On certain Opial-type results in Cesàro spaces of vector-valued functions

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Abstract. Given a Banach space $X$, we consider Cesàro spaces $\text{Ces}_p(X)$ of $X$-valued functions over the interval $[0,1]$, where $1 \leq p < \infty$. We prove that if $X$ has the Opial/uniform Opial property, then certain analogous properties also hold for $\text{Ces}_p(X)$. We also prove a result on the Opial/uniform Opial property of Cesàro spaces of vector-valued sequences.

1 Introduction

Let us begin by recalling the definitions of Cesàro sequence and function spaces. For $1 \leq p < \infty$, the Cesàro sequence space $\text{ces}_p$ is defined as the space of all sequences $a = (a_n)_{n \in \mathbb{N}}$ of real numbers such that

$$\|a\|_{\text{ces}_p} := \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |a_i| \right)^p \right)^{1/p} < \infty.$$ 

$\|\cdot\|_{\text{ces}_p}$ defines a norm on $\text{ces}_p$. Leibowitz [13] and Jagers [11] proved that $\text{ces}_1 = \{0\}$ and $\text{ces}_p$ is separable and reflexive for $1 < p < \infty$. In [5] it was proved that for any $p \in (1, \infty)$, the space $\text{ces}_p$ is not isomorphic to $\ell^q$ for any $q \in [1, \infty]$.

The Cesàro function space $\text{Ces}_p$ on $[0,1]$ is defined in an analogous way as the space of all measurable functions $f : [0,1] \to \mathbb{R}$ such that

$$\|f\|_{\text{Ces}_p} := \left( \int_{0}^{1} \left( \frac{1}{t} \int_{0}^{t} |f(s)| \, ds \right)^p \, dt \right)^{1/p} < \infty,$$

where, as usual, two functions are identified if they agree a.e. $\|\cdot\|_{\text{Ces}_p}$ defines a norm on $\text{Ces}_p$.

Here are some basic results on Cesàro function spaces:

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1. Introduction

(1) Ces$_1$ is the weighted Lebesgue space $L^1_w[0,1]$, where $w(t) := \log(1/t)$.

(2) Ces$_p$ is a subspace of $L^p[0,a]$ for every $p \in [1,\infty)$ and every $a \in (0,1)$, but not for $a = 1$.

(3) Ces$_p$ is separable and nonreflexive for every $p \in [1,\infty)$.

(4) For $1 < p < \infty$ one has $L^p[0,1] \subseteq$ Ces$_p$ and $\|f\|_{\text{Ces}_p} \leq q\|f\|_p$ for all $f \in L^p[0,1]$, where $q$ is the conjugated exponent to $p$.

These and further results are collected in [3, Theorem 1]. Also, by [3, Theorem 7], for $p \in (1,\infty)$ the space Ces$_p$ is not isomorphic to $L^q[0,1]$ for any $q \in [1,\infty]$. For further information on Cesàro function spaces see [2–4] and references therein. For more information on Cesàro sequence spaces see, for example, the introduction of [3] and references therein. Results on more general types of Cesàro function spaces, where the space $L^p$ appearing implicitly in the definition of Ces$_p$ is replaced by a more general function space, can be found for example in [14,15].

Now consider a real Banach space $X$. $X$ is said to have the fixed point property (resp. weak fixed point property) if for every closed and bounded (resp. weakly compact) convex subset $C \subset X$, every nonexpansive mapping $F : C \to C$ has a fixed point (where $F$ is called nonexpansive if $\|F(x) - F(y)\| \leq \|x - y\|$ for all $x, y \in C$).

A bounded, closed, convex subset $C \subseteq X$ is said to have normal structure provided that for each subset $B \subseteq C$ which contains at least two elements there exists a point $x \in B$ such that

$$\sup_{y \in B} \|x - y\| < \text{diam}(B),$$

where diam$(B)$ denotes the diameter of $B$. The space $X$ itself is said to have normal structure if every bounded, closed, convex subset of $X$ has normal structure. It is well known that if $C$ is weakly compact and has normal structure, then every nonexpansive mapping $F : C \to C$ has a fixed point (see e.g. [8, Theorem 2.1]), thus spaces with normal structure have the weak fixed point property. For example, every compact, convex set has normal structure (see e.g. [20, p.119]) and hence all finite-dimensional spaces possess normal structure. Also, every space which is uniformly convex in every direction has normal structure (see e.g. [20, Corollary 5.6]). An example of a Banach space which fails the weak fixed point property is $L^1[0,1]$ (see [1]).

The space $X$ is said to have the Opial property if

$$\limsup_{n \to \infty} \|x_n\| < \limsup_{n \to \infty} \|x_n - x\|$$

holds for every weakly null sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ and every $x \in X \setminus \{0\}$ (one could as well use lim inf instead of lim sup or assume from the beginning that both limits exist).
1. Introduction

This property was first considered by Opial in [18] (starting from the Hilbert spaces as canonical example) to provide a result on iterative approximations of fixed points of nonexpansive mappings. It is shown in [18] that the spaces \( \ell^p \) for \( 1 \leq p < \infty \) enjoy the Opial property, whereas \( L^p[0,1] \) for \( 1 < p < \infty, p \neq 2 \) fails to have it. Note further that every Banach space with the Schur property (i.e. weak and norm convergence of sequences coincide) trivially has the Opial property. Also, \( X \) is said to have the nonstrict Opial property if it fulfils the definition of the Opial property with “\( \leq \)” instead of “\( < \)” ([22], in [7] it is called weak Opial property). It is known that every weakly compact convex set in a Banach space with the Opial property has normal structure (see e.g. [20, Theorem 5.4]) and thus the Opial property implies the weak fixed point property.

