The Kepler map in the three-body problem

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Abstract

The Kepler map was derived by Petrosky (1986) and Chirikov and Vecheslavov (1986) as a tool for description of the long-term chaotic orbital behaviour of the comets in nearly parabolic motion. It is a two-dimensional area-preserving map, describing the motion of a comet in terms of energy and time. Its second equation is based on Kepler’s third law, hence the title of the map. Since 1980s the Kepler map has become paradigmatic in a number of applications in celestial mechanics and atomic physics. It represents an important kind of general separatrix maps. Petrosky and Broucke (1988) used refined methods of mathematical physics to derive analytical expressions for its single parameter. These methods became available only in the second half of the 20th century, and it may seem that the map is inherently a very modern mathematical tool. With the help of the Jacobi integral I show that the Kepler map, including analytical formulae for its parameter, can be derived by quite elementary methods. The prehistory and applications of the Kepler map are considered and discussed.
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1 Introduction

The Kepler map was discovered in 1986 by physicists, but in application to dynamical astronomy. The first publications belonged to Petrosky (1986)
and Chirikov and Vecheslavov (1986); they were followed very soon by many other contributions, where, on one hand, the initial results were further developed and described in much greater detail (Petrosky and Broucke, 1988; Vecheslavov and Chirikov, 1988; Chirikov and Vecheslavov, 1989; Emelyanenko, 1990) and, on the other hand, the same mathematical construction was derived in application to problems in atomic physics (Casati et al., 1987; Gontis and Kaulakys, 1987; Casati et al., 1988; Borgonovi et al., 1988; Jensen et al., 1988).

Petrosky (1986) and Chirikov and Vecheslavov (1986) derived the Kepler map as a tool for description of the chaotic motion of the comets in near-parabolic orbits. The model consists in the assumption that the main perturbing effect of a planet is concentrated when the comet is close to the perihelion of its orbit. This effect is defined by the phase of encounter with the planet.

Today the Kepler map is known and used to describe dynamics in several different settings of a hierarchical three-body problem: in the external restricted planar (Petrosky, 1986; Petrosky and Broucke, 1988) and strongly non-planar (Emelyanenko, 1990) problems in cometary dynamics; as well as in the abstract Sitnikov problem, where the tertiary moves perpendicular to the orbital plane of the main binary. (Urminsky and Heggie (2008) considered a variant of the Sitnikov problem and derived a map, which is in fact the Kepler map; see Eqs. (11) in their paper.)

The Kepler map has a single parameter. Its analytical formula was first given (in the restricted planar three-body problem) by Petrosky (1986), but only in a simplified form of asymptotics for large values of the pericentre distance of the cometary orbit, and the deduction was not attached. The latter was provided by Petrosky and Broucke (1988). They used refined methods of mathematical physics to derive analytical expressions for the parameter. These methods became available only in the second half of the 20th century, and it may seem that the map is inherently a very modern mathematical tool. However, in the present paper I show that the Kepler map, including analytical formulae for its parameter, can be derived by quite elementary methods. What is more, the asymptotics for its parameter can be obtained by a method, which is much simpler than that used in (Petrosky and Broucke, 1988). I discuss the prehistory of the Kepler map and its current applications, and demonstrate that the necessary tools for the derivation of the Kepler map have become available already in the middle of 19th century.
2 Elementary derivation of the Kepler map

Let us consider the motion of a comet in the planar restricted three-body problem Sun–Jupiter–comet. We choose an inertial Cartesian coordinate system with the origin at the mass centre of the Sun and Jupiter. The motion of a comet with the coordinates \((x, y)\) is described by the differential equations

\[
\ddot{x} = \nu \frac{x_S - x}{r_{13}^3} + \mu \frac{x_J - x}{r_{23}^3}, \\
\ddot{y} = \nu \frac{y_S - y}{r_{13}^3} + \mu \frac{y_J - y}{r_{23}^3},
\]

