Markov chains in random environment with applications in queueing theory and machine learning

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Abstract

We prove the existence of limiting distributions for a large class of Markov chains on a general state space in a random environment. We assume suitable versions of the standard drift and minorization conditions. In particular, the system dynamics should be contractive on the average with respect to the Lyapunov function and large enough small sets should exist with large enough minorization constants. We also establish that a law of large numbers holds for bounded functionals of the process. Applications to queueing systems and to machine learning algorithms are presented.

1 Introduction

Markov chains in stationary random environments (MCREs) with a general (not necessarily countable) state space feature in several branches of applied probability. Rough volatility models of mathematical finance (see [5, 6]), queueing models with non-i.i.d. service and interarrival times (see [3] and Section 3 below) and sequential Monte Carlo methods (see Section 4 below) are prominent examples. It seems that existing studies on the ergodic theory of MCREs (such as [9, 10, 16, 17]) impose conditions that exclude the treatment of relevant models from the above list of applications.

The article [7], introducing new tools, managed to establish the existence of limiting laws and ergodic theorems for certain classes of MCREs which satisfy suitable versions of the standard drift and minorization conditions of Markov chain theory (as presented e.g. in [13]).

Assumption 2.2 of [7], however, severely restricted the scope of applications by requiring that the system dynamics is contractive whatever the state of the random environment is. The present study aims to remove this restriction: we require only that process dynamics is contractive on the average, in the sense of Assumption 2.3 below.

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In Section 2 our main results are stated in an abstract framework. Two applications are worked out in detail in Sections 3 and 4. In Section 3 we study a queuing model, where service times are not i.i.d. In Section 4 we treat the stochastic gradient Langevin dynamics with stationary data, a sampling algorithm with important applications in machine learning, see [18][11]. Proofs are presented in Section 5.

Notations and conventions. Let \( \mathbb{R}_+ := \{ x \in \mathbb{R} : x \geq 0 \} \). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. We denote by \( \mathbb{E}[X] \) the expectation of a random variable \( X \). For \( 1 \leq p < \infty \), \( L^p \) is used to denote the usual space of \( p \)-integrable real-valued random variables and \( \|X\|_p \) stands for the \( L^p \)-norm of a random variable \( X \).

We fix a standard Borel space \( (\mathcal{X}, \mathcal{B}) \). The set of probability Borel measures on \( (\mathcal{X}, \mathcal{B}) \) are denoted by \( \mathcal{M}_1 \). The total variation metric on \( \mathcal{M}_1 \) is defined by

\[
d_{TV}(\mu_1, \mu_2) = \|\mu_1 - \mu_2\| \quad (\mu_1, \mu_2 \in \mathcal{M}_1),
\]

where \( |\mu_1 - \mu_2| \) denotes the total variation of the signed measure \( \mu_1 - \mu_2 \).

For \( \mu_1, \mu_2 \in \mathcal{M}_1 \), let \( \mathcal{C}(\mu_1, \mu_2) \) denotes the set of probability measures on \( \mathcal{B} \otimes \mathcal{B} \) such that its respective marginals are \( \mu_1 \) and \( \mu_2 \). Then, \( d_{TV}(\mu_1, \mu_2) \) can be expressed as the twice of the optimal transportation cost, between \( \mu_1 \) and \( \mu_2 \), that is

\[
\frac{1}{2} d_{TV}(\mu_1, \mu_2) = \inf_{\nu \in \mathcal{C}(\mu_1, \mu_2)} \int_{\mathcal{X} \times \mathcal{X}} \mathbb{1}_{x \neq y} (dx, dy). \tag{1}
\]

In the sequel, we employ the convention that \( \sum_{k=0}^l = 0 \) and \( \prod_{k=0}^l = 1 \) whenever \( k, l \in \mathbb{Z}, k > l \).

## 2 Main results

Let \( (\mathcal{Y}, \mathcal{A}) \) be a measurable space and \( Y : \mathbb{Z} \times \Omega \to \mathcal{Y} \) a strongly stationary \( \mathcal{A} \)-valued stochastic process which we interpret as the environment which influences the evolution of our main process of interest \( X \) below. We consider a parametric family of stochastic kernels, that is a map \( Q : \mathcal{Y} \times \mathcal{X} \times \mathcal{B} \to [0, 1] \), where for all \( B \in \mathcal{B} \) the function \( (y, x) \mapsto Q(y, x, B) \) is \( \mathcal{A} \otimes \mathcal{B} \)-measurable and for all \( (y, x) \in \mathcal{Y} \times \mathcal{X} \), \( B \mapsto Q(y, x, B) \) is a Borel probability measure on \( \mathcal{B} \).

We assume that we are given the \( \mathcal{X} \)-valued process \( X_t, t \in \mathbb{N} \) such that \( X_0 = x_0 \in \mathcal{X} \) is fixed and

\[
\mathbb{P}(X_{t+1} \in B \mid \mathcal{F}_t) = Q(Y_t, X_t, B) \quad \mathbb{P}-a.s., \quad t \in \mathbb{N},
\]

where the filtration is

\[
\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t; Y_s, s \leq t), \quad t \in \mathbb{N}.
\]

Let \( \mu_t \in \mathcal{M}_1 \) denote the law of \( X_t \) for \( t \in \mathbb{N} \).

We aim to study the ergodic properties of \( X_t \) and the convergence of \( \mu_t \) to a limiting law as \( t \to \infty \) under various assumptions.

**Definition 2.1.** Let \( P : \mathcal{X} \times \mathcal{B} \to [0, 1] \) be a probabilistic kernel. For a bounded measurable function \( \phi : \mathcal{X} \to \mathbb{R} \), we define

\[
[P\phi](x) = \int_{\mathcal{X}} \phi(z)P(x, dz), \quad x \in \mathcal{X}.
\]

This definition makes sense for any non-negative measurable \( \phi \), too.
Theorem 2.7. Under Assumptions 2.2, 2.3, 2.4 and 2.6, there exists a probability law \( \mu \) such that, for all \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \),

\[
[Q(y)V](x) \leq \gamma(y)V(x) + K(y).
\]

Furthermore, we may and will assume that \( K(.) \geq 1 \).

Assumption 2.2. (Drift condition) Let \( V : \mathcal{X} \to \mathbb{R}_+ \) be a measurable function. We assume that there are measurable functions \( K, \gamma : \mathcal{Y} \to (0, \infty) \) such that, for all \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \),

\[
\inf_{x \in \gamma^{-1}([0, R(y)])} Q(y, x, A) \geq (1 - \alpha(y)) \kappa(y, A), \quad \text{where} \quad R(y) = \frac{2K(y)}{\epsilon \gamma(y)}
\]

and \( V^{-1}([0, R(y)]) \neq \emptyset \).

Remark 2.5. If there is \( x \in \mathcal{X} \) with \( V(x) = 0 \) then \( V^{-1}([0, R(y)]) \neq \emptyset \) automatically holds.

The following easily verifiable condition controls the tail distribution of \( \alpha(Y_0) \) which will play a very important role in our convergence estimates.

Assumption 2.3. (Long-time contractivity condition) We assume that

\[
\bar{\gamma} := \limsup_{n \to \infty} \mathbb{E}^{1/n} \left( K(Y_0) \prod_{k=1}^{n} \gamma(Y_k) \right) < 1.
\]

The next assumption stipulates the existence of suitable “small sets”. It corresponds to Assumption 2.5 in [7] but we need a different formulation here.

Assumption 2.4. (Minorization condition) Let \( \lambda(\cdot), K(\cdot) \) be as in Assumption 2.2. We assume that for some \( 0 < \theta < 1^{1/2} - 1 \), there is a measurable function \( \alpha : \mathcal{Y} \to (0, 1) \) and a probability kernel \( \kappa : \mathcal{Y} \times \mathbb{B} \to [0, 1] \) such that, for all \( y \in \mathcal{Y} \) and \( A \in \mathbb{B} \),

\[
\inf_{x \in \gamma^{-1}([0, R(y)])} Q(y, x, A) \geq (1 - \alpha(y)) \kappa(y, A), \quad \text{where} \quad R(y) = \frac{2K(y)}{\epsilon \gamma(y)}
\]

and \( V^{-1}([0, R(y)]) \neq \emptyset \).

Remark 2.5. If there is \( x \in \mathcal{X} \) with \( V(x) = 0 \) then \( V^{-1}([0, R(y)]) \neq \emptyset \) automatically holds.

The following easily verifiable condition controls the tail distribution of \( \alpha(Y_0) \) which will play a very important role in our convergence estimates.

Assumption 2.6. (Thin tail condition) Let us assume that, exists \( 0 < \theta < 1 \) such that

\[
\lim_{n \to \infty} \mathbb{E}^{1/n} \left( \alpha(Y_0)^n \right) = 0.
\]

Now come the main results of the present paper: with the above presented assumptions, the law of \( X_t \) converges to a limiting law as \( t \to \infty \), moreover, bounded functionals of \( X_t \) admit ergodic behavior provided that \( Y_t \) is ergodic.

Theorem 2.7. Under Assumptions 2.2, 2.3, 2.4 and 2.6 there exists a probability law \( \mu_* \), such that \( \mu_N \to \mu_* \) in total variation as \( N \to \infty \).

More precisely, for any \( 1/2 < \lambda < 1 \), there exist \( c(\lambda), \nu(\lambda) > 0 \) such that

\[
\nu_{\YY}(\mu_N, \mu_\ast) \leq 2 \sum_{n=\mathbb{N}}^{\infty} \mathbb{E} \left( \max_{0 \leq k \leq [n^{1/3}]} \alpha(Y_k)^{[n^{1/3}]-1} \right) \gamma(\lambda^{-1})^{[n^{1/3}]^2-n^{1/3}+cn^{2/3}e^{-\nu_{\YY}}}
\]

holds for all \( N \in \mathbb{N} \).
Remark 2.8. To help deciphering the expression (??), we remark that setting \( \lambda = \frac{3}{4} \), (??) is easily seen to be dominated by
\[
2 \sum_{n=N}^{\infty} \left[ \mathbb{E} \left( \max_{0 \leq k \leq n} \alpha(Y_k)^{n^{1/2}} \right) + n^{2/3} e^{-n^{1/3}} \right]
\]
for \( N \geq 216 \), with suitable constants \( c, \nu > 0 \). In the particular case where \( \alpha(\cdot) \) is constant, the latter expression is of the order \( \sum_{n=N}^{\infty} n^{2/3} e^{-n^{1/3}} \).

Theorem 2.9. Let Assumptions 2.2, 2.3, 2.4 and 2.6 be in force. If \( Y \) is ergodic, then for any bounded and measurable \( \Phi: \mathcal{X} \to \mathbb{R} \)
\[
\frac{\Phi(X_1) + \ldots + \Phi(X_N)}{N} \to \int_{\mathcal{X}} \Phi(z) \mu_*(dz), \quad N \to \infty
\]
holds in \( L^p \), \( 1 \leq p < \infty \).

Remark 2.10. Since \( \Phi \) is bounded, convergence in (8) takes place in probability iff it happens in \( L^p \) for all \( 1 \leq p < \infty \). We preferred the current formulation of Theorem 2.9 since we obtain \( L^p \) rates during the proofs, see also Remark 5.13. These rates, however, have too complicated expressions to be stated here.

3 A queuing model

We consider a single-server queuing model where customers are numbered by \( n \in \mathbb{N}^+ \). The time between the arrival of customers \( n + 1 \) and \( n \) is described by the random variable \( \epsilon_{n+1} \), for each \( n \in \mathbb{N} \). The service time for customer \( n \) is given by the random variable \( Y_n \), for \( n \in \mathbb{Z} \).

The waiting time \( W_n \) of customer \( n \) satisfies the Lindley recursion
\[
W_{n+1} = W_n + Y_n - \epsilon_{n+1}, \quad n \in \mathbb{N}, \tag{3}
\]
with \( W_0 := 0 \) (since customer 0 does not need to wait at all). When \( (Y_n)_{n \in \mathbb{Z}} \) and \( (\epsilon_n)_{n \in \mathbb{N}^+} \) are i.i.d. sequences independent of each other then \( W_n \) is a (general state space) Markov chain whose ergodic properties have been extensively studied. Here we are interested in a more general setting where the process \( (Y_n)_{n \in \mathbb{Z}} \) is assumed merely stationary.

