Global higher integrability for minimisers of convex functionals with \((p,q)\)-growth

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Functionals with \((p,q)\)-growth

\[
\min_{u \in W^{1,p}_{g}(\Omega, \mathbb{R}^m)} \mathcal{F}(u) \quad \text{where} \quad \mathcal{F}(u) = \int_{\Omega} F(x, Du) \, dx.
\]

Here \(F(x, z)\) is a convex functional with controlled \((p, q)\)-growth in \(z\) satisfying a natural uniform \(\alpha\)-Hölder condition in \(x\).
Functionals with \((p,q)\)-growth

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\min_{u \in \mathcal{W}_{g}^{1,p}(\Omega, \mathbb{R}^{m})} \mathcal{F}(u) \quad \text{where} \quad \mathcal{F}(u) = \int_{\Omega} F(x, Du) \, dx.
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Here \(F(x, z)\) is a convex functional with controlled \((p, q)\)-growth in \(z\) satisfying a natural uniform \(\alpha\)-Hölder condition in \(x\).

**Goal:** Obtain conditions on the gap \(q - p\) that guarantee \(\mathcal{W}^{1,q}(\Omega)\)-regularity of minimisers of \(\mathcal{F}(\cdot)\).
Related results in the local autonomous case

many results due to Acerbi, Baroni, Colombo, Diening, Fusco, Giaquinta, Harjulehto, Hästö, Marcellini, Mingione, Ruzicka, . . . .
In particular

Carozza, Kristensen, Passarelli di Napoli ('13): natural growth conditions, \( q < \frac{np}{n-1} \)

Schäffner ('19): controlled growth conditions, \( q < p \left(1 + \frac{2}{n-1}\right) \)

de Filippis, K., Kristensen ('20): controlled duality growth conditions, \( q < \frac{np}{n-2} \)
Growth conditions

- natural growth: $|z|^p \lesssim F(x, z) \lesssim |z|^q$
- controlled growth: $|z|^{p-2} |\xi|^2 \lesssim F_{zz}(x, z) \xi \cdot \xi \lesssim |z|^{q-2} |\xi|^2$
- controlled duality growth:
  
  $|z|^{p-2} |\xi|^2 \lesssim F_{zz}(x, z) \xi \cdot \xi \lesssim |\partial_z F(x, z)|^{\frac{q-2}{q-1}} |\xi|^2$
Related results in the local non-autonomous case

**Esposito, Leonetti, Mingione (’04):** $q \leq \frac{(n+\alpha)p}{n}$ is necessary condition, see also Fonseca, Malý, Mingione (’04), Balci, Diening, Surnachev (’20)

**Esposito, Leonetti, Mingione (’04):** controlled growth, $q < \frac{(n+\alpha)p}{n}$ for many examples including double-phase, $p(x)$

**Esposito, Leonetti, Petricca (’19):** extension to functionals satisfying additional assumption on the $x$-dependence

few global results both in the autonomous and non-autonomous case (Byun, Oh (’17), Bulíček, Maringová, Stroffolini, Verde (’18))
Growth conditions

Let $0 < \alpha \leq 1$. $F(x, z)$ is measurable in $x$, continuously differentiable in $z$ and moreover satisfies

$$\nu (\mu^2 + |z|^2 + |w|^2)^{\frac{p-2}{2}} \leq \frac{F(x, z) - F(x, w) - \langle \partial_z F(x, w), z - w \rangle}{|z - w|^2}$$

(H1)

$$|F(x, z)| \lesssim (1 + |z|^2)^{\frac{q}{2}}$$

(H2)

$$|F(x, z) - F(y, z)| \leq \Lambda |x - y|^\alpha \left(1 + |z|^2\right)^{\frac{q}{2}}.$$  

(H3)

for some $\mu, \nu, \Lambda > 0$, all $z, w \in \mathbb{R}^{n \times m}$ and almost every $x, y \in \Omega$, where $1 \leq p \leq q$. 
An additional assumption on the $x$-dependence

