Anomalous Action of QCD from the General Quark Propagator

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Abstract

The anomalous action of the chiral effective theory to $O(p^4)$ is investigated by generalizing the consideration in [1] with including the wave function part in the general quark propagator. It is found that the QCD dynamics dependence of the Wess-Zumino term is explicit and this dependence means that the QCD chiral symmetry must be broken dynamically in the low energy region for dynamically generating the Wess-Zumino term. In addition, we found that the next to leading order anomalous action in the momentum expansion of the chiral perturbation theory is gauge invariant and QCD dynamics dependent.
I. INTRODUCTION

Due to the quark confinement in the low energy region, the strong interaction processes cannot be calculated in the framework of conventional perturbation theory in which the expansion is in terms of the coupling constant $g_s$. In this sense, to describe the low energy dynamics of QCD, some effective theories, based on the Weinberg’s folk theorem[^2] that any quantum theory that at sufficiently low energy and large distances looks Lorentz invariant and satisfied the cluster decomposition principle will also at sufficiently low energy look like a quantum field theory, was proposed. Among all these effective theories, chiral perturbation theory[^2, 3] got great triumphes in describing the hadron processes with pseudoscalar mesons.

In principle, there are two ways to obtain the chiral perturbation theory. One is the phenomenological construction[^2, 3], in which, chiral symmetry $SU(3)_R \times SU(3)_L$ and its breaking $SU(3)_R \times SU(3)_L \rightarrow SU(3)_V$ of QCD was considered, the pseudoscalar mesons emerge from the chiral symmetry breaking as Goldstone bosons and live in the coset space of subgroup of original chiral symmetry. The lagrangian is expressed in terms of the order of small quantities such as the ratio of external momenta of mesons to the chiral symmetry breaking scale, and in front of each independent structure term, a coefficient is given. It is believed that the QCD dynamics are incorporated in the unknown coefficients and these coefficients can be determined from the underlying QCD in principle, but in practice they are determined phenomenologically due to our few knowledge about the nonperturbative calculation. In this approach, the pseudoscalar mesons get mass through the Gell-Mann, Oakes and Renner relations[^4] by adding the quark masses to the chiral effective theory which breaks the chiral symmetry explicitly. In fact, this method had been used to describe the hadron physics before the appearance of QCD[^5].

Another approach is to obtain the chiral perturbation theory from QCD[^6], in which, the relation between the phenomenological lagrangian and QCD was established. It is found that the coefficients of the phenomenological lagrangian can be expressed in terms of the Green functions of quarks so that they can be regarded as the fundamental QCD definitions of the coefficients of the phenomenological lagrangian. Then, unlike the phenomenological construction in which the coefficients were determined phenomenologically, in the QCD approach these coefficients can be evaluated numerically from the first principle of QCD as
long as we have found method to compute all related Green’s functions of QCD. As the first approximation, the explicit results were given in \[7\], which can be shown further to be equivalent to the computation based on a gauge invariant, nonlocal, dynamical model (GND model) \[8\], in which all the coefficients of the chiral effective lagrangian were expressed in terms of the quark self-energy and the numerical results based on the quark self-energy determined in some models were also given, these numerical results coincide with the experimental ones. In fact, there was another approach by using the anomaly method\[9\], but the signs of the numerical results of coefficients \(L_7\) and \(L_8\) predicted in this method are different from the phenomenological results. In addition, it seems that all the coefficients in this anomaly approach is independent of the QCD dynamics. In fact, it is shown\[7\] that the anomalous contributions are cancelled by some parts of the normal contributions and the cancellation leaves the quark self-energy dependent contributions to the coefficients of the chiral lagrangian.

What we mentioned above are mainly the normal part of the chiral lagrangian. Besides the normal part, there is anomalous part with odd intrinsic parity of the chiral lagrangian which is due to the negative parity of the pseudoscalar mesons. The anomalous part of the chiral lagrangian was first given in\[10\] by integrating the consistent condition of anomaly in the language of current algebra. Latter, this lagrangian was constructed geometrically\[11\] with photon as an external field. Its topological meaning was studied in detail by several groups\[12, 13\]. Like the normal part of the chiral effective theory, the quantum corrections do not renormalize the coefficient of the Wess-Zumino term since it is the leading order of the anomalous action. For the loop corrections, it is found that they are gauge invariant and give rise the higher order anomaly\[14\] since the Wess-Zumino term is not gauge invariant, the explicit form of the next to leading order were give in\[15\]. This does not mean that the nonrenormalization theorem is violated, because only the summation of all the orders of the effective theory is equivalent to the underlying theory. For the anomalous part of the chiral effective theory, as for the normal part, among all these constructions, the QCD effects were hidden in some constants such as the pion decay constants, so that the explicit effects of the QCD dynamics were also not clear. Because of all these reasons, it is interesting to investigate the explicit QCD dynamics dependence of the anomalous action. In our previous work\[1\], in the fame work of GND model, we found that the anomalous terms come from both dynamics dependent and independent sources, after some cancellations, the Wess-
Zumino term is yielded and it is found that the coefficient relates to the quark self-energy. In addition, when the quark self-energy vanishes, the anomalous action also vanishes, this means that the anomalous action is a QCD dynamics dependent quantity. In fact, before our work, the dependence of strong dynamics of Wess-Zumino term was investigated in the constituent quark model by introducing a constituent quark masses $M_Q$ and it is found that the Wess-Zumino term vanishes when the strong interaction was switched off by set $M_Q = 0$. But in all their approximations, the hard constituent quark masses $M_Q$ will cause wrong bad ultraviolet behavior. The anomalous section of the effective action and its applications were reviewed in

In fact, the quark propagator in GND quark model is first order approximation of the dynamical perturbation theory, and when one switches off the external sources, only the quark self-energy effect is considered. Generally speaking, the inverse of quark propagator takes the form

$$S^{-1}(k) = iA(k^2)\gamma - B(k^2)$$  \hspace{1cm} (1.1)

where $B(k^2)$ arises from quark self-energy and the quark condensation which indicates the chiral symmetry breaking if

$$\{S^{-1}(p), \gamma_5\} = -2\gamma_5B(p^2) \neq 0$$  \hspace{1cm} (1.2)

and $A(k^2)$ is quark wave function part. In the high energy region, their explicit forms can be determined order by order explicitly through the ordinary perturbation theory, but we cannot get their analytic forms without approximation in the infrared region yet since we do not know how to deal with nonperturbation theory.

As we mentioned before, in leading order of dynamical perturbation theory $A(k^2) = 1$, in addition, Schwinger-Dyson equation tells us that $A(k^2) = 1$ only happens in Landau gauge with bare gluon-quark vertex and bare gluon propagator in the kernel of the integration equation. So that, to respect real original QCD effects, we have to include the $A(k^2)$ effect. In the previous work, it is for simplicity of computation, we take the approximation $A(k^2) = 1$. The main purpose of present paper is to overcome this shortcoming and investigate the effect of $A(k^2)$.

