Gross features of finite nuclei at finite temperatures

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Abstract

A simple expression is obtained for the low temperature behavior of the energy and entropy of finite nuclei for \(20 \leq A \leq 250\). The dependence on \(A\) of these quantities is for the most part due to the presence of the asymmetry energy.

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In nuclei the nuclear force between the nucleons is short-ranged which leads to saturation of the binding energy, \(E(A,Z)\), per nucleon. Empirically it is known that for the stable isotopes

\[
\frac{E(A,Z)}{A} \approx \alpha
\]  

(1)

where \(\alpha \approx 8\) MeV and \(A \geq 20\). This fundamental gross property of nuclei is well accounted for by the semi-empirical mass formula [1, 2, 3] which provides a simple parametrization of the binding energy per nucleon for all known nuclei. Qualitatively it is also consistent with the simple Fermi gas model prediction [4, 5]

\[
\frac{\mathcal{E}(A)}{A} = \frac{3}{5} \epsilon_f
\]  

(2)

where \(\mathcal{E}\) is the ground state energy and \(\epsilon_f\) is the Fermi energy which is constant as long as the particle density remains constant. It is interesting to note that at low temperatures \((T < T_f)\) the excitation energy in the Fermi gas model is given by [4, 6]

\[
\frac{\mathcal{E}(A,T)}{A} = \frac{3}{5} \epsilon_f + \frac{\pi^2 T^2}{4 \epsilon_f} \quad \text{or}
\]

\[
= a + bT^2
\]  

(3)

(4)

where \(a\) and \(b\) are constant again as long as the particle density remains constant. Although the Fermi gas model may be an oversimplified model, nonetheless it underscores the relevance of independent particle (or quasi-particle) methods in nuclear structure physics. Mean field methods have been used throughout the periodic table, both at zero temperature and at finite but low temperatures, and typically yield a \(T^2\) behaviour of the energy density. This suggests that perhaps at low but finite temperatures a simple scaling relation might also exist for the excitation energy, \(\mathcal{E}(A,T)\), of finite nuclei.

In order to test the validity of equation (4) we have made use of a finite temperature extension of the semi-empirical mass formula [7]. The following form for the temperature dependent binding energy [3, 8] has been assumed

\[
E(A,Z,T) = \alpha(T)A + \beta(T)A^{2/3} + (\gamma(T) - \eta(T)\left(\frac{4t^2}{A^{1/3}} + 4|\zeta|\right))
\]

\[+ \kappa(T)\left(\frac{Z^2}{A^{1/3}}(1 - \frac{0.7636}{Z^{2/3}} - \frac{2.29 \kappa^2(T)}{(0.8076)^2 A^{1/3}}) + \delta(T)f(A,Z)A^{-3/4}\right)
\]  

(5)
where $A = N + Z$, $t_\zeta = \frac{1}{2}(Z - N)$ and $f(A, Z) = (-1, 0, +1)$ for (even-even, even-odd, odd-odd) nuclei. Here 1 is the volume energy, 2 is the surface energy, 3 is the asymmetry energy, 4 is the Coulomb energy and 5 is the pairing energy contribution to temperature dependent binding energy. In this parametrization the temperature dependence of the contributions to the Coulomb energy term which arise from exchange and surface effects \cite{3, 9} was ignored. Also, no attempt has been made to include shell effects in the finite temperature expression.

The excitation energy per particle is given by

$$\frac{E(A, Z, T)}{A} = \frac{E(A, Z, T) - E(A, Z, 0)}{A}.$$  \hspace{1cm} (6)

At $T = 0$ the coefficients are given by \cite{3} $\alpha(0) = -16.11$ MeV, $\beta(0) = 20.21$ MeV, $\gamma(0) = 20.65$ MeV, $\eta(0) = 48.00$ MeV, and $\kappa(0) = 0.8076$ MeV obtained from a fit to the experimental nuclear ground state energies of 488 odd mass nuclei. The $T = 0$ coefficient for the pairing term is taken as $\delta(0) = 33.0$ MeV \cite{8}.

To obtain the temperature dependence of the coefficients, the available experimental information about the excited states of nuclei throughout the periodic table was used to determine the partition function of each nucleus in the canonical ensemble

$$Z(A, Z, T) = \sum_i^n g_i \exp(-\beta E_i) + \int_{E_n}^{E_{max}} dE \, g_{A,Z}(E) \exp(-\beta E)$$ \hspace{1cm} (7)

where $g_i = 2j_i + 1$ is the spin degeneracy factor and $E_i$ the excitation energy of the $i$th state of the nucleus, and $\beta = 1/T$. The sum in the first term of equation (7) runs over the experimentally measured (discrete) excited states.

Since the experimentally known spectrum in most cases is only sufficient to allow the accurate determination of $Z$ for very low temperatures ($T \ll 1$ MeV), it is necessary to supplement the experimentally known spectrum with an appropriate approximation to the continuum $g_{A,Z}(E)$. For this purpose, the fits obtained in \cite{10} were used. For sufficiently large energies, the usual Fermi gas expression for the total density of states (i.e. including the spin degeneracy) is used:

$$g_{A,Z}(E) = \frac{\sqrt{\pi} \exp(2\sqrt{a_{A,Z} U})}{12 \sqrt{a_{A,Z} U^{5/4}}}. \hspace{1cm} (8)$$

Here $a_{A,Z}$ is the level density parameter and $U = E - P(N) - P(Z)$, where $P(N)$ and $P(Z)$ are the pairing corrections for neutron number $N$ and proton number $Z$ respectively \cite{10}. 

