ODEs with Preisach operator under the derivative and with discontinuous in time right-hand side

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Abstract. We consider ordinary differential equations with a Preisach operator under the derivative. A special case when the right-hand side has discontinuities in time is studied. We present theorems about the existence and uniqueness of solutions. We also prove a theorem which describes the behavior of a solution at the points of discontinuity of the right-hand side.

1. Introduction

This paper is devoted to a special kind of ordinary differential equation with a Preisach operator. This type of equation has been used in various areas of science to describe systems with hysteresis. The Preisach operator was originally used to describe magnetization phenomena in physics [6], but was understood to have a more generic nature, so that it can be applied in various areas [14]. Recent applications can be found, for example, in hydrology [9, 10, 11] and economics [12, 13].

Differential equations with a derivative of a Preisach operator and a smooth right-hand side have been studied in [1]. The operator introduces singularities in the equation, which can lead to non-uniqueness of solutions and a necessity to consider singular points in a special way. These questions have been addressed in [1]. In this paper we extend the class of right-hand sides of such equations to those containing discontinuities in time. Such equations naturally arise, for example, in hydrological models where discontinuous terms describe the rainfall [11]. These discontinuities lead to another set of singular points, compared to those of [1]. Such points are analyzed below.

The results of this paper provide a mathematical background for an algorithm to solve equations with a Preisach operator under the derivative, see companion paper [2]. This algorithm takes into account points of discontinuity of the right-hand side, so it can be viewed as an extended version of an algorithm published earlier [3]. The companion paper [2] also contains several numerical experiments that demonstrate the use of the algorithm for a particular hydrological problem.

The paper is organised as follows. In section 2 we describe the equation that we are interested in and give a definition of a solution of such an equation. Section 3 contains a brief definition of a Preisach operator and describes its properties that are necessary to formulate the results of the paper. Section 4 contains the main results about existence, uniqueness and extendability of solutions, and about the behavior of solutions at singular points. Then in section 5 we provide a simple explanation of how to come up with the main formula (4) which describes the behavior
Let \((\tau_i)\) be any interval that does not contain points from \(\mathbb{R}\) from which the Preisach nonlinearity is constructed. Consider a family \(\alpha, \beta\) of relays \((\alpha, \beta)\) with threshold values \(\alpha \leq \beta\). The plane of all thresholds \(\alpha, \beta\) with a density \(\mu(\alpha, \beta)\) is referred to as the Preisach plane, and each point above the line \(\beta = \alpha\) in this plane corresponds to a single relay from this family.

Introduce a probability measure on the half-plane \(\{\alpha \leq \beta\}\) with a density \(\mu(\alpha, \beta)\), and a measurable function \(\omega_0(\alpha, \beta)\) which we call the initial state of the family \(\mathcal{R}\). Then for any initial state \(\omega_0(\alpha, \beta)\) and any continuous input \(x(t)\) for \(t \geq t_0\) we define the function

\[
y(t) = \mathcal{P} x(t) = \int_{\alpha \leq \beta} \mu(\alpha, \beta) R_{\alpha, \beta}[t_0, \omega_0(\alpha, \beta)] x(t) \, d\alpha \, d\beta.
\]
This model is called the Preisach nonlinearity (or, when convenient, Preisach model or Preisach operator). Here $y(t)$ is the output of the model with initial state $\omega_0$ and input $x(t)$.

Throughout this paper, we assume the following properties of the measure density $\mu$:

1. $\mu$ is zero outside some strip $0 \leq \beta - \alpha \leq d$,
2. The function $\mu = \mu(\alpha, \beta)$ is continuously differentiable in the strip $0 \leq \beta - \alpha \leq d$,
3. There is a strip $0 \leq \beta - \alpha \leq d_1$ with $d_1 \leq d$ where $\mu$ is positive.

The same assumptions are used in [1], where differential equations with a smooth right-hand side are considered.

For $t > t_0$ the function

$$\omega(\alpha, \beta) = R_{\alpha, \beta}[t_0, \omega_0(\alpha, \beta)]x(t).$$

is interpreted as the state of the Preisach nonlinearity at the moment $t$.

