Noncommutativity Effects in FRW Scalar Field Cosmology

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Abstract

We study effects of noncommutativity on the phase space generated by a non–minimal scalar field which is conformally coupled to the background curvature in an isotropic and homogeneous FRW cosmology. These effects are considered in two cases, when the potential of scalar field has zero and nonzero constant values. The investigation is carried out by means of a comparative detailed analysis of mathematical features of the evolution of universe and the most probable universe wave functions in classically commutative and noncommutative frames and quantum counterparts. The influence of noncommutativity is explored by the two noncommutative parameters of space and momentum sectors with a relative focus on the role of the noncommutative parameter of momentum sector. The solutions are presented with some of their numerical diagrams, in the commutative and noncommutative scenarios, and their properties are compared. We find that impose of noncommutativity in the momentum sector causes more ability in tuning time solutions of variables in classical level, and has more probable states of universe in quantum level. We also demonstrate that special solutions in classical and allowed wave functions in quantum models impose bounds on the values of noncommutative parameters.

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1 Introduction

Scalar fields are an integral part of modern models in particle physics [1], and recently play very important roles in cosmology and have become a powerful tool to build cosmological models as well. They have key role in some of these models as current models of early cosmological inflation [2], or, in the viability of scalar field models as favorite candidates for dark matter [3]. Scalar field cosmological models have extensively been studied in the literatures, see, e.g., Refs. [4] and references therein. In the simplest interactions, a scalar field is coupled to gravity. In many cosmological models, scalar fields present degrees of freedom and appear as dynamical variables of corresponding phase space, where this point can be regarded as relevance of noncommutativity in these models.

The proposal of noncommutativity concept between space–time coordinates was introduced first by Snyder [5], and about twenty years ago, a mathematical theory, nowadays known as noncommutative geometry (NCG), has begun to take shape [6] based on this concept. In the last decade, study and investigation of physical theories in the noncommutative (NC) frame, like string and M–theory [7, 8], has caused a renewed interest on noncommutativity in the classical and quantum fields.

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In particular, a novel interest has been developed in considering the NC classical and quantum cosmology. In these studies, the influence of noncommutativity has been explored by the formulation of a version of NC cosmology in which a deformation of minisuperspace [9]–[12] or, of phase space [13] is required instead of space–time deformation. From qualitative point of view, noncommutativity in the configuration space leads to general effects, however, a non–trivial noncommutativity in momentum sector introduces distinct effects in what concern with the behavior of dynamical variables.

Our purpose in this work is to build a NC scenario for the Friedmann–Robertson–Walker (FRW) cosmology including matter field via a deformation achieved by the Moyal product [7] in classical and quantum level. We introduce effects of noncommutativity by two parameters, namely \( \theta \) and \( \beta \), which are the NC parameters corresponding to space and momentum sectors, respectively. Then, we will show that impose of noncommutativity in the momentum sector causes more ability in tuning time solutions of variables in classical level, and has more probable states of universe in quantum level.

The work is organized as follows. In Section 2, we specify a model and inspect it in the classical version within the commutative and NC frames. Section 3 considers the quantum version of this model by investigating universe wave functions and compares their properties in the commutative and NC frames. A brief conclusion is presented in the last section.

## 2 The Classical Model

We consider a classical model consisted of a cosmological system that is presented by a four-dimensional action with a non–minimally coupled scalar field to gravity in a FRW universe. To specify the NC effects of the model, we first treat the commutative version, and then the NC one in the following subsections.

### 2.1 Commutative Phase Space

A general action for a non–minimally coupled scalar field can be described by

\[
A = \int \sqrt{-g} \left[ f(\phi)R - \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi) \right] d^4x, \tag{1}
\]

where \( g \) is the determinant of the metric \( g_{\mu\nu} \), \( R \) is the Ricci scalar, \( V(\phi) \) and \( f(\phi) \) are potential and coupling functions of the scalar field, respectively. We assume a homogeneous scalar field, that is \( \phi = \phi(t) \), and the following FRW metric of the minisuperspace

\[
ds^2 = -N^2(t)dt^2 + a^2(t) \left( \frac{dr^2}{1-kr^2} + r^2d\Omega^2 \right), \tag{2}
\]

where \( N(t) \) is a lapse function, \( a(t) \) is a scale factor and \( k \) specifies geometry of the universe. Substituting the metric (2) in action (1), one obtains the Lagrangian density

\[
\mathcal{L} = 6af\left( kN - \frac{\dot{a}^2}{N} \right) - 6\frac{a^2\dot{a}f}{N} + a^3N\left( \frac{\dot{\phi}^2}{2N^2} - V \right), \tag{3}
\]

where total time derivative terms have been neglected.

