GENERATING THE SPIN MAPPING CLASS GROUP BY DEHN TWISTS

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Abstract. Let \( \phi \) be a \( \mathbb{Z}/2\mathbb{Z} \)-spin structure on a closed oriented surface \( \Sigma_g \) of genus \( g \geq 4 \). We determine a generating set of the stabilizer of \( \phi \) in the mapping class group of \( \Sigma_g \) consisting of Dehn twists about an explicit collection of \( 2g + 1 \) curves on \( \Sigma_g \). If \( g = 3 \) then we determine a generating set of the stabilizer of an odd \( \mathbb{Z}/4\mathbb{Z} \)-spin structure consisting of Dehn twists about a collection of 6 curves.

1. Introduction

For some \( r \geq 2 \), a \( \mathbb{Z}/r\mathbb{Z} \)-spin structure on a closed surface \( \Sigma_g \) of genus \( g \) is a cohomology class \( \phi \in H^1(UT \Sigma_g, \mathbb{Z}/r\mathbb{Z}) \) which evaluates to one on the oriented fibre of the unit tangent bundle \( UT \Sigma_g \to \Sigma_g \) of \( \Sigma_g \). Such a spin structure exists for all \( r \) which divide \( 2g - 2 \). If \( r \) is even, then it reduces to a \( \mathbb{Z}/2\mathbb{Z} \)-spin structure on \( \Sigma_g \).

A \( \mathbb{Z}/2\mathbb{Z} \)-spin structure on \( \Sigma_g \) has a parity, either even or odd. Thus there is a notion of parity for all \( \mathbb{Z}/r\mathbb{Z} \)-spin structures with \( r \) even. If \( \phi, \phi' \) are two \( \mathbb{Z}/r\mathbb{Z} \)-spin structures on \( \Sigma_g \) so that either \( r \) is odd or \( r \) is even and the parities of \( \phi, \phi' \) coincide, then there exists an element of the mapping class group \( \text{Mod}(\Sigma_g) \) of \( \Sigma_g \) which maps \( \phi \) to \( \phi' \). Hence the stabilizers of \( \phi \) and \( \phi' \) in \( \text{Mod}(\Sigma_g) \) are conjugate.

Spin structures naturally arise in the context of abelian differentials on \( \Sigma_g \). The moduli space of such differentials decomposes into strata of differentials whose zeros are of the same order and multiplicity. Understanding the orbifold fundamental group of such strata requires some understanding of their projection to the mapping class group. If the orders of the zeros of the differentials are all multiples of the same number \( r \geq 2 \), then this quotient group preserves a \( \mathbb{Z}/r\mathbb{Z} \)-spin structure \( \phi \) on \( \Sigma_g \). Hence the orbifold fundamental groups of components of strata relate to stabilizers \( \text{Mod}(\Sigma_g)[\phi] \) of spin structures \( \phi \) on \( \Sigma_g \).

To make such a relation explicit we define

**Definition 1.** A **curve system** on a closed surface \( \Sigma_g \) is a finite collection of smoothly embedded simple closed curves on \( \Sigma_g \) which are non-contractible and mutually not freely homotopic, and such that any two curves from this collection intersect transversely in at most one point.

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A curve system defines a \textit{curve diagram} which is a finite graph whose vertices are the curves from the system and where two such vertices are connected by an edge if the curves intersect.

\textbf{Definition 2.} A curve system on $\Sigma_g$ is \textit{admissible} if it decomposes $\Sigma_g$ into a collection of topological disks and if its curve diagram is a tree.

Using a construction of Thurston and Veech (see [Lei04] for a comprehensive account), admissible curve systems on $\Sigma_g$ give rise to abelian differentials on $\Sigma_g$, and the component of the stratum and hence the equivalence class of a spin structure (if any) it defines can be read off explicitly from the combinatorics of the curve system. This makes it desirable to investigate the subgroup of the mapping class group generated by Dehn twists about the curves of an admissible curve system.

The main goal of this article is to present a systematic study of stabilizers of suitably chosen curves in the spin mapping class group $\text{Mod}(\Sigma_g)[\varphi]$ and to use this information to build generators for this group by induction over subsurfaces. As a main application we obtain the following.

For $g \geq 3$ let $C_g$ and $V_g$ be the collections of $2g + 1$ nonseparating simple closed curves on a closed surface $\Sigma_g$ of genus $g$ shown in Figure 1. We show

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (C) at (0,0) {$C_g$};
  \node (V) at (3,0) {$V_g$};
  \draw (C) -- (V);
\end{tikzpicture}
\caption{Figure 1}
\end{figure}

\textbf{Theorem 3.} \hspace{1em} (1) Let $\varphi$ be an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure on a closed surface $\Sigma_g$ of genus $g \geq 3$. Then $\text{Mod}(\Sigma_g)[\varphi]$ is generated by the Dehn twists about the curves from the curve system $C_g$.

(2) Let $\varphi$ be an even $\mathbb{Z}/2\mathbb{Z}$-spin structure on a closed surface $\Sigma_g$ of genus $g \geq 4$. Then $\text{Mod}(\Sigma_g)[\varphi]$ is generated by the Dehn twists about the curves from the curve system $V_g$.

That the spin mapping class group can be generated by finitely many finite products of Dehn twists is due to Hirose. In [Hi02] he found for any genus $g \geq 2$ a generating set for the stabilizer of an even $\mathbb{Z}/2\mathbb{Z}$-spin structure by finitely many finite products of Dehn twists, and the stabilizer of an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure is treated in [Hi05].

For surfaces of genus $g \geq 5$, Calderon [Cal20] and Calderon and Salter [CS19] identified the image of the orbifold fundamental group of most components of strata...
in the mapping class group by constructing a different but equally explicit generating set for the spin mapping class group. Earlier Salter (Theorem 9.5 of \cite{Sa19}) obtained a partial result by identifying for \( g \geq 5 \) a finite generating set of a finite index subgroup of the spin mapping class group by Dehn twists. Walker \cite{W09,W10} obtained some information on the image of the orbifold fundamental group of some strata of quadratic differentials in the mapping class group using completely different tools.

Theorem \( \text{3} \) does not construct generators for the stabilizer of an even \( \mathbb{Z}/2\mathbb{Z} \)-spin structure on a surface of genus \( g = 2, 3 \). Namely, in these cases there is no admissible curve system with the property that the Dehn twists about the curves from the system stabilize an even \( \mathbb{Z}/2\mathbb{Z} \)-spin structure and such that the Dehn twists about these curves generate a finite index subgroup of the mapping class group. This corresponds to a classification result of Kontsevich and Zorich \cite{KZ03}. There is no component of a stratum of abelian differentials with a single zero on a surface of genus 2 and even spin structure. On a surface \( \Sigma_3 \) of genus 3, the component of the stratum of abelian differentials with two zeros of order two and even spin structure is hyperelliptic and hence the projection of its orbifold fundamental group to \( \text{Mod}(\Sigma_3) \) commutes with a hyperelliptic involution and is of infinite index.

Our results can be used to construct an explicit finite set of generators of the stabilizer of a \( \mathbb{Z}/r\mathbb{Z} \)-spin structure for any \( r \leq 2g - 2 \) and any closed surface \( \Sigma_g \), given by Dehn twists, positive powers of Dehn twists and products of Dehn twists about two simple closed curves forming a bounding pair. Potentially they can also be used inductively to find generators by Dehn twists about curves from an admissible curve system. We carry this program only out in a single case, which is the odd \( \mathbb{Z}/4\mathbb{Z} \)-spin structure on a surface of genus 3.

Consider the system \( \mathcal{E}_6 \) of simple closed curves on the surface \( \Sigma_3 \) of genus 3 shown in Figure 2 which is of particular relevance for the understanding of the stratum of abelian differentials with a single zero on \( \Sigma_3 \) \cite{LM14}. We show

\[
\text{Theorem 4.} \quad \text{The subgroup of \( \text{Mod}(\Sigma_3) \) generated by the Dehn twists about the curves from the curve system } \mathcal{E}_6 \text{ equals the stabilizer of an odd } \mathbb{Z}/4\mathbb{Z} \text{-spin structure on } \Sigma_3.\n\]

The strategy for the proofs of the main results is as follows.

For some \( r \geq 2 \) let us consider an arbitrary \( \mathbb{Z}/r\mathbb{Z} \)-spin structure \( \varphi \) on a compact oriented surface \( S \) of genus \( g \geq 2 \), perhaps with boundary. Following \cite{H189} and \cite{Sa19}, the spin structure can be viewed as a \( \mathbb{Z}/r\mathbb{Z} \)-valued function on oriented
closed curves on \( S \) which assumes the value one on the oriented boundary of an embedded disk in \( S \). Changing the orientation of the curve changes the value of \( \varphi \) on the curve to its negative \([HJ89, Sa19]\).

Define a graph \( CG_1^+ \) as follows. Vertices are nonseparating simple closed curves \( c \) on \( S \) with \( \varphi(c) = \pm 1 \), and two such vertices \( d, e \) are connected by an edge if \( d, e \) can be realized disjointly and if furthermore, \( S - (d \cup e) \) is connected. Thus \( CG_1^+ \) is a subgraph of the curve graph of \( S \). The stabilizer \( \text{Mod}(S)[\varphi] \) of \( \varphi \) in the mapping class group of \( S \) acts on \( CG_1^+ \) as a group of simplicial automorphisms.

In Section 2 we show that for any \( g \geq 3 \) and \( r \leq 2g - 2 \) the graph \( CG_1^+ \) is connected. We also note that for an odd \( \mathbb{Z}/2\mathbb{Z} \)-spin structure on a surface of genus \( g = 2 \), this is not true. In Section 3 we verify that the action of the group \( \text{Mod}(S)[\varphi] \) on the graph \( CG_1^+ \) is transitive on vertices.

For a vertex \( c \) of \( CG_1^+ \) we are then led to describing the intersection of \( \text{Mod}(S)[\varphi] \) with the stabilizer of \( c \) in \( \text{Mod}(S) \). Most important is the understanding of the intersection of \( \text{Mod}(S)[\varphi] \) with the so-called disk pushing subgroup, namely the kernel of the natural homomorphism of the stabilizer of \( c \) to the mapping class group of the surface obtained from \( S - c \) by capping off the two distinguished boundary components of \( S - c \). This is also carried out in Section 3.

In Section 4 we specialize further to a \( \mathbb{Z}/2\mathbb{Z} \)-spin structure \( \varphi \). We find a presentation of \( \text{Mod}(S)[\varphi] \) as a quotient of a \( \mathbb{Z}/2\mathbb{Z} \)-extension of the free product of two copies of the stabilizer of a vertex of \( CG_1^+ \), amalgamated over the stabilizer of an edge of \( CG_1^+ \). This is used to prove the first part of Theorem 3 with an argument by induction on the genus \( g \) of the closed surface \( \Sigma_g \).

The proof of the second part of Theorem 3 uses similar methods and is contained in Section 5. A variation of these arguments yield the proof of Theorem 4 in Section 6.

The appendix contains a technical variation of the main result of Section 2 which is used in Section 5. Its proof follows along exactly the same line as the proof of the main result of Section 2.

This work is inspired by the article \([Sa19]\) of Salter. However, aside from some simple constructions using curves and Proposition 4.9 of \([Sa19]\), our approach uses different methods.

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2. Graphs of curves with fixed spin value

In this section we consider a compact surface $S$ of genus $g \geq 2$, with or without boundary. For a number $r \geq 2$ we introduce $\mathbb{Z}/r\mathbb{Z}$-spin structures on $S$ and use these structures to define various subgraphs of the curve graph of $S$. Of primary interest is a graph $\mathcal{G}_1$ whose vertices are nonseparating simple closed curves with spin value $\pm 1$ and where two such curves are connected by an edge if they can be realized disjointly. We then study connectedness of this graph.

Small genus of the surface may cause the graph $\mathcal{G}_1$ to have few edges. This problem leads us to proceed in two steps. In Proposition 2.12 we show connectedness of $\mathcal{G}_1$ for surfaces of genus $g \geq 3$ and $r = 2, 4$, taking advantage of some special properties of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ spin structures. Proposition 2.16 shows connectedness of $\mathcal{G}_1$ for surfaces of genus $g \geq 4$ and all $r$, taking advantage of sufficiently large complexity of the underlying surface. These results are used in Section 3 to study the stabilizer of a spin structure in the mapping class group of $S$.

This section is divided into 5 subsections. We begin with summarizing some information on spin structures. Each of the remaining subsections is devoted to the investigation of a specific subgraph of the curve graph of $S$ defined by a spin structure $\varphi$ on $S$.

2.1. Spin structures. The following is taken from [HJ89], see Definition 3.1 of [Sa19]. For its formulation, denote by $\iota$ the symplectic form on $H_1(S, \mathbb{Z})$.

**Definition 2.1** (Humphries-Johnson). For a number $r \geq 2$, a $\mathbb{Z}/r\mathbb{Z}$-spin structure on $S$ is a $\mathbb{Z}/r\mathbb{Z}$-valued function $\varphi$ on isotopy classes of oriented simple closed curves on $S$ with the following properties.

1. (Twist linearity) Let $c, d$ be oriented simple closed curves and let $T_c$ be the left Dehn twist about $c$; then
   
   $\varphi(T_c(d)) = \varphi(d) + \iota(d, c)\varphi(c) \pmod{r}$.

2. (Normalization) $\varphi(\zeta) = 1$ for the oriented boundary $\zeta$ of an embedded disk $D \subset S$.

As an additional property, one obtains that whenever $c^{-1}$ is obtained from $c$ by reversing the orientation, then $\varphi(c^{-1}) = -\varphi(c)$ (Lemma 2.2 of [HJ89]).

Humphries and Johnson [HJ89] (see Theorem 3.5 of [Sa19]) also give an alternative description of spin structures. Namely, for some choice of a hyperbolic metric on $S$ let $UTS$ be the unit tangent bundle of $S$. It can be viewed as the quotient of the complement of the zero section in the tangent bundle of $S$ by the multiplicative group $(0, \infty)$ and hence it does not depend on the metric.

The Johnson lift of a smoothly embedded oriented simple closed curve $c$ on $S$ is simply the closed curve in $UTS$ which consists of all unit tangents of $c$ defining the given orientation. The following is Theorem 2.1 and Theorem 2.5 of [HJ89] as formulated in Theorem 3.5 of [Sa19].
Theorem 2.2 (Humphries-Johnson). Let \( S \) be a compact surface and let \( \zeta \) be the oriented fibre of the unit tangent bundle \( UTS \to S \). A cohomology class \( \psi \in H^1(UTS, \mathbb{Z}/r\mathbb{Z}) \) with \( \psi(\zeta) = 1 \) determines a \( \mathbb{Z}/r\mathbb{Z} \)-spin structure via \( \alpha \to \psi(\tilde{\alpha}) \) where \( \alpha \) is an oriented simple closed curve on \( S \) and \( \tilde{\alpha} \) is its Johnson lift. This determines a 1-1 correspondence between \( \mathbb{Z}/r\mathbb{Z} \)-spin structures and \( \{ \psi \in H^1(UTS, \mathbb{Z}/r\mathbb{Z}) \mid \psi(\zeta) = 1 \} \).

There is another interpretation as follows; we refer to p.131 of [Hai95] for more information on this construction. Given a number \( r \geq 2 \) which divides \( 2g - 2 \), an application of the Gysin sequence for the Euler class of \( UTS \) yields a short exact sequence
\[
0 \to \mathbb{Z}/r\mathbb{Z} \to H^1(UTS, \mathbb{Z}/r\mathbb{Z}) \to H_1(S, \mathbb{Z}/r\mathbb{Z}) \to 0.
\]
By covering space theory, an \( r \)-th root of the tangent bundle of \( S \), viewed as a complex line bundle for some fixed complex structure, is determined by a homomorphism \( H_1(UTS, \mathbb{Z}/r\mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z} \) whose composition with the inclusion \( \mathbb{Z}/r\mathbb{Z} \to H^1(UTS, \mathbb{Z}/r\mathbb{Z}) \) is the identity and therefore

Proposition 2.3. There is a natural one-to-one correspondence between the \( r \)-th roots of the canonical bundle of \( S \) and splittings of the sequence (1).

A \( \mathbb{Z}/2\mathbb{Z} \)-spin structure on a compact surface \( S \) of genus \( g \) with empty or connected boundary has a parity which is defined as follows.

A geometric symplectic basis for \( H_1(S, \mathbb{Z}) \) is a system \( a_1, b_1, \ldots, a_g, b_g \) of simple closed curves on \( S \) such that \( a_i \), \( b_i \) intersect in a single point and that \( a_i \cup b_i \) is disjoint from \( a_j \cup b_j \) for \( i \neq j \). Then the parity of the spin structure \( \varphi \) equals
\[
\text{Arf}(\varphi) = \sum_i (\varphi(a_i) + 1)(\varphi(b_i) + 1) \in \mathbb{Z}/2\mathbb{Z}.
\]
This does not depend on the choice of the geometric symplectic basis.

2.2. The graph of nonseparating curves with vanishing spin value. The curve graph \( CG \) of \( S \) is the graph whose vertices are essential (that is, neither nullhomotopic nor homotopic into the boundary) simple closed curves in \( S \) and where two such curves are connected by an edge if they can be realized disjointly. We can use the spin structure \( \varphi \) to introduce various subgraphs of \( CG \) and study their properties. One of the main technical ingredients to this end is the following result of Salter (Corollary 4.3 of [Sa19]).

Lemma 2.4 (Salter). Let \( \Sigma \subset S \) be an embedded one-holed torus. Then there exists a simple closed curve \( c \subset \Sigma \) with \( \varphi(c) = 0 \).

Denote by \( CG_0 \subset CG \) the complete subgraph of the curve graph whose vertex set consists of nonseparating curves \( c \) with \( \varphi(c) = 0 \). Note that this is well defined, that is, it is independent of the choice of an orientation of \( c \). As a fairly easy consequence of Lemma 2.4 we obtain
Lemma 2.5. Let $\varphi$ be a spin structure on a closed surface of genus $g \geq 3$. Then $\mathcal{CG}_0$ is connected.

Proof. We use the following result of Masur-Schleimer [MS06], see Theorem 1.2 of [Put08]. Let $\mathcal{SG} \subset \mathcal{CG}$ be the complete subgraph whose vertex set consists of separating simple closed curves; then $\mathcal{SG}$ is connected. Note that this requires that $g \geq 3$.

Let $a, b$ be vertices of $\mathcal{CG}_0$. Choose simple closed curves $\hat{a}, \hat{b}$ which intersect $a, b$ in a single point; such curves exist since $a, b$ are nonseparating. Then the boundary $c, d$ of a tubular neighborhood of $a \cup \hat{a}$ and $b \cup \hat{b}$, respectively, is a separating simple closed curve which decomposes $S$ into a one-holed torus containing $a, b$ and a surface of genus $g - 1 \geq 2$ with boundary.

Connect $c$ to $d$ by an edge path $(c_i)_{0 \leq i \leq k} \subset \mathcal{SG}$ (here $c = c_0$ and $d = c_k$). Construct inductively an edge path $(a_i) \subset \mathcal{CG}_0$ connecting $a = a_0$ to $b = a_k$ such that for each $i$, $a_i$ is disjoint from $c_i$, as follows. Put $a_0 = a$ and assume that we constructed already such a path for some $j < k$. Then $a_j$ is disjoint from $c_j$.

