STABILITY AND STABILIZATION FOR THE THREE-DIMENSIONAL NAVIER-STOKES-VOIGT EQUATIONS WITH UNBOUNDED VARIABLE DELAY

Vu Manh Toi
Faculty of Computer Science and Engineering, Thuyloi University
175 Tay Son, Dong Da, Hanoi, Vietnam

(Communicated by Cristina Pignotti)

Abstract. We consider the 3D Navier-Stokes-Voigt equations in a bounded domain with unbounded variable delay. We study the stability of stationary solutions by the classical direct method, and by an appropriate Lyapunov functional. We also give a sufficient condition of parameters for the polynomial stability of the stationary solution in a special case of unbounded variable delay. Finally, when the condition for polynomial stability is not satisfied, we stabilize the stationary by using the finite Fourier modes and by internal feedback control with a support large enough.

1. Introduction. Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with smooth boundary $\partial \Omega$. The three-dimensional Navier-Stokes-Voigt equations with unbounded variable delay is the following system:

\[
\begin{aligned}
\frac{\partial}{\partial t} (u - \alpha^2 \Delta u) - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f + G(u(t - \rho(t))) & \text{in } \Omega \times \mathbb{R}^+, \\
\nabla \cdot u &= 0 & \text{in } \Omega \times \mathbb{R}^+, \\
u
\end{aligned}
\]

\[
\begin{aligned}
 u &= 0 & \text{on } \partial \Omega \times \mathbb{R}^+, \\
 u(x, \theta) &= \phi(\theta), \theta \in (-\infty, 0], x \in \Omega.
\end{aligned}
\]

Here $u = u(x,t)$ represents the velocity of the fluid, called the filtered velocity, $p$ denotes the pressure, $\phi$ is the initial datum, $\alpha > 0$ is a scale parameter with dimension of length and $\nu > 0$ is the kinematic viscosity. The function $f \in V'$ is a non-delayed external force and $G : \mathbb{R}^3 \to \mathbb{R}^3$ is a Lipschitz function with $G(0) = 0$, i.e., there exists $L_g > 0$ such that

$$
\|G(u) - G(v)\|_{\mathbb{R}^3} \leq L_g \|u - v\|_{\mathbb{R}^3}, \forall u, v \in \mathbb{R}^3,
$$

where $V'$, the dual space of $V$, is defined in the Section 2 below. The function $\rho \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies $\rho_* = \sup_{t \geq 0} \rho'(t) < 1$.

The Navier-Stokes-Voigt equations was first introduced by Oskolkov in [30] as a model of motion of certain linear viscoelastic fluids. It is interesting to observe that the inviscid version of the Navier-Stokes-Voigt equations (1), i.e., when $\nu = 0$, coincides with the simplified Bardina model of turbulent flows. The system (1) was also proposed by Cao, Lunasin and Titi in [7] as a regularization of the 3D Navier-Stokes...
equations for the sake of direct numerical simulations, for small value of $\alpha$. The presence of the regularizing term $-\alpha^2 \Delta_t u$ in (1) leads to the global well-posedness of (1) both forward and backward in time, even in the case of three dimension. However, it changes the parabolic character of the limit Navier-Stokes equations, so the Navier-Stokes-Voigt system behaves like a damped hyperbolic system. The Navier-Stokes-Voigt equations are known as $\alpha$-models in fluid mechanics (see e.g., [26]), but it has attractive advantage over other $\alpha$-models in that one does not need to impose any additional artificial boundary condition (besides the Dirichlet boundary conditions) to obtain the global well-posedness. The Navier-Stokes-Voigt equations was also applied in image inpainting (see [19]).

The existence, long-time behavior and regularity of solutions to the 3D Navier-Stokes-Voigt equations without delays in bounded domains or unbounded domains satisfying the Poincaré inequality have attracted the attention of many mathematicians [2, 3, 6, 15, 16, 20, 21, 30, 31, 32]. There are many results involving PDEs in fluid mechanics with delay (see e.g., [1, 8, 9, 10, 11, 12, 27, 28, 29]). However, all the results that deal with finite delay (constant delay, bounded variable delay or bounded distributed delay) in the phase spaces $C([-h, 0]; X)$ and $L^2(-h, 0; X)$, with suitable Banach space $X$, or infinite distributed delay in

$$C_\gamma(X) = \left\{ \phi \in C((-\infty, 0]; X) : \lim_{\theta \to -\infty} e^{\gamma \theta} \|\phi(\theta)\|_X \text{ exists} \right\} (\gamma > 0).$$

In this paper, base on the idea in [25], we study the behavior of solutions to the 3D Navier-Stokes-Voigt equations with unbounded variable delay in the phase space $Y = \left\{ \phi \in BCL_{-\infty}(H) : \phi(0) \in V \right\}$ with $BCL_{-\infty}(H)$ is defined in the Section 2. More precisely, we study the existence, uniqueness of weak solution, and stationary solution as well as the stability of stationary solution (in the sense of local stability, asymptotic stability and polynomial stability) by several different approaches. Beside that, we use finite Fourier modes and by internal feedback control with a support large enough to stabilize the stationary solution when the sufficient condition for polynomial stability does not hold (in this case, the stationary solution may not be polynomial stable).

The rests of the paper is organized as follows. In Section 2, for convenience of the reader, we recall some results on function spaces and the results on the existence and unique of weak solutions to (1). In Section 3, we first prove the existence and uniqueness of weak stationary solutions, and then the stability of stationary solutions by using some different methods are studied here. Section 4 is to use the finite Fourier modes and by internal feedback control with a support large enough to stabilize the stationary solution when the condition for polynomial stability does not hold. The last section is to give the proof of Theorem 2.2.

2. Preliminaries. To study the theory of incompressible fluid, we denote

$$V = \left\{ u \in (C_0^\infty(\Omega))^3 : \nabla \cdot u = 0 \right\}.$$  

Denote also by $H$ and $V$, the closures of $V$ in the $(L^2(\Omega))^3$ and $(H^1(\Omega))^3$, respectively. Then $H$ and $V$ are Hilbert spaces with inner products given by

$$(u, v) = \int_\Omega u(x) \cdot v(x) dx$$  

and $$(u, v) = \sum_{i, j=1}^3 \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx,$$
respectively, and the associated norms

\[ |u| = (u, u)^{\frac{1}{2}} \text{ and } \|u\| = ((u, u))^{\frac{1}{2}}. \]

We use \( \| \cdot \| \) for the norm in \( V' \) and \( \langle \cdot, \cdot \rangle_{V', V} \) for the dual pairing between \( V \) and \( V' \). Let \( P : (L^2(\Omega))^3 \to H \) be the Helmholtz-Leray projector, and denote by \( A = -PA\Delta \) the Stokes operator such that \( \langle Au, v \rangle_{V', V} = ((u, v)) \), with domain \( D(A) = (H^2(\Omega))^3 \cap V \). The Stokes operator \( A \) is a positive self-adjoint operator with compact inverse. Hence there exists a complete orthonormal set of eigenfunctions \( \{w_j\}_{j=1}^\infty \subset H \) such that \( Aw_j = \lambda_j w_j \) and

\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to +\infty \text{ as } j \to \infty. \]

We have the following Poincaré type inequalities

\[ \|w\|^2 \geq \lambda_1 |w|^2 \quad \text{for all } w \in V. \] (2)

Using (2), we have the following estimate

\[ \|w\|^2 \geq d_0 (|w|^2 + \alpha^2 \|w\|^2) \quad \text{for all } w \in V. \] (3)

with

\[ d_0 = \frac{\lambda_1}{1 + \alpha^2 \lambda_1}. \]

Following the classical notation for Navier-Stokes equations, for every \( u, v \in V \), we write \( B(u, v) = P[(u \cdot \nabla)v] \). The bilinear operator \( B \) can be extended continuously from \( V \times V \) with value in \( V' \). We have that \( B \) satisfies the following properties

\[ \langle B(u, v), w \rangle_{V', V} = -\langle B(u, w), v \rangle_{V', V}, \forall u, v, w \in V, \] (4)

and in particular,

\[ \langle B(u, v), v \rangle_{V', V} = 0, \forall u, v \in V. \]

