On M-Algebras, the Quantisation of Nambu-Mechanics, and Volume Preserving Diffeomorphisms

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Abstract: M-branes are related to theories on function spaces $\mathcal{A}$ involving $M$-linear non-commutative maps from $\mathcal{A} \times \cdots \times \mathcal{A}$ to $\mathcal{A}$. While the Lie-symmetry-algebra of volume preserving diffeomorphisms of $T^M$ cannot be deformed when $M > 2$, the arising $M$-algebras naturally relate to Nambu’s generalisation of Hamiltonian mechanics, e.g. by providing a representation of the canonical $M$-commutation relations, $[J_1, \cdots, J_M] = i\hbar$. Concerning multidimensional integrability, an important generalisation of Lax-pairs is given.

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1. Introduction

Generalizing fundamental concepts, such as Lie algebras or Hamiltonian dynamics, may have quite divers merits; it can lead to new, interesting possibilities, – or reassure oneself of our present notions. While the result that volume preserving diffeomorphisms of toroidal $M$-branes, as a Lie-symmetry algebra, cannot be deformed (if $M > 2$) is of the latter nature – the following ideas appear to be worthwhile pursuing:

— Using a $^*$-deformation of the algebra of functions on some $M$-dimensional manifold for representing the $M$-linear analogue to Heisenberg’s commutation relations that Nambu [1] encountered in multi-Hamiltonian dynamics.
— Generalizing the Jacobi identity for Lie algebras to a (2-bracket) identity involving $2M - 1$ elements of a vectorspace $V$ for which an antisymmetric $M$-linear map ($M$-commutator) from $V \times \cdots \times V$ to $V$ is defined (in a dynamical context, an identity involving $M$, rather than 2, of the $M$-commutators, may be preferred).
— A potential relevance of $M$-algebras to the quantisation of space-time.

Perhaps most importantly (on a concrete, practical level), an explicit example is given (the multidimensional diffeomorphism-invariant integrable field theories found in [2]) for the usefulness (envisioned some time ago [3]) of generalizing Lax-pairs to -triples, . . . .

2. M-algebras from M-branes

A relativistic M-brane moving in D-dimensional space time may be described, in a light-cone gauge, by the VDiff$\Sigma$-invariant sector of ([4])

$$H = \frac{1}{2} \int_{\Sigma} d^M \varphi \left( \vec{p}^2 + g \right)$$

(1)

where $g$ is the determinant of the $M \times M$ matrix $(g_{rs}) := (\nabla_r x^i \nabla_s x_i)_{r,s=1\cdots M}$, $x^i$ and $p_i$ ($i = 1, \cdots, D - 2 =: d$) are canonically conjugate fields, and $\rho$ is a fixed non-dynamical density on the $M$-dimensional parameter-manifold $\Sigma$ ($M = 1$ for strings, $M = 2$ for membranes, . . . ). Generators of VDiff$\Sigma$, the group of volume-preserving diffeomorphisms of $\Sigma$ (resp. the component connected to the identity), are represented by

$$K := \int_{\Sigma} f^r p_i \partial_r x^i d^M \varphi$$

(2)

with $\nabla_r f^r = 0$. $g$ may be written as

$$g = \sum_{i_1 < i_2 < \cdots < i_M} \{x_{i_1}, \cdots, x_{i_M}\}\{x^{i_1}, \cdots, x^{i_M}\},$$

(3)

where the ‘Nambu-bracket’ $\{\cdots\}$ is defined for functions $f_1, \cdots, f_M$ on $\Sigma$ as

$$\{f_1, \cdots, f_M\} := \epsilon^{r_1\cdots r_M} \partial_{r_1} f_1 \cdots \partial_{r_M} f_M.$$
This trivial, but important observation suggests to consider Hamiltonians

\[ H_\lambda := \frac{1}{2} \text{Tr} \left( \vec{P}^2 \pm \sum_{i_1 < \cdots < i_M} |X_{i_1}, \cdots, X_{i_M}|_\lambda^2 \right), \]  

(5)

resp.

\[ H_\lambda = \frac{1}{2} \sum_{i=1}^d \beta (P_i, P_i) + \frac{1}{2} \sum_{i_1 < \cdots < i_M} \beta \left( [X_{i_1}, \cdots, X_{i_M}]_\lambda, [X_{i_1}, \cdots, X_{i_M}]_\lambda \right), \]

(6)

where \( X^i \) and \( P_i \) are elements of (possibly finite dimensional, \( \lambda \)-dependent) vectorspaces \( V \) on which antisymmetric \( M \)-linear maps \([, \cdots, ]_\lambda : V \times \cdots \times V \to V\) are defined, and \( \beta \) a positive definite hermitean form, preferably invariant with respect to some analogue of volume preserving diffeomorphisms (cp. (2)).

With

\[ [T_{a_1}, \cdots, T_{a_M}]_\lambda = f^a_{a_1 \cdots a_M} (\lambda) T_a \]

(7)

and

\[ \beta(T_a, T_b) = \delta^a_b \]

(8)

for some (possibly \( \lambda \)-dependent) basis \( \{T_a\}_{a=1}^{\dim V} \) of \( V \), i.e.

\[ f^a_{a_1 \cdots a_M} (\lambda) = \beta(T_a, [T_{a_1}, \cdots, T_{a_M}]_\lambda), \]

(9)

(6) reads

\[ H_\lambda = \frac{1}{2} p^*_i p_i + \frac{1}{2} (f^a_{a_1 \cdots a_M} (\lambda))^* f^b_{b_1 \cdots b_M} (\lambda) \]

\[ \frac{1}{M!} x^*_{i_1 a_1} \cdots x^*_{i_M a_M} x_{i_1 b_1} \cdots x_{i_M b_M}, \]

(10)

while (1) may be written as

\[ H = \frac{1}{2} p^*_i p_i + \frac{1}{2} (g^a_{a_1 \cdots a_M})^* g^b_{b_1 \cdots b_M} \]

\[ \frac{1}{M!} x^*_{i_1 a_1} \cdots x^*_{i_M a_M} x_{i_1 b_1} \cdots x_{i_M b_M}; \]

(11)

\[ g^a_{a_1 \cdots a_M} := \int_{\Sigma} Y^*_\alpha \{Y_{\alpha_1}, \cdots, Y_{\alpha_M}\} \rho^M \varphi \]

(12)

is defined with respect to some orthonormal basis of functions (on \( \Sigma \)) satisfying

\[ \int_{\Sigma} Y^*_\alpha Y_\beta \rho^M \varphi = \delta^\alpha_\beta \]

\[ \alpha, \beta = 1 \cdots \infty \]

(13)

(even for real \( x_i \), it is often convenient to take a complex basis).

