Noise sensitivity from fractional query algorithms and the axis-aligned Laplacian

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Abstract

We introduce the notion of classical fractional query algorithms, which generalize decision trees in the average-case setting, and can potentially perform better than them. We show that the limiting runtime complexity of a natural class of these algorithms obeys the non-linear partial differential equation $\min_k \frac{\partial^2 u}{\partial x_k^2} = -2$, and that the individual bit revealment satisfies the Schramm-Steif bound for Fourier weight, connecting noise sensitivity with PDEs. We discuss relations with other decision tree results.

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1 Introduction

1.1 Decision trees and Boolean functions

A decision tree is an adaptive algorithm for determining the value of a function \( f : \{-1, 1\}^n \rightarrow \mathbb{R} \) given an unknown input \( x \in \{-1, 1\}^n \). At each step, the algorithm (possibly randomly) chooses an index \( i \in [n] \), and queries the value of the bit \( x_i \). We do not require that the algorithm always calculate \( f(x) \) exactly; it can stop running and output some value in \( \mathbb{R} \) even if it has not queried enough bits to fix the value of \( f(x) \). In general, the goal of the decision tree is to read as few bits as possible. Due to their simplicity as a computational model, decision trees have been studied extensively, especially in the worst-case setting, where the complexity of a tree is defined as the maximum expected number of queries it makes over all inputs \( x \in \{-1, 1\}^n \); see the excellent survey by Buhrman and de Wolf [6] for an exposition.

In this paper we will investigate the average-case setting, where the input \( x \) is drawn from some distribution \( \mu \) on \( \{-1, 1\}^n \). Two common complexity measures in this case are the expected number of queries that the tree makes, and the maximum probability that it reads any particular bit. In this setting, the complexity of a decision tree is tied with central notions in the analysis of Boolean functions, such as their variance and influences (see Section 2.1 for a review of Boolean functions). One result in this vein is given by Schramm and Steif [18], and was originally used to show quantitative noise sensitivity for percolation crossing events. It relates the probability of a tree with 2-bit queries (called \textit{revealment}) to the Fourier mass at level \( k \) of the function:

\[ \text{Theorem 1 (Theorem 1.8 in [18]). Let } f : \{-1, 1\}^n \rightarrow \mathbb{R} \text{ have Fourier representation } f(x) = \sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} x_i, \text{ and let } T \text{ be a decision tree calculating } f \text{ when the input is uniform. Set } \delta = \max_i \mathbb{P}[T \text{ reads bit } i]. \text{ Then for every } k \in \mathbb{N}, \text{ the Fourier coefficients of } f \text{ satisfy } \sum_{|S| = k} \hat{f}(S)^2 \leq \delta k \|f\|_2^2. \] (1)

Two vectors \( x, y \in \{-1, 1\}^n \) are said to be \( \rho \)-correlated if \( x \) and \( y \) are uniform in \( \{-1, 1\}^n \) and \( \mathbb{E}[x_i y_i] = \rho \) for all \( i \in [n] \). A sequence of functions \( f_m : \{-1, 1\}^{n_m} \rightarrow \{-1, 1\} \) is said to be noise sensitive with respect to noise \( \varepsilon_m > 0 \), if, when \( x, y \in \{-1, 1\}^{n_m} \) are \( (1-\varepsilon_m) \)-correlated, then \( \lim_{m \to \infty} \mathbb{E}[f(x)f(y)] - \mathbb{E}[f(x)]^2 = 0 \). For monotone functions, if \( \delta(f_m) \to 0 \), then Theorem 1, together with a theorem of Benjamini, Kalai and Schramm connecting Fourier coefficients and noise sensitivity [2, Theorem 1.5], gives quantitative bounds on how small the noise \( \varepsilon_m \) can be, depending on how quickly \( \delta(f_m) \to 0 \).

Another useful inequality was given by O’Donnell, Saks, Schramm and Steif [14], and relates the revealment probabilities to the function’s influences. The influence \( \text{Inf}_i(f) \) of the \( i \)-th bit is the probability that flipping the \( i \)-th bit changes \( f \)'s value when the input is uniform. The theorem states:

\[ \text{Theorem 2 (Theorem 1.1 in [14]). Let } f : \{-1, 1\}^n \rightarrow \{-1, 1\}. \text{ Let } T \text{ be a decision tree calculating } f \text{ when the input is uniform and set } \delta_i = \mathbb{P}[T \text{ reads bit } i]. \text{ Then } \text{Var}(f) \leq \sum_{i=1}^n \delta_i \text{Inf}_i(f). \] (2)

Inequality (2) can be used to show that functions which have small decision trees (i.e. make a small number of queries) must have an influential variable.

The above two theorems show why it is beneficial to find decision trees which query each bit with as small probability as possible. However, a classical theorem by Benjamini, Schramm and Wilson [3] states that the revealment \( \delta \) cannot be too small.

\[ \text{Theorem 3 (Theorem 2 in [3]). Let } f : \{-1, 1\}^n \rightarrow \{-1, 1\}. \text{ Let } T \text{ be a decision tree calculating } f \text{ which can err with probability } w. \text{ Let } \delta = \max_i \mathbb{P}[T \text{ reads bit } i]. \text{ Then } w \geq \frac{1}{8} \text{Var}(f) - \frac{1}{4} \delta^2. \]
1.2 Our results

Our main results are summarized as follows.

- Generalizing the notion decision trees, we define fractional query algorithms, where a query does not return the value of the bit, but rather reveals some information on what the bit is likely to be. Repeated fractional queries reveal more information, turning the bits into time-dependent processes $X_i(t)$ taking values in $[-1, 1]$. This class of algorithms contains the class of decision trees, but can potentially contain algorithms which have better run-time complexity. Our framework allows us to formulate the problem of decision tree complexity in the language of optimal control theory.

- We prove Theorem 1 for fractional query algorithms, and show that weaker versions of Theorem 2 and Theorem 3 also apply.

- We show that for a natural class of fractional query algorithms, the limiting run-time complexity obeys a partial differential equation corresponding to a simple dynamic programming principle.

1.2.1 Fractional query algorithms

Let $\mu$ be a product measure on $\{-1, 1\}^n$. There are two equivalent views for average-case decision tree inputs. In the first, an input $x$ is chosen according to $\mu$, and the decision tree simply queries its bits. In the second, no input is chosen beforehand, and whenever the decision tree queries a bit, that bit is randomly set to $\pm 1$ according to $\mu$. In this case, the input to the decision tree can be seen as a stochastic process $X(t) \in \{-1,*,1\}^n$. The process starts at $X(0) = (*,\ldots,*), where the *'s indicate that none of the bits have been read yet. At time $t$, the decision tree picks a (possibly random) index $i_t$ to query, and the $i_t$-th coordinate is set to $\pm 1$ with probabilities according to $\mu$. The process stops at time $\tau$, once the partial input $X(\tau)$ determines $f$ (or once the algorithm guesses the value of $f$, if it is allowed to make errors). The probability of reading a bit is given by $E\left[ X_i(\tau)^2 \right]$ ($*$'s are treated as 0), and the expected number of queries $C$ is equal to $C = E\sum_{i=1}^n X_i(\tau)^2$.

Building on the second view, in our framework the random input bits do not have to be completely queried, but rather can be queried fractionally. At time $t$, the algorithm chooses a bit $i_t$ and makes an $\varepsilon$-query on it; this causes the $i_t$-th coordinate to be randomly updated: $X_{i_t}(t+1) = X_{i_t}(t) \pm \varepsilon$, each with probability 1/2. Thus, the input stochastic process $X(t)$ does not have to take values in $\{-1,*,1\}^n$ as is the case for decision trees, but rather can take arbitrary values in the continuous cube, $X(t) \in [-1,1]^n$. Like ordinary decision trees, the process continues in this way until some time $\tau$, when the algorithm decides that it has enough information to guess the value of $f$ from the partial input $X(\tau)$. The idea behind this type of algorithm is that by using partial queries, the algorithm can get a sense of what the input will be, and avoid completely revealing bits which are likely to be useless. When the cost of making a partial query is appropriately defined, the complexity of fractional query algorithms yields bounds on properties of Boolean functions in a similar fashion to decision trees. We give the formal definitions below.

**Definition 4** (Axis-aligned jump process). Let $\varepsilon = 2^{-k}$ for some integer $k > 0$. An axis-aligned jump process with jump size $\varepsilon$ is a discrete-time martingale $X(t) \in [-1,1]^n \cap 2^{-k} \mathbb{Z}^n$ defined as follows\(^3\). At time $t$, if $X(t) \notin \{-1,1\}^n$, a direction $i_t \in [n]$ with $X_{i_t}(t) \in (-1,1)$ is chosen according to a direction choosing strategy $S$ (see below for more details). The new position $X(t+1)$ is then updated in a martingale fashion: for all $j \neq i_t$, we have $X_j(t+1) = X_j(t)$, and for $i_t$ we have $X_{i_t}(t+1) = X_{i_t}(t) + \varepsilon$, each with probability 1/2.

If $X(t) \in \{-1,1\}^n$, then $X(t)$ stays put, i.e. $X(t+1) = X(t)$. We denote this final value as $X(\infty)$. It is not hard to see that $X(\infty)$ distributes according to the product measure whose mean is $X(0)$.

\(^3\)Forcing the jumps to be of size $\varepsilon$ makes the analysis of these algorithms simpler. Allowing the jumps to be arbitrarily small moves the analysis into the continuous time domain, which introduces problems of measurability. See also Section 1.3. We will later take the limit of $\varepsilon \to 0$, in a controlled fashion.
Remark 5. There are several types of direction choosing strategies. The simplest type are deterministic Markov strategies, where the direction in which to go is only determined by the current position. We can thus write $i_t = S (X (t))$, for some function $S : [−1, 1]^n \to [n]$. In the most general setting, we can let the chosen direction depend explicitly on the time $t$, on the history of the trajectory, and on additional randomness which is independent of future decisions. We can also allow “lazy” strategies, which sometimes do not move $X (t)$ at all (this will be useful for later analysis).

Definition 6 (Axis-aligned query algorithm). Let $X (t)$ be an axis-aligned jump process with $X (0) = x_0$. An axis-aligned query algorithm is a triple $Q = (X (t), \tau, A)$, where $\tau$ is an $X (t)$-adapted stopping time, and $A : [−1, 1]^n \to \mathbb{R}$ is some function. The output of the algorithm is $A (X (\tau))$. We treat $X (\infty)$ as the input to the function $f$ under the product distribution whose mean is $x_0$, and say that the algorithm has 0 error if $A (X (\tau)) = f (X (\infty))$ almost surely. The individual bit revealments are given by
\[
\delta_i := \mathbb{E} \left[ (X_i (\tau) - X_i (0))^2 \right] = \text{Var} (X_i (\tau)),
\]
and the total cost of algorithm is given by
\[
C (x_0, Q) := \sum_{i=1}^n \delta_i.
\]
We write $C (Q) = C (0, Q)$ for the cost of the algorithm on the uniform measure. We denote the set of all 0-error axis-aligned algorithms by $S$, and the set of all 0-error decision trees by $D$. Decision trees are just axis-aligned algorithms, where the jump size is $\varepsilon = 1$.

For an example of how the process $X (t)$ might look like, see Figure 1, which shows some sample paths of the best axis-aligned algorithm for calculating the OR function on two bits.

Figure 1: As will be shown in Section 5, the fastest axis-aligned algorithm for computing the $n$-bit OR function always updates the largest bit. Here are three sample paths for this algorithm, with $n = 2$ and $\varepsilon = 2^{-7}$. The first two images correspond to a function value of 1, while the last image corresponds to a value of $-1$.

