Iterative technique for a beam lying on a transversely isotropic foundation

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Abstract. The effect of the foundation heterogeneity on the mechanical behaviour of a beam on Vlasov soils is discussed. According to a refined Vlasov soil model, the static problem of beams lying on transversely isotropic soils can be solved by an iterative method. In this paper, based on the energy variational principle, the differential equations for beams under an axial force on refined Vlasov foundations are derived. The methods for solving the internal forces and deformations of beams lying on refined elastic foundations are given. Additionally, an equation for the attenuation parameters is also established, and the characteristic parameters of the refined model are solved by iterative technique. Numerical results show that the foundation heterogeneity have a influence on the deformations and internal forces of the beam-soil system. Moreover, relatively accurate characteristic parameters can be obtained through the iterative process. The refined Vlasov model has broad application prospects.

1. Introduction
The calculation and analysis of the interaction between foundations and structures in engineering are almost impenetrable. Research in this field is not only of great significance for mechanics, but also promotes the development of structural engineering and geotechnical engineering. It is of great significance to be able to calculate and analyse these models accurately and reasonably. However, due to its mechanical and mathematical complexities, it has always been a difficult problem in relation to the contact problems of solid mechanics. The analysis of structures on elastic foundations includes the mechanical analysis of the structures and selection of foundation models [1-3]. The Vlasov foundation model is a kind of two-parameter model. According to the Vlasov model, the mechanical properties of elastic foundations can be represented and characterized by two or more physical parameters.

The mechanical analysis of structures on elastic foundations is one of the research contents of soil-structure interaction (SSI) [4-5]. A lot of research has been done on the issue. Gedikli [6] studied eigenvalue problems of the Euler Bernoulli beams resting on Vlasov foundations. Höller et al. [7-8] carried out a semi-analytical thermoelastic multiscale analysis. Miao et al. [9] obtained the closed-form solution of beams resting on Pasternak elastic models.

The research results on soil-structure interaction are becoming increasingly mature. However, previous studies on soil-structure interaction are not perfect. The simple Winkler model obviously can not accurately describe the mechanical characteristics of actual complex soils. For a traditional Vlasov foundation, the soil is considered isotropic and homogeneous. However, the soil is closer to a transversely isotropic elastic body. The progress in this field has been slow. Fabrikant [10] solved the contact problems for transversely isotropic bodies using a generalized images method.
Besides, previous studies rarely considered the influence of axial forces on beams. In practical engineering, the situation of a foundation beam subjected to axial forces is common. Moreover, the foundation beam is often simplified to be free at both ends. However, the boundary conditions in the actual situation are not always free, and the two ends are often clamped.

As another example, a two-parameter foundation adopts two characteristic constants to represent the reaction modulus and shear modulus of physical models. However, the attenuation parameter needs to be estimated. Yang [11] thought that there was no suitable method to calculate the attenuation parameters. Jones and Xenophontos [12] made an experimental study on two-parameter foundation models. Yue [13] used an iterative procedure to analyse the bending behaviour of a beam on the Gibson foundation. Vallabhan and Das [14-16] studied and iteratively calculated the attenuation parameters of beams on Vlasov foundations.

The subject of this paper is the analysis of elastically supported beams. Such conditions of support can be found in a large variety of engineering problems. However, in other problems, the concept of beam and foundation is more of an abstract nature. All these problems are related through an affinity in the mathematical formulation. Hence, considering the transverse isotropy of foundation soils, the static problem of beams on Vlasov soils is analysed systematically in this study. The effect of transverse isotropy on the mechanical behaviour of beams resting on foundations is investigated. An iterative technique is proposed to determine the value of the attenuation parameter.

2. Potential energy function

A brief account of the beam-soil system is given here. The beam resting on a transversely isotropic foundation is studied in Figure 1. The thickness of beam is $h$, length is $L$, and depth of soil is $H$. An axial force applied to a beam is $F$. The friction between the beam and foundation is ignored.