(see, e.g., Szebehely (1967)), where

\[
\begin{align*}
    r_{13}^2 &= (x_S - x)^2 + (y_S - y)^2, \\
    r_{23}^2 &= (x_J - x)^2 + (y_J - y)^2,
\end{align*}
\]

(2)

\[
\begin{align*}
    x_S &= -\mu \cos(t - t_0), \\
    y_S &= -\mu \sin(t - t_0), \\
    x_J &= \nu \cos(t - t_0), \\
    y_J &= \nu \sin(t - t_0),
\end{align*}
\]

(3)

where \(r_{13}\) and \(r_{23}\) are the distances Sun–comet and Jupiter–comet, respectively; \((x_S, y_S)\) and \((x_J, y_J)\) are the coordinates of the Sun and Jupiter, respectively; \(\mu\) is the mass of Jupiter, \(\nu = 1 - \mu\) is the mass of the Sun. We set the length unit to be equal to the constant Sun–Jupiter distance, the mass unit equal to the sum of the Solar and Jovian masses, and the time unit equal to \(1/(2\pi)\) of the period of the orbital motion of Jupiter.

Let us expand the right-hand sides of Eqs. (1) in power series of \(\mu\), retaining the first-order terms only:

\[
\begin{align*}
    \ddot{x} &= -\frac{x}{r^3} + \mu F(x, y, t, t_0), \\
    \ddot{y} &= -\frac{y}{r^3} + \mu G(x, y, t, t_0),
\end{align*}
\]

(5)
with \( r = (x^2 + y^2)^{1/2} \),

\[
F(x, y, t, t_0) = [x - \cos(t - t_0)]r^{-3} + 3x[x\cos(t - t_0) + y\sin(t - t_0)]r^{-5} + \\
+ [\cos(t - t_0) - x]\{[x - \cos(t - t_0)]^2 + [y - \sin(t - t_0)]^2\}^{-3/2}, \quad (6)
\]

\[
G(x, y, t, t_0) = [y - \sin(t - t_0)]r^{-3} + 3y[x\cos(t - t_0) + y\sin(t - t_0)]r^{-5} + \\
+ [\sin(t - t_0) - y]\{[x - \cos(t - t_0)]^2 + [y - \sin(t - t_0)]^2\}^{-3/2}; \quad (7)
\]

see, e.g., \cite{Liu1994, Zhou2000}. The quantity \( t_0 \) is the initial epoch. It is chosen in such a way, that the comet is at the perihelion of its orbit when \( t = 0 \). Designating the phase angle of Jupiter at \( t = 0 \) as \( g \), one has \( t_0 = -g \).

The constant energy \( E \) of the unperturbed orbital motion is

\[
E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{r} = -\frac{1}{2a}, \quad (8)
\]

where \( a \) is the semi-major axis of the cometary orbit. For the perturbed motion the energy is

\[
E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1 - \mu}{r_{13}} - \frac{\mu}{r_{23}} = -\frac{1}{2a}, \quad (9)
\]

\cite{Szebehely1967}, and it is not constant. Then from Eqs. (5) one has

\[
\dot{E} = \mu[\dot{x}(t)F(x(t), y(t), t, t_0) + \dot{y}(t)G(x(t), y(t), t, t_0)]. \quad (10)
\]

The increment of the energy \( E \) for one cometary orbital period is given by the integral \cite{Liu1994, Zhou2000}:

\[
\Delta E = \mu \int_{-\infty}^{+\infty} [\dot{x}(t)F(t, g) + \dot{y}(t)G(t, g)]dt, \quad (11)
\]

where \( g = -t_0 \). \( \Delta E \) is a \( 2\pi \)-periodic function of \( g \). It is anti-symmetric with respect to \( g = \pi \).