In literatures, $Z(k^2) = 1/A(k^2)$ and $\Sigma(k^2) = B(k^2)/A(k^2)$ are called the quark dressing function and mass function respectively. In the language of renormalization, the quark
propagator (1.1) depends on the renormalization point $\mu$, i.e.

$$S^{-1}(\mu, k) = iA(\mu, k^2)\not{k} - B(\mu, k^2)$$  \hfill (1.3)

and in the standard momentum subtraction scheme

$$Z(\mu, \mu^2) = 1 \hfill (1.4)$$

$$\Sigma(\mu^2) = B(\mu, \mu^2)/A(\mu, \mu^2) = m(\mu) \hfill (1.5)$$

And the scale $\mu$ characterized the mode of chiral symmetry breaking. As was shown in [19], when the scale is large enough which makes QCD in the perturbation region, quantum corrections cannot break the chiral symmetry dynamically if the current quark mass which breaks the chiral symmetry explicitly was not introduced. While in the low energy, the chiral symmetry is broken due to the quark condensation and the mixed quark-gluon condensation. In case that the current quark mass is introduced, the chiral symmetry will be broken explicitly due to the small light quark masses when scale $\mu$ is large, but there is no Wess-Zumino anomaly generated. When $\mu$ becomes so small that the theory is in the nonperturbative region, the dynamical chiral symmetry breaking term becomes larger than current quark mass and the chiral symmetry is broken dynamically. Considering this, at high energy region, the renormalized quark propagator can be related to the bare one through the wave function renormalization constant $Z_2$ through

$$S^{\text{bare}}(c, k) = Z_2(\mu, c)S(\mu, k) \hfill (1.6)$$

where $c$ is a small constant which is induced in a regularization such as $\epsilon$ in dimensional regularization and describes the pole behavior of $Z_2$. Since the physical or the renormalized propagator $S(\mu, k)$ is finite, the bare propagator should be negative power of $c$ which is needed to cancel the infinities arise from the loop integral. Then, for a sufficiently small $c$, the relations between the wave function renormalization $Z_2$ and the quark dressing function $Z$ at different renormalization point can be yielded as

$$\frac{Z_2(\mu_1, c)}{Z_2(\mu_2, c)} = \frac{Z(\mu_2, c)}{Z(\mu_1, c)} \hfill (1.7)$$

$$\Sigma(k^2) = \Sigma(\mu_1, k^2) = \Sigma(\mu_2, k^2) \hfill (1.8)$$

which means that mass function must be renormalization point independent and a change of the renormalization point is just an overall rescaling of $Z(\mu, k^2)$ by a momentum independent
constant. Then, the measure of the nonperturbative physics is the deviation of $Z(\mu, k^2)$ from 1 and the difference of $\Sigma(k^2)$ from the renormalized quark mass $m(\mu)$. In this paper we will not distinguish the concepts of quark dressing function and the wave function renormalization constant and call $A(k^2)$ as the wave function renormalization constant.

Formally, the functions $A(k^2)$ and $B(k^2)$ are constrained by the Schwinger-Dyson equation. Although Schwinger-Dyson equation may give a strong constraint on the dress function $A(k^2)$ and quark self-energy $B(k^2)$, it cannot be solved exactly since it is a recurrent integral equation. Based on some approximations, special forms of $A(k^2)$ and $B(k^2)$ were got from the Schwinger-Dyson equation in Landau gauge, axial gauge, and covariant gauge.

In this paper, we will study the anomalous action based on the general quark propagator without discussing the implications of the dynamical chiral symmetry breaking which have been discussed in and also the amplitude of the scale $\mu$, we only consider the theory in the nonperturbative region. For the convenience of our discussion, we will introduce the vector, axial-vector, scalar and pseudoscalar sources like what was done in the normal section of the chiral effective lagrangian. The anomalous action we will investigate is the part of the action proportional to odd number of Levi-Civita tensor $\epsilon_{\mu\nu\alpha\beta}$.

This paper was organized as follows: In section II, we will discuss the meaning of chiral symmetry by considering the relation between the chiral effective lagrangian and the effective action we are interested in and give chiral invariant general quark propagator. In section III, we shall calculate the Wess-Zumino term from the general quark propagator in section II and investigate the effects of QCD dynamics in the anomalous section of the chiral effective lagrangian. Our conclusions and outlooks are presented in Section IV.

II. VARIATIONS OF THE EXTERNAL SOURCES UNDER CHIRAL TRANSFORMATION.

In this section, we shall analyze some transformation properties of the external sources under chiral rotation that we will use in the following calculations. The explicit action which is proportional to the inverse of full propagator is presented when we switch off the external sources by comparing the Gasser-Leutwyler lagrangian.

It is found that the introduction of some external sources in the study of the chiral
perturbation is convenient. Following ref. [3], in four dimensional Euclidean space-time, we consider pure quark kinetic part of bare QCD action in the presence of external scalar, pseudoscalar, vector and axial-vector sources

\[ S[\psi, \bar{\psi}, J] = \int d^4x \bar{\psi} \mathcal{D}\psi \]

\[ = \int d^4x [\bar{\psi} \mathcal{D}\psi + \bar{\psi} J(x) \psi] \tag{2.1} \]

where the external sources are defined as

\[ J(x) = -i\hat{\psi}(x) - i\hat{\phi}(x) \gamma_5 - s(x) + ip(x) \gamma_5 \tag{2.2} \]

and quark mass matrix can be extracted from the scalar external source \( s \). In chiral perturbation theory, the power counting of the vector \( v_\mu \) and axial-vector \( a_\mu \) sources are both \( \mathcal{O}(p) \) while scalar \( s \) and pseudoscalar \( p \) sources are both \( \mathcal{O}(p^2) \). For the following convenience, we rewrite the Dirac operator \( \mathcal{D} \) as

\[ \mathcal{D} \equiv \nabla - s + ip\gamma_5 \quad \nabla \equiv \gamma^\mu \nabla_\mu \quad \nabla_\mu \equiv \partial_\mu - iv_\mu - ia_\mu \gamma_5 = -\nabla_\mu^\dagger \tag{2.3} \]

Formally, the generating functional of QCD is

\[ Z[J] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \mathcal{D}A_\mu e^{\int d^4x [\mathcal{L}(\psi, \bar{\psi}, \Psi, \bar{\Psi}, A_\mu) + \bar{\psi} J \psi]} \tag{2.4} \]

where \( \mathcal{L}(\psi, \bar{\psi}, \Psi, \bar{\Psi}, A_\mu) \) is lagrangian of QCD with \( \psi, \Psi \) and \( A_\mu \) are light, heavy quarks and gluon fields respectively. By integrating out the quark and gluon fields and integrating in the pseudoscalar mesons, one can formally rewrite the QCD generating functional as

\[ Z[J] = \int \mathcal{D}U e^{S_{GL[U,J]}} \tag{2.5} \]

where \( S_{GL[U,J]} \) is the chiral effective lagrangian of pseudoscalar mesons with external sources. In principle, \( S_{GL[U,J]} \) consists normal part and anomalous part due to the intrinsic parity of pseudoscalar mesons, that is

\[ S_{GL[U,J]} = S_{normal[U,J]} + S_{anomaly[U,J]} \tag{2.6} \]

the normal part \( S_{normal[U,J]} \) was given in [3] to \( \mathcal{O}(p^4) \) and the leading anomalous part \( S_{anomaly[U,J]} \) can be found in [10, 11] and next to leading part was given in [15].

Then, the action \( 2.1 \) which we are interested relates the generating functional through

\[ Z[J] = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\bar{\Psi} e^{S_{eff}[\psi, \bar{\psi}, U, J]} \tag{2.7} \]
By comparing (2.5) and (2.7), we yield the relation between the chiral effective action (2.5) and the effective action (2.7)

\[ e^{S_{GL}[U,J]} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S_{\text{eff}}[\psi, \bar{\psi}, U, J]} \] (2.8)

Because the massless QCD is invariant under chiral transformation, the action \( S_{\text{eff}}[\psi, \bar{\psi}, U, J] \) is required to be invariant under the following local \( SU(3)_R \times SU(3)_L \) chiral transformation

\[
\begin{align*}
\psi(x) &\rightarrow \psi'(x) = [V_L(x)P_L + V_R(x)P_R] \psi(x) \\
J(x) &\rightarrow J'(x) = [V_L(x)P_R + V_R(x)P_L][J(x) + \phi_x] [V_L^\dagger(x)P_L + V_R^\dagger(x)P_R] \\
U(x) &\rightarrow U'(x) = V_R(x)U(x)V_L^\dagger(x)
\end{align*}
\] (2.9)

where \( V_L(x) \) and \( V_R(x) \) are the left and right chiral rotation matrices and \( P_{R,L} \) are the project operators \( \frac{1}{2}(1 \pm \gamma_5) \).

Mathematically, the pseudoscalar field \( U(x) \) has the decomposition \( U(x) = \Omega^2(x) \), under local chiral transformation \( SU(3)_R \times SU(3)_L \), \( \Omega(x) \) transforms as

\[ \Omega(x) \rightarrow \Omega'(x) = h^\dagger(x)\Omega(x)V_L^\dagger(x) = V_R(x)\Omega(x)h(x) \] (2.10)

and the gauge group \( h(x) \) can be determined by \( V_L, V_R \) and \( \Omega(x) \). The symmetry \( h(x) \) is the so called hidden local symmetry, its gauge bosons can be identified with the vector mesons through the so called hidden local symmetry method[27, 28].

