The Volume of Black Holes

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Abstract

We propose a definition of volume for stationary spacetimes. The proposed volume is independent of the choice of stationary time-slicing, and applies even though the Killing vector may not be globally timelike. Moreover, it is constant in time, as well as simple: the volume of a spherical black hole in four dimensions turns out to be just $\frac{4}{3}\pi r^3$. We then consider whether it is possible to construct spacetimes that have finite horizon area but infinite volume, by sending the radius to infinity while making discrete identifications to preserve the horizon area. We show that, in three or four dimensions, no such solutions exist that are not inconsistent in some way. We discuss the implications for the interpretation of the Bekenstein-Hawking entropy.
I. INTRODUCTION

Time and space are often regarded as being interchanged across a black hole horizon; the interior of a Schwarzschild black hole, for example, can usefully be thought of as a collapsing universe. Moreover, what one means by the volume of space depends on how spacetime is split into space and time; spatial volume is not a slicing-invariant quantity. Hence, at first sight, it does not seem to make any sense to talk about the volume of a black hole.

This is unsatisfactory because one of the most celebrated facts about quantum gravity is that the entropy is vastly reduced from what it would have been in quantum field theory. The Bekenstein-Hawking entropy is equal to $\frac{\text{area}}{4l_p^2}$, which, it is proclaimed, is numerically much less than $\frac{\text{volume}}{l_p^3}$. This begs the question: what volume?

Now, questions of thermodynamics typically require thermal equilibrium, and, geometrically, “equilibrium” means that the spacetime possesses a symmetry under time translation i.e. there exists a timelike Killing vector. Suppose we have such a timelike Killing vector. Can one determine a volume in this more restricted setting?

Remarkably, the answer is yes. In this paper, we will show that, if the spacetime admits a Killing vector that is timelike in some region then it is possible to define a thermodynamically meaningful notion of volume. This is so even in the absence of a global timelike Killing vector; indeed, the presence of a horizon implies that the Killing vector becomes null there and, for nonextremal black holes, spacelike across it. Nevertheless, the volume that we define is not only constant in time, but also independent of the choice of stationary time slice (with the one proviso that the asymptotic form of the metric also be preserved).

Armed with a working definition of volume, an interesting next question is: are there families of spacetimes whose horizons have bounded area but whose volume can be arbitrarily large? For example, one might try to send the mass of a black hole to infinity, while simultaneously making discrete identifications on the spacetime to preserve the horizon area. Were such a construction to exist, it would be more than a curious fact: as we will argue, it would strongly suggest that the Bekenstein-Hawking entropy counts only the number of entangled states – rather than the total number of states – in the quantum gravity Hilbert space. However, we will be able to show that, at least in three or four spacetime dimensions, no such families of spacetimes exist. We interpret this as evidence that the Bekenstein-Hawking entropy might not be entanglement entropy.

II. THE VOLUME OF A BLACK HOLE

For illustration, we will have in mind nonrotating black holes; the final formula, though, requires only stationarity and applies equally to rotating black holes. Consider then a spacetime with a horizon and a line element of the form

$$ds^2 = -\alpha(r)dt_s^2 + \frac{dr^2}{\alpha(r)} + r^2 d\Sigma_{D-2}^2(x),$$

where $d\Sigma_{D-2}^2$ can be taken to be the line element of a maximally symmetric $D-2$-dimensional space. For instance, $\alpha$ could be

$$\alpha(r) = \frac{2\Lambda}{(D-1)(D-2)}r^2 + \eta - \frac{2M}{r^{D-3}}.$$  

One could also consider adding charge. When $\Lambda \geq 0$, $\eta$ is $+1$ but, in AdS, $\eta$ can also be $0$ or $-1$, corresponding to black holes with flat or negatively curved horizons. The horizon is at $r_+$ where $r_+$ is the largest root of $\alpha(r) = 0$. The time coordinate, $t_s$, is static time; the metric is invariant...
not only under $t_s \to t_s + c$, but also under $t_s \to -t_s$. However, the coordinate breaks down at the horizon, as evidenced by the divergence of $g_{rr}$ and $g_{tt}$. To continue through the horizon, one defines a new coordinate, $t$; static time is then expressed as $t_s(t, r, \vec{x})$. In order for $\partial_t$ to remain a Killing vector, the metric must be independent of $t$. Writing

