Deviation from Bimaximal Mixing and Leptonic CP Phases in $S_4$ Family Symmetry and Generalized CP

Cai-Chang Li*, Gui-Jun Ding†

Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China

Abstract

The lepton flavor mixing matrix having one row or one column in common with the bimaximal mixing up to permutations is still compatible with the present neutrino oscillation data. We provide a thorough exploration of generating such a mixing matrix from $S_4$ family symmetry and generalized CP symmetry $H_{\text{CP}}$. Supposing that $S_4 \times H_{\text{CP}}$ is broken down to $Z_3^{ST2SU} \times H_{\text{CP}}^\nu$ in the neutrino sector and $Z_4^{TST2U} \times H_{\text{CP}}^l$ in the charged lepton sector, one column of the PMNS matrix would be of the form $(1/2, 1/\sqrt{2}, 1/2)^T$ or $(1/\sqrt{2}, 1/2, 1/2)^T$, and the Dirac CP is conserved or maximally broken while both Majorana CP phases are trivial. The phenomenological implications of the remnant symmetry $K^{(TST2,T^2U)}_4 \times H_{\text{CP}}^\nu$ in the neutrino sector and $Z_2^{SU} \times H_{\text{CP}}^l$ in the charged lepton sector are also studied. One row of PMNS matrix is determined to be $(1/2, 1/2, 1/\sqrt{2})^T$, and the Dirac CP phase is trivial or maximal as well. Two models based on $S_4$ family symmetry and generalized CP are constructed to implement these model independent predictions enforced by remnant symmetry. The correct mass hierarchy among the charged leptons is achieved. The vacuum alignment and higher order corrections are discussed.

*E-mail: lcc0915@mail.ustc.edu.cn
†E-mail: dingggj@ustc.edu.cn
1 Introduction

The neutrino flavor mixing and neutrino oscillation have been firmly established so far. The standard three flavor neutrino oscillation relates the flavor eigenstates of neutrinos to the mass eigenstates through the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix. This matrix is a $3 \times 3$ unitary matrix and can be parameterized by three mixing angles $\theta_{12}, \theta_{13}, \theta_{23}$, one Dirac type CP violating phase $\delta_{\text{CP}}$ similar to the quark sector and two additional Majorana phases $\alpha_{21}, \alpha_{31}$ if neutrinos are Majorana particles. Recently the last lepton mixing angle $\theta_{13}$ has been precisely measured to be about $9^\circ$ [1–5]. This discovery pushes neutrino oscillation experiments into a new era of precise determination of the lepton mixing angles and neutrino mass squared differences, and it also opens up new windows to probe leptonic CP violation. Although we still don’t have convincing evidence for lepton CP violation, the current global fit to the neutrino oscillation data indicates nontrivial values of the Dirac type CP phase [6–8]. The present T2K data already exclude values of $\delta_{\text{CP}}$ between $0.14\pi \sim 0.87\pi$ at the 90% confidence level [9,10]. Furthermore, several long-baseline neutrino oscillation experiments such as LBNE [11], LBNO [12–16] and Hyper-Kamiokande [17] are proposed to measure CP violation. Study of neutrino mixing including the CP violating phase would allow us to distinguish different flavor models.

In view of the fantastic experimental program of observing lepton CP violation and the fundamental role played by CP violation, it is crucial to be able to predict CP phases. The idea of combining flavor symmetry with generalized CP symmetry is a very interesting approach to predict both flavor mixing angles and CP phases from symmetry principle. The concept of generalized CP transformations has been put forward about thirty years ago. CP invariance at high energy scale and its subsequent breaking lead to nontrivial constraints on the fermion mass matrices [18,19]. It is somewhat tricky to include the generalized CP symmetry in the presence of a family symmetry. Generally the generalized CP transformation must be subject to the so-called consistency condition, which implies that the generalized CP transformation corresponds to an automorphism of the family symmetry group [20–22]. Furthermore, it is shown that that physical CP transformations always have to be class-inverting automorphisms of family symmetry group [23]. As a result, the conventional CP transformation $\varphi \mapsto \varphi^*$ can not be consistently defined, but rather a non-trivial transformation in flavor space is needed.

Generalized CP symmetry together with family symmetry can give us interesting phenomenological predictions. The simplest example is the so-called $\mu - \tau$ reflection symmetry which is a combination of the canonical CP transformation and the $\mu - \tau$ exchange symmetry. The invariance of the light neutrino mass matrix under $\mu - \tau$ reflection in the charged lepton diagonal basis leads to maximal atmospheric mixing angle $\theta_{23}$ and maximal Dirac CP phase $\delta_{\text{CP}}$ with $\delta_{\text{CP}} = \pm \pi/2$ [24,26]. The phenomenological implications of the generalized CP symmetry has been analyzed within the context of popular $A_4$ [27], $S_4$ [22,28,30] and $T'$ [31] family symmetries. By breaking the full symmetry down to $Z_2 \times CP$ in the neutrino sector, the TM$_1$ and TM$_2$ mixing patterns in which the first and the second columns of the tri-bimaximal mixing is kept respectively, can be exactly produced. The Dirac CP phase $\delta_{\text{CP}}$ is predicted to be conserved or maximally broken. Furthermore, the generalized CP has been extended to $\Delta(48)$ [32,33], $\Delta(96)$ [34] and $\Delta(6n^2)$ series [35] family symmetries as well. Some new mixing textures compatible with the experimental data are found, in particular CP phases can be neither vanishing nor maximal. A number of interesting models with definite predictions for CP phases have been constructed. There are other approaches to dealing with family symmetry and CP violation [36–38].

Besides the well-known tri-bimaximal mixing, the bimaximal (BM) mixing can also be naturally derived from the $S_4$ family symmetry [39,41]. Since $\theta_{13}$ is not so small as expected...
and $\theta_{12}$ is not maximal, the BM pattern has been ruled out. However, the scheme with only one row or one column of the BM mixing preserved is still viable. In the present work, we shall assume $S_4$ family symmetry and generalized CP symmetry which is then spontaneously broken down to $Z_2 \times CP$ in the neutrino sector or the charged lepton sector. As a consequence, only one column or one row of the BM mixing is preserved and the PMNS matrix deviates from BM pattern. Moreover, the concrete forms of the deviation from the BM mixing are constrained by the remnant symmetry, the corresponding predictions for the mixing angles and CP phases are investigated in a model independent way. Furthermore two models realizing these scenarios are built.

The paper is organized as follows. In section 2, we present the basic concept of generalized CP symmetry and model independent approach of predicting lepton flavor mixing from remnant symmetry. The deviation from BM mixing induced by a rotation between two generation neutrinos or a rotation between two generation charged lepton fields is investigated in section 3, and the corresponding phenomenological predictions for the lepton mixing parameters are discussed. In section 4, the model-independent implications of the symmetry breaking pattern of $S_4 \rtimes H_{CP}$ into $Z_2^{ST2SU} \times H_{CP}^\nu$ in the neutrino sector and $Z_2^{ST2U} \times H_{CP}^l$ in the charged lepton sector are analyzed. In this case, one column of the PMNS matrix is determined to be $(1/2, 1/\sqrt{2}, 1/2)^T$ or $(1/\sqrt{2}, 1/2, 1/2)^T$, and the Dirac CP phase is trivial or maximal. In section 5, the phenomenological implications of the remnant symmetry $K_{4}^{ST2,T^2U} \rtimes H_{CP}^\nu$ in the neutrino sector and $Z_2^{SU} \times H_{CP}^l$ in the charged lepton sector are studied. The resulting PMNS matrix has a row of form $(1/2, 1/2, 1/\sqrt{2})^T$, and the Dirac CP phase is also trivial or maximal. In section 6, we construct a $S_4$ model with generalized CP symmetry, where the symmetry breaking pattern of section 4 is implemented. The model reproducing all aspects of the general results of section 5 is presented in section 7. Section 8 is devoted to our conclusion. The group theory of $S_4$ and the Clebsch-Gordan coefficients in our basis are collected in Appendix A. Finally the scenario of residual symmetry $Z_2 \times H_{CP}^l$ in the neutrino sector and $K_{4} \rtimes H_{CP}^l$ in the charged lepton sector is discussed in Appendix B.

2 Basic framework

We now consider a theory which is invariant under both family symmetry $S_4$ and generalized CP at high energy scale. For a field multiplet $\varphi(x)$ in a irreducible representation $\mathbf{r}$ of $S_4$, it transforms under the action of $S_4$ as

$$\varphi(x) \xrightarrow{g} \rho_\mathbf{r}(g)\varphi(x), \quad g \in S_4,$$

where $\rho_\mathbf{r}(g)$ is the representation matrix for the element $g$ in the representation $\mathbf{r}$. The generalized CP transformation on $\varphi$ is defined as

$$\varphi(x) \xrightarrow{CP} X_\mathbf{r}\varphi^\ast(t, -x),$$

where $X_\mathbf{r}$ is the generalized CP transformation, and it is a unitary matrix. The obvious action of CP on the possible spinor indices has been suppressed. One subtle point that we should treat with care is that the family symmetry and the generalized CP must be compatible with each other. The following consistency condition has to be fulfilled [19–22],

$$X_\mathbf{r}\rho_\mathbf{r}(g)X_\mathbf{r}^{-1} = \rho_\mathbf{r}(g'), \quad g, g' \in S_4,$$

which maps one element $g$ into another element $g'$. For the faithful representation $\mathbf{r} = 3, 3'$, the representation matrices of no two elements are identical. As a consequence, the mapping
of $g \to g'$ is bijective, and then the consistency equation of Eq. (2.3) will define a unique mapping of the family symmetry group $S_4$ to itself. Hence the generalized CP transformation $X_r$ corresponds an automorphism of $S_4$.

There is a complete classification for the automorphism group of the symmetric group $S_n$, which is summarized in Table 1. We see that both center and outer automorphism group of $S_n$ are trivial except $n = 2, 6$. Therefore all the automorphisms of $S_4$ are inner automorphisms, and can be generated by group conjugation. Now we consider the representative automorphism $\sigma_{STST^2} : (S, T, U) \to (S, ST, SU)$, where $\sigma_h$ is defined as $\sigma_h : g \to hg\text{ }h^{-1}$ for any $h, g \in S_4$. The corresponding generalized CP transformation denoted by $X_r^0$ should satisfy the following consistency equations:

$$X_r^0\rho_r^*(S)\left(X_r^0\right)^{-1} = \rho_r(\sigma_{STST^2}(S)) = \rho_r(S),$$

$$X_r^0\rho_r^*(T)\left(X_r^0\right)^{-1} = \rho_r(\sigma_{STST^2}(T)) = \rho_r(ST),$$

$$X_r^0\rho_r^*(U)\left(X_r^0\right)^{-1} = \rho_r(\sigma_{STST^2}(U)) = \rho_r(SU).$$

(2.4)

Given the representation matrices listed in Table 10, we see that the following relations are satisfied for any irreducible representations $r$ of $S_4$,

$$\rho_r^*(S) = \rho_r(S), \quad \rho_r^*(T) = \rho_r(ST), \quad \rho_r^*(U) = \rho_r(SU).$$

(2.5)

Therefore $X_r^0$ is determined to be a unity matrix up to an overall phase,

$$X_r^0 = 1.$$  

(2.6)

For a given solution $X_r$ of Eq. (2.3), we can easily check that $\rho_r(h)X_r$ is also a solution for any $h \in S_4$. Since $\rho_r(h)X_r$ maps one group element $g$ into $hg\text{ }h^{-1} \equiv \sigma_h(g')$ \footnote{We have $(\rho_r(h)X_r)\rho_r^*(g)\left(\rho_r(h)X_r\right)^{-1} = \rho_r(h)(X_r\rho_r^*(g)X_r^{-1})\rho_r^{-1}(h) = \rho_r(h)\rho_r(g')\rho_r^{-1}(h) = \rho_r(hg'h^{-1})$.} the inner automorphism is equivalent to a family symmetry transformation. As a consequence, the generalized CP transformation compatible with the $S_4$ family symmetry is of the form

$$\rho_r(h)X_r^0 = \rho_r(h), \quad h \in S_4.$$  

(2.7)

In particular we see that the canonical CP transformation with $\rho_r(1) = X_r^0 = 1$ is allowed. Therefore all coupling constants would be constrained to be real in a $S_4$ model with imposed CP symmetry.

Being similar to the paradigm of family symmetry, the imposed symmetry is $S_4 \rtimes H_{CP}$ at high energy, which is broken down to different residual symmetry subgroups $G_{\nu} \rtimes H_{CP}^{\nu}$ and $G_l \rtimes H_{CP}^l$ in the neutrino and the charged lepton sectors respectively. The misalignment between $G_{\nu} \rtimes H_{CP}^{\nu}$ and $G_l \rtimes H_{CP}^l$ leads to particular predictions for mixing angles and CP phases. The basic procedure of predicting lepton flavor mixing from remnant symmetries
in a model independent way has been stated clearly in Refs. \[27, 33, 34\]. In the following, we briefly review the most important points which will be exploited later. Without loss of generality, the three generations of left-handed lepton doublets are assigned to be a $S_4$ triplet $\mathbf{3}$. The irreducible representation $\mathbf{3}^\prime$ is distinct from $\mathbf{3}$ in the overall sign of the generator $U$, therefore the same results are obtained if the lepton doublet fields are embedded into $\mathbf{3}^\prime$ instead of $\mathbf{3}$. Firstly, invariance under the residual symmetries $G_l$ and $G_\nu$ implies

\[
\rho_3^T(g_l) m_3^\dagger m_3 \rho_3(g_l) = m_\nu^\dagger m_\nu, \quad g_l \in G_l \\
\rho_3^T(g_\nu) m_\nu \rho_3(g_\nu) = m_\nu, \quad g_\nu \in G_\nu, \tag{2.8}
\]

where the charged lepton mass matrix $m_\nu$ is given in the convention in which the right-handed (left-handed) fields are on the right-hand (left-hand) side of $m_\nu$. Furthermore, the neutrino and the charged lepton mass matrices are subject to the constraint of residual CP symmetry as follows,

\[
X^T_{\nu3} m_\nu X_{\nu3} = m_\nu^*, \quad X_{\nu3} \in H^\nu_{CP}, \\
X_{l3}^\dagger m_\nu \rho_3 m_\nu^\dagger X_{l3} = \left(m_\nu^\dagger m_\nu\right)^*, \quad X_{l3} \in H^l_{CP}. \tag{2.9}
\]

The remnant family symmetry should be consistent with remnant CP symmetry, and hence the following consistency equations should be fulfilled,

\[
X_{\nu3} \rho_3^T(g_\nu) X_{\nu}^{-1} = \rho_3(g_\nu), \quad g_\nu, g_\nu' \in G_\nu, \\
X_{l3} \rho_3^T(g_l) X_{l3}^{-1} = \rho_3(g_l), \quad g_l, g_l' \in G_l. \tag{2.10}
\]

We can obtain the most general form of $m_\nu$ and $m_\nu^\dagger m_\nu$ from the invariant requirements of Eq. (2.8) and Eq. (2.9), then diagonalize them, and finally we can determine the lepton mixing matrix $U_{PMNS}$. Last but not least, generally we have many possible choices for the residual symmetry subgroups. However, if the residual family symmetries are taken to be another pair of subgroups $G'_\nu$ and $G'_l$ which are conjugate to $G_\nu$ and $G_l$,

\[
G'_\nu = h G_\nu h^{-1}, \quad G'_l = h G_l h^{-1}, \quad h \in S_4. \tag{2.11}
\]

The lepton mixing matrix would be predicted to be of the same form as that in $G_\nu$, $G_l$ case \[27, 33, 34\]. As a result, we only need to analyze few independent residual family symmetries not related by group conjugation and the compatible remnant CP. We assume that the light neutrinos are Majorana particles, and hence the remnant family symmetry $G_\nu$ in the neutrino sector must be $K_4$ or $Z_2$ subgroups. The case that $S_4 \times H_{CP}$ is broken down to $Z_2 \times H^\nu_{CP}$ in the neutrino sector and $Z_3 \times H^l_{CP}$ in the charged lepton sector has been comprehensively studied \[22, 28\]. One column of the PMNS matrix is then determined to be proportional to $(2, -1, -1)^T$ or $(1, 1, 1)^T$, i.e. the so-called TM$_1$ and TM$_2$ mixing patterns can be produced exactly. Besides the $Z_3$ subgroup, the residual family symmetry $G_l$ in the charged lepton sector can be $Z_4$ or $K_4$ subgroups of $S_4$ \footnote{Choosing $G_l$ to be a non-abelian subgroup would lead to a degenerate mass spectrum.}. For example, the choice $G_l = Z^T_{4ST} (\text{or } G_l = K_4^{(S,U)})$ and $G_\nu = K_4^{(TST^2,T^2U)}$ leads to BM mixing no matter whether the generalized CP is included or not. In order to be in accordance with experimental data, we degrade $G_\nu$ from $K_4$ to $Z_2$ or $G_l$ from $Z_4(K_4)$ to $Z_2$ such that only one column or one row of the BM mixing matrix is fixed. After the generalized CP transformation defined in Eq. (2.7) is taken into account further, the resulting lepton mixing matrix $U_{PMNS}$ is found to depend on only one free real parameter. In the following, we shall firstly investigate the phenomenological predictions of preserving one column or one row of BM mixing, which may
originate from a $2 \times 2$ rotation in the neutrino or the charged lepton sector. Furthermore, the $S_4$ family symmetry together with the generalized CP is imposed onto the theory, and then lepton flavor mixing arising from the symmetry breaking into different residual subgroups in the neutrino and the charged lepton sectors are discussed in section 4 and section 5. We find that the PMNS matrix has one column or one row in common with BM mixing up to permutations, and moreover the CP phases are predicted to take definite values because of the constraint of generalized CP symmetry.

3 Deviation from bimaximal mixing

In a particular phase convention, the BM mixing matrix $U_{BM}$ is of the following form \[42\]

\[
U_{BM} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}}
\end{pmatrix},
\]

which leads to the three lepton mixing angles

\[
\theta_{12}^{BM} = \theta_{23}^{BM} = 45^\circ, \quad \theta_{13}^{BM} = 0^\circ.
\]

Comparing with the latest global fitting results \[6–8\], we see that rather large corrections are needed to be compatible with the experimental data. In the following, we shall consider the minimal modifications for simplicity. The additional rotation of the 1-2, 1-3 or 2-3 generation of charged leptons or neutrinos in the BM basis would be considered. As a consequence, one column or one row of BM mixing would be retained. Similar deviation from tri-bimaximal mixing has been widely studied \[43–47\]. First of all, we discuss the case of a extra 1-2 rotation in the charged lepton sector. The PMNS mixing matrix is obtained by multiplying the BM matrix $U_{BM}$ by a 1-2 rotation matrix in the left-hand side as follows:

\[
U_{PMNS} = \begin{pmatrix}
\cos \theta & -\sin \theta e^{-i\delta} & 0 \\
\sin \theta e^{i\delta} & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix} U_{BM} = \frac{1}{2} \begin{pmatrix}
\sqrt{2} \cos \theta - \sin \theta e^{-i\delta} & -\sqrt{2} \cos \theta - \sin \theta e^{-i\delta} & \sqrt{2} \sin \theta e^{-i\delta} \\
\cos \theta + \sqrt{2} \sin \theta e^{i\delta} & \cos \theta - \sqrt{2} \sin \theta e^{i\delta} & -\sqrt{2} \cos \theta \\
1 & 1 & \sqrt{2}
\end{pmatrix},
\]

where $\theta$ and $\delta$ are real free parameters, and their values can be fitted by the experimental data. Then the three mixing angles read as

\[
\sin^2 \theta_{13} = \frac{1}{2} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1}{2} + \frac{\sqrt{2} \sin 2\theta \cos \delta}{3 + \cos 2\theta}, \quad \sin^2 \theta_{23} = 1 - \frac{2}{3 + \cos 2\theta}.
\]

We see that the atmospheric and reactor mixing angles are related with each other by

\[
\sin^2 \theta_{23} = \frac{1}{2} - \frac{1}{2} \tan^2 \theta_{13}.
\]

Hence $\theta_{23}$ is constrained to lie in the first octant, i.e. $\theta_{23} < \frac{\pi}{4}$. The Jarlskog invariant $J_{CP}$ is given by

\[
J_{CP} = \frac{\sin 2\theta \sin \delta}{8\sqrt{2}}.
\]
Then the Dirac CP phase $\delta_{CP}$ in the standard parameterization \[48\] is
\[
\sin \delta_{CP} = \frac{\sin 2\theta \sin \delta}{|\sin 2\theta| \sin 2\theta_{12}} = \frac{(3 + \cos 2\theta) \sin 2\theta \sin \delta}{|\sin 2\theta| \sqrt{(3 + \cos 2\theta)^2 - 8 \sin^2 2\theta \cos^2 \delta}}.
\]

For the value of $\delta = 0$, the above mixing parameters are simplified into
\[
\sin^2 \theta_{13} = \frac{1}{2} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1}{2} + \frac{\sqrt{2} \sin 2\theta}{3 + \cos 2\theta}, \quad \sin^2 \theta_{23} = \frac{2 \cos^2 \theta}{3 + \cos 2\theta}, \quad \sin \delta_{CP} = 0, \quad (3.7)
\]
where CP is preserved. In the case of $\delta = \pi/2$, we have
\[
\sin^2 \theta_{13} = \frac{1}{2} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1}{2}, \quad \sin^2 \theta_{23} = \frac{2 \cos^2 \theta}{3 + \cos 2\theta}, \quad \sin \delta_{CP} = \pm 1, \quad (3.8)
\]
where CP is maximally violated. Since the rotation of 2-3 generation of charged leptons gives a vanishing $\theta_{13}$, we turn to investigate an additional rotation of 1-3 generations. We can obtain the PMNS mixing matrix by multiplying the BM matrix by a 1-3 rotation matrix in the left-hand side as
\[
U_{PMNS} = \begin{pmatrix}
\cos \theta & 0 & -\sin \theta e^{-i\delta} \\
0 & 1 & 0 \\
\sin \theta e^{i\delta} & 0 & \cos \theta
\end{pmatrix} U_{BM}
\]
\[
= \frac{1}{2} \begin{pmatrix}
\sqrt{2} \cos \theta & -\sin \theta e^{-i\delta} & -\sqrt{2} \cos \theta - \sin \theta e^{-i\delta} \\
\cos \theta + \sqrt{2} \sin \theta e^{i\delta} & 1 & \cos \theta - \sqrt{2} \sin \theta e^{i\delta} \\
\cos \theta & 0 & \sqrt{2} \cos \theta
\end{pmatrix}. \quad (3.9)
\]

We can straightforwardly extract the lepton mixing angles as follows
\[
\sin^2 \theta_{13} = \frac{1}{2} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1}{2} + \frac{\sqrt{2} \sin 2\theta \cos \delta}{3 + \cos 2\theta}, \quad \sin^2 \theta_{23} = \frac{2 \cos^2 \theta}{3 + \cos 2\theta}, \quad (3.10)
\]
which leads to the relation between the atmospheric and reactor mixing angles,
\[
\sin^2 \theta_{23} = \frac{1}{2} + \frac{1}{2} \tan^2 \theta_{13}. \quad (3.11)
\]
which implies $\theta_{23} > \pi/4$ and $\theta_{23}$ is in the second octant. The Jarlskog invariant reads as
\[
J_{CP} = -\frac{\sin 2\theta \sin \delta}{8\sqrt{2}}, \quad (3.12)
\]
and then the Dirac CP phase is given by
\[
\sin \delta_{CP} = -\frac{(3 + \cos 2\theta) \sin 2\theta \sin \delta}{|\sin 2\theta| \sqrt{(3 + \cos 2\theta)^2 - 8 \sin^2 2\theta \cos^2 \delta}}. \quad (3.13)
\]
We perform numerical analysis by scanning the free parameters $\theta$ and $\delta$ in the regions of $-\pi < \theta \leq \pi$ and $-\pi < \delta \leq \pi$. The correlations and the possible allowed values of the mixing parameters are obtained, as shown in Fig. 1. We see that there is a strong correlation between $\sin^2 \theta_{23}$ and $\sin \theta_{13}$, which is given in Eq. (3.10) and Eq. (3.11). Note that the allowed regions of the mixing parameters are rather large although only two free parameters $\theta$ and $\delta$ are involved. Furthermore, we take into account the current bounds for three neutrino mixing angles presented in Ref. [7], then the values of the mixing parameters would shrink to quite small areas. It is remarkable that the Dirac CP phase $\delta_{CP}$ is constrained to be in
Figure 1: Correlations among mixing angles ($\sin \theta_{13}, \sin^2 \theta_{12}, \sin^2 \theta_{23}$) and CP parameters ($J_{CP}, \delta_{CP}$) for additional rotations of 1-2 and 1-3 generation of charged leptons in the BM basis. In the first panel, the results of $\sin^2 \theta_{23}$ vs. $\sin \theta_{13}$ for 1-2 and 1-3 rotations are shown in solid line and dashed line respectively. The pink regions in the last three subfigures are the predictions for $\sin^2 \theta_{12}$, $J_{CP}$ and $\delta_{CP}$ with respect to $\sin \theta_{13}$ if both $\theta$ and $\delta$ vary in the range of $-\pi$ to $\pi$. The black areas in the fourth panel denote the allowed region by the experimental data of three mixing angles for 1-2 rotation and the blue areas for 1-3 rotation. In the second and third subfigures, the allowed regions for 1-2 and 1-3 rotations coincide. The red stars represent the best fit values in $S_4$ family symmetry combined with generalized CP.