In terms of the nonlinear field \( \Omega(x) \), we can introduce the rotated external fields

\[ J_\Omega(x) = [\Omega(x)P_R + \Omega^\dagger(x)P_L][J(x) + \phi_x][\Omega^\dagger(x)P_L + \Omega(x)P_R] \]

\[ \equiv -i\phi_\Omega(x) - i\phi_\Omega(x)\gamma_5 - s_\Omega(x) + ip_\Omega(x)\gamma_5 \] (2.11)

Replacing all the external fields with the rotated ones and defining

\[ D_\Omega = \nabla_\Omega - s_\Omega + ip_\Omega\gamma_5, \quad \nabla_\Omega^\mu \equiv \partial^\mu - iv_\Omega^\mu - ia_\Omega^\mu\gamma_5 \] (2.12)

then, under chiral transformation \( SU(3)_R \times SU(3)_L \), one can easily prove the transformations

\[ J_\Omega(x) \rightarrow J'_\Omega(x) = h^\dagger(x)[J_\Omega(x) + \phi_x]h(x) \]

\[ D_\Omega \rightarrow D'_\Omega = h^\dagger(x)D_\Omega h(x) \] (2.13)
From above transformations, one can conclude that the local chiral transformation can be realized through hidden local symmetry, once the theory keeps the hidden local symmetry, it preserves the chiral symmetry automatically. That is, by introducing the $\Omega$ field, the chiral symmetry can be reflected through the hidden local symmetry. In fact, even in the presence of massive quark, by replacing the external fields with the rotated ones, the local chiral symmetry can be preserved.\cite{32}

For the convenience in our discussion of the Wess-Zumino term, we extend $\Omega(x)$ into five dimensional space with

$$\Omega(t,x) = \Omega(1,x) \quad \Omega(0,x) = 1$$

and define the $t$ dependent rotated sources

$$J_{\Omega}(t,x) \equiv -i\phi \Omega(t,x) - i\phi \Omega(t,x)\gamma_5 - s\Omega(t,x) + ip\Omega(t,x)\gamma_5$$

we can verify

$$s_{\Omega}(t,x) = \frac{1}{2}[\Omega(t,x)[s(x) - ip(x)]\Omega(t,x) + \Omega^\dagger(t,x)[s(x) + ip(x)]\Omega^\dagger(t,x)]$$

$$p_{\Omega}(t,x) = \frac{i}{2}[\Omega(t,x)[s(x) - ip(x)]\Omega(t,x) - \Omega^\dagger(t,x)[s(x) + ip(x)]\Omega^\dagger(t,x)]$$

$$v_{\Omega}^\mu(t,x) = \frac{1}{2}[\Omega^\dagger(t,x)[v_{\mu}(x) + a_{\mu}(x) + i\partial_{\mu}]\Omega(t,x)$$

$$\quad + \Omega(t,x)[v_{\mu}(x) - a_{\mu}(x) + i\partial_{\mu}]\Omega^\dagger(t,x)]$$

$$a_{\Omega}^\mu(t,x) = \frac{1}{2}[\Omega^\dagger(t,x)[v_{\mu}(x) + a_{\mu}(x) + i\partial_{\mu}]\Omega(t,x)$$

$$\quad - \Omega(t,x)[v_{\mu}(x) - a_{\mu}(x) + i\partial_{\mu}]\Omega^\dagger(t,x)]$$

Now consider infinitesimal transformation at parameter space $t \rightarrow t + \delta t$, correspondingly

$$\Omega(t,x) \rightarrow \Omega(t + \delta t,x) = (1 + i\delta t\lambda^a\pi^a(x))\Omega(t,x) = \Omega(t,x)(1 + i\delta t\lambda^a\pi^a(x))$$

$$\Omega^\dagger(t,x) \rightarrow \Omega^\dagger(t + \delta t,x) = (1 - i\delta t\lambda^a\pi^a(x))\Omega^\dagger(t,x) = \Omega^\dagger(t,x)(1 - i\delta t\lambda^a\pi^a(x))$$

we can easily prove

$$\delta J_{\Omega}(t,x) = \{i\delta t\lambda^a\pi^a(x)\gamma_5, J_{\Omega}(t,x) + \bar{\phi}_x\}$$

$$\delta a_{\Omega}^\mu(t,x) = -\frac{\delta t}{2}[\frac{\delta U}{\delta t} U^\dagger, v_{\Omega}^\mu(t,x)]$$

$$\delta \nabla_{\Omega}^\mu(t,x) = \frac{i\delta t}{2}\frac{\delta U}{\delta t} U^\dagger(t,x), a_{\Omega}^\mu(t,x)]$$
And in addition
\[ \delta \nabla_\Omega^2(t, x) = -\frac{\delta t}{2} \left\{ [\lambda^a \pi^a(x), a_\Omega^\mu(t, x)] \nabla_\Omega^\mu(t, x) + \nabla_\Omega^\mu(t, x) [\lambda^a \pi^a(x), a_\Omega^\mu(t, x)] \right\} \] (2.27)

For the dimensional extended Dirac operator \( D_\Omega(t, x) \)
\[ D_\Omega(t, x) \equiv \nabla_\Omega^t s_\Omega(t, x) + ip_\Omega(t, x) \gamma_5 = \partial + J_\Omega(t, x) \] (2.28)
we have
\[ \delta D_\Omega(t, x) = \left\{ i \delta t \lambda^a \pi^a \gamma_5, D_\Omega(t, x) \right\} = \frac{1}{2} \left\{ \frac{\delta U(t, x)}{\delta t} U^\dagger(t, x) \gamma_5 \delta t, D_\Omega(t, x) \right\} \] (2.29)
Equivalently
\[ \frac{\partial D_\Omega(t, x)}{\partial t} = \frac{1}{2} \left\{ \frac{\partial U(t, x)}{\partial t} U^\dagger(t, x) \gamma_5, D_\Omega(t, x) \right\} \] (2.30)
\[ \frac{\partial D_\Omega^\dagger(t, x)}{\partial t} = \frac{1}{2} \left\{ \frac{\partial U^\dagger(t, x)}{\partial t} U(t, x) \gamma_5, D_\Omega^\dagger(t, x) \right\} \]
\[ = -\frac{1}{2} \left\{ U^\dagger(t, x) \frac{\partial U(t, x)}{\partial t} \gamma_5, D_\Omega^\dagger(t, x) \right\} \] (2.31)

By now, the transformation properties of the external sources were derived from the classical lagrangian (2.1), we want to include the QCD dynamics effects next. These effects can be calculated perturbatively at high energy region, however, at low energy region, the quark-gluon coupling \( g_s \) is strong and it is not suitable to regard it as an expansion parameter. In this sense, we do not know how to get the forms of the wave function renormalization constant \( A(k^2) \) and the quark self-energy \( B(k^2) \) although formally they satisfy the Schwinger-Dyson equation. Since we have the conclusion that the Wess-Zumino term is an intrinsic property of the non-Abelian gauge theory both phenomenologically[10] and geometrically[11], we naively want to know whether we can get the Wess-Zumino term when the QCD non-Abelian gauge interaction was included in the action (1.1) and explore the contributions of the wave function renormalization constant \( A(k^2) \) and self-energy \( B(k^2) \).

