THE KURANISHI MAP FOR VECTOR BUNDLES ON CERTAIN PRODUCTS OF CURVES

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Abstract. We describe deformations of vector bundles on surfaces that are a product of two smooth projective curves. We explicitly describe the Kuranishi map around unstable vector bundles and compare the homologies of the Kuranishi spaces of stable and unstable deformations.

Keywords: Deformation theory, Kuranishi map, stable bundle, homology, Kuranishi deformation space.

MSC: Primary 32G08; Secondary 32G05, 14J60.

1. Moduli

We study deformation spaces of vector bundles over complex surfaces that are products of two smooth projective curves. We take the product \( \Sigma := \Sigma_1 \times \Sigma_2 \) of general curves of genera \( 1 < g_1 < g_2 \), so that the surface \( \Sigma \) is of general type. Kuranishi deformations of bundles on such surfaces are particularly manageable because they can be expressed in terms of deformations of bundles on each of the individual curves. In fact, we assume throughout that \( \text{Pic}(\Sigma) \) is generated by the box products of pullbacks of line bundles from \( \Sigma_i \). This property, called Pic-independence [Fu], holds for a very general choice of \( (X,Y) \) in \( \mathcal{M}_{g_1} \times \mathcal{M}_{g_2} \), i.e. for all \( (X,Y) \) outside countably many proper subvarieties of \( \mathcal{M}_{g_1} \times \mathcal{M}_{g_2} \), see [I, Prop. 3.19] and [BL, Th. 11.5.1].

Fix the smooth type of a rank 2 vector bundle \( E \) on \( \Sigma_1 \times \Sigma_2 \), or equivalently, its Chern classes. Here we will consider only the case of trivial determinant, hence \( c_1(E) = 0 \). For each holomorphic bundle \( E \) with the smooth type of \( E \), we discuss the deformation space of \( E \).

The Kodaira–Spencer theory of deformations implies that any Zariski tangent vector of the local moduli space of holomorphic vector bundles at \( E \) can be interpreted as an element of \( H^1(\Sigma, \text{End } E) \), where \( \text{End } E \) is the endomorphism bundle of \( E \), see [CWy]. The Kuranishi map associated to \( E \) can be written as

\[
\Psi_E : H^1(\Sigma, \text{End } E) \to H^2(\Sigma, \text{End } E)
\]

\[
w \mapsto [w, w]
\]

and has the following properties [DK, Prop. 6.4.3]:

(i) \( \Psi_E \) and its derivative both vanish at 0, and a versal deformation of \( E \) is parametrized by the complex space \( \Psi_E^{-1}(0) \).

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(ii) The two-jet of $\Psi_E$ at the origin is given by the combination of cup product and bracket:

$$H^1(\text{End } E) \otimes H^1(\text{End } E) \rightarrow H^2(\text{End}_0 E).$$

**Definition 1.** We call the germ of $\Psi_{E}^{-1}(0)$ the **Kuranishi deformation space** of $E$ and denote it by $K(\Sigma, E)$. We denote by $K^s(\Sigma, E)$ the subset of $K(\Sigma, E)$ consisting of stable rank 2 bundles.

We will omit the first entry $\Sigma$ when it is clear from the context, and add a subscript $K_{c_2}(\Sigma, E)$ and $K_{c_2}^s(\Sigma, E)$ respectively when we want to fix the value of the second Chern class.

**Remark 2.** For a fixed choice of $E$ it may happen that $K^s(\Sigma, E)$ is empty (see App. A). In such a case we will see in Secs. 3 and 7 that the Kuranishi map reduces to a single component, labelled $i$) in both cases. Furthermore, in such a case, it then also happens that the target vanishes, so that the Kuranishi deformation space of such a bundle is the entire vector space $H^1(\text{End } E)$.

We prove:

**Theorem (9).** Let $\Sigma_1$ and $\Sigma_2$ be Pic-independent smooth projective curves. Let $E$ be a bundle on $\Sigma_1 \times \Sigma_2$ with trivial determinant and with second Chern number $c_2 > 8g_1g_2$. Then, for $0 < q < c_2$ we have

$$H_q(K_{c_2}(\Sigma, E), K_{c_2}^s(\Sigma, E)) = 0.$$  

2. Background

We recall the basic concept of stability used here.

**Definition 3.** Let $X$ be a compact Kähler complex manifold and $E \to X$ a holomorphic vector bundle. Then if $\dim X = n > 1$, the degree of $E$ is

$$\deg E = \int_X c_1(E) \wedge \omega^{n-1},$$

where $\omega \in H^2(X, \mathbb{Z})$. The form $\omega$ is called a polarization of $E$. The definition of degree depends on the chosen polarization. The slope of $E$ is $\mu(E) = \deg(E)/\text{rank}(E)$. Then $E$ is slope stable (resp. semistable) if $\mu(E') < \mu(E)$ (resp. $\mu(E') \leq \mu(E)$) for every proper subbundle $0 \to E' \to E$.

The most natural choice for the product of curves is to take a polarization such that $\omega$ is a class of type $(1,1)$, making the contributions of $\Sigma_1$ and $\Sigma_2$ well balanced.

