A LINEAR ESTIMATION TO THE NUMBER OF ZEROS FOR ABELIAN INTEGRALS IN A KIND OF QUADRATIC REVERSIBLE CENTERS OF GENUS ONE*

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Abstract In this paper, using the method of Picard-Fuchs equation and Riccati equation, we consider the number of zeros for Abelian integrals in a kind of quadratic reversible centers of genus one under arbitrary polynomial perturbations of degree \( n \), and obtain that the upper bound of the number is \( 2 \left( \lfloor n/2 \rfloor + \lfloor n+1 \rfloor / 2 + 1 \right) \) (\( n \geq 1 \)), which linearly depends on \( n \).

Keywords Abelian integral, quadratic reversible center, weakened Hilbert’s 16th problem, limit cycle.

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1. Introduction and Main Result

Consider
\[
\dot{x} = \frac{H_y(x, y)}{\mu(x, y)} + \varepsilon P(x, y), \quad \dot{y} = -\frac{H_x(x, y)}{\mu(x, y)} + \varepsilon Q(x, y),
\]
and
\[
\dot{x} = \frac{H_y(x, y)}{\mu(x, y)}, \quad \dot{y} = -\frac{H_x(x, y)}{\mu(x, y)},
\]
where \( \varepsilon (0 < \varepsilon \ll 1) \) is a real parameter, \( H_y(x, y)/\mu(x, y), H_x(x, y)/\mu(x, y), P(x, y), Q(x, y) \) are all polynomials of \( x \) and \( y \), with \( \max \{ \deg(P(x, y)), \deg(Q(x, y)) \} = n \) and \( \max \{ \deg(H_y(x, y)/\mu(x, y)), \deg(H_x(x, y)/\mu(x, y)) \} = m \). We suppose that the system (1.2) is an integrable system, it has at least one center. The function \( H(x, y) \) is a first integral with the integrating factor \( \mu(x, y) \), that is, we can define a continuous family of periodic orbits
\[
\{ \Gamma_h \} \subset \{(x, y) \in \mathbb{R}^2 : H(x, y) = h, h \in \Delta \},
\]
which are defined on a maximal open interval \( \Delta = (h_1, h_2) \). The problem to be studied in this paper is: for any small number \( \varepsilon \), how many limit cycles in the system (1.1) can be bifurcated from the family of periodic orbits \( \{ \Gamma_h \} \). It is well known that in any compact region of the periodic orbits, the number of limit cycles

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of the number of zeros for associated Abelian integrals $A(h)$,

$$A(h) = \int_{\Gamma_h} \mu(x, y) [Q(x, y) \, dx - P(x, y) \, dy], \quad h \in \Delta. \quad (1.3)$$

A) If the system (1.2) is a Hamiltonian system, i.e., the $\mu(x, y)$ is constant, then the $H(x, y)$ is a polynomial of $x$ and $y$ with $\text{deg}(H(x, y)) = m + 1$. Finding the least upper bound $Z(m, n)$ of the number of isolated zeros for Abelian integrals $A(h)$ is an important and difficult problem, where the upper bound $Z(m, n)$ only depends on $m$, $n$, and does not depend on the specific forms of $H(x, y)$, $P(x, y)$, and $Q(x, y)$. This problem is called the weakened Hilbert’s 16th problem, it is also called the Hilbert-Arnold problem [1], which has been studied diffusely, such as, for the following system

$$\dot{x} = y, \quad \dot{y} = -g(x) + \varepsilon f(x)y, \quad \quad (1.4)$$

Atabaigi [2] showed that the upper bound of the number of zeros is 3 when $g(x) = x^3(x - 1)^2$ and $f(x) = \alpha + \beta x + \gamma x^2$; Moghim et al. [16] showed that the upper bound is 2 when $g(x) = e^{-x}(1 - e^{-x})$ and $f(x) = \alpha + \beta x + \gamma x^2$. For the system (1.1), Rebollo-Perdomo et al. [18] showed the upper bound when $H(x, y) = y(x^2 - 1)$; for other specially planar systems, researchers obtain plentiful important results [3, 5, 12, 15, 20], and more specific situations can be found in the books [4, 7], the review article [13], and the references therein.

