Abstract

We take advantage of the combinatorial interpretations of many sequences of polynomials of binomial type to define a sequence of symmetric functions corresponding to each sequence of polynomials of binomial type. We derive many of the results of Umbral Calculus in this context including a Taylor’s expansion and a binomial identity for symmetric functions. Surprisingly, the delta operators for all the sequences of binomial type correspond to the same operator on symmetric functions.

Les suites de fonctions symétriques de type binomial

On s’appuie ici sur les interprétations combinatoires de nombreuses suites de polynômes de type binomial pour définir une suite de fonctions symétriques associée à chaque suite de polynômes de type binomial. On retrouve dans ce cadre, de nombreux résultats du calcul ombral, en particulier une version de la formule de Taylor et la formule d’identité du binôme pour les fonctions symétriques. On s’aperçoit que les opérateurs différentiels de degré un pour toutes les suite de polynômes de type binomial correspondent à un opérateur unique sur les fonction symétriques.

Dedicated to the memory of
Rabbi Selig Starr

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I Linear Sequences

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Although it is well known that many sequence of polynomials of binomial type \( p_n(x) \) enumerate the number of functions from an \( n \)-element set to an \( x \)-element set enriched with a certain type of structure on each block. There we show that if one allow pseudospecies then every sequence of polynomials of binomial type is of this form. Moreover, by counting the enriched functions more carefully we define sequence of symmetric functions of binomial type \( p_n(y) \). Using these methods we rederive many classical results of the theory of symmetric functions, and of umbral calculus as well as a few new ones. We define a shift operator and prove the accompanying binomial theorem, and then classify the set of operators invariant under it. The algebra of shift-invariant operators turns out to naturally isomorphic to the dual Hopf algebra of symmetric functions. Finally, we extend all these ideas to bases \( p_{\lambda}(y) \) through the use of genera—a generalization of Joyal’s species.

It is thought that the ideas in this paper would make an excellent introduction to umbral calculus, species, and symmetric functions for a beginning graduate student in combinatorics.
1.1 Combinatorial Interpretations

To devise an Umbral Calculus on symmetric functions we must first study the combinatorial properties of
symmetric function. The linear sequence of symmetric functions associated to each sequence of polynomials
of binomial type is closely related to the sets of functions which these polynomials enumerate.

Classically, a sequence of polynomials \( p_n(x) \) (with \( \deg(p_n(x)) = n \)) is said to be of binomial type if
\[
p_n(x + a) = \sum_{k=0}^{n} \binom{n}{k} p_k(a)p_{n-k}(x).
\]
(1)

In this case, \( q_n(x) = p_n(x)/n! \) is said to be a sequence of divided powers since it obeys the identity
\[
q_n(x + a) = \sum_{k=0}^{n} q_k(a)q_{n-k}(x).
\]

For example,

| Example Number | A Species | B Name of Function Enriched by A | C Sequence of Polynomials of Binomial Type Enumerated by B | D Operator Associated with C | E Operator Conjugate to C | F Linear Sequence of Symmetric Functions of Binomial Type Enumerated by B | G Linear Sequence of Symmetric Functions of Divided Powers Enumerated by B | H Exponential Generating Function for F |
|----------------|-----------|----------------------------------|----------------------------------------------------------|-----------------------------|---------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|
| 1              | Deg : \( E \rightarrow \{E\} \) | Injection : \( E \rightarrow \emptyset \) if \(|E| = 0\), Injection : \( E \rightarrow E \) if \(|E| = 1\), and Injection : \( E \rightarrow \emptyset \) if \(|E| > 1\) | \( x^n = (x-1)(x-2)\cdots(x-n+1) \) | Lin : \( E \rightarrow \) complete orderings of the set \( E \) | \( \log(1+D) \) | \( \sum_{\lambda \vdash n} m_{\lambda} \) | \( \sum_{\mu \in P^*} \prod_{k=1}^{\lambda-n} y_{\mu_k} \) | \( \exp \left( t \sum_{y \in X} y \right) \) | \( \prod_{y \in X} (1 + yt) \) | \( \prod_{y \in X} (1 - yt)^{-1} \) |
| 2              | Inj : \( E \rightarrow \emptyset \) if \(|E| = 0\), \( E \rightarrow E \) if \(|E| = 1\), and \( E \rightarrow \emptyset \) if \(|E| > 1\) | Lin : \( E \rightarrow \) complete orderings of the set \( E \) | \( (x)_n = x(x-1)\cdots(x-n+1) \) | \( h_n(y) = \sum_{\lambda \vdash n} m_{\lambda} \) | \( \log(1+D) \) | \( \prod_{y \in X} (1 + yt) \) | \( \prod_{y \in X} (1 - yt)^{-1} \) | 

1. the powers \( x^n \),
2. the lower factorial \( (x)_n \),
3. the upper factorial \( (x)^n \)
Table 2: Examples 4–6

|   | Example Number | 4 | 5 | 6 |
|---|----------------|---|---|---|
| A | Species        | $F = \exp(T): E \mapsto \text{rooted forests on the set } E$ | $\exp(\text{Lin}): E \mapsto \text{assemblies of linear orders on the set } E$ | $F_1 = \exp(\text{Alg}) : E \mapsto \text{rooted forests of trees of length at most one on the set } E$ |
| B | Name of a Function Enriched by A | Reluctant Function | Laguerre Function | Inverse-Abel Function |
| C | Sequence of Polynomials of Binomial Type Enumerated by B | $A_n(x) = x(x + n)^{n-1}$ | $L_n(-x) = \sum_{k=1}^{n} \binom{n-1}{k-1} x^k$ | $\mu_n(x) = \sum_{k=1}^{n} \binom{n-k}{k} x^k$ |
| D | Operator Associated with C | $D^{-1}$ | $D/(D - 1)$ | $D^{-1}$ |
| E | Operator Conjugate to C | $D^{-1}$ | $D/(D - 1)$ | $D^{-1}$ |
| F | Linear Sequence of Symmetric Functions of Binomial Type Enumerated by B | $A_n(y) = \frac{1}{n!} \sum_{\lambda \triangleright n} \prod_{i} \lambda_i^{\lambda_i-2} \prod_{i} \lambda_i^{\lambda_i-1}$ | $L_n(y) = \frac{1}{n!} \sum_{\lambda \triangleright n} \prod_{i} \lambda_i^{\lambda_i-2} \prod_{i} \lambda_i^{\lambda_i-1}$ | $\mu_n(y) = \frac{1}{n!} \sum_{\lambda \triangleright n} \prod_{i} \lambda_i^{\lambda_i-2} \prod_{i} \lambda_i^{\lambda_i-1}$ |
| H | Exponential Generating Function for F | $\exp \left( \sum_{y \in X} \frac{yt}{1-yt} \right)$ | $\exp \left( \sum_{y \in X} \frac{yt}{1-yt} \right)$ | $\exp \left( \sum_{y \in X} \frac{yt}{1-yt} \right)$ |

4. the Abel polynomials $A_n(x)$,
5. the LaGuerre polynomials $L_n(-x)$, and
6. the inverse-Abel polynomials $\mu_n(x)$

are all sequences of binomial type. We show that they all have similar combinatorial interpretations. In fact, we show that this is in a sense typical of all sequences of binomial type. They all count the number of functions from an $n$-element set $N$ to an $x$-element set $X$ which are enriched with a “structure” of some sort on each fiber where the fibers $f^{-1}(y)$ are the inverse images of elements of the range of the function. In Figure 0 and each of the following figures, we display a typical enriched function from the $n$-element set

Figure 0: Typical Enriched Function

```
•  •  ⇒ •y1
•  •  ⇒ •y2
•  •  ⇒ •y3
•  •  ⇒ •y4
```

on the left to the $x$-element set on the right. Note that we occasionally allow $x$ to be infinite.
Let us reformalize the above ideas in the language of species.

**Definition 1.1 (Species)** Given a type of structure (e.g., rooted trees), its species $S$ is the functor from the category $\text{Sets}$ of finite sets and bijections to itself. For any finite set $E$, we say that the members of the set $S[E]$ are $S$-structures, and for any bijection $f : E \rightarrow F$, we describe the function $S[f]$ as a relabeling of $S$-structures. For this paper, we need to assume that there is only one structure on the empty set and there is at least one structure on every one element set; that is, $|S[\emptyset]| = 1$ and $|S[\{0\}]| \neq 0$.

We define the sum of two species $S_1$ and $S_2$ on $E$ by their disjoint union

$$(S_1 + S_2)[E] = S_1[E] \cup S_2[E].$$

Similarly, we define the product of $S_1$ and $S_2$ on $E$ to be set of quadruples

$$(S_1S_2)[E] = \left\{ (E_1, E_2, A_1, A_2) : E_1 \text{ and } E_2 \text{ are disjoint, } E_1 \cup E_2 = E, \text{ and } A_i \in S_i[E_i] \right\}.$$

In other words, we divide $E$ into two parts and place an $S_1$-structure on one part, and an $S_2$-structure on the other.

Next, we define the exponentiation of a species $S$ on $E$ to be

$$\exp(S)[E] = \left\{ (\phi, (A_B)_{B \in \phi}) : \phi \text{ is a partition of } S, (A_B)_{B \in \phi} \text{ is a sequence of } S \text{ structures one on each block of } \phi \right\}.$$

That is, we divide $E$ into a number of parts and place a $S$ structure on each part.

Similarly, we define the composition of $S_1$ with $S_2$ on $E$ to be

$$\exp(S)[\pi] = \left\{ (\phi, (A_B)_{B \in \phi}, C) : \phi \text{ is a partition of } S, (A_B)_{B \in \phi} \text{ is a sequence of } S \text{ structures one on each block of } \phi, \text{ and } C \text{ is an } S \text{ structure on } \phi \right\}.$$

That is, we divide $E$ into a number of parts, place a $S_2$-structure on each part, and finally place a $S_1$-structure on the parts themselves.

Finally, we define the derivative $S'$ of a species $S$ on $E$ to be the value of $S$ on a set $E \cup \{\infty\}$ with one more element.

Now, these enriched functions which we are counting are merely pairs $(f, (a_y)_{y \in X})$ where $f : N \rightarrow X$ and $a_y \in S[f^{-1}(y)]$.

**Example 1.1.1: (Powers of $x$)** The powers $x^n$ are the canonical example of a sequence of binomial type. They count all of the functions $f : N \rightarrow X$, since each of the $n$ members of $N$ can be mapped independently onto any of the $x$ members of $X$. 
Figure 1: Typical Function
\[
\begin{align*}
\bullet \rightarrow \bullet & \Rightarrow \bullet y_1 \\
\bullet \rightarrow \bullet & \Rightarrow \bullet y_2 \\
\bullet \rightarrow \bullet & \Rightarrow \bullet y_3 \\
\rightarrow & \Rightarrow \bullet y_4
\end{align*}
\]

Figure 2: Typical Injection
\[
\begin{align*}
\bullet & \Rightarrow \bullet y_1 \\
\rightarrow & \Rightarrow \bullet y_2 \\
\bullet & \Rightarrow \bullet y_3 \\
\bullet & \Rightarrow \bullet y_4 \\
\rightarrow & \Rightarrow \bullet y_5 \\
\rightarrow & \Rightarrow \bullet y_6
\end{align*}
\]

These ordinary functions can be brought under the umbrella of the preceding discussion by observing that they can be thought of as functions enriched by the degenerate species: \( \text{Deg} : E \mapsto \{ E \} \). Thus, on each fiber there is only one possible “structure.”

**Example 1.1.2: (Lower Factorial)** The lower factorial \((x)_n = x(x-1) \cdots (x-n+1) = n! \binom{x}{n}\) counts the number of injections of \(N\) into \(X\). Obviously, the first member of \(N\) can be mapped in \(x\) different way. This leaves \(x-1\) choices for the second, and so on.

Next, note that an injection is a function enriched with the species
\[
\text{Inj} : E \mapsto \left\{ \begin{array}{ll}
\{ \emptyset \} & \text{if } |E| = 0, \\
E & \text{if } |E| = 1, \text{ and} \\
\emptyset & \text{if } |E| > 1.
\end{array} \right.
\]

Thus, there are no structures available for “illegal” fibers containing two or more points.

