Quadratic algebras for three dimensional non degenerate superintegrable systems with quadratic integrals of motion

Y. Tanoudis† and C. Daskaloyannis‡
Mathematics Department
Aristotle University of Thessaloniki
54124 Thessaloniki - Greece
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Abstract

The three dimensional superintegrable systems with quadratic integrals of motion have five functionally independent integrals, one among them is the Hamiltonian. Kalnins, Kress and Miller have proved that in the case of non degenerate potentials there is a sixth quadratic integral, which is linearly independent of the other integrals. The existence of this sixth integral imply that the integrals of motion form a ternary parafermionic-like quadratic Poisson algebra with five generators. We show that in all the non degenerate cases (with one exception) there are at least two subalgebras of three integrals having a Poisson quadratic algebra structure, which is similar to the two dimensional case.

1 Introduction

In classical mechanics, a superintegrable or completely integrable is a Hamiltonian system with a maximum number of integrals. Two well known examples are the harmonic oscillator and the Coulomb potential. In the $N$-dimensional space the superintegrable system has $2N - 1$ integrals, one among them is the Hamiltonian.

A compilation of the known three dimensional superintegrable potentials with quadratic integrals and their integrals with quadratic integrals of motion can be found in [2]. Kalnins,
Kress and Miller\cite{1, 3} studying the Darboux equations in three dimensions have classified the complex systems. In fact Kalnins, Kress and Miller studied non degenerate potentials (depending on 4 parameters)\cite{1} and the degenerate potentials (depending on 3 parameters)\cite{3}. The real systems, i.e the systems possessing a real potential can be found in the Evans seminal paper \cite{2}. One of the results of the paper \cite{1} is the so called ”5 to 6” Theorem, which states that any three dimensional non degenerate superintegrable system with quadratic integrals of motion has always a sixth integral $F$ that is linearly independent but not functionally dependent regarding the set of five integrals $A_1, A_2, B_1, B_2, H$. This statement leads to the result that any three dimensional superintegrable non degenerate system form a parafermionic-like Poisson algebra of special character, whose the definition is given in Section \cite{2}. In Section \cite{3} we show that the special character means that the structure of this algebra is such that contains one at least subalgebra.

The degenerate superintegrable systems lead to non linear deviations of the quadratic algebras. Two between them are special cases of non degenerate potentials, which have a sixth integral of motion but their algebras are not quadratic ones. The algebra of the degenerate systems is under current investigation.

\section{Parafermionic-like Poisson algebras}

The Universal Enveloping Algebra $U(g)$ of the Lie algebra $g$ with generators $x_1, x_2, \ldots, x_n$ satisfy the relations

\[
[x_i, x_j] = x_i x_j - x_j x_i = \sum_m c^m_{ij} x_m
\]

The generators satisfy the obvious ternary (trilinear) relations

\[
T(x_i, x_j, x_k) \equiv [x_i, [x_j, x_k]] = \sum_n d^m_{ijk} x_n, \quad \text{where} \quad d^m_{ijk} = \sum_m c^m_{im} c^m_{jk}
\]

Generally a ternary algebra is an associative algebra $A$ satisfying whose the generators satisfy relations like the following one

\[
T(x_i, x_j, x_k) = \sum_n d^m_{ijk} x_n
\]

where $T : A \otimes A \otimes A \rightarrow A$ is a trilinear map. If this trilinear map is defined as in eq. (1) the corresponding algebra is an example of the triple Lie algebras, which were introduced by Jacobson \cite{4} in 1951. At the same time Green \cite{5} was introduced the parafermionic algebra as an associative algebra, whose operators $f^\dagger_i, f_i$ satisfy the ternary relations:

\[
\begin{align*}
[f_k, [f^\dagger_\ell, f_m]] &= 2\delta_{kl} f_m \\
[f_k, [f^\dagger_\ell, f^\dagger_m]] &= 2\delta_{kl} f^\dagger_m - 2\delta_{km} f^\dagger_\ell \\
[f_k, [f_\ell, f_m]] &= 0
\end{align*}
\]
We call parafermionic Poisson algebra the Poisson algebra satisfying the ternary relations:

$$\{ x_i, \{ x_j, x_k \}_P \}_P = \sum_m c_{ijk}^m x_m$$

which is the classical Poisson analogue of the Lie triple algebra \( \mathfrak{h} \).

The quadratic parafermionic Poisson algebra is a Poisson algebra satisfying the relations:

$$\{ x_i, \{ x_j, x_k \}_P \}_P = \sum_{m,n} d_{ijk}^{mn} x_m x_n + \sum_m c_{ijk}^m x_m$$

A classical superintegrable system with quadratic integrals of motion on a two dimensional manifold possesses has two functionally independent integrals of motion \( A \) and \( B \), which are in involution with the Hamiltonian \( H \) of the system:

$$\{ H, A \}_P = 0, \quad \{ H, B \}_P = 0$$

the Poisson bracket \( \{ A, B \}_P \) is different to zero and it is generally an integral of motion cubic in momenta, therefore it could not be in general a linear combination of the integrals \( H, A, B \). Generally if we the Poisson brackets of the integrals of motion \( \{ A, \{ A, B \}_P \}_P, \{ \{ A, B \}_P, B \}_P \) are not linear functions of the integrals of motion, therefore they don’t close in a Lie Poisson algebra with three generators. If we consider all the nested Poisson brackets of the integrals of motion, generally they don’t close in an Poisson Lie algebra structure.

All the known two dimensional superintegrable systems with quadratic integrals of motion the have a common structure \([6, 7, 8, 9]\):

$$\{ H, A \}_P = 0, \quad \{ H, B \}_P = 0, \quad \{ A, B \}_P \neq 0 \quad \{ A, B \}_P^2 = 2F(A, H, B)$$

$$\{ A, \{ A, B \}_P \}_P = \frac{\partial F}{\partial B}, \quad \{ B, \{ A, B \}_P \}_P = -\frac{\partial F}{\partial A}$$

(2)

where \( F = F(A, B, H) \) is a cubic function of the integrals of motion

$$F(A, B, H) = \alpha A^3 + \beta B^3 + \gamma A^2 B + \delta AB^2 + (\epsilon_0 + \epsilon_1 H) A^2 + (\zeta_0 + \zeta_1 H) B^2 +$$

$$+ (\eta_0 + \eta_1 H) AB + (\theta_0 + \theta_1 H + \theta_2 H^2) A +$$

$$+ (\kappa_0 + \kappa_1 H + \kappa_2 H^2) B + (\lambda_0 + \lambda_1 H + \lambda_2 H^2 + \lambda_3 H^3)$$

(3)

where the greek letters are constants.

Therefore any two dimensional superintegrable system correspond to some parafermionic-like quadratic Poisson Algebra with two generators. In fact the two dimensional systems can be classified in six classes by classifying the corresponding parafermionic-like quadratic Poisson algebras \([8]\). By classifying the correponding associative algebras all the two dimensional superintegrable systems with quadratic integrals of motion are classified too \([9]\).
3 The structure of the Poisson algebra of three dimensional potentials

The known three-dimensional superintegrable systems defined on a flat space are described by the Hamiltonian

\[ H = \frac{1}{2} (p_x^2 + p_y^2 + p_z^2) + V(x, y, z) \]  

were initially studied by Evans [2]. Kalnins, Kress, and Miller in [1] have classified all the three dimensional superintegrable systems with quadratic integrals of motion. These potentials are distinguished to the non degenerate ones, which are potentials which depend on four parameters, i.e. they are linear combination of four potentials. The degenerate potential depend on less than four parameters. One of the general results in [1] is the so called “5 to 6” theorem:

**5→6 Theorem:** Let \( V \) be a nondegenerate potential (depending on 4 parameters) corresponding to a conformally flat space in 3 dimensions

\[ ds^2 = g(x, y, z)(dx^2 + dy^2 + dz^2) \]

that is superintegrable and there are 5 functionally independent constants of the motion \( \mathcal{L} = \{ S_\ell : \ell = 1, \cdots 5 \} \) There is always a 6th quadratic integral \( S_6 \) that is functionally dependent on \( \mathcal{L} \), but linearly independent

By studying all the known non degenerate potentials, we can prove the following theorem:

**Proposition 1** In the case of the non degenerate with quadratic integrals of motion, on a conformally flat manifold, the integrals of motion satisfy a parafermionic-like quadratic Poisson Algebra with 5 generators which described from the following:

\[
\{ S_i, \{ S_j, S_k \} \}_P = \sum_{mn} d^{m,n}_{i,j,k} S_m S_n + \sum_m c^{m}_{i,j,k} S_m
\]  

(5)

In all the three dimensional superintegrable systems with quadratic integrals with 4 or 3 parameters, the integrals of motion satisfy a ”special” form of the Poisson Parafermionic-like algebra [3].

