Veering of Rayleigh–Lamb waves in orthorhombic materials

Andrea Nobili
Department of Engineering Enzo Ferrari, University of Modena and Reggio Emilia, Modena, Italy

Barış Erbaş
Department of Mathematics, Eskisehir Technical University, Eskisehir, Turkey

Cesare Signorini
Institute of Construction Materials, Technical University Dresden, Dresden, Germany

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Abstract
We analyse veering of Rayleigh–Lamb waves propagating in a plane of elastic symmetry for a thin orthotropic plate. We demonstrate that veering results from interference of partial waves in a similar manner as it occurs in systems composed of one-dimensional (1D) structures, such as beams or strings. Indeed, in the neighbourhood of a veering point, the system may be approximated by a pair of interacting tout strings, whose wave speed is the geometric average of the phase and group velocity of the relevant partial wave at the veering point. This complementary pair of partial waves provides the coupling terms in a form compatible with a action–reaction principle. We prove that veering of symmetric waves near the longitudinal bulk wave speed repeats itself indefinitely with the same structure. However, the dispersion behaviour of Rayleigh–Lamb waves are richer than that of 1D systems, and this reflects also on the veering pattern. In fact, the interacting tout string model fails whenever the dispersion branch is not guided by either partial wave. This often occurs when neighbouring veering points interact and partial waves no longer provide guiding curves.

Keywords
Veering, Rayleigh–Lamb waves, orthorhombic elastic materials

1. Introduction
Rayleigh–Lamb (RL) waves in thin plates have long attracted great attention in view of their theoretical and practical importance. They encompass a large array of important phenomena, such as dispersion, localization, and interference. Despite their apparent simplicity, a satisfactory understanding of the underlying physics has only been gained in fairly recent times [1, §8.1.5.11]. This understanding is
especially valuable because it provides, among many assets, the foundation for consistent asymptotic reduced theories for shell, plates, and beams [2–5]. Consideration of anisotropic features adds considerable complications, and yet it possesses relevant practical importance, as well illustrated in the classical monograph [6]. As an example of such complications, following Tovstik and Tovstik [7], we mention that Kirchhoff–Love and Timoshenko–Reissner plate models fail to be consistent with the outcomes of the three-dimensional (3D) theory for an orthotropic material. The recent review paper [8] accounts for the many contributions appearing in the literature that investigate specific features of RL propagation. For example, Hussain and Ahmad [9] pointed out that orthotropy is attached to special points possessing zero-group velocity, which pave the way to anomalous dispersion, that is, situations where the energy flows in the direction opposite to that of propagation for the wave train. Equally, Zadeh and Sorokin [10] illustrate the effect of curvature on the waveguide properties.

Wave coupling occurs in multiple instances, such as in reduced models, for example, strings, beams, and rods [11,12] or between different propagation modes, as it is the case for torsional and bending waves [13]. Coupling of waves takes up many different forms (for instance through mode conversion and localization) [14], among which veering is especially remarkable because it is associated with rapid divergence of the propagation branches in the neighbourhood of the veering point, alongside eigenvector inversion. This peculiar behaviour may be most easily explained in coupled oscillators, where tuning the coupling device brings the specific propagation features of either in a veering condition. Mace and Manconi [15,16] study veering in discrete conservative elastic systems under a framework for the analysis thereof. They distinguish between weak and strong coupling and introduce the concept of uncoupled block system.

In this paper, we investigate veering in a continuous system, namely for RL waves. In this situation, matter is complicated by the presence of multiple wave modes (branches) and internal coupling. Nonetheless, we can show that the concept of partial waves still works as a building block for both the dispersion pattern and the interference thereof. After developing the classical governing equations and travelling wave solution for orthorhombic media, respectively in section 2 and, partial waves are introduced and analysed in section 4. In section 5, they are shown to guide RL modes, and their intersection defines the veering points and the form of the interacting systems (section 6). Finally, conclusions are drawn in section 7.

2. Governing equations

Let us consider an infinite thin plate of thickness $2h$, made of linear elastic homogeneous material with orthorhombic material symmetry (Figure 1). A Cartesian orthogonal frame is introduced $\{O,x\}$, where $x = [x_1,x_2,x_3]$ is the coordinate vector such that $x_3$ is a direct axis of even order (i.e. the plane $(x_1,x_2)$ is a mirror plane) and $x_1$ is directed along a symmetry axis for the material, as by Royer and Dieulesaint [17]. Also, in this frame, the strip lower/upper boundaries are located, respectively, at $x_2 = \pm h$. For convenience, Voigt’s (or matrix) notation is adopted throughout [18, p. 134],

\[
\begin{align*}
(11) &\leftrightarrow 1, (22) \leftrightarrow 2, (33) \leftrightarrow 3, (23) = (32) \leftrightarrow 4, (13) = (31) \leftrightarrow 5, (12) = (21) \leftrightarrow 6.
\end{align*}
\]

The elastic constants are gathered in the stiffness matrix [17, equation (3.64)]

![Figure 1. A free infinite orthotropic thin plate in plane strain.](image-url)
which is not a rank-2 tensor, for it lacks the transformation property thereof. Cubic symmetry, that is considered by Solie and Auld [19], may be retrieved upon taking \( c_{12} = c_{13} = c_{23}, \ c_{11} = c_{22} = c_{33}, \) and \( c_{44} = c_{55} = c_{66}. \) Isotropic materials are a special case of cubic symmetry with

\[
c_{11} = \lambda + 2\mu, \quad c_{12} = \lambda, \quad c_{66} = \mu,
\]

where \( \mu > 0 \) and \( \lambda > -2/3\mu \) are Lamé elastic constants. We recall that positiveness of the strain energy density demands

\[
c_{11}, c_{22}, c_{55}, c_{66} > 0, \quad c_{11}c_{22} - c_{12}^2 > 0,
\]

So thus we may define the generalized Young modulus

\[
E_c = c_{11} - c_{12}^2 = \frac{E_1}{1 - \nu_{13}\nu_{31}},
\]

where technical (or engineering) moduli [20] are introduced in the last equality. In an isotropic material, \( E_c \) reduces to the Young modulus in plane strain \( E = E_1/(1 - \nu^2). \) We recall that in an anisotropic plate, the bending stiffness within the Kirchhoff theory is given by \( D_x = E_c I \) and \( D_y = \nu_{31}D_x/\nu_{13}, \) wherein \( I = 2h^3/3 \) is the second moment of inertia [3].