**Example 1.1.3: (Upper Factorial)** A disposition is a function enriched with the species \( \text{Lin} \) of linear orders; it is a function with a linear ordering on each of its fibers.

To count the number of dispositions, observe that if \(N = \{1, 2, 3, \ldots, n\}\), then 1 can be mapped to any of the \(x\) members of \(X\). The same is true of 2; however, if \(f(1) = f(2)\), then we must also choose the order of 1 and 2. Hence, there are total of \(x+1\) choices. Regardless of which choice we take, there are \(x+2\) choices for 3, and so on. By a simple induction, the number of dispositions is given by the upper factorial \((x)_n^n = x(x+1)(x+2) \cdots (x+n-1)\).
Figure 3: Typical Disposition

\[ \begin{array}{c}
\bullet < \bullet < \bullet \\
\Rightarrow \bullet y_1 \\
\end{array} \]

\[ \begin{array}{c}
\bullet \\
\Rightarrow \bullet y_2 \\
\end{array} \]

\[ \begin{array}{c}
\bullet < \bullet \\
\Rightarrow \bullet y_3 \\
\end{array} \]

\[ \begin{array}{c}
\bullet \\
\Rightarrow \bullet y_4 \\
\end{array} \]

Figure 4: Typical Reluctant Function

\[ \begin{array}{c}
\bullet < \bullet < \bullet < \bullet \\
\Rightarrow \bullet y_1 \\
\end{array} \]

\[ \begin{array}{c}
\bullet < \bullet < \bullet \\
\Rightarrow \bullet y_2 \\
\end{array} \]

\[ \begin{array}{c}
\bullet < \bullet < \bullet \\
\Rightarrow \bullet y_3 \\
\end{array} \]

\[ \begin{array}{c}
\bullet < \bullet < \bullet \\
\Rightarrow \bullet y_4 \\
\end{array} \]

Figure 5: Typical Laguerre Function

\[ \begin{array}{c}
\bullet < \bullet < \bullet < \bullet \\
\Rightarrow \bullet y_1 \\
\end{array} \]

\[ \begin{array}{c}
\bullet < \bullet < \bullet \\
\Rightarrow \bullet y_2 \\
\end{array} \]

\[ \begin{array}{c}
\bullet < \bullet < \bullet \\
\Rightarrow \bullet y_3 \\
\end{array} \]

\[ \begin{array}{c}
\bullet < \bullet < \bullet \\
\Rightarrow \bullet y_4 \\
\end{array} \]
Example 1.1.4: (Abel Polynomials) We see that the Abel polynomials \( A_n(x) = x(x + n)^{n-1} \) counts the number of functions enriched by the species \( F \) of labeled rooted forests. Such functions are called reluctant functions.

Example 1.1.5: (Laguerre Polynomials) Next, the Laguerre polynomials

\[
L_n(-x) = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{n!}{k!} x^k
\]

count the number of Laguerre functions. These are functions from \( N \) to \( X \) enriched with the species \( \exp(\text{Lin}) \) of collections of linear orders on each fiber. In a sense, the Laguerre functions are related to the dispositions in the same way that functions are related to injection. We will soon see the importance of this relationship.

Example 1.1.6: (Inverse-Abel Polynomials) Finally, we observe that the inverse-Abel polynomials \( \mu_n(x) = \sum_{k=0}^{n} \binom{n-k}{k} (\frac{n}{k}) x^k \) counts the number of functions enriched by the species \( F_1 \) of forests of rooted trees of length at most one.

1.2 Symmetric Functions

To study these combinatorial relations more closely, we count enriched functions according to the size of the their fibers.

Suppose we represent an (enriched) function \( f : N \to X \) by the product

\[
\bar{f} = \prod_{i \in N} f(i).
\]
1.2 Symmetric Functions

For instance, the function depicted in Figure 1 is represented by \(y_1^3 y_2^2 y_4\). We represent a collection \(\mathcal{F}\) of enriched functions by the sum of the representations of the functions,

\[
\tilde{\mathcal{F}} = \sum_{f \in \mathcal{F}} \prod_{i \in \mathbb{N}} f(i).
\]

Thus, if \(\mathcal{F}\) is invariant under permutation of \(X\), then \(\tilde{\mathcal{F}}\) is a symmetric function. For example, if \(\mathcal{S}\) is a species and \(\mathcal{S}\) is the collection of all \(\mathcal{S}\)-enriched functions from \(N\) to \(X = \{y_1, y_2, \ldots\}\), then \(\tilde{\mathcal{F}}\) is a linear sequence symmetric functions \(p_n(y)\).

This linear sequence is said to be of binomial type, and its sister sequence \(p_n(y)/n!\) is said to be a linear sequence of divided powers. To some extent, the sequence of divided powers enumerates the set of functions from an “unlabeled” set \(N\) to a labeled set \(X\). Both linear sequences are said to be associated with the species \(\mathcal{S}\), and related to the sequence of polynomials of binomial type \(p_n(x)\) which enumerates the number of \(\mathcal{S}\)-enriched functions from an \(n\)-element set to an \(x\)-element set.

The homomorphism \(\text{proj}\) makes this relationship quite explicit. It is defined by setting the variables \(y_1, \ldots, y_x\) equal to one, and setting all other variables equal to zero. Thus, \(\text{proj} p_n(y) = p_n(x)\).

Note that \(\text{proj}\) is characterized by its action on the monomial symmetric functions

\[
\text{proj} m_\lambda(y) = \frac{(x)_\ell(\lambda)}{\prod_{i \geq 0} \text{mult}_i \lambda_i!}
\]

where

1. The monomial symmetric function is given by the sum

\[
m_\lambda(y) = \sum_{\alpha} y_1^{\alpha_1} y_2^{\alpha_2} \ldots
\]

over all distinct permutations \(\alpha\) of the linear partition \(\lambda\).

2. A linear partition \(\lambda\) is a nonincreasing infinite sequence, \((\lambda_i)_{i \geq 1}\), of nonnegative integers which is eventually zero; for example, \(\lambda = (17, 2, 2, 1, 0, 0, \ldots)\) is a linear partition. A linear partition \(\lambda\) is said to be a partition of \(n\) if the sum of its parts is \(n\), and we write \(\lambda \vdash n\). The set of all linear partitions is denoted by \(\mathcal{P}\). Partitions may be compared in at least three ways.

   (a) For any partition \(\lambda\) and vector \(\alpha\), if \(\alpha_i \leq \lambda_i\) for all \(i\), then we write \(\alpha \preceq \lambda\), and we denote by \(\lambda - \alpha\) their vector difference \((\lambda - \alpha)_i = \lambda_i - \alpha_i\). Similarly, we denote by \(\lambda + \mu\) their vector sum.

   (b) Each nonzero \(\lambda_i\) is called a part \(\lambda_i\). In the above example, the multiset of parts of \(\lambda\) is \(\{1, 2, 2, 17\}\). The number of parts of \(\lambda\) is denoted \(\ell(\lambda)\), and the number of parts of \(\lambda\) equal to \(i\) is denoted \(\text{mult}_i(\lambda)\). If the parts of \(\mu\) form a submultiset of \(\lambda\), that is to say if \(\text{mult}_i(\mu) \leq \text{mult}_i(\lambda)\) for all \(i\); then we write \(\mu \subseteq \lambda\). We denote by \(\lambda \setminus \mu\) the partition whose multiset of parts is the difference between the multisets of parts for \(\lambda\) and \(\mu\). Similarly, we denote by \(\lambda \cup \mu\) the partition whose multiset of parts in the union of the multiset of parts for \(\lambda\) and \(\mu\).

A linear partition is said to have distinct parts if its multiset of parts is, in fact, a set. The set of all linear partitions with distinct parts is denoted by \(\mathcal{P}^*\).
(c) Finally, the Ferrers diagram of a partition \( \lambda \) is defined to be the set of ordered pairs \((i, j)\) such that \(1 \leq j \leq \lambda_i\). If the Ferrers diagram of \( \mu \) is a subset of the Ferrers diagram of \( \lambda \), we write \( \mu \subseteq \lambda \), and we denote by \( \lambda/\mu \) the set difference between the two Ferrers diagrams.

\[ \text{proj is obviously a homomorphism for the algebra of symmetric functions } \Lambda \text{ to the algebra of polynomials } C[x]. \]

The monomial symmetric functions are the simplest known basis for \( \Lambda \). They allow us to give an explicit formula for \( p_n(y) \) in terms of its species.

**Proposition 1.2** The linear sequence of divided powers associated with the species \( S \) is given by the sum

\[ q_n(y) = \sum_{\lambda \vdash n} \left( \prod_i \frac{a_{\lambda_i}}{\lambda_i!} \right) m_\lambda(y) \]

over partitions \( \lambda \) of the nonnegative integer \( n \) where \( a_i \) is the number of \( S \)-structures on an \( i \) element set.

The associated linear sequence of binomial type is thus

\[ p_n(y) = \sum_{\lambda \vdash n} \left( \frac{n!}{\lambda_1! \lambda_2! \cdots} \right) \left( \prod_i a_{\lambda_i} \right) m_\lambda(y) \]

where \( \binom{n}{\lambda} = \frac{n!}{\lambda_1! \lambda_2! \cdots} \). By assumption, \( a_0 = 1 \) and \( a_1 \geq 1 \).

By projecting back to the polynomials, we derive a new result concerning polynomials which gives the coefficients of one sequence of binomial type in terms of the lower factorials.

**Corollary 1.3** Let \( p_n(x) \) be the sequence of polynomials of binomial type associated with the species \((a_n)_{n \geq 1}\). Then

\[ p_n(y) = \sum_{\lambda \vdash n} \binom{n}{\lambda} \left( \prod_i \frac{a_{\lambda_i}}{\text{mult}_i(\lambda)!} \right) (x)_{\ell(\lambda)}. \]

**Proof:** This is a direct application of Proposition 1.2 bearing in mind that the polynomial analog of the monomial symmetric function is \( \text{proj} m_\lambda(y) = (x)_{\ell(\lambda)}/\prod_i \text{mult}_i(\lambda)! \).

Let \( C \) be the category of complex numbers along with “maps” from each complex number \( z \) to itself. Then the map \# : \( E \mapsto |E| \) is a functor from the category \( \text{Sets} \) to the category \( C \). We have seen that the linear sequence \( p_n(y) \) is completely determined by the composition of functors \( Q = \# \circ S \). We will therefore call any functor \( Q : \text{Sets} \to C \) a quasi-species regardless of whether or not its splits into the composition of \# and a species. In this case, \( a_n \) above will refer to the value of \( Q \) on an \( n \)-element set. We define the
linear sequence of divided powers associated with a quasi-species via Proposition \[1.2\]. However, only in the case of species is there a clear combinatorial interpretation as above.

NB: The examples below in this and the ensuing sections (through §4) are continuations of the examples in §1.1. They are numbered according to their section and their subject. For instance, Example \[1.2.1\] is the continuation of Example \[1.1.1\].

**Example 1.2.1: (Powers Symmetric Function)** The set of all functions from \(N\) to \(X\) is represented by the linear sequence of symmetric functions of binomial type

\[
\text{Id}_n(y) = (y_1 + y_2 + \cdots)^n = \sum_{\lambda \vdash n} \binom{n}{\lambda} m_\lambda(y)
\]

whose sister sequence of divided powers is \((y_1 + y_2 + \cdots)^n/n!\). Thus, the symmetric functions \(\text{Id}_n(y)\)

Table 3: Powers Symmetric Function \(\text{Id}_n(y)\)

| \(\text{Id}_0(a, b, c)\) | 1 |
| \(\text{Id}_1(a, b, c)\) | \(a + b + c\) |
| \(\text{Id}_2(a, b, c)\) | \(2ab + (a^2 + b^2)\) |
| \(\text{Id}_3(a, b, c)\) | \((a^3 + b^3 + c^3) + 3(a^2b + a^2c + ba^2 + bc^2 + ca^2 + cb^2) + 6abc\) |

represents the powers \(x^n\).