The range of $\pm [2.5, \pi]$ and $[-0.6, 0.6]$ for 1-2 and 1-3 rotations respectively. For comparison, the theoretical predictions of the generalized CP symmetry discussed in section 5 are also shown in Fig. 1.

Then we study the deviation from BM mixing induced by a rotation in the neutrino sector. Since the rotation of 1-2 generations leads to $\theta_{13} = 0$, we do not discuss this scenario. Firstly we consider the case that the neutrino mass matrix is rotated between 1-3 generations in the BM basis. The PMNS matrix is obtained by multiplying the BM matrix $U_{BM}$ by a 1-3 rotation matrix in the right-hand side as follows:

$$U_{PMNS} = U_{BM} \begin{pmatrix} \cos \theta & 0 & \sin \theta e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin \theta e^{i\delta} & 0 & \cos \theta \end{pmatrix}$$
which leads to the following lepton mixing angles:

\[
\sin^2 \theta_{13} = \frac{1}{2} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{2}{3 + \cos 2\theta}, \quad \sin^2 \theta_{23} = \frac{1}{2} - \frac{\sqrt{2} \sin 2\theta \cos \delta}{3 + \cos 2\theta}.
\]  

(3.15)

Therefore \(\theta_{12}\) is constrained to satisfy \(\sin^2 \theta_{12} \geq \frac{1}{2}\) which is obviously not compatible with the experimental data [6–8]. Next we consider the rotation of 2-3 generation of neutrinos. The PMNS matrix is given by

\[
\begin{pmatrix}
\begin{pmatrix}
\sqrt{2} \cos \theta & -\sqrt{2} \\
\cos \theta + \sqrt{2} \sin \theta e^{i\delta} & 1 - \sqrt{2} \cos \theta + \sin \theta e^{-i\delta}
\end{pmatrix} & \begin{pmatrix}
\sqrt{2} \sin \theta e^{-i\delta} \\
\cos \theta - \sqrt{2} \sin \theta e^{i\delta}
\end{pmatrix}
\end{pmatrix}
\]

(3.14)

The three mixing angles are expressed in terms of \(\theta\) and \(\delta\) as

\[
\sin^2 \theta_{13} = \frac{1}{2} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1 + \cos 2\theta}{3 + \cos 2\theta}, \quad \sin^2 \theta_{23} = \frac{1}{2} - \frac{\sqrt{2} \sin 2\theta \cos \delta}{3 + \cos 2\theta}.
\]  

(3.17)

For the measured value of \(1.77 \times 10^{-2} \leq \sin^2 \theta_{13} \leq 2.97 \times 10^{-2}\), we obtain \(0.485 \leq \sin^2 \theta_{12} \leq 0.491\) which is outside of the experimentally preferred 3\(\sigma\) range [6–8]. Consequently this case doesn’t agree with the experimental data as well. In short summary, simple perturbative rotation to the BM mixing in the neutrino sector is not viable, the observed values of \(\theta_{12}\) and \(\theta_{13}\) can not be produced simultaneously. It is notable that agreement with the experimental data can be achieved if permutations of rows and columns are allowed. There are altogether six possible permutations represented by permutation matrices as follows

\[
\begin{align*}
P_{123} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & P_{132} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & P_{213} &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
P_{231} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & P_{312} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & P_{321} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{align*}
\]  

(3.18)

If we perform both 2-3 rotation of neutrino and exchanges of rows and columns, the following PMNS matrix can be obtained

\[
\begin{pmatrix}
\begin{pmatrix}
\sqrt{2} \cos \theta + \sin \theta e^{-i\delta} & 1 \\
-\sqrt{2} \sin \theta e^{-i\delta} & \sqrt{2} \cos \theta - \sqrt{2} \sin \theta e^{i\delta}
\end{pmatrix} & \begin{pmatrix}
1 & 0 \\
-\sqrt{2} \sin \theta e^{-i\delta} & 1 \cos \theta + \sqrt{2} \sin \theta e^{i\delta}
\end{pmatrix}
\end{pmatrix}
\]

(3.19)
The three mixing angles read as

\begin{align}
\sin^2 \theta_{13} &= \frac{1}{8} (3 - \cos 2\theta - 2\sqrt{2} \sin \theta \cos \delta), \\
\sin^2 \theta_{12} &= \frac{2}{5 + \cos 2\theta + 2\sqrt{2} \sin \theta \cos \delta}, \\
\sin^2 \theta_{23} &= \frac{2 + 2 \cos \theta}{5 + \cos 2\theta + 2\sqrt{2} \sin \theta \cos \delta}.
\end{align}

The following relation is found

\begin{equation}
4 \sin^2 \theta_{12} \cos^2 \theta_{13} = 1.
\end{equation}

\footnote{The permutation of rows and columns of the PMNS matrix corresponds to exchange of charged lepton masses and neutrino masses respectively.}
The Jarlskog invariant $J_{CP}$ is given by

$$J_{CP} = -\frac{\sin 2\theta \sin \delta}{8\sqrt{2}},$$

(3.22)

The Dirac CP phase $\delta_{CP}$ is determined to be

$$\sin \delta_{CP} = -\frac{(5 + \cos 2\theta + 2\sqrt{2} \sin 2\theta \cos \delta) \sin 2\theta \sin \delta}{|\cos \theta| \sqrt{2} (3 + \cos 2\theta + 2\sqrt{2} \sin 2\theta \cos \delta) \left[(3 - \cos 2\theta)^2 - 8 \sin^2 2\theta \cos^2 \delta\right]}$$

(3.23)

Similar to perturbative rotation from the charged lepton sector discussed above, the numerical results are presented in Fig. 2. For reactor mixing angle $\theta_{13}$ in the $3\sigma$ region from global fits [6–8], $\theta_{23}$ is constrained to be smaller than 45$^\circ$, and $\delta_{CP}$ is in the range of $\pm [2, \pi]$. The case of $\theta_{23}$ in the second octant can be accounted for by exchanging the second and the third rows in Eq. (3.19). The resulting PMNS matrix is of the form:

$$U_{PMNS} = P_{321}U_{BM} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & e^{-i\delta} \sin \theta \\ 0 & -e^{i\delta} \sin \theta & \cos \theta \end{pmatrix} P_{231}$$

$$= \frac{1}{2} \begin{pmatrix} \sqrt{2} \cos \theta + \sin \theta e^{-i\delta} & 1 & \cos \theta - \sqrt{2} \sin \theta e^{i\delta} \\ -\sqrt{2} \cos \theta + \sin \theta e^{-i\delta} & 1 & \cos \theta + \sqrt{2} \sin \theta e^{i\delta} \\ -\sqrt{2} \sin \theta e^{-i\delta} & \sqrt{2} & -\sqrt{2} \cos \theta \end{pmatrix},$$

(3.24)

The predictions for $\theta_{12}$ and $\theta_{13}$ are the same as those in Eq. (3.20), while $\theta_{23}$ becomes its complementary angle, i.e.

$$\sin^2 \theta_{13} = \frac{1}{8} (3 - \cos 2\theta - 2\sqrt{2} \sin 2\theta \cos \delta),$$
$$\sin^2 \theta_{12} = \frac{2}{5 + \cos 2\theta + 2\sqrt{2} \sin 2\theta \cos \delta},$$
$$\sin^2 \theta_{23} = \frac{3 - \cos 2\theta + 2\sqrt{2} \sin 2\theta \cos \delta}{5 + \cos 2\theta + 2\sqrt{2} \sin 2\theta \cos \delta}. \quad (3.25)$$

Compared with the one in Eq. (3.23), the Dirac CP phase $\delta_{CP}$ becomes its supplementary angle,

$$\sin \delta_{CP} = -\frac{(5 + \cos 2\theta + 2\sqrt{2} \sin 2\theta \cos \delta) \sin 2\theta \sin \delta}{|\cos \theta| \sqrt{2} (3 + \cos 2\theta + 2\sqrt{2} \sin 2\theta \cos \delta) \left[(3 - \cos 2\theta)^2 - 8 \sin^2 2\theta \cos^2 \delta\right]}.$$ 

(3.26)

It is straightforward to get numerical results for this case, as shown in Fig. 3 $\delta_{CP}$ is constrained to be in the range $[-1.1, 1.1]$. Note that the above predictions can be tested if the three lepton mixing angles are measured more precisely by future experiments. In the following, we shall show a lepton mixing matrix with one column or one row in common with BM mixing can be achieved from $S_4$ family symmetry, and $\delta_{CP}$ is predicted to take specific values $0$, $\pi$ or $\pm \pi/2$ after generalized CP is imposed.

### 4 Lepton flavor mixing from remnant symmetry $Z_2^{ST^2SU} \times H_{CP}^\nu$ in the neutrino sector and $Z_4^{TST^2U} \times H_{CP}^l$ in the charged lepton sector

The full symmetry $S_4 \times H_{CP}$ is broken to $Z_4^{TST^2U} \times H_{CP}^l$ in charged lepton sector. The remnant CP symmetry $H_{CP}^l$ should be compatible with the remnant family symmetry $Z_4^{TST^2U}$.
Therefore the consistency equation similar to Eq. (2.3) must be fulfilled as follows,

\[ X_{ir} \rho_r^t(TST^2U)X_{ir}^{-1} = \rho_r(g'), \quad g' \in Z_{4}^{TST^2U}. \] (4.1)

It is straightforward to check that only 8 choices out of the 24 CP transformations in Eq. (2.7) are acceptable

\[ H_{CP}^{l} = \{ \rho_r(1), \rho_r(S), \rho_r(U), \rho_r(SU), \rho_r(TST^2), \rho_r(T^2ST), \rho_r(TST^2U), \rho_r(T^2STU) \}. \] (4.2)

As already explained in above, the remnant family symmetry \( Z_{4}^{TST^2U} \) imposes the following constraints on the charged lepton mass matrix,

\[ \rho_3^t(TST^2U)m_i^tm_i \rho_3(TST^2U) = m_i^tm_i, \] (4.3)

From Appendix A we find the representation matrix \( \rho_3(TST^2U) = \text{diag}(i, 1, -i) \) is diagonal. Therefore the hermitian combination \( m_i^tm_i \) is a diagonal matrix, i.e.

\[ m_i^tm_i = \text{diag}(m_e^2, m_\mu^2, m_\tau^2), \] (4.4)

where \( m_e, m_\mu \) and \( m_\tau \) represent the electron, muon and tau masses respectively. Hence lepton flavor mixing totally arises from the neutrino sector in this case. The invariance under the action of residual CP symmetry \( H_{CP}^{l} \) implies

\[ X_{i3}^\dagger m_i^tm_i X_{i3} = \left(m_i^tm_i\right)^* . \] (4.5)

For the case of \( X_{ir} = \rho_r(1), \rho_r(TST^2U), \rho_r(S), \rho_r(T^2STU) \), Eq. (4.5) is automatically satisfied, and no new constraint is generated. For the remaining values \( X_{ir} = \rho_r(U), \rho_r(TST^2), \rho_r(SU), \rho_r(T^2ST) \), the residual CP invariant condition of Eq. (4.5) leads to the constraint \( m_e = m_\tau \) which is not consistent with the measured different masses of electron and tau.

Now we turn to the neutrino sector. The present 3\( \sigma \) confidence level ranges of the magnitude of the elements of the leptonic mixing matrix are fitted to be [6]:

\[ ||U_{PMNS}||_{3\sigma} = \begin{pmatrix} 0.801 \rightarrow 0.845 & 0.514 \rightarrow 0.580 & 0.137 \rightarrow 0.158 \\ 0.225 \rightarrow 0.517 & 0.441 \rightarrow 0.699 & 0.614 \rightarrow 0.793 \\ 0.246 \rightarrow 0.529 & 0.464 \rightarrow 0.713 & 0.590 \rightarrow 0.776 \end{pmatrix} \] (4.6)
If the $S_4$ family symmetry is broken to $G_\nu = K_4^{(TST^2,T^2U)} = \{1, TST^2, T^2U, ST^2SU\}$, the lepton mixing matrix is of BM form which is ruled out by the measured values $\theta_{12}$ and $\theta_{13}$. Inspired by our previous work on family symmetry and generalized CP \cite{22,27,28,32–34}, we break $S_4$ to $Z_2$ subgroup with remnant CP symmetry in the neutrino sector, and then only one column of the PMNS matrix would be fixed. For the case of $G_\nu = Z_2^{ST^2}$, the resulting PMNS matrix has one column of the form $(0, -1/\sqrt{2}, 1/\sqrt{2})^T$. It is obviously not compatible with the data at $3\sigma$ level, since no element of the PMNS matrix could be zero as shown in Eq. (4.6). As a consequence, we choose $G_\nu = Z_2^{ST^2SU}$ which enforces one column of the mixing matrix being $(1/\sqrt{2}, 1/2, 1/2)^T$.\footnote{For $G_\nu = Z_2^{T^2U}$, one column of the PMNS matrix is $(-1/\sqrt{2}, 1/2, 1/2)^T$, where the minus sign can be removed by the rephasing freedom of the charged lepton fields. In addition, we note $SZ_2^{T^2U}S^{-1} = Z_2^{ST^2SU}$ and $SZ_4^{ST^2U}S^{-1} = Z_4^{ST^2U}$. Hence the lepton mixing matrix would be predicted to be of the same form as the $G_\nu = Z_2^{ST^2SU}$ case.} The remnant CP symmetry $H_{CP}^\nu$ has to be consistent with the remnant family symmetry $Z_2^{ST^2SU}$. That is to say, its element $X_{\nu\tau}$ should satisfy the consistency equation

$$X_{\nu\rho}(ST^2SU)X_{\nu\tau}^{-1} = \rho_{\nu}(ST^2SU).$$ (4.7)

One can easily check that only 4 CP transformations are admissible,

$$H_{CP}^\nu = \{\rho_{\nu}(1), \rho_{\nu}(ST^2SU), \rho_{\nu}(TST^2), \rho_{\nu}(T^2U)\}.$$ (4.8)

Invariance of the light neutrino mass matrix under both residual family symmetry $Z_2^{ST^2SU}$ and residual CP symmetry $H_{CP}^\nu$ leads to

$$m_{\nu} = \alpha \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \epsilon \begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix},$$ (4.11)

where the four free parameters $\alpha, \beta, \gamma$ and $\epsilon$ are generally complex, and they are further constrained by the remnant CP invariant condition of Eq. (4.9) to be real or purely imaginary. This neutrino mass matrix $m_{\nu}$ is essentially simplified into a $2 \times 2$ matrix, after performing a unitary transformation $U_{BMR}$ with

$$U_{BMR} = P_{312}U_{BM} = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \\ 1 & -1 \end{pmatrix},$$ (4.12)

where $U_{BM}$ is the BM mixing matrix given in Eq. (3.1). In other words, we have

$$m_{\nu} = U_{BM}^Tm_{\nu}U_{BM} = \begin{pmatrix} \alpha - 2\beta + \sqrt{2}\gamma & 0 & 0 \\ 0 & \alpha - 2\beta - \sqrt{2}\gamma & 2\epsilon \\ 0 & 2\epsilon & -\alpha - 4\beta \end{pmatrix}.$$ (4.13)

In the following, we continue to study the remaining constraint of Eq. (4.8) which is the invariant condition of the neutrino mass matrix under the residual CP symmetry. The four possible $X_{\nu\tau}$ shown in Eq. (4.8) lead to two different phenomenological predictions.
(I) $X_{\nu} = \rho_r(1)$, $X_{\nu} = \rho_r(ST^2SU)$

In this case, the residual CP invariant requirement of Eq. (4.10) implies that all the four parameters $\alpha$, $\beta$, $\gamma$ and $\epsilon$ are real. Hence the light neutrino mass matrix $m'_{\nu}$ is a real symmetric matrix and it can be diagonalized by a simple unitary matrix as follows

$$U'_{\nu}^T m'_{\nu} U'_{\nu} = \text{diag}(m_1, m_2, m_3), \quad \text{with} \quad U'_{\nu} = R(\theta) P_{231} K_{\nu}, \quad (4.14)$$

where $K_{\nu}$ is a unitary diagonal matrix with entries $\pm 1$ or $\pm i$, which renders the light neutrino masses $m_{1,2,3}$ positive. The rotation matrix $R(\theta)$ is of the form

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}, \quad (4.15)$$

with the rotation angle $\theta$ given by

$$\tan 2\theta = \frac{-4\epsilon}{2\alpha + 2\beta - \sqrt{2}\gamma}. \quad (4.16)$$

The light neutrino masses are

$$m_1 = \frac{1}{2} \left| 6\beta + \sqrt{2}\gamma + \text{sign}((2\alpha + 2\beta - \sqrt{2}\gamma) \cos 2\theta) \sqrt{16\epsilon^2 + (2\alpha + 2\beta - \sqrt{2}\gamma)^2} \right|,$$
$$m_2 = |\alpha - 2\beta + \sqrt{2}\gamma|,$$
$$m_3 = \frac{1}{2} \left| 6\beta - \sqrt{2}\gamma - \text{sign}((2\alpha + 2\beta - \sqrt{2}\gamma) \cos 2\theta) \sqrt{16\epsilon^2 + (2\alpha + 2\beta + \sqrt{2}\gamma)^2} \right|. \quad (4.17)$$

We see that the three neutrino masses are determined by four real parameters. As a consequence, the measured mass squared differences $\delta m^2 \equiv m_3^2 - m_2^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$ can be easily accommodated, and the neutrino mass spectrum can be either normal ordering (NO) or inverted ordering (IO). Since the light neutrino masses are less constrained in this approach, the unitary matrix $U'_{\nu}$ is pinned down up to permutation of its columns. In other words, a permutation matrix shown in Eq. (3.18) can be multiplied in the right hand side of $U'_{\nu}$.

In this case we choose $P_{231}$ to achieve better agreement with experimental data. Recalling that the charged lepton matrix is diagonal, we obtain that the lepton mixing matrix $U_{PMNS}$ is predicted to be of the form

$$U_{PMNS} = U_{BMR} U'_{\nu} = \frac{1}{2} \begin{pmatrix} \sin \theta + \sqrt{2} \cos \theta & 1 & \cos \theta - \sqrt{2} \sin \theta \\ -\sqrt{2} \sin \theta & \sqrt{2} & -\sqrt{2} \cos \theta \\ \sin \theta - \sqrt{2} \cos \theta & 1 & \cos \theta + \sqrt{2} \sin \theta \end{pmatrix} K_{\nu}, \quad (4.18)$$

up to row and column permutations. The lepton mixing angles and CP phases can be read out as

$$\sin^2 \theta_{13} = \frac{1}{8} (3 - \cos 2\theta - 2\sqrt{2} \sin 2\theta), \quad \sin^2 \theta_{12} = \frac{2}{5 + \cos 2\theta + 2\sqrt{2} \sin 2\theta},$$
$$\sin^2 \theta_{23} = \frac{4 \cos^2 \theta}{5 + \cos 2\theta + 2\sqrt{2} \sin 2\theta}, \quad \sin \delta_{CP} = \sin \alpha_{21} = \sin \alpha_{31} = 0, \quad (4.19)$$

where the PDG convention for the lepton mixing angles and CP phases is adopted \cite{48}, $\delta_{CP}$ is the Dirac CP phase, $\alpha_{21}$ and $\alpha_{31}$ stand for the Majorana CP phases. We see that CP is

\footnote{Note that the equality $K_{\nu} P_{ijk} = P_{ijk} K'_{\nu}$ is fulfilled, where $K'_{\nu} = P_{ijk}^{-1} K_{\nu} P_{ijk}$ is also a diagonal phase matrix, and it can be obtained by reordering the non-zero elements of $K_{\nu}$.}
\[ \sin^2 \theta_{13} \frac{1}{8} (3 - \cos 2\theta - 2\sqrt{2} \sin 2\theta) \]
\[ \sin^2 \theta_{12} \frac{2}{5 + \cos 2\theta + 2\sqrt{2} \sin 2\theta} \]
\[ \sin^2 \theta_{23}(\theta_{23} < \pi/4) \frac{4 \cos^2 \theta}{5 + \cos 2\theta + 2\sqrt{2} \sin 2\theta} \]
\[ \sin^2 \theta_{23}(\theta_{23} > \pi/4) \frac{3 - \cos 2\theta + 2\sqrt{2} \sin 2\theta}{5 + \cos 2\theta + 2\sqrt{2} \sin 2\theta} \]
\[ \theta_{bf} \]
\[ \chi^2_{\text{min}} \]

Table 2: The predictions for the lepton mixing angles and their best fitting values in case I. The number before (in) the parenthesis denotes the best fitting value for \( \theta_{23} < \pi/4 \) (\( \theta_{23} > \pi/4 \)). The 1\( \sigma \) and 3\( \sigma \) bounds for the mixing angles are taken from Ref. [7].

conserved in this case since the neutrino mass matrix is real. In contrast with the general phenomenological analysis of section 3, the CP phases are predicted to be of definite value 0 of \( \pi \) due to the generalized CP symmetry. The three mixing angles \( \theta_{12}, \theta_{13} \) and \( \theta_{23} \) depend on single parameter \( \theta \), hence they are strongly correlated with each other

\[ 4 \cos^2 \theta_{13} \sin^2 \theta_{12} = 1, \quad \sin^2 \theta_{23} = \frac{1}{3} + \frac{\tan \theta_{13}}{9} \left( \tan \theta_{13} \pm 2\sqrt{6 - 2 \tan^2 \theta_{13}} \right). \quad (4.20) \]

The correlations between different mixing angles are displayed in Fig. 4 and Fig. 5. For the measured value of the reactor angle \( \sin^2 \theta_{13} = 0.0234 \) [7], we have \( \sin^2 \theta_{12} \simeq 0.256 \) and \( \sin^2 \theta_{23} \simeq 0.420 \) (or \( \sin^2 \theta_{23} \simeq 0.252 \)), which are in accordance with the experimentally preferred regions. In order to see quantitatively to which extent this mixing pattern can account for the present experimental data, we perform a conventional \( \chi^2 \) analysis, as shown in Table 2. The minimum of the \( \chi^2 \) is \( \chi^2_{\text{min}} = 9.865 \) for NO and 10.454 for IO. Hence excellent agreement with experimental data can be achieved. Furthermore, the best fitting of value of \( \theta_{23} \) is \( \sin^2 \theta_{23}(\theta_{bf}) \simeq 0.42 \). As a result, \( \theta_{23} \) in the first octant is favored. This point can also be seen clearly from Fig. 5. The present neutrino oscillation data can not tell us whether \( \theta_{23} \) is larger than or smaller than 45°. As the case stands, the scenario of \( \theta_{23} \) in the second octant can also be accommodated by this mixing texture by interchanging its second and third rows,

\[ U'_{PMNS} = P_{132} U_{PMNS} = \frac{1}{2} \begin{pmatrix} \sin \theta + \sqrt{2} \cos \theta & 1 \cos \theta - \sqrt{2} \sin \theta \\ \sin \theta - \sqrt{2} \cos \theta & 1 \cos \theta + \sqrt{2} \sin \theta \\ -\sqrt{2} \sin \theta & \sqrt{2} & -\sqrt{2} \cos \theta \end{pmatrix} \]. \quad (4.21) \]

The predictions for \( \theta_{13}, \theta_{12} \) and CP phases remain the same as those given by Eq. (4.19), while \( \theta_{23} \) becomes its complementary angle, i.e.

\[ \sin^2 \theta_{23} = \frac{3 - \cos 2\theta + 2\sqrt{2} \sin 2\theta}{5 + \cos 2\theta + 2\sqrt{2} \sin 2\theta}. \quad (4.22) \]

Its relation with \( \theta_{13} \) becomes

\[ \sin^2 \theta_{23} = \frac{2}{3} - \frac{\tan \theta_{13}}{9} \left( \tan \theta_{13} \pm 2\sqrt{6 - 2 \tan^2 \theta_{13}} \right). \quad (4.23) \]
Figure 4: Correlations between $\sin \theta_{13}$ and $\sin^2 \theta_{12}$ in case I. The best fitting value is marked with a red star, and the points for $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$ are labelled with a cross to guide the eye. The shown 1$\sigma$ and 3$\sigma$ ranges for the mixing angles are taken from Ref. [7]. Since the 1$\sigma$, 3$\sigma$ bounds of $\sin \theta_{13}$ are slightly different for NO and IO, the results for NO and IO cases almost coincide such they can hardly be distinguished.