Obvious questions are:
1) Does there exist a ‘natural’ sequence of finite dimensional vectorspaces \( V_n \) with basis \( \{ T^{(n)}_a \} \) and antisymmetric maps \( F_n : V_n \times \cdots \times V_n \to V_n \) such that for each \((M + 1)\)-tuple \((a a_1 \cdots a_M)\)

\[
\lim_{n \to \infty} f_{a_1 \cdots a_M}^a (\lambda_n) = g_{a_1 \cdots a_M}^a.
\]

(14)

2) For which \( M \) do there exist finite dimensional analogues of (2), \( K(n) \), leaving (10)\( \lambda_n \) invariant, such that, as \( n \to \infty \), the full invariance under volume-preserving diffeomorphisms is recovered?

3) What about \( \lambda \)-deformations with infinite dimensional \( V \)'s?

Let us look at the case of a \( M \)-torus, \( \Sigma = T^M \):

Choosing \( Y_{\vec{m}} = e^{i \vec{m} \varphi} \), \( \vec{m} = (m_1, \cdots, m_M) \in \mathbb{Z}^M \), \( \rho \equiv 1 \),

one gets

\[
g_{\vec{m}_1 \cdots \vec{m}_M} = i^M (\vec{m}_1, \cdots, \vec{m}_M) \delta_{\vec{m}_1 + \cdots + \vec{m}_M}^\rho
\]

(16)

where \( (\vec{m}_1, \cdots, \vec{m}_M) \in \mathbb{Z}^M \) denotes the determinant of the corresponding \( M \times M \) Matrix (an element of \( GL(M, \mathbb{Z}) \)).

Consider now the following ‘\(*M\)-product’ (a deformation of the ordinary commutative product of \( M \) functions \( f_1, \cdots, f_M \) on \( \Sigma \)):

\[
(f_1 \cdots f_M)_* := f_1 \cdots f_M + \sum_{m=1}^{\infty} (-i)^m \frac{1}{m!} \left( \frac{e^{i \lambda M}}{\lambda M} \right)^m \epsilon_{r_1 \cdots r_{M}}^m \epsilon_{r_{M+1} \cdots r_M}^m \frac{\partial^m f_1}{\partial \varphi_{r_1} \cdots \partial \varphi_{r_m}} \cdots \frac{\partial^m f_M}{\partial \varphi_{r_{M+1}} \cdots \partial \varphi_{r_M}}.
\]

(17)

One then finds that

\[
(Y_{\vec{m}_1} \cdots Y_{\vec{m}_M})_* = \sqrt{\omega}^{- (\vec{m}_1, \cdots, \vec{m}_M)} Y_{\vec{m}_1 + \cdots + \vec{m}_M} = e^{i \sum \vec{m}}.
\]

(18)

Defining

\[
[f_1, \cdots, f_M]_* := \sum_{\sigma \in S_M} (\text{sign } \sigma) (f_{\sigma 1} \cdots f_{\sigma M})_*
\]

(19)

to simply be the antisymmetrized \( *M \)-product, one gets

\[
[T_{\vec{m}_1}, \cdots, T_{\vec{m}_M}] = \frac{-i}{2 \pi \Lambda} \sin (2 \pi \Lambda (\vec{m}_1, \cdots, \vec{m}_M)) T_{\vec{m}_1 + \cdots + \vec{m}_M}
\]

(20)

with \( \Lambda := \frac{\lambda}{2 \pi M!} \) and \( T_{\vec{m}} := \lambda^{- \frac{1}{\lambda M}} Y_{\vec{m}} \).

For \( M > 1 \) arbitrary (but fixed), let \( V \) denote the vectorspace (over \( \mathbb{C} \)) generated by \( \{ T_{\vec{m}} \}_{\vec{m} \in \mathbb{Z}^M} \), \( V^\Lambda \) denote \( (V, *) \) and \( A^{\Lambda} \) denote \( (V, [\cdots])_* \).
The hermitean form $\beta$ (cp. (8),(9)),

$$\beta(T_{\vec{m}}, T_{\vec{n}}) = \delta_{\vec{m} \vec{n}}^\Lambda; \quad \beta(c_i X_i, d_j X_j) = c_i^* d_j \beta(X_i, X_j),$$

will have the important property ('invariance') that (for $X_i = x_{im} T_{\vec{m}}$ with $x_{im}^* = x_{i-\vec{m}}$)

$$\beta(X, [X_{i_1}, \ldots, X_{i_M}]) = -\beta(X_{i_r}, [X_{i_1}, \ldots, X_{i_{r-1}}, X, X_{i_{r+1}}, \ldots, X_{i_M}]),$$

as

$$\beta(T_{\vec{m}}, [T_{\vec{m}_1}, \ldots, T_{\vec{m}_M}]) = \frac{-i}{2\pi \Lambda} \delta_{\vec{m}_1,+,\ldots,\vec{m}_M} \sin(2\pi \Lambda (\vec{m}_1, \ldots, \vec{m}_M)).$$

For rational $\Lambda = \frac{\tilde{N}}{N}$ (assuming $N$ and $\tilde{N} < N$ having no common divisor $> 1$) both $A^\Lambda$ and $M^\Lambda$ may be divided by an ideal of finite codimension, namely (using the periodicity of the structure-constants) the vectorspace I generated by all elements of the form $T_{\vec{m}} - T_{\vec{m}+N}$ (anything). One thus arrives at considering (for arbitrary odd $N$)

$$V^{(N)} := \langle T_{\vec{m}} | m_r = -\frac{N-1}{2}, \ldots, +\frac{N-1}{2} \rangle_{C} \quad r = 1 \cdots M \quad (21)$$

with a $*_M$ product on $V^{(N)}$ defined just as in (18):

$$(T_{\vec{m}_1} \cdots T_{\vec{m}_M})_* := \frac{-i N}{2\pi \tilde{N} M!} \omega^{-\frac{1}{2}} (\vec{m}_1, \ldots, \vec{m}_M) T_{\vec{m}_1,+,\ldots,\vec{m}_M} (\text{mod } N)$$

$$\omega = e^{4\pi i \frac{\tilde{N}}{N}}, \quad (22)$$

and a corresponding alternating product,

$$[T_{\vec{m}_1}, \ldots, T_{\vec{m}_M}]_* = \frac{-i N}{2\pi \tilde{N}} \sin\left(2\pi \frac{\tilde{N}}{N} (\vec{m}_1, \ldots, \vec{m}_M)\right) T_{\vec{m}_1,+,\ldots,\vec{m}_M} (\text{mod } N) \quad (23)$$

The 'structure constants' of the alternating finite dimensional $M$-algebras

$$A^\Lambda_N := (V^{(N)}, [,\ldots, ]_*), \quad f^{(N)}_{\vec{m}_1,\ldots,\vec{m}_M} := \frac{-i N}{2\pi \tilde{N}} \sin\left(2\pi \frac{\tilde{N}}{N} (\vec{m}_1, \ldots, \vec{m}_M)\right) \cdot \delta_{\vec{m}_1,+,\ldots,\vec{m}_M} (\text{mod } N) \quad (24)$$

satisfy (14) (up to an $N$ and $\mathbb{Z}_N^M$-independent rescaling of the generators, resp. factors of $i$, which anyway drop out in (10) and (11); $n = N^M$, $f^{(N)} \overset{\Lambda}{=} f(\lambda_n), \vec{m} \in \mathbb{Z}_N^M, V^{(N)} = V_{n=N^3}$, and $\lim_{N \to \infty} V^{(N)} = V$).

