The vacuum conservation theorem

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Abstract
A version of the vacuum conservation theorem is proved which does not assume the existence of a time function nor demands stronger properties than the dominant energy condition. However, it is shown that a stronger stable version plays a role in the study of compact Cauchy horizons.

1 Introduction
Hawking’s vacuum conservation theorem [5] states that in a spacetime which admits a time function and which satisfies the dominant energy condition, the stress-energy tensor vanishes on a compact domain delimited by two locally achronal hypersurfaces $S_1$ and $S_2$ provided it vanishes on $S_1$ (or $S_2$). Roughly speaking, if a space region is empty at one time it will remain so at later times provided no matter-energy flows in from outside. A more detailed proof can be found in [6, Sect. 4.3] and a simplified argument was obtained by Carter in [1].

In this work we wish to remove the assumption on the existence of a time function. This is desirable since the domain under consideration is often of the form $\overline{D^+(S_1)}$ where the horizon $S_2 := H^+(S)$ might form precisely due to the presence of closed timelike curve behind it, where a time function cannot exist.

Remark 1.1. Hawking was aware of this limitation in his theorem and in fact on [6, p. 298] he sketched a possible more involved proof which however demanded a slightly stronger energy condition [5, point (5), p. 293]. Furthermore, in the new argument Hawking still assumes that a time function of the globally hyperbolic set Int$D(S)$ extends up to the boundary $H^+(S)$ remaining $C^1$, which should not be expected in general due to the lack of regularity of $H^+(S)$.

In this work we shall retain Carter’s simplification and we shall reach the desired conclusion without invoking the existence of a time function. Also, we shall not need to strengthen the energy condition.

2 The dominant energy condition
We recall that a spacetime $(M, g)$ is a paracompact, time oriented Lorentzian manifold of dimension $N + 1 \geq 2$. The metric has signature $(-, +, \cdots, +)$. In

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our terminology the lightlike, timelike or causal vectors are non-zero. A null (non-spacelike) vector is lightlike (resp. causal) or zero. We assume that the metric satisfies the Einstein equations with cosmological constant, and that the stress-energy tensor satisfies:

**Definition 2.1.** The dominant energy condition. On every tangent space $T_pM$ the linear endomorphism

$$v^\mu \to -T^\mu_\nu v^\nu$$

sends the future-directed non-spacelike cone into itself.

When $v^\mu$ is timelike and normalized $-T^\mu_\nu v^\nu$ represents the energy-momentum flow which the condition demands to be causal, roughly speaking energy cannot travel faster than light. Physically, the dominant energy condition is nothing more than that since the constraint for $v$ null follows by continuity from that for $v$ timelike.

Let us recall some basic consequences of the dominant energy condition. As observed by Carter a non-trivial endomorphism (1) cannot send a future-directed timelike vector into the zero vector. Indeed, let us consider a small perturbation

$$v^\mu + \delta v^\mu \to -T^\mu_\nu (v^\nu + \delta v^\nu) = -T^\mu_\nu \delta v^\nu.$$

Clearly, if the rank of $T$ is non-zero then we can find a $\delta v^\nu$ with non-vanishing image, and altering its sign if necessary, we obtain an image vector which is not a future-directed non-spacelike vector. The contradiction proves that only null vectors can be sent to the zero vector.

Since the scalar product of two future-directed causal vectors is non-positive and vanishes if and only if they are proportional and lightlike we see that, for any two future-directed causal vectors $v$ and $w$

$$T(v, w) \geq 0,$$

equality holding only if at least one of the vectors is lightlike. In particular, the dominant energy condition (DEC) implies the weak energy condition (WEC) which demands that for every timelike vector $v$ (and hence by continuity, for every null $v$) $T(v, v) \geq 0$, and the WEC implies the null energy condition (NEC) also called null convergence condition according to which $T(v, v) \geq 0$ for every null vector $v$.