**Remark 4.** Moduli spaces of vector bundles correspond to moduli spaces of instantons via the celebrated Kobayashi–Hitchin correspondence. In 1978 Atiyah and Jones [AJ] conjectured that the inclusion of the moduli space of instantons of charge $k$ into the moduli space of all connections induces isomorphisms in homology and homotopy in degrees less than $k/2$. The conjecture was proved for SU(2) instantons on $S^4$ [BHMM], for SU($n$) instantons $S^4$ [Ti], for ruled surfaces [HM], and for rational surfaces [Ga]; it remains open in all other cases. Other results on the stable topology of
moduli spaces include \([\text{Ta, CW, Sa}]\). Our result here contributes to this type of topological questions, by making the technical step of verifying the topological implications of any particular choice of stability unnecessary.

We will apply Kirwan’s techniques for removing subvarieties of high codimension, while preserving isomorphisms in homology through a range.

In \([\text{Ki, Cor. 6.4}]\) Kirwan proved the following result for a quasi-projective variety \(X\). Let \(\mu\) be a non-negative integer such that every \(x_0 \in X\) has a neighbourhood in \(X\) which is analytically isomorphic to an open subset of
\[
\{x \in \mathbb{C}^N | f_1(x) = \cdots = f_M(x) = 0\}
\]
for some integers \(N, M\), and holomorphic functions \(f_i\) depending on \(x_0\) with \(M \leq \mu\). If \(Y\) is a closed subvariety of codimension \(k\) in \(X\), then for \(q < k - \mu\),
\[
H^q(X, X - Y) = 0 = H^q(X, X - Y).
\]
(1)

The same result applies, with an identical proof, with \(X\) a complex space instead of a quasi-projective variety, and we shall use the complex version in our calculations.

3. The Kuranishi map around a split bundle

Consider the surface \(\Sigma = \Sigma_1 \times \Sigma_2\) and let \(L = L_1 \boxtimes L_2\) be a line bundle over \(\Sigma\) of type \((m, n)\). In this section we study the Kuranishi map near a split bundle \(E = L \oplus L^{-1}\).

We carry out the study for the case of \(c_1 = 0\), other cases are just notationally heavier, but do not bring any real extra difficulty. Our aim is to describe explicitly points whose Kuranishi deformation space that have obstructed deformation theory. At such points we investigate the nature of the singularity in comparison to the dimension of the set of stable bundles.

Given the splitting \(E = L \oplus L^{-1}\), we have that the Kuranishi space at \(E\) is contained in
\[
H^1(\Sigma, \text{End } E) = H^1(\Sigma, L^2) \oplus H^1(\Sigma, \mathcal{O}) \oplus H^1(\Sigma, \mathcal{O}) \oplus H^1(\Sigma, L^{-2}).
\]
We identify the directions as:

**Notation 5.** \(T_u = H^1(\Sigma, L^2)\), \(T_o = H^1(\Sigma, \mathcal{O})\) and \(T_s = H^1(\Sigma, L^{-2})\).

Observe that \(T_u\) and \(T_o\) contain unstable deformations of \(E\) whereas \(T_s\) is the direction of stable deformations. We consider the components of the Kuranishi map at \(E\):

\[i)\] \(H^1(\Sigma, L^2) \otimes H^1(\Sigma, \mathcal{O}) \rightarrow H^2(\Sigma, L^2)\),

\[ii)\] \(H^1(\Sigma, L^2) \otimes H^1(\Sigma, L^{-2}) \rightarrow H^2(\Sigma, \mathcal{O})\),

\[iii)\] \(H^1(\Sigma, \mathcal{O}) \otimes H^1(\Sigma, L^{-2}) \rightarrow H^2(\Sigma, L^{-2})\).

We could also list the component \(H^1(\Sigma, \mathcal{O}) \otimes H^1(\Sigma, \mathcal{O}) \rightarrow H^2(\Sigma, \mathcal{O})\), but it is not needed for our analysis, since \(H^2(\Sigma, \mathcal{O})\) poses a bounded number of obstructions and \(H^1(\Sigma, \mathcal{O})\) does not depend on the choice of the point \(E\) and accordingly does not play an interesting role for this task. We will use the local study to estimate the codimension of the set of unstable bundles. This section serves as a warm-up for the more general calculation that will be carried out in Sec. 4 around nonfiltrable bundles.
Observe that the Kuranishi map around any bundle $E$ will have another component with target space $H^2(\Sigma, \mathcal{O})$. We may regard equations coming from this component as unavoidable equations. Thus, in a sense these represent a minimal number of equations that would be present overall in case we also allowed the complex structure of the base space to vary. Our choice here is to fix the complex structure of the surface and vary only that of vector bundles, disregarding this component.

**Proposition 6.** Assume $c_2 > 8g_1g_2$. Then removing the set of split points does not change the homology of the Kuranishi space $\mathcal{K}_{c_2}(\Sigma, E)$ in dimension smaller than $c_2$.

**Proof.** Recall that in this case $c_2 = -2mn$. We assume that $c_2 > 8g_1g_2$.

1. We claim that $H^2(\Sigma, L^2) = 0$ for large $m$ and hence the first component can be ignored, since it will produce no equations for the local model at $E$. In fact, $H^2(\Sigma, L^2) = H^2(\Sigma_1, L_1) \otimes H^0(\Sigma_2, L_2) \oplus H^1(\Sigma_1, L_1) \otimes H^1(\Sigma_2, L_2) = H^0(\Sigma_1, L_1) \otimes H^2(\Sigma_2, L_2)$ and $H^2(\Sigma_1, L_1) = H^2(\Sigma_2, L_2) = 0$. Furthermore, if $m = c_1(L) > 2g(X) - 2$, then $H^1(X, L) = 0$, producing no obstructions for large $c_2$.

