Stability of the Yang-Mills heat equation for finite action *

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November 2, 2017

Abstract

The existence and uniqueness of solutions to the Yang-Mills heat equation over domains in \( \mathbb{R}^3 \) was proven in a previous paper for initial data lying in the Sobolev space of order one-half, which is the critical Sobolev index for this equation. In the present paper the stability of these solutions will be established. The variational equation, which is only weakly parabolic, and has highly singular coefficients, will be shown to have unique strong solutions up to addition of a vertical solution. Initial data will be taken to be in Sobolev class one-half. The proof relies on an infinitesimal version of the ZDS procedure: one solves first an augmented, strictly parabolic version of the variational equation and then adds to the solution a function which is vertical along the original path. Energy inequalities and Neumann domination techniques will be used to establish apriori initial behavior for solutions.

*Key words and phrases. Yang-Mills, heat equation, weakly parabolic, variational equation, gauge groups, Gaffney-Friedrichs inequality, Neumann domination.

2010 Mathematics Subject Classification. Primary; 35K58, 35K65, Secondary; 70S15, 35K51, 58J35.
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1 Introduction

Denote by $K$ a compact Lie group with Lie algebra $\mathfrak{k}$. Let

$$A(t, x) \equiv \sum_{j=1}^{3} A_j(t, x) dx^j$$

be a $\mathfrak{k}$ valued 1-form on $\mathbb{R}^3$ for each $t \geq 0$. Its curvature is given at time $t$ by $B(t) = dA(t) + A(t) \wedge A(t)$. Here $d$ denotes the spatial exterior derivative. The Yang-Mills heat equation is the weakly parabolic non-linear equation given by

$$\partial A/\partial t = -d^*_A B(t), \quad t > 0,$$

where $d^*_A$ denotes the gauge covariant exterior co-derivative. Ignoring the terms on the right side of (1.2) which are quadratic and cubic in $A(t)$, one finds the linear expression $-d^* d A$. Since $d^* d$ is only a portion of the Laplacian, $-\Delta = d^* d + dd^*$, on 1-forms, the equation (1.2) is only weakly parabolic.

The existence and uniqueness of solutions to (1.2) has already been investigated in [2, 3] and [6] for the initial value problem over $\mathbb{R}^3$ as well as for the initial-boundary value problem over a bounded convex region in $\mathbb{R}^3$. In the following, $M$ will denote either all of $\mathbb{R}^3$ or the closure of a convex bounded open subset of $\mathbb{R}^3$ with smooth boundary. It was shown in [6] that for any connection form $A_0 \in H_{1/2}(M; \Lambda^1 \otimes \mathfrak{k})$ the equation (1.2) has a solution with initial value $A_0$. It was also shown that the degree of regularity of the solution depends not only on the regularity of $A_0$ but also on some algebraic features of $A_0$ which are intimately connected to the gauge invariance of the equation. The key distinction in regularity properties of solutions is best understood from the following notion of a strong solution.

By a strong solution of (1.2) over an interval $(0, T)$ we mean a function $A : (0, T) \to \{\mathfrak{k} \text{ valued 1-forms on } M\}$ such that, for each point $t \in (0, T)$, there holds

a) $A(t)$ is in $H_1(M; \Lambda^1 \otimes \mathfrak{k})$

b) $B(t)$ is in $H_1(M; \Lambda^2 \otimes \mathfrak{k})$

c) Equation (1.2) holds.

In addition, some continuity properties as a function of $t$ are assumed. Condition a) allows one to define $B(t)$ while condition b) allows one to give meaning
to the right side of Equation (1.2). But it can easily happen that condition a) fails while condition b) holds. In this case we refer to the solution as an almost strong solution. Of course one must interpret the derivative that occurs in $B(t)$ as a weak derivative in this case. For example, if $K$ is the circle group and we identify its Lie algebra with $i\mathbb{R}$, then $\sqrt{1-t}A$ is real valued, and if $A_0$ is exact, say $A_0 = \sqrt{-1}d\lambda$ for some real valued function $\lambda$ on $M$, then the function $A(t) \equiv A_0$ is a solution to (1.2) because, in this commutative case, the curvature is simply given by $B(t) = dA(t)$, which is $\sqrt{-1}d^2\lambda$ and which is zero in a weak sense, no matter how irregular $A_0$ is. Thus in this example condition a) can fail even though condition b) always holds. For a general compact Lie group $K$ the same phenomenon occurs: Let $g : M \to K$ be a function and let $A_0 = g^{-1}dg$. Then the (weak) curvature of $A_0$ is zero and the function $A(t) \equiv A_0$ is a solution in some sense to (1.2), no matter how irregular $A_0$ is. In this case the flow of the equation does not smooth the initial value $A_0$. For a general connection form $A_0 \in H_{1/2}(M; \Lambda^1 \otimes \mathfrak{k})$ it appears to be impossible to separate out a part which propagates without smoothing, as in this example, from a part which is smoothed by the equation, without destroying gauge invariance of the initial value problem (1.2). It was shown in [6] that for any initial connection form $A_0 \in H_{1/2}(M; \Lambda^1 \otimes \mathfrak{k})$, the equation (1.2) always has a solution in a generalized sense. The solution may not have the regularity required by condition a) but does have the regularity required by condition b). It was also shown that there exists a gauge function $g$ such that the gauge transform

$$A_0^g \equiv g^{-1}A_0g + g^{-1}dg$$

(1.3)

is indeed the initial value of a strong solution. The main result of [6] may thus be stated succinctly as “any connection form $A_0 \in H_{1/2}(M; \Lambda^1 \otimes \mathfrak{k})$ is, after gauge transformation, the initial value of a strong solution”. Uniqueness also holds when properly formulated.

The goal of the present paper is to prove the analogous theorem for the variational equation. Along a solution $A(\cdot)$ to (1.2), the variational equation is given by

$$-\partial v(t)/\partial t = d_A^*d_Av + [v \lhd B].$$

(1.4)

Here $v(t)$ is, for each $t \geq 0$, a $\mathfrak{k}$ valued 1-form on $M$. The last term represents an interior product. The $t$ dependence in $A(t)$, $B(t)$ and $v(t)$ on the right is suppressed. The second order derivative terms that appear on the right side of (1.4) are $d^*dv$, and consequently the variational equation is only weakly
parabolic, as was (1.2). The solutions $A(t, x)$ to (1.2) that are of interest to us have a strong singularity at $t = 0$. Therefore the linear equation (1.4), in addition to being only weakly parabolic, has highly singular coefficients.

There are functions in the initial data space for the variational equation which are not smoothed by the flow of the equation, but they can be singled out in a gauge invariant manner, unlike the circumstance for (1.2). They are the vertical vectors at $A_0$, i.e. the tangent vectors to the orbit of the gauge group through $A_0$. Such a vector can be represented as $v_1 = d_{A_0} \alpha$ for some $\mathfrak{g}$ valued function $\alpha$ on $M$. The solution to (1.4) with this initial value is simply given by $v(t) = d_{A(t)} \alpha$ and experiences no smoothing under the flow. The main theorem of this paper asserts that, for any element $v_0 \in H^{1/2}(M; \Lambda^1 \otimes \mathfrak{g})$, there is a generalized solution to (1.4) with initial value $v_0$ and in addition, there is also a strong solution $v(t)$ such that

$$\lim_{t \downarrow 0} (v(t) - v_0) \text{ is vertical at } A_0. \quad (1.5)$$

In short, we will prove that any vector $v_0 \in H^{1/2}$ is the initial value of a strong solution modulo vertical vectors. This is the precise infinitesimal analog of the main theorem of [6], which asserts existence of strong solutions to (1.2) modulo gauge transformations. In the case that $M \neq \mathbb{R}^3$, boundary conditions must be imposed on the solution $v(t)$ for $t > 0$ in the discussion above. These will be discussed in Section 3.

The existence proof for the Yang-Mills heat equation itself, given in [2] and [6], relied on a method that goes back in one form or another to Zwanziger [20], Donaldson [5] and Sadun [17]. To prove existence of solutions to the variational equation we are going to use an infinitesimal version of the ZDS procedure. The infinitesimal version of the ZDS procedure introduced in this paper has proven to be advantageous over other methods that naturally present themselves for the problems at hand: In the infinite dimensional manifold of connection forms over $\mathbb{R}^3$ (with some Sobolev restrictions) the tangent space at a point $A$ decomposes into vertical and horizontal subspaces in a gauge invariant way. The vertical vectors play a distinguished role, as already noted. Moreover the horizontal component of any solution to (1.4) propagates by a strictly parabolic equation. Consequently, techniques that rely on projection into these two subspaces can be expected to be useful. But the use of these projections entails use of the Green functions for gauge covariant Laplacians under Dirichlet or Neumann boundary conditions. There does not appear to be a useful Poincaré inequality for these Laplacians in
the case of Neumann boundary conditions. Consequently, useful bounds on their Green operators are hard to get in this important case. Even for a compact manifold without boundary Green operators exist only for irreducible connections. This class of connections is open in a Sobolev topology $H_k$ for sufficiently large $k$, and has been used in the works [1, 13], for example, in their analysis of the quotient space

\[ \mathcal{Y} = \left\{ \text{connection forms} \right\} / \text{gauge group} \]  

(1.6)

and in [14]. But we wish to deal with connections in Sobolev class $1/2$, where restriction to irreducible connections is not feasible. Fortuitously, the infinitesimal version of the ZDS procedure circumvents this problem.

For a solution $A(\cdot)$ to the Yang-Mills heat equation define

\[ \rho(A) := \int_0^1 s^{-1/2} \| B(s) \|^2_2 ds. \]  

(1.7)

This is a gauge invariant functional of the initial data. It plays a fundamental role, both technically and conceptually, because it captures in a gauge invariant manner the notion of $H_{1/2}$ initial data. To understand how this happens, it is illuminating to compute its value when the gauge group $K$ is the circle group. One finds in this case that it reduces to the $H_{1/2}$ norm of the initial value $A_0$ when $A_0$ is in Coulomb gauge, i.e., $d^\ast A_0 = 0$. As is well known, the space \( \{ A_0 : d^\ast A_0 = 0 \} \) constitutes a section of the bundle $\mathcal{Y}$ when $K$ is the circle group. This example is discussed further in Remark [2, 21]. If $K$ is not commutative this method of identifying solutions modulo gauge transformations with some section for the quotient space $\mathcal{Y}$ does not play well. It is well understood that if $K$ is not commutative there is no good analog for the Coulomb gauge. Problems associated with the Gribov ambiguity enter [19, 14, 20]. But it is the quotient space that plays the role of the configuration space for the classical Yang-Mills field. Our objective, when $K$ is not commutative, is to make the quotient space into a complete, infinite dimensional, Riemannian manifold, which in some suitable sense consists of $H_{1/2}$ connection forms on $\mathbb{R}^3$ modulo the corresponding gauge group. This will be carried out in [7]. The functional $\rho(A)$ will play a fundamental role in this procedure by determining, in a gauge invariant way, which initial data are to be regarded as being “in” $H_{1/2}$ when $K$ is not commutative. Our use of the term “action” for $\rho(A)$ is motivated by the fact that if $A_0 \in H_{1/2}(\mathbb{R}^3)$ then it has an extension to a slab in Minkowski space which makes a finite contribution to the magnetic component of the Lagrangian.
2 Statement of results

2.1 Strong and almost strong solutions

$M$ will denote either $\mathbb{R}^3$ or the closure of a bounded convex open subset of $\mathbb{R}^3$ with smooth boundary. $K$ will denote a compact Lie group with Lie algebra $\mathfrak{k}$. We will always take $K$ to be a subgroup of the orthogonal resp. unitary group of a finite dimensional real resp. complex inner product space $\mathcal{V}$. We can identify $\mathfrak{k}$ with a real subspace of $\text{End} \mathcal{V}$. We denote by $\langle \cdot, \cdot \rangle$ an $\text{Ad} K$ invariant inner product on $\mathfrak{k}$. The induced norm on $\mathfrak{k}$ is equivalent to the operator norm of $\mathfrak{k} \subset \text{End} \mathcal{V}$ since $\mathfrak{k}$ is finite dimensional.

We will assume as given a time dependent, $\mathfrak{k}$ valued 1-form $A(t)$ on $M$:

$$A(t)(x) = \sum_{j=1}^{3} A_j(x, t) \, dx^j,$$

where each $A_j$ is a $\mathfrak{k}$ valued function on $M \times [0, \infty)$. $W_1(M; \Lambda^p \otimes \mathfrak{k})$ will denote the set of those $p$-forms in $L^2(M)$ whose weak first derivatives are in $L^2(M)$. We will usually write $W_1$ when the order, $p$, is clear from the context. $H_1(M; \Lambda^p \otimes \mathfrak{k})$ will denote the set of $\mathfrak{k}$-valued $p$-forms in $W_1$ which satisfy the boundary conditions specified in Notation 3.1.

If $M \neq \mathbb{R}^3$ then $A(t)$ will always be assumed to satisfy the boundary conditions $A(t)\text{norm} = 0$ when Neumann boundary conditions are under discussion and $A(t)\text{tan} = 0$ when Dirichlet boundary conditions are under discussion. We will write $A(t) \in H_1(M)$ in all three cases. Its $H_1$ norm is given by

$$\|A(t)\|_{H_1}^2 = \int_M \left( \sum_{j=1}^{3} |\partial_j A(t, x)|^2_{\Lambda^1 \otimes \mathfrak{k}} + |A(t, x)|^2_{\Lambda^1 \otimes \mathfrak{k}} \right) dx. \quad (2.1)$$

**Definition 2.1** A **strong solution** to the Yang-Mills heat equation over $(0, \infty)$ is a continuous function

$$A(\cdot) : (0, \infty) \to L^2(M; \Lambda^1 \otimes \mathfrak{k}) \subset \{ \mathfrak{k}-valued \, 1\text{-forms on } M \} \quad (2.2)$$

such that

a) $A(t) \in H_1$ for all $t \in (0, \infty)$ and $A(\cdot) : (0, \infty) \to H_1$ is continuous,

b) $B(t) := dA(t) + A(t) \wedge A(t) \in H_1$ for each $t \in (0, \infty)$,

c) the strong $L^2(M)$ derivative $A'(t) \equiv dA(t)/dt$ exists on $(0, \infty)$, and

$A'(\cdot) : (0, \infty) \to L^2(M)$ is continuous,

d) $A'(t) = -d_{A(t)}^*B(t)$ for each $t \in (0, \infty)$. \quad (2.3)
In [6] it was proven that for some connection forms $A_0$ in $H_{1/2}(M; \Lambda^1 \otimes \mathfrak{g})$ there is a strong solution $A(\cdot)$ to (2.3) over $(0, \infty)$ which converges to $A_0$ in $H_{1/2}$ as $t \downarrow 0$. Here $H_{1/2}$ refers to the Sobolev norm that interpolates between $L^2$ and the $H_1$ norm given in (2.1). $A(\cdot)$ extends to a continuous function on $[0, \infty)$ into $H_{1/2}$ and therefore into $L^3(M)$ by Sobolev. The initial values $A_0$ which are permitted in this theorem include, up to gauge transformation, all connection forms in $H_{1/2}$. In this paper we will make use only of the properties listed in Definition 2.1 and such further properties as are explicitly spelled out. In particular $A(0)$ need not be in $L^3$ in most of this paper.

We will be concerned with existence, uniqueness and properties of solutions to the variational equation (1.4) along such a strong solution to (2.3). The spatial derivatives of $A(t)$ enter into the coefficients of the variational equation and can have bad singularities near $t = 0$. The existence of solutions to the variational equation is jeopardized by these singularities. The singular nature of this initial behavior of $A(\cdot)$ was studied in [6] and much of the information derived there will be needed in this paper.

The initial behavior of a strong solution to the Yang-Mills heat equation is deducible in large part from the following gauge invariant condition, which will often be a key hypothesis.

**Definition 2.2** Let $1/2 \leq a < 1$. A strong solution to the Yang-Mills equation (2.3) over $(0, \infty)$ has finite $a$-action if

$$
\rho_A(t) \equiv (1/2) \int_0^t s^{-a} \|B(s)\|_{L^2(M)}^2 ds < \infty
$$

for some $t > 0$ (and therefore for all $t < \infty$ because $s \mapsto \|B(s)\|_2$ is non-increasing). In the important case $a = 1/2$ we will simply say that $A$ has finite action.

**Notation 2.3** In addition to the gauge invariant condition (2.4) we will also need the following gauge invariant condition on $A(\cdot)$. For each $s \in [0, \infty)$ the function

$$[0, \infty) \ni t \mapsto A(t) - A(s) \text{ is continuous into } L^3(M; \Lambda^1 \otimes \mathfrak{g}).$$

(2.5)

This is strictly weaker than the assumption that $A(\cdot)$ is continuous as a function from $[0, \infty)$ into $L^3$ because (2.5) can hold even if $A(t) \notin L^3(M)$ for
any \( t \geq 0 \). This is a relevant issue only in case \( M = \mathbb{R}^3 \). Although continuity of \( A(\cdot) \) into \( L^3 \) was proved in \([6]\) under the condition that the initial value \( A_0 \) is in \( H_{1/2}(\mathbb{R}^3) \), we will remove this restriction on the initial data in \([7]\) in order to incorporate instanton sections into these structures. Only \( (2.3) \) will survive. In this paper two results will require that \( A(t) \in L^3 \) for some \( t > 0 \) (namely in Sections \( 7.3 \) and \( 7.4 \)) and this condition will be made explicit where used. All other results are independent of these.

**Notation 2.4** We continue to use the notation from \([2]\) for the exterior and interior commutator products, given by \( [u \wedge v] = \sum_{I,J} [u_I, v_J] dx^I \wedge dx^J \) when \( u \equiv \sum_i u_i dx^i \) and \( v \equiv \sum_j v_j dx^j \) are \( \text{End} \mathcal{V} \) valued forms, while \( \langle w, [u \cdot v] \rangle_{\Lambda^p \otimes \mathfrak{k}} = \langle [u \wedge w], v \rangle_{\Lambda^{p+r} \otimes \mathfrak{k}} \) for all \( w \in \Lambda^p \otimes \mathfrak{k} \) when degree \( u = p \) and degree \( v = p + r \). Then \( d_A u = du + [A \wedge u] \) and \( d_A^* u = d^* u + [A \cdot u] \).

**Definition 2.5** The variational equation for the Yang-Mills heat equation \( (2.3) \) is

\[-v'(s) = d^*_A(s) d_A(s) v(s) + [v(s) \cdot B(s)]. \tag{2.6}\]

**Notation 2.6** (Gauge invariant Sobolev norms) Although there is no gauge invariant Sobolev norm for a gauge potential \( A \), there are gauge invariant Sobolev norms for variations of \( A \). For any connection form \( A \) over \( M \) that lies in \( W_1(M) \) define

\[
\partial^A_j \omega = \partial_j \omega + [A_j, \omega] = (\partial_j + \text{ad} A_j) \omega \tag{2.7}
\]

for a \( \mathfrak{k} \) valued \( p \)-form \( \omega \). Ignoring boundary conditions for the moment we define

\[
\| \omega \|^2_{H^1_A} = \int_M \left( |\partial^A_j \omega(x)|^2_{\Lambda^1 \otimes \mathfrak{k}} + |\omega(x)|^2_{\Lambda^1 \otimes \mathfrak{k}} \right) dx. \tag{2.8}
\]

This is the gauge invariant \( H_1 \) norm on forms that we will use in most of this paper. The corresponding \( H_b \) norms are given by

\[
\| \omega \|^b_{H^b_A} \equiv \| \omega \|^b_{H^b(M)} = \|(1 - \Delta_A)^{b/2} \omega\|_{L^2(M)}, \quad b \geq 0, \tag{2.9}
\]

where \( \Delta_A \) denotes the Bochner Laplacian on \( \mathfrak{k} \)-valued 1-forms over \( M \). The precise domain of this gauge covariant operator will be explained in Notation \( 3.1 \). These norms are gauge invariant in the sense that

\[
\| \omega^g \|^2_{H^1_A} = \| \omega \|^2_{H^1_A} \tag{2.10}
\]

for any sufficiently regular gauge function \( g : M \to K \). Here \( A^g \) is defined in \([1.3]\) and \( \omega^g = g^{-1} \omega g \).
Given a strong solution $A(\cdot)$ to the Yang-Mills heat equation and a number $T \in (0, \infty)$ we will write $A = A(T)$ and use this connection form to define gauge invariant Sobolev norms on forms, as in (2.8). We will see in Lemma 7.6 that these Sobolev norms are equivalent for different $T$. But in Section 3 we will make a choice of $T$ that is well adapted for use in the contraction principle.

**Definition 2.7** A strong solution to the variational equation along $A(\cdot)$ over $[0, \infty)$ is a continuous function

$$v : [0, \infty) \to L^2(M; \Lambda^1 \otimes \mathfrak{k}) \quad (2.11)$$

such that

a) $v(t) \in H^A_1$ for all $t \in (0, \infty)$ and $v(\cdot) : (0, \infty) \to H^A_1$ is continuous, \hfill (2.12)

b) $d_{A(t)}v(t) \in H^A_1$ for each $t \in (0, \infty)$, \hfill (2.13)

c) the strong $L^2(M)$ derivative $v'(t) \equiv dv(t)/dt$ exists on $(0, \infty)$, \hfill (2.14)

d) The variational equation (2.6) holds on $(0, \infty)$. \hfill (2.15)

A function $v(\cdot)$ satisfying all of the preceding conditions except a) will be called an almost strong solution. In this case the spatial exterior derivative $dv(t)$, which enters into the definition of $d_{A(t)}v(t)$, must be interpreted as a weak derivative. It can happen that for some $t > 0$, the weak exterior derivative $d_{A(t)}v(t)$ is in $W^1$, as required by (2.13), even though $v(t)$ is not in $W^1(M)$. This is, typically, a manifestation of the identity $d^2\lambda = 0$, which holds in a generalized sense even if $d\lambda$ is not in $W^1$. This was already pointed out in the introduction.

**Definition 2.8** (Vertical solutions) A vertical solution to the variational equation along $A(\cdot)$ is a function of the form

$$z(t) = d_{A(t)}\alpha, \quad 0 < t < \infty \quad (2.16)$$

for some element $\alpha \in H^A_1(M; \mathfrak{k})$. Recall the standard terminology: A $\mathfrak{k}$-valued 1- form $\omega \in L^2(M; \Lambda^1 \otimes \mathfrak{k})$ is horizontal at a connection form $A$ if $d^*_A\omega = 0$. The horizontal 1-forms at $A$ form a closed subspace of $L^2(M; \Lambda^1 \otimes \mathfrak{k})$ because $d^*_A$ is a closed operator on $L^2$. For each $t$ the 1-form $d_{A(t)}\alpha$ is clearly orthogonal to the horizontal subspace at $A(t)$.  

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Lemma 2.9 (Vertical solutions) Let $A(\cdot)$ be a strong solution to (2.3) over $(0, \infty)$ of finite action and satisfying (2.5). Let $\alpha \in H^A_1(M; \mathfrak{k})$. Define $z(t)$ as in (2.16). Then $z(\cdot)$ is an almost strong solution to (2.6). It is a strong solution if and only if $d_{A(t_0)}\alpha \in H^A_1$ for some $t_0 > 0$.

The proof will be given in Section 7.1.

Theorem 2.10 Assume that $1/2 \leq a < 1$ and $1/2 \leq b < 1$. Let $A(\cdot)$ be a strong solution to the Yang-Mills heat equation over $(0, \infty)$ with finite action and satisfying (2.3). Let $v_0 \in H^A_b(M; \Lambda^1 \otimes \mathfrak{k})$. Then

1. There exists an almost strong solution $v(\cdot)$ to the variational equation (2.6) over $[0, \infty)$ with initial value $v_0$.
2. For each real number $\tau > 0$ there exists a vertical almost strong solution $d_{A(t)}\alpha_\tau$ such that the function

$$v_\tau(t) \equiv v(t) - d_{A(t)}\alpha_\tau, \quad t \geq 0$$

(2.17)

is a strong solution to the variational equation with initial value $v_0 - d_{A(0)}\alpha_\tau$. Moreover

$$\sup_{0 \leq t \leq 1} \|v(t) - v_\tau(t)\|_2 \to 0 \quad \text{as} \quad \tau \downarrow 0.$$  

(2.18)

3. If $\|A(t)\|_{L^1(M)} < \infty$ for some $t > 0$ then

$$v : [0, \infty) \to H^A_b$$

(2.19)

is continuous.

4. Strong solutions are unique when they exist.

The proof will be given in Section 8.

Remark 2.11 Theorem 2.10 is the precise infinitesimal analog of the main theorem of [6] since a vertical vector is the infinitesimal analog of a gauge transformation.

Remark 2.12 The assertion (2.19) shows that the almost strong solution $v$ is continuous at $t = 0$ as a function into $H^A_b$. One should expect that the strong solution $v_\tau$ is continuous at $t = 0$ as a function into $H^A_b$ also and not just into $L^2$. But my techniques fail to produce this result in the doubly critical case $a = 1/2, b = 1/2$. The issue may be a conceptual one, related to
the nature of the gauge group associated to $a = 1/2$, rather than a matter of technique. The critical gauge group $G_{3/2}$ just barely fails to be a Hilbert manifold. See for example [6, Remark 5.21] for a discussion of the breakdown of smoothness of this gauge group, which is associated to initial data in $H_{1/2}^A$. This remark applies also to Theorem 2.15. We will actually prove that for $1/2 \leq b < 1$ the strong solution $v_\tau$ is a continuous function on $[0, \infty)$ into $L^\rho(M; \Lambda^1 \otimes \mathfrak{t})$ for $2 \leq \rho < 3$. But for $b = 1/2$ the expected continuity of $v_\tau$ into $H_{1/2}^A$ implies continuity into $L^3$, by Sobolev, which we fail to achieve. Continuity into $L^\rho$ will be proved in Lemma 7.2 along with a strengthening of (2.18) to allow Lebesgue power $\rho$ with $2 \leq \rho < 3$.

**Remark 2.13** (The horizontal component) The failure of $v_\tau(t)$ to be continuous into $H_{b}^A$ at $t = 0$ is entirely due to the poor behavior of the vertical component of $v_\tau(t)$. Even if $v_0$ is horizontal at $A_0$, the solution $v(t)$ rapidly acquires a large vertical component. By contrast, the horizontal component of $v(t)$ is well behaved. Let $\bar{v}(t)$ denote the horizontal component of $v(t)$ at $A(t)$. Then $\bar{v}(t)$ satisfies its own differential equation, independent of the vertical component. Moreover it relates to the variation of the action function $\rho_A(t)$ very well. This will be developed in a sequel to this paper, [7].

### 2.1.1 Solutions of finite action

**Definition 2.14** ($b$-action) Let $0 \leq b < 1$. A solution $v$ to the variational equation (2.6) has **finite strong $b$-action** if, for some number $\tau > 0$, there holds

$$\int_0^\tau s^{-b} \left( \| \nabla^A v(s) \|^2_{L^2(M)} + \| v(s) \|^2_{L^2(M)} \right) ds < \infty. \tag{2.20}$$

The integrand is gauge invariant.

**Theorem 2.15** Assume that $1/2 \leq a < 1$ and $1/2 \leq b < 1$ and that max$(a,b) > 1/2$. Let $c = \min(a,b)$. Let $A(\cdot)$ be a strong solution to the Yang-Mills heat equation over $(0, \infty)$ with finite $a$-action such that $\| A \|^2_{L^2(M)} < \infty$. Let $v_0 \in H_0^A(M; \Lambda^1 \otimes \mathfrak{t})$, wherein either $M = \mathbb{R}^3$ or else Neumann boundary conditions (3.4) hold. Then the strong solution $v_\tau$, constructed in Theorem 2.10 for $\tau > 0$, has finite strong $c$-action in the sense of Definition 2.14.

This will be proved in Sections 7.4 and 8.

See Remark 2.12 for a discussion of the failure of Theorem 2.15 in the doubly critical case $a = b = 1/2$. 

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Remark 2.16 Theorem 2.15 is the only theorem in this paper in which Dirichlet boundary conditions fail to be encompassed by our techniques.

2.2 The infinitesimal ZDS procedure

Remark 2.17 The Zwanziger, Donaldson, Sadun \cite{20, 5, 17} (ZDS) method for proving existence of solutions to the Yang-Mills heat equation consists in modifying the equation so as to make it strictly parabolic and then recovering a solution to the Yang-Mills heat equation itself from the solution to the modified equation by making a time dependent gauge transformation. See \cite{2} or \cite{6} for a more detailed description. In this paper we are going to use an infinitesimal version of the ZDS procedure. To this end we first modify the variational equation (2.6) by adding a term that makes it strictly parabolic.

Definition 2.18 The augmented variational equation for a time dependent $k$ valued 1-form $w(t)$ over $M$ is

$$-w'(t) = (d^*A + dA^*)w + [w \lrcorner B].$$

(2.21)

Here $A(\cdot)$ is a solution to (2.3). The time dependence of $A, B$ and $w$ on the right side is suppressed.

Theorem 2.19 Suppose that $A(\cdot)$ is a strong solution to the Yang-Mills heat equation over $[0, \infty)$ of finite action. Let $0 < b < 1$. Assume that $v_0 \in H^A_b(M; \Lambda^1 \otimes \mathfrak{k})$. Then there exists a continuous function $w : [0, \infty) \rightarrow H^A_b$

such that $w(0) = v_0$ and

a) $w$ is a strong solution to the augmented variational equation \((2.21)\) over $[0, \infty)$ satisfying the boundary conditions \((3.3)\) resp. \((3.4)\) in case $M \neq \mathbb{R}^3$,

b) $t^{1-b}\|w(t)\|_{H^A_b}^2 \rightarrow 0$ as $t \downarrow 0$.

The solution is unique under the preceding conditions. Moreover

c) $\int_0^T s^{-b}\|w(s)\|_{H^A_b}^2 ds \leq \gamma_T\|v_0\|_{H^A_b}^2$  

(2.22)

for all $T \geq 0$ and for some constant $\gamma_T$ depending only on $T$ and $\rho_A(T)$.  

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Parts a) and b) of this theorem will be proven in Section 3. Part c) will be proven in Section 4.

In the infinitesimal ZDS procedure we recover the solution \( v \) to (2.6) by simply adding on to \( w \) an appropriate vertical correction as follows.

**Theorem 2.20** (Recovery theorem) Assume that \( 1/2 \leq a < 1 \) and \( 1/2 \leq b < 1 \). Let \( A \) be a solution to the YM heat equation (2.3) over \([0, \infty)\) with finite a-action. Let \( v_0 \in H_b^A(M) \). Let \( w(s) \) be the strong solution to the augmented variational equation (2.21) over \([0, \infty)\) with initial value \( v_0 \), satisfying the conclusions a) and b) of Theorem 2.19. Define

\[
v(t) = w(t) + d_{A(t)} \int_0^t d_{A(s)}^* w(s) ds \quad \text{for } 0 \leq t < \infty.
\]

(2.23)

Then \( v(\cdot) \) is an almost strong solution to the variational equation with initial value \( v_0 \).

Let \( \tau > 0 \) and define

\[
v_\tau(t) = w(t) + d_{A(t)} \int_\tau^t d_{A(s)}^* w(s) ds \quad \text{for } 0 < t < \infty.
\]

(2.24)

Then \( v_\tau \) is a strong solution to the variational equation over \((0, \infty)\). Let

\[
\alpha_\tau = \int_0^\tau d_{A(s)}^* w(s) ds.
\]

(2.25)

Then \( \alpha_\tau \in H_1^A \) and the function

\[
t \mapsto d_{A(t)} \alpha_\tau, \quad 0 < t < \infty
\]

(2.26)

is an almost strong vertical solution to the variational equation. Moreover

\[
v_\tau(t) = v(t) - d_{A(t)} \alpha_\tau \quad \text{for } 0 < t < \infty \quad \text{and}
\]

\[
v_\tau(t) \text{ converges to } v_0 - d_{A_0} \alpha_\tau \quad \text{in } L^2(M) \quad \text{as } t \downarrow 0.
\]

(2.27)

(2.28)

If \( \|A\|_3 < \infty \) then

\[
v : [0, \infty) \to H_b^A(M; \Lambda^1 \otimes \mathfrak{g})
\]

(2.29)

is continuous. In particular, \( \|v(t) - v_0\|_{H_b^A} \to 0 \) as \( t \downarrow 0 \).
We will refer to the second term on the right in (2.23) as the vertical correction to the solution \( w \) of the augmented variational equation (2.21). It is not by itself a solution to the variational equation because \( d A(t) \) is applied to a time dependent function.

The proof of Theorem 2.20 will be given in Section 8.