If $a_j$ also is disjoint from $c_{j+1}$ then define $a_{j+1} = a_j$. Otherwise $a_j$ is contained in the same component $\Sigma$ of $S - c_j$ as $c_{j+1}$. Choose a one-holed torus $T \subset S - \Sigma$. Such a torus exists since $c_j$ decomposes $S$ into two surfaces of positive genus with connected boundary. By Lemma 2.4 this torus contains a nonseparating simple closed curve $a_{j+1}$ with $\varphi(a_{j+1}) = 0$, and this curve is disjoint from both $a_j$ and $c_{j+1}$. This yields the induction step. □

Remark 2.6. The proof of Lemma 2.5 extends with a bit more care to compact surfaces of genus at least 3 with connected boundary. We expect that the Lemma also holds true for $g = 2$.

2.3. The graph of nonseparating curves with spin value $\pm 1$ on a surface of genus 2. Define $\mathcal{CG}_1$ to be the complete subgraph of $\mathcal{CG}$ of all nonseparating simple closed curves $c$ on $S$ with $\varphi(c) = \pm 1$. Note that this condition does not depend on the orientation of $c$ and hence it is indeed a condition on the vertices of $\mathcal{CG}$. In this subsection we discuss the special case $g = 2$.

Proposition 2.7. Let $\varphi$ be an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure on a closed surface $S$ of genus 2. Then any two simple closed nonseparating curves $c, d$ on $S$ with $\varphi(c) = \varphi(d) = 1$ intersect.

Proof. Let $\varphi$ be a $\mathbb{Z}/2\mathbb{Z}$-spin structure on $S$. Let $c$ be a nonseparating simple closed curve on $S$ with $\varphi(c) = 1$. Assume that there is a nonseparating simple closed curve $d$ with $\varphi(d) = 1$ which is disjoint from $c$. As a surface of genus two does not admit bounding pairs, the surface $S - (c \cup d)$ is a four-holed sphere. Thus there exists a simple closed separating curve $e$ which decomposes $S$ into two one-holed tori $T_1, T_2$ such that $c \in T_1, d \in T_2$.

Denoting by $\iota$ the mod two homological intersection form on $H_1(S, \mathbb{Z}/2\mathbb{Z})$, there are two nonseparating simple closed curves $v \subset T_1, w \subset T_2$ so that

$$\iota(v, c) = 1 = \iota(w, d) \text{ and } \iota(w, c) = \iota(v, d) = 0.$$
The curves \(a_1 = c, b_1 = v, a_2 = d, b_2 = w\) define a geometric symplectic basis for \(H_1(S, \mathbb{Z})\). Since \(\varphi(a_1) = \varphi(a_2) = 1\), the formula (2) for the Arf invariant shows that \(\varphi\) is even as claimed. \(\square\)

2.4. \(\mathbb{Z}/r\mathbb{Z}\)-spin structures for \(r = 2, 4\) on a surface of genus \(g \geq 3\). In this subsection we study the graph \(CG_1\) for a \(\mathbb{Z}/r\mathbb{Z}\)-spin structure on a surface of genus \(g \geq 3\) for \(r = 2, 4\). We begin with evoking a result of Salter [Sa19].

Namely, let \(c, d\) be disjoint simple closed curves on the compact surface \(S\). Let \(\epsilon\) be an embedded arc in \(S\) connecting \(c\) to \(d\) whose interior is disjoint from \(c \cup d\). A regular neighborhood \(\nu\) of \(c \cup \epsilon \cup d\) is homeomorphic to a three-holed sphere. Two of the boundary components of \(\nu\) are the curves \(c, d\) up to homotopy. We choose an orientation of \(c, d\) in such a way that \(\nu\) lies to the left. The third boundary component \(c + \epsilon d\), oriented in such a way that \(\nu\) is to its right, satisfies \([c + \epsilon d] = [c] + [d]\) where \([c]\) denotes the homology class of the oriented curve \(c\). The following is Lemma 3.13 of [Sa19].

Lemma 2.8 (Salter). \(\varphi(c + \epsilon d) = \varphi(c) + \varphi(d) + 1\).

As a consequence, if \(r = 2, 4\) then the boundary of any embedded pair of pants \(P \subset S\) contains a simple closed curve \(c\) with \(\varphi(c) = \pm 1\). To use this fact for our purpose we introduce another graph related to simple closed curves on surfaces.

Definition 2.9. Let \(S\) be a compact surface of genus \(g \geq 2\). The graph of nonseparating pairs of pants \(\mathcal{NS}\) is the graph whose vertices are pairs of pants in \(S\) whose boundary consists of three pairwise distinct nonseparating simple closed curves and where two such pair of pants are connected by an edge if their intersection consists of precisely one boundary component.

By the preceding remark, the graph \(\mathcal{NS}\) can be used to find paths in the graph \(CG_1\) provided we can show that it is connected. To this end we evoke an observation of Putman (Lemma 2.1 of [Put08]) which we refer to as the Putman trick in the sequel.

Lemma 2.10 (Putman). Let \(G\) be a graph which admits a vertex transitive isometric action of a finitely generated group \(\Gamma\) and let \(v\) be a vertex of \(G\). If for each element \(s\) of a finite generating set \(S\) of \(\Gamma\), the vertex \(v\) can be connected to \(sv\) by an edge path in \(G\), then \(G\) is connected.

We apply the Putman trick to show

Lemma 2.11. The graph of nonseparating pairs of pants is connected.

Proof. If \(P \subset S\) is a nonseparating pair of pants, then \(S - P\) is a connected surface of genus \(g - 2\) with three distinguished boundary components. Thus the pure mapping class group \(P\text{Mod}(S)\) of \(S\) acts transitively on the vertices of \(\mathcal{NS}\). As a consequence, it suffices to show that there exists a generating set \(S\) of \(P\text{Mod}(S)\) and a nonseparating pair of pants \(P \in \mathcal{NS}\) which can be connected to its image \(\psi(P)\) by an edge path in \(\mathcal{NS}\) for every element \(\psi \in S\).
Now $\text{PMOD}(S)$ can be generated by Dehn twists $T_{c_i}$ about the collection of simple closed curves $c_0, \ldots, c_k$ shown in Figure 3 (see Section 4.4 of \cite{FM12}).

Furthermore, the simple closed curves $c_0, c_1, c_3$ are nonseparating and bound a pair of pants $P$. This pair of pants is stabilized by all elements of $S$ with the exception of the Dehn twists about the simple closed curves $c_2$ and $c_4$.

As the genus of $S$ is at least 3, for $i = 2, 4$ the complement $\hat{S}$ in $S$ of the union of $P$ with $c_i$ is a surface of genus $g - 2 \geq 1$ with two distinguished boundary components, one of which, say the curve $c$, is a boundary component of $P$.

The surface $\hat{S}$ contains a nonseparating simple closed curve $d$. As in Lemma 2.8, choose an embedded arc $\epsilon \subset \hat{S}$ connecting $c$ to $d$. A small neighborhood of $c \cup \epsilon \cup d$ is a nonseparating pair of pants $\hat{P}$ whose intersection with $P$ equals the curve $c$. As $\hat{P}$ is disjoint from $c_i$, it is left fixed by the Dehn twist $\psi_i$ about $c_i$ and hence $\hat{P}, \psi_i(P)$ is a path of length three in $\mathcal{NS}$ connecting $P$ to $\psi_i(P)$. Lemma 2.11 now is an immediate consequence of Lemma 2.10. □

The following is now an easy consequence of Lemma 2.11.

**Proposition 2.12.** Let $r = 2, 4$ and let $\varphi$ be a $\mathbb{Z}/r\mathbb{Z}$ spin structure on a compact surface $S$ of genus $g \geq 3$, with or without boundary. Then the graph $\mathcal{CG}_1$ is connected.

**Proof.** Let $c, d$ be nonseparating simple closed curves with $\varphi(c) = \pm 1, \varphi(d) = \pm 1$. Choose nonseparating pairs of pants $P, Q$ containing $c, d$ in their boundary. By Lemma 2.11, we can connect $P$ to $Q$ by a path in the graph $\mathcal{NS}$, say the path $(P_i)$ with $P_0 = P$ and $P_k = Q$.

For each $i$ let $c_i$ be a boundary component of $P_i$ with $\varphi(c_i) = \pm 1$ and such that $c_0 = c, c_k = d$. Then for each $i$, either $c_i = c_{i+1}$ or $c_i$ and $c_{i+1}$ are disjoint. Thus $(c_i)$ is a path in $\mathcal{CG}_1$ connecting $c$ to $d$. This shows the proposition. □
2.5. $\mathbb{Z}/r\mathbb{Z}$-spin structures on a surface of genus $g \geq 4$. In this subsection we investigate the graph $CG_1$ on a surface of genus $g \geq 4$ for an arbitrary $r \geq 2$. To show connectedness we use the following auxiliary graph $\mathcal{PS}$. The vertices of $\mathcal{PS}$ are pairs of disjoint separating curves $(c, d)$ which each decompose $S$ into a surface of genus $g - 1$ and a one-holed torus. Thus $S - (c \cup d)$ is the disjoint union of two one-holed tori and a surface of genus $g - 2$. Two such pairs $(c_1, d_1)$ and $(c_2, d_2)$ are connected by an edge if up to renaming, $c_1 = c_2$ and $d_2$ is disjoint from $c_1, d_1$. Then $S - (c_1 \cup d_1 \cup d_2)$ is the disjoint union of a surface of genus $g - 3$ with at least three holes and three one-holed tori. In particular, the graph $\mathcal{PS}$ is only defined if the genus of $S$ is at least three.

We use the Putman trick to show

**Lemma 2.13.** For a compact surface $S$ of genus $g \geq 4$, perhaps with boundary, the graph $\mathcal{PS}$ is a connected Mod$(S)$-graph.

**Proof.** The mapping class group Mod$(S)$ of the surface $S$ clearly acts on $\mathcal{PS}$, furthermore this action is vertex transitive. Namely, for any two vertices $(a_1, b_1)$ and $(a_2, b_2)$ of $\mathcal{PS}$, the complement $S - (a_i \cup b_i)$ is the union of two one-holed tori and a surface of genus $g - 2$ with $k + 2$ boundary components where $k \geq 0$ is the number of boundary components of $S$. Hence there exists $\varphi \in$ Mod$(S)$ with $\varphi(a_1, b_1) = (a_2, b_2)$.

Consider again the curve system $H$ shown in Figure 3 with the property that the Dehn twists about these curves generate the mapping class group. Choose a pair of disjoint separating simple closed curves $(a, b)$ which decompose $S$ into a surface of genus $g - 1$ and a one-holed torus $X(a), X(b)$ and such that a curve $c \in H$ intersects at most one of the curves $a, b$. If it intersects one of the curves $a, b$, then this intersection consists of precisely two points. For example, we can choose $a$ to be the boundary of a small neighborhood of $c_1 \cup c_2$, and $b$ to be the boundary of a small neighborhood of $c_5 \cup c_6$.

Now let $c \in H$ and let $T_c$ be the left Dehn twist about $c$. If $c$ is disjoint from $a \cup b$, then $T_c(a, b) = (a, b)$ and there is nothing to show. Thus assume that $c$ intersects $a$.

The image $T_c(a)$ of $a$ is a separating simple closed curve contained in a small neighborhood $Y$ of $X(a) \cup c$. By assumption on $c$, this surface is a two-holed torus disjoint from $b$. As $g \geq 4$, the genus of $S - (Y \cup X(b))$ is at least one and hence there is a separating curve $e \subset S - (Y \cup X(b))$ which decomposes $S - (Y \cup X(b))$ into a one-holed torus and a surface $S'$. But this means that $(a, b)$ can be connected to $T_c(a, b) = (T_c(a), b)$ by the edge path $(a, b) \to (e, b) \to (T_c(a, b))$. As the roles of $a$ and $b$ can be exchanged, the lemma now follows from the Putman trick. □

We shall use another auxiliary graph which is defined as follows.

**Definition 2.14.** Let $S$ be a compact surface $S$ of genus $g \geq 1$ with two distinguished boundary components $A_1, A_2$. The *nonseparating arc graph* is the graph whose vertices are isotopy classes of embedded arcs in $S$ connecting $A_1$ to $A_2$. The endpoints of an arc may move freely along the boundary circles $A_1, A_2$ in such an
isotopy class. Two such arcs $\epsilon_1, \epsilon_2$ are connected by an edge if $\epsilon_1, \epsilon_2$ are disjoint and $S - (\epsilon_1 \cup \epsilon_2)$ is connected.

We apply the Putman trick to show

**Lemma 2.15.** The nonseparating arc graph $A(A_1, A_2)$ on a compact surface $S$ of genus $g \geq 1$ with two distinguished boundary components $A_1, A_2$ is connected.

**Proof.** Clearly the pure mapping class group $P\text{Mod}(S)$ of $S$ acts transitively on the vertices of $A(A_1, A_2)$, so it suffices to show that there exists a generating set $S$ of $P\text{Mod}(S)$ and an arc $\epsilon \in A(A_1, A_2)$ which can be connected to its image $\psi(\epsilon)$ by an edge path in $A(A_1, A_2)$ for every element $\psi \in S$.

There exists two disjoint arcs $\epsilon_1, \epsilon_2$ connecting $A_1$ to $A_2$ such that $\epsilon_1 \cup \epsilon_2$ projects to an essential nonseparating simple closed curve in the surface obtained from $S$ by capping off the boundary components $A_1, A_2$. Furthermore, we may assume that $\epsilon_1$ intersects one of the curves shown in Figure 3, say the curve $c_1$, in a single point and is disjoint from the remaining curves, and $c_1$ is disjoint from $\epsilon_2$.

Then $T_{c_1} \epsilon_1 = \epsilon_1$ for $i \geq 2$, and $\epsilon_1$ can be connected to $T_{c_1} (\epsilon_1)$ by the edge path $\epsilon_1, \epsilon_2, T_{c_1} \epsilon_1$. By the Putman trick this implies that $A(A_1, A_2)$ is connected. \hfill $\square$

We are now ready to show

**Proposition 2.16.** Let $\varphi$ be an $r$-spin structure ($r \geq 2$) on a compact surface $S$ of genus $g \geq 4$. Then the graph $CG_1$ is connected.

**Proof.** Let $S$ be a compact surface of genus $g \geq 2$ and consider the graph $PS$. To each of its vertices, viewed as a disjoint pair $(c, d)$ of separating simple closed curves, we associate in a non-deterministic way a vertex $A(c, d)$ of $CG_1$ as follows.

Denote by $\Sigma_c, \Sigma_d$ the one-holed torus bounded by $c, d$. If one of the tori $\Sigma_c, \Sigma_d$ contains a simple closed curve $a$ with $\varphi(a) = \pm 1$ then define $A(c, d) = a$.

Now assume that none of the tori $\Sigma_c, \Sigma_d$ contains a simple closed curve $a$ with $\varphi(a) = \pm 1$. By Lemma 2.3 there are simple closed nonseparating curves $a \subset \Sigma_c, b \subset \Sigma_d$ so that $\varphi(a) = 0 = \varphi(b)$. Since the tori $\Sigma_c, \Sigma_d$ are disjoint, the pair $(a, b)$ is nonseparating, that is, $S - (a \cup b)$ is connected. Choose an embedded arc $\epsilon$ in $S$ connecting $a$ to $b$. By Lemma 2.8 the curve $A(c, d) = a +, b$ satisfies $\varphi(a +, b) = \pm 1$, furthermore it is nonseparating.

Let $a$ be any vertex of $CG_1$ and let $b$ be any simple closed curve which intersects $a$ in a single point. Such a curve exists since $a$ is nonseparating. Then a tubular neighborhood of $a \cup b$ is a torus containing $a$. Let $c$ be the boundary curve of this torus and choose a second separating simple closed curve $d$ so that $(c, d) \in PS$.

Let $e \in CG_1$ be another vertex. Construct as above a vertex $(p, q) \in PS$ so that $e$ is contained in the one-holed torus cut out by $p$. Connect $(c, d)$ to $(p, q)$ by an edge path $(c_i, d_i)_{0 \leq i \leq k}$ in $PS$. We use this edge path to construct an edge path
Define $a_0 = a$ and by induction, let us assume that we constructed already the path $(a_j)_{0 \leq j \leq j_i}$ for some $i \geq 0$. We distinguish two cases.

Case 1: One of the tori $\Sigma_{c_i}, \Sigma_{d_j}$ contains a curve $f$ with $\varphi(f) = \pm 1$.

By construction, in this case we may assume by renaming that $f = a_j \subset \Sigma_{c_i}$.

If $c_i \in \{c_{i+1}, d_{i+1}\}$ then define $a_{j+1} = a_{j+1} = a_{j_i} = \Lambda(c_{i+1}, d_{i+1})$ and note that this is consistent with the requirements for the induction step.

Thus we may assume now that $c_i \notin \{c_{i+1}, d_{i+1}\}$. If one of the tori $\Sigma_{c_{i+1}}, \Sigma_{d_{i+1}}$, say the torus $\Sigma_{c_{i+1}}$, contains a curve $h$ with $\varphi(h) = \pm 1$, then as $\Sigma_{c_i}$ is disjoint from $\Sigma_{c_{i+1}}$, the curve $h$ is disjoint from $a_j$, and we can define $a_{j+1} = h = a_{j+1} = \Lambda(c_{i+1}, d_{i+1})$.

Thus assume that neither $\Sigma_{c_{i+1}}$ nor $\Sigma_{d_{i+1}}$ contains such a curve. Since $\Sigma_{c_i}$ and $\Sigma_{c_{i+1}}, \Sigma_{d_{i+1}}$ are pairwise disjoint, we can find an embedded arc $\epsilon$ in $S - \Sigma_{c_i}$ connecting a simple closed curve $u \subset \Sigma_{c_{i+1}}$ with $\varphi(u) = 0$ to a curve $h \subset \Sigma_{d_{i+1}}$ with $\varphi(h) = 0$. We then can define $a_{j+1} = u + \epsilon h = \Lambda(c_{i+1}, d_{i+1}) = a_{j+1}$.

Case 2: None of the tori $\Sigma_{c_i}, \Sigma_{d_j}$ contains a curve $f$ with $\varphi(f) = \pm 1$.

In this case there are simple closed curves $f \subset \Sigma_{c_i}, h \subset \Sigma_{d_j}$ with $\varphi(f) = \varphi(h) = 0$, and there is an embedded arc $\epsilon$ connecting $f$ to $h$ so that

$$a_{j_i} = \Lambda(c_{i+1}, d_{i+1}) = f + \epsilon h.$$  

Assume without loss of generality that $d_i = d_{i+1}$.

Let us in addition assume for the moment that the arc $\epsilon$ is disjoint from $c_{i+1}$. If furthermore there exists a simple closed curve $u \subset \Sigma_{c_{i+1}}$ with $\varphi(u) = \pm 1$, then this curve is a choice for $\Lambda(c_{i+1}, d_{i+1})$ which is disjoint from $a_j$, and we are done.