$$H_N = \frac{1}{2} \sum_{i=1}^{M} x_{i_1 - \vec{m}_1} \cdots x_{i_M - \vec{m}_M} x_{i_1} \cdots x_{i_M} \delta_{\vec{m}_1,+,\ldots,\vec{m}_M} (\text{mod } N) \quad (25)$$

could therefore be considered as a finite-dimensional analogue of (1).
3. Multidimensional Commutation Relations

Before turning to questions of symmetry, let me discuss in a little more detail the \(*\)-algebras \(\mathbb{M}^A\), with defining relations (cp. (18); note the slight change of notation/normalisation)

\[
(T_{\vec{m}_1} \cdots T_{\vec{m}_M})_* = \omega^{-\frac{1}{2}} (\vec{m}_1, \cdots, \vec{m}_M) T_{\vec{m}_1+\cdots+\vec{m}_M} (*),
\]

and as vectorspaces generated by basis-elements \(T_{\vec{m}}, \vec{m} \in S\) (where \(S = \mathbb{Z}^M\), \(S = (\mathbb{Z}_N)^M\), or any combination thereof – in the M-brane context, depending on whether \(\Sigma = T^M\), resp. a fully, or partially, discretized M-torus).

First of all note, that for any \(M\) elements, \(A_1, \cdots, A_M \in V\), any even permutation \(\sigma \in S_M\) (the symmetric group in \(M\) objects), and any choice of \(S\) (even \(\mathbb{R}^M\)),

\[
(A_1 \cdots A_M)_* = (A_{\sigma(1)} \cdots A_{\sigma(M)}) \quad \text{(sign } \sigma = +),
\]

and that \(E := T_0\) acts as a ‘unity’ in the sense that if one of the \(A_r\) is equal to \(T_0\), the \(*\)-product becomes commutative (i.e. independent of the order of its \(M\) entries).

Using \(E\), one may identify \(T_{(m=\pm|m|,0,\ldots,0)}\) with the \(|m|\)-th power of \(E_{\pm 1} := T_{(\pm 1,0,\ldots,0)}\),

\[
T_{(m,0,\ldots,0)} = (((((E \cdots E E_{\pm 1}), \cdots E E_{\pm 1}))))_*,
\]

so that one may wonder whether \(\mathbb{M}^A\) can be thought of as being generated by

\[
E = T_0, \quad E_{\pm 1} = T_{(\pm 1,0,\ldots,0)}, \quad E_{\pm M} = T_{(0,\ldots,0, \pm 1)}.
\]

This is indeed the case: Let \(\mathbb{F}^M\) be the free (non associative) \(M\)-algebra generated by \(2M + 1\) elements \(E, E_{\pm 1}, \cdots, E_{\pm M}\); define arbitrary powers \((E_r)^m\) of the generating elements as in (27) (from now on \(E_{-r}^-m =: E_r^-m\), a notation that will be justified via (29)), and let

\[
E_{\vec{m}} := E_{m_1}^m E_{m_2}^{m_2} \cdots E_{m_M}^{m_M}.
\]

Divide \(\mathbb{F}^M\) by the ideal generated by elements

\[
E_{\vec{m}'} E_{\vec{m}''} \cdots E_{\vec{m}^{(M)}} = \omega^{2\gamma(\vec{m}', \vec{m}'', \cdots, \vec{m}^{(M)})} \cdot E_{\vec{m}'+\cdots+\vec{m}^{(M)}}
\]

where \(\omega = e^{4\pi i A}\) and

\[
2\gamma(\vec{m}', \cdots, \vec{m}^{(M)}) := (m_1 \cdot m_2 \cdot \cdots \cdot m_M) - (\vec{m}', \vec{m}'', \cdots, \vec{m}^{(M)}) - \sum_{r=1}^M M \prod_{s=1}^M m_s^{(r)}
\]

This quotient then is isomorphic to \(\mathbb{M}^A\), as can be seen by defining

\[
T_{\vec{m}} := \omega^{\frac{1}{2} m_1 m_2 \cdots m_M} E_{-m_1}^{m_1} E_{-m_2}^{m_2} \cdots E_{-m_M}^{m_M},
\]
which (due to (29) being zero in $\mathbb{F}^A/I$) satisfies (18) (with $Y$ standing for $T$).

Note that
\[
E_2^{m_2} E_1^{m_1} E_3^{m_3} \cdots E_M^{m_M} = \omega^{m_1 m_2 \cdots m_M} \cdot E_1^{m_1} E_2^{m_2} \cdots E_M^{m_M},
\]

in particular:
\[
E_2 E_1 E_3 \cdots E_M = \omega E_1 E_2 \cdots E_M
\]

(while any even permutation does not alter the product, cp. (26)).

In order to get a feeling for (29)/(30) it may be instructive to consider $M = 3$: due to (29),
\[
\begin{align*}
(E_1^{n_1} E_2^{n_2} E_3^{n_3})(E_1^{l_1} E_2^{l_2} E_3^{l_3})(E_1^{k_1} E_2^{k_2} E_3^{k_3}) &= E_1^{n_1+l_1+k_1} E_2^{n_2+l_2+k_2} E_3^{n_3+l_3+k_3} \\
&\quad \cdot \omega^{n_1 l_2 k_3 + n_2 l_1 k_3 + n_3 l_1 k_2} \\
&\quad \cdot \sqrt{\omega}^{n_1 l_2 (l_3 + k_3) + n_2 (l_1 l_3 + k_3) + n_3 (l_1 l_2 + k_1 l_2)} \\
&\quad \cdot \sqrt{\omega}^{n_1 n_2 (l_3 + k_3) + n_1 n_3 (l_2 + l_3) + n_2 n_3 (l_1 + l_2)}
\end{align*}
\]

The general rule (30) can hence be stated as follows:

Consider all possible triples (resp. $M$-tuples) containing powers of each of the $E_r (r = 1 \cdots M)$ exactly once. If the ‘contraction’ picks out exactly one factor from each of the 3 (resp. $M$) factors in (34) it does not contribute if they are already in the correct order, modulo even permutations (cp. 26), (like $E_1^{n_1} E_2^{l_2} E_3^{l_3}$, or $E_2^{n_2} E_3^{l_2} E_1^{l_1}$), while they contribute a factor $\omega^{(\text{product of the $E$-powers})}$, when they are not in the correct (modulo even permutation) order (like $E_2^{n_2} E_1^{l_1} E_3^{l_3}$). Contractions entirely within one of the factors don’t contribute, while mixed contractions (involving at least 2, but not all, of the factors in (34)), all contribute a factor $\sqrt{\omega}^{(\text{product of the $E$-powers})}$, irrespective of their order.