2. Here, $\mathcal{O}$ is fixed by the choice of $\Sigma$ and this is precisely the part that poses $h^2(\Sigma, \mathcal{O})$ obstructions. These are unavoidable, but are a bounded number depending only on the topology of $\Sigma$ and small once we compare to the deformation space with large $c_2$.

3. This component $T_o \otimes T_s \to H^2(\Sigma, L^{-2})$ can give a number of obstructions that grows with $m$. Here we wish to remove the set $T_o$ which contains the directions of unstable deformations, in the case when obstructions are present. Since

\[ T_s = \left( H^1(\Sigma_1, L_1^{-2}) \otimes H^0(\Sigma_2, L_2^{-2}) \right) \oplus \left( H^0(\Sigma_1, L_1^{-2}) \otimes H^1(\Sigma_2, L_2^{-2}) \right), \]

\[ T_o = \left( H^0(\Sigma_1, \mathcal{O}) \otimes H^1(\Sigma_2, \mathcal{O}) \right) \oplus \left( H^1(\Sigma_1, \mathcal{O}) \otimes H^0(\Sigma_2, \mathcal{O}) \right), \]

and $H^0(\Sigma_1, \mathcal{O}) = H^0(\Sigma_2, \mathcal{O}) = \mathbb{C}$ and $H^0(\Sigma_1, L_1^{-2}) = 0$ for large $m$, we have $T_s = H^1(\Sigma_1, L_1^{-2}) \otimes H^0(\Sigma_2, L_2^{-2})$, and $T_o = H^1(\Sigma_1, \mathcal{O}) \oplus H^1(\Sigma_2, \mathcal{O})$.

Therefore

\[ T_o \otimes T_s = \left( H^1(\Sigma_1, \mathcal{O}) \oplus H^1(\Sigma_2, \mathcal{O}) \right) \otimes \left( H^1(\Sigma_1, L_1^{-2}) \otimes H^0(\Sigma_2, L_2^{-2}) \right) \]

\[ = H^1(\Sigma_1, \mathcal{O}) \otimes H^1(\Sigma_1, L_1^{-2}) \otimes H^0(\Sigma_2, L_2^{-2}) \]

\[ \oplus H^1(\Sigma_2, \mathcal{O}) \otimes H^1(\Sigma_1, L_1^{-2}) \otimes H^0(\Sigma_2, L_2^{-2}) \]

\[ \simeq H^1(\Sigma_1, L_1^{-2}) \otimes H^1(\Sigma_1, \mathcal{O}) \otimes H^0(\Sigma_2, L_2^{-2}) \]

\[ \oplus H^1(\Sigma_1, L_1^{-2}) \otimes H^1(\Sigma_2, \mathcal{O}) \otimes H^0(\Sigma_2, L_2^{-2}), \]

where we may change the order, since the product is taken over $\mathbb{C}$. Furthermore, $H^2(\Sigma, L^{-2}) = H^2(\Sigma_1, L_1^{-2}) \otimes H^1(\Sigma_2, L_2^{-2})$, so the obstruction
map can be written as
\[
\left( H^1(\Sigma_1, L_1^{-2}) \otimes H^0(\Sigma_2, L_2^{-2}) \otimes H^1(\Sigma_1, \mathcal{O}) \right) \oplus \\
\left( H^1(\Sigma_1, L_1^{-2}) \otimes H^0(\Sigma_2, L_2^{-2}) \otimes H^1(\Sigma_2, \mathcal{O}) \right) \\
\rightarrow H^1(\Sigma_1, L_1^{-2}) \otimes H^1(\Sigma_2, L_2^{-2}).
\] (2)

The first coordinates map by the identity, therefore it suffices to study the zero set of the map
\[
\left( H^0(\Sigma_2, L_2^{-2}) \otimes H^1(\Sigma_1, \mathcal{O}) \right) \oplus \left( H^0(\Sigma_2, L_2^{-2}) \otimes H^1(\Sigma_2, \mathcal{O}) \right) \rightarrow H^1(\Sigma_2, L_2^{-2}),
\]
that is,
\[
H^0(\Sigma_2, L_2^{-2}) \otimes \left( H^1(\Sigma_1, \mathcal{O}) \oplus H^1(\Sigma_2, \mathcal{O}) \right) \rightarrow H^1(\Sigma_2, L_2^{-2}).
\] (3)

Claim. The topology of \( \mathbb{K}_{c_2}(\Sigma, E) \cap (A_o \oplus T_s) \) remains unchanged upon removing \( T_o \) up to real dimension \( c_2 \).

We set \( V_1(m) := H^1(\Sigma_1, L_1^{-2}), V_0(n) := H^0(\Sigma_2, L_2^{-2}) \) and
\[
A := T_o = H^1(\Sigma_1, \mathcal{O}) \oplus H^1(\Sigma_2, \mathcal{O})
\]
and we want to identify the zero set of the Kuranishi map (2). Setting \( K = V_0(n) \otimes A \) as the zero set of the quadratic cup-product map (3), the zero set of the Kuranishi map (2) in \( V_1(m) \otimes V_0(n) \otimes A \) becomes
\[
K^{\nu_1} = K \times_A K \times_A \cdots \times_A K
\]
\( \nu_1 = \dim_{\mathbb{C}} V_1(m) \) times.