Otherwise cut $S$ open along the simple closed curve $h \subset \Sigma_{d_i} = \Sigma_{d_{i+1}}$ and let $H_1, H_2$ be the two boundary components of $S - h$. By renaming, assume without loss of generality that $\epsilon$ connects the boundary component $H_1$ to the curve $f$, i.e. it leaves the curve $h$ from the side corresponding to $H_1$. Now note that $M = S - h - \epsilon - \Sigma_{c_i}$ is a connected surface of genus $g - 2 \geq 2$ with two distinguished boundary circles, one of which is the curve $H_2$, and $M \supset \Sigma_{c_{i+1}}$. Therefore there exists an embedded arc $\epsilon' \subset M$ connecting $H_2$ to a simple closed curve $u \subset \Sigma_{c_{i+1}}$ with $\varphi(u) = 0$. Define $a_{j_{i+1}} = h + \epsilon' u$ and note that this definition is consistent with all requirements. This construction completes the induction step under the additional assumption that arc $\epsilon$ is disjoint from $c_{i+1}$.

We are left with the case that $\epsilon$ is not disjoint from $\Sigma_{c_{i+1}}$. Cut $S$ open along $f \cup h$ and note that the resulting surface $Z$ has genus $g - 2 \geq 2$ and four distinguished boundary components, say the components $F_1, F_2, H_1, H_2$. Assume that $\epsilon$ connects $F_1$ to $H_1$. 


Consider the nonseparating arc graph $A(F_1, H_1)$ in $Z$ of arcs connecting $F_1$ to $H_1$. By Lemma 2.15, this graph is connected. Let $e_i$ be a path in $A(F_1, H_1)$ which connects $e$ to an arc $e'$ disjoint from $\Sigma_{c+1}$. For any two consecutive of such arcs, say the arcs $e_j, e_{j+1}$, the surface $Z - (e_1 \cup e_2)$ is connected and hence we can find a disjoint arc $\delta_j$ connecting $F_2$ to $H_2$. The curves $f + e, h, f + \delta_j, h, f + e_{j+1}$ are disjoint and yield a path in $CG_1$ connecting $f + e$ to a curve $f + e'$ which is disjoint from $\Sigma_{c+1}$. We then can apply the construction for the case that the arc connecting $f$ to $h$ is disjoint from $\Sigma_{c+1}$. This completes the proof of the proposition. \[\square\]

For technical reasons we need a stronger version of Proposition 2.12 and Proposition 2.16. Consider a $\mathbb{Z}/r\mathbb{Z}$-spin structure $\varphi$ on a compact surface $S$ of genus $g$ (with or without boundary) for an arbitrary number $r \geq 2$. We introduce another graph $CG^+_1$ as follows. The vertices of $CG^+_1$ coincide with the vertices of $CG_1$. Any two such vertices $c, d$ are connected by an edge if $c, d$ are disjoint and if furthermore $S - (c \cup d)$ is connected. Thus $CG^+_1$ is obtained from $CG_1$ by removing some of the edges. In particular, if $CG^+_1$ is connected then then same holds true for $CG_1$. We use connectedness of $CG_1$ to establish connectedness of $CG^+_1$.

**Lemma 2.17.** If the genus $g$ of $S$ is at least 3 then the graph $CG^+_1$ is connected provided that $CG_1$ is connected.

**Proof.** Let $c, d \in CG_1$ be two vertices which are connected by an edge in $CG_1$ and which are not connected by an edge in $CG^+_1$. This means that $c, d$ are disjoint, and $S - (c \cup d)$ is disconnected. We have to show that $c, d$ can be connected in $CG^+_1$ by an edge path.

To this end recall that $c, d$ are nonseparating and therefore the disconnected surface $S - (c \cup d)$ has two connected components $S_1, S_2$. The surface $S_1$ has genus $g_1 \geq 1$ and at least two boundary components, and the surface $S_2$ has genus $g_2 = g - g_1 - 1 \geq 0$ and at least two boundary components.

Choose a simple closed curve $d_i \subset S_1$ ($i = 1, 2$) which bounds with $c \cup d$ a pair of pants $P_i$. Write $\Sigma_i = S_i - P_i$; the genus of $\Sigma_i$ equals $g_i$. Glue $P_1$ to $P_2$ along $c \cup d$ so that the resulting surface $\Sigma_0$ is a two-holed torus containing $c \cup d$ in its interior. Choose a nonseparating simple closed curve $e \subset \Sigma_0$ which intersects both $c, d$ in a single point. Since $\varphi(e) = \pm 1$ we have $\varphi(T\varphi) = \varphi(e) \pm 1$ where $T\varphi$ is the left Dehn twist about $e$. Thus via replacing $e$ by $T^k\varphi e$ for a suitable choice of $k \in \mathbb{Z}$ we may assume that $\varphi(e) = 1$. In other words, we may assume that $e$ is a vertex of $CG_1$.

Assume for the moment that $g_2 \geq 1$. By Lemma 2.4 there exist simple closed curves $a \subset \Sigma_1, b \subset \Sigma_2$ with $\varphi(a) = \varphi(b) = 0$. Connect $a$ to $b$ by an embedded arc $\epsilon$ which is disjoint from $c \cup e$ (and crosses through the curve $d$). The curve $a + e$ satisfies $\varphi(a + e) = 1$, and it is disjoint from both $c$ and $e$. Moreover, the surfaces $S - (c \cup a + e)$ and $S - (e \cup a + e)$ are connected. As a consequence, $c$ can be connected to $e$ by an edge path in $CG^+_1$ of length two which passes through $a + e$.

By symmetry of this construction, $e$ can also be connected to $d$ by an edge path in $CG^+_1$ and hence $c$ can be connected to $d$ by such a path. This completes the proof in the case that the genus $g_2$ of $S_2$ is positive.
If the genus of $S_2$ vanishes then the genus of $S_1$ equals $g_1 = g - 1 \geq 2$. Any nonseparating simple closed curve in $S_1$ forms with both $c, d$ a nonseparating pair. To find such a curve $e$ with $\phi(e) = 1$, note that $S_1$ contains two disjoint one-holed tori $T_1, T_2$, and by Lemma 2.4 there are embedded simple closed curves $a_i \in T_i$ which satisfy $\phi(a_i) = 0$. Then for any arc $\epsilon$ in $S_1$ connecting $a_1$ to $a_2$, the curve $e = a_1 + \epsilon a_2$ is nonseparating, and it is connected with both $c, d$ by an edge in $CG^{+}_1$. This is what we wanted to show. 

Proposition 2.12, Proposition 2.16 and Lemma 2.17 together show Corollary 2.18. Let $\phi$ be a $\mathbb{Z}/r\mathbb{Z}$-spin structure on a closed surface $\Sigma$ of genus $g \geq 3$. Then the graph $CG^{+}_1$ is connected.

3. The action of $\text{Mod}(S)[\phi]$ on geometrically defined graphs

In this section we consider an arbitrary $\mathbb{Z}/r\mathbb{Z}$-spin structure $\phi$ on a compact surface $S$ of genus $g \geq 3$, possibly with boundary, for some number $r \geq 2$. Our goal is to gain some information on the stabilizer $\text{Mod}(S)[\phi]$ of $\phi$ through its action on the graph $CG^{+}_1$ introduced in Section 2.

We begin with some information on the stabilizer of a spin structure $\phi$ on a compact surface $S$ with boundary. Fix a boundary component $C$ of $S$. Denote by $P_C \text{Mod}(S)$ the subgroup of the mapping class group $\text{Mod}(S)$ of $S$ which fixes the boundary component $C$. Note that as we allow that a mapping class in $P_C \text{Mod}(S)$ exchanges boundary components of $S$ different from $C$, the group $P_C \text{Mod}(S)$ coincides with the pure mapping class group of $S$ only if the boundary of $S$ consists of one or two components.

Write $P_C \text{Mod}(S)[\phi]$ to denote the stabilizer of $\phi$ in $P_C \text{Mod}(S)$. This is a subgroup of $P_C \text{Mod}(S)$ of finite index. Let $\Sigma$ be the surface obtained from $S$ by attaching a disk to $C$. There is an embedding $S \to \Sigma$ which induces a surjective homomorphism

$$\Pi : P_C \text{Mod}(S) \to \text{Mod}(\Sigma).$$

By a result of Johnson, extending earlier work of Birman (see Section 4.2.5 of [FM12]), there is an exact sequence

$$1 \to \mathbb{Z} \to \ker(\Pi) \xrightarrow{\Sigma} \pi_1(\Sigma) \to 1$$

where $\mathbb{Z}$ is the infinite cyclic central subgroup of $P_C \text{Mod}(S)$ generated by the Dehn twists about $C$ and where $\pi_1(\Sigma)$ is a so-called point pushing group.

For the formulation of the following lemma, recall that the integral homology $H_1(\Sigma, \mathbb{Z})$ of a compact surface $\Sigma$ of genus $g \geq 2$, possibly with boundary, is a free abelian group $\mathbb{Z}^h$ for some $h \geq 4$. In fact, $h = 2g$ if the boundary of $\Sigma$ is empty or connected, and in this case this group is generated by the homology classes of nonseparating simple closed curves on $\Sigma$. If the boundary of $\Sigma$ is disconnected, then it is still true that $H_1(\Sigma, \mathbb{Z})$ is generated by simple closed possibly peripheral curves.
For $m \geq 1$ let $\Lambda_m \subset \pi_1(S)$ be the subgroup defined by the exact sequence
\[ 0 \to \Lambda_m \to \pi_1(S) \to H_1(S, \mathbb{Z}/m\mathbb{Z}) \to 0. \]
If $\zeta : \pi_1(S) \to H_1(S, \mathbb{Z})$ denotes the natural surjective projection, then $\Lambda_m$ is the
preimage under $\zeta$ of the lattice in $H_1(S, \mathbb{Z})$ generated by $m$ times the simple loop generators, and it is a subgroup of $\pi_1(S)$ of finite index. Using the notations from
the previous paragraph we have

**Lemma 3.1.** Assume that the boundary circle $C$ is equipped with the orientation induced from the orientation of $S$.

1. If $\varphi(C) = -1$ then $\Upsilon(\ker \Pi \cap P_\Sigma \text{Mod}(S)[\varphi]) = \pi_1(\Sigma)$.
2. If $\varphi(C) = 1$, then $\Upsilon(\ker \Pi \cap P_\Sigma \text{Mod}(S)[\varphi]) = \Lambda_m$ where $m = r/2$ if $r$ is
   even, and $m = r$ otherwise.

**Proof.** Choose a basepoint $p$ for $\pi_1(\Sigma)$ in the interior of the attached disk. Let
$\alpha \subset \Sigma$ be a simple nonseparating loop through the basepoint $p$. Up to homotopy,
the oriented boundary of a tubular neighborhood of $\alpha$ consists of two simple closed
curves $c_1, c_2$ which enclose the circle $C$. In other words, together with $C$ the curves
$c_1, c_2$ bound a pair of pants $P$ in $S$. We equip the curves $c_i$ with the orientation as
boundary curves of $P$.

By Proposition 3.8 of [Sa19], we have
\[ \varphi(C) + \varphi(c_1) + \varphi(c_2) = -1 \]
and hence if $\varphi(C) = -1$ then $\varphi(c_1) + \varphi(c_2) = 0$.

Let as before $T_d$ be the left Dehn twist about a simple closed curve $d$. Let
$\beta \subset S$ be an oriented simple closed curve which crosses through the pair of pants
$P$. As $c_1, c_2$ are disjoint, we have $\iota(T_{c_1}^{-1}(\beta), c_1) = \iota(\beta, c_1)$ and therefore Definition
2.1 shows that
\[ \varphi(T_{c_1}T_{c_2}^{-1}(\beta)) = \varphi(T_{c_2}^{-1}(\beta)) + \iota(\beta, c_1)\varphi(c_1) \]
\[ = \varphi(\beta) + \iota(\beta, c_1)\varphi(c_1) + \iota(\beta, c_2)\varphi(c_2). \]

On the other hand, as $c_1 + c_2$ is homologous to the boundary curve $C$, the homolog-
ical intersection number fulfills $\iota(\beta, c_1 + c_2) = 0$. Hence from (5) we conclude that
if $\varphi(C) = -1$ then $\varphi(T_{c_1}T_{c_2}^{-1}(\beta)) = \varphi(\beta)$. Since $\beta$ was an arbitrary simple closed
curve, this shows that $T_{c_1}T_{c_2}^{-1} \in P_\Sigma \text{Mod}(S)[\varphi]$. But $T_{c_1}T_{c_2}^{-1}$ is just
the point-pushing map about $\alpha$ and therefore $\alpha$ is contained in $\Upsilon(P_\Sigma \text{Mod}(S)[\varphi])$.
We refer to [FM12] for a comprehensive discussion of the various versions of the
Birman exact sequence.

As the point pushing group $\pi_1(\Sigma)$ is generated by point pushing maps along
simple loops, this shows the first part of the lemma.

To show the second part of the lemma, assume now that $\varphi(C) = 1$. Equation
(5) shows that $\varphi(c_1) + \varphi(c_2) = -2$ and hence by Formula (6) we have
\[ \varphi(T_{c_1}T_{c_2}^{-1}(\beta)) = \varphi(\beta) + \iota(\beta, c_1)\varphi(c_1) + \iota(\beta, c_2)(\varphi(c_1) + 2). \]
Now let us assume that the oriented simple closed curve $\beta$ crosses a single time through $c_1$, say when it enters $P$. Then $\iota(\beta, c_1) = -1, \iota(\beta, c_2) = 1$ and hence
\begin{equation}
\varphi(T_{c_1}T_{c_2}^{-1}(\beta)) = \varphi(\beta) - \varphi(c_1) + \varphi(c_1) + 2 = \varphi(\beta) + 2.
\end{equation}
Using this formula $r/2$ times if $r$ is even, and $r$ times if $r$ is odd, we conclude that the point pushing map about $\alpha$ is not contained in $\text{Mod}(S)[\varphi]$, but it is the case for its $r/2$-th power or $r$-th power, respectively. Namely, putting $m = r/2$ if $r$ is even and $m = r$ otherwise, it follows from the above discussion that we have $\varphi((T_{c_1}T_{c_2}^{-1})^m(\beta)) = \varphi(\beta)$ for every simple closed curve $\beta$ which either is disjoint from $P$ or which crosses through $P$ precisely once. As such curves span the first homology of $S$, we conclude that the pull-back of $\varphi$ under $(T_{c_1}T_{c_2}^{-1})^m$ coincides with $\varphi$ on a collection of simple closed curves which span $H_1(S, \mathbb{Z})$. Corollary 2.6 of [HJ89] then shows that indeed, $(T_{c_1}T_{c_2}^{-1})^m \in P_C\text{Mod}(S)[\varphi]$. Moreover, by equation (7), we know that $(T_{c_1}T_{c_2}^{-1})^k \notin P_C\text{Mod}(S)[\varphi]$ if $k$ is not a multiple of $m$.

On the other hand, by Lemma 3.15 of [Sn19], Dehn twists about separating simple closed curves in $S$ are contained in $\text{Mod}(S)[\varphi]$. As the commutator subgroup of $\pi_1(\Sigma)$ is generated by simple closed separating curves, and for each such curve $\alpha$ both Dehn twists $T_{c_1}, T_{c_2}$ about the boundary curves of a tubular neighborhood of $\alpha$ as above are contained in $P_C\text{Mod}(S)[\varphi]$, this yields the second part of the lemma.

Consider again an arbitrary compact surface $S$ of genus $g \geq 2$, equipped with a $\mathbb{Z}/r\mathbb{Z}$-spin structure $\varphi$ for some $r \geq 2$. We use Lemma 3.1 to analyze the action of $\text{Mod}(S)[\varphi]$ on the graph $CG^+_1$. We begin with the investigation of the stabilizer of a vertex $c$ of $CG^+_1$ in $\text{Mod}(S)[\varphi]$. As $\text{Mod}(S)[\varphi]$ is a subgroup of $\text{Mod}(S)$ of finite index, the stabilizer $\text{Stab}(c)[\varphi]$ of $c$ in $\text{Mod}(S)[\varphi]$ is a subgroup of finite index of the stabilizer $\text{Stab}(c)$ of $c$ in $\text{Mod}(S)$.

The group $\text{Stab}(c)$ can be described as follows. Cut $S$ open along $c$. The result is a surface $\Sigma^2$ of genus $g - 1$ with two distinguished boundary components $C_1, C_2$. These components are equipped with an orientation as subsets of the oriented boundary of $\Sigma^2$. To simplify notations, let $\text{Mod}(\Sigma^2)$ be the subgroup of the mapping class group of $\Sigma^2$ which preserves the subset $C_1 \cup C_2$ of the boundary. We allow that an element of $\text{Mod}(\Sigma^2)$ exchanges $C_1$ and $C_2$. The stabilizer $\text{Stab}(c)$ of $c$ in the mapping class group $\text{Mod}(S)$ of $S$ can be identified with the quotient of the group $\text{Mod}(\Sigma^2)$ by the relation $T_{C_1}T_{C_2}^{-1} = 1$ where $T_{C_1}$ denotes the left Dehn twist about the boundary circle $C_1$ (Theorem 3.18 of [FM12]). In short, we have
$$\text{Stab}(c) = \text{Mod}(\Sigma^2)/\mathbb{Z}.$$ 

The infinite cyclic subgroup of $\text{Stab}(c)$ generated by the Dehn twist about $c$ is central. The quotient group $\text{Stab}(c)/\mathbb{Z}$ can naturally be identified with the mapping class group $\text{Mod}(\Sigma^2)$ of a surface of genus $g - 1$ with two punctures and perhaps with boundary if the boundary of $S$ is non-trivial. We refer to [FM12] for a comprehensive discussion of these facts.

Let $\Sigma$ be the surface obtained from $\Sigma_2$ by forgetting the punctures. Alternatively, $\Sigma$ is obtained from $\Sigma^2$ by attaching a disk to each boundary component. The group
Mod(Σ₂) = Stab(c)/Z fits into the Birman exact sequence

\[ 1 \to \pi_1(C(Σ, 2)) \to \text{Stab}(c)/Z \to \text{Mod}(Σ) \to 1 \]

where \( \pi_1(C(Σ, 2)) \) is the surface braid group, that is, the fundamental group of the configuration space of two unordered distinct points in \( Σ \). In particular, \( \pi_1(C(Σ, 2)) \) is a normal subgroup of \( \text{Stab}(c)/Z = \text{Mod}(Σ₂) \).

The surjective homomorphism

\[ \theta : \text{Stab}(c) \to \text{Stab}(c)/Z = \text{Mod}(Σ₂) \]

restricts to a homomorphism \( \text{Stab}(c)[ϕ] \to \text{Mod}(Σ₂) \). The next proposition gives some first information on its image under the assumption that \( ϕ \) is a \( \mathbb{Z}/2\mathbb{Z} \)-spin structure and \( ϕ(c) = 1 \).

**Proposition 3.2.** Let \( ϕ \) be a \( \mathbb{Z}/2\mathbb{Z} \)-spin structure on \( S \) and let \( c \) be a simple closed curve with \( ϕ(c) = 1 \). Then \( ρ(\pi_1(C(Σ, 2))) \subseteq \theta(\text{Stab}(c)[ϕ]) \).