To exhibits the form of the action include QCD non-Abelian gauge interaction and its dynamical effects, following the method given in ref.[8], let us turn to the relation (2.8) between the Gasser-Leutwyler action \( S_{GL} \) and the effective action \( S_{eff} \). By using the rotated basis, this relation can be rewritten as
\[ e^{S_{GL}[\psi, J]} = \frac{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S_{GL}[\psi, \bar{\psi}, U, J]} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S[\psi, \bar{\psi}, J]} - \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S[\psi, \bar{\psi}, J]}} {\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S_{GL}[\psi, \bar{\psi}, U, J]} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S[\psi, \bar{\psi}, J]}} = \mathcal{N} \frac{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S_{GL}[\psi, \bar{\psi}, U, J, \Omega]} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S[\psi, \bar{\psi}, J, \Omega]} - \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S_{GL}[\psi, \bar{\psi}, U, J, \Omega]} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S[\psi, \bar{\psi}, J, \Omega]}} {\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S_{GL}[\psi, \bar{\psi}, U, J, \Omega]} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S[\psi, \bar{\psi}, J, \Omega]}} \] (2.32)
where $S[\psi, \bar{\psi}, J]$ was defined by (2.1) and $\mathcal{N} = \det[\partial + J]$. The anomaly due to the variation of the integration measure was cancelled between the denominator and the numerator. The relation (2.32) means that, formally, we can write $S_{\text{eff}}$ as

$$S_{\text{eff}}[\psi_\Omega, \bar{\psi}_\Omega, 1, J_\Omega] = S[\psi_\Omega, \bar{\psi}_\Omega, J_\Omega] + S_{\text{int}}[\psi_\Omega, \bar{\psi}_\Omega, J_\Omega]$$

(2.33)

where $S[\psi_\Omega, \bar{\psi}_\Omega, J_\Omega]$ is the action defined in (2.1) by substituting relative quantities with the rotated ones. $S_{\text{int}}$ are the terms represent effects of color gauge interaction which are proportional to $\alpha_s$ at the leading order, it includes the fermion self-energy term and the wave function renormalization constant which are caused by integrating out the heavy fermion and gluon degree of freedom in the underling QCD and integrating in the local pseudoscalar fields $U(x)$. We formally write the interaction part $S_{\text{int}}$ as

$$S_{\text{int}}[\psi_\Omega, \bar{\psi}_\Omega, J_\Omega] \sim \int d^4 x \bar{\psi}_\Omega(x) \{ [A(\partial^2) - 1](\partial \bar{\psi}_\Omega) + B(\partial^2) \} \psi_\Omega(x)$$

(2.34)

Where we have taken the minimal extension by including in effect of quark wave function renormalization corrections into the $S_{\text{int}}$ and

$$\tilde{J}_\Omega \equiv -i\bar{\psi}_\Omega(t, x) - i\psi_\Omega(t, x)\gamma_5$$

(2.35)

It should be noticed that (2.34) is the most general form when the dynamical effects are independent of external sources. In the presence of external sources, the general effective action (2.34) due to the color gauge interaction can be decomposed in terms of its general spinor structure which will introduce many more other functions besides the present $A(\partial^2)$ and $B(\partial^2)$, these extra functions will cause much more computation difficulties and we will leave the discussion of them in future. Now as first step beyond original quark self-energy and reduce the difficulties of the computation, we only consider one more extra function: quark wave function renormalization function $A(\partial^2)$ but leaves the discussion of most general propagator elsewhere. In (2.34), we select the dynamical dependence coefficients of the differential operator and the vector source are the same because we need the variety of vector sources to compensate the variety induced by the differential operator under the local chiral transformation, the coefficients of the vector source and the axial-vector source are also the same as we considered the combinations

$$V_{R,\mu}^\Omega = \frac{1}{2}[v_\mu^\Omega + a_\mu^\Omega]$$

$$V_{L,\mu}^\Omega = \frac{1}{2}[v_\mu^\Omega - a_\mu^\Omega]$$

(2.36)
and the Parity invariance of QCD induced property

\[ P : V_{R,\mu}^\Omega \leftrightarrow V_{L,\mu}^\Omega \]  

(2.38)

Combining this with (2.33), we yield the effective action as

\[ S_{\text{eff}}[\psi_\Omega, \bar{\psi}_\Omega, 1, J_\Omega] \sim \int d^4x \bar{\psi}_\Omega(x) \left\{ A(\partial^2)[\bar{\psi} + \bar{J}_\Omega] + s_\Omega - ip_\Omega \gamma_5 + B(\partial^2) \right\} \psi_\Omega(x) \]  

(2.39)

Substituting (2.39) into (2.32), we get the full expression of \( S_{\text{GL}} \) as

\[ S_{\text{GL}}[U, J] \sim \ln \det \left\{ A(\partial^2)[\bar{\psi} + \bar{J}_\Omega] + s_\Omega - ip_\Omega \gamma_5 + B(\partial^2) \right\} \]

\[ - \ln \det[\bar{\psi} + J_\Omega] + \ln \det[\bar{\psi} + J] \]  

(2.40)

It should be noticed that there is no factors \( A(\partial^2) \) and \( B(\partial^2) \) in the last two determinants, this is because these two factors originate from the color interaction and the last two determinants are from the free quark lagrangian.

As we discussed before, the local chiral symmetry can be reflected by the hidden local symmetry through the introduction of the rotated sources, but it is obvious that the action (2.40) cannot preserve the chiral symmetry because

\[ h^\dagger(x)A(-\partial^2)h(x) = A[-h^\dagger(x)\partial^2h(x)] = A[-[\partial_\mu + h^\dagger(x)\partial_\mu h(x)]^2] \]  

(2.41)

Similarly for \( B(-\partial^2) \). This means that the chiral transformation always induces an extra term in \( A(-\partial^2) \) which makes the theory change under chiral transformation. To compensate this extra term we consider the operator \( A(-\nabla_\Omega^2) \) in stead of \( A(-\partial^2) \) in the action with \( \nabla_\Omega^\mu = \partial^\mu - iv_\Omega^\mu \). Because, under local chiral transformation

\[ \nabla_\Omega^\mu \rightarrow \nabla_\Omega'^\mu = h^\dagger(x)\nabla_\Omega^\mu h(x) \]  

(2.42)

that is, the variation of the vector source compensates the variation induce by the action of the differential operator on the local phase factor, we have

\[ A(-\nabla_\Omega^2) \rightarrow A(-\nabla_\Omega'^2) = h^\dagger(x)A(-\nabla_\Omega^2)h(x) \]  

(2.43)

which indicates that \( A(-\nabla_\Omega^2) \) transforms homogenously under chiral transformation. Using the same analysis on \( B(-\partial^2) \), we finally get the local chiral invariant action as

\[ S_{\text{GL}}[U, J] \sim \ln \det \left\{ A(-\nabla_\Omega^2)[\bar{\psi} + \bar{J}_\Omega] + s_\Omega - ip_\Omega \gamma_5 + B(-\nabla_\Omega^2) \right\} \]

\[ - \ln \det[\bar{\psi} + J_\Omega] + \ln \det[\bar{\psi} + J] \]  

(2.44)
Now let’s make some comments on this action: When we do not consider the wave function renormalization, i.e., set $A = 1$, we arrive at the GND quark model and its anomaly had been analyzed in\cite{1}. If the QCD dynamical effects were switched off, that is, $A = 1$ and $B = 0$, the first two terms of this action cancelled each other and the left term is independent of the pseudoscalar fields, then one cannot yield the Wess-Zumino term in this case. As shown in\cite{1}, the scalar source $s$ and pseudoscalar source $p$ do not contribute to the leading order anomaly, so that we will not consider these two sources in the following, therefore, our final action can be written as

$$S_{GL}[U, \tilde{J}] = \ln \det[D'_\Omega] - \ln \det[\tilde{\varphi} + \tilde{J}_\Omega] + \ln \det[\varphi + J]$$  \hspace{1cm} (2.45)

with

$$D'_\Omega = A(-\nabla^2_\Omega)\nabla_\Omega - B(-\nabla^2_\Omega)$$  \hspace{1cm} (2.46)

III. CALCULATION OF ANOMALOUS EFFECTIVE LAGRANGIAN WITH THE GENERALIZED FERMION PROPAGATOR

In this section, based on the action (2.45), we will calculate the anomalous section of chiral effective action to the leading order, $O(p^4)$. Since the difference of (2.45) with that in our previous work \cite{1} is quark wave function renormalization function $A$, the questions are interested become that: is there any changes due to arbitrary $A$ for the leading order anomalous action? Does this anomalous action has its QCD dynamics dependence? Is there higher order anomalous action and where is it from? What is the meaning of higher order anomalous action?