**Proposition 2** In all the known cases (with only one exception) the non degenerate systems we can choose beyond the Hamiltonian \( H \) four functionally independent integrals of motion \( A_1, B_1, A_2, B_2, \) and one additional quadratic integral of motion \( F \), such that all the integrals of motion are linearly independent. These integrals satisfy a Poisson parafermionic-like algebra [3]. The ”special” form of the algebra defined by the integrals \( A_1, B_1, A_2, B_2 \) is characterized by two cubic functions

\[ F_1 = F_1(A_1, A_2, B_1, H) \quad F_2 = F_2(A_1, A_2, B_2, H) \]
and satisfy the relations:

\[
\begin{align*}
\{A_1, A_2\}_P &= \{A_1, B_2\}_P = \{A_2, B_1\}_P = 0, \\
\{A_1, B_1\}_P^2 &= 2F_1(A_1, A_2, H, B_1) = \text{cubic function} \\
\{A_2, B_2\}_P &= 2F_2(A_1, A_2, H, B_2) = \text{cubic function} \\
\{A_i, \{A_i, B_i\}_P\} &= -\frac{\partial F_i}{\partial B_i}, \quad \{B_i, \{A_i, B_i\}_P\} = -\frac{\partial F_i}{\partial A_i}, \\
\{\{A_1, B_1\}_P, B_2\}_P &= \{A_1, \{B_1, B_2\}_P\}_P, \quad \{\{A_2, B_2\}_P, B_1\}_P = -\{A_2, \{B_1, B_2\}_P\}_P
\end{align*}
\]

(6)

If we put

\[
C_1 = \{A_1, B_1\}_P, \quad C_2 = \{A_2, B_2\}_P, \quad D = \{B_1, B_2\}_P,
\]

the relations (6) imply the following ones:

\[
\begin{vmatrix}
\{C_1, B_2\}_P C_1 - \frac{\partial F_1}{\partial A_2} C_2 - \frac{\partial F_1}{\partial B_1} D = \{C_2, B_1\}_P C_2 - \frac{\partial F_2}{\partial A_1} C_1 + \frac{\partial F_2}{\partial B_2} D = 0 \\
\{C_1, D\}_P - \frac{\partial F_1}{\partial A_2} \{A_2, D\}_P = \{A_1, D\}_P - \frac{\partial F_2}{\partial A_1} \{A_2, D\}_P - \frac{\partial F_1}{\partial B_2} \\
\frac{\partial F_2}{\partial A_1} \{A_2, D\}_P - \frac{\partial F_1}{\partial B_2}
\end{vmatrix}
\]

(7)

\[
\{C_1, C_2\}_P = \frac{\frac{\partial F_1}{\partial B_1} \frac{\partial F_2}{\partial A_1} C_1 + \frac{\partial F_1}{\partial A_2} \frac{\partial F_2}{\partial B_2} C_2 + \frac{\partial F_1}{\partial B_1} \frac{\partial F_2}{\partial B_2} D}{C_1 C_2}
\]

Schematically the structure of the above algebra is described by the following "Π" shape

\[
\begin{array}{c}
\quad A_1 \quad - \quad A_2 \\
| \quad | \quad | \\
B_2 \quad | \quad B_1 \\
\end{array}
\]

(8)

where with dashed line represented the vanishing of Poisson bracket whereas the other brackets between the integrals are non vanishing Poisson brackets.

It is important to notice that the integrals \(A_1, B_1\) satisfy a parafermionic-like quadratic Poisson algebra similar to the algebra as in two dimensional case (2). The corresponding structure function to the two dimensional one (3) can be written as:

\[
F_1(A_1, B_1, H, A_2) = \alpha_1 A_1^3 + \beta_1 B_1^3 + \gamma_1 A_1^2 B_1 + \delta_1 A_1 B_1^2 + (\epsilon_{01} + \epsilon_{11} H + \epsilon_{21} A_2) A_1^2 + \\
(\zeta_{01} + \zeta_{11} H + \zeta_{21} A_2) B_1^2 + (\eta_{01} + \eta_{11} H + \eta_{21} A_2) A_1 B_1 + \\
(\theta_{01} + \theta_{11} H + \theta_{21} H^2 + \theta_{31} A_2 + \theta_{41} A_2^2 + \theta_{51} A_2 H) A_1 + \\
(\kappa_{01} + \kappa_{11} H + \kappa_{21} H^2 + \kappa_{31} A_2 + \kappa_{41} A_2^2 + \kappa_{51} A_2 H) B_1 + \\
\lambda_0 + \lambda_{11} H + \lambda_{21} H^2 + \lambda_{31} H^3 + \lambda_{41} A_2 + \lambda_{51} A_2^2 + \lambda_{61} A_2^3 + \\
\lambda_{71} A_2 H + \lambda_{81} A_2^2 H + \lambda_{91} A_2 H^2
\]

(9)

The pair \(A_2, B_2\) forms also a parafermionic-like algebra with the corresponding structure function \(F_2(A_2, B_2, H, A_1)\), which has a similar form as in (3).
4 Non degenerate Potentials

In this section we give explicitly the form of the quadratic algebras for the non degenerate systems given by Kalnins, Kress and Miller (KKM)\[1\] and Evans (Ev) \[2\]. The full algebra is given after some definitions.

• KKM Potential $V_1$ This potential which is also referred as harmonic oscillator potential get six, known, linearly independent integrals $H, A_1, A_2, B_1, B_2, F$ and studied in \[1\], \[2\].

\[
H = p_x^2 + p_y^2 + p_z^2 + \delta(x^2 + y^2 + z^2) + \frac{\alpha_1}{x^2} + \frac{\alpha_2}{y^2} + \frac{\alpha_3}{z^2}
\]

\[
A_1 = p_x^2 + \delta x^2 + \frac{\alpha_1}{x^2}, \quad A_2 = p_z^2 + \delta z^2 + \frac{\alpha_3}{z^2}
\]

\[
B_1 = J_x^2 + k x^2 + \frac{\alpha_2 x^2}{y^2} + \frac{\alpha_3 y^2}{x^2}, \quad B_2 = J_z^2 + \frac{\alpha_3 y^2}{z^2} + \frac{\alpha_2 z^2}{y^2}
\]

and

\[
F = J_x^2 + J_y^2 + J_z^2 + \frac{\alpha_2(x^2 + z^2)}{y^2} + \frac{\alpha_3 x^2(y^2 + z^2)}{x^2 z^2}
\]

where,

\[
J_x = yp_z - zp_y \quad J_y = xp_z - zp_x \quad J_z = xp_y - yp_x \quad J^2 = J_x^2 + J_y^2 + J_z^2
\]

The above integrals satisfy the following relations:

\[
\{A_1, A_2\} = \{A_1, B_2\} = \{A_2, B_1\} = \{B_1, F\} = \{B_2, F\} = 0
\]

This relation corresponds to a complicated diagram, having five "Π" structures as in (8)

\[
\begin{align*}
A_1 & \quad A_2 \\
\mid & \quad \mid \\
B_2 & \quad B_1 \\
\mid & \quad \mid \\
\downarrow & \quad \downarrow \\
\downarrow & \\
F &
\end{align*}
\]