**Proof.** Let \( \pi_1(PC(Σ, 2)) \) be the intersection of the fibre of the Birman exact sequence \([8]\) with the subgroup of \( \text{Mod}(Σ₂) \) which fixes each of the two distinguished punctures. Following Section 4.2.5 of [FM12], the group \( \pi_1(PC(Σ, 2)) \) can be described as follows.

Let \( C_1, C_2 \) be the distinguished boundary components of the surface \( Σ² = S - c \). Let \( Σ¹ \) be the surface obtained from \( Σ² \) by attaching a disk to the boundary circle \( C_1 \). Let \( P\text{Stab}(c) \) and \( P\text{Mod}(Σ²) \) be the index two subgroup of \( \text{Stab}(c) \) and \( \text{Mod}(Σ²) \) which preserves each of the two boundary components \( C_1, C_2 \) of \( S - c \). The inclusion \( Σ² \to Σ¹ \) induces a surjective homomorphism

\[ Ξ : P\text{Stab}(c)/Z \to \text{Mod}(Σ¹)/Z \]

where as before \( \text{Mod}(Σ¹) \) is required to fix the boundary component \( C_2 \) of \( Σ¹ \) and where the group \( Z \) acts as the group of Dehn twists about \( c \) and about \( C_2 \). The kernel \( \ker(Ξ) \) of this homomorphism is isomorphic to \( \pi_1(Σ¹) \) (see [FM12] for more information on this version of the Birman exact sequence).

The spin structure \( ϕ \) pulls back to a spin structure \( \hat{ϕ} \) on \( Σ² \). Since \( ϕ \) is a \( \mathbb{Z}/2\mathbb{Z} \)-spin structure on \( S \) and \( ϕ(c) = 1 \), the value of \( \hat{ϕ} \) on each of the two boundary circles \( C_1, C_2 \) coincides with the value of a spin structure on the boundary of an embedded disk. This implies that \( \hat{ϕ} \) induces a spin structure \( ϕ' \) on \( Σ¹ \). Or, equivalently, \( \hat{ϕ} \) is the pull-back of a spin structure \( ϕ' \) on \( Σ¹ \) via the inclusion \( Σ² \to Σ¹ \). By Lemma 3.1, the group \( \ker(Ξ) = \pi_1(Σ¹) \) stabilizes \( \hat{ϕ} \), that is, we have \( \ker(Ξ) \subseteq \text{Mod}(Σ²)[\hat{ϕ}] \).

Apply Lemma 3.1 a second time to the homomorphism \( \text{Mod}(Σ¹)/Z \to \text{Mod}(Σ) \) where \( Σ \) is obtained from \( Σ¹ \) by attaching a disk to \( C_2 \). As the group \( \pi_1(PC(Σ, 2)) \) can be described as the quotient by its center \( \mathbb{Z}/2 \) of the kernel of the homomorphism \( P\text{Mod}(Σ²) \to \text{Mod}(Σ) \) which is obtained by applying the Birman exact sequence twice, first to a map which caps off the boundary component \( C_1 \), followed by the map which caps off \( C_2 \), this shows that \( \pi_1(PC(Σ, 2)) \subseteq \theta(\text{Stab}(c)[ϕ]) \). As exchanging \( C_1 \) and \( C_2 \) also preserves \( \hat{ϕ} \) the proposition follows. □
We are now ready to give a complete description of the stabilizer in $\text{Mod}(S)[\varphi]$ of a nonseparating simple closed curve $c$ on $S$ with $\varphi(c) = 1$ where as before, $\varphi$ is a $\mathbb{Z}/2\mathbb{Z}$-spin structure on a compact surface $S$ of genus $g \geq 3$.

Cut $S$ open along $c$ and write $\Sigma^2 = S - c$. The spin structure $\varphi$ of $S$ pulls back to a $\mathbb{Z}/2\mathbb{Z}$-spin structure $\hat{\varphi}$ on $\Sigma^2$. Denote as before by $\Sigma$ the surface of genus $g - 1$ with empty or connected boundary obtained from $\Sigma^2$ by capping off the two distinguished boundary components. We have

**Proposition 3.3.** The $\mathbb{Z}/2\mathbb{Z}$-spin structure $\varphi$ on $S$ induces a $\mathbb{Z}/2\mathbb{Z}$-spin structure $\varphi_c$ on $\Sigma$ whose parity coincides with the parity of $\varphi$. If $\Pi : \text{Stab}(c)/\mathbb{Z} \to \text{Mod}(\Sigma)$ denotes the surjective homomorphism induced by the inclusion $S - c \to \Sigma$ then

$$\Pi^{-1}\text{Mod}(\Sigma)[\varphi_c] = \text{Stab}(c)[\varphi]/\mathbb{Z}.$$ 

**Proof.** As $\varphi$ is a $\mathbb{Z}/2\mathbb{Z}$-spin structure, the value of $\varphi$ on a boundary circle of $S - c$ corresponding to a copy of $c$ coincides with the value of a $\mathbb{Z}/2\mathbb{Z}$-spin structure on the boundary of a disk. Thus $\varphi$ induces a spin structure $\varphi_c$ on $\Sigma$.

To compare the parities of the spin structures $\varphi$ and $\varphi_c$, assume that $\Sigma$ is obtained from $S - c$ by attaching disks $D_1, D_2$ to the two boundary components of $S$ which correspond to the two copies of $c$. Choose a geometric symplectic basis $a_1, b_1, \ldots, a_{g-1}, b_{g-1}$ for $\Sigma$, consisting of simple closed oriented curves which do not intersect the disks $D_1, D_2$. Then $a_1, b_1, \ldots, a_{g-1}, b_{g-1}$ can be viewed as a system of curves in $\Sigma^2 = \Sigma - (D_1 \cup D_2)$ which maps to a curve system with the same properties in $S$ by the map $\Sigma^2 \to S$. This curve system can be extended to a geometric symplectic basis for $S$ containing the curve $c$, equipped with any orientation. As $\varphi(c) = 1$ we have $\varphi(c) + 1 = 0$. The claim now follows from the fact that $\varphi_c(u) = \varphi(\hat{u})$ for $u \in \{a_1, b_1, \ldots, a_{g-1}, b_{g-1}\}$ where $\hat{u}$ is the image of $u$ under the inclusion $\Sigma^2 \to S$, together with the formula (2) for the Arf invariant.

We are left with showing that $\text{Stab}(c)[\varphi]/\mathbb{Z} = \Pi^{-1}\text{Mod}(\Sigma)[\varphi_c]$. Observe first that as $\varphi_c$ is induced from $\varphi$, we have $\Pi\text{Stab}(c)[\varphi]/\mathbb{Z} \subset \text{Mod}(\Sigma)[\varphi_c]$.

To show that in fact equality holds let $\Sigma_2$ be the surface obtained from $S - c$ by replacing the boundary components by punctures. The group $\text{Stab}(c)[\varphi]/\mathbb{Z}$ can be identified with a subgroup $\Gamma_c$ of $\text{Mod}(\Sigma_2)$. We view the punctures of $\Sigma_2$ as marked points $p_1, p_2$ in $\Sigma$.

Let $\theta$ be any diffeomorphism of $\Sigma$ which preserves $\varphi_c$. Then $\theta$ is isotopic to a diffeomorphism of $\Sigma$ which equals the identity on a disk $D \subset \Sigma$ containing both points $p_1, p_2$. Thus $\theta$ lifts to a diffeomorphism $\theta'$ of $\Sigma_2$ which preserves the pull-back of $\varphi_c$ to a spin structure on $\Sigma_2$.

The boundary circle $\partial D$ of $D$ can be viewed as a simple closed curve in $S - c$. Via the projection $S - c \to S$ which identifies the two distinguished boundary components of $S - c$, the curve $\partial D$ projects to a separating simple closed curve in $S$ which decomposes $S$ into a one-holed torus $T$ containing $c$ and a surface of genus $g - 1$ with connected boundary. The diffeomorphism $\theta'$ lifts to a diffeomorphism $\Theta$ of $S$ which is the identity on $T$. 
Then $\Theta^*\varphi$ is a spin structure on $S$ which defines the same function on $H_1(S,\mathbb{Z})$ as $\varphi$. Using once more the result of Humphries and Johnson [HJ89] (see Theorem 3.9 of [Sa19]), this implies that $\Theta$ stabilizes $\varphi$. As $\Theta$ projects to the mapping class of $\Sigma$ defined by the diffeomorphism $\theta$, this shows surjectivity of the homomorphism $\Pi : \text{Stab}(\varphi)/\mathbb{Z} \rightarrow \text{Mod}(\Sigma)[\varphi]$.

On the other hand, by Proposition 3.2 the kernel of the homomorphism $\Pi$ also is contained in $\text{Stab}(\varphi)/\mathbb{Z}$. Together this completes the proof of the proposition. $\square$

The next observation uses Proposition 4.9 of [Sa19]. For its formulation, recall from Section 2 the definition of the graph $CG_1^+$. Its vertices are nonseparating simple closed curves with prescribed value $\pm 1$ of the spin structure. The graph $CG_1^+$ is well defined if the genus $g$ of $S$ is at least two although it may not have edges.

Note that in the statement of Proposition 3.4 we allow that the surface $S$ has non-empty boundary, and we consider $\mathbb{Z}/r\mathbb{Z}$-spin structures where $r$ may be larger than $2g-2$. This is crucial for an inductive approach towards higher spin structures via cutting surfaces open along separating simple closed curves, and it is used in the proof of Theorem 4.

**Proposition 3.4.** Let $\varphi$ be a $\mathbb{Z}/r\mathbb{Z}$-spin structure on a compact surface $S$ of genus $g \geq 2$ with empty or connected boundary. Then for any two directed edges $e_1, e_2$ of the graph $CG_1^+$ there exists a mapping class $\zeta \in \text{Mod}(S)[\varphi]$ with $\zeta(e_1) = e_2$. In particular, the action of $\text{Mod}(S)[\varphi]$ on $CG_1^+$ is vertex transitive.

**Proof.** The proof consists of an adjustment of the argument in the proof of Proposition 4.9 of [Sa19].

Recall that a geometric symplectic basis for $S$ is a set $\{a_1, b_1, \ldots, a_{2g}, b_{2g}\}$ of simple closed curves on $S$ such that $a_i, b_i$ intersect in a single point, and $a_i \cup b_i$ is disjoint from $a_j \cup b_j$ for $j \neq i$.

A vertex of $CG_1^+$ is a simple closed curve $c$ on $S$ with $\varphi(c) = \pm 1$. In the sequel we always orient such a vertex $c$ in such a way that $\varphi(c) = 1$. For a given directed edge $e$ of $CG_1^+$ with ordered endpoints $c, d$, we aim at constructing a geometric symplectic basis $B(e)$ such that $a_1 = c, a_2 = d, \varphi(a_i) = 0$ for $i \geq 3$, $\varphi(b_i) = 0$ for $i \leq g-1$ and $\varphi(b_g) = 0$ or 1 as predicted by the parity of $\varphi$. If such a basis $B(e_1), B(e_2)$ can be found for any two directed edges $e_1, e_2$ of $CG_1^+$ with ordered endpoints $c_1, d_1$ and $c_2, d_2$, then there exists a diffeomorphism $\zeta$ of $S$ which maps $B(e_1)$ to $B(e_2)$ and maps $c_1, d_1$ to $c_2, d_2$. The pullback $\zeta^*\varphi$ of $\varphi$ is a spin structure on $S$ whose values on $B(e_1)$ coincide with the values of $\varphi$. By a result of Humphries and Johnson [HJ89], see Theorem 3.9 of [Sa19], this implies that $\zeta^*\varphi = \varphi$ and hence the isotopy class of $\zeta$ is contained in $\text{Mod}(S)[\varphi]$ and maps the directed edge $e_1$ to the directed edge $e_2$.

To simplify further, choose any geometric symplectic basis $B = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$
for \( S \) with \( \alpha_1 = c, \alpha_2 = d \). A small tubular neighborhood of \( \alpha_i \cup \beta_i \) is a one-holed torus \( T_i \) embedded in \( S \). By Lemma [2.4] for all \( i \geq 3 \) we may replace \( \alpha_i \) by an oriented simple closed curve in \( T_i \), again denoted by \( \alpha_i \), which satisfies \( \varphi(\alpha_i) = 0 \).

Assume that \( \beta_i (i = 1, 2) \) is oriented in such a way that \( \iota(\beta_i, \alpha_i) = 1 \) where \( \iota \) is the symplectic form. As \( \varphi(T_{\alpha_i}(\beta_i)) = \varphi(\beta_i) + 1 \), via perhaps replacing \( \beta_i \) by its image under a suitably chosen power of a Dehn twist about \( \alpha_i \) we may assume that \( \varphi(\beta_i) = 0 \). Therefore for the construction of a geometric symplectic basis \( B(\epsilon) \) with the required properties, it suffices to modify successively the curves \( \beta_i (i \geq 3) \) while keeping \( \alpha_j \) (\( j \geq 1 \)) and \( \beta_k \) for \( k < i \) fixed such that \( \varphi \) assumes the prescribed values on the modified curves.

We follow the proof of Proposition 4.9 of [Sa19]. For \( 1 \leq i \leq g \) let \( \delta_i \) be the boundary curve of the torus \( T_i \) which is a small tubular neighborhood of \( \alpha_i \cup \beta_i \), equipped with the orientation as an oriented boundary circle of \( S - T_i \) (\( i \geq 1 \)). By homological coherence (Proposition 3.8 of [Sa19]), we have \( \varphi(\delta_i) = 1 \) for all \( i \).

Thus if \( \epsilon \) is an embedded arc in \( S \) connecting \( \beta_3 \) to \( \delta_1 \) whose interior is disjoint from \( \alpha_3 \) and all \( \delta_j \) for \( j \neq 3 \), then \( \varphi(\beta_3 + \epsilon \delta_1) = \varphi(\beta_3) + 2 \). Moreover, \( \beta_3 + \delta_1 \) is disjoint from \( \delta_j \) for all \( j \neq 3 \).

A small tubular neighborhood of \( \alpha_3 \cup (\beta_3 + \epsilon \delta_1) \) is a one-holed torus \( \hat{T}_3 \) disjoint from the tori \( T_i \) for \( i \neq 3 \). Thus we can repeat this construction with an arc connecting \( \beta_3 + \epsilon \delta_1 \) to \( \delta_1 \) which is disjoint from \( \alpha_3 \) and whose interior is disjoint from all \( \delta_j \) for \( j \neq 3 \). Repeating further if necessary, we can find a simple closed curve \( \beta'_j \) intersecting \( \alpha_3 \) in a single point and disjoint from the curves \( \delta_j \) for \( j \neq 3 \) so that \( \varphi(\beta'_j) \in \{0, 1\} \).

Let \( \delta'_j \) be the boundary of a tubular neighborhood of \( \alpha_3 \cup \beta'_j \). Then \( \delta'_j \) is disjoint from all the curves \( \delta_j \) for \( j \neq 3 \). As in the proof of Proposition 4.9 of [Sa19], repeat this procedure with the curve \( \beta_4 \) and the curves \( \delta_1, \delta_2, \delta_3, \ldots, \delta_g \). In finitely many steps we can change the geometric symplectic basis \( B \) to a geometric symplectic basis \( B' = \{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta'_2, \ldots, \alpha_g, \beta'_g\} \) which fulfills \( \varphi(\beta'_j) = 0 \) or 1 for all \( 3 \leq j \leq g \).

It remains to further alter \( \beta'_j \) for \( 3 \leq j \leq g - 1 \) to a nonseparating simple closed curve \( \beta''_j \) with \( \varphi(\beta''_j) = 0 \), and to alter \( \beta''_g \) to a simple closed curve \( \beta''_g \) with \( \varphi(\beta''_g) = 0 \) or 1 depending on the parity of the \( \mathbb{Z}/r\mathbb{Z} \)-spin structure \( \varphi \). This construction is carried out in detail in the proof of Proposition 4.9 of [Sa19] and will not be presented here as it would require the introduction of a significant amount of new notation. It takes place in a subsurface of \( S \) of genus \( g - 2 \) which is disjoint from \( \alpha_1, \beta_1, \alpha_2, \beta_2 \) and contains \( \alpha_i, \beta_i \) for \( 3 \leq i \leq g \). The resulting geometric symplectic basis has the properties we are looking for.

\( \square \)

**Remark 3.5.** The proof of Proposition 3.4 can also be used to show the following. Under the assumption of the proposition, let \( c, d \subset S \) be two nonseparating simple closed curves with \( \varphi(c) = \varphi(d) = 0 \); then there exists some \( \zeta \in \text{Mod}(S)[\varphi] \) with \( \zeta(c) = d \). In fact, this case is more explicitly covered by Proposition 4.2 and Proposition 4.9 of [Sa19].
The next statement is an extension of Proposition 3.4 to surfaces with more than one boundary component under some restrictions on the spin structure.

**Corollary 3.6.** Let $\varphi$ be a $\mathbb{Z}/r\mathbb{Z}$-spin structure on a compact surface $S$ of genus $g \geq 2$ with non-empty boundary which is induced from a spin structure $\varphi'$ on a compact surface $\Sigma$ of genus $g$ with empty or connected boundary by an inclusion $S \to \Sigma$ which maps each boundary component of $S$ to the boundary of an embedded disk in $\Sigma$. Then for any two vertices $c, d$ of $\text{CG}_1^+$ there exists a mapping class $\zeta \in \text{Mod}(S)[\varphi]$ with $\zeta(c) = d$. In particular, the action of $\text{Mod}(S)[\varphi]$ is transitive on the vertices of $\text{CG}_1^+$.

**Proof.** Let $\Psi : S \to \Sigma$ be the natural embedding. Let $c, d$ be vertices of the graph $\text{CG}_1^+$ for the spin structure $\varphi$ on $S$. Then $c, d$ are nonseparating simple closed curves and hence their images $\Psi(c), \Psi(d)$ are nonseparating simple closed curves on $\Sigma$. Furthermore, as $\varphi$ is the pull-back of a spin structure $\varphi'$ on $\Sigma$, we have $\varphi'(\Psi(c)) = \varphi'(\Psi(d)) = 1$.

By Proposition 3.4, there exists a mapping class $\theta \in \text{Mod}(\Sigma)(\varphi')$ which maps $\Psi(c)$ to $\Psi(d)$. We can choose a diffeomorphism of $\Sigma$ representing $\theta$ which equals the identity on each component of $\Sigma - S$. Thus there exists a lift $\Theta$ of $\theta$ to a mapping class of $S$. This mapping class is contained in $\text{Mod}(S)[\varphi]$, and it maps the simple closed curve $c$ to a simple closed curve $d'$ whose image under $\Psi$ is isotopic to $\Psi(d)$.

Using once more the Birman exact sequence, this implies that there exists a mapping class $\beta$ in the kernel of the homomorphism $\text{Mod}(S) \to \text{Mod}(\Sigma)$ which maps $d'$ to $d$. But by an iterated application of Lemma 3.1, this kernel is contained in $\text{Mod}(S)[\varphi]$ and hence $c$ can be mapped to $d$ by an element of $\text{Mod}(S)[\varphi]$. \qed

The augmented Teichmüller space $\overline{T}(S)$ of the compact surface $S$ is the union of the Teichmüller space with so-called boundary strata. Each of these boundary strata is defined by a non-empty system $C$ of pairwise disjoint essential simple closed curves. The stratum defined by such a curve system can be thought of as the Teichmüller space of the surface obtained from $S$ by shrinking each component of $C$ to a node. In other words, such a stratum is a complex manifold which is naturally biholomorphic to the Teichmüller space of the surface obtained by cutting $S$ open along the components of $C$ and replacing each boundary component of the resulting bordered surface by a puncture.

