RANK-LEVEL DUALITY AND CONFORMAL BLOCK DIVISORS

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Abstract. We describe new relations among conformal block divisors in Pic(\(\overline{M}_{0,n}\)). These relations appear from various rank-level dualities of conformal blocks on \(\mathbb{P}^1\) with \(n\) marked points.

1. Introduction

Let \(\mathfrak{g}\) be a simple Lie algebra, \(\mathfrak{h}\) a Cartan subalgebra and \(\ell\) a positive integer. Consider an \(n\)-tuple \(\vec{\lambda} \in P_\ell(\mathfrak{g})^n\), where \(P_\ell(\mathfrak{g})\) denotes the set of dominant integral weights of level \(\ell\) (see Section 2). Corresponding to this data, the moduli space of stable curves of genus 0 with \(n\) marked points \(\overline{M}_{0,n}\) carries a vector bundle \(\mathcal{V}_{\vec{\lambda}}(\mathfrak{g}, \ell)\). These vector bundles are known as conformal block bundles. The fibers of the conformal block bundles are referred to as conformal blocks. The rank \(\text{rk} \mathcal{V}_{\vec{\lambda}}(\mathfrak{g}, \ell)\) of a conformal block bundle \(\mathcal{V}_{\vec{\lambda}}(\mathfrak{g}, \ell)\) is given by the famous Verlinde formula. More generally, one can define conformal block bundles on the moduli stacks of genus \(g\) curves with \(n\) marked points, but in this paper we consider the case \(g = 0\). We refer the reader to [TUY] and [T] for more details.

Inspired by the work of Fakhruddin [F], we are interested in studying the conformal block divisors (divisors associated to the top exterior power of the conformal block bundles) to get geometric informations about \(\overline{M}_{0,n}\). The F-conjecture and the Mori Dream Space conjecture assert that the nef cone of \(\overline{M}_{0,n}\) is polyhedral. In [F], Fakhruddin showed that conformal block bundles are globally generated, hence their first Chern classes are all nef. Since it is known that many conformal block divisors are extremal, one may naturally hope that the cone of conformal block divisors is also polyhedral (this was first asked by Fakhruddin in [F]). But on the contrary, we possibly have an enormous supply of non-zero conformal block divisors in the nef cone of \(\overline{M}_{0,n}\). A very natural interesting question is: Are there identities between non-zero conformal block divisors?

The main motivation of the present paper is to produce new relations among conformal block divisors from rank-level duality isomorphisms. Rank-level duality is a duality between conformal blocks on \(\mathbb{P}^1\) with \(n\)-marked points (with chosen coordinates around the marked points) associated to two different Lie algebras. T. Nakanishi and A. Tsuchiya proved (see [NT]) that on \(\mathbb{P}^1\), certain conformal blocks of \(\mathfrak{sl}(r)\) at level \(s\) are dual to conformal blocks of \(\mathfrak{sl}(s)\) at level \(r\). Rank-level duality statements between conformal blocks for symplectic groups was proved by T. Abe [A]. The author in [MB] used diagram automorphisms and T. Abe’s result to produce new symplectic rank-level dualities. The author proved a rank-level duality for odd orthogonal groups in [M1].

2010 Mathematics Subject Classification. Primary 14E30, 17B67 Secondary 81T40.

The author was partially supported by NSF grant DMS-0901249.
Remark 1.1. The coordinate free description of the rank-level duality map on $\overline{M}_{0,n}$ is not straightforward. In Section 4 we discuss a conjectural coordinate free rank-level duality map on $\overline{M}_{0,n}$ and explain the geometry of the relations described in Theorem 1.3. The proof of the main result of this paper Theorem 1.3 is independent of the conjectural description.

We now state the main result of this paper. Let $h^*$ be the dual to a Cartan subalgebra of $h$ of $g$ and $Q$ be the root lattice of $(g,h)$. For $\Lambda \in h^*$, we define a subset $P^\Lambda_\ell(g) := \{\lambda \in P_\ell(g)|\Lambda - \lambda \in Q\}$. Consider a conformal embedding (see Section 3 for a definition) $\phi: g_1 \oplus g_2 \to g$, where $g_1$, $g_2$ and $g$ are simple Lie algebras with Dynkin multi-index $\ell = (\ell_1, \ell_2)$. We extend it to a map of affine Lie algebras $\hat{\phi}: \hat{g}_1 \oplus \hat{g}_2 \to \hat{g}$. Consider a level one integrable highest weight module $H_\Lambda(g)$, and restrict it to $\hat{g}_1 \oplus \hat{g}_2$. The module $H_\Lambda(g)$ decomposes into irreducible integrable $\hat{g}_1 \oplus \hat{g}_2$-modules of level $\ell = (\ell_1, \ell_2)$ as follows:

