FINITENESS OF THE HOFER-ZEHNDER CAPACITY OF NEIGHBORHOODS
OF SYMPLECTIC SUBMANIFOLDS

GUANGCUN LU

ABSTRACT. We use the minimal coupling procedure of Sternberg and Weinstein and our
pseudo-symplectic capacity theory to prove that every closed symplectic submanifold in
any symplectic manifold has an open neighborhood with finite (π₁-sensitive) Hofer-Zehnder
symplectic capacity. Consequently, the Weinstein conjecture holds near closed symplectic
submanifolds in any symplectic manifold.

1. INTRODUCTION AND RESULTS

The existence of periodic orbits of Hamiltonian flows near a closed symplectic submani-
fold was recently studied by several authors, (cf. [2, 6, 11, 23, 35]). This question is closely
related to the following:

Question 1.1. Does every compact symplectic submanifold B in any symplectic manifold
(M, ω) have a neighborhood with finite (π₁-sensitive) Hofer-Zehnder symplectic capacity?

A positive answer to Question 1.1 has some nice applications in symplectic topology
and Hamiltonian dynamics. For example, it directly implies the existence of (contractible)
Hamiltonian periodic orbits on a generic hypersurface near the symplectic submanifold.

For a closed symplectic submanifold B of a geometrically bounded symplectic manifold
(M, ω), if (M, ω) is symplectically aspherical, that is, ω|π₂(M) = 0 and c₁(TM)|π₂(M) =
0, Cieliebak, Ginzburg and Kerman [2] used symplectic homology to show that for a smooth
function H : M → ℝ which attains an isolated minimum on B, the levels \{H = ε\} carry
contractible periodic orbits for a dense set of small values ε > 0. Under the same assump-
tions, it was proved by Ginzburg and Gürel in [6] that for a sufficiently small neighborhood
U of B, there is a constant C = C(U) such that the Hamiltonian flow of every smooth
function H supported in U with min_B H > C has a nontrivial contractible periodic orbit of

period less than or equal to one. In [25, Corollary 1.3], Macarini showed that for a closed symplectic submanifold $B$ of any geometrically bounded symplectic manifold $(M, \omega)$, there exists a neighborhood $U$ of $B$ in $M$ such that if $H$ is a proper Hamiltonian on $U$ and constant on $B$, then $H$ has periodic orbits with contractible projection on $B$ on almost all energy levels. Recently, Schlenk [35] generalized Hofer’s energy-capacity inequality and used it to prove that for a closed submanifold $B$ in geometrically bounded and stably strongly semi-positive symplectic manifolds, if either $\dim B < \text{codim} B$, or $\dim B = \text{codim} B$ and $B$ is not Lagrangian, then $B$ has a small open neighborhood with finite ($\pi_1$-sensitive) Hofer-Zehnder capacity. In [23], the author introduced the pseudo symplectic capacity theory and used it to answer affirmably Question 1.1 for any closed symplectic submanifold of codimension two in any symplectic manifold.

Very recently, motivated by these and a very useful formula for the Hofer-Zehnder symplectic capacity of the product of a symplectic manifold and the standard symplectic ball, (cf. [4, 8, 15, 16, 23, 30]), Kerman [11] proposed the following.

**Question 1.2.** Let $B$ be a closed symplectic submanifold of a symplectic manifold $(M, \omega)$, and let $U_R$ be a symplectic tubular neighborhood of $B$ with (sufficiently small) radius $R$. Is the Hofer-Zehnder capacity of $U_R$ equal to $\pi R^2$?

Here $U_R$ is defined as follows. Let $\pi : (E, \sigma_E) \to B$ be the symplectic normal bundle of $B$ in $(M, \omega)$, and $J_E \in \mathcal{J}(E, \sigma_E)$, where $\mathcal{J}(E, \sigma_E)$ is the space of all complex structures on the vector bundle $E \to B$ compatible with $\sigma_E$, (cf. [29, page 69]). Denote by $g_E = \sigma_E \circ (1 \times J_E)$ the Hermitian metric on $E$. Then with a Hermitian connection $\nabla$ on $E$, one can extend a canonical fiberwise 1-form $\alpha_E$ on $E \setminus B$ defined by $\alpha_E(b, v)(\cdot) = g_E(v, v)^{-1}\sigma_E(b)(v, \cdot)$ to a genuine 1-form $\alpha$ on $E \setminus B$. In [11], $U_R = \{(b, v) \in E | g_E(b)(v, v) \leq R^2\}$ equipped with the symplectic form $\omega|_B + d(g_E(v, v)\alpha)$ was called a symplectic tubular neighborhood of $B$.

Clearly, by Weinstein’s symplectic neighborhood theorem, Question 1.2 implies Question 1.1 since $U_R$ is symplectomorphic to a neighborhood of $B$ in $(M, \omega)$ for $R > 0$ small enough. Using Floer homology, Kerman [11] affirmably answered Question 1.2 for the closed symplectic submanifold $B$ of dimension $2m$ and codimension $2k$ in a geometrically bounded, symplectically aspherical manifold $(M, \omega)$ whose unit normal bundle $S(E)$ is homologically trivial in degree $2m$ in the sense that $H_{2m}(S(E), \mathbb{Z}_2) = H_{2m}(B, \mathbb{Z}_2) \oplus$
$H_{2(m-k)+1}(B, \mathbb{Z}_2)$. Actually Questions 1.1, 1.2 may be viewed as special cases of a general question: how to compute the symplectic capacities of a symplectic fibration equipped with a compatible symplectic form, which was proposed by C. Viterbo to the author during his visit to Institut des Hautes Études Scientifiques (IHES) in Spring 1999.

The proof of in [23, Theorem 1.24], which was partially motivated by [1, 31], consists of three steps. The first step is to construct a suitable projective bundle over the submanifold and a symplectic form on the total space of the projective bundle such that the symplectic submanifold has a neighborhood symplectomorphic to that of the zero section in the projective bundle. Then we proved that the total space of the projective bundle is symplectically uniruled with respect to the chosen symplectic form. Finally, the desired result follows from the properties of our pseudo symplectic capacity developed in that paper. Since we there considered the symplectic submanifold $B$ of codimension two, its symplectic normal bundle has real rank 2, and thus may be viewed a (complex) line bundle. The projectivized bundle of the sum of the latter and trivial complex line bundle is a $\mathbb{C}P^1$-bundle, and it is not hard to construct a symplectic form on the total space of it for which this $\mathbb{C}P^1$-bundle is symplectically uniruled and naturally contains $B$ as a symplectic submanifold with the symplectic normal bundle being isomorphic to that of $B$ in the original symplectic manifold. For the symplectic submanifolds of higher codimension, the construction of the expected symplectic form on that kind of projective bundles needs be elaborated. This can be completed with the construction of the minimal coupling by Sternberg and Weinstein. To the knowledge of the author Polterovich, [31] first used the coupling form in the study of symplectic topology. It had been furthermore used in [32] (also see [33]) and [3].

To state our results, we need to review several notions. Firstly, for the conveniences of the readers, we recall the definition of the $\pi_1$-sensitive Hofer-Zehnder capacity $c_{HZ}$. It was introduced by the author [15, 16] (denoted by $\bar{C}_{HZ}$), and Schwarz [36] independently. Recall that in [9], a smooth real function $H$ on a symplectic manifold $(M, \omega)$ is called admissible if there exist an nonempty open subset $U$ and a compact subset $K \subset M \setminus \partial M$ such that

(a) $H|_U = 0$ and $H|_{M \setminus K} = \max H$;
(b) $0 \leq H \leq \max H$;
(c) $\dot{x} = X_H(x)$ has no nonconstant fast periodic solutions.
Here $X_H$ is defined by $i_{X_H} \omega = dH$, and “fast” means “of period less than 1”. Denote by $\mathcal{H}_{ad}(M, \omega)$ the set of admissible Hamiltonians on $(M, \omega)$. The Hofer-Zehnder symplectic capacity of $(M, \omega)$ is defined by $c_{HZ}(M, \omega) = \sup \{ \max H \mid H \in \mathcal{H}_{ad}(M, \omega) \}$. If the condition (c) in the definition of the admissibility is replaced by the condition $(c)^0$, $\dot{x} = X_H(x)$ has no nonconstant fast and contractible periodic solutions, the corresponding function $H$ is said to be $0$-admissible. Let $\mathcal{H}^0_{ad}(M, \omega)$ be the set of 0-admissible Hamiltonians on $(M, \omega)$.

Then the $\pi_1$-sensitive Hofer-Zehnder capacity of $(M, \omega)$ is defined by

$$
(1.1) \quad c^0_{HZ}(M, \omega) = \sup \{ \max H \mid H \in \mathcal{H}^0_{ad}(M, \omega) \}.
$$

It always holds that $\mathcal{W}_G \leq c_{HZ}(M, \omega) \leq c^0_{HZ}(M, \omega)$ for them and the Gromov symplectic width $\mathcal{W}_G$.

Recall that a symplectic fibration $\Pi : M \rightarrow B$ with symplectic fibre $(F, \sigma)$ is a fibration whose structure group is a subgroup of $\text{Symp}(F, \sigma)$. In this case, for any local trivialization $\Phi : \Pi^{-1}(U) \rightarrow U \times F$ and any $b \in U$, there exists a natural symplectic form $\sigma_b = \Phi_b^* \sigma$ on the fiber $F_b$ which is independent of the choice of the local trivialization $\Phi$, where $\Phi_b : F_b \rightarrow F$ is the restriction of $\Phi$ to $F_b$ followed by the projection onto $F$. A symplectic form $\omega$ on $M$ is said to be compatible with the symplectic fibration $\Pi$ if $\sigma_b = i_b^* \omega$ for each inclusion $i_b : F_b \hookrightarrow M$ of the fibre, that is, each fibre $(F_b, \sigma_b)$ is a symplectic submanifold of $(M, \omega)$.

Let $\pi : (E, \bar{\omega}) \rightarrow B$ be a $2n$-dimensional symplectic vector bundle over a compact symplectic manifold $(B, \beta)$ (with or without boundary). It may be naturally viewed as a symplectic fibration with symplectic fibre $(\mathbb{R}^{2n}, \omega_{std})$, and hence each fibre $E_b$ carries a natural symplectic structure $(\omega_{std})_{b}$ (cf. Section 3 for details). Throughout this paper, we use $\omega_{std}$ (or $\omega_{std}^{(n)}$ if necessary) to denote the standard symplectic form on $\mathbb{R}^{2n}$. With a $\bar{\omega}$-compatible complex structure $J_E \in \mathcal{J}(E, \bar{\omega})$, one gets a Hermitian structure $(\bar{\omega}, J_E, g_{J_E})$ on $E$. Denote by $F : U(E) \rightarrow B$ the bundle of unitary frames of $E$. It is a principal $U(n)$-bundle, and $E$ is an associated bundle, $E = U(E) \times_{U(n)} \mathbb{C}^n$, (see (3.18) for an explicit identification). Hereafter, the unitary group $U(n)$ acts on $\mathbb{C}^n$ via

$$
(1.2) \quad U \cdot (z_1, \cdots, z_n) = (z_1, \cdots, z_n)U \quad \forall U = X + iY \in U(n).
$$

This action is Hamiltonian with respect to $\omega_{std}$, and has the moment map

$$
(1.3) \quad \mu_{U(n)} : \mathbb{C}^n \rightarrow \mathfrak{u}(n), \quad z = (z_1, \cdots, z_n) \mapsto i \frac{z^t \bar{z}}{2}
$$
after identifying the Lie algebra \( u(n) = T_e U(n) \) with its dual \( u(n)^* \) via the inner product \( (X, Y) = \text{Tr}(\bar{X}^t Y) \). Denote by

\[
D_\varepsilon(E) = U(E) \times_{U(n)} B^{2n}(\varepsilon)
\]

the open disk bundle of radius \( \varepsilon > 0 \). Here \( B^{2n}(\varepsilon) = \{ z \in \mathbb{C}^n \mid |z| < \varepsilon \} \). Let \( \mathcal{A}(U(E)) \) be the affine space of all connection \( (u(n)-\text{value}) \)-1-forms on \( U(E) \). By the construction due to Sternberg and Weinstein, (cf. Theorem 3.2(8)), for each \( A \in \mathcal{A}(U(E)) \) there are \( 0 < \varepsilon_0 = \varepsilon_0(F, A) \leq 1 \) and a canonical symplectic form \( \bar{\omega}_A \) in \( D_{\varepsilon_0}(E) \) such that

\[
\bar{\omega}_A|_{D_{\varepsilon_0}(E)_b} = (\omega_{\text{std}})_b|_{D_{\varepsilon_0}(E)_b} \quad \forall b \in B,
\]

and that the symplectic normal bundle of the zero section \( 0_E \) in \( (D_{\varepsilon_0}(E), \bar{\omega}_A) \) is \( (E, \bar{\omega}) \). For each \( t > 0 \) let

\[
\bar{\omega}_A^t := \pi^* \beta + t(\bar{\omega}_A - \pi^* \beta).
\]

It restricts to \( t(\omega_{\text{std}})_b \) on each fibre \( D_{\varepsilon_0}(E)_b \), and for every \( 0 < \varepsilon < \varepsilon_0 \) there is \( t_0 = t_0(A, \varepsilon) > 0 \) such that \( \bar{\omega}_A^t \) is also symplectic in \( D_{\varepsilon}(E) \) for each \( 0 < t < t_0(A, \varepsilon) \), the reader may refer to Theorems 3.3(v). The pair \( (D_{\varepsilon}(E), \bar{\omega}_A^t) \) is our symplectic tubular neighborhood of \( B \). Our main result is the following theorem.