Due to (32), all ‘monomials’ are proportional to one of the elements $E_{\bar{m}}$ (cp. (28)) – which therefore form a basis (with the convention $E_{\bar{0}} \equiv E$). Note that $2\pi M! \Lambda = \lambda \to 0$ is a ‘classical limit’ (resp. $\lambda \neq 0$ a ‘quantisation’ of the classical Nambu-structure) as, formally,
\[
[\ln E_1, \ln E_2, \cdots, \ln E_M] = i \lambda E.
\]

Having obtained this relation, one could of course start with objects $\ln E_r =: J_r$, $[J_1, J_2, \cdots, J_M] = i \lambda E$, and derive generalized ‘Hausdorff-formulae’ for products involving the $e^{i m_r J_r}$.

Of course, (35) cannot be true in any $M$-algebra containing only finite linear combinations of the basis-elements $E_{\bar{m}}$, as $T_0 = E$ never appears on the r.h.s. of (20); this is similar to the fact that the canonical commutation relations of ordinary quantum mechanics, $[q, p] = i \hbar I$, cannot hold for trace-class operators. (35) may be justified by formally expanding $\ln E_r = -\sum_{n_r=1}^{\infty} \sum_{k_r=0}^{n_r} \binom{n_r}{k_r} (-1)^{k_r} n_r^{k_r} E_r^{k_r}$, using
\[
[E_1^{k_1}, E_2^{k_2}, \cdots, E_M^{k_M}] = \frac{M!}{2} (1 - \omega^{k_1 \cdots k_M}) E_1^{k_1} \cdots E_M^{k_M}
\]
and then resumming recursively, after the first step obtaining
\[
\frac{M!}{2} \ln E_1 \cdots \ln E_M = \frac{M!}{2} \sum_{r > 1} \frac{\ln (E_1 \omega_k^{x \cdots k_M})}{\ln E_r} E_k^{E_r} = \frac{M!}{2} (\ln \omega) \cdot E,
\] (36)
as formally,
\[
\sum_{n_r=1}^{\infty} \sum_{k_r=1}^{n_r} \left( \frac{(-1)^{k_r}}{n_r} \right) k_r E_r^k = E_r \cdot \sum_{n'=0}^{\infty} (E - E_r)^{n'} = E.
\]

4. Breakdown of Conventional Symmetries

Let us now discuss the question, whether theories like (5) or (6) can have symmetries reminiscent of volume preserving diffeomorphisms; in particular whether the generators (2) may be ‘translated’ to finite dimensional analogues. For simplicity, consider again \( \Sigma = T^M \).

As \( f^r = \partial_s \omega^r = \epsilon^r \omega_{1 \cdots r-1} \partial_s \omega_{r-2} \) for non-constant (divergence-free) vector-fields on \( T^M \), (2) may be written in the form
\[
K_{r_1 \cdots r_{M-2}} = \int d^M \varphi \omega_{r_1 \cdots r_{M-2}} \{ p_i, x^i, \varphi^{r_1}, \cdots, \varphi^{r_{M-2}} \},
\] (37)
resp., in Fourier-components,
\[
K^\tilde{r}_{r_1 \cdots r_{M-2}} = \sum_{\tilde{m}, \tilde{n} \in \mathbb{Z}^M} \delta_{\tilde{m}+\tilde{n}} p_{\tilde{m}} x_{\tilde{n}} \{ \tilde{m}, \tilde{n}, \tilde{e}_{r_1}, \cdots, \tilde{e}_{r_{M-2}} \}.
\] (38)
(where \( \tilde{e}_r \) denotes the unit vector in the \( r \)-direction).

Suppose the deformed theory was invariant under transformations that are still generated in a conventional way by phase-space functions of the form
\[
K^{\tilde{r}} = \sum_{\tilde{m}, \tilde{n} \in S} c_{\tilde{m}+\tilde{n}} \delta_{\tilde{m}+\tilde{n}} c_{\tilde{m}+\tilde{n}}.
\] (39)

Using \( [x_{\tilde{m}\tilde{n}}, p_{\tilde{j}\tilde{n}}] = \delta_{\tilde{i}j} \delta_{\tilde{m}+\tilde{n}}, \) while leaving open whether \( S = \mathbb{Z}^M \) or \( S = (\mathbb{Z}_N)^M \) as well as (independently) whether \( \delta \) is defined \( \text{mod} \ N \), or not, one has
\[
[K^{\tilde{r}}, K^{\tilde{r}'}] = \sum_{\tilde{m}, \tilde{n} \in S} p_{\tilde{m}} x_{\tilde{n}} \delta_{\tilde{m}+\tilde{n}} c_{\tilde{m}+\tilde{n}}
\approx c_{\tilde{m}+\tilde{n}} = \sum_{\tilde{k} \in S} \left( \delta_{\tilde{k}\tilde{m}} - \delta_{\tilde{k}\tilde{n}} \right) c_{\tilde{m}+\tilde{n}} \tilde{c}_{\tilde{k}} - \left( \tilde{c} \leftrightarrow \tilde{c}' \right),
\] (40)
\footnote{For \( M = 2 \), this question was already considered in [4] and answered positively.}
while $\dot{K}^r = 0$ would require $c_{\vec{m}\vec{n}} = - c_{\vec{n}\vec{m}}$ and

$$
\sin (2\pi \Lambda (\vec{a}_1, \cdots, \vec{a}_M)) \sin (2\pi \Lambda (\vec{a}_1 + \cdots + \vec{a}_M, \vec{a}_2', \cdots, \vec{a}_M'))
\cdot c_{\vec{a}_1, \cdots, \vec{a}_i, \cdots, \vec{a}_M', \vec{a}_1'} \cdot x_i x_{i'} x_{i_1} x_{i_1'} \cdots x_{i_M} \vec{a}_M x_{i_M} \vec{a}_M' = 0
$$

(41)

(where for (41) consistency of the $\delta$-functions used in (39) and (25)$_\Lambda$ with the index set $S$ was assumed).

