THE HOMOTOPY TYPES OF $Sp(2)$-GAUGE GROUPS OVER CLOSED, SIMPLY-CONNECTED FOUR-MANIFOLDS

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Abstract. We determine the number of distinct fibre homotopy types for the gauge groups of principal $Sp(2)$-bundles over a closed, simply-connected four-manifold.

1. Introduction

Let $X$ be a pointed topological space, $G$ a topological group and $P \longrightarrow X$ a principal $G$-bundle. The gauge group $G(P)$ of $P$ is the group of $G$-equivariant automorphisms of $P$ that fix $X$. The topology of gauge groups is of interest due to its connections with various moduli spaces [1, 2] and Donaldson Theory [7].

Considerable effort has gone into determining the homotopy types of gauge groups for specific groups $G$ and spaces $X$. Typically, $G$ and $X$ are chosen because of their interest in geometry or physics. In general, Crabb and Sutherland [4] showed that, even if there are infinitely many inequivalent classes of principal $G$-bundles $P$ over $X$, there are only finitely many distinct homotopy types for their gauge groups. Precise enumerations of the homotopy types have been made in the following cases: $SU(2)$-bundles over $S^4$ [13] or over a closed, simply-connected four-manifold [14]; $SU(3)$-bundles over $S^4$ [9] or over a closed, simply-connected four-manifold [21]; $SU(5)$-bundles over $S^4$ [22]; $SO(3)$-bundles over $S^4$ [11]; and $Sp(2)$-bundles over $S^4$ [20]. Substantial information has also been obtained for $SU(4)$-bundles over $S^4$ [6]; $Sp(3)$-bundles over $S^4$ [5]; and $G_2$-bundles over $S^4$ [12].

In this paper we consider the homotopy types of the gauge groups of principal $Sp(2)$-bundles over a closed, simply-connected four-manifold. It is well known that the principal $Sp(2)$-bundles over a closed simply-connected four-manifold $M$ are classified by their second Chern class, which can take any integer value. Let $P_k \longrightarrow M$ be the principal $Sp(2)$-bundle classified by the integer $k$, and let $G_k(M)$ be its gauge group. We say that $G_k(M)$ is fibre homotopy equivalent to $G_\ell(M)$ if there is a homotopy commutative diagram

$$
\begin{align*}
G_k(M) \simeq & \longrightarrow G_\ell(M) \\
\downarrow & \downarrow \\
G & .
\end{align*}
$$

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For integers $a$ and $b$ let $(a, b)$ be their greatest common denominator.

**Theorem 1.1.** Let $M$ be a closed, simply-connected four-manifold. The following hold:

(a) if $(40, k) = (40, \ell)$ then $G_k(M)$ is fibre homotopy equivalent to $G_\ell(M)$ when localized rationally or at any prime;

(b) if $G_k(M)$ is fibre homotopy equivalent to $G_\ell(M)$ then $(40, k) = (40, \ell)$.

Two comments should be made. First, existing statements enumerating the homotopy types of gauge groups are phrased in terms of homotopy equivalence rather than fibre homotopy equivalence. A fibre homotopy equivalence between $G_k(M)$ and $G_\ell(M)$ is stronger than a homotopy equivalence, so part (a) implies a corresponding homotopy equivalence statement. In part (b) the stronger condition of fibre homotopy equivalence is used. In Theorem 7.3 we show that if $G_k(M)$ is homotopy equivalent to $G_\ell(M)$ then $(20, k) = (20, \ell)$. However, it is not clear if this can be improved to $(40, k) = (40, \ell)$.

Second, an interesting feature of the proof of Theorem 1.1 (b) involves showing that the boundary map in a fibration involving the classifying space $BG_k(M)$ has “order” 40 (what is meant by “order” is described precisely following Proposition 3.2). This is interesting because the analogous boundary map in the case of gauge groups for principal $SU(2)$ or $SU(3)$-bundles over $M$ has its “order” vary depending on whether $M$ is Spin or non-Spin. In the $Sp(2)$-case the “order” is independent of the existence of a Spin structure. One wonders if this holds more generally.

## 2. Framing the problem

This section summarizes what is known and what remains to be proved in Theorem 1.1. In general, let $G$ be a simply-connected, simple compact Lie group and let $M$ be a closed, simply-connected four-manifold. Since $[M, BG] \cong \mathbb{Z}$ the principal $G$-bundles over $M$ are classified by the second Chern class of $M$. Let $P_k \to M$ be a principal $G$-bundle whose second Chern class is $k \in \mathbb{Z}$.

Let $G_k(M)$ be the gauge group of $P_k$.

The following decomposition was proved for the Spin case in [19] and for the non-Spin case in [17].

**Theorem 2.1.** Let $G$ be a simply-connected, simple compact Lie group and let $M$ be a closed, simply-connected four-manifold with $H^2(M; \mathbb{Z})$ of rank $d \geq 1$.

(a) If $M$ is Spin then there is an integral homotopy equivalence

$$G_k(M) \cong G_k(S^4) \times \prod_{i=1}^{d} \Omega^2 G.$$  

(b) If $M$ is non-Spin then there is an integral homotopy equivalence

$$G_k(M) \cong G_k(\mathbb{C}P^2) \times \prod_{i=1}^{d-1} \Omega^2 G.$$
Theorem 2.1 implies that to count the number of distinct homotopy types of the gauge groups $G_k(M)$ it suffices to do so in the special cases $G_k(S^4)$ and $G_k(\mathbb{C}P^2)$. Further, at odd primes there is another decomposition, proved in [19].

**Theorem 2.2.** Let $G$ be a simply-connected, simple compact Lie group. Localization rationally or at an odd prime $p$. Then there is a homotopy equivalence

$$G_k(\mathbb{C}P^2) \simeq G_k(S^4) \times \Omega^2 G.$$  

Theorem 2.2 implies that the only possible difference between the number of distinct homotopy types of the gauge groups $G_k(S^4)$ and $G_k(\mathbb{C}P^2)$ occurs at the prime 2. The homotopy types $G_k(S^4)$ for $G = Sp(2)$ were determined in [20].

**Theorem 2.3.** Let $G = Sp(2)$. The following hold:

(a) if $G_k(S^4) \simeq G_\ell(S^4)$ then $(40, k) = (40, \ell)$;

(b) if $(40, k) = (40, \ell)$ then $G_k(S^4) \simeq G_\ell(S^4)$ when localized rationally or at any prime.

Collectively, Theorems 2.1, 2.2, and 2.3 imply that the only case we need to consider to complete the proof of Theorem 1.1 is that of $G_k(\mathbb{C}P^2)$ at the prime 2. Resolving this case occupies the remainder of the paper.

3. A METHOD FOR COUNTING THE HOMOTOPY TYPES OF GAUGE GROUPS

Return to the case of any simply-connected, simple compact Lie group $G$, let $M = S^4$ or $\mathbb{C}P^2$ and let $Map_k(M, BG)$ be the component of the space of continuous unbased maps from $M$ to $BG$ which contains the map inducing $P_k$. Similarly, let $Map_k^*(M, BG)$ be the space of continuous pointed maps from $M$ to $BG$ which contains the map inducing $P_k$. Observe that there is a fibration $Map_k^*(M, BG) \to Map_k(M, BG) \xrightarrow{ev} BG$, where $ev$ evaluates a map at the basepoint of $M$. Let $BG_k(M)$ be the classifying space of $G_k(M)$. By [11, 8], there is a homotopy equivalence $BG_k(M) \simeq Map_k(M, BG)$. The evaluation fibration therefore determines a homotopy fibration sequence

$$G \xrightarrow{\partial_k} Map_k^*(M, BG) \rightarrow BG_k(M) \xrightarrow{ev} BG$$

which defines the map $\partial_k$. In the case $M = S^4$, write $\partial_k$ for $\partial_k$.

The evaluation fibration satisfies a naturality property. The fact that the pinch map $\mathbb{C}P^2 \xrightarrow{\pi} S^4$ to the top cell induces an isomorphism $[S^4, BG] \cong [\mathbb{C}P^2, BG]$ implies that it induces a one-to-one correspondence between the components of $Map_k(S^4, BG)$ and $Map_k(\mathbb{C}P^2, BG)$ and the
components of \( \text{Map}_k^*(S^4, BG) \) and \( \text{Map}_k^*(\mathbb{C}P^2, BG) \). Moreover, it is well known that there is a homotopy equivalence \( \text{Map}_k^*(S^4, BG) \simeq \text{Map}_0^*(S^4, BG) \) for every \( k \), and by [18] there is also a compatible homotopy equivalence \( \text{Map}_k^*(\mathbb{C}P^2, BG) \simeq \text{Map}_0^*(\mathbb{C}P^2, BG) \). Therefore, writing \( \Omega^3_k \) for \( \text{Map}_0^*(S^4, BG) \), we obtain a homotopy fibration diagram

\[
\begin{array}{cccc}
G & \xrightarrow{\partial_k} & \Omega^3_0 G & \xrightarrow{\text{ev}} & BG \\
\downarrow & & \downarrow \pi^* & & \downarrow \\
G & \xrightarrow{\bar{\partial}_k} & \text{Map}_0^*(\mathbb{C}P^2, BG) & \xrightarrow{\text{ev}} & BG.
\end{array}
\]  

The key to understanding the homotopy types of \( G_k(S^4) \) and \( G_k(\mathbb{C}P^2) \) is understanding the homotopy classes \( \partial_k \) and \( \bar{\partial}_k \). The adjoint of \( \partial_k \) can be identified. Let \( i: S^3 \to G \) be the inclusion of the bottom cell and let \( 1: G \to G \) be the identity map. Lang [15] identified the homotopy class of the triple adjoint of \( \partial_k \) as follows.

