Connective structures for principal gerbes.

Introduction.

Let $N$ be a manifold, $H$ a Lie group and $P$ a $H$-principal bundle defined over $N$, a $P$-gerbe defined over $N$ bounded by the sheaf of automorphisms of $P$. In this paper we define the fundamental notions of the differential geometry of $P$-gerbes, that is the notions of connective structure curving holonomy and characteristic classes. The interest of such definitions is to provide a geometric action in string theory. In classical physics, the variational functional of the evolution of a particle in a phase space is a function of the holonomy of a gauge connection. In string theory, the action is given by the holonomy of a Deligne connective structure, when the gauge group is the circle. It is natural to provide the definition of non abelian holonomy in order to describe the action when the gauge group is non commutative.

Acknowledgements.

The authors want to thank Pierre Deligne for helpful corrections, and Dusa McDuff and Johan Dupont for helpful discussions.

Notations.

Let $U_{i_1}, \ldots, U_{i_p}$ be open subsets of a manifolds $N$, and $C$ a presheaf defined on $N$. We will denote by $U_{i_1 \ldots i_p}$ the intersection of $U_{i_1}, \ldots, U_{i_p}$. If $e_{i_1}$ is an object of $C(U_{i_1})$, $e_{i_1}^{i_2 \ldots i_p}$ will be the restriction of $e_{i_1}$ to $U_{i_1 \ldots i_p}$. For a map $h : e \rightarrow e'$ between two objects of $C(U_{i_1} \ldots i_p)$, we denote by $h^{i_{p+1} \ldots i_n}$ the restriction of $h$ to a morphism between $e^{i_{p+1} \ldots i_n} \rightarrow e'^{i_{p+1} \ldots i_n}$.

Definition.

Let $N$ be a manifold, $H$ a Lie group and $P \rightarrow N$ a $H$-principal bundle defined on $N$. A $P$-gerbe is a gerbe bounded by the sheaf of automorphisms of $P$. More precisely it is defined as follows:

To each open subset $U$ of $N$ we associate a category $C_P(U)$. The group of automorphisms of an object of $C_P(U)$ is the group of automorphisms of the restriction of $P$ to $U$. We suppose that the following conditions are satisfied:

Gluing conditions for objects.

Let $U$ be an open subset of $N$, $(U_i)_{i \in I}$ an open cover of $U$ and $e_i$ an object of $C_P(U_i)$. Suppose given an arrow $u_{ij} : e_j^{i} \rightarrow e_j^{i}$ between the respective restrictions of $e_{i}$ and $e_{j}$ to $U_i \cap U_j$ such that $u_{ij}^{i_1 i_2} u_{ij}^{i_3 i_4} = u_{ij}^{i_1 i_3} u_{ij}^{i_2 i_4}$. Then there exists an object $e_U$ of $C_P(U)$ whose restriction to $U_i$ is $e_i$. 
Gluing conditions for arrows.

Let $e$ and $e'$ be two objects of $C_P(U)$. The correspondence defined on the category of open subsets of $U$ by $V \rightarrow \text{Hom}(e|_V, e'|_V)$ is a sheaf of sets, where $e|_V$ and $e'|_V$ are the respective restrictions of $e$ and $e'$ to $V$.

We suppose that there exists an open cover of $N (U_i)_{i \in I}$ such that the category $C_P(U_i)$ is not empty, and objects of $C_P(U_i)$ are isomorphic.

An example of $P$-gerbe is defined as follows: Let $G$ be a Lie group and $H$ a closed normal subgroup of $G$. The quotient $G/H$ is a Lie group. We suppose that the projection $G \rightarrow G/H$ has local sections. Consider a $G/H$-bundle $p_{G/H}: P \rightarrow N$, defined on the manifold $N$. That is a locally trivial bundle whose transition functions is defined by the trivialization $(U_i, u_{ij})$, $u_{ij}: U_i \cap U_j \rightarrow G$ defined by the coordinate changes:

$$U_i \cap U_j \times G/H \rightarrow U_i \cap U_j \times G/H$$

$$(x, y) \rightarrow (x, yu_{ij}(x))$$

The functions $u_{ij}$ verify the following property: $u_{i i_1 i_2} u_{i_1 i_2 i_3} u_{i_2 i_3 i_1}(x)$ are in the center of $H$ to insure the $G/H$-bundle to be well-defined as the $H$-bundle $p_H$ whose transition functions are defined by:

$$U_i \cap U_j \times H \rightarrow U_i \cap U_j \times H$$

$$(x, y) \rightarrow (x, u_{ij}^{-1}(x)yu_{ij}(x))$$

Let $\mathcal{H}$ be the Lie algebra of $H$ and $\text{Ad}$ the adjoint representation. We can define the locally trivial $\mathcal{H}$-bundle $p_{\mathcal{H}}$ over $N$ whose transition functions are defined by:

$$U_i \cap U_j \times \mathcal{H} \rightarrow U_i \cap U_j \times \mathcal{H}$$

$$(x, y) \rightarrow (x, \text{Ad}(u_{ij}^{-1})(x))(y))$$

**Proposition.**

Let $U$ be an open subset of $N$, we denote by $C_H(U)$ the category of $G$-principal bundles whose quotient by $H$ is the restriction of $p_{G/H}$ to $U$. A morphism between a pair of objects $e$ and $e'$ of $C_H(U)$ is a morphism of $G$-bundles which cover the identity of the restriction of $p_{G/H}$ to $U$. The correspondence defined on the category of open subsets of $N$ by $U \rightarrow C_H(U)$ is a gerbe bounded by the sheaf of automorphisms of $p_H$.

**Proof.**

Gluing property for arrows:

Let $U$ be an open subset of $N$, and $(U_i)_{i \in I}$ an open cover of $U$. Consider an object $e_i$ of $C_H(U_i)$, and a map $u_{ij} : e_j \rightarrow e_i'$ such that $u_{i_1 i_2 i_3} u_{i_2 i_3 i_1} = u_{i_1 i_3 i_2}$.
The definition of bundle implies the existence of a $G$-bundle $e$ whose restriction to $U_i$ is $e_i$. Since the quotient of $e_i$ by $H$ is the restriction of $p_{G/H}$ to $U_i$, we deduce that the quotient of $e$ by $H$ is the restriction of $p_{G/H}$ to $U$.

Gluing condition of arrows.

Let $e$ and $e'$ be a pair of objects of $C_H/U$, the correspondence defined on the category of open subsets of $U$ by $V \rightarrow Hom(e|_V, e'|_V)$ is a sheaf of sets, since it is the sheaf of morphisms between two bundles. The bundles $e|_V$ and $e'|_V$ are the respective restrictions of $e$ and $e'$ to $V$.

Consider a trivialization $(U_i, u_{ij})$ of $p_{G/H}$. The bundle $U_i \times G$ is an element of $C_H(U_i)$, thus $C_H(U_i)$ is not empty, and for each object $e$ and $e'$ of $C_H(U)$, the restrictions of $e$ and $e'$ to $U_i \cap U$ are isomorphic to the trivial bundle $U_i \cap U \times G$ by an isomorphism whose projection to $U_i \cap U \times G/H$ is the identity.

Let $e$, be an object of $C_H(e)$, and $f$ an automorphism of $e$, The restriction $f_i$ of $f$ to the restriction of $e$ to $U_i \cap U$ is an automorphism of the trivial bundle $U_i \cap U \times G$ which projects to the identity on $U_i \cap U \times G/H$. We deduce that $f_i$ is defined by a map $f_i' : U_i \cap U_j \rightarrow H$. On $U_i \cap U_j \cap U$, we have $f_j = u_{ij}^{-1}f_i u_{ij}$. This implies that $f$ is a section of $p_H$.

**The classifying cocycle of a principal gerbe.**

Let $(U_i)_{i \in I}$ be an open cover of $N$ such that the category $C_p(U_i)$ is not empty and the objects of $C_p(U_i)$ are isomorphic. Consider for each $i$, an object $e_i$ of $C_p(U_i)$, and arrow $u_{ij} : e_i \rightarrow e'_j$. We can define the automorphism of $e_i'_{ij} : c_{i_1i_2i_3} = u_{i_1i_2}^{-1}u_{i_2i_3}^{-1}u_{i_1i_3}$.

**Proposition.**

The family of maps $c_{i_1i_2i_3}$ is a non commutative Čech 2-cocycle.