Remark 7. The Fourier representation of $f : \{-1, 1\}^n \to \mathbb{R}$ as a multilinear polynomial $f (x) = \sum_{S \subseteq [n]} \hat{f} (S) \prod_{i \in S} x_i$ allows us to extend $f$’s domain to the continuous cube $[-1, 1]^n$. This gives two natural choices for the output function $A$:

1. $A (x) = f (x)$. For any given stopping time $\tau$, this choice minimizes the $L^2$ error $\mathbb{E} \left[ (f (X (\infty)) - A (X (\tau)))^2 \right]$ (to see this, differentiate the error by the Fourier coefficients of $A$).

2. $A (x) = \text{sign} (f (x))$ (with arbitrary $\pm 1$ values when $f (x) = 0$). When $f$ takes only the values $\pm 1$ on $\{-1, 1\}^n$, for any given stopping time $\tau$, this choice minimizes the error probability $\mathbb{P} [A (X (\tau)) \neq f (X (\infty))]$. 


By mimicking their coordinate-by-coordinate updates, axis-aligned algorithms are a natural generalization of decision trees. However, our revealment results apply to a larger class of algorithms, defined below, which use more general processes to model their input. For example, the process $X(t)$ can be driven by Brownian motion, where only one coordinate can move at a given time. These processes are common in optimal control theory (see e.g. [13]).

**Definition 8** (Fractional query algorithm). Let $\mu$ be a product measure with mean $x_0$ from which the inputs are drawn. A fractional query algorithm on inputs distributed by $\mu$ is a triplet $Q := (X(t), \tau, A)$ such that:

1. $X(t) \in [-1, 1]^n$ is a martingale, $X(0) = x_0$, and $X(\infty) \sim \mu$. The time parameter can be either discrete or continuous.
2. $\tau$ is an $X(t)$-adapted stopping time.
3. $A$ is a function $A : [-1, 1]^n \rightarrow \mathbb{R}$.
4. For every $S \subseteq [n]$, the stochastic process $(\prod_{i \in S} X_i(t))$ is a martingale.

The output of the algorithm is $A(X(\tau))$. The revealment and cost of fractional query algorithms are the same as in equations (3) and (4). We denote the set of all fractional query algorithms by $Q$.

**Remark 9.** Using the fact that the expected value of the quadratic variation of a martingale is equal to its variance, we can also write the cost as

$$C(x_0, Q) = \mathbb{E} \sum_{i=1}^{n} [X_i]_{\tau},$$

where $[X_i]_{t}$ is the quadratic variation of $X_i$ at time $t$. When $X(t)$ is a discrete-time process, this means that the cost of the algorithm is given by the expected sum of squares of its jumps.

**Remark 10.** Condition (4) ensures that the individual coordinates possess “enough independence”, and is needed for one of our theorems later on. It is satisfied for all axis-aligned processes.

### 1.2.2 Revealment bounds

The class of fractional query algorithms $Q$ contains the class of decision trees $D$, and so in general, for every $i \in [n]$, $\inf_{Q \in Q} \delta_i \leq \inf_{D \in D} \delta_i$. We show that several classical results concerning decision tree revealment hold also for fractional query algorithms. We start with Theorem 1 by Schramm and Steif.

**Theorem 11.** Let the inputs be drawn from the uniform distribution. Let $f : \{-1, 1\}^n \rightarrow \mathbb{R}$, let $(X(t), \tau, f)$ be a fractional query algorithm for $f$ with $L^2$ error bounded by $\varepsilon$:

$$\mathbb{E} \left[ (f(X(\infty)) - f(X(\tau)))^2 \right] \leq \varepsilon^2.
$$

Denote $\delta = \max_{i \in [n]} \mathbb{E} \left[ X_i(\tau)^2 \right]$. Then for every $k \in [n]$,

$$\sum_{|S|=k} \hat{f}(S)^2 \leq \left( \sqrt{\mathbb{E} (f(X(\tau)))^2} \sqrt{k} \delta + \varepsilon \right)^2.
$$

In particular, if $(X(t), \tau, f)$ has 0 error, we recover Schramm and Steif’s original bound:

$$\sum_{|S|=k} \hat{f}(S)^2 \leq \delta k \|f\|_2^2.
$$

Next, using the computations in the proof of the above theorem, we show that when restricted to bounded-degree functions, a weaker version of the OSSS-inequality (2) applies for fractional query algorithms.
Theorem 12. Let the inputs be drawn from the uniform distribution. Let \( f : \{-1,1\}^n \rightarrow \mathbb{R} \) have degree bounded by \( d \), let \( (X(t), \tau, f) \) be a 0-error fractional query algorithm for \( f \), and let \( \delta_i = \mathbb{E}\left[X_i(\tau)^2\right] \). Then
\[
\text{Var}(f) \leq d \sum_i \delta_i \mathbb{E}\left[\|\partial_i f\|^2\right]. \tag{5}
\]

This improves Corollary 3.4 in [14], which had the same result for decision trees but with the leading constant \( d \) replaced by “maximum depth of a decision tree” (the depth is always larger than the degree). Proving Theorem 2 in full generality remains an open problem; see Section 1.3.

Finally, we remark that for 0-error algorithms, the bound in Theorem 3 holds also for fractional query algorithms.

Theorem 13. Let \( f : \{-1,1\}^n \rightarrow \{-1,1\} \) and let \( (X(t), \tau, f) \) be a fractional query algorithm which calculates \( f \) exactly. Let \( \delta = \max_i \mathbb{E}\left[X_i(\tau)^2\right] \). Then
\[
\delta \geq \sqrt{\frac{\text{Var}(f)}{2n}}.
\]

In fact, this theorem stems from a similar result about function certificates (see Section 3.3), whose proof is basically identical to that in [3]. This gives hope that fractional query algorithm can achieve better revealment bounds if they are allowed to make errors.

The proofs of the above theorems are found in Section 3.

1.2.3 Limits of axis-aligned algorithms

When considering axis-aligned algorithms, it is natural to ask what happens in the limit of \( \varepsilon \rightarrow 0 \). By Remark 9, for a fixed \( \varepsilon \), each step taken by the process increases the cost by \( \varepsilon^2 \), and so the task is to minimize the expected runtime. Intuitively, we can extend the notion of axis-aligned processes to start at any \( \varepsilon \in [-1,1]^n \), rather than just dyadic starting positions. If we define \( u_\varepsilon(x) : [-1,1]^n \rightarrow \mathbb{R} \) to be the cost of the best axis-aligned algorithm which starts at \( x \), then \( u_\varepsilon \) should satisfy
\[
\min_k \frac{\partial^2}{\partial x_k^2} u_\varepsilon = -2 \tag{6}
\]
where \( \varepsilon \) is the unit vector in direction \( i \). Taking \( \varepsilon \rightarrow 0 \), the underlying processes \( X(t) \) should converge to a continuous time process, whose cost \( u(x) \) at every starting point \( x \) is the best possible among all axis-aligned algorithms. Equation (6) suggests that \( u \) should satisfy \( \min_k \frac{\partial^2}{\partial x_k^2} u = -2 \) (see Remark 26).

This intuition can be made precise in the framework of viscosity solutions to partial differential equations (see Section 2.2. The main difficulty is that we have no assurance about the differentiability of neither \( u_\varepsilon \) nor \( u \)). This is the main content of Section 4, which culminates in the following theorem.

Theorem 14. Let \( f : \{-1,1\}^n \rightarrow \{-1,1\} \) be non-constant, and let \( u(x) = \lim_{\varepsilon \to 0} \inf_{Q_\varepsilon} C(x,Q_\varepsilon) \), where \( \varepsilon \) is of the form \( 2^{-k} \) and the infimum is taken over all axis-aligned algorithms with jump size \( \varepsilon \) starting at \( x \). Then \( u \) is the unique continuous viscosity solution to the Dirichlet boundary-value problem
\[
\begin{cases}
\Delta_1 u = -2 & x \in (-1,1)^n \\
u = F & x \in \partial(-1,1)^n,
\end{cases}
\]
where \( \Delta_1 \) is the nonlinear operator \( \min_k \frac{\partial^2}{\partial x_k^2} \), and \( F \) is obtained by recursively solving the Dirichlet problem on the \((n-1)\) dimensional facets for the appropriate restrictions of \( f \).
As a possible application of this theorem, consider the class of transitive Boolean functions (see Section 2.1 for a definition). For such functions, the individual bit revealments can be made equal under the uniform distribution, and a query algorithm $Q$ yields $\delta = C(Q) / n$. Theorem 11 then gives

$$\sum_{|S|=k} \hat{f}(S)^2 \leq \frac{u(0)}{n} k \|f\|_2^2,$$

allowing us to obtain noise sensitivity bounds by solving PDEs.

Using this framework, in Section 5 we calculate $u(x)$ exactly for the OR function between two bits, and show that as $n \to \infty$, fractional query algorithms need to query only 1 bit in expectation in order to calculate the OR function on $n$ bits (whereas decision trees need to query 2 bits in expectation). This is the best separation we have found so far between decision trees and fractional query algorithms; we do not know whether an asymptotic separation is possible (see Section 1.3). We also propose a heuristic, which might prove useful in analyzing recursive functions.

1.2.4 Fractional random-turn games

Finally, in Section 6 we show that random-turn games, introduced by Peres, Schramm, Sheffield and Wilson in [16], yield natural axis-aligned algorithms. However, these algorithms do not necessarily minimize the cost $C(Q)$.

1.3 Open questions

1. We have so far been able to neither prove nor disprove that fractional query algorithms can run asymptotically faster than decision trees. Does there exist a global constant $M$, so that for every function $f : \{-1, 1\}^n \to \{-1, 1\}$, $\inf_{Q \in Q} C(Q) \geq M \inf_{D \in D} (D)$? We find both possible answers to be exciting: if fractional query algorithms can run asymptotically faster (say, on transitive monotone functions), then Theorem 11 gives improved noise sensitivity estimates; and if fractional query algorithms are equivalent to decision trees, then Theorem 14 allows us to obtain decision tree lower bounds by solving partial differential equations.

2. Show that for functions $f : \{-1, 1\}^n \to \{-1, 1\}$, Theorem 2 applies to fractional query algorithms. The problem, as is alluded to in [14], is that the inequality (2) is at heart an $L^1$ bound, while the fractional query costs are in essence $L^2$ variables: since $X(\tau)$ is a martingale, the cost $\mathbb{E}X_i(\tau)^2$ incurred by bit $i$ is equal to the expected quadratic variation $\mathbb{E}[X_i]$. It is possible to define instead the cost by the total variation of $X_i(\tau)$, and get an analogous inequality for this alternate cost, but this would undeniably rule out a large class of processes $X(t)$ which have infinite total variation (such as Brownian motion).

3. Can Theorem 3 be extended to fractional query algorithms which are allowed to make errors?

4. Let $S$ be the set of axis-aligned algorithms. Show that these are the best algorithms possible, i.e. that $\inf_{Q \in Q} C(Q) = \inf_{S \in S} C(S)$.

5. Show that when the boundary conditions are recursively defined by Boolean functions as in Theorem 14, the limiting value function $u$ of axis-aligned algorithms is twice continuously differentiable (thereby eliminating the need to use viscosity solutions, and making life easier for the working mathematician).