**Lemma 3.1.** The adjoint of \( G \xrightarrow{\partial_k} \Omega^3_0 G \) is homotopic to the Samelson product \( S^3 \wedge G \xrightarrow{(k, 1)} G \). Consequently, the linearity of the Samelson product implies that \( \partial_k \simeq k \circ \bar{\partial}_1 \). \( \square \)

The following proposition, proved in [21], uses the order of \( \bar{\partial}_1 \) to estimate the number of homotopy types of the gauge groups \( G_k(\mathbb{C}P^2) \).

**Proposition 3.2.** Suppose that the map \( G \xrightarrow{\bar{\partial}_1} \Omega^3_0 G \) has order \( m \). If \( (m, k) = (m, \ell) \) then \( G_k(\mathbb{C}P^2) \) is homotopy equivalent to \( G_k(\mathbb{C}P^2) \) when localized rationally or at any prime. \( \square \)

**Remark 3.3.** In fact, the proof of Proposition 3.2 in [21] shows a stronger result: if \( (m, k) = (m, \ell) \) then \( G_k(\mathbb{C}P^2) \) is fibre homotopy equivalent to \( G_\ell(\mathbb{C}P^2) \) when localized rationally or at any prime.

With this we can prove Theorem 1.1(a).

**Proof of Theorem 1.1 (a).** By Theorem 2.1 it suffices to prove the statement for the special cases \( M = S^4 \) and \( M = \mathbb{C}P^2 \). The \( M = S^4 \) case is the statement of Theorem 2.3 By [20] the map \( \text{Sp}(2) \xrightarrow{\partial_1} \Omega^3_0 \text{Sp}(2) \) has order 40. Part (a) now follows immediately from Proposition 3.2 and Remark 3.3 \( \square \)

While this proves the statement in Theorem 1.1(a) there is a more subtle question that needs to be addressed. The proof depended only on the order of \( \text{Sp}(2) \xrightarrow{\partial_1} \Omega^3_0 \text{Sp}(2) \) while not taking into account the composite \( \text{Sp}(2) \xrightarrow{\partial_1} \Omega^3_0 \text{Sp}(2) \xrightarrow{\pi^*} \text{Map}_0^*(\mathbb{C}P^2, B\text{Sp}(2)) \). Since \( G_k(\mathbb{C}P^2) \) is the homotopy fibre of \( \bar{\partial}_1 = \pi^* \circ \partial_1 \), it is more natural to consider properties of this map. Notice, though, that it makes no sense to talk about the order of the map \( \bar{\partial}_1 \) since \( \text{Map}_0^*(\mathbb{C}P^2, BG) \) need not be an \( H \)-space. Instead, the factorization of \( \bar{\partial}_1 \) through \( \partial_1 \) lets us consider the “order” of \( \bar{\partial}_k \), by which we mean the least integer \( n \) such that the composite \( G \xrightarrow{\partial_1} \Omega^3_G \xrightarrow{n} \Omega^3_0 G \to \text{Map}_0^*(\mathbb{C}P^2, BG) \) is null.
potentially, it is possible that the “order” of $\partial_1$ is 20 instead of 40, in which case it is more reasonable to expect Theorem 1.1 (a) having a g.c.d. condition involving 20 rather than 40.

In Theorem 5.7 we show that the “order” of $\partial_1$ is 40. This also plays an important role in proving Theorem 1.1 (b). The build-up to Theorem 5.7 requires several calculations which occupy the next two sections.

4. Preliminary information for the “order” of $\partial_1$

Recall that $H_*(Sp(2); \mathbb{Z}) \cong \Lambda(x_3, x_7)$ where $x_i$ has degree $i$. So $Sp(2)$ can be given a CW-structure with three cells, one in each of the dimensions 3, 7 and 10. Let $i: S^3 \to Sp(2)$ be the inclusion of the bottom cell. Let $A$ be the 7-skeleton of $Sp(2)$, so $A$ has two cells, one each in dimensions 3 and 7. Let $j: A \to Sp(2)$ be the skeletal inclusion. There are homotopy cofibrations

$$S^6 \xrightarrow{f_1} S^3 \to A$$
$$S^9 \xrightarrow{f_2} A \to Sp(2)$$

for some maps $f_1$ and $f_2$. Following Toda’s notation [23] for the homotopy groups of spheres, it is well known that $f_1 = \nu'$, where $\nu'$ represents a generator of $\pi_6(S^3) \cong \mathbb{Z}/12\mathbb{Z}$ (see [16], for example). Let $\nu_4$ represent the integral generator of $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$, and by definition let $\nu_m = \Sigma^{m-4}\nu_4$ for $m \geq 5$. In particular, $\nu_4^2 = \nu_4 \circ \nu_7$. James [10] showed that $\Sigma f_2$ factors as the composite $S^{10} \xrightarrow{2\nu_4^2} S^4 \to \Sigma A$.

Lemma 4.1. There is a homotopy cofibration sequence

$$S^3 \xrightarrow{i} Sp(2) \to S^7 \vee S^{10} \xrightarrow{\Sigma \nu'} g \to S^4$$

where $g = 2\nu_4^2$.

Proof. Define the space $C$ and the map $\delta$ by the homotopy cofibration sequence

(3)

$$S^3 \xrightarrow{i} Sp(2) \to C \xrightarrow{\delta} S^4.$$ 

In [20] it was shown that $C$ is homotopy equivalent to $S^7 \vee S^{10}$; to also identify the homotopy class of $\delta$ we give an alternative description of the homotopy type of $C$.

Let $q$ be the composite $A \xrightarrow{j} Sp(2) \to S^7$ and notice that $q$ is the pinch map to the top cell. From this composite we obtain a homotopy pushout diagram

(4)

for some map $g$. James’ factorization of $\Sigma f_2$ as $S^{10} \xrightarrow{2\nu_4^2} S^4 \to \Sigma A$ therefore implies that the difference $2\nu_4^2 - g$ lifts to the homotopy fibre $F$ of $S^4 \to \Sigma A$. For dimensional reasons, this lift
factors through the 10-skeleton of $F$, which by a Serre spectral sequence calculation, is $S^7$. The composite $S^7 \hookrightarrow F \rightarrow S^4$ is homotopic to $\Sigma\nu'$, so $2\nu'^2 - g$ factors as the composite $S^{10} \xrightarrow{h} S^7 \xrightarrow{\Sigma\nu'} S^4$ for some map $h$. By [23], $\pi_{10}(S^7) \cong \mathbb{Z}/12\mathbb{Z}$, generated by $\nu_7$, so $h = s \cdot \nu_7$ for some $s \in \mathbb{Z}/12\mathbb{Z}$, implying that $2\nu'^2 - g \simeq \Sigma\nu' \circ (s \cdot \nu_7)$. By [23], the Hopf invariant $H: \pi_{10}(S^4) \rightarrow \pi_{10}(S^7)$ is an injection. But as $\Sigma\nu' \circ \nu_7 \simeq \Sigma(\nu' \circ \nu_6)$, we obtain $H(\Sigma\nu' \circ \nu_7) = 0$. Thus $\Sigma\nu' \circ \nu_7$ is null homotopic, and hence $g \simeq 2\nu'^2$.

The homotopy cofibration sequence $S^9 \xrightarrow{f_2} A \rightarrow Sp(2) \rightarrow S^{10}$ has a coaction $\psi: Sp(2) \rightarrow Sp(2) \vee S^{10}$. Composing with the standard map $Sp(2) \rightarrow S^7$ we obtain a composite $\phi: Sp(2) \xrightarrow{\psi} Sp(2) \vee S^{10} \rightarrow S^7 \vee S^{10}$. This coaction lets us use a Mayer-Vietorus like argument to produce a homotopy cofibration $Sp(2) \xrightarrow{\phi} S^7 \vee S^{10} \xrightarrow{\Sigma\nu' \circ g} S^4$ from the pushout in the middle square of [4]. Finally, notice that as $\phi \circ i$ is null homotopic, $\phi$ extends to a map $\epsilon: C \rightarrow S^7 \vee S^{10}$. As $\phi_*$ is an epimorphism, so is $\epsilon_*$. Comparing Poincaré series, $\epsilon_*$ must therefore be an isomorphism and so $\epsilon$ is a homotopy equivalence. Hence, up to homotopy, there is a cofibration sequence $S^3 \xrightarrow{i} Sp(2) \xrightarrow{\phi} S^7 \vee S^{10} \xrightarrow{\Sigma\nu' \circ g} S^4$ where $g = 2\nu'^2$. □

By [20], there is a homotopy commutative diagram
\[
\begin{array}{ccc}
Sp(2) & \xrightarrow{\partial_1} & \Omega^3_0 Sp(2) \\
S^7 \vee S^{10} & \xrightarrow{a+b} & \\
\end{array}
\]
where $a$ represents a generator of $\pi_7(\Omega^3_0 Sp(2)) \cong \pi_{10}(Sp(2)) \cong \mathbb{Z}/8\mathbb{Z}$, and $b$ represents some class in $\pi_{10}(\Omega^3_0 Sp(2)) \cong \pi_{13}(Sp(2)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Since $b$ has order at most 4, we immediately obtain the following.