Furthermore, there exists a positive constant \( c_0 \) such that (see [24, Lemma 2.1])

\[ \|B(u, v)\|_{V'} \leq c_0 |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} \|w\|, \forall u, v, w \in V \] (5)

which implies that

\[ \|B(u, v)\|_v \leq c_0 |u|^{1/2} \|u\|^{1/2} \|v\|, \forall u, v \in V. \] (6)

We now aim to establish well-posedness and stability results for the 3D Navier-Stokes-Voigt equations with unbounded variable delay in the following phase space which was introduced in [25]:

\[ BCL_{-\infty}(H) = \left\{ \varphi \in C((-\infty, 0]; H) : \lim_{\theta \to -\infty} \varphi(\theta) \text{ exists in } H \right\}. \]

Note that it is a Banach space endowed with the norm

\[ \|\varphi\|_{BCL_{-\infty}(H)} = \sup_{\theta \in (-\infty, 0]} |\varphi(\theta)|. \]

We can rewrite the three-dimensional Navier-Stokes-Voigt equations (1) in the following functional form

\[
\begin{cases}
\frac{d}{dt}(u + \alpha^2 Au) + \nu Au + B(u, u) = Pf + PG(u(t - \rho(t))), \forall t > 0,
\quad \\
u(\theta) = \phi(\theta), \theta \in (-\infty, 0].
\end{cases}
\] (7)
\textbf{Definition 2.1.} Let $T > 0$ and $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$. A weak solution to (7) in $(-\infty, T]$ is a vector function $u \in C((-\infty, T]; H) \cap C([0, T]; V)$ such that $\frac{du}{dt} \in L^2(0, T; V)$, $u(\theta) = \phi(\theta), \theta \leq 0$ and
\[
\frac{d}{dt} \left((u(t), \varphi) + \alpha^2((u(t), \varphi))\right) + \nu((u(t), \varphi)) + \langle B(u(t), u(t)), \varphi \rangle_{V', V} = \langle f, \varphi \rangle_{V', V} + \langle G(u(t - \rho(t)), \varphi)\rangle, \quad \text{for all } \varphi \in V \text{ and a.e. } t \in (0, T).
\]

for all $\varphi \in V$ and a.e. $t \in (0, T)$.

We have the following result. The proof of this theorem is given in the Appendix.

\textbf{Theorem 2.2.} Let $f \in V'$ and $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$ given. Then there exists a unique weak solution to (7).

\section{Stability of stationary solutions.}

The stationary equation of (7) is the following form
\[
\nu Au + B(u, u) = Pf + PG(u).
\]
A weak solution to (8) is a function $u_\infty \in V$ such that
\[
\nu((u_\infty, \varphi)) + \langle B(u_\infty, u_\infty), \varphi \rangle_{V', V} = \langle f, \varphi \rangle_{V', V} + \langle G(u_\infty), \varphi \rangle, \quad \forall \varphi \in V.
\]

\textbf{Theorem 3.1.} Assume that $\nu > L_\nu \lambda_1^{-1}$ then there exists at least one weak solution $u_\infty$ to (8) satisfying
\[
\|u_\infty\| \leq \frac{\|f\|_*}{\nu - L_\nu \lambda_1^{-1}}.
\]
In addition, if the following condition holds
\[
(\nu - L_\nu \lambda_1^{-1})^2 > c_0 \lambda_1^{-1/4} \|f\|_*,
\]
where $c_0$ is the constant in (6), then the stationary solution to (7) is unique.

\textbf{Proof.} The proof is same as in [13, Theorem 3.1]. So we omit it here. \hfill \Box

To study the stability of the stationary solution to (7), we first recall the following definition of the stability (see e.g., [14]).

\textbf{Definition 3.2.} A stationary solution $u_\infty$ to (7) is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$ satisfies $\|\phi - u_\infty\|_{BCL_{-\infty}(H)} + \alpha^2\|\phi(0) - u_\infty\| < \delta$, then the solution $u(\cdot, \phi)$ to (7) exists for all $t \geq 0$ and satisfies $|u(t, \phi) - u_\infty| < \varepsilon$ for any $t \geq 0$.

A stationary solution $u_\infty$ to (7) is said to be asymptotically stable if it is stable and the solution $u(\cdot, \phi)$ to (7) exists for all $t \geq 0$ and satisfies
\[
\lim_{t \to \infty} (|u(t, \phi) - u_\infty|^2 + \alpha^2\|u(t, \phi) - u_\infty\|^2) = 0.
\]

\subsection{Local stability via a directly approach.}

\textbf{Theorem 3.3.} If $\nu > L_\nu \lambda_1^{-1}$ then there exists at least one weak solution $u_\infty$ to (8) satisfying (9). Moreover, if
\[
2\nu \geq \frac{2c_0 \lambda_1^{-1/4} \|f\|_*}{\nu - L_\nu \lambda_1^{-1}} + \frac{(2 - \rho_*)L_\nu \lambda_1^{-1}}{1 - \rho_*},
\]
then the solution $u_\infty$ is unique, and for any $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$, the solution $u$ to (7) satisfies

$$
|u(t) - u_\infty|^2 + \alpha^2 \|u(t) - u_\infty\|^2 \\
\leq C \left( |\phi(0) - u_\infty|^2 + \alpha^2 \|\phi(0) - u_\infty\|^2 + \|\phi - u_\infty\|_{L^2(-\rho(0), 0; H)}^2 \right), \forall t \geq 0,
$$

(11)

where $C = \max\{1, L_g/(1 - \rho_s)\}$. That is, the stationary solution $u_\infty$ to (7) is stable.

**Proof.** We first see that all assumptions of Theorem 3.1 are satisfied. Thus, the existence and uniqueness of weak stationary solution $u$ satisfying (9) are obtained from Theorem 3.1. Let $w = u - u_\infty$. Then it satisfies

$$
\frac{d}{dt}(w + \alpha^2 Aw) + \nu Aw + B(w, w) + B(u_\infty, w) + B(w, u_\infty) = PG(u(t - \rho(t)) - PG(u_\infty).
$$

(12)

Here we have used the fact that

$$
B(u, w) - B(u_\infty, w) = B(w, w) + B(u_\infty, w) + B(w, u_\infty).
$$

Multiplying (12) by $w$, integrating over $\Omega$, using (5), and the Young inequality, we deduce

$$
\frac{1}{2} \frac{d}{dt}(\|w\|^2 + \alpha^2 \|w\|^2) + \nu \|w\|^2 = -\langle B(w, u_\infty), w \rangle_{V^*, V} + \langle (G(u(t - \rho(t)) - G(u_\infty), w \rangle \\
\leq c_0 \|w\|^{1/2} \|w\|^3/2 \|u_\infty\| + L_g \|u(t - \rho(t)) - u_\infty\| \|w\| \\
\leq \left( c_0 \lambda_1^{-1/4} \|u_\infty\| + \frac{L_g \lambda_1^{-1}}{2} \right) \|w\|^2 + \frac{L_g}{2} \|w(t - \rho(t))\|^2.
$$

Hence

$$
\frac{d}{dt}(\|w\|^2 + \alpha^2 \|w\|^2) \leq \left( \frac{2c_0 \lambda_1^{-1/4} \|f\|_*}{\nu - L_g \lambda_1^{-1}} + \frac{L_g \lambda_1^{-1}}{2} - 2\nu \right) \|w\|^2 + L_g \|w(t - \rho(t))\|^2.
$$

(13)

Taking $\eta = s - \rho(s) = \tau(s)$ then

$$
\int_0^t |w(s - \rho(s))|^2 ds \leq \frac{1}{1 - \rho_s} \int_{-\rho(0)}^t |w(\eta)|^2 d\eta.
$$

Thus

$$
|w(t)|^2 + \alpha^2 \|w(t)\|^2 \leq |w(0)|^2 + \alpha^2 \|w(0)\|^2 + \frac{L_g}{1 - \rho_s} \int_{-\rho(0)}^0 |\phi(s) - u_\infty|^2 ds \\
+ \left( \frac{2c_0 \lambda_1^{-1/4} \|f\|_*}{\nu - L_g \lambda_1^{-1}} + \frac{(2 - \rho_s) L_g \lambda_1^{-1}}{1 - \rho_s} - 2\nu \right) \int_0^t \|w(s)\|^2 ds.
$$

Using the condition (10) we obtain (11). \hfill \Box

**Remark 1.** As in the case of 2D Navier-Stokes equations [25], a sufficient condition to obtain exponential stability is that $\rho(t)$ is bounded.
3.2. Stability and asymptotic stability via the construction of Lyapunov functionals. We have the following result.

