Criteria for univalence and quasiconformal extension of harmonic mappings in terms of the Schwarzian derivative

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Abstract. We prove that if the Schwarzian norm of a given complex-valued locally univalent harmonic mapping \( f \) in the unit disk is small enough, then \( f \) is, indeed, globally univalent in the unit disk and can be extended to a quasiconformal mapping in the extended complex plane.

Mathematics Subject classification. 31A05, 30C55, 30C62.

Keywords. Harmonic mapping, Schwarzian derivative, Univalence criterion, Quasiconformal extension.

Introduction. In 1949, Nehari [9] proved that if a locally univalent analytic function \( \varphi \) in the unit disk \( \mathbb{D} \) satisfies

\[
\sup_{z \in \mathbb{D}} |S(\varphi)(z)| (1 - |z|^2)^2 \leq 2, \tag{0.1}
\]

then \( \varphi \) is globally univalent in \( \mathbb{D} \). Here, \( S(\varphi) \) denotes the Schwarzian derivative of \( \varphi \) defined by

\[
S(\varphi) = \left( \frac{\varphi''}{\varphi'} \right)' - \frac{1}{2} \left( \frac{\varphi''}{\varphi'} \right)^2.
\]

Ahlfors and Weill [1] generalized Nehari’s criterion of univalence by proving that if such a function \( \varphi \) satisfies \( \|S(\varphi)\| \leq 2t \) for some \( t < 1 \), then \( \varphi \) is injective in \( \mathbb{D} \) and has a \( K \)-quasiconformal extension to \( \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \), where \( K = (1 + t)/(1 - t) \).

Let now \( f \) be a complex-valued locally univalent harmonic mapping in the unit disk. By considering the complex conjugate, if needed, we can assume that \( f \) is sense-preserving. This is, \( f = h + \overline{g} \) where \( h \) and \( g \) are analytic functions.

This research was supported by Grants Fondecyt 1110160 and 1110321, Chile. The second author is also supported by Academy of Finland Grant 268009 and by Spanish MINECO Research Project MTM2012-37436-C02-02.
in \( \mathbb{D} \) such that \( h \) is locally univalent and the (second complex) dilatation \( \omega = g'/h' \) is an analytic function mapping the unit disk into itself. The following definition for the Schwarzian derivative \( S_f \) of such functions \( f \) was presented in [5]:

\[
S_f = S(h) + \frac{\overline{\omega}}{1 - |\omega|^2} \left( \frac{h''}{h'} \omega' - \omega'' \right) - \frac{3}{2} \left( \frac{\overline{\omega} \omega'}{1 - |\omega|^2} \right)^2,
\]

(0.2)

where \( S(h) \) is the classical Schwarzian derivative of \( h \), as in (0.1).

The main purpose in this paper is to prove that there exists a constant \( \delta_0 > 0 \) such that if the locally univalent harmonic mapping \( f \) in the unit disk has Schwarzian norm

\[
\|S_f\| = \sup_{z \in \mathbb{D}} |S_f(z)| (1 - |z|^2)^2 \leq \delta_0,
\]

then \( f \) is one-to-one in \( \mathbb{D} \). We will also see that if \( \|S_f\| \leq \delta_0 t \) for some \( t < 1 \), then \( f \) has a quasiconformal extension to \( \hat{\mathbb{C}} \).

1. Background.

1.1. The Schwarzian derivative. As was mentioned in the introduction, every harmonic mapping \( f \) in the unit disk \( \mathbb{D} \) can be written as \( f = h + g \) with \( h \) and \( g \) analytic in \( \mathbb{D} \). This decomposition is unique up to an additive constant (see [3, p. 7]). We refer the reader to the book [3] for a comprehensive treatment on harmonic mappings.

Lewy [8] proved that a harmonic mapping in the unit disk is locally univalent if and only if its Jacobian is different from zero. In terms of the decomposition \( f = h + g \), the Jacobian \( J_f \) of \( f \) equals \( |h'|^2 - |g'|^2 \). Thus, locally univalent harmonic mappings in \( \mathbb{D} \) are either sense-preserving if \( J_f > 0 \) or sense-reversing if \( J_f < 0 \). Note that any analytic function is a sense-preserving harmonic mapping. Also, that a harmonic function \( f = h + g \) is sense-preserving if and only if \( h \) is locally univalent and the dilatation \( \omega = g'/h' \) maps the unit disk into itself. It is obvious that \( f \) is sense-preserving if and only if \( \overline{f} \) is sense-reversing. In this paper, we will consider harmonic mappings which are sense-preserving in the unit disk. For this kind of mappings, the Schwarzian derivative is given by (0.2). It is clear that if \( f \) is analytic, then \( S_f \) coincides with the classical definition of the Schwarzian derivative given by (0.1).