We restrict our considerations to the case of conformally coupled scalar field model [12, 14]. The general reason for selecting such a scalar field is that it allows exact solutions in simple cases, as those discussed along this work, and it is rich enough to be useful as a probe for the significant modifications that NCG introduces in classical and quantum cosmologies. That is, we set \( f(\phi) = 1/(2\kappa) - \xi\phi^2/2 \), where \( \kappa = 8\pi G/c^4 \) and \( \xi \) is the non–minimal coupling parameter that represents a direct coupling between the scalar field and curvature, and has an arbitrary value. Obviously, the case \( \xi = 0 \) is the minimally coupling situation, however, as mentioned, we consider the conformal coupling case, i.e. \( \xi = 1/6 \), and employ the unites \( \hbar = 1 = c \) and \( \kappa = 3 \).

Based on these assumptions and rescaling the scalar field as
\[ \chi = a\phi / \sqrt{2} \]

the Lagrangian (3) reads
\[ \mathcal{L} = kNa - \frac{a\dot{a}^2}{2} + \frac{a\dot{\chi}^2}{2} - \frac{kN\chi^2}{a} - a^3 NV. \]  

Thus, the corresponding Hamiltonian is
\[ \mathcal{H} = N \left( -\frac{p_a^2}{4a} + \frac{p_{\chi}^2}{4a} - ka + \frac{k\chi^2}{a} + a^3 V \right), \]

where \( p_a \) and \( p_{\chi} \) are the canonical conjugate momenta. For the conformal time gauge selection, namely \( N = a \), one gets
\[ \mathcal{H} = -\frac{p_a^2}{4} + \frac{p_{\chi}^2}{4} - ka^2 + k\chi^2 + a^4 V. \]

Then, the Hamilton equations are
\[ \dot{a} = \{a, \mathcal{H}\} = -\frac{1}{2}p_a, \]
\[ p_a = \{p_a, \mathcal{H}\} = 2ka - 4a^3 V + \sqrt{2}\chi a^2 V', \]
\[ \dot{\chi} = \{\chi, \mathcal{H}\} = \frac{1}{2}p_{\chi}, \]
\[ p_\chi = \{p_\chi, \mathcal{H}\} = -2k\chi - \sqrt{2}a^3 V', \]

where the prime denotes derivative with respect to \( \phi \).

In this work, in order to proceed further, we simply treat two special cases for the potential function, namely when there is no potential and when there is a non-zero constant value potential, \( V = V_0 \).

In free potential case, solutions to equations (7), corresponding to the values of index curvature, with the Hamiltonian constraint, \( \mathcal{H} \approx 0 \), are as follows
\[ k = 1 : \left\{ \begin{array}{l} a(t) = A_1 \cos t + A_2 \sin t \quad \text{and} \quad \chi(t) = B_1 \cos t + B_2 \sin t, \\end{array} \right. \]  

with constraint: \( A_1^2 + A_2^2 = B_1^2 + B_2^2 \),  

\[ k = -1 : \left\{ \begin{array}{l} a(t) = A_3 e^t + A_4 e^{-t} \quad \text{and} \quad \chi(t) = B_3 e^t + B_4 e^{-t}, \end{array} \right. \]  

with constraint: \( A_3 A_4 = B_3 B_4 \),  

\[ k = 0 : \left\{ \begin{array}{l} a(t) = A_5 t + A_6 \quad \text{and} \quad \chi(t) = B_5 t + B_6, \end{array} \right. \]  

with constraint: \( A_5^2 = B_5^2 \),

where \( A_i \)'s and \( B_i \)'s are constants of integration.

We will compare these solutions with their NC analogues in the next section, where we will also discuss the case of non-zero constant potential along with its NC correspondent.

### 2.2 Noncommutative Phase Space

Noncommutativity in classical physics is described by the Moyal product law (shown by the \( \ast \) notation in below) between two arbitrary functions of phase space variables, namely \( \zeta^a = (x^i, p^j) \) for \( i = 1, \cdots, l \) and \( j = l + 1, \cdots, 2l \), as \([7]\)
\[ (f \ast g)(\zeta) = \exp \left[ \frac{1}{2} \alpha^{ab} \partial_a^{(1)} \partial_b^{(2)} \right] f(\zeta_1)g(\zeta_2) \bigg|_{\zeta_1 = \zeta_2 = \zeta}, \]  

\[ (f \ast g)(\zeta) = \exp \left[ \frac{1}{2} \alpha^{ab} \partial_a^{(1)} \partial_b^{(2)} \right] f(\zeta_1)g(\zeta_2) \bigg|_{\zeta_1 = \zeta_2 = \zeta}, \]
such that

\[
(a_{ab}) = \begin{pmatrix}
\theta_{ij} & \delta_{ij} + \sigma_{ij} \\
-\delta_{ij} - \sigma_{ij} & \beta_{ij}
\end{pmatrix},
\]  
(12)

where \(a, b = 1, 2, \ldots, 2l\), \(\theta_{ij}\) and \(\beta_{ij}\) are assumed to be elements of real and antisymmetric matrices, \(\sigma_{ij}\) is a symmetric matrix (which can be written as a combination of \(\theta_{ij}\) and \(\beta_{ij}\)), and dimension of the classical phase space is \(2l\). The deformed or modified Poisson brackets are defined as

\[
\{f, g\}_\alpha = f \ast g - g \ast f,
\]  
(13)

and also the modified Poisson brackets of variables are

\[
\{x_i, x_j\}_\alpha = \theta_{ij}, \quad \{x_i, p_j\}_\alpha = \delta_{ij} + \sigma_{ij} \quad \text{and} \quad \{p_i, p_j\}_\alpha = \beta_{ij}.
\]  
(14)

