Wong-Zakai Approximation for SDEs Driven by
$G$–Brownian Motion

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Abstract: In this paper, we build the equivalence between rough differential equations driven by the lifted $G$-Brownian motion and the corresponding Stratonovich type SDE through the Wong-Zakai approximation. The quasi-surely convergence rate of Wong-Zakai approximation to $G$–SDEs with mesh-size $\frac{1}{n}$ in the $\alpha$-Hölder norm is estimated as $(\frac{1}{n})^{\frac{1}{2}\alpha}$ -$. As corollary, we obtain the quasi-surely continuity of the above RDEs with respect to uniform norm.

Key words: $G$-expectation, rough paths, Wong-Zakai approximation, quasi-surely continuity

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1 Introduction

$G$-expectation theory is introduced by Peng in [14] [15] [16]. It is widely used as a helpful tool for financial problems concerning model uncertainty, ambiguous volatility [4] [19]. Also, it provides a coexistence framework for a set of mutually singular martingale measures (quasi-surely instead of almost surely). However, according to probability uncertainty of the $G$–expectation theory, it is hard to simulated the $G$–Brownian motion in a probabilistic way. For example, how to numerically generate a random variable sharing the same “distribution” with $G$–Brownian motion is still an important problem in $G$–expectation theory. In this paper, authors provide a way of calculating $G$–SDEs pathwisely.

Wong and Zakai first build the approximation of Stratonovich SDEs by a sequence of ODEs in [20] [21]. In their papers, they give general sufficient conditions under which solutions of ODEs converge to SDEs of Stratonovich’s kind.
As well-known, these conditions work well in one-dimensional case. For the high dimensional case, the approximation sensitively depends on how well $W_n$ converges to $W$, see Stroock and Varadhan [18] for the high dimensional case. Lyons establishes the rough path theory in his seminal work [12], and uses Wong-Zakai theorem to get the equivalence between RDE solutions and SDE solutions. The numerical analysis of Taylor’s expansion kind(Euler or Milstein’s) approximation for SDEs driven by fractional Brownian Motion is studied in [3] pathwisely and [1] in the sense of expectation. Recently, Kelly and Melbourne [11] provide an approximation to SDEs through rough path method by constructing smooth and càdlàg approximations with smooth flows to (lifted)Brownian motion. Also, Hairer and Pardoux [7] give a positive answer to the Wong-Zakai kind approximation to SPDEs by the fast-developing regularity structure method(nontrivially including rough paths as examples). All these results imply rough path is a helpful tool for studying Wong-Zakai kind approximation, and then further obtaining numerical results concerning SDEs driven by $G$-Brownian motion.

To build the equivalence between rough differential equations(RDEs) driven by lifted $G$–Brownian motion and $G$–SDEs, firstly one needs to show solutions for these RDEs are quasi-continuous. To be more precisely, suppose $Y_t(\omega)$ solves the following RDE driven by the lifted $G$-Brownian motion $B$,

$$dY_t = f(Y)dB.$$ 

One needs the quasi-surely continuity with respect to uniform topology of the RDE solution $Y_t$ to conclude $Y_t \in L_G(\Omega_T)$, while rough path theory built on a stronger topology, the $p$-variation or $\alpha$-Hölder topology. Another problem is that, because of nonlinearity of $G$-expectation $\mathbb{E}_G$, $L_G^2$ is a true Banach space instead of Hilbert space, there is no Wiener Chaos theory in $G$–framework and even bounded convergence theorem fails, which are important tools to obtain the convergence of piecewise linearized $G$–Brownian motion to the lifted $G$–Brownian motion under rough path metric in the sense of expectation.

$G$-Brownian motion is first lifted as a geometric rough path under $p$-variation norm in [10]. In that paper, authors also establish the Euler-Maruyama approximation for $G$–SDEs. Later, authors of this paper start from the view of Gubinelli [8, 9], and lift the $G$-Brownian motion by Kolmogorov’s theorem (rough paths version in $G$-framework) in [17]. Also, $G$-Stratonovich integral is introduced there. Followed by that work, in this paper, we establish the equivalence between RDE solutions and $G$-Stratonovich SDE solutions by applying Wong-Zakai schemes and the universal continuity for rough paths. Firstly, we show that solutions of ODEs driven by piecewisely linearized $G$-Brownian motion actually belong to the random variable space in $G$-framework. The method is also applicable to $G$–ODEs driven by other piecewisely linearized $G$–martingales. Secondly we use the Wong-Zakai argument to show the convergence of ODEs solutions to Stratonovich SDEs solutions. Thirdly, by applying rough path
results, we prove these ODEs solutions also converge to RDEs solutions in $G$-framework. To prove the convergence of piecewisely linearized $G$–Brownian motion to the lifted $G$–Brownian motion under rough path topology, we do a direct calculation instead of classical methods from Wiener Chaos expansion, which is also applicable to many martingales in $G$–framework. Furthermore, the quasi-surely convergence rate of Wong-Zakai approximation is calculated by rough paths results.

The paper is organized as the following. In Section 2, we recall some basic notations in $G$-expectation theory and rough path theory. Then in Section 3, we give the main results of this paper. Firstly, we prove the Wong-Zakai theorem in $G$-framework. Secondly, we estimate the convergence rate by the continuity theorem of the Itô-Lyons mapping.

2 Preliminaries about $G$-expectation and Rough Path

In this part, we review some definitions and conclusions on $G$-expectation and rough path theory. See lecture notes as [5, 6, 13, 14, 16] for details.

2.1 The rough path theory

Denote by $\mathbb{R}^m \otimes \mathbb{R}^n$ the algebraic tensor of two Euclidean spaces. For any path on some interval $[0, T]$ with values in a $\mathbb{R}^d$, its $\alpha$-Hölder norm(semi-norm) is defined by

$$\|X\|_\alpha = \sup_{0 \leq s < t \leq T} \frac{|X_{s,t}|}{|t-s|^{\alpha}},$$

where $X_{s,t} := X_t - X_s$, for any path $X$.

Denote $C^\alpha([0, T], \mathbb{R}^d)$ as the space of paths with finite $\alpha$-Hölder norm and values in $\mathbb{R}^d$. Similarly, a mapping $X$ from $[0, T]^2$ to $\mathbb{R}^d \otimes \mathbb{R}^d$ is attached with norm

$$\|X\|_{2\alpha} = \sup_{0 \leq s \neq t \leq T} \frac{|X_{s,t}|}{|t-s|^{2\alpha}},$$

whenever it’s finite.

