Characteristic polynomial of anti-adjacency matrix of directed cyclic friendship graph

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Abstract. Graph theory has some applications. One of them is used to do social network analysis as in Facebook with each person as nodes and every like, share, comment, tag as edges. Usually a network can be represented by a graph. Analyzing a network is the same as analysing the structure of a graph. A structure of a graph can be analysed through some matrix representations such as anti-adjacency matrix. The entries of the anti-adjacency matrix of a directed graph represent the presence or absence of directed arcs from a vertex to the others. This paper is about the properties of the anti-adjacency matrix of directed cyclic friendship graph, those are the characteristic polynomial and the eigenvalues of the anti-adjacency matrix. The method used to obtain the general form of the coefficients of the characteristic polynomial is by adding up the determinant values of all directed induced subgraphs, cyclic or acyclic, of the directed cyclic friendship graph. Furthermore, the methods used to find the eigenvalues of the anti-adjacency matrix of directed cyclic friendship graph are the substitution method and factorization method. The result of this research are the general form of the coefficients of the characteristic polynomial and the general form of the eigenvalues of the anti-adjacency matrix depend on the number of triangles in directed cyclic friendship graph.

1. Introduction
Graph theory is a branch of mathematics that has many applications in various fields of science including statistics, biology, computer science and others. One of the applications of graph theory is used to do social network analysis as in Facebook with each person as nodes and every like, share, comment, tag as edges[1]. Graph Theory was first introduced by Leonhard Euler (1707-1783) a Swiss mathematician, to solve a problem known as the Konigsberg Seven Bridges. Nowadays some researches in graph theory are devoted to the matrix representation of various kinds of graphs.

Firmansyah (2014) conducted one of the studies on the anti-adjacency matrix of several classes of acyclic graphs [2]. Then Wildan (2015) discussed about the coefficients of characteristic polynomial of the anti-adjacency matrix of directed cyclic graphs in general up to the fifth coefficients [3]. That research motivated me to find some more about characteristic polynomial of anti-adjacency matrix of a specific graph, that is the friendship graph. The friendship graph can be cyclic by giving the right direction in each of its arcs. This paper discusses the polynomial characteristics and eigenvalues of anti-adjacency matrix of directed cyclic friendship graph. The directed cyclic friendship graph in this paper is obtained by adding clockwise direction in each arcs.
2. Preliminaries

This section consists of some definitions, theorems, and lemmas required to prove the theorems in the main result.

2.1. Definitions

**Definition 1** An anti-adjacency matrix of a directed graph $G$ with the set of vertices $V(G) = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ is an $n \times n$ matrix $B = [b_{ij}]$, where $b_{ij} = 0$, if there exists a directed edge from $v_i$ to $v_j$ and $b_{ij} = 1$ otherwise.[4]

**Definition 2** A friendship graph $F_n$ is a graph which consists of $n$ triangles with a common vertex [5]. Friendship graph $F_4$ is shown in Figure 1 below.

![Figure 1. Friendship graph $F_4$.](image_url)

In this paper directed cyclic friendship graph is obtained by adding clockwise direction in each arcs contained in $F_n$. The general form of a directed cyclic friendship graph is shown in Figure 2.

![Figure 2. Directed cyclic friendship graph $F_n$.](image_url)

The anti-adjacency matrix of directed cyclic friendship graph, has $(2n + 1) \times (2n + 1)$ entries defined as the following:

$$
    b_{ij} = \begin{cases} 
    0, & \{i = 1, j = 2k\}; \{i = 2k + 1, j = 1\}; \{i = 2k, j = 2k + 1\}, k \leq n \in \mathbb{N} \\
    1, & \text{others.}
    \end{cases}
$$

Figure 3 showsthe general form of the anti-adjacencymatrix of a directed cyclic friendship graph.
2.2. Theorems

**Theorem 1** [4] Let $G$ be a directed acyclic graph, with $V(G) = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ and $B$ be the anti-adjacency of $G$, then $\det(B) = 1$ if $G$ has a Hamiltonian path and $\det(B) = 0$ otherwise.