The effect of the $x_{i\vec{m}}$-factors in (41) is to make the product $\sin \cdot \sin \cdot c$, symmetric under any interchange $\vec{a}_r \leftrightarrow \vec{a}_r'$, as well as any simultaneous interchange $\vec{a}_r \leftrightarrow \vec{a}_s$, $\vec{a}_r' \leftrightarrow \vec{a}_s'$. Choosing, e.g., $\vec{a}_r' = \vec{a}_r (r = 1 \cdots M)$, with $\sin (2\pi \Lambda (\vec{a}_1 \cdots \vec{a}_M)) \neq 0$, (41) requires that

$$
\sum_{\sigma \in S_M} c_{\vec{a}_{\sigma 1} + 2(\vec{a}_{\sigma 2} + \cdots + \vec{a}_{\sigma M}), \vec{a}_{\sigma 1}} = 0.
$$

(42)

This condition is insensitive to any alteration of the deformation: replacing the sine-function in (41) (resp. (25)$_\Lambda$, $\cdots$) by any other function of the determinant will still result in (42) as a necessary condition. Apart from $M = 2$ ($c_{\vec{a}_1 + 2\vec{a}_2, \vec{a}_1} + c_{\vec{a}_2 + 2\vec{a}_1, \vec{a}_2} = 0$ is trivially satisfied by any odd function) (42) is not satisfied by

$$
c_{\vec{m}\vec{n}} = \sin (2\pi \Lambda (\vec{m}, \vec{n}, \vec{k}_1, \cdots, \vec{k}_{M-2}))
$$

(43)

- - nor would (40) be a linear combination of the generators (39), for such a $c_{\vec{m}\vec{n}}$; for $M = 3$, e.g., one would obtain

$$
\approx c_{\vec{m}\vec{n}}(\vec{l}\vec{l}'; \vec{k}\vec{k}') = \sin (2\pi \Lambda (\vec{l}, \vec{l}', \vec{k} + \vec{k}'))
\cdot \sin (2\pi \Lambda \left(\left(\vec{m} - \vec{n}, \vec{k} - \vec{k}' + \vec{l} - \vec{l}' - \vec{l} + \vec{l}'\right) / 2 \right))
\cdot \sin (2\pi \Lambda \left(\vec{m} - \vec{n}, \vec{k} + \vec{k}' - \vec{l} + \vec{l}'\right) / 2)
$$

(44)

- - which means that the algebra closes only for $\vec{k}' = \vec{k}$ (for $\Lambda = \frac{1}{N}$ this would give $N^3$ closed Lie algebras, each of dimension $N^3$; in fact, each consisting of $N$ copies of $gl(N)$).

- In any case, if $c_{\vec{m}\vec{n}}$ was a function of $(\vec{m}_1 \vec{n}_1 \vec{k}_1, \cdots, \vec{k}_{M-2})$, one could let $\vec{a}_2, \vec{a}_3, \cdots \vec{a}_M$ differ only in the (irrelevant') $\vec{k}_1, \cdots, \vec{k}_{M-2}$ directions and obtain

$$
f ((2M - 2)\vec{a}_2, \vec{a}_1, \cdots)) = (M - 1) f ((2\vec{a}_1, \vec{a}_2, \cdots)) = 0,
$$

(45)

which eliminates all $c_{\vec{m}\vec{n}}$ that are non-linear functions of the determinant.

Interestingly, $c_{\vec{m}\vec{n}} = (\vec{m}, \vec{n}, \text{something})$ if $M > 2$ is suggested by yet another consideration: replacing $\{p_i, x_i, \varphi^3, \cdots, \varphi^M\}$ (cp. (37); for notational simplicity taking $r_1 = 3, \cdots, r_{M-2} = M$) by

$$
[P_i, X_i, \ln E_3, \cdots, \ln E_M],
$$

(46)
(with \( P_i = p_{i\bar{n}}T_{\bar{m}}, X_i = x_{i\bar{m}}T_{\bar{n}} \)) formally expanding the logarithms in a power series, using (20), and then (formally) summing again, one obtains something proportional to

\[ p_{i\bar{n}} x_{i\bar{n}} T_{\bar{m}+\bar{n}} \cdot (m_1 n_2 - m_2 n_1) \tag{47} \]

\[
\begin{align*}
[P_i, X_i, \ln E_3, \ldots, \ln E_M] & = p_{i\bar{n}} x_{i\bar{n}} (-)^{M-2} \sum_{n_3=1}^{\infty} \left( \sum_{k_3=0}^{n_3} \sum_{n_M=1}^{\infty} \sum_{k_M=0}^{n_M} \frac{(n_3)}{(k_3)} \frac{(-)^{k_3+\cdots+k_M}}{n_3 \cdots n_M} \right) \\
& \quad \cdot [T_{\bar{m}}, T_{\bar{n}}, E_3^k, \ldots, E_M^k] \\
& \sim \sum \cdots \sin (2\pi \lambda (\bar{m}, \bar{n}, k_3 \bar{e}_3, \ldots, k_M \bar{e}_M)) \cdot T_{\bar{m}+\bar{n}+\bar{k}} \\
& \sim \sum \cdots \left( \sqrt{\omega}^{k_3-\cdots-k_M} z - \sqrt{\omega}^{-k_3-\cdots-k_M} z \right) \left( \sqrt{\omega} \prod_{r=1}^{M} (m_r + n_r + k_r) \right) \\
& \quad \cdot E_1^{m_1+n_1} E_2^{m_2+n_2} E_3^{m_3+n_3+k_3} \cdots E_M^{m_M+n_M+k_M} \\
& \sim \sum \cdots \left( \ln \left( \sqrt{\omega}^{k_3-\cdots-k_M} z + \prod_{r \neq 3} (m_r + n_r + k_r) \right) \right) \\
& \quad - \ln \left( \sqrt{\omega}^{-k_3-\cdots-k_M} z + \prod_{r \neq 3} (\cdots) \right) \cdot \sqrt{\omega}^{(m_3+n_3) \prod_{r \neq 3} (\cdots)} \\
& \quad \cdot E_1^{m_1+n_1} E_2^{m_2+n_2} E_3^{m_3+n_3} E_4^{m_4+n_4+k_4} \cdots E_M^{m_M+n_M+k_M} \\
& \quad \left( z := (\bar{m}, \bar{n}, \bar{e}_3, \ldots, \bar{e}_M) = (m_1 n_2 - m_2 n_1) \right) \\
& \sim (\ln \omega) p_{i\bar{n}} x_{i\bar{n}} z (\bar{m}, \bar{n}) \sqrt{\omega} \prod_{r=1}^{M} (m_r + n_r) \cdot E_1^{m_1+n_1} \cdots E_M^{m_M+n_M} \\
& \sim (m_1 n_2 - m_2 n_1) p_{i\bar{n}} x_{i\bar{n}} (\ln \omega) \cdot T_{\bar{m}+\bar{n}} \\
& \quad \text{where (for } r > 3) - \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{(n)}{(k)} \frac{(-)^{k}}{n} k \cdot E_r^k \cdot (\omega^{-})^k = E \text{ was used.}
\end{align*}
\]

However, \( c_{\bar{m}\bar{n}} = (\bar{m}, \bar{n}, \text{ anything}) \) does not satisfy (41). Moreover, even if one considers more general deformations of the Hamiltonian, i.e. replacing the sine-function in (41) by an arbitrary odd (power-series) function \( f \) of the determinant, the corresponding condition,

\[ f (\bar{a}_1, \ldots, \bar{a}_M) f (\bar{a}_1 + \cdots + \bar{a}_M, \bar{a}_2', \ldots, \bar{a}_M') \cdot (\bar{e}', \bar{a}_1', \cdots) = 0 \]

\[ + (M \cdot 2^M - 1) \text{ permutations,} \tag{49} \]