The simplest way to study physical theories within the NCG is replacement of the Moyal product with the ordinary multiplication. Actually, where variables of classical phase space obey the usual linear, non–canonical, transformation

\[
\text{with some conditions on the NC parameters, which we will specify for our model in below.}
\]

Although, for a compatible extension, the transformation (15) must have an inverse and this imposes Poisson brackets represented by relations (14) and (16) must be considered as distinct relations.

In geometrical language, the usual Poisson brackets are mapped to the modified Poisson brackets (15) can be considered as an extension of the classical mechanics to the NC classical mechanics.

Furthermore, let \(\mathcal{H} = \mathcal{H}(x_i, p_i)\) be the Hamiltonian of a system in the commutative case, we shift the canonical variables through (15) and assume that the functional form of the Hamiltonian in the NC case is still the same as the commutative one, i.e.

\[
\mathcal{H}_{\text{nc}} \equiv \mathcal{H}(x'_i, p'_i) = \mathcal{H}\left(x_i - \frac{1}{2} \theta_{ij} p^j, p_i + \frac{1}{2} \beta_{ij} x^i\right).
\]  
(17)

This function is defined on the commutative space and therefore, the equations of motion for unprimed variables are obviously \(\dot{x}^i = \partial \mathcal{H}_{\text{nc}}/\partial p_i\) and \(\dot{p}^i = -\partial \mathcal{H}_{\text{nc}}/\partial x_i\). Evidently, the effects due to the noncommutativity arise by terms including the parameters \(\theta_{ij}\) and \(\beta_{ij}\).

Let us now proceed to study the behavior of the model in a phase space with deformed Poisson brackets (16) such that the minisuperspace variables do not commute with each other. For our model, the two gravitational degrees of freedom are the minisuperspace coordinates \((a, \chi)\). The corresponding phase space variables are \((x^1, x^2, p^1, p^2) = (a, \chi, p_a, p_\chi)\), where \(p_a\) and \(p_\chi\) are the linear minisuperspace momenta. We assume the NC parameters

\[
\theta^{12} \equiv \theta \geq 0 \quad \text{and} \quad \beta^{12} \equiv 4\beta \geq 0,
\]  
(18)

where \(\sigma^{12} = \theta \beta\), where \(\theta\) and \(\beta\) are constants. Thus, after making the transformations

\[
a \to a - \frac{\theta}{2} p_\chi, \quad p_a \to p_a + 2 \beta \chi, \quad \chi \to \chi + \frac{\theta}{2} p_a \quad \text{and} \quad p_\chi \to p_\chi - 2 \beta a,
\]  
(19)
in (6), the NC Hamiltonian will be

\[ H_{\text{nc}} = -\frac{1}{4} k\theta^2 (p_a^2 - p_\chi^2) + (k\theta - \beta)(x p_\chi + a p_a) + (\beta^2 - k)(a^2 - \chi^2) + (a - \frac{1}{2} \theta p_a)^4 \tilde{V}, \]  

(20)

where \( \tilde{V} = V(\tilde{\phi}) \) is a modified potential with \( \tilde{\phi} = \sqrt{2}(x + \theta p_\chi)/(a - \theta p_a/2) \). Besides, the inverse transformation of (19) exists when its determinant is not zero, that is when \( \theta \beta \neq 1 \). Now, the equations of motion for the NC Hamiltonian are

\[
\begin{align*}
\dot{t} &= -\frac{1}{2}(1 - k\theta^2) p_a + (k\theta - \beta) \chi + \frac{1}{\sqrt{2}} \theta (a - \frac{1}{2} \theta p_a)^3 \tilde{V}', \\
p_a &= 2(k - \beta^2) a + (k - \theta) x p_\chi - 4(a - \frac{1}{2} \theta p_a)^3 V + \sqrt{2}(x + \frac{1}{2} \theta p_a)(a - \frac{1}{2} \theta p_a)^2 \tilde{V}', \\
\dot{\chi} &= (k\theta - \beta) a + \frac{1}{2}(1 - k\theta^2) x p_\chi - 2\theta(a - \frac{1}{2} \theta p_a)^3 V + \frac{1}{\sqrt{2}} \theta (x + \frac{1}{2} \theta p_a)(a - \frac{1}{2} \theta p_a)^2 \tilde{V}', \\
p_\chi &= (\beta - k\theta) p_a + 2(\beta^2 - k) \chi - \sqrt{2}(a - \frac{1}{2} \theta p_a)(\tilde{V})',
\end{align*}
\]

(21)

where \( \tilde{V}' = d\tilde{V}/d\tilde{\phi} \). Comparing equations (21) with (7) shows that, in general, equations of motion are coupled in the NC case and for \( \theta = 0 = \beta \), equations (21) reduce to (7) as expected.

