is infinite-dimensional.

We will now check that the supremum of the sequence $(\lambda_k)_{k \in \mathbb{N}}$ is $\lambda$ itself. Assume, to the contrary, that there existed some $\mu < \lambda$ such that $\lambda_k \leq \mu$ for every $k \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$, we would have $P_{\mu|T_k} = P_{\mu}(P_{\lambda_k} - P_{\lambda_{k-1}}) = (P_{\lambda_k} - P_{\lambda_{k-1}}) = T_k$, so $P_{\mu}(g_k) = g_k$ and $P_{\mu}$ would be an isomorphism on $G$. But then $P_{\mu}U^{-1}$ would be an isomorphism on $F$, where $U^{-1} = I - U^{-1}K$ is a compact perturbation of the identity, so $P_{\mu}$ would be upper semi-Fredholm on $F \subseteq M$, contradicting our assumption that $P_{\mu|M}$ is strictly singular.

This means, in turn, that $(x_n(f))_{n \in \mathbb{N}}$ is a null sequence for every $f \in C_0([0, \lambda], \mathcal{X})$, because each $x_k$ is supported on $I_k$ and $\|x_k\| < 2$, so
and we can define a projection
\[ Q: C_0([0, \lambda], X) \rightarrow C_0([0, \lambda], X) \]
as \[ Q(f)(\gamma) = x_k(f)g_k(\gamma) \] if \( \gamma \in I_k \), whose range is clearly \( G \). Then \( G \) is complemented in \( C_0([0, \lambda], X) \), and then so is \( U^{-1}(G) = F \subseteq M \), which proves that \( C_0([0, \lambda], X) \) is subprojective in this case too.

A Banach space \( X \) has the Schur property when every weakly convergent sequence in \( X \) is convergent. Bourgain and Delbaen [4] obtained two separable \( L_\infty \)-spaces \( X_{bd} \) and \( Y_{bd} \) which admit Schauder bases and satisfy the following properties:

- \( X_{bd} \) has the Schur property, hence it is hereditarily \( \ell_1 \); and
- \( Y_{bd} \) is hereditarily reflexive and \( Y_{bd}^* \) is isomorphic to \( \ell_1 \).

To study these spaces, we need the following folklore result. We include a proof for the convenience of the reader.

**Proposition 2.10.** Every infinite-dimensional separable \( L_\infty \)-space \( X \) has a quotient isomorphic to \( c_0 \).

**Proof.** Note that \( X^* \) contains a sequence \( (x_n^*)_{n \in \mathbb{N}} \) equivalent to the unit vector basis of \( \ell_1 \). Since \( X \) is separable, passing to a subsequence we can assume that \( (x_n^*)_{n \in \mathbb{N}} \) is weak*-convergent and, subtracting the limit, that \( (x_n^*)_{n \in \mathbb{N}} \) is weak*-null.

We consider the operator \( T: X \rightarrow c_0 \) defined as \( T(x) = (x_n^*(x))_{n \in \mathbb{N}} \). Since its conjugate operator \( T^* \) takes the unit vector basis of \( \ell_1 \) to the sequence \( (x_n^*)_{n \in \mathbb{N}} \), \( T^* \) is an injection, hence \( T \) is a surjection.

In Proposition 2.10 we can replace “\( X \) separable” by “the unit ball of \( X^* \) is weak* sequentially compact” [4, Chapter XIII].

The next result for \( X_{bd} \) shows that an analogue of Proposition 2.8 for hereditarily-\( \ell_1 \) spaces is not valid without further hypothesis.

**Proposition 2.11.** The spaces \( X_{bd} \) and \( Y_{bd} \) are neither subprojective nor superprojective.

**Proof.** The spaces \( X_{bd} \) and \( Y_{bd} \) are not subprojective because \( \ell_1 \) or a reflexive space cannot contain an infinite-dimensional \( L_\infty \)-space, and being an \( L_\infty \)-space is inherited by complemented subspaces.

