Dynamics of coupled parametric oscillators beyond coupled Ising spins

Leon Bello,1 Marcello Calvanese Strinati,2 Emanuele G. Dalla Torre,2 and Avi Pe’er1

1Department of Physics and BINA Center of Nanotechnology, Bar-Ilan University, 52900 Ramat-Gan, Israel
2Department of Physics, Bar-Ilan University, 52900 Ramat-Gan, Israel

(Dated: February 12, 2019)

Coupled parametric oscillators were recently employed as simulators of artificial Ising networks, with the potential to solve computationally hard minimization problems. We present a detailed study of the simplest coupled oscillator system - two degenerate parametric oscillators, exploring the entire phase diagram in terms of pump power, coupling nature (dissipative or energy conserving) and strength, both analytically and experimentally in a radio-frequency experiment. When the coupling between the oscillators includes a conservative component, we predict and observe a new dynamical regime, unique to parametric oscillators, where the spin-1/2 description does not apply. In a wide range of parameters near the oscillation threshold the oscillators never synchronize (in contrast to coupled spin-1/2), but show persistent, full-scale, coherent beats, whose frequency reflects the coupling strength. This new beating regime is a unique manifestation of coherent dynamics in coupled oscillators in general, and a building block for coherent Ising machines in particular.

Introduction The history of parametric oscillators probably traces back to the XIX century with studies by Michael Faraday on the crispations of wine glasses [1]. In modern physics, the optical parametric oscillator (OPO) is widely known due to its applications in classical and quantum optics. Below the oscillation threshold, the OPO generates squeezed vacuum [2–5], with applications in precise metrology [6–9], micro- and nano-electromechanical systems [10–13], quantum information [14–17] and quantum communications [18, 19]. Above threshold, an OPO is the primary source of coherent light at wavelengths that are not laser accessible.

The working mechanism of a degenerate parametric oscillator is the well known period doubling instability [20]. In contrast to the lasing instability, the gain in a parametric oscillator depends on the phase of the oscillation, relying on the coherent nonlinear coupling between the pump field (at frequency ωp) and the oscillation (at exactly ωp/2) to amplify a single quadrature component of the oscillation field while attenuating the other quadrature. The phase of the amplified quadrature can acquire two distinct values, which give rise to two solutions that are relatively shifted by one period of the pump. Each solution breaks the time-translational symmetry of the pump, and thus an OPO is the simplest example of a classical discrete (Floquet) time crystal [21–30].

Borrowing the common terminology from condensed matter systems, one can refer to a parametric oscillator as a classical two-level system (spin-1/2, or Ising spin). Based on this analogy, it has been recently suggested that coupled parametric oscillators can be used to simulate chains or networks of Ising spins [31–39]. The basic idea relies on the inherent mode competition and positive feedback within the oscillators to find the most efficient coupled-mode oscillation, which reflects the ground-state configuration of the corresponding Ising model (under certain assumptions). The idea is to implement a set of optical parametric oscillators to realize many independent spin-1/2 systems, where the coupling of the field between the oscillators reflects the coupling between spins, giving rise to a coupled network of spins. This coherent Ising machine (CIM) can simulate the spin dynamics and aims at calculating the ground state of the corresponding Ising model, thereby solving minimization problems that cannot be solved on a classical computer.

In this letter, we consider the simplest case of two coupled degenerate parametric oscillators with both energy-preserving and energy-dissipating coupling. We show that the Ising-type interaction arises from the dissipative terms, which give preference to either in-phase (“ferromagnetic”) or anti-phase (“anti-ferromagnetic”) synchronization of the oscillators. An energy-preserving coupling on the other hand, induces a unique coherent dynamics, where the oscillators do not synchronize, but rather display coherent everlasting beats for a wide range of the parameter space. These oscillations may have implications for the operation of CIMs, as we discuss below.

Furthermore, the coherent transfer of energy between parametric oscillators (beating) is of interest in itself. Coherent dynamics in coupled oscillators (for which beating is a signature mark) is a fundamental principle of wave physics that was widely studied. However, such coherent beating is almost always a transient phenomenon that decays due to decoherence, dissipation and non-linear effects. Remarkably, the parametric interaction in the oscillators acts to preserve this fragile coherence, displaying a steady state of everlasting, full-scale coherent beats.