$$\bigoplus_{(\lambda,\mu) \in \hat{B}(\Lambda)} \tilde{m}^\Lambda_{\lambda,\mu} H_\Lambda(g_1) \otimes H_\mu(g_2) \simeq H_\Lambda(g),$$

where $\hat{B}(\Lambda)$ is a finite set parameterizing the components of the decomposition and $\tilde{m}^\Lambda_{\lambda,\mu}$ is the multiplicity of the component $H_\Lambda(g_1) \otimes H_\mu(g_2)$. It is important to point out that for $(\lambda,\mu) \in \hat{B}(\Lambda)$, it is necessary that $(\lambda,\mu) \in P^\Lambda_\ell(g_1) \times P^\Lambda_\ell(g_2)$. We refer the reader to Section 4 for more details. For any subset $A = \{a_1, \ldots, a_i\}$ of $\{1, \ldots, n\}$ and $\tilde{\lambda} \in P_\ell(g)^n$, we denote the $i$-tuple $\{\lambda_{a_1}, \ldots, \lambda_{a_i}\}$ by $\tilde{\lambda}_A$.

Remark 1.2. The following assumptions are motivated by rank-level duality of conformal blocks. All conformal embeddings for which rank-level duality hold are known satisfy these assumptions. Examples include the embeddings $\mathfrak{sl}(r) \oplus \mathfrak{sl}(s) \to \mathfrak{sl}(rs)$ and $\mathfrak{sp}(2r) \oplus \mathfrak{sp}(2s) \to \mathfrak{so}(4rs)$. We refer the reader to [11] for more details.

Consider $\tilde{\lambda} \in P_1(g)^n$, $\tilde{\lambda} \in P_{\ell_1}(g_1)^n$ and $\tilde{\mu} \in P_{\ell_2}(g_2)^n$ such that the following holds:

1. $\text{rk} V_{\tilde{\lambda}}(g, 1) = 1$, $\text{rk} V_{\tilde{\lambda}}(g_1, \ell_1) = \text{rk} V_{\tilde{\mu}}(g_2, \ell_2)$, and $(\lambda_i, \mu_i) \in \hat{B}(A_i)$ for $1 \leq i \leq n$.
2. For every partition $A \cup A^c = \{1, \ldots, n\}$ such that $|A|, |A^c| \geq 2$ and every $(\lambda, \mu) \in \hat{B}(\Lambda)$

$$\text{rk} V_{\tilde{\lambda}_A,\lambda}(g_1, \ell_1) = \text{rk} V_{\tilde{\mu}_{A^c},\mu}(g_2, \ell_2),$$

where $\Lambda$ is the unique weight in $P_1(g)$ such that $\text{rk} V_{\tilde{\lambda}_{A,\lambda}}(g, 1) = 1$.

3. Given any $\Lambda \in P_1(g)$ such that $\text{rk} V_{\tilde{\lambda}_{A,\lambda}}(g, 1) = 1$, there exists a bijection $f_\lambda$ between $P^\Lambda_{\ell_1}(g_1)$ and $P^\Lambda_{\ell_2}(g_2)$ such that $(\lambda, \mu) \in \hat{B}(\Lambda)$, where $\mu = f_\lambda(\lambda)$. The main result of the paper is the following (These assumptions hold for all the known rank-level dualities):

Theorem 1.3. With the above assumptions, the following relation among conformal blocks divisors holds in $\text{Pic}(\overline{M}_{0,n})$:

$$c_1(V_{\tilde{\lambda}}(g_1, \ell_1)) + c_1(V_{\tilde{\mu}}(g_2, \ell_2)) = \text{rk} V_{\tilde{\lambda}}(g_1, \ell_1) \left\{ c_1(V_{\tilde{\lambda}}(g, 1)) + \sum_{j=1}^n n^\Lambda_{\lambda_j, \mu_j} \psi_j \right\}$$

$$- \sum_{i=2}^{\frac{|\pi|}{2}} \epsilon_i \left\{ \sum_{A \subseteq \{1, \ldots, n\}} b_{A,A^c} D_{A,A^c} \right\},$$

where $\psi_j = -\frac{1}{2}\pi_j$ and $D_{A,A^c} = \text{deg} D_{A,A^c}$.
where \([D_{A,A^c}]\) denotes the class of the boundary divisor corresponding to the partition \(A \cup A^c = \{1, \ldots, n\}\), \(e_i = \frac{1}{2}\) if \(i = n/2\) and one otherwise, \(\psi_j\) is the \(j\)-th psi class, \(n_{\lambda,j}^{A^c}\) and \(b_{A,A^c}\) are non-negative integers as defined in Section 3 and Section 5.1 respectively.

