PALEY TYPE INEQUALITY OF THE FOURIER TRANSFORM ON
THE HEISENBERG GROUP

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Abstract. A paley type inequality for the Fourier transform on $H^p(\mathbb{H}^n)$, the Hardy space on the Heisenberg group, is obtained for $0 < p \leq 1$.

1. Introduction

The study of Hardy spaces has been originated during the 1910’s in the setting of Fourier series and complex analysis in one variable. In 1972, Fefferman and Stein [5] introduced Hardy spaces $H^p$ by mean of maximal function

$$f^*(x) = \sup_{r>0} |f * \phi_r(x)|$$

where $\phi$ belongs to $S$, the Schwartz space of rapidly decreasing smooth functions satisfying $\int \phi(x) dx = 1$. The delation $\phi_r$ is given by $\phi_r(x) = r^{-n} \phi(x/r)$. We say that a tempered distributions $f \in S'$ is in $H^p$ if $f^*$ is in $L^p$.

Using the maximal function above, Coifman [4] showed that any $f$ in $H^p$ can be represented as a linear combination of atoms, that is

$$f = \sum_{k=1}^{\infty} \beta_k a_k, \quad \beta_k \in \mathbb{C},$$

where the $a_k$ are atoms and the sum converges in $H^p$. Moreover,

$$\|f\|_{H^p} \approx \inf \left\{ \sum_{k=1}^{\infty} |\beta_k|^p : \sum_{k=1}^{\infty} \beta_k a_k \text{ is a decomposition of } f \text{ into atoms} \right\}.$$ 

It has been shown that the study of some analytic problems on $H^p(\mathbb{R}^n)$ is summed up to investigate some properties of these atoms, and therefore the problems become quite simple. In 1980, Taibleson and Weiss [17] gave the definition of molecules belonging to $H^p$, and showed that every molecule is in $H^p$ with continuous embedding map. By the atomic decomposition and the molecule characterization, the proof of $H^p$ boundedness of the operators on Hardy space becomes easier. The theory of $H^p$ have been extensively studied in [7] and [6].

In the setting of the euclidian case, Hardy’s inequality for Fourier transform asserts that for all $f \in H^p(\mathbb{R}^n) \ 0 < p \leq 1$.

$$\int_{\mathbb{R}^n} \frac{|\hat{f}(\xi)|^p}{|\xi|^{n(2-p)}} d\xi \leq \|f\|_{H^p(\mathbb{R}^n)}^p, \quad 0 < p \leq 1$$

where $H^p(\mathbb{R}^n)$ indicates the real Hardy space. Hardy’s type inequality for Fourier transform has been extensively studied in [16]. Kanjin [13] proved Hardy’s inequalities for Hermit and Laguerre expansions for functions in $H^1$ and for Hankel transform [12]. In connection with properties of regularity of the spherical means on $\mathbb{C}^n$, Thangavelu [18] proved a Hardy’s inequality for special Hermit functions. These standard inequalities for

Key words and phrases. Hardy-Littlewood inequality; Heisenberg group.
higher dimensional has been studied in [14]. Recently, an extension has been given by [1], the latter establish a Hardy’s type inequality associated with the Hankel transform for over critical exponent $\sigma > \sigma_0 = 2 - p$. We point out here that the result obtained for Hardy’s inequality for the Hankel transform improves the work of Kanjin [12] in which he proved the result for $\sigma_0 = 2 - p$. Although, in [2, 3, 15] extended this form of this inequality to Laguerre hypergroup and its dual.

In this paper we are interested in the Heisenberg group $\mathbb{H}^n$ is the Lie group with underlying manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ and multiplication $(z, t). (z', t') = (z + z', t + t' + 2Im(z.z'))$, where $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. If we identify $\mathbb{C}^n \times \mathbb{R}$ with $\mathbb{R}^{2n+1}$ by $z_j = x_j + ix_{j+n}$, $j = 1, \ldots, n$, then the group law can be rewritten as

$$(x_1, x_2, \ldots, x_{2n}, t). (y_1, y_2, \ldots, y_{2n}, t') = (x_1 + y_1, \ldots, x_n + y_n, t + t' - 2 \sum_{j=1}^{n} (x_jy_{j+n} - y_jx_{j+n})).$$

The reverse element of $u = (z, t)$ is $u^{-1} = (-z, -t)$ and we write the identity of $\mathbb{H}^n$ as $0 = (0, 0)$.