In free potential case, equations (21) read

\[
\begin{align*}
\dot{t} &= -\frac{1}{2}(1 - k\theta^2) p_a + (k\theta - \beta) \chi, \\
p_a &= 2(k - \beta^2) a + (k - \theta) x p_\chi, \\
\dot{\chi} &= (k\theta - \beta) a + \frac{1}{2}(1 - k\theta^2) x p_\chi, \\
p_\chi &= (\beta - k\theta) p_a + 2(\beta^2 - k) \chi.
\end{align*}
\]

(22)

By eliminating momenta variables, one gets

\[
\begin{align*}
\dot{a} &= -m^2 a + 2(k \theta - \beta) \dot{\chi}, \\
\dot{\chi} &= -m^2 \chi + 2(k \theta - \beta) \dot{a},
\end{align*}
\]

(23)

with the Hamiltonian constraint

\[
(\beta - 1)(a^2 - \chi^2) = \text{constant if } 1 - k\theta^2 = 0, \]

or

\[
m^2 (a^2 - \chi^2) - (\dot{a}^2 - \dot{\chi}^2) = 0 \quad \text{if } 1 - k\theta^2 \neq 0, \]

(24)

where \( m^2 \equiv k(1 - \beta^2) \). It is noticeable that when \( k = 0 \), the NC parameter \( \theta \) is removed from equations (23), that is, in the spatially flat FRW universe, the motion equations are affected only by the NC parameter \( \beta \).

If one takes \( \Delta \equiv (1 - k\theta^2)(\beta^2 - k) \), real solutions of equations (23) can be written as

\[
\Delta > 0 : \begin{cases} 
    a(t) &= e^{(k\theta - \beta)t}(A_1 \sinh \sqrt{\Delta} t + B_1 \cosh \sqrt{\Delta} t) + e^{-(k\theta - \beta)t}(C_1 \sinh \sqrt{\Delta} t + D_1 \cosh \sqrt{\Delta} t), \\
    \chi(t) &= e^{(k\theta - \beta)t}(A_1 \sinh \sqrt{\Delta} t + B_1 \cosh \sqrt{\Delta} t) - e^{-(k\theta - \beta)t}(C_1 \sinh \sqrt{\Delta} t + D_1 \cosh \sqrt{\Delta} t),
\end{cases}
\]

(25)

and

\[
\Delta < 0 : \begin{cases} 
    a(t) &= e^{(k\theta - \beta)t}(A_2 \sin \sqrt{-\Delta} t + B_2 \cos \sqrt{-\Delta} t) + e^{-(k\theta - \beta)t}(C_2 \sin \sqrt{-\Delta} t + D_2 \cos \sqrt{-\Delta} t), \\
    \chi(t) &= e^{(k\theta - \beta)t}(A_2 \sin \sqrt{-\Delta} t + B_2 \cos \sqrt{-\Delta} t) - e^{-(k\theta - \beta)t}(C_2 \sin \sqrt{-\Delta} t + D_2 \cos \sqrt{-\Delta} t),
\end{cases}
\]

(26)

where \( A_i \)'s, \( B_i \)'s, \( C_i \)'s and \( D_i \)'s are constants of integration.

\footnote{Note that, the condition \( 1 - k\theta^2 = 0 \) is possible only when \( k = 1 \) and hence \( \theta = 1 \).}
The quantity \( \Delta \) is always positive when \( k = 0, -1 \), hence, in general, solutions (25) are in the form of
\[
\sum_{j=1}^{2} \left( b e^{\delta_j t} + c e^{-\delta_j t} \right) ,
\]
with \( b \) and \( c \) as new constants, and
\[
\delta_1 \equiv k\theta - \beta + \sqrt{\Delta} \quad \text{and} \quad \delta_2 \equiv k\theta - \beta - \sqrt{\Delta} .
\]
Comparing solutions (27) with its commutative analogues (9) and (10) shows that for \( k = -1 \), solutions are still hyperbolic, but with an extra coefficient in the exponent that depends on the NC parameters and gives enough room for better adjustments. In the case \( k = 0 \), one has \( \delta_1 = 0 \) and \( \delta_2 = -2\beta \), and hence, the time dependence of solutions have been modified from linear in commutative case to hyperbolic in the NC case, where the former solution is not suitable for an accelerating universe, though the latter one is capable to be adjusted with the observed accelerated expansion. Also, in the late time, solution (27) describes a de Sitter universe for which its cosmological constant is written in terms of the NC parameter as \( \Lambda/3 = 4\beta^2 \).