A (level-2)rough path on some interval $[0, T]$ with values in $\mathbb{R}^d$ includes a continuous path $X : [0, T] \rightarrow \mathbb{R}^d$, along with its “iterated integration” part $X : [0, T]^2 \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$, which satisfies “Chen’s identity”

$$X_{s,t} - X_{s,u} - X_{u,t} = X_{s,u} \otimes X_{u,t},$$

and the above Hölder continuity. In the sequel, suppose $\alpha \in (\frac{1}{3}, \frac{1}{2})$ for the need of rough integral.
**Definition 2.1.** For a fixed $\alpha$, the space of rough paths $\mathcal{C}^\alpha([0,T],\mathbb{R}^d)$ on $[0,T]$ consists of pairs $(X,\dot{X})$ satisfying “Chen’s identity” \[ \int_0^T \langle Y, X \rangle d\xi \leq C(\|Y\|_\alpha^\beta \|X\|_\alpha^\beta + \|\dot{X}\|_\alpha^\beta) \] and the condition of finite $\alpha$-Hölder norm and $2\alpha$-Hölder norm respectively for $X$ and $\dot{X}$. For any $X := (X, \dot{X}) \in \mathcal{C}^\alpha([0,T],\mathbb{R}^d)$, define its semi-norm as the following
\[ \|X\|_{\mathcal{C}^\alpha} := \|X\|_\alpha + (\|\dot{X}\|_{2\alpha})^{\frac{1}{2}}. \]

Also, we use the following metric,
\[ \varrho(X, \dot{X}) := \|X - \dot{X}\|_\alpha + \|\dot{X} - \ddot{X}\|_{2\alpha}. \]

**Definition 2.2.** A path $Y \in \mathcal{C}^\alpha([0,T],\mathbb{R}^m)$ is said to be controlled by a given path $X \in \mathcal{C}^\alpha([0,T],\mathbb{R}^d)$, if there exists $Y' \in \mathcal{C}^\alpha([0,T],\mathcal{L}(\mathbb{R}^d,\mathbb{R}^m))$, such that the remainder term
\[ R^Y_{s,t} := Y^\prime_{s,t} - Y^\prime_{s,t}, \]

satisfies $\|R^Y\|_{2\alpha} < \infty$.

Denote the collection of controlled rough paths by $\mathcal{D}^{2\alpha}_X([0,T],\mathbb{R}^m)$. In addition, $Y'$ is called the Gubinelli derivative of $Y$. For $(Y, Y') \in \mathcal{D}^{2\alpha}_X([0,T],\mathbb{R}^m)$, we define its semi-norm by $\|Y, Y'\|_{X,2\alpha} := \|Y'\|_\alpha + \|R^Y\|_{2\alpha}$. For example, given any $F \in C^\alpha_b(\mathbb{R}^d,\mathbb{R}^m)$, the set of bounded functions from $\mathbb{R}^d$ to $\mathbb{R}^m$ with bounded derivatives up to order 2, one can easily check that $(Y, Y') := (F(X), DF(X)) \in \mathcal{D}^{2\alpha}_X([0,T],\mathbb{R}^m)$.

**Theorem 2.3. (Lyons, Gubinelli)** Suppose $X \in \mathcal{C}^\alpha([0,T],\mathbb{R}^d)$, and $(Y, Y') \in \mathcal{D}^{2\alpha}_X([0,T],\mathcal{L}(\mathbb{R}^d,\mathbb{R}^m))$. Then the following compensated Riemann sum converges.
\[ \int_0^T Y dX := \lim_{|P| \to 0} \sum_{(s,t) \in P} (Y_s X_{s,t} + Y'_{s,t} X_{s,t}), \] (2)

where $P$ are partitions of $[0,T]$, with modulus $|P| \to 0$. Furthermore, one has the bound with $K$ depending only on $\alpha$,
\[ \left\| \int_s^t Y dX - Y_s X_{s,t} - Y'_{s,t} X_{s,t} \right\| \leq K(\|Y\|_\alpha^\beta \|\dot{X}\|_{2\alpha} + \|\dot{X}\|_{2\alpha} \|Y'\|_\alpha^\beta |t-s|^{3\alpha}). \] (3)

Furthermore, one has the following continuity of the Itô-Lyons map which is also known as universal limit theorem.

**Theorem 2.4. (Lyons, Gubinelli)** Suppose $f \in C^\alpha_b(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^d,\mathbb{R}^m))$ and $X \in \mathcal{C}^\alpha([0,T],\mathbb{R}^d)$, there exists a unique solution $(Y, f(Y)) \in \mathcal{D}^{2\alpha}_X([0,T],\mathbb{R}^m)$ solving the following RDE,
\[ Y_t = \xi + \int_0^t f(Y_s) dX_s, \quad t \leq T. \]

Furthermore, if $\tilde{Y}$ solves another RDE driven by signal $\tilde{X}$ with initial value $\tilde{\xi}$, Assuming $\|\dot{X}\|_{\mathcal{C}^\alpha}, \|\dot{\tilde{X}}\|_{\mathcal{C}^\alpha} \leq K$ for some constant $K$, one has the following local Lipschitz estimate on some subinterval $[0,T_0]$,
\[ \|Y - \tilde{Y}\|_{\mathcal{C}^\alpha} \leq C(\|\xi - \tilde{\xi}\| + \|X - \tilde{X}\|_{\mathcal{C}^\alpha}), \] with $C$ and $T_0$ depending on $K, \alpha$ and $f$. 

4
2.2 The $G$-expectation theory

For simplicity, most of the following results are defined in one-dimensional case, but the extension to multidimensional case is almost trivial. Let $\Omega$ be a given set and $\mathcal{H}$ be a linear space of real valued functions on $\Omega$ containing constants. One calls a functional on $\mathcal{H}$ a sublinear function if the following four properties are satisfied:

\begin{itemize}
  \item $\hat{\mathbb{E}}[c] = c, \quad \forall c \in \mathbb{R}$;
  \item $\hat{\mathbb{E}}[X_1] \geq \hat{\mathbb{E}}[X_2]$ if $X_1 \geq X_2$;
  \item $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X], \quad \lambda \geq 0 \quad X \in \mathcal{H}$;
  \item $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y], \quad \forall X, Y \in \mathcal{H}$.
\end{itemize}

