LOWER BOUNDS FOR MOMENTS OF THE DERIVATIVE OF THE RIEMANN ZETA FUNCTION

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Abstract. We establish in this paper sharp lower bounds for the $2k$-th moment of the derivative of the Riemann zeta function on the critical line for all real $k \geq 0$.

Mathematics Subject Classification (2010): 11M06

Keywords: lower bounds, moments, derivative, Riemann zeta function

1. Introduction

It is an important subject in analytical number theory to investigate moments of the Riemann zeta function $\zeta(s)$ on the critical line as they can be applied to study the maximum size of the zeta function as well as primes in short intervals via zero density estimates. We denote the $2k$-th moment of $\zeta(s)$ on the critical line by

$$M_k(T) = \int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2k} dt.$$ 

The study on $M_k(T)$ dates back to the work of G.H. Hardy and J. E. Littlewood [13], who established an asymptotic formula for $M_1(T)$. In [18], A. E. Ingham established an asymptotic formula for $M_2(T)$. No other asymptotic formulas are known for $M_k(T)$ except for the trivial case $k = 0$. Despite of this, J. P. Keating and N. C. Snaith [21] made precise conjectured formulas for $M_k(T)$ for all real $k \geq 0$ by drawing analogues with the random matrix theory. Using multiple Dirichlet series, A. Diaconu, D. Goldfeld and J. Hoffstein [10] also obtained the same conjectured formulas. More precise asymptotic formulas with lower order terms were conjectured by J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith in [5].

Owing much to the work in [1, 2, 6–8, 14–17, 26–31, 34, 35], we now have sharp upper and lower bounds for $M_k(T)$ of the conjectured order of magnitude for all $k \geq 0$ with some of them being valid under the truth of the Riemann hypothesis (RH).

Among the many methods applied in the above work, we point out notably a simple and powerful method developed by Z. Rudnick and K. Soundararajan [31, 32] towards establishing sharp lower bounds for moments of families of $L$-functions, a method of K. Soundararajan [35] and its refinement by A. J. Harper [14] to derive sharp upper bounds for moments of families of $L$-functions under the generalized Riemann hypothesis (GRH). We note also an upper bounds principle developed by M. Radziwiłł and K. Soundararajan in [27] for establishing upper bounds for moments of families of $L$-functions as well as its dual lower bounds principle developed by W. Heap and K. Soundararajan in [10].

Similar to $M_k(T)$, it is also interesting to study moments of the derivatives of $\zeta(s)$ on the critical line. For integers $l \geq 1$, let

$$I_{k,l}(T) = \int_T^{2T} |\zeta^{(l)}(\frac{1}{2} + it)|^{2k} dt.$$ 

An asymptotic formula for $I_{1,l}(T)$ is also given in the above mentioned work of A. E. Ingham [18]. In [4], J. B. Conrey obtained an asymptotic formula for $I_{2,l}(T)$. Also, in connection with the random matrix theory, J.B. Conrey, M.O. Rubinstein and N.C. Snaith [9, Conjecture 1] conjectured that

$$I_{k,1}(T) \sim a_k b_k T (\log T)^{k^2 + 2k},$$

for some explicit constants $a_k, b_k$.

Under RH, M. B. Milinovich [22] established essentially upper bounds of the correct order of magnitude for $I_{k,l}(T)$ for positive integers $k, l$. His result was further improved to yield optimal upper bounds by A. Ivić [19] for $I_{k,2}(T)$ for positive integers $k$. The methods employed in [22] and [19] allow one to deduce upper bounds for $I_{k,l}(T)$ from the corresponding ones for $M_k(T)$. As sharp upper bounds for $M_k(T)$ are known for all $k \geq 0$ under RH from the
Lemma 2.1. 

Let \( k \) be a real-valued function that is continuously differentiable on the interval \([0, \infty)\). The proof of Theorem 1.1 also makes use of some approaches there as well. 

The lower bound principle. 