The structure functions are given by the relations:

\[
\{A_1, B_1\} = C_1, \quad \{A_2, B_2\} = C_2, \quad \{B_1, B_2\} = D, \quad \{F, A_1\} = L, \quad \{F, A_2\} = M, \quad \{F, B_2\} = N
\]

\[
C_1^2 = 2F_1, \quad C_2^2 = 2F_2, \quad L^2 = 2F_3, \quad M^2 = 2F_4, \quad N^2 = 2F_5
\]
The full algebra is given by the following relations:

\[ F_1 = -8(A_1^2(\alpha_2 + B_1) + A_1B_1(A_2 - H) + \delta B_1^2 + \alpha_1((A_1 + A_2 - H)^2 - 4\alpha_2\delta)) \]

\[ F_2 = -8((A_1 + A_2 - H)(\alpha_3 A_1 + A_2(\alpha_3 + B_2) - \alpha_3 H) + \delta B_2^2 + \alpha_2(A_2^2 - 4\alpha_3\delta)) \]

\[ F_3 = -8\left((\alpha_2 + \alpha_3 + F)A_1^2 + (B_2 - F)HA_1 + (B_2 - F)^2\delta + \alpha_1((A_1 - H)^2 - 4(\alpha_2 + \alpha_3 + B_2)\delta)\right) \]

\[ F_4 = -8(\alpha_1 + \alpha_2 + \alpha_3 + F)A_2^2 + 8(2\alpha_3 - B_1 + F)HA_2 - 8\alpha_3 H^2 + 8(-(B_1 - F)^2 + 4\alpha_1\alpha_3 + 4\alpha_2\alpha_3 + 4\alpha_3\beta_1)\delta \]

\[ F_5 = 8(-\alpha_2(B_1 + B_2 - F)^2 + \alpha_1(4\alpha_2\alpha_3 - B_2^2) - B_1(\alpha_3 B_1 + B_2 B_1 + B_2 - F)) \]

The full algebra is given by the following relations:

\[ \{A_1, \{A_1, B_1\}\} = \frac{\partial F_1}{\partial B_1} = -8A_1(A_1 + A_2 - H) - 16\delta B_1 \]

\[ \{A_2, \{A_2, B_2\}\} = \frac{\partial F_2}{\partial B_2} = -8A_2(A_2 + A_1 - H) - 16\delta B_2 \]

\[ \{\{A_1, F\}, A_1\} = -\frac{\partial F_3}{\partial F} = 8A_1(A_1 - H) - 16\delta(B_2 - F) \]

\[ \{\{A_2, F\}, A_2\} = -\frac{\partial F_4}{\partial F} = 8A_2(A_2 - H) - 16\delta(B_1 - F) \]

\[ \{\{A_1, F\}, A_2\} = \{\{A_2, F\}, A_1\} = 8A_1 A_2 + 16\delta(B_1 + B_2 - F) \]

\[ \{\{A_1, B_1\}, F\} = \{\{A_1, F\}, B_1\} = \{\{A_2, F\}, B_1\} = \{\{B_1, B_2\}, F\} = 0 \]

\[ \{\{A_2, B_2\}, B_1\} = \{\{B_1, B_2\}, A_2\} = 8A_1(B_1 - F) + 8(B_1 + B_2 - F)(A_2 - H) \]

\[ \{\{A_1, B_1\}, B_2\} = \{\{A_1, B_1\}, B_2\} = 8A_2(B_2 - F) + 8(B_1 + B_2 - F)(A_1 - H) \]

\[ \{B_1, \{B_1, B_2\}\} = \frac{\partial F_5}{\partial B_1} = -8(B_1 + 2B_2 - F + 2\alpha_2)B_1 - 16(\alpha_1 + \alpha_2)B_2 + 16\alpha_2 F \]

\[ \{\{B_1, B_2\}, B_2\} = \frac{\partial F_5}{\partial B_2} = -8(B_2 + 2B_1 - F + 2\alpha_2)B_2 - 16(\alpha_2 + \alpha_3)B_1 + 16\alpha_2 F \]

\[ \{\{A_1, F\}, F\} = \frac{\partial F_4}{\partial A_1} = -16\alpha_1(A_1 - H) - 16(\alpha_2 + \alpha_3)A_1 - 8(2A_1 - H)F - 8B_2 H \]

\[ \{\{A_2, F\}, F\} = \frac{\partial F_4}{\partial A_2} = -16\alpha_3(A_2 - H) - 16(\alpha_1 + \alpha_2)A_2 - 8(2A_2 - H)F - 8B_1 H \]

\[ \{\{A_1, B_1\}, B_1\} = \frac{\partial F_1}{\partial A_1} = -16\alpha_1(A_1 + A_2 - H) - 16\alpha_2 A_1 - 8(2A_1 - A_2)B_1 + 8B_1 H \]

\[ \{\{A_1, B_2\}, B_2\} = \frac{\partial F_2}{\partial A_2} = -16\alpha_3(A_1 + A_2 - H) - 16\alpha_2 A_2 - 8(2A_2 - A_1)B_2 + 8B_2 H \]

\[ \{\{A_1, B_1\}, F\} = \{\{A_1, F\}, B_1\} = -16(\alpha_1 + \alpha_2)A_1 - 16\alpha_1 A_2 - 8A_1 B_1 + 8A_2 B_2 - 8F(A_1 + A_2 - H) + (16\alpha_1 - 8B_2)H \]

\[ \{\{A_2, B_2\}, F\} = \{\{A_2, F\}, B_2\} = -16A_2(\alpha_2 + \alpha_3) - 16\alpha_3 A_1 - 8A_2 B_2 + 8A_1 B_1 - 8F(A_1 + A_2 - H) + (16\alpha_3 - 8B_1)H \]
The second algebra which expand in terms of \( C_1, C_2, D \) with coefficients any linear combination of integrals \( A_1, A_2, B_1, B_2, F, H \) is:

\[
\{C_1, C_2\} = -8A_2C_1 + 8A_1C_2 + 16\delta D, \quad \{C_1, D\} = 8(F - B_1 - B_2)C_1 - 8(2\alpha_1 + B_1)C_2 - 8\alpha_1 D
\]

\[
\{C_2, D\} = 8(B_2 + 2\alpha_3)C_1 + 8(B_1 + B_2 - F)C_2 - 8\alpha_2 D
\]

The relation between \( B_1, B_2, C_1, C_2, M, L, F \) is:

\[C_1 + C_2 + M + L = 0\]

- **KKM Potential \( V_{\Pi} \)** This potential get six, known, linearly independent integrals \( H, A_1, A_2, B_1, B_2, F \).

\[
H = p_x^2 + p_y^2 + p_z^2 + \alpha(x^2 + y^2 + z^2) + \frac{\beta(x - iy)}{(x + iy)^2} + \frac{\gamma}{(x + iy)^2} + \frac{\delta}{z^2}
\]

\[
A_1 = p_x^2 + \alpha z^2 + \frac{\delta}{z^2}, \quad A_2 = J_z^2 + \frac{2ixy(\gamma - 2\beta)}{(x + iy)^2} + \frac{2\gamma x^2}{(x + iy)^2}
\]

\[
B_1 = J_x^2 + z^2(\beta(4xy(y - ix) + z^2(\gamma + iy)) + \gamma(x + iy)(2x(x + iy) + z^2)) + \delta(x - iy)(x + iy)^4
\]

\[
B_2 = (p_x + ip_y)^2 + \alpha(x + iy)^2 - \frac{\beta}{(x + iy)^2}, \quad F = (J_y - iJ_x)^2 + \frac{\delta(x + iy)^4 - \beta z^4}{z^2(x + iy)^2}
\]

The integrals of motion satisfy the relations

\[
\{A_1, A_2\} = \{A_1, B_2\} = \{A_2, B_1\} = \{B_1, F\} = \{B_2, F\} = 0
\]