**Theorem 2** [3] Let $P(B(G)) = \lambda^n + b_1 \lambda^{n-1} + b_2 \lambda^{n-2} + \cdots + b_{n-1} \lambda + b_n$ be a characteristic polynomial of an anti-adjacency matrix $B(G)$ of a directed graph, let $B(\langle U \rangle_{\text{acyclic}})_{ij}$ be the determinant of an anti-adjacency matrix of a directed acyclic induced subgraph with $i$ vertices and $j_1 = 1, 2, \ldots, w_1$; where $w_1$ is the number of a directed acyclic-induced subgraphs $\langle U \rangle_{\text{acyclic}}$ with $i$-vertices of directed cyclic graph $G$, and let $B(\langle U \rangle_{\text{acyclic}})_{ij}$ be the determinant of the anti-adjacency matrix of directed cyclic induced subgraph with $i$-vertices and $j_2 = 1, 2, \ldots, w_2$, where $w_2$ is the number of directed cyclic induced subgraph $\langle U \rangle_{\text{acyclic}}$ with $i$ vertices of directed cyclic graph $G$, then $b_i = (-1)^i \left( \sum_{j_1=1}^{w_1} B(\langle U \rangle_{\text{acyclic}})_{ij_1} \right) + \sum_{j_2=1}^{w_2} B(\langle U \rangle_{\text{acyclic}})_{ij_2}$, where $i = 1, 2, \ldots, n$.

**Theorem 3** [3] Let $P(B(G)) = \lambda^n + b_1 \lambda^{n-1} + b_2 \lambda^{n-2} + \cdots + b_{n-1} \lambda + b_n$ be a characteristic polynomial of an anti-adjacency matrix $B(G)$ of a directed graph with $n$ vertices and $m$ arcs, then $b_1 = -n$ and $b_2 = m$.

**Theorem 4** [6] Let $B$ be an anti-adjacency matrix of directed cycle graph $C_n^\pi$, then $\det(B(C_n^\pi)) = n - 1$. 

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**Figure 3.** General form of anti-adjacency matrix of directed cyclic friendship graph $F_n^\pi$.
2.3. Lemmas
The proofs of Lemma 1 up to Lemma 6 are related to the types of directed acyclic or cyclic induced subgraphs of $F_n$. The following figures show the various type of the induced subgraph of $F_n$.

Figure 4 consists of the types of directed acyclic subgraphs.

![Diagram of directed acyclic subgraphs](image)

**Figure 4.** The types of directed acyclic induced subgraphs.
Figure 5 consists of the types of directed cyclic subgraphs.

**C1**

**C2**

**C3**

**C4**

*Figure 5. The types of directed cyclic induced subgraphs.*

**Lemma 1** Let $B$ be an anti-adjacency matrix of directed acyclic subgraph type A1. Then, $\det(B) = 1$.

*Proof.* Let $B$ be an anti-adjacency matrix of directed acyclic subgraph type A1. This type is a directed acyclic induced subgraph which contain a Hamiltonian path. Hence, according to Theorem 1 $\det(B) = 1$. 
Lemma 2 Let $B_2, B_3, B_4$, respectively, be an anti-adjacency matrix of directed acyclic subgraph type $A_2, A_3, A_4$. Then, $\det(B_2) = \det(B_3) = \det(B_4) = 0$.

Proof. Let $B_2, B_3, B_4$, respectively, be an anti-adjacency matrix of directed acyclic subgraph type $A_2, A_3, A_4$. Each of these types is a directed acyclic induced subgraph that does not contain a Hamiltonian path. Hence, according to Theorem 1 $\det(B_2) = \det(B_3) = \det(B_4) = 0$.

Lemma 3 Let $B$ be an anti-adjacency matrix of cyclic subgraph type $C_1$. Then, $\det(B) = 2$.

Proof. Let $B$ be an anti-adjacency matrix of cyclic subgraph type $C_1$. Subgraph type $C_1$ is a directed cyclic induced subgraph in the form of $\overline{C_3}$. Hence, according to Theorem 4 $\det(B) = 2$.

Lemma 4 Let $B_2, B_3, B_4$, respectively, be an anti-adjacency matrix of directed cyclic subgraph type $C_2, C_3, C_4$. Then, $\det(B_2) = \det(B_3) = \det(B_4) = 0$.

Proof. Let $B_2, B_3, B_4$, respectively, be an anti-adjacency matrix of directed cyclic subgraph type $C_2, C_3, C_4$. Hence, by using the row reduction process, $\det(B_2) = \det(B_3) = \det(B_4) = 0$.

Lemma 5 Let $G$ is a directed cyclic friendship graph, then number of directed acyclic induced subgraphs with 3 vertices which contains a Hamiltonian path as in type $A_1$ is $n(n-1) = n^2 - n$.