3.1. The lower bound principle. We may assume that \( k > 0 \) and \( T \) is a large number throughout the proof. We also point out that the explicit constants involved in estimations using \( \ll \) or the big-\( O \) notations in the proof depend on \( k \) only and are uniform with respect to \( p \) and \( T \).

We follow the ideas of A. J. Harper in \([14]\) to define for a large number \( M \) depending on \( k \) only,

\[
\alpha_0 = 0, \quad \alpha_j = \frac{2^{j-1}}{(\log \log T)^2} \quad \forall \ j \geq 1, \quad J = J_{k,T} = \max\{j : \alpha_j \leq 10^{-M}\}.
\]
We denote $\ell_j := \lceil e^{2k\alpha_j^{-3/4}} \rceil$ for $1 \leq j \leq J$ and divide the interval $(0, T^\alpha_j]$ into disjoint subintervals $I_j = (T^{\alpha_{j-1}}, T^{\alpha_j}]$, $1 \leq j \leq J$. We define for any real number $\ell$ and any $x \in \mathbb{R}$,

$$E_\ell(x) = \sum_{j=0}^{\ell} \frac{x^j}{j!}.$$ 

We also define for any real number $\alpha$ and any $1 \leq j \leq J$,

$$P_j(s) = \sum_{p \in I_j} \frac{1}{p^s}, \quad N_j(s, \alpha) = E_{\ell_j}(\alpha P_j(s)), \quad N(s, \alpha) = \prod_{j=1}^{J} N_j(s, \alpha).$$

We deduce from [11] (3.1) and Lemma 2.1 that for any large number $N$, we can take $T$ large enough so that

(3.1) \[ P_1(1) \leq \frac{1}{N}\ell_1, \quad P_j(1) \leq \min\left(10, \frac{1}{N}\ell_j\right), \quad 2 \leq j \leq J. \]

We denote $\Omega(n)$ for the number of prime powers dividing $n$ and $g(n)$ for the multiplicative function given on prime powers by $g(p^r) = 1/r!$ and define functions $b_j(n), 1 \leq j \leq J$ such that $b_j(n) = 0$ or $1$ and that $b_j(n) = 1$ only when $\Omega(n) \leq \ell_j$ and all the prime factors of $n$ are from the interval $I_j$. We then have

$$N_j(s, \alpha) = \sum_{n_j} \frac{\alpha^{\Omega(n_j)}}{g(n_j)} b_j(n_j)\frac{1}{n_j^s}, \quad 1 \leq j \leq J.$$

It follows from [11], Section 3.1 that each $N_j(s, \alpha)$ is a short Dirichlet polynomial of length at most $T^{\alpha_j\lceil e^{2k\alpha_j^{-3/4}} \rceil}$ and that $N(s, \alpha)$ is also a short Dirichlet polynomial of length at most $T^{40e^2k10^{-M/4}}$.

We now write for simplicity that

(3.2) \[ N(s, \alpha) = \sum_{n} \frac{a_\alpha(n)}{n^s}. \]

We note that $a_\alpha(n) \neq 0$ only when $n = \prod_{1 \leq j \leq J} n_j$ such that $b_j(n_j) = 1$, in which case we have

$$a_\alpha(n) = \prod_{n_j} \frac{\alpha^{\Omega(n_j)}}{g(n_j)} b_j(n_j).$$

We combine [11] (2.1), (3.3) to see that for all $n \geq 3$,

(3.3) \[ a_\alpha(n) \leq e^{\frac{|\alpha|\log n}{\log 2} (1+O(\frac{1}{\log n}))} \text{ and } a_k(n) = 0 \text{ when } n > T^{40e^2k10^{-M/4}}. \]

Moreover, we note that [11] (3.4) implies that for $\Re(s) \geq -1/\log T$ and $T$ large enough,

(3.4) \[ |N(s, \alpha)| \ll e^{\frac{|\alpha|\log T}{\log 2} (1+O(\frac{1}{\log T}))} T^{40e^2k10^{-M/4}+1/10} \log T. \]

In the proof of Theorem 1.1, we need the following lower bounds principle of W. Heap and K. Soundararajan [16] for our case.

