A Linear Programming Inequality with Applications to Concentration of Measure

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Abstract

We prove an elementary yet useful inequality bounding the maximal value of certain linear programs. This leads directly to a bound on the martingale difference for arbitrarily dependent random variables, providing a generalization of some recent concentration of measure results. The linear programming inequality may be of independent interest.

1 Introduction

1.1 Background

Over the past decade there has been a flurry of new concentration of measure inequalities; we refer the reader to [4] for an in-depth survey, or [2, 3, 5] for some more recent advances.

In [2] the martingale difference method was employed in a novel way to obtain a general concentration inequality for dependent random variables, with respect to the (unweighted) Hamming metric. At the core of that approach lies a certain linear programming inequality associated with bounding martingale differences [2, Theorem 4.8]. In this paper, we give a considerably simpler proof of a rather more general result, extending it to the weighted Hamming metrics. The applications to measure concentration are immediate (culminating in Corollary 3.3); additionally, it is hoped that the linear programming inequality and the technique employed for proving it will find further applications.

Since the main focus of this paper is the inequality in Theorem 2.5, we forgo a detailed discussion of measure concentration and how our bound relates to existing results. Such a discussion may be found in [2, 3].

1.2 Notational conventions

Throughout this paper, $S$ will denote a finite set. Random variables are capitalized ($X$), specified sequences (words) are written in lowercase ($x \in S^n$), the shorthand $X_{i:j}^n \doteq (X_i, \ldots, X_j)$ is used for all sequences, and word concatenation is denoted using the multiplicative notation: $x_i^j x_{j+1}^k = x_i^k$. Similarly, if $w \in \mathbb{R}^n$ and $1 \leq k \leq \ell \leq n$, then $w_{k}^{\ell} \doteq (w_k, \ldots, w_\ell) \in \mathbb{R}^{\ell-k+1}$.

We use the indicator variable $\mathbf{1}_{\{\cdot\}}$ to assign 0-1 truth values to the predicate in $\{\cdot\}$. The ramp function is defined by $(z)_+ = z \mathbf{1}_{\{z > 0\}}$. The positive reals are denoted by $\mathbb{R}_+ \doteq (0, \infty)$. 

1
The probability \( P \) and expectation \( E \) operators are defined with respect the measure space specified in context.

## 2 Linear programming inequality

We begin with a natural generalization of some of the definitions in [2]. Fix a finite set \( S \), \( n \in \mathbb{N} \) and \( w \in \mathbb{R}^n_+ \). Then

