THE NUMBER OF PRIME FACTORS OF INTEGERS WITH DENSE DIVISORS

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Abstract. We show that for integers \( n \), whose ratios of consecutive divisors are bounded above by an arbitrary constant, the normal order of the number of prime factors is

\[ C \log \log n, \]

where

\[ C = (1 - e^{-\gamma})^{-1} = 2.280... \]

and \( \gamma \) is Euler’s constant. We explore several applications and resolve a conjecture of Margenstern about practical numbers.

1. Introduction

We say that a positive integer \( n \) is \( t \)-dense if the ratios of consecutive divisors of \( n \) do not exceed \( t \). Let \( D(x, t) \) denote the set of \( t \)-dense integers \( n \leq x \) and write \( D(x, t) = |D(x, t)| \). Let \( \omega(n) \) (resp. \( \Omega(n) \)) be the number of prime factors of \( n \), counted without (resp. with) multiplicity.

Theorem 1 gives the average and normal order of \( \omega(n) \) and \( \Omega(n) \) for the \( t \)-dense integers. We write \( \log_2 x \) for \( \log \log x \) and define

\[ E(x, t) := C \log_2 x - (C - 1) \log_2 t, \quad C := (1 - e^{-\gamma})^{-1} = 2.280291... \]

Theorem 1. Let \( \xi(x) \to \infty \). Uniformly for \( x \geq t \geq 2 \),

\[
\frac{\sum_{n \in D(x, t)} \omega(n)}{D(x, t)} = E(x, t) + O(1)
\]

(1)

and

\[
\left| \left\{ n \in D(x, t) : |\omega(n) - E(x, t)| > \xi(x) \sqrt{\log_2 x} \right\} \right| \ll \frac{D(x, t)}{\xi(x)^2}.
\]

(2)

These results also hold with \( \Omega \) in place of \( \omega \).

Note that \( \log_2 x \leq E(x, t) \leq C \log_2 x + O(1) \), as \( x \geq t \geq 2 \). If \( t = x \), then \( E(x, t) = \log_2 x \) and \( D(x, t) = [1, x] \cap \mathbb{N} \), so Theorem 1 contains the well-known results about the average and normal order of \( \omega(n) \) on \( \mathbb{N} \). If \( t \geq 2 \) is constant, then \( E(x, t) = C \log_2 x + O(1) \), so that the average and normal order of \( \omega(n) \) for \( t \)-dense integers \( n \) is \( C \log_2 n \).

The \( t \)-dense integers are a special case of a family of integer sequences that arise as follows. Let \( \theta \) be an arithmetic function. Let \( \mathcal{B} = \mathcal{B}_\theta \) be the

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set of positive integers containing \( n = 1 \) and all those \( n \geq 2 \) with prime factorization \( n = p_1^{a_1} \cdots p_k^{a_k}, \) \( p_1 < p_2 < \ldots < p_k, \) which satisfy
\[
p_i \leq \theta(p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}}) \quad (1 \leq i \leq k).
\] (3)

We write \( \mathcal{B}(x) = \mathcal{B} \cap [1, x] \) and \( B(x) = |\mathcal{B}(x)|. \) When \( \theta(n) = nt, \) then \( \mathcal{B} \) is the set of \( t \)-dense integers \([6, 10, 14]\). If \( \theta(n) = \sigma(n) + 1, \) where \( \sigma(n) \) is the sum of the positive divisors of \( n, \) then \( \mathcal{B} \) is the set of practical numbers \([2, 6, 7, 8, 9, 10, 14]\), i.e. integers \( n \) such that every \( m \leq n \) can be expressed as a sum of distinct positive divisors of \( n. \)

We will derive Corollaries 1 through 3 from Theorem 1 in Section 2.

**Corollary 1.** Assume \( \theta \) satisfies \( \max(2, n) \leq \theta(n) \leq nf(n), \) where \( f \) is non-decreasing. Let \( \xi(x) \to \infty. \) We have
\[
\sum_{n \in \mathcal{B}(x)} \omega(n) \overline{B(x)} = C \log_2 x \left\{ 1 + O \left( \left( \frac{\log f(x)}{\log_2 x} \right)^{1/3} \right) \right\}
\] and
\[
\left| \left\{ n \in \mathcal{B}(x) : |\omega(n) - C \log_2 x| > \xi(x) \sqrt{\log_2 x} \right\} \right| \ll B(x) \frac{\log f(x)}{\xi(x)^2}.
\] These results also hold with \( \Omega \) in place of \( \omega. \)

Conjecture 5 of Margenstern \([2]\) proposes that, for practical numbers, \( \sum_{n \in \mathcal{B}(x)} \omega(n) \sim \mu x/(\log x)^{\eta}, \) for some constants \( \mu > 0 \) and \( 1/2 < \eta < 1. \)
The estimate (4) disproves this conjecture, since \( B(x) \sim c_0 x/\log x \) by \([14, \text{Thm. 1.2}], \) if \( \max(2, n) \leq \theta(n) \ll n(\log 2n)/(\log_2 3n)^{1+\epsilon} \) for \( n \geq 1. \)
Corollary 2 shows that almost all large practical numbers \( n \) have about \( C \log_2 n \) prime factors.

**Corollary 2.** If \( \theta \) satisfies \( \max(2, n) \leq \theta(n) \ll n(\log 2n)^{o(1)}, \) then the average and normal order of \( \omega(n) \) on \( \mathcal{B} \) is \( C \log_2 n. \) That is, as \( x \to \infty, \)
\[
\sum_{n \in \mathcal{B}(x)} \omega(n) \sim \sum_{n \in \mathcal{B}(x)} C \log_2 n
\] and all but \( o(B(x)) \) integers \( n \in \mathcal{B}(x) \) satisfy
\[
\omega(n) = (1 + o(1))C \log_2 n.
\] These results also hold with \( \Omega \) in place of \( \omega. \)

Let \( \tau(n) \) be the number of positive divisors of \( n. \)

**Corollary 3.** If \( \theta \) satisfies \( \max(2, n) \leq \theta(n) \ll n(\log 2n)^{o(1)}, \) then
\[
(\log x)^{C \log 2 - o(1)} \leq \sum_{n \in \mathcal{B}(x)} \tau(n) \overline{B(x)} \ll (\log x)^{e \log 2},
\] as \( x \to \infty, \) and all but \( o(B(x)) \) integers \( n \in \mathcal{B}(x) \) satisfy
\[
\tau(n) = (\log n)^{C \log 2 + o(1)} = (\log n)^{1.580577...+o(1)}.
\]
Conjecture 4 of Margenstern [2] says that, in the case of practical numbers,
\[ \sum_{n \in B(x)} \tau(n) \sim \nu x (\log x)^\delta, \]
for constants \( 1/2 < \nu, \delta < 1 \). The estimate (8) implies \( \delta \in [C \log 2 - 1, e \log 2 - 1] = [0.580..., 0.884...], \) since \( B(x) \asymp x / \log x \). In [17], we prove this conjecture with \( \delta \)

The next two corollaries are improvements to the lower bounds of Theorems 1 and 3 of [5]. The proofs of both of these theorems rely on the fact that almost all \( n \in B \) satisfy \( \Omega(n) < (e + \varepsilon) \log_2 n \). With Corollary 2, this can be improved to \( \Omega(n) < (C + \varepsilon) \log_2 n \), under the assumption \( \theta(n) \ll n(\log n)^{o(1)} \). In the lower bound of [5, Thm. 1] for the count of practical numbers that are also shifted primes, which has an exponent of \( (e + 1) \log(e + 1) - e \log(e) + 1 + \varepsilon = 3.16470... \), we can replace \( e \) by \( C \) to get 3.01711....