The corresponding "\( \Pi \)" shape diagram is the same as in (10). We can define

\[
\{A_1, B_1\} = C_1, \quad \{A_2, B_2\} = C_2, \quad \{B_1, B_2\} = D, \quad \{F, A_1\} = L, \quad \{F, A_2\} = M, \quad \{F, B_2\} = N
\]

\[
C_1 = 2F_1, \quad C_2 = 2F_2, \quad L^2 = 2F_3, \quad M^2 = 2F_4, \quad N^2 = 2F_5
\]

\[
F_1 = -8(\alpha(A_2 - A_1)^2 + A_1H(A_2 - B_1 - 2\delta) +
+\delta(H^2 - 4\alpha(A_2 + B - \gamma)) + A_2^2(B_1 + \beta - \gamma + \delta))
\]

\[
F_2 = 8(\beta(-B_2^2 + (A_1 - H)^2 - 4\alpha\beta) - A_2(B_2^2 + 4\alpha\beta) + \gamma(B_2(B_2 + H - A_1) + 4\alpha\beta - \alpha\gamma^2)
\]

\[
F_3 = 8\beta A_1^2 + 8B_2FA_1 - 8F_2\alpha - 8B_2^2\delta - 32\alpha\beta\delta
\]

\[
F_4 = 8(\beta A_2^2 - (F + \gamma)^2 + 2\beta(B_1 + 2\delta)) A_2 + (B_1 + F)(B_1\beta + F(\gamma - \beta)) - (\gamma - 2\beta)^2\delta
\]

\[
F_5 = -8\left((\beta - \gamma + \delta)B_2^2 - H(F + \gamma)B_2 + F^2\alpha - H^2\beta + B_1(B_2^2 + 4\alpha\beta) +
+2F\alpha\gamma + \alpha((\gamma - 2\beta)^2 + 4\beta\delta)\right)
\]

The full algebra is given by the following relations:

\[
\{A_1, \{B_1, B_2\}\} = \{\{A_1, B_1\}, B_2\} = 8A_1B_2 - 16\alpha F
\]
The second algebra which expand in terms of $C_1, C_2, D$ with coefficients any linear combination of integrals $A_1, A_2, B_2, B, F, H$

\[
\{C_1, C_2\} = -8B_2C_1 + 8(A_1 - H)C_2 + 8(H - A_1)D
\]

\[
\{C_1, D\} = -8B_2C_1 - 8HC_2 + 8(H - A_1)D, \quad \{C_2, D\} = 8B_2C_2 - 8B_2D
\]

The relation between $B_1, B_2, F, A_2, A_1, H, M, D$ is:

\[
(B_2 + F)C_1 + (-A_2 + B_1 + H + 2\delta)C_2 + (A_1 + A_2 - B_1 - H)D + (A_1 + 2\alpha)M = 0
\]

- **KKM Potential** $V_{III}$

\[
H = p_x^2 + p_y^2 + p_z^2 + \alpha(x^2 + y^2 + z^2) + \frac{\beta}{(x + iy)^2} + \frac{\gamma z}{(x + iy)^3} + \frac{\delta(x^2 + y^2 - 3z^2)}{(x + iy)^4}
\]

\[
A_1 = J^2 + \frac{\beta(x + iy)(\beta(x + iy)(2x(x + iy) + z^2) + \gamma z(x^2 + y^2 + z^2) + \delta(-4ixy(x + iy)^2 - 2z^2(x^2 + y^2) - 3z^4))}{(x + iy)^4}
\]
The full algebra is given by the following relations:

\[ A_2 = (J_2 - iJ_1)^2 + \frac{z(\gamma x + i\gamma y - 4\delta z)}{(x + iy)^2}, \quad B_1 = (p_x + ip_y)^2 + \alpha(x + iy)^2 - \frac{\delta}{(x + iy)^2} \]

\[ B_2 = J_3(J_2 - iJ_1) + \frac{(x + iy)(-4\beta z(x + iy) + \gamma(2x(x + iy) - 3z^2)) - 8\delta z(x^2 + y^2 - z^2)}{4(x + iy)^3} \]

\[ F = p_z(p_x + ip_y) + \frac{4\alpha z(x + iy)^4 - \gamma(x + iy) + 4\delta z}{4(x + iy)^3} \]

The integrals of motion satisfy the following relations:

\[ \{A_1, A_2\} = \{A_1, B_2\} = \{A_2, B_1\} = \{B_1, F\} = 0 \]

These relations correspond to the diagram

\[
\begin{align*}
A_1 & - - - A_2 \\
| & \quad | \\
| & \quad | \\
B_2 & B_1 \\
| & \quad | \\
F &
\end{align*}
\]

If we put

\[ \{A_1, B_1\} = C_1, \quad \{A_2, B_2\} = C_2, \quad \{B_1, B_2\} = D, \quad \{F, A_1\} = L, \quad \{F, A_2\} = M \]

and

\[ C_1^2 = 2F_1, \quad C_2^2 = 2F_2, \quad M^2 = 2F_3 \]

then

\[ F_1 = -8(\alpha A_2^2 - \beta B_1) + A_2(2\alpha \beta - B_1) + \alpha(\beta - 2\delta)^2 - \delta H^2 + B_1^2(\delta - \beta) + A_1(B_1^2 + 4\alpha \delta) \]

\[ F_2 = 2A_2^3 + 4\beta A_2^2 - \gamma A_1 + 2\gamma B_2(\beta - 2\delta) + 8\delta B_2^2 + \frac{1}{2}A_2(4\gamma B_2 - \gamma^2 + 4(\beta - 2\delta)^2 + 16\delta A_1) \]

\[ F_3 = 2A_2 B_1^2 - 2F_1 \gamma B_1 - \frac{\alpha \gamma^2}{2} + 8F^2 \delta + 8A_2 \alpha \delta \]

The full algebra is given by the following relations:

\[ \{\{A_1, B_1\}, F\} = \{\{A_1, F\}, B_1\} = -8B_1 F - 4\alpha \gamma \]

\[ \{\{A_2, B_2\}, B_1\} = \{\{B_1, B_2\}, A_2\} = 16\delta F - 2\gamma B_1 \]

\[ \{\{A_2, F\}, B_1\} = \{\{B_2, F\}, A_2\} = \{B_1, \{B_1, B_2\}\} = 0 \]

\[ \{A_1, \{A_1, B_1\}\} = \frac{\partial F_1}{\partial B_1} = -16(A_1 - \beta + \delta)B_1 + 8(A_2 + 8\beta)H \]

\[ \{\{A_1, F\}, A_2\} = \{\{A_2, F\}, A_1\} = 2(4B_2 - \beta)B_1 + 8(A_2 + \beta)F \]
\[
\{A_2, B_2\}, F = \{A_2, F\}, B_2 = -2(3A_2 + \beta)B_1 + 2\gamma F - 4\delta H
\]
\[
\{A_1, \{B_1, B_2\}\} = \{A_1, B_1\}, B_2 = 8B_1B_2 - 2\gamma B_1 + 8(A_2 + \beta)F
\]
\[
\{A_1, F\}, B_2 = \{B_2, F\}, A_1 = -4(A_1 - \beta + \delta)B_1 + 2(A_2 + \beta)H
\]
\[
\{A_2, F\}, A_2 = -\frac{\partial F_1}{\partial F} = 2\gamma B_1 - 16\delta F, \quad \{A_2, F\}, F = \frac{\partial F_1}{\partial A_2} = 2B_1^2 + 8\alpha \delta
\]
\[
\{A_2, B_2\} = \frac{\partial F_2}{\partial A_2} = 6A_2^2 + 8\beta A_2 + 2\gamma B_2 + 8\delta A_1 + 2\beta^2 + 8\delta^2 - 8\beta \delta
\]
\[
\{B_1, B_2\}, F = -2B_1^2 - 8\alpha \delta, \quad \{B_2, F\}, B_1 = 2B_1^2 + 8\alpha \delta, \quad \{B_2, F\}, F = 2B_1 F + \alpha \gamma
\]
\[
\{A_1, B_1\}, B_1 = \frac{\partial F_1}{\partial A_1} = -8B_1^2 - 32\alpha \delta, \quad \{A_2, B_2\} = \frac{\partial F_2}{\partial B_2} = 2\gamma A_2 + 2\gamma(\beta - 2\delta) + 16B_2 \delta
\]
\[
\{B_2, F\}, B_2 = (-2B_2 + \frac{\gamma}{2})B_1 - 2(A_2 + \beta)F, \quad \{A_1, F\}, A_1 = 2(4B_2 - \gamma)H + 16(A_1 + \delta - \beta)F
\]
\[
\{A_1, F\}, F = -8F^2 + 2B_1 H - 12\alpha A_2 - 4\alpha \beta, \quad \{B_1, B_2\}, B_2 = 6A_2 B_1 + 2\beta B_1 - 2F \gamma + 4H \delta
\]
The second algebra which expand in terms of \(C_1, C_2, D\) with coefficients any linear combination of integrals \(A_1, A_2, B_2, B_2, F, H\)

