Seven-dimensional forest fires

Daniel Ahlberg, Hugo Duminil-Copin, Gady Kozma
and Vladas Sidoravicius

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Abstract

We show that in high dimensional Bernoulli percolation, removing from a thin infinite cluster a much thinner infinite cluster leaves an infinite component. This observation has implications for the van den Berg-Brouwer forest fire process, also known as self-destructive percolation, for dimension high enough.

1 Introduction

Think about the open vertices of supercritical percolation as if they were trees, and about the infinite cluster as a forest. Suddenly a fire breaks out and the entire forest is cleared. New trees then start growing randomly. When can one expect a new infinite cluster to appear? The surprising conjecture in [vdBB04] is that in the two-dimensional case, even if the original forest were extremely thin, still a considerable amount of trees must be added to create a new infinite cluster. Heuristically, the conjecture claims that the infinite cluster might occupy a very low proportion of vertices but they sit in a way that separates the remaining finite clusters by gaps that cannot be easily bridged. This conjecture is still open. See [vdBB04, vdBBV08, vdBdL09] for connections to other models of forest fires and more.

Let us define the model formally, in three steps. The model was originally introduced as a site percolation model, but we will define it for bonds, as some of the auxiliary results we need have only been proved for bond percolation. We are given a graph $G$, a probability $p \in [0,1]$ (“the original density”) and a probability $\varepsilon \in [0,1]$ (“the recovered density”). Let $\mathbb{P}_p$ be the Bernoulli bond percolation measure on $G$ with parameter $p$.

1. Assign independent uniformly distributed values from $[0,1]$ to the edges of $G$. Let $\omega_p \in \{0,1\}^{E(G)}$ denote the set of edges with value at most $p$. The configuration $\omega_p$ is distributed as $\mathbb{P}_p$, and a cluster refers
to a maximal connected component of edges. It will be of importance below that as \( p \) ranges over \([0, 1]\), we obtain a simultaneous coupling of Bernoulli configurations on \( G \) such that \( \omega_{p_1} \subset \omega_{p_2} \) when \( p_1 \leq p_2 \).

2. Let \( \tilde{\mathbb{P}}_p \) be the law of the configuration \( \tilde{\omega}_p \) constructed as follows: for any edge \( e \),

\[
\tilde{\omega}_p(e) = \begin{cases} 
\omega_p(e) & \text{if } e \text{ is in a finite cluster of } \omega_p, \\
0 & \text{otherwise}.
\end{cases}
\]

3. Let \( \tilde{\mathbb{P}}_{p,\varepsilon} \) be the law of \( \tilde{\omega}_{p,\varepsilon} \) where \( \tilde{\omega}_{p,\varepsilon} \) is defined as follows: for any edge \( e \), \( \tilde{\omega}_{p,\varepsilon}(e) = \max\{\tilde{\omega}_p(e), \omega'_\varepsilon(e)\} \), where \( \omega'_\varepsilon \) is a percolation configuration with edge-weight \( \varepsilon \) which is independent of \( \omega_p \).

We can now define our property of interest.

**Definition** We say that the graph \( G \) recovers from fires if for every \( \varepsilon > 0 \), there exists \( p > p_c(G) \) such that \( \tilde{\mathbb{P}}_{p,\varepsilon} \) has an infinite connected component, with probability 1. We say that \( G \) site-recovers from fires if the analogous definitions for site percolation hold.

In \([\text{vdBB04}]\) the authors showed that a binary tree site-recovers from fires and conjectured that \( \mathbb{Z}^2 \) lattice does not site-recover from fires. The binary tree is an example of a non-amenable graph, that is, a graph in which the boundary of a (finite) set of vertices is comparable in size to the set itself. Recovery from fires, both in edge and site sense, was proven in \([\text{AST13}]\) for a large class of non-amenable transitive graphs. Our result concerns hyper-cubic lattices.