The notion of uniform Opial property was introduced by Prus in [19]: \( X \) is said to have the uniform Opial property if for every \( c > 0 \) there is some \( r > 0 \) such that

\[
1 + r \leq \liminf_{n \to \infty} \|x_n - x\|
\]

holds for every \( x \in X \) with \( \|x\| \geq c \) and every weakly null sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \) with \( \liminf\|x_n\| \geq 1 \). In [19] it was proved that a Banach space is reflexive and has the uniform Opial property if and only if it has the so-called property \((L)\) (see [19] for the definition), and that \( X \) has the fixed point property whenever \( X^* \) has property \((L)\).

A modulus corresponding to the uniform Opial property was defined in [16]:

\[
r_X(c) := \inf \left\{ \liminf_{n \to \infty} \|x_n - x\| - 1 \right\} \quad \forall c > 0,
\]

where the infimum is taken over all \( x \in X \) with \( \|x\| \geq c \) and all weakly null sequences \( (x_n)_{n \in \mathbb{N}} \) in \( X \) with \( \liminf\|x_n\| \geq 1 \) (if \( X \) has the Schur property, we agree to set \( r_X(c) := 1 \) for all \( c > 0 \)). Then \( X \) has the uniform Opial property if and only if \( r_X(c) > 0 \) for every \( c > 0 \).

Here we will use instead the following equivalent formulation of the uniform Opial property ([12, Definition 3.1]): \( X \) has the uniform Opial property if and only if for every \( \varepsilon > 0 \) and every \( R > 0 \) there is some \( \eta > 0 \) such that

\[
\eta + \liminf_{n \to \infty} \|x_n\| \leq \liminf_{n \to \infty} \|x_n - x\|
\]

holds for all \( x \in X \) with \( \|x\| \geq \varepsilon \) and every weakly null sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \) with \( \limsup\|x_n\| \leq R \).

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Note that one always has \( \limsup\|x_n\| \leq \limsup\|x_n - x\| + \|x\| \leq 2\limsup\|x_n - x\| \) if \( (x_n)_{n \in \mathbb{N}} \) converges weakly to zero, since the norm is weakly lower semicontinuous. In general, the constant 2 is the best possible. Consider, for example, in the space \( c \) of all convergent sequences (with sup-norm) the weak null sequence \((2e_n)_{n \in \mathbb{N}}\) (where \( e_n \) is the sequence whose \( n \)-th entry is 1 and all other entries are 0) and \( x = (1,1,1,\ldots) \).
2. Opial properties in Cesàro sums

In [10] the author defined a modulus for this formulation in the following way:

$$\eta_X(\varepsilon, R) := \inf \left\{ \liminf_{n \to \infty} \|x_n - x\| - \liminf_{n \to \infty} \|x_n\| \right\} \quad \forall \varepsilon, R > 0,$$

where the infimum is taken over all $x \in X$ with $\|x\| \geq \varepsilon$ and all weakly null sequences $(x_n)_{n \in \mathbb{N}}$ in $X$ with $\limsup_{n \to \infty} \|x_n\| \leq R$. Thus $X$ has the uniform Opial property if and only if $\eta_X(\varepsilon, R) > 0$ for all $\varepsilon, R > 0$ (see also [10, Lemma 1.1] for a more precise connection between the moduli $r_X$ and $\eta_X$).

In [10] the author studied Opial properties in infinite $\ell^p$-sums and also some analogous results for Lebesgue-Bochner spaces of vector-valued functions (these spaces cannot have the usual Opial property, as even $L^p[0,1]$ for $1 < p < \infty$, $p \neq 2$ does not enjoy this property, but if one replaces weak convergence by pointwise weak convergence (almost everywhere), then some “Opial-like” results for Lebesgue-Bochner spaces can be established, see [10] for the detailed formulations and proofs).

The purpose of this paper is to prove some results analogous to those of [10] for Cesàro spaces of vector-valued functions and Cesàro sums. We will start with the latter.

2 Opial properties in Cesàro sums

Given a sequence $(X_n)_{n \in \mathbb{N}}$ of Banach spaces and $p \in (1, \infty)$, we define the $p$-Cesàro sum $\left[ \bigoplus_{n \in \mathbb{N}} X_n \right]_{ces_p}$ of $(X_n)_{n \in \mathbb{N}}$ as the space of all sequences $x = (x_n)_{n \in \mathbb{N}}$ with $x_n \in X_n$ for each $n$ such that $(\|x_n\|)_{n \in \mathbb{N}} \in ces_p$, equipped with the norm

$$\|x\|_{ces_p} := \|((\|x_n\|)_{n \in \mathbb{N}})\|_{ces_p} = \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} \|x_i\| \right)^p \right)^{1/p}.$$

In [6] it was proved that $ces_p$ has the uniform Opial property for every $p \in (1, \infty)$. In [21, Theorem 1] Saejung proved that $ces_p$ can be regarded as a subspace of the $\ell^p$-sum $X_p := \left[ \bigoplus_{n \in \mathbb{N}} \ell^1(n) \right]_p$ (where $\ell^1(n)$ denotes the $n$-dimensional space with $\ell^1$-norm) via the isometric embedding $T : ces_p \to X_p$ defined by

$$(Ta)(n) := \frac{1}{n} (a_1, \ldots, a_n) \quad \forall n \in \mathbb{N}, \forall a \in ces_p.$$

In [21, Theorem 7] it is proved that the $\ell^p$-sum of any sequence of finite-dimensional spaces has the uniform Opial property (see also [20, Example 4.23 (2.)] and Corollary 3.14 in [10]). Thus Saejung obtains a new proof that $ces_p$ has the uniform Opial property ([21, Corollary 9]).