**Theorem 1.3.** Let \( B \) be any closed symplectic submanifold in any symplectic manifold \( (M, \omega) \) and let \( \pi : (E, \bar{\omega}) \to B \) be the symplectic normal bundle of \( B \) in \( (M, \omega) \). For a given \( J_E \in J(E, \bar{\omega}) \), let \( F : U(E) \to B \) be the corresponding bundle of unitary frames. Then for a connection 1-form \( A \in \mathcal{A}(U(E)) \) and \( \bar{\omega}_A^t \) in (1.6) with \( \beta = \omega|_B \), it holds that

\[
c_{HZ}(D_{\varepsilon}(E), \bar{\omega}_A^t) \leq c_{HZ}^0(D_{\varepsilon}(E), \bar{\omega}_A^t) \leq \pi t \varepsilon^2
\]

for any \( 0 < \varepsilon < \varepsilon_0(F, A) \) and \( 0 < t < t_0(\varepsilon, A) \). Furthermore

\[
W_G(D_{\varepsilon}(E), \bar{\omega}_A^t) = c_{HZ}(D_{\varepsilon}(E), \bar{\omega}_A^t) = c_{HZ}^0(D_{\varepsilon}(E), \bar{\omega}_A^t) = \pi t \varepsilon^2
\]

for sufficiently small \( \varepsilon > 0 \) and \( t > 0 \). So for any \( \varepsilon > 0 \) there exists an open neighborhood \( W \) of \( B \) in \( M \) such that

\[
W_G(W, \omega) = c_{HZ}(W, \omega) = c_{HZ}^0(W, \omega) < \varepsilon.
\]
From the proof of Theorem 1.3, it is easily seen that for the above
\((U_R, \omega|_B + d(g_E(v,v)\alpha))\) there exist \(0 < \epsilon \ll \epsilon\) and a symplectic embedding from
\((D_\epsilon(E), \tilde{\omega}_A^t)\) into \((U_R, \omega|_B + d(g_E(v,v)\alpha))\) which maps the zero section \(0_E\) onto \(0_E\). Conversely, there exist \(0 < r \ll R\) and a symplectic embedding from \((U_r, \omega|_B + d(g_E(v,v)\alpha))\) into \((D_\epsilon(E), \tilde{\omega}_A^t)\) mapping the zero section \(0_E\) onto \(0_E\).

It is well known that the Hofer-Zehnder capacity and pseudo symplectic capacity are closely related to the famous Weinstein conjecture in [40], (cf. [9, 23]). Every compact smooth hypersurface \(S\) in a symplectic manifold \((M, \omega)\) determines a distinguished line bundle \(TS \supset L_S \rightarrow S\) whose fiber at \(x \in S\) is given by \(T_xS \cap (T_xS)^\omega\). A closed characteristic of \(S\) is an embedded circle \(P \subset S\) satisfying \(TP = L_S|_P\). The hypersurface \(S\) is said to be of contact type if there exists a Liouville vector field \(X\) (i.e., \(L_X\omega = \omega\)) in some neighborhood of it which is transversal to \(S\) everywhere. Weinstein [40] conjectured that every hypersurface \(S\) of contact type in symplectic manifolds carries a closed characteristic. After Viterbo [39] first proved it in the standard Euclidean symplectic space this conjecture was proved in many symplectic manifolds, (cf. [5, 23, 35] and the references therein for a detailed description of the progress on this question). As a direct consequence of (1.9), we get the following corollary.

**Corollary 1.4.** Let \(W\) be a neighborhood of \(B\) as in (1.9). Then for every smooth function
\(H : M \rightarrow \mathbb{R}\) supported in \(W\) and with \(\max H - \min H > c^\circ_{HZ}(W)\), its Hamiltonian flow has a nontrivial contractible periodic orbit of period less than or equal to one. Moreover, for a compact hypersurface \(S\) contained in \(W\) and every thickening of \(S\) in \(W\), \(\psi : S \times (-1, 1) \rightarrow W\) there is a closed characteristic on \(S_t := \psi(S \times t)\) for almost every \(t \in (-1, 1)\). In particular, the Weinstein conjecture holds near a closed symplectic submanifold \(B\) in any symplectic manifold \((M, \omega)\).

So Corollary 1.4 generalizes the corresponding results in [2, 6, 11, 23, 25]. The proof from (1.9) to Corollary 1.4 is standard, see [10, 26, 38]. Another direct consequence of (1.9) is the following corollary.

**Corollary 1.5.** Let \((N, \sigma)\) be any closed symplectic manifold. Then there is an open neighborhood \(U\) of the zero section in the twisted cotangent bundle \((T^*N, \omega_{can} + \pi^*\sigma)\) such that \(c^\circ_{HZ}(U, \omega_{can} + \pi^*\sigma) < \infty\).
This implies the existence of contractible periodic orbits of a charge on the symplectic manifold $N$ subject to the magnetic field $\sigma$ on almost every sufficiently small energy level. A more general result was recently obtained in [35] by a different method. The reader may also find some related interesting results in [11]. Equation (1.9) also implies some nonsqueezing phenomenon in symplectic geometry which was first discovered by Gromov in his celebrated paper [7]. For example, it implies that a given standard symplectic ball $(B^{2n}(r), \omega_{\text{std}})$ cannot be symplectically embedded in a small neighborhood of a closed symplectic submanifold in any symplectic manifold of dimension $2n$.

The organization of the paper is as follows. In the next section we first review the minimal coupling procedure by Sternberg and Weinstein following [37, Appendix], and then prove the main result Theorem 2.5 therein. In Section 3, we shall construct the symplectic forms on the total space of the projectivized bundle of the sum of the symplectic normal bundle and the trivial line bundle, and study their properties. Theorems 3.2, 3.3 summarize our main results in that section. The main result Theorem 1.3 will be proved in Section 4 after main results Theorems 4.4-4.5 in that section.

2. Minimal Coupling and Symplectic Reduction

In this section, we first briefly review how to use the minimal coupling form procedure of Sternberg and Weinstein to construct the symplectic structures on the associated bundle, and then point out some properties of such symplectic structures. Those properties are needed in our arguments and are easily proved. Our main reference is [37, Appendix].

Let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$, and let $\langle \cdot, \cdot \rangle$ denote the pairing between the dual Lie algebra $\mathfrak{g}^*$ and $\mathfrak{g}$. For a Hamiltonian $G$-space $(M, \omega)$, let $\mu_G : M \to \mathfrak{g}^*$ denote the Ad$^*$-equivariant moment map. The famous Marsden-Weinstein reduction theorem is shown as follows.

**Theorem 2.1.** Suppose that 0 is a regular value of $\mu_G$ and that the group $G$ acts freely and properly on $\mu_G^{-1}(0)$. Then $\mu_G^{-1}(0)$ is a coisotropic submanifold in $(M, \omega)$ and the corresponding isotropic foliation is given by the orbits of $G$. Moreover, there exists a unique symplectic form $\omega_G$ on the Marsden-Weinstein quotient $M//G := \mu_G^{-1}(0)/G$ such that $q_G^*\omega_G = \omega|_{\mu_G^{-1}(0)}$, where $q_G : \mu_G^{-1}(0) \to M//G$ is the quotient projection.
A symplectic submanifold $N \subset M$ is called a Hamiltonian $G$-subspace of $(M, \omega)$ if $g \cdot N = \{ g \cdot x | x \in N \} \subset N$ for each $g \in G$. Clearly, for a Hamiltonian $G$-subspace $N \subset M$ of $(M, \omega)$, the moment map of the $G$-action on $(M, \omega)$ restricts to that of the induced $G$-action on $(N, \omega|_N)$. By Theorem 2.1, one easily gets the following proposition.

**Proposition 2.2.** For a Hamiltonian $G$-subspace $N \subset M$ of $(M, \omega)$, if $0$ is also a regular value of $\mu_G|_N$ and $G$ also acts freely and properly on $(\mu_G|_N)^{-1}(0)$, then the restriction of $\omega_G$ to $N/G = (\mu_G|_N)^{-1}(0)/G$ is exactly the unique symplectic structure from symplectic reduction of the Hamiltonian $G$-subspace $(N, \omega|_N)$.

Now let $\pi_P : P \to B$ be a principal $G$-bundle over a compact symplectic manifold $(B, \beta)$, and let $\pi_2 : P \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the natural projection to the second factor. Every connection 1-form $A \in \mathcal{A}(P)$ yields a corresponding minimal coupling form on $P \times \mathfrak{g}^*$ defined by

\begin{equation}
\delta_A = \pi_P^* \beta - d\langle \pi_2, A \rangle.
\end{equation}

By the transformation properties of connections, $\langle \pi_2, A \rangle$, and hence $\delta_A$, is $G$-invariant under the diagonal action

\begin{equation}
g \cdot (p, \zeta) = (g^{-1} \cdot p, \text{Ad}(g)^* \zeta).
\end{equation}

Moreover, the injection $\iota : P \to P \times \mathfrak{g}^*$ given by $\iota(p) = (p, 0)$ is $G$-equivariant and $\iota^* \delta_A = \pi_P^* \beta$. It is also easily checked that $\delta_A$ is nondegenerate at all points of $\iota(P)$.

**Theorem 2.3.** Under the above assumptions, there exists a neighborhood $W$ of zero in $\mathfrak{g}^*$ (which depends on $A$ and on the principal bundle $\pi_P$, and can also be required to be invariant under the coadjoint action of $G$) such that the following hold.

(i) The minimal coupling form $\delta_A$ restricts to a symplectic form on $P \times W$.

(ii) The action of $G$ on $P \times \mathfrak{g}^*$ given by (2.2) is Hamiltonian with respect to $\delta_A$ on $P \times W$ and has the moment map, $-\pi_2 : P \times \mathfrak{g}^* \to \mathfrak{g}^*$.

(iii) For two different connection 1-forms $A_1$ and $A_2$ on $P$, let $W_1$ (resp., $W_2$) be the corresponding neighborhood of zero in $\mathfrak{g}^*$ such that the minimal coupling form $\delta_{A_1}$ (resp., $\delta_{A_2}$) is symplectic on $P \times W_1$ (resp., on $P \times W_2$). Then there are smaller neighborhoods of zero in $\mathfrak{g}^*$, $W_1^* \subset W_1$ and $W_2^* \subset W_2$ such that there
exists a symplectomorphism from \((P \times W_1^*, \delta_{A_1})\) onto \((P \times W_2^*, \delta_{A_2})\) that not only commutes with the action of \(G\) but also restricts to the identity on \(P \times \{0\}\).

**Theorem 2.4.** Under the assumptions of Theorem 2.3, furthermore assume that \((F, \sigma)\) is a Hamiltonian \(G\)-space with moment map \(\mu^F_G : F \to g^*\) satisfying
\[(2.3) \quad \mu^F_G(F) \subset \mathcal{W}.
\]
Then the diagonal action of \(G\) on \(W := (P \times W) \times F\) is Hamiltonian and the corresponding moment map is given by
\[(2.4) \quad \mu^W_G : W \to g^*, \quad (p, g^*, f) \mapsto \mu^F_G(f) - g^*.
\]
Moreover, there exists a symplectic form \(\omega_A\) on the total space of the associated fibre bundle \(\pi_F : M := P \times_G F \to B\) such that the map \((F, \sigma) \mapsto (M, \omega_A)\) is a symplectic embedding, that is, \(\omega_{A|M_B} = \sigma_b\) for any \(b \in B\), where \(\sigma_b\) is the symplectic form on \(M_B\) obtained from the symplectic fibration \(M \to B\) with symplectic fibre \((F, \sigma)\). Consequently, each fibre is a symplectic submanifold.

Indeed, it is easy to check that 0 is a regular value of \(\mu^W_G\), and that
\[\mathcal{Y} : P \times F \to (\mu^W_G)^{-1}(0), \quad (p, f) \mapsto (p, \mu^F_G(f), f)
\]
is a \(G\)-equivariant diffeomorphism. Moreover, \(G\) acts freely on \((\mu^W_G)^{-1}(0)\). The Marsden-Weinstein reduction procedure yields a unique symplectic form \(\omega_A'\) on the quotient \((\mu^W_G)^{-1}(0)/G\) whose pullback under the quotient projection \(\Pi_W : (\mu^W_G)^{-1}(0) \to (\mu^W_G)^{-1}(0)/G\) is equal to the restriction of \(\delta_A \oplus \sigma\) to \((\mu^W_G)^{-1}(0)\). Let \(\mathcal{Y} : P \times_G F \to (\mu^W_G)^{-1}(0)/G\) be the diffeomorphism induced by \(\mathcal{Y}\). Then the symplectic form \(\omega_A = \mathcal{Y}^* \omega_A'\) satisfies the desired requirements. Note that shrinking \(\mathcal{W}\) while preserving (2.3), one obtains the same symplectic form \(\omega_A\) on \(P \times_G F\). The following theorem summarizes some related properties of the above constructions. Their proofs are easy, (cf. [22]).

**Theorem 2.5.** Under the assumptions of Theorem 2.4, the following properties hold.

(i) Let \((F^\sharp, \sigma^\sharp)\) be another Hamiltonian \(G\)-space of dimension \(\dim F\) with moment map satisfying (2.3). If there exists a symplectic embedding \(\varphi : (F^\sharp, \sigma^\sharp) \to (F, \sigma)\) which commutes with the Hamiltonian actions of \(G\) on \((F^\sharp, \sigma^\sharp)\) and \((F, \sigma)\), that
is, $g \cdot \varphi(x) = \varphi(g \cdot x)$ for all $g \in G$ and $x \in F^2$, then the bundle embedding
$\varphi_P : M^2 := P \times_G F^2 \to M$ induced by $\varphi$ is a symplectic embedding from $(M^2, \omega_A^\#)$ to $(M, \omega_A)$, where $\omega_A^\#$ is the symplectic form on $M^2$ constructed as above. Furthermore, if $F_0$ is a Hamiltonian $G$-subspace of $(F, \sigma)$, then $P \times_G F_0$ is a symplectic submanifold in $(M, \omega_A)$.

(ii) If $F$ is a vector space and $\mu_G^F(0) = 0$, then the zero section $Z_0 := P \times_G \{0\} \subset M$ is a symplectic submanifold. More precisely, $\omega_A|_{Z_0} = \pi^*_{F\beta}|_{Z_0}$. Consequently, $Z_0$ is a symplectic submanifold in $(M, \omega_A)$ and the symplectic normal space $(T_{(b,0)}Z_0)^{\omega_A}$ at any point $(b,0) \in Z_0$ is exactly the symplectic vector space $(M_b, \sigma_b(0))$ (because $T_{(b,0)}M = T_{(b,0)}Z_0 \oplus T_0M_b = T_{(b,0)}Z_0 \oplus M_b$).