In an orthorhombic material, several bulk wave speeds are defined [18],

\[
c_1 = \sqrt{\frac{c_{11}}{\rho}} \quad c_2 = \sqrt{\frac{c_{22}}{\rho}} \quad c_{SV} = \sqrt{\frac{c_{66}}{\rho}} \quad c_{SH} = \sqrt{\frac{c_{55}}{\rho}},
\]

Respectively, bulk longitudinal along \( x_1 \) and along \( x_2 \) and transverse shear vertical (SV) and shear horizontal (SH) wave speed. To such speeds, in analogy to the longitudinal wave speed for beams, we add the combination [21]

\[
c_c = \sqrt{\frac{E_c}{\rho}} < c_1.
\]

Strain \( \epsilon \) is small and it is related to the displacement field \( u = [\xi_1, \xi_2, \xi_3] \) through the usual linear relations \( \epsilon_{ij} = (\xi_{i,j} + \xi_{j,i})/2, \quad i, j \in \{1, 2, 3\}, \) where a suffix comma denotes differentiation with respect to the relevant space variable, for example, \( u_{i,1} = \partial u_i / \partial x_1, \) and summation over twice repeated subscripts is assumed. We recall that \( \gamma_{ij} = 2\epsilon_{ij}, i \neq j \) is the engineering shear strain. The stress \( \sigma \) is related to strain through Hook’s constitutive law

\[
\sigma = C\epsilon.
\]

The equilibrium equations, in the absence of body forces, read (superposed dots denote time differentiation) valid for orthorhombic materials

\begin{align*}
\begin{align}
&c_{11}\dot{u}_{1,1} + c_{55}\dot{u}_{1,3} + c_{66}\dot{u}_{1,2} + (c_{12} + c_{66})\dot{u}_{2,12} + (c_{13} + c_{55})\dot{u}_{3,13} = \rho\ddot{u}_1, \\
&c_{66}\dot{u}_{2,11} + c_{44}\dot{u}_{2,33} + c_{22}\dot{u}_{2,22} + (c_{12} + c_{66})\dot{u}_{1,12} + (c_{23} + c_{44})\dot{u}_{3,23} = \rho\ddot{u}_2, \\
&c_{55}\dot{u}_{3,11} + c_{44}\dot{u}_{3,22} + c_{33}\dot{u}_{3,33} + (c_{13} + c_{55})\dot{u}_{1,13} + (c_{23} + c_{44})\dot{u}_{2,23} = \rho\ddot{u}_3.
\end{align}
\end{align}
3. Waves in unbounded media

Christoffel equations are obtained plugged into the equilibrium equations (7) travelling wave solutions [22]

\[ u_i(x_1, x_2, x_3, t) = A \exp[i(k \cdot x - \omega t)], \]

where \( A = [A_i] \) is the polarization vector, \( k = [k_i] \) is the wave vector, \( \omega \) is the wave frequency, \( i \) is the imaginary unit, that is, \( i^2 = -1 \), and a dot denotes the scalar product between vectors. We restrict attention to waves propagating in the sagittal plane \((x_1, x_2)\), which contains the surface normal and the propagation direction (wave vector) [18, §5.1]. Consequently, we take \( k_3 = 0 \) and no dependence on \( x_3 \) (i.e. plane strain). We introduce the ratio \( \Lambda = k_2/k_1 \), which corresponds to the tangent of the angle of wave propagation to the \( x_1 \)-axis.

The general solution of the RL dispersion problem may be constructed from a superposition of simple waves, named partial waves [1,19]. Partial waves travel along the plate (along \( x_1 \)) with the same wavenumber \( k_1 = k > 0 \), while bouncing back and forth at the plate boundaries. Their interaction is generally induced by the boundary conditions and determine the dispersion pattern. Here, \( v = \omega/k \) is the phase velocity along \( x_1 \). In order to determine the wave vector, we introduce the dimensionless space and time co-ordinates

\[ \{\xi_1, \xi_2, \tau\} = \{h^{-1}x_1, h^{-1}x_2, T^{-1}t\}. \]

Having let the reference time \( T = h/c_{SV} \) in terms of the body shear wave speed \( c_{SV} \). In this framework, we define the dimensionless velocities

\[ V_1 = c_1/c_{SV}, \quad V_2 = c_2/c_{SV}, \quad V_5 = c_{SH}/c_{SV}, \quad V_c = c_c/c_{SV} < V_1. \]

Hereinafter, with a slight abuse of notation, a subscript comma indicates partial differentiation with respect to the relevant dimensionless variables, that is, \( u_{3,1} = \partial u_3/\partial \xi_1 \). Besides, for the sake of compactness, we may sometimes drop the explicit indication of functional dependence, for example, we may write \( u_i \) instead of \( u_i(\xi_1, \xi_2, \tau) \). The equilibrium equations for plane motions (7) become (cfr.) [9]

\[ \frac{c_{11}}{c_{66}}u_{1,11} + u_{1,22} + \left( \frac{c_{12}}{c_{66}} + 1 \right) u_{2,12} = u_{1,\tau \tau}, \]

\[ u_{2,11} + \frac{c_{22}}{c_{66}}u_{2,22} + \left( \frac{c_{12}}{c_{66}} + 1 \right) u_{1,12} = u_{2,\tau \tau}. \]