Several properties of this operator \( S_f \) are the following.

(i) \( S_f \equiv 0 \) if and only if \( f = \alpha T + \beta \overline{T} \), where \( |\alpha| \neq |\beta| \) and \( T \) is a Möbius transformation

\[
T(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0.
\]

(ii) Whenever \( f \) is a sense-preserving harmonic mapping and \( \phi \) is an analytic function such that the composition \( f \circ \phi \) is well-defined, the Schwarzian derivative of \( f \circ \phi \) can be computed using the chain rule

\[
S_{f \circ \phi} = S_f(\phi) \cdot (\phi')^2 + S\phi.
\]
(iii) For any affine mapping \( L(z) = az + b \bar{z} \) with \( |a| \neq |b| \), we have that \( S_{L \circ f} = S_f \). Note that \( L \) is sense-preserving if and only if \( |b| < |a| \).

The Schwarzian norm \( \|S_f\| \) of a sense-preserving harmonic mapping \( f \) in the unit disk is defined by

\[
\|S_f\| = \sup_{z \in \mathbb{D}} |S_f(z)| \cdot (1 - |z|^2)^2.
\]

It is easy to check (using the chain rule again and the Schwarz-Pick lemma) that \( \|S_{f \circ \sigma}\| = \|S_f\| \) for any automorphism of the unit disk \( \sigma \). For further properties of \( S_f \) and the motivation for this definition, see [5].

1.2. An affine and linear invariant family. In [10, 11] Pommerenke studied the so-called linear invariant families; that is, families of locally univalent holomorphic functions \( \varphi \) in the unit disk normalized by the conditions \( \varphi(0) = 1 - \varphi'(0) = 0 \) and which are closed under the transformation

\[
\Phi_\zeta(z) = \frac{\varphi\left( \frac{\zeta + z}{1 + \overline{\zeta}z} \right) - \varphi(\zeta)}{(1 - |\zeta|^2)\varphi'(\zeta)}, \quad \zeta \in \mathbb{D}.
\]

Let \( \mathcal{F} \) be a family of sense-preserving harmonic mappings \( f = h + \overline{g} \) in \( \mathbb{D} \), normalized with \( h(0) = g(0) = 0 \) and \( h'(0) = 1 \). The family is said to be affine and linear invariant if it is closed under the two operations of Koebe transform and affine change:

\[
K_\zeta(f)(z) = \frac{f\left( \frac{z + \zeta}{1 + \overline{\zeta}z} \right) - f(\zeta)}{(1 - |\zeta|^2)h'(\zeta)}, \quad |\zeta| < 1,
\]

and

\[
A_\varepsilon(f)(z) = \frac{f(z) - \varepsilon f(\overline{z})}{1 - \varepsilon g'(0)}, \quad |\varepsilon| < 1.
\]

We refer the reader to the paper [12], where Sheil-Small offers a deep study of affine and linear invariant families \( \mathcal{F} \) of harmonic mappings in \( \mathbb{D} \).

Using that the Schwarzian derivative for harmonic mappings satisfies the chain rule and is invariant under affine changes \( af + b\overline{f} \), \( |a| \neq |b| \), it is easy to show that the family \( \mathcal{F}_\lambda \) of sense-preserving harmonic mappings \( f = h + \overline{g} \) in \( \mathbb{D} \) with \( h(0) = g(0) = 0 \), \( h'(0) = 1 \), and \( ||S_f|| \leq \lambda \) is affine and linear invariant. We let \( \mathcal{F}_\lambda^0 = \{ f \in \mathcal{F}_\lambda : g'(0) = 0 \} \). These two families are studied in [2], where it is shown in particular that \( \mathcal{F}_\lambda^0 \) is a compact family of harmonic mappings with respect to the topology of uniform convergence on compact subsets of \( \mathbb{D} \). In that paper, the following notation was used: an analytic function in the unit disk \( \omega \) with \( \omega(\mathbb{D}) \subset \mathbb{D} \) is said to belong to \( \mathcal{A}_\lambda^0 \) (resp. \( \mathcal{A}_\lambda \)) if there exists a harmonic mapping \( f = h + \overline{g} \in \mathcal{F}_\lambda^0 \) (resp. \( \mathcal{F}_\lambda \)) with dilatation \( \omega \). The quantity

