Quantum Fisher Information of Entangled Coherent States in a Lossy Mach-Zehnder Interferometer

Xiaoxing Jing, Jing Liu, Wei Zhong, and Xiaoguang Wang
Zhejiang Institute of Modern Physics, Department of Physics, Zhejiang University, Hangzhou 310027, China
E-mail: xgwang@zimp.zju.edu.cn

Abstract. We give an analytical result for the quantum Fisher information of entangled coherent States in a lossy Mach-Zehnder Interferometer recently proposed by J. Joo et al. [Phys. Rev. Lett. 107, 083601(2011)]. For small loss of photons, we find that the entangled coherent state can surpass the Heisenberg limit. Furthermore, The formalism developed here is applicable to the study of phase sensitivity of multipartite entangled coherent states.

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1. INTRODUCTION

Precision measurements are important across all fields of science and technology. By employing quantum features like entanglement and squeezing, quantum metrology promises enhancing precision and has drawn a lot of attention in the last decade [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. Quantum metrology deals with the ultimate precision limits in estimation procedures, taking into account the constraints imposed by quantum mechanics, and allows one to gain advantages over purely classical approaches [1, 2, 3, 4]. As a key component of the quantum metrology theory, quantum parameter estimation has many applications in experiments, such as the detection of gravitational radiation [12, 13], quantum frequency standards [15, 14, 16], clock synchronization [17, 18], to name a few.

Quantum Fisher information (QFI) is another significant concept in quantum metrology and has been studied widely [20, 21, 22, 23, 24, 25, 26, 19, 27, 28, 29]. As an extension of the classical Fisher information in statistics and information theory, QFI plays a paramount role in quantum estimation theory. In quantum metrology theory, these two concepts are linked by the quantum Cramér-Rao inequality [21, 22],

$$\text{var}(\phi) \geq \frac{1}{\nu F},$$

where \(\text{var}(\phi)\) is the variance of an unbiased estimator \(\phi\) of a parameter \(\varphi\), \(\nu\) represents the number of repeated experiments and \(F\) is the QFI of the parameter. The inverse of the QFI provides the lower bound of the error of the estimation.

In this paper, we consider a fundamental parameter estimation task in which the parameter \(\varphi\) is generated by some unitary dynamics \(U = \exp(-i\varphi H)\). This kind of parameter estimation task is common in many experimental setups such as Mach-Zehnder interferometers and Ramsey interferometers. Based on a recent expression of QFI [31], we show that the QFI of \(\varphi\) for a unitary parameterized dynamics is the mean variance of \(H\) over the eigenstates minus the transition terms of \(H\). Next we take a two dimension case as our interest. The eigenvalues and eigenstates of a general \(2 \times 2\) density matrix have been given in terms of its determinant, difference between diagonal elements and phase of off-diagonal elements. For integrity we also give the eigenvalues and eigenstates for a density operator on a nonorthogonal basis of two dimensions.

While exact results and analytical solutions are known for noiseless situations, the determination of the ultimate precision limit in the presence of noise is still a challenging problem in quantum mechanics. Recently, J. Joo et al. studied the entangled coherent states in a Mach-Zehnder interferometer under perfect and lossy conditions [5]. They found the entangled coherent states (ECS) can reach better precision in comparison to N00N, “bat”, and “optimal” states in both conditions. In lossy conditions, they modeled the particle loss by fictitious beam splitters and adopted a numeric strategy to calculate the QFI of the ECS. Utilizing our formula we give an analytic expression of the QFI. We find that even in a lossy condition, the ECS can still surpass the Heisenberg limit.

This paper is organized as follows. In Sec. II, we give a brief review of the QFI
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and obtain an explicit formula of the QFI for a family of density matrices parameterized through a unitary dynamics. In Sec. III, we give the eigenvalues and eigenstates of a 2-dimensional density matrix in terms of its determinant, difference between diagonal elements and phase of off-diagonal elements. We also generalize the eigen problem in a nonorthogonal basis. Afterward, in Sec. IV, we apply our result to the ECS in a lossy Mach-Zehnder interferometer and get an analytical expression of the QFI. Finally, the conclusion is given in Sec. V.