Set $X_j, X_{j+n}$, and $T$ is a basis for the left invariant vector fields on $\mathbb{H}^n$. The corresponding complex vector fields are

$$Z_j = \frac{1}{2}(X_j - iX_{j+n}) = \frac{\partial}{\partial z_j} + iz_j \frac{\partial}{\partial t}, \quad Z_j = \frac{1}{2}(X_j + iX_{j+n}) = \frac{\partial}{\partial z_j} - iz_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, n.$$

The Heisenberg group is a connected, simply connected nilpotent Lie group. We define one-parameter dilations on $\mathbb{H}^n$, for $R > 0$, by $\rho_R(z, t) = (Rz, R^2t)$. These dilations are group automorphisms and the Jacobian determinant is $R^Q$, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$. We will denoted by $f_\mu(z, t) = \rho^{-Q}f((z, t)_\mu)$ the dilated of the function $f$ defined on $\mathbb{H}^n$.

A homogeneous norm on $\mathbb{H}^n$ is given by

$$|(z, t)|_{\mathbb{H}^n} = (|z|^4 + 4t^2)^{1/4},$$

With this norm, we define the Heisenberg ball centered at $u = (z, t)$ of radius $r$, i.e., the set

$$B(u, r) = \{v \in \mathbb{H}^n : |uv^{-1}|_{\mathbb{H}^n} < R\},$$

and we denote by $B_R = B(0, R) = \{v \in \mathbb{H}^n : |v|_{\mathbb{H}^n} < R\}$ the open ball centered at $0$, the identity element of $\mathbb{H}^n$, with radius $R$. The volume of the ball $B(u, R)$ is $C_Q R^Q$, where $C_Q$ is the volume of the unit ball $B_1$.

The Haar measure $dV$ on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ which is denoted by $dzd\zeta dt$.

Let $J = (j_1, j_2, j_0) \in \mathbb{Z}^n_+ \times \mathbb{Z}^n_+ \times \mathbb{Z}_+$, where $\mathbb{Z}_+$ the set of all nonnegative integers, we set $h(J) = |j_1| + |j_2| + 2j_0$, where, if $j = (j_1, \ldots, j_n)$, then $|j_1| = \sum_{k=1}^{n} j_k$. If $P(z, t) = \sum_j a_j(z, t)^j$ is a polynomial where $(z, t)^j = z^j t^j$, then we call $\max\{|h(J) : a_j \neq 0\}$ the homogeneous degree of $P(z, t)$. The set of all polynomials whose homogeneous degree $\leq s$ is denoted by $\mathcal{P}_s$. Schwartz space on $\mathbb{H}^n$ write as $\mathcal{S}(\mathbb{H}^n)$.

Fix $\lambda > 0$, let $\mathcal{H}_\lambda$ be the Bargmann’s space:

$$\mathcal{H}_\lambda = \left\{ F \text{ holomorphic on } \mathbb{C}^n : \|F\|^2 = \left(\frac{2\lambda}{\pi}\right)^n \int_{\mathbb{C}^n} |F(\zeta)|^2 e^{-2\lambda|\zeta|^2} d\zeta < \infty \right\}.$$ 

Then, $\mathcal{H}_\lambda$ is a Hilbert space and the monomials

$$F_{a, \lambda}(\zeta) = \sqrt{\frac{(2\lambda)^{n|\alpha|}}{\alpha!}} \zeta^\alpha, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n,$$
form an orthonormal basis for $\mathcal{H}_\lambda$, where $\alpha! = \alpha_1!\alpha_2!...\alpha_n!$, $|\alpha| = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\zeta^\alpha = \zeta_1^{\alpha_1}\zeta_2^{\alpha_2}...\zeta_n^{\alpha_n}$. Suppose $W_{k,\lambda}$ and $W^+_{k,\lambda}$ are the closed operators on $\mathcal{H}_\lambda$ such that