In the inertial coordinate system that we have chosen (that with the origin at the mass centre of the Sun and Jupiter), the Jacobi integral is

\[
\dot{x}^2 + \dot{y}^2 - \frac{2(1 - \mu)}{r_{13}} - \frac{2\mu}{r_{23}} - 2(x\dot{y} - y\dot{x}) = \text{const} \quad (12)
\]
Let us derive an analytical expression for the increment of the angular momentum $D$ per one orbital revolution of the tertiary. We choose the angular momentum, because in the case of the energy the analytical calculation is too complicated to achieve the result; however, as we have just seen, the result must be the same. The angular momentum is

$$D = x\dot{y} - y\dot{x},$$

(13)

and its time derivative

$$\dot{D} = x\ddot{y} - y\ddot{x}.$$  

(14)

Substituting Eqs. (1) for $\ddot{y}$ and $\ddot{x}$, one has

$$\dot{D} = \nu \frac{xy_s - x_s y}{r_{13}^2} + \mu \frac{x y_1 - x_1 y}{r_{23}^2},$$

(15)

where

$$r_{13}^2 = \mu^2 + r^2 - 2(x_s x + y_s y),$$

$$r_{23}^2 = \nu^2 + r^2 - 2(x_1 x + y_1 y),$$

(16)

i.e., $\dot{D}$ is the sum of four terms:

$$\dot{D} = A + B + C + D$$

(17)

with

$$A = \nu \frac{xy_s}{r_{13}^2}, \quad B = -\nu \frac{x_s y}{r_{13}^2}, \quad C = \mu \frac{xy_1}{r_{23}^2}, \quad D = -\mu \frac{x_1 y}{r_{23}^2}.$$  

(18)

It is sufficient to find $A$ and $B$, because

$$C = -A(\nu \to -\mu), \quad D = -B(\nu \to -\mu).$$

(19)

Let us write down the well-known elementary formulae for the unperturbed parabolic motion:

$$r = q(1 + u^2), \quad x = q(1 - u), \quad y = 2qu, \quad t = \kappa \left(u + \frac{u^3}{3}\right),$$

(20)

$$u = \left(\tau + (1 + \tau^2)^{1/2}\right)^{1/3} + \left(\tau - (1 + \tau^2)^{1/2}\right)^{1/3}, \quad \tau = \frac{3}{2\kappa} t,$$

(21)

where $\kappa = (2q^3)^{1/2}$, the eccentric anomaly $u = \tan \frac{f}{2}$, and $q$ and $f$ are the perihelion distance and the true anomaly, respectively. Note that we
consider solely the case of prograde orbits here; analysis of the retrograde case is analogous.

We use exact relations (20) for substitutions in calculating the increment of the angular momentum. Thus we follow a standard approach for deriving the energy increments in the separatrix maps (Chirikov, 1979). So, inserting Eqs. (20) in Eqs. (18), we find

\[
\begin{align*}
A &= -\mu \nu q \frac{(1 - u^2) \sin \left[ \kappa \left( u + \frac{u^3}{3} \right) - t_0 \right]}{r_{13}^3}, \\
B &= 2\mu \nu q \frac{u \cos \left[ \kappa \left( u + \frac{u^3}{3} \right) - t_0 \right]}{r_{13}^3}. \\
\end{align*}
\]

(22)

Combining Eqs. (16), (3), and (4) and inserting Eqs. (20) in the resulting expressions, we find an expression for \( r_{13} \) to substitute in the denominators in Eqs. (22):

\[
\begin{align*}
r_{13}^2 &= \mu^2 + q^2 (1 + u^2)^2 + 2\mu q \left\{ (1 - u^2) \cos \left[ \kappa \left( u + \frac{u^3}{3} \right) - t_0 \right] + 2u \sin \left[ \kappa \left( u + \frac{u^3}{3} \right) - t_0 \right] \right\}.
\end{align*}
\]

(23)

