INTERSECTION FORMS AND THE GEOMETRY OF LATTICE CHERN-SIMONS THEORY

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Abstract

We show that it is possible to formulate Abelian Chern-Simons theory on a lattice as a topological field theory. We discuss the relationship between gauge invariance of the Chern-Simons lattice action and the topological interpretation of the canonical structure. We show that these theories are exactly solvable and have the same degrees of freedom as the analogous continuum theories.

In the continuum, Chern-Simons theory [1,2] is a topological field theory [3,4]. It is exactly solvable and is related to interesting topological structures such as the Witten invariants of three-manifolds and the Jones polynomial and other knot invariants for links embedded in three-manifolds. Its canonical formalism also has an interesting relationship with conformal field theory in two dimensions. As a model for physical phenomena its U(1)

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(and sometimes U(N)) versions are proposed as the infrared limit of phenomenological theories of the fractionally quantized Hall effect[5] and other theories of anyons, including speculations about the mechanism for high $T_c$ superconductivity[6].

In recent years there have been efforts to write Chern-Simons theory on a lattice[7-11]. This gives an automatic ultraviolet regularization of the field theory and is important for condensed matter and statistical physics applications. It also makes well-defined the correspondence between matter-coupled Chern-Simons theory and a theory of anyons [11].

Generally, it is difficult to formulate lattice Chern-Simons theory in a natural way as one lacks geometrical principles which would replace the general covariance of a topological field theory in the continuum.

Nevertheless, we shall show in this Letter that when lattice Chern-Simons theory is formulated so that it is both local and gauge invariant on the lattice (here the term “local” means that the canonical structure is such that a link commutes nontrivially only with links with whom it shares a common site), we obtain an exactly solvable model which shares many of the features of its continuum kin and can be regarded as a topological field theory on the lattice.

To begin, we review some of the features of continuum U(1) Chern-Simons theory on the space $\mathcal{M}^3 = R^1 \otimes \Sigma^g$, where $\Sigma^g$ is a compact oriented two-dimensional Riemann surface with genus $g$. The action is

$$S = \int_{\mathcal{M}^3} d^3x \left( \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + A_\mu j^{\mu} \right) \quad (1)$$

The vector density $j^{\mu}(x) = \int d\tau \sum_i \frac{d}{d\tau} r_i^{\mu}(\tau) \delta^3(x - r_i(\tau))$ corresponds to a collection of Wilson loops parametrized by $r_i^{\mu}(\tau)$ and $\partial_\mu j^{\mu} = 0$. As is usual in gauge theory, the action
is linear in the temporal component of the gauge field and the field equation resulting from its variation enforces the constraint

\[ B(x) + \frac{2\pi}{k} j^0(x) = 0 \]  

where \( B = dA \) (restricted to \( \Sigma^g \)) is the magnetic flux. It is useful to think of the remaining first order action, \( \int dt \int_{\Sigma^g} \left( -\frac{k}{2\pi} A \wedge \dot{A} - A \cdot j \right) \) as already being cast in phase space, which is the set of gauge connection 1-forms on \( \Sigma^g \). From this we obtain the canonical commutator

\[
\left[ A_i(x), A_j(y) \right] = \frac{2\pi i}{k} \epsilon_{0ij} \delta^2(x - y)
\]  

which has the property that, if we consider two distinct oriented curves, \( C \) and \( C' \) on \( \Sigma^g \),

\[
\left[ \int_C A, \int_{C'} A \right] = \frac{2\pi i}{k} \nu[C, C']
\]  

where \( \nu[C, C'] \) is the number of right-handed minus the number of left-handed intersections of \( C \) and \( C' \). (A right-handed intersection occurs when, if we move along the positive direction of \( C \), \( C' \) crosses from right to left.) Thus, the commutator gives a representation of the intersection form for Wilson loops. For closed curves \( C \) and \( C' \), \( \nu[C, C'] \) is a topological invariant and if either \( C \) or \( C' \) is contractible, \( \nu[C, C'] \) must vanish. Therefore, there is a nontrivial commutator only for homologically nontrivial curves. If \( \alpha_j, \beta_j, j = 1, \ldots, g \), are a canonical set of closed curves on \( \Sigma^g \) generating its first homology group then \( \nu[\alpha_i, \alpha_j] = \nu[\beta_i, \beta_j] = 0, \nu[\alpha_i, \beta_j] = \delta_{ij} \). Upon imposition of the constraint (2) the integrals of \( A \) over the 1-cycles \( \alpha_i, \beta_i \) are the only remaining degrees of freedom and, modulo the remaining symmetry under large gauge transformations, they form the reduced phase space\[^2\].