Remark 2.21 (Significance of \( \rho_A(\tau) \)) The functional \( \rho_A(\tau) \) defined in (2.4) is a gauge invariant function of the initial data \( A_0 \) for the Yang-Mills heat equation for each \( \tau > 0 \). It therefore descends to a function on the quotient space \( \mathcal{Y} \), heuristically defined in (1.6). Its significance can be understood by computing its value in case \( K = S^1 \) when \( A_0 \) lies in the section of this bundle corresponding to the Coulomb gauge. In this case, after multiplying by \( \sqrt{-1} \), we can take \( A(t) \) to be a real valued 1-form on \( \mathbb{R}^3 \) for each \( t \geq 0 \). Since the magnetic field is now given by \( B(t) = dA(t) \), the Yang-Mills heat equation reduces to the Maxwell heat equation \( \partial A(t)/\partial t = -d^*dA(t) \). The identity \((\partial/\partial t)d^*A(t) = d^*(\partial A(t)/\partial t) = -d'(d^*dA(t)) = 0\) shows that if the initial data \( A_0 \) in the Coulomb gauge, i.e., \( d^*A_0 = 0 \), then so is \( A(t) \). The Coulomb gauge space is therefore invariant under the Maxwell heat flow. Moreover the Maxwell heat equation reduces to \( \partial A(t)/\partial t = \Delta A(t) \) for functions \( A(t) \) in the Coulomb gauge because \(-\Delta A(t) = (d^*d + dd^*)A(t) = d^*dA(t) \). Hence if \( A_0 \) is in Coulomb gauge then the solution to the Maxwell heat equation is simply given by \( A(t) = e^{t\Delta}A_0 \). We can compute \( \rho_A(\tau) \) easily in this case: Since \( d^*A(t) = 0 \), we have \( \| B(t) \|_2^2 = \| dA(t) \|_2^2 + \| d^*A(t) \|_2^2 = (\Delta A(t), A(t)) = (\Delta e^{2t\Delta}A_0, A_0) \). Therefore, using the spectral theorem and the identity \( \int_0^\infty k^{-1/2}e^{-2kx}dx = c_1x^{1/2} \), we find, for \( a = 1/2 \)

\[
\rho_A(\tau) = \int_0^\tau t^{-1/2}\| B(t) \|_2^2 dt \\
= \int_0^\tau t^{-1/2}(\Delta e^{2t\Delta}A_0, A_0) dt \\
= c_1((\Delta)^{1/2}A_0, A_0) - \int_\tau^\infty t^{-1/2}(\Delta e^{2t\Delta}A_0, A_0) dt \\
= c_1\| A_0 \|^2_{H^1/2} + O(\tau^{-1/2}\| e^{\tau \Delta}A_0 \|^2_2)
\]

Thus \( \rho_A(\tau) \) gives the \( \dot H^1/2 \) norm of \( A_0 \) in the Coulomb gauge, exactly for \( \tau = \infty \) and qualitatively for finite \( \tau > 0 \). In this example gauge transformations are given by \( A_0 \mapsto A_0 + d\lambda \), with \( \lambda \) a real valued function on \( \mathbb{R}^3 \). Since the Coulomb gauge space is orthogonal to the exact 1-forms it provides a
section for the quotient space \( \{\text{all } A_0\}/\{\text{exact } A_0\} \). Thus \( \rho_A(\tau) \) descends to a function on the quotient space for each \( \tau \), while at the same time giving the \( H_{1/2} \) norm, locally, of the lift to the Coulomb section.

3 Solutions for the augmented variational equation

In this section we will prove existence and uniqueness of solutions to the augmented variational equation (2.21) over a short time interval. The space \( M \) on which the initial data sits will be \( \mathbb{R}^3 \) or a bounded region in \( \mathbb{R}^3 \) with smooth boundary. In the latter case we will impose Dirichlet or Neumann boundary conditions on the solution. A standard procedure for analyzing the equation (2.21) consists in separating out the second order terms in (2.21) and writing the differential equation as an equivalent integral equation, whose solution is then established by a contraction principle. But this procedure, as stated, would lead to the equation \( \frac{d}{dt}w(t) = \Delta w(t) + K_0(t)w(t) \), where \( \Delta := -(d^*d + dd^*) \) is the Laplacian on \( \mathfrak{f} \) valued 1-forms and \( K_0(t) \) is a first order differential operator. Unfortunately the coefficients in \( K_0(t) \) depend on the gauge potential \( A(t) \) and its derivatives, which become highly singular as \( t \downarrow 0 \). As a result, the bounds needed to show that the related integral operator is a contraction in the relevant space fail. Instead, we are going to separate out a gauge covariant version of the Laplacian. Let \( T > 0 \) and let \( A = A(T) \). Denote by \( \Delta_A \) the Bochner Laplacian defined in Section 2. Then we may write (2.21) in the form \( \frac{d}{dt}w(t) = \Delta_A w(t) + K(t)w(t) \), where \( K(t) \) is a first order differential operator whose coefficients depend on the difference \( \alpha(t) := A(t) - A(T) \) and its covariant derivative. We will see that the required bounds on \( K(t) \) depend on bounds on \( \alpha(t), 0 < t \leq T \), which in turn depend on \( T \) being small. We have thereby a circumstance in which the unperturbed linear operator \( \Delta_A \) itself depends on the time needed for contraction.

In order to carry this out it will be necessary to have detailed information about the nature of the singularity of \( A(t) \) near \( t = 0 \). The required initial behavior bounds on \( A(t) \) and its derivatives will be derived in Section 3.2, after the overall strategy is explained in Section 3.1.
### 3.1 Path space and the integral equation

**Notation 3.1** \( A(t) \) will continue to denote a strong solution to the Yang-Mills heat equation (2.3) over \([0, \infty)\) with curvature \( B(t) \). The gauge covariant exterior derivatives and co-derivatives that we need to use were informally described in Notation 2.4. We will elaborate here on their domains. Suppose that \( A \) is a \( k \)-valued connection form on the closure \( M \) of a region in \( \mathbb{R}^3 \) and lying in \( W^1(M; \Lambda^1 \otimes \mathfrak{k}) \). In case \( M = \mathbb{R}^3 \) we define \( d_A \) as the closure in \( L^2(\mathbb{R}^3; \Lambda^p \otimes \mathfrak{k}) \) of the operator \( C^\infty_c(\mathbb{R}^3; \Lambda^p \otimes \mathfrak{k}) \ni \omega \mapsto d\omega + [A \wedge \omega] \). In case \( M \) is the closure of a bounded open set in \( \mathbb{R}^3 \) there are two versions of \( d_A \) of interest to us. Corresponding to Dirichlet boundary conditions (aka relative boundary conditions, \([15]\)), \( d_A \) will denote the closure in \( L^2(M; \Lambda^p \otimes \mathfrak{k}) \) of the operator \( C^\infty(M^{int}; \Lambda^p \otimes \mathfrak{k}) \ni \omega \mapsto d\omega + [A \wedge \omega] \). This is the minimal version of \( d_A \). Corresponding to Neumann boundary conditions (aka absolute boundary conditions, \([15]\)), \( d_A \) will denote the closure in \( L^2(M; \Lambda^p \otimes \mathfrak{k}) \) of the operator \( C^\infty(M; \Lambda^p \otimes \mathfrak{k}) \ni \omega \mapsto d\omega + [A \wedge \omega] \). This is the maximal version of \( d_A \). In all three cases \( d_A^* \) denotes the Hilbert space adjoint. The Hodge Laplacian on \( k \)-valued \( p \)-forms over \( M \) is

\[
\Delta_A = \sum_{j=1}^3 (\partial_j^A)^2,
\]

where \( \partial_j^A \) is defined in (2.7). By the Bochner-Weitzenböck formula we may write for a \( \mathfrak{k} \) valued 1-form

\[
(d_A^*d_A + d_Ad_A^*)\omega = -\Delta_A\omega + [\omega \llcorner B].
\]

In all cases of interest to us, when using this formula, the curvature \( B \) will be bounded. Consequently the operator \( \omega \mapsto [\omega \llcorner B] \) is a bounded operator on \( L^2 \). We can therefore take the closed version of (3.1) to be given by (3.2). Its domain is the same as that of \( d_A^*d_A + d_Ad_A^* \). Then \( \Delta_A \) is a self-adjoint operator with this domain. Thus if \( M \neq \mathbb{R}^3 \) then the Bochner Laplacian has a Dirichlet version and Neumann version on \( k \) valued 1-forms. On 0-forms both Laplacians are given by \( d_A^*d_A \).

Further discussion of the operators \( d_A, d_A^* \) and their associated boundary conditions may be found in [2, Section 3]. In the present paper we will need some specific information about the boundary conditions, especially in the
case of Neumann boundary conditions. If $\omega$ is a $k$ valued 1-form over $M$ then its location in one of the following domains implies the boundary condition indicated.

Form domain of Dirichlet Laplacian $\Delta_A$ : $\omega_{\tan} = 0$ \hspace{1cm} (3.3)
Form domain of Neumann Laplacian $\Delta_A$ : $\omega_{\text{norm}} = 0$ \hspace{1cm} (3.4)
Domain of Neumann Laplacian $\Delta_A$ : $\omega_{\text{norm}} = 0$, $(d_A\omega)_{\text{norm}} = 0$ \hspace{1cm} (3.5)

It might be useful to note that if $A_{\text{norm}} = 0$, which is the case of interest in dealing with Neumann boundary conditions for the variational equation, then the pair of conditions in (3.5) is equivalent to the pair of conditions $\omega_{\text{norm}} = 0, (d\omega)_{\text{norm}} = 0$ because $[A \wedge \omega]_{\text{norm}} = 0$ when both $A_{\text{norm}} = 0$ and $\omega_{\text{norm}} = 0$.

**Notation 3.2** Choose $T \in (0, \infty)$ and define

$$A = A(T), \quad \alpha(t) = A(t) - A(T), \quad 0 \leq t \leq T. \hspace{1cm} (3.6)$$

Then

$$A(t) = A + \alpha(t), \quad 0 \leq t \leq T. \hspace{1cm} (3.7)$$

We are going to use $\Delta_A$ as the “unperturbed” Laplacian in most of this paper.

**Lemma 3.3** Define a multiplication operator $M(t)$ on $k$ valued 1-forms by

$$M(t)\omega = \sum_{j=1}^{3} (ad \alpha_j(t))^2 \omega + [\text{div}_A \alpha(t), \omega] - 2[\omega \lrcorner B(t)]. \hspace{1cm} (3.8)$$

Denote by $K(t)$ the first order differential operator given by

$$K(t)\omega = 2[(\alpha(t) \cdot \nabla^A)\omega] + M(t)\omega, \hspace{1cm} (3.9)$$

where $[\alpha(t) \cdot \nabla^A \omega] = \sum_{j=1}^{3} [\alpha_j(t), \partial_j^A \omega]$. Then the augmented variational equation (2.21) can be written

$$w'(t) = \Delta_A w(t) + K(t)w(t) \hspace{1cm} (3.10)$$
Proof. In view of Notation 3.2 we may write $\partial_j A(t) = \partial_j A + ad \alpha_j(t)$. Suppressing $t$ on the right we therefore find.

$$\Delta_A(t) \omega = \sum_{j=1}^{3} (\partial_j A + ad \alpha_j)(\partial_j A + ad \alpha_j) \omega$$

$$= \sum_{j=1}^{3} (\partial_j A)^2 \omega + \sum_j \{\partial_j A[\alpha_j, \omega] + [\alpha_j, \partial_j A \omega]\} + \sum_j (ad \alpha_j)^2 \omega$$

$$= \Delta_A \omega + [div_A \alpha, \omega] + 2 \sum_j [\alpha_j, \partial_j A \omega] + \sum_j (ad \alpha_j)^2 \omega.$$

Hence, in view of the Bochner-Weitzenböck formula (3.2) we may write the augmented variational equation (2.21) as

$$w'(t) = -(d_A^* A d_A + d_A d_A^*) w(t) - [w(t) \ll B(t)]$$

$$= \Delta_A(t) w(t) - 2[w(t) \ll B(t)]$$

$$= \Delta_A w(t) + 2[\alpha \cdot \nabla A w(t)]$$

$$+ [div_A \alpha, w(t)] + \sum_j (ad \alpha_j)^2 w(t) - 2[w(t) \ll B(t)]$$

$$= \Delta_A w(t) + K(t) w(t).$$

Remark 3.4 (Strategy) Informally, the differential equation (3.10) together with the initial condition $w(0) = w_0$, is equivalent to the integral equation

$$w(t) = e^{t \Delta_A} w_0 + \int_0^t e^{(t-s) \Delta_A} \left(K(s)w(s)\right) ds. \quad (3.11)$$

We will first prove the existence and uniqueness of solutions to the integral equation (3.11). These are so-called “mild” solutions. It will then be necessary to show that the mild solution is actually a strong solution to (2.21). To this end we will establish bounds on the operator $K(t)$ which will allow us to prove Hölder continuity of $w(\cdot)$ and $K(\cdot)$ on intervals $[\tau, T]$, with $\tau > 0$, and thereby make applicable a general theorem [16, Theorem 11.44], ensuring that the mild solution is a strong solution. The required bounds on $K(t)$ will be derived by a common method for the three cases $M = \mathbb{R}^3$, or $M \neq \mathbb{R}^3$ with Neumann or Dirichlet boundary conditions. The three cases
will be encoded into appropriate Sobolev spaces, defined by (2.9) in terms of the associated Laplacians. In all three cases the associated Laplacian is given by (3.1) with \( A = A \) and with appropriate boundary conditions. \( H^1_A \) is the form domain of \( \Delta_A \). Thus if \( M \neq \mathbb{R}^3 \) then a form \( \omega \in W_1(M) \) is in the Neumann version of \( H^1_A(M) \) if and only if \( \omega_{\text{norm}} = 0 \) and is in the Dirichlet version of \( H^1_A(M) \) if and only if \( \omega_{\tan} = 0 \). This defines two distinct notions of \( H^1_A \) in case \( M \neq \mathbb{R}^3 \). See [2, Remark 4.10] for further discussion of these domains.

**Remark 3.5** (Marini boundary conditions.) For the solution \( A(\cdot) \) there is a third kind of boundary condition that was studied in [2] and [3]. It consists in setting the normal component of the curvature of \( A(t) \) to zero on \( \partial M \) for \( t > 0 \). This kind of boundary condition was first used by A. Marini [9, 10, 11, 12] for the four dimensional elliptic Yang-Mills boundary value problem. It will be used in a future work [8] for showing that the initial value of a finite action solution to the Yang-Mills heat equation over \( \mathbb{R}^3 \) is, upon restriction to a bounded region \( M \), an allowable initial value for a solution to the Yang-Mills heat equation over \( M \). Marini boundary conditions are forced in this context. This extension of the present work will be derived from Neumann boundary conditions in [8]. This kind of localization theorem seems indispensable for use of the Yang-Mills heat equation as a regularization tool in local quantum field theory in order to take into account that signals do not propagate faster than the speed of light.

**Notation 3.6** (Path space) Let \( 0 \leq b < 1 \) and let \( 0 < T < \infty \). Define

\[
Q_T^{(b)} = \left\{ w \in C\left([0, T]; H^A_b(M; \Lambda^1 \otimes \xi)\right) \cap C\left((0, T]; H^1_A(M; \Lambda^1 \otimes \xi)\right) : \limsup_{t \downarrow 0} t^{(1-b)/2} \|w(t)\|_{H^1_A} = 0 \right\}. \tag{3.12}
\]

For \( w \in Q_T^{(b)} \) define

\[
|w|_t = \sup_{0 < s \leq t} s^{(1-b)/2} \|w(s)\|_{H^A_b(M)}, \quad 0 \leq t \leq T. \tag{3.13}
\]

Then

\[
\|w(s)\|_{H^1_A} \leq s^{(b-1)/2} |w|_t, \quad 0 < s \leq t \leq T. \tag{3.14}
\]

The space \( Q_T^{(b)} \) is a Banach space in the norm

\[
\|w\|_{Q_T^{(b)}} = |w|_T + \sup_{0 \leq s \leq T} \|w(s)\|_{H^A_b}. \tag{3.15}
\]
Theorem 3.7 Assume that \( A(\cdot) \) is a strong solution to the Yang-Mills heat equation over \([0, \infty)\) with finite action. Suppose that \( 0 < b < 1 \) and that \( w_0 \in H^b_0(M) \). Then the integral equation (3.11) has a unique solution in the path space \( Q^{(b)}_T \) for a sufficiently small \( T \) depending on \( A(\cdot) \). Moreover

\[
\|w\|_{Q^{(b)}_T} \leq c_{b,T} \|w_0\|_{H^b_0}
\]  
(3.16)

for some constant \( c_{b,T} \) depending only on \( b, T \) and \( \rho_A(T) \), where \( \rho_A(t) \) is defined in (2.4) with \( a = 1/2 \).

The proof will be given in the next three sections.

3.2 Initial behavior of \( A \)

The initial behavior of various \( L^p(M) \) norms of \( A(t), B(t) \) and their time and space derivatives will be needed to prove bounds for the operator \( K(t) \) defined in (3.9), and later for establishing bounds on the solution \( w(t) \) to the augmented variational equation. These in turn will be needed to recover the desired solution to the variational equation from \( w(\cdot) \). In this subsection we are going to derive the required initial behavior bounds for \( A \) and its derivatives. They extend the initial behavior bounds derived in [6] and will be used frequently throughout the rest of this paper. Many of these are not gauge invariant bounds. But their proofs depend on the gauge invariant bounds derived in [6].

We reiterate that \( M \) can be chosen to be all of \( \mathbb{R}^3 \) or to be a bounded subset with Dirichlet or Neumann boundary conditions as in Section 3.1.

The next lemma summarizes some of the initial behavior bounds for \( A \) established in [6]. Recall the notation from (2.4): \( \rho_A(t) = (1/2) \int_0^t s^{-a} \|B(s)\|_2^2 ds \).

Definition 3.8 (Standard dominating function) By a standard dominating function we mean a continuous function \( C : [0, \infty)^2 \to [0, \infty) \) which is zero at \((0,0)\) and non-decreasing in each variable. On the right hand side of each of the following bounds is a function of time and of \( A \) of the form \( C(t, \rho_A(t)) \) for some standard dominating function \( C(\cdot, \cdot) \). All of the bounds are gauge invariant.

Lemma 3.9 Let \( 1/2 \leq a < 1 \). If \( A(\cdot) \) is a strong solution to the Yang-Mills heat equation over \((0, \infty)\) with finite \( a \)-action then there exist standard
dominating functions $C_j$ such that

$$\sup_{0<s\leq t} s^{1-a} \|B(s)\|_2^2 \leq C_1(t, \rho_A(t))$$ (3.17)

$$\sup_{0<s\leq t} s^{2-a} \|A'(s)\|_2^2 \leq C_2(t, \rho_A(t))$$ (3.18)

$$\sup_{0<s\leq t} s^{2-a} \|B(s)\|_2^6 \leq C_3(t, \rho_A(t))$$ (3.19)

$$\sup_{0<s\leq t} s^{3-a} \|B'(s)\|_2^2 \leq C_4(t, \rho_A(t))$$ (3.20)

$$\sup_{0<s\leq t} s^{3-a} \|A'(s)\|_2^2 \leq C_5(t, \rho_A(t))$$ (3.21)

$$\sup_{0<s\leq t} s^{(3/2)-a} \|B(s)\|_2^3 \leq C_6(t, \rho_A(t))$$ (3.22)

$$\sup_{0<s\leq t} s^{(5/2)-a} \|A'(s)\|_2^3 \leq C_7(t, \rho_A(t))$$ (3.23)

$$\int_0^t s^{1-a} \|A'(s)\|_2^2 ds \leq C_8(t, \rho_A(t))$$ (3.24)

$$\int_0^t s^{2-a} \|A'(s)\|_2^2 ds \leq C_9(t, \rho_A(t))$$ (3.25)

$$\int_0^t s \|A'(s)\|_2^3 ds \leq C_{10}(t, \rho_A(t))$$ (3.26)

$$\int_0^t s^{1-a} \|B(s)\|_2^2 ds \leq C_{11}(t, \rho_A(t))$$ (3.27)

$$\int_0^t s^{2-a} \|B'(s)\|_2^2 ds \leq C_{12}(t, \rho_A(t))$$ (3.28)

$$\int_0^t s^{3-a} \|B'(s)\|_2^2 ds \leq C_{13}(t, \rho_A(t))$$ (3.29)

$$\int_0^t \|B(s)\|_2^3 ds \leq C_{14}(t, \rho_A(t))$$ (3.30)

$$\sup_{0<s\leq t} s^{3/2} \|A'(s)\|_\infty \to 0 \text{ as } t \downarrow 0$$ (3.31)

$$\sup_{0<s\leq t} s \|B(s)\|_\infty \to 0 \text{ as } t \downarrow 0$$ (3.32)

$$\int_0^t s \|B(s)\|_\infty^2 ds < \infty$$ (3.33)

**Proof.** The inequalities (3.17) - (3.21) and (3.24), (3.25), (3.27), (3.28).
Inequalities involve $L^3$ norms and follow by interpolation thus: For $0 < s \leq t$ one has

$$s^{(3/2) - a} \| B(s) \|_3^2 \leq \left( s^{(1-a)/2} \| B(s) \|_2 \right) \left( s^{(2-a)/2} \| B(s) \|_6 \right) \leq (C_1 C_3)^{1/2} |s - t|,$$

$$s^{(5/2) - a} \| A'(s) \|_3^2 \leq \left( s^{(2-a)/2} \| A'(s) \|_2 \right) \left( s^{(3-a)/2} \| A'(s) \|_6 \right) \leq (C_2 C_5)^{1/2} |s - t|,$$

by (3.17), (3.19) and then (3.18), (3.21). The inequalities (3.22) and (3.23) follow. Interpolation also shows that

$$\int_0^t s \| A'(s) \|_3^2 ds \leq \int_0^t s^{a-1/2} \left( s^{(1-a)/2} \| A'(s) \|_2 \right) \left( s^{(2-a)/2} \| A'(s) \|_6 \right) ds$$

$$\leq t^{a-1/2} \left( \int_0^t s^{1-a} \| A'(s) \|_2^2 ds \right)^{1/2} \left( \int_0^t s^{2-a} \| A'(s) \|_6^2 ds \right)^{1/2}$$

$$\leq t^{a-1/2} C_8^{1/2} C_9^{1/2},$$

which is (3.26) with $C_1(t, \rho_A(t)) = t^{a-1/2} (C_8 C_9)^{1/2}$. Similarly,

$$\int_0^t \| B(s) \|_3^2 ds \leq \int_0^t (s^{a-1/2}) \left( s^{1-a/2} \| B(s) \|_2 \right) \left( s^{(1-a)/2} \| B(s) \|_6 \right) ds$$

$$\leq t^{a-1/2} \left( \int_0^t s^{-a} \| B(s) \|_2^2 ds \right)^{1/2} \left( \int_0^t s^{1-a} \| B(s) \|_6^2 ds \right)^{1/2}$$

$$\leq t^{a-1/2} (2 \rho_A(t))^{1/2} C_{11}(t, \rho_A(t))^{1/2},$$

which establishes (3.30).

Note: Whereas all the bounds in Lemma 3.9 are gauge invariant, the only gauge invariant bounds in the next theorem are (3.36), (3.37), (3.41) and (3.44). In most of the non-gauge invariant inequalities a non-gauge invariant condition is imposed on $A(T)$ for some $T > 0$, but the last quantifier is omitted to save space.

**Theorem 3.10** Let $1/2 \leq a < 1$. Assume that $A$ is a strong solution to the Yang-Mills heat equation (2.3) over $(0, \infty)$ with finite $a$-action. Then
$L^6$ inequalities.

1. $s^{-a}\| A(s) - A(r) \|_6^2 \leq C_{21}(r, \rho_A(r)), \ 0 < s \leq r < \infty.$ \ (3.36)

2. $\int_0^t s^{-a}\| A(s) - A(t) \|_6^2 ds \leq C_{22}(t, \rho_A(t)), \ 0 \leq t < \infty.$ \ (3.37)

3. $a_t := \sup_{0 < s \leq t} s^{(1-a)/2}\| A(s) \|_6 \to 0 \text{ as } t \downarrow 0 \text{ if } \| A(T) \|_6 < \infty$ \ (3.38)

4. $\hat{a}_t := \int_0^t s^{-a}\| A(s) \|_6^2 ds \leq \infty \text{ if } \| A(T) \|_6 < \infty$ \ (3.39)

$L^3$ inequalities. Assume that $1/2 \leq a < 1$ and $\| A(T) \|_6 < \infty$. Then

5. $\int_0^t s^{1-2a}\| d^*A(s) - d^*A(t) \|_3^2 d s \leq (1-a)^{-2}a_t^2C_9(t, \rho_A(t))$ \ (3.40)

6. $t^{1/2}\| d^*_A(A(t) - A(T)) \|_3 \leq C_{24}(T, \rho_A(T)), \ 0 < t \leq T \text{ if } a = 1/2$ \ (3.41)

7. $\int_0^t s^{(1/2)-a}\| d(A(s) - A(t)) \|_3^2 d s \leq C_{32}(t, \rho_A(t)) + a_t^2C_{33}(t, \rho_A(t))$ \ (3.42)

8. $\limsup_{t \downarrow 0} t^{(3-2a)/4}\| dA(t) \|_3 = 0, \text{ and } \limsup_{t \downarrow 0} t^{1/2}\| dA(t) \|_3 = 0$ \ (3.43)

$L^\infty$ inequality.

9. $t^{1/2}\| A(t) - A(T) \|_\infty \leq C_{25}(T, \rho_A(T)), \ 0 < t \leq T \text{ if } a = 1/2$ \ (3.44)

$L^2$ inequalities. Assume that $1/2 \leq a < 1$ and $\| A(T) \|_6 < \infty$. Then

10. $\int_0^t s^{-a}\| d^*A(s) - d^*A(t) \|_2^2 d s \leq a_t^2C_{40}(t, \rho_A(t))$ \ (3.45)

11. $\int_0^t s^{-a}\| dA(s) - dA(t) \|_2^2 d s \to 0 \text{ as } t \downarrow 0.$ \ (3.46)

12. $\int_0^T s^{-a}\| dA(s) \|_2^2 d s < \infty \text{ if } A(T) \in H_1$ \ (3.47)

13. $\sup_{0 < s \leq t} s^{1-a}\| d^*A(s) - d^*A(t) \|_2^2 \to 0 \text{ as } t \downarrow 0.$ \ (3.48)

14. $\sup_{0 < s \leq t} s^{1-a}\| dA(s) - dA(t) \|_2^2 \to 0 \text{ as } t \downarrow 0.$ \ (3.49)
$H_1$ inequalities. Assume that $\|A(T)\|_6 < \infty$ and that $M = \mathbb{R}^3$. Then

15. $\infty > \int_0^t s^{-a} \|\nabla(A(s) - A(t))\|_2^2 ds \to 0$ as $t \downarrow 0$. \hspace{1cm} (3.50)

16. $\infty > \sup_{0<s \leq t} s^{(1-a)} \|\nabla(A(s) - A(t))\|_2^2 \to 0$ as $t \downarrow 0$ \hspace{1cm} (3.51)

17. $\int_0^T s^{-a} \|\nabla A(s)\|_2^2 ds < \infty$ if $\|\nabla A(T)\|_2 < \infty$ \hspace{1cm} (3.52)

18. $\limsup_{t \downarrow 0} t^{(1-a)/2} \|\nabla A(t)\|_2 = 0$ if $\|\nabla A(T)\|_2 < \infty$. \hspace{1cm} (3.53)

Remark 3.11 The inequality (3.43) has no analog for $d^* because $\|d^*A(t)\|_3$ need not be finite for any $t > 0$ under our hypotheses. For example let $K = S^1$ and take $A_0 = d\lambda$ with $d^*d\lambda \in L^2(\mathbb{R}^3)$ but $d^*d\lambda \notin L^3(\mathbb{R}^3)$. Then $A(t) := d\lambda \in H_1$ for all $t \geq 0$ and is a strong solution to the Yang-Mills heat equation. In this case $dA(t) = 0$ but $d^*A(t) \notin L^3(\mathbb{R}^3)$ for any $t \geq 0$. This is a pure gauge solution. \hspace{1cm} (3.40) and (3.41) show that “pure gauge contributions” to differences, such as $\|d^*A(s) - d^*A(t)\|_3$, cancel to some degree.

The proof of Theorem 3.10 depends on the following special case of the generalized Hardy’s inequality \cite[Theorem 6.1.4]{18}.

Lemma 3.12 (Hardy’s inequality.) Let $g : (0, \infty) \to \mathbb{R}$ be locally integrable. Suppose that $0 < T < \infty$. Define

$$G(t) = \int_t^T g(s) ds, \quad 0 < t \leq T. \hspace{1cm} (3.54)$$

Let $-\infty < \beta < 1$. Then

$$\int_0^T t^{-\beta} G(t)^2 dt \leq \frac{4}{(1-\beta)^2} \int_0^T s^{2-\beta} g(s)^2 ds, \hspace{1cm} (3.55)$$

and, if $h : (0, \infty) \to [0, \infty)$ is differentiable, then

$$\int_0^T s^{-\beta} \left( h(s) - h(T) \right)^2 ds \leq \frac{4}{(1-\beta)^2} \int_0^T s^{2-\beta} h'(s)^2 ds. \hspace{1cm} (3.56)$$
Proof. We will derive the inequality (3.55) from the Generalized Hardy inequality, [18, Equ. (6.1.31)]. Suppose that \( f : (0, \infty) \to \mathbb{R} \) is locally integrable and vanishes off a bounded interval. Define

\[
F(t) = \int_t^\infty s^{-1} f(s) \, ds, \quad t > 0
\]

(3.57)

Let \( \alpha > 0 \). The generalized Hardy inequality [18, Equ. (6.1.31)], with \( \theta = 0 \) and \( p = 2 \), asserts that

\[
\int_0^\infty t^{2\alpha-1} F(t)^2 dt \leq \alpha^{-2} \int_0^\infty s^{2\alpha-1} f(s)^2 ds,
\]

(3.58)

as one sees from [18, Equ. (6.1.29)], with \( \theta = 0 \), because, in the notation of [18], we have \( (K_{\alpha,0}f)(t) = t^{\alpha} F(t) \).

Now in (3.57) let \( f(s) = sg(s) \) for \( 0 < s \leq T \) and let \( f(s) = 0 \) for \( s > T \). Then \( F(t) = G(t) \) for \( 0 < t \leq T \) and \( F(t) = 0 \) for \( t > T \). The two integrands in (3.58) are therefore zero off the interval \((0,T]\) and the integrals really extend over the interval \((0,T]\). Put \( \alpha = (1 - \beta)/2 \). Then \( \alpha > 0 \) and \( 2\alpha - 1 = -\beta \). (3.58) now reduces to (3.55).

For the proof of (3.56) choose \( g(s) = h'(s) \). Then \( G(t) = h(T) - h(t) \) and (3.56) follows. \( \blacksquare \)

Proof of Theorem 3.10.

Proof of 1. For \( 0 < s \leq r \) we may write \( A(s) - A(r) = -\int_s^r A'(\sigma) d\sigma \). Hence

\[
\|A(s) - A(r)\|_6^2 \leq \left( \int_s^r \|A'(\sigma)\|_6 d\sigma \right)^2
\]

\[
\leq \left( \int_s^r \sigma^{(a-2)/2} \|A'(\sigma)\|_6 d\sigma \right)^2
\]

\[
\leq \int_s^r \sigma^{a-2} d\sigma \int_0^r \sigma^{2-a} \|A'(\sigma)\|_6^2 d\sigma
\]

\[
\leq (1-a)^{-1}(s^{a-1} - r^{a-1})C_9(r, \rho_A(r)).
\]

Therefore \( s^{1-a}\|A(s) - A(r)\|_6^2 \leq C_9(r, \rho_A(r))/(1-a) \), which proves (3.36).

Proof of 2. Choose \( T = t \) in (3.56) and let \( h(s) = \|A(s) - A(t)\|_6 \). Then \( h(t) = 0 \) and \( |h'(s)| \leq \|A'(s)\|_6 \). Choosing \( \beta = a \), we have \( s^{2-\beta}h'(s)^2 \leq \)
\( s^{2-a}\|A'(s)\|_6^2 \). (3.56) now shows that
\[
\int_0^t s^{-a}\|A(s) - A(t)\|_6^2 ds \leq \frac{4}{(1-a)^2} \int_0^t s^{2-a}\|A'(s)\|_6^2 ds. \quad (3.59)
\]
From this and (3.25) we see that (3.37) holds with \( C_{22} = 4(1-a)^2 C_9. \)

**Proof of 3.** It follows from (3.36) that
\[
s^{(1-a)/2}\|A(s)\|_6 \leq \sqrt{C_{21}(r, \rho_A(r))} + s^{(1-a)/2}\|A(r)\|_6. \quad (3.60)
\]
Take \( r = T \) to conclude that if \( \|A(T)\|_6 < \infty \) then \( \|A(s)\|_6 < \infty \) for \( 0 < s \leq T \). Now choose a small \( r \) and observe that if \( 0 < t \leq r \) then
\[
a_t = \sup_{0<s\leq t} s^{(1-a)/2}\|A(s)\|_6 \leq \sqrt{C_{21}(r, \rho_A(r))} + t^{(1-a)/2}\|A(r)\|_6.
\]
Hence \( \lim_{t \to 0} a_t \leq \sqrt{C_{21}(r, \rho_A(r))} \), which is small for small \( r \) because \( C_{21} \) is a standard dominating function. This proves (3.38).