6. Show that for axis-aligned algorithms with jump size $\varepsilon \to 0$, there is a sense in which the underlying process $X(t)$ converges to Brownian motion, where only one coordinate can move at a time (see e.g. [9, 13] for an exact formulation of this framework).
1.4 Related fractional work

In the context of axis-aligned algorithms, Jacka, Warren and Windridge [9] inspect the minimum time for continuous processes to compute the value of 3-majority. While this can be seen as a problem in optimal control theory, most work in the literature focus on different payoff models. See Section 1.1 in [9] for more detail.

Fractional processes appear in other areas of Boolean analysis. In [7], Chattopadhyay, Hatami, Hosseini and Lovett construct a random walk \( X(t) \) so that

\[
|E[f(X(\tau)) - Ef]| \leq \varepsilon
\]

for functions with bounded Fourier tails. In some sense, they approach the problem from a different viewpoint than us: in order for (7) to be meaningful, they require that \( E[X_i(\tau)^2] \) is large, whereas our theorems are useful when \( E[X_i(\tau)^2] \) is small. This stresses the difference between noise sensitive functions and functions with bounded Fourier tails.

Axis-aligned processes can be seen as a type of single player “tug-of-war” game. The influential work by Peres, Schramm, Sheffield and Wilson [17] considers a two-player version. There, a dynamic programming equation also leads to a PDE, involving the infinity Laplacian \( \Delta_\infty \) rather than the axis-aligned Laplacian \( \Delta^\dagger \).

Algorithms which make fractional queries have already been investigated in the quantum setting, by allowing gates to impart fractional phases (or, equivalently, by running a Hamiltonian for a small amount of time). It has been shown that if errors are allowed, quantum fractional query algorithms are not much more powerful than quantum discrete query algorithms. See e.g. [4, 11].

The process \( X(t) \) can be seen as a noisy version of the true input \( X(\infty) \), where more confidence is given to coordinates with larger absolute value. A different model based on noisy inputs is given by Ben-David and Blais in [1]. They consider decision trees which can make noisy queries, and pay a cost of \( \gamma^2 \) in order to get a random bit \( \tilde{x}_i \in \{-1,1\} \) that is \( \gamma \)-correlated with the true input \( x_i \).

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2 Background and notation

We denote the standard basis of \( \mathbb{R}^n \) by \( \{e_1, \ldots, e_n\} \). For a set \( \Omega \subseteq \mathbb{R}^n \), we denote its boundary by \( \partial \Omega \).

2.1 Boolean functions

For a general introduction to Boolean functions, see [15]; in what follows, we provide a brief overview of the required background and notation.

A distribution \( \mu \) over \( \{-1,1\}^n \) is a product measure if, for \( x \sim \mu \), the bits \( x_i \) are all independent. Denote by \( \mu_{1/2} \) the uniform measure on \( \{-1,1\}^n \). For every function \( f: \{-1,1\}^n \to \mathbb{R} \), its expectation and variance are given by

\[
Ef = E_{y \sim \mu_{1/2}} f(y) \quad \text{and} \quad \text{Var}(f) = E(f - Ef)^2.
\]

Every function \( f: \{-1,1\}^n \to \mathbb{R} \) may be uniquely written as a sum of monomials:

\[
f(y) = \sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} y_i,
\]
where \([n] = \{1, \ldots, n\}\), and \(\hat{f}(S)\) are known as the Fourier coefficients. By Parseval’s identity, the squared norm of a function is given by
\[
\|f\|_2^2 := \mathbb{E}[f^2] = \sum_{S \subseteq [n]} \hat{f}(S)^2,
\]
while the variance of a function is given by the sum of its Fourier weights of level greater than 0:
\[
\operatorname{Var}(f) = \sum_{S \neq \emptyset} \hat{f}(S)^2.
\]
Equation (8) may be used to extend a function’s domain from the discrete hypercube \([-1, 1]^n\) to real space \(\mathbb{R}^n\). We call this the harmonic extension, and denote it also by \(f\). Under this notation, \(f(0) = \mathbb{E}f\). The derivative of a function \(f\) in direction \(i\) is defined as
\[
\partial_i f \left( y \right) = \frac{f \left( y^{-i}1 \right) - f \left( y^{i-1}1 \right)}{2},
\]
where \(y^{-i}a\) has \(a\) at coordinate \(i\), and is identical to \(y\) at all other coordinates. The Fourier representation of the derivative is given by
\[
\partial_i f = \sum_{S \ni i} \hat{f}(S) \prod_{j \in S, j \neq i} x_j.
\]
(9)
A function is called monotone if \(f(x) \leq f(y)\) whenever \(x_i \leq y_i\), for all \(i \in [n]\). A function is called transitive if for all \(i, j \in [n]\), there exists a permutation \(\pi\) on the \(n\) bits with \(\pi(i) = j\) such that for all \(x \in \{-1, 1\}^n\), \(f(x) = f(\pi(x))\).

### 2.2 Viscosity solutions

**Definition 15.** For a twice-differentiable function \(u : \mathbb{R}^n \to \mathbb{R}\), define the axis-aligned Laplacian \(\Delta_i\) by
\[
\Delta_i u = \min_k \frac{\partial^2 u}{\partial x_k^2}.
\]
In this work, we will need to apply the operator \(\Delta_i\) to functions \(u\) which are continuous, but might not necessarily be twice- or even once-differentiable. One natural framework for this is that of viscosity solutions. For a good introduction, see [8]. We will require only the basic definitions.

**Definition 16.** Let \(g : \mathbb{R}^n \to \mathbb{R}\). The function \(u : \mathbb{R}^n \to \mathbb{R}\) is called a

1. viscosity subsolution to the equation \(\Delta_i u = g\), if for every smooth function \(\varphi : \mathbb{R}^n \to \mathbb{R}\) such that \(\varphi - u\) has a minimum at point \(x_0\), we have \(\Delta_i \varphi(x_0) \geq g(x_0)\);
2. viscosity supersolution to the equation \(\Delta_i u = g\), if for every smooth function \(\varphi : \mathbb{R}^n \to \mathbb{R}\) such that \(\varphi - u\) has a maximum at point \(x_0\), we have \(\Delta_i \varphi(x_0) \leq g(x_0)\);
3. viscosity solution to the equation \(\Delta_i u = g\) if it is both a viscosity subsolution and a viscosity supersolution.

**Remark 17.** Suppose \(\varphi(x)\) is such that \(\varphi - u\) has a minimum at \(x_0\) but \(\Delta_i \varphi(x_0) < g(x_0)\). Then for \(\varepsilon > 0\) small enough, the function \(\varphi'(x) := \varphi(x) + \varepsilon \|x - x_0\|^2\) also satisfies \(\Delta_i \varphi'(x_0) < g(x_0)\), and has a strict minimum at \(x_0\). When checking whether \(u\) is a viscosity solution to \(\Delta_i u = g\), we can therefore restrict ourselves to test functions \(\varphi\) with strict minima / maxima at \(x_0\).
3 Classical revealment bounds

3.1 Theorem 11

The proof follows the lines of the original proof by Schramm and Steif. Some additional calculations must be made when using the language of fractional query algorithms.

Proof of Theorem 11. Let $g : \{-1, 1\}^n \rightarrow \mathbb{R}$ be defined as $g(x) = \sum_{|S|=k} \hat{f}(S) \prod_{i \in S} x_i$. Let $Z(t) = f(X(\infty)) - f(X(t))$ be the error of the algorithm as a function of time. Since $X(\infty)$ is uniform on $\{-1, 1\}^n$, by Plancherel’s inequality, we have, for all $t$,

$$\|g\|_2^2 \leq \mathbb{E} \left[ g(X(\infty)) (Z(t) + f(X(t))) \right].$$

By the Cauchy-Schwarz inequality, for $t = \tau$,

$$\mathbb{E} [g(X(\infty)) (Z(\tau))] \leq \sqrt{\mathbb{E} g(X(\infty))^2} \sqrt{\mathbb{E} Z(\tau)^2} \leq \|g\|_2 \varepsilon. \quad (10)$$

Since $f(X(\tau))$ is determined by $X(\tau)$, we have

$$\mathbb{E} [g(X(\infty)) f(X(\tau))] = \mathbb{E} [\mathbb{E} [g(X(\infty)) f(X(\tau)) | X(\tau)]]$$

$$= \mathbb{E} [f(X(\tau)) \mathbb{E} [g(X(\infty)) | X(\tau)]]$$

$$\overset{\text{Property } (4)}{=} \mathbb{E} [f(X(\tau)) \mathbb{E} [g(X(\tau))]]$$

$$\overset{\text{Cauchy-Schwarz}}{\leq} \sqrt{\mathbb{E} f(X(\tau))^2} \sqrt{\mathbb{E} g(X(\tau))^2}. \quad (11)$$

To bound the term $\mathbb{E} [g(X(\tau))^2]$, we show how the harmonic extension is related to an interpolation function. Let $x \in [-1, 1]^n$. For any function $h : \{-1, 1\}^n \rightarrow \mathbb{R}$ with Fourier decomposition $h(y) = \sum_{S} \hat{h}(S) \prod_{i \in S} y_i$, let $h_x : \{-1, 1\}^n \rightarrow \mathbb{R}$ be defined by mapping $y_i \mapsto \sqrt{1 - x_i^2} y_i + x_i$ for every $i$:

$$h_x(y) = \sum_{S} \hat{h}(S) \prod_{i \in S} \left( \sqrt{1 - x_i^2} y_i + x_i \right). \quad (12)$$

Note that the expected value of $h_x$ is given by $\mathbb{E} h_x = \hat{h}_x(\emptyset) = \sum_{S} \hat{h}(S) \prod_{i \in S} x_i$. This is exactly the same as inputting the vector $x$ into the harmonic extension of $h$. Thus

$$h(x) = \hat{h}_x(\emptyset).$$

Taking $h = g$ and $x = X(\tau)$ in (12), we have

$$\mathbb{E} \left[ g(X(\tau))^2 \right] = \mathbb{E} \left[ \hat{g}(X(\tau))(\emptyset)^2 \right]$$

$$\overset{\text{Parseval}}{=} \mathbb{E} \left[ \|g(X(\tau))\|^2 - \sum_{|S|>0} \hat{g}(X(\tau))(S)^2 \right].$$

By definition of the 2-norm,

$$\mathbb{E} \|g_X(\tau)\|_2^2 = \mathbb{E}_{X(\tau)} \mathbb{E}_y \left[ g_X(\tau)(y)^2 \right]$$

$$= \mathbb{E}_{X(\tau)} \mathbb{E}_y \sum_{S,T \subseteq [n]} \hat{g}(S) \hat{g}(T) \prod_{i \in S} \left( \sqrt{1 - X_i(\tau)^2} y_i + X_i(\tau) \right) \prod_{j \in T} \left( \sqrt{1 - X_j(\tau)^2} y_j + X_j(\tau) \right).$$
If \( i \in S \cap T \), then
\[
\mathbb{E}_y \left[ \left( \sqrt{1 - X_i (\tau)^2} y_i + X_i (\tau) \right)^2 \right] = \mathbb{E}_y \left[ \left( 1 - X_i (\tau)^2 \right) y_i^2 + X_i (\tau)^2 - 2 y_i X_i (\tau) \sqrt{1 - X_i (\tau)^2} \right] \overset{y_i^2 = 1, \mathbb{E} y_i = 0}{=} 1.
\]

