KATO’S INEQUALITY WHEN $\Delta u$ IS A MEASURE

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Abstract. We extend the classical Kato’s inequality in order to allow functions $u \in L^1_{\text{loc}}$ such that $\Delta u$ is a Radon measure. This inequality has been recently applied by Brezis, Marcus, and Ponce [5] to study the existence of solutions of the nonlinear equation $-\Delta u + g(u) = \mu$, where $\mu$ is a measure and $g : \mathbb{R} \to \mathbb{R}$ is an increasing continuous function.

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1. Introduction and main result

Let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded open subset. The classical Kato’s inequality (see [8]) states that given any function $u \in L^1_{\text{loc}}(\Omega)$ such that $\Delta u \in L^1_{\text{loc}}(\Omega)$, then $\Delta u^+ \in \mathcal{D}'(\Omega)$ and the following holds:

$$\Delta u^+ \geq \chi_{\{u \geq 0\}} \Delta u \quad \text{in} \; \mathcal{D}'(\Omega).$$

Our main result in this paper (see Theorem 1.1 below) extends (1) to the case $\Delta u \in \mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ denotes the space of Radon measures on $\Omega$. In other words, $\mu \in \mathcal{M}(\Omega)$ if and only if, for every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ such that

$$|\int_{\Omega} \varphi \, d\mu| \leq C_\omega \|\varphi\|_{\infty}, \forall \varphi \in C^\infty_0(\omega).$$

We first recall that any $\mu \in \mathcal{M}(\Omega)$ can be uniquely decomposed as a sum of two Radon measures on $\Omega$ (see e.g. [7]): $\mu = \mu_d + \mu_c$, where

$$\mu_d(A) = 0 \quad \text{for any Borel measurable set} \; A \subset \Omega \quad \text{such that} \; \text{cap}(A) = 0,$$

$$|\mu_c|(\Omega \setminus F) = 0 \quad \text{for some Borel measurable set} \; F \subset \Omega \quad \text{such that} \; \text{cap}(F) = 0.$$

Here, cap denotes the Newtonian ($W^{1,2}$) capacity of a set. We observe that $\mu_d$ and $\mu_c$ are singular with respect to each other. This decomposition is the analog of the classical Radon-Nikodym Theorem, but with respect to cap. Clearly, $(\mu_d)^+ = (\mu^+)_d$ and $(\mu_c)^+ = (\mu^+)_c$.

Using the above notation, we can now state our main result:

**Theorem 1.1.** Let $u \in L^1_{\text{loc}}(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. Then, $\Delta u^+ \in \mathcal{M}(\Omega)$, and the following holds:

$$\Delta u^+_d \geq \chi_{\{u \geq 0\}} (\Delta u)_d \quad \text{on} \; \Omega,$$

$$(-\Delta u^+_c) = (-\Delta u)_c^+ \quad \text{on} \; \Omega.$$

Note that the right-hand side of (2) is well-defined because $u$ is quasicontinuous. More precisely, if $u \in L^1_{\text{loc}}(\Omega)$ and $\Delta u \in \mathcal{M}(\Omega)$, then there exists $\tilde{u} : \Omega \to \mathbb{R}$ quasicontinuous such that $u = \tilde{u}$ a.e. in $\Omega$ (see [1] and also [4, Lemma 1]). In (2), we then identify $u$ with its quasicontinuous representative. It is easy to see that $\chi_{\{u \geq 0\}}$ is locally integrable in $\Omega$ with respect to the measure $|(\Delta u)_d|$.
The proof of \eqref{2} requires a theorem of Boccardo, Gallouët, and Orsina \cite{2}, which says that a Radon measure \( \mu \) is diffuse (i.e. \( \mu_c = 0 \)) if and only if \( \mu \in L^1_{\text{loc}}(\Omega) + \Delta \left[ H^1_{\text{loc}}(\Omega) \right] \). Identity \eqref{3} relies on (and in fact is equivalent to) the “inverse” maximum principle, recently established by Dupaigne and Ponce \cite{6} (see Theorem \ref{thm:inverse_max_principle} below).