\[
\{C_1, C_2\} = 8B_1 C_2 - 8(A_2 + \beta)D, \quad \{C_1, D\} = -8B_1 D, \quad \{C_2, D\} = -4\delta C_1
\]
The relation between \(B_1, B_2, A_2, C_1, C_2, D\)

\[
B_1 C_2 - \frac{8F \delta C_2}{\gamma} - A_2 D - D \beta - \frac{C_1 \gamma}{4} + 2D \delta + \frac{4A_2 C_1 \delta}{\gamma} - \frac{8B_2 D \delta}{\gamma} = 0
\]

• KKM Potential \(V_v\)

\[
H = p_x^2 + p_y^2 + p_z^2 + \alpha(4x^2 + y^2 + z^2) + \beta x + \frac{\gamma}{(y + iz)^2} + \frac{\delta(y - iz)}{(y + iz)^2}
\]
\[
A_1 = p_x^2 + 4\alpha x^2 + \beta x, \quad A_2 = J_1^2 + \frac{2y(\gamma y + iz(\gamma - 2\delta))}{(y + iz)^2}
\]
\[
B_1 = J_2 p_z - J_3 p_y + \frac{\beta}{4}(y^2 + z^2) + x(\alpha(y^2 + z^2) - \frac{2\gamma y}{(y + iz)^3} + \frac{\delta - \gamma}{(y + iz)^2})
\]
\[
B_2 = (p_z - ip_y)^2 + \frac{\delta - \alpha(y + iz)^4}{(y + iz)^2}
\]
\[
F = (p_z - ip_y)(J_2 + iJ_3) - \frac{1}{4}(y + iz)^2(4\alpha x + \beta) - \frac{\delta x}{(y + iz)^2}
\]
The intergrals satisfy the equation:

\[
\{A_1, A_2\} = \{A_1, B_2\} = \{A_2, B_1\} = \{A_2, F\} = \{B_1, F\} = 0
\]
The corresponding diagram is the same as in the case of the potential KKM \( V_{III} \) see (II).

\[ \{A_1, B_1\} = C_1, \quad \{A_2, B_2\} = C_2, \quad \{B_1, B_2\} = D, \quad \{F, A_1\} = L, \quad \{F, A_2\} = M, \quad \{F, B_2\} = N \]

\[ C_1^2 = 2F_1, \quad C_2^2 = 2F_2, \quad M^2 = 2F_4 \]

\[ F_1 = 8(\gamma(B_1(A_2 + B_1 - H) - \alpha \gamma) + \delta(-B_1^2 + (A_2 - H^2) + 4\alpha \gamma) - 4\alpha \delta^2 - A_1(B_1^2 + 4\alpha \delta)) \]

\[ F_2 = \frac{1}{2}(4A_2^2 - 8A_2^2H - 16\alpha B_2^2 + 4\beta H B_2 - \beta^2(A_1 - \gamma + \delta) + 4A_2(H^2 - \beta B_2 - 4\alpha(A_1 - \gamma + \delta)) \]

\[ F_3 = 2A_2B_1 + 2F \beta B_1 - 8F^2 \alpha + \frac{\beta^2 \delta}{2} + 8A_2 \alpha \delta \]

The full algebra is given by the following relations:

\[
\{\{A_2, B_2\}, B_2\} = \frac{\partial F_2}{\partial A_2} = -8B_2^2 - 32\alpha \delta \\
\{\{A_2, B_2\}, F\} = \{\{A_2, F\}, B_2\} = -8B_2F - 4\beta \delta \\
\{A_1, \{A_1, B_1\}\} = \frac{\partial F_1}{\partial B_1} = -16\alpha B_1 - 2\beta(A_1 - H) \\
\{A_1, \{B_1, B_2\}\} = \{\{A_1, B_1\}, B_2\} = -16\alpha F + 2\beta B_2 \\
\{\{B_1, F\}, A_1\} = \{\{A_1, F\}, B_2\} = \{\{B_1, B_2\}, B_2\} = 0 \\
\{\{A_1, F\}, A_2\} = \{\{A_2, F\}, A_1\} = 8B_1B_2 + 8(A_1 - H)F \\
\{\{A_2, B_2\}, B_1\} = \{\{B_1, B_2\}, A_2\} = 8B_1B_2 + 8(A_1 - H)F \\
\{A_2, \{A_2, B_2\}\} = \frac{\partial F_2}{\partial B_2} = -16(A_2 + \delta - \gamma)B_2 + 8\gamma(A_1 - H) \\
\{\{A_1, B_1\}, F\} = \{\{A_1, F\}, B_1\} = -2(3A_1 - H)B_2 - 2\beta F + 4\alpha \gamma \\
\{\{A_2, F\}, B_1\} = \{\{B_1, F\}, A_2\} = -4A_2B_2 + 2\gamma A_1 + 4(\gamma - \delta)B_2 - 2\gamma H \\
\{A_1, B_1\}, B_1\} = \frac{\partial F_1}{\partial A_1} = 2H^2 + 2(3A_1 - 4H)A_1 - 8\alpha A_2 - 2\beta B_1 + 8\alpha(\gamma - \delta) \\
\{\{A_2, F\}, A_2\} = 16A_2F + 8\gamma B_1 + 16(\delta - \gamma)F, \quad \{\{B_1, F\}, F\} = 2B_2F + \beta \delta \\
\{\{A_1, F\}, A_1\} = -\frac{\partial F_3}{\partial F} = 16A_2F - 2\beta B_2, \quad \{\{A_1, F\}, F\} = \frac{\partial F_3}{\partial A_1} = 2B_2^2 + 8\alpha \delta \\
\{\{B_1, B_2\}, F\} = \{\{B_1, F\}, B_2\} = 2B_2^2 + 8\alpha \delta, \quad \{\{B_1, F\}, B_1\} = -2B_1B_2 - 2A_1F + 2FH \\
\{\{A_2, F\}, F\} = -8F^2 + 2\gamma B_2 + 12\delta A_1 - 4\delta H, \quad \{B_1, \{B_1, B_2\}\} = 2(3A_1 - H)B_2 + 2\beta F - 4\alpha \gamma \\
\]
\[ H = p_x^2 + p_y^2 + p_z^2 + \alpha(4x^2 + y^2 + z^2) + \beta x + \frac{\gamma}{y^2} + \frac{\delta}{z^2} \]

\[ A_1 = p_x^2 + 4\alpha x^2 + \beta x, \quad A_2 = p_y^2 + \alpha y^2 + \frac{\gamma}{y^2}, \quad B_1 = J_2 p_z + \alpha z^2 + \frac{\beta z^2}{4} - \frac{\delta x}{z^2} \]

\[ B_2 = J_1^2 + \frac{\gamma z^2}{y^2} + \frac{\delta y^2}{z^2}, \quad F = p_y J_3 - \alpha x y^2 - \frac{\beta y^2}{4} + \frac{\gamma x}{y^2} \]

The above integrals satisfy the relations:

\[ \{A_1, A_2\} = \{A_1, B_2\} = \{A_2, B_1\} = 0 \]

These relations correspond to the diagram:

\[ \begin{array}{c}
A_1 \\
\downarrow \\
B_2 \\
\downarrow \\
B_1 \\
\end{array} \begin{array}{c}
\quad A_2 \\
\quad F \\
\end{array} \]