There is $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $x \in \Omega$ there is $\hat{y} \in \overline{B_\varepsilon(x)} \cap \Omega$ such that

$$F(\hat{y}, z) \leq F(y, z) \quad \forall y \in \overline{B_\varepsilon(x)} \cap \Omega, \quad z \in \mathbb{R}^{n \times m}. \quad (H4)$$

previously considered by Esposito, Leonetti, Petricca (’19) and similar to conditions of Zhikov (’95)
Examples

(I) $\mathcal{F}_3(u) = \int_{\Omega} |Du|^p + a(x)|Du|^q \, dx$ where $0 \leq a(x) \in C^{0,\alpha}(\Omega)$,

(II) $\mathcal{F}_3(u) = \int_{\Omega} |Du|^{p(x)} \, dx$ where $p \leq p(x) \leq q$,

(III) $\mathcal{F}_4(u) = \int_{\Omega} |Du|^p + |a_{\alpha,\beta}(x)D_i u^\alpha D_j u^\beta|^q \, dx$ where $a_{\alpha,\beta}(\cdot) \in C^{0,\alpha}(\Omega)$ and for all $x \in \Omega$ and $\xi \in \mathbb{R}^{n \times m}$,

(IV) $\mathcal{F}_8(u) = \int_{\Omega} F(x, Du) \, dx$ where $F(x, z) = h(a(x), z)$ where

(i) $t \to h(t, z)$ is increasing

(ii) $h(x, z)$ is convex in the second argument

(iii) $a(x) \in C(\overline{\Omega})$

(iv) $F(x, z)$ satisfies (H1)-(H3)
Pointwise minimisers

\[ u \in W_{g}^{1,1}(\Omega) \] is a (pointwise) minimiser of \( F(\cdot) \) in the class \( W_{g}^{1,p}(\Omega) \) if it holds that \( F(x, Du) \in L^{1}(\Omega) \) and

\[ \int_{\Omega} F(x, Du) \, dx \leq \int_{\Omega} F(x, Du + D\phi) \, dx \]

for all \( \phi \in W_{0}^{1,1}(\Omega) \).
Relaxed minimisers

\( u \in W_g^{1,p}(\Omega) \) is a \( W^{1,q} \)-relaxed minimiser of \( \mathcal{F}(\cdot) \) in the class \( W_g^{1,p}(\Omega) \) if \( u \) minimises the relaxed functional

\[
\overline{\mathcal{F}}(v) = \inf_{(v_j) \subset Y} \left\{ \liminf_{j \to \infty} \int_{\Omega} F(Dv_j) : v_j \rightharpoonup v \text{ weakly in } X \right\}
\]

amongst all \( v \in X = W_g^{1,p}(\Omega) \) where \( Y = W_g^{1,q}(\Omega) \).

Serrin (’61), Marcellini (’86), Zhikov (’95), Buttazzo, Belloni (’95), Fonseca, Malý (’97), Foss (’01), Acerbi, Bouchitté, Fonseca (’03), Schmidt (’08), …
Lavrentiev gap functional

Consider a topological space $X$ of weakly differentiable functions and a dense subspace $Y \subset X$.

$$
\bar{F}_X = \sup \{ G : X \rightarrow [0, \infty] : G \text{ slsc, } G \leq F \text{ on } X \} \\
\bar{F}_Y = \sup \{ G : X \rightarrow [0, \infty] : G \text{ slsc, } G \leq F \text{ on } Y \}
$$

Define the Lavrentiev gap functional for $u \in X$ as

$$
\mathcal{L}(u, X, Y) = \begin{cases} 
\bar{F}_Y(u) - \bar{F}_X(u) & \text{if } \bar{F}_X(u) < \infty \\
0 & \text{else}
\end{cases}
$$

In this talk: $X = W^{1,p}_g(\Omega), \ Y = W^{1,q}_g(\Omega)$. 

