Abstract. By the so-called Main Theorem of Recursive Analysis, every computable real function is necessarily continuous. We wonder whether and which kinds of hypercomputation allow for effective evaluation of also discontinuous functions. More precisely the present work considers the following three super-Turing notions of real function computability:

- relativized computation; specifically given oracle access to the Halting Problem \(0 \) or its jump \(0^{\text{'}\text{'}\text{'}}\);
- encoding input \( \mathbb{R} \rightarrow \mathbb{R} \) and/or output \( y = f(x) \) in weaker ways also related to the Arithmetic Hierarchy;
- non-deterministic computation.

It turns out that any \( f: \mathbb{R} \rightarrow \mathbb{R} \) computable in the first or second sense is still necessarily continuous whereas the third type of hypercomputation does provide the required power to evaluate, for instance, the discontinuous Heaviside function.

1 Motivation

What does it mean for a Turing Machine, capable of operating only on discrete objects, to compute a real number \( x \):

- To determine its binary expansion, i.e., to decide \( A \) with \( x = \sum_{n=2}^{\infty} \frac{1}{2^n} \)?
- To compute a sequence \( (q_n) \) of rational numbers eventually converging to \( x \)?
- To compute a fast convergent sequence \( (q_n) \) for \( x \), i.e. with \( \lim_{n \to \infty} q_n \)?
- To approximate \( x \) from below, i.e., to compute \( (q_n) \) such that \( x = \sup q_n \)?

All these notions make sense in being closed under arithmetic operations like addition and multiplication. In fact they are well-known equivalent to variants studied in literature (e.g. [Tur35], [BH02], [Tur37], [Wei01]) in order.

Now what does it mean for a Turing Machine \( M \) to compute a real function \( f: \mathbb{R} \rightarrow \mathbb{R} \)? Most naturally it means that \( M \) realizes effective evaluation \( x \mapsto f(x) \) in that, upon input of \( x \in \mathbb{R} \) given in one of the above ways, it outputs \( y = f(x) \) also in one (not necessarily the same) of the above ways.
And, again, many possible combinations have already been investigated. For instance the standard notion of real function computation in Recursive Analysis \cite{Grz57,PER89,Ko91,Wei01} refers (or is equivalent) to input and output given according to \eqref{eq:input-output}. Here, the Main Theorem of Computable Analysis implies that any computable \( f \) will necessarily be continuous \cite[Theorem 4.3.1]{Wei01}.

We are interested in ways of lifting this restriction, that is, in the following Question 1.

 Does hypercomputation in some sense permit the computational evaluation of (at least certain) discontinuous real functions?

That is related to the Church-Turing Hypothesis: A Turing Machine’s ability to simulate every physical process would imply all such processes to behave continuously\footnote{G. Leibniz was convinced of (\textit{Natura non facit saltus}) but which we nowadays know to be violated for instance by the Quantum Hall Effect awarded a Nobel Prize in 1985. Since this (nor any other) discontinuous physical process cannot be simulated on a classical Turing Machine, it constitutes a putative candidate for a system capable of realizing hypercomputation.}

1.1 Summary

The standard (and indeed the most general) way of turning a Turing Machine into a hypercomputer is to grant it access to an oracle like, say, the Halting Problem \( \overline{0} \) or its iterated jumps like \( \overline{0}^{(d)} \) in Kleene’s Arithmetic Hierarchy. However regarding computational evaluation of real functions, closer inspection in Section 3.1 reveals that this Main Theorem relies solely on information rather than recursion theoretic arguments and therefore requires continuity also for oracle-computable real functions with respect to input and output of form \( \overline{Cn} \).

(For the special case of an \( \overline{0} \) (oracle, this had been observed in \cite[Theorem 16]{Ho99}).)

A second idea, applicable to real but not to discrete computability, changes the input and output representation for \( x \) and \( y = f(x) \) from \( \overline{Cn} \) to a weaker form like, say, \( c_n \). This relates to the Arithmetic Hierarchy, too, however in a different way: Computing \( x \) in the sense of \( c_n \) is equivalent to computing \( x \) in the sense of \( \overline{0^{(d)}} \) relative (i.e., given access) to the Halting Problem \( \overline{0} \) and thus suggests to write \( \overline{0} = c_n \). Most promisingly, the Main Theorem \cite[Corollary 3.2.12]{Wei01} which requires continuity of \( (w \mid (w \in \overline{0} \iff w \in \overline{0})) \) for computable real functions applies to \( \overline{0} \) but not to \( \overline{0} \) because the latter lacks the technical property of admissibility.

It therefore came as quite a surprise when Brattka and Hertling established that any \( \overline{0} \) (computable \( f \) (that is, with respect to input \( x \) and output \( f(x) \) encoded according to \( c_n \)) still satisfies continuity; see \cite[Exercise 4.1.13]{Wei01}.)

Section 3.2 contains an extension of this and a series of related results. Specifically we manage to prove that continuity is necessary for \( \overline{0} \) (computability of \( f \); here, \( \subseteq \overline{0} \subseteq \overline{0} \) denote the first levels of an entire hierarchy of real number representations explained in Lemma 5.
which emerge naturally from the Real Arithmetic Hierarchy of Weihrauch and Zheng [ZW01].

In Section 4, we closer investigate the two approaches to real function hypercomputation. Specifically it is established (Section 4.1) that the hierarchy of real number representation actually does yield a hierarchy of weakly computable real functions. Furthermore a comparison of both oracle-supported and weakly computable (and hence necessarily continuous) real functions in Section 4.2 reveals a relativized version of the Effective Weierstraß Theorem to fail.

Our third approach to real hypercomputation (Section 5) finally allows the Turing Machines under consideration to behave non-deterministically. Remarkably and in contrast to the classical (Type-1) theory, this does significantly increase their principal capabilities. For example, all quasi-strongly $\{Q \vert$ analytic functions in the sense of Chadzelek and Hotz [CH99]$\}$ and in particular many discontinuous real functions now become computable as well as conversion among the aforementioned representations $c_n$ and $b_{1/2}$.

2 Arithmetic Hierarchy and Reals

In [Ho99], Ho observed an interesting relation between computability of a real number $x$ in the respective senses of $c_n$ and $c_{\text{rec}}$ in terms of oracles: $x = \lim_{n} q_n$ for an (eventually convergent) computable rational sequence $(q_n)$ of a fast convergent rational sequence computable with oracle $0$, that is, a sequence $(p_m)$ $Q$ recursive in $0$ with $x = \lim_{m} p_m$. This suggests to use $0$ synonymously for $c_n$; and denoting by $1 \mathbb{R} = \mathbb{R}_{c}$ the set of reals computable in the sense of Recursive Analysis (that is with respect to $0$), it is therefore natural to write, in analogy to Kleene’s classical Arithmetic Hierarchy, $2 \mathbb{R}$ for the set of all $x \in 2 \mathbb{R}$ computable with respect to $0$. Weihrauch and Zheng extended these considerations and obtained for instance [ZW01, Corollary 7.3] the following characterization of $1 \mathbb{R}$: $x$ is computable in the sense of $0$ defined as follows:

$0$: $x = \lim_{n} \lim_{j} q_{n,j}$ for some computable rational sequence $(q_n)$

where $h_{i:n} n$ denotes some fixed computable pairing or, more generally, tupling function. Similarly, $1 \mathbb{R}$ contains all $x \in 2 \mathbb{R}$ computable with respect to $0$, whereas $2 \mathbb{R}$ includes all $x \in 2 \mathbb{R}$ computable in the sense of $0$ defined as follows:

$0$: $x = \sup_{n} \inf_{j} q_{n,j}$ for some computable rational sequence $(q_n)$.

To $2 \mathbb{R}$ belongs for instance the radius of convergence $r = \lim_{n} \sup_{n} a_n = 1$ of a computable power series $\sum_{n=0}^{\infty} a_n x^n$ [ZW01, Theorem 6.2]. Moreover we take from [ZW01, Definition 7.1 and Corollary 7.3] the following
Definition 2 (Real Arithmetic Hierarchy). Let \( d = 0; 1; 2; \ldots \):

\[
\begin{align*}
(d) & : d + 1 \mathbb{R} \text{ consists of all } x \in \mathbb{R} \text{ of the form } x = \sup \inf \cdots \quad \\
& \quad \text{for a computable rational sequence } (q_n), \\
& \text{ where } = \sup \text{ or } = \inf \text{ depending on } d \text{'s parity;}
\end{align*}
\]

\[
\begin{align*}
(d) & : d + 1 \mathbb{R} \text{ similarly for } x = \inf \sup \cdots \quad \\
& \quad \text{for a computable rational sequence } (q_n).
\end{align*}
\]

(For an extension to levels beyond \( d \) see [Bar03]...)

The close analogy between the discrete and this real variant of the Arithmetic Hierarchy is expressed in [ZW01] by a variety of elegant results like, e.g.,

Fact 3. a) \( x \in d \mathbb{R} \) if deciding its binary expansion is in \( d \).
b) \( x \) is computable with respect to \((d)\)
   i there is a \( f(d) \) (computable fast convergent rational sequence for \( x \).
c) \( x \) is computable with respect to \((d)\)
   i \( x \) is the supremum of a \( f(d) \) (computable rational sequence.
d) \( d \mathbb{R} = d \mathbb{R} \setminus d \mathbb{R} \).
e) \( d \mathbb{R} \cap d \mathbb{R} \).

Proof. a) Theorem 7.8, b+c) Lemma 7.2, d) Definition 7.1, and e) Theorem 7.8 in [ZW01], respectively.

2.1 Type-2 Theory of Effectivity

Specifying an encoding formalizes how to feed some general form of input like graphs or integers into a Turing Machine with fixed alphabet. Already in the discrete case, the complexity of a problem usually depends heavily on the chosen encoding; e.g., numbers in unary versus binary. This dependence becomes even more important when dealing with objects from a continuum like the set of reals and their computability. While Recursive Analysis usually considers one particular encoding for \( \mathbb{R} \), the Type-2 Theory of Effectivity (TTE) due to Weihrauch provides (a convenient formal framework for studying and comparing) a variety of encodings for different universes. Formally speaking, a representation for \( \mathbb{R} \) is a partial surjective mapping \( \mathbb{R} \to \mathbb{R} \); and an infinite string \( 2 \) dom \( ( \) is regarded as an \( (\text{name for the real number} \) x = ( ).

In this way, \( (\text{name for the real number} \) x = ( ) \) (computing a real function \( f : \mathbb{R} \to \mathbb{R} \) means to compute a transformation on infinite strings \( F \) such that any \( \{ \text{name e for} x = ( ) \text{ gets transformed to a} \{ \text{name e} = F( ) \text{ for} f(x) = y, \text{ that is, satisfying} ( ) = y\} \text{ of [We01]. Section 3}.\text{Converting} \{ \text{name e} \to \{ \text{name e thus is mentioned to} ( ) \text{computability of id} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{, and is called reducibility}\} \text{[We01]. Definition 2.3.2}.\text{Computationally equivalence, that)}

\( ^\dagger \) indicated by the symbol "\( ^\dagger \)" whose absence here generally refers to total functions
\( ^\dagger \) we use this notation instead of [We01]'s \( ( ) \text{computability to stress its connection (but not to be confused) with} [ ! ] \text{computability appearing in Section 4.2}.\text{Computationally equivalence, that}
is mutual reducibility and is denoted by \( \preceq \) whereas \( \preceq \) means but 6.