**Lemma 3.2.** With notations as above, we have for $0 < k \leq 1/2$,

(3.5) \[ \int_1^T -\zeta'\left(\frac{1}{2} + it\right)N\left(\frac{1}{2} + it, k - 1\right)N\left(\frac{1}{2} - it, k\right)dt \ll \left(\int_1^T |\zeta'\left(\frac{1}{2} + it\right)|^{2k}dt\right)^{1/2} \left(\int_1^T |\zeta'\left(\frac{1}{2} + it\right)|^2N\left(\frac{1}{2} + it, k - 1\right)^2dt\right)^{(1-k)/2} \times \left(\prod_{j=1}^{J} \int_1^T \left|N_j\left(\frac{1}{2} + it, k\right)\right|^2 + |Q_j\left(\frac{1}{2} + it, k\right)|^{2r_1}dt\right)^{k/2}. \]

Also, we have for $k > 1/2$,

(3.6) \[ \int_1^T -\zeta'\left(\frac{1}{2} + it\right)N\left(\frac{1}{2} + it, k - 1\right)N\left(\frac{1}{2} - it, k\right)dt \leq \left(\int_1^T |\zeta'\left(\frac{1}{2} + it\right)|^{2k}dt\right)^{1/2} \left(\prod_{j=1}^{J} \int_1^T \left|N_j\left(\frac{1}{2} + it, k\right)\right|^2 + |Q_j\left(\frac{1}{2} + it, k\right)|^{2r_1}dt\right)^{k/2} \frac{2k-1}{\ell_j}. \]

Here the implied constants in (3.5) and (3.6) depend on $k$ only, and we define

$$Q_j(s, k) = \left(\frac{64 \max(2, k + 3/2)P_j(s)}{\ell_j}\right)^{1/2}.$$
with \(r_k = 2 + [1/k]\) for \(0 < k \leq 1/2\) and \(r_k = 1 + [2k/(2k-1)]\) for \(k > 1/2\).

We skip the proof of the above lemma as it can be established similar to those of [11, Lemma 3.2-3.3]. We deduce from the above lemma that in order to establish Theorem 1.1, it suffices to prove the following three propositions.

**Proposition 3.3.** With notations as above, we have for \(k > 0\),
\[
(3.7) \quad \int_1^T -\zeta'(\frac{1}{2} + it)N(\frac{1}{2} + it, k-1)N(\frac{1}{2} - it, k)dt \gg T(\log T)^{k^2+1}.
\]

**Proposition 3.4.** With notations as above, we have for \(0 < k \leq 1/2\),
\[
(3.8) \quad \int_1^T |\zeta'(\frac{1}{2} + it)|^2 |N(\frac{1}{2} + it, k-1)|^2 dt \ll T(\log T)^{k^2+2}.
\]

**Proposition 3.5.** With notations as above, we have for \(k > 0\),
\[
\int_1^T \prod_{i=1}^S (|N(\frac{1}{2} + it, k)|^2 + |Q_i(\frac{1}{2} + it, k)|^{2\epsilon_i})dt \ll T(\log T)^{k^2}.
\]

We shall omit the proof of Proposition 3.5 as it is similar to that of [11, Proposition 3.5], upon making use of Lemma 2.2. In the rest of the paper, we shall prove the remaining two propositions.

### 3.6. Proof of Proposition 3.3

The proof is based on the approaches used in Section 5 of [3] and Section 5 of [24]. We denote the left side expression in (3.7) by \(S_1\) and apply Cauchy’s residue theorem to deduce that
\[
S_1 = \frac{1}{2\pi i} \int_C -\zeta'(s)N(s, k-1)N(1-s, k)\,ds,
\]
where \(C\) consists of line segments from \(\frac{1}{2} + it\) to \(\kappa + it\), then from \(\kappa + it\) to \(\kappa + iT\) and lastly from \(\kappa + iT\) to \(\frac{1}{2} + iT\), where \(\kappa = 1 + (\log T)^{-1}\).