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space. By Theorem 1.6 in Chapter 3 of [16], we know that if $X = (X_1, \cdots, X_d)$ is $G$-normally distributed, $u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}^d$, is the unique viscosity solution of the following $G$-heat equation:

$$
\partial_t u - G(D_x^2 u) = 0, \quad u(0, x) = \varphi(x),
$$

with function $G$ defined as above. Conversely, fixed any monotonic, sublinear function $G(\cdot) : \mathbb{R}_d \to \mathbb{R}$, one could construct the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Let $\Omega = \mathcal{C}_0^0(\mathbb{R}_+^T, \mathbb{R}^d)$, the space of real valued continuous paths $(\omega)_t \geq 0$ vanishing at the origin. Denote the coordinate process by $B_t$ and $u^{\varphi(\cdot)}(t, x)$ the unique viscosity solution to the $G$-heat equation (4) with initial function $\varphi$. Define $L_{ip}(\Omega_T) := \{\varphi(B_{t_1 \wedge T}, \cdots, B_{t_k \wedge T}) : k \in \mathbb{N}, \quad 0 \leq t_1, \cdots, t_k \in [0, \infty), \varphi \in \mathcal{C}_0^0, L_{ip}(\mathbb{R}^{k \times d})\}$ for any $T > 0$ and $\hat{\mathbb{E}}(\Omega) := \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n)$. We define a mapping $\hat{\mathbb{E}}$ from $L_{ip}(\Omega)$ to $\mathbb{R}$ by defining $\hat{\mathbb{E}}[\varphi(B_t)] := u^{\varphi(\cdot)}(t, 0)$ and recursively solving the $G$-heat equation for general elements:

$$
\hat{\mathbb{E}}[\varphi(B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})] := \hat{\mathbb{E}}[\varphi^{t_1 - t_{n-1}}(B_{t_1}, \cdots, B_{t_{n-1}} - B_{t_{n-2}})],
$$

where $\varphi^{t_1 - t_{n-1}}(x_1, \cdots, x_{n-1}) := u^{\varphi(x_1, \cdots, x_{n-1})}(t_n - t_{n-1}, 0)$. One can check that $\hat{\mathbb{E}}[\cdot]$ is well defined and it is a sublinear expectation on $L_{ip}(\Omega)$. For each $p \geq 1$, $L^p_G(\Omega_T)$ denotes the completion of the linear space $L_{ip}(\Omega_T)$, under norm $\| \cdot \|_{L^p_G} := \{\hat{\mathbb{E}}[\| \cdot \|^p]\}^{\frac{1}{p}}$. Here is a description of $\hat{\mathbb{E}}$ from [2].

**Theorem 2.5.** Assume $\Gamma$ is a bounded, convex and closed subset of $\mathbb{R}^{d \times d}$, which represents function $G$, i.e.,

$$
G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}(A \gamma'), \text{ for } A \in \mathbb{S}_d.
$$

Denote the Wiener measure by $\mathbb{P}_0$. Then, for any time sequence $0 = t_0 < t_1 < \cdots < t_k$, the $G$-expectation has the following representation
\[ \hat{E}[\varphi(B_{t_0,t_1},...,B_{t_{k-1},t_k})] = \sup_{a \in \mathcal{A}} E_{P_0}[\varphi(\int_0^{t_1} a_s dB_s, ..., \int_{t_{k-1}}^{t_k} a_s dB_s)] \]
\[ = \sup_{P_a \in \mathcal{P}} E_{P_a}[\varphi(B_{t_0,t_1},...,B_{t_{k-1},t_k})], \]

where \( \mathcal{A}^\Gamma \) is the set of progressively measurable processes with values in \( \Gamma \) and \( \mathcal{P}^\Gamma \) is the set of laws of \( \int_0^t a_s dB_s \) with \( a \in \mathcal{A}^\Gamma \) under Wiener measure. Furthermore, \( \mathcal{P}^\Gamma \) is tight.

According to this theorem, one could extend \( \hat{E} \) from \( L^p_{G} \) to any Borel measurable random variable by defining
\[ \| \cdot \|_{L^p} := \sup_{P_a \in \mathcal{P}^\Gamma} \hat{E}_{P_a}[^{\cdot}]. \]

Next, we introduce the capacity corresponding to the \( G \)-expectation and give the description of \( L^p_{G} \). Define
\[ \hat{c}(A) := \sup_{P \in \mathcal{P}^\Gamma} P(A), \text{ for } A \in \mathcal{B}(\Omega_T). \]

**Definition 2.6.** A property is said to hold “quasi-surely” (q.s.) with respect to \( \hat{c} \), if it holds true outside a \( \hat{c} \)-polar set (Borel set with capacity 0), and is denoted by \( \hat{c}^{-q.s.} \).

**Definition 2.7.** A process \( Y \) on \([0,T]\) is said to be a quasi-surely modification of another process \( X \), if for any \( t \in [0,T] \)
\[ Y_t = X_t, \text{ \( \hat{c}^{-q.s.} \).} \]

If a property stands true \( \hat{c}^{-q.s.} \), then for any \( P \in \mathcal{P}^\Gamma \), it holds true \( P-a.s. \). By the definition of \( L^p_{G} \), we do not distinguish two random variables if they are equal outside a polar set.

**Definition 2.8.** Equip the space \( \Omega_T \) with the uniform topology. A mapping \( X \) on \( \Omega_T \) with values in \( \mathbb{R} \) is said to be quasi-continuous if for any \( \varepsilon > 0 \), there exists an open set \( O \), with \( \hat{c}(O) < \varepsilon \) such that \( X \) is continuous in \( O^\complement \).

**Definition 2.9.** One says that \( X : \Omega_T \to \mathbb{R} \) has a quasi-continuous version if there exists a quasi-continuous function \( Y \), such that \( X = Y, \hat{c}^{-q.s.} \).

**Theorem 2.10.** One has the following representation for \( L^1_{G} \),
\[ L^1_{G}(\Omega_T) = \{ X \in \mathcal{B}(\Omega_T) : X \text{ has a quasi-continuous version, } \lim_{n \to \infty} \|X|_{\{X|_{\cdot} > n\}}\|_{L^1} = 0 \}. \]

Denote \( M^{P_0}_{G}(0,T) \) the collection of processes with form
\[ \eta_t(\omega) = \sum_{i=0}^{N-1} \xi_i(\omega)1_{[t_i,t_{i+1})}(t), \]
for a partition \( \{0 = t_0 < \ldots < t_N = T\} \) and \( \xi_i \in L_{\text{Lip}}(\Omega_{t_i}), i = 0, \ldots, N - 1 \). Then denote by \( M_G^{0}(0, T) \) the completion of \( M_G^{0}(0, T) \) under norm \( \| \cdot \|_{M_G^{0}} := \{ \hat{E}\int_0^T |\eta_t|^p \text{d}s \}^\frac{1}{p} \). One can define stochastic integrals for elements in \( M_G^{0}(0, T) \) (see [13], [15], [16]). Define the quadratic variation processes \( \langle B \rangle_t \) of \( G \)-Brownian motion by \( \langle B \rangle_t := \lim_{n \to \infty} \sum_{i} (B(t_{i+1}) - B(t_i))^2 \), where \( \Pi_n \) is a sequence of partitions with mesh size converging to null. It can be shown that \( \sigma^2 \leq \frac{d\langle B \rangle_t}{dt} \leq \hat{\sigma}^2, \hat{\epsilon}-\text{q.s.} \), where \( \sigma = \sqrt{-\hat{E}[{-B^2_t}]} \) and \( \hat{\sigma} = \sqrt{\hat{E}[B^2]} \).