B) If the system (1.2) is an integrable non-Hamiltonian system, then the $\mu(x, y)$ is not constant. When $H(x, y)$ or $\mu(x, y)$ are not polynomials, the research work of the associated Abelian integrals $A(h)$ becomes much more difficult. Thus, researchers consider this problem by starting from the simplest case, namely $m$ is low. For the specific case of $m = 2$, people conjecture that the upper bound $Z(2, n)$ of the number of zeros for associated Abelian integrals $A(h)$ linearly depends on $n$. For the system (1.1), Novikov et al. [17] showed that the upper bound of the number of zeros is $7n/4 + 9$ when $H(x, y) = x^2y(1 - x - y)$ and $\mu(x, y) = y$; Sui et al. [19] showed the upper bound when $H(x, y) = x^2 + y^2$ and $\mu(x, y) = 1/(x^2 + 1)^m$. Unfortunately, this conjecture is still far from being solved. For quadratic reversible centers of genus one, in reference [6], Gautier et al. showed that there are essentially 22 types in the classification, divided into $(r1)-(r22)$ specifically. For the linear dependence of the upper bound of the number of zeros, case $(r1)$ was studied in [21]; case $(r2)$ is a Hamiltonian system; cases $(r3)-(r6)$ were studied in [14]; cases $(r9)$, $(r13)$, $(r17)$, and $(r19)$ were studied in [11]; cases $(r11)$, $(r16)$, $(r18)$, and $(r20)$ were studied in [10]; cases $(r12)$ and $(r21)$ were studied in [9]; case $(r10)$ was studied in [8]. All of these upper bounds linearly depend on $n$. In this paper, we consider the case $(r22)$, and obtain that the upper bound is $2[(n + 1)/2 + [n/2] + 2 (n \geq 1)$. Our result shows that the upper bound linearly depends on $n$.

The quadratic reversible type $(Q^R_n)$ has the form as follows:

$$\dot{z} = -iz + az^2 + 2|z|^2 + b\tilde{z}, \quad z = x + yi,$$

or

$$\dot{x} = (a + b + 2)x^2 - (a + b - 2)y^2 + y, \quad \dot{y} = -x[1 - 2(a - b)y].$$

Let $\dot{x} = 1 - 2(a - b)y, \quad \dot{y} = x, \quad dt = -2(a - b)dt$. Using $x, y, t$ instead of $\tilde{x}, \tilde{y}, \tau$, thus we obtain a new system

$$\dot{x} = -xy, \quad \dot{y} = -\frac{a + b + 2}{2(a - b)}y^2 + \frac{a + b - 2}{8(a - b)^3}x^2 - \frac{b - 1}{2(a - b)^3}x - \frac{a - 3b + 2}{8(a - b)^3}. \quad (1.5)$$
From the system (1.5), when \((a, b) = (-2, 0)\), we can get the case \((r22)\) as follows:

\[
(r22) \quad \dot{x} = -xy, \quad \dot{y} = \frac{1}{24} x^2 - \frac{1}{24} x. \tag{1.6}
\]

The system (1.6) is an integrable non-Hamiltonian quadratic system. It has a center \((1, 0)\), a family of periodic orbits \(\{\Gamma_h\} (-1/2^5 < h < 0)\), an integral curve \(x = 0\) (see Figure 1), and a first integral as follows:

\[
H(x, y) = \frac{1}{2} y^2 + \frac{1}{25} x^2 - \frac{1}{24} x = h, \quad h \in \left(-\frac{1}{25}, 0\right), \tag{1.7}
\]

with an integrating factor \(\mu(x, y) = 1/x\).

![Figure 1. The periodic orbits of the system (r22)](image)

In this paper, our main result is the following theorem.

**Theorem 1.1.** If \(P(x, y)\) and \(Q(x, y)\) are any polynomials of \(x\) and \(y\), then the upper bound of the number of zeros of Abelian integrals \(A(h)\) for the system \((r22)\) depends linearly on \(n\). Concretely, the upper bound is \(2 \left(\frac{n+1}{2}\right) + \left\lfloor\frac{n}{2}\right\rfloor + 2\) for \(n \geq 1\); and the upper bound is 0 for \(n = 0\).