It is natural to limit the set of possible initial states to those defined by

$$\omega_0(\alpha, \beta) = \begin{cases} 
0, & \text{if } \alpha + \beta > \eta_0(\beta - \alpha) + 2x_0, \\
1, & \text{if } \alpha + \beta \leq \eta_0(\beta - \alpha) + 2x_0.
\end{cases}$$

where a function $\eta_0 : [0, d] \to \mathbb{R}$ satisfies

$$\eta_0(0) = 0, \quad |\eta_0(\xi_1) - \eta_0(\xi_2)| \leq |\xi_1 - \xi_2|.$$ 

We will identify this function $\eta_0$ with the initial state $\omega_0$. Then for any initial state $\eta_0 = \eta_0(\xi)$ and any continuous input $x : [t_0, \infty) \to \mathbb{R}$ the variable state $\eta(t) = \eta(t; \xi)$ at any moment $t \geq t_0$ is defined. We will refer to this function $\eta(t)$ as to the state of the Preisach nonlinearity.

Geometrically this definition of a state means that the relays with parameter values $(\alpha, \beta)$ below the curve $L(t) = \{\alpha + \beta = \eta(t; \beta - \alpha) + 2x(t)\}$ are switched on, and the relays that correspond to points above this curve are switched off, see figure 1. Then the output $y(t)$ of the operator equals the measure of the shaded set $S(t) = \{\alpha + \beta \leq \eta(t; \beta - \alpha) + 2x(t)\}$.

The evolution of the Preisach state admits a simple geometrical interpretation [8], see figure 2. Here the input $x(t)$ moves along the horizontal line and controls the point on the diagonal.
\( \alpha = \beta \) immediately above itself. When the input increases, this point on the diagonal drags the horizontal line and shades the domain below this line and above the diagonal, see figure 2(a). When the input decreases, the diagonal point drags the vertical line, and ‘clears’ everything to the right of this line and above the diagonal, see figure 2(b).

The situations depicted in figure 2 are important for the subsequent text. Namely, we will say that the Preisach state has a horizontal last segment, if it looks like in the figure 2(a), and a vertical last segment, if it looks like in the figure 2(b).

The Preisach operator with a measure density satisfying (p1)–(p3) has some important properties, which were proved in [1] and which are required for the proofs in section 6. For convenience, they are reproduced below.

**Proposition 1.** Suppose that the output \( y(t) \) of Preisach operator \( \mathcal{P}[\eta(t)]x(t) \) is increasing (decreasing) on the interval \([t_1, t_2]\). Then the input \( x(t) \) is also increasing (decreasing) on the same interval.

**Proposition 2.** Suppose that the output \( y(t) \) of Preisach operator \( \mathcal{P}[\eta(t)]x(t) \) is increasing (decreasing) and absolutely continuous on the interval \([t_1, t_2]\). Then the input \( x(t) \) is also absolutely continuous on the same interval.

### 4. Main Results

Let

\[
\begin{align*}
\Omega &= \{(t, x) : t \notin \mathbb{T} \land F(t, x) = 0 \land F_t(t, x) = 0\}, \\
\Omega_1 &= \{(t, x) : t \in \mathbb{T} \land F(t - 0, x)F(t + 0, x) = 0\},
\end{align*}
\]

where \( F_t \) denotes \( \frac{\partial F}{\partial t} \).

**Definition 2.** A triple \((t_0, x_0, \eta_0)\) is called admissible, if \((t_0, x_0) \notin \tilde{\Omega} = \Omega \cup \Omega_1\), and the initial state \( \eta_0 \) has the following property: if \( F(t_0, x_0) = 0, F_t(t_0, x_0) > 0 \), then \( \eta_0 \) has a vertical last segment, and if \( F(t_0, x_0) = 0, F_t(t_0, x_0) < 0 \), then \( \eta_0 \) has a horizontal last segment. If \( t_0 \in \mathbb{T} \), then \( \eta_0 \) has a nonzero last segment (horizontal or vertical).
Theorem 1. For any admissible initial data, the initial value problem (1)–(2) has a unique solution on some interval \( t_0 \leq t < t_1 \).