The following condition is standard: in a stable system service times should be shorter on the average than inter-arrival times.

Assumption 3.1. The sequence of \( \mathbb{R}_+ \)-valued inter-arrival times \( \epsilon_n \), \( n \geq 1 \) is i.i.d. and we have
\[
\mathbb{E}[Y_0] < \mathbb{E}[\epsilon_1].
\]

Assumption 3.2. For some \( M > 0 \), the sequence of service times is included in a strict sense \( [0, M] \)-valued stationary process \( Y_n, n \in \mathbb{Z} \) which is independent of \( (\epsilon_n)_{n \geq 1} \). There is \( \eta > 0 \) such that the limit
\[
\Gamma(\alpha) := \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{E} e^{\alpha Y_n + \ldots + Y_0}
\]
exists for all \( \alpha \in (-\eta, \eta) \) and \( \Gamma \) is differentiable on \( (-\eta, \eta) \).
Remark 3.3. The above assumption is clearly inspired by the Gärtner-Ellis theorem hence sufficient conditions for its fulfillment can be deduced from the literature about large deviation principles. For instance, if \( Y_n = \phi(Z_n) \) for some bounded measurable \( \phi : \mathbb{R}^m \to \mathbb{R} \) and an \( \mathbb{R}^m \)-valued geometrically ergodic Markov chain \( Z_n, n \in \mathbb{Z} \) started from its invariant distribution then (4) holds true for some \( \eta > 0 \), see Theorem 4.1 of [11] for a precise formulation. Thus Theorem 3.7 is applicable to a large class of models.

We also mention a non-Markovian example: let \( Y_t = \sum_{i=0}^{\infty} a_i \zeta_i \), where \( \zeta_i \), \( i \in \mathbb{Z} \) are independent and identically distributed \( \mathbb{R}_+ \)-valued random variables, \( \sum_{i=-\infty}^{\infty} |a_i| < \infty \). Assumption (3.2) is satisfied for this process by Theorem 2.1 of [4].

Under suitable conditions, it is possible to relax the boundedness condition on the process \( Y \) in Assumption 3.2. Due to the tedious technicalities this is not pursued here.

Now, we turn to the verification of Assumptions 2.2 and 2.3 under the previous two conditions.

Lemma 3.4. Let Assumptions 3.1 and 3.2 be in force. Then there exists \( \bar{\alpha} > 0 \) such that for

\[
\begin{align*}
V(w) &:= e^{\bar{\alpha}w} - 1, \ w \geq 0 \\
\gamma(y) &:= \mathbb{E}[e^{\bar{\alpha}(y-\epsilon_j)}], \ y \geq 0 \\
K &:= 1 + e^{\bar{\alpha}M},
\end{align*}
\]

holds for all \( y \in [0, M], \ w \in \mathbb{R}_+ \), where \( Q \) is defined as

\[
Q(y, w, A) := \mathbb{P}\left[(w + y - \epsilon_j)_+ \in A, \ y \in [0, M], \ w \in \mathbb{R}_+, A \in \mathcal{B}([0, \infty))\right].
\]

Furthermore,

\[
\bar{\gamma} := \limsup_{n \to \infty} \mathbb{E}^{1/n}\left(K \prod_{k=1}^{n} \gamma(Y_k)\right) < 1.
\]

Proof. Define \( \lambda(\alpha) := \Gamma(\alpha) + \ln(\mathbb{E}[e^{-\alpha \eta}]) \). The functions

\[
\lambda_n(\alpha) := \frac{1}{n} \ln \mathbb{E}\left[e^{\alpha \sum_{j=1}^{n}(Y_{j-1} - \epsilon_j)}\right], \ \alpha \in (-\eta, \eta), \ n \in \mathbb{N}^+
\]

are finite and differentiable. They are also clearly convex. Define

\[
\psi_n(\alpha) := \mathbb{E}\left[\frac{e^{\alpha \sum_{j=1}^{n}(Y_{j-1} - \epsilon_j)} - 1}{\alpha}, \ \alpha \in (0, \eta), \ n \in \mathbb{N}^+.
\]

By the Lagrange mean value theorem and measurable selection, there exists a random variable \( \xi_n(\alpha) \in [0, \alpha] \) such that

\[
\psi_n(\alpha) = \mathbb{E}\left[\left(\sum_{j=1}^{n}(Y_{j-1} - \epsilon_j)\right) e^{\xi_n(\alpha) \sum_{j=1}^{n}(Y_{j-1} - \epsilon_j)}\right].
\]
which completes the proof.

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also holds:

Lemma 3.6. 

Assumption 3.5 automatically holds.

Here

which is uniformly bounded in \( \alpha \in (0, \eta) \) (for \( n \) fixed). Hence reverse Fatou’s lemma shows that

\[
\limsup_{\alpha \to 0^+} \psi_n(\alpha) \leq \mathbb{E} \left[ \sum_{j=1}^{n} (Y_{j-1} - \epsilon_j) \right] = n \mathbb{E} [Y_0 - \epsilon_1].
\]

This implies that, for all \( n \geq 1 \), \( \lambda_n'(0) = \lim_{\alpha \to 0^+} \psi_n(\alpha) \leq \mathbb{E} [Y_0 - \epsilon_1]. \)

Since \( \lambda_n'(0) \to \lambda'(0) \) for \( \alpha \in (-\eta, \eta) \) by Assumption 3.2, it follows from Theorem 25.7 of [14] that also \( \lambda_n'(0) \to \lambda'(0) \) hence \( \lambda'(0) < 0 \) by Assumption 3.1. By Corollary 25.5.1 of [14], differentiability of \( \lambda \) implies its continuous differentiability, too. Hence from \( \lambda(0) = 0 \) and \( \lambda'(0) < 0 \) we obtain that there exists \( \bar{\alpha} > 0 \) satisfying

\[
\lim_{n \to \infty} \frac{1}{n} \ln \mathbb{E} [e^{\bar{\alpha}(Y_0 + \ldots + Y_n) \sum \epsilon_i + \ldots + \epsilon_n}] < 0.
\]

Now we define the Lyapunov function \( V(w) := e^{\bar{\alpha}w}, w \geq 0 \) and choose \( \gamma(y) := \mathbb{E} [e^{\bar{\alpha}(Y_1 + \ldots + Y_n)}], y \geq 0 \). Notice that

\[
[Q(y)V](w) = \mathbb{E} [V([w + y - \epsilon_1], y)] \leq \mathbb{E} [e^{\bar{\alpha}(w+y-\epsilon_1)}] + 1 \leq \gamma(y)e^{\bar{\alpha}w} + 1 \leq \gamma(y)V(w) + \gamma(M) + 1,
\]

so (5) holds with \( K \) as defined above. By (5), the long-time contractvity condition also holds:

\[
\limsup_{n \to \infty} \mathbb{E}^{1/n} [K \gamma(Y_1) \ldots \gamma(Y_n)] < 1,
\]

which completes the proof. \( \square \)

Now we present another assumption on the inter-arrival times which will be needed to show the minorization condition. Notice that, for unbounded \( \epsilon_1 \), Assumption 3.5 automatically holds.

**Assumption 3.5.** One has \( P(\epsilon_1 \geq \tau) > 0 \) for

\[
\tau := M + \frac{4(1 + e^{\bar{\alpha}M})}{(1 - \bar{\alpha}^2) \gamma(0)}.
\]

Now let us turn to the verification of the minorization condition under the assumption above.

**Lemma 3.6.** Let Assumption 3.5 be in force. Choose \( \epsilon := (1/\sqrt{\gamma')^2} - 1)/2 \). Then there is \( \alpha \in (0, 1) \) such that, for all \( y \in [0, M] \) and \( A \in \mathcal{B} (\mathbb{R}_+) \),

\[
\inf_{w \in [0, R(y)]} Q(y, w, A) \geq (1 - \alpha) \delta_\theta(A), \text{ where } R(y) = \frac{2K(y)}{\epsilon \gamma(y)}
\]

and \( \delta_\theta \) is the one-point mass concentrated on 0.
Proof. Note that $R(y) \leq R := \frac{2(1+e^{3\alpha})}{\gamma(0)}$.

$$Q(y, w, A) = \mathbb{P}\left([w + y - \epsilon_1]_+ \in A\right) \geq \mathbb{P}\left([w + y - \epsilon_1]_+ = 0\right) \delta_0(A) = (1 - \mathbb{P}(w + y - \epsilon_1 > 0)) \delta_0(A) \geq (1 - \mathbb{P}(R + M - \epsilon_1 > 0)) \delta_0(A)$$

so we may set $\alpha := \mathbb{P}(R + M - \epsilon_1 > 0) < 1$, see Assumption 3.5. 

Theorem 2.7 allows us to deduce that the queuing system in consideration converges to a stationary state and an ergodic theorem is valid. Theorem 3.7 below opens the door for the statistical analysis of such systems.

**Theorem 3.7.** Under Assumptions 3.1, 3.2 and 3.5 there exists a probability $\mu$ on $\mathcal{B}(\mathbb{R}_+)$ such that, for all $0 < \varrho < 1/3$

$$d_{TV}(\text{Law}(W_n), \mu) \leq c_1(\varrho) \sum_{j=n}^{\infty} e^{-c_2(\varrho)j^{1/3-\varrho}},$$

for some $c_1(\varrho), c_2(\varrho) > 0$. Furthermore, if $(Y_n)_{n \in \mathbb{N}}$ is ergodic, then for an arbitrary measurable and bounded $\Phi : \mathbb{R}_+ \to \mathbb{R}$,

$$\frac{\Phi(W_0) + \ldots + \Phi(W_{n-1})}{n} \to \int_{[0,\infty)} \Phi(z) \mu(dz), \quad (8)$$

in $L^p$ for all $1 \leq p < \infty$.

**Proof.** According to Lemma 3.4 and 3.6 conditions 2.2, 2.3, 2.4 and 2.6 are satisfied hence by Theorem 2.7, for any $1/2 < \lambda < 1$, exists $c(\lambda), \nu(\lambda) > 0$ and a probability $\mu$ on $\mathcal{B}([0, \infty))$ such that

$$d_{TV}(\text{Law}(W_n), \mu) \leq 2 \sum_{k=n}^{\infty} \left[ \alpha^{1/3} - 1 + \tilde{c}(\lambda - 1)k^{1/3} + c k^{2/3} e^{-\nu k^{1/3}} \right],$$

where $\alpha$ is as in Lemma 3.6.

Clearly, the $k$th term in the above sum is of the order $O(e^{-c_0 k^{1/3-\varrho}})$ for arbitrarily small $\varrho > 0$ and for some $c_0 = c_0(\varrho) > 0$, hence we obtain the claimed convergence rate. By Theorem 2.9 the second part of this theorem holds also true.

**Remark 3.8.** It is known that $\text{Law}(W_n)$ converges to a limiting distribution under rather mild conditions, see Example 14.1 on page 189 of [3]. Details of this approach seem to be available only in Russian, see [22]. However, as far as we know, Theorem 3.7 above is the first result providing a rate of convergence and a law of large numbers in this setting.

4 Stochastic gradient Langevin algorithm

We consider, for some $\lambda > 0$,

$$\theta_{n+1} = \theta_n - \lambda H(\theta_n, Y_n) + \sqrt{\lambda} \xi_{n+1},$$

where $\xi_{n+1}$ is a sequence of independent random variables with mean zero and variance one.
where \( \xi_n, n \geq 1 \) is an independent sequence of standard \( d \)-dimensional Gaussian random variables, \( Y_n, n \in \mathbb{Z} \) is a \( \mathbb{R}^m \)-valued strict sense stationary process and \( H : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d \) a measurable function. We assume that \( \theta_0, (Y_n)_{n \in \mathbb{Z}} \) and \( (\xi_n)_{n \in \mathbb{N}} \) are independent.