Then, expanding the right-hand sides of Eqs. (22) in power series in \( \mu \), taking into account that \( q \gg 1 \), we obtain in the first order of \( \mu \):

\[
\begin{align*}
A + C &= -\frac{3\mu}{2q^4} \frac{(1 - u^2) \sin \left[ \kappa \left( u + \frac{u^3}{3} \right) - t_0 \right]}{(1 + u^2)^5}, \\
B + D &= \frac{3\mu}{q^4} \frac{u \cos \left[ \kappa \left( u + \frac{u^3}{3} \right) - t_0 \right]}{(1 + u^2)^5}.
\end{align*}
\]

(24)

From Eqs. (17) and (20) one can find the angular momentum increment per an orbital revolution of the comet. As follows from the Jacobi integral (12), the angular momentum increment is equal to the energy increment. So, the energy increment is given by the integral

\[
\Delta E = \kappa \int_{-\infty}^{+\infty} (A + B + C + D)(1 + u^2)du.
\]

(25)

To evaluate it, first of all we define the functions

\[
I_n^0(x) = \int_{-\infty}^{+\infty} \frac{1}{(1 + u^2)^n} \cos \left[ x \left( u + \frac{u^3}{3} \right) \right] du,
\]

6
\[ I^1_n(x) = \int_{-\infty}^{+\infty} \frac{u}{(1 + u^2)^n} \sin \left[ x \left( u + \frac{u^3}{3} \right) \right] \, du, \]

\[ I^2_n(x) = \int_{-\infty}^{+\infty} \frac{u^2}{(1 + u^2)^n} \cos \left[ x \left( u + \frac{u^3}{3} \right) \right] \, du. \]

(26)

Two of them, \( I^0_n \) and \( I^1_n \), were introduced by Petrosky and Broucke (1988) in a different designation. The following recurrent relations

\[ I^1_{n+1}(x) = \frac{x}{2n} I^0_{n-1}(x), \]

\[ 2n I^0_{n+1}(x) = (2n - 1) I^0_n(x) + x I^1_n(x), \]

\[ I^2_n(x) = I^0_{n-1}(x) - I^0_n(x), \]

\[ \frac{dI^0_n(x)}{dx} = -\frac{2}{3} I^1_n(x) - \frac{1}{3} I^1_{n-1}(x), \]

\[ \frac{dI^1_n(x)}{dx} = -\frac{2}{3} I^0_n(x) + \frac{1}{3} I^0_{n-1}(x) + \frac{1}{3} I^0_{n-2}(x) \]

(27)

are valid for these functions (the 1st, 2nd, 4th, and 5th of them were deduced and used by Petrosky and Broucke (1988) in other designations; see appendix in their paper).

From Eq. (25) we find

\[ \Delta E = W(q) \sin t_0, \]

(28)

where

\[ W(q) = \frac{3\mu}{2^{1/2} q^{5/2}} \left[ I^0_0(\kappa) + 2 I^1_0(\kappa) - I^2_0(\kappa) \right] = \]

\[ \frac{3\mu}{2^{1/2} q^{5/2}} \left[ 2 I^0_1(\kappa) + 2 I^1_1(\kappa) - I^2_3(\kappa) \right], \]

(29)

where \( \kappa = (2q^3)^{1/2} \). This coefficient, if divided by 4, coincides with the corresponding coefficient found by Petrosky and Broucke (1988); see the last equation in the appendix in their paper. Most probably, the deviation of factor 4 is due to a misprint in their paper, because the final asymptotic results, compared below, coincide completely.

As demonstrated by Petrosky and Broucke (1988), some of the terms in Eq. (29) can be expressed through the modified Bessel functions of the second
kind and the Airy functions, because

\[
I_0^0(x) = 3^{-1/2} K_{1/3} \left( \frac{2}{3} x \right) = \pi x^{-1/3} \text{Ai} \left( x^{2/3} \right), \quad I_1^1(x) = 3^{-1/2} K_{2/3} \left( \frac{2}{3} x \right),
\]

(30)

where

\[
K_\nu(x) = \sec \left( \frac{1}{2} \nu \pi \right) \int_0^\infty \cos(x \sinh t) \cosh(\nu t) dt,
\]

\[
\text{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos \left( xt + \frac{t^3}{3} \right) dt
\]

(31)

by definition, see [Abramowitz and Stegun, 1970; Petrosky and Broucke, 1988]. However, the $W(q)$ coefficient has not yet been completely expressed through known special functions. Petrosky (1986) and Petrosky and Broucke (1988) found a formula for the asymptotics of $W(q)$ at $q \to \infty$. Its derivation is given in the appendix of [Petrosky and Broucke, 1988]. It is rather complicated and involves, in particular, approximate analytical solution of an ancillary differential equation and approximate numerical evaluation of an integral.