$$W_{k,\lambda}F_{\alpha,\lambda} = (2(\alpha_k + 1)\lambda)^{1/2}F_{\alpha+e_k,\lambda},$$
$$W^+_{k,\lambda}F_{\alpha,\lambda} = (2\alpha_k\lambda)^{1/2}F_{\alpha-e_k,\lambda},$$

for $\lambda > 0$, and

$$W_{k,\lambda} = W^+_{k,-\lambda},$$
$$W^+_{k,\lambda} = W_{k,-\lambda},$$

for $\lambda < 0$,

where $e_k = (0, ..., 1, ..., 0) \in \mathbb{Z}^n$ with the 1 in the $k$-th position. Then

$$\prod_\lambda (z, t) = exp^{i\lambda t} exp^{(-z, W_\lambda + \pi, W^+_\lambda)}$$

is an irreducible unitary representation of $\mathbb{H}^n$ on $\mathcal{H}_\lambda$, where $z, W_\lambda = \sum_{k=1}^n z_k W_{k,\lambda}$.

The group Fourier transform of $f \in L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)$ is an operator-valued function defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{H}^n} f(z, t) \prod_\lambda (z, t)dV.$$  \hfill (1.2)

Obviously, $\|\mathcal{F}(f)(\lambda)\| \leq \|f\|_{L^1}$. Here, $\| - \|$ denotes the operator norm. Similar as in $\mathbb{R}^n$, for $f \in L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)$, we have the following Plancherel and inversion formulas :

$$\|f\|^2 = \frac{2^{n-1}}{\pi^{n+1}} \int_R \|\mathcal{F}(f)(\lambda)\|^2_{HS}|\lambda|^n d\lambda, \quad f \in L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n),$$

$$\int_R tr\left(\prod_\lambda (z, t)\mathcal{F}(f)(\lambda)\right)|\lambda|^nd\lambda = \frac{(2\pi)^{n+1}}{4^n}f(u)$$ \hfill (1.4)

where $tr$ is the canonical semifinite trace and $\| - \|_{HS}$ denotes the Hilbert-Schmidt norm.

For $(\lambda, m, \alpha) \in \mathbb{R}^* \times \mathbb{Z}^n \times \mathbb{Z}_n^+$, where $\mathbb{R}^* = \mathbb{R}\{0\}$, we use the notations

$$m^+_i = \max\{m_i, 0\}, \quad m^-_i = \min\{m_i, 0\},$$
$$m^+_\iota = (m^+_1, m^+_2, ..., m^+_n), \quad m^-_\iota = (m^-_1, m^-_2, ..., m^-_n).$$

The partial isometry operator $W^m_\alpha(\lambda)$ on $\mathcal{H}_\lambda$ [1] by

$$W_{k,\lambda}(\alpha)F_{\beta,\lambda} = (-1)^{|m^+\iota|} \delta_{\alpha+m^+,\beta}F_{\alpha+m^-,\lambda}, \quad \text{for } \lambda > 0;$$
$$W^m_\alpha(\lambda) = [W^m_{\alpha}(\lambda)]^*, \quad \text{for } \lambda < 0.$$

Thus $\{W^m_\alpha(\lambda) : m \in \mathbb{Z}^n, \alpha \in \mathbb{Z}_n^+\}$ is an orthonormal basis for the Hilbert-Schmidt operators on $\mathcal{H}_\lambda$. Given a function $f \in L^2(\mathbb{H}^n)$ such that

$$f(z, t) = \sum_{m, \alpha} f_m(r_1, ..., r_n, t)e^{i(\alpha_1\theta_1 + ... + \alpha_n\theta_n)}, \quad \text{where } z_j = r_j e^{i\theta_j},$$

then

$$\mathcal{F}(f)(\lambda) = \sum_{m, \alpha} R_f(\lambda, m, \alpha)W^m_\alpha(\lambda),$$

where

$$R_f(\lambda, m, \alpha) = \int_{\mathbb{H}^n} f_m(r_1, ..., r_n, t)e^{iat_1\ell^{m_1}_\alpha(2|\lambda|^2r_1^2)... \ell^{m_n}_\alpha(2|\lambda|^2r_n^2)dV},$$