We need to find some inequalities implied by the dominant energy condition [6]. Observe that if $V^\mu$ is future-directed timelike and normalized and $A^\mu$ is orthogonal to it then, with $|A| = (A_\alpha A^\alpha)^{1/2}$, the vector $V^\mu \pm A^\mu / |A|$ is lightlike thus

$$T^\mu_\nu V_\nu (V_\mu \pm A_\nu / |A|) \geq 0,$$

which can be rewritten

$$|T(V, A)| \leq |A| T(V, V).$$

Similarly, if $B^\mu$ is another vector orthogonal to $V^\mu$, $V^\mu + B^\mu / |B|$ is lightlike thus

$$T^\mu_\nu (V_\mu + B_\mu / |B|)(V_\nu + A_\nu / |A|) \geq 0.$$
\[ T(V, V) + T(V, B)/|B| + T(V, A)/|A| + T(A, B)/(|A| |B|) \geq 0. \]  \hfill (3)

We are going to obtain an inequality involving just \( T(A, A) \), thus we can redefine \( A \rightarrow -A \) if necessary to make \( T(V, A) \leq 0 \). Replacing \( B = A \) and \( B = -A \) in Eq. 3 gives

\[ |T(A, A)| \leq T(V, V) |A|^2. \]

From this inequality we obtain the two inequalities

\[ T\left( \frac{A}{\sqrt{2|A|}} + \frac{B}{\sqrt{2|B|}}, \frac{A}{\sqrt{2|A|}} + \frac{B}{\sqrt{2|B|}} \right) \leq T(V, V) \left( 1 + A^\mu B_\mu/(|A| |B|) \right), \]

\[ -T\left( \frac{A}{\sqrt{2|A|}} - \frac{B}{\sqrt{2|B|}}, \frac{A}{\sqrt{2|A|}} - \frac{B}{\sqrt{2|B|}} \right) \leq T(V, V) \left( 1 - A^\mu B_\mu/(|A| |B|) \right). \]

Summing (and considering the equation so obtained also for \( B \rightarrow -B \))

\[ |T(A, B)| \leq T(V, V) |A| |B|. \]  \hfill (4)

Any tensor \( D_\mu \nu \) whose contractions with \( V^\alpha \) vanishes can be written as the sum of \( N^2 \) terms of the form \( A_\mu B_\nu \) thus

\[ |T^{\mu\nu} D_\mu \nu| \leq N^2 \sqrt{D_{\alpha\beta} D^{\alpha\beta}} T(V, V). \]  \hfill (5)

Equations (2), (4), prove that in any orthonormal frame in which \( e_0 \) is timelike

\[ |T^{ab}| \leq T^{00}, \quad a, b = 0, 1, \cdots, N. \]

\section{Vacuum conservation theorem}

We are ready to prove a first vacuum conservation result.

**Theorem 3.1.** Let \((M, g)\) be a spacetime which satisfies the dominant energy condition. Let us consider an open relatively compact region \( U \) of spacetime delimited by two locally achronal hypersurfaces \( S_1 \) and \( S_2 \), possibly with edge, such that \( \bar{U} \) is the union of integral connected segments (possibly degenerate to a point) of a \( C^1 \) future-directed timelike vector field \( V \) which have past endpoint in \( S_1 \) and future endpoint in \( S_2 \). Then the stress energy tensor vanishes on \( S_1 \) iff it vanishes on \( S_2 \) iff it vanishes on \( \bar{U} \).

**Proof.** Let us first give the proof for \( S_1 \) and \( S_2 \) continuously differentiable. Let us normalize \( V, V^\alpha V_\alpha = -1 \). Let \((t, p) \rightarrow \varphi_t(p)\) be the map which assigns to \((t, p)\) the endpoint of the curve \( x : [0, t] \rightarrow M, x(0) = p, \dot{x} = V \), wherever it exists. By well known results on the dependence on initial condition and external parameters of ordinary differential equations \( \text{[4]} \) this flow map \( \varphi \) is smooth on an open subset of \( \mathbb{R} \times M \) where the map \((t, p) \rightarrow \varphi_t(p)\) makes sense. For every \( t \) the map \( \varphi_t \) is actually a local diffeomorphism wherever it is defined \( \text{[7]} \) Theor. 2.9,
By assumption every point of $\bar{U}$ can be reached from $S_1$ following an integral line of $V$. Let $t: U \to [0, +\infty)$ be the function defined through: $t(q)$ is such that $\varphi_t(x) = q$ for some $x \in S_1$. By the implicit function theorem $t$ has the same regularity of $S_1$ thus $C^1$, and by construction $dt[V] = 1$. We stress that we are not claiming that $t$ is a time function, just that it is well defined over $\bar{U}$.