We apply (3.4) and the estimation (see [12, (20)])
\[
\zeta'(s) \ll \begin{cases} 
(1 + |t|)^{(1-\Re(s))/2+\epsilon}, & 0 \leq \Re(s) \leq 1, \\
(1 + |t|^\epsilon), & \Re(s) \geq 1,
\end{cases}
\]
to see that the integral is bounded by \(O(T^{1-\epsilon})\) on the horizontal edges of the contour. We thus conclude that
\[
(3.9) \quad S_1 = S_{1,R} + O(T^{1-\epsilon}),
\]
where
\[
(3.10) \quad S_{1,R} = \frac{1}{2\pi i} \int_{\kappa+i}^{\kappa+iT} -\zeta'(s)N(s, k-1)N(1-s, k)\,ds + O(T^{1-\epsilon}).
\]

To evaluate \(S_{1,R}\), we define the Dirichlet convolution \(f \ast g\) for two arithmetic functions \(f(k), g(k)\) by
\[
f \ast g(k) = \sum_{m+n=k} f(m)g(n).
\]
Using this notation and that given in (3.2), we apply Lemma 2.2 to evaluate \(S_{1,R}\) in (3.10) to see that
\[
S_{1,R} = \frac{T-1}{2\pi} \sum_n \frac{(\log \ast a_{k-1})(n) \cdot a_k(n)}{n} + O\left(\left(\sum_{n=1}^\infty \frac{(\log \ast a_{k-1})(n)^2}{n^{2\kappa-1}}\right)^{1/2} \left(\sum_{n=1}^\infty \frac{a_k(n)^2}{n^{1-2\kappa}}\right)^{1/2}\right).
\]
We apply the estimations given in (3.3) to see that for \(T\) large enough,
\[
\sum_{n=1}^\infty \frac{a_k(n)^2}{n^{1-2\kappa}} \ll e^{4k\log T/\log \log T} \sum_{n \leq T^{4k^{-2}10^{-M/4}}} \frac{1}{n^{1-2\kappa}} \ll T^{1-\epsilon}.
\]
Moreover, we have that
\[
(\log \ast a_{k-1})(n) \leq \log n \sum_{n \leq T^{4k^{-2}10^{-M/4}}} |a_{k-1}(n)| \leq T^{1/2-\epsilon} \log n.
\]
It follows from the above that
\[
\sum_{n=1}^\infty \frac{(\log \ast a_{k-1})(n)^2}{n^{2\kappa-1}} \ll T^{1-2\epsilon} \sum_{n=1}^\infty \frac{\log^2 n}{n^{2\kappa-1}} \ll T^{1-\epsilon},
\]
where the last estimation above follows from the bound that (see \([14] (16)\)) uniformly for \(\sigma > 1\) and any integer \(i \geq 0\),

\[
\sum_{n=1}^{\infty} \frac{\log^i n}{n^\sigma} \ll \frac{1}{(\sigma - 1)^{i+1}}.
\]

We then conclude from the above discussions that

\[
S_{1,R} = \frac{T - 1}{2\pi} \sum_{n,m} \frac{a_{k-1}(m)ak(mn)(\log n)}{mn} + O(T^{1-\varepsilon})
\]

\[= \frac{T - 1}{2\pi} \sum_{n} \log n \sum_{m} \frac{a_{k-1}(m)ak(mn)}{m} + O(T^{1-\varepsilon}).\]

It remains to estimate the last expression above. To do so, we may assume that \(n = \prod_{j=1}^{\mathcal{J}} n_j\) with \(b_j(n_j) = 1\) for \(1 \leq j \leq \mathcal{J}\). Then the inner sum of the last expression above becomes

\[
\sum_{m} \frac{a_{k-1}(m)ak(mn)}{m} = \prod_{j=1}^{\mathcal{J}} \left( \sum_{m_j} \frac{k^{\Omega(n_j m_j)}(k-1)^{\Omega(m_j)}}{g(n_j m_j)g(m_j)} b_j(n_j m_j) b_j(m_j) \right)
\]

\[= \prod_{j=1}^{\mathcal{J}} \left( \sum_{m_j} \frac{k^{\Omega(n_j m_j)}(k-1)^{\Omega(m_j)}}{g(n_j m_j)g(m_j)} b_j(n_j m_j) \right),\]

where the last equality above follows by noting that \(b_j(n_j m_j) = 1\) implies that \(b_j(m_j) = 1\) for all \(1 \leq j \leq \mathcal{J}\).