(iii) For any compact symplectic submanifold $B^\circ \subset B$, (since the connection form $A$ can always restrict to a connection form on the restriction principal bundle $P^\circ := P|_{B^\circ}$), the corresponding minimal coupling form $\delta_A^\circ = \pi^*_{P\beta} - d\langle \pi_2, A|_{P^\circ}\rangle$ on $P^\circ \times g^\#$ is equal to the restriction of the coupling form $\delta_A$ on $P \times g^\#$ to $P^\circ \times g^\#$. By shrinking $W$, assume that $\delta_A$ (resp., $\delta_A^\circ$) is nondegenerate on $P \times W$ (resp., $P^\circ \times W$). Then $M^\circ := (P|_{B^\circ}) \times_G F$ is also a symplectic submanifold in $(M, \omega_A)$ and $\omega_A|_{M^\circ}$ is exactly equal to the symplectic form constructed in the above method from the restriction of the connection form $A$ on the principal $G$ subbundle $P^\circ \to B^\circ$.

(iv) In the trivial principal bundle $P = B \times G$, there exists a canonical flat connection defined by $\mathcal{H}^f_{(u)} = \text{Ker}(\Pi_{2\ast}) : T_u(B \times G) \to T_yG$, where $\Pi_2 : B \times G \to G$ is the projection on the second factor. The corresponding connection form is given by $A_{\text{can}} := \Pi^*_2 \theta$. Here $\theta$ is the canonical left invariant $g$-valued 1-form defined by $\theta(a)(\tilde{X}(a)) = X \in g$ for $a \in G$, $X \in g$, where $\tilde{X}$ is the unique left invariant vector field on $G$ which has value $X$ at $e$. Then the symplectic form $\omega_{A_{\text{can}}}$ on $P \times_G F = B \times F$ is equal to $\beta \oplus \sigma$. Here the condition (2.3), of course, has been assumed, but the present $W$ depends merely on $G$ itself.

(v) Let $A_1$ and $A_2$ be two connection forms on $P$. Suppose that $\mu^G_F(F)$ is contained in the intersection of the open subsets $W^*_{\alpha}$ and $W^*_{\beta}$ in Theorem 2.3(iii). Then there exists a bundle isomorphism $\Phi : P \times_G F \to P \times_G F$ which sends $\omega_{A_1}$ to $\omega_{A_2}$, i.e., $\Psi^* \omega_{A_2} = \omega_{A_1}$. 


Remark 2.6. For a Hamiltonian \(G\)-space \((F, \sigma)\) with moment map \(\mu^F_G : F \to \mathfrak{g}^*\), there always exists \(\varepsilon_0 = \varepsilon_0(W, F, \sigma, G)\) such that \(\varepsilon^2 \mu^F_G(F) \subset W\) for any \(\varepsilon \in (0, \varepsilon_0]\). Since the \(G\)-action is also Hamiltonian with respect to \(\varepsilon^2 \sigma\) and the corresponding moment \(\mu^F_G = \varepsilon^2 \mu_G : F \to \mathfrak{g}^*\), one can always obtain a family of deformedly equivalent symplectic forms \(\{\omega_{\varepsilon(n)} \mid 0 < \varepsilon \leq \varepsilon_0\}\) on \(P \times_G F\). Here the reason that we use \(\varepsilon^2 \sigma\) (and thus \(\varepsilon^2 \mu^F_G\)), instead of \(\varepsilon \sigma\) (and \(\varepsilon \mu^F_G\)), will be seen in next section.

3. SYMPLECTIC FORMS ON PROJECTIVE BUNDLES

In this section, we shall construct two families of symplectic forms on the disk bundle and projective bundle, and give their properties. Firstly, a \(2n\)-dimensional symplectic vector bundle \(\pi : (E, \bar{\omega}) \to B\) may be naturally viewed as a symplectic fibration with symplectic fibre \((\mathbb{R}^{2n}, \omega_{\text{std}})\). Let \(\bar{\omega}_{\text{std}}\) denote the standard skew-symmetric bilinear map on \(\mathbb{R}^{2n}\). Taking any \(J_E \in \mathcal{J}(E, \bar{\omega})\) and setting \(g_{J_E} : E \times E \to \mathbb{R}, g_{J_E}(u, v) = \bar{\omega}(u, J_E v)\), one gets a Hermitian structure \((\bar{\omega}, J_E, g_{J_E})\) on \(E\), (cf. [29]). Then one can choose an open cover \(\{U_\alpha\}_{\alpha \in \Lambda}\) of \(B\) such that for each \(\alpha \in \Lambda\), there exists a unitary trivialization

\[
U_\alpha \times \mathbb{R}^{2n} \to E\big|_{U_\alpha} : (b, v) \mapsto \Phi_\alpha(b)v
\]

satisfying \(\Phi_\alpha^* J_E = J_{\text{std}}, \Phi_\alpha^* \bar{\omega} = \omega_{\text{std}}, \text{ and } \Phi_\alpha^* g_J = g_{\text{std}}\). Here \(g_{\text{std}}\) and \(J_{\text{std}}\) are the standard inner product and complex structure on \(\mathbb{R}^{2n}\), respectively. As a symplectic fibration \(E \to B\), each fibre \(E_b\) carries a natural symplectic structure

\[
(\omega_{\text{std}})_b := (\Phi_\alpha(b)^{-1})^* \omega_{\text{std}} \quad \text{(if } b \in U_\alpha)\]

to satisfy \((\omega_{\text{std}})_b|_{T_b E_b} = \bar{\omega}_b\).

The Lie algebra \(u(n) = \{X \in M_n(\mathbb{C}) \mid \bar{X}^t = -X\}\) of \(U(n)\) carries an invariant (real) inner product defined by \((X, Y) = \text{Tr}(X^t Y)\). By Riesz theorem, for each \(f \in u(n)^*\), there exists a unique \(\zeta_f \in u(n)\) such that \(\langle f, \xi \rangle = (\xi, \zeta_f)\) for any \(\xi \in u(n)\). Since \(\zeta_{Ad_g^{-1}f} = Ad_g^{-1}\zeta_f\) for any \(f \in u(n)\), and

\[
u(n)^* \to u(n), f \mapsto \zeta_f
\]

is a real vector space isomorphism, we may identify \(u(n)^*\) with \(u(n)\) and \(Ad_g^*\) with \(Ad_g^{-1}\). It is under such identifications that the moment map of the action of \(U(n)\) on \((\mathbb{C}^n, \omega_{\text{std}})\) in (1.2) is given by (1.3).
Let $J_{FS}$ be the standard complex structure on the $n$-dimensional complex projective space $\mathbb{C}P^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$. The standard action of $U(n + 1)$ on $\mathbb{C}^{n+1}$ as in (1.2) induces a natural one on $\mathbb{C}P^n$:

$$U \cdot [z_0, \cdots, z_n] = [(z_0, \cdots, z_n)U]$$

for any $[z_0, \cdots, z_n] \in \mathbb{C}P^n$ and $U \in U(n + 1)$. Let $\omega_{FS}$ be the unique $U(n+1)$-invariant Kähler form on $\mathbb{C}P^n$ associated with the Fubini-Study metric which has integration $\pi$ on $\mathbb{C}P^1 \subset \mathbb{C}P^n$. The action in (3.4) is Hamiltonian with respect to $\omega_{FS}$, and in homogeneous coordinates $[z_0, \cdots, z_n]$, the corresponding moment map $\mu'_{U(n+1)} : \mathbb{C}P^n \to u(n+1)^*$ is given by

$$\langle \mu'_{U(n+1)}([z]), \zeta \rangle = \frac{i}{2} \sum_{j,k} \zeta_{jk} \bar{z}_j z_k - \frac{1}{2} \sum_j |z_j|^2.$$  

Identifying $u(n+1)^*$ with $u(n+1)$ via the isomorphism as in (3.3), the moment map $\mu'_{U(n+1)} : \mathbb{C}P^{n+1} \to u(n+1)$ is written as

$$\mu'_{U(n+1)}([z]) = \frac{i}{2} \sum_j |z_j|^2.$$  

Consider the Lie group inclusion homomorphism

$$\varphi : U(n) \to U(n + 1), \ U \mapsto \left( \begin{array}{cc} U & 0 \\ 0 & 1 \end{array} \right),$$

we easily get the following proposition.

**Proposition 3.1.** The action of $U(n)$ on $\mathbb{C}P^n$ via

$$U \cdot [z_0, \cdots, z_n] = [(z_0, \cdots, z_{n-1})U, z_n]$$

is Hamiltonian and the moment map $\mu''_{U(n)} : \mathbb{C}P^n \to u(n)$ is given by

$$\mu''_{U(n)} : \mathbb{C}P^n \to u(n), \ z \mapsto \frac{i}{2} \frac{Z^t \bar{Z}}{\sum_{l=0}^n |z_l|^2}.$$  

Here $Z = (z_0, \cdots, z_{n-1})$ and $u(n)^*$ has been identified with $u(n)$ via (3.3).

Using the action in (3.8), one can form a $\mathbb{C}P^n$-bundle

$$\mathcal{P} : U(E) \times_{U(n)} \mathbb{C}P^n \to B,$$
which is exactly the projective bundle $P(E \oplus \mathbb{C})$. Denote by

\begin{equation}
Z_0 := P(\{0\} \oplus \mathbb{C}) \quad \text{and} \quad Z_\infty := P(E \oplus \{0\})
\end{equation}

the zero section and divisor at infinity respectively. The symplectic embedding

\begin{equation}
\varphi : (B^{2n}(1), \omega_{\text{std}}) \rightarrow (\mathbb{C}P^n, \omega_{\text{FS}}), \; z \mapsto [z, \sqrt{1 - |z|^2}]
\end{equation}

satisfies

\begin{equation}
\varphi(U \cdot z) = [U \cdot z, \sqrt{1 - |z|^2}] = U \cdot \varphi(z),
\end{equation}

and hence induces an embedding

\begin{equation}
\varphi_P : U(E) \times_{U(n)} B^{2n}(1) \rightarrow U(E) \times_{U(n)} \mathbb{C}P^n, \; [f, z] \mapsto [f, \varphi(z)]
\end{equation}

that maps the zero sections $0_E \subset D_1(E)$ onto $Z_0 \subset P(E \oplus \mathbb{C})$. Here $D_1(E)$ is defined by (1.4), i.e., $D_1(E) = U(E) \times_{U(n)} B^{2n}(1)$.

**Theorem 3.2.** For each given connection 1-form $A$ on $F : U(E) \rightarrow B$, there exist a small $\varepsilon_0 = \varepsilon_0(F, A) > 0$ and two smooth families of (compatible) symplectic forms $\{\omega_{A\varepsilon} | 0 < \varepsilon < \varepsilon_0\}$ on $P(E \oplus \mathbb{C})$ such that the following properties hold:

1. The maps $(B^{2n}(1), \varepsilon^2 \omega_{\text{std}}) \hookrightarrow (D_1(E), \omega_{A\varepsilon})$ and $(\mathbb{C}P^n, \varepsilon^2 \omega_{\text{FS}}) \hookrightarrow (P(E \oplus \mathbb{C}), \Omega_{A\varepsilon})$ are all symplectic embeddings.

2. The map $\varphi_P$ in (3.14) is a symplectic embedding from $(D_1(E), \omega_{A\varepsilon})$ to $(P(E \oplus \mathbb{C}), \Omega_{A\varepsilon})$.

3. The zero section $0_E \subset E$ is a symplectic submanifold in $(D_1(E), \omega_{A\varepsilon})$ with symplectic normal bundle $(E, \varepsilon^2 \omega) \rightarrow B$. More precisely, for any $b \in B$,

\begin{equation}
\omega_{A\varepsilon}|_{D_1(E)b} = \varepsilon^2(\omega_{\text{std}})_b \quad \text{and} \quad \Omega_{A\varepsilon}|_{P(E \oplus \mathbb{C})b} = \varepsilon^2(\omega_{\text{FS}})_b,
\end{equation}

where $(\omega_{\text{std}})_b$ is defined by (3.2), and $(\omega_{\text{FS}})_b$ comes from $P(E \oplus \mathbb{C})$ being a symplectic fibration with symplectic fibre $(\mathbb{C}P^n, \omega_{\text{FS}})$. So the symplectic forms $\omega_{A\varepsilon}$ and $\Omega_{A\varepsilon}$ are compatible with the symplectic fibrations.

4. The zero section $Z_0$ and the divisor $Z_\infty$ at infinity are symplectic submanifolds in $(P(E \oplus \mathbb{C}), \Omega_{A\varepsilon})$.

---

1Hereafter the subscript $A\varepsilon$ does not mean to multiply $A$ by $\varepsilon$!
(5°) If \( F_0 \) is a Hamiltonian \( G \)-subspace of \( (B^{2n}(1), \omega_{\text{std}}) \) (resp., \( (\mathbb{CP}^n, \omega_{FS}) \)), then \( U(E) \times_{U(n)} F_0 \) is a symplectic submanifold in \( (D_1(E), \omega_{A \epsilon}) \) (resp., \( (P(E \oplus \mathbb{C}), \Omega_{A \epsilon}) \)).

(6°) For any compact symplectic submanifold \( B^o \subset B \), let \( U(E)^o = U(E)|_{B^o} \) and \( \{\omega_{A \epsilon}^o \mid 0 < \epsilon < \varepsilon_0^o \} \) (resp., \( \{\Omega_{A \epsilon}^o \mid 0 < \epsilon < \varepsilon_0^o \} \) be the corresponding family of symplectic forms on \( D_1(E)^o := U(E)^o \times_{U(n)} B^{2n}(1) \) (resp., \( P(E \oplus \mathbb{C})^o := U(E)^o \times_{U(n)} \mathbb{CP}^n \)) obtained by the restriction connection form \( A|_{U(E)^o} \). Then \( \omega_{A \epsilon}^o \) (resp., \( \Omega_{A \epsilon}^o \)) is equal to the restriction of \( \omega_{A \epsilon} \) (resp., \( \Omega_{A \epsilon} \)) to \( D_1(E)^o := U(E)^o \times_{U(n)} B^{2n}(1) \) (resp., \( P(E \oplus \mathbb{C})^o := U(E)^o \times_{U(n)} \mathbb{CP}^n \)) for each \( 0 < \epsilon < \varepsilon < \varepsilon_0 \).