**Theorem 3.4.** If \( \nu > L_g \lambda_1^{-1} \) then there exist at least one weak solution \( u_\infty \) to (8) satisfying (9). In addition, if \( \nu \geq \frac{c_0 \lambda_1^{-1/4}}{\nu - L_g \lambda_1^{-1}} \) + \( \frac{L_g \lambda_1^{-1}}{\sqrt{1 - \rho_*}} \) then the stationary solution \( u_\infty \) is unique, stable and satisfies

\[
|u(t) - u_\infty|^2 + \alpha^2 \|u(t) - u_\infty\|^2 \leq |\phi(0) - u_\infty|^2 + \alpha^2 \|\phi(0) - u_\infty\|^2
\]

\[
+ \frac{L_g}{\sqrt{1 - \rho_*}} \|\phi - u_\infty\|^2_{L^2(-\rho(0), 0; H)} \quad \forall t \geq 0,
\]

for any solution \( u \) to (7) respect to the initial data \( \phi \in BCL_{-\infty}(H) \) with \( \phi(0) \in V \).

Furthermore, if \( \nu > \frac{c_0 \lambda_1^{-1/4}}{\nu - L_g \lambda_1^{-1}} \) + \( \frac{L_g \lambda_1^{-1}}{\sqrt{1 - \rho_*}} \) then \( u_\infty \) is asymptotically stable.

**Proof.** Note that \( \sqrt{1 - \rho_*} < 1 \) since \( \rho_* \in (0, 1) \). So, all assumptions of Theorem 3.1 are satisfied. Therefore, we obtain the existence and uniqueness of weak stationary solutions \( u_\infty \) satisfying (9). Now, let us set \( w(t) = u(t) - u_\infty \) then it satisfies

\[
\frac{d}{dt}(w(t) + \alpha^2 Aw(t)) = -\nu Aw(t) - B(w(t), w(t)) - B(u_\infty, w(t)) - B(w(t), u_\infty)
\]

\[
+ P(G(u(t - \rho(t))) - G(u_\infty)),
\]

with initial condition \( w(\theta) = \phi(\theta) - u_\infty, \theta \in (-\infty, 0] \).

For any \( \phi \in BCL_{-\infty}(H) \) with \( \phi(0) \in V \), for any \( t > 0 \) we consider \( \mathcal{L} \) given by

\[
\mathcal{L}(t, \phi) = |\phi(0) - u_\infty|^2 + \alpha^2 \|\phi(0) - u_\infty\|^2 + \frac{L_g}{\sqrt{1 - \rho_*}} \int_{-\rho(t)}^0 |\phi(s) - u_\infty|^2 ds.
\]

Then, for any solution \( u(\cdot, \phi) \) of (7) respect to initial data \( \phi \in BCL_{-\infty}(H) \) with \( \phi(0) \in V \), we have

\[
\mathcal{L}(t, u_t) = |u(t) - u_\infty|^2 + \alpha^2 \|u(t) - u_\infty\|^2 + \frac{L_g}{\sqrt{1 - \rho_*}} \int_{t-\rho(t)}^t |u(s) - u_\infty|^2 ds.
\]

By (7) and (5) we have

\[
\frac{d}{dt} \mathcal{L}(t, u_t) = 2 \left\langle \frac{d}{dt}(w(t) + \alpha^2 Aw(t)), w(t) \right\rangle_{V', V}
\]

\[
+ \frac{L_g}{\sqrt{1 - \rho_*}} |w(t)|^2 - \frac{L_g (1 - \rho(t))}{\sqrt{1 - \rho_*}} |w(t - \rho(t))|^2
\]

\[
= -2\nu |w(t)|^2 - 2 \langle B(w(t), u_\infty), w(t) \rangle_{V', V} + 2 \langle G(u(t - \rho(t))), w(t) \rangle + \frac{L_g}{\sqrt{1 - \rho_*}} |w(t)|^2
\]

\[
- \frac{L_g (1 - \rho(t))}{\sqrt{1 - \rho_*}} |w(t - \rho(t))|^2.
\]

Thus, by the Cauchy inequality, the Poincaré inequality (2) and the bound (9), we get from the above inequality that

\[
\frac{d}{dt} \mathcal{L}(t, u_t) \leq -2 \left( \nu - \frac{c_0 \lambda_1^{-1/4}}{\nu - L_g \lambda_1^{-1}} - \frac{L_g \lambda_1^{-1}}{\sqrt{1 - \rho_*}} \right) |w(t)|^2.
\]

(15)
Integrating (15) from 0 to \( t \), for any \( t \geq 0 \), we deduce
\[
\mathcal{L}(t, u_t) + 2 \left( \nu - \frac{c_0 \lambda^{-1/4}}{\nu - L_g \lambda_1^{-1/4}} \right) \int_0^t \|u(s)\|^2 ds \leq \mathcal{L}(0, u_0). \tag{16}
\]
From the definition of \( \mathcal{L}(t, u_t) \) we have
\[
\mathcal{L}(t, u_t) = \|u(t) - u_\infty\|^2 + \alpha^2 \|\phi(0) - u_\infty\|^2 + \frac{L_g}{\sqrt{1 - \rho_*}} \|\phi - u_\infty\|_{L^2(-\rho(0), 0; H)}^2.
\]
and
\[
\mathcal{L}(0, u_0) = \|\phi(0) - u_\infty\|^2 + \alpha^2 \|\phi(0) - u_\infty\|^2 + \frac{L_g}{\sqrt{1 - \rho_*}} \|\phi - u_\infty\|_{L^2(-\rho(0), 0; H)}^2.
\]
So, (16) becomes
\[
\|u(t) - u_\infty\|^2 + \alpha^2 \|\phi(0) - u_\infty\|^2 + \frac{L_g}{\sqrt{1 - \rho_*}} \|\phi - u_\infty\|_{L^2(-\rho(0), 0; H)}^2.
\]
Therefore, if \( \nu \geq \frac{c_0 \lambda^{-1/4} \|f\|_r}{\nu - L_g \lambda_1^{-1/4}} + \frac{L_g \lambda_1^{-1}}{\sqrt{1 - \rho_*}} \) then the stationary solution \( u_\infty \) is stable and satisfies (14). If \( \nu > \frac{c_0 \lambda^{-1/4} \|f\|_r}{\nu - L_g \lambda_1^{-1/4}} + \frac{L_g \lambda_1^{-1}}{\sqrt{1 - \rho_*}} \) then from (17) we have
\[
2d_0 \left( \nu - \frac{L_g \lambda_1^{-1}}{\sqrt{1 - \rho_*}} \right) \int_0^\infty \left( \|u(t) - u_\infty\|^2 + \alpha^2 \|u(t) - u_\infty\|^2 \right) dt \leq \|\phi(0) - u_\infty\|^2 + \alpha^2 \|\phi(0) - u_\infty\|^2 + \frac{L_g}{\sqrt{1 - \rho_*}} \|\phi - u_\infty\|_{L^2(-\rho(0), 0; H)}^2.
\]
By the continuity of \( u(t) \) in \( V \) we deduce \( \lim_{t \to \infty} (\|u(t) - u_\infty\|^2 + \alpha^2 \|u(t) - u_\infty\|^2) = 0 \), i.e., the stationary solution \( u_\infty \) is asymptotically stable.

**Remark 2.** In Theorem 3.3 using the direct method, the stationary solution \( u_\infty \) to (7) is stable if \( \nu \geq \frac{c_0 \lambda^{-1/4} \|f\|_r}{\nu - L_g \lambda_1^{-1/4}} + \frac{L_g \lambda_1^{-1}}{2(1 - \rho_*)} \). Since \( \frac{(2 - \rho_*) L_g \lambda_1^{-1}}{2(1 - \rho_*)} > \frac{L_g \lambda_1^{-1}}{\sqrt{1 - \rho_*}} \) for \( \rho_* \in (0, 1) \) we can see that Theorem 3.4 improves Theorem 3.3.

