Inter-Condensate Tunneling in Bose-Einstein Condensates with Feshbach Resonances

Eddy Timmermans\textsuperscript{1}, Paolo Tommasini\textsuperscript{1}, Robin Côté\textsuperscript{1}, Mahir Hussein\textsuperscript{2,1}

and Arthur Kerman\textsuperscript{3}

\textsuperscript{1}Institute for Atomic and Molecular Physics
Harvard-Smithsonian Center for Astrophysics 60 Garden Street, Cambridge, MA 02138

\textsuperscript{2} Instituto de Física, Universidade de São Paulo, C.P. 66318, CEP 05315-970 São Paulo, Brazil

\textsuperscript{3} Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology
Cambridge, MA 02139

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Abstract

Recently, effects of Feshbach resonances in atom-atom interactions were observed by varying the external magnetic field of an atomic Bose-Einstein condensate (BEC). We point out that the quasi-bound molecules created in the intermediate state of the resonance can form a second, molecular condensate. The many-body state of the system is then a hybrid atomic/molecular condensate with inter-condensate tunneling of atom pairs. A sudden variation of the magnetic field results in oscillations of the number of atoms and molecules in their respective condensates, providing a signature of this novel
type of quantum tunneling.

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Recently, Ketterle’s group reported evidence for magnetic-field-induced Feshbach resonances \[1\] in the interparticle interactions of atomic-trap Bose-Einstein condensates \[2\]-\[4\]. Near resonance, the two-body interactions are characterized by an effective interaction strength that can be tuned by varying the magnetic field, suggesting the construction of condensates with tunable interaction strength \[3\]. We present a theoretical interpretation of the condensate Feshbach resonances that predicts the formation of a molecular condensate that is coupled to the atomic condensate by quantum tunneling of atom pairs.

**I. THE BINARY COLLISION RESONANCE**

In an external magnetic field, the hyperfine induced spin flip of one atom in the presence of a second atom can bind the interacting atoms by bringing them to a spin configuration of higher energy. A resonance occurs if the potential of the bound channel supports an eigenstate that is nearly degenerate with the incident atoms \[6\]-\[10\].

The detuning of the resonance, $\epsilon$, is the energy difference of the quasi-bound molecular state and the continuum of the incident atoms, pictured in Fig.(1). The energy difference $\Delta$ between the continua of the initial and quasi-bound channels, is the energy gain of a single spin flipped atom in the magnetic field $B$. When $B$ takes on its resonant value, $B_0$, the energy of the quasi-bound molecule lines up with the continuum of the incident atoms so that $\Delta$ equals the binding energy of the bound state, $\Delta = E_b$. Near resonance, $\Delta \approx E_b + (\partial \Delta / \partial B) \times [B - B_0]$, and one controls the detuning by varying the magnetic field, $\epsilon = \Delta - E_b \approx (\partial \Delta / \partial B) \times [B - B_0]$.

In the many-body Hamiltonian, the spin flips to quasi-bound molecules are described by

$$ H_{FR} = \alpha \int d^3r \ \hat{\psi}_m^\dagger(r) \hat{\psi}_a(r) \hat{\psi}_a(r) + h.c. , $$

(1)