(7°) If \( P \) is the trivial principal bundle \( P = B \times U(n) \) and \( A \) is taken as the connection form \( A_{\text{can}} \) of the canonical flat connection in \( P \) given by Theorem 2.5(iv), then for each \( 0 < \epsilon < \varepsilon_0(P, A_{\text{can}}) \), the symplectic form \( \omega_{A_{\text{can}}^o} \) (resp., \( \Omega_{A_{\text{can}}^o} \)) on \( P \times_{U(n)} B^{2n}(1) = B \times B^{2n}(1) \) (resp., \( P \times_{U(n)} \mathbb{CP}^n = B \times \mathbb{CP}^n \)) is equal to \( \beta \oplus (\varepsilon^2 \omega_{\text{std}}) \) (resp., \( \beta \oplus (\varepsilon^2 \omega_{FS}) \)) for each \( 0 < \epsilon < \varepsilon_0(P, A_{\text{can}}) \).

(8°) For each connection 1-form \( A \) on \( U(E) \), let \( \varepsilon_0 = \varepsilon_0(F, A) > 0 \) be as above. Then there exists a unique symplectic form \( \bar{\omega}_A \) on \( D_{\varepsilon_0}(E) \) such that for each \( 0 < \epsilon < \varepsilon_0 \).

\[
\psi_{\epsilon} : (D_1(E), \omega_{A \epsilon}) \to (D_{\varepsilon_0}(E), \bar{\omega}_A), \ (b, v) \mapsto (b, \varepsilon v)
\]

is a symplectomorphism.

(9°) For any two different connection forms \( A_1 \) and \( A_2 \) on \( F : U(E) \to B, 0 < \epsilon \ll \min\{\varepsilon_0(F, A_1), \varepsilon_0(F, A_2)\} \), and \( 0 < \delta \leq 1 \), there exist fibre bundle isomorphisms

\[
\varphi_{\epsilon}^\delta : D_{\delta}(E) \to D_{\varepsilon}(E) \quad \text{and} \quad \Phi_{\epsilon} : P(E \oplus \mathbb{C}) \to P(E \oplus \mathbb{C})
\]

such that \( (\varphi_{\epsilon}^\delta)^* \omega_{A_2 \epsilon} = \omega_{A_1 \epsilon} \) and \( \Phi_{\epsilon}^* \Omega_{A_2 \epsilon} = \Omega_{A_1 \epsilon} \).

**Proof.** The proofs can be obtained by Theorem 2.5 directly. We only outline them. By the explicit equivalence between \( U(E) \times_{U(n)} \mathbb{CP}^n \to B \) and \( E \to B \) given by

\[
\Xi : [p, (z_1, \ldots, z_n)] \mapsto z_1 v_1 + \cdots + z_n v_n,
\]

where \( p = (v_1, \ldots, v_n) \) is a unitary frame of \( E_b \), for any \( \epsilon > 0 \), we can write

\[
D_{\epsilon}(E) = \{(b, v) \in E \mid g_{J\epsilon}(v, v) < \epsilon^2\}.
\]

For a given connection \( A \) on the principal bundle \( F : U(E) \to B \), let \( W = W(A) \) be the largest open neighborhood of zero in \( u(n)^* \) so that the corresponding minimal coupling
form \( \delta_A \) is symplectic in \( U(E) \times W \). Since \( Cl(B^{2n}(1)) \) and \( C P^n \) are compact, by (3.6) and Proposition 3.1, we can choose \( 0 < \varepsilon_0 = \varepsilon_0(A,F) \leq 1 \) such that

\[
(3.20) \quad \varepsilon^2 \mu_{U(n)}(B^{2n}(1)) \subset W \quad \text{and} \quad \varepsilon^2 \mu''_{U(n)}(C P^n) \subset W
\]

for all \( \varepsilon \in (0, \varepsilon_0) \). (For example, ones can take \( \varepsilon_0 \) to be the supremum of \( \varepsilon > 0 \) satisfying the inclusion relations in (3.20).) Now applying Theorem 2.4 to the cases \( P = U(E) \) and \( (F, \sigma) = (B^{2n}(1), \varepsilon^2 \omega_{std}) \) or \( (C P^n, \varepsilon^2 \omega_{FS}) \) we immediately get (1°).

Note that (3.13) means that the symplectic embedding \( \varphi \) in (3.12) commutes with the Hamiltonian actions in (1.2) and (3.8). Condition (2°) follows from Theorem 2.5(i).

The conclusion in (3°) is easily derived from Theorem 2.5(ii). It precisely says \( (T_{(b,0)}(0_E))^{\omega_{A\varepsilon}} = (E_b, \varepsilon^2 \tilde{\omega}_b) \) for any \( b \in B \).

To see (4°), note that \( C P^{n-1} = \{ [z_0, \cdots, z_{n-1}, 0] \in C P^n \} \subset C P^n \) is a symplectic submanifold in \( (C P^n, \omega_{FS}) \) that is invariant under the action in (3.8). By Theorem 2.5(i), \( Z_\infty \) is a symplectic submanifold in \( (P(E \oplus \mathbb{C}), \Omega_{A\varepsilon}) \). Since the symplectic embedding \( \varphi_p \) in (2°) maps the zero section \( 0_E \) onto \( Z_0 = P(\{0\} \oplus \mathbb{C}) \), \( Z_0 \) is a symplectic submanifold in \( (P(E \oplus \mathbb{C}), \Omega_{A\varepsilon}) \).

Theorem 3.2(5°) is a direct consequence of the final conclusion of Theorem 2.5(i). Theorem 3.2(6°), (7°) follows from Theorem 2.5(iii), (iv), respectively. Condition (8°) can easily be obtained by Theorem 2.5(i). Finally, we prove (9°). Let \( W_1^* \subset W(A_1) \) and \( W_2^* \subset W(A_2) \) be open neighborhoods of zero in \( u(n)^* \) such that \( (U(E) \times W_1^*, \delta_{A_1}) \) is symplectomorphic to \( (U(E) \times W_2^*, \delta_{A_2}) \). Then for \( 0 < \varepsilon \ll \min\{ \varepsilon_0(F,A_1), \varepsilon_0(F,A_2) \} \), it holds that \( \varepsilon^2 \mu_{U(n)}(Cl(B^{2n}(1))) \subset W^* \) and \( \varepsilon^2 \mu''_{U(n)}(C P^n) \subset W^* \). Under this case Theorem 2.5(v) gives the desired results directly. Theorem 3.2 are proved.

Since \( \omega_{A\varepsilon} \) is compatible with symplectic fibration, by Theorem 3.2(3°), we can write

\[
(3.21) \quad \omega_{A\varepsilon} = \pi^* \beta + \tau_{A\varepsilon}, \quad \tau_{A\varepsilon}|_{0_E} = 0,
\]

\[
\tau_{A\varepsilon}|_{D_1(E)_b} = \varepsilon^2(\omega_{std})_b \quad \forall b \in B.
\]

Similarly, by Theorem 3.2(4°), we easily get that \( \Omega_{A\varepsilon}|_{Z_0} = \beta \) after identifying \( Z_0 \equiv 0_E \equiv B \). It follows that we can also write

\[
(3.22) \quad \Omega_{A\varepsilon} = \mathcal{P}^* \beta + \Gamma_{A\varepsilon}, \quad \Gamma_{A\varepsilon}|_{Z_0} = 0,
\]

\[
\Gamma_{A\varepsilon}|_{P(E \oplus \mathbb{C})_b} = \varepsilon^2(\omega_{FS})_b \quad \forall b \in B.
\]

\(^2\)Here the choice of \( \varepsilon_0(A,F) \) shows that it is not canonical.
Note that the almost complex structure $J$ on $P(E \oplus \mathbb{C})$ constructed by (4.15) is not necessarily $\Omega_{A \epsilon}$-tamed. However, we can show that for sufficiently small $t > 0$, the closed 2-form $P^* \beta + t \Gamma_{A \epsilon}$ is also symplectic and tame this $J$ (see Lemma 4.2). Hence we are led to the following strengthened version of Theorem 3.2, whose precise statement is needed in the proof of Theorem 4.5.

**Theorem 3.3.** Under the assumption of Theorem 3.2, for each $0 < \epsilon < \epsilon_0$, there exists a small $t_0 = t_0(A, \epsilon) > 0$ such that for each $0 < t < t_0$, the form

$$\omega_{A \epsilon}^t := \pi^* \beta + t \tau_{A \epsilon} \quad \text{(resp.,} \quad \Omega_{A \epsilon}^t := P^* \beta + t \Gamma_{A \epsilon} \)$$

is a symplectic form on $D_1(E)$ (resp. $P(E \oplus \mathbb{C})$), where $\tau_{A \epsilon}$ and $\Gamma_{A \epsilon}$ are given by (3.21) and (3.22), respectively. Moreover, they also satisfy the following properties.

(i) The map $(B^{2n}(1), t \varepsilon^2 \omega_{\text{std}}) \hookrightarrow (D_1(E), \omega_{A \epsilon}^t)$ and $(\mathbb{C}P^n, t \varepsilon^2 \omega_{FS}) \hookrightarrow (P(E \oplus \mathbb{C}), \Omega_{A \epsilon}^t)$ are symplectic embeddings.

(ii) The map $\varphi_P$ in (3.14) is a symplectic embedding from $(D_1(E), \omega_{A \epsilon}^t)$ into $(P(E \oplus \mathbb{C}), \Omega_{A \epsilon}^t)$.

(iii) The zero section $0_E \subset E$ is a symplectic submanifold in $(D_1(E), \omega_{A \epsilon}^t)$ with symplectic normal bundle $(E, t \varepsilon^2 \omega) \rightarrow B$.

(iv) The zero section $Z_0 = P(\{0\} \oplus \mathbb{C})$ is a symplectic submanifold in $(P(E \oplus \mathbb{C}), \Omega_{A \epsilon}^t)$.

(v) Let the closed two-form $\omega^t_A$ be defined by (1.6). Then for any $0 < \epsilon < \epsilon_0(A, F)$ and $0 < t < t_0(A, \epsilon)$, $\omega^t_A$ is symplectic in $D_\epsilon(E)$ and

$$\psi_\epsilon : (D_1(E), \omega_{A \epsilon}^t) \rightarrow (D_\epsilon(E), \omega^t_A), \quad (b, v) \mapsto (b, \epsilon v)$$

is also a symplectomorphism.

(vi) For any compact symplectic submanifold $B^\circ \subset B$ of codimension zero, as in Theorem 3.2(6°), let $\omega_{A \epsilon}^\circ$ (resp., $\Omega_{A \epsilon}^\circ$) be the symplectic form on $D_1(E)^\circ$ (resp., $P(E \oplus \mathbb{C})^\circ$) for $0 < \epsilon < \epsilon_0^\circ$. As in (3.23), there exist a small $t_0^\circ(A, \epsilon) > 0$ and two families of symplectic forms,

$$\omega^t_{A \epsilon} = (\pi|_{D_1(E)^\circ})^* \beta + t \tau_{A \epsilon}^\circ \quad \text{and} \quad \Omega^t_{A \epsilon} = (P|_{P(E \oplus \mathbb{C})^\circ})^* \beta + t \Gamma_{A \epsilon}^\circ$$

for $0 < t < t_0^\circ(A, \epsilon)$, where $\omega^\circ_{A \epsilon} = (\pi|_{D_1(E)^\circ})^* \beta + \tau_{A \epsilon}^\circ$ and $\Omega^\circ_{A \epsilon} = (P|_{P(E \oplus \mathbb{C})^\circ})^* \beta + \Gamma_{A \epsilon}^\circ$. Then for each $0 < \epsilon < \min\{\epsilon_0^\circ, \epsilon_0\}$ and $0 < t < \min\{t_0, t_0^\circ\}$, the
symplectic form $\omega^t_{A\varepsilon}$ (resp., $\Omega^t_{A\varepsilon}$) is equal to the restriction of $\omega^t_{A\varepsilon}$ (resp., $\Omega^t_{A\varepsilon}$) to $D_1(E)^0$ (resp., $P(E \oplus \mathbb{C})^0$).

(vii) Under the assumptions of Theorem 3.2(9°), for any $0 < t < \min\{t_0(A_1, \varepsilon), t_0(A_2, \varepsilon)\}$ and $\delta \in (0, 1]$, $\omega^t_{A\varepsilon}$ and $\Omega^t_{A\varepsilon}$ are all symplectomorphisms.

(viii) If $P = B \times U(n)$ and $A = A_{\text{can}}$ are as in Theorem 3.2(7°), then for any $0 < \varepsilon < \varepsilon_0(P, A_{\text{can}})$, one can take $t_0(A_{\text{can}}, \varepsilon) = \infty$ and $\bar{\omega}^t_{A_{\text{can}}} = \beta \oplus (t \varepsilon^2 \omega_{\text{std}})$ for each $t > 0$.