**Example 1.2.2: (Elementary Symmetric Function)** The set of all injections from \(N\) to \(X\) is represented by the lower factorial symmetric function

\[
(y)_n = n!e_n(y)
\]

whose sister sequence of divided powers \(e_n(y)\) otherwise known as the *elementary symmetric function* is (by Proposition \[1.2\]) given by the sum

\[
e_n(y) = \sum_{\mu \in P^n, \mu \not= \emptyset} \prod_{k=1}^{n} y_{\mu_k}
\]

over all linear partitions \(\mu\) with distinct parts.
Table 4: Elementary Symmetric Function $e_n(y)$

| $e_0(a, b, c)$ | 1 |
| $e_1(a, b, c)$ | $a + b + c$ |
| $e_2(a, b, c)$ | $ab + ac + bc$ |
| $e_3(a, b, c)$ | $abc$ |

Table 5: Complete Symmetric Function $h_n(y)$

| $h_0(a, b, c)$ | 1 |
| $h_1(a, b, c)$ | $a + b + c$ |
| $h_2(a, b, c)$ | $ab + ac + bc + a^2 + b^2 + c^2$ |
| $h_3(a, b, c)$ | $a^3 + b^3 + c^3 + a^2b + a^2c + b^2a + b^2c + ca^2 + cb^2 + abc$ |

Example 1.2.3: (Complete Symmetric Function) Dually, the set of all unlabeled dispositions from $N$ to $X$ is represented by the complete symmetric function $h_n(x)$. By Proposition 1.2, the complete symmetric function is given by the sum

$$h_n(y) = \sum_{\rho \in \mathcal{P}} \prod_{k=1}^{n} y_{\rho_k}$$

over all linear partitions $\rho$. Similarly, the set of all dispositions is represented by the upper factorial symmetric function $(y)^n = n!h_n(y)$.

Example 1.2.4: (Abel Symmetric Function) The linear sequence of symmetric functions of binomial type for the set of Abel functions from $N$ to $X$ is given by the Abel symmetric function

$$A_n(y) = n! \sum_{\lambda \vdash n} \left( \prod_{i} \frac{\lambda_i^{\lambda_i - 2}}{(\lambda_i - 1)!} \right) m_\lambda(y).$$

Table 6: Abel Symmetric Function $A_n(y)$

| $A_0(a, b, c)$ | 1 |
| $A_1(a, b, c)$ | $a + b + c$ |
| $A_2(a, b, c)$ | $2(ab + ac + bc) + 2(a^2 + b^2 + c^2)$ |
| $A_3(a, b, c)$ | $9(a^3 + b^3 + c^3) + 6(a^2b + a^2c + b^2a + b^2c + ca^2 + cb^2) + 6abc$ |
1.3 Generating Functions

We also wish to derive the exponential generation function for linear sequences of symmetric functions of binomial type. Suppose that \( S \) is a (quasi-)species such that \( S[E] \) has \( a_i \) elements where \( i \) is the number of elements in the set \( E \). By assumption, \( a_0 = 1 \) and \( a_1 \geq 1 \). The generating function of the species \( S \) is defined to be

\[
Gen[S](t) = \sum_{i=0}^{\infty} a_i t^i / i!.
\]

Notice that all the operations among species which we have defined are held equally by their generating functions. That is,

\[
\begin{align*}
Gen[S_1](t) + Gen[S_2](t) &= Gen[S_1 + S_2](t) \\
Gen[S_1](t)Gen[S_2](t) &= Gen[S_1S_2](t) \\
\exp(Gen[S_2](t)) &= Gen[\exp(S)](t) \\
Gen[S_1](Gen[S_2](t)) &= Gen[S_1(S_2)](t) \\
dGen[S](t)/dt &= Gen[S'](t)
\end{align*}
\]

The linear sequence of polynomials for a given species may be expressed in terms of this generating function.

**Theorem 1.4** Let \( p_n(y) \) be the linear sequence of polynomials associated with the quasi-species \( S = (a_i)_{i \geq 1} \). Then \( p_n(y) \) is given implicitly by the generating function

\[
\sum_{n=0}^{\infty} p_n(y)t^n/n! = \prod_{y \in X} Gen[S](yt).
\]

**Proof:** Consider the following sequence of equalities:

\[
\begin{align*}
\sum_{n=0}^{\infty} p_n(y)t^n/n! &= \frac{1}{n!} \sum_{n=0}^{\infty} \sum_{f: \{1,2,\ldots,n\} \rightarrow X} \prod_{i=1}^{n} f(i)t \\
&= \frac{1}{n!} \sum_{n=0}^{\infty} \sum_{\pi \vdash N} \sum_{f: \{1,2,\ldots,n\} \rightarrow X \text{ enriched by } S_{\pi}} \prod_{y \in X} (yt)^{|\pi_y|} \\
&= \sum_{n=0}^{\infty} \sum_{\pi \vdash N} \prod_{y \in X} a_{|\pi_y|} (yt)^{|\pi_y|} / |\pi_y|! \\
&= \sum_{n=0}^{\infty} \sum_{\alpha \vdash x} a_{\alpha y} (yt)^{\alpha y} / \alpha y!
\end{align*}
\]
\[
\sum_{\alpha \in \pi} \prod_{y \in X} a_{\alpha,y} (yt)^{\alpha,y} / \alpha! = \prod_{y \in X} \sum_{k=0}^{\infty} a_k (yt)^k / k! = \prod_{y \in X} \text{Gen}[S](yt) \tag{6}
\]

where:

- in equation (3) and equation (4), \( \pi \) is a sequence of \( x \) (possibly empty) disjoint sets indexed by \( X \) whose union is \( N \),
- in equation (2), \( f \) is a function from \( \{1, 2, \ldots, n\} \) to \( X \) enriched by the species \( S \); in equation (3), \( f \) is in addition required to have a fiber structure (kernel) which corresponds to \( \pi \), and
- in equation (5) and equation (6), \( \alpha \) is a sequence of nonnegative integers which total to \( n \).

Thus, by the remarks prior to Example 1.2.1,

**Corollary 1.5** The sequence of polynomials of binomial type associated with the quasi-species \( S \) is given implicitly by the generating function

\[
\sum_{n=0}^{\infty} p_n(x) t^n / n! = \text{Gen}[S](t)^x = \exp(x \log \text{Gen}[S](t)). \tag{8}
\]

Note that \( \text{Gen}[S](t) \) has a logarithm since \( a_0 = 1 \). Now, by a classical result of Umbral Calculus (see [11]), \( p_n(x) \) is the conjugate sequence of polynomials of binomial type for the delta operator \( \log(\text{Gen}[S](D)) \) where \( D \) is the derivative, and is associated and basic for the compositional inverse of that delta operator.

Conversely, the sequence of polynomials of binomial type conjugate to the delta operator \( g(D) \) enumerates functions enriched by the quasi-species with generating function \( \exp(g(t)) \).

Now, we see that counting enriching function enriched with *collections* of \( S \)-structures as opposed to counting functions enriched with merely a single \( S \)-structure is tantamount in terms of polynomials to Umbral composition with the lower factorial sequence.

In general, let \( p_n(y) \) and \( q_n(y) \) be the linear sequences of symmetric functions which enumerate functions enriched with the species \( S \) and \( \exp(T) \) (collections of \( T \)-structures) respectively. Next, let \( p_n(x) = \text{proj} p_n(y) \) and \( q_n(x) = \text{proj} q_n(y) \) be their polynomial counterparts. Now, suppose \( r_n(x) \) is the Umbral composition of \( p_n(x) \) with \( q_n(x) \), and that \( r_n(y) \) is the corresponding linear sequence of symmetric functions. Then \( r_n(y) \) is associated with the species \( S(T) \). That is, it enumerates functions with a collection of \( T \)-structures on
1.3 Generating Functions

each fiber and a single $S$-structure which unites the $S$-structures. All of this will be better understood later through the use of transfer operators.

**Example 1.3.1: (Powers of $x$)** The generating function for the degenerate species is $e^t$, and the logarithm of $e^t$ is $t$. Thus, the generating function for its associated linear sequence of symmetric functions of binomial type is

$$\exp \left( t \sum_{i \geq 0} y_i \right),$$

and the generating function for its sequence of polynomials of binomial type is $e^{xt}$. Thus, $x^n$ is the conjugate graded sequence for $D$. $D$ is its own compositional inverse, so $Dx^n = nx^n$.

**Example 1.3.2: (Lower Factorial)** The generating function for the species $\text{Inj}$ is $1 + t$. The compositional inverse of $\log(1 + t)$ is $e^t - 1$, so the forward difference operator $e^D - 1$ is the delta operator for $(x)_n$. That is, $(x + 1)_n - (x)_n = n(x)_{n-1}$. By equation (7), the generating function for the elementary symmetric function is given by

$$\prod_{n=1}^{\infty} (1 + y_n t) = \sum_{n=0}^{\infty} e_n(y)t^n.$$

**Example 1.3.3: (Upper Factorial)** The generating function for the species $\text{Lin}$ of linear orders is $1/(1 - t)$. The compositional inverse of $\log(1/(1 - t))$ is $1 - e^{-t}$, so the backward difference operator $1 - e^{-D}$ is the delta operator for $(x)^n$. That is, $(x)^n - (x - 1)^n = (x)^{n-1}$. By equation (8), the generating function for the complete symmetric function is given by

$$\prod_{n=1}^{\infty} (1 - y_n t)^{-1} = \sum_{n=0}^{\infty} h_n(y)t^n.$$

**Example 1.3.4: (Abel Function)** Let $T$ be the species of rooted trees. A tree with its root removed is a rooted forest, and a collection of rooted trees is a rooted forest. Thus, by the theory of species

$$\text{Gen}[T](x) = x \text{Gen}[F](x),$$

and

$$\text{Gen}[F](x) = \exp(\text{Gen}[T](x)).$$

Hence, the compositional inverse of the generating function for $F$ is $x/e^x$. Now, $A_n(x)$ is the associated sequence of the Abel operator $De^{-D}$, so as we claimed earlier $A_n(x)$ counts function enriched with forests of rooted trees.

**Example 1.3.5: (Laguerre Polynomials)** The generating function for the species $\exp(\text{Lin})$ of collections of linear orders is $\exp(x/(1 - x))$. Thus, the Laguerre symmetric functions are given by the generating
Table 7: Laguerre Symmetric Function $L_n(y)$

| $L_0(a, b, c)$ | 1           |
| $L_1(a, b, c)$ | $a + b + c$ |
| $L_2(a, b, c)$ | $3(ab + ac + bc) + 2(a^2 + b^2 + c^2)$ |
| $L_3(a, b, c)$ | $7(a^3 + b^3 + c^3) + 9(a^2b + a^2c + ab^2 + ac^2 + bc^2) + 6abc$ |

function

$$\sum_{n=0}^{\infty} L_n(y) \frac{t^n}{n!} = \prod_{y \in X} \exp \left( \frac{yt}{1 - yt} \right) = \exp \left( \sum_{y \in X} \frac{yt}{1 - yt} \right).$$

Also, the Laguerre polynomials $L_n(-x)$ form the conjugate sequence for the delta operator $D/(1 - D)$. The compositional inverse of this operator is $D/(1 + D)$. Thus, the Laguerre polynomials are associated and basic for the Weirstrass operator.

Example 1.3.6: (Inverse-Abel Polynomials) The generating function for the species $T_1$ of trees of length at most one is $xe^x$ since there are exactly $n$ such trees on every set of size $n$. Thus, the generating function for the species $F_1 = \exp(T_1)$ of forests of such trees is $\exp(xe^x)$. Hence, the generating function

Table 8: Inverse-Abel Symmetric Function $\mu_n(y)$

| $\mu_0(a, b, c)$ | 1           |
| $\mu_1(a, b, c)$ | $a + b + c$ |
| $\mu_2(a, b, c)$ | $2(ab + ac + bc) + 2(a^2 + b^2 + c^2)$ |
| $\mu_3(a, b, c)$ | $3(a^3 + b^3 + c^3) + 6(a^2b + a^2c + ab^2 + ac^2 + bc^2) + 6abc$ |

for the inverse-Abel symmetric functions is

$$\sum_{n=0}^{\infty} \mu_n(y) \frac{t^n}{n!} = \prod_{y \in X} \exp \left( yte^{yt} \right) = \exp \left( \sum_{y \in X} yte^{yt} \right).$$

1.4 The Symmetric Derivative

We define a derivation on the ring of symmetric functions. (Up to a constant) this operator fills the role of the basic delta operator simultaneously for each linear sequence of symmetric functions associated with a sequence of polynomials of binomial type.

