Characterization and parameterization of the singular manifold of a simple 6-6 Stewart platform

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Abstract  
This paper presents a study of the characterization of the singular manifold of the six-degree-of-freedom parallel manipulator commonly known as the Stewart platform. We consider a platform with base vertices in a circle and for which the bottom and top plates are related by a rotation and a contraction. It is shown that in this case the platform is always in a singular configuration and that the singular manifold can be parameterized by a scalar parameter.

1 Introduction

The Stewart platform is a parallel manipulator with six degrees of freedom [1]. We will use the (standard) variables \( x, y, z, \text{pitch}, \text{roll} \) and \( \text{yaw} \), where \( x, y \) and \( z \) are the coordinates of the centre of the top platform, and \( \text{pitch}, \text{roll} \) and \( \text{yaw} \) denote the Euler angles defining the inclination of this platform with respect to the bottom platform, see Figure 1.

The aim of this paper is to study the singular manifold which is defined by the physical configurations for which it will not be possible to determine the position of the platform uniquely by fixing the lengths of the legs. This is a well-known problem in parallel manipulators[1].

The solution to the forward kinematics problem naturally divides into two cases, namely, a singular and a non-singular. In the non-singular case we recall the work[2] of Ji and Wu and show that there are 8 possible isolated singular solutions that correspond to the same legs lengths. In the singular case we extend the previous analysis and show how to obtain, for a given set of length legs, a set of singular solutions all of them parameterized by a scalar parameter. These solutions are a continuous curves in position space.
and in rotation space in which the platform moves without changing the values of the leg lengths. This fully characterize the singular manifold and shows that the platform is, in this case, completely singular.

Spatial rotations in three dimensions can be parameterized \[2, 3\] using both Euler angles \((\phi, \theta, \psi)\) and unit quaternions \(q = (q_0, q_1, q_2, q_3), \|q\| = 1.\) A unit quaternion may be described as a vector in \(\mathbb{R}^4\)

\[
q = (q_0, q_1, q_2, q_3), \quad (1)
\]

\[
q^T q = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1. \quad (2)
\]

The rotation matrix is given by

\[
R = \begin{pmatrix}
2q_0^2 - 1 + 2q_1^2 & 2q_1q_2 - 2q_0q_3 & 2q_0q_2 + 2q_1q_3 \\
2q_1q_2 + 2q_0q_3 & 2q_0^2 - 1 + 2q_2^2 & 2q_2q_3 - 2q_0q_1 \\
2q_1q_3 - 2q_0q_2 & 2q_0q_1 + 2q_2q_3 & 2q_3^2 - 1 + 2q_1^2
\end{pmatrix}. \quad (3)
\]

Consider the Stewart platform shown in Figure 1. As shown there, the two coordinate systems \(O\) and \(O'\) are fixed to the base and the mobile platforms. The platform geometry can be described by vectors \(L_i, i = 1, 2, \ldots, 6,\) defined by \(L_i = P + T_i - B_i, i = 1, 2, \ldots, 6,\) where \(B_i\) and \(T_i\) are the base and top vertices coordinates, respectively, and \(P\) is the center point of the top plate. We assume that these points are related by

\[
T_i = \mu A B_i, \quad i = 1, 2, \ldots, 6, \quad (4)
\]

where \(A\) is a \(3 \times 3\) orthogonal matrix \((A^T A = I,\) where \(I\) is the \(3 \times 3\) identity matrix\) and \(\mu \in [0, 1[\) is called the rescaling factor. The coordinates of the base vertices are given by

\[
B_i = (x_i, y_i, 0), \quad i = 1, 2, \ldots, 6. \quad (5)
\]
Given the position $\mathbf{P} = (x, y, z)$ and the transformation matrix $R$ between the two coordinate systems, the leg vectors may be written as

$$
\mathbf{L}_i = \mathbf{T}_i - \mathbf{B}_i + \mathbf{P},
$$

(6)

$$
= (\mu RA - I)\mathbf{B}_i + \mathbf{P}, \quad i = 1, 2, \ldots, 6.
$$

(7)