Note that the factor \(b_j(n_j m_j)\) restricts \(m_j\) to have all prime factors in \(I_j\) such that \(\Omega(n_j m_j) \leq \ell_j\). If we remove this restrictions on \(\Omega\), then the sum over \(m_j\) becomes

\[
\sum_{m_j} \frac{1}{m_j} \frac{k^{\Omega(n_j m_j)}(k-1)^{\Omega(m_j)}}{g(n_j m_j)g(m_j)} = \prod_{p \in I_j} \left( 1 + \frac{k(k-1)}{p} + O\left(\frac{1}{p^2}\right) \right) \prod_{p \in I_j} \frac{k^{\ell_j} + k^{\ell_j+1}(k-1)^2}{(l_{i,j} + 1)l_{i,j}^2 p_{i,j}} + \cdots .
\]

We recast the last product above as

\[
\prod_{p \in I_j} \frac{k^{\ell_j} + k^{\ell_j+1}(k-1)^2}{(l_{i,j} + 1)l_{i,j}^2 p_{i,j}} = \frac{k^{\Omega(n_j)}}{g(n_j)} \prod_{p \in I_j} \left( 1 + O\left(\frac{1}{p}\right) \right),
\]

and we note that each factor in the last product above is positive.

Using Rankin’s trick by noticing that \(2^{\Omega(n_j m_j) - \ell_j} \geq 1\) if \(\Omega(n_j m_j) > \ell_j\), we see that the error introduced this way does not exceed

\[
\sum_{m_j} \frac{1}{m_j} \frac{k^{\Omega(n_j m_j)}(1 - k^{\Omega(m_j)})}{g(n_j m_j)g(m_j)} 2^{\Omega(n_j m_j) - \ell_j}
\]

\[= 2^{\Omega(n_j) - \ell_j} \sum_{m_j} \frac{1}{m_j} \frac{k^{\Omega(n_j m_j)}2^{\Omega(m_j)}(1 - k^{\Omega(m_j)})}{g(n_j m_j)g(m_j)}
\]

\[\leq 2^{\Omega(n_j) - \ell_j/2} \frac{k^{\Omega(n_j)}}{g(n_j)} \prod_{p \in I_j} \left( 1 + \frac{k(k-1)}{p} + O\left(\frac{1}{p^2}\right) \right) \prod_{p \in I_j} \left( 1 + O\left(\frac{1}{p}\right) \right),
\]

where the last estimation above follows from \([3.1]\).
It follows that we may write

\[
\sum_{m} \frac{a_{k-1}(m)a_k(mn)}{m} = \prod_{p \in \mathcal{P} \leq 1} \left( 1 + \frac{k(k-1)}{p} + O\left(\frac{1}{p^2}\right) \right)
\]

\[
\times \prod_{j=1}^{\mathcal{J}} \left( 1 + f_j(n_j) \right) \frac{k^{\Omega(n_j)}}{g(n_j)} \prod_{p \in I_j} \left( 1 + O\left(\frac{1}{p}\right) \right) \left( 1 + \frac{k(k-1)}{p} + O\left(\frac{1}{p^2}\right) \right)^{-1}
\]

\[
= \prod_{p \in \mathcal{P} \leq 1} \left( 1 + \frac{k(k-1)}{p} + O\left(\frac{1}{p^2}\right) \right) \prod_{j=1}^{\mathcal{J}} \left( 1 + f_j(n_j) \right) \frac{k^{\Omega(n_j)}}{g(n_j)} \prod_{p \in I_j} \left( 1 + O\left(\frac{1}{p}\right) \right),
\]

where

\[
|f_j(n_j)| \leq 2^{\Omega(n_j) - \ell_j/2}.
\]