### 3.3. Polynomial stability: The proportional delay case.

The purpose of this section is to analyze the stability of stationary solutions to (7) in the special case of unbounded variable delay, namely \( \rho(t) = (1 - q) t, q \in (0, 1) \). We will give a sufficient condition to obtain the polynomial stability of stationary solution for the 3D Navier-Stokes-Voigt equations. To do this, we first recall the following pantograph equation and technical lemma that will be used in this framework (see [4, 22, 23]):
\[
y'(t) = ay(t) + by(qt), \quad \forall t \geq 0, \ q \in (0, 1).
\]
The following lemma can be deduced from Lemmas 3.4-3.5 in [4].

**Lemma 3.5.** [4, Lemma 3.6 (i)] Let \( a < 0, b > 0 \) and \( q \in (0, 1) \). Suppose \( h \in C(\mathbb{R}_+, \mathbb{R}_+) \) satisfies
\[
D^+ h(t) \leq ah(t) + bh(qt), \quad t \geq 0,
\]
with \( h(0) > 0 \) and where \( D^+ h \) denotes the Dini derivative of \( h \) at \( t \) in the following sense
\[
D^+ h = \lim_{\delta \downarrow 0} \sup_{\delta > 0} \frac{h(t + \delta) - h(t)}{\delta}.
\]
Then there exist $C = C(a, b, q) > 0$ such that

$$h(t) \leq Ch(0)(1 + t)^\gamma, \quad \forall t \geq 0,$$

where $\mu$ obeys $a + bq^\gamma = 0$.

We now have the following result.

**Theorem 3.6.** Consider (7) with $\rho(t) = (1 - q)t$ for $q \in (0, 1)$. If $\nu > L_g\lambda_1^{-1}$ then there exist at least one weak solution $u_\infty$ to (8) satisfying (9). And, if

$$\nu > \frac{c_0\lambda_1^{-1/4}\|f\|_*}{\nu - L_g\lambda_1^{-1}} + \frac{(2 + \alpha^2\lambda_1)L_g\lambda_1^{-1}}{2},$$

then the stationary solution $u_\infty$ is unique and for any solution $u$ to (7) respect to the initial data $\phi \in \mathcal{BCL}_{\infty}(H)$ with $\phi(0) \in V$, we have

$$|u(t) - u_\infty|^2 + \alpha^2\|u(t) - u_\infty\|^2 \leq C (|\phi(0) - u_\infty|^2 + \alpha^2\|\phi(0) - u_\infty\|^2) (1 + t)^\gamma, \quad \forall t \geq 0,$$

where

$$\gamma = \log \left( \frac{2\nu - \frac{c_0\lambda_1^{-1/4}\|f\|_*}{\nu - L_g\lambda_1^{-1}} - L_g\lambda_1^{-1}}{(1 + \alpha^2\lambda_1)L_g\lambda_1^{-1}} \right) < 0. \tag{19}$$

**Proof.** The existence and uniqueness of $u_\infty$ is obtained from Theorem 3.1. As in the proof of Theorem 3.3, we deduce the same estimate as (13) by taking $\rho(t) = (1 - q)t$,

$$\frac{d}{dt}(|w(t)|^2 + \alpha^2\|w(t)\|^2) \leq \left( \frac{2c_0\lambda_1^{-1/4}\|f\|_*}{\nu - L_g\lambda_1^{-1}} + L_g\lambda_1^{-1} - 2\nu \right)\|w(t)\|^2 + L_g\|w(qt)\|^2, \tag{20}$$

where $w(t) = u(t) - u_\infty$. By using (18) and (3) we have from (20) that

$$\frac{d}{dt}(|w(t)|^2 + \alpha^2\|w(t)\|^2) \leq d_0 \left( \frac{2c_0\lambda_1^{-1/4}\|f\|_*}{\nu - L_g\lambda_1^{-1}} + L_g\lambda_1^{-1} - 2\nu \right) (|w(t)|^2 + \alpha^2\|w(t)\|^2)
+ L_g (|w(qt)|^2 + \alpha^2\|w(qt)\|^2).
$$

Setting $h(t) = |w(t)|^2 + \alpha^2\|w(t)\|^2$ we obtain that

$$h'(t) \leq d_0 \left( \frac{2c_0\lambda_1^{-1/4}\|f\|_*}{\nu - L_g\lambda_1^{-1}} + L_g\lambda_1^{-1} - 2\nu \right) h(t) + L_g h(qt)$$

with $h(0) = |w(0)|^2 + \alpha^2\|w(0)\|^2 = |\phi(0) - u_\infty|^2 + \alpha^2\|\phi(0) - u_\infty\|^2$. Applying Lemma 3.5 we conclude that there exists a positive constant $C$ depending on $L_g, \nu, \alpha, \lambda_1, \|f\|_*$ such that

$$h(t) \leq Ch(0)(1 + t)^\gamma,$$

where $\gamma$ satisfies

$$d_0 \left( \frac{2c_0\lambda_1^{-1/4}\|f\|_*}{\nu - L_g\lambda_1^{-1}} + L_g\lambda_1^{-1} - 2\nu \right) + L_g \gamma = 0.$$

That is, $\gamma$ is given by (19). This completes the proof. \qed
4. Stabilization result for stationary solution. In this section, we consider the polynomial stability of the stationary solution $u_\infty$ to (7) $\rho(t) = (1 - q)t$ where $q \in (0, 1)$ is a constant. From Theorem 3.6, we see that the stationary solution $u_\infty$ might not be polynomial stable when condition (18) is not satisfied, i.e.,

$$
\nu \leq \frac{c_0\lambda_1^{-1}||f||^4}{\nu - L_g\lambda_1^{-1}} + \frac{(2 + \alpha^2\lambda_1)L_g\lambda_1^{-1}}{2}.
$$

Here, the condition $\nu > L_g\lambda_1^{-1}$ is always assumed to ensure the existence of stationary solution $u_\infty$ of (7) (see Theorem 3.1). Therefore, (21) is equivalent to

$$
\nu \leq L_g\lambda_1^{-1} + \frac{\alpha^2L_g}{4} + \sqrt{\frac{\alpha^4L_g^2}{16} + c_0\lambda_1^{-1/4}||f||^4}.
$$

We will use finite Fourier modes and by internal feedback control with a support large enough to stabilize $u_\infty$ in the sense of the polynomial stability.

4.1. Stabilization by finite Fourier modes. For $\mu > 0$, we consider the following system:

$$
\begin{align*}
\frac{d}{dt}(u + \alpha^2Au) + \nu Au + B(u, u) &= Pf - PG(uqt) \\
-u\sum_{j=1}^{N}(u - u_\infty, w_j)w_j, \forall t > 0,
\end{align*}
$$

where $w_j$ is the j-th eigenfunction for Stokes operator $A$.

By the same as in Theorem 2.2 we can prove that (22) has a unique solution. We now have the following result.