2. QFI AND PARAMETER ESTIMATION FOR UNITARY DYNAMICS

2.1. Brief Review of Quantum Fisher Information

In this section, we briefly review the calculation of the QFI. For a parameterized quantum states \( \rho_\varphi \), a widely used version of QFI \( F_\varphi \) is defined as \cite{21,22}

\[
F_\varphi := \text{tr}( \rho_\varphi L^2 ),
\]

where the symmetric logarithmic derivative (SLD) operator \( L \) is determined by

\[
\partial_\varphi \rho_\varphi = \frac{1}{2} [ L \rho_\varphi + \rho_\varphi L ].
\]

Consider a density operator \( \rho_\varphi \) on a \( N \)-dimensional system (\( N \) can be infinite). The corresponding spectrum decomposition is given by

\[
\rho_\varphi = \sum_{i=1}^{M} p_i |\psi_i\rangle \langle \psi_i |,
\]

where \( p_i \) is the eigenvalue and \( |\psi_i\rangle \) is the eigenstate, and \( M \leq N \), implying that there are \( N - M \) zero eigenvalues. With the decomposition of the density matrix one can directly obtain the element of the SLD operator from Eq. (3) as

\[
\langle \psi_k | L | \psi_l \rangle = \frac{2 \langle \psi_k | \partial_\varphi \rho_\varphi | \psi_l \rangle}{p_l + p_k}.
\]

Notice that the matrix element of \( L \) is not defined when \( p_l + p_k = 0 \).

It turns out that the QFI is completely determined in the support of \( \rho_\varphi \), that is, the space spanned by those eigenvectors corresponding to the nonvanishing eigenvalues. It can be expressed as \cite{31}

\[
F_\varphi = \sum_{i=1}^{M} \frac{1}{p_i} (\partial_\varphi p_i)^2 + \sum_{i=1}^{M} 4 p_i \langle \partial_\varphi |\psi_i\rangle \langle \psi_i | \partial_\varphi \psi_i \rangle
\]

\[
- \sum_{i=1}^{M} \sum_{j=1}^{M} \frac{8 p_i p_j}{(p_i + p_j)} |\langle \psi_i | \partial_\varphi \psi_j \rangle|^2.
\]

For the special case of a pure state (\( M = 1 \)), Eq. (6) reduces to

\[
F(\psi_1) = 4[\langle \partial_\varphi \psi_1 | \partial_\varphi \psi_1 \rangle - |\langle \psi_1 | \partial_\varphi \psi_1 \rangle|^2].
\]

Using this form of the QFI for pure states, we can rewrite Eq. (6) as
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\[ F_\varphi = \sum_{i=1}^{M} \frac{1}{p_i} (\partial_\varphi p_i)^2 + \sum_{i=1}^{M} p_i F(\psi_i) \]

\[ - \sum_{i \neq j} \frac{8p_ip_j}{(p_i + p_j)} |\langle \psi_i | \partial_\varphi \psi_j \rangle|^2. \]  

(8)

It is clear that the first term can be regarded as the classical contribution \[30, 31, 22\], and the second term as the mean QFI over the eigenstates. The third term can be regarded as a sum of harmonic mean of transition terms.

There are several similar formulas in the literature where the summation in the last term runs over all the eigenstates, as long as \( p_i + p_j \neq 0 \). Eq. (8) have some advantages over them both in analytical and numerical calculations since \( i, j \) are symmetric and one only need to find the non-vanishing eigenstates of \( \rho_\varphi \).

2.2. QFI for unitary parameterized dynamics

In quantum estimation theory, the most fundamental parameter estimation task is to estimate a small parameter \( \varphi \) generated by some unitary dynamics

\[ U = \exp(-i\varphi H). \]  

(9)

Here \( H \) is a Hermitian operator and can be regarded as the generator of parameter \( \varphi \). This form of parameterization process is typical in interferometers. For instance, in a Ramsey interferometer \( H \) can be a collective angular momentum operator \( J_n \) [27], which can be viewed as a generator of SU(2). In Mach-Zehnder interferometers, denoting \( a_i \) and \( a_i^\dagger \) (i=1,2) as the annihilation and creation operators for \( i \)th mode, then \( H \) can be (1) the photon number difference between two modes: \( a_1^\dagger a_1 - a_2^\dagger a_2 \) [32], (2) the number operator in one mode: \( a_2^\dagger a_2 \) [5, 18], (3) the number operator to the \( k \)th power: \( (a_2^\dagger a_2)^k \), in a nonlinear interferometer [6].