\( \bar{e}' = \sum_{r=1}^{M} (\bar{a}_r + \bar{a}_r') \), can never be satisfied by any non-linear function \( f \) – as on can see, e.g., by choosing \( \bar{a}_r = \mu_r \bar{a}_r \). Supposing \( f(x) = \alpha x + \beta x^{2n+1} = \cdots \), and denoting \( (\bar{a}_1, \cdots, \bar{a}_M) \) by \( z \), \( \prod_{r=1}^{M} \mu_r \) by \( \mu \), the terms \( \mu_1, \alpha z \beta (\mu z)^{2n+1} \), e.g., (occurring only twice, with the same sign) could never cancel.
The preceding arguments possibly suffice to prove that, independent of the above dynamical context, the Lie algebra of volume-preserving diffeomorphisms of $T^{M>2}$ does not possess any non-trivial deformations.*

5. Rigidity of Canonical Nambu-Poisson Manifolds

For the multilinear antisymmetric map (4), and $2M - 1$ arbitrary functions $f_1, \cdots, f_{2M-1}$, one has (cp. [5]):

\[
\{\{f_M, f_1, \cdots, f_{M-1}\}, f_{M+1}, \cdots, f_{2M-1}\} + \{f_M, \{f_{M+1}, f_1, \cdots, f_{M-1}\}, f_{M+2}, \cdots, f_{2M-1}\} + \cdots + \{f_M, \cdots, f_{2M-2}, \{f_{2M-1}, f_1, \cdots, f_{M-1}\}\} = \{\{f_M, \cdots, f_{2M-1}\}, f_1, \cdots, f_{M-1}\}. \tag{50}
\]

Takhtajan [5], stressing its relevance for time-evolution in Nambu-mechanics [1], named (50) ‘Fundamental Identity (FI)’, and defined a ‘Nambu-Poisson-manifold of order $M$’ to be a manifold $X$ together with a multilinear antisymmetric map $\{\cdots\}$ satisfying (50) and the Leibniz-rule

\[
\{f_1\tilde{f}_1, f_2, \cdots, f_M\} = f_1\{\tilde{f}_1, f_2, \cdots, f_M\} + \{f_1, \cdots, f_M\} \tilde{f}_1 \tag{51}
\]

for functions $f_r : X \rightarrow \mathbb{R}$ (or $\mathbb{C}$).

Without (51), i.e. just demanding (50) for an antisymmetric $M$ linear map: $V \times \cdots \times V \rightarrow V$, $V$ some vectorspace, Takhtajan defines a ‘Nambu-Lie-gebra’ [5], – also called ‘Filippov [6] Lie algebra’ [7]). I would like to point out various other identities satisfied by canonical Nambu-Poisson brackets (4), and show that all of them – including (50)! – do not allow deformations (of certain natural type), if $M > 2$.

At least from a non-dynamical point of view, all identities involving Nambu-brackets obtained from antisymmetrizing the product of two determinants formed from $2M$ $M$-vectors,

\[
(a_1 \cdots a_M)(\bar{a}_{M+1} \cdots \bar{a}_{2M}) \tag{52}
\]

with respect to $M + 1$ of the $\bar{a}_\alpha (\alpha = 1 \cdots 2M)$ should be treated on an equal footing. For $M = 3$, e.g., one has – apart from

\[
(a \bar{b} \bar{c}_1)(\bar{c}_2 \bar{c}_3 \bar{c}_4) - (a \bar{b} \bar{c}_2)(\bar{c}_3 \bar{c}_4 \bar{c}_1) + (a \bar{b} \bar{c}_3)(\bar{c}_4 \bar{c}_1 \bar{c}_2) - (a \bar{b} \bar{c}_4)(\bar{c}_1 \bar{c}_2 \bar{c}_3) = 0, \tag{53}
\]

which gives rise to (50)$_{M=3}$ for functions $f \in T^3$ – also

\[
(a \bar{c}_{1} \bar{c}_2)(\bar{c}_3 \bar{c}_4 \bar{b}) = 0, \tag{54}
\]

* M. Bordemann has informed me that apparently an even more general statement of this nature has recently been proven in [19].
yielding the following 6-term identity (FI)_6 (which can of course also be proven by using just the definition (4), \( \{f, g, h\} = \epsilon_{\alpha\beta\gamma} \partial_\alpha f \partial_\beta g \partial_\gamma h \), rather than (54); i.e. not necessarily specifying the manifold \( X \)):

\[
\{\{f, f_1, f_2\} f_3, f_4\} = 0
\]

(55)
as well as the 4-term identity (\( \widetilde{\text{FI}} \)),

\[
\{\{f, f_1, f_2\}, g, f_3\} + \{\{f, f_2, f_3\}, g, f_1\} + \{\{f, f_3, f_1\}, g, f_2\} = - \{f, g, \{f_1, f_2, f_3\}\}
\]

(56)

- - each of which is independent of \((50)_{M=3} \) (while any 2 of the 3 identities yield the 3\textsuperscript{rd}).

Naively, one would think that (56) should follow from \((50)_{3} \) alone, as (54) follows from (53) (perhaps one should note that for general \( M \), a theorem concerning vector invariants [8] states, that any (!) vector-bracket identity is an algebraic consequence of \((\vec{a}[1 \vec{a}[2] \cdots \vec{a}M)(\vec{a}_{M+1} \cdots \vec{a}_{2M}) = 0 \);

however, in the proof of (56) via vector-bracket identities, one in particular needs (54) for the special case \( \vec{a} = \vec{b} \) – which cannot be stated as an identity between functions on \( X \).)

Curiously (with respect to a statistical approach to \( M \)-branes), vector-bracket identities (‘Basis Exchange Properties’ [9]) also play an important role in combinatorial geometry.

¿From an aesthetic point of view, the most natural quadratic identity for (4) is

\[
\sum_{\sigma \in S_{2M-1}} (\text{sign } \sigma) \{\{f_{\sigma 1}, \cdots, f_{\sigma M}\} f_{\sigma M+1}, \cdots, f_{\sigma 2M-1}\} = 0.
\]

(57)

For \( M = 3 \), e.g., one could see this to be a consequence of \((50)_{3} \) and (56) by grouping the 10 distinct terms in (57) according to whether \( \{f_{\sigma 1}, f_{\sigma 2}, f_{\sigma 3}\} \) contains both \( f_4 \) and \( f_5 \) (3 terms, ‘type A’), only one of them (3 ‘B-terms’ and 3 ‘C-terms’) or none of them (1 term, ‘type D’); for the B (resp. C)-terms one can use (56) while (50) for the A-terms, to get \( \pm \{f_4, f_5, \{f_1 f_2 f_3\}\} \) for each of the 4 types, and for the B and C-terms with a sign opposite to the one obtained from the D (and A) term(s). (57) (taken without the derivation-requirement) is a beautiful generalisation of Lie-algebras \((M = 2) \), and has recently started to attract the attention of mathematicians – mostly under the name of \((M - 1)\)-ary Lie algebras [10 - 13].