**Theorem 1.** For \( d \) sufficiently large, \( \mathbb{Z}^d \) recovers from fires.

Here and below, \( \mathbb{Z}^d \) refers to the \( \mathbb{Z}^d \) nearest neighbour lattice. The main property of \( \mathbb{Z}^d \) that we will use is that \( \mathbb{P}_p, (0 \leftrightarrow B(0, r)) \leq Cr^{-2} \) (see below for a discussion on this condition, and also for the notations). This was proved in \([\text{KN11}]\) based on results of Hara, van der Hofstad & Slade \([\text{HvdHS03}, \text{Har08}]\). These establish the necessary estimate for \( d \) sufficiently large (19 seems to be enough, though this can be improved) and also for stretched-out lattices in \( d > 6 \). The number 6 is actually meaningful and is the limit of the technique involved, lace expansion. Our proof easily extends to stretched-out 7-dimensional lattice (hence the title of the article), but for simplicity we will prove the theorem only for nearest-neighbour percolation in \( d \) sufficiently high. In fact, our proof provides further information in the supercritical percolation regime. Recall the common notation \( \mathcal{C}_\infty(\omega_p) \) for the infinite cluster of edges present in \( \omega_p \).
Theorem 2. For every \( \varepsilon > 0 \) and \( d \) sufficiently large, there exists \( p > p_c \) such that \( \omega_{p_c + \varepsilon} \setminus C_{\infty}(\omega_p) \) contains an infinite cluster almost surely.

Theorem 1 is clearly a corollary of Theorem 2. Another consequence is that for every \( \varepsilon > 0 \), the critical probability for percolation on the random graph obtained from \( \mathbb{Z}^d \) by removing a sufficiently ‘thin’ supercritical percolation cluster is almost surely at most \( p_c + \varepsilon \). Theorem 2 and the last statement cannot possibly hold for (site) percolation on \( \mathbb{Z}^2 \), since an infinite cluster cuts space up into finite pieces.

**Proof sketch** We will show that for every \( \varepsilon > 0 \), there exists some \( p > p_c \) such that when removing the infinite cluster of \( p \)-percolation from \( (p_c + \varepsilon) \)-percolation, the remainder still percolates. The proof proceeds by a renormalization procedure.

1. We first choose \( \ell \in \mathbb{N} \) sufficiently large such that for any \( L \geq \ell \), connectivity properties of boxes of size \( L^2 \times \ell^{d-2} \) in \( (p_c + \varepsilon) \)-percolation behave like \((1 - \eta)\)-percolation on a coarse grain lattice for some small \( \eta \). This is a standard application of Grimmett-Marstrand [GM90] and renormalization theory.

2. We then use the fact that the one-arm exponent in high dimensions is 2 to note that for any \( L \), only a small number \( M \) of vertices in a box of size \( L^2 \times \ell^{d-2} \) can connect to distance \( L \) in critical percolation.

3. Picking \( L \) sufficiently large, one can argue that these \( M \) points do not alter the connectivity properties of boxes of size \( L^2 \times \ell^{d-2} \) for \((p_c + \varepsilon)\)-percolation. In particular, the coarse grain percolation still behaves like \((1 - \eta)\)-percolation even after removing that small number of vertices.

4. We now pick \( p \) sufficiently close to \( p_c \) that the behaviour (for \( \omega_p \)) at scale \( L \) is not altered by moving from \( p_c \) to \( p \). Since there are less sites in \( C_{\infty}(\omega_p) \) than sites connected to distance \( L \) in \( \omega_p \), this \( p \) gives the result.

