AN OBSTRUCTIONS TO NON-NEGATIVE CURVATURE ON BUNDLES WITH PULL-BACK CONNECTIONS

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Abstract. Motivated to study the geometry of the exotic spheres constructed in [5], we derive a necessary condition for non-negative sectional curvature in certain total spaces of Riemannian submersions with totally geodesic fibers. In particular, we derive a condition on the metric of the original base space and prove that the bundles in [5] and [1] have sections of negative curvature.

1. Introduction

Recall from [2] that, for a Riemannian submersion with totally geodesic fibers $\pi : P \rightarrow N$, the vertical sectional curvatures of a plane $X \wedge U$ in $P$ is given by $\text{sec}_P(X, U) = |A^\perp_X U|^2$, where $A^\perp$ is the dual, with respect to $g_P$, of the integrability tensor of $\pi$. In particular, a necessary condition for positive curvature on $P$ is the non-degenerancy of $A$. In this case, $\pi : P \rightarrow N$ is called a fat bundle.

By the other hand, [4] proves that, there exists a universal bundle $V_k \rightarrow G_k$ with connection $\omega_0$ such that, if $\pi : (P, g_P) \rightarrow (M, g_M)$ is a $O(k)$ or $U(k)$ principal bundle with connection 1-form $\omega$, then, there is a bundle map $\Phi : P \rightarrow V_k$ such that $\Phi^*\omega_0 = \omega$. In fact, $V_k \rightarrow G_k$ are exactly the respective Grassmanians. In particular, the universal principal $U(1) = S^1$-bundle is fat.

In this work we assume that the degenerancy of the $A$-tensor in a bundle $P' \rightarrow M$ is given in a controlled way and prove a necessary condition for non-negative sectional curvature on $P'$. In fact, we prove the following

Theorem 1.1. Let $f^*P \rightarrow M$ be the pull-back of a fat Riemannian submersion with a connection metric with induced connection. Then, if $M$ is compact and $f^*P$ has non-negative sectional curvature, $f^{-1}(a)$ is a totally geodesic submanifold of $M$ for every regular point $a \in N$ of $f$.

At the end, we prove that, for any metric in $S^8$ or $S^{10}$, the bundles with connections as in Theorem 4.3 of [5], whose total spaces admit isometric actions with exotic spheres as quotients, have sections of negative curvature.

Here, we confound the terms submersion and bundle, when we discuss connection metrics, since they are known to be equivalent ([2]). We follow the notation and conventions of [2].

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2. Graphics and Pull-back Bundles

We start with a characteriza-tion of the normal space of a graph and its second fundamental form. Then, we connect it to the study of the metric defined by the standard embedding of a pull-back bundle. Afterward, we show that connection metric with the pull-backed connection can be realized in this fashion, as far as \( M \) is compact, then we prove Theorem 2.5.

Let \( (M, g_M) \) and \( (N, g_N) \) be Riemannian manifolds and \( f : M \to N \) be a smooth map. We consider the graph of \( f \), \( \Gamma_f \), as the subset

\[
\Gamma_f = \{(x, y) \in M \times N \mid y = f(x)\}.
\]

Let \( F : M \to \Gamma_f \) be the diffeomorphism defined by \( F(x) = (x, f(x)) \) and observe that, for \( \nu \Gamma_f \), the normal space to \( F \), \( F^*(\nu \Gamma_f) = f^*TN \). An isomorphism is given by the map \( (\Xi_N)_{\nu} : T_{f(x)}N \to \nu \Gamma_f \) defined by \( \Xi_N(Y) = (-df(Y), Y) \), where \( g_M(df(X), Y) = g_N(X, dfY) \) being \( T(M \times N) = TM \times TN \) identified in the natural way. We also define the isomorphism \( \Xi : TM \oplus f^*TN \to F^*T(M \times N) \) as \( \Xi(X, Y) = df(X) + \Xi_N(Y) \).