**Proof of 4.** By (3.37) we have
\[
\left( \int_0^t s^{-a}\|A(s)\|_6^2 ds \right)^{1/2} \leq \sqrt{C_{22}(t, \rho_A(t))} + \left( \int_0^t s^{-a}\|A(t)\|_6^2 ds \right)^{1/2}.
\]
The last term is finite because \( \|A(t)\|_6 < \infty \) by (3.36) and \( a < 1 \).

**Proof of 5.** The Yang-Mills heat equation \( A' = -d^*_A B \) together with the Bianchi identity show that \( d^*_A A' = -(d^*_A)^2 B = 0 \). Hence
\[
d^* A'(s) = -[A(s) \lrcorner A'(s)]. \quad (3.61)
\]
In (3.56) choose \( \beta = 2a - 1 \), \( T = t \) and \( h(s) = \|d^* A(s) - d^* A(t)\|_3 \). From (3.61) we find \( |h'(s)| \leq \|d^* A'(s)\|_3 \leq \| [A(s) \lrcorner A'(s)] \|_3 \leq c\|A(s)\|_6\|A'(s)\|_6. \) Therefore
\[
s^{2-a}h'(s)^2 \leq c^2 \left( s^{1-a}\|A(s)\|_6^2 \right)^{1/2} \left( s^{2-a}\|A'(s)\|_6^2 \right)^{1/2}
\leq c^2 a_t^2 \left( s^{2-a}\|A'(s)\|_6^2 \right).
\]
Then (3.56), with \( \beta = 2a - 1 \) and \( T = t \), gives
\[
\int_0^t s^{-2a} \| d^* A(s) - d^* A(t) \|_3^2 ds \leq (1 - a)^{-2} \int_0^t a_t^2 s^{2-a} \| A'(s) \|_6^2 ds \\
\leq (1 - a)^{-2} a_t^2 C_9(t, \rho_A(t))
\]
by (3.25). This proves (3.40).

**Proof of 6.** The derivation of (3.61) shows that
\[
0 = d^*_{A(s)} A'(s) = d^*_{A} A'(s) + [\alpha(s) \wedge A'(s)].
\]
Since \( A'(s) = \alpha'(s) \) we may write the previous identity as
\[
d^*_{A} \alpha'(s) = -[\alpha(s) \wedge A'(s)]. \tag{3.62}
\]
Using \( \alpha(T) = 0 \) we therefore find
\[
d^*_{A} \alpha(t) = \int_t^T [\alpha(s) \wedge A'(s)] ds. \tag{3.63}
\]
Hence
\[
\| d^*_{A} \alpha(t) \|_3 \leq c \int_t^T \| \alpha(s) \|_6 \| A'(s) \|_6 ds \\
= c \int_t^T s^{-1/2} \left( s^{-1/4} \| \alpha(s) \|_6 \right) \left( s^{3/4} \| A'(s) \|_6 \right) ds \\
\leq c t^{-1/2} \left( \int_t^T s^{-1/2} \| \alpha(s) \|_6^2 ds \right)^{1/2} \left( \int_t^T s^{3/2} \| A'(s) \|_6^2 ds \right)^{1/2} \\
\leq c t^{-1/2} C_{22}(T, \rho_A(T))^{1/2} C_9(T, \rho_A(T))^{1/2} \tag{3.64}
\]
in view of (3.37) and (3.25). This proves (3.61).

**Proof of 7.** The identities \( B' = d_A A' = dA' + [A \wedge A'] \) show that
\[
dA' = B' - [A \wedge A']. \tag{3.65}
\]
Let \( h(s) = \| d(A(s) - A(t)) \|_3 \). Then
\[
|h'(s)| \leq \| dA'(s) \|_3 \leq \| B'(s) \|_3 + c \| A(s) \|_6 \| A'(s) \|_6. \tag{3.66}
\]

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Chose $\beta = a - (1/2)$ in (3.56) and observe that

$$(1/2)s^{2-\beta} |h'(s)|^2 \leq s^{5/2-a} \|B'(s)\|^2 \leq s^{5/2-a}c^2 s^{a-(1/2)}(s^{1-a} \|A(s)\|^2) \leq s^{5/2-a} \|B'(s)\|^2 + c^2 s^{a-(1/2)}a_t^2 (s^{2-a} \|A'(s)\|^2).$$

Now

$$(\int_0^t s^{(5/2-a)} \|B'(s)\|^2 ds)^{1/2} \leq (\int_0^t s^{2-a} \|B'(s)\|^2 ds)^{1/2},$$

and

$$(\int_0^t s^{3-a} \|B'(s)\|^2 ds)^{1/2} \leq C_{12}(t, \rho_A(t))^{1/2} C_{13}(t, \rho_A(t))^{1/2}. \quad (3.67)$$

Therefore

$$2 \int_0^t s^{(1/2-a)} \|dA(s) - dA(t)\|^2 \leq \frac{4}{((3/2) - a)^2} \int_0^t s^{(5/2-a)} |h'(s)|^2 ds \leq \frac{16}{(3 - 2a)^2} \left\{ s^{5/2-a} \|B'(s)\|^2 + c^2 t^{a-(1/2)}a_t^2 (s^{2-a} \|A'(s)\|^2) \right\} ds \leq \frac{16}{(3 - 2a)^2} \left\{ C_{31}(t, \rho_A(t)) + c^2 t^{a-(1/2)}a_t^2 C_{39}(t, \rho_A(t)) \right\}$$

wherein we have used (3.25). This proves (3.42).

**Proof of 8.** Since $dA(t) = B(t) - (1/2)[A(t) \wedge A(t)]$ it suffices to show that

$$t^{(3-2a)/4} \left( \|B(t)\|_3 + \|[A(t) \wedge A(t)]\|_3 \right) \to 0, \quad \text{as} \quad t \downarrow 0. \quad (3.68)$$

By interpolation we have

$$\left( t^{(3-2a)/4} \|B(t)\|_3 \right)^2 \leq \left( t^{(1-a)/2} \|B(t)\|_2 \right) \left( t^{(2-a)/2} \|B(t)\|_6 \right).$$

The two factors on the right go to zero as $t \downarrow 0$, the first by (3.17) and the second by (3.19). Further, $t^{4-a} \|A(t) \wedge A(t)\|_3 \leq c t^{(1-a)/2} \|A(t)\|_6 \to 0$ by (3.39). Since $(3 - 2a)/4 \geq (1 - a)$ for $1/2 \leq a < 1$, the limit (3.68).
holds. Finally, the second limit in (3.43) holds because \((1/2) \geq (3 - 2a)/4\) for \(1/2 \leq a < 1\).

**Proof of 9.** By (3.31) there is a constant \(k_t\) for each \(t \in (0,T]\) such that \(\|A'(s)\|_\infty \leq s^{-3/2}k_t\) for \(0 < s \leq t\) and \(k_t \to 0\) as \(t \downarrow 0\). Hence, writing \(\alpha(t) = A(t) - A(T)\), we have

\[
\|\alpha(t)\|_\infty = \| \int_t^T A'(s)ds \|_\infty
\leq \int_t^T \|A'(s)\|_\infty ds
\leq \int_t^T s^{-3/2}k_T ds
= t^{-1/2}(1 - (t/T)^{1/2})2k_T
\]

This proves (3.44). Actually \(t^{1/2}\|\alpha(t)\|_\infty \to 0\) as \(t \downarrow 0\) as one sees from the inequalities \(t^{1/2}\|\alpha(t)\|_\infty \leq s^{-3/2}k_r ds + t^{1/2} \int_T^T s^{-3/2}k_T ds\). The lim sup of the left is then at most \(2k_r\), which is small for small \(r > 0\).

**Proof of 10.** We will apply the Hardy inequality (3.56) with \(T = t\) to the function \(h(s) = \|d^*A(s) - d^*A(t)\|_2\). We have \(|h'(s)| \leq \|d^*A'(s)\|_2 = \|[A(s) \cup A'(s)]\|_2\) by (3.61). Hence \(s^{2-a}h'(s)^2 \leq s^{2-a}\|[A(s) \cup A'(s)]\|_2^2\). Therefore, by Hardy’s inequality (3.56) with \(\beta = a\) we find

\[
\int_t^s s^{-a}\|d^*(A(s) - A(t))\|_2^2 ds \leq \frac{4}{(1-a)^2} \int_0^t s^{2-a}\|[A(s) \cup A'(s)]\|_2^2 ds
\leq \frac{4}{(1-a)^2} \int_0^t s^{2-a}\|[A(s)]_2^2\|A'(s)\|_3^2 ds
\]

The integral can be estimated by

\[
\int_0^t s^{2-a}\|[A(s)]_2^2\|A'(s)\|_3^2 ds \leq \int_0^t \left(s^{1-a}\|[A(s)]_2^2\right) \left(s\|[A'(s)]_3^2\right) ds
\leq a^2 \int_0^t s\|[A'(s)]_3^2 ds
\leq a^2 C_{10}(t, \rho_A(t))
\]

by (3.38) and (3.26). This proves (3.45).
Proof of 11. Let \( h(s) = \|d(A(s) - A(t))\|_2 \). Then \( |h'(s)| \leq \|dA'(s)\|_2 = \|B'(s) - [A(s) \wedge A'(s)]\|_2 \) by (3.65). From Hardy’s inequality (3.56) with \( \beta = a \) and \( T = t \) we find

\[
\int_0^t s^{-a} \|d(A(s) - A(t))\|_2^2 ds \\
\leq \frac{4}{(1-a)^2} \int_0^t s^{2-a} \|B'(s) - [A(s) \wedge A'(s)]\|_2^2 ds \tag{3.70}
\]

We see from (3.28) that \( \int_0^t s^{2-a} \|B'(s)\|_2^2 ds < \infty \). Moreover \( \|A(s) \wedge A'(s)\|_2 \leq c\|A(s)\|_6 \|A'(s)\|_3 \). So the bound (3.69) shows that \( \int_0^t s^{2-a} \|B'(s) - [A(s) \wedge A'(s)]\|_2^2 ds < \infty \) also. Combining these bounds we find

\[
\int_0^t s^{-a} \|d(A(s) - A(t))\|_2^2 ds \leq \frac{8}{(1-a)^2} \left( C_{12}(t, \rho_A(t)) + c^2 a^2 C_{10}(t, \rho_A(t)) \right). \tag{3.71}
\]

The assertion (3.46) follows.

Proof of 12. From (3.46) with \( t = T \) we have

\[
\int_0^T s^{-a} \|dA(s) - dA(T)\|_2^2 ds < \infty \tag{3.72}
\]

By assumption, \( A(T) \in H_1 \), which is defined in (2.1). Therefore \( \|dA(T)\|_2 < \infty \). Since \( a < 1 \) we have \( \int_0^T s^{-a} \|dA(T)\|_2^2 ds < \infty \). Hence (3.72) implies (3.47).

Proof of 13. Integrating (3.61) over the interval \([s, t]\) we find

\[
d^* A(s) - d^* A(t) = \int_s^t [A(\sigma) \perp A'(\sigma)] d\sigma. \tag{3.73}
\]

Hence

\[
\|d^* A(s) - d^* A(t)\|_2 \leq \int_s^t \| [A(\sigma) \perp A'(\sigma)] \|_2 d\sigma \\
\leq c \int_s^t \|A(\sigma)\|_6 \|A'(\sigma)\|_3 d\sigma.
\]
But

\[
\left( \int_s^t \| A(\sigma) \|_6 \| A'(\sigma) \|_3 d\sigma \right)^2
= \left\{ \int_s^t \sigma^{(a-1)/2} \left( \sigma^{-a/2} \| A(\sigma) \|_6 \right) \left( \sigma^{1/2} \| A'(\sigma) \|_3 \right) d\sigma \right\}^2
\leq s^{a-1} \left( \int_s^t \sigma^{-a} \| A(\sigma) \|_6^2 d\sigma \right) \left( \int_s^t \sigma \| A'(\sigma) \|_3^2 d\sigma \right)
\leq s^{a-1} \left( \int_0^t \sigma^{-a} \| A(\sigma) \|_6^2 d\sigma \right) \left( \int_0^t \sigma \| A'(\sigma) \|_3^2 d\sigma \right)
\leq s^{a-1} \hat{a}_t \ C_{10}(t, \rho_A(t))
\]

by (3.39) and (3.26). Therefore

\[
\sup_{0 < s \leq t} s^{1-a} \| d^* A(s) - d^* A(t) \|_2^2 \leq c^2 \hat{a}_t \ C_{10}(t, \rho_A(t))
\] (3.75)

This goes to zero as \( t \downarrow 0 \). This proves (3.48).

Proof of 14. From the identity (3.65) we find

\[
s^{(1-a)/2} \| d(A(s) - A(t)) \|_2
\leq s^{(1-a)/2} \| B(s) - B(t) \|_2 + s^{(1-a)/2} \int_s^t \| [A(\sigma) \wedge A'(\sigma)] \|_2 d\sigma.
\] (3.76)

Since \( \| [A(\sigma) \wedge A'(\sigma)] \|_2 \leq c \| A(\sigma) \|_6 \| A'(\sigma) \|_3 \), (3.74) shows that the second term in (3.76) is bounded by \((c \hat{a}_t C_{10})^{1/2}\) and therefore goes to zero uniformly in \( s \leq t \) as \( t \downarrow 0 \). The first term on the right in (3.76) is at most \( s^{1-a} \| B(s) \|_2 + t^{1-a} \| B(t) \|_2 \) which is bounded by \( 2C_1(t, \rho_A(t)) \) in accordance with (3.17). This completes the proof of (3.49).

Proof of 15-16. Over \( \mathbb{R}^3 \) we have the identity

\[
\| \nabla \omega \|_2^2 = || d\omega \|_2^2 + || d^* \omega \|_2.
\] (3.77)

Consequently (3.50) follows immediately from (3.45) and (3.46) while (3.51) follows immediately from (3.48) and (3.49).

Proof of 17-18. It follows from (3.51) with \( t = T \) that if \( \| \nabla A(T) \|_2 < \infty \) then (3.52) holds since \( a < 1 \).
Now take $t = T$ in (3.51). It follows that if $\|\nabla A(T)\|_2 < \infty$ then $\|\nabla A(r)\|_2 < \infty$ for $0 < r \leq T$. Hence an argument similar to that in the proof of (3.38) shows that (3.53) follows from (3.51).

This completes the proof of Theorem 3.10. ■

3.3 Estimates for the integral equation

Throughout this subsection we will assume that $A(\cdot)$ is a strong solution to the Yang-Mills heat equation over $[0, \infty)$ with finite action.

**Lemma 3.13** For each $T > 0$ there is a constant $\mu_T$, depending only on $T$ and $\rho_A(T)$, such that

$$t^{1/2}\|K(t)\omega\|_2 \leq \mu_T\|\omega\|_{H^1_T}, \quad 0 < t \leq T$$

(3.78)

and $\mu_T \to 0$ as $T \downarrow 0$.

(3.79)

Let $0 < \tau < T$. There is a constant $m_\tau$, depending only on $\tau, T$ and $\rho_A(T)$, such that

$$\|(K(t) - K(r))\omega\|_2 \leq (t - r)^{3/4}m_\tau\|\omega\|_{H^1_T}, \quad \tau \leq r \leq t \leq T.$$  

(3.80)

**Proof.** We need to prove bounds of the form (3.78) and (3.80) for each of the four operators that appear in (3.9). Taking the terms in the multiplication operator $M(t)$ first we have

$$t^{1/2}\|[\alpha_j(t), [\alpha_j(t), \omega]]\|_2 \leq c^2t^{1/2}\|\alpha_j(t)\|_6^2\|\omega\|_6$$

$$\leq c^2C_{21}(T, \rho_A(T))\|\omega\|_6$$

(3.81)

by (3.36). Furthermore, in view of (3.21) and (3.36) with $a = 1/2$ we have

$$\|[\alpha_j(t), [\alpha_j(t), \omega]] - [\alpha_j(t), [\alpha_j(t), \omega]]\|_2$$

$$\leq c^2\|\alpha_j(t) - \alpha_j(r)\|_6\|\alpha_j(t)\|_6 + \|\alpha_j(r)\|_6\|\omega\|_6$$

$$\leq 2c^2\left(\|A_j(t) - A_j(r)\|_6\max_{r \leq s \leq T}\|\alpha_j(s)\|_6\right)\|\omega\|_6$$

$$\leq 2c^2\left(\int_r^t \|A'(s)\|_6 ds \left(\tau^{-1/2}C_{21}(T, \rho_A(T))\right)^{1/2}\|\omega\|_6ight)$$

$$\leq 2c^2\left(\int_r^t s^{-5/4}C_5(T, \rho_A(T))^{1/2}\left(\tau^{-1/2}C_{21}(T, \rho_A(T))\right)^{1/2}\|\omega\|_6ight)$$

$$\leq 2c^2\tau^{-5/4}(t - r)C_5(T, \rho_A(T))^{1/2}\left(\tau^{-1/2}C_{21}(T, \rho_A(T))\right)^{1/2}\|\omega\|_6$$

$$= (t - r)^{-3/2}C_{40}(T, \rho_A(T))\|\omega\|_6.$$  

(3.82)
Concerning the second term in (3.8) we have
\[
t^{1/2} \| [\text{div}_{\mathbf{A}} \alpha(t), \omega] \|_2 \leq c t^{1/2} \| \text{div}_{\mathbf{A}} \alpha(t) \|_3 \| \omega \|_6
\leq c C_{24}(T, \rho_A(T)) \| \omega \|_6
\] (3.83)
by (3.41). Furthermore, using (3.62), (3.36) and (3.21), we find
\[
\| [\text{div}_{\mathbf{A}} (\alpha(t) - \alpha(r)), \omega] \|_2 = \| \int_r^t [\text{div}_{\mathbf{A}} \alpha'(s), \omega] ds \|_2
\leq c \int_r^t \| \text{div}_{\mathbf{A}} \alpha'(s) \|_3 ds \| \omega \|_6
\leq c^2 \int_r^t \| \alpha(s) \|_6 \| A'(s) \|_6 ds \| \omega \|_6
\leq c^2 \int_r^t s^{-1/4} s^{-5/4} ds \ (C_{21}C_5)^{1/2} \| \omega \|_6
\leq c^2 (t - r)^{-3/2} (C_{21}C_5)^{1/2} \| \omega \|_6.
\]
The third term in (3.8) is easily estimated by
\[
t^{1/2} \| [\omega \downarrow B(t)] \|_2 \leq c t^{1/2} \| B(t) \|_3 \| \omega \|_6
\leq c C_6(T, \rho_A(T))^{1/2} \| \omega \|_6
\] by (3.22) with \( a = 1/2 \). Furthermore
\[
\| [\omega \downarrow (B(t) - B(r))] \|_2 \leq c \| B(t) - B(r) \|_3 \| \omega \|_6.
\]
We will show that there is a standard dominating function \( C_{42} \) such that
\[
\| B(t) - B(r) \|_3 \leq (t - r)^{3/4} \tau^{-5/4} C_{42}(T, \rho_A(T)), \quad 0 < \tau \leq r \leq t \leq T.
\] (3.84)
For the proof of (3.84) apply Hölder’s inequality to find

\[ \|B(t) - B(r)\|_3 \leq \int_r^t \|B'(s)\|_3 ds \]

\[ \leq \left( \int_r^t 1^{4/3} ds \right)^{3/4} \left( \int_r^t \|B'(s)\|_3^4 ds \right)^{1/4} \]

\[ \leq (t - r)^{3/4} \left( \int_r^t \|B'(s)\|_2^2 \|B'(s)\|_6^2 ds \right)^{1/4} \]

\[ \leq (t - r)^{3/4} \left( C_4 \int_r^t s^{-5/2} \|B'(s)\|_2^2 ds \right)^{1/4} \]

\[ \leq (t - r)^{3/4} \left( C_4 \tau^{-5} \int_r^t s^{5/2} \|B'(s)\|_2^2 ds \right)^{1/4} \]

\[ \leq (t - r)^{3/4} \left( C_4 \tau^{-5} C_{15}(T, \rho_A(T)) \right)^{1/4} \]

by (3.20) with \( a = 1/2 \) and \( t = T \) and by (3.29).

Thus the multiplication operator \( M(t) \) satisfies

\[ t^{1/2} \|M(t)\omega\|_2 \leq q_T \|\omega\|_6, \quad 0 < t \leq T \quad \text{and} \]

\[ \|(M(t) - M(r))\omega\|_2 \leq (t - r)^{3/4} q_\tau, T \|\omega\|_6, \quad 0 < \tau \leq r \leq t \leq T \]

for some constants \( q_T \) and \( q_\tau, T \) which are majorized by dominating functions of \( T \) and \( \rho_A(T) \) for each \( \tau > 0 \).

The differential operator term in (3.9) can be dominated as follows.

\[ t^{1/2} \|\alpha(t) \cdot \nabla^A \omega\|_2 \leq c t^{1/2} \|\alpha(t)\|_\infty \|\nabla^A \omega\|_2 \]

\[ \leq c C_{25}(T) \|\omega\|_{H^1_A} \]

by (3.44). Furthermore, in view of (3.31) we have

\[ \|[(\alpha(t) - \alpha(r)) \cdot \nabla^A \omega]\|_2 \leq c \|\alpha(t) - \alpha(r)\|_\infty \|\nabla^A \omega\|_2 \]

\[ \leq c \int_r^t \|A'(s)\|_\infty ds \|\nabla^A \omega\|_2 \]

\[ \leq c \int_r^t s^{-3/2} ds \cdot C_{15}^{1/2} \|\nabla^A \omega\|_2 \]

\[ \leq c (t - r)^{-3/2} \cdot C_{15}^{1/2} \|\nabla^A \omega\|_2. \]

This completes the proof of the lemma. ■
Remark 3.14 Three of the terms in the operator $K(t)$ have been shown above to be Hölder continuous of order one and the fourth one of order $3/4$. But higher order initial behavior estimates developed in [4] show that $\|B'(t)\|_6$ is bounded on $[\tau, T]$, from which it would follow that all four terms are Hölder continuous of order one. However we will not need this improvement in this paper.

Lemma 3.15 (Free propagation) Let $0 \leq b < 1$ and suppose that $w_0 \in H_b$. Then, for some constants $c_b$ and $\gamma_b$ there holds

$$e^{2t}c_b^2\|w_0\|_{H_b}^2 \geq t^{1-b}\|e^{t\Delta A}w_0\|_{H^A}^2 \to 0 \text{ as } t \downarrow 0 \quad \text{and} \quad (3.85)$$

$$\int_0^T t^{-b}\|e^{t\Delta A}w_0\|_{H^A}^2 dt \leq c^2T\gamma_b^2\|w_0\|_{H_b}^2. \quad (3.86)$$

In particular, the function $t \mapsto e^{t\Delta A}w_0$ lies in $Q^{(b)}_T$.

Proof. The proof of (3.85) and (3.86) relies only on the spectral theorem and a computation that may be found in [6, Lemma 3.4]. The continuity of $t \mapsto e^{t\Delta A}w_0$ on $[0, T]$ into $H^A_b$ and on $(0, T]$ into $H^A_1$ is clear. The condition (3.12) follows from (3.85).

Remark 3.16 If $L$ is a non-negative self-adjoint operator on a Hilbert space and $D = L^{1/2}$ then

$$\|D^\alpha e^{-tL}\| \leq c_\alpha t^{-\alpha/2}, \quad t > 0, \quad \alpha \geq 0, \quad (3.87)$$

for some constant $c_\alpha$, as follows from the spectral theorem and the inequality $\sup_{\lambda > 0} \lambda^{\alpha/2} e^{-t\lambda} = t^{-\alpha/2} \sup_{\sigma > 0} \sigma^{\alpha/2} e^{-\sigma}$. Here $\lambda \geq 0$ is a spectral parameter for $L$. The case of interest for us will be $L = 1 - \Delta_A$ acting on $L^2(M; \Lambda^1 \otimes \mathfrak{t})$.

Remark 3.17 The following identity, which arises frequently, is listed here for convenience. Let $\mu$ and $\nu$ be real numbers with $\mu < 1$ and $\nu < 1$. Then

$$\frac{1}{t} \int_0^t (t - s)^{-\mu}s^{-\nu} ds = t^{-\mu-\nu} C_{\mu,\nu} \quad (3.88)$$

for some finite constant $C_{\mu,\nu}$. This follows from the substitution $s = t\sigma$. 

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Lemma 3.18 Suppose that $0 < b < 1$ and $w \in Q_T^{(b)}$. Let

$$ (Yw)(t) = \int_0^t e^{(t-s)\Delta} K(s)w(s)ds, \quad 0 \leq t \leq T \tag{3.89} $$

If $0 \leq r \leq 1$ then

$$ \| (Yw)(t) \|_{H^r} \leq c_r \int_0^t (t-s)^{-r/2} s^{-1/2} \| w(s) \|_{H^r} ds \mu_t, \quad 0 \leq t \leq T \tag{3.90} $$

and

$$ \| (Yw)(t) \|_{H^r} \leq c_r b^{(b-r)/2} |w|_T \mu_T, \quad 0 \leq t \leq T. \tag{3.91} $$

Proof. By (3.87), (3.78), (3.14) and (3.88) we have

\[
\| (Yw)(t) \|_{H^r} = \int_0^t \| D^r e^{-(t-s)\Delta} K(s)w(s) \|_{2\to 2} ds \\
\leq \int_0^t \| D^r e^{-(t-s)\Delta} \|_{2\to 2} \| K(s)w(s) \|_{2\to 2} ds \\
\leq \int_0^t c_r (t-s)^{-r/2} s^{-1/2} \| w(s) \|_{H^r} ds \mu_t \tag{3.92} \\
\leq \int_0^t c_r (t-s)^{-r/2} s^{-1/2} s^{(b-1)/2} ds \| w \|_t \mu_t \\
= \int_0^t c_r (t-s)^{-r/2} s^{(b/2)-1} ds \| w \|_t \mu_t \\
= c_r C_r/(b/2) t^{(b-r)/2} \| w \|_t \mu_t. \tag{3.93} 
\]

This proves both (3.90) and (3.91). The condition $b > 0$ is needed in the fourth line. $lacksquare$

3.4 Existence and uniqueness of mild solutions

Proof of Theorem 3.7 Define

$$ (Zw)(t) = e^{t\Delta} w_0 + (Yw)(t), \quad w \in Q_T^{(b)}. \tag{3.94} $$

We will show that, for sufficiently small $T$, $Z$ is a contraction mapping on a closed subset of the Banach space $Q_T^{(b)}$ invariant under $Z$. Take $r = b$ in
and then \( r = 1 \) to find
\[
\|(Yw)(t)\|_{H^b_0} \leq c_{b,b}\mu T|w|_T \quad (3.95)
\]
\[
\|(Yw)(t)\|_{H^b_1} \leq c_{1,b}\mu T|w|_T t^{(b-1)/2}. \quad (3.96)
\]
Therefore
\[
\sup_{0< t \leq T} \|(Yw)(t)\|_{H^b_0} \leq c_{b,b}\mu T|w|_T \quad \text{and} \quad (3.97)
\]
\[
\sup_{0< t \leq T} t^{(1-b)/2}\|(Yw)(t)\|_{H^b_1} \leq c_{1,b}\mu T|w|_T. \quad (3.98)
\]
Hence, in view of the definition \((3.15)\), we find
\[
\|Yw\|_{\mathcal{Q}^{(b)}_T} \leq c_5\mu T|w|_T, \quad (3.99)
\]
where \( c_5 = c_{b,b} + c_{1,b} \). We may choose \( T > 0 \) so small that \( c_5\mu T \leq 1/2 \). Then
\[
\|Yw\|_{\mathcal{Q}^{(b)}_T} \leq (1/2)|w|_T \leq (1/2)\|w\|_{\mathcal{Q}^{(b)}_T}. \quad (3.100)
\]

\( Yw \) is easily seen to have the appropriate continuity properties to lie in \( \mathcal{Q}^{(b)}_T \). \( Y \) is therefore a contraction in the Banach space \( \mathcal{Q}^{(b)}_T \) with contraction constant \( 1/2 \).

Concerning the freely propagated term in \((3.94)\), Lemma \(3.15\) shows that it lies in \( \mathcal{Q}^{(b)}_T \). Moreover, the inequality \( \|e^{t\Delta}w_0\|_{H^b_0} \leq \|w_0\|_{H^b_0} \), together with \((3.85)\), shows that \( t^{(1-b)/2}\|e^{t\Delta}w_0\|_{H^b_1} \leq e^Tc_b\|w_0\|_{H^b_0} \) for \( 0 < t \leq T \). Hence \( \|e^{(\cdot)\Delta}w_0\|_{\mathcal{Q}^{(b)}_T} \leq (1 + e^Tc_b)\|w_0\|_{H^b_0} \). Let \( \epsilon_b(T) = 1 + e^Tc_b \). Then the operator \( Z \) has the bounds
\[
\|Zw\|_{\mathcal{Q}^{(b)}_T} \leq \epsilon_b(T)\|w_0\|_{H^b_0} + (1/2)\|w\|_{\mathcal{Q}^{(b)}_T}, \quad \text{and} \quad (3.101)
\]
\[
\|Zw_1 - Zw_2\|_{\mathcal{Q}^{(b)}_T} \leq (1/2)\|w_1 - w_2\|_{\mathcal{Q}^{(b)}_T}. \quad (3.102)
\]
when \( w_1(0) = w_2(0) = w_0 \) because the freely propagated terms in \( Zw_j \) are the same and therefore cancel.

Let \( W = \{w \in \mathcal{Q}^{(b)}_T : w(0) = w_0\} \). This is a closed subset of \( \mathcal{Q}^{(b)}_T \) because of the presence of the second term in the norm, defined in \((3.15)\). It is invariant under \( Z \) because \((Zw)(0) = w(0) = w_0 \). Thus \( Z \) is a contraction on \( W \) and therefore has a unique fixed point in \( W \). By \((3.101)\) a fixed point under \( Z \) satisfies \( \|w\|_{\mathcal{Q}^{(b)}_T} = \|Zw\|_{\mathcal{Q}^{(b)}_T} \leq \epsilon_b(T)\|w_0\|_{H^b_0} + (1/2)\|w\|_{\mathcal{Q}^{(b)}_T} \), from which \((3.10)\) follows by subtraction. This completes the proof of Theorem \(3.7\).
3.5 Mild solutions are strong solutions

Remark 3.19 (Strategy) Typically, a solution to the integral equation (3.11) will be a strong solution if the integrand \( K(s)w(s) \) is Hölder continuous as a function of \( s \) into a suitable Banach space of functions on \( M \). In our circumstances the coefficient operator \( K(s) \) has a singularity at \( s = 0 \). For \( s > 0 \) it is more regular but by no means smooth. It will be necessary to use detailed information developed in Section 3.2 concerning the behavior of the connection form \( A(s) \) near and away from \( s = 0 \). We will show first that any mild solution \( w \) is Hölder continuous away from \( t = 0 \) as a function into \( H^{A}\). The main theorem of this section asserts that, for any mild solution \( w \), the function \( K(s)w(s) \) is a Hölder continuous function into \( L^2(M) \) on any finite interval \( [\tau, T] \) when \( \tau > 0 \). We will then use this to show that the solution to (3.11) is actually a strong solution for \( t > 0 \). We will prove this for \( w_0 \in H_b(M) \) whenever \( 0 < b < 1 \).

Theorem 3.20 Suppose that \( A(\cdot) \) is a strong solution to the Yang-Mills heat equation over \([0, \infty)\) of finite action, Let \( w \) be a solution to the integral equation (3.11) lying in \( Q_{T}^{(b)} \) for some \( b \in (0, 1) \). Then \( w \) is a strong solution to (3.10) over \((0, T]\).

The proof depends on the following lemmas.