For \( k = 1 \) geometry, one has \( \Delta = (1-\theta^2)(\beta^2-1) \), which can be positive or negative or zero, depends on the different choices of the NC parameters. That is, when \((\theta > 1, \beta < 1) \) or \((\theta < 1, \beta > 1) \), \( \Delta \) is positive. Hence, we have solutions (25), however, in this case the NC parameters have upper and lower bounds. When \( \Delta < 0 \), existence of oscillating solutions is provided when both NC parameters simultaneously have a lower bound, namely \( \theta \) and \( \beta > 1 \), or an upper bound, namely \( \theta \) and \( \beta < 1 \). The commutative solutions (8) are oscillating with the period of \( 2\pi \) and constant amplitudes, whereas the NC solutions (26) have the period of \( 2\pi/\sqrt{-\Delta} \) and varying amplitudes with factor \( e^{i(\theta-\beta)t} \). Note that, the choice \( \theta = \beta \), leads to a constant amplitude as the commutative case, but with a period of \( 2\pi/[1-\theta^2] \). The case \( \Delta = 0 \) is possible when \( \theta = 1 \) or when \( \beta = 1 \), where solutions are again hyperbolic. The case \( k = 1 \) and \( \theta = 1 \), resembles the condition \( 1-k\theta^2 = 0 \) with arbitrary \( \beta \), however, with the value of \( \beta = 1 \) one gets trivial constant solutions.

Now, we treat a non–zero positive constant potential \( V = V_o \). Since, in general, the motion equations are not sufficiently simple to be solved, we restrict ourself to the case \( k = 0 \) and \( \beta = 0 \). Thus, after a little algebra, equations (21) give
\[
\dot{a}^2 = V_o(a - v_o\theta)^4 + d \quad \text{and} \quad \chi(t) = \chi_o + v_o t - \theta \dot{a} ,
\]
where \( \chi_o, d \) and \( v_o \equiv p_\chi/2 \) are constants and the Hamiltonian constraint gives \( d = v_o^2 \).

By setting \( \theta = 0 \), one obtains the commutative equations as
\[
\dot{a}^2 = V_o a^4 + v_o^2 \quad \text{and} \quad \chi(t) = \chi_o + v_o t .
\]
Solutions of \( a(t) \) are in terms of the Jacobi elliptic functions, i.e.
\[
a(t) = \pm D \text{ sn} \left( i\sqrt{V_o}Dt, i \right) ,
\]
with \( D \equiv [v_o/(i\sqrt{V_o})]^{1/2} \) and the initial condition \( a(0) = 0 \). These two solutions are periodic functions with respect to the time, and for \( V_o < 0 \), they qualitatively behave as sine functions with amplitude \( |D| \). For \( V_o > 0 \), solution (30) qualitatively behaves as a tangent function and its diagram is plotted in Fig. 1 (left) for numerical values \( v_o = 1, \chi_o = 2, \theta = 2 \) and \( V_o = 1 \). Obviously, physical solution corresponds to the positive part of this diagram.

Equations (28) and (29) show that the effect due to the noncommutativity is a shift in the scale factor with a magnitude \( |v_o\theta| \) that changes the zero point of the scale factor. This effect can remove a negative scale factor when the constant potential is negative. That is, in such a case, the corresponding NC solution is \( v_o\theta \pm D \text{ sn} \left( i\sqrt{V_o}Dt, i \right) \), where by choosing \( v_o\theta \geq |D| \) with \( v_o > 0 \), then the resulted scale factor will be positive. Clearly, plot of \( a(t) \) in the NC case is the same as the commutative one with a vertical transfer of magnitude.
The linear behavior of the scalar field in the commutative case is altered by an additional term $-\theta \dot{a}$. Two plots of the scalar field in the NC case are sketched in Fig. 1 (right) for numerical values $v_o = 1$, $\chi_o = 2$ and $\theta = 2$, one with $V_0 = -1$ (solid line) and one with $V_0 = +1$ (dashed line). Note that, in a positive potential, the scalar field goes to $\pm \infty$ for very late time (when $t$ goes to infinity), whereas, in a negative potential, it goes to $+\infty$ monotonically.

In terms of the cosmic time, $d\tau = adt$, the first equation in (29) may have a solution, for $V_0 > 0$, as
\[
a(\tau) \propto \left[ \sinh \left(2\sqrt{V_0} \tau\right) \right]^{1/2}.
\]
(31)
The behavior of such a scale factor in the late cosmic time, i.e. $\tau \gg 0$, is
\[
a(t) \propto \frac{1}{t_1 - t},
\]
(32)
where $t_1$ is a constant such that if $\tau \to \infty$, then $t \to t_1$. Hence, in this limit, the scalar field in the NC case grows proportion to $\theta$ as $\theta(t_1 - t)^{-2}$, and deviation from the linear time dependence becomes very large, as one can recognize it in Fig. 1 (left).