So the length for each $i$-leg is given by

$$
\mathbf{L}_i^T \mathbf{L}_i = ((\mu RA - I)\mathbf{B}_i + \mathbf{P})^T ((\mu RA - I)\mathbf{B}_i + \mathbf{P})
$$

(8)

Given $\mathbf{q}$, $\mathbf{A}$ and $\mathbf{P}$ the leg lengths are given by

$$
L_i = \sqrt{((\mu RA - I)\mathbf{B}_i + \mathbf{P})^T ((\mu RA - I)\mathbf{B}_i + \mathbf{P})}.
$$

(9)

2 Forward kinematics

In the forward kinematics the six leg lengths $L_i$, $i = 1, 2, \ldots, 6$, are given, while $R$ and $\mathbf{P}$ are unknown. Let $\mathbf{e}_x = (1, 0, 0)$, $\mathbf{e}_y = (0, 1, 0)$, $\mathbf{e}_z = (0, 0, 1)$ and expand (8), then one gets,

$$
L_i^2 = \mathbf{P}^T \mathbf{P} + \mathbf{B}_i^T ((\mu(\mathbf{RA})^T - I)(\mu RA - I)) \mathbf{B}_i
$$

$$
+ 2\mathbf{B}_i^T (\mu(\mathbf{RA})^T - I)\mathbf{P},
$$

(10)

or

$$
L_i^2 = \mathbf{P}^T \mathbf{P} + 2x_i (\mathbf{e}_x^T (\mu(\mathbf{RA})^T \mathbf{P} - \mathbf{P})) + 2y_i (\mathbf{e}_y^T (\mu(\mathbf{RA})^T \mathbf{P} - \mathbf{P}))
$$

$$
- 2\mu (x_i^2 (\mathbf{e}_x^T \mathbf{RA} \mathbf{e}_x) + x_i y_i (\mathbf{e}_x^T \mathbf{RA} \mathbf{e}_y + \mathbf{e}_y^T \mathbf{RA} \mathbf{e}_x)
$$

$$
+ y_i^2 (\mathbf{e}_y^T \mathbf{RA} \mathbf{e}_y)) + (1 + \mu^2)(x_i^2 + y_i^2).
$$

(11)

Define $\mathbf{w} = (w_1, w_2, w_3, w_4, w_5, w_6)$ as

$$
w_1 = \mathbf{P}^T \mathbf{P},
$$

(12)

$$
w_2 = 2\mu \mathbf{e}_x^T ((\mathbf{RA})^T \mathbf{P} - \mathbf{P}),
$$

(13)

$$
w_3 = 2\mu \mathbf{e}_y^T ((\mathbf{RA})^T \mathbf{P} - \mathbf{P}),
$$

(14)

$$
w_4 = -2\mu \mathbf{e}_x^T \mathbf{RA} \mathbf{e}_x,
$$

(15)

$$
w_5 = -2\mu ((\mathbf{e}_x^T \mathbf{RA} \mathbf{e}_y + \mathbf{e}_y^T \mathbf{RA} \mathbf{e}_x),
$$

(16)

$$
w_6 = -2\mu \mathbf{e}_y^T \mathbf{RA} \mathbf{e}_y
$$

(17)

and $\mathbf{d} = (d_1, d_2, d_3, d_4, d_5, d_6)$, where

$$
d_i = L_i^2 - (1 + \mu^2)(x_i^2 + y_i^2), \quad i = 1, 2, \ldots, 6.
$$

(18)

Then relation (11) can be written as a linear system with the form

$$
Q \mathbf{w} = \mathbf{d},
$$

(19)
where the matrix $Q$ is given by

$$Q = \begin{bmatrix}
1 & x_1 & y_1 & x_1 y_1 & y_1^2 \\
1 & x_2 & y_2 & x_2 y_2 & y_2^2 \\
1 & x_3 & y_3 & x_3 y_3 & y_3^2 \\
1 & x_4 & y_4 & x_4 y_4 & y_4^2 \\
1 & x_5 & y_5 & x_5 y_5 & y_5^2 \\
1 & x_6 & y_6 & x_6 y_6 & y_6^2
\end{bmatrix}. \tag{20}$$

Note that if the base points are all different and belong to a conic section then $\det Q = 0$. The matrix given by (20) corresponds to the well known Braikenridge-Maclaurin construction.