**Theorem 4.1.** If $\nu > L_g\lambda_1^{-1}$ then there exists at least one weak solution $u_\infty$ to (8) satisfying (9). Moreover, if

$$
\mu > \frac{27c_0^4\lambda_1^{-1}||f||^4}{32\nu^3(\nu - L_g\lambda_1^{-1})^3} + \frac{L_g(1 + \alpha^2\lambda_1)}{2}
$$

and

$$
\lambda_{N+1} > \left(\frac{27c_0^4\lambda_1^{-1}||f||^4}{16\nu^3(\nu - L_g\lambda_1^{-1})^3} + L_g(1 + \alpha^2\lambda_1)\right)^{-1} (\nu - L_g\lambda_1^{-1})^{-1},
$$

then for any solution $u$ to (22) respect to the initial data $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$ we have for all $t \geq 0$

$$
|u(t) - u_\infty|^2 + \alpha^2||u(t) - u_\infty||^2 \leq C (||\phi(0) - u_\infty||^2 + \alpha^2||\phi(0) - u_\infty||^2) (1 + t)^{\gamma_0},
$$

where

$$
\gamma_0 = \log_q \left(\frac{\lambda_{N+1}^{-1}}{\lambda_{N+1}^{-1}} \left(\frac{27c_0^4\lambda_1^{-1}||f||^4}{16\nu^3(\nu - L_g\lambda_1^{-1})^3} + L_g(1 + \alpha^2\lambda_1)\right)\right) < 0.
$$

**Proof.** We put $z = u - u_\infty$ then we have

$$
\begin{align*}
\frac{d}{dt}(z + \alpha^2Az) + \nu Az + B(z, z) + B(z, u_\infty) + B(u_\infty, z) \\
= Pf(G(uqt)) - G(u_\infty)) - \mu \sum_{j=1}^{N}(z, w_j)w_j, \forall t > 0,
\end{align*}
$$

where $z(\theta) = \phi(\theta) - u_\infty, \theta \in (-\infty, 0]$. 


Multiplying the first equation in (26) by $z$ and integrating over $\Omega$, using (4)-(5) and the Young inequality, we deduce that

$$\frac{1}{2} \frac{d}{dt} \left( |z(t)|^2 + \alpha^2 \|z(t)\|^2 \right) + \nu \|z\|^2$$

\[ \leq \nu \|z\|^2 + \left( \frac{27c_0^4 \lambda_1^{-1} \|f\|^4}{16\nu^3(\nu - L_g \lambda_1^{-1})^4} + L_g(1 + \alpha^2 \lambda_1) \right) |z(t)|^2 \]

\[ + \frac{L_g}{2(1 + \alpha^2 \lambda_1)} |z(t)|^2 - \mu \sum_{j=1}^{N} |(z(t), w_j)|^2. \]

Since

$$|z(t)|^2 = \sum_{j=1}^{N} |(z(t), w_j)|^2 + \sum_{j=N+1}^{\infty} |(z(t), w_j)|^2 \leq \sum_{j=1}^{N} |(z(t), w_j)|^2 + \lambda_{N+1}^{-1} |z(t)|^2$$

then by using (9), we have

\[
\frac{d}{dt} \left( |z(t)|^2 + \alpha^2 |z(t)|^2 \right) \leq \left\{ -\nu + \lambda_{N+1}^{-1} \left( \frac{27c_0^4 \lambda_1^{-1} \|f\|^4}{16\nu^3(\nu - L_g \lambda_1^{-1})^4} + L_g(1 + \alpha^2 \lambda_1) \right) \right\} |z(t)|^2 \\
\quad + \left\{ -2\mu + \frac{27c_0^4 \lambda_1^{-1} \|f\|^4}{16\nu^3(\nu - L_g \lambda_1^{-1})^4} + L_g(1 + \alpha^2 \lambda_1) \right\} \sum_{j=1}^{N} |(z(t), w_j)|^2 \\
\quad + \frac{L_g}{1 + \alpha^2 \lambda_1} \left( |z(t)|^2 + \alpha^2 |z(t)|^2 \right). \tag{27}
\]

Using conditions (23)-(24) and (3), we obtain from (27) that

\[
\frac{d}{dt} \left( |z(t)|^2 + \alpha^2 |z(t)|^2 \right) \leq d_0 \left\{ -\nu + \lambda_{N+1}^{-1} \left( \frac{27c_0^4 \lambda_1^{-1} \|f\|^4}{16\nu^3(\nu - L_g \lambda_1^{-1})^4} + L_g(1 + \alpha^2 \lambda_1) \right) \right\} |z(t)|^2 + \alpha^2 |z(t)|^2 \\
\quad + \frac{L_g}{1 + \alpha^2 \lambda_1} \left( |z(t)|^2 + \alpha^2 |z(t)|^2 \right). \]

Applying Lemma 3.5 with $h(t) = |z(t)|^2 + \alpha^2 |z(t)|^2$ and $h(0) = |z(0)|^2 + \alpha^2 |z(0)|^2 = |\phi(0) - u_\infty|^2 + \alpha^2 |\phi(0) - u_\infty|^2$, one gets (25) with $\gamma_0$ satisfying

$$d_0 \left\{ -\nu + \lambda_{N+1}^{-1} \left( \frac{27c_0^4 \lambda_1^{-1} \|f\|^4}{16\nu^3(\nu - L_g \lambda_1^{-1})^4} + L_g(1 + \alpha^2 \lambda_1) \right) \right\} + \frac{L_g}{1 + \alpha^2 \lambda_1} \gamma_0 = 0.$$

This completes the proof. \qed

**Remark 3.** Theorem 4.1 showed that stationary solution $u_\infty$ can be polynomial stable by using finite Fourier modes $N$ if the parameter $\mu$ is large enough so that (23) holds, $\lambda_{N+1}$ satisfies (24) and when any fixed parameters of system, $\nu$ satisfies

$$L_g \lambda_1^{-1} < \nu \leq L_g \lambda_1^{-1} + \frac{\alpha^2 L_g^2}{4} + \sqrt{\frac{\alpha^4 L_g^2}{16} + c_0 \lambda_1^{-1/4} \|f\|_\ast}.$$
Obviously, when $\nu > L_0 \lambda_1^{-1} + \frac{\alpha^2 L_0^2}{4} + \sqrt{\frac{\alpha^4 L_0^4}{16} + c_0 \lambda_1^{-1/4} ||f||_*}$, then by Theorem 3.6 the stationary $u_\infty$ to (7) is unique and polynomial stable without finite Fourier modes.

4.2. **Stabilization by internal feedback control with a support large enough.**

With the stationary $u_\infty$ to (7), we consider the following system:

$$
\frac{\partial}{\partial t} (u - \alpha^2 \Delta u) - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f + G(u(\gamma t)) - k_1 \omega (u - u_\infty) \quad \text{in } \Omega \times \mathbb{R}_+,
$$

$$
\nabla \cdot u = 0 \quad \text{in } \Omega \times \mathbb{R}_+,
$$

$$
u u = 0 \quad \text{on } \partial \Omega \times \mathbb{R}_+ ,
$$

$$
\begin{cases}
\frac{d}{dt}(u + \alpha^2 Au) + \nu Au + B(u, u) = Pf + PG(u(\gamma t)) - kP(1_\omega (u - u_\infty)), \\
u u = \phi(\theta), \theta \in (-\infty, 0], x \in \Omega,
\end{cases}
$$

(28)

Let $A_\omega$ be the Stokes operator defined on $\Omega_\omega := \Omega \setminus \omega$, i.e.,

$$
\langle A_\omega u, \psi \rangle = \int_{\Omega_\omega} \nabla u \cdot \nabla \psi dx, \forall \psi \in V_\omega, \quad D(A_\omega) = (H^2(\Omega_\omega))^3 \cap V_\omega,
$$

where $V_\omega$ is the closure in $H^1(\Omega_\omega)$ of $V_\omega := \{u \in (C^0_0(\Omega_\omega))^3 : \nabla \cdot u = 0 \text{ in } \Omega_\omega \}$. We denote the first eigenvalue of $A_\omega$ by $\lambda_1^*(\omega)$, i.e.,

$$
\lambda_1^*(\omega) = \inf \left\{ \int_{\Omega_\omega} |\nabla u|^2 dx : \int_{\Omega_\omega} |u|^2 dx = 1 \right\}
$$

$$
= \inf \left\{ \langle A_\omega u, u \rangle_{\omega} : |u|^2_{\omega} = 1 \right\},
$$

where $\langle \cdot, \cdot \rangle_\omega$ and $| \cdot |_\omega$ are the scalar and norm respectively for $H_\omega$, the closure of $V_\omega$ in $(L^2(\Omega_\omega))^3$.

We recall the following result.

**Lemma 4.2.** [5] For each $\varepsilon > 0$ there is $k_0 = k_0(\varepsilon) > 0$ such that

$$
(\nu Au + kP(1_\omega u), u) \geq (\nu \lambda_1^*(\omega) - \varepsilon)|u|^2,
$$

(29)

for all $u \in V$ and $k \geq k_0$.

