ON FINITE BARELY NON-ABELIAN $p$-GROUPS

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Abstract. We will classify the finite barely non-abelian $p$-groups.

1. Introduction

It is an important theme to determine the structure of a group by using its subgroup in the group theory. Let $p$ be prime and let $G$ be a finite $p$-group. An old group-theoretic result of Rèdei [3] proved that if every proper subgroup of $G$ is abelian then either abelian or minimal non-abelian. Blackburn [1] showed that if every proper subgroup of $G$ is generated by two elements then $G$ is either metacyclic or a 3-group of maximal class with a few exceptions. Minimal non-abelian $p$-groups have been investigated in recent years; see [6], [7] and [8].

We will say that a $p$-group $G$ is barely non-abelian if it satisfies the following conditions: (1) every proper subgroup of $G$ is abelian and (2) if $H_0 \subset H \subset G$ are subgroups, where $H$ is cyclic and $H_0$ is normal in $G$, then $G/H_0$ is abelian.

For $p = 2$, this class of groups naturally came up in [2]. The main result of [2] relies on the classification of barely non-abelian 2-groups; see [2, Proposition 4.6]. The proof of [2, Proposition 4.6] depends, in turn, on a result of Rèdei; see [3]. The purpose of this paper is to classify barely non-abelian $p$-groups for every prime $p$. Our main result is as follows.

Theorem 1.1. A non-abelian $p$-group $G$ is barely non-abelian if and only if $|G| = p^3$ or $G$ is isomorphic to $M(p^k)$, where $k \geq 4$.

The remainder of this paper will be devoted to proving this theorem. Our proof will be entirely elementary; we will not appeal to Rèdei’s theorem. In particular, for $p = 2$ we will give a new elementary proof of [2, Proposition 4.6].

2. Barely non-abelian $p$-groups

In this section, we introduce a barely non-abelian $p$-group $G$ and investigate the properties of $G$. 

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Definition 2.1. We call a finite non-abelian $p$-group $G$ barely non-abelian $p$-group if it satisfies the following conditions:

1. every proper subgroup of $G$ is abelian,
2. if $H_0 \subsetneq H \subseteq G$ are subgroups, where $H$ is cyclic and $H_0$ is normal in $G$, then $G/H_0$ is abelian.

Example 2.2. Let $p = 2$. $Q_8$, $D_8$ and $M(2n)$ $(n \geq 4$ a power of 2) are barely non-abelian 2-groups. We define the group $M(2n)$ as the semidirect product of $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, where the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}/n\mathbb{Z}$ by sending 1 to $\frac{n}{2} + 1$. Equivalently,

\[ M(2n) = \{ r, s \mid r^n = s^2 = 1, \ sr = r^{n/2+1}s \}. \]

Note that $M(8)$ is the dihedral group $D_8$; see [2, Proposition 4.6].

Let $p$ be an odd prime. There are two barely non-abelian $p$-groups of order $p^3$.

\[ G_1 = \langle r, s \mid r^{p^2} = s^p = 1, \ sr = r^{p+1}s \rangle, \]
\[ G_2 = \langle r, s \mid r^p = s^p = c^p = 1, \ rc = cr, \ sc = cs, \ sr = crs \rangle. \]

We know that the barely non-abelian $p$-groups of order $p^3$ is isomorphic to the semidirect product of $\mathbb{Z}/p^2\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$ for $G_1$ and $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$ for $G_2$, respectively.

Now, we define the group $M(p^n)$ as the semidirect product of $\mathbb{Z}/p^{n-1}\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$, that is,

\[ M(p^n) = \{ r, s \mid r^{p^n-1} = s^p = 1, \ sr = r^{p^{n-2}+1}s \}, \]

where $n \geq 4$. Note that $M(p^3) = G_1$.

Lemma 2.3. (a) Every proper subgroup of $M(p^n)$ is abelian.
(b) Every proper quotient of $M(p^n)$ is abelian.
(c) $M(p^n)$ is barely non-abelian for any $n \geq 4$.