1. \( K_n \) denotes the set of all functions \( \kappa : S^n \to \mathbb{R} \) (and \( K_0 = \mathbb{R} \))
2. the weighted Hamming metric on \( S^n \times S^n \) is defined by
   \[
   d_w(x, y) = \sum_{i=1}^{n} w_i \mathbb{I}_{\{x_i \neq y_i\}}
   \]
3. for \( \varphi \in K_n \), its Lipschitz constant with respect to \( d_w \), denoted by \( \| \varphi \|_{\text{Lip},w} \), is defined to be the smallest \( c \) for which
   \[
   |\varphi(x) - \varphi(y)| \leq c d_w(x, y)
   \]
   for all \( x, y \in S^n \); any \( \varphi \) with \( \| \varphi \|_{\text{Lip},w} \leq c \) is called \( c \)-Lipschitz
4. for \( v \in [0, \infty) \), define \( \Phi_{w,n}^+= \) the set of all \( \varphi \) such that \( \| \varphi \|_{\text{Lip},w} \leq 1 \) and
   \[
   0 \leq \varphi(x) \leq v + \sum_{i=1}^{n} w_i, \quad x \in S^n;
   \]
   we omit the \(+\) superscript when \( v = 0 \), writing simply \( \Phi_{w,n} \)
5. the marginal projection operator \((\cdot)'\) takes \( \kappa \in K_n \) to \( \kappa' \in K_{n-1} \) by
   \[
   \kappa'(y) = \sum_{x_1 \in S} \kappa(x_1 y), \quad x \in S^{n-1};
   \]
   for \( n = 1 \), \( \kappa' \) is the scalar \( \kappa' = \sum_{x_1 \in S} \kappa(x_1) \)
6. for \( y \in S \), the \( y \)-section operator \((\cdot)_y \) takes \( \kappa \in K_n \) to \( \kappa_y \in K_{n-1} \) by
   \[
   \kappa_y(x) = \kappa(xy), \quad x \in S^{n-1};
   \]
   for \( n = 1 \), \( \kappa_y(\cdot) \) is the scalar \( \kappa(y) \)
7. the functional \( \Psi_{w,n} : K_n \to \mathbb{R} \) is defined by \( \Psi_{w,0}(\cdot) = 0 \) and
   \[
   \Psi_{w,n}(\kappa) = w_1 \sum_{x \in S^n} (\kappa(x))_+ + \Psi_{w_2,n-1}(\kappa');
   \]
   when \( w_i = 1 \) we omit it from the subscript, writing simply \( \Psi_n \)
8. the finite-dimensional vector space \( K_n \) is equipped with the inner product
   \[
   \langle \kappa, \lambda \rangle = \sum_{x \in S^n} \kappa(x) \lambda(x)
   \]
9. two norms are defined on \( \kappa \in K_n \): the \( \Phi_w \)-norm,
   \[
   \| \kappa \|_{\Phi,w} = \sup_{\varphi \in \Phi_{w,n}} |\langle \kappa, \varphi \rangle| \]
   and the \( \Psi_w \)-norm,
   \[
   \| \kappa \|_{\Psi,w} = \max_{s=\pm 1} \Psi_{w,n}(s \kappa).
   \]
Remark 2.1. For the special case $w_i \equiv 1$, $d_w$ is the unweighted Hamming metric used in [2]. It is straightforward to verify that $\Phi_w$-norm and $\Psi_w$-norm satisfy the vector-space norm axioms for any $w \in \mathbb{R}^n_+$; this is done in [2] for $w_i \equiv 1$. Since we will not be appealing to any norm properties of these functionals, we omit the proof. Note that for any $y \in S$, the marginal projection and $y$-section operators commute; in other words, for $\kappa \in K_{n+2}$, we have $(\kappa')_{y} = (\kappa_{y})'$ $\in K_n$ and so we can denote this common value by $\kappa_y' \in K_n$:

$$
\kappa_y'(z) = \sum_{x_1 \in S} \kappa_y(x_1 z) = \sum_{x_1 \in S} \kappa(x_1 z y), \quad z \in S^n.
$$

The main result of this section is

**Theorem 2.2.** For all $w \in \mathbb{R}^n_+$ and all $\kappa \in K_n$, we have

$$
\|\kappa\|_{\Phi_w} \leq \|\kappa\|_{\Psi_w}.
$$

**Remark 2.3.** We refer to [6] – more properly, to [9], from which the former immediately follows – as a linear programming inequality for the reason that $F(\cdot) = \langle \kappa, \cdot \rangle$ is a linear function being maximized over the finitely generated, compact, convex polytope $\Phi_{w,n} \subset \mathbb{R}^{S^n}$. We make no use of this simple fact and therefore forgo its proof, but see [2, Lemma 4.4] for a proof of a closely related claim. The term “linear programming” is a bit of a red herring since no actual LP techniques are being used; for lack of an obvious natural name, we have alternatively referred to [6] in previous papers and talks as the “$\Phi$-norm bound” or the “$\Phi$-$\Psi$ inequality.”