It is the property (4), that the symplectic structure gives the intersection form for loops on \( \Sigma^g \), which we shall discover on the lattice, as a consequence of lattice gauge
invariance and locality. We work in continuum time and a square spatial lattice with spacing 1. We begin by fixing some notation. The forward and backward shift operators are

\[ S^i f(x) = f(x + \hat{i}) , \quad S_i^{-1} f(x) = f(x - \hat{i}) , \]

respectively, and forward and backward difference operators are

\[ d^i f(x) = f(x + \hat{i}) - f(x) , \quad d_i = S_i - 1 , \quad \hat{d}_i f(x) = f(x) - f(x - \hat{i}) , \quad \hat{d}_i = 1 - S_i^{-1} = S_i^{-1} d_i . \]

Summation by parts on a lattice takes the form (neglecting surface terms)

\[ \sum_x f(x) d^i g(x) = - \sum_x \hat{d}_i f(x) g(x) \]

by virtue of the lattice Leibniz rule

\[ d^i (fg) = f d^i g + d^i f S_i g = f d_i g + S_i (\hat{d}_i fg) \]

(no sum on \( i \)).

The components \( A_i(x) \) of the gauge field are real-valued functions on the links specified by the pair \([x, \hat{i}]\), \( A_0 \) is a function on lattice sites, the magnetic field \( B(x) = d_1 A_2(x) - d_2 A_1(x) \) is a function on plaquettes where \( x \) labels the plaquette with corners \( x, x + \hat{1}, x + \hat{1} + \hat{2}, x + \hat{2} \), and the electric field \( F_0 = \dot{A}_i - d_i A_0 \) is a function on links.

A gauge invariant, local, nondegenerate Chern-Simons term was found in ref. [11] :

\[ S = \int dt \sum_x \left( \frac{k}{2\pi} A_0(x, t) e^{ix_i d_i A_j(x, t)} - \frac{k}{4\pi} A_i(x, t) K_{ij} \dot{A}_j(x, t) + A_\mu(x, t) j^\mu(x, t) \right) \]

(5)

where \( i, j = 1, 2 \), with \( j_\mu \) a conserved current, \( \partial_0 j_0 - \hat{d}_i j_i = 0 \) and

\[ K_{ij} = -\frac{1}{2} \left( \begin{array}{rr} S_2 - S_2^{-1} & -(1 + S_2^{-1} + S_1 + S_2^{-1} S_1) \\ -1 + S_1^{-1} + S_2 + S_1^{-1} S_2 & S_1^{-1} - S_1 \end{array} \right) \]

\[ = -\frac{1}{2} \left( \begin{array}{rr} d_2 + \hat{d}_2 & -2 - 2d_1 + 2\hat{d}_2 + \hat{d}_2 d_1 \\ 2 + 2d_2 - 2\hat{d}_1 - \hat{d}_1 d_2 & -d_1 - \hat{d}_1 \end{array} \right) \]

(6)

Analogous to (3), \( K_{ij}^{-1} \) determines the symplectic structure on the phase space which is the space of functions \( A_i(x) \) from the links of the lattice to \( R^1 \),

\[ [A_i(x), A_j(y)] = -\frac{2\pi i}{k} K_{ij}^{-1}(x - y) \]

(7)

\[ \dagger \] The divergence of a vector field is correctly transcribed onto the lattice through backward differencing.
Also, the equation of motion for $A_0$ gives the gauge constraint

$$
\epsilon^{ij} d_i A_j(x) + \frac{2\pi}{k} f^0(x) = 0
$$

(8)

This associates the magnetic flux in the plaquette with corners $x, x + \hat{1}, x + \hat{1} + \hat{2}, x + \hat{2}$ with the charge at $x$. It was shown in [11] that (7) gives Wilson line-sums the lattice analog of the algebra (4),

$$
\left[ \sum_{[x,i] \in C} dx_i A_i(x), \sum_{[y,j] \in C'} dy_j A_j(y) \right] = \frac{2\pi i}{k} \nu(C, C')
$$

(9)

where the sums are over lattice curves (connected assemblies of oriented links) and $dx_i = \pm 1$ according to whether the link $[x,i]$ is backward or forward directed.

