Regularization by transport noises for 3D MHD equations

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Abstract We consider the problem of regularization by noises for the three-dimensional magnetohydrodynamical (3D MHD) equations. It is shown that in a suitable scaling limit, the multiplicative noise of transport type gives rise to bounds on the vorticity fields of the fluid velocity and magnetic fields. As a result, if the noise intensity is big enough, then the stochastic 3D MHD equations admit a pathwise unique global solution for large initial data with high probability.

Keywords magnetohydrodynamical equation, vorticity, well-posedness, regularization by noises, transport noise

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1 Introduction

The topic of regularization by noises for ordinary differential equations (ODEs) and partial differential equations (PDEs) has attracted a lot of attention in the past two decades. There are mainly two different types of noises: the additive noise and the multiplicative noise. The additive noise has been very successful in improving the well-posedness of finite-dimensional ODEs [17, 37] (and there are many extensions to stochastic differential equations (SDEs) with the nondegenerate multiplicative noise [47, 51, 52] among others); by the method of characteristics, one can solve linear transport equations perturbed by the transport noise [18, 23]. The nondegenerate additive noise was also used to regularize abstract stochastic evolution equations on the infinite-dimensional Hilbert space [12–14], but the nonlinearity treated therein is not of the advection form in fluid dynamics. The 3D Navier-Stokes equations driven by the additive noise have been studied in [11, 28, 29], yielding some regularity results not available in the deterministic theory.

Concerning the multiplicative noise, we focus on the one of transport type modeling perturbations due to some background motion (see [8, 41] for some early results). Later on, the transport noise was shown to regularize some inviscid models, including the point vortex model of 2D Euler equation and modified surface quasi-geostrophic (mSQG) equations [24, 40], and point charges of 1D Vlasov-Poisson...
equations [15]. In the recent works [21, 26, 30, 39], it was shown that linear transport equations and Euler type equations perturbed by transport noises converge, under a suitable scaling of the noises, to the corresponding (deterministic) parabolic equations; moreover, the limiting equations have a large viscosity coefficient if the noise intensity is big enough. We have applied this idea to the vorticity form of 3D Navier-Stokes equations [27], obtaining long-term well-posedness for large initial data with high viscosity coefficient if the noise intensity is big enough. We have also applied this idea to the vorticity form of 3D MHD equations [38], obtaining long-term well-posedness for smooth initial data. Global well-posedness of the Cauchy problem for a 3D incompressible MHD type system was proved in [38] for smooth initial data (see also [1, 48] for the related results). There are also lots of studies on the stochastic 2D and/or 3D MHD equations (see [4, 43, 46, 50]). According to the classical theory of 3D MHD equations, the vorticity (and in particular, its direction field) plays a key role in the well-posedness results (see, e.g., [35]).

The purpose of this paper is to apply the idea in [27] to the 3D MHD equations on the torus \( T^3 = \mathbb{R}^3 / \mathbb{Z}^3 \):

\[
\begin{align*}
\partial_t u + u \cdot \nabla u - SB \cdot \nabla B + \nabla (p + |B|^2/2) &= \text{Re}^{-1} \Delta u, \\
\partial_t B + u \cdot \nabla B - B \cdot \nabla u &= \text{Rm}^{-1} \Delta B, \\
\nabla \cdot u &= 0, \quad \nabla \cdot B = 0,
\end{align*}
\]

which describe the motion of an electrically conductive fluid in a magnetic field. Here, \( u = (u_1, u_2, u_3) \) is the fluid velocity field, \( B = (B_1, B_2, B_3) \) is the magnetic field, \( \text{Re} \) is the Reynolds number, \( \text{Rm} \) is the magnetic Reynolds number, and \( S = M^2 / \text{ReRm} \) with \( M \) being the Hartman number. Similar to the theory of 3D Navier-Stokes equations (see [16, 44]), for any \( L^2 \)-initial data \((u_0, B_0)\), there exist global weak solutions to (1.1) (whose uniqueness is open), while if \((u_0, B_0)\) has \( H^1 \)-regularity, then one can show the local existence of a unique strong solution (whose global existence remains open). Global well-posedness of the Cauchy problem for a 3D incompressible MHD type system was proved in [38] for smooth initial data which are close enough to the equilibrium state \((x, 0, 0)\) (see also [1, 48] for the related results). There are also lots of studies on the stochastic 2D and/or 3D MHD equations (see [4, 43, 46, 50]). According to the classical theory of 3D MHD equations, the vorticity (and in particular, its direction field) plays a key role in the well-posedness results (see, e.g., [35]).

We are concerned with the vorticity formulation of (1.1). To this end, let

\[ \xi = \nabla \times u, \quad \eta = \nabla \times B \]

be the vorticity fields of the velocity and magnetic vector fields, respectively; conversely, \( u \) (resp. \( B \)) can be expressed by \( \xi \) (resp. \( \eta \)) via the Biot-Savart law. As in [35, (2.1)], the above system can be rewritten as

\[
\begin{align*}
\partial_t \xi + \mathcal{L}_u \xi - S \mathcal{L}_B \eta &= \text{Re}^{-1} \Delta \xi, \\
\partial_t \eta + \mathcal{L}_u \eta - \mathcal{L}_B \xi - 2T(B, u) &= \text{Rm}^{-1} \Delta \eta,
\end{align*}
\]

where \( \mathcal{L}_u \xi := u \cdot \nabla \xi - \xi \cdot \nabla u \) is the Lie derivative, and

\[ T(B, u) = \begin{pmatrix}
\partial_2 B \cdot \partial_3 u - \partial_3 B \cdot \partial_2 u \\
\partial_3 B \cdot \partial_1 u - \partial_1 B \cdot \partial_3 u \\
\partial_1 B \cdot \partial_2 u - \partial_2 B \cdot \partial_1 u
\end{pmatrix} \]

with \( \partial_i = \frac{\partial}{\partial x_i}, \ i = 1, 2, 3 \). The exact values of \( \text{Re}, \text{Rm} \) and \( S \) are not important in our analysis below, and thus we assume that they are all equal to 1 for simplicity. To write the system (1.2) in a more compact form, we introduce the notation \( \Phi = (\xi, \eta)^* \) and denote the nonlinear part by

\[ b(\Phi, \Phi) = \begin{pmatrix}
b_1(\Phi, \Phi) \\
b_2(\Phi, \Phi)
\end{pmatrix} = \begin{pmatrix}
\mathcal{L}_u \xi - \mathcal{L}_B \eta \\
\mathcal{L}_u \eta - \mathcal{L}_B \xi - 2T(B, u)
\end{pmatrix}; \]

then the system (1.2) can be simply written as

\[ \partial_t \Phi + b(\Phi, \Phi) = \Delta \Phi. \]
We introduce a few notations of functional spaces. For $s \in \mathbb{R}$, let $H^s(\mathbb{T}^3, \mathbb{R}^3)$ be the usual Sobolev space of vector fields on $\mathbb{T}^3$, endowed with the norm $\| \cdot \|_{H^s}$; $H^0(\mathbb{T}^3, \mathbb{R}^3)$ coincides with $L^2(\mathbb{T}^3, \mathbb{R}^3)$. For simplicity, we assume in this paper that the vector fields have the zero mean, and define the spaces

$$
H = \left\{ \Phi = (\xi, \eta)^* : \xi, \eta \in L^2(\mathbb{T}^3, \mathbb{R}^3), \nabla \cdot \xi = \nabla \cdot \eta = 0, \int_{\mathbb{T}^3} \xi dx = \int_{\mathbb{T}^3} \eta dx = 0 \right\},
$$

$$
V = \{ \Phi = (\xi, \eta)^* \in H : \xi, \eta \in H^1(\mathbb{T}^3, \mathbb{R}^3) \}.
$$

We write $\langle \cdot, \cdot \rangle$ or $\langle \cdot, \cdot \rangle_{L^2}$ for the inner product in $H$ with the corresponding norm $\| \cdot \|_H = \| \cdot \|_{L^2}$; $\| \cdot \|_V = \| \cdot \|_{H^1}$ is the norm in $V$. Note that in the periodic case, we do not have to distinguish the spaces for the velocity and the magnetic fields. It is well known that the system (1.3) admits a unique local solution for general initial data $\Phi_0 = (\xi_0, \eta_0) \in H$; moreover, there exists a small $\tau_0 > 0$ such that if $\| \Phi_0 \|_H \leq \tau_0$, then the unique solution exists globally in time.

Inspired by [27] (see in particular [27, Subsection 1.2] therein), we perturb the vorticity equation (1.3) by the transport noise, i.e.,

$$
\partial_t \Phi + b(\Phi, \Phi) = \Delta \Phi + \Pi(\dot{W} \cdot \nabla \Phi), \tag{1.4}
$$

where $\Pi$ is the Leray projection operator, $\dot{W}$ denotes the formal time derivative of a spatially divergence-free noise $W(t, x)$ on $\mathbb{T}^3$, and

$$
\Pi(\dot{W} \cdot \nabla \Phi) = \begin{pmatrix} \Pi(\dot{W} \cdot \nabla \xi) \\ \Pi(\dot{W} \cdot \nabla \eta) \end{pmatrix}. \tag{1.5}
$$

Note that the same noise acts on both equations, in accordance with the interpretation that it represents turbulent small scales; the equation for velocity describes large fluid scales coupled by the transport mechanism of the velocity small scales, and both velocity components act on the equation for the magnetic field. We apply Leray’s projection $\Pi$ to the noise terms to make them divergence-free; otherwise, the equations are in general not meaningful since the other quantities are all divergence-free. We remark that since the noise $W$ is spatially divergence-free, the stochastic 3D MHD equation (1.4) has the same energy estimate as the deterministic system (1.3); this implies that the stochastic equation (1.4) has unique local solutions for general initial data, while the solutions are global for initial data satisfying $\| \Phi_0 \|_H \leq \tau_0$, where $\tau_0$ is the same parameter as the one at the end of the previous paragraph. It seems that at first glance, the transport noise has no regularizing effect on the 3D MHD system; however, following [27], we show that the noise enhances dissipation in a suitable scaling limit, and thus leads to long-term well-posedness for big initial data with large probability.

We use the same noise as in [27] to perturb the equations, i.e.,

$$
W(t, x) = \frac{C_\nu}{\| \theta \|_{\ell^2}} \sum_{k \in \mathbb{Z}_0^3} \sum_{\alpha=1}^2 \theta_k \sigma_{k, \alpha}(x) W^{k, \alpha}_t.
$$

Here, for some $\nu > 0$, $C_\nu = \sqrt{3\nu/2}$ denotes the intensity of the noise; $\mathbb{Z}_0^3$ is the collection of nonzero lattice points and $\theta \in \ell^2 := \ell^2(\mathbb{Z}_0^3)$, the space of square summable real sequences indexed by $\mathbb{Z}_0^3$. Next, $\{ \sigma_{k, \alpha} \}_{k \in \mathbb{Z}_0^3, \alpha = 1, 2}$ are divergence-free vector fields on $\mathbb{T}^3$ and $\{ W^{k, \alpha} \}_{k \in \mathbb{Z}_0^3, \alpha = 1, 2}$ are independent standard complex Brownian motions (see Subsection 2.1 for their precise definitions). Now, the stochastic equations studied in this paper have the following precise form:

$$
d\Phi + b(\Phi, \Phi) dt = \Delta \Phi dt + \frac{C_\nu}{\| \theta \|_{\ell^2}} \sum_{k, \alpha} \theta_k \Pi(\sigma_{k, \alpha} \cdot \nabla \Phi) \circ dW^{k, \alpha}, \tag{1.6}
$$

where $\sum_{k, \alpha}$ stands for $\sum_{k \in \mathbb{Z}_0^3} \sum_{\alpha=1}^2$, $\Pi(\sigma_k \cdot \nabla \Phi)$ is understood as in (1.5) and $\circ$ means that we are making use of the Stratonovich stochastic differentiation. We always consider those $\theta \in \ell^2$ having only finitely many nonzero components, and assume that

$$
\theta_k = \theta_l \quad \text{for all } |k| = |l|. \tag{1.7}
$$
We can rewrite (1.6) in the Itô form as
\[ d\Phi + b(\Phi, \Phi)dt = [\Delta\Phi + S_0(\Phi)]dt + \frac{C_\nu}{\|\theta\|_{L^2}^2} \sum_{k,\alpha} \theta_k \Pi(\sigma_{k,\alpha} \cdot \nabla\Phi) dW^{k,\alpha}, \tag{1.8} \]
where the Stratonovich-Itô corrector is given by
\[ S_0(\Phi) = \frac{C_\nu^2}{\|\theta\|_{L^2}^2} \sum_{k,\alpha} \theta_k^2 \Pi[\sigma_{k,\alpha} \cdot \nabla\Pi(\sigma_{-k,\alpha} \cdot \nabla\Phi)]. \]

It is a symmetric second-order differential operator on vector fields.