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This parametrization is obtained by means of a saddle point approximation, and is probably only valid up to $T \approx 6$ MeV. However, in the region up to $T \approx 4$ MeV, this parametrization is probably acceptable.

At lower energies, a suitable fit to the nuclear energy level density can be obtained to the form

$$g_{A,Z}(E) = \frac{\sqrt{2\pi\sigma}}{\tau} \frac{\exp(E - E_0)}{\tau},$$

(9)

with $\sigma$ the spin-dependence parameter. Values for the parameters $a_{A,Z}$, $\tau$, $E_0$, and $\sigma$, as well as the respective regions where (8) and (9) should be used, for a large number of nuclei can be found in [10].

The nuclei used to determine the aforementioned coefficients can therefore be divided into three groups: Nuclei where sufficient discrete states are known to allow the use of the discrete spectrum at low energies and the Fermi gas expression (8) at higher energies. Nuclei where the discrete spectrum does not extend high enough for (8) to be valid. For these nuclei, the discrete spectrum is used for low excitation energies, followed by the exponential form (9) for intermediate energies and finally the Fermi gas expression (8) at high energies. Nuclei where very little of the discrete spectrum is known. In these cases, (9) is used for the low- and intermediate- excitation portions of the spectra and (8) for the highly excited part. All three groups are spread across the whole periodic table.

The lower bound $E_n$ on the integral in (7) is taken to be the energy at which (8) should become valid (from [10]) for the first case above, 80% of the largest discrete energy level for the second case above, and zero for the third case. For temperatures up to $T \approx 4$ MeV, the upper bound $E_{max} \approx 3$ GeV was used.

The coefficients in the mass formula have been determined by a least squares fit of (6) to the ensemble average of the excitation energy

$$\mathcal{E}(A, Z, T) = -\frac{\partial}{\partial \beta} \ln Z(A, Z, T)$$

(10)

FIG. 1: Quadratic fit of the calculated binding energy of Xe ($A=130$) using equation (5). The fit to the Xe data is representative of the fits obtained for other nuclei.
determined from a total of 313 nuclei in the mass region $22 \leq A \leq 250$ for temperatures $T \leq 4$ MeV. The temperature dependence of the six coefficients is given in reference [7]. This finite temperature parametrization has been used to identify the pairing phase transition in symmetric nuclear matter [11]. The same techniques have also been successfully employed to identify the remnants of the pairing phase transition in finite nuclei [12].

The excitation energy, $\mathcal{E}(A, Z, T)$, has been calculated for a number of stable isotopes in the mass region $20 \leq A \leq 250$ using the temperature dependent coefficients determined in reference [7]. For each stable isotope, the constant $b$ in equation (4) was determined from a quadratic fit of $\mathcal{E}(A, Z, T)$ versus $T$. These fits were in general excellent, as shown in figure 1 with only occasional small deviations from the quadratic fit occurring at low temperature. This is not surprising given that the continuum contributions are given by Fermi gas expressions. However, this alone in no way guarantees any simple dependence of $b$ on $A$. Note also that no attempt has been made in the fits to take into account the presence of low temperature collective to non-collective phase transitions [12, 13]. The coefficient $b$ was then plotted as a function of $A$, as shown in figure 2. As can be seen in figures 2 and 3 there is a relatively simple dependence of $b$ on $A$. In order to determine the effect on $b$ of the different terms in the temperature dependent binding energy (5), various combinations of the five terms were plotted against $T$ then quadratically fitted to obtain $b$ for each $A$ and finally these points were linearly fitted, as shown in figures 2 and 3. Although only 20 stable nuclei were used these were chosen at random to cover the range from $A=20$ to $A=250$. In figure 3 the equations for the fits are $b = -2.032 \times 10^{-4}A + 0.152$ when all terms of equation (5) are used and $b = -1.617 \times 10^{-5}A + 0.140$ if term 3 of equation (5) is excluded. The gradient is small in both cases but especially small when term 3 is excluded, with $b$ gradually varying from 0.141 to 0.137 for nuclei larger than $A=40$. Surface effects become significant in nuclei smaller than $A=40$ and the pairing effects are most likely overestimated. This most probably accounts for the deviation of the corresponding points from the fit in figure 3 and subsequently prompted the exclusion of these points from the linear fit. These results demonstrate that the asymmetry term (i.e. term 3) is largely responsible for the $A$ dependence of $b$. 
FIG. 2: The resultant \( b \) from various combinations of terms in equation (5) has been plotted against \( A \), thereby illustrating the linearity of all five cases.

FIG. 3: Linear fit of the \( b \) resulting from equation (5) and equation (5) without term 3 has been plotted.

Furthermore in the Canonical ensemble the specific heat is given by

\[
C = \frac{\partial E}{\partial T} = T \frac{\partial S}{\partial T}.
\]  \hfill (11)

At low \( T \) up to an additive constant the entropy per particle, \( S \), for finite nuclei in the mass range \( 20 \leq A \leq 250 \) is therefore given simply by

\[
S = \frac{1}{2} bT \]  \hfill (13)

with \( b = -2.032 \times 10^{-4} A + 0.152 \).

It is interesting to note that the simple Fermi gas model does not take properly into account that the nucleus is composed of protons and neutrons and that therefore the effects of the Coulomb force must be considered. Were only the nuclear force present one might expect \( b \) to be independent of \( A \). The presence of the Coulomb force gives rise to the asymmetry term which at \( T=0 \) leads to a displacement of the island of stability away from nuclei with \( N = Z \) and at finite temperature is largely responsible for an \( A \) dependence in \( b \).
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