In the next sections we will show that one can obtain the rotation matrix $R$ and the position $P$ in terms of the solution $w = (w_1, w_2, \ldots, w_6)$ of the linear system given by (19). The solution to the forward kinematics problem naturally divides into two cases, namely, a non-singular case where $\det Q \neq 0$ and a singular case where $\det Q = 0$.

In the singular case, we obtain for a given set of length legs, $L_1, L_2, \ldots, L_6$, a singular solution parameterized by a scalar parameter. These solutions are curves in position space and in rotation space in which the platform moves without changing the values of the leg lengths.

### 2.1 Non-singular case

In the case where the six base vertices are not on a conic section, one gets $\det Q \neq 0$, and so the solution of (19), $w = (w_1, w_2, w_3, w_4, w_5)$, can be obtained from

$$w = Q^{-1}d. \tag{21}$$

The first three equations (12), (13) and (14) determines the rotation parameters, namely, $q$, and the last three (15), (16) and (17) the position $P = (x, y, z)$.

To determine the rotation parameters consider the equations

\begin{align*}
w_4 &= -2\mu \left(2q_1^2 - 2q_0^2 - 1\right), \tag{22} \\
w_5 &= -8\mu q_1 q_2, \tag{23} \\
w_6 &= -2\mu \left(2q_2^2 - 2q_0^2 - 1\right), \tag{24}
\end{align*}

which are obtained from (15), (16) and (17), respectively. Eliminating $q_0$, one gets,

\begin{align*}
q_1^2 - q_2^2 &= -(w_4 - w_6)/(4\mu), \tag{25} \\
q_1 q_2 &= -w_5/(8\mu). \tag{26}
\end{align*}
Let
\[ \alpha = \frac{w_4 - w_6}{4\mu}, \quad \beta = \frac{-w_5}{8\mu}. \] (27)

Then the above equations can be written as
\[ q_1^4 + \alpha q_1^2 - \beta^2 = 0, \] (28)
\[ q_2^4 - \alpha q_2^2 - \beta^2 = 0. \] (29)

So,
\[ q_1^2 = \frac{-\alpha + \gamma}{2}, \] (30)
\[ q_2^2 = \frac{\alpha + \gamma}{2}, \] (31)

where
\[ \gamma = \sqrt{\alpha^2 + 4\beta^2}. \] (32)

Substituting yields
\[ q_3^2 = \frac{1}{2} + \frac{w_4}{4\mu} - \frac{\alpha + \gamma}{2}, \] (33)
\[ q_0^2 = \frac{1}{2} - \frac{w_4}{4\mu} + \frac{\alpha - \gamma}{2}. \] (34)

Assuming \( q_0 \geq 0 \) and that (33) and (31) have two roots each, then, \( q_1 \) is determined by (23). Consequently, we have a total of four different quaternions. These are
\[ s_1 = (\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3), \] (35)
\[ s_2 = (\bar{q}_0, \bar{q}_1, \bar{q}_2, -\bar{q}_3), \] (36)
\[ s_3 = (\bar{q}_0, -\bar{q}_1, -\bar{q}_2, \bar{q}_3), \] (37)
\[ s_4 = (\bar{q}_0, -\bar{q}_1, -\bar{q}_2, -\bar{q}_3), \] (38)

where \((\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3)\) are the roots.

