APPROXIMATION PROPERTIES OF \((p,q)\)-GAMMA OPERATORS

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Abstract. In this paper, we introduce a new analogue of Gamma operators and we call it as \((p,q)\)-Gamma operators which is a generalization of \(q\)-Gamma operators. Moments of these operators is estimated. And some other results of these operators are studied by means of modulus of continuity and Peetre \(K\)-functional. Then, some theorems concerned with the rate of convergence and the weighted approximation for these operators are also obtained. Finally, a Voronovskaya asymptotic formula is also presented.

1. Introduction and definitions

Let \(f\) be a function defined on \([0, \infty)\) and satisfy the following growth condition:

\[
|f(x)| \leq Me^{-\beta x} \quad (M > 0; \beta \geq 0; x \to \infty). \tag{1}
\]

In [18], Zeng investigated and studied some approximation properties of the following sequence of linear positive operators (named Gamma operators)

\[
G_n(f;x) = \frac{1}{x^n \Gamma(n)} \int_0^\infty f \left( \frac{t}{n} \right) t^{n-1} e^{-\frac{x}{t}} \, dt. \tag{2}
\]

During the last decade, the wide application of \(q\)-calculus in the field of approximation theory has led to the discovery of new generalizations of classical operators. For more comprehensive details, the readers should look through the references material [1], [14], [15]. In [3], Cai introduced and studied \(q\)-analogue of Gamma operators as follows:

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Definition A. For \( f \in C[0, \infty) \) satisfies (1), \( q \in (0, 1) \) and \( n \in \mathbb{N} \), the \( q \)-Gamma operators \( G_{n,q}(f;x) \) is defined by

\[
G_{n,q}(f;x) = \frac{1}{x^n \Gamma_q(n)} \int_0^\infty f(t) t^{n-1} E_q \left( -\frac{qt}{x} \right) \, dt.
\]

Stancu generalization of \( G_{n,q}(f;x) \) was discussed and studied in [12]. Nowadays, with the rapid development of the post-quantum calculus, which called the \((p,q)\)-calculus for short, the generalizations of several classical operators have been studied intensively (Such as [4, 5, 9]) since Mursaleen et al. first defined and constructed \((p,q)\)-Bernstein operators[10] and \((p,q)\)-Bernstein-Stancu operators[11]. The \((p,q)\)-calculus has been used in many other fields, such as Lie group theory, CAGD, physical sciences(see [7], [16]). Let us recall the basic notations of \((p,q)\)-calculus which can be found in [15].

For any fixed real number \( p > 0, q > 0 \), the \((p,q)\)-integers \([m]_{p,q}\) are defined by

\[
[m]_{p,q} = p^{m-1} + p^{m-2}q + p^{m-3}q^2 + \ldots + pq^{m-2} + q^{m-1} = \begin{cases} \frac{p^m-q^m}{p-q}, & p \neq q \neq 1; \\ mp^{m-1}, & p = q \neq 1; \\ [m], & p = q = 1, \end{cases}
\]

where \([m]_q\) denotes the \(q\)-integers and \( m = 0, 1, 2, \ldots \). Also \((p,q)\)-factorial is defined as follows:

\[
[m]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q}\cdots[m]_{p,q}, & m \geq 1; \\ 1, & m = 0. \end{cases}
\]

Now, we introduce two types of \((p,q)\)-analogues of exponential function \( e_{p,q}(x) \) and \( E_{p,q}(x) \) (see [2]):

\[
e_{p,q}(x) = \sum_{m=0}^\infty \frac{p^{m(m-1)} x^m}{[m]_{p,q}!}, x \in \mathbb{R}, |p| < 1 \text{ and } |q| < 1;
\]

\[
E_{p,q}(x) = \sum_{m=0}^\infty \frac{q^{m(m-1)} x^m}{[m]_{p,q}!}, x \in \mathbb{R}, |p| < 1 \text{ and } |q| < 1.
\]

Let \( f \) be an arbitrary function. The improper \((p,q)\)-integral of \( f(x) \) on \([0, \infty)\) is defined as (see [13])

\[
\int_0^\infty f(x) \, dp_q x = (p-q) \sum_{i=-\infty}^\infty \frac{q^i}{p^{i+1}} f \left( \frac{q^i}{p^{i+1}} \right), 0 < \frac{q}{p} < 1.
\]