Theorem 2. For any admissible initial data the solution of initial value problem (1)–(2) can be uniquely extended to the maximal interval \( t_0 \leq t < T \) (finite or infinite) such that \( (t, x(t)) \in \mathbb{R}^2 \setminus \tilde{\Omega} \) for all \( t \) from this interval. If this interval is finite \( (T < \infty) \), then either \( x(t) \to +\infty \), \( x(t) \to -\infty \) or \( x(t) \to x_\ast \) as \( t \to T - 0 \), and \( (T, x_\ast) \in \Omega \) in the latter case.

The proofs of these two theorems are obtained by considering equivalent modified problems with smooth right hand sides \( \tilde{F} \) on intervals \( (\tau_i, \tau_i+1) \), and then applying the corresponding existence, uniqueness and extension theorems from [1].

If \( (T, x_\ast) \in \Omega_1 \), it does not necessarily mean that the solution cannot be uniquely continued past that point. Here we omit the analysis of such points for simplicity.

We should also note that the triple \( (t, x(t), \eta(t)) \) remains admissible along the trajectory of a solution \( x(t), t \in [t_0, T) \).

Let \( (t_0, x_0, \eta_0) \) be an admissible initial data, and \( x: [0, T) \to \mathbb{R} \) be a solution of the initial value problem (1)–(2). Then for any \( t \in (t_0, T), t \not\in T : x(t) \) has finite left and right derivatives. For \( t_\ast \in T \) such that \( F(t_\ast - 0, x_\ast)F(t_\ast + 0, x_\ast) > 0, x(t) \) also has finite one-sided derivatives. All these facts are proved by considering modified problems with smooth right-hand sides and applying Theorem 4.3 from [1] which gives values for \( x'(t) \). The same is true for the left derivative of \( x \) at the points \( t_\ast \in T \) such that \( F(t_\ast - 0, x_\ast)F(t_\ast + 0, x_\ast) < 0 \). The right derivative at such points, however, is infinite, and the behavior of the solution to the right of such a time point is described by the following theorem.

Theorem 3. Let \( t_\ast \in T \) be such that \( F(t_\ast - 0, x_\ast)F(t_\ast + 0, x_\ast) < 0 \). Then the solution \( x(t) \) has an infinite right derivative at \( t_\ast \), and there exist \( \delta > 0 \) and \( C > 0 \) such that for any \( t_\ast \leq t < t_\ast + \delta \) the following inequality holds:

\[
\left| (x(t) - x_\ast) - \text{sgn}(F_\ast) \sqrt{2(t - t_\ast)} \frac{|F_\ast|}{\mu_\ast} \right| \leq C(t - t_\ast),
\]

where

\[
F_\ast = F(t_\ast + 0, x_\ast), \quad \mu_\ast = \mu(x_\ast, x_\ast), \quad x_\ast = x(t_\ast).
\]

The same inequality holds at the moment \( t_0 \), if \( t_0 \in T \), \( F(t_\ast + 0, x_0) > 0 \) and \( \eta_0 \) has a vertical last segment (or \( F(t_\ast + 0, x_0) < 0 \) and \( \eta_0 \) has a horizontal last segment).

The last theorem gives explicit formulas for the constants \( \delta \) and \( C \).

Theorem 4. Let \( t_\ast \in T \) be such that \( F(t_\ast - 0, x_\ast)F(t_\ast + 0, x_\ast) < 0 \). Denote by \( \Delta x \) the nonzero length of the last segment of the Preisach state \( \eta(t_\ast) \). Denote by \( (t_\ast, T) \) an interval that does not contain any points from \( T \) such that there exists a constant \( C_1 \) which bounds the first derivatives of \( F \) and \( \mu \):

\[
\left| \frac{\partial F}{\partial t} \right|, \left| \frac{\partial F}{\partial x} \right|, \left| \frac{\partial \mu}{\partial \alpha} \right|, \left| \frac{\partial \mu}{\partial \beta} \right| \leq \frac{C_1}{2}.
\]

for all \( (t, x) \in [t_\ast, T] \times [x_\ast - \Delta x, x_\ast + \Delta x] \). Introduce constants

\[
\tilde{C} = \frac{4\sqrt{3}C_1}{\mu_\ast^2} (\mu_\ast + F_\ast),
\]

\[
\delta_\ast = \min \left\{ T - t_\ast, \Delta x, \frac{F_\ast}{2C_1}, \frac{\mu_\ast}{\mu_\ast}, \frac{6F_\ast}{2C_1}, \frac{\mu_\ast}{\mu_\ast} \right\},
\]

\[
\tilde{\delta} = \frac{\delta_\ast^2 \mu_\ast}{6F_\ast} \leq \delta_\ast.
\]

Then Theorem 3 holds with \( \tilde{\delta} \) and \( \tilde{C} \).
It is possible to use Theorem 4 to obtain guaranteed error estimates for numerical methods for solving differential equations with a Preisach operator under the derivative and a discontinuous in $t$ right-hand side. However, such estimates lie beyond the scope of this paper, and obtaining them will require additional work.