**Remark 1.4.** Since \(\text{rk}V_{\Lambda}(\mathfrak{g}, 1) = 1\), by factorization (see [TUY]) we know that for every partition \(A \cup A^c = \{1, 2, \ldots, n\}\) there exists a unique \(\Lambda \in P_1(\mathfrak{g})\) such that \(\text{rk}V_{\Lambda}(\mathfrak{g}, 1) = 1\). The weight \(\Lambda\) depends on the partition but for convenience we do not include it in the notation.

1.1. **Idea of proof.** The main part of this project was to formulate the relations among conformal block divisors. A conceptual and geometric approach to the relations in conjecturally outlined in Section 5. Our main tool in the proof of Theorem 1.3 is Fakhruddin’s formula for Chern classes of conformal block divisors. We rewrite Fakhruddin’s formula with psi classes by using a Lemma in [FG]. Once this is done, the rest of the proof is a direct calculation to show that the left hand side of the relation in Theorem 1.3 is same as the right hand side.

1.2. **Acknowledgments.** I thank Prakash Belkale for helpful discussions during the preparation of this manuscript and for pointing out the twisting by psi classes. I would also like to thank Angela Gibney and Han-Bom Moon for useful discussions on the birational geometry of \(\overline{M}_{0,n}\).

2. **Basic Notations from Lie Theory**

Let \(\mathfrak{g}\) be a simple Lie algebra over \(\mathbb{C}\) and \(\mathfrak{h}\) a Cartan subalgebra of \(\mathfrak{g}\). The root system associated to \((\mathfrak{g}, \mathfrak{h})\) is denoted by \(Q\). The Cartan Killing form \((\cdot, \cdot)\) on \(\mathfrak{g}\) is normalized such that \((\theta, \theta) = 2\), where \(\theta\) is the longest root. We identify \(\mathfrak{h}\) with \(\mathfrak{h}^*\) using \((\cdot, \cdot)\). We define the affine Lie algebra \(\hat{\mathfrak{g}}\) to be

\[
\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}c,
\]

where \(c\) belongs to the center of \(\hat{\mathfrak{g}}\) and the Lie bracket is given as follows:

\[
[X \otimes f(t), Y \otimes g(t)] = [X, Y] \otimes f(t)g(t) + (X, Y) \text{Res}_{t=0}(gdf).c,
\]

where \(X, Y \in \mathfrak{g}\) and \(f(t), g(t) \in \mathbb{C}((t))\). Let \(X(n) = X \otimes t^n\) and \(X = X(0)\) for any \(X \in \mathfrak{g}\) and \(n \in \mathbb{Z}\). The finite dimensional Lie algebra \(\mathfrak{g}\) can be realized as a subalgebra of \(\hat{\mathfrak{g}}\) under the identification of \(X\) with \(X(0)\).

The finite dimensional irreducible modules of \(\mathfrak{g}\) are parametrized by the set of dominant integral weights \(P_+ \subset \mathfrak{h}^*\). Let \(V_\lambda(\mathfrak{g})\) denote the irreducible module of highest weight \(\lambda \in P_+\). We fix a positive integer \(\ell\) which we call the level. The set of dominant integral weights of level \(\ell\) is defined as follows:

\[
P_\ell(\mathfrak{g}) := \{\lambda \in P_+ | (\lambda, \theta) \leq \ell\}.
\]

For each \(\lambda \in P_\ell(\mathfrak{g})\) there is a unique irreducible integrable highest weight \(\hat{\mathfrak{g}}\)-module \(H_\lambda(\mathfrak{g})\) which satisfies the following properties:

1. \(V_\lambda(\mathfrak{g}) \subset H_\lambda(\mathfrak{g})\),
2. The central element \(c\) of \(\hat{\mathfrak{g}}\) acts by the scalar \(\ell\).
3. Conformal Embeddings

Let $\mathfrak{s}, \mathfrak{g}$ be two simple Lie algebras and $\phi: \mathfrak{s} \to \mathfrak{g}$ an embedding of Lie algebras. Let $(\cdot, \cdot)_s$ and $(\cdot, \cdot)_g$ denote the normalized Cartan killing forms such that the the length of the longest root is 2. We define the Dynkin index of $\phi$ to be the unique integer $d_\phi$ satisfying

$$(\phi(x), \phi(y))_g = d_\phi(x, y)_s$$

for all $x, y \in \mathfrak{s}$. When $\mathfrak{s} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is semisimple, we define the Dynkin multi-index of $\phi = \phi_1 \oplus \phi_2: \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$ to be $d_\phi = (d_{\phi_1}, d_{\phi_2})$.