We can define

\[ \{A_1, B_1\} = C_1, \quad \{A_2, B_2\} = C_2 \]

and

\[ C_1^2 = 2F_1, \quad C_2^2 = 2F_2 \]

where

\[ F_1 = -8\alpha B_1^2 + 2(A_1 + A_2 - H)(A_1(A_1 + A_2 - H) - \beta B_1) - \frac{\delta}{2}(16\alpha A_1 + \beta^2) \]

\[ F_2 = -8(\alpha B_2 + \gamma A_1^3 - 2A_1 H(B_2 + 2\gamma) + A_1(-2\gamma H + A_2(B_2 + 2\gamma)) + A_2^2(B_2 + \gamma + \delta) + \gamma(H^2 - 4\alpha \delta)) \]

The full parafermionic-like algebra is given by the following relations:

\[ \{\{B_1, F\}, A_1\} = 0 \]

\[ \{\{B_1, B_2\}, B_2\} = -8(B_1 + F)B_2 - 16B_1 \gamma - 16F \delta \]

\[ \{\{A_1, F\}, A_2\} = \{\{A_2, F\}, A_1\} = -16\alpha F - 2\beta A_2 \]

\[ \{B_1, \{B_1, B_2\}\} = 2(3A_1 + A_2 - H)B_2 - 8B_1 F + 4\delta A_2 \]

\[ \{\{B_2, F\}, B_1\} = 2(3A_1 + B_2 + 2\delta)A_2 - 8B_1 F - 2B_2 H \]

\[ \{A_1, \{A_1, B_1\}\} = \frac{\partial F_1}{\partial B_1} = -16\alpha B_1 - 2\beta(A_1 + A_2 - H) \]
The second algebra which expand in terms of $\{C_1, C_2, D\}$ with coefficients any linear combination of integrals $A_1, A_2, B_2, F, H$

\[
\{C_1, C_2\} = -8A_2C_1 + 2\beta C_2 + 16\alpha D
\]

\[
\{C_1, D\} = -8FC_1 + 2(3A_1 + A_2 - H)C_2 - 2\beta D, \quad \{C_2, D\} = 8(B_2 + 2\gamma)C_1 + 8FC_2 - 8A_2 D
\]

The relation between $C_1, C_2, A_1, B_1, B_2, F, M$

\[
\frac{1}{4}(4B_2 + 4F + 8\gamma)C_1 + \frac{1}{4}(-2A_1 - 4B_1 + 4F)C_2 + \\
+\frac{1}{4}(4A_1 - 4H + \beta)D + \frac{1}{4}(-4B_1 + 4B_2 + 8\delta)M = 0
\]

• KKM Potential $V_{\text{tot}}$

\[
H = p_x^2 + p_y^2 + p_z^2 + \alpha(-2(x-iy)^3 + 4\alpha(x^2+y^2) + z^2) + \beta(-3(x-iy)^2 + 2(x+iy)) + \gamma(x-iy) + \frac{\delta}{z^2}
\]

\[
A_1 = (p_x - ip_y)^2 + 4(x - iy)(\alpha(x - iy) + \beta), \quad A_2 = p_z^2 + \alpha z^2 + \frac{\delta}{z^2}
\]
\( B_1 = J_3(p_x - ip_y) - \)
\(-\frac{i}{4}(p_x + ip_y)^2 + \frac{i}{4}(3\alpha x^4 + 4x^3(\alpha + \beta - 3\alpha iy) + 4xy(\alpha(3iy^2 + y - 2i)) - 3\beta y) + \)
\(+ 2\gamma ix(i + y) - x^2(2\alpha(y(2i + 9y) - 2) + 4\beta(3iy - 2) + \gamma) + y(y - 2i)(3\alpha y^2 + 2iy(\alpha + 2\beta) + \gamma) \)
\[ B_2 = (J_2 + iJ_1)p_x + z^2(\alpha(x - iy) + \beta) - \frac{\delta(x - iy)}{z^2} \]
\[ F = (J_2 + iJ_1)^2 + z^2(3\alpha x^2 - 3\alpha y^2 + 2\beta - 4iy(\alpha + \beta) + x(4\beta - 6\alpha iy) - \gamma) + \delta((x - iy)^2 + 4iy) \]

The integrals satisfy the relations
\[ \{A_1, A_2\} = \{A_1, B_2\} = \{A_2, B_1\} = 0 \]

The "Π shape diagram corresponding to the above relations is given by \[ (12) \].
\[ \{A_2, B_2\}, F\} = 8(2A_2 + A_1 - H)A_2 + 8(A_1 + 2\beta - \gamma)B_2 - 32\alpha\delta \]

\[ \{A_1, B_1\}, F\} = i(24A_1A_2 + 32(\alpha + \beta)B_2 - 8(2\beta + \gamma)A_2 - 16\alpha F) \]

\[ \{A_2, B_2\}, B_1\} = \{B_1, B_2\}, A_2\} = i(2A_1A_2 + 8\alpha B_2 - 4\alpha F - 2\gamma A_2) \]

\[ \{A_1, F\}, B_2\} = \{\{B_2, F\}, A_1\} = i(8A_1A_2 + 32\alpha B_2 - 16\alpha F - 8\gamma A_2) \]

\[ \{B_1, B_2\}, F\} = -4(2iA_2 - 4B_1 + 2iB_2 - iH)A_2 + 2i(2B_2 - F)A_1 + +4i(2\beta - \gamma)B_2 + 2i(\gamma - 2\beta)F + 16i\alpha\delta + 16i\beta \delta \]

\[ \{A_1, B_1\}, B_1\} = \frac{\partial F_1}{\partial A_1} = -6A_1^2 + 8\gamma A_1 - 32\alpha iB_1 + 8\beta A_2 - 8\beta H - 2\gamma^2 \]

\[ \{\{A_1, F\}, F\} = 32(2A_2 + A_1 - H)A_2 + 32A_1B_2 + 32(2\beta - \gamma)B_2 - 128\alpha\delta \]

\[ \{A_2, F\}, F\} = 16(4A_2 + A_1 + 4iB_1 + 2B_2 - 2H)A_2 + (A_1 + 2\beta - \gamma)F + -64\delta(2\alpha + \beta) \]

\[ \{B_1, F\}, B_2\} = -4i(2A_2 + 4B_2 - H)A_2 - 4i(2B_2 - F)A_1 + 8i(H + \beta)B_2- -4\beta iF + 16i\alpha + \beta \]

\[ \{B_2, F\}, B_1\} = -8(2B_1 + iB_2)A_2 + 6i(F - 2B_2)A_1 + 4i(2H + \gamma)B_2 - 2\gamma iF \]

\[ \{B_1, F\}, B_1\} = -8(A_1 + A_2 - H)B_2 + 6A_1F + 16iA_2B_1 + 8\beta B_2 - 2(\gamma + 2\beta)F \]

\[ \{A_2, B_2\}, B_2\} = \frac{\partial F_2}{\partial A_2} = 4A_1A_2 + 8B_2\beta, \quad \{A_1, \{A_1, B_1\}\} = \frac{\partial F_1}{\partial B_1} = -32i\beta^2 - 32iA_1\alpha \]

\[ \{A_2, F\}, B_1\} = \{\{B_1, F\}, A_2\} = i(4A_1A_2 + 8A_1B_2 + 32(2\alpha + 2\beta - \gamma)B_2 - 8(\alpha + \beta)F - 4\gamma A_2) \]

The second algebra which expand in terms of \(C_1, C_2, D\) with coefficients any linear combination of integrals \(A_1, A_2, B_2, F, H\)

\[ \{C_1, C_2\} = 16\alpha iC_2, \quad \{C_1, D\} = 8\beta C_2 - 16\alpha iD, \quad \{C_2, D\} = -2A_2C_1 \]

There exist an relation between \(A_1, A_2, C_1, C_2, B_2, D, H\)

\[ -i\frac{1}{\beta\gamma}(\beta A_2 - 2\alpha B_2)C_1 - \frac{1}{\beta\gamma}(2\alpha A_2 + \beta A_1 - 2\alpha H - \beta\gamma)C_2 + i\frac{4}{\beta\gamma}(\alpha A_1 + \beta^2)D = 0 \]