Saejung’s embedding idea directly generalises to $ces_p$-sums. For a given sequence $(X_n)_{n \in \mathbb{N}}$ of Banach spaces we consider the mapping $S$ from the
3. Opial-type properties in Cesàro spaces of vector-valued functions

Cesàro sum \( \left( \bigoplus_{n \in \mathbb{N}} X_n \right)_{ces} \) to \( \left( \bigoplus_{n \in \mathbb{N}} (X_1 \oplus_1 \cdots \oplus_1 X_n) \right)_p \) defined by

\[
(Sx)(n) := \frac{1}{n}(x_1, \ldots, x_n) \quad \forall n \in \mathbb{N}, \forall x \in \left( \bigoplus_{n \in \mathbb{N}} X_n \right)_{ces}.
\]

Then \( S \) is an isometric embedding.

In [10, Proposition 3.11] the author proved that for any \( 1 \leq p < \infty \) the \( \ell^p \)-sum of any family of Banach spaces with the Opial property/nonstrict Opial property has again the Opial property/nonstrict Opial property. Thus, via the above embedding, we obtain the following result.

**Proposition 2.1.** Let \( p \in (1, \infty) \). If \( (X_n)_{n \in \mathbb{N}} \) is a sequence of Banach spaces such that each \( X_n \) has the Opial property (nonstrict Opial property), then \( \left( \bigoplus_{n \in \mathbb{N}} X_n \right)_{ces} \) also has the Opial property (nonstrict Opial property).

It was also proved in [10, Theorem 3.13] that, for any family \( (X_i)_{i \in I} \) of Banach spaces and every \( 1 \leq p < \infty \), the sum \( \left( \bigoplus_{i \in I} X_i \right)_p \) has the uniform Opial property if

\[
\inf_{\eta_{X_J}} \{ \eta_{X_J}(\varepsilon, R) : J \subseteq I \text{ finite} \} > 0 \quad \forall \varepsilon, R > 0,
\]

where \( X_J := \left( \bigoplus_{i \in J} X_i \right)_p \) for \( J \subseteq I \).

Using this together with the above embedding, we obtain the following result.

**Proposition 2.2.** Let \( p \in (1, \infty) \) and \( (X_n)_{n \in \mathbb{N}} \) be a sequence of Banach spaces. Put \( Y_m := \left( \bigoplus_{n=1}^m (X_1 \oplus_1 \cdots \oplus_1 X_n) \right)_p \) for each \( m \in \mathbb{N} \). If

\[
\inf_{\eta_{Y_m}} \{ \eta_{Y_m}(\varepsilon, R) : \varepsilon, R > 0 \},
\]

then \( \left( \bigoplus_{n \in \mathbb{N}} X_n \right)_{ces} \) has the uniform Opial property.

Note that this implies in particular that \( \left( \bigoplus_{n \in \mathbb{N}} X_n \right)_{ces} \) has the uniform Opial property if each \( X_n \) has the Schur property.

3 Opial-type properties in Cesàro spaces of vector-valued functions

Now we consider Cesàro spaces of vector-valued functions on \([0, 1]\). Usually, for a given Banach space \( X \) and a Köthe function space \( E \) (see for instance [17] for the definition), one considers the Köthe-Bochner space \( E(X) \) of all (equivalence classes of) \( X \)-valued Bochner-measurable functions \( f \) such that \( \|f(\cdot)\| \in E \), endowed with the norm \( \|f\|_{E(X)} := \|f(\cdot)\|_E \) (this includes the Lebesgue-Bochner spaces for \( E = L^p \)). Such spaces have been intensively studied (see for example the collection of results in [17]). However, Cesàro
function spaces are not Köthe spaces in the usual sense, since they are not contained in $L^1$ (see point (2) in the introduction). But the Cesàro spaces, like Köthe spaces, satisfy the important monotonicity property: if $f \in \text{Ces}_p$ and $g : [0,1] \rightarrow \mathbb{R}$ is measurable with $|g(t)| \leq |f(t)|$ a.e., then $g \in \text{Ces}_p$ and \( \|g\|_{\text{Ces}_p} \leq \|f\|_{\text{Ces}_p} \).

Therefore, given a Banach space $X$, we can still define the space $\text{Ces}_p(X)$ of all (equivalence classes of) Bochner-measurable functions $f : [0,1] \rightarrow X$ such that $f(\cdot) \in \text{Ces}_p$, equipped with the norm $\|f\|_{\text{Ces}_p(X)} := \|\|f(\cdot)\||_{\text{Ces}_p}$.

We will now prove a result for sequences of functions in $\text{Ces}_p(X)$ which are pointwise a.e. convergent to zero with respect to the weak topology of $X$, where $X$ is assumed to have the nonstrict Opial property. The result is similar to the one obtained in [10, Proposition 4.1] for Lebesgue-Bochner spaces. The proof also makes use of similar techniques.