Examining this a little shows that what the proof really needs is that the one-arm exponent is bigger than 1 i.e. that

\[
P_{p_c}(0 \leftrightarrow \partial B(0, r)) \leq r^{-1-c} \quad c > 0.
\]

(the number of points removed in the second renormalization step will no longer be bounded independently of \( L \), but would still be too small to block the cluster of the boxes at scale \( \ell \)). This is interesting as it is conjectured to hold also below 6 dimensions. While nothing is proved, simulations hint
that it might hold for $\mathbb{Z}^5$ [AS94, §2.7]. On the other hand, let us note that in $\mathbb{Z}^3$ this probability is larger than $c_r r^{-1}$ (this is well-known but we are not aware of a precise reference – compare to [vdBK85, (3.15)] and [Kes82, Theorem 5.1]). Hence, the approach used here has no hope of working in $\mathbb{Z}^3$ (though, of course, this does not preclude the possibility that $\mathbb{Z}^3$ does recover from fires). We remarks that a similar renormalization technique was recently used in [GHK13], also under the assumption that the one-arm exponent is bigger than 1.

**Notations** Identify $\mathbb{Z}^2$ with the subgraph of $\mathbb{Z}^d$ of points with the $d-2$ last coordinates equal to 0. Let $S_\ell = \{x \in \mathbb{Z}^d : |x_i| \leq \ell \forall i \geq 3\}$ be the two-dimensional slab of height $2\ell + 1$. We will also use the following standard notations: For a subgraph $G$ of $\mathbb{Z}^d$, we say that $x$ is connected to $y$ in $G$ if they are in the same connected component of $G$. We denote it by $x \xleftarrow{G} y$ or simply $x \xleftarrow{} y$ when the context is clear. We also use the notation $x \xleftarrow{} \infty$ to denote the fact that $x$ is contained in an infinite connected component.

Let $|| \cdot \||_\infty$ be the infinity norm on $\mathbb{R}^d$ defined by

$$||x||_\infty = \max\{|x_i| : i = 1, \ldots, d\}.$$ 

We consider the hypercubic lattice $\mathbb{Z}^d$ for some large but fixed $d$. For $\ell, L > 0$, define the ball $B_x(L) = \{y \in \mathbb{Z}^d : ||y - x||_\infty \leq L\}$ and let $\partial B_x(L)$ be its inner vertex boundary.

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## 2 Proof

From now on, $d$ is fixed and is large enough. For $x \in \mathbb{Z}^2$, let $\mathcal{A}(x, \ell, L, M)$ be the event that there are less than $M$ sites $y$ in the $(6L + 1) \times (6L + 1) \times (2\ell + 1)^{d-2}$ box $S_\ell \cap B_x(3L)$ that are connected to a site at distance $L$ from themselves. Note that we do not assume that this connection is contained in the slab $S_\ell$, the connection may be anywhere in $B_y(L)$.

**Lemma 3.** Let $\eta > 0$ and $\ell > 0$. There exists $M > 0$ such that for any integer $L$, there exists $p > p_c$ such that

$$\mathbb{P}_p(\mathcal{A}(x, \ell, L, M)) \geq 1 - \eta.$$
Proof. By [KN11], there exists $C > 0$ such that (for large enough $d$)
\[ P_{p_c}(0 \leftrightarrow \partial B_0(n)) \leq \frac{C}{n^2}. \tag{1} \]
Choose $M$ in such a way that $\frac{49(2\ell+1)^d - 2C}{M} < \eta$. For any integer $L$, Markov’s inequality implies
\[ P_{p_c}(\{y \in S_\ell \cap B_x(3L) : y \leftrightarrow \partial B_y(L)\} \geq M) \leq \frac{1}{M} \sum_{y \in S_\ell \cap B_x(3L)} P_{p_c}(y \leftrightarrow \partial B_y(L)). \]
By (1) and the choice of $M$, the right-hand side is thus strictly smaller than $\eta$. By choosing $p$ close enough to $p_c$, we obtain that
\[ P_{p_c}(\{y \in S_\ell \cap B_x(3L) : y \leftrightarrow \partial B_y(L)\} \geq M) \leq \eta. \]