The \((p,q)\)-Gamma function of the second kind was defined in [13] as follows

\[
\gamma_{p,q}(z) = \int_0^\infty q^z x^{z-1} e_{p,q}(x) \, dp_q x, \Re(z) > 0.
\]
Meantime, the \((p,q)\)-Gamma function fulfills the following relation
\[
\gamma_{p,q}(z + 1) = [z]_{p,q} \gamma_{p,q}(z),
\]
moreover, for any nonnegative integer \(n > 0\), the following relation holds
\[
\gamma_{p,q}(n + 1) = [n]_{p,q}!.
\]

Now, we construct \((p,q)\)-Gamma operators using the \((p,q)\)-Gamma function of the second kind preserving linear functions as:

**Definition 1.** For \(f \in C[0, \infty)\) satisfies (1), \(0 < q < p \leq 1\) and \(n \in \mathbb{N}\), the \((p,q)\)-analogue of Gamma operators (2) are defined as
\[
G_n^{p,q}(f;x) = \frac{1}{x^n \gamma_{p,q}(n)} \int_0^\infty f \left( \frac{q^n t}{[n]_{p,q}} \right) q^{\frac{n(n-1)}{2}} t^{n-1} e_{p,q} \left( -\frac{pt}{x} \right) \, d_{p,q} t.
\]

This paper is organized as follows: In Section 1, we give some basic notations and the definition of \((p,q)\)-Gamma operators. In Section 2, we present basic lemmas and estimate the moments of the operators. In Section 3, we present a direct result of \((p,q)\)-Gamma operators in terms of first and second order modulus of continuity. In Section 4, we deal with the rate of convergence. In Section 5, we study the weighted approximation of the \((p,q)\)-Gamma operators. In Section 6, we obtain Voronovskaja type asymptotic formula.

### 2. Auxiliary results

In this section, in order to prove our main results, we first establish some useful lemmas.

**Lemma 1.** Let \(0 < q < p \leq 1\), \(x \in [0, \infty)\) and \(k = 0, 1, \cdots\), we have
\[
G_n^{p,q}(t^k;x) = \frac{x^k q^{-\frac{k(k-1)}{2}} [n+k-1]_{p,q}!}{[n]_{p,q}[n-1]_{p,q}!}.
\]

**Proof.** Direct computation gives
\[
G_n^{p,q}(t^k;x) = \frac{1}{x^n \gamma_{p,q}(n)} \int_0^\infty \left( \frac{q^n t}{[n]_{p,q}} \right)^k q^{\frac{n(n-1)}{2}} t^{n-1} e_{p,q} \left( -\frac{pt}{x} \right) \, d_{p,q} t
\]
\[
= \frac{x^k}{[n]_{p,q}^k \gamma_{p,q}(n)} \int_0^\infty q^{\frac{n(n-1)}{2} + nk} \left( \frac{t}{x} \right)^{n+k-1} e_{p,q} \left( -\frac{pt}{x} \right) \, d_{p,q} \left( \frac{t}{x} \right)
\]
\[
= \frac{x^k q^{-\frac{k(k-1)}{2}}}{[n]_{p,q}^k \gamma_{p,q}(n)} \int_0^\infty q^{\frac{(n+k)(n+k-1)}{2}} \left( \frac{t}{x} \right)^{n+k-1} e_{p,q} \left( -\frac{pt}{x} \right) \, d_{p,q} \left( \frac{t}{x} \right)
\]
\[
= \frac{x^k q^{-\frac{k(k-1)}{2}} [n+k-1]_{p,q}!}{[n]_{p,q}[n-1]_{p,q}!}.
\]
Lemma 1 is proved. \(\Box\)
**Lemma 2.** Let $0 < q < p \leq 1$, $x \in [0, \infty)$, we have

$$G_n^{p,q}(1;x) = 1, \quad G_n^{p,q}(t;x) = x, \quad G_n^{p,q}(t^2;x) = \left(1 + \frac{p^n}{q[n]_{p,q}}\right)x^2,$$

(3)

$$G_n^{p,q}(t^3;x) = \left(1 + \frac{p^n([2]_{p,q} + q)}{q^2[n]_{p,q}} + \frac{p^{2n}}{q^3[n]^3_{p,q}}\right)x^3,$$

$$G_n^{p,q}(t^4;x) = \left(1 + \frac{(q^2 + q[2]_{p,q} + [3]_{p,q})p^n}{q^3[n]_{p,q}} + \frac{(2)[p,q][3]_{p,q} + q[3]_{p,q} + q^2[2]_{p,q})p^{2n}}{q^5[n]^2_{p,q}} + \frac{[2]_{p,q}[3]_{p,q}p^3n}{q^6[n]^3_{p,q}}\right)x^4.$$