We define the symmetric derivative as follows.

$$Dp(y_1, y_2, \ldots) = \lim_{\epsilon \to 0} \frac{p(\epsilon, y_1, y_2, \ldots) - p(0, y_1, y_2, \ldots)}{\epsilon} = \frac{dp(\epsilon, y_1, y_2, \ldots)}{d\epsilon}. \quad (9)$$
In other words, we “add” a new variable, take the derivative with respect to it, and set it equal to zero.

For calculations involving $D$, it is useful to note how $D$ behaves with respect to the monomial symmetric functions.

$$D m_{\lambda}(y) = \begin{cases} m_{\lambda \cup \{1\}}(y) & \text{if } 1 \text{ is a part of } \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

**Example 1.4.1: (Powers Symmetric Function)** The symmetric derivative acts on $Id_n(y)$ as the derivative acts on the powers of $x$.

$$Di_d(y) = d Id_{n-1}(y).$$

**Example 1.4.2: (Lower Factorial Symmetric Function)** Also,

$$De_n(y) = e_{n-1}(y),$$

so

$$D(y)_n = n(y)_{n-1}.$$

Hence, in this case the symmetric derivative plays the role of the forward difference operator rather than that of the ordinary derivative. (We could also have derived this using the fact $e_n(y) = m_{(1^n)}(y)$.)

**Example 1.4.3: (Upper Factorial Symmetric Function)** Also,

$$Dh_n(y) = h_{n-1}(y),$$

so $D(y)^n = n(y)^{n-1}$. Again, $D$ play the role of the backwards difference operator.

We can use equation (10) to calculate the symmetric derivative of the Schur function. The **Schur function** is given by the determinant

$$s_{\lambda}(y) = \det [h_{\lambda_i-i+j}(y)]_{i,j=1}^{\infty}. \tag{11}$$

Now, since $D$ is a derivation,

$$Ds_{\lambda}(y) = \sum_{n=1}^{\infty} \det [h_{\lambda_i-i+j-\delta_{n}}(y)]_{i,j=0}^{\infty}$$

$$= \sum_{n \in \lambda} s_{\lambda \cup \mu(n-1)}(y)$$

where the last sum is over the distinct parts of the partition $\lambda$. For example,

$$Ds_{5221}(y) = s_{4221}(y) + s_{5211}(y) + s_{522}(y)$$

By analogous reasoning, for $\mu \subseteq \lambda$, the **skew Schur function**

$$s_{\lambda/\mu} = \det [h_{\lambda_i-i+j}](y)$$

has a derivative of

$$Ds_{\lambda/\mu}(y) = \sum_{n \in \lambda} s_{(\lambda \cup \mu(n-1))/\mu}(y).$$
Considering the combinatorial interpretation, we see that the only contribution arises from functions in which the new variable has a fiber of size one. This fiber can contain any of the \( n \) elements of \( N \), and any of the \( a_1 \) many structures allowed for one-element sets. Thus,

\[
D_p(y) = na_1p_{n-1}(y),
\]
or equivalently,

\[
\epsilon(D^m p_n) = n!a_1^n \delta_{nm}
\]
where \( \epsilon \) acts on a symmetric function \( p(y) \) by mapping all of the variables to zero: \( \epsilon p(y) = p(0, 0, \ldots) \); \( \epsilon \) is the symmetric analog of evaluation at zero since

\[
\epsilon p(y) = [\text{proj} p(y)]_{x=0}.
\]

Hence, \( a_1^{-1} D \) plays the role of the basic delta operator. This is true even in the case of linear sequences of symmetric functions arising from quasi-species.

**Example 1.4.4:** *(Abel Symmetric Function)* \( DA_n(y) = nA_{n-1}(y) \).

**Example 1.4.5:** *(Laguerre Symmetric Function)* \( DL_n(y) = nL_{n-1}(y) \).

**Example 1.4.6:** *(Inverse-Abel Symmetric Function)* \( D\mu_n(y) = n\mu_{n-1}(y) \).
1.5 The Iterated Symmetric Derivative

Note further that if we define the \textit{symmetric} $i$-th derivative
\[ D_i m_{\lambda}(y) = \begin{cases} i! m_{\lambda \setminus i}(y) & \text{if } i \text{ is a part of } \lambda, \\ 0 & \text{otherwise}, \end{cases} \]
then
\[ D_i p_n(y) = (n)_i a_i p_{n-i}(y) \]
for any linear sequence of symmetric functions of binomial type $p_n(y)$. In particular,
\[ D_i p_n(y) = a_i i! D_i p_n(y) / a_i^i. \]
Nevertheless, $D_i$ is not a multiple of $D^i$.

More intuitively, $D_i p(y)$ is calculated by introducing a new variable $y_0$, and then differentiating $p_{y_0,y_1,y_2,...}$
$i$ times with respect to $y_0$ and setting $y_0$ to zero.

Except when $i = 1$, $D_i$ is not a derivation. For example,
\[ D_2 m_{(1)}(y)^2 = D_2 \left( 2m_{(11)}(y) + m_{(2)}(y) \right) = 1 \]
whereas
\[ 2m_{(1)}(y) \left( D_2 m_{(1)}(y) \right) = 0. \]

Note that $D_i$ is $D_{s_i}(y)$ in the notation of A. Lascoux [7].

**Example 1.5.1:** (Powers Symmetric Function) This is the action of the Iterated derivative on the Powers Symmetric Function. $D_i d_n(y) = (n)_i d_{n-i}(y)$.

**Example 1.5.2:** (Lower Factorial Symmetric Function) We would also like to compute the action of the Iterated derivative on the elementary and complete symmetric functions.
\[ D_i e_n(y) = D_i(y)_n / n! = (n)_i \delta_{i,1} y_{n-i} / n! = \delta_{i,1} e_{n-i}(y). \]
Thus, $D_i e_n(y) = 0$ except in the case of the ordinary symmetric derivative where $D_1 e_n(y) = e_{n-1}(y)$.

**Example 1.5.3:** (Upper Factorial Symmetric Function) Similarly,
\[ D_i h_n(y) = D_i(y)^n / n! = (n)_i i! (y)^{n-i} / n! = i! h_{n-i}(y). \]
1.6 The Binomial Theorem

We next wish to devise an analog of the usual binomial identity for linear sequences of polynomials of binomial type (equation (1)). However, we must first determine the appropriate analog of the shift operator $E^a$. We define the symmetric shift operator $E^a$ by

$$E^a = \sum_{n=0}^{\infty} a^n D_n / n!.$$  

Hence,

$$E^a m_\lambda(y) = \sum_{n=0}^{\infty} a^n m_{\lambda/n}(y) = m_\lambda(a, y_1, y_2, \ldots),$$

and thus

$$E^a p(y) = p(a, y_1, y_2, \ldots).$$  \hspace{1cm} (13)

Considering equation (13), this is a very appropriate definition of the symmetric shift operator. Note that by equation (13), $E^a$ is an isomorphism of the field of symmetric functions. However, note that its inverse is not $E^{-a}$. In fact, $E^a E^b \neq E^{a+b}$ except in the case where $a$ or $b$ equals zero, for

$$E^a E^b m_{(11)}(y) = m_{(11)}(y) + (a + b)m_{(1)}(y) + ab$$

whereas

$$E^{a+b} m_{(11)}(y) = m_{(11)}(y) + (a + b)m_{(1)}(y).$$

Also, in consideration of equation (13), the operators $E^a$ all commute. For any set $S$, we define the shift $E^S$ to be the composition in any order of the various shifts $E^a$ for all $a \in S$. In the case of an infinite set $S$, this definition makes sense via the use of inverse limits. Thus, for $z = \{z_1, z_2, \ldots\},$

$$E^z p(y) = p(z \cup y).$$

Returning to our theory of linear sequence of symmetric functions of binomial type. We can now derive the binomial identity.

**Theorem 1.6 (Binomial Theorem)** If $p_n(y)$ is a linear sequence of symmetric functions of binomial type, then for all complex numbers $a$

$$p_n(a, y_1, y_2, \ldots) = \sum_{k=0}^{n} \binom{n}{k} p_k(a, 0, 0, \ldots)p_{n-k}(y).$$  \hspace{1cm} (14)
1.6 The Binomial Theorem

Figure 9: Binomial Theorem

\[
\begin{array}{c}
\spadesuit \\
\spadesuit
\Rightarrow \bullet_2 \\
\bullet
\Rightarrow \bullet_1 \\
\bullet
\Rightarrow \bullet_3 \\
\bullet
\Rightarrow \bullet_y
\end{array}
\]

**Proof:** The left side of equation (14) enumerates the set of enriched functions from \( N \) to \( X \cup \{a\} \). The right side of equation (14) counts the number of ways to choose a subset of \( N \) (represented in Figure 9 by spades \( \spadesuit \)) and map it to \( a \) with an enriched function (represented by a thin arrow \( \rightarrow \)) while mapping the remainder of \( N \) (represented by dots \( \bullet \)) into \( X \) with another enriched function (represented by a thick arrow \( \Rightarrow \)). Obviously, these two sets of enriched functions are identical. \( \blacksquare \)

**Corollary 1.7** If \( p_n(y) \) is a linear sequence of symmetric functions of binomial type, and \( y \) and \( z \) are sets of variables, then

\[
p_n(y \cup z) = \sum_{k=0}^{n} \binom{n}{k} p_{n-k}(y)p_k(z).
\]

**Proof:** By iterating Theorem 2 \( m \)-times, we have

\[
p_n(z_1, \ldots, z_m, y) = \sum_k \binom{n}{k} p_{n-k}(y) \sum_{k_1, \ldots, k_m} \binom{k}{k_1, \ldots, k_m} \prod_{i=1}^{m} p_{k_i}(z_i, 0, 0, \ldots).
\]

where the inner sum is over sequences of nonnegative integers summing to \( k \). However, this sum may be computed by equation (15) itself (with \( m \) having a value of one less).

\[
\sum_{k_1, \ldots, k_m} \binom{k}{k_1, \ldots, k_m} \prod_{i=1}^{m} p_{k_i}(z_i, 0, 0, \ldots) = p_k(z_1, \ldots, z_m, 0, 0, \ldots)
\]

Thus,

\[
p_n(z_1, \ldots, z_m, y) = \sum_{k=0}^{n} \binom{n}{k} p_k(z_1, \ldots, z_m, 0, 0, \ldots)p_{n-k}(y).
\]

Our result now follows for infinitely many variables by the usual technique of inverse limits. \( \blacksquare \)

**Example 1.6.1:** (Powers Symmetric Function) Applying the symmetric version of the binomial theorem to \( \text{Id}_n(y) \) we rediscover the usual binomial theorem

\[
(a + y_1 + y_2 + \cdots)^n = \sum_{k=0}^{n} \binom{n}{k} a^k(y_1 + y_2 + \cdots)^{n-k}.
\]
**Example 1.6.2:** *(Lower Factorial Symmetric Function)* Applying the binomial theorem to the lower factorial symmetric function, we derive the following identity held by the elementary symmetric function:

\[ e_n(a, y_1, y_2, \ldots) = e_n(y_1, y_2, \ldots) + ae_{n-1}(y_1, y_2, \ldots) , \]

or more generally

\[ e_n(z_1, z_2, \ldots, y_1, y_2, \ldots) = \sum_{k=0}^{n} e_k(y_1, y_2, \ldots) e_{n-k}(z_1, z_2, \ldots) . \]

**Example 1.6.3:** *(Upper Factorial Symmetric Function)* Applying the binomial theorem to the upper factorial symmetric function, we derive the following identity held by the complete symmetric function:

\[ h_n(a, y_1, y_2, \ldots) = \sum_{k=0}^{\infty} a^k h_{n-k}(y_1, y_2, \ldots) . \]

We are also interested in the shifts of the other important symmetric functions. The *power sum symmetric function* \( \text{pow}_n(y) \) is defined by

\[ \text{pow}_n(y) = m_{(n)}(y) = \sum_{i \geq 1} y_i^n \quad \text{(16)} \]

Clearly, \( \text{pow}_n(y \cup z) = \text{pow}_n(y) + \text{pow}_n(z) \). Note that the kernel of the homomorphism \( \text{proj} \) from the symmetric functions to the polynomials is described very simply in terms of the power sum symmetric function. Since \( \text{proj} \text{pow}_n(y) = x \), the kernel of \( \text{proj} \) is the algebra generated by the symmetric functions \( \text{pow}_m(y) - \text{pow}_n(y) \) where \( m \) and \( n \) are distinct integers.