For a set $S \subset \mathbb{Z}^d$, let $\omega_S$ be the configuration obtained from $\omega$ by closing the edges adjacent to a site in $S$. Let $\mathcal{B}(x, \ell, L, M)$ be the event that for any set $S$ of $M$ sites contained in $B_x(3L)$, $\omega_S$ contains
\begin{itemize}
  \item a cluster crossing from $\partial B_x(L)$ to $\partial B_x(3L)$ contained in the slab $S_\ell$,
  \item a unique cluster in the box $S_\ell \cap B_x(3L)$ of radius larger than $L$.
\end{itemize}

Lemma 4. Let $\eta > 0$ and $\varepsilon > 0$. There exists $\ell > 0$ such that for any $M > 0$, there is $L > 0$ so that
\[ P_{p_c+\varepsilon}(\mathcal{B}(x, \ell, L, M)) \geq 1 - \eta. \]

Proof. For a given $\ell$ and $L$ denote by $E = E(x, \ell, L)$ the event that:
\begin{enumerate}
  \item There is a crossing from $\partial B_x(L)$ to $\partial B_x(3L)$ in $S_\ell$.
  \item There is exactly one cluster in $S_\ell \cap B_x(3L)$ of radius larger than $L$.
\end{enumerate}
Shortly, the event $E$ is just $\mathcal{B}$ without the set $S$, or if you want $\mathcal{B}$ is the event that $E$ occurred in $\omega^S$ for all $S$ with $|S| \leq M$.
We claim that for $\ell$ sufficiently large, $P_{p_c+\varepsilon}(\neg E) \leq \exp(-cL)$ for some $c = c(\varepsilon, \ell) > 0$ independent of $L$. Finding such an $\ell$ is a standard exercise in renormalization theory, but let us give a few details nonetheless. Call a box of side-length $2\ell+1$ good if it contains crossings between opposite faces in all directions, and if all clusters of diameter at least $\frac{1}{4}\ell$ connect inside the box. By choosing $\ell$ large, we can require that a box is good with arbitrarily high probability (see e.g. the appendix of [BBHIK08]). Considering such boxes centered around the sites in $\ell\mathbb{Z}^2$. The events that these boxes are good are 2-dependent (in the sense of [LSS97] i.e. any box is independent
of all boxes not neighbouring it), and hence by [LSS97], if the probability that a box is good is sufficiently large, then the good boxes stochastically dominate two-dimensional percolation at density, say, \( \frac{\theta}{10} \). Now, a cluster of good boxes contains a cluster in the underlying percolation, since the crossings of adjacent boxes must intersect. This means that if either of the conditions in the definition of \( E \) fail, then there is a cluster of bad boxes with at least \( L/l \) boxes. But the probability for that, from Peierls’ argument, is at most \( (4/10)^{L/l} \cdot (6L/l)^2 \). This shows the claim.

Fix \( M > 0 \). Let \( F_M \) be the set of configurations in \( B_x(3L) \) for which there exists \( S \subseteq B_x(3L) \) with \( |S| = M \) and \( \omega^S \not\in E \). We have

\[
\mathbb{P}_{p_c+\varepsilon}(F_M) \leq \sum_{S \subseteq B_x(3L), |S| = M} \mathbb{P}_{p_c+\varepsilon}(\omega^S \not\in E)
\leq \sum_{S \subseteq B_x(3L), |S| = M} (1 - p_c - \varepsilon)^{-2dM} \mathbb{P}_{p_c+\varepsilon}(\neg E)
\leq (1 - p_c - \varepsilon)^{-2dM} (6L + 1)^d \mathbb{P}_{p_c+\varepsilon}(\neg E)
\leq (1 - p_c - \varepsilon)^{-2dM} (6L + 1)^d \exp(-cL).
\]

For \( L \) large enough, this quantity is smaller than \( \eta \). The lemma follows from the fact that if \( \omega \in \mathcal{B}(x, \ell, L, M) \), then there exists \( S \subseteq B_x(3L) \) with \( |S| = M \) and \( \omega^S \not\in E \), i.e. \( \omega \in F_M \).