We may work backwards from (9), and begin by considering it a primitive requirement that $K^{-1}$ is anti-Hermitean, nondegenerate and gives the symplectic structure in (9). In particular, if $C$ in (9) is a homologically trivial closed loop and $C'$ is an open curve, then $\nu[C, C']$ is an integer, the number of times $C$ links each endpoint of $C'$, with sign determined by the orientations of $C$ and $C'$. This means that $K^{-1}$ must obey

$$
\hat{d}_i = d_j \epsilon_{jk} K^{-1}_{ki}, d_i = -K^{-1}_{ij} \epsilon_{jk} \hat{d}_k
$$

(10)

This turns out to be identical to the condition on $K$ that one obtains by requiring that the action (5) is invariant under the gauge transform $A_i \rightarrow A_i + d_i \chi$, $A_0 \rightarrow A_0 + \dot{\chi}$, or equivalently that $\frac{k}{2\pi} B(x)$ is the generator of static gauge transforms,

$$
\frac{k}{2\pi} \left[ \sum_x \chi(x) B(x), A_i(y) \right] = -i d_i \chi(y)
$$

(11)

Thus, the local form of the gauge constraint in (8) as well as gauge invariance of the action are equivalent conditions to the topological invariance and integer-valuedness of $\nu[C, C']$. 
in (9) when either \( C \) or \( C' \) is a closed contour which is the boundary of a set of plaquettes. Pictorially, gauge invariance, together with the locality of the symplectic structure \( K^{-1} \) guarantee that curves which touch but do not penetrate are counted as zero intersection:

Conversely, these conditions, together with the requirement that \( K^{-1} \) allows interactions of links with only those other links that share a common site, and that it have a local inverse, may be summarized by precisely the condition (10) (which we found by requiring gauge invariance), and makes \( K^{-1} \) the kernel that counts intersections of two lattice curves. The remaining structure of \( K^{-1} \) is the normalization, and is fixed by requiring that (9) should hold for the simple crossings.
This, together with the Gauss Law\textsuperscript{†} determines $K^{-1}$ and $K$ uniquely as that given in equation (6). Several results follow for partial intersection numbers:

These would be ambiguous in the continuum and on the lattice they follow uniquely from our other requirements on $K$.

Note that the Kernel in the action, $S_{CS} = \int dt \sum_x A_\mu(x) D_{\mu\nu}(x-y) A_\nu(y)$ here can be parametrized as $D_{\mu\nu} = T_{\mu\lambda} \epsilon_{\lambda\sigma\nu} d_\sigma$ and that the requirement of gauge invariance is precisely $dT = d$ (where $d_0 = \hat{d}_0 = \partial_0$), the analog of (10). When we require that $T$ is a local matrix, with a local inverse, (the determinant is a monomial in shifts) this $T^{-1}$-matrix is the kernel which counts the intersections of surfaces with curves on a 3-dimensional lattice. This, in turn implies that (as we shall see in the following) the effective action for Chern-Simons theory coupled to currents is related to the linking numbers of trajectories.

To solve the model (3), we first consider a lattice with trivial first homology group,

\textsuperscript{†} In fact, there are exactly four such $K$’s, corresponding to the four possible ways of assigning a plaquette to one of its corners. Gauss’ Law in these cases would appear as e.g. $B + \frac{2\pi}{k} S_1 j_0 = 0$. 