If $\mathfrak{g}$ is simple, we define for any level $\ell$ and a dominant weight $\lambda$ of level $\ell$ the conformal anomaly $c(\mathfrak{g}, \ell)$ and the trace anomaly $\Delta_\lambda(\mathfrak{g}, \ell)$ as

$$c(\mathfrak{g}, \ell) = \frac{\ell \dim \mathfrak{g}}{g^* + \ell} \quad \text{and} \quad \Delta_\lambda(\mathfrak{g}, \ell) = \frac{(\lambda, \lambda + 2\rho)}{2(g^* + \ell)},$$

where $g^*$ is the dual Coxeter number of $\mathfrak{g}$ and $\rho$ denotes the half sum of positive roots. If $\mathfrak{g}$ is semisimple, we define the conformal anomaly and trace anomaly by taking sum of the conformal anomalies over all simple components.

**Definition 3.1.** Let $\phi = (\phi_1, \phi_2): \mathfrak{s} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$ be an embedding of Lie algebras with Dynkin multi-index $\ell = (\ell_1, \ell_2)$. We define $\phi$ to be a conformal embedding $\mathfrak{s}$ in $\mathfrak{g}$ at level $k$ if $c(\mathfrak{g}_1, \ell_1k) + c(\mathfrak{g}_2, \ell_2k) = c(\mathfrak{g}, k)$.

It is shown in [KW] that the above equality only holds if $k = 1$. Many familiar and important embeddings are conformal. For a complete list of conformal embeddings we refer the reader to [BB]. Next we recall an important property that characterizes conformal embeddings.

Let $\phi: \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$ be an embedding of Lie algebra with Dynkin multi-index $(\ell_1, \ell_2)$. We extend the map $\phi$ to a map of affine Lie algebras in the obvious way:

$$\hat{\phi}: \hat{\mathfrak{g}}_1 \oplus \hat{\mathfrak{g}}_2 \to \hat{\mathfrak{g}}.$$ 

We consider $\Lambda \in \mathcal{P}_1(\mathfrak{g})$ and let $V_\Lambda(\mathfrak{g})$ denote the highest weight irreducible module of the Lie algebra $\mathfrak{g}$. We restrict $V_\Lambda(\mathfrak{g})$ to $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. The $\mathfrak{g}$-module $V_\Lambda(\mathfrak{g})$ decomposes into direct sum of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ modules as follows

$$V_\Lambda(\mathfrak{g}) \simeq \bigoplus_{(\lambda, \mu) \in B(\Lambda)} m_{\lambda, \mu} V_\lambda(\mathfrak{g}_1) \otimes V_\mu(\mathfrak{g}_2),$$

where $m_{\lambda, \mu}$ is the multiplicity of the component $V_\lambda(\mathfrak{g}_1) \otimes V_\mu(\mathfrak{g}_2)$ and $B(\Lambda)$ is a finite set. Similarly for $\Lambda \in \mathcal{P}_1(\mathfrak{g})$, we consider the highest weight integrable irreducible $\hat{\mathfrak{g}}$-module $\mathcal{H}_\Lambda(\mathfrak{g})$ and restrict it to $\hat{\mathfrak{g}}_1 \oplus \hat{\mathfrak{g}}_2$. The module $\mathcal{H}_\Lambda(\mathfrak{g})$ decomposes into $\hat{\mathfrak{g}}_1 \oplus \hat{\mathfrak{g}}_2$ as follows

$$\mathcal{H}_\Lambda(\mathfrak{g}) \simeq \bigoplus_{(\lambda, \mu) \in \hat{B}(\Lambda)} \tilde{m}_{\lambda, \mu} \mathcal{H}_\lambda(\mathfrak{g}_1) \otimes \mathcal{H}_\mu(\mathfrak{g}_2).$$

Since the integrable modules are infinite dimensional, the set $|\hat{B}(\Lambda)|$ could be infinite. It is easy to see that $B(\Lambda) \subseteq \hat{B}(\Lambda)$. In most cases $B(\Lambda)$ is strictly contained in $\hat{B}(\Lambda)$. We recall the following from [KW]:

1. An embedding is conformal if and only if $\hat{B}(\Lambda)$ is finite for all level one weights $\Lambda$. 
(2) Let \( g_1 \oplus g_2 \rightarrow g \) be a conformal embedding. If \((\lambda, \mu) \in \bar{B}(\Lambda)\), then \( \Delta_\lambda(g_1, \ell_1) + \Delta_\mu(g_2, \ell_2) - \Delta_\Lambda(g, 1) \) is a non-negative integer \( n^\Lambda_{\lambda, \mu} \). Furthermore the difference of trace anomalies \( n^\Lambda_{\lambda, \mu} \) is zero if and only if \((\lambda, \mu) \in \bar{B}(\Lambda)\).

We only consider conformal embeddings for the rest of this paper.

## 4. Preliminaries on \( \bar{M}_{0,n} \) and the Chern Class Formula of Fakhruddin

In this section we set up notations and conventions for various important divisors on \( \bar{M}_{0,n} \). We also recall the Chern class formula from \([F]\) and rewrite it with psi-classes on \( \bar{M}_{0,n} \) using a result of \([FG]\).