For the other part, Proposition 2.11 implies that \( X_{bd} \) is not superprojective, and for \( Y_{bd} \) (and also for \( X_{bd} \)) we can apply Proposition 2.10 to obtain a surjection \( T: Y_{bd} \rightarrow c_0 \). The kernel of \( T \) cannot be contained in any infinite-codimensional complemented subspace \( M \), because \( T \) would be an isomorphism on the complement of \( M \) and \( Y_{bd} \) does not contain copies of \( c_0 \).

Note that \( Y_{bd}^* \simeq \ell_1 \) is subprojective, but \( X_{bd}^* \simeq C([0, 1])^* \) is not.
3. Sufficient conditions for superprojectivity

An operator $T: X \to Y$ is said to be unconditionally converging if there is no subspace $M$ of $X$ isomorphic to $c_0$ such that the restriction $T|_M$ is an isomorphism. We denote the sets of unconditionally converging and weakly compact operators from $X$ into $Y$ by $U(X, Y)$ and $W(X, Y)$, respectively.

**Definition.** A Banach space $X$ has property (V) if $U(X, Y) \subseteq W(X, Y)$ for every Banach space $Y$; i.e. if every non-weakly compact operator $T: X \to Y$ is an isomorphism on a subspace of $X$ isomorphic to $c_0$.

It is well known that $C(K)$ spaces have property (V), and it is not difficult to see that property (V) is inherited by quotients. Property (V) relates to superprojectivity because of the following result.

**Theorem 3.1.** Let $X$ be a Banach space with property (V) such that $X^*$ is hereditarily $\ell_1$. Then $X$ is superprojective.

**Proof.** Let $M$ be an infinite-codimensional subspace of $X$. Then $(X/M)^*$ contains a copy of $\ell_1$, so $X/M$ admits an infinite-dimensional separable quotient. Indeed, either $X/M$ has a quotient isomorphic to $c_0$ or it contains a copy of $\ell_1$ [19], in which case it has a quotient isomorphic to $\ell_2$. By passing to that further quotient, we can assume that $X/M$ itself is separable. However, $X^*$ is hereditarily $\ell_1$, so $X/M$ is not reflexive, and the quotient map $Q_M$ is not weakly compact. By property (V), there exists a subspace $A$ of $X$ isomorphic to $c_0$ such that $Q_M|_A$ is an isomorphism, where $Q_M(A) \simeq c_0$ is complemented because $X/M$ is separable. Then $X/M = Q_M(A) \oplus B$, hence $X = A \oplus Q_M^{-1}(B)$ and $M \subseteq Q_M^{-1}(B)$, so $X$ is superprojective.

**Remark.** In the proof of Theorem 3.1 we need $X^*$ to be hereditarily $\ell_1$ to ensure the existence of separable quotients. If this fact can be guaranteed for other reasons (e.g., $X$ separable) we can replace “$X^*$ hereditarily-$\ell_1$” by the weaker condition “$X$ does not admit infinite-dimensional reflexive quotients”.

Following Pietsch [31, 3.2.7], we define $Sp(U^{-1} \circ K)$ as the class of spaces $X$ satisfying that $U(X, Y) \subseteq K(X, Y)$ for every Banach space $Y$. This class admits a characterisation in terms of property (V) and the Schur property. Let us first state an auxiliary result.

**Proposition 3.2.** Let $X$ be a Banach space. Then the following are equivalent:

(i) $X^*$ has the Schur property;

(ii) $X$ has the DPP and contains no copies of $\ell_1$;

(iii) $W(X, Y) \subseteq K(X, Y)$ for every Banach space $Y$.

**Proof.** For the equivalence between (i) and (ii), we refer to [8] Theorem 3].
For (iii), assume that \( X^* \) has the Schur property, and take \( T \in W(X, Y) \); then \( T^* \in W(Y^*, X^*) = K(Y^*, X^*) \), hence \( T \in K(X, Y) \). Conversely, if there exists a weakly null sequence \( (x^*_n)_{n \in \mathbb{N}} \) in \( X^* \) that is not norm null, then the operator \( T: X \to c_0 \) given by \( T(x) = (x^*_n(x))_{n \in \mathbb{N}} \) is weakly compact but not compact.