The correlation among different mixing angles are plotted in Fig. 5. Although $\theta_{23} > \pi/4$ is slightly disfavored for NO, the global minimum of $\chi^2$ is somewhat large about 27.807. In the case of IO mass spectrum, the best fitting values are

$$\sin^2 \theta_{13} = 0.0242, \quad \sin^2 \theta_{12} = 0.256, \quad \sin^2 \theta_{23} = 0.579, \quad \chi^2_{\text{min}} = 10.086.$$  

(4.24)

It is notable that the Dirac CP $\delta_{CP}$ is predicted to be conserved here. The present experiments have very low sensitivity to leptonic CP. T2K has recently reported a weak indication for $\delta_{CP}$ around $3\pi/2$ [9]. Analysis of the SuperKamiokande atmospheric neutrino data gives preferable range $(1.2 \pm 0.5) \pi$ [49]. The global analysis of all oscillation data gives $\delta_{CP} = 1.39^{+0.38}_{-0.27}\pi(1\sigma)$ for NO and $\delta_{CP} = 1.31^{+0.29}_{-0.33}\pi(1\sigma)$ for IO and no restriction appears at 3$\sigma$ level [7]. Hence conserved CP with $\delta_{CP} = 0, \pi$ can be accommodated by both present data and global analysis. Future long baseline neutrino experiments LBNE [11], LBNO [12–16] and Hyper-Kamiokande [17] are designed to measure the Dirac phase. If the signal of leptonic CP violation is discovered, our model would be ruled out. In addition, the predictions for the atmospheric mixing angle $\theta_{23}$ can be tested by future atmospheric neutrino oscillation experiments such as the India-based Neutrino Observatory.

(II) $X_{\nu r} = \rho_r(T^2U), X_{\nu r} = \rho_r(TST^2)$

In this case, the parameters $\alpha$, $\beta$ and $\gamma$ are constrained to be real while $\epsilon$ is pure imaginary. Then the light neutrino mass matrix $m'_{\nu}$ given by Eq. (4.13) is diagonalized by a unitary transformation $U'_{\nu}$ with

$$U'_{\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -i \sin \theta & i \cos \theta \end{pmatrix} K_{\nu},$$

(4.25)
Figure 5: Correlations between $\sin^2 \theta_{23}$ and $\sin \theta_{13}$, $\sin^2 \theta_{12}$ in case I, where the solid lines and dashed lines represent the results for $\theta_{23} < \pi/4$ and $\theta_{23} > \pi/4$ respectively. The best fitting value is marked with a red star, and the points for $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$ are labelled with a cross to guide the eye. The shown 1$\sigma$ and 3$\sigma$ ranges for the mixing angles are taken from Ref. [7]. Note that the 1$\sigma$ bounds of $\sin^2 \theta_{23}$ for NO and IO are different. Because $\theta_{23}$ in the second octant with NO mass spectrum is disfavored [7], the corresponding results are not plotted here.

where the parameters $\theta$ satisfies

$$\tan 2\theta = \frac{4i\epsilon}{6\beta + \sqrt{2}\gamma}.$$  \hfill (4.26)

The light neutrino mass eigenvalues $U'^T m'_\nu U'_\nu = \text{diag} (m_1, m_2, m_3)$ take the form

$$m_1 = |\alpha - 2\beta + \sqrt{2}\gamma|,$$

$$m_2 = \frac{1}{2} \left| 2\alpha + 2\beta - \sqrt{2}\gamma - \text{sign}((3\sqrt{2} \beta + \gamma) \cos 2\theta) \sqrt{(6\beta + \sqrt{2}\gamma)^2 - 16\epsilon^2} \right|,$$

$$m_3 = \frac{1}{2} \left| 2\alpha + 2\beta - \sqrt{2}\gamma + \text{sign}((3\sqrt{2} \beta + \gamma) \cos 2\theta) \sqrt{(6\beta + \sqrt{2}\gamma)^2 - 16\epsilon^2} \right|. \hfill (4.27)$$

Note that the factor $K_\nu$ in Eq. (4.25) is also a diagonal phase matrix with non-vanishing elements $\pm 1$ or $\pm i$ such that the eigenvalues $m_{1,2,3}$ are positive. Note that the lepton mixing
matrix is only determined up to permutations of rows and columns, since both neutrino masses and charged lepton masses are almost unconstrained in the present framework. By considering all the $6 \times 6 = 36$ possible permutations, we find the PMNS matrix which can “best” describe the experimental data is

$$U_{PMNS} = P_{213}U_{BMR}U'_{\nu} = \frac{1}{2} \begin{pmatrix} \sqrt{2} & -\sqrt{2} \cos \theta & -\sqrt{2} \sin \theta \\ 1 & \cos \theta - i \sqrt{2} \sin \theta & \sin \theta + i \sqrt{2} \cos \theta \\ 1 & \cos \theta + i \sqrt{2} \sin \theta & \sin \theta - i \sqrt{2} \cos \theta \end{pmatrix} K_\nu. \quad (4.28)$$

The lepton mixing angles and CP phases are predicted to be

$$|\sin \delta_{CP}| = 1, \quad \sin \alpha_{21} = \sin \alpha_{31} = 0, \quad \sin^2 \theta_{13} = \frac{1}{2} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1+\cos 2\theta}{3+\cos 2\theta}, \quad \sin^2 \theta_{23} = \frac{1}{2}. \quad (4.29)$$

We see that the Dirac CP is maximally violated with $\delta_{CP} = \pm \frac{\pi}{2}$ whereas Majorana CP is preserved. The atmospheric mixing angle $\theta_{23} = \pi/4$ is maximal, and the solar angle $\theta_{12}$ and the reactor mixing angle $\theta_{13}$ are related by

$$\sin^2 \theta_{12} = \frac{1}{2} - \frac{1}{2} \tan^2 \theta_{13}. \quad (4.30)$$

For the $3\sigma$ interval $0.0176 \leq \sin^2 \theta_{13} \leq 0.0298$ from global data fit [7], $\theta_{12}$ is predicted to vary in the range of $0.485 \leq \sin^2 \theta_{12} \leq 0.491$ which is not compatible with the experimental data. Therefore the observed values of $\theta_{12}$ and $\theta_{13}$ can not be achieved simultaneously in this case. The best fitting values are presented in Table 3. Notice that the minimum values of the $\chi^2$ functions are rather large: 115.354 for NO, 111.464 for IO. The reason is that the solar angle $\theta_{12}$ is constrained to be much larger than its measured value for $\theta_{13}$ in the $3\sigma$ region, as has been explained.

In short, perturbatively modified BM mixing by a simple rotation in the neutrino sector, which is discussed in section 3, can be realized by breaking the $S_4$ family symmetry into a $Z_2$ subgroup instead of $K_4$ in the neutrino sector. By further extending the $S_4$ family symmetry to include compatible generalized CP symmetry, the phase $\delta$ of the perturbative rotation cannot take arbitrary value. We have definite predictions for the leptonic CP phases: the Dirac CP phase $\delta_{CP}$ is fully conserved or maximally violated while both Majorana CP phases are conserved. We note that case I with conserved $\delta_{CP}$ is preferred over case II with maximal $\delta_{CP}$. In the following section, we investigate how to realize the perturbatively modified BM mixing by a simple rotation in the charged lepton sector from $S_4$ and generalized CP.

| Analytic Expression | Best Fit for NO | Best Fit for IO |
|---------------------|----------------|----------------|
| $\sin^2 \theta_{13}$ | $\frac{1}{2} \sin^2 \theta$ | 0.0247 | 0.0252 |
| $\sin^2 \theta_{12}$ | $\frac{1+\cos 2\theta}{3+\cos 2\theta}$ | 0.487 | 0.487 |
| $\sin^2 \theta_{23}$ | $\frac{1}{2}$ | 1/2 | 1/2 |
| $\theta_{bf}$ | — | 0.224 | 0.226 |
| $\chi^2_{\text{min}}$ | — | 115.354 | 111.464 |

Table 3: The predictions for the lepton mixing angles and their best fitting values in case II. The $1\sigma$ and $3\sigma$ bounds for the mixing angles are taken from Ref. [7].
5 Lepton flavor mixing from remnant symmetries $K_4^{(TST^2, T^2 U)} \times H_{CP}^\nu$ in the neutrino sector and $Z_2^{SU} \times H_{CP}^L$ in the charged lepton sector

Remembering that BM mixing is generated if the $S_4$ family symmetry is broken down to $K_4^{(TST^2, T^2 U)}$ in the neutrino sector and to $K_4^{(SU)}$ in the charged lepton sector, no matter whether generalized CP is imposed. Motivated by the previous section, we would like to break $S_4$ into $Z_2^{SU}$ instead of $K_4^{(SU)}$ and then the remnant symmetries in the neutrino and the charged lepton sectors are $K_4^{(TST^2, T^2 U)} \times H_{CP}^\nu$ and $Z_2^{SU} \times H_{CP}^L$ respectively. In this scenario, only one row of the PMNS matrix can be fixed because of the residual $Z_2^{SU}$. For $K_4^{(TST^2, T^2 U)} \times H_{CP}^\nu$ to be a well-defined symmetry, the element $X_{\nu r}$ of $H_{CP}^\nu$ must satisfy the consistence conditions:

$$X_{\nu r} \rho_r^*(h) X_{\nu r}^{-1} = \rho_r(h'), \quad \text{with} \quad h, h' \in K_4^{(TST^2, T^2 U)}.$$  

(5.1)

We find that the residual CP transformation $X_{\nu r}$ can take 8 possible values,

$$H_{CP}^\nu = \{ \rho_r(1), \rho_r(TST^2), \rho_r(T^2 U), \rho_r(ST^2 SU), \rho_r(S), \rho_r(T^2 ST), \rho_r(ST^2 U), \rho_r(T^2 SU) \}.$$  

(5.2)

The light neutrino mass matrix $m_\nu$ is constrained by the residual family symmetry $K_4^{(TST^2, T^2 U)}$ and the residual CP symmetry $H_{CP}^\nu$ as

$$\rho_3^T(h) m_\nu \rho_3(h) = m_\nu, \quad h \in K_4^{(TST^2, T^2 U)},$$

(5.3)

$$X_{\nu 3}^T m_\nu X_{\nu 3} = m_\nu^*, \quad X_{\nu 3} \in H_{CP}^\nu.$$  

(5.4)

Eq. (5.3) constrains the light neutrino mass matrix to be of the form

$$m_\nu = a \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} + c \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

(5.5)

which can be diagonalized by a unitary matrix $U_\nu$, i.e.

$$U_\nu^T m_\nu U_\nu = \text{diag} \left( a + 2b - \sqrt{2}c, a + 2b + \sqrt{2}c, -a + 4b \right),$$

(5.6)

where

$$U_\nu = \frac{1}{2} \begin{pmatrix} 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \\ 1 & 1 & \sqrt{2} \end{pmatrix}.$$  

(5.7)

Note that $U_\nu$ is fixed up to column permutations since the order of the eigenvalues of $m_\nu$ in Eq. (5.6) is not determined. The residual CP symmetry invariant condition of Eq. (5.4) implies that the three parameters $a$, $b$ and $c$ are all real for $X_{\nu r} = \rho_r(1), \rho_r(TST^2), \rho_r(T^2 U), \rho_r(ST^2 SU)$. In the case of $X_{\nu r} = \rho_r(S), \rho_r(T^2 ST), \rho_r(ST^2 U), \rho_r(T^2 SU)$, $a$, $b$ are real and $c$ is pure imaginary. As a consequence, the neutrino masses are partially degenerate ($m_1 = m_2$). Therefore this case is not viable, and it will be ignored in the following.

---

\footnote{For the residual symmetry $G_1 = Z_2^S$, one row of the PMNS matrix would be $(1/\sqrt{2}, -1/\sqrt{2}, 0)^T$ which doesn’t match with the experimental data. $G_1 = Z_2^U$ leads to the same lepton mixing pattern as that of $G_1 = Z_2^{SU}$, because we have conjugation relations $(TST^2) Z_2^{SU} (TST^2)^{-1} = Z_2^{SU}$ and $(TST^2) K_4^{(TST^2, T^2 U)} (TST^2)^{-1} = K_4^{(TST^2, T^2 U)}$.}
The original symmetry $S_4 \rtimes H_{CP}$ is broken down to $Z_2^{SU} \rtimes H_{CP}^I$. The elements of $H_{CP}^I$ fulfill
\[ X_{tr} \rho_{tr}^*(SU) X_{tr}^{-1} = \rho_{tr}(SU). \] (5.8)
It is easy to check that only four generalization CP transformations are acceptable,
\[ H_{CP}^I = \{ \rho_{tr}(TST^2), \rho_{tr}(TST^2U), \rho_{tr}(T^2ST), \rho_{tr}(T^2STU) \}. \] (5.9)
We are able to construct the hermitian combination $m_{l}^{\dagger} m_{l}$ of the charged lepton mass matrix from its invariance under the residual symmetry $Z_2^{SU} \rtimes H_{CP}$,
\[ \rho_{3}^*(SU) m_{l}^{\dagger} m_{l} \rho_{3}(SU) = m_{l}^{\dagger} m_{l}, \]
\[ X_{l3}^* m_{l}^{\dagger} m_{l} X_{l3} = (m_{l}^{\dagger} m_{l})^*. \] (5.10)
Two distinct phenomenological predictions arise for the four possible generalized CP transformations in Eq. (5.9).

(III) $X_{tr} = \rho_{tr}(TST^2), \rho_{tr}(TST^2U)$

The most general $m_{l}^{\dagger} m_{l}$ satisfying Eq. (5.10) is of the following form
\[ m_{l}^{\dagger} m_{l} = \begin{pmatrix} \alpha & (1 + i)\beta & i\epsilon \\ (1 - i)\beta & \gamma & (1 + i)\beta \\ -i\epsilon & (1 - i)\beta & \alpha \end{pmatrix}, \] (5.11)
where $\alpha$, $\beta$, $\gamma$ and $\epsilon$ are real. In order to diagonalize this mass matrix $\mathcal{M}_l \equiv m_{l}^{\dagger} m_{l}$, we firstly perform a unitary transformation $\mathcal{M}_l' = W_l^{\dagger} \mathcal{M}_l W_l$ with
\[ W_l = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & -i \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \] (5.12)
Then it is easy to check
\[ \mathcal{M}_l' = W_l^{\dagger} \mathcal{M}_l W_l = \begin{pmatrix} \gamma & \sqrt{2}(1 + i)\beta & 0 \\ \sqrt{2}(1 - i)\beta & \alpha + \epsilon & 0 \\ 0 & 0 & \alpha - \epsilon \end{pmatrix}, \] (5.13)
which is further diagonalized by a simple unitary matrix
\[ U_l^{\dagger} \mathcal{M}_l' U_l = \text{diag} (m_{e}^2, m_{\mu}^2, m_{\tau}^2), \] (5.14)
where $U_l'$ is given by
\[ U_l' = \begin{pmatrix} e^{i\pi} \cos \theta & e^{i\pi} \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \] (5.15)
up to rephasings and column permutations, and the angle $\theta$ is specified by
\[ \tan 2\theta = \frac{4\beta}{\alpha + \epsilon - \gamma}. \] (5.16)
The charged lepton masses are
\[ m_e^2 = \frac{1}{2} \left[ \alpha + \epsilon + \gamma - \text{sign} \left( (\alpha + \epsilon - \gamma) \cos(2\theta) \right) \sqrt{16\beta^2 + (\alpha + \epsilon - \gamma)^2} \right], \]
Table 4: The predictions for the lepton mixing angles and their best fitting values in case III. The number before (in) the parenthesis denotes the best fitting value for \( \theta_{23} < \pi/4 \) (\( \theta_{23} > \pi/4 \)). The 1\( \sigma \) and 3\( \sigma \) bounds for the mixing angles are taken from Ref. [7].

| Analytic Expression | Best Fit for NO | Best Fit for IO |
|---------------------|----------------|----------------|
| \( \sin^2 \theta_{13} \) | \( \frac{1}{2} \sin^2 \theta \) | 0.0250 (0.0248) | 0.0253 (0.0253) |
| \( \sin^2 \theta_{12} \) | \( \frac{1}{2} - \frac{\sqrt{2} \sin 2\theta}{3+\cos 2\theta} \) | 0.342 (0.343) | 0.341 (0.341) |
| \( \sin^2 \theta_{23} (\theta_{23} < \pi/4) \) | \( \frac{1+\cos 2\theta}{3+\cos 2\theta} \) | 0.487 | 0.487 |
| \( \sin^2 \theta_{23} (\theta_{23} > \pi/4) \) | \( \frac{2}{3+\cos 2\theta} \) | 0.513 | 0.513 |
| \( \theta_{bf} \) | — | 0.225 (0.224) | 0.227 (0.227) |
| \( \chi^2_{\text{min}} \) | — | 6.938 (9.890) | 4.288 (4.409) |

\[
n^2_{\mu} = \frac{1}{2} \left[ \alpha + \epsilon + \gamma + \text{sign} \left( (\alpha + \epsilon - \gamma) \cos(2\theta) \right) \sqrt{16\beta^2 + (\alpha + \epsilon - \gamma)^2} \right],
\]

\[
n^2_{\tau} = \alpha - \epsilon.
\]

In the end, the lepton flavor mixing matrix takes the form

\[
U_{PMNS} = U_{\alpha}^T W^T_{\tau} \nu
\]

\[
= K_t \frac{1}{2} \left( \begin{array}{ccc}
- \sin \theta - \sqrt{2} \cos \theta & - \sin \theta + \sqrt{2} \cos \theta & \sqrt{2} \sin \theta \\
\cos \theta - \sqrt{2} \sin \theta & \cos \theta + \sqrt{2} \sin \theta & - \sqrt{2} \cos \theta \\
1 & 1 & \sqrt{2}
\end{array} \right) K_{\nu},
\]

where \( K_t = \text{diag}(e^{-i\pi/4}, e^{-i\pi/4}, e^{i\pi/2}) \) and \( K_{\nu} = \text{diag}(1, 1, -i) \) are unitary diagonal matrices. Note that the phases in \( K_t \) are unphysical and they can be absorbed by field redefinition. The lepton mixing parameters can be straightforwardly extracted as follows

\[
\sin \delta_{CP} = \sin \alpha_{21} = \sin \alpha_{31} = 0, \\
\sin^2 \theta_{13} = \frac{1}{2} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1}{2} - \frac{\sqrt{2} \sin 2\theta}{3+\cos 2\theta}, \quad \sin^2 \theta_{23} = \frac{1+\cos 2\theta}{3+\cos 2\theta}.
\]

We see that the CP is conserved and the mixing angles are closely related with each other,

\[
\sin^2 \theta_{12} = \frac{1}{2} \pm \tan \theta_{13} \sqrt{1 - \tan^2 \theta_{13}}, \quad 2 \cos^2 \theta_{13} \cos^2 \theta_{23} = 1.
\]

The measured values of reactor mixing angle \( \sin^2 \theta_{13} = 0.0234 \) fixes the parameter \( \theta \approx 12.494^\circ \), and then the other two mixing angles are determined to be \( \sin^2 \theta_{12} \approx 0.347, \sin^2 \theta_{23} \approx 0.488 \) which are in the experimentally allowed regions. The correlations among the mixing angles are plotted in Fig. [6] and Fig. [7]. We see that the predictions for the lepton mixing angles agree rather well with their measured values for certain values of the parameter \( \theta \). The best fitting results of this mixing pattern is listed in Table [4], the global minimum of the \( \chi^2 \) is quite small: 4.288 for IO and 6.938 for NO spectrum. Hence this mixing pattern can describe the experimental data very well. From Eq. (5.20), we have \( \sin^2 \theta_{23} = 1 - 1/(2 \cos^2 \theta_{13}) < 1/2 \), namely \( \theta_{23} \) is in the first octant, as can be seen from Fig. [7]. Exchanging the second and the thirds rows of the PMNS matrix in Eq. (5.18), we obtain a new mixing texture

\[
U_{PMNS}' = K_t \frac{1}{2} \left( \begin{array}{ccc}
- \sin \theta - \sqrt{2} \cos \theta & - \sin \theta + \sqrt{2} \cos \theta & \sqrt{2} \sin \theta \\
\cos \theta - \sqrt{2} \sin \theta & \cos \theta + \sqrt{2} \sin \theta & - \sqrt{2} \cos \theta \\
1 & 1 & \sqrt{2}
\end{array} \right) K_{\nu},
\]
Figure 6: Correlations between sin $\theta_{13}$ and sin$^2 \theta_{12}$ in case III. The best fitting value is marked with a red star, and the points for $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$ are labelled with a cross to guide the eye. The shown $1\sigma$ and $3\sigma$ ranges for the mixing angles are taken from Ref. [7].

where $K'_l = \text{diag}(e^{-i\pi/4}, e^{i\pi/4}, e^{-i\pi/4})$ and $K_\nu = \text{diag}(1, 1, -i)$. The corresponding lepton mixing parameters read

$$
\sin \delta_{CP} = \sin \alpha_{21} = \sin \alpha_{31} = 0,
$$

$$
\sin^2 \theta_{13} = \frac{1}{2} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1}{2} - \frac{\sqrt{2} \sin 2\theta}{3 + \cos 2\theta}, \quad \sin^2 \theta_{23} = \frac{2}{3 + \cos 2\theta}, \quad (5.22)
$$

which can also accommodate the best fit values of the mixing angles, as is showed in Table 4.