Proof. (a) Let $S$ be a proper subgroup of $M(p^n)$. If $S$ contains the index $p$ subgroup $\langle r \rangle$, then $S = \langle r \rangle$ and hence $S$ is abelian. If not, let $S_0 = S \cap \langle r \rangle$. Then $S_0 \subseteq \langle r^p \rangle$ is central in $M(p^n)$. Hence $S/S_0 \subseteq M(p^n)/\langle r \rangle \cong \mathbb{Z}/p\mathbb{Z}$, that is, $S/S_0$ is cyclic. Thus $S$ is abelian, as desired.
(b) Assume $M(p^n)/N$ is not abelian for some non-trivial normal subgroup $N$ of $M(p^n)$. Then $N$ cannot contain $r^{p^{n-2}}$. Otherwise,

\[ (sN)(rN) = srN = r^{p^{n-2}+1}sN = (rN)(sN). \]

Hence we have

\[ N \cap \langle r \rangle = \{1\}. \]

Since $\langle r \rangle$ has an index $p$ in $M(p^n)$, this implies that $|N| = p$. Moreover, $N$ and $\langle r \rangle$ are complementary normal subgroups in $M(p^n)$. Thus $M(p^n) \cong N \times \langle r \rangle$ is abelian. It is a contradiction.
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Theorem 2.4. Suppose $G$ be a barely non-abelian $p$-group of order $\geq p^4$.

(a) The center $Z(G)$ has index $p^2$ in $G$.

(b) If $S$ is a proper subgroup of $G$, then $[S : (S \cap Z(G))] \leq p$.

(c) $x^p \in Z(G)$ for every $x \in G$.

Let $G'$ be the commutator subgroup of $G$.

(d) $G' \subset Z(G)$.

(e) $|G'| = p$. In the sequel we shall denote the non-identity element of $G'$ by $c$, that is, $G' = \langle c \rangle$ of order $p$.

(f) If $x \in G$ is an element of order $n \geq p^2$ then $x^{n/p} \in G'$.

(g) $G$ is generated by two elements $r$ and $s$ such that $rs = csr$.

Proof. (a) Let $H$ be a subgroup of index $p$ in $G$; see, e.g., [4, 5.3.1(ii)]. Choose $g \in G \setminus H$; applying [4, 5.3.1(ii)] once again, we can find a subgroup $H' \subset G$ such that $g \in H'$ and $[G : H'] = p$. Since $G$ is a barely non-abelian group, both $H$ and $H'$ are abelian. Thus every $x \in H \cap H'$ commutes with $g$ and with every element of $H$. Since $H$ and $g$ generate $G$, we conclude that $x \in Z(G)$, i.e.,

\begin{equation}
H \cap H' \subset Z(G).
\end{equation}

Since $G$ is non-abelian,

\begin{equation}
|G : Z(G)| \geq p^2;
\end{equation}

see, e.g., [5, 6.3.4]. On the other hand, since $[G : H] = [G : H'] = p$, it is easy to see that

\begin{equation}
|G : (H \cap H')| = p^2.
\end{equation}

Part (a) now follows from (2.3-2.5). For future reference we remark that our argument also shows that

\begin{equation}
H \cap H' = Z(G).
\end{equation}

(b) By [4, 5.3.1(ii)], $S$ is contained in a subgroup $H$ of index $p$. By (2.6), $Z(G) = H' \cap H'$, where $H'$ is another subgroup of $G$ of index $p$. Then $S \cap Z(G) = S \cap H'$, and the latter clearly has index $\leq p$ in $S$.

(c) Apply part (b) to the cyclic group $S = \langle x \rangle$.

(d) Follows from the fact that the factor group $G/Z(G)$ has order $p^2$ and, hence, is abelian.