We have the following result.

**Theorem 4.3.** If $\nu > L_0 \lambda_1^{-1}$ then there exists at least one weak stationary solution $u_\infty$ to (7) satisfying (9). Moreover, if

$$
\lambda_1^*(\omega) > \frac{c_0 \lambda_1^{-1/4} ||f||_*}{\nu(\nu - L_0 \lambda_1^{-1})} + \frac{L_0}{2\nu} \left( 1 + \frac{1 + \alpha^2 \lambda_1}{\lambda_1} \right),
$$

(30)

then there exists a weak solution $u$ to (28) respect to the initial data $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$, and $k \geq k_0$ large enough, such that for all $t \geq 0$,

$$
|u(t) - u_\infty|^2 + \alpha^2 ||u(t) - u_\infty||^2 \leq C (||\phi(0) - u_\infty||^2 + \alpha^2 ||\phi(0) - u_\infty||^2) (1 + t)^{\gamma_1},
$$

(31)
where
\[
\gamma_1 = \log_q \left( \frac{2 \nu \lambda_1^* (\omega) - 2 c_0 \lambda_1^{-1/4} \| f \|_* - L_g}{\nu - (1+ \alpha^2 \lambda_1) L_g \lambda_1^{-1}} \right) < 0. \tag{32}
\]

**Proof.** The existence of a weak solution is to proved by same as in Theorem 2.2. So we only prove the estimate (31). Setting \( z = u - u_\infty \) then we have
\[
\frac{d}{dt} (z + \alpha^2 Az) + \nu A z + B(z, z) + B(z, u_\infty) + B(u_\infty, z) = P(G(u qt)) - G(u_\infty)) - kP(1, \omega z), \quad \forall t > 0.
\]
Multiplying this equation by \( z \) and integrating over \( \Omega \), using (6), (29) and the Poincaré inequality (2), the Lipschitz condition of \( G \) and the Cauchy inequality, we deduce for \( k \geq k_0 \) large enough that
\[
\frac{1}{2} \frac{d}{dt} \left( |z(t)|^2 + \alpha^2 |z(t)|^2 \right) \leq - \left( \nu \lambda_1^* (\omega) - c_0 \lambda_1^{-1/4} \| u_\infty \| \| z \|^2 + L_g |z| z(t) \right) |z(t)|^2
\]
\[
\leq - \left( \nu \lambda_1^* (\omega) - \epsilon - c_0 \lambda_1^{-1/4} \| u_\infty \| - \frac{L_g}{2} \right) |z(t)|^2 + \frac{L_g}{2} |z(t)|^2.
\]
Thus, using estimate (9) with note that \( \epsilon \) is arbitrary small enough, we obtain that
\[
\frac{d}{dt} \left( |z(t)|^2 + \alpha^2 |z(t)|^2 \right) \leq - \left( 2 \nu \lambda_1^* (\omega) - \frac{2 c_0 \lambda_1^{-1/4} \| f \|_*}{\nu - L_g \lambda_1^{-1}} - L_g \right) |z(t)|^2 + L_g |z(t)|^2.
\]
Using conditions (30) and the estimate (3) we obtain that
\[
\frac{d}{dt} \left( |z(t)|^2 + \alpha^2 |z(t)|^2 \right) \leq - d_0 \left( 2 \nu \lambda_1^* (\omega) - \frac{2 c_0 \lambda_1^{-1/4} \| f \|_*}{\nu - L_g \lambda_1^{-1}} - L_g \right) \left( |z(t)|^2 + \alpha^2 |z(t)|^2 \right)
\]
\[
+ L_g \left( |z(t)|^2 + \alpha^2 |z(t)|^2 \right).
\]
Applying Lemma 3.5 with \( h(t) = |z(t)|^2 + \alpha^2 |z(t)|^2 \) and \( h(0) = |z(0)|^2 + \alpha^2 |z(0)|^2 = |\phi(0) - u_\infty|^2 + \alpha^2 |\phi(0) - u_\infty|^2 \), one gets (31) with \( \gamma_1 \) satisfying
\[
-d_0 \left( 2 \nu \lambda_1^* (\omega) - \frac{2 c_0 \lambda_1^{-1/4} \| f \|_*}{\nu - L_g \lambda_1^{-1}} - L_g \right) + L_g \gamma_1 = 0,
\]
i.e., \( \gamma_1 \) is given by (32). This completes the proof. \( \square \)

**Remark 4.** From definition of \( \lambda_1^* (\omega) \) and using the Poincaré inequality, one can see that \( \lambda_1^* (\omega) \) can be made arbitrarily large by making the annular domain \( \Omega_\omega = \Omega \setminus \omega \) thin enough. Therefore, it follows from Theorem 4.3 that the stationary solution \( u_\infty \) is polynomially stabilizable if \( \Omega_\omega \) is sufficiently thin.

5. **Appendix: Proof of Theorem 2.2. Step 1. Existence.** We first prove the existence of weak solution to (7) by using the Galerkin method.

**Step 1.1. The Galerkin approximation.** Let \( P_n : H \rightarrow \text{span}\{w_1, \ldots, w_n\} \), where \( \{w_j\}_{j=1}^{\infty} \) is the basis of all the eigenfunctions of the Stokes operator \( A \) which is orthonormal in \( H \) and orthogonal in \( V \). We consider the approximation solution
In particular, for any $t > 0$ and $u^n(t)$ the required to satisfy the following system of differential equations with infinite delay:

$$\begin{cases}
\frac{d}{dt} (u^n, w_j) + \alpha^2 ((u^n, w_j)) + \nu ((u^n, w_j)) + \langle B(u^n, u^n), w_j \rangle_{\nu', \nu} \\
u^n(\theta) = P_n \phi(\theta), \theta \in (-\infty, 0].
\end{cases} \tag{33}$$

The above system of ordinary functional differential equations with infinite delay fulfills the conditions for the existence and uniqueness of local solutions (see [18, Theorem 1.1, p. 36] or [17]). Hence, we conclude that the approximation solution fulfills the conditions for the existence and uniqueness of local solutions (see [18, Theorem 1.1, p. 36] or [17]). Next, we will obtain a priori estimates and ensure that the solutions $u^n$ exists in the whole interval $[0, T]$. Step 1.2. A priori estimates. Multiplying each equation in (33) by $\gamma_{n,j}(t), j = 1, \ldots, n$, summing up, using (4), the Lipschitz condition of $G$ and the Cauchy inequality, noting that $|u^n(t)| \leq \|u^n\|_{BCL_{-\infty}(H)}$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u^n\|^2 + \alpha^2 \|u^n\|^2 + \nu \int_0^t \|u^n(s)\|^2 ds \leq \|f_t\| \|u^n\| + L \|u^n\|_{BCL_{-\infty}(H)} \|u^n(t)\|$$

$$\leq \nu \|u^n\|^2 + \frac{1}{2\nu} \|f\|^2 + L \|u^n\|^2_{BCL_{-\infty}(H)}. \tag{34}$$

Here, we have denoted $u_t$, the function defined on $(-\infty, 0]$ by $u_t(\theta) = u(t + \theta)$, $\theta \in (-\infty, 0]$. We deduce from (34) that

$$\|u^n(t)\|^2 + \alpha^2 \|u^n(t)\|^2 + \nu \int_0^t \|u^n(s)\|^2 ds \leq \|u^n(0)\|^2 + \alpha^2 \|u^n(0)\|^2 + \frac{t}{\nu} \|f\|^2 + 2L \int_0^t \|u^n\|^2_{BCL_{-\infty}(H)} ds.$$

In particular, for any $t > 0$

$$\sup_{-t < \theta \leq 0} \|u^n(t + \theta)\|^2 + \alpha^2 \|u^n(t)\|^2 \leq \|\phi\|^2_{BCL_{-\infty}(H)} + \alpha^2 \|\phi(0)\|^2 + \frac{t}{\nu} \|f\|^2 + 2L \int_0^t \left( \|u^n\|^2_{BCL_{-\infty}(H)} + \alpha^2 \|u^n(s)\|^2 \right) ds.$$