3 The Quantum Model

We proceed to quantize the cosmological model given by the action (1) in the case of free potential, such that the canonical quantization of the phase space leads to the Wheeler–DeWitt (WD) equation, $\hat{H}\Psi = 0$, where $\hat{H}$ is the Hamiltonian operator and $\Psi$ is a wave function of universe. For arguments about the quantization based on WD equation in the FRW universe including matter fields, see, e.g., Refs. [15]. We employ the usual canonical transition from classical to quantum mechanics via the generalized Dirac quantization from the Poisson brackets to the quantum commutators, i.e. $\{\} \to -i[\]$. Then, as the classical approach, we investigate the commutative and NC frames in the following subsections.

3.1 Commutative Frame

As usual, the operator form of Hamiltonian (6) can be acquired by the replacements $p_a \to -i\partial_a$ and $p_\chi \to -i\partial_\chi$. Assuming a particular factor ordering, the corresponding WD equation, for $V = 0$, is
\[
\left[ \frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial \chi^2} + 4k(\chi^2 - a^2) \right] \Psi(a, \chi) = 0.
\]
(33)
Considering the following change of variables
\[
a = \rho \cosh \varphi \quad \text{and} \quad \chi = \rho \sinh \varphi,
\]
(34)
equation (33) reads
\[
\left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} - 4k\rho^2 \right) \Psi(\rho, \varphi) = 0.
\]
(35)
We assume solutions of equation (35) as a product ansatz
\[
\Psi(\rho, \varphi) = \psi(\rho)e^{2i\alpha\varphi},
\]
(36)
where $\alpha$ is a real constant. Substitution of (36) in equation (35) leads to
\[
\psi'' + \frac{\psi'}{\rho} + 4\left( \frac{\alpha^2}{\rho^2} - k\rho^2 \right) \psi = 0,
\]
(37)
where the prime denotes derivative with respect to $\rho$. The well-defined eigenfunctions of equation (37), considering boundary conditions, can be written as
\[
\psi_\alpha(\rho) \propto \begin{cases} 
J_{i\alpha}(-k\rho^2) & \text{for } k = -1 \\
\cos(2\alpha \ln \rho) & \text{for } k = 0 \\
K_{i\alpha}(k\rho^2) & \text{for } k = 1,
\end{cases}
\] (38)
where the functions $J_{i\nu}$ and $K_{i\nu}$ are the first and modified second kind of the Bessel functions, respectively. Hence, the wave packet corresponding to (38) is
\[
\Psi(\rho, \varphi) = \int_{-\infty}^{+\infty} C_\alpha \psi_\alpha(\rho)e^{2i\alpha\varphi}d\alpha,
\] (39)
where $C_\alpha$ can be taken \([9, 10]\) to be a shifted Gaussian weight function with constants $b$ and $c$, i.e. $e^{-(b(\alpha-c))^2}$.

Fig. 2 shows plots of the probability $|\Psi|^2$ for $k = 0$ and $k = 1$ with values $b = 1 = c$, in the range of $0 < \rho < 10$ and $-5 < \varphi < 5$. The corresponding plot for $k = -1$ is qualitatively similar to the case $k = 1$.

### 3.2 Noncommutative Frame

We assume that, in a general NC quantum phase space, the coordinate operators of the FRW minisuperspace and their generalized momenta obey the star deformed Heisenberg algebra, like the ones in noncommutative quantum mechanics, as [16]
\[
[\hat{x}_i, \hat{x}_j]_\alpha = i\theta_{ij}, \quad [\hat{x}_i, \hat{p}_j]_\alpha = i(\delta_{ij} + \sigma_{ij}) \quad \text{and} \quad [\hat{p}_i, \hat{p}_j]_\alpha = i\beta_{ij}.
\] (40)
The notations and definitions are the same as in the NC classical model. Now, the corresponding noncommutative WD equation can be written by replacing operator product with the Moyal product, namely
\[
\hat{H}_{nc}(\hat{x}_i, \hat{p}_i) \ast \Psi(x, p) = 0.
\] (41)
It is well-known in noncommutative quantum mechanics [16] that the original phase space and its symplectic structure can be modified to reformulate relations in the commutative algebra when the new variables, as in (15), are introduced. Hence, the original equation (41) for these new variables reads [17]
\[
\hat{H}_{nc} \left( \hat{x}_i - \frac{1}{2}\theta_{ij}\hat{p}_j, \hat{p}_i + \frac{1}{2}\beta_{ij}\hat{x}_j \right) \Psi(x, p) = 0.
\] (42)
Therefore, the noncommutative WD equation corresponding to relation (20), with $\theta \beta \neq 1$ and for $V = 0$, can be written as
\[
\left[ (1-k\theta^2)(\partial_a^2 - \partial_\chi^2) - 4i(k\theta - \beta)(\chi \partial_a + a \partial_\chi) + 4(\beta^2 - k)(a^2 - \chi^2) \right] \Psi(a, \chi) = 0,
\] (43)
that, with the change of variables (34), reads
\[
\left[ (1-k\theta^2)\left( \frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\rho \partial \rho} - \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \right) - 4i(k\theta - \beta) \frac{\partial}{\partial \varphi} - 4(k - \beta^2)\rho^2 \right] \Psi = 0.
\] (44)
If $1 - k\theta^2 \neq 0$, by the ansatz (36), equation (44) reduces to
\[
\psi'' + \frac{\psi'}{\rho} + 4 \left( \frac{\alpha^2}{\rho^2} - \epsilon \rho^2 + 2\gamma \alpha \right) \psi = 0,
\] (45)
where $\epsilon \equiv (k - \beta^2)/(1 - k\theta^2)$ and $\gamma \equiv (k\theta - \beta)/(1 - k\theta^2)$. Before discussing solutions of (45), we enumerate a few features of this equation. Firstly, it reduces to equation (37) when one sets $\epsilon = k$ and $\gamma = 0$, where a trivial case is $\theta = 0 = \beta$, as expected. Secondly, it is interesting that, in the two following cases where the noncommutativity is still present, equation (45) again reduces to equation (37). These two cases are $k = 1$ with $\theta = \beta \neq 1$, and $k = 0$ with $\beta = 0$ and arbitrary $\theta$. In another words, in these cases, general form of the NC wave functions are the same as the commutative solutions (38). Finally, in the case $k = 0$, solutions to equation (45) does not depend on the $\theta$ parameter, which is also a common feature in the classical model.