We realize experimentally a pair of coupled parametric oscillators using parametrically driven RF resonators with a tunable coupling. Our experimental findings agree with the solution of an analytical model that accounts for the coupling strength and phase, periodic drive, gain, losses, nonlinearities, and coupling with energy-preserving and dissipative components. Our main finding is that, depending on the relation between the two coupling components, two distinct oscillation regimes exist:
(i) When the dissipative component of the coupling dominates, the system displays the expected behaviour of coupled spin-1/2, which is the working principle of the two-oscillator CIM [31]. The system oscillates parametrically at exactly half the pump frequency and, near threshold, its configuration corresponds to the lowest energy state of an Ising model; (ii) When the energy-preserving coupling dominates, the system displays a richer phenomenology: When the pump frequency is twice the bare-oscillator frequency, the system exhibits periodic beats that never decay or lose coherence. Only when the pump power is raised further, beyond a higher nonlinear threshold, the oscillators synchronize to the Ising solution. The beating regime, which is unique to parametric oscillators and cannot be observed in coupled lasers, represents a trajectory in phase space that visits periodically all the possible spin configurations and may have implications for the operation of CIMs. This novel regime, in which the system is not amenable to the description of Ising spins, is the main subject of our analysis.

Theoretically, we first study the coupled system by resorting to a linear stability analysis, based on Floquet’s theorem [40–42], which allows us to characterize all the parametric instabilities of the system without nonlinearities. We then address in detail by a multi-scale analysis [43] the degenerate case (the pump frequency is twice that of the bare-oscillators, which allows us to determine analytically the phase diagram of the coupled OPOs including nonlinearities. We find four major phases of oscillation (see Fig. 1): (i) A below-threshold stable phase of no oscillation (the regime of squeezed noise). (ii) An Ising phase slightly above threshold with two possible steady synchronized oscillations, corresponding to the lowest-energy configuration of two coupled Ising spins. This CIM phase exists only when the coupling is dominated by the dissipative component. (iii) Further above threshold, a synchronized phase with four possibilities of steady oscillation, behaving as two uncoupled spins. (iv) An extended region near threshold, where the oscillators show periodic exchange of energy between them (coherent beating) with a non-universal envelope (beat) frequency. This beating behavior, which appears only when the energy-preserving component of the coupling dominates, was not addressed in standard analyses of CIMs, and differs from the usual response of parametric oscillators, whose frequency is dictated by the pump only. The existence of the beating phase near threshold suggests an alternative route to the Ising regime: In addition to the standard direct transition from sub-threshold squeezed noise to the Ising state (see arrow A on Fig. 1, right panel), the oscillators may also cross first into the coherent beating phase and only then reach the synchronized Ising regime (arrow B), which may be of interest in the dynamical analysis of CIMs.

**Theoretical model** We study a system of two degenerate single-mode parametric oscillators, with equal gain and loss terms, coupled via energy-preserving and energy-dissipating terms, in the presence of pump-depletion nonlinearity. We analytically model our system by a set of classical equations of motion:

\[
\begin{align*}
\ddot{x}_{1} + \Omega_{1}^{2}(t) x_{1} + \omega_{0} g \dot{x}_{1} - \omega_{0}(r - \alpha) \dot{x}_{2} &= 0 \\
\ddot{x}_{2} + \Omega_{2}^{2}(t) x_{2} + \omega_{0} g \dot{x}_{2} + \omega_{0}(r + \alpha) \dot{x}_{1} &= 0.
\end{align*}
\]

Here, \(x_{1}\) and \(x_{2}\) represent the oscillation amplitudes, the resonant frequency \(\Omega_{1,2}(t)\) is parametrically modulated in time as \(\Omega_{1}^{2}(t) = \omega_{0}^{2}[1 + h(x_{1}) \sin \gamma t]\) and \(\Omega_{2}^{2}(t) = \omega_{0}^{2}[1 + h(x_{2}) \sin (\gamma t + \phi)]\) with \(\omega_{0}\) being the bare resonant frequency of the oscillators, \(\gamma\) the pump frequency and \(\phi\) the relative phase between the pumps. The quantities \(h(x) = h(1 - \beta x^{2})\) represent the normalized pump power, where \(\beta\) accounts for the pump depletion nonlinearity when the oscillation is substantial; \(g\) is the intrinsic loss and \(r\) and \(\alpha\) represent the energy-preserving and energy-dissipating coupling terms, respectively.

If \(x_{1,2}\) are sufficiently small, the nonlinearity can be neglected (\(\beta = 0\), which is valid near and below the oscillation threshold, allowing us to diagonalize Eq. (1) by introducing the two eigenmodes \(x_{\pm}(t) = x_{1}(t) + c_{\pm}(r, \alpha) x_{2}(t)\), where the coefficients \(c_{\pm}(r, \alpha)\) are determined by the values of \(r\) and \(\alpha\). The stability analysis of the system can then be carried out by means of a perturbative approach based on Floquet’s theorem. We discuss here the main results, referring the interested reader to Ref. [44] for a detailed discussion.