The rest part of this paper is structured as follows. In Section 2, we seek a simple expression of Abelian integrals \(A(h)\), prove Proposition 2.1. In Section 3, we study relations among functions \(J_m(h)\) and their derivatives \(J'_m(h)\) for \(m = 0, 1\); relation between \(J_0(h)\) and \(J_1(h)\), obtain two Picard-Fuchs equations and a variable coefficient first order linear ordinary differential equation. In Section 4, we consider relation between \(J(h)\) and \(J_1(h)\), obtain a variable coefficient first order linear ordinary differential equation. Finally, we prove Theorem 1.1 using the method of Picard-Fuchs equation and Riccati equation. In Section 5, we give a short conclusion.

**2. Simple Expression of Abelian Integrals \(A(h)\)**

In this section, we give a simple expression of Abelian integrals \(A(h)\), obtaining Proposition 2.1.
We suppose $P(x, y) = \sum_{0 \leq i+j \leq n} a_{i,j} x^i y^j$ and $Q(x, y) = \sum_{0 \leq i+j \leq n} b_{i,j} x^i y^j$. From (1.3), the Abelian integrals $A(h)$ in Theorem 1.1 have the form

$$A(h) = \oint_{\Gamma_h} x^{-1} \left( \sum_{0 \leq i+j \leq n} b_{i,j} x^i y^j dx - \sum_{0 \leq i+j \leq n} a_{i,j} x^i y^j dy \right), \quad h \in \left( -\frac{1}{2^5}, 0 \right),$$

where $x^{-1}$ is an integrating factor.

For conciseness, we introduce functions $I_{i,j}(h)$ as follows:

$$I_{i,j}(h) = \oint_{\Gamma_h} x^{-1} y^j dx,$$

where $i = -1, 0, 1, \ldots, n - 1, n$; $j = 0, 1, 2, \ldots, n, n + 1$, and $0 \leq i + j \leq n$. When $j = 1$, we write $I_{i,1}(h)$ as $J_i(h)$.

Note that

$$\oint_{\Gamma_h} x^{i-1} y^{j+1} dy = \frac{\oint_{\Gamma_h} x^{i-1} dy^{j+1}}{j + 1} = \frac{1 - i}{j + 1} \oint_{\Gamma_h} x^{i-1} y^{j+1} dx = \frac{1 - i}{j + 1} I_{i-1,j+1}(h).$$

Thus, $A(h)$ can be written as

$$A(h) = \sum_{0 \leq i \leq n, \atop 0 \leq j \leq n, \atop 0 \leq i+j \leq n} b_{i,j} I_{i,j}(h) + \sum_{0 \leq i \leq n, \atop 0 \leq j \leq n, \atop 0 \leq i+j \leq n} a_{i,j} \frac{i - 1}{j + 1} I_{i-1,j+1}(h) = \sum_{i,j} \tilde{b}_{i,j} I_{i,j}(h),$$

where $\tilde{b}_{i,j} = b_{i,j} + i/j a_{i+1,j-1}$.

The following Proposition 2.1 gives a simple expression of Abelian integrals $A(h)$.

**Proposition 2.1.** The Abelian integrals $A(h)$ can be expressed as

$$A(h) = \begin{cases} \frac{1}{h} \left[ \alpha(h) J_0(h) + \beta(h) J_1(h) \right], & (n \geq 1), \\ \gamma(h) J_{-1}(h), & (n = 0), \end{cases}$$

where $\alpha(h)$, $\beta(h)$, and $\gamma(h)$ are polynomials of $h$ with $\deg(\alpha(h)) \leq [(n + 1)/2]$, $\deg(\beta(h)) \leq [n/2]$, for $n \geq 1$; $\deg(\gamma(h)) = 0$, for $n = 0$.

**Proof.** Since the periodic orbits $\Gamma_h$ are symmetric about $x$-axis, thus, $I_{i,j}(h) = 0$ as $j$ is even, so, we only need to consider the case of $j$ is odd.

From (1.7), we obtain

$$y \frac{\partial y}{\partial x} + 2Cx - 2C = 0,$$

where $C = 1/2^5$.

Multiplying the equality (2.3) by $x^i y^{j-2} dx$ and integrated it over $\Gamma_h$, we see that

$$\frac{i}{j} I_{i,j}(h) = 2C \left[ I_{i+2,j-2}(h) - I_{i+1,j-2}(h) \right],$$

where $j = 1, 3, 5, \ldots, 2[n/2] + 1$. We restrict $i = -1, 0, 1, 2, \ldots, n - 1$, and $0 \leq i + j \leq n$. 