**Lemma 4.2.** There is a homotopy commutative diagram
\[
\begin{array}{ccc}
Sp(2) & \xrightarrow{\partial_1} & \Omega^3_0 Sp(2) \\
S^7 \vee S^{10} & \xrightarrow{p_1} & S^7 \xrightarrow{4a} \Omega^3_0 Sp(2) \\
\end{array}
\]
where $p_1$ is the pinch map onto the left wedge summand. □

5. **Determining the “order” of $\Omega_1$**

Start with the homotopy cofibration sequence
\[(5)\quad S^3 \xrightarrow{\eta} S^2 \rightarrow CP^2 \rightarrow S^4 \xrightarrow{\Sigma\eta} S^3\]
and the homotopy fibration
\[(6)\quad S^3 \rightarrow Sp(2) \rightarrow S^7.\]
Applying the pointed mapping space functor $\text{Map}^*(\ ,\ ,\ )$ with the spaces in the left variable coming from $[3]$ and the spaces in the right variable coming from $[9]$, and writing $\text{Map}^*(S^i, X)$ as $\Omega^i X$, we obtain a homotopy fibration diagram

$$
\begin{array}{c}
\Omega^2 S^3 \xrightarrow{(\Sigma\gamma)^*} \Omega^4 S^3 \xrightarrow{} \text{Map}^*(\mathbb{C}P^2, S^3) \xrightarrow{} \Omega^2 S^3 \\
\Omega^3 Sp(2) \xrightarrow{(\Sigma\gamma)^*} \Omega^4 Sp(2) \xrightarrow{} \text{Map}^*(\mathbb{C}P^2, Sp(2)) \xrightarrow{} \Omega^2 Sp(2) \\
\Omega^3 S^7 \xrightarrow{(\Sigma\gamma)^*} \Omega^4 S^7 \xrightarrow{} \text{Map}^*(\mathbb{C}P^2, S^7) \xrightarrow{} \Omega^2 S^7 \\
\end{array}
$$

(7)

For the remainder of the section, localize all spaces and maps at 2.

**Lemma 5.1.** There is an isomorphism $\pi_3(\text{Map}^*(\mathbb{C}P^2, Sp(2))) \cong \mathbb{Z}$ and the map $\Omega^3 Sp(2) \to \text{Map}^*(\mathbb{C}P^2, Sp(2))$ induces $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ on $\pi_3$.

**Proof.** Apply $\pi_3$ to (7), using the homotopy group information for spheres in [23] and $Sp(2)$ in [10], to obtain a diagram of exact sequences

$$
\begin{array}{c}
\mathbb{Z}/4\mathbb{Z} \xrightarrow{(\Sigma\gamma)^*} \mathbb{Z}/2\mathbb{Z} \xrightarrow{a} \pi_3(\text{Map}^*(\mathbb{C}P^2, S^3)) \xrightarrow{b} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\eta^*} \mathbb{Z}/4\mathbb{Z} \\
\mathbb{Z} \xrightarrow{c} \pi_3(\text{Map}^*(\mathbb{C}P^2, Sp(2))) \xrightarrow{d} \mathbb{Z}/2\mathbb{Z} \xrightarrow{e} \mathbb{Z} \\
\mathbb{Z} \xrightarrow{g} \pi_3(\text{Map}^*(\mathbb{C}P^2, S^7)) \xrightarrow{h} 0 \\
0 \xrightarrow{0} 0 \\
\end{array}
$$

(8)

where the maps $a$ through $g$ are to be determined.

First, consider the map $a$. By [23], $\pi_3(\Omega^4 S^3) \cong \mathbb{Z}/4\mathbb{Z}$ and $\pi_3(\Omega^4 S^3) \cong \mathbb{Z}/2\mathbb{Z}$ are generated by the adjoints of $\nu'$ and $\nu' \circ \eta$ respectively. As $(\Sigma\gamma)^*$ precomposes with $\eta$, this map is onto. Hence $a = 0$. Second, consider the map $b$. By [23], $\pi_3(\Omega^2 S^3) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the adjoint of $\eta^3$, so $\eta^*$ sends this to the adjoint of $\eta^3$. But this equals $2\nu'$, which generates $\pi_3(\Omega^3 S^1)$, so the map $b$ is an injection. Hence exactness along the top row in (8) implies that $\pi_3(\text{Map}^*(\mathbb{C}P^2, S^3)) \cong 0$. This then implies that the map $f$ is an injection.

Third, exactness along the bottom row in (8) implies that $g$ is an isomorphism. Fourth, by [10], the map $\pi_3(\Omega^4 Sp(2)) \to \pi_3(\Omega^4 S^7)$ is $\mathbb{Z} \xrightarrow{d} \mathbb{Z}$, so in (8) we have $c = 0$ and $e = 4$.

Finally, consider the middle row in (8). By exactness, either $\pi_3(\text{Map}^*(\mathbb{C}P^2, Sp(2)))$ is isomorphic to $\mathbb{Z}$ with $d = 2$ or it is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with $d$ being the inclusion of the integral summand. But as $f$ is an injection and $\pi_3(\text{Map}^*(\mathbb{C}P^2, S^7)) \cong \mathbb{Z}$, it must be the first possibility that holds. Hence $\pi_3(\text{Map}^*(\mathbb{C}P^2, Sp(2))) \cong \mathbb{Z}$ and the map $\Omega^4 Sp(2) \to \text{Map}^*(\mathbb{C}P^2, Sp(2))$ induces $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ on $\pi_3$. 

□
Lemma 5.2. There is an isomorphism \( \pi_6(\text{Map}^*(\mathbb{CP}^2, Sp(2))) \cong \mathbb{Z}/8\mathbb{Z} \), and the map \( \Omega^1 Sp(2) \to \text{Map}^*(\mathbb{CP}^2, Sp(2)) \) induces an isomorphism on \( \pi_6 \).

Proof. By [16], \( \pi_6(\Omega^1 Sp(2)) \cong 0 \) and \( \pi_6(\Omega^2 Sp(2)) \cong 0 \), so when \( \pi_6 \) is applied to the homotopy fibration in the middle row of (13) we obtain an isomorphism \( \pi_6(\Omega^1 Sp(2)) \to \pi_6(\text{Map}^*(\mathbb{CP}^2, Sp(2))) \).

By [16], \( \pi_6(\Omega^1 Sp(2)) \cong \mathbb{Z}/8\mathbb{Z} \).

Lemma 5.3. The following hold:

(a) there is an isomorphism \( \pi_9(\text{Map}^*(\mathbb{CP}^2, Sp(2))) \cong \mathbb{Z}/4\mathbb{Z} \);

(b) the map \( \Omega^1 Sp(2) \to \text{Map}^*(\mathbb{CP}^2, Sp(2)) \) sends the order 4 generator of \( \pi_9(\Omega^1 Sp(2)) \cong \mathbb{Z}/4\mathbb{Z} \) to an order 4 generator of \( \pi_9(\text{Map}^*(\mathbb{CP}^2, Sp(2))) \).

Proof. Apply \( \pi_9 \) to the top two rows of (13) and use the homotopy group information for spheres in [23] and \( Sp(2) \) in [16] to obtain a diagram of exact sequences

\[
\begin{array}{cccccccc}
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & a & \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & b & \pi_9(\text{Map}^*(\mathbb{CP}^2, S^3)) & c & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & d & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \\
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & i & \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & j & \pi_9(\text{Map}^*(\mathbb{CP}^2, Sp(2))) & k & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \ell & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \\
\end{array}
\]

where the maps \( a \) through \( \ell \) are to be determined. Note the two maps labelled \( e \) are the same, that is, they are both \( \pi_9(\Omega^1 S^3) \to \pi_9(\Omega^1 Sp(2)) \).

We aim to show that \( j \) is onto, it is an injection on the \( \mathbb{Z}/4\mathbb{Z} \) summand and it is trivial on the \( \mathbb{Z}/2\mathbb{Z} \) summand. These properties then imply that \( \pi_9(\text{Map}^*(\mathbb{CP}^2, Sp(2))) \cong \mathbb{Z}/4\mathbb{Z} \) and that the map \( \Omega^1 Sp(2) \to \text{Map}^*(\mathbb{CP}^2, Sp(2)) \) sends the order 4 generator of \( \pi_9(\Omega^1 Sp(2)) \cong \mathbb{Z}/4\mathbb{Z} \) to an order 4 generator of \( \pi_9(\text{Map}^*(\mathbb{CP}^2, Sp(2))) \).

First, by [16], the map \( S^3 \to Sp(2) \) induces an isomorphism on \( \pi_{11} \) and \( \pi_{12} \), implying that \( h \) and \( e \) respectively are isomorphisms. In fact, the generators of \( \pi_{11}(Sp(2)) \) and \( \pi_{12}(Sp(2)) \) are chosen as the images of those from \( \pi_{11}(S^3) \) and \( \pi_{12}(S^3) \) respectively, so \( e \) and \( h \) may be taken to be identity maps. As well, Theorem 5.1 and relation (5.1) of [16] imply that the map \( f \) is \( 2 \oplus id \), where \( id \) is the identity map.