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A regularity result

**Theorem (L.K. ’20)**

Suppose \( \Omega \) is a Lipschitz domain. Let \( g \in W^{1+\alpha,q}(\Omega) \). Suppose \( F(x,\cdot) \) satisfies (H1)-(H3) with \( 1 < p \leq q < \frac{(n+\alpha)p}{n} \). If \( u \) is a relaxed minimiser of \( \mathcal{F}(\cdot) \) in the class \( W^{1,p}_g(\Omega) \), then \( u \in W^{1,q}(\Omega) \). Moreover for any \( 0 \leq \beta < \alpha \) there is \( \gamma > 0 \) such that

\[
\|u\|_{W^{1,\frac{np}{n-\beta}}(\Omega)} \lesssim \left(1 + \mathcal{F}(u) + \|g\|_{W^{1+\alpha,q}(\Omega)}\right)^\gamma. \tag{1}
\]

Suppose that in fact \( \Omega \) is a \( C^{1,\alpha} \)-domain and \( g \in W^{1+\max(\alpha,\frac{1}{q}),q}(\Omega) \). Suppose \( F(x,\cdot) \) satisfies in addition (H4) and let \( u \) be a pointwise minimiser of \( \mathcal{F}(\cdot) \) in the class \( W^{1,p}_g(\Omega) \). Then \( u \in W^{1,q}(\Omega) \) and the estimate (1) holds.
Outline of the proof: an a-priori estimate

\[ \mathcal{F}_\varepsilon(u) = \int_\Omega F(x, Du) + \varepsilon |Du|^q \, dx \]

**Lemma (L.K. '20)**

Suppose \( \Omega \) is a Lipschitz domain. Let \( 1 \leq p \leq q < \frac{(n+\alpha)p}{n} \),
\( g \in W^{1+\alpha, q}(\Omega) \). Let \( v_\varepsilon \) be the minimiser of \( \mathcal{F}_\varepsilon(\cdot) \) in the class \( W^{1, q}_g(\Omega) \). Then for any \( 0 \leq \beta < \alpha \) there is \( \gamma > 0 \) such that the estimate

\[ \| v_\varepsilon \|_{W^{1, \frac{np}{n-\beta}}(\Omega)} \lesssim \left( 1 + \mathcal{F}_\varepsilon(v_\varepsilon) + \| g \|_{W^{1+\alpha, q}(\Omega)} \right)^\gamma \]

holds, with the implicit constant independent of \( \varepsilon \) and \( \gamma \).
\( \mathcal{F}_\varepsilon(\cdot) \) and the relaxed functional

**Lemma (Marcellini (’89))**

Let \( g \in W^{1,q}(\Omega) \). Suppose \( F(x, z) \) satisfies (H1) and (H2). Suppose \( u \) is a relaxed minimiser of \( \mathcal{F}(\cdot) \) in the class \( W^{1,p}_g(\Omega) \) and \( u_\varepsilon \) is the pointwise minimiser of \( \mathcal{F}_\varepsilon(\cdot) \) in the class \( W^{1,q}_g(\Omega) \). Then \( \mathcal{F}_\varepsilon(u_\varepsilon) \to \mathcal{F}(u) \) as \( \varepsilon \to 0 \). Moreover up to passing to a subsequence \( u_\varepsilon \to u \) in \( W^{1,p}(\Omega) \).

allows to pass to the limit in the a-priori estimate for relaxed minimisers.
A remark on Besov-spaces

\[ B_{q, p}^s(\Omega) = B_{q, p}^s(\Omega, \mathbb{R}^m) = [W^{1, p}(\Omega, \mathbb{R}^m), L^p(\Omega, \mathbb{R}^m)]_{s, q} \]

Let \( D \) be a set generating \( \mathbb{R}^n \), star-shaped with respect to 0. For \( s \in (0, 1), \ p \in [1, \infty] \), consider

\[ [v]_{s, p, \Omega}^p = \sup_{h \in D \setminus \{0\}} \int_{\Omega_h} \left( \frac{|v_h(x) - v(x)|}{|h|^s} \right)^p \, dx. \]

\[ C_1 \| v \|_{B_{\infty, p}^s(\Omega)} \leq \| v \|_{L^p(\Omega)} + [v]_{s, p, \Omega} \leq C_2 \| v \|_{B_{\infty, p}^s(\Omega)}. \]