Let \( K_a \) be the fiber over a point \( a \in A \). We want to remove \( A \), that is, the zero section of the fiber product \( K \).

Note that if \( K_a = 0 \) for some \( a \), then \( T_o \) is an irreducible component of \( \mathbb{K}_{c_2}(\Sigma, E) \cap (T_o \oplus T_s) \) and in that case \( \mathbb{K}_{c_2}(\Sigma, E) \) has a totally unstable component.

We have avoided this case by demanding in our hypothesis that we work in the complement of unstable components. Therefore, in our case \( K_a \) is never zero.

Now, observe that in Eq. (3) we have that \( T_o \) appears in codimension \( h^0(\Sigma_2, L_2^{-2}) \) and the target gives us \( h^1(\Sigma_2, L_2^{-2}) \) equations. So, to have a meaningful bound here, all we need is that the number of equations be less than the codimension. But, by Riemann–Roch we have:
\[
h^0(\Sigma_2, L_2^{-2}) - h^1(\Sigma_2, L_2^{-2}) = -2n - g_2 + 1 > 0.
\]

Since \( K_a \) is never zero, we can then promote this to the bounds we require in Eq. (2), using the fact that \( \nu_1 = 2m - m_1 + 1 \). So, we have that \( T_o \) appears in the Kuranishi map (2) in codimension \( \nu_1 \dim K_a = \nu_1 \nu_0 = (2m - g_1 + 1)(-2n - g_2 + 1) \) and the corresponding neighbourhood is defined by \( h^1(\Sigma_1, L_1^{-2}) h^1(\Sigma_2, L_2^{-2}) \) equations.

Thus, we obtained the following bound on codimension of \( T_o \) minus number of defining equations, to apply estimate (1):
\[
h^1(\Sigma_1, L_1^{-2})(h^0(\Sigma_2, L_2^{-2}) - h^1(\Sigma_2, L_2^{-2})) = (2m - g_1 + 1)(-2n - g_2 + 1) > c_2(E)
\]
whenever \( m >> 0 \) is large enough to make \( h^0(\Sigma_1, L_1^{-2}) = 0 \).
Note that the last inequality holds when both $2m - g_1 + 1 > 0$ and $-2n - g_2 + 1 > 0$, hence $c_2 = -2mn > 2(2g_1 - 2)(2g_2 - 2)$. Thus, it suffices to require $c_2 > 8g_1g_2$. □

4. The Kuranishi map around a nonfiltrable bundle

We work with a general surface $\Sigma = \Sigma_1 \times \Sigma_2$ with $1 < g_1 < g_2$. In such a case $\Sigma$ is a minimal surface of general type with irregularity $q = g_1 + g_2$, and geometric genus $p_g = g_1g_2$, $K^2 = 4(g_1 - 1)(g_2 - 1)$.

Here $L = L_1 \boxtimes L_2$ is a line bundle over $\Sigma$ of type $(m, n)$ over $\Sigma_1 \times \Sigma_2$. We now study the Kuranishi map near an unstable bundle, in the most general (and most frequent) case, namely, when $E$ is not an extension of line bundles, instead when $E$ is only an extension of a rank one torsion free sheaf $F$ by a destabilizing line bundle $L$. See [Fr, Ch. 2, Prop. 5] for the proof that over any smooth projective surface, a rank 2 bundle $E$ can be written as such an extension. Hence

$$0 \to L \to E \to F \to 0$$

and

$$0 \to F \to F^{\vee\vee} \to Q \to 0$$

with $Q$ supported at points and $F^{\vee\vee} = L^{-1}$. The tangent space to $\mathfrak{g}_L$ deformations of $E$ is

$$\text{Ext}^1(L, L) \oplus \text{Ext}^1(L, F) \oplus \text{Ext}^1(F, L) \oplus \text{Ext}^1(F, F)$$

and there is a map

$$\mathbb{K}(E) \hookrightarrow \text{Ext}^1(L, L) \oplus \text{Ext}^1(L, F) \oplus \text{Ext}^1(F, L) \oplus \text{Ext}^1(F, F).$$

Since $\text{Ext}^1(L, L) = H^1(\mathcal{O})$ the space of $\mathfrak{g}_L$ deformations of $E$ is

$$\text{Ext}^1(L, F) \oplus \text{Ext}^1(F, L) \oplus \text{Ext}^1(F, F).$$

Considering only $\mathfrak{g}_L$ deformations, we may write the map as

$$\mathbb{K}(E) \hookrightarrow T_s \oplus T_u \oplus T_o$$

where we set the notation:

$$T_s : = \text{Ext}^1(L, F) = H^1(L^{-1} \otimes F),$$

$$T_u : = \text{Ext}^1(F, L)$$

$$T_o : = \text{Ext}^1(F, F) = H^1(\mathcal{O}) \oplus \Gamma \mathcal{E}xt^1(F, F).$$