Writing the vector \( (X, Y) \in T(M \times N) \) as a column, one can verify that

\[
\Xi^{-1} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} (1 + df\dagger df)^{-1} df\dagger (1 + df\dagger df)^{-1} \\ df(1 + df\dagger df)^{-1} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.
\]

In particular, for \( pr_{\nu \Gamma_f} \), the orthogonal projection to \( \nu \Gamma_f \), we have

**Lemma 2.1.** For \( (X, Y), (X', Y') \in T(M \times N) \),

\[
g_M \times g_N \left( pr_{\nu \Gamma_f} \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} X' \\ Y' \end{pmatrix} \right) = g_N((1 + df\dagger df)^{-1}(Y - df(X)), Y' - df(X')).
\]

**Proof.** The lemma follows by noticing that

\[
pr_{\nu \Gamma_f} = \Xi pr_{\nu \Gamma_f \times T_{f(x)}N} \Xi^{-1} = \begin{pmatrix} 0 & -df\dagger \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -df(1 + df\dagger df)^{-1} & (1 + df\dagger df)^{-1} \end{pmatrix},
\]

and \( df(1 + df\dagger df)^{-1} = (1 + df\dagger df)^{-1}df \).

Let \( \Pi(X, Y) = Y - df(X) \in f^*TN \) and \( O = (1 + df\dagger df)^{-1} \), so \( pr_{\nu \Gamma_f} = \Pi^\dagger OP \Pi \).

Let \( \tilde{X} = (X, Y) \) and \( \tilde{X}' = (X', Y') \) be two vector fields in \( \Gamma_f \) arbitrarily extended to \( M \times N \). Noticing that \( Y = df(X) \) and \( Y' = df(X') \), we have that, for \( \nabla, \nabla^M \) and \( \nabla^N \), the covariant derivatives associated to \( g_M \times g_N, g_M \) and \( g_N \),

\[
\nabla_{\bar{X}} \tilde{X}' = (\nabla^N_{\bar{X}} X', \nabla^M_{\bar{X}} Y').
\]

In particular, we can think of \( \nabla^N_{dfX} dfX' = \nabla^N_{dfX} Y' \) as a well-defined vector-field, being the second fundamental form, \( \Pi_f \), of \( \Gamma_f \) completely determined by

\[
d^2f(X, X') = \Pi(\nabla_{dfX} df(X')) = \nabla^N_{dfX} dfX' - df(\nabla^M_{\bar{X}} X').
\]

In fact,

\[
\Pi_f(\tilde{X}, \tilde{X}') = pr_{\nu \Gamma_f} \nabla_{\bar{X}} \tilde{X}' = \Pi^\dagger OP \Pi \nabla_{\bar{X}} \tilde{X}' = \Pi^\dagger Op^2(X, X').
\]

It follows that \( d^2f \) is a symmetric \((2,1)\)-tensor.

Now, let \( \pi : (P, g_P) \to (N, g_N) \) be a Riemannian submersion. We define \( \pi_f : f^*P \to M \), the pull-back of \( \pi \) as the manifold \( f^*P = \{(x, p) \in M \times P \mid f(x) = \pi(p)\} \) endowed with the restriction of the projection to the first coordinate. We write \( \langle \cdot, \cdot \rangle \) for \( g_M \times g_P \) and consider \( f^*P \) as a submanifold of \( M \times P \). We denote by
In particular, $V_T$ vanishes when restricted to $H_M$.

**Proposition 2.2.** Let $\tilde{\pi} = (\text{id} \times \pi) : M \times P \to M \times N$. Then, the restriction of $\tilde{\pi}$ to $f^*P \to \Gamma_f$ is a Riemannian submersion. Furthermore, the derivative of $\tilde{\pi}$ induces an isometry $\nu_{(x,p)}P \to \nu_{(x,f(x))}\Gamma_f$ for every $(x,p) \in f^*P$.