We borrow from TTE also two ways of constructing new representations from giving ones: The conjunction \( \wedge \) of and \( \wedge \) is the least upper bound with respect to \( \leq \) [Wei01, Lemma 3.3.8] and for (nively or countably many) representations \( A_i \), their product \( \prod_{i=1}^{\infty} A_i \) denotes a natural representation for the set \( \bigcap_{i=1}^{\infty} A_i \) [Wei01, Definition 3.3.2]. In particular, in order to encode \( x \in \mathbb{R} \) as a rational sequence \( (q_n) \in \mathbb{Q}^\infty \), we (often implicitly) refer to the representation \( \mathbb{Q}^\infty \) due to [Wei01, Definition 3.1.2.4 and Lemma 3.3.16].

2.2 Arithmetic Hierarchy of Real Representations

Observe that (the characterizations due to Fact 3 of) each level of the Real Arithmetic Hierarchy gives rise not only to a notion of computability for real numbers but also canonically to a representation for \( \mathbb{R} \); for instance let

- encode (arbitrary!) \( x \in \mathbb{R} \) as a fast convergent rational sequence \( (q_n) \);
- encode \( x \in \mathbb{R} \) as a rational sequence \( (q_n) \) with supremum \( x = \sup_n q_n \);
- encode \( x \in \mathbb{R} \) as a rational sequence \( (q_n) \) with limit \( x = \lim_n q_n \);
- encode \( x \in \mathbb{R} \) as \( (q_n) \in \mathbb{Q}^\infty \) such that \( x = \sup_{n_1} \inf_{n_2} \cdots \inf_{n_d} q_{n_1;\cdots;n_d} \);
- encode \( x \in \mathbb{R} \) as \( (q_n) \in \mathbb{Q}^\infty \) such that \( x = \inf_{n_1} \sup_{n_2} \cdots \sup_{n_d} q_{n_1;\cdots;n_d} \).

As already pointed out, the first three of them are already known and used in TTE as \( \llbracket \cdot \rrbracket \), \( \llbracket \cdot \rrbracket \), and \( \llbracket \cdot \rrbracket \), respectively [Wei01, Section 4.1]. In general one obtains, similar to Definition 4, a hierarchy of real representations as follows:

Definition 4. Let \( (0) = \llbracket \cdot \rrbracket \), \( (0) = \llbracket \cdot \rrbracket \), \( (0) = \llbracket \cdot \rrbracket \), and \( (c_n) \), respectively [Wei01, Section 4.1]. Now set \( d \in \mathbb{N} \):

\[
A^{(d)}(\text{name for } x \in \mathbb{R} \text{ is } (\text{name for a} \text{ sequence } (q_n))) \quad \mathbb{Q} \text{ such that }
\]

\[
x = \sup_{n_1} \inf_{n_2} \cdots \inf_{n_d} q_{n_1;\cdots;n_d} \] .

\[
A^{(d)}(\text{name for } x \in \mathbb{R} \text{ is a sequence } (q_n)) \quad \mathbb{Q} \text{ such that }
\]

\[
x = \inf_{n_1} \sup_{n_2} \cdots \sup_{n_d} q_{n_1;\cdots;n_d} \] .

Regarding Fact 3, one may see \( \llbracket \cdot \rrbracket \) and \( \llbracket \cdot \rrbracket \) as the first and second Jump of \( \llbracket \cdot \rrbracket \), respectively; same for \( \llbracket \cdot \rrbracket \) and \( \llbracket \cdot \rrbracket \).

Results from [ZW01] about the Real Arithmetic Hierarchy are easily rephrased in terms of these representations. Fact 3d) for example translates as follows:

\[
x \text{ is (computable if it is both (computable and (computable.}
\]

Observe that this is a non-uniform claim whereas closer inspection of the proofs in particular of Lemma 3.2 and Lemma 3.3 in [ZW01] reveals them to hold fully uniformly so that we have
Lemma 5. Moreover, the uniformity of Ziegler, Lemma 3.2] yields the following interesting Scholium:

Let $\sim^0_0$ denote the representation encoding $x \in \mathbb{R}$ as $(q_n)_{\mathbb{Q}}$ with $x = \liminf q_n$; and $\sim_0^0$ similarly with the additional requirement that $q_n < x$ for infinitely many $n$.

Then it holds $\sim^0_0 \sim_0^0 \sim^0_0 \sim^0_0$ (being the trivial direction).

3 Computability and Continuity

Recursive Analysis has established as folklore that any computable real function is continuous. More precisely, computability of a partial function from/to infinite strings $f : \mathbb{R} \rightarrow \mathbb{R}$ requires continuity with respect to the Cantor Topology of $\mathbb{R}$; and this requirement carries over to functions $f : (A; \leq_A) \rightarrow (B; \leq_B)$ where input $a \in A$ and output $b = f(a)$ are encoded by respective admissible representations and $\leq_A$. Roughly speaking, this property expresses that the mappings $f : A \rightarrow B$ satisfy a certain compatibility condition with respect to the topologies $A / B$ involved. For $A = B = \mathbb{R}$, the (standard) representation (for example) is admissible [Wei01, Lemma 4.1.1.1], thus recovering the folklore claim.

Now in order to treat and non-trivially investigate computability also of discontinuous real functions $f : \mathbb{R} \rightarrow \mathbb{R}$, there are basically two ways out: Either enhance the underlying Type-2 Machine model or resort to non-admissible representations. It turns out that for either choice, at least the straightforward approaches fail:

- extending Turing Machines with oracles as well as
- considering weakened representations for $\mathbb{R}$.

3.1 Type-2 Oracle Computation

Specifically concerning the first approach, most results in Computable Analysis relativize. Specifically, we make the following Observations:

Observation 7. Let $O$ be arbitrary. Replace in [Wei01, Definition 2.1.1] the Turing Machine $M$ by $M^O$, that is, one with oracle access to $O$. This Type-2 Computability in $O$ still satisfies

- closure under composition [Wei01, Theorem 2.1.12];
- computability of string functions requires continuity [Wei01, Theorem 2.2.3];
- same for computable functions on represented spaces with respect to admissible representations [Wei01, Corollary 3.2.12].

A scholium is a note amplifying a proof or course of reasoning, as in mathematics.
In particular, the Main Theorem of Recursive Analysis [Wei01, Theorem 4.3.1] relativizes.

A strengthening and counterpart to Observation 7b), we have

Lemma 8. For a partial function on infinite strings $f : \mathbb{N} \to \mathbb{N}$, the following are equivalent:

There exists an oracle $O$ such that $f$ is computable relative to $O$;

- $f$ is Cantor-continuous and $\text{dom}(f)$ is a $G\delta$ set.

Compare this with Type-1 Theory (that is, computability on finite strings) where every function $f : \mathbb{N} \to \mathbb{N}$ is recursive in some appropriate $O$.

Proof (Lemma 8). If $f$ is recursive in $O$, then it is also continuous by Observation 7b), that is, the relativized version of [Wei01, Theorem 2.2.3]. Furthermore, the relativization of [Wei01, Theorem 2.2.4] reveals $\text{dom}(f)$ to be a $G\delta$ set.

Conversely suppose that continuous $f$ has $G\delta$ domain. Then $f = h_\uparrow$ for some monotone total function $h : \mathbb{N} \to \mathbb{N}$ according to [Wei01, Theorem 2.3.7.2] where, by [Wei01, Definition 2.1.10.2], $h_\uparrow : \mathbb{N} \to \mathbb{N}$ denotes the (existing and unique) extension of $h$ from $\mathbb{N}$ to $\mathbb{N}$. A classical Type-1 function on finite strings, this $h$ is recursive in a certain oracle $O$.

The relativization of [Wei01, Lemma 2.1.11.2] then asserts also $h_\uparrow$ to be computable in $O$.

The conclusion of this subsection is that oracles do not increase the computational power of a Type-2 Machine sufficiently in order to handle also discontinuous functions. So let us proceed to the second approach to real hypercomputation:

### 3.2 Weaker Representations for Reals

In the present section we are interested in relaxations of the standard representation for single reals and their effect on the computability of function evaluation $x \mapsto f(x)$. Since, with the exception of one of the ones introduced in Definition 2, it is admissible with respect to the usual Euclidean topology on $\mathbb{R}$, Lemma 4.1.4, Example 4.1.14.1, the relativized Main Theorem (Observation 4) is not applicable. Hence, chances are good for evaluation $x \mapsto f(x)$ to become computable even for discontinuous $f : \mathbb{R} \to \mathbb{R}$; and indeed we have the following

Example 9. Heaviside's function

$$
\text{h} : \mathbb{R} \to \mathbb{R}; \quad x \mapsto 0 \text{ for } x \leq 0; \quad x \mapsto 1 \text{ for } x > 0
$$

is both ($\uparrow$, $\downarrow$) computable and ($\uparrow$, $\downarrow$) computable.

Proof. Given $(q_n)$ $Q$ with $x = \sup_n q_n$, exploit ($\uparrow$, $\downarrow$) computability of the restriction $h_{\uparrow, Q} : Q \to \mathbb{R}$ to obtain $p_n = h(q_n)$. Then indeed, $(p_n)$ $Q$.
Suppose for a start that Heaviside's function, in spite of its discontinuity at $x = 0$, is computable by some Type-2 machine $M$. Feed the rational sequence $q_0 = 2^n$, a valid (name for $x$) to this $M$. By presumption it will then spit out a sequence $(p_n)$ of the input. We may therefore feed $q_0$ as the initial part of the input.

Now re-use $M$ in order to evaluate $h$ at $x = q_0 > 0$ (encoded as the rational sequence $(q_0)$) with $p_0, p_2, \ldots, p_n$ for $y = h(x) = 0$; in particular, $p_2 = 2^n$ for $y = 1$. Up to output of $p_2$, $M$ has executed a finite number $N = 2^N$ of operations and in particular read at most the initial part $p_0, p_2, \ldots, p_{n-2}$ of the input.