This algorithm is called “stochastic gradient Langevin dynamics” (SGLD). Suggested by [18], it has recently become widely used for sampling from high-dimensional probability distributions. More precisely, let \( U : \mathbb{R}^d \to \mathbb{R} \) be differentiable with derivative \( h = \nabla U \) such that \( h(\theta) = E[H(\theta, Y_0)] \). For \( \lambda \) small and \( n \) large, \( \text{Law}(\theta_n) \) is expected to be close to the probability defined by

\[
\pi(A) = \frac{\int_A e^{-U(\theta)} d\theta}{\int_{\mathbb{R}^d} e^{-U(\theta)} d\theta}, \quad A \in \mathcal{B}(\mathbb{R}^d),
\]

see e.g. [18 [1]]. The literature on SGLD is abundant but practically all studies assume that \( Y_n, n \in \mathbb{Z} \) are i.i.d. For the case where the step size \( \lambda_n \) is a decreasing, it has been shown in [19] that, under suitable assumptions, the averages

\[
D_n := \frac{\phi(\theta_1) + \ldots + \phi(\theta_n)}{n}
\]

converge almost surely to \( D := \int_{\mathbb{R}^d} \phi(z) \pi(dz) \). In the case of fixed \( \lambda \), [15] estimated the \( L^2 \) distance between \( D_n \) and \( D \).

In the present article we also keep \( \lambda \) fixed, but we establish a novel result: the SGLD recursion converges to a limiting law \( \mu(\lambda) \) (in total variation) and \( D_n \) tends to \( \int_{\mathbb{R}^d} \phi(z) \mu(\lambda)(dz) \) in \( L^p \), \( 1 \leq p < \infty \). As far as we know this ergodic property has not yet been pointed out, even in the case of i.i.d. \( Y_n, n \in \mathbb{Z} \). We can now prove it for a broad class of stationary processes \( Y_n, n \in \mathbb{Z} \). We think of \( Y_n \) as an observed data sequence. As these are rarely i.i.d. in practice, Theorem 4.6 below formulates strong theoretical support for the use of SGLD with possibly dependent data.

The following standard dissipativity condition is required, see e.g. [12].

**Assumption 4.1.** There is a measurable \( \Delta : \mathbb{R}^m \to \mathbb{R} \) and \( b \geq 0 \) such that, for all \( \theta \in \mathbb{R}^d \) and \( y \in \mathbb{R}^m \),

\[
(H(\theta, y), \theta) \geq \Delta(y)|\theta|^2 - b.
\]

Furthermore, \( E[\Delta(Y_0)] > 0 \). We may and will assume that \( \Delta \) is a bounded function.

**Assumption 4.2.** There is \( \eta > 0 \) such that the limit

\[
\Gamma(\alpha) := \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{E}e^{n(\Delta(Y_1) + \ldots + \Delta(Y_n))}
\]

exists for all \( \alpha \in (-\eta, \eta) \) and \( \Gamma \) is continuously differentiable on \((-\eta, \eta)\).

**Assumption 4.3.** There exist \( K_1, K_2, K_3 \) such that

\[
|H(\theta, y)| \leq K_1|\theta| + K_2|y| + K_3.
\]

Note that Assumption 4.3 holds, in particular, if \( H \) is Lipschitz-continuous.

**Assumption 4.4.** \( Y_0 \) is bounded, say, \( |Y_0| \leq M \) a.s.

**Remark 4.5.** Boundedness of \( Y_0 \) could be relaxed to assuming only \( E[e^{\beta|Y_0|^2}] < \infty \) for some \( \beta > 0 \). This relaxation leads to a weaker rate estimate through rather tedious technicalities hence we prefer not to treat it here.
It turns out that the law of $\theta_n$ tends to a limit as $n \to \infty$ and ergodic averages converge to the expectation under the limit law.

**Theorem 4.6.** Let $\lambda > 0$ be small enough. Under Assumptions 4.1, 4.2, 4.3 and 4.4 there exists a probability law $\mu$ such that, for all $0 < \varrho < 1/3$,

$$||\text{Law}(W_n) - \mu||_{TV} \leq c_1(\varrho) \sum_{j=m}^{\infty} e^{-c_2(\varrho)j^{2/3-q}},$$

for some $c_1(\varrho), c_2(\varrho) > 0$. Moreover, for arbitrary bounded measurable $\Phi : \mathbb{R}^d \to \mathbb{R}$,

$$\frac{\Phi(\theta_1) + \ldots + \Phi(\theta_n)}{n} \to \int_{\mathbb{R}^d} \Phi(x) \mu(dx),$$

as $n \to \infty$ in $L^p$ for all $p \geq 1$.

**Remark 4.7.** The convergence rates given by the above theorem are not sharp enough for practical purposes. However, Theorem 4.6 provides a universal ergodic property for the stochastic gradient Langevin dynamics, irrespective of dependencies in the data stream (as long as they satisfy Assumption 4.2). No result of this calibre has heretofore been available in the related literature.

**Proof of Theorem 4.6.** Choose $V(\theta) := |\theta|^2, \theta \in \mathbb{R}^d$ and define

$$Q(y, \theta, A) := \mathbb{P}(\theta - \lambda H(\theta, y) + \sqrt{\lambda} \xi_1 \in A), \quad y \in \mathbb{R}^m, \theta \in \mathbb{R}^d, A \in \mathcal{A}(\mathbb{R}^d).$$

Then we have

$$[Q(y)V](\theta) = \mathbb{E}[V(\theta - \lambda H(\theta, y) + \sqrt{\lambda} \xi_1)]$$

$$= \lambda \mathbb{E}[|\xi_1|^2 + \lambda^2|H(\theta, y)|^2 + |\theta|^2]$$

$$+ 2\sqrt{\lambda} \mathbb{E}[\theta - \lambda H(\theta, y), \xi_1] - 2\lambda \langle \theta, H(\theta, y) \rangle$$

$$\leq \lambda(d + 2b) + 3\lambda^2[K_1^2|\theta|^2 + K_2^2|y|^2 + K_3^2] + (1 - 2\lambda \Delta(y))|\theta|^2$$

so Assumption 2.2 holds with $K(y) := \lambda(d + 2b) + 3\lambda^2 K_2^2 + 3\lambda^2 K_2^2 |y|^2, \gamma(y) := 1 + 3\lambda^2 K_1^2 - 2\lambda \Delta(y)$. Note that, due to the boundedness of $\Delta$, $\gamma(y) \geq 0$ for all $y$ for $\lambda$ small enough, in fact $\gamma(y) \geq \bar{\gamma} > 0$ for some $\bar{\gamma}$. By Assumption 4.1 for $\lambda$ small enough, $\mathbb{E}[3\lambda^2 K_1^2 - 2\lambda \Delta(Y_0)] < 0$.

Arguments similar to those in the preceding section show that, when $\lambda$ is small enough,

$$\limsup_{n \to \infty} \frac{1}{n} \ln \mathbb{E}[e^{\lambda \sum_{i=1}^{\infty} \langle \lambda \xi_1^2 - 2\Delta(Y_i) \rangle}] < 0.$$ 

Noting $1 + x \leq e^x$ this implies

$$\bar{\gamma} := \limsup_{n \to \infty} E^{1/n}[K(Y_0)\gamma(Y_1)\ldots\gamma(Y_n)] \leq$$

$$\limsup_{n \to \infty} E^{1/n}[\gamma(Y_1)\ldots\gamma(Y_n)]^\lambda(d + 2b) + 3\lambda^2 K_2^2 + 3\lambda^2 K_2^2 M^2 =$$

$$\limsup_{n \to \infty} E^{1/n}[\gamma(Y_1)\ldots\gamma(Y_n)] < 1$$

hence Assumption 2.3 also holds.
Let \( 0 < \varepsilon < 1/\gamma^{1/2} - 1 \), \( R(y) := \frac{2K(y)}{\varepsilon \gamma(y)} \), define \( C(y) := \{ \theta \in \mathbb{R}^d : |\theta|^2 \leq R(y) \} \) and set
\[
\kappa(y, A) := \frac{\text{Leb}(C(y) \cap A)}{\text{Leb}(C(y))}, \quad A \in \mathcal{B}(\mathbb{R}^d).
\]

Denoting \( f(\theta) := \exp\{-|\theta|^2/2\}/(2\pi)^{d/2}, \theta \in \mathbb{R}^d \), for each \( y \in \mathbb{R}^m, |y| \leq M, \theta \in C(y) \) and \( A \in \mathcal{B}(\mathbb{R}^d) \)
\[
Q(y, \theta, A) = \mathbb{P}(\theta - \lambda H(\theta, y) + \sqrt{\lambda} \xi_1 \in A) \geq \mathbb{P}(\theta - \lambda H(\theta, y) + \sqrt{\lambda} \xi_1 \in C(y) \cap A)
\]
\[
\geq \int_{\mathbb{R}^d} \mathbb{1}_{\theta - \lambda H(\theta, y) + \sqrt{\lambda} \xi_1 \in A} f(w) \, dw
\]
\[
= \frac{1}{\lambda^{d/2}} \int_{C(y) \cap A} f\left( \frac{z - \theta + \lambda H(\theta, y)}{\sqrt{\lambda}} \right) \, dz
\]
\[
\geq \frac{\text{Leb}(C(y))}{(2\pi \lambda)^{d/2}} \exp\left( - \frac{\max_{z \in C(y)} |z - \theta + \lambda H(\theta, y)|^2}{2\lambda} \right) \kappa(y, A).
\]

Note that, for \(|y| \leq M\) and \( \theta, z \in C(y) \), we have
\[
\frac{1}{2\lambda} |z - \theta + \lambda H(\theta, y)|^2 \leq \frac{(2 + \lambda K_1)^2}{\lambda} \frac{2K(y)}{\varepsilon \gamma(y)} + \lambda(K_2 M + K_3)^2
\]
\[
\leq \frac{2(2 + \lambda K_1)^2}{\varepsilon \gamma} \left[ \frac{d + 2b + 3\lambda(K_3^2 + K_3^2 M^2)}{\varepsilon \gamma} \right] + \lambda(K_2 M + K_3)^2.
\]

Clearly, we can choose \( \lambda \) small enough such that
\[
\frac{1}{2\lambda} |z - \theta + \lambda H(\theta, y)|^2 \leq \frac{9(d + 2b)}{\varepsilon \gamma} + 1.
\]

According to our previous estimate for \( Q(y, \theta, A) \), for \( \lambda \) small enough, we have
\[
Q(y, \theta, A) \geq \frac{\text{Leb}(C(y))}{(2\pi \lambda)^{d/2}} \exp\left( - \frac{9(d + 2b)}{\varepsilon \gamma} - 1 \right) \kappa(y, A)
\]
\[
\geq \tilde{c} e^{-\varepsilon/\gamma} \kappa(y, A)
\]
for suitable \( \tilde{c}, \tilde{\varepsilon} > 0 \) depending on \( b, d, M \) and \( \sup_{y \in \mathbb{R}^m} |\Delta(y)| \).

Obviously, there exists \( 0 < \varepsilon < 1/\gamma^{1/2} - 1 \) such that \( \tilde{c} e^{-\varepsilon/\gamma} < 1 \) which proves that Assumption 2.3 and 2.6 holds with \( \alpha := 1 - \tilde{c} e^{-\varepsilon/\gamma} \). We thus get that the claimed convergence rate holds by Theorem 2.7.

## 5 Proofs

In this section, we gathered the proofs of Theorem 2.7 and 2.9. For \( R > 0 \), denote by \( c(R) \) the set of mappings from \( \mathcal{X} \) into \( \mathcal{X} \) whose restriction to \( V^{-1}([0, R]) \) is constant. Through this section, \( \varepsilon > 0 \) and \( R(y) \) will be as in Assumption 2.4.

### 5.1 Preliminary lemmas and notations

The following random mapping representation of \( Q \) will play a crucial role in the proofs. It generalizes the idea of Lemma 6.1 in [7].
Lemma 5.1. There exists a sequence of measurable functions \( T_t : \mathcal{Y} \times \mathcal{X} \times \Omega \to \mathcal{X} \), \( t \in \mathbb{Z} \) such that

\[
P(\{ \omega \in \Omega \mid T_t(y, x, \cdot) \in A \}) = Q(y, x, A)
\]

for all \( t \in \mathbb{Z} \), \( y \in \mathcal{Y} \), \( x \in \mathcal{X} \), \( A \in \mathcal{B} \) and there are events \( J_t(y) \in \mathcal{Y} \), for all \( t \in \mathbb{Z} \), \( y \in \mathcal{Y} \) such that

\[
J_t(y) \subset \{ \omega \in \Omega \mid T_t(y, \cdot, \omega) \in c(R(y)) \} \text{ and } P(J_t(y)) \geq 1 - a(y) \tag{9}
\]

Furthermore, the sigma-algebras \( \sigma(T_t(y, x, \cdot), x \in \mathcal{X}, y \in \mathcal{Y}) \), \( t \in \mathbb{Z} \) are independent.