**Theorem 3.1.** Let $1 \leq p < \infty$ and let $X$ be a Banach space with the nonstrict Opial property. Let $(f_n)_{n \in \mathbb{N}}$ be a bounded sequence in $\text{Ces}_p(X)$ such that $(f_n(t))_{n \in \mathbb{N}}$ converges weakly to zero for almost every $t \in [0,1]$. Suppose further that there exists a $g \in \text{Ces}_p$ such that $\|f_n(t)\| \rightarrow g(t)$ a.e. Let $f \in \text{Ces}_p(X)$ and $\varphi(t) := \lim inf_{n \rightarrow \infty} \|f_n(t) - f(t)\|$ for $t \in [0,1]$. Then

\[
2^{p-1} \int_0^1 \frac{1}{t^p} \left( \left( \int_0^t \varphi(s) \, ds \right)^p - \left( \int_0^t g(s) \, ds \right)^p \right) \, dt \leq 2^{p-1} \limsup_{n \rightarrow \infty} \|f_n - f\|_{\text{Ces}_p(X)} - \limsup_{n \rightarrow \infty} \|f_n\|_{\text{Ces}_p(X)}. \tag{3.1}
\]

In particular,

\[
\limsup_{n \rightarrow \infty} \|f_n\|_{\text{Ces}_p(X)} \leq 2^{1-1/p} \limsup_{n \rightarrow \infty} \|f_n - f\|_{\text{Ces}_p(X)} \quad \forall f \in \text{Ces}_p(X). \tag{3.2}
\]

**Proof.** Using the identification of $\text{Ces}_1$ with $L^1_w[0,1]$ from [3, Theorem 1] (where $w(t) = \log(1/t)$) the assertion for $p = 1$ easily follows from [10, Proposition 4.1]. We will therefore assume $p > 1$.

So let $f \in \text{Ces}_p(X)$. Without loss of generality, we may assume that $\lim_{n \rightarrow \infty} \|f_n\|_{\text{Ces}_p(X)}$ and $\lim_{n \rightarrow \infty} \|f_n - f\|_{\text{Ces}_p(X)}$ exist and also that $\|f_n(t)\| \rightarrow g(t)$ and $f_n(t) \rightarrow 0$ weakly for every $t \in [0,1]$.

Since $M := \sup_{n \in \mathbb{N}} \|f_n\|_{\text{Ces}_p(X)} < \infty$ it follows from Fatou’s Lemma that $\varphi \in \text{Ces}_p$.

Let us put

\[
a := \|\varphi\|_{\text{Ces}_p} - \|g\|_{\text{Ces}_p} = \int_0^1 \frac{1}{t^p} \left( \left( \int_0^t \varphi(s) \, ds \right)^p - \left( \int_0^t g(s) \, ds \right)^p \right) \, dt.
\]

Since $X$ has the nonstrict Opial property we have $\varphi(t) \geq g(t)$ for every $t \in [0,1]$. Hence $a \geq 0$.

Let $0 < \varepsilon < 1$. Denote by $\lambda$ the Lebesgue measure on $[0,1]$. 6
3. Opial-type properties in Cesàro spaces of vector-valued functions

The equi-integrability of finite subsets of $L^1$ enables us to find a $0 < \tau < \varepsilon$ such that for every measurable set $A \subseteq [0, 1]$ one has

$$\lambda(A) \leq \tau \Rightarrow \int_A \left( \frac{1}{t} \int_0^t F(s) \, ds \right)^p \, dt \leq \varepsilon \quad \forall F \in \{ \| f(\cdot) \|, \varphi, g \}.$$  \hfill (3.3)

Next we choose $0 < \theta < \tau$ such that

$$\frac{n\theta}{1-p} \left( \left( 1 - \frac{\tau}{3} \right) \left( 1 - \frac{\tau}{3} \right) \right)^{p-1} \left( \int_0^{1-\frac{\tau}{3}} F(s) \, ds \right)^{p-1} \leq \varepsilon \quad \hfill (3.4)$$

for $F \in \{ \varphi, g \}$ and then, again by equi-integrability, we find $\delta > 0$ such that for every measurable subset $D \subseteq [0, 1 - \frac{\tau}{3}]$ one has

$$\lambda(D) \leq \delta \Rightarrow \int_D F(s) \, ds \leq \theta \quad \forall F \in \{ \| f(\cdot) \|, \varphi, g \} \quad \hfill (3.5)$$

(remember that $\text{Ces}_p|_{[0,b]} \subseteq L^1[0,b]$ for every $b \in (0,1)$).

Now we apply Egorov’s theorem (cf. [9, Theorem A, p.88]) to find a measurable set $C \subseteq [0, 1]$ with $\lambda(\{0,1\} \setminus C) \leq \delta$ such that $\| f_n(t) \| \to g(t)$ uniformly in $t \in C$. It follows that

$$\lim_{n \to \infty} \int_{\{0,1\} \cap C} \| f_n(s) \| \, ds = \int_{\{0,1\} \cap C} g(s) \, ds \quad \forall t \in [0, 1].$$

Thus we can apply Egorov’s theorem once more to deduce that there exists a measurable set $F \subseteq [0, 1)$ with $\lambda([0,1] \setminus F) \leq \tau/3$ such that

$$\lim_{n \to \infty} \int_{\{0,1\} \cap C} \| f_n(s) \| \, ds = \int_{\{0,1\} \cap C} g(s) \, ds \quad \text{uniformly in } t \in F.$$ 

Put $B := F \cap \left[ \frac{\tau}{3}, 1 - \frac{\tau}{3} \right]$. Then $\lambda(\{0,1\} \setminus B) \leq \lambda([0,1] \setminus F) + 2\tau/3 \leq \tau$ and

$$\lim_{n \to \infty} \int_B \left( \frac{1}{t} \int_{\{0,1\} \cap C} \| f_n(s) \| \, ds \right)^p \, dt = \int_B \left( \frac{1}{t} \int_{\{0,1\} \cap C} g(s) \, ds \right)^p \, dt. \quad \hfill (3.6)$$