**Proposition 3.3.** A Banach space \( X \) belongs to \( Sp(U^{-1} \circ K) \) if and only if it has property (V) and its dual \( X^* \) has the Schur property.

**Proof.** Property (V) for \( X \) is equivalent to \( U(X, Y) \subseteq W(X, Y) \) for every \( Y \), and \( X^* \) being Schur is equivalent to \( W(X, Y) \subseteq K(X, Y) \) for every \( Y \) by Proposition 3.2, which gives the desired result.

**Corollary 3.4.** Every Banach space in \( Sp(U^{-1} \circ K) \) is superprojective.

**Proof.** It is enough to observe that spaces with the Schur property are hereditarily \( \ell_1 \) and apply Proposition 3.3 and Theorem 3.1.

**Corollary 3.5.** A Banach space whose dual is isometric to \( \ell_1(\Gamma) \) belongs to \( Sp(U^{-1} \circ K) \), hence it is superprojective.

**Proof.** The dual \( \ell_1(\Gamma) \) has the Schur property, and the space itself has property (V) by [23, Corollary].

Note that, when \( K \) is scattered, \( C(K)^* \) is isometric to \( \ell_1(K) \) [11, Theorem 14.24], and that the space \( Y_{bd} \) shows that in the previous Corollary we cannot replace “dual isometric” by “dual isomorphic”.

The next results highlight the interest of the class \( Sp(U^{-1} \circ K) \) by showing its stability under quotients, \( c_0 \)-sums and projective tensor products.

**Proposition 3.6.** The class \( Sp(U^{-1} \circ K) \) is stable under passing to quotients.

**Proof.** Suppose that \( X \) belongs to \( Sp(U^{-1} \circ K) \) and \( Q: X \to Z \) is a surjective operator. Given \( T \in U(Z, Y) \) we have \( TQ \in U(X, Y) \). Then \( TQ \in K(X, Y) \), hence \( T \in K(Z, Y) \).

**Proposition 3.7.** Given a sequence \( (X_n)_{n \in \mathbb{N}} \) of spaces in \( Sp(U^{-1} \circ K) \), the space \( c_0(X_n) = \{ (x_n)_{n \in \mathbb{N}} : x_n \in X_n, \ (\|x_n\|)_{n \in \mathbb{N}} \in c_0 \} \) belongs to \( Sp(U^{-1} \circ K) \).

**Proof.** In the case \( X_n = X \) for all \( n \), it was proved by Cembranos [5, Teorema 2] that \( c_0(X_n) \) has property (V) when each \( X_n \) does, and the proof is valid when the spaces \( X_n \) are different. Moreover \( c_0(X_n)^* \equiv \ell_1(X_n^*) \) has the Schur property when each \( X_n^* \) does.

**Theorem 3.8.** If the spaces \( X \) and \( Y \) belong to \( Sp(U^{-1} \circ K) \), then so does \( X \hat{\otimes}_\pi Y \).
Proof. This is a consequence of two stability results for projective tensor products. Ryan \cite[Corollary 3.4]{32} proved that if $X^*$ and $Y^*$ have the Schur property then $(X \hat{\otimes}_\pi Y)^*$ also has the Schur property. Moreover, if $X^*$ is Schur then $X$ contains no copies of $\ell_1$ by Proposition \ref{prop:3.2}, so any bounded sequence in $X$ must contain a weakly Cauchy subsequence. Since weakly Cauchy sequences in $Y^*$, which is Schur, must converge, this means that $L(X,Y^*) = K(X,Y^*)$, and it follows from a result of Emmanuele and Hensgen \cite[Theorem 2]{10} that if $X$ and $Y$ have property (V) and $L(X,Y^*) = K(X,Y^*)$, then $X \hat{\otimes}_\pi Y$ has property (V).