$$dt_s = \frac{\partial t_s}{\partial t}(t, r, \vec{x}) dt + \frac{\partial t_s}{\partial r}(t, r, \vec{x}) dr + \nabla t_s(t, r, \vec{x}) \cdot d\vec{x},$$

we see that $\partial_t$ is a Killing vector if and only if the transformation takes the form

$$t_s = \lambda t + f(r, \vec{x}).$$

Here $\lambda$ is a constant, which we can take to be positive to preserve the orientation of time. In fact, our definition of volume will require that $\lambda$ be restricted to 1:

$$t_s \equiv t + f(r, \vec{x}).$$

When $\eta \neq 0$, $\lambda$ can be set to 1 by demanding a fixed asymptotic form of the metric. There remains an enormous class of time-slicings, since each choice of the almost arbitrary function $f(r, \vec{x})$ defines a different time slice. To reduce clutter, we will take $f(r, \vec{x})$ to be a function only of $r$. We then express the line element in the new time coordinate as

$$ds^2 = -\alpha(r)dt^2 - 2\alpha(r)f' dt dr + dr^2 \left(1 - \alpha(r)f'^2\right) + r^2 d\Sigma^2_{D-2}(\vec{x}).$$

By choosing $f$ so that $f'$ is real and such that $g_{rr}$ stays positive and finite, one obtains a stationary slicing that extends through the horizon. Note that, although $\partial_t$ may become spacelike across the horizon, the normal to a surface of constant $t$ is – thanks to the off-diagonal term – everywhere timelike; such surfaces constitute bona fide spatial sections. We would like to define an invariant measure on these sections. First note that if we were to take the volume to be the proper volume of a hypersurface of constant $t$, we would not get an invariant volume because $g_{rr}$ manifestly depends on the choice of time slice through its dependence on $f$. Indeed, by considering a slicing that is nearly lightlike, one can arrange for the proper three-volume to be as close to zero as one wants. If this were the right notion of volume, it would not at all be clear that holography entails a reduction in the degrees of freedom.

Instead, observe that the determinant of the spacetime metric

$$-\det g_{(D)} = + \left(\frac{2}{\alpha(r)} - \alpha(r)f'^2\right) + r^2 d\Sigma^2_{D-2}(\vec{x}).$$

has no dependence on the time-slicing: $f'(r)$ drops out and $\lambda$ has been set to 1. This suggests the following definition of spatial volume. Consider the differential spacetime volume

$$dV_D(t) = \int_{t}^{t+dt} dt' \int dr \int d^{D-2}x \sqrt{-g_{(D)}}.$$

While the combination $d^Dx \sqrt{-g_{(D)}}$ is slicing-invariant (in fact, coordinate-invariant), $dV_D$ is not, because the limits on the integral are defined in terms of a time coordinate. However, if the time coordinate is of the form $t_s = \lambda t + f(r, \vec{x})$ – that is, if $\partial_t$ is a Killing vector – then the integrand is time-independent with time appearing in $dV_D$ only through the multiplicative factor, $dt$.

We therefore propose that

$$V_{\text{space}} = \frac{dV_D}{dt}.$$


Equivalently,

\[ V_{\text{space}} \equiv \int d^{D-1}x \sqrt{-g(D)} . \]  

(10)

In other words, if, rather than using \( \sqrt{g(D-1)} \) as the measure, one uses \( \sqrt{-g(D)} \) instead, then two things happen. First, the volume is constant in time for all choices of Killing time since the integrand is time-independent. And second, the integral is invariant under stationary time slices.

Here is why. Imagine the spacetime integral, (8), as a Riemann sum of little strips, each of coordinate length \( dt \), lined up side-by-side from \( r = 0 \) to \( r = r_+ \). According to (5), a particular constant-time slice merely shifts these strips up or down along the orbit of the Killing vector in an \( f \)-dependent manner. But the metric is unchanged under such shifts. Hence the integral is invariant even though different time-slicings correspond to integration over different spacetime regions. After dividing out by \( dt \), we therefore obtain an invariant spatial volume. Nor is this construction affected by the nature – timelike, spacelike, or null – of the Killing vector.

Actually, this is also the notion of volume that appears in thermodynamics. To see this, write the partition function as

\[ Z = \exp(-F/\beta) = \exp \left( - \int d^{D}x \sqrt{-g(D)} \mathcal{L} \right) . \]

(11)

Now the inverse temperature, \( \beta \), is the period, \( \int d\tau \), where \( \tau \) is a complexified time coordinate. Notice: there is no \( \sqrt{-g_{tt}} \) factor in \( \beta \). Suppose the field is constant in \( \tau \). Then the free energy is

\[ F = \int d^{D-1}x \sqrt{-g(D)} \mathcal{L} . \]

(12)

If the system is extensive, the free energy is proportional to the volume. We see that this is not inconsistent with regarding \( \int d^{D-1}x \sqrt{-g(D)} \) as the volume.