Proof. We firstly prove that $\Omega^t_{A\varepsilon} = \mathcal{P}^* \beta + t \Gamma_{A\varepsilon}$ are symplectic forms on $P(E \oplus \mathbb{C})$ for sufficiently small $t > 0$. For any $x \in P(E \oplus \mathbb{C})$ let

$$V_x := \text{Ker} d\mathcal{P}(x) \subset T_x P(E \oplus \mathbb{C}).$$

Then it is equal to $T_x P(E \oplus \mathbb{C})_b$, where $b = \mathcal{P}(x)$. Since each fibre is a symplectic submanifold, the subspace of $T_x P(E \oplus \mathbb{C})$,

$$\mathcal{H}_x := (V_x)^{\Omega_{A\varepsilon}} = \{X \in T_x P(E \oplus \mathbb{C}) \mid \Omega_{A\varepsilon}(X, Y) = 0 \forall Y \in V_x\}$$

$$= \{X \in T_x P(E \oplus \mathbb{C}) \mid \Gamma_{A\varepsilon}(X, Y) = 0 \forall Y \in V_x\}$$

is not only symplectic, but also a horizontal complement of $V_x$, that is,

$$T_x P(E \oplus \mathbb{C}) = \mathcal{H}_x \oplus V_x$$

is a symplectic direct sum decomposition and the projection

$$d\mathcal{P}(x) : \mathcal{H}_x \to T_b B$$

is a bijection. By Lemma 4.2, we have an almost complex structure $J$ on $P(E \oplus \mathbb{C})$ and a small $\bar{t} > 0$ such that $\Omega^t_{A\varepsilon}(X, JX) > 0$ for any nonzero $X \in TP(E \oplus \mathbb{C})$ and $t \in (0, \bar{t}]$. So, such $\Omega^t_{A\varepsilon}$ must be nondegenerate. Then the desired $t_0$ can be taken as the supremum of $\bar{t} > 0$ for which all $\Omega^t_{A\varepsilon}$ are nondegenerate for any $t \in (0, \bar{t}]$. The conclusions for $\omega^t_{A\varepsilon}$ can be proved in the same way (by constructing an almost complex structure on the compact bundle $Cl(D_1(E))$ as in (4.15)). Of course we can shrink $t_0$ if necessary.
Conditions (i), (iii), (iv) and (viii) are obvious. To see (ii), note that $\varphi^*_{\mathcal{P}}(P^*\beta) = \pi^*(\varphi^*_P\beta)$. So $\varphi^*_{\mathcal{P}}\Omega_{A\varepsilon} = \omega_{A\varepsilon}$ if and only if $\varphi^*_{\mathcal{P}}\Gamma_{A\varepsilon} = \tau_{A\varepsilon}$. The desired conclusion follows immediately.

To prove (v), we still use $\pi$ to denote the bundle projections from $D_1(E)$ and $D_\varepsilon(E)$ to $B$. Then $\pi \circ \psi_\varepsilon = \pi$, and hence $d\pi \circ d\psi_\varepsilon = d\pi$. In particular, we have

\begin{equation}
\psi_\varepsilon^*(\pi^*\beta) = \pi^*\beta
\end{equation}

For any $x \in D_1(E)$ and $y \in D_\varepsilon(E)$ let

\begin{equation}
\begin{align*}
V_x &:= \text{Ker}(d\pi(x)), & H_x &:= (V_x)^{\omega_{A\varepsilon}}, \\
V_y &:= \text{Ker}(d\pi(y)), & H_y' &:= (V_y')^{\bar{\omega}_A}.
\end{align*}
\end{equation}

Then $T_xD_1(E) = H_x \oplus V_x$ and $T_yD_\varepsilon(E) = H_y' \oplus V_y'$. Clearly, $d\psi_\varepsilon(x)(V_x) = V_x'$. Since $\psi_\varepsilon^*\bar{\omega}_A = \omega_{A\varepsilon}$, one easily derives that $d\psi_\varepsilon(x)(H_x) = H_x'$. These imply that

\begin{equation}
\psi_\varepsilon^*(\bar{\omega}_A - \pi^*\beta) = \omega_{A\varepsilon} - \pi^*\beta.
\end{equation}

Hence (3.31) and (3.33) together show that $\psi_\varepsilon^*\bar{\omega}_A = \omega_{A\varepsilon}^t$ for any $t$. But $\omega_{A\varepsilon}^t$ is symplectic in $D_1(E)$ for any $t \in (0, t_0)$. Hence $\bar{\omega}_A$ is symplectic in $D_\varepsilon(E)$ for each $0 < t < t_0(A, \varepsilon)$. The desired conclusion is proved.

Condition (vi) easily follows from Theorem 3.2(6) and the proof of (v) above.

Finally, since $\pi \circ \varphi_\varepsilon = \pi$ and $\mathcal{P} \circ \Phi_\varepsilon = \mathcal{P}$, as in the proof of (v) ones can derive (vii) from Theorem 3.2(9) easily. $\square$

4. PROOF OF MAIN RESULT

We briefly review the definition of the pseudo symplectic capacity of the Hofer-Zehnder type introduced by the author in [23]. See [17, 19, 20, 18] for more estimates and applications.

For a connected symplectic manifold $(M, \omega)$ of dimension at least 4 and two nonzero homology classes $\alpha_0, \alpha_\infty \in H_*(M; \mathbb{Q})$, we say that a smooth function $H : M \to \mathbb{R}$ is $(\alpha_0, \alpha_\infty)$-admissible (resp., $(\alpha_0, \alpha_\infty)^0$-admissible) if there exist two compact submanifolds $P$ and $Q$ of $M$ with connected smooth boundaries and of codimension zero such that the following condition groups (1), (2), (3), (4), (5), and (6) (resp., (1), (2), (3), (4), (5), and (6)) hold:

(1) $P \subset \text{Int}(Q)$ and $Q \subset \text{Int}(M)$;
(2) $H|_P = 0$ and $H|_{M \setminus \text{Int}(Q)} = \max H$;

(3) $0 \leq H \leq \max H$;

(4) There exist chain representatives of $\alpha_0$ and $\alpha_\infty$, still denoted by $\alpha_0$ and $\alpha_\infty$, such that $\text{supp}(\alpha_0) \subset \text{Int}(P)$ and $\text{supp}(\alpha_\infty) \subset M \setminus Q$;

(5) There are no critical values in $(0, \varepsilon) \cup (\max H - \varepsilon, \max H)$ for a small $\varepsilon = \varepsilon(H) > 0$;

(6) The Hamiltonian system $\dot{x} = X_H(x)$ on $M$ has no nonconstant fast periodic solutions;

(6°) The Hamiltonian system $\dot{x} = X_H(x)$ on $M$ has no nonconstant contractible fast periodic solutions.

Let $\mathcal{H}_{ad}(M, \omega; \alpha_0, \alpha_\infty)$ and $\mathcal{H}_{ad}^0(M, \omega; \alpha_0, \alpha_\infty)$ denote the sets of $(\alpha_0, \alpha_\infty)$-admissible functions and $(\alpha_0, \alpha_\infty)^0$-admissible ones, respectively. In [23], we defined

$$
C^{(2)}_{HZ}(M, \omega; \alpha_0, \alpha_\infty) := \sup \{ \max H \mid H \in \mathcal{H}_{ad}(M, \omega; \alpha_0, \alpha_\infty) \},
$$

$$
C^{(2)}_{HZ}(M, \omega; \alpha_0, \alpha_\infty) := \sup \{ \max H \mid H \in \mathcal{H}_{ad}^0(M, \omega; \alpha_0, \alpha_\infty) \}.
$$

They were, respectively, called the pseudo symplectic capacity of the Hofer-Zehnder type and the $\pi_1$-sensitive pseudo symplectic capacity of the Hofer-Zehnder type. In particular, we get a genuine symplectic capacity

$$
C^{(2)}_{HZ}(M, \omega) := C^{(2)}_{HZ}(M, \omega; \text{pt, pt})
$$

and a $\pi_1$-sensitive symplectic capacity

$$
C^{(2)}_{HZ}(M, \omega) := C^{(2)}_{HZ}(M, \omega; \text{pt, pt}).
$$

We also showed in [23, Lemma 1.4] that there exist the following relations among them, the usual Hofer-Zehnder capacity $c_{HZ}$ and the $\pi_1$-sensitive Hofer-Zehnder capacity $c^0_{HZ}$:

$$
C^{(2)}_{HZ}(M, \omega) = c_{HZ}(M, \omega) \quad \text{and} \quad C^{(2)}_{HZ}(M, \omega) = c^0_{HZ}(M, \omega)
$$

if a symplectic manifold $(M, \omega)$ is either closed or satisfies the condition that for each compact subset $K \subset M \setminus \partial M$, there exists a compact submanifold $W \subset M$ with connected boundary and of codimension zero such that $K \subset W$.

For a closed symplectic manifold $(M, \omega)$ and $\alpha_0, \alpha_\infty \in H_*(M; \mathbb{Q})$, let

$$
\text{GW}_g(M, \omega; \alpha_0, \alpha_\infty) \in (0, +\infty]
$$
be the infimum of the \( \omega \)-areas \( \omega(A) \) of the homology classes \( A \in H_2(M; \mathbb{Z}) \) for which the Gromov-Witten invariant \( \Psi_{A, g, m+2}(C; \alpha_0, \alpha_\infty, \beta_1, \cdots, \beta_m) \neq 0 \) for some homology classes \( \beta_1, \cdots, \beta_m \in H_*(M; \mathbb{Q}) \) and \( C \in H_*(\overline{M}_{g,m+2}; \mathbb{Q}) \) and an integer \( m \geq 1 \). Here for a given class \( A \in H_2(M; \mathbb{Z}) \), the Gromov-Witten invariant of genus \( g \) and with \( m + 2 \) marked points is a homomorphism

\[
\Psi_{A, g, m+2} : H_*(\overline{M}_{g,m+2}; \mathbb{Q}) \times H_*(M; \mathbb{Q})^{m+2} \to \mathbb{Q},
\]

the reader may refer to [14, 21] for details. (In the latter paper, we used the cohomology and denoted by \( \text{GW} \) the GW-invariants. It is easily translated into the homology while \( M \) is a closed manifold.) We also define

\[
\text{GW}(M, \omega; \alpha_0, \alpha_\infty) := \inf \{ \text{GW}_g(M, \omega; \alpha_0, \alpha_\infty) | g \geq 0 \} \in [0, +\infty].
\]

Based on [14], we proved in [23, Theorem 1.10] that

\[
\begin{align*}
C_{HZ}^{(2)}(M, \omega; \alpha_0, \alpha_\infty) & \leq \text{GW}(M, \omega; \alpha_0, \alpha_\infty), \\
C_{HZ}^{(2)}(M, \omega; \alpha_0, \alpha_\infty) & \leq \text{GW}_0(M, \omega; \alpha_0, \alpha_\infty)
\end{align*}
\]

for any closed symplectic manifold \( (M, \omega) \) of dimension \( \dim M \geq 4 \) and homology classes \( \alpha_0, \alpha_\infty \in H_*(M; \mathbb{Q}) \setminus \{0\} \). The following proposition lists partial results in [23, Proposition 1.7], which are needed in the following arguments.

**Proposition 4.1.** Let \( W \subset \text{Int}(M) \) be a smooth compact submanifold of codimension zero and with connected boundary such that the homology classes \( \alpha_0, \alpha_\infty \in H_*(M; \mathbb{Q}) \setminus \{0\} \) have representatives supported in \( \text{Int}(W) \) and \( \text{Int}(M) \setminus W \), respectively. Denote by \( \tilde{\alpha}_0 \in H_*(W; \mathbb{Q}) \) and \( \tilde{\alpha}_\infty \in H_*(M \setminus W; \mathbb{Q}) \) the nonzero homology classes determined by them. Then

\[
C_{HZ}^{(2)}(W, \omega; \tilde{\alpha}_0, pt) \leq C_{HZ}^{(2)}(M, \omega; \alpha_0, \alpha_\infty)
\]

and, in particular, one has

\[
C_{HZ}^{(2)}(W, \omega; \alpha_0, pt) \leq C_{HZ}^{(2)}(M, \omega; \alpha_0, \alpha_\infty)
\]

for any \( \alpha \in H_*(M; \mathbb{Q}) \setminus \{0\} \) with representative supported in \( \text{Int}(M) \setminus W \). If the inclusion \( W \hookrightarrow M \) induces an injective homomorphism \( \pi_1(W) \to \pi_1(M) \), then

\[
C_{HZ}^{(2)}(W, \omega; \tilde{\alpha}_0, pt) \leq C_{HZ}^{(2)}(M, \omega; \alpha_0, \alpha_\infty)
\]
and corresponding to (4.11), one has

\[ c_{HZ}^0(W, \omega) = C_{HZ}^{(2\omega)}(W, \omega) \leq C_{HZ}^{(2\omega)}(M, \omega; pt, \alpha). \]

We now construct a class of almost complex structures on \( P(E \oplus \mathbb{C}) \). The readers will see why we need Theorem 3.3. Note that every given Riemannian metric \( P(E \oplus \mathbb{C}) \) naturally restricts to a natural Riemannian metric \( g_b \) on the fibre \( P(E \oplus \mathbb{C})_b \) for each \( b \in B \). Using the standard method, (cf. [29, page 64]) one gets a compatible almost complex structure \( J_b \in \mathcal{J}(P(E \oplus \mathbb{C})_b, (\omega_{FS})_b) \). Clearly, \( J_b \) smoothly depends on \( b \in B \). Take another compatible almost complex structure \( J_B \in \mathcal{J}(B, \beta) \). By (3.30), we can obtain its horizontal lift \( J_B \) to \( \mathcal{H} \):

\[ (\tilde{J}_B)_x : \mathcal{H}_x \to \mathcal{H}_x, \ X \mapsto (dP(x))^{-1} \circ (J_B)_b \circ dP(x)(X), \]

where \( b = P(x) \). Then using the decomposition (3.29), we may define an almost complex structure \( J \) on \( P(E \oplus \mathbb{C}) \) as follows:

\[ J_x : T_x P(E \oplus \mathbb{C}) = \mathcal{H}_x \oplus \mathcal{V}_x \to T_x P(E \oplus \mathbb{C}), \]

\[ X^h \oplus X^v \mapsto (\tilde{J}_B)_x(X^h) \oplus (J_b)_x(X^v). \]

**Lemma 4.2.** \( J \) is \( \Omega_{\epsilon A_{\epsilon}}^1 \)-tamed for sufficiently small \( t > 0 \).

**Proof.** For any nonzero \( X = X^h + X^v \in T_x P(E \oplus \mathbb{C}) = \mathcal{H}_x \oplus \mathcal{V}_x \), the direct computation yields

\[
\Omega_{\epsilon A_{\epsilon}}^1(X, JX) = \Omega_{\epsilon A_{\epsilon}}^1(X^h + X^v, (\tilde{J}_B)_x X^h + (J_b)_x X^v) \\
= \mathcal{P}^* \beta(X^h + X^v, (\tilde{J}_B)_x X^h + (J_b)_x X^v) \\
+ t\Gamma_{\epsilon A_{\epsilon}}(X^h + X^v, (\tilde{J}_B)_x X^h + (J_b)_x X^v) \\
= \beta(dP(x)(X^h), dP(x)((\tilde{J}_B)_x X^h)) \\
+ t\Gamma_{\epsilon A_{\epsilon}}(X^h, (\tilde{J}_B)_x X^h) + t\varepsilon(\omega_{FS})_b(X^v, (J_b)_x X^v) \\
+ t\Gamma_{\epsilon A_{\epsilon}}(X^h, (J_b)_x X^v) + t\Gamma_{\epsilon A_{\epsilon}}(X^v, (\tilde{J}_B)_x X^h) \\
= \beta(dP(x)(X^h), (J_b)_b dP(x)X^h) \\
+ t\Gamma_{\epsilon A_{\epsilon}}(X^h, (\tilde{J}_B)_x X^h) + t\varepsilon(\omega_{FS})_b(X^v, (J_b)_x X^v).
\]

Here \( \Gamma_{\epsilon A_{\epsilon}}(X^h, (J_b)_x X^v) = 0 \) and \( \Gamma_{\epsilon A_{\epsilon}}(X^v, (\tilde{J}_B)_x X^h) = 0 \) are because of \( (J_b)_x X^v \in \mathcal{V}_x, (\tilde{J}_B)_x X^h \in \mathcal{H}_x \) and the equality above (3.29), that is, \( \mathcal{H}_x = \{ X \in T_x P(E \oplus \mathbb{C}) | \Gamma_{\epsilon A_{\epsilon}}(X, Y) = 0 \ \forall Y \in \mathcal{V}_x \} \). Since \( P(E \oplus \mathbb{C}) \) is compact, and by (3.30) the projection
$d\mathcal{P}(x)$ is a bijection, it easily follows that for a given $\epsilon \in (0,1)$, there exists a $0 < \bar{t} \leq t_0$ such that
\begin{equation}
\beta(d\mathcal{P}(x)(X^h), (J_B)_b d\mathcal{P}(x)X^h) + t \Gamma_{\mathcal{A}_\epsilon}(X^h, (\bar{J}_B)_x X^h) \\
\geq (1 - \epsilon) \beta(d\mathcal{P}(x)(X^h), (J_B)_b d\mathcal{P}(x)X^h)
\end{equation}
for all $t \in (0, \bar{t}]$ and any $X \in TP(E \oplus \mathbb{C})$. Note that $J_b$ is compatible with $(\omega_{FS})_b$. The desired claim is proved.