**Lemma 3.21** Let \( 0 < \alpha < 1 \) and \( 0 < T < \infty \). Let \( -L \) be a non negative self-adjoint operator on a Hilbert space and let \( D = (1 - L)^{1/2} \). There is a constant \( e_{T,\alpha} \) such that

\[
\|D(e^{\epsilon L} - 1)e^{\delta L}\| \leq e^{\alpha} \delta^{-\frac{1}{2} - \alpha} e_{T,\alpha} \quad \forall \epsilon > 0 \quad \text{and} \quad \forall \delta \in (0, T]. \tag{3.103}
\]

**Proof.** With the help of the operator inequality

\[
\|D(e^{\epsilon L} - 1)e^{\delta L}\| \leq \|D^{-2\alpha}(e^{\epsilon L} - 1)\| \|D^{1+2\alpha}e^{\delta L}\|
\]

it suffices to make estimates of the two norms. By the spectral theorem, \( \|D^{-2\alpha}(e^{\epsilon L} - 1)\| \) is at most the supremum over \( x \in [0, \infty) \) of \((1 + x)^{-\alpha}(1 - e^{-\epsilon x}) = (1 + e^{-1}y)^{-\alpha}(1 - e^{-y}) = e^\alpha(\epsilon + y)^{-\alpha}(1 - e^{-y}) \leq e^\alpha y^{-\alpha}(1 - e^{-y}) \leq e^\alpha c_\alpha \), wherein we have put \( y = \epsilon x \). The second norm, writing \( c = (1/2) + \alpha \), is at most the supremum over \( x \in [0, \infty) \) of \((1 + x)^c e^{-\delta x} = (1 + \delta^{-1}y)^c e^{-y} = \delta^{-c}(\delta + y)^c e^{-y} \leq \delta^{-c}(T + y)^c e^{-y} \leq \delta^{-c} e_{T,\alpha} \) for \( 0 < \delta \leq T \), wherein we have put \( y = \delta x \). \( \blacksquare \)
Lemma 3.22 (Hölder continuity of $\rho$) Let $0 < \tau < T < \infty$. Suppose that $0 < b < 1$ and that $w \in Q_T^{(b)}$. Let $0 < \alpha < 1/2$. Define

$$\rho(t) = (Yw)(t) = \int_0^t e^{(t-s)\Delta} K(s) w(s) ds.$$ (3.104)

There is a constant $c_5$ depending only on $\alpha, \tau$ and $T$ such that

$$\|\rho(t) - \rho(r)\|_{H_1^A} \leq c_5 (t - r)^\alpha \mu_T |w|_T, \text{ for } \tau \leq r < t < T,$$ (3.105)

where $\mu_T$ is defined in Lemma 3.13.

Proof. Choosing $r$ and $t$ as in (3.105), we may write

$$\rho(t) - \rho(r) = \int_0^r \left( e^{(t-s)\Delta} - e^{(r-s)\Delta} \right) K(\sigma) w(\sigma) d\sigma + \int_r^t e^{(t-s)\Delta} K(\sigma) w(\sigma) d\sigma.$$ Therefore

$$\|\rho(t) - \rho(r)\|_{H_1^A} \leq \int_0^r \left\| \left( e^{(t-s)\Delta} - e^{(r-s)\Delta} \right) K(\sigma) w(\sigma) \right\|_{H_1^A} d\sigma + \int_r^t \left\| e^{(t-s)\Delta} K(\sigma) w(\sigma) \right\|_{H_1^A} d\sigma$$

$$\leq \int_0^r \left\| \left( e^{(t-s)\Delta} - e^{(r-s)\Delta} \right) \right\|_{2 \to H_1^A} \left\| K(\sigma) w(\sigma) \right\|_2 d\sigma + \int_r^t \left\| e^{(t-s)\Delta} \right\|_{2 \to H_1^A} \left\| K(\sigma) w(\sigma) \right\|_2 d\sigma.$$

$$\leq (t - r)^\alpha \int_0^r (r - \sigma)^{-\frac{b}{2} - \alpha} \left\| K(\sigma) w(\sigma) \right\|_2 d\sigma \cdot e_{T, \alpha} + \int_r^t (t - \sigma)^{-1/2} \left\| K(\sigma) w(\sigma) \right\|_2 d\sigma \cdot c_1.$$ (3.106)

The bound in the first line in (3.106) comes from (3.103) with $\delta = r - \sigma$ and $\epsilon = t - r$, while the bound in the second line comes from the spectral theory bound (3.87).

By (3.78) and (3.14) we have

$$\|K(s)w(s)\|_2 \leq s^{-1/2} \mu_T \|w(s)\|_{H_1^A} \leq \mu_T s^{-1/2} s^{(b-1)/2} |w|_T.$$ (3.107)
Insert the bound (3.107) into (3.106) to find
\[
\|\rho(t) - \rho(r)\|_{H_1^\alpha} \leq \left\{ (t - r)^\alpha \int_0^r (r - \sigma)^{-\frac{1}{2} - \alpha} \sigma^{(b/2) - 1} d\sigma \cdot e_{T,\alpha} \right. \\
+ \int_r^t (t - \sigma)^{-1/2} \sigma^{(b/2) - 1} d\sigma \cdot c_1 \right\} \mu_T \|w\|_T.
\]
(3.108)
(3.109)

The integral in line (3.108), which is finite because \(\alpha < 1/2\) and \(b > 0\), is at most \(r^{\frac{1}{2} - \alpha} \cdot \text{Const.} \leq r^{\frac{1}{2} - \alpha} \cdot \text{Const.}\) by (3.88). The integral in line (3.109) is at most
\[
\tau^{(b/2) - 1} \int_r^t (t - \sigma)^{-1/2} d\sigma = 2\tau^{(b/2) - 1}(t - r)^{1/2}.
\]
(3.110)

Since \((t - r)^{1/2} \leq (t - r)^\alpha \cdot \text{constant on } [\tau, T]\) the assertion (3.105) follows. ■

**Lemma 3.23** (Hölder continuity of \(w(\cdot)\)) Suppose that \(w\) is a solution to the integral equation (3.11) lying in \(Q_T^{(b)}\) for some \(b \in (0, 1)\). Let \(0 < \alpha < 1/2\) and let \(0 < \tau < T < \infty\). Then there is a constant \(c_6\), depending only on \(\alpha, \tau, T, A, w(0)\) and \(|w|_T\) such that
\[
\|w(t) - w(r)\|_{H_1^\alpha} \leq c_6(t - r)^\alpha \quad \text{for} \quad \tau \leq r \leq t \leq T
\]
(3.111)

**Proof.** For any function \(w_0 \in H_1^\alpha\) the function \(t \mapsto e^{t \Delta w_0}\) is differentiable on the interval \([\tau, \infty)\) into \(H_1^\alpha\) and therefore locally Hölder of order \(\alpha\). Since the second term on the right in (3.11) has been shown in Lemma 3.22 to be Hölder continuous of order \(\alpha\) on the interval \([\tau, T]\) the lemma follows. ■

**Lemma 3.24** (Hölder continuity of \(K(\cdot)w(\cdot)\)) Let \(w(\cdot)\) be a solution of the integral equation (3.11) lying in \(Q_T^{(b)}\) for some \(b \in (0, 1)\). Let \(\tau > 0\) and let \(0 < \alpha < 1/2\). \(K(s)w(s)\) is Hölder continuous on \([\tau, T]\) of order \(\alpha\) as a function into \(L^2\)

**Proof.** If \(\tau \leq r < t \leq T\) then, in view of (3.80) (3.78) and (3.111) we have
\[
\|K(t)w(t) - K(r)w(r)\|_2 \leq \|(K(t) - K(r))w(t)\|_2 + \|K(r)(w(t) - w(r))\|_2 \\
\leq (t - r)^{3/4}\|w(t)\|_{H_1^\alpha} m_\tau + \mu_T \tau^{-3/2}\|w(t) - w(r)\|_{H_1^\alpha} \\
\leq (t - r)^{3/4}\tau^{-1/2}\|w\|_T m_\tau + \mu_T \tau^{-1/2}(t - r)^\alpha c_6.
\]
wherein we have used \( t^{(b-1)/2} \leq \tau^{-1/2} \) for \( t \geq \tau \) in the last line. ■

**Proof of Theorems 3.20 and 2.19.** We can apply Theorem 11.44 in [16] over the interval \([\tau, T]\) for any \( \tau > 0 \) because we now know, in view of Lemma 3.24, that the forcing function \( K(s)w(s) \) is Hölder continuous on this interval as a function into \( L^2(M; \Lambda^1 \otimes \mathfrak{k}) \). The strong time derivative \( w'(s) \) of the function \([\tau, T] \ni s \mapsto w(s) \in L^2(M)\) therefore exists, \( w(s) \) is in the domain of \( \Delta_A \), and both \( w'(s) \) and \( \Delta_A w(s) \) are Hölder continuous of order \( \alpha \) on \([\tau, T]\) into \( L^2(M; \Lambda^1 \otimes \mathfrak{k}) \). Moreover the equation (3.10) holds for each \( t \in [\tau, T] \). This proves Theorem 3.20.

For the uniqueness of strong solutions asserted in Theorem 2.19 observe that the hypotheses of Theorem 2.19 imply that \( w(\cdot) \) lies in \( Q_T^{(b)} \) for any \( T > 0 \). We can now apply an argument similar to that used in the proof of [6, Theorem 3.30] to conclude that \( w(\cdot) \) satisfies the integral equation (3.11). Uniqueness now follows from Theorem 3.7.

To extend \( w(\cdot) \) to a solution over all of \([0, \infty)\) observe that

\[
\rho_A(t_0, t_0 + t) := \frac{1}{2} \int_{t_0}^{t_0 + t} (s - t_0)^{-1/2} \|B(s)\|_2^2 \, ds \leq \rho_A(t)
\]

for all \( t_0 \geq 0 \) and all \( t \geq 0 \) because \( \|B(s)\|_2^2 \) is nonincreasing. Therefore if one starts the existence theorem at some time \( t_0 \geq 0 \) then the short time \( T \) needed to make \( c_5 \mu_T \leq 1/2 \) in the proof of contractivity of \( Y \) in Section 3.4 can be chosen independently of \( t_0 \) because \( \mu_T \) depends monotonically only on \( T \) and \( \rho_A(t_0, t_0 + T) \). Having proven existence up to time \( t_0 \geq T \) one can therefore continue the solution up to time \( t_0 + (T/2) \) by applying the short time existence theorem to \( w(t_0 - (T/2)) \). This completes the proof of Theorem 2.19, Parts a) and b). ■

### 4 Finite \( b \)-action for the augmented equation

For \( 0 \leq b < 1 \) either of the following two conditions gives a measure of the singular behavior of \( w(t) \) near \( t = 0 \).

\[
\|w(t)\|^2_{H^1_A} = o(t^{b-1}) \quad \text{as} \quad t \downarrow 0.
\]

\[
\int_0^T t^{-b} \|w(t)\|^2_{H^1_A} \, dt < \infty \quad \text{for some} \quad T \in (0, \infty).
\]
Both conditions in (4.1) and (4.2) are gauge invariant. Neither one implies the other. The existence theorem, Theorem 3.7, shows that the solution to the augmented variational equation (2.21) satisfies (4.1) if $b \in (0,1)$ and $w_0 \in H_b^A$. In this section we will prove that if $b \in [1/2,1)$ then the solution also satisfies (4.2).

**Theorem 4.1** Assume that $a(\cdot)$ is a strong solution of the Yang-Mills heat equation over $[0,\infty)$ of finite action. Suppose that $1/2 \leq b < 1$, $0 < T < \infty$ and that $w_0 \in H_b^A$, where $A = A(T)$ as in Section 3. If $w(\cdot)$ is the mild solution of the augmented variational equation (2.21) with initial value $w_0$ and lying in $Q_T^{(b)}$, then for sufficiently small $T$ there holds

$$\int_0^T s^{-b}\|w(s)\|^2_{H^A}ds \leq \gamma_T \|w_0\|^2_{H_b^A} < \infty$$  \hspace{1cm} (4.3)

for some constant $\gamma_T$ depending only on $T$ and $\rho_A(T)$.

We are going to use the following abstract action bound lemma from [6].

**Lemma 4.2** Let $L$ be a non-negative self-adjoint operator on a Hilbert space $H$. Suppose that $\alpha, \mu, b$ are real numbers such that

$$0 \leq \alpha \leq 1,$$  \hspace{1cm} (4.4)

$$0 \leq \mu \leq b < 1,$$  \hspace{1cm} (4.5)

$$\delta \equiv 1 - \alpha - \mu \geq 0,$$  \hspace{1cm} (4.6)

Then there is a constant $C_{\alpha,\mu}$, depending only on $\alpha$ and $\mu$, such that for any measurable function $g : (0,T) \rightarrow H$ there holds

$$\int_0^T t^{-b}\left\| \int_0^t s^{-\mu}L^\alpha e^{(t-s)L}g(s)ds \right\|^2 dt \leq T^{2\delta} \int_0^T s^{-b}\|g(s)\|^2 ds \cdot C_{\alpha,\mu}$$  \hspace{1cm} (4.7)

when the right side is finite. Moreover $C_{\alpha,0} \leq 1$.

**Proof.** See [6] Theorem 3.19] .

**Proof of Theorem 4.1** Define

$$g(s) = s^{1/2}\left(K(s)w(s)\right), \hspace{1cm} 0 < s \leq T.$$  \hspace{1cm} (4.8)
Then
\[ \|g(s)\|_2 \leq \|w(s)\|_{H^1_T} \mu_T \] (4.9)
by (3.78). Let us write \( L = 1 - \Delta_A \) and \( D = L^{1/2} \). With \( Yw \) defined as in (3.89) we have
\[
\|(Yw)(t)\|_{H^1_T} = \left\| \int_0^t e^{(t-s)A}s^{-1/2}g(s)ds \right\|_{H^1_T} \\
= \left\| \int_0^t D(e^{(t-s)(1-L)}s^{-1/2}g(s)ds \right\|_2 \\
= \left\| e^{t} \int_0^t s^{-1/2}L^{1/2}e^{-(t-s)L}e^{-s}g(s)ds \right\|_2. 
\] (4.10)

We can apply Lemma 4.2 with \( H = L^2(M; \Lambda^1 \otimes \mathfrak{p}) \), \( \alpha = 1/2 \) and \( \mu = 1/2 \). Then (4.7) holds with \( \delta = 0 \). Hence
\[
\int_0^T t^{-b} \|(Yw)(t)\|_{H^1_T}^2 dt = \int_0^T t^{-b}e^{2t} \left\| \int_0^t e^{-(t-s)L}e^{-s}g(s)ds \right\|_2^2 dt \\
\leq e^{2T} \int_0^T s^{-b}\|e^{-s}g(s)\|_2^2 ds \cdot C_{\alpha,\mu} \\
\leq e^{2T} \mu_T^2 \int_0^T s^{-b}\|w(s)\|_{H^1_T}^2 ds C_{\alpha,\mu} 
\] (4.11)
for \( 1/2 \leq b < 1 \). Choose \( T \) sufficiently small so that \( e^{2T} \mu_T^2 C_{\alpha,\mu} \leq 1/4 \) and also \( c_5 \mu_T \leq 1/2 \), as in Section 3.4. Then
\[
\left( \int_0^T t^{-b}\|(Yw)(t)\|_{H^1_T}^2 dt \right)^{1/2} \leq \frac{1}{2} \left( \int_0^T s^{-b}\|w(s)\|_{H^1_T}^2 ds \right)^{1/2} 
\] (4.12)
and also (3.100) holds.

We can now adapt the final step of the proof of Theorem 3.7 for our present purpose by simply changing the norm on the space \( Q_T^{(b)} \) defined in Notation 3.6 thus: Replace the norm (3.15) by
\[
\|w\|_{Q_T^{(b)}} = \left( \int_0^T s^{-b}\|w(s)\|_{H^1_T}^2 ds \right)^{1/2} + \|w\|_{Q_T^{(b)}} 
\] (4.13)
and strengthen the condition (3.12) by requiring also \( \|w\|_{\hat{Q}_{T}^{(b)}} < \infty \). Then, by (4.12) and (3.100), \( Y \) is a contraction in the resulting Banach space, \( \hat{Q}_{T}^{(b)} \) with contraction constant 1/2. Moreover, if \( w_0 \in H^A_b \) then the freely propagated term \( e^{t\Delta A}w_0 \) in the integral equation (3.11) lies in this space by (3.86). Hence the integral equation (3.11) has a unique solution in \( \hat{Q}_{T}^{(b)} \). Since \( \hat{Q}_{T}^{(b)} \subset Q_{T}^{(b)} \), the unique solution in \( \hat{Q}_{T}^{(b)} \) is the same as the unique solution in \( Q_{T}^{(b)} \). The inequality (4.3) now follows in the same way as (3.16). \( \square \)

5 Initial behavior of solutions to the augmented variational equation

5.1 Pointwise and integral identities

For a solution \( w(\cdot) \) to the augmented variational equation (2.21) on some interval there are two quantities whose behavior near \( t = 0 \) will largely determine the short time behavior of the solution to the variational equation (2.6) itself. Define

\[
\psi(s) = d^*_A(s)w(s) \quad \text{and} \quad \zeta(s) = d^*_A d_A w(s) + [w(s)] B(s).
\]

\( \psi(s) \) measures the deviation of \( w(s) \) from horizontal at \( A(s) \). The augmented variational equation may be written

\[
-w'(s) = \zeta(s) + d_A \psi(s).
\]

**Lemma 5.1** (Pointwise identities) If \( w \) is a solution to the augmented variational equation (2.21) on some interval then

(Order)

(1) \( d^*_A \zeta(s) = [w \cdot A'] \) \( \quad (5.4) \)

(1) \( (d/ds)\psi(s) = -d^*_A d_A \psi + 2[A' \cdot w] \). \( \quad (5.5) \)

(2) \( -w''(s) = \left( d^*_A d_A + d_A d^*_A \right) w' \)

\[ + \left\{d^*_A[A' \wedge w] + [A' \cdot d_A w] + d_A [A' \cdot w] + [A', d^*_A w] \right\} + [w \cdot B]' \]. \( \quad (5.6) \)
Proof. Using the identity \( d_A^*[\omega \wedge B] = [d_A \omega \wedge B] - [\omega \wedge d_A^* B] = [d_A \omega \wedge B] + [\omega \wedge A'] \), we may apply \( d_A^* \) to \( \zeta \) to find \( d_A^* \zeta = [B \wedge d_A w] + [d_A w \wedge B] + [w \wedge A'] = [w \wedge A'] \), which is (5.4).

For the proof of (5.5) differentiate the definition of \( \psi \) to find
\[
(d/ds) \psi(s) = (d/ds)(d_A^*(s) w(s))
= d_A^* w' + [A' \wedge w]
= -d_A^*(\zeta(s) + d_A \psi) + [A' \wedge w]
= -[w \wedge A'] - d_A^* d_A \psi + [A' \wedge w],
\]
which proves (5.5).

Differentiate (2.21) with respect to \( s \) to find (5.6).

Lemma 5.2 (Integral identities) Denote by \( L_A \) the gauge covariant Hodge Laplacian given by
\[
-L_A = d_A^* d_A + d_A d_A^*,
\]
(not to be confused with the Bochner Laplacian given by (3.1).) If \( w \) is a strong solution to the augmented variational equation (2.21) on some interval then

(Order)
\[
(0) \quad \frac{d}{ds} \|w(s)\|^2_2 + 2 \left\{ \|d_A w(s)\|^2_2 + \|d_A^* w(s)\|^2_2 \right\}
= -2(B(s), [w(s) \wedge w(s)]),
\]
\[ (5.8) \]
\[
(1) \quad \frac{d}{ds} \left\{ \|d_A(s) w(s)\|^2_2 + \|d_A^*(s) w(s)\|^2_2 \right\} + \|w'(s)\|^2_2 + \|L_A(s) w(s)\|^2_2
= 2 \left\{ ([A' \wedge w], d_A w) + ([A' \wedge w], d_A^* w) \right\} + \|w \wedge B\|^2_2
\]
\[ (5.9) \]
\[
(2) \quad \frac{d}{ds} \|w'(s)\|^2_2 + 2 \left\{ \|d_A(s) w'(s)\|^2_2 + \|d_A^*(s) w'(s)\|^2_2 \right\}
= -2 \left\{ ([A' \wedge w], d_A w') + ([A' \wedge w], d_A^* w') \right\}
+ \left( [A' \wedge d_A w] + [A', d_A^* w], w' \right) + ([w \wedge B]' w'),
\]
\[ (5.10) \]

Proof. From (2.21) we find
\[
(1/2)(d/ds) \|w(s)\|^2_2 = ((d/ds) w(s), w(s))
= (-d_A^* d_A w - d_A d_A^* w - [w \wedge B], w)
= -\|d_A w\|^2_2 - \|d_A^* w\|^2_2 - (B, [w \wedge w]),
\]
46
which is (5.8).

For ease in reading define \( g(s) = \left( w(s) \right)_B B(s) \). Then we may write (2.21) as \( w' = L_A w - g \). For the proof of (5.9) observe first that

\[
(w', L_A w) = (w', w' + g) = \| w' \|^2 + (w', g), \quad \text{while also} \quad (5.11)
\]

\[
(w', L_A w) = (L_A w - g, L_A w) = \| L_A w \|^2 - (g, L_A w) = \| L_A w \|^2 - (g, w' + g) = \| L_A w \|^2 - \| g \|^2 - (g, w'). \quad (5.12)
\]

Adding (5.11) to (5.12) gives

\[
2(w', L_A w) = \| w' \|^2 + \| L_A w \|^2 - \| g \|^2. \quad (5.13)
\]

Hence

\[
\frac{1}{2} \frac{d}{ds} \left\{ \| d_A(s) w(s) \|^2 + \| d_A^*(s) w(s) \|^2 \right\} = \left\{ \left( \frac{d}{ds} (d_A w), d_A w \right) + \left( \frac{d}{ds} (d_A^* w), d_A^* w \right) \right\} = \left\{ ([A' \wedge w, d_A w] + [A' \rhd w, d_A^* w]) + (w', d_A^* d_A w) + (w', d_A d_A^* w) \right\} = \left\{ ([A' \wedge w, d_A w] + [A' \rhd w, d_A^* w]) - (w', L_A w) \right\}. \quad (5.14)
\]

Replace the last term in (5.14) by (5.13) to find (5.9).

To prove the second order identity (5.10) use (5.6) to see that

\[
\frac{1}{2} (d/ds) \| w'(s) \|^2 = (w'', w') = \left( \frac{d}{ds} (d_A^* w) \right), \quad (g', w') - \left\{ \left\{ d_A^*[A' \wedge w] + [A' \rhd d_A w] + d_A[A' \rhd w] + [A', d_A^* w] \right\}, w' \right\}.
\]

Hence

\[
\frac{1}{2} (d/ds) \| w'(s) \|^2 = \| d_A w' \|^2 + \| d_A^* w' \|^2 \]

\[
= - \left\{ \left\{ ([A' \wedge w], d_A w') + ([A' \rhd w], d_A^* w') \right\} + \left\{ [A' \rhd d_A w] + [A', d_A^* w], w' \right\} \right\} - (g', w'),
\]

which is (5.10).
We will use these identities to derive differential inequalities and then, from these, derive information about initial behavior of solutions with the help of the following lemma.

**Lemma 5.3** Suppose that \( f, g, h \) are nonnegative continuous functions on \((0, t]\) and that \( f \) is differentiable. Suppose also that
\[
\frac{d}{ds}f(s) + g(s) \leq h(s), \quad 0 < s \leq t \tag{5.15}
\]
Let \(-\infty < b < 1\) and assume that
\[
\int_0^t s^{-b}f(s)ds < \infty \tag{5.16}
\]
Then
\[
t^{1-b}f(t) + \int_0^t s^{(1-b)}g(s)ds \leq \int_0^t s^{(1-b)}h(s)ds + (1-b) \int_0^t s^{-b}f(s)ds. \tag{5.17}
\]
If equality holds in (5.15) then equality holds in (5.17).

**Proof.** See [6, Lemma 4.8] for a proof. \(\blacksquare\)

**Remark 5.4** (Gaffney-Friedrichs-Sobolev inequality) The main technique in the next few subsections will be based on the Gaffney-Friedrichs inequality, which asserts, for our convex subset of \(\mathbb{R}^3\), that for any integer \(p \geq 1\) and any \(k\)-valued \(p\)-form \(\omega\) (satisfying appropriate boundary conditions) there holds
\[
(1/2)\|\omega\|^2_{H^1_t} \leq \left\{ \|d^*_A\omega\|^2_2 + \|d_A\omega\|^2_2 + \lambda(B)\|\omega\|^2_2 \right\} \tag{5.18}
\]
for any \(k\)-valued connection form \(A \in W^1_t(M; \Lambda^1 \otimes \mathfrak{b})\) with curvature \(B\). Here we have written
\[
\lambda(B) = 1 + \gamma\|B\|_2^4, \tag{5.19}
\]
where \(\gamma \equiv (27/4)\kappa^6c^4\) is a constant depending only on a Sobolev constant \(\kappa\) for \(M\) and the commutator bound \(c \equiv \sup\{\|\text{ad} x\|_{t \to t} : \|x\|_t \leq 1\}\). The \(H^1_t\) norm is defined in Notation 2.6

Usually we will use the Sobolev bound that follows from this:
\[
\|\omega\|^2_6 \leq \kappa^2 \left\{ \|d^*_A\omega\|^2_2 + \|d_A\omega\|^2_2 + \lambda(B)\|\omega\|^2_2 \right\}. \tag{5.20}
\]
These inequalities allow us to make good use of the Bianchi identity, which usually simplifies one of the terms on the right side of (5.18) and (5.20). See [2] Theorem 2.17, Remark 2.18 and Eqn.(4.31)] for the derivation of these inequalities.
5.2 Initial behavior of $w$, order 1

In Section 3 we proved existence and uniqueness of strong solutions to the augmented variational equation for initial value in $H_b^A$ with $0 < b < 1$. We want to derive more detailed information about the short time behavior of derivatives of the solution. All of our bounds on derivatives will be dominated by the following gauge invariant functional of the solution.

**Definition 5.5** Let $0 \leq b < 1$. The $b$-action of a function $w : [0, \infty) \to \{t \text{ valued 1-forms on } M\}$ up to time $t$ is

$$
\|w\|_t^2 = \begin{cases} 
\int_0^t s^{-b} \|\nabla^A (s) w(s)\|_2^2 ds & \text{if } M = \mathbb{R}^3 \\
\int_0^t s^{-b} \left( \|\nabla^A (s) w(s)\|_2^2 + \|w(s)\|_2^2 \right) ds & \text{if } M \text{ is bounded.}
\end{cases}
$$

(5.21)

A strong solution $w$ to the augmented variational equation has finite $b$-action if

$$
\|w\|_t < \infty \text{ for all } t > 0.
$$

(5.22)

In previous sections we have used the Sobolev norms given by $\|\omega\|_{H^1_b}^2 = \|\nabla^A (T) \omega\|_2^2 + \|\omega\|_2^2$ rather than the varying norms used in the integrands in (5.21). The notion “finite strong $b$-action” was defined in Definition 2.14 by the condition

$$
\int_0^\tau s^{-b} \|w(s)\|_{H^1_b}^2 ds < \infty \text{ for some } \tau > 0.
$$

(5.23)

This differs from the notion of finite $b$-action given in Definition 5.5 in two ways: Most importantly, the additive $L^2$ norm, which is present in the integrand in (5.23), is absent from (5.21) when $M = \mathbb{R}^3$. Secondly, there is the distinction between use of $A(s)$ versus fixed $A(T)$. This is not a significant distinction because these norms are equivalent, uniformly for $0 \leq s \leq \tau$, by virtue of Lemma 7.6 and our standing assumption (2.5). When $M \neq \mathbb{R}^3$ the presence of the term $\|w(s)\|_2^2$ is essential for use in Sobolev inequalities for dominating the $L^6$ norm because none of our boundary conditions requires $w$ to be zero on $\partial M$. In this case (5.23) is equivalent to (5.22). When $M = \mathbb{R}^3$, however, this added term is not needed for bounding $L^6$ norms and (5.23) is strictly stronger than (5.22) when $M = \mathbb{R}^3$. We will use the action norm (5.21) extensively to bound $L^6$ norms and no other $L^p$ norms. It will be used
in [7] as a gauge invariant Riemannian metric on a space of solutions to the Yang-Mills heat equation.

Since a strong solution to the augmented variational equation is a continuous function on \((0, \infty)\) into \(H^A_t\), it follows that \(\|w\|_t < \infty\) for all \(t > 0\) if \(\|w\|_t < \infty\) for some \(t > 0\). It was shown in (4.3) that (5.23) holds for any mild solution to the augmented variational equation lying in \(Q^{(b)}_T\), at least when \(1/2 \leq b < 1\). In particular \(\|w\|_t < \infty\) also, for all \(t > 0\) if its initial value lies in \(H^A_t\).

In this section we are going to let \(b \in [0, 1)\) and take as a hypothesis that our solution \(w\) has finite \(b\)-action in the sense of (5.22). Whether \(M\) is bounded or not we have the easily verified bounds

\[
\int_0^t s^{-b} \left(\|d_{A(s)}w(s)\|^2_2 + \|d^*_{A(s)}w(s)\|^2_2\right) ds \leq 4\|w\|^2_t \tag{5.24}
\]

\[
\int_0^t s^{-b} \|w(s)\|^2_6 ds \leq \kappa_6^2\|w\|^2_t. \tag{5.25}
\]

Our goal in this section is to establish bounds on the initial behavior of \(w\) and its derivatives entirely in terms of the action \(\|w\|_t\). The artificial decomposition (3.10) and the associated estimates will not be used in this section or any further in this paper.

**Theorem 5.6 (Initial behavior of \(w\), order 1).** Let \(0 \leq b < 1\). Suppose that \(A(\cdot)\) is a strong solution to the Yang-Mills heat equation over \([0, \infty)\) of finite action. Let \(w(\cdot)\) be a strong solution to the augmented variational equation (2.21), not necessarily lying in \(Q^{(b)}_T\), but with finite \(b\)-action in the sense of (5.22). Let \(\psi(s) = d^*_{A(s)}w(s)\) and \(\zeta(s) = d^*_{A}d_Aw(s) + [w(s), \gamma B(s)]\) as in (5.2) and (5.22). Then there are standard dominating functions \(C_j\) such that

\[
t^{1-b} \left\{\|d_A(t)w(t)\|^2_2 + \|d^*_A(t)w(t)\|^2_2\right\} + \int_0^t s^{1-b} \left\{\|w'(s)\|^2_2 + \|L_{A(s)}w(s)\|^2_2\right\} ds
\]

\[
\leq C_{87}(t, \rho_A(t))\|w\|^2_t \tag{5.26}
\]

\[
\int_0^t s^{-b} \left\{\|d^*_A d_A w(s)\|^2_2 + \|d_A w(s)\|^2_6 + \|d_A \psi(s)\|^2_2\right.
\]

\[
\left. + \|\psi(s)\|^2_6 + \|d\psi(s)\|^2_2 + \|\zeta(s)\|^2_2\right\} ds \leq C_{88}(t, \rho_A(t))\|w\|^2_t, \tag{5.27}
\]

50
where $L_{A(s)}$ is the Hodge Laplacian, defined in (3.7). Moreover the following interpolation bounds hold.

\[
\int_0^t s^{(1/2)-b} \| \psi(s) \|_3^2 ds < \infty \quad \text{if } 0 \leq b < 1 \quad \text{and} \quad (5.28)
\]

\[
\int_0^t \| \psi(s) \|_3 ds = O(t^{(2b+1)/4}) \quad \text{if } 0 \leq b < 1. \quad (5.29)
\]

The proof depends on the following lemmas.

**Lemma 5.7** (Differential inequality, order 1) Suppose that $A(\cdot)$ is a strong solution to the Yang-Mills heat equation over $(0, \infty)$ with finite action and that $w$ is a strong solution to (2.21) over $(0, \infty)$. Then

\[
\frac{d}{ds} \left\{ \| d_{A(s)} w(s) \|_2^2 + \| d^*_{A(s)} w(s) \|_2^2 + \| L_{A(s)} w(s) \|_2^2 \right\} + \| w'(s) \|_2^2 + \| L_{A(s)} w(s) \|_2^2 \\
\leq 2c \| A'(s) \|_3 \| w(s) \|_6 \left( \| d_{A(s)} w(s) \|_2 + \| d^*_{A(s)} w(s) \|_2 \right) + c^2 \| w(s) \|_6^2 \| B(s) \|_3^2.
\]

(5.30)

**Proof.** It suffices to show that the right side of (5.9) is bounded by the right side of (5.30). But

\[
2 \left\{ ([A' \wedge w], d_{A(s)} w) + ([A' \odot w], d^*_{A(s)} w) \right\} + \| w \odot B \|_2^2 \\
\leq 2c \left\{ \| A'(s) \|_3 \| w(s) \|_6 \| d_{A(s)} w(s) \|_2 + \| A'(s) \|_3 \| w(s) \|_6 \| d^*_{A(s)} w(s) \|_2 \right\} \\
+ c^2 \| w(s) \|_6^2 \| B(s) \|_3^2,
\]

which is (5.30). \qed

The term $\| L_{A(s)} w(s) \|_2^2$ in line (5.26) contains second derivatives of $w$. We wish to use these second derivatives to estimate $L^6$ norms of the first derivatives of $w$. However the cross terms in the expansion of $\| L_{A(s)} w(s) \|_2^2$ will have to be separated out first and controlled before we can use the Gaffney-Friedrichs-Sobolev inequality (5.20). The next lemma is aimed at this.