For the case of a general function $f : \mathbb{R} \to \mathbb{R}$ with discontinuity at some $x > 0$, let $y = f(x) \in \mathbb{R}$. By proceeding to an appropriate subsequence of $(x_n)$, we may suppose without loss of generality that $x_n \to x$. Then there is a rational double sequence $(q_{n,j})$ such that $q_{n,j} \to x_n$; thus $q_{n,j} \to 0^+$. We may therefore feed
(q, n) as a (name e in order to evaluate f at x and obtain in turn a (name e (p, n) Q for y. As before, p, is output after having only read some finite initial part (q, n) of the input. Then

for n \(N\) reveals this very initial part to also be the start of a valid (name e for x = n) whereas

\[2^{M+2} < y \updownarrow \downarrow \downarrow \downarrow p, \quad j f(\mathbf{x}) \updownarrow f(\mathbf{x}) \downarrow y \downarrow 2^{M+2} \uparrow p, \quad f(\mathbf{x}) \downarrow 2^{M}\]

shows that \((p, n)\) is not a valid initial part of a (name e for \(f(\mathbf{x})\); contradiction.

b) We prove ( 0 ) (uncomputability of the unipotent Heaviside Function

\[\mathbb{H}: 0 \uparrow 1; \quad 0 < x \uparrow 0\]

as a prototype lacking lower semi-continuity.

Consider again the (name e \(q, n = 2^n\) for \(x = 0\) which the hypothetical Type-2 M machine transforms into a (name e for \(y = \mathbb{H}(x) = 1\), that is, a sequence \(p, n\) Q with sup \(p, n = y\); in particular \(p, n = 1/2\) for some \(M \cdot 2^n\) gets output having read only \((q, n)\) for some \(n \cdot 2^n\). The latter finite segment is also the initial part of a valid (name e for \(x = q, n > 0\) whereas \((p, n)\) has sup \(y\) and thus is not the initial part of a valid (name e for \(y = \mathbb{H}(x) = 0\); contradiction.

This proof for the case \(\mathbb{H}\) carries over to an arbitrary \(f: \mathbb{R} \rightarrow \mathbb{R}\) just like in a), that is, by replacing \(q, n = 2^n\) with rational approximations to a general sequence \(x, 2^n\) \(\mathbb{R}\) witnessing violated lower semi-continuity of \(f\) in that \(f(\lim x, n) \uparrow \lim \inf f(x, n))\).

As in a) and b), we treat for notational simplicity the case of \(f: \mathbb{R} \rightarrow \mathbb{R}\) violating monotonicity in that \(f(0) = 1\) and \(f(1) = 0\); the general case can again be handled similarly. Feed the (name e \((q, n) = (0; 0; 0; \cdots)\) for \(x = 0\) into a m machine which be presumption produces a sequence \((p, n)\) Q with sup \(p, n = 1\) and in particular \(p, n = 1\) for some \(M \cdot 2^n\). Up to output of \(p, n\), only \((q, n)\) has been read for some \(n \cdot 2^n\). Now consider the rational sequence \((q, n)\) consisting of \(N\) zeros followed by an infinity of 1s, that is, a valid (name e for \(x = 1\). This new input will cause the m machine to output a sequence \((p, n)\) Q coinciding with \((p, n)\) for \(m \cdot M\) in particular \(p, n = 1\) contradicting that \((p, n)\) is supposed to satisfy sup \(p, n = f(\mathbf{x}) = 0\).

d) Suppose that, in spite of its discontinuity at \(x = 0\), \(\mathbb{H}\) be ( 0 ) uncomputable by some Type-2 M machine M.

Consider the sequence \(q^{(1)} = q^{(1)} = 1\) which is by definition a valid 0 (name e for \(f^{(1)} = \lim q^{(1)}\)). So upon input of \(q^{(1)}\), M will generate a corresponding sequence \(p^{(1)}\) Q as a 0 (name e for \(y^{(1)} = \mathbb{H}(x^{(1)}) = 0\), that is, satisfying \(\lim p^{(1)} = 0\); in particular, \(p^{(1)} = 0\) for some \(n \cdot 2^n\). Up to this output, M has read only a finite initial part of the input \(q^{(1)}\), say, up to \(n \cdot n\).
Next consider the sequence $q^{(2)}$ defined by $q^{(2)}_n = 1$ for $n = n_1$ and $q^{(2)}_n = \frac{1}{2}$ for $n > n_1$: a valid name for $x^{(2)} = \frac{1}{2}$ which $M$ by presumption transforms into a sequence $p^{(2)}$ with $\lim_{m \to \infty} p^{(2)}_m = y^{(2)} = \lim_{n \to \infty} x^{(2)} = 0$; in particular, $q^{(2)}_n = \frac{1}{2}$ for some $m_2 > m_1$. However, due to $M$'s deterministic behavior and since $q^{(1)}$ and $q^{(2)}$ initially coincide, it still holds $p^{(2)}_n = \frac{1}{2}$.

Now by repeating the above argument we obtain a sequence of sequences $q^{(k)}$, each constant for $n = n_k$ of value (and thus a valid name for) $x^{(k)} = 2^{k+1}$ and transformed by $M$ into a sequence $p^{(k)}$ satisfying $p^{(k)}_m = \frac{1}{2}$ for $i = 1, \ldots, k$ with strictly increasing $(n_k), (m_k) \in N$. The ultimate sequence $q^{(1)}$, well-defined by $q^{(1)}_n = q^{(k)}_n$ for $n = n_k$ (and in fact the limit of the sequence of sequences $q^{(k)}$ with respect to Baire's Topology), therefore converges to (and is hence a valid name for) $x^{(1)} = 0$; and it gets mapped by $M$ to a sequence $q^{(1)}$ satisfying $q^{(1)}_m = \frac{1}{2}$ for in infinitely many $m$ contradicting that a valid name for $y^{(1)} = \lim_{m} x^{(1)} = 1$ should have $\lim_{m} = 1$.

Being only information-theoretic, the above arguments obviously relativize.

The main result of the present section is an extension of Fact 10 to one level up on the hierarchy of real representations from Definition 4. This suggests similar claims to hold for the entire hierarchy and might not be as surprising any more as Fact 10d) in [BH02]; nevertheless, already this additional step makes proofs significantly more involved.

**Theorem 11 (First Main Theorem of Real Hypercomputation).**

Consider $f : \mathbb{R} \to \mathbb{R}$.

a) If $f$ is $(0! 0! 0!)$-computable, then it is lower semi-continuous.

b) If $f$ is $(0! 0! 0^*)$-computable, then it is monotonically increasing.

c) If $f$ is $(0! 0^*)$-computable, then it is continuous.

The claims remain valid under oracle-supported computation.

We point out that the proofs of Fact 10 proceed by constructing an input for which a presumed machine $M$ fails to produce the correct output. They differ however in the 'length' of these constructions: for Claims (a) to (c), the counterexample inputs are obtained by running $M$ for a finite number of steps on a single, fixed argument whereas in the proof of Claim (d), $M$ is repeatedly started on an adaptively extended sequence of arguments. The latter argument may thus be considered as of length $\omega$, the first infinite ordinal. Our proof of Theorem 11d) will be even longer and is therefore put into the following subsection.

### 3.3 Proof of Theorem 11

As in the proof of Fact 10, we treat the special case of the stepped Heaviside Function $h$ for reasons of notational convenience and clarity of presentation; the according arguments can be immediately extended to the general case.
Claim 12. \( h : \mathbb{R} \to \mathbb{R} \) is not \((\{0!\}!\{0\})\) computable.

Proof. Suppose a Type-2 Machine \( M \) \((\{0!\}!\{0\})\) computes \( h \). In particular, upon input of \( x^{(1)} = 1 \) in form of the sequence \( q^{(1)}_n \) with \( q^{(1)}_1 = 1 \), \( M \) will output a rational double sequence \( p^{(1)}_k = \left( p^{(1)}_k \right)^{\mu\lambda} \) with \( 0 = y^{(1)} = h(x^{(1)}) = \sup_k \inf_{\mu\lambda} p^{(1)}_k \). Observe that \( p^{(1)}_1 = \frac{1}{3} \) for some \( \mu\lambda \). When writing \( p^{(1)}_1 = \frac{1}{3} \), \( M \) has only read a finite part of \( (q^{(1)}_n) \), say, up to \( n_1 \).

Now consider \( x^{(2)} = \frac{1}{2} \), given by way of the sequence \( q^{(2)}_n \) with \( q^{(2)}_n = 1 \) for \( n < n_1 \) and \( q^{(2)}_n = \frac{1}{2} \) for \( n \geq n_1 \). Then, too, \( M \) will output a double sequence \( p^{(2)}_k \) with \( 0 = \frac{1}{3} = \sup_k \inf_{\mu\lambda} p^{(2)}_k \). Observe that, similarly, some \( p^{(2)}_1 = \frac{1}{3} \) is output having read only a finite part of \( (q^{(2)}_n) \), say, up to \( n_2 \). Moreover, as \( q^{(1)} \) and \( q^{(2)} \) coincide up to \( n_1 \) and since \( M \) operates deterministically, \( p^{(2)}_1 = p^{(1)}_1 = \frac{1}{3} \).

Continuing this process with \( x^{(k)} = 2^{k+1} \) for \( k = 3, 4, \ldots \) as indicated in Figure 1, eventually yields a rational sequence \( q^{(1)}_n \) with \( \lim_n q^{(1)}_n = \lim_{n \to \infty} \frac{1}{2} = x^{(1)} = 0 \), upon input of which \( M \) outputs a double sequence \( p^{(1)}_k \) such that \( p^{(1)}_1 = \frac{1}{3} \) for all \( k = 1, 2, \ldots \). In particular, \( y^{(1)} = \sup_k \inf_{\mu\lambda} p^{(1)}_k = \frac{1}{3} \) whereas \( h(x^{(1)}) = 1 \) : contradiction.

Notice that the above proof involves one-dimensionally indexed sequences \( (q_n) \) for input and two-dimensionally indexed ones \( (p_{k\mu\lambda}) \) for output. We now proceed a step further in proof difficulty, namely involving two-dimensionally indexed indices for both input and output in order to establish Item b).

Claim 13. Let \( f : \mathbb{R} \to \mathbb{R} \) violate monotonicity in that \( f(0) = 1 \) and \( f(1) = 0 \). Then, \( f \) is not \((\{0!\}!\{0\})\) computable.

Proof. We construct a \((\{0!\}!\{0\})\) (namely for \( x = 0 \)) from an iteratively defined sequence of initial segments of \( 0 \) (namely for \( x = 1 \)).
Start with \( q_{ij}^{(1)} = 1 \) for all \( i,j \). Then, \( q^{(1)} = (q_{ij}^{(1)}) \) is obviously a \( 0 \) (name e for \( x = 1 \)) and thus yields by presumption, upon input to \( M \), a \( 0 \) (name \( p_{k'}^{(1)} \) for \( f(1) = 0 \), that is, with \( 0 = \sup_k \inf_{q_{ij}^{(1)}} \). In particular, \( p_{i,j}^{(1)} \frac{1}{3} \) for some \( e' \).