Given \( \Phi_0 \in H \), we denote by \( \Phi(t; \Phi_0, \nu, \theta) \) the unique local solution to (1.8) with the maximal time of existence \( \tau = \tau(\Phi_0, \nu, \theta) \). Recall that \( \nu > 0 \) measures the noise intensity. We write \( B_H(K) \) for the ball in \( H \) centered at the origin with the radius \( K > 0 \). Here is the main result of our paper.

**Theorem 1.1.** Given \( K > 0 \) and small \( \varepsilon > 0 \), there exist big \( \nu > 0 \) and \( \theta \in \ell^2 \) such that
\[ P(\tau(\Phi_0, \nu, \theta) = +\infty) > 1 - \varepsilon \quad \text{for all } \Phi_0 \in B_H(K). \]

Equivalently, with large probability uniformly over \( \Phi_0 \in B_H(K) \), the unique solution to (1.8) with initial data \( \Phi_0 \) exists globally in time.

For the sake of readers’ understanding, we briefly describe here the ideas of the proof. Similar to [27], we take a sequence \( \{\theta^N\}_{N \geq 1} \subset \ell^2 \) satisfying
\[ \theta_k^N = \frac{1}{|k|^\kappa} \{N \leq |k| \leq 2N\}, \quad k \in \mathbb{Z}_0^3, \tag{1.9} \]
where \( \kappa > 0 \) is any fixed parameter. It is obvious that
\[ \lim_{N \to \infty} \frac{\|\theta^N\|_{\ell^\infty}}{\|\theta^N\|_{\ell^2}} = 0; \tag{1.10} \]
more importantly, for any smooth divergence-free vector field \( v \) on \( \mathbb{T}^3 \), it was proved in [27, Section 5] that
\[ \lim_{N \to \infty} S_{\theta^N}(v) = \frac{3}{5} \nu \Delta v \quad \text{holds in } L^2(\mathbb{T}^3, \mathbb{R}^3). \tag{1.11} \]

Consider the following sequence of stochastic 3D MHD equations:
\[ d\Phi_N + b(\Phi_N, \Phi_N)dt = [\Delta\Phi_N + S_0(\Phi_N)]dt + \frac{C_\nu}{\|\theta^N\|_{L^2}^2} \sum_{k,\alpha} \theta_k^N \Pi(\sigma_{k,\alpha} \cdot \nabla\Phi_N) dW^{k,\alpha}. \tag{1.12} \]

In fact, we introduce a suitable cut-off in the nonlinear part \( b(\Phi_N, \Phi_N) \) (see Section 2 for details). Thanks to the limit (1.10) and \( L^2 \)-bounds on the solutions, we can show that the martingale part in (1.12) will vanish in the weak sense. Furthermore, the key result (1.11) implies that the limit equation is
\[ \partial_t \Phi + b(\Phi, \Phi) = \left(1 + \frac{3}{5} \nu\right) \Delta\Phi. \tag{1.13} \]

This equation has an enhanced dissipation which comes from the noise intensity \( \nu \); hence, (1.13) is much better posed than the deterministic system (1.3). Theorem 1.1 will be proved by using the fact that the solutions to (1.12) are close to those of (1.13).

The rest of this paper is organized as follows. In Section 2, we first introduce the explicit choices of the vector fields \( \{\sigma_{k,\alpha}\}_{k \in \mathbb{Z}_0^3, \alpha=1,2} \) and the complex Brownian motions \( \{W^{k,\alpha}\}_{k \in \mathbb{Z}_0^3, \alpha=1,2} \), and then we prove the well-posedness of stochastic 3D MHD equations with a cut-off. The proof of Theorem 1.1 is given in Section 3, where we first prove a scaling limit result (i.e., Theorem 3.1) which is a crucial step for proving the main result. Finally in Appendix A, we present the sketched proof of the key limit (1.11).
2 Notations and global well-posedness of stochastic 3D MHD equations with a cut-off

In Subsection 2.1, we first introduce the definitions of divergence-free vector fields \( \{ \sigma_{k, \alpha} \}_{k \in \mathbb{Z}_0^3, \alpha = 1, 2} \) and the complex Brownian motions. In Subsection 2.2, we apply the Galerkin approximation and the compactness method to prove the existence of weak solutions to the stochastic 3D MHD equation (2.3) with a cut-off; then we show that the pathwise uniqueness holds for the system, and thus, by the Yamada-Watanabe type result, we conclude that (2.3) admits a unique strong solution.

2.1 Notations

This part is taken from the beginning of [27, Section 2]. Recall that \( \mathbb{Z}_0^3 = \mathbb{Z}^3 \setminus \{0\} \) is the nonzero lattice points; let \( \mathbb{Z}_0^3 = \mathbb{Z}_+^3 \cup \mathbb{Z}_-^3 \) be a partition of \( \mathbb{Z}_0^3 \) such that
\[
\mathbb{Z}_+^3 \cap \mathbb{Z}_-^3 = \emptyset, \quad \mathbb{Z}_+^3 = -\mathbb{Z}_-^3.
\]
Let \( L_0^2(\mathbb{T}^3, \mathbb{C}) \) be the space of complex valued square integrable functions on \( \mathbb{T}^3 \) with the zero average; it has the complete orthonormal system (CONS):
\[
e_k(x) = e^{2\pi i k \cdot x}, \quad x \in \mathbb{T}^3, \quad k \in \mathbb{Z}_0^3,
\]
where \( i \) is the imaginary unit. For any \( k \in \mathbb{Z}_0^3 \), let \( \{ a_{k,1}, a_{k,2} \} \) be an orthonormal basis of \( k^\perp := \{ x \in \mathbb{R}^3 : k \cdot x = 0 \} \) such that \( \{ a_{k,1}, a_{k,2}, \frac{k}{|k|} \} \) is right-handed. The choice of \( \{ a_{k,1}, a_{k,2} \} \) is not unique. For \( k \in \mathbb{Z}_0^3 \), we define \( a_{k, \alpha} = a_{-k, \alpha}, \alpha = 1, 2 \). Now we can define the divergence-free vector fields
\[
\sigma_{k, \alpha}(x) = a_{k, \alpha} e_k(x), \quad x \in \mathbb{T}^3, \quad k \in \mathbb{Z}_0^3, \quad \alpha = 1, 2.
\] (2.1)
Then \( \{ \sigma_{k,1}, \sigma_{k,2} : k \in \mathbb{Z}_0^3 \} \) is a CONS of the subspace \( H_C \subset L_0^2(\mathbb{T}^3, \mathbb{C}^3) \) of square integrable and divergence-free vector fields with the zero mean. A vector field
\[
v = \sum_{k, \alpha} v_{k, \alpha} \sigma_{k, \alpha} \in H_C
\]
has real components if and only if \( \overline{v_{-k, \alpha}} = v_{-k, \alpha} \).

Next, we introduce the family \( \{ W^{k, \alpha} : k \in \mathbb{Z}_0^3, \alpha = 1, 2 \} \) of complex Brownian motions. Let
\[
\{ B^{k, \alpha} : k \in \mathbb{Z}_0^3, \alpha = 1, 2 \}
\]
be a family of independent standard real Brownian motions; then the complex Brownian motions can be defined as
\[
W^{k, \alpha} = \begin{cases} B^{k, \alpha} + i B^{-k, \alpha}, & k \in \mathbb{Z}_1^3, \\ B^{-k, \alpha} - i B^{k, \alpha}, & k \in \mathbb{Z}_0^3. \end{cases}
\]
Note that \( \overline{W^{k, \alpha}} = W^{-k, \alpha} \) (\( k \in \mathbb{Z}_0^3, \alpha = 1, 2 \)), and they have the following quadratic covariation:
\[
[W^{k, \alpha}, W^{l, \beta}]_t = 2 t \delta_{k,-l} \delta_{\alpha, \beta}, \quad k, l \in \mathbb{Z}_0^3, \quad \alpha, \beta \in \{1, 2\}. \] (2.2)

In the following, for a vector field \( X \) in \( H^1(\mathbb{T}^3, \mathbb{R}^3) \), we write \( \mathcal{L}_X \) for the adjoint operator of the Lie derivative \( \mathcal{L}_X \); for any \( H^1 \)-vector fields \( Y \) and \( Z \), \( \langle \mathcal{L}_X Y, Z \rangle_{L^2} = -\langle Y, \mathcal{L}_X^* Z \rangle_{L^2} \). If \( X \) is divergence-free, one has \( \mathcal{L}_X Y = X \cdot \nabla Y + (\nabla X)^\ast Y \), where for \( i = 1, 2, 3 \), \( ((\nabla X)^\ast Y)_i = Y \cdot \partial_i X \).

2.2 Global well-posedness of (2.3)

Due to the nonlinear terms in (1.8), we can only prove the existence of local solutions for general initial data. Therefore, we need a cut-off technique as below. Let \( R > 0 \) be fixed and \( f_R \in C_b^1(\mathbb{R}_+, [0, 1]) \) be
Proof. We omit the time variable to save notations. In the Stratonvich form, (2.3) reads as (see (1.6))

\[ d\Phi + f_R(\Phi)b(\Phi, \Phi)dt = [\Delta \Phi + S_\theta(\Phi)]dt + \frac{C_v}{\|\theta\|^{\ell^2}} \sum_{k,\alpha} \theta_k \Pi(\sigma_{k,\alpha} \cdot \nabla \Phi) dW^{k,\alpha}, \tag{2.3} \]

where \( f_R(\Phi) = f_R(\|\Phi\|_{H^{-\delta}}) \) for some fixed \( \delta \in (0, 1/2) \). We remark that if the \( H^{-\delta} \)-norm of \( \Phi \) does not attain the threshold \( R \), then the cut-off function \( f_R(\Phi) \) can be dropped and (2.3) reduces to (1.8).

Thanks to the cut-off, we can show that for any initial data \( \Phi_0 \in H \), the above system (2.3) admits a pathwise unique global solution, strong in the probabilistic sense and weak in the analytic sense. First of all, we explain what we mean by a solution to (2.3).

**Definition 2.1.** Given a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) and a family of independent \((\mathcal{F}_t)\)-complex Brownian motions \(\{W^{k,\alpha}\}_{k \in \mathbb{Z}^n, \alpha = 1, 2} \) defined on \(\Omega\), we say that an \((\mathcal{F}_t)\)-progressively measurable process \(\Phi = (\xi, \eta)^* \) is a strong solution to (2.3) with the initial condition \(\Phi_0 = (\xi_0, \eta_0)^* \) if it has trajectories of the class \(L^\infty(0, T; H) \cap L^2(0, T; V)\) in \(C([0, T], H^{-\delta})\), and for any divergence-free vector field \(v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)\), \(\mathbb{P}\)-a.s., the following identities hold for all \(t \in [0, T]\):

\[
\langle \xi, v \rangle = \langle \xi_0, v \rangle + \int_0^t \langle \xi_s, v \rangle ds + \int_0^t \langle \xi_s, \Delta v + S_\theta(v) \rangle ds - \frac{C_v}{\|\theta\|^{\ell^2}} \sum_{k,\alpha} \theta_k \int_0^t \langle \xi_s, \sigma_{k,\alpha} \cdot \nabla v \rangle dW^{k,\alpha}_s, \tag{2.4} \]

\[
\langle \eta, v \rangle = \langle \eta_0, v \rangle + \int_0^t \langle \eta_s, \Delta v + S_\theta(v) \rangle ds - \frac{C_v}{\|\theta\|^{\ell^2}} \sum_{k,\alpha} \theta_k \int_0^t \langle \eta_s, \sigma_{k,\alpha} \cdot \nabla v \rangle dW^{k,\alpha}_s. \tag{2.5} \]

Our main result in this part can be stated as follows.