(i) For \( i = 0 \), that is, \((i, j) = (0, 1), (0, 3), \cdots, (0, 2 \lfloor (n + 1)/2 \rfloor - 1)\), from (2.4), we obtain
\[
I_{2,j-2}(h) = I_{1,j-2}(h).
\tag{2.5}
\]
From (2.5), let \( j = 3 \), one has
\[
J_2(h) = J_1(h).
\tag{2.6}
\]
(ii) For \( i \neq 0 \), that is, \((i, j) \neq (0, 1), (0, 3), \cdots, (0, 2 \lfloor (n + 1)/2 \rfloor - 1)\), from (2.4), we have
\[
I_{i,j}(h) = \frac{2jC}{i} [I_{i+2,j-2}(h) - I_{i+1,j-2}(h)],
\tag{2.7}
\]
which indicates that \( I_{i,j}(h) \) can be expressed in terms of \( I_{i+2,j-2}(h) \) and \( I_{i+1,j-2}(h) \). Then step by step, since \( j \) is a positive odd number, we use \((j - 1)/2\) times (2.7) and obtain that \( I_{i,j}(h) \) can be written as a linear combination of \( J_k(h) (k = -1, 0, \cdots) \) and \( I_0,j(h) (j = 1, 3, \cdots, 2 \lfloor n/2 \rfloor - 1) \) with the form
\[
I_{i,j}(h) = \begin{cases} 
J_i, & (i \neq 0, j = 1), \\
\sum_{k=0}^{\lfloor i/2 \rfloor} c_{(i,j), i+ \frac{i-1}{2} + k} J_{i+ \frac{i-1}{2} + k}(h), & (i \geq 1, j \geq 3), \\
\sum_{k=0}^{\lfloor j/2 \rfloor} c_{(-1,j), 1+ \frac{j-3}{2} + k} J_{1+ \frac{j-3}{2} + k}(h) + d_{(-1,j),0}I_{0,j-2}(h), & (i = -1, 3 \leq j \leq 2 \lfloor n/2 \rfloor + 1),
\end{cases}
\tag{2.8}
\]
where \( c_{(i,j), i+ (j-1)/2 + k} \) represents the coefficient obtained when \( I_{i,j}(h) \) generates \( J_{i+(j-1)/2+k}(h) \), and \( d_{(-1,j),0} \) represents the coefficient obtained when \( I_{-1,j}(h) \) generates \( I_{0,j-2}(h) \), they are all real number.

From (2.1) and (2.8), we have
\[
A(h) = A_0(h) + A_1(h) + A_2(h) + A_3(h),
\tag{2.9}
\]
where
\[
A_0(h) = \sum_{k=1}^{\lfloor n/2 \rfloor} \tilde{b}_{0,2k-1}I_{0,2k-1}(h) + \sum_{k=1}^{\lfloor n/2 \rfloor} \tilde{b}_{-1,2k+1}d_{(-1,2k+1),0}I_{0,2k-1}(h),
\]
\[
A_1(h) = \tilde{b}_{-1,1}J_{-1}(h) + \sum_{k=1}^{n-1} \tilde{b}_{k,1}J_k(h),
\]
\[
A_2(h) = \sum_{1 \leq i \leq n, 4 \leq i+j \leq n} \tilde{b}_{i,j} \sum_{k=0}^{(j-1)/2} c_{(i,j), i+(j-1)/2 + k} J_{i+(j-1)/2+k}(h),
\]
\[
A_3(h) = \sum_{i=-1, 2 \leq i+j \leq n} \tilde{b}_{-1,i} \sum_{k=0}^{(j-3)/2} c_{(-1,j), 1+(j-3)/2 + k} J_{1+(j-3)/2+k}(h).
\]

For \( A_0(h) \), we can get
\[
A_0(h) = \sum_{k=1}^{\lfloor n/2 \rfloor} c_{0,2k-1} I_{0,2k-1}(h),
\tag{2.10}
\]
where \( e_{0,2k-1} = \tilde{b}_{0,2k-1} + \tilde{b}_{-1,2k+1}d_{(1,2k+1),0} \) \((k = 1, 2, \ldots, n/2)\), for \( n \) is even;
\( e_{0,2k-1} = \tilde{b}_{0,2k-1} + \tilde{b}_{-1,2k+1}d_{(-1,2k+1),0} \) \((k = 1, 2, \ldots, (n-1)/2)\), \( e_{0,2k-1} = \tilde{b}_{0,2k-1} \) \((k = (n+1)/2)\), for \( n \) is odd.