Here we show that the asymptotics can be derived in a much more straightforward and simple way. First of all, using the 1st, 2nd, 3rd, and 4th recurrent relations in list (27), we reduce Eq. (29) to the form

\[
W(q) = \frac{3\mu}{2^{1/2} q^{5/2} \kappa} \left[ 2I_0^0(\kappa) + 36I_1^1(\kappa) - 18I_2^2(\kappa) + 24 \frac{dI_0^0(\kappa)}{d\kappa} \right].
\]

(32)

Then we take the asymptotic expressions for $I_n^0(x)$ ($n = 0, 1, 2$) at $x \to \infty$ from the papers [Heggie, 1975; Roy and Haddow, 2003], where the corresponding integral was evaluated using the method of steepest descents (the only complication in using this method was that the saddle points of the exponent under integral are situated at the poles of the integrand). These expressions are

\[
I_0^0(x) \simeq I_0^1(x) \simeq -I_0^2(x) \simeq \frac{\pi^{1/2}}{120} x^{5/2} \exp \left( -\frac{2}{3} x \right).
\]

(33)

Finally, we arrive at

\[
W(q) \simeq 2^{1/4} \pi^{1/2} \mu q^{-1/4} \exp \left( -\frac{(2q)^{3/2}}{3} \right),
\]

(34)
in complete agreement with formula (3.16a) in [Petrosky and Broucke, 1988].
(Except that the minus sign is obviously lacking under the exponent in eq. (3.16a) in [Petrosky and Broucke, 1988], due to a misprint. Note that the same coefficient given in [Petrosky, 1986] is $2\pi$ times greater; this is apparently a misprint.)

3 The Kepler map: limits for application

As [Petrosky, 1986] discovered, if one writes down the expression for the energy increment together with the expression for the increment of Jupiter’s phase angle $g$ (following from Kepler’s third law) on the time interval “between two consecutive perihelion passages” (as it is usually stated), one obtains a two-dimensional area-preserving map

$$
E_{i+1} = E_i + W(q) \sin g_i,
$$

$$
g_{i+1} = g_i + 2\pi |2E_{i+1}|^{-3/2},
$$

where the subscript $i$ denotes the current number of the perihelion passage, $g_i = -t_0$. The coefficient $W(q)$ is given by formulae (32) and (34), if $\mu \ll 1$ and $q \gg 1$.

In fact, there is an inconsistency here: instead of formulation “between two consecutive perihelion passages”, it is correct to say that the energy increment in Eqs. (35) is taken between two consecutive aphelion passages, while the phase increment is indeed taken between perihelion passages. In atomic physics, this inconsistency was pointed out by Nauenberg [1990], who derived a more complicated map without this asynchronism. The asynchronism can be as well removed without construction of a separate map, but by means of a simple procedure of synchronization, described for the case of ordinary separatrix maps in [Shevchenko, 1998b, 2000].

By means of substitution $E = Wy$, $g = x$, map (35) is reducible to

$$
y_{i+1} = y_i + \sin x_i,
$$

$$
x_{i+1} = x_i + \lambda |y_{i+1}|^{-3/2},
$$

where $\lambda = 2^{-1/2} \pi W^{-3/2}$. The $y$ variable has the meaning of the normalized orbital energy of the comet, and $x$ is the normalized time.