**Corollary 4.** Fix a nonzero integer \( h \) and assume \( \theta \) satisfies
\[
\max(2, n) \leq \theta(n) \ll n(\log 2n)^{o(1)}, \quad \theta(mn) \ll m^{O(1)} \theta(n) \quad (n, m \in \mathbb{N}).
\]

We have
\[
\frac{x}{(\log x)^{3.01712}} \ll_h \big| \{ p \leq x : p \text{ prime, } p - h \in B \} \big| \ll_h \frac{x}{(\log x)^2},
\]
where \( h \) is not divisible by \( \prod_{p \leq \theta(1)} p \) in the lower bound.

Similarly, in the lower bound of [5, Thm. 3] for the count of twin practical numbers, which has an exponent of \( 2 + 4e \log 2 + \varepsilon = 9.53667... \), we can replace \( e \) by \( C \) to get 8.32230...

**Corollary 5.** Fix a nonzero integer \( h \) and assume \( \theta \) satisfies (10). We have
\[
\frac{x}{(\log x)^{8.32231}} \ll_h \big| \{ n \leq x : n \in B, n + h \in B \} \big| \ll_h \frac{x}{(\log x)^2}.
\]
For the lower bound, assume that (i) \( n \in B \) and \( m \leq 3n/|h| \) imply \( mn \in B \), and (ii) if \( \theta(1) < 3 \), then \( h \in 2\mathbb{Z} \) if \( \theta(2) \geq 3 \), and \( h \in 4\mathbb{Z} \) if \( \theta(2) < 3 \).

Conditions (i) and (ii) in Corollary 5 are satisfied by the practical numbers and by the 2-dense integers for any nonzero even integer \( h \), and by the \( t \)-dense integers for any nonzero integer \( h \), provided \( t \geq 3 \).

The \( \varphi \)-practical numbers [4, 12] are integers \( n \) such that \( x^n - 1 \) has a divisor in \( \mathbb{Z}[x] \) of every degree up to \( n \). Although not an example of a set \( B_0 \), they are a superset of \( B_0_1 \) with \( \theta_1(n) = n + 1 \), and a subset of \( B_0_2 \) with \( \theta_2(n) = n + 2 \). Therefore, Corollaries 1 through 5 also apply to the \( \varphi \)-practical numbers, provided \( h \) is odd in the lower bound of Corollary 4, while \( h \) is even in the lower bound of Corollary 5.

Theorem 1 is a consequence of Theorems 2 and 3. Theorem 2 gives an estimate for
\[
D_q(x) = D_q(x, t) := \{|n \in D(x, t) : q|n\},
\]
when \( q \) has a bounded number of prime factors. As in the case \( q = 1 \) (see [14, Thm. 1.3]), the main term contains the function \( d(v) \), which is defined
by \( d(v) = 0 \) for \( v < 0 \) and
\[
d(v) = 1 - \int_{0}^{v-1} \frac{d(u)}{u+1} w\left(\frac{v-u}{u+1}\right) \, du \quad (v \geq 0),
\]
where \( w(u) \) denotes Buchstab’s function.

**Theorem 2.** Let \( k \in \mathbb{N} \cup \{0\} \) be fixed. Uniformly for \( x \geq 1, t \geq 2, q \in \mathbb{N} \) with \( \Omega(q) = k, \ v = \log x / \log t \), we have
\[
D_q(x, t) = xd(v)\eta_{q,t} \left\{ 1 + O_k \left( \frac{1}{\log xt} + \frac{\log 2q \log qt}{\log^2 xt} \right) \right\} + O(1),
\]
where
\[
q^{-1} \ll \eta_{q,t} \ll k q^{-1}. \tag{14}
\]

**Corollary 6.** Let \( k \in \mathbb{N} \cup \{0\} \) be fixed. Uniformly for \( x \geq 1, t \geq 2, q \in \mathbb{N} \) with \( \Omega(q) = k \), we have
\[
D_q(x, t) = c_q x \log xt \left\{ 1 + O_k \left( \frac{1}{\log xt} + \frac{\log 2q \log qt}{\log^2 xt} \right) \right\} + O(1),
\]
where
\[
c_q = c_{q,t} = C\eta_{q,t} \log t, \quad q^{-1} \log t \ll c_q \ll k q^{-1} \log t, \tag{16}
\]
and
\[
D_q(x, t) \ll_k \frac{x \log t}{q \log xt}. \tag{17}
\]

The estimates (15) and (16) follow from (12) and (14), since \( d(v) = C(v+1)^{-1} \{1 + O \left( (v+1)^{-2} \right) \} \) by [13, Thm. 1]. The upper bound (17) follows from (13), because \( D_q(x, t) = 0 \) if \( q > x \).

Theorem 3 gives estimates for \( c_q \) when \( q \) is a prime or a product of two primes. These estimates are needed to derive Theorem 1 from Theorem 2.

**Theorem 3.** Let \( p \leq q \) be primes. The constant factor in (15) satisfies
\[
c_q := c_1 = C(\log t - \gamma) + O\left( e^{-\sqrt{\log t}} \right), \tag{18}
\]
\[
qc_q = Cc_q \left\{ 1 + O \left( \frac{1}{\log q} + \frac{\log^2 t}{\log^2 q} \right) \right\}, \tag{19}
\]
\[
qc_q = c_q + C \log q + O \left( \exp \left( -\sqrt{\log t} \right) \right) \quad (q \leq t), \tag{20}
\]
\[
pqc_{pq} = C^2 c_\theta \left\{ 1 + O \left( \frac{1}{\log p} + \frac{\log^2 t}{\log^2 p} + \frac{\log^2 p}{\log^2 q} \right) \right\}, \tag{21}
\]
\[
pqc_{pq} = (Cc_\theta + C^2 \log p) \left\{ 1 + O \left( \frac{1}{\log q} + \frac{\log^2 t}{\log^2 q} + e^{-\sqrt{\log t}} \right) \right\} \quad (p \leq t), \tag{22}
\]
\[
pqc_{pq} = c_\theta + C \log pq + O \left( e^{-\sqrt{\log t}} \right) \quad (p \leq q \leq t). \tag{23}
\]
In Section 2 we derive Corollaries 1, 2 and 3 from Theorem 1. Section 3 contains several lemmas, used in the proofs of Theorems 2 and 3, about members of $B$ that are multiples of a natural number $q$. The proof of Theorem 2 is given in Section 4. In Section 5 we establish Theorem 3 with the help of Corollary 6, which is a consequence of Theorem 2. Finally, in Section 6 we apply Theorems 2 and 3 to prove Theorem 1.

