REALIZATION OF TROPICAL CURVES IN ABELIAN SURFACES

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Abstract. We construct algebraic curves in abelian surfaces starting from tropical curves in real tori. We give a necessary and sufficient condition for a tropical curve in a real torus to be realizable by an algebraic curve in an abelian surface. When the condition is satisfied, the number of algebraic curves can be computed by a combinatorial formula. This gives us an algebraic-tropical correspondence theorem for abelian surfaces analogous to Mikhalkin’s correspondence theorem for toric surfaces. In other words, the number of algebraic curves passing through generic points in an abelian surface can be computed purely combinatorially via tropical curves.

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1. Introduction

Let $B^\times$ be a punctured disk on the complex plane. An analytic family of holomorphic curves in $\mathbb{P}_C^n \times B^\times$ gives rise to tropical curves in $\mathbb{R}^n$, which are balanced piecewise linear weighted graphs. One idea of tropical geometry is to use such combinatorial gadgets to study holomorphic curves (cf. [18, 7, 11]). However, not all tropical curves arise from holomorphic curves. The problem of determining whether a tropical curve arises from holomorphic curves is called the tropical realization problem, also known as the tropical lifting problem.

For toric surfaces, it is a consequence of Mikhalkin’s correspondence theorem [17] that all 3-valent tropical curves in $\mathbb{R}^2$ are realizable. The papers [24, 27] generalized the correspondence theorem to general toric varieties in the case of rational curves. In [6], these were further generalized and the correspondence theorem was proved for regular (but not necessarily 3-valent) tropical curves satisfying certain technical conditions. Finally, in [21], it was proved that all regular tropical curves in $\mathbb{R}^n$ are realizable.

In all these studies, the crucial assumption is that the tropical curves are regular. On the side of holomorphic curves, this corresponds to the condition that the cohomology group in which the obstructions to deforming the curves lie is zero, so that the obstructions automatically vanish. As a consequence, the deformation theory of tropical curves is directly comparable to that of holomorphic curves.

In this paper, we aim to study the realization problem for the case of abelian surfaces, or more generally two dimensional complex tori. Although these varieties are not toric, they are closely related to toric varieties through Mumford’s construction of degenerating families of complex tori [19]. Holomorphic curves in a Mumford’s family of complex tori give rise to tropical curves in a real torus, see Subsection 2. It is natural to ask whether a given tropical curve in a real torus arise from a holomorphic curve. We give a necessary and sufficient condition to this realization problem in the case of 3-valent tropical curves. Moreover, when the tropical curve is realizable, we can compute the number of associated holomorphic curves via a simple combinatorial formula.

The most serious difficulty in this study is the fact that any tropical curve in a real torus is not regular. This affects the problem in almost all aspects, and our study presents quite different picture compared to the previous studies cited above. Explicitly, there are three major points we have to consider.

1. Given a tropical curve on a real torus, there may not be a corresponding degenerate holomorphic curve.
2. Even if there is a degenerate curve, it may not have a first order deformation.
(3) When a first order deformation exists, we need to check that obstructions to deforming it to higher orders vanish.

In the realization problem, one usually starts with constructing a singular holomorphic curve in the central fiber of a degenerating family of a given ambient space in a way that the singular curve reflects properties of a given tropical curve. The central fiber is usually a union of toric varieties glued along toric divisors, and the singular curve is constructed by gluing simple local pieces whose geometry is determined by the combinatorics of the vertices of the tropical curve. The regularity of the tropical curve ensures the transversality in this gluing process. Therefore, the point (1) above did not pose a major problem in the previous studies, at least when the tropical curve is 3-valent. In particular, for a regular tropical curve, there is always a corresponding singular curve. In the situation of this paper, since tropical curves are not regular, we need to be more careful about the gluing. In fact, it turns out that there are tropical curves for which a corresponding singular curve does not exist. However, by studying the gluing condition carefully, we can deduce a simple necessary and sufficient condition for the existence of a corresponding singular curve. This will be done in Theorem 3.8.

After constructing a singular curve on the central fiber of a degenerating family of an ambient space, one tries to deform it to a general fiber of the family. If this is possible, since the singular curve reflects properties of the tropical curve, one can relate tropical curves to holomorphic curves on the original ambient space. Although the central fiber and the curve on it are singular, working in the log smooth deformation theory [13, 12], the regularity of the tropical curve ensures that the obstruction to deforming the singular curve vanish. Thus, the points (2) and (3) were absent in the previous studies, too.

In the present case, even the existence of a first order deformation is not obvious. We need to check the vanishing of the obstructions to such a deformation, and to do this we use a method in [22] which allows us to reduce the calculation of the obstruction class to that of suitable residues, see Proposition 4.3. We combine it with a trick which reduces the situation to the ‘standard’ one, where the curves are represented by linear equations, see Subsection 4.3. Using these arguments, we can explicitly calculate the obstruction to the first order deformation, and we will see it always vanishes in Proposition 4.10.

As for the point (3), when the cohomology group relevant to the obstruction does not vanish, this point can be usually rather difficult to solve. Namely, for each non-negative integer $k$, given a $k$-th order deformation of a map, the obstruction to deforming it one step further depends nonlinearly on the lower order terms. Fortunately, in the present situation we can make use of results in [22] related to
the classical notion of semiregularity [16]. Theorems in [22] claim that when an
immersion from a reduced curve into a smooth complex surface satisfies the so-called
semiregularity condition, then it is unobstructed in the sense that any first order
deformation of it can be extended to arbitrary higher order. When the surface has
trivial canonical sheaf, then the semiregularity condition is automatically satisfied.
The problem is that here we deal with relative situations and the varieties are
singular, but it turns out that the study in [22] can be applied with some more
calculation. Here, the assumption (\*) below plays an important role. Combined
again with the trick reducing the situation to the standard ones mentioned above,
we will be able to show the unobstructedness of the deformation. This will be done
in Subsection 4.7.

We now give the precise statements of our results. We work in the analytic
category, since we will depend on the result in [22] whose proof partly depends on
transcendental method. Let $\Lambda = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}$ be a
$2 \times 2$ integer matrix with nonzero
determinant. Let $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$ be nonzero complex numbers. We denote by
$k = \mathbb{C}\{t\}[t^{-1}]$ the field of convergent series on a punctured disc around the origin
of the complex plane.

Let $B^\times$ be a small punctured disc and consider the product $(\mathbb{C}^\times)^2 \times B^\times$. It has
the natural group structure by the fiberwise multiplication. Let $\Lambda$ be a subgroup
of it isomorphic to $\mathbb{Z}^2$ which is a discrete subgroup on each fiber. We take the
quotient of $(\mathbb{C}^\times)^2 \times B^\times$ by $\Lambda$ to make a family $X^\times$ of compact surfaces over $B^\times$,
but in this paper we assume that $\Lambda$ is of the following special form.

\[(\ast) \quad \Lambda \text{ is generated by } \lambda_1 := (\alpha_{11} t^{n_{11}}, \alpha_{12} t^{n_{12}}) \text{ and } \lambda_2 := (\alpha_{21} t^{n_{21}}, \alpha_{22} t^{n_{22}}).\]

Such a family can be compactified to a family $X$ over the disc $B$, see [19]. We call
such a family a Mumford family. We can also obtain a family of compact surfaces
when $\alpha_{ij} t^{n_{ij}}$ is replaced by a more general series like $\alpha_{ij} t^{n_{ij}} + \alpha_{ij,1} t^{n_{ij}+1} + \cdots$, but
in general the associated surface may not contain any curve.

Let $N = \mathbb{Z}^2$ and $S = N_\mathbb{R}/\Lambda$. A parametrized algebraic curve in $X^\times$ consists of a finite field extension $k \hookrightarrow k'$, a smooth projective curve $C$ over $k'$ and a
map $f : C \to X_{k'}^\times$, where $X_{k'}^\times$ is the base change of $X^\times$ corresponding to the
field extension. A parametrized tropical curve in $S$ consists of a metric graph $\Gamma$
and a $\mathbb{Z}$-affine immersion $h : \Gamma \to S$ that is balanced at every vertex of $\Gamma$. We
assume moreover that the image of every vertex of $\Gamma$ is a rational point. Given a
parametrized algebraic curve $f : C \to X_{k'}^\times$ whose closure in some compactification
$X_{k'}^\times$ is an irreducible surface, we can associate with it a parametrized tropical curve
$h : \Gamma \to S$ in $S$, see Section 2.
On the other hand, a parametrized tropical curve $h: \Gamma \to S$ is called \textit{realizable} if it comes from a parametrized algebraic curve in $\mathcal{X}$ via the above tropicalization process. Our main goal is to find necessary and sufficient conditions for realizability.

Given a parametrized tropical curve $h: \Gamma \to S$, we can lift it to a $\Lambda$-periodic tropical curve $\tilde{h}: \tilde{\Gamma} \to N_\mathbb{R}$. Let $\Delta \subset N_\mathbb{R}$ be a parallelogram fundamental domain of the $\Lambda$-action with four sides $B_1, \ldots, B_4$ such that $\lambda_1(B_3) = B_1$ and $\lambda_2(B_4) = B_2$. We assume that $\tilde{h}(\tilde{\Gamma})$ intersects the boundary $\partial \Delta$ transversally, i.e. the intersections occur away from the vertices of $\tilde{\Gamma}$ and the corners of $\partial \Delta$.

Let $\tilde{e}_1, \ldots, \tilde{e}_l$ (resp. $\tilde{f}_1, \ldots, \tilde{f}_m$) be the edges of $\tilde{\Gamma}$ whose image intersects $B_1$ (resp. $B_2$). Let $(a_i, b_i)$ (resp. $(c_j, d_j)$) be the weight vector (i.e. derivative) of $\tilde{h}$ at $\tilde{e}_i$ (resp. $\tilde{f}_j$) pointing from the inside to the outside of $\Delta$.

The weight $w_e$ of an edge $e$ of $\Gamma$ is the multiplicity of the weight vector. Let $\delta$ be the greatest common divisor of the weights of all edges of $\Gamma$. The weight $w_v$ of a 3-valent vertex $v$ of $\Gamma$ is the norm of the cross product of the weight vectors of any two of the three edges connected to $v$. It is independent of the choice by the balancing condition.

We introduce the following quantity

$$\sigma := \prod_{i=1}^l \left( \frac{\alpha_{12} a_i}{\delta} \frac{b_i}{\delta} \right) \cdot \prod_{j=1}^m \left( \frac{\alpha_{22} c_j}{\delta} \frac{d_j}{\delta} \right).$$

It follows from the balancing condition that $\sigma$ does not depend on the choice of the fundamental domain $\Delta \subset N_\mathbb{R}$.

\textbf{Theorem 1.1.} Let $h: \Gamma \to S$ be a 3-valent parametrized tropical curve in $S$. It is realizable by a parametrized algebraic curve in $X$ if and only if the quantity $\sigma$ is equal to $(-1)^{\sum w_v / \delta}$.

Given Theorem 1.1, we are curious to count how many parametrized algebraic curves in $X$ can give rise to a given realizable parametrized tropical curve in $S$. In order to obtain a finite count, we need to fix some constraints to bring down the dimension of moduli spaces.

Let $h: \Gamma \to S$ be a 3-valent realizable parametrized tropical curve of genus $g$ passing through $g$ rational points $p_1, \ldots, p_g \in S$. Assume that the preimages $h^{-1}(p_i)$ lie in the interiors of the edges of $\Gamma$, and the constraints make the tropical curve rigid. Let $\Gamma'$ denote the graph $\Gamma$ subdivided by $h^{-1}(p_i)$, and let $v_i$ be the vertex of $\Gamma'$ with $h(v_i) = p_i$. Let $x_1, \ldots, x_g$ be $k$-rational points in $X$, possibly after a finite base field extension, with $\tau(x_i) = p_i$. 

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Let $\tilde{e}_1, \ldots, \tilde{e}_l$ (resp. $\tilde{f}_1, \ldots, \tilde{f}_m$) be the edges of $\tilde{\Gamma}$ whose image intersects $B_1$ (resp. $B_2$). Let $(a_i, b_i)$ (resp. $(c_j, d_j)$) be the weight vector (i.e. derivative) of $\tilde{h}$ at $\tilde{e}_i$ (resp. $\tilde{f}_j$) pointing from the inside to the outside of $\Delta$.

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We define a map

\[
G: \text{Map}(V(\Gamma'), N) \longrightarrow \bigoplus_{e \in E(\Gamma')} (N/N_e) \oplus \bigoplus_{i=1}^{g} N
\]

\[
\phi \longmapsto \left( (\phi(\partial^+ e) - \phi(\partial^- e))_e, (\phi(v_i))_i \right)
\]

where \( V(-) \) denotes the set of vertices, \( E(-) \) denotes the set of edges, \( N_e \) denotes the sublattice generated by the weight vector of \( e \), and \( \partial^- e, \partial^+ e \) denote the two endpoints of an edge \( e \) according to a fixed orientation. Consider the tensor product of \( G \) with \( \mathbb{C}^* \) over \( \mathbb{Z} \):

\[
G_{\mathbb{C}^*}: \text{Map}(V(\Gamma'), N_{\mathbb{C}^*}) \longrightarrow \bigoplus_{e \in E(\Gamma')} (N/N_e)_{\mathbb{C}^*} \oplus \bigoplus_{i=1}^{g} N_{\mathbb{C}^*}.
\]

**Theorem 1.2.** The number of parametrized algebraic curves in \( X \) (up to equivalence given by base field extension) passing through \( x_1, \ldots, x_g \) that give rise to the 3-valent realizable parametrized tropical curve \( h: \Gamma \to S \) is equal to the product

\[
|\ker G_{\mathbb{C}^*}| \cdot \prod_{e \in E(\Gamma)} w_e.
\]

**Related works.** The realization problem in tropical geometry has been studied extensively in the literature. Besides the works [17, 24, 27, 20] already mentioned in the beginning, various interesting results are obtained in the works by Shustin [25], Speyer [26], Brugallé-Shaw [4], Katz-Payne [15], Katz [14], Bogart-Katz [3], Gathmann-Schmitz-Winstel [8]. We would also like to mention the work of Hallo-Rose [10] regarding the enumeration of tropical curves in tropical abelian varieties. For the counting problem of Gromov-Witten type, there is a beautiful result by Bryan and Leung [5]. Contrary to Gromov-Witten type invariants, our counting is sensitive to the complex structure. Thus, it gives a combinatorial counting of curves on the family of complex tori satisfying the condition (*) above.

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2. Tropicalization of curves in complex tori

In this section, we give precise definitions concerning tropical curves and describe the tropicalization procedure for curves in complex tori. We use the setup in the introduction.

**Definition 2.1.** A metric graph $\Gamma$ is a metric space whose underlying topological space is a graph, and the metric on each edge is modeled on an interval $[0, l]$ for $l \in (0, +\infty]$. A vertex of $v$ of $\Gamma$ is called an infinite vertex if any neighborhood of $v$ has infinite length, otherwise it is called a finite vertex. We assume that all the infinite vertices are 1-valent. We denote by $V(\Gamma)$ the set of vertices of $\Gamma$, by $V_\infty(\Gamma)$ the set of infinite vertices of $\Gamma$, and by $E(\Gamma)$ the set of edges of $\Gamma$. For each vertex $v$ of $\Gamma$, we denote by $e_{v,1}, e_{v,2}, \ldots$ the edges of $\Gamma$ connected to $v$. For each edge $e$ of $\Gamma$, we denote by $e^c$ its interior, i.e. the edge minus the two endpoints.

**Definition 2.2.** A parametrized tropical curve in $\mathbb{N}_\mathbb{R}$ consists of a metric graph $\Gamma$ and a continuous immersion $h: \Gamma \setminus V_\infty(\Gamma) \to \mathbb{N}_\mathbb{R}$ satisfying the following conditions:

1. The image of every finite vertex in $\Gamma$ is a rational point.
2. For each finite vertex $v$ of $\Gamma$ and each edge $e$ connected to $v$, the restriction $h|_{e^c}$ is linear with integer derivative $w_{v,e} \in \mathbb{N}$, defined with respect to the unit tangent vector pointing from $v$ to $e$. We call $w_{v,e}$ the weight vector of the edge $e$ at $v$.
3. (Balancing condition) For each finite vertex $v$, we require that
   $$\sum_{e \ni v} w_{v,e} = 0$$
   where the sum is over all edges connected to $v$.

For easier notation, we will denote the parametrized tropical curve above by $h: \Gamma \to \mathbb{N}_\mathbb{R}$, omitting minus $V_\infty(\Gamma)$.

**Definition 2.3.** A parametrized tropical curve in $S = \mathbb{N}_\mathbb{R}/\Lambda$ consists of a metric graph $\Gamma$ without infinite vertices whose underlying graph is finite and a continuous immersion $h: \Gamma \to S$ satisfying the same conditions as in Definition 2.2.

**Definition 2.4.** For an edge $e$ with an endpoint $v$ of a parametrized tropical curve, the weight of $e$, denoted by $w_e$, is the multiplicity of the integral vector $w_{v,e}$. For any vertex $v$, we denote by $\gamma_v$ the greatest common divisor of the weights of all the edges connected to $v$. For a 3-valent vertex $v$, the weight of $v$, denoted by $w_v$, is the norm of the cross product of the weight vectors of any two of the three edges connected to $v$. It is independent of the choice of the two edges by the balancing condition. For a 2-valent vertex $v$, we define its weight $w_v$ to be 1.
Now we explain how to obtain a tropical curve on a real two dimensional torus given a suitable family of holomorphic curves in a degenerating family of complex tori. Since this is not the main theme of this paper (our purpose is to construct a family of holomorphic curves from a given tropical curve), we will be brief. See [24, Section 6] for more about this construction.

Let \( X \to B \) be a Mumford family of degenerating complex tori of dimension two, see [19]. Here \( B \) is a small disk in \( \mathbb{C} \). It is determined by a polyhedral decomposition of a real two dimensional torus. Its central fiber \( X_0 \) is a reduced union of toric surfaces glued along toric divisors. Suppose we have a family of irreducible holomorphic curves \( \psi^\times : C^\times \to X^\times \) over \( B^\times \), where \( B^\times \) is the punctured disc \( B \setminus \{0\} \), and assume that the closure of the image of \( \psi^\times \) in \( X \) is an irreducible variety. In particular, it gives a holomorphic curve \( C_0 \) on the central fiber \( X_0 \). By [24, Proposition 6.2], after some toric blow-ups if necessary (note that for each point of \( X_0 \), there is a neighborhood of it in \( X \) which is canonically isomorphic to a toric variety, so toric blow-ups make sense), we can assume that \( C_0 \) is torically transverse in the sense of [24, Definition 4.1]. This modification of \( X \) gives a refinement \( \mathcal{D} \) of the original polyhedral decomposition of the real two dimensional torus. Then each component of \( C_0 \) is a torically transverse curve in an irreducible component of \( X_0 \) which is a toric surface, and its intersection multiplicity with the toric divisors determines a tropical curve with one vertex whose underlying graph is a subgraph of \( \mathcal{D} \). These pieces can be glued into a global tropical curve on the real torus.

Remark 2.5. In this construction, the curve \( C_0 \) can be highly singular and may have geometric genus lower than that of the curve on the generic fiber. Also, it may have singular points on toric divisors of components of \( X_0 \). Even in such a case, the above construction gives a tropical curve on the torus. This is because the weights of edges and the balancing condition at vertices are determined by topological data as follows.