Let us introduce the vector field

$$X^\mu = -e^{-Ct}T^\mu_\nu V^\nu,$$

which is future-directed timelike wherever $T \neq 0$ or zero. Let us denote with $\mu$ the usual volume form. Let us consider the non-negative integrals

$$I_k = \int_{S_k} i_X \mu.$$

If $X \neq 0$ at a point $r \in S_k$ then $X$ is timelike and hence transverse to $S_k$, thus the integral is strictly positive. As a consequence $I_k = 0$ iff $X = 0$ on $S_k$ iff $T = 0$ on $S_k$.

Recalling the identity $div X \mu = div X \mu$ we obtain from Stokes theorem

$$I_2 - I_1 = \int_U div X \mu.$$

Using $T^\mu_\nu = 0$

$$div X = e^{-Ct}T^\mu_\nu [C t_\mu V_\nu - V_\mu_\nu].$$

Since

$$[C t_\mu V_\nu - V_\mu_\nu]V^\mu V^\nu = -C,$$

the matrix in square bracket can be written

$$C t_\mu V_\nu - V_\mu_\nu = -C V_\mu V_\nu + V_\mu A_\nu + B_\mu V_\nu + D_{\mu\nu},$$

for some continuous tensor fields $A, B, D$ such that their contractions with $V$ vanish. This fact can be most easily understood passing to an orthonormal base with $e_0 = V$. In the same way one easily sees that $A^\alpha A_{\alpha}, B^\beta B_{\beta} \geq 0$ and $D^{\mu\nu} D_{\mu\nu} \geq 0$.

On every relatively compact open set $O$ which admits a coordinate system the contraction $T^{\mu\nu} [C t_\mu V_\nu - V_\mu_\nu]$ can be made non-positive by choosing a sufficiently large value of $C$, and non-negative by choosing a sufficiently large value of $-C$. Indeed, by the dominant energy condition and by using the continuity of $A, B, C, V$, on the compact set $\bar{O}$ we have that there is a constant $K > 0$ such that $(N + 1)$ is the spacetime dimension

$$|T^{\mu\nu} V_\mu A_\nu| \leq |A| T(V, V) \leq KT(V, V),$$

$$|T^{\mu\nu} D_{\mu\nu}| \leq N^2 \sqrt{D_{\alpha\beta} D^{\alpha\beta}} T(V, V) \leq KT(V, V),$$

where an equation analogous to the first one holds also for $B$. 

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Since \( \bar{U} \) can be covered by a finite number of relatively compact coordinated neighborhoods we have on \( \bar{U} \), \((\text{sgn}C)\text{div}X \leq 0\). Observe that actually by choosing \(|C|\) sufficiently large the equality case is excluded unless \( T = 0 \) at the point under consideration. Thus for \( C > 0 \) large enough

\[
0 \leq I_2 \leq I_1
\]

where the second equality holds only if \( T = 0 \) on \( U \) and hence \( \bar{U} \). Thus if we take \( C \) large enough we have that \( T = 0 \) on \( S_1 \) implies \( T = 0 \) on \( \bar{U} \) and hence in \( S_2 \).

Analogously, for \( -C > 0 \) large enough

\[
0 \leq I_1 \leq I_2
\]

where the second equality holds only if \( T = 0 \) on \( U \) and hence \( \bar{U} \). Thus if we take \( -C \) large enough we have that \( T = 0 \) on \( S_2 \) implies \( T = 0 \) on \( \bar{U} \) and hence in \( S_1 \).