The key technical lemma is a decomposition of $\Psi_{w,n}(\cdot)$ in terms of $y$-sections, proved in [2] for the case $w_i \equiv 1$:

**Lemma 2.4.** For all $n \geq 1$, $w \in \mathbb{R}^n_+$ and $\kappa \in K_n$, we have

$$
\Psi_{w,n}(\kappa) = \sum_{y \in S} \left[ \Psi_{w_{n-1},n-1}(\kappa_y) + w_n \left( \sum_{x \in S^{n-1}} \kappa_y(x) \right) \right].
$$

**Proof.** We proceed by induction on $n$. To prove the $n = 1$ case, recall that $S^0$ is the set containing a single (null) word and that for $\kappa \in K_1$, $\kappa_y \in K_0$ is the scalar $\kappa(y)$. Thus, by definition of $\Psi_{w,1}(\cdot)$, we have

$$
\Psi_{w,1}(\kappa) = w_1 \sum_{y \in S} \kappa(y),
$$

which proves (4) for $n = 1$.

Suppose the claim holds for some $n = \ell \geq 1$. Pick any $w \in \mathbb{R}^{\ell+1}_+$ and $\kappa \in K_{\ell+1}$ and examine

$$
\sum_{y \in S} \left[ \Psi_{w,\ell,1}(\kappa_y) + w_{\ell+1} \left( \sum_{x \in S^\ell} \kappa_y(x) \right) \right] + w_{\ell+1} \left( \sum_{x \in S^\ell} \kappa_y(x) \right)
$$

$$
= \sum_{y \in S} \left[ \left( w_1 \sum_{x \in S^\ell} (\kappa_y(x))_+ + \Psi_{w_{\ell-1},\ell}(\kappa_y') \right) + w_{\ell+1} \left( \sum_{x \in S^\ell} \kappa_y(x) \right) \right] + w_{\ell+1} \left( \sum_{x \in S^\ell} \kappa_y(x) \right)
$$

$$
= \sum_{y \in S} \left[ \Psi_{w_{\ell-1},\ell-1}(\kappa'_y) + w_{\ell+1} \left( \sum_{u \in S^{\ell-1}} \kappa'_y(u) \right) \right] + w_{\ell+1} \left( \sum_{z \in S^{\ell+1}} (\kappa(z))_+ \right)
$$

(7)
where the first equality follows from the definition of $\Psi_{w,\ell}$ in (2) and the second one from the easy identities
\[
\sum_{y \in S} \sum_{x \in S^t} (\kappa_y(x))_+ = \sum_{x \in S^{t+1}} (\kappa(z))_+
\]
and
\[
\sum_{x \in S^t} \kappa_y(x) = \sum_{u \in S^{t-1}} \kappa'_y(u).
\]

On the other hand, by definition we have
\[
\Psi_{w,\ell+1}(\kappa) = w_1 \sum_{z \in S^{t+1}} (\kappa(z))_+ + \Psi_{w,\ell+1}(\kappa').
\]

To compare the r.h.s. of (7) with the r.h.s. of (8), note that the $w_1 \sum_{z \in S^{t+1}} (\kappa(z))_+$ term is common to both and
\[
\sum_{y \in S} \left[ \Psi_{w,\ell-1}(\kappa'_y) + w_{\ell+1} \left( \sum_{u \in S^{t-1}} \kappa'_y(u) \right)_+ \right] = \Psi_{w,\ell+1}(\kappa')
\]
by the inductive hypothesis. This establishes (9) for $n = \ell + 1$ and proves the claim. \qed

Our main result, Theorem 2.2, is an immediate consequence of

**Theorem 2.5.** For all $n \geq 1$, $w \in \mathbb{R}^n_{++}$, $v \in [0, \infty)$ and $\kappa \in K_n$, we have
\[
\sup_{\varphi \in \Phi_{w,n}} \langle \kappa, \varphi \rangle \leq \Psi_{w,n}(\kappa) + v \left( \sum_{x \in S^n} \kappa(x) \right)_+.
\]