By independence of the coordinates of \( y \), indices in \( S \cap T \) therefore do not contribute to the product \( \prod_{i \in S} (\ldots) \prod_{j \in T} (\ldots) \), and we are left with a product of the form \( \prod_{i \in S \setminus T} (\ldots) \prod_{j \in T \setminus S} (\ldots) \). Since \( \mathbb{E} y_i = 0 \) for all \( i \in [n] \) and \( \mathbb{E} \prod_{i \in A} X_i (\tau) = 0 \) for all \( A \subseteq [n] \), the coefficient of \( \hat{g} (S) \hat{g} (T) \) is 0 if \( S \neq T \), and 1 if \( S = T \). Thus
\[
\mathbb{E} \left\| \hat{g}_{X(\tau)} \right\|_2^2 = \sum_{|S| \leq [n]} \hat{g} (S)^2,
\]
and we get
\[
\mathbb{E} \left[ g (X (\tau))^2 \right] \leq \sum_{|S| = k} \hat{g} (S)^2 - \mathbb{E} \sum_{|S| > 0} \hat{g}_{X(\tau)} (S)^2
\]
by definition of \( g \)
\[
\leq \sum_{|S| = k} \left( \hat{g} (S)^2 - \mathbb{E} \hat{g}_{X(\tau)} (S)^2 \right).
\]

(13)

Since \( g \) itself does not have Fourier coefficients of frequencies larger than \( k \), the only sets \( S \) with \( |S| = k \) in the Fourier decomposition of \( g_{X(\tau)} \) can come from choosing the factor \( \sqrt{1 - X_i (\tau)^2} \) every time in the product in (12). Thus, for \( |S| = k \),
\[
\hat{g} (S)^2 - \hat{g}_{X(\tau)} (S)^2 = \hat{g} (S)^2 \left( 1 - \prod_{i \in S} \left( \sqrt{1 - X_i (\tau)^2} \right)^2 \right)
\]
\[
= \hat{g} (S)^2 \left( 1 - \prod_{i \in S} (1 - X_i (\tau)^2) \right)
\]
Weierstrass inequality
\[
\leq \hat{g} (S)^2 \left( 1 - \left( 1 - \sum_{i \in S} X_i (\tau)^2 \right) \right)
\]
\[
\leq \hat{g} (S)^2 \sum_{i \in S} X_i (\tau)^2.
\]

(14)

Plugging this back into (13), we get
\[
\mathbb{E} \left[ g (X (\tau))^2 \right] \leq \sum_{|S| = k} \hat{g} (S)^2 \sum_{i \in S} \mathbb{E} X_i (\tau)^2
\]
\[
\leq \sum_{|S| = k} \hat{g} (S)^2 k \delta = \|g\|_2^2 k \delta.
\]

Together with (10) and (11), this yields the desired result.

\( \square \)

3.2 Theorem 12

Proof of Theorem 12. Since both the left-hand side and the right-hand side of (5) are invariant to shifts of the type \( f \mapsto f + c \), we can assume without loss of generality that \( \mathbb{E} f = 0 \).
Let $k \in \mathbb{N}$. As in the proof of Theorem 11, define $g_k : \{-1, 1\}^n \to \mathbb{R}$ by
\[
g_k(x) = \sum_{|S| = k} \hat{f}(S) \prod_{i \in S} x_i.
\]
By equations (11), (13) and (14) in the proof of Theorem 11, using the fact that $f$ is computed exactly, we have
\[
\left( \|g_k\|_2^2 \right)^2 \leq \|f\|_2^2 \sum_{|S| = k} \hat{g}_k(S)^2 \sum_{i \in S} \mathbb{E}X_i(\tau)^2 = \|f\|_2^2 \sum_{i = 1}^n \delta_i \sum_{S \ni i, |S| = k} \hat{f}(S)^2.
\]
Supposing that $f$ is a degree-$d$ polynomial, we can sum the above for all $k = 1, \ldots, d$. Recalling that
\[
\partial_i f = \sum_{S \ni i} \hat{f}(S) \chi_{S \setminus \{i\}} \text{(see (9))}, \quad \text{we get } \|\partial_i f\|_2^2 = \sum_{S \ni i} \hat{f}(S)^2,
\]
and so
\[
\sum_{k = 1}^d \left( \|g_k\|_2^2 \right)^2 \leq \|f\|_2^2 \sum_{i = 1}^n \delta_i \|\partial_i f\|_2^2. \quad (15)
\]
The left hand side gives an upper bound to $\|f\|_2^4$:
\[
\left( \|f\|_2^2 \right)^2 = \left( \sum_{k = 1}^d \|g_k\|_2^2 \right)^2 \leq d \sum_{k = 1}^d \left( \|g_k\|_2^2 \right)^2,
\]
and plugging this into (15) gives
\[
\|f\|_2^2 \leq d \sum_{i = 1}^n \delta_i \|\partial_i f\|_2^2.
\]
Since we assume that $\mathbb{E}f = 0$, we have $\text{Var}(f) = \|f\|_2^2$, and the result follows.

### 3.3 Theorem 13

The proof does not use the fact that the processes are allowed to take fractional values in $[-1, 1]^n$, and instead uses a result about certificates. For $x \in \{-1, 1\}^n$, an $f(x)$-certificate is a minimal set $S \subseteq [n]$ such that for all $y \in \{-1, 1\}^n$ with $x_i = y_i$ for $i \in S$, we have $f(x) = f(y)$. In other words, the value of $f$ does not change if we change the bits outside of $S$. Theorem 13 is actually a special case of the following average-case certificate complexity result, whose proof follows that of Theorem 3 in [3].

**Lemma 18.** For every $x \in \{-1, 1\}^n$, let $\Gamma_x$ be a probability distribution on $2^{[n]}$ supported on the set of $f(x)$-certificates, and let $S_x \sim \Gamma_x$. Let $\delta_i = \mathbb{P}_{x \sim \text{Unif}(\{-1, 1\}^n)} [i \in S_x]$ and $\delta = \max_i \delta_i$. Then
\[
\delta \geq \sqrt{\frac{\text{Var}(f)}{2n}}.
\]

**Proof.** Let $x, y \in \{-1, 1\}^n$ be independent and uniformly random. Let $N = |S_x \cap S_y|$. Then
\[
\mathbb{P}[N > 0] \leq \mathbb{E}[N] = \sum_{i = 1}^n \delta_i^2 \leq n\delta^2.
\]
Let $z \in \{-1, 1\}^n$ be defined by
\[
z_i = \begin{cases} x_i & i \in S_x, \\ y_i & \text{otherwise}. \end{cases}
\]
Clearly, \( f(x) = f(z) \), since both \( x \) and \( z \) are identical on the same \( f(x) \)-certificate \( S_x \). If \( N = 0 \), then \( z \) is also identical with \( y \) on \( S_y \), and so \( f(y) = f(z) \), implying that \( f(x) = f(y) \). We then have
\[
1 = P[N = 0] + P[N > 0] \\
\leq P[f(x) = f(y)] + n\delta^2 \\
= P[f(x) = 1]^2 + (1 - P[f(x) = 1])^2 + n\delta^2 \\
= 1 - \frac{1}{2} \text{Var}(f) + n\delta^2,
\]
yielding the result. \( \square \)

Proof of Theorem 13. If a fractional query algorithm calculates \( f \) exactly, it must completely reveal all the bits of some \( f(X(\infty)) \) certificate. Hence, for every \( x \in \{-1, 1\}^n \), every fractional query algorithm induces a probability distribution \( \Gamma_x \) on \( f(x) \) certificates, and \( \mathbb{E}\left[ X_i(\tau)^2 \right] \geq P_{x \sim \text{Unif}(-1,1)^n} [i \in S_x] \). \( \square \)

4 Dynamic programming and the axis-aligned Laplacian

In this section, we show that the cost \( u_\varepsilon \) of axis-aligned query algorithms converges as \( \varepsilon \to 0 \) (Proposition 23). We show that that \( u \) is Lipschitz, and so the convergence is uniform (Lemma 21 and Lemma 24). This allows us to prove that \( \Delta_1 u = -2 \) (Theorem 27). Uniqueness follows from a general comparison principle on the axis-aligned Laplacian (Theorem 29).

We first redefine axis-aligned processes to allow starting at non-dyadic points \( x \in [-1, 1]^n \). In order to keep the process inside \([-1, 1]^n \) at all times, and still preserve its martingale nature, this forces some changes to the update rule near the boundary of the cube.

Definition 19 (Axis-aligned jump process). Let \( \varepsilon > 0 \). An axis-aligned jump process with jump size \( \varepsilon \) is a discrete-time martingale \( X(t) \in [-1, 1]^n \). At time \( t \), if \( X(t) \notin \{-1, 1\}^n \), a direction \( i_t \in [n] \) with \( X_i(t) \in (-1, 1) \) is chosen according to a direction choosing strategy \( S \). Let \( a_t, b_t \) be defined as follows.

1. If the distance between \( X_{i_t}(t) \) to both endpoints \( \{-1,1\} \) is at least \( \varepsilon \), i.e. \( X_{i_t}(t) + \varepsilon < 1 \) and \( X_{i_t}(t) - \varepsilon > -1 \), then \( a_t = X_{i_t}(t) - \varepsilon \) and \( b_t = X_{i_t}(t) + \varepsilon \).
2. If \( X_{i_t}(t) + \varepsilon > 1 \), then \( a_t = 1 - 2\varepsilon \) and \( b_t = 1 \).
3. If \( X_{i_t}(t) - \varepsilon < -1 \), then \( a_t = -1 \) and \( b_t = -1 + 2\varepsilon \).

The new position \( X(t+1) \) is then updated in a martingale fashion: for all \( j \neq i_t \), we have \( X_j(t+1) = X_j(t) \), and for \( i_t \) we have
\[
X_{i_t}(t+1) = \begin{cases} a_t \text{ with probability } \frac{b_t - X_{i_t}(t)}{2\varepsilon} \\ b_t \text{ with probability } \frac{X_{i_t}(t) - a_t}{2\varepsilon} \end{cases}.
\]

We will deal only with 0-error axis-aligned query algorithms. We therefore always set \( \tau = \inf \{ t > 0 \mid f(X(t)) \in \{-1,1\} \} \) and \( A(x) = f(x) \). Each direction choosing strategy \( S \) thus determines an axis-aligned query algorithm.

For a given strategy \( S \), denote its expected cost when starting at \( X(0) = x \) by \( C(x, S) = \sum_{i=1}^n \text{Var}(X_i(\tau)) \). For \( x \in [-1, 1]^n \), let \( u_\varepsilon : [-1, 1]^n \to \mathbb{R} \) be the best possible cost over all axis-aligned algorithms:
\[
u_\varepsilon(x) = \inf_S C(x, S).
\]

Under strategy \( S \), the cost when starting at \( x \in [-1, 1]^n \) can be broken up into two parts: the cost for taking a single step, given by \( \mathbb{E} \| X(1) - x \|_2^2 \), and the cost of continuing the strategy starting at \( X(1) \), given by
\( \mathbb{E} C (X (1), S') \), where \( S' \) is some other strategy related to \( S \), which incorporates the knowledge that \( t = 1 \) and \( X (0) = x \). Thus

\[
C (x, S) = \mathbb{E} \left[ \| X (1) - X (0) \|_2^2 + C (X (1), S') \right].
\]

The right-hand side is a convex combination of choices of directions, and the strategy \( S \) is always improved by picking the minimal direction. It is then clear that in order to minimize the cost, it is enough to consider only deterministic Markov strategies. Since each Markov strategy only has finitely many directions to choose from, there must in fact exist an optimal strategy.