Consider the map \( a \). Explicitly, \( a \) is the map \( \pi_9(\Omega^1 S^3) \to \pi_9(\Omega^4 S^3) \). Taking adjoints, \( a \) sends a map \( S^{12} \to S^{3} \) to the composite \( S^{13} \xrightarrow{\eta_2} S^{11} \xrightarrow{\xi} S^{3} \). By [23], the generators for \( \pi_{12}(S^3) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) are \( \eta_4 \) and \( \mu_3 \) and the generators for \( \pi_{13}(S^3) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) are \( \epsilon' \) and \( \eta_3 \mu_4 \), of orders 4 and 2 respectively. Thus \( a(\eta_4 \epsilon_4) = \eta_3 \epsilon_4 \eta_3 \) and \( a(\mu_3) = \mu_3 \eta_3 \eta_2 \), neither of which are immediately generators of \( \pi_{13}(S^3) \). In general, two maps \( u : S^m \to S^n \) and \( v : S^{m'} \to S^{n'} \) satisfy \( \Sigma^u \circ \Sigma^v \simeq \pm \Sigma^u \circ \Sigma^v \). In our case, we obtain \( \epsilon_5 \eta_1 = \eta_6 \epsilon_6 \) and \( \mu_5 \eta_4 = \eta_6 \mu_6 \), where the signs have disappeared since all composites are of order 2. Thus after two suspensions the image of \( a \) consists of \( \eta_6 \epsilon_6 \eta_1 = \eta_6^2 \epsilon_7 \) and \( \mu_5 \eta_4 = \eta_6 \mu_6 \). By [23] Lemma 6.6, \( \eta_6^2 \epsilon_7 = 2 \Sigma^2 \epsilon' \) and \( \Sigma^2 \epsilon' \) has order 4. Moreover, by [23] Theorem 7.3, the elements \( \eta_6 \mu_6 \) and \( \Sigma^2 \epsilon' \) represent distinct generators.
in $\pi_{13}(S^5)$. Thus after two suspensions $a$ induces an injection, implying that $a$ itself must be an injection.

Let $x, y$ be generators for each of the summands in $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Since $a$ is an injection and consists of elements of order 2, the image is generated by $2sx + ty$ and $2s'x + t'y$ for $s, s', t, t' \in \mathbb{Z}/2\mathbb{Z}$. The injectivity of $a$ implies that at least one of $t$ or $t'$ is equal to 1. As $f = 2 \oplus id$, the image of $f \circ a$ is $ty$ and $t'y$. The commutativity of the leftmost square in (9) then implies that the image of $i$ is $ty$ and $t'y$. Therefore the image of $i$ in $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is trivial when projected to $\mathbb{Z}/4\mathbb{Z}$ and onto when projected to $\mathbb{Z}/2\mathbb{Z}$. Thus, exactness along the bottom row of (9) implies that $j$ is an injection on the $\mathbb{Z}/4\mathbb{Z}$ summand and is trivial on the $\mathbb{Z}/2\mathbb{Z}$ summand.

Consider the map $\ell$. Since $h$ and $e$ are identity maps, it is equivalent to determine $d$. Explicitly, $d$ is the map $\pi_9(\Omega^2S^3) \to \pi_9(\Omega^3S^3)$. Taking adjoints, $d$ sends a map $S^{11} \to S^3$ to the composite $S^{12} \mapright{\eta_3} S^{11} \mapright{\eta_1} S^3$. By [23], $\pi_{11}(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $\eta_3$, and as above, $\pi_{12}(S^3) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is generated by $\eta_3\eta_4$ and $\mu_3$. Observe that $d(\epsilon_3) = \epsilon_3\eta_{11}$, so as before, after two suspensions the image of $d$ is $\epsilon_3\eta_{11} = \eta_5\epsilon_6$, which is nontrivial. Therefore after two suspensions $d$ induces an injection, implying that $d$ itself is an injection. Hence $\ell$ is an injection, implying that $k$ is the zero map and $j$ is onto. \qed

In what follows we work with $\text{Map}^*(\mathbb{C}P^2, B\text{Sp}(2))$, and note that the pointed exponential law implies that $\Omega\text{Map}^*(\mathbb{C}P^2, B\text{Sp}(2)) \simeq \text{Map}^*(\mathbb{C}P^2, \text{Sp}(2))$. By Lemma 5.1, $\pi_4(\text{Map}^*(\mathbb{C}P^2, B\text{Sp}(2))) \cong \mathbb{Z}$. Let $\gamma: S^4 \to \text{Map}^*(\mathbb{C}P^2, B\text{Sp}(2))$ represent a generator. The second statement in Lemma 5.1 implies that there is a homotopy commutative diagram

\[
\begin{array}{ccc}
S^4 & \xrightarrow{\theta} & \Omega^3\text{Sp}(2) \\
\downarrow 2 & & \downarrow 2 \\
S^4 & \xrightarrow{\gamma} & \text{Map}^*(\mathbb{C}P^2, B\text{Sp}(2))
\end{array}
\]

where $\theta$ represents a generator of $\pi_4(\Omega^3\text{Sp}(2)) \cong \mathbb{Z}$.

**Lemma 5.4.** The composite $S^7 \xrightarrow{\nu_4} S^4 \xrightarrow{\gamma} \text{Map}^*(\mathbb{C}P^2, B\text{Sp}(2))$ has order 8.

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
S^7 & \xrightarrow{\nu_4} & S^4 & \xrightarrow{\theta} & \Omega^3\text{Sp}(2) \\
\downarrow 4 & & \downarrow 2 & & \downarrow 2 \\
S^7 & \xrightarrow{\nu_4} & S^4 & \xrightarrow{\gamma} & \text{Map}^*(\mathbb{C}P^2, B\text{Sp}(2)).
\end{array}
\]

The left square homotopy commutes since $\nu_4$ has Hopf invariant one. The right square homotopy commutes by (10). We claim that the upper direction around this diagram is a map of order 2. This would imply that $\gamma \circ \nu_4 \circ 4$ has order 2, implying in turn that $\gamma \circ \nu_4$ has order 8.
It remains to show that the upper direction around the diagram has order 2. By Lemma 5.2, the map \( \Omega^3 Sp(2) \to \text{Map}^*(\mathbb{C}P^2, BSp(2)) \) induces an isomorphism on \( \pi_7 \), implying that it suffices to show that the map \( \theta \circ \nu_4 \) has order 2. If \( c: S^7 \to Sp(2) \) is the adjoint of \( \theta \), it is equivalent to show that \( c \circ \nu_7 \) has order 2. Consider the composite

\[
S^{10} \overset{2 \nu_7}{\longrightarrow} S^7 \overset{\nu_4}{\longrightarrow} Sp(2) \overset{q}{\longrightarrow} S^7.
\]

By [16], \( q \) induces an isomorphism on \( \pi_{10} \), so it suffices to show that \( q \circ c \circ \nu_7 \) has order 2. By [23], \( \pi_{10}(S^7) \cong \mathbb{Z}/8\mathbb{Z} \) is generated by \( \nu_7 \), and by [16], \( q \circ c \simeq \pm 4 \). Thus \( q \circ c \circ \nu_7 \simeq \pm 4 \nu_7 \) has order 2. □

Recall that the map \( S^{10} \overset{g}{\longrightarrow} S^4 \) in Lemma 4.1 is \( 2\nu_4^2 \).

**Proposition 5.5.** The composite \( S^{10} \overset{g}{\longrightarrow} S^4 \overset{\gamma}{\longrightarrow} \text{Map}^*(\mathbb{C}P^2, BSp(2)) \) has order 2. □

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
S^{10} & \overset{2 \nu_7}{\longrightarrow} & S^7 \\
\downarrow{\lambda} & & \downarrow{\nu_4} \\
\Omega^3 Sp(2) & \overset{j}{\longrightarrow} & \text{Map}^*(\mathbb{C}P^2, BSp(2))
\end{array}
\]

(11)

where \( j \) is the usual map and \( \lambda \) will be defined momentarily. By Lemma 5.2, \( \pi_7(\text{Map}^*(\mathbb{C}P^2, BSp(2))) \cong \mathbb{Z}/8\mathbb{Z} \) so Lemma 5.4 implies that \( \gamma \circ \nu_4 \) represents a generator. Lemma 5.2 also states that the map \( \Omega^3 Sp(2) \to \text{Map}^*(\mathbb{C}P^2, BSp(2)) \) induces an isomorphism on \( \pi_7 \). Thus \( \gamma \circ \nu_4 \) lifts to a map \( \lambda: S^7 \to \Omega^3 Sp(2) \) which represents a generator of \( \pi_7(\Omega^3 Sp(2)) \cong \mathbb{Z}/8\mathbb{Z} \) and makes (11) homotopy commute.

Mimura and Toda chose a generator of \( \pi_{10}(Sp(2)) \cong \mathbb{Z}/8\mathbb{Z} \) they called \([\nu_7] \). Let \( c: S^{10} \to Sp(2) \) be the triple adjoint of \( \lambda \), so \( c \) is also a generator of \( \pi_{10}(Sp(2)) \). Then \( c \simeq u \cdot [\nu_7] \), where \( u \in \mathbb{Z}/8\mathbb{Z} \) is a unit. In [10] it is also shown that \([\nu_7] \nu_{10} \) represents the order 4 generator in \( \pi_{13}(Sp(2)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). Thus \( c \circ \nu_{10} \) also has order 4. Taking adjoints, \( \lambda \circ \nu_7 \) has order 4. Now reconsider (11). By Lemma 5.3 the map \( \Omega^3 Sp(2) \to \text{Map}^*(\mathbb{C}P^2, BSp(2)) \) sends the order 4 generator of \( \pi_{10}(\Omega^3 Sp(2)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) to an order 4 generator of \( \pi_{10}(\text{Map}^*(\mathbb{C}P^2, BSp(2))) \). Thus \( j \circ \lambda \circ \nu_7 \) has order 4, implying that \( j \circ \lambda \circ 2 \nu_7 \) has order 2. The homotopy commutativity of (11) therefore implies that \( \gamma \circ \nu_4 \circ 2 \nu_7 \) has order 2. That is, \( \gamma \circ g \) has order 2. □

We will also need to know how \( S^7 \overset{2 \nu_7}{\longrightarrow} S^4 \) composes with \( \gamma \).

**Lemma 5.6.** The composite \( S^7 \overset{2 \nu_7}{\longrightarrow} S^4 \overset{\gamma}{\longrightarrow} \text{Map}^*(\mathbb{C}P^2, BSp(2)) \) has order at most 2.