Suppose the initial state \( \rho_0 \) has already been decomposed as

\[ \rho_0 = \sum_i p_i |\phi_i \rangle \langle \phi_i |. \]  

(10)

Here we assume \( \rho_0 \) is independent of \( \varphi \). After the unitary rotation, \( \rho_\varphi \) can be decomposed as

\[ \rho_\varphi = \sum_i p_i |\psi_i \rangle \langle \psi_i |, \]  

(11)

with

\[ |\psi_i \rangle = e^{-i\varphi H} |\phi_i \rangle. \]  

(12)

Substituting Eq. (11) into Eq. (8) leads to the QFI given by

\[ F_\varphi = 4 \left[ \sum_{i=1}^{M} p_i (\Delta H_i)^2 - \sum_{i \neq j} \frac{2p_ip_j}{p_i + p_j} |H_{ij}|^2 \right], \]  

(13)
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where

\[(\Delta H_i)^2 = \langle \phi_i | H^2 | \phi_i \rangle - \langle \phi_i | H | \phi_i \rangle^2,\]

and

\[| H_{ij} |^2 = |\langle \phi_i | H | \phi_j \rangle|^2,\]

are the variance and transition probability of \( H \) in the eigenstates of \( \rho_0 \). Since \( p_i \) is independent of \( \varphi \), the classical contribution vanishes. The first term in Eq. (13) is the mean variance of \( H \) over the eigenstates, while the second term is a sum of transition probability of \( H \) with a harmonic mean weight.

If \( \rho_0 \) is a pure state, we can take \( p_i = \delta_{i1} \), then

\[F_\varphi = 4(\Delta H_1)^2;\]

(16)

if \( \rho_0 \) only has two nonzero components, we take \( p_1p_2 \neq 0 \) and \( p_i = 0 \) when \( i > 2 \), then

\[F_\varphi = 4p_1(\Delta H_1)^2 + 4p_2(\Delta H_2)^2 - 16p_1p_2|H_{12}|^2.\]

(17)

In the following, we take \( M = 2 \) as our main interest.

3. EIGEN PROBLEM OF A Nonorthogonal 2 \times 2 Density Matrix

According to Eq. (8) and Eq. (13), we only need to find the non-vanishing eigenstates of the density operator rather than all its eigenstates. However, it is generally not feasible to get the analytical diagonalization of \( \rho_\varphi \). In that case, one has to resort to numeric methods or decompose the density operator into a nonorthogonal basis and use the convexity of QFI.

In this paper, we develop a systematic routine to find the eigenvalues and eigenstates of a density operator of rank 2 and apply it to an interesting scenario. Let us consider a 2 \times 2 density operator \( \tilde{\rho} \) on a nonorthogonal basis

\[\tilde{\rho} = a|\Psi_1\rangle\langle \Psi_1 | + b|\Psi_1\rangle\langle \Psi_2 | + b^*|\Psi_2\rangle\langle \Psi_1 | + d|\Psi_2\rangle\langle \Psi_2 |,\]

(18)

where \(|\Psi_1\rangle,|\Psi_2\rangle\) are normalized states and \( a, d \) are real numbers due to the hermiticity of density operator. The special case when \(|\Psi_1\rangle\) and \(|\Psi_2\rangle\) are orthogonal is discussed in Appendix A. In order to get the eigenvalues and eigenvectors of \( \tilde{\rho} \), we first recast it into an orthogonal basis (one can also solve the eigen problem in the original nonorthogonal basis, see Appendix B.)

Denoting \( p = \langle \Psi_1 | \Psi_2 \rangle \), we introduce a new set of basis by the Gram-Schmidt procedure [33]

\[|\Phi_1\rangle = |\Psi_1\rangle,\]

\[|\Phi_2\rangle = \frac{1}{\sqrt{1 - |p|^2}}(|\Psi_2\rangle - p|\Psi_1\rangle),\]

which are orthonormal. Through the inverse transformation: \(|\Psi_1\rangle = |\Phi_1\rangle, |\Psi_2\rangle = \sqrt{1 - |p|^2}|\Phi_2\rangle + p|\Phi_1\rangle\), the density matrix in the new basis reads

\[\tilde{\rho} = \begin{pmatrix} a + bp^* + b^*p + d|p|^2 & (b + dp)\sqrt{1 - |p|^2} \\ (b^* + dp^*)\sqrt{1 - |p|^2} & d(1 - |p|^2) \end{pmatrix}.\]