Since $M = \sup_{n \in \mathbb{N}} \| f_n \|_{\text{Ces}_p(X)} < \infty$, we can find a subsequence $(n_k)_{k \in \mathbb{N}}$ of indices such that all the limits involved in the following calculations exist. We have

$$\lim_{n \to \infty} \| f_n \|_{\text{Ces}_p(X)}^p = \lim_{k \to \infty} \left( \int_B \left( \frac{1}{t} \int_0^t \| f_{n_k}(s) \| \, ds \right)^p \, dt + \int_{\{0,1\} \setminus B} \left( \frac{1}{t} \int_0^t \| f_{n_k}(s) \| \, ds \right)^p \, dt \right)$$

$$\leq \sum_{k \to \infty} \left( \int_B \left( \frac{1}{t} \int_{\{0,1\} \cap C} \| f_{n_k}(s) \| \, ds \right)^p \, dt + \int_{\{0,1\} \setminus B} \left( \frac{1}{t} \int_{\{0,1\} \cap C} \| f_{n_k}(s) \| \, ds \right)^p \, dt \right)$$

$$+ \lim_{k \to \infty} \int_{\{0,1\} \setminus B} \left( \frac{1}{t} \int_0^t \| f_{n_k}(s) \| \, ds \right)^p \, dt.$$
3. Opial-type properties in Cesàro spaces of vector-valued functions

where we have used the inequality \((a + b)^p \leq 2^{p-1}(a^p + b^p)\) for \(a, b \geq 0\), which is due to the convexity of the function \(t \mapsto t^p\).

From (3.6) it now follows that

\[
\lim_{n \to \infty} \left\| f_n \right\|_{\text{Ces}^p(X)}^p \leq 2^{p-1} \left( \int_B \left( \frac{1}{t} \int_{[0,t]\cap C} g(s) \, ds \right)^p \, dt \right) + \lim_{k \to \infty} \left( \frac{1}{t} \int_0^t \left\| f_{n_k}(s) \right\| \, ds \right)^p \, dt \right) + \lim_{k \to \infty} \int_{[0,1] \setminus B} \left( \frac{1}{t} \int_0^t \left\| f_{n_k}(s) \right\| \, ds \right)^p \, dt .
\] (3.7)

Because of \(\lambda([0, 1] \setminus B) \leq \tau\) and (3.3) we have

\[
\int_{[0,1] \setminus B} \left( \frac{1}{t} \int_0^t \left\| f(s) \right\| \, ds \right)^p \, dt \leq \varepsilon .
\]

Thus by the triangle inequality for \(L^p\) we get

\[
\lim_{k \to \infty} \left( \int_{[0,1] \setminus B} \left( \frac{1}{t} \int_0^t \left\| f_{n_k}(s) \right\| \, ds \right)^p \, dt \right)^{1/p} \leq \lim_{k \to \infty} \left( \int_{[0,1] \setminus B} \left( \frac{1}{t} \int_0^t \left\| f_{n_k}(s) - f(s) \right\| \, ds \right)^p \, dt \right)^{1/p} + \varepsilon^{1/p} .
\]

It follows that

\[
\lim_{k \to \infty} \int_{[0,1] \setminus B} \left( \frac{1}{t} \int_0^t \left\| f_{n_k}(s) \right\| \, ds \right)^p \, dt \leq \lim_{k \to \infty} \int_{[0,1] \setminus B} \left( \frac{1}{t} \int_0^t \left\| f_{n_k}(s) - f(s) \right\| \, ds \right)^p \, dt
\]

\[
+ \lim_{k \to \infty} \left( \int_{[0,1] \setminus B} \left( \frac{1}{t} \int_0^t \left\| f_{n_k}(s) - f(s) \right\| \, ds \right)^p \, dt \right)^{1/p} + \varepsilon^{1/p} \right)^p
\]

\[
- \int_{[0,1] \setminus B} \left( \frac{1}{t} \int_0^t \left\| f_{n_k}(s) - f(s) \right\| \, ds \right)^p \, dt .
\]

Put \(L := M + \|f\|_{\text{Ces}^p(X)} + 1\). Since \(|a^p - b^p| \leq pL^{p-1}|a - b|\) for all \(a, b \in [0, L]\) (mean-value theorem) we obtain

\[
\lim_{k \to \infty} \int_{[0,1] \setminus B} \left( \frac{1}{t} \int_0^t \left\| f_{n_k}(s) \right\| \, ds \right)^p \, dt \leq \lim_{k \to \infty} \int_{[0,1] \setminus B} \left( \frac{1}{t} \int_0^t \left\| f_{n_k}(s) - f(s) \right\| \, ds \right)^p \, dt + pL^{p-1} \varepsilon^{1/p} .
\] (3.8)
3. Opial-type properties in Cesàro spaces of vector-valued functions

Next we define \( h(s) := (1 - p)^{-1}(s^n - s/3^{1-p}) \) for \( s \geq 0 \).

Recall that \( B \subseteq [\frac{\tau}{3}, 1 - \frac{\tau}{3}] \), \( \lambda([0, 1] \setminus C) \leq \delta \) and \( \theta < \tau \). Thus it follows from (3.5) that

\[
\int_B \left( \frac{1}{t} \int_{[0,t]} \|f(s)\| \, ds \right)^p \, dt \leq \tau^p \int_B \frac{1}{t^p} \, dt \leq \tau^p \int_{\frac{\tau}{3}}^1 \frac{1}{t^p} \, dt = h(\tau).
\]

Hence

\[
\lim_{k \to \infty} \left( \int_B \left( \frac{1}{t} \int_{[0,t]} \|f_{n_k}(s)\| \, ds \right)^p \, dt \right)^{1/p}
\leq \lim_{k \to \infty} \left( \int_B \left( \frac{1}{t} \int_{[0,t]} \|f_{n_k}(s) - f(s)\| \, ds \right)^p \, dt \right)^{1/p} + h(\tau)^{1/p}.
\]

Using the same trick as before we now obtain

\[
\lim_{k \to \infty} \int_B \left( \frac{1}{t} \int_{[0,t]} \|f_{n_k}(s)\| \, ds \right)^p \, dt
\leq \lim_{k \to \infty} \int_B \left( \frac{1}{t} \int_{[0,t]} \|f_{n_k}(s) - f(s)\| \, ds \right)^p \, dt + ph(\tau)^{1/p} A^{p-1},
\]

where \( A := M + \|f\|_{Ces^p} + K^{1/p} \) and \( K := \sup_{s \in [0,1]} h(s) \).