where $\hat{\psi}_m, \hat{\psi}_m^\dagger (\hat{\psi}_a, \hat{\psi}_a^\dagger)$ are the annihilation and creation field operators of the molecules (atoms). The $\alpha$-parameter in Eq.(1) is the transition matrix element proportional to the overlap of the molecular continuum and bound state wave functions.
The $H_{FR}$ interaction is responsible for the Feshbach resonance and gives a dispersive contribution to the effective interaction strength. To find the off-resonant detuning dependence we determine the energy shift of two atoms confined to a volume $\Omega$. If the initial state $|\text{ini}\rangle$ of energy $E_{\text{ini}}$ has both atoms in the same momentum state, then the interatomic interaction, $[\lambda_a/2 \int d^3r \hat{\psi}_a^\dagger(r) \hat{\psi}_a(r) \hat{\psi}_a(r) \hat{\psi}_a(r)]$, contributes a shift $\Delta E_a = \lambda_a/\Omega$ in first order perturbation. If we denote the quasi-bound state of energy $E_{\text{int}}$ by $|\text{int}\rangle$, then the second order shift $\Delta E_{FR}$ due to $H_{FR}$ is given by $\Delta E_{FR} = |\langle \text{int} | H_{FR} | \text{ini} \rangle|^2 / [E_{\text{ini}} - E_{\text{int}}] = -[2\alpha^2/\epsilon]/\Omega$. Thus, $\Delta E = \Delta E_a + \Delta E_{FR} = \lambda_{\text{eff}}/\Omega$, where the effective interaction strength is equal to $\lambda_{\text{eff}} = \lambda_a - 2\alpha^2/\epsilon$.

II. FESHBACH-RESONANCE INTERACTIONS IN THE CONDENSATE

From momentum conservation, it follows that all quasi-bound molecules created from condensate atoms of vanishing momentum also occupy the same zero-momentum center-of-mass state, giving a first indication that the quasi-bound molecules form a condensate. A more insightful argument follows from the Heisenberg equations $i\hbar \dot{\hat{\psi}}_a = [\hat{\psi}_a, \hat{H}]$, $i\hbar \dot{\hat{\psi}}_m = [\hat{\psi}_m, \hat{H}]$, where $\hat{H}$ is the Hamiltonian. The expectation value of these operator equations gives equations of motion for the condensate fields, $\phi_m = \langle \hat{\psi}_m \rangle$ and $\phi_a = \langle \hat{\psi}_a \rangle$. In the mean-field approximation (e.g. $\langle \hat{\psi}_a(r) \hat{\psi}_a(r) \rangle \approx \phi_a^2(r)$), this procedure yields two coupled equations in place of the usual single condensate Gross-Pitaevskii equation [12]- [13]:

\[
\begin{align*}
    i\hbar \dot{\phi}_m &= \left[ -\frac{\hbar^2 \nabla^2}{4M} + \epsilon + \lambda_m n_m + \lambda n_a \right] \phi_m + \alpha \phi_a^2, \\
    i\hbar \dot{\phi}_a &= \left[ -\frac{\hbar^2 \nabla^2}{2M} + \lambda_a n_a + \lambda n_m \right] \phi_a + 2\alpha \phi_a^* \phi_m, 
\end{align*}
\]

(2)

where $M$ denotes the mass of a single atom, $n_m$ and $n_a$ represent the condensate densities, $n_m = |\phi_m|^2$ and $n_a = |\phi_a|^2$, and $\lambda_m$, $\lambda_a$ and $\lambda$ represent the strength of the molecule-molecule, atom-atom and molecule-atom interactions. The $\alpha$-terms that couple the equations describe tunneling of pairs of atoms between the $\phi_m$ and $\phi_a$-fields. In particular, $\phi_m$ acquires a source term, $\alpha \phi_a^2$, so that its value cannot remain zero if $\phi_a$ is finite: the tunneling creates
III. OFF-RESONANT STATICS

Provided the condensate lifetime exceeds the time required by the system to reach its equilibrium, a discussion of the ‘ground state’ of the many-body system is meaningful. The equations that describe the static system are similar to Eqs.(2) with the time derivatives replaced by the chemical potential: $i\hbar \dot{\phi}_a \rightarrow \mu \phi_a$ and $i\hbar \dot{\phi}_m \rightarrow 2\mu \phi_m$, where the chemical potential of the molecules is twice the chemical potential of the atoms, in accordance with the condition for chemical equilibrium.