For \( A_2(h) \), the maximum number of \( i + (j - 1)/2 + k \) for \( k = 0, 1, \ldots, (j - 1)/2 \)
is \( i + (j - 1)/2 + (j - 1)/2 = i + j - 1 = n - 1 \), and the minimum number is
\[ 1 + (3 - 1)/2 + 0 = 2, \]
that is, \( \{i + (j - 1)/2 + k\} = \{2, 3, 4, \ldots, n - 2, n - 1\} \).

For \( A_3(h) \), the maximum number of \( 1 + (j - 3)/2 + k \) for \( k = 0, 1, \ldots, (j - 3)/2 \)
is \( 1 + (j - 3)/2 + (j - 3)/2 = j - 2 = 2[n/2] - 1 \leq n - 1 \), and the minimum number
is \( 1 + (3 - 3)/2 + 0 = 1 \), that is, \( \{i + (j - 3)/2 + k\} = \{1, 2, 3, \ldots, 2[n/2] - 1\} \).

Suppose that \( A_4(h) := A_1(h) + A_2(h) + A_3(h) \), so \( A_4(h) \) is a linear combination
of \( J_{-1}(h), J_1(h), J_2(h), \ldots, J_{n-1}(h) \). Thus, we have
\begin{align*}
A_4(h) = \tilde{b}_{-1,1}J_{-1}(h) + \sum_{k=1}^{n-1} e_k J_k(h),
\end{align*}
(2.11)
where \( e_k \in \mathbb{R} \) \((k = 1, 2, 3, \ldots, n - 1)\).

From (2.9), we have
\begin{align*}
A(h) = A_0(h) + A_4(h).
\end{align*}
(2.12)

Again, from (1.7), we have
\begin{align*}
\frac{1}{2} y^2 + Cx^2 - 2Cx = h.
\end{align*}
(2.13)

Multiplied the equality (2.13) by \( x^{i-1}y^{j-2}dx \) and integrated it over \( \Gamma_h \), we see
\begin{align*}
\frac{1}{2} I_{i,j}(h) = hI_{i,j-2}(h) - CI_{i+2,j-2}(h) + 2CI_{i+1,j-2}(h).
\end{align*}
(2.14)

(i) For \( i \neq 0 \), let \( j = 3 \), by the equality (2.7), the equality (2.14) becomes
\begin{align*}
(i + 3)CJ_{i+2}(h) = ihJ_i(h) + (2i + 3)CJ_{i+1}(h), (i \neq 0).
\end{align*}
(2.15)

A) For \( i \geq 3 \), we can rewrite equality (2.15) as
\begin{align*}
hJ_i(h) = \frac{i - 2}{(i + 1)C} h^2 J_{i-2}(h) + \frac{2i - 1}{i + 1} hJ_{i-1}(h),
\end{align*}
which indicates that \( hJ_i(h) \) can be expressed in terms of \( h^2 J_{i-2}(h) \) and \( hJ_{i-1}(h) \).
Then step by step, moreover \( J_2(h) = J_1(h) \), we obtain that \( hJ_i(h) \) can be written
as a linear combination of \( J_0(h) \) and \( J_1(h) \) with polynomial coefficients of
\( h \).
\begin{align*}
hJ_i(h) = \alpha_i(h)J_0(h) + \beta_i(h)J_1(h),
\end{align*}
where \( \alpha_i(h) \) and \( \beta_i(h) \) are polynomials of \( h \) with \( \deg(\alpha_i(h)) = 0, \deg(\beta_i(h)) \leq [(i + 1)/2] \).