Note that $T_s$ contains deformations towards stable bundles, whereas both $T_u$ and $T_o$ give directions of unstable deformations. $T_o$ is the direction corresponding to varying the holomorphic structure of $L$ and a lengthy but straightforward calculation shows that its splitting is given by

$$\text{Ext}^1(F, F) = \Gamma \mathcal{E}xt^1(F, F) \oplus H^1(\mathcal{O}).$$

The components of the Kuranishi map are

1) $\text{Ext}^1(F, L) \otimes \text{Ext}^1(F, F) \to \text{Ext}^2(F, L)$

2) $\text{Ext}^1(L, F) \otimes \text{Ext}^1(F, L) \to \text{Ext}^2(F, F) = H^2(\mathcal{O})$

3) $\text{Ext}^1(F, F) \otimes \text{Ext}^1(L, F) \to \text{Ext}^2(L, F) = H^2(L^{-1} \otimes F) = H^2(L^{-2}).$

In this case we have that $c_2 = -mn + l(Q)$ and we proceed to discuss the topology when $c_2$ goes to infinity which may happen if either $-mn$ or $l(Q)$
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... goes to infinity. We will see that following the argument takes us to the same maps and bounds obtained in Sec. 3 in either case. We now study each component of the Kuranishi map.

i) Consider the map \( \text{Ext}^1(F, L) \otimes \text{Ext}^1(F, F) \rightarrow \text{Ext}^2(F, L) \).

We will use the fact \( \text{Ext}^1(F, F) = H^1(\mathcal{O}) \oplus \text{Ext}^1_{\text{loc}}(F, F) \), proved in Lem. 12 of App. B; together with the fact that \( \text{Ext}^1_{\text{loc}}(F, F) = \Gamma \text{Hom}(F, Q) = \text{Hom}(F, Q) \) maps to \( \text{Ext}^1(F, F) \) since we have the short exact sequence

\[ 0 \rightarrow F \rightarrow F^{\vee \vee} \rightarrow Q \rightarrow 0. \]

Taking \( \text{Hom}(., L) \), we obtain

\[ \text{Ext}^1(Q, L) \rightarrow H^1(L^2) \rightarrow \text{Ext}^1(F, L) \rightarrow \text{Ext}^2(Q, L) \rightarrow H^2(L^2) \rightarrow \text{Ext}^2(F, L) \rightarrow 0. \]

The map appearing in i) can be described by the following three components

\[
\begin{align*}
\text{Ext}^2(Q, L) & \otimes \text{Hom}(F, Q) \\
\uparrow & \\
\text{Ext}^1(F, L) & \otimes \text{Ext}^1(F, F) \\
\downarrow & \\
\text{Ext}^1(F^{\vee \vee}, L) & \otimes \text{Ext}^1(F, F^{\vee \vee})
\end{align*}
\]

The diagram commutes, so \( \text{Ext}^1(F, F) \) kills \( \text{im} \text{Ext}^1(F^{\vee \vee}, L) \) and pairs with \( \text{Ext}^2(Q, L) \). We are interested in a neighbourhood of \( [E] \), that is, the class of \( E \) in \( \text{Ext}^1(F, L) \). Consider the image of \( [E] \) inside \( \text{Ext}^2(Q, L) \). Recall that the non-zero fibers of \( Q \) are one-dimensional, because \( Q \) is a quotient of \( \mathcal{O} \). Since \( E \) is a vector bundle, it maps to non-zero values in the fibres of \( \text{Ext}^2(Q, L) \) at all points in the \( \text{supp}(Q) \). Hence, by acting on \( \text{im}[E] \in \text{Ext}^2(Q, L) \) with elements of \( \text{Hom}(Q, Q) \), we obtain any element of \( \text{Ext}^2(Q, L) \). It follows that \( \text{im} \text{Hom}(F, Q) \otimes [E] = \text{Ext}^2(Q, L) \otimes \text{im} \text{Hom}(F, Q) \) in \( \text{Ext}^2(F, L) \), where \( \text{im} \) denotes the respective image by the map depicted in diagram 4. Moreover, as \( [E] \) varies, the zero set of the Kuranishi map projected to \( \text{Ext}^1(F, F^{\vee \vee}) \) forms a vector bundle over its image with fibre \( \text{Ker}(., \otimes [E]) \). In other words, near \( [E] \) the quadratic part of the Kuranishi map comes entirely from the pairing

\[ \text{im} \text{Ext}(L^{-1}, L) \otimes \text{Ext}^1(L^{-1}, L^{-1}) \rightarrow \text{im} \text{Ext}^2(L^{-1}, L) \]

which is the same as the one for the case of a split bundle case, described in Sec. 3, and the same conclusions apply. Hence, we arrive at the same map as in the case i) of Sec. 3:

\[ H^1(\Sigma, L^2) \otimes H^1(\Sigma, \mathcal{O}) \rightarrow H^2(\Sigma, L^2). \]

ii) Considering the map \( \text{Ext}^1(L, F) \otimes \text{Ext}^1(F, L) \rightarrow H^2(\mathcal{O}) \), we see that the target space \( H^2(\mathcal{O}) \) is fixed, so this component imposes the expected bounded number of obstructions, which are negligible for large \( c_2 \).

iii) Finally we consider the map

\[ \text{Ext}^1(F, F) \otimes \text{Ext}^1(L, F) \rightarrow \text{Ext}^2(L, F) = H^2(L^{-2}). \]