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such as the latticized open plane $R^2$. We represent the commutator (5) by choosing the functional variables

$$B(x) = \frac{2\pi i}{k} \frac{\partial}{\partial \lambda(x)} , \quad \lambda(x) = \frac{1}{d \cdot \hat{d}} \left( \hat{d} \cdot A - \frac{\hat{d}K^{-1}d}{2d \cdot \hat{d}} B(x) \right)$$

(12)

Then, a wave-function which solves the constraint (8) is

$$\Psi_{\text{phys}}[\lambda, j_\mu, t] = \exp \left( i \sum_x \lambda(x) j_0(x, t) \right) \tilde{\Psi}[j_\mu, t]$$

(13)

The Hamiltonian is

$$H = \sum_x j_i(x) A_i(x) = \sum_x \left( j \times \hat{d} \frac{1}{d \cdot \hat{d}} B(x) + j \cdot d \frac{1}{d \cdot \hat{d}} \hat{d} \cdot A \right)$$

(14)

where we have used the identity $\delta_{ij} = \epsilon_{ikl} \frac{1}{d \cdot \hat{d}} \epsilon_{jil} + d_i \frac{1}{d \cdot \hat{d}} \hat{d}_j$. The Schrödinger equation

$$i \frac{\partial}{\partial t} \Psi_{\text{phys}}[\lambda, j_\mu, t] = H \Psi_{\text{phys}}[\lambda, j_\mu, t]$$

(15)

is solved by

$$\Psi_{\text{phys}} = \exp \left( i \sum_x \lambda(x) j_0(x, t) - \frac{2\pi i}{k} \int_{-\infty}^{t} \sum \left( j \times \hat{d} \frac{1}{d \cdot \hat{d}} j^0 + j \cdot d \frac{1}{d \cdot \hat{d}} \hat{d} \cdot j^0 \right) \right)$$

(16)

The trajectory of a particle is a piecewise linear lattice curve consisting of instantaneous hoppings in spatial directions between lattice sites and temporal segments representing the particle at rest on a particular site. The second term in the phase of the wave-function in (16) is a topological invariant and can be interpreted as the linking number of periodic lattice trajectories. To see this, first consider adding a closed spacelike curve to $j^\mu$, the perimeter of a plaquette. This is described by the change in current

$$\delta j^0(x, t) = 0 \quad , \quad \delta j^1(x, t) = -\hat{d}_2 \delta(x - a) \delta(t) \quad , \quad \delta j^2(x, t) = \hat{d}_1 \delta(x - a) \delta(t)$$
which has the property that $\hat{d} \times \delta j(x,t) = d \cdot \hat{d} \delta(x - a) \delta(t)$, implying that the phase of the wavefunction changes by $\frac{2\pi}{k} \theta(t) j^0(a,0)$, which is $\frac{2\pi}{k}$ times the linking number of the lattice curves with the plaquette. If we add a time-like plaquette, for example

$$\delta j^0(x,t) = \theta(t - t_1) \theta(t_2 - t) \hat{d}_1 \delta(x - a), \quad \delta j^1(x,t) = (\delta(t - t_1) - \delta(t - t_2)) \delta(x - a)$$

The phase changes by

$$\frac{2\pi}{k} \int_{t_1}^{t_2} dt j^2(a,t)$$

which is also $2\pi/k$ times an integer, the number of times the lattice curves link the time-like plaquette. To get the latter result we have neglected the ‘self-linking’ of the space-like plaquette with itself. Such self-linking numbers are not well-defined here but require further regularization. (Here, self-linking numbers involve ill-defined products of the form $\theta(t) \delta(t)$.)

In general, a lattice current due to a charged particle which hops from lattice site $x_{p-1}$ to $x_p$ along links $C_p$ at time $t_p$ has the form

$$j^0(x,t) = \sum_p \delta(x - x_p) \theta(t_{p+1} - t) \theta(t - t_p)$$

$$j^i(x,t) = \sum_p \sum_{[y_p, i_p] = \text{links in } C_p} \delta(x - y_p) dy_i \delta(t - t_p) \tag{17}$$

A general conserved current may be written as a sum of currents of the form (17) $j_\mu = \sum_\alpha j_\mu^\alpha$. The quadratic self-interaction terms for currents in (16) are of necessity ambiguous, because they are in fact the self-linking number of the trajectories – the latter is a topological invariant of framed links, and the ambiguity can only be lifted by introducing a framing.
The cross terms of (16) between two trajectories \( \alpha, \beta \) are also ambiguous unless the instantaneous hoppings in spatial directions of the two lattice curves occur at distinct times. When this condition is met, the cross term is well defined. The \( \alpha, \beta \) cross term may be combined with the \( \beta, \alpha \) cross term, and calculated in terms of a lattice angle function. Lattice angle functions have been discussed in earlier works [8,11], and are usually defined as a contour sum as 
\[
\theta_C(x, y) = 2\pi \sum_C d\ell \frac{d}{d\ell} \delta(\ell - y)
\]
and are multivalued functions of position in that they depend on windings of the contour \( C \) around the point \( y \).