**Proof.** From Lemma 1, we get the first and second equalities of (3) easily. Using $[n + m]_{p,q} = q^n[n]_{p,q} + p^n[m]_{p,q}$, we have

$$G_n^{p,q}(t^2;x) = \frac{[n + 1]_{p,q}x^2}{q[n]_{p,q}} = \left(1 + \frac{p^n}{q[n]_{p,q}}\right)x^2.$$  

Next,

$$G_n^{p,q}(t^3;x) = \frac{[n + 1]_{p,q}[n + 2]_{p,q}x^3}{q^3[n]^2_{p,q}} = \frac{(q[n]_{p,q} + p^n)(q^2[n]_{p,q} + p^n[2]_{p,q})x^3}{q^3[n]^2_{p,q}}$$

$$= \left(1 + \frac{p^n([2]_{p,q} + q)}{q^2[n]_{p,q}} + \frac{p^{2n}}{q^3[n]^3_{p,q}}\right)x^3.$$  

Finally,

$$G_n^{p,q}(t^4;x) = \frac{[n + 1]_{p,q}[n + 2]_{p,q}[n + 3]_{p,q}x^4}{q^6[n]^3_{p,q}}$$

$$= \frac{(q[n]_{p,q} + p^n)(q^2[n]_{p,q} + p^n[2]_{p,q})(q^3[n]_{p,q} + p^n[3]_{p,q})x^4}{q^6[n]^3_{p,q}}$$

$$= \left(1 + \frac{(q^2 + q[2]_{p,q} + [3]_{p,q})p^n}{q^3[n]_{p,q}} + \frac{([2]_{p,q}[3]_{p,q} + q[3]_{p,q} + q^2[2]_{p,q})p^{2n}}{q^5[n]^2_{p,q}} + \frac{[2]_{p,q}[3]_{p,q}p^3n}{q^6[n]^3_{p,q}}\right)x^4.$$  

Lemma 2 is proved. □
LEMMA 3. Let $0 < q < p \leq 1$, $x \in [0, \infty)$, we have

$$G_n^{p,q}(t-x;x) = 0;$$
$$A(x) := G_n^{p,q}((t-x)^2;x) = \frac{p^n}{q^n[n]_{p,q}} x^2;$$

$$G_n^{p,q}((t-x)^4;x) = \frac{p^n(p-q)}{q^3[n]_{p,q}} x^4 + \frac{[2]_{p,q}[3]_{p,q} q + q^2[2]_{p,q} - 4q^2}{q^5[n]_{p,q}} \frac{p^n}{q^3[n]_{p,q}} x^4 + \frac{[2]_{p,q}[3]_{p,q} q^3}{q^6[n]_{p,q}} x^4.$$ (4)

Proof. Because $G_n^{p,q}(t-x;x) = G_n^{p,q}(t;x) - x$, $G_n^{p,q}((t-x)^2;x) = G_n^{p,q}(t^2;x) - 2G_n^{p,q}(t;x) + x^2$ and $G_n^{p,q}((t-x)^4;x) = G_n^{p,q}(t^4;x) - 4xG_n^{p,q}(t^3;x) + 6x^2G_n^{p,q}(t^2;x) - 4x^3G_n^{p,q}(t;x) + x^4$, and from Lemma 2, we obtain Lemma 3 easily. \(\Box\)

REMARK 1. The sequences $(p_n)$, $(q_n)$ satisfying $0 < q_n < p_n < 1$ such that $p_n \to 1$, $q_n \to 1$ and $p_n^n \to a$, $q_n^n \to b$, $[n]_{p_n,q_n} \to \infty$ as $n \to \infty$ where $0 \leq a, b < 1$, then:

1. $\lim_{n \to \infty} [n]_{p_n,q_n} G_n^{p_n,q_n}((t-x)^2;x) = ax^2$;
2. $\lim_{n \to \infty} [n]_{p_n,q_n} G_n^{p_n,q_n}((t-x)^4;x) = 0$.