When \( D = C_\rho(\theta, n) \) is a cone, \( C_1, C_2 \) are independent of the choice of \( n \).
As $\Omega$ is Lipschitz, it satisfies the **uniform cone property**: there are $\rho_0, \theta_0 > 0$ and a map $\mathbf{n}: \mathbb{R}^n \to S^{n-1}$ such that for every $x \in \mathbb{R}^n$

$$C_{\rho_0}(\theta_0, \mathbf{n}(x)) \subset \{ h \in \mathbb{R}^n : |h| \leq \rho_0, (B_{3\rho_0}(x) \setminus \Omega) + h \subset \mathbb{R}^n \setminus \Omega \}.$$

Based on an argument by Savaré introduce

$$T_h v = \phi v_h + (1 - \phi) v.$$
Suppose $u_\varepsilon = v_\varepsilon + g$ is a minimiser of $\mathcal{F}_\varepsilon(\cdot)$. Consider

$$\sup_{h \in C_{\rho_0}(\theta_0, n(x_0))} \frac{\mathcal{F}_\varepsilon(T_h \tilde{v}_\varepsilon + g) - \mathcal{F}_\varepsilon(v_\varepsilon + g)}{|h|^{\alpha}}.$$
Suppose \( u_\varepsilon = v_\varepsilon + g \) is a minimiser of \( \mathcal{F}_\varepsilon(\cdot) \). Consider

\[
\sup_{h \in C_{\rho_0}(\theta_0, n(x_0))} \frac{\mathcal{F}_\varepsilon(T_h \tilde{v}_\varepsilon + g) - \mathcal{F}_\varepsilon(v_\varepsilon + g)}{|h|^{\alpha}}.
\]

lower bound: use the Euler-Lagrange equation for \( v_\varepsilon \)
Assume $g = 0$. Set $\Delta_h \nu = \nu - \nu_h$.

\[
\begin{align*}
\mathcal{F}_\varepsilon(T_h \tilde{\nu}_\varepsilon) - F_\varepsilon(\nu_\varepsilon) &= \int_{\Omega} F_\varepsilon(x, T_h D\tilde{\nu} + D\phi(\tilde{\nu}_h - \nu)) - F_\varepsilon(x, T_h D\tilde{\nu}) \, dx \\
&\quad + \int_{\Omega} F_\varepsilon(x, T_h D\tilde{\nu}) - F_\varepsilon(x, D\tilde{\nu}) \, dx = A_1 + A_2.
\end{align*}
\]

\[
|A_1| \leq (\Lambda + \varepsilon) \int_{\Omega} |D\phi \Delta_h \tilde{\nu}| (1 + |T_h D\tilde{\nu}|^2 + |D\phi \Delta_h \tilde{\nu}|^2)^{\frac{q-1}{2}} \, dx
\]
\[
\lesssim |h| (1 + \|D\nu\|_{L^q(\Omega)}^q + \|g\|_{W^{1+\alpha,q}(\Omega)}^q).
\]
\[ |A_2| \leq \int_{B_{2\rho_0}(x_0)} \phi(F_\varepsilon(x, D\tilde{v}_h) - F_\varepsilon(x, D\tilde{v})) \, dx \]

\[ = \int_{B_{3\rho_0}(x_0)} \phi(x - h)F_\varepsilon(x - h, D\tilde{v}) - \phi(x)F_\varepsilon(x - h, D\tilde{v}) \, dx \]

\[ + \int_{B_{2\rho_0}(x_0)} \phi(x)(F_\varepsilon(x - h, D\tilde{v}) - F_\varepsilon(x, D\tilde{v})) \, dx \]

