The logarithmic mean method for Nonlinear Conductivities at Cell Faces

Xinxin Jia, Xiaoling Sun*, Lei WANG, Hao Zhang, Xiangchun Li, Zongrui Hao, Liya Duuan

Institute of Oceanographic Instrumentation, Qilu University of Technology (Shandong Academy of Sciences), Qingdao, 266061, China.

E-mail: upcjxx@163.com. *Corresponding author’s e-mail: jiaxx@qlu.edu.cn

Abstract: In this work, logarithmic mean method, which is based on the assumption that the diffusion coefficients profile is linear between nodes, is developed to calculate diffusion coefficients at cell interfaces. The new method is evaluated by solving one dimension steady diffusion equation and convection diffusion equation with strongly non-linear diffusion coefficients. Results indicate that the logarithmic mean method can achieve high precision as the Kirchhoff integral mean method even if the number of nodes is small, while the harmonic mean method needs more nodes to achieve the same accuracy. Furthermore, the logarithmic mean method only needs the diffusion coefficients of nodes, the disadvantage of Kirchhoff method which needs concrete expression of diffusion coefficients and numerous integral operation thus can be overcome.

1. Introduction

The conservation equations governing flow and heat transfer all include diffusion term, whether they are heat conduction equations or convection diffusion equations and even N-S equations. As for practical engineering problems, the diffusion coefficients are not always constant, sometimes they may change rapidly. Some of the changes are attributable to variable thermodynamic properties, such as heat conductivity, and others are attributable to flow state, such as turbulent diffusivity. The diffusion coefficients expressions can be provided in variable thermodynamic properties problems, while the turbulent diffusivity expressions which usually depend on governing equations cannot be provided. In the discrete process of convection diffusion equations, the diffusion coefficients are defined only on the nodes, so the diffusion coefficients at interfaces can be obtained from interpolation of nodes. Patankar [1] once analyzed physical significance of arithmetic mean and harmonic mean, and he strongly recommended the harmonic mean method. Actually, the arithmetic mean method just focuses on a local description of diffusion coefficients at interfaces, while the harmonic mean method reflects continuity of diffusion flux at interfaces. The harmonic mean method by its nature is a mathematical description of series thermal resistance. When step change of thermodynamic properties happens, the logical solution can be obtained through harmonic mean method. Voller and Swaminathan [2] proposed an integral mean method which is based on Kirchhoff transformation. The integral mean method is a precise algorithm, but analytic expressions of diffusion coefficients is needed and numerous integral operations are involved. When expressions of diffusion coefficients are complicated or there are no analytic expressions, there is even more work to do in the integral mean method and which will consume more computing resources. The linear interpolation of evaluated variables to deal with evaluated variables dependent diffusion coefficients is proposed by Z.L. Liu [3], but this method also needs the analytic...
expressions of diffusion coefficients. In addition, as for the convection diffusion problems in cylindrical coordinate system, the generalized diffusion coefficients are in connection with radial coordinate \([4\sim 9]\), even though the diffusion coefficients are constant, the temperature profile is not linear, because the continuity of diffusion flux is not continuity.

In this article, first, based on the assumption that the diffusion coefficients profile is linear between nodes, the logarithmic mean method is thus developed. Second, computing performance of logarithmic mean method and other interpolation methods are compared by solving one dimension steady diffusion equation and convection diffusion equation with strongly non-linear diffusion coefficients.

2. Logarithmic mean method

For non-linear heat conduction problems, heat conductivity is a function of evaluated variable. The evaluated variables are defined on the nodes, and that is the same with the heat conductivities. In the process of computing discrete heat conduction equations coefficients, the heat conductivities at interfaces must be established beforehand. In a general way, the diffusion coefficients at interfaces can be obtained from interpolation of nodes. In the field of non-linear diffusion problems numerical calculation, there is no conclusion that which interpolation method is reasonable. And the aim of this section is to develop a new interpolation method to compute the heat conductivities at interfaces accurately and efficiently.