**Fact 20.** There is a Markov strategy \( S \) such that for every \( x \in [-1, 1]^n \), \( u_\varepsilon (x) = C (x, S) \). For \( x \in [-1 + \varepsilon, 1 - \varepsilon]^n \), we have

\[
u_\varepsilon (x) = \min_i \frac{u_\varepsilon (x + \varepsilon e_i) + u_\varepsilon (x - \varepsilon e_i)}{2} + \varepsilon^2.
\]

We can construct an equivalent, recursive formulation of the cost function \( u_\varepsilon \). In this variant, we are given a function \( F : \partial [-1, 1]^n \rightarrow \mathbb{R} \) defined on the boundary of the hypercube. We again run a process \( X (t) \) while choosing directions using a strategy \( S \), but the goal is now to minimize the running-cost plus the exit-cost \( F \). More formally, let \( \sigma = \inf \{ t > 0 \mid X (t) \in \partial [-1, 1]^n \} \) be the first time that the process \( X (t) \) hits the boundary of the hypercube. The cost of the strategy \( S \) in this model is

\[
C_F (x, S) = \sum_{i=1}^{n} \text{Var} (X_i (\sigma)) + EF (X (\sigma)).
\]

When \( F \) is defined recursively on the facets of the hypercube as the best cost among all axis-aligned algorithms, then minimizing \( C_F (x, S) \) is equivalent to minimizing \( C (x, S) \) (i.e. start with \( F (x) = 0 \) for all \( x \in \{-1, 1\}^n \), then iteratively set the value of \( F \) for all lines, squares, cubes, etc. by considering axis-aligned algorithms with jump size \( \varepsilon \)). In this case, the cost \( u_\varepsilon \) equals \( F \) on the boundary. This formulation is useful for proving the following lemma.

**Lemma 21.** Let \( n > 0 \) be an integer and let \( L_n = \frac{n + 3}{2} \). Then \( u_\varepsilon \) is \( L_n \)-Lipschitz under the one-norm, i.e. \( |u_\varepsilon (x) - u_\varepsilon (y)| \leq \frac{n + 3}{2} \| x - y \|_1 \) for all \( x, y \in [-1, 1]^n \).

**Proof.** By induction on the dimension. For \( n = 1 \), if \( f \) is constant then \( u_\varepsilon = 0 \). Otherwise, there is only one strategy: move in a martingale fashion until you reach the endpoints. Thus

\[
u_\varepsilon (x) = \text{Var} (X (\tau)) = \mathbb{E} X (\tau)^2 - \mathbb{E} X (0)^2 = 1 - x^2.
\]

We then have

\[
|u_\varepsilon (x) - u_\varepsilon (y)| = |1 - x^2 - (1 - y^2)| = |x^2 - y^2| = |x - y| |x + y| \leq 2 |x - y|.
\]

Now assume that the claim is true for any two points on the same \( n - 1 \)-dimensional facet of the hypercube, and let \( x, y \in [-1, 1]^n \). We prove the claim for \( n \) dimensions by considering the different relations that \( x \) and \( y \) can have.

1. Suppose both \( x, y \) are on the boundary \( \partial [-1, 1]^n \). If they are on the same facet, the claim follows by the induction hypothesis. If they are on two adjacent facets, we can assume without loss of generality that \( x = (1, x_2, x_3, \ldots, x_n) \) and \( y = (y_1, 1, y_3, \ldots, y_n) \). Then the point \( z = (1, 1, x_3, \ldots, x_n) \) shares a facet with \( x \) and with \( y \) and satisfies \( \| x - y \|_1 = \| x - z \|_1 + \| y - z \|_1 \). We then have

\[
|u_\varepsilon (x) - u_\varepsilon (y)| \leq |u_\varepsilon (x) - u_\varepsilon (z)| + |u_\varepsilon (z) - u_\varepsilon (y)| \\
\leq L_{n-1} \| x - z \|_1 + L_{n-1} \| z - y \|_1 \\
= L_{n-1} \| x - y \|_1.
\]

If \( x \) and \( y \) are on opposite facets, then \( \| x - y \|_1 \geq 2 \), and since \( 0 \leq u_\varepsilon \leq n \), we have

\[
|u_\varepsilon (x) - u_\varepsilon (y)| \leq n \leq \frac{n}{2} \| x - y \|_1 \leq L_n \| x - y \|_1.
\]
2. Suppose that $x \in (-1, 1)^n$ and $y$ is its projection onto the boundary of the hypercube in some direction $j \in [n]$. Without loss of generality, we can assume $j = 1$ and $y = (1, x_2, \ldots, x_n)$. If $u_x(y) = 0$, let $T$ be the strategy that always picks direction 1, and let $X(t)$ be the axis-aligned jump process which starts at $x$ and is moved by $T$. Then
\[
C(x, T) = \frac{1+x}{2} u_x(y) + \frac{1-x}{2} u_x(-1, x_2, \ldots x_n) + 1 - x_1^2
\]
\[
\text{for } u_x \leq n-1 \text{ on the boundary.}
\]
\[
\leq \frac{1-x}{2} (n-1) + 1 - x_1^2
\]
\[
\begin{align*}
\langle |x| \leq 1 \rangle & \leq n + 3 \frac{2}{2} (1-x) = L_n \|x-y\|_1. \\
\end{align*}
\]
Thus
\[
|u_x(x) - u_x(y)| = u_x(x) \leq C(x, T) \leq L_n \|x-y\|_1.
\]
If $u_x(y) \neq 0$, then let $S$ be an optimal strategy, and let $Y(t)$ be the corresponding axis-aligned jump process which starts at $y$. Let $T$ be a strategy which repeats the choices that $S$ makes on $y$’s facet if $x$ in the interior:
\[
T(x_1, x_2, \ldots, x_n) = \begin{cases} 
S(1, x_2, \ldots, x_n) & x \in (-1, 1)^n \\
S(x) & x \in \partial [-1, 1]^n.
\end{cases}
\]
Let $X(t)$ be the corresponding axis-aligned jump process which starts at $x$. Couple $X(t)$ and $Y(t)$ so that they move together, and let $\tau$ be the first time that $Y(t)$ hits a facet $j \in 2, \ldots, n$. Since $u_x(y) \neq 0$, we have $\tau < \infty$. Necessarily, $X(t)$ will also hit the same facet at this time (see Figure 2a), and so using the boundary condition formulation, we have $C(x, T) = \sum_{i=1}^n \text{Var}(X_i(\tau)) + \text{Eu}_x(X(\tau))$ and $C(y, S) = \sum_{i=1}^n \text{Var}(Y_i(\tau)) + \text{Eu}_x(Y(\tau))$. The processes $X(t)$ and $Y(t)$ are coupled and only move in directions $2, \ldots, n$, and so we have
\[
C(x, T) - C(y, S) = \text{Eu}_x(X(\tau)) - \text{Eu}_x(Y(\tau)).
\]
Since $S$ is optimal but $T$ might be non-optimal, we get
\[
u_x(x) - u_x(y) \leq C(x, T) - C(y, S)
\]
\[
= \text{Eu}_x(X(\tau)) - \text{Eu}_x(Y(\tau))
\]
\[
\leq L_{n-1} \|X(\tau) - Y(\tau)\|_1 = L_{n-1} \|x-y\|_1.
\]
We use a similar technique to get the opposite inequality. Let $S$ be an optimal strategy, and let $X(t)$ be the corresponding axis-aligned jump process which starts at $x$. Let $Y(t)$ start at $y$ and be moved by a strategy $T$ defined as follows. The process $Y(t)$ and strategy $T$ are coupled with $X(t)$ and $S$ so that whenever $S$ moves $X(t)$ in directions $2, \ldots, n$, $T$ moves $Y(t)$ in the same direction; when $S$ moves $X(t)$ in direction 1, $T$ does nothing. Let $\tau$ be the first time that $X(t)$ reaches the boundary of the hypercube. After time $\tau$, $T$ moves $Y(t)$ optimally. The coordinates $2, \ldots, n$ of $X(t)$ and $Y(t)$ are always the same for $t \leq \tau$ (see Figure 2b). Since $S$ is optimal but $T$ might be non-optimal, we have
\[
u_x(y) - u_x(x) \leq C(y, T) - C(x, S)
\]
\[
= \sum_{i=1}^n \left( \text{Var}(Y_i(\tau)) - \text{Var}(X_i(\tau)) \right) + \text{Eu}_x(Y(\tau)) - \text{Eu}_x(X(\tau))
\]
\[
= -\text{Var}(X_1(\tau)) + \text{Eu}_x(Y(\tau)) - \text{Eu}_x(X(\tau))
\]
\[
\leq \text{Eu}_x(Y(\tau)) - \text{Eu}_x(X(\tau)) \leq L_n \|X(\tau) - Y(\tau)\|_1,
\]
where for the last inequality we use the fact that at time $\tau$ both points are on the boundary of the hypercube. Since $\|X(\tau) - Y(\tau)\|_1 = 1 - X_1(\tau)$, we have $\text{E} \|X(\tau) - Y(\tau)\|_1 = 1 - x = \|x-y\|_1$, and the result follows.
3. Suppose that \( x, y \in (-1, 1)^n \) differ by only one coordinate. Let \( S \) be an optimal Markov strategy which moves \( X(t) \). Let \( T \) be the following strategy, which is coupled with \( X(t) \):

\[
T = \begin{cases} 
    S(X(t)) & \text{if both } X(t) \text{ and } Y(t) \text{ are in the interior of the hypercube} \\
    S(Y(t)) & \text{otherwise.} 
\end{cases}
\]

Let \( \tau \) be the first time that either \( X(t) \) or \( Y(t) \) hit the boundary of the hypercube. Up until time \( \tau \) they move together, and so the difference in running cost between them is 0, and the distance between them stays constant. Since \( S \) is optimal but \( T \) might be non-optimal, and since for \( t > \tau \) both processes move according to an optimal strategy, we have

\[
u_\varepsilon (y) - u_\varepsilon (x) \leq C(y, Y) - C(x, S) \\
= E u_\varepsilon (Y(\tau)) - E u_\varepsilon (X(\tau)) \\
\leq L_n E \|X(\tau) - Y(\tau)\|_1 = L_n \|x - y\|_1 ,
\]

where for the last inequality, we use the fact that at time \( \tau \), either both points are on the same facet, in which case the desired inequality follows from the induction hypothesis, or one point is the projection of the other, in which case the desired inequality follows from Item (2) above. Switching the roles of \( x \) and \( y \) gives the opposite inequality.

4. Finally, for any two general points \( x, y \in [-1, 1]^n \), consider the points \( z_i = (x_1, \ldots, x_{n-i}, y_{n-i+1}, \ldots, y_n) \) for \( i = 1, \ldots, n \). Set also \( z_0 = x \). Then

\[
|u_\varepsilon (x) - u_\varepsilon (y)| \leq \sum_{i=1}^n |u_\varepsilon (z_i) - u_\varepsilon (z_{i-1})| \\
\leq \sum_{i=1}^n L_n |z_i - z_{i-1}| \\
\leq L_n \|x - y\|_1 .
\]

\[\square\]

Figure 2: 2a. \( X(t) \) follows the strategy of \( Y(t) \) until \( Y(t) \) hits a facet. At time \( \tau \), both are on the same facet. 2b. \( Y(t) \) follows the strategy of \( X(t) \). At time \( \tau \), both processes are on the boundary.
Remark 22. The Lipschitz constant in Lemma 21 is tight up to an additive constant: consider the function

\[ f(x) = \begin{cases} 1 & x_1 = 1 \\ \prod_{i=2}^{n} x_i & x_1 = -1. \end{cases} \]

Then \( u_\varepsilon(1,0,\ldots,0) = 0 \) while \( u_\varepsilon(-1,0,\ldots,0) = n - 1 \).