Using Fenchel Nielsen coordinates, the augmented Teichmüller space can be equipped with a natural topology. For this topology, the usual Teichmüller space embeds into $\overline{T}(S)$ as an open dense subset. Furthermore, the inclusion of the Teichmüller space of a punctured surface defined by the curve system $C$ onto a boundary stratum of $\overline{T}(S)$ also is an embedding. We refer to [Wol10] for a detailed description and for a discussion of the following

**Theorem 3.7.** The augmented Teichmüller space $\overline{T}(S)$ is a non locally compact stratified space. The mapping class group $\text{Mod}(S)$ of $S$ acts on $\overline{T}(S)$, with quotient the Deligne Mumford compactification of the moduli space of curves of genus $g$. 


Fix again a $\mathbb{Z}/2\mathbb{Z}$-spin structure $\varphi$ on a surface $S$ of genus $g \geq 2$. Define the spin Teichmüller space $T_{\text{spin}}(S)$ to be the Teichmüller space of $S$ together with this spin structure. The group $\text{Mod}(S)[\varphi]$ acts on $T_{\text{spin}}(S)$ as a group of biholomorphic transformations, with quotient the spin moduli space $M_{\varphi} = T(S)/\text{Mod}(S)[\varphi]$.

We can define an augmented spin Teichmüller space $T_{\text{spin}}(S)$ as the union of spin Teichmüller space with all strata of augmented Teichmüller space which are defined by systems of nonseparating simple closed curves $c$ on $S$ with $\varphi(c) = 1$. Equipped with the subspace topology, this is a subspace of $\overline{T}(S)$ which is invariant under the action of the spin mapping class group. As a corollary of the discussion in this section, we have

**Corollary 3.8.** The quotient $T_{\text{spin}}(S)/\text{Mod}(S)[\varphi]$ is a partial bordification of the spin moduli space $T_{\text{spin}}(S)/\text{Mod}(S)[\varphi]$. Its boundary contains the spin moduli space of the same parity on a surface of genus $g-1$ with two marked points (punctures) as an open dense subset.

**Remark 3.9.** Corollary 3.8 can be thought of as describing a specific subset of a Deligne Mumford compactification of the moduli space of curves with a fixed spin structure. Such a Deligne Mumford compactification was constructed by Cornalba [Co89].

4. **Structure of the spin mapping class group of odd parity**

The goal of this section is to prove the first part of Theorem 3. We begin with some additional information on the spin mapping class group. Fix a $\mathbb{Z}/r\mathbb{Z}$-spin structure $\varphi$ on a closed surface $\Sigma_g$ of genus $g$ for some $r \geq 2$. For a simple closed curve $c$ on $\Sigma_g$ with $\varphi(c) = \pm 1$, this spin structure restricts to a spin structure on the surface $\Sigma_{g-1}^2$ of genus $g-1$ with two boundary circles $c_1, c_2$ obtained by cutting $\Sigma_g$ open along $c$. We denote this spin structure again by $\varphi$. Define the group $\Gamma_{g-1}^2$ to be the following quotient of the spin mapping class group $\text{Mod}(\Sigma_{g-1}^2)[\varphi]$.

The group $\text{Mod}(\Sigma_{g-1}^2)[\varphi]$ contains a rank two free abelian central subgroup generated by the $r$-th powers of the left Dehn twists $T_{c_1}, T_{c_2}$ about the boundary circles $c_1, c_2$ of $\Sigma_{g-1}^2$. Define $\Gamma_{g-1}^2 = \text{Mod}(\Sigma_{g-1}^2)[\varphi]/\mathbb{Z}$ where the infinite cyclic subgroup $\mathbb{Z}$ is generated by $T_{c_1}^r, T_{c_2}^{-r}$. Then $\Gamma_{g-1}^2$ is isomorphic to the stabilizer in $\text{Mod}(\Sigma_g)[\varphi]$ of the curve $c$. Note that up to isomorphism, the group $\Gamma_{g-1}^2$ does not depend on the vertex $c \in C_G$. Namely, by Proposition 3.4 the stabilizers in $\text{Mod}(\Sigma_g)[\varphi]$ of nonseparating simple closed curves $c$ with $\varphi(c) = \pm 1$ are all conjugate and hence isomorphic.

Observe that the group $\Gamma_{g-1}^2$ is an infinite cyclic central extension of a finite index subgroup of the mapping class group of a surface $\Sigma_{g-1,2}$ of genus $g-1$ with two punctures. Thus it makes sense to talk about its action on isotopy classes of essential curves on the surfaces $\Sigma_{g-1,2}$ and $\Sigma_{g-1}^2$. The map $\Sigma_{g-1}^2 \to \Sigma_{g-1,2}$ which contracts each boundary component to a puncture defines a bijection on such isotopy classes.
We have

**Proposition 4.1.** Let \( \varphi \) be a \( \mathbb{Z}/r\mathbb{Z} \)-spin structure on a closed surface \( \Sigma_g \) of genus \( g \geq 3 \). There is a commutative diagram

\[
\begin{array}{ccc}
\Gamma_g^2 & \xrightarrow{\iota_1} & \Gamma_g^2 * \Gamma_g^2 \times \mathbb{Z}/2\mathbb{Z} \\
\downarrow{\iota_2} & & \downarrow{\rho} \\
\text{Mod}(\Sigma_g)[\varphi] & &
\end{array}
\]

where the homomorphisms \( \iota_1, \iota_2 \) are inclusions, and the homomorphism \( \rho \) is surjective. The subgroup \( \Lambda \) of \( \Gamma_g^2 \) is the stabilizer in \( \Gamma_g^2 \) of a nonseparating simple closed curve \( d \) on \( \Sigma_g^2 \). The group \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \Gamma_g^2 \times \mathbb{Z}/2\mathbb{Z} \) by exchanging the two factors, and it acts as an automorphism on \( \Lambda \).

**Proof.** Fix a pair of nonseparating simple closed disjoint curves \( c, d \) on \( \Sigma_g \) with \( \varphi(c) = \varphi(d) = \pm 1 \) which are connected by an edge in the graph \( \mathcal{G}^+_1 \), that is, so that \( \Sigma_g - (c \cup d) \) is connected. Let \( \Gamma_c, \Gamma_d \subset \text{Mod}(\Sigma_g)[\varphi] \) be the stabilizers of \( c, d \) in the spin mapping class group of \( \Sigma_g \). By Corollary 3.6, these groups are naturally isomorphic to the group \( \Gamma_g^2 \), and they intersect in the index two subgroup \( \Lambda = \Gamma_c \cap \Gamma_d \) of the stabilizer of \( c \cup d \) in \( \text{Mod}(\Sigma_g)[\varphi] \). As \( \text{Mod}(\Sigma_g)[\varphi] \) is isomorphic to the group \( \mathbb{Z}/2\mathbb{Z} \), the stabilizer of \( c \cup d \) in \( \text{Mod}(\Sigma_g)[\varphi] \) is a \( \mathbb{Z}/2\mathbb{Z} \) extension of \( \Gamma_c \cap \Gamma_d \), where the generator \( \Phi \) of \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \Lambda \) by exchanging \( c \) and \( d \). This involution extends to an involution of \( \Gamma_c * \Gamma_d \), exchanging the two subgroups \( \Gamma_c, \Gamma_d \).

By the universal property of free amalgamated products, there is a homomorphism

\( \rho: \Gamma = \Gamma_c * \Gamma_d \times \mathbb{Z}/2\mathbb{Z} \to \text{Mod}(\Sigma_g)[\varphi] \).

All we need to show is that \( \rho \) is surjective, that is, that \( \rho(\Gamma) = \text{Mod}(\Sigma_g)[\varphi] \).

As \( \text{Mod}(\Sigma_g)[\varphi] \) acts transitively on the vertices of the graph \( \mathcal{G}^+_1 \), for this it suffices to show that its subgroup \( \rho(\Gamma) \) acts transitively on the vertices of \( \mathcal{G}^+_1 \) as well. Namely, by construction, the stabilizer of the vertex \( c \) of \( \mathcal{G}^+_1 \) in \( \rho(\Gamma) \) coincides with its stabilizer in \( \text{Mod}(\Sigma_g)[\varphi] \). As \( \rho(\Gamma) \) is a subgroup of \( \text{Mod}(\Sigma_g)[\varphi] \), this then implies equality.

To show transitivity of the action of \( \rho(\Gamma) \) on the vertices of \( \mathcal{G}^+_1 \) let \( v \in \mathcal{G}^+_1 \) be any vertex. By Corollary 2.18, the graph \( \mathcal{G}^+_1 \) is connected and hence we can find an edge path \( (c_i) \subset \mathcal{G}^+_1 \) connecting \( c_0 = c \) to \( c_k = v \). We also may assume that \( c_1 = d \).

By the assumption \( \varphi(d) = \pm 1 \), for one of the two boundary components \( d_1, d_2 \) of \( \Sigma_g - d \), equipped with the orientation as a boundary component of \( \Sigma_g - d \), say the component \( d_1 \), we have \( \varphi(d_1) = -1 \). Thus we can attach a disk \( D \) to \( c_1 \) and obtain a surface \( \Sigma' \) with spin structure \( \varphi' \) which induces the spin structure \( \varphi \) on \( \Sigma_g - d \). As a consequence, the restriction of \( \varphi \) to \( \Sigma_g - d \) fulfills the hypothesis in Corollary 3.6. As \( c = c_0 \) and \( c_2 \) are nonseparating simple closed curves in \( \Sigma_g - d \) with \( \varphi(c) = \varphi(c_2) = \pm 1 \), Corollary 3.6 shows that there exists an element
\[ \Psi_1 \in \Gamma_d \subset \rho(\Gamma) \text{ such that } \Psi_1(c) = c_2. \] Then the stabilizer of \( c_2 \) in \( \text{Mod}(\Sigma_g)[\varphi] \) equals \( \Psi_1 \Gamma \Psi_1^{-1} \) and hence it is contained in \( \rho(\Gamma) \). Thus we can apply Corollary 3.6 to \( \Psi_1 \Gamma \Psi_1^{-1} \) and find an element \( \Psi_2 \in \rho(\Gamma) \) which maps \( c_1 \) to \( c_3 \). Proceeding inductively and using the fact that \( \Gamma_c \) is conjugate to \( \Gamma_d \) in \( \rho(\Gamma) \) by the generator of the subgroup \( \mathbb{Z}/2\mathbb{Z} \), this completes the proof of the proposition. \( \square \)

Recall from the introduction the definition of an admissible curve system on a closed surface \( \Sigma_g \) of genus \( g \geq 2 \). The mapping class group of \( \Sigma_g \) naturally acts on the family of all admissible curve systems on \( \Sigma_g \). Recall also that the curve diagram of an admissible curve system is a finite tree.

Since the curve diagram of an admissible curve system \( \mathcal{C} \) is connected, each curve \( c \in \mathcal{C} \) intersects at least one other simple closed curve on \( \Sigma_g \) transversely in a single point and hence it is nonseparating.

We need some technical information on admissible curve systems. To this end let \( \mathcal{C} \) be any admissible curve system on an oriented surface \( S \). We require that the boundary of \( S \) is empty, but we allow for the moment that \( S \) has punctures. For admissibility, we require that all complementary components of \( \mathcal{C} \) are either topological disks or once punctured topological disks.

The union \( \bigcup \{ c \mid c \in \mathcal{C} \} \) is an embedded graph \( G \) in \( S \) whose vertices are the intersection points between the curves from \( \mathcal{C} \). Choose a basepoint \( x \in G \) which is contained in the interior of an edge of \( G \). This edge is contained in a simple closed curve \( c_0 \in \mathcal{C} \) which defines a distinguished vertex \( v_0 \) in the curve diagram of \( \mathcal{C} \).

Construct inductively a family \( L \) of homotopy classes of loops in \( G \) based at \( x \) as follows. Let \( L_0 \) be the family consisting of the two based loop which go once around the simple closed curve \( c_0 \in \mathcal{C} \) containing \( x \) in either direction. Assume by induction that for some \( k \geq 1 \) we defined a system of based loops \( L_{k-1} \). Let \( \{ c_{k_1}, \ldots, c_{k_s} \} \subset \mathcal{C} \) be the curves in \( \mathcal{C} \) whose distance in the curve diagram to the distinguished vertex \( v_0 \) equals \( k \). Define

\[ L_k = \{ T_{\pm 1} c_{k_u} d \mid u \leq s, d \in L_{k-1} \} \]

and let \( L = L_b \) where \( b \geq 1 \) is the maximal distance of a vertex in the curve diagram of \( \mathcal{C} \) to the distinguished vertex \( v_0 \).

The following appears implicitly in [PV96] and explicitly as Lemma 9.3 of [Sa19].

**Lemma 4.2.** The loops from the system \( L \) generate the fundamental group \( \pi_1(S, x) \) of \( S \).

As a consequence we obtain (see Lemma 9.4 of [Sa19]).

**Lemma 4.3.** Let \( \mathcal{C} \) be an admissible curve system on a surface \( S \), possibly with punctures. Let \( p \) be a puncture of \( S \) and assume that there are two curves \( c_1, c_2 \in \mathcal{C} \) which bound a once punctured annulus, with \( p \) as puncture. Then the subgroup \( \Gamma \) of \( \text{Mod}(S) \) generated by the Dehn twists about the curves from the curve system \( \mathcal{C} \) contains the kernel of the homomorphism \( \text{Mod}(S) \to \text{Mod}(\Sigma) \) where \( \Sigma \) is obtained from \( S \) by forgetting \( p \).
For a closed surface $\Sigma_g$ of genus $g \geq 2$ consider the system $S_g$ of $3g - 2$ simple closed curve on $\Sigma_g$ shown in Figure 4.

The following is well known but hard to locate in the literature.

**Lemma 4.4.** For $g = 2$, Dehn twists about the curves from the curve system $S_g$ generate the stabilizer of an odd spin structure in $\text{Mod}(\Sigma_2)$.

**Proof.** $S_2$ is just a chain of 4 curves which are invariant under the hyperelliptic involution. The subgroup $\Gamma$ of $\text{Mod}(\Sigma_2)$ generated by the Dehn twists about these curves fixes one of the Weierstrass points, that is, one of the fixed points of the hyperelliptic involution.

Replacing the Weierstrass point by a small disk shows that we may view $\Gamma$ as a subgroup of the quotient by its center of the symmetric mapping class group of a surface of genus 2 with connected boundary, which equals the braid group in 5 strands (Theorem 9.2 of [FM12]). This group is known to be generated by the Dehn twists about the curves from the curve system $S_2$ (see Section 9.1 of [FM12]) and hence $\Gamma \subset \text{Mod}(\Sigma_2)$ equals the stabilizer of one of the Weierstrass points. This Weierstrass point defines a $\mathbb{Z}/2\mathbb{Z}$-spin structure on $\Sigma_2$ whose stabilizer in $\text{Mod}(\Sigma_2)$ equals the stabilizer of the Weierstrass point. This shows the lemma. □

**Lemma 4.5.** The Dehn twists about the curves from the system $S_g$ preserve an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure on $\Sigma_g$.

**Proof.** There exists a cyclic subgroup $G$ of the diffeomorphism group of $\Sigma_g$ of order $g - 1$ which preserves $S_g$ and acts freely on $\Sigma_g$ as a group of rotations about the center curve $c_0$. The group $G$ cyclically permutes the complementary components of $S_g$.

As a consequence, the curve system $S_g$ descends to a curve system on a closed surface $\Sigma_2$ of genus 2. The curve diagram of this system is just a line segment.
of length 4 and hence the Dehn twists about these curves preserve an odd spin
structure on $\Sigma_2$ (see Lemma 4.4). This spin structure lifts to a spin structure on
$\Sigma_g$ which is invariant under the Dehn twist about the curves from $S_g$. The parity
of this spin structure is odd, as can also easily be checked explicitly using the formula
(2). This is what we wanted to show.

We use Lemma 4.3 and Proposition 4.1 to show

**Proposition 4.6.** Let $\varphi$ be an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure on a surface $\Sigma_g$ of genus
$g \geq 2$. Then the group $\text{Mod}(\Sigma_g)[\varphi]$ is generated by the Dehn twists about the curves
from the curve system $S_g$.

**Proof.** Lemma 4.5 shows that the subgroup $\Gamma$ of $\text{Mod}(\Sigma_g)$ generated by the Dehn
twists about the curves from the curve system $S_g$ is a subgroup of $\text{Mod}(\Sigma_g)[\varphi]$. We
have to show that it coincides with $\text{Mod}(\Sigma_g)[\varphi]$. We proceed by induction on the genus. By Lemma 4.4, the claim of the propo-
sition holds true for $g = 2$. Thus let us assume that the proposition is known for
some $g - 1 \geq 2$. Consider the curve system $S_g$ on a surface of genus $g$. Using the
labeling from Figure 4, let $a_1$ be the simple closed curve on $\Sigma_g$ which intersects the
curve $c_1$ in a single point and is disjoint from any other curve from $S_g$. We know
that $\varphi(a_1) = 1$. We aim at showing that $\Gamma \cap \text{Stab}(a_1) = \text{Mod}(\Sigma_g)[\varphi] \cap \text{Stab}(a_1)$.

To this end cut $\Sigma_g$ open along $a_1$. The resulting surface is a surface $\Sigma'_g$ of genus
$g - 1$ with two boundary components. Replace these two boundary components by
punctures and let $\Sigma_{g-1,2}$ be the resulting twice punctured surface. As before, the
spin structure $\varphi$ descends to a spin structure, again denoted by $\varphi$, on the surface
$\Sigma'_{g-1}$ obtained by closing the punctures, and to a spin structure on $\Sigma_{g-1,2}$. The
curve system $S_g$ descends to the curve system $S_{g-1}$ on $\Sigma_{g-1}$.

By induction hypothesis, the Dehn twists about the curves from the curve system
$S_{g-1}$ generate the spin mapping class group $\text{Mod}(\Sigma_{g-1})[\varphi]$. On the other hand, we
can apply Lemma 4.3 to each of the two punctures of $\Sigma_{g-1,2}$ as each of these two
punctures is contained in a once punctured annulus bounded by two curves from the
restriction of $S_g$ to $\Sigma_{g-1,2}$. We conclude that the point pushing maps about
these punctures are contained in the group $\Gamma \cap \text{Stab}(a_1)$. As a consequence, the
group $\Gamma \cap \text{Stab}(a_1)$ surjects onto the index two subgroup of $\text{Mod}(\Sigma_{g-1,2})$ which
fixes each of the two punctures.

We have to show that there also is an element of $\Gamma \cap \text{Stab}(a_1)$ which exchanges
the two boundary components of $\Sigma_g - a_1$. For this it suffices to find an element of
$\Gamma$ which fixes the curves $c_1, c_2$ and exchanges $d_1, d_2$.