A particular solution of equation (45) can be written in terms of the Whittaker functions, $W_{\mu,\nu}$ and $M_{\mu,\nu}$, as

$$\psi_\alpha (\rho) = \rho^{-1} \left[ A_\alpha M_{\mu,\nu} \left( 2\sqrt{\epsilon} \rho^2 \right) + B_\alpha W_{\mu,\nu} \left( 2\sqrt{\gamma} \rho^2 \right) \right], \quad (46)$$

where $A_\alpha$ and $B_\alpha$ are superposition constants, $\mu \equiv \alpha \gamma/\sqrt{\epsilon}$ and $\nu \equiv i\alpha$. The both Whittaker functions, even in classically forbidden regions, are convergent when $\epsilon$ is negative\(^3\) that can obviously occur for $k = 0$ and $k = -1$. Also, $\epsilon$ can be negative in the case $k = 1$ for special values of the NC parameters, we will discuss these situations in below.

In the case $k = 0$ for which $\epsilon = -\beta^2$, one can write solution (46) with one of the Whittaker functions, e.g., $M_{\mu,\nu}$ and gets its corresponding wave packet as

$$\Psi (\rho, \varphi) = \rho^{-1} \int_{-\infty}^{\infty} e^{-b(a-c)^2} M_{\alpha,i\alpha} \left( 2i\beta \rho^2 \right) e^{2ia\varphi} d\alpha. \quad (47)$$

Fig. 3 shows the probability $|\Psi|^2$ corresponding to (47) for $\beta = 1/10$ and the Gaussian constants $b = 1 = c$. The comparison of this figure with the left diagram in Fig. 2 identifies that, although the most probable state of a commutative universe is around $\varphi = 0$, the most probable state related to a NC universe is shifted for $\varphi$ less than zero. Also, another difference in those two figures is the displacement of peaks to greater values of $\rho$ in a NC universe with respect to a commutative one. By consecutive diagrams, it can be shown that this difference is also more manifested when $\beta$ gets smaller values.

As mentioned before, the case $\epsilon < 0$ can also be attained when $k = 1$, for which $\epsilon = (1 - \beta^2)/(1 - \theta^2)$. This quantity is negative when one of the inequalities ($\theta < 1$ and $\beta > 1$) or ($\theta > 1$ and $\beta < 1$) are held. The first index of the Whittaker functions, for both of these inequalities, is also a pure imaginary number. When $\theta \beta = 1$, wave packets resemble those of the case $k = 0$, but this choice violates the existence of the inverse transformation for (19). On the other hand, plots of wave packets for $\theta \beta \neq 1$, in general, show that they are not suitable for description of a real universe. However, choosing special values, such as considering the first inequality with $\theta = 0$ and $\beta$ very large values that $\beta^2 \pm 1 \approx \beta^2$ or considering the second inequality with $\beta = 0$ and $\theta$ very large values that $\theta^2 \pm 1 \approx \theta^2$, one can get $\mu \approx i\alpha$ and hence again, construct well–behaved wave packets in a closed universe, which their corresponding plots are qualitatively similar to Fig. 3. For instance, Fig. 3 does also illustrate a similar plot of probability for $\beta \approx 0$ and $\theta \approx 10$ in this case, where the visible contrast between such a figure and its commutative analogue, Fig. 2 (right), implies that the probability of expansion in a NC universe is more than a commutative counterpart. Also, the most probable state of a commutative universe is around $\varphi = 0$ and $\rho = 0$, but the most probable state related to a NC universe is shifted for $\varphi$ less than zero and $\rho$ greater than zero. Besides, there is displacement of peaks to greater values of $\rho$ in a NC universe with respect to a commutative one. This effect is more manifested when $\theta$ gets greater values. Hence, there are different possible universes (states) from which, our present universe could be evolved and tunneled in the past from one state to another.\(^4\)

The above discussion, in the latter paragraph, is completely fulfilled when $k = -1$, except that there is no bound on the NC parameters, for $\epsilon$ is always negative in this case.