### 3 Main Result

#### 3.1 Wong-Zakai approximation in \( G \)-framework

In this part, we consider piece-wise linearized approximation to \( G \)-SDEs. Suppose \( B \) is the \( d \)-dimensional \( G \)-Brownian motion on \([0, 1] \). \( \{t_j^{(n)}\}_{j=0}^n \) is the partition with mesh size \( 1/n \) and \( B_t^{(n)} \) is the piece-wise linearization of \( G \)-Brownian motion according to \( \{t_j^{(n)}\}_{j=0}^n \). Consider the following ODEs with initial condition \( y_0 \in \mathbb{R}^m \)

\[
dY_t^{(n)} = f(Y_t^{(n)})dB_t^{(n)} + g(Y_t^{(n)})d\langle B \rangle_t + h(Y_t^{(n)})dt. \tag{6}
\]

According to our description of \( \langle B \rangle \) in Section 2, it is clear that for any \( n \), \( Y_t^{(n)} \) can be defined pathwisely. The following lemma proves that \( Y_t^{(n)}(\omega) \) is indeed quasi-continuous as a function on \( \Omega_T \).

**Lemma 3.1.** Assume \( f, g, h \in C^1_b \). For fixed \( n \), one has \( Y_t^{(n)} \in L^2_G(\Omega_{t_{j+1}}^{(n)}, t_{j+1}) \), for any \( t \in [t_j^{(n)}, t_{j+1}^{(n)}] \).

**Proof.** By induction, one only needs to show for any \( n \geq 1, j = 0, \ldots, n-1 \), \( Y_t^{(n)} \in L^1_G(\Omega_{t_{j+1}}^{(n)}, t_{j+1}) \). In the following proof, we omit \( (n) \) in \( Y^{(n)} \) and \( t_j^{(n)} \) for simplicity, i.e. suppose \( Y_t \) solves the following \( G \)-stochastic ODE pathwisely

\[
Y_t = Y_j + \frac{B_{t_j, t_{j+1}}}{t_{j+1} - t_j} \int_{t_j}^t f(Y_s)ds + \int_{t_j}^t g(Y_s)d\langle B \rangle_s + \int_{t_j}^t h(Y_s)ds, \quad t \in [t_j, t_{j+1}],
\tag{7}
\]

and \( Y_{t_j} \in L^2_G(\Omega_{t_j}) \). Consider the discretization of (7),

\[
y_t^{(m)} = y_{k-1}^{(m)} + \frac{B_{t_{j+1} - t_j}}{t_{j+1} - t_j} f(y_{k-1}^{(m)})(t - \tau_{k-1}^{(m)}) + g(y_{k-1}^{(m)})B_{\tau_{k-1}^{(m)}, t_{j+1}}^{(m)} + h(y_{k-1}^{(m)})(t - \tau_{k-1}^{(m)}), \quad t \in [\tau_{k-1}^{(m)}, \tau_{k}^{(m)}],
\tag{8}
\]

where \( \{\tau_k^{(m)}\}_{k=0}^m \) is the partition of \([t_j, t_{j+1}] \) with mesh-size \( \frac{1}{m} \), and \( y_{k}^{(m)} := y_{\tau_k^{(m)}}, \quad y_{\tau_k^{(m)}} := Y_{t_j}. \)
By taking expectation one obtains

\[ Y_{t_{j+1}} - \hat{y}_{t_{j+1}} = \sum_{i=0}^{n-1} \left( B_{t_{j+1} - t_j} \int_{t_{j+1} - t_j}^{t_{j+1}} (f(Y_s) - f(y^{(m)}_s))ds \right) \]

\[ + \int_{t_{j+1}}^{t_{j+1}} (g(Y_s) - g(y^{(m)}_s))d\langle B \rangle_s + \int_{t_{j+1}}^{t_{j+1}} (h(Y_s) - h(y^{(m)}_s))ds \]

\[ = \sum_{i=0}^{n-1} \left( \frac{B_{t_{j+1} - t_j}}{t_{j+1} - t_j} \int_{t_{j+1} - t_j}^{t_{j+1}} (f(Y_s) - f(y^{(m)}_s))ds + \int_{t_{j+1}}^{t_{j+1}} (g(Y_s) - g(y^{(m)}_s))d\langle B \rangle_s \right) \]

\[ + \int_{t_{j+1}}^{t_{j+1}} (h(Y_s) - h(y^{(m)}_s))ds + \frac{B_{t_{j+1} - t_j}}{t_{j+1} - t_j} \int_{t_{j+1}}^{t_{j+1}} (f(y^{(m)}_s) - f(y^{(m)}_s))ds \]

\[ + \int_{t_{j+1}}^{t_{j+1}} (g(y^{(m)}_s) - g(y^{(m)}_s))d\langle B \rangle_s + \int_{t_{j+1}}^{t_{j+1}} (h(y^{(m)}_s) - h(y^{(m)}_s))ds \right). \]

By the uniform bound \(|\langle B \rangle_t| \leq \bar{a}^2 t\) and a simple calculation, one has

\[ \int_{t_{j+1}}^{t_{j+1}} |y^{(m)}_s - y^{(m)}_t|ds \leq C(1 + |Y_{t_{j+1}}|)e^{C(1 + \frac{|B_{t_{j+1} - t_j}|}{t_{j+1} - t_j})(t_{j+1} - t_j)(\tau^{(m)}_s - \tau^{(m)}_i)^2}. \]  \hfill (9)

Here and from here on \( C \) is a generic constant. According to (9) and Lipschitz-ness for \( f, g, h \), one can simply get that

\[ |Y_{t_{j+1}} - \hat{y}_{t_{j+1}}| \leq C(1 + |Y_{t_{j+1}}|)e^{C(1 + \frac{|B_{t_{j+1} - t_j}|}{t_{j+1} - t_j})(t_{j+1} - t_j)} \frac{1}{m}(t_{j+1} - t_j). \]

Notice that \( t_{j+1} \) can be replaced by any \( t \in [t_j, t_{j+1}] \). By Gronwall’s inequality, one has the following inequality,

\[ |Y_{t_{j+1}} - \hat{y}_{t_{j+1}}| \leq C(1 + |Y_{t_{j+1}}|)e^{C((t_{j+1} - t_j) + |B_{t_{j+1} - t_{j+1}}|)}. \]

By taking expectation one obtains

\[ \mathbb{E}|Y_{t_{j+1}} - \hat{y}_{t_{j+1}}| \leq \frac{C}{m^2}, \]

which implies our result since \( y^{(m)}_t \in L^2_G(\Omega_{t_{j+1}}) \), for any \( t \in [t_j, t_{j+1}] \). \hfill \square

Now we give the Wong-Zakai approximation in \( G \)-framework. The proof is an adaptation of the classical case in \( G \)-framework. Considering the length of the proof, we leave it in the appendix.
Theorem 3.3. Suppose $Y_t^{(n)}$ solves ODEs \( \dot{t} \) – quasi surely, and $X_t$ solves the following $G$–SDE of Stratonovich’s kind,

$$X_t = x_0 + \int_0^t f(s, X_s) \circ dB_s + \int_0^t g(s, X_s) d(B)_s + \int_0^t h(s, X_s) ds.$$

with $f \in C^2_b, g, h \in C^1_b$. Then for any $t \in [0, 1], Y_t^{(n)}$ converges to $X_t$ in $L^2_b$-norm sense. Furthermore, for any $t \in [0, 1]$, one has the following inequality,

$$\hat{E}[(Y_t^{(n)} - X_t)^2] \leq K \left( \frac{1}{\sqrt{n}} \right).$$

where $K$ depends on $f, g, h$ and $\bar{\sigma}$.