While antiplane motion is governed by

\[ \frac{c_{55}}{c_{66}}u_{3,11} + \frac{c_{44}}{c_{66}}u_{3,22} = u_{3,\tau \tau}. \]

We shall look for solutions in the form of plane harmonic waves

\[ u_i(\xi_1, \xi_2, \tau) = U_i(\xi_2) \exp i(K\xi_1 - \Omega \tau), \quad i \in \{1, 2, 3\}, \]

where \( K = k_1h \) and \( \Omega = \omega T > 0 \) are the dimensionless wavenumber and angular frequency. With these definitions, \( V = \Omega/K = v/c_{SV} \). Antiplane motions have the general form

\[ U_3(\xi_2) = a_1\lambda_{SH}^1 \sinh (\lambda_{SH} \xi_2) + a_2 \cosh (\lambda_{SH} \xi_2), \quad \lambda_{SH}^2 = \frac{c_{66}}{c_{44}}K^2(V_5^2 - V^2), \]

where \( a_1 \) and \( a_2 \) are arbitrary constants, and the solution has been written in a form independent of the sign chosen for \( \lambda_{SH} \).
The equilibrium equations for the in-plane motion (10) may be cast in terms of a single fourth-order Ordinary Differential Equation (ODE) in, say, $U_1(\xi_2)$, which lends the bi-quadratic characteristic equation for $\lambda$

$$a_2 \lambda^4 - a_1 K^2 \lambda^2 + a_0 K^4 = 0,$$

where (cfr.(21) with $c_{66} V^2 = \rho c_R^2$)

$$a_2 = c_{22} c_{66},$$
$$a_1 = c_{11} c_{22} \left(1 - \frac{V^2}{V_F^2}\right) + c_{66}^2 \left(1 - V^2\right) - (c_{12} + c_{66})^2$$
$$a_0 = c_{11} c_{66} \left(1 - V^2\right) \left(1 - \frac{V^2}{V_F^2}\right).$$

The coefficient $a_1$ is the generalization to orthorhombic materials of the coefficient $B$ of Solie and Auld [19]. Clearly, the sign of $\lambda$ is immaterial, and therefore, without loss of generality, we restrict attention to the pair of solutions of equation (13) with positive real part (this amounts to selecting a branch cut for the square root)

$$\Lambda_{1,2} = K \Lambda_{1,2}, \quad \Re(\Lambda_{1,2}) > 0,$$

being

$$\Lambda_{1,2}^2 = \frac{a_1 - \sqrt{\Delta}}{2a_2}, \quad \Delta = a_1^2 - 4a_0 a_2.$$  

Physically, $\Lambda_{1,2}$ represent the ratio between longitudinal and transversal wavenumbers, that is, $\tan \beta$, where $\beta$ is the angle of wave propagation to the $x$ axis. In particular, whenever $\Lambda_{1,2} = 0$ an infinite plane wave-front propagating indefinitely is possible, that is, a bulk wave. In the isotropic case, we have that the discriminant $\Delta = \mu^2 (\lambda + \mu)^2 K^2 V^2$ is always positive and, as expected, $\Lambda_{1,2}(0) = 1$, because standing waves propagate equally in either direction. When $\Delta < 0$, $\Lambda_{1,2}$ becomes a complex conjugated pair describing evanescent waves. The expressions for $\Lambda_{1,2}$ represent a generalization to orthorhombic materials of equation (17) of Solie and Auld [19]. Similar to these, the smallest solution (in terms of absolute value) of (15) corresponds to quasi-longitudinal waves (qP), while the largest gives quasi-shear waves (qSV). For large values of $V$, we get

$$\Lambda_1 = -V^2, \quad \Lambda_2 = -\frac{V^2}{V_F^2}.$$  

It is expedient to introduce the auxiliary quantity

$$V_s^2 = \left(1 + V_F^2 \right)^{-1} \left(V_c^2 - \frac{2 c_{12}}{c_{22}}\right),$$

which allows rewriting $a_1$ such that its sign may be easily determined

$$a_1 = c_{66}^2 \left(1 + V_F^2\right) \left(V_s^2 - V^2\right).$$

We point out that, in general, $V_s^2$ may be positive, negative or zero. Indeed, the condition

$$c_{66} \equiv \sqrt{c_{11} c_{22} - c_{12}},$$

warrants that $V_s^2 \geq 0$. Besides, if $c_{12} \geq 0$, we have $0 < V_s^2 < V_F^2 < V_2^2$ and $a_1 > 0$ provided that $V^2 > V_s^2$. In the isotropic case, the inequality (18) is strictly satisfied and we have

$$V_s^2 = 2 \left[1 - \left(1 + V_F^2\right)^{-1}\right].$$
We observe that according to equation (19), it is $1 < V_s < V_1$. With the usual restriction on the Lamé constants, it is further seen that $2\sqrt{2/7} < V_a < \sqrt{2}$. Hereinafter, to fix ideas, we shall assume that

$$1 < V_s < V_c < V_1$$

Holds also in the orthorhombic case, which is usually the case for real orthorhombic materials.