**Lemma 5.8** (Cross terms) Suppose that $A$ is a solution to the Yang-Mills heat equation over $(0, \infty)$ with finite action. Let $0 \leq b < 1$. If $w$ is a function
(not necessarily a solution) with finite $b$-action then there exists a standard dominating function $C_{93}$ such that

$$
\int_0^t s^{1-b} \left( \|d_A^* d_A w(s)\|_2^2 + \|d_A d_A^* w(s)\|_2^2 \right) ds
\leq 2 \int_0^t s^{1-b} \|L_{A(s)} w(s)\|_2^2 ds + C_{93}(t, \rho_A(t)) \|w\|_1^2.
$$

(5.31)

The proof depends on the following lemma.

**Lemma 5.9 (Cross term inequality)** Let $0 \leq b < 1$ and let $s > 0$. Suppose that $w(s)$ lies in the domains of both $d_{A(s)}^* d_{A(s)}$ and $d_{A(s)} d_{A(s)}^*$. Let

$$
(U(s) = 2(d_A w(s), [B(s), d_A^* w(s)]).
$$

(5.32)

Then

$$
\|d_A^* d_A w(s)\|_2^2 + \|d_A d_A^* w(s)\|_2^2 = \|L_{A(s)} w(s)\|_2^2 - U(s).
$$

(5.33)

Moreover

$$
s^{1-b} \|U(s)\| \leq (1/2)s^{1-b} \|d_A^* d_A w(s)\|_2^2
\leq c^2 \left( s^{3/2} \|B(s)\|_2^2 + \left( s^{1/2} \|d_A^* w(s)\|_2^2 \right) + \left( s^{1/2} \|d_A w(s)\|_2^2 \right) \right)
\leq (1/2)c^2 s^{1-b} \|w\|_6^2 \|B\|_3^3 + \left( s\lambda(B(s))/2 \right) \left( s^{-b} \|d_A w(s)\|_2^2 \right).
$$

(5.34)

(5.35)

(5.36)

**Proof.** Expand $\|L_{A(w)}\|_2^2$ to find

$$
\|L_{A(w)}\|_2^2 = (d_A^* d_A w + d_A d_A^* w, d_A^* d_A w + d_A d_A^* w)
= \|d_A^* d_A w\|_2 + \|d_A d_A^* w\|_2^2 + 2(d_A^* d_A w, d_A d_A^* w).
$$

The last term is $2(d_A w, d_A^* d_A w)$, which is $2(d_A w, [B(s), d_A^* w])$, by the Bianchi identity. This gives (5.33) in view of the definition of $U(s)$ in (5.32). By Hölder’s inequality we now find

$$
s^{1-b} \|U(s)\| = 2s^{1-b} \left( \|d_A w(s), [B(s), d_A^* w(s)]\| \right)
\leq 2cs^{1-b} \|B(s)\|_6 \|d_A^* w(s)\|_2 \|d_A w(s)\|_3
\leq 2c \left( s^{3/4} \|B(s)\|_6 \right) \left( s^{-b/2} \|d_A^* w(s)\|_2 \right) \left( s^{(1/4) - (b/2)} \|d_A w(s)\|_3 \right)
\leq c^2 \left( s^{3/2} \|B(s)\|_6^2 \right) \left( s^{1/2} \|d_A^* w(s)\|_2^2 \right) + s^{(1/2) - b} \|d_A w(s)\|_3^2.
$$

(5.37)
The first of these two terms is the first term in line (5.35). The second term in line (5.37) can be dominated by interpolation between $L^2$ and $L^6$ thus:

\[
\begin{align*}
s^{(1/2-b)}\|d_Aw(s)\|_2^2 & \leq \left(\kappa s^{-b/2}\|d_Aw(s)\|_2\right) \left(s^{(1-b)/2} \kappa^{-1}\|d_Aw(s)\|_6\right) \\
& \leq (1/2)\kappa^2 s^{-b}\|d_Aw(s)\|_2^2 + (1/2)s^{1-b}\kappa^{-2}\|d_Aw(s)\|_6^2. \\
\end{align*}
\]

The first term in line (5.38) is the second term in line (5.35). We can dominate the second term in line (5.38) by applying the Gaffney-Friedrichs-Sobolev inequality (5.21) to the 2-form $\omega = d_Aw(s)$. We find

\[
(1/2)s^{1-b}\kappa^{-2}\|d_Aw(s)\|_6^2 \\
\leq (1/2)s^{1-b}\left(\|d_A^*d_Aw(s)\|_2^2 + \|d_Ad_Aw(s)\|_2^2 + \lambda(B(s))\|d_Aw(s)\|_2^2\right) \\
\leq (1/2)s^{1-b}\|d_A^*d_Aw(s)\|_2^2 + \lambda(B(s))\|d_Aw(s)\|_2^2 + \left(1/2\right)s\lambda(B(s))\left(s^{-b}\|d_Aw(s)\|_2^2\right). \\
\leq (1/2)s^{1-b}\|d_A^*d_Aw(s)\|_2^2 + (1/2)s^{1-b}c^2\|w(s)\|_0^2\|B(s)\|_2^3 + \left(1/2\right)s\lambda(B(s))\left(s^{-b}\|d_Aw(s)\|_2^2\right). \\
\]

The three terms on the right are the terms that appear in lines (5.34) and (5.36). This completes the proof of Lemma 5.9. ■

**Proof of Lemma 5.8.** From (5.33) we see that

\[
\begin{align*}
\int_0^t s^{1-b}\left(\|d_A^*d_Aw(s)\|_2^2 + \|d_Ad_A^*w(s)\|_2^2\right)ds \\
& \leq \int_0^t s^{1-b}\|L_A(s)w(s)\|_2^2ds + \int_0^t s^{1-b}|U(s)|ds \\
& \leq \int_0^t s^{1-b}\|L_A(s)w(s)\|_2^2ds + \left(1/2\right)\int_0^t s^{1-b}\|d_A^*d_Aw(s)\|_2^2ds \\
& + \int_0^t \left\{c^2\left(s^{3/2}\|B(s)\|_6\right)^2\left(s^{-b}\|d_A^*w(s)\|_2^2\right) + \left(\kappa^2/2\right)\left(s^{-b}\|d_Aw(s)\|_2^2\right)ight. \\
& \left. + (1/2)c^2s^{1-b}\|w(s)\|_0^2\|B(s)\|_2^3 + \left(s\lambda(B(s))/2\right)\left(s^{-b}\|d_Aw(s)\|_2^2\right)\right\}ds. \\
\end{align*}
\]

The second term in line (5.39) cancels with half of one term on the left. It suffices to show, therefore, that the integral of each of the four terms in the
These four integrals add to at most 5.3 with \( f, g, h \) in accordance with Lemma 3.9 with \( a = \frac{1}{2} \). All four integral factors are dominated by \( \|w\|_2 \) by (5.24) and (5.25). This concludes the proof of (5.31).

Proof of Theorem 5.6. For the proof of (5.26) we need only apply Lemma 5.3 with \( f, g, h \) chosen to match up with the differential inequality (5.30). Thus we take \( f(s) = \|d_A(s)w(s)\|_2^2 + \|d^*_A(s)w(s)\|_2^2 \), take \( g(s) = \|w'(s)\|_2^2 + \|L_A(s)w(s)\|_2^2 \) and take \( h(s) \) to be the entire right hand side of (5.30). We find from (5.17) that

\[
\begin{align*}
& t^{1-b} \left\{ \|d_A(t)w(t)\|_2^2 + \|d^*_A(t)w(t)\|_2^2 \right\} + \int_0^t s^{1-b} \left\{ \|w'(s)\|_2^2 + \|L_A(s)w(s)\|_2^2 \right\} ds \\
& \leq \int_0^t s^{1-b} \left\{ 2c\|A'(s)\|_3 \|w(s)\|_6 \left( \|d_A(s)w(s)\|_2 + \|d^*_A(s)w(s)\|_2 \right) \\
& \quad + c^2 \|w(s)\|_2^2 \|B(s)\|_3^2 \right\} ds \\
& + (1-b) \int_0^t s^{-b} \left\{ \|d_A(s)w(s)\|_2^2 + \|d^*_A(s)w(s)\|_2^2 \right\} ds.
\end{align*}
\] (5.41)

The integrals in lines (5.41) and (5.42) add to at most

\[
\begin{align*}
& (2c)(\sup_{0<s\leq t} s\|A'(s)\|_3) \left( \int_0^t s^{-b} \|w(s)\|_6^2 ds \right)^{1/2} \\
& \quad \left( \int_0^t s^{-b} \left( \|d_A(s)w(s)\|_2 + \|d^*_A(s)w(s)\|_2 \right)^2 ds \right)^{1/2} \\
& + c^2 \sup_{0<s\leq t} s\|B(s)\|_3^2 \int_0^t s^{-b} \|w(s)\|_6^2 ds.
\end{align*}
\] (5.44)
The two suprema in (5.44) are bounded by standard dominating functions, in accordance with Lemma 3.9, while the integral factors are dominated by \( \|w\|_t^2 \) by (5.24) and (5.25). The integral in (5.43) is also dominated by \( \|w\|_t^2 \). This completes the proof of (5.26).

For the proof of (5.27) observe that the inequality (5.31) combined with (5.26) shows that the integral of the first and third terms in (5.27) is finite and in fact dominated by an \( A \) dependent multiple of \( \|w\|_t \).

The second term in line (5.27) is integrable by the GFS inequality (5.20) because

\[
\kappa^{-2}\|d_A w(s)\|_6^2 \leq \|d_A^* d_A w(s)\|_2^2 + \|(d_A)^2 w(s)\|_2^2 + \lambda(B(s))\|d_A w(s)\|_2^2,
\]

(5.45)

which implies

\[
\kappa^{-2} \int_0^t s^{1-b}\|d_A w(s)\|_6^2 ds \leq \int_0^t \left\{ s^{1-b}\|d_A^* d_A w(s)\|_2^2 + s^{1-b}\| [B(s) \wedge w(s)] \|_2^2 + s\lambda(B(s))\left(s^{-b}\|d_A w(s)\|_2^2\right)\right\} ds.
\]

(5.46)

The first term on the right in (5.46) is integrable since it is equal to the first term on the left in (5.27), whose integrability has already been proven. The second term in (5.46) is at most \( c^2 \left(s\|B(s)\|_3^2\right) \left(s^{-b}\|w(s)\|_6^2\right)\), which, in view of (5.42) (with \( a = 1/2 \)) is a bounded function times an integrable function, as is the third term also.

The fourth term in (5.27) is integrable by an application of the ordinary Sobolev inequality. Indeed, since \( d_A^* w(s) \) is a 0-form Sobolev’s inequality shows that \( \kappa^{-2}\|d_A^* w(s)\|_6^2 \leq \|d_A d_A^* w(s)\|_2^2 + \|d_A^* w(s)\|_2^2 \), from which the integrability of \( s^{1-b}\|\psi(s)\|_6^2 \) follows because, upon multiplication by \( s^{1-b} \), the first term on the right is integrable and the second term is bounded, by (5.26), and therefore integrable.

The fifth term in (5.27) differs only slightly from the third term because \( \|d\psi(s)\|_2 \leq \|d_A \psi(s)\|_2 + \| [A(s) \wedge \psi(s)] \|_2 \).

But

\[
\int_0^t s^{1-b}\| [A(s) \wedge \psi(s)] \|_2^2 ds \leq c^2 \int_0^t \|A(s)\|_6^2 s^{1-b}\|\psi(s)\|_3^2 ds
\]

\[
\leq c^2 \sup_{0 < s \leq t} (s^{1/2}\|A(s)\|_6^2) \int_0^t s^{(1/2)-b}\|\psi(s)\|_3^2 ds,
\]

(5.47)
\[
\int_0^t s^{(1/2)-b} \| \psi(s) \|_3^2 ds \leq \int_0^t (s^{-b/2} \| \psi(s) \|_2)(s^{(1-b)/2} \| \psi(s) \|_6) ds \\
\leq \left( \int_0^t s^{-b} \| \psi(s) \|_2^2 ds \right)^{1/2} \left( \int_0^t s^{1-b} \| \psi(s) \|_6^2 ds \right)^{1/2}.
\] (5.48)

The first factor in (5.48) is finite because \( w \) has finite \( b \)-action. The second factor already appears as the fourth term on the left in (5.27) and is therefore finite. The supremum in line (5.47) is finite by virtue of (3.38).

Concerning the sixth term in (5.27), the augmented variational equation (2.21) shows that \( \zeta(s) = -w'(s) - d_A \psi(s) \). But \( \int_0^t s^{1-b} \left( \| w'(s) \|_2^2 + \| d_A \psi(s) \|_2^2 \right) ds < \infty \) by (5.26) and (5.27) (third term). This completes the proof of (5.27).

The inequality (5.28) follows from (5.48). Finally, the Schwarz inequality shows that

\[
\int_0^t \| \psi(s) \|_3 ds \leq \left( \int_0^t s^{b-(1/2)} ds \right)^{1/2} \left( \int_0^t s^{(1/2)-b} \| \psi(s) \|_2^2 ds \right)^{1/2} = o(t^{(b+(1/2))/2}),
\]

which is (5.29). This completes the proof of Theorem 5.6.

5.3 Initial behavior of \( w \), order 2

**Theorem 5.10 (Initial behavior, order 2).** Suppose that \( A(\cdot) \) is a strong solution to the Yang-Mills heat equation over \([0, \infty)\) with finite action. Let \( 0 \leq b < 1 \). Let \( w(\cdot) \) be a strong solution to the augmented variational equation (2.21) along \( A(\cdot) \) with finite \( b \)-action in the sense of (5.22). Then there are standard dominating functions \( C_j \) such that, for \( 0 < t < \infty \), there holds

\[
t^{2-b} \| w'(t) \|_2^2 + \int_0^t s^{2-b} \left\{ \| d_{A(s)} w'(s) \|_2^2 + \| d^*_{A(s)} w'(s) \|_2^2 \right\} ds \leq \| w \|_4^2 C_{83}(t, \rho_A(t))
\]

and

\[
\int_0^t s^{2-b} \| w'(s) \|_6^2 ds \leq \| w \|_4^2 C_{91}(t, \rho_A(t)).
\]

(5.49) (5.50)
Define \( \psi \) and \( \zeta \) as in (5.1) and (5.2). The following integral bounds on the third order derivatives of \( w \) hold.

\[
\int_0^T s^{2-b} \left( \|d^*_A d_A \psi(s)\|^2 + \|d^*_A \zeta(s)\|^2 + \|d_A \zeta(s)\|^2 \right) ds < \infty.
\]  

(5.51)

Moreover

\[
\int_0^t s^{2-b} \|d_A \psi(s)\|^2 ds < \infty, \quad 0 \leq b < 1,
\]  

(5.52)

\[
\int_0^t s^{2-b} \|d_A^* d_A w(s)\|^2 ds < \infty, \quad 0 \leq b < 1,
\]  

(5.53)

\[
\int_0^t s^{2-b} \|\zeta(s)\|^2 ds < \infty, \quad 0 \leq b < 1.
\]  

(5.54)

The following interpolation consequences hold.

\[
\int_0^T \|\psi(s)\|_q^2 ds < \infty, \quad q^{-1} = (1/2) - (b/3), \quad 0 \leq b < 1.
\]  

(5.55)

\[
\int_0^T s^{1/2} \left( \|d_A \psi(s)\|^2 + \|d_A^* d_A w(s)\|^2 + \|\zeta(s)\|^2 + \|d \psi(s)\|^2 \right) ds < \infty,
\]  

(5.56)

\[
r^{-1} = (2/3) - (b/3), \quad (1/2) \leq b < 1.
\]  

\[
\int_0^T s^{(3/2)-b} \left( \|d_A \psi(s)\|_3^2 + \|d_A^* d_A w(s)\|_3^2 \right.
\]

\[
+ \|\zeta(s)\|_3^2 + \|d \psi(s)\|_3^2 \left) ds < \infty, \quad 0 \leq b < 1.
\]  

(5.57)

\[
\int_0^T \|d_A \psi(s)\|_\rho ds < \infty \text{ if } 2 \leq \rho < 3, \quad 1/2 \leq b < 1
\]  

(5.58)

**Remark 5.11** We will show in the next section by different methods that

\[
\int_0^t s^{(3/2)-b} \|\psi(s)\|_\infty^2 ds < \infty, \text{ which, interestingly, holds even though (5.57) just barely fails to give this inequality because Sobolev just barely fails to give control of } \|\psi(s)\|_\infty \text{ by } \|d_A \psi(s)\|_3.
\]  

The proof of Theorem 5.10 depends on the following lemmas.
Lemma 5.12 (Differential inequality) Suppose that $w(\cdot)$ is a strong solution to the augmented variational equation (2.21) on the interval $(0, T)$. Then

$$
\frac{d}{ds} \|w'(s)\|_2^2 + \left( \|d_{A(s)}w'(s)\|_2^2 + \|d'_{A(s)}w'(s)\|_2^2 \right) 
\leq c_1 \|A'(s)\|_3^2 \left( \|w(s)\|_6^2 + 2\kappa^2 (\|d_Aw\|_2^2 + \|d_A^* w\|_2^2) \right) 
+ c_2 \|B(s)\|_3^2 \|w'(s)\|_6^2 + c_3 s^{-1} \|w'(s)\|_2^2 
+ c_5 s^{1/2} \|B'(s)\|_2^2 \|w(s)\|_6^2
$$

(5.59)

for some constants $c_j$ that depend only on a Sobolev constant $\kappa$ and the commutator bound $c$.

Proof. Our strategy will be to bound the terms on the right side of the integral identity (5.10). In our use of Hölder’s inequality we will be forced to use $\|w'(s)\|_6$ as a frequent factor. By the Gaffney-Friedrichs-Sobolev inequality this has the same degree of singularity (as $s \downarrow 0$) as some of the terms on the left side of (5.10). We will arrange the estimates in such a way as to allow cancellation of some of these singular terms.

We are going to use Hölder’s inequality repeatedly to show that the right side of the identity (5.10) is at most

$$
2c^2 \|A'\|_3^2 \|w\|_6^2 + (1/2) \left( \|d_{A}w'\|_2^2 + \|d'_{A} w'\|_2^2 \right)
$$

(5.60)

$$
+ 4c^2 \kappa^2 \|A'\|_3^2 \left( \|d_Aw\|_2^2 + \|d_A^* w\|_2^2 \right) + (1/4) \kappa^{-2} \|w'\|_6^2
$$

(5.61)

$$
+ (1/4)(12c\kappa^{3/2}) \|B\|_2^2 \|w'\|_2^2 + (1/8) \kappa^{-2} \|w'\|_6^2
$$

(5.62)

$$
+ 8(c\kappa)^2 s^{1/2} \|B'\|_2^2 \|w\|_6^2 + 2^{-5} \kappa^{-2} s^{-1} \|w'\|_2^2 + (1/8) \kappa^{-2} \|w'\|_6^2.
$$

(5.63)

Here, as below, we have suppressed the argument $s$ in all functions.

To bound the first two terms on the right side of (5.10) observe that

$$
2|([A' \wedge w], d_{A}w') + ([A' \wedge w], d'_{A} w')| 
\leq 2 \left( \| [A' \wedge w] \|_2 \|d_{A}w'\|_2 + \| [A' \wedge w] \|_2 \|d'_{A} w'\|_2 \right)
\leq 2c \|A'\|_3 \|w\|_6 \left( \|d_{A}w'\|_2 + \|d'_{A} w'\|_2 \right)
\leq (1/2) \left( 2c \|A'\|_3 \|w\|_6 \right)^2 + (1/2) \left( \|d_{A}w'\|_2^2 + \|d'_{A} w'\|_2^2 \right).
$$

(5.64)
This contributes the line (5.60) in our bound of the right side of (5.10).

To bound the second pair of terms on the right side on (5.10), we use $2ab \leq (2\kappa a)^2 + (b/2\kappa)^2$ twice, with $\kappa$ equal to the Sobolev constant in (5.20), to find

$$2|([A' \cdot d_A w] + [A', d_A^* w], w')| \leq 2c\|A'\|_3\left(\|d_A w\|_2 + \|d_A^* w\|_2\right)\|w'\|_6$$
$$\leq (2ck\|A'\|_3^2)\left(\|d_A w\|_2^2 + \|d_A^* w\|_2^2\right) + (1/4)\kappa^{-2}\|w'\|_6^2. \quad (5.65)$$

This contributes the line (5.61) in our bound of the right side of (5.10).

The last term in (5.10) is $2([w' \cdot B] + [w \cdot B'], w')$. These two terms must be estimated in different ways. To estimate $2([w' \cdot B], w')$ we can use the interpolation $\|f\|_4 \leq \|f\|_2^{1/4}\|f\|_6^{3/4}$ to find

$$2|([w' \cdot B], w')| \leq 2c\|B\|_2\|w'\|_4^2$$
$$\leq \left(12ck^{3/2}\|B\|_2\|w'\|_2^{1/2}\right)\left((1/6)\kappa^{-3/2}\|w'\|_6^{3/2}\right)^{4/3}$$
$$\leq (1/4)^3 \left(12ck^{3/2}\|B\|_2\|w'\|_2^{1/2}\right)^{4} + (3/4)\left((1/6)\kappa^{-3/2}\|w'\|_6^{3/2}\right)^{4/3}$$
$$\leq (1/4)^3(12ck^{3/2})^4\|B\|_2^4\|w'\|_2^2 + (1/8)\kappa^{-2}\|w'\|_6^2 \quad (5.66)$$

because $(3/4)(1/6)^{4/3} \leq 1/8$. This contributes the line (5.62) in our bound of the right side of (5.10).

Our estimate of the final term $2([w \cdot B'], w')$ appearing in (5.10) will have an explicit $s$ dependence. We have

$$2|([w \cdot B'], w')| \leq 2c\|B'\|_2\|w\|_6\|w'\|_3$$
$$= \left(4c^2\kappa s^{1/4}\|B'\|_2\|w\|_6\right)\left((1/2)\kappa^{-1}s^{-1/4}\|w'\|_3\right)^2$$
$$\leq (1/2)^2 \left(4c^2\kappa s^{1/4}\|B'\|_2\|w\|_6\right)^2 + (1/2)^2 \left((1/2)\kappa^{-1}s^{-1/4}\|w'\|_3\right)^2$$
$$= 8(c\kappa)^2s^{1/2}\|B'\|^2_2\|w\|_6^2 + (1/8)\kappa^{-2}s^{-1/2}\|w'\|_3^2$$
$$\leq 8(c\kappa)^2s^{1/2}\|B'\|^2_2\|w\|_6^2 + (1/8)\kappa^{-2}\{s^{-1}\|w'\|_2^2 + \|w'\|_6^2\}. \quad (5.67)$$

In the last line we have used $s^{-1/2}\|w'\|_3^2 \leq (s^{-1/2}\|w'\|_2)\|w'\|_6 \leq (1/4)s^{-1}\|w'\|_2 + \|w'\|_6^2$.

This completes the proof that the right hand side of (5.10) is dominated by the sum of the four lines (5.60) - (5.63).
Notice that the last term in each of the three lines (5.61)-(5.63) is a multiple of $\|w\|^2_6$ and they add to $(1/2)\kappa^{-2}\|w'(s)\|^2_6$. From the Gaffney-Friedrichs-Sobolev inequality (5.20) we find

$$(1/2)\kappa^{-2}\|w\|^2_6 \leq (1/2)\left(\|d_Aw'\|^2_2 + \|d^*_A w'\|^2_2\right) + (1/2)\lambda(B)\|w'\|^2_2.$$  

Therefore, adding the last term in line (5.60) to the last terms in the next three lines, we find that the sum of the last terms in all four lines is at most

$$\left(\|d_Aw'\|^2_2 + \|d^*_A w'\|^2_2\right) + (1/2)\lambda(B)\|w'\|^2_2. \tag{5.68}$$

The term $\left(\|d_Aw'\|^2_2 + \|d^*_A w'\|^2_2\right)$ appears on the left side of (5.10) with a factor of 2. We can therefore cancel this term with half of its multiple on the left to find that the left side of (5.59) is bounded by the remaining terms in the lines (5.60) - (5.63) plus $(1/2)\lambda(B)\|w'\|^2_2$. These add to the right side of (5.59). This completes the proof of (5.59). □

Lemma 5.13 (Interpolation bounds) Suppose that $f : [0, t) \to \{functions on M\}$.

Assume that $1/2 \leq b \leq 3/2$. Let $r^{-1} = (2/3) - (b/3)$. Then $2 \leq r \leq 6$ and

$$\int_0^t s^b \|f(s)\|^2 ds \leq \left(\int_0^t s^{1/2} \|f(s)\|^2 ds\right)^{(3/2) - b} \left(\int_0^t s^{3/2} \|f(s)\|^2 ds\right)^{b - (1/2)}, \tag{5.69}$$

$$\int_0^t s^{1/2} \|f(s)\|^2 ds \leq \left(\int_0^t s^{-b} \|f(s)\|^2 ds\right)^{(3/2) - b} \left(\int_0^t s^{2-b} \|f(s)\|^2 ds\right)^{b - (1/2)}, \tag{5.70}$$

$$\int_0^t s^{-1/2} \|f(s)\|^2 ds \leq \left(\int_0^t s^{-b} \|f(s)\|^2 ds\right)^{(3/2) - b} \left(\int_0^t s^{1-b} \|f(s)\|^2 ds\right)^{b - (1/2)}. \tag{5.71}$$

Assume that $0 \leq b \leq 1$. Let $q^{-1} = (1/2) - (b/3)$. Then $2 \leq q \leq 6$ and

$$\int_0^T \|f(s)\|^2 ds \leq \left(\int_0^T s^{-b} \|f(s)\|^2 ds\right)^{1-b} \left(\int_0^T s^{1-b} \|f(s)\|^2 ds\right)^b. \tag{5.72}$$
Moreover, if $\rho \geq 2$ and $1/\rho > (1/2) - (b/3)$ then
\[
\int_0^T \|f(s)\|_\rho ds \leq C_{b,\rho} \left( \int_0^t s^{-b}\|f(s)\|_2^2 ds \right)^{\alpha/2} \left( \int_0^t s^{-2b}\|f(s)\|_6^2 ds \right)^{\beta/2},
\]
with a finite constant $C_{b,\rho}$ and non-negative constants $\alpha, \beta$.

Assume that $-\infty < b < 2$. Then
\[
\int_0^T s^{(3/2) - b}\|f(s)\|_3^2 ds \leq \left( \int_0^T s^{-b}\|f(s)\|_2^2 ds \right)^{1/2} \left( \int_0^T s^{-2b}\|f(s)\|_6^2 ds \right)^{1/2}.
\]

**Proof.** All of these inequalities are consequences of the interpolation inequality
\[
\|f(s)\|_\rho^2 \leq \left( \|f(s)\|_2^\frac{\alpha}{2} \right)^\alpha \left( \|f(s)\|_6^\frac{\beta}{2} \right)^\beta,
\]
\[
\alpha = (3/\rho) - (1/2), \quad \beta = (3/2) - (3/\rho),
\]
which is valid for $2 \leq \rho \leq 6$. (5.75) implies that, for any real number $\gamma$,
\[
s^{\gamma + \beta}\|f(s)\|_\rho^2 \leq \left( s^{\gamma}\|f(s)\|_2^\frac{\alpha}{2} \right)^\alpha \left( s^{1+\gamma}\|f(s)\|_6^\frac{\beta}{2} \right)^\beta.
\]
Integrate this inequality over $(0, t)$ and use Hölder’s inequality to find
\[
\int_0^t s^{\gamma + \beta}\|f(s)\|_\rho^2 ds \leq \left( \int_0^t s^\gamma\|f(s)\|_2^\frac{\alpha}{2} ds \right)^\alpha \left( \int_0^t s^{1+\gamma}\|f(s)\|_6^\frac{\beta}{2} ds \right)^\beta.
\]
All four inequalities in the statement of the lemma now result from proper choice of $\gamma$. Thus:

To prove (5.69), (5.70) and (5.71) take $\rho = r$ and $b = 2 - 3r^{-1}$ to find $\beta = b - (1/2)$ in all three cases. Choose $\gamma = 1/2$ to find $\gamma + \beta = b$, from which (5.69) follows. Choose $\gamma = 1 - b$ to find $\gamma + \beta = 1/2$, from which (5.70) follows. And choose $\gamma = -b$ to find $\gamma + \beta = -1/2$, from which (5.71) follows.

To prove (5.72) replace $\rho$ by $q$ in (5.75) - (5.77) and take $b = (3/2) - 3q^{-1}$ to find $\beta = b$. Choose $\gamma = -b$ to find $\gamma + \beta = 0$, from which (5.72) follows.

For the proof of (5.73) observe that by (5.75) we have
\[
\int_0^\tau \|f(s)\|_\rho ds \leq \int_0^\tau s^{(b-1)\alpha + (b-2)\beta/2} \left( s^{1-b}\|f(s)\|_2 \right)^\alpha \left( s^{2-b}\|f(s)\|_6 \right)^\beta ds
\]
\[
\leq \left( \int_0^\tau s^{b-1-\beta} ds \right)^{1/2} \left( \int_0^\tau s^{-b}\|f(s)\|_2^2 ds \right)^{\alpha/2} \left( \int_0^\tau s^{-2b}\|f(s)\|_6^2 ds \right)^{\beta/2},
\]
(5.78)
wherein we have used Hölder’s inequality with the three powers 2, \((2/\alpha)\), \((2/\beta)\). But \(b - \beta = b - \{(3/2) - (3/\rho)\} = 3\{(1/\rho) - [(1/2) - (b/3)]\}\). Hence the first integral in (5.78) is finite if and only if \((1/\rho) - [(1/2) - (b/3)] > 0\). This proves (5.73).

To prove (5.74) choose \(\rho = 3\), giving \(\alpha = \beta = 1/2\), and choose \(\gamma = 1 - b\) in (5.77).

**Proof of Theorem 5.10.** Starting with the differential inequality (5.59), we may apply Lemma 5.3, choosing \(f(s) = ||w'(s)||^2\), \(g(s) = ||d_A(s)w'(s)||^2 + ||d^*_A(s)w'(s)||^2\) and \(h(s)\) equal to the entire right hand side of (5.59). Replace the number denoted \(b\) in Lemma 5.3 by \(b - 1\), with our present meaning of \(b\). Then (5.17) shows that

\[
t^2b||w'(t)||^2 + \int_0^t s^{2-b}\left\{|d_A(s)w'(s)||^2 + ||d^*_A(s)w'(s)||^2\right\}ds \\
\leq (2-b) \int_0^t s^{1-b}||w'(s)||^2ds \\
+ \int_0^t s^{2-b}\left\{c_1||A'(s)||^2\left(||w(s)||^2 + 2\kappa^2(||d_Aw||^2 + ||d^*_A||^2)\right) \\
+ c_2||B(s)||^4||w'(s)||^2 + c_3 s^{-1}||w'(s)||^2 + c_4||w'(s)||^2 \\
+ c_5 s^{1/2}||B'(s)||^2||w(s)||^2\right\}ds.
\]

The line (5.79) is finite by (5.26) and is bounded by an \(A\) dependent multiple of \(||w||^2\). This justifies use of Lemma 5.3.

We need to show now that the integrals in lines (5.80) through (5.82) are all finite and dominated by an \(A\) dependent multiple of \(||w||^2\). The sum of these lines is bounded by

\[
c_1\left(\sup_{0<s\leq t} s^2||A'(s)||^2\right) \int_0^t s^{2-b}\left(||w(s)||^2 + 2\kappa^2(||d_Aw||^2 + ||d^*_A||^2)\right)ds \\
+ c_2\left(\sup_{0<s\leq t} s||B(s)||^4\right) \int_0^t s^{1-b}||w'(s)||^2ds + c_3 \int_0^t s^{1-b}||w'(s)||^2ds \\
+ c_4 \int_0^t s^{1-b}||w'(s)||^2ds + c_5 \sup_{0<s\leq t} \left(s^{5/2}||B'(s)||^2\right) \int_0^t s^{2-b}||w(s)||^2ds.
\]

The three suprema in these three lines are all dominated by a standard bounding function of \(t, \rho_A(t)\), in accordance with Lemma 3.9 with \(a = 1/2\).
The integral in line (5.83) is at most \(5\kappa^2\|w\|_T^2\) by (5.24) and (5.25). Therefore line (5.83) is dominated by an \(A\) dependent multiple of \(\|w\|_T^2\), as required for (5.49).

In line (5.84) both integrals are appropriately dominated in accordance with the first order estimate (5.26). The first term in line (5.85) is also dominated in accordance with (5.26). The second integral in that line is at most \(\kappa^2\|w\|_T^2\). This completes the proof of (5.49).