### 4.1. Divisors on \( \bar{M}_{0,n} \)

The moduli space \( \bar{M}_{0,n} \) is stratified. A vital codimension \( k \) stratum is an irreducible component of the locus of curves with at least \( k \) nodes. The boundary divisors on \( \bar{M}_{0,n} \) is composed of vital codimension 1-strata labeled by \( D_{A, A^c} \), where \( A \cup A^c = \{1, \ldots, n\} \) and \(|A|, |A^c| \geq 2\). We have the following identification \( D_{A, A^c} = D_{A^c, A} \). We denote by \([D_{A, A^c}]\) the linear equivalence class of \( D_{A, A^c} \) in \( \text{Pic}(\bar{M}_{0,n}) \).

The \( i \)-th psi class \( \psi_i \) on \( \bar{M}_{0,n} \) is defined to be the first Chern class of the line bundle \( \mathbb{L}_i \), where \( \mathbb{L}_i \) is the line bundle on \( \bar{M}_{0,n} \) whose fiber over the point \((C, p_1, \ldots, p_n)\) is the cotangent space \( T_{p_i}(C) \). We recall the following Lemma from \([FG]\).

**Lemma 4.1.** The psi classes \( \psi_i \) for \( 1 \leq i \leq n \), have the following expression in terms of boundary divisors in \( \text{Pic}(\bar{M}_{0,n}) \).

\[
\psi_i = \sum_{\substack{A \subset \{1, 2, \ldots, n\} \\
2 \leq |A| \leq n-2}} \frac{(n - |A|)(n - |A| - 1)}{(n - 1)(n - 2)} [D_{A, A^c}]
\]

### 4.2. Chern classes of conformal blocks

N. Fakhruddin’s gave the following formula for the first Chern classes of conformal block bundles. We refer the reader to \([F]\) for more details:

**Proposition 4.2.** Let \( g \) be a simple Lie algebra, \( \ell \) a positive integer and consider an \( n \)-tuple \( \bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in P_\ell(g)^n \). Then

\[
c_1(V_{\bar{\lambda}}(g, \ell)) = \sum_{i=2}^{\frac{n}{2}} \epsilon_i \sum_{A \subset \{1, 2, \ldots, n\}} \frac{\rk V_{\bar{\lambda}}(g, \ell)}{(n - 1)(n - 2)} \left( (n - i)(n - i - 1) \sum_{a \in A} \Delta_{\lambda_a}(g, \ell) + i(i - 1) \sum_{a' \in A^c} \Delta_{\lambda_{a'}}(g, \ell) \right)
\]

\[
- \left( \sum_{\lambda \in P_\ell(g)} \Delta_\lambda(g, \ell) \cdot \rk V_{\bar{\lambda}}(g_1, \ell_1) \cdot \rk V_{\bar{\lambda}_{A^c, A^c}}(g, \ell) \right) [D_{A, A^c}],
\]

where \([D_{A, A^c}]\) denotes the class of the boundary divisor corresponding to the partition \( A \cup A^c = \{1, \ldots, n\} \), \( \epsilon_i = \frac{1}{2} \) if \( i = n/2 \) and one otherwise.

We rewrite Fakhruddin’s formula using psi classes and Lemma 4.1.
Proposition 4.3.

\[ c_1(\mathcal{V}_X(\mathfrak{g}, \ell)) = \text{rk} \mathcal{V}_X(\mathfrak{g}, \ell) \left( \sum_{j=1}^{n} \Delta_{\lambda_j}(\mathfrak{g}, \ell) \psi_j \right) - \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} \sum_{A \subseteq \{1,2,\ldots,n\}} \epsilon_i \left\{ \sum_{\lambda \in P_{\ell}(\mathfrak{g})} \Delta_{\lambda}(\mathfrak{g}, \ell) \cdot \text{rk} \mathcal{V}_{\lambda,\lambda}(\mathfrak{g}, \ell) \cdot \text{rk} \mathcal{V}_{\lambda,\lambda^*}(\mathfrak{g}, \ell) \right\} [D_{A,A^c}], \]

where \( \psi_j \) is the j-th \( \psi \) class, \([D_{A,A^c}]\) denotes the class of the boundary divisor \( D_{A,A^c} \), \( \epsilon_i = \frac{1}{2} \) if \( i = n/2 \) and one otherwise.

5. Rank-level duality and geometry of relations

Conformal embeddings give rise to maps of conformal blocks associated to stable \( n \)-pointed curves of genus \( g \) with chosen coordinates. These maps are known as rank-level duality maps. We refer the reader to [M1] for more details.

We are interested in studying the conformal block divisors on \( \overline{M}_{0,n} \). So we need a coordinate free description of rank-level duality maps. The coordinate free description is not straightforward.