Our consideration is: If there is anomalous action, it should be evaluated through the direct calculation of the action (2.45). To investigate the QCD dynamics effects in the anomalous action, one can set $A(k^2) = 1$ and $B(k^2) = 0$ in the action (2.45) which mean that the strong interaction effects were switched off. In addition, if there is anomalous action by the total calculation of (2.45), this anomalous action can be evaluated through the terms in the r.h.s. of (2.45), so we need to calculate the anomalous action from every term in the r.h.s.of (2.45). Since the first term in the r.h.s.of (2.45) is QCD dynamics dependent, we expect the exact form of the Wess-Zumino term and the Bardeen anomaly will impose some
constraints on the behavior of the strong dynamics dependent function $A(k^2)$ and $B(k^2)$ even though this constraint may be rude.

A. Direct Calculation of Anomalous Action Based on the Effective Action (2.45).

To investigate the anomalous action in the effective action (2.45), we make an inverse rotation of (2.13) which makes the last two terms of (2.45) cancel each other and the pseudoscalar fields in the second term rotate to the first term, that is

$$S_{GL}[U, \tilde{J}] = \ln \det \{ \tilde{A}(-\nabla^2_\Omega) \nabla - \hat{B}(-\nabla^2_\Omega) \}$$

where

$$\tilde{A}(-\nabla^2_\Omega) = [\Omega^\dagger(t, x)P_R + \Omega(t, x)P_L]A(-\nabla^2_\Omega)[\Omega(t, x)P_R + \Omega^\dagger(t, x)P_L]$$

$$\hat{B}(-\nabla^2_\Omega) = [\Omega^\dagger(t, x)P_R + \Omega(t, x)P_L]B(-\nabla^2_\Omega)[\Omega(t, x)P_R + \Omega^\dagger(t, x)P_L]$$

Considering identities $\ln \det A = \text{Tr} \ln A$ and $\delta \text{Tr} \ln A = \text{Tr} \delta A A^{-1}$, we have

$$S[U, \tilde{J}] - S[1, \tilde{J}] = \int_0^1 dt \text{Tr} \left[ \frac{\partial \hat{D}'_\Omega(t)}{\partial t} \hat{D}'_\Omega(t)^{-1} \right]$$

where the trace Tr is around the color, configure, flavor and Lorentz space and

$$\hat{D}'_\Omega(t) = \tilde{A}(-\nabla^2_\Omega) \nabla - \hat{B}(-\nabla^2_\Omega)$$

It should be noticed that we introduced a term $S[1, \tilde{J}]$ in (3.4) which would not change our conclusion on the Wess-Zumino term since it is independent of the pseudoscalar fields.

Since the anomalous effective lagrangian is proportional to the odd number of Levi-Civita tensor $\epsilon^{\mu\nu\alpha\beta}$, i.e., odd number of $\gamma_5$ in the trace of Lorentz matrices, in the following we only concentrate on this kind of terms. After the trace around configure space, we have the anomalous action as

$$S[U, \tilde{J}] - S[1, \tilde{J}] = \int_0^1 dt \int d^4x \int \frac{d^4k}{(2\pi)^4} \text{tr}_{c,f,l} \left[ \frac{\partial \hat{D}'_\Omega(t)}{\partial t} \hat{D}'_\Omega(t)^{-1} \right]$$

the subindex $\epsilon$ means we are only interested in the terms consist odd number of Levi-Civita tensors and $\partial \to ik + \partial$ indicates that after the trace on the configure space, the differential operator $\partial$ should be replaced by $ik + \partial$, that is

$$\hat{D}'_{\Omega, \partial \to ik + \partial} = \tilde{A}[-(\nabla_\Omega + ik)^2][\nabla + ik] - \hat{B}_\Omega[-(\nabla_\Omega + ik)^2]$$
The next problem we have to solve is how to deal with the function \( \hat{A}[-(\bar{\nabla}_\Omega + ik)^2] \) and \( \hat{B}[-(\bar{\nabla}_\Omega + ik)^2] \). To illustrate this problem, let us focus on \( \hat{A}[-(\bar{\nabla}_\Omega + ik)^2] \), to deal with this function, one has to deal with function \( A[-(\bar{\nabla}_\Omega + ik)^2] \) at first. To make its power counting explicit, we expand it around \( k^2 \), then using Taylor expansion, we have

\[
A[-(ik + \bar{\nabla}_\Omega)^2] = A[k^2 - 2ik \cdot \bar{\nabla}_\Omega - \bar{\nabla}^2_\Omega] = A(k^2) + \sum_{n=1}^{\infty} A^{(n)}(k^2) \frac{(-1)^n}{n!} (2ik \cdot \bar{\nabla}_\Omega + \bar{\nabla}^2_\Omega)^n
\]

\[
\equiv A(k^2) + A(k_\mu, \bar{\nabla}_\Omega) \quad (3.8)
\]

where \( A \) is a function of external source and its chiral counting is equal or higher than \( O(p) \), explicitly, it is

\[
A(k_\mu, \bar{\nabla}_\Omega) = \sum_{n=1}^{\infty} A^{(n)}(k^2) \frac{(-1)^n}{n!} (2ik \cdot \bar{\nabla}_\Omega + \bar{\nabla}^2_\Omega)^n \quad (3.9)
\]

So that

\[
\hat{A}[-(\bar{\nabla}_\Omega + ik)^2] = A(k^2) + \hat{A}(k_\mu, \bar{\nabla}_\Omega) \quad (3.10)
\]

with

\[
\hat{A}(k_\mu, \bar{\nabla}_\Omega) = [\Omega^\dagger(t, x)P_R + \Omega(t, x)P_L]A(k_\mu, \bar{\nabla}_\Omega)[\Omega(t, x)P_R + \Omega^\dagger(t, x)P_L] \quad (3.11)
\]

Similarly, we can write

\[
B[-(ik + \nabla_\Omega)^2] = B(k^2) + B(k_\mu, \bar{\nabla}_\Omega) \quad (3.12)
\]

with

\[
B(k_\mu, \bar{\nabla}_\Omega) = \sum_{n=1}^{\infty} B^{(n)}(k^2) \frac{(-1)^n}{n!} (2ik \cdot \bar{\nabla}_\Omega + \bar{\nabla}^2_\Omega)^n \quad (3.13)
\]

Then

\[
\hat{B}[-(\bar{\nabla}_\Omega + ik)^2] = \hat{B}(k^2) + \hat{B}(k_\mu, \bar{\nabla}_\Omega) \quad (3.14)
\]

with

\[
\hat{B}(k^2) = B(k^2)[U^\dagger(t, x)P_R + U(t, x)P_L] \quad (3.15)
\]

\[
\hat{B}(k_\mu, \bar{\nabla}_\Omega) = [\Omega^\dagger(t, x)P_R + \Omega(t, x)P_L]B(k_\mu, \bar{\nabla}_\Omega)[\Omega^\dagger(t, x)P_R + \Omega(t, x)P_L] \quad (3.16)
\]
Then, we can formally write

$$\dot{D}_\Omega(t) = A(k^2)i\hat{k} - B(k^2)[U^\dagger(t, x)P_R + U(t, x)P_L] + \mathcal{E}[\nabla^\mu_\Omega, a^\mu_\Omega] \quad (3.17)$$

with

$$\mathcal{E}[\nabla^\mu_\Omega, a^\mu_\Omega] = \tilde{A}(k_\mu, \nabla^\mu_\Omega)i\hat{k} + \{A(k^2) + \tilde{A}(k_\mu, \nabla^\mu_\Omega)\} \nabla - \tilde{B}(k_\mu, \nabla^\mu_\Omega) \quad (3.18)$$