For the proof of (5.50) we need only apply the Gaffney-Friedrichs-Sobolev inequality (5.20), which shows that \(\kappa^{-2}\|w'(s)\|_6^2 \leq \|d_A(s)w'(s)\|_2^2 + \|d_A^*(s)w'(s)\|_2^2 + (1 + \gamma\|B(s)\|_2^2)\|w'(s)\|_2^2\). Upon multiplication by \(s^{2-b}\) the inequality (5.49) shows that the first two terms are integrable over \((0, t)\). The last term is \(s(1 + \gamma\|B(s)\|_2^2)\) times \(s^{1-b}\|w'(s)\|_2^2\), which is the product of a bounded factor, in accordance with Lemma 3.9 and an integrable factor, in accordance with (5.26). This proves (5.50).

The remaining inequalities in the theorem will be derived from (5.49) and (5.50) with the help of the GFS inequality and interpolation. We need the following identities. As in (5.2) we write \(\zeta(s) = d_A^*d_Aw(s) + [w(s), B(s)]\). And as in (5.3) we have \(w'(s) = -\zeta(s) - d_A\psi(s)\). Using the identity \(d_A^*\zeta(s) = [w(s), A'(s)]\) from (5.4), and the Bianchi identity, we may apply \(d_A^*\) and \(d_A\) to this equation to find

\[-d_A^*d_A\psi = d_A^*w' + [w, A'] \quad \text{and} \quad -d_A\zeta = d_Aw' + [B, \psi].\]

We assert that

\[\int_0^T s^{2-b}\left(\|w(s), A'(s)\|_2^2 + \|B(s), \psi(s)\|_2^2\right)ds < \infty.\]

Indeed

\[\int_0^T s^{2-b}\left(\|w(s), A'(s)\|_2^2 + \|B(s), \psi(s)\|_2^2\right)ds\]

\[\leq c^2 \int_0^T s^{2-b}\left(\|A'(s)\|_2^2\|w(s)\|_6^2 + \|B(s)\|_2^2\|\psi(s)\|_6^2\right)ds\]

\[\leq c^2 \left(\sup_{0 < s \leq T} s^2\|A'(s)\|_2^2\right) \int_0^T s^{-b}\|w(s)\|_6^2 ds\]

\[+ c^2 \left(\sup_{0 < s \leq T} s\|B(s)\|_2^2\right) \int_0^T s^{1-b}\|\psi(s)\|_6^2 ds.\]
The two suprema are finite by (3.23) and (3.22) (with \( a = 1/2 \)), respectively. The two integrals are finite by the finite action assumption ((5.21) is finite) and (5.27), respectively. This proves (5.88). But the left hand side of (5.51) is equal to

\[
\int_0^T s^{2-b} \left( \|d'_A w' + [w \perp A'] \|^2_2 + \| [w \perp A'] \|^2_2 + \|d_A w' + [B, \psi] \|^2_2 \right) ds. \tag{5.89}
\]

It follows from (5.49) and (5.88) that this integral is finite. This proves (5.51).

Concerning the \( L^6 \) bounds (5.52) - (5.54) observe first that

\[
\int_0^t s^{2-b} \lambda(B(s)) \left( \|d_A(s) \psi(s) \|^2_2 + \| \zeta(s) \|^2_2 \right) ds \\
\leq \left( \sup_{0 < s \leq t} s \lambda(B(s)) \right) \int_0^t s^{1-b} \left( \|d_A(s) \psi(s) \|^2_2 + \| \zeta(s) \|^2_2 \right) ds \\
< \infty \tag{5.90}
\]

by (3.17) and (5.27).

The \( L^6 \) bound (5.54) for \( \zeta \) now follows from the GFS inequality (5.20) together with (5.51) and (5.90). The \( L^6 \) bound (5.52) follows from the GFS inequality (5.20), (5.51) and the additional equality

\[
\int_0^t s^{2-b} \|d_A d_A \psi(s) \|^2_2 ds = \int_0^t s^{2-b} \| [B(s), \psi(s)] \|^2_2 ds, \tag{5.91}
\]

which is finite by (5.88).

The inequality (5.53) can be deduced more easily from what has already been proven than from another application of the Gaffney-Friedrichs-Sobolev inequality. We have \( d'_A d_A w = \zeta - [w \perp B] \). Since \( \zeta \) satisfies the inequality (5.54) we need only show that \( [w \perp B] \) does also. But

\[
\int_0^t s^{2-b} \|w(s) \perp B(s) \|^2_0 ds \leq c^2 \int_0^t s^{2-b} \|w(s)\|^2_0 \|B(s)\|^2_\infty ds \tag{5.92}
\]

\[
\leq c^2 \left( \sup_{0 < s \leq t} s^2 \|B(s)\|^2_\infty \right) \int_0^t s^{-b} \|w(s)\|^2_0 ds \tag{5.93}
\]

\[
< \infty \tag{5.94}
\]

because the supremum is finite by (3.32) and \( w \) has finite \( b \)-action. This completes the proof of (5.53).
For the proof of the interpolation inequalities (5.55) choose \( f(s, x) = |\psi(s, x)| \) in (5.72) to find
\[
\int_0^T \|\psi(s)\|_q^2 ds \leq \left( \int_0^T s^{-b} \|\psi(s)\|_2^2 ds \right)^1 - b \left( \int_0^T s^{1-b} \|\psi(s)\|_6^2 ds \right)^b.
\] (5.95)

The first factor is finite because \( w \) has finite \( b \) action. The second factor is finite by (5.27). This proves (5.55).

To prove (5.56) Put \( f(s, x) = |d_A \psi(s, x)| \) in (5.70) to find
\[
\int_0^T s^{1/2} \|d_A \psi(s)\|_r^2 ds
\leq \left( \int_0^T s^{1-b} \|d_A \psi(s)\|_2^2 ds \right)^{(3/2) - b} \left( \int_0^T s^{2-b} \|d_A \psi(s)\|_6^2 ds \right)^{b - (1/2)}.
\]
The first factor is finite by (5.27). The second factor is finite by (5.32).

The same argument applies to the second and third terms in (5.56) because each of them satisfy the same \( L^2 \) initial behavior bounds (5.26) as \( d_A \psi \) and the same \( L^6 \) initial behavior bounds (5.32) - (5.34).

Concerning the fourth term in (5.56), observe that with \( q^{-1} = r^{-1} - (1/6) = (1/2) - (b/3) \), we have
\[
\int_0^T s^{1/2} \| [A(s), \psi(s)] \|_q^2 ds \leq c^2 \int_0^T s^{1/2} \| A(s) \|_6^2 \|\psi(s)\|_6^2 ds
\leq c^2 \left( \sup_{0<s\leq T} s^{1/2} \| A(s) \|_6^2 \right) \int_0^T \|\psi(s)\|_6^2 ds,
\]
which is finite by (3.38) and (5.55). Therefore the fourth term in (5.56) differs from the first term by a finite amount. This completes the proof of (5.56).

To prove (5.57) choose \( f(s, x) = |d_{A(s)} \psi(s, x)| \) in (5.74). Then (5.74) together with (5.27) and (5.32) show that \( \int_0^T s^{(3/2) - b} \|d_{A(s)} \psi(s)\|_3^2 ds < \infty \). The same argument applies to the second and third terms in (5.57).

Concerning the fourth term in (5.57), since \( \|d\psi(s)\|_3 \leq \|d_{A(s)} \psi(s)\|_3 + \)
\[ \| [A(s), \psi(s)] \|_3, \text{ it suffices to observe that} \]
\[ \int_0^T s^{(3/2)-b} \| [A(s), \psi(s)] \|^2_3 ds \leq c^2 \int_0^T s^{(3/2)-b} \| A(s) \|^2_6 \| \psi(s) \|^2_6 ds \]
\[ \leq c^2 \left( \sup_{0 < s \leq T} s^{1/2} \| A(s) \|^2_6 \right) \int_0^T s^{1-b} \| \psi(s) \|^2_6 ds \]
\[ < \infty \]
by (3.38) and (5.27). This completes the proof of (5.57).

In regard to (5.58), choose \( f(s) = d_{A(s)} \psi(s) \) in (5.73) to find that for \( 0 < b \leq 1 \) and \( \rho \geq 2 \) we have
\[ \int_0^T \| d_{A(s)} \psi(s) \|_\rho ds < \infty \text{ if } 1/\rho > (1/2) - (b/3), \]
(5.96)
in view of (3.27) and (3.52). In particular if \( 1/2 \leq b < 1 \) then the restriction on \( \rho \) is satisfied if \( 2 \leq \rho < 3 \). This proves (5.58).

### 5.4 High \( L^p \) bounds for \( \psi \)

Our energy methods typically establish \( L^p \) bounds on functions for \( 2 \leq p \leq 6 \). For larger values of \( p \) we will use the following Neumann domination method.

**Theorem 5.14** Let \( 1/2 \leq a < 1 \) and \( 0 < b < 1 \). Suppose that \( A(\cdot) \) is a strong solution to the Yang-Mills heat equation over \( M \) with finite a-action and satisfying the Neumann boundary conditions \( A(t) \text{norm} = 0, B(t) \text{norm} = 0 \) for \( t > 0 \) in case \( M \neq \mathbb{R}^3 \). Let \( w \) be a strong solution to the augmented variational equation (2.21) with finite b-action in the sense of (5.22) and satisfying Neumann boundary conditions (3.5) in case \( M \neq \mathbb{R}^3 \). Define \( \psi(t) = d_{A(t)}^* w(t) \) for \( t > 0 \) as in (5.1). Then there are standard dominating functions \( C_p \) such that
\[ \int_0^T t^{(3/2)-b-(3/p)} \| \psi(t) \|^2_\rho dt \leq C_p(T, \rho A(T))\| w \|^2_T < \infty, \quad 6 \leq p \leq \infty \]
(5.97)
for all \( T < \infty \).

**Remark 5.15** Dirichlet boundary conditions are noticeably absent in the allowed hypothesis. This arises from the failure of Lemma 5.17 in this case.
The proof depends on the following five lemmas. The first lemma is taken from [6].

**Lemma 5.16** (Neumann domination with averaging) Let $0 < T < \infty$. Suppose that $M \subseteq \mathbb{R}^N$ is the closure of an open set with smooth boundary and that $A : (0, T] \rightarrow C^1(M; \Lambda^1 \otimes \mathfrak{k})$ is a time dependent 1-form on $M$ which is continuous in the time variable. Let $0 \leq p \leq n$. Let $\omega : (0, T) \rightarrow C^2(M; \Lambda^p \otimes \mathfrak{k})$ be a time dependent, $\mathfrak{k}$ valued, $p$-form on $M$ which is continuously differentiable in the time variable and satisfies the equation

$$\omega'(s, x) = \sum_{j=1}^{N} (\nabla_j A(s))^2 \omega(s, x) + h(s, x),$$

where $h \in C((0, T] \times M; \Lambda^p \otimes \mathfrak{k})$. Assume also that if $M \neq \mathbb{R}^N$ then

$$\nabla_n|\omega(s, x)|^2 \leq 0, \quad 0 < s < T, \quad x \in \partial M,$$

where $n$ is the outward drawn unit normal. Denote by $\Delta_N$ the Laplacian on real valued functions over $\mathbb{R}^N$ if $M = \mathbb{R}^N$ or the Neumann Laplacian on real valued functions if $M \neq \mathbb{R}^N$. Then

$$|\omega(t, x)| \leq t^{-1} \int_0^t e^{(t-s)\Delta_N}|\omega(s, \cdot)|ds(x)$$

$$+ t^{-1} \int_0^t e^{(t-s)\Delta_N} s|h(s, \cdot)|ds(x).$$

**Proof.** See [6, Proposition 4.21]. □

**Lemma 5.17** (Normal derivative) Suppose that $M \neq \mathbb{R}^3$ and that $A(\cdot)$ is a strong solution to the Yang-Mills heat equation over $(0, \infty)$ satisfying Neumann boundary conditions. Let $w$ be a strong solution to (2.21) satisfying Neumann boundary conditions [3,5]. Define $\psi(s) = d^n_{A(s)}w(s)$ for $s > 0$ as in (5.1). Then

$$\nabla_n|\psi(s, x)|^2 = 0 \quad \text{for all } s > 0 \text{ and } x \in \partial M.$$

**Proof.** We are assuming now that $A(t)_{\text{norm}} = 0$ and $B(t)_{\text{norm}} = 0$ for all $t > 0$ and that

$$w(t)_{\text{norm}} = 0 \quad \text{on } \partial M \quad \text{for all } t > 0$$

$$\left( d_{A(t)}w(t) \right)_{\text{norm}} = 0 \quad \text{on } \partial M \quad \text{for all } t > 0.$$
We will show that these four conditions imply that the solution \( w(\cdot) \) to the augmented variational equation (2.21) satisfies
\[
(d_{A(t)}\psi(t))_{\text{norm}} = 0 \quad \text{for all } t > 0. \tag{5.104}
\]
We may write (2.21) as
\[
(d_A\psi(t)) = -\left\{ w'(t) + d^*_A d_A w + [w \cdot B] \right\}. \tag{5.102}
\]
It suffices to show that each of the three terms in braces has normal component zero. Differentiating (5.102) with respect to \( t \) we see that \( w'(t)_{\text{norm}} = 0 \). Since \( B(t)_{\text{norm}} = 0 \), we also have \([w(t) \cdot B(t)]_{\text{norm}} = 0 \) (for any \( w \)). It follows from (5.103) and [2, Eq. (3.20)] that \((d^*_A d_A w(t))_{\text{norm}} = 0 \). This proves (5.104).

It follows now from (5.104) that
\[
\nabla |\psi(t,x)|^2 = \langle (d_A \psi(t,x))_{\text{norm}}, \psi(t,x) \rangle + \langle \psi(t,x), (d_A \psi(t,x))_{\text{norm}} \rangle = 0.
\]

**Lemma 5.18** *(Pointwise bound)* If \( M = \mathbb{R}^3 \), or if \( M \neq \mathbb{R}^3 \) but Neumann boundary conditions hold for \( A \) and \( w \), then
\[
|\psi(t,x)| \leq t^{-1} \int_0^t e^{(t-s)\Delta N} |\psi(s,\cdot)| ds (x) + t^{-1} \int_0^t e^{(t-s)\Delta N} 2s \left| A'(s) \cdot w(s) \right| ds (x). \tag{5.105}
\]

**Proof.** Take \( \omega(s) = \psi(s) \) in (5.98). Then (5.5) shows that (5.98) holds with \( h(s) = 2[A'(s) \cdot w(s)] \). Lemma 5.17 shows that the condition (5.99) holds when \( \omega = \psi \). We may therefore apply Lemma 5.16 to see that (5.105) holds.

**Lemma 5.19** *(Weighted estimate)* There is a standard dominating function \( C_{50} \) such that
\[
\int_0^T s^{(5/2)-a-b} \| [A'(s) \cdot w(s)] \|_2^2 \leq C_{50}(T, \rho_A(T)) \| w \|_T^2 \quad \forall T \geq 0. \tag{5.106}
\]

**Proof.**
\[
\int_0^T s^{(5/2)-a-b} \| [A'(s) \cdot w(s)] \|_2^2 ds \leq c^2 \int_0^T s^{(5/2)-a-b} \| A'(s) \|_3^2 \| w(s) \|_6^2 ds \\
\leq c^2 \sup_{0 < s \leq T} \left( s^{(5/2)-a} \| A'(s) \|_3^2 \right) \int_0^T s^{-b} \| w(s) \|_6^2 ds.
\]

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The supremum is finite by (3.23). The last integral is at most \( \kappa_2^2 \| w \|^2_T \) by (5.25). □

**Lemma 5.20 (A convolution inequality)** Let \( 0 \leq c < 1 \) and \( 0 < T < \infty \). Suppose that \( \alpha \) and \( \beta \) are non-negative functions on \((0, T]\) such that

\[
\alpha(t) \leq \frac{1}{t} \int_0^t (t - s)^{-c} \beta(s) \, ds \quad \text{for} \quad 0 < t \leq T. \tag{5.107}
\]

Then for any real number \( b_1 < 2c + 1 \) there holds

\[
\int_0^T t^{b_1} \alpha(t)^2 \, dt \leq \gamma \int_0^T s^{b_1 - 2c} \beta(s)^2 \, ds \tag{5.108}
\]

for some constant \( \gamma \) depending only on \( b_1 \) and \( c \).

**Proof.** This is [6, Lemma 4.24]. □

**Proof of Theorem 5.14** From (5.105) we find

\[
\| \psi(t) \|_p \leq t^{-1} \int_0^t \| e^{(t-s)\Delta_N} (|\psi(s, \cdot)| + 2s \| [A'(s, \cdot) \cdot w(s, \cdot)] \|) \|_p \, ds \\
\leq t^{-1} \int_0^t \| e^{(t-s)\Delta_N} \|_{2-p} \left( \| \psi(s) \|_2 + 2s \| [A'(s, \cdot) \cdot w(s)] \|_2 \right) \, ds \\
\leq t^{-1} \int_0^t c_1 (t - s)^{-\left(3/4\right) + \left(3/2p\right)} \left( \| \psi(s) \|_2 + 2s \| [A'(s, \cdot) \cdot w(s)] \|_2 \right) \, ds. \tag{5.109}
\]

In Lemma 5.20 choose \( c = (3/4) - (3/2p) \) and choose \( b_1 = (3/2) - b - (3/2p) \). Then \( b_1 - 2c = -b < 0 \). Take \( \alpha(t) = \| \psi(t) \|_p \) and \( \beta(s) = c_1 \left( \| \psi(s) \|_2 + 2s \| [A'(s, \cdot) \cdot w(s)] \|_2 \right) \) in Lemma 5.20. Then (5.108) and (5.109) show that

\[
\int_0^T t^{(3/2) - b - (3/2p)} \| \psi(t) \|_p^2 \, dt \leq \gamma \int_0^T s^{-b} \beta(s)^2 \, ds. \tag{5.110}
\]

But, by (5.21), \( \int_0^T s^{-b} \| \psi(s) \|^2 ds \leq 4 \| w \|^2_T < \infty \) because \( w \) has finite \( b \)-action. Moreover, since \( A \) has finite \( a \)-action and therefore finite \( (1/2) \) action, we can put \( a = 1/2 \) in (5.106) to conclude that \( \int_0^T s^{-b} \beta(s)^2 \, ds < \infty \). This completes the proof of (5.97). □
6 Recovery of \( v \) from \( w \): the differential equation

6.1 \( v \) and \( v_\tau \) satisfy the variational equation

We intend to recover a solution to the variational equation (2.6) from a solution \( w \) to the augmented variational equation (2.21). The two functions \( v(t) \), given in (2.23) and \( v_\tau(t) \) given in (2.24) will be shown to be solutions to the variational equation. They differ just by the choice of the lower limit \( \tau \) in (2.24). The behavior of these two functions differ considerably, even for fixed \( t > 0 \), because in (2.23) the integrand comes close to the singular point at \( s = 0 \) whereas in (2.24) it does not.

Let \( \psi(s) = d_{A(s)}^*w(s) \) again, as in (5.1), and let \( \tau \geq 0 \). Define

\[
v_\tau(t) := w(t) + d_{A(t)} \int_{\tau}^{t} \psi(s)ds, \quad 0 < t < \infty,
\]

and let

\[
\alpha = \int_{0}^{\tau} \psi(s)ds, \quad \tau > 0.
\]

Clearly

\[
v_0(t) = v_\tau(t) + d_{A(t)}\alpha \quad \text{for} \quad \tau > 0.
\]

The solution \( v(t) \) defined in (2.23) coincides with \( v_0(t) \). We will use the notation \( v_0(t) \) in this section so as to be able to treat the cases \( \tau = 0 \) and \( \tau > 0 \) as simultaneously as possible. We are going to show that

a) \( v_\tau(t) \) is an almost strong solution if \( \tau = 0 \). Moreover it has the correct initial value \( v_0 \),

b) \( v_\tau(t) \) is a strong solution if \( \tau > 0 \). It differs from the almost strong solution \( v_0(\cdot) \) by the vertical solution \( d_{A(t)}\alpha \). We will see in Section 7.1 that Lemma 2.9 is applicable.

The second term in (6.1) is clearly vertical at \( A(t) \) for each \( t \in (0, \infty) \). It is the vertical correction to \( w \) needed to convert the solution, \( w \), of (2.21) to a solution of (2.6). But the second term is not a vertical solution itself because the scalar factor \( \int_{\tau}^{t} \psi(s)ds \) is not independent of \( t \).

In this subsection we are going to show, at an algebraic level, that \( v_\tau(\cdot) \) satisfies the variational equation over \((0, \infty)\) for all \( \tau \in [0, \infty) \). In Section
we will show that $v_\tau(\cdot)$ is an almost strong solution if $\tau = 0$ and a strong solution if $\tau > 0$. In Section 7.1 we will show that both solutions converge to their correct initial values in the $L^2$ sense. In Section 7.3 we will show that $v_0(t)$ converges to its initial value $v_0$ in the sense of $H^1_b$, as asserted in Theorem 2.10.

**Theorem 6.1** ($v$ and $v_\tau$ solve the variational equation) Suppose that $w(s)$ is a solution to the augmented variational equation (2.21) on $(0, \infty)$. Let $\psi(s) = d^*_A w(s)$ again as in (5.1). Fix $\tau \geq 0$ and define $v_\tau$ by (6.1). Then

$$-v_\tau'(t) = d^*_A d_A v_\tau(t) + [v_\tau(t) \cdot B(t)], \quad 0 < t < \infty. \quad (6.4)$$

Let $\zeta(s) = d^*_A d_A w(s) + [w(s) \cdot B(s)]$ as in (5.2). Then $v_\tau$ can also be represented as

$$v_\tau(t) = w(\tau) - \int_\tau^t \left( \zeta(s) + [A(s) - A(t), \psi(s)] \right) ds, \quad 0 < t < \infty. \quad (6.5)$$

**Proof.** Let

$$\eta(t) = \int_\tau^t \psi(s) ds, \quad 0 < t < \infty. \quad (6.6)$$

Then $v_\tau(t) = w(t) + d_A(t) \eta(t)$ by (6.1). We have

$$\frac{d}{dt} d_A(t) \eta(t) = [A', \eta] + d_A \eta'$$

$$= -[d^*_A B, \eta] + d_A d^*_A w$$

because $A' = -d^*_A B$ by the Yang-Mills heat equation. Hence, suppressing $t$ in places, we find

$$-v_\tau'(t) = -w'(t) - \frac{d}{dt} d_A(t) \eta(t)$$

$$= \left\{ (d^*_A d_A + d_A d^*_A) w + [w \cdot B] \right\} - d_A d^*_A w + [d^*_A B, \eta]$$

$$= d^*_A d_A w + [w \cdot B] + [d^*_A B, \eta]$$

$$= d^*_A d_A (v_\tau - d_A \eta) + [(v_\tau - d_A \eta) \cdot B] + [d^*_A B, \eta]$$

$$= d^*_A d_A v_\tau + [v_\tau \cdot B] + \left\{ -d_A^*(d_A)^2 \eta - [d_A \eta \cdot B] + [d^*_A B, \eta] \right\}. \quad (6.7)$$

By the Bianchi identity we have $d_A^*(d_A)^2 \eta = d_A^*[B, \eta] = [d^*_A B, \eta] - [d_A \eta \cdot B]$. The expression in braces in line (6.7) is therefore zero. This proves (6.4).
From (5.3) we see that \( d_A \psi = -\{ w' + \zeta \} \). Hence
\[
\begin{align*}
d \int_\tau^t \psi(s) ds &= \int_\tau^t \left( d_{A(s)} \psi(s) - [A(s), \psi(s)] \right) ds \\
&= - \int_\tau^t \left( \{ w'(s) + \zeta(s) \} + [A(s), \psi(s)] \right) ds \\
&= w(\tau) - w(t) - \int_\tau^t \left( \zeta(s) + [A(s), \psi(s)] \right) ds.
\end{align*}
\]
Thus
\[
\begin{align*}
d \int_\tau^t \psi(s) ds &= w(\tau) - w(t) - \int_\tau^t \left( \zeta(s) + [A(s), \psi(s)] \right) ds \quad \text{and (6.8)} \\
d_{A(t)} \int_\tau^t \psi(s) ds &= w(\tau) - w(t) - \int_\tau^t \left( \zeta(s) + [A(s) - A(t), \psi(s)] \right) ds. \quad (6.9)
\end{align*}
\]
Add \( w(t) \) to (6.9) to find (6.5), given the definition (6.1). This completes the proof of the theorem. \( \blacksquare \)

Remark 6.2 The two representations of \( v_\tau(t) \) given in (6.1) and (6.5) differ in the following crucial way. The second derivatives of \( w(s) \) in the integrand in (6.5) are \( d_{A(s)}^* d_{A(s)} w(s) \) whereas the second derivatives of \( w \) that appear in (6.1) (after moving \( d_{A(t)} \) under the integrand and shifting time parameter to \( s \)) are \( d_{A(s)} d_{A(s)}^* w(s) \). In order to compute \( H_1 \) norms we will have to use the Gaffney-Friedrichs inequality, which requires computing exterior derivatives of \( v_\tau(t) \) and its coderivatives. For computing exterior derivatives the representation (6.1) will allow us to use the Bianchi identity, while (6.5) will allow us to compute coderivatives using the adjoint Bianchi identity.

6.2 Strong solutions vs. almost strong solutions

Theorem 6.3 Let \( 0 < b < 1 \). Choose \( T \in (0, \infty) \) and let \( A = A(T) \). Suppose that \( w \) is the solution to the augmented variational equation (2.21) constructed in Theorem 2.19. Define \( v_\tau(t) \) by (6.1) again. Then
\[
\begin{align*}
v_\tau(t) &\in H_1^A \quad \forall \ t \in (0, \infty) \quad \text{if } \tau > 0. \quad (6.10) \\
d_{A(t)} v_\tau(t) &\in H_1^A \quad \forall \ t \in (0, \infty) \quad \text{if } \tau \geq 0. \quad (6.11)
\end{align*}
\]
In particular, \( v_\tau(t) \) is a strong solution to the variational equation over \( (0, \infty) \) if \( \tau > 0 \) and is an almost strong solution if \( \tau = 0 \).
The proof depends on the following lemma.

**Lemma 6.4** Assume that $0 < b < 1$ and $0 \leq t < \infty$. Define again $\eta(t)$ by (6.6). Then

\[
\|\eta(t)\|_2 < \infty \quad \forall \tau \geq 0 \quad \forall t \geq 0 \quad (6.12)
\]

\[
\|d_A(t)\eta(t)\|_2 < \infty \quad \forall \tau \geq 0 \quad \text{if } t > 0 \quad (6.13)
\]

\[
\|d_A d_2(t)\eta(t)\|_2 < \infty \quad \forall \tau \geq 0 \quad \text{if } t > 0 \quad (6.14)
\]

\[
\|[B(t), \eta(t)]\|_2 < \infty \quad \forall \tau \geq 0 \quad \text{if } t > 0 \quad (6.15)
\]

\[
\|d_A[B(t), \eta(t)]\|_2 < \infty \quad \forall \tau \geq 0 \quad \text{if } t > 0 \quad (6.16)
\]

\[
\|d_A^* [B(t), \eta(t)]\|_2 < \infty \quad \forall \tau \geq 0 \quad \text{if } t > 0 \quad (6.17)
\]

\[
\|d_A^* \int_{\tau}^{t} \zeta(s)ds\|_2 < \infty \quad \forall \tau > 0 \quad \text{if } t > 0 \quad (6.18)
\]

\[
\|d_A^* \int_{\tau}^{t} [A(t) - A(s), \psi(s)]ds\|_2 < \infty \quad \forall \tau > 0 \quad \text{if } t > 0 \quad (6.19)
\]

**Proof.** We will write integrals over $[\tau, t]$ or $[t, \tau]$ as if $t \geq \tau$ with no loss of generality. Let $t_1 = \max(t, \tau)$. The proof of (6.12) follows from the inequalities

\[
\|\eta(t)\|_2 \leq \int_0^{t_1} \|\psi(s)\|_2 dl \leq \left(\int_0^{t_1} s^b ds\right)^{1/2} \left(\int_0^{t_1} s^{-b} \|\psi(s)\|_2^2 ds\right)^{1/2} \leq s_1^{(b+1)/2} \left(\int_0^{t_1} s^{-b} \|d_A^*(s) w(s)\|_2^2 ds\right)^{1/2} \leq s_1^{(b+1)/2} 2 \|w\|_{t_1} < \infty
\]

for all $t \geq 0$ and $\tau \geq 0$ by (5.24). To prove (6.13) and the remaining inequalities we take $t > 0$. Then

\[
\|d_A(t)\eta(t)\|_2 \leq \int_{\tau}^{t} \|d_A(t)\psi(t)\|_2 dl
\]

\[
\leq \int_{\tau}^{t} \|d_A(s)\psi(s)\|_2 dl + \int_{\tau}^{t} \|A(t) - A(s)\|_6 \|\psi(s)\|_3 dl
\]

\[
\leq \int_0^{t_1} \|d_A(s)\psi(s)\|_2 dl + \left(\int_{\tau}^{t} s^{-a} \|A(t) - A(s)\|_6^2 dl\right)^{1/2} \left(\int_0^{t_1} s^a \|\psi(s)\|_3^2 dl\right)^{1/2} < \infty \quad \text{for all } t > 0 \quad \text{and } \tau \geq 0 \quad \text{if } 0 < b < 1
\]

because the first term is finite by (5.27) (since $b > 0$), while the second term is a product of an integral over $[\tau, t]$ (or $[t, \tau]$), which is finite when $\tau > 0$.
because it excludes a neighborhood of \( s = 0 \), and is finite for \( \tau = 0 \) by (3.37). The second factor is finite by (5.28) because \( a \geq 1/2 \geq (1/2) - b \) for all \( b \in [0, 1) \).

**Proof of (6.14).** For the proof of (6.14) we have

\[
d_A d_{A(t)} \eta(t) = d_{A(t)} d_{A'(t)} \eta(t) + [(A(T) - A(t)) \wedge d_{A(t)} \eta(t)] = [B(t), \eta(t)] + [(A(T) - A(t)) \wedge d_{A(t)} \eta(t)].
\]

Therefore

\[
\|d_A d_{A(t)} \eta(t)\|_2 \leq c\|B(t)\|_{\infty}\|\eta(t)\|_2 + c\|A(T) - A(t)\|_{\infty}\|d_{A(t)} \eta(t)\|_2.
\]

The two \( L^\infty \) norms are finite by (3.32) and (3.44) respectively. (6.14) now follows from (6.12) and (6.13).

**Proof of (6.15), (6.16) and (6.17).** Since \( \|B(t)\|_{\infty} < \infty \) by (3.32), the inequality (6.15) follows from the inequality \( \|B(t), \eta(t)\|_2 \leq c\|B(t)\|_{\infty}\|\eta(t)\|_2 \) and from (6.12). The identities

\[
d_A [B(t), \eta(t)] = d_{A(t)} [B(t), \eta(t)] + [(A(T) - A(t)) \wedge [B(t), \eta(t)]],
\]

\[
d_A^*[B(t), \eta(t)] = d_{A(t)}^*[B(t), \eta(t)] + [(A(T) - A(t)) \wedge [B(t), \eta(t)]],
\]

are similar and have similar bounds. Thus

\[
\|d_{A(t)} \eta(t) \wedge B(t)\|_2 \leq c\|B(t)\|_{\infty}\|d_{A(t)} \eta(t)\|_2,
\]

which is finite by (3.32) and (6.13). Moreover

\[
\|A'(t)\|_{\infty} < \infty \quad \text{and} \quad \|A(T) - A(t)\|_{\infty} < \infty
\]

by (3.31) and (3.44). Therefore the remaining three terms in the lines (6.20) and (6.21) have finite \( L^2 \) norms by (6.12) and (6.15).

**Proof of (6.18).** To prove (6.18) we may write

\[
\|d_A^* \int_\tau^t \zeta(s) ds\|_2 = \| \int_\tau^t \left( d_A^* \zeta(s) + [(A(T) - A(s)) \wedge \zeta(s)] \right) ds \|_2
\]

\[
\leq \int_\tau^t \|d_A^* \zeta(s)\|_2 ds + c \int_\tau^t \|A(T) - A(s)\|_6 \|\zeta(s)\|_3 ds
\]

\[
\leq \int_\tau^t \|d_A^* \zeta(s)\|_2 ds + c \left( \int_\tau^t s^{-a} \|A(T) - A(s)\|_6^2 ds \right)^{1/2} \left( \int_\tau^t s^a \|\zeta(s)\|_3^2 ds \right)^{1/2}
\]

(6.22)

(6.23)
The integral over \((0,t)\) is finite by (3.37). Since the interval \([\tau,t]\) is bounded away from zero the integrals over \([\tau,t]\) in lines (6.22) and (6.23) are finite by (5.51) and (5.57), respectively. This proves (6.18).