Consider a conformal embedding \( \mathfrak{g}_1 \oplus \mathfrak{g}_2 \rightarrow \mathfrak{g} \) of Dynkin multi-index \((\ell_1, \ell_2)\). Let \( \tilde{\Lambda} = (\Lambda_1, \ldots, \Lambda_n) \in P_1(\mathfrak{g})^n \) and \( \tilde{\lambda} = (\lambda_1, \ldots, \lambda_n) \), \( \tilde{\mu} = (\mu_1, \ldots, \mu_n) \) be such that \((\lambda_i, \mu_i) \in B(\Lambda_i)\) for \( 1 \leq i \leq n \). We would like to show the following:

Conjecture 5.1. There exists a map \( \alpha \) between the following vector bundles on \( \overline{M}_{0,n} \) such that after choosing coordinates this map coincides with the rank-level duality map

\[ \alpha : \mathcal{V}_{\tilde{\lambda}}(\mathfrak{g}_1, \ell_1) \otimes \mathcal{V}_{\tilde{\mu}}(\mathfrak{g}_2, \ell_2) \otimes (\bigotimes_{i=1}^{n} L_{\lambda_i - \mu_i}) \rightarrow \mathcal{V}_{\tilde{\Lambda}}(\mathfrak{g}, 1), \]

where \( n^{\Lambda_i}_{\lambda_i, \mu_i} \)'s are difference of trace anomalies as defined in Section 3.

We hope to get back to the proof of this conjecture in a separate paper which uses techniques from the coordinate free description of vertex algebras developed by [BF].

Remark 5.2. The proof of Theorem 1.3 does not depend on Conjecture 5.1. The above Conjecture is only a motivation for the formulation of Theorem 1.3.

5.1. Behavior on boundary divisors. Consider a boundary divisor \( D_{A,A^c} \) given by the partition \( A \cup A^c = \{1, \ldots, n\} \). Let \( \tilde{\Lambda} \in P_1(\mathfrak{g})^n \) be such that rank of \( \mathcal{V}_{\tilde{\Lambda}}(\mathfrak{g}, 1) \) is one. Hence by factorization of conformal blocks there exists an unique \( \Lambda \in P_1(\mathfrak{g}) \) such that the following holds:

\[ \text{rk} \mathcal{V}_{\Lambda \Lambda}(\mathfrak{g}, 1) = 1 \quad \text{rk} \mathcal{V}_{\Lambda^* \Lambda^*}(\mathfrak{g}, 1) = 1. \]

Consider \( \tilde{\lambda} = (\lambda_1, \ldots, \lambda_n) \), \( \tilde{\mu} = (\mu_1, \ldots, \mu_n) \) such that \((\lambda_i, \mu_i) \in B(\Lambda_i)\) for all \( 1 \leq i \leq n \).

Definition 5.3. For a partition \( A \cup A^c = \{1, \ldots, n\} \), we define \( b_{A,A^c} \) to be the following non-negative integer:

\[ b_{A,A^c} = \sum_{(\lambda, \mu) \in B(\Lambda)} n^{\Lambda}_{\lambda, \mu} \cdot \text{rk} \mathcal{V}_{\Lambda \Lambda}(\mathfrak{g}_1, \ell_1) \cdot \text{rk} \mathcal{V}_{\Lambda^* \Lambda^*}(\mathfrak{g}_1, \ell_1), \]
where \( n^A_{\lambda,\mu} \) are the difference of trace anomalies.

It is easy to observe that \( b_{A,A^c} = b_{A^c,A} \) for any partition \( A \cup A^c = \{1, \ldots, n\} \).

**Remark 5.4.** The integer \( b_{A,A^c} \) depends on the choice of \( \Lambda \) for a given boundary divisor \( A \cup A^c \) and the weights \( \overline{\lambda} \). It can be shown that the integer \( b_{A,A^c} \) is the order of vanishing of the rank-level duality map on \( D_{A,A^c} \).

The above discussion tells us that whenever we have a rank-level duality isomorphism, the line bundle \( \det V_{\lambda}(g_1, \ell_1) \otimes \det V_{\mu}(g_2, \ell_2) \) is isomorphic to the following:

\[
(\det V_{\lambda}(g_1, 1) \otimes \bigotimes_{i=1}^n \mathbb{L}_{i}^{\otimes n^A_{\lambda_i,\mu_i}})^{\otimes r_k V_{\lambda}(g_1, \ell_1)} \boxtimes O_{\mathcal{M}_{0,n}} \left( - \sum_{i=2}^{\frac{n}{2}} \epsilon_i \sum_{A \subseteq \{1, \ldots, n\}} |A| \neq i b_{A,A^c} D_{A,A^c} \right),
\]

where \( \epsilon_i = \frac{1}{2} \) if \( i = n/2 \) and one otherwise. Hence the Chern classes of the line bundles should be equal. Our formulation of Theorem 1.3 is a consequence of the above discussion.