When the energy-dissipative coupling dominates, \(\alpha > r\), there is only one parametric resonance at \(\gamma = 2\omega_{0}\). The two eigenmodes \(x_{\pm}\) have different thresholds \(h_{th,\pm} = \cdots\).
2g±√α²−r². Therefore, by increasing h above the lower threshold, one can selectively excite \( x_- \), and for higher h, also \( x_+ \). The two modes are excited independently and oscillate with the same frequency (\( γ/2 \), with an exponential time dependence: \( x_±(t) \sim e^{(h−h_{th})t/4} \cos(γt/2) \). This is the standard case for CIMs.

In contrast, when energy-preserving coupling dominates \( r > α \), the system displays a richer phase diagram with three distinct parametric resonances at \( γ = 2ω_0 \) and \( γ = 2ω_0 \pm ω_0 \sqrt{r^2−α^2} \), depending on the relative phase of the pumps \( φ \). Specifically, when \( φ = 0 \), only the resonance at \( γ = 2ω_0 \) can be excited, whereas for \( φ = π \) only the resonances at \( γ = 2ω_0 \pm ω_0 \sqrt{r^2−α^2} \) exist (for a generic \( 0 < φ < π \), all three resonances are found). For the resonance at \( γ = 2ω_0 \), both eigenmodes \( x_± \) are excited simultaneously, leading to full scale beats above the threshold \( h_{th} \): \( x_±(t) \sim e^{rω_0t/√(γ^2−α^2)/2} e^{(h−h_{th})t/4} \cos(γt/2) \). When \( φ = π \), the modes \( x_± \) are independently excited (at \( γ = 2ω_0 \pm ω_0 \sqrt{r^2−α^2} \), respectively), and parametric amplification occurs without beats. We therefore see (Fig. 1) that at \( r = α \) the system undergoes a transition from a CIM to a coherent beating behaviour.

Intuitively, the beating phase can be understood from the well known analysis of coupled harmonic oscillators: When two oscillators of frequency \( ω_0 \) are coupled in an energy-preserving manner, the oscillation frequency modes split into a symmetric mode at a lower frequency \( ω_0 − δ \) and an anti-symmetric mode at \( ω_0 + δ \) (where \( δ \) reflects the coupling strength). As opposed to laser oscillators that can oscillate on any frequency or phase, parametric oscillators are restricted in phase and frequency by the pump. Thus, when pumped at \( γ = 2ω_0 \), parametric oscillators cannot oscillate on any single coupled mode, since the mode frequency is now shifted away from the pump. Nevertheless, simultaneous oscillation of both modes is still possible, which now act as non-degenerate signal \( ω_0 + δ \) and idler \( ω_0 − δ \) of the combined oscillator. As a consequence, energy-preserving coupling of parametric oscillators generates tunable two-mode squeezed vacuum and parametric light, where the frequency separation between the modes can be tuned by varying the strength of the coupling without any change to the internal geometry/structure of the individual oscillators.

We now expand the analysis further above the threshold (beyond the linear Floquet analysis) by incorporating the nonlinearity \( β \neq 0 \). For brevity, we focus on degenerate pumping at \( γ = 2ω_0 \), where the system displays richer physics. To describe analytically the oscillation in this nonlinear regime, we resort to a multiple-scale perturbative expansion [43], also known as the slow-varying envelope approximation in non-linear optics. For simplicity, we focus on the case of \( φ = 0 \).

The fast time scale of the oscillator is associated with the carrier frequency \( t = 2π/ω_0 \) and the loss \( g \) is the small expansion parameter of the theory, allowing us to identify the slow time scale as \( τ = gt \). We therefore write \( x_1(t, τ) = A(τ)e^{iω_0t} + A^*(τ)e^{-iω_0t} \) and \( x_2(t, τ) = B(τ)e^{iω_0t} + B^*(τ)e^{-iω_0t} \), where \( A \) and \( B \) are the complex amplitudes of \( x_1 \) and \( x_2 \), respectively. Normalizing the pump and the coupling in Eq. (1) with respect to the loss as \( h = h/g, r = r/g \) and \( α = α/g \), and defining the dimensionless time as \( \tilde{τ} = ω_0τ \), the long-time dynamics is captured by the set of ODEs [44]:

\[
\frac{\partial A}{\partial \tilde{τ}} = \frac{\tilde{h}}{4} A^* - \frac{\tilde{h}}{4} \left( 3|A|^2 A^* - A^3 \right) - \frac{A}{2} \frac{\tilde{r} - \tilde{α}}{2} B = 0
\]

\[
\frac{\partial B}{\partial \tilde{τ}} = \frac{\tilde{h}}{4} B^* - \frac{\tilde{h}}{4} \left( 3|B|^2 B^* - B^3 \right) - \frac{B}{2} \frac{\tilde{r} + \tilde{α}}{2} A = 0
\]

(2)