Lemma 6 Let $G$ is a directed cyclic friendship graph, then the number of directed cyclic induced subgraphs of type $C_1$ is $n$.

Lemma 7 If $\lambda = y + \frac{2n+1}{3}$ is substituted into the equation $\lambda^3 - (2n+1)\lambda^2 + (3n)\lambda - (n^2 + n) = 0$, then it will form an equation $y^3 - \frac{4n^2-5n+1}{3} y + \frac{-16n^3+3n^2-12n-2}{27} = 0$.

Lemma 8 If $y = z + \frac{c}{z}$ is substituted into the equation $y^3 - \frac{4n^2-5n+1}{3} y + \frac{-16n^3+3n^2-12n-2}{27} = 0$, then it will form an equation $z^3 + \frac{9c-4n^2+5n-1}{3} z + \frac{c^3}{3} z^3 + \frac{-16n^3+3n^2-12n-2}{27} = 0$.

Lemma 9 If $c = \frac{4n^2-5n+1}{9}$, and $u = z$ are substituted into the equation $u^2 + \frac{9c-4n^2+5n-1}{3} u^4 + \frac{9c-4n^2+5n-1}{3} c^2 z^2 + c^3 + \frac{-16n^3+3n^2-12n-2}{27} z^3 = 0$, then it will form an equation $u^2 + \frac{-16n^3+3n^2-12n-2}{27} u + \frac{(4n^2-5n+1)c^3}{9} = 0$.

3. Main Result
In this section we discuss about the theorems which are related to the main result. In Theorem 1, we discuss about the coefficients $b_i, 1 \leq i \leq 2n+1$, of the characteristic polynomial of the anti-adjacency matrix of directed cyclic friendship graph $\overline{F_n}$. In Theorem 2, we discuss about the eigenvalues of its anti-adjacency matrix.

The method used to obtain the general form of the coefficients of the characteristic polynomial is by adding up the determinant values of all directed induced subgraphs, cyclic or acyclic. We know from Lemmas section there are directed induced subgraphs that the determinant is not equal to zero. So, we can focus our research for directed cyclic induced subgraphs in type $C_1$, and for directed acyclic induced subgraphs in type $A_1$. 


Theorem 5 Let \( B(\overline{F}_n^+) \) be an anti-adjacency matrix of directed cyclic friendship graph, then the characteristic polynomial is:

\[
P(B(\overline{F}_n^+)) = \lambda^{2n+1} + (-1)(2n+1)\lambda^{2n} + (3n)\lambda^{2n-1} + (-1)(n^2 + n)\lambda^{2n-2}, \text{ for } n \geq 1.
\]  

(1)

Proof.

\( \diamond \) For \( 1 \leq i \leq 2 \)

Because \( P(B(\overline{F}_n^+)) = \lambda^{2n+1} + b_1 \lambda^{2n} + b_2 \lambda^{2n-1} + \cdots + b_{2n} \lambda + b_{2n+1} \) is the characteristic polynomial of the anti-adjacency matrix of directed cyclic friendship graph with \( 2n + 1 \) vertices and \( 3n \) arcs, then according to Theorem 3, \( b_1 = (-1)(2n+1) \) and \( b_2 = 3n \).

\( \diamond \) For \( i = 3 \)

- Acyclic parts

Let \( B \) be the anti-adjacency matrix of directed acyclic induced subgraphs of directed cyclic friendship graph \( \overline{F}_n^+ \). According to Lemma 5, there are \((n^2 - n)\) directed acyclic induced subgraphs, each of them consists of a Hamiltonian path and according to Lemma 1, \( \det(B) = 1 \). Hence, based on Theorem 2 we conclude that \( \sum_{j=1}^3 |B(\langle U \rangle_{acyclic})^{(j)}_3| = (n^2 - n)(1) = n^2 - n \).

- Cyclic parts

Let \( B \) be the anti-adjacency matrix of cyclic directed cyclic induced subgraphs of a directed cyclic friendship graph \( \overline{F}_n^+ \). According to Lemma 6, there are \((n)\) directed cyclic induced subgraphs like type C1 and according to Lemma 3, \( \det(B) = 2 \). Hence, based on Theorem 2 we conclude that \( \sum_{j=2}^3 |B(\langle U \rangle_{cyclic})^{(j)}_3| = (n)(2) = 2n \).

So, we conclude that \( b_3 = (-1)^3(n^2 - n + 2n) = (-1)(n^2 + n) \).