(IV) $X_{lr} = \rho_r(T^2ST), \rho_r(T^2STU)$

In this case, the charged lepton mass matrix is determined to be

$$
m_l^\dagger m_l = \begin{pmatrix}
\alpha & (1 - i)\beta & i\epsilon \\
(1 + i)\beta & \gamma & (1 + i)\beta \\
-i\epsilon & (-1 - i)\beta & \alpha
\end{pmatrix}, \quad (5.23)
$$

where $\alpha$, $\beta$, $\gamma$ and $\epsilon$ are all real parameters. After performing the unitary transformation $W_l$, we have

$$
W_l^\dagger m_l^\dagger m_l W_l = \begin{pmatrix}
\gamma & \sqrt{2}(1 + i)\beta & 0 \\
\sqrt{2}(1 - i)\beta & \alpha + \epsilon & 0 \\
0 & 0 & \alpha - \epsilon
\end{pmatrix}. \quad (5.24)
$$

Following the procedures stated in case III, we find that the unitary transformation $U_l$ diagonalizing this charged lepton mass matrix is of the form

$$
U_l = \frac{1}{\sqrt{2}} \begin{pmatrix}
-i \sin \theta & i \cos \theta & -i \\
\sqrt{2} e^{\frac{3i}{4}} \cos \theta & \sqrt{2} e^{\frac{3i}{4}} \sin \theta & 0 \\
-\sin \theta & \cos \theta & 1
\end{pmatrix}, \quad (5.25)
$$
which yields \( U_l^T m_i^T m_i U_l = \text{diag}(m_e^2, m_\mu^2, m_\tau^2) \). The angle \( \theta \) fulfills

\[
\tan 2\theta = \frac{4\beta}{\alpha + \epsilon - \gamma}.
\]

The charged lepton masses are found to be of the same form as those in Eq. (5.17). Combining the unitary transformations \( U_\nu \) and \( U_l \) from neutrino and charged lepton sectors, we obtain the predictions for the PMNS matrix:

\[
U_{PMNS} = \frac{1}{2} \begin{pmatrix}
\sqrt{2} \cos \theta + i \sin \theta & -\sqrt{2} \cos \theta + i \sin \theta & -\sqrt{2} \sin \theta \\
\sqrt{2} \sin \theta - i \cos \theta & -\sqrt{2} \sin \theta - i \cos \theta & \sqrt{2} \cos \theta \\
1 & 1 & i\sqrt{2}
\end{pmatrix} e^{i\frac{\pi}{4}},
\]

up to independent permutations of rows and columns. Therefore the lepton mixing angles and CP phases are predicted to be

\[
\sin^2 \theta_{13} = \frac{1}{2} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1}{2}, \quad \sin^2 \theta_{23} = \frac{1 + \cos 2\theta}{3 + \cos 2\theta}, \quad |J_{CP}| = \frac{1}{8\sqrt{2}} |\sin 2\theta|,
\]
Analytic Expression | Best Fit for NO | Best Fit for IO
---|---|---
$\sin^2 \theta_{13}$ | $\frac{1}{2} \sin^2 \theta$ | 0.0235 (0.0233) | 0.0240 (0.0240)
$\sin^2 \theta_{12}$ | $\frac{1}{2}$ | $\frac{1}{2} \left( \frac{1}{2} \right)$ | $\frac{1}{2} \left( \frac{1}{2} \right)$
$\sin^2 \theta_{23} (\theta_{23} < \pi/4)$ | $\frac{1 + \cos 2 \theta}{3 + \cos 2 \theta}$ | 0.488 | 0.488
$\sin^2 \theta_{23} (\theta_{23} > \pi/4)$ | $\frac{2}{3 + \cos 2 \theta}$ | 0.512 | 0.512
$|\sin \delta_{CP}|$ | 1 | 1 (1) | 1 (1)
$|J_{CP}|$ | $\frac{1}{8\sqrt{2}} |\sin 2 \theta|$ | 0.0375 (0.0372) | 0.0378 (0.0378)
$|\sin \alpha_{21}|$ | $\frac{2 \sqrt{2} (2 \sin \theta \cos \theta + 3 \sin 2 \theta)}{(3 + \cos 2 \theta)^2}$ | 0.583 (0.581) | 0.589 (0.589)
$|\sin \alpha_{31}|$ | $\frac{2 \sqrt{2} \sin \theta}{3 + \cos 2 \theta}$ | 0.306 (0.305) | 0.310 (0.310)
$\theta_{bf}$ | — | 0.219 (0.217) | 0.221 (0.221)
$\chi^2_{\text{min}}$ | — | 129.945 (132.715) | 127.612 (127.727)

Table 5: The predictions for the lepton mixing angles and their best fitting values in case IV. The number before (in) the parenthesis denotes the best fitting value for $\theta_{23} < \pi/4$ ($\theta_{23} > \pi/4$). The 1σ and 3σ bounds for the mixing angles are taken from Ref. [7].

$$|\sin \delta_{CP}| = 1,$$  
$$|\sin \alpha_{21}| = \left| \frac{2 \sqrt{2} (2 \sin \theta \cos \theta + 3 \sin 2 \theta)}{(3 + \cos 2 \theta)^2} \right|,$$  
$$|\sin \alpha_{31}| = \left| \frac{2 \sqrt{2} \sin \theta}{3 + \cos 2 \theta} \right| \quad (5.28)$$

where $J_{CP}$ is the well-known Jarlskog invariant. We see that this mixing pattern predicts maximal Dirac CP phase, and both Majorana CP phases are non-trivial functions of the parameters $\theta$. Furthermore, the solar angle $\theta_{12}$ is maximal which is completely ruled out beyond 3σ confidence level. The best fitting results are listed in Table 5. Although the measured values of $\theta_{13}$ and $\theta_{23}$ can be accommodated very well, this texture is not viable due to its unrealistic prediction for $\theta_{12}$, even if the freedom of permutating the rows and columns of the PMNS matrix is taken into account.

## 6 Model predicting one column of BM mixing with $S_4$ and generalized CP

In this section we will construct a model based on $S_4$ family symmetry and generalized CP symmetry $H_{CP}$. The auxiliary symmetry $Z_4 \times Z_2$ is introduced to disentangle the flavon fields associated with the neutrino sector from those associated with the charged lepton sector and to eliminate unwanted dangerous operators. By a judicious choice of flavons, the symmetry breaking pattern of $S_4 \times H_{CP}$ into $Z_2^{STSU} \times H_{CP}^\nu$ and $Z_4^{STSU} \times H_{CP}^l$ is explicitly realized, whose phenomenological consequence has been analyzed in a model independent way in section 4. As a result, the interesting mixing textures in Eq. (4.18) and Eq. (4.28) are reproduced exactly in this model. This model is formulated in the context of supersymmetry. We assign the three generations of left-handed lepton doublets $l$ and of right-handed neutrino $\nu^c$ to $S_4$ triplet 3. The right-handed charged leptons $e^c$, $\mu^c$ and $\tau^c$ are singlet states of $S_4$, and they transform as 1, 1’ and 1 respectively. The matter fields, flavon fields, driving fields and their transformation properties under the family symmetry $S_4 \times Z_4 \times Z_2 \times U(1)_R$ are
summarized in Table 6.

### 6.1 Vacuum alignment

The issue of vacuum alignment is handled with the help supersymmetric driving field mechanism [50]. This approach utilizes a global $U(1)_R$ continuous symmetry which contains the discrete $R$–parity as a subgroup. The flavon and Higgs fields are uncharged under $U(1)_R$, the matter fields carry $R$ charge +1 and the driving fields $\zeta^0$, $\rho^0$, $\varphi^0_T$, $\eta^0$ and $\varphi^0_S$ carry two units of $R$ charge. Consequently all terms in the superpotential either contain two matter superfields or one driving field. The leading order (LO) driving superpotential $w_d$ invariant under the family symmetry $S_4 \times Z_4 \times Z_2$ is of the form

$$w_d = w^l_d + w^r_d,$$

where $w^l_d$ and $w^r_d$ are responsible for the LO vacuum alignment of the charged lepton sector and neutrino sector respectively, and they can be expressed as

$$w^l_d = f_1(\rho^0(\varphi_T \varphi_T)\phi_2) + f_2(\rho^0(\varphi_T \phi)\phi_1) + f_3(\rho^0(\varphi_T \phi)\phi_2) + f_4(\varphi^0_T(\varphi_T \varphi_T)\phi_3) + f_5(\varphi^0_T(\varphi_T \varphi_T)\phi_3) + f_6(\varphi^0_T(\varphi_T \varphi_T)\phi_3),$$

$$w^r_d = g_1(\eta^0 \eta)\phi_1 + g_2(\eta^0 \eta)\phi_2 + g_3(\eta^0 \eta)\phi_3 + g_4(\varphi^0_S \varphi_S)\phi_1 + g_5(\varphi^0_S \varphi_S)\phi_3 + g_6(\varphi^0_S \varphi_S)\phi_3,$$

where the subscripts 1, 2, 3 etc stand for contractions into the corresponding $S_4$ irreducible representations. Note that the terms proportional to $f_4$, $f_6$ and $g_6$ give null contributions because of the antisymmetric contractions $(3 \otimes 3)_3$ and $(3' \otimes 3')_3$. As we require the theory invariant under the generalized CP transformations defined in Eq. (2.7), all couplings $f_i$ and $g_i$ would be real. The driving field is assumed to have vanishing vacuum expectation value (VEV). In the limit of unbroken supersymmetry, the vacuum configuration is fixed by the vanishing $F$–term of the driving field. For the vacuum alignment of the charged lepton sector, we have

$$\frac{\partial w^l_d}{\partial \phi_1} = 2f_1(\varphi_T^2 - \varphi_T \varphi_T) + \sqrt{3}f_2(\varphi_T \phi_1 + \varphi_T \phi_3) + 2f_3(\phi_2^2 - \phi_1 \phi_3) = 0,$$

$$\frac{\partial w^l_d}{\partial \phi_2} = 3f_1(\varphi_T^2 + \varphi_T^2) + f_2(\varphi_T \phi_3 - 2\varphi_T \phi_2 + \varphi_T \phi_1) + \sqrt{3}f_3(\phi_2^2 + \phi_3^2) = 0,$$

$$\frac{\partial w^l_d}{\partial \varphi_T} = f_5(\varphi_T \phi_2 + \varphi_T \phi_1) = 0,$$

$$\frac{\partial w^l_d}{\partial \varphi_T} = f_5(\varphi_T \phi_1 - \varphi_T \phi_3) = 0,$$

$$\frac{\partial w^l_d}{\partial \varphi_T} = -f_5(\varphi_T \phi_3 + \varphi_T \phi_2) = 0.$$
This set of equations are satisfied by the alignment:

$$\langle \varphi_T \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} v_T, \quad \langle \phi \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v_\phi, \quad v_\phi = -\frac{f_2 + \sqrt{f_2^2 - 12f_1f_3}}{2\sqrt{3}f_3} v_T. \quad (6.5)$$

The VEVs $v_\phi$ and $v_T$ are naturally of the same order of magnitude, since they are related through the couplings $f_1$, $f_2$ and $f_3$ which are expected to have absolute values of order one. To reproduce the observed hierarchy among the charged lepton masses, we choose

$$\frac{v_\phi}{\Lambda} \sim \frac{v_T}{\Lambda} \sim \lambda^2,$$

where $\lambda \approx 0.23$ is the Cabibbo angle [48]. The $F-$term conditions of the neutrino sector are

$$\frac{\partial w_d^\nu}{\partial \eta_1^1} = g_1 \xi \eta_1 + g_2 (\eta_2^2 - \eta_1^2) + 2g_3 (\varphi^2_{S_2} - \varphi_{S_1} \varphi_{S_3}) = 0,$$

$$\frac{\partial w_d^\nu}{\partial \eta_2^2} = g_1 \xi \eta_2 + 2g_2 \eta_1 \eta_2 + \sqrt{3} g_3 (\varphi^2_{S_1} + \varphi^2_{S_3}) = 0,$$

$$\frac{\partial w_d^\nu}{\partial \varphi^0_{S_1}} = g_4 \xi \varphi_{S_3} + g_5 (\sqrt{3} \eta_2 \varphi_{S_1} - \eta_1 \varphi_{S_3}) = 0,$$

$$\frac{\partial w_d^\nu}{\partial \varphi^0_{S_2}} = g_4 \xi \varphi_{S_2} + 2g_5 \eta_1 \varphi_{S_2} = 0,$$

$$\frac{\partial w_d^\nu}{\partial \varphi^0_{S_3}} = g_4 \xi \varphi_{S_1} + g_5 (\sqrt{3} \eta_2 \varphi_{S_3} - \eta_1 \varphi_{S_1}) = 0. \quad (6.7)$$

It is then straightforward to work out the most general solutions to these equations. Disregarding the ambiguity caused by $S_4$ family symmetry transformations we find three possible non-trivial solutions. The first one is given by

$$\langle \xi \rangle = v_\xi, \quad \langle \eta \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_\eta, \quad \langle \varphi_S \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_S, \quad (6.8)$$

with

$$v_\eta = -\frac{g_4 v_\xi}{2g_5}, \quad v_S^2 = \frac{g_4 (2g_1 g_5 + g_2 g_4)}{8g_3 g_5^2} v_\xi^2, \quad (6.9)$$

where $v_\xi$ is undetermined and generally complex. Given the representation matrices in Appendix A, it is easy to check that this vacuum breaks the $S_4$ family symmetry to $Z^4_{SU_2}$. The second solution is

$$\langle \xi \rangle = v_\xi, \quad \langle \eta \rangle = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} v_\eta, \quad \langle \varphi_S \rangle = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} v_S, \quad (6.10)$$

with

$$v_\eta = \frac{g_4 v_\xi}{4g_5}, \quad v_S^2 = -\frac{g_4 (2g_1 g_5 + g_2 g_4)}{16g_3 g_5^2} v_\xi^2. \quad (6.11)$$

The residual family symmetry $Z^4_{SU_2}$ is preserved by this alignment. The third takes the form

$$\langle \xi \rangle = v_\xi, \quad \langle \eta \rangle = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} v_\eta, \quad \langle \varphi_S \rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_S, \quad (6.12)$$
where

$$v_\eta = -g_4 v_\xi, \quad v_\xi^2 = \frac{g_4 (g_1 g_5 - g_2 g_4)}{4 g_3 g_5^2} v_\eta^2, \quad v_\xi \text{ undetermined},$$  \hspace{1cm} (6.13)

We see that the two VEVs $v_\eta$ and $v_\xi$ share the same phase modulo $\pi$, while the phase difference between $v_S$ and $v_\xi$ is 0, $\pi$ for $g_3 g_1 (g_1 g_5 - g_2 g_4) > 0$ or $\pm \pi/2$ for $g_3 g_1 (g_1 g_5 - g_2 g_4) < 0$. Since the phase of $v_\xi$ can always be absorbed by lepton fields, we could take $v_\xi$ to be real without loss of generality. Consequently $v_\eta$ is real, and $v_S$ is either real or pure imaginary depending on the combination $g_3 g_1 (g_1 g_5 - g_2 g_4)$ being positive or negative.

We find that the symmetry $S_4 \times H_{CP}$ is broken to $Z_2^{ST2SU} \times H_{CP}^\nu$ by the VEVs of $\xi$, $\eta$ and $\phi_S$, where the remnant CP symmetry $H_{CP}^\nu = \{\rho_3(1), \rho_3(ST2SU)\}$ for real $v_S$ and $H_{CP}^\nu = \{\rho_3(T^2U), \rho_3(TST2)\}$ for pure imaginary $v_S$. In order to derive the interesting mixing textures of Eq. (4.18) and Eq. (4.28) discovered in section 4, we shall choose the third solution in this work. Furthermore, the three VEVs $v_\xi$, $v_\eta$ and $v_S$ are expected to be of the same order of magnitude without fine tuning among the parameters $g_i (i = 1, 2, 3, 4, 5)$. As usual, we shall take them to be of the same order as the VEVs of charged lepton sector flavons, i.e.

$$\frac{v_\xi}{\Lambda} \sim \frac{v_\eta}{\Lambda} \sim \frac{v_S}{\Lambda} \sim \lambda^2.$$  \hspace{1cm} (6.14)

### 6.2 The structure of the model

The superpotential for the charged lepton masses is

$$w_\ell = \frac{y_\ell}{\Lambda} \tau^c(l \varphi_T)_1 h_d + \frac{y_{\mu_1}}{\Lambda^2} \mu^c(l \varphi_T)_3 h_d + \frac{y_{\mu_2}}{\Lambda^2} \mu^c(l \varphi_T)_3 h_d + \frac{y_{\mu_3}}{\Lambda^2} \mu^c(l \varphi_T)_3 h_d + \sum_{i=1}^4 \frac{y_{e_i}}{\Lambda^3} e^c(l O_i)_1 h_d + \ldots,$$

\hspace{1cm} (6.15)

where

$$O = \{\varphi_T \varphi_T \varphi_T, \varphi_T \varphi_T \varphi_T, \varphi_T \varphi_T \varphi_T, \varphi_T \varphi_T \varphi_T\}.$$  \hspace{1cm} (6.16)

Notice that all possible $S_4$ contractions should be considered. Dots stand for higher dimensional operators corrections which we will be discussed later. All the Yukawa couplings are real because of the generalized CP symmetry. Substituting the flavon VEVs in Eq. (6.5), we find the charged lepton mass matrix is diagonal with

$$m_e = \begin{vmatrix} y_e & v_\eta^3 \\ \frac{v_\eta}{\Lambda^3} & v, \end{vmatrix} v_d, \quad m_\mu = \begin{vmatrix} y_{\mu_1} & v_\xi^2 - y_{\mu_2} v_\phi & v_\phi^2 \\ \frac{v_\xi}{\Lambda^2} & y_{\mu_2} & y_{\mu_3} v_\phi \\ \frac{v_\phi}{\Lambda^2} & \frac{v_\phi}{\Lambda^2} & \frac{v_\phi^2}{\Lambda^2} \end{vmatrix} v_d, \quad m_\tau = \begin{vmatrix} y_\tau & v_\eta^2 \\ \frac{v_\eta}{\Lambda^2} & v, \end{vmatrix} v_d,$$

\hspace{1cm} (6.17)

where $v_d = \langle h_d \rangle$, $y_e$ stands for the total result of all the different contributions of the $y_{e_i}$ terms. For $v_\phi \sim v_T \sim \lambda^2 \Lambda$, the mass hierarchies of the charged lepton are obtained, i.e.

$$m_e : m_\mu : m_\tau \simeq \lambda^4 : \lambda^2 : 1.$$  \hspace{1cm} (6.18)

As the representation matrix of the element $T^2STU$ is diagonal $\rho_3(T^2STU) = \text{diag} (-i, 1, i)$, we have $\rho_3(T^2STU) m_1 m_1 m_2 = m_1 m_2$. This is to say, a remnant symmetry $Z_4^{T^2STU}$ is preserved by $m_1 m_2$ which is relevant to lepton flavor mixing, although the $S_4$ family symmetry is completely broken by the VEVs of $\varphi_T$ and $\phi$ with $T^2STU \langle \varphi_T \rangle = -i \langle \varphi_T \rangle$ and $T^2STU \langle \phi \rangle = -i \langle \phi \rangle$.

The light neutrino masses are generated via type-I seesaw mechanism. The LO superpotential responsible for neutrino masses is

$$w_\nu = \frac{y_1}{\Lambda} \xi (\nu^c)_1 h_u + \frac{y_2}{\Lambda} ((\nu^c)^2 \eta)_1 h_u + \frac{y_3}{\Lambda} ((\nu^c)_3 \varphi_S)_1 h_u + M(\nu^c \nu^c)_1,$$  \hspace{1cm} (6.19)
where again all couplings are real due to the invariance under the generalized CP. The last term is the Majorana mass term for the right-handed neutrinos,

\[ m_M = M \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} . \]  

(6.20)

With the vacuum alignment of \( \xi, \eta \) and \( \varphi_S \) in Eq. (6.12), we find the Dirac mass matrix is of the following form,

\[ m_D = y_1 v_u \xi \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + y_2 v_u \eta \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} + y_3 v_u \varphi_S \begin{pmatrix} 0 & 1 & -\sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & -1 & 0 \end{pmatrix} . \]  

(6.21)

The light neutrino mass matrix is given by the see-saw formula

\[ m_\nu = -m_D^T m_M^{-1} m_D \]

\[ = \alpha \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \epsilon \begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} , \]  

(6.22)

where

\[ \alpha = \left( \frac{8}{3} y^2 - 8x^2 - 1 \right) m_0 , \quad \beta = \left( 2x - 2x^2 + \frac{1}{3} y^2 \right) m_0 , \quad \gamma = -\sqrt{2} y^2 m_0 , \quad \epsilon = -6 y x m_0 , \]  

(6.23)

with

\[ x = \frac{y_2 v_u}{y_1 v_\xi} , \quad y = \frac{y_3 v_u}{y_1 v_\xi} , \quad m_0 = \frac{y_1 v_\xi^2 v_u}{\Lambda^2 M} . \]  

(6.24)

Note that the phase of \( v_\xi \) can be factorized out as an overall phase of \( m_\nu \) and therefore it can be absorbed by field redefinition. Accordingly Eq. (6.13) implies that the VEVs \( v_\xi \) and \( v_\eta \) are real while \( v_S \) is real for \( g_3 g_4 (g_1 g_5 - g_2 g_4) > 0 \) and pure imaginary for \( g_3 g_4 (g_1 g_5 - g_2 g_4) < 0 \).

In case of real \( v_S \), all the four parameters \( \alpha, \beta, \gamma \) and \( \epsilon \) are real. The VEVs of the flavon \( \xi, \eta \) and \( \varphi_S \) break the \( S_4 \) family symmetry to \( Z_2^{ST^2 SU} \) and break the generalized CP to \( H_{C_P}^\prime = \{ \rho_S(1), \rho_S(ST^2 SU) \} \) in the neutrino sector. Hence the case I discussed in section 4 is exactly reproduced here. The lepton flavor mixing matrix is of the form shown in Eq. (4.18), and the predictions for light neutrino masses and mixing parameters are presented in Eqs (4.17,4.19). On the other hand, if \( v_S \) is pure imaginary, \( \alpha, \beta \) and \( \gamma \) are real while \( \epsilon \) is an imaginary parameter. The remnantsymmetry in the neutrino sector would be \( Z_2^{ST^2 SU} \times H_{C_P}^\prime \) with \( H_{C_P}^\prime = \{ \rho_S(T^2 U), \rho_S(TST^2) \} \). Therefore this case is identical to the case II of section 4 and the analytical expressions for PMNS matrix and lepton mixing parameters are displayed in Eqs (4.28,4.29).