The above theorems were formulated under the assumption that the Preisach measure density $\mu$ satisfies properties (p1)–(p3). It should be noted that in important applications this may not be the case. For example, in the wedge model for hysteresis in hydrology (see [4, 5]) the measure density $\mu(\alpha, \beta)$ is equal to zero for any $\alpha > 0$. However, the results of this paper can still be applied to the wedge model after a simple modification. Namely, if there exists a set $S \subset \mathbb{R}$ such that properties (p1)–(p3) hold for any $(\alpha, \beta) \in \{\alpha, \beta \in S \land \alpha \leq \beta\}$, then the definition of an admissible triple $(t_0, x_0, \eta_0)$ can be extended to include a requirement that $x_0 \in S$, and all the theorems will hold on an interval $[t_0, T)$ such that $x(t) \in S$ for all $t$ from this interval.

5. Simple derivation of the main formula (4)

Consider the problem (1)–(2) with admissible triple $(t_0, x_0, \eta_0)$ such that $\eta_0 \in T$, $F(t_0 + 0, x_0) > 0$ and the initial state $\eta_0$ has a vertical last segment. We will construct a local approximation of the solution $x(t)$ in a vicinity of point $t_0$.

First of all, we rewrite the equation (1) in the following form:

$$y'(t) = Q(x(t))x'(t) = F(t, x(t)), \quad (9)$$

where $Q(x) = \frac{\partial \mu}{\partial x}$. Note that the output $y(t)$ of the Preisach operator is increasing in some vicinity of $t_0$ due to $F(t_0 + 0, x_0) > 0$, and, according to Proposition 1, the input $x(t)$ is also increasing in this vicinity. Thus the Preisach state $\eta(t)$ has a horizontal last segment for $t > t_0$ in a vicinity of $t_0$. Then $Q(x)$ is the measured length of this segment, which has endpoints $(x_0, x)$ and $(x, x)$ in Preisach plane. At the moment $t_0$ this horizontal segment has zero length, thus creating a singularity.

In order to find an approximation we write the Taylor series for $Q(x)$ and $F(t, x)$ at the point $(t_0, x_0)$ for $t > t_0$ (i.e. after the jump of $F(t, x)$ had occurred). Namely,

$$F(t, x) = F(t_0 + 0, x_0) + o(t - t_0) + o(x - x_0) \approx F_0,$$

and

$$Q(x) = 0 + \mu(x_0, x_0)(x - x_0) + o(x - x_0) \approx \mu_0(x - x_0).$$

where $F_0 = F(t_0 + 0, x_0)$ and $\mu_0 = \mu(x_0, x_0)$. This allows us to approximate (9) by

$$\mu_0 z(t)z'(t) \approx F_0,$$

where $z(t) = x(t) - x_0$. Integrating both sides and solving for $z$ gives

$$z(t) \approx \sqrt{2(t - t_0) \frac{F_0}{\mu_0}}. \quad (10)$$

This is exactly the approximation from (4). Theorems 3 and 4 give a means by which this formula can be interpreted.

The computer algorithm in the companion paper [2] uses approximation (10) to make a special step when crossing a discontinuity point. It uses a standard numerical ODE solver prior to the jump, then a single time-step is made in accordance with (10), after which the standard ODE solver can continue once more with new initial conditions.
6. Proof of theorems

6.1. Proof of Theorem 1
If \( t_0 \notin \mathbb{T} \), then the function \( F(t,x) \) is smooth in the vicinity of the point \((t_0,x_0)\), and local existence and uniqueness follow from the corresponding theorem for a smooth right-hand side (see Theorem 4.1 from [1]).