The shift of the monomial symmetric function is given by

\[ m_\lambda(y \cup z) = \sum_{\mu \cup \nu = \lambda} m_\mu(y)m_\nu(z) . \]

Finally, we can compute the shift of the Schur function. The Schur function can be defined by equation (11) or by the ratio of anti-symmetric functions in \( n \) variables

\[ s_\lambda(y) = a_{\lambda + \delta}(y)/a_{\delta}(y) \]

where \( a_\mu(y) = \det \left[ y_i^{\lambda_j+n-j} \right] \) and \( \delta = (n, n-1, \ldots, 2, 1) \). This definition is consistent with changes of variables, since the ratio is commutes with the map taking one of the variables to zero. After taking inverse limits, this definition is equivalent to equation (11). Hence, \( a_\delta(y) \) is simply the Vandermonde determinant.

\[ a_\delta(y) = \prod_{1 \leq i < j \leq n} (y_i - y_j) . \]
Thus, 

$$E^a a_\delta(y) = (a - y_1)(a - y_2) \cdots (a - y_n) a_\delta(y).$$

Expanding by minors, we have

$$E^a s_\lambda(y) = \det \begin{bmatrix} a^{\lambda_1+n+1} & a^{\lambda_2+n} & \cdots & a^{\lambda_n+1} \\ y_1^{\lambda_1+n+1} & y_1^{\lambda_2+n} & \cdots & y_1^{\lambda_n+1} \\ y_2^{\lambda_1+n+1} & y_2^{\lambda_2+n} & \cdots & y_2^{\lambda_n+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_n^{\lambda_1+n+1} & y_n^{\lambda_2+n} & \cdots & y_n^{\lambda_n+1} \end{bmatrix}$$

$$= \sum_{j=1}^{n+1} (-1)^j a^{n-j+\lambda_j}(y) a_{\delta+(\lambda_1, \lambda_2, \ldots, \lambda_i-1, \lambda_{i+1}+1, \lambda_{i+2}+1, \ldots, \lambda_{n+1}+1)}(y)$$

$$= \left( \prod_{i=1}^{n+1} \frac{a}{a - y_i} \right) \left( \sum_{j=1}^{n+1} a^{\lambda_i-j} s_{\lambda_1, \lambda_2, \ldots, \lambda_i-1, \lambda_{i+1}+1, \lambda_{i+2}+1, \ldots, \lambda_{n+1}+1}(y) \right) \sum_{\mu} a^{\lambda_i-|\mu|} s_{\mu}(y)$$

where the sum is over partitions $\mu \subseteq \lambda$ such that the difference of there Ferrers diagrams $\lambda - \mu$ is a collection of disjoint horizontal strips.

This fact leads immediately to the definition of the Schur function in terms of tableaux, and thence to the identities

$$s_\lambda(y \cup z) = \sum_{\mu \subseteq \lambda} s_{\lambda/\mu}(y)s_{\mu}(y)$$

$$s_{\lambda/\mu}(y \cup z) = \sum_{\mu \subseteq \lambda} s_{\lambda/\mu}(y)s_{\mu/\nu}(y).$$

Also, we see that the polynomial analog of the Schur function $\text{projs}_\lambda(y)$ is the number of standard tableaux of shape $\lambda$ utilizing the alphabet $1, 2, \ldots, x$.

There is more work yet to be done regarding the shift operator. For example,

**Open Problem 1.8** What is the relation between the action of the shift operator $E^a$ over the ring of symmetric function and the theory of Baxter algebras?

### 1.7 Shift-Invariant Operators

Notice that the symmetric derivatives commute with each other and with the symmetric shift operator. These are essentially the only linear operators which do so.
THEOREM 1.9 (Classification of Shift-Invariant Operators) Let $\theta$ be a linear operator on symmetric functions. $E^a\theta = \theta E^a$ for all $a$ if and only if $\theta$ is a complex formal power series in the iterated derivatives $D_i$.

Proof: (If) Immediate.

(Only If) Let $b_{\lambda\mu}$ be the coefficients of the action of $\theta$ on the monomial symmetric functions

$$\theta m_\lambda(y) = \sum_{\mu \in P} b_{\lambda\mu} m_\mu(y).$$

Then

$$\theta E^a m_\lambda(y) = \theta \sum_{i=0}^\infty a^i m_{\lambda \setminus i}(y)$$
$$= \sum_{\mu \in P} \sum_{i=0}^\infty a^i b_{\lambda \setminus i,\mu} m_\mu(y)$$
$$E^a \theta m_\lambda(y) = E^a \sum_{\mu \in P} b_{\lambda\mu} m_\mu(y)$$
$$= \sum_{\mu \in P} \sum_{i=0}^\infty a^i b_{\lambda\mu} m_{\mu \setminus i}(y).$$

We equate coefficients of $a^i m_\mu(y)$, and determine that

$$b_{\lambda \setminus i,\mu} = b_{\lambda,\mu \cup (i)}.$$

Thus, $b_{\lambda\mu}$ must equal zero unless $\mu \subseteq \lambda$. Moreover, $b_{\lambda\mu}$ depends only on $\lambda \setminus \mu$.

Hence, the spaces of linear shift invariant operators is contained in the span of the operators

$$\theta_{\nu} : m_\lambda(y) \mapsto m_{\lambda \setminus \nu}(y).$$

Thus, all shift-invariant operators are formal power series in terms of the symmetric derivatives, since

$$\theta_{\nu} = \prod_{i=0}^\infty \frac{D_{\nu_i}}{\nu_i!}.$$

PORISM 1.10 The iterated derivatives $D_i$ are algebraically independent.
1.7 Shift-Invariant Operators

Let us write $D_\lambda$ for the product of derivatives $\theta_\lambda = \prod D_{\lambda_i}/\lambda_i!$. The $D_\lambda$ form a basis for the module of linear shift-invariant operators. Clearly, we have the following identity

$$\epsilon D_\lambda m_\nu(y) = \delta_{\lambda\nu}. \tag{17}$$

Thus, the operator $\epsilon D_\lambda$ is equal to the classical operator $\langle h_\lambda(y) |$ where $\langle | \rangle$ is the inner product defined by

$$\langle h_\lambda(y) | m_\lambda(y) \rangle = \delta_{\lambda\mu}. \tag{18}$$

**Theorem 1.11 (Expansion Theorem)** Let $\theta$ be a linear shift invariant operator. Then

$$\theta = \sum_{\lambda \in \mathcal{P}} (\epsilon \theta m_\lambda(y)) D_\lambda. \tag{19}$$

**Theorem 1.12 (Taylor’s Theorem)** Let $p(y)$ be a symmetric function. Then

$$p(y) = \sum_{\lambda \in \mathcal{P}} (\epsilon D_\lambda p(y)) m_\lambda(y). \tag{20}$$

Also, we have the symmetric analog of Roman’s identity for formal power series of binomial type.

**Theorem 1.13 (Roman’s Identity)** Let $p_n(y)$ be a linear sequence of symmetric functions of binomial type, and let $\theta$ and $\phi$ be shift invariant linear operators. Then

$$\epsilon \theta \phi p_n(y) = \sum_{k=0}^n \binom{n}{k} (\epsilon \theta p_k(y))(\epsilon \phi p_{n-k}(y)). \tag{21}$$

**Proof:** It suffices to consider the case $\theta = D_{b_1}^1 D_{b_2}^2 \cdots$ and $\phi = D_{c_1}^1 D_{c_2}^2 \cdots$. However, then both sides of equation (21) equal zero unless

$$n = (b_1 + c_1) + 2(b_2 + c_2) + \cdots$$

in which case both sides equal

$$n!a_1^{b_1+c_1}a_2^{b_2+c_2}$$

where $(a_i)_{i \geq 0}$ is the quasi-species associated with $p_n(y)$. \hfill \Box

Thus, we have the following generalization of the symmetric version of binomial theorem.

**Corollary 1.14** Let $p_n(y)$ be a linear sequence of symmetric functions of binomial type, and let $\theta$ be a shift invariant linear operator. Then

$$\theta p_n(y) = \sum_{k=0}^n \binom{n}{k} (\epsilon \theta p_k(y))p_{n-k}(y). \tag{22}$$

\hfill \Box
1.8 Coalgebras

The foregoing theory may be profitably be recast in terms of coalgebras. Let $\Lambda_y$ represent the algebra of symmetric functions in the variables $y$. $\Lambda_y$ is isomorphic for any choice of variables $y$, so we write $\Lambda$ for the algebra of symmetric functions in any set of variables.

Now, $\Lambda_x \otimes \Lambda_y$ is isomorphic to the algebra of formal power series invariant under permutations of $x$ or $y$. Thus, $\Lambda_{x \cup y}$ is naturally a subalgebra of $\Lambda_x \otimes \Lambda_y$. Hence, $E = E^y$ may be thought of as a map from $\Lambda$ to $\Lambda \otimes \Lambda$, and $\epsilon = \epsilon_y$ is a map from $\Lambda_y = \Lambda$ to the complex numbers.

From this point of view, we arrive at a result due to L. Geissinger [4].

**Theorem 1.15** The symmetric functions $\Lambda$ form a commutative Hopf algebra when they are equipped with

- the shift operator $E : \Lambda \to \Lambda \otimes \Lambda$ as the comultiplication,
- the evaluation operator $\epsilon : \Lambda \to \mathbb{C}$ as the counit,
- the usual multiplication $\mu : \Lambda \otimes \Lambda \to \Lambda$,
- the inclusion $\iota : \mathbb{C} \to \Lambda$ as the unitary map, and
- the classical involution $\omega : \Lambda \to \Lambda$ as the antipode.

**Proof:** (Commutative Algebra) $\Lambda$ is well known to be an algebra. That is, it obeys the commutative diagrams in Figures 10–12 where $\tau$ is the commutation map $p(x) \otimes q(x) \mapsto q(x) \otimes p(x)$.

![Figure 10: Associativity](image)

(Cocommutative Coalgebra) Cocommutivity follows from commutivity once we observe that $\Lambda$ is a bialgebra; however, for now we observe this directly from Figure 13 since $p(x \cup y) = p(y \cup x)$. To actually, show that $\Lambda$ is a coalgebra refer to Figures 14 and 15. Coassociativity merely means that $p(x \cup (y \cup z)) = p((x \cup y) \cup z)$, and the counitary property means that $p(y_1, y_2, \ldots, 0, 0, \ldots) = p(y_1, y_2, \ldots)$.

(Bialgebra) There are four requirements one of which is represented by Figure 16. They are all satisfied
1.8 Coalgebras

Figure 11: Unitary Property
\[ C \otimes \Lambda \sim \Lambda \otimes \Lambda \sim \Lambda \otimes C \]

Figure 12: Commutativity
\[ \Lambda \otimes \Lambda \sim \Lambda \otimes \Lambda \]

Figure 13: Cocommutivity
\[ \Lambda_y \otimes \Lambda_x \rightarrow \Lambda_x \otimes \Lambda_y \]

Figure 14: Coassociativity
\[ \Lambda_x \otimes \Lambda_y \rightarrow \Lambda_x \otimes \Lambda_y \]

Figure 15: Counitary Property
\[ C \otimes \Lambda_y \sim \Lambda_x \otimes \Lambda_y \]

Figure 16: Bialgebra
\[ \Lambda_x \otimes \Lambda_x \rightarrow \Lambda_x \otimes \Lambda_y \otimes \Lambda_y \]
since as we have already observed $E^y$ and $\epsilon_y$ are both algebra homomorphism.