- **KKM potential** \(V_{\text{ext}}\) This potential is somehow exceptional because is the only one, where the integrals don't satisfy a "\(\Pi\)" shape diagram.

\[
H = p_x^2 + p_y^2 + p_z^2 + (x + iy)\alpha + (\frac{3}{4}(x + iy)^2 + \frac{z}{4})\beta + +((x + iy)^3 + \frac{3}{4}z(x + iy) + \frac{1}{16}(x - iy))\gamma + +\frac{5}{16}(x + iy)^4 + \frac{3}{8}z(x + iy)^2 + \frac{1}{16}(x^2 + y^2 + z^2)\delta
\]

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The integrals of motion are given by:

\[ A_1 = (p_x + ip_y)^2 + \frac{1}{16}(x + iy)(2\gamma + \delta(x + iy)) \]

\[ A_2 = p_z(p_x + ip_y) + \frac{1}{16}(2\beta iy - 3\gamma y^2 + \gamma z + \delta x^3 - \delta iy^3 + \delta iyz + 3x^2(1 + i\delta) + x(2\beta + 6\gamma iy - 3\delta y^2 + \delta z)) \]

\[ B_1 = J_2p_z - J_3py_i - (J_3p_x - J_1p_y) - \frac{1}{16i}(-8\delta x^5 - 8(3\gamma + 5i\delta) x^4 - 4(-20\delta y^2 + 24i\gamma y + 4\beta + z\delta)) x^3 + (80i\delta y^3 + 144\gamma y^2 - 2i(24\beta + 6z\delta + \delta) y - 16\alpha + \gamma) x^2 + (-40\delta y^4 + 96i\gamma y^3 + 4(12\beta + 3z\delta + \delta)y^2 - 2i(16\alpha + 3\gamma)y + 4z(2\beta + z\delta)) x - 24\delta y^3 \gamma + y^2(16\alpha + 7\gamma) + z(16\alpha + (8z - 1)\gamma) - 8\gamma y^5 \delta + 2i\gamma(2z - 1)(2\beta + z\delta) + 2i\gamma^3(8\beta + 2z\delta + \delta)) \]

\[ B_2 = i(J_3p_z + iJ_1p_y + iJ_2p_x + \frac{i}{4}p_2^2 - \frac{1}{16i}(x^2 + 2i\gamma x - y^2 - z) \]

\[ (3\delta x^2 + 8\gamma x + 6iy\delta x + 4\beta + 8i\gamma - 3y^2 \delta + \delta z) \]

\[ F = \frac{1}{4}(4J_1^2 + 8i(J_2 - p_x)J_1 - 4J_2^2 + p_2^2 + 4J_2(p_x - ip_y) + p_z(p_z - 12iJ_3)) + \frac{1}{64}((16z - 3)\delta x^4 + 8(6z - 1)\gamma x^3 + 2(8z - 1)\beta + 8i\gamma x^2 + 2z(16\alpha + (8z + 3)\gamma)x + y^4(16z + 5)\delta + z(4\beta + z\delta) - 4i\beta y^3(4(3z\gamma + \gamma) + x(16z + 3)\delta) - y^2(6(16z + 1)\delta x^2 + 24(6z + 1)\gamma x + 4(8z + 3)\beta + (6z(2z + 1) - 1)\delta) + iy(4(16z - 1)\delta x^3 + 144z\gamma x^2 + 8(8z\beta + \beta + z(3z + 1)\delta)x + 16(2z\alpha + \alpha) + (2z(8z + 7) - 1)\gamma) \]

This system satisfies the relations

\[ \{A_1, A_2\} = \{A_1, B_2\} = \{B_1, F\} = 0 \]

and the corresponding diagram is

\[ A_1 \rightarrow A_2 \]

\[ \downarrow \]

\[ B_2 \quad B_1 \]

\[ \downarrow \]

\[ F \]

The next relations correspond to the above diagram

\[ \{A_2, B_2\} = C_2, \quad C_2^2 = 2F_2 \]

\[ F_2 = \frac{1}{2}A_1^3 + \frac{\beta}{16}A_1^2 + \frac{\beta^2}{512}A_1 - \frac{\gamma}{32}A_1A_2 - \frac{\beta\gamma}{512}A_2 - \frac{\gamma^2}{512}B_2 - \frac{\delta}{32}A_1B_2 - \frac{\delta}{128}A_2^2 \]

The full algebra is given by the following relations:

\[ \{\{B_1, A_2\}, B_2\} = \frac{1}{8}(16A_1 + \beta)A_1 - \frac{\gamma}{16}A_2 \]
\[
\{A_1, \{A_1, B_2\}\} = \{A_1, \{A_1, B_1\}\} = \{A_1, F, B_2\} = 0
\]

\[
\{F, B_2, A_2\} = \frac{1}{8}(16A_1 - \beta)A_2 + \frac{\alpha}{2}A_1 - \frac{3\gamma}{8}B_2 - \frac{\gamma}{32}H
\]

\[
\{A_1, F, B_2\} = \{\{B_2, F\}, A_1\} = \frac{1}{4}(16A_1 + \beta)A_1 - \frac{\gamma}{8}A_2
\]

\[
\{F, A_1, A_2\} = \{F, A_2, A_1\} = -\frac{\gamma}{128}(16A_1 + \beta) - \frac{\delta}{16}A_2
\]

\[
\{B_1, B_2, A_1\} = \{B_2, \{A_1, B_1\}\} = -\frac{\gamma}{16}A_1 - \frac{\delta}{32}A_2 + \frac{\beta\gamma}{256}
\]

\[
\{F, B_2, B_2\} = -4A_2^2 - \frac{1}{64}(16\alpha - \gamma)A_2 - 8A_1B_2 + \frac{\gamma}{16}B_1
\]

\[
\{F, A_1, A_1\} = -\frac{\delta}{8}A_1 - \frac{\gamma^2}{128}, \{F, B_1, A_1\} = \frac{\gamma}{8}A_1 + \frac{\delta}{16}A_2 + \frac{\beta\gamma}{128}
\]

\[
\{F, A_1, B_1\} = \frac{1}{32}(384A_1 + 8\beta + \delta)A_2 + \frac{1}{32}(48\alpha + \gamma)A_1 - \frac{\gamma}{4}B_2 - \frac{\delta}{16}B_1 - \frac{\gamma}{16}H + \frac{\beta\gamma}{512} + \frac{\alpha\beta}{32}
\]

\[
\{A_2, B_2, B_2\} = \frac{\delta F_2}{\partial A_2} = -\frac{\gamma}{32}A_1 - \frac{\delta}{64}A_2 - \frac{\beta\gamma}{512}. \{A_2, \{A_2, B_2\}\} = \{A_2, \{A_2, B_2\}\} = \frac{\delta}{32}A_1 - \frac{\gamma^2}{512}
\]

\[
\{B_1, A_2, A_2\} = \frac{\gamma}{16}A_1 + \frac{\delta}{32}A_2 + \frac{\beta\gamma}{256}, \{B_1, A_2, A_1\} = \{A_2, \{A_1, B_1\}\} = \frac{\delta}{16}A_1 + \frac{\gamma^2}{256}
\]

\[
\{A_2, B_2, B_1\} = -\frac{1}{128}(384A_1 + 32\beta + \delta)A_1 + \frac{3\delta}{32}B_2 + \frac{\gamma}{8}A_2 + \frac{\delta}{128}H - \frac{1}{2048}(8\beta^2 + \gamma^2) + \frac{\alpha\gamma}{128}
\]

\[
\{B_1, B_2, A_2\} = -\frac{1}{128}(128A_1 + 16\beta + \delta)A_1 + \frac{3\delta}{32}B_2 + \frac{\gamma}{16}A_2 + \frac{\delta}{128}H - \frac{1}{2048}(8\beta^2 + \gamma^2) + \frac{\alpha\gamma}{128}
\]

\[
\{A_1, B_1, F\} = -\frac{1}{32}(384A_1 + 8\beta + \delta)A_2 - \frac{1}{32}(48\alpha + \gamma)A_1 + \frac{\gamma}{4}B_2 - \frac{\delta}{16}B_1 + \frac{\gamma}{16}H - \frac{\beta\gamma}{512} - \frac{\alpha\beta}{32}
\]