For notational simplicity, we assume \( C_0 \) is an embedded curve. Let \( p \in C_0 \) be a point on a toric divisor \( D \) of some component of \( X_0 \). Take the intersection of a neighborhood \( U_p \) of \( p \) in \( X \) and the fiber \( X_t \) of \( X \) over \( t \in B \) with \( |t| \) small. Taking \( U_p \) appropriately, \( U_p \cap X_t \) is homeomorphic to \( C \times \Delta \), where \( C = S^1 \times (-1,1) \) is a cylinder and \( \Delta \) is a disc. On the other hand, the intersection \( X_0 \cap U_p \) is homeomorphic to \( E \times \Delta \), where \( E \) is the union of two cones obtained by collapsing the circle \( S^1 \times \{0\} \) of \( C \) to a point. Moreover, \( X_0 \cap U_p \) is a deformation retract of \( U_p \) and the set \( \{e\} \times \Delta \), where \( e \) is the apex of \( E \), is an open subset of the toric divisor \( D \). Let \( a \) be the homology class of the intersection of the image of \( \psi^\times \) and \( X_t \cap U_p \), where \( H_1(X_t \cap U_p, \mathbb{Z}) \cong H_1(C, \mathbb{Z}) \cong \mathbb{Z} \). This \( a \) does not depend on \( t \) when \( |t| \) is sufficiently small. There may be several connected components
in $\text{Im} \psi^\times \cap (X_t \cap U_p)$, but it does not matter. Clearly, $a$ is equal to the (total) intersection multiplicity of each branch of $C_0$ at $p$ with $D$. This shows that the weights of edges of the associated tropical curve are well-defined. Moreover, the balancing condition is the requirement needed for the claim that each component of $C_0$ is a cycle without boundary.

3. Construction of pre-log curves in the special fiber

The goal of this section is to study the existence of pre-log curves associated with a given 3-valent parametrized tropical curve $h: \Gamma \to S$.

Without loss of generality, we assume that $\Gamma$ is connected. Let $\mathcal{D}$ be a polyhedral subdivision of $S$ whose 1-skeleton contains the image $h(\Gamma)$. We denote by $\mathcal{D}^0$ and $\mathcal{D}^1$ respectively the set of 0-cells and 1-cells of $\mathcal{D}$. Since finite base field extensions rescale the lattice $N \subset N_\mathbb{R}$, up to a finite base field extension, we can assume that $\mathcal{D}$ is integral, i.e. $\mathcal{D}^0$ contains only lattice points. Also, for later purpose, we assume that the lattice length of the image of each edge $e$ of $\Gamma$ is an integral multiple of its weight $w_e$. The preimage $h^{-1}(\mathcal{D}^0)$ induces a subdivision of $\Gamma$, which we denote by $\Gamma$ again. Note that now $\Gamma$ can have 2-valent vertices.

The subdivision $\mathcal{D}$ induces a degeneration $\mathcal{X}$ of $X$ over a small disc. For every nonnegative integer $n$, we denote $X_n := \mathcal{X}/(t^{n+1})$, where $t$ is a coordinate on the disc. In particular, $X_0 = \mathcal{X}_s$ is the special fiber of $\mathcal{X}$. For $\rho \in \mathcal{D}^0$, we denote by $X_{0,\rho}$ the corresponding irreducible component of $X_0$, which is a toric variety over $\mathbb{C}$. We denote by $\partial X_{0,\rho}$ its toric boundary. For $\sigma \in \mathcal{D}^1$, we denote by $X_{0,\sigma}$ the corresponding 1-stratum of $X_0$.

The subdivision $\mathcal{D}$ also induces a $\Lambda$-periodic polyhedral subdivision $\tilde{\mathcal{D}}$ of $N_\mathbb{R}$. Thus we obtain a toric variety $\tilde{\mathcal{X}}$ over $\mathbb{C}[t]$ of infinite type whose restriction over the disc is a covering of $\mathcal{X}$. Let $\tilde{X}_0$ denote the special fiber of $\tilde{\mathcal{X}}$.

Definition 3.1. A pre-log curve in $X_0$ associated with the parametrized tropical curve $h: \Gamma \to S$ consists of a proper algebraic curve $C_0$ over $\mathbb{C}$ and a map $\varphi_0: C_0 \to X_0$ satisfying the following conditions:

1. The curve $C_0$ has at worst nodal singularities.
2. The dual intersection graph of $C_0$ is isomorphic to $\Gamma$.
3. All irreducible components of $C_0$ are isomorphic to $\mathbb{P}^1_\mathbb{C}$.
4. For all $v \in V(\Gamma)$, we have $\varphi_0(C_{0,v}) \subset X_{0,h(v)}$, where $C_{0,v}$ denotes the corresponding irreducible component of $C_0$.
5. For all $e \in E(\Gamma)$, we have $\varphi_0(p_e) \in X_{0,h(e)} \setminus \partial X_{0,h(e)}$, where $p_e$ denotes the corresponding node of $C_0$. 


(6) For all $v \in V(\Gamma)$, we have
\[ (\varphi_0|_{C_0,v})^*(\partial X_{0,h(v)}) = \sum_{e \ni v} w_e \cdot p_e. \]

Our strategy for the construction of pre-log curves is to glue local pieces together, taking care of boundary conditions. Let us begin by studying the boundary values of local pieces.

Let $T$ be the metric graph consisting of one finite vertex $v$, $n$ infinite vertices and $n$ edges $e_1, \ldots, e_n$ connecting the infinite vertices to the finite vertex. Let $g: T \to \mathbb{N}_\mathbb{R}$ be a parametrized tropical curve in $\mathbb{N}_\mathbb{R}$ with $g(v) = 0 \in \mathbb{N}_\mathbb{R}$. For $i = 1, \ldots, n$, let $w_i$ be the weight of the edge $e_i$. Write $w_{v,e_i} = w_i(p_i, q_i) \in \mathbb{N}$, presented using the standard basis of $\mathbb{N}_\mathbb{R} = \mathbb{Z}^2$.

Let $Y$ denote the (non-proper) toric variety over $\mathbb{C}$ associated with the fan given by the image $g(T)$.

\textbf{Definition 3.2.} A \textit{pre-log curve} in $Y$ associated with the tropical curve $g: T \to \mathbb{N}_\mathbb{R}$ consists of a map $\varphi_0: \mathbb{P}^1 \to Y$ such that the pullback of the toric divisor corresponding to the ray $g(e_i)$ is exactly one point with multiplicity $w_i$ for $i = 1, \ldots, n$.

Now we restrict to the case where $T$ contains exactly three edges. Moreover, let $g_0: T \to \mathbb{N}_\mathbb{R}$ be the special case of $g$ where the three weight vectors are equal to $(1, 0), (0, 1)$ and $(-1, -1)$ respectively. Let $Y_0$ denote the (non-proper) toric variety over $\mathbb{C}$ associated with the fan given by the image $g_0(T)$, which is isomorphic to $\mathbb{P}^2$ minus three points.

The image $g(T)$ is equal to the image of the composition of $g_0$ and the linear map given by the matrix
\[ L = \begin{pmatrix} w_1 p_1 & w_2 p_2 \\ w_1 q_1 & w_2 q_2 \end{pmatrix}. \]
The linear map given by $L$ induces a map
\[ \Phi_L: Y_0 \to Y \]
between toric surfaces. Note that the weight $w_v$ of the vertex $v$ is equal to $|\det L|$ (cf. Definition 2.4).

Let $(e_1^\vee, e_2^\vee)$ be the dual standard basis of the dual space $\mathbb{N}^\vee$. Let $x_1, x_2$ be the corresponding rational functions on $Y_0$. Consider the vectors in $\mathbb{N}^\vee$ with coordinates
\[ (-q_1, p_1), \quad (-q_2, p_2), \quad (-q_3, p_3). \]
They are primitive integer vectors in the annihilator subspaces $g(e_1)^\perp$, $g(e_2)^\perp$ and $g(e_3)^\perp$ respectively. Let $X_1, X_2$ and $X_3$ be the corresponding rational functions on $Y$. 
Lemma 3.3. The pullbacks of the functions $X_1, X_2$ and $X_3$ by the map $\Phi_L$ are equal to

\[
\frac{\det L}{x_2^{w_1}}, \quad \frac{-\det L}{x_1^{w_2}}, \quad \left(\frac{x_1}{x_2}\right)^{\frac{\det L}{w_3}}
\]

respectively.

Proof. We have

\[
\begin{pmatrix} w_1p_1 & w_1q_1 \\ w_2p_2 & w_2q_2 \end{pmatrix} \begin{pmatrix} -q_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -w_2p_2q_1 + w_2q_2p_1 \end{pmatrix} = \left(\frac{0}{\det L}\right).
\]

So the pullback of $X_1$ by the map $\Phi_L$ is equal to $\frac{\det L}{x_2^{w_1}}$. Similar computations apply to the other two functions. \[\square\]

Lemma 3.4. Let $\gamma$ be the greatest common divisor of $w_1, w_2, w_3,$ and write

\[w_i = \gamma w'_i, \quad i = 1, 2, 3.\]

For any pre-log curve $\phi_0: \mathbb{P}^1 \to Y$ associated with the tropical curve $g: T \to N_\mathbb{R}$, let $\mu_i$ be the value of the function $X_i$ at the intersection point of $\phi_0(\mathbb{P}^1)$ with the toric divisor corresponding to the ray $g(e_i)$. Then we have the relation

\[\mu_1^{w'_1} \mu_2^{w'_2} \mu_3^{w'_3} = (-1)^{\frac{w_1}{\gamma}}.\]

Conversely, for any numbers $\nu_1, \nu_2, \nu_3$ satisfying the relation

\[\nu_1^{w'_1} \nu_2^{w'_2} \nu_3^{w'_3} = (-1)^{\frac{w_1}{\gamma}},\]

there is a pre-log curve $\phi_0: \mathbb{P}^1 \to Y$ associated with the tropical curve $g: T \to N_\mathbb{R}$ with $\mu_i = \nu_i$, for $i = 1, 2, 3$.

Proof. Let $\psi_0: \mathbb{P}^1 \to Y_0$ be a pre-log curve associated with the tropical curve $g_0: T \to N_\mathbb{R}$. Its image is given by an equation of the form

\[\beta_1 x_1 + \beta_2 x_2 - 1 = 0,\]

where $\beta_1$ and $\beta_2$ are nonzero complex numbers. It intersects the toric boundary of $Y_0$ at three points with coordinates

\[
(x_1, x_2) = \left(\frac{1}{\beta_1}, 0\right), \quad (x_1, x_2) = \left(0, \frac{1}{\beta_2}\right), \quad \left(\frac{x_1}{x_2}, x_2\right) = \left(0, \frac{-\beta_1}{\beta_2}\right).
\]
Note that the composition $\phi_0 := \Phi_L \circ \psi_0 : \mathbb{P}^1 \to Y$ is a pre-log curve associated with the tropical curve $g : T \to N_\mathbb{R}$, and every pre-log curve in $Y$ associated with $g$ arises in this way (cf. [24, Lemma 5.3]). By Lemma 3.3, we have

\[
\mu_1 = \beta_2 \frac{\det L}{w_1}, \quad \mu_2 = \beta_1 \frac{\det L}{w_2}, \quad \mu_3 = \left( -\frac{\beta_2}{\beta_1} \right) \frac{\det L}{w_3}.
\]

Thus we have

\[
\begin{align*}
\mu_1 \mu_2 \mu_3 &= \beta_2 \beta_1 \left( -\frac{\beta_2}{\beta_1} \right) \frac{\det L}{w_3} \frac{w_1}{w_2} \frac{w_2}{w_3} = \beta_2 \beta_1 \left( -\frac{\beta_2}{\beta_1} \right) \frac{\det L}{w_3} \frac{w_3}{w_1} = \left( -1 \right) \frac{\det L}{w_1} = \left( -1 \right) \frac{w_v}{w_1}.
\end{align*}
\]

Conversely, let $\nu_1, \nu_2, \nu_3$ be complex numbers satisfying the relation

\[
\nu_1 \nu_2 \nu_3 = \left( -1 \right) \frac{w_v}{w_1}.
\]

By the discussion above, in order to obtain a pre-log curve in $Y$ associated with the tropical curve $g : T \to N_\mathbb{R}$ with $\mu_i = \nu_i$ for $i = 1, 2, 3$, it suffices to prove that $(\nu_1, \nu_2, \nu_3)$ can be written in the form of Eq. (3.5). Take complex numbers $\beta_1$ and $\beta_2$ so that

\[
\begin{align*}
\nu_1 &= \beta_2 \frac{\det L}{w_1}, \\
\nu_2 &= \beta_1 \frac{\det L}{w_2},
\end{align*}
\]

Then

\[
\nu_3 = \zeta_3 \left( -\frac{\beta_2}{\beta_1} \right) \frac{\det L}{w_3},
\]

where $\zeta_3$ is a $w'_3$-th root of unity. Note that we can multiply $\beta_2$ by any $\frac{w_v}{w_1}$-th root of unity $\zeta_1$, and multiply $\beta_1$ by any $\frac{w_v}{w_2}$-th root of unity $\zeta_2$ without changing $\nu_1$ and $\nu_2$. So it suffices to find $\zeta_1$ and $\zeta_2$ such that

\[
\left( \frac{\zeta_1}{\zeta_2} \right)^{\frac{\det L}{w_3}} = \zeta_3^{-1}.
\]

Write

\[
\zeta_1 = e^{2\pi i \frac{w_1}{w_v}}, \quad \zeta_2 = e^{2\pi i \frac{w_2}{w_v}}, \quad \zeta_3 = e^{2\pi i \frac{w_3}{w_v}},
\]

where $l, m, n$ are integers. Substituting these into Eq. (3.6), we see that it is enough to have

\[
\begin{align*}
lw_1 \cdot \det L + mw_2 \cdot \det L &\equiv -\frac{n}{w_3} \mod 1,
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
lw_1 - mw_2 &\equiv -\text{sign} \left( \det L \right) \cdot n \gamma \mod w_3,
\end{align*}
\]

where

\[
\text{sign} \left( \det L \right) = \begin{cases} 1 & \text{if } \det L > 0, \\
-1 & \text{if } \det L < 0.
\end{cases}
\]
Since $\gamma$ is the greatest common divisor of $w_1, w_2, w_3$, we can find integers $l, m, n$ so that Eq. (3.7) holds, completing the proof of the lemma. \hfill \Box

Now we begin the construction of pre-log curves associated with a given 3-valent parametrized tropical curve $h: \Gamma \to S$.

We first recall some notations from Section 1. We lift $h$ to a $\overline{\mathcal{X}}$-periodic tropical curve $\tilde{h}: \tilde{\Gamma} \to N_{\mathbb{R}}$. Let $\Delta \subset N_{\mathbb{R}}$ be a parallelogram fundamental domain of the $\overline{\mathcal{X}}$-action with four sides $B_1, \ldots, B_4$, such that $\mathcal{X}_1(B_3) = B_1$, $\mathcal{X}_2(B_4) = B_2$, and that $\tilde{h}(\tilde{\Gamma})$ intersects the boundary $\partial \Delta$ transversally. Let $\tilde{e}_1, \ldots, \tilde{e}_i$ (resp. $\tilde{f}_1, \ldots, \tilde{f}_m$) be the edges of $\tilde{\Gamma}$ whose images intersect $B_1$ (resp. $B_2$). Let $(a_i, b_i)$ (resp. $(c_j, d_j)$) be the weight vector of $\tilde{e}_i$ (resp. $\tilde{f}_j$) pointing from the inside to the outside of $\Delta$. Let us define a complex number $\sigma$ by

$$\sigma := \prod_{i=1}^{l} \left( \frac{a_i}{\alpha_i^{12\delta}} \frac{b_i}{\alpha_i^{11\delta}} \right) \cdot \prod_{j=1}^{m} \left( \frac{c_j}{\alpha_j^{22\delta}} \frac{-d_j}{\alpha_j^{21\delta}} \right),$$

here $\delta$ is the greatest common divisor of all edge weights of $\Gamma$.

**Theorem 3.8.** There exists a pre-log curve associated with $h: \Gamma \to S$ if and only if $\sigma = (-1)^{\sum_{v \in \Gamma[0]} w_v / \delta}$.

**Proof.** We introduce an ordering $v_1, \ldots, v_n$ on $V(\Gamma)$ as follows: We start with any 3-valent vertex $v_1 \in V$. By subdividing $\Gamma$ if necessary, we can assume that the image of each edge connected to $v_1$ does not intersect the image of $\partial \Delta$ under the covering map $N_{\mathbb{R}} \to S$. Assume we have constructed a sequence $v_1, \ldots, v_i$. Then we choose $v_{i+1} \in V(\Gamma) \setminus \{v_1, \ldots, v_i\}$ such that $v_{i+1}$ is connected to $v_j$ for some $j = 1, \ldots, i$.

For $i = 1, \ldots, n$, let $\Gamma_i \subset \Gamma$ be the closed subgraph given by the union of the vertices $v_i, \ldots, v_n$ and all the edges between them. If $v_i$ is a 3-valent vertex, let $g_{v_i}: T \to N_{\mathbb{R}}$ be the parametrized tropical curve in $N_{\mathbb{R}}$ such that the finite vertex of $T$ maps to $0 \in N_{\mathbb{R}}$, and the weight vectors associated with the three infinite edges of $T$ are equal to those associated with the three edges of $\Gamma$ connected to $v_i$.

The case when $v_i$ is 2-valent is easy and we omit it.

Let $\psi_{v_i}: P_{v_i} = \mathbb{P}^1 \to X_{0, h(v_i)} \subset X_0$ be a pre-log curve associated with the tropical curve $g_{v_i}: T \to N_{\mathbb{R}}$. Let $C_{0,n} := P_{v_n}$ and $\phi_n := \psi_{v_n}$. Suppose we have constructed $\phi_{i+1}: C_{0,i+1} \to X_0$ for some $i \geq 2$ such that the dual graph of $C_{0,i+1}$ is $\Gamma_{i+1}$. We construct $\phi_i: C_{0,i} \to X_0$ as follows.

Let $C_{0,i}$ be the gluing of $C_{0,i+1}$ and $P_{v_i} := \mathbb{P}^1$ such that the dual graph of $C_{0,i}$ is $\Gamma_i$. By the construction of the ordering $v_1, \ldots, v_n$, there exists at least one edge connected to $v_i$ that does not belong to $\Gamma_i$. Therefore, we can extend the map
\( \phi_{i+1} : C_{0,i+1} \to X_0 \) to a map \( \phi_i : C_{0,i} \to X_0 \) such that the restriction \( \psi_{v_i} := \phi_i|_{P_{v_i}} \) is a pre-log curve in \( X_{0,h(v_i)} \) associated with the tropical curve \( g_{v_i} : T \to NR \).

We iterate this from \( i = n \) to \( i = 2 \). Finally, let \( C_{0,1} \) be the gluing of \( C_{0,2} \) and \( P_{v_i}:=\mathbb{P}^1 \) such that the dual graph of \( C_{0,1} \) is \( \Gamma_1 = \Gamma \). We need to extend the map \( \phi_2 : C_{0,2} \to X_0 \) to a map \( \phi_1 : C_{0,1} \to X_0 \) such that the restriction \( \psi_{v_{1}} := \phi_1|_{P_{v_{1}}} \) is a pre-log curve in \( X_{0,h(v_{1})} \) associated with the tropical curve \( g_{v_{1}} : T \to NR \).

For \( i = 2, \ldots, n \) and \( j = 1, 2, \ldots, \), let \( \mu_{v_i,j} \) be the coordinate of the intersection point between \( \psi_{v_i}(P_{v_i}) \) and the stratum \( X_{0,h(e_{v_i,j})} \), as in Lemma 3.4.

**Lemma 3.9.** Let \( \nu_1, \nu_2, \nu_3 \) be the coordinates of the intersection points between \( \phi_2(C_{0,2}) \) and the three boundary components of \( X_{0,h(v_{1})} \). We have

\begin{equation}
(3.10)
\sigma \cdot \nu_1 w_{v_1,1}/\delta \nu_2 w_{v_1,2}/\delta \nu_3 w_{v_1,3}/\delta = \prod_{i=2}^{n} (-1)^{w_{v_i}/\delta},
\end{equation}

where \( w_{v_i,j} \) is short for \( w_{e_{v_i,j}} \), the weight of the \( j \)-th edge connected to the vertex \( v_i \).