**Theorem 2.2.** Assume that \(\Phi_0 = (\xi_0, \eta_0)^* \in H, T > 0\) and \(\theta \in \ell^2\) verifies the symmetry property (1.7). Then there exists on the interval \([0, T]\) a pathwise unique strong solution \(\Phi = (\xi, \eta)^* \) to (2.3) in the sense of Definition 2.1. Moreover, there are a constant \(C_{\|\Phi_0\|_{H^{\delta}}, T} > 0\) independent of \(\nu > 0\) and \(\theta \in \ell^2\) such that

\[
\mathbb{P}\text{-a.s., } \|\Phi\|_{L^\infty(0, T; H)} \vee \|\Phi\|_{L^2(0, T; V)} \leq C_{\|\Phi_0\|_{H^{\delta}}, T}. \tag{2.6} \]

To show the existence of weak solutions to the above equations, we first prove an \textit{a priori} estimate on the solutions. In the following, we make frequent use of the Sobolev embedding inequality

\[
\|\varphi\|_{L^K} \leq c_{\nu} \|\varphi\|_{H^s} \quad \text{with } \frac{1}{q} = \frac{1}{2} - \frac{s}{3}, \quad s < \frac{3}{2}. \tag{2.7} \]

We also need the interpolation inequality: for any \(s_0 < s < s_1\),

\[
\|\varphi\|_{H^r} \leq \|\varphi\|_{H^{s_1}}^{(s_1 - s)/(s_1 - s_0)} \|\varphi\|_{H^{s_0}}^{(s - s_0)/(s_1 - s_0)}. \]

Moreover, we write \(C\) (without a subscript) for generic positive constants independent of the key parameters such as \(\delta\) and \(R\).

**Lemma 2.3.** Let \(\Phi_0 \in H\). Then there exists a \(C_{\delta, R} > 0\) such that, \(\mathbb{P}\)-a.s.,

\[
\|\Phi_t\|_{L^2}^2 + \int_0^t \|\Phi_s\|_{H^1} ds \leq \|\Phi_0\|_{L^2}^2 + C_{\delta, R} T \quad \text{for all } t \in [0, T]. \]

**Proof.** We omit the time variable to save notations. In the Stratonovitch form, (2.3) reads as (see (1.6))

\[
d\Phi + f_R(\Phi)b(\Phi, \Phi)dt = \Delta \Phi dt + \frac{C_v}{\|\theta\|^{\ell^2}} \sum_{k,\alpha} \theta_k \Pi(\sigma_{k,\alpha} \cdot \nabla \Phi) \circ dW^{k,\alpha}. \]
By the Stratonovich calculus,
\[ d\|\Phi\|^2_{L^2} = 2\langle \Phi, -F_R(\Phi)b(\Phi, \Phi) + \Delta \Phi \rangle dt + \frac{2C_V}{\|\theta\|^2} \sum_{k, \alpha} \theta_k\langle \Phi, \Pi(\sigma_{k, \alpha} \cdot \nabla \Phi) \rangle \circ dW^{k, \alpha}. \]

Note that the two components of \( \Phi = (\xi, \eta)^* \) are divergence-free; by integration by parts,
\[ \langle \Phi, \Pi(\sigma_{k, \alpha} \cdot \nabla \Phi) \rangle = \langle \xi, \Pi(\sigma_{k, \alpha} \cdot \nabla \xi) \rangle + \langle \eta, \Pi(\sigma_{k, \alpha} \cdot \nabla \eta) \rangle = \langle \xi, \sigma_{k, \alpha} \cdot \nabla \xi \rangle + \langle \eta, \sigma_{k, \alpha} \cdot \nabla \eta \rangle = 0, \]

since \( \sigma_{k, \alpha} \) is also divergence-free. Therefore, the above equation reduces to
\[ d\|\Phi\|^2_{L^2} = -2F_R(\Phi)\langle \Phi, b(\Phi, \Phi) \rangle dt - 2\|\nabla \Phi\|^2_{L^2} dt. \tag{2.8} \]

Recalling the expression of \( b(\Phi, \Phi) \), we have
\[ \langle \Phi, b(\Phi, \Phi) \rangle = \langle \xi, b_1(\Phi, \Phi) \rangle + \langle \eta, b_2(\Phi, \Phi) \rangle. \]

We estimate the two quantities separately.

**Step 1.** We have
\[ \langle \xi, b_1(\Phi, \Phi) \rangle = \langle \xi, L_\alpha \xi \rangle - \langle \xi, L_B \eta \rangle =: I_1 + I_2. \tag{2.9} \]

As \( u \) is divergence-free, we have \( I_1 = -\langle \xi, \xi \cdot \nabla u \rangle_{L^2} \) and thus, by Hölder’s inequality,
\[ |I_1| \leq ||\xi||^2_{L^2} ||\nabla u||_{L^2} \leq C ||\xi||^3_{L^2} \leq C ||\xi||^{3/2}_{H^{1/2}}. \]

Using the interpolation inequality with \( s_0 = -\delta, s_1 = 1 \) and \( s = 1/2 \), we obtain
\[ |I_1| \leq C ||\xi||^{3/2(1+\delta)}_{H^{-\delta}} ||\xi||^{3(1+\delta)/2(1+\delta)}_{H^{1/2}} \leq \varepsilon ||\xi||^2_{H^1} + C_{\delta, \varepsilon} ||\eta||^{6/(1-2\delta)}_{H^{-\delta}}, \tag{2.10} \]

where in the last step we have used the inequality \( ab \leq \frac{a^q}{p} + \frac{b^p}{q} \) for \( a, b \geq 0, p = 4(1+\delta)/(6\delta + 3) \) and \( q = 4(1+\delta)/(1-2\delta) \); \( \varepsilon > 0 \) is some fixed constant to be determined later.

Next, we deal with \( I_2 \):
\[ I_2 = -\langle \xi, B \cdot \nabla \eta \rangle + \langle \eta, \xi \cdot \nabla B \rangle =: I_{2,1} + I_{2,2}. \]

By Hölder’s inequality with exponents \( \frac{1}{2} + \frac{1}{p} + \frac{1}{2} = 1, \]
\[ |I_{2,1}| \leq ||\nabla \eta||_{L^{p}} ||\xi||_{L^2} ||B||_{L^{p}} \leq C ||\eta||_{H^{1}} ||\xi||_{H^{1/2}} ||B||_{H^{1}} \leq C ||\eta||_{H^{1}} ||\xi||_{H^{1/2}} ||\eta||_{L^{p}}. \]

Using the interpolation inequality with \( s = 1/2, s = 0 \), while \( s_0 = -\delta \) and \( s_1 = 1 \) are fixed, we have
\[ |I_{2,1}| \leq C ||\eta||_{H^{1}} ||\xi||_{H^{-\delta}} ||\xi||^{1/(2(1+\delta))}_{H^{1}} ||\xi||^{(1+\delta)/2(1+\delta)}_{H^{1}} ||\eta||^{1/(1+\delta)}_{H^{-\delta}} ||\eta||^{6/(1+\delta)}_{H^{-\delta}} \]
\[ = C ||\xi||^{1/(2(1+\delta))}_{H^{-\delta}} ||\xi||^{(1+\delta)/2(1+\delta)}_{H^{1}} ||\eta||^{1/(1+\delta)}_{H^{-\delta}} ||\eta||^{(1+\delta)/2(1+\delta)}_{H^{1}}. \]

Now applying the inequality \( abcd \leq \frac{a^p}{p} + \frac{b^q}{q} + \frac{c^r}{r} + \frac{d^s}{s} \) \( (a, b, c, d \geq 0) \) with
\[ p = \frac{12(1+\delta)}{1-2\delta}, \quad q = \frac{4(1+\delta)}{1+2\delta}, \quad r = \frac{6(1+\delta)}{1-2\delta}, \quad s = \frac{2(1+\delta)}{1+2\delta}, \]

we obtain
\[ |I_{2,1}| \leq \varepsilon ||\xi||^2_{H^{1}} + \varepsilon ||\eta||^2_{H^{1}} + C_{\delta, \varepsilon} (||\xi||^{6/(1-2\delta)}_{H^{-\delta}} + ||\eta||^{6/(1-2\delta)}_{H^{-\delta}}). \tag{2.11} \]

In the same way,
\[ |I_{2,2}| \leq ||\xi||_{L^q} ||\eta||_{L^p} ||\nabla B||_{L^q} \leq C ||\xi||_{L^q} ||\eta||_{L^p} \leq C ||\xi||_{H^{1/2}} ||\eta||_{H^{1/2}}^2 \]
\[ \leq C ||\xi||^{1/(2(1+\delta))}_{H^{-\delta}} ||\xi||^{(1+\delta)/2(1+\delta)}_{H^{1}} ||\eta||^{1/(1+\delta)}_{H^{-\delta}} ||\eta||^{(1+\delta)/2(1+\delta)}_{H^{1}}. \]

The right-hand side can be estimated similarly as for \( |I_{2,1}| \), and thus we have
\[ |I_2| \leq |I_{2,1}| + |I_{2,2}| \leq 2\varepsilon (||\xi||^2_{H^{1}} + ||\eta||^2_{H^{1}}) + C_{\delta, \varepsilon} (||\xi||^{6/(1-2\delta)}_{H^{-\delta}} + ||\eta||^{6/(1-2\delta)}_{H^{-\delta}}). \]
Combining this result with (2.9) and (2.10) leads to
\[ |⟨ξ, b_1(Φ, Φ)⟩| \leq 3ε(‖ξ‖_{H^1}^2 + ‖η‖_{H^1}^2) + C_{δ,ε}(‖ξ‖_{H^{-δ}}^{6/(1−2δ)} + ‖η‖_{H^{-δ}}^{6/(1−2δ)}). \tag{2.12} \]

**Step 2.** Now we turn to estimate
\[ \langle η, b_2(Φ, Φ) \rangle = \langle η, L_u η \rangle - \langle η, L_B η \rangle - 2⟨η, T(B, u)⟩ =: J_1 + J_2 + J_3. \]
The arguments are similar to those in Step 1. We have $J_1 = −⟨η, η \cdot ∇u⟩_{L^2}$, and thus,
\[ |J_1| \leq ‖η‖_{L^2}^2 ‖∇u‖_{L^3} \leq C‖η‖_{L^2}^2 ‖ξ‖_{L^3} \leq C‖ξ‖_{H^{1/2}} ‖η‖_{H^{1/2}}. \]
Repeating the estimate for $|I_2,2|$ yields
\[ |J_1| \leq ε(‖ξ‖_{H^1}^2 + ‖η‖_{H^1}^2) + C_{δ,ε}(‖ξ‖_{H^{-δ}}^{6/(1−2δ)} + ‖η‖_{H^{-δ}}^{6/(1−2δ)}). \]
Next, the term $J_2$ can be treated in the same way as $I_2$ and we have
\[ |J_2| \leq 2ε(‖ξ‖_{H^1}^2 + ‖η‖_{H^1}^2) + C_{δ,ε}(‖ξ‖_{H^{-δ}}^{6/(1−2δ)} + ‖η‖_{H^{-δ}}^{6/(1−2δ)}). \]
Finally, the definition of $T(B, u)$ implies
\[ |J_3| \leq C‖ξ‖_{L^3} ‖∇B‖_{L^3} ‖∇u‖_{L^3} \leq C‖ξ‖_{H^1}^2 ‖ξ‖_{H^{1/2}} \leq C‖ξ‖_{H^1} ‖ξ‖_{H^{1/2}}, \]
and hence, this last quantity can also be treated as $|I_2,2|$. Summarizing these estimates, we obtain
\[ |⟨η, b(Φ, Φ)⟩| \leq 5ε(‖ξ‖_{H^1}^2 + ‖η‖_{H^1}^2) + C_{δ,ε}(‖ξ‖_{H^{-δ}}^{6/(1−2δ)} + ‖η‖_{H^{-δ}}^{6/(1−2δ)}). \]