An equivalent statement of Theorem \ref{thm:main} is the following:

\begin{corollary}
Let \( u \in L^1_{\text{loc}}(\Omega) \) be such that \( \Delta u \in M(\Omega) \). Then, \( \Delta |u| \in M(\Omega) \), and the following holds:

\begin{align}
(\Delta |u|)_d & \geq \text{sgn}(u) (\Delta u)_d \quad \text{on } \Omega, \\
(\Delta |u|)_c &= -|\Delta u|_c \quad \text{on } \Omega.
\end{align}

\end{corollary}

Here, \( \text{sgn}(t) = 1 \) for \( t > 0 \), \( \text{sgn}(t) = -1 \) for \( t < 0 \), and \( \text{sgn}(0) = 0 \).

\begin{remark}
A slight modification of the proof of Theorem \ref{thm:main} shows that
\begin{align}
(\Delta u^+)_d & \geq \chi_{[u>0]}(\Delta u)_d \quad \text{on } \Omega.
\end{align}

In other words, we can replace the set \([u \geq 0]\) in \eqref{2} by \([u > 0]\) and still get the same result.
\end{remark}

Here is a simple consequence of \eqref{6}:

\begin{corollary}
Let \( u \in L^1_{\text{loc}}(\Omega) \) be such that \( \Delta u \in M(\Omega) \). If \( u \geq 0 \) a.e. in \( \Omega \), then
\begin{align}
(\Delta u)_d & \geq 0 \quad \text{on the set } [u = 0].
\end{align}
\end{corollary}

\section{Proof of \eqref{2} in Theorem \ref{thm:main}}

We start with the following:

\begin{lemma}
Assume \( \mu \in M(\Omega) \) is a diffuse measure with respect to \( \text{cap} \) (i.e. \( \mu_c = 0 \) on \( \Omega \)). Let \( (v_n) \) be a sequence in \( L^\infty(\Omega) \cap H^1(\Omega) \) such that \( \|v_n\|_\infty \leq C \) and \( v_n \rightharpoonup v \) in \( H^1 \). Then,
\begin{align}
(8) \quad v_n & \rightharpoonup v \quad \text{in } L^1_{\text{loc}}(\Omega; d\mu).
\end{align}

Equivalently, there exists a subsequence \( (v_{n_k}) \) converging to \( v \) \( |\mu|-\text{a.e. in } \Omega \).
\end{lemma}

\begin{proof}
Without loss of generality, we may assume that \( |\mu|(\Omega) < \infty \). By Theorem 2.1 of Boccardo, Gallouët, and Orsina \cite{2}, we know that \( \mu = f - \Delta g \) in \( \mathcal{D}'(\Omega) \), for some \( f \in L^1(\Omega) \) and \( g \in H^1(\Omega) \). Using a standard density argument, we conclude that
\begin{align}
(9) \quad \int_\Omega w \phi \, d\mu &= \int_\Omega w \phi f + \int_\Omega \nabla g \cdot \nabla (w \phi), \quad \forall \phi \in C^\infty_0(\Omega), \quad \forall w \in L^\infty \cap H^1.
\end{align}

By assumption, the sequence \( (|v_n - v|) \) is bounded in \( H^1(\Omega) \) and, by Rellich’s theorem, \( |v_n - v| \to 0 \) in \( L^2(\Omega) \). Thus,
\begin{align}
(10) \quad |v_n - v| & \to 0 \quad \text{in } H^1.
\end{align}

Given \( \varepsilon > 0 \), let \( \omega \subset \subset \Omega \) be such that \( |\mu|(\Omega \setminus \omega) < \varepsilon \). We then fix \( \varphi_0 \in C^\infty_0(\Omega) \) so that \( 0 \leq \varphi_0 \leq 1 \) in \( \Omega \) and \( \varphi_0 = 1 \) on \( \omega \). Applying \eqref{2} with \( w = |v_n - v| \) and \( \varphi = \varphi_0 \).
we have
\[ \int_{\Omega} |v_n - v| \, d\mu \leq \int_{\omega} |v_n - v| \, d\mu + 2C|\mu|_{\Omega}\omega \]
\[ \leq \int_{\Omega} |v_n - v| \varphi \, d\mu + 2C\varepsilon = \int_{\Omega} |v_n - v| \varphi_0 f + \int_{\Omega} \nabla g \cdot \nabla (|v_n - v| \varphi_0) + 2C\varepsilon. \]
By (10), we know that \( \int_{\Omega} \nabla g \cdot \nabla (|v_n - v| \varphi_0) \to 0 \) as \( n \to \infty \). Since \( (v_n) \) is bounded in \( L^\infty \) and \( v_n \to v \) in \( L^2 \), we have \( v_n \to v \) with respect to the weak* topology of \( L^\infty \); thus, \( \int_{\Omega} |v_n - v| \varphi_0 f \to 0 \). We conclude that \( \limsup_{n \to \infty} \int_{\Omega} |v_n - v| \, d\mu \leq 2C\varepsilon \).