**Proof.** Let \( e \) be an edge of \( \Gamma \) not connected to \( v_1 \), with endpoints \( v_i \) and \( v_{v_r} \). Assume that \( e = e_{v_i,j} = e_{v_r,j'} \). If \( h(e) \) does not intersect the image of \( \partial \Delta \) under the covering map \( \tilde{N}_{\mathbb{R}} \to S \), we have \( \mu_{v_i,j} \mu_{v_r,j'} = 1 \). Otherwise, the product \( \mu_{v_i,j} \mu_{v_r,j'} \) takes into account the coordinate changes induced by the action of \( \Lambda \) on \( X \). More precisely, let \( \tilde{e} \) be the representative of \( e \) in \( \tilde{\Gamma} \) that intersects the boundary \( B_1 \) or \( B_2 \) of \( \Delta \), and let \( w_e(p, q) \in N \) be the weight vector of \( e \) pointing from the inside to the outside of \( \Delta \). In the case where \( \tilde{e} \) intersects \( B_1 \), we have \( \mu_{v_i,j} \mu_{v_r,j'} = \alpha_{12}^{w_e(p, q)} \alpha_{21}^{-w_e(q, p)} \). In the case where \( \tilde{e} \) intersects \( B_2 \), we have \( \mu_{v_i,j} \mu_{v_r,j'} = \alpha_{22}^{w_e(p, q)} \alpha_{21}^{-w_e(q, p)} \). Taking the product of \( \mu_{v_i,j} \mu_{v_r,j'} \) over all edges of \( \Gamma \) except \( e_{v_1,1}, e_{v_1,2}, e_{v_1,3} \), we obtain

\[
\prod_{i=2}^{n} \prod_{j} \mu_{v_i,j} w_{v_i,j}/\delta = \sigma \cdot \nu_1 w_{v_1,1}/\delta \nu_2 w_{v_1,2}/\delta \nu_3 w_{v_1,3}/\delta.
\]

On the other hand, if \( v_i \) is 2-valent, we have \( \mu_{v_1,1} \mu_{v_1,2} = 1 \). If \( v_i \) is 3-valent, by Lemma 3.4, we have \( \mu_{v_1,1} w_{v_1,1}/\delta \mu_{v_1,2} w_{v_1,2}/\delta \mu_{v_1,3} w_{v_1,3}/\delta = (-1)^{w_{v_1}/\delta} \). Therefore, we have

\[
\prod_{i=2}^{n} \prod_{j} \mu_{v_i,j} w_{v_i,j}/\delta = \prod_{i=2}^{n} (-1)^{w_{v_i}/\delta}.
\]

Combining the two equations above, we obtain Eq. (3.10).

Let \( \gamma_{v_1} \) be the greatest common divisor of the edge weights \( w_{v_1,1}, w_{v_1,2} \) and \( w_{v_1,3} \). By Lemma 3.4, we can fill in the pre-log curve \( \psi_{v_1} : P_{v_1} \to X_{0,h(v_1)} \) if and only if the numbers \( \nu_1, \nu_2, \nu_3 \) in Lemma 3.9 satisfy the equality

\begin{equation}
(3.11)
\nu_1 w_{v_1,1}/\gamma_{v_1} \nu_2 w_{v_1,2}/\gamma_{v_1} \nu_3 w_{v_1,3}/\gamma_{v_1} = (-1)^{w_{v_1}/\gamma_{v_1}}.
\end{equation}
Therefore, by Lemma 3.9, the existence of a pre-log curve associated with $h: \Gamma \to S$ implies $\sigma = (-1)^{\sum w_\nu / \delta}$. This proves one direction in the statement of Theorem 3.8.

Now for the other direction, let us assume that $\sigma = (-1)^{\sum w_\nu / \delta}$ holds, and we would like to fill in the last piece $\psi_{v_i}: P_{v_i} \to X_{0,h(v_i)}$. Lemma 3.9 implies that the numbers $\nu_1, \nu_2, \nu_3$ satisfy the equality

$$
\nu_1^{\gamma_{v_1}/\gamma_1} \nu_2^{w_{v_1,2}/\gamma_1} \nu_3^{w_{v_1,3}/\gamma_1} = \zeta \cdot (-1)^{w_{v_1}/\gamma_1},
$$

for some $\gamma_{v_1}/\delta$-th root of unity $\zeta$.

**Lemma 3.13.** We can modify the maps $\psi_{v_i}: P_{v_i} \to X_{0,h(v_i)}$ for $i = 2, \ldots, n$, so that the equality (3.11) holds.

**Proof.** Let

$$
\gamma_{v_1}/\delta = \epsilon_1^{m_1} \cdots \epsilon_k^{m_k}
$$

be its decomposition into primes. Accordingly, the multiplier $\zeta$ factors as

$$
\zeta = \zeta_1 \cdots \zeta_k,
$$

where $\zeta_j$ is an $\epsilon_j^{m_j}$-th root of unity.

Consider the prime $\epsilon_j$. Since $\delta$ is the greatest common divisor of all the edge weights of $\Gamma$, there is a vertex $u$ of $\Gamma$ and an edge $e$ connected to $u$ such that $w_e/\delta$ is not divisible by $\epsilon_j$. We choose $u$ such that there is a injective path $P$ in $\Gamma$ from $v_1$ to $u$ consisting of edges whose weights are all divisible by $\delta \epsilon_j$.

Let $u_1 = v_1, u_2, u_3, \ldots, u_{l-1}, u_l = u$ be the vertices on the path $P$. Let $e_1, \ldots, e_{l-1}$ be the edges on the path $P$. Reordering the numbering of the edges connected to the vertices $u_1, \ldots, u_l$ if necessary, we can assume that

$$
e_1 = e_{u_1,1} = e_{u_2,1}, \quad e_2 = e_{u_2,2} = e_{u_3,1}, \quad \ldots, \quad e_{l-1} = e_{u_{l-1},2} = e_{u_l,1}.
$$

We already have pre-log curves $\psi_{u_i}: P_{u_i} \to X_{0,h(u_i)}$ for $i = 2, \ldots, l$. We denote by $\mu_1, \mu_2, \ldots$ the coordinates of the intersection points between $\psi_{u_i}(P_{u_i})$ and the boundary components of $X_{0,h(u_i)}$ as in Lemma 3.4.

Now we are going to modify the values $\mu_{u_2,1}, \mu_{u_2,2}$ along the path $P$ in a consistent way so that the power $\nu_1^{\gamma_{v_1}/\gamma_1}$ gets multiplied by $\zeta_j^{-1}$.

For any integer $z$, let $p(z)$ denote the power of $\epsilon_j$ in $z$, i.e. $z/\epsilon_j^{p(z)}$ is an integer indivisible by $\epsilon_j$. Note that $u_1 = v_1, \gamma_{v_1} = \gamma_{u_1}, \nu_1 = \mu_{u_2,1}$ and $w_{v_1,1} = w_{u_2,1}$. So we want to change $\mu_{u_2,1}$ so that the power $\mu_{u_2,1}^{w_{u_2,1}/\gamma_{u_1}}$ is multiplied by $\zeta_j^{-1}$. Also note that $\zeta_j^{-1}$ is an $\epsilon_j^{p(z)/\delta}$-th root of unity, and

$$
p(w_{u_2,1}/\gamma_{u_1}) + m_j = p(w_{u_2,1}/\gamma_{u_1}) + p(\gamma_{u_1}/\delta) = p(w_{u_2,1}/\delta).
$$

Therefore, it suffices to multiply $\mu_{u_2,1}$ by a suitable $\epsilon_j^{p(w_{u_2,1}/\delta)}$-th root of unity, which we denote by $\xi_1$. 


If \( u_2 \) is 2-valent, for the pre-log curve \( \psi_{u_2} : P_{u_2} \to X_{0,h(u_2)} \) to exist, we need to have \( \mu_{u_2,1}\mu_{u_2,2} = 1 \). In order to keep this equality while we multiply \( \mu_{u_2,1} \) by \( \xi_1 \), it suffices to multiply \( \mu_{u_2,2} \) by \( \xi_2 := \xi_1^{-1} \). If \( u_2 \) is 3-valent, for the pre-log curve \( \psi_{u_2} : P_{u_2} \to X_{0,h(u_2)} \) to exist, by Lemma 3.4, the coordinates \( \mu_{u_2,1}, \mu_{u_2,2}, \mu_{u_2,3} \) must satisfy the equation

\[
\begin{align*}
\mu_{u_2,1}^{\omega_{u_2,1}/\gamma_{u_2}} \mu_{u_2,2}^{\omega_{u_2,2}/\gamma_{u_2}} \mu_{u_2,3}^{\omega_{u_2,3}/\gamma_{u_2}} &= (-1)^{\omega_{u_2}/\gamma_{u_2}}.
\end{align*}
\]

Multiplying \( \mu_{u_2,1} \) by \( \xi_1 \) changes the left hand side by \( \xi_1^{\omega_{u_2,1}/\gamma_{u_2}} \). Since \( \xi_1 \) is an \( \epsilon_j^{p(\omega_{u_2,1}/\delta)} \)-th root of unity, and we have the equalities

\[
\begin{align*}
p(\omega_{u_2,1}/\delta) - p(\omega_{u_2,1}/\gamma_{u_2}) &= p(\gamma_{u_2}/\delta), \\
p(\gamma_{u_2}/\delta) + p(\omega_{u_2,2}/\gamma_{u_2}) &= p(\omega_{u_2,2}/\delta),
\end{align*}
\]

in order to keep the equality (3.14), it suffices to multiply \( \mu_{u_2,2} \) by a \( \epsilon_j^{p(\omega_{u_2,2}/\delta)} \)-th root of unity, which we denote by \( \xi_2 \).

Since \( \mu_{u_2,2}\mu_{u_3,1} \) is a fixed constant as in the proof of Lemma 3.9, \( \mu_{u_3,1} \) gets multiplied by \( \xi_2^{-1} \). Next we need to change \( \mu_{u_3,2} \) for the pre-log curve \( \psi_{u_3} : P_{u_3} \to X_{0,h(u_3)} \) continue to exist.

Iterating this process for \( j = 3, \ldots, l - 1 \), we multiply \( \mu_{u_1,1} \) by some \( \epsilon_j^{p(\omega_{u_1,1}/\delta)} \)-th root of unity, and \( \mu_{u_1,2} \) by some \( \epsilon_j^{p(\omega_{u_1,2}/\delta)} \)-th root of unity.

When we reach \( j = l \), consider the equation

\[
\begin{align*}
\mu_{u_1,1}^{\omega_{u_1,1}/\gamma_{u_1}} \mu_{u_1,2}^{\omega_{u_1,2}/\gamma_{u_1}} \mu_{u_1,3}^{\omega_{u_1,3}/\gamma_{u_1}} &= (-1)^{\omega_{u_1}/\gamma_{u_1}}.
\end{align*}
\]

Note that \( \mu_{u_1,1} \) is multiplied by a \( \epsilon_j^{p(\omega_{u_1-1,2}/\delta)} \)-th root of unity. By the choice of the vertex \( u_1 = u_1, \gamma_{u_1}/\delta \) is not divisible by \( \epsilon_j \). So we have

\[
\begin{align*}
p(\omega_{u_1-1,2}/\delta) - p(\omega_{u_1,1}/\gamma_{u_1}) &= p(\gamma_{u_1}/\delta) = 0,
\end{align*}
\]

hence the power \( \mu_{u_1,1}^{\omega_{u_1,1}/\gamma_{u_1}} \) in (3.15) is not changed. In other words, the equality (3.15) holds without any changes to \( \mu_{u_1,2} \) or \( \mu_{u_1,3} \).

To summarize, we have changed the values \( \mu_{u_1,1}, \mu_{u_1,2} \) along the path \( \mathcal{P} \) in a consistent way so that the power \( \nu_{u_1,1}^{\omega_{u_1,1}/\gamma_{u_1}} \) is multiplied by \( \zeta_j^{-1} \). Repeating this procedure for each \( \zeta_j, j = 1, \ldots, n \), we obtain the desired equality (3.11), completing the proof of Lemma 3.13. \( \Box \)

Finally we conclude the proof of Theorem 3.8 by Lemma 3.13 and Lemma 3.4. \( \Box \)
4. Deformations to general fibers

In this section, we study deformations of pre-log curves constructed in Theorem 3.8. We use the formulation of logarithmic deformation theory [13, 12]. The crucial point is the calculation of the obstruction based on the local study of semiregular subvarieties developed in [22].

4.1. Description of the dual of obstruction class. Let $\varphi_0 : C_0 \to X_0$ be a pre-log curve in $X_0$ associated with a given connected at most 3-valent parametrized tropical curve $h : \Gamma \to S$ as in Section 3. We use the notations in the beginning of Section 3. The toroidal scheme $\mathcal{X}$ induces a smooth log structure on $X_0$. We equip $C_0$ with a natural log structure so that the map $\varphi_0$ extends to a log morphism (cf. [24, Proposition 7.1] and [9, Proposition 4.23]). Let

$$N = \varphi_0^*\Theta_{X_0}/\Theta_{C_0}$$

be the logarithmic normal sheaf of $\varphi_0$, where $\Theta_{X_0}$ and $\Theta_{C_0}$ denote the logarithmic tangent sheaves on $X_0$ and $C_0$ respectively.

A flag of the graph $\Gamma$ is a pair $(v, e)$ consisting of a vertex $v$ and an edge $e$ connected to $v$. Let $F(\Gamma)$ denote the set of flags of $\Gamma$.

**Lemma 4.1.** Consider $H^0(C_0, N^\vee \otimes \omega_{C_0})$, where $\omega_{C_0}$ denotes the dualizing sheaf of $C_0$. It is in one-to-one correspondence with maps

$$u : F(\Gamma) \to N^\vee_C, \quad (v, e) \mapsto u_{v,e}$$

satisfying the following conditions:

1. We have $u_{v,e} \in w_{v,e}^\perp \subset N^\vee_C$.
2. For every edge $e$ of $\Gamma$, let $v, v'$ be the two endpoints of $e$. We have

$$u_{v,e} + u_{v',e} = 0.$$

3. For every vertex $v$ of $\Gamma$, we have

$$\sum_{e \ni v} u_{v,e} = 0,$$

where the sum is taken over all the edges connected to $v$.

**Proof.** Let $\tilde{C}_0 \to C_0$ be the normalization map. Then $\omega_{C_0}$ is isomorphic to the pushforward of the sheaf of 1-forms on $\tilde{C}_0$ with logarithmic poles at the points lying over the nodes of $C_0$, such that the sum of residues at the two points lying over every node of $C_0$ is zero (cf. [1, §10.2] and [2, §II.6]). Let $\psi$ be an element in $H^0(C_0, N^\vee \otimes \omega_{C_0})$. Let $v$ be a vertex of $\Gamma$ and $e_1, e_2, \ldots$ the edges connected

A 3-valent tropical curve we start with may acquire 2-valent vertices due to subdivisions in the process of construction of pre-log curves.
to \( v \). We denote by \( C_{0,v} \) the corresponding irreducible component of \( C_0 \), and by \( p_{e_1}, p_{e_2}, \ldots \) the corresponding nodes. Since \( \mathcal{X} \) is the quotient of the (infinite type) toric variety \( \widetilde{\mathcal{X}} \), we have \( \Theta_{X_0} \simeq N \otimes \mathcal{O}_{X_0} \). So the restriction \( \psi|_{C_{0,v}} \) is a differential form on \( C_{0,v} \) with values in \( N^\vee \), and with possible poles at \( p_{e_1}, p_{e_2}, \ldots \). We define \( u_{v,e_i} \) to be the vector valued residue of \( \psi \) at \( p_{e_i} \), for \( i = 1, 2, \ldots \).

Since the fiber of \( \Theta_{C_0} \subset \varphi^*_0 \Theta_{X_0} \) at \( p_{e_i} \) is generated by the weight vector \( w_{v,e_i} \in N \), the condition (1) is satisfied. The condition (2) follows from the explicit description of the dualizing sheaf \( \omega_{C_0} \) given above. The condition (3) follows from the residue theorem.

Conversely, the same reasoning shows that any map \( u: F(\Gamma) \rightarrow N^\vee \) satisfying the conditions (1-3) determines uniquely an element in \( H^0(C_0, N^\vee \otimes \omega_{C_0}) \). So we have proved the lemma. \( \square \)

**Proposition 4.2.** The space \( H^0(C_0, N^\vee \otimes \omega_{C_0}) \) is of dimension one.

**Proof.** Since the graph \( \Gamma \) is at most 3-valent, Lemma 4.1(3) implies that for every vertex \( v \), any one of the vectors \( \{u_{v,e_i}\} \) determines the other vectors uniquely. Together with the other conditions in Lemma 4.1, by the connectedness of \( \Gamma \), we see that the vector \( u_{v,e} \) at one single flag \( (v,e) \) determines the map \( u: F(\Gamma) \rightarrow N^\vee \) completely. Thus, the space \( H^0(C_0, N^\vee \otimes \omega_{C_0}) \) is of dimension at most one.

In order to construct a nonzero element of \( H^0(C_0, N^\vee \otimes \omega_{C_0}) \), we use standard coordinates on \( N \simeq \mathbb{Z}^2 \) and the dual coordinates on \( N^\vee \). For each flag \( (v,e) \) of \( \Gamma \), if the weight vector \( w_{v,e} \) has coordinates \( (a,b) \), we set \( u_{v,e} = (-b,a) \). This gives a nonzero element, completing the proof. \( \square \)

**4.2. The pairing between the obstruction class and its dual.** Now we turn to the calculation of the obstruction classes via Čech cohomology. First, we introduce a representation of Čech 1-cocycles on nodal curves with coefficients in invertible sheaves in a way suited to our purpose. Let \( C_0 \) be a complete curve with at worst nodal singularity. Let \( \{U_i\}_{i \in I} \) be an open covering of \( C_0 \). We take \( U_i \) so that it is homeomorphic either to a disk or to the union of two disks attached at the centers. When \( U_i \) is the union of disks, let \( U_{i,1}, U_{i,2} \) be the irreducible components. These are locally closed subsets of \( C_0 \). We assume that for each node of \( C_0 \), there is a unique open subset \( U_i \) which contains it. Also, we assume that any intersection \( U_i \cap U_j \) is homeomorphic to a disk if it is not empty.

Let \( \mathcal{L} \) be an invertible sheaf on \( C_0 \). We associate a local meromorphic section \( \xi_i \) of \( \mathcal{L}|_{U_i} \) with each \( U_i \) when \( U_i \) is a disk, and the pair of meromorphic sections \( \xi_{i,j} \) of \( \mathcal{L}|_{U_{i,j}} \), \( j = 1, 2 \), with each \( U_i \) when \( U_i \) is the union of disks, in the following way. Namely, let \( V_k, V_j \) be locally closed subsets as above, that is, each of them is one of \( U_i \) if \( U_i \) is a disk, or \( U_{i,j} \) if \( U_i \) is the union of disks. Let \( \xi_{i,e} \) and \( \xi_{i,e} \) be the associated
meromorphic sections on them. Then we require that $\xi'_k - \xi'_l$ is holomorphic when the intersection $V_k \cap V_l$ is an open subset of $C_0$.

By construction, for each $V_k$, there is a unique open subset in $\{U_i\}_{i \in I}$ associated with it. We write it as $U_{V_k}$. Note that if $V_k \cap V_l$ is an open subset, then $V_k \cap V_l = U_{V_k} \cap U_{V_l}$. It follows that the differences $\xi'_k - \xi'_l$ give a Čech 1-cocycle $\xi$ with values in $\mathcal{L}$ associated with the covering $\{U_i\}_{i \in I}$. Namely, if the intersection $U_{V_k} \cap U_{V_l}$ is not empty, then associate the section $\xi'_k - \xi'_l$ with it. Note that any non-empty intersection $U_i \cap U_j$ is written in this form.

Now let $\psi$ be an element of $H^0(C_0, \mathcal{L}^\vee \otimes \omega_{C_0})$, the dual space of $H^1(C_0, \mathcal{L})$. Then one can show that the pairing between the class $\xi = \{\xi'_k\}$ and $\psi$ is calculated as follows. Namely, let $\{q_\alpha\}_{\alpha \in A} \subset C_0$ be the set of poles of the local sections $\xi_i$ which are not the nodes of $C_0$. For each $\alpha$, we choose a locally closed subset $V_{q_\alpha}$ which contains $q_\alpha$, and denote by $\xi_\alpha(= \xi'_k)$ the corresponding local section. Let $\{p_i\}$ be the set of nodes of $C_0$. By assumption, it is contained in the unique open subset $U_i$. Let $p_{i,1}, p_{i,2}$ be the corresponding points on the irreducible components $U_{i,1}, U_{i,2}$. These also have the associated local sections $\xi_{i,1}$ and $\xi_{i,2}$. If $\xi'_l$ is the given local meromorphic section on some $V_l$, the fiberwise pairing between $\xi'_l$ and $\psi$ gives a meromorphic 1-form $\psi(\xi'_l)$ on $V_l$. Then we have the following.