3. Local approximation

Let $C_B[0, \infty)$ be the space of all real-valued continuous bounded functions $f$ on $[0, \infty)$, endowed with the norm $\|f\| = \sup_{x \in [0, \infty)} |f(x)|$. The first-order and second-order modulus of continuities, the Peetre’s $K$-functional of the function $f \in C_B[0, \infty)$ are defined by for $\delta > 0$

$$\omega(f;\delta) = \sup_{0 < t \leq \delta} \sup_{x \in [0, \infty)} |f(x+t) - f(x)|,$$
$$\omega_2(f;\delta) = \sup_{0 < t \leq \delta} \sup_{x \in [0, \infty)} |f(x+2t) - 2f(x+h) + f(x)|,$$
$$K_2(f;\delta) = \inf_{g \in C_B[0, \infty)(f;\omega_2(f;\sqrt{\delta}))},$$

where $C_B^2[0, \infty) := \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$. By [6], there exists an absolute constant $M > 0$ such that

$$K_2(f;\delta) \leq M\omega_2(f;\sqrt{\delta}).$$ (5)

LEMMA 4. For $f \in C_B[0, \infty)$, we have

$$|G_n^{p,q}(f;x)| \leq \|f\|.$$  

Proof. The proof of this lemma is obvious. \(\Box\)
Theorem 1. Let $f \in C_B[0, +\infty)$, $0 < q < p \leq 1$, then for every $x \in [0, \infty)$ and $n > 2$ we have

$$|G_n^{p,q}(f;x) - f(x)| \leq C \omega_2 \left(f, \sqrt{A(x)}\right),$$

where $C$ is some positive constant and $A(x)$ is given by equation (4).

Proof. For all $g \in C_B^2[0, \infty)$, using the Taylor’s expansion for $x \in [0, \infty)$, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

Applying the operators $G_n^{p,q}$ to both sides of the equality above and using Lemma 3, we get

$$|G_n^{p,q}(g;x) - g(x)| = \left|G_n^{p,q} \left( \int_x^t (t-u)g''(u)du; x \right) \right| \leq G_n^{p,q} \left( \left| \int_x^t (t-u)g''(u)du \right|; x \right)$$

$$\leq G_n^{p,q} \left( \|g''\| (t-x)^2; x \right) \leq A(x)\|g''\|.$$

By Lemma 4, we have

$$|G_n^{p,q}(f;x) - f(x)| \leq |G_n^{p,q}(f-g;x) - (f-g)(x)| + |G_n^{p,q}(g;x) - g(x)|$$

$$\leq 2\|f-g\| + A(x)\|g''\|.$$

Lastly, taking infimum on both side of the equality above over all $g \in C_B^2[0, \infty)$

$$|G_n^{p,q}(f;x) - f(x)| \leq 2K_2 (f;A(x))$$

for which we have the desired result by (5). This completes the proof of Theorem 1. □

Theorem 2. Let $0 < \gamma \leq 1$ and $E$ be any bounded subset of the interval $[0, \infty)$. If $f \in C_B[0, \infty)$ is locally in Lip($\gamma$), i.e., the condition

$$|f(x) - f(t)| \leq L|x-t|^{\gamma}, t \in E \quad \text{and} \quad x \in [0, \infty)$$

holds, then, for each $x \in [0, \infty)$, we have

$$|G_n^{p,q}(f;x) - f(x)| \leq L \left\{ (A(x))^{\gamma} + 2(d(x;E))^{\gamma} \right\},$$

where $L$ is a constant depending on $\gamma$ and $f$; and $d(x;E)$ which is the distance between $x$ and $E$ is defined by

$$d(x;E) = \inf\{|t-x| : t \in E\}.$$

Proof. From the properties of infimum, at least an point $t_0$ exists in the closure of $E$, that is $t_0 \in E$, such that