**Proof.** We first observe that $\nu_{(x,p)}$ is an isometry when restricted to $T_xM \times T_pP$ and $\nu_{(x,p)}f^*P \subset T_xM \times H_p$. Since $\text{id} \times d\pi_p$ is an isometry when restricted to $T_xM \times H_p$, and it sends $(T_xM \times H_p) \cap T_{(x,p)}f^*P$ to $TT_f$, it sends vectors orthogonal to $Tf^*P$ to $\nu \Gamma_f$ in an isometric way. Now, the first statement follows by noticing that $\tilde{\nu}X = (X, \nu_p(d\pi_x(X))) \in T_{(x,p)}f^*P$ and is orthogonal to $\{0\} \times V_p = V_{(x,p)}$. But,

$$||((X, \nu_p(d\pi_x(X))))||_{T_xM \times T_pP}^2 = ||X||_{T_xM}^2 + ||d\pi_x(X)||_{T_pP}^2 = ||\tilde{\nu}X, \nu_p(d\pi_x(X))||_{T_xM \times T_pN}^2.$$

For the rest of the paper, we assume that the fibers of $\pi$ are totally geodesic, and, first reduce the proof of Theorem 

**Proposition 2.3.** If $M$ is compact, then, there exists a metric $g_M$ on $M$, $\epsilon > 0$ and an isometry $(f^*P, g_M) \to (f^*P, g'_M \times g'_p)$, where $g'_M$ is the connection metric associated with $\epsilon g_M$ and the connection in $\pi$. Furthermore, given a $\in N$, for any submanifold $L \subset f^{-1}(a) \subset M$, the second fundamental form of $L$ on the metrics $g_M$ and $g'_M$ coincide.

**Proof.** Since connection metrics are specified by the connection and the metric in the base, according to what was discussed so far, for the first statement, it is sufficient to prove that there is a metric $g'_M$ and $\epsilon > 0$ such that, the graph of $f : (M, g'_M) \to (N, \epsilon g_N)$ is isometric to $(M, g_M)$. Write $d\tilde{f}$ for the dual of $df$ with respect to $g_M$ and $g_N$ and let $\epsilon > 0$ be a real number such that $1 - \epsilon d\tilde{f}d\tilde{f}$ is still positive definite. Then, for $g'_M = (1 - \epsilon d\tilde{f}d\tilde{f})g_M$ we have that the metric in the graph of $f : (M, g'_M) \to (N, \epsilon g_N)$ is

$$F^*(g'_M \times \epsilon g_N) = g'_M + \epsilon f^*g_N = g_M - \epsilon f^*g_N + \epsilon f^*g_N = g_M$$

as desired. Now, let $X, Y$ be vector fields of $L$. We observe that, for $Z$, a vector field of $M$ restricted to $L$ we have

$$g'_M(X, Z) = g_M(X, Y) - \epsilon g_N(dfX, dfZ) = g_M(X, Y)$$

since the entire tangent bundle of $L$ is contained in the kernel of $df$. In particular, the normal bundles and the intrinsic metrics of $L$ coincide for both metrics. It proves the assertion, since the second fundamental form of a fixed submanifold just depends on the normal projection - with effect, recall that $\Pi(X, Y) = \frac{1}{2}[X, Y]$. 

Proceeding to the second fundamental form of the pull-back bundle, we write $A(X, Y) = \frac{1}{2}p_M[p_HX, p_HY]$, for the $A$-tensor of $\pi$ and $g_P(A_X^U, Y) = g_p(U, A(X, Y))$. We have

$$A(X, Y) = \frac{1}{2}p_M[p_HX, p_HY]$$
Proposition 2.4. If $\tilde{X} = (X, Y)$, $\tilde{X}' = (X', Y')$ are vector fields on $f^*P$, then
d$\tilde{\pi}\Pi(\tilde{X}, \tilde{X}') = \Pi^0O(d^2 f(X, X') + \Lambda(Y, Y'))$,
where $\Lambda(Y, Y') = (0, d\pi(-A_Y Y' - A_Y, Y))$.

Proof. From [2], we have
d$\pi\nabla^g Y' = \nabla^N_{d\pi^* Y} - d\pi(A^g_Y, Y + A^g_Y, Y')$.
So, for $\tilde{Z} = (X'', Y'')$, a normal vector and $\tilde{g}_{M \times N} = g_M \times g_N$,
$\langle \Pi(\tilde{X}, \tilde{Y}), \tilde{Z} \rangle = \langle \pi(f_t U), \tilde{Z} \rangle$
g$s_m \times N(\pi(f_t U), \tilde{Z}) = g_M \times N(\Pi^0O(d^2 f(X, X') + \Lambda(Y, Y'))\tilde{Z})$.
The following lemma culminates in Theorem [1.1]

Lemma 2.5. If $f^*P \to M$ has non-negative curvature, then
$A(\mathcal{LO}d^2 f(X, X), \mathcal{L}df Z) = 0$.