So that

$$\dot{D}_\Omega(t)^{-1} = \left\{A(k^2)i\hat{k} - B(k^2)[U^\dagger(t, x)P_R + U(t, x)P_L] + \mathcal{E}[\nabla^\mu_\Omega, a_\mu]\right\}^{-1}$$

$$= \left\{-A(k^2)i\hat{k} - B(k^2)[U^\dagger(t, x)P_R + U(t, x)P_L]\right\}^{-1}$$

$$\times \left\{1 + \mathcal{E}[k_\mu, \nabla^\mu_\Omega](A(k^2)i\hat{k} - B(k^2)[U^\dagger(t, x)P_R + U(t, x)P_L])^{-1}\right\}^{-1}$$

$$= \frac{-A(k^2)i\hat{k} - B(k^2)[U^\dagger(t, x)P_R + U(t, x)P_L]}{A^2(k^2)k^2 + B^2(k^2)}$$

$$\times \sum_{m=0}^{\infty} (-1)^m \left\{\mathcal{E}[\nabla^\mu_\Omega, a^\mu_\Omega] - A(k^2)i\hat{k} - B(k^2)[U(t, x)P_R + U^\dagger(t, x)P_L]\right\}^{-m} \quad (3.19)$$

By now, the calculation becomes straightforward although complex. Explicit calculation gives

$$S[U, \tilde{J}] - S[1, \tilde{J}] = \frac{-N_cC}{48\pi^2} \varepsilon_{\mu\nu\alpha\beta} \int_0^1 dt \int d^4x$$

$$\times \text{tr} \left[U^\dagger(t, x)\frac{\partial U(t, x)}{\partial t}L_\mu(t, x)L_\nu(t, x)L_\alpha(t, x)L_\beta(t, x)\right] \quad (3.20)$$

where $N_c$ is the number of color, $L_\mu(t, x) = U^\dagger(t, x)\partial_\mu U(t, x)$ and

$$C = \frac{3}{\pi^2} \int d^4k \left[\frac{4A^4(k^2)B^6(k^2)}{[A^2(k^2)k^2 + B^2(k^2)]^5} - \frac{8A^4(k^2)B^5(k^2)B'(k^2)k^2}{[A^2(k^2)k^2 + B^2(k^2)]^5} + \frac{8A^3(k^2)A'(k^2)B^6(k^2)k^2}{[A^2(k^2)k^2 + B^2(k^2)]^5}\right]$$

$$= -12 \int_0^\infty dk^2 \left[\frac{A^2(k^2)B^4(k^2)k^2}{[A^2(k^2)k^2 + B^2(k^2)]^3} \frac{d}{dk^2} \frac{B^2(k^2)}{[A^2(k^2)k^2 + B^2(k^2)]}\right] \quad (3.21)$$

This constant shows the QCD dynamics dependence of the Wess-Zumino term.

Comparing this conclusion with that given in [1], it is seen that when we set $A(q^2) = 1$ which means the leading order of dynamical perturbation theory, the present result is the same as that of [1]. From the present expression (3.20), it is clear that the Wess-Zumino term arrives from the action (2.45) which depends on the dynamics of QCD, when the dynamics of QCD is switched off by taking $A(k^2) = 1$ and $B(k^2) = 0$, the coefficient $C$ also vanishes, this means that the Wess-Zumino term is an intrinsic property of QCD and is a result of the strong interaction. Note that the QCD dynamics dependence of Wess-Zumino term has

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also been investigated in \[29\] based on the GCM model, but our result here has a factor \(k^2A(k^2)/[k^2A(k^2) + B^2(k^2)]\) difference with the one in \[29\].

In the case of dynamical chiral symmetry breaking, i.e. \(B(k^2) \neq 0\), the pseudoscalar mesons emerge as the Goldstone bosons and there will be anomalous processes which can be described by the Wess-Zumino term \[10\]. In this case, one should have \(C = 1\) to recover the exact coefficient of Wess-Zumino term. We find that as long as the functions \(A(k^2)\) and \(B(k^2)\) behave as

\[
t \equiv \frac{A^2(k^2)k^2}{B^2(k^2)} \rightarrow \begin{cases} 
\infty & k^2 \to \infty \\
0 & k^2 \to 0
\end{cases}
\]  

(3.22)

The coefficient \(C\) can explicitly be integrated out

\[
C = -12 \int_0^\infty dt \frac{t}{[1 + t]^2} \frac{d}{dt} \frac{1}{[1 + t]} = 1 \tag{3.23}
\]

which does coincide with the exact coefficient of Wess-Zumino term.

Although relation (3.22) seems like a constraint on the behavior of the function \(B(k^2)/A(k^2)\), while it is very crude. From the explicit calculations given in the literatures, we can extract the ratio

\[
\frac{B(k^2)/A(k^2)}{k^2} \rightarrow \frac{4m_D^2}{k^4} \ln^\gamma(-\frac{k^2}{\mu^2}), \quad \gamma = \frac{12}{33 - 2n} \tag{3.24}
\]

\[
\frac{B(k^2)/A(k^2)}{k^2} \rightarrow \begin{cases} 
[1 - \frac{3}{4} + \frac{3}{4} t + \mathcal{O}(t^2)]^{1/2}, & k^2 \to 0; \\
[1 - \frac{1}{2} + \frac{3}{4} t + \mathcal{O}(t^2)]^{1/2}, & k^2 \to \infty
\end{cases} \tag{3.25}
\]

\[
\frac{B(k^2)/A(k^2)}{k^2} \rightarrow \begin{cases} 
\frac{1}{k^2} \Phi(2, \frac{5}{2}, -\frac{k^2}{\beta M^2}) + \frac{c_2}{M^2} (\frac{k^2}{\beta M^2})^{-3/2} \Phi(\frac{1}{2}, -\frac{1}{2}, -\frac{k^2}{\beta M^2}), & k^2 \to 0; \\
\frac{1}{k^2} \Phi(2, \frac{5}{2}, -\frac{k^2}{\beta M^2}) + \frac{c_2}{M^2} (\frac{k^2}{\beta M^2})^{-3/2} \Phi(\frac{1}{2}, -\frac{1}{2}, -\frac{k^2}{\beta M^2}), & k^2 \to \infty
\end{cases} \tag{3.26}
\]

where (3.24) was given in \[17\] by using the Landau gauge, eq.(3.25) is from the conclusion given in \[22\] based on the covariant gauge and eq.(3.26) was the result from the axial-gauge \[21\]. In (3.24), \(n\) and \(m_D\) are the number of flavors and constant dynamical quark mass respectively. In (3.27), \(\lambda\) is a constant due to the ghost self-energy at zero point and \(t \propto k^2\). \(\Phi\) is the confluent hypergeometric function, \(\beta \propto \alpha_s\) and \(M\) is the renormalization point in (3.26). After some basic algebra, it is seen that the above three results (3.24,3.26) all satisfy the condition eq.(3.22).

To rewrite the four dimensional Wess-Zumino term (3.20) in five dimensional space-time
with four dimension space-time boundary, we use the same trick as
\[
\frac{\partial}{\partial t} \text{tr} f \{ L_i(t,x)L_j(t,x)L_k(t,x)L_l(t,x)L_m(t,x) \} \epsilon^{ijklm}
\]
\[= 5 \frac{\partial}{\partial x^n} \text{tr} f \{ U^\dagger(t,x)U(t,x) L_i(t,x)L_j(t,x)L_k(t,x)L_l(t,x) \} \epsilon^{ijklm} \tag{3.27}
\]
where \( \epsilon^{ijklm} \) is a totally antisymmetric tensor. This relation yields
\[
\int_Q d\Sigma^{ijklm} \text{tr} f \{ L_i(1,x)L_j(1,x)L_k(1,x)L_l(1,x)L_m(1,x) \}
\]
\[= \int d\Sigma^{ijklm} \int_0^1 dt \frac{\partial}{\partial t} \text{tr} f \{ L_i(t,x)L_j(t,x)L_k(t,x)L_l(t,x)L_m(t,x) \}
\]
\[= 5 \int d^4x \int_0^1 dt \epsilon^{\mu\nu\alpha\beta} \text{tr} f \{ U^\dagger(t,x) \frac{\partial U(t,x)}{\partial t} L_{\mu}(t,x)L_{\nu}(t,x)L_{\alpha}(t,x)L_{\beta}(t,x) \} \tag{3.28}
\]
From this and using \( C = 1 \) we can get the standard Wess-Zumino term \[10\]
\[
\Gamma^-[U] = - \frac{N_c}{240\pi^2} \int_Q d\Sigma^{ijklm} \text{tr} f \{ L_i(1,x)L_j(1,x)L_k(1,x)L_l(1,x)L_m(1,x) \} \tag{3.29}
\]
In conclusion, it is seen that the Wess-Zumino term is an intrinsic property of QCD and depends on the QCD dynamics. All the QCD dynamics effects in the Wess-Zumino term can be collected into the coefficient \( C \) which was given by \[3.21\] and, when the coefficient approaches to unit, one arrives at the exact Wess-Zumino term. When one switches off the QCD dynamics, the coefficient will vanish, which means a vanishing anomalous action and no anomalous pseudoscalar processes occur.