**Proof.** We will prove the claim by induction on $n$. For $n = 1$, pick any $w_1 \in \mathbb{R}_+$, $v \in [0, \infty)$ and $\kappa \in K_1$. Since by construction any $\varphi \in \Phi_{w_1,1}^+$ is $w_1$-Lipschitz with respect to the discrete metric on $S$, $\varphi$ must be of the form
\[
\varphi(x) = \hat{\varphi}(x) + \hat{v}, \quad x \in S,
\]
where $\hat{\varphi} : S \to [0, w_1]$ and $0 \leq \hat{v} \leq v$ (in fact, we have the explicit value $\hat{v} = (\max_{x \in S} \varphi(x) - w_1)_+$). Therefore,
\[
\langle \kappa, \varphi \rangle = \langle \kappa, \hat{\varphi} \rangle + \hat{v} \sum_{x \in S} \kappa(x).
\]

The first term in the r.h.s. of (10) is clearly maximized when $\hat{\varphi}(x) = w_1 \mathbbm{1}_{\{\kappa(x) > 0\}}$ for all $x \in S$, which shows that it is bounded by $\Psi_{w,1}(\kappa)$. Since the second term in the r.h.s. of (10) is bounded by $v \left( \sum_{x \in S} \kappa(x) \right)_+$, we have established (9) for $n = 1$.

Now suppose the claim holds for $n = \ell$, and pick any $w \in \mathbb{R}_{++}^{\ell+1}$, $v \in [0, \infty)$ and $\kappa \in K_{\ell+1}$. By the reasoning given above (i.e., using the fact that $0 \leq \varphi \leq v + \sum_{i=1}^{\ell+1} w_i$ and that $\varphi$ is $1$-Lipschitz with respect to $d_w$), any $\varphi \in \Phi_{w,\ell+1}^{++}$, must be of the form $\varphi = \hat{\varphi} + \hat{v}$, where $\hat{\varphi} \in \Phi_{w,\ell+1}$ and $0 \leq \hat{v} \leq v$. Thus we write $\langle \kappa, \varphi \rangle = \langle \kappa, \hat{\varphi} \rangle + \hat{v} \sum_{x \in S^{\ell+1}} \kappa(x)$ and decompose
\[
\langle \kappa, \hat{\varphi} \rangle = \sum_{y \in S} \langle \kappa_y, \hat{\varphi}_y \rangle,
\]

where $\sum_{y \in S} (\kappa_y) = \sum_{x \in S^{t+1}} (\kappa(z))_+$ and $\sum_{u \in S^{t-1}} \kappa'_y(u)$.
making the obvious but crucial observation that
\[ \tilde{\varphi} \in \Phi_{w,\ell+1} \implies \tilde{\varphi}_y \in \Phi_{w',\ell+1}. \]

Then it follows by the inductive hypothesis that
\[ \langle \kappa, \tilde{\varphi}_y \rangle \leq \Psi_{w',\ell+1}(\kappa) + w_{\ell+1} \left( \sum_{x \in S^\ell} \kappa(x) \right) + \sum_{x \in S^\ell+1} \kappa(x) \geq \Psi_{w,\ell+1}(\kappa). \]

Applying Lemma 2.4 to (12), we have
\[ \sum_{y \in S} \langle \kappa_y, \tilde{\varphi}_y \rangle \leq \sum_{y \in S} \left[ \Psi_{w',\ell}(\kappa_y) + w_{\ell+1} \left( \sum_{x \in S^\ell} \kappa_y(x) \right) \right] = \Psi_{w,\ell+1}(\kappa). \]

This, combined with (11) and the trivial bound
\[ \tilde{v} \sum_{x \in S^\ell+1} \kappa(x) \leq v \left( \sum_{x \in S^\ell+1} \kappa(x) \right) \]
proves the claim for \( n = \ell + 1 \) and hence for all \( n \).

## 3 Applications to concentration of measure

This section assumes some familiarity with the notion of measure concentration; see the References section (in particular, [4, 5]) for introductory and survey material. Briefly, we shall concern ourselves with the metric probability space \((S^n, d_w, P)\) where \( S \) is a finite set, \( w \in \mathbb{R}_+^n \), \( d_w \) is the weighted Hamming metric defined in (1) and \( P \) is a (possibly non-product) probability measure on \( S^n \). For random variables \( f : S^n \to \mathbb{R} \), our goal is to bound \( P\{|f - Ef| > t\} \).