It is useful to study the constraints on the model imposed by the observed values of the mass-squared splitting \( \delta m^2 = m_2^2 - m_3^2 \), \( \Delta m^2 = m_3^2 - (m_1^2 + m_2^2)/2 \) and the reactor mixing angle \( \theta_{13} \). As the light neutrino mass matrix effectively depends on three real (imaginary) parameters \( x, y, m_0 \), their values can be completely fixed. Given the best fitting results \( \delta m^2 = 7.54 \times 10^{-5} \text{eV}^2 \), \( \Delta m^2 = 2.43 \times 10^{-3}(-2.38 \times 10^{-3}) \text{eV}^2 \) and \( \sin^2 \theta_{13} = 0.0234(0.0240) \) for NO (IO) neutrino mass spectrum from Ref. [7], the possible solutions for \( x, y \) and the corresponding predictions for the light neutrino masses, the lepton mixing angles, CP phases and the effective mass \( |m_{\beta \beta}| \) of neutrinoless double-beta decay are collected in Table [7]. In the case that both \( x \) and \( y \) are real, there are other solutions predicting \( \theta_{23} = 30.137^\circ \) which is out of the 3σ range, and they are not included in Table [7]. It is remarkable that the absolute
values of the light neutrino masses are fixed in the present model. We find that the light neutrino mass spectrum can be either NO or IO in case I while it can only be NO in case II. Regarding the sum of the light neutrino masses, the latest Planck result is $\sum m_\nu < 0.23eV$ at 95% confidence level \(^{[51]}\). This bound is saturated for all the solutions except the second one which gives $m_1 + m_2 + m_3 \simeq 0.238eV$ close to the upper bound. Furthermore, the effective mass $|m_{\beta\beta}|$ can take the values 12.650 meV, 33.044 meV, 22.821 meV, 48.936 meV and 19.192 meV in this model. The most stringent upper limit on $|m_{\beta\beta}|$ from GERDA \(^{[52]}\), EXO-200 \(^{[53,54]}\) and KamLAND-ZEN \(^{[55]}\) is $|m_{\beta\beta}| < (120-250)meV$. Hence our predictions for $|m_{\beta\beta}|$ are compatible with present experimental measurements. Our model could be directly tested by future neutrinoless double-beta decay experiments such as nEXO which is expected to have the mass sensitivity of 5 ~ 11 meV \(^{[56]}\).

Higher dimensional operators, suppressed by additional powers of the cutoff scale $\Lambda$, can be added to the leading terms studied above. As a result, the LO predictions would be modified. The subleading operators can be obtained by inserting $\Psi_\nu$ into the LO terms in all possible ways and by extracting the $S_4$ invariants\(^7\) where $\Psi_l = \{\phi, \varphi_T\}$ and $\Psi_\nu = \{\xi, \eta, \varphi_S\}$. The auxiliary symmetry $Z_4 \times Z_2$ prevents insertion of one power of the flavons $\Psi_l$ or $\Psi_\nu$. Obviously the higher order terms in the driving superpotential $w_l$ induce shifts in the vacuum of $\Psi_l$ and $\Psi_\nu$ at relative order $\lambda^4$. The neutrino mass matrix $m_\nu$ and the charged lepton mass matrix $m_l$ receive corrections from both the higher order terms of the superpotential $w_\nu, w_l$ and the shifted VEVs. Detailed and straightforward calculations demonstrate that each entry of $m_\nu$ and $m_l$ is corrected by terms of relative order $\lambda^4$. Hence the subleading contributions to the lepton masses and mixing parameters are suppressed by $\langle \Phi_\nu \rangle^2/\Lambda^2 \sim \lambda^4$ with respect to LO results and thus they can be ignored.

### 7 Model predicting one row of BM mixing with $S_4$ and generalized CP

Since the previous model naturally gives rise to the case I and case II of section \(^4\) in the following we shall present an explicit model realization for the remaining case III and case IV investigated in section \(^5\). The model is also based on $S_4$ family symmetry and generalized CP, which is supplemented by $Z_5 \times Z_6$. The flavon fields and driving fields are properly arranged such that $S_4 \times H_{CP}$ is broken to $K_4^{(TST^2,T^2U)} \times H_{CP}$ with $H_{CP}^\nu = \{\rho_\nu(T^{2U}), \rho_\nu(TST^2), \rho_\nu(ST^2SU)\}$ in the neutrino sector at leading order, and the flavor symmetry preserved by the charged lepton mass matrix $m_l \dagger m_l$ is $K_4^{(SU)}$. As a result, the lepton flavor mixing is predicted to be of the BM form at leading order. Furthermore, the next-to-leading-order (NLO) corrections break the remnant symmetry down to $Z_2^{SU} \times H_{CP}$ in the charged lepton sector. Consequently the resulting PMNS matrix has

\(^7\)The driving superpotential also receives corrections from terms $(\eta^0\Phi_l^l)_1$ and $(\varphi^0_S\Phi_l^l)_1$.
where all couplings $f_i$ and $g_i$ are real due to the imposed generalized CP symmetry. In the charged lepton sector, the equations for the vanishing of the derivatives of $w_d$ with respect to each component of the driving fields are as follows:

$$\frac{\partial w_d}{\partial \xi} = M^2 \xi + f_1(2\varphi_{T1}\varphi_{T1} + \varphi_{T2}^2) = 0,$$

$$\frac{\partial w_d}{\partial \rho} = f_2 \xi^2 + f_3(\eta_1^2 + \eta_2^2) = 0,$$

$$\frac{\partial w_d}{\partial \eta} = f_4(\varphi_{T1}\varphi_3 + \varphi_{T2}\varphi_2 + \varphi_{T3}\varphi_1) = 0,$$

$$\frac{\partial w_d}{\partial \eta_1} = f_5 \eta_1 + \sqrt{3} f_6(\varphi_{T1}\varphi_1 + \varphi_{T2}\varphi_3) = 0,$$

$$\frac{\partial w_d}{\partial \eta_2} = f_5 \eta_2 + f_6(\varphi_{T1}\varphi_3 - 2\varphi_{T2}\varphi_2 + \varphi_{T3}\varphi_1) = 0,$$

$$\frac{\partial w_d}{\partial \varphi_{T1}} = f_7(\varphi_{T1}\varphi_2 + \varphi_{T3}\varphi_1) = 0,$$

$$\frac{\partial w_d}{\partial \varphi_{T2}} = f_7(\varphi_{T1}\varphi_1 - \varphi_{T3}\varphi_3) = 0,$$

$$\frac{\partial w_d}{\partial \varphi_{T3}} = -f_7(\varphi_{T2}\varphi_3 + \varphi_{T3}\varphi_2) = 0,$$

$$\frac{\partial \omega_d}{\partial \kappa} = M^2 + f_8(2\varphi_1\varphi_3 + \varphi_2^2) = 0.$$  (7.2)

Table 8: The particle contents and their transformation properties under the family symmetry $S_4 \times Z_5 \times Z_6$ and $U(1)_R$, where $\omega_5 = e^{2i\pi/5}$ and $\omega_6 = e^{2i\pi/6}$.

one row of the form $(1/2, 1/2, 1/\sqrt{2})$ which is exactly the third row of the BM mixing pattern, and agreement with experimental data can be achieved. As we shall show below, the general model independent results of case III and case IV can be naturally reproduced in this model. The involved fields and their transformation rules under the family symmetry are summarized in Table 8. We start to explore the vacuum structure of the model in the following section.

7.1 Vacuum alignment

The most general flavon superpotential invariant under the symmetry of the model is

$$w_d = M_\xi \xi_1 \xi + f_1 \xi_1 (\varphi_{T1}\varphi_{T1})_1 + f_2 \xi_1^2 + f_3 \xi_1 (\eta_1 \eta_1 + \eta_2^2) + f_4 \xi_1 (\varphi_{T1}\varphi_3 + \varphi_{T2}\varphi_2 + \varphi_{T3}\varphi_1) + f_5 \xi_1 \xi_1 \xi_1$$

where all couplings $f_i$ and $g_i$ are real due to the imposed generalized CP symmetry. In the charged lepton sector, the equations for the vanishing of the derivatives of $w_d$ with respect to each component of the driving fields are as follows:

$$\frac{\partial w_d}{\partial \xi} = M_\xi \xi + f_1(2\varphi_{T1}\varphi_{T1} + \varphi_{T2}^2) = 0,$$

$$\frac{\partial w_d}{\partial \rho} = f_2 \xi^2 + f_3(\eta_1^2 + \eta_2^2) = 0,$$

$$\frac{\partial w_d}{\partial \eta} = f_4(\varphi_{T1}\varphi_3 + \varphi_{T2}\varphi_2 + \varphi_{T3}\varphi_1) = 0,$$

$$\frac{\partial w_d}{\partial \eta_1} = f_5 \eta_1 + \sqrt{3} f_6(\varphi_{T1}\varphi_1 + \varphi_{T2}\varphi_3) = 0,$$

$$\frac{\partial w_d}{\partial \eta_2} = f_5 \eta_2 + f_6(\varphi_{T1}\varphi_3 - 2\varphi_{T2}\varphi_2 + \varphi_{T3}\varphi_1) = 0,$$

$$\frac{\partial w_d}{\partial \varphi_{T1}} = f_7(\varphi_{T1}\varphi_2 + \varphi_{T3}\varphi_1) = 0,$$

$$\frac{\partial w_d}{\partial \varphi_{T2}} = f_7(\varphi_{T1}\varphi_1 - \varphi_{T3}\varphi_3) = 0,$$

$$\frac{\partial w_d}{\partial \varphi_{T3}} = -f_7(\varphi_{T2}\varphi_3 + \varphi_{T3}\varphi_2) = 0,$$

$$\frac{\partial \omega_d}{\partial \kappa} = M_\kappa^2 + f_8(2\varphi_1\varphi_3 + \varphi_2^2) = 0.$$  (7.2)
We find that there are only two solutions (up to $S_4$ transformations) for above equations. The first one is

$$\langle \xi \rangle = v_\xi, \quad \langle \eta \rangle = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) v_\eta, \quad \langle \varphi_T \rangle = \left( \begin{array}{c} 1 + i \\ 0 \\ i - 1 \end{array} \right) v_T, \quad \langle \phi \rangle = \left( \begin{array}{c} i - 1 \\ 0 \\ 1 + i \end{array} \right) v_\phi. \quad (7.3)$$

The second solution is of the form

$$\langle \xi \rangle = v_\xi, \quad \langle \eta \rangle = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) v_\eta, \quad \langle \varphi_T \rangle = \left( \begin{array}{c} i - 1 \\ 0 \\ 1 + i \end{array} \right) v_T, \quad \langle \phi \rangle = \left( \begin{array}{c} 1 + i \\ 0 \\ i - 1 \end{array} \right) v_\phi, \quad (7.4)$$

where $v_\xi$ is undetermined for both solutions, and the VEVs $v_\xi, v_\eta, v_T$ and $v_\phi$ are related by

$$v_\eta^2 = -\frac{f_2}{f_3} v_\xi^2, \quad v_T^2 = \frac{M_\xi v_\xi}{4 f_1}, \quad v_\phi = \frac{f_5 v_\xi v_\eta}{4 \sqrt{3} f_6 v_T}, \quad (7.5)$$

with

$$v_\xi^3 = -\frac{3 f_3 f_6^2 M_\xi M_\kappa^2}{f_1 f_2 f_5^2 f_8}. \quad (7.6)$$

Hence the VEV $v_\xi$ is fixed to be

$$v_\xi = -\left( \frac{3 f_3 f_6^2 M_\xi M_\kappa^2}{f_1 f_2 f_5^2 f_8} \right)^{1/3}, \quad \left( \frac{3 f_3 f_6^2 M_\xi M_\kappa^2}{f_1 f_2 f_5^2 f_8} \right)^{1/3} e^{i \pi/3}, \text{ or } \left( \frac{3 f_3 f_6^2 M_\xi M_\kappa^2}{f_1 f_2 f_5^2 f_8} \right)^{1/3} e^{5i \pi/3}. \quad (7.7)$$

In the present paper, we shall concentrate on the fist solution, i.e. the case of real $v_\xi$. The other two options of complex $v_\xi$ would not be considered. Accordingly the VEVs $v_\eta, v_T$ and $v_\phi$ would be real or pure imaginary. If $v_\eta, v_T$ and $v_\phi$ are all real parameters, this can be achieved for $f_2 f_3 < 0$ and $f_1 M_\xi v_\xi > 0$, the residual CP symmetry preserved by the vacuum of Eq. (7.3) is $H_{CP}^1 = \{ \rho_r(TST^2), \rho_r(TST^2U) \}$. If $v_\eta$ is real and $v_T, v_\phi$ are pure imaginary, this can be realized for $f_2 f_3 < 0$ and $f_1 M_\xi v_\xi < 0$, another two of the 24 generalized CP symmetries are preserved with $H_{CP}^1 = \{ \rho_r(T^2ST), \rho_r(T^2STU) \}$. On the other hand, the generalized CP symmetry $H_{CP}^1$ will be completely broken for imaginary $v_\eta$ no matter $v_T, v_\phi$ are real or imaginary. Furthermore, since the different VEVs are related via dimensionless couplings in Eq. (7.5), these VEVs are expected to have the same order of magnitude which we choose to be $\lambda^2$. It is easy to check that the first vacuum in Eq. (7.3) breaks $S_4$ to $Z_2^{SU}$ subgroup and the second one in Eq. (7.4) breaks $S_4$ to $Z_2^U$. Because the remnant symmetries $Z_2^{SU}$ and $Z_2^U$ are conjugate to each other, as shown in the footnote 6 both types of vacuum give rise to the same PMNS matrix. Consequently we shall choose the first vacuum for demonstration in the following model.

In the neutrino sector, the vacuum is determined by $F$–term conditions associated with the driving fields $\sigma^0$ and $\varphi_S^0$,

$$\frac{\partial w_d}{\partial \sigma_1^0} = g_1 \rho \sigma_1 + g_2 (\sigma_2^2 - \sigma_1^2) + 2 g_3 (\varphi_S^2 - \varphi_S \varphi_S) = 0,$$

$$\frac{\partial w_d}{\partial \sigma_2^0} = g_1 \rho \sigma_2 + 2 g_2 \sigma_1 \sigma_2 + \sqrt{3} g_3 (\varphi_S^2 + \varphi_S) = 0,$$

$$\frac{\partial w_d}{\partial \varphi_S} = g_4 \rho \varphi_S + g_5 (\sqrt{3} \sigma_2 \varphi_S - \sigma_1 \varphi_S) + 2 g_6 \varphi_S \varphi_S = 0,$$

$$\frac{\partial w_d}{\partial \varphi_S^0} = g_4 \rho \varphi_S^0 + 2 g_5 \sigma_1 \varphi_S^0 + g_6 (\varphi_S^2 - \varphi_S) = 0,$$

$$\frac{\partial w_d}{\partial \varphi_S} = g_4 \rho \varphi_S + g_5 (\sqrt{3} \sigma_2 \varphi_S - \sigma_1 \varphi_S) + 2 g_6 \varphi_S \varphi_S = 0,$$

$$\frac{\partial w_d}{\partial \varphi_S^0} = g_4 \rho \varphi_S^0 + 2 g_5 \sigma_1 \varphi_S^0 + g_6 (\varphi_S^2 - \varphi_S) = 0,$$
with operators, which will be studied in section 7.3. From the view of symmetry and its breaking, consequently the hermitian combination

\[ m = v_{\rho} \sqrt{g_4 (2g_1 g_5 + g_2 g_4)} / g_3, \]

where dots represent higher dimensional operators which we will consider later. After the VEVs obey the relations

\[ v_{\sigma} = g_4 v_{\rho} / 4g_5, \quad v_S = v_{\rho} / 4g_5 \sqrt{-g_4 (2g_1 g_5 + g_2 g_4) / g_3}, \]

with \( v_{\rho} \) undetermined. The vacuum alignment in Eq. (7.9) is invariant under the action of both the \( TST^2 \) and \( T^2U \) elements of \( S_4 \), consequently it breaks the \( S_4 \) family symmetry to Klein four \( K_4 \) subgroup. Furthermore, since all couplings \( g_i \) are real, then Eq. (7.10) implies that \( v_{\sigma} \) and \( v_{\rho} \) have the same phase up to \( \pi \), and the phase difference between \( v_{\rho} \) and \( v_S \) is 0, \( \pi \) or \( \pm \pi / 2 \) determined by the sign of \( g_3 g_4 (2g_1 g_5 + g_2 g_4) \). Similar to previous model, we expect a common order of magnitude for all the VEVs which is taken to be \( \lambda^2 \). \[ \text{7.2 Leading order results} \]

The charged lepton masses are described by the following superpotential

\[ w_l = y_{\tau} \mu^c (l \phi_T) T h_d + y_{\mu} \Lambda^2 \mu^c \phi (l \phi_T) T h_d + \ldots, \]

where dots represent higher dimensional operators which we will consider later. After the electroweak and flavor symmetries breaking by the VEVs shown in Eq. (7.3), we obtain a charged lepton mass matrix as follows

\[ m_l = \begin{pmatrix} 0 & 0 & 0 \\ \frac{(1+i) y_{\mu} v_{\rho} v_{\phi}}{\Lambda^2} & 0 & \frac{(1+i) y_{\mu} v_{\rho} v_{\phi}}{\Lambda^2} \\ \frac{(1+i) y_{\tau} v_{\rho} v_{\phi}}{\Lambda^2} & 0 & \frac{(1+i) y_{\tau} v_{\rho} v_{\phi}}{\Lambda^2} \end{pmatrix} v_d. \]

Consequently the hermitian combination \( m_l^T m_l \) of is of the form

\[ m_l^T m_l = 2 \begin{pmatrix} y_{\tau}^2 \frac{|v_{\tau}|^2}{\Lambda^4} + y_{\mu}^2 \frac{|v_{\mu}|^2}{\Lambda^4} & 0 & i \left( -y_{\tau}^2 \frac{|v_{\tau}|^2}{\Lambda^4} + y_{\mu}^2 \frac{|v_{\mu}|^2}{\Lambda^4} \right) \\ 0 & 0 & 0 \\ i \left( y_{\tau}^2 \frac{|v_{\tau}|^2}{\Lambda^4} - y_{\mu}^2 \frac{|v_{\mu}|^2}{\Lambda^4} \right) & 0 & y_{\tau}^2 \frac{|v_{\tau}|^2}{\Lambda^4} + y_{\mu}^2 \frac{|v_{\mu}|^2}{\Lambda^4} \end{pmatrix} v_d^2. \]

which is exactly diagonalized by the unitary matrix \( W_l \) in Eq. (5.12),

\[ W_l^T m_l^T m_l W_l = \text{diag}(m_{\mu}^2, m_{\mu}^2, m_{\tau}^2), \]

with

\[ m_{\epsilon}^2 = 0, \quad m_{\mu}^2 = 4y_{\mu}^2 \frac{|v_{\epsilon} v_{\phi}|^2}{\Lambda^4} v_d^2, \quad m_{\tau}^2 = 4y_{\tau}^2 \frac{|v_{\tau}|^2}{\Lambda^2} v_d^2. \]

Note that the correct mass hierarchy between muon and tau is generated for \( v_{\epsilon}/\Lambda \sim v_{\tau}/\Lambda \sim v_{\phi}/\Lambda \sim \lambda^2 \). The electron is massless at LO and its mass is generated by higher dimensional operators, which will be studied in section 7.3. From the view of symmetry and its breaking,
although the VEVs of $\xi$, $\eta$, $\varphi_T$ and $\phi$ leave $Z^S_2$ invariant, the remnant symmetry of $m^T_i m_i$ is $K^{(S,U)}_4$. In other words, we have $\rho_3(S)m^T_i m_i \rho_3(S) = m^T_i m_i$ and $\rho_3(U)m^T_i m_i \rho_3(U) = m^T_i m_i$. The enhancement of the remnant symmetry from $Z^S_2$ to $K^{(S,U)}_4$ is because that $|v_\phi|^2$ and $|v_\rho|^2$ instead of $v_T$ and $v_\phi$ are involved in $m^T_i m_i$.

Now we come to the neutrino sector. The LO superpotential of for the neutrino masses is

$$w_\nu = y_\nu^c l_1 h_u + y_1 \rho (\nu^c \nu) l_1 + y_2 (\nu^c \nu^c)_{2 \sigma} l_1 + y_3 ((\nu^c \nu^c)_{3 \varphi} S \varphi S)_1,$$  \hspace{1cm} (7.16)$$

where the first term is Dirac mass term and the last three are Majorana mass terms. The generalized CP symmetry constrains all the couplings to be real. The flavons $\rho$, $\sigma$ and $\varphi_S$ get VEVs shown in Eq. (7.9), and then the Dirac and right-handed Majorana neutrino mass matrices read as

$$m_D = yv_u \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad m_M = y_1 v_{\rho} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + y_2 v_{\sigma} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} + y_3 v_S \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \hspace{1cm} (7.17)$$

The light neutrino Majorana mass matrix is given by the seesaw relation

$$m_\nu = -m_D^T m_M^{-1} m_D = a \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} + c \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \hspace{1cm} (7.18)$$

where the three parameters $a$, $b$ and $c$ are

$$a = \frac{[-3y_1^2 v_\rho^2 + 2 (6y_2^2 v_\sigma^2 + y_3^2 v_S^2)] y_2^2 v_u^2}{3(y_1 v_\rho - 4y_2 v_\sigma) \left((y_1 v_\rho + 2y_2 v_\sigma)^2 - 2y_3^2 v_S^2 \right)},$$

$$b = \frac{[3y_2^2 v_\sigma (y_1 v_\rho + 2y_2 v_\sigma) - y_3^2 v_S^2] y_2^2 v_u^2}{3(y_1 v_\rho - 4y_2 v_\sigma) \left((y_1 v_\rho + 2y_2 v_\sigma)^2 - 2y_3^2 v_S^2 \right)},$$

$$c = \frac{y_3 y_2^2 v_S v_u^2}{(y_1 v_\rho + 2y_2 v_\sigma)^2 - 2y_3^2 v_S^2}. \hspace{1cm} (7.19)$$

We see that the light neutrino mass matrix $m_\nu$ in Eq. (7.18) is of the same form as that in Eq. (5.5), and it is the most general neutrino matrix invariant under $K^{(TST^2,T^2U)}_4$ subgroup. Hence $m_\nu$ is exactly diagonalized by the unitary transformation $U_\nu$ shown in Eq. (5.7),

$$K_\nu^T U_\nu^T m_\nu U_\nu K_\nu = \text{diag}(m_1, m_2, m_3), \hspace{1cm} (7.20)$$

where $K_\nu$ is a diagonal phase matrix which sets $m_i (i = 1, 2, 3)$ to be positive. The masses of three neutrinos are

$$m_1 = \left| a + 2b - \sqrt{2}c \right|, \quad m_2 = \left| a + 2b + \sqrt{2}c \right|, \quad m_3 = \left| a - 4b \right|. \hspace{1cm} (7.21)$$

Here the VEVs of $\rho$, $\sigma$ and $\varphi_S$ breaks both $S_4$ family symmetry and generalized CP in the neutrino sector. From the vacuum alignment of section 7.1 we know that the remnant family symmetry is $K^{(TST^2,T^2U)}_4$. Since the phase of $v_\rho$ can be factored out from $m_\nu$, $v_\rho$ can be taken to be real. As a consequence, $v_\sigma$ is real and $v_S$ can be real or purely imaginary. If $v_S$ is imaginary, this can be realized for $g_3 g_4 (2g_1 g_S + g_2 g_4) > 0$. We find the generalized CP symmetry $H_{CP}$ is broken to $H_{CP} = \{ \rho_T(S), \rho_T(ST^2U), \rho_T(T^2ST), \rho_T(T^2SU) \}$ in the neutrino sector. The parameters $a$, $b$ are real while $c$ is purely imaginary. Therefore the light neutrino masses would be partially degenerate with $m_1 = m_2$ which is not consistent with the precisely measured $\delta m^2 = m_2^2 - m_1^2 \approx 7.54 \times 10^{-5} \text{eV}^2 \neq 0$. If $v_S$ is real, this
scenario can be achieved for $g_3g_4(2g_1g_5 + g_2g_4) < 0$. The residual CP symmetry is $H_{CP} = \{\rho_r(T^2U), \rho_r(TST^2), \rho_r(ST^2SU)\}$ in this case. Then all the three parameters $a$, $b$ and $c$ are real. The phenomenological constraints of $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$ can be easily satisfied by properly choosing the values of $a$, $b$ and $c$. Either NO or IO neutrino mass spectrum is allowed. Hence we shall assume $\nu_S$ is real in the following.