**Proposition 4.3.** [22] The pairing between the classes $\xi$ and $\psi$ is given by the following formula:

$$\langle \psi, \xi \rangle = \sum \text{res}_{q_\alpha} \psi(\xi_\alpha) + \sum \text{res}_{p_{i,j}} \psi(\xi_{i,j}).$$

Here $\text{res}_p \phi$ denotes the residue of a meromorphic 1-form $\phi$ at the point $p$. \hfill $\square$

There are some remarks regarding this formula.

**Remark 4.4.**

1. By the assumption, the sum does not depend on the choice of $\xi_\alpha$ associated with a pole $q_\alpha$.

2. Since a section of the dualizing sheaf $\omega_{C_0}$ may have a logarithmic singularity at each node of $C_0$, there can be a contribution to the sum even when $\xi_{i,j}$ is holomorphic at $p_{i,j}$.

3. It can be proven that this definition descends to the natural nondegenerate pairing between $H^0(C_0, \mathcal{L}^\vee \otimes \omega_{C_0})$ and $H^1(C_0, \mathcal{L})$. In particular, a class $\xi$ in $H^1(C_0, \mathcal{L})$ vanishes if and only if the pairing gives zero for any class $\psi$ in $H^0(C_0, \mathcal{L}^\vee \otimes \omega_{C_0})$. It also follows that, any class in $H^1(C_0, \mathcal{L})$ can be represented by some set of local sections $\{\xi'_k\}$ as above. See [22] for details.

4. In the present paper, since $H^0(C_0, \mathcal{L}^\vee \otimes \omega_{C_0})$ is one dimensional for the sheaf $\mathcal{L} = \mathcal{N}$ relevant to us, a class $\xi \in H^1(C_0, \mathcal{N})$ vanishes if and only if its coupling with a generator $\psi$ of $H^0(C_0, \mathcal{N}^\vee \otimes \omega_{C_0})$ vanishes. The class $\psi$ is described explicitly below.
Let \( \{a_1, a_2\} \) be any basis of \( N^+ \) and \( x, y \) be the corresponding functions on \( \widetilde{X} \). Let \( \frac{dx}{x} \wedge \frac{dy}{y} \) be a 2-form relative to the base on \( \widetilde{X} \). This is holomorphic on the complement of the central fiber, and has logarithmic poles along the singular locus of the central fiber. Also, it descends to \( X \) since it is invariant under the group action. We write this form by \( \psi_X \) and its restriction to \( \widetilde{X}_0 \) by \( \psi_{X_0} \). By Proposition 4.2, it follows that the pullback of \( \psi_{X_0} \) to \( C_0 \) gives a generator \( \psi \) of the group \( H^0(C_0, N^+ \otimes \omega_{C_0}) \).

4.3. **Standard tropical curve with two vertices.** To calculate the obstructions to deforming, we manipulate our tropical curve a bit. First, by applying base changes and introducing 2-valent vertices, and further refining the subdivision of the 2-torus we can assume the following:

- For every 3-valent vertex, all the neighboring vertices are 2-valent.
- The integral length of the image of any edge is equal to its weight. In other words, each edge is modeled on the interval \([0, 1]\) in terms of Definition 2.1.

Let \( h_s : \Gamma_s \setminus V_\infty(\Gamma_s) \to \mathbb{R}^2 \) be a tropical curve defined as follows. Namely, the connected open graph \( \Gamma_s \setminus V_\infty(\Gamma_s) \) has two vertices \( V_1, V_2 \), where \( V_1 \) is 3-valent and \( V_2 \) is 2-valent. In particular, there is only one bounded edge \( E \) connecting \( V_1 \) and \( V_2 \). All the edges have weight one. The map \( h_s \) sends \( V_1 \) to the origin, and \( V_2 \) to \((1, 0)\). The unbounded edges attached to \( V_1 \) have weight vectors \((0, 1)\) and \((-1, -1)\). These conditions, together with the balancing condition, determine \( h_s \). We call this tropical curve \((\Gamma_s, h_s)\) with two vertices **standard**.

Now take a pair of adjacent vertices \( v_1, v_2 \) of \( \Gamma \) and assume \( v_1 \) is 3-valent. Then the restriction of \( h \) to a neighborhood \( U_{v_1v_2} \) of the edge \( e_{v_1v_2} \) connecting \( v_1 \) and \( v_2 \) gives a tropical curve \( h_{v_1v_2} : \Gamma_{v_1v_2} \to \mathbb{R}^2 \), determined uniquely up to integral affine transformations on \( \mathbb{R}^2 \) in a natural manner. Here \( \Gamma_{v_1v_2} \) is an open graph with two vertices obtained from \( U_{v_1v_2} \) by extending edges other than \( e_{v_1v_2} \) to infinity.

Such a tropical curve \( h_{v_1v_2} : \Gamma_{v_1v_2} \to \mathbb{R}^2 \) is obtained from the standard tropical curve in the following way. Namely, let \( \Phi : \mathbb{R}^2 \to \mathbb{R}^2 \) be an integral affine map which sends \( h_s(V_1) = (0, 0) \) to \( h_{v_1v_2}(v_1) \). Regarding \( h_{v_1v_2}(v_1) \) as the origin, we can think of \( \Phi \) as a linear map. Then \( \Phi \) is determined by the condition that it sends the vectors \((0, 1)\) and \((-1, -1)\) to the weight vectors of the unbounded edges \( e_1 \) and \( e_2 \) attached to \( v_1 \). The balancing condition forces that the weight vector of \( E \), which is \((1, 0)\), is sent to the weight vector of \( e_{v_1v_2} \), and we have the identity \( \Phi \circ h_s = h_{v_1v_2} \).

The map \( \Phi \) induces a similar map on the side of holomorphic curves. Namely, put \( h_{v_1v_2}(\Gamma_{v_1v_2}) \) on the affine plane \( \mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3 \) and take the closure \( \overline{U}_{h_{v_1v_2}(\Gamma_{v_1v_2})} \).
of the cone over it. It has a natural structure of an incomplete fan whose 1-dimensional cones are the rays spanned by vertices of \( h_{v_1 v_2}(\Gamma_{v_1 v_2}) \) and the rays in the intersection \( \overline{\mathcal{C}}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})} \cap \mathbb{R}^2 \times \{0\} \). Let \( \overline{\mathcal{Y}}_{\mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})}} \) be the associated toric variety. It has an open toric subvariety \( \mathcal{Y}_{\mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})}} \) corresponding to the edge \( e_{v_1 v_2} \). Namely, \( \mathcal{Y}_{\mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})}} \) is obtained from \( \overline{\mathcal{Y}}_{\mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})}} \) by removing toric divisors corresponding to the rays in \( \overline{\mathcal{C}}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})} \cap \mathbb{R}^2 \times \{0\} \). We choose \( \mathcal{Y}_{\mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})}} \) rather than \( \overline{\mathcal{Y}}_{\mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})}} \) since it has a natural embedding into \( \mathcal{X} \), but \( \overline{\mathcal{Y}}_{\mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})}} \) does not. It also has a natural toric map \( \mathcal{Y}_{\mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})}} \to \mathbb{C} \). Let \( Y_{0,v_1 v_2} \) be the fiber over \( 0 \in \mathbb{C} \).

Let \( \mathcal{C}_s \) be the cone in \( \mathbb{R}^3 \) obtained from \( (\Gamma_s, h_s) \) by the same construction as \( \mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})} \). Let \( \mathcal{Y}^s_{v_1 v_2} \) be the associated toric variety as above and \( Y_{0,v_1 v_2} \) be the central fiber. Then \( \mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})} \) is obtained from \( \mathcal{C}_s \) by the integral linear transformation

\[
\begin{pmatrix}
  a & b & 0 \\
  c & d & 0 \\
  0 & 0 & 1
\end{pmatrix},
\]

where \( \Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Therefore, there is a natural toric map \( P_\Phi : \mathcal{Y}^s_{v_1 v_2} \to \mathcal{Y}_{\mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})}} \).

Then any pre-log curve \( \phi_0 : D_0 \to Y_{0,v_1 v_2} \) of type \( (\Gamma_{v_1 v_2}, h_{v_1 v_2}) \) is obtained as the image of a pre-log curve \( \psi_{0,v_1 v_2} : D_0 \to Y^s_{0,v_1 v_2} \) of type \( (\Gamma_s, h_s) \) by the map \( P_\Phi \). Precisely, since we deleted some toric divisors from \( \overline{\mathcal{Y}}_{\mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})}} \), \( \phi_0 \) is not a pre-log curve of type \( (\Gamma_{v_1 v_2}, h_{v_1 v_2}) \) but the restriction of it to an open subset.

Since \( \Gamma_{v_1 v_2} \) has two vertices and one bounded edge, \( D_0 \) has two components \( D_{0,1}, D_{0,2} \) and one node \( p \). Let \( Y_{0,1} \) and \( Y_{0,2} \) be the components of \( Y_{0,v_1 v_2} \) and assume that the component \( D_{0,i} \) is mapped to \( Y_{0,i} \) by \( \phi_0 \), for \( i = 1, 2 \). Similarly, let \( Y^s_{0,i} \) be the components of \( Y^s_{0,v_1 v_2} \) to which \( D_{0,i} \) is mapped by \( \psi_{0,v_1 v_2} \). The toric variety \( \mathcal{Y}_{\mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})}} \) is isomorphic to \( \text{Spec} \mathbb{C}[x, y, z^{\pm 1}, t]/(xy - t^m) \), where \( m \) is the weight of the bounded edge \( e_{v_1 v_2} \) of \( \Gamma_{v_1 v_2} \). The functions \( x, y, z \) and \( t \) are characters of the torus acting on \( \mathcal{Y}_{\mathcal{C}_{h_{v_1 v_2}(\Gamma_{v_1 v_2})}} \). Similarly, \( \mathcal{Y}^s_{v_1 v_2} \) is isomorphic to \( \text{Spec} \mathbb{C}[X, Y, Z^{\pm 1}, t]/(XY - t) \). These functions can be taken so that the map \( P_\Phi \) induces maps

\[
x \mapsto X^m Z^R, \quad y \mapsto Y^m Z^{-R}, \quad z \mapsto Z^{R'}, \quad t \mapsto t
\]

on these functions. Here \( R \) and \( R' \) are integers.

On the other hand, we can take an affine neighborhood \( V \) of \( p \) in \( D_0 \) which is isomorphic to \( \text{Spec} \mathbb{C}[s, \frac{1}{s - \alpha}, u]/su \), where \( \alpha \) is a non-zero complex number. Taking
X, Y and s, u suitably, we can choose a lift $\psi_{0,v_1v_2}$ of $\phi_0$ so that the map $\psi_0$ gives

$$X \mapsto s, \ Y \mapsto u, \ Z \mapsto a + bs,$$

and $\phi_0$ gives

$$x \mapsto s^m(a + bs)^R, \ y \mapsto a^{-R}u^m, \ z \mapsto (a + bs)^R,$$

where $a, b$ are non-zero complex numbers such that $-\frac{a}{b} = \alpha$.

Let $\varphi_k : C_k \rightarrow X_k = \mathcal{X} \times_{\text{Spec} \mathbb{C}[t]} \text{Spec} \mathbb{C}[t]/t^{k+1}$ be a deformation of $\varphi_0 : C_0 \rightarrow X_0$. Then the restriction of $C_k$ to the topological space underlying $V$ is isomorphic to $\text{Spec} \mathbb{C}[s, \frac{1}{s-\alpha}, u, t]/(su - t, t^{k+1})$, and $\varphi_k$ induces a map

$$x \mapsto s^m(a + bs)^R + tf(s, u, t), \ y \mapsto u^m(a + bs)^{-R} + t\tilde{g}(s, u, t), \ z \mapsto (a + bs)^R + t\tilde{h}(s, u, t),$$

where $f, g, h$ are functions in $\mathbb{C}[s, \frac{1}{s-\alpha}, u, t]/(su - t, t^{k+1})$. We have the following lifting property.

**Proposition 4.5.** The map $\varphi_k|_V$ lifts to a map $\tilde{\psi}_k|_V : C_k|_V \rightarrow Y_{k,v_1v_2}$ so that the equality $\varphi_k|_V = P_\Phi \circ \tilde{\psi}_k|_V$ holds. Here $Y_{k,v_1v_2} = Y_{v_1v_2} \times_{\text{Spec} \mathbb{C}[t]} \text{Spec} \mathbb{C}[t]/t^{k+1}$.

**Proof.** It suffices to prove that the map $x \mapsto s^m(a + bs)^R + tf(s, u, t), y \mapsto u^m(a + bs)^{-R} + t\tilde{g}(s, u, t), z \mapsto (a + bs)^R + t\tilde{h}(s, u, t)$ lifts to some map

$$X \mapsto s + tF(s, u, t), \ Y \mapsto u + tG(s, u, t), \ Z \mapsto a + bs + tH(s, u, t),$$

so that

$$(s + tF(s, u, t))^m(a + bs + tH(s, u, t))^R = s^m(a + bs)^R + tf(s, u, t),$$

$$(u + tG(s, u, t))^m(a + bs + tH(s, u, t))^{-R} = u^m(a + bs)^{-R} + t\tilde{g}(s, u, t),$$

and

$$(a + bs + tH(s, u, t))^R = (a + bs)^R + t\tilde{h}(s, u, t).$$

Here $F, G, H$ are analytic functions. It also suffices to prove the existence of a lift on an analytic neighborhood of the node of $D_0$, since away from the image of it the map $P_\Phi$ is unramified and once a lift is constructed around the node, clearly it extends to the remaining part by analytic continuation.

Then we can take an $R'$-th root of $(a + bs)^R + t\tilde{h}(s, u, t)$ which is $a + bs$ modulo $t$, and it determines $H$. Dividing $s^m(a + bs)^R + tf(s, u, t)$ by $(a + bs + tH(s, u, t))^R$, we obtain a function of the form $s^m + tf(s, u, t)$ for some analytic function $\tilde{f}$. Similarly, dividing $u^m(a + bs)^{-R} + t\tilde{g}(s, u, t)$ by $(a + bs + tH(s, u, t))^{-R}$, we obtain $u^m + t\tilde{g}(s, u, t)$ for some analytic function $\tilde{g}$. Expanding $\frac{1}{s-\alpha}$ around $s = 0$, we regard $\tilde{g}$ and $\tilde{h}$ as polynomial of $t$ whose coefficients are series of $s$ and $u$.

Since $xy = t^m$, we have $(s^m + tf(s, u, t))(u^m + t\tilde{g}(s, u, t)) = t^m$. It follows that

$$t(s^m\tilde{g}(s, u, t) + u^m\tilde{f}(s, u, t)) = 0.$$
From this, we see that every monomial of $\tilde{f}$ contains $s^m$ as a factor, after rewriting $t = su$ if necessary. Similarly, every monomial of $\tilde{g}$ contains $u^m$ as a factor. Thus, we can write $\tilde{f}(s, u, t) = s^m f_1(s, u, t)$ for some $f_1$. Then we have

$$s^m + t\tilde{f}(s, u, t) = s^m (1 + t f_1(s, u, t)) = s^m (1 + tF_1(s, u, t))^m$$

for some $F_1$. Similar claim holds for $u^m + t\tilde{g}(s, u, t)$. This proves the proposition. □

The case where both vertices $v_1, v_2$ are 2-valent has the similar and easier description. We omit the details for this case.

4.4. Lifting defining equations for curves to $Y_s$. In [?], the obstructions to deforming maps between varieties are calculated, where the target space is a fixed smooth complex manifold, the domain has dimension one less than that of the target, and the map is locally an embedding. The calculation of the obstruction in this paper is based on it, but we have to take care of the fact that $X_0$ is singular and the complex structures deform in the family $X$. Also, the map $\varphi_0$ is not locally an embedding in general. To overcome these points, we need to reduce the calculation to the standard case introduced in the previous subsection. In this subsection, we prepare some setup required for it.

Let $\{U_i\}_{i \in I}$ be an open covering of $C_0$ as in Subsection 4.2. By taking these open subsets small enough, we can assume that $\varphi_0|_{U_i}$ is an injection and that the restriction $\varphi_0|_{U_i \cap U_j} : U_i \cap U_j \to X_0$ is an embedding. We can also assume that if both $U_i$ and $U_j$ contain nodes, then $U_i \cap U_j = \emptyset$ unless $U_i = U_j$. We also take an open covering $\{W_k\}_{k \in I'}$, $I' \subset I$, of $X_0$ so that for each $U_i$, its image $\varphi_0(U_i)$ is contained in $W_i$. Moreover, we take each $W_i$ to be biholomorphic to a cylinder (that is, the product of two discs) when $U_i$ does not contain a node, and the normal crossing union of two cylinders (that is, isomorphic to a suitable neighborhood of the origin in $\{xy = 0\} \subset \mathbb{C}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{C}\}$) when $U_i$ contains a node. By shrinking $W_i$ if necessary, we can also assume $\varphi_0(U_i)$ is the zero locus of an analytic function on $W_i$.

Assume that an $n$-th order deformation $\varphi_n : C_n \to X_n$ of the pre-log curve $\varphi_0 : C_0 \to X_0$ exists, where $n \geq 0$. Let $U_{i,n}$ denote the ringed space which is the restriction of $C_n$ to $U_i$. Also, let $W_{i,n}$ denote the ringed space which is the restriction of $X_n$ to $W_i$. Let $F_{i,n} = 0$ be the defining equation of the image $\varphi_n(U_{i,n})$ in $W_{i,n}$. When $U_i$ contains a node of $C_0$, then the ring of functions on $W_{i,n}$ contains the ring $\mathbb{C}[x, y, z^{\pm 1}, t]/(xy - t^m, t^{n+1})$ using the notation in the previous subsection, which is the ring of functions of a toric open subset containing $W_{i,n}$. Here $m$ is a positive integer which is equal to the weight of the edge of $\Gamma$ connecting the
vertices corresponding to the two components given by \( x = 0 \) and \( y = 0 \) in \( X_0 \). The function \( F_{i,n} \) can be chosen from this subring.

For the calculation of the obstruction below, we need to regard the function \( F_{i,n} \), which is defined over \( \mathbb{C}[t]/t^{n+1} \), as a function defined over \( \mathbb{C}[t]/t^{n+2} \). This is done by a splitting of the exact sequence

\[
0 \to \mathbb{C}[x, y, z^{\pm 1}]/(xy) \to \mathbb{C}[x, y, z^{\pm 1}, t]/(xy-t^m, t^{n+2}) \to \mathbb{C}[x, y, z^{\pm 1}, t]/(xy-t^m, t^{n+1}) \to 0.
\]

There is an obvious splitting \( \mathbb{C}[x, y, z^{\pm 1}, t]/(xy-t^m, t^{n+1}) \to \mathbb{C}[x, y, z^{\pm 1}, t]/(xy-t^m, t^{n+2}) \). Namely, given an element \( F(x, y, z, t) \) in \( \mathbb{C}[x, y, z^{\pm 1}, t]/(xy-t^m, t^{n+1}) \), represent it in a way which does not contain \( xy \) or \( t^{n+1} \), and regard the result as an element of \( \mathbb{C}[x, y, z^{\pm 1}, t]/(xy-t^m, t^{n+2}) \).