Proof. We follow the proof of Lemma 6.1 in [7]. So, let \( U_n \) and \( \varepsilon_n \), \( n \in \mathbb{Z} \) be sequences of i.i.d. uniform random variables on \([0, 1]\) independent of each other. Without loss of generality, we may assume that \( \{U_t, \varepsilon_t : t \in \mathbb{Z}\} \) is independent of \( Y_t \), \( t \in \mathbb{Z} \). The case of countable \( \mathcal{X} \) is easy hence omitted. In the case of \( \mathcal{X} \) uncountable we can also assume (by the Borel isomorphism theorem) that \( \mathcal{X} = \mathbb{R} \) and \( \mathcal{B}(\mathbb{R}) \) is the standard Borel \( \sigma \)-algebra of \( \mathbb{R} \).

For \( y \in \mathcal{Y} \), \( x \in \mathcal{X} \) and \( A \in \mathcal{B}(\mathbb{R}) \), let

\[
q(y, x, A) := \frac{1}{a(y)}[Q(y, x, A) - (1 - a(y))\kappa(y, A)]I_{V(x) \leq R(y)} + Q(y, x, A)I_{V(x) > R(y)}
\]

and define

\[
T_t(y, x, \omega) = \kappa^{-1}(y, \varepsilon_t)I_{U_t \leq 1 - a(y)}I_{V(x) \leq R(y)} + q^{-1}(y, x, \varepsilon_t)(1 - I_{U_t \leq 1 - a(y)}I_{V(x) \leq R(y)}),
\]

where

\[
\begin{align*}
\kappa^{-1}(y, z) & := \inf\{r \in \mathbb{Q} \mid k(y, (-\infty, r]) \geq z\} \quad \text{and} \\
q^{-1}(y, x, z) & := \inf\{r \in \mathbb{Q} \mid q(y, x, (-\infty, r]) \geq z\},
\end{align*}
\]

\( z \in \mathbb{R} \) are the pseudoinverses of the corresponding cumulative distribution functions.

Obviously, \( x \mapsto T_t(y, x, \omega) \) is constant on \( V^{-1}([0, R(y)]) \) whenever \( U_t \leq 1 - a(y) \), this implies [9] with \( J_t(y) := \{ \omega \mid U_t(\omega) \leq 1 - a(y) \} \). Furthermore, for all \( r \in \mathbb{R} \), \( t \in \mathbb{Z} \) and for any fixed \( y \in \mathcal{Y} \) and \( x \in \mathcal{X} \)

\[
P(\{ \omega \in \Omega \mid T_t(y, x, \cdot) \leq r \}) = I_{V(x) \geq R(y)}P(q^{-1}(y, x, \varepsilon_t) \leq r)
\]

\[
+ I_{V(x) < R(y)}[a(y)P(q^{-1}(y, x, \varepsilon_t) \leq r) + (1 - a(y))P(\kappa^{-1}(y, \varepsilon_t) \leq r)].
\]

By the definition of the pseudoinverse, we can write

\[
P(\kappa^{-1}(y, \varepsilon_t) \leq r) = P(\kappa(y, (-\infty, r']) \geq \varepsilon_t, r' \in \mathbb{Q} \cap (r, \infty))
\]

\[
= P(\kappa(y, (-\infty, \infty]) \geq \varepsilon_t) = \kappa(y, (-\infty, \infty])
\]

and similarly

\[
P(q^{-1}(y, x, \varepsilon_t) \leq r) = q(y, x, (-\infty, r])
\]

hence

\[
P(\{ \omega \in \Omega \mid T_t(y, x, \cdot) \leq r \}) = Q(y, x, (-\infty, r])
\]

as we desired.

It remains only to show that \( T_t \) is measurable with respect to sigma algebras \( \mathcal{Y} \otimes \mathcal{B}(\mathbb{R}) \otimes \sigma(\{U_t, \varepsilon_t : t \in \mathbb{Z}\}) \) and \( \mathcal{B}(\mathbb{R}) \). Indeed, \( T_t \) is a composition of measurable functions. The claimed independence of the sigma-algebras clearly holds too.
We drop the dependence of the mappings \( T_t \) on \( \omega \) in the notation and will simply write \( T_t(y) x := T_t(x, y, \cdot) \). For \( s \in \mathbb{Z} \) and \( x \in \mathcal{X} \), define the family of auxiliary processes
\[
  Z_{s,t}^x = x, \quad Z_{s,t+1}^x = T_{t+1}(y_t) Z_{s,t}^x, \quad t \geq s, \tag{10}
\]
where \( y = (\ldots, y_{-1}, y_0, y_1, \ldots) \in \mathcal{Y}^\mathbb{Z} \) is a fixed trajectory. Let \( \mathcal{G}_t := \sigma(e_i, U_i, i \leq t) \) and \( \mathcal{G}_t^+ := \sigma(e_i, U_i, i > t), t \in \mathbb{Z} \). Clearly, \( \mathcal{G}_t \) is independent of \( \mathcal{G}_t^+ \) and \( Z_{s,t}^x \) is adapted to \( \mathcal{G}_t \) moreover the process \( Z_{s,t}^x, t \geq s \) heavily depends on the choice of \( y \).

In the sequel, \( S : \mathcal{Y}^\mathbb{Z} \to \mathcal{Y}^\mathbb{Z} \) stands for the usual left shift operation i.e.
\[
  (Sy)_j = y_{j+1}, \quad j \in \mathbb{Z} \tag{11}
\]
and \( n \geq 1 \) is an arbitrary natural number.

**Lemma 5.2.** For \( M, N \in \mathbb{N}^+ \) and \( \lambda \in (0, 1) \), we define the sets
\[
  B_{N,M}^{\lambda} := \left\{ y \in \mathcal{Y}^{\mathbb{Z}} \mid \prod_{i=1}^{[N/M]} \gamma (Y_{k\lfloor N/M \rfloor + i}) \leq \tilde{\gamma}^{[N/M]}, \quad k = 0, 1, \ldots, [N/1]^{M-1} - 1 \right\}.
\]
Then \( Y \) and its shifted copies fall into \( B_{N,M}^{\lambda} \) with large probability. More precisely, there exist \( c, \nu > 0 \) such that
\[
  \mathbb{P} \left( (S^k Y) \in B_{N,M}^{\lambda} \right) \geq 1 - cN^{1-1/M} e^{-\nu N^{1/M}}
\]
holds.

**Proof.** The second inequality easily follows from the first one. By the union bound and the strong stationarity of \( Y_t, t \in \mathbb{Z} \), we can write
\[
  \mathbb{P} \left( \bigcup_{k=0}^{n-1} (S^k Y) \notin B_{N,M}^{\lambda} \right) \leq \sum_{k=0}^{n-1} \mathbb{P} \left( (S^k Y) \notin B_{N,M}^{\lambda} \right) = n \mathbb{P} \left( Y \notin B_{N,M}^{\lambda} \right) \leq cnN^{1-1/M} e^{-\nu N^{1/M}}.
\]

In order to prove the first inequality, we use the union bound and the strong stationarity of \( Y_t, t \in \mathbb{Z} \) again.
\[
  \mathbb{P} \left( Y \notin B_{N,M}^{\lambda} \right) \leq \sum_{k=0}^{[N/1]^{M-1}-1} \mathbb{P} \left( \prod_{i=1}^{[N/1]} \gamma (Y_{k\lfloor 1 \rfloor + i}) \geq \tilde{\gamma}^{[N/1]} \right)
  \leq N^{1-1/M} \mathbb{P} \left( \prod_{i=1}^{[N/1]} \gamma (Y_{i}) \geq \tilde{\gamma}^{[N/1]} \right)
  \leq N^{1-1/M} \mathbb{E} \left( \prod_{i=1}^{[N/1]} \gamma (Y_{i}) \right).
\]

By Assumption (23) exists \( \tilde{c} > 0 \) such that \( \mathbb{E} (K(B) \prod_{i=1}^{n} \gamma (Y_{i})) \leq \tilde{c} \tilde{\gamma}^{[N/1]} \), \( n \in \mathbb{N} \) moreover \( K(.) \geq 1 \). With this, we get
\[
  \mathbb{E} \left( \prod_{i=1}^{[N/1]} \gamma (Y_{i}) \right) \leq \tilde{c} \tilde{\gamma}^{[N/1]} \leq \frac{\tilde{c}}{\tilde{\gamma}^{1/M}} \mathbb{E}^{[N/1]}
\]
and
\[
  \mathbb{P} \left( Y \notin B_{N,M}^{\lambda} \right) \leq \frac{\tilde{c}}{\mathbb{E}^{[N/1]}} N^{1-1/M} \tilde{\gamma}^{1/M}
\]
thus the first inequality holds with \( c = \tilde{c} / \sqrt{\tilde{\gamma}^{1+1/M}} \) and \( \nu = \frac{1-\lambda}{2} \log \tilde{\gamma} \).

\[ \square \]
Let $N \in \mathbb{N}^+$ and $y \in \mathcal{Y}$ be such that $S' y \in B^\lambda_{[N^{1/6},\delta]}$, $t = 0, \ldots, [N^{1/6}] - 1$. For $t \in \{t = 1, \ldots, [N^{1/6}] - [N^{1/6}]^\delta\}$, exists $q \in \{0, 1, \ldots, [N^{1/6}]^\delta - [N^{1/6}]^\beta\}$ and $r \in \{0, 1, \ldots, [N^{1/6}] - 1\}$ for which $t = q[N^{1/6}] + r$ holds. So, for $k \in \{0, 1, \ldots, [N^{1/6}]^\delta - 1\}$, we have

$$\prod_{i=1}^{[N^{1/6}]} (S'y)_{k[N^{1/6}] + t} = \prod_{i=1}^{[N^{1/6}]} (y_{t + k[N^{1/6}] + r}) = \prod_{i=1}^{[N^{1/6}]} (y_{(k + q)[N^{1/6}] + l + r}) = \prod_{i=1}^{[N^{1/6}]} (y_{(k + q)[N^{1/6}] + l + r}) < \gamma^\lambda_{N^{1/6}}$$

hence $S'y \in B^\lambda_{[N^{1/6},\delta,\gamma]}$. Thus we arrive at the following important remark.

**Remark 5.3.** If for some $N \in \mathbb{N}^+$ and $y \in \mathcal{Y}$, $S'y \in B^\lambda_{[N^{1/6},\delta]}$, $t = 0, \ldots, [N^{1/6}] - 1$, then $S'y \in B^\lambda_{[N^{1/6},\delta,\gamma]}$, $t = 1, \ldots, [N^{1/6}]^\delta - [N^{1/6}]^\beta$ holds, as well.

**Lemma 5.4.** For $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $k, l \in \mathbb{Z}$, $l < k$, we have

$$[Q(y_{k-1}) \ldots Q(y_{l})V](x) \leq V(x) \prod_{r=l}^{k-1} \gamma(y_r) + \sum_{r=l}^{k-1} K(y_r) \prod_{j=r+1}^{k-1} \gamma(y_j).$$

**Proof.** We prove by induction. Let $x \in \mathcal{X}$ and $l \in \mathbb{Z}$ be arbitrary and fixed. For $k = l + 1$, we have

$$[Q(y_l)V](x) \leq \gamma(y_l)V(x) + K(y_l). \quad (12)$$

which holds by Assumption 2.2.