**Proof.** First observe that the homotopy fibration \( \Omega^3 Sp(2) \to \text{Map}^*(\mathbb{C}P^2, BSp(2)) \to \Omega Sp(2) \) implies that there is an exact sequence \( \pi_5(\Omega^3 Sp(2)) \to \pi_5(\text{Map}^*(\mathbb{C}P^2, BSp(2))) \to \pi_5(\Omega Sp(2)) \). By [16], \( \pi_5(\Omega^3 Sp(2)) \cong \pi_5(\Omega Sp(2)) \cong 0 \), implying that \( \pi_5(\text{Map}^*(\mathbb{C}P^2, BSp(2))) \cong 0 \).
Next, by [23], $2\nu' = r_4^2$. Therefore the composite $S^7 \xrightarrow{\partial_1} S^7 \xrightarrow{\Sigma \nu'} S^4 \xrightarrow{\gamma} \text{Map}^*_0(\mathbb{C}P^2, B\text{Sp}(2))$ factors through the composite $S^5 \xrightarrow{\eta} S^4 \xrightarrow{\gamma} \text{Map}^*_0(\mathbb{C}P^2, B\text{Sp}(2))$. The latter is null homotopic since $\pi_5(\text{Map}^*_0(\mathbb{C}P^2, B\text{Sp}(2))) \cong 0$, and hence $\gamma \circ \Sigma \nu' \circ 2$ is null homotopic. Consequently, $\gamma \circ \Sigma \nu'$ has order at most 2.

**Theorem 5.7.** The composite $Sp(2) \xrightarrow{\partial_1} \Omega^4_0 Sp(2) \xrightarrow{4} \Omega^4_0 Sp(2) \to \text{Map}^*_0(\mathbb{C}P^2, B\text{Sp}(2))$ is non-trivial.

**Proof.** By Lemma 4.1 there is a homotopy commutative diagram

$$
\begin{array}{ccc}
Sp(2) & \xrightarrow{\partial_1} & \Omega^4_0 Sp(2) \\
\downarrow & & \downarrow 4 \\
S^7 \vee S^{10} & \xrightarrow{p_1} & S^7 \xrightarrow{4a} \Omega^4_0 Sp(2)
\end{array}
$$

and by Lemma 4.1 there is a homotopy cofibration $Sp(2) \to S^7 \vee S^{10} \xrightarrow{\Sigma \nu' - g} S^4$. Aiming for a contradiction, suppose that the composite $Sp(2) \xrightarrow{\partial_1} \Omega^4_0 Sp(2) \xrightarrow{4} \Omega^4_0 Sp(2) \to \text{Map}^*_0(\mathbb{C}P^2, B\text{Sp}(2))$ is null homotopic. Then there is an extension

$$
\begin{array}{ccc}
S^7 \vee S^{10} & \xrightarrow{p_1} & S^7 \xrightarrow{4a} \Omega^4_0 Sp(2) \\
\downarrow \Sigma \nu' - g & & \downarrow \\
S^4 & \xrightarrow{\theta} & \text{Map}^*_0(\mathbb{C}P^2, B\text{Sp}(2))
\end{array}
$$

(12)

for some map $\theta$.

We claim that $\theta$ must represent a generator of $\pi_4(\text{Map}^*_0(\mathbb{C}P^2, B\text{Sp}(2))) \cong \mathbb{Z}$. Restrict (12) to the $S^7$ summand of $S^7 \vee S^{10}$. Since $a$ represents a generator of $\pi_7(\Omega^3 Sp(2)) \cong \mathbb{Z}/8\mathbb{Z}$, the map $4a$ has order 2. By Lemma 5.2 the map $\Omega^4_0 Sp(2) \to \text{Map}^*_0(\mathbb{C}P^2, B\text{Sp}(2))$ induces an isomorphism on $\pi_7$. Thus the upper direction around (12) has order 2 when restricted to $S^7$. Therefore the homotopy commutativity of the diagram implies that $\theta \circ \Sigma \nu'$ has order 2. If $\theta$ did not represent a generator of $\pi_4(\text{Map}^*_0(\mathbb{C}P^2, B\text{Sp}(2))) \cong \mathbb{Z}$, then as we are localized at 2, we must have $\theta \simeq 2t \cdot \gamma$ for some integer $t$. But then $\theta \circ \Sigma \nu' \simeq (t\gamma) \circ 2 \circ \Sigma \nu' \simeq (t\gamma) \circ \Sigma \nu' \circ 2$, where the right homotopy holds since $\Sigma \nu'$ is a suspension. By Lemma 5.4 $\gamma \circ \Sigma \nu' \circ 2$ is null homotopic, implying that $\theta \circ \Sigma \nu'$ is null homotopic, a contradiction. Hence $\theta$ represents a generator of $\pi_4(\text{Map}^*_0(\mathbb{C}P^2, B\text{Sp}(2))) \cong \mathbb{Z}$, so $\theta \simeq \pm \gamma$.

Now restrict (12) to the $S^{10}$ summand of $S^7 \vee S^{10}$. The upper direction around the diagram is null homotopic, implying that $\theta \circ g \simeq \pm \gamma \circ g$ is null homotopic. But this contradicts Proposition 5.5. Hence the composite $Sp(2) \xrightarrow{\partial_1} \Omega^4_0 Sp(2) \xrightarrow{4} \Omega^4_0 Sp(2) \to \text{Map}^*_0(\mathbb{C}P^2, B\text{Sp}(2))$ must be nontrivial. \qed
6. Preliminary information for Theorem 1.1 (b)

Now we turn to Theorem 1.1 (b). A key ingredient is Theorem 7.5 which states that if there is a homotopy equivalence $G_k(\mathbb{C}P^2) \simeq G_\ell(\mathbb{C}P^2)$ (rather than a fibre homotopy equivalence) then $(20, k) = (20, \ell)$. The build-up to Theorem 7.5 takes most of the next two sections.

In this section we establish 2-primary properties of certain homotopy sets related to $Sp(2)$ and $G_k(\mathbb{C}P^2)$. Localize all spaces and maps at 2. Three sequences of spaces will be used. The first is the homotopy fibration sequence

$$\Omega S^7 \xrightarrow{\delta} S^3 \xrightarrow{j} Sp(2) \longrightarrow S^7.$$  

The second is the homotopy cofibration sequence

$$S^6 \xrightarrow{\nu'} S^3 \xrightarrow{i} A \xrightarrow{q} S^7 \xrightarrow{\Sigma
\nu'} S^4,$$

where $\nu'$ is the attaching map for the top cell of $A$, $i$ is the inclusion of the bottom cell and $q$ is the pinch map to the top cell. Toda’s notation is used for $\nu'$; it represents a generator of $\pi_6(S^3) \cong \mathbb{Z}/4\mathbb{Z}$.

Applying $\text{Map}^\ast(\cdot, BSp(2))$ to the homotopy cofibration sequence

$$S^3 \xrightarrow{\eta} S^2 \longrightarrow \mathbb{C}P^2 \longrightarrow S^4 \xrightarrow{\Sigma\eta} S^3.$$  

provides the third sequence of spaces, the homotopy fibration sequence

$$\Omega^2 Sp(2) \xrightarrow{(\Sigma\eta)^\ast} \Omega^3 Sp(2) \longrightarrow \text{Map}^\ast(\mathbb{C}P^2, BSp(2)) \longrightarrow \Omega Sp(2) \xrightarrow{\eta^\ast} \Omega^2 Sp(2).$$

To simplify notation, let $N = \text{Map}^\ast(\mathbb{C}P^2, BSp(2))$. Applying $[A, \ ]$ to the homotopy fibration sequence (15) gives an exact sequence

$$[A, \Omega^2 Sp(2)] \xrightarrow{(\Sigma\eta)^\ast} [A, \Omega^3 Sp(2)] \longrightarrow [A, N] \longrightarrow [A, \Omega Sp(2)] \xrightarrow{\eta^\ast} [A, \Omega^2 Sp(2)].$$

The goal of this section is to calculate $[A, N]$. The starting point is the following result of Choi, Hirato and Mimura [3].

**Lemma 6.1.** There is an isomorphism $[A, \Omega^3 Sp(2)] \cong \mathbb{Z}/8\mathbb{Z}$ and a representative of the generator is the composite $A \xrightarrow{s} Sp(2) \xrightarrow{\delta_1} \Omega_0^3 Sp(2)$. □

Given Lemma 6.1 to calculate $[A, N]$ in (16) we determine the image of $(\Sigma\eta)^\ast$ and the kernel of $\eta^\ast$. To do so we first collect some information on the homotopy groups of $S^3$, $S^7$ and $Sp(2)$. For $m \geq 3$, let $\eta_m: S^{m+1} \longrightarrow S^m$ be $\Sigma^{m-2}\eta$.

**Lemma 6.2** (Toda [23]). In the relevant dimensions the groups $\pi_i(S^3)$ are:

| $i$ | 5 | 6 | 8 | 9 | 10 |
|---|---|---|---|---|---|
| 2-component | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | 0 | 0 |
| generator | $\eta_3^2$ | $\nu'$ | $\nu'\eta_3^2$ | – | – |
Further, by [23, Lemma 5.7] the composite \( S^8 \xrightarrow{\Sigma^2 \nu'} S^5 \xrightarrow{\eta^5} S^3 \) is null homotopic.

In the relevant dimensions the groups \( \pi_i(S^7) \) are:

| \( i \) | 7 | 10 | 11 |
|-------|---|----|----|
| 2-component | \( \mathbb{Z} \) | \( \mathbb{Z}/8\mathbb{Z} \) | 0 |
| generator | \( \nu_7 \) | \( \nu_7 \) | \( \nu_7 \) |

Further, \( 2 \cdot \nu'_7 \cong \Sigma^4 \nu' \).