Since

$$\|u^n\|^2_{BCL_{-\infty}(H)} + \alpha^2 \|u^n(t)\|^2$$

$$= \max \left\{ \sup_{-t < \theta \leq 0} |u^n(t + \theta)|^2 + \alpha^2 \|u^n(t)\|^2; \sup_{\theta \leq -t} |u^n(t + \theta)|^2 + \alpha^2 \|u^n(t)\|^2 \right\}$$

$$\leq \max \left\{ \sup_{-t < \theta \leq 0} |u^n(t + \theta)|^2 + \alpha^2 \|u^n(t)\|^2; \|\phi\|^2_{BCL_{-\infty}(H)} + \alpha^2 \|u^n(t)\|^2 \right\},$$

we have

$$\|u^n\|^2_{BCL_{-\infty}(H)} + \alpha^2 \|u^n(t)\|^2 \leq 2\|\phi\|^2_{BCL_{-\infty}(H)} + \alpha^2 \|\phi(0)\|^2 + \frac{t}{\nu} \|f\|^2 + 2L \int_0^t \left( \|u^n\|^2_{BCL_{-\infty}(H)} + \alpha^2 \|u^n(s)\|^2 \right) ds. \tag{35}$$
Applying the integral form of Gronwall inequality to (35) we obtain
\[ \|u^n_t\|^2_{BCL_{-\infty}(H)} + \alpha^2 \|u^n(t)\|^2 \leq \left( 2\|\phi\|^2_{BCL_{-\infty}(H)} + \alpha^2\|\phi(0)\|^2 + \frac{t}{\nu}\|f\|^2 \right) e^{2L\nu t}. \]
Thus, \( \{u^n_t\} \) is uniformly bounded in \( L^\infty(0, T; BCL_{-\infty}(H)) \) and \( \{u^n\} \) is uniformly bounded in \( L^\infty(0, T; V) \).

From (33), using (6), the Poincaré inequality (2), the Lipschitz condition of \( G \), we obtain the uniform boundedness of \( \{\partial_t (u^n + \alpha^2 Au^n)\} \) in \( L^2(0, T; V') \). It follows that \( \{\partial_t u^n\} \) is uniformly bounded in \( L^2(0, T; V) \).

From the above boundedness, we conclude that there exist a subsequence of \( \{u^n\} \) (up to relabeling the index) such that
\[ u^n \rightharpoonup u \text{ in } L^\infty(0, T; V), \]
\[ u^n \rightarrow u \text{ in } L^2(0, T; V), \]
\[ \partial_t u^n \rightharpoonup \partial_t u \text{ in } L^2(0, T; V). \]

Since \( \{u^n\} \) is uniformly bounded \( L^2(0, T; V) \), \( \{\partial_t u^n\} \) is also uniformly bounded in \( L^2(0, T; H) \) with noting that \( V \) is compactly embedded in \( (L^3(\Omega))^3 \) and \( (L^3(\Omega))^3 \) is continuously embedded in \( H \) then by the Aubin-Lions compactness lemma, we have \( u^n \rightarrow u \) strongly in \( L^2(0, T; (L^3(\Omega))^3) \). This and together the above convergences and boundedness, we can pass to the limit in \( L^2(0, T; V') \) of the nonlinear term \( B(u^n, u^n) \).

**Step 1.3. Passing to the limit of the nonlinear function related to delay term.**
Since \( u^n, u \in L^\infty(0, T; V) \) and \( \partial_t u^n, \partial_t u \in L^2(0, T; V) \), we can see that \( u^n, u \in C([0, T]; V) \). So, \( \{h_n(t) = \|u^n(t)\|\} \) is a family of continuous real functions on closed, bounded interval \([0, T] \). On one hand, by the uniform boundedness of \( \{u^n\} \) in \( L^\infty(0, T; V) \), \( \{h_n(t)\} \) is uniformly bounded in \([0, T]\). On the other hand, for any \( t, s \in [0, T] \),
\[ |h_n(t) - h_n(s)| \leq \|u^n(t) - u^n(s)\| = \left| \int_s^t \partial_t u^n(\tau)d\tau \right| \]
\[ \leq \int_s^t \|\partial_t u^n(\tau)\|d\tau \]
\[ \leq |t - s|^{1/2}\|\partial_t u^n\|_{L^2(0, T; V)}. \]
Since \( \{\partial_t u^n\} \) is uniformly bounded in \( L^2(0, T; V) \), we see that \( \{h_n(t)\} \) is equicontinuous on \([0, T]\). Therefore, by applying the Arzelà-Ascoli lemma we deduce that
\[ u^n \rightarrow u \text{ in } C([0, T]; V). \]
In particular,
\[ u^n \rightarrow u \text{ in } C([0, T]; H). \]
(36)
We now have that (see [25])
\[ P_n \phi \rightharpoonup \phi \text{ in } BCL_{-\infty}(H) \text{ as } n \rightarrow \infty. \]
(37)
Using (36) and (37) we have
\[ u^n_t \rightarrow u_t \text{ in } BCL_{-\infty}(H), \quad \forall t \in [0, T]. \]
(38)
Indeed, we have
\[
\|u^n_t - u_t\|_{BCL^{-\infty}(H)} = \sup_{\theta \leq 0} |u^n(t + \theta) - u(t + \theta)| = \max \left\{ \sup_{\theta \leq -t} |u^n(t + \theta) - u(t + \theta)| ; \sup_{-t \leq \theta \leq 0} |u^n(t + \theta) - u(t + \theta)| \right\} \leq \max \left\{ \|P \phi - \phi\|_{BCL^{-\infty}(H)} ; \|u^n - u\|_{C([0,T];H)} \right\} \to 0.
\]
Using (38) and the Lipschitz condition of \(G\), one can pass to the limit of \(G(u^n(t - \rho(t)))\).

Hence, we finish the proof of the existence of solutions to (7).

**Step 2. The uniqueness and dependence continuous on the initial datum.**

Let \(u\) and \(v\) be two solutions to (7) with respect to the initial datum \(\phi\) and \(\psi\) in \(BCL^{-\infty}(H)\) with \(\phi(0), \psi(0) \in V\). We denote \(w = u - v\), then \(w\) satisfies
\[
\begin{align*}
\frac{d}{dt}(w + \alpha^2 Aw) + \nu Aw + B(w, u) + B(v, w) &= P\left(G(u(t - \rho(t))) - G(v(t - \rho(t)))\right), t > 0, \\
w(\theta) = \phi(\theta) - \psi(\theta), \ \theta \in (-\infty, 0].
\end{align*}
\]
Taking the inner product of the first equation in (39) with \(w\), using (6), Lipschitz condition of \(G\) and the Young inequality, we have
\[
\frac{1}{2} \frac{d}{dt}(|w|^2 + \alpha^2 \|w\|^2) + \nu \|w\|^2 = -\langle B(w, u), w \rangle_{V', V} + |G(u(t - \rho(t))) - G(v(t - \rho(t)))| w|
\leq c_0 |w|^{1/2} \|w\|^{3/2} \|u\| + L_g \|w\|_{BCL^{-\infty}(H)} |w|
\leq \frac{\nu}{2} \|w\|^2 + \frac{27c_0^4}{32\nu^3} \|u\|^4 \|w\|^2 + L_g \|w\|^2_{BCL^{-\infty}(H)}.
\]
Thus
\[
\frac{d}{dt}(|w|^2 + \alpha^2 \|w\|^2) \leq \frac{27c_0^4}{32\nu^3} \|u\|^4 \|w\|^2 + 2L_g \|w\|^2_{BCL^{-\infty}(H)}.
\]
This implies that
\[
|w(t)|^2 + \alpha^2 \|w(t)\|^2 \leq |w(0)|^2 + \alpha^2 \|w(0)\|^2 + \frac{27c_0^4}{16\nu^3} \int_0^t \|u(s)\|^4 |w(s)|^2 ds
\]
\[+ 2L_g \int_0^t \|w_s\|^2_{BCL^{-\infty}(H)} ds.
\]
Hence, as in the proof of (35), we deduce that
\[
\|w_t\|^2_{BCL^{-\infty}(H)} + \alpha^2 |w(t)|^2 
\leq 2 \|\phi - \psi\|^2_{BCL^{-\infty}(H)} + \alpha^2 |\phi(0) - \psi(0)|^2 + \int_0^t \left( \frac{27c_0^4}{16\nu^3} \|u(s)\|^4 + 2L_g \right) \left( \|w_s\|^2_{BCL^{-\infty}(H)} + \alpha^2 \|w(s)\|^2 \right) ds.
\]
Here we have used the fact that
\[
|w(s)|^2 \leq \|w_s\|^2_{BCL^{-\infty}(H)} \leq \|w_s\|^2_{BCL^{-\infty}(H)} + \alpha^2 |w(s)|^2, \ 0 \leq s \leq t \leq T.
\]
By applying the Gronwall inequality to (40) on $[0, T]$, we obtain that
\[
\|u_t - v_t\|_{B^{CL}_{-\infty}(H)}^2 + \alpha^2 \|u(t) - v(t)\|^2
\leq \left( 2\|\phi - \psi\|^2_{B^{CL}_{-\infty}(H)} + \alpha^2 \|\phi(0) - \psi(0)\|^2 \right) \exp \left( \frac{27\alpha^4}{16\nu^2} \int_0^T \|u(s)\|^4 \, ds + 2L_2T \right).
\]
Since $u \in L^\infty(0, T; V)$ then we complete the proof.