In the following, when giving numerical results, we shall consider steel as a prototype for isotropic materials

$$\lambda = 115\text{GPa}, \quad \mu = 77\text{GPa},$$

and carbon-epoxy composite for orthorhombic materials

$$c_{11} = 55.15\text{GPa}, \quad c_{22} = 18.38\text{GPa}, \quad c_{66} = 9.00\text{GPa}, \quad c_{12} = 4.60\text{GPa}.$$

Table 1 gathers the dimensionless speeds for both materials. Figure 2 shows that $\Lambda_{1,2}^2$ are monotonic decreasing functions of $V$ that are concave downwards, that is, $d^2\Lambda_{1,2}/dV^2 < 0$. They possess the simple zero $\Lambda_{1,2}^2(1) = \Lambda_{1,2}^2(V_1) = 0$ and, consequently, $V = 1$ and $V = V_1$ are branch points for the square root in $\Lambda_{1,2}$, respectively. It follows that the relevant derivatives $d\Lambda_{1,2}/dV(V_1)$ turn unbounded. Obviously, $\Lambda_{1,2}$ are both real for $V < 1$, respectively purely imaginary and real for $1 < V < V_1$ and both purely imaginary for $V > V_1$. For future purposes, we determine

$$\Lambda_{1}^2(V_1) = -(1 + V_c^{-2})(V_1^2 - V_s^{-2}).$$

The solution of the equilibrium equations (10) is

$$\begin{bmatrix} U_1(\xi_2) \\ U_2(\xi_2) \end{bmatrix} = \mathbf{G} \varphi$$

where $\varphi = [\varphi_e, \varphi_o]$ is separated in the even and odd amplitudes, respectively. $\varphi_e = [e_1, e_2]$ and $\varphi_o = [o_1, o_2]$, while

$$\mathbf{G} = \begin{bmatrix} \cosh(\lambda_1 \xi_2) & \cosh(\lambda_2 \xi_2) \\ \omega_2 \sinh(\lambda_1 \xi_2) & \omega_1 \sinh(\lambda_2 \xi_2) \end{bmatrix} \begin{bmatrix} \lambda_1^{-1} \sinh(\lambda_1 \xi_2) & \lambda_2^{-1} \sinh(\lambda_2 \xi_2) \\ \omega_2 \cosh(\lambda_1 \xi_2) & \omega_1 \cosh(\lambda_2 \xi_2) \end{bmatrix}.$$ 

having let the dimensionless functions of $V^2$ (cfr., (21) equation (17))

$$\alpha_{1,2}(V) = \frac{c_{66}}{c_{12} + c_{66}} \left( \Lambda_{1,2} + \frac{V_1^2 - V_1^2}{\Lambda_{1,2}} \right).$$

It is worth noticing that $\alpha_1(V)$ blows up for $V \to 1$, for then $\Lambda_1(V) \to 0$ as $\sqrt{V - 1}$. Conversely, as $V \to V_1$, it is $\Lambda_2(V_1) \to 0$ and yet $\alpha_2(V_1) \to 0$, while

$$\alpha_1(V_1) = \left( 1 + \frac{c_{12}}{c_{66}} \right)^{-1} \Lambda_1(V_1)$$

Table 1. Dimensionless wave speeds for steel (21) and carbon-epoxy (22).

| Speed | Steel | Carbon-epoxy |
|-------|-------|--------------|
| $V_s$ | 1.24  | 1.92         |
| $V_c$ | 1.69  | 2.45         |
| $V_1$ | 1.87  | 2.48         |
| $V_o$ | 1.87  | 1.43         |
| $V_R$ | 0.93  | 0.96         |
is purely imaginary in view of (23) and of the inequalities (20). We observe that the Rayleigh function may be written in a symmetric form in terms of $a_i$ and $L_i$, $i \in \{1, 2\}$, as

$$R(V^2) = \begin{vmatrix} u_{11} & u_{12} \\ -u_{21} & -u_{22} \end{vmatrix} = s_1 - s_2,$$

having let

$$\xi_{11}(V) = 1 + \frac{c_{22}}{c_{12}} \alpha_1 \Lambda_1, \quad \xi_{12}(V) = 1 + \frac{c_{22}}{c_{12}} \alpha_2 \Lambda_2,$$

$$\xi_{21}(V) = \iota (\Lambda_1 - \alpha_1), \quad \xi_{22}(V) = \iota (\Lambda_2 - \alpha_2),$$

and, clearly,

$$s_1 = \xi_{11} \xi_{22}, \quad s_2 = \xi_{12} \xi_{21}.$$

4. Partial waves

RL waves emerge from consideration of the plate boundary conditions (BCs). In particular, when Mindlin’s BCs are considered, either the micro-chain (MC) conditions,

$$\sigma_{22} = 0 \quad \text{and} \quad u_1 = 0,$$

or the lubricated rigid support (LRS) conditions

$$\sigma_{12} = 0 \quad \text{and} \quad u_2 = 0,$$

RL waves collapse into partial waves. In standard practice, symmetric and antisymmetric (flexural) RL waves are discussed separately: they are obtained splitting the problem in its even and odd part with respect to $\xi_2$, see [1,19]. This separation holds also for partial waves. For symmetric LRS and antisymmetric MC we have

$$(\alpha_2 \lambda_1 - \alpha_1 \lambda_2) \sinh \lambda_1 \sinh \lambda_2 = 0,$$

while for symmetric MC and antisymmetric LRS, it is

$$(\alpha_2 \lambda_1 - \alpha_1 \lambda_2) \cosh \lambda_1 \cosh \lambda_2 = 0.$$
and it corresponds to a family of P modes. Therefore, P modes bounce back and forth at the plate boundaries with an integer number, \( m \), of half wavelengths occurring in between. Accordingly, they appear in the same (opposite) fashion at the plate boundaries, that is, they are symmetric (antisymmetric), when \( m \) is even (odd). Antisymmetric waves repeat periodically every two thickness cycles. In particular, the P mode \( m = 0 \) describes a plane wave with speed \( V = V_1 \), that is, it gives bulk longitudinal waves. Symmetric and antisymmetric P modes possess the eigenforms \( f_{oP} = (0, 1) \) and \( f_{aP} = (0, 1) \), respectively. Similarly, the second set of solutions

\[
\lambda_1 = \nu 1/2n\pi, \quad n \in \{0, 1, 2, \ldots \}. \quad (30)
\]

Provides a family of SV modes, which may equally be even or odd according to the parity of \( n \). Symmetric and antisymmetric SV modes possess the eigenforms \( f_{oSV} = (1, 0) \) and \( f_{aSV} = (1, 0) \), respectively. In the terminology of Mace and Manconi [16], partial waves describe the uncoupled-blocked systems and their spectra (equations (29) and (30)) form the skeleton of the eigenvalues, wherein the wavenumber \( K \) acts as variable parameter.