2. Proof of Corollaries 1, 2 and 3

Lemma 1. We have

$$D(x, t) \ll \frac{x \log t}{\log xt} \quad (x > 1/t, t \geq 2),$$

$$D(x, t) \gg \frac{x \log t}{\log xt} \quad (x \geq 1, t \geq 2),$$

$$D(x/q, t) - D(q/t, t) \leq D_q(x, t) \leq D(x/q, qt) \quad (x \geq 0, t \geq 2, q \geq 1).$$

Proof. The first two estimates follow from [6, Thm. 1].

If $m \in D(x/q, t)$ and $m > q/t$, then $mq \in D_q(x, t)$. This shows that $D(x/q, t) - D(q/t, t) \leq D_q(x, t)$.

If $n \in D(x, t)$ and $q|n$, we write $n = qm$ and observe that $m \in D(x/q, qt)$. Thus, $D_q(x, t) \leq D(x/q, qt)$. □

Proof of Corollary 1. We first show (5). For $1 \leq n \leq x$, we have $\max(2, n) \leq \theta(n) \leq nf(n) \leq n f(x)$. Thus, $B(x) \subset D(x, f(x))$. If $f(x) \leq x$, (2) yields

$$\left| \left\{ n \in B(x) : |\omega(n) - E(x, f(x))| > \frac{\xi(x)}{2} \sqrt{\log x} \right\} \right| \ll \frac{D(x, f(x))}{\xi(x)^2}. $$

The assumption $\theta(n) \geq \max(2, n)$ implies $B(x) \gg x/\log x$, by [14, Thm. 1.2]. By Lemma 1,

$$\frac{D(x, f(x))}{\xi(x)^2} \ll \frac{x \log f(x)}{\xi(x)^2 \log x} \ll B(x) \frac{\log f(x)}{\xi(x)^2}. $$

The result being trivial if $\log f(x) > \xi(x)^2$, we may assume $\log f(x) \leq \xi(x)^2$, so that

$$|E(x, f(x)) - C \log_2 x| = |(C - 1) \log_2 f(x)| \leq \frac{\xi(x)}{2} \sqrt{\log_2 x},$$

for $x \geq x_0$. Thus, (5) holds if $f(x) \leq x$. If $f(x) > x$, then $\xi(x)^2 \geq \log f(x) > \log x$, so that $|\omega(n) - C \log_2 x| > \frac{\xi(x)}{2} \sqrt{\log_2 x}$ implies $\omega(n) > \xi(x) > \sqrt{\log x}$. The result now follows from Nicolas’ Theorem [3], an asymptotic estimate for the quantity $|\{n \leq x : \Omega(n) = k\}|$, which easily implies

$$|\{n \leq x : \Omega(n) \geq y \log_2 x\}| \ll \frac{x}{(\log x)^y \log 2 - 1}, \quad (24)$$

uniformly for $x \geq 2$ and $y \geq 2 + \delta$, for any fixed $\delta > 0$.
Next, we show that (5) implies (4). From (24) we have
\[ |\{ n \leq x : \omega(n) > 6 \log_2 x \}| \leq |\{ n \leq x : \Omega(n) > 6 \log_2 x \}| \ll \frac{B(x)}{\log^2 x}. \quad (25) \]
Since \( \omega(n) \leq \Omega(n) \ll \log n \), the contribution to \( \sum_{n \in B(x)} \omega(n) \) from \( n \) with \( \omega(n) > 6 \log_2 x \) is \( \ll B(x)/\log x \), while the contribution from \( n \) with \( \omega(n) \leq 6 \log_2 x \) and \( |\omega(n) - C \log_2 x| \leq \xi(x)/\sqrt{\log x} \) is
\[ \ll (6 \log_2 x) B(x) \frac{\log f(x)}{\xi(x)^2}, \]
by (5). The contribution to \( \sum_{n \in B(x)} \omega(n) \) from \( n \) with \( \omega(n) \leq 6 \log_2 x \) and \( |\omega(n) - C \log_2 x| \leq \xi(x)/\sqrt{\log x} \) is
\[ B(x) \left( 1 + O \left( \frac{\log f(x)}{\xi(x)^2} + \frac{1}{\log^2 x} \right) \right) C \log_2 x \left( 1 + O \left( \frac{\xi(x)}{\sqrt{\log_2 x}} \right) \right). \]
If \( \log f(x) \leq \log_2 x \), (4) now follows with \( \xi(x) = (\log f(x))^{1/3}(\log_2 x)^{1/6} \). If \( \log f(x) > \log_2 x \), (4) follows directly from (25). The argument works the same with \( \Omega(n) \) in place of \( \omega(n) \). \( \square \)

**Proof of Corollary 2.** Assume \( \max(2, n) \leq \theta(n) \ll n(\log 2n)^{o(1)} \). Define \( f(x) = \max_{n \leq x} \theta(n)/n \), so that \( f \) is non-decreasing and \( f(x) = (\log x)^{o(1)} \) as \( x \to \infty \), that is \( \log f(x) = o(\log_2 x) \). The relation (6) follows from (4). Choosing \( \xi(x) = (\log f(x) \log_2 x)^{1/4} \) in (5) yields (7). \( \square \)

**Lemma 2.** Let \( \varepsilon > 0 \). For \( 2 \leq \alpha \leq 4 - \varepsilon \) we have
\[ \sum_{\Omega(n) \geq \alpha \log_2 x} \tau(n) \ll x (\log x)^{\alpha(\log 2 - \log \alpha + 1) - 1}. \]

**Proof.** This is a variation of Exercise 05 in [1]. Write \( y^{\Omega(n)} = \sum_{d|n} f(d) \), so that \( f(n) \) is multiplicative and \( f(p^k) = y^k(1 - 1/y) \) for \( k \geq 1 \), by Möbius inversion. For \( 0 \leq y \leq 2 - \varepsilon \),
\[ \sum_{n \leq x} \tau(n)y^{\Omega(n)} = \sum_{n \leq x} \tau(n) \sum_{d|n} f(d) \leq \sum_{d \leq x} f(d) \tau(d) \sum_{m \leq x/d} \tau(m) \]
\[ \leq x \log x \sum_{d \leq x} f(d) \tau(d)/d \leq x \log x \sum_{p^k \leq x} f(p^k) \tau(p^k)/d \]
\[ = x \log x \prod_{p \leq x} \sum_{k \geq 0} f(p^k) \tau(p^k)/p^k \ll x (\log x)^{2y - 1}. \]
If \( 1 \leq y \leq 2 - \varepsilon \), we get
\[ \sum_{\Omega(n) \geq \alpha \log_2 x} \tau(n) \ll x (\log x)^{2y - 1}. \]
The result now follows with \( y = \alpha/2 \). \( \square \)
Proof of Corollary 3. Since $2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)}$ for all $n \geq 1$, the estimate (9) and the lower bound in (8) follow at once from (7). For the upper bound in (8), we write
\[
\sum_{n \in B(x)} \tau(n) \leq B(x)^{\log_2 x} = B(x)(\log x)^{e \log 2}
\]
and
\[
\sum_{n \in B(x)} \tau(n) \ll x(\log x)^{e \log 2 - 1} \ll B(x)(\log x)^{e \log 2},
\]
by Lemma 2 with $\alpha = e$.