**Step 3.** Thanks to the estimates in Steps 1 and 2, we deduce from (2.8) that
\[ d‖Φ‖_{L^2}^2 \leq 2f_R(Φ)d\varepsilon(‖ξ‖_{H^1}^2 + ‖η‖_{H^1}^2) + C_{δ,ε}(‖ξ‖_{H^{-δ}}^{6/(1−2δ)} + ‖η‖_{H^{-δ}}^{6/(1−2δ)})dt - 2‖∇Φ‖_{L^2}^2 dt \]
\[ \leq 16ε(‖ξ‖_{H^1}^2 + ‖η‖_{H^1}^2)dt + C_{δ,ε}(R + 1)^{6/(1−2δ)} dt - 2‖∇Φ‖_{L^2}^2 dt, \]
where we have used $0 \leq f_R \leq 1$ and $f_R|_{[t_1,∞)} \equiv 0$. The Poincaré inequality implies that for some constant $C_0 > 0$,
\[ ‖∇Φ‖_{L^2}^2 \geq C_0‖Φ‖_{H^1}^2 = C_0(‖ξ‖_{H^1}^2 + ‖η‖_{H^1}^2). \]
Therefore, taking $ε = C_0/16$ leads to
\[ d‖Φ‖_{L^2}^2 + C_0‖Φ‖_{H^1}^2 dt \leq C_{δ,ε}(R + 1)^{6/(1−2δ)} dt. \]
This gives us the desired estimate. \hfill \square

With the above *a priori* estimate in hand, it is standard to apply the Galerkin approximation to show the existence of weak solutions to (2.3) (see, for example, [27, Section 3]). Here, we sketch the main steps. Let $H_N$ be the finite-dimensional subspace of $H$ spanned by the fields $\{σ_k,α : |k| \leq N, α = 1, 2\}$. Denote by $Π_N : H \to H_N$ the orthogonal projection, and define
\[ b_N(ϕ_N) = f_R(ϕ_N)Π_N b(ϕ, ϕ_N), \quad C^{k,α}_N(ϕ_N) = Π_N(σ_k,α \cdot ∇ϕ_N), \quad ϕ_N ∈ H_N. \]
Consider the finite-dimensional SDE on $H_N$:
\[ dϕ_N(t) = [−b_N(ϕ_N(t)) + Δϕ_N(t)]dt + \frac{C_0}{θ} θ^{k,α} \sum_{k,α} θ_{k,α} C^{k,α}_N(ϕ_N(t)) \circ dW^{k,α}_t, \quad ϕ_N(0) = Π_NΦ_0. \tag{2.13} \]
The *a priori* estimate in Lemma 2.3 tells us that for any $N \geq 1$, P-a.s., for all $t ≤ T$,
\[ ‖ϕ_N(t)‖_{L^2} + \int_0^t ‖ϕ_N(s)‖_{H^1}^2 ds \leq ‖ϕ_N(0)‖_{L^2} + C_{δ,R,T} ≤ ‖Φ_0‖_{L^2} + C_{δ,R,T}. \tag{2.14} \]
Therefore, there exists a subsequence \( \{ \phi_{N_i} \}_{i \geq 1} \) converging weakly-\* in \( L^\infty(\Omega, L^\infty(0, T; L^2)) \) and weakly in \( L^2(\Omega, L^2(0, T; H^1)) \). To show the weak existence of solutions to (2.3), we use the classical compactness argument as in [22]; here, we follow more closely [27, Section 3].

Let \( Q_N \) be the law of the process \( \phi_N(\cdot) \) \((N \geq 1)\). By (2.14) and using the equation (2.13), it is not difficult to show (see, e.g., [27, Corollary 3.5]) that there exists a constant \( C > 0 \) such that

\[
\sup_{N \geq 1} \left[ \int_0^T \| \phi_N(t) \|_{H^1}^2 dt + \int_0^T \int_0^T \frac{\| \phi_N(t) - \phi_N(s) \|^2_{H^2}}{|t-s|^{1+2\gamma}} ds dt \right] \leq C,
\]

where \( \gamma \in (0, 1/2) \) is fixed; moreover, for any \( p \) big enough, there is \( C_p > 0 \) such that

\[
\sup_{N \geq 1} \left[ \int_0^T \| \phi_N(t) \|^p_{L^p} dt + \int_0^T \int_0^T \frac{\| \phi_N(t) - \phi_N(s) \|^4_{H^2}}{|t-s|^{7/3}} ds dt \right] \leq C_p.
\]

Recall the compact embeddings (see [45])

\[
\begin{align*}
L^2(0, T; V) \cap W^{\gamma, 2}(0, T; H^{-\delta}) &\subset L^2(0, T; H), \\
L^p(0, T; H) \cap W^{1/3, 4}(0, T; H^{-\delta}) &\subset C([0, T], H^{-\delta}),
\end{align*}
\]

where \( W^{\alpha,p}(0, T; H^{-\delta}) \), for some \( \alpha \in (0, 1) \) and \( p > 1 \), is the time fractional Sobolev space. The above uniform bounds imply that the family \( \{ Q_{N_N} \}_{N \geq 1} \) is tight in \( L^2(0, T; H) \) and \( C([0, T], H^{-\delta}) \). Now, the Prohorov theorem (see [7, p. 59, Theorem 5.1]) implies that there exists a subsequence \( \{ Q_{N_{N_i}} \}_{i \geq 1} \) which converges weakly to some probability measure \( Q \) supported on \( L^2(0, T; H) \cap C([0, T], H^{-\delta}) \). Moreover, by the Skorohod representation theorem (see [7, p. 70, Theorem 6.7]), there exist a new probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) and a sequence of stochastic processes \( \{ \tilde{\phi}_{N_i} \}_{i \geq 1} \) and \( \tilde{\phi} \) defined on \( \tilde{\Omega} \), such that

(i) \( \tilde{\phi}_{N_i} \), has the law \( Q_{N_i} \) for all \( i \geq 1 \), and \( \tilde{\phi} \) has the law \( Q \);

(ii) \( \tilde{\phi}_{N_i} \) converges as \( i \to \infty \) to \( \tilde{\phi} \) in the topology of \( L^2(0, T; H) \cap C([0, T], H^{-\delta}) \).

Since \( \tilde{\phi}_{N_i} \) has the same law \( Q_{N_i} \) as \( \phi_{N_i} \), and the latter enjoys the pathwise estimate (2.14), one can deduce that the limit \( \tilde{\phi} \) also enjoys

\[
\| \tilde{\phi} \|_{L^\infty(0, T; H)} \vee \| \tilde{\phi} \|_{L^2(0, T; V)} \leq C \| \phi_0 \|_{H^{\gamma, \delta, R, T}}.
\]

Having these preparations in mind and writing (2.13) in the weak form, one can pass to the limit in the nonlinear terms and prove that \( \tilde{\phi} \) solves (2.3) in the weak sense. The details are omitted here; we only give a sketched proof of the fact that (2.3) enjoys the pathwise uniqueness among those solutions satisfying the bounds (2.6).

**Proof of the pathwise uniqueness of (2.3).** First, we remark that thanks to the bounds (2.6), it is enough to require that \( v \in H^1(\mathbb{T}^3, \mathbb{R}^3) \) in the equations (2.4) and (2.5); indeed, we have

\[
\int_0^t \langle \xi_s, \Delta v + S_\theta(v) \rangle ds = -\int_0^t \langle \nabla \xi_s, \nabla v \rangle ds - \frac{C_2^2}{\| \theta \|^2_{L^2}} \sum_{k, \alpha} \theta_k^2 \int_0^t \langle \sigma_{k, \alpha}, \nabla \xi_s, \Pi(\sigma_{k, \alpha}, \nabla v) \rangle ds,
\]

similarly for the corresponding term in (2.5). Here, we write \( \langle \cdot, \cdot \rangle \) for the duality between \( H^1(\mathbb{T}^3) \) and \( H^{-1}(\mathbb{T}^3) \), which are used to denote \( \mathbb{R}^3 \)-valued functions and distributions with \( d = 3 \) or \( d = 6 \). Moreover, one can show that the assumptions of [42, p. 72, Theorem 2.13] are verified, and thus we can apply the Itô formula [42, (2.5.3)]. In the following, we omit the time variable to simplify notations.

Let \( \Phi_i = (\xi_i, \eta_i)^\ast \), \( i = 1, 2 \) be two solutions to (2.3) on the same filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) with the same initial condition \( \Phi_0 = (\xi_0, \eta_0)^\ast \) and the same family of Brownian motions \( \{ W_{k, \alpha} \}_{k \in \mathbb{Z}_3, \alpha = 1, 2} \) satisfying

\[
\text{P-a.s.,} \quad \| \Phi_i \|_{L^\infty(0, T; H)} \vee \| \Phi_i \|_{L^2(0, T; V)} \leq C \| \phi_0 \|_{H^{\gamma, \delta, R, T}}, \quad i = 1, 2.
\]

Then it holds in the distribution sense that for \( i = 1, 2 \),

\[
d\Phi_i = -f_R(\Phi_i)b(\Phi_i, \Phi_i) dt + [\Delta \Phi_i + S_\theta(\Phi_i)] dt + \frac{C_2}{\| \theta \|^2_{L^2}} \sum_{k, \alpha} \theta_k^2 \Pi(\sigma_{k, \alpha}, \nabla \Phi_i) dW_{k, \alpha}.
\]
Let $\Phi = (\xi, \eta)^* := \Phi_1 - \Phi_2 = (\xi_1 - \xi_2, \eta_1 - \eta_2)^*$; then
\[
\mathrm{d}\Phi = -[f_R(\Phi_1)b(\Phi_1, \Phi_1) - f_R(\Phi_2)b(\Phi_2, \Phi_2)]\mathrm{d}t + |\Delta \Phi + S_0(\Phi)|\mathrm{d}t
+ \frac{C_v}{\|\theta\|_{L^2}} \sum_{k,\alpha} \theta_k \Pi(\sigma_{k,\alpha} \cdot \nabla \Phi) \mathrm{d}W^{k,\alpha}.
\]

By the Itô formula (see [42, (2.5.3)]),
\[
d\|\Phi\|^2_{L^2} = -2\langle \Phi, f_R(\Phi_1)b(\Phi_1, \Phi_1) - f_R(\Phi_2)b(\Phi_2, \Phi_2)\rangle \mathrm{d}t + 2\langle \Phi, \Delta \Phi + S_0(\Phi)\rangle \mathrm{d}t
+ \frac{2C_v}{\|\theta\|_{L^2}} \sum_{k,\alpha} \theta_k^2 \Pi(\sigma_{k,\alpha} \cdot \nabla \Phi) \mathrm{d}W^{k,\alpha}.
\]

The definition of $S_0(\Phi)$ leads to
\[
\langle \Phi, \Delta \Phi + S_0(\Phi)\rangle = -\|\nabla \Phi\|^2_{L^2} - \frac{C_v^2}{\|\theta\|^2_{L^2}} \sum_{k,\alpha} \theta_k^2 \Pi(\sigma_{k,\alpha} \cdot \nabla \Phi) \|\Phi\|^2_{L^2}.
\]

Moreover, since $\sigma_{k,\alpha}$ and the components $\xi$ and $\eta$ of $\Phi$ are all divergence-free, we have
\[
\langle \Phi, \Pi(\sigma_{k,\alpha} \cdot \nabla \Phi)\rangle = \langle \Phi, \sigma_{k,\alpha} \cdot \nabla \Phi\rangle = 0.
\]