## 6. Proof of Theorem 1.3

In this section we give a complete proof of Theorem 1.3. For convenience we recall the assumptions on \( \overline{\lambda}, \bar{\mu} \) in Theorem 1.3. They are as follows:

1. \( r_k V_{\lambda}(g_1, 1) = 1 \), \( r_k V_{\lambda}(g_1, 1) = r_k V_{\overline{\lambda}}(g_2, \ell_2) \), and \( (\lambda_i, \mu_i) \in \overline{\lambda}(A_i) \) for \( 1 \leq i \leq n \).
2. For every partition \( A \cup A^c = \{1, \ldots, n\} \) such that \( |A|, |A^c| \geq 2 \) and every \( (\lambda, \mu) \in \overline{\lambda}(L) \)

\[
r_k V_{\lambda,\mu}(g_1, \ell_1) = r_k V_{\overline{\lambda},\mu}(g_2, \ell_2),
\]

where \( \lambda \) is the unique weight in \( P_1(g) \) such that \( r_k V_{\lambda,\mu}(g, 1) = 1 \).

3. Given any \( \lambda \in P_1(g) \) such that \( r_k V_{\lambda}(g, 1) = 1 \), there exists a bijection \( f_{\lambda} \) between \( P_1^A(g_1) \) and \( P_1^A(g_2) \) such that \( (\lambda, \mu) \in \overline{\lambda}(L) \), where \( \mu = f_{\lambda}(\lambda) \).

**Lemma 6.1.** For any \( \lambda \in P_1(g_1) \setminus P_1^A(g_1) \),

\[
r_k V_{\lambda,\mu}(g_1, \ell_1) = 0.
\]

**Proof.** If \( \lambda \in P_1(g_1) \setminus P_1^A(g_1) \), then \( \lambda + \sum_{\alpha \in A} \lambda_a \) is not in the root lattice. Hence it does not have any invariants. By the third assumption, we identity \( P_1^A(g_1) \) and \( P_1^A(g_1) \) via the given bijection \( f_{\lambda} \). We use Proposition 4.3 Lemma 6.1 and write the sum of \( c_1(\overline{\lambda}(g_1, \ell_1)) \) and \( c_1(\overline{\lambda}(g_2, \ell_2)) \) as follows:

\[
r_k V_{\lambda}(g_1, \ell_1) \left( \sum_{j=1}^n (\Delta_{\lambda}(g_1, \ell_1) + \Delta_{\mu}(g_2, \ell_2)) \psi_j \right) - \sum_{i=2}^{\frac{n}{2}} \epsilon_i \sum_{A \subseteq \{1,2,\ldots,n\}} |A| \neq 2 r_k V_{\lambda,\mu}(g_1, \ell_1), r_k V_{\overline{\lambda},\mu}(g_2, \ell_2) \right)[D_{A,A^c}],
\]

where \( \epsilon_i = \frac{1}{2} \) if \( i = n/2 \) and one otherwise, and \( \mu \) is of the form \( f_{\lambda}(\lambda) \).
Lemma 6.2. The following equality holds:

\[ \sum_{A \subseteq \{1, \ldots, n\}} \sum_{\lambda \in P^A_{1} (g_1) \atop |A|=i} \Delta_{A} (g,1) \ rk V_{\bar{\lambda}_{A, \lambda}} (g_1, \ell_1) \ rk V_{\bar{\lambda}_{A_1, \lambda}} (g_1, \ell_1) = \ rk V_{\bar{\lambda}} (g_1, \ell_1) \ \sum_{A \subseteq \{1, \ldots, n\}} \sum_{\lambda \in P^A_{1} (g_1) \atop |A|=i} \Delta_{A} (g,1) \ rk V_{\bar{\lambda}_{A, \lambda}} (g,1) \ rk V_{\bar{\lambda}_{A_1, \lambda}} (g,1) \]

Proof. By our assumption \( \text{rk} V_{\bar{\lambda}} (g,1) = 1 \), there exists a unique \( \Lambda \in P_1 (g) \) (depending on the partition) for every partition \( A \cup A^c \) of \( \{1, \ldots, n\} \) such that \( \text{rk} V_{\bar{\lambda}_{A, \lambda}} (g,1) = 1 \). We rewrite the left hand side as follows:

\[ \sum_{A \subseteq \{1, \ldots, n\}} \sum_{\lambda \in P^A_{1} (g_1) \atop |A|=i} \Delta_{A} (g,1) \ rk V_{\bar{\lambda}_{A, \lambda}} (g_1, \ell_1) \ rk V_{\bar{\lambda}_{A_1, \lambda}} (g_1, \ell_1) = \ \sum_{A \subseteq \{1, \ldots, n\}} \sum_{\lambda \in P^A_{1} (g_1) \atop |A|=i} \Delta_{A} (g,1) \left( \sum_{\lambda \in P^A_{1} (g_1)} \ rk V_{\bar{\lambda}_{A, \lambda}} (g_1, \ell_1) \ rk V_{\bar{\lambda}_{A_1, \lambda}} (g_1, \ell_1) \right) \]

\[ = \ rk V_{\bar{\lambda}} (g_1, \ell_1) \ \sum_{A \subseteq \{1, \ldots, n\}} \sum_{\lambda \in P^A_{1} (g_1) \atop |A|=i} \Delta_{A} (g,1) \ rk V_{\bar{\lambda}_{A, \lambda}} (g_1,1) \ rk V_{\bar{\lambda}_{A_1, \lambda}} (g_1,1) \]