and $\ell^{m}_\alpha$ is the Larguerre function of type $|m|$ and degree $|\alpha|$. 
Let $P$ be a polynomial in $z, \overline{z}, t$ on $\mathbb{H}^n$, and we define the difference-differential operator $\Delta_P$ acting on the Fourier transform of $f \in L^1 \cap L^2(\mathbb{H}^n)$ by

$$\Delta_P \left( \sum_{m, \alpha} R_f(\lambda, m, \alpha) W^m_\alpha(\lambda) \right) = \sum_{m, \alpha} R_{Pf}(\lambda, m, \alpha) W^m_\alpha(\lambda),$$

namely, $\Delta_P F(f)(\lambda) = P(\widehat{f})(\lambda)$. In [2] and [10], the authors gave the explicit expressions for $\Delta_t, \Delta_z, \Delta_{\overline{z}}$. For convenience, we shall write $\Delta^{(j,t)} = \Delta^J$.

The paper is organized as follows. In the Second section we give an appropriate definition of atoms and investigate the atoms characterization of Hardy spaces $H^p(\mathbb{H}^n)$ for $0 < p \leq 1$. In the last section we state and prove our main result:

**Theorem 1.1.** Let $0 < p \leq 1$, and $s \geq J = [Q(1/p-1)]$, the greatest integer not exceeding $Q(1/p - 1)$. Then for any $f \in H^p(\mathbb{H}^n)$ the Fourier transform of $f$ satisfies the following Hardy’s type inequality

$$\int_{\mathbb{R}} \frac{||F(f)(\lambda)||_{L^p_{\mathbb{H}^n}}^p}{((2|\alpha| + n)|\lambda|)} |\lambda|^n d\lambda \leq C(p, n) ||f||_{H^p(\mathbb{H}^n)}^p, \quad (1.5)$$

provided that

$$\frac{Q}{2}(2-p) \leq \sigma < \frac{Q}{2} + p\left(\frac{J+1}{2}\right) \quad (1.6)$$

where $C(p,n)$ depend only on $p$ and $n$.

Finally, we mention that $C$ will be always used to denote a suitable positive constant that is not necessarily the same in each occurrence.

2. Atomic decomposition for $H^p(\mathbb{H}^n)$

Now we state the definition of atomic Hardy spaces in the setting of the Heisenberg group $H^p(\mathbb{H}^n)$, $0 < p \leq 1$. To this end, we introduce the following kind of atoms, which is closely related to the Haar measure $dV$.

**Definition 2.1.** Let $0 < p \leq 1 \leq q < \infty$, $p \neq q$, $s \in \mathbb{Z}$ and $s \geq J = [Q(1/p - 1)]$. (Such an ordered triple $(p, q, s)$ is called admissible). A $(p, q, s)$-atom centered at $x_0 \in \mathbb{H}^n$ is a function $a \in L^q(\mathbb{H}^n)$, supported on a ball $B(x_0, R) \subset \mathbb{H}^n$ with centre $x_0 = (z_0, t_0)$ and satisfying the following

(i) $||a||_{L^q(\mathbb{H}^n)} \leq |B(0, r)|^{\frac{q}{p} - \frac{1}{q}},$ a.e,

(ii) $\int_{\mathbb{H}^n} a(x) P(x) dV(x) = 0,$ for every $P \in \mathcal{P}_s$.

Here, (i) means that the size condition of atoms, and (ii) is called the cancelation moment condition.

A characterization of $H^p(\mathbb{H}^n)$ is included in the following statements.

**Proposition 2.1.** Let $0 < p \leq 1$. If $\{a_k\}_{k=0}^\infty$ is a sequence of $p$-atoms, and $\{\lambda_k\}_{k=0}^\infty$ is a sequence of complex numbers with

$$\left( \sum_{k=0}^\infty |\lambda_k|^p \right)^{1/p} < \infty,$$

then $\sum_{k=0}^\infty \lambda_k a_k$ converges in $H^p(\mathbb{H}^n)$ and

$$\left\| \sum_k \lambda_k a_k \right\|_{H^p(\mathbb{H}^n)} \leq C(p, n) \left( \sum_k |\lambda_k|^p \right)^{1/p}.$$
Conversely, if \( f \in H^p(\mathbb{H}^n) \) there exists a sequence \( \{a_k\}_{k=0}^\infty \) of \( p \)-atoms, and a sequence \( \{\lambda_k\}_{k=0}^\infty \) of complex numbers such that
\[
f = \sum_k \lambda_k a_k \quad \text{and} \quad \left( \sum_k |\lambda_k|^p \right)^{1/p} \leq C(p, n) \|f\|_{H^p(\mathbb{H}^n)},
\]
where \( C(p, n) \) depends on \( p \) and \( n \).