Combining the unitary transformation $W_l$ and $U_\nu K_\nu$ from the charged lepton and the neutrino sectors, we obtain the lepton mixing matrix

$$U_{PMNS} = W_l^\dagger U_\nu K_\nu = \frac{1}{2} \begin{pmatrix} -\sqrt{2} & \sqrt{2} & 0 \\ e^{-i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} & \sqrt{2}e^{i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} & \sqrt{2}e^{-i\frac{\pi}{4}} \end{pmatrix} K_\nu = K_l U_{BM} K'_\nu,$$  

(7.22)

where $K_l$ and $K'_\nu$ are diagonal phase matrices:

$$K_l = \text{diag}(-1, e^{-i\frac{\pi}{4}}, e^{i\frac{\pi}{4}}), \quad K'_\nu = \text{diag}(1, 1, -i)K_\nu.$$  

(7.23)

The unphysical phase matrix $K_l$ can absorbed by lepton fields. Therefore the lepton mixing matrix is BM mixing at LO. In the following section, we shall analyze the higher order corrections needed to modify the BM mixing in order to obtain an acceptable lepton mixing pattern.

### 7.3 Next-to-leading-order corrections

In brief, at leading order the model gives rise to a vanishing electron mass ($m_e = 0$) and the BM mixing pattern leading to $\theta_{13} = 0^\circ$ and $\theta_{12} = \theta_{23} = 45^\circ$ which obviously don’t match with the experimental measurements. Therefore the next-to-leading-order (NLO) corrections are crucial to achieve agreement with the present data. We will demonstrate in the following that a non-zero electron mass and realistic mass hierarchies among the charged lepton are obtained after the NLO contributions are included. In addition, the LO remnant symmetry $K_{\text{R(SU)}}$ of $m^m m_l$ is further broken down to $Z_2^{SU}$ such that the symmetry breaking patterns of case III and case IV discussed in section 5 are realized and the resulting PMNS matrix is of the form of Eq. (5.18) or Eq. (5.27). We first start with the corrections to the flavon superpotential $w_d$ in Eq. (7.1) which determines the vacuum alignment. The symmetry allowed NLO terms including the driving fields $\xi^0, \rho^0, \zeta^0, \eta^0, \varphi_T^0$ and $\kappa^0$ are

\[
\delta w_d = f_{9}\xi^0\xi(\varphi_T\varphi_T)_{1}/\Lambda + f_{10}\rho^0(\varphi_T(\varphi_T)_{3})_{1}/\Lambda + f_{11}\rho^0(\varphi_T(\varphi_T)_{3})_{1}/\Lambda + f_{12}\xi^0(\eta(\varphi_T\varphi_T)_{2})_{1}/\Lambda + f_{13}\xi^0(\eta(\varphi_T\varphi_T)_{2})_{1}/\Lambda + f_{14}\xi^0(\eta(\varphi_T\varphi_T)_{2})_{1}/\Lambda + f_{15}\xi^0(\eta(\varphi_T\varphi_T)_{2})_{1}/\Lambda + f_{16}\xi^0(\eta(\varphi_T\varphi_T)_{2})_{1}/\Lambda + f_{17}\xi^0(\eta(\varphi_T\varphi_T)_{2})_{1}/\Lambda + f_{18}\xi^0(\eta(\varphi_T\varphi_T)_{2})_{1}/\Lambda + f_{19}\xi^0(\eta(\varphi_T\varphi_T)_{2})_{1}/\Lambda + f_{20}\xi^0(\eta(\varphi_T\varphi_T)_{2})_{1}/\Lambda + f_{21}\xi^0(\eta(\varphi_T\varphi_T)_{2})_{1}/\Lambda + f_{22}\kappa^0(\varphi_T(\varphi_T\varphi_T)_{3})_{1}/\Lambda.
\]

(7.24)

We see that they are suppressed by one of power of $1/\Lambda$ with respect to the LO terms in Eq. (7.1). The new vacuum configuration is obtained by searching for the zeros of the $F-$terms of $w_d + \delta w_d$ with respect to the driving fields $\xi^0, \rho^0, \zeta^0, \eta^0, \varphi_T^0$ and $\kappa^0$. To the first order in $1/\Lambda$ expansion, the LO vacuum alignment of the charged lepton sector is modified into

\[
\langle \xi \rangle = v_\xi + \delta v_\xi, \quad \langle \eta \rangle = \begin{pmatrix} v_\eta + \delta v_{\eta 1} \\ 0 \\ 0 \end{pmatrix}, \quad \langle \varphi_T \rangle = \begin{pmatrix} (1 + i)(v_T + \delta v_{T 1}) \\ \delta v_{T 2} \\ (i - 1)(v_T + \delta v_{T 3}) \end{pmatrix}, \quad \langle \phi \rangle = \begin{pmatrix} i(v_\phi + \delta v_{\phi 1}) \\ -i\delta v_{\phi 2} \\ (1 + i)(v_\phi + \delta v_{\phi 3}) \end{pmatrix}.
\]

(7.25)
The shifts $\delta v_\xi, \delta v_\eta, \delta v_T$ and $\delta v_\phi$ are solved to be

$$
\delta v_\xi = X \frac{v_\xi}{\Lambda}, \quad \delta v_\eta_1 = (X - \frac{f_{10}}{2f_{12}f_{20}}) \frac{M_e v_\eta}{\Lambda}, \\
\delta v_\eta_2 = \delta v_{T_2} = 0, \quad \delta v_{T_1} = \delta v_{T_3} = (X - \frac{f_6 M^2}{2f_{6}^2}) \frac{M_\xi v_T}{2\Lambda}, \\
\delta v_{\phi_1} = \delta v_{\phi_3} = \frac{3f_6 (f_2 - f_{20} - f_{12} - f_{19})}{2f_{12}f_{20}} \frac{M_e v_{\phi_0}}{\Lambda}, \quad \delta v_{\phi_2} = -\frac{\sqrt{3}(f_{15} + f_{16}) v_\rho^2 v_{\phi_0}}{f_{rev_\Lambda}},
$$

(7.26)

where $X$ is a real parameter of order one with

$$
X = \frac{[f_2 f_5 (2f_8 (f_{13} + f_{14}) - 3f_6 f_{21}) + f_3^2 f_8 f_{10} + 3f_6^2 (f_2 f_{20} - f_3 f_{19})] M^2_\xi - f_1 f_2 f_5^2 f_9 M^2}{3f_1 f_2 f_5^2} \frac{M^2_\xi}{\Lambda}
$$

(7.27)

Notice that the shifts of the vacuum are suppressed by $\lambda^2$ compared with the LO VEVs, and the structure of the LO vacuum of the flavons $\eta$ and $\varphi_T$ is unchanged by the NLO corrections. Because the NLO driving superpotential $\delta w'_d$ only contain the charged lepton flavon fields $\xi, \eta, \varphi_T$ and $\phi$, hence their VEVs still preserve the $Z^S_{2SU}$ subgroup even at NLO. Indeed the vacuum in Eq. (7.25) is the most general form which is compatible with the residual family symmetry $Z^S_{2SU}$ in the charged lepton sector.

In the same way, the subleading corrections to the flavon superpotential of $\rho, \sigma$ and $\varphi_S$ are of the form

$$
\delta w'_\nu = (\sigma^0 \xi \varphi_T \Psi^2_\nu)_1 / \Lambda^2 + (\sigma^0 \phi^2 \Psi^2_\nu)_1 / \Lambda^2 + (\varphi^0 S \xi \varphi_T \Psi^2_\nu)_1 / \Lambda^2 + (\varphi^0 S \phi^2 \Psi^2_\nu)_1 / \Lambda^2.
$$

(7.28)

where $\Psi_\nu = \{\rho, \sigma, \varphi_S\}$ denotes the neutrino flavon fields, and the real coupling constant in front of each term has been omitted. The resulting contributions to the $F$–terms of the driving fields $\sigma^0$ and $\varphi^0_S$ are suppressed by $(\xi, \varphi_T)/\Lambda^2 \sim \langle \phi \rangle^2 / \Lambda \sim \lambda^4$ with respect to the LO terms in Eq. (7.1). Hence they induce shifts in the VEVs of $\rho, \sigma$ and $\varphi_S$ at relative order $\lambda^4$. After some straightforward algebra, the new VEVs can be written as

$$
\langle \rho \rangle = v_\rho, \quad \langle \sigma \rangle = \left(1 + \epsilon_1 \lambda^4 \right) v_\sigma, \quad \langle \varphi_S \rangle = \left(1 + \epsilon_3 \lambda^4 \right) v_\sigma,
$$

(7.29)

where $v_\rho$ remains undetermined, and the coefficients $\epsilon_i (i = 1, 2, \ldots, 5)$ are unspecified constants with absolute value of order one. In the following we study the subleading corrections to the LO mass matrices from both the modified vacuum and higher dimensional operators in the Yukawa superpotential $w_l$ and $w_\nu$. In the neutrino sector, the subleading operators are obtained by adding to each term of $w_\nu$ the factor of $\xi \varphi_T$ or $\phi^2$ in all possible ways, i.e.

$$
\delta w_\nu = (\nu^\dagger \xi \varphi_T)_1 h_u / \Lambda^2 + (\nu^\dagger \phi^2)_1 h_u / \Lambda^2 + (\nu^\dagger \nu^\dagger \xi \varphi_T \Psi_\nu)_1 / \Lambda^2 + (\nu^\dagger \nu^\dagger \phi^2 \Psi_\nu)_1 / \Lambda^2.
$$

(7.30)

In addition to these corrections, we have to consider the ones from $w_\nu$ in Eq. (7.16) with the deviations of the VEVs at NLO, as shown in Eq. (7.29). Eventually we find that the neutrino mass matrix is corrected by terms of relative order $\lambda^4$ in every entry. As a result, the lepton mixing parameters acquire corrections of order $\lambda^4$ which can be safely neglected.

The NLO operators contributing to the charged lepton masses are given by

$$
\delta w_l = y_{e_1} e^\xi (l(\eta v)^3)_1 h_d / \Lambda^3 + y_{e_2} e^\xi (l(\varphi_T \varphi_T)_3)_1 h_d / \Lambda^3 + y_{e_3} e^\xi ((l \varphi_T)_2 (\phi \phi)_2)_1 h_d / \Lambda^3 + y_{e_4} e^\epsilon ((l \varphi_T)_3 (\phi \phi)_3)_1 h_d / \Lambda^3 + y_{e_5} e^\epsilon ((l \eta \varphi_T)_3 (\phi \phi)_3)_1 h_d / \Lambda^3
$$

(7.31)

$$
+ y_{e_6} e^\epsilon ((l \eta)_3 (\phi \phi)_3)_1 h_d / \Lambda^3 + y_{e_7} e^\epsilon ((l \eta)_3 (\phi \phi)_3)_1 h_d / \Lambda^3 + y_{e_8} e^\epsilon ((l \eta)_3 (\phi \phi)_3)_1 h_d / \Lambda^3 + y_{e_9} e^\epsilon ((l \eta)_3 (\phi \phi)_3)_1 h_d / \Lambda^3
$$

(7.32)

$$
+ y_{e_{10}} e^\epsilon ((l \eta)_3 (\phi \phi)_3)_1 h_d / \Lambda^3 + y_{e_{11}} e^\epsilon ((l \eta)_3 (\phi \phi)_3)_1 h_d / \Lambda^3 + y_{e_{12}} e^\epsilon ((l \eta)_3 (\phi \phi)_3)_1 h_d / \Lambda^3 + y_{e_{13}} e^\epsilon ((l \eta)_3 (\phi \phi)_3)_1 h_d / \Lambda^3
$$

(7.33)

34
+ y_m \mu^c ((l \phi) 2 (\varphi_T \varphi_T) 2) \frac{h_d}{\Lambda^3} + y_m \mu^c ((l \phi) 3 (\varphi_T \varphi_T) 3) \frac{h_d}{\Lambda^3}

\text{The charged lepton mass matrix is obtained by inserting the shifted vacuum alignment of Eq. (7.25) into the LO operators plus the contribution of these higher dimensional operators evaluated with the LO VEVs of Eq. (7.3). We find that the charged lepton mass matrix including NLO corrections takes the following form}

\begin{equation}
\begin{aligned}
m_l \simeq & \begin{pmatrix}
(1 + i) a_1 v_T v_\phi^2 / \Lambda^3 & 4 i y_{e_2} v_\xi v_\phi^2 / \Lambda^3 & (1 - i) a_1 v_T v_\phi^2 / \Lambda^3 \\
(1 + i) y_\mu v_\xi v_\phi / \Lambda^2 & - i b_1 v_\xi v_\phi v_\phi^2 / (\Lambda^3 v_T) & (1 - i) y_\mu v_\xi v_\phi / \Lambda^2 \\
(1 - i) y_\tau v_T / \Lambda & 0 & (1 + i) y_\tau v_T / \Lambda \\
\end{pmatrix} v_d ,
\end{aligned}
\end{equation}

\text{where both } a_1 \text{ and } b_1 \text{ are real parameters,}

\begin{equation}
a_1 = 4 \sqrt{3} y_e + y_{e_4} + y_{e_4} v_{\nu e / v_T} = 4 \sqrt{3} y_{e_3} + y_{e_4} + 4 \sqrt{3} y_{e_1} \frac{f_5}{f_7},
\end{equation}

\begin{equation}
b_1 = \frac{v_T}{v_\xi v_\phi v_\phi^2} (y_\mu v_\xi \delta v_\phi^2 \Lambda + 8 y_\mu v_\xi v_\phi^2 / \Lambda^3) = \frac{2 y_\mu f_5}{3 f_7} - \frac{3 y_{\mu} f_{15} + f_{16}}{f_7}. \end{equation}

\text{In order to diagonalize the charged lepton mass matrix } m_l^\dagger m_l, \text{ it is helpful to use the unitary transformation } W_l \text{ in Eq. (5.12) first, i.e.}

\begin{equation}
W_l^\dagger m_l^\dagger m_l W_l \simeq \begin{pmatrix}
16 y_{e_2} \frac{v_\xi^2 |v_T|^4}{\Lambda^4} + b_1 \frac{v_\xi^2 |v_\phi|^2 |v_\phi|^4}{v_T / \Lambda^3} & - (1 + i) \sqrt{2} b_1 y_\mu \frac{v_\xi^2 |v_\phi|^2 |v_\phi|^2}{v_T / \Lambda^3} & 0 \\
(1 - i) \sqrt{2} b_1 y_\mu \frac{v_\xi^2 |v_\phi|^2 |v_\phi|^2}{v_T / \Lambda^3} & 4 y_\mu^2 \frac{v_\xi^2 |v_\phi|^2}{\Lambda^4} & 0 \\
0 & 0 & 4 y_\tau^2 \frac{v_\tau v_T^2}{\Lambda^4} \\
\end{pmatrix} v_d ,
\end{equation}

\text{From Eq. (7.5) and Eq. (7.7), we see that } v_\tau^2 \text{ is real since } v_\xi \text{ is chosen to be real, while the VEV } v_T \text{ can be real or pure imaginary depending on the sign of the product } f_2 f_3 f_8. \text{ In case of } f_2 f_3 f_8 < 0, v_T \text{ is real such that the above mass matrix } W_l^\dagger m_l^\dagger m_l W_l \text{ is of the same form as the one shown in Eq. (5.13). As a consequence, the case III analyzed in section 5 is realized. Then the lepton mixing matrix is of the form}

\begin{equation}
U_{PMNS} = K_l \frac{1}{2} \begin{pmatrix}
- \sin \theta - \sqrt{2} \cos \theta & - \sin \theta + \sqrt{2} \cos \theta & \sqrt{2} \sin \theta \\
\cos \theta - \sqrt{2} \sin \theta & \cos \theta + \sqrt{2} \sin \theta & - \sqrt{2} \cos \theta \\
1 & 1 & \sqrt{2} \\
\end{pmatrix} K_\nu ,
\end{equation}

\text{with } K_l = \text{diag}(e^{-i \pi / 4}, e^{-i \pi / 4}, e^{i \pi / 4}) \text{ and } K_\nu = \text{diag}(1, 1, -i). \text{ The parameter } \theta \text{ is}

\begin{equation}
tan 2 \theta = - \frac{b_1}{y_\mu v_T / \Lambda} \frac{v_\mu^2}{v_\tau / \Lambda} .
\end{equation}

\text{The lepton mixing angle are given by}

\begin{equation}
\sin^2 \theta_{13} = \frac{1}{2} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1}{2} - \frac{\sqrt{2} \sin 2 \theta}{3 + \cos 2 \theta}, \quad \sin^2 \theta_{23} = \frac{1 + \cos 2 \theta}{3 + \cos 2 \theta},
\end{equation}

\text{and both Dirac CP phase and Majorana CP phases are trivial. Very good agreement with the experimental data can be achieved (see Table [3]). The charged lepton masses are determined to be}

\begin{equation}
m_e \simeq 4 \frac{y_{e_2} v_\xi v_\phi^2}{\Lambda^3} v_d, \quad m_\mu \simeq 2 \frac{y_\mu v_\xi v_\phi}{\Lambda^2} v_d, \quad m_\tau \simeq 2 \frac{y_\tau v_T}{\Lambda} v_d .
\end{equation}
Table 9: The particle contents and their transformation property under the family symmetry $S_4 \times Z_5 \times Z_6$ and $U(1)_R$ with $\omega_5 = e^{2i\pi/5}$ and $\omega_6 = e^{2i\pi/6}$.

The electron mass is generated at NLO level, and realistic charged lepton mass hierarchy $m_e : m_\mu : m_\tau \simeq \lambda^4 : \lambda^2 : 1$ is produced.

For the case of $f_2 f_3 f_8 > 0$, $\nu_T$ is pure imaginary. Compared with Eq. (5.24), the mass matrix $W_l^l m_i^l m_l W_i$ in Eq. (7.34) has the most general form consistent with residual symmetry $Z_2^{SU} \times H_{CP}^{O}$ with $H_{CP}^{O} = \{\rho_1 (T^2 ST), \rho_1 (T^2 STU)\}$. Hence this is identical to the case IV discussed in the general analysis of section 5. Thus the PMNS matrix is of the form given by Eq. (5.27), i.e.

$$U_{PMNS} = \frac{1}{2} \begin{pmatrix} \sqrt{2} \cos \theta + i \sin \theta & -\sqrt{2} \cos \theta + i \sin \theta & -\sqrt{2} \sin \theta \\ \sqrt{2} \sin \theta - i \cos \theta & -\sqrt{2} \sin \theta - i \cos \theta & \sqrt{2} \cos \theta \\ 1 & 1 & -i\sqrt{2} \end{pmatrix} e^{i \frac{\pi}{4}}, \quad (7.39)$$

with

$$\tan 2\theta = -\frac{b_1}{y_\mu} \frac{v^2_\eta}{v_T \Lambda}. \quad (7.40)$$

The predictions for lepton mixing angles and CP phases are presented in Eq. (5.28). Although the measured values of $\theta_{13}$ and $\theta_{23}$ can be accounted for, $\theta_{12}$ is predicted to be maximal which is beyond the $3\sigma$ region. Hence the scenario of $f_2 f_3 f_8$ being negative is preferred in the present model. For both mixing patterns shown in Eq. (7.35) and Eq. (7.39), the atmospheric mixing angle $\theta_{23}$ fulfills

$$\sin^2 \theta_{23} = \frac{1 + \cos 2\theta}{3 + \cos 2\theta} = \frac{1}{1 + \sec^2 \theta} \leq \frac{1}{2}. \quad (7.41)$$

As a result, $\theta_{23}$ deviates from maximal mixing and it lies in the first octant in this model.

### 7.4 A variant model for $\theta_{23} > \pi/4$

Since the octant of $\theta_{23}$ is not known so far, we would like to minimally modify this model to accommodate the situation of $\theta_{23} > 45^\circ$. The family symmetry is still $S_4 \times Z_5 \times Z_6$. The fields and their assignments are listed in Table 8. Comparing with the fields of previous model shown in Table 8, we see that the two models only differ in the transformation properties of $\mu^c$ and $\tau^c$. Because both flavon fields and driving fields are kept intact, the vacuum is unchanged. Then the LO vacuum configuration is given in Eqs. (7.3) and (7.9), and the NLO VEVs are given by Eqs. (7.25) and (7.29). The neutrino sector remains the same, and the effective superpotential for the charged lepton masses including both LO and NLO contributions, is of the following form:

$$w_l = y_\tau \tau^c (l \phi)_1 h_d/\Lambda + y_\mu \mu^c (l \eta \varphi_T)_3 1 h_d/\Lambda^2 + y_\tau \tau^c (l \eta \varphi_T)_3 1 h_d/\Lambda^2$$
$$+ y_\tau \tau^c \xi (l (\eta \varphi_T)_3 1 h_d/\Lambda^3 + y_\tau \tau^c \xi (l (\eta \varphi_T)_3 1 h_d/\Lambda^3 + y_\tau \tau^c (l \varphi_T)_2 \phi \phi_2 1 h_d/\Lambda^3$$
$$+ y_\tau \tau^c (l \varphi_T)_3 (\phi \phi)_3 1 h_d/\Lambda^3 + y_\tau \tau^c (l \varphi_T)_3 (\phi \phi)_3 1 h_d/\Lambda^3 + y_\tau \tau^c (l \varphi_T)_3 1 h_d/\Lambda^3$$

36
+y_{\mu 2} \mu c((l\phi)_{2}(\phi\phi)_{2})_{1}h_{d}/\Lambda^{3} + y_{\mu 3} \mu c((l\phi)_{3}(\phi\phi)_{3})_{1}h_{d}/\Lambda^{3} + y_{\mu 4} \mu c((l\phi)_{4}(\phi\phi)_{4})_{1}h_{d}/\Lambda^{3} + y_{\tau 2} \tau c((l\phi)_{2}(\phi\phi)_{2})_{1}h_{d}/\Lambda^{3} + y_{\tau 3} \tau c((l\phi)_{3}(\phi\phi)_{3})_{1}h_{d}/\Lambda^{3} + y_{\tau 4} \tau c((l\phi)_{4}(\phi\phi)_{4})_{1}h_{d}/\Lambda^{3}.