![Figure 2](image-url) Illustration to the iterative construction employed in the proof of Claim 13

Until output of \( p_{i,j}^{(1)} \), \( M \) has read only finitely many entries of \( q^{(1)} \); say, up to \( i_1 \) and \( j_1 \), that \( s \) covered in Figure 2 by the light gray rectangle. Now consider \( q^{(2)} \) defined as in this figure: Since \( \inf_j q_{ij}^{(2)} = 0 \) for \( i = i_1 \) and \( \inf_j q_{ij}^{(2)} = 1 \) for \( i > i_1 \), \( \sup_i \inf_j q_{ij}^{(2)} = 1 \), that is, this is again valid \( 0 \) (name e for \( x = 1 \)); and again, \( M \) will by presumption convert \( q^{(2)} \) into a \( 0 \) (name \( p^{(2)} \) for \( f(1) = 0 \). In particular, \( p_{2,j}^{(2)} \frac{1}{3} \) for some \( e' \) and, being a deterministic machine, \( M \)'s operation on the initial part (dark gray) on which input \( q^{(1)} \) will not have generated the same \( e \) initial output, namely \( p_{i,j}^{(2)} = p_{i,j}^{(1)} \frac{1}{3} \).

Again, until output of \( p_{i,j}^{(1)} \), \( M \) has read only a finite part of \( q^{(2)} \), of, say, up to \( i_2 > i_1 \) (light gray). By now considering input \( q^{(3)} \) with \( \inf_j q_{ij}^{(3)} = 0 \) for \( i = i_2 \) as in Figure 2, we arrive at \( p^{(3)} \) and \( p_{i,j}^{(3)} \) \( \frac{1}{3} \) and so on with \( i_j q_{ij}^{(4)} \) \( p^{(4)} \); \( i_j \) and so on.

Finally observe that continuing these arguments eventually leads to a rational double sequence \( q^{(\infty)} = (q_{ij}^{(\infty)}) \) which has \( \inf_j q_{ij}^{(\infty)} = 0 \) for \( i = i_1 = 1 \) and is therefore a valid \( 0 \) (name e for \( x = 0 \) (rather than \( x = 1 \)) but gets mapped by \( M \) to \( p^{(1)} = (p_{k}^{(1)}) \) with \( \inf_k p_{k}^{(1)} = p_{k}^{(1)} \frac{1}{3} \) for all \( k \). Since \( f(0) = 1 \), this contradicts our presumption that \( M \) maps \( 0 \) (name e for \( x = 0 \) (name e for \( f(x) \)).

The above proofs involving \( 0 \) and \( 0 \) proceed by constructing an infinite sequence of inputs \( q^{(1)}; q^{(2)}; \ldots; q^{(1)} \) (each possibly a multi-indexed sequence of...
its own). Finally asserting Claim c) involving $\omega_1$, we will extend this method from length $\omega_1$, the first infinite ordinal, to an even longer one.

Claim 14. $h : \mathbb{R} \rightarrow \mathbb{R}$ is not $(\omega_1 \rightarrow \omega_1)$-computable.

Proof. Outwit a Type-2 Machine $M$, presumed to realize this computation, as follows:

i) Take $q^{(1)}$ to be the constant double sequence 1, i.e., $q^{(1)}_{ij} = 1$ for all $i,j$. Being a $\omega_0$-name for 1, it is by presumption mapped to a $\omega_0$-name $p^{(1)}$ for $h(1) = 0$, that is, satisfying $\lim \lim p^{(1)}_{k,r} = 0$. In particular, at most every column $n \neq k$ contains an entry $\neq 0$ with $p^{(1)}_{k,r} = 0$. Until output of the first such $p^{(1)}_{k,r}$, $M$ has read only a finite part of $q^{(1)}$, say, up to $i_1,j_1$.

![Fig. 3. The first infinitely long iterative construction employed in the proof of Claim 13.](image)

ii) Observe that this Argument i) equally applies to the scaled input sequence $2^m q^{(1)}$ for any $m$. So define $q^{(2)}_{ij} = q^{(1)}_{ij}$ for $j < j_1$ (i.e., inherit the initial part of $q^{(1)}$) and $q^{(2)}_{ij} : = \frac{1}{2}$ for $j > j_1$. Now upon input of this $q^{(2)}$, $M$ will output $p^{(2)}$ with, again, in infinitely many $p^{(2)}_{k,r} = \frac{1}{2}$, the first one, $(k_2,r'_2)$, say, after having read $q^{(2)}$ only up to some $(i_2,j_1)$. Further, one of $M$'s determinism in places $p^{(2)}_{k,r} = p^{(1)}_{k,r} = \frac{1}{2}$.

By repeating for $m = 2, 3, \ldots$, we eventually obtain, similarly to the proof of Claim 13, an input sequence $q^{(1)}$ with $q^{(1)}$ with $\lim \lim q^{(1)}_{ij} = 0$, that is, a valid $\omega_0$-name for $x = 0$ (rather than 1). This is mapped by $M$ to $p^{(1)}$ with $p^{(1)}_{k,r} = \frac{1}{2}$ for all $m$. On the other hand, $p^{(1)}$ is by presumption a $\omega_0$-name for $x(0) = 1$. Therefore, there are infinitely many $m$ with $p^{(1)}_{k,r} = \frac{1}{2}$ for some $r > n$ and $p^{(1)}_{k,r} = \frac{1}{2}$; see the grey columns in the right part of Figure 3.
iii) Since this gives no contradiction yet, we proceed by considering the first such column $m$ containing an entry $\frac{1}{3}$ as well as an entry $\frac{2}{3}$. Take the initial part of the input $q^{(\omega)}$ up to $(i, j)$, say, depicted in grey in the left part of Figure 4 that $M$ has read until output of both of them; extend it with $\frac{1}{3}$s in top direction and with $1$s to the right. Feed this $0$ (name for $x = 1$) into $M$ until output of an entry $p_{k'}$, $\frac{1}{3}$ in some column $k$ beyond $m$. Then repeat extending to the right with $1$s replaced by $\frac{1}{3}$s for a second entry $p_{k'}$, $\frac{1}{3}$.

![Figure 4](image-url)

Fig. 4. Second infinitely long iterative construction employed in the proof of Claim 14.

More generally, proceed similarly as in ii) and extend $q^{(\omega)}_{i, j}$ to the right in such a way with some $0$ (name for $q^{(\omega)}$) for $x = 0$ as to obtain another column $m^0$ with both entries $\frac{1}{3}$ and $\frac{2}{3}$; see the middle part of Figure 4. Again, $M$ outputs the latter two entries having read only a finite part; say, up to $(i_0, j)$. Now extend this part, too, with $\frac{1}{3}$ in top direction and with another $q^{(\omega)}_{i_0, j}$ obtained, again, as in ii) for a third column $m_0^0$ with both entries $\frac{1}{3}$ and $\frac{2}{3}$; and so on.

This eventually leads to an input $q^{(2\omega)}$ which, due to the extensions to the top, represents a $0$ (name for $q^{(2\omega)}$) for $x = \frac{1}{3}$ and is thus mapped by presumption to a $0$ (name for $p^{(2\omega)}$) for $x = \frac{1}{3}$. In particular, almost every column $n$ of $p^{(2\omega)}$ has almost every entry $\frac{1}{3}$ while maintaining in infinitely many columns $n$ with preceding entries $\frac{1}{3}$ and $\frac{2}{3}$; see the right part of Figure 4. This asserts the existence of infinitely many columns $n$ in $p^{(2\omega)}$ containing $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{1}{3}$ in order. And again, already a finite initial part of $q^{(2\omega)}$ up to some $(i_2, j_2)$ gives rise to the first such triple.

iv) Notice that the arguments in iii) similarly yield the existence of an appropriate, scaled counterpart $\frac{1}{2}q^{(2\omega)}$ of $q^{(2\omega)}$, of some $\frac{1}{4}q^{(4\omega)}$, and so on, all leading to output containing in infinitely many columns $n$ with alternating triples.
as above. We now construct input \( q^{(3)} \) leading to output \( p^{(2)} \) containing an infinity of columns, each with four entries \( \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \text{ and } \frac{2}{3} \).

To this end, take the initial part of \( q^{(2)} \) leading to output of the first column \( n \) with alternating triple in the above sense; then extend \( q \) with the initial part of the scaled version \( \frac{1}{2} q^{(3)} \) leading to another column \( n \) with such a triple; and so on. Observing that, due to the scaling, the thus obtained \( q^{(3)} \) contains a \( \Omega \) (namely \( x = 0 \), almost every column of the output \( p^{(3)} \) representing \( h(0) = 1 \) contains entries \( \frac{2}{3} \) in addition to the infinitely many columns with triples as above; see the left part of Figure 5.

Fig. 5. Third, fourth, and fifth in infinitely long iterative construction employed in the proof of Claim 14

v) Our next step is a \( \Omega \) (namely \( x = \frac{1}{4} \)) giving rise to \( p^{(4)} \) with infinitely many columns containing alternating quintuples. This is obtained by repeating the argument in iv) to obtain initial segments of (variants of) \( q^{(3)} \), stacking them horizontally in order to obtain an in infinity of columns with alternating quadruples; while extending in top direction with \( \frac{1}{4} \); see the middle part of Figure 5. This forces \( M \) to output a \( \Omega \) (namely \( x = \frac{1}{4} \)) for \( h \left( \frac{1}{4} \right) = 0 \) and thus in almost every column every entry being \( \frac{1}{4} \), thus extending the alternating quadruples to quintuples.

vi) Noticing that the vertical extension in v) was similar to step iii), we now take a step similar to iv) based on horizontally stacked initial parts of scaled counterparts of \( q^{(4)} \) in order to obtain a \( \Omega \) (namely \( x = \frac{1}{8} \)) containing in infinitely many alternating six-tuples. Then again construct a \( \Omega \) (namely \( x = \frac{1}{8} \)) by horizontally stacking initial segments of (variants of) \( q^{(5)} \) while extending them vertically with \( \frac{1}{8} \) and so on.

Now for the bottom line: By proceeding the above construction, one eventually obtains a rational double sequence \( q^{(i)} \) with \( \lim_{j \to \infty} q^{(i)}_{i,j} = 0 \) for all \( i \).
is, a \(0\) (name for \(x = 0\) | mapped by \(M\) to some \(p^{(r)}\) containing (in infinitely many) columns \(\# k\) with in infinitely many alternating entries \(\frac{1}{2}\) and \(\frac{3}{4}\) | contradicting that, for \(0\)-names \(p = (p_k, r)\), \(\lim p_k\) is required to exist for every \(k\).

### 4 Hierarchies of Hypercomputable Real Functions

The present section investigates and compares the first levels of the two hierarchies of hypercomputable real functions induced by the two approaches to real function hypercomputation considered in Section 3: based on oracles and based on weakened encodings.