**Proof.**

Let $c_{i_1i_2i_3} = u_{i_1i_3}^{-1}u_{i_2i_3}^{-1}u_{i_1i_2}u_{i_3}$ on $U_{i_1i_2i_3}$, we have

$c_{i_1i_2i_3}'c_{i_1i_3i_4} = c_{i_2i_3i_4}c_{i_1i_2i_4}$

**Connective structure on $H$-gerbes.**

**Definition.**

Consider a gerbe $C_p$ defined on a manifold $N$ whose band is $L$, the sheaf of automorphisms of the principal bundle $P \rightarrow N$. A **connective structure** on $C_p$, is a correspondence which associates to each object $e_U$ of $C_p(U)$ an affine space $Co(e_U)$, called the torsor of connections, which is a subset of the set of $p_{H[U]}$-valued 1-forms defined on $U$, where $p_{H[U]}$ is the restriction of $p_H$ to $U$.

The following properties are supposed to be satisfied by this assignment:

1. The correspondence $e_U \rightarrow Co(e_U)$ is functorial with respect to restrictions to smaller subsets.
2. For every isomorphism $h : e_U \rightarrow e'_U$ between objects of $C_p(U)$, there exists an isomorphism of torsors $h^* : Co(e_U) \rightarrow Co(e'_U)$ compatible with the composition of morphisms of $C_p(U)$, and the restrictions to smaller subsets.
(iii)- For each morphism $g$ of the object $e_U$ of $C_P(U)$, and $\nabla_{e_U}$ a connection of $Co(e_U)$,

$$g^*\nabla_{e_U} = \text{Ad}(g^{-1})(\nabla_{e_U}) + g^{-1}dg$$

For each open subset $U$, we define $C'_P(U)$ to be the category whose objects are pair of objects $(e_U, \nabla_{e_U})$, where $\nabla_{e_U}$ is an element of $Co(e_U)$. A morphism $f : (e_U, \nabla_{e_U}) \to (e'_U, \nabla'_{e_U})$ is $\nabla'_{e_U} - u^*(\nabla_{e_U})$ where $u : e_U \to e'_U$ is a morphism of $C(U)$. We suppose the correspondence $U \to C'_P(U)$ to be a gerbe $\bullet$

**The classifying cocycle of a connective structure.**

Let $(U_i, h_{ij})_{i \in I}$ a trivialization of $p_H$ such that $C_P(U_i)$ is not empty, and $e_i$ an object of $C_P(U_i)$ and $u_{ij}$ a morphism between $e_i$ and $e_j$. Consider an element $\alpha_i$ of $Co(e_i)$. We define $c_{i_1i_2i_3}$ to be $u_{i_1i_2}^{-1}u_{i_1i_3}^{-1}u_{i_2i_3}^{-1}u_{i_1i_2}$. On $U_i \cap U_j$, we can define the $\mathcal{H}$-valued form $\alpha_{ij} = \alpha_i^j - u_{ij}^*\alpha_i^j$.

The Cech boundary of the $p_H$ 1-cocycle $\alpha_{ij}$ is:

$$u_{i_1i_2}^*\alpha_{i_2i_3} - \alpha_{i_1i_3} + \alpha_{i_1i_2}$$

$$u_{i_1i_3}^*(\alpha_{i_3} - c_{i_1i_2i_3}^*\alpha_{i_3})$$

$$= \text{Ad}(h_{i_1i_3}^{-1})(\alpha_{i_3} - \text{Ad}(c_{i_1i_2i_3}^{-1})(\alpha_{i_3}) + c_{i_1i_2i_3}^{-1}d(c_{i_1i_2i_3}))$$

We have used the fact that on the trivialization $U_i \cap U_j \times \mathcal{H}$, let $\alpha$ and $\alpha'$ be two elements of $Co(e_i)$, $u_{ij}^*(\alpha - \alpha')$ is transformed in $\text{Ad}(h_{ij}^{-1})(\alpha - \alpha')$ by the transition functions of $p_H$, since $\alpha - \alpha'$ is an element of the vector space of the affine space $Co(e_i)$.

**Example.**

Consider a normal subgroup $H$ of a Lie group $G$, and a $G/H$-bundle $p_{G/H}$ over the manifold $N$. We have defined a gerbe $C_H$ at page 2. We define the connective structure $Co$ on $C_H$ as follows: for each open subset $U$ of $N$, and an object $e_U$ of $C_H(U)$, $Co(e_U)$ is the set of 1-forms $\theta : U \to C(\mathcal{H})$ where $C(\mathcal{H})$ is the center of $\mathcal{H}$. This definition is natural since if the center of $H$ is trivial, then the gerbe $C_H$ is trivial. The characteristic classes defined below are also trivial.