**B. Anomalous Action from Each Term of the r.h.s. of Effective Action (2.45).**

In above subsection, we have proved that there is leading order anomaly in the action (2.45), now let’s investigate the imaginary part of the action originates from every term of (2.45).

Let us look at the anomalous action from \( \Gamma^-_1 \) from the first term of (2.45). By using the same method as that used in above subsection, explicit calculation shows that the anomalous action \( \Gamma^-_{1,D} \) due to the QCD dynamics of the first term of (2.45) is
\[
\Gamma^-_{1,D} = 4iC_1 \epsilon_{\mu\nu\alpha\beta} \int_0^1 dt \int d^4x \frac{\partial}{\partial t} \text{tr} \left[ \nabla_\mu \nabla_\nu \nabla_\alpha a_\beta + \nabla_\mu a_\nu a_\alpha a_\beta \right]_\Omega
\]
\[+ 4iC_2 \epsilon_{\mu\nu\alpha\beta} \int_0^1 dt \int d^4x \frac{\partial}{\partial t} \text{tr} \left[ \nabla_\mu \nabla_\nu \nabla^2 a_\beta \right]_\Omega
\]
\[+ 4iC_3 \epsilon_{\mu\nu\alpha\beta} \int_0^1 dt \int d^4x \frac{\partial}{\partial t} \text{tr} \left[ \nabla_\mu \nabla_\nu \nabla_\alpha a_\beta - \nabla_\mu a_\nu a_\alpha a_\beta \right]_\Omega \tag{3.30}
\]
where the index $\Omega$ means that all the external sources are rotated sources and depend on the chiral angle $\Omega$. And the definitions of the constants $C_i$ are defined as

$$
C_1 = \int \frac{d^4k}{(2\pi)^4} \frac{A^2(k^2)B^4(k^2)}{[A^2(k^2)k^2 + B^2(k^2)]^2} \frac{d}{dk^2} \frac{B^2(k^2)}{[A^2(k^2)k^2 + B^2(k^2)]} 
$$

(3.31)

$$
C_2 = \int \frac{d^4k}{(2\pi)^4} \frac{2A^4(k^2)B^2(k^2)}{[A^2(k^2)k^2 + B^2(k^2)]^2} \frac{d}{dk^2} \frac{B^2(k^2)}{[A^2(k^2)k^2 + B^2(k^2)]} 
$$

(3.32)

$$
C_3 = \int \frac{d^4k}{(2\pi)^4} \frac{A^6(k^2)k^4}{[A^2(k^2)k^2 + B^2(k^2)]^2} \frac{d}{dk^2} \frac{B^2(k^2)}{[A^2(k^2)k^2 + B^2(k^2)]} 
$$

(3.33)

Using (2.23) and (2.20), we can prove relations

$$
\epsilon_{\mu\nu\alpha\beta} \frac{\partial}{\partial t} \text{tr} \left[ \nabla_\mu \nabla_\nu \nabla_\alpha a_\beta \right]_{\Omega} = \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} \text{tr} \left\{ \frac{\partial U}{\partial t} \left[ 2a_\mu \nabla_\nu \nabla_\alpha a_\beta + \nabla_\mu \nabla_\nu a_\alpha a_\beta - a_\mu \nabla_\nu a_\alpha \nabla_\beta - \nabla_\mu a_\nu a_\alpha a_\beta + a_\mu a_\nu \nabla_\alpha \nabla_\beta + 2 \nabla_\mu \nabla_\nu a_\alpha a_\beta + 2 \nabla_\mu a_\nu a_\alpha \nabla_\beta + a_\mu a_\nu \nabla_\alpha \nabla_\beta \right] \right\} 
$$

(3.34)

$$
\epsilon_{\mu\nu\alpha\beta} \frac{\partial}{\partial t} \text{tr} \left[ \nabla_\mu a_\nu a_\alpha a_\beta \right]_{\Omega} = \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} \text{tr} \left\{ \frac{\partial U}{\partial t} \left[ 2a_\mu a_\nu a_\alpha a_\beta + 2 \nabla_\mu a_\nu a_\alpha a_\beta + a_\mu a_\nu \nabla_\alpha \nabla_\beta - \nabla_\mu a_\nu a_\alpha a_\beta - a_\mu \nabla_\nu a_\alpha a_\beta + \nabla_\mu \nabla_\nu a_\alpha a_\beta + \nabla_\mu \nabla_\nu a_\alpha a_\beta \right] \right\} 
$$

(3.35)

we get the anomalous action with form

$$
\Gamma_{1,D}^- = 2N_c C_1 \epsilon_{\mu\nu\alpha\beta} \int_0^1 dt \int d^4x \text{tr} \left[ \frac{\partial U}{\partial t} U^\dagger \right] 
$$

$$
\times \left\{ 2 \nabla_\mu \nabla_\nu a_\alpha a_\beta + 2 a_\mu a_\nu a_\alpha a_\beta - 2 \nabla_\mu a_\nu a_\alpha a_\beta + 2 \nabla_\mu a_\nu a_\alpha a_\beta + 2 \nabla_\mu a_\nu a_\alpha a_\beta + 2 a_\mu a_\nu a_\alpha a_\beta + 2 a_\mu a_\nu a_\alpha a_\beta \right\} 
$$

(3.36)

By using the behavior (3.32), we can exactly get

$$
C_1 = C_2 = \frac{1}{3} C_3 = \frac{-1}{12} \frac{1}{16\pi^2} 
$$

(3.37)

With this equation, Anomalous action with external gauge fields can be yielded

$$
\Gamma_{1,D}^- = \frac{-1}{32\pi^2} \epsilon_{\mu\nu\alpha\beta} \int_0^1 dt \text{tr} \left[ \frac{\partial U}{\partial t} U^\dagger \right] 
$$

$$
\times \left\{ V_\mu V_\alpha a_\beta + \left\{ \frac{2i}{3} a_\mu a_\nu, V_\alpha a_\beta \right\} + \frac{4}{3} d_\mu a_\nu a_\alpha a_\beta + \frac{8i}{3} a_\mu V_\nu a_\alpha a_\beta + \frac{4}{3} a_\mu a_\nu a_\alpha a_\beta \right\} \right\} 
$$

(3.38)
where
\[ V^\Omega_{\mu\nu} = \partial_\mu v^\Omega_\nu - \partial_\nu v^\Omega_\mu - i[v^\Omega_\mu, v^\Omega_\nu] \] (3.39)
\[ d^\mu a^\alpha = \partial_\mu a^\alpha - i[v^\alpha_\mu, a^\alpha] \] (3.40)
which has the same form as the gauge anomaly by replacing \( U \) with the gauge transformation \( g \).

From the above relations such as (3.30) and (3.36), we see that the quark dynamics plays an important role in the form of anomalous action, if there is no quark condensation, there is no anomalous action as mentioned in [1]. The role of quark wave function renormalization function \( A(k^2) \) is passive which play the key role only with quark self energy \( B(k^2) \) in the combination of \( \Sigma(k^2) = B(k^2)/A(k^2) \). In addition, the conclusion (3.38) means that the first part of (2.45) contributes to the imaginary part of the action (2.45).