(19)
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The determinant of this density matrix, expectation value of $\sigma_3$ and off-diagonal phase read

$$\det(\tilde{\rho}) = (1 - |p|^2)(ad - |b|^2),$$
$$\langle \sigma_3 \rangle_{\tilde{\rho}} = 1 - 2d(1 - |p|^2),$$
$$e^{i\chi} = \frac{b+dp}{|b+dp|}. \quad (20)$$

According to appendix A, the eigenvalues and eigenstates of $\tilde{\rho}$ can be expressed in terms of $\det(\tilde{\rho})$, $\langle \sigma_3 \rangle_{\tilde{\rho}}$ and $\tilde{\tau}$. For clarity, we denote the eigenvalues and eigenstates as $\tilde{\lambda}_\pm$ and $|\tilde{\lambda}_\pm\rangle$ correspondingly. The values of $\tilde{\lambda}_\pm$ are

$$\tilde{\lambda}_\pm = \frac{1 \pm \sqrt{1 - 4\det(\tilde{\rho})}}{2}, \quad (21)$$

and the eigenstates read

$$|\tilde{\lambda}_+\rangle = \tilde{v}_+ e^{i\chi} |\Phi_1\rangle + \tilde{v}_- |\Phi_2\rangle,$$
$$|\tilde{\lambda}_-\rangle = -\tilde{v}_- e^{i\chi} |\Phi_1\rangle + \tilde{v}_+ |\Phi_2\rangle, \quad (22)$$

where

$$\tilde{v}_\pm = \left( \frac{\sqrt{1 - 4\det(\tilde{\rho})} \pm \langle \sigma_3 \rangle_{\tilde{\rho}}}{2 \sqrt{1 - 4\det(\tilde{\rho})}} \right)^{\frac{1}{2}}. \quad (23)$$

Hence the density matrix can be decomposed as

$$\tilde{\rho} = \sum_{i=\pm} \tilde{\lambda}_i |\tilde{\lambda}_i\rangle \langle \tilde{\lambda}_i|. \quad (24)$$

Alternatively, one can transform the eigenstates back to the nonorthogonal basis,

$$|\tilde{\lambda}_+\rangle = \left( \frac{p\tilde{v}_-}{\sqrt{1 - |p|^2}} - \frac{\tilde{v}_-}{\sqrt{1 - |p|^2}} \right) |\Psi_1\rangle + \frac{\tilde{v}_+}{\sqrt{1 - |p|^2}} |\Psi_2\rangle,$$
$$|\tilde{\lambda}_-\rangle = \left( -\frac{p\tilde{v}_+}{\sqrt{1 - |p|^2}} - \frac{\tilde{v}_+}{\sqrt{1 - |p|^2}} \right) |\Psi_1\rangle + \frac{\tilde{v}_-}{\sqrt{1 - |p|^2}} |\Psi_2\rangle. \quad (25)$$

4. QFI OF ECS IN A LOSSY MACH-ZEHNDER INTERFEROMETER

4.1. Reformulation of the Density Matrix of ECS in A Lossy Mach-Zehnder Interferometer

In a recent paper [5], the author analyzed the QFI of an entangled coherent state(ECS) in the Mach-Zehnder interferometer. The main idea of their proposition is as follows. A coherent state $|\alpha/\sqrt{2}\rangle$ and a coherent state superposition(CSS)

$$|CSS\rangle = \mathcal{N}_\alpha (|\alpha/\sqrt{2}\rangle + |-\alpha/\sqrt{2}\rangle), \quad (26)$$

are fed into the first 50:50 beam splitter of the Mach-Zehnder interferometer and become the ECS, 

\[ |\text{ECS}\rangle_{1,2} = \mathcal{N}_\alpha [|\alpha\rangle_1|0\rangle_2 + |0\rangle_1|\alpha\rangle_2], \]

where 

\[ \mathcal{N}_\alpha = 1/\sqrt{2(1 + e^{-|\alpha|^2})} \]

is the normalized coefficient. Then a parameter is imprinted in one of the mode by a unitary phase shift \( U(\varphi) \). They modeled particle loss in the realistic scenario by two fictitious beam splitters \( B_{1,3}^T, B_{2,4}^T \) with the same transmission coefficient \( T \). When \( T = 1 \), the interferometer has no photon loss. Here the subscript 3, 4 represent the environment modes. After tracing out the environment modes, they got the density matrix of the original mode \( \rho_{12} \).