B) For \( i = 2 \), by the equality (2.6), one has
\begin{align*}
hJ_2(h) = hJ_1(h).
\end{align*}

C) For \( i = 1 \), \( hJ_1(h) \) can also be a linear combination of \( J_0(h) \) and \( J_1(h) \) as
\begin{align*}
hJ_1(h) = hJ_1(h).
\end{align*}

D) For \( i < 0 \), from the equality (2.15), we obtain
\begin{align*}
hJ_i(h) = -\frac{2i + 3}{i} CJ_{i+1}(h) + \frac{i + 3}{i} CJ_{i+2}(h).
\end{align*}
(2.16)
From the equality (2.16), let \( i = -1 \), we have
\[
hJ_{-1}(h) = CJ_0(h) - 2CJ_1(h).
\]
As a consequence, all \( hJ_i(h) \) \((i = -1, 1, 2, \ldots, n - 1)\) can be written as a linear combination of \( J_0(h) \) and \( J_1(h) \) with polynomial coefficients of \( h \),
\[
hJ_i(h) = \alpha_i(h)J_0(h) + \beta_i(h)J_1(h), \quad (2.17)
\]
where \( \alpha_i(h) \) and \( \beta_i(h) \) are polynomials of \( h \) with \( \deg(\alpha_i(h)) = 0, \deg(\beta_i(h)) \leq [(i + 1)/2] \), for \( i \geq 1 \); and \( \deg(\alpha_i(h)) = 0, \deg(\beta_i(h)) = 0 \), for \( i = -1 \).
Substituting these formulae into \( hA_4(h) \), we obtain
\[
hA_4(h) = \hat{\alpha}_1(h)J_0(h) + \hat{\beta}_1(h)J_1(h), \quad (2.18)
\]
where \( \hat{\alpha}_1(h) \) and \( \hat{\beta}_1(h) \) are polynomials of \( h \) with \( \deg(\hat{\alpha}_1(h)) = 0, \deg(\hat{\beta}_1(h)) \leq [n/2] \), for \( n \geq 2 \); and \( \deg(\hat{\alpha}_1(h)) = 0, \deg(\hat{\beta}_1(h)) = 0 \), for \( n = 0, 1 \).
(ii) For \( i = 0 \), from (2.14) and (2.5), we have
\[
hI_{0,j}(h) = \begin{cases} 
\hbar J_0(h), & (j = 1), \\
2\hbar^2I_{0,j-2}(h) + 2\hbar hI_{1,j-2}(h), & (3 \leq j \leq 2 \left[ \frac{n+1}{4} \right] - 1). 
\end{cases} \quad (2.19)
\]
From equalities (2.8) and (2.17), we get
\[
hI_{1,j-2}(h) = \hat{\alpha}_2(h)J_0(h) + \hat{\beta}_2(h)J_1(h), \quad (j \geq 3), \quad (2.20)
\]
where \( \hat{\alpha}_2(h) \) and \( \hat{\beta}_2(h) \) are polynomials of \( h \) with \( \hat{\alpha}_2(h) = 0 \) and \( \deg(\hat{\beta}_2(h)) \leq (j-1)/2 \).
From equalities (2.19) and (2.20), and then step by step, we have
\[
hI_{0,j}(h) = \hat{\alpha}_3(h)J_0(h) + \hat{\beta}_3(h)J_1(h), \quad (2.21)
\]
where \( \hat{\alpha}_3(h) \) and \( \hat{\beta}_3(h) \) are polynomials of \( h \) with \( \deg(\hat{\alpha}_3(h)) \leq (j + 1)/2, \deg(\hat{\beta}_3(h)) \leq (j - 1)/2 \) for \( j \geq 3 \); and \( \deg(\hat{\alpha}_3(h)) = 1, \hat{\beta}_3(h) = 0 \) for \( j = 1 \).
From (2.10) and (2.21), we obtain
\[
hA_0(h) = e_{0,1}hJ_0(h) + e_{0,3}hI_{0,3}(h) + \cdots + e_{0,2[\frac{n+1}{4}]-1}hI_{0,2[\frac{n+1}{4}]-1}(h)
\]
\[
= \hat{\alpha}_4(h)J_0(h) + \hat{\beta}_4(h)J_1(h), \quad (2.22)
\]
where \( \hat{\alpha}_4(h) \) and \( \hat{\beta}_4(h) \) are polynomials of \( h \) with \( \deg(\hat{\alpha}_4(h)) \leq [(n + 1)/2], \deg(\hat{\beta}_4(h)) \leq [(n - 1)/2] \) and \( \deg(\hat{\alpha}_4(h)) = 1, \hat{\beta}_4(h) = 0 \) for \( n = 1, 2 \).
For \( n \geq 1 \), we suppose that \( J(h) := hA(h) \), from (2.12), (2.18) and (2.22), we have
\[
hA(h) = J(h) = \alpha(h)J_0(h) + \beta(h)J_1(h), \quad (2.23)
\]
where \( \alpha(h) \) and \( \beta(h) \) are polynomials of \( h \) with \( \deg(\alpha(h)) \leq [(n + 1)/2], \deg(\beta(h)) \leq [n/2] \).
For \( n = 0 \), from (2.1), we obtain \( A(h) = \gamma(h)J_{-1}(h) \), where \( \gamma(h) = -a_{0,0} \), and \( \deg(\gamma(h)) = 0 \).
3. Picard-Fuchs Equations and Riccati Equation