Consequently,
\[
d\|\Phi\|^2_{L^2} = -2\langle \Phi, f_R(\Phi_1)b(\Phi_1, \Phi_1) - f_R(\Phi_2)b(\Phi_2, \Phi_2)\rangle \mathrm{d}t - 2\|\nabla \Phi\|^2_{L^2} \mathrm{d}t. \tag{2.20}
\]

It remains to estimate the first term on the right-hand side of (2.20). We have
\[
|\langle \Phi, f_R(\Phi_1)b(\Phi_1, \Phi_1) - f_R(\Phi_2)b(\Phi_2, \Phi_2)\rangle|
\leq |\langle f_R(\Phi_1) - f_R(\Phi_2)\rangle(\Phi, b(\Phi_1, \Phi_1))| + |\langle f_R(\Phi_2)\rangle(\Phi, b(\Phi_1, \Phi_1) - b(\Phi_2, \Phi_2))|
\leq: J_1 + J_2. \tag{2.21}
\]

First, it is clear that
\[
|\langle f_R(\Phi_1) - f_R(\Phi_2)\rangle| \leq \|f_R\|_{L^\infty} \|\Phi_1\|_{H^{-\frac{s}{2}}} \|\Phi_2\|_{H^{-\frac{s}{2}}} \leq C \|\Phi_1 - \Phi_2\|_{H^{-\frac{s}{2}}} \leq C \|\Phi\|_{L^2}, \tag{2.22}
\]

and by definition,
\[
\langle \Phi, b(\Phi_1, \Phi_1)\rangle = \langle \xi, \mathcal{L}_{u_1} \xi_1 - \mathcal{L}_{B_1} \eta_1 \rangle + \langle \eta, \mathcal{L}_{u_1} \eta_1 - \mathcal{L}_{B_1} \xi_1 - 2T(B_1, u_1) \rangle;
\]

thus
\[
J_1 \leq C \|\Phi\|_{L^2} (|\langle \xi, \mathcal{L}_{u_1} \xi_1 \rangle| + |\langle \xi, \mathcal{L}_{B_1} \eta_1 \rangle| + |\langle \eta, \mathcal{L}_{u_1} \eta_1 \rangle| + |\langle \eta, \mathcal{L}_{B_1} \xi_1 \rangle| + |\langle \eta, T(B_1, u_1) \rangle|). \tag{2.23}
\]

We demonstrate how to estimate the first term on the right-hand side and the others can be treated in a similar way. We have
\[
J_{1,1} := C \|\Phi\|_{L^2} (|\langle \xi, \mathcal{L}_{u_1} \xi_1 \rangle| \leq C \|\Phi\|_{L^2} (|\langle \xi, u_1 \cdot \nabla \xi_1 \rangle| + C \|\Phi\|_{L^2} (|\langle \xi, \xi_1 \cdot \nabla u_1 \rangle| =: J_{1,1,1} + J_{1,1,2}.
\]

By Hölder’s inequality with the exponent $\frac{1}{\frac{1}{2} + \frac{1}{2}} = 1$,
\[
J_{1,1,1} \leq C \|\Phi\|_{L^2} \|\xi\|_{L^1} \|\xi_{1^1} u_1\|_{L^1} \|\nabla \xi_1\|_{L^2} \leq C \|\Phi\|_{L^2} (|\langle \xi, u_1 \cdot \nabla \xi_1 \rangle| + C \|\Phi\|_{L^2} (|\langle \xi, \xi_1 \cdot \nabla u_1 \rangle| \leq C \|\Phi\|_{L^2} \|\nabla \xi_1\|_{L^2}.
\]

where we have used the Sobolev embedding inequality (2.7). Moreover, applying the interpolation inequality and the Poincaré inequality, we have
\[
J_{1,1,1} \leq C \|\Phi\|_{L^2} \|\xi\|_{L^1} \|\xi_{1^1} u_1\|_{L^1} \|\nabla \xi_1\|_{L^2} \leq C \|\Phi\|_{L^2} \|\xi\|_{H^{1/2}} \|\xi_{1^1} u_1\|_{H^{1/2}} \|\nabla \xi_1\|_{L^2}.
\]

(2.24)
By Young’s inequality with the exponent $\frac{1}{4} + \frac{3}{4} = 1$, for $\varepsilon > 0$ small enough, we obtain
\[
J_{1.1.1} \leq \varepsilon\|\nabla \xi\|_{L^2}^2 + C_{\varepsilon}\|\Phi\|_{L^2}\|\xi\|_{L^2}^{4/3}\|\nabla \xi\|_{L^2}^{1/3} \leq \varepsilon\|\nabla \xi\|_{L^2}^2 + C_{\varepsilon}\|\Phi\|_{L^2}\|\nabla \xi\|_{L^2}^{4/3},
\]
since by (2.19), $\xi_t$ is almost surely bounded in $L^\infty(0, T; L^2)$. Next, we turn to estimate $J_{1.1.2}$. By Hölder’s inequality with the exponent $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$, we have
\[
J_{1.1.2} \leq C\|\Phi\|_{L^2}\|\xi\|_{L^2}\|\xi_t\|_{L^2} \leq C\|\Phi\|_{L^2}\|\xi\|_{H^{1/2}}\|\xi_t\|_{H^{1/2}} \leq C\|\Phi\|_{L^2}\|\xi_t\|_{L^2}\|\nabla \xi_t\|_{L^2} \leq C\|\Phi\|_{L^2}\|\xi_t\|_{L^2}\|\nabla \xi_t\|_{L^2},
\]
which is the same as the right-hand side of (2.24). Thus, similarly as above, we have
\[
J_{1.1.2} \leq \varepsilon\|\nabla \xi\|_{L^2}^2 + C_{\varepsilon}\|\Phi\|_{L^2}\|\nabla \xi\|_{L^2}^{4/3}.
\]
Summarizing the above arguments, we obtain
\[
J_{1.1} \leq 2\varepsilon\|\nabla \xi\|_{L^2}^2 + C_{\varepsilon}\|\Phi\|_{L^2}\|\nabla \xi\|_{L^2}^{4/3}. \tag{2.25}
\]
Proceeding as above for other terms in (2.23), we finally get
\[
J_t \leq n_1\varepsilon\|\nabla \Phi\|_{L^2}^2 + C_{\varepsilon}\|\Phi\|_{L^2}^2(\|\nabla \Phi_1\|_{L^2}^{4/3} + \|\nabla \Phi_2\|_{L^2}^{4/3} + \|\nabla \Phi_3\|_{L^2}^{4/3}),
\]
where $n_1$ is some integer and $\|\nabla \Phi\|_{L^2}^2 = \|\nabla \xi\|_{L^2}^2 + \|\nabla \eta\|_{L^2}^2$. The estimate of $J_2$ in (2.21) is similar, and thus we finally see that for some $n \in \mathbb{N}$,
\[
|\langle \Phi, f_R(\Phi_1)b(\Phi_1, \Phi_3) - f_R(\Phi_2)b(\Phi_2, \Phi_3) \rangle| \leq n_2\|\nabla \Phi\|_{L^2}^2 + C_{\varepsilon}\|\Phi\|_{L^2}^2(\|\nabla \Phi_1\|_{L^2}^{4/3} + \|\nabla \Phi_2\|_{L^2}^{4/3} + \|\nabla \Phi_3\|_{L^2}^{4/3}).
\]

Taking $\varepsilon = 1/n$ and substituting this estimate into (2.20), we arrive at
\[
d\|\Phi\|_{L^2}^2 \leq n\|\Phi\|_{L^2}^2(\|\nabla \Phi_1\|_{L^2}^{4/3} + \|\nabla \Phi_2\|_{L^2}^{4/3} + \|\nabla \Phi_3\|_{L^2}^{4/3})dt.
\]
The bounds in (2.19) imply that the quantity in the parentheses on the right-hand side is integrable with respect to $t \in [0, T]$. Since $\Phi_0 = 0$, Gronwall’s inequality yields, P-a.s., $\Phi_t = 0$ for all $t \in [0, T]$. Thus we have proved the pathwise uniqueness property of (2.3).

\section{Scaling limit and proofs of the main result}

Recall the sequences $\{\theta^N\}_{N \geq 1} \subset \ell^2$ defined in (1.9); we consider the following sequence of stochastic 3D MHD equations with a cut-off:
\[
d\Phi^N + f_R(\Phi^N)b(\Phi^N, \Phi^N)dt = [\Delta \Phi^N + S_{\theta^N}(\Phi^N)]dt + \frac{C_\nu}{\|\theta^N\|_{\ell^2}} \sum_{k, \alpha} \theta_k^N \Pi(\sigma_{k, \alpha} \cdot \nabla \Phi^N) dW^{k, \alpha} \tag{3.1}
\]
subject to the initial data $\Phi^N_0 = (\xi^N_0, \eta^N_0)^* \in B_{H^s}(K)$. For any $T > 0$, Theorem 2.2 implies that there exists a pathwise unique strong solution $\Phi^N_t = (\xi^N_t, \eta^N_t)^*$ to (3.1) satisfying
\[
\|\Phi^N\|_{L^\infty(0, T; H)} \vee \|\Phi^N\|_{L^2(0, T; V)} \leq C_{K, \delta, R, T}, \tag{3.2}
\]
where $C_{K, \delta, R, T}$ is some constant independent of $\nu$ and $N$; furthermore, for any divergence-free test vector field $v$, the following equalities hold P-a.s., on $[0, T]$:
\[
\langle \xi^N_t, v \rangle = \langle \xi^N_0, v \rangle + \int_0^t f_R(\Phi^N_s)[(\xi^N_s, \nabla \Phi^N_s v) - (\eta^N_s, \nabla \Phi^N_s v)]ds.
\]
estimates to (2.15) and (2.16). Therefore, the family

\[ \mathcal{C} \]

new probability space \( \hat{\Omega} \), corresponding to the solution \( \hat{\Phi} \)

in the weak sense the deterministic

solution to the unique global solution

With these results in mind, we can prove the following lemma.

The first main result of this section is the next limit theorem.

**Theorem 3.1** (Scaling limit). Fix any \( K > 0 \) and \( T > 0 \), and assume that the initial data \( \Phi^N_0 \) converge weakly in \( H \) to some \( \Phi_0 \). Then we can find big \( \nu \) and \( R \) such that as \( N \to \infty \), the solutions \( \Phi^N \) converge in probability to the unique global solution \( \Phi \) to the deterministic 3D MHD equations

\[ \partial_t \Phi + b(\Phi, \Phi) = \left( 1 + \frac{3}{5} \nu \right) \Delta \Phi. \]  

(3.5)

Moreover, denoting \( \| \cdot \|_X = \| \cdot \|_{C([0,T],H^{-\delta})} \lor \| \cdot \|_{L^2(0,T;H)} \lor \| \cdot \|_{L^2(0,T;H^\delta)} \lor \| \cdot \|_{L^\infty(0,T;H^\delta)} \), for any \( \varepsilon > 0 \), we have

\[ \lim_{N \to \infty} \sup_{\Phi_0 \in B_{H^\delta}(K)} P(\| \Phi^N_0(\cdot, \Phi_0) - \Phi(\cdot, \Phi_0) \|_X > \varepsilon) = 0. \]  

(3.6)

where we write \( \Phi^N(\cdot, \Phi_0) \) (resp. \( \Phi(\cdot, \Phi_0) \)) for the unique solution to (3.1) (resp. (3.5)) with the initial condition \( \Phi_0 \).