If $g = 3$ then consider the hyperelliptic involution of the surface $\Sigma_2$ obtained by
cutting $\Sigma_3$ open along the simple closed curve $a_1$ and removing the punctures. This
element can be represented as an explicit word in the Dehn twists about the curve
$c_2, c_0, c_4, c_3$ (or, rather, their projection to $\Sigma_2$). The mapping class $\psi$, viewed as an
element of the mapping class group of $\Sigma_3$, preserves the curves $c_1$ and exchanges $d_1$
and $d_2$. 

For $g \geq 4$ the same argument can be used. Namely, the element $\psi$ still acts as an involution on $\Sigma_g$ which preserves the curves $c_1, c_2$ and exchanges $d_1$ and $d_2$. However, this involution does not preserve the curve system $S_g$.

To summarize, we showed so far that $\Gamma$ surjects onto $\text{Stab}(a_1)[\varphi]/\mathbb{Z}$. Thus to show that $\Gamma \cap \text{Stab}(a_1) = \text{Mod}(\Sigma_g)[\varphi] \cap \text{Stab}(a_1)$ it suffices to show that $\Gamma$ contains the square $T_{a_1}^2$ of the Dehn twist about $a_1$. For an application of Proposition 4.1, we have to show furthermore that $\Gamma$ contains an involution $\Psi$ which exchanges the curve $a_1$ with a curve disjoint from $a_1$. We show first that $\Gamma$ contains an involution which maps $a_1$ to $a_2$.

To this end consider again first the case $g = 3$. The curve system $S_3$ contains a curves system $E_6 \subset S_3$ obtained from $S_3$ by deleting the curve $d_2$. This is the curve system shown in Figure 2 in the introduction. By Theorem 1.4 of [Ma00], there exists an explicit word $c(E_6)$ in the Dehn twists about the curves from the system $E_6$, the image of the so-called Garside element of the Artin group of type $E_6$, which acts as a reflection on the curve diagram of $E_6$ exchanging the curves $c_1$ and $c_3$. Then this reflection exchanges $a_1$ and $a_2$ and hence it has the desired properties.

As before, this reasoning extends to any $g \geq 4$. Namely, the element $c(E_6)$, viewed as an element of the mapping class group of $\Sigma_g$, still acts as an involution on $\Sigma_g$ which exchanges $a_1$ and $a_2$ and preserves the subsurface of $\Sigma_g$ filled by the curves $c_1, c_2, c_0, c_4, c_3, d_2$.

For an application of Proposition 4.1 we are left with showing that the square of the Dehn twist about $a_1$ is contained in $\Gamma$. By the above discussion, we know that $\Gamma \cap \text{Stab}(a_1)$ surjects onto $\text{Mod}(\Sigma_{g-1,2})[\varphi]$. In particular, $\Gamma$ contains $T_{a_1}^2$, viewed as an element of $\text{Stab}(a_1) \subset \text{Mod}(\Sigma_g)$. Since $a_1$ is the image of $a_2$ under an involution contained in $\Gamma$, it follows that $T_{a_1}^2 \in \Gamma$.

To summarize, we showed that $\Gamma \cap \text{Stab}(a_1) = \text{Mod}(\Sigma_g)[\varphi] \cap \text{Stab}(a_1)$, furthermore $\Gamma$ contains an involution $\Psi$ which exchanges $a_1$ and $a_2$. Proposition 4.1 now shows that $\Gamma = \text{Mod}(\Sigma_g)[\varphi]$. This completes the proof of the Proposition.

\[\square\]

We use Proposition 4.6 as the base case for the proof of Theorem 4 from the introduction. The curve system $C_g$ is shown in Figure 1 in the introduction. Note that we have $C_3 = S_3$.

**Theorem 4.7.** Let $\varphi$ be an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure on a surface $\Sigma_g$ of genus $g \geq 3$. Then the group $\text{Mod}(\Sigma_g)[\varphi]$ is generated by the Dehn twists about the curves from the curve system $C_g$.

**Proof.** The curve system $C_g$ is obtained from the curve system $S_g$ by deleting the curves $d_3, \ldots, d_{g-1}$. Let $\Gamma$ be the subgroup of $\text{Mod}(\Sigma_g)[\varphi]$ generated by the Dehn twists about the curves from the curve system $C_g$. By Proposition 4.6, it suffices to show that the Dehn twists $T_{d_i}$ for $i = 3, \ldots, g - 1$ are contained in $\Gamma$. Moreover, as $D_3 = C_3$, we may assume that $g \geq 4$.

Let $a_i$ be the simple closed curve which intersects $c_{2i-1}$ in a single point and does not intersect any other curve from $S_g$. We first claim that $T_{a_i}^2 \in \Gamma$. 

To show the claim consider the subsurface $\Sigma^1_2$ of $\Sigma_g$ which is filled by the curves $a_1, c_0, c_1, c_2, d_1, d_2$. This is a surface of genus 2 with connected boundary. The curves $d_1, d_2$ bound a one-holed annulus containing the boundary circle $C$ of $\Sigma^1_2$.

By homological coherence (Proposition 3.8 of [Sa19]), we have $\varphi(C) = 1$. Thus the spin structure $\varphi$ descends to a spin structure on $\Sigma^1_2$, on the surface $\Sigma_{2,1}$ obtained from $\Sigma^1_2$ by replacing the boundary component by a puncture and on the surface $\Sigma_2$ obtained from $\Sigma_{2,1}$ by forgetting the puncture, again denoted by $\varphi$. The curves of the curve system $C_g$ which are contained in $\Sigma^1_2$ define a curve system $\mathcal{F}$ on $\Sigma^1_2$ which descends to a curve system on $\Sigma_2$. The curve diagram of this system is just a line segment of length 4. By Lemma 4.3, the Dehn twists about the curves from $\mathcal{F}$ project onto $\text{Mod}(\Sigma_2)[\varphi]$.

On the other hand, $\mathcal{F}$ also contains two simple closed curves which enclose the boundary component of $\Sigma_{2,1}$. It now follows from Lemma 4.3 that the subgroup of $\text{Mod}(\Sigma_{2,1})$ generated by the Dehn twists about the curves from $\mathcal{F}$ equals $\text{Mod}(\Sigma_{2,1})[\varphi]$. In particular, this group contains $T_{a_1}^2$ and therefore $T_{a_1}^2 \in \Gamma$.

We claim next that $T_{a_2}^2 \in \Gamma$. To this end consider the subsurface $\Sigma^1_3$ of $\Sigma_g$ which is filled by the system of curves $G = \{c_1, c_2, c_0, c_4, c_3, d_1, d_2, d_3\}$. This is a surface of genus 3 with connected boundary. The curves $d_1, d_3$ bound a one-holed annulus containing the boundary circle $A$ of $\Sigma^1_3$.

The subsurface $\Sigma^1_3$ of $\Sigma_g$ contains the curves $c_1, c_2, c_0, c_4, c_3, d_2$ whose curve diagram is the Dynkin diagram of type $E_6$ (see Figure 2 in the introduction). There is an involution of $\Sigma^1_3$ which fixes the curves $c_0, d_2$ and exchanges $c_2, c_4$ and $a_1, a_2$. By Theorem 1.4 of [Ma00], this involution is contained in the subgroup of the mapping class group of $\Sigma^1_3$ which is generated by the Dehn twists about the curves $c_1, c_2, c_0, c_4, c_3, d_2$. As a consequence, there is an element of $\Gamma$ which exchanges $a_1$ and $a_2$. This implies that $T_{a_2}^2 \in \Gamma$.

By the chain relation for Dehn twists of surfaces (see p.108 of [FM12]), we have $(T_{a_3} T_{c_0} T_{c_4})^3 = T_{d_2} T_{d_3}$. Since $T_{d_2} \in \Gamma$, we conclude that $T_{d_3} \in \Gamma$.

Now repeat this argument, replacing the curves $c_j$ by $c_{j+2}$ and the curve $a_i$ by $a_{i+1}$ where the first step discussed above is the case $i = 1$. In finitely many such steps we find that indeed $T_{d_i} \in \Gamma$ for all $i$. This is what we wanted to show. \hfill \Box

5. Structure of the spin mapping class group of even parity

The goal of this section is to prove the second part of Theorem 3. Our strategy is to reduce this result to the first part of Theorem 3 by a change of parity construction.

Consider for the moment an arbitrary $\mathbb{Z}/r\mathbb{Z}$-spin structures $\varphi$ on a compact surface $S$ of genus $g \geq 4$. In the appendix we introduce a graph $\mathcal{G}_2^+$ whose vertices are ordered pairs $(a, b)$ of nonseparating simple closed curves which intersect in a single point and hence they fill a one-holed torus $T(a, b)$. Furthermore, it is required that $\varphi(a) = 2$ and $\varphi(b) = 0$. The spin structure on $S$ restricts to a spin structure $\hat{\varphi}$ on $\Sigma(a, b) = S - T(a, b)$. 

By homological coherence (Proposition 3.5 of [Sa19]), if we orient the boundary circle $c$ of $\Sigma(a,b)$ as the oriented boundary of $\Sigma(a,b)$ then we have $\varphi(c) = 1$. Thus if $r = 2$ then $\varphi$ descends to a spin structure $\dot{\varphi}$ on the surface $\Sigma$ obtained from $\Sigma(a,b)$ by capping off the boundary. This spin structure $\dot{\varphi}$ has a parity, either even or odd.

**Lemma 5.1.** A $\mathbb{Z}/2\mathbb{Z}$-spin structure $\varphi$ on $S$ induces a $\mathbb{Z}/2\mathbb{Z}$-spin structure $\dot{\varphi}$ on the surface $\Sigma$ whose parity is opposite to the parity of $\varphi$.

**Proof.** Choose a geometric symplectic basis $a_1, b_1, \ldots, a_{2g-1}, b_{2g-1}$ for $\Sigma$. This basis then lifts to a curve system on the surface $\Sigma(a,b) = S - T(a,b)$. Using the inclusion $\Sigma(a,b) \to S$, this basis can be extended to a geometric symplectic basis of $S$ by adding $a, b$. As $\varphi(a) = \varphi(b) = 0$, the parity of $\varphi$ is opposite to the parity of $\dot{\varphi}$. □

The next observation is an analog of Proposition 3.4. Note that we only require $g \geq 3$ here.

**Proposition 5.2.** Let $\varphi$ be a $\mathbb{Z}/r\mathbb{Z}$-spin structure on a compact surface $S$ of genus $g \geq 3$ with empty or connected boundary. Then for any two vertices $c, d$ of the graph $CG_2^+$ there exists a mapping class $\zeta \in \text{Mod}(S)[\varphi]$ with $\zeta(c) = d$. In particular, the action of $\text{Mod}(S)[\varphi]$ is transitive on the vertices of the graph $CG_2^+$.

**Proof.** The proof is very similar to the proof of Proposition 3.4 and will be omitted. □

Consider again a $\mathbb{Z}/r\mathbb{Z}$-spin structure $\varphi$ on a closed surface $\Sigma_g$ of genus $g \geq 3$. Let $c$ be a separating simple closed curve on $\Sigma_g$ which is the boundary of a small neighborhood of a torus $\Sigma_c^1$ and a surface $\Sigma_{g-1}^1$ of genus $g - 1$ with connected boundary. The spin structure restricts to a spin structure on $\Sigma_c^1$. If $r$ is even then this spin structure has a parity, and this parity is odd.

Since $c$ is separating, the group $\text{Mod}(\Sigma_{g-1}^1)[\varphi] \times \text{Mod}(\Sigma_c^1)[\varphi]$ contains a rank two free abelian central subgroup generated by the left Dehn twists $T_{c_1}, T_{c_2}$ about the boundary circles $c_1, c_2$ of $\Sigma_{g-1}^1, \Sigma_c^1$. Define

$$\Gamma_{g-1,2}^2 \cong \text{Mod}(\Sigma_{g-1}^1)[\varphi] \times \text{Mod}(\Sigma_c^1)[\varphi] / \mathbb{Z},$$

where the infinite cyclic subgroup $\mathbb{Z}$ is generated by $T_{c_1} T_{c_2}^{-1}$. Then $\Gamma_{g-1,2}^2$ is isomorphic to the stabilizer in $\text{Mod}(\Sigma_g)[\varphi]$ of the curve $c$. Note that up to isomorphism, the group $\Gamma_{g-1,2}^2$ does not depend on $c$ since by Proposition 5.2 the stabilizers in $\text{Mod}(\Sigma_g)[\varphi]$ of vertices of $CG_2^+$ are all conjugate and hence isomorphic.

Observe that the group $\Gamma_{g-1,2}^2$ is an infinite cyclic central extension of the product of a finite index subgroup of the mapping class group of a surface $\Sigma_{g-1,1}$ of genus $g - 1$ with one puncture and a once punctured torus $\Sigma_{1,1}$. Thus it makes sense to talk about its action on isotopy classes of essential curves on the surfaces $\Sigma_{g-1,1}$ and $\Sigma_{1,1}$. The map $\Sigma_{g-1}^1 \coprod \Sigma_c^1 \to \Sigma_{g-1,1} \coprod \Sigma_{1,1}$ which contracts each boundary component to a puncture defines a bijection on such isotopy classes.

The following observation is the analog of Proposition 4.1.
Proposition 5.3. Let \( \varphi \) be a \( \mathbb{Z}/r\mathbb{Z} \)-spin structure on a closed surface \( \Sigma_g \) of genus \( g \geq 4 \). There is a commutative diagram

\[
\begin{array}{ccc}
\Gamma^2_{g-1,2} & \xrightarrow{\iota_1} & \Gamma^2_{g-1,2} \ast_A \Gamma^2_{g-1,2} \times \mathbb{Z}/2\mathbb{Z} \\
& \iota_2 \downarrow & \downarrow \rho \\
& & \text{Mod}(\Sigma_g)\langle \varphi \rangle
\end{array}
\]

where the homomorphisms \( \iota_1, \iota_2 \) are inclusions, and the homomorphism \( \rho \) is surjective. The subgroup \( A \) of \( \Gamma^2_{g-1,2} \) is the stabilizer in \( \Gamma^2_{g-1,2} \) of a separating simple closed curve \( d \) on \( \Sigma^2_{g-1} \) which is defined by a vertex of the graph \( \mathcal{C}\mathcal{G}^+_2 \). The curve \( d \) decomposes \( \Sigma^2_{g-1} \) into a one-holed torus and a surface of genus \( g-2 \) with two boundary components. The group \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \Gamma^2_{g-1,2} \ast_A \Gamma^2_{g-1,2} \) by exchanging the two factors, and it acts as an automorphism on \( A \).

**Proof.** Fix a pair of vertices of the graph \( \mathcal{C}\mathcal{G}^+_2 \) which are connected by an edge. These two vertices then determine a pair of disjoint separating simple closed curves \( c, d \) on \( \Sigma_g \) which cut from \( \Sigma_g \) a one-holed torus each. These tori are disjoint. Let \( \Gamma_c, \Gamma_d \subset \text{Mod}(\Sigma_g)\langle \varphi \rangle \) be the stabilizers of \( c, d \) in the spin mapping class group of \( \Sigma_g \). By Corollary 3.6 these groups are naturally isomorphic to the group \( \Gamma^2_{g-1,2} \ast \mathbb{Z}/2\mathbb{Z} \) and they intersect in the index two subgroup \( A = \Gamma_c \cap \Gamma_d \) of the stabilizer of \( c \cup d \) in \( \text{Mod}(\Sigma_g)\langle \varphi \rangle \) consisting of all elements which preserve both \( c, d \) individually. The full stabilizer of \( c \cup d \) in \( \text{Mod}(\Sigma_g)\langle \varphi \rangle \) is a \( \mathbb{Z}/2\mathbb{Z} \) extension of \( \Gamma_c \cap \Gamma_d \), where the generator \( \Phi \) of \( \mathbb{Z}/2\mathbb{Z} \) acts as involution on \( A = \Gamma_c \cap \Gamma_d \) exchanging \( c \) and \( d \). This involution extends to an involution of \( \Gamma_c \ast_A \Gamma_d \) exchanging the two subgroups \( \Gamma_c, \Gamma_d \).

By the universal property of free amalgamated products, there is a homomorphism

\[ \rho : \Gamma = \Gamma_c \ast_A \Gamma_d \ast_N \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Mod}(\Sigma_g)\langle \varphi \rangle. \]

All we need to show is that \( \rho \) is surjective, that is, that \( \rho(\Gamma) = \text{Mod}(\Sigma_g)\langle \varphi \rangle \).

As \( \text{Mod}(\Sigma_g)\langle \varphi \rangle \) acts transitively on the vertices of the graph \( \mathcal{C}\mathcal{G}^+_2 \), for this it suffices to show that its subgroup \( \rho(\Gamma) \) acts transitively on the vertices of \( \mathcal{C}\mathcal{G}^+_2 \) as well. Namely, by construction, the stabilizer of the vertex \( c \) of \( \mathcal{C}\mathcal{G}^+_2 \) in \( \rho(\Gamma) \) coincides with its stabilizer in \( \text{Mod}(\Sigma_g)\langle \varphi \rangle \). As \( \rho(\Gamma) \) is a subgroup of \( \text{Mod}(\Sigma_g)\langle \varphi \rangle \), this then implies equality.

To show transitivity of the action of \( \rho(\Gamma) \) on the vertices of \( \mathcal{C}\mathcal{G}^+_2 \) let \( v \in \mathcal{C}\mathcal{G}^+_2 \) be any vertex. By Proposition A.3 the graph \( \mathcal{C}\mathcal{G}^+_2 \) is connected and hence we can find an edge path \( (e_i) \subset \mathcal{C}\mathcal{G}^+_2 \) connecting \( c_0 = e \) to \( c_k = v \). We also may assume that \( c_1 = d \).

By Proposition 5.2 there exists an element \( \Psi_1 \in \Gamma_d \subset \rho(\Gamma) \) such that \( \Psi_1(c_0) = c_2 \). Then the stabilizer of \( c_2 \) in \( \text{Mod}(\Sigma_g)\langle \varphi \rangle \) equals \( \Psi_1 \Gamma_c \Psi_1^{-1} \) and hence it is contained in \( \rho(\Gamma) \). Thus we can apply Corollary 3.6 to \( \Psi_1 \Gamma_c \Psi_1^{-1} \) and find an element \( \Psi_2 \in \rho(\Gamma) \) which maps \( c_1 \) to \( c_3 \). Proceeding inductively and using the fact that \( \Gamma_c \) is conjugate to \( \Gamma_d \) in \( \rho(\Gamma) \) by the generator of the subgroup \( \mathbb{Z}/2\mathbb{Z} \), this completes the proof of the proposition. \( \square \)
For a surface $S$ of genus $g \geq 3$ consider the following system $U_g$ of $3g - 2$ simple closed curve on $S$. Note that for $g = 3$, the system $S_3$ is just a chain of 7 curves which are invariant under a hyperelliptic involution. It follows from Lemma 4.5 and Lemma 5.1 that the Dehn twists about these curves preserve an even $\mathbb{Z}/2\mathbb{Z}$-spin structure on $\Sigma_g$.

We use Lemma 4.3 and Proposition 4.1 to show

**Proposition 5.4.** Let $\varphi$ be an even $\mathbb{Z}/2\mathbb{Z}$-spin structure on a surface $\Sigma_g$ of genus $g \geq 4$. Then the group $\text{Mod}(\Sigma_g)[\varphi]$ is generated by the Dehn twists about the curves from the curve system $U_g$.