From (3.7) and Fatou’s Lemma it follows that

\[
\lim_{n \to \infty} \left\| f_n \right\|_{Ces^p(X)}^p
\leq 2^{p-1} \lim_{k \to \infty} \int_B \left( \frac{1}{t} \int_{[0,t] \cap C} \|f_{n_k}(s) - f(s)\| \, ds \right)^p \, dt
+ 2^{p-1} \lim_{k \to \infty} \int_B \left( \frac{1}{t} \int_{[0,t] \cap C} \|f_{n_k}(s)\| \, ds \right)^p \, dt + 2^{p-1} \int_B \left( \frac{1}{t} \int_{[0,t] \cap C} g(s) \, ds \right)^p \, dt
- 2^{p-1} \int_B \left( \frac{1}{t} \int_{[0,t] \cap C} \liminf_{k \to \infty} \|f_{n_k}(s) - f(s)\| \, ds \right)^p \, dt
+ \lim_{k \to \infty} \int_{[0,1] \setminus B} \left( \frac{1}{t} \int_0^t \|f_{n_k}(s)\| \, ds \right)^p \, dt.
\]

Combining this with (3.8) and (3.9) we obtain (by using \( x^p + y^p \leq (x + y)^p \)
3. Opial-type properties in Cesàro spaces of vector-valued functions

for \( x, y \geq 0 \)

\[
\lim_{n \to \infty} \|f_n\|_{\text{Ces}_p(X)}^p \leq 2^{p-1} \lim_{k \to \infty} \|f_n - f\|_{\text{Ces}_p(X)}^p + 2^{p-1} p\theta \tau^1 A^{p-1} + p L^{p-1} \varepsilon^{1/p} + 2^{p-1} \int_B \left( \frac{1}{t} \int_{[0,t] \cap C} g(s) \, ds \right)^p \, dt \\
- 2^{p-1} \int_B \left( \frac{1}{t} \int_{[0,t] \cap C} \liminf_{k \to \infty} \|f_n(k) - f(s)\| \, ds \right)^p \, dt,
\]

thus

\[
\lim_{n \to \infty} \|f_n\|_{\text{Ces}_p(X)}^p \leq 2^{p-1} \lim_{n \to \infty} \|f_n - f\|_{\text{Ces}_p(X)}^p + 2^{p-1} p\theta \tau^1 A^{p-1} + p L^{p-1} \varepsilon^{1/p} + 2^{p-1} \int_B \left( \frac{1}{t} \int_{[0,t] \cap C} g(s) \, ds \right)^p \, dt.
\]

(3.10)

Since \( \lambda([0, 1] \setminus C) \leq \delta \) it follows from (3.5) that for \( F \in \{g, \varphi\} \) and \( t \in (0, 1 - \frac{7}{12}) \) we have

\[
\left| \frac{1}{t} \int_0^t F(s) \, ds - \frac{1}{t} \int_{[0,t] \cap C} F(s) \, ds \right| \leq \frac{\theta}{t}
\]

and hence

\[
\left| \left( \frac{1}{t} \int_0^t F(s) \, ds \right)^p - \left( \frac{1}{t} \int_{[0,t] \cap C} F(s) \, ds \right)^p \right| \leq \frac{\theta}{t^p} \left( \frac{1}{t} \int_0^t F(s) \, ds \right)^{p-1}.
\]

Since \( B \subseteq \left[ \frac{7}{12}, 1 - \frac{7}{12} \right] \) it follows that

\[
\left| \int_B \left( \frac{1}{t} \int_0^t F(s) \, ds \right)^p \, dt - \int_B \left( \frac{1}{t} \int_{[0,t] \cap C} F(s) \, ds \right)^p \, dt \right| \\
\leq \int_B \frac{\theta}{t^p} \left( \int_0^t F(s) \, ds \right)^{p-1} \, dt \\
\leq p\theta \left( \int_0^{1-\frac{7}{12}} F(s) \, ds \right)^{p-1} \left( 1 - \frac{7}{3} \right)^{1-p} - \left( \frac{7}{3} \right)^{1-p} \right) \\
= p\theta \left( \int_0^{1-\frac{7}{12}} F(s) \, ds \right)^{p-1} \frac{1}{1-p} \left( (1-\frac{7}{3})^{1-p} - \left( \frac{7}{3} \right)^{1-p} \right).
\]

Thus it follows from (3.4) that for \( F \in \{g, \varphi\} \) one has

\[
\left| \int_B \left( \frac{1}{t} \int_0^t F(s) \, ds \right)^p \, dt - \int_B \left( \frac{1}{t} \int_{[0,t] \cap C} F(s) \, ds \right)^p \, dt \right| \leq \varepsilon.
\]

(3.11)
Since \( \lambda([0,1] \setminus B) \leq \tau \) we also have
\[
\left| \int_B \left( \frac{1}{t} \int_0^t F(s) \, ds \right)^p \, dt - \int_0^1 \left( \frac{1}{t} \int_0^t F(s) \, ds \right)^p \, dt \right| \leq \varepsilon \quad (3.12)
\]
for \( F \in \{g, \varphi\} \), by (3.3).