Acknowledgments. The author would like to thank the reviewers for the helpful comments and suggestions, which helped to improve the presentation of the paper.

REFERENCES

[1] C. T. Anh and D. T. P. Thanh, Existence and long-time behavior of solutions to Navier-Stokes-Voigt equations with infinite delay, *Bull. Korean Math. Soc.*, 55 (2018), 379–403.
[2] C. T. Anh and P. T. Trang, Pull-back attractors for three-dimensional Navier-Stokes-Voigt equations in some unbounded domains, *Proc. Roy. Soc. Edinburgh Sect. A*, 143 (2013), 223–251.
[3] C. T. Anh and P. T. Trang, On the regularity and convergence of solutions to the 3D Navier-Stokes-Voigt equations, *Comput. Math. Appl.*, 73 (2017), 601–615.
[4] J. A. D. Appleby and E. Buckwar, Sufficient conditions for polynomial asymptotic behaviour of the stochastic pantograph equation, in *Proc. 10‘th Coll. Qualitative Theory of Diff. Equ.*, *Electron. J. Qual. Theory Differ. Equ.*, 2016 (2016), 1–32.
[5] V. Barbu and C. Lefter, Internal stabilizability of the Navier-Stokes equations. Optimization and control of distributed systems, *Systems Control Lett.*, 48 (2003), 161–167.
[6] L. C. Berselli and L. Bisconti, On the structural stability of the Euler-Voigt and Navier-Stokes-Voigt models, *Nonlinear Anal.*, 75 (2012), 117–130.
[7] C. Cao, D. D. Holm and E. S. Titi, On the Clark-\(\alpha\) model of turbulence: Global regularity and long-time dynamics, *J. Turbul.*, 6 (2005), 11 pp.
[8] T. Caraballo and X. Han, Stability of stationary solutions to 2D-Navier-Stokes models with delays, *Dyn. Partial Differ. Equ.*, 14 (2017), 271–297.
[9] T. Caraballo and X. Han, A survey on Navier-Stokes models with delays: Existence, uniqueness and asymptotic behavior of solutions, *Discrete Contin. Dyn. Syst. Ser. S*, 8 (2015), 1079–1101.
[10] T. Caraballo, A. M. Márquez-Durán and J. Real, Pullback and forward attractors for a 3D LANS-\(\alpha\) model with delay, *Discrete Contin. Dyn. Syst.*, 15 (2006), 559–578.
[11] T. Caraballo, A. M. Márquez-Durán and J. Real, Asymptotic behaviour of the three-dimensional \(\alpha\)-Navier-Stokes model with delays, *J. Math. Anal. Appl.*, 340 (2008), 410–423.
[12] T. Caraballo and J. Real, Attractors for 2D-Navier-Stokes models with delays, *J. Differential Equations*, 205 (2004), 271–297.
[13] T. Caraballo and J. Real, Asymptotic behaviour of two-dimensional Navier-Stokes equations with delays, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 459 (2003), 3181–3194.
[14] T. Caraballo, J. Real and L. Shaikhet, Method of Lyapunov functionals construction in stability of delay evolution equations, *J. Math. Anal. Appl.*, 334 (2007), 1130–1145.
[15] M. Coti Zelati and C. G. Gal, Singular limits of Voigt models in fluid dynamics, *J. Math. Fluid Mech.*, 17 (2015), 233–259.
[16] J. García-Luengo, P. Marín-Rubio and J. Real, Pullback attractors for three-dimensional non-autonomous Navier-Stokes-Voigt equations, *Nonlinearity*, 25 (2012), 905–930.
[17] J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.*, 21 (1978), 11–41.
[18] Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Infinite Delay*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1991.
[19] M. J. Holst, M. A. Ebrahimi and E. Lunasin, The Navier-Stokes-Voight model for image inpainting, *IMA J. Appl. Math.*, 78 (2013), 869–894.
[20] V. K. Kalantarov, B. Levant and E. S. Titi, Gevrey regularity for the attractor of the 3D Navier-Stoke-Voight equations, *J. Nonlinear Sci.*, 19 (2009), 133–152.
[21] V. K. Kalantarov and E. S. Titi, Global attractors and determining modes for the 3D Navier-Stokes-Voigt equations, *Chin. Ann. Math. Ser. B*, 30 (2009), 697–714.
[22] T. Kato, Asymptotic behavior of solutions of the functional differential equation \(y'(x) = ay(\lambda x) + by(x)\), in *Delay and Functional Differential Equations and Their Applications*, Proc. Conf., Park City, Utah, Academic Press, New York, (1972), 197–217.

[23] T. Kato and J. B. McLeod, The functional-differential equation \(y'(x) = ay(\lambda x) + by(x)\), *Bull. Amer. Math. Soc.*, **77** (1971), 891–937.

[24] J. S. Linshiz and E. S. Titi, Analytical study of certain magnetohydrodynamic-\(\alpha\) models, *J. Math. Phys.*, **48** (2007), 065504, 28 pp.

[25] L. Liu, T. Caraballo and P. Marín-Rubio, Stability results for 2D Navier-Stokes equations with unbounded delay, *J. Differential Equations*, **265** (2018), 5685–5708.

[26] E. Lunasin, M. Holst and G. Tsogtgerel, Analysis of a general family of regularized Navier-Stokes and MHD models, *J. Nonlinear Sci.*, **20** (2010), 523–567.

[27] P. Marín-Rubio, A. M. Márquez-Durán and J. Real, Three dimensional system of globally modified Navier-Stokes equations with infinite delays, *Discrete Contin. Dyn. Syst. Ser. B*, **14** (2010), 655–673.

[28] P. Marín-Rubio, J. Real and A. M. Márquez-Durán, On the convergence of solutions of globally modified Navier-Stokes equations with delays to solutions of Navier-Stokes equations with delays, *Adv. Nonlinear Stud.*, **11** (2011), 917–927.

[29] P. Marín-Rubio, A. M. Márquez-Durán and J. Real, Pullback attractors for globally modified Navier-Stokes equations with infinite delays, *Discrete Contin. Dyn. Syst.*, **31** (2011), 779–796.

[30] A. P. Oskolkov, The uniqueness and solvability in the large of boundary value problems for the equations of motion of aqueous solutions of polymers, *Zap. Naučn. Sem. Leningrad. Otdel. Math. Inst. Steklov.*, (LOMI), **38** (173), 98–136.

[31] Y. Qin, X. Yang and X. Liu, Averaging of a 3D Navier-Stokes-Voight equation with singularly oscillating forces, *Nonlinear Anal. RWA*, **13** (2012), 893–904.

[32] G. Yue and C. Zhong, Attractors for autonomous and nonautonomous 3D Navier-Stokes-Voight equations, *Discrete Contin. Dyn. Syst. Ser. B*, **16** (2011), 985–1002.

Received address: toivm@tlu.edu.vn