3. Multiples of $q$ in $B$

In this section we develop some general identities for sets $B$, defined by (3), with
\[
\theta : \mathbb{N} \to \mathbb{R} \cup \{\infty\}, \quad \theta(1) \geq 2, \quad \theta(n) \geq P^+(n) \quad (n \geq 2),
\]
where $P^+(n)$ denotes the largest prime factor of $n$. Let
\[
\Phi(x, y) = 1_{x \geq 1} + |\{2 \leq n \leq x : P^-(n) > y\}|,
\]
where $P^-(n)$ denotes the smallest prime factor of $n$. Let
\[
\psi(n) := \begin{cases} 1 & \text{if } n \in B \\ 0 & \text{else.} \end{cases}
\]
and define
\[
\lambda_n(s) := \frac{\psi(n)}{n^s} \prod_{p \leq \theta(n)} \left(1 - \frac{1}{p^s}\right), \quad \lambda_n := \lambda_n(1),
\]
\[
\mu_n(s) := \sum_{p \leq \theta(n)} \frac{\log p}{p^s - 1} - \log n, \quad \mu_n := \mu_n(1).
\]

Lemma 3. Let $\theta$ satisfy (26) and let $q_1 \leq q_2 \leq \ldots \leq q_k$ be primes. For $x \geq 0$,
\[
\sum_{n \geq 1} \psi(n) \Phi \left( \frac{x}{n}, \theta(n) \right) = \lfloor x \rfloor \quad (27)
\]
and
\[
\sum_{n \geq 1} \psi(n) \Phi \left( \frac{x}{n}, \theta(n) \right) = \sum_{\theta(n) \geq q_k} \psi(n) \Phi \left( \frac{x}{nq_k}, \theta(n) \right). \quad (28)
\]

Proof. The relation (27) is [15, Lemma 3]. We will show (28). Every $m \in q_1 \cdots q_k \mathbb{N}$ factors uniquely as $m = nr$ where $n \in B$ and $P^-(r) > \theta(n)$ if $r > 1$. If $q_1 \nmid n$ then $\theta(n) < q_1$. If $q_1 | n$, let $j$ be the largest index such that
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\( q_1 \cdots q_j \mid n \), so that \( q_{j+1} > \theta(n) \) if \( j < k \). We count all integer multiples of \( q_1 \cdots q_k \) up to \( x \) according to \( j \) and \( n \):

\[
\left\lfloor \frac{x}{q_1 \cdots q_k} \right\rfloor = \sum_{\theta(n) < q_1} \psi(n) \Phi \left( \frac{x}{n q_1 \cdots q_k}, \theta(n) \right) + \sum_{j=1}^{k-1} \sum_{\theta(n) < q_{j+1}} \psi(n) \Phi \left( \frac{x}{n q_{j+1} \cdots q_k}, \theta(n) \right) + \sum_{q_1 \cdots q_k \mid n} \psi(n) \Phi \left( \frac{x}{n}, \theta(n) \right).
\]

(29)

We can now establish (28) by induction on \( k \). When \( k = 1 \), (29) and (27) yield (28). For the inductive step, we write the inner sum of (29) as

\[
\sum_{\theta(n) < q_j + 1} \psi(n) \Phi \left( \frac{x}{n q_{j+1} \cdots q_k}, \theta(n) \right) = \sum_{\theta(n) \geq q_j + 1} \psi(n) \Phi \left( \frac{x}{n q_{j+1} \cdots q_k}, \theta(n) \right) - \sum_{\theta(n) \geq q_j + 1} \psi(n) \Phi \left( \frac{x}{n q_{j+1} \cdots q_k}, \theta(n) \right).
\]

Thus, the sum over \( j \) in (29) is telescoping and the result follows from (27).

\[ \square \]

**Lemma 4.** Let \( \theta \) satisfy (26) and let \( q_1 \leq q_2 \leq \ldots \leq q_k \) be primes. For \( \Re(s) > 1 \) we have

\[
\sum_{n \geq 1} \lambda_n(s) = 1
\]

(30)

and

\[
\sum_{n \geq 1} \lambda_n(s) = \frac{1}{q_k} \sum_{\theta(n) \geq q_k} \lambda_n(s).
\]

(31)

Both relations hold at \( s = 1 \) if \( B(x) = o(x) \).

**Proof.** The relation (30) is [16, Lemma 1] when \( \Re(s) > 1 \) and [15, Theorem 1] when \( s = 1 \). The proof of (31) mirrors that of (28). We first assume \( \Re(s) > 1 \). Every \( m \in q_1 \cdots q_k \mathbb{N} \) factors uniquely as \( m = nr \) where \( n \in \mathcal{B} \) and \( P^-(r) > \theta(n) \) if \( r > 1 \). If \( q_1 \nmid n \) then \( \theta(n) < q_1 \). If \( q_1 \mid n \), let \( j \) be the largest index such that \( q_1 \cdots q_j \mid n \), so that \( q_{j+1} > \theta(n) \) if \( j < k \). We rearrange
the terms of the Dirichlet series \( \sum_{q_1 \cdots q_k | m} m^{-s} \) according to \( n \) and \( j \). After dividing by \( \zeta(s) \), this shows that, for \( \text{Re}(s) > 1 \),

\[
\frac{1}{(q_1 \cdots q_k)^s} = \sum_{\theta(n) < q_1} \frac{\lambda_n(s)}{(q_1 \cdots q_k)^s} + \sum_{j=1}^{k-1} \sum_{\theta(n) < q_{j+1}} \frac{\lambda_n(s)}{(q_{j+1} \cdots q_k)^s} + \sum_{q_1 \cdots q_k | n} \lambda_n(s).
\]

We establish (31) by induction on \( k \). When \( k = 1 \), the result follows from applying (30) to the first sum of (32). For the inductive step, note that the inner sum in (32) is

\[
\sum_{\theta(n) < q_{j+1}} \frac{\lambda_n(s)}{(q_{j+1} \cdots q_k)^s} = \sum_{q_1 \cdots q_j | n} \frac{\lambda_n(s)}{(q_j \cdots q_k)^s} - \sum_{\theta(n) \geq q_{j+1}} \frac{\lambda_n(s)}{(q_{j+1} \cdots q_k)^s},
\]

by the inductive hypothesis. Thus, the sum over \( j \) in (32) is a telescoping sum and the result follows from (30).