The definition (14) together with equations (29) and (30) show that P and SV modes may be written as a function of \( K^2 \) and \( \nu^2 \). The dimensionless group velocity of such partial waves is given by

\[
V_{g,1,2}(\nu) = \frac{d\nu}{dK} = V - \frac{\Lambda_{1,2}}{d\Lambda_{1,2}/d\nu}(\nu)
\]

wherein the last term is the reciprocal of the logarithmic derivative. In particular

\[
V_{g1}(1) = 1, \quad V_{g1}(V_1) = V_1 \left( 1 - \frac{1 - V^2/V_1^2}{1 - V^2/(V_1^2 - 1) + \left(1 + V^2/(V_1^2 - V^2)\right)} \right) < V_1, \quad (32)
\]

and clearly \( V_{g2}(V_1) = V_1 \). In the isotropic case, it is \( V_{g1}(V_1) = V_1^{-1} \). In light of the fact that \( \Lambda_{1,2}^2 \) are decreasing functions of \( \nu \) and observing that equation (31) may be rewritten as

\[
V_{g1,2}(\nu) = V - 2\frac{\Lambda_{1,2}^2}{d\Lambda_{1,2}^2/d\nu}.
\]

It is easily proved that, for \( \nu < 1 \), we have \( V_g > V \), that is propagation is anomalous, for waves move slower than the wave packet as ripples in a pond. However, there are no partial wave branches in that region. For \( 1 < \nu < V_1 \), there are only SV wave branches describing SV waves moving faster than the wave packet, that is, \( V > V_{g1} \). Finally, for \( \nu > V_1 \), both P and SV waves move faster than the wave packet.

P and SV modes frequency spectra for a steel plate are plotted in Figure 3. It clearly appears that P modes with \( m \geq 1 \) asymptote bulk longitudinal waves from above and similarly SV modes asymptote bulk SV waves from above. Indeed, writing \( \Lambda_2 = K\Lambda_2 \) and considering the limit \( K \rightarrow +\infty \) along any curve (29), demands that \( \Lambda_2 \rightarrow 0 \), which in turn requires \( \nu \rightarrow V_1^+ \), for the product \( K\Lambda_2 \) to yield a finite purely imaginary number. A similar argument shows that \( \nu \rightarrow 1^+ \) in the limit \( K \rightarrow +\infty \) for SV modes (30).

5. RL waves

For a free plate, we have the BCs

\[
\sigma_{22} = \sigma_{12} = \sigma_{32} = 0, \quad \text{at} \ x_2 = \pm h,
\]

that, introducing the constitutive law (6), become

\[
\frac{c_{12}}{c_{66}} u_{1,1} + \frac{c_{22}}{c_{66}} u_{2,2} = 0, \quad u_{1,2} + u_{2,1} = 0, \quad \text{and} \ u_{3,2} = 0, \quad \text{at} \ \xi_2 = \pm 1. \quad (33)
\]
5.1. SH waves

As already pointed out, in orthorhombic materials, SH waves are decoupled from SV and P waves. Enforcing the last of the BCs (33) on the general solution (12) lends the dispersion relation

\[ \sinh \left( \frac{2l}{3} \right) = 0, \]

whose solutions are

\[ l = \frac{p^2}{C_0}p = 2, 3, \ldots \]

Besides, we have that \( a_1 = 0 \); hence, \( U_3(\xi_2) \) is an even function of \( \xi_2 \). The frequency spectrum for SH waves is plotted in Figure 4. It is worth observing that, for large values of \( K \), the spectrum curves tend to the SH bulk wave velocity \( V = V_5 \).

5.2. Symmetric waves

Consideration of symmetric waves lends the homogeneous algebraic system

\[
S(K, \Omega) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

Figure 3. Partial waves for steel: even (odd) P modes (dotted, black) and odd (even) SV modes (dashed, red) in the left (right) panel. The Rayleigh wave line spectrum is also shown (dash-dotted, blue).

Figure 4. Frequency spectrum (34) for SH waves in a carbon-epoxy plate.
where we have let the matrix

$$S(K, \Omega) = \begin{bmatrix} \zeta_{11} \cosh \lambda_1 & \zeta_{12} \cosh \lambda_2 \\ -\zeta_{21} \sinh \lambda_1 & -\zeta_{22} \sinh \lambda_2 \end{bmatrix}.$$ 

This matrix may be rewritten in hermitian form

$$S_h(K, \Omega) = \begin{bmatrix} \zeta_{11} \cosh \lambda_1 & \zeta_{12} \cosh \lambda_2 \\ \zeta_{21} \sinh \lambda_1 & -\zeta_{22} \sinh \lambda_2 \end{bmatrix}.$$ 

Demanding that non-trivial solutions of the system (35) exist provides the dispersion relation (cfr., equation (8.1.54)))

$$d_s(K^2, \Omega^2) = 0,$$

with

$$d_s(K^2, \Omega^2) = s_1 \coth \lambda_1 - s_2 \coth \lambda_2.$$ 

The frequency spectrum of a plate made of carbon-epoxy composite is plotted in Figure 5. We observe that odd SV modes are obtained by setting $S_{11} = 0$ and even P modes by putting $S_{22} = 0$, where $S_{ij}$ denotes the $(i,j)$-element of the matrix $S$ of equation (35).