Let us now evaluate the volume for some simple spacetimes. For a four-dimensional spherically symmetric black hole, we find that the volume takes a satisfyingly familiar form:

\[ V_{\text{spherical hole}} = \int_{0}^{r_+} dr \int_{0}^{\pi} d\theta \int_{0}^{2\pi} d\phi \sqrt{-\text{det}g} = \frac{4}{3} \pi r_+^3 . \]

(13)

It is amusing that this is precisely the proper three-volume of flat Euclidean space. A slicing in which the constant-time hypersurfaces are flat is given by Painlevé coordinates, for which the line element takes the form

\[ ds^2 = -\alpha(r)dt^2 - 2\sqrt{1 - \alpha(r)} dt \, dr + dr^2 + r^2 d\Sigma^2_{D-2} . \]

(14)

These coordinates have already proven their utility in tunneling calculations [1]. But, more generally, such coordinates might not be defined globally because the square root is required to remain real. So the volume (10) should not be thought of as the proper volume of a slice with \( g_{rr} = 1 \) as no such slice may exist.

The quantity \( 4\pi r_+^3 / 3 \) is precisely in accord with what an observer in the time-independent region would consider to be the black hole volume. To an internal observer, however, \( r \) is more like a time coordinate, and the result would presumably not be interpreted as a volume. This is not surprising: it is well-known that the thermodynamic properties of a black hole exist only from an outside point of view.

Finally, it should be clear that the above arguments did not rely on the initial time coordinate, \( t_0 \), being static. Only stationarity – the existence of a somewhere timelike Killing vector – is necessary. So the same volume formula applies to rotating black holes. One finds that the volume of a four-dimensional Kerr black hole is

\[ V_{\text{Kerr hole}} = \frac{4}{3} \pi r_+ (r_+^2 + a^2) . \]

(15)
III. FINITE AREA BUT INFINITE VOLUME?

As an elementary example, consider D-dimensional Rindler space, with $D > 2$. In Cartesian coordinates, an observer moving with constant acceleration in the positive $X^1$ direction, has a future Rindler horizon described by the light-sheet $T = X^1$. The light-sheet has infinite extent in the $X^i$ directions, for $i = 2 \ldots D - 1$, so the horizon has infinite area. The volume of the spacetime behind the Rindler horizon is also intuitively infinite. However, if we now make a toroidal compactification of all the transverse directions,

$$X_i \sim X_i + L_i, \quad i = 2 \ldots D - 1$$  \hspace{1cm} (16)

the horizon area becomes finite: $A = \Pi_i L_i$. (The compactification does not imply a dimensional reduction; the $L_i$ could be chosen to be enormous compared with the $D$-dimensional Planck length.) However, because $X^1$ is not identified, spatial sections behind the horizon are noncompact and intuitively have infinite volume. Thus this would appear to be an example of a spacetime with a horizon of finite area and infinite volume. However, it has been shown that Rindler space with all but one spatial direction compactified is inconsistent \cite{2}. The formal proof of this claim consists of demonstrating a contradiction between finite entropy and the two-dimensional Poincare group, $ISO(1, 1)$.

To find other spacetimes with this property, we note that Rindler space is the infinite mass limit of a nonextremal black hole. Thus, in general, what we would like to do is to take a spacetime with a horizon, then send the radius of the horizon to infinity while making discrete identifications to keep the area finite as the radius is sent to infinity. More precisely, we would like to quotient by groups that have the following properties:

(i) The group must be a subgroup of the isometry group of the spatial section of the horizon. This is necessary so that the quotient space has a well-defined metric.

(ii) The group must act freely on the spacetime. Otherwise, we would introduce singularities. However, we may allow a fixed point to occur at a point that is already singular as singularities are not formally part of the manifold.

(iii) The fundamental domain must not have any cycles whose length vanishes during the process of simultaneously blowing up the horizon radius and quotienting by the groups. This is because, if there were cycles of vanishing length, the gravity description could not be trusted; winding modes of strings winding around the vanishing cycle would become lighter than momentum modes. This is a restrictive requirement. It implies that the identifications have to act democratically in all dimensions along the horizon. Otherwise, the directions in which they do act would be forced to become vanishingly small to preserve the area as the radius is increased.