If \( B(x) = o(x) \), the validity of (31) at \( s = 1 \) follows from (28), in much the same way that the validity of (30) at \( s = 1 \) follows from (27), which is demonstrated in the proof of [15, Thm. 1].

**Lemma 5.** Let \( \theta \) satisfy (26) and let \( q_1 \leq q_2 \leq \ldots \leq q_k \) be primes. For \( \text{Re}(s) > 1 \) we have

\[
\sum_{n \geq 1} \lambda_n(s) \mu_n(s) = 0 \tag{33}
\]

and

\[
\sum_{n \geq 1 \atop q_1 \cdots q_k | n} \lambda_n(s) \mu_n(s) = \frac{1}{q_k^s} \sum_{\theta(n) \geq q_k} \lambda_n(s) (\mu_n(s) - \log q_k). \tag{34}
\]

**Proof.** Differentiate (30) and (31) with respect to \( s \). □

4. **Proof of Theorem 2**

The following estimate for \( \Phi(x, y) \), which differs from the one we used in [14], simplifies the proof of Theorem 2.

**Lemma 6.** Uniformly, for \( x \geq 0 \), \( y \geq 2 \), we have

\[
\Phi(x, y) = 1_{x \geq 1} + x \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) + \frac{x}{\log y} \left\{ w(u) - e^{-\gamma} - \frac{u}{x} \right\} + O \left( \frac{e^{-u/3}}{\log y} \right)
\]

\[
= 1_{x \geq 1} + x \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) + O \left( \frac{xe^{-u/3}}{\log y} \right),
\]

where \( u = \frac{\log \max(1, x)}{\log y} \) and \( w(u) \) is Buchstab’s function.
Lemma 8. Assume $B(x) = B_t(x)$ is the counting function of a set $B_t \subset \mathbb{N}$ that depends on the parameter $t$. Assume

$$B(x) = x \int_1^\infty \frac{B(y)}{y^2 \log yt} \left( e^{-y} - w \left( \frac{\log x/y}{\log yt} \right) \right) dy + R(x) \quad (x \geq 1),$$

such that the integrals

$$\alpha_t := e^{-\gamma} \int_1^\infty \frac{B(y)}{y^2 \log yt} dy, \quad \beta_t := -\frac{1}{\log t} \int_1^\infty \frac{R(y)}{y^2} dy$$

converge. Then

$$B(x) = x \eta_d(v) + O \left\{ 1 + x \beta_t (v + 1)^{-3.03} + R(x) + I(x) + J(x) \right\},$$

where $v = \log x / \log t$, $d(v)$ is given by (11), $\eta_t = \alpha_t + \beta_t$,

$$I(x) = \frac{x}{\log xt} \int_x^\infty R(y) \frac{dy}{y^2}, \quad J(x) = \frac{x}{(\log xt)^{3.03}} \int_1^x R(y) (\log yt)^{2.03} \frac{dy}{y^2}.$$  

Proof. We follow the second half of the proof of [14, Thm. 1.3]. The only modification needed is the use of the improved estimates

$$(v + 1)d(v) = C + O((v + 1)^{-2.03}), \quad (v \geq 0), \quad (35)$$

and

$$(v + 1)^2 d'(v) = -C + O((v + 1)^{-2.03}), \quad (v \geq 0). \quad (36)$$

The estimate (35) is a consequence of [13, Cor. 6], while (36) follows from inserting (35) in the proof of [13, Cor. 5]. In [14], we used slightly weaker estimates for simplicity, with an exponent of $-2$ instead of $-2.03$ in the error terms. In the proof of Theorem 2, the improved exponent will save a factor of $\log_2 x$ (when estimating the contribution from $R_2(x)$ to $J(x)$). $\square$

For $n \in \mathbb{N}$ with prime factorization $n = p_1 \cdots p_k$, where $p_1 \leq \ldots \leq p_k$, define

$$F(n) := \max_{1 \leq j \leq k} p_j^2 p_{j+1} \cdots p_k.$$  

Lemma 7. Assume $B(x) = B_t(x)$ is the counting function of a set $B_t \subset \mathbb{N}$ that depends on the parameter $t$. Assume
Proof. Note that \( n \in D_m(x, t) \) if and only if \( n \leq x \), \( m | n \) and \( F(n) \leq nt \).
Also, \( m | n \) implies \( F(m) \leq F(n) \). Thus, if \( D_m(x, t) \neq 0 \) and \( n \in D_m(x, t) \),
then \( m \leq n \leq x \) and \( F(m) \leq F(n) \leq nt \leq xt \), so \( x \geq \max(m, F(m)/t) \). \( \Box \)

**Lemma 9.** Let \( n \geq 2 \) with prime factorization \( n = p_1 \cdots p_k \), \( p_1 \leq \ldots \leq p_k \).
If \( F(n) \leq x \), then \( p_{k-j+1} \cdots p_k \leq x^{1-2^{-j}} \) for \( 1 \leq j \leq k \).

**Proof.** We use induction on \( j \). When \( j = 1 \), the claim is that \( p_k \leq x^{1/2} \),
which follows from \( p_k^2 \leq F(n) \leq x \), for all \( k \geq 1 \). Assume now that the
claim is correct for some \( j \in \mathbb{N} \) and all \( k \geq j \). Let \( k \geq j + 1 \). We have
\[
p_k \cdots p_{k-j+1} \cdots p_k \leq F(n) \leq x.
\]
Thus, if \( p_{k-j}p_{k-j+1} \cdots p_k \geq x^{1-2^{-j-1}} \), then \( p_{k-j} \leq x^{2^{-j-1}} \).
By the inductive hypothesis,
\[
p_{k-j}(p_{k-j+1} \cdots p_k) \leq x^{2^{-j-1}}x^{1-2^{-j}} = x^{1-2^{-j-1}},
\]
for all \( k \geq j + 1 \). \( \Box \)

**Lemma 10.** Let \( k \geq 0 \) be fixed. For \( m \geq 1 \) with \( \Omega(m) = k \), \( t \geq 2 \) and
\( x \geq \max(m, F(m)/t) \), we have
\[
\frac{x \log t \log 2m}{m \log xt \log 2x} \gg_k 1.
\]

**Proof.** This is obvious if \( m = 1 \) or if \( t \geq x^{2-k} \). If \( m \geq 2 \) and \( t < x^{2-k} \), then
\( F(m) \leq xt \) and Lemma 9 imply \( m \leq (xt)^{1-2^{-k}} < x^{1-4^{-k}} \ll_k x/(\log xt)^2 \),
from which the claim follows. \( \Box \)

**Proof of Theorem 2.** In the remainder of this paper, we write \( D_q(x) \) for
\( D_q(x, t) \) and \( D(x) \) for \( D(x, t) \). We will show by induction on \( k \geq 0 \) that, for
\( k = \Omega(q) \), the estimates (12), (13) and (14) hold and that, for primes \( r \) with
\( r \geq P^+ \), we have
\[
\sum_{\substack{n \in \mathcal{D}_q \\text{ s.t. } n \geq r/2 \atop n \geq r/t}} \Phi(x/n, nt) - \frac{x}{r} \sum_{\substack{n \in \mathcal{D}_q \\text{ s.t. } n \geq r/2 \atop n \geq r/t}} \lambda_n \ll 1 + \frac{x \log t \log qr \log qt}{qr \log^3 xt} =: R_2(x, qr),
\]
for \( x \geq \max(qr, F(qr)/t) \). Note that (13) and (35) imply
\[
D_q(x) \ll_k x d(v) \eta_{q,t} \geq_k \frac{x \log t}{q \log xt}, \quad (x \geq 1, q \geq 1, t \geq 2),
\]
since \( D_q(x) = 0 \) if \( q > x \).