A beam with unit width is taken as the research object, giving a typical plane strain problem. The potential energy function $V$ is

$$V = V_b + V_s + V_q$$

where $V_b = \text{strain energy in the beam}$, $V_s = \text{strain energy in the transversely isotropic soil}$, and $V_q = \text{external potential energy function}$.

$$V_b = \frac{1}{2} \int_0^L EI (w')^2 \, dx$$

$$V_s = \frac{b}{2} \int_{-\infty}^{\infty} \int_0^H \left( \sigma_x \varepsilon_x + \sigma_z \varepsilon_z + \tau_{xz} \gamma_{xz} \right) \, dx \, dz$$

$$V_q = -\int_0^L q(x)w \, dx - \frac{F}{2} \left( \frac{d^2 w}{dx^2} \right)^2 \bigg|_0^L$$

where $E$ is the modulus of elasticity, $w$ is the deflection, $I$ denotes the moment of inertia of the cross section of the beam, $b$ denotes the section width, $\sigma_x, \sigma_z, \tau_{xz}$ and $\varepsilon_x, \varepsilon_z, \gamma_{xz}$ are the stress and strain components of the transversely isotropic soils, respectively [14-16]. $EI$ denotes the flexural stiffness, and $q(x)$ and $P$ are the general loadings.
Figure 1. The beam lying on a transversely isotropic foundation.

Only consider the continuity of vertical deformations at the contact between the structure and foundation. It is worth noting that \( w_s \) is considered to be close to 0 and the attenuation function \( \varphi(z) \) describes the variation of vertical deformation \( w_s(x, z) \) along the depth, i.e.,

\[
 w_s(x, z) = \overline{w_s}(x)\varphi(z)
\]

where \( \overline{w_s}(x) \) is the vertical displacement of the transversely isotropic soil surface. The deflections of the beam and foundation have the following relationship.

\[
 w(x) = \overline{w_s}(x)
\]

Due to long-term deposition, natural soils are often distributed in layers and present transverse isotropic properties. They are generally isotropic parallel to the bedding direction, but anisotropic perpendicular to the bedding direction due to the difference in mineral composition and physical properties of the deposit. Therefore, transversely isotropy is a closer description of the properties of real soils than the homogeneous isotropy. In the process of foundation design, if these foundations are considered as homogeneous isotropic foundations, it may cause uneven settlements of building foundations, misestimate the deformations of structures, and undermine the safety of buildings.

The relationship between stress and strain in the transversely isotropic medium is:

\[
\begin{align*}
\sigma_x &= C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{13}\varepsilon_z + \tau_{xy} = C_{66}\gamma_{xy} \\
\sigma_y &= C_{12}\varepsilon_x + C_{11}\varepsilon_y + C_{13}\varepsilon_z + \tau_{yz} = C_{44}\gamma_{yz} \\
\sigma_z &= C_{13}\varepsilon_x + C_{13}\varepsilon_y + C_{33}\varepsilon_z + \tau_{zx} = C_{44}\gamma_{zx}
\end{align*}
\]

where \( \sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx} \) and \( \varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \) represent the stress and strain components at a point in the transversely isotropic medium, respectively, and \( C_y \) are the material parameters of a transversely isotropic foundation. There is the following relationship between the parameters \( C_y \) and engineering constants \( (\mu_{s1}, E_{s1}, \mu_{s2}, E_{s2}, G_s) \):
where \( \mu_{s1}, E_{s1} \) and \( \mu_{s2}, E_{s2} \) are the Poisson's ratio and elastic modulus in the isotropic direction, and in the direction perpendicular to the isotropic direction, respectively, and \( G_s \) is the shear modulus. \( \lambda \) and \( N \) are the physical constants.

Substituting the constitutive equations and geometric equations for foundations into the potential energy (3), equation (3) can be rewritten as:

\[
V_s = \frac{1}{2} \int_{-\infty}^{\infty} \left[ k \bar{w}_s^2 + G_p \left( \frac{d\bar{w}_s}{dx} \right)^2 \right] dx
\]

\[
G_p = \int_{0}^{H} C_{44} b \phi^2 dz
\]

\[
k = \int_{0}^{H} C_{33} b \left( \frac{d\phi}{dz} \right)^2 dz
\]

where \( k, G_p \) are the characteristic parameters of a transversely isotropic model. That is, \( k \) and \( G_p \) are the reaction modulus and shear modulus, respectively.