If \( t_0 \in \mathbb{T} \), then consider the modified problem (1')–(2) on a sufficiently small interval \([t_0,t_0+\varepsilon]\) with a smooth right-hand side function \(F\). According to definition 2, \(F(t_0,x_0) \neq 0\), which implies that the initial data will be admissible for this modified problem, so it also has a unique solution. It remains to note that any solution of problem (1')–(2) on the interval \([t_0,t_0+\varepsilon]\) is a solution of the original system (1)–(2) on the same interval and vice versa, and therefore the original problem also has a unique local solution.

6.2. Proof of Theorem 2
Consider the interval \([t_0,\tau_1]\), where \(\tau_1 > t_0\), \(\tau_1 \in \mathbb{T}\) is a “jump point” of \(g(t)\) closest to \(t_0\). On this interval the problem (1)–(2) is equivalent to the modified problem (1')–(2). According to the theorem on the extension of solutions for equations with a smooth right-hand side (see Theorem 4.2 from [1]), the problem (1')–(2) has a unique solution that can be extended to a maximal interval \(t_0 \leq t < T\), and there are the following options for \(T\):

(i) \(\bar{T} \leq \tau_1\), and \(x(t) \to \pm \infty\) or \(x(t) \to x_*\) as \(t \to T - 0\), where \((\bar{T},x_*) \in \Omega \subset \bar{\Omega}\). In this case we can set \(T = \bar{T}\), and \([t_0,T]\) will be the maximal interval to which the solution of (1)–(2) can be extended.

(ii) \(\bar{T} > \tau_1\). Then either \((\tau_1,x(\tau_1)) \in \bar{\Omega}\), in which case we can set \(T = \tau_1\), or \((\tau_1,x(\tau_1)) \notin \bar{\Omega}\). In the latter case we consider the initial value problem problem with initial data \(\tau_1, x(\tau_1)\) and \(\eta(\tau_1)\) on the interval \([\tau_1,\tau_2]\), where \(\tau_2\) is the next jump point, and repeat the same reasoning as above.

If this process stops on some interval \([\tau_i,\tau_{i+1}]\), then the statement of the theorem will hold with corresponding \(T = \bar{T}\). If it doesn’t stop, or if the last interval \([\tau_k,\infty)\) is infinite and for that last interval we get \(\bar{T} = \infty\), then the statement of the theorem will hold with \(T = \infty\).

6.3. Proof of Theorems 3 and 4
Consider the case when \(t_* \in \mathbb{T}\), \(F(t_* - 0, x_*) < 0\), \(F(t_* + 0, x_*) > 0\). This implies that the state \(\eta_* = \eta(t_*)\) of the Preisach nonlinearity has a vertical last segment of length \(\Delta x\).

If \(F(t_* + 0, x_*) > 0\) then \(y'(t) > 0\) in some interval \(t_* \leq t < t_1\). Then \(y(t)\) is increasing on this interval, and according to Proposition 1 this implies that \(x(t)\) is also increasing on the same interval.

Introduce constants \(C_1, \bar{C}, \delta_*, \delta\) given by equations (5)–(8) as specified in Theorem 4. The interval \((t_*,t_* + \delta)\) does not contain points from \(\mathbb{T}\), which implies that if we replace \(g(t)\) with its continuous extension from this interval to \(\mathbb{R}\), then the modified problem will have the same solution. So, in the following text, the function \(F\) will be assumed to be continuously differentiable on \(\mathbb{R}^2\).

Denote

\[ \ddot{x}(t,\varepsilon) = x_* + \sqrt{2(t-t_*)} \frac{F_* + C_1 \varepsilon}{\mu_* - C_1 \varepsilon}. \]

Select an arbitrary \(\delta \in (0,\bar{\delta})\) and let \(\varepsilon = \sqrt{\frac{2F_*}{\mu_*}, \delta < \varepsilon < \delta_*}\).