The method of martingale differences has been used to prove concentration of measure results since the work of Hoeffding, Azuma, and McDiarmid; see the exposition and references in [2, 3]. Let \((S^n, d_w, P)\) be as defined above and associate to it the (canonical) random process \((X_i)_{1 \leq i \leq n}, X_i \in S\), satisfying
\[ P\{X \in A\} = P(A) \]
for any \( A \subset S^n \).

For \( 1 \leq i \leq n \), \( f : S^n \to \mathbb{R} \) and \( y_i^1 \in S^i \), define the martingale difference
\[ V_i(f; y_i^1) = E[f(X) | X_i^1 = y_i^1] - E[f(X) | X_i^{i-1} = y_i^{i-1}]. \]

Let
\[ \bar{V}_i(f) \doteq \max_{y_i^1 \in S^i} |V_i(f; y_i^1)| \]
and
\[ D^2(f) \doteq \sum_{i=1}^n \bar{V}_i^2(f). \]

Then Azuma’s inequality [4] states that
\[ P\{|f - Ef| > t\} \leq 2 \exp\left(-t^2 / 2D^2(f)\right) \]
In [2] and [3], a technique was developed for bounding the martingale difference $V_i(f; y)$ in terms of the Lipschitz constant of $f$ and mixing properties of the measure $P$. To this end, we introduce the so-called $\eta$-mixing coefficients (see discussion ibid. regarding the appearance of these coefficients in earlier work of Marton [6] and Samson [7]).

For $1 \leq i < j \leq n$ and $x \in S^i$, let

$$L(X^n_j | X^n_i = x)$$

be the law (distribution) of $X^n_j$ conditioned on $X^n_i = x$. For $y \in S^{i-1}$ and $z, z' \in S$, define

$$\eta_{ij}(y, z, z') = \|L(X^n_j | X^n_i = yz) - L(X^n_j | X^n_i = yz')\|_{TV},$$

(17)

where $\|\cdot\|_{TV}$ is the total variation norm, defined here, for a signed measure $\tau$ on a finite space $\mathcal{X}$ by

$$\|\tau\|_{TV} = \frac{1}{2} \sum_{x \in \mathcal{X}} |\tau(x)|.$$

Additionally, define

$$\bar{\eta}_{ij} = \max_{y \in S^{i-1}} \max_{z, z' \in S} \eta_{ij}(y, z, z').$$

The main application of Theorem 2.5 to measure concentration is the following bound on the martingale difference:

**Theorem 3.1.** Let $S$ be a finite set, and let $(X_i)_{1 \leq i \leq n}, X_i \in S$ be the random process associated with the measure $P$ on $S^n$. Let $\Delta_n$ be the upper-triangular $n \times n$ matrix defined by $(\Delta_n)_{ij} = 1$ and

$$(\Delta_n)_{ij} = \bar{\eta}_{ij}$$

(18)

for $1 \leq i < j \leq n$. Then, for all $w \in \mathbb{R}_+^n$ and $f : S^n \to \mathbb{R}$, we have

$$\sum_{i=1}^n \bar{V}^2_i(f) \leq \|f\|_{\text{Lip}, w}^2 \|\Delta_n w\|_2^2$$

(19)

where $\bar{V}^2_i(f)$ is defined in (16).