We now investigate how to get the Wess-Zumino term from the the anomalous action (3.36). The Wess-Zumino term is the pure pseudoscalar mesons lagrangian, so that we should switch off the external fields by setting \( \tilde{J} = 0 \). In this case, there are integrable conditions
\[ V^\Omega_{\mu\nu} = i[a^\mu_\Omega, a^\nu_\Omega] \] (3.41)
\[ d^\mu a^\nu = d^\nu a^\mu \] (3.42)
\[ a^\mu_\Omega = \frac{i}{2} \Omega^\dagger(t, x)[\partial_\mu U(t, x)]\Omega(t, x) \] (3.43)
Then, substitute the above three relations to the non-Abelian anomaly (3.38), we get
\[ \Gamma_{1,D}[U] = -\frac{N_c}{48\pi^2} \int_0^1 dt \int d^4x \epsilon_{\mu\nu\alpha\beta} \times \text{tr}_f \{ U^\dagger(t, x) \partial_\mu U(t, x) \} \text{tr}_f \{ \partial_\nu L_\mu(t, x) L_\alpha(t, x) L_\beta(t, x) \} \] (3.44)
where \( L_\mu(t, x) \equiv U^\dagger(t, x) \partial_\mu U(t, x) \).

By now, can we say that the leading order anomalous action of (2.45) comes from its first part? As argued in [10], the anomalous of term of the effective action is not chiral invariant which seems conflict with the argument that the first term of (2.45) is chiral invariant, does this means that our calculation is wrong? The answer is NO! This is because, we only considered the imaginary part of the first term of (2.45) that depends on the QCD dynamical effects, if one switch off these effects in the first term of (2.45) through \( A(k^2) = 1 \).
and $B(k^2) = 0$, one can easily find that there is another term which dependent on the rotated sources which will give rise the anomaly obviously but with a negative sign. Its contribution $\Gamma_{1,0}^-$ can be calculated by using the Fujikawa’s path integral method which is shown below. Then, one totally has

$$\Gamma_1^- = \Gamma_{1,0}^- + \Gamma_{1,D}^-$$  \hspace{1cm} (3.45)$$

So that, totally, the leading order imaginary section of the action induced by the QCD dynamical effects in the first term of (2.45) cancels that induced by the rotated sources and leaves the next to leading order anomalous action, and, the next to leading order anomalous action is chiral invariant.

Next, let us investigate the anomalous $\Gamma_2^-$ induced by the last two terms of (2.45). This anomaly can be evaluated by the Fujikawa’s method [34], i.e.,

$$\Gamma_2^- = -\text{Tr} \ln[\vartheta + \bar{J}_\Omega] + \text{Tr} \ln[\vartheta + \bar{J}]$$

$$= -\int_0^1 dt \text{Tr}\left\{ \frac{\partial \bar{J}_\Omega(t)}{\partial t} [\vartheta + \bar{J}_\Omega(t)] \right\}$$

$$= -\lim_{\Lambda \to \infty} \int_0^1 dt \text{Tr}\left\{ \frac{\partial U}{\partial t} U^\dagger \gamma_5 \exp\left[ \frac{[\vartheta + \bar{J}_\Omega]^2}{\Lambda^2} \right] \right\}$$  \hspace{1cm} (3.46)$$

After the trace on configure space, expanding the exponent, taking the limit $\Lambda \to \infty$, keeping the $O(p^4)$ terms and the terms proportional to $\epsilon_{\mu\nu\alpha\beta}$, we finally get the result

$$\Gamma_2^- = -\text{Tr} \ln[\vartheta + J_\Omega] + \text{Tr} \ln[\vartheta + J]$$

$$= -\frac{N_c}{32\pi^2} \epsilon_{\mu\nu\alpha\beta} \int_0^1 dt \int d^4 x \text{Tr} \left\{ \frac{\partial U}{\partial t} U^\dagger \times \left[ V_{\mu\nu} V_{\alpha\beta} + \frac{4}{3} d_\mu a_\nu d_\alpha a_\beta + \frac{2i}{3} \{ V_{\mu\nu}, a_\alpha a_\beta \} + \frac{8i}{3} a_\mu V_{\alpha\beta} a_\nu + \frac{4}{3} a_\mu a_\nu a_\alpha a_\beta \right] \right\}_\Omega$$  \hspace{1cm} (3.47)$$

which is the famous Bardeen anomaly [30]. After using the integrability condition (3.41-3.43), we can get the Wess-Zumino term as mentioned above.

In summary, from the r.h.s. of (2.45), the anomalous action is

$$\Gamma^- = \Gamma_{1,0}^- + \Gamma_{1,D}^- + \Gamma_2^-$$  \hspace{1cm} (3.48)$$

with $\Gamma_{1,D}^- = \Gamma_2^- = -\Gamma_{1,0}^-$.  

IV. CONCLUSIONS AND OUTLOOKS

In conclusion, start from the general quark propagator, after chiral rotation, we get the anomalous section of the chiral effective action with external sources. For which we can
yield the Wess-Zumino term after switch off the external fields.

The anomalous action we obtained depends on the QCD dynamics closely, when we switch off the QCD dynamics, the Wess-Zumino term will vanish. That means that the anomalous processes described by the Wess-Zumino term is a QCD processes. To investigate which term the anomaly arise from, we explicitly calculated the anomaly induced by each term of action (2.45). We found that the first term of (2.45) can induce QCD dynamics dependent and independent anomalous action but with a sign difference then the leading order anomaly vanishes, this means that the first term of (2.45) only contribute to the next to leading order anomaly and the next to leading anomaly is chiral invariant. We also found that the last two terms of (2.45) also contribute to anomaly and with the same sign of Wess-Zumino term when we switch off the external sources.

Then, similar to the conclusion given in [1], one observes two kinds of alternative cancellations: One cancellation is regarded as that the anomalous action arising from the QCD dynamical independent section of the first term of (2.45) is cancelled by the anomaly due to the dynamical dependent section of the first term of (2.45) and leaves the anomalous action from the second term of (2.45). In this cancellation, the anomalous action is found to be QCD dynamics independent. An alternative cancellation is that the anomalous induced by the QCD dynamics independent section of the first term of (2.45) was cancelled by the second term of (2.45) which is also QCD dynamics independent. As a consequence, one obtains the anomalous action from the QCD dynamics dependent section of the first term of (2.45). In such a cancellation, the dependence of the Wess-Zumino term on the QCD can be seen explicitly. By considering the totally calculation given in the first subsection of section III we believe that the second cancellation is more reasonable.

The QCD dynamics dependence of the Wess-Zumino term reflects its scale dependence. At low energy region, the dynamical mass is dominate and chiral symmetry is broken dynamically. While at high energy region, the chiral symmetry is explicitly broken due to the small quark mass, and the chiral symmetry is preserved when the quark mass is neglected, then, there will be no anomalous meson process. This situation is very similar to the chiral anomaly studied in ref. [35] based on the loop regularization [36], in which both the massless and massive QCD will be anomaly free when the sliding energy scale $\mu_s$ is large enough.

In our calculation, we dealt with the fermion determinant explicitly by using the expansion of the QCD dependent functions $A(k^2)$ and $B(k^2)$ around the external sources. Besides
this method, the anomalous action from the term depending on the QCD dynamics can also be evaluated with Fujikawa’s path integral method\cite{34}, the anomalous action based on the general quark propagator can also be calculated with this method. But in this method, the behavior of the quark self-energy and the wave function renormalization constant should be determined under the limit $\tau \to \infty$ with $\tau$ as the regulator introduced in this method, this makes the effects of the dynamical functions obscure.

Generally speaking, the quark propagator with external sources is much more complex than that we considered here. Explicitly, it should be Lorentz and gauge covariant\cite{31} and can be decomposed in the form of the Lorentz structure, i.e., the summation of the vector, axial-vector, scalar, pseudoscalar and tensor terms. It is noticed that the conclusion is much more complex and the physical meanings of the coefficients still deserve to be investigated, we shall discuss it elsewhere.

Besides the anomalous action with pseudoscalar mesons, this method can be extended to the processes with resonances such as the vector mesons and axia-vector mesons are incorporated into the chiral effective theory by the hidden local symmetry method\cite{27, 28}. The leading gauge invariant anomalous action with external gauge fields was constructed in\cite{13} by adopting the topological method with considering the t’Hooft matching condition.

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