3.2 The Convergence Rate under Uniform Norm for Wong-Zakai Approximation

In this part, we will calculate the quasi-surely convergence rate for the Wong-Zakai approximation by rough path theory. According to Theorem 2.4, one needs to calculate $g(B, B^{(n)})$. Compared with the proof in [10], we give the Kolmogorov criterion for rough path distance and then we no longer need the partition to be dyadic.

Theorem 3.4. For fixed $q \geq 2, \beta > \frac{1}{q}$, assume $X(\omega) : [0, T] \to \mathbb{R}^d$ and $X(\omega) : [0, T]^2 \to \mathbb{R}^{d \times d}$ are processes with $X_t \in L^q_G(\Omega_T), \mathbb{X}_{s,t} \in L^q_G(\Omega_T), \forall s, t \in [0, T]$, and satisfy relation [14] quasi-surely. If for any $s, t \in [0, T]$, one has bounds

$$\|X_{s,t}\|_{L^q_G} \leq C|t - s|^\beta, \quad \|X_{s,t}\|_{L^q_G} \leq C|t - s|^{2\beta},$$

for some constant $C$. Then for all $\alpha \in [0, \beta - \frac{1}{2})$, $(X, \mathbb{X})$ has a quasi-surely modification, also denoted as $(X, \mathbb{X})$, and there exist $K_\alpha \in L^q_G, \mathbb{K}_\alpha \in L^q_G$ such that for any $s, t \in [0, T]$, one has inequalities

$$|X_{s,t}| \leq K_\alpha|t - s|^\alpha, \quad |\mathbb{X}_{s,t}| \leq \mathbb{K}_\alpha|t - s|^{2\alpha}, \quad \dot{c} - q.s..$$

Specially, if $\beta - \frac{1}{q} > \frac{1}{2}$, then quasi-surely $X = (X, \mathbb{X}) \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$, for any $\alpha \in \left( \frac{1}{q}, \beta - \frac{1}{2} \right)$.

Proof. The proof is given in [17].

Theorem 3.4. (Kolmogorov criterion for rough path distance in $G$-framework)
Suppose $X, \mathbf{X}$ satisfy the moment condition as the above theorem. Let $\Delta X = \mathbf{X} - X$ and $\Delta \mathbf{X} = \mathbf{X} - \mathbf{X}$ and assume that for some $\varepsilon > 0$, one has bounds,

$$\|\Delta X_{s,t}\|_{L^q_G} \leq C\varepsilon|t - s|^\beta, \quad \|\Delta \mathbf{X}_{s,t}\|_{L^q_G} \leq C\varepsilon|t - s|^{2\beta}.$$

Then there exists a constant $M$, depending on $C, \alpha, \beta, q$, such that

$$\|\|\Delta X\|_{L^q_G} \leq M\varepsilon, \quad \|\|\Delta \mathbf{X}\|_{2\alpha L^q_G} \leq M\varepsilon.$$

In particular, if $\beta - \frac{1}{q} > \frac{1}{2}$, then for any $\alpha \in \left( \frac{1}{q}, \beta - \frac{1}{2} \right)$, $\|\|\mathbf{X} - \mathbf{X}\|_{\mathcal{C}^\alpha}\|_{L^q_G} \leq M\varepsilon.$
Proof. By the same argument as the above theorem, there exists \( \Delta K_\alpha \in L^0_G \), such that
\[
\frac{|\Delta X_{s,t}|}{|t-s|^\alpha} \leq \varepsilon \Delta K_\alpha,
\]
so it comes to the first inequality. For the second inequality, by Chen’s identity, one has the following estimate,
\[
|\Delta X_{s,t}| \leq \sum_{i=0}^{N-1} |\Delta X_{\tau_i, \tau_{i+1}}| + \sum_{i=0}^{N-1} |X_{s, \tau_{i+1}} X_{\tau_{i+1}, \tau_{i+2}} - \tilde{X}_{s, \tau_i} \tilde{X}_{\tau_i, \tau_{i+1}}|,
\]
\[
\leq |\Delta K_\alpha| |t-s|^{2\alpha \varepsilon} + \Delta K_\alpha \varepsilon |t-s|^{2\alpha} + \Delta K_\alpha \varepsilon |t-s|^{2\alpha},
\]
with symbols adapted from the above proof. Then the bound for \( \|\|\Delta X\|\|_{L^q} \frac{\alpha}{2} \) follows.

\[ \square \]

Denote \( B^{strat} = (B, B^{strat}) \), where \( B^{strat} = \int_s^t B_{s,r} \circ dB_r \). In the following we give a direct calculation of the quasi-surely convergence rate for \( B^{(n)} \) to \( B^{strat} \) under \( L^p_G \) metric, which is also applied in the classical case.

**Proposition 3.5.** Fix \( \alpha \in (\frac{1}{2}, 1) \). Suppose \( T = 1 \), \( B^{(n)} \) be the piecewise linearization as before, and \( B^{(n)} \) be the natural enhancement of \( B^{(n)} \), i.e. \( B^{strat}_{s,t} = \int_s^t B_{s,r} \circ dB_r \). Then for any \( \theta < \frac{1}{2} - \alpha \) and \( q \geq 2 \), \( B^{(n)} \) converges to \( B^{strat} \) under \( \alpha - \text{H"older rough norm in the } \mathbb{L}^q \) sense. Furthermore, one has the following inequalities,
\[
\|\|B^{strat} - B^{(n)}\|\|_{\theta, \mathbb{L}^q} \leq K(\frac{1}{n})^\theta, \tag{14}
\]
\[
\theta_\alpha(B^{strat}, B^{(n)}) \leq M(\frac{1}{n})^\theta, \quad \hat{c} - q.s., \tag{15}
\]
where \( K \) depends on \( \hat{\sigma} \) and \( M \) depends on \( \hat{\sigma} \) and the path \( \omega \).