The total potential energy function is:

\[
V = \frac{1}{2} \int_{0}^{L} EI (w')^2 dx + \frac{1}{2} \int_{-\infty}^{\infty} \left[ k \bar{w}_s^2 + G_p \left( \frac{d\bar{w}_s}{dx} \right)^2 \right] dx - \int_{0}^{L} q(x)wdx - \frac{F}{2} \int_{0}^{L} \left( \frac{d\bar{w}_s}{dx} \right)^2 dx
\]

3. Governing equations and iterative solution

By minimizing the function \( V \) with respect to \( w, \bar{w}_s, \) and \( \phi \), we obtain the following equation:

\[
\delta V = EI \int_{0}^{L} \frac{d^3w}{dx^3} \delta w dx + EI \frac{d^2w}{dx^2} \int_{0}^{L} \frac{d^2w}{dx^2} \delta w dx - EI \frac{d^3w}{dx^3} \Bigg|_{0}^{L} - \int_{-\infty}^{0} G_p \frac{d^2\bar{w}_s}{dx^2} \delta \bar{w}_s dx - \int_{0}^{L} G_p \frac{d^2\bar{w}_s}{dx^2} \Bigg|_{0}^{L} - \int_{0}^{L} G_p \frac{d^2\bar{w}_s}{dx^2} \delta \bar{w}_s dx
\]

\[
+ \int_{-\infty}^{0} k \bar{w}_s \delta \bar{w}_s dx + \int_{0}^{L} k \bar{w}_s \delta \bar{w}_s dx + \int_{0}^{L} k \bar{w}_s \delta \bar{w}_s dx + G_p \frac{d\bar{w}_s}{dx} \int_{0}^{L} \delta \bar{w}_s dx
\]

\[
+ G_p \frac{d\bar{w}_s}{dx} \delta \bar{w}_s \Bigg|_{0}^{L} + \int_{0}^{L} \left( -m \frac{d\phi}{dz} + n \phi \right) \delta \phi dz + m \frac{d\phi}{dz} \delta \phi \Bigg|_{0}^{L}
\]

\[
- \int_{0}^{L} q(x) \delta w dx - F \frac{d\bar{w}_s}{dx} \delta \bar{w}_s \Bigg|_{0}^{L} + F \int_{0}^{L} \frac{d^2w}{dx^2} \delta w dx = 0
\]

where \( \bar{w}_s1 \) and \( \bar{w}_s2 \) are the vertical deformations of the soil surface on two sides of a beam [13-16].
The governing domain equation for a beam \( 0 < x < L \) is established by mathematical derivation:

\[
EI \frac{d^4 w}{dx^4} + (F - G_p) \frac{d^2 w}{dx^2} + kw = q(x)
\]  

(17)

The governing domain equations for the foundation surface outside the beam are also obtained. Left side \( x < 0 \):

\[
-G_p \frac{d^2 w_1}{dx^2} + kw_1 = 0
\]

(18)

Right side \( x > L \):

\[
-G_p \frac{d^2 w_2}{dx^2} + kw_2 = 0
\]

(19)

In the same way, the governing equation for \( \varphi \) is derived.

\[
\frac{d^2 \varphi}{dz^2} - \left( \frac{\gamma}{H} \right)^2 \varphi = 0
\]

(20)

where \( \varphi(z) \) is a attenuation function. The boundary conditions are \( \varphi(0) = 1 \) and \( \varphi(H) = 0 \), and the expression for \( \varphi \) is derived.

\[
\frac{n}{m} = \left( \frac{\gamma}{H} \right)^2 = \frac{C_{44}}{C_{33}} \int_0^{+\infty} \frac{(dw/dx)^2 dx}{w^2 dx}
\]

(22)

In addition, the boundary conditions for the transversely isotropic foundation beam are derived using equation (14).

\[
EI \frac{d^4 w}{dx^4} \left[ \frac{d\delta w}{dx} \right]_0^L - EI \frac{d^4 w}{dx^4} \left[ \frac{d\delta w}{dx} \right]_0^L - F \frac{d\delta w}{dx} \left[ \frac{d\delta w}{dx} \right]_0^L = 0
\]

(23)

The analytical solution to the equation (17) can be expressed as

\[
w(x) = e^{\eta x} \left[ C_1 \sin(\chi x) + C_2 \cos(\chi x) \right] + e^{-\eta x} \left[ C_3 \sin(\chi x) + C_4 \cos(\chi x) \right]
\]

(24)

where \( C_1, C_2, C_3, C_4 \) are four constants determined form the specified boundary conditions at the beam ends. The parameters \( \eta \) and \( \chi \) are coefficients derived from the theory of differential equations.