$$d(x;E) = |t_0 - x|.$$
Using the triangle inequality, we have
\[ |G_n^{p,q}(f;x) - f(x)| \leq G_n^{p,q}(|f(t) - f(x)|;x) \]
\[ \leq G_n^{p,q}(|f(t) - f(t_0)|;x) + G_n^{p,q}(|f(t_0) - f(x)|;x) \]
\[ \leq L(G_n^{p,q}(|t - t_0|^\gamma;x) + G_n^{p,q}(|t_0 - x|^\gamma;x)) \]
\[ \leq L(G_n^{p,q}(|t - x|^\gamma;x) + 2|t_0 - x|^\gamma) \]
Choosing \( a_1 = \frac{2}{\gamma} \) and \( a_2 = \frac{2}{2 - \gamma} \) and using the well-known Hölder inequality
\[ |G_n^{p,q}(f;x) - f(x)| \leq L \left\{ \left( G_n^{p,q}(|t - x|^\gamma_1;x) \right)^{\frac{1}{a_1}} \left( G_n^{p,q}(|1|^\gamma_2;x) \right)^{\frac{1}{a_2}} + 2|t_0 - x|^\gamma \right\} \]
\[ \leq L \left\{ \left( G_n^{p,q}((t - x)^2;x) \right)^{\frac{\gamma}{2}} + 2|t_0 - x|^\gamma \right\} \]
\[ \leq L \left\{ A^\gamma(x) + 2(d(x;E))^\gamma \right\} \]
This completes the proof. \( \square \)

4. Rate of convergence

Let \( B_{x^2}[0,\infty) \) be the set of all functions \( f \) defined on \((0,\infty)\) satisfying the condition \(|f(x)| \leq M_f(1 + x^2)\), where \( M_f > 0 \) is a constant depending only on \( f \). Let \( C_{x^2}[0,\infty) \) denote the subset of all continuous functions in \( B_{x^2}[0,\infty) \). Let \( C_{x^2}[0,\infty) \) be the subset of all functions \( f \in C_{x^2}[0,\infty) \) with the norm \( \|f\|_{x^2} = \sup_{x \in [0,\infty)} \frac{|f(x)|}{1 + x^2} \) and
\[ C_{x^2}[0,\infty) = \left\{ f \in C_{x^2}[0,\infty) : \lim_{x \to \infty} \frac{|f(x)|}{1 + x^2} < \infty \right\} . \]
Meantime, we denote the modulus of continuity on \( f \) on the closed interval \([0,a]\), \( a > 0 \) by
\[ \omega_a(f, \delta) = \sup_{|t-x| \leq \delta} \sup_{t,x \in [0,a]} |f(t) - f(x)| . \]
Obviously, for the function \( f \in C_{x^2}[0,\infty) \), the modulus of continuity \( \omega_a(f, \delta) \) tends to zero as \( \delta \to 0^+ \). Now we give the theorem about the rate of convergence theorem for the operators \( G_n^{p,q}(f;x) \).

**Theorem 3.** Let \( f \in C_{x^2}[0,\infty) \), \( 0 < q < p \leq 1 \) and \( \omega_{a+1}(f, \delta) \) be its modulus of continuity on the finite interval \([0,a+1] \subset [0,\infty)\), where \( a > 0 \). Then, for every \( n > 2 \),
\[ \|G_n^{p,q}(f;x) - f(x)\|_{C[0,a]} \leq 4M_f(1 + a^2)A(a) + 2\omega_{a+1} \left( f, \sqrt{A(a)} \right) . \]

**Proof.** For all \( x \in [0,a] \) and \( t > a + 1 \), we easily have \((t-x)^2 > (t-a)^2 \geq 1\), therefore,
\[ |f(t) - f(x)| \leq |f(t)| + |f(x)| \leq M_f(2 + x^2 + t^2) = M_f(2 + x^2 + (x - t - x)^2) \]
\[ \leq M_f(2 + 3x^2 + 2(x - t)^2) \leq M_f(4 + 3x^2)(t - x)^2 \leq 4M_f(1 + a^2)(t - x)^2, \]
and for all \( x \in [0, a] \), \( t \in [0, a + 1] \) and \( \delta > 0 \), we have
\[
|f(t) - f(x)| \leq \omega_{a+1}(f, |t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right) \omega_{a+1}(f, \delta).
\] (7)

From (6) and (7), we get
\[
|f(t) - f(x)| \leq 4M_f(1 + a^2)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right) \omega_{a+1}(f, \delta).
\]