Note that the almost complex structure $J$ in (4.15) satisfies $d\mathcal{P} \circ J = J_B \circ d\mathcal{P}$. That is, the projection $\mathcal{P}$ is holomorphic with respect to the almost complex structures $J$ and $J_B$. In terms of [27, Definition 2.8], we say that the almost complex structure $J$ on $P(E \oplus \mathbb{C})$ is fibred with respect to $J_B$. For such an almost complex structure $J$ on $P(E \oplus \mathbb{C})$, any $J$-holomorphic curve in $P(E \oplus \mathbb{C})$ representing a homology class of fiber must sit entirely in a fiber. By the following remark, each $J_b$ can be chosen as a canonical one such that for the corresponding almost complex structure $J$, the projection $\mathcal{P} : (P(E \oplus \mathbb{C}), J) \to (B, J_B)$ is an almost complex fibration with fibre $(\mathbb{C}P^n, J_{FS})$ in the sense of [1, Definition 6.3.A].

Remark 4.3. The above almost complex structure $J_b \in J(P(E \oplus \mathbb{C})_b, (\omega_{FS})_b)$ can be chosen as canonical one as done in [1]. Namely, a Hermitian vector bundle may yield a canonical (almost) complex structure on each fibre of its projective bundle. For every $J_E \in J(E, \bar{\omega})$, one has a corresponding almost complex structure on $P(E)$, $J_{can}$ which restricts to an almost complex structure $J_{can,b}$ on $P(E)_b = P(E_b)$ for each $b \in B$.

Recall that in [23, Definition 1.14], a closed symplectic manifold $(M, \omega)$ is called $g$-symplectic uniruled if the Gromov-Witten invariants
\begin{equation}
\Psi_{A, g, m+2}(C; pt, \alpha, \beta_1, \cdots, \beta_m) \neq 0
\end{equation}
for some homology classes $A \in H_2(M; \mathbb{Z})$, $\alpha, \beta_1, \cdots, \beta_m \in H_*(M; \mathbb{Q})$ and $C \in H_*(\overline{M}_{g,m+2}; \mathbb{Q})$ and an integer $m \geq 1$. In particular, if $C$ can be chosen as the class of a point, $(M, \omega)$ is said to be strong $g$-symplectic uniruled.

Theorem 4.4. Under the assumptions of Theorem 3.3, for the symplectic forms $\Omega^t_{\mathcal{A}_\epsilon}$ in (3.23), it holds that every symplectic manifold $(P(E \oplus \mathbb{C}), \Omega^t_{\mathcal{A}_\epsilon})$ is strong 0-symplectic uniruled. More precisely, the GW-invariant
\begin{equation}
\Psi_{L,0,3}(pt; pt, Z_0, Z_\infty) = 1,
\end{equation}
where \( L \) denotes the class of the line in the fiber of \( P(E \oplus \mathbb{C}) \).

We put off its proof to the end of this paper.

**Theorem 4.5.** Under the assumptions of Theorem 4.4, the pseudo symplectic capacity

\[
C_{HZ}^{(2\varepsilon)}(P(E \oplus \mathbb{C}), \Omega^t_{A_\varepsilon}; pt, Z_0) \leq \pi t \varepsilon^2
\]

for each \( 0 < \varepsilon < \varepsilon_0(A, \mathcal{F}) \) and \( 0 < t < t_0(A, \varepsilon) \). In particular, the \( \pi_1 \)-sensitive Hofer-Zehnder capacity

\[
c_{HZ}^{\varepsilon}(D_\varepsilon(E), \bar{\omega}^t_A) = c_{HZ}^{\varepsilon}(D_1(E), \omega^t_{A_\varepsilon}) \leq \pi t \varepsilon^2
\]

for each \( 0 < \varepsilon < \varepsilon_0(\mathcal{F}, A) \) and \( 0 < t < t_0(A, \varepsilon) \). Furthermore, for a given connection form \( A' \) on \( U(E) \) which is flat near some symplectically embedded ball in \( (B, \beta) \) of radius being the symplectic radius of \( (B, \beta) \), and each

\[
0 < \varepsilon \ll \min\{\varepsilon_0(\mathcal{F}, A), \varepsilon_0(\mathcal{F}, A'), \varepsilon_0(P, A_{con}), \sqrt{\frac{W_G(B, \beta)}{\pi}}\},
\]

it holds that

\[
c(D_\varepsilon(E), \bar{\omega}^t_A) = c(D_1(E), \omega^t_{A_\varepsilon}) = \pi t \varepsilon^2
\]

for each \( 0 < t < \min\{t_0(A', \varepsilon), t_0(A, \varepsilon)\} \) and \( c = W_G, c_{HZ} \), and \( c_{HZ}^{\varepsilon} \).

**Proof.** Step 1. Proving (4.21). Firstly, note that (4.9) and (4.19) directly yields (4.20) because \( \Omega^t_{A_\varepsilon}(L) = \pi t \varepsilon^2 \). Next, by Theorem 3.3(v), we have

\[
c_{HZ}^{\varepsilon}(D_\varepsilon(E), \bar{\omega}^t_A) = c_{HZ}^{\varepsilon}(D_1(E), \omega^t_{A_\varepsilon})
\]

for any \( 0 < \varepsilon < \varepsilon_0(A, \mathcal{F}) \) and \( 0 < t < t_0(A, \varepsilon) \). By the exact homotopy sequence of the fibration, we can easily derive that the embedding \( \varphi_\delta : D_\delta(E) \to P(E \oplus \mathbb{C}) \) induces an injective homomorphism \( \pi_1(D_\delta(E)) \to \pi_1(P(E \oplus \mathbb{C})) \) for any \( 0 < \delta < 1 \). Applying (4.13) to \( W_\delta := \varphi_\delta(Cl(D_\delta(E))) \), we get

\[
C_{HZ}^{\varepsilon}(W_\delta, \Omega^t_{A_\varepsilon}) \leq C_{HZ}^{\varepsilon}(P(E \oplus \mathbb{C}), \Omega^t_{A_\varepsilon}; pt, Z_0) \leq \pi t \varepsilon^2
\]

for any \( 0 < \varepsilon < \varepsilon_0(\mathcal{F}, A) \) and \( 0 < t < t_0(A, \varepsilon) \). Here the second inequality comes from (4.20). Note that (4.4) and Theorem 3.3(ii) imply

\[
c_{HZ}^{\varepsilon}(Cl(D_\delta(E)), \omega^t_{A_\varepsilon}) = C_{HZ}^{\varepsilon}(Cl(D_\delta(E)), \omega^t_{A_\varepsilon}) = C_{HZ}^{\varepsilon}(W_\delta, \Omega^t_{A_\varepsilon}),
\]
for each $0 < \epsilon < \epsilon_0(\mathcal{F}, A)$ and $0 < t < t_0(A, \epsilon)$. Setting $\delta \to 1$ and using the definition of $c_{HZ}^\delta$, we obtain that
\begin{equation}
(4.27) \quad c_{HZ}^\delta(D_1(E), \omega_{A^\epsilon}) \leq \pi t \epsilon^2
\end{equation}
for each $0 < \epsilon < \epsilon_0(\mathcal{F}, A)$ and $0 < t < t_0(A, \epsilon)$. Now the desired (4.21) follows from (4.24) and (4.27) directly.

**Step 2. Proving (4.23).** Let $\dim B = 2k$. Take a symplectic embedding $\Upsilon : (B^{2k}(r), \omega_{\text{std}}) \to (B, \beta)$, where $r = \mathcal{W}_G(B, \beta)$ is the Gromov symplectic radius. Since $U(E)|_{\mathcal{T}(B^{2k}(r))}$ can be trivialized, for any small $\epsilon > 0$, we may choose a connection form $A'$ on $U(E)$ such that it is flat near $\Delta := Cl(\Upsilon(B^{2k}(r - \epsilon)))$. By Theorem 3.3(vii), for any $0 < \epsilon \ll \min\{\epsilon_0(\mathcal{F}, A), \epsilon_0(\mathcal{F}, A')\} \leq 1$, $0 < t < \min\{t_0(A, \epsilon), t_0(A', \epsilon)\}$, and $\delta \in (0, 1]$, we have a symplectomorphism
\begin{equation}
(4.28) \quad \varphi^\delta_\epsilon : (D_\delta(E), \omega_{A^\epsilon}) \to (D_\delta(E), \omega_{A^\epsilon}'),
\end{equation}
which is also a bundle isomorphism. Note that $\Delta$ has the same dimension as $B$. From the first paragraph of the proof of Theorem 3.2, it is easily observed that $\epsilon_0^\delta(A'|_{Cl(\Delta)}, \mathcal{F}|_{U(E)|_{Cl(\Delta)}})$ can be taken as $\epsilon_0(A', \mathcal{F})$. Hence, Theorem 3.3(vi) implies that $(D_1(E)|_{\Delta}, \omega_{A^\epsilon}'|_{\Delta}) = (D_1(E)|_{\Delta}, \omega_{A^\epsilon}'|_{\Delta})$ is a symplectic submanifold in $(D_1(E), \omega_{A^\epsilon}')$ for each $0 < \epsilon < \epsilon_0(A', \mathcal{F})$ and $0 < t < t_0(A', \epsilon)$. So for any symplectic capacity $c$ (including $c_{HZ}^\delta$), it holds that
\begin{equation}
(4.29) \quad c(D_1(E)|_{\Delta}, \omega_{A^\epsilon}'|_{\Delta}) \leq c(D_1(E), \omega_{A^\epsilon}').
\end{equation}
By Theorem 3.3(v), the symplectomorphism
\begin{equation}
(4.30) \quad \psi_\epsilon : (D_1(E), \omega_{A^\epsilon}') \to (D_\epsilon(E), \tilde{\omega}_{A^\epsilon}', (b, v) \mapsto (b, \epsilon v)
\end{equation}
maps $(D_1(E)|_{\Delta}, \omega_{A^\epsilon}'|_{\Delta}) = (D_1(E)|_{\Delta}, \omega_{A^\epsilon}'|_{\Delta})$ onto $(D_\epsilon(E)|_{\Delta}, \tilde{\omega}_{A^\epsilon}'|_{\Delta})$. We get
\begin{equation}
(4.31) \quad c(D_1(E)|_{\Delta}, \omega_{A^\epsilon}'|_{\Delta}) = c(D_\epsilon(E)|_{\Delta}, \tilde{\omega}_{A^\epsilon}'|_{\Delta})
\end{equation}
for $0 < \epsilon < \epsilon_0(\mathcal{F}, A')$ and $0 < t < t_0(A', \epsilon)$. Therefore (4.28), (4.29), and (4.31) yields
\begin{equation}
(4.32) \quad c(D_1(E), \omega_{A^\epsilon}') = c(D_1(E), \omega_{A^\epsilon}') \geq c(D_1(E)|_{\Delta}, \omega_{A^\epsilon}'|_{\Delta}) = c(D_\epsilon(E)|_{\Delta}, \tilde{\omega}_{A^\epsilon}'|_{\Delta})
\end{equation}
for any $0 < \varepsilon \ll \min\{\varepsilon_0(F, A), \varepsilon_0(F, A')\} \leq 1$ and $0 < t < \min\{t_0(A, \varepsilon), t_0(A', \varepsilon)\}$.