It is natural to look at the behavior of \( u_\varepsilon \) as \( \varepsilon \to 0 \). Is there a “limiting strategy” in some sense, that works for smaller and smaller step-sizes? At the very least, there is a limiting cost:

**Proposition 23.** There exists a function \( u : [-1,1]^n \to \mathbb{R} \) such that for every \( x \in [-1,1]^n \), \( u_\varepsilon(x) \to u(x) \) as \( \varepsilon \to 0 \) dyadically (i.e. we consider numbers of the form \( \varepsilon = 2^{-k} \) as \( k \to \infty \)).

**Proof.** By the recursive construction of \( u_\varepsilon \), it suffices to prove this for \( x \in (-1,1)^n \). For every such \( x \), we have \( u_{\varepsilon/2}(x) \leq u_\varepsilon(x) \): an \( \varepsilon \)-strategy \( S \) can be coupled with an \( \varepsilon/2 \)-strategy in the following manner. If \( S \) picks coordinate \( i \) at time \( t \), so that \( X_i(t+1) \in \{a,b\} \), the simulating strategy can repeatedly pick coordinate \( i \) until \( X_i \) takes one of the values \( \{a,b\} \); the randomness can be coupled so that the same value is reached. Since the processes are martingales, the escape probabilities and the expected sum square of jumps are equal. So \( u_\varepsilon(x) \) is non-negative and monotone decreasing as \( \varepsilon = 2^{-k} \to 0 \), and must converge. \( \square \)

In fact, the following lemma is a consequence of the Arzela-Ascoli theorem and Lemma 21:

**Lemma 24.** \( \|u_\varepsilon - u\|_\infty \to 0 \) as \( \varepsilon \to 0 \) dyadically. The function \( u \) is \( L_n \)-Lipschitz.

The function \( u \) can always be bounded from above by considering strategies for \( u_\varepsilon \):

**Fact 25.** For every \( x \in (-1,1)^n \), every dyadic \( \alpha > 0 \) small enough, and every direction \( i \),

\[ u(x) \leq \frac{u(x + \alpha e_i) + u(x - \alpha e_i)}{2} + \alpha^2. \]

**Proof.** Let \( 0 < \varepsilon \leq \alpha \) be dyadic, and consider the following strategy for \( u_\varepsilon \) when starting at \( x \): always choose to go in direction \( i \) until you reach either \( x - \alpha e_i \) or \( x + \alpha e_i \) (we assume \( \alpha \) is small enough so that both points are in the interior \( (-1,1)^n \)); afterwards, continue optimally. There is equal probability of hitting either \( x - \alpha e_i \) or \( x + \alpha e_i \), and the running cost for doing so is \( \alpha^2 \). Thus

\[ u(x) \leq u_\varepsilon(x) \leq \frac{u_\varepsilon(x + \alpha e_i) + u_\varepsilon(x + \alpha e_i)}{2} + \alpha^2. \]

Taking the limit \( \varepsilon \to 0 \) gives the result. \( \square \)

**Remark 26.** The intuition behind the operator \( \Delta_i = \min_k \frac{\partial^2 u}{\partial x_i^2} \) is as follows. Consider the dynamic programming equation (16):

\[ u_\varepsilon(x) = \min_i u_\varepsilon(x + \varepsilon e_i) + u_\varepsilon(x + \varepsilon e_i) + \varepsilon^2. \]

Since there are only finitely many indices, there is an index \( k \) which appears infinitely many times as \( \varepsilon \to 0 \). For this particular \( k \), taking the limit \( \varepsilon \to 0 \), we have

\[ -2 = \lim_{\varepsilon \to 0} \frac{u_\varepsilon(x + \varepsilon e_k) - 2u_\varepsilon(x) + u_\varepsilon(x + \varepsilon e_k)}{\varepsilon^2}. \]

If we could replace \( u_\varepsilon \) in the above equation by \( u \), and if we knew that \( u \) was twice-differentiable, the right hand side would equal the second derivative of \( u \), and we would get \( \frac{\partial^2 u}{\partial x_i^2} = -2 \). Since the index \( k \) was chosen as the minimum, we would hope to reach the following partial differential equation for \( u \):

\[ \Delta_i u := \min_k \frac{\partial^2 u}{\partial x_k^2} = -2. \]
**Theorem 27.** Either \( u \equiv 0 \), or it satisfies \( \Delta_1 u = -2 \) for all \( x \in (-1,1)^n \) in the viscosity sense.

The proof uses standard techniques (see e.g [5, Chapter 3]), and relies on the uniform convergence property proved above.

**Proof.** We start by showing that \( u \) is a viscosity subsolution (recall Definition 16). Let \( x_0 \in (-1,1)^n \), and let \( \varphi : [-1,1]^n \to \mathbb{R} \) be a smooth function so that \( \varphi - u \) has a minimum at \( x_0 \), with \( \varphi(x_0) = u(x_0) \). Let \( k \) be the direction for which \( \frac{\partial^2}{\partial k^2} \varphi(x_0) \) is minimal. By Fact 25, for every dyadic \( \alpha \) small enough we have

\[
\varphi(x_0) = u(x_0)
\]

Fact 25 \( u(x_0 + \alpha e_k) + u(x_0 - \alpha e_k) \geq 2 \)  
\[
\leq \frac{\varphi(x_0 + \alpha e_k) + \varphi(x_0 - \alpha e_k)}{2} + \alpha^2
\]

\[
= \varphi(x_0) + \alpha^2 \frac{1}{2} \frac{\partial}{\partial x_k^2} \varphi(x_0) + o(\alpha^2) + \alpha^2.
\]

Rearranging, dividing by \( \alpha^2 \), and taking the limit \( \alpha \to 0 \) gives

\[
\frac{\partial^2}{\partial x_k^2} \varphi(x_0) \geq -2.
\]

Since \( k \) was chosen to be the direction for which \( \frac{\partial^2}{\partial k^2} \varphi(x_0) \) is minimal, we get that

\[
\Delta_1 \varphi(x_0) \geq -2,
\]

which means that \( u \) is a subsolution to the Dirichlet problem. Note that we did not uniform convergence of \( u_\varepsilon \) to \( u \) to prove this; we only used the monotone pointwise convergence of \( u_\varepsilon \) to \( u \), which is used in Fact 25. The dynamic programming equation for \( u_\varepsilon \) makes it relatively easy to bound from above.

For showing that \( u \) is a supersolution, let \( \varphi : [-1,1]^n \to \mathbb{R} \) be a smooth function so that \( u - \varphi \) has a strict minimum at \( x_0 \). Denote \( \psi_\varepsilon = u_\varepsilon - \varphi \), and \( \psi = \lim_{\varepsilon \to 0} \psi_\varepsilon = u - \varphi \). By Lemma 24, we also have uniform convergence of \( \psi_\varepsilon \) to \( \psi \):

\[
\|\psi_\varepsilon - \psi\|_\infty = \|u_\varepsilon - \varphi - (u - \varphi)\|_\infty = \|u_\varepsilon - u\|_\infty \to 0.
\]

Let \( x_\varepsilon = \text{argmin}_{x \in [-1,1]^n} \{\psi_\varepsilon\} \); the minimum exists since \( \psi_\varepsilon \) is continuous for every \( \varepsilon \). By definition, for all \( x \in [-1,1]^n \),

\[
u_\varepsilon(x) - \varphi(x) \geq u_\varepsilon(x_\varepsilon) - \varphi(x_\varepsilon).
\]

(17) We’ll now show that \( x_\varepsilon \to x_0 \), where \( x_0 \) is the minimizer of \( \psi(x) = u(x) - \varphi(x) \). Suppose not. Since \([-1,1]^n\) is compact, there exists a subsequence, which we still call \( x_\varepsilon \), which converges to some \( z \neq x_0 \). Now, since \( x_0 \) is a strict minimizer of \( \psi \), denote

\[
0 < \Delta := \psi(z) - \psi(x_0).
\]

(18) Since \( \psi_\varepsilon \to \psi \) uniformly, for small enough \( \varepsilon \) we have:

\[
\psi_\varepsilon(x_\varepsilon) < \psi(x_\varepsilon) + \frac{\Delta}{3}
\]

(19)

and

\[
\psi_\varepsilon(x_\varepsilon) > \psi(x_\varepsilon) - \frac{\Delta}{3}.
\]

(20) Since \( \psi \) is continuous and \( x_\varepsilon \to z \), for small enough \( \varepsilon \) we have

\[
\psi(x_\varepsilon) > \psi(z) - \frac{\Delta}{3}.
\]

(21)
Now, on one hand,

$$\psi_\varepsilon (x_\varepsilon) \overset{(20)}{> } \psi (x_\varepsilon) - \frac{\Delta}{3} \overset{(21)}{> } \psi (z) - \frac{2\Delta}{3}.$$

On the other hand,

$$\psi_\varepsilon (x_0) \overset{(19)}{< } \psi (x_0) + \frac{\Delta}{3} \overset{(18)}{=} \psi (z) - \Delta + \frac{\Delta}{3} = \psi (z) - \frac{2\Delta}{3} \overset{(22)}{< } \psi_\varepsilon (x_\varepsilon).$$

Thus $$\psi_\varepsilon (x_0) \overset{(19)}{< } \psi_\varepsilon (x_\varepsilon),$$ a contradiction to the minimality of $$x_\varepsilon.$$

For small enough $$\varepsilon,$$ since $$x_0 \in (-1, 1)^n$$ and $$x_\varepsilon \rightarrow x_0,$$ we have that $$x_\varepsilon \pm \varepsilon e_i \in (-1, 1)^n$$ for all directions $$i \in [n].$$ By the dynamic programming equation (16), for every such $$\varepsilon,$$ the value of function $$u_\varepsilon$$ evaluated at $$x_\varepsilon$$ is given by

$$-\varepsilon^2 = \min_i \frac{u_\varepsilon (x_\varepsilon \pm \varepsilon e_i) + u_\varepsilon (x_\varepsilon - \varepsilon e_i)}{2} - u_\varepsilon (x_\varepsilon).$$

Since there are only finitely many directions $$i \in [n],$$ there is some direction $$k \in [n]$$ which appears infinitely many times as the minimizer in the above equation. Restricting ourselves just to those $$\varepsilon$$’s, we have

$$-\varepsilon^2 = \frac{u_\varepsilon (x_\varepsilon + \varepsilon e_k) + u_\varepsilon (x_\varepsilon - \varepsilon e_k)}{2} - u_\varepsilon (x_\varepsilon)$$

$$= \frac{1}{2} (u_\varepsilon (x_\varepsilon + \varepsilon e_k) - u_\varepsilon (x_\varepsilon)) + \frac{1}{2} (u_\varepsilon (x_\varepsilon - \varepsilon e_k) - u_\varepsilon (x_\varepsilon)). \quad (23)$$

By applying (17), the first expression in the parenthesis on the right hand side can be bounded by:

$$u_\varepsilon (x_\varepsilon + \varepsilon e_k) - u_\varepsilon (x_\varepsilon) = u_\varepsilon (x_\varepsilon + \varepsilon e_k) - \varphi (x_\varepsilon + \varepsilon e_k) - u_\varepsilon (x_\varepsilon) + \varphi (x_\varepsilon + \varepsilon e_k)$$

$$\geq u_\varepsilon (x_\varepsilon) - \varphi (x_\varepsilon) - u_\varepsilon (x_\varepsilon) + \varphi (x_\varepsilon)$$

$$= \varphi (x_\varepsilon + \varepsilon e_k) - \varphi (x_\varepsilon).$$

Similarly,

$$u_\varepsilon (x_\varepsilon - \varepsilon e_k) - u_\varepsilon (x_\varepsilon) \geq \varphi (x_\varepsilon - \varepsilon e_k) - \varphi (x_\varepsilon).$$

Plugging this back into (23), we have

$$-\varepsilon^2 \geq \frac{\varphi (x_\varepsilon + \varepsilon e_k) + \varphi (x_\varepsilon - \varepsilon e_k)}{2} - \varphi (x_\varepsilon)$$

$$= \frac{1}{2} \varepsilon^2 \frac{\partial^2 \varphi (x_\varepsilon)}{\partial x_k^2} + o (\varepsilon^2).$$

Dividing by $$\varepsilon^2,$$ taking the limit $$\varepsilon \rightarrow 0$$ gives us

$$\frac{\partial^2 \varphi (x_0)}{\partial x_k^2} \leq -2.$$ 

Since this is true for some $$k,$$ it is true in particular for the smallest second derivative of $$\varphi.$$ Thus

$$\Delta_1 \varphi (x_0) \leq -2,$$

which means that $$u$$ is a supersolution to the Dirichlet boundary problem.