We apply the above estimation to see that

\[
\sum_{n} \frac{\log n}{n} \sum_{m} \frac{a_{k-1}(m)a_k(mn)}{m} = \prod_{p \in \mathcal{P} \leq 1} \left( 1 + \frac{k(k-1)}{p} + O\left(\frac{1}{p^2}\right) \right).
\]

(3.12)

\[
\sum_{n=\Pi_j n_j} \log n \prod_{j=1}^{\mathcal{J}} \left( 1 + f_j(n_j) \right) \frac{k^{\Omega(n_j)}}{g(n_j)} b_j(n_j) \prod_{p \in I_j} \left( 1 + O\left(\frac{1}{p}\right) \right)
\]

Note that we have

\[
\sum_{n=\Pi_j n_j} \log n \prod_{j=1}^{\mathcal{J}} \left( 1 + f_j(n_j) \right) \frac{k^{\Omega(n_j)}}{g(n_j)} b_j(n_j) \prod_{p \in I_j} \left( 1 + O\left(\frac{1}{p}\right) \right)
\]

\[
= \sum_{n=\Pi_j n_j} \prod_{j=1}^{\mathcal{J}} \left( 1 + f_j(n_j) \right) \frac{k^{\Omega(n_j)}}{g(n_j)} b_j(n_j) \prod_{p \in I_j} \left( 1 + O\left(\frac{1}{p}\right) \right) \left( \sum_j \log n_j \right)
\]

\[
= \sum_{j'=1}^{\mathcal{J}} \prod_{j=1}^{\mathcal{J}} \left( 1 + f_j(n_j) \right) \frac{k^{\Omega(n_j)}}{g(n_j)} b_j(n_j) \prod_{p \in I_j} \left( 1 + O\left(\frac{1}{p}\right) \right) \left( \sum_j \log n_j \right)
\]

\[
\times \prod_{n_{j'}, n_{j'}} \left( 1 + f_{j'}(n_{j'}) \right) \frac{k^{\Omega(n_{j'})}}{g(n_{j'})} b_{j'}(n_{j'}) \prod_{p \in I_{j'}} \left( 1 + O\left(\frac{1}{p}\right) \right).
\]

We denote \( \mathcal{N}_j, 1 \leq j \leq \mathcal{J} \) for the set of integers \( n_j \) such that \( n_j \) is divisible only by primes \( p \in I_j \). We estimate the last sum of the last expression above by observing that \( 1 - 2^{\Omega(n_j) - \ell_j/2} \leq 0 \) when \( \Omega(n_j) \geq \ell_j/2 \), so that

\[
\sum_{n_j} \frac{\log n_j}{n_j} \left( 1 + f_j(n_j) \right) \frac{k^{\Omega(n_j)}}{g(n_j)} b_j(n_j) \prod_{p \in I_j} \left( 1 + O\left(\frac{1}{p}\right) \right) \geq \sum_{n_j \in \mathcal{N}_j} \frac{\log n_j}{n_j} \left( 1 - 2^{\Omega(n_j) - \ell_j/2} \right) \frac{k^{\Omega(n_j)}}{g(n_j)} \prod_{p \in I_j} \left( 1 + O\left(\frac{1}{p}\right) \right).
\]

We further observe that

\[
\sum_{n_j \in \mathcal{N}_j} \frac{\log n_j}{n_j} \left( 1 - 2^{\Omega(n_j) - \ell_j/2} \right) \frac{k^{\Omega(n_j)}}{g(n_j)} \prod_{p \in I_j} \left( 1 + O\left(\frac{1}{p}\right) \right)
\]