The proof for the Lipschitz regularity of \( S_1 \) and \( S_2 \) is as above, there are just two critical steps. The first is the construction of function \( t \). We cannot define it starting the flow from \( S_1 \) since, as this hypersurface is only Lipschitz, \( t \) would not be \( C^1 \). However, in a neighborhood of \( p \in S_1 \) one can introduce coordinates \( \{x^\mu\} \) such that \( \partial_0 = V \), and \( \{x^i\} \) are induced from the \( V \)-projection on a smooth spacelike hypersurface \( \Sigma \) transverse to \( V \). Then \( S_1 \) can be locally expressed as a Lipschitz graph \( x_0 = h(x) \) which can be approximated by a smooth graph by Whitney approximation theorem [8, Theor. 6.21]. Through a partition of unity it is possible to patch together these local graphs into a smooth hypersurface \( S'_1 \) with the property that every \( V \)-integral curve which intersect \( S_1 \) intersects \( S'_1 \) and conversely. Then the definition of \( t \) is as above but starting the flow from \( S'_1 \).

The other delicate step concerns the application of the Gauss-Greens (divergence) theorem which, fortunately, has indeed be generalized to Lipschitz domains and even Lipschitz fields [2, Sect. 5.8] [12]. In this theorem the boundary term \( \int_B i_{X^\mu} \), \( B \subset S_k \), can be calculated expressing \( S_k \) as a local graph and extending the expression of the integral in terms of a \( C^1 \) function \( h \) to the Lipschitz case. More in detail, if \( X = a(x^0, x) \partial_0 + b^i(x^0, x) \partial_i \)

\[
i_{X^\mu}|_{S_k} \equiv \sqrt{-|g(h(x), x)|} i_{X^0} dx^0 \wedge \cdots \wedge dx^N |_{S_k}
= \sqrt{-|g(h(x), x)|} (a + \partial_0 h) dx^1 \wedge \cdots \wedge dx^N,
\]

Observe that \( \partial_0 h \) is \( L_{\text{loc}}^1 \) by the Lipschitzness of \( h \), thus the integral \( \int_B i_{X^\mu} \) is well defined.

**Remark 3.2.** The physical meaning of the integral \( I_k \) can be understood as follows. Let us denote with \( \hat{S}_k \) the open subset of \( S_k \) in which \( X \neq 0 \) hence timelike. Let us endow \( \hat{S}_k \), \( k = 1, 2 \), with the following volume form

\[
\nu = \frac{1}{-g(X, u)} i_{X^\mu}
\]
where \( n \) is a \( C^0 \) future-directed causal field normal to \( S_k \) (possibly lightlike). Here \( \nu \) is evaluated just on the tangent space to \( \hat{S}_k \). This choice of volume is independent of the transverse field \( X \) but it depends on the scale of \( n \) (this multiplying factor could be naturally fixed through normalization if \( n \) were timelike). We have

\[
I_k = \int_{S_k} i_X \mu = \int_{\hat{S}_k} i_X \mu = \int_{\hat{S}_k} [-g(X, n)] \nu = \int_{\hat{S}_k} T(V, n) \nu. \tag{6}
\]

On should be very careful in using the last expression when \( S_k \) is partly lightlike for one can easily miss the difference between \( \hat{S}_k \) and \( S_k \).

**Corollary 3.3.** Let \( S \) be a locally achronal compact topological hypersurface possibly with edge and let \( \overline{D^+(S)} \) be compact. Then the stress energy tensor vanishes on \( S \) iff it vanishes on \( H^+(S) \) iff it vanishes on \( \overline{D^+(S)} \).

**Proof.** Let \( V \) be any smooth normalized future-directed timelike field, and let \( S_1 := S \) and \( S_2 := H^+(S) \). The assumption of Theorem 3.1 are satisfied thus the desired conclusion follows.

The usual vacuum conservation theorem states that if the stress-energy tensor vanishes on \( S \) then it vanishes on \( D^+(S) \) and on the horizon \( H^+(S) \). Actually, the previous result establishes a converse implication which is of particular interest namely that if \( T \neq 0 \) somewhere on \( S \) then \( T \neq 0 \) at some point \( p \in H^+(S) \). Observe that we cannot conclude that \( T(n, n) \neq 0 \) at \( p \), for it can be that \( T^\mu_\nu n^\nu \propto n^\mu \) as for some stress-energy tensor of pure aligned radiation (Type II [6, Sect. 4.3]).