Of particular interest is the off-resonant ($\epsilon > 0$) limit: although $\epsilon$ is tuned close enough to observe resonance effects, its value still exceeds the kinetic energy and interaction energies ($\lambda_m n_m, \lambda n_m, \lambda_a n_a$ and $\lambda n_a$). As a consequence, $\epsilon >> \mu$, and $n_m << n_a$. The $\phi_m$-equation yields $\phi_m \approx -\alpha n_a / \epsilon$ (where we take $\phi_a$ to be real) and the $\phi_a$-equation yields a Gross-Pitaevskii equation with the effective binary collision interaction strength:

$$\mu \phi_a = \left[ -\frac{\hbar^2 \nabla^2}{2M} + \left( \lambda_a - \frac{2\alpha^2}{\epsilon} \right)n_a \right] \phi_a . \quad (3)$$

Note that equation (3) by itself does not describe the appearance of the small molecular condensate, a feature that could have important applications [14].

In the off-resonant limit we can also understand the increase in loss rate that served as a signal to detect the Feshbach resonance. As the fraction of molecules remains low, we can assume that the atomic condensate decays mostly due to inelastic atom-atom collisions $\dot{n}_a = -c_{aa} n_a^2$, and the molecular condensate due to molecule-atom collisions, $\dot{n}_m = -c_{ma} n_a n_m \approx -n_a^2 c_{ma} [\alpha / \epsilon]^2$, where $c_{aa}$ and $c_{mm}$ represent the corresponding rate coefficients. If we count each $a$-atom as one and each $m$-molecule as two condensate particles, the condensate particle density is $n = n_a + 2n_m$, and the condensate loss is described by $\dot{n} = \dot{n}_a + 2\dot{n}_m \approx -n^2 (c_{aa} + 2c_{am} n [\alpha / \epsilon]^2)$.

The observed Feshbach resonances create quasi-bound molecules of high vibrational quantum number. The single particle lifetime of such loosely bound molecules, limited
due to collisions with atoms and other molecules, is substantially lower than the lifetime of the individual atoms. Estimates for the alkali-dimer rate coefficients give $10^{-11} - 10^{-9}$ $\text{cm}^3/\text{sec}$\textsuperscript{[15]}, compared to $c_{aa} \sim 10^{-14}\text{cm}^3\text{sec}^{-1}$. Thus, a purely molecular condensate of density $10^{14}\text{cm}^{-3}$ does not live longer than $10^{-3}$ seconds. In contrast, the molecular condensate in an off-resonant BEC survives much longer by compensating for the loss of molecules with atom pairs that tunnel in from the atomic condensate.

IV. DYNAMICS

The dynamics of a condensate in a time varying magnetic field, giving a time-dependent detuning $\epsilon(t)$, is particularly interesting. A proper description requires an adjustment of Eqs.(2) to account for condensate loss.

A formal treatment of the loss includes the channels of all chemical reactions that remove particles from the condensate. The two-body collision channels can be eliminated in perturbation theory, modifying the equations of motion (Eqs.(2)) in a simple and predictable way: the interaction strengths become ‘absorptive’ with an imaginary part that determines the loss rates. In the $\phi_m$-equation, for example, $\lambda \rightarrow \lambda - i\hbar c_{ma}/2$ and $\lambda_m \rightarrow \lambda_m - i\hbar c_{mm}/2$, where $c_{mm}$ is the rate coefficient accounting for molecule-molecule collisions.

In the off-resonant limit, the atomic density variations can be neglected and the atomic condensate serves as a reservoir of atoms. The appropriate approximations to the $\phi_m$-equation of Eqs.(2), gives

$$i\hbar \dot{\phi}_m = \left[\epsilon(t) + \lambda n_a - i\hbar \frac{\gamma_m}{2}\right] \phi_m + \alpha \phi_a^2(t),$$

where $\gamma_m$ is the molecule loss-rate: $\gamma_m = c_{ma}n_a$. In the same off-resonant limit, $\phi_a \approx \sqrt{n_a} \exp[-i\lambda_a n_a t/\hbar]$.