By Schwarz’s inequality and Lemma 3, we have
\[
|G_n^{p,q}(f; x) - f(x)| \leq G_n^{p,q}(|f(t) - f(x)|; x)
\]
\[
\leq 4M_f(1 + a^2)G_n^{p,q}((t-x)^2; x) + G_n^{p,q}\left(\left(1 + \frac{|t-x|}{\delta}\right); x\right) \omega_{a+1}(f, \delta)
\]
\[
\leq 4M_f(1 + a^2)\frac{1}{\delta}\sqrt{G_n^{p,q}((t-x)^2; x)} + \omega_{a+1}(f, \delta)
\]
\[
\leq 4M_f(1 + a^2)A(x) + \omega_{a+1}(f, \delta)\left(1 + \frac{1}{\delta}\sqrt{A(x)}\right)
\]
\[
\leq 4M_f(1 + a^2)A(a) + \omega_{a+1}(f, \delta)\left(1 + \frac{1}{\delta}\sqrt{A(a)}\right).
\]

By taking \( \delta = \sqrt{A(a)} \), we get the proof of Theorem 3. \( \square \)

As is known, if \( f \) is not uniformly continuous on the interval \((0, \infty)\), the usual first modulus of continuity \( \omega(f; \delta) \) does not tend to zero as \( \delta \to 0 \). For every \( f \in C_{x^2}^0[0, \infty) \), we would like to take a weighted modulus of continuity \( \Omega(f; \delta) \) which tends to zero as \( \delta \to 0 \).

Let \( \Omega(f; \delta) = \sup_{0 < h \leq \delta, x \geq 0} \frac{|f(x+h) - f(x)|}{1 + (x+h)^2} \), for every \( f \in C_{x^2}^0[0, \infty) \).

The weighted modulus of continuity \( \Omega(f; \delta) \) was defined by Yuksel and Ispir in [17]. It is known that \( \Omega(f; \delta) \) has the following properties:

(i) \( \Omega(f; \delta) \) is a monotone increasing function of \( \delta \);

(ii) For each \( f \in C_{x^2}^0[0, \infty) \), \( \lim_{\delta \to 0^+} \Omega(f; \delta) = 0 \);

(iii) For each \( m \in \mathbb{N} \), \( \Omega(f; m\delta) \leq m\Omega(f; \delta) \);

(iv) For each \( \lambda \in \mathbb{R}^+ \), \( \Omega(f; \lambda\delta) \leq (1 + \lambda)\Omega(f; \delta) \).

**THEOREM 4.** Let \( f \in C_{x^2}^0[0, \infty) \) and the sequences \((p_n), (q_n)\) satisfying \( 0 < q_n < p_n < 1 \) such that \( p_n \to 1, q_n \to 1, [n]_{p_n,q_n} \to \infty \) as \( n \to \infty \), then there exists a positive integer \( N \in \mathbb{N}_+ \) such that for all \( n > N \), the inequality
\[
\sup_{x \in [0, \infty)} \frac{|G_n^{p_n,q_n}(f; x) - f(x)|}{(1 + x^2)^{\frac{3}{2}}} \leq 10\Omega\left(f; \frac{1}{\sqrt{[n]_{p_n,q_n}}}\right)
\]
holds.
Proof. For \( t > 0, x \in (0, \infty) \) and \( \delta > 0 \), by the definition of \( \Omega(f; \delta) \) and the above property (iv), we get
\[
|f(t) - f(x)| \leq (1 + (x + |x - t|)^2) \Omega(f; |t - x|)
\]
\[
\leq 2(1 + x^2) \left( 1 + \left( \frac{|t - x|}{\delta} \right)^2 \right) \Omega(f; \delta).
\]

Using \( \lim_{n \to \infty} p_n = \lim_{n \to \infty} q_n = 1 \), \( \lim_{n \to \infty} [n]_{p_n,q_n} = \infty \) and Lemma 3, there exists a positive integer \( N \in \mathbb{N}_+ \) such that for all \( n > N \),
\[
G^n_{p_n,q_n}((t - x)^2; x) \leq \frac{2(1 + x^2)}{[n]_{p_n,q_n}}, \quad (8)
\]
\[
G^n_{p_n,q_n}((t - x)^4; x) \leq x^4. \quad (9)
\]

Since \( G^n_{p_n,q_n} \) is linear and positive, we have
\[
|G^n_{p_n,q_n}(f; x) - f(x)| \leq 2(1 + x^2) \Omega(f; \delta) \left\{ 1 + G^n_{p_n,q_n}((t - x)^2; x) \right\} + G^n_{p_n,q_n} \left( (1 + (t - x)^2) \frac{|t - x|}{\delta}; x \right). \quad (10)
\]