Proof. We recall that, given three vectors $X, Z, U$, for $R$, the (4,0)-curvature tensor, we have
$R(X, tU + Z, tU + Z, X) = t^2 R(X, U, U, X) + 2t R(X, U, Z, X) + R(X, Z, Z, X)$
In particular, if $R(X, U, U, X) = 0, R(X, tU + Z, tU + Z, X) = 0$ for all $t$ if and only if
$R(X, Z, Z, X) = 0$ and

(2.4)
$R(X, U, Z, X) = 0$.

By the other hand, writing $R_M, R_N$ and $R_P$ for the (4,0)-curvature tensors of
$M, N$ and $P$, respectively, Gauss formula together with Proposition [2.4] gives us
$R(\tilde{X}', \tilde{X}', \tilde{X}', \tilde{X}') = R_M(X', X, X, X') + R_P(Y', Y, Y', Y')$
$+ g_N(O(d^2 f(X, X) - 2A^g_{X'} Y) + \Pi^0O(d^2 f(X', X') + \Lambda(Y, Y'))$
$- g_N(O(d^2 f(X, X') + \Lambda(Y, Y')) + \Pi^0O(d^2 f(X', X') + \Lambda(Y, Y'))$.

Where $\tilde{X}, \tilde{X}'$ and $\tilde{Z}$ are as in Proposition [2.4] Observing that, if $X \in \ker d^f,$
$\tilde{X} = (X, 0)$ is horizontal in $f^*P$, and taking $\tilde{U} = (0, U)$ a vertical vector and
$\tilde{Z} = (Z, \mathcal{L}df Z)$ an horizontal vector, we conclude that

(2.5)
$R(\tilde{U}, \tilde{X}, \tilde{X}, \tilde{U}) = 0$

and

(2.6)
$R(\tilde{U}, \tilde{X}, \tilde{X}, \tilde{Z}) = g_N(O(d^2 f(X, X), d^2 f(0, Z) + \Lambda(U, \mathcal{L}df Z))$
$= -g_P(A(\mathcal{LO}df Z, \mathcal{LO}d^2 f(X, X)), U)$

So, condition [2.4] holds for every $U$ if and only if $A(\mathcal{LO}df Z, \mathcal{LO}d^2 f(X, X)) = 0$. □

Proof of Theorem [1.1]: Now, for $\pi(p) = a \in N$ and $\pi$ fat, the map $\Xi_N : T_a N \to V_p$
defined by
$Y \mapsto A(\mathcal{L}df Z, \mathcal{LO}d^2 f(X, X), \mathcal{L}df Y)$
is surjective if and only if $d^2 f(X, X) \neq 0$. In particular, $\dim \ker \Xi_N \geq \dim N - \dim \mathcal{V}$
if and only if $d^2 f(X, X) = 0$. By the other hand, condition [2.4] implies
that \( df(T_x M) \subset \ker \Xi_X \), so, if \( a \) is a regular point \( \dim \ker \Xi_X \geq \dim df(T_x M) = \dim N > \dim N - \dim V \).

Since, in a regular point, we always can choose an extension \( \bar{X} \) of \( X \) such that \( df \bar{X} = 0 \), we have
\[
d^2f(X, X) = \nabla^N_{dfX} df \bar{X} - df(\nabla^M_X \bar{X}) = -df(\text{pr}_{(\ker df)^\perp} \nabla^M_X \bar{X})
\]
We conclude the proof by observing that \( \text{pr}_{(\ker df)^\perp} \nabla^M_X \bar{X} \) is the second fundamental form of \( f^{-1}(a) \) and that \( df \) is injective on \( (\ker df)^\perp \).