The axis-aligned Laplacian is a non-linear operator. It is, however, monotone in the Hessian $$\nabla^2 u.$$

**Fact 28.** Let $$u, v$$ be twice-differentiable functions. If $$\nabla^2 u \geq \nabla^2 v$$ (i.e. the matrix $$\nabla^2 u - \nabla^2 v$$ is positive semidefinite), then $$\Delta_1 u \geq \Delta_1 v.$$
Proof. If \( \nabla^2 (u - v) \) is positive semidefinite, then for all \( i \in [n] \),
\[
0 \leq \langle e_i, \nabla^2 (u - v) e_i \rangle = \frac{\partial^2}{\partial x_i^2} u - \frac{\partial^2}{\partial x_i^2} v.
\]
In particular, for \( i^* = \arg\min_i \frac{\partial^2}{\partial x_i^2} u \) we get \( \Delta_i u \geq \frac{\partial^2}{\partial x_{i^*}^2} v \geq \Delta_i v \).
\[\square\]

This gives hope that solutions to the Dirichlet boundary-value problem \( \Delta_i u (x) = f (x) \) are unique. This is indeed true, if the function \( f \) does not change sign, and follows from a general comparison principle.

**Theorem 29.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain. Let \( f : \Omega \to \mathbb{R} \) be continuous with \( \sup_{\Omega} f < 0 \). Suppose that \( u_1, u_2 \) are continuous functions such that \( \Delta_i u_1 \leq f \leq \Delta_i u_2 \) in \( \Omega \) and \( u_1 \geq u_2 \) on \( \partial \Omega \). Then \( u_1 \geq u_2 \) in \( \Omega \).

Theorems 27 and 29 immediately yield Theorem 14 as a corollary. The proof of Theorem 29 follows the scheme of Lu and Wang [12]. We start by showing a strict comparison principle.

**Lemma 30.** Let \( f_1, f_2 : \Omega \to \mathbb{R} \) be continuous functions such that \( f_1 < f_2 \). Let \( u_1, u_2 \) be continuous functions such that \( \Delta_i u_1 \leq f_1 \leq \Delta_i u_2 \leq f_2 \) in \( \Omega \) and \( u_1 \geq u_2 \) on \( \partial \Omega \). Then \( u_1 \geq u_2 \) in \( \Omega \).

**Proof.** Suppose for the sake of contradiction that there exists \( x^* \in \Omega \) such that \( u_1 (x^*) < u_2 (x^*) \). Let \( \varepsilon > 0 \), define \( M_\varepsilon = \max_{x,y \in \Omega} u_2 (x) - u_1 (y) - \frac{1}{2\varepsilon} \| x - y \|^2 \), and let \( x_\varepsilon \) and \( y_\varepsilon \) be the maximizers. By Lemma 3.1 of [8], \( \lim_{\varepsilon \to 0} M_\varepsilon = \max_{\Omega} \{ u_2 - u_1 \} > 0 \), and \( \lim_{\varepsilon \to 0} \| x_\varepsilon - y_\varepsilon \|^2 = 0 \). By Lemma 3.2 of [8], there exist symmetric \( n \times n \) matrices \( X \) and \( Y \) with \( X \leq Y \) such that:

1. There exists a sequence of smooth functions \( \varphi_k \) and points \( x_k \) such that \( \varphi_k (x_k) \) has a local minimum at \( x_k \), \( x_k \to x_\varepsilon \), and \( \nabla^2 \varphi_k (x_k) \to X \).

2. There exists a sequence of smooth functions \( \psi_k \) and points \( y_k \) such that \( \psi_k (y_k) \) has a local maximum at \( y_k \), \( y_k \to y_\varepsilon \), and \( \nabla^2 \psi_k (y_k) \to Y \).

By Definition 16 of viscosity solutions, \( f_2 (x_k) \leq \Delta_i \varphi_k (x_k) \) and \( f_1 (y_k) \geq \Delta_i \psi_k (y_k) \) for every \( k \). Taking the limit \( k \to \infty \), since \( \min_i X_{ii} \), \( f_1 \) and \( f_2 \) are all continuous functions, we have
\[
f_2 (x_\varepsilon) \leq \min_i X_{ii} \leq \min_i Y_{ii} \leq f_1 (y_\varepsilon) . \tag{24}
\]
Since \( \Omega \) is bounded and the maximum of \( u_2 - u_1 \) is obtained in the interior of \( \Omega \), there are subsequences of \( x_\varepsilon \) and \( y_\varepsilon \), which we also denote \( x_\varepsilon \) and \( y_\varepsilon \), which converge to some \( x_0 \in \Omega \). Sending \( \varepsilon \to 0 \), continuity of \( f_1, f_2 \) together with equation (24) then give
\[
f_2 (x_0) \leq f_1 (x_0) ,
\]
contradicting the assumption of the lemma.
\[\square\]

**Proof of Theorem 29.** For any \( 0 < \delta < 1 \), define \( u_\delta = (1 - \delta) u_2 - \delta \max_{\partial \Omega} |u_2| \). Since \( \Delta_i \alpha f = \alpha \Delta_i f \) for all \( \alpha > 0 \), we have \( \Delta_i u_\delta = (1 - \delta) \Delta_i u_2 \geq (1 - \delta) f \geq f \geq \Delta_i u_1 \). Also, \( u_\delta \leq u_2 \leq u_1 \) on \( \partial \Omega \). By Lemma 30, \( u_1 \geq u_\delta \) in \( \Omega \). Sending \( \delta \to 0 \) gives the desired result.
\[\square\]

## 5 An example: The OR function

### 5.1 n bits
Let \( f (x) \) be the \( n \)-bit OR function, which returns 1 if and only if one of the bits \( x_i \) is equal to 1. This is a symmetric function - it depends only on the number of bits in the input. There is thus essentially only
one decision tree algorithm: read bits at random until the value of the function is computed. Every bit has value 1 with probability 1/2, so apart from the \( n \)-th read bit, every bit has a probability of 1/2 of ending the computation. Thus, as \( n \to \infty \), the number of bits queried tends towards a geometric random variable with parameter 2, and the expected number of bits queried is 2.

For fractional query algorithms, there are many more algorithms to choose from. Since the OR function needs only a single bit to be set to 1, the natural algorithm is to always update the largest bit. Updating smaller bits is intuitively wasteful, because either they reach the value \(-1\) (in which case the other bits need to be evaluated anyway), or they take longer to reach the value 1 than the largest bit. This intuition holds true in the axis-aligned algorithm setting.

**Theorem 31.** Let \( \varepsilon = 2^{-k} \), and let \( S_{\text{max}} : [-1,1]^n \to [n] \) be given by \( S_{\text{max}}(x) = \text{argmax}_i(x) \). Then \( S_{\text{max}} \) is optimal, i.e. for all other decision strategies \( T \), we have \( C(S_{\text{max}}) \leq C(T) \).

**Proof.** Given a strategy \( T \) with process \( X(t) \), define a new strategy \( Q = Q(T) \) with process \( Y(t) \), which runs according to \( T \) up to the first time \( t_1 \) that \( T \) doesn’t update the largest bit. Denote a largest bit by \( i_0 \) and let \( \alpha = X_{i_0}(t_1) \). \( Q \) does update this bit, and then makes the same choices that \( T \) would make assuming the largest bit wasn’t updated. It continues so until the time \( t_2 \) where \( T \) finally updates the original bit, or another bit of value \( \alpha \). From this point on, it again runs according to \( T \).

The claim is that \( Q \) always does better than \( S \) in terms of expected runtime: until time \( t_1 \), both strategies give identical processes. During the interval \( t_1 < t < t_2 \), \( T \) cannot finish before \( Q \) does: in order to get \( f(x) = -1 \), all bits need to be updated, and in particular bit \( i_0 \); in order to get \( f(x) = 1 \), a bit with value \( \alpha \) needs to be updated. After time \( t_2 \), if \( Q \) didn’t already finish, we have that \( X(t) \) and \( Y(t) \) are permutations of each other, and so in expectation \( Q \) and \( T \) perform the same. Thus \( C(Q(T)) \leq C(T) \).

Let \( S^* \) be an optimal strategy. The strategy \( S_m = Q^m(S^*) \) is identical with \( S_{\text{max}} \) for the first \( m \) steps. We thus have

\[
C(S_{\text{max}}) \leq C(S_m) + P[\text{algorithm runs longer than } m \text{ steps}] \cdot n = C(S^*) + P[\text{algorithm runs longer than } m \text{ steps}] \cdot n.
\]

The probability that the algorithm runs longer than \( m \) steps goes to 0 as \( m \to \infty \), and we have

\[
C(S_{\text{max}}) \leq C(S^*),
\]

so picking the maximum is also an optimal strategy.

**Remark 32.** For continuous-time processes, Jacka, Warren and Windridge [9, Section 6] prove that when \( n = 2 \), picking the maximum entry is the optimal strategy in a stronger sense: this strategy actually stochastically dominates all other strategies. They conjecture that the same is true for general \( n \); this conjecture is strengthened by the above result (continuous-time processes harbor subtle difficulties of measurability when there are two variables with the same values).

**Proposition 33.** The best fractional query algorithm needs to query only one bit: \( C(S^*) \to 1 \) as \( n \to \infty \).

**Proof.** We give a strategy \( S \) whose cost tends to 1 as \( n \to \infty \). Let \( \varepsilon = 1/n \). Suppose that at time \( t \), all coordinates are equal to \(-t/n\). For every coordinate \( i \), run \( X_i \) until it exits the interval \([-t/n, 1]\). We call this a single iteration. At the end of the iteration, either we have found a bit whose value is 1, or all bits are set to \(-t+1/n\). The probability for a single bit to exit at 1 is \( \frac{1}{n+t+1} \), and the probability to exit at \(-t+1/n\) is \( \frac{n+t}{n+t+1} \). The probability that all coordinates exit at \(- (t+1)/n \) is then

\[
\left( 1 - \frac{1}{n+t+1} \right)^n \leq \left( 1 - \frac{1}{3n} \right)^n \leq e^{-1/3}.
\]

After \( t \) iterations, either all bits have value \(-t/n\), or one bit has value 1 and all other bits have value \(-t/n\) or \(- (t-1)/n \). If we stop after \( t \) iterations, the expected cost is then bounded by

\[
c_t \leq 1 + \sum_{i=1}^{n} \left( \frac{t}{n} \right)^2 = 1 + \frac{t^2}{n}.
\]
Denoting $p = e^{-1/3}$, the expected cost of the algorithm then satisfies
\[
C ( S ) \leq \sum_{i=1}^{n} (1 - p) p^t \left( 1 + \frac{t^2}{n} \right) \leq 1 + \frac{1}{n} \sum_{i=1}^{n} (1 - p) p^t t^2.
\]
The last term on the right hand side goes to 0 as $n \to \infty$, and we get the desired result.