To estimate the second term of (10), applying the Cauchy-Schwartz inequality and \((x + y)^2 \leq 2(x^2 + y^2)\), we have
\[
G^n_{p_n,q_n} \left( (1 + (t - x)^2) \frac{|t - x|}{\delta}; x \right) \leq \sqrt{2} \left( G^n_{p_n,q_n} \left( 1 + (t - x)^4; x \right) \right)^{\frac{1}{2}} \left( G^n_{p_n,q_n} \left( \frac{(t - x)^2}{\delta^2}; x \right) \right)^{\frac{1}{2}}.
\]

By (8) and (9),
\[
G^n_{p_n,q_n} \left( (1 + (t - x)^2) \frac{|t - x|}{\delta}; x \right) \leq \frac{2(1 + x^2)^{\frac{3}{2}}}{\delta [n]_{p_n,q_n}}.
\]
Taking \( \delta = \frac{1}{\sqrt{[n]_{p_n,q_n}}} \), we can obtain
\[
|G^n_{p_n,q_n}(f; x) - f(x)| \leq 10(1 + x^2)^{\frac{5}{2}} \Omega \left( f; \frac{1}{\sqrt{[n]_{p_n,q_n}}} \right).
\]

The proof is completed. \( \square \)

5. Weighted approximation

Now, we obtain the weighted approximation theorem as follows:

Theorem 5. Let the sequences \((p_n), (q_n)\) satisfying \(0 < q_n < p_n \leq 1\) such that \(p_n \to 1, q_n \to 1, [n]_{p_n,q_n} \to \infty\) as \(n \to \infty\). Then for \(f \in C^0_{\infty}(0, \infty)\), we have
\[
\lim_{n \to \infty} \|G^n_{p_n,q_n}(f) - f\|_{L^2} = 0.
\]
Since $G_{p_n,q_n}^n(1;x) = 1$, $G_{p_n,q_n}^n(t;x) = x$, (11) holds for $k = 0, 1$.

By (3), we have,
\[
\left\|G_{p_n,q_n}^n(t^2;x) - x^2\right\|_{x^2} = \sup_{x \in [0,\infty)} \frac{1}{1 + x^2} \left|G_{p_n,q_n}^n(t^2;x) - x^2\right| = \sup_{x \in [0,\infty)} \frac{x^2}{1 + x^2} \left|\frac{p_n^n}{q_n^n} x\right|,
\]
which implies that
\[
\lim_{n \to \infty} \left\|G_{p_n,q_n}^n(t^2;x) - x^2\right\|_{x^2} = 0.
\]
Thus the proof is completed. \(\square\)

Now, we present a weighted approximation theorem for function in $C_{x^2}[0,\infty)$.

**Theorem 6.** Let the sequences $(p_n)$, $(q_n)$ satisfying $0 < q_n < p_n \leq 1$ such that $p_n \to 1$, $q_n \to 1$, $[n]_{p_n,q_n} \to \infty$ as $n \to \infty$. For every $f \in C_{x^2}[0,\infty)$ and $\alpha > 0$, we have
\[
\lim_{n \to \infty} \sup_{x \in [0,\infty)} \frac{|G_{p_n,q_n}^n(f;x) - f(x)|}{(1 + x^2)^{1+\alpha}} = 0.
\]

**Proof.** Let $x_0 \in [0,\infty)$ be arbitrary but fixed. Then
\[
\sup_{x \in [0,\infty)} \frac{|G_{p_n,q_n}^n(f;x) - f(x)|}{(1 + x^2)^{1+\alpha}} \leq \sup_{x \in [0,x_0]} \frac{|G_{p_n,q_n}^n(f;x) - f(x)|}{(1 + x^2)^{1+\alpha}} + \sup_{x \in (x_0,\infty)} \frac{|G_{p_n,q_n}^n(f;x) - f(x)|}{(1 + x^2)^{1+\alpha}}
\]
\[
\leq \|G_{p_n,q_n}^n(f;x) - f(x)\|_{C[0,x_0]} + M_f \sup_{x \in (x_0,\infty)} \frac{|G_{p_n,q_n}^n((1+t^2);x)|}{(1 + x^2)^{1+\alpha}} + \sup_{x \in (x_0,\infty)} \frac{|f(x)|}{(1 + x^2)^{1+\alpha}}.
\]
(12)