For the special case that the detuning undergoes a sudden shift to a value $\epsilon_f$ and remains constant thereafter, we find

$$\phi_m(t) = \phi_\infty \exp\left[\frac{-i}{\hbar} (2\lambda_a n_a)\right] +$$
\[ \phi_0 - \phi_\infty \times \exp \left( \frac{-i}{\hbar} \left( \epsilon_f + \lambda n_a \right) t \right) \exp \left( \frac{-\gamma_m t}{2} \right) , \]  

(5)

where \( \phi_0 \) is the initial field, \( \phi_0 = \phi_m(t = 0) \), and \( \phi_\infty \) is the static off-resonant field value,

\[ \phi_\infty = -\alpha n_a / \left[ \epsilon_f + (\lambda - 2\lambda a)n_a - i\hbar \gamma_m/2 \right] \approx -\alpha n_a / \epsilon_f . \]

Note that the molecular density has an oscillating contribution \( \sim 2|\phi_0 - \phi_\infty| \phi_\infty^* \cos \left( \left[ \epsilon_f + \lambda n_a - 2\lambda a n_a \right] t / \hbar \right) \exp \left( -\gamma_m t / 2 \right) . \)

The molecules that appear and disappear during the oscillations tunnel in and out from the atomic condensate so that the atomic density has a contribution that oscillates out of phase with twice the amplitude. The oscillation is a genuine quantum tunneling effect and stems from the interference of the propagating initial field amplitude with the propagating amplitude of atom-pair tunneling. In general, the oscillations appear if the detuning was changed at a rate \( |\dot{\epsilon}| \) that exceeds \( |\epsilon \gamma_m| \), or \( |\dot{B}/(B - B_0)| \gg \gamma_m \). In the opposite limit, \( |\dot{\epsilon}/\epsilon| < \gamma_m \), the system adiabatically follows its ground state. Experimentally, the oscillation can be observed, for example, by illuminating the BEC with light that is resonant with a transition of the quasi-bound molecule. The intensity of the image will be modulated at the frequency of the quantum tunneling oscillations.

Near resonance, the nonlinear density variations cannot be neglected, but the dynamical behavior remains qualitatively similar: a sudden shift of detuning results in out-of-phase oscillations of the atomic and molecular condensate densities, as shown in Fig.(2). The oscillations are damped and the condensate density decays on a longer time scale.

V. CONCLUSIONS

In an atomic BEC, the molecules created in the intermediate state of a Feshbach resonance form a second condensate that is coherent with the atomic condensate field. Pairs of atoms tunnel between the condensates. We have discussed a clear signature of this novel type of quantum tunneling: a sudden change in the external magnetic field results in oscillations of the number of atoms and molecules in their respective condensates.

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Figure Captions

Fig.1: Schematic representation of the molecular potentials of the incident and intermediate state channels. The energy difference of the continuum levels, $\Delta$, is the sum of the binding energy $E_b$ of the quasi-bound state and the ‘detuning’ $\epsilon$.

Fig.2: Plot of the particle densities: the total condensate density, $n = n_a + 2n_m$, in full line, the atomic density $n_a$ in dashed line and the molecular density $n_m$ in dash-dotted line. The oscillatory behavior of the atomic and molecular densities is a signature of quantum tunneling. The calculation is for a homogeneous BEC that was initially in equilibrium at density $n = 10^{14} cm^{-3}$ when the detuning experienced a sudden shift from $\epsilon = 50\lambda n$ to $\epsilon = 2\lambda n$. The order of magnitude of the interaction parameters, $\lambda n = \lambda_mn = \lambda_an = \alpha\sqrt{2n} = 10^5$ Hz, and of the decay parameters, $c_{ma} = c_{mm} = 5 \times 10^{-10} cm^3 sec^{-1}$ (while neglecting the atomic decay) are realistic.