\[ \lesssim |h|^\alpha \left( 1 + \|Dv\|_{L^q(\Omega)}^q + \|g\|_{W^{1+\alpha, q}(\Omega)}^q \right). \]
A family of cubes $\{ K_i \}_{i \in I}$ is called a Whitney-Besicovitch-covering (WB-covering) of $\Omega$ if there is a triple $(\delta, M, \varepsilon)$ of positive numbers such that

$$\bigcup_{i \in I} \frac{1}{1 + \delta} K_i = \bigcup_{i \in I} K_i = \Omega$$

$$\sum_{i \in I} \chi_{K_i} \leq M$$

$$K_i \cap K_j \neq \emptyset \Rightarrow |K_i \cap K_j| \geq \varepsilon \max(|K_i|, |K_j|).$$

**Kislyakov, Kruglyak (’05):** if $\Omega$ has non-empty complement a WB-covering exists. Moreover there is a partition of unity $\{\psi_i\}$ adapted to this cover.
Outline of the proof: constructing approximating sequences

Lemma (L.K. '20)

Let $\Omega$ be a domain. Suppose $1 < p \leq q < \frac{(n+\alpha)p}{n}$. Given $u \in W^{1,p}(\Omega)$, $\varepsilon \in (0, 1/(1 + 1/6)^n)$ and $m \geq 1$ write

$$u_{\varepsilon} = \sum_{i \in I} u \ast \phi_{\varepsilon \delta_i} \psi_i$$

with $\delta_i = |K_i|^m/n$.

Assume that $F(x, \cdot)$ satisfies (H1)-(H3) and (H4). Then if $m$ is sufficiently large, up to passing to a subsequence if necessary, $u_{\varepsilon} \in W^{1,p}_{u}(\Omega) \cap W^{1,q}_{\text{loc}}(\Omega)$, $u_{\varepsilon} \rightarrow u$ in $W^{1,p}(\Omega)$ and

$$\int_{\Omega} F(x, Du_{\varepsilon}) \, dx \rightarrow \int_{\Omega} F(x, Du) \, dx$$

as $\varepsilon \searrow 0$. 
Some details of the proof of Lemma 4 I

\[ Du_\varepsilon = \sum_{i \in I} Du \star \phi_\varepsilon \delta_i \psi_i + \sum_{i \in I} u \star \phi_\varepsilon \delta_i \otimes D \psi_i = A_1 + A_2. \]
Some details of the proof of Lemma 4.1

\[ Du_\varepsilon = \sum_{i \in I} Du \ast \phi_{\varepsilon \delta_i} \psi_i + \sum_{i \in I} u \ast \phi_{\varepsilon \delta_i} \otimes D\psi_i = A_1 + A_2. \]

\[ \int_{\Omega} F(x, Du_\varepsilon) \, dx \lesssim \int_{\Omega} F(x, A_1) \, dx + \int_{\Omega} |A_2|(1 + |A_1|^{q-1} + |A_2|^{q-1}) \]

\[ \lesssim \sum_{i \in I} \int_{\Omega} F(x, Du \ast \phi_{\varepsilon \delta_i}) \psi_i \, dx + \int_{\Omega} |A_2|(1 + |A_1|^{q-1} + |A_2|^{q-1}) \]
Some details of the proof of Lemma 4 II

Set $G_\varepsilon(x, z) = \min_{y \in B_\varepsilon(x) \cap \Omega} F(y, z)$. Let $|z| \leq C\varepsilon^{-\frac{n}{p}}$ and $\varepsilon \in (0, \varepsilon_0)$.