Since $W \ll 1$ usually (see Eq. (34)), one has $\lambda \gg 1$. This means that chaos in the motion of comets is not adiabatic. In particular, the Kepler map can be locally approximated by the standard map with good accuracy.
One iteration of the Kepler map corresponds to one orbital revolution of the comet, and this means that the map time unit, corresponding to one iteration, is not constant. The increment of real time per iteration is 
\[ \Delta x_{i+1} = x_{i+1} - x_i. \]

In the considered model, the pericentre distance \( q \) is set to be constant. This was justified in (Liu and Sun, 1994): they showed that the variation of \( q \), at each return of a comet, if \( q \gg 1 \), affects the value of \( \Delta E \) only in the second order of \( \mu \). According to (Petrosky and Broucke, 1988), the higher order harmonics in \( \Delta E \) are exponentially small with \( q \) with respect to the first harmonic.

If \( q > 1 \), as in the case considered above, then the comet does not cross the orbit of Jupiter. If \( q < 1 \), the orbit of Jupiter is crossed and \( \Delta E \) as a function of \( g \) has two singularities with \( |\Delta E| \to \infty \); see (Zhou et al., 2000, 2002).

### 4 The Kepler map as a general separatrix map

The Kepler map is an example of a general separatrix map. In its model, the separatrix (the \( y = 0 \) line) separates the bound and unbound states of motion.

As distinct from the Kepler map, the well-known ordinary separatrix map has a logarithmic, with respect to the energy, increment in phase. To ensure a direct comparison with the Kepler map (36), let us write down the ordinary separatrix map in the form adopted in (Shevchenko, 1998):

\[
\begin{align*}
y_{i+1} &= y_i + \sin x_i, \\
x_{i+1} &= x_i + \lambda \ln |y_{i+1}| + c,
\end{align*}
\]

(37)

where \( \lambda \) and \( c \) are parameters. In the perturbed pendulum model of non-linear resonance, \( y \) denotes the normalized relative pendulum’s energy, \( x \) is normalized time.

Consider a map similar to map (37), but with a power-law phase increment instead of the logarithmic one:

\[
\begin{align*}
y_{i+1} &= y_i + \sin x_i, \\
x_{i+1} &= x_i + \lambda |y_{i+1}|^{-\gamma},
\end{align*}
\]

(38)
or, in an equivalent form commonly used,

\[
  w_{i+1} = w_i + W \sin \tau_i,
  \\
  \tau_{i+1} = \tau_i + \nu |w_{i+1}|^{-\gamma}.
\] (39)

Map (39) has two parameters, \(W\) and \(\nu\), instead of the single parameter \(\lambda\) in map (38); apart from the \(\gamma\) parameter. The two-parameter map (39) is reducible to the one-parameter map (38) with \(\lambda = \nu |W|^{-\gamma}\) by means of the substitution \(w = W y, \tau = x\).

A number of mechanical and physical models are described by maps (38) and (39) with rational values of \(\gamma\). The values of \(\gamma = 1/4\) and \(1/3\) correspond to the Markeev maps (Markeev, 1995, 1994) for the motion near the separatrices of resonances in two degenerate cases; \(\gamma = 1/2\) gives the “\(\hat{L}\)-map” (Zaslavsky et al., 1991) for the motion of a non-relativistic particle in the field of a wave packet; this value of \(\gamma\) also gives a map for the classical Morse oscillator driven by time-periodic force (Abdullaev, 2006); \(\gamma = 1\) gives the Fermi map (Zaslavsky and Chirikov, 1964; Lichtenberg and Lieberman, 1992) for the Fermi acceleration mechanism for cosmic rays; \(\gamma = 3/2\) gives the Kepler map for a number of astronomical and physical applications; \(\gamma = 2\) gives the “ultrarelativistic map” (Zaslavsky et al., 1991) for the motion of a relativistic particle in the field of a wave packet.

5 Applications of the Kepler map in dynamical astronomy

Major modern domains of application of the Kepler map in dynamical astronomy are as follows.