**Proof.** We observed above that the subgroup $\Gamma$ of $\text{Mod}(\Sigma_g)$ generated by the Dehn twist about the curves from the curve system $U_g$ is a subgroup of $\text{Mod}(\Sigma_g)[\varphi]$. We have to show that it coincides with $\text{Mod}(\Sigma_g)[\varphi]$.

To this end we proceed by induction on the genus, beginning with genus 4. Let $a$ be the separating simple closed curve which intersects $c_3$ in two points and is disjoint from the remaining curves from the system $U_4$. It decomposes $U_4$ into a one holed torus $\Sigma_1^1$ containing the curves $c_1, c_2$, and a surface $\Sigma_3^1$ of genus 3 with connected boundary which contains the curve system $S_3$. As we are looking at a $\mathbb{Z}/2\mathbb{Z}$-spin structure we know that the pair $(c_1, c_2)$ and hence the curve $a$ defines a vertex in $CG^+_2$. The spin structure $\varphi$ induces a spin structure on $\Sigma_3^1$ and $\Sigma_1^1$, again denoted by $\varphi$. It also induces a spin structure on the closed surface $\Sigma_3$ of genus 3 obtained from $\Sigma_3^1$ by capping off the boundary, again denoted by $\varphi$.

It is well known that the mapping class group of one holed tori is generated by a pair of Dehn twists about simple closed curves which intersect in a single point. Thus we have $\text{Mod}(\Sigma_1^1) \subset \Gamma \cap \text{Stab}(a)$.

On the other hand, by Proposition 4.6, the Dehn twists about the curves from the system $S_3$ generate the spin mapping class group $\text{Mod}(\Sigma_3)[\varphi]$ of $\Sigma_3$. Thus the projection of $\Gamma$ to $\text{Mod}(\Sigma_3)[\varphi]$ is surjective.
To apply Proposition 5.3 we have to show that the point pushing group of \( \text{Mod}(\Sigma_{3,1})[\varphi] \) is contained in the projection of \( \Gamma \cap \text{Stab}(a) \). We use once more Lemma 4.3 to this end.

Consider the curves \( c_0, c_7, c_6, d_1, c_5 \) which define a curve system on the surface \( \Sigma_{3,1} \) whose curve diagram is the Dynkin diagram \( D_5 \). By Theorem 1.5 of [Ma00], there exists an explicit word in the Dehn twists about these curves which defines the product \( T_{a_4}^3 T_{a_4}' \) where \( a_4 \) is simple closed curve in \( \Sigma_3^1 \) which intersects \( c_4 \) in a single point and is disjoint from all other curves and where \( a_4' \) is the simple closed curve which bounds together with \( a_4 \) a once punctured annulus in \( \Sigma_3^1 \).

On the other hand, the chain relation [FM12] yields that \( T_{a_4}' T_{a_4} = (T_{c_1} T_{c_2} T_{c_3})^4 \) since \( T_{a_4}, T_{a_4}' \) commute we deduce that \( (T_{a_4}')^{-2} T_{a_4}^{-2} T_{a_4} = T_{a_4} T_{a_4}'^{-1} \in \Gamma \). As a consequence, the group \( \Gamma \cap \text{Stab}(a) \) contains the point pushing map \( T_{a_4} T_{a_4}'^{-1} \) about the based loop \( \alpha \) in \( \Sigma_3 \) which is homotopic to the common projection of \( a_4, a_4' \). Lemma 4.3 now shows that \( \Gamma \cap \text{Stab}(a) \) contains indeed the point pushing group of \( \text{Mod}(\Gamma_{3,1})[\varphi] \).

By Proposition 5.3, we are left with finding an element \( \Psi \in \Gamma \) which maps \( a \) to a curve disjoint from \( a \). However, the curve system \( U_4 \) contains a subsystem consisting of the curves \( c_i \) for \( i = 0, \ldots, 7 \). The Dehn twists about these curves are well known to generate the stabilizer of a Weierstrass point in the hyperelliptic mapping class group, that is, the subgroup of the mapping class group which commutes with a hyperelliptic involution [FM12], and compare Lemma 4.4. This group is isomorphic to the quotient of the Artin braid group in 2\( g+1 \) strands by its center, and it contains an element \( \psi \) which maps \( a \) to a disjoint curve, e.g. the boundary of a small neighborhood of \( c_0 \cup c_5 \). The proposition for \( g = 4 \) now follows from Proposition 5.3.

By induction, let us now assume that the proposition is known for some \( g - 1 \geq 4 \). Consider the curve system \( U_g \) on a surface of genus \( g \). Using the labeling from Figure 5, let \( a_7 \) be the simple closed curve on \( \Sigma_g \) which intersects the curve \( c_7 \) in a single point and is disjoint from any other curve from \( U_g \). We know that \( \varphi(a_7) = 1 \). We aim at showing that \( \Gamma \cap \text{Stab}(a_7) = \text{Mod}(\Sigma_g)[\varphi] \cap \text{Stab}(a_7) \).

To this end cut \( \Sigma_g \) open along \( a_7 \). The resulting surface is a surface \( \Sigma_{g-1}^2 \) of genus \( g - 1 \) with two boundary components. Replace these two boundary components by punctures and let \( \Sigma_{g-1,2} \) be the resulting twice punctured surface. As before, the spin structure \( \varphi \) descends to a spin structure, again denoted by \( \varphi \), on the surface \( \Sigma_{g-1} \) obtained by closing the punctures, and to a spin structure on \( \Sigma_{g-1,2} \). The curve system \( U_g \) descends to the curve system \( U_{g-1} \) on \( \Sigma_{g-1} \).

By induction hypothesis, the Dehn twists about the curves from the curve system \( U_{g-1} \) generate the spin mapping class group \( \text{Mod}(\Sigma_{g-1})[\varphi] \). On the other hand, we can apply Lemma 4.3 to each of the two punctures of \( \Sigma_{g-1,2} \) as each of these two punctures is contained in a once punctured annulus bounded by two curves from the restriction of \( U_g \) to \( \Sigma_{g-1,2} \). We conclude that the point pushing maps about these punctures are contained in the group \( \Gamma \cap \text{Stab}(a_7) \). As a consequence, the group \( \Gamma \cap \text{Stab}(a_7) \) surjects onto \( \text{Mod}(\Sigma_{g-1,1}) \).
To summarize, we showed that $\Gamma$ surjects onto $\text{Stab}(a_7)[\varphi]/\mathbb{Z}$ where $\mathbb{Z}$ is the intersection of $\text{Mod}(\Sigma_g)[\varphi]$ with the infinite cyclic group of Dehn twists about $a_7$. Thus to show that $\Gamma \cap \text{Stab}(a) = \text{Mod}(\Sigma_g)[\varphi] \cap \text{Stab}(a)$ it suffices to show that $\Gamma$ contains the square $T^2_{a_7}$ of the Dehn twist about $a_7$ as well as an involution $\Psi$ which exchanges $a_7$ with a simple closed curve disjoint from $a_7$.

To find an involution $\Psi$ as required, consider first the case $g = 4$. The curve system $\mathcal{U}_4$ contains a curve system $\mathcal{E}_6 \subset \mathcal{U}_4$ consisting of the curves $c_7, c_6, c_0, d_1, c_5, c_4$. By Theorem 1.4 of [Ma00], there exists an explicit word $c(E_6)$ in the Dehn twists about the curves from the system $\mathcal{E}_6$, the image of the so-called Garside element of the Artin group of type $E_6$, which acts as a reflection on the curve diagram of $\mathcal{E}_6$ exchanging the curves $c_7$ and $c_4$. Then this reflection maps $a_7$ to a disjoint curve $a'_7$ and hence it has the desired properties.

This reasoning extends to any $g \geq 5$. Namely, the element $c(E_6)$, viewed as an element of the mapping class group of $\Sigma_g$, still acts as an involution on $\Sigma_g$ which maps $a_7$ to a disjoint curve $a'_7$ and preserves the subsurface of $\Sigma_g$ filled by the curves $c_7, c_6, c_0, d_1, c_5, c_4$. Thus there always exists an involution $\Psi \in \Gamma$ which maps $a_7$ to a disjoint curve $a'_7$.

For an application of Proposition 4.4, we are left with showing that the square of the Dehn twist about $a_7$ is contained in $\Gamma$. By the above discussion, we know that $\Gamma \cap \text{Stab}(a_7)$ surjects onto $\text{Mod}(\Sigma_{g-1,2})[\varphi]$. In particular, $\Gamma$ contains $T^2_{a'_7}$, viewed as an element of $\text{Stab}(a_7) \subset \text{Mod}(\Sigma_g)$. Since $a_7$ is the image of $a'_7$ under an involution contained in $\Gamma$, it follows that $T^2_{a'_7} \in \Gamma$.

To summarize, we showed that $\Gamma \cap \text{Stab}(a_7) = \text{Mod}(\Sigma_g)[\varphi] \cap \text{Stab}(a_7)$, furthermore $\Gamma$ contains an involution $\Psi$ which exchanges $a_7$ and $a'_7$. Proposition 4.4 now shows that $\Gamma = \text{Mod}(\Sigma_g)[\varphi]$. This completes the proof of the Proposition. \hfill $\square$

We use Proposition 4.6 as the base case for the proof of the second part of Theorem 3 from the introduction. The curve system $\mathcal{V}_g$ is defined as in the Theorem 3. Note that we have $\mathcal{V}_3 = \mathcal{U}_3$.

**Theorem 5.5.** Let $\varphi$ be an even $\mathbb{Z}/2\mathbb{Z}$-spin structure on a surface $\Sigma_g$ of genus $g \geq 4$. Then the group $\text{Mod}(\Sigma_g)[\varphi]$ is generated by the Dehn twists about the curves from the curve system $\mathcal{V}_g$.

**Proof.** The curve system $\mathcal{V}_g$ is obtained from the curve system $\mathcal{U}_g$ by deleting the curves $d_2, \ldots, d_{g-2}$.

Let $\Gamma$ be the subgroup of $\text{Mod}(\Sigma_g)[\varphi]$ generated by the Dehn twists about the curves from the curve system $\mathcal{V}_g$. By Proposition 4.6, it suffices to show that the Dehn twists $T_{d_i}$ for $i = 2, \ldots, g - 2$ are contained in $\Gamma$.

To see that $T_{d_{g-2}} \in \Gamma$, note that $d_{g-2}$ is the image of $d_1$ under the hyperelliptic involution of the surface of genus 3 with connected boundary filled by the curves $d_1, d_{g-2}, c_0, c_1, c_2, c_3, c_4$.\hfill $\square$
Consider the surface $S$ filled by $c_1, \ldots, c_6, d_1, d_2, d_{g-2}$. This is a surface of genus 4 with connected boundary. The union of the system $V_g$ with the curve $d_{g-2}$ intersects $S$ in a curve system of type $U_4$. By Proposition 5.4 and what we have proved so far, the stabilizer of the surface $S$ in the group $\Gamma$ surjects onto the spin mapping class group of the surface obtained from $S$ by capping off the boundary. In particular, if we denote by $e_1, e_3$ the nonseparating simple closed curves which intersect $c_4$ in a single point, do not intersect any other curve and form a bounding pair, then $T_{e_1}T_{e_2}^{-1} \in \Gamma$.

Now by Theorem 1.4 of [Ma00], the stabilizer in $\Gamma$ of the surface of genus 3 with two boundary components obtained from $S$ by removing the one-holed torus $T$ filled by $c_1, c_2$ contains a half-twist which exchanges the two boundary components of the surface. Let $S'$ be the surface obtained from $S - T$ by replacing two boundary components by punctures. Lemma 5.3 applied to the Dehn twists about the curves $e_1, e_2$ enclosing the boundary component of $T$, shows that the subgroup of the mapping class group of $S'$ which is the point pushing group of the puncture corresponding to the boundary of $T$ is contained in the projection of the stabilizer of $S - T$ in $\Gamma$. But then the same holds true for the point pushing group of the second puncture of $S'$.

This shows that we have $T_{d_2}T_{d_{g-2}}^{-1} \in \Gamma$. As $T_{d_{g-2}} \in \Gamma$, we conclude that the same holds true for $T_{d_2}$. To generate the remaining twists about the curves $d_i$ we argue as in the proof of Theorem 4.7, using the Dehn twists $T_{d_1}$ and $T_{d_2}$. $\square$

6. Generating the $\mathbb{Z}/4\mathbb{Z}$-spin mapping class group in genus 3

The goal of this section is to prove Theorem 4 from the introduction. Our strategy is similar to the strategy used in Section 4. We first introduce one more graph of curves which will be useful to this end.

Consider an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure $\varphi$ on a surface $\Sigma_3$ of genus 3. A separating simple closed curve $a$ on $\Sigma_3$ decomposes $\Sigma_3$ into a one-holed torus $T$ and a surface $\Sigma_1^1$ of genus 2 with connected boundary. By homological coherence (Proposition 3.15 of [Sa19]), we have $\varphi(a) = 1$. In particular, $\varphi$ induces a spin structure on the surface $\Sigma_1^1$ which has a parity. Define $a$ to be odd if this parity is odd. Note that a vertex of the graph $G_2^1$ defined in the appendix and used in Section 5 defines a separating simple closed curve which is even, that is, it is not odd.

Let $\mathcal{O}S$ be the graph whose vertices are odd separating simple closed curves on $(\Sigma_3, \varphi)$ and where two such curves are connected by an edge if they are disjoint. Let $\Phi$ be a $\mathbb{Z}/4\mathbb{Z}$-spin structure on $\Sigma_3$ whose $\mathbb{Z}/2\mathbb{Z}$-reduction equals $\varphi$. The stabilizer $\text{Mod}(\Sigma_3)[\varphi]$ and its subgroup $\text{Mod}(\Sigma_3)[\Phi]$ act on $\mathcal{O}S$ as a group of simplicial automorphisms. The following observation is similar to Proposition 3.4. It uses some special properties of $\mathbb{Z}/4\mathbb{Z}$-spin structures.

**Lemma 6.1.** (1) The group $\text{Mod}(\Sigma_3)[\Phi]$ acts transitively on the vertices of $\mathcal{O}S$.

(2) Let $a \in \mathcal{O}S$ be any vertex. Then the stabilizer of $a$ in $\text{Mod}(\Sigma_3)[\Phi]$ acts transitively on the edges of $\mathcal{O}S$ issuing from $a$. 
Proof. A vertex \( a \) of \( OS \) decomposes \( \Sigma_3 \) into a one-holed torus \( T \) and a surface \( \Sigma_3 - T \) of genus 2 with connected boundary and odd spin structure. Since the parity of the spin structure of \( \varphi \) on \( \Sigma_3 \) is odd, the torus \( T \) contains a simple closed curve \( c \) with \( \varphi(c) = 1 \) and hence \( \Phi(c) = \pm 1 \). Via perhaps changing the orientation of \( c \) we may assume that \( \Phi(c) = 1 \), furthermore there is a simple closed curve \( d \) in \( T \) which intersects \( c \) in a single point and satisfies \( \Phi(d) = 0 \).

By homological coherence (Proposition 3.15 of [Sa19]), if we orient \( a \) as the oriented boundary of the surface \( V = \Sigma_3 - T \) then we have \( \Phi(a) = 1 \). Since the spin structure induced on \( V \) is odd, a geometric symplectic basis for \( V \), ordered), a tubular neighborhood \( T' \) of %\( a_1 \cup b_1 \)% is an embedded bordered torus in \( V \). Choose an orientation for \( a_1 \) so that \( \Phi(a_1) = 1 \). After perhaps replacing \( b_1 \) by its image under a multiple of a Dehn twist about \( a_1 \) we may assume that \( \Phi(b_1) = 0 \).

Consider the pair of curves \( a_2, b_2 \). Since the spin structure on \( V \) is odd, we have \( \varphi(a_2) = \varphi(b_2) = 0 \) and hence \( \Phi(a_2), \Phi(b_2) \in \{0,2\} \). Our goal is to modify \( a_2, b_2 \) so that \( \Phi \) vanishes on the modified curves. Thus assume without loss of generality that \( \Phi(a_2) = 2 \). Connect \( a_2 \) to the boundary curve \( a \) of \( V \) by an embedded arc \( \epsilon \) which is disjoint from \( T' \) and \( b_2 \), and connect \( b_2 \) to the boundary \( \delta \) of \( T' \) by an embedded arc \( \eta \) which is disjoint from \( \epsilon \) and \( a_2 \). Since \( \Phi(a) = 1 \) for the orientation as a boundary curve of \( V \), we obtain that \( \Phi(a_2 + \epsilon, a) = 0 \), furthermore this curve is disjoint from \( T' \) and intersects \( b_2 \) in a single point. Replace \( a_2 \) by \( a_2 + \epsilon, a \). Similarly, if \( \Phi(b_2) = 2 \) then we replace \( b_2 \) by \( b_2 + \eta, \delta \). This process yields a geometric symplectic basis for \( \Sigma_3 \) consisting of simple closed curves disjoint from \( a \).

Given any other odd separating curve \( a' \) on \( \Sigma_3 \) we can find in the same way a geometric symplectic basis for \( \Sigma_3 \) consisting of curves disjoint from \( a' \). Then there is a mapping class which maps \( a \) to \( a' \) and identifies the geometric symplectic bases in such a way that the values of \( \Phi \) on these curves match up. By the result of Humphries and Johnson [HJ89], this implies that this mapping class is contained in \( \text{Mod}(\Sigma_3)[\Phi] \). In other words, there is an element of \( \text{Mod}(\Sigma_3)[\Phi] \) which maps \( a \) to \( a' \). This shows the first part of the lemma.

The proof of the second part of the lemma is completely analogous but easier and will be omitted. \( \square \)

**Lemma 6.2.** The graph \( OS \) is connected.

*Proof.* Consider the curve system \( C_3 \) on the surface \( \Sigma_3 \). Using the labels for the curves shown in Figure 4, there is an odd separating simple closed curve \( a \) which intersects the curve \( c_2 \) in two points and is disjoint from the remaining curves from the system \( C_3 \). Using the Putman trick, Theorem 4.7 and the first part of Lemma 6.1, all we need to show is that the curve \( a \) can be connected to \( T_{c_2}(a) \) by an edge path in \( OS \).

However, the curve \( a' \) which intersects the curve \( c_4 \) in two points and is disjoint from the remaining curves from the system \( C_3 \) is separating and odd, and it is disjoint from both \( a \) and \( T_{c_2}(a) \). Thus \( a, a', T_{c_2}(a) \) is an edge path in \( OS \) which connects \( a \) to \( T_{c_2}(a) \). \( \square \)
Using the labels from Figure 2 from the introduction, let \( d \) be the separating simple closed curve on \( \Sigma_3 \) which intersects the curve \( c_2 \) in two points and is disjoint from the remaining curves from the system \( \mathcal{E}_6 \). We show

**Lemma 6.3.** The subgroup \( \Gamma \) of \( \text{Mod}(\Sigma_3) \) which is generated by the Dehn twists about the curves from the curve system \( \mathcal{E}_6 \) equals the stabilizer \( \text{Mod}(\Sigma_3)[\Phi] \) of an odd \( \mathbb{Z}/4\mathbb{Z} \)-spin structure \( \Phi \) on \( \Sigma_3 \) if and only if its intersection with \( \text{Stab}(d) \) coincides with \( \text{Stab}(d) \cap \text{Mod}(\Sigma_3)[\Phi] \).