**Proof.** To show inequality (14), according to Theorem 3.4 by taking \( \beta = \frac{1}{2} - \theta \), it suffices to show
\[
\|\Delta B^{strat}_{s,t}\|_{L^q} \leq K(\frac{1}{n})^\theta (t-s)^{\frac{1}{2} - \theta}, \quad \|\Delta B_{s,t}^{(n)}\|_{\mathbb{L}^q} \leq K(\frac{1}{n})^\theta (t-s)^{1 - 2\theta}, \tag{16}
\]
for any \( q \geq 2 \), where \( \Delta B^{(n)} = B - B^{(n)} \) and \( \Delta B = B - B^{strat} \). For the first inequality, if \( t_i^{(n)} \leq s < t \leq t_{i+1}^{(n)} \), for some \( 0 \leq i \leq n-1 \), one has
\[
\Delta B_{s,t}^{(n)} = B_{s,t} + \frac{t-s}{t_{i+1}^{(n)} - t_i^{(n)}} B_{t_i^{(n)}, t_{i+1}^{(n)}}.
\]
It follows that
\[
\mathbb{E}|\Delta B_{s,t}^{(n)}|^q \leq 2^q [\mathbb{E}|B_{s,t}|^q + (\frac{t-s}{t_{i+1}^{(n)} - t_i^{(n)}})^q \mathbb{E}|B_{t_i^{(n)}, t_{i+1}^{(n)}}|^q]
\]
\[
\leq C_q \hat{\sigma}^q [t-s]^q + |t-s|^q |t_{i+1}^{(n)} - t_i^{(n)}|^\frac{q}{n}\]
\[
\leq C_q \hat{\sigma}^q [t-s]^q + q^q (\frac{1}{n})^\theta.
\]
where \( C_q \) depends only on \( q \). If \( t_i \leq s < t_{i+1} < \ldots < t_k < t \leq t_{k+1} \), for some \( 0 \leq i < k \leq n \), one has the following inequalities, according to Theorem 2.5, and similar results for \( \Delta \).

For the second inequality in (16), suppose \( t_{i_0}^{(n)} \leq \tau_0 := \tau_1 < \ldots < \tau_{k-1} < t =: \tau_k \leq t_{j_0}^{(n)} \), for some \( 0 \leq i_0 < j_0 \leq n \) and \( k \geq 2 \) (the following is trivial if \( k = 1 \)), where \( \tau_i := t_{i_0+i_0}^{(n)}, i \geq 1 \) for convenience. According to Chen’s identity, one has the following identity,

\[
\Delta \mathbb{B}^{(n)}_{s,t} = \sum_{i=0}^{k-1} \Delta \mathbb{B}^{(n)}_{\tau_i,\tau_{i+1}} + \sum_{0 \leq i < j \leq k-1} (B_{\tau_i,\tau_{i+1}} \otimes B_{\tau_j,\tau_{j+1}} - B_{\tau_i,\tau_{j+1}} \otimes B_{\tau_{r_i},\tau_{j+1}}). \tag{17}
\]

For elements \( \mathbb{B}^{l,t,(n)}_{\tau_i,\tau_{i+1}} \) in the diagonal of matrix \( \Delta \mathbb{B}^{(n)}_{s,t} \), one has

\[
\mathbb{B}^{l,t,(n)}_{\tau_i,\tau_{i+1}} = \int_{\tau_i}^{\tau_{i+1}} B_{\tau_i,t}^{(n)} dB_{\tau_i,t}^{(n)} = \frac{(B_{\tau_i,t}^{(n+1)} - B_{\tau_i,t}^{(n)})^2}{2}, \quad i = 1, \ldots, k - 2,
\]

Then the first sum in (17) on the diagonal is

\[
\sum_{i=0}^{k-1} \Delta \mathbb{B}^{(n)}_{\tau_i,\tau_{i+1}} = \mathbb{E}^{strat}_{s,\tau_1} - \mathbb{E}^{strat}_{\vartheta,\tau_1} + \mathbb{E}^{strat}_{\vartheta,\tau_{k-1}} - \mathbb{E}^{strat}_{t_{k-1},t}. \tag{18}
\]

According to Theorem 2.5 one has the following inequalities,

\[
\mathbb{E}^{(n)}_{s,\tau_1} |q| \leq C_q\widehat{\sigma}^q |\tau_1 - s| \leq C_q\widehat{\sigma}^q (\frac{1}{n})^q, \tag{19}
\]

\[
\mathbb{E}^{(n)}_{s,\tau_1} |q| \leq C_q\overline{\sigma}^q |\tau_1 - s| \leq C_q\overline{\sigma}^q (\frac{1}{n})^q, \tag{20}
\]

and similar results for \( \mathbb{E}^{m,t,(n)}_{\tau_{k-1},t} \) and \( \mathbb{E}^{(n)}_{\tau_{k-1},t} \), which implies our desired estimate.

For elements \( \mathbb{E}^{l,m,(n)}_{\tau_i,\tau_{i+1}} \) off the diagonal of matrix \( \Delta \mathbb{E}^{(n)}_{s,t} \), notice that

\[
\mathbb{E}^{l,m,(n)}_{\tau_i,\tau_{i+1}} = \frac{1}{2} \left( \int_{\tau_i}^{\tau_{i+1}} B_{\tau_i,t}^l dB_{\tau_i,t}^m - \int_{\tau_i}^{\tau_{i+1}} B_{\tau_i,t}^m dB_{\tau_i,t}^l \right).
\]

By applying B-D-G inequality in \( G \)-framework and Hőlder inequality, it follows that

\[
\mathbb{E}^{l,m,(n)}_{\tau_i,\tau_{i+1}} - \mathbb{E}^{m,l,(n)}_{\tau_i,\tau_{i+1}} \leq C_q \left( \mathbb{E} \left[ \sum_{i=1}^{k-2} \int_{\tau_i}^{\tau_{i+1}} (B_{\tau_i,t}^m)^2 dB_{\tau_i,t}^l \right]^\frac{q}{2} + \mathbb{E} \left[ \sum_{i=1}^{k-2} \int_{\tau_i}^{\tau_{i+1}} (B_{\tau_i,t}^l)^2 dB_{\tau_i,t}^m \right]^\frac{q}{2} \right)
\]

\[
\leq C_q \left( \sum_{i=1}^{k-2} \int_{\tau_i}^{\tau_{i+1}} (B_{\tau_i,t}^m)^2 dB_{\tau_i,t}^l + \overline{\sigma}^q |t - s| \frac{1}{n^2} \right)
\]

\[
\leq C_q \overline{\sigma}^q |t - s| \frac{1}{n^2} \tag{21}
\]
As for the second sum in (17), note that $B_{\tau_i}^{(n)} = B_{\tau_i}$, $i = 1, ..., k - 1$, and one obtains

$$
\sum_{0 \leq i < j \leq k - 1} (B_{\tau_i, \tau_{i+1}} B_{\tau_j, \tau_{j+1}} - B_{\tau_i}^{(n)} B_{\tau_j}^{(n)})
$$

$$
= (B_{s, t} - B_{s, \tau_i}^{(n)}) B_{\tau_i, \tau_{k-1}} + B_{\tau_1, \tau_{k-1}} (B_{\tau_{k-1}, t} - B_{\tau_{k-1}, t}^{(n)}) + B_{s, \tau_i} B_{\tau_{k-1}, t} - B_{s, \tau_i}^{(n)} B_{\tau_{k-1}, t}
$$

$$
= (B_{s}^{(n)} - B_{s}) B_{\tau_1, \tau_{k-1}} + B_{\tau_1, \tau_{k-1}} (B_{t} - B_{t}^{(n)}) + B_{s, \tau_i} B_{\tau_{k-1}, t} - B_{s, \tau_i}^{(n)} B_{\tau_{k-1}, t}. \quad (22)
$$