When \( U_i \) does not contain a node of \( C_0 \), the ring of functions of \( W_{i,n} \) contains the ring \( \mathbb{C}[x, \frac{1}{x}, z^{\pm 1}, t]/(t^{n+1}) \) or \( \mathbb{C}[y, \frac{1}{y}, z^{\pm 1}, t]/(t^{n+1}) \). Assume it contains the former. Consider the exact sequence

\[
0 \to \mathbb{C}[x, \frac{1}{x}, z^{\pm 1}]/ \to \mathbb{C}[x, \frac{1}{x}, z^{\pm 1}, t]/(t^{n+2}) \to \mathbb{C}[x, \frac{1}{x}, z^{\pm 1}, t]/(t^{n+1}) \to 0.
\]

Again, there is an obvious splitting \( \mathbb{C}[x, \frac{1}{x}, z^{\pm 1}, t]/(t^{n+1}) \to \mathbb{C}[x, \frac{1}{x}, z^{\pm 1}, t]/(t^{n+2}) \) given by writing an element of \( \mathbb{C}[x, \frac{1}{x}, z^{\pm 1}, t]/(t^{n+1}) \) so that it does not contain \( t^{n+1} \), and simply regard it as an element of \( \mathbb{C}[x, \frac{1}{x}, z^{\pm 1}, t]/(t^{n+2}) \). In this case, the defining equation of \( \varphi_0(U_i) \) in \( W_i \) is not contained in \( \mathbb{C}[x, \frac{1}{x}, z^{\pm 1}, t]/(t^{n+1}) \) in general, since at the image \( \varphi_0(p) \) of some point \( p \in U_i \), there may be several branches of \( \varphi_0(C_0) \), and if \( U_i \) is small, \( \varphi_0(U_i) \) is contained in one of the branches of them. Therefore, the defining equation of \( \varphi_0(U_i) \) and also of \( \varphi_n(U_{i,n}) \) will not be a polynomial of \( x, z \) and \( t \), but a polynomial of \( t \) whose coefficients are analytic functions of \( x \) and \( z \). However, in this case too, the obvious extension of the above splitting clearly gives a unique lift of the defining function to \( \mathbb{C}[t]/t^{n+2} \).

We note that there is some subtlety here, and this is the place where the assumption \((*)\) in the introduction plays a role. Namely, there will be several maps \( P_\Phi : Y_{v_1v_2}^s \to \mathcal{X} \) introduced in the previous subsection whose image contain \( \varphi_0(U_i) \). The toric coordinates \( x, y \) and \( z^{\pm 1} \) which can be pulled back to \( Y_{v_1v_2}^s \) depend on the choice of the vertices \( v_1 \) and \( v_2 \) of \( \Gamma \). Since the above splitting process contains a choice of explicit representative of an element, it can depend on the choice of these coordinates. However, under the assumption \((*)\), the coordinate change between these coordinate functions does not depend on the parameter \( t \). From this, it follows that the splitting above is well-defined, that is, independent of the choice of \( P_\Phi : Y_{v_1v_2}^s \to \mathcal{X} \). This point does not appear in the case where \( U_i \) contains a node, since in this case there is a unique \( P_\Phi : Y_{v_1v_2}^s \to \mathcal{X} \) whose image contains \( \varphi_0(U_i) \).
4.5. A representative of the obstruction cocycle. Let $F_{i,n} = 0$ be the defining equation of the image $\varphi_n(U_{i,n})$ in $W_{i,n}$. Similarly, let $F_{j,n} = 0$ be the defining equation of the image $\varphi_n(U_{j,n})$ in $W_{j,n}$, and assume that $U_i \cap U_j \neq \emptyset$. On the intersection $W_{i,n} \cap W_{j,n}$, these are related by

$$F_{i,n} = G_{ij,n} F_{j,n},$$

where $G_{ij,n}$ is a holomorphic function defined over $\mathbb{C}[t]/t^{n+1}$ whose reduction over $\mathbb{C}[t]/t$ is non-vanishing. Regarding these functions as defined over $\mathbb{C}[t]/t^{n+2}$ as in the previous subsection, we obtain the difference

$$t^{n+1} \nu_{ij,n+1} = F_{i,n} - G_{ij,n} F_{j,n}$$

on $W_{i,n+1} \cap W_{j,n+1}$. Here $\nu_{ij,n+1}$ can be regarded as a holomorphic function on $W_{i,0} \cap W_{j,0}$.

Assume that there is another $W_k$ containing $\varphi_0(U_k)$ such that $U_i \cap U_j \cap U_k \neq \emptyset$, and define $F_{k,n}$, $G_{jk,n}$ and $G_{ki,n}(= G_{ik,n}^{-1})$ as above. These determine $\nu_{jk,n+1}$, $\nu_{ki,n+1}$ etc. on the relevant intersections.

**Lemma 4.6.** On $W_{i,0} \cap W_{j,0} \cap W_{k,0} \cap \varphi_0(U_i)$, the identity

$$\nu_{ik,n+1} = \nu_{ij,n+1} + G_{ij,0} \nu_{jk,n+1}$$

holds.

**Proof.** This is proved in [16] when the target space is a fixed complex manifold, see also [22]. For relative deformations with smooth fibers, this is proved in [23]. In the present case, we need to take care of the fact that $X_0$ is singular and the complex structures in the family $\mathcal{X}$ vary. However, since the deformation of the complex structure is rather special in the present case, the proof is close to the case with a fixed target.

When any of the open subsets $U_i, U_j, U_k$ does not contain a node of $C_0$, then the proof is the same as the case with a fixed target. This is because in this case we can take coordinates on $W_{i,n+1}, W_{j,n+1}, W_{k,n+1}$ such that the coordinate transformations between them do not depend on the deformation parameter $t$.

Therefore, assume $U_i$ contains a node of $C_0$. In this case, by the way we take the open covering $\{U_i\}$, $U_j$ and $U_k$ do not contain a node when $U_i \cap U_j \cap U_k \neq \emptyset$. As in the previous subsection, the ring of functions on $W_{i,n}$ contains the ring $\mathbb{C}[x, y, z^\pm, t]/(xy - t^m, t^{n+1})$, which is the ring of functions of a toric open subset containing $W_{i,n}$. Here $m$ is a positive integer. Let $F_{i,n} = 0$ be the defining equation of the image $\varphi_n|_{U_{i,n}}$ in $W_{i,n}$ as above. Then we can assume $F_{i,n}$ belongs to $\mathbb{C}[x, y, z^\pm, t]/(xy - t^m, t^{n+1})$. 
Assume that the images of $U_j$ and $U_k$ are contained in the component given by $y = 0$. Let $F_{i,n}|_{W_{i,n} \cap W_{j,n} \cap W_{k,n}}$ be the restriction of $F_{i,n}$ to $W_{i,n} \cap W_{j,n} \cap W_{k,n}$. It is obtained by substituting $y = \frac{t_m}{x}$ to $F_{i,n}$. When we regard $F_{i,n}$ as a function defined over $\mathbb{C}[t]/t^{n+2}$, the difference between it and the function obtained from $F_{i,n}|_{W_{i,n} \cap W_{j,n} \cap W_{k,n}}$ is the sum of the terms which contain the factor $t^{n+1}$ when we substitute $y = \frac{t_m}{x}$. Let $H_{i,n+1}$ be the sum of these terms. Then we have

$$t^{n+1} \nu_{i,k,n+1} = H_{i,n+1} + F_{i,n}|_{W_{i,n} \cap W_{j,n} \cap W_{k,n}} - G_{i,k,n}F_{k,n}$$

$$= t^{n+1} \nu_{i,n+1} + G_{i,n,j}F_{j,n} + G_{i,n,j}F_{j,n} - G_{i,j}F_{j,n} - G_{i,k,n}F_{k,n}$$

$$= t^{n+1} \nu_{i,n+1} + t^{n+1} \nu_{i,n+1} + (G_{i,n} - G_{i,k,n}-G_{i,j}F_{j,n})F_{k,n}$$

which is an equation over $\mathbb{C}[t]/t^{n+2}$. Since

$$G_{i,n}G_{j,n} \equiv G_{i,n} \mod t^{n+1},$$

we have

$$(G_{i,n}G_{j,n} - G_{i,k,n})F_{k,n} \equiv (G_{i,n} - G_{i,k,n})F_{k,n} \mod t^{n+2}.$$  

Since $F_{k,0} = 0$ on $C_0$, we have the claim. The case when $U_j$ or $U_k$ contains a node of $C_0$ is similar. 

Note that the pull back of the set of functions $\{G_{i,n}\}$ to $C_0$ is the set of transition functions for the normal sheaf of $\varphi_0$. Thus, the lemma shows that the set of functions $\{\nu_{i,n+1}\}$, when pulled back to $C_0$, behaves as a Čech 1-cocycle with values in the normal sheaf of $\varphi_0$. Note that away from the nodes, the usual normal sheaf and the log normal sheaf are canonically identified. By construction, the Čech 1-cocycle $\{\nu_{i,n+1}\}$ is the obstruction cocycle to deforming $\varphi_n$ to a map over $\mathbb{C}[t]/t^{n+2}$ (see [16]).

However, we do not work with $\{\nu_{i,n+1}\}$. Instead, we construct another cocycle using the local covering in Subsection 4.3 which also represents the obstruction to deforming $\varphi_n$ to a map over $\mathbb{C}[t]/t^{n+2}$. We use the same notation as in Subsection 4.3. Let $v_1, v_2$ be adjacent vertices of $\Gamma$ and let $e_{v_1v_2}$ be the edge connecting them. Let $\mathcal{Y}^{e_{v_1v_2}}_{h_{v_1v_2}(\Gamma_{v_1v_2})}$ be the open subset of the toric variety associated with the fan given by the closure $\mathcal{C}_{h_{v_1v_2}(\Gamma_{v_1v_2})}$ of the cone over $h_{v_1v_2}(\Gamma_{v_1v_2})$. Let $\mathcal{Y}^{e_{v_1v_2}}_{v_1v_2}$ be the open subset of the toric variety associated with the closure $\mathcal{C}_s$ of the cone over a standard tropical curve with two vertices $(\Gamma_s, h_s)$. The cone $\mathcal{C}_{h_{v_1v_2}(\Gamma_{v_1v_2})}$ is obtained from $\mathcal{C}_s$ by a linear map $\Phi$. The variety $\mathcal{Y}^{e_{v_1v_2}}_{h_{v_1v_2}(\Gamma_{v_1v_2})}$ is isomorphic to $\text{Spec} \mathbb{C}[x, y, z^\pm 1, t]/(xy - t^m)$. Let $Y_{0,v_1v_2}$, $Y_{0,s,v_1v_2}$ be the central fibers of $\mathcal{Y}^{e_{v_1v_2}}_{h_{v_1v_2}(\Gamma_{v_1v_2})}$ and $\mathcal{Y}^{e_{v_1v_2}}_{v_1v_2}$, respectively. The map $\Phi$ induces a map $P_{\Phi}|_{Y^{e_{v_1v_2}}_{0,v_1v_2}}: Y^{e_{v_1v_2}}_{0,v_1v_2} \to Y^{e_{v_1v_2}}_{0,v_1v_2}$. 


Let $p$ be the node of $C_0$ corresponding to the edge $e_{v_1v_2}$. Let $(C_{0,v_1} \cup C_{0,v_2})^o$ be the open subset of $C_{0,v_1} \cup C_{0,v_2}$ given by $(C_{0,v_1} \cup C_{0,v_2})^o = (C_{0,v_1} \cup C_{0,v_2}) \cap \varphi_0^{-1}(\mathcal{T}_{\mathbb{R}^n_{v_1v_2}(\mathbb{R}^n_{v_1v_2}))}$. As we mentioned in Subsection 4.3, the restriction $\varphi_0|_{(C_{0,v_1} \cup C_{0,v_2})^o}$ lifts to a map $\psi_{0,v_1v_2}|(C_{0,v_1} \cup C_{0,v_2})^o : (C_{0,v_1} \cup C_{0,v_2})^o \to Y^s_{0,v_1v_2}$. Let $U_i$ be the part of the open covering of $C_0$ which contains the node $p$. Assume there are other open subsets $U_j$ and $U_k$ of $C_0$ not containing a node, and whose images by $\varphi_0$ are contained in the component of $Y^s_{0,v_1v_2}$ given by $y = 0$. Assume $U_i \cap U_j \cap U_k \neq \emptyset$. Recall we take $U_j$ and $U_k$ small enough so that we can assume $\varphi_0|_{U_j}$ and $\varphi_0|_{U_k}$ are isomorphisms of open subsets isomorphic to $U_j$ and similar for $U_k$. There are unique connected components $W^s_{j,v_1v_2}$ and $W^s_{k,v_1v_2}$ of $\psi_{0,v_1v_2}|(C_{0,v_1} \cup C_{0,v_2})^o$ which contain the images of $\psi_{0,v_1v_2}|_{U_j}$ and $\psi_{0,v_1v_2}|_{U_k}$, respectively.

Assume there is an $n$-th order deformation $\varphi_n : C_n \to X_n$ of the map $\varphi_0$. Let $\varphi_n|_{(C_{0,v_1} \cup C_{0,v_2})^o} : \varphi_n|_{U_{i,n}}$, etc. be the restrictions of $\varphi_n$. By Proposition 4.5, $\varphi_n|_{(C_{0,v_1} \cup C_{0,v_2})^o}$ lifts to a map $\psi_{n,v_1v_2}|(C_{0,v_1} \cup C_{0,v_2})^o$, which is a deformation of $\psi_{0,v_1v_2}$.

We fix an equation $\widetilde{F}_{i,n} = 0$ defining the image of $\psi_{n,v_1v_2}|_{U_{i,n}}$. Let $F_{j,n} = 0$ and $F_{k,n} = 0$ be defining equations of the images of $\varphi_n|_{U_{j,n}}$ and $\varphi_n|_{U_{k,n}}$ in $W^s_{j,n}$ and $W^s_{k,n}$, respectively. Then the pull back $P_\varphi|_{W^s_{j,n,v_1v_2}} = F_{j,n} = 0$ and $P_\varphi|_{W^s_{k,n,v_1v_2}} = F_{k,n} = 0$ give defining equations of the images of $\psi_{n,v_1v_2}|_{U_{j,n}}$ and $\psi_{n,v_1v_2}|_{U_{k,n}}$. Here $W^s_{j,n,v_1v_2}$ and $W^s_{k,n,v_1v_2}$ are the restriction of $Y^s_{v_1v_2}/(\mathbb{R}^{n+2})$, seen as an analytic ringed space, to $W^s_{j,v_1v_2}$ and $W^s_{k,v_1v_2}$, respectively.

By the same calculation as in Lemma 4.6, regarding these functions as defined over $\mathbb{C}[t]/(t^{n+2})$, we obtain local sections $U_{i,j,n+1}, U_{k,n+1}, U_{j,k,n+1}$ of the normal sheaf of $\psi_0$ which satisfy the relation as in Lemma 4.6. Note that on the intersection of any two of the open subsets $W_i, W_j, W_k$, the map $P_\varphi$ is an isomorphism onto the image. Therefore, the above sections are naturally considered as sections of the normal sheaf (or the log normal sheaf, since away from nodes they are canonically isomorphic) of $\varphi_0$.

Apply this construction to all edges of $\Gamma$ and open subsets of $\{U_i\}$. That is, let $v, v'$ be any pair of adjacent vertices of $\Gamma$. We have the associated toric degeneration $\mathcal{Y}_{\mathbb{R}^{n+1}_{v,v'}}$ and a map $P_{\varphi'} : \mathcal{Y}_{\mathbb{R}^{n+1}_{v,v'}} \to \mathcal{Y}_{\mathbb{R}^{n+1}_{v,v'}}$ from a toric degeneration constructed from a standard tropical curve with two vertices. Let $P_{\varphi'} : Y_{0,vv'} \to Y_{0,vv'}$ be the induced map on the central fibers. Let $\{U'_i\}$ be the subset of the open covering $\{U_i\}$ of $C_0$ whose image by $\varphi_0$ is contained in $Y_{0,vv'}$. By the above construction, on each $U'_i$, the map $\varphi_n$ lifts to a map $\psi_{n,vv'}|_{U'_{i,n}} : U'_{i,n} \to Y^s_{n,vv'}$. We can construct an $(n + 1)$-th order deformations of $\psi_{n,vv'}|_{U'_{i,n}}$ as in the proof of Lemma 4.6, and by
taking the differences of the defining equations, we obtain local sections of the log normal sheaf of \( \varphi_0 \). Note that for any pair of open subsets \( U, U' \) in \( \{ U_i \} \) such that \( U \cap U' \) is non-empty, there is some \( \mathcal{N}_{h_{vv'}}(\Gamma_{vv'}) \) which contains both of the images of \( U \) and \( U' \). Therefore, for any intersection of two open subsets of \( \{ U_i \} \), we obtain a section by this construction. This construction might a priori depends on the choices made, namely, the choice of \( \mathcal{N}_{h_{vv'}}(\Gamma_{vv'}) \) which contains \( U \) and \( U' \), and the choice of a lift \( \psi_{n,vv'} \) after fixing \( \mathcal{N}_{h_{vv'}}(\Gamma_{vv'}) \). However, it is not the case and we have the following.

**Lemma 4.7.** The local sections of the log normal sheaf \( \mathcal{N} \) of \( \varphi_0 \) obtained by this construction gives a Čech 1-cocycle on \( C_0 \) associated with the covering \( \{ U_i \} \).

**Proof.** We only need to check that for any pair \( U_i, U'_i \) of intersecting open subsets in \( \{ U_i \} \), the associated section depends neither on the choice of \( P_0 : \mathcal{N}_{0,vv'}(\Gamma_{vv'}) \) by which the map \( \varphi_0 \) is lifted, nor on the choice of the lift of \( \varphi_0 \) after choosing \( \mathcal{N}_{h_{vv'}}(\Gamma_{vv'}) \). In fact, once this is checked, Lemma 4.6 claims that the cocycle condition is satisfied.

For each \( \mathcal{N}_{h_{vv'}}(\Gamma_{vv'}) \), we choose a lift \( \psi_{n,vv'}|_{(C_{n,v} \cup C_{n,v'})} \) of \( \varphi_n|_{(C_{n,v} \cup C_{n,v'})} \) as above. When an open subset \( U_i \) contains a node, there is a unique \( \mathcal{N}_{h_{vv'}}(\Gamma_{vv'}) \) which contains its image. Therefore, in this case there is no choice for \( P_0 \). Assume \( U_i \) does not contain a node and its image is contained in \( \mathcal{N}_{h_{vv'}}(\Gamma_{vv'}) \) and \( \mathcal{N}_{h_{vv'}}(\Gamma_{vv'}) \). Let \( P_0 : \mathcal{Y}_{vv'} \to \mathcal{N}_{h_{vv'}}(\Gamma_{vv'}) \) and \( P_{0'} : \mathcal{Y}_{vv'} \to \mathcal{N}_{h_{vv'}}(\Gamma_{vv'}) \) be the maps from toric varieties associated with standard tropical curves with two vertices. Let \( Y_0 \) and \( Y_0' \) be the central fibers of \( \mathcal{Y}_{vv'} \) and \( \mathcal{Y}_{vv'} \). Similarly, let \( \mathcal{Y}_{vv'} \) and \( \mathcal{Y}_{vv'} \) be the central fibers of \( \mathcal{Y}_{vv'} \) and \( \mathcal{Y}_{vv'} \). In this case, recall that we take \( U_i \) small enough so that there is an open subset \( W_i \) containing \( \varphi_0(U_i) \) such that its inverse images in \( Y_{vv'} \) and \( Y_{vv'} \) consist of disjoint open subsets isomorphic to \( W_i \). There is a unique open subset \( W_{i,vv'} \) among these such that the image \( \psi_{0,vv'}(U_i) \) is contained in \( W_{i,vv'} \). Similarly, there is a unique \( W_{vv'} \) which contains \( \psi_{0,vv'}(U_i) \).

Let \( F_{i,n} = 0 \) be the defining equation of \( \varphi_n(U_i) \) in \( W_{i,n} \), here \( U_{i,n} \) is the restriction of the analytic ringed space \( C_n \) to \( U_i \), and \( W_{i,n} \) is the restriction of \( X_n \) to \( W_i \). The open subsets \( W_{i,n,vv'} \) and \( W_{i,n,vv'} \) are defined similarly. Then we take the defining equations of the images of \( \psi_{n,vv'}|_{U_{i,n}} \) and \( \psi_n|_{U_{i,n},vv'} \) by \( P_{0}|_{W_{i,n,vv'}} \) and \( P_{0'}|_{W_{i,n,vv'}} \), respectively. Recall that the process of regarding a function over \( \mathbb{C}[t]/t^{n+1} \) as a function over \( \mathbb{C}[t]/t^{n+2} \) discussed in Subsection 4.4 does not depend on the choice of \( \mathcal{N}_{h_{vv'}} \) or \( \mathcal{N}_{h_{vv'}} \). Explicitly, this means that if
we denote by $F$ and $F'$ the functions $P_{\Phi|_{W_{t,n+1}}}$, $F_i$, and $P_{\Phi|_{W_{t,n+1}}}$, $F_i$ regarded as defined over $\mathbb{C}[t]/t^{n+2}$, we have $(P_{\Phi|_{W_{t,n+1}}})^*F = (P_{\Phi|_{W_{t,n+1}}})^*F'$ as functions on $W_{t,n+1}$.