In this section, we give two relations among functions \( J_m(h) \) and their derivatives \( J'_m(h) \) for \( m = 0, 1 \); a relation between \( J_0(h) \) and \( J_1(h) \), obtaining two Picard-Fuchs equations and a variable coefficient first order linear ordinary differential equation.

The following two lemmas give two relations among functions \( J_m(h) \) and their derivatives \( J'_m(h) \) for \( m = 0, 1 \).

**Lemma 3.1.** The functions \( J_m(h) \) for \( m = 0, 1 \) satisfy the following Picard-Fuchs equation

\[
\begin{pmatrix}
J_0(h) \\
J_1(h)
\end{pmatrix} = \begin{pmatrix}
2h & 2C \\
0 & h + C
\end{pmatrix}
\begin{pmatrix}
J'_0(h) \\
J'_1(h)
\end{pmatrix}.
\]

**Proof.** By (1.7), we have \( y^2 = 2h - 2Cx^2 + 4Cx \), \( \partial y/\partial h = 1/y \), and \( ydy = (2C - 2Cx)dx \). Since \( J_1(h) = \frac{1}{y} x^{i-1}ydx \), \( J'_1(h) = \frac{1}{y} x^{i-1}/ydx \). Thus

\[
J_i(h) = \int_{\Gamma_h} x^{i-1}y^2dx = \int_{\Gamma_h} x^{i-1}(2h - 2Cx^2 + 4Cx)dx/y,
\]

\[
= 2hJ'_i(h) - 2CJ'_{i+2}(h) + 4CJ'_{i+1}(h),
\]

and

\[
iJ_i(h) = \int_{\Gamma_h} ix^{i-1}ydx = \int_{\Gamma_h} ydx = -\int_{\Gamma_h} x^{i-1}2C - 2Cx dydx/y,
\]

\[
= 2CJ'_{i+2}(h) - 2CJ'_{i+1}(h).
\]

From (3.2) and (3.3), we have

\[
(i + 1)J_i(h) = 2hJ'_i(h) + 2CJ'_{i+1}(h).
\]

By (3.4), let \( i = 0, 1 \) respectively, we obtain

\[
J_0(h) = 2hJ'_0(h) + 2CJ'_1(h),
\]

\[
J_1(h) = hJ'_1(h) + CJ'_2(h).
\]

From (2.6), one has

\[
J'_2(h) = J'_1(h).
\]

From three simultaneous equations (3.5)–(3.7), it follows that

\[
J_0(h) = 2hJ'_0(h) + 2CJ'_1(h),
\]

\[
J_1(h) = (h + C)J'_1(h).
\]

From equalities (3.8) and (3.9), we obtain (3.1). \( \square \)

**Lemma 3.2.** The functions \( J_m(h) \) for \( m = 0, 1 \) satisfy the following Picard-Fuchs equation

\[
\begin{pmatrix}
J'_0(h) \\
J'_1(h)
\end{pmatrix} = \frac{1}{B(h)} \begin{pmatrix}
h + C & -2C \\
0 & 2h
\end{pmatrix}
\begin{pmatrix}
J_0(h) \\
J_1(h)
\end{pmatrix},
\]

where \( B(h) = 2h(h + 1/2^2) \).

**Proof.** It can be calculated directly from Lemma 3.1. \( \square \)
Lemma 3.3. \( J_i \left( -1/2^5 \right) = 0 \) \((i = 0, 1); J_i(h) < 0 \) \((i = -1, 0, 1)\), when \( h \in \left( -1/2^5, 0 \right) \).

Since \( J_i(h) = \int_{-h}^{h} x^{i-1} y dx \). The proof only requires some simple calculations, so it is omitted.