Lemma 6.3 (Mimura-Toda [16]). In the relevant dimensions, the groups \( \pi_i(Sp(2)) \) are:

| \( i \) | 4 | 5 | 6 | 7 | 8 | 9 | 13 |
|-------|---|---|---|---|---|---|----|
| 2-component | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) | 0 | \( \mathbb{Z} \) | 0 | 0 | \( \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) |

Further, the map \( S^5 \xrightarrow{\eta_5} S^4 \) induces an isomorphism \( \pi_4(Sp(2)) \xrightarrow{\cong} \pi_5(Sp(2)) \).

We first aim towards Proposition 6.6, which describes the map \( \eta^* \) in (16).

Lemma 6.4. The map \( S^3 \xrightarrow{i} A \) induces an isomorphism \( [A, \Omega Sp(2)] \xrightarrow{i^*} [S^3, \Omega Sp(2)] \).

Proof. The homotopy cofibration (14) induces an exact sequence of groups

\[
[S^7, \Omega Sp(2)] \xrightarrow{q^*} [A, \Omega Sp(2)] \xrightarrow{i^*} [S^3, \Omega Sp(2)] \xrightarrow{(\nu')^*} [S^6, \Omega Sp(2)].
\]

Identifying the first, third and fourth terms using Lemma 6.3 we obtain an exact sequence

\[
0 \xrightarrow{q^*} [A, \Omega Sp(2)] \xrightarrow{i^*} \mathbb{Z}/2\mathbb{Z} \xrightarrow{(\nu')^*} \mathbb{Z}.
\]

Since \( \mathbb{Z} \) is torsion-free, \( (\nu')^* \) must be the zero map. The lemma now follows.

Lemma 6.5. The map \( S^3 \xrightarrow{i} A \) induces an isomorphism \( [A, \Omega^2 Sp(2)] \xrightarrow{i^*} [S^3, \Omega^2 Sp(2)] \).

Proof. The homotopy cofibration (14) induces an exact sequence of groups

\[
[S^7, \Omega^2 Sp(2)] \xrightarrow{q^*} [A, \Omega^2 Sp(2)] \xrightarrow{i^*} [S^3, \Omega^2 Sp(2)] \xrightarrow{(\nu')^*} [S^6, \Omega^2 Sp(2)].
\]

Identifying the first, third and fourth terms using Lemma 6.3 we obtain an exact sequence

\[
0 \xrightarrow{q^*} [A, \Omega^2 Sp(2)] \xrightarrow{i^*} \mathbb{Z}/2\mathbb{Z} \xrightarrow{(\nu')^*} 0.
\]

Therefore \( i^* \) is an isomorphism.

Proposition 6.6. The map \( \Omega Sp(2) \xrightarrow{\eta^*} \Omega^2 Sp(2) \) induces an isomorphism \( [A, \Omega Sp(2)] \xrightarrow{\eta^*} [A, \Omega^2 Sp(2)] \).

Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
[A, \Omega Sp(2)] & \xrightarrow{a} & [S^3, \Omega Sp(2)] \\
\downarrow{b} & & \downarrow{d} \\
[A, \Omega^2 Sp(2)] & \xrightarrow{c} & [S^3, \Omega^2 Sp(2)].
\end{array}
\]
Here, $a$ and $c$ are induced by $i^*$ while $b$ and $c$ are induced by $\eta^*$. By Lemma 6.4 $a$ is an isomorphism; by Lemma 6.5 $c$ is an isomorphism. Observe that $d$ is the map $\pi_4(\text{Sp}(2)) \to \pi_5(\text{Sp}(2))$ induced by precomposing with $\nu^*: S^5 \to S^4$, which is an isomorphism by Lemma 6.3. Thus $d$ is an isomorphism. The commutativity of (17) therefore implies that $b$ is an isomorphism as well, proving the lemma. □

Next, we aim towards Proposition 6.14 which describes the map $(\Sigma \eta)^*$ in (16).

**Lemma 6.7.** The map $S^3 \xrightarrow{i} A$ induces an isomorphism $[A, \Omega^2 S^3] \xrightarrow{i^*} [S^3, \Omega^2 S^3] \cong \mathbb{Z}/2\mathbb{Z}$.

**Proof.** The homotopy cofibration (14) induces an exact sequence of groups

$$[S^7, \Omega^2 S^3] \xrightarrow{\nu} [A, \Omega^2 S^3] \xrightarrow{i^*} [S^3, \Omega^2 S^3] \xrightarrow{(\nu')^*} [S^6, \Omega^2 S^3].$$

Identifying the first, third and fourth terms using Lemma 6.2 we obtain an exact sequence

$$0 \xrightarrow{\nu} [A, \Omega^2 S^3] \xrightarrow{i^*} \mathbb{Z}/2\mathbb{Z} \xrightarrow{(\nu')^*} \mathbb{Z}/2\mathbb{Z}.$$

The generator of $\pi_5(S^3)$ is $\eta_2^2$, so $(\nu')^*(\eta_2^2)$ is the composite $S^8 \xrightarrow{\nu^2} S^5 \xrightarrow{\eta_2^2} S^3$, which is null homotopic by Lemma 6.2. Thus $(\nu')^*$ is the zero map. The lemma now follows. □

**Lemma 6.8.** The map $S^3 \xrightarrow{i} A$ induces multiplication by 4 in the map $\mathbb{Z} \cong [A, \Omega^4 S^7] \xrightarrow{i^*} [S^3, \Omega^4 S^7] \cong \mathbb{Z}$.

**Proof.** The homotopy cofibration (14) induces an exact sequence of groups

$$[S^7, \Omega^4 S^7] \xrightarrow{\nu} [A, \Omega^4 S^7] \xrightarrow{i^*} [S^3, \Omega^4 S^7] \xrightarrow{(\nu')^*} [S^6, \Omega^4 S^7].$$

Identifying the first, third and fourth terms using Lemma 6.2 we obtain an exact sequence

$$0 \xrightarrow{\nu} [A, \Omega^4 S^7] \xrightarrow{i^*} \mathbb{Z} \xrightarrow{(\nu')^*} \mathbb{Z}/8\mathbb{Z}.$$

Thus $[A, \Omega^4 S^7]$ is isomorphic to the kernel of $(\nu')^*$. To determine this kernel, observe that the generator of $[S^3, \Omega^4 S^7] \cong \pi_7(S^7) \cong \mathbb{Z}$ is $i_7$ while the generator of $[S^6, \Omega^4 S^7] \cong \mathbb{Z}/8\mathbb{Z}$ is $\nu_7$. By Lemma 6.2, $2\nu_7 \simeq \Sigma^4 \nu'$ so we obtain $(\Sigma^4 \nu')^*(i_7) = i_7 \circ \Sigma^4 \nu' \simeq \Sigma^4 \nu' \simeq 2\nu_7$. Therefore the kernel of $(\nu')^*$ is isomorphic to $\mathbb{Z}$ and is generated by $4i_7$. □

**Lemma 6.9.** The map $S^3 \xrightarrow{j} \text{Sp}(2)$ induces an isomorphism $[A, \Omega^2 S^3] \xrightarrow{j^*} [A, \Omega^2 \text{Sp}(2)]$.

**Proof.** The maps $S^3 \xrightarrow{i} A$ and $S^3 \xrightarrow{j} \text{Sp}(2)$ induce a commutative diagram

$$
\begin{array}{ccc}
[A, \Omega^2 S^3] & \xrightarrow{a} & [A, \Omega^2 \text{Sp}(2)] \\
\downarrow b & & \downarrow d \\
[S^3, \Omega^2 S^3] & \xrightarrow{c} & [S^3, \Omega^2 \text{Sp}(2)].
\end{array}
$$

By Lemma 6.7, the map $b$ is an isomorphism. By (16), $j$ induces an isomorphism $\pi_5(S^3) \to \pi_5(\text{Sp}(2))$, so the map $c$ is an isomorphism. By Lemma 6.3, $d$ is an isomorphism. Hence in (16) the maps $b$, $c$ and $d$ are isomorphisms, implying that $a$ is as well. □
Lemma 6.10. The fibration connecting map $\Omega S^7 \xrightarrow{\delta} S^3$ induces the zero map on the homotopy sets $[A, \Omega^4 S^7] \xrightarrow{\delta_*} [A, \Omega^3 S^3]$.

Proof. By Lemma 6.8 $[A, \Omega^4 S^7] \cong \mathbb{Z}$ and $\mathbb{Z} \cong [A, \Omega^4 S^7] \xrightarrow{i^*} [S^3, \Omega^4 S^7] \cong \mathbb{Z}$ is multiplication by 4. Therefore, if $\epsilon$ is a generator of $[A, \Omega^4 S^7]$, then there is a homotopy commutative diagram

$$\begin{array}{ccc}
S^3 & \xrightarrow{E^4} & \Omega^4 S^7 \\
\downarrow & & \downarrow \\
A & \xrightarrow{i} & \epsilon
\end{array}$$

where $E^4$ is the four-fold suspension. Consider $\delta_*(\epsilon)$, which is the composite $A \xrightarrow{\epsilon} \Omega^4 S^7 \xrightarrow{\Omega^3 \delta} \Omega^3 S^3$. By (19), $\Omega^3 \delta \circ \epsilon \circ i$ is homotopic to the composite $S^3 \xrightarrow{E^4} \Omega^4 S^7 \xrightarrow{i} \Omega^4 S^7 \xrightarrow{\Omega^3 \delta} \Omega^3 S^3$. But the latter composite is null homotopic since $\pi_6(S^3) \cong \mathbb{Z}/4\mathbb{Z}$. Thus $\Omega^3 \delta \circ \epsilon$ extends through the cofibre of $i$ to a map $S^7 \xrightarrow{j} \Omega^3 S^3$. But by Lemma 6.2 $\pi_{10}(S^3) = 0$. Thus $\Omega^3 \delta \circ \epsilon$ is null homotopic. That is, $\delta_*(\epsilon) = 0$. As $\epsilon$ generates $[A, \Omega^4 S^7]$, this implies that $\delta_*$ is the zero map. \(\square\)

Corollary 6.11. The map $S^3 \xrightarrow{j} Sp(2)$ induces a monomorphism $[A, \Omega^3 S^3] \xrightarrow{j_*} [A, \Omega^3 Sp(2)]$.