Corollary 3.9. Let $X_1, \ldots, X_n$ be spaces belonging to $Sp(U^{-1} \circ K)$. Then $X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n$ is superprojective.

Note that $c_0 \hat{\otimes}_\pi c_0$ is not an $\mathcal{L}_\infty$-space because $(c_0 \hat{\otimes}_\pi c_0)^{**}$ fails the DPP \cite[Corollary 11]{16}, and it was proved in \cite{13} that $C(K) \hat{\otimes}_\pi C(L)$ is subprojective when $K$ and $L$ are countable compact.

We do not know if $C(K,X)$ is superprojective when $K$ is scattered and $X$ is superprojective, but the following result gives a partial positive answer. Recall that a Banach space $X$ has property (u) when for every weakly Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ there exists a weakly unconditionally Cauchy series $\sum_{i=1}^\infty y_i$ so that $(x_n - \sum_{i=1}^n y_i)_{n \in \mathbb{N}}$ is weakly null. Banach spaces with an unconditional basis have property (u) \cite[Theorem 3]{29}.

Proposition 3.10. Suppose that $K$ is a scattered compact and $X$ is a Banach space with property (u) such that $X^*$ has the Schur property. Then $C(K,X)$ belongs to $Sp(U^{-1} \circ K)$, and so it is superprojective.

Proof. $X$ contains no copies of $\ell_1$ by Proposition \ref{prop:3.2}. Since $K$ is scattered and $X$ has property (u), $C(K,X)$ has property (V) \cite[Theorem 3]{6}. Moreover $C(K,X)^* \equiv \ell_1(K,X^*)$ has the Schur property, hence $C(K,X)$ belongs to $Sp(U^{-1} \circ K)$ and it is superprojective by Theorem \ref{thm:3.3}. \qed

Pełczyński proved \cite[Proposition 2]{30} that a Banach space with property (u) and containing no copies of $\ell_1$ has property (V), so the condition on $X$ in Proposition \ref{prop:3.10} implies $X \in Sp(U^{-1} \circ K)$.

3.1. The Hagler space. In \cite{21}, a Banach space $JH$ is constructed such that $JH$ is separable and hereditarily $c_0$ and $JH^*$ is nonseparable and has the Schur property, hence it is hereditarily $\ell_1$. $JH$ also has property (S), which is defined as follows.

Definition. A Banach space $X$ has property (S) if every weakly null, non-norm null sequence in $X$ has a subsequence equivalent to the unit vector basis of $c_0$. 

Note that $JH$ is subprojective by Proposition 2.7. Also, $JH^*$ is not separable, so $JH$ cannot admit an unconditional basis.

**Proposition 3.11.** The space $JH$ belongs to $Sp(U^{-1} \circ K)$, hence $JH$ and $JH \otimes \pi JH$ are superprojective. Moreover $JH^*$ is subprojective.

**Proof.** Let us first see that $JH$ belongs to $Sp(U^{-1} \circ K)$. Let $T: JH \rightarrow Y$ be a non-compact operator, and let $(y_n)_{n \in \mathbb{N}}$ be a bounded sequence in $JH$ such that $(T(y_n))_{n \in \mathbb{N}}$ has no convergent subsequence. Since $JH$ contains no copies of $\ell_1$ and has property (S), passing to subsequences and taking $u_n := y_{2n} - y_{2n-1}$, we can assume that $(u_n)_{n \in \mathbb{N}}$ is equivalent to the unit vector basis of $c_0$ and $(T(u_n))_{n \in \mathbb{N}}$ is a seminormalised basic sequence, and then, since $\sum_{n \in \mathbb{N}} T(u_n)$ is weakly conditionally Cauchy, the sequence $(T(u_n))_{n \in \mathbb{N}}$ is equivalent to the unit vector basis of $c_0$ [9 Corollary V.7]. Thus $T|_{[u_n]}$ is an isomorphism, hence $JH$ belongs to $Sp(U^{-1} \circ K)$ and Corollary 3.4 implies that $JH$ is superprojective.