**Proof of (6.19).** We have the identities

\[
\begin{align*}
  d_A^* \int_{\tau}^{t} [A(t) - A(s), \psi(s)] ds &= \int_{\tau}^{t} d_A^*[A(t) - A(s), \psi(s)] ds \\
  &= \int_{\tau}^{t} \left( [d_A^*(A(t) - A(s)), \psi(s)] + [(A(t) - A(s)) \cdot d_A \psi(s)] \right) ds.
\end{align*}
\]

Therefore

\[
\begin{align*}
  \|d_A^* \int_{\tau}^{t} [A(t) - A(s), \psi(s)] ds\|_2 &\leq c \int_{\tau}^{t} \left( \|d_A^*(A(t) - A(s))\|_3 \|\psi(s)\|_6 + \|A(t) - A(s)\|_3 \|d_A \psi(s)\|_6 \right) ds \\
  &\leq c \left( \int_{\tau}^{t} s^{b-1} \|d_A^*(A(t) - A(s))\|_3^2 ds \right)^{1/2} \left( \int_{0}^{t} s^{1-b} \|\psi(s)\|_6^2 ds \right)^{1/2} \\
  &\quad + c \left( \int_{\tau}^{t} s^{b-2} \|A(t) - A(s)\|_3^2 ds \right)^{1/2} \left( \int_{\tau}^{t} s^{2-b} \|d_A \psi(s)\|_6^2 ds \right)^{1/2}.
\end{align*}
\]

The \(A\) integrals are finite because the interval \([\tau,t]\) excludes a neighborhood of zero. The first \(\psi\) integral is finite by (5.27). The second \(\psi\) integral is finite because it differs from (5.52) by at most \(\int_{\tau}^{t} s^{2-b} \|A(T) - A(s)\|_\infty^2 \|\psi(s)\|_6^2 ds\), while \(\|A(T) - A(s)\|_\infty\) is bounded over \([\tau,t]\) in accordance with (3.44). \(\blacksquare\)

**Proof of Theorem (6.3).** Let \(\tau > 0\). In order to prove that \(v_\tau(t) \in H^A_1\) for \(t > 0\) it suffices, by the Gaffney-Friedrichs inequality, to show that \(v_\tau(t), d_A v_\tau(t)\) and \(d_A^* v_\tau(t)\) are all in \(L^2(M)\) for each \(t > 0\). Now (6.1) and (6.5) show that

\[
\begin{align*}
  d_A v_\tau(t) &= d_A w(t) + d_A d_A(t) \int_{\tau}^{t} \psi(s) ds, \\
  d_A^* v_\tau(t) &= d_A^* w(\tau) - d_A^* \int_{\tau}^{t} (\zeta(s) + [A(s) - A(t), \psi(s)]) ds.
\end{align*}
\]

Since \(w(t)\) and \(w(\tau)\) are both in \(H^A_1\) for \(t > 0\) and \(\tau > 0\) we need only address the second term in each line. But the second term in line (6.24) is in \(L^2(M)\) by (6.14) and the second term in line (6.25) is in \(L^2(M)\) by (6.18).
and (6.19). \( v_\tau(t) \) itself is in \( L^2(M) \) by (6.1) and (6.13). Hence \( v_\tau(t) \in H^A_1 \) for each \( t > 0 \) when \( \tau > 0 \). In order to show that it is strong solution to the variational equation (2.6), we only need to show that \( d_{A(t)}v_\tau(t) \) is in \( H^A_1 \) for each \( t > 0 \) since Theorem 6.1 already shows that it satisfies the variational equation informally. From (6.1) and the Bianchi identity we see that

\[
d_{A(t)}v_\tau(t) = d_{A(t)}w(t) + [B(t), \eta(t)].
\]

(6.26)

The first term is in \( H^A_1 \) because \( w \) is a strong solution by Theorem 2.19. The second term is in \( H^A_1 \) by the Gaffney-Friedrichs inequality in view of (6.15), (6.16) and (6.17). Therefore \( v_\tau(\cdot) \) is a strong solution when \( \tau > 0 \). But the last argument shows that \( d_{A(t)}v_\tau(t) \in H^A_1 \) for any \( \tau \geq 0 \) because (6.15), (6.16) and (6.17) all hold for any \( \tau \geq 0 \). Therefore \( v_\tau(t) \) is an almost strong solution even for \( \tau = 0 \). This completes the proof of Theorem 6.3.

7 Recovery of \( v \) from \( w \): initial value

We have shown in Section 6 that the functions \( v \) and \( v_\tau \), defined in (2.23) and (2.24), are respectively almost strong and strong solutions to the variational equation over \((0, \infty)\). In Section 7.1 we will show that both take on their correct initial values in the sense of \( L^\rho(M; \Lambda^1 \otimes \mathfrak{t}) \) convergence for \( 2 \leq \rho < 3 \).

We will show in Section 7.3 that the almost strong solution attains its initial value in the stronger sense of \( H^A_b \) convergence. And in Section 7.4 we will show that the strong solution has finite \( b \)-action. For the latter two results we will have to use some non-gauge invariant techniques along with the non-gauge invariant hypothesis that \( \| A(s) \|_3 < \infty \) for some \( s > 0 \).

7.1 Initial values in the \( L^\rho \) sense

It will be convenient to write (2.25) in the form

\[
\alpha_\tau = \int_0^\tau \psi(s)ds,
\]

(7.1)

where \( \psi(s) = d^*_{A(s)}w(s) \) since we will make extensive use of the initial behavior of \( \psi(s) \) and some of its derivatives that has been established in Section 5.
Theorem 7.1 Assume that $1/2 \leq a < 1$ and $1/2 \leq b < 1$. Suppose that $2 \leq \rho < 3$ and let $\tau > 0$. Denote by $w$ the strong solution to the augmented variational equation constructed in Theorem 2.19. Define $v(t)$ and $v_\tau(t)$ by (2.23) and (2.24) respectively. Then

\begin{align}
\|v_\tau(t) - (v_0 - d_{A_0} \alpha_\tau)\|_\rho & \to 0 \quad \text{as} \quad t \downarrow 0 \quad \text{and} \\
\sup_{0 \leq t \leq 1} \|v_\tau(t) - v(t)\|_\rho & \to 0 \quad \text{as} \quad \tau \downarrow 0.
\end{align}

(7.2) (7.3)

Since $v_\tau$ is a strong solution to the variational equation, by Theorem 6.3, it is continuous on $(0, \infty)$ into $H^1_A$ and therefore also into $H^1_b$ for $0 \leq b \leq 1$. Theorem 7.1 is properly concerned, therefore, only with the behavior of $v_\tau$ at $t = 0$.

Lemma 7.2 (Continuity of the vertical correction) Let $1/2 \leq a < 1$ and $1/2 \leq b < 1$. Let $0 < T < \infty$ and let $A = A(T)$. Suppose that $2 \leq \rho < 3$. Then

\begin{align}
\alpha_\tau & \in L^p(M; \mathfrak{T}) \quad \text{for} \quad 2 \leq p < \infty, \\
\alpha_\tau & \in H^1_A(M; \mathfrak{T}), \\
\sup_{0 < t \leq 1} \|d_{A(t)} \alpha_\tau\|_\rho & \to 0 \quad \text{as} \quad \tau \downarrow 0, \\
t, \tau \mapsto d_{A(t)} \alpha_\tau & \in L^p(M; \Lambda^1 \otimes \mathfrak{T}) \quad \text{is continuous on } [0, \infty)^2.
\end{align}

(7.4) (7.5) (7.6) (7.7)

Proof. From (5.38) we see that for $1/2 \leq b < 1$ we have

\begin{align}
\int_0^\tau \|d_{A(s)} \psi(s)\|_\rho ds & < \infty \quad \text{if} \quad 2 \leq \rho < 3 \quad \text{and} \\
\int_0^\tau \|d_{A(s)} \psi(s)\|_\rho ds & \to 0 \quad \text{as} \quad \tau \downarrow 0 \quad \text{if} \quad 2 \leq \rho < 3.
\end{align}

(7.8) (7.9)

For any number $p \in [6, \infty)$ there is a number $\rho \in [2, 3)$ such that $p^{-1} = \rho^{-1} - (1/3)$. There is therefore a Sobolev constant $\kappa_p$ such that $\|\psi(s)\|_p \leq \kappa_p \left(\|d_{A(s)} \psi(s)\|_\rho + \|\psi(s)\|_2\right)$. (Actually, the term $\|\psi(s)\|_2$ is not needed if $M = \mathbb{R}^3$.) Hence, in view of (7.3), we have

\begin{align}
\int_0^\tau \|\psi(s)\|_p ds & < \infty
\end{align}

(7.10)
since \( \int_0^\tau \| \psi(s) \|^2 ds < \infty \), as shown in (6.12). Since (7.10) holds also for \( p = 2 \), it holds for all \( p \in [2, \infty) \). This proves (7.4).

For all \( \tau \geq 0 \) and \( t \geq 0 \) we can write

\[
\|d_{A(t)}\alpha_\tau\|_\rho = \int_0^\tau d_{A(s)}\psi(s)ds + \int_0^\tau [A(t) - A(s), \psi(s)]ds. \tag{7.11}
\]

Hence

\[
\|d_{A(t)}\alpha_\tau\|_\rho \leq \int_0^\tau \|d_{A(s)}\psi(s)\|_\rho ds + \int_0^\tau [A(t) - A(s), \psi(s)]ds\|_\rho \\
\leq \int_0^\tau \|d_{A(s)}\psi(s)\|_\rho ds + c \sup_{0 < s \leq \tau} \|A(t) - A(s)\|_3 \int_0^\tau \|\psi(s)\|_\rho ds, \tag{7.12}
\]

where \( \rho^{-1} = 3^{-1} + p^{-1} \). The first integral is finite by (7.3). Since \( \rho < 3 \) we have \( p < \infty \) and therefore the integral in (7.12) is finite by (7.10). The supremum in line (7.12) is finite by (2.5).

It follows from (7.8) and (7.12) that

\[
d_{A(t)}\alpha_\tau \in L^p(M), \quad t \geq 0, \quad \tau \geq 0. \tag{7.13}
\]

In particular, for \( t = T \) and \( \rho = 2 \) we can conclude that (7.5) holds. The last estimate in line (7.12) also shows that \( \sup_{0 \leq t \leq 1} \| \int_0^T [A(t) - A(s), \psi(s)]ds \|_\rho \to 0 \) as \( \tau \downarrow 0 \), which together with (7.9) shows that (7.6) holds.

It remains to prove the joint continuity (7.7). For \( 0 \leq \tau_0 \leq \tau \) we have

\[
\|d_{A(t)}\alpha_\tau - d_{A(t_0)}\alpha_{\tau_0}\|_\rho \leq \|(d_{A(t)} - d_{A(t_0)})\alpha_\tau\|_\rho + \|d_{A(t_0)}(\alpha_\tau - \alpha_{\tau_0})\|_\rho \\
\leq c\|A(t) - A(t_0)\|_3\|\alpha_\tau\|_p + \left\| \int_{\tau_0}^\tau \left( d_{A(s)}\psi(s) + [A(t_0) - A(s), \psi(s)] \right)ds \right\|_\rho \\
\leq c\|A(t) - A(t_0)\|_3 \int_{\tau_0}^\tau \|\psi(s)\|_p ds + \int_{\tau_0}^\tau \|d_{A(s)}\psi(s)\|_\rho \\
\quad + c \sup_{t_0 \leq s \leq \tau} \|A(t_0) - A(s)\|_3 \int_{\tau_0}^\tau \|\psi(s)\|_p ds.
\]

All three terms go to zero as \( |t - t_0| + |\tau - \tau_0| \to 0 \). This concludes the proof of the lemma. \( \blacksquare \)

**Remark 7.3** (Larger \( \rho \) from larger \( b \)) The restriction on \( \rho \) specified in (5.96) allows larger \( \rho \) for larger \( b \). In order for this to yield larger \( \rho \) in (7.6) and
when \( b > 1/2 \) it seems unavoidable to assume also that \( a > 1/2 \), so as to allow larger \( \rho \) on the left side of \((7.12)\). For example if \( a > 1/2 \) then one can use \( \|A(t) - A(s)\|_q \) in \((7.12)\) for some \( q > 3 \), allowing a finite value of \( p \) even if \( \rho \geq 3 \). We won’t pursue the arithmetic needed for this because we don’t foresee a need for this extension. It might be of some interest to note that the vital condition \((7.10)\), which we have derived from a Sobolev inequality, also follows from the high \( L^p \) bound \((5.97)\) for \( b \geq 1/2 \) because \((3/2) - b - (3/p) < 1 \). But the use of \((5.97)\) disallows Dirichlet boundary conditions. Perhaps of some ultimate importance is the fact that for \( \rho = 2 \) we have \( p = 6 \) in \((7.12)\) and in this case we can use the simple energy bound \((5.27)\), which already implies that \( \int_0^\tau \|\psi(s)\|_6 ds < \infty \) for \( b > 0 \).

**Proof of Theorem** \((7.1)\). By \((2.23)\) we have \( v(t) = w(t) + dA(t)\alpha_t \). Both terms are continuous functions of \( t \in [0, \infty) \) into \( L^\rho(M; \Lambda^1 \otimes \mathfrak{k}) \), the first because \( H A_b \hookrightarrow L^2 \cap L^3 \) is continuous, and the second by \((7.7)\). Thus \( v(t) \) is a continuous function into \( L^\rho(M; \Lambda^1 \otimes \mathfrak{k}) \). Since, by \((2.17)\), \( v_\tau(t) = v(t) - dA(t)\alpha_\tau \), \( v_\tau \) is also a continuous function into \( L^\rho(M; \Lambda^1 \otimes \mathfrak{k}) \) by \((7.7)\). This proves \((7.2)\). Moreover \((7.6)\) proves \((7.3)\).

**Proof of Lemma** \((7.9)\). (Vertical solutions) We wish to show that the function \( z(t) := dA(t)\alpha \) is an almost strong solution of the variational equation when \( \alpha \) is an element of \( H^A_1(M; \mathfrak{k}) \). Note, by the way, that this hypothesis is satisfied by the elements \( \alpha_\tau \) defined in \((7.1)\), as we see from \((7.5)\). Under our present hypothesis \( z(t) \in L^2(M; \Lambda^1 \otimes \mathfrak{k}) \), although it need not be in \( H^A_1 \) for any \( t > 0 \). But by the Bianchi identity,

\[
dA(t)z(t) = [B(t), \alpha],
\]

which we will show is in \( H^A_1 \). First notice that the computation that gives \((7.14)\) involves second derivatives of \( \alpha \), which may only exist as distributions. The second order derivatives that appear give \( d^2 \alpha \) which is zero in the distribution sense. Thus although the right side of \((7.14)\) is a well defined function, the equation has to be interpreted in the distribution sense. Now the identity \( d^*_{A(t)}[B(t), \alpha] = [d^*_{A(t)}B(t), \alpha] - [dA(t)\alpha \atical B(t)] \), together with \((7.14)\) and the Yang-Mills heat equation, \( -A'(t) = d^*_{A(t)}B(t) \), yields

\[
d^*_{A(t)}dA(t)z(t) = -[A'(t), \alpha] - [z(t) \atical B(t)],
\]

\[
dA(t)dA(t)z(t) = [B(t) \land z(t)].
\]
Since $B(t)$ and $A'(t)$ are both bounded for each $t > 0$ by (3.32) and (3.31), and since $\alpha \in L^2(M; \mathfrak{f})$ and $z(t) \in L^2(M; A^1 \otimes \mathfrak{f})$, the right sides of (7.14), (7.15) and (7.16) are all in $L^2$. The Gaffney-Friedrichs inequality now shows that $d_A(t)z(t)$ is in $H^1(t)$. At the same time, the definition $z(t) = d_A(t)\alpha$ shows that $z(t)$ is a solution to the variational equation because $-z'(t) = -[A'(t), \alpha] = d_A d_A z + [z, \mathcal{A}]$, as follows from (7.15). Hence $z(t)$ is a vertical, almost strong solution to the variational equation.

In case $z(t_0) \in H^1$ for some $t_0 > 0$ then the identity $z(t) = z(t_0) + [A(t) - A(t_0), \alpha]$ shows that $z(t)$ will also be in $H^1$ if $[A(t) - A(t_0), \alpha] \in H^1$. But for $t > 0$ the latter is in $H^1$ by another Gaffney-Friedrichs argument: Let $\beta = A(t) - A(t_0)$. Then $d_A^\beta \in L^3$ by (3.41), $d_A^\beta \in L^3$ by (3.43) since $A \wedge \beta \in L^6 \cdot L^6 \subset L^3$, and $\beta \in L^\infty$ by (3.44). In the meanwhile $d_A^\beta \in L^2$ and $\alpha \in L^6 \cap L^2$. The product rule now shows that $[\beta, \alpha], d_A[\beta, \alpha]$ and $d_A^\beta[\beta, \alpha]$ are all in $L^2$. Therefore $[\beta, \alpha] \in H^1$.

Thus $z(\cdot)$ is a strong solution if and only if $z(t_0) \in H^1$ for some $t_0 > 0$. This completes the proof of Lemma 2.9.

7.2 A non-gauge-invariant representation of $v_\tau$

The representations of $v$ and $v_\tau$ given in (2.23) and (2.24) capture the infinitesimal analog of the ZDS procedure. The next theorem gives another, highly non-gauge invariant representation of both. It will be needed to prove that $v(t)$ converges to $v_0$ in the $H^1$ norm as $t \downarrow 0$ and to prove that $v_\tau$ has finite $b$-action for $\tau > 0$.

**Theorem 7.4** (A non-gauge-invariant representation of $v_\tau(t)$.) Suppose that $w(s)$ is a solution to the augmented variational equation (2.21) on $(0, \infty)$. Let $\psi(s) = d_{A(s)}^* w(s)$ again as in (5.1). Fix $\tau \geq 0$ and define $v_\tau$ by (6.1). Let $P^\perp$ be the projection in $L^2(M; A^1 \otimes \mathfrak{f})$ onto the orthogonal complement of the null space of $d^*$. Define

$$w(s) = P^\perp w(s). \quad (7.17)$$

Then $v_\tau$ is also given by

$$v_\tau(t) = w(t) + \tilde{w}(\tau) - \hat{w}(t)$$

$$+ \int_\tau^t \left( [A(t), \psi(s)] - P^\perp \left( [A(s), \psi(s)] \right) \right) ds \quad (7.18)$$

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for \( t > 0 \) and \( \tau \geq 0 \). The spatial derivatives of the integrand in (7.18) are given by

\[
\begin{align*}
\frac{d\left( [A(t), \psi(s)] - P^\perp \left( \zeta(s) + [A(s), \psi(s)] \right) \right)}{ds} &= d[A(t), \psi(s)] \quad (7.19) \\
\frac{d^*\left( [A(t), \psi(s)] - P^\perp \left( \zeta(s) + [A(s), \psi(s)] \right) \right)}{ds} &= d^*[A(t) - A(s), \psi(s)] - [w(s) \cdot \Delta A'(s)] + [A(s) \cdot \Delta \zeta(s)]. \quad (7.20)
\end{align*}
\]

**Proof.** Apply the projection \( P^\perp \) to (6.8). \( P^\perp \) is the identity operator on exact 1-forms since these span the orthogonal complement of the kernel of \( d^* \). Thus we have

\[
\begin{align*}
\frac{d}{ds} \int_\tau^t \psi(s) ds &= \hat{\omega}(\tau) - \hat{\omega}(t) - \int_\tau^t P^\perp \left( \zeta(s) + [A(s), \psi(s)] \right) ds \\
\text{and therefore} \\
d_{A(t)} \int_\tau^t \psi(s) ds &= \hat{\omega}(\tau) - \hat{\omega}(t) + \int_\tau^t \left( [A(t), \psi(s)] - P^\perp \left( \zeta(s) + [A(s), \psi(s)] \right) \right) ds.
\end{align*}
\]

Add \( w(t) \) to find (7.18).

The identity (7.19) follows immediately from the identity \( dP^\perp = 0 \) on any 1-form. For the proof of (7.20) we can use the identity \( d^*P^\perp \omega = d^* \omega \) and (5.4) to find

\[
\begin{align*}
\frac{d^*P^\perp}{ds} \left( [A(s), \psi(s)] \right) &= d^* \left( \zeta + [A, \psi] \right) \\
&= d^* \zeta + d^*[A, \psi] \\
&= d^*_A \zeta - [A \cdot \Delta \zeta(s)] + d^*[A, \psi] \\
&= [w(s) \cdot \Delta A'(s)] - [A(s) \cdot \Delta \zeta(s)] + d^*[A(s), \psi(s)].
\end{align*}
\]

This completes the proof of the theorem. □

**Remark 7.5** In addition to the three representations of \( v_\tau \), (6.1), (6.5) and (7.18), there is a fourth representation of \( v(t) \) that has a more gauge invariant structure than (7.18). Let \( A = A(T) \) as before and denote by \( P_A \) the projection in \( L^2(M; \Lambda \otimes \mathfrak{t}) \) onto the null space of \( d^*_A \). Much like in the derivation
of (7.18) from (6.8) we can deduce that
\[ d_{A(t)} \int_t^\tau \psi(s) ds \]
\[ = \hat{w}(\tau) - \hat{w}(t) + \int_t^\tau \left\{ [A(t) - A, \psi(s)] - \frac{1}{2} \partial A \right\} ds \]
where \( \hat{w}(t) = P_{A}^\perp w(t) \). This representation has the advantage that only differences \( A(t) - A \) occur, making the representation gauge invariant. But the analog of the identity (7.19) fails because \( d_{A} P_{A}^\perp \omega = [B(T), G_{A} d^{*}_{A} \omega] \) on 1-forms \( \omega \), where \( G_{A} = (d^{*}_{A} d_{A})^{-1} \phi \) is the Green operator on \( \mathfrak{k} \) valued scalars. Attempts to use this Green operator in our context have not been successful.

### 7.3 Initial value of the almost strong solution in the \( H_{b}^A \) sense

Our techniques in Sections 7.3 and 7.4 are going to rely on using the non-gauge invariant Sobolev space \( H_{1}^{0} \) defined by
\[ \| \omega \|_{H_{1}^{0}}^{2} = \int_{M} \left( \sum_{j=1}^{3} | \partial_j \omega(x) |_{\Lambda^1 \otimes \mathfrak{k}}^{2} + | \omega(x) |_{\Lambda^1 \otimes \mathfrak{k}}^{2} \right) dx. \] (7.22)

All results in the preceding sections have made (usually unavoidable) use of the gauge invariant Sobolev norm \( H_{A}^{1} \). In order to transfer information from preceding sections to the present two sections it will be necessary to show equivalence of these two norms. Under our standing assumption, that \( A(\cdot) \) is a strong solution to the Yang-Mills heat equation, these two norms are automatically equivalent because \( A \equiv A(T) \in L^{6} \cap L^{2} \subset L^{3} \), and the next lemma assures that the two norms are equivalent when \( A \in L^{3} \). The equivalence of these norms is therefore not an issue for this paper. But we plan to use a weaker notion of solution in \( \mathbb{R}^{3} \) in order to include sections of instantons into the Yang-Mills configuration space. The condition \( A(T) \in L^{3}(\mathbb{R}^{3}) \) will be disallowed. We therefore wish to keep track of exactly where the condition \( A(T) \in L^{3}(\mathbb{R}^{3}) \) is used in this paper. The hypothesis that \( A \in L^{3}(M) \) in the theorems of Sections 7.3 and 7.4 are therefore purposefully made explicit even though they already are implied by the assumption that \( A(\cdot) \) is a strong solution. The condition that \( A \in L^{3}(M) \) has not been used
in any previous part of this paper. This discussion is of substance only if $M = \mathbb{R}^3$ because $L^6(M) \subset L^3(M)$ when $M$ is bounded while $A(T)$ will always be in $L^6(M)$ in [7].

The equivalence of $H_1$ norms is based on the following elementary lemma.

**Lemma 7.6** Suppose that $A_1$ and $A_2$ are two connection forms over a region $\mathcal{R} \subset \mathbb{R}^3$ lying in $W_1^1(\mathcal{R})$. Then for all $\omega$ in the domains of the following operators one has

$$\|\nabla A_1 \omega\|_2^2 \leq C \|\nabla A_2 \omega\|_2^2$$  \hspace{1cm} if $\mathcal{R} = \mathbb{R}^3$  \hspace{1cm} (7.23)

$$\|\nabla A_1 \omega\|_2^2 + \|\omega\|_2^2 \leq C \left( \|\nabla A_2 \omega\|_2^2 + \|\omega\|_2^2 \right)$$  \hspace{1cm} if $\mathcal{R} = \mathbb{R}^3$ or $M$,  \hspace{1cm} (7.24)

where

$$C = 1 + (1 + c \kappa_6 \|A_1 - A_2\|_{L^3(\mathcal{R})})^2$$  \hspace{1cm} (7.25)

and $\kappa_6$ is a Sobolev constant. In particular, the norms $H_1^0(M)$ and $H_1^A(M)$ are equivalent if $A \in L^3(M)$ or if $A(s) \in L^3(M)$ for some $s > 0$.

**Proof.** In case $\mathcal{R} = \mathbb{R}^3$ we have

$$\|\nabla A_1 \omega\|_2 \leq \|\nabla A_2 \omega\|_2 + \|(A_1 - A_2)\omega\|_2$$

$$\leq \|\nabla A_2 \omega\|_2 + c \|(A_1 - A_2)\|_3 \|\omega\|_6$$

$$\leq \|\nabla A_2 \omega\|_2 + c \|(A_1 - A_2)\|_3 \kappa_6 \|\nabla A_2 \omega\|_2,$$  \hspace{1cm} (7.26)

where $\kappa_6$ is a Sobolev constant for which $\|\omega\|_6 \leq \kappa_6 \|\nabla A_2 \omega\|_2$ $\forall \omega \in C^\infty_c(\mathbb{R}^3)$. Square (7.26) to find (7.23). Add $\|\omega\|_2^2$ to the square of (7.26) to find (7.24) (without the initial term 1) in case $\mathcal{R} = \mathbb{R}^3$. The same proof applies in case $\mathcal{R} = M$ but one needs the additional term $\|\omega\|_2^2$ from the start to use the Sobolev inequality $\|\omega\|_6^2 \leq \kappa_6^2 \left( \|\nabla A_2 \omega\|_2^2 + \|\omega\|_2^2 \right)$ for Neumann or Dirichlet boundary conditions.

In particular if $A \in L^3(M)$ then, choosing $A_1 = 0$ and $A_2 = A$ and vice versa, we see that $H_1^0$ and $H_1^A$ are equivalent. Moreover if $A(s) \in L^3(M)$ then so is $A(T) \in L^3(M)$ because $A(T) - A(s) \in L^3(M)$ by (2.5). Therefore $H_1^0$ and $H_1^A$ are equivalent. ■

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**Theorem 7.7** Let $1/2 \leq a < 1$ and $1/2 \leq b < 1$. Assume that $\|A(s)\|_{L^3(M)} < \infty$ for some $s > 0$. Suppose that $w$ is a solution to $$(2.21)$$ with finite $b$-action. Define $v(t)$ by $$(2.23)$$. Then

$$\|v(t) - v_0\|_{H_b^A} \to 0 \quad \text{as} \quad t \downarrow 0. \quad (7.27)$$

Furthermore $v$ is a continuous function into $H_b^A$ on all of $[0, \infty)$.

For the proof of continuity of $v$ at $t = 0$ we need to show that the second term in $(2.23)$, i.e. the vertical correction, converges to zero in $H_b^A$, since $w(t)$ converges to $v_0$ in $H_b^A$ by Theorem 3.7. Proof of continuity at $t = 0$ is more delicate than at $t > 0$ and will be proved in the next theorem.

**Theorem 7.8** (The vertical correction) Assume that $\|A(T)\|_{L^3(M)} < \infty$. Let $1/2 \leq a < 1$ and $1/2 \leq b < 1$. Suppose that $w$ is a solution to $(2.21)$ with finite $b$-action. Then

$$\|d_{A(t)} \int_0^t \psi(s)ds\|_{H_b^A} \to 0 \quad \text{as} \quad t \downarrow 0. \quad (7.28)$$

In particular the left hand side of $(7.28)$ is finite for $0 < t < \infty$.

The proof depends on the following lemmas.

**Lemma 7.9** (Riesz avoidance) Let $0 \leq b \leq 1$. Define $p$ in the interval $[5/6, 2]$ by

$$p^{-1} = 2^{-1} + (1 - b)/3. \quad (7.29)$$

If $\omega$ is a $\mathfrak{k}$ valued 1-form in $L^2(M)$ with $d\omega \in L^p$ and $d^*\omega \in L^p$ then $\omega \in H_b$. There is a Sobolev constant $c_p$ such that

$$\|\omega\|_{H_b^p} \leq c_p \left(\|d^*\omega\|_p + \|d\omega\|_p + \|\omega\|_2\right). \quad (7.30)$$

**Proof.** This is a slightly simplified version of [6, Lemma 6.17]

**Lemma 7.10** Assume that $1/2 \leq a < 1$ and $1/2 \leq b < 1$. Let $p^{-1} = (5/6) - (b/3)$ as in $(7.29)$. Write $\zeta(s) = d_{A(s)}^*d_{A(s)}w(s) + [w(s) \wedge B(s)]$ as in
Then for any $t \geq 0$ we have

$$
\int_0^t \| [w(s) A'(s)] \|_p ds < \infty, \quad (7.31)
$$

$$
\int_0^t \| [A(s) \zeta(s)] \|_p ds < \infty \quad \text{and} \quad (7.32)
$$

$$
\int_0^t \| \zeta(s) \|_2 ds < \infty. \quad (7.33)
$$

Define $\eta(t) = \int_0^t \psi(s) ds$. Then

$$
\int_0^t \| d^*[A(s) - A(t), \psi(s)] \|_p ds \to 0 \quad \text{as} \quad t \downarrow 0, \quad (7.34)
$$

$$
\int_0^t \| d[A(t), \psi(s)] \|_p ds \to 0 \quad \text{as} \quad t \downarrow 0 \quad \text{and} \quad (7.35)
$$

$$
\| [B(t), \eta(t)] \|_p \to 0 \quad \text{as} \quad t \downarrow 0. \quad (7.36)
$$

**Proof.** Let $r^{-1} + 6^{-1} = p^{-1}$. Then $r^{-1} = (2/3) - (b/3)$ and

$$
\int_0^t \| [w(s) A'(s)] \|_p ds \leq c \int_0^t \| w(s) \|_6 \| A'(s) \|_r ds
$$

$$
\leq c \left( \int_0^t s^{-b} \| w(s) \|_6^2 ds \right)^{1/2} \left( \int_0^t s^b \| A'(s) \|_r^2 ds \right)^{1/2}. \quad (7.31)
$$

The first factor is finite because $w$ is assumed to have finite $b$-action. Using the interpolation inequality (5.69) with $f(s, x) = |A'(s, x)|$ we see that the second factor is also finite in view of the initial behavior bounds (3.24) and (3.25) with $a = 1/2$. Here, as elsewhere, we are using the fact that $A$ has finite $(1/2)$-action if it has finite $a$-action for some $a \in [1/2, 1)$. This proves (7.31).

For the proof of (7.32) choose $r$ as above. Then

$$
\int_0^t \| [A(s) \zeta(s)] \|_p ds \leq \int_0^t \| A(s) \|_6 \| \zeta(s) \|_r ds
$$

$$
\leq c \left( \int_0^t s^{-1/2} \| A(s) \|_6^2 ds \right)^{1/2} \left( \int_0^t s^{1/2} \| \zeta(s) \|_r^2 ds \right)^{1/2}. \quad (7.32)
$$

The first factor is finite by (3.39), with $a = 1/2$. The second factor is finite by the initial behavior bound (5.56). This proves (7.32).
For the proof of (7.33) we have
\[
\int_0^t \| \zeta(s) \|_2^2 ds \leq \left( \int_0^t s^{b-1} ds \right)^{1/2} \left( \int_0^t s^{1-b} \| \zeta(s) \|_2^2 ds \right)^{1/2}
\]
which is finite when \( b > 0 \) by (5.27). This proves (7.33).
To prove (7.34) let \( \alpha(s) = A(s) - A(t) \) and let \( q^{-1} = p^{-1} - 3^{-1} = (1/2) - (b/3) \).