We have \( H^2(L^{-2}) = H^1(\Sigma_1, L_1^{-2}) \otimes H^1(\Sigma_2, L_2^{-2}) \) which does not vanish. We now try to remove \( T_0 \). The argument is analogous to the one in the Sec. 3, namely this component gives obstructions to the Kuranishi map, and is
defined on $T_o \otimes T_s$ where $T_s$ contains deformations toward stable bundles. We want to remove $T_o$ from the the Kuranishi space at $E$. The spaces $T_o$ and $T_s$ are determined by the following exact sequences

$$
0 \to \text{Hom}(L, Q) \to T_s \to \text{H}^1(O) \to 0
$$

$$
0 \to \text{H}^1(O) \to T_o \to \Gamma \text{Ext}^1(F, F) \to 0.
$$

**Claim.** $\text{Hom}(L, Q)$ gets killed under the pairing.

**Proof.** First consider the map $\text{Hom}(L, Q) \otimes \text{H}^1(O) \to \text{Ext}^2(L, F)$. Given that $\text{H}^1(O) = \text{Ext}^1(L, L)$ the previous map factors through $\text{Ext}^1(L, Q)$ and the latter is zero because $\text{Ext}^1(L, Q) = 0$ since $L$ is locally free and $\text{H}^1(\text{Hom}(L, Q)) = 0$ because $Q$ is supported at points. There remains the map

$$
\text{Hom}(L, Q) \otimes \Gamma \text{Ext}^1(F, F) \to \text{Ext}^2(L, F).
$$

The fact that this map is zero follows from the following stronger statement.

**Claim.** $T_s = \Gamma \text{Ext}^1(F, F)$ gets killed under the pairing.

**Proof.** We want to show that the map

$$
\text{Ext}^1(L, F) \otimes \Gamma \text{Ext}^1(F, F) \to \text{Ext}^2(L, F)
$$

is zero. But $\Gamma \text{Ext}^1(F, F) \simeq \Gamma \text{Hom}(F, Q) = \text{Hom}(F, Q)$ hence the domain becomes $\text{Ext}^1(L, F) \otimes \text{Hom}(F, Q)$ and the map factors through $\text{Ext}^1(L, Q)$ which is zero, because $\text{Ext}^1(L, Q) = 0$.

The third part of the Kuranishi map in this case gets reduced to the following:

$$
\text{H}^1(O) \otimes \text{H}^1(L^{-2}) \to \text{Ext}^2(L, F) = \text{H}^2(L^{-2}),
$$

but this is the same as part iii in the case when $E$ was an extension of line bundles. The remaining part of the argument proceeds as in Sec. 3, and once again we obtain the same bounds.

In conclusion, we have showed that each component of the Kuranishi map gives the same bounds in this case for the set of unstable points, as we had obtained in Sec. 3, and accordingly, we obtain a result analogous to Prop. 6.

**Proposition 7.** Assume $c_2 > 8g_1g_2$. Then, removing the set of unstable bundles does not change the homology of the Kuranishi space $\mathbb{K}_{c_2}(\Sigma, E)$ in dimension smaller than $c_2$.

**Remark 8.** Note that our result is independent of the choice of polarization. Fix a polarization $\omega = (\alpha, \beta) \in \text{H}^2(\Sigma, \mathbb{Z})$ choose $L$ which is destabilizing for $E$, then if $L := L_1 \boxtimes L_2 \to \Sigma$ is a line bundle of type $(m, n)$, we have

$$
\deg(L) = \alpha m + \beta n \geq 0.
$$

This conclusion holds whether we are in the setting of Sec. 3 or Sec. 4 for any choice of $\omega$.

Note that in Sec. 4 we have studied the neighborhoods of nonfiltrable bundles, which also includes the case of filtrable bundles when $Q = \emptyset$ occurs, and the same bounds apply, since after all, the bounds coincide with those of Sec. 3.
5. THE TOPOLOGY OF THE KURANISHI SPACE

Our main result here concerns a neighborhood of an unstable bundle $E$. Note that since stability is an open condition, our homological statement is trivially true for a stable bundle $E$. Otherwise, assume that $E$ has a destabilizing bundle of degree $(m, n)$. Since here $c_1 = 0$ and $c_2 = -2mn > 0$ one of $m, n$ must be positive. We will assume that $m$ is positive.

Observe that $H_0(\mathbb{K}_{c_2}(\Sigma, E), \mathbb{K}_{c_2}^s(\Sigma, E)) = \mathbb{Z}$ if the deformation space of $E$ contains no stable bundles, and 0 otherwise, because both $\mathbb{K}_{c_2}(\Sigma, E)$ and $\mathbb{K}_{c_2}^s(\Sigma, E)$ are path connected.

**Theorem 9.** Let $\Sigma_1$ and $\Sigma_2$ be Pic-independent smooth projective curves. Let $E$ be bundle on $\Sigma_1 \times \Sigma_2$ with trivial determinant and with second Chern number $c_2 > 8g_1g_2$. Then, for $0 < q < c_2$ we have

$$H_q(\mathbb{K}_{c_2}(\Sigma, E), \mathbb{K}_{c_2}^s(\Sigma, E)) = 0.$$ 

**Proof.** The Kuranishi map around $E$ was studied in Prop. 6 for the case of split bundles and in Prop. 7 for the case of nonfiltrable bundles; in either case it has 3 components which we labelled $\iota, \iota \iota$ and $\iota \iota \iota$.