From (3.11) and (3.12) we obtain
\[
\left| \int_B \left( \frac{1}{t} \int_{[0,t] \cap C} F(s) \, ds \right)^p \, dt - \int_0^1 \left( \frac{1}{t} \int_0^t F(s) \, ds \right)^p \, dt \right| \leq 2\varepsilon
\]
for \( F \in \{g, \varphi\} \).

Together with (3.10) this implies
\[
\lim_{n \to \infty} \|f_n\|_{\text{Ces}_p(X)}^p \\
\leq 2^{p-1} \lim_{n \to \infty} \|f_n - f\|_{\text{Ces}_p(X)}^p + 2^{p-1} p h(\tau)^{1/p} A^{p-1} + p L^{p-1} \varepsilon^{1/p} \\
+ 2^{p-1} \int_0^1 \left( \left( \frac{1}{t} \int_0^t g(s) \, ds \right)^p - \left( \frac{1}{t} \int_0^t \varphi(s) \, ds \right)^p \right) \, dt + 2^{p-1} 4\varepsilon.
\]
Hence by definition of \( a \) we have
\[
\lim_{n \to \infty} \|f_n\|_{\text{Ces}_p(X)}^p \\
\leq 2^{p-1} \lim_{n \to \infty} \|f_n - f\|_{\text{Ces}_p(X)}^p \\
+ 2^{p-1} p h(\tau)^{1/p} A^{p-1} + p L^{p-1} \varepsilon^{1/p} - 2^{p-1} a + 2^{p-1} \varepsilon.
\]
Since \( h(\tau) \to 0 \) for \( \tau \to 0 \) and \( \tau < \varepsilon \), we obtain for \( \varepsilon \to 0 \)
\[
\lim_{n \to \infty} \|f_n\|_{\text{Ces}_p(X)}^p \\
\leq 2^{p-1} \lim_{n \to \infty} \|f_n - f\|_{\text{Ces}_p(X)}^p - 2^{p-1} a
\]
and we are done. \( \square \)

Note that the assumptions that \( X \) has the nonstrict Opial property and that \( (f_n(t))_{n \in \mathbb{N}} \) converges weakly to zero a.e. were only used to ensure that \( a \geq 0 \), which is only needed to conclude (3.2) from (3.1). In other words, (3.1) is also valid without these two assumptions.

We have the following Corollary in the case that \( X \) even has the Opial property (compare with [10, Corollary 4.2]).

**Corollary 3.2.** Let \( 1 \leq p < \infty \) and let \( X \) be a Banach space with the Opial property. Let \( (f_n)_{n \in \mathbb{N}} \) be a bounded sequence in \( \text{Ces}_p(X) \) such that \( (f_n(t))_{n \in \mathbb{N}} \) converges weakly to zero for almost every \( t \in [0,1] \). Suppose further that there exists a \( g \in \text{Ces}_p \) such that \( \|f_n(t)\| \to g(t) \) a.e. Then
\[
\limsup_{n \to \infty} \|f_n\|_{\text{Ces}_p(X)} < 2^{1-1/p} \limsup_{n \to \infty} \|f_n - f\|_{\text{Ces}_p(X)} \quad \forall f \in \text{Ces}_p(X) \setminus \{0\}.
\]
3. Opial-type properties in Cesàro spaces of vector-valued functions

Proof. Let a be defined as in the previous proof. Since X has the Opial property we have \( \varphi(t) \geq g(t) \) for every \( t \in [0, 1] \) and even “\( > \)” if \( f(t) \neq 0 \), which by assumption happens on a set of positive measure. Thus \( a > 0 \) and hence the desired inequality follows from Theorem 3.1.

Concerning the uniform Opial property, we also have the following analogue of [10, Theorem 4.3] for Cesàro function spaces (the proof also uses similar techniques).

Theorem 3.3. Let \( 1 \leq p < \infty \) and let \( X \) be a Banach space with the uniform Opial property. Let \( M, R > 0 \) and \( f \in \text{Ces}_p(X) \setminus \{0\} \). Then there exists \( \eta > 0 \) such that the following holds: whenever \( (f_n)_{n \in \mathbb{N}} \) is a sequence in \( \text{Ces}_p(X) \) with \( \sup_{n \in \mathbb{N}} \|f_n\|_{\text{Ces}_p(X)} \leq R \) such that \( (f_n(t))_{n \in \mathbb{N}} \) converges weakly to zero and \( \lim_{n \to \infty} \|f_n(t)\| \leq M \) for almost every \( t \in [0, 1] \), then

\[
\limsup_{n \to \infty} \|f_n\|_{\text{Ces}_p(X)} + \eta \leq 2^{1-1/p} \limsup_{n \to \infty} \|f_n - f\|_{\text{Ces}_p(X)}.
\]

Proof. Fix \( 0 < \tau < \|f\|_{\text{Ces}_p(X)} \) and put \( A := \{s \in [0, 1] : \|f(s)\| \geq \tau\} \). If \( \lambda(A) = 0 \), then we would obtain \( \|f\|_{C_p(X)} \leq \int_0^1 \|f\|_{C_p(X)} \, dt = \tau \). Thus we must have \( \lambda(A) > 0 \). Let \( w := \eta_X(\tau, M) \).

Define \( A_t := A \cap [0, t] \) for \( t \in [0, 1] \). Then \( \lambda(A_t) \rightarrow \lambda(A) \) for \( t \rightarrow 1 \) and hence we can find \( t_0 \in (0, 1) \) such that \( \lambda(A_{t_0}) \geq \lambda(A)/2 \) for \( t \in [t_0, 1] \).