\(^2\)The case $k = -1$ with $\theta = -\beta$ is not valid, for our assumption (18).

\(^3\)For properties of the Whittaker functions and their indices, see, e.g., Refs. [18].

\(^4\)Such a kind of arguments can also be found in, e.g., Ref. [10].
For when $\epsilon$ is positive, the Whittaker term of $M_{\mu,\nu}$, in classically forbidden region, is divergent. Besides, the both Whittaker terms, when $\epsilon > 0$, are proportional to $\exp(-\sqrt{\epsilon} \rho^2)$, and hence, are quickly damped as $\rho$ grows.

We should mention that $\epsilon$ also vanishes for the case $k = 1$ with $\beta = 1$, which leads to $\gamma = -1/(1 + \theta)$. In this case, solution of equation (45) can be considered as

$$\psi_\alpha(\rho) \propto J_{i\alpha}(2i\gamma \sqrt{\alpha} \rho).$$

(48)

This solution, as can easily be checked, is not well defined for constructing wave packets and hence, we do not consider it.

The condition $1 - k\theta^2 = 0$ is equivalent to $k = 1$ and $\theta = 1$, for which equation (44) reduces to

$$\left[ i(1 - \beta) \frac{\partial}{\partial \varphi} + (1 - \beta^2) \rho^2 \right] \Psi = 0.$$

(49)

Its solution is

$$\Psi = R(\rho)e^{i(1+\beta)\varphi \rho^2},$$

(50)

where $R(\rho)$ is any differentiable function. The probability of this wave function is independent of the $\beta$ parameter.

4 Conclusions

In this work, we have carried out an investigation into the role of NCG in cosmological scenario by introducing a NC deformation in the algebra of phase space variables almost along the same lines proposed in Ref. [12]. The space is generated by a conformal scalar field when is non–minimally coupled to geometry whose background is the FRW metric, in all three cases $k = -1, 0, 1$. The noncommutativity has been introduced in the both, space and momentum, sectors with more attention on the role of the NC parameter $\beta$. The investigation has been carried out by means of a comparative analysis of the evolution of universe and the most probable universe wave functions in classically commutative and NC frames and quantum counterparts.

In absence of the scalar field potential function, the classical exact solutions have been obtained in both commutative and NC frames. Contrary to the commutative case, the NC solutions can be regulated with the aid of both NC parameters. Especially, in spatially flat universes, the major player in tuning solutions is the $\beta$ parameter, where it can be employed to adjust time dependent solutions with the observational data, e.g., the accelerating expansion rate. Interestingly, we have found that the existence of particular solutions imposes bounds on the values of NC parameters.

In the presence of a non–zero constant potential and absence of $\beta$, the time dependence of the scalar field changes contrary to the scale factor which is just shifted by a constant value proportional to the $\theta$ parameter. This shift can remove negative scale factors when negative constant potentials are used. The scalar field deviation from linear time dependence of the commutative case, for a positive constant potential, becomes more apparent in the late cosmic time.

One expects that when noncommutativity effects are turned on in the quantum scenario they should introduce significant modifications. To see this, the corresponding quantum cosmology has also been considered and the exact solutions, through the WD equation in the commutative and NC frames, are obtained in order to investigate differences in the solutions. As in the case of classical model, the $\beta$ parameter is the only one that is responsible for NC effects in spatially flat universes. Comparing the numerical diagrams of the probability in commutative and NC cases, especially in the case of positive and negative curvature indexes, shows that expansion in a NC universe is more probable, depending on the values of the NC parameters, than a commutative counterpart. In the flat case, existence of universes with more possible states invokes smaller values of $\beta$ in NC frames. Also, existence of more probable states and transferring peaks of probability to greater values of $\rho$, defined in (34), in the NC frames with respect to the commutative ones are intensified when $\beta$ gets
smaller values. Also, the latter property can qualitatively be apparent in the closed and open cases when, for instance, $\beta = 0$ with $\theta$ sufficiently large values, or $\theta = 0$ with $\beta$ sufficiently large values.

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Figure 1: The scale factor in the commutative case for $V_0 = 1$ (left) and the scalar field in the NC case (right) for $V_0 = -1$ (solid line) and $V_0 = 1$ (dashed line), all plots with numerical values $v_o = 1$, $\chi_o = 2$ and $\theta = 2$.

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**Figure 2**: The probability $|\Psi|^2$ in the commutative case for $k = 0$ (left) and $k = 1$ (right), both with the Gaussian constants $b = 1 = c$.

**Figure 3**: The probability $|\Psi|^2$ in the NC case for $k = 0$, $\beta = 1/10$ and the Gaussian constants $b = 1 = c$. 