#### 4.1 Weakly Computable Real Functions

For every \((\quad)!\quad)\) (computable function \(f: A \rightarrow B\), one may obviously replace representation for \(A\) by a stronger one and for \(B\) by a weaker one while maintaining computability of \(f\):

\[
f(\quad)!\quad)^{\text{com putable } \uparrow}; \quad (\quad)!\quad)^{\text{com putable}}.
\]

However if both are made weaker then \((\quad)!\quad)^{\text{com putability}} of \(f\) may in general be violated. For \(= = \leq\), though, we have seen in Example 9 that the implication \((\quad)!\quad) (\quad)!\quad)^{\text{com putability}} does hold at least for the case of \(f\) being Heaviside's function. By the following result, it holds in fact for every \(f\):

**Theorem 15 (Second Main Theorem of Real Hypercomputation).**

Consider \(f: \mathbb{R} \rightarrow \mathbb{R}\).

- a) If \(f\) is \((\quad)!\quad)^{\text{com putable}}, then it is also \((\quad)!\quad)^{\text{com putable}}.
- b) If \(f\) is \((\quad)!\quad)^{\text{com putable}}, then it is also \((\quad)!\quad)^{\text{com putable}}.
- c) If \(f\) is \((\quad)!\quad)^{\text{com putable}}, then it is also \((\quad)!\quad)^{\text{com putable}}.
- d) If \(f\) is \((\quad)!\quad)^{\text{com putable}}, then it is also \((\quad)!\quad)^{\text{com putable}}.
- e) If \(f\) is \((\quad)!\quad)^{\text{com putable}}, then it is also \((\quad)!\quad)^{\text{com putable}}.

The claims remain valid under oracle-supported computation.

As a consequence, we obtain the following partial strengthening of Lemma 4:

**Corollary 16.**

It holds \(\quad\leq\mathfrak{t}\quad \leq\mathfrak{t}\), \(0\leq\mathfrak{t}\quad \leq\mathfrak{t}\), \(0\leq\mathfrak{t}\quad \leq\mathfrak{t}\), \(0\leq\mathfrak{t}\quad \leq\mathfrak{t}\), where \(\quad\mathfrak{t}\quad\) denotes continuous reducibility of representations [Wei01, Def. 2.3.2].

**Proof.** The positive claims follow from Lemmas 4 and 5. For a negative claim like \(\quad0\quad\mathfrak{t}\quad\mathfrak{t}\quad0\quad\) suppose the contrary. Then by Lemmas 4, with the help of some appropriate oracle \(O\), one can convert \(0\) (names to \(0\) (names. As Heaviside's function \(h\) is \((\quad)!\quad)^{\text{com putable}} by Example 9 and Theorem 15, composition with the presumed conversion implies \((\quad)!\quad)^{\text{com putability}} of \(h\) relative to \(0\) | contradicts Theorem 11).
Proof (Theorem 15a). Let $f$ be $(0 \mid 0)(\text{computable and } x \text{ given by a } 0)(\text{name } e)$, that is, a rational sequence $q = (q_i)$ with $x = \lim_i \lim_j q_{i,j}$. For each $i$, compute by assumption from the $0(\text{name } e \text{ for } q_i) = (q_{i,j})$, of $x_i = \lim_j q_{i,j}$, a $0(\text{name } e \text{ for } f(x_i))$ that is, a sequence $p_i = (p_{i,j})$, with $f(x_i) = \lim_j p_{i,j}$.

Continuity of $f$ due to Fact 15-2 asserts

$$\lim_i \lim_j p_{i,j} = \lim_i f(x_i) = f \lim_i x_i = f \lim_i \lim_j q_{i,j} = f(x)$$

this sequence $p$ to be a $0(\text{name } e \text{ for } y = f(x))$.

Where the last proof exploits Fact 15-2, the next one relies on Theorem 15b:

Proof (Theorem 15b). A $(0)(\text{name } e \text{ for } x \in R$ is a rational sequence $a = (a_i)$ with $x = \lim_i \lim_j a_{i,j}$. For each $i$, exploit $(0)(\text{computability of } f$ to obtain, from the $(0)(\text{name } e \text{ for } a_i)$, of $x_i = \lim_j \lim_k a_{i,j,k}$, a sequence $p_{i,j}$, with $\lim_j \lim_k p_{i,j,k}$, as $(0)(\text{name } e \text{ for } f(x_i))$. Similarly, to case d), this sequence $p$ constitutes a $(0)(\text{name } e \text{ for } y = f(x))$ by continuity of $f$ due to Theorem 15b.

Proof (Theorem 15b). Let $f$ be $(0 \mid 0)(\text{computable}). Its $(0 \mid 0)(\text{computationally by assumption to evaluate } f(q_i))$ for each $i$ up to error $2^{-n}$; that is, obtain $p_n = 2 \ Q$ with $f(q_i) = 2^n$. Since $f$ is continuous by Fact 15a, it follows $f(x) = \lim_n p_n = \lim_n p_n$, so that $(p_n)$ is a $(0)(\text{name } e \text{ for } y = f(x))$.

It is interesting that the latter proof works in fact uniformly in $f$, i.e., we have:

Scholium 17. The apply operator $C(R \times \mathbb{R}) (f; x) \ y = f(x)$ is $(0 \mid 0)(\text{computable})$.

Similarly, Theorem 15b) follows from Lemma 15 below together with the observation that every $(0 \mid 0)(\text{computationally a } f \text{ has a computation } \text{ for } 0)(\text{name } e)$.

[0:200] Corollary 5.1(2) and Theorem 3.7] here, $(0 \mid 0)$ denotes a natural representation for the space $\text{LSC}(R)$ of lower semicontinuous functions $f: R \to R$ considered in [0:200]. Specifically, a $(0 \mid 0)(\text{name } e \text{ for } a \in \text{R})$ is an enumeration of all rational tuples $(a_i; b; c)$ such that $c < m \in f(a_i; b)$; the latter making sense as a lower semicontinuous function attains its minimum (though not necessarily its maximum) on any compact set. $(0 \mid 0)$ indeed is a representation for $\text{LSC}(R)$ because different lower semicontinuous functions give rise to different such collections $f(a_i; b; c) \infty$; cf. [0:200], Lemma 3.3]

Lemma 18. $\text{LSC}(R \times \mathbb{R}) (f; x) \ y = f(x)$ is $(0 \mid 0)(\text{computationally})$.

Proof. Let $(a_k; b_k; c_k)$ denote the given $(0 \mid 0)(\text{name } e \text{ for } x \times \mathbb{R}) (R)$ and $(q_i)$, the given $(0)(\text{name } e \text{ for } x \times \mathbb{R}) \ 0 \text{ our goal is to } 0)(\text{compute } y = f(x))$. Define the sequence $p = (p_k)$, $Q \ (f+1) \ g$ by

\[ \begin{align*}
q_k &< m \quad c_m : m \quad k \quad (a_m; b_m) \quad (a_k; b_k) \\
q_{k+1} &< m + 1 \quad \text{ otherwise }
\end{align*} \]

From the given information, one can obviously compute $p$. Moreover, this sequence satisfies
\[ \lim \inf (y; 1) \]

Let \( y > 0 \) be arbitrary. Since \( f \) is lower semi-continuous, its preimage \( f^{-1}(y; 1) \) is an open set and therefore contains an entire ball around \( x \). In fact, the center of this ball may be chosen as rational and its diameter of the form \( 2^{-L} \) for some \( L \geq 2 \); formally (see Figure 6):

\[ 9K \leq L \leq 2 N : x \leq 2 \left( a_k ; b_k \right) \quad \left[ a_k ; b_k \right] f^{-1}(y; 1) \quad ^{(*)} \]

where we have exploited that every rational pair \((a, b)\) occurs in the list representing the \([\ldots] \) (nam e.M. onceover, as it consists of all rational triples \((a, b, c)\) with \( c < m \inf [a, b] \),

\[ 9M \leq K : \left[ a_k ; b_k \right] = \left[ a_M ; b_M \right] \quad ^{(*)} \leq y \quad 2 \]

with \((*)\) a consequence of \([a_k ; b_k] f^{-1}(y; 1) \) in Equation 2. Finally,

\[ \lim \inf \{a_k \} = 2 \left( a_k ; b_k \right) \quad 9N : 8n \leq N : q_n \leq 2 \left( a_k ; b_k \right) \]

So putting things together, for each \( n \leq N, \quad L^0, \) and \( k \leq M \), we either have \( p_{a_k ; b_k} = +1 \) \( y \leq 2 \); or we are in the first case of Equation 4, thus:
- \( q_n \leq 2 \left( a_k ; b_k \right) \) with \( a_k \leq a_k \), \( a_k \geq 2 \),
- \( q_n \leq 2 \left( a_k ; b_k \right) \) by Equation 4,
- \( a_k ; b_k \) by Equation 4 due to \( \leq L^0 \); see Equation 2.
- So \( a_k ; b_k \) by Equation 4,
- implying \( p_{a_k ; b_k} = q_n \) \( y \leq 2 \) by Equations 4 and 5 since \( k \leq M \).

Summarizing, it holds \( p_{a_k ; b_k} \leq 2 \) for all \((k, n) \leq N \) not belonging to the finite set \( f_0 ; f_1 ; \ldots ; N \) \( 1g f_0 ; f_1 ; \ldots ; L^0 \) \( 1g f_0 ; f_1 ; \ldots ; M \) \( 1g \) of exceptions. Consequently \( \lim \inf y \geq 2 \); even \( \lim \inf y \) because \( y > 0 \) was arbitrary.

\[ \text{Fig. 6. Nesting of some rational intervals of dyadic length contained in } f^{-1}(y; 1). \]

The parameters are chosen in such a way that, whenever \((a_k ; b_k) \) meets some other \((a_k ; b_k) \) of length \( b_k \), \( a_k \geq 2 \), for \( L^0 \geq L + 2 \), then \([a_k ; b_k] \) is entirely contained within the larger \([a_k ; b_k] \).
Lemma 20. Let $f: \mathbb{R} \to \mathbb{R}$ be lower semi-continuous and $x_n \to x$. Suppose that for some $y_n \in (x_n, x_{n+1})$, there exists $n_0$ such that $f(x_n) < y_n < f(x_{n+1})$ for all $n \geq n_0$. Then $f(x) = \liminf_{n \to \infty} y_n$.

Indeed, since the $y_n$ are in particular all rational pairs $(a_n, b_n)$ and these intervals are dense in $\mathbb{R}$, there exists every $\epsilon > 0$ and $N$ such that for all $n \geq N$, $|y_n - f(x)| < \epsilon$. For $\delta > 0$ sufficiently small, there exists $n_0$ such that $|y_n - f(x)| < \delta$ for all $n \geq n_0$. Therefore, for $\epsilon = \delta$, we have $\liminf_{n \to \infty} y_n = f(x)$.

Concluding, we have $\liminf_{n \to \infty} y_n = y$. Although $p$ may attain the value $+\infty$, this can easily be overcome by proceeding to $p_m = p_n$ for $p_n \leq 1$ and $p_m = m$ if $0 < p_n < 1$ because this transformation $p \mapsto p_m$ is defined by the first case in Equation (1) and thus agrees with some $c_n < \inf(a_m, b_m)$.