We observe that the first branch of the spectrum rests in the sector $V < 1$, where $\lambda_1, \lambda_2$ are real numbers, and therefore, for large values of $K$, we have $\coth \lambda_1, \lambda_2 \to 1$ and the solution of (37) tends to Rayleigh wave speed equation. Consequently, for this branch, SV modes cannot act as guiding curves, that is, the spectrum branches do not follow any of the SV modes (30) (see,(19, §4)) for a different take on the concept of guiding curve). In contrast, for all the other branches of the RL frequency spectrum, SV modes are guidelines in the short-wave high-frequency (SWHF) regime. This occurs because such branches rest in the sector $1 < V < V_1$ where $\lambda_1$ is purely imaginary and $\lambda_2$ is real; as $K$ grows larger, $\coth \lambda_1$ oscillates wildly unless (30) holds, while $\coth \lambda_2 \to 1$. Then, equation (37) is satisfied provided that $s_2 \to 0$, which occurs for $V \to 1^+$. We thus proved that a definite SWHF limit exists provided that the spectrum branches follow odd SV modes and their phase velocity asymptotes the shear bulk wave speed from above. A similar analysis reveals that, in the sector $V > V_1$, P modes act as guiding curves.

We conclude that when the wavelength becomes very small compared to the plate thickness, only the first spectrum branch is independent of the boundary pair and behaves like only one existed. We can
then interpret SV (P) modes as the perturbation of shear (longitudinal) bulk waves which take into account the pair of boundaries.

We further emphasize that the concept of guiding curve is strictly related to the idea of weak coupling in the sense developed by Mace and Manconi [16]. Indeed, for a weakly coupled system, the spectrum branches quickly collapse onto partial waves outside the close neighbourhood of the veering points.

The long-wave low-frequency (LWLF) approximation of the first symmetric spectrum branch reveals that the system behaviour is equivalent to longitudinal vibrations of a beam-plate with Young’s modulus $E_c$ [23]

$$E_c k^2 = \rho \omega^2.$$ 

5.3. Antisymmetric waves

Consideration of antisymmetric (flexural) waves demands taking the odd part for $\sigma_{yy}$ and the even part for $\sigma_{xy}$ in equation (33), and it gives the homogeneous algebraic system

$$\mathbf{A}(K^2, \Omega^2) \begin{bmatrix} o_1 \\ o_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$ \hspace{1cm} (38)

where

$$\mathbf{A}(K^2, \Omega^2) = \begin{bmatrix} \zeta_{11} \sinh \lambda_1 & \zeta_{12} \sinh \lambda_2 \\ -\zeta_{21} \cosh \lambda_1 & -\zeta_{22} \cosh \lambda_2 \end{bmatrix}.$$ 

This matrix may be rewritten in hermitian form

$$\mathbf{A}_h(K^2, \Omega^2) = \begin{bmatrix} \zeta_{11} \sinh \lambda_1 & \zeta_{12} \sinh \lambda_2 \\ -\zeta_{21} \cosh \lambda_1 & -\zeta_{22} \cosh \lambda_2 \end{bmatrix}.$$ \hspace{1cm} (39)

The corresponding dispersion relation $d_o = 0$ is (cfr. [1] equation (8.1.59))

$$d_o(K^2, \Omega^2) = s_1 \tanh \lambda_1 - s_2 \tanh \lambda_2.$$ \hspace{1cm} (40)

The frequency spectrum for flexural waves in a carbon-epoxy plate is shown in Figure 6. We observe that even SV modes are obtained by setting $A_{11} = 0$ and odd P modes by putting $A_{22} = 0$. Once again, the first branch rests in the sector $V < 1$ and therefore, it asymptotes Rayleigh waves in the SWHF...
approximation. Branches in the sector $1 < V < V_1$ are guided by SV modes in the SWHF limit and their phase speed tends to the shear bulk wave speed from above; the argument going as in the symmetric case.

The long-wave low-frequency (LWLF) approximation of the first flexural spectrum branch is given by

$$D_{11} k^4 - 2 h \rho w^2 = 0, \quad D_{11} = E\nu I_{11}, \quad I_{11} = \frac{2}{3} h^3,$$

corresponding to flexural vibrations of an orthotropic Kirchhoff beam-plate with flexural rigidity $D_{11}$ and second moment of inertia $I_{11}$.

6. Internal veering

In classical veering, as illustrated by Mace and Manconi [15], the dispersion relation emerges setting to zero the determinant of an hermitian matrix whose off-diagonal terms are small. Indeed, diagonal terms represent the dispersion relation of some mechanically well-defined 1D systems, while off-diagonal terms are expression of the coupling among these. The hermitian nature of the matrix comes from the action-reaction principle. In classical veering, therefore, each system is clearly identifiable at the beginning and likewise are its dispersion modes. Besides, the position of each mechanical system is defined by its own displacement degree of freedom. Veering brings a rotation of the polarization vector from one system to the other. RL dispersion curves exhibit a different form of veering, which we name internal after the observation that it originates from the interaction of SV and P partial waves. In case of internal veering, the definition of the interacting modes is not so straightforward. Also, polarization rotation occurs differently.

6.1. Symmetric waves

We now describe the essential features of internal veering with respect to symmetric modes for a free plate. Veering occurs when even P and odd SV modes intersect, that is veering points are located by solving the pair of transcendental equations

$$S_{11} = S_{22} = 0. \quad (41)$$

This amounts to letting in turn $\epsilon_2 = 0$ and then $\epsilon_1 = 0$. With respect to the terminology developed by Mace and Manconi [15], this has no correspondence to either the uncoupled-blocked system or to the uncoupled disconnected system. Indeed, as already pointed out, P and SV modes emerge from considering Mindlin’s micro-chain or lubricated wall boundary conditions. Therefore, the interacting systems share the same kinematical description but differ by the boundary conditions. The off-diagonal entries are coupling terms, and they correspond to odd P and even SV modes. In the hermitian writing of equation (36), a form of action–reaction principle is preserved.