Proof. The homotopy fibration (13) induces an exact sequence $[A, \Omega^4 S^7] \xrightarrow{\delta_*} [A, \Omega^3 S^3] \xrightarrow{j_*} [A, \Omega^3 Sp(2)]$. By Lemma 6.10 $\delta_*$ is the zero map. Hence $j_*$ is a monomorphism. \(\square\)

Lemma 6.12. The map $\Omega^2 S^3 \xrightarrow{(\Sigma \eta)^*} \Omega^3 S^3$ induces an injection $[A, \Omega^2 S^3] \xrightarrow{(\Sigma \eta)^*} [A, \Omega^3 S^3]$.

Proof. By Lemma (13) $[A, \Omega^2 S^3] \cong \mathbb{Z}/2\mathbb{Z}$ and there is an isomorphism $[A, \Omega^2 S^3] \xrightarrow{j_*} [S^3, \Omega^2 S^3]$. Therefore, if $\epsilon$ is a generator of $[A, \Omega^2 S^3]$, then there is a homotopy commutative diagram

$$\begin{array}{ccc}
S^3 & \xrightarrow{\pi_3^*} & \Omega^2 S^3 \\
\downarrow & & \downarrow \\
A & \xrightarrow{i} & \epsilon
\end{array}$$

where $\pi_3^*$ is the adjoint of $\eta_3^*$. Consider the composite $A \xrightarrow{\epsilon} \Omega^2 S^3 \xrightarrow{(\Sigma \eta)^*} \Omega^3 S^3$. By (20), $(\Sigma \eta)^* \circ \epsilon \circ i$ is homotopic to the composite $S^3 \xrightarrow{\pi_3^*} \Omega^2 S^3 \xrightarrow{(\Sigma \eta)^*} \Omega^3 S^3$, which in turn is homotopic to the adjoint of $\eta_3^*$. Therefore, as $\eta_3^*$ is nontrivial so is $(\Sigma \eta)^* \circ \epsilon$. Hence $[A, \Omega^2 S^3] \xrightarrow{j_*} [A, \Omega^3 S^3]$ is an injection because it is nonzero on the generator of $[A, \Omega^2 S^3] \cong \mathbb{Z}/2\mathbb{Z}$. \(\square\)

Lemma 6.13. The map $\Omega^2 S^3 \xrightarrow{(\Sigma \eta)^*} \Omega^3 S^3$ induces an injection $[A, \Omega^2 Sp(2)] \xrightarrow{(\Sigma \eta)^*} [A, \Omega^3 Sp(2)]$.

Proof. The map $\Omega^2 X \xrightarrow{(\Sigma \eta)^*} \Omega^3 X$ is natural with respect to maps $X \xrightarrow{} Y$, so applying this to the case $BS^3 \xrightarrow{H} BSp(2)$ and taking homotopy sets $[A, \ ]$ gives a commutative diagram

$$\begin{array}{ccc}
[A, \Omega^2 S^3] & \xrightarrow{a} & [A, \Omega^3 S^3] \\
\downarrow b & & \downarrow d \\
[A, \Omega^2 Sp(2)] & \xrightarrow{c} & [A, \Omega^3 Sp(2)]
\end{array}$$

(21)
Here, $a$ and $c$ are induced by $(\Sigma \eta)^*$ while $b$ and $c$ are induced by $j_*$. By Lemma 6.9, $b$ is an isomorphism and, by Corollary 6.11, $d$ is an injection. By Lemma 6.12, $a$ is an injection. Therefore the commutativity of (21) implies that $c$ is also an injection.

Now we identify the injection in Lemma 6.13.

**Proposition 6.14.** The injection $[A, \Omega^2 Sp(2)] \xrightarrow{(\Sigma \eta)^*} [A, \Omega^3 Sp(2)]$ in Lemma 6.13 is the injection $\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/8\mathbb{Z}$.

**Proof.** By Lemmas 6.7 and 6.9, $[A, \Omega^2 Sp(2)] \cong \mathbb{Z}/2\mathbb{Z}$, and by Lemma 6.1, $[A, \Omega^3 Sp(2)] \cong \mathbb{Z}/8\mathbb{Z}$. Since $[A, \Omega^2 Sp(2)] \longrightarrow [A, \Omega^3 Sp(2)]$ is an injection, the lemma follows. \qed

Finally, we calculate $[A, N]$.

**Proposition 6.15.** There is an isomorphism of sets $[A, N] \cong \mathbb{Z}/4\mathbb{Z}$ and the map $\Omega^3_0 Sp(2) \longrightarrow N$ induces an epimorphism $\mathbb{Z}/8\mathbb{Z} \cong [A, \Omega^3 Sp(2)] \longrightarrow [A, N] \cong \mathbb{Z}/4\mathbb{Z}$.

**Proof.** By Propositions 6.6 and 6.14 the exact sequence in (16) simplifies to an exact sequence
\[
0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/8\mathbb{Z} \cong [A, \Omega^3 Sp(2)] \longrightarrow [A, N] \longrightarrow 0.
\]
The proposition follows immediately. \qed

7. The proof of Theorem 1.1 (b)

In this section we prove Theorem 1.1 (b). A key ingredient is Theorem 7.5, which To simplify notation, let $N_k = \text{Map}_k^*(\mathbb{C}P^2, BSp(2))$. Applying $[A, -]$ to (2) in the $G = Sp(2)$ case we obtain a commutative diagram
\[
\begin{array}{c}
[A, Sp(2)] \xrightarrow{(\partial_k)_*} [A, \Omega^3_0 Sp(2)] \xrightarrow{\partial_1} [A, B\mathcal{G}_k(S^4)] \xrightarrow{\partial_2} [A, BSp(2)] \\
\end{array}
\]
Since $A$ is connected, there is an isomorphism $[A, N_0] \cong [A, N]$, where $N = \text{Map}_*^*(\mathbb{C}P^2, BSp(2))$ and $[A, N] \cong \mathbb{Z}/4\mathbb{Z}$ as well.

**Lemma 7.1.** The image of $(\partial_k)_*$ is $\mathbb{Z}/(4/(4, k))\mathbb{Z}$.

**Proof.** By Lemma 6.1, $[A, \Omega^3_0 Sp(2)] \cong \mathbb{Z}/8\mathbb{Z}$ and the composite $A \xrightarrow{s} Sp(2) \xrightarrow{\partial_1} \Omega^3_0 Sp(2)$ represents a generator. By Lemma 6.1, $\partial_k \cong k \cdot \partial_1$, so the image of $(\partial_k)^*$ in (22) is $\mathbb{Z}/(8/(8, k))\mathbb{Z}$. By Proposition 6.15, the map $[A, \Omega^3_0 Sp(2)] \longrightarrow [A, N_k]$ is reduction mod 4. Thus the commutativity of the left square in (22) implies that the image of $(\partial_k)^*$ is $\mathbb{Z}/(4/(4, k))\mathbb{Z}$. \qed

**Lemma 7.2.** There is an isomorphism $[A, BSp(2)] \cong 0$. 

Proof. The dimension of $A$ is 7 and the map $BSp(2) \to BSp(\infty)$ induced by the standard inclusion of $Sp(2)$ into $Sp(\infty)$ is 10-connected. Therefore $[A, BSp(2)] \cong [A, BSp(\infty)]$. But $[A, BSp(\infty)]$ is $\tilde{K}_Sp(A)$, the reduced symplectic $K$-theory of $A$. Since $\tilde{K}_Sp(S^{4m-1}) = 0$ for every $m \geq 1$, applying $\tilde{K}_Sp$ to the homotopy cofibration $S^3 \to A \to S^7$ shows that $\tilde{K}_Sp(A) \cong 0$. □

**Proposition 7.3.** There is an isomorphism of sets $[A, BG_k(CP^2)] \cong \mathbb{Z}/(4, k) \mathbb{Z}$. 

**Proof.** By Lemma 7.2 the exact sequence in the bottom row of (22) is 

$$[A, S^0Sp(2)] \xrightarrow{\partial} [A, \Omega^2_0Sp(2)] \to [A, BG_k(CP^2)] \to 0.$$ 

Thus $[A, BG_k(CP^2)]$ is the cokernel of $(\partial_k)^*$. By Lemma 7.1 this cokernel is $\mathbb{Z}/(4, k) \mathbb{Z}$. □

What we would like to say is that a homotopy equivalence $G_k(CP^2) \simeq G_{\ell}(CZ^P)$ implies that there is a bijection of sets $[A, BG_k(CP^2)] \cong [A, BG_{\ell}(CP^2)]$. This would imply that $(4, k) = (4, \ell)$ and prove Theorem 1.1 (b). However, since $A$ is not a suspension, and possibly not even a $co$-$H$-space, the desired implication is not immediate through adjunction. Instead, a different argument is needed.