(Hopf Algebra) Finally, we must show that

$$\theta \omega_y E^y p(x) = \theta \omega_x E^y p(x) = \epsilon_x(p(x))$$

where $\theta$ is the evaluation of $x$ at $y$ and $\omega_y$ is the classical involution of $\Lambda_y$. However, this follows immediately from the following series of equalities

\[
1 = \prod_{y \in X} (1 - yt) \left( \prod_{y \in X} (1 - yt)^{-1} \right) \\
= \left( \sum_{n \geq 0} (-1)^n e_n(y) t^n \right) \left( \sum_{n \geq 0} h_n(y) \right) \\
= \sum_{n \geq 0} \sum_{i=0}^{n} (-1)^i e_i(y) h_{n-i}(y) t^n \\
\delta_{n,0} = \sum_{i=0}^{n} (-1)^i e_i(y) h_{n-i}(y) t^n \\
= \sum_{i=0}^{n} (-1)^i (\theta \omega_x \epsilon_i(x)) h_{n-i}(y) t^n \\
= \theta \omega_y E^y h_n(x)
\]

In this notation, an operator $\theta$ is shift-invariant if and only if $(\theta \otimes \mathbb{1}) E^x = E^x \theta$

The dual Hopf algebra $\Lambda^*$ is generated by the maps $m^*_\mu : m_\mu(y) \rightarrow \delta_{\lambda\mu}$. Thus, by equation $(17)$ $m^*_\lambda = \epsilon D_\lambda$. Therefore, automorphisms of the algebra of shift-invariant operators correspond to automorphisms of the coalgebra of symmetric functions. These coalgebra maps are called Transfer operators.

It is best to compare the study of symmetric functions by coalgebra methods, to the study of polynomials also by coalgebra methods. In that case, we are dealing with the usual multiplication along with the comultiplication map

$$\Delta x^n = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k},$$

the counitary map of evaluation at zero, and the antipodal map of substitution by $-x$. However, this Hopf algebra is seen to be a homomorphic image of the Hopf algebra of symmetric functions as follows: Define $\text{proj}_x$ to act on the variables $x$ and set exactly $x$ of them equal to 1, and the remainder to zero (where $x$ is a new variable). As noted before $\text{proj}_x$ is an algebra homomorphism from the symmetric functions to the polynomials. Moreover, it is easy to see that $\text{proj}$ satisfies all of the necessary relations—for example it satisfies

$$\Delta \text{proj}_x = (\text{proj}_x \otimes \text{proj}_y) \Delta$$
and equation (12)—so that \( \text{proj} \) is a Hopf algebra homomorphism.

However, although the umbral calculus of polynomials and of symmetric functions is best thought of in terms of coalgebras, we have no coalgebra version of the logarithmic algebra.

**Open Problem 1.16** Is the logarithmic algebra of \([11]\) naturally a coalgebra?

If so, then perhaps in analogy to polynomials we have \( \mu(X^n \otimes X^m) = \binom{n+m}{n} X^{n+m} \). Would this then be the incidence coalgebra for the poset of hybrid sets? \([14]\)

**Part II**

**Full Sequences**

**1.9 Introduction**

Since we require \( \deg(p_n(x)) = n \), every sequence of polynomials \( p_n(x) \) is a basis for the ring of polynomials. Clearly that is not the case for linear sequences of symmetric functions. Since linear sequences of symmetric functions are indexed by a single integer, there are insufficiently many. By the fundamental theorem of symmetric functions, the linear sequences of elementary and complete symmetric symmetric functions are both transcendence bases. However, this is not true in general since the sequence \( \text{Id}_n(y) = (y_1 + y_2 + \cdots)^n \) used in Example 1.2.1 is not a transcendence basis.

**1.10 Full Sequences**

Thus, we are led to discuss larger sequences of symmetric functions which are indexed by partitions and span the space of symmetric functions.

**Definition 1.17** (Full Sequence) Order the partitions in reverse lexicographical order as in \([13]\). Then define the exact degree of a nonzero homogeneous symmetric function \( p(y) \) of degree \( n \) to be the largest partition \( \lambda \vdash n \) such that the coefficient \( c_\lambda = \epsilon \Delta_\lambda p(y) \) is nonzero in the Taylor expansion (Theorem 1.10).

\[
p(y) = \sum_{\lambda \in \mathcal{P}} c_\lambda m_\lambda(y).
\]

A full sequence then is a sequence \( (p_\lambda(y))_{\lambda \in \mathcal{P}} \) of homogeneous symmetric functions indexed by partitions \( \lambda \) such that each \( p_\lambda(y) \) is exactly of degree \( \lambda \).
When \( p_\lambda(y) \) is a full sequence and \( \alpha \) is a vector of nonnegative integers with finite support, then we denote by \( p_\alpha(y) \) the symmetric function \( p_\mu(y) \) where \( \mu \) is the partition formed by arranging the members of \( \alpha \) in weakly descending order. Moreover, we will adopt this convention for any sequence indexed by partitions.

Clearly, every full sequence is a basis for the algebra of symmetric functions. Moreover, the popular bases mentioned above—Schur \( s_\lambda(y) \) and monomial \( m_\lambda(y) \)—are full sequences. Moreover, the linear sequences of complete \( h_n(y) \), elementary \( e_n(y) \), and power \( pow_n(y) \) symmetric functions can be extended to full sequences via the product rules:

\[
e_\lambda(y) = \prod_i e_{\lambda_i}(y)
\]

\[
h_\lambda(y) = \prod_i h_{\lambda_i}(y)
\]

\[
pow_\lambda(y) = \prod_i pow_{\lambda_i}(y)
\]

By the fundamental theorem of symmetric functions, these are bases for the module of symmetric functions. In fact, it can be seen that \( h_\lambda(y) \) and \( e_\lambda(y) \) are full sequences.

To determine the analog of \( e_\lambda(y) \), \( h_\lambda(y) \), and \( pow_\lambda(y) \) in the theory of polynomials, we apply the homomorphism \( proj \).

\[
proj e_\lambda(y) = \prod_{(i,j) \in \lambda} (x - i + 1)/i
\]

\[
proj h_\lambda(y) = \prod_{(i,j) \in \lambda} (x + i + 1)/i
\]

\[
proj pow_\lambda(y) = x^{\ell(\lambda)}.
\]

Now, since \( E^n \) is an isomorphism, we can also calculate the shift of these full sequences.

\[
e_\lambda(y \cup z) = \sum_\alpha e_\alpha(y)e_{\lambda-\alpha}(z)
\]

where the sum is over all vectors \( \alpha \) of nonnegative integers, and similarly

\[
h_\lambda(z_1, z_2, \ldots, y_1, y_2, \ldots) = \sum_\alpha h_\alpha(z)h_{\lambda-\alpha}(y)
\]

or

\[
h_\lambda(a, y_1, y_2, \ldots) = \sum_\alpha a^{[\alpha]}h_{\lambda-\alpha}(y)
\]

where the sum is over all vectors \( \alpha \) of nonnegative integers, and finally

\[
pow_\lambda(y \cup z) = \sum_{\mu \subseteq \lambda} \prod_i \left( \frac{\text{mult}_i(\lambda)}{\text{mult}_i(\mu)} \right) pow_\mu(y) pow_{\lambda/\mu}(z).
\]
Equations (20) and (21) are typical of full sequences of divided powers. Let \( q_\lambda(y) \) be a full sequence with
\[
q_\lambda(y \cup z) = \sum_\alpha q_\alpha(y)q_{\lambda-\alpha}(z).
\]
Then \( q_\lambda(y) \) is a full sequence of divided powers. Note that \( q_n(y) \) is a linear sequence of divided powers. Conversely, note that if \( q_n(y) \) is a linear sequence of divided powers and their product \( q_\lambda(y) = \prod q_\lambda_i(y) \) is a full sequence, then it is a full sequence of divided powers. However, that is not necessarily the only extension of \( q_n(y) \) to a full sequence, and the product may not be full. For example,
\[
\prod_{i \geq 1} \text{Id}_{\lambda_i}(y) = \text{Id}_{|\lambda|}(y).
\]

Similarly, if \( p_\lambda(y) \) is a full sequence obeying
\[
p_\lambda(y \cup z) = \sum_\alpha \binom{\lambda}{\alpha}p_\alpha(y)p_{\lambda-\alpha}(z)
\]
where \( \binom{\lambda}{\alpha} = \frac{\lambda!}{\alpha!(\lambda-\alpha)!} \) and \( \alpha! = \prod \alpha_i! \), then \( p_\lambda(y) \) is a full sequence of binomial type. Its linear subsequence \( p_{(n)}(y) \) is a linear sequence of binomial type.

Clearly, if \( p_\lambda(y) = \lambda!q_\lambda(y) \), then \( p_\lambda(y) \) is a full sequence of binomial type if and only if \( q_\lambda(y) \) is a full sequence of divided powers.

1.11 Transfer Operators and Adjoints

There is a unique map which sends any full sequence to another full sequence. In particular, we are interested in maps which send one full sequence of divided powers (or equivalently binomial type) onto another. Such maps we call transfer operators.

**Theorem 1.18**

1. The image of an full sequence of divided powers (resp. binomial type) under any transfer operator is another transfer operator.

2. A linear operator on symmetric functions is a transfer operator if and only if it is a homogeneous coalgebra isomorphism.

**Proof:** Coalgebra maps preserve the symmetric shift \( E^n \). We can define the adjoint of any linear map on symmetric functions or shift-invariant operators in a straightforward fashion; we are most interested in the adjoints of transfer operators (that is, coalgebra maps), and of automorphisms of the algebra of shift-invariant operators.
**Definition 1.19** Let $\theta$ be a linear map on the space of shift-invariant operators, and let $\phi$ be a linear map on the space of symmetric functions. Then $\text{adj}(\theta)$ and $\text{adj}(\phi)$ are defined by the relations

\[
\epsilon(\theta f(D))p(y) = \epsilon f(D) (\text{adj}(\theta)p(y))
\]
\[
\epsilon(\text{adj}(\phi)f(D))p(y) = \epsilon f(D) (\phi p(y))
\]

for all shift-invariant operators $f(D)$ and symmetric functions $p(y)$.

By Theorems 1.11 and 1.12, these adjoints are well defined. For example, the adjoint of the operator $f(D)$ on symmetric functions is the operator which multiplies shift-invariant operators by $f(D)$.

These adjoints are adjoints in the usual sense with respect to the action on the Hopf algebra of symmetric functions of the Hopf algebra of shift-invariant operators which has been identified with the dual of the Hopf algebra of symmetric functions.

Thus, $\text{adj}(\text{adj}(\theta)) = \theta$ and $\text{adj}(\text{adj}(\phi)) = \phi$. Moreover the adjoint of any transfer operator is an automorphism of the algebra of symmetric functions and visa versa.

**Proposition 1.20** The adjoint is an automorphism between the group of transfer operators, and the group of automorphisms of the algebra of shift-invariant operators.

We can make this connection more explicit.