As $x \in 2 \{a_m, b_m\} = \{a_m, b_m\}$.

In order to obtain a similar uniform claim yielding Theorem 15, recall that every $(\ldots, \ldots)$ computable function $f: \mathbb{R} \to \mathbb{R}$ is necessarily both monotone increasing and lower semicontinuous (Fact 14). This suggests Definition 19. Let $MLSC(\mathbb{R})$ denote the class of all monotone increasing, lower semicontinuous functions $f: \mathbb{R} \to \mathbb{R}$. A $(\ldots, \ldots)$ name $e$ for $f \in MLSC(\mathbb{R})$ is an enumeration of the set $\{f(a); c \in \mathbb{Q}^2 : c < f(a)\}$.

Lemma 20. a) Distinct $f, g \in MLSC(\mathbb{R})$ have different sets $\{f(a); c \in \mathbb{Q}^2 : c < f(a)\}$ according to Definition 15, that is, $(\ldots, \ldots)$ constitutes a well-defined representation.

b) A function $f \in MLSC(\mathbb{R})$ is $(\ldots, \ldots)$ computable if it has a computable $(\ldots, \ldots)$ name $e$.

c) Let $f \in MLSC(\mathbb{R})$, $(a_k, c_k)$ with $f(a); c \in \mathbb{Q}^2 : c < f(a)g = f(a_k, c_k)$. Then, the rational sequence $p$ defined by

$$p_{k,m,r} = \begin{cases} m \land a_m & \text{if } a_k < c_q < a_k + 2 \\ +1 & \text{otherwise} \end{cases}$$

satisfies $\liminf_{n \to \infty} f = f(x) = y$.

d) Therefore, the apply operator $MLSC(\mathbb{R}) \to \mathbb{R}$ $(f, x) \mapsto f(x)$ is $(\ldots, \ldots)$ computable.

Proof. a) Let $f, g \in MLSC(\mathbb{R})$ with $f \leq g$, that is, $\limsup_{x \to x_0} f(x) < g(x_0)$ for some $x_0$. There exists some $c_0 \in \mathbb{Q}$ with $f(x_0) < c_0 < g(x_0)$. Being monotone increasing and lower semicontinuous, their pre-images $f^{-1}(c_0)$ and $g^{-1}(c_0)$ are open intervals $f^{-1}(c_0)$ and $g^{-1}(c_0)$, respectively. As $x_0$ belongs to the second but not to the rest, we have $x_0 < x_f < x_0$ and therefore $x_0 < a_0 < x_f$ for some $a_0 \in \mathbb{Q}$. Then $a_0 \in \{a_0, b_0\}$. Thus, $a_0 \in \mathbb{Q}$ yields $c_0 < g(a_0)$ whereas $a_0 \notin \{x_f, b_0\}$ as $a_0 \notin f(a_0)$. 

Theorem 15. Let $f: \mathbb{R} \to \mathbb{R}$ be lower semi-continuous and $x_n \to x$. Suppose that for some $y_n \in (x_n, x_{n+1})$, there exists $n_0$ such that $f(x_n) < y_n < f(x_{n+1})$ for all $n \geq n_0$. Then $f(x) = \liminf_{n \to \infty} y_n$.

Indeed, since the $y_n$ are in particular all rational pairs $(a_n, b_n)$ and these intervals are dense in $\mathbb{R}$, there exists every $\epsilon > 0$ and $N$ such that for all $n \geq N$, $|y_n - f(x)| < \epsilon$. For $\delta > 0$ sufficiently small, there exists $n_0$ such that $|y_n - f(x)| < \delta$ for all $n \geq n_0$. Therefore, for $\epsilon = \delta$, we have $\liminf_{n \to \infty} y_n = f(x)$. 

Concluding, we have $\liminf_{n \to \infty} y_n = y$. Although $p$ may attain the value $+\infty$, this can easily be overcome by proceeding to $p_m = p_n$ for $p_n \leq 1$ and $p_m = m$ if $0 < p_n < 1$ because this transformation $p \mapsto p_m$ is defined by the first case in Equation (1) and thus agrees with some $c_n < \inf(a_m, b_m)$.

As $x \in 2 \{a_m, b_m\} = \{a_m, b_m\}$.
Let $M$ denote a Type-2 machine $(\ldots)(\text{computing } f : 2 \to \mathbb{L}SC(R))$. Evaluating $f$ at a $2 \to Q$ by simulating $M$ on the $(\ldots)(\text{name } (a;\alpha;\alpha;\ldots))$ for a thus yields a $(\ldots)(\text{name } f(a))$ which is (equivalent to) a list of all $2 \to Q$ with $c < f(a)$ \[ \text{naming } Q. \] So dovetailing this simulation for all a $2 \to Q$ yields the desired $(\ldots)(\text{name } f)$.

Conversely, knowing a $(\ldots)(\text{name } (a_k;\alpha_k))$ for $2 \to \mathbb{L}SC(R)$ and given an increasing sequence $(q_i)$ $Q$ with $x = \sup q_i$, let

\[
p_n = c_n \text{ if } a_n < q_p; \quad p_n = 1 \text{ otherwise .}
\]

Then, in the first case, $p_n = c_n < f(a_n)$ \[ \text{f for } (q_i) \] $f(x) = :y$ by monotonicity, and $p_n = 1$ $y$ in the second; hence $\sup p_n = y$. To see $\sup p_n = y$, $x$ arbitrary $> 0$ and consider the open half-interval $f^{-1}(y + \varepsilon)$ containing $x$ and thus also some rational $a = a_K 2 (x; x), K 2 N$. Further one $q_i$, $x$ yields some $e N \to 2 N$ such that $q_i 2 (a_K ;x)$ for all $n \in N$. And finally there exists $M N$ with $a_M = a_K$ and $q_i f(a_M)$. Together this asserts $\exists q_i > a_K = a_M$ because $M N$ and thus $p_n = q_i$, $f(a_K) > y$ due to $\forall 2 f^{-1}(y + \varepsilon)$. For arbitrary.

To see the reverse inequality $\liminf y^n$, take arbitrary $' 2 N$. There exists $k 2 N$ with $a_k < x < a_k + 2^l$ and, because of $\liminf q_i = x$, also $n 2 N$ with $a_k < q_i < a_k + 2^l$. We therefore have in finitely many triples $(n,k;')$ for which $p_{n,k;'}$ agrees with a certain $c_m < f(a_m)$ \[ \text{f for } (a_k) \] $f(x) = y$.

Given a $(\ldots)(\text{name } x$, one can obtain a sequence $(q_i)$ $Q$ with $x = \liminf q_i$ by virtue of Scholium \[ \text{b}. \] From this, the sequence $p$ $Q$ with $\liminf = f(x)$ according to c) is obviously computable and yields, again by Scholium \[ \text{b}, \] \[ \text{a} \] $(\ldots)(\text{name } y = f(x))$.

Concluding this subsection, the classes of $(\ldots)(\text{computable real functions } f : R \to R)$ form, for $d = 0, 1, \ldots$ respectively, a hierarchy. By Fact \[ \text{b}, \] this hierarchy is strict as can be seen from the constant functions $f(x)$ $c$ with $2 \to c, d + 1 R$.

### 4.2 Arithmetic Weierstrass Hierarchy

Section \[ \text{b} \] established the sequence $(\ldots)$ of increasingly weaker representations for $R$ to yield the strict hierarchy of $(\ldots)(\text{computable}$, $(\ldots)$
com putable, and \( f \) \((\text{com putable functions} f : [0;1])\) \( R \). We now compare these classes with those induced by the other kind of realhypercomputation suggested in Section 2; relative to the Halting Problem \( H = \omega \), and its iterated jumps \( \omega, \ldots \).

Such a computable makes sense because both weakly and oracle-computable real functions are necessarily continuous according to Fact 10d)/Theorem 11c) and Lemma 8, respectively.

**Proof.**

 paretheseclasseswiththoseinducedbytheotherkindofrealhypercomputation to \( a \) \( f \) \((\text{from allowing the fast convergent sequence (}) \; \text{or} \; \text{sequence of rational polynomials (}) \)

The aforementioned other approach to continuous real hypercomputation arises from allowing the fast convergent sequence \( \{P_n\} \) to be computable in \( \omega \) or \( \omega \). The \( f \text{computable} f : [0;1] \) \( R \) have in fact already been characterized by Ho as Claim a) of the following

**Lemma 22.**

\( a \) To a real function \( f : [0;1] \) \( R \), there exists a \( P \) \((\text{computable sequence of rational polynomials (}) \text{such that}) \)

\[ \text{Lemma 22.} \quad \text{a) To a real function} f : [0;1] \text{!} \quad \text{R} \text{!} \quad \text{there exists a} \quad \text{computable sequence of rational polynomials} P_n \quad \text{such that} \quad \text{f} \quad \text{is} \quad \text{computable} \]

\[ \text{b) For an arbitrary oracle A, the sequence (of discrete degrees and numerators/denominators of the coefficients of) } (P_n) \quad \text{in } Q \quad \text{such that} \quad \text{A}(\text{computable}) \]

\[ \text{c) To a real function} f : [0;1] \text{!} \quad \text{R} \quad \text{there exists a} \quad \text{computable sequence of rational polynomials} P_n \text{\text{such that}} \]

\[ \text{Notice the similarity of Claim s a + c) to Fact 3b).} \]

**Proof.**

\( a \) See [Ho99, Theorem 16].
Lemma 22. By the Fact 21, their respective ground-levels coincide. Our next result compares equality of real numbers.

Let $f, g : [0,1] \to \mathbb{R}$ be computable. Then $f = g$ if and only if there exists a $k \in \mathbb{N}$ such that $|f(x) - g(x)| < 2^{-k}$ for all $x \in [0,1]$.

Theorem 23. a) Let $f : [0;1]$ be computable (in the sense of Lemma 22a). Then, $f$ is $(\leq^0, \leq^0)$-computable.

b) Let $f : [0;1]$ be $(\leq^0, \leq^0)$-computable. Then, $f$ is $(\leq^0, \leq^0)$-computable.

c) There is a $(\geq^0, \geq^0)$-computable function $f : [0;1]$ such that $f(x) > x$ for all $x \in [0,1]$. The idea to c) is that every $(\leq^0, \leq^0)$-computable function $f : [0;1]$ has a modulus of uniform continuity that is recursive in $\delta$. Whereas a $(\leq^0, \leq^0)$-computable function $f$, although uniform, is not necessarily recursive in $\delta$.

Before proceeding to the proof, we first provide a tool which turns out to be useful in the sequel. It is well-known in Recursive Analysis that, although equality of real numbers is undecidable due to the Main Theorem, inequality is at least semi-decidable. The following lemma generalizes this to $0$ and to $(0,0)$-computable functions:

Lemma 24. a) Let $f : R \to [0;\infty)$. Then $f$ is $(\leq^0, \leq^0)$-computable.

b) Let $f : R \to (0, \infty)$. Then $f$ is $(\leq^0, \leq^0)$-computable.
c) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be \( (0, 1) \) computable. Then the property
\[
(a,b;c;m) \rightarrow Q^3 \quad \text{for } 8 \times 2 \quad [a;b] : c \quad 2 = f(x) = c + 2
\]
is decidable relative to \( \emptyset \).