Now Theorem 3.3(viii) implies that for any $0 < \eta < \varepsilon_0(P, A_{can})$ and $t > 0$, the symplectic manifold $(D_\eta(E)|_\Delta, \omega^t_{A|\Delta})$ can be identified with

$$(4.33)\quad (\Delta \times B^{2n}(\eta), \beta \oplus (t\omega^{(n)}_{std})) \approx (Cl(B^{2k}(r - \varepsilon)) \times B^{2n}(\eta), \omega^{(k)}_{std} \oplus (t\omega^{(n)}_{std})).$$

So for $0 < t < W_G(B, \beta)/\pi \eta^2$, it holds that

$$c(D_\eta(E)|_\Delta, \omega^t_{A|\Delta}) = c(\Delta \times B^{2n}(\eta), \beta \oplus (t\omega^{(n)}_{std}))$$

$$= c(Cl(B^{2k}(r - \varepsilon)) \times B^{2n}(\eta), \omega^{(k)}_{std} \oplus (t\omega^{(n)}_{std}))$$

$$\geq W_G(Cl(B^{2k}(r - \varepsilon)) \times B^{2n}(\eta), \omega^{(k)}_{std} \oplus (t\omega^{(n)}_{std}))$$

$$(4.34)\quad \geq \pi t \eta^2.$$

Taking $\eta = \varepsilon \ll \min\{\varepsilon_0(F, A), \varepsilon_0(F, A'), \varepsilon_0(P, A_{can})\}$, (4.32) and (4.34) yield that

$$(4.35)\quad c(D_1(E), \omega^t_{A_{can}}) \geq \pi t \varepsilon^2$$

for any $0 < t < \min\{t_0(A', \varepsilon), t_0(A, \varepsilon), W_G(B, \beta)/\pi \varepsilon^2\}$. Note that we can choose $\varepsilon > 0$ so small that $W_G(B, \beta)/\pi \varepsilon^2 > 1$ and that we can also assume that $t_0(A', \varepsilon)$ and $t_0(A, \varepsilon)$ are no more than $1$. The desired (4.23) immediately follow from (4.21) and (4.35). \qed

**Proof of Theorem 1.3.** Clearly, (1.7) and (1.8) directly follow from (4.21) and (4.23), respectively. To get (1.9), for a given $\varepsilon > 0$, we choose $t > 0$ so small that $\pi t \varepsilon^2 < \varepsilon$. Consider the (disk) bundle isomorphism

$$(4.36)\quad \Theta_{t, \varepsilon} : D_{\sqrt{t} \varepsilon}(E) \to D_\varepsilon(E), \quad (b, v) \mapsto (b, \frac{1}{\sqrt{t}} v).$$

Then $\Theta^s_{t, \varepsilon} \omega^t_A$ restricts to $(\omega_{std})_b$ on each fibre $D_{\sqrt{t} \varepsilon}(E)_b$, and the zero section $0_E$ is also a symplectic submanifold in $(D_{\sqrt{t} \varepsilon}(E), \Theta^s_{t, \varepsilon} \omega^t_A)$. Hence, the zero section $0_E$ has the symplectic normal bundle $(E, \omega)$ in $(D_{\sqrt{t} \varepsilon}(E), \Theta^s_{t, \varepsilon} \omega^t_A)$. By the Weinstein’s symplectic neighborhood theorem, there exists $0 < \eta \ll \varepsilon$ such that $(D_{\sqrt{\tau} \varepsilon}(E), \Theta^s_{t, \varepsilon} \omega^t_A)$ and thus $(D_\eta(E), \omega^t_A)$ $\subset (D_\varepsilon(E), \omega^t_A)$ is symplectomorphic to an open neighborhood $W$ of $B$ in $M$. Then (1.9) easily follows from this and (1.8). \qed

**Proof of Theorem 4.4.** Since the Gromov-Witten invariants are symplectic deformation invariants, it suffices to prove that (4.19) holds on $(P(E \oplus \mathbb{C}), \Omega^t_{A_C})$ for sufficiently small $t > 0$. To this end, we need to construct a suitable $\Omega^t_{A_C}$-tamed almost complex
structure on $P(E \oplus \mathbb{C})$. Using the standard complex structure $i$ and symplectic structure on $\mathbb{C}$, we get a complex structure $J_E \oplus i$ on $E \oplus \mathbb{C}$ and the corresponding Hermitian structure. As in Remark 4.3, it in turn yields a complex structure on each fibre $P(E \oplus \mathbb{C})_b$ of $P(E \oplus \mathbb{C})$, still denoted by $J_{\text{can},b}$. One now has obvious Kähler identification $(P(E \oplus \mathbb{C})_b, (\omega_{FS})_b, J_{\text{can},b}) \equiv (\mathbb{C}^n, \omega_{FS}, J_{FS})$. From it and any compatible almost complex structure $J_B \in \mathcal{J}(B, \beta)$ we can, as in (4.15), form an almost complex structure $J$ on $P(E \oplus \mathbb{C})$ which is $\Omega^t_{AE}$-tamed for some sufficiently small $0 < t < t_0(A, \varepsilon)$. Moreover, the projection $\mathcal{P} : (P(E \oplus \mathbb{C}), J) \to (B, J_B)$ is an almost complex fibration with fibre $(\mathbb{C}^n, J_{FS})$ by the final sentence above Remark 4.3. It is well-known that $J_{FS} \in \mathcal{J}_{\text{reg}}(\mathbb{C}^n, \omega_{FS})$, (cf. [28, Prop.7.4.3]). Let $[\mathbb{C}P^1] \subset H_2(\mathbb{C}P^n, \mathbb{Z})$ denote the class of the line $\mathbb{C}P^1 \subset \mathbb{C}P^n$. It was proved in [34] that the Gromov-Witten invariant $\Psi_{[\mathbb{C}P^1, 0, 3]}^{(\mathbb{C}P^n, \omega_{FS})}(pt; pt, pt, [\mathbb{C}P^{n-1}]) = 1$, where $pt$ always denotes the class of a single point for different spaces, and $[\mathbb{C}P^{n-1}] \in H_{2n-2}(\mathbb{C}P^n, \mathbb{Z})$ is the class of the hyperplane $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$.

Note that $Z_0$ (resp., $Z_\infty$) restricts to a single point (resp., the hyperplane $\mathbb{C}P^{n-1}$) on each fiber $P(E \oplus \mathbb{C})_b \equiv \mathbb{C}^n$. If $L \in H_2(P(E \oplus \mathbb{C}), \mathbb{Z})$ denotes the class of the line in the fiber, for the fibered almost complex structure $J$ just constructed and a given point $x \in P(E \oplus \mathbb{C})$, every $J$-holomorphic sphere of class $L$ in $P(E \oplus \mathbb{C})$ passing $x$ must sit in the fiber $P(E \oplus \mathbb{C})_b$ at $b = \mathcal{P}(x) \in B$. Therefore, there exists such a unique curve passing $x$ and generically intersecting with $Z_0$ and $Z_\infty$. As expected, we arrive at (4.19) for $(P(E \oplus \mathbb{C}), \Omega^t_{AE})$. Actually, by [1, Proposition 6.3.B], the $J$ is regular for the class $L$. So if $\mathcal{M}(P(E \oplus \mathbb{C}), L, J)$ is the space of all $J$-holomorphic spheres in $P(E \oplus \mathbb{C})$ representing the class $L$, then $\mathcal{M}(P(E \oplus \mathbb{C}), L, J)/\text{PSL}(2, \mathbb{R})$ is a compact smooth manifold of dimension $\dim \mathbb{R}B + 2 \text{rank}_\mathbb{R}E - 6$, and that the dimension condition

$$\deg[pt] + \deg[Z_0] + \deg[Z_\infty] = \dim \mathbb{R}P(E \oplus \mathbb{C}) + 2c_1(L) = \dim \mathbb{R}B + 2 \text{rank}_\mathbb{R}E + 2$$

(4.37)

is satisfied. It easily follows that the Gromov-Witten invariant (in the sense of [34]) $\Psi_{L,0,3}(pt; pt, [Z_0], [Z_\infty]) = 1$.

Since our [23] follows [14], the GW-invariant in (4.19) is one constructed by Liu-Tian in [14] (also see [21] for details). Let $\Psi^\text{vir}_{L,0,3}(pt; pt, [Z_0], [Z_\infty])$ denote this GW-invariant constructed with the virtual moduli cycles. In the remainder of the paper we shall show that
it is also equal to 1. (Namely, while some GW-invariant can be simultaneously defined in the methods in [14, 34], they agree.) Instead of using the method in [18], we use the method of proof of [23, Proposition 7.6] to prove that the GW-invariant in the sense of the general definition is also equal to 1. Let \( \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C}), L, J) \) be the space of all equivalence classes of all 3-pointed stable \( J \)-maps of genus zero and of class \( L \) in \( P(E \oplus \mathbb{C}) \). Then each stable map \( [f] \in \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C}), L, J) \) has an image set contained in a single fiber of \( P(E \oplus \mathbb{C}) \) since the image set is connected. Assume that \( \text{Im}(f) \subset P(E \oplus \mathbb{C}) \). This \( [f] \) may naturally be viewed as an element of \( \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C})_b, L_b, J_{\text{can},b}) \), where \( L_b \) is the class of the line in \( P(E \oplus \mathbb{C})_b \equiv \mathbb{C}P^n \). Since \( \Omega^{i_b}_{2\epsilon}|_{P(E \oplus \mathbb{C})_b} = t^2(\omega_{FS})_b \) and \( L_b \) is indecomposable with respect to \( (\omega_{FS})_b \), by the proof of [23, Proposition 7.6], it is easily seen that \( \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C})_b, L_b, J_{\text{can},b}) \) is a stratified smooth compact manifold (using the regularity of \( J_{\text{can},b} \)). More precisely,

\[
\bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C})_b, L_b, J_{\text{can},b}) = \bigcup_{i=1}^{4} \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C})_b, L_b, J_{\text{can},b})_i,
\]

where each stratum \( \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C})_b, L_b, J_{\text{can},b})_i \) is a smooth manifold, and

\[
\begin{align*}
\dim \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C})_b, L_b, J_{\text{can},b})_1 &= 2n + 2(n + 1), \\
\dim \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C})_b, L_b, J_{\text{can},b})_2 &= 2n + 2(n + 1) - 4, \\
\dim \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C})_b, L_b, J_{\text{can},b})_3 &= 2n + 2(n + 1) - 6, \\
\dim \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C})_b, L_b, J_{\text{can},b})_4 &= 2n + 2(n + 1) - 6.
\end{align*}
\]

So \( \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C}), L, J) = \bigcup_{b \in B} \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C})_b, L_b, J_{\text{can},b}) \) is also a stratified smooth compact manifold (with correct dimension because of the regularity of \( J \)). That is,

\[
\bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C}), L, J) = \bigcup_{i=1}^{4} \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C}), L, J)_i,
\]

where \( \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C}), L, J)_i = \bigcup_{b \in B} \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C})_b, L_b, J_{\text{can},b})_i, i = 1, 2, 3, 4. \) Since each stable map \( [f] \in \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C}), L, J) \) has no free components, we have the following claim.

**Claim 4.6.** A virtual moduli cycle of \( \bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C}), L, J) \) can be taken as

\[
\bar{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C}), L, J) \to \mathcal{B}_{0,3,L}^{P(E \oplus \mathbb{C})}, [f] \mapsto [f].
\]
Once it is proved, then almost repeating the proof of [23, Proposition 7.6], we can obtain the desired (4.19):

$$\Psi^{vir}_{L,0,3}(pt; pt, [Z_0], [Z_\infty]) = \Psi_{L,0,3}(pt; pt, [Z_0], [Z_\infty]) = 1.$$  

In order to prove Claim 4.6, as in the proof of [23, Proposition 7.6], each $[f] \in \overline{M}_{0,3}(P(E \oplus \mathbb{C}), L, J)$ must be one of the following four cases.

(i) The domain $\Sigma = \mathbb{C}P^1$, $z_i$, $i = 1, 2, 3$, are three distinct marked points on $\Sigma$, and $f : \Sigma \to P(E \oplus \mathbb{C})$ is a $J$-holomorphic map of class $L$.

(ii) The domain $\Sigma$ has exactly two components $\Sigma_1 = \mathbb{C}P^1$ and $\Sigma_2 = \mathbb{C}P^1$ which have a unique intersecting point. $f|_{\Sigma_1}$ is nonconstant and $\Sigma_1$ only contains a marked point. $f|_{\Sigma_2}$ is constant and $\Sigma_2$ contains two marked points.

(iii) The domain $\Sigma$ has exactly two components $\Sigma_1 = \mathbb{C}P^1$ and $\Sigma_2 = \mathbb{C}P^1$ which have a unique intersecting point. $f|_{\Sigma_1}$ is nonconstant and $\Sigma_1$ contains no marked point. $f|_{\Sigma_2}$ is constant and $\Sigma_2$ contains three marked points.

(iv) The domain $\Sigma$ has exactly three components $\Sigma_1 = \mathbb{C}P^1$, $\Sigma_2 = \mathbb{C}P^1$, and $\Sigma_3 = \mathbb{C}P^1$. $\Sigma_1$ and $\Sigma_2$ (resp., $\Sigma_2$ and $\Sigma_3$) has only an intersecting point, and $\Sigma_1$ and $\Sigma_3$ has no intersecting point. $f|_{\Sigma_1}$ is nonconstant and $\Sigma_1$ contains no a marked point. $f|_{\Sigma_2}$ is constant and $\Sigma_2$ contains a marked point. $f|_{\Sigma_3}$ is constant and $\Sigma_3$ contains two marked points.

Note that in each case, the nonconstant $f|_{\Sigma_1}$ is simple, and thus somewhere injective. It follows that the automorphism group $Aut(f)$ of $f$ is trivial though $[f]$ might contain many representatives. Let $B^{P(E \oplus \mathbb{C})}_{k,p}$ be the set of equivalence classes of all $3$-pointed stable $L^{k,p}$-maps in $P(E \oplus \mathbb{C})$ which represent class $L$, have genus $0$ and domains that belong to the four types above. For $[f] \in \overline{M}_{0,3}(P(E \oplus \mathbb{C}), L, J)$, let $\tilde{U}_\delta(f, H)$ be the local uniformizer near $[f] \in B^{P(E \oplus \mathbb{C})}_{k,p}$ as constructed in [12, Section 2] (also see [21, Section 2] for details). Since $Aut(f)$ is trivial, [12, Lemma 2.6] showed that the natural projection $\tilde{U}_\delta(f, H) \to B^{P(E \oplus \mathbb{C})}_{k,p}$ gives rise to a homeomorphism to an open neighborhood of $[f]$ in $B^{P(E \oplus \mathbb{C})}_{k,p}$. This suggests that some open neighborhood $W$ of $\overline{M}_{0,3}(P(E \oplus \mathbb{C}), L, J)$ in $B^{P(E \oplus \mathbb{C})}_{k,p}$ might carry a stratified Banach manifold structure.