When \( k = 0 \), \( q = 1 \), (12) is [14, Eq. (13)], (13) is [14, Thm. 1.3] and (14)
is [14, Eq. (6)]. To show (37) for \( k = 0 \) and \( q = 1 \), assume that \( r \) is prime
and \( x \geq \max(r, r^2/t) \). Equations (27) and (30) show that

\[
\sum_{n \in \mathcal{D}_{n \geq r/t}} \Phi(x/rn, nt) - \frac{x}{r} \sum_{n \in \mathcal{D}_{n \geq r/t}} \lambda_n
\]

\[
= -\{x/r\} - \sum_{n \in \mathcal{D}_{n < r/t}} \Phi(x/rn, nt) + \frac{x}{r} \sum_{n \in \mathcal{D}_{n < r/t}} \lambda_n
\]

\[
\ll 1 + D(r/t) + \sum_{n \in \mathcal{D}_{n < r/t}} \frac{x}{rn \log nt} \exp \left( -\frac{\log xt/r}{3 \log nt} \right),
\]

by the second estimate in Lemma 6. We have \( D(r/t) \ll r \log t/(t \log r) \) by Lemma 1, so \( D(r/t) \ll R_2(x, r) \) follows from

\[
\frac{r}{\log^2 r \log rt} \ll \frac{xt/r}{\log^3 xt} \times \frac{xt/r}{\log^3 (xt/r)}
\]

since \( xt/r \geq \sqrt{xt} \). This holds because \( r \leq xt/r \). Finally, the last sum in (39) is \( \ll R_2(x, r) \) by Lemma 1. Thus, (37) holds for \( k = 0 \).

For the inductive step, assume that (13) (and hence (38)) and (14) hold for \( q \in \mathbb{N} \) with \( \Omega(q) = k \) for some \( k \geq 0 \). If \( k \geq 1 \), assume that (37) holds for \( \Omega(q) = k - 1 \). Let \( r \) be a prime with \( r \geq P^+(q) \) and write \( m = qr \). We note that in the remainder of this proof, all implied constants in the \( \ll \) and big-O notation may depend on \( k \).

We estimate the first sum in (28) with Lemma 6 and apply (31) to get

\[
D_m(x) + \sum_{n \in \mathcal{D}_m} \frac{x}{n \log nt} \left\{ w \left( \frac{\log x/n}{\log nt} \right) - e^{-\gamma} - \frac{n^2 t}{x} \right|_{n^2 \leq \frac{x}{t}} + O \left( \frac{e^{-\log x/n}}{\log nt} \right) \right\}
\]

\[
= \sum_{n \in \mathcal{D}_q} \Phi(x/rn, nt) - \frac{x}{r} \sum_{n \in \mathcal{D}_q} \lambda_n.
\]

(40)

The contribution from the last two terms in the first sum in (40) is

\[
\ll \tilde{R}_1(x) := \frac{x \log mt}{m (\log xt)^2},
\]

by Lemma 1. In the second application of this argument we will be able to replace \( \tilde{R}_1(x) \) by the smaller

\[
R_1(x) := \frac{x \log t}{m (\log xt)^2}.
\]

The error from applying Abel summation to the remaining terms of the first sum in (40) is also \( \ll \tilde{R}_1(x) \), since \( w(u) - e^{-\gamma} \ll e^{-u} \) and \( w'(u) \ll e^{-u} \).
Thus,

\[ D_m(x) = x \int_1^\infty \frac{D_m(y)}{y^2 \log yt} \left( e^{-\gamma} - w \left( \frac{\log x/y}{\log yt} \right) \right) dy + O(R_1(x)) \]

\[ + \sum_{n \in \mathcal{D}_q \atop n \geq r/t} \Phi(x/ri, ni) - \frac{x}{r} \sum_{n \in \mathcal{D}_q \atop n \geq r/t} \lambda_n. \]  

(41)

If \( x/r < \max(q, F(q)/t, r/t) \), that is \( x < \max(m, F(m)/t) \), then Lemma 8 shows that the first sum in (41) vanishes, while the second sum is \( \ll \frac{x}{m} \min\left( 1, \frac{\log t}{\log m} \right) \), by the inductive hypothesis (38) and since \( \log r \leq \log m \leq \log r^{k+1} \ll_k \log r \).

Define

\[ R_2(x) := \begin{cases} \frac{x}{m} \min\left( 1, \frac{\log t}{\log m} \right) & \text{if } x < \max(m, F(m)/t) \\ 1 + \frac{x \log t \log m \log nt}{m \log^3 xt} & \text{if } x \geq \max(m, F(m)/t). \end{cases} \]

Thus,

\[ D_m(x) = x \int_1^\infty \frac{D_m(y)}{y^2 \log yt} \left( e^{-\gamma} - w \left( \frac{\log x/y}{\log yt} \right) \right) dy + R(x) \]  

holds with \( R(x) \ll \tilde{R}_1(x) + R_2(x) \) when \( x < \max(m, F(m)/t) \).

Assume now that \( x \geq \max(m, F(m)/t) \). If \( k = 0 \) and \( q = 1 \), we have already shown that the last row of (41) is \( \ll R_2(x) \).

If \( k \geq 1 \), write \( q = uv \) where \( v = P^+(q) \). Equations (28) and (31) show that the last row of (41) equals

\[ S - T = (U - V), \]

say. Now \( x \geq \max(m, F(m)/t) \) implies \( x/r \geq \max(q, F(q)/t, r/t) \). The inductive hypothesis (37) yields

\[ S - T \ll 1 + \frac{(x/r) \log t \log q \log qt}{q \log^3(x/rt)} \ll 1 + \frac{x \log t \log m \log mt}{m \log^3 xt} = R_2(x), \]

since \( xt/r \geq \sqrt{xt} \). The second estimate in Lemma 6 and (38) show that

\[ U - V \ll D_q(r/t) + \sum_{n \in \mathcal{D}_q \atop n < r/t} \frac{x}{nr \log nt} \exp\left( -\frac{\log nt}{3 \log nt} \right) \]

\[ \ll \frac{r^2 \log t}{mt \log r} + R_2(x) \ll R_2(x), \]
where the last assertion is implied by

\[
\frac{r}{\log r \log m \log mt} \ll \frac{xt/r}{\log^3(xt/r)^1},
\]

which holds because \(mt > m \geq r\) and \(r \leq xt/r\). As the last row of (41) is \(\ll R_2(x)\), we have established that (37) holds for \(\Omega(q) \leq k\) and that (42) holds with \(R(x) \ll \tilde{R}_1(x) + R_2(x)\).