By (13)–(22), one obtains

$$
\|\Delta B_{n, t}^{(n)}\| \leq C_q (\sigma^2 + \sigma) \left( \frac{1}{n} \right) t - s \left( \frac{4}{n} \right) |t - s|^{1-2\theta}.
$$

By the randomness of $q$, one can get the following inequality by Theorem 3.4

$$
\|q_o(B_{strat}^{(n)}, B^{(n)})\| \leq K \left( \frac{1}{n} \right)^{\theta}.
$$

Then (19) follows by a classical Borel-Cantelli argument in G-framework. Indeed, for any $\theta < \frac{1}{2} - \alpha$, one may choose $q > 2$ and $\theta < \theta' < \frac{1}{2} - \alpha$, such that $q(\theta' - \theta) > 2$. It is clear that

$$
\hat{c} \left( \bigcap_{M=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} \{ q_o(B_{strat}^{(n)}, B^{(m)}) > M \frac{1}{m^{\theta'}} \} \right)
$$

$$
\leq \hat{c} \left( \bigcup_{n=1}^{\infty} \bigcup_{m \geq n} \{ q_o(B_{strat}^{(n)}, B^{(m)}) > \frac{1}{m^{\theta'}} \} \right) = \hat{c}(\limsup_{m} \{ q_o(B_{strat}^{(n)}, B^{(m)}) > \frac{1}{m^{\theta'}} \}).
$$

Note that

$$
\hat{c}(q_o(B_{strat}^{(n)}, B^{(m)}) > \frac{1}{m^{\theta'}}) \leq \left( \frac{1}{m^{\theta'}} \right)^q = \frac{1}{m^{\theta q(\theta' - \theta)}}.
$$

According to Borel-Cantelli lemma in G-framework (see Lemma 5 in [2]), one obtains

$$
\hat{c} \left( \bigcap_{M=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} \{ q_o(B_{strat}^{(n)}, B^{(m)}) > M \frac{1}{m^{\theta}} \} \right) = 0,
$$

which implies the desired result.

Here is the main result of this section.

**Theorem 3.6.** Suppose $f, g, h \in C^3_b$, and $Y^{(n)}$ defined as in (6). Also, suppose $X$ solves the following G-Stratonovich SDE

$$
X_t = x_0 + \int_0^t f(s, X_s) \circ dB_s + \int_0^t g(s, X_s) d(B)_s + \int_0^t h(s, X_s) ds,
$$
and $Y$ solves the following RDE driven by G-Stratonovich rough paths,

$$dY_t = f(Y_t)dB_{\text{strat}}^t + g(Y_t)d\langle B \rangle_t + h(Y_t)dt,$$

with initial condition $x_0$. Then for any $\theta < \frac{1}{2} - \alpha$, one has the following inequality,

$$\|Y - Y^{(n)}\|_\alpha \leq M(\omega) \frac{1}{n^{\theta}}, \quad \hat{c} - q.s..$$

In particular, $X = Y, \hat{c} - q.s.,$ and

$$\|X - Y^{(n)}\|_\alpha \leq M(\omega) \frac{1}{n^{\theta}}, \quad \hat{c} - q.s..$$

Proof. Apply Theorem 2.4 and Lemma 3.5, and one obtains the first inequality. The particular part follows from Theorem 3.2 and the fact that $X, Y$ are quasi-surely continuous with respect to $t$. \hfill $\square$

The following corollary implies the continuity of solutions of RDEs driven by lifted G-Brownian motion with respect to uniform norm on the canonical space.

**Corollary 3.7.** $Y$ is RDE solutions defined as the above theorem, then for any $t < T$, $Y_t$ has a quasi-continuous version.

Proof. It follows by $Y_t \in L_G(\Omega_t)$ and the representation of $L_G(\Omega_t)$, i.e. Theorem 2.10. \hfill $\square$

4 Appendix

**PROOF of THEOREM 3.2**

Without loss of generality, we suppose $g, h = 0,$ for simplicity. In the following, the constant $K$ may be different from line to line. Consider the Maruyama’s approximation to G-Stratonovich SDE (10),

$$dX_t^{(n)} = X_{j-1}^{(n)} + f(X_{j-1}^{(n)})B_{t^{(n)}_{j-1},t} + \frac{1}{2}Df(X_{j-1}^{(n)})f(X_{j-1}^{(n)})\langle B \rangle_{t^{(n)}_{j-1},t}, \quad t \in [t^{(n)}_{j-1}, t^{(n)}_j],$$

where $X_{j-1}^{(n)} = X_{t^{(n)}_{j-1}}, j = 1...n.$

By Maruyama’s approximation in G-framework, one actually have

$$\hat{E}[\sup_{t \in [0,T]} |X_t - X_t^{(n)}|^2] \leq K\left( \frac{1}{n} \right)$$

(i.e. Theorem 7 in Part 3 of [10] with a little extension), so one only needs to show

$$\hat{E}(Y_t^{(n)} - X_t^{(n)})^2 \leq K \frac{1}{\sqrt{n}}, \quad \forall t \in [0,1].$$

(25)
By Taylor’s expansion, for (27), one has
\[ \mathbb{E}(Y_t^{(n)} - Y_j^{(n)})^2 \leq \frac{K}{n}, \]
\[ \mathbb{E}(X_t^{(n)} - X_j^{(n)})^2 \leq \frac{K}{n}. \]

It suffices to prove (25) for \( t = t_j^{(n)} \). Firstly, one has the following identity,
\[ Y_j^{(n)} - X_j^{(n)} = (Y_{j-1}^{(n)} - X_{j-1}^{(n)}) + (f(Y_{j-1}^{(n)}) - f(X_{j-1}^{(n)}))B_{t_j^{(n)},t_j^{(n)}}^{(n)} \quad (26) \]
\[ + \int_{t_j^{(n)}}^{t_j^{(n)}} (f(Y_s^{(n)}) - f(Y_{j-1}^{(n)}))ds \frac{B_{t_j^{(n)},t_j^{(n)}}^{(n)} - t_{j-1}^{(n)}}{t_j^{(n)} - t_{j-1}^{(n)}} \quad (27) \]
\[ - \frac{1}{2}Df(X_{j-1}^{(n)})f(X_{j-1}^{(n)})(B_{t_j^{(n)},t_j^{(n)}}^{(n)}). \quad (28) \]