Proof. a) is standard; c) follows from b) which is established as follows:
By lower semi-continuity of \( f \) reveals the mapping oracle, it thus becomes (since \( f \) is computable).

\[
\begin{align*}
&b) \text{ Let } a,b \in \mathbb{R}^+ \text{ and } f(x) = c,\quad x \in [a,b]. \\
c) \text{ Let } h : \mathbb{N} \rightarrow \mathbb{R} \text{ denote a computable function.}
\end{align*}
\]

\[
\text{Let } X = \{ x \in [a,b] : f(x) > c \text{ and } f(x) \text{ is computable} \}.
\]

By Theorem 11a), if \( h(x) \) exceeds \( x \) into the Type-2 Machine computing \( f(x) \), then \( f(x) \) is computable.

Proof (Theorem 23).

\[
\begin{align*}
a) & \text{ Let } (P_n) \text{ denote a computable sequence converging uniformly to } f. \text{ Let } x \in [0;1] \text{ be given as the limit of a sequence } (q_n) \rightarrow x. \text{ Then } P_n \rightarrow f_n(x) \text{ eventually converges to } f(x). \\
b) & \text{ Let } x \in [0;1] \text{ be given by (an equivalently) its decimal form of two rational sequences } (a_n) \text{ and } (b_n) \text{ with } f(x) = \lim_n a_n/b_n. \text{ There exists a rational sequence } (c_n) \text{ forming a } 0 \text{ sequence } f_n(x) = \lim_n c_n, \text{ satisfying } c \geq f(x) \text{ for all } m, \text{ and by virtue of Lemma 24c), such a sequence can be found with the help of a } \emptyset \text{ oracle.}
\end{align*}
\]

This reveals that \( f \) is \( \emptyset \)-recursive in the sense of [Ho99, Corollary 4.2] and thus, similarly to [Ho99, Section 4] and [E.R.89, Theorem 1.11] to obtain the counter-example

\[
\begin{align*}
f(x) = \begin{cases} 
\frac{1}{2^n} & \text{if } x = \frac{k}{2^n}, \text{ for some } k, n \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

(7)

(taken as a non-overlapping superposition of scaled translates of such pulses) to be \( C^1 \).

By Theorem 14, a sequence \( \{ 2^n \} \) is \( (0, 1) \) computable; in fact even uniformly in \( n : \lim_n q_n \). One can then establish the existence of a sequence \( \{ p_{n,M} \} \) converging uniformly to \( f(x) \) for all \( M \), \( N \) obtained by the disjoint supports of the terms of \( f(x) \) in Equation 1. Therefore \( \lim_{n,M} p_{n,M} = f(x) \), thus establishing \( (0, 1) \) computability of \( f \).

Suppose \( f \) was \( (0, 1) \) computable. Then, by virtue of [Ho99, Lemma 15],
It has a \(\theta^0\)(recursive modulus of uniform continuity; cf. Definition 6.2.6.2). In particular given \(n, N\), one can \(\theta^0\)(compute \(n, N\) such that \(x = \frac{m}{2}\) and \(y = \frac{x}{2}\) satisfy \(2^n f(x) f(y) = \theta a_m j\) contradicting that \((a_m)\) has no \(\theta^0\)(recursive modulus of continuity.

\[\Box\]

5 Type-2 Nondeterminism

Concerning the two kinds of real hypercomputation considered so far based on oracle-support and weak real number encodings that is recall that the according proofs of Fact [10] and Theorem [11] crucially rely on the underlying Turing machines to behave deterministically. This raises the question whether nondeterminism might yield the additional power necessary for evaluating discontinuous real functions like Heaviside's.

In the discrete (i.e., Type-1) setting where any computation is required to terminate, there are many possible choices of a nondeterministic machine and of course be simulated by a deterministic one; however already here subject to the important condition that all paths of the nondeterministic computation indeed terminate, cf. STV89. In contrast, a Type-2 computation realizes a transformation from \(\mathbb{N}\) to \(\mathbb{N}\) strings and is therefore a generally non-terminating process. Therefore, nondeterminism here involves an infinite number of guesses which turns out cannot be simulated by a deterministic Type-2 machine.

We also point out that nondeterminism has already before been revealed not only a useful but indeed the most natural concept of computation on \(\![\Box]\). More precisely, Büchi extended Finite Automata from \(\mathbb{N}\) to \(\mathbb{N}\) strings and proved that there, as opposed to deterministic, nondeterministic ones are closed under complement [Tho90] and thus the appropriate model of computation. Since automata and Turing machines constitute the bottom and top levels, respectively, of Chomsky's Hierarchy of classical languages \(\mathcal{L}\) (Type-1 setting), we suggest that over \(\mathbb{N}\) strings (Type-2 setting) both their respective counterparts, that is Büchi Automata and Type-2 Machines be considered nondeterministically; compare Figure 7.

The concept of nondeterministic computation of a function \(f : \mathbb{N} \to \mathbb{N}\) (as opposed to a decision problem) is taken from the famous Immerman-Szelepscenyi Theorem in computational complexity; cf. for instance Pap94.
the paragraph preceding Theorem 7.6: For $x \in \text{dom}(f)$, some computing paths of the according machine $M$ may fail by leading to rejecting states, as long as

1) there is an accepting computation of $M$ on $x$ and
2) every accepting computation of $M$ on $x$ yields the correct output $f(x)$.

This notion extends straightforwardly from Type-1 to the Type-2 setting:

Definition 25. Let $A$ and $B$ be sets with respective representations $A \downarrow B$. A function $f : A \rightarrow B$ is called nondeterministically (computable if some nondeterministic one-way Turing Machine $M$

\{ upon input of any $(a, b, c)$ for some $a \in \text{dom}(f)$,
\{ has a computation which outputs $a$ (name for $b = f(a)$) and
\{ every infinite computation of $M$ on $a$ outputs $a$ (name for $b = f(a)$).

This definition is sensible insofar as it leads to closure under composition:

Observation 26. Let $f : A \rightarrow B$ be nondeterministically (computable and $g : B \rightarrow C$ be nondeterministically (computable. Then, $g \circ f : A \rightarrow C$ is nondeterministically (computable.

A subtle point in Definition 25, the nondeterministic machine may ‘withdraw’ a guess as long as it does so within finite time.

Example 27 (‘Deciding’ the Arithmetic Hierarchy). Let $P \rightarrow N$ be recursive,

$$A = \{ x \in 2^\omega : 3xj \in \omega \}$$

on (or below) level $2k$ of Kleene’s Arithmetic Hierarchy. Then the function $\lambda_x : P \rightarrow N$ is nondeterministically computable:

Observe that $x \in \omega$ and $y \in \omega$ implies there exists $z \in \omega$ such that $x + y = z$. So given $x \in \omega$, let $M$ output \"0\" and then verify, while continuously splitting out blanks \"0\", that $\lambda_x(x) = 1$ indeed holds. To this end, the machine starts ‘guessing’ the values off = $(f_1, \ldots, f_k)$ restricted to $f_0; \ldots; f_n$ for $n = 1; 2; \ldots$ Simultaneously by means of dove-tailing, $M$ tries all $y \in 2^\omega$ and aborts in case that the assertion $\lambda_x(y) = f_0(y_1); \ldots; f_n(y_n)$ fails. Now if $x \in A$, then an appropriate $f$ exists, and ultimately ‘found’ by $M$, and leads to infinite execution; whereas if $x \notin A$, then $M$ will eventually terminate for any guessed $f$.

Since $f_0 y \in A$ and $2k + 1$, a machine $M$ can output \"0\" and then similarly verify $\lambda_x(x) = 0$. The main machine $M$, upon input of $x \in \omega$, nondeterministically chooses to proceed either like $M^+$, or like $M$. Its computation satisfies the requirements of Definition 25.

This condition is slightly stronger than the one required in Definition 14.
The power of nondeterministic computation permits conversion forth and back among representations on the Real Arithmetic Hierarchy from Definition 2:

Theorem 28 (Third Main Theorem of Real Hypercomputation). For each \( d = 0; 1; 2; \ldots \), the identity \( R^3 \times \mathbb{R} \) is nondeterministically \( (d+1)! \)-computable. It is furthermore nondeterministically \( (d+1)! \)-computable.

Proof. Consider first the case \( d = 0 \). Let \( x \in \mathbb{R} \) be given by a sequence \( (q_n) \) eventually converging to \( x \). Then, there exists a fast convergent Cauchy subsequence \( (q_{n_k}) \), that is, satisfying

\[
8 k \cdot j q_{n_k} - q_{n_{k+1}} j \leq (d+1)! n_k^{-(d+2)} \tag{8}
\]

and thus forming a \( (d+1)! \)-name for \( x \). To find this subsequence, guess iteratively for each \( k \in \mathbb{N} \) some \( n_{k+1} > n_k \) and check whether it complies with Inequality (8) for the (finitely many) \( k \); if it does not, we may abort this computation in accordance with Definition 2.

For \( d = 1 \), let \( x = \lim_{n \to \infty} x_n \) with \( x_n = \lim_{m \to \infty} q_{n+m} \). Then apply the case \( d = 0 \) to convert for each \( n \) the \( (d+1)! \)-name \( (q_{n+m}) \) of \( x_n \in \mathbb{R} \) into an according \( (d+1)! \)-name, that is, a sequence \( p_{n,m} \) satisfying \( j k p_{n,m} j \leq 2^n \). Its diagonal \( (p_{n,n}) \) then has \( k p_{n,n} j k x_n j 2^n \) and is thus a \( (d+1)! \)-name for \( x \). Higher levels \( d \) can be treated similarly by induction.

For \( (d+1)! \)-computability, let \( x \in (0; 2) \) be given by a fast convergent sequence \( (q_n) \). We guess the leading digit \( b_2 \) \( 0; 1 \) for \( x \)'s binary expansion \( b_2 \); in case \( b_2 = 0 \), check whether \( x > 1 \) a \( (d+1)! \)-name for \( x \) and if so, abort; similarly in case \( b_2 = 1 \), abort if it turns out that \( x < 1 \). Otherwise (that is, proceeding while simultaneously continuing the above \( \text{semi-decision process via dove-tailing} \) replace \( x \) by \( 2(x - b_2) \) and repeat guessing the next bit.

It is also instructive to observe how, in the case of non-unique binary expansion (i.e., for dyadic \( x \)), nondeterministic in the above \( (d+1)! \)-computability generates, in accordance with the third requirement of Definition 2, both possible expansions.