First, consider the case when $\mathbb{K}_{c_2}(\Sigma, E)$ contains no stable bundles. Then the domains of the components $\iota \iota$ and $\iota \iota \iota$ are both zero, because $T_1 = 0$. So, there remains only component $\iota$ whose target $H^2(\Sigma, L^2) = H^1(\Sigma_1, L^2_1) \otimes H^1(\Sigma_2, L^2_2)$ vanishes whenever $2m = \deg(L^2) > 2g_1 - 2$. Therefore, if $m > g_1 - 1$ the Kuranishi map is identically zero in this case, so that $\mathbb{K}_{c_2}(\Sigma, E)$ is a vector space and the homology statement follows immediately.

Second, consider the case when the deformation space of $E$ contains stable bundles. We use the results of Sec. 3 and 4 to conclude that the codimension of the set of unstable bundles is large and grows with $c_2$. Observe that the Kuranishi space is defined in a neighborhood of a singular point $E$ by at most $H^2(\text{End}_0 E)$ equations. The codimension requires fine estimates, which we carried out separately for each component of the Kuranishi map. In fact, we showed in Prop. 6 and 7 that the difference of codimension and number of local defining equations is bounded below by $c_2$. Then, an application of Eq. (1), combined with the estimate on the codimension of the set of unstable bundles for each component of the Kuranishi map, shows that removing unstable points does not change the homology of the Kuranishi deformation space in degrees $0 < q < c_2$. So that $\mathbb{K}_{c_2}(\Sigma, E)$ and $\mathbb{K}_{c_2}^s(\Sigma, E)$ have the same homology in this range, as we wished to show.

We finish with a comment about the bounds on $c_2$. For the cases when the deformation space contains stable bundles, we have required $m > 2g_1 - 2$ for components $\iota$ and $\iota \iota$ to obtain $H^1(\Sigma_1, L^1_1) = 0$, and then for component $\iota \iota \iota$ we asked for $m$ large enough to make $H^0(\Sigma_1, L^1_1^{-2}) = 0$, which produces the same bound on $m$. Note that the bound on $m$ needed in case the deformation space has only unstable bundles is smaller, so it does not influence the result. Since $c_2 = -2mn$ and the situation is symmetric in $g_1, g_2$, we could instead have required $-n > 2g_2 - 2$. So, the optimal bound to consider is $c_2 = -2mn > 2(2g_1 - 2)(2g_2 - 2)$. Hence, it is enough to require $c_2 > 8g_1g_2$. 

\[\square\]
**Acknowledgements.** E. Ballico was partially supported by MIUR and GN-SAGA of INdAM (Italy). F. Rubilar was supported by Becas Doctorado Nacional Conicyt Folio 21170589. E. Gasparim and F. Rubilar were also supported by the Vicerrectoría de Investigación y Desarrollo Tecnológico (UCN Chile). B. Suzuki was supported by the ANID-FAPESP cooperation 2019/13204-0.

**APPENDIX A. THE UNSTABLE COMPONENT**

In this section we show that an unstable bundle may have an entire Zariski open neighborhood consisting of unstable bundles. This is relevant to our topological estimates, because for such a bundle $E$ the subset $\mathbb{K}^s(\Sigma, E)$ is empty. Alternatively, we may rephrase this in the language of stacks, as follows. The moduli stack of rank 2 vector bundles on a projective surface may have entire components of large dimension consisting only of unstable bundles. Hence, the stack of all bundles may be reducible for all (arbitrarily large) values of $c_2$, even though for large $c_2$ stable bundles will be contained in a definite irreducible component, by results of Gieseker–Li \cite{GL} and O’Grady \cite{OG}. For basic properties of stacks see \cite{Go}.

Let $X$ be a smooth and connected projective surface. Fix an ample line bundle $H$ on $X$, a line bundle $R$ on $X$ (it will give our $c_1(E)$) and a line bundle $L$ on $X$ such that $2L \cdot H > R \cdot H$ and $h^0(K_X \otimes R \otimes (L^{\otimes 2})^\vee) = 0$. Note that for any fixed $H, R$ these conditions are satisfied if we take $L = H^{\otimes t}$ with $t$ sufficiently large. Fix $c_2 \in \mathbb{Z}$ such that $c_2 \geq -L^2 + L \cdot R$. Let $A(R, L, c_2)$ be the set of all isomorphism classes of rank 2 bundles $E$ which fit in an exact sequence

$$0 \to L \to E \to I_Q \otimes R \otimes L^\vee \to 0$$

with $Q$ a finite subset of $X$ with $\mathfrak{g}(Q) = c_2 + L^2 - L \cdot R$. Since $2L \cdot H > R \cdot H$, each $E \in A(R, L, c_2)$ is slope $H$-unstable and (5) is its unique destabilizing filtration. In particular no two non-proportional extensions (5), not even when associated to different sets $Q, Q'$, give isomorphic bundles.

**Remark 10.** Take $c_2 = -L^2 + L \cdot R$. In this case $A(R, L, c_2)$ is the set of all isomorphism classes of extensions of $R \otimes L^\vee$ by $L$. If $h^1((L^{\otimes 2})^\vee \otimes R) = 0$, then $A(R, L, c_2) = \{ L \otimes R \otimes L^\vee \}$.