To calculate the QFI of \( \rho_{12} \), they adopted numerical methods and truncated the coherent state at \( n = 15 \). Using the approach developed in Sec. (II) and Sec. (III), we can give the analytical expression of the QFI. In the following, we reformulate the density operator in a form as Eq. (18).

First, we denote the density operator before phase shift and particle loss as 

\[ \rho_{\text{in}} = |\text{ECS}\rangle_{1,2}|0\rangle_3|0\rangle_4 \langle 0|_4 \langle 0|_3 \langle \text{ECS}|_{1,2}. \]

In the interferometer, \( \rho_{\text{in}} \) suffers both particle loss and phase shift before exiting the second 50:50 beam splitter. The phase accumulation \( U(\varphi) = e^{-i\varphi|\alpha|^2} \) and the particle loss process, indicated by the fictitious beam splitters \( B_{1,3}^T, B_{2,4}^T \), are interchangeable \([23, 34]\). Here \( B_{1,3}^T \) and \( B_{2,4}^T \) satisfy the relation \([35]\)

\[ B_{1,2}^T|\alpha_1\rangle_1|\alpha_2\rangle_2 = |\alpha_1\sqrt{T} + \alpha_2\sqrt{R}\rangle_1|\alpha_1\sqrt{R} - \alpha_2\sqrt{T}\rangle_2. \]

Thus the final reduced density operator can be written as 

\[ \rho_{1,2} = \text{Tr}_{3,4}(B_{1,3}^T B_{2,4}^T U \rho_{\text{in}} U^\dagger B_{1,3}^{T\dagger} B_{2,4}^{T\dagger}) \]

\[ = \text{Tr}_{3,4}(U B_{1,3}^T B_{2,4}^T \rho_{\text{in}} B_{2,4}^{T\dagger} B_{1,3}^{T\dagger} U^\dagger). \]

The authors in Ref. \([5]\) use the expression (30). To apply our result in Sec. (II) and Sec. (III), we take the expression (31).

Second, the phase accumulation operator can be brought forward further, i.e., 

\[ \rho_{1,2} = U \text{Tr}_{3,4}(B_{1,3}^T B_{2,4}^T \rho_{\text{in}} B_{2,4}^{T\dagger} B_{1,3}^{T\dagger}) U^\dagger \]

\[ = U \tilde{\rho}_{1,2} U^\dagger, \]

where 

\[ \tilde{\rho}_{1,2} = \text{Tr}_{3,4}(B_{1,3}^T B_{2,4}^T \rho_{\text{in}} B_{2,4}^{T\dagger} B_{1,3}^{T\dagger}). \]

That is, in such a lossy situation, the phase shift is still a unitary process for \( \tilde{\rho}_{1,2} \). Therefore we can calculate the QFI of \( \rho_{1,2} \) by finding the decomposition of \( \tilde{\rho}_{1,2} \). With the denotation of

\[ \alpha' = \alpha\sqrt{T} \]

\[ \beta' = \alpha\sqrt{1 - T} = \alpha\sqrt{R} \]
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\(\tilde{\rho}_{1,2}\) can be specifically calculated as
\[
\tilde{\rho}_{1,2} = \mathcal{N}_\alpha^2|\alpha',0\rangle\langle\alpha',0| + e^{-|\beta'|^2}|0,\alpha\rangle\langle0,\alpha'| + e^{-|\beta|^2}|0,\alpha\rangle\langle0,\alpha'| + |0,\alpha\rangle\langle0,\alpha'|.
\] (34)

We can see \(\tilde{\rho}_{1,2}\) has the same form of Eq. (18). In the next subsection we show the decomposition of \(\tilde{\rho}_{1,2}\) and calculate the QFI.