**Definition.**

A **curving** of a connective structure $Co$ is a correspondence

$$D(e_U, : Co(e_U) \to D(e_U)$$

where $D(e_U)$ is an affine space whose underlying vector space is a set of $p_H$ valued 2-forms which satisfies the following property:

(i)- For each morphism $h : e'_U \to e_U$, $(D(e_U, \nabla)) = D(e'_U, h^*\nabla)$. 

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(ii) If $\alpha$ is a $p_{\mathcal{H}|U}$ 1-form on $U$ such that $\nabla + \alpha$ is an element of $Co(e_U)$, then

$$D(e_U, \nabla + \alpha) = D(e_U, \nabla) + d\alpha + \alpha \wedge \alpha$$

The assignment $e_U \to D(e_U, \nabla)$ is compatible with the restrictions to smaller subsets.

**Characteristic classes.**

**Definition.**

A polynomial function of degree $l$ $F : \mathcal{H}^l \to \mathbb{R}$ is said to be invariant if and only if for every $h \in H$ $F(\text{Ad}(h)) = F$.

Let $C_P$ be a gerbe bounded by $P$ endowed with the connective structure $C_P'$, consider $[\Omega]$ the cohomology class of the classifying cocycle $\Omega$ of $C_P'$ identified with a DeRham 3-form using the Cech-DeRham isomorphism. For every invariant polynomial $P$ of degree $l$ we can define the 3l-form $P(\Omega)$ by $P \circ \Lambda^l L$. The cohomology classes of the forms $P(\Omega)$ are the characteristic classes of the curving.

**Holonomy of non abelian gerbes.**

Let $C_P$ be a principal gerbe defined over a manifold $N$ endowed with a connective structure $Co$ and a curving $Cur$. Let $l : N_2 \to N$ be a differentiable map whose domain is the compact surface $N_2$. We can pull-back the gerbe $C_P$, $Co$ and $Cur$ to $N_2$ by using $l$. Let $(U_i)_{i \in I}$ be an open covering of $N_2$ such that $(l^*C_P)(U_i)$ is not empty. Consider an object $e_i$ of $(l^*(C_P)(U_i))$, $\nabla_i$ an element of $l^*(Co)(U_i)$, and $L_i$ the curving of $\nabla_i$. Since $N_2$ is a surface, $L_i$ is exact. We can set $L_i = d(L'_i)$. On $U_{i_1i_2}$ we have $L_{i_2} - L_i = d(\nabla_i - u_{i_1i_2}^{-1}\nabla_{i_2}) + (\nabla_{i_1} - u_{i_1i_2}^{-1}\nabla_{i_2})\wedge(\nabla_i - u_{i_1i_2}^{-1}\nabla_{i_2})$, the form $(\nabla_{i_1} - u_{i_1i_2}^{-1}\nabla_{i_2})\wedge(\nabla_i - u_{i_1i_2}^{-1}\nabla_{i_2}) = d(L'_{i_1i_2})$.

This implies that $L'_{i_2} - L'_{i_1} = \nabla_{i_1} - u_{i_1i_2}^{-1}\nabla_{i_2} + L'_{i_1i_2} + d(L'_{i_1i_2})$.

This implies that the Cech boundary $\delta(h_{i_1i_2})$ of $h_{i_1i_2} = \nabla_{i_1} - u_{i_1i_2}^{-1}\nabla_{i_2} + L'_{i_1i_2}$ is a 2-chain of closed forms. We set $\delta(h_{i_1i_2}) = d(C_1_{i_1i_2})$.

The chain $C_{i_1i_2} + \delta(h_{i_1i_2})$ is a 2-chain of constant $\mathcal{H}$-functions. Since $N_2$ is a surface, we can find a cover such that this chain is a cocycle. It suffices to find an open cover $(U_l)_{l \in I}$ such that $U_{i_1i_2i_3i_4}$ is empty and $U_l$ is a 1-Eilenberg-Mclane space. Thus using the Cech-DeRham isomorphism, we identify this chain to a $l^*(p_{\mathcal{H}})$ 2-form $H$. We define

$$\text{Hol}(N_2, C_P, Co) = \exp(\int_{N_2} H)$$

**Reference.**

J.L Brylinski, Loops spaces, Characteristic Classes and Geometric Quantization, Progr. Math. 107, Birkhauser, 1993.