For the relation between \( J_0(h) \) and \( J_1(h) \), assume that \( U(h) := J_0(h)/J_1(h) \), we obtain the following corollary.

Corollary 3.1. The function \( U(h) \) satisfies the following variable coefficient first order linear ordinary differential equation

\[
B(h)U'(h) = (C - h)U(h) - 2C, \tag{3.11}
\]

where \( B(h) = 2h(h + 1/2^5) \).

Proof. Using Lemma 3.2, and differentiated both sides of \( U(h) \) with respect to \( h \), we obtain (3.11). \( \square \)

4. The Number of Zeros for Abelian Integrals \( A(h) \)

In this section, we give a relation between function \( J(h) \) and \( J_1(h) \), obtaining a variable coefficient first order linear ordinary differential equation. Finally, we prove Theorem 1.1 using the method of Picard-Fuchs equation and Riccati equation.

For the relation between \( J(h) \) and \( J_1(h) \), assume that \( V(h) := J(h)/J_1(h) \), we obtain the following lemma.

Lemma 4.1. For \( n \geq 1 \), the function \( V(h) \) satisfies the following variable coefficient first order linear ordinary differential equation

\[
B(h)\alpha V'(h) = D(h)V(h) + G(h), \tag{4.1}
\]

where \( D(h) = B(h)\alpha' + (C - h)\alpha \), \( G(h) = B(h)\alpha(h)\beta'(h) - B(h)\alpha'(h)\beta(h) - (C - h)\alpha(h)\beta(h) - 2C\alpha^2(h) \). Thus, \( \deg(D(h)) \leq [(n + 1)/2] + 1 \), and \( \deg(G(h)) \leq [(n + 1)/2] + [n/2] + 1 \).

Proof. Using the equality (2.23) and Corollary 3.1, differentiated both sides of \( V(h) \) with respect to \( h \), we obtain (4.1).

We use \( \sharp A(h) \) to denote the number of zeros of Abelian integrals \( A(h) \) in \( \Delta \), and we need the following lemma.

Lemma 4.2 ([14]). The smooth functions \( W(h) \), \( \phi(h) \), \( \psi(h) \), \( \xi(h) \), and \( \eta(h) \) satisfy the following Riccati equation

\[
\eta(h)W'(h) = \phi(h)W^2(h) + \psi(h)W(h) + \xi(h),
\]

then

\[
\sharp W(h) \leq \sharp \eta(h) + \sharp \xi(h) + 1.
\]

Lemma 4.2 is Lemma 5.3 in [14], and the proof can be found in [14], so it is omitted.

Finally, we complete the proof of Theorem 1.1 using the method of Picard-Fuchs equation and Riccati equation.
Proof. Using the equality (2.23), Proposition 2.1, Lemma 4.1 and Lemma 4.2, therefore
\[ \sharp A(h) = \sharp J(h) = \sharp V(h) \leq \sharp B(h) + \sharp \alpha(h) + \sharp G(h) + 1. \]

For \( n \geq 1 \), since \( \deg(\alpha(h)) \leq [(n + 1)/2] \), \( \deg(G(h)) \leq [(n + 1)/2] + [n/2] + 1 \), noticing that \( B(h) = 2h \left( h + 1/2^2 \right) \) and there is no zero in \( (-1/2^5, 0) \), we obtain
\[ \sharp A(h) \leq \left[ \frac{n + 1}{2} \right] + \left( \left[ \frac{n + 1}{2} \right] + \left[ \frac{n}{2} \right] + 1 \right) + 1 = 2 \left[ \frac{n + 1}{2} \right] + \left[ \frac{n}{2} \right] + 2. \]

For \( n = 0 \), since \( A(h) = \gamma(h)J_{-1}(h) \), where \( \deg(\gamma(h)) = 0 \), \( J_{-1}(h) < 0 \), we have \( \sharp A(h) = 0. \)

5. Conclusion

In this paper, we study the linear estimation to the number of zeros for Abelian integrals in the quadratic reversible system \((r^{22})\) under arbitrary polynomial perturbations of degree \( n \), according to the method of Picard-Fuchs equation and Riccati equation. At the same time, we prove that the upper bound of the number is \( 2 \left( [n + 1]/2 \right) + [n/2] + 2 \) \( (n \geq 1) \). Our result shows that the upper bound depends linearly on \( n \).

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