This completes the proof of the lemma. \( \square \)

7. Rank level duality in type A

In this section we apply our relation on conformal divisors of type A and derive information about ranks of certain conformal blocks. Let \( Y_{r,s} \) denote the set of Young diagrams with at most \( r \) rows and \( s \) columns. For \( \lambda = (s \geq \lambda_1 \geq \cdots \geq \lambda_r) \in Y_{r,s} \), one can associate an irreducible \( \text{GL}_r \) module \( V_{\lambda} \). Two Young diagrams \( \lambda_1 \) and \( \lambda_2 \) define the same \( \text{SL}_r \) representation
if $\lambda_1^I - \lambda_2^I$ is a constant independent of $i$. It is clear that for $\lambda \in Y_{r,s}$, the corresponding dominant integral weight $\lambda \in P_s(\mathfrak{sl}(r))$.

We consider the embedding $\mathfrak{sl}(r) \oplus \mathfrak{sl}(s) \to \mathfrak{sl}(rs)$ induced by tensor product $\mathbb{C}^r$ with $\mathbb{C}^s$. It is known that the embedding is conformal with Dynkin multi-index $(s, r)$. The branching rule in [ABI] tells us $(\lambda, \lambda^T) \in B(\omega_{|\lambda|})$, where $|\lambda|$ denotes the number of boxes in the Young diagram of $\lambda$ and $\omega_{|\lambda|}$ denotes the $|\lambda|$-th fundamental weight of $\mathfrak{sl}(rs)$.

Consider $\lambda \in Y_{rs}$ such that $\sum_{i=1}^n |\lambda_i| = rs$. Let $\lambda^T = (\lambda_1^T, \ldots, \lambda_n^T)$ and $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$, where $\Lambda_i = \omega_{|\lambda_i|}$. It is known that $\text{rk} \mathcal{V}_\Lambda(\mathfrak{sl}(rs), 1) = 1$. Let $D_{A,A^c}$ be a boundary given by the partition $A \cup A^c$ and $\Lambda \in P_1(\mathfrak{sl}(rs))$ be such that $\text{rk} \mathcal{V}_{\Lambda,A}(\mathfrak{sl}(rs), 1) = 1$. With the above notation, we have the following result about rank of conformal blocks:

**Corollary 7.1.** If $\lambda \in \tilde{B}(\Lambda) \setminus B(\Lambda)$, then at least one of the bundles $\mathcal{V}_{\Lambda,A}(\mathfrak{sl}(r), s)$ or $\mathcal{V}_{\Lambda,A^c,\Lambda}(\mathfrak{sl}(r), s)$ is zero.

**Proof.** It follows from the rank-level duality in [NT] that the above conditions satisfy the axioms of Theorem 1.3. Since $\sum_{i=1}^n |\lambda_i| = rs$, the following is shown in [BGM]:

$$c_1(\mathcal{V}_\Lambda(\mathfrak{sl}(r), s)) = c_1(\mathcal{V}_{\Lambda^T}(\mathfrak{sl}(s), r)) = 0.$$  

Further Proposition 5.2 in [F] tells us $c_1(\mathcal{V}_{\Lambda}(\mathfrak{sl}(rs), 1)) = 0$. We also observe that the difference of trace anomaly $n_{\Lambda_i, \Lambda_i^T}^A = 0$. The proof now follows directly from Theorem 1.3. \qed

**Remark 7.2.** The vanishing statement in Corollary 7.1 is new and is different from known vanishing results about conformal blocks in the existing literature.

**Remark 7.3.** In [BGM], the author along with P. Belkale and A. Gibney proved a critical level symmetry $c_1(\mathcal{V}_\Lambda(\mathfrak{sl}(r+1), \ell)) = c_1(\mathcal{V}_{\Lambda^T}(\mathfrak{sl}(\ell+1), r))$, where the sum of the boxes of the Young diagram of $\lambda = (\lambda_1, \ldots, \lambda_n)$ is $(r+1)(\ell+1)$. The level rank symmetry in [BGM] was conjectured in a weaker form by A. Gibney. The relations among the conformal block divisors in this work are different from the critical level symmetries in [BGM].

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