To determine the position, consider the equations
\[ u^T = 2\mu e_x^T((RA)^T - I), \] (39)
\[ v^T = 2\mu e_y^T((RA)^T - I). \] (40)

Thus
\[ p^T p = w_1, \] (41)
\[ u^T p = w_2, \] (42)
\[ v^T p = w_3. \] (43)
Obviously (42) and (43) represent two planes and their intersection is a line with equation given by
\[
P = r_0 + tr_1,
\]
where \(t\) is the parameter of the line. The vectors \(r_0\) and \(r_1\) are given by
\[
r_0 = \frac{(v^Tv)w_2 - (u^Tv)w_3}{(u^Tu)(v^Tv) - (u^Tv)^2}\ u - \frac{-(u^Tv)w_2 + (u^Tu)w_3}{(u^Tu)(v^Tv) - (u^Tv)^2}\ v,
\]
\[
r_1 = \frac{u \times v}{||u \times v||}.
\]
The line (44) intersects the sphere (41) at two points \(P_\pm\) given by
\[
P_\pm = r_0 \pm t^*r_1,
\]
where
\[
t^* = \sqrt{w_1 - r_0^Tr_0}.
\]
Note that in order to \(P_\pm\) exist one should have
\[
w_1 \geq r_0^Tr_0.
\]
So, both \(R\) and \(P\) are found, and totally they have eight possible different solutions for a given set of leg lengths.

### 2.2 Singular case

In this case, we assume that all points belong to a circle \(x_i^2 + y_i^2 = 1\) (we can assume \(r = 1\) without loss of generality), \(i = 1, 2, \ldots, 6\). In this case the matrix
\[
Q = \begin{pmatrix}
1 & x_1 & y_1 & x_1^2 & x_1y_1 & 1 - x_1^2 \\
1 & x_2 & y_2 & x_2^2 & x_2y_2 & 1 - x_2^2 \\
1 & x_3 & y_3 & x_3^2 & x_3y_3 & 1 - x_3^2 \\
1 & x_4 & y_4 & x_4^2 & x_4y_4 & 1 - x_4^2 \\
1 & x_5 & y_5 & x_5^2 & x_5y_5 & 1 - x_5^2 \\
1 & x_6 & y_6 & x_6^2 & x_6y_6 & 1 - x_6^2
\end{pmatrix}
\]
is singular, that is, \(\det Q = 0\) and in fact, if all points are different and belong to a conic section the rank of \(Q\) is five (corresponding to the Braikenridge-Maclaurin construction). This will be the case if \(x_i^2 + y_i^2 = 1, i = 1, 2, \ldots, 6\), and \((x_i, y_i) \neq (x_j, y_j)\) for \(i \neq j, i, j = 1, 2, \ldots, 6\).

This fact enables us to explicitly compute the \(LU\) factorization of the matrix \(Q\) in terms of the coordinate of the vertices of the base \((x_i, y_i),\)
$i = 1, 2, \ldots, 6$. These expressions are too big to be shown here but a script for the Maxima computer algebra system[4] is available upon request to the author.

So the linear system $Qw = d$ can be put into the form

$$Uw = L^{-1}d,$$  \hspace{1cm} (52)

where $\det L = 1$ and $U$ is a matrix with rank 5. The solution of (52) is given in terms of a solution $(w_2, w_3, w_4, w_5, w_6)$ which depends on the value of $w_1$, which we take to be a free parameter. Notice that any other quantity could be used for this purpose, although expression (41) suggests that $w_1$ is the good choice. So the expressions given by (30), (31), (33) and (34) can be used to determine the values of the quaternion $q$, the rotation matrix, and the point $P$ as a function of the free parameter $w_1$.

3 Conclusions

The singular manifold of a Stewart platform is defined by the physical configurations for which it will not be possible to determine the position of the platform uniquely by fixing the lengths of the legs. By considering a simple Stewart platform, for which the base vertices are in a circle (although the result naturally holds for any conic section) and the bottom and top plates are related by a rotation and a contraction, it was shown that the platform is always in a singular configuration. It was also shown how to characterize the singular manifold in this case and how it can be parameterized by a scalar parameter.

References

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