3. PROOF OF THE MAIN RESULT

Now we are in a position to give the proof of the main result. First we state the following proposition which has its own interest.

**Proposition 3.1.** For all \((z, t) \in \mathbb{H}^n\) the function \( \prod_\lambda (z, t) \) satisfies
\[
\prod_\lambda (z, t) = \sum_{2k+\ell \leq J} \omega_{k,\ell}(\lambda, n) z^k t^\ell + R_\theta(z, t), \quad 0 < \theta < 1,
\]
where
\[
R_\theta(z, t) = \sum_{2k+\ell = J+1} \frac{(i\lambda t)^k}{k!} \frac{(z.W_\lambda - \overline{z}.W_\lambda^+)^\ell}{\ell!}.
\]
Here \( \omega_{k,\ell}(\lambda, n) \) are functions expressed by mean of \( \lambda, n \).

Set \( \mathcal{H}_N^\lambda \) be the subspace of \( \mathcal{H}_\lambda \) spanned by \( \{W_\alpha^N(\lambda) : |\alpha| \leq N\} \). Remark that (see [9, 11]) \( z.W_\lambda - \overline{z}.W_\lambda^+ \) is bounded from \( \mathcal{H}_N^\lambda \) to \( \mathcal{H}_{N+1}^\lambda \) and whose bound \( < (2|\alpha| + n)|\lambda|^{1/2}|z| \). Then
\[
R_\theta(z, t) \leq C \sum_{2k+\ell = J+1} \omega_{k,\ell} \left( (2|\alpha| + n)|\lambda| \right)^{k+\ell} z^k t^\ell.
\]

**Proof of Theorem 1.1.** Let \( f = \sum_{k=0}^\infty \beta_k a_k \in H^p(\mathbb{H}^n) \), being element of \( H^p(\mathbb{H}^n) \) where \( a_k \) are atoms. Since \( 0 < p \leq 1 \) it follows
\[
\int_\mathbb{R} \frac{||\mathcal{F}(f)(\lambda)\|_{H^p}\rchi^p}{((2|\alpha| + n)|\lambda|)^p} |\lambda|^n d\lambda \leq C \sum_{k=0}^\infty |\beta_k|^p \int_\mathbb{R} \frac{||\mathcal{F}(a_k)(\lambda)\|_{H^p}\rchi^p}{((2|\alpha| + n)|\lambda|)^p} |\lambda|^n d\lambda.
\]
In order to prove Theorem 1.1, it is enough to prove,
\[
\int_\mathbb{R} \frac{||\mathcal{F}(a_k)(\lambda)\|_{H^p}\rchi^p}{((2|\alpha| + n)|\lambda|)^p} |\lambda|^n d\lambda \leq C.
\]
This follows as \( f = \sum_{k=0}^\infty \beta_k a_k \) implies \( \mathcal{F}(a_k)(\lambda) \leq \left| \sum_k \beta_k \mathcal{F}(a_k)(\lambda) \right|^{p} \leq \sum_{k=0}^\infty |\beta_k|^p |\mathcal{F}(a_k)(\lambda)|^{p} \)
and hence
\[
\int_\mathbb{R} \frac{||\mathcal{F}(f)(\lambda)\|_{H^p}\rchi^p}{((2|\alpha| + n)|\lambda|)^p} |\lambda|^n d\lambda \leq C \sum_{k=0}^\infty |\beta_k|^p \int_\mathbb{R} \frac{||\mathcal{F}(a_k)(\lambda)\|_{H^p}\rchi^p}{((2|\alpha| + n)|\lambda|)^p} |\lambda|^n d\lambda
\leq C \left\{ \sum_{k=0}^\infty |\beta_k|^p \right\}^{1/p} \leq C \|f\|_{H^p(\mathbb{H}^n)}.
\]
Let us now take $\gamma$ an arbitrary nonnegative real number, and decomposing the left hand side of (3.3) as