### 4.2. Calculation of the ECS’s QFI

In order to find the decomposition of \(\tilde{\rho}_{1,2}\), we set \(|\Psi_1\rangle = |\alpha',0\rangle, |\Psi_2\rangle = |0,\alpha\rangle\) correspondingly. Comparing Eq. (34) with Eq. (18), we can find the determinant of this density matrix, expectation value of \(\sigma_3\) and off-diagonal phase as
\[
\det(\tilde{\rho}_{1,2}) = \mathcal{N}_\alpha^4(1 - e^{-2|\alpha'|^2})(1 - e^{-2|\beta'|^2}),
\]
\[
\langle\sigma_3\rangle_{\tilde{\rho}_{1,2}} = 1 - 2\mathcal{N}_\alpha^2 + 2\mathcal{N}_\alpha e^{-2|\alpha'|^2},
\]
\[
e^{i\tilde{\varphi}} = 1.
\] (35)

According to the preceding section, we can find the eigenvalues as
\[
\tilde{\lambda}_\pm = \frac{1}{2} \pm \frac{\sqrt{2}e^{-|\alpha|^2} + e^{-2|\alpha'|^2} + e^{-2|\beta'|^2}}{2 + 2e^{-|\alpha|^2}},
\] (36)
and
\[
\tilde{\nu}_\pm = \frac{1}{2} \pm \frac{e^{-|\alpha|^2} + e^{-2|\alpha'|^2}}{2\sqrt{2}e^{-|\alpha|^2} + e^{-2|\alpha'|^2} + e^{-2|\beta'|^2}}.
\] (37)

Next we analyze the parametrization procedure. The unitary operator on \(\tilde{\rho}_{1,2}\) reads
\[
U(\varphi) = \exp(-i\varphi \hat{a}_1^\dagger \hat{a}_2),
\] (38)
i.e., the generator of \(\varphi\) is \(H = \hat{a}_1^\dagger \hat{a}_2\). According to Eq. (17), we only need to calculate the variance of \(H\) in \(|\tilde{\lambda}_\pm\rangle\) and the transition probability of \(H\) between \(|\tilde{\lambda}_\pm\rangle\). Since \(H|\Psi_1\rangle = 0\), we choose Eq. (25) for convenience.

The variance in \(|\tilde{\lambda}_+\rangle\) is
\[
\Delta H_1^2 = \langle\tilde{\lambda}_+|((\hat{a}_1^\dagger \hat{a}_2)^2)|\tilde{\lambda}_+\rangle - (\langle\tilde{\lambda}_+|\hat{a}_1^\dagger \hat{a}_2|\tilde{\lambda}_+\rangle)^2
= \frac{\tilde{\nu}_-^2}{1 - p^2}(|\alpha|^2^2 + |\alpha'|^2^2) - \frac{\tilde{\nu}_+^2}{1 - p^2}(|\alpha'|^4).
\] (39)

Similarly, the variance in \(|\tilde{\lambda}_-\rangle\) is
\[
\Delta H_2^2 = \frac{\tilde{\nu}_+^2}{1 - p^2}(|\alpha|^2^2 + |\alpha'|^2^2) - \frac{\tilde{\nu}_-^2}{1 - p^2}(|\alpha'|^4),
\] (40)
and the transition term is
\[
|H_{12}|^2 = (\frac{\tilde{\nu}_- \tilde{\nu}_+}{1 - p^2}|\alpha'|^2)^2.
\] (41)

Utilizing above expressions and based on Eq. (17), we can obtain the QFI of \(\rho_{1,2}\) as
\[
F = 4\mathcal{N}_\alpha^2|\alpha|^2 T [1 + \mathcal{G}(T, \alpha)],
\] (42)
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where

\[ G(T, \alpha) = \frac{(N_\alpha^2 - 1)e^{-2|\alpha|^2T} + N_\alpha^2e^{-2|\alpha|^2R} + 2N_\alpha^2e^{-|\alpha|^2}}{1 - e^{-2|\alpha|^2T}}|\alpha|^2T. \]

Notice that \( N_\alpha \) satisfies the relation

\[ 2N_\alpha^2e^{-|\alpha|^2} = 1 - 2N_\alpha^2, \]

then \( G(T, \alpha) \) can be rewritten as

\[ G(T, \alpha) = |\alpha|^2T \left[ 1 - N_\alpha^2 - \frac{N_\alpha^2(1 - e^{-2|\alpha|^2R})}{1 - e^{-2|\alpha|^2T}} \right]. \]

Introduce the total average photon number \( \bar{n} = \langle a_1^\dagger a_1 + a_2^\dagger a_2 \rangle \), and it is easy to find that in this case

\[ \bar{n} = 2N_\alpha^2|\alpha|^2, \]

then \( G(T, \alpha) \) can be further written into

\[ G(T, \alpha) = T \left[ |\alpha|^2 \left( 2 + \frac{\bar{n}}{2} - \frac{\bar{n}}{2} - \frac{\bar{n}}{2}e^{-2|\alpha|^2R} \right) \right]. \]

and the QFI (43) can be finally simplified as

\[ F = \bar{n}T \left[ 2 + \left( 2|\alpha|^2 - \bar{n} - \frac{\bar{n}}{2}e^{-2|\alpha|^2R} \right) T \right]. \]

The QFI is only determined by the total average photon number \( \bar{n} \) and the transmission coefficient \( T \).