Lemma. Under these conditions the solution \(x(t)\) exists and is increasing on the interval \([t_*,t_* + \delta]\), and for any \(t \in [t_*,t_* + \delta]\)

\[ x_* \leq x(t) \leq \ddot{x}(t,\varepsilon) \leq x_* + \varepsilon. \]  \(11\)
Proof of the lemma will be given after the proof of the theorem. The function \( \bar{x}(t, \varepsilon) \) is differentiable with respect to \( \varepsilon \). This implies

\[
|\bar{x}(t, \varepsilon) - \bar{x}(t, 0)| \leq \varepsilon \max_{0 \leq \theta \leq \varepsilon} \left| \frac{\partial \bar{x}}{\partial \varepsilon}(t, \theta) \right|, \quad \text{where } \bar{x}(t, 0) = x_* + \sqrt{2(t - t_*) \frac{F_*}{\mu_*}}.
\]

Using the inequalities

\[
\frac{\mu_*}{2} < \mu_* - C_1 \varepsilon \leq \mu_*, \quad F_* \leq F_* + C_1 \varepsilon < \frac{3F_*}{2},
\]

we write the following series of relations

\[
\left| \frac{\partial \bar{x}}{\partial \varepsilon} \right| = \sqrt{\frac{(t - t_*)(\mu_* - C_1 \varepsilon)}{2(F_* + C_1 \varepsilon)}} \cdot \frac{C_1(\mu_* + F_*)}{(\mu_* - C_1 \varepsilon)^2}
\leq \sqrt{\frac{(t - t_*)\mu_*}{2F_*}} \cdot \frac{4}{\mu_*^2} \cdot C_1(\mu_* + F_*)
= \frac{2\sqrt{2C_1}}{\mu_* \sqrt{\mu_* F_*}} (\mu_* + F_*) \sqrt{t - t_*} = C_2 \sqrt{t - t_*}.
\]

Finally, taking into account that \( \varepsilon = \sqrt{\delta / \mu_*} \),

\[
x(t) \leq \bar{x}(t, \varepsilon) \leq \bar{x}(t, 0) + C_2 \sqrt{t - t_*} \varepsilon = \bar{x}(t, 0) + C_2 \sqrt{\delta / \mu_*} \sqrt{t - t_*}
\leq \bar{x}(t, 0) + C_2 \sqrt{\frac{6F_*}{\mu_*} \cdot \delta} = x_* + \sqrt{2(t - t_*) \frac{F_*}{\mu_*}} + C\delta,
\]

where

\[
C = C_2 \sqrt{\frac{6F_*}{\mu_*}} = \frac{4\sqrt{3}C_1}{\mu_*^2} (\mu_* + F_*) = \tilde{C}.
\]

Note that this inequality holds for each \( t_* \leq t \leq t_* + \delta \), so we can write it for \( t = t_* + \delta \):

\[
x(t_* + \delta) \leq x_* + \sqrt{2\delta \frac{F_*}{\mu_*}} + \tilde{C}\delta,
\]

which holds for all \( 0 \leq \delta < \tilde{\delta} \). Replacing \( \delta \) with \( t - t_* \), \( t_* \leq t < t_* + \tilde{\delta} \), we get

\[
x(t) \leq x_* + \sqrt{2(t - t_*) \frac{F_*}{\mu_*}} + \tilde{C}(t - t_*) \quad \text{for all } t_* \leq t < t_* + \tilde{\delta}, \quad (12)
\]

which provides us with the upper bound for \( x(t) \). To get the lower bound, we use a series of similar calculations, so for the rest of the proof we give only the most important equations. First, we introduce a function \( \bar{x}(t, \varepsilon) \):

\[
\bar{x}(t, \varepsilon) = x_* + \sqrt{2(t - t_*) \frac{F_* - C_1 \varepsilon}{\mu_* + C_1 \varepsilon}},
\]
and prove that $x(t) \geq x(t, \varepsilon)$. Then we obtain an estimate for the difference $x(t, \varepsilon) - x(t, 0)$:

$$|x(t, \varepsilon) - x(t, 0)| \leq \varepsilon \max_{0 \leq \theta \leq \varepsilon} \left| \frac{\partial x}{\partial \varepsilon} (t, \theta) \right| \leq C_3 \delta,$$

where $C_3 = \frac{3 \sqrt{2} C_1}{\mu_*^2} (\mu_* + F_*) < \tilde{C}$.