We follow some of the arguments below the proof of Lemma 2.3. Let \( Q^N \) be the law of \( \{\Phi^N_t\}_{t \in [0,T]} \) \( (N \geq 1) \). Using the uniform bounds (3.2) and the equations (3.3) and (3.4), we can prove the similar estimates to (2.15) and (2.16). Therefore, the family \( \{Q^N\}_{N \geq 1} \) of laws is tight on \( L^2(0,T;H) \) and on \( C([0,T],H^{-\delta}) \). Thus, by the Prohorov theorem we can find a subsequence \( \{Q^N_i\}_{i \geq 1} \) converging weakly to some probability measure \( Q \), supported on \( L^2(0,T;H) \) and on \( C([0,T],H^{-\delta}) \). Applying the Skorohod theorem yields a probability space \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}) \) and a sequence of processes \( \{\hat{\Phi}^N_i\}_{i \geq 1} \) and \( \hat{\Phi} \) defined on \( \hat{\Omega} \) such that

(a) \( \hat{\Phi} \) is distributed as \( Q \) and \( \hat{\Phi}^N \) is distributed as \( Q^N_i \) for all \( i \geq 1 \);

(b) \( \hat{\Phi} \)-a.s., \( \hat{\Phi}^N \) converges as \( i \to \infty \) to \( \hat{\Phi} \) in the topology of \( L^2(0,T;H) \) and on \( C([0,T],H^{-\delta}) \).

We remark that for every \( N \geq 1 \), if we consider \( Q^N \) together with the laws of the Brownian motions \( \{W^k \in [k \in Z^d, \alpha, \beta] \} \), we see that we can find on the new probability space \( \hat{\Omega} \) a sequence of Brownian motions \( \{W^k \in [k \in Z^d, \alpha, \beta] \} \) such that for any \( \alpha \geq 1 \), \( \hat{\Phi}^N \) and \( \hat{W}^N \) satisfy the equations (3.3) and (3.4). Let \( \hat{\Phi}^N \) and \( \hat{\Phi}^N \) be the velocity field and the magnetic field on the new probability space \( \hat{\Omega} \), corresponding to the solution \( \Phi^N_i = (\xi^N_i, \eta^N_i) \); then

\( \langle \xi^N_t, v \rangle = \langle \xi^N_t, \eta^N_t \rangle + \int_0^t f_R((\Phi^N_s))[(\xi^N_s, \xi^N_s, v)] - \xi^N_t v) ds \]

\[ + \int_0^t (\xi^N_s, \Delta v) + S_N(v)(v) ds - \frac{C}{\| \Phi^N_t \|} \sum_{k, \alpha} \theta_k^N \int_0^t (\xi^N_s, \sigma_{k, \alpha} v) dW_s^{N, k, \alpha}, \]  

(3.7)

\( \langle \eta^N_t, v \rangle = \langle \eta^N_t, \eta^N_t \rangle + \int_0^t f_R((\Phi^N_s))[(\eta^N_s, \eta^N_s, v)] - \xi^N_t v) ds \]

\[ + \int_0^t (\eta^N_s, \Delta v) + S_N(v)(v) ds - \frac{C}{\| \Phi^N_t \|} \sum_{k, \alpha} \theta_k^N \int_0^t (\eta^N_s, \sigma_{k, \alpha} v) dW_s^{N, k, \alpha}. \]  

(3.8)

With these results in mind, we can prove the following lemma.

**Lemma 3.2.** Let \( \Phi_0 \) be the weak limit of the initial data \( \{\Phi^N_0\}_{N \geq 1} \); then the limit process \( \hat{\Phi} \) solves in the weak sense the deterministic 3D MHD equations with a cut-off:

\[ \partial_t \hat{\Phi} + f_R(\hat{\Phi})b(\hat{\Phi}, \hat{\Phi}) = \left( 1 + \frac{3}{5} \nu \right) \Delta \hat{\Phi}, \quad \hat{\Phi}_{|t=0} = \Phi_0. \]  

(3.9)
Proof. Let \( \hat{u} \) and \( \hat{B} \) be the velocity field and the magnetic field associated with the limit process \( \hat{\Phi} = (\hat{\xi}, \hat{\eta}) \). Thanks to the item (b) above, it is standard to prove the convergence of the terms on the first line of (3.7). Moreover, by the key limit (1.11), it is clear that, P-a.s., as \( i \to \infty \),

\[
\int_0^t (\hat{\xi}_{N,i}, \Delta v + S_{N,i}(v))ds \to \left(1 + \frac{3}{5} \nu \right) \int_0^t (\hat{\xi}_s, \Delta v)ds
\]

in the topology of \( C([0,T], \mathbb{R}) \).

Next, we deal with the martingale part: by the Itô isometry,

\[
\hat{E} \left[ \frac{C_p}{\|\theta^{N_i}\|_2^2} \sum_{k,\alpha} \theta_k^{N_i} \int_0^t (\hat{\xi}_{N,i}^k, \sigma_{k,\alpha} \cdot \nabla v) d\hat{W}_{s,N_i,k,\alpha}^2 \right] \\
= \hat{E} \left[ \frac{C_p^2}{\|\theta^{N_i}\|_2^2} \sum_{k,\alpha} (\theta_k^{N_i})^2 \int_0^t |(\hat{\xi}_{N,i}^k, \sigma_{k,\alpha} \cdot \nabla v)|^2 ds \right] \\
\leq C_p^2 \|\theta^{N_i}\|_2^2 \hat{E} \int_0^t \sum_{k,\alpha} |(\hat{\xi}_{N,i}^k, \sigma_{k,\alpha} \cdot \nabla v)|^2 ds.
\]

Note that \( \hat{\Phi}^{N_i} = (\hat{\xi}^{N_i}, \hat{\eta}^{N_i}) \) has the same law as the solution \( \Phi^{N_i} \) to (3.1), and thus it fulfills the estimate (3.2). We have

\[
\sum_{k,\alpha} |(\hat{\xi}_{N,i}^k, \sigma_{k,\alpha} \cdot \nabla v)|^2 = \sum_{k,\alpha} |(\nabla v)^* \hat{\xi}_{N,i}^k, \sigma_{k,\alpha}|^2 \leq \| (\nabla v)^* \hat{\xi}_{N,i}^k \|^2_{L^2} \leq C_{K,\delta,R,T} \| \nabla v \|^2_{L^2},
\]

where in the second step we have used the fact that \( \{\sigma_{k,\alpha}\}_{k,\alpha} \) is an orthonormal family in \( L^2(\mathbb{T}^3, \mathbb{R}^3) \). Consequently,

\[
\hat{E} \left[ \frac{C_p}{\|\theta^{N_i}\|_2^2} \sum_{k,\alpha} \theta_k^{N_i} \int_0^t (\hat{\xi}_{N,i}^k, \sigma_{k,\alpha} \cdot \nabla v) d\hat{W}_{s,N_i,k,\alpha} \right]^2 \\
\leq C_p^2 C_{K,\delta,R,T} \| \nabla v \|^2_{L^2} \frac{\|\theta^{N_i}\|^2_{L^2}}{\|\theta^{N_i}\|^2_{L^2}}.
\]

The right-hand side vanishes as \( i \to \infty \) due to the limit (1.10), and thus the martingale part tends to 0 in the mean square sense. Summarizing the above arguments, we have proved

\[
\langle \hat{\xi}_t, v \rangle = \langle \xi_0, v \rangle + \int_0^t f_R(\hat{\Phi}_s)[\langle \hat{\xi}_s, \mathcal{L}_u^*, v \rangle - \langle \hat{\eta}_s, \mathcal{L}_{B_s}^*, v \rangle]ds + \left(1 + \frac{3}{5} \nu \right) \int_0^t (\hat{\xi}_s, \Delta v)ds.
\]

In the same way, we can prove that (3.8) converges to the following equation:

\[
\langle \hat{\eta}_t, v \rangle = \langle \eta_0, v \rangle + \left(1 + \frac{3}{5} \nu \right) \int_0^t (\hat{\eta}_s, \Delta v)ds + \int_0^t f_R(\hat{\Phi}_s)[\langle \hat{\eta}_s, \mathcal{L}_u^*, v \rangle - \langle \hat{\xi}_s, \mathcal{L}_{B_s}^* v \rangle - 2T(\hat{B}_s, \hat{u}_s, v)]ds.
\]

This finishes the proof. \( \square \)

Next, we give a short proof of the following well-known result.

Lemma 3.3. Given \( K > 0 \), there exists big \( \nu = \nu(K) > 0 \) such that for all \( \Phi_0 \in B_H(K) \), the deterministic 3D MHD equation (3.5) admits a unique global solution \( \Phi(\cdot, \Phi_0) \); moreover, there exist some \( C_K > 0 \) and \( \lambda = \lambda(K) > 0 \) such that for any \( \Phi_0 \in B_H(K) \), it holds that

\[
\| \Phi(t, \Phi_0) \|_{L^2} \leq C_K e^{-\lambda t}, \quad t \geq 0.
\]

Proof. The proof is simpler than that of Lemma 2.3. We define \( \nu_1 = 1 + \frac{3}{5} \nu \); then

\[
\frac{d}{dt} \| \Phi \|^2_{L^2} = -2\langle \Phi, b(\Phi, \Phi) \rangle - 2\nu_1 \| \nabla \Phi \|^2_{L^2}.
\]

More precisely,

\[
\frac{d}{dt} \left( \| \xi \|^2_{L^2} + \| \eta \|^2_{L^2} \right) = -2\langle \xi, b_1(\Phi, \Phi) \rangle + \langle \eta, b_2(\Phi, \Phi) \rangle - 2\nu_1 (\| \nabla \xi \|^2_{L^2} + \| \nabla \eta \|^2_{L^2}).
\]

(3.11)
We estimate separately the first two quantities on the right-hand side.

**Step 1.** By the definition of $b_1(\Phi, \Phi)$, we have
\[
\langle \xi, b_1(\Phi, \Phi) \rangle = \langle \xi, \mathcal{L}_u \xi \rangle - \langle \xi, \mathcal{L}_B \eta \rangle =: I_1 + I_2.
\]
First, since $u$ is divergence-free, we have $I_1 = -\langle \xi, \xi \cdot \nabla u \rangle$, and by Hölder’s inequality, the Sobolev embedding and the interpolation inequality,
\[
|I_1| \leq \|\xi\|_{L^3}^2 \|\nabla u\|_{L^3} \leq C \|\xi\|_{H^{1/2}}^3 \leq C \|\xi\|_{L^2}^{3/2} \|\xi\|_{H^{1/2}}^{3/2} \leq \varepsilon \|\xi\|_{H^1}^2 + C^4 \varepsilon^{-3} \|\xi\|_{L^2}^6,
\]
where $\varepsilon > 0$ is a small constant. Next,
\[
I_2 = -\langle \xi, B \cdot \nabla \eta \rangle + \langle \xi, \eta \cdot \nabla B \rangle =: I_{2,1} + I_{2,2}.
\]
For $I_{2,1}$, again by Hölder’s inequality,
\[
|I_{2,1}| \leq \|\xi\|_{L^3} \|B\|_{L^6} \|\nabla \eta\|_{L^2} \leq C \|\xi\|_{H^{1/2}} \|B\|_{H^1} \|\eta\|_{H^1} \leq C \|\xi\|_{L^3}^{1/2} \|\xi\|_{L^6}^{1/2} \|\eta\|_{L^2} \|\eta\|_{H^1}.
\]
By Young’s inequality with exponents $\frac{1}{12} + \frac{1}{4} + \frac{1}{6} + \frac{1}{2} = 1$, we obtain
\[
|I_{2,1}| \leq \varepsilon \left(\|\xi\|_{H^1}^2 + \|\eta\|_{H^1}^2\right) + C^4 \varepsilon^{-3} \left(\|\xi\|_{L^2}^6 + \|\eta\|_{L^2}^6\right).
\]
Similarly, for $I_{2,2}$, we have
\[
|I_{2,2}| \leq \|\xi\|_{L^3} \|\eta\|_{L^3} \|\nabla B\|_{L^2} \leq C \|\xi\|_{L^3} \|\eta\|_{L^3} \|\eta\|_{H^1} \leq C \|\xi\|_{L^3}^{1/2} \|\xi\|_{H^{1/2}} \|\eta\|_{L^2} \|\eta\|_{H^1},
\]
and thus it admits the same estimate as $|I_{2,1}|$. In summary, we obtain
\[
|\langle \xi, b_1(\Phi, \Phi) \rangle| \leq 3\varepsilon \left(\|\xi\|_{H^1}^2 + \|\eta\|_{H^1}^2\right) + C^4 \varepsilon^{-3} \left(\|\xi\|_{L^2}^6 + \|\eta\|_{L^2}^6\right).
\]