Finally, other choices of $\psi_{n,v'}$ after fixing $\mathcal{Y}_{h,v'(F_{vv'})}$ differ from the given one only by covering transformations. Since a covering transformation $\delta$ satisfies $P_\phi = P_\phi \circ \delta$, this does not change the local section.

Let $\{\nu_{ij,n}\}$ be the Čech 1-cocycle on $C_0$ with values in $N$ obtained in this way. Now we check the following.

**Proposition 4.8.** The cohomology class of $\{\nu_{ij,n}\}$ is the obstruction to deforming $\varphi_n$ to a map over $\mathbb{C}[t]/t^{n+2}$.

**Proof.** Assume that the cohomology class of $\{\nu_{ij,n}\}$ vanishes. This means that there is a section $\eta_i$ of the restriction $N|_{U_i}$ of the log normal sheaf of $\varphi_0$ such that $\eta_i - \eta_j = \nu_{ij,n}$ on each $U_i \cap U_j$.

Note that on each $U_i$, the map $\psi_{0,v'}|_{U_i}$ can be promoted to a map between log analytic varieties, with respect to the restriction of the fixed log structure on $C_0$ and the toric log structure on $Y_{v'}$. In fact, there is a unique such enhancement up to isomorphisms since $\psi_{0,v'}$ is an embedding (see [24] for the classification of log structures in a similar situation). Then it is easy to see that the log normal sheaves of $\psi_{0,v'}|_{U_i}$ and of $\varphi_0|_{U_i}$ are canonically isomorphic. Thus, we can regard $\eta_i$ as a section of the log normal sheaf $N_{\psi_{0,v'}|_{U_i}}$ of $\psi_{0,v'}|_{U_i}$.

When $U_i$ does not contain a node, we have an exact sequence

$$0 \to O_{W_{i,0,v'}} \to O_{W_{i,v'}}(\psi_{0,v'}(U_i)) \to N_{\psi_{0,v'}}|_{U_i} \to 0.$$ 

Here $O_{W_{i,0,v'}}(\psi_{0,v'}(U_i))$ is the sheaf of analytic functions on $W_{i,0,v'}$ which can have a first order pole along $\psi_{0,v'}(U_i)$. Since we take $W_i$ to be biholomorphic to a cylinder, we have $H^1(W_{i,0,v'}, O_{W_{i,0,v'}}) = 0$ by the Poincaré’s Lemma for holomorphic functions. Thus, a section of $N_{\psi_{0,v'}}|_{U_i}$ can be lifted to a section in $O_{W_{i,0,v'}}(\psi_{0,v'}(U_i))$.

When $U_i$ contains a node, we have a similar sequence, but now $W_{i,0,v'}$ is the normal crossing union of two cylinders. If $W_{i}^1$ and $W_{i}^2$ are irreducible components of $W_{i,0,v'}$, we have an exact sequence

$$0 \to O_{W_{i,v'}} \to (i_1)_* O_{W_{i}^1} \oplus (i_2)_* O_{W_{i}^2} \to (i_1)_* O_{W_{i}^1 \cap W_{i}^2} \to 0,$$

where $i_1: W_{i}^1 \to W_{i,0,v'}$, $i_2: W_{i}^2 \to W_{i,0,v'}$ and $i_1: W_{i}^1 \cap W_{i}^2 \to W_{i,0,v'}$ are inclusions. Since $H^1(W_{i,0,v'}, (i_1)_* O_{W_{i}^1} \oplus (i_2)_* O_{W_{i}^2}) \cong H^1(W_{i}^1, O_{W_{i}^1}) \oplus H^1(W_{i}^2, O_{W_{i}^2})$ and $H^1(W_{i}^1, O_{W_{i}^1})$ vanishes by the Poincaré’s Lemma, $H^1(W_{i,0,v'}, (i_1)_* O_{W_{i}^1} \oplus$
(\nu)_{0} also vanishes. Moreover, the map \( H^{0}(W_{i,0,v}', (i_{1})_{*}O_{W_{i}^{2}} \oplus (i_{2})_{*}O_{W_{2}^{2}}) \to H^{0}(W_{i,0,v}', (i_{12})_{*}O_{W_{i}^{2} \cap W_{2}^{2}}) \) is a surjection since a holomorphic function on \( W_{i}^{2} \cap W_{2}^{2} \) can be extended to both \( W_{i}^{2} \) and \( W_{2}^{2} \) since they are cylinders. Therefore, we have \( H^{1}(W_{i,0,v}', O_{W_{i}^{2}}) = 0 \) and it follows that a section of \( \mathcal{N}_{\psi_{0,v'}}| U_{i} \) can be lifted to a section in \( O_{W_{i}^{2}}(\psi_{0,v'}(U_{i})) \).

Let \( h_{i} \) be an element in \( O_{W_{i}^{2}}(\psi_{0,v'}(U_{i})) \) which lifts \( \eta_{i} \). When \( \varphi_{0}(U_{i}) \) is contained in another \( \mathcal{Y}_{\mathcal{C}_{\nu_{1},\nu_{2}}(\tau_{\nu_{1},\nu_{2}})} \), we can take the lift of \( \eta_{i} \) compatibly in the sense that under the canonical isomorphism between \( O_{W_{i}^{2}}(\psi_{0,v'}(U_{i})) \) and \( O_{W_{i}^{2}}(\psi_{0,v'}(U_{i})) \), these lifts coincide. Here \( W_{i,0,v'}^{2} \) is a lift of \( W_{i} \) to \( \mathcal{Y}_{\mathcal{C}_{\nu_{1},\nu_{2}}(\tau_{\nu_{1},\nu_{2}})} \).

By these calculations, it follows that if \( F_{i,n} = 0 \) is the defining equation of the image of \( \psi_{n,v'}| U_{i,n} \), regarding it as an equation over \( \mathbb{C}[t]/t^{n+2} \), the modified equation \( F_{i,n} - t^{n+1}h_{i} = 0 \) gives the image of a local deformation \( \psi_{n+1,v'}| U_{i,n+1} \) of \( \psi_{n,v'}| U_{i,n} \). By construction, this has the property that the images of the composition \( P_{\Phi} \circ \psi_{n+1,v'}| U_{i,n+1} \) glue on \( X_{n+1} \) so that it gives an \( (n+1) \)-th order deformation of the given \( \varphi_{n} \). In other words, if the cohomology class of \( \{ \nu_{ij,n} \} \) vanishes, an \( (n+1) \)-th order deformation of \( \varphi_{n} \) exists.

Conversely, it is easy to see that if the cohomology class of \( \{ \nu_{ij,n} \} \) does not vanish, then an \( (n+1) \)-th order deformation of \( \varphi_{n} \) does not exist. \( \square \)

By this result, combined with Proposition 4.3, we conclude that the calculation of the obstruction to deforming \( \varphi_{n} \) can be entirely done on \( \mathcal{Y}_{v'}^{s} \), as if these are coordinate neighborhoods of \( \mathcal{X} \). The merit of this is that since the lift \( \psi_{0,v'}: U_{\alpha} \to \mathcal{Y}_{v'}^{s} \) is an embedding, the study of [22] can be applied. In the following, we apply this to explicit situations.

4.6. Existence of first order deformations. In this subsection, we prove that the obstruction to the existence of first order deformations of \( \varphi_{0} \) vanishes. To calculate the obstruction, we use Proposition 4.8. Namely, let \( P_{\Phi}: \mathcal{Y}_{v'}^{s} \to \mathcal{Y}_{\mathcal{C}_{\nu_{1},\nu_{2}}(\tau_{\nu_{1},\nu_{2}})} \) be a branched covering as before. We fix a lift \( \psi_{0,v'}| (C_{0,v} \cup C_{0,v})^{c} \to \mathcal{Y}_{v'}^{s} | (C_{0,v} \cup C_{0,v})^{c} \). For any open subset \( U_{i} \) contained in \( (C_{0,v} \cup C_{0,v})^{c} \), we take a first order local deformation of \( \psi_{0,v'}| U_{i} \) by regarding the defining equation of the image as an equation over \( \mathbb{C}[t]/t^{2} \) as before, and construct the obstruction cocycle \( \{ \nu_{ij,0} \} \) by taking their differences.

To see whether the class \( \{ \nu_{ij,0} \} \) is zero or not, we use Proposition 4.3. Recall that by pushing out \( \{ \nu_{ij,0} \} \) by \( P_{\Phi} \), it gives an \( \mathcal{N} \)-valued cocycle. Here \( \mathcal{N} \) is the log normal sheaf of \( \varphi_{0} \). To apply Proposition 4.3, we need to represent \( \{ (P_{\Phi})_{*}\nu_{ij,0} \} \) as in Subsection 4.2. Namely, we need to take a set of meromorphic sections \( \{ \xi_{k} \} \) of \( \mathcal{N} \) on locally closed subsets \( \{ V_{k} \} \) associated with the covering \( \{ U_{i} \} \) so that their
differences on the intersection coincide with the cocycle \( \{(P_{\Phi})_{*}\nu_{j,0}\} \). Then we calculate the residue of the pairing between each \( \xi_k \) and the generator \( \psi \) of the space \( H^0(C_0, N^\vee \otimes \omega_{C_0}) \).

Such a calculation can also be lifted to the standard case introduced in Subsection 4.3. The map \( P_{\Phi} \) induces a map \( P_{\Phi}^* \) from \( N^\vee \) to \( N_{\psi_0,\nu_0}^\vee \). For some section \( \xi_k \) of \( N \) on a locally closed subset of \( C_0 \), if we have a section \( \eta'_k \) of \( N_{\psi_0,\nu_0} \) such that \( P_{\Phi}(\eta'_k) = \xi_k \), obviously the following holds.

**Lemma 4.9.** We have the equality \( (P_{\Phi}^*\psi, \eta'_k) = (\psi, \xi_k') \) of local meromorphic sections of \( \omega_{C_0} \). Here \((, , )\) on the left hand side is the fiberwise pairing between \( N_{\psi_0,\nu_0}^\vee \otimes \omega_{C_0} \) and \( N_{\psi_0,\nu_0}^\vee \), and \((, )\) on the right hand side is the fiberwise pairing between \( N^\vee \otimes \omega_{C_0} \) and \( N \). \( \square \)

Therefore, we can calculate the pairing between the obstruction class \( \xi_k \) and \( \psi \) using first order deformations of \( \psi_0,\nu_0 \).

Let \( v \) and \( v' \) be adjacent vertices of \( \Gamma \). We assume that \( v \) is 3-valent and \( v' \) is 2-valent. The case where both of the vertices are 2-valent is easier, see the proof of Proposition 4.10 below. Take a pre-log curve \( \psi_{0,\nu}: (C_0, \nu_0)^0 \rightarrow Y_{0,\nu}^* \) as above.

Let \( U_i \) be the unique open subset taken in Subsection 4.2 which contains the node \( C_{0,v} \cap C_{0,v'} \). Using the coordinates \( X, Y, Z \) on \( Y_{0,\nu}^* \) introduced in Subsection 4.3, the image of \( \psi_{0,\nu} \) is given by the equation

\[
X + aY + bZ + c = 0, \quad t = 0,
\]

where \( a, b, c \) are nonzero complex numbers and \( t = XY \). A first order deformation of it is given by regarding the equation \( X + aY + bZ + c = 0 \) as defined over \( \mathbb{C}[t]/t^2 \), and forgetting \( t = 0 \), as we discussed in Subsection 4.4.

On an open subset \( U_j \) of \( C_0 \) which does not contain a node, a first order deformation is also obtained by regarding a defining equation \( F_{j,0} = 0 \) of the image as an equation over \( \mathbb{C}[t]/t^2 \). If the intersection \( U_i \cap U_j \) is non-empty, we can take \( \{X, Z, t\} \) or \( \{Y, Z, t\} \) as a coordinate system on the open subset \( W_{j,1,\nu'}^* \) of \( Y_{1,\nu'}^* \). Assume that \( \{X, Z, t\} \) is a coordinate system on it. Then \( F_{j,0} \) is given by \( G(X, Z)(X + bZ + c) \) on \( U_i \cap U_j \), where \( G(X, Z) \) is an invertible function. Over \( \mathbb{C}[t]/t^2 \), the equation \( X + aY + bZ + c = 0 \) can be written as \( X + a_X^0 + bZ + c = 0 \) on \( W_{1,\nu'}^* \cap W_{j,1,\nu'}^* \), and the equation \( F_{j,0} = 0 \) on \( W_{j,1,\nu'}^* \) does not change its form. Then the difference \( X + a_X^0 + bZ + c - G^{-1}F_{j,0} = a_X \) can be regarded as a local section of the normal sheaf of \( \psi_{0,\nu} \) on \( U_i \cap U_j \), as in Lemma 4.6. Explicitly, it is identified with the local section \( \partial Z \) of the normal sheaf. In particular, the pushout of it by \( P_{\Phi} \) gives a part of the obstruction cocycle.

From these calculations, we have the following.
Proposition 4.10. The pairing between the obstruction class and \( \psi \) in Proposition 4.3 vanishes. In particular, there is a first order deformation of \( \varphi_0 \).

Proof. We use the notation in the above paragraph. Let \( U_{i,1} \) and \( U_{i,2} \) be the locally closed subsets associated with \( U_i \) as in Subsection 4.2, and assume \( U_{i,1} \cap U_{j} \) is nonempty so that \( U_i \cap U_j = U_{i,1} \cap U_j \). Take \( \frac{\partial}{\partial z} \), considered as a meromorphic section of the normal sheaf of \( \psi_{0,vv'} \) on the locally closed subset \( U_{i,1} \). Take the zero section on \( U_j \). By the above calculation, these sections on the locally closed subsets give a representative of the obstruction class in the sense of Subsection 4.2. Therefore, it suffices to see that the pairing of it with the pull back of \( \psi \) vanishes.

Recall that this pairing is the sum of local contributions of residues. The pull back of \( \psi \) by \( P_\Phi \) is of the form \( q dX \wedge dz \), where \( q \) is a non-zero integer. Then the fiberwise pairing between \( P_\Phi^* \psi \) and \( \frac{\partial}{\partial z} \), which gives a local meromorphic section of \( \omega_{C_0} \), has a pole of order two and has no residue at the node. Therefore, each local contribution to the pairing is zero, and the pairing itself vanishes.

Although we calculated in the case where one of the vertices \( v_1, v_2 \) is 3-valent, the case where both of them are 2-valent is similar. In fact, the same calculation as above shows that on locally closed subsets associated with an open subset containing a node corresponding to an edge connecting 2-valent vertices, we can take the zero section as the representative of the obstruction cocycle. Namely, the defining equation of the image of the curve is of the form \( Z + c = 0 \) in this case, and it does not change its form when it is considered over \( \mathbb{C}[t]/t^2 \). This proves the proposition. \( \square \)

4.7. Vanishing of the obstructions to deforming at higher order. Assuming an \( n(\geq 1) \)-th order deformation \( \varphi_n : C_n \to X_n \) of the pre-log curve \( \varphi_0 : C_0 \to X_0 \) exists, we are going to compute the obstruction to deforming it to the \( (n+1) \)-th order.

Let \( \{ U_i \}_{i \in I} \) be an open covering of \( C_0 \) as in the previous subsection. Let \( U_{i,n} \) denote the ringed space which is the restriction of \( C_n \) to \( U_i \). Also, let \( W_{i,n} \) denote the ringed space which is the restriction of \( X_n \) to \( W_i \). As in the previous subsection, based on Proposition 4.8, we do all the calculation in the standard case. We take a pair \( v, v' \) of adjacent vertices of \( \Gamma \), and also \( Y_{vv'} \), \( \psi_{0,vv'} \) etc. as before. Let \( W_{i,n,vv'}^s \) be the chosen lift of \( W_{i,n} \) in \( Y_{vv'}^s \).

By Proposition 4.5, the map \( \varphi_n \) locally factors through a map to \( Y_{vv'}^s \). Let \( \psi_{i,n} : U_{i,n} \to W_{i,n,vv'}^s \) be such a map on \( U_{i,n} \). Although \( \psi_{i,n} \) depends on the choice of \( v, v' \), we drop the subscript \( vv' \) from the notation \( \psi_{i,n} \) and \( F_{i,n} \), etc. below for simplicity, since by Lemma 4.7, the choice of \( v, v' \) plays little role in the following
argument. Let \( F_{i,n} = 0 \) be the defining equation of the image of \( \psi_{i,n} \) in \( W_{i,n,v,v'}^s \). Similarly, let \( F_{j,n} = 0 \) be the defining equation of the image of \( \psi_{j,n} \) in \( W_{j,n,v,v'}^s \).

On the intersection \( W_{i,n,v,v'}^s \cap W_{j,n,v,v'}^s \), these are related by

\[
F_{i,n} = G_{i,j,n} F_{j,n},
\]

where \( G_{i,j,n} \) is a holomorphic function defined over \( \mathbb{C}[t]/t^{n+1} \) whose reduction over \( \mathbb{C}[t]/t \) is non-vanishing. Recall that by regarding these functions as defined over \( \mathbb{C}[t]/t^{n+2} \), we obtain the difference

\[
t^{n+1} \nu_{i,j,n+1} = F_{i,n} - G_{i,j,n} F_{j,n}
\]

on \( W_{i,n+1,v,v'}^s \cap W_{j,n+1,v,v'}^s \). Here \( \nu_{i,j,n+1} \) can be regarded as a holomorphic function on \( W_{i,n,v,v'}^s \cap W_{j,0,v,v'}^s \), and the pull back of these to \( C_0 \) constitute the obstruction cocycle to deforming \( \varphi_n \).

To see the vanishing of the cohomology class of \( \{ \nu_{i,j,n+1} \} \), we apply Proposition 4.3. To do so, we need to represent \( \{ \nu_{i,j,n+1} \} \) by sections over locally closed subsets as in the previous subsection.

We can write the function \( F_{i,n} \), regarded as a function over \( \mathbb{C}[t]/t^{n+2} \), in the form

\[
F_{i,n} = F_{i,0} \exp(f_{i,n+1}),
\]

where \( f_{i,n+1} \) is a function on \( W_{i,n+1,v,v'}^s \) which can have poles along \( \{ F_{i,0} = 0 \} \) (the function \( F_{i,0} \) is also regarded as a function on \( W_{i,n+1,v,v'}^s \) by repeatedly applying the argument in Subsection 4.4), and is zero when reduced over \( \mathbb{C}[t]/t \). Similarly we define \( f_{j,n+1} \). Let \( f_i(n + 1) \) and \( f_j(n + 1) \) be the coefficients of \( t^{n+1} \) in \( f_{i,n+1} \) and \( f_{j,n+1} \), respectively. These can be naturally considered as meromorphic functions on \( W_{i,0,v,v'}^s \) and \( W_{j,0,v,v'}^s \).

Assume that \( U_i \) contains a node. By definition, we have

\[
t^{n+1} \nu_{i,j,n+1} = F_{i,0} \exp(f_{i,n+1}) - G_{i,j,n} F_{j,0} \exp(f_{j,n+1}).
\]

The function \( F_{i,0} \) is of the form \( X + aY + bZ + c = 0 \) as in Subsection 4.6. We assume that the image of \( U_j \) is contained in the irreducible component of \( Y_{0,v,v'}^s \) (the central fiber of \( Y_{v,v'}^s \)) given by \( Y = 0 \).