**Remark 3.2.** Since $\bar{V}_i(f)$ and $\|f\|_{\text{Lip}, w}$ are both homogeneous functionals of $f$ (in the sense that $T(af) = |a|T(f)$ for $a \in \mathbb{R}$), there is no loss of generality in taking $\|f\|_{\text{Lip}, w} = 1$. Additionally, since $V_i(f; y)$ is translation-invariant (in the sense that $V_i(f; y) = V_i(f + a; y)$ for all $a \in \mathbb{R}$), there is no loss of generality in restricting the range of $f$ to $[0, \text{diam}_{d_w}(S^n)]$. In other words, it suffices to consider $f \in \Phi_{w,n}$. Since essentially this result (for $w_i \equiv 1$) is proved in [2] in some detail, we only give a proof sketch here, highlighting the changes needed for general $w$. We also remark that the extension of this result to countable $S$ is quite straightforward, along the lines of [2, Lemma 6.1].

**Proof.** It was shown in Section 5 of [2] that if $d_w$ is the unweighted Hamming metric (that is, $w_i \equiv 1$) and $f : S^n \to \mathbb{R}$ is 1-Lipschitz with respect to $d_w$, then

$$\bar{V}_i(f) \leq 1 + \sum_{j=i+1}^n \bar{\eta}_{ij}.$$

(20)
This was done by showing that for $1 \leq i \leq n$ and $y \in S^i$, there is a $g_i : S^n \to \mathbb{R}$ (whose explicit construction, depending on $y$ and $\mathbf{P}$, is given in [2, Eq. (5.2)]), such that for all $f : S^n \to \mathbb{R}$, we have

$$|V_i(f; y)| \leq |\langle g_i, f \rangle|.$$  \hspace{1cm} (21)

It was additionally shown in the course of proving [2, Theorem 5.1] that

$$\langle g_i, f \rangle = \langle T_y g_i, T_y f \rangle,$$

where the operator $T_y : K_n \to K_{n-i+1}$ is defined by

$$(T_y h)(x) = h(yx), \quad \text{for all } x \in S^{n-i+1}.$$  \hspace{1cm} \tag{22}

Appealing to [2, Theorem 4.8] – the $w_i \equiv 1$ special case of Theorem 2.5 proved here – we get

$$\langle T_y g_i, T_y f \rangle \leq \Psi_n(T_y g_i).$$  \hspace{1cm} (23)

It is shown in [2, Theorem 5.1] that the form of $g_i$ implies that

$$\Psi_n(T_y g_i) \leq 1 + \sum_{j=i+1}^{n} \bar{\eta}_{ij},$$

establishing (20). To generalize (20) to $w_i \not\equiv 1$, we use the fact that if $f \in K_n$ is 1-Lipschitz with respect to $d_w$, then $T_y f \in K_{n-i+1}$ is 1-Lipschitz with respect to $d_w$. Thus, applying Theorem 2.5, we get

$$\langle T_y g_i, f \rangle \leq \Psi_{w,n-i+1}(T_y g_i).$$  \hspace{1cm} (24)

It follows directly from the definition of $\Psi_{w,n}$ and the calculation in [2, Theorem 5.1] that

$$\tilde{V_i}(f) \leq w_i + \sum_{j=i+1}^{n} w_j \bar{\eta}_{ij} \leq \sum_{j=1}^{n} (\Delta_n)_{ij} w_j = (\Delta_n w)_i.$$  \hspace{1cm} (25)

Squaring and summing over $i$, we obtain (19).

\begin{proof}
\end{proof}

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\textbf{Corollary 3.3.} Let $S$ be a finite set and $\mathbf{P}$ a measure on $S^n$, for $n \geq 1$. For any $w \in \mathbb{R}_{+}^n$ and $f : S^n \to \mathbb{R}$, we have

$$\mathbf{P}\{|f - \mathbf{E}_f| > t\} \leq 2 \exp\left(-\frac{t^2}{2 \|f\|_{\text{Lip},w}^2 \|w\|_2^2 \|\Delta_n\|_2^2}\right)$$

where $\|\Delta_n\|_2$ is the $\ell_2$ operator norm of the matrix defined in (13).

\begin{proof}
Since by definition of the $\ell_2$ operator norm, $\|\Delta_n w\|_2 \leq \|\Delta_n\|_2 \|w\|_2$, the claim follows immediately via (16) and (19).
\end{proof}
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