\[
R_\lambda = \max_{\omega \in \mathcal{A}_\lambda^0} |\omega'(0)| = \sup_{\omega \in \mathcal{A}_\lambda} ||\omega^*||,
\]
where
\[ \| \omega^* \| = \sup_{z \in \mathbb{D}} \frac{\left| \omega'(z) \right| \cdot (1 - |z|^2)}{1 - |\omega(z)|^2}, \]

was shown to play a distinguished role in the analysis offered in [2].

Perhaps at this point we should point out that according to the first of the properties for the Schwarzian derivative mentioned in the previous section, we have that the family \( F_0 \) consists only of functions of the form \( f = \alpha T + \beta T \), where \( |\alpha| > |\beta| \) and \( T \) is a Möbius transformation. Therefore, all dilatations in \( \mathcal{A}_0 \) are constant functions.

2. Main Results. The following lemma will be important for our purposes.

**Lemma 2.1.** As before, let \( R_\lambda = \max_{\omega \in \mathcal{A}_\lambda^0} |\omega'(0)| \). Then
\[
\lim_{\lambda \to 0^+} R_\lambda = 0.
\]

**Proof.** Since \( \mathcal{F}^0_{\lambda_1} \subset \mathcal{F}^0_{\lambda_2} \) whenever \( 0 < \lambda_1 < \lambda_2 \), we have \( 0 \leq R_{\lambda_1} \leq R_{\lambda_2} \) as well. Therefore, we conclude that there exists \( \lim_{\lambda \to 0^+} R_\lambda \) and it remains to check that this limit equals 0.

Consider an arbitrary positive number \( \lambda \). By the definition of \( R_\lambda \) and the compacity of \( \mathcal{F}^0_\lambda \), we see that for each such \( \lambda \) there is a harmonic mapping \( f_\lambda \in \mathcal{F}^0_{\lambda} \) with dilatation \( \omega_\lambda \) satisfying \( |\omega'_\lambda(0)| = R_\lambda \). Since for a given \( \rho > 0 \) the family \( \{ f_\lambda : \lambda \leq \rho \} \subset \mathcal{F}^0_\rho \) and \( \mathcal{F}^0_\rho \) is compact, we get that there is a function \( f_0 \in \cap_{\rho > 0} \mathcal{F}^0_\rho \) with dilatation \( \omega_0 \) such that \( f_\lambda \to f_0 \) as \( \lambda \to 0 \) uniformly on compact subsets in the unit disk (hence \( \omega'_\lambda(0) \to \omega'_0(0) \) as \( \lambda \to 0 \) too).

Obviously, \( \cap_{\rho > 0} \mathcal{F}^0_\rho = \mathcal{F}^0_0 \) and the dilatations of functions in \( \mathcal{F}^0_0 \) are constants, thus \( 0 = \omega'_0(0) = \lim_{\lambda \to 0} R_\lambda \).

We now state and prove the main theorems in this paper.

**Theorem 2.2.** There exists \( \delta_0 > 0 \) such that if \( \| S_f \| \leq \delta_0 \), then \( f \) is univalent.

**Proof.** For any real number \( \lambda > 0 \), we have that if \( f = h + \overline{g} \in \mathcal{F}_\lambda \), the Schwarzian norm of \( h \) is bounded by [5, Thm. 6]. Hence (see [10]),
\[
\sup_{z \in \mathbb{D}} \left| \frac{h''(z)}{h'(z)} \right| \cdot (1 - |z|^2) \leq K_1
\]
for some constant \( K_1 > 0 \). Moreover, by using \( \omega \) to denote the dilatation of \( f \), we have (see, for instance, [4]) that there exists another positive constant \( K_2 \) such that
\[
\sup_{z \in \mathbb{D}} \frac{|\omega''(z)| \cdot (1 - |z|^2)^2}{1 - |\omega(z)|^2} \leq K_2 \| \omega^* \| \leq K_2 R_\lambda.
\]