Theorem 28 implies that nondeterministic in the computability of real functions is largely independent of the representation under consideration. In striking contrast to the classical case (Corollary 16), where the e\text{activity} subtleties arising from different encodings had confused already Turing himself [Tur37].

Corollary 29. a) With respect to nondeterministic \( d \)-reduction \( \leq_d \), it holds

\[
b_2 \leq n < n_0 < n_0 < n_0 < n_0 < \ldots
\]

b) The entire Real Arithmetic Hierarchy of Weihrauch and Zheng is nondeterministically computable.

Proof. a) follows from Lemma \( \text{a} \) and Theorem 28.

b) Let \( x \in R \) for some \( d \in \mathbb{N} \). Then, \( x \in R \) is \( (d+1)! \)-computable by Definition 2; hence nondeterministically \( (d+1)! \)-computable by a. Alternatively combine Example 27 with Fact 2a).
In particular, this kind of hypercomputation allows for nondeterministic (evaluation of Heaviside's function by appending to the ( ! ) (computation in Example 5) a conversion from \( \mathbb{N} \) back to \( \mathbb{N} \). Section 5.1 establishes many more real functions, both continuous and discontinuous ones, to be nondeterministically computable, too.

5.1 Nondeterministic and Analytic Computation

We now show that Type-2 nondeterminism includes the algebraic so-called BCSS-model of real number computation due to Blum, Cucker, Shub, and Smale \([BSS89, BCSS98]\) employed for instance in Computational Geometry \([PS85, Section 1.4]\). As a matter of fact, nondeterministic real hypercomputation even covers all quasi-strongly \( \mathbb{Q} \) analytic functions \( f: \mathbb{R}^d \rightarrow \mathbb{R} \) in the sense of Chadzelek and Hotz \([CH99, Definition 5]\). The latter can be considered a synthesis of the Type-2 (i.e., infinite approximate) and the BCSS (i.e., finite exact) model of real number computation. Its computational power admits an elegant characterization (see Lemma 31b+c) in terms of the following Definition 30.

**Definition 30.** A \( (n) \) for \( x \in \mathbb{R} \) is some \( (q_n) \) such that

\[
9N 8n N : \ j_q \cdot x \cdot j 2^n : \tag{9}
\]

The encoding sequence of rational approximations must thus converge fast with the exception of some initial segment of finite yet unknown length. It corresponds to (computation by an Inductive Turing Machine in the sense of Bur04) which is roughly speaking a Type-2 Machine but whose output tape(s) need not be one-way \([Ae01, Section 2.1]\) provided that the contents of every cell ultimately stabilizes.

**Lemma 31.**

a) It holds \( \leq_H \leq_0 \).

b) A function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is \( (H \! \! \! H) \) (computable if it is computable by a quasi-strongly \( \mathbb{Q} \) (analytic machine.

c) By \( \leq_H \) (computation is equivalent to \( \leq_H \) (computability.

d) The class of \( (H \! \! \! H) \) (computable functions is closed under computable.

The above claims relativize.

**Proof.**

a) is immediate.

b) Observe that the robustness of the program \( \text{required in [CH99, top of p.157]} \) amounts to the argument \( x \in \mathbb{R} \) if being accessible by rational approximations \( q_n \), \( Q \) of error \( j_q \cdot x \cdot j 2^n \), that is, in terms of a \( (n) \) name. The output \( y = f(x) \) on the other hand proceeds by way of two sequences \( (p_n) \) \( (m) \) \( Q \) such that \( m \leq 0 \) and \( j_p \cdot y \cdot j = m \) holds for all sufficiently large. By effectively proceeding to an appropriate subsequence, we can w.l.o.g. suppose \( m = 2^n \), hence \( (p_n) \) is \( (n) \) for \( y = f(x) \).

c) By a), every \( (H \! \! \! H) \) (computation is \( (H \! \! \! H) \) (computable, too.

For the converse implication, take the Type-2 Machine \( M \) computing \( \text{name for } x \in \mathbb{R} \) to \( (n) \) (name for \( y = f(x) \)). Let \( (p_n) \) satisfy Equation (9) for some unknown \( N \).
Example 33. Let us illustrate Proposition 32a) with the following

Proof. It is also \((k,q,n,\ldots)\) that \(M\) has output only finitely \((\text{say } M \vdash \exists N)\) in any

\(P_{\text{m}} \subseteq 2 \cdot Q\). In that case, restart \(M\) on \((q,E)\), presuming \(N = 1\) while, again,

checking this presum ption consistent with \((9)\), but this time throw away the

first \(M\) elements of the sequence printed by \(M\). Continue analogously for

\(N = 2; 3; \ldots\).

We claim that this yields output of \(f\) (name for) \(y\). Since \((q,E)\) is a valid \((\text{nam e for})\) \(y\), a feasible \(N\) will eventually be found. Before that happens, the several partial

runs of \(M\) have produced only finitely \((\text{say } M \vdash \exists N)\) any rational

numbers \(p_{\text{m}}\); and after that, the \(P_{\text{m}}\) run generates by presum ption a valid \((\text{name for})\) \(y\). Out of this sequence \((p_{\text{m}})\), the \(r\)st \(M\) entries

may have been exchanged by outputs of previous simulation trials; however

according to Definition \(\text{H}\), the representation \(\text{H}^\alpha\) is immune against such

name modi  cations.

d) Quasi-strongly \((Q\text{ (analytic functions are closed under composition according to CH99, Lemma 2)})\) now apply b+c.

A BCSS \((\text{or, equivalently, an } R\{0\})\) machine \(M\) is permitted to store a finite

number of arbitrary real constants \(r_1; \ldots; r_n\) \(\text{CH99, Instruction 1(b) in Table } 1\)
on p.154] and use it for instance to solve the Hal ting or any other \text{xed discrete}

problem \(\text{CSSS}\), Example 6]. Slightly correcting \(\text{CH99, Theorem 3}\), \(M\)'s

simulation by a rational \(R\{0\}\) machine thus requires knowledge of \(r = (r_1; \ldots; r_n)\) \(2 \cdot R^k\); e.g., by virtue of oracle access to \(0 = \text{fbin}(n) = \lfloor \lg f_0; \lg^2 \text{nat enc of a } k_{\text{bin}}\text{name}\).

\(2 f_0; \lg^2 \text{ of } r)\) \(\text{com p}\) by \(\text{BV99}\) for the case of simulating \((\text{semi})\) decidability.

Proposition 32. a) A function \(f : R \rightarrow R\) computable by a BCSS \((\text{machine})\)

with constants \(r \cdot R^k\) is also \((\text{name for})\) \(\{\text{computable relative to } r\}.

b) Every \((\ldots)\) computable function \(f : R \rightarrow R\) is also \((\text{name for})\) \(\{\text{computable}.

c) Let \(f : R \rightarrow R\) be \((\text{name for})\) \(\{\text{computable relative to some oracle } O\}

in \((\text{Kleene's})\) Arithmetic Hierarchy. Then \(f\) is nondeterministically \(\text{Type-2} \\text{computable}.

Proof. a) See (the proof of) \(\text{CH99, Theorem 3}\).

b) Combine Lemma \(\text{S}\) and c).

c) The nondeterministic simulation can answer queries to \(O\) due to Example \(P\).

As \(n \rightarrow n \rightarrow 0\) by Corollary \(\text{P}\) and Lemma \(\text{S}\), the claim follows. \(\text{\Box}\)

Let us illustrate Proposition \(\text{S}\) a) with the following

Example 33. Heaviside's Function \(h : R \rightarrow f_0; \lg^2\) is trivially BCSS \(\text{computable}.

It is also \((\text{name for})\) \(\{\text{computable by means of conservative branching} : G \text{\ then}

\(x \cdot R\) by virtue of \((q_i)\) \(Q\) with \(s\) and unknown \(N \rightarrow 2\). \(N\), \(let p_n = 0\) if

\(q_n = 2^n\) and \(p_n = 1\) otherwise.

Indeed if \(x = 0\) then, for all \(N, q_n \rightarrow 2^n\) and thus \(p_n = 0 = f(x)\). If on the
other hand $x > 0, x > 2^M$ for some $M \geq 2 N$; then, for all $n \geq M x f M + 1; N g, q_n > 2^n$ so $p_n = 1 = f(x)$.

Of course the class of nondeterministic Type-2 Machines (and thus also that of the nondeterministically computable real functions) is still only countably in finite most (even constant) functions $f : R \rightarrow R$ actually remain infeasible to this kind of real hypercomputation.

6 Conclusion

Recursive Analysis is often criticized for being unable, due to its Main Theorem, to non-trivially treat discontinuous functions. Although one can in Type-2 Theory devise sensible computability notions for, say, generalized (and in particular discontinuous) functions as for instance in [AW 03], evaluation $x \rightarrow f(x)$ of an $L^2$ function or a distribution $f$ at a point $x \in R$ does not make sense here already mathematically. Regarding the Main Theorem's connection to the Church-Turing Hypothesis indicated in the introduction, the present work has investigated whether and which models of hypercomputation allows for lifting that restriction.

A first idea, relativized computation on oracle Turing Machines, was ruled out right away. A second idea, computation based on weakened encodings of real numbers, renders evaluation $x \rightarrow h(x)$ of Heaviside's function, for instance ($\mu 0 \times \mu$) computable. The drawback of this notion of real hypercomputation: it lacks closure under composition.

Example 34. Let $f : R \rightarrow R$, $f(0) = 0$ and $f(x) = 1$ for $x \neq 0$. Let

$g(x) = x$. Then both $f$ and $g$ are ($\mu 0 \times \mu$) computable but their composition $g \circ f : 0 \rightarrow 0, 0 \neq 0 \rightarrow 1$ lacks lower semi-continuity.

Requiring both argument and value $y = f(x)$ to be encoded in the same way, say, $0$, or $00$), asserts closure under both composition and negation $f \rightarrow f$; and the prerequisites of the Main Theorem applies only to the case ($\mu 0 \times \mu$). Surprisingly, ($\mu 0 \rightarrow 0$) computability and ($\mu 0 \rightarrow 0$) computability still require continuity! These results extend to ($\mu d \rightarrow d$) computability for arbitrary $d$, although already the step from $d = 1$ to $d = 2$ made proofs significantly more involved.

These claims immediately relativize, that is, even a mixture of oracle support and weak real number encodings does not allow for hypercomputational evaluation of discontinuous functions. This is due to the purely information-theoretic nature of the argument employed, specifically: the deterministic behavior of the Turing $M$ machines under consideration.

So we have finally looked at nondeterminism as a further way of enhancing the underlying machine model beyond Turing's barrier. Over the Type-2 setting of infinite strings, this parallels Büchi's well-established generalization of finite automata to so-called regular languages. While the practical realizability of Type-2 nondeterminism is admittedly even more questionable than that of
classical NP-machines, it does yield an elegant notion of hypercomputation with nice closure properties and invariant under various encodings.

A precise characterization of the class of nondeterministically computable real functions will be subject of future work.

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