Unfortunately, all identities (50), (55)–(57), can be shown to be rigid, in the following sense: assuming that

\[
[T_{\vec{m}_1}, \cdots, T_{\vec{m}_M}]_\lambda = g_\lambda ((\vec{m}_1, \cdots, \vec{m}_M)) T_{\vec{m}_1 + \cdots + \vec{m}_M}
\]

(58)

with \( g_\lambda(x) \) a smooth odd function proportional to \( x + \lambda^n c.x^n \) as \( \lambda \to 0 \) \((n > 1)\) any of the above identities will require the constant \( c \) to be equal to zero (I have proved this

*I would like to thank W. Soergel for mentioning refs. [10]/[11] to me and J.L. Loday for sending me a copy of [10] and [12]; also, I would like to express my gratitude to R. Chatterjee and L. Takhtajan for sending me their papers on Nambu Mechanics (cp. [5]).
only for $M = 3$, and in the case of (57) – the a priori most promising case – for general $M > 2$).

Concerning
\[
g_{\lambda} \left( (\vec{a}, \vec{b}, \vec{c}_1) \right) g_{\lambda} \left( (\vec{a} + \vec{b} + \vec{c}_1, \vec{c}_2, \vec{c}_3) \right) + g_{\lambda} \left( (\vec{a}, \vec{b}, \vec{c}_2) \right) g_{\lambda} \left( (\vec{a} + \vec{b} + \vec{c}_2, \vec{c}_3, \vec{c}_1) \right) + g_{\lambda} \left( (\vec{a}, \vec{b}, \vec{c}_3) \right) g_{\lambda} \left( (\vec{a} + \vec{b} + \vec{c}_3, \vec{c}_1, \vec{c}_2) \right) \]
\[
= g_{\lambda} ((\vec{c}_1, \vec{c}_2, \vec{c}_3)) g_{\lambda} ((\vec{c}_1 + \vec{c}_2 + \vec{c}_3, \vec{a}, \vec{b})) ,
\]
\[ (59) \]
i.e. the deformation of $(50)_{M=3}$, one could assume $z := (\vec{c}_1, \vec{c}_2, \vec{c}_3) \neq 0$, $\vec{a} = \sum_{r=1}^{3} \alpha_r \vec{c}_r$, $\vec{b} = \sum_{r=1}^{3} \beta_r \vec{c}_r$, such that $g(y) := \tilde{g}_{\lambda}(y) := g_{\lambda}(zy)$ must satisfy
\[
g \left( \alpha_2 \beta_3 - \alpha_3 \beta_2 \right) \cdot g \left( 1 + \alpha_1 + \beta_1 \right)
+ \text{cyclic permutations}
\]
\[
= g \left( 1 \right) \cdot g \left( \alpha_2 \beta_3 - \alpha_3 \beta_2 + \text{cycl.} \right)
\]
\[ (60) \]
for all $\alpha_r, \beta_r$; which is clearly impossible for any nonlinear $g$ of the required form, (e.g., as in next to lowest order in $\lambda$ the terms $\alpha_1 (\alpha_2 \beta_3)^{n>1}$ appear only once).

Similarly, the deformation of (56) is impossible due to the analogous requirement
\[
g \left( \alpha_3 \right) g \left( \beta_2 - \beta_1 + (\alpha_1 \beta_2 - \alpha_2 \beta_1) \right) + \text{cycl.}
\]
\[
= - g \left( 1 \right) g \left( (\alpha_1 \beta_2 - \alpha_2 \beta_1) + \text{cycl.} \right) .
\]
\[ (61) \]
Finally, concerning possible deformations of (57), let $\left( \vec{a}_1, \cdots, \vec{a}_M \right) \neq 0$, and
\[
\vec{a}_{M+\bar{r}} = \sum_{s=1}^{M} \alpha^{(p)}_s \vec{a}_s \left( \bar{r} = 1, \cdots, M - 1; \right);
\]
then
\[
g \left( 1 + \alpha_1^{(1)} + \cdots + \alpha_1^{(M-1)} \right) \cdot g \left( \begin{array}{c}
1 \\
0 \\
\vdots \\
0
\end{array} \right)
\]
\[
\alpha^{(1)} \cdots \alpha^{(M-1)}
\]
\[
=: [1]
\]
e.g., contains (in next to lowest order in $\lambda$) a term $\alpha_1^{(1)} \cdot \alpha_1^{(2)} \cdot [1]$ (of total degree $(M + 1)$ in the $\alpha^{(p)}_s$), which cannot appear anywhere else (in the same order in $\lambda$), – in contradiction to the assumption that (57) should hold for $[\cdots]_{\lambda}$ (cp. (58)) replacing the curly bracket (4).
6. A Remark on Generalized Schild Actions

Consider

\[ S := -\int d\phi^0 d^M \phi f(G), \quad (62) \]

where \( G := (-)^M \text{det} (G_{\alpha\beta}), \ G_{\alpha\beta} := \frac{\partial x^\mu}{\partial \phi^\alpha} \frac{\partial x^\nu}{\partial \phi^\beta} \eta_{\mu\nu}, \ \eta_{\mu\nu} = \text{diag} (1, -1, \ldots, -1), \]
\( \alpha, \beta = 0, \ldots, M \) and \( f \) some smooth monotonic function like \( G^\gamma \) (\( \gamma = 1 \text{ resp.} \frac{1}{2} \)) corresponding to a generalized Schild-, resp. Nambu-Goto, action for \( M \)-branes). Apart from a few subtleties (like \( \gamma = 1 \) allowing for vanishing \( G \), while \( \gamma = \frac{1}{2} \) does not) actions with different \( f \) are equivalent, in the sense that the equations of motion,

\[ \partial_\alpha \left( f'(G) G^{\alpha\beta} \partial_\beta x^\mu \right) = 0 \quad \mu = 0 \ldots D - 1 \quad (63) \]

are easily seen to imply

\[ \partial_\alpha G = 0 \quad \alpha = 0, \ldots, M \quad (64) \]

(just multiply (63) by \( \partial_\alpha x_\mu \) and sum) – unless \( f(G) = \text{const.} \sqrt{G} \), in which case (62) is fully reparametrisation invariant and a parametrisation may be assumed in which \( G = \text{const.} \) (such that (63) becomes proportional to \( \partial_\alpha (G^{\alpha\beta} \partial_\beta x^\mu) \) also in this case). Due to

\[ G = \sum_{\mu_1 < \ldots < \mu_{M+1}} \{ x^{\mu_1}, \ldots, x^{\mu_{M+1}} \} \{ x_{\mu_1}, \ldots, x_{\mu_{M+1}} \} \quad (65) \]

(63) may be written as (cp. [14] for strings, and [15] for membranes, in the case of \( \gamma = 1 \) resp. \( \frac{1}{2} \))

\[ \{ f'(G)\{ x^{\mu_1}, \ldots, x^{\mu_{M+1}} \}, x_{\mu_2}, \ldots, x_{\mu_{M+1}} \} = 0, \quad (66) \]

whose deformed analogue (note the similarity between \( G = \text{const.} \) and condition (3.9) of [16])

\[ [ [ x^{\mu_1}, \ldots, x^{\mu_{M+1}} ] , x_{\mu_2}, \ldots, x_{\mu_{M+1}} ] = 0 \quad (67) \]

looks very suggestive when thinking about space-time quantization in \( M \)-brane theories.