In order to prove Theorem 1 and 2, we will use Lemma 4 to construct an infinite cluster at density \( p_c + \varepsilon \), and Lemma 3 to make sure that the infinite cluster present at the lower density \( p \) does not interfere too much with this construction.

Proof of Theorems 1 and 2. Recall the notations \( \omega_p, \tilde{\omega}_p \) and \( \omega'_p \) from page 1. We need to show that for any \( \varepsilon > 0 \), there exists \( p > p_c \) such that \( \tilde{\omega}_{p,\varepsilon} \) has an infinite component. Note that \( (\omega_{p,\varepsilon} \cup \omega'_{p}) \setminus \mathcal{C}_\infty(\omega_p) \) is stochastically dominated by \( \tilde{\omega}_{p,\varepsilon} \). Thus, it suffices to show that for every \( \varepsilon > 0 \), there is \( p > p_c \) such that \( \omega_{p,\varepsilon} \setminus \mathcal{C}_\infty(\omega_p) \) contains an infinite component. That is, Theorem 1 follows from Theorem 2, and it suffices to prove the latter.

Let therefore \( \varepsilon > 0 \). Fix \( \eta > 0 \) such that \( 1 - 2\eta \) exceeds the critical parameter for any 8-dependent percolation on vertices of \( \mathbb{Z}^2 \). Define successively \( \ell, M, L \) and \( p_0 \) as follows. Fix \( \ell = \ell(\varepsilon, \eta) > 0 \) as defined in Lemma 4. Pick \( M = M(\eta, \ell) > 0 \) as defined in Lemma 3. This defines \( L = L(\eta, \varepsilon, \ell, M) > 0 \) by Lemma 4, and then \( p = p(\eta, \ell, M, L) > p_c \) by Lemma 3.

Let \( \mathbf{P} \) denote the joint law of \( (\omega_p, \omega_{p,\varepsilon}) \) under the increasing coupling described above. A site \( x \in L\mathbb{Z}^2 \) is said to be *good* if \( \omega_p \in \mathcal{A}(x, \ell, L, M) \) and \( \omega_{p,\varepsilon} \in \mathcal{B}(x, \ell, L, M) \). By definition,

\[
\mathbf{P}[\mathcal{A}(x, \ell, L, M) \cap \mathcal{B}(x, \ell, L, M)] \geq 1 - 2\eta.
\]
Since these events depend on edges in $B_x(4L)$ only, the site percolation (on $L\mathbb{Z}^2$) thus obtained is 8-dependent. As a consequence, there exists an infinite cluster of good sites on the coarse grained lattice $L\mathbb{Z}^2$.

On the event that there exists an infinite cluster of good sites on the coarse grained lattice, there exists an infinite path in $\omega_{p_c+\varepsilon} \setminus C_\infty(\omega_p)$. Indeed, by induction, consider a path of adjacent good sites $x_1, \ldots, x_n$. Consider $C_i$ to be a cluster in

$$[\omega_{p_c+\varepsilon} \setminus C_\infty(\omega_p)] \cap [B_{x_i}(3L) \setminus B_{x_i}(L)]$$

of radius larger than $L$. By the definition of $\mathcal{A}$ there are at most $M$ sites in the box $S_i \cap B_{x_i}(3L)$ connected to distance $L$ in $\omega_p$. Hence the same box also contains no more than $M$ sites in $C_\infty(\omega_p)$ since any site connected to infinity must be connected to distance $L$. Using the definition of $\mathcal{B}$ with $S$ being exactly $C_\infty(\omega_p) \cap S_i \cap B_{x_i}(3L)$ we see that $\omega_{p_c+\varepsilon} \setminus C_\infty(\omega_p)$ contains a crossing cluster for the box $S_i \cap B_{x_i}(3L)$ with all the properties listed before Lemma 4. In particular, the uniqueness property ensures two such crossing clusters in two neighbouring boxes must intersect. The result follows readily.

\[\square\]

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