**Theorem 1.21** Let $\theta$ be a transfer operator which maps the linear sequence of binomial type $p_n(y)$ associated with the quasi-species $(a_n)$ to the linear sequence of binomial type $q_n(y)$ associated with the quasi-species $(b_n)$. Let $\text{adj}(\theta) D_n/n! = \sum_{\lambda \vdash n} c_\lambda D_\lambda$. Then

\[
b_n = \sum_{\lambda \vdash n} c_\lambda \prod_{i \geq 1} a_{\lambda_i}.
\]

**Proof:** Consider the following sequence of equalities.

\[
b_n = \epsilon D_n q_n(y)/n! = \epsilon D_n \theta p_n(y)/n!
\]
\[
= \epsilon \text{adj}(\theta) D_n p_n(y)/n!
\]
\[
= \sum_{\lambda \vdash n} c_\lambda \epsilon D_\lambda p_n(y)
\]
\[
= \sum_{\lambda \vdash n} c_\lambda \prod_{i \geq 1} a_{\lambda_i}.
\]
**Theorem 1.22** Let $\theta$ be the linear operator on shift-invariant operators given by

$$\theta D_\lambda = \sum_{\mu \in \mathcal{P}} c_{\lambda \mu} D_\mu,$$

and let $\phi$ be the linear operator on symmetric functions given by

$$\phi m_\lambda(y) = \sum_{\mu \in \mathcal{P}} d_{\lambda \mu} m_\mu(y).$$

Then $\theta$ and $\phi$ are adjoint if and only if the matrices $(c_{\lambda \mu})_{\lambda, \mu \in \mathcal{P}}$ and $(d_{\lambda \mu})_{\lambda, \mu \in \mathcal{P}}$ are transposes of each other. That is, $\text{adj}(\theta) = \phi$ if and only if $c_{\lambda \mu} = d_{\mu \lambda}$ for all partitions $\lambda$ and $\mu$.

**Proof:** Consider the following series of equalities.

$$c_{\lambda \mu} = \epsilon \sum_{\nu \in \mathcal{P}} c_{\lambda \nu} D_\nu m_\mu(y)$$

$$= \epsilon \text{adj}(\phi) D_\lambda m_\mu(y)$$

$$= \epsilon D_\lambda \phi m_\mu(y)$$

$$= \epsilon D_\lambda \sum_{\nu \in \mathcal{P}} d_{\mu \nu} m_\nu(y)$$

$$= d_{\mu \lambda}. \quad \Box$$

The classical involution $\omega$ of the symmetric functions is the transfer operator which sends the elementary symmetric function to the complete symmetric function and back again. By Theorem 1.22, the coefficients of $\text{adj}(\omega)$ are given by the coefficients of the forgotten symmetric function $f_\lambda(y) = \omega m_\lambda(y)$. Thus, by [2, Theorem 8(ii)],

**Corollary 1.23** The adjoint of the classical involution $\omega$ of the symmetric functions is given by the involution

$$\text{adj}(\omega) D_n = \sum_{\lambda \vdash n} \frac{n!}{\prod_i i^{\lambda_i} \lambda_i!} D_\lambda. \quad \Box$$

### 1.12 Genera

We have left unanswered the question of what these full sequences of binomial type enumerate. Surely, they do not count functions enriched with mere species, for they do not possess enough structure. Indeed, we define a generalization of the species called the genus (plural: genera) whose enriched functions are counted by full sequences.
**Definition 1.24** (Genera) A genus is a functor $G$ from the category $\text{Part}$ of sets with partitions to the category of sets. For any partition $\pi$ of a finite set $E$, we say that the members of the set $[\pi]$ are $G$-structures, and for any bijection $f: E \to F$, we describe the function $G[f]: G[\pi] \to G[f(\pi)]$ as a relabeling of $G$-structures. For this paper, we need to assume that there is only one structure on the empty partition and there is at least one structure on partitions of a one element set; that is, $|G[\emptyset]| = 1$ and $|G[\{\{0\}\}]| \neq 0$.

Note that this definition is equivalent to the partitionals of [18]; however, it differs in the multiplication, exponentiation, and composition to be defined later. Species are then be seen as the restriction of genera to set partitions consisting of only one block.

The equivalence class of $G$ is thus determined by the number $G_\lambda$ of $G$-structures on set partitions $\pi \vdash S$ of type $\lambda$. (The type of $\pi$ is the integer partition $\lambda$ which corresponds to the size of its blocks. For example, if $\pi = \{\{a, b, c\}, \{d, e\}, \{f\}, \{g\}\}$, then the type of $\pi$ would be $\lambda = (3211)$.) The generating function of $G$ is defined to be the formal power series

$$\text{Gen}[G](y) = \sum_{\lambda \in \mathcal{P}} a_\lambda m_\lambda(y)/\lambda!.$$  

As with species, again we consider the category $C$ of complex numbers. Any functor from $\text{Part}$ to $C$ is called a quasi-genus. Note that any genus $G$ can be extended a quasi-genus $Q$ by composing with $\#$. We will perform computations with quasi-genera as if they were actual genera. In this case, $a_\lambda$ above will refer to the value of $Q$ on a partition of type $\lambda$.

Conversely, any genus can be restricted to a species by composing on the other side with the functor $\iota$ from the category $\mathcal{S}$ to the category $\text{Part}$ defined by $\iota: E \mapsto \{E\}$.

The generating functions for a genus $G$ and for its associated species $S = G \circ \iota$ are simply related.

$$\text{Gen}[G](t, 0, 0, \ldots) = \text{Gen}[S](t).$$  

(See Example 1.3 for a converse.)

We define the sum of two genera $G_1$ and $G_2$ on a partition $\pi \vdash S$ by their disjoint union

$$(G_1 + G_2)[\pi] = G_1[\pi] \cup G_2[\pi]$$  

so that

$$\text{Gen}[G_1](y) + \text{Gen}[G_2](y) = \text{Gen}[G_1 + G_2](y).$$
In other words, a $G_1 + G_2$-structure is either a $G_1$-structure or a $G_2$-structure. With respect to this addition, the set of all quasi-genera is a complex vector space. Hence, we can consider the following pseudo-basis:

$$G^{(\lambda)}[\pi] = \begin{cases} 1 & \text{if } \pi \text{ is of type } \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

The additive identity then is the genus with no structures whatsoever on any partition.

Similarly, we define the product of $G_1$ and $G_2$ on $\pi \vdash S$ to be set of quadruples

$$(G_1, G_2)[\pi] = \{ (S_1, S_2, A_1, A_2): S_1 \text{ and } S_2 \text{ are disjoint}, S_1 \cup S_2 = S, \text{ and } A_i \in G_i[\pi_i] \text{ where } \pi_i \text{ is the restriction of } \pi \text{ to } S_i. \}$$

In other words, as in Figure 18 in which each of the horizontal lines represents a block of $\pi \vdash S$, we divide $S$ in half—right and left, and place a $G_1$-structure on the restriction of $\pi$ to one half, and a $G_2$-structure on the restriction of $\pi$ to the other half.

Figure 18: Product of Genera

**Proposition 1.25** Let $G_1$ and $G_2$ be any two genera, then $Gen[G_1](y)Gen[G_2](y) = Gen[G_1G_2](y)$.

**Proof:** By linearity, it will suffice to consider the case $G_1 = G^{(\lambda)}$ and $G_2 = G^{(\mu)}$. Now, let $\pi = \{\pi_1, \pi_2, \ldots\} \vdash S$ be of type $\nu$ where the blocks $\pi_i$ are listed in decreasing order of size. We must count the number of ways to divide split $S$ into $S_1$ and $S_2$ such that $\pi$ restricted to $S_1$ and $S_2$ is of type $\lambda$ and $\mu$ respectively. Suppose further that $S_1 \cap \pi_i$ contains $\alpha_i$ elements and $S_2 \cap \pi_i$ contains $\beta_i$ elements. Then $\alpha$ and $\beta$ must be permutations of $\lambda$ and $\mu$ respectively, and $\alpha + \beta = \nu$. Subject to these conditions then there are $\nu!/\lambda!\mu!$ choices for $S_1$ and $S_2$. Hence,

$$G^{(\lambda)}G^{(\mu)} = \sum_{\nu \in \mathcal{P}} \sum_{\alpha + \beta = \nu} \frac{\nu!}{\lambda!\mu!} G^{(\nu)},$$

where $\alpha$ is a permutation of $\lambda$, $\beta$ is a permutation of $\mu$. 


On the other hand,
\[
\frac{m_\lambda(y) m_\mu(y)}{\lambda! \mu!} = \sum_{\alpha \text{ is a permutation of } \lambda} \sum_{\beta \text{ is a permutation of } \mu} y^{\alpha+\beta} / \lambda! \mu! \\
= \sum_{\nu \in \mathcal{P}} \sum_{\alpha+\beta=\nu} \frac{\nu!}{\lambda! \mu!} m_\nu(y). 
\]

Thus, the multiplicative identity is the genus $G^{(0)}$ which has one structure on the unique partition of the empty set, but has no structures for any other partitions.

Next, we define the exponentiation of a genus $G$ on $\pi \vdash S$ to be

\[
\exp(G)[\pi] = \left\{ (\phi, (A_B)_{B \in \phi}) : \phi \text{ is a partition of } S, (A_B)_{B \in \theta} \text{ is a sequence indexed by the blocks of } \phi \text{ for each block } B \text{ of the partition} \right\} \\
\left\{ \phi, A_B \in G[\pi_B] \text{ where } \pi_B \text{ is the restriction of } \pi \text{ to } B. \right\}
\]

That is, we divide $S$ into a number of parts and place a $G$ structure on the restriction of $\pi$ to each part. In

Figure 19: Exponentiation and Composition of Genera

Figure 19, the horizontal lines represent the blocks of $\pi$ and the circled areas represent the blocks $\phi$. Since $\exp(G)$ is equivalent to $\sum_{n \geq 0} G^n / n!$, we immediately have

\[
\exp \left( \text{Gen}(G)(y) \right) = \text{Gen} \left( \exp(G) \right)(y).
\]
Similarly, we define the composition of $G_1$ with $G_2$ on $\pi \vdash S$ to be

$$\exp(G)[\pi] = \{ (\phi, (A_B)_{B \in \phi}, C) : \phi \text{ is a partition of } S, (A_B)_{B \in \phi} \text{ is a sequence indexed by the blocks of } \phi \text{ for each block } B \text{ of the partition } \phi, A_B \in G[\pi_B] \text{ where } \pi_B \text{ is the restriction of } \pi \text{ to } B, \text{ and } C \in G_1[\zeta] \text{ where } \zeta \text{ is the partition of the blocks of } \phi \text{ defined by the equivalence relation } B_1 \sim B_2 \text{ if and only if for every } P \in \pi, \text{ we have } |B_1 \cap P| = |B_2 \cap P| \}$$

That is, we divide $S$ into a number of parts, place a $G_2$-structure on the restriction of $\pi$ to each parts. Finally we consider the partition of the parts of $S$ classified according to their intersections with the blocks of $\pi$ (as shown in in Figure [12] by wavy lines), and we place a $G_1$-structure on it.

Recall that the plethystic composition $p \circ q(y)$ of two symmetric functions $p(y)$ and $q(y)$ is defined as follows. We express $g(y)$ as a sum of monomials $g(y) = \sum_{\alpha} c_\alpha y^\alpha$, and introduce a new set of variables $z$ defined by $\prod_{i=1}^{\ell}(1 + z_i t) = \prod_{\alpha}(1 + x^\alpha t)^{c_\alpha}$. Finally, we define $p \circ q(y) = p(z)$.

The generating function of a composition of genera can be expressed in term of plethystic composition.

**Theorem 1.26** Let $G_1$ and $G_2$ be any two genera, then $Gen[G_1(G_2)](y) = (Gen[G_1] \circ Gen[G_2])(y)$.

**Proof**: By linearity, it will suffice to consider the case $G_1 = G^{(\lambda)}$. Now, let $G = G_2 = \sum_{\mu \in P} c_\mu G^{(\mu)}$.