When \( T = R = 1/2 \), the QFI reduces into

\[ F = \bar{n} + \frac{\bar{n}}{2}(|\alpha|^2 - \bar{n}) \geq \bar{n}. \]

The last inequality is due to the fact that \(|\alpha|^2 \geq \bar{n}\) with the equal sign holds in the limit of \(|\alpha|^2 \to \infty\). Since \( F \) decreases monotonically with the transmission coefficient, the ECS can surpass the shot noise limit as long as \( T > \frac{1}{2} \); when \( T = 1 - R = 1 \), i.e., there is no particle loss in the interferometer, the QFI can be simplified as

\[ F = \bar{n} \left( 2 + 2|\alpha|^2 - \bar{n} \right), \]

and due to \(|\alpha|^2 \geq \bar{n}\), we have

\[ F \geq \bar{n}^2 + 2\bar{n}. \]

There is a debate over the ultimate scaling of the phase sensitivity for states with a fluctuating number of particles [36]. There are two candidates in the literature: the so-called Hofmann limit \( \delta \varphi \sim 1/\sqrt{\bar{n}^2} \), and the Heisenberg limit \( \delta \varphi \sim 1/\bar{n} \). Here we will show that the ECS can surpass the Heisenberg limit and Hofmann limit, even in the presence of particle loss.
From inequality (47), one can find that the QFI without particle loss is greater than $\overline{n^2}$, next we will show it is also greater than $\overline{n^4}$. The average of $n^2 = (a_1^\dagger a_1 + a_2^\dagger a_2)^2$ does not change after the first beam splitter. Then it is easy to find

$$\overline{n^2} = \langle \text{ECS}|_{1,2}(a_1^\dagger a_1 + a_2^\dagger a_2)^2|\text{ECS}\rangle_{1,2},$$

$$= 2N_\alpha^2 \left[ |\alpha|^2 + |\alpha|^4 \right]$$

$$= (1 + |\alpha|^2) \overline{\pi},$$

and compare with the QFI, we have

$$F = 2\overline{n^2} - \overline{\pi^2} = \overline{n^2} + \Delta(n),$$

where $\Delta(n)$ is the variance of the photon number. It is clear that $F$ is larger than both of $\overline{n^2}$ and $\overline{\pi^2}$.

Figure 1 shows the variation of QFI with the increase of $R$. Points A, B and C represent the intersection with the Hofmann limit, Heisenberg limit and shot noise limit respectively. The corresponding reflection coefficients read $R_A = 0.03$, $R_B = 0.07$ and $R_C = 0.52$. From this figure, one can find that when $R < R_A$, the ECS can always surpass the Hofmann limit, and for $R < R_B$, the precision is still better than the Heisenberg limit. This indicates that the precision is robust and overcomes the Heisenberg limit with a small loss of photons within $R_B$. If the precision is only required in the range of shot noise limit, then this interferometer can tolerate a loss of half photons.

The ECS is very useful and robust for quantum metrology [37, 38]. Our formula gives an easy approach to the determination of the QFI of ECS and one doesn’t have to resort to numeric methods.
5. Conclusion

We have derived an explicit formula for the QFI for a large class of states in which the parameter is introduced by a unitary dynamics $U = \exp(-iH\varphi)$. We pointed out that the QFI in this scenario is the mean variance of $H$ over the eigenstates minus weighted cross terms. Finally, we analyzed the QFI of a density matrix with $M = 2$ and apply our result into an entangled coherent state in a Mach-Zehnder interferometer, which was proposed in a recent paper [5].

We have found the analytical expression of the QFI for the ECS when there is particle loss. We find that even in the lossy condition, the ECS can still surpass the Heisenberg limit. The formalism developed here can be applicable to the study of more complicated states, such as the reduced two-mode mixed state when the total multi-mode system is in a multipartite entangled coherent states.

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Note added: After the submission of our manuscript, we notice that the authors in Ref. [39] do a relevant work and have a similar conclusion.