We can now calculate the phase diagram of Eq. (2) in the \( h/h_{th} \) vs. \( r \) plane (see Fig. 1), using tools of nonlinear dynamics [20] to determine the number of fixed points and their stability. Below the threshold \( h < h_{th} \), a unique stable fixed point exists at \( A = B = 0 \) (the origin). Above the threshold (\( h > h_{th} \)) the origin is unstable and two situations are encountered (assuming \( \tilde{α} \neq 0 \)): For \( \tilde{r} < \tilde{α} \), two stable fixed points correspond to the correct ground states of two Ising spins - the oscillators synchronize together with a fixed amplitude and phase. For larger \( \tilde{h} \), two additional stable points correspond to the two other Ising configurations, as discussed in the analysis of CIMs [31]. For \( \tilde{r} > \tilde{α} \), one first finds a stable limit cycle, which manifests itself as beats in the time evolution of \( A \) and \( B \), and only for larger \( \tilde{h} \), the phase with two or four stable fixed points appears. If \( \tilde{α} = 0 \) the CIM phase does not exist at all. For \( φ > 0 \), the width of the limit cycle region gradually decreases, eventually vanishing at \( φ = π \) (see [44] for details).
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tion threshold, just before synchronization, and after synchro-

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in top panels).

Experimental methods: Since the dynamics described
here is coherent and purely classical, it is suitable to
realize the coupled parametric oscillators in a radio-
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The coupled parametric oscillators are realized with
two ring RF resonators (see Fig. 2) of 50 cm long coaxial

cables with repetition rate of roughly 85 MHz. Each
resonator includes: (i) an RF frequency mixer (Mini-

Circuits ZX50-10-S+) pumped at 170 MHz by an RF
synthesizer acting as the nonlinear parametric amplifier,
(ii) a broadband (regular) low-noise amplifier with gain
of approximately 15 dB (Mini-Circuits ZX60-P105LN),

which compensates for the losses of the cavity, (iii) a -

15 dB coupler for the resonator output (Mini-Circuits
ZFDC-15-5), and (iv) a tunable attenuator to electroni-
cally tune the overall gain of the oscillator. The cou-
ping between the parametric oscillators is achieved with
a fixed power splitter and a couple of tunable attenua-
tors to control the effective coupling. The oscillators are
pumped by two phase-locked synthesizers, allowing us to
control of the relative phase between the pumps.

Since we aim primarily at demonstrating the properties
of the beating phase (limit cycle) with energy-preserving
coupling, we focus experimentally on \(\alpha = 0\) and moni-
tor the field emitted from the parametric oscillators for

various values of the pump power \(h\) with respect to the
oscillation threshold \(h_{th}\) and various coupling strengths

\(r\), determined by the beat frequency at threshold. Our
results are shown in Fig. 3(a)-(d). The left plots show
the experimental results, while the right panels show the



corresponding theoretical solution, obtained by numeri-
cally solving Eq. (1). The latter plots are overlapped by

the oscillation envelopes \(2|A(gt)|, 2|B(gt)|\) (orange and
black), computed by solving the slow-varying Eq. (2).
For pumping slightly above threshold, both oscillators
demonstrate a regular, nearly sinusoidal beating envelope
over a carrier signal at half the pump frequency, which
matches the cavity resonance at 87 MHz [Fig. 3(a),(b)].

As we further increase the pump power, the period of the
beats increases and their shape becomes elongated and
pear-shaped [Fig. 3(c),(d)], until finally diverging at the
transition to a synchronized steady-state (not shown).

In Fig. 3(e), we show the flow of Eq. (2) as \(B_R \equiv \Re[B] vs. A_R \equiv \Re[A]\), for three different cases: slightly above

the oscillation threshold, where all fixed points are sad-

dle points and the limit cycle is nearly a perfect circle

around the origin, corresponding to perfect beats; just

before synchronization, where the limit cycle becomes

sharper and the beats assume an asymmetric shape; af-



After sync.
study of two coupled degenerate parametric oscillators, 
explored in an RF experiment, analytically and numerically. 
A single parametric oscillator is the prototype example of a discrete time crystal, which spontaneously breaks the symmetry associated with the time-periodicity of the pump, analogous to an Ising spin. Although naively, one would expect this to hold also when several parametric oscillators are coupled together, our study reveals a much richer phase diagram with a new limit-cycle phase, where the oscillators perform coherent beats that never decay or decohere when the coupling contains a significant energy-preserving component. This beating phase represents a new class of coherent dynamics that was not previously considered within the vastly researched subject of coupled oscillators and is unique to coupled parametric oscillators, demonstrating a new aspect of their coherent link to the pumping field and to each other.

We are grateful to J. Avron, I. Bonamassa, C. Conti, N. Davidson, I. Gershenzon, D. A. Kessler, R. Lifshitz, Y. Michael and C. Tradonsky for fruitful discussions. A. P. acknowledges support from ISF grant No. 46/14 and BSF-NSF grant No. 2017743. M. C. S. acknowledges support from ISF, grants No. 231/14 and 1452/14.

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