**Step 2.** Now we estimate
\[
\langle \eta, b_2(\Phi, \Phi) \rangle = \langle \eta, \mathcal{L}_u \eta \rangle - \langle \eta, \mathcal{L}_B \xi \rangle - 2\langle \eta, T(B, u) \rangle =: J_1 + J_2 + J_3.
\]
The first term $J_1 = -\langle \eta, \eta \cdot \nabla u \rangle$ can be treated as follows:
\[
|J_1| \leq \|\eta\|_{L^3}^2 \|\nabla u\|_{L^3} \leq C \|\xi\|_{L^3} \|\eta\|_{L^3}^2 \leq \varepsilon \left(\|\xi\|_{H^1}^2 + \|\eta\|_{H^1}^2\right) + C^4 \varepsilon^{-3} \left(\|\xi\|_{L^2}^6 + \|\eta\|_{L^2}^6\right).
\]
The second term $J_2$ is similar to $I_2$, and we have
\[
|J_2| \leq 2\varepsilon \left(\|\xi\|_{H^1}^2 + \|\eta\|_{H^1}^2\right) + C^4 \varepsilon^{-3} \left(\|\xi\|_{L^2}^6 + \|\eta\|_{L^2}^6\right).
\]
Finally, by the definition of $T(B, u)$, we have
\[
|J_3| = 2\|\eta, T(B, u)\| \leq C \|\eta\|_{L^3} \|\nabla B\|_{L^2} \|\nabla u\|_{L^3} \leq C \|\xi\|_{L^3} \|\eta\|_{L^3}^2 \leq \varepsilon \left(\|\xi\|_{H^1}^2 + \|\eta\|_{H^1}^2\right) + C^4 \varepsilon^{-3} \left(\|\xi\|_{L^2}^6 + \|\eta\|_{L^2}^6\right).
\]
Summarizing these estimates leads to
\[
|\langle \eta, b_2(\Phi, \Phi) \rangle| \leq 4\varepsilon \left(\|\xi\|_{H^1}^2 + \|\eta\|_{H^1}^2\right) + C^4 \varepsilon^{-3} \left(\|\xi\|_{L^2}^6 + \|\eta\|_{L^2}^6\right)
\]
for another constant $C > 0$ independent of $\varepsilon$.

**Step 3.** Substituting the estimates in Steps 1 and 2 into (3.11), we arrive at
\[
\frac{d}{dt} \left(\|\xi\|_{L^2}^2 + \|\eta\|_{L^2}^2\right) \leq -2\nu_1 \left(\|\nabla \xi\|_{L^2}^2 + \|\nabla \eta\|_{L^2}^2\right) + 14\varepsilon \left(\|\xi\|_{H^1}^2 + \|\eta\|_{H^1}^2\right) + C^4 \varepsilon^{-3} \left(\|\xi\|_{L^2}^6 + \|\eta\|_{L^2}^6\right);
written in a more compact form, one has
\[
\frac{d}{dt} \left\| \Phi \right\|_{L^2}^2 \leq -2\nu_1 \left\| \nabla \Phi \right\|_{L^2}^2 + 14\varepsilon \left\| \Phi \right\|_{H^1}^2 + C_1^2\varepsilon^{-3} \left\| \Phi \right\|_{L^2}^6.
\]
Choosing \( \varepsilon = \nu_1/14 \) gives us
\[
\frac{d}{dt} \left\| \Phi \right\|_{L^2}^2 \leq -\nu_1 \left\| \nabla \Phi \right\|_{L^2}^2 + \nu_1 \left\| \Phi \right\|_{L^2}^2 + C_1^4\nu_1^{-3} \left\| \Phi \right\|_{L^2}^6 \\
\leq -(4\pi^2 - 1)\nu_1 \left\| \Phi \right\|_{L^2}^2 + C_1^4\nu_1^{-3} \left\| \Phi \right\|_{L^2}^6,
\]
(3.12)
where \( C_1 > 0 \) is another constant independent of \( \nu_1 = 1 + \frac{2}{5}\nu \) and in the second step we have used the Poincaré inequality on \( \mathbb{T}^3 \). This differential inequality can be solved explicitly and for \( \left\| \Phi_0 \right\|_{L^2} \leq K \), if \( \nu_1 = 1 + \frac{2}{5}\nu \) is big enough such that
\[
\nu_1 \geq \frac{C_1 K}{\sqrt{\pi}},
\]
then one has
\[
\left\| \Phi(t, \Phi_0) \right\|_{L^2} \leq \frac{2^{1/4}}{K_0} - \left( 4\pi^2 - 1 \right)\nu_1 t/2 \quad \text{for all} \ t > 0.
\]
Thus we obtain (3.10) with \( C_K = 2^{1/4} K \) and \( \lambda = (4\pi^2 - 1)\nu_1/2 > 0 \).

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. For fixed \( K > 0 \), we choose \( \nu = \nu(K) > 0 \) as in Lemma 3.3; if we take \( R = C_K + 1 \), then the solution \( \Phi \) obtained in Lemma 3.2 will not attain the cut-off threshold, and thus it coincides with the unique global solution \( \Phi(\cdot, \Phi_0) \) to (3.5). We conclude that the limit law \( Q = \delta_{\Phi(\cdot, \Phi_0)} \), which is uniquely determined. The tightness of the family \( \{Q^N\}_{N \geq 1} \) yields that the whole sequence converges weakly to \( \delta_{\Phi(\cdot, \Phi_0)} \) as \( N \to \infty \). As the limit is deterministic, we see that a sequence of processes \( \{\Phi^N\}_{N \geq 1} \) converges in probability to \( \Phi(\cdot, \Phi_0) \). This proves the first assertion of Theorem 3.1.

Next, we turn to proving the second assertion of Theorem 3.1. We follow the idea of proof of [27, Theorem 1.4] and argue by contradiction. Suppose that there exists an \( \varepsilon_0 > 0 \) small enough such that
\[
\limsup_{N \to \infty} \sup_{\Phi_0 \in B_H(K)} \mathbb{P}(\left\| \Phi^N(\cdot, \Phi_0) - \Phi(\cdot, \Phi_0) \right\|_X > \varepsilon_0) = 0,
\]
where we have denoted \( \left\| \cdot \right\|_X = \left\| \cdot \right\|_{L^2(0, T; H)} \vee \left\| \cdot \right\|_{C([0, T], H^{-1})} \). Recall that \( \Phi^N(\cdot, \Phi_0) \) is the pathwise unique solution to (3.1) with the initial condition \( \Phi_0 \in B_H(K) \), while \( \Phi(\cdot, \Phi_0) \) is the unique global solution to the deterministic 3D MHD equation (3.5) with the initial condition \( \Phi_0 \). Then we can find a subsequence of integers \( \{N_i\}_{i \geq 1} \) and \( \Phi^N_0 \in B_H(K) \) such that (choosing \( \varepsilon_0 \) even smaller if necessary)
\[
\mathbb{P}(\left\| \Phi^N(\cdot, \Phi^N_0) - \Phi(\cdot, \Phi^N_0) \right\|_X > \varepsilon_0) > 0.
\]
(3.13)
For each \( i \geq 1, \) let \( Q^N_i \) be the law of \( \Phi^N(\cdot, \Phi^N_0) \). Since \( \{\Phi^N_0\}_{i \geq 1} \) is contained in the ball \( B_H(K) \), there exists a subsequence of \( \{\Phi^N_i\}_{i \geq 1} \) (not relabeled for simplicity) converging weakly to some \( \Phi_0 \in B_H(K) \).

Similar to the discussion at the beginning of this section, we can show that the family \( \{Q^N_i\}_{i \geq 1} \) is tight on \( \mathcal{X} = L^2(0, T; H) \cap C([0, T], H^{-1}) \), and hence, up to a subsequence, \( Q^N_i \) converges weakly to some probability measure \( Q \) supported on \( \mathcal{X} \). As a result, by Skorokhod’s representation theorem, we can find a new probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) \) and a sequence of processes \( \{\tilde{\Phi}^N_i\}_{i \in \mathbb{N}} \) defined on \( \tilde{\Omega} \) such that for each \( i \in \mathbb{N} \), \( \tilde{\Phi}^N_i \) has the same law \( Q^N_i \) as \( \Phi^N_i(\cdot, \Phi^N_0) \), and \( \tilde{P}\)-a.s., \( \tilde{\Phi}^N_i \) converges as \( i \to \infty \) to some \( \tilde{\Phi} \) strongly in \( \mathcal{X} \). As before, the limit \( \tilde{\Phi} \) solves the deterministic 3D MHD equation (3.5) with the initial condition \( \Phi_0 \). From this, we conclude that \( \tilde{\Phi} = \Phi(\cdot, \Phi_0) \), and thus, as \( i \to \infty \), \( \tilde{\Phi}^N_i \) converges in \( \mathcal{X} \) to \( \tilde{\Phi}(\cdot, \Phi_0) \) in probability, i.e., for any \( \varepsilon > 0 \),
\[
\lim_{i \to \infty} \tilde{\mathbb{P}}(\left\| \tilde{\Phi}^N_i - \tilde{\Phi}(\cdot, \Phi_0) \right\|_X > \varepsilon) = 0.
\]
(3.14)
Note that \( \tilde{\Phi}^N_i \overset{d}{\sim} Q^N_i \), and (3.13) implies
\[
\tilde{\mathbb{P}}(\left\| \tilde{\Phi}^N_i - \tilde{\Phi}(\cdot, \Phi^N_0) \right\|_X > \varepsilon_0) \geq \varepsilon_0 > 0.
\]
(3.15)
We have the triangle inequality
\[ \|\Phi^N - \Phi(\cdot, \Phi_0)\|_{L^2} \leq \|\Phi^N - \Phi(\cdot, \Phi_0)\|_{L^2} + \|\Phi(\cdot, \Phi_0) - \Phi(\cdot, \Phi_0^N)\|_{L^2}. \]  
(3.16)

Recall that \(\|\Phi_0^N\|_{L^2} \leq K, i \geq 1\). Lemma 3.3 implies that \(\{\Phi(\cdot, \Phi_0^N)\}_{i \geq 1}\) is bounded in \(L^\infty(0, T; H)\). This estimate and the first inequality in (3.12) further imply that the family is bounded in \(L^2(0, T; V)\). One can also show its boundedness in \(W^{1,2}(0, T; H^{-2})\) by using the equation (3.5). Then by the embedding results similar to those in (2.17), the family \(\{\Phi(\cdot, \Phi_0^N)\}_{i \geq 1}\) is sequentially compact in \(X = L^2(0, T; H) \cap C([0, T], H^{-\delta})\). Therefore, up to a subsequence, \(\Phi(\cdot, \Phi_0^N)\) converges in \(X\) to some \(\Phi\) which can be shown to solve (3.5) since \(\Phi_0^N\) converges weakly to \(\Phi_0\). In other words, \(\Phi = \Phi(\cdot, \Phi_0)\) and \(\|\Phi(\cdot, \Phi_0^N) - \Phi(\cdot, \Phi_0)\|_{L^2} \to 0\) as \(i \to \infty\). Combining this result with (3.14)–(3.16), we get a contradiction. 

Finally we can provide the following proof of Theorem 1.1.

**Proof of Theorem 1.1.** We divide the proof into three steps.

**Step 1.** We fix \(K > 0\), and choose \(\nu\) and \(R\) as in the proof of Theorem 3.1; we know that the unique global solution to (3.5) satisfies
\[ \|\Phi(\cdot, \Phi_0)\|_{C([0, T], H^{-\delta})} \leq \|\Phi(\cdot, \Phi_0)\|_{C([0, T], H^{-\delta})} \leq C_K = R - 1 \quad \text{for all } \Phi_0 \in B_H(K), \]  
(3.17)

Moreover, by the exponential decay (3.10), we can take \(T > 1\) big enough such that
\[ \|\Phi(\cdot, \Phi_0)\|_{L^2(T-1, T; H)} \leq r_0/2 \quad \text{for all } \Phi_0 \in B_H(K), \]  
(3.18)

where \(r_0\) is the small number mentioned at the end of the paragraph involving the stochastic 3D MHD equation (1.4) (see (1.6) for the more precise form); the latter admits a unique global solution for any initial condition \(\Phi_0 \in B_H(r_0)\). Note that \(r_0\) is independent of \(\theta \in \ell^2\) and \(\nu > 0\). Without loss of generality, we can assume \(r_0 \leq 1\).