\[
= - \frac{d}{ds} \left( \sum_{n_j \in \mathcal{N}_j} \frac{1}{n_j^{1+s}} \left( 1 - 2^{\Omega(n_j) - \ell_j/2} \right) \frac{k^{\Omega(n_j)}}{g(n_j)} \prod_{p \in I_j} \left( 1 + O\left(\frac{1}{p}\right) \right) \right) \bigg|_{s=0}.
\]
Upon writing
\[
\sum_{n_j \in N_j} \frac{1}{n_j^{1+s}} g(n_j) \prod_{p \in I_j \setminus p|n_j} \left( 1 + O\left( \frac{1}{p} \right) \right) = \prod_{p \in I_j} \left( 1 + \frac{k}{p^{1+s}} \left( 1 + O\left( \frac{1}{p} \right) \right) + \frac{k^2}{2p^{2(1+s)}} \left( 1 + O\left( \frac{1}{p} \right) \right) + \cdots \right),
\]
we deduce that
\[
- \frac{d}{ds} \left( \sum_{n_j \in N_j} \frac{1}{n_j^{1+s}} g(n_j) \prod_{p \in I_j \setminus p|n_j} \left( 1 + O\left( \frac{1}{p} \right) \right) \right) \bigg|_{s=0} = \prod_{p \in I_j} \left( 1 + \frac{k}{p} + O\left( \frac{1}{p^2} \right) \left( \sum_{p \in I_j} \frac{k \log p}{p} + O\left( \frac{1}{p^2} \right) \right) \right) \left( 1 + O\left( \frac{1}{p} \right) \right).
\]
Note also that we have
\[
- \frac{d}{ds} \left( \sum_{n_j \in N_j} \frac{2^{\Omega(n_j)-\ell_j/2}}{n_j^{1+s}} g(n_j) \prod_{p \in I_j \setminus p|n_j} \left( 1 + O\left( \frac{1}{p} \right) \right) \right) \bigg|_{s=0} = 2^{-\ell_j/2} \prod_{p \in I_j} \left( 1 + \frac{2k}{p} + O\left( \frac{1}{p^2} \right) \left( \sum_{p \in I_j} \frac{2k \log p}{p} + O\left( \frac{1}{p^2} \right) \right) \right) \leq 2^{-\ell_j/4} \prod_{p \in I_j} \left( 1 + \frac{k}{p} + O\left( \frac{1}{p^2} \right) \left( \sum_{p \in I_j} \frac{k \log p}{p} + O\left( \frac{1}{p^2} \right) \right) \right).
\]
It follows that
\[
\sum_{n_j \in N_j} \frac{\log n_j}{n_j} \left( 1 - 2^{\Omega(n_j) - \ell_j/2} \right) \frac{2^{\Omega(n_j)}}{g(n_j)} \prod_{p \in I_j \setminus p|n_j} \left( 1 + O\left( \frac{1}{p} \right) \right) \geq (1 - 2^{-\ell_j/4}) \prod_{p \in I_j} \left( 1 + \frac{k}{p} + O\left( \frac{1}{p^2} \right) \right) \left( \sum_{p \in I_j} \frac{k \log p}{p} + O\left( \frac{1}{p^2} \right) \right).
\]
We apply similar arguments as above to see that we have
\[
\sum_{n_j \in N_j} \frac{1}{n_j} \left( 1 + f_j(n_j) \right) \frac{2^{\Omega(n_j)}}{g(n_j)} b_j(n_j) \prod_{p \in I_j \setminus p|n_j} \left( 1 + O\left( \frac{1}{p} \right) \right) \geq \sum_{n_j \in N_j} \frac{1}{n_j} \left( 1 - 2^{\Omega(n_j) - \ell_j/2} \right) \frac{2^{\Omega(n_j)}}{g(n_j)} \prod_{p \in I_j \setminus p|n_j} \left( 1 + O\left( \frac{1}{p} \right) \right) \geq (1 - 2^{-\ell_j/4}) \prod_{p \in I_j} \left( 1 + \frac{k}{p} + O\left( \frac{1}{p^2} \right) \right).
\]
We then conclude that
\[
\sum_{n \in I_j \setminus \prod_{j=1}^J n_j} \frac{\log n}{n} \prod_{j=1}^J \left( 1 - 2^{\Omega(n_j) - \ell_j/2} \right) \frac{2^{\Omega(n_j)}}{g(n_j)} b_j(n_j) \prod_{p \in I_j \setminus p|n_j} \left( 1 + O\left( \frac{1}{p} \right) \right) \geq \prod_{j=1}^J \left( 1 - 2^{-\ell_j/4} \right) \prod_{p \in \bigcup_{j=1}^J I_j} \left( 1 + \frac{k}{p} + O\left( \frac{1}{p^2} \right) \right) \left( \sum_{p \in \bigcup_{j=1}^J I_j} \frac{k \log p}{p} + O\left( \frac{1}{p^2} \right) \right).
\]
We apply the above estimation into \(5.12\) and apply \(8.5\), \(5.11\) together with Lemma \(2.4\) to conclude that
\[
S_1 \gg T(\log T)^{k^2+1}.
\]
This completes the proof of the proposition.
3.7. Proof of Proposition 3.4. We denote the left side expression in (3.8) by $S_2$ and apply Cauchy’s integral formula for derivatives to see that