**Remark 11.** Assume $c_2 + L^2 - R \cdot L > 0$, then for any $Q \subset X$ with $\mathfrak{g}(Q) = c_2 + L^2 - L \cdot R$, the set of all locally free $E$ fitting into (5) is isomorphic to a non-empty Zariski open subset of a projective space of positive dimension because any non-empty finite subset $A$ of $X$ has the Cayley–Bacharach property with respect to $R \otimes (L^{\otimes 2})^\vee$ by our choice of $L$. Since the set of all subsets of $X$ with cardinality $c_2 + L^2 - L \cdot R$ is a variety of dimension $2(c_2 + L^2 - L \cdot R)$ and each $E \in A(R, L, c_2)$ fits in a unique extension (5), up to a scalar multiple, we get $\dim(A(R, L, c_2)) \geq 2(c_2 + L^2 - L \cdot R)$. Note that $Q$ is supported on zero dimensional scheme.

Now we reverse the data, i.e. we assume only that $c_1(R) = c_1(E)$ and $c_2 \in \mathbb{Z}$. We fix an integer $a > 0$. Fix an integer $t > 0$ such that $h^0(K_X \otimes R \otimes (H^{\otimes t})^\vee) = 0$ and $-t^2 H^2 + tH \cdot R \geq c_2 + a$. Take $L := H^{\otimes t}$. The set $A(R, L, c_2)$ has dimension at least $2a$. 

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Appendix B. Some homological algebra

The results of this appendix are used in Sec. 4.

**Lemma 12.** Given that $L$ is a line bundle and $Q$ is a zero dimensional sheaf such $0 \to F \to L^{-1} \to Q \to 0$, there exists a canonical splitting

$$\text{Ext}^1(F, F) = \Gamma(\text{Ext}^1(F, F)) \oplus \mathbb{H}^1(O).$$

**Proof.** Since $Q$ is supported at points, $\text{Hom}(Q, L^{-1}) = \text{Ext}^1(Q, L^{-1}) = 0$. Applying $\mathcal{H}om(Q, \cdot)$ to the short exact sequence

$$0 \to F \to L^{-1} \to Q \to 0,$$

we obtain

$$0 \to \mathcal{H}om(Q, Q) \to \text{Ext}^1(Q, F) \to 0$$

and

$$0 \to \text{Ext}^1(Q, Q) \to \text{Ext}^2(Q, F) \to \text{Ext}^2(Q, L^{-1}) \to \text{Ext}^2(Q, Q) \to 0.$$

The first part implies $\text{Ext}^1(Q, F) = \text{Hom}(Q, Q) = \mathbb{C}$. For the second part, the last map is an isomorphism, so the penultimate map is zero and it follows that $\text{Ext}^2(Q, F) \cong \text{Ext}^1(Q, Q)$. Consequently,

$$\text{Ext}^2(Q, F) = \text{Ext}^1(Q, Q) = \mathbb{C}^2.$$

Since $\text{Ext}^1(L^{-1}, F) = \text{Ext}^2(L^{-1}, F) = \mathcal{H}om(Q, F) = 0$, applying $\mathcal{H}om(\cdot, F)$ to the short exact sequence (6) we obtain

$$0 \to \text{Ext}^1(F, F) \to \text{Ext}^2(Q, F) \to 0,$$

and combining with the result of (7) gives

$$\text{Ext}^1(F, F) \cong \text{Ext}^1(Q, Q) = \mathbb{C}^2. \quad (8)$$

On the other hand, given that $\text{Ext}^1(F, F) = \text{Ext}^2(Q, F)$ and $\mathcal{H}om(F, Q) = \text{Ext}^1(Q, Q)$ the diagram

$$
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
\mathcal{H}om(F, Q) & \text{Ext}^1(F, F) & \text{Ext}^1(F, L^{-1}) \\
\| & \| & \|
\end{array}
\begin{array}{ccc}
\text{Ext}^1(Q, Q) & \cong & \text{Ext}^2(Q, F) \\
\downarrow & 0 & \downarrow \\
\text{Ext}^2(Q, L^{-1}) & \text{Ext}^2(Q, Q) & 0 \\
\end{array}
$$

gives

$$\mathcal{H}om(F, Q) \cong \text{Ext}^1(F, F).$$

Thus,

$$\text{Hom}(F, Q) = \Gamma \text{Ext}^1(F, F). \quad (9)$$

The results of (8) and (9) are then plugged into the left lower corner of the following diagram

$$
\begin{array}{cccc}
\text{Hom}(Q, Q) & \to & \text{Ext}^1(Q, F) & \to & 0 & \to & \text{Ext}^1(Q, Q) \\
\| & \downarrow & \downarrow & & \downarrow & \downarrow & \| \\
\text{Hom}(L^{-1}, Q) & \to & \text{Ext}^1(L^{-1}, F) & \to & \text{Ext}^1(L^{-1}, L^{-1}) & \to & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \| \\
\text{Hom}(F, Q) & \to & \text{Ext}^1(F, F) & \to & \text{Ext}^1(F, L^{-1}) & \to & \text{Ext}^1(F, Q) \\
\| & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \| \\
\text{Ext}^1(Q, Q) & \cong & \text{Ext}^2(Q, F) & \to & \text{Ext}^2(Q, L^{-1}) & \cong & \text{Ext}^2(Q, Q)
\end{array}
$$
and a little diagram chase together with (9) yields
\[ \text{Ext}^1(F, F) = \text{Ext}^1(L^{-1}, L^{-1}) \oplus \text{Hom}(F, Q) = H^1(O) \oplus \Gamma(\text{Ext}^1(F, F)). \]

\[ \square \]

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