Let us write \( F_{i,0,X} = X + bZ + c \). Dividing the equation above by \( F_{i,0,X} \exp(f_{j,n+1}) \), we have

\[
\frac{t^{n+1} \nu_{i,j,n+1}}{F_{i,0,X}} = \frac{F_{i,0}}{F_{i,0,X}} \exp(f_{i,n+1} - f_{j,n+1}) - G_{i,j,n} \frac{F_{j,0}}{F_{i,0,X}}
\]

on \( W_{i,n+1,v,v'}^s \cap W_{j,n+1,v,v'}^s \). Note that since the function \( \exp(f_{j,n+1}) \) is of the form \( 1 + t(\cdots) \), dividing by it does not affect the left hand side as the equation is defined over \( \mathbb{C}[t]/t^{n'+2} \). Also note that \( G_{i,j,n} \frac{F_{j,0}}{F_{i,0,X}} \) is a holomorphic function on \( W_{i,n+1,v,v'}^s \cap W_{j,n+1,v,v'}^s \).
Now we have

$$\frac{F_{i,0}}{F_{i,0,X}} = 1 + \frac{aY}{F_{i,0,X}} = 1 + \frac{at}{XF_{i,0,X}}.$$  

Furthermore,

$$1 + \frac{at}{XF_{i,0,X}} = \exp(\log(1 + \frac{at}{XF_{i,0,X}})) = \exp(\sum_{l=1}^{n+1} (-1)^{l+1} \frac{(at)^l}{l} \frac{XF_{i,0,X}}{X}).$$

Therefore, we have

$$\frac{F_{i,0}}{F_{i,0,X}} \exp(f_{i,n+1} - f_{j,n+1}) = \exp(\tilde{f}_{i,n+1} - f_{j,n+1}),$$

where $\tilde{f}_{i,n+1} = f_{i,n+1} + \sum_{l=1}^{n+1} (-1)^{l+1} \frac{(at)^l}{l} \frac{XF_{i,0,X}}{X}.$

Recall that the function $f_{i,n+1}$ belongs to $\mathbb{C}[X,Y,Z^{\pm 1},t,\frac{1}{F_{i,0}}]/(XY - t^{n+1})$.

Substituting $Y = \frac{X}{t}$ and expanding

$$\frac{1}{F_{i,0}} = \frac{1}{F_{i,0,X}} \left(1 + \frac{at}{XF_{i,0,X}}\right) = \frac{1}{F_{i,0,X}} \sum_{l=0}^{n+1} (-1)^l \left(\frac{at}{XF_{i,0,X}}\right)^l,$$

we write

$$\tilde{f}_{i,n+1} = \sum_{l=1}^{n+1} t^l \tilde{f}_i(l),$$

where $\tilde{f}_i(l)$ belongs to $\mathbb{C}[X, \frac{1}{X}, Z^{\pm 1}, \frac{1}{F_{i,0,X}}]$. Now we have the following.

**Lemma 4.11.** When neither $U_i$ nor $U_j$ does not contain a node, the identity

$$f_{i}(n + 1) - f_{j}(n + 1) = \nu_{n+1,ij} + \kappa$$

holds on $W_{i,0,vv}^* \cap W_{j,0,vv}^*$, here $\kappa$ is a holomorphic function. When one of them, say $U_i$, contains a node, then the identity

$$\tilde{f}_{i}(n + 1) - f_{j}(n + 1) = \nu_{n+1,ij} + \kappa$$

holds on $W_{i,X,vv}^* \cap W_{j,0,vv}^*$. Here $W_{i,X,vv}^*$ is the irreducible component of $W_{i,0,vv}^*$ given by $Y = 0$.

**Proof.** This was proved in [22] in the case of deformations of maps from curves into a fixed surface. However, in the above setting, all the relevant coordinate transformations do not depend on the deformation parameter $t$ (here we use the assumption $(\ast)$ in the introduction as in Subsection 4.4). Therefore, the proof for the case with a fixed target space is valid without change. $\square$
The Poincaré residue of the 2-form $\frac{\nu_{ij,n+1}}{F_{i,0}} P^*_{f} \psi$ along $\{F_{i,0} = 0\} \cap W_{i,0,vv'}^s \cap W_{f,0,vv'}^s$ is defined by the pullback of the 1-form $\tilde{\zeta}_{ij}$ on $W_{i,0,vv'}^s \cap W_{f,0,vv'}^s$ satisfying
\begin{equation}
\frac{\nu_{ij,n+1}}{F_{i,0}} P^*_{f} \psi = \zeta_{ij,n+1} \wedge \frac{dF_{i,0}}{F_{i,0}}
\end{equation}
to $U_i \cap U_j$. From this definition, it is clear that the pullback of $\zeta_{ij,n+1}$ to $U_i \cap U_j$ coincides with the fiberwise pairing between $\nu_{ij,n+1}$ and $P^*_{f} \psi$ (recall that $\nu_{ij,n+1}$ is naturally considered as a local section of the normal sheaf $N$ on $C_0$).

Lemma 4.6 implies that the set of sections $\{\zeta_{ij,n+1}\}$ gives a Čech 1-cocycle on $C_0$ with values in $\omega_{C_0}$. The cohomology class of $\{\zeta_{ij,n+1}\}$ is the one obtained by the pairing $H^0(C_0, N \otimes \omega_{C_0}) \otimes H^1(C_0, N) \to H^1(C_0, \omega_{C_0})$, and by Propositions 4.2 and 4.8, we have the following.

**Proposition 4.13.** The vanishing of the cohomology class of $\zeta_{ij,n+1}$ is equivalent to the vanishing of the obstruction to deforming $\varphi_n$. \hfill $\square$

Let $C_{0,v}$ be any irreducible component of $C_0$. The restrictions of $\{\zeta_{ij,n+1}\}$ to $C_{0,v}$ give a Čech 1-cocycle on it with respect to the restriction of the open covering $\{U_i\}$ of $C_0$. By Remark 4.4 (3) applied to the case $L = \omega_{C_0,v}$, such a cocycle can be represented by the differences of local meromorphic sections $\{\zeta_n\}$ on $C_{0,v}$. Moreover, the cohomology class defined by the Čech 1-cocycle only depends on the residues of $\zeta_{ij,n+1}$. In particular, if we know such a representative of the class $[\zeta_{ij,n+1}]$, it is easy to see whether $[\zeta_{ij,n+1}]$ is zero or not.

In our case, we do not have an obvious candidate for the meromorphic sections $\zeta_{i,n+1}$, but instead we have constructed a meromorphic closed 2-form $f_i(n+1)P^*_{f} \psi$ or $\tilde{f}_i(n+1)P^*_{f} \psi$ on each open subset $W_{i,0,vv'}^s$ or $W_{f,0,vv'}^s$. Note that the image of the map $\psi_{0,vv'}$ is a union of linear curves, as in Subsection 4.6. Consider the irreducible component $Y_{X,vv'}^s$ of $Y_{vv'}^s$ defined by $Y = 0$. Fix a metric on $Y_{X,vv'}^s$. Let $\psi_{0,vv'}(C_{0,v})$ be the component of the image of $\psi_{0,vv'}$ contained in $Y_{X,vv'}^s$. By the exponential map, we can identify a disk bundle of radius $\delta$ of the normal bundle of $\psi_{0,vv'}(C_{0,v})$ with a tubular neighborhood $N_{\delta}(\psi_{0,vv'}(C_{0,v}))$ of $\psi_{0,vv'}(C_{0,v})$. The boundary $\partial N_{\delta}(\psi_{0,vv'}(C_{0,v}))$ has a structure of a circle bundle over $C_{0,v}$, and let $\pi_{\delta}: \partial N(\psi_{0,vv'}(C_{0,v})) \to C_{0,v}$ be the projection.

If $W_{i,0,vv'}^s$ is contained in $Y_{X,vv'}^s$, we can integrate the 2-form $f_i(n+1)P^*_{f} \psi$ along the fibers of $\pi_{\delta}$. Let $\eta_{i,\delta} = \frac{1}{2\pi i} \int_{\delta} f_i(n+1)P^*_{f} \psi$ denote the integration along the fibers. This gives a closed 1-form on $U_i$. In the case of $f_i(n+1)P^*_{f} \psi$, we can also integrate along the fibers away from the node.
$p \in U_i$ of $C_0$, since there $\tilde{f}_i(n + 1)$ may have poles along the fiber. Again, we denote by $\eta_{i,\delta}$ the integration along the fibers. This is a closed 1-form on $U_{i,X} \setminus B_p$, here $B_p$ is a small closed disc around $p$.

The differences of these closed 1-forms $\eta_{ij,n+1,\delta} = \eta_{i,\delta} - \eta_{j,\delta}$ on $U_i \cap U_j$ gives a Čech 1-cocycle on each irreducible component of $C_0$ with values in closed 1-forms. Also, on each irreducible component $C_{0,v}$ of $C_0$, we have an isomorphism

$$H^1(C_{0,v}, \mathcal{C}^1) \cong H^2(C_{0,v}, \mathcal{C}) \cong H^1(C_{0,v}, \omega_{C_{0,v}}),$$

where $\mathcal{C}^1$ is the sheaf of closed $C^\infty$ 1-forms and $\omega_{C_{0,v}}$ is the canonical sheaf.

We review the argument in [22] comparing the class $[\eta_{ij,n+1,\delta}]$ and the obstruction class $[\zeta_{ij,n+1}]$. See [22] for the full details. First, the sections $\eta_{i,\delta}$ can be seen as the analogue of the representation of classes in $H^1(C_0, \omega_{C_0})$ by meromorphic sections on locally closed subsets as in Subsection 4.2. In particular, the cohomology class of $\{\eta_{ij,n+1,\delta}\}$ depends only on the value of the integral of $\eta_{i,\delta}$ along a contour $\gamma_p$ encircling $B_p$ in the positive direction, for each node $p$ of $C_0$. The value of this contour integral is the analogue of the residues of the meromorphic sections discussed in Subsection 4.2. In particular, the following analogue of Proposition 4.3 holds.

**Proposition 4.14.*** The cohomology class defined by $\eta_{i,\delta}$ on each component $C_{0,v}$ can be identified with the sum of the values of the contour integrals around nodes of $C_0$ contained in $C_{0,v}$ under the identification $H^1(C_{0,v}, \mathcal{C}^1) \cong H^2(C_{0,v}, \mathcal{C}) \cong H^0(C_{0,v}, \mathcal{C})^\vee \cong \mathbb{C}$. \hfill $\square$

Note that the contour integral equals to the integral of the 2-form $f_i(n + 1)P^*_\varphi \psi$ or $\tilde{f}_i(n + 1)P^*_\varphi \psi$ along the torus $\pi_{\delta}^{-1}(\gamma_p)$. By the Stokes’ theorem, this integral does not depend on $\delta$. Therefore, for any $\delta$, $\eta_{i,\delta}$ defines the same cohomology class.

On the other hand, on the intersection $U_i \cap U_j$, the fiberwise integral

$$\frac{1}{2\pi i} \int_{\delta} (f_i(n + 1) - f_j(n + 1))P^*_\varphi \psi = \frac{1}{2\pi i} \int_{\delta} \zeta_{ij,n+1} \land \frac{dF_{i,0}}{F_{i,0}}$$

converges to $\zeta_{ij,n+1}$ as $\delta$ goes to zero. Combining this with the observation in the above paragraph, it follows that the cohomology classes defined by $\eta_{ij,n+1,\delta}$ and by $\zeta_{ij,n+1}$ coincide on each $C_{0,v}$.

Note that while $\eta_{ij,n+1,\delta}$ defines a cohomology class on each component of $C_0$, $\zeta_{ij,n+1}$ determines a cohomology class on $C_0$. However, recalling that the cohomology class of $\zeta_{ij,n+1}$ is determined only by the residues at the nodes and the contour integrals of $\eta_{i,\delta}$ correspond to the residues, one can see that $\eta_{i,\delta}$ also determines a cohomology class on $C_0$, and it is equal to the one determined by $\zeta_{ij,n+1}$. Again, analogous to Propositions 4.3, we have the following.
Proposition 4.15. The class of $H^1(C_0, \omega_{C_0})$ determined by $\eta_{i,\delta}$ is identified with the sum of all the contour integrals around the nodes of $C_0$ under the identification $H^1(C_0, \omega_{C_0}) \cong H^0(C_0, \mathcal{O}_{C_0}) \cong \mathbb{C}$. 

Then, by Proposition 4.13, we have the following.

Corollary 4.16. The obstruction to deforming $\varphi_n$ vanishes if and only if the sum of all the contour integrals of $\eta_{i,\delta}$ around the nodes of $C_0$ vanishes.

Thus, we have reduced the problem of proving the existence of deformations to checking the vanishing of the sum of the contour integrals of $\eta_{i,\delta}$. Note that at each node of $C_0$, there are two contour integrals associated with it, corresponding to the branches of $C_0$ at the node. Thus, to see the vanishing of the obstruction, it suffices to see that the sum of each pair of contour integrals associated with each node vanishes. We will prove this in the rest of this section.

4.7.1. Vanishing of the sum of integrals. Recall that the value of the integral along a contour $\gamma_p$ around a node is equal to the integral of the two form $\int_{i,n+1} \Psi_{i,n+1} + \sum_{l=1}^{n+1} \frac{(-1)^{l+1}}{l} (\frac{at}{XF_{i,l}i})^l$. Here $F_{i,0} \exp(f_{i,n+1}) = 0$ is the defining equation of the image of $\psi_{i,n} : U_{i,n} \rightarrow Y^{n+1}_{X_{i,n}}$, and $F_{i,0} = X + aY + bZ + c$ is a function defining the image of $\psi_{i,0}$. Also, $F_{i,0,X} = X + bZ + c$. Moreover, $P_{\Psi}^\Psi \Psi$ is given by $\frac{dx}{\sqrt{X}} \wedge \frac{dz}{\sqrt{Z}}$ up to a multiplicative constant. For notational simplicity, we put this constant to be one.

First we calculate the contribution from the latter part of $\int_{i,n+1}$.

Lemma 4.17. The integration of the two form $(\frac{at}{XF_{i,l}i})^l \psi_{i,n+1}$ along the torus $\pi_{\delta}^{-1}(\gamma_p)$ is given by $(-1)^{l+1} \left( \frac{2l - 1}{l} \right) \frac{4\pi^2 \beta_0 \alpha^l}{c^2}$.

Proof. We can take $\tilde{Z} = X + bZ + c$ as one of the coordinates on the component of $Y_{i,v,v'}$ given by $Y = 0$. Then the image of $\psi_{0,v,v'}$ coincides with the $X$-axis in the $(X, \tilde{Z})$-plane. Recall that we fix a metric on $Y_{i,v,v'}$ and take a tubular neighborhood $N_\delta(\psi_{0,v,v'}(U_{i,X}))$ of a part of the image of $\psi_{0,v,v'}$. By taking the metric suitably, we can assume that around the point $(X, \tilde{Z}) = (0, 0)$, $N_\delta(\psi_{0,v,v'}(U_{i,X}))$ is the product of the $X$-axis and a disc on the $\tilde{Z}$-axis, and the projection $\pi_{\delta} : N_\delta(\psi_{0,v,v'}(U_{i,X})) \rightarrow U_{i,X}$ is compatible with this product structure.

We have

\begin{align*}
(\frac{at}{XF_{i,l}i})^l P^\Psi_{\Psi} & = (\frac{at}{X\tilde{Z}})^l \frac{dX}{\tilde{Z}} \wedge (\frac{dZ}{\tilde{Z}(X-X-c)}) \\
& = - \frac{\beta_0 \gamma l}{X^{l+1} \tilde{Z}(X+c)} \frac{dX}{\sqrt{X}} \wedge \frac{dZ}{\sqrt{Z}} \\
& = - \frac{\beta_0 \gamma l}{X^{l+1} \tilde{Z}(X+c)} \sum_{m=0}^{\infty} (\frac{2}{X+c})^m dX \wedge d\tilde{Z}.
\end{align*}
To the integration around $\tilde{Z} = 0$, the term $m = l - 1$ contributes, and the integration is equal to $-\frac{2\pi ba't}{X^{l+1}(X+c)}dX$.

We have

$$\frac{-2\pi ba't}{X^{l+1}(X+c)}dX = -\frac{2\pi ba't}{c^4X^{l+1}} \sum_{m=0}^{\infty} (-1)^m \left(\frac{t}{m} \right) \left(\frac{X}{c^4}\right)^m.$$  

To the integration along $\gamma_p$, the term $m = l$ contributes, and the integration is equal to $(-1)^{l+1} \left(\frac{2l-1}{l}\right) \frac{4\pi^2 ba't}{c^4}$.

On the component of $Y_{0,s}$ given by $X = 0$, we have the similar integration. In this case, we need to integrate a 2-form

$$\tilde{f}_{i,n+1} \Phi P_{\Phi}^* \psi = -(f_{i,n+1} + \sum_{l=1}^{n+1} \frac{(-1)^{l+1}}{l} \left(\frac{t}{Y_{F_{i,0},Y}}\right)^l dY \wedge dZ,$$

where $F_{i,0,Y} = aY + bZ + c$, along a 2-torus which projects onto a contour $\gamma_p'$ encircling the point $p$. The overall negative sign comes from the relation $\frac{dX}{X} = -\frac{dY}{Y}$.

By the similar calculation as above, we can prove the following.

**Lemma 4.18.** The integration of the two form $(\frac{at}{Y_{F_{i,0},Y}})^l P_{\Phi}^* \psi$ along the torus $(\pi_\delta')^{-1}(\gamma_p')$ is given by $(-1)^l \left(\frac{2l-1}{l}\right) \frac{4\pi^2 ba't}{c^4}$. Here $\pi_\delta'$ is the projection from a circle bundle over $\gamma_p'$ as in the case of $\pi_\delta$. \qed

Thus, contributions from the terms $\sum_{l=1}^{n+1} \frac{(-1)^{l+1}}{l} \left(\frac{at}{X_{F_{i,0},X}}\right)^l$ and $\sum_{l=1}^{n+1} \frac{(-1)^{l+1}}{l} \left(\frac{t}{Y_{F_{i,0},Y}}\right)^l$ cancel. Therefore, we need to show that the sum of the integrations of $\tilde{f}_{i,n+1} \Phi P_{\Phi}^* \psi$ along the 2-tori over $\gamma_p$ and over $\gamma_p'$ cancel. This can be seen by direct calculation as in the above lemmas, but there is a much simpler geometric argument as follows.

**Lemma 4.19.** The sum of the integrations of $\tilde{f}_{i,n+1} \Phi P_{\Phi}^* \psi$ along the 2-tori over $\gamma_p$ and over $\gamma_p'$ is zero.

**Proof.** Recall that the function $f_{i,n+1}$ belongs to $\mathbb{C}[X, Y, Z^{\pm 1}, t, \frac{1}{F_{i,0}}]/(XY - t, t^{n+1})$. By regarding it as an element in $\mathbb{C}[X, Y, Z^{\pm 1}, t, \frac{1}{F_{i,0}}]/(XY - t)$, the 2-form $f_{i,n+1} \Phi P_{\Phi}^* \psi$ gives a meromorphic 2-form on $Y_{\nu,\psi}^\nu$. In the space $Y_{\nu,\psi}^\nu$, the 2-tori $\pi_\delta^{-1}(\gamma_p)$ and $(\pi_\delta')^{-1}(\gamma_p')$ are homologous, but their orientations are opposite, because the orientations of the contours $\gamma_p$ and $\gamma_p'$ are opposite. Since $f_{i,n+1} \Phi P_{\Phi}^* \psi$ is a closed 2-form and we can take an oriented 3-manifold $F$ in $Y_{\nu,\psi}^\nu$ with $\partial F = \pi_\delta^{-1}(\gamma_p) \cup (\pi_\delta')^{-1}(\gamma_p')$ such that $f_{i,n+1} \Phi P_{\Phi}^* \psi$ does not diverge on $F$, the claim follows from the Stokes’ theorem. \qed
Combining Lemmas 4.18 and 4.19 with Corollary 4.16 and also with Proposition 4.10, we finally have the following.

**Theorem 4.20.** Any pre-log curve constructed in Theorem 3.8 can be deformed to a generic fiber of $\mathcal{X}$.

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5. **Multiplicity of enumeration: Proof of Theorem 1.2**

In this section, we study the enumeration of parametrized algebraic curves giving rise to a fixed realizable parametrized tropical curve, and we give a proof of Theorem 1.2.

Let $h: \Gamma \to S$ be a 3-valent realizable parametrized tropical curve of genus $g$ passing through $g$ rational points $p_1, \ldots, p_g \in S$. By Theorem 4.20, this is equivalent to assuming that the condition of Theorem 3.8 is satisfied. Assume that the preimages $h^{-1}(p_i)$ lie in the interior of the edges of $\Gamma$, and these constraints make the tropical curve rigid. Let $\Gamma'$ denote the graph $\Gamma$ subdivided by $h^{-1}(p_i)$, and let $v_i$ be the vertex of $\Gamma'$ with $h(v_i) = p_i$. Let $\mathcal{D}$ be the polyhedral subdivision of $S$ in Section 3. Up to refining the subdivision $\mathcal{D}$, we can assume that each $h(v_i)$ is a 0-cell of $\mathcal{D}$. The subdivision $\mathcal{D}$ induces a subdivision of $\Gamma$ which we denote by $\tilde{\Gamma}$.