On the assumption that the heat conductivities profile is linear between nodes. Considering the thermal conduction between node P and E displayed in Fig. 1, defining a local coordinate system based on node P, one dimension steady source-free diffusion equation can be written as

\[
\frac{d}{dx} \left( a + bx \right) \frac{d\phi}{dx} = 0
\]

the boundary conditions are defined as

\[
\begin{align*}
    x = 0, \phi &= \phi_p \\
    x = \delta, \phi &= \phi_e
\end{align*}
\]

Solving Eq. (1) with definite conditions (Eq. (2)), the heat flux transferred from node P to E is obtained:

\[
Q_e = A_x \frac{\left( \phi_e - \phi_p \right)}{\ln\left( \left( a + b \delta \right)/a \right)} b
\]

Where the variable \( a \) and \( b \) can be expressed by the heat conductivity of node P and E, which are defined as \( \Gamma_p \) and \( \Gamma_e \). Then the Eq. (3) is rewritten as

\[
Q_e = A_x \frac{\left( \phi_e - \phi_p \right)}{\delta} \frac{\left( \Gamma_e - \Gamma_p \right)}{\ln\left( \Gamma_e/\Gamma_p \right)}
\]

According to the definition formula of heat conductivity, the heat conductivity at interface \( e \) is displayed as

\[
\Gamma_e = \frac{\Gamma_e - \Gamma_p}{\ln\left( \Gamma_e/\Gamma_p \right)}
\]

Then Eq. (5) is logarithmic mean formula for variable thermodynamic property problems, while for constant thermodynamic property problems the heat conductivity is reduced to \( \Gamma_p \) or \( \Gamma_e \).

For radial coordinate diffusion or conduction problems, the heat conduction area is not a constant,
the product of conductivity and conduction area as a variable calculated as equation (5).

3. Result & Discussion
To verify the validity of logarithmic mean method (L_M), calculations compared with the arithmetic mean method (A_M), harmonic mean method (H_M) and Kirchhoff transformation method (K_M) were performed for the test problems about heat conduction and convective diffusion with variable thermodynamic property.

3.1 The thermal conductivity problems
The key features of the proposed approach can be demonstrated by considering the following steady one-dimensional heat diffusion equation

$$\frac{d}{dx} \left( \Gamma \frac{dT}{dx} \right) = 0 \quad (6)$$

The solution is dependent on the expression of heat conductivity.

For Problem 1, the heat conductivity is specified by \( \Gamma = \Gamma_0(1+bT)^m \) and the boundary is defined as

\[
T_{|x=0} = T_0 = 0 \\
T_{|x=L} = T_L = 1
\]  

(7)

Solving Eq. (6) with above definite conditions, thus the analytic solution is obtained:

\[
(1+bT)^{m+1} = (1+bT_0)^{m+1} + \frac{(1+bT_L)^{m+1} - (1+bT_0)^{m+1}}{L} x
\]  

(8)

In programming, consider a case where \( b = 1 \), we take the number of nodes N=22. Calculating above problem by employing different interpolation method (as mentioned above), Fig. 2 and 3 shows the relative errors of computed results for different value.

As we can see from Fig. 2, for arithmetic mean method, the maximum relative error is more than 10%, and the average relative error is not less than 5%, which indicates that the calculation error for arithmetic mean method is maximum among all interpolation methods. The relative error for Kirchhoff transform method is zero, and for logarithmic mean method, the relative error is nearly zero. For harmonic mean method, there are big errors near both ends of the computational domain, and the sign symbols of errors for all nodes are different. When the number of nodes is 82, the relative errors are small, while are far larger than those of logarithmic mean method. Fig. 3 shows distribution of relative errors for \( m=10 \). We can reached the same conclusions as Fig. 2.

For problem 2, the heat conductivity is specified by \( \Gamma = 1/(1+T^3) \) and the boundary is defined as
4

\[ \begin{align*}
T|_{x=0} &= T_0 = 0 \\
T|_{x=L} &= T_L = 2
\end{align*} \tag{9} \]

The Problem 2 is designed by Voller [10]. Calculating above problem by employing different interpolation method (as mentioned above), the numerical solution and analytic solution are displayed in Fig. 4.