(7.42)

These terms lead to a charged lepton mass matrix as follows

\[
m_{l} \simeq \begin{pmatrix}
-(1+i) a_{1} v_{\nu} v_{\phi}^{2}/\Lambda^{3} & 4 i y_{\nu e} v_{e} v_{\phi}^{2}/\Lambda^{3} & (1-i) a_{1} v_{\nu} v_{\phi}^{2}/\Lambda^{3} \\
(1-i) y_{\nu \mu} v_{\nu} v_{T}/\Lambda^{2} & 0 & -(1+i) y_{\nu \mu} v_{\nu} v_{T}/\Lambda^{2} \\
(1+i) y_{\nu \tau} v_{\nu} /\Lambda & -i y_{\tau} \delta v_{\phi} /\Lambda & (i-1) y_{\tau} v_{\nu} /\Lambda
\end{pmatrix} v_{d},
\]

(7.43)

where parameter \(a_{1}\) is defined in Eq. (7.33). After performing the unitary transformation \(W_{l}\), we have

\[
W_{l}^{\dagger} m_{l}^{\dagger} W_{l} \simeq \begin{pmatrix}
y_{e}^{2} |\delta v_{\phi 2}|^{2}/\Lambda^{2} + 16 y_{e 2}^{2} |\xi|^{2} |v_{T}|^{4}/\Lambda^{6} & -(1+i) \sqrt{2} y_{e 2} v_{\phi} \delta v_{\phi 2} /\Lambda^{2} & 0 \\
-(1-i) \sqrt{2} y_{e 2}^{2} \delta v_{\phi 2} /\Lambda^{2} & 4 y_{e 2}^{2} |v_{\phi}|^{2}/\Lambda^{2} & 0 \\
0 & 0 & 4 y_{e 2}^{2} |v_{\phi} v_{T}|^{2}/|v_{H}|^{2}/\Lambda^{4}
\end{pmatrix} v_{d}^{2}.
\]

(7.44)

In case of \(v_{T} > 0\) which can be realized for \(f_{2} f_{3} f_{8} < 0\), the product \(v_{\phi} \delta v_{\phi 2}^{*}\) would be real. After some simple algebra, we find \(m_{l}^{\dagger} m_{l}\) is diagonalized by the unitary matrix \(U_{l}\),

\[
U_{l} = \frac{1}{\sqrt{2}} \begin{pmatrix}
-i \sin \theta & -i & i \cos \theta \\
\sqrt{2} e^{\frac{i \pi}{4}} \cos \theta & 0 & \sqrt{2} e^{\frac{i \pi}{4}} \sin \theta \\
(-\sin \theta & 1 & \cos \theta
\end{pmatrix},
\]

(7.45)

where

\[
\tan 2\theta = -\frac{\delta v_{\phi 2}}{v_{\phi}} = \frac{\sqrt{3}(f_{15} + f_{16}) v_{H}^{2}}{f_{T} v_{T} \Lambda}.
\]

(7.46)

The mass eigenvalues of the charged lepton are

\[
m_{e} \simeq 4 \left| y_{e 2} v_{e} v_{\phi}^{2}/\Lambda^{3} \right| v_{d}, \quad m_{\mu} \simeq 2 \left| y_{\mu} v_{\nu} v_{T}/\Lambda^{2} \right| v_{d}, \quad m_{\tau} = 2 \left| y_{\tau} v_{\phi} \Lambda \right| v_{d}.
\]

(7.47)

As the neutrino sector stays the same, the PMNS matrix is predicted to be

\[
U_{PMNS} = K_{l} \frac{1}{2} \begin{pmatrix}
-\sin \theta - \sqrt{2} \cos \theta & -\sin \theta + \sqrt{2} \cos \theta & \sqrt{2} \sin \theta \\
1 & 1 & \sqrt{2} \\
\cos \theta - \sqrt{2} \sin \theta & \cos \theta + \sqrt{2} \sin \theta & -\sqrt{2} \cos \theta
\end{pmatrix} K_{\nu},
\]

(7.48)

with \(K_{l} = \text{diag}(e^{-i \frac{\pi}{4}}, e^{i \frac{\pi}{4}}, e^{-i \frac{\pi}{4}})\) and \(K_{\nu} = \text{diag}(1,1,-i)\). We see that this PMNS matrix is related to the corresponding one in Eq. (7.35) of previous model by exchanging its second and third rows. The lepton mixing angles follow immediately

\[
\sin^{2} \theta_{13} = \frac{1}{2} \sin^{2} \theta, \quad \sin^{2} \theta_{12} = \frac{1}{2} - \frac{\sqrt{2} \sin 2\theta}{3 + \cos 2\theta}, \quad \sin^{2} \theta_{23} = \frac{2}{3 + \cos 2\theta} \geq \frac{1}{2}.
\]

(7.49)

Obviously \(\theta_{23} \geq \pi/4\) and it is in the second octant. If the VEV \(v_{T}\) is imaginary which corresponds to \(f_{2} f_{3} f_{8} > 0\), the product \(v_{\phi} \delta v_{\phi 2}^{*}\) become pure imaginary. Along similar lines, the PMNS matrix is found to take the form:

\[
U_{PMNS} = \frac{1}{2} \begin{pmatrix}
\sqrt{2} \cos \theta + i \sin \theta & -\sqrt{2} \cos \theta + i \sin \theta & -\sqrt{2} \sin \theta \\
1 & 1 & \sqrt{2} \\
\sqrt{2} \sin \theta - i \cos \theta & -\sqrt{2} \sin \theta - i \cos \theta & \sqrt{2} \cos \theta
\end{pmatrix} e^{i \frac{\pi}{4}},
\]

(7.50)
which leads to
\[
\sin^2 \theta_{13} = \frac{1}{2} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1}{2}, \quad \sin^2 \theta_{23} = \frac{2}{3 + \cos 2\theta},
\]  
(7.50)
and maximal Dirac CP phase. For the best fitting value $\sin^2 \theta_{13} = 0.0234$, we obtain $\sin^2 \theta_{23} \simeq 0.512$. However, $\theta_{12}$ is equal to 45$^\circ$ which is drastically different from its experimental value. Hence the case of $f_2 f_3 f_8 > 0$ in both present model and previous model is disfavored.

8 Summary and conclusions

Although the BM mixing pattern has already been ruled out by experiment data, the scheme of keeping one column or one row of BM mixing is viable. We perform a comprehensive analysis of how to naturally realize this scheme from $S_4$ family symmetry and generalized CP symmetry in this paper. Furthermore, two models with $S_4$ family symmetry and generalized CP are constructed to implement the model independent results enforced by remnant symmetry.

We firstly study the deviation from BM mixing which originates from a rotation between two generation of neutrinos or charged leptons. The phenomenological predictions for the lepton mixing angles and Dirac CP phase are discussed in detail. In this approach, all mixing parameters depend on two real parameters $\theta$ and $\delta$ while the Majorana CP phases are indeterminate. For an additional rotation of 1-2 or 1-3 generation of charged leptons in the BM basis, good agreement with experiment data can be achieved, and the Dirac CP phase $\delta_{CP}$ is constrained to be in the range of $\pm [2.5, \pi]$ or $[\pi, 0]$ respectively, after the present 3$\sigma$ bounds of mixing angles from global data analysis are taken into account. For rotations in the neutrino sector, the measured values of the lepton mixing angles can not be accommodated. With the help of independent permutations of rows and columns of PMNS matrix, interesting mixing patterns shown in Eq. (3.19) and Eq. (3.24) are found. The Dirac CP phase is in the range of $\pm [2, \pi]$ and $[-1.1, 1.1]$ respectively. Note that $\delta_{CP}$ can vary within a quite wide range.

Since the BM mixing can be derived if we impose $S_4$ family symmetry and spontaneously break it down to $G_\nu = K_4^{ST^2 T^2(U)}$ in the neutrino sector and to $G_l = Z_4^{ST^2 U}$ or $G_l = K_4^{(SU)}$ in the charged lepton sector. It is easy to see that one column of the BM matrix would be retained if we degrade $G_\nu$ from $K_4$ to $Z_2$ subgroup, and one row of the BM mixing would be preserved once $G_l$ is degraded from $K_4$ (or $Z_4$) to $Z_2$. In order to have definite predictions for the leptonic CP violating phases, we extend the $S_4$ family symmetry to include generalized CP symmetry. The phenomenological implications of the symmetry breaking of $S_4 \times H_{CP}$ into $Z_2^{ST^2 SU} \times H_{CP}$ in the neutrino sector and $Z_4^{ST^2 U} \times H_{CP}$ in the charged lepton sector are investigated in a model independent way. We find that the resulting PMNS matrix has one column of the form $(1/2, 1/\sqrt{2}, 1/2)^T$ or $(1/\sqrt{2}, 1/2, 1/2)^T$, the Dirac CP phase $\delta_{CP}$ is predicted to be conserved or maximally broken, and both Majorana CP phases are trivial. Note that the scenario of maximal $\delta_{CP}$ is not preferred because the observed values of $\theta_{12}$ and $\theta_{13}$ can not be produced simultaneously in this case. Our prediction for $\delta_{CP}$ can be directly tested by future long baseline neutrino oscillation experiments LBNE, LBNO and Hyper-Kamiokande. If signal of leptonic CP violation is discovered, our proposal would be ruled out.

It is usually assumed the remnant symmetry in the neutrino sector is $Z_2 \times CP$ in the context of family symmetry combined with generalized CP. In this work, we also consider another situation that $Z_2 \times CP$ is preserved in the charged lepton sector instead of in the
neutrino sector. The lepton flavor mixing arising from the remnant symmetry $K^{(TST^2,T^2)}_4 \times H_{\nu}^{CP}$ in the neutrino sector and $Z^{SU}_2 \times H_{\nu}^{CP}$ in the charged lepton sector is explored. One row of PMNS matrix is fixed to be $(1/2, 1/2, 1/\sqrt{2})$ or $(1/2, 1/2, -i/\sqrt{2})$, and the Dirac CP is fully conserved or maximally broken as well. The experimental data can be accommodated very well in the CP conserved case while the case of maximal $\delta_{CP}$ is disfavored due to its unrealistic prediction of maximal $\theta_{12}$.

Inspired by the above fascinating results, we construct a model based on $S_4 \times H_{\nu}^{CP}$ which is spontaneously broken down to $Z^{ST^2SU}_2 \times H_{\nu}^{CP}$ in the neutrino sector and $Z^{TST^2U}_4 \times H_{\nu}^{CP}$ in the charged lepton sector by the VEVs of flavons. The PMNS matrix is really found to be of the form predicted from remnant symmetry in section 4. Depending on the coefficient $g_3g_4(g_1g_5 - g_2g_4)$ being positive or negative, $\delta_{CP}$ can be trivial or maximal. At leading order, the light neutrino mass matrix effectively contains only three real parameters which can be fixed by the measured values of the mass-squared difference $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$ and the reactor angle $\theta_{13}$. As a consequence, the light neutrino masses are completely determined. The predictions for the effective mass $|m_{\beta\beta}|$ are safely below the present upper limit, and yet they are within the future sensitivity of planned neutrinoless double-beta decay experiments.

Moreover, we present another model and its variant where the BM mixing is realized at LO. After the NLO corrections are included, the charged lepton mass hierarchy is obtained and the BM mixing is corrected by the effect of charged lepton diagonalization. One row of PMNS matrix is determined to be $(1/2, 1/2, 1/\sqrt{2})$ or $(1/2, 1/2, -i/\sqrt{2})$, and all the general model independent predictions for lepton flavor mixing in section 5 are naturally reproduced. The Dirac CP phase $\delta_{CP}$ is trivial $0, \pi$ for $f_2f_3f_8 < 0$ and maximal $\pm \pi/2$ for $f_2f_3f_8 > 0$.

In the past years, family symmetry and generalized CP symmetry has been shown to be a very powerful and promising framework to predict lepton mixing angles and CP violating phases. It is intriguing to extend this approach to the quark sector to understand the established CP violation at $B-$factory and strong CP problem.

**Acknowledgements**

This work is supported by the National Natural Science Foundation of China under Grant Nos. 11275188 and 11179007.
Table 10: The representation matrices of the generators $S$, $T$ and $U$ for the five irreducible representations of $S_4$ in our working basis.

### Appendix

#### A Group theory of $S_4$ and Clebsch-Gordan coefficients

$S_4$ is a symmetric group of degree four, and it is a good candidate for a family symmetry to realize the tri-bimaximal and BM mixing. Hence $S_4$ has been widely studied in the literature. For the sake of being self-contained, in the following we shall present our convention for the $S_4$ group, the working basis and the associated Clebsch-Gordan coefficients.

$S_4$ group can be generated by three generators $S$, $T$ and $U$ obeying the relations

$$
S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = (STU)^4 = 1.
$$

Note that the chosen generators $\tilde{S}$ and $\tilde{T}$ of Ref. 39 are related to our generators $S$, $T$ and $U$ via $\tilde{S} = ST^2SU$ and $\tilde{T} = T^2STU$ or vice versa $S = \tilde{T}^2$, $T = \tilde{T}\tilde{S}$, $U = \tilde{S}\tilde{T}^2\tilde{T}$. It is straightforward to check that the multiplication rules $\tilde{T}^4 = \tilde{S}^2 = (\tilde{ST})^3 = (\tilde{T}\tilde{S})^3 = 1$ are satisfied. The 24 group elements can be divided into the five conjugacy classes as follows:

$$
1C_1 = \{1\}, \\
3C_2 = \{S, TST^2, T^2ST\}, \\
6C_2' = \{U, TU, SU, T^2U, STSU, ST^2SU\}, \\
8C_3 = \{T, ST, TS, STS, T^2, ST^2, T^3S, ST^2S\}, \\
6C_4 = \{STU, TSU, T^2SU, ST^2U, TST^2U, T^2STU\},
$$

where $kC_n$ denotes a conjugacy class with $k$ elements and the subscript $n$ is the order of its elements. Since the number of conjugacy class is equal to the number the number of irreducible representation, $S_4$ has five irreducible representations: two singlet representations $1$ and $1'$, one doublet representation $2$ and two triplet representations $3$ and $3'$. Note that both $3$ and $3'$ are faithful representations of $S_4$. Our choice for the representation matrices of the generators $S$, $T$ and $U$ are listed in Table 10. For the three-dimensional representation $3$, the representation matrices for the elements are as follows:

$$
1C_1 : 1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
3C_2 : S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
$$

$$
TST^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix},
$$

$$
T^2ST = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

Table 10: The representation matrices of the generators $S$, $T$ and $U$ for the five irreducible representations of $S_4$ in our working basis.
Table 11: Character table of $S_4$. We give an example of the elements for each class in the last column.

$$
\begin{array}{c|cccc|c}
    & \chi_1 & \chi'_1 & \chi_2 & \chi'_2 & \chi_3 & \chi'_3 \text{ Example} \\
1C_1 & 1 & 1 & 2 & 3 & 3 & 1 \\
3C_2 & 1 & 1 & 2 & -1 & -1 & S \\
6C'_2 & 1 & -1 & 0 & -1 & 1 & U \\
8C_3 & 1 & 1 & -1 & 0 & 0 & T \\
6C_4 & 1 & -1 & 0 & 1 & -1 & STU \\
\end{array}
$$

For the $3'$ representation, the matrices representing the elements of $1C_1$, $3C_2$ and $8C_3$ are the same as those listed above for the representation $3$, while they are the opposite for $6C'_2$ and $6C_4$. The reason is that the generator $U$ changes its sign in $3$ and $3'$ representations, the elements in $1C_1$, $3C_2$ and $8C_3$ contain an even number of $U$, while those in $6C'_2$ and $6C_4$ contain an odd number of $U$. Character of an element is the trace of its representation matrix. The character table of $S_4$ group can be easily obtained, as shown in Table 11. The Kronecker products between various irreducible representations follow immediately:
$1 \otimes R = R \otimes 1 = R, \quad 1' \otimes 1' = 1, \quad 1' \otimes 2 = 2, \quad 1' \otimes 3 = 3', \quad 1' \otimes 3' = 3,$

$2 \otimes 2 = 1 \oplus 1' \oplus 2, \quad 2 \otimes 3 = 2 \otimes 3' = 3 \oplus 3',$

$3 \otimes 3 = 3' \otimes 3' = 1 \oplus 2 \oplus 3 \oplus 3', \quad 3 \otimes 3' = 1' \oplus 2 \oplus 3 \oplus 3'.$

(A.3)

where $R$ denotes any $S_4$ irreducible representation. In the following, we shall present the Clebsch-Gordan (CG) coefficients in our basis. We use $\alpha_i$ to indicate the elements of the first representation of the product and $\beta_i$ to indicate those of the second representation. We first report the CG coefficients associated with the singlet representation $1'$:

$1' \otimes 1' = 1 \sim \alpha \beta$

$1' \otimes 2 = 2 \sim \left( \begin{array}{c} \alpha \beta_2 \\ -\alpha \beta_1 \end{array} \right)$

$1' \otimes 3 = 3' \sim \left( \begin{array}{c} \alpha \beta_1 \\ \alpha \beta_2 \\ \alpha \beta_3 \end{array} \right)$

$1' \otimes 3' = 3 \sim \left( \begin{array}{c} \alpha \beta_1 \\ \alpha \beta_2 \\ \alpha \beta_3 \end{array} \right)$

(A.4)

The CG coefficients for the products involving the doublet representation $2$ are the following ones:

$2 \otimes 2 = 1 \oplus 1' \oplus 2 \quad \text{with} \quad \begin{cases} 1 \sim \alpha_1 \beta_1 + \alpha_2 \beta_2 \\ 1' \sim \alpha_1 \beta_2 - \alpha_2 \beta_1 \\ 2 \sim \left( \begin{array}{c} \alpha_2 \beta_2 - \alpha_1 \beta_1 \\ \alpha_1 \beta_2 + \alpha_2 \beta_1 \end{array} \right) \\ 3 \sim \left( \begin{array}{c} \sqrt{3} \alpha_2 \beta_3 - \alpha_1 \beta_1 \\ 2 \alpha_1 \beta_2 \\ \sqrt{3} \alpha_1 \beta_1 - \alpha_1 \beta_3 \end{array} \right) \end{cases}$

$2 \otimes 3 = 3 \oplus 3' \quad \text{with} \quad \begin{cases} 3 \sim \left( \begin{array}{c} \sqrt{3} \alpha_1 \beta_3 + \alpha_2 \beta_1 \\ -2 \alpha_2 \beta_2 \\ \sqrt{3} \alpha_1 \beta_1 + \alpha_2 \beta_3 \end{array} \right) \end{cases}$

$2 \otimes 3' = 3 \oplus 3' \quad \text{with} \quad \begin{cases} 3 \sim \left( \begin{array}{c} \sqrt{3} \alpha_1 \beta_3 - \alpha_1 \beta_1 \\ -2 \alpha_2 \beta_2 \\ \sqrt{3} \alpha_1 \beta_1 + \alpha_2 \beta_3 \end{array} \right) \end{cases}$

(A.5)
Finally the CG coefficients involving the three-dimensional representations 3 and 3’ are as follows:

\[
\begin{align*}
3 \otimes 3 &= 3' \otimes 3' = 1 \oplus 2 \oplus 3 \oplus 3' \\
3 \otimes 3 &= 1' \oplus 2 \oplus 3 \oplus 3'
\end{align*}
\]

\[
\begin{align*}
1 &\sim \alpha_1 \beta_3 + \alpha_2 \beta_2 + \alpha_3 \beta_1 \\
2 &\sim \left( \frac{2 \alpha_2 \beta_2 - \alpha_1 \beta_3 - \alpha_3 \beta_1}{\sqrt{3}(\alpha_1 \beta_1 + \alpha_3 \beta_3)} \right) \\
3 &\sim \left( \begin{array}{c}
\alpha_1 \beta_2 - \alpha_2 \beta_1 \\
\alpha_3 \beta_1 - \alpha_1 \beta_3 \\
\alpha_2 \beta_3 - \alpha_3 \beta_2
\end{array} \right) \\
3' &\sim \left( \begin{array}{c}
\alpha_1 \beta_2 - \alpha_2 \beta_1 \\
\alpha_3 \beta_1 - \alpha_1 \beta_3 \\
\alpha_2 \beta_3 - \alpha_3 \beta_2
\end{array} \right)
\end{align*}
\]

Note that all the CG coefficients are real. The group structure of $S_4$ has been studied comprehensively in Ref. [57]. It has nine $Z_2$ subgroups, four $Z_3$ subgroups, three $Z_4$ subgroups, four $K_4 \cong Z_2 \times Z_2$ subgroups, four $S_3$ subgroups, three $D_4$ subgroups and the alternating group $A_4$ as a subgroup. In the present work, we focus on the Abelian subgroups as the remnant symmetry, which can be expressed in terms of the generators $S$, $T$ and $U$ as follows:

- **$Z_2$ subgroups**

  \[
  \begin{align*}
  Z_2^{STSU} &= \{1, ST^2SU\}, \\
  Z_2^{TU} &= \{1, TU\}, \\
  Z_2^{STSU} &= \{1, STSU\}, \\
  Z_2^{U} &= \{1, U\}, \\
  Z_2^{SU} &= \{1, SU\}, \\
  Z_2^{STU} &= \{1, T^2ST\}, \\
  Z_2^{STSU} &= \{1, TSTS\}, \\
  Z_2^{TST^2} &= \{1, TST^2\}.
  \end{align*}
  \]

  (A.7)

  The former six $Z_2$ subgroups are related to each other by group conjugation, and the latter three subgroups are conjugate to each other as well.

- **$Z_3$ subgroups**

  \[
  \begin{align*}
  Z_3^{ST} &= \{1, ST, T^2S\}, \\
  Z_3^{STSU} &= \{1, ST, ST^2S\}, \\
  Z_3^{T} &= \{1, T, T^2\}, \\
  Z_3^{TST} &= \{1, TS, ST^2\}.
  \end{align*}
  \]

  (A.8)

  All the above $Z_3$ subgroups are conjugate to each other.

- **$Z_4$ subgroups**

  \[
  \begin{align*}
  Z_4^{TST^2U} &= \{1, TST^2U, S, T^2STU\}, \\
  Z_4^{ST^2U} &= \{1, ST^2U, TST^2, T^2SU\},
  \end{align*}
  \]

  \footnote{Here $D_4$ is the symmetry group of the square, and its order is eight. Its mathematical definition is $D_4 = \langle r, s | r^4 = s^2 = (rs)^2 = 1 \rangle$.}


\[ Z_{TSU}^4 = \{1, TSU, T^2ST, STU\} , \]  

which are related with each under group conjugation.