Spherical horizons

Consider first spacetimes whose horizons, when sliced using stationary time, are spheres. These have the isometry group $O(D - 1)$. We need a family of discrete subgroups of arbitrarily high order, so that, by quotienting with groups of ever larger order, we can keep the area bounded even as the radius diverges. For $D > 3$, there are two infinite families of discrete subgroups of $O(D - 1)$: the cyclic and the dihedral groups. The cyclic groups, $C_n$, have order $n$ and are isomorphic to $Z_n$. They act by identifying points in the azimuthal direction: $\phi \sim \phi + 2\pi/n$. $C_n$ does not act freely because, for example, it leaves the poles of the two-sphere fixed, in violation of requirement (ii). The dihedral groups, $D_n$, have order $2n$, and are isomorphic to $Z_2 \times Z_n$. They are nonabelian and act freely. However, both $C_n$ and $D_n$ essentially act mainly along the azimuth. The fundamental domain, after modding out by $D_n$, can be regarded as a wedge extending down from the pole to
the equator, much like a segment of an orange. As the radius of the sphere becomes ever greater, the width of the segment must vanish to preserve the area, thus violating requirement (iii).

In three dimensions, stationary sections of the horizon are just circles. So here we need subgroups of $O(2)$. Obviously, we can mod out by $Z_n$. There are two spacetimes with horizons in three dimensions: the BTZ black hole and three-dimensional de Sitter space. For de Sitter space modding out by $Z_n$ in the angular direction results in a conical singularity at $r = 0$. (After appropriate relabelings, this can be regarded as a Schwarzschild-de Sitter space, without an identification.)

Finally, consider the BTZ black hole with mass $M_0 > 0$. The line element is

$$ds^2 = -(r^2/l^2 - 8GM_0)dt^2 + \frac{dr^2}{r^2/l^2 - 8GM_0} + r^2d\phi^2.$$  \hspace{1cm} (17)

The horizon is a circle which we can think of as a real line modded out by $Z$. The entropy is just

$$S_0 = \frac{\pi l}{2G}\sqrt{2GM_0}.$$ \hspace{1cm} (18)

We now make the identification

$$\phi \sim \phi + \frac{2\pi}{n},$$ \hspace{1cm} (19)

which has the effect of changing the grading of the original identification of the real line. Now let $M \to sM_0$ and define $n = \sqrt{\lfloor s \rfloor}$. Then, as $s \to \infty$, $S \to S_0$. We see that $M$ can be made arbitrarily large so long as $n$ is increased suitably, without causing the entropy to diverge.

However, there is a problem here. Although discrete identifications can be performed on the horizon, the uniqueness of the volume breaks down because there is nothing to fix $\lambda$ to 1; time can be rescaled. Indeed, after making the identification, if one defines the new variables

$$\phi' \equiv n\phi \quad r' \equiv r/n \quad t' \equiv nt,$$ \hspace{1cm} (20)

then the metric becomes that of a BTZ black hole with mass $M' = M_0/n^2$, and no identification on the horizon. Thus area and volume cannot be separately adjusted.

We have considered spherically-symmetric horizons, and shown that there are no finite area and infinite volume solutions. We could also have tried quotienting nonspherically-symmetric spacetimes such as the Kerr black hole or Taub-NUT space. However, their isometry groups are just subgroups of those of a sphere, and hence they also do not yield finite area and infinite volume quotients.

Flat horizons

Flat horizons exist in Rindler space, which we have already rejected, and in AdS. The AdS black brane solutions have the line element

$$ds^2 = -\left(\frac{r^2}{l^2} - \frac{2GM}{r^{D-3}}\right)dt^2 + \frac{dr^2}{r^2/l^2 - 2GM} + \frac{r^2}{l^2} \sum_{i=1}^{D-2} dx_i^2.$$ \hspace{1cm} (21)

The isometry group of the stationary slices is just $E(D - 2)$ i.e. $ISO(D - 1)$. The lattice groups are discrete subgroups with no fixed points. Thus we can make a toroidal identification on the horizon:

$$x_i \sim x_i + L.$$ \hspace{1cm} (22)