To see that $JH^*$ is subprojective, let $M$ be a subspace of $JH^*$. As $JH$ is separable and $JH^*$ is Schur, we can find a sequence $(x_n^*)_{n \in \mathbb{N}}$ in $M$ equivalent to the unit vector basis of $\ell_1$ which is weak* -convergent to some $x_0^* \in X^*$. Let $y_n^* := x_n^* - x^*$; by a remark of Johnson and Rosenthal [17], Lemma 3.1.19 we can find a bounded sequence $(y_n)_{n \in \mathbb{N}}$ in $X$ such that $y_n^*(y_i) = \delta_{i,n}$. Passing to a subsequence we can assume that $(y_n)_{n \in \mathbb{N}}$ is weakly Cauchy, hence $(y_{2n} - y_{2n-1})_{n \in \mathbb{N}}$ is weakly null. We denote $z_n^* := y_{2n}^*$ and $z_n := y_{2n} - y_{2n-1}$. Since $JH$ has property (S), we can assume that $(z_n)_{n \in \mathbb{N}}$ is equivalent to the unit vector basis of $c_0$.

We consider the operators $A: X \rightarrow c_0$ and $B: c_0 \rightarrow X$ defined by $A(x) = (z_n^*(x))_{n \in \mathbb{N}}$ and $Bc_n = z_n$. Then $P = BA$ is a projection on $X$ and $R(P^*) \subseteq M + (x_0)$, so $M$ contains a subspace complemented in $X^*$.

The dual $JH^*$ is not superprojective because it contains $\ell_1$.

**Proposition 3.12.** Let $K$ be a scattered compact. Then both $C(K, JH) \equiv C(K) \otimes_x JH$ and $C(K) \otimes_\pi JH$ belong to $Sp(U^{-1} \circ K)$, hence they are superprojective.

**Proof.** It was proved by Knaust and Odell [24] Theorem 2.1] that property (S) implies property (u). Since $JH^*$ has the Schur property, Proposition 3.10 implies $C(K, JH) \in Sp(U^{-1} \circ K)$.

The result for $C(K) \otimes_\pi JH$ follows from Theorem 3.8.

### 3.2. The Schreier space.

The Schreier space $S$ is defined as the space of all scalar sequences $x = (x_i)_{i \in \mathbb{N}}$ satisfying

$$\|x\|_S := \sup \left\{ \sum_{i=1}^p |x_{n_i}| : p \leq n_1 < \cdots < n_p \right\} < \infty.$$ 

It satisfies the following properties:
(a) The unit vector basis is an unconditional basis for $S$.
(b) $S$ is a subspace of $C(\omega^\omega)$; as such, it is hereditarily $c_0$.
(c) $S$ fails the DPP [8, Comments after Theorem 5]. Hence $S^*$ is not Schur (Proposition 3.2) and $S$ does not belong to $Sp(U^{-1} \circ K)$.

**Proposition 3.13.** The space $S$ is subprojective and superprojective, and its dual $S^*$ is subprojective but not superprojective.

**Proof.** $S$ is subprojective by Proposition 2.7. It is also separable, admits no infinite-dimensional reflexive quotients [27, Theorem B and Corollary 1.10], contains no copies of $\ell_1$, and satisfies property (u) because it has an unconditional basis. Thus $S$ has property (V), and Theorem 3.1 implies that $S$ is superprojective.

Its dual space $S^*$ has an unconditional basis and, since $S$ admits no infinite-dimensional reflexive quotient, $S^*$ contains no reflexive subspace. Thus $S^*$ is hereditarily $\ell_1$, hence it is subprojective by Proposition 2.8 and it is not superprojective by Proposition 2.1.

Note that $S \notin Sp(U^{-1} \circ K)$ because $S^*$ is not Schur. This is confirmed by the following result.

**Proposition 3.14.** The projective tensor products $S \hat{\otimes}_\pi S$ and $S \hat{\otimes}_\pi \ell_p$ ($1 < p < \infty$) are not superprojective.