We also introduce incidence conditions on $\mathcal{X}$ as in [24, Section 3]. Namely, let $x_1, \ldots, x_g$ be general points on $\tilde{\mathcal{X}}$. Recall that $h: \Gamma \to S$ is the quotient of a $\Lambda$-periodic tropical curve $\tilde{h}: \tilde{\Gamma} \to \mathbb{R}$. Choose inverse images $\tilde{p}_1, \ldots, \tilde{p}_g \in \mathbb{R}$ of $p_1, \ldots, p_g \in S$. Each ray in $\mathbb{R} \times \mathbb{R}$ spanned by $(\tilde{p}_i, 1)$ gives a 1-parameter subgroup acting on $\tilde{\mathcal{X}}$. The intersection between the closure of the orbit of $x_i$ by this group and the central fiber $\tilde{X}_0$ of $\tilde{\mathcal{X}}$ is a point in the interior (that is, complement of the toric divisors) of the component $\tilde{X}_{\tilde{p}_i}$ (see [24, ]). The images of these orbit closures in $\mathcal{X}$ give incidence conditions on $\mathcal{X}$. We denote these orbit closures again by $x_1, \ldots, x_g$ to save letters. These are sections of $\mathcal{X} \to B$.

We fix an orientation for every edge of $\Gamma$. We obtain induced orientations for the edges of the subdivisions. For each edge $e$ of $\Gamma$, or any of its subdivisions, let $\partial^- e$ and $\partial^+ e$ denote the two endpoints according to the fixed orientation, and let $N_e \subset N$ denote the sublattice generated by the weight vector of $e$. We introduce the following maps of lattices.

$$F: \text{Map}(V(\Gamma), N) \longrightarrow \bigoplus_{e \in E(\Gamma)} (N/N_e)$$

$$\phi \longmapsto \left( \phi(\partial^+ e) - \phi(\partial^- e) \right)_e,$$
\[ G: \text{Map}(V(\Gamma'), N) \longrightarrow \bigoplus_{e \in E(\Gamma')} (N/N_e) \oplus \bigoplus_{i=1}^{g} N \]
\[ \phi \longmapsto \left( \left( \phi(\partial^+ e) - \phi(\partial^- e) \right)_e, \left( \phi(v_i) \right)_i \right), \]
\[ \tilde{F}: \text{Map}(V(\tilde{\Gamma}), N) \longrightarrow \bigoplus_{e \in E(\tilde{\Gamma})} (N/N_e) \]
\[ \phi \longmapsto \left( \left( \phi(\partial^+ e) - \phi(\partial^- e) \right)_e \right), \]
\[ \tilde{G}: \text{Map}(V(\tilde{\Gamma}), N) \longrightarrow \bigoplus_{e \in E(\tilde{\Gamma})} (N/N_e) \oplus \bigoplus_{i=1}^{g} N \]
\[ \phi \longmapsto \left( \left( \phi(\partial^+ e) - \phi(\partial^- e) \right)_e, \left( \phi(v_i) \right)_i \right). \]

For any abelian group \( A \), we define
\[ F_A := F \otimes 1_A: \text{Map}(V(\Gamma), N) \otimes A \longrightarrow \bigoplus_{e \in E(\Gamma)} (N/N_e) \otimes A. \]
We define \( G_A, \tilde{F}_A \) and \( \tilde{G}_A \) similarly.

**Lemma 5.1.** We have
\[ \text{rk Ker } F = g - 1 + \text{rk Coker } F, \]
where \( \text{rk} \) denotes the rank of abelian group.

**Proof.** We have
\[ \text{rk Ker } F = 2|V(\Gamma)| - |E(\Gamma)| + \text{rk Coker } F, \]
(5.2) Since \( \Gamma \) is 3-valent, we have
\[ 3|V(\Gamma)| = 2|E(\Gamma)|. \]
By computing the Euler characteristic of \( \Gamma \), we have
\[ |V(\Gamma)| - |E(\Gamma)| = 1 - g. \]
Taking the difference of the previous two equations gives
\[ 2|V(\Gamma)| - |E(\Gamma)| = g - 1. \]
Substituting it into Eq. (5.2), we obtain the result. \( \square \)
Proposition 5.3. Let $\varphi_0 : C_0 \to X_0$ be any pre-log curve associated with the tropical curve $h : \hat{\Gamma} \to S$, and we equip it with a log structure as in Section 4. Let $N = \varphi_0^* \Theta_{X_0}/\Theta_{C_0}$ be the logarithmic normal sheaf of $\varphi_0$. The derived global section $R\Gamma(N)$ is quasi-isomorphic to the complex

$$\bigoplus_{v \in V(\Gamma)} N_C \xrightarrow{F_C} \bigoplus_{e \in E(\Gamma)} (N/N_e)_C$$

concentrated in degrees 0 and 1. In particular, we have $H^0(C_0, N) \simeq \text{Ker } F_C$ and $H^1(C_0, N) \simeq \text{Coker } F_C$.

Proof. For every 3-valent vertex $v$ of $\hat{\Gamma}$, we have

$$\Theta_{C_0}|_{C_0,v} = T_{C_0,v} \otimes \mathcal{O}_{C_0,v}(-3) = \mathcal{O}_{C_0,v}(2) \otimes \mathcal{O}_{C_0,v}(-3) = \mathcal{O}_{C_0,v}(-1).$$

Since $C_0,v$ is rational, $R\Gamma(\mathcal{O}_{C_0,v}(-1)) \simeq 0$. So we have

$$R\Gamma(N|_{C_0,v}) \simeq R\Gamma(\varphi_0^* \Theta_{X_0}|_{C_0,v}) \simeq N_C.$$

For every 2-valent vertex $v$ of $\hat{\Gamma}$, we have

$$\Theta_{C_0}|_{C_0,v} = T_{C_0,v} \otimes \mathcal{O}_{C_0,v}(-2) = \mathcal{O}_{C_0,v}(2) \otimes \mathcal{O}_{C_0,v}(-2) = \mathcal{O}_{C_0,v}.$$

Moreover, the map $\Theta_{C_0}|_{C_0,v} \to \varphi_0^* \Theta_{X_0}|_{C_0,v} \simeq N_C \otimes \mathcal{O}_{C_0,v}$ is given by the weight vector of an edge $e$ connected to $v$. So we have

$$R\Gamma(N|_{C_0,v}) \simeq (N/N_e)_C.$$

Similarly, for every edge $e$ of $\hat{\Gamma}$, we have

$$R\Gamma(N|_{C_0,v}) \simeq (N/N_e)_C.$$

Therefore, it follows from the following exact triangle

$$R\Gamma(N) \longrightarrow \bigoplus_{v \in V(\hat{\Gamma})} R\Gamma(N|_{C_0,v}) \longrightarrow \bigoplus_{e \in E(\hat{\Gamma})} R\Gamma(N|_{C_0,v}) \xrightarrow{+1}$$

that $R\Gamma(N)$ is quasi-isomorphic to the complex

$$\bigoplus_{v \in V(\Gamma)} N_C \xrightarrow{F_C} \bigoplus_{e \in E(\Gamma)} (N/N_e)_C$$

concentrated in degrees 0 and 1. $\Box$

Corollary 5.4. We have $\text{rk Coker } F = 1$ and $\text{rk Ker } F = g$.

Proof. The first equality follows from Proposition 4.2, Serre duality and Proposition 5.3. The second equality follows from the first by Lemma 5.1. $\Box$

Remark 5.5. We can also obtain Corollary 5.4 purely combinatorially, for example by considering dual spaces, as in the proof of Proposition 4.2.
Remark 5.6. Note that the kernel $\text{Ker} F_R$ (resp. $\text{Ker} \tilde{F}_R$) describes infinitesimal deformations of the tropical curve $h: \Gamma \to S$ (resp. $h: \tilde{\Gamma} \to S$). Recall that the constraints $h(v_i) = p_i$ make the tropical curve rigid by our assumption. Hence the evaluation map

$$\text{Ker} F_R \rightarrow \bigoplus_{i=1}^g (N/N_{e_i})_R$$

$$\phi \mapsto \left( \left( \phi(\partial^e_{e_i}) \right)_i \right)$$

is injective, where $e_i$ denotes the edge of $\Gamma$ containing $v_i$. Then it follows from the dimension counting in Corollary 5.4 that the map above is in fact an isomorphism.

We deduce that the evaluation map

$$\text{Ker} \tilde{F}_R \rightarrow \bigoplus_{i=1}^g N_R$$

$$\phi \mapsto \left( \left( \phi(v_i) \right)_i \right)$$

is surjective.

Recall that we constructed incidence conditions $x_1, \ldots, x_g$, which are sections of $X \rightarrow B$ such that the limit $x_i(0) \in X_0$ lies in the smooth locus of $X_0$. Let $\text{PreLog}$ denote the set of pre-log curves associated with the tropical curve $h: \tilde{\Gamma} \to S$. Let $\text{PreLog}' \subset \text{PreLog}$ denote the subset consisting of pre-log curves $\varphi_0: C_0 \rightarrow X_0$ together with marked points $s_i \in C_{0,v_i}$ for $i = 1, \ldots, g$ such that $\varphi_0(s_i) = x_i(0)$.

Proposition 5.7. The set $\text{PreLog}'$ has a natural structure of a $(\text{Ker} G_{\mathbb{C}^*})$-torsor. In particular, there exists exactly $|\text{Ker} G_{\mathbb{C}^*}|$ isomorphism classes of pre-log curves $\varphi_0: C_0 \rightarrow X_0$ associated with the tropical curve $h: \tilde{\Gamma} \to S$ together with marked points $s_i \in C_{0,v_i}$ satisfying $\varphi_0(s_i) = x_i(0)$ for $i = 1, \ldots, g$.

Proof. Let us first prove that the set $\text{PreLog}'$ is nonempty.

For every vertex $v$ of $\tilde{\Gamma}$, we have $N_{\mathbb{C}^*}$ acting on the toric variety $X_{0,h(v)}$. If for each edge $e \in E(\tilde{\Gamma})$ with endpoints $v$ and $v'$, the $N_{\mathbb{C}^*}$-action on $X_{0,h(v)}$ and the $N_{\mathbb{C}^*}$-action on $X_{0,h(v')}$ coincide on the stratum $X_{0,h(v')} = X_{0,h(v)} \cap X_{0,h(v')}$, then from one pre-log curve associated with the tropical curve $h: \tilde{\Gamma} \to S$, we obtain other pre-log curves via the actions. In other words, $\text{Ker} \tilde{F}_{\mathbb{C}^*}$ acts naturally on the set $\text{PreLog}$.

We identify all the domain curves $C_0$ in the set $\text{PreLog}$. For $i = 1, \ldots, g$, we choose a closed point $s_i$ in the open stratum $C_{0,v_i}$ associated with $v_i$. Since $v_i$ is 2-valent, the open stratum $C_{0,v_i}^\circ$ is $\mathbb{P}^1_{\mathbb{C}}$ minus two points. Hence the choice of
\[ \Phi : H^0(C_0, \mathcal{N}) \longrightarrow \bigoplus_{i=1}^g \Theta_{X_0}((\varphi_0(s_i)))/ f_{0*}(\Theta_{C_0}(s_i)) \]

given by evaluation of the sections of \( \mathcal{N} \) at the marked points \( s_i \) is an isomorphism.
Proof. The range of $\Psi$ is isomorphic to $\bigoplus_{i=1}^{g} (N/N_{e_{i}})_{\mathbb{C}}$, where $e_{i}$ denotes the edge of $\Gamma$ containing $v_{i}$. By Proposition 5.3, the domain of $\Psi$ is isomorphic to $\text{Ker } F_{\mathbb{C}}$. So we can identify the map $\Psi$ with

$$\text{Ker } F_{\mathbb{C}} \rightarrow \bigoplus_{i=1}^{g} (N/N_{e_{i}})_{\mathbb{C}}$$

$$\phi \mapsto \left(\left(\phi(\overline{e_{i}})\right)_{i}\right).$$

Therefore, $\Psi$ is an isomorphism by Remark 5.6.

For every $n \in \mathbb{Z}_{\geq 0}$, let $k_{n}$ denote the ring $\mathbb{C}[t]/(t^{n+1})$.

**Proposition 5.9.** Let $\varphi_{0}: C_{0} \rightarrow X_{0}$ be a pre-log curve associated with the tropical curve $h: \hat{\Gamma} \rightarrow S$, equipped with a log structure. Assume we have $s_{i} \in C_{0,v_{i}}$ for $i = 1, \ldots, g$ such that $\varphi_{0}(s_{i}) = x_{i}(0)$. Then for every $n \in \mathbb{Z}_{\geq 0}$, there exists a unique pointed curve $\left(C_{n}, \left(s_{1}^{n}, \ldots, s_{g}^{n}\right)\right)$ over $k_{n}$ extending $\left(C_{0}, (s_{1}, \ldots, s_{g})\right)$, and a $k_{n}$-morphism $\varphi_{n}: C_{n} \rightarrow X_{n}$ extending $\varphi_{0}: C_{0} \rightarrow X_{0}$, such that $\varphi_{n} \circ s_{i}^{n} = x_{i}$ for $i = 1, \ldots, g$.

**Proof.** The proposition holds for $n = 0$ by assumption. Now assuming the claim holds for $n$, let us prove it for $n + 1$. By Theorem 4.20, there exists a $k_{n+1}$-curve $C_{n+1}$ extending $C_{n}$ and a $k_{n+1}$-morphism $f_{n+1}: C_{n+1} \rightarrow X_{n+1}$ extending $\varphi_{n}: C_{n} \rightarrow X_{n}$. By deformation theory, once one extension exists, the set of all extensions is a torsor over $H^{0}(C_{n}, N)$. Note that the deformations of a marked point $s_{i}$ from order $n$ to $n + 1$ is a torsor over $\Theta C_{0}(s_{i}) \simeq \mathbb{C}$. Hence the set of all extensions $f_{n+1}: C_{n+1} \rightarrow X_{n+1}$ together with the marked points is a torsor over $H^{0}(C_{n}, N) \oplus \bigoplus_{i=1}^{g} \Theta C_{0}(s_{i})$. Consider the evaluation map

$$\Psi \oplus f_{0*}: H^{0}(C_{0}, N) \oplus \bigoplus_{i=1}^{g} \Theta C_{0}(s_{i}) \rightarrow \bigoplus_{i=1}^{g} \left(\Theta X_{0}(\varphi_{0}(s_{i}))/f_{0*}(\Theta C_{0}(s_{i})) \oplus f_{0*}(\Theta C_{0}(s_{i}))\right).$$

It is an isomorphism by Lemma 5.8. Hence there exists a unique extension $f_{n+1}: C_{n+1} \rightarrow X_{n+1}$ together with marked points $s_{i}^{n+1}$ such that $f_{n+1} \circ s_{i}^{n+1} = x_{i}$ for $i = 1, \ldots, g$, completing the proof. \qed

To conclude, Proposition 5.7 shows that there are exactly $|\text{Ker } G_{\mathbb{C}}|$ isomorphism classes of pre-log curves $\varphi_{0}: C_{0} \rightarrow X_{0}$ associated with the tropical curve $h: \hat{\Gamma} \rightarrow S$ together with marked points $s_{i} \in C_{0,v_{i}}$ satisfying $\varphi_{0}(s_{i}) = x_{i}(0)$ for $i = 1, \ldots, g$. Then for each such pre-log curve $\varphi_{0}: C_{0} \rightarrow X_{0}$, by [24, Proposition 7.1] (see also [9, Proposition 4.23]), the number of ways to equip it with a log structure is equal to $\prod_{e \in E(\Gamma)} w_{e}$, the product of all the edge weights. Once the log structure is fixed, by Proposition 5.9, there exists a unique deformation to the generic fiber satisfying the constraints $x_{i}$. Therefore, the total number of parametrized algebraic curves in
$X$ associated with the realizable tropical curve $h: \Gamma \to S$ satisfying the constraints $x_i$ is equal to the product

$$|\text{Ker} \ G_{\mathbb{C}^*}| \cdot \prod_{e \in E(\Gamma)} w_e.$$ 

So we have achieved the proof of Theorem 1.2.

REFERENCES

[1] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths. Geometry of algebraic curves. Volume II, volume 268 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[2] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. Compact complex surfaces, volume 4 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 2004.

[3] Tristram Bogart and Eric Katz. Obstructions to lifting tropical curves in surfaces via intersection theory. arXiv preprint arXiv:1110.0533, 2011.

[4] Erwan Brugallé and Kristin M Shaw. Obstructions to approximating tropical curves in surfaces via intersection theory. arXiv preprint arXiv:1409.0533, 2014.

[5] Jim Bryan and Naichung Conan Leung. Generating functions for the number of curves on abelian surfaces. Duke Math. J., 99(2):311–328, 1999.

[6] Man-Wai Cheung, Lorenzo Fantini, Jennifer Park, and Martin Ulirsch. Faithful realizability of tropical curves. arXiv preprint arXiv:1410.4152, 2014.

[7] Andreas Gathmann. Tropical algebraic geometry. Jahresber. Deutsch. Math.-Verein., 108(1):3–32, 2006.

[8] Andreas Gathmann, Kirsten Schmitz, and Anna Lena Winstel. The realizability of curves in a tropical plane. arXiv preprint arXiv:1307.5686, 2013.

[9] Mark Gross. Tropical geometry and mirror symmetry, volume 114 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2011.

[10] Lars Halvard Halle and Simon C. F. Rose. Tropical count of curves on abelian varieties. Commun. Number Theory Phys., 11(1):219–248, 2017.

[11] Ilia Itenberg, Grigory Mikhalkin, and Eugenii Shustin. Tropical algebraic geometry, volume 35 of Oberwolfach Seminars. Birkhäuser Verlag, Basel, second edition, 2009.

[12] Fumiharu Kato. Log smooth deformation theory. Tohoku Math. J. (2), 48(3):317–354, 1996.

[13] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[14] Eric Katz. Lifting tropical curves in space and linear systems on graphs. Adv. Math., 230(3):853–875, 2012.

[15] Eric Katz and Sam Payne. Realization spaces for tropical fans. In Combinatorial aspects of commutative algebra and algebraic geometry, volume 6 of Abel Symp., pages 73–88. Springer, Berlin, 2011.
[16] Kunihiko Kodaira and Donald Spencer. A theorem of completeness of characteristic systems of complete continuous systems. *American Journal of Mathematics*, 81:477–500, 1959.

[17] Grigory Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *J. Amer. Math. Soc.*, 18(2):313–377, 2005.

[18] Grigory Mikhalkin. Tropical geometry and its applications. In *International Congress of Mathematicians. Vol. II*, pages 827–852. Eur. Math. Soc., Zürich, 2006.

[19] David Mumford. An analytic construction of degenerating abelian varieties over complete rings. *Compositio Math.*, 24:239–272, 1972.

[20] Takeo Nishinou. Correspondence theorems for tropical curves. *arXiv preprint arXiv:0912.5090*, 2009.

[21] Takeo Nishinou. Graphs and obstruction theory for algebraic curves. *arXiv preprint arXiv:1503.06435*, 2015.

[22] Takeo Nishinou. Obstructions to deforming maps from curves to surfaces. *arXiv preprint arXiv:1901.11239*, 2018.

[23] Takeo Nishinou. Deformations of hypersurfaces on families of varieties. *preprint*, 2020.

[24] Takeo Nishinou and Bernd Siebert. Toric degenerations of toric varieties and tropical curves. *Duke Math. J.*, 135(1):1–51, 2006.

[25] E. Shustin. A tropical approach to enumerative geometry. *Algebra i Analiz*, 17(2):170–214, 2005.

[26] David E Speyer. Uniformizing tropical curves I: genus zero and one. *arXiv preprint arXiv:0711.2677*, 2007.

[27] Ilya Tyomkin. Tropical geometry and correspondence theorems via toric stacks. *Math. Ann.*, 353(3):945–995, 2012.

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