\[
d \times d\theta_C(x, y) = 2\pi \delta(x - y) \tag{18a}
\]
\[
\theta_C(x, y) - \theta_C(x, y) = 2\pi \sum_z \omega(CC'^{-1}, z) \tag{18b}
\]
where \( \omega(CC'^{-1}, z) \) is the winding number of the closed curve \( CC'^{-1} \) around the point \( z \).

Unlike the continuum angle function, which satisifies the further identity
\[
\theta_C(x, y) - \theta_C(y, x) = \pi + 2\pi \nu[C, C'] \tag{19a}
\]
this angle-function has the property that
\[
\theta_C(x, y) - \theta_C(y, x) = \pi + 2\pi \nu[C, C'] + \xi(x - y) \tag{19b}
\]
The last term in (16) is precisely the defect function \( \xi(x - y) \) for the lattice angle function which the first term in (16) calculates. Thus the result for the cross terms \( \alpha, \beta \) and \( \beta, \alpha \) together is proportional to the total change in angle between the curves accumulated during the time evolution,
\[
\frac{1}{k} \int dt \frac{d}{dt} \tilde{\theta}(x(t) - x'(t)) , \quad x \in \alpha, \ x' \in \beta
\]
(Note that the result is independent of \( C \).) where \( \tilde{\theta} \) is an “improved” lattice angle function, in that it satisfies (19a) as well as (18ab). Thus for a periodic trajectory of \( N \) particles
on the lattice the phase of the wave-function is the linking number of trajectories and the wave-function therefore carries a 1-dimensional unitary representation of the $N^{th}$-order braid group of the plane where braiding is constrained to follow links of the lattice. This implies that particles coupled to the lattice Chern-Simons theory are anyons with statistics parameter $1/k$. Furthermore, aside from this phase the theory is trivial. The Hilbert space contains only one state.

A more complicated situation arises when the lattice has a nontrivial first homology group. Here, for simplicity we shall set $j_\mu = 0$ and consider the example of a toroidal lattice where the gauge fields have periodic boundary conditions

$$A_i(x_1 + N_1, x_2) = A_i(x_1, x_2) \ , \ A_i(x_1, x_2 + N_2) = A_i(x_1, x_2)$$

Here, the gauge group is U(1) and we require that the gauge transformation obeys the boundary condition

$$\chi(x_1 + N_1, x_2) = \chi(x_1, x_2) + 2\pi m_1 \ , \ \chi(x_1, x_2 + N_2) = \chi(x_1, x_2) + 2\pi m_2$$

where $m_i$ are integers. We can choose the canonical generators of the first homology and the gauge invariant canonical variables as

$$q = \sum_n A_1(n, 0) \ , \ p = -\frac{k}{2\pi} \sum_n A_2(0, n)$$

which obey the algebra

$$[q, p] = i$$

The large gauge transforms $q \to q + 2\pi m_1$, $p \to p - km_2$ are generated by $m_1$ and $m_2$ operations of the unitary operators $g_1 = e^{2\pi ip}$, $g_2 = e^{ikq}$, respectively. These operators
obey the algebra

\[ g_1 g_2 = g_2 g_1 e^{2\pi i k} \] (23)

The wave-function should carry a unitary representation of this algebra. Here we are assuming that \( k = k_1/k_2 \) is a rational number. It is straightforward to construct a representation of (23). Assume that we find an eigenvector \( \psi_\theta \) of \( g_1 \) such that

\[ g_1 \psi_\theta = e^{i\theta} \psi_\theta . \] (24)