Note that $x(t, 0) = \bar{x}(t, 0)$, so we obtain the inequality

$$x(t) \geq x_* + \sqrt{2(t - t_*) \frac{F_*}{\mu_*}} - \tilde{C}(t - t_*)$$

for all $t_* \leq t < t_* + \delta$, which together with (12) proves (4) in the case when $F(t_* + 0, x_*) > 0$. The case when $F(t_* + 0, x_*) < 0$ can be considered in the same way with the appropriate modification of functions $\bar{x}$ and $x$. Thus, the statement of Theorem 3 holds with constants $\delta$ and $\tilde{C}$, so both Theorem 3 and Theorem 4 are proved.

**Proof of Lemma.** The proof consists of two parts. Firstly, we suppose that the solution $x(t)$ is defined and is increasing on the interval $[t_*, t_* + \delta]$, and that $x(t) \leq x_* + \varepsilon$ for all $t$ from this interval, and prove that (11) holds under these assumptions. Secondly, we prove that these initial assumptions were correct.

Introduce a continuous increasing function $\varphi$ defined on $[x_*, x_* + \varepsilon]$:

$$\varphi(x) = y_* + \text{mes}_{\mu}\{ (\alpha, \beta) : x_* \leq \alpha \leq x, \alpha \leq \beta \leq x \},$$

where $y_* = y(t_*)$. If $x(t)$ is increasing on $[t_*, t_* + \delta]$ and $x(t) \leq x_* + \varepsilon < x_* + \Delta x$, and $\eta_*$ has a vertical last segment of size $\Delta x$, then the output $y$ of the Preisach nonlinearity equals $y = \varphi(x)$. The function $\varphi(x)$ is increasing on $[x_*, x_* + \varepsilon]$, therefore it has an inverse function $\psi(y) = \varphi^{-1}(y)$ defined on $[y_*, \varphi(x_* + \varepsilon)]$, and the equation

$$\psi'(t) = F(t, \psi(y))$$

holds for all $t \in [t_*, t_* + \delta]$.

Let $\tilde{F} = F_* + C_1 \varepsilon$. Then we get the following inequality for the function $F(t, x)$:

$$F(t, x) \leq F_* + (t - t_*) \max |F_1| + (x - x_*) \max |F_x| \leq F_* + \frac{C_1}{2} \delta + \frac{C_1}{2} \varepsilon$$

$$< F_* + C_1 \varepsilon = \tilde{F}$$

for all $(t, x) \in [t_*, t_* + \delta] \times [x_*, x_* + \varepsilon]$. In the same way we obtain an inequality for $\mu(\alpha, \beta)$:

$$\mu(\alpha, \beta) \geq \mu_* - (\alpha - x_*) \max \left| \frac{\partial \mu}{\partial \alpha} \right| + (\beta - x_*) \max \left| \frac{\partial \mu}{\partial \beta} \right| \geq \mu_* - C_1 \varepsilon,$$  \hspace{1cm} (13)

for $\alpha, \beta \in [x_*, x_* + \varepsilon]$.

Let $\tilde{y}(t)$ be a solution of the equation

$$\tilde{y}'(t) = \tilde{F} = F_* + C_1 \varepsilon,$$

with initial data $\tilde{y}(t_*) = y(t_*) = y_*$. Then, according to the theorem on differential inequalities (see, for example, Theorem 4.1 in [15]), $y(t_*) = \tilde{y}(t_*)$ and $F \leq \tilde{F}$ imply that $y(t) \leq \tilde{y}(t)$ for $t \in [t_*, t_* + \delta]$. The function $y$ has the following explicit form:

$$\tilde{y} = y_* + (F_* + C_1 \varepsilon)(t - t_*).$$
Thus, we obtain the inequality
\[ y(t) \leq y_*(F_* + C_1\epsilon)(t - t_*) \quad \text{for all } t_* \leq t \leq t_* + \delta. \tag{14} \]

Now we return to the \( x \) variable, and to do so we need a lower estimate for the function \( \varphi(x) \). Let \( \tilde{\varphi}(x) \) be a continuous increasing function defined for all \( x \geq x_* \) as
\[ \tilde{\varphi}(x) = y_* + \frac{1}{2}(\mu_* - C_1\epsilon)(x - x_*)^2. \]