**Proof.** The dual space of $S \hat{\otimes}_\pi S$ can be identified with $L(S, S^*)$. By [20, Corollary 3.5] it is enough to show that that there is a non-compact operator in $L(S, S^*)$.

Given $x = (x_i)_{i \in \mathbb{N}} \in S$, we denote the decreasing rearrangement of $(|x_i|)_{i \in \mathbb{N}}$ by $x^d = (x^d_i)_{i \in \mathbb{N}}$. Note that, for each $n \in \mathbb{N}$, $x^d_n + \cdots + x^d_{2n-1} \leq \|x^d\|_S$, so $x^d_{2n-1} \leq \|x^d\|_S/n$ and

$$
\|x\|_2^2 = \|x^d\|_2^2 \leq 2(\sum 1/n^2)\|x^d\|_S^2 \leq 2(\sum 1/n^2)\|x\|_S^2,
$$

which means that $S \subseteq \ell_2$ and the natural inclusion $J: S \rightarrow \ell_2$ is a bounded operator, and then $J^* J: S \rightarrow S^*$ is not compact.

The proof for $S \hat{\otimes}_\pi \ell_p$ is similar.

Observe that the previous argument does not apply to $S \hat{\otimes}_\pi c_0$. We do not know if $S \hat{\otimes}_\pi c_0$ is superprojective.

### 3.3. The predual of the Lorentz spaces $d(w, 1)$

Given $p \geq 1$ and a non-increasing sequence of positive numbers $w = (w_n)_{n \in \mathbb{N}}$, we consider the space $d(w, p)$ of all sequences of scalars $x = (a_i)_{i \in \mathbb{N}}$ for which

$$
\|x\| = \sup \left( \sum_{n=1}^{\infty} |a_{\pi(n)}|^p w_n \right)^{1/p} < \infty,
$$

where the supremum is taken over all permutations $\pi$ of $\mathbb{N}$. Then $d(w, p)$ endowed with $\|\cdot\|$ is a Banach space [20, Section 3a]. To exclude trivial cases ($\ell_p$ or $\ell_\infty$) and normalise the vectors we assume
that $\lim_n w_n = 0$, $\sum_n w_n = \infty$ and $w_1 = 1$. In this case $d(w, p)$ is called a Lorentz sequence space [26, Definition 4.e.1].

The unit vector basis $(e_n)_{n \in \mathbb{N}}$ is a symmetric basis for $d(w, 1)$ and its biorthogonal sequence $(e_n^*)_{n \in \mathbb{N}}$ is a symmetric basis for the predual $d(w, 1)_*$ of $d(w, 1)$. In particular, $d(w, 1)_*$ contains no copies of $\ell_1$.

**Proposition 3.15.** The space $d(w, 1)$ is subprojective and its predual $d(w, 1)_*$ is superprojective.

**Proof.** The space $d(w, 1)$ is hereditarily $\ell_1$ [26, Proposition 4.e.3], hence subprojective by Proposition 2.8.

Since $d(w, 1)_*$ has an unconditional basis, it satisfies property (u) [29, Theorem 3], and $d(w, 1)_*$ does not contain copies of $\ell_1$ because $d(w, 1)$ is separable. Then $d(w, 1)_*$ has property (V) and Theorem 3.1 implies that it is superprojective.

**Proposition 3.16.** The space $d(w, 1)$ fails the Schur property, so $d(w, 1)_* \not\in Sp(U^{-1} \circ K)$.

**Proof.** Note that $(e_n)_{n \in \mathbb{N}}$ is a symmetric basis in $d(w, 1)$ and

$$\lim_{n \to \infty} \frac{\|e_1 + \cdots + e_n\|}{n} = \lim_{n \to \infty} \frac{w_1 + \cdots + w_n}{n} = 0,$$

so $(e_n)_{n \in \mathbb{N}}$ is a normalised weakly null sequence.

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SUBPROJECTIVITY AND SUPERPROJECTIVITY OF BANACH SPACES

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