Since $|f(x)| \leq M_f(1 + x^2)$, we have $\sup_{x \in (x_0,\infty)} \frac{|f(x)|}{(1 + x^2)^{1+\alpha}} \leq \frac{M_f}{(1 + x_0^2)^{\alpha}}$. Let $\varepsilon > 0$ be arbitrary. We can choose $x_0$ to be so large that
\[
\frac{M_f}{(1 + x_0^2)^{\alpha}} < \varepsilon.
\]
(13)

In view of Lemma 2, while $x \in (x_0,\infty)$, we obtain
\[
M_f \lim_{n \to \infty} \frac{|G_{p_n,q_n}^n((1+t^2);x)|}{(1 + x^2)^{1+\alpha}} = M_f \frac{(1 + x^2)}{(1 + x^2)^{1+\alpha}} = M_f \frac{(1 + x^2)}{(1 + x^2)^{\alpha}} \leq \frac{M_f}{(1 + x_0^2)^{\alpha}} < \varepsilon.
\]
Using Theorem 3, we can see that the first term of the inequality (12), implies that

$$\|G_{n}^{p_{n},q_{n}}(f;x)-f(x)\|_{C[0,x_{0}]} < \varepsilon, \text{ as } n \to \infty.$$ \hspace{1cm} (14)

Combining (12)-(14), we get the desired result. \hspace{1cm} □

6. **Voronovskaja type theorem**

In this section, we give a Voronovskaja-type asymptotic formula for $G_{n}^{p_{n},q_{n}}(f;x)$ by means of the second and fourth central moments.

**Theorem 7.** The sequences $(p_{n}), (q_{n})$ satisfying $0 < q_{n} < p_{n} \leq 1$ such that $p_{n} \to 1, q_{n} \to 1$ and $p_{n}^{a} \to a, q_{n}^{b} \to b, |n|_{p_{n},q_{n}} \to \infty$ as $n \to \infty$ where $0 \leq a, b < 1$. For $f \in C_{B}^{2}[0,\infty)$, the following equality holds

$$\lim_{n \to \infty} [n]_{p_{n},q_{n}} (G_{n}^{p_{n},q_{n}}(f;x) - f(x)) = \frac{a}{2} f''(x)x^{2},$$

for every $x \in [0,A], A > 0$.

**Proof.** Let $x \in [0,\infty)$ be fixed. In order to prove this identity, we use Taylor’s expansion

$$f(t) - f(x) = (t-x)f'(x) + (t-x)^{2} \left( \frac{f''(x)}{2} + \theta(t,x) \right),$$

where $\theta(t,x)$ is bounded and $\lim_{t \to x} \theta(t,x) = 0$. By applying the operator $G_{n,q_{n}}(f;x)$ to the above relation, we obtain

$$G_{n}^{p_{n},q_{n}}(f;x) - f(x) = f'(x)G_{n}^{p_{n},q_{n}}((t-x);x) + \frac{1}{2} f''(x)G_{n}^{p_{n},q_{n}}((t-x)^{2};x)$$

$$+ G_{n}^{p_{n},q_{n}}(\theta(t,x)(t-x)^{2};x)$$

$$= \frac{1}{2} f''(x)G_{n}^{p_{n},q_{n}}((t-x)^{2};x) + G_{n}^{p_{n},q_{n}}(\theta(t,x)(t-x)^{2};x).$$

Since $\lim_{t \to x} \theta(t,x) = 0$, then for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|t-x| < \delta$ implies $|\theta(t,x)| < \varepsilon$ for all fixed $x \in [0,\infty)$ where $n$ is large enough. While if $|t-x| \geq \delta$, then $|\theta(t,x)| \leq \frac{M}{\delta^{2}}(t-x)^{2}$, where $M > 0$ is a constant. Using Remark 1, we have

$$\lim_{n \to \infty} [n]_{p_{n},q_{n}} G_{n}^{p_{n},q_{n}}((t-x)^{2};x) = ax^{2}$$

and

$$[n]_{p_{n},q_{n}} G_{n}^{p_{n},q_{n}}(\theta(t,x)(t-x)^{2};x) \leq \varepsilon [n]_{p_{n},q_{n}} G_{n}^{p_{n},q_{n}}((t-x)^{2};x)$$

$$+ \frac{M}{\delta^{2}} [n]_{p_{n},q_{n}} G_{n}^{p_{n},q_{n}}((t-x)^{4};x) \to 0 \text{ as } n \to \infty.$$
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