$$\int_{\mathbb{R}} \frac{\|F(a_k)(\lambda)\|_{HS}^p}{(2|\alpha| + n)|\lambda|} |\lambda|^n d\lambda = \int_{0 < |\lambda| \leq \gamma} \frac{\|F(a_k)(\lambda)\|_{HS}^p}{(2|\alpha| + n)|\lambda|} |\lambda|^n d\lambda$$

$$+ \int_{|\lambda| > \gamma} \frac{\|F(a_k)(\lambda)\|_{HS}^p}{(2|\alpha| + n)|\lambda|} |\lambda|^n d\lambda$$

$$:= S_1 + S_2.$$ 

To estimate $S_1$ we may use Proposition 3.1 and cancelation property of atoms. Hence, by the cancelation property of atom,

$$F(a_k)(\lambda) = \int_{\mathbb{R}^n} \left[ \sum_{2k+\ell = J} \omega_{k,\ell}(\lambda, n) \ z^k t^\ell + R_\theta(z, t) \right] a(z, t) \ dV(z, t).$$

Now with the help of properties (i), (ii) for $a(p, \infty, s)$-atoms of $H^p(\mathbb{H}^n)$ together with Proposition 3.1 we get

$$F(a_k)(\lambda) \leq C \sum_{2k+\ell = J+1} \omega_{k,\ell} \left( (2|\alpha| + n)|\lambda| \right)^{k+\frac{p}{2}} \int_{B(o, R)} z^k t^\ell |B(0, R)|^{-\frac{1}{p}} \ dV(z, t)$$

$$\leq C \sum_{2k+\ell = J+1} \omega_{k,\ell} R^{Q(p-1)+p(2k+\frac{p}{2})} \left( (2|\alpha| + n)|\lambda| \right)^{k+\frac{p}{2}}.$$ 

Integrating with respect to the measure $d\gamma_n(\lambda) = |\lambda|^n d\lambda$ over the domain $0 \leq |\lambda| \leq \gamma$, we obtain

$$S_1 = \int_{0 < |\lambda| \leq \gamma} \frac{\|F(a_k)(\lambda)\|_{HS}^p}{(2|\alpha| + n)|\lambda|} |\lambda|^n d\lambda$$

$$\leq C \sum_{2k+\ell = J+1} \omega_{k,\ell} R^{Q(p-1)+p(2k+\frac{p}{2})} \int_{0 < |\lambda| \leq \gamma} \left( (2|\alpha| + n)|\lambda| \right)^{p(k+\frac{p}{2})-\sigma} |\lambda|^n d\lambda$$

$$\leq 2C \sum_{\ell = 0}^{J+1} \omega_{\ell} R^{Q(p-1)+p(J+1+\frac{p}{2})} \int_{0}^{\gamma} \left( (2|\alpha| + n)|\lambda| \right)^{p(\frac{J+1}{2})-\sigma} |\lambda|^n d\lambda.$$

That is

$$S_1 \leq C \ R^{Q(p-1)+p(J+1+\frac{p}{2})} \gamma^{p(\frac{J+1}{2})-\sigma}, \ \forall \ell = 0, 1, ..., J + 1, \ (3.4)$$

provided that $p(\frac{J+1}{2}) + \frac{Q}{2} - \sigma > 0$, which follows from the inequality (1.6).

Now to estimate $S_2$, we may apply Hölder’s inequality for $q = \frac{p}{2}$ and Plancherel formula. Thus, we immediately obtain

$$S_2 \leq \left( \int_{\mathbb{R}} \left( \frac{\|F(a_k)(\lambda)\|_{HS}^p}{(2|\alpha| + n)|\lambda|} \right)^\frac{p}{2} |\lambda|^n d\lambda \right)^\frac{2}{p} \left( \int_{|\lambda| > \gamma} \left( (2|\alpha| + n)|\lambda| \right)^{\frac{2p}{p-2}} |\lambda|^n d\lambda \right)^\frac{p-2}{p}$$