Appendix A. Eigenvalues and Eigenstates of A $2 \times 2$ Density Matrix

A general $2 \times 2$ density matrix $\rho$ is given in the form

$$
\rho = \begin{pmatrix}
\eta & \xi e^{i\tau} \\
\xi e^{-i\tau} & 1 - \eta
\end{pmatrix}.
$$

(A.1)

For this matrix to represent a physical state, one condition must be met: the determinant of $\rho$ must be positive, i.e., $\det(\rho) = \eta(1 - \eta) - \xi^2 \geq 0$ (this inequality implies $\eta \geq 0$, thus fulfill the positivity requirement of density matrix). Here $\xi > 0$, $\tau \in [0, 2\pi)$ are real numbers due to the Hermiticity of density matrix.

The eigenvalues of $\rho$ can be easily calculated as

$$
\lambda_{\pm} = \frac{1 \pm \sqrt{1 - 4\det(\rho)}}{2},
$$

(A.2)

and the corresponding normalized eigenvectors read

$$
|\lambda_+\rangle = \left(v_+e^{i\tau}, v_-\right)^T,
$$

$$
|\lambda_-\rangle = \left(-v_-e^{i\tau}, v_+\right)^T,
$$

(A.3)

with

$$
v_\pm = \left(\frac{\sqrt{1 - 4\det(\rho)} \pm \langle \sigma_3 \rangle}{2\sqrt{1 - 4\det(\rho)}}\right)^{\frac{1}{2}},
$$

(A.4)
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Here $\sigma_3$ is a Pauli matrix and $\langle \sigma_3 \rangle = \text{Tr}(\rho \sigma_3) = 2\eta - 1$.

We can see that the eigenvalues and eigenvectors of $\rho$ are fully determined by $\text{det}(\rho)$, $\langle \sigma_3 \rangle$ and $\tau$.

Appendix B. An equivalent way to solve the eigen problem of density operator in nonorthogonal basis

In this appendix we provide an equivalent way to solve the eigen problem of Eq. (18). Instead of recasting $\tilde{\rho}$ into an orthonormal basis, we assume the eigenvector as

$$|\phi\rangle = c_1|\Psi_1\rangle + c_2|\Psi_2\rangle.$$ (B.1)

Then the eigen equation reads

$$\tilde{\rho}|\phi\rangle = \lambda|\phi\rangle,$$ (B.2)

specifically (in the basis of $|\Psi_{1,2}\rangle$),

$$\begin{pmatrix} a + bp^* & ap + b \\ b^* + cp^* & b^*p + c \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$ (B.3)

i.e., we need find the eigenvalues and eigenvectors of the left matrix. One can easily find the trace and determinant are the same as those of Eq. (19), thus the eigenvalues are equal according to Eq. (A.2).

The eigenvectors can also be easily calculated as

$$|\phi_1\rangle = P_{11}|\Psi_1\rangle + P_{21}|\Psi_2\rangle,$$

$$|\phi_2\rangle = P_{12}|\Psi_1\rangle + P_{22}|\Psi_2\rangle,$$ (B.4)

with the normalized conditions

$$|P_{11}|^2 + |P_{21}|^2 + 2\text{Re}(pP_{11}^*P_{21}) = 1,$$

$$|P_{12}|^2 + |P_{22}|^2 + 2\text{Re}(pP_{12}^*P_{22}) = 1,$$ (B.5)

where $\text{Re}$ stands for real component. After some straightforward calculation, we can find

$$P_{11} = \tilde{v}_+ e^{i\tilde{\tau}} - \frac{\tilde{v}_- p}{\sqrt{1 - |p|^2}},$$

$$P_{21} = \frac{\tilde{v}_-}{\sqrt{1 - |p|^2}},$$

$$P_{12} = -\tilde{v}_- e^{i\tilde{\tau}} - \frac{\tilde{v}_+ p}{\sqrt{1 - |p|^2}},$$

$$P_{22} = \frac{\tilde{v}_+}{\sqrt{1 - |p|^2}},$$ (B.6)

where $e^{i\tilde{\tau}}$ and $\tilde{v}_\pm$ are defined in Eq. (20) and Eq. (23), i.e., the eigenstates in Eq. (B.4) are actually the same with Eq. (25).
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This method is a routine way to solving eigen problem. However, taking account of the normalization condition Eq. (B.5), it is quite tedious in calculation. We hope the method in the main text can offer some convenience when dealing with similar problems.

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