**Step 2.** Now we consider the approximating equation (3.1), but with the same initial condition \(\Phi_0\) as in (3.5). Given \(T > 0\) as in (3.18) and arbitrary small \(\varepsilon > 0\), Theorem 3.1 implies that there exists an \(N_0 = N_0(K, \nu, R, \varepsilon) \in \mathbb{N}\) such that for all \(N \geq N_0\), the pathwise unique strong solution \(\Phi^N(\cdot, \Phi_0)\) to (3.1) satisfies that for all \(\Phi_0 \in B_H(K),\)
\[ P(\|\Phi^N(\cdot, \Phi_0) - \Phi(\cdot, \Phi_0)\|_{L^2} \leq r_0/2) \geq 1 - \varepsilon, \]  
(3.19)

where \(\|\cdot\|_{L^2} = \|\cdot\|_{L^2(0, T; H)} \vee \|\cdot\|_{C([0, T], H^{-\delta})}.\) In the sequel, we fix such an \(N \geq N_0\). Combining this with (3.17), we deduce
\[ P(\|\Phi^N(\cdot, \Phi_0)\|_{C([0, T], H^{-\delta})} < R) \geq 1 - \varepsilon. \]

Defining the stopping times \(\tau^N_R = \inf\{t > 0 : \|\Phi^N(t, \Phi_0)\|_{H^{-\delta}} > R\} \) (inf \(\emptyset = T\), we have
\[ P(\tau^N_R > T) \geq 1 - \varepsilon. \]

For any \(t \leq \tau^N_R\), we have
\[ f(R(\Phi^N(t, \Phi_0))) = f_R(\|\Phi^N(t, \Phi_0)\|_{H^{-\delta}}) = 1, \]
and thus \(\{\Phi^N(t, \Phi_0)\}_{t \leq \tau^N_R}\) is a solution to the following equation without a cut-off:
\[ d\Phi^N + b(\Phi^N, \Phi^N)dt = [\Delta \Phi^N + S_{\mu \kappa}(\Phi^N)]dt + \frac{C_{\nu}}{\|\sqrt{\nu}\|_{L^2}} \sum_{\kappa, \alpha} \theta^N_{\kappa} \Pi(\sigma_{\kappa, \alpha} \cdot \nabla \Phi^N) dW^\kappa_{\alpha}. \]  
(3.20)

To sum up, with probability greater than \(1 - \varepsilon\), uniformly in \(\Phi_0 \in B_H(K),\) this equation admits a pathwise unique solution on \([0, T]\).

**Step 3.** Finally, let \(\Omega_{N, \varepsilon}\) be the event on the left-hand side of (3.19). Then \(P(\Omega_{N, \varepsilon}) \geq 1 - \varepsilon.\) On the event \(\Omega_{N, \varepsilon},\) the triangle inequality yields
\[ \|\Phi^N(\cdot, \Phi_0)\|_{L^2(T-1, T; H)} \leq \|\Phi^N(\cdot, \Phi_0) - \Phi(\cdot, \Phi_0)\|_{L^2(T-1, T; H)} + \|\Phi(\cdot, \Phi_0)\|_{L^2(T-1, T; H)} \]
\[ \leq \frac{r_0}{2} + \frac{r_0}{2} = r_0, \]
where in the second step we have used (3.18). This inequality holds for all $\omega \in \Omega_{N,\epsilon}$. As a result, for any $\omega \in \Omega_{N,\epsilon}$, there exists a $t = t(\omega) \in [T - 1, T]$ such that
\[
\|\Phi^N(t(\omega), \Phi_0, \omega)\|_{L^2} \leq r_0.
\]
Therefore, restarting the equation (3.20) at time $t(\omega)$ with the initial condition $\Phi^N(t(\omega), \Phi_0, \omega)$, we conclude that the solution extends to all $t > t(\omega)$ for every $\omega \in \Omega_{N,\epsilon}$. This completes the proof of Theorem 1.1 for $\nu > 0$ taken as above and $\theta = \theta^N \in \ell^2$.

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Appendix A

In this appendix, we provide a heuristic proof of the key limit (1.11) in a special case; the full proof is quite long and the interested reader is referred to [27, Section 5].

First, recall that for a divergence-free smooth vector field \( v \),

\[
S_\theta(v) = \frac{C_\nu^2}{\|\theta\|^2} \sum_{k, \alpha} \frac{\theta_k^2}{\alpha} \Pi[\sigma_{k, \alpha} \cdot \nabla \Pi(\sigma_{-k, \alpha} \cdot \nabla v)],
\]

where \( \Pi \) is the Leray projection operator. Let \( \Pi^\perp \) be the operator which is orthogonal to \( \Pi \). Then we have

\[
S_\theta(v) = \frac{C_\nu^2}{\|\theta\|^2} \sum_{k, \alpha} \frac{\theta_k^2}{\alpha} \Pi[\sigma_{k, \alpha} \cdot \nabla (\sigma_{-k, \alpha} \cdot \nabla v)] - \frac{C_\nu^2}{\|\theta\|^2} \sum_{k, \alpha} \frac{\theta_k^2}{\alpha} \Pi[\sigma_{k, \alpha} \cdot \nabla \Pi^\perp (\sigma_{-k, \alpha} \cdot \nabla v)].
\]

It is not difficult to show that (see [27, (2.4)])

\[
\frac{C_\nu^2}{\|\theta\|^2} \sum_{k, \alpha} \frac{\theta_k^2}{\alpha} \Pi[\sigma_{k, \alpha} \cdot \nabla (\sigma_{-k, \alpha} \cdot \nabla v)] = \nu \Pi(\Delta v) = \nu \Delta v;
\]

thus, if we define

\[
S_{\theta^N}^+(v) = \frac{C_\nu^2}{\|\theta\|^2} \sum_{k, \alpha} \frac{\theta_k^2}{\alpha} \Pi[\sigma_{k, \alpha} \cdot \nabla \Pi^\perp (\sigma_{-k, \alpha} \cdot \nabla v)],
\]

then it suffices to prove that for \( \theta^N \) defined in (1.9),

\[
\lim_{N \to \infty} S_{\theta^N}^+(v) = \frac{2}{5} \nu \Delta v \quad \text{holds in } L^2(\mathbb{T}^3, \mathbb{R}^3). \tag{A.1}
\]

Below we prove a weaker form of the limit in a particular case.

Recall that for a general vector field \( X \), formally,

\[
\Pi^\perp X = \nabla \Delta^{-1} \text{div}(X).
\]

On the other hand, if \( X = \sum_{l \in \mathbb{Z}^3} X_l e_l \) and \( X_l \in \mathbb{C}^3 \), then

\[
\Pi^\perp X = \sum_l \frac{l \cdot X_l}{|l|^2} le_l = \nabla \left[ \frac{1}{2 \pi i} \sum_l \frac{l \cdot X_l}{|l|^2} e_l \right].
\]

We take a special vector field

\[
v = \sigma_{l,1} + \sigma_{l,2} = (a_{l,1} + a_{l,2}) e_l,
\]

where \( a_{l,1} \) and \( a_{l,2} \) are defined in Subsection 2.1. Using any of the equalities above, one can prove (see [27, Corollary 5.3])

\[
S_{\theta^N}^+(v) = -\frac{6\pi^2 \nu}{\|\theta^N\|^2} \sum_{l, \beta} \sum_{\alpha=1}^2 \left| l \right|^2 \Pi \left\{ \left[ \sum_{k} (\theta_k^N)^2 \sin^2(\zeta_{k,l})(a_{l,\beta} \cdot (k - l)) \frac{k - l}{|k - l|^2} \right] e_l \right\}
\]

\[
\sim -\frac{6\pi^2 \nu}{\|\theta^N\|^2} \sum_{l, \beta} \sum_{\alpha=1}^2 \left| l \right|^2 \Pi \left\{ \left[ \sum_{k} (\theta_k^N)^2 \sin^2(\zeta_{k,l})(a_{l,\beta} \cdot k) \frac{k}{|k|^2} \right] e_l \right\},
\]

where \( \zeta_{k,l} \) is the angle between the vectors \( k \) and \( l \), and \( \sim \) means the difference between the two quantities vanishing as \( N \to \infty \). The complex conjugate \( \bar{v} \) of \( v \) is divergence-free, and hence

\[
(S_{\theta^N}^+(v), \bar{v})_{L^2} \sim -\frac{6\pi^2 \nu}{\|\theta^N\|^2} \sum_{l, \beta} \sum_{\alpha=1}^2 \left| l \right|^2 \left\{ \left[ \sum_{k} (\theta_k^N)^2 \sin^2(\zeta_{k,l})(a_{l,\beta} \cdot k) \frac{k}{|k|^2} \right] e_l (a_{l,1} + a_{l,2}) e_{-l} \right\}_{L^2}
\]

\[
= -\frac{6\pi^2 \nu}{\|\theta^N\|^2} \sum_{l, \beta} \sum_{\alpha=1}^2 \left( (\theta_k^N)^2 \sin^2(\zeta_{k,l})(a_{l,\beta} \cdot k)(a_{l,\beta'} \cdot k) \right)_{|k|^2}.
\]
Recall that \( \{a_{l,1}, a_{l,2}, \frac{1}{|l|}\} \) is an orthonormal system of \( \mathbb{R}^3 \). By symmetry, the terms with \( \beta \neq \beta' \) vanish, and thus,
\[
\langle S^\perp_{\theta^N}(v), \bar{v} \rangle_{L^2} \sim -\frac{6\pi^2 \nu}{\|\theta^N\|_{L^2}^2} |l|^2 \sum_{\beta=1}^2 \sum_k (\theta^N_k)^2 \sin^2(\angle_{k,l}) \frac{(a_{l,\beta} \cdot k)^2}{|k|^2}
\]
\[
= -\frac{6\pi^2 \nu}{\|\theta^N\|_{L^2}^2} |l|^2 \sum_k (\theta^N_k)^2 \sin^4(\angle_{k,l}),
\]
where we have used
\[
\sum_{\beta=1}^2 (a_{l,\beta} \cdot k)^2 \frac{1}{|k|^2} = 1 - \left( \frac{k}{|k|} \cdot \frac{l}{|l|} \right)^2 = \sin^2(\angle_{k,l}).
\]
Now, approximating the sums by integrals and computing it in spherical variables yield
\[
\frac{1}{\|\theta^N\|_{L^2}^2} \sum_k (\theta^N_k)^2 \sin^4(\angle_{k,l}) \sim \int_{\{N \leq |x| \leq 2N\}} \frac{\sin^4(\angle_{x,l})}{|x|^2} \frac{dx}{|x|^2} = \int_{\{N \leq |x| \leq 2N\}} \frac{dx}{|x|^2} \int_0^\pi \sin^5 \psi d\psi \int_0^{2\pi} d\varphi
\]
\[
= \frac{1}{2} \int_0^\pi \sin^5 \psi d\psi = \frac{8}{15},
\]
Thus, as \( N \to \infty \),
\[
\langle S^\perp_{\theta^N}(v), \bar{v} \rangle_{L^2} \to -6\pi^2 \nu |l|^2 \cdot \frac{8}{15} = \frac{16}{5} \pi^2 \nu |l|^2 = \frac{2}{5} \nu \langle \Delta v, \bar{v} \rangle_{L^2},
\]
since \( \Delta v = -4\pi^2 |l|^2 v = -4\pi^2 |l|^2 (\sigma_{l,1} + \sigma_{l,2}). \)