We need to estimate \(I(x)\) and \(J(x)\) from Lemma 7. For this purpose we may assume that \(x \geq \max(m, F(m)/t)\). If not, Lemma 8 shows that \(D_m(x) = 0\), so that (12) holds since the main term in (12) is absorbed by the error terms. We find that \(I(x) + J(x) \ll \tilde{R}_1(x) + R_2(x)\) and \(\beta_t \ll \frac{\log m}{m \log \Gamma}\) so that \(x \beta_t(v + 1)^{-3} \ll R_2(x)\).

The conclusion of Lemma 7 is that

\[
D_m(x) = x \eta_{m,t} d(v) + O(\tilde{R}_1(x) + R_2(x)). \tag{43}
\]

The lower bound in Lemma 1 yields \(\eta_{m,t} \gg m^{-1}\). Lemma 11 shows that \(\eta_{m,t} \ll k m^{-1}\). Together with (43), Lemmas 8 and 10, this implies

\[
D_m(x) \ll k \frac{x \log t}{m \log xt}, \quad (x \geq 1, t \geq 2, \Omega(m) \leq k + 1). \tag{44}
\]

Running through this proof a second time with this upper bound replacing the one in Lemma 1, we can replace \(\tilde{R}_1(x)\) by \(R_1(x)\) to obtain

\[
D_m(x) = x \eta_{m,t} d(v) + O(R_1(x) + R_2(x)),
\]

where \(1/m \ll \eta_{m,t} \ll k 1/m\). This shows that (12) and (14) hold with \(q\) replaced by \(m = qr\).

To see that (12) implies (13), note that the term \(O(1)\) is acceptable by Lemma 10, provided \(x \geq \max(m, F(m)/t)\). If \(x < \max(m, F(m)/t)\), then \(D_m(x) = 0\) and \(x < F(m) \leq m^2\), so (13) (with \(q\) replaced by \(m\)) follows from \(\log x \leq 2 \log m\). This completes the proof of Theorem 2. \(\square\)

**Lemma 11.** Assume that (43) holds for \(m\) with \(\Omega(m) \leq k + 1\), and that \(\eta_{q,t} \ll q^{-1\gamma} \Omega(q) \leq k\). Then \(\eta_{m,t} \ll k m^{-1}\) for \(\Omega(m) = k + 1\).

**Proof.** Let \(m = qr\) where \(r\) is prime and \(r \geq P^+(q)\). Equation (34) yields

\[
\sum_{n \geq 1 \atop qr|m} \lambda_n(s)\mu_n(s) = \frac{1}{rs} \sum_{n \geq r/t \atop q|r/n} \lambda_n(s)(\mu_n(s) - \log r).
\]

As in [16], we let \(s = 1 + 1/\log^2 N\) and split the sum on the left according to \(n \leq N\) and \(n > N\). As \(N \to \infty\), the contribution from \(n \leq N\) converges to (see [16, Lemma 3]) \(\sum_{qr|n} \lambda_n\mu_n\), while the contribution from \(n > N\) converges to (see [16, Lemma 4]) \(-c_{qr}(1 - e^{-\gamma})\), by (43), where \(\eta_{qr,t} = Cc_{qr} \log t\). Applying the same reasoning to the sum on the right-hand side,
we get
\[
\sum_{n \geq 1 \atop q \nmid n} \lambda_n \mu_n - c_q (1 - e^{-\gamma}) = \frac{1}{r} \left( \sum_{n \geq r/t \atop q \nmid n} \lambda_n (\mu_n - \log r) - c_q (1 - e^{-\gamma}) \right).
\]

Since \( \mu_n \ll \log t \), we obtain
\[
rc_q \ll S \log t + S \log r + c_q, \quad S := r \sum_{n \geq 1 \atop q \nmid n} \lambda_n = \sum_{n \geq r/t \atop q \nmid n} \lambda_n,
\]
by (31). Now \( \eta_{q,t} \ll q^{-1} \) and (44) holds with \( q \) in place of \( m \). Thus, \( c_q \ll q^{-1} \log t \) and \( S = \sum_{n \geq r/t \atop q \nmid n} \lambda_n \ll q^{-1} \min(1, \log t/\log r) \). This shows that \( rc_q \ll q^{-1} \log t \), which is the desired result.

\[\square\]

5. PROOF OF THEOREM 3

Proof of (18), (19) and (20). The estimate (18) is [16, Cor. 3]. Equation (34), with \( k = 1 \) and \( q_1 = q \) prime, is
\[
\sum_{n \geq 1 \atop q \nmid n} \lambda_n(s) \mu_n(s) = \frac{1}{q^s} \sum_{n \geq q/t} \lambda_n(s) (\mu_n(s) - \log q).
\]
(45)

As in [16], we let \( s = 1 + 1/\log^2 N \) and split the sum on the left according to \( n \leq N \) and \( n > N \). As \( N \to \infty \), the contribution from \( n \leq N \) converges to (see [16, Lemma 3]) \( \sum_{n \geq 1 \atop q \nmid n} \lambda_n \mu_n \), while the contribution from \( n > N \) converges to (see [16, Lemma 4]) \( -c_q (1 - e^{-\gamma}) \), by Corollary 6. Applying the same reasoning to the right-hand side of (45), we get
\[
\sum_{n \geq 1 \atop q \nmid n} \lambda_n \mu_n - c_q (1 - e^{-\gamma}) = \frac{1}{q} \left( \sum_{n \geq q/t} \lambda_n (\mu_n - \log q) - c_q (1 - e^{-\gamma}) \right).
\]

With the estimate [16, Lemma 13]
\[
\mu_n = \log t - \gamma + O \left( e^{-\sqrt{\log n}} \right),
\]
(46)
we obtain
\[
qc_q (1 - e^{-\gamma}) = S \log q + c_q (1 - e^{-\gamma}) + O \left( S e^{-\sqrt{\log \max(q,t)}} \right),
\]
(47)
where
\[
S := \sum_{n \geq q/t} \lambda_n = q \sum_{n \geq 1 \atop q \nmid n} \lambda_n,
\]
by (31). If \( q \leq t \), then \( S = 1 \) by (30), so that (20) follows from (47).