(e) Since $G$ is a non-abelian $p$-group, it has an element $r$ of order $n \geq p^2$. Let $H = \langle r \rangle$ and $H_0 = \langle r^{n/p} \rangle$ be cyclic subgroups of $G$ of orders $n$ and $p$ respectively. By part (c), $r^{n/p} = \left(r^{n/p} \right)^p \in Z(G)$ and hence $H_0 = \langle r^{n/p} \rangle \subseteq Z(G)$. Then $H_0$ is normal in $G$. Since $G$ is a barely non-abelian group, $G/H_0$ is abelian. In other words,

\begin{equation}
G' \subset H_0.
\end{equation}
Thus \(|G'| \leq |H_0| = p\). On the other hand, since \(G\) is non-abelian, \(|G'| \neq 1\).

Thus \(G'\) has exactly \(p\) elements, as claimed.

(f) By (2.7), \(x^{n/p} \in G'\).

(g) Choose two non-commuting elements \(r\) and \(s\) in \(G\). Since \(G\) is a barely non-abelian group, these elements generate \(G\). By part (e), \(rsr^{-1}s^{-1} \in G' = \langle c \rangle\). Without loss of generality, we may assume let \(rs = csr\), as desired. □

We now proceed to give a complete list of barely non-abelian \(p\)-groups.

**Theorem 2.5.** Let \(G\) be a barely non-abelian \(p\)-group. Then \(G\) is one of the following groups:

(a) \(|G| = p^3\),

(b) \(M(p^k)\), where \(k \geq 4\).

**Proof.**

(a) follows from Example 2.2.

(b) Write \(G = \langle r, s \rangle, G' = \langle c \rangle\), and \(sr = crs\). Denote the orders of \(r\) and \(s\) by \(n\) and \(m\) respectively. We may assume without loss of generality that \(p\) is an odd prime, \(|G| \geq p^4\) and \(n \geq m\). Let \(n = m = p\). Then \(G/G'\) is an abelian group of order \(\leq p^2\). Hence \(|G| \leq p^2|G'| = p^3\).

Now let \(n \geq m \geq p^2\). By Theorem 2.4(c), we have \(r^{n/p} \in G'\), where the order of \(r^{n/p}\) is \(p\). We may assume that \(G' = \langle r^{n/p} \rangle\).

By Theorem 2.4(c) once again, \(s^{m/p} \in G'\). Then there exists a positive integer \(t\) such that \(s^{m/p} = (r^{n/p})^t\). Let \(\tilde{s} = (r^{n/m})^{-1}s\). We claim that

\[
\tilde{s}^{m/p} = 1.
\]

We now consider two cases.

Case I: \(m < n\).

\[
\tilde{s}^{m/p} = \left((r^{n/m})^{-t}\right)^{m/p} = \left(r^{n/p}\right)^{-t}s^{m/p} = 1,
\]

where \(r^{n/m} \in Z(G)\), as claimed.

Case II: \(m = n\). Then

\[
\tilde{s}^p = \left((r^{n/m})^{-t}\right)^p = \left(r^{-t}s\right)^p = c^{pt}s^p(r^{n/p})^t,
\]

where \(s^p\) and \((r^{-1})^p\) are in \(Z(G)\). Hence

\[
\tilde{s}^{m/p} = \left(c^{pt}s^p(r^{n/p})^t\right)^{m/p^2} = c^{mpt}s^{p}r^{n/p} = (r^{n/p})^{-t} = 1,
\]

where \(s^{m/p} = \left(r^{n/p}\right)^t\) because \(n = m\). This proves the claim. Now observe that \(G = \langle r, s \rangle = \langle r, \tilde{s} \rangle\) and \(r\tilde{s}r^{-1}\tilde{s}^{-1} = r\tilde{s}r^{-1}\tilde{s}^{-1} = c\), where \(c = r^{n/p}\). Thus we may replace \(s\) by \(\tilde{s}\). By (2.8), \(\tilde{s}\) has order \(\leq m/p\). After repeating this process a finite number of times, we may assume \(m = p\).
Thus $G$ is generated by elements $r$ and $s$ such that $r^n = s^p = 1$ and $sr = r^{n/p+1}s$, where $n \geq p^3$ is a power of $p$. This completes the proof of Theorem 2.5. □

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