\( \diamond \) For \( 4 \leq i \leq 2n + 1 \)

- Acyclic parts

This part is related to the subgraphs of type A4. Every subgraph of type A4 with \( i \) vertices, for \( 4 \leq i \leq 2n + 1 \), does not contain any Hamiltonian path. Hence, according to Lemma 2 and Theorem 2 \( \sum_{j=1}^3 |B(\langle U \rangle_{acyclic})^{(j)}_i| = 0 \).

- Cyclic parts

These parts are related to the subgraphs of types C2, C3, C4. Let \( B_2, B_3, B_4 \), respectively be an anti-adjacency matrix of directed cyclic induced subgraphs that form directed cyclic friendship graphs \( \overline{F}_n^+ \), according to Lemma 4, \( \det(B_2) = \det(B_3) = \det(B_4) = 0 \). Hence, according to Theorem 2 \( \sum_{j=1}^3 |B(\langle U \rangle_{cyclic})^{(j)}_i| = 0 \).

So, we conclude that \( b_i = (-1)^i(0) = 0 \), for \( 4 \leq i \leq 2n + 1 \).

The methods used to find the eigenvalues of the anti-adjacency matrix of directed cyclic friendship graph are the substitution method for the complex eigenvalues and factorization method for the real eigenvalues.

Theorem 6 If \( P(B(\overline{F}_n^+)) = \lambda^{2n+1} - (2n+1)\lambda^{2n} + (3n)\lambda^{2n-1} - (n^2 + n)\lambda^{2n-2} \) is the characteristic polynomial of the anti-adjacency matrix of directed cyclic friendship graph, then

a) \( \lambda_m = 0 \) for \( 1 \leq m \leq 2n - 2 \).
b) \( \lambda_m = \frac{9u \sqrt[3]{8} e^{i \frac{4\pi}{3} (0 + 4k\pi)}}{9ue^{i (0 + 4k\pi) \frac{2n+1}{3}}} \)

with \( u = \frac{1}{54} (16n^3 - 3n^2 + 12n + 2 + 3\sqrt{96n^5 - 111n^4 + 108n^3 - 24n^2 + 12n}) \)

and \( k = 0, 1, 2 \) for \( 2n - 1 \leq m \leq 2n + 1 \)

**Proof.**

a) Set \( P(B(F_n^m)) = \lambda^{2n+1} - (2n + 1)\lambda^{2n} + (3n)\lambda^{2n-1} - (n^2 + n)\lambda^{2n-2} = 0 \)

Factorize \( P(B(F_n^m)) \) with \( \lambda^{2n-2} \), so we get the following:

\[
(\lambda^{2n-2}) (\lambda^3 - (2n + 1)\lambda^2 + (3n)\lambda - (n^2 + n)) = 0
\]

if and only if

\[
\lambda^2 = 0
\]

Or

\( \lambda^3 - (2n + 1)\lambda^2 + (3n)\lambda - (n^2 + n) = 0 \)

From (2) we can see that there are \((2n - 2)\) solutions with each of the value \( \lambda = 0 \).

b) From (3), with a little manipulation we find other roots of

\[
\lambda^3 - (2n + 1)\lambda^2 + (3n)\lambda - (n^2 + n) = 0
\]

Substitute \( \lambda = y + \frac{2n + 1}{3} \)

into equation (3). By applying **Lemma 7**, equation (3) can be stated as follow

\[
y^3 - \frac{(4n^2 - 5n + 1)}{3} y + \frac{-16n^3 + 3n^2 - 12n - 2}{27} = 0
\]

(4)

Substitue \( y = z + \frac{c}{2} \)

(6)

into equation (5) with \( c \) is a constant that will be determined later. By applying **Lemma 8**, equation (5) can be stated as follow

\[
z^3 + \frac{9c - 4n^2 + 5n - 1}{3} z + \frac{9c - 4n^2 + 5n - 1}{3z} - c + \frac{c^3}{27} + \frac{-16n^3 + 3n^2 - 12n - 2}{27} = 0
\]

(7)

Multiple with \( z^3 \) for both side of equation (7) we get

\[
z^6 + \frac{9c - 4n^2 + 5n - 1}{3} z^4 + \frac{9c - 4n^2 + 5n - 1}{3z^2} c z^2 + c^3 + \frac{-16n^3 + 3n^2 - 12n - 2}{27} z^3 = 0
\]

(8)