3. An Application to the Study of Exotic Spheres

Given a pull-back bundle \( f^*P \to M \) with the induced connection, Theorem 1.1 gives an obstruction for the problem of finding a metric in \( M \) such that the respective connection metric has non-negative curvature. I.e., for any such metric, the regular level sets of \( f \) must be totally geodesic. Assuming then, that a metric satisfy this condition, we have

**Lemma 3.1.** Let \( c : [0, 1] \to N \) is a curve such that \( c(0) \) is a singular point and \( c(t) \) is regular for all \( t > 0 \). Then, the regular level sets \( f^{-1}(c(t)) \) are isometric to \( f^{-1}(c(1)) \) for \( t > 0 \) and there is a totally geodesic embedding \( f^{-1}(c(1)) \hookrightarrow f^{-1}(c(0)) \).

From this lemma, we conclude that singular level sets must have, at least the dimension of near fibers. In particular it implies that there is no metric on \( S^8 \) or \( S^{10} \) such that the bundles \((-h\eta_8)^*S^7\) or \((-hb_{10})^*S^7\) have connection metrics with non-negative curvature, where \( \eta : S^3 \to S^2 \) and \( h : S^7 \to S^4 \) are the Hopf bundles and \( \eta_8 \) and \( b_{10} \) are analytic suspensions of \( \eta \) and a generator of \( \pi_6S^3 \) given in [?]. By the other hand, these bundles admit \( S^3 \) actions (which are isometric for many metrics and connections) with quotients exotic spheres.

**Proof of Lemma 3.1.** Write \( L = f^{-1}(c(1)) \) and \( S = f^{-1}(c(0)) \). From [3], we now that the curve \( c \) induces isometries \( \phi_t : f^{-1}(c(1)) \to f^{-1}(c(t)) \). So we have an embedding \( i : (0, 1) \times L \to M \). Now, we can find an embeddin \( \tilde{i} : [0, 1] \times L \to M \) by taking \( \tilde{i}(0, x) = \lim_{t \to 0} i(t, x) \). It defines a continuos bijective map \( \tilde{i}_0 : L \to M \), smoothnes is guaranteed by extending the vector fields \( X(t) = d\tilde{i}_{(t, 0)}(0, X) \) and geodesics doesn’t go out of \( L \) since both exponential and distance maps are continuos and
\[
0 = \lim_{t \to 0} (d(\gamma_X(t)(s), S)) = d(\gamma_X(0)(s), S),
\]
being \( \gamma_X(s) \) the geodesic of the vector \( X \) at instant \( s \).

We also would like to remark that, fot the Hopf-bundles \( \eta, h \) and \( H : S^{15} \to S^8 \), the proof of Theorem 1.1 also demonstrates that:

**Theorem 3.2.** Let \( f^*P \to M \) be a submersion induced by one of the Hopf-bundles above with the induced connection. Then, if \( \text{rank} df_x > 1 \) for all \( x \) and \( f^*P \) has non-negative sectional curvature, then, any level set of \( f \) is totally geodesic.

An example of map that violates this condition is the geodesic two fold \( \rho_2 : S^n \to S^n \) starting at \( c_0 \in S^n \), defined by
\[
\cos t c_0 + \sin t X \mapsto \cos 2t c_0 + \sin 2t X,
\]
where \( X \in T_{e_0}S^n \). In the general case of a geodesic \( k \)-fold, the condition given by Lema 2.5 can also be verified. We believe that, in certain sense, these are the only maps in the case where \( N \) is a round sphere, in the sense that, if an analytic map gives a connection metric with non-negative curvature, that it is a composition of a (non-Riemannian) submersion with a map like this.

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References

[1] Puettmann T. Rigas A. Durán, C. An infinite family of gromoll-meyer spheres. *Archiv der Mathematik*, 95:269–282, 2010.
[2] Walshap G. Gromoll, D. *Metric Foliations and Curvature*. Birkhäuser Verlag, Basel, 2009.
[3] D. Johnson and L. Whitt, Totally geodesic foliations. *J. Diff. Geo.*, 15:225–235, 1980.
[4] S. Narasimham and S. Ramanan. Existence of universal connections. *American journal of Mathematics*, 83:563–572, 1961.
[5] L. Sperança. On explicit constructions of exotic spheres. *preprint*, arXiv, 2012.