In general, we have
\[
S = \frac{e^{-\gamma} c_q}{\log q} \left( 1 + O \left( \frac{1}{\log q} + \frac{\log^2 t}{\log^2 q} \right) \right), \quad (q \geq 2, t \geq 2).
\]

(48)
If \( q \leq t \), this holds because \( S = 1 \) and \( c_q \asymp \log t \). If \( q \geq t \), (48) follows from applying Abel summation to \( S = \sum_{n \geq q/t} \lambda_n \), together with Mertens’ formula and Corollary 6 (with \( q = 1 \)). Combining (48) with (47) proves (19).

Proof of (21), (22) and (23). Equation (34), with \( k = 2 \) and \( q_1 = p \leq q_2 = q \) is

\[
\sum_{n \geq q/t}^{n \geq 1} \lambda_n(s) \mu_n(s) = \frac{1}{q^s} \sum_{n \geq q/t}^{n \geq 1} \lambda_n(s)(\mu_n(s) - \log q).
\] (49)

As in [16], we let \( s = 1 + 1/\log^2 N \) and split the sum on the left according to \( n \leq N \) and \( n > N \). As \( N \to \infty \), the contribution from \( n \leq N \) converges to (see [16, Lemma 3]) \( \sum_{n \geq 1} \lambda_n \mu_n \), while the contribution from \( n > N \) converges to (see [16, Lemma 4]) \( -c_{pq}(1 - e^{-\gamma}) \), by Corollary 6. Applying the same reasoning to the right-hand side of (49), we get

\[
\sum_{n \geq q/t}^{n \geq 1} \lambda_n \mu_n - c_{pq}(1 - e^{-\gamma}) = \frac{1}{q} \left( \sum_{n \geq q/t}^{n \geq 1} \lambda_n(\mu_n - \log q) - c_p(1 - e^{-\gamma}) \right).
\]

We estimate \( \mu_n \) with (46) to obtain

\[
qc_{pq}(1 - e^{-\gamma}) = T \log q + c_p(1 - e^{-\gamma}) + O \left( \frac{T}{\log \max(pt,q)} \right),
\] (50)

where

\[
T := \sum_{n \geq q/t}^{n \geq 1} \lambda_n = q \sum_{n \geq 1}^{n \geq q/t} \lambda_n,
\]

by (31).

If \( p \leq q \leq t \), then

\[
T = \sum_{n \geq q/t}^{n \geq 1} \lambda_n = \frac{1}{p} \sum_{n \geq p/t}^{n \geq 1} \lambda_n = \frac{1}{p} \sum_{n \geq 1}^{n \geq q/t} \lambda_n = \frac{1}{p},
\]

by (31) and (30). The estimate (23) now follows from (50) with \( T = 1/p \) and \( c_p \) estimated by (20).

In general, we have

\[
T = \frac{e^{-\gamma}c_p}{\log q} \left( 1 + O \left( \frac{1}{\log q} + \frac{\log^2 pt}{\log^2 q} \right) \right), \quad (q \geq p \geq 2, t \geq 2).
\] (51)

If \( q < pt \), this is implied by \( T \ll c_p/\log q \), which follows from (17) (with \( q \) replaced by \( p \)). If \( q \geq pt \), we estimate \( \lambda_n \) with Mertens’ formula and use Abel summation and the estimate (15). The contribution from the first two error terms in (15) is clearly acceptable, while the term \( O(1) \) contributes

\[
\ll \int_{q/t}^{\infty} dy y^{-2} \log yt \leq \frac{t}{q \log q} \cdot \frac{c_p}{\log q} \cdot \frac{pt}{q \log t} \ll \frac{c_p}{\log q} \cdot \frac{\log^2 pt \log^2 q}{\log^2 q}.
\]

Now substitute (51) into (50) and estimate \( c_p \) with (19) to get (21), and with (20) to get (22). \( \square \)
6. Proof of Theorem 1

Proof of (1). We have
\[
\sum_{n \in \mathcal{D}(x)} \omega(n) = \sum_{p \leq x} D_p(x) + xD(v) \sum_{p \leq x} \eta_{p,t} \left( 1 + O \left( \frac{\log p}{\log x} \right) \right),
\]
by (13). The contribution from the error term is \(\ll xD(v) \log x\), since \(\eta_{p,t} \ll 1/p\) by (14). Now \(\eta_{p,t} \log t = c_p\) and \(\eta_{1,t} \log t = c_0\), by Corollary 6. With (18), (19) and (20), we find that
\[
\sum_{p \leq x} \eta_{p,t} = \frac{1}{C \log t} \sum_{p \leq x} c_p = \frac{c_0}{C \log t} (E(x, t) + O(1)) = \eta_{1,t} (E(x, t) + O(1)).
\]
The result now follows from (13) with \(q = 1\), that is \(D(x) = xD(v) \eta_{1,t} (1 + O(1/\log x))\).

To see that (1) remains valid when \(\omega\) is replaced by \(\Omega\), note that
\[
\sum_{n \in \mathcal{D}(x)} (\Omega(n) - \omega(n)) = \sum_{n \in \mathcal{D}(x)} \sum_{k \geq 2} \sum_{p \mid n} 1 = \sum_{p \leq x} D_p^k(x) \ll x \log \frac{t}{\log t} \ll D(x),
\]
by Lemma 1.

Proof of (2). We have
\[
\sum_{n \in \mathcal{D}(x)} \omega(n)^2 = \sum_{p, q \leq x} D_{pq}(x) + O \left( \sum_{p \leq x} D_p(x) \right),
\]
where \(p, q\) run over primes. The last term is \(\ll D(x) \log x\), by (1). Thus, (13) yields
\[
\sum_{n \in \mathcal{D}(x)} \omega(n)^2 = O(D(x) \log x) + xD(v) \sum_{p, q \leq x} \eta_{pq,t} \left( 1 + O \left( \frac{\log pq}{\log x} \right) \right).
\]
Since \(\eta_{pq,t} \ll 1/pq\), the contribution from the error term is \(\ll xD(v) \log x \ll D(x) \log x\). With (21), (22) and (23), we find that
\[
\sum_{p, q \leq x} \eta_{pq,t} = \frac{1}{C \log t} \sum_{p, q \leq x} c_{pq} = \frac{c_0}{C \log t} (E(x, t) + O(1)) \eta_{1,t} (E(x, t) + O(2)) \eta_{1,t} (E(x, t)^2 + O(\log_2 x)).
\]
Combining this with (1) and (13) (with \(q = 1\)), we get
\[
\sum_{n \in \mathcal{D}(x)} (\omega(n) - E(x, t))^2 \ll D(x) \log x,
\]
which implies (2). This estimate remains valid if \(\omega(n)\) is replaced by \(\Omega(n)\), since
\[
\sum_{n \in \mathcal{D}(x)} (\Omega(n) - \omega(n))^2 \leq \sum_{p \neq q \leq x} \sum_{k \geq 2} D_{pq^k}(x) + \sum_{p \leq x} \sum_{k \geq 2} 2k D_{p^k}(x) \ll D(x),
\]
by Lemma 1. \qed
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