$$\leq C \|F(a_k)\|_{L^p}^p \left( \int_{|\lambda| > \gamma} \left( (2|\alpha| + n)|\lambda| \right)^{\frac{2p}{p-2}} |\lambda|^n d\lambda \right)^\frac{p-2}{p}$$

$$\leq 2C \|F(a_k)\|_{L^p}^p \left( \int_{\gamma}^{\infty} \left( (2|\alpha| + n)|\lambda| \right)^{\frac{2p}{p-2}} |\lambda|^n d\lambda \right)^\frac{p-2}{p}$$

$$\leq C \|F(a_k)\|_{L^{\frac{p}{2}}}^p \left( \int_{|\lambda| > \gamma} |\lambda|^n d\lambda \right)^\frac{Q(2-p)-\sigma}{p}.$$
provided that \(\frac{Q}{2}(2 - p) - \sigma < 0\), which is a consequence of the left hand side of (1.6). Thanks to Plancherel’s formula for Laguerre Fourier transform it follows

\[
\|F(a_k)\|_{L^2}^2 = \|a_k\|_{L^2(H^n)}^2 = \int_{H^n} |a_k(z, t)|^2 \, dV(z, t)
\leq |B(0, R)|^{1 - \frac{2}{p}}
\leq C \, R^{-Q(\frac{2}{p} - p)}.
\]

That is

\[
\|F(a_k)\|_{L^2}^p \leq C \, R^{-Q(\frac{2}{p} - p)}
\]

and hence,

\[
S_2 \leq C \, R^{-Q(\frac{2}{p} - p)} \gamma \frac{Q}{2}(2 - p) - \sigma.
\] (3.5)

However, to prove that \(S_1 + S_2 \leq C\), we shall discuss the cases \(0 < R < 1\) and \(R \geq 1\). Hence, in order to deal with the case \(0 < R < 1\), we need more precise estimates, so we consider the set \(\Gamma_{\gamma}\); the collection of all numbers \(\gamma\) satisfying

\[
\Gamma_{\gamma} = \left\{ \gamma > 0, \frac{Q}{2}(2 - p) \log(R) \leq \log(\gamma) \leq \frac{Q(1 - p) - p(J + 1)}{p(J + 1)} + \frac{Q}{2} - \sigma \right\}.
\]

We mention that the collection \(\Gamma_{\gamma}\) above is an nonempty set if and only if

\[
\frac{Q}{2}(2 - p) \log(R) < \frac{Q(1 - p) - p(J + 1)}{p(J + 1)} + \frac{Q}{2} - \sigma
\]

which is a different formulation of the hand side of (1.6), that is \(\frac{Q}{2}(2 - p) \leq \sigma\).

Now let us choose \(\gamma \in \Gamma_{\gamma}\) and using the fact that \(\frac{Q}{2} + p\frac{(J + 1)}{2} - \sigma > 0\) together with the right hand side of (1.6) it follows that

\[
S_1 \leq C \, R^{Q(p - 1) + p(J + 1)} \gamma^p \frac{(J + 1)}{2} + \frac{Q}{2} - \sigma,
\] (3.6)

Also, with the same choose of \(\gamma \in \Gamma_{\gamma}\) and under the condition \(\frac{Q}{2}(2 - p) < \sigma\), together with the help of the left hand side of (1.6) we obtain

\[
S_2 \leq C.
\] (3.7)

Combining (3.6) and (3.7) we obtain

\[
S_1 + S_2 \leq C \quad \text{for} \quad 0 < R < 1.
\] (3.8)

Now, to deal with the case \(R \geq 1\), we may take

\[
\gamma = R^{Q(1 - p) - p(J + 1)} \frac{(J + 1)}{2} + \frac{Q}{2} - \sigma
\] (3.9)

so, using the fact that \(R \geq 1\), we obtain

\[
\gamma \leq R^{\frac{Q}{2}(2 - p)} \frac{Q}{2}(2 - p) - \sigma.
\] (3.10)

which leads to

\[
S_1 + S_2 \leq C \quad \text{for} \quad R \geq 1.
\] (3.11)

Hence, to prove (3.3), it is enough to combine (3.8) and (3.11). The proof of the main theorem is completed.

**Acknowledgements.** This project was supported by King Saud University, Deanship of Scientific Research, College of Science Research Center.
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