**Induction step:** Operators $V \mapsto [Q(y)V]$, $y \in \mathcal{Y}$ are linear, monotone and for $V \equiv 1$, $[Q(y)V] \equiv 1$, $y \in \mathcal{Y}$ hence by Assumption 2.2 we can write

$$[Q(y_k) \ldots Q(y_{l})V](x) = [Q(y_k)(Q(y_{k-1}) \ldots Q(y_{l})V)](x) \leq$$

$$[Q(y_k)V](x) \prod_{r=l}^{k-1} \gamma(y_r) + \sum_{r=l}^{k-1} K(y_r) \prod_{j=r+1}^{k-1} \gamma(y_j) \leq$$

$$V(x) \prod_{r=l}^{k-1} \gamma(y_r) + \sum_{r=l}^{k-1} K(y_r) \prod_{j=r+1}^{k} \gamma(y_j)$$

which completes the proof. \hfill \Box

Let $N \in \mathbb{N}^+$, $\lambda \in (1/2, 1)$ be fixed and $P_1, P_2 : \Omega \rightarrow \mathcal{X}$ arbitrary $\mathcal{G}_0$-measurable random variables, which may depend on $y$. Furthermore, in the remaining part of this subsection, we assume that $y \in B^\lambda_{N^{1/3}}$. Our purpose will be to prove that, with a large probability $Z^y_{0,N^3} = Z^y_{0,N^3}$ for $N$ large enough. In other words, a coupling between the processes $Z^y_{0,N^3}$ and $Z^y_{0,N^3}$ is realized.

First, we are going to prove that the process $\overline{Z}_t := \left(\overline{r}_{0,t}^\lambda, \overline{r}_{0,t}^\nu\right)$, $t \in \mathbb{N}$ visits the sets $\overline{B}(y)$ frequently enough, where

$$\overline{B}(y) = \{(x_1, x_2) \in \mathcal{X}^2 \mid V(x_1) + V(x_2) \leq R(y)\}, \ y \in \mathcal{Y}. \quad (13)$$
Let us define the successive visiting times

$$\sigma_0 := 0, \quad \sigma_{k+1} := \min\{i > \sigma_k \mid Z_i \in \overline{D}(y_i)\}, \quad k \in \mathbb{N}$$

that are obviously \((\xi_t)_{t \in \mathbb{N}^*}\text{-stopping times.}\)

**Lemma 5.5.** For the tail distribution of \(\sigma_N\), we have

$$p(\sigma_N > N^3) < \alpha(N^2) \frac{\sqrt{N}}{2} \sum_{k=0}^{N-1} E[V(Z_{0,kN^2+1}^l) + V(Z_{0,kN^2+1}^l)].$$

**Proof.** If \(\sigma_N > N^3\), then exists \(k \in \{0, \ldots, N-1\}\) for which \(Z_{kN^2+l} \not\in \overline{D}(||y_{kN^2+l}||)\), \(l = 1, \ldots, N^2\). Thus we can write

$$p(\sigma_N > N^3) \leq \sum_{k=0}^{N-1} \sum_{l=1}^{N^2} p(Z_{kN^2+l} \not\in \overline{D}(y_{kN^2+l})).$$

We estimate a general term of the latter sum. For typographical reasons, we will write \(a := kN^2\) and \(b := N^2\). By the tower rule, we have

$$p\left(\bigcap_{l=1}^{b} \{Z_{a+l} \not\in \overline{D}(y_{a+l})\}\right) = E\left[\prod_{l=1}^{b} \mathbb{1}_{\{Z_{a+l} \not\in \overline{D}(y_{a+l})\}}\right] =$$

$$E\left[E\left[\mathbb{1}_{\{Z_{a+b} \not\in \overline{D}(y_{a+b})\}} \prod_{l=1}^{b} \mathbb{1}_{\{Z_{a+l} \not\in \overline{D}(y_{a+l})\}}\right]\right].$$

By Assumption \(\mathcal{H}\) we can write

$$E\left[V(Z_{0,a+b}^l) + V(Z_{0,a+b}^l) \mathbb{1}_{\{Z_{a+b} \not\in \overline{D}(y_{a+b})\}} + V(T_{a+b}(y_{a+b})Z_{0,a+b}^l) \mathbb{1}_{\{Z_{a+b} \not\in \overline{D}(y_{a+b})\}}\right] =$$

$$E\left[V(T_{a+b}(y_{a+b})Z_{0,a+b}^l) + V(T_{a+b}(y_{a+b})Z_{0,a+b}^l) \mathbb{1}_{\{Z_{a+b} \not\in \overline{D}(y_{a+b})\}} + [Q(y_{a+b})V](Z_{0,a+b}^l) + [Q(y_{a+b})V](Z_{0,a+b}^l) \right] \leq$$

$$\gamma(y_{a+b})\left[V(Z_{0,a+b}^l) + V(Z_{0,a+b}^l) \right] + 2K(y_{a+b}).$$

On the other hand, if \(Z_{a+b} \not\in \overline{D}(y_{a+b}),\) then

$$V(Z_{0,a+b}^l) + V(Z_{0,a+b}^l) > R(y_{a+b})$$

which immediately implies that

$$2K(y_{a+b}) < \varepsilon \gamma(y_{a+b})(V(Z_{0,a+b}^l) + V(Z_{0,a+b}^l)).$$

Recall that \(0 < \varepsilon < 1/\sqrt{\gamma}\) thus we have

$$E\left[V(Z_{0,a+b}^l) + V(Z_{0,a+b}^l) \mathbb{1}_{\{Z_{a+b} \not\in \overline{D}(y_{a+b})\}}\right] <$$

$$\frac{\gamma(y_{a+b})}{\sqrt{\gamma}} \left[V(Z_{0,a+b}^l) + V(Z_{0,a+b}^l)\right].$$

14
This argument can clearly be iterated and leads to
\[
\mathbb{P}\left(\bigcap_{i=1}^b \{Z_{\sigma^i} \notin \overline{\mathcal{D}}(y_{\sigma^i})\}\right) \leq \prod_{i=1}^{b-1} \frac{\gamma(Y_{\sigma^{i+1}})}{\sqrt{\gamma^b R(Y_{\sigma^b})}} \times \mathbb{E}\left[V(Z_{0,\sigma^1}^{p_1,y}) + V(Z_{0,\sigma^1}^{p_0,y})\right].
\]
Taking into account that \(y \in B_{N^3}^{1,3}\), \(R(y_{\sigma+b}) = \frac{2E(y_{\sigma+b})}{\gamma(U(y_{\sigma+b}))}\) and \(K(\cdot) \geq 1\), hence we can write
\[
\prod_{i=1}^b \gamma(y_{\sigma^i}) = \prod_{i=1}^{N^2} \gamma(y_{kN^2+i}) = \prod_{i=0}^{N-1} \prod_{j=1}^N \gamma(y_{(kN^2+i)N+j}) < \tilde{\gamma}^N^2
\]
moreover
\[
\frac{\prod_{i=1}^{b-1} \gamma(y_{\sigma^i})}{\sqrt{\gamma^b R(y_{\sigma^b})}} \leq \frac{1 - \sqrt{\gamma}}{2} \times \frac{(\lambda - \frac{1}{2})N^2}{2}.
\]
Finally, we sum up for \(k = 0, 1, \ldots, N-1\) and get
\[
\mathbb{P}\left(\sigma_N > N^3\right) < \tilde{\gamma}^{(\lambda - \frac{1}{2})N^2} \times \frac{1 - \sqrt{\gamma}}{2} \times \frac{\tilde{\gamma}^N^2}{2} \sum_{k=0}^{N-1} \mathbb{E}\left[V(Z_{0,kN^2+1}^{p_1,y}) + V(Z_{0,kN^2+1}^{p_0,y})\right]
\]
which completes the proof. \(\square\)

**Lemma 5.6.** For the coupling probability, we have the following estimate.

\[
\mathbb{P}\left(Z_{0,N^3}^{p_1,y} \neq Z_{0,N^3}^{p_0,y}\right) \leq \left(\max_{0 \leq k < N^3} \alpha(y_k)\right)^{N-1}
\]

\[
+ \tilde{\gamma}^{(\lambda - \frac{1}{2})N^2} \times \frac{1 - \sqrt{\gamma}}{2} \times \frac{\tilde{\gamma}^N^2}{2} \sum_{k=0}^{N-1} \mathbb{E}\left[V(Z_{0,kN^2+1}^{p_1,y}) + V(Z_{0,kN^2+1}^{p_0,y})\right]
\]

**Proof.** For typographical reasons, we will write \(\sigma(N)\) instead of \(\sigma_N\) in this proof. Recall that, \(T_s(y) : \mathcal{X} \to \mathcal{X}\) is constant on \(V^{-1}([0, R(y)])\), \(t \in \mathbb{Z}\), \(y \in \mathcal{Y}\) with probability at least \(1 - \alpha(y)\). By the definition of \(\overline{\mathcal{D}}(y)\), \(Z_1 \in \overline{\mathcal{D}}(y)\) implies that \(Z_{0_i}^{p_i,y} \in V^{-1}([0, R(y)])\) for \(i = 1, 2\), moreover \(Z_{0_1}^{p_1,y} = Z_{0_2}^{p_0,y}\) with probability at least \(1 - \alpha(y_1)\).

Let us introduce the abbreviation \(M_N = \max_{0 \leq k < N^3} \alpha(y_k)\) for a moment. We can write
\[
\mathbb{P}\left(Z_{0,N^3}^{p_1,y} = Z_{0,N^3}^{p_0,y}, \sigma_N \leq N^3\right) \leq \mathbb{P}\left(U_{\sigma(j)+1} > 1 - \alpha(y_{\sigma(j)}); j = 1, \ldots, N-1\right) \leq \mathbb{P}\left(U_{\sigma(j)+1} > 1 - M_N; j = 1, \ldots, N-1\right) \leq
\]
\[
\mathbb{E}\left[\prod_{j=1}^{N-2} \mathbb{I}\left[U_{\sigma(j)+1} > 1 - M_N\right] \mathbb{I}\left\{U_{\sigma(N-1)+1} > 1 - M_N\right\}\right].
\]

Clearly, \(U_{\sigma(N-1)+1}\) is independent of \(\mathcal{G}_{\sigma(N-1)}\) so
\[
\mathbb{P}\left(U_{\sigma(N-1)+1} > 1 - M_N\right) = \mathbb{P}\left(U_{\sigma(N-1)+1} > 1 - M_N\right) = \max_{0 \leq k < N^3} \alpha(y_k).
\]
Iteration of this argument leads to the following estimation.

\[ P \left( Z_{0,N^3}^{p,y} = Z_{0,N^3}^{p,y}, \sigma_N \leq N^3 \right) \leq \left( \max_{0 \leq k < N^3} \alpha(y_k) \right)^{N-1} \]

By Lemma 5.7, we have

\[ P \left( Z_{0,N^3}^{p,y} = Z_{0,N^3}^{p,y} \right) \leq P \left( Z_{0,N^3}^{p,y} = Z_{0,N^3}^{p,y}, \sigma_N \leq N^3 \right) + P \left( \sigma_N > N^3 \right) \]

\[ \leq \left( \max_{0 \leq k < N^3} \alpha(y_k) \right)^{N-1} + \gamma(\lambda-\beta)N^3 \sum_{k=0}^{N-1} E \left[ V(x_{k,N^3}, y) \right] + V(x_{0,N^3}, y) \]

which completes the proof.