Now, let’s compute the number of $G^{(\lambda)}(G)$ structures on the partition $\pi = \{ \pi_1, \pi_2, \ldots \} \vdash S$ of type $\nu$ where the blocks $\pi_i$ are listed in decreasing order of size. We must divide $S$ into exactly $|\lambda|$ parts which intersect with $\pi$ in exactly $\ell(\lambda)$ different ways with the $i$th way occurring exactly $\nu_i$ times. Let $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(\ell(\lambda))}$ represent these various intersection types. These intersection types must all be distinct. However, two sets of intersection types differing only by the rearrangement of $\alpha^{(i)}$ and $\alpha^{(j)}$ corresponding to $\lambda_i = \lambda_j$ are not considered distinct; this caveat will be denoted (*). Next, let $S_1, S_2, \ldots, S_{\ell(\lambda)}$ be the $\ell(\lambda)$ blocks of intersection type $\alpha^{(i)}$. That is, $S_1 \cap \pi_k = \alpha^{(i)}_k$ and the $S_{\ell(\lambda)}$ form a partition of $S$. Then $\nu = \lambda_1 \alpha^{(1)} + \cdots + \lambda_{\ell(\lambda)} \alpha^{(\ell(\lambda))}$. Each $S_{\ell(\lambda)}$ is then endowed with a $G$ structure in one of $c_\mu$ way where $\mu^{(i)}$ is the partition formed by sorting the vector $\alpha^{(i)}$. Hence,

$$G^{(\lambda)}(G) = \sum_{\nu} \sum_{\mu^{(i)} \in P} \sum_{(\ast) \lambda_1 \alpha^{(1)} + \lambda_2 \alpha^{(2)} + \cdots = \nu} \prod_{i=1}^{\ell(\lambda)} \left( \frac{c_{\mu^{(i)}}}{\theta^{i}} \right)^{\lambda_i} G^{(\nu)} \nu! / \lambda!$$

where the $H^{(1)}(y)$ are a set of genera of which $c_{\mu}/\mu!$ equals $G^{(\mu)}$ for each partition $\mu \in P$ and $\theta$ is the linear operator $\theta : m_{\nu}(y) \mapsto \nu^! m_{\nu}(y)$. The conclusion now follows immediately. ■

Thus, the compositional identity is the genus $G^{(1)}$ which has one structure on the unique partition of any one element set, and no structures otherwise.
Finally, we define the $n^{th}$ iterated derivative $D_n G$ of a genus $G$ on a partition $\pi$ to be the set consisting of $G$-structures on a partition $\pi \cup \{\{\infty_1, \ldots, \infty_n\}\}$ containing a new block of size $n$ along with a complete ordering of this new block. Thus, $D_n G(\lambda) = n! G(\lambda^{(n)})$ so that

$$D_n \text{Gen}[G](y) = \text{Gen}[D_n G](y).$$

By considering species and their generating functions, we immediately observe that $\text{pow}_n \circ \text{pow}_m(y) = \text{pow}_{nm}(y)$ from which we can deduce that general composition is associative.

Having now completely defined an adequate generalization of species. We can now consider function enriched by it.

**Definition 1.27 (Generic Functions)** Let $S$ and $X$ be sets, and let $\pi$ be a partition of $S$. $f$ is called a generic function or a function enriched by the genus $G$ if it is a function from $S$ to $X$ which is equipped with a $G$-structure for the restriction of $\pi$ to each of its fibers.

![Figure 20: Typical Generic Function](image)

**Theorem 1.28** A full sequence of symmetric functions $p_\lambda(y)$ is of binomial type if and only if it enumerates the generic functions from a set $S$ equipped with a partition $\pi \vdash S$ of type $\lambda$ to the set $X = \{y_1, y_2, \ldots\}$ for some genus (or quasi-genus) $G$.

Proof: (If) The left side of equation (22) enumerates the set of generic functions from $\pi \vdash S$ to $X_1 \cup X_2$. The right side of equation (22) counts the number of ways to divide $S$ into two parts, and map the parts to $X_1$ and $X_2$ respectively with generic functions. The sum is over the types of the resulting restrictions of the partition $\pi$. Obviously, these two set of generic functions are identical.
(Only If) Define the numbers $G_{\lambda, \alpha}$ by the identity

$$D_n p_\lambda(y) = \sum_\alpha \binom{\lambda}{\alpha} G_{\lambda, \alpha} p_{\lambda-\alpha}(y)$$

Thus,

$$E^z p_\lambda(y) = \sum_\alpha \binom{\lambda}{\alpha} z^{|\alpha|} G_{\lambda, \alpha} |\alpha|! p_{\lambda-\alpha}(y).$$

However, by equation (22), $z^{|\alpha|} G_{\lambda, \alpha}/|\alpha|!$ does not depend on $\lambda$. Hence, $G_{\lambda, \alpha} = G_\alpha$ does not depend on $\lambda$. Moreover, it is invariant under permutation of $\alpha$. Thus, $G_\alpha$ is a quasi-genus, and $p_\lambda(y)$ enumerates its generic functions.

Restricting our attention to species, we have

**Corollary 1.29** A linear sequence of symmetric functions $p_n(y)$ is of binomial type if and only if it enumerates the functions from $N$ to $X = \{y_1, y_2, \ldots\}$ enriched by some species (or quasi-species) $S$.

The two examples of full sequences of binomial type given are associated with fairly simple genera.

**Example 1.1** $e_\lambda(y)$ enumerates functions enriched with a genus $G$ with a single structure for any partition consisting solely of one element blocks, and no structures otherwise.

**Example 1.2** $h_\lambda(y)$ enumerates functions enriched with a genus $G$ with $\lambda!$ structures on any partition of type $\lambda$. For example, $G[\pi]$ could be the set of posets which completely order the blocks of $\pi$ yet leave elements of distinct blocks incomparable.

As we have explained about the products of most linear sequences of symmetric functions of binomial type form similar full sequences.

**Example 1.3** Let $p_n(y)$ be the linear sequence of symmetric functions of binomial type associated with the quasi-species $S = (a_n)_{n \geq 0}$. Then if full, $p_\lambda(y) = \prod_i p_{\lambda_i}(y)$ is the full sequence of symmetric functions of binomial type associated with the quasi-genus $G = (a_\lambda)_{\lambda \in P}$ where $a_\lambda = \prod_i a_{\lambda_i}$. That is, each $G$ structure on $\pi$ is a sequence of $S$-structures—one for each block of $\pi$. The generating function for $G$ is

$$\text{Gen}(G)(y) = \prod_{y \in X} \text{Gen}(S)(y).$$

Conversely, we have noted that every full sequence of symmetric function of binomial type restricts to a similar linear sequence.
Example 1.4 Let $p_\lambda(y)$ be the full sequence of symmetric functions of binomial type associated with the quasi-genus $G = (G_\lambda)_{\lambda \in \mathcal{P}}$. Then $p_n(y) = p_{(n)}(y)$ is the linear sequence associated with the quasi-species $S = G \circ \iota$. That is, $S_n = S_{(n)}$, and for any set $E$, $S[E] = G[E]$. (See Figure 17.)

Thus, derivatives of a full sequence of symmetric functions of binomial type are now easy to compute.

Proposition 1.30 Let $p_\lambda(y)$ be the full sequence of symmetric functions of binomial type associated with the quasi-genus $(G_\lambda)_{\lambda \in \mathcal{P}}$. Its symmetric derivative is

$$Dp_\lambda(y) = \sum_{i \geq 1} iG_{(1)} \text{mult}_i(\lambda)p_\mu(y)$$

where $\mu = \lambda \setminus i \cup \{i - 1\}$, and more generally

$$D_n p_\lambda(y) = n! \sum_{|\alpha| = n} \binom{\lambda}{\alpha} G_\alpha p_{\lambda - \alpha}(y)$$

Proof: To compute the $n$th iterated derivative, we distinguish an element of the range of the generic functions, and count generic functions for which the inverse image of that element contains exactly $n$ points.

Figure 21: Contribution to the Derivative of a Full Sequence of Binomial Type

Here, we sum over how these $n$ points are arranged within the block structure of the domain of the generic function. □

Corollary 1.31 Let $p_\lambda(y)$ be the full sequence of symmetric functions of binomial type associated with the quasi-genus $G = (G_\lambda)_{\lambda \in \mathcal{P}}$. Its shift is

$$E^x p_\lambda(y) = \sum_{\alpha} \binom{\lambda}{\alpha} x^{|\alpha|} G_\alpha p_{\lambda - \alpha}(y).$$
From Corollary 1.31 and equation (22), we have the following explicit formula for a full sequence of symmetric functions of binomial type.

**Corollary 1.32** The full sequence of symmetric functions of binomial type \( p_\lambda(y) \) associated with the quasi-genus \( (G_\lambda)_\lambda \in P \) is given by the formula

\[
p_\lambda(x, y, z, \ldots) = \sum_{\lambda=\alpha+\beta+\gamma+\cdots} G_\alpha G_\beta G_\gamma \cdots x^{[\alpha]} y^{[\beta]} z^{[\gamma]} \cdots.
\]

Thus, its evaluation is given by

\[
\epsilon p_\lambda(y) = \delta_{\lambda,(0)}.
\]

Also, we see that the definition of a full sequence was not really as strict as one may have thought.

**Corollary 1.33** If \( p_\lambda(y) \) is associated with a quasi-genus, and is a basis for \( \Lambda \), then it is a full sequence.

Now, by iterating Proposition 1.30, we have

**Corollary 1.34** Let \( p_\lambda(y) \) be the full sequence of symmetric functions of binomial type associated with the quasi-genus \( (G_\lambda)_\lambda \in P \). Then

\[
D_\mu p_\lambda(y) = \sum_{|\alpha^{(i)}|=\mu_i} \binom{\lambda}{\alpha^{(1)}, \alpha^{(2)}, \ldots} G_{\alpha^{(1)}} G_{\alpha^{(2)}} \cdots p_{\lambda-\alpha^{(1)}-\alpha^{(2)}}(y)
\]

where each \( \alpha^{(i)} \) is a partition, and \( \binom{\lambda}{\alpha^{(1)}, \alpha^{(2)}, \ldots} \) is given by the product of multinomial coefficients

\[
\binom{\lambda}{\alpha^{(1)}, \alpha^{(2)}, \ldots} = \prod_{i \geq 1} \binom{\lambda_i}{\alpha^{(1)}_i - \alpha^{(2)}_i - \cdots, \alpha^{(1)}_i, \alpha^{(2)}_i, \ldots} = \frac{\lambda!}{(\lambda - \alpha^{(1)}_i - \alpha^{(2)}_i - \cdots)!\alpha^{(1)}_i!\alpha^{(2)}_i! \cdots}.
\]

Finally, we compute the coefficients of transfer operators in terms of genera.

**Theorem 1.35** Let \( p_\lambda(y) \) be the full sequence of binomials type associated with the quasi-genus \( G = (G_\lambda)_\lambda \in P \). Suppose \( \theta \) is the transfer operator given by

\[
\text{adj}(\theta)D_n/n! = \sum_{\lambda \vdash n} \epsilon c_\lambda D_\lambda.
\]

Then \( \theta p_\lambda(y) \) is the full sequence of binomial type associated with the quasi-genus \( (H_\lambda)_\lambda \in P \) where

\[
H_\lambda = \sum_{\mu \vdash n} \epsilon c_\mu \sum_M \binom{\lambda}{M} G_{M_1} G_{M_2} \cdots
\]

where \( M \) is a matrix of nonnegative integers whose column sums are \( \lambda \), whose row sums are \( \mu \), and whose \( i \)th row is \( M_i \) and \( \binom{\lambda}{M} = \lambda! / \prod_{i,j} M_{ij}! \).
Proof: Consider the following sequence of equalities.

\[
H_\lambda = \epsilon \sum_{|\alpha| = n} \binom{\lambda}{\alpha} H_{\alpha \lambda - \alpha}(y)
\]

\[
= \epsilon D_n q_\lambda(y)/n!
\]

\[
= \epsilon D_n \theta p_\lambda(y)/n!
\]

\[
= \epsilon \text{adj}(\theta) D_n p_\lambda(y)/n!
\]

\[
= \epsilon \sum_{\mu \vdash n} c_\mu D_\mu p_\lambda(y)
\]

\[
= \epsilon \sum_{\mu \vdash n} c_\mu \sum_{|\alpha^{(1)}| = \mu_1} \binom{\lambda}{\alpha^{(1)}, \alpha^{(2)}, \ldots} G_{\alpha^{(1)}} G_{\alpha^{(2)}} \cdots p_{\lambda - \alpha^{(1) - \alpha^{(2)} - \ldots}}(y)
\]

\[
= \sum_{\mu \vdash n} c_\mu \sum_{M} \binom{\lambda}{M} G_{M_1} G_{M_2} \cdots \blacksquare
\]

When one takes \( \lambda = (n) \), Theorem 1.35 specializes to Theorem 1.21.
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