$$G_\varepsilon(x, z) \geq \delta F(x, z) - \delta \Lambda(1 + |z|^2)^{\frac{q}{2}} + (1 - \delta)|z|^p$$

$$\geq \delta F(x, z) - \delta \Lambda C^{q-p} \varepsilon^{\alpha + \frac{n(p-q)}{p}} |z|^p + (1 - \delta)|z|^p - \delta(\Lambda \varepsilon_0^{\alpha} + 1)$$

$$\geq \delta F(x, z) + (1 - \delta - \Lambda C^{q-p} \varepsilon_0^{\alpha + \frac{n(p-q)}{p}}) |z|^p - \Lambda (\varepsilon_0^{\alpha} + 1).$$

$$G_\varepsilon(x, Du \star \phi_\varepsilon) = F(\hat{y}, Du \star \phi_\varepsilon) \leq \int_{B_1} F(\hat{y}, Du(y))\phi_\varepsilon(x - y) \, dy$$

$$\leq (F(\cdot, Du(\cdot)) \star \phi_\varepsilon)(x)$$
Some details of the proof of Lemma 4 III

Set $\Theta = 1 + n \left( \frac{1}{q} - \frac{1}{p} \right)$ if $p < n$, $\Theta = \frac{n}{q}$ if $p \geq n$.

$$\int_{\Omega} |A_1|^{q-1} |A_2| \, dx$$

$$\lesssim C^q \sum_{j \in I} (\varepsilon \| D\psi_j \|_{L^q(\Omega_j)})^{-\frac{n}{p} \left( 1 - \frac{p}{q} \right)} \sum_{l_j} \varepsilon^{\Theta} \| D\psi_i \|_{L^\infty(\Omega_i)} 1_{\psi_i} \|_{L^q(\Omega)}$$

$$\lesssim C^q \varepsilon^\tau \sum_{i \in I} \| D\psi_i \|_{L^\infty(\Omega_i)} 1_{\psi_i} \|_{L^q(\Omega)}$$

with $\tau = \Theta - \frac{n(q-1)}{p} \left( 1 - \frac{p}{q} \right) > 0$
Outline of the proof: the special case of the ball

Let $\frac{1}{2} < s < 1$ and assume $F(x, z) = F(z)$.

$$u^s(x) = \begin{cases} su(x/s) & \text{in } sB_1 \\ |x|g(x/|x|) & \text{in } B_1 \setminus sB_1. \end{cases}$$

Then $u^s \to u$ in $W^{1,p}(B_1)$, $u^s \in W^{1,q}$ near $\partial B_1$, $u^s = g$ on $\partial B_1$ and

$$\mathcal{F}(u^s) \to \mathcal{F}(u) \quad \text{as } s \to 1.$$
Outline of the proof: Regularising near the boundary

\[
F^s(x, z) = F \left( \psi^{-1}_s(x), zD\psi_s(\psi^{-1}_s(x)) \right)
\]

\[
\mathcal{F}^s(u) = \int_{\Omega} F^s(x, Du) \, dx
\]

\[
u^s(x) = u(\psi^{-1}_s(x)).
\]
Proposition (L.K. ’20)

Suppose $\Omega$ is a Lipschitz domain and $g \in W^{1+\frac{1}{q},q}(\Omega)$. Let $X = W^{1,p}_g(\Omega)$ endowed with the weak topology. Suppose that $1 < p \leq q < \frac{(n+\alpha)p}{n}$ and that $F(\cdot, \cdot)$ satisfies (H1)-(H3) and (H4). Then with the choice $Y = W^{1,q}_{\text{loc}}(\Omega)$,

$$\mathcal{L}(\cdot, X, Y) = 0 \text{ on } X.$$ 

If in fact $1 < p \leq q < \min(p + 1, (1 + \frac{\alpha}{(\alpha+1)n})p)$, then $\mathcal{L}(\cdot, X, Y) = 0$ on $X$ with the choice $Y = W^{1,q}_g(\Omega)$. 
Idea: Decompose $\Omega$ into star-shaped Lipschitz domains and apply the regularisation from the case of the ball.

\[
\int_{\Omega} F(x, Du_{\varepsilon}) \, dx
\]

\[
= \int_{\Omega} F(x/s, Du_{\varepsilon}) + \int_{\Omega} F(x, Du_{\varepsilon}) - F(x/s, Du_{\varepsilon}) \, dx.
\]

Now change coordinates in the first term and estimate the second term similar to before.
Thanks for your attention!