**Lemma 5.7.** Let Assumption 2.6 be in force. Then, for every \( N \in \mathbb{N}^+ \) and \( 1 \leq p < \infty \),

\[ \sum_{N=1}^{\infty} \max_{0 \leq k < [N^{1/M}]} \alpha(Y_k)^{[N^{1/M}]^{-1}} \left\|_{p} < \infty. \right. \]

More precisely, there exists \( c, \nu, \beta > 0 \) depending only on \( M, p \) and \( \theta \) such that

\[ \left\| \max_{0 \leq k < [N^{1/M}]} \alpha(Y_k)^{[N^{1/M}]^{-1}} \right\|_{p} \leq c E \left[ \alpha(Y_0)^{[N^\beta]} \right]^{\frac{1}{\theta p}}, \quad N \in \mathbb{N}^+. \]

**Proof.** Let \( \beta = \frac{1}{M(1-\theta)} \). For sufficiently large \( N \in \mathbb{N}^+ \), \( \frac{1}{p} \frac{[N^{1/M}]}{[N^{1/M-1}]} > 1 \) hence by Jensen’s inequality and the strong stationarity of \( Y_t \), \( t \in \mathbb{Z} \), we have

\[ \left\| \max_{0 \leq k < [N^{1/M}]} \alpha(Y_k)^{[N^{1/M}]^{-1}} \right\|_{p} \leq E \left[ \max_{0 \leq k < [N^{1/M}]} \alpha(Y_k)^{[N^\beta]} \right] \leq N E \left[ \alpha(Y_0)^{[N^{\beta}]} \right]^{\frac{1}{\theta p}}, \quad N \in \mathbb{N}^+. \]

hence we obtain

\[ \left\| \max_{0 \leq k < [N^{1/M}]} \alpha(Y_k)^{[N^{1/M}]^{-1}} \right\|_{p} \leq N \frac{[N^{1/M}]^{-1}}{[N^{1/M-1}]} \left( E \left[ \alpha(Y_0)^{[N^{\beta}]} \right] \right)^{\frac{1}{\theta p}}, \quad N \in \mathbb{N}^+. \]

Taken into consideration that \( \lim_{N \to \infty} \frac{[N^{1/M}]^{-1}}{[N^{1/M-1}]} = 0 \) and \( \lim_{N \to \infty} \frac{[N^{1/M}]^{-1}}{[N^{1/M-1}]} = 1 \), we conclude that there exists \( c, \nu > 0 \) depending on \( M, p \) and \( \theta \) for which

\[ \left\| \max_{0 \leq k < [N^{1/M}]} \alpha(Y_k)^{[N^{1/M}]^{-1}} \right\|_{p} \leq c E \left[ \alpha(Y_0)^{[N^{\beta}]} \right]^{\frac{1}{\theta p}}, \quad N \in \mathbb{N}^+ \]

holds.

By Assumption 2.6, for some \( 0 < \theta' < 1 \), \( E \left[ \alpha(Y_0)^{N} \right]^{\frac{1}{\theta'}} \to 0 \) as \( N \to \infty \) and trivially the same holds for any \( 0 < \theta < \theta' \). Let us fix some \( \theta \in (0, \theta') \). Then, by the previous point, we have the estimate

\[ \left\| \max_{0 \leq k < [N^{1/M}]} \alpha(Y_k)^{[N^{1/M}]^{-1}} \right\|_{p} \leq c \left( E \left[ \alpha(Y_0)^{[N^{\beta}]} \right] \right)^{\frac{1}{\theta p}}, \quad N \in \mathbb{N}^+, \]

where \( c, \nu, \beta > 0 \) depends only on \( M, p \) and \( \theta \) thus for sufficiently large \( n \in \mathbb{N}^+ \),

\[ \sum_{N=n}^{\infty} \left\| \max_{0 \leq k < [N^{1/M}]} \alpha(Y_k)^{[N^{1/M}]^{-1}} \right\|_{p} < c \sum_{N=n}^{\infty} \frac{1}{2^{(N^{1/M})^{-\theta}}} < \infty \]

16
which proves that
\[ \sum_{N=1}^{\infty} \max_{0 \leq k < [N^{1/3}]^{1/3}} \alpha(Y_k^{[N^{1/3}-1]})_p < \infty. \]

\[ \square \]

5.2 Pointwise convergence of kernels

Let us introduce the sequence of probabilistic kernels \( \mu_k(\cdot, \cdot) : Y \times \mathcal{B} \rightarrow [0, 1] \), \( k \in \mathbb{N} \) such that for any fixed \( y \in Y \),
\[ \mu_0(y, \cdot) = \delta_{x_0}, \]
\[ \mu_n(y, \cdot) = \text{Law}(Z_{0,n}^{x_0, S^{-n}y}). \] (16)

In this point, for typical variation distance, we can write
\[ \text{According to the optimal transportation cost characterization of the total variation distance, we give an estimation for } \]
distance which proves that
\[ \text{falls into } A \in \mathcal{B} \]
\[ \text{We prove that under Assumptions 2.2, 2.3 and 2.4, for Law(} Y \text{) a.s. } y \in Y, \text{ we define the following sets.} \]

**Lemma 5.8.** For \( n \in \mathbb{N}^+ \) and \( 1/2 < \lambda < 1 \), we define the following sets.
\[ A_n^\lambda = \left\{ y \in Y \mid \frac{d_{TV}(\mu_n(y, \cdot), \mu_{n+1}(y, \cdot))}{2} < \left( \max_{0 \leq k < [n^{1/3}]^{1/3}} \alpha(y_k) \right)^{[n^{1/3}]^{1/3}} + \tilde{\gamma}(\lambda^{-1})n^{1/3} - n^{1/3} \right\} \]

Then \( Y \) falls into \( A_n^\lambda \) with large probability. More precisely, there exist \( c, \nu > 0 \) for which
\[ \mathbb{P}(Y \in A_n^\lambda) \geq 1 - cn^{2/3}e^{-m^{1/3}}. \]

**Proof.** According to the optimal transportation cost characterization of the total variation distance, we can write
\[ \frac{1}{2} d_{TV}(\mu_n(y, \cdot), \mu_{n+1}(y, \cdot)) = \inf_{\kappa \in \mathcal{E}(\mu_n(y, \cdot), \mu_{n+1}(y, \cdot))} \int \mathbb{I}_{x \neq y} \kappa(dx, dy) \]
\[ \leq \mathbb{P}(Z_{x_0, n}^{x_0, S^{-n}y} \neq Z_{0,n}^{x_0, S^{-n}y}) = \mathbb{P}(Z_{x_0,n}^{x_0, S^{-n}y} \neq Z_{0,n}^{x_0, S^{-n}y}) \]
\[ \leq \mathbb{P}(Z_{0,n}^{x_0, S^{-n}y} \neq Z_{0,n}^{x_0, [n^{1/3}]^{1/3}}). \]

If \( y \in Y \) such that \( S^{-n+1}y \in B_n \), then we can apply Lemma 5.6 since \( x_0 \) is \( \mathcal{G}_0 \)-measurable, and obtain
\[ \mathbb{P}(Z_{x_0,n}^{x_0, S^{-n+1}y} \neq Z_{0,n}^{x_0, S^{-n+1}y}) \leq \left( \frac{1}{\sqrt{2}} \sum_{k=0}^{[n^{1/3}]^{1/3}-1} \mathbb{E}[V(Z_{0,k}^{x_0, S^{-n+1}y}) + V(Z_{0,k}^{x_0, [n^{1/3}]^{1/3}+1})]. \]
By Lemma 5.4 and Assumption 2.2 and the tower rule, for \(0 \leq k < \lfloor n^{1/3} \rfloor\), we have

\[
E\left[ V(Z_{0,k[n^{1/3}]^{j+1}}^{\gamma}) + V(Z_{0,[k[n^{1/3}]^{j+1}]}^{\gamma}) \right] = \\
E\left[ Q(y_{0,[k[n^{1/3}]^{j+1}]}) \ldots Q(y_{0,[k[n^{1/3}]^{j+1}]}) \right](x_0) + \\
E\left[ Q(y_{0,[k[n^{1/3}]^{j+1}]}) \ldots Q(y_{0,[k[n^{1/3}]^{j+1}]}) \right](T_0(y_{0}−n)x_0) \leq \\
(V(x_0) + E(V(T_0(y_{0}−n)x_0)))\prod_{r=−n+1}^{k[n^{1/3}]^{j+1}−n+1} γ(y_r) + 2\sum_{r=−n+1}^{k[n^{1/3}]^{j+1}−n+1} K(y_r) \prod_{j=−r+1}^{k[n^{1/3}]^{j+1}−n+1} γ(y_j) \leq \\
V(x_0) \prod_{r=−n+1}^{k[n^{1/3}]^{j+1}−n+1} γ(y_r) + 2\sum_{r=−n+1}^{k[n^{1/3}]^{j+1}−n+1} K(y_r) \prod_{j=−r+1}^{k[n^{1/3}]^{j+1}−n+1} γ(y_j)
\]

where we have taken into account that \(K(.) \geq 1\).

By the Markov inequality and the strong stationarity of \(Y_t\), \(t \in \mathbb{Z}\), we can write

\[
P\left( Y \in \left\{ y \in \mathcal{Y} \left| \sum_{k=0}^{\lfloor n^{1/3} \rfloor−1} E\left[ V(Z_{0,k[n^{1/3}]^{j+1}}^{\gamma}) + V(Z_{0,[k[n^{1/3}]^{j+1}]}^{\gamma}) \right] \right| \geq \frac{2s−n^{1/3}}{1−\sqrt{r}} \right) \leq \\
c_1 \gamma^{n^{1/3}−1/2} \sum_{k=0}^{\lfloor n^{1/3} \rfloor−1} \left[ V(x_0) + \sqrt{\gamma} (1 + \sqrt{\gamma}) \gamma^{n^{1/3}−1/2} + 2\sum_{r=0}^{k[n^{1/3}]^{j+1}−n+1} \sqrt{\gamma} \right] \leq \\
c_1 (V(x_0) + [n^{1/3}]) \gamma^{n^{1/3}},
\]

where \(c_1\) is chosen such that \(E(K(Y)) \prod_{r=1}^{\lfloor n^{1/3} \rfloor−1} γ(y_r) \leq c_1 \gamma^{n^{1/3}}, n \in \mathbb{N}\) holds.

So, by Lemma 5.2 and our previous considerations, exists \(c_1, c_2, v' > 0\) such that

\[
P\left( Y \notin A^{\lambda}_n \right) \leq c_1 (V(x_0) + [n^{1/3}]) \gamma^{n^{1/3}} + c_v n^{2/3} e^{-v'n^{1/3}}.
\]

Clearly, there exists \(c > 0\) such that, for \(v = \min(−\log \gamma, v')\),

\[
c_1 (V(x_0) + [n^{1/3}]) \gamma^{n^{1/3}} + c_v n^{2/3} e^{-v'n^{1/3}} \leq cn^{2/3} e^{-m^{1/3}}
\]

holds which completes the proof.

The next lemma is a crucial ingredient of the proof both of Theorem 2.7 and 2.9.
Lemma 5.9. Under Assumption 2.6, there exists $1 < p < \infty$, such that
\[
\sum_{n=0}^{\infty} \left\| d_{TV}(\mu_n(y,\cdot),\mu_{n+1}(y,\cdot)) \right\|_p < \infty.
\]

Proof. According to Lemma 5.8 there exist $c, \nu > 0$ such that $\mathbb{P}(Y \in A_n^1) \geq 1 - cn^{2/3} e^{-m^{1/3}}$. So, we obtain the following upper bound for the general term
\[
\|d_{TV}(\mu_n(y,\cdot),\mu_{n+1}(y,\cdot))\|_p \leq \|d_{TV}(\mu_n(y,\cdot),\mu_{n+1}(y,\cdot))I_{Y \in A_n^1}\|_p + 2 \mathbb{P}(Y \not\in A_n^1)
\]
\[
\leq 2 \left[ \max_{0 \leq k < \lfloor m^{1/3} \rfloor} \alpha(Y_k) |\nu|^{1/3} - 1 \right] + \tilde{\gamma} (\lambda - \frac{1}{\nu}) |\nu|^{1/3} |n^{1/3} + cn^{2/3} e^{-m^{1/3}} \right], \quad n \in \mathbb{N}^+
\]
which, by Lemma 5.7, has a finite sum.
\[\square\]

We notice that the sequence of expected total variation distances has a finite sum, that is
\[
\sum_{n=0}^{\infty} \mathbb{E}(d_{TV}(\mu_n(y,\cdot),\mu_{n+1}(y,\cdot))) < \infty
\]
which implies the following.