Let $(K_0, \Omega_0)$ define the position of a veering point, that is, it is a solution of (41). To fix ideas, we consider veering points on the line $\Omega_0 = V_1 K_0$ as in Figure 5, which arise from the interaction between odd SV modes and the first P mode (i.e. bulk longitudinal waves $m = 0$). We observe that this choice appears most unfavourable, for we are right at the branch point for $\lambda_2$. $K_0$ may be simply obtained by solving equation (30) with $V = V_1$ (for bulk longitudinal waves we have $\lambda_2 (V_1) = 0$)

$$K_0 \lambda_1 (V_1) = \frac{1}{2} (1 + 2 n) \pi, \quad n \in \mathbb{N},$$

and making use of equation (23), we get

$$K_0^2 = \frac{\left( \frac{1}{2} + n \right)^2 \pi^2}{4 (1 + V_2^{-2}) (V_1^2 - V_2^2)}. \quad (42)$$
We observe that $K_0$ is real, provided that $V_* < V_1$, as we already assumed in equation (20). Besides, it is easy to see that

$$dV = \frac{d\Omega}{K} - \frac{\Omega}{K^2} dK,$$

and when $(K, \Omega)$ lies on a curve $V = \text{const}$ we get

$$dV = 0 \iff V = V_g,$$

That is the phase velocity equals the group velocity. Expanding in Taylor series the matrix $S_h$ about the veering point is possible, despite the branch point singularity for the square root in $\Lambda_2(V_1)$, because, as already mentioned, the dependence on the lambdas is really through their square, which is the reason by which the sign of the lambdas is immaterial. Indeed we find, at leading order,

$$(S_{h0} + dS_h)[K, \Omega, K_0, \Omega_0] = \begin{bmatrix} q_{11} dK + r_{11} d\Omega & p_{12} \\ p_{21} & q_{22} dK + r_{22} d\Omega \end{bmatrix},$$

where, after tedious manipulations,

$$r_{11} = -(1)^n \xi_1(V_1) \xi_2(V_1) \frac{d\Lambda_1}{dV}(V_1),$$

$$r_{22} = -(1)^n \frac{c_{22} + c_{66}}{c_{21} c_{12} + c_{66}} \xi_2(V_1) \left( \frac{d\Lambda_2}{dV}(V_1) - 2 \frac{c_{66}}{c_{12}} V_1 \right),$$

and

$$p_{12} = -p_{21} = \omega s_2(V_1).$$

In (46b), the derivative of $\Lambda_2$ appears that is bounded. Making use of equations (23,26,30), we see that

$$\xi_1(V_1) = 1 - \frac{c_{22} + c_{66}}{c_{21} c_{12} + c_{66}} (V_*^2 - V_1^2),$$

$$\xi_2(V_1) = 1,$$

$$\xi_2(V_1) = - \frac{c_{22} + c_{66}}{c_{21} c_{12} + c_{66}} \sqrt{(1 + V_2^2)(V_*^2 - V_2^2)} = - \frac{c_{12}}{c_{21} + c_{66}} \frac{(1 + n)\pi}{K_0},$$

which are functions of $V_1$ alone.

In general, $dK$ and $d\Omega$ are arbitrary quantities, however, when moving along an SV/P partial wave, we have, respectively,

$$d(K \Lambda_{1,2}) = 0,$$

hence we get the connection $d\Omega = V_{g1,2} dK$ that, substituted into the expansion for $dS_{11}$ ($dS_{22}$), yields the result

$$\frac{q_{ii}}{r_{ii}} = -V_{g_i}, \quad i \in \{1, 2\}.$$

We observe that $dK^2 = 2K dK$ and $d\Omega^2 = 2\Omega d\Omega$, thus $d\Omega^2/dK^2 = V V_g$ is the product of the phase and group velocities. Then, recalling that the dispersion relation is a transcendental function of $K^2$ and $\Omega^2$, we prefer to write (cfr.,(15) equation (16)))
\[ (S_{h0} + dS_h)[K^2, \Omega^2, V_1] = \begin{bmatrix} \frac{\partial}{\partial \Omega} (d\Omega^2 - c_1^2 dK^2) & \psi_2(V_1) \\ -\psi_2(V_1) & \frac{\partial}{\partial \Omega} (d\Omega^2 - c_1^2 dK^2) \end{bmatrix}, \]  

where, making use of equation (44) for \( c_2 \),

\[ c_1 = \sqrt{V_1 V_{g1}}, \quad c_2 = \sqrt{V_1 V_{g2}} = V_1. \]  

Consequently, we deduce that, in the neighbourhood of a veering point and within a leading term Taylor approximation, the system behaves like a pair of interacting tout strings, whose wave speeds \( c_1 \) and \( c_2 \) are the geometric mean of the relevant phase and group velocities. In particular, for all the countable infinite number of veering points on the line \( V = V_1 \), the tout strings wave speeds are the same and therefore veering repeats itself periodically. In general, we can say that the frequency spectra of the even P and odd SV partial waves define the envelope of the wave speed field for a pair of tout strings whose properties are frequency dependent.