$$S_2 = \int_1^{T} |\zeta'(\frac{1}{2} + it)|^2 |N(\frac{1}{2} + it, k - 1)|^2 dt = \int_1^{T} \left| \frac{1}{2\pi i} \int_{C_1} \frac{\zeta(\frac{1}{2} + \alpha + it)}{\alpha^2} d\alpha \right|^2 |N(\frac{1}{2} + it, k - 1)|^2 dt,$$

where $C_1$ denotes the positively oriented circle in the complex plane centered at 0 of radius $R = (\log T)^{-1}$. We then apply the Cauchy-Schwarz inequality to the integral over $\alpha$ above to deduce that

$$S_2 \leq \frac{1}{2\pi} \int_1^{T} \left| \int_{C_1} \frac{1}{\alpha^2} d\alpha \right| \left| \int_{C_1} |\zeta(\frac{1}{2} + \alpha + it)|^2 |N(\frac{1}{2} + it, k - 1)|^2 dt \right|$$

(3.13)

$$\leq \frac{1}{2\pi} R^{-2} \max_{|\alpha|=R} \int_1^{T} |\zeta(\frac{1}{2} + \alpha + it)|^2 |N(\frac{1}{2} + it, k - 1)|^2 dt.$$

We denote the last integral above by $I$ and we fix an $\alpha = \beta + i\gamma$ with $\beta, \gamma \in \mathbb{C}$ such that $|\alpha| = R$ to estimate it. Without loss of generality, we may assume that $\beta \leq 0$ and we apply Cauchy’s residue theorem to deduce that

$$I = \frac{1}{2\pi i} \int_{C_2} |\zeta(s)|^2 |N(s - \alpha, k - 1)|^2 ds,$$

where $C_2$ consists of line segments from $\frac{1}{2} + \beta + (1 + \gamma)i$ to $\frac{1}{2} + (1 + \gamma)i$, then from $\frac{1}{2} + (1 + \gamma)i$ to $\frac{1}{2} + (T + \gamma)i$ and lastly from $\frac{1}{2} + (T + \gamma)i$ to $\frac{1}{2} + \beta + (T + \gamma)i$.

The integration on the on the horizontal edges of the contour can be estimated to be $O(T^{1-\varepsilon})$ using (3.3) and the convexity bound for $\zeta(s)$ (see [20] Exercise 3, p. 100) that asserts

$$\zeta(s) \ll (1 + |s|)^{\frac{\kappa-1}{2}+\varepsilon}, \quad 0 \leq \Re(s) \leq 1,$$

We then deduce that

$$I = I_R + O(T^{1-\varepsilon}),$$

where

$$I_R = \frac{1}{2\pi i} \int_{1+\gamma}^{T+\gamma} |\zeta(\frac{1}{2} + it)|^2 |N(\frac{1}{2} + it - \alpha, k - 1)|^2 dt.$$

We now apply arguments similar to the proof of [16] Proposition 2 to deduce that for $T$ large enough,

$$I_R \ll T(\log T)^{k^2}.$$

We apply the above estimation in (3.13) to conclude that

$$S_2 \ll (T(\log T)^{k^2+2}.$$

This completes the proof of the proposition.

Acknowledgments. P. G. is supported in part by NSFC grant 11871082.

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