Using the estimate (13) we get
\[ \varphi(x) = y_* + \int_{x_*}^{x} \int_{\alpha}^{x} \mu(\alpha, \beta)d\beta d\alpha \geq y_* + (\mu_* - C_1\epsilon) \int_{x_*}^{x} \int_{\alpha}^{x} d\beta = y_* + \frac{1}{2}(\mu_* - C_1\epsilon)(x - x_*)^2 = \tilde{\varphi}(x) \]
for all \( x \in [x_*, x_* + \epsilon] \). Consider a continuous increasing extension of function \( \varphi \) such that the inequality \( \varphi(x) \geq \tilde{\varphi}(x) \) holds for all \( x \geq x_* \). Then both \( \varphi \) and \( \tilde{\varphi} \) will have inverse functions \( \psi \) and \( \tilde{\psi} \) defined for \( y \geq y_* \), and \( \psi \leq \tilde{\psi} \). The function \( \tilde{\psi} = \tilde{\varphi}^{-1} \) can be written explicitly as
\[ \tilde{\psi}(y) = x_* + \sqrt{\frac{2(y - y_*)}{\mu_* - C_1\epsilon}}. \]

Then, applying \( \psi \) to both sides of (14), we obtain
\[ x(t) = \psi(y(t)) \leq \psi(y_* + (F_* + C_1\epsilon)(t - t_*)) \leq \tilde{\psi}(y_* + (F_* + C_1\epsilon)(t - t_*)) = x_* + \sqrt{\frac{2(t - t_*)}{\mu_* - C_1\epsilon}} F_* + C_1\epsilon = \bar{x}(t, \epsilon) \]
for all \( t \in [t_*, t_* + \delta] \). Thus, under our initial assumptions the statement of the lemma holds.

Note that, due to the choice of \( C_1 \) and \( \epsilon \), \( F_* + C_1\epsilon < \frac{1}{2}F_* \), \( \mu_* - C_1\epsilon > \frac{1}{2}\mu_* \), and thus \( \bar{x}(t, \epsilon) < x_* + \sqrt{\frac{\delta^2 F_*}{\mu_*}} = x_* + \epsilon \).

Now we have to show that our assumptions were correct. We assumed that \( x(t) \leq x_* + \epsilon \). If this is not true, then there exists a time moment \( t \) such that \( t_* < t \leq t_* + \delta \) and \( x(t) > x_* + \epsilon \). In this case the continuity of \( x \) implies that there exists a moment \( t_1 \) such that for \( t_* \leq t \leq t_1 \): \( x(t) \leq x_* + \epsilon \) and \( x(t_1) = x_* + \epsilon \). Then if we repeat all the above steps for the interval \([t_*, t_1]\), we will get the same inequality \( x(t) \leq \bar{x}(t, \epsilon) \) for all \( t \in [t_*, t_1] \). But \( \bar{x}(t_1, \epsilon) < x_* + \epsilon \), and \( x(t_1) = x_* + \epsilon \). This contradiction proves that \( x(t) < x_* + \epsilon \) for all \( 0 \leq t \leq \delta \).

We also assumed that \( x(t) \) increases on \([t_*, t_* + \delta]\). If this is not true, then there exists a local maximum \( t_1 \in (t_*, t_* + \delta] \), such that \( x(t) \) increases on \([t_*, t_1]\) and \( x'(t_1) = 0 \), and thus \( F(t_1, x(t_1)) = 0 \). But we already proved that \( x(t_1) < x_* + \epsilon \), which means that function \( F(t, x) \) is bounded by \( F(t, x) > F_* - C_1\epsilon > \frac{1}{2}F_* > 0 \). This contradiction shows that \( x(t) \) increases on \([t_*, t_* + \delta]\).

The last assumption was that the local solution \( x(t) \) at the point \((t_*, x_*)\) can be continued to the interval \([t_*, t_* + \delta]\). If this is not true, then according to Theorem 2 there exists a moment \( t_1 \in (t_*, t_* + \delta] \) such that either \( x(t) \to +\infty \), or \( x(t) \to \bar{x} \) as \( t \to t_1 - 0 \) where \((t_1, \bar{x}) \in \hat{\Omega} \). But neither of these can be true, because \( x(t) \) is bounded by \( x(t) \leq x_* + \epsilon \), it does not have a local maximum on the interval \((t_1, t_* + \delta]\), and this interval does not contain points from \( \mathbb{T} \).

This completes the proof of the lemma. \( \Box \)
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