By Taylor’s expansion, for (27), one has
\[ \int_{t_{j-1}^{(n)}}^{t_j^{(n)}} (f(Y_s^{(n)}) - f(Y_{j-1}^{(n)}))ds \frac{B_{t_j^{(n)},t_j^{(n)}}^{(n)} - t_{j-1}^{(n)}}{t_j^{(n)} - t_{j-1}^{(n)}} \]
\[ = \int_{t_{j-1}^{(n)}}^{t_j^{(n)}} f'(Y_{j-1}^{(n)})(Y_s^{(n)} - Y_{j-1}^{(n)})ds \frac{B_{t_j^{(n)},t_j^{(n)}}^{(n)} - t_{j-1}^{(n)}}{t_j^{(n)} - t_{j-1}^{(n)}} \]
\[ = \int_{t_{j-1}^{(n)}}^{t_j^{(n)}} Df(Y_{j-1}^{(n)}) \int_{t_{j-1}^{(n)}}^{s} f(Y_r^{(n)})drds \frac{B_{t_j^{(n)},t_j^{(n)}}^{(n)} - t_{j-1}^{(n)}}{t_j^{(n)} - t_{j-1}^{(n)}}. \quad (29) \]

where \( Y_{j-1}^{(n)} = Y_{j-1}^{(n)} + \theta(Y_{j-1}^{(n)} - Y_{j-1}^{(n)}), \theta \in (0, 1). \)

By subtracting (28) from (29) and inserting terms, one could obtain that
\[ Y_j^{(n)} - X_j^{(n)} = (Y_{j-1}^{(n)} - X_{j-1}^{(n)}) + (f(Y_{j-1}^{(n)}) - f(X_{j-1}^{(n)}))B_{t_j^{(n)},t_j^{(n)}}^{(n)} \]
\[ + \frac{1}{2}[Df(Y_{j-1}^{(n)})f(Y_{j-1}^{(n)}) - Df(X_{j-1}^{(n)})f(X_{j-1}^{(n)})](B_{t_j^{(n)},t_j^{(n)}}^{(n)} - t_{j-1}^{(n)} - (B_{t_j^{(n)},t_j^{(n)}}^{(n)} - t_{j-1}^{(n)}) \]
\[ + \int_{t_{j-1}^{(n)}}^{t_j^{(n)}} (s - t_j^{(n)})Df(Y_{j-1}^{(n)})f(Y_{j-1}^{(n)})ds \frac{B_{t_j^{(n)},t_j^{(n)}}^{(n)} - t_{j-1}^{(n)} - (B_{t_j^{(n)},t_j^{(n)}}^{(n)} - t_{j-1}^{(n)}) \]
\[ + \int_{t_{j-1}^{(n)}}^{t_j^{(n)}} \int_{t_{j-1}^{(n)}}^{s} Df(Y_{j-1}^{(n)})f(Y_{j-1}^{(n)})]drds \frac{B_{t_j^{(n)},t_j^{(n)}}^{(n)} - t_{j-1}^{(n)} - (B_{t_j^{(n)},t_j^{(n)}}^{(n)} - t_{j-1}^{(n)}) \]

Denote the above six terms as \( \varepsilon_l^{(n,j)} \), \( l = 1..6 \). Firstly, by \( f \in \mathcal{C}_b^2 \), it is clear that
\[ \mathbb{E}|\varepsilon_1^{(n,j)}|^2 \leq \frac{K}{n} \]
\[ \mathbb{E}|\varepsilon_2^{(n,j)}|^2 \leq \frac{K}{n} \]
\[ \mathbb{E}|\varepsilon_3^{(n,j)}|^2 \leq \frac{K}{n} \]
\[ \mathbb{E}|\varepsilon_4^{(n,j)}|^2 \leq \frac{K}{n} \]
\[ \mathbb{E}|\varepsilon_5^{(n,j)}|^2 \leq \frac{K}{n} \]
\[ \mathbb{E}|\varepsilon_6^{(n,j)}|^2 \leq \frac{K}{n} \]
Secondly, by Lipschitzness of \( f \) one has
\[
\hat{E}|\varepsilon^{(n,j)}_i|^2 + \hat{E}|\varepsilon^{(n,j)}_2|^2 + \hat{E}|\varepsilon^{(n,j)}_3|^2 + 2\hat{E}(\varepsilon^{(n,j)}_1 \varepsilon^{(n,j)}_3) \leq (1 + \frac{K}{n})\hat{E}|Y^{(n)}_{j-1} - X^{(n)}_{j-1}|^2.
\]

By Lemma 3.1, \( Y^{(n)}_{k-1} \in L^2(\Omega_{t_{k-1}^n}) \), which is independent from \( B^{(n)}_{t_{j-1}^n} \) and \( \langle B \rangle^{(n)}_{t_{j-1}^n} \), so one gets
\[
\hat{E}(\varepsilon^{(n,j)}_1 \varepsilon^{(n,j)}_3) = 0, \quad l = 2, 4.
\]

Also, note that
\[
\hat{E}|B^{(n)}_{t_{j-1}^n}([B^{(n)}_{t_{j-1}^n}])^{\otimes 2} - \langle B \rangle^{(n)}_{t_{j-1}^n}| = \hat{E}||B^{(n)}_{t_{j-1}^n} \int_{t_{j-1}^n}^{t_j^3} B^{(n)}_{t_{j-1}^n} dB_r|| \leq \hat{E}(\varepsilon^{(n,j)}_3)^2,
\]
\[
\hat{E}|\langle B \rangle^{(n)}_{t_{j-1}^n}([B^{(n)}_{t_{j-1}^n}])^{\otimes 2} - \langle B \rangle^{(n)}_{t_{j-1}^n}| \leq K(\frac{1}{n})^2,
\]
and one could obtain
\[
|\hat{E}(\varepsilon^{(n,j)}_4 \varepsilon^{(n,j)}_1)| \leq K(\frac{1}{n})^2, \quad l = 2, 3.
\]

As for other intersection terms concerning \( \varepsilon^{(n,j)}_5, \varepsilon^{(n,j)}_6 \), one can apply Hölder’s inequality directly by noticing \( \hat{E}|\varepsilon^{(n,j)}_1|^2 \) bounded and obtain
\[
\hat{E}(\varepsilon^{(n,j)}_k \varepsilon^{(n,j)}_l) \leq K(\frac{1}{n})^2, \quad l = 1...6, k = 5, 6.
\]

Finally, one gets
\[
\hat{E}|Y^{(n)}_{j} - X^{(n)}_{j}|^2 \leq (1 + \frac{K}{n})\hat{E}|Y^{(n)}_{j-1} - X^{(n)}_{j-1}|^2 + K(\frac{1}{n})^2
\]
\[
\leq \sum_{i=0}^{j-1} (1 + \frac{K}{n})^i K(\frac{1}{n})^2 \leq \frac{K}{\sqrt{n}}.
\]

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