Let us introduce two modifications of the dominant energy condition.

**Definition 3.4.** The **stable dominant energy condition** \[3\]. On every tangent space \( T_p M \) the linear endomorphism \( (1) \) sends the future-directed casual cone into the future-directed timelike cone.

The **weakened stable dominant energy condition**. At every point \( T = 0 \) or the stable dominant energy condition holds, which is equivalent to: the linear endomorphism \( (1) \) sends the future-directed casual cone into the future-directed timelike cone plus the zero vector \[4\].

The stable dominant energy condition is preserved under small perturbations of the endomorphism, which means that the source content of spacetime is not on the verge of violating the dominant energy condition. Unfortunately, it might seem a too strong condition as it excludes the vacuum case, that is, under this condition \( T \neq 0 \) at every point. Observe, that we might want to impose the stable dominant energy condition only where a source is really present. This

\[\footnote{That the former characterization implies the latter is clear. For the converse, if at the given point \( T = 0 \) or if no causal vector is sent to zero we have finished. Thus assume \( T \neq 0 \) and there is a f.d. causal vector \( w \) which has zero image. Since \( T \neq 0 \) there is a f.d. causal vector \( v \) such that \( u^\alpha := -T^\alpha_\nu v^\nu \neq 0 \), necessarily f.d. timelike by assumption, then \( 0 = T(w, v) = -g(w, u) \) which is a contradiction since it must be positive.}
leads ut to the weakened stable dominant energy condition which contemplates the vacuum case.

In the four dimensional spacetime case and in the classification of [6, Sect. 4.3], this condition allows only Type I stress-energy tensors (namely the diagonal ones) where the energy density is larger than the absolute values of the principal pressures, $|p_i| < \rho$ or $p_i = \rho = 0$ (observe that types III and IV are excluded by the null energy condition).

Clearly, under the weakened stable dominant energy condition the implication $T \neq 0 \Rightarrow T(n, n) \neq 0$ on the horizon holds true, thus\(^2\).

**Proposition 3.5.** Under the assumptions of Corollary 3.3 and under the weakened stable dominant energy condition $T \neq 0$ somewhere on $S$, implies $T(n, n) \neq 0$ somewhere on the horizon $H^+(S)$.

This result will allow us to show that under the weakened stable dominant energy condition compact Cauchy horizon cannot form [10], a result which seems significative for a final resolution of the weak/strong cosmic censorship conjecture.

Physically speaking, in absence of quantum effects the weakened stable dominant energy condition seems reasonable in all those phenomena in which it is known that the source of spacetime is not just coherent radiation, particularly in the study of the development of future singularities. Less convincing is its validity at the beginning of the Universe since the Higgs field may still stay in its non-degenerate minimum so that, as all the particles would be massless, the only source content would be radiation. In this case if $S$ is a present time partial Cauchy hypersurface the radiation would emerge from the boundary $H^-(S)$ which would play the role of Big Bang null hypersurface. For more on this picture of the beginning of the Universe and its advantages the reader is referred to [9][11].

### 4 Conclusions

We have proved a version of the vacuum conservation theorem which has some good features: (a) it involves causal rather than just spacelike hypersurfaces, (b) it holds for Lipschitz hypersurfaces, (c) it does not assume the existence of a time function, (d) it includes a converse preservation claim according to which if the stress-energy tensor does not vanish on $S$ then it does not vanish on $H(S)$. The application of this theorem to the study of compact Cauchy horizons has been briefly commented.

\(^2\)Hawking [6, p. 293] claims a similar result but his proof and claim seem incorrect. He assumes a weaker form of energy condition, which does not exclude the possibility $T^{ab} = \alpha n^a n^b$, where $n$ is a lightlike vector, and then he applies a version of the conservation theorem which he has not really proved. Probably he used Eq. [7] missing the difference between $\hat{S}$ and $S$, and that between $V$ and $n$. 
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