**Proof.** Since \( \Gamma \) is a subgroup of \( \text{Mod}(\Sigma_3)[\Phi] \), the condition is clearly necessary, so we have to show sufficiency. Thus assume that \( \Gamma \cap \text{Stab}(d) = \text{Mod}(\Sigma_3)[\Phi] \cap \text{Stab}(d) \).

Consider again the graph \( OS \). Lemma 6.2 shows that \( OS \) is connected. Moreover, by Lemma 6.1, the group \( \text{Mod}(\Sigma_3)[\Phi] \) acts transitively on the directed edges of \( OS \) as a group of simplicial automorphisms. The curve \( d \) is odd and hence a vertex of \( OS \).

By Theorem 1.4 of \([Ma00]\), the group \( \Gamma \) contains an involution which induces a reflection in the curve diagram of the curve system \( \mathcal{E}_6 \) at the edge connecting the vertices \( c_0 \) and \( c_3 \). It maps the simple closed curve \( d \) to the separating simple closed curve \( d' \) which intersects \( c_4 \) in two points and is disjoint from all other curves from the system. Since \( d \) is odd, the same is true for \( d' \).

We use this as follows. Let \( e \) be any vertex of \( OS \) and let \( d = d_0, d_1, d_2, \ldots, d_m = e \) be an edge path in \( OS \) which connects \( d \) to \( e \). We may assume that \( d_1 = d' \). Since there exists an element of \( \Gamma \) which maps \( d \) to \( d' \), the stabilizer of \( d' \) in \( \Gamma \) is conjugate to the stabilizer of \( d \) and hence by our assumption, it coincides with the stabilizer of \( d' \) in \( \text{Mod}(\Sigma_3)[\Phi] \). In particular, by the second part of Lemma 6.1, there exists an element of \( \Gamma \) which fixes \( d' \) and maps \( d_1 \) to \( d_2 \). Arguing inductively as in the proof of Proposition 4.1, we conclude that \( \Gamma \) acts transitively on the odd separating curves in \( \Sigma_3 \). As \( \Gamma \) is a subgroup of \( \text{Mod}(\Sigma_3)[\Phi] \) and furthermore the stabilizer of a vertex in \( \Gamma \) coincides with its stabilizer in \( \text{Mod}(\Sigma_3)[\Phi] \), it has to coincide with \( \text{Mod}(\Sigma_3)[\Phi] \). The lemma follows. \( \square \)

Our final goal is to show that the group \( \Gamma \) fulfills the assumption in Lemma 6.3 which completes the proof of Theorem 4. We proceed in two steps.

Let \( a_1, a_5 \) be the nonseparating simple closed curves on \( \Sigma_3 \) which intersect \( c_1, c_5 \) in a single point and are disjoint from the remaining curves from the system \( \mathcal{E}_6 \). We have \( \Phi(a_j) = \pm 1 \), in particular, by Lemma 3.13 of \([Sa19]\), the intersection of \( \text{Mod}(\Sigma_3)[\Phi] \) with the infinite cyclic group of Dehn twists about the curve \( a_j \) is generated by \( T^{4}_{a_j} \).

**Lemma 6.4.** For \( j = 1, 5 \), the group \( \Gamma \) contains \( T^{4}_{a_j} \).

**Proof.** Consider the subsystem \( D^j_5 \) \((j = 1, 5)\) obtained from the curve system \( \mathcal{E}_6 \) by removing the curve \( c_j \). By Theorem 1.3 (d) of \([Ma00]\), the mapping class \( T^{4}_{a_j} \) can be represented as an explicit word in the Dehn twists about the curves from this curve system. Thus we have \( T^{4}_{a_j} \in \Gamma \). \( \square \)
Lemma 6.4 is used in the proof of the final step towards Theorem 4

Lemma 6.5. The stabilizer in $\Gamma$ of the curve $d$ coincides with the stabilizer of $d$ in Mod($\Sigma_3$)|$\Phi$.

Proof. Let $T$ be the one-holed torus component of $\Sigma_3 - d$. The stabilizer Stab($d$)|$\Phi$ of $d$ in Mod($\Sigma_3$)|$\Phi$ is the quotient of the product of two subgroups $G_1, G_2$ by an infinite cyclic central subgroup. The group $G_1$ is the group of all isotopy classes of diffeomorphisms of $\Sigma_3$ which fix the bordered surface $S = \Sigma_3 - T$ pointwise and preserve the spin structure $\Phi$. It is isomorphic to the subgroup of the mapping class group of a one-holed torus which preserves the spin structure $\Phi$. The group $G_2$ is the group of all isotopy classes of diffeomorphisms of $\Sigma_3$ which fix $T$ pointwise and preserve the spin structure $\Phi$. The center of Stab($d$)|$\Phi$ is generated by a Dehn twist $T_d$ about $d$.

Consider the curve system $\mathcal{A}_4 \subset \mathcal{E}_6$ which consists of the curves $c_0, c_3, c_4, c_5$. It is contained in the subsurface $\Sigma_2^1 = \Sigma_3 - T$ of $\Sigma_3$ of genus 2 which is bounded by $d$. The Dehn twists about these curves generate a subgroup $\mathcal{A}(\mathcal{A}_4)$ of $\Gamma \cap G_1$ which is isomorphic to the braid group in five strands (see [FM12] or [Ma00] for the last statement). By Theorem 1.4 of [Ma00], the Dehn twist $T_d$ can be represented as an explicit word in the Dehn twists about the curves from the curve system $\mathcal{A}(\mathcal{A}_4)$. In particular, we have $T_d \in \Gamma$.

Let $\Sigma_{2,1}$ be the surface obtained from $\Sigma_2^1 = \Sigma_3 - T$ by replacing the boundary component by a puncture, and let $\Sigma_2$ be obtained from $\Sigma_{2,1}$ by forgetting the puncture. Let $\varphi$ be the $\mathbb{Z}/2\mathbb{Z}$-reduction of the spin structure $\Phi$. The spin structure $\varphi$ induces an odd spin structure on $\Sigma_{2,1}$ and $\Sigma_2$, again denoted by $\varphi$. By Lemma 4.4, the subgroup $\mathcal{A}(\mathcal{A}_4)$ of $\Gamma \cap G_1$ surjects onto the spin mapping class group $\text{Mod}(\Sigma_2)$. Consequently the restriction of the puncture forgetful homomorphism $G_2 \to \text{Mod}(\Sigma_2)|\varphi$ to $\Gamma \cap G_2$ is surjective.

By homological coherence, if we orient $d$ as the oriented boundary of the surface $\Sigma_3 - T$, then we have $\Phi(d) = 1$. Thus by Lemma 3.4, the intersection of the point pushing group $\pi_1(\Sigma_2)$ with the stabilizer of $\Phi$ in Mod($\Sigma_{2,1}$) is the preimage of the sublattice $\Lambda$ of $H_1(\Sigma_2, \mathbb{Z})$ generated by squares of primitive homology classes of oriented simple closed curves under the natural homomorphism $\pi_1(\Sigma_2) \to H_1(\Sigma_2, \mathbb{Z})$. Or, equivalently, it equals the kernel of the surjective homomorphism $\pi_1(\Sigma_2) \to H_1(\Sigma_2, \mathbb{Z}/2\mathbb{Z})$. In particular, Mod($\Sigma_{2,1}$)|$\Phi$ $\cap$ $\pi_1(\Sigma_2)$ contains the commutator subgroup of $\pi_1(\Sigma_2)$.

We claim first that the square of the point pushing map along a simple closed curve $\alpha$ with $\Phi(\alpha) = \pm 1$ is contained in $\Gamma$. To this end note that as $\Phi(\alpha) = \pm 1$ if and only if we have $\varphi(\alpha) = 1$ where $\varphi$ is the $\mathbb{Z}/2\mathbb{Z}$-reduction of $\Phi$, the group Mod($\Sigma_2$)|$\varphi$ and hence $\Gamma$ acts transitively on these curves. Thus by equivariance, it suffices to verify this claim for a single such curve.

Consider again the simple closed curve $a_5 \subset \Sigma_{2,1}$ with $\Phi(a_5) = \pm 1$ which intersects $c_5$ in a single point and is disjoint from all other curves from the curve system $\mathcal{E}_6$. Let $a'$ be the simple closed curve which bounds with $a_5$ and the boundary circle
of \( \Sigma_2 \), a pair of pants, that is, \( a_5 \) and \( a' \) bound a holed annulus in \( \Sigma_2 \). By the chain relation in the mapping class group (see [FM12]), we have
\[
(T_{a_0} T_{c_3} T_{c_4})^6 = T_{a_5} T_{a'} = \zeta \in \Gamma.
\]
On the other hand, Lemma 6.4 shows that \( T_{a_5}^4 \in \Gamma \). As \( T_{a_5} \) and \( T_{a'} \) commute, we have \( T_{a_5}^{-4} \zeta^2 = T_{a_5}^{-2} T_{a'}^2 \in \Gamma \), and this is just the square of the point pushing transformation (via replacing the boundary circle \( C \) by a puncture) along \( a_5 \). Thus the square of the point pushing transformation about \( a_5 \) is contained in \( \Gamma \), which is what we wanted to show.

Now the sublattice \( \Lambda \subset H_1(\Sigma_2, \mathcal{Z}) \) is additively generated by elements of the form \( 2b \) where \( b \) is an oriented simple closed curve with \( \varphi(b) = 1 \) and hence we conclude that \( \Gamma \cap \pi_1(\Sigma_2) \) surjects onto \( \Lambda \).

We are left with showing that the point pushing map along any element in the commutator subgroup of \( \pi_1(\Sigma_2) \) is contained in \( \Gamma \). The commutator subgroup of \( \pi_1(\Sigma_2) \) is generated by separating simple closed curves. Note that \( \text{Mod}(\Sigma_2)[\varphi] \) acts transitively on separating simple closed curves in \( \Sigma_2 \). Namely, as the parity of \( \varphi \) is odd, the formula (2) for the Arf invariant shows that any separating simple closed curve \( c \) on \( \Sigma_2 \) decomposes \( \Sigma_2 \) into two one holed tori \( T_1, T_2 \) such that up to exchanging \( T_1 \) and \( T_2 \), there is a geometric symplectic basis \( \alpha_1, \beta_1 \) for \( T_1 \) with \( \varphi(\alpha_1) = 1, \varphi(\beta_1) = 0 \), and a geometric symplectic basis \( \alpha_2, \beta_2 \) for \( T_2 \) with \( \varphi(\alpha_2) = \varphi(\beta_2) = 0 \). Then transitivity of the action of \( \text{Mod}(\Sigma_2)[\varphi] \) on separating simple closed curves follows once again from [HJ89]. Thus it suffices to show the following: there exists a separating simple closed curve \( e \) in \( \Sigma_2 \) such that the point pushing map along \( e \) in \( \Sigma_2 \) is contained in \( \Gamma \).

Now by Theorem 1.4 of [Ma00], the Dehn twist about the separating simple closed curve \( d' \) which intersects \( c_4 \) in two points and is disjoint from the remaining curves from \( \mathcal{E}_3 \) is contained in \( \Gamma \). This separating curve is odd in the sense described above. The second separating curve which bounds together with the boundary circle \( C \) and \( d' \) a pair of pants is the boundary of a tubular neighborhood of \( c_0 \cup c_1 \). As the Dehn twists about \( c_0, c_1 \) are contained in \( \Gamma \), the same holds true for the Dehn twist about that curve. We conclude that the point pushing maps about separating simple closed curves is contained in \( \Gamma \).

To summarize, the quotient of \( \Gamma \cap G_2 \) by the infinite cyclic group of Dehn twists about the boundary curve \( d \) contains a generating set for the point pushing subgroup of \( G_2/\mathcal{Z} \) and hence it contains this point pushing subgroup. As \( \Gamma \cap G_2 \) surjects onto the quotient \( G_2/\mathcal{Z} \) by the point pushing subgroup, we conclude that \( \Gamma \) surjects onto \( G_2/\mathcal{Z} \). But \( \Gamma \) contains the infinite cyclic center of \( G_2 \) and hence \( \Gamma \cap G_2 = G_2 \).

To complete the proof of the lemma we are left with showing that the subgroup \( G_1 \) of \( \text{Mod}(\Sigma_3)[\Phi] \) is contained in \( \Gamma \). But conjugation by the involution in \( \Gamma \) which acts as an involution on the curve diagram of the curve system \( \mathcal{E}_3 \) and exchanges \( c_1 \) and \( c_5 \) and \( c_2 \) and \( c_4 \) maps \( G_1 \) to a subgroup of \( G_2 \) and hence to a subgroup of \( \Gamma \). Thus \( G_1 \subset \Gamma \) as well and we conclude that indeed, \( \Gamma \cap \text{Stab}(d) = \text{Mod}(\Sigma_3)[\Phi] \cap \text{Stab}(d) \).
Remark 6.6. Theorem 4 classifies connected components of the preimage in the Teichmüller space of abelian differentials of the odd component of the stratum of abelian differentials on a surface $\Sigma_3$ of genus 3 with a single zero. Those components correspond precisely to odd $\mathbb{Z}/4\mathbb{Z}$-spin structures on $\Sigma_3$.

Remark 6.7. The results in this article give a general recipe for finding generators of spin mapping class groups. This recipe is motivated by the recent work on compactifications of strata of abelian differentials in [BCGGM18] and the goal to obtain a topological interpretation of this compactification.

Appendix A. Additional graphs of nonseparating curves with fixed spin value

In this appendix we complement the main result in Section 2 by studying connectedness of some additional geometrically defined graphs related to spin structures. The proofs do not use new ideas. We use the assumptions and notations from Section 2.

We begin with adding more constraints to the graph $CG^+_1$. Define a graph $CG^{++}_1$ as follows. Vertices of $CG^{++}_1$ are ordered pairs $(c,d)$ of nonseparating simple closed curves $c,d$ such that $\varphi(c) = \pm 1, \varphi(d) = 0$ and that $c,d$ intersect in a single point. Then $c \cup d$ fills a one-holed torus $T(c,d) \subset S$. Two such pairs $(c,d), (c',d')$ are connected by an edge if and only if the tori $T(c,d)$ and $T(c',d')$ are disjoint. We use Corollary 2.18 to show

Lemma A.1. For $g \geq 4$ the graph $CG^{++}_1$ is connected.

Proof. Let $(a,b), (c,d)$ be two vertices in the graph $CG^{++}_1$. Then $a,c$ are vertices in the graph $CG^+_1$. Connect $a = a_0$ to $c = a_k$ by an edge path $(a_i)$ in $CG^+_1$; this is possible by Corollary 2.18. Our goal is to construct inductively a path $(c_j, d_j) \subset CG^{++}_1$ connecting $(a,b)$ to $(c,d)$ which passes through vertices $(c_{j_i}, d_{j_i})$ with $c_{j_i} = a_i$.

To this end observe that if the curve $b$ is disjoint from $a_1$, then we can find a curve $d_1$ which intersects $a_1$ in a single point and is disjoint from $(a,b)$. In particular, $a \cup b$ is disjoint from $c_1 \cup d_1$.

We can not expect in general that $\varphi(d_1) = 0$. However, as before, there exists some $k \in \mathbb{Z}$ such that $\varphi(T^{k}_{a_1}(d_1)) = 0$. Define $c_1 = a_1$ and $d_1 = T^{k}_{c_1}(d_1)$ and note that $d_1$ is disjoint from $a \cup b$ and intersects $c_1$ in a single point. Thus the pair $(c_1, d_1)$ is a vertex in $CG^{++}_1$ which is connected to $(a,b)$ by an edge.

Let us now assume that $b$ is not disjoint from $a_1$. Since $b$ intersects $a$ in a single point, it determines a vertex in the nonseparating arc graph $A(A_1, A_2)$ of $S - a$; here $A_1, A_2$ are the two boundary components of $S - a$ which glue back to $a$. Denote this arc by $b_0$.

Connect $b_0$ to an arc $b'$ disjoint from $a_1$ by an edge path $(b_i)$ in $A(A_1, A_2)$. This is possible by Lemma 2.15. Cut $S - a$ open along $b_0$. The result is a surface of
genus $g - 1 \geq 3$ with connected boundary, and $S - (b \cup b_1)$ is a surface of genus $g - 2 \geq 2$ with two boundary components.

A surface of genus at least 2 contains a nonseparating curve $u$ with $\varphi(u) = 1$, and in fact it contains a pair $(u,v) \in \mathcal{C}_1$. In other words, there exists a vertex of $\mathcal{C}_1$ which is disjoint from $a, b, b_1$. Connect $(a,b)$ to $(a,b_1)$ by the edge path $(a,b) \to (u,v) \to (a,b_1)$ and proceed by induction. □

Define a graph $\mathcal{D}$ as follows. Vertices are ordered pairs $(x,y)$ where $x$ is a vertex in $\mathcal{C}_1$ and where $y$ is a disjoint simple closed nonseparating curve with $\varphi(y) = 0$. Two such pairs are connected by an edge if they can be realized disjointly. The following observation is a straightforward application of Lemma A.1 and the tools used so far. Its proof will be omitted.

**Lemma A.2.** For $g \geq 4$ the graph $\mathcal{D}$ is connected.

Define now a graph $\mathcal{C}_2$ as follows. Vertices are pairs $(x,y)$ where $x$ is a nonseparating simple closed curve on $S$ with $\varphi(x) = 2$ and where $y$ is a simple closed curve with $\varphi(y) = 0$ intersecting $x$ in a single point. Two such vertices are connected by an edge of length one if and only if they can be realized disjointly.

We use the above constructions to show

**Proposition A.3.** For $g \geq 4$ the graph $\mathcal{C}_2$ is connected.

**Proof.** Given a pair of disjoint simple closed curves $(c,d)$ with $\varphi(c) = \pm 1$ and $\varphi(d) = 0$, cut $S$ open along $c,d$ and denote the boundary components of the resulting surface by $C_1, C_2, D_1, D_2$. For one of the two choices of $C_1, C_2$, say for $C_1$, the curve $c' + \epsilon, d$ defined by any embedded arc $\epsilon$ connecting $C_1$ to either of $D_1, D_2$ satisfies $\varphi(c' + \epsilon, d) = \pm 2$.

As a consequence, to any vertex $(c,d) \in \mathcal{D}$ we can associate in a non-deterministic way a vertex in $\mathcal{C}_2$ by replacing the simple closed curve $a$ with $\varphi(a) = \pm 1$ in the pair which defines a vertex of $\mathcal{C}_1$ to the simple closed curve component of the pair which defines a vertex in $\mathcal{D}$.

Adjacent vertices may not give rise to disjoint curves, but this issue can be resolved using a path in the nonseparating arc graph. Using the fact that the surface obtained by removing from $S$ a torus and cutting the resulting surface open along a nonseparating simple closed curve has genus at least 2, we find for any two such arcs a disjoint curve $e$ with $\varphi(e) = \pm 1$. Connect $b$ to this curve with a disjoint arc. □
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