**Lemma 7.4.** For any $k \in \mathbb{Z}$ we have $\pi_3(BG_k(CP^2)) \cong 0$.

**Proof.** There is a homotopy fibration

$$(23) \quad \text{Map}^*_k(CP^2, BSp(2)) \to BG_k(CP^2) \to BSp(2)$$

and the homotopy cofibration sequence $S^3 \xrightarrow{q} S^2 \to CP^2 \to S^4$ induces a homotopy fibration sequence

$$(24) \quad \Omega^3_0Sp(2) \to \text{Map}^*_k(CP^2, BSp(2)) \to \Omega Sp(2) \xrightarrow{\pi_3} \Omega^2 Sp(2).$$

Apply $\pi_3$ to (24). By Lemma 6.3 $\pi_3(\Omega^3_0Sp(2)) \cong \pi_6(Sp(2)) \cong 0$, and $\pi_3(\eta^*)$ is the same as the map $\pi_4(Sp(2)) \xrightarrow{(\Sigma \eta^*)} \pi_5(Sp(2))$, which is an isomorphism. Hence $\pi_3(\text{Map}^*_k(CP^2, BSp(2))) \cong 0$. Now apply $\pi_3$ to (23). By connectivity, $\pi_3(BSp(2)) \cong 0$ so as $\pi_3(\text{Map}^*_k(CP^2, BSp(2))) \cong 0$ we obtain $\pi_3(BG_k(CP^2)) \cong 0$. □

Consider the homotopy cofibration sequence $S^3 \to A \xrightarrow{q} S^7 \xrightarrow{\Sigma \nu'} S^4$. Applying $[\ , BG_k(CP^2)]$ gives an exact sequence of pointed sets

$$(25) \quad [S^4, BG_k(CP^2)] \xrightarrow{(\Sigma \nu')^*} [S^7, BG_k(CP^2)] \xrightarrow{q^*} [A, BG_k(CP^2)] \to [S^3, BG_k(CP^2)].$$

By Lemma 7.4 $\pi_3(BG_k(CP^2)) \cong 0$, so we really have an exact sequence of pointed sets

Note that $[S^4, BG_k(CP^2)]$ and $[S^7, BG_k(CP^2)]$ are groups and $(\Sigma \nu')^*$ is a group homomorphism, but $[A, BG_k(CP^2)]$ is only a set since $A$ may not be a co-$H$-space, and $q^*$ is therefore only a map of
sets. So exactness and the fact that \( q^* \) is an epimorphism implies that there is a bijection between the set \([A, B\mathcal{G}_k(\mathbb{C}P^2)]\) and the group coker \((\Sigma \nu')^*\).

Now we prove Theorems 7.5 and 1.1 (b), which are stated integrally.

**Theorem 7.5.** If \( \mathcal{G}_k(\mathbb{C}P^2) \simeq \mathcal{G}_\ell(\mathbb{C}P^2) \) then \((20, k) = (20, \ell)\).

*Proof.* By Theorems 2.2 and 2.3 (a) it suffices to prove the 2-components of the g.c.d. conditions. That is, it suffices to prove that if there is a 2-local homotopy equivalence \( \mathcal{G}_k(\mathbb{C}P^2) \simeq \mathcal{G}_\ell(\mathbb{C}P^2) \) then \((4, k) = (4, \ell)\).

Let \( e : \mathcal{G}_k(\mathbb{C}P^2) \to \mathcal{G}_\ell(\mathbb{C}P^2) \) be a homotopy equivalence. Then for any path-connected space \( X \) there is an isomorphism \( [\Sigma X, B\mathcal{G}_k(\mathbb{C}P^2)] \cong [X, \mathcal{G}_k(\mathbb{C}P^2)] \) which is natural for maps \( \Sigma X \to \Sigma Y \). Applying this to (20), together with the map \( e \), gives a commutative diagram

\[
\begin{array}{c}
[S^4, B\mathcal{G}_k(\mathbb{C}P^2)] \\
\cong
\end{array}
\begin{array}{c}
[S^7, B\mathcal{G}_k(\mathbb{C}P^2)] \\
\cong
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
[A, B\mathcal{G}_k(\mathbb{C}P^2)] \\
\rightarrow
\end{array}
\begin{array}{c}
0
\end{array}
\]

(26)

where the top and bottom row are exact and the vertical isomorphisms are induced by adjunction and applying \( e_* \). In the top row, \([A, B\mathcal{G}_k(\mathbb{C}P^2)]\) bijects with coker \((\Sigma \nu')^*\), and in the bottom row, \([A, B\mathcal{G}_\ell(\mathbb{C}P^2)]\) bijects with coker \((\Sigma \nu')^*\). Therefore there is a bijection between \([A, B\mathcal{G}_k(\mathbb{C}P^2)]\) and \([A, B\mathcal{G}_\ell(\mathbb{C}P^2)]\).

By Proposition 7.3 \([A, B\mathcal{G}_k(\mathbb{C}P^2)] \cong \mathbb{Z}/(4, k)\mathbb{Z}\) and similarly \([A, B\mathcal{G}_\ell(\mathbb{C}P^2)] \cong \mathbb{Z}/(4, \ell)\mathbb{Z}\). Therefore the bijection between \([A, B\mathcal{G}_k(\mathbb{C}P^2)]\) and \([A, B\mathcal{G}_\ell(\mathbb{C}P^2)]\) implies that \((4, k) = (4, \ell)\). \(\square\)

*Proof of Theorem 1.1 (b).* By Theorem 2.1 it suffices to prove the statement for the special cases \( M = S^4 \) and \( M = \mathbb{C}P^2 \). The \( M = S^4 \) case is the statement of Theorem 2.3. Thus we are reduced to proving that if \( \mathcal{G}_k(\mathbb{C}P^2) \) is fibre homotopy equivalent to \( \mathcal{G}_\ell(\mathbb{C}P^2) \) then \((40, k) = (40, \ell)\).

Let \( u_k \) and \( v_k \) be the 2-component of the integers \((20, k)\) and \((40, k)\) respectively. Observe that \( u_k \) can be 0, 1, 2 or 4. If \( u_k < 4 \), then \( v_k = u_k \) and so \((40, k) = (20, k)\); otherwise \( v_k \) is either 4 or 8.

Since \( \mathcal{G}_k(\mathbb{C}P^2) \) is fibre homotopy equivalent to \( \mathcal{G}_\ell(\mathbb{C}P^2) \) the two are also homotopy equivalent. Theorem 7.5 therefore implies that \((20, k) = (20, \ell)\). In particular, this implies that \( u_k = u_\ell \). If \( u_k < 4 \) then \((20, k) = (40, k)\) and similarly \((20, \ell) = (40, \ell)\). Therefore \((40, k) = (40, \ell)\), as asserted. If \( u_k = 4 \) then \( v_k = v_\ell \) and we obtain \((40, k) = (40, \ell)\), as asserted. It remains, then, to deal with the case when \( v_k = 8 \) and \( u_\ell = 4 \) or vice-versa.

Assume that \( v_k = 8 \) and \( v_\ell = 4 \) (the other case is similar). We will show that \( \mathcal{G}_k(\mathbb{C}P^2) \) cannot be fiber homotopy equivalent to \( \mathcal{G}_\ell(\mathbb{C}P^2) \). Note that \( k = 8K \) and \( \ell = 4L \) where \( K \) is some number and \( L \) is an odd number. Write \( L = 2L' + 1 \), giving \( \ell = 4L = 8L' + 4 \). The definition of \( \mathcal{G}_k \) as \( \pi^* \circ \partial_k \)
and Lemma 3.1 imply that
\[ \overline{\partial}_k = \pi^* \circ \partial_k \simeq \pi^* \circ 8K \circ \partial_1 \quad \overline{\partial}_\ell = \pi^* \circ \partial_\ell \simeq \pi^* \circ (8L' + 4) \circ \partial_1. \]

Localize at 2. By [20] the map \( \partial_1 \) has order 8. Therefore \( \overline{\partial}_k \) is null homotopic and \( \overline{\partial}_\ell \simeq \pi^* \circ 4 \partial_1 \simeq \pi^* \circ \partial_4 \). Let \( h: G_k(CP^2) \to G_\ell(CP^2) \) be a fibre homotopy equivalence. Then there is a homotopy commutative diagram
\[
\begin{array}{ccc}
G_k(CP^2) & \xrightarrow{r_k} & Sp(2) & \xrightarrow{\overline{\eta}_k} & \text{Map}^*(\mathbb{C}P^2, BSp(2)) \\
\downarrow h & & \downarrow & & \downarrow \\
G_\ell(CP^2) & \xrightarrow{r_\ell} & Sp(2) & \xrightarrow{\overline{\eta}_\ell} & \text{Map}^*(\mathbb{C}P^2, BSp(2))
\end{array}
\]
where both rows are fibrations. Since \( \overline{\eta}_k \) is null homotopic, the map \( r_k \) has a right homotopy inverse \( s: Sp(2) \to G_k(CP^2) \). The homotopy commutativity of the square therefore implies that \( r_\ell \circ h \circ s \) is a right homotopy inverse for \( r_\ell \). This implies that \( \overline{\partial}_\ell \) is null homotopic. However, \( \overline{\partial}_\ell \simeq \overline{\partial}_4 \) and Theorem 5.7 implies that \( \overline{\partial}_4 \) is nontrivial, a contradiction. Therefore the case \( v_k = 8 \) and \( v_\ell = 4 \) cannot occur.

Hence in all cases, the fibre homotopy equivalence between \( G_k(CP^2) \) and \( G_\ell(CP^2) \) implies that \( (40, k) = (40, \ell) \).

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