Hence, using (0.2) and the triangle inequality, we see that for any such function \( f = h + \overline{g} \),
\[
\| S(h) \| \leq \lambda + K_1 R_\lambda + K_2 R_\lambda + \frac{3}{2} R_\lambda^2.
\] (2.1)
Now, using Lemma 2.1 and the fact that \( R_\lambda \) increases with \( \lambda \), we have that there exists a unique solution \( \delta_0 \), say, of the equation

\[
\lambda + K_1 R_\lambda + K_2 R_\lambda + \frac{3}{2} R_\lambda^2 = 2.
\]

This implies by (2.1) that if \( \lambda \leq \delta_0 \), then \( \|S(h)\| \leq 2 \). In other words, the analytic part \( h \) of any function \( f = h + \overline{g} \in \mathcal{F}_{\delta_0} \) is univalent by the classical Nehari criterion of univalence.

To prove that not only \( h \) but the function \( f = h + \overline{g} \) itself is univalent whenever \( f \in \mathcal{F}_{\delta_0} \), we proceed as follows. By the affine invariance property of \( \mathcal{F}_{\delta_0} \), we see that for any \( a \in \mathbb{D} \) the function \( f_a = f + \overline{a}f \) belongs to \( \mathcal{F}_{\delta_0} \) as well. It is easy to check that if \( f_a = h_a + \overline{a}g \), then \( h_a = h + \overline{a}g \). Thus, the functions \( h + \overline{a}g \) are also univalent for all \( |a| < 1 \). Since \( f \) is sense-preserving, an application of the Hurwitz theorem gives that \( h + \overline{a}g \) is indeed univalent for all \( |a| \leq 1 \).

Now, assume (in order to get a contradiction) that \( f = h + \overline{g} \) is not univalent in the unit disk. Therefore, there exist two different points \( z_1, z_2 \in \mathbb{D} \) such that \( f(z_1) = f(z_2) \). We have two possibilities:

1. \( h(z_1) = h(z_2) \). In this case, it follows that \( g(z_1) = g(z_2) \). This is a contradiction to the fact that the function \( h + g \) is univalent in the unit disk.

2. \( h(z_1) \neq h(z_2) \). Denote \( \theta = \arg \{h(z_1) - h(z_2)\} \in [0, 2\pi) \). We are assuming that \( f(z_1) = f(z_2) \), so that \( h(z_1) - h(z_2) = g(z_2) - g(z_1) \). Thus,

\[
e^{-i\theta}(h(z_1) - h(z_2)) = e^{-i\theta}(g(z_2) - g(z_1))
\]

is a positive real number. Therefore, by taking complex conjugates, we obtain

\[
e^{-i\theta}(g(z_2) - g(z_1)) = e^{i\theta}(g(z_2) - g(z_1))
\]

and as a consequence we get

\[
h(z_1) - h(z_2) = e^{2i\theta}(g(z_2) - g(z_1)),
\]

which contradicts the fact that the function \( h + e^{2i\theta}g \) is univalent in \( \mathbb{D} \). This ends the proof of the theorem. \( \square \)

We would like to point out that by finding an upper bound for the quantity \( R_\lambda \) in terms of \( \lambda \), one could give an estimate of the value \( \delta_0 \) in the previous theorem. Unfortunately, so far we are not able to obtain such an upper bound.

The next result is related to a criterion for the existence of quasiconformal extensions of harmonic mappings in terms of their Schwarzian norm.

**Theorem 2.3.** Let \( f \) be a sense-preserving harmonic mapping in the unit disk with \( \|S_f\| \leq \delta_0 t \) for some \( t < 1 \), where \( \delta_0 \) is as in Theorem 2.2. Assume, in addition, that the dilatation \( \omega_f \) of \( f \) satisfies

\[
\|\omega_f\|_\infty = \sup_{z \in \mathbb{D}} |\omega_f(z)| < 1.
\]

Then \( f \) can be extended to a quasiconformal map in \( \hat{\mathbb{C}} \).
Before proving this second theorem, we would like to stress that the hypotheses \(|\omega|_\infty < 1\) cannot be removed as the following example shows.