- **\(K_4\)** subgroups

\[
\begin{align*}
K_{4}^{(S,ST^2)} &= Z_2^S \times Z_2^{ST^2} = \{1, S, ST^2, T^2ST\}, \\
K_{4}^{(S,U)} &= Z_2^S \times Z_2^U = \{1, S, U, SU\}, \\
K_{4}^{(TS^2,T^2U)} &= Z_2^{TS^2} \times Z_2^{TU} = \{1, TS^2, T^2U, ST^2SU\}, \\
K_{4}^{(T^2ST, TU)} &= Z_2^{T^2ST} \times Z_2^{TU} = \{1, T^2ST, TU, STSU\},
\end{align*}
\]

where \(K_{4}^{(S,ST^2)}\) is a normal subgroup of \(S_4\), and the other three \(K_4\) subgroups are conjugate to each other.

**B The general analysis of \(S_4\) breaking to \(Z_2\) in neutrino sector and to \(K_4\) in charged lepton sector with remnant CP**

In this appendix, we shall analyze the last scenario in which \(S_4 \times H_{CP}\) is broken down to \(Z_2 \times H_{CP}'\) in the neutrino sector and \(K_4 \times H_{CP}'\) in the charged lepton sector. Since \(S_4\) has nine \(Z_2\) subgroups given by Eq. (A.7) and four \(K_4\) subgroups in Eq. (A.10), there are \(9 \times 4 = 36\) possible preserved remnant family symmetry in the neutrino and the charged lepton sectors, the remnant CP symmetry is fixed by the consistency condition. Reminding that the different residual subgroups \((Z_2, K_4)\) related by group conjugation lead to the same prediction for lepton flavor mixing, it is sufficient to only consider five representative cases: \((G_\nu, G_l) = (Z_2^{ST^2SU}, K_{4}^{(S,U)}), (Z_2^{ST^2SU}, K_{4}^{(S,ST^2)}), (Z_2^{T^2ST}, K_{4}^{(S,U)}), (Z_2^S, K_{4}^{(S,ST^2)}), (Z_2^U, K_{4}^{(S,U)})\). For the last two cases \((G_\nu, G_l) = (Z_2^S, K_{4}^{(S,ST^2)}), (Z_2^U, K_{4}^{(S,U)})\), one column of the PMNS matrix is determined to be \((0,0,1)^T\) which is obviously not compatible with the experimental data shown in Eq. (4.6). For the centered two cases \((G_\nu, G_l) = (Z_2^{ST^2SU}, K_{4}^{(S,U)}), (Z_2^{T^2ST}, K_{4}^{(S,U)})\), the PMNS matrix would have one column of form \((0, -1/\sqrt{2}, 1/\sqrt{2})^T\) which also can not be accommodated by the present data. For the remaining case \((G_\nu, G_l) = (Z_2^{ST^2SU}, K_{4}^{(SU)})\), one column of \(U_{PMNS}\) would be \((1/\sqrt{2}, 1/2, 1/2)^T\). We shall demonstrate that the corresponding lepton flavor mixing matrix is of the same form as that predicted by \((G_\nu, G_l) = (Z_2^{ST^2SU}, Z_4^{T^2STU})\) discussed in section 4. Agreement with experimental data could be achieved.

The residual family symmetry \(G_l = K_{4}^{(SU)}\) constraints the charged lepton mass matrix as follows,

\[
\rho_3^A(S)m^\dagger_l m_l \rho_3(S) = m^\dagger_l m_l , \quad \rho_3^A(U)m^\dagger_l m_l \rho_3(U) = m^\dagger_l m_l . \tag{B.1}
\]

The most general mass matrix \(m^\dagger_l m_l\) satisfying these conditions is determined to be

\[
\begin{pmatrix}
m_{11} & 0 & \text{im}_{13} \\
0 & m_{22} & 0 \\
-\text{im}_{13} & 0 & m_{11}
\end{pmatrix}, \tag{B.2}
\]

where \(m_{11}, m_{13}\) and \(m_{22}\) are real parameters. We can diagonalize this \(m^\dagger_l m_l\) by a unitary matrix \(V_l\) with

\[
V_l^\dagger m^\dagger_l m_l V_l = \text{diag}(m_{22}, m_{11} - m_{13}, m_{11} + m_{13}) , \tag{B.3}
\]
where
\[ V_l = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & i \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \] (B.4)

Because the order of the charged lepton masses is undetermined in our framework, \( V_l \) can undergo rephasing and permutations of its column vectors. The mass matrix \( m_{l_i}^\dagger m_{l_i} \) in Eq. (B.2) is further constrained by the residual CP symmetry \( H_{CP}^l \) which should be consistent with the residual family symmetry \( K_4^{(SU)} \),

\[ X_{l\rho}^S(S)X_{l\rho}^{-1} = \rho(S), \quad X_{l\rho}^S(U)X_{l\rho}^{-1} = \rho(U), \quad S, U \in K_4^{(SU)}. \] (B.5)

Then we find the residual CP symmetry \( H_{CP}^l \) is

\[ H_{CP}^l = \{ \rho(1), \rho(S), \rho(U), \rho(SU), \rho(T^2ST), \rho(TST^2), \rho(TST^2U), \rho(T^2STU) \}. \] (B.6)

The invariance under the action of the remnant CP symmetry \( H_{CP}^l \) implies that

\[ X_{l3}^\dagger m_{l_i}^\dagger m_{l_i} X_{l3} = \left( m_{l_i}^\dagger m_{l_i} \right)^*. \] (B.7)

The constraint \( m_{13} = 0 \) arises for the case of \( X_{l\rho} = \rho(1), \rho(S), \rho(U), \rho(SU) \). As a consequence, the charged lepton masses are partially degenerate, and thus this case is not phenomenologically viable. For the remaining values \( X_{l\rho} = \rho(T^2ST), \rho(TST^2), \rho(TST^2U), \rho(T^2STU) \), we can straightforwardly check that the remnant CP invariance condition of Eq. (B.7) is fulfilled by the charged lepton mass matrix \( m_{l_i}^\dagger m_{l_i} \) in Eq. (B.2). Hence no new constraints are generated.

Now we come to the neutrino sector with the remnant symmetry \( Z_2^{ST^2SU} \times H_{CP}^\nu \). From section 4, we know that the most general neutrino mass matrix invariant under \( G_\nu = Z_2^{ST^2SU} \) is of the form

\[ m_\nu = \alpha \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \epsilon \begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix}. \] (B.8)

The residual CP symmetry is \( H_{CP}^\nu = \{ \rho(1), \rho(ST^2SU), \rho(T^2SU), \rho(TST^2) \} \). In the following, we shall present the predictions for the lepton mixing matrix for different residual CP transformations.

\( \bullet \) \( X_{l\rho} = \rho(1), \rho(ST^2SU) \)

The four parameters \( \alpha, \beta, \gamma \) and \( \epsilon \) are real, neutrino mass matrix \( m_\nu \) is diagonalized by the unitary matrix \( U_\nu \) given in Eq. (4.18),

\[ U_\nu = \frac{1}{2} \begin{pmatrix} \sin \theta + \sqrt{2} \cos \theta & 1 & \cos \theta - \sqrt{2} \sin \theta \\ -\sqrt{2} \sin \theta & \sqrt{2} & -\sqrt{2} \cos \theta \\ \sin \theta - \sqrt{2} \cos \theta & 1 & \cos \theta + \sqrt{2} \sin \theta \end{pmatrix} P_{312}, \] (B.9)

where the column permutation \( P_{312} \) is performed since the neutrino mass order is indeterminate, and the phase factor \( K_\nu \) is left out. Including the contribution \( V_l \) from the charged lepton sector, the lepton mixing matrix \( U_{PMNS} \) is of the form

\[ U_{PMNS} = V_l^\dagger U_\nu \]
\[ \begin{pmatrix} \sqrt{2} & -\sqrt{2} \cos \theta \\ e^{i\pi/4} & e^{i\pi/4} \cos \theta + \sqrt{2} e^{-i\pi/4} \sin \theta \\ e^{-i\pi/4} & e^{-i\pi/4} \cos \theta + \sqrt{2} e^{i\pi/4} \sin \theta \end{pmatrix} \]

\[ = \frac{1}{2} \begin{pmatrix} \sqrt{2} & -\sqrt{2} \cos \theta & -\sqrt{2} \sin \theta \\ e^{i\pi/4} \cos \theta + \sqrt{2} e^{-i\pi/4} \sin \theta & e^{i\pi/4} \sin \theta - \sqrt{2} e^{-i\pi/4} \cos \theta \\ e^{-i\pi/4} \cos \theta + \sqrt{2} e^{i\pi/4} \sin \theta & e^{-i\pi/4} \sin \theta - \sqrt{2} e^{i\pi/4} \cos \theta \end{pmatrix} \]

\[ = \text{diag} \left( 1, e^{i\pi/4}, e^{-i\pi/4} \right) \frac{1}{2} \begin{pmatrix} \sqrt{2} & -\sqrt{2} \cos \theta & -\sqrt{2} \sin \theta \\ 1 & \cos \theta - i\sqrt{2} \sin \theta & \sin \theta + i\sqrt{2} \cos \theta \\ 1 & \cos \theta + i\sqrt{2} \sin \theta & \sin \theta - i\sqrt{2} \cos \theta \end{pmatrix} P_{231}. \quad (B.10) \]

It is identical with the lepton mixing matrix in Eq. (4.28) besides the phases factor diag\( (1, e^{i\pi/4}, e^{-i\pi/4}) \) which can be absorbed by the charged lepton fields. Hence we conclude that the PMNS matrix is predicted to be of the same form as that of case II.

- \( X_{\nu} = \rho_T(T^2 U), \rho_T(TST^2) \)

Residual CP invariance condition implies that \( \alpha, \beta \) and \( \gamma \) are real while \( \epsilon \) is purely imaginary. In this case, the the neutrino diagonalization matrix \( U_\nu \) is given by,

\[ U_\nu = \frac{1}{2} \begin{pmatrix} 1 & \cos \theta - i\sqrt{2} \sin \theta & \sin \theta + i\sqrt{2} \cos \theta \\ \sqrt{2} & -\sqrt{2} \cos \theta & -\sqrt{2} \sin \theta \\ 1 & \cos \theta + i\sqrt{2} \sin \theta & \sin \theta - i\sqrt{2} \cos \theta \end{pmatrix} P_{231}. \quad (B.11) \]

where the permutation matrix \( P_{231} \) is multiplied from the right. The PMNS matrix is given by

\[ U_{PMNS} = V_i^\dagger U_\nu \]

\[ = \frac{1}{2} \begin{pmatrix} 1 & \cos \theta - i\sqrt{2} \sin \theta & \sin \theta + i\sqrt{2} \cos \theta \\ -\sqrt{2} \sin \theta & \sqrt{2} \cos \theta & -\sqrt{2} \sin \theta \\ e^{i\pi/4} (\sin \theta - \sqrt{2} \cos \theta) & e^{i\pi/4} (\cos \theta + \sqrt{2} \sin \theta) & e^{-i\pi/4} (\cos \theta - \sqrt{2} \sin \theta) \end{pmatrix} \]

\[ = \text{diag} \left( 1, e^{i\pi/4}, e^{-i\pi/4} \right) P_{231} \frac{1}{2} \begin{pmatrix} \sin \theta + \sqrt{2} \cos \theta & 1 & \cos \theta - \sqrt{2} \sin \theta \\ -\sqrt{2} \sin \theta & \sqrt{2} & -\sqrt{2} \cos \theta \\ \sin \theta - \sqrt{2} \cos \theta & 1 & \cos \theta + \sqrt{2} \sin \theta \end{pmatrix}, \quad (B.12) \]

where the phase \( \text{diag} \left( 1, e^{i\pi/4}, e^{-i\pi/4} \right) \) can be absorbed into charged lepton fields, and the permutation \( P_{231} \) can be removed by reordering the eigenvalues of \( m_1^2 m_1 \) in Eq. (B.3). Therefore this PMNS matrix coincides with Eq. (4.18) which is the lepton mixing matrix of case I discussed in section 4.

In short, the symmetry breaking pattern of \( S_4 \times H_{CP} \) into \( Z_2^{ST^2 SU} \times H_{CP}^\nu \) in the neutrino sector and \( K_4^{(SU)} \times H_{CP}^l \) in the charged lepton sector leads to the same predictions for the lepton flavor mixing as the remnant symmetries \( Z_2^{ST^2 SU} \times H_{CP}^\nu \) of neutrino sector and \( Z_4^{ST^2 U} \times H_{CP}^l \) of charged lepton sector.
References

[1] K. Abe et al. [T2K Collaboration], Phys. Rev. Lett. 107, 041801 (2011) [arXiv:1106.2822 [hep-ex]].

[2] P. Adamson et al. [MINOS Collaboration], Phys. Rev. Lett. 107, 181802 (2011) [arXiv:1108.0015 [hep-ex]]; Phys. Rev. Lett. 110, no. 25, 251801 (2013) [arXiv:1304.6335 [hep-ex]].

[3] Y. Abe et al. [DOUBLE-CHOOZ Collaboration], Phys. Rev. Lett. 108, 131801 (2012) [arXiv:1112.6353 [hep-ex]]; Phys. Rev. D 86, 052008 (2012) [arXiv:1207.6632 [hep-ex]]; Physics Letters B, Volume 735, 30 July 2014, Pages 51-56 [arXiv:1401.5981 [hep-ex]].

[4] F. P. An et al. [DAYA-BAY Collaboration], Phys. Rev. Lett. 108, 171803 (2012) [arXiv:1203.1669 [hep-ex]]; Chin. Phys. C 37, 011001 (2013) [arXiv:1210.6327 [hep-ex]].

[5] J. K. Ahn et al. [RENO Collaboration], Phys. Rev. Lett. 108, 191802 (2012) [arXiv:1204.0626 [hep-ex]].

[6] M. C. Gonzalez-Garcia, M. Maltoni, J. Salvado and T. Schwetz, JHEP 1212, 123 (2012) [arXiv:1209.3023 [hep-ph]].

[7] F. Capozzi, G. L. Fogli, E. Lisi, A. Marrone, D. Montanino and A. Palazzo, arXiv:1312.2878 [hep-ph].

[8] D. V. Forero, M. Tortola and J. W. F. Valle, arXiv:1405.7540 [hep-ph].

[9] K. Abe et al. [T2K Collaboration], Phys. Rev. Lett. 112, 061802 (2014) [arXiv:1311.4750 [hep-ex]].

[10] K. Abe et al. [T2K Collaboration], Phys. Rev. Lett. 112, 181801 (2014) [arXiv:1403.1532 [hep-ex]].

[11] C. Adams et al. [LBNE Collaboration], arXiv:1307.7335 [hep-ex]; M. Bass et al. [LBNE Collaboration], arXiv:1311.0212 [hep-ex].

[12] D. Autiero, J. Aysto, A. Badertscher, L. B. Bezrukov, J. Bouchez, A. Bueno, J. Busto and J. -E. Campagne et al., JCAP 0711, 011 (2007) [arXiv:0705.0116 [hep-ph]].

[13] A. Rubbia, arXiv:1003.1921 [hep-ph].

[14] D. Angus et al. [LAGUNA Collaboration], arXiv:1001.0077 [physics.ins-det].

[15] A. Rubbia [LAGUNA Collaboration], Acta Phys. Polon. B 41, 1727 (2010).

[16] S. K. Agarwalla et al. [LAGUNA-LBNO Collaboration], JHEP 1405, 094 (2014) [arXiv:1312.6520 [hep-ph]].

[17] K. Abe, T. Abe, H. Aihara, Y. Fukuda, Y. Hayato, K. Huang, A. K. Ichikawa and M. Ikeda et al., arXiv:1109.3262 [hep-ex]; E. Kearns et al. [Hyper-Kamiokande Working Group Collaboration], arXiv:1309.0184 [hep-ex].
[18] G. Ecker, W. Grimus and W. Konetschny, Nucl. Phys. B 191 (1981) 465; G. Ecker, W. Grimus and H. Neufeld, Nucl. Phys. B 247 (1984) 70; G. Ecker, W. Grimus and H. Neufeld, J. Phys. A 20 (1987) L807; H. Neufeld, W. Grimus and G. Ecker, Int. J. Mod. Phys. A 3, 603 (1988).

[19] W. Grimus and M. N. Rebelo, Phys. Rept. 281, 239 (1997) [arXiv:9506272[hep-ph]].

[20] M. Holthausen, M. Lindner and M. A. Schmidt, JHEP 1304, 122 (2013) [arXiv:1211.6953 [hep-ph]].

[21] F. Feruglio, C. Hagedorn and R. Ziegler, JHEP 1307, 027 (2013) [arXiv:1211.5560 [hep-ph]].

[22] G. -J. Ding, S. F. King, C. Luhn and A. J. Stuart, JHEP 1305, 084 (2013) [arXiv:1303.6180 [hep-ph]].

[23] M. -C. Chen, M. Fallbacher, K. T. Mahanthappa, M. Ratz and A. Trautner, Nucl. Phys. B 883, 267 (2014) [arXiv:1402.0507 [hep-ph]].

[24] P. F. Harrison and W. G. Scott, Phys. Lett. B 535, 163 (2002) [hep-ph/0203209]; P. F. Harrison and W. G. Scott, Phys. Lett. B 547, 219 (2002) [hep-ph/0210197]; P. F. Harrison and W. G. Scott, Phys. Lett. B 594, 324 (2004) [hep-ph/0403278].

[25] W. Grimus and L. Lavoura, Phys. Lett. B 579, 113 (2004) [hep-ph/0305309]; W. Grimus and L. Lavoura, [arXiv:1207.1678]; P. M. Ferreira, W. Grimus, L. Lavoura and P. O. Ludl, JHEP 1209, 128 (2012) [arXiv:1206.7072].

[26] Y. Farzan and A. Y. .Smirnov, JHEP 0701, 059 (2007) [hep-ph/0610337].

[27] G. -J. Ding, S. F. King and A. J. Stuart, JHEP 1312 (2013) 006 [arXiv:1307.4212].

[28] C. -C. Li and G. -J. Ding, Nucl. Phys. B 881, 206 (2014) [arXiv:1312.4401 [hep-ph]].

[29] F. Feruglio, C. Hagedorn and R. Ziegler, Eur. Phys. J. C 74, 2753 (2014) [arXiv:1303.7178 [hep-ph]].

[30] C. Luhn, Nucl. Phys. B 875, 80 (2013) [arXiv:1306.2358 [hep-ph]].

[31] I. Girardi, A. Meroni, S. T. Petcov and M. Spinrath, JHEP 1402, 050 (2014) [arXiv:1312.1966 [hep-ph]].

[32] G. -J. Ding and Y. -L. Zhou, [arXiv:1312.5222 [hep-ph]].

[33] G. -J. Ding and Y. -L. Zhou, JHEP 1406, 023 (2014) [arXiv:1404.0592 [hep-ph]].

[34] G. -J. Ding and S. F. King, Phys. Rev. D 89, 093020 (2014) [arXiv:1403.5846 [hep-ph]].

[35] S. F. King and T. Neder, [arXiv:1403.1758 [hep-ph]].

[36] G. C. Branco, J. M. Gerard and W. Grimus, Phys. Lett. B 136, 383 (1984); I. de Medeiros Varzielas and D. Emmanuel-Costa, Phys. Rev. D 84, 117901 (2011) [arXiv:1106.5477 [hep-ph]]; I. de Medeiros Varzielas, D. Emmanuel-Costa and P. Leser, Phys. Lett. B 716, 193 (2012) [arXiv:1204.3633 [hep-ph]]; I. de Medeiros Varzielas, JHEP 1208, 055 (2012) [arXiv:1205.3780 [hep-ph]]; G. Bhattacharyya, I. de Medeiros Varzielas and P. Leser, Phys. Rev. Lett. 109, 241603 (2012) [arXiv:1210.0545 [hep-ph]]; I. P. Ivanov and L. Lavoura, Eur. Phys. J. C 73, 2416 (2013) [arXiv:1302.3656 [hep-ph]]; I. de Medeiros Varzielas and D. Pidt, J. Phys. G 41, 025004 (2014) [arXiv:1307.0711 [hep-ph]].
[37] K. S. Babu and J. Kubo, Phys. Rev. D 71, 056006 (2005) [hep-ph/0411226]; K. S. Babu, K. Kawashima and J. Kubo, Phys. Rev. D 83, 095008 (2011) [arXiv:1103.1664 [hep-ph]].

[38] M. -C. Chen and K. T. Mahanthappa, Phys. Lett. B 681, 444 (2009) [arXiv:0904.1721 [hep-ph]]; A. Meroni, S. T. Petcov and M. Spinrath, Phys. Rev. D 86, 113003 (2012) [arXiv:1205.5241 [hep-ph]].

[39] G. Altarelli, F. Feruglio and L. Merlo, JHEP 0905, 020 (2009) [arXiv:0903.1940 [hep-ph]].

[40] D. Meloni, JHEP 1110, 010 (2011) [arXiv:1107.0221 [hep-ph]].

[41] G. -J. Ding and Y. -L. Zhou, Nucl. Phys. B 876, 418 (2013) [arXiv:1304.2645 [hep-ph]].

[42] V. D. Barger, S. Pakvasa, T. J. Weiler and K. Whisnant, Phys. Lett. B 437, 107 (1998) [hep-ph/9806387].

[43] C. H. Albright and W. Rodejohann, Eur. Phys. J. C 62, 599 (2009) [arXiv:0812.0436 [hep-ph]]; C. H. Albright, A. Dueck and W. Rodejohann, Eur. Phys. J. C 70, 1099 (2010) [arXiv:1004.2798 [hep-ph]].

[44] X. -G. He and A. Zee, Phys. Rev. D 84, 053004 (2011) [arXiv:1106.4359 [hep-ph]].

[45] B. Wang, J. Tang and X. -Q. Li, Phys. Rev. D 88, 073003 (2013) [arXiv:1303.1592 [hep-ph]].

[46] Y. Shimizu and M. Tanimoto, arXiv:1405.1521 [hep-ph].

[47] S. T. Petcov, arXiv:1405.6006 [hep-ph].

[48] J. Beringer et al. [Particle Data Group Collaboration], Phys. Rev. D 86 (2012) 010001.

[49] A. Himmel [Super-Kamiokande Collaboration], AIP Conf. Proc. 1604, 345 (2014) [arXiv:1310.6677 [hep-ex]].

[50] G. Altarelli and F. Feruglio, Nucl. Phys. B 741, 215 (2006) [hep-ph/0512103].

[51] P. A. R. Ade et al. [Planck Collaboration], arXiv:1303.5076 [astro-ph.CO].

[52] M. Agostini et al. [GERDA Collaboration], Phys. Rev. Lett. 111 (2013) 12, 122503 [arXiv:1307.4720 [nucl-ex]].

[53] M. Auger et al. [EXO Collaboration], Phys. Rev. Lett. 109 (2012) 032505 [arXiv:1205.5608 [hep-ex]].

[54] J. B. Albert et al. [EXO-200 Collaboration], Nature 510 (2014) 229-234 [arXiv:1402.6956 [nucl-ex]].

[55] A. Gando et al. [KamLAND-Zen Collaboration], Phys. Rev. Lett. 110 (2013) 062502 [arXiv:1211.3863 [hep-ex]].

[56] [ Delia Tosi on behalf of the EXO Collaboration, “The search for neutrino-less double-beta decay: summary of current experiments,” arXiv:1402.1170 [nucl-ex].

[57] G. -J. Ding, Nucl. Phys. B 827, 82 (2010) [arXiv:0909.2210 [hep-ph]].