It is easy to see that this satisfies all the requisite properties. After identification, the topology of the stationary slices is now $T^{D-2}$. But since $\eta = 0$ in $\Box$, $\lambda$ cannot be set to 1. So, again, just as with the BTZ black hole (which can be regarded as a special case of (21)), the volume cannot be invariantly defined.
Hyperbolic horizons

AdS also has black hole solutions with hyperbolic horizons:

\[
    ds^2 = -\left(\frac{r^2}{l^2} - 1 - \frac{2GM}{r^{D-3}}\right)dt^2 + \frac{dr^2}{r^2/l^2 - 1 - \frac{2GM}{r^{D-3}}} + r^2 d\Sigma_{D-2}^2.
\]  

(23)

Here \(d\Sigma_{D-2}^2\) is the line element of a unit hyperbolic space, \(H^{D-2}\), a noncompact Riemannian manifold with constant unit negative curvature (i.e. “Euclidean” anti-de Sitter space). The isometry group of \(H^{D-2}\) is \(O(1, D-2)\), which is just the Lorentz group.

Consider \(D = 4\). Hawking’s uniqueness theorem \(\text{[3]}\) on horizon topology does not apply to AdS black holes; indeed, \(H^2\) has infinitely many topologically inequivalent compactifications \(\text{[4]}\). One might hope that some of these might lead to finite area and infinite volume spacetimes. However, the global Gauss-Bonnet theorem says that the integral of the Ricci scalar is related to the Euler characteristic, \(\chi\), of the horizon:

\[
    \frac{1}{4\pi} \int RdA = \chi = 2 - 2g.
\]

(24)

In two dimensions, compact oriented surfaces without boundaries or punctures are topologically characterized by the genus, \(g\). Thus we find that the area is bounded from below:

\[
    A = 4\pi (g - 1) r_+^2 \geq 4\pi r_+^2.
\]

(25)

We see that, irrespective of the compactification, the area becomes infinite as the radius is sent to infinity.

In conclusion, we have shown that, in three or four dimensions, there are no classes of spacetimes that have bounded horizon area but unbounded volume. It would be interesting to see whether this no-go theorem can be extended to higher dimensions. Two loopholes in higher dimensions are that asymptotically flat black geometries can have horizons with more complicated topologies such as \(S^1 \times S^2\) \(\text{[5]}\), and that higher-dimensional hyperbolic horizons are not subject to the Gauss-Bonnet theorem.

IV. DISCUSSION

We have seen that one cannot construct a family of spacetimes that have horizons of bounded area but unbounded volume. It is intriguing that in each case where this might have worked, something went wrong: either there was a cycle of vanishing length (Schwarzschild black holes), or there was a conical singularity (de Sitter space), or the definition of volume became ambiguous (AdS branes and BTZ), or there was a conflict with symmetries (Rindler space), or the area itself diverged (hyperbolic horizons). Perhaps there is a deeper reason why such a construction may be impossible.

One deeper reason might be a conflict with holography. The statistical interpretation of Bekenstein-Hawking entropy, or gravitational entropy, remains contentious \(\text{[6]}\). One school of thought holds that this entropy enumerates all possible gravitational degrees of freedom within the volume enclosed by the area. That is, it counts the total number of states in the quantum gravity Hilbert space. An alternative interpretation is that gravitational entropy counts only entangled states \(\text{[8]}\). The fact that the entropy scales as the area has two very different implications from these two perspectives. The first implies that quantum gravity is highly nonlocal, with far fewer degrees of freedom than a local quantum field theory would have had. In contrast, the second
interpretation implies that quantum gravity is local, locality being precisely the reason that the field deep inside the hole is not entangled with the field outside. Which of these two interpretations is correct is not immediately obvious because we are unable to count the quantum gravity Hilbert states directly. For example, in string theory, the counting of microstates \[8\] is typically done in a dual picture for which the string coupling is weak, leaving the gravitational interpretation of the states unclear (though recent work attempts a more direct approach \[9\]).

Now, entanglement entropy is indifferent to the volume of space. Indeed, its most appealing feature is that the entropy-area relation appears quite naturally. From an entanglement entropy perspective, there appears to be no reason why finite area but infinite volume solutions should not exist. In fact, their existence would have been evidence in support of entanglement entropy; the alternate interpretation – that the total number of Hilbert states in an infinite volume is finite – would then have seemed hard to believe. However, the fact that no such solutions exist suggests – assuming that the underlying reason is holographic – that volume and entropy are not independent. The Bekenstein-Hawking entropy may really be counting all the quantum gravity Hilbert states.

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