The parameters of soils in elastic foundation models are generally determined by experiments or experiences. Generally, in the foundation calculations, the value of Winkler parameter \( k \) (modulus of subgrade reaction) is determined according to experiences after consulting relevant tables. However, the iterative procedure is adopted to obtain the attenuation parameter \( \gamma \) in this study. The iterative steps are as follows: a) Initialize the attenuation parameter. b) Obtain the attenuation function using equation (21). c) Calculate the characteristic parameters form equations (11) and (12). d) Obtain the corresponding deformations by solving equation (17). e) Determine the new attenuation parameter according to equation (22). f) Repeat the above steps until the difference is less than a specified tolerance. If the initial value is close to the exact value, the convergence speed is faster.
4. Numerical analysis

Example 1 has been solved to verify the correctness of theoretical derivation and numerical calculation. The modulus of elasticity $E_s = 2.6 \times 10^7 \text{ N/m}^2$, Poisson’s ratio $\mu_s = 0.32$, depth $H = 2 \text{ m}$. Other parameters can be found in literature [5]. The numerical results of the physical model after degradation are in good agreement with those of other studies.

Example 2. The static problem of a beam lying on a transversely isotropic foundation is analysed. Length $L = 10 \text{ m}$, height $h = 0.25 \text{ m}$, width $b = 1 \text{ m}$, Poisson’s ratio $\mu = 0.18$, and modulus of elasticity $E = 2.4 \times 10^{10} \text{ N/m}^2$. The concentrated force is $P = 420 \text{ kN}$, and the axial force is $F = 4000 \text{ N}$, and the uniformly distributed loading is $q = 380 \text{ N/m}^2$. The parameters of the transversely isotropic foundation are as follows. The depth of soil is $H = 2.5 \text{ m}$, modulus of elasticity and Poisson’s ratio are $E_{s1} = 47.04 \text{ MPa}$, $E_{s2} = 35.28 \text{ MPa}$, $\mu_{s1} = 0.3$, $\mu_{s2} = 0.3$, and the shear modulus is $G_s = 18 \text{ MPa}$.

In Table 1, the second row shows the characteristic parameters and deflection of the beam on a transversely isotropic foundation when both ends of the beams are free. And the third row shows the corresponding values when one end of the beam is clamped and the other is fixed. In Table 1, $w$ refers to the displacement at the center. It is interesting to note that the different boundary conditions of the elastic foundation beams have little effect on the two characteristic parameters and attenuation parameters of transversely isotropic two-parameter foundation models. The Vlasov model requires an estimation of the attenuation parameter $\gamma$. The refined model proposed in this paper automatically obtains a consistent value of $\gamma$. The deformation and rotation angle of the beam with free ends are shown in Figures 2 and 3. In addition, Figures 4, 5, and 6 illustrate the bending moment, shear force of a beam and reaction force of the foundation. The trends of these graphs are the same as those in reference [5], thus verifying the accuracy and reliability of the mathematical model.

Table 1. Numerical results of beams lying on transversely isotropic elastic foundations in Example 2.

| Parameters | $\eta$ | $w$ (mm) | $\gamma$ | $k \left(10^7 \text{ N/m}^3 \right)$ | $G_s \left(10^7 \text{ N/m} \right)$ |
|------------|--------|----------|----------|---------------------------------|---------------------------------|
| Both ends are free. | | 3.785 | 0.5286 | 2.1507 | 1.2856 |
| One is clamped, the other is fixed. | | 3.697 | 0.5286 | 2.1507 | 1.2856 |

Figure 2. Deformation of the refined foundation beam.
Figure 3. Angle of the refined foundation beam.

Figure 4. Bending moment of the refined foundation beam.

Figure 5. Shear force of the refined foundation beam.
Figure 6. Reaction force of the transversely isotropic soil.

5. Conclusions
The research on soil-structure interaction (SSI) plays an important role in structural and geotechnical engineering. Considering a foundation composed of transversely isotropic soils, a modified elastic model is proposed. The following conclusions could be drawn:

1) The heterogeneity of the transversely isotropic soil affects the deflections, rotation angles, bending moments, shear forces of beams, and reaction modulus and shear modulus of the foundation models, which should be considered in practice.

2) Regarding the refined physical model, the characteristic parameters and mechanical responses can be calculated iteratively. The characteristic parameters, such as reaction modulus and shear modulus, are not unique for a certain type of soil. They depend on factors such as the depth of the foundation, the stiffness of the beam and foundation.

3) With the diversification of structural design types and the continuous enhancement of application requirements, the mechanical performance of beams on elastic foundations is becoming increasingly important. By considering the elastic-plastic characteristics of foundations, some more accurate solutions can be obtained.

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