Put \( \theta := \int_{t_0}^1 \|f\|_{C_p(X)} \, dt \) and \( \nu := \min\{(w^p\lambda(A)\theta/2)^{1/p}, 2^{1-1/p}(3R + 1)\} \).

Next we define \( \omega := 2^{1-1/p}(3R + 1) - (2^{p-1}(3R + 1)^p - \nu^p)^{1/p} \) and finally \( \eta := \min\{\omega, 1\} \).

Now let \( (f_n)_{n \in \mathbb{N}} \) be as above. Without loss of generality we may assume that \( g(t) := \lim_{n \to \infty} \|f_n(t)\| \leq M \) and \( f_n(t) \rightarrow 0 \) weakly for every \( t \in [0, 1] \).

Let \( \varphi(t) := \liminf_{n \to \infty} \|f_n(t) - f(t)\| \) for all \( t \in [0, 1] \). Then we have \( \varphi \geq g \) and the definition of \( \eta_X \) implies that even \( \varphi(s) - g(s) \geq \eta_X(\tau, M) = w \) for all \( s \in A \).

Using the relation \( (a - b)^p \leq a^p - b^p \) for \( a \geq b \geq 0 \) we obtain

\[
\left( \int_0^t \varphi(s) \, ds \right)^p - \left( \int_0^t g(s) \, ds \right)^p \geq \left( \int_0^t (\varphi(s) - g(s)) \, ds \right)^p \geq w^p \lambda(A_t)^p
\]

for every \( t \in [0, 1] \).

Theorem 3.1 now implies that

\[
2^{p-1} \limsup_{n \to \infty} \|f_n\|_{\text{Ces}_p(X)} - \limsup_{n \to \infty} \|f_n\|_{\text{Ces}_p(X)}^p \\
\geq 2^{p-1} \int_0^1 \frac{\lambda(A_t)^p}{\nu^p} \, dt \geq 2^{p-1} w^p \int_{t_0}^1 \frac{\lambda(A_t)^p}{\nu^p} \, dt \geq w^p \frac{\lambda(A)^p}{2} \theta \geq \nu^p, \quad (3.13)
\]
Proof. We also put \( \theta := \int_{t_0}^1 t^p \, dt \) and \( \nu := \min\{(\omega p^2\theta/2)^{1/p}, 2^{1-1/p}(3R+1)\} \), as well as \( \omega := 2^{1-1/p}(3R+1) - (2^{p-1}(3R+1)^p - \nu^p)^{1/p} \) and finally \( \eta := \min\{\omega, 1\} \).

Now let \( (f_n)_{n \in \mathbb{N}} \in \text{Ces}_p(X) \) and \( f \in L'(\{0, 1\}, X) \) be as above. We assume without loss of generality that \( g(t) := \lim_{n \to \infty} \|f_n(t)\| \leq M \) and \( f_n(t) \to 0 \) weakly for every \( t \in [0, 1] \).

Finally, we have the following analogue of \cite[Theorem 4.4]{10} (we denote by \( L^p([0, 1], X) \) the \( L^p \)-Bochner space).

**Theorem 3.4.** Let \( 1 < p < \infty \) and let \( X \) be a Banach space with the uniform Opial property. Let \( p < r \leq \infty \) and \( \varepsilon, M, K, R > 0 \). Then there exists \( \eta > 0 \) such that the following holds: whenever \( (f_n)_{n \in \mathbb{N}} \) is a sequence in \( \text{Ces}_p(X) \) with \( \sup_{n \in \mathbb{N}} \|f_n\|_{\text{Ces}_p(X)} \leq R \) such that \( (f_n(t))_{n \in \mathbb{N}} \) converges weakly to zero and \( \lim_{n \to \infty} \|f_n(t)\| \leq M \) for almost every \( t \in [0, 1] \) and \( f \in L'(\{0, 1\}, X) \subseteq L^p([0, 1], X) \subseteq \text{Ces}_p(X) \) is such that \( \|f\|_r \leq K \) and \( \|f\|_{\text{Ces}_p(X)} \geq \varepsilon \), then

\[
\limsup_{n \to \infty} \|f_n\|_{\text{Ces}_p(X)} + \eta \leq 2^{1-1/p} \limsup_{n \to \infty} \|f_n - f\|_{\text{Ces}_p(X)}.
\]
Let $A := \{ s \in [0,1] : \| f(s) \| \geq \tau \}$. Since $\varepsilon \leq \| f \|_{\text{Ces}_p(X)} \leq q \| f \|_p$ (see (4) on page 2) we can proceed analogously to the proof of [10, Theorem 4.4] to show that $\lambda(A) \geq Q$. Let $A_t := A \cap [0,t]$ for $t \in [0,1]$. We have $\lambda(A) - \lambda(A_{t_0}) = \lambda(A \cap [t_0,1]) \leq 1 - t_0 = Q/2$ and hence $\lambda(A_t) \geq \lambda(A_{t_0}) \geq Q/2$ for $t \in [t_0,1]$.

As in the previous proof we can now use Theorem 3.1 to conclude

$$2^{p-1} \limsup_{n \to \infty} \| f_n - f \|_{\text{Ces}_p(X)}^p - \limsup_{n \to \infty} \| f_n \|_{\text{Ces}_p(X)}^p \geq \nu^p$$

and from this obtain, also as in the previous proof, that

$$\limsup_{n \to \infty} \| f_n \|_{\text{Ces}_p(X)} \leq 2^{1-1/p} \limsup_{n \to \infty} \| f_n - f \|_{\text{Ces}_p(X)} - \eta.$$
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