**Example.** Consider the sense-preserving harmonic mapping \(f = z + \overline{g}\), where \(g'\) equals the lens-map \(\ell_\alpha\), \(0 < \alpha \leq 1\), defined by

\[
\ell_\alpha(z) = \frac{\ell(z)^\alpha - 1}{\ell(z)^\alpha + 1}, \quad z \in \mathbb{D},
\]

with \(\ell(z) = (1 + z)/(1 - z)\). Note that \(\ell_1\) equals the identity in the unit disk and that \(|\ell_\alpha|_\infty = 1\) for all \(0 < \alpha \leq 1\). In [7], it is explicitly checked that \(|\ell_\alpha^*| = \alpha\).

Bearing in mind (0.2), that the dilatation of \(f\) is \(\ell_\alpha\), and that

\[
\sup_{z \in \mathbb{D}} \frac{|\ell''_\alpha(z)|}{1 - |\ell_\alpha(z)|^2} (1 - |z|^2)^2 \leq K_2 ||\ell^*_\alpha|| = K_2 \alpha
\]

for some absolute constant \(K_2\), we have

\[
||S_f|| = \sup_{z \in \mathbb{D}} \left| \frac{\ell_\alpha(z) \ell''_\alpha(z)}{1 - |\ell_\alpha(z)|^2} + \frac{3}{2} \left( \frac{\ell_\alpha(z) \ell'_\alpha(z)}{1 - |\ell_\alpha(z)|^2} \right)^2 \left(1 - |z|^2\right)^2 \right|
\]

\[
\leq \sup_{z \in \mathbb{D}} \frac{|\ell''_\alpha(z)|}{1 - |\ell_\alpha(z)|^2} (1 - |z|^2)^2 + \frac{3}{2} \sup_{z \in \mathbb{D}} \left| \frac{\ell'_\alpha(z)}{1 - |\ell_\alpha(z)|^2} \right| (1 - |z|^2)^2
\]

\[
\leq K_2 \alpha + \frac{3}{2} \alpha^2.
\]

Therefore, by choosing any \(\alpha\) small enough, we obtain \(||S_f|| \leq \delta_0 t\) for any given \(0 < t < 1\). On the other hand, the function \(f\) is not quasiconformal since its (second complex) dilatation coincides with \(\ell_\alpha\) and \(|\ell_\alpha|_\infty = 1\).

We now prove Theorem 2.3.

**Proof.** Since we are assuming that \(||S_f|| \leq \delta_0 t\) for some \(t < 1\), we have that \(f \in \mathcal{F}_{\delta_0 t}\). By arguing as in the proof of the previous theorem and using again that if \(\lambda_1 \leq \lambda_2\) then \(R_{\lambda_1} \leq R_{\lambda_2}\), we get

\[
||S(h)|| \leq \delta_0 t + K_1 R_{\delta_0 t} + K_2 R_{\delta_0 t} + \frac{3}{2} R_{\delta_0 t}^2
\]

\[
\leq \delta_0 t + K_1 R_{\delta_0} + K_2 R_{\delta_0} + \frac{3}{2} R_{\delta_0}^2
\]

\[
< \delta_0 + K_1 R_{\delta_0} + K_2 R_{\delta_0} + \frac{3}{2} R_{\delta_0}^2 = 2,
\]

so that \(||S(h)|| \leq 2s\) for some \(s < 1\). This shows (by the Ahlfors-Weill theorem) that the analytic part \(h\) of every function \(f = h + \overline{g}\) in the family \(\mathcal{F}_{\delta_0 t}\) can be extended to a \(K_s\)-quasiconformal function in \(\widehat{\mathbb{C}}\), where \(K_s = (1 + s)/(1 - s)\). Using again that the family \(\mathcal{F}_{\delta_0 t}\) is invariant under affine transformations, we get that not only \(h\) but also \(h + ag\) (where \(a \in \overline{\mathbb{D}}\)) has a \(K_s\)-quasiconformal extension to \(\widehat{\mathbb{C}}\). By arguing as in the proof of [6, Thm. 2], we conclude that \(f\) itself has a \(K\)-quasiconformal extension for an appropriate value of \(K\). \(\square\)
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Received: 20 October 2014

Revised: 26 November 2014