- Highly eccentric motion in the restricted planar three-body problem without crossings of orbits of planets (Petrosky, 1986; Petrosky and Broucke, 1988).
- Highly eccentric motion in the restricted non-planar three-body and four-body problems with crossings of orbits of planets (Chirikov and Vecheslavov, 1986; Vecheslavov and Chirikov, 1988; Chirikov and Vecheslavov, 1989).
- Mean-motion resonances in the perturbed highly eccentric motion (Petrosky, 1986; Malyshkin and Tremaine, 1999; Pan and Sari, 2004).
Chaotic diffusion in the dynamics of comets and meteor streams (Emel'yanenko, 1990, 1992; Liu and Sun, 1994; Zhou and Sun, 2001, Zhou et al., 2000, 2002; Malyshkin and Tremaine, 1999).

• The Sitnikov problem (Urminsky and Heggie, 2008).

The Kepler map was invented as a tool for exploring the chaotic dynamics of particles in the perturbed highly elongated orbits. This is already clear from the titles of the pioneering works:

Petrosky (1986): “Chaos and cometary clouds in the Solar system”;
Chirikov and Vecheslavov (1986): “Chaotic dynamics of comet Halley”;
Sagdeev and Zaslavsky (1987): “Stochasticity in the Kepler problem and a model of possible dynamics of comets in the Oort cloud”;
Petrosky and Broucke (1988): “Area-preserving mappings and deterministic chaos for nearly parabolic motion”;
Chirikov and Vecheslavov (1989): “Chaotic dynamics of comet Halley”.

Since the discovery of the Kepler map by Petrosky (1986) and Chirikov and Vecheslavov (1986), the most important generalization of the Kepler map (already performed heuristically in a first approximation by Chirikov and Vecheslavov (1986)) has been an introduction of a “non-harmonic” Kepler map, where the energy increment is a truncated series of Fourier harmonics in the phase variable, or it is a tabulated periodic function, which may have singularities. This allows one to describe the cometary motion with $q$ close to one and even less than one. In the planar circular restricted three-body problem Sun–planet–comet, Liu and Sun (1994) derived a non-harmonic Kepler map describing the dynamical evolution of comets in near-parabolic orbits under the perturbation of a planet, when $q$ can be close to 1. Zhou et al. (2000) generalized this approach for the planet-crossing case, when $q$ can be less than 1.

6 Applications of the Kepler map in physics

Major modern domains of application of the Kepler map in physics are as follows.

• Classical chaotic ionization of hydrogen atoms in a microwave field (Gontis and Kaulakys, 1987; Casati et al., 1988; Jensen et al., 1988, 1991).
• Generalizations of the Kepler map for multi-frequency fields (Kaulakys and Vilutis, 1999).

• Hydrogen atoms driven by microwave with arbitrary polarization (Pakoński and Zakrzewski, 2001).

• The “synchronized” Kepler map (Nauenberg, 1990; Pakoński and Zakrzewski, 2001).

Similar to the astronomical applications, the Kepler map was invented as a tool for exploring a chaotic behaviour, as it is clear from the contents of the pioneering works, which appeared practically in the same time as in astronomy:

Gontis and Kaulakys (1987): “Stochastic dynamics of hydrogenic atoms in the microwave field: modelling by maps and quantum description”;

Casati et al. (1987): “Exponential photonic localization for the hydrogen atom in a monochromatic field”; Casati et al. (1988): “Hydrogen atom in monochromatic field: chaos and dynamical photonic localization”.

In the astronomical and physical papers by Petrosky (1986), Chirikov and Vecheslavov (1989), Sagdeev and Zaslavsky (1987), Petrosky and Broucke (1988), Chirikov and Vecheslavov (1989), Gontis and Kaulakys (1987), Casati et al. (1987), Casati et al. (1988), Jensen et al. (1988), the Kepler map was derived almost simultaneously in astronomy and physics, by means of calculating the increments of energy and phase. However, it should be noted that the map derivation in the problem of the hydrogen atom in the microwave field is much simpler than in the restricted three-body problem in celestial mechanics, because the adopted potential model is much simpler. The energy increment in the former problem is expressed usually through the Anger functions.