Taking \( \varepsilon > 0 \) arbitrarily small, (3) follows.

Given \( k > 0 \), we denote by \( T_k : \mathbb{R} \to \mathbb{R} \) the truncation operator, i.e. \( T_k(s) = s \) if \( s \in [-k, k] \) and \( T_k(s) = \text{sgn}(s) k \) if \( |s| > k \). Recall the following standard inequality (see e.g. [4] Lemma 1):

**Lemma 2.2.** Assume \( u \in L^1_{\text{loc}}(\Omega) \) and \( \Delta u \in \mathcal{M}(\Omega) \). Then, \( T_k(u) \in H^1_{\text{loc}}(\Omega) \), \( \forall k > 0 \); moreover, given \( \omega \subset \subset \omega' \subset \subset \Omega \), there exists \( C > 0 \) such that
\[ \int_{\omega} |\nabla T_k(u)|^2 \leq k \left( \int_{\omega'} |\Delta u| + C \int_{\omega'} |u| \right). \]

Another ingredient to prove (2) is our next result, which extends Lemma 2 in [3]:

**Proposition 2.1.** Let \( \Phi : \mathbb{R} \to \mathbb{R} \) be a \( C^1 \)-convex function such that \( 0 \leq \Phi' \leq 1 \) on \( \mathbb{R} \). If \( u \in L^1_{\text{loc}}(\Omega) \) and \( \Delta u \in \mathcal{M}(\Omega) \), then
\[ \Delta \Phi(u) \geq \Phi'(u)(\Delta u)_d - (\Delta u)_c^- \quad \text{in } D'(\Omega). \]

**Proof.** Without loss of generality, we shall assume that \( \Phi \in C^2 \) and \( \Phi'' \) has compact support in \( \mathbb{R} \). The general case can be easily deduced by approximation (note that since \( \Phi \) is convex and \( \Phi' \) is uniformly bounded, both limits \( \Phi'(\pm \infty) \) exist and are finite). We may also assume that \( u \in L^1(\Omega) \) and \( \int_{\Omega} |\Delta u| < \infty \).

For every \( x \in \Omega \), define \( u_n(x) = \rho_n \ast u(x) = \int_{\Omega} \rho_n(x - y)u(y) \, dy \), where \( \rho_n \) is a family of radial mollifiers such that \( \text{supp} \rho_n \subset B_{1/n} \). Since \( \Phi'' \geq 0 \) in \( \mathbb{R} \), we have
\[ \Delta \Phi(u_n) = \Phi'(u_n) \Delta u_n + \Phi''(u_n) |\nabla u_n|^2 \geq \Phi'(u_n) \Delta u_n \quad \text{in } \Omega. \]

Let \( \varphi \in C_0^\infty(\Omega) \) with \( \varphi \geq 0 \). We multiply both sides of the inequality above by \( \varphi \) and integrate by parts. For every \( n \geq 1 \) such that \( d(\text{supp} \varphi, \partial \Omega) > 1/n \), we have
\[ \int_{\Omega} \Phi(u_n) \Delta \varphi \geq \int_{\Omega} \Phi'(u_n) \varphi \Delta u_n \]
\[ = \int_{\Omega} \left\{ \rho_n \ast [\Phi'(u_n) \varphi] \right\} \Delta u \geq \int_{\Omega} \left\{ \rho_n \ast [\Phi'(u_n) \varphi] \right\} (\Delta u)_d - \int_{\Omega} (\rho_n \ast \varphi)(\Delta u)_c^-. \]

Clearly,
\[ \int_{\Omega} \Phi(u_n) \Delta \varphi \to \int_{\Omega} \Phi(u) \Delta \varphi \quad \text{and} \quad \int_{\Omega} (\rho_n \ast \varphi)(\Delta u)_c^- \to \int \varphi(\Delta u)_c^- \]

We now establish the following:

**Claim.** \( \rho_n \ast [\Phi'(u_n) \varphi] \to \Phi'(u) \varphi \) in \( H^1(\Omega) \).
In fact, since \( \rho_n * [\Phi'(u_n) \varphi] \to \Phi'(u) \varphi \) in, say, \( L^1(\Omega) \) and since \( \varphi \) has compact support in \( \Omega \), it suffices to show that \( (\Phi'(u_n)) \) is bounded in \( H^1_{\text{loc}}(\Omega) \). Let \( M > 0 \) be such that \( \text{supp} \Phi'' \subset [-M,M] \). Then,

\[
\nabla \Phi'(u_n) = \Phi''(u_n) \nabla u_n = \Phi''(u_n) \nabla T_M(u_n) \quad \text{in} \quad \Omega.
\]

Let \( \omega \subset \subset \omega' \subset \subset \Omega \). For \( n \geq 1 \) sufficiently large, it follows from (11) that

\[
\int_\omega |\nabla \Phi'(u_n)|^2 \leq \Vert \Phi'' \Vert_\infty \int_\omega |\nabla T_M(u_n)|^2 \leq CM \left( \int_{\omega'} |u_n| + \int_{\omega'} |\Delta u_n| \right) \leq CM \left( \int_\Omega |u| + \int_\Omega |\Delta u| \right),
\]

for some constant \( C > 0 \) independent of \( n \).

In view of the previous claim, we can now apply Lemma 2.1 above with \( v_n = \rho_n * [\Phi'(u_n) \varphi] \) and \( \mu = (\Delta u)_d \) to conclude that

\[
(\Phi'(u_n)) \to \Phi'(u) \varphi \quad \text{in} \quad (\Delta u)_d.
\]

Combining (13) and (14) yields

\[
\int_\Omega \Phi(u) \Delta \varphi \geq \int_\Omega \Phi'(u) \varphi (\Delta u)_d - \int_\Omega \varphi \Delta u^-_c, \quad \forall \varphi \in C^\infty_0(\Omega) \quad \text{with} \quad \varphi \geq 0 \quad \text{in} \quad \Omega,
\]

which is precisely (12).

**Proof of (2).** Let \( (\Phi_n) \) be a sequence of smooth convex functions in \( \mathbb{R} \) such that \( \Phi_n(t) = t \) if \( t \geq 0 \) and \( \Phi_n(t) \leq 1/n \) if \( t < 0 \). In particular, \( 0 \leq \Phi' \leq 1 \) in \( \mathbb{R} \). It follows from the previous proposition that

\[
\Delta \Phi_n(u) \geq \Phi_n'((\Delta u)_d - (\Delta u)_c^-) \quad \text{in} \quad \mathcal{D}(\Omega).
\]

As \( n \to \infty \), we get

\[
\Delta u^+ \geq \chi_{\{|u| \geq 0\}} (\Delta u)_d - (\Delta u)_c^- \quad \text{in} \quad \mathcal{D}(\Omega).
\]

In particular, \( \Delta u^+ \in \mathcal{M}(\Omega) \). Taking the diffuse part from both sides of (15), we conclude that (2) holds.

3. **Proof of (3) in Theorem 1.1**

Identity (3) relies on the following:

**Theorem 3.1** ("Inverse" maximum principle [6]). Let \( u \in L^1_{\text{loc}}(\Omega) \) be such that \( \Delta u \in \mathcal{M}(\Omega) \). If \( u \geq 0 \) a.e. in \( \Omega \), then

\[
(-\Delta u)_c^+ \geq 0 \quad \text{on} \quad \Omega.
\]

To complete the proof of Theorem 1.1, we now present:

**Proof of (3).** From the proof of (2), we already know that \( \Delta u^+ \) is a Radon measure on \( \Omega \). Applying the “inverse” maximum principle to \( u^+ \), we have \( (-\Delta u^+)_c \geq 0 \) on \( \Omega \). Since \( u^+ - u \geq 0 \) a.e. in \( \Omega \), it also follows from Theorem 3.1 above that \( (-\Delta u^+)_c \geq (-\Delta u)_c^- \) on \( \Omega \). Thus,

\[
(-\Delta u^+)_c \geq (-\Delta u)_c^+ \quad \text{on} \quad \Omega,
\]

which gives the “\( \geq \)” in (3). The reverse inequality just follows by taking the concentrated part from both sides of (15). In fact,

\[
(-\Delta u^+)_c \leq (\Delta u)_c^- = (-\Delta u)_c^+ \quad \text{on} \quad \Omega.
\]
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