The identity
\[
d^* [\alpha(s), \psi(s)] = \alpha(s) \langle d\psi \rangle + [d^* \alpha(s), \psi]
\]
allows the bounds
\[
\int_0^t \| d^* [\alpha(s), \psi(s)] \|_p ds \leq c \int_0^t \left( \| \alpha(s) \|_6 \| d\psi(s) \|_r + \| d^* \alpha(s) \|_3 \| \psi \|_q \right) ds
\]
\[
\leq c \left( \int_0^t s^{-1/2} \| \alpha(s) \|_6^2 ds \right)^{1/2} \left( \int_0^t s^{1/2} \| d\psi(s) \|_r^2 ds \right)^{1/2}
\]
\[
+ c \left( \int_0^t \| d^* \alpha(s) \|_3^2 ds \right)^{1/2} \left( \int_0^t \| \psi \|_q^2 ds \right)^{1/2}
\]
(7.38)

The four factors are finite by (3.37) (with \( a = 1/2 \)), (5.56), (3.40) and (5.55), respectively. This proves (7.34).

The proof of (7.35) resembles, partly, the preceding proof. Similar to the bounds in (7.38), we have
\[
\int_0^t \| d[A(t), \psi(s)] \|_p ds \leq \int_0^t \left( \| [A(t) \wedge d\psi(s)] \|_p + \| dA(t), \psi(s) \|_p \right) ds
\]
\[
\leq c \int_0^t \left( \| A(t) \|_6 \| d\psi(s) \|_r + \| dA(t) \|_3 \| \psi(s) \|_q \right) ds
\]
\[
\leq c \| A(t) \|_6 \left( \int_0^t s^{-1/2} ds \right)^{1/2} \left( \int_0^t s^{1/2} \| d\psi(s) \|_r^2 ds \right)^{1/2}
\]
\[
+ c \| dA(t) \|_3 t^{1/2} \left( \int_0^t \| \psi(s) \|_q^2 ds \right)^{1/2}
\]
(7.39)

(7.40)
The last integral in line (7.39) is finite by (5.56) while the first two factors are \( \| A(t) \|_6 O(t^{1/4}) \), which goes to zero as \( t \downarrow 0 \) in accordance with (3.38). In line (7.40) the product of the first two factors goes to zero by (3.43) while the last integral is finite by (5.55). This proves (7.35).
For the proof of (7.36) we have
\[ \| [B(t), \eta(t)] \|_p \leq c \| B(t) \|_6 \| \eta(t) \|_r. \]
Now
\[ \| \eta(t) \|_r \leq \int_0^t \| \psi(s) \|_r ds \]
\[ \leq \left( \int_0^t s^{1/2} ds \right)^{1/2} \left( \int_0^t s^{-1/2} \| \psi(s) \|_2^2 ds \right)^{1/2}. \]
The second factor is bounded for \( t \in [0, T] \) by (5.71). The first factor is \( O(t^{3/4}) \). Since, by (3.19), \( t^{3/4} \| B(t) \|_6 = o(1) \) as \( t \downarrow 0 \) the assertion (7.36) follows.

**Lemma 7.11** Assume that \( 1/2 \leq a < 1 \) and \( 1/2 \leq b < 1 \). Let \( p^{-1} = (5/6) - (b/3) \) as in (7.29). Let \( t_0 > 0 \). Then
\[ \| \int_0^t d[A(t) - A(t_0), \psi(s)]ds \|_p + \| \int_0^t d[A(t_0), \psi(s)]ds \|_p \]
\[ \| \int_0^t d^*[A(t) - A(t_0), \psi(s)]ds \|_p + \| \int_0^t d^*[A(t_0) - A(s), \psi(s)]ds \|_p \to 0 \]
as \( t \to t_0 \).

**Proof.** Just as in the inequalities leading to (7.40) we have
\[ \int_0^t \| d[A(t) - A(t_0), \psi(s)]ds \|_p + \int_0^t \| d^*[A(t) - A(t_0), \psi(s)]ds \|_p \]
\[ \leq 4c \| A(t) - A(t_0) \|_6 t^{1/4} \left( \int_0^t s^{1/2} \| d\psi(s) \|_2^2 ds \right)^{1/2} \]
\[ + c \left( \| d(A(t) - A(t_0)) \|_3 + \| d^*(A(t) - A(t_0)) \|_3 \right) t^{1/2} \left( \int_0^t \| \psi(s) \|_6^2 ds \right)^{1/2} \]
Now \( \| A(t) - A(t_0) \|_6 \to 0 \) as \( t \to t_0 \) because of (3.21). The two \( L^3 \) norms in the last line are also continuous in \( t \) at \( t_0 \neq 0 \), as we can derive from the two identities (3.61) and \( B'(s) = d_A A'(s) = dA'(s) + [A(s) \wedge A'(s)] \). Indeed we find
\[ \| d^*(A(t) - A(t_0)) \|_3 \leq \left| \int_{t_0}^t c \| A(s) \|_6 \| A'(s) \|_6 ds \right| \quad \text{and} \]
\[ \| d(A(t) - A(t_0)) \|_3 \leq \left| \int_{t_0}^t B'(s) \|_3 ds \right| + \left| \int_{t_0}^t c \| A(s) \|_6 \| A'(s) \|_6 ds \right| \]
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As long as \( t \) and \( t_0 \) stay away from zero all of these integrals are finite, by (3.21) and (3.38), for the first line and part of the second line, and by (3.28) and (3.29) for the rest of the second line, and go to zero as \( t \to t_0 \). To deal with the second term in (7.41) just replace the integral over \([0, t]\) in (7.39) and (7.40) by the integral over \([t_0, t]\) and replace \( A(t) \) by \( A(t_0) \) everywhere.

Use (3.43) to see that \( \|dA(t_0)\|_3 < \infty \). It follows that the second term in (7.41) goes to zero. Similarly, one need only replace the integrals over \([0, t]\) in (7.38) by integrals over \([t_0, t]\) and put \( t = t_0 \) in the definition of \( \alpha(s) \) to see that the fourth term in (7.41) goes to zero as \( t \to t_0 \).

Remark 7.12 It’s interesting to observe that one cannot replace \( d \) by \( d^* \) in the inequality (7.35) because \( \|d^*A(t)\|_3 \) can be identically infinite under our hypothesis - that \( A(\cdot) \) has finite action. It’s tempting to believe that some power counting scheme could unify the many estimates in this paper. But it would have to take into account the fact that \( dA(t) \) and \( d^*A(t) \) have very different behavior, in spite of both being first derivatives of \( A \).

Proof of Theorem 7.8. By the representation (7.18) with \( \tau = 0 \) we may write the vertical correction for \( \tau = 0 \) as

\[
d_A(t) \int_0^t \gamma(t, s) ds = \hat{w}(0) - \hat{w}(t) + \int_0^t \gamma(t, s) ds \quad 0 < t < \infty, \tag{7.45}
\]

where

\[
\gamma(t, s) = [A(t), \psi(s)] - P^\perp \left( \zeta(s) + [A(s), \psi(s)] \right). \tag{7.46}
\]

Since \( w(t) \) is continuous into \( H_0^3 \) so is \( \hat{w}(t) \) because \( P^\perp : H_0^3 \to H_0^3 \) is continuous by [6, Lemma 6.10]. Therefore \( \|\hat{w}(0) - \hat{w}(t)\|_{H_0^3} \to 0 \) as \( t \downarrow 0 \). With a view toward applying Lemma 7.9 observe that (7.19) and (7.20) give

\[
d \int_0^t \gamma(t, s) ds = \int_0^t d[A(t), \psi(s)] ds \tag{7.47}
\]

\[
d^* \int_0^t \gamma(t, s) ds = \int_0^t d^*[A(t) - A(s), \psi(s)] ds \tag{7.48}
\]

\[
+ \int_0^t \left( - [w(s) \cup A'(s)] + [A(s) \cup \zeta(s)] \right) ds. \tag{7.49}
\]

By Lemma 7.9 it suffices to show that the \( L^p(M) \) norm of each of these three integrals goes to zero as \( t \downarrow 0 \) as well as \( \| \int_0^t \gamma(t, s) ds \|_2 \). But
\[ \int_0^t \|d[A(t), \psi(s)]\|_p ds \to 0 \text{ by (7.35) and } \int_0^t \|d^*[A(t) - A(s), \psi(s)]\|_p ds \to 0 \text{ by (7.34).} \]

(7.31) and (7.32) show that the line (7.49) also goes to zero in \(L^p(M)\) norm. Furthermore,

\[ \int_0^t \|\gamma(t, s)\|_2 ds \leq \int_0^t \left( \|A(t), \psi(s)\|_2 + \|\zeta(s)\|_2 + \|A(s), \psi(s)\|_2 \right) ds \]

\[ \leq c \|A(t)\|_6 \int_0^t \|\psi(s)\|^2 ds + \int_0^t \|\zeta(s)\|^2 ds + c \int_0^t \|A(s)\|_6 \|\psi(s)\|^2 ds \]

\[ \leq c \|A(t)\|_6 o(t^{1/4}) + \int_0^t \|\zeta(s)\|^2 ds \]

\[ + c \left( \int_0^t s^{b-(1/2)} \|A(s)\|^2 ds \right)^{1/2} \left( \int_0^t s^{(1/2)-b} \|\psi(s)\|^2 ds \right)^{1/2} \]

by (5.29) (for \(b \geq 0\)). All three terms go to zero as \(t \downarrow 0\), the first by (3.38), the second by (7.33) and the third by (4.39) and (5.28). This completes the proof that the vertical correction goes to zero in \(H^0_b\) norm.

Now \(A \in L^3(M)\) as assumed in the statement of the theorem. Therefore, by Lemma 7.6 the \(H^1_b\) norm is equivalent to the \(H^0_b\) norm. By interpolation between \(H^1_b\) and \(L^2\) it follows that the \(H^b_b\) norm is equivalent to the \(H^0_b\) norm. This concludes the proof of (7.28). ■

**Proof of Theorem 7.7.** Continuity of \(v\) at \(t = 0\) has been proved in Theorem 7.8. Suppose then that \(t_0 > 0\) and \(t > 0\). We need to show that the differences of the integral term in (7.45) go to zero in \(H^0_b\) norm. Taking differences at \(t\) and \(t_0\) in (7.47) - (7.49) we find

\[ d \int_0^t \gamma(t, s) ds - d \int_0^{t_0} \gamma(t_0, s) = \int_0^t d[A(t) - A(t_0), \psi(s)] ds \]

\[ + \int_0^t d[A(t_0), \psi(s)] ds \]

\[ d^* \int_0^t \gamma(t, s) ds - d^* \int_0^{t_0} \gamma(t_0, s) ds = \int_0^t d^*[A(t) - A(t_0), \psi(s)] ds \]

\[ + \int_0^t d^*[A(t_0) - A(s), \psi(s)] ds \]

\[ + \int_0^t \left( - [w(s) \cdot A'(s)] + [A(s) \cdot \zeta(s)] \right) ds. \]
To apply Lemma 7.9 we need to show that each of the five integrals on the right go to zero in $L^p(M)$ norm as $t \to t_0$. This is clear for the integral in line (7.54) because of (7.31) and (7.32). The $L^p(M)$ norm of each of the remaining four integrals goes to zero as $t \to t_0$ by the corresponding assertion in Lemma 7.11. Finally, to complete the application of Lemma 7.9 we need to show that $\| \int_0^t \gamma(t,s)ds - \int_0^{t_0} \gamma(t_0,s)ds \|_2 \to 0$ as $t \to t_0 \neq 0$. The cancelations and estimates needed are similar to the preceding difference computations but a little simpler. We omit the details. ■

7.4 Finite action of the strong solution

**Theorem 7.13** (*Finite action of the strong solution*) Suppose that $1/2 \leq a < 1$ and $1/2 \leq b < 1$ and that $w$ is a strong solution of the augmented variational equation (2.21) with finite strong b-action in the sense that (5.23) holds. Assume that $A \in L^3(M)$. Suppose also that $\max(a,b) > 1/2$. Let $c = \min(a,b)$ and let $\tau > 0$. Define $v_\tau$ by (2.24). Then

$$\int_0^\tau t^{-c} \left( \| \nabla^A v_\tau(t) \|_2^2 + \| v_\tau(t) \|_2^2 \right) dt < \infty.$$  (7.55)

For the proof we are once again going to use the non-gauge invariant representation of $v_\tau$ given in (7.18). Let

$$u_\tau(t) = \int_{\tau}^t \gamma(t,s)ds,$$  (7.56)

where $\gamma(t,s)$ is defined in (7.46). Then, by (7.18), we have

$$v_\tau(t) = w(t) + \hat{w}(\tau) - \hat{w}(t) + u_\tau(t).$$  (7.57)

The finite strong b-action of the first three terms in (7.57) will be easily established at the end of this section using the equivalence of norms discussed at the beginning of Section 7.3. That equivalence will depend on use of the condition $A \in L^3(\mathbb{R}^3)$ in case $M = \mathbb{R}^3$. The proof of finite strong b-action of $u_\tau$ in the $H^1_0$ norm does not depend on this condition but has a lengthy proof. We will focus on this first. In the next theorem we will establish finite strong b-action of the fourth term in (7.57) while merely assuming finite b-action for $w$.  

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Theorem 7.14 Suppose that $1/2 \leq a < 1$ and $1/2 \leq b < 1$ and that $w$ is a solution to the augmented variational equation (2.21) with finite $b$-action in the sense of (5.22). Assume also that $\max(a,b) > 1/2$. Let $c = \min(a,b)$ and let $\tau > 0$. Then
\[
\int_{0}^{\tau} t^{-c} \|u_{\tau}(t)\|_{H^0_1}^2 dt < \infty. \tag{7.58}
\]

Remark 7.15 (Strategy) We will use the Gaffney-Friedrichs inequality (5.18) with connection form zero. It suffices to prove, then, that
\[
\int_{0}^{\tau} t^{-c} \left( \|d^* u_{\tau}(t)\|_2^2 + \|du_{\tau}(t)\|_2^2 + \|u_{\tau}(t)\|_2^2 \right) dt < \infty. \tag{7.59}
\]

The following proposition addresses each of these three terms.

Proposition 7.16 Under the hypotheses of Theorem 7.14 and with $\gamma(t,s)$ defined in (7.46) we have
\[
\int_{0}^{\tau} t^{-c} \left( \int_{t}^{\tau} \|d^* \gamma(t,s)\|_2^2 ds \right)^2 dt < \infty \tag{7.60}
\]
\[
\int_{0}^{\tau} t^{-c} \left( \int_{t}^{\tau} \|d\gamma(t,s)\|_2^2 ds \right)^2 dt < \infty \tag{7.61}
\]
\[
\int_{0}^{\tau} t^{-a} \left( \int_{t}^{\tau} \|\gamma(t,s)\|_2^2 ds \right)^2 dt < \infty. \tag{7.62}
\]

The proof depends on the following three lemmas and corollary.

Lemma 7.17 Let $1/2 \leq a < 1$ and $0 \leq b < 1$. If $w$ is a solution to the augmented variational equation (2.21) with finite $b$-action in the sense of (5.22) then (7.62) holds.

Proof. From (7.46) we see that
\[
\|\gamma(t,s)\|_2 \leq \| [A(t), \psi(s)] \|_2 + \| \zeta(s) \|_2 + \| [A(s), \psi(s)] \|_2 \tag{7.63}
\]
It suffices to prove (7.62) for each of these terms. Now
\[
\int_{0}^{\tau} t^{-a} \left( \int_{t}^{\tau} \| [A(t), \psi(s)] \|_2 ds \right)^2 dt \leq c^2 \int_{0}^{\tau} t^{-a} \| A(t) \|_6^2 dt \left( \int_{0}^{\tau} \| \psi(s) \|_3 ds \right)^2,
\]
which is finite by (3.39) and (5.29) for all \(a \in [1/2, 1)\) and all \(b \in [0, 1)\).

Concerning the second term, observe that \(\int_0^\tau s^{2-a}\|\zeta(s)\|_2^2 ds < \infty\) because \(\int_0^\tau s^{1-b}\|\zeta(s)\|_2^2 ds < \infty\) in accordance with (5.27), while \(2 - a \geq 1 - b\) for all \(a \in [1/2, 1)\) and all \(b \in [0, 1)\). Therefore Hardy’s inequality (3.55) with \(\beta = a\) shows that (7.62) holds in this range for the second term in (7.63). Finally,

\[
\left(\int_0^\tau \| [A(s), \psi(s)] \|_2 ds\right)^2 \leq \left(\int_0^\tau c\|A(s)\|_6\|\psi(s)\|_6 ds\right)^2 \leq c^2 \int_0^\tau s^{-1/2}\|A(s)\|_6^2 ds \int_0^\tau s^{1/2}\|\psi(s)\|_6^2 ds,
\]

which is bounded by (3.39) and (5.28) for all \(a \in [1/2, 1)\) and all \(b \in [0, 1)\). Therefore the third term in (7.63) makes a finite contribution to (7.62). \(\blacksquare\)

**Lemma 7.18** Let \(1/2 \leq a < 1\) and \(0 \leq b < 1\). If \(w\) is a solution to the augmented variational equation (2.21) with finite \(b\)-action in the sense of (5.22) then

\[
\int_0^\tau s^{2-b}\|d[A(s), \psi(s)]\|_2^2 ds < \infty \quad \text{and} \quad (7.64)
\]

\[
\int_0^\tau s^{2-b}\| [A(s) \cdot \zeta(s)] - [w(s) \cdot A'(s)] \|_2^2 ds < \infty. \quad (7.65)
\]

**Proof.** From the identity \(d[A, \psi] = [dA, \psi] - [A \wedge d\psi]\) we find

\[
\|d[A(s), \psi(s)]\|_2 \leq c\|dA(s)\|_3\|\psi(s)\|_6 + c\|A(s)\|_6\|d\psi(s)\|_3.
\]

It suffices, therefore, to prove that

\[
\int_0^\tau s^{2-b}\left(\|dA(s)\|_3^2\|\psi(s)\|_6^2 + \|A(s)\|^2_6\|d\psi(s)\|^2_3\right) ds < \infty. \quad (7.66)
\]

But

\[
\int_0^\tau s^{2-b}\|dA(s)\|_3^2\|\psi(s)\|_6^2 ds = \int_0^\tau \left(s\|dA(s)\|_3^2\right)\left(s^{1-b}\|\psi(s)\|^2_6\right) ds.
\]

The first factor in the integrand is bounded, by (3.43), and the second is integrable, by (5.27). Similarly,

\[
s^{2-b}\|A(s)\|^2_6\|d\psi(s)\|^2_3 = \left(s^{1/2}\|A(s)\|^2_6\right)\left(s^{3/2-b}\|d\psi(s)\|^2_3\right),
\]

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wherein the first factor is bounded, by (3.38), and the second factor is
integrable, by (5.57). This proves (7.64).

For the proof of (7.65) we have the bound

\[
\| - \left[ w(s) \right] A'(s) + [A(s) \right] \zeta(s) \|_2 \\
\leq c \| w(s) \|_6 \| A'(s) \|_3 + c \| A(s) \|_6 \| \zeta(s) \|_3.
\]  

(7.67)

Therefore it suffices to show that

\[
\int_0^{\tau} s^{2-b} \left\{ \| w(s) \|_6^2 \| A'(s) \|_3^2 + \| A(s) \|_6^2 \| \zeta(s) \|_3^2 \right\} ds
\]

is finite. The first term may be written \((s^{-b} \| w(s) \|_6^2)(s^2 \| A'(s) \|_3^2)\), which is an integrable function times a bounded function by respectively (4.3) and (3.23) (for all \(a \geq 1/2\)). The second term may be written \((s^{1/2} \| A(s) \|_6^2)(s^{3/2-b} \| \zeta(s) \|_3^2)\), which is a bounded function (by (3.38)) times an integrable function by (5.57). This completes the proof of the lemma. \(\blacksquare\)

**Corollary 7.19** Let \(1/2 \leq a < 1\) and \(0 \leq b < 1\). If \(w\) is a solution to the augmented variational equation (2.21) with finite \(b\)-action in the sense of (5.22) then

\[
\int_0^{\tau} t^{-b} \left( \int_t^{\tau} \| d[A(s), \psi(s)] \|_2^2 ds \right)^2 dt < \infty \quad \text{and} \quad (7.68)
\]

\[
\int_0^{\tau} t^{-b} \left( \int_t^{\tau} \| - [w \right] A' + [A \right] \zeta(s) \|_2^2 ds \right)^2 dt < \infty. \quad (7.69)
\]

**Proof.** By Hardy’s inequality, (3.55), the left hand side of (7.68) is at most

\[
\frac{4}{(1-b)^2} \int_0^{\tau} s^{2-b} \| d[A(s), \psi(s)] \|_2^2 ds,
\]

(7.70)

which has been shown to be finite in Lemma 7.18. By the same argument, (7.69) follows from (7.65). \(\blacksquare\)

**Lemma 7.20** Suppose that \(1/2 \leq a < 1\) and \(1/2 \leq b < 1\). Assume that \(\max(a, b) > 1/2\). Let \(c = \min(a, b)\). If \(w\) is a solution to the augmented variational equation (2.21) with finite \(b\)-action in the sense of (5.22) then

\[
\int_0^{\tau} t^{-c} \left( \int_t^{\tau} \| d^*[A(t) - A(s), \psi(s)] \|_2^2 ds \right)^2 dt < \infty \quad \text{and} \quad (7.71)
\]

\[
\int_0^{\tau} t^{-c} \left( \int_t^{\tau} \| d[A(t) - A(s), \psi(s)] \|_2^2 ds \right)^2 dt < \infty. \quad (7.72)
\]

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Therefore

\[
\|d^s[\alpha(s, t), \psi(s)]\|_2 \leq c\|d^s\alpha(s, t)\|_2\|\psi(s)\|_\infty + c\|\alpha(s, t)\|_6\|d\psi(s)\|_3 
\]  
(7.74)

\[
\|d[\alpha(s, t), \psi(s)]\|_2 \leq c\|d\alpha(s, t)\|_2\|\psi(s)\|_\infty + c\|\alpha(s, t)\|_6\|d\psi(s)\|_3.
\]

The statements (3.45) and (5.46) show that \(\int_0^s t^{-a}\|d^\#\alpha(s, t)\|_2^2 dt\) is bounded for \(s \in [0, T]\), where \(d^\# = d\) or \(d^s\). It will suffice to prove only (7.74).

We need to show that each of the two terms on the right side of (7.74) makes a finite contribution to (7.71). For the first term we have

\[
\int_0^\tau t^{-c}\left(\int_t^\tau \|d^s\alpha(s, t)\|_2\|\psi(s)\|_\infty ds\right)^2 dt
\]

\[
\leq \int_0^\tau t^{-c}\left(\int_t^\tau s^{b-(3/2)}\|d^s\alpha(s, t)\|_2^2 ds\right)\left(\int_0^\tau s^{(3/2)-b}\|\psi(s)\|_\infty^2 ds\right) dt
\]

\[
= \int_0^\tau s^{b-(3/2)}\left(\int_0^s t^{-c}\|d^s\alpha(s, t)\|_2^2 dt\right) ds\left(\int_0^\tau s^{(3/2)-b}\|\psi(s)\|_\infty^2 ds\right) 
\]  
(7.75)

The last factor in (7.75) is finite for any \(b \in [0, 1)\) by (5.97) with \(p = \infty\). Moreover \(\int_0^\tau t^{-c}\|d^s\alpha(s, t)\|_2^2 dt\) is bounded for \(0 \leq s \leq \tau\) because \(c \leq a\). Therefore if \(b > 1/2\) then the right hand side of (7.75) is finite. If \(b = 1/2\) then \(c = 1/2\) and \(a > 1/2\). In this case we may write \(t^{-c} = t^{-a-(1/2)}t^{-a}\) to find, since \(b - (3/2) = -1\) that the integral is at most

\[
\int_0^\tau s^{-1}s^{a-(1/2)}\left(\int_0^s t^{-a}\|d^s\alpha(s, t)\|_2^2 dt\right) ds\left(\int_0^\tau s^{(3/2)-b}\|\psi(s)\|_\infty^2 ds\right) < \infty.
\]

The contribution to (7.71) of the second term in (7.74) can be estimated by the same use of Schwarz’ inequality and reversal of order of integration as for the first term, giving

\[
\int_0^\tau t^{-c}\left(\int_t^\tau \|\alpha(s, t)\|_6\|d\psi(s)\|_3 ds\right)^2 dt
\]

\[
\leq \int_0^\tau s^{b-(3/2)}\left(\int_0^s t^{-c}\|\alpha(s, t)\|_6^2 dt\right) ds\left(\int_0^\tau s^{(3/2)-b}\|d\psi(s)\|_3^2 ds\right). 
\]  
(7.76)

**Proof.** Unlike the inequalities in Corollary 7.19 we cannot use Hardy’s inequality now because the integrands in (7.71) and (7.72) depend on \(t\). Let \(\alpha = \alpha(s, t) = A(s) - A(t)\). Then, similar to the identity (7.37), we have

\[
d[\alpha, \psi(s)] = [d\alpha, \psi(s)] - [\alpha \wedge d\psi(s)].
\]  
(7.73)
Since \( \int_{t}^{\infty} t^{-a} \| \alpha(s, t) \|^2 dt \) is bounded for \( s \in (0, \tau) \), by (3.37), and since \( \int_{0}^{\tau} s^{(3/2)-b} \| d\psi(s) \|^2 ds < \infty \) by (5.57), the same argument used for the first term applies to the second term as well.

**Proof of Proposition 7.16.** We see from (7.46), (7.19) and (7.20) that

\[
d\gamma(t, s) = d[A(t), \psi(s)] \quad \text{and} \quad d^\ast \gamma(t, s) = d^\ast[A(t) - A(s), \psi(s)] - [w(s) \downarrow A'(s)] + [A(s) \downarrow \zeta(s)].
\]

(7.78)

Therefore, combining (7.69) and (7.71) we find (7.60). By combining (7.72) with (7.68) we find (7.61) because

\[
d[A(t), \psi(s)] = d[A(t) - A(s), \psi(s)] + d[A(s), \psi(s)].
\]

In applying Corollary 7.19 one should observe that \( c \leq b \).

Therefore all assertions in Proposition 7.16 have been proved.

**Proof of Theorem 7.14.** Since \( c \leq a \) the inequality (7.62) holds for \( a \) replaced by \( c \). Therefore all three integrals in (7.59) have been shown to be finite. This proves (7.59) and therefore (7.58).

**Proof of Theorem 7.13.** In the representation (7.51) of the strong solution \( v_r(t) \) the first term \( w(t) \) has finite strong b-action for \( H^A \) by assumption and therefore finite strong b-action for \( H^0_1 \) by the equivalence of norms. Here we are using the hypothesis \( A \in L^3(M) \), which is a substantial hypothesis in case \( M = \mathbb{R}^3 \) but is automatic when \( M \) is bounded because \( A \) is always in \( L^6(M) \), being a strong solution to the Yang-Mills heat equation. Since \( c \leq b \), the first term also has finite strong c-action for \( H^0_1 \).

Concerning the second term, note that \( w(\tau) \in H^A_1 = H^0_1 \). Since \( P^\perp \) is continuous on \( H^0_1 \) it follows that \( \hat{w}(\tau) \in H^0_1 \). Therefore \( \int_{0}^{\tau} s^{-c} \| \hat{w}(\tau) \|_{H^0_1}^2 ds = \| \hat{w}(\tau) \|_{H^0_1}^2 \int_{0}^{\tau} s^{-c} ds < \infty \).

The strong c-action of the third term is bounded by

\[
\| P^\perp \|_{H^0_1 \rightarrow H^0_1}^2 \int_{0}^{\tau} s^{-c} \| w(s) \|_{H^0_1}^2 ds,
\]

which is finite by (5.23) since the \( H^0_1 \) norm is equivalent to the \( H^A_1 \) norm and \( c \leq b \).

The fourth term in (7.57) has been shown to have finite strong c-action for \( H^0_1 \) in Theorem 7.14 even under the weaker condition that \( w \) merely has finite b-action in the sense of (5.22).

Hence \( \int_{0}^{\tau} t^{-c} \| v_r(t) \|_{H^0_1}^2 dt < \infty \). Finally we can shift back to the Sobolev norm \( H^A_1 \) using once more the equivalence of norms under the hypothesis that \( A \in L^3(\mathbb{R}^3) \). This completes the proof of Theorem 7.13. ■
8 Proofs of the main theorems

8.1 Existence

Proof of Theorem 2.20 (recovery) and part of Theorem 2.10 (existence). Since $v$ and $v_\tau$, defined in (2.23) and (2.24) respectively, are both given by (6.1) with $\tau = 0$ and $\tau > 0$, respectively, we already know from Theorem 6.1 that they are solutions to the variational equation (2.6), at least at the level of informal computation. In Theorem 6.3 we proved that $v(t)$ is an almost strong solution and $v_\tau(t)$ is a strong solution.

Concerning the initial values of these two solutions, Theorem 7.1 shows that both the almost strong solution $v$ and the strong solution $v_\tau$ converge to their initial values in the sense of $L^\rho(M; \Lambda^1 \otimes \mathfrak{g})$ for $2 \leq \rho < 3$. (2.18) follows from (7.6). Theorem 7.7 shows that the almost strong solution $v(t)$ is continuous on $[0, \infty)$ into $H^A_B$, as required in (2.28), and in particular converges to its correct initial value $v_0$ in $H^A_B$ norm. This completes the proof of Theorem 2.10 Parts 1., 2. and 3. Moreover the extension from $L^2$ to $L^\rho$ discussed in Remark 2.12 has also been proved.

All of these proofs are proofs about the functions constructed in the recovery theorem, Theorem 2.20, which is now proved.

Of these two theorems it remains only to prove uniqueness (Part 4. of Theorem 2.10) This will be proved in Section 8.2

Proof of Theorem 2.15. (Finite action for $\tau > 0$) The proof that the strong solution $v_\tau(\cdot)$ constructed by the ZDS procedure has finite strong c-action when $c = \min(a, b)$ and $\max(a, b) > 1/2$ is restated in Theorem 7.13 and proved in Section 7.4

8.2 Uniqueness

Parts 1., 2, and 3. of Theorem 2.10 asserting the existence of solutions $v$ to the variational equation and their properties, have been proved in the proof of Theorem 2.20. It remains to prove the uniqueness of strong solutions.

As in the case of the Yang-Mills heat equation itself, the coefficients in the variational equation (2.6) are too singular for the standard proof of uniqueness, based on Gronwall’s inequality, to be applicable. We will instead adapt the proof used in [6]. It suffices to prove uniqueness in the largest class of
solutions of interest to us. Accordingly we will assume that $a = 1/2$ and $b = 1/2$.

**Theorem 8.1** Suppose that $a = b = 1/2$ and that $A(\cdot)$ is a strong solution of the Yang-Mills heat equation of finite action. Let $v_1$ and $v_2$ be two strong solutions of the variational equation along $A(\cdot)$ with finite $(1/2)$-action in the sense that \((5.22)\) holds for both. If $v_1(0) = v_2(0)$ then $v_1(t) = v_2(t)$ for all $t \geq 0$.

**Proof.** Let $v_1$ and $v_2$ be two solutions of finite $(1/2)$-action. Then so is $v \equiv v_1 - v_2$. We want to show that if $v(0) = 0$ then $v(t) = 0$ for all $t \geq 0$. Denote by $B(t)$ the curvature of $A(t)$. We have then, for $t > 0$,

\[
(d/dt)\|v(t)\|_2^2 = 2(v'(t), v(t)) \\
= 2(-d_A^*d_A v(t), v(t)) - 2([v(t) \cdot B(t)], v(t)) \\
= -2\|d_A v(t)\|_2^2 - 2([v(t) \cdot B(t)], v(t)) \\
\leq 2c\|B(t)\|_{\infty}\|v(t)\|_2^2.
\]

Let $\beta(t) = 2c\|B(t)\|_{\infty}$ and $f(t) = \|v(t)\|_2^2$. Then

\[
f'(t) \leq \beta(t)f(t), \quad t > 0.
\] (8.1)

We know from \((5.33)\) that

\[
\int_0^T t\beta(t)^2 dt < \infty
\] (8.2)

for any $T \in (0, \infty)$ when $A(\cdot)$ has finite action. Hence the proof in [6, Theorem (7.20)] that $f(t) \equiv 0$ if $f(0) = 0$ goes through once we have shown that

\[
f(t) = o(t^{1/2}).
\] (8.3)

It will be shown in [7, Theorem 5.5] that if $v$ has finite $(1/2)$-action then

\[
\int_0^T s^{1/2}\|v'(s)\|_2^2 ds < \infty
\] (8.4)
Thus, if \( v(0) = 0 \) then

\[
\|v(t)\|_2 \leq \int_0^t \|v'(s)\|_2 ds \\
\leq \left( \int_0^t s^{-1/2} ds \right)^{1/2} \left( \int_0^t s^{1/2} \|v'(s)\|_2^2 ds \right)^{1/2} \\
= t^{1/4} \sqrt{2} \left( \int_0^t s^{1/2} \|v'(s)\|_2^2 ds \right)^{1/2}.
\]

This proves (8.3) and concludes the proof of uniqueness asserted in Part 4. of Theorem 2.10. This completes the proof of Theorem 2.10. □

9 Bibliography

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