Of course, at least one bit needs to be queried.

5.2 2 bits

Below we give a derivation of $u ( x )$ for the 2-bit parity function. In fact, it suffices to calculate the value of $u ( x )$ on the diagonal $x_1 = x_2$: for all other points, we know what the optimal direction is, and the value of $u$ is just a linear combination of the value on the diagonal and the value on the boundary (which is 0). For example, for $x_2 > x_1$, we have
\[
u ( x_1, x_2 ) = \frac{x_2 - x_1}{1 - x_1} \left( 0 + (1 - x_2)^2 \right) + \frac{1 - x_2}{1 - x_1} \left( g ( x_1 ) + (x_2 - x_1)^2 \right),
\]
where $g ( x ) = u ( x, x )$. To obtain $g$, denote $g_x = u_x ( x, x )$. For axis-aligned processes with jump size $\varepsilon$, the optimal strategy always picks the largest entry, and this gives a recurrence relation for $g_x$, with the following strategy. Suppose that $X ( t ) = ( x, x )$. Update coordinate 1, until either $X ( t ) = (1, x)$, or $X ( t ) = (x - \varepsilon, x)$. Then, update coordinate 2, until either $X ( t ) = (x - \varepsilon, 1)$, or $X ( t ) = (x - \varepsilon, x - \varepsilon)$. A short calculation gives the relation
\[
g_x ( x ) - g_x ( x - \varepsilon ) \frac{\varepsilon}{\varepsilon} + \frac{\varepsilon g_x ( x )}{(1 - x)^2} + \frac{2g_x ( x )}{1 - x} = 2 (1 - x + \varepsilon) + \frac{1 - x + \varepsilon}{1 - x}.
\]
This suggests that $g ( x )$ satisfies the differential equation
\[
g' ( x ) + g ( x ) \frac{2}{1 - x} = 2 (1 - x).
\]
Formally, we have not shown that $\frac{g_x ( x ) - g_x ( x - \varepsilon )}{\varepsilon}$ converges to $g' ( x )$ as $\varepsilon \to 0$, or indeed that $g$ is even differentiable. However, due to the uniqueness guaranteed by Theorem 14, if we find a continuous function $v$ which satisfies the boundary conditions for OR and has $\Delta_1 v = -2 \in (-1,1)^n$, then necessarily $u = v$.

The solution to the differential equation (26) (with appropriate boundary conditions) is given by
\[
g ( x ) = 2 (1 - x)^2 \log 2 + 4x \log (1 - x) - 2x^2 \log (1 - x) - 2 \log (1 - x).
\]
The function $u ( x_1, x_2 )$ obtained by plugging in (27) into (25) is given in Figure 3. It can be shown to satisfy $\Delta_1 u ( x_1, x_2 ) = -2$ as needed.

Some sample paths of $X ( t )$ for $\varepsilon = 2^{-7}$ are given in Figure 1.

5.3 A heuristic for the OR of two functions

Let $n = n_1 + n_2$ and $g_i : \{-1, 1\}^{n_i} \to \{-1, 1\}$ be two given functions. Let $f ( x_1, x_2 ) = OR ( g_1 ( x_1 ), g_2 ( x_2 ) )$, where $x_i \in \{-1, 1\}^{n_i}$. For a given $\varepsilon > 0$, what is the optimal direction-choosing strategy $S$ for $f$?

The following may be a useful heuristic for choosing whether to update $g_1$ or to update $g_2$. Let $S_1$ and $S_2$ be the optimal strategies for $g_1$ and $g_2$, and let $u_1 ( x )$ and $u_2 ( y )$ be the costs for running $S_1$ on $x$ and $S_2$ on $y$, respectively. Consider the (definitely non-optimal) strategy which chooses a function $g_i$ and completely evaluates it using $S_i$. The expected cost for doing so is $u_i ( x_i )$. If eventually $g_i = 1$ (which happens with
Figure 3: The limiting cost \( u(x, y) \) for the OR function on two bits.

probability \( \frac{1 + g_i(x_i)}{2} \), then the algorithm is finished. Otherwise, the other function needs to be computed. In total, the cost for picking \( g_i \) this way is, for \( \{i, j\} = \{1, 2\} \),

\[
c(g_i) = u_i(x_i) + 1 - \frac{g_i(x_i)}{2} u_j(x_j).
\]

A short calculation shows that this simple strategy should then pick \( i = 1 \) (i.e. \( c(g_1) \leq c(g_2) \)) only if

\[
\frac{u_1(x_1)}{1 + g_1(x_1)} \leq \frac{u_2(x_2)}{1 + g_2(x_2)}.
\]

This suggests a heuristic for functions of the form \( f = OR(g_1, g_2) \): at each step, pick either \( g_1 \) or \( g_2 \) according to the condition (28), and update a single bit according to \( S_1 \) or \( S_2 \).

One possible application for this heuristic is the iterated majority function. Let \( k \in \mathbb{N} \) and \( n = 3^k \). The iterated majority function \( f_k \) is recursively defined as follows:

\[
f_k = \begin{cases} 
\text{maj} (x_1, x_2, x_3) & k = 1 \\
\text{maj} (f_{k-1}(x_1, \ldots, x_{3^{k-1}}), f_{k-1}(x_{3^{k-1}+1}, \ldots, x_{2 \cdot 3^{k-1}}), f_{k-1}(x_{2 \cdot 3^{k-1}+1}, \ldots, x_{3^k})) & \text{otherwise},
\end{cases}
\]

where \( \text{maj} : \{-1, 1\}^3 \rightarrow \{-1, 1\} \) returns the most frequent bit in its input. The function \( f_k \) can be represented by a complete ternary tree of depth \( k \), where the leaves are the input bits \( x_1, \ldots, x_n \), and each internal node has value equal to the majority of its three children. The value of \( f_k \) is the value of the root.

The best decision tree complexity of iterated majority is still unknown. The simplest non-trivial algorithm is as follows: Pick two random subtrees, and recursively compute their value. If they are equal, the algorithm terminates. If not, then the third subtree needs to be computed as well. The expected number of bits queried with this algorithm is \( 2.5^k \). However, better algorithms exist, which do not evaluate entire subtrees at once.

For example, the algorithm given by Jayram, Kumar and Sivakumar [10, Appendix B] requires reading only approximately \( \left( \frac{13 + \sqrt{713}}{16} \right)^n \approx 2.4813^k \) bits on average, under the uniform input. This algorithm recursively reads one random subtree, but can jump between the two remaining subtrees until the function value has been determined. Once a single subtree has been read, the iterated majority turns into either an OR (if the subtree’s value was 1) or an AND function (if the subtree’s value was -1) between the two remaining subtrees. It is then possible to apply the above heuristic to choose which of the two trees to update. Using easier-to-compute criterion

\[
\frac{n - \|x_1\|^2}{1 + g_1(x_1)} \leq \frac{n - \|x_2\|^2}{1 + g_2(x_2)}
\]
rather than (28), we have performed numerical simulations of this strategy for \( k = 1, \ldots, 9 \), yielding an estimated average number of bits of order \( \leq 2.45^k \).

6 Fractional random-turn games and the influence process

Consider the following random-turn two-player game, introduced by Peres, Schramm, Sheffield and Wilson in [16]. Let \( f : \{-1,1\}^n \to \mathbb{R} \), and let \( x(0) = (0, \ldots, 0) \in \mathbb{R}^n \). At each time \( t \in \mathbb{N} \), a coin is flipped. If the result is heads, player I picks an index \( i \) according to some strategy and sets \( x_i(t) = 1 \). If the result is tails, player II picks an index \( i \) and sets \( x_i(t) = -1 \). After \( n \) turns, we have \( x(n) \in \{-1,1\}^n \). Player I then gains \( f(x) \), while player II loses \( f(x) \). The goal of each player is to maximize their expected payoff, and the value of the game is \( v = \mathbb{E}[f(x(n))] \), when both players play optimally.

**Theorem 34** (Theorem 2.1 in [16]). Let \( \mu_{1/2} \) be the uniform measure on the hypercube. The value of a random-turn game is \( \mathbb{E}_{x \sim \mu_{1/2}} f(x) \). Moreover, any optimal strategy for one of the players is also an optimal strategy for the other player.

**Lemma 35** (Lemma 3.1 in [16]). Let \( f : \{-1,1\}^n \to \{-1,1\} \) be monotone. Let \( T \) be the set of bits that have already been fixed at time \( t \), and let \( f|_T \) be the restriction of \( f \) to those bits. Then a move is optimal if and only if it picks a variable \( x_i \) such that \( \text{Inf}_i(f|_T) \) is maximal.

In the language of decision trees, attempting to maximize the expected gain gives rise to a decision tree which always picks a variable with maximal influence.

These results can be generalized to fractional query algorithms by considering fractional random-turn two-player games. In these games, if the coin flip turns up heads, player I picks an index \( i \) and sets \( x_i(t) = x_i(t) + \varepsilon \), and if the result is tails, player II picks an index \( i \) and sets \( x_i(t) = x_i(t) - \varepsilon \), with the constraint that once a bit reaches the values \( \pm 1 \), it cannot be picked any more. The following can be proved using the same proof techniques as in [16].

**Theorem 36.** Let \( \varepsilon = 1/M \) for some integer \( M > 0 \). Let \( p \in [-1,1]^n \cap \frac{1}{M} \mathbb{Z}^n \). The value of a fractional random-turn game with \( x(0) = p \) is \( \mathbb{E}_{x \sim \mu_{M}} f(x) = f(p) \). Moreover, any optimal strategy for one of the players is also an optimal strategy for the other player.

**Lemma 37.** Let \( \varepsilon = 1/M \) for some integer \( M > 0 \), and let \( f : \{-1,1\}^n \to \{-1,1\} \) be monotone. A move is optimal if and only if it selects a bit with maximal \( \partial_i f(x(t)) \).

When both players use the same optimal strategy, \( x(t) \) is an axis-aligned jump process. However, it does not necessarily minimize the cost \( \mathbb{E}\|x(\tau)\|^2 \). For example, for decision trees, Simkin [19] showed that the influence-based decision tree for iterated majority on 9 bits reads 396/64 bits in expectation, while the best tree reads only 393/64 bits. For fractional algorithms, Jacka, Warren and Windridge [9] considered evaluating the value of 3-majority, when \( X(t) \) is 3-dimensional Brownian motion, and only one entry can be moved at a time. They show that the strategy which always moves the middle bit takes the least amount of time in expectation. Since \( \mathbb{E} B(t)^2 = t \) for Brownian motion, this is the same as minimizing the cost of a fractional query algorithm. However, the derivatives of 3-majority \( f(x_1, x_2, x_3) = \frac{1}{2} (x_1 + x_2 + x_3 - x_1x_2x_3) \) are given by, for \( \{i, j, k\} = \{1, 2, 3\} \),

\[
\partial_i f(x_1, x_2, x_3) = \frac{1}{2} - \frac{1}{2} x_j x_k.
\]

When \( 0 < x_1 < x_2 < x_3 \), we have \( x_1x_2 < x_1x_3 < x_2x_3 \), and so \( \partial_1 f > \partial_2 f > \partial_3 f \). In this case the influence-based process would pick the largest entry, not the middle one.

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