Operating \( g_2 \ell \) times gives another eigenvector of \( g_1 \) with eigenvalue \( e^{i(\theta + 2\pi k \ell)} \),

\[ g_1 g_2^\ell \psi_\theta = g_2^\ell g_1 e^{2\pi i k \ell} \psi_\theta = e^{\theta + 2\pi i k \ell} g_2^\ell \psi_\theta \] (25)

When \( \ell = k_2 \) we obtain the original eigenvalue. Thus we conclude that the algebra (23) may be represented by \( k_2 \times k_2 \) unitary matrices and that \( g_2^{k_2} \psi_\theta \xi = e^{i\xi} \psi_\theta \xi \). Thus, the representation is specified by two angles \( \theta \) and \( \xi \). In the Schrödinger polarization the wavefunction is a function of \( q \) and (24) implies

\[ \psi_{\theta \xi}(q) = \sum_{n \in \mathbb{Z}} e^{(n+\theta/2\pi)iq} \psi_{\theta \xi}(n) \] (26)

and

\[ g_2^\ell \psi_{\theta \xi}(q) = \sum_{n \in \mathbb{Z}} e^{(n+\theta/2\pi+k\ell)iq} \psi_{\theta \xi}(n) \] (27)

When \( \ell = k_2 \) we obtain \( \psi_{\theta \xi}(n) = e^{i\xi} \psi_{\theta \xi}(n+k_1) \). Thus, for each \( \ell \) there are \( k_1 \) independent coefficients in the expansion (26). The Hilbert space is \( k_1 k_2 \)-dimensional. This construction can be generalized to lattices with more complicated homology. For a lattice with genus \( g \), the dimension of the Hilbert space turns out to be \((k_1 k_2)^g\). This dimension is the same
as that of the Hilbert space in the continuum theory on the space \( R^1 \times \Sigma^g \) where \( \Sigma^g \) is a Riemann surface of genus \( g \).

In conclusion, we have shown that the present version of lattice Chern-Simons theory, where the form of the action is deduced from the requirements of locality of the action and canonical structure, as well as from gauge invariance, recovers the topological features of continuum Chern-Simons theory. An interesting feature is the introduction of a new and practically unique lattice kernel which can be regarded as an intersection density and which is responsible for the topological nature of the theory. We showed that the lattice model is exactly solvable. The solution resembles the solution of the analogous continuum Chern-Simons theory.

The wave-functions carry a representation of the Braid group. Note that, since particles are constrained to occupy lattice sites, their motion is more restricted than in the continuum, and it is possible to form clusters of particles such that not every braiding operation is allowed. Thus, the braid group on the lattice must differ from the one in the continuum by certain constraints. It would be interesting to investigate this further.

It is intriguing to us that the Hilbert space of lattice Chern-Simons theory on the torus is identical to that in the continuum. In previous work on the continuum theory [2] the Hilbert space was related to quantization of the moduli space of the torus, i.e. the space of metrics on the torus modulo diffeomorphisms. On the lattice there are no diffeomorphisms or metrics and no concept of moduli space. Yet we find a quantum theory on the lattice virtually identical to that in the continuum.
References

1. A.S.Schwarz, Lett. Math. Phys. 2(1978), 247; Comm. Math. Phys. 67 (1979), 1.

2. E.Witten, Comm. Math. Phys. 121 (1989), 351.

3. E.Witten, Comm. Math. Phys. 117 (1988), 353.

4. D.Birmingham, M.Blau, M.Rakowski and G.Thompson, ‘Topological Field Theory’,
   to be published in Physics Reports, 1992.

5. T.H.Hansson, S.Kivelson and S.-G.Zhang, Phys. Rev.Lett. 62 (1989), 82; G. Moore
   and N. Read, Nucl. Phys. B360 (1991), 362.

6. A. Fetter, R. Hanna and R. Laughlin, Phys. Rev. B40, 8745 (1989).

7. J. Fröhlich and P. Marchetti, Comm. Math. Phys. 116 (1988), 127.

8. M. Lüscher, Nucl. Phys. B326 (1989), 557.

9. V.F. Müller, Z.Phys.C47:301-310,1990.

10. D.Eliezer, G.W.Semenoff and S.S.C.Wu, Mod. Phys. Lett. A, in press, 1991.

11. D.Eliezer and G.W.Semenoff, Ann. Phys. (N.Y.), in press, 1992.