Substitue \( c = \frac{4n^2 - 5n + 1}{9} \), and \( u = z^3 \) into equation (8). By applying **Lemma 9**, equation (8) can be stated as follow

\[
u^2 + \frac{-16n^3 + 3n^2 - 12n - 2}{27} u + \frac{4n^2 - 5n + 1}{9} \frac{1}{3} u^3 = 0
\]

By using formula \( \frac{-b + \sqrt{b^2 - 4ac}}{2a} \) we get

\[
u = \frac{1}{54} (16n^3 - 3n^2 + 12n + 2 + 3\sqrt{96n^5 - 111n^4 + 108n^3 - 24n^2 + 12n})
\]

(9)

Because \( u = z^3 \), according [6] by written \( z = re^{i\theta} \) we can transform equation (9) into:

\[
u = z^3 = (re^{i\theta})^3 = r^3 e^{i3\theta}
\]

It can be seen from equation (9) that \( u \) is real \((96n^5 - 111n^4 + 108n^3 - 24n^2 + 12n) \) for \( n \geq 1 \), so is \( z^3 \), then we conclude

\( r^3 = u \) and \( 3\theta = 0 + 2k\pi \)

or
\( r = \frac{3}{\sqrt[3]{u}} \text{ and } \theta = \frac{2k\pi}{3} \) \hspace{1cm} (10)

From equation (10) according [6] we obtain:

\[ z_k = \frac{3}{\sqrt[3]{u}} e^{i \frac{2k\pi}{3}} \] \hspace{1cm} (11)

With \( k = 0,1,2 \). From equation (11) and (6) we get

\[ y = \frac{3}{\sqrt[3]{u}} e^{i \frac{2k\pi}{3}} + \frac{4n^2 - 5n + 1}{9\sqrt[3]{u} e^{i \frac{2k\pi}{3}}} \]

That can be simplified to

\[ y = \frac{9u^3}{\sqrt[3]{u}} e^{i \frac{4k\pi}{3}} + \frac{(4n^2 - 5n + 1)(\sqrt[3]{u})^2}{9ue^{i\frac{2k\pi}{3}}} \] \hspace{1cm} (12)

From equation (12) and (4) we get:

\[ \lambda = \frac{9u^3}{\sqrt[3]{u}} e^{i \frac{4k\pi}{3}} + \frac{(4n^2 - 5n + 1)(\sqrt[3]{u})^2}{9ue^{i\frac{2k\pi}{3}}} + \frac{2n + 1}{3} \]

4. Conclusion

Let \( P \left( B(F_n) \right) = \lambda^{2n+1} + b_1\lambda^{2n} + b_2\lambda^{2n-1} + \cdots + b_{n-1}\lambda + b_n \) be a characteristic polynomial of the anti-adjacency matrix \( B(F_n) \) of the directed cyclic friendship graph. So, we obtained that the general form of the coefficients of the characteristic polynomial and the general form of the eigenvalues of the anti-adjacency matrix depends on the number of triangles in directed cyclic friendship graph.

The general form of the coefficients of characteristic polynomial and the eigenvalues of the anti-adjacency matrix of directed cyclic friendship graph \( F_n \) can be seen in Table 1 and Table 2.

**Table 1.** Coefficients of characteristic polynomial of anti-adjacency matrix of directed cyclic friendship graph \( F_n \).

| Index | \( b_i \) |
|-------|-----------|
| \( i = 1 \) | \((-1)(2n + 1)\) |
| \( i = 2 \) | 3n |
| \( i = 3 \) | \((-1)(n^2 + n)\) |
| \( 4 \leq i \leq 2n + 1 \) | 0 |

**Table 2.** The eigenvalues of anti-adjacency matrix of directed cyclic friendship graph \( F_n \).

| Index | \( \lambda_m \) | Parameter |
|-------|----------------|-----------|
| \( 1 \leq m \leq 2n - 2 \) | 0 | - |
| \( 2n - 1 \leq m \leq 2n + 1 \) | \[\frac{9u^3}{\sqrt[3]{u}} e^{i \frac{4k\pi}{3}} + \frac{(4n^2 - 5n + 1)(\sqrt[3]{u})^2}{9ue^{i\frac{2k\pi}{3}}} + \frac{2n + 1}{3}\] | \( u = \frac{1}{54} \) \((16n^3 - 3n^2 + 12n + 2 + 3\sqrt{96n^4 - 111n^3 + 108n^2 - 24n + 12n})\) and \( k = 0,1,2 \) |

**Open problem** For the result can be done to find whether there is any relation between the coefficient with the structure of the graph or there is relation between eigenvalues with the structure of the graph.

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