\[
\{F, \{B_1, A_2\}\} = -\frac{1}{128}(384A_1 + 512B_2 + 128H + \delta - 24\beta)A_1 - \frac{1}{128}(8F + H) -
- \frac{1}{32}(8\beta + 96\delta)B_2 + \frac{\beta}{16}H - \frac{1}{32}(256A_2 + 48\alpha + 96\gamma)A_2 -
- \frac{1}{4096}(\gamma^2 - 256\alpha^2)
\]

\[
\{\{A, B\}, B_1\} = -\frac{1}{128}(A_1 + 512B_2 + 128H + \delta + 8\beta)A_1 + \frac{1}{32}(256A_2 + 48\alpha + \gamma)A_2 -
- \frac{1}{32}(8\beta + \delta)B_2 - \frac{1}{128}(8\beta - \delta)H - \frac{\delta}{16}F
\]

\[
\{F, \{B_2, A_2\}\} = \frac{1}{64}(384A_1 + 8\beta + \delta)A_2 + \frac{1}{64}(48\alpha + \gamma)A_1 - \frac{\gamma}{8}B_2 - \frac{\delta}{32}B_1 - \frac{\gamma}{32}H + \frac{\beta\gamma}{1024} + \frac{\alpha\beta}{64}
\]

\[
\{\{F, A_2\}, B_2\} = -\frac{1}{64}(256A_1 + 16\beta + \delta)A_2 - \frac{1}{64}(16\alpha + \gamma)A_1 - \frac{\gamma}{4}B_2 - \frac{\delta}{32}B_1 - \frac{\beta\gamma}{1024} - \frac{\alpha\beta}{64}
\]
\[
\{F, \{B_1, B_2\}\} = -\{F, B_2\}, B_1 = -\frac{1}{128}(256B_1 - 16\alpha + \gamma)A_1 - \frac{\gamma}{8}F - \frac{\beta}{8}B_1 - \frac{1}{128}(128A_2 - \gamma + 12\alpha)H - \frac{1}{32}(384A_2 + 48\alpha + 5\gamma)B_2
\]

\[
\{F, \{F, B_2\}\} = -\frac{1}{32}(32A_1 + 320B_2 + 384F - 48H + \beta)A_1 - \frac{\beta}{4}F - \frac{1}{32}(16\alpha + \gamma)B_1 - \frac{1}{8}(192B_2 + 64H + \beta)B_2 - \frac{1}{2}H^2 + \frac{\beta}{32}H
\]

\[
\{F, \{F, A_1\}\} = -\frac{1}{64}(128A_1 + 512B_2 + 128H + 8\beta + \delta)A_1 + \frac{1}{64}(\delta - 8\beta)H - \frac{\delta}{8}F + \frac{1}{16}(256A_2 + 48\alpha + \gamma)A_2 - \frac{1}{16}(8\beta + \delta)B_2 + \frac{1}{2048}(256\alpha^2 - \gamma^2)
\]

\[
\{\{A_1, B_1\}, B_1\} = \frac{1}{64}(384A_1 + 32\beta + \delta)A_1 - \frac{\gamma}{4}A_2 - \frac{3\delta}{16}B_2 - \frac{\delta}{64}H + \frac{1}{1024}(8\beta^2 + \gamma^2) - \frac{\alpha\gamma}{64}
\]

\[
\{B_1, \{B_1, A_2\}\} = \frac{1}{64}(384A_1 + 8\beta + \delta)A_2 + \frac{1}{64}(\gamma + 48\alpha)A_1 - \frac{\gamma}{8}B_2 - \frac{\gamma}{32}H + \frac{\delta}{32}B_1 + \frac{\alpha\beta}{64} + \frac{\beta\gamma}{1024}
\]

\[
\{B_1, \{B_2, B_2\}\} = -\frac{1}{128}(256A_2 + 16\alpha + \gamma)A_1 - \frac{\beta}{8}A_2 - \frac{\gamma}{8}B_2 - \frac{\delta}{64}B_1 + \frac{\beta\gamma}{2048} - \frac{\alpha\beta}{128}
\]

\[
\{F, \{F, A_2\}\} = -\frac{1}{64}(384A_1 + 1536B_2 + 128H + 16\beta + \delta)A_2 + \frac{1}{64}(3\gamma - 16\alpha)H - \frac{1}{16}F - \frac{1}{32}(128B_1 + 16\alpha + \gamma)A_1 - \frac{1}{32}(8\beta + \delta)B_1 - \frac{1}{16}(48\alpha + 3\gamma)B_2 - \frac{\delta\gamma}{1024} - \frac{\alpha\beta}{64}
\]

\[
\{F, A_2\}, A_2 \} = -\frac{1}{64}(128A_1 + 32\beta - \delta)A_1 + \frac{\gamma}{8}A_2 + \frac{\delta}{16}B_2 + \frac{\delta}{64}H - \frac{1}{1024}(8\beta^2 - \gamma^2) + \frac{\alpha\gamma}{64}
\]

The second algebra which expand in terms of $C_1, C_2, D, L$ with coefficients any linear combination of integrals $A_1, A_2, B_2, H$

\[
\{C_1, C_2\} = 0, \ {C_1, D} = -\frac{\gamma}{8}C_2 - \frac{\delta}{64}L, \ {C_2, D} = \frac{\gamma}{16}C_2 + \frac{\delta}{128}L
\]

\[
\{C_1, L\} = -\frac{\delta}{8}C_2, \ {C_2, L} = \frac{\delta}{16}C_2
\]
\[
\{D, L\} = \left( -\frac{A_1\delta^2}{2(16\alpha - \gamma)\gamma} + \frac{2B_2\delta^2}{(16\alpha - \gamma)\gamma} + \frac{H\delta^2}{16\alpha - \gamma} + \frac{4A_2\delta}{32A_2\beta} + \frac{4A_1\beta\delta}{(16\alpha - \gamma)\gamma} - \frac{16B_2\beta\delta}{4H\beta\delta} \right) - \frac{(16\alpha - \gamma)\gamma}{16(16\alpha - \gamma)\gamma} - \frac{16\alpha - \gamma}{16\alpha - \gamma} + \frac{4A_2\delta}{32A_2\beta} \right) C_2 + \left( -\frac{2A_1\delta^2}{(16\alpha - \gamma)\gamma} - \frac{\gamma\delta}{16\alpha - \gamma} + \frac{16\alpha - \gamma}{16\alpha - \gamma} + \frac{16A_1\beta\delta}{(16\alpha - \gamma)\gamma} + \frac{\beta\gamma}{16\alpha - \gamma} \right) D + \left( -\frac{2(16\alpha - \gamma)}{16\alpha - \gamma} + \frac{16(16\alpha - \gamma)}{16\alpha - \gamma} + (16\alpha - \gamma)\gamma - \frac{4A_2\delta\beta}{A_2\delta^2} - \frac{8A_1\beta}{16(16\alpha - \gamma)} + \frac{\gamma^2}{16(16\alpha - \gamma)} + \frac{2(16\alpha - \gamma)\gamma}{16\alpha - \gamma} + \frac{\alpha\gamma}{16\alpha - \gamma} + \frac{A_1\delta}{16\alpha - \gamma} \right) L
\]

The relation between \(C_1, C_2\) is:

\[C_1 + 2C_2 = 0\]

5 Conclusions

The three dimensional non degenerate systems of Kalnins, Kress and Miller \[1\] satisfy a parafermionic like quadratic Poisson algebra. All the algebras have at least a two-dimensional like parafermionic quadratic Poisson subalgebra. All the systems with one exception the have at least two subalgebras forming a special "Π" structure. Each subalgebra coreesponds to classical superintegrable system possessing two Hamiltonians.

There is no results yet about the quantum superintegrable systems as also a a compact general classification theory for three dimensionl superintegrable potentials. The structure of the corresponding Poisson algebras for the degenerate states is under investigation.

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