Fig. 4 indicates that there is nearly no difference between the analytic solution and the numerical solution obtained through logarithmic mean method or Kirchhoff transform method. While for arithmetic mean method and harmonic mean method, there is big error between the numerical solution and the analytic solution. Especially for arithmetic mean method, the calculation error is significant.

3.2 The convective diffusion Problem

The calculation performance of logarithmic mean method for convective diffusion equation is demonstrated by following problem

\[ \Phi \frac{dT}{dx} = \frac{d}{dx} \left( \Gamma \frac{dT}{dx} \right) \tag{10} \]

with \( T(0) = 0, \ T(2) = 1 \). The numerical solutions for \( \Phi = 0.01 \) and \( \Phi = 0.1 \) are obtained through employing different interpolation method. Fig. 4 exhibits the comparison between numerical solution and analytic solution.
As we can see from Fig. 5 and 6, flow velocity have no effect on numerical solution solved by different interpolation method. Logarithmic mean method achieves high calculation accuracy, just like Kirchhoff transform method. Only when the number of nodes is not less than 82, the harmonic mean method can reach the same accuracy as logarithmic mean method.

4. Conclusions
This article presented a new interpolation method (logarithmic mean method) to compute the heat conductivities at interfaces. The performance of logarithmic mean method was demonstrated by solving heat conduction and convective diffusion problem with variable thermodynamic property. Results showed that the calculation accuracy of logarithmic mean method is as high as Kirchhoff transform method. To reach the same accuracy as logarithmic mean method does, the harmonic mean method needs more nodes. In a sense, logarithmic mean method can save large storage space. Besides, logarithmic mean method just needs the diffusion coefficients on nodes, which avoids numerous integral operation. Compared with Kirchhoff transform method, logarithmic mean method can save plenty of computing time. Thus, we can consider using logarithmic mean method instead of Kirchhoff transform method to calculate vortex viscosity at cell face in turbulent flow simulation which would consume a great deal of computing resources.

Acknowledgments
This work was provided by National Natural Science Foundation of China (No. 61702308), Natural Science Funds of Shandong Academy of Sciences(kjhz2018-11), Shandong Key Contribution Projects of Science and Technology(2018YFH0705).

References
[1] S.V. Patankar, 1978, A numerical method for conduction in composite materials, flow in irregular geometries and conjugate heat transfer[J], Proceeding of Sixth International Heat Transfer Conference, Toronto, Canada, Vol.3, pp.297-302.
[2] V.R. Voller, C.R. Swaminathan, 1993, Treatment of discontinuous thermal conductivity in control-volume solutions of phase-change problems[J], Numerical Heat Transfer, Part B., Vol.24, pp.161-180.
[3] Z.L. Liu, C.F. Ma, 2003, A new numerical interpolation method for diffusion coefficients at interfaces[J], Journal of Engineering Thermophysics, 2003, Vol.24(6), pp.1031-1033.
[4] W.Q. Tao, 2001, Numerical Heat Transfer [M], second edition, Xi'an Jiaotong University Press, Xi' an, pp.88-99.
[5] Novikov D S, Kiselev V G. Effective medium theory of a diffusion - weighted signal[J]. NMR in Biomedicine, 2010, 23(7): 682-697.
[6] Tateishi A A, Ribeiro H V, Lenzi E K. The role of fractional time-derivative operators on anomalous diffusion[J]. Frontiers in Physics, 2017, 5: 52.
[7] Sun G, Zhang Y, Sun W, et al. Multi-scale prediction of the effective chloride diffusion coefficient of concrete[J]. Construction and Building Materials, 2011, 25(10): 3820-3831.
[8] Van der Ven A, Thomas J C, Xu Q, et al. Linking the electronic structure of solids to their thermodynamic and kinetic properties[J]. Mathematics and computers in simulation, 2010, 80(7): 1393-1410.
[9] K.C. Chang, U.J. Payne, 1992, Numerical treatment of diffusion coefficient at interfaces[J], Numerical Heat Transfer, Part A, Vol.21, pp.363-376.
[10] V.R. Voller, 2001, Numerical treatment of rapidly changing and discontinuous conductivities[J], International J of Heat and Mass Transfer, Vol.44, pp.4553-4556.