7 Prehistory of the Kepler map

The second equation of the Kepler map is based on Kepler’s third law, hence the title of the map. The third law was published in 1619 in the fifth book of Harmonices Mundi (Kepler, 1619) — continuation of Mysterium Cosmographicum. A short note on this law appeared already in 1618, in A short
Could the Kepler map have been discovered much earlier than at the end of 20th century? To derive analytical expressions for the energy parameter of the Kepler map, Petrosky (1986) and Petrosky and Broucke (1988) used refined methods of mathematical physics, such as construction of new canonical variables by means of the Lie algebraic formalism (the Hori method), some elements of the KAM theory, a method of reduction of a Fourier series with a small denominator to the Fourier integral in the form of the Cauchy integral, a method of embedding the small denominator in an analytic function through a suitable analytic continuation, consideration of conditions for determining the Riemann sheet of the analytic continuation, analogies with scattering theory in quantum mechanics. These methods became available in the 20th century, and mostly in the sixties of the 20th century. However, the Kepler map as a mathematical construction, put aside from the way of its original derivation, is elementary.

The Kepler map was derived as an answer to the question, what is the long-term orbital behaviour of comets in highly eccentric orbits subject to perturbations from planets. It cannot be said that this question became actual only with apparition of the Halley comet in 1986. The highly unpredictable motion of comets is a long-standing problem in dynamical astronomy.

As shown by Valsecchi (2007), Andrey Lexell (1777a,b, 1778a,b) introduced the modern understanding of the dynamics of small Solar system bodies already in 1777–1778 (this understanding implies taking into account, first of all, the effects of resonances and encounters with planets). Later on, LeVerrier (1844, 1848, 1857), exploring the motion of the comet Lexell, discovered essential sensitivity of the trajectory to the initial conditions: the trajectory changed qualitatively upon small variations of the initial data; this was a manifestation of the phenomenon called “dynamical chaos” now (Valsecchi, 2007). Thus the scientific grounds for exploration of the new phenomenon became actual already in the middle of the 19th century. On the other hand, a mathematical derivation of a formula for the energy increment in the Kepler map (the major problem in constructing this map) could have been accomplished since 1836, when the Jacobi integral was discovered.

It is well known that a mathematically simple setting of a problem and a simple formulation of its solution do not at all imply a simple way of solving the problem. What is more, arriving at simple formulae does not at
all always require simple analytical calculations. An example can be given, when derivation of a line-sized formula required gigabytes of computer memory consumption (Shevchenko, 2008). However, such a simple mathematical construction as the Kepler map, as we have seen above, could have well been derived, because the appropriate scientific grounds and tools had become available, some 250 years after the formulation by Kepler of the third law of planetary motion, in contrast to 400 years in reality. Nevertheless the opportunity was utterly blocked by the scientific paradigm of Laplacian determinism.

8 Conclusions

The Kepler map was derived in 1980s by Petrosky (1986) and Chirikov and Vecheslavov (1986) in order to describe the long-term chaotic orbital behaviour of comets in nearly parabolic motion. Since that time this map has become paradigmatic in a number of applications in celestial mechanics and atomic physics.

We have shown that the Kepler map, including analytical formulae for its parameter, can be derived by quite elementary methods. Though discovered so recently, it could well be derived already in the middle of the 19th century. A strict mathematical derivation for the energy increment could have appeared since 1836, when the Jacobi integral was discovered.

The key word in the titles of all the pioneering papers on the Kepler map is “chaos”, i.e., dynamical chaos. This could not have been a subject of a scientific study earlier than in the second half of the 20th century. When dynamical chaos had become a central subject of studies in nonlinear dynamics, the Kepler map was immediately derived, together with other general separatrix maps (Fermi map, ordinary separatrix map, Markeev maps).

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