Corollary 5.10. For Law$(Y)$ - a.s. $\mu_n(y,\cdot), n \in \mathbb{N}$ is Cauchy and hence convergent in the metric space $(\mathcal{M}_1, d_{TV})$. Let us denote this pointwise limit by $\mu(y,\cdot)$. \[\square\]

5.3 Ergodicity of $\Phi(Z_t^y)$

Let $N \geq 1$ be an arbitrary natural number and $y \in \mathcal{Y}$. Let us define the truncated process
\[
W_t(y) := \left[ \Phi(Z_{0,t}^y) - \mathbb{E} \left( \Phi(Z_{0,t}^y) \right) \right] 1_{t \leq \lfloor N^{1/p} \rfloor}, \quad t \in \mathbb{N}.
\]

We will use the results of Section 6. For $p \geq 1$, introduce the quantities $M_p(W) = \sup_{t \in \mathbb{N}} \|W_t\|_p$ and
\[
\Gamma_p(W) = \sum_{\tau=1}^{\infty} \gamma_p(W, \tau),
\]
where $\gamma_p(W, \tau) = \sup_{\tau \geq \tau} \|W_t - \mathbb{E}(W_t | \mathcal{F}_{\tau+1}^+)\|_p$, $\tau \geq 1$. If $\tau > \lfloor N^{1/p} \rfloor$, then $\gamma_p(W, \tau) = 0$ thus $\Gamma_p(W)$ is finite which means that $W_t, t \in \mathbb{N}$ is L-mixing of order $p$ with respect to $(\mathcal{F}_t, \mathcal{F}_t^+), t \in \mathbb{N}$. According to Lemma 5.2, for $p \geq 2$, we have the estimate
\[
\left\| \frac{1}{N} \sum_{t=1}^{\lfloor N^{1/p} \rfloor} W_t \right\|_p \leq C_p M_p^{1/2}(W) \sqrt{\frac{\Gamma_p(W)}{N}},
\]
where $C_p$ is a constant that does not depend either on $N$ or on $W$.

Let us consider the estimate
\[
\Gamma_p(W) = \sum_{\tau=1}^{\infty} \gamma_p(W, \tau) \leq 2 \sqrt{N} \|\Phi\|_\infty + \sum_{\tau=\lfloor N^{1/p} \rfloor + 1}^{\lfloor N^{1/p} \rfloor} \gamma_p(W, \tau)
\]
and for $s, t \in \mathbb{N}, t \geq s$ introduce the auxiliary process
\[
\overline{W}_{s,t} := \left[ \Phi(Z_{s,t}^y) - \mathbb{E} \left( \Phi(Z_{0,t}^y) \right) \right] 1_{t \leq \lfloor N^{1/p} \rfloor}.
\]
Note that, $W_{x,t}$ is measurable with respect to $\mathcal{G}^+_t$ moreover

$$W_t - \bar{W}_{x,t} = \Phi\left(Z_{x_0,y}^t\right) - \Phi\left(Z_{x,t}^r\right)$$

(21)

which will be important later.

For $[N^{1/6}]^3 < \tau \leq [N^{1/6}]^6$, there exists $q, r \in \{0, 1, \ldots, [N^{1/6}]^3\}$ such that

$$\tau = q[N^{1/6}]^3 + r,$$

where $q \geq 1$. By our previous observation, $\bar{W}_{x-[N^{1/6}]^6, t}$ is measurable with respect to $\mathcal{G}^+_{t-[N^{1/6}]^3}$ moreover $\mathcal{G}^+_{t-[N^{1/6}]^3} \subseteq \mathcal{G}^+_t$ because $q \geq 1$ and thus $\bar{W}_{x-[N^{1/6}]^6, t}$ is $\mathcal{G}^+_x$-measurable.

By Lemma 6.1 and (21), we can write

$$\gamma_p(W, \tau) = \max_{\tau \leq [N^{1/6}]^6} \left\| W_t - \mathbb{E}(W_t | \mathcal{G}_{t-}^+ \right\|_p \leq \max_{\tau \leq [N^{1/6}]^6} 2 \left\| W_t - \bar{W}_{[N^{1/6}]^6, t} \right\|_p$$

$$= 2 \left\| W_{t+[N^{1/6}]^6} - \bar{W}_{t,[N^{1/6}]^6} \right\|_p$$

$$= \max_{0 < t \leq [N^{1/6}]^6} 2 \left\| \Phi(Z_{x_0, y}^t) - \Phi(Z_{x, t}^r) \right\|_p$$

$$\leq 4|\Phi|_\infty \max_{0 < t \leq [N^{1/6}]^6} \mathbb{P}\left(Z_{x_0, y}^t \neq Z_{x, t}^r \right) .$$

We substitute this back into (20) and we obtain the following upper bound

$$\left\| \frac{1}{N} \sum_{i=1}^N W_i \right\|_p \leq 2C_p|\Phi|_\infty \left( \frac{1}{\sqrt{N}} + 2 \max_{0 < t \leq [N^{1/6}]^6} \mathbb{P}\left(Z_{x_0, y}^t \neq Z_{x, t}^r \right) \right)^{1/2} .$$

(22)

**Lemma 5.11.** For $N \in \mathbb{N}^+$, $1/2 < \lambda < 1$ and $0 < t \leq [N^{1/6}]^6 - [N^{1/6}]^3$, let us define the following sets.

$$C_N^\lambda = \left\{ Y \in \mathcal{B} | \mathbb{P}\left(Z_{x_0, y}^t \neq Z_{x, t}^r \right) < \left( \max_{0 \leq k < [N^{1/6}]} \alpha(y_{k+1}) \right)^{[N^{1/6}]^3} + \varphi(\lambda - 1) \right\} .$$

Then there exist $c, \nu > 0$ such that

$$\mathbb{P}\left(Y \in C_N^\lambda \right) \geq 1 - cN^{7/6}e^{-\nu N^{1/6}} ,$$

where $C_N = \bigcap_{t=1}^{[N^{1/6}]^6} C_N^\lambda$.

**Proof.** Let $N \in \mathbb{N}^+$ and $0 < t \leq [N^{1/6}]^6 - [N^{1/6}]^3$ be arbitrary and fixed. We have the following identities

$$\mathbb{P}\left(Z_{x_0, y}^t \neq Z_{x, t}^r \right) = \mathbb{P}\left(Z_{x_0, y}^t \neq Z_{x, t}^r \right) = \mathbb{P}\left(Z_{x_0, y}^t \neq Z_{x, t}^r \right),$$

where $Z' = Z_{x_0, y}$ and $Z'' = Z_{x, t}^r$.

If $y \in \mathcal{B}$ such that $S'y \in B_{[N^{1/6}]^3}^\lambda$, $t = 0, 1, \ldots, [N^{1/6}] - 1$, then by Remark 5.3 $S'y \in B_{[N^{1/6}]^3}^\lambda$, $0 < t \leq [N^{1/6}]^6 - [N^{1/6}]^3$, furthermore $Z''$ is $\mathcal{G}_0$-measurable hence we can apply Lemma 6.6 thus we obtain

$$\mathbb{P}\left(Z_{x_0, y}^t \neq Z_{x_0, y}^t \right) \leq \left( \max_{0 \leq k < [N^{1/6}]} \alpha(y_{k+1}) \right)^{[N^{1/6}]^3} + \varphi(\lambda - 1) \right\} .$$

$$\left[ V(Z_0, k[N^{1/6}]^3 - 1) + V(Z_0, k[N^{1/6}]^3 + 1) \right] .$$

20
By Lemma 5.4 and the tower rule, for $0 \leq k < \lfloor N^{1/6} \rfloor$, we have

$$E \left[ V \left( Z_{0, k}^{x, y} \right) \right] = E \left[ \left( Q \left( y_{k+k} \right) \ldots Q \left( y_{k+1} \right) \right) \left( Z_{0, k}^{x, y} \right) \right] = E \left( Q \left( y_{k+k} \right) \ldots Q \left( y_{k+1} \right) \right) V \left( x_{k+k} \right) + E \left( Q \left( y_{k+k} \right) \ldots Q \left( y_{k+1} \right) \right) V \left( x_{k+1} \right).
$$

By the Markov inequality and the strong stationarity of $Y_t$, we can write

$$P \left( Y \in \mathcal{Y} \right) \leq \frac{E \left[ V \left( Z_{0, k}^{x, y} \right) \right]}{\gamma \left( 1 + \gamma \right)} \sum_{k=0}^{\lfloor N^{1/6} \rfloor} E \left[ \prod_{r=0}^{k} \gamma \left( Y_r \right) \right] + 2 \sum_{k=0}^{\lfloor N^{1/6} \rfloor} E \left[ K \left( Y_r \right) \prod_{j=r+1}^{k} \gamma \left( Y_r \right) \right].$$

So, by Lemma 5.2 and our previous considerations, exists $c_1, c_2, \nu > 0$ such that

$$P \left( \bigcup_{i=1}^{\lfloor N^{1/6} \rfloor} Y \notin C_{i, r} \right) \leq \sum_{i=1}^{\lfloor N^{1/6} \rfloor} P \left( Y \notin C_{i, r}, \bigcap_{j=r}^{\lfloor N^{1/6} \rfloor} S' y \notin B_{i, r} \right) + P \left( \bigcup_{r=0}^{\lfloor N^{1/6} \rfloor} S' y \notin B_{i, r} \right) \leq c_1 N \left( V \left( x_0 \right) + \left| N^{1/6} \right| \right) + c_2 N e^{-\nu \left| N^{1/6} \right|}.$$

Clearly, there exists $c > 0$ such that, for $\nu = \min \left(-\log \gamma, \nu' \right)$,

$$c_1 N \left( V \left( x_0 \right) + \left| N^{1/6} \right| \right) + c_2 N e^{-\nu \left| N^{1/6} \right|} \leq c N^{7/6} e^{-\nu N^{1/6}}.$$
holds which completes the proof.

Finally, we arrive at the following important result which will play a central role in the proof of Theorem 2.9

**Lemma 5.12.** There exists \( \tilde{c}(p, \tilde{\gamma}, \lambda) > 0 \) depending only on \( p, \tilde{\gamma} \) and \( \lambda \) such that

\[
\mathbb{E}^{1/p} \left[ \left\| \frac{1}{N} \sum_{i=1}^{N} W_i(Y) \right\|_p \right] \leq \tilde{c}(p, \tilde{\gamma}, \lambda) \| \Phi \|_{\infty} \left( N^{-1/4} + \max_{0 \leq k < N^{1/6}} \| a(Y_k) \|_{N^{1/6} - 1} \right)^{1/2}.
\]

**Proof.** Without the loss of generality, we may assume that \( p \geq 2 \). Clearly on \( C_N^1 \)

\[
\max_{0 < r \leq [N^{1/6} - [N^{1/6}]]} \mathbb{P} \left( Z_{r,1+\lceil N^{1/6} \rceil} \neq Z_{r+1,\lceil N^{1/6} \rceil} \right) \leq \max_{0 < r \leq [N^{1/6} - [N^{1/6}]]} \left( \max_{0 \leq k < \lceil N^{1/6} \rceil} a(Y_{k+r}) \right)^{[N^{1/6}] - 1} + \tilde{\gamma} \left( \lambda^{-1/2} \right)^{N^{1/6} - [N^{1/6}]}
\]

holds, hence by (22), we can write

\[
\mathbb{E}^{1/p} \left[ \left\| \frac{1}{N} \sum_{i=1}^{N} W_i(Y) \right\|_p \right] \leq \mathbb{E}^{1/p} \left[ \left\| \frac{1}{N} \sum_{i=1}^{N} W_i(Y) \right\|_p \right] + \mathbb{E}^{1/p} \left[ \left\| \frac{1}{N} \sum_{i=1}^{N} W_i(Y) \right\|_p \right] \leq 2\| \Phi \|_{\infty} \left[ C_p \left( \frac{1}{\sqrt{N}} + 2\tilde{\gamma} \left( \lambda^{-1/2} \right)^{N^{1/6} - [N^{1/6}]} \right)^{1/2} \right. \\
\left. + \mathbb{P} \left( Y \notin C_N^1 \right)^{1/p} \right]
\]

The square root function is subadditive hence by Lemma 5.11 exists \( \tilde{c}(p, \tilde{\gamma}, \lambda) \) depending only on \( p, \tilde{\gamma} \) and \( \lambda \) such that

\[
C_p \left( \frac{1}{\sqrt{N}} + 2\tilde{\gamma} \left( \lambda^{-1/2} \right)^{N^{1/6} - [N^{1/6}]} \right) + \max_{0 \leq k < [N^{1/6}]} \| a(Y_k) \|_{N^{1/6} - 1} \leq \tilde{c}(p, \tilde{\gamma}, \lambda) \frac{1}{2} \left( N^{-1/4} + \max_{0 \leq k < [N^{1/6}]} \| a(Y_k) \|_{N^{1/6} - 1} \right)^{1/2}.
\]