Letting

\[ \Delta K = K^2 - K_0^2, \quad \Delta \Omega = \Omega^2 - \Omega_0^2, \]

we write the approximate dispersion relation (cfr. [15], equation (14))

\[ \Delta \Omega^2 - (c_1^2 + c_2^2) \Delta K \Delta \Omega + c_1^2 c_2^2 \Delta K^2 - 4\Omega_0^2 \eta^2 = 0, \]  

with

\[ \eta^2 = \frac{s_2(V_1)^2}{r_1 r_2}. \]  

The approximation (51) may be obtained directly operating a Taylor expansion of the dispersion relation (37) up to second order terms. The solution of equation (51) provides two branches, named upper and lower,

\[ \Delta \Omega = \frac{1}{2} (c_1^2 + c_2^2) \Delta K \left( 1 \pm \sqrt{1 - 4 \frac{c_1^2 c_2^2 - 4\Omega_0^2 \eta^2/\Delta K^2}{(c_1^2 + c_2^2)^2}} \right). \]  

It should be emphasized that the string model expansion (49) is consistent inasmuch as \( \Omega - \Omega_0 \) and \( K - K_0 \) are small, so that a leading term approximation is meaningful. Assuming \( \Omega - \Omega_0 \sim K - K_0 \sim \epsilon \ll 1 \), we have, for the solution set of equation (51),

\[ 4\Omega_0^2 \eta^2 - 4\Omega_0^2 \left[ 1 - (c_1^2 + c_2^2) V_1^{-1} + c_1^2 c_2^2 V_1^{-2} \right] \epsilon, \]

where the stands for “same order as”. Whence, using equation (50), we demand

\[ \eta^2 \sim (V_{g1} - 1)(V_1 - 1) \epsilon, \]

which is independent of the veering point under consideration, that is, independent of \( n \). Therefore, here we require \( \eta^2 \) to be small while \( 4\Omega_0^2 \eta^2 \) may be, and generally is, large. This approach is at variance with that developed by Mace and Manconi [15]. As an example, for steel we have \( \eta^2 \approx 0.28125 \) and for carbon-epoxy \( \eta^2 \approx 0.0163 \). The smallness of \( \eta^2 \) sets the size of neighbourhood where the tout string approximation is meaningful, regardless of the veering point under scrutiny.

### 6.2. Numerical results

Figure 7 plots the simple approximation (53) for a carbon-epoxy composite plate at the veering point corresponding to the SV mode \( n = 1 \). The same approximation is repeated in Figure 8 for the SV mode
\( n = 3 \) and, as anticipated, the same behaviour is matched. In general, consideration of the leading term alone (string model) appears surprisingly accurate, even in the large, inasmuch as the spectrum branches are well represented (guided) by the corresponding partial waves. For instance, moving along the lower branch in Figure 7, we veer from longitudinal bulk waves (P mode \( m = 0 \)) to the SV mode \( n = 1 \). In contrast, the upper branch is generally not well described by either partial wave until the close neighbourhood of veering is reached. For this reason, the Taylor expansion method, as here described, is doomed to provide poor accuracy there, no matter how many terms in the expansion. The reason by which the spectrum is not guided by the SV mode on reaching the veering point along the upper branch may be ascribed to the presence of yet another veering point, so that the two interact (see Figure 5). In fact, moving along the upper branch in Figure 7, we see that the spectrum behaves in between a P and an SV mode until a point \( \gamma = \gamma_c \) is reached, where \( S_{11} = S_{12} = 0 \). Beyond this point, the spectrum approaches the even P mode \( m = 0 \), and the approximation is excellent again. It should be emphasized that this departure from the guiding curve is not possible in systems of 1D elements, wherein dispersion is bound to a number of dispersion curves.

**7. Conclusion**

We analyse RL waves travelling in a plane of material symmetry for an orthorhombic layer. Emphasis is placed on \textit{veering}, that is a coupling phenomenon by which wave branches exchange their role in close

\[ \text{Figure 7. Approximation (57) near the veering point } n = 1 \text{ for a plate made of carbon-epoxy composite (dashed, red) superposed onto the frequency spectrum of symmetric waves (solid, black).} \]

\[ \text{Figure 8. Approximation (57) near the veering point } n = 3 \text{ for a plate made of carbon-epoxy composite (dashed, red) superposed onto the frequency spectrum of symmetric waves (solid, black).} \]
proximity to their intersection point (the veering point). Physically, this amounts to destructive wave interference taking place at the veering point (that is a point of no propagation) and constructive interference occurring in its close neighbourhood. Interference occurs in such a way that the “emerging” wavemodes are swapped compared to the “incoming” modes. We first recall that RL modes are themselves originating from interference of partial waves (here named P and SV modes), which express waves complying with special boundary conditions allowing for no mode conversion. In this sense, partial waves appear “more fundamental” than RL modes, for it is precisely their combination through the boundary conditions which originates the latter. Indeed, this mechanism is apparent in the frequency spectrum of RL waves, wherein partial waves take up the role of guiding waves, in the sense that they bound the propagation curves. We show here that the same mechanism stands at the ground of veering. Indeed, veering points for symmetric (antisymmetric) RL modes corresponds to intersection points for even P/odd SV (odd P/even SV) partial waves. This situation can be compared with veering in two-dimensional systems, wherein eigenmodes pertaining to either mechanical system (considered independent or uncoupled) interact by means of the coupling device. In the case of RL modes, asymptotic analysis reveals that interaction occurs in the form of a pair of tout strings whose wave speed are the geometric mean of the relevant wave phase and group velocities. An approximation dispersion relation is obtained whose range of validity depends on the strength of the coupling. Numerical results show that the quality of the approximation is good inasmuch as interaction among neighbouring veering points does not occur. Indeed, this interaction weakens the role of partial waves as guiding waves.

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ORCID iD
Andrea Nobili https://orcid.org/0000-0002-9657-5903

Notes
1. In Solie and Auld’s study [19], reference is made to the minus and to the plus solutions, which however correspond to the smallest and to the largest only inasmuch as \( a_1 > 1 \).

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