7. Multidimensional Integrable Systems from \( \mathcal{M} \)-algebras

Several ideas used in the context of integrable systems are based on bilinear operations. Our problems to extend results about low (especially 1+1) dimensional integrable field theories to higher dimensions may well rest on precisely this fact. Already some time ago, attempts were made to overcome this difficulty by generalizing Lax-pairs to -triples ([3], p. 72) and Hirota’s bilinear equations for ‘\( \tau \)-functions’ [17] to multilinear equations ([3], p. 107-111).

At that time, good examples were lacking, and – not being an exception to the rule that generalisations involving the number of dimensions (of one sort or an other) are usually hindered by implicitly low dimensional point(s) of view – the proposed generalisation of
Hirota-operators may have still been too naive; while hoping to come back to the question of multidimensional $\tau$-functions in the near future, I would now like to give an example ($M > 3$ will then be obvious) for an equation of the form

$$\dot{\mathcal{L}} = \frac{1}{\rho} \{ \mathcal{L}, \mathcal{M}_1, \mathcal{M}_2 \}$$

(68)

being equivalent to the equations of motion of a compact 3 dimensional manifold $\hat{\Sigma} \subset \mathbb{R}^4$ (described by a time-dependent 4-vector $x^i(\varphi^1, \varphi^2, \varphi^3, t)$), moving in such a way that its normal velocity is always equal to the induced volume density $\sqrt{\mathcal{I}}$ (on $\hat{\Sigma}$) divided by a fixed non-dynamical density $\rho(\varphi)$ (‘the’ volume density of the parameter manifold):

$$\begin{align*}
\dot{x}_1 &= \frac{1}{\rho} \{ x_2, x_3, x_4 \} \\
\dot{x}_2 &= -\frac{1}{\rho} \{ x_3, x_4, x_1 \} \\
\dot{x}_3 &= \frac{1}{\rho} \{ x_4, x_1, x_2 \} \\
\dot{x}_4 &= -\frac{1}{\rho} \{ x_1, x_2, x_3 \}.
\end{align*}$$

(69)

With the curly bracket defined as before (cp. (4)), it will be an immediate consequence of (68) that

$$Q_n := \int_{\hat{\Sigma}} d^3 \varphi \rho(\varphi) \mathcal{L}^n$$

(70)

is time-independent (for any $n$).

In [2] evolution-equations of the form (69) (in any number of dimensions) were shown to correspond to the diffeomorphism invariant part of an integrable Hamiltonian field theory (as well as to a gradient flow); one way to solve such equations is to note ([18], [2]) that the time at which the hypersurface will pass a point $\vec{x}$ in space will simply be a harmonic function.

In any case, the (a) form of $(\mathcal{L}, \mathcal{M}_1, \mathcal{M}_2)$ that will yield the equivalence of (69) with (68) is:

$$\begin{align*}
\mathcal{L} &= (x_1 + ix_2)\frac{1}{\lambda} + (x_3 + ix_4)\frac{1}{\mu} + \mu(x_3 - ix_4) - \lambda(x_1 - ix_2) \\
\mathcal{M}_1 &= \frac{\mu}{2}(x_3 - ix_4) - \frac{1}{2\mu}(x_3 + ix_4) \\
\mathcal{M}_2 &= \frac{\lambda}{2}(x_1 - ix_2) + \frac{1}{2\lambda}(x_1 + ix_2)
\end{align*}$$

(71)

(involving two spectral parameters, $\lambda$ and $\mu$). Surely, this observation will have much more elegant formulations, and conclusions.
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References

[1] Y. Nambu; Phys. Rev. D7 # 8 (1973) 2405.
[2] M. Bordemann, J. Hoppe; ‘Diffeomorphism Invariant Integrable Field Theories and Hypersurface Motions in Riemannian Manifolds’; ETH-TH/95-31, FR-THEP-95-26.
[3] J. Hoppe; ‘Lectures on Integrable Systems’; Springer-Verlag 1992.
[4] J. Hoppe; ‘Quantum Theory of a Massless Relativistic Surface’; MIT Ph. D. thesis 1982 and Elem. Part. Res. J. (Kyoto) 80 (1989) 145.
[5] L. Takhtajan; Comm. Math. Phys. 160 (1994) 295.
   R. Chatterjee; ‘Dynamical Symmetries and Nambu Mechanics’; Stony Brook preprint 1995.
   R. Chatterjee, L. Takhtajan; ‘Aspects of Classical and Quantum Nambu Mechanics’
   (1995; to appear in Lett. Math. Phys.).
[6] V.T. Filippov; ‘n-ary Lie algebras’; Sibirsiki Math. J. 24 # 6 (1985) 126 (in russian).
[7] P. Lecomte, P. Michor, A. Vinogradov; ‘n-ary Lie and Associative Algebras’; preprint 1994.
[8] H. Weyl; ‘The Classical Groups’; 2nd edition, Princeton University Press.
[9] N. White; ‘Theory of Matroids’; Cambridge University Press 1987.
[10] J.L. Loday; ‘La renaissance des opérades’; in Séminaire Bourbaki, exposé 792, Novembre 1994.
[11] V. Ginzburg, M.M. Kapranov; Duke Math. J. 76 (1994) 203.
[12] Ph. Hanlon, M. Wachs; ‘On Lie k-Algebras; preprint 1993.
[13] A.V. Gnedbaye; C.R. Acad. Sci. Paris, t. 321, Série I, p. 147, 1995.
[14] A. Schild; Phys. Rev. D16 (1977) 1722.
[15] A. Sugamoto; Nucl. Phys. B215 [FS7] (1983) 381.
[16] S. Doplicher, K. Fredenhagen, J. Roberts; Comm. Math. Phys. 172 (1995) 187.
[17] R. Hirota; ‘Direct methods of finding solutions of nonlinear evolution equations’, in Lect. Notes in Math. 515, Springer-Verlag 1976.
[18] J. Hoppe; Phys. Lett. B335 (1994) 41.
[19] P. Lecomte, C. Roger; ‘Rigidité de l’algèbre de Lie des champs de vecteurs unimodulaires’; Université IGD Lyon 1, preprint (1995).