Finally, we obtain the desired upper bound

\[
\mathbb{E}^{1/p} \left[ \left\| \frac{1}{N} \sum_{i=1}^{N} W_i(Y) \right\|_p \right] \leq \tilde{c}(p, \tilde{\gamma}, \lambda) \| \Phi \|_{\infty} \left( N^{-1/4} + \max_{0 \leq k < [N^{1/6}]} \| a(Y_k) \|_{N^{1/6} - 1} \right)^{1/2}.
\]
which completes the proof. \qed

5.4 Proof of Theorem 2.7

For \( A \in \mathcal{B} \) arbitrary and any decomposition of \( A \) into disjoint and measurable sets \( A = \cup_i A_i \), we have

\[
\sum_i |\mathbb{E}(\mu_n(Y, A_i)) - \mathbb{E}(\mu_{n+1}(Y, A_i))| \leq \mathbb{E} \left[ \sum_i |\mu_n(Y, A_i) - \mu_{n+1}(Y, A_i)| \right]
\leq \mathbb{E}(d_{TV}(\mu_n(Y, \cdot), \mu_{n+1}(Y, \cdot))).
\]

Taking supremum on the left-hand side, we get

\[
d_{TV}(\mathbb{E}(\mu_n(Y, \cdot)), \mathbb{E}(\mu_{n+1}(Y, \cdot))) \leq \mathbb{E}(d_{TV}(\mu_n(Y, \cdot), \mu_{n+1}(Y, \cdot))).
\]

As easily seen, \( \mu_n(A) = \mathbb{E}(\mu_n(Y, A)) \) holds for \( A \in \mathcal{B} \), so we infer that

\[
d_{TV}(\mu_n, \mu_{n+1}) \leq \mathbb{E}(d_{TV}(\mu_n(Y, \cdot), \mu_{n+1}(Y, \cdot))).
\]

Then it follows from Corollary 5.10 that

\[
\sum_{n=1}^{\infty} d_{TV}(\mu_n, \mu_{n+1}) < \infty,
\]

hence \( \mu_n, n \in \mathbb{N} \) is a Cauchy sequence in the complete metric space \((\mathcal{M}, d_{TV})\). Hence it converges to some probability \( \mu_* \) as \( n \to \infty \). The claimed convergence rate follows from (17) with \( p = 1 \):

\[
d_{TV}(\mu_N, \mu_*) \leq \sum_{n=N}^{\infty} d_{TV}(\mu_n, \mu_{n+1}) \leq \sum_{n=N}^{\infty} \mathbb{E}(d_{TV}(\mu_n(Y, \cdot), \mu_{n+1}(Y, \cdot)))
\leq 2 \sum_{n=N}^{\infty} \mathbb{E} \left[ \max_{0 \leq k \leq n^{1/3}} a(Y_k) |n^{1/3} - 1| + \gamma(n^{1/3} - 1) \right].
\]

Only remains to prove that \( \mu_* \) and \( \mu_* \) coincide. It is clear that for every \( A \in \mathcal{B} \),

\[
\mu_*(A) = \lim_{n \to \infty} \mu_n(A) = \lim_{n \to \infty} \mathbb{E}(\mu_n(Y, A)) = \mathbb{E}(\lim_{n \to \infty} \mu_n(Y, A)) = \mathbb{E}(\mu_*(Y, A)) = \mu_*(A),
\]

hence \( \mu_* = \mu_* \).

\qed

5.5 Proof of Theorem 2.9

Let \( N \geq 1 \) arbitrary integer, \( 1 \leq p < \infty \) and consider the following estimate.

\[
\left\| \frac{1}{N} \sum_{t=1}^{N} \Phi(Z_{0,t}^{x,Y}) - \int_X \Phi(z) \mu_*(dz) \right\|_p \leq \left\| \frac{1}{N} \sum_{t=1}^{N-1} \int_X \Phi(z) [\mu_*(S^t Y, dz) - \mu_*(dz)] \right\|_p
\]

\[
+ \left\| \frac{1}{N} \sum_{t=1}^{N} \phi(z) (\mu_t - \mu_*)(S^{t-1} Y, dz) \right\|_p
\]

\[
+ \left\| \frac{1}{N} \sum_{t=1}^{N} \phi(z) (Z_{0,t}^{x,Y}) - \int_X \Phi(z) \mu_t(S^{t-1} Y, dz) \right\|_p
\]

(23)
The stochastic process \( Y \) is strongly stationary and ergodic hence the left shift \( S : \mathcal{Y}^Z \to \mathcal{Y}^Z \) is an ergodic endomorphism of the probability space \((\mathcal{Y}^Z, \mathcal{A}^\otimes, \text{Law}(Y))\), moreover \( \mathcal{Y}^Z \ni y \to \int_x \Phi(z) \mu_\gamma(y, dz) \) is obviously in \( L^1 \) hence Birkhoff’s ergodic theorem implies that

\[
\frac{1}{N} \sum_{t=0}^{N-1} \int_x \Phi(z) \mu_\gamma(S^t Y, dz) \to \int_x \Phi(z) \mu_\gamma(dz), \quad N \to \infty,
\]

almost surely and also in \( L^p \) due to Lebesgue’s dominated convergence theorem. By the strong stationary property of \( Y \) again, for the second term, we have

\[
\left\| \frac{1}{N} \sum_{t=1}^{N} \int_x \Phi(z) (\mu_t - \mu_\gamma)(S^{t-1} Y, dz) \right\|_p \leq \frac{\|\Phi\|_\infty}{N} \sum_{t=1}^{N} \|d_{TV}(\mu_t(\cdot, \cdot), \mu_\gamma(\cdot, \cdot))\|_p
\]

\[
\leq \frac{\|\Phi\|_\infty}{N} \sum_{t=1}^{N} \sum_{n=2}^{\infty} \|d_{TV}(\mu_n(\cdot, \cdot), \mu_{n+1}(\cdot, \cdot))\|_p,
\]

which is a Cèsaro sum and due to Lemma 5.9 the general term tends to zero thus we obtain

\[
\left\| \frac{1}{N} \sum_{t=1}^{N} \int_x \Phi(z) (\mu_t - \mu_\gamma)(S^{t-1} Y, dz) \right\|_p \to 0, \quad N \to \infty.
\]

Finally, due to the definition of \( \mu_t(\cdot, \cdot) \), for any fixed \( y \in \mathcal{Y}^Z \), the law of \( Z^{\varphi_n}_{0,t} \) equals to \( \mu_\gamma(S^{t-1} Y, \cdot) \) hence for the last term, we have

\[
\left\| \frac{1}{N} \sum_{t=1}^{N} \left( \Phi\left(Z^{\varphi_n}_{0,t}\right) - \int_x \Phi(z) \mu_\gamma(S^{t-1} Y, dz) \right) \right\|_p \leq \frac{2N-\lfloor N^{1/6} \rfloor}{N} \|\Phi\|_\infty + \mathbb{E}^{1/p} \left[ \mathbb{E} \left( \frac{1}{N} \sum_{t=1}^{N} \left( \Phi\left(Z^{\varphi_n}_{0,t}\right) - \int_x \Phi(z) \mu_\gamma(S^{t-1} Y, dz) \right)^p \right) \sigma(Y) \right]
\]

\[
\leq \frac{12}{N^{1/6}} \|\Phi\|_\infty + \mathbb{E}^{1/p} \left( \left\| \frac{1}{N} \sum_{t=1}^{N} W_t(Y) \right\|_p \right).
\]

According to Lemma 5.12 exists \( \tilde{c}(p, \tilde{\gamma}, \lambda) > 0 \) such that

\[
\mathbb{E}^{1/p} \left( \left\| \frac{1}{N} \sum_{t=1}^{N} W_t(Y) \right\|_p \right) \leq \tilde{c}(p, \tilde{\gamma}, \lambda) \|\Phi\|_\infty \left( N^{-1/4} + \max_{0 \leq k < \lfloor N^{1/6} \rfloor} \alpha(Y_k) \left( \lfloor N^{1/6} \rfloor - 1 \right)^{-1/2} \right)
\]

hence we obtain

\[
\left\| \frac{1}{N} \sum_{t=1}^{N} \left( \Phi\left(Z^{\varphi_n}_{0,t}\right) - \int_x \Phi(z) \mu_\gamma(S^{t-1} Y, dz) \right) \right\|_p \leq \frac{12}{N^{1/6}} \|\Phi\|_\infty + \tilde{c}(p, \tilde{\gamma}, \lambda) \|\Phi\|_\infty \left( N^{-1/4} + \max_{0 \leq k < \lfloor N^{1/6} \rfloor} \alpha(Y_k) \left( \lfloor N^{1/6} \rfloor - 1 \right)^{-1/2} \right),
\]

where by Lemma 5.7 the upper bound tends to zero as \( N \to \infty \).
To sum up,

\[ \left\| \frac{1}{N} \sum_{t=1}^{N} \Phi(X_t) - \int_{\mathcal{F}} \Phi(z) \mu_\phi(dz) \right\|_p \to 0, N \to \infty \]

because the laws of \( X_t \) and \( Z^\gamma_{\omega_t} \) coincide. This completes the proof of Theorem 2.9.

**Remark 5.13.** Birkhoff’s ergodic theorem does not provide upper bound for the difference between time and space averages hence we have convergence rate for every term in (23) except for the first one. However, in the ideal case this term is of the order \( 1/\sqrt{N} \) and this can be shown for \( Y \) with suitably favourable ergodic properties.

### 6 Appendix

For the reader’s convenience, we recall a concept of mixing defined in [8] which was used in some of the estimations above. Let \( \mathcal{G}_t, t \in \mathbb{N} \) be an increasing sequence of sigma-algebras and let \( \mathcal{G}^*_t, t \in \mathbb{N} \) be a decreasing sequence of sigma-algebras such that, for each \( t \in \mathbb{N}, \mathcal{G}_t \) is independent of \( \mathcal{G}^*_t \).

Let \( W_t, t \in \mathbb{N} \) be a real-valued stochastic process. For each \( p \geq 1 \), introduce

\[ M_p(W) := \sup_{t \in \mathbb{N}} E^{1/p}[|W_t|^p]. \]

For each process \( W \) such that \( M_1(W) < \infty \) define, for each \( p \geq 1 \),

\[ \gamma_p(W, \tau) := \sup_{t \geq \tau} E^{1/p}[|W_t - E[W_t|\mathcal{G}^*_t]|^p], \tau \in \mathbb{N}, \Gamma_p(W) := \sum_{\tau=0}^{\infty} \gamma_p(W, \tau). \]

For some \( p \geq 1 \), the process \( W \) is called \( L \)-mixing of order \( p \) with respect to \( (\mathcal{G}_t, \mathcal{G}^*_t), t \in \mathbb{N} \) if it is adapted to \( (\mathcal{G}_t)_{t \in \mathbb{N}} \) and \( M_p(W) < \infty, \Gamma_p(W) < \infty \). We say that \( W \) is \( L \)-mixing if it is \( L \)-mixing of order \( p \) for all \( p \geq 1 \).

We recall Lemma 2.1 of [8].

**Lemma 6.1.** Let \( \mathcal{G} \subset \mathcal{F} \) be a sigma-algebra, \( X, Y \) random variables with \( E^{1/p}[|X|^p] < \infty, E^{1/p}[|Y|^p] < \infty \) with some \( p \geq 1 \). If \( Y \) is \( \mathcal{G} \)-measurable then

\[ E^{1/p}[|X - E[X|\mathcal{G}]|^p] \leq 2E^{1/p}[|X - Y|^p] \]

holds. \( \Box \)

Finally, a trivial consequence of Theorem 1.1 of [8] is formulated.

**Lemma 6.2.** For an \( L \)-mixing process \( W \) of order \( p \geq 2 \) satisfying \( E[W_t] = 0, t \in \mathbb{N}, E^{1/p}\left[ \sum_{i=1}^{N} W_t^p \right] \leq C_p N^{1/2} M_p^{1/2}(W) \Gamma_p^{1/2}(W), \)

holds for each \( N \geq 1 \) with a constant \( C_p \) that does not depend either on \( N \) or on \( W \). \( \Box \)
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