Regardless of these, we still adopt the Liu-Tian construction method in [12, 14] to get a system of bundles

$$\tilde{(\mathcal{E}^I, \mathcal{W}^I)} = \{(\tilde{\mathcal{E}}^I_J, \tilde{\mathcal{W}}^I_J), \tilde{\pi}_I, \tilde{\Pi}_I, \Gamma_I, \tilde{\pi}^j_I, \tilde{\Pi}^j_I, \lambda^j_I : J \subset I \in \mathcal{N}\)
(44) \((\tilde{\mathcal{E}}, \tilde{\mathcal{V}}) = \{(\tilde{E}_I, \tilde{V}_I), \tilde{\pi}_I, \tilde{\pi}_J, \tilde{\Pi}_I, \tilde{\Pi}_J, \tilde{p}_I, \Gamma_I \mid J \subset I \in \mathcal{N}\}\)

as in [12, 14]. Then by restrictions, we get a system of stratified smooth Banach bundles as in [21, (3.74)]:

\[(45) (\tilde{\mathcal{E}}^*, \tilde{\mathcal{V}}^*) = \{(\tilde{E}_I^*, \tilde{V}_I^*), \pi_I, \tilde{\pi}_J, \tilde{\Pi}_I, \tilde{\Pi}_J, \tilde{p}_I, \Gamma_I \mid J \subset I \in \mathcal{N}\}.\]

Now following the idea of the proof of [12, Theorem 4.1] (see [21, (3.75) and (3.76)]), we have the obvious pullback stratified smooth Banach bundle system

\[(46) (\mathcal{P}_I^* \tilde{\mathcal{E}}^*, \tilde{\mathcal{V}}^* \times \mathcal{B}_\eta(\mathbb{R}^q)) = \{(\mathcal{P}_I^* \tilde{E}_I^*, \tilde{V}_I^* \times \mathcal{B}_\eta(\mathbb{R}^q)), \pi_I, \tilde{\pi}_J, \Pi_I, \tilde{\Pi}_J, p_I, \Gamma_I \mid J \subset I \in \mathcal{N}\},\]

and its global section \(\Psi = \{\Psi_I \mid I \in \mathcal{N}\},\)

\[(47) \Psi_I : \tilde{\mathcal{V}}_I^* \times \mathcal{B}_\eta(\mathbb{R}^q) \to \mathcal{P}_I^* \tilde{\mathcal{E}}_I^*, (\tilde{x}_I, t) \mapsto (\tilde{\partial}_I)_I(\tilde{x}_I) + \sum_{i=1}^{n_3} \sum_{j=1}^{q_i} t_{ij}(\tilde{s}_{ij})_I(\tilde{x}_I),\]

where \(t = \{t_{ij} \mid 1 \leq j \leq q_i, 1 \leq i \leq n_3\} \in \mathbb{R}^q\). Clearly, \(\Psi_I(\tilde{x}_I, 0) = 0\) for any zero \(\tilde{x}_I\) of \((\tilde{\partial}_I)_I\) in \(\tilde{V}_I\). By [21, Theorem 3.14 and Corollary 3.15], we get a small \(\eta > 0\) and a residual subset \(\mathcal{B}_\eta(\mathbb{R}^q) \subset \mathcal{B}_\eta(\mathbb{R}^q)\) such that for each \(t \in \mathcal{B}_\eta(\mathbb{R}^q)\), the global section \(\Psi^{(t)}_I = \{\Psi^{(t)}_I \mid I \in \mathcal{N}\}\) of the bundle system \((\tilde{\mathcal{E}}^*, \tilde{\mathcal{V}}^*)\) is transversal to the zero section, where \(\Psi^{(t)}_I : \tilde{\mathcal{V}}_I^* \to \tilde{\mathcal{E}}_I^*, \tilde{x}_I \mapsto \Psi_I(\tilde{x}_I, t)\). So the set \(\tilde{\mathcal{M}}_I^t := (\Psi^{(t)}_I)^{-1}(0)\) is a stratified smooth Banach manifold of dimension \(\dim B + 4n + 2\). It also holds that

(A) the stratified Banach manifold \(\tilde{\mathcal{M}}_I^t\) has no strata of codimension odd, and each stratum of \(\tilde{\mathcal{M}}_I^t\) of codimension \(r\) is exactly the intersection of \(\tilde{\mathcal{M}}_I^t\) and the stratum of \(\tilde{V}_I^t\) of codimension \(r\) for \(r = 0, \ldots, \dim B + 4n + 2\);

(B) the family \(\tilde{\mathcal{M}}^t = \{\tilde{\mathcal{M}}_I^t \mid I \in \mathcal{N}\}\) is compatible in the sense that for any \(J \subset I \in \mathcal{N}\),

\[(48) \tilde{\pi}_J^t : (\tilde{\pi}_J^t)^{-1}(\tilde{\mathcal{M}}_I^t) \to \text{Im}(\tilde{\pi}_J^t) \subset \tilde{\mathcal{M}}_I^t\]
is a continuous and stratified smooth open embedding;

(C) for each $I \in \mathcal{N}$ and any two $t, t' \in B^\text{res}_\eta (\mathbb{R}^q)$, the cornered stratified Banach manifolds $\widetilde{M}^t_I$ and $\widetilde{M}^{t'}_I$ are cobordant, and thus maps $\tilde{\pi}_I : \widetilde{M}^t_I \to \mathcal{W}$ and $\tilde{\pi}_I : \widetilde{M}^{t'}_I \to \mathcal{W}$ are also cobordant.

Since $|\Gamma_I| = 1$ for any $I \in \mathcal{N}$, the formal summations

$$C^t := \sum_{I \in \mathcal{N}} \{ \tilde{\pi}_I : \widetilde{M}^t_I \to \mathcal{W} \} \quad \forall t \in B^\text{res}_\eta (\mathbb{R}^q),$$

a family of cobordant singular cycles in $\mathcal{W}$, are virtual moduli cycles in $\mathcal{W}$ constructed by Liu-Tian method (cf. [12, 13, 14, 21]). As explained [12, page 65], the summation precisely means that on the overlap of two pieces of $C^t$, we only count them once. Since all $\tilde{\pi}_I$ and $\tilde{\pi}_I'$ are stratified smooth open embeddings, $\{(\tilde{\pi}_I, M^t_I) | I \in \mathcal{N}\}$ is actually a compatible coordinate chart cover of a compact, stratified smooth Banach manifold $\bigcup_{I \in \mathcal{N}} \tilde{\pi}_I(M^t_I) \subset \mathcal{W}$ of dimension $\dim B + 4n + 2$ and without strata of codimension one.

Note that each $(\tilde{\partial}_J)_I$ is essentially the Cauchy-Riemann operator $\partial_J$ and that $J$ is regular with respect to the class $L$. It is not hard to prove that $0 \in B^\text{res}_\epsilon (\mathbb{R}^q)$ and $\bigcup_{I \in \mathcal{N}} \tilde{\pi}_I(M^0_I) = \overline{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C}), L, J)$. This proved Claim 4.6. \hfill \Box

**Remark 4.7.** As suggested in the above proof, so far $\mathcal{E} \to \mathcal{W}$ might be a stratified Banach bundle on a stratified Banach manifold. The Cauchy-Riemann operator $\partial_J$ is a Fredholm section (restricting each stratum) and has $\overline{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C}), L, J)$ as zero set of it. The original arguments of transversality and gluing may yield finitely many continuous stratified smooth sections $s_i : \mathcal{W} \to \mathcal{E}, i = 1, \cdots, m$ such that the section

$$\Phi : \mathcal{W} \times B^\text{res}_\eta (\mathbb{R}^m) \to \Pi^*_\tau \mathcal{E}, (\tau, t) \mapsto \tilde{\partial}_J \tau + t_1 s_1(\tau) + \cdots + t_m s_m(\tau)$$

is transversal to the zero section for $\eta > 0$ small enough. Consequently, for generic small $t \in B^\text{res}_\eta (\mathbb{R}^m)$, the section $\Phi_t : \mathcal{W} \to \mathcal{E}, \tau \mapsto \tilde{\partial}_J \tau + t_1 s_1(\tau) + \cdots + t_m s_m(\tau)$ is transversal to the zero section. In particular, since $J$ is regular, the section $\Phi_0 = \tilde{\partial}_J$ is transversal to the zero section. So we do not need to use Liu-Tian method as above and can construct a desired perturbation cycle of $\overline{\mathcal{M}}_{0,3}(P(E \oplus \mathbb{C}), L, J)$, which is cobordant to virtual moduli cycles constructed by Liu-Tian method. These are, in detail, explained and developed in general abstract settings in [24].
Acknowledgements. This work was partially supported by the NNSF 10371007 of China and the Program for New Century Excellent Talents of the Education Ministry of China. This work began during the author’s visit at the Institut des Hautes Études Scientifiques (IHES) in Winter 2004 and was completed during the author’s visit to the Abdus Salam International Centre for Theoretical Physics in Summer 2005. The author cordially thanks Professors J. P. Bourguignon and D.T. Lê for their invitations and for the financial support and hospitality of their institutions, and Professor C. Viterbo for his invitation to join Symplectic Geometry Seminar in École Polytechnique during the author’s visit to IHES in Winter 2004. The author also thanks Paul Biran, Leonardo Macarini, Felix Schlenk and Ramadas Ramakrishnan for some related discussions. The author would like to thank Professor Leonid Polterovich for mailing me his lovely book [33] and for very valuable improvement suggestions, and he would like to thank Ely Kerman for sending me his preprint. Thanks are also due to the anonymous referee, whose comments and corrections improved the exposition.

REFERENCES

[1] P. Biran, Lagrangian barriers and symplectic embeddings, Geometric and Functional Analysis 11(2001), no.3, 407-464.
[2] K. Cieliebak, V. L. Ginzburg and E. Kerman, Symplectic homology and periodic orbits near symplectic submanifolds, Commentarii Mathematici Helvetici 79(2004), no.3, 554-581.
[3] M. Entov, K-area, Hofer metric and geometry of conjugacy classes in Lie groups, Inventiones Mathematicae 146 (2001), 93–141.
[4] A. Floer, H. Hofer and C. Viterbo, The Weinstein conjecture in $P \times \mathbb{C}^d$, Mathematische Zeitschrift 203 (1989), no.3, 469-482.
[5] V. L. Ginzburg, The Weinstein conjecture and the theorems of nearby and almost existence, The breadth of symplectic and Poisson geometry, 139–172, Progr. Math., vol. 232, Birkhäuser Boston, Massachusetts, 2005, pp. 139-172.
[6] V. L. Ginzburg and B.Z. Gürel, Relative Hofer-Zehnder capacity and periodic orbits in twisted cotangent bundles, Duke Mathematical Journal 123(2004), no.1, 1-47.
[7] M. Gromov, Pseudoholomorphic curves in symplectic manifolds, Inventiones Mathematicae 82 (1985), no.2, 307-347.
[8] H. Hofer and C. Viterbo, The Weinstein conjecture in the presence of holomorphic spheres, Communications on Pure and Applied Mathematics 45 (1992), no.5, 583-622.
[9] H. Hofer and E. Zehnder, A new capacity for symplectic manifolds, Analysis et cetera ( P. Rabinowitz and E. Zehnder, eds.), Academic Press, Massachusetts, 1990, pp. 405-427.
[10] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser Advanced Texts: Basel Textbooks, Birkhäuser, Basel, 1994.

[11] E. Kerman, *Squeezing in Floer theory and refined Hofer-Zehnder capacities of sets near symplectic submanifolds*, Geometry and Topology 9(2005), no. 40, 1775-1834.

[12] G. Liu and G. Tian, *Floer homology and Arnold conjecture*, Journal of Differential Geometry 49(1998), no.1, 1-74.

[13] Gang Liu and Gang Tian, *On the Equivalence of Multiplicative Structures in Floer Homology and Quantum Homology*, Acta Mathematicae Sinica, English Series 15(1999), no.1, 53-80.

[14] G. Liu and G. Tian, *Weinstein Conjecture and GW Invariants*, Communications in Contemporary Mathematics 2(2000), no.4, 405-459.

[15] G. C. Lu, *Weinstein conjecture on some symplectic manifolds containing the holomorphic spheres*, Kyushu Journal of Mathematics 52(1998), no.2, 331-351.

[16] G. C. Lu, *Correction to: “Weinstein conjecture on some symplectic manifolds containing the holomorphic spheres”*, Kyushu Journal of Mathematics 54(2000), no.1, 181-182.

[17] G. C. Lu, *Symplectic capacities of toric manifolds and combinatorial inequalities*, Comptes Rendus Mathématique. Académie des Sciences. Paris 334(2002), no.10, 889–892.

[18] G. C. Lu, *An extension of Biran’s Lagrangian barrier theorem*, Proceedings of the American Mathematical Society 133(2005), no. 5, 1563-1567.

[19] G. C. Lu, *Corrigendum to the note: “Symplectic capacities of toric manifolds and combinatorial inequalities”*, Comptes Rendus Mathématique. Académie des Sciences. Paris 340(2005), no.10, 751–754.

[20] G. C. Lu, *Symplectic capacities of toric manifolds and related results*, Nagoya mathematical Journal 181(2006), no.1, 149-184.

[21] G. C. Lu, *Virtual moduli cycles and Gromov-Witten invariants of noncompact symplectic manifolds*, Communication in Mathematical Physics 261(2006), no. 1, 43-131.

[22] G. C. Lu, *Finiteness of Hofer-Zehnder symplectic capacity of neighborhoods of symplectic submanifolds*, http://arxiv.org/abs/math.SG/0510172 V2, 31 Oct 2005.

[23] G. C. Lu, *Gromov-Witten invariants and pseudo symplectic capacities*, to appear in Israel Journal of Mathematics 156, or http://arxiv.org/abs/math.SG/0103195.

[24] G. C. Lu and G. Tian, *Constructing virtual Euler cycles and class*, http://arxiv.org/abs/math.SG/0605290.

[25] L. Macarini, *Hofer-Zehnder semicapacity of cotangent bundles and symplectic submanifolds*, http://arxiv.org/abs/math.SG/0303230.

[26] L. Macarini and F. Schlenk, *A refinement of the Hofer-Zehnder theorem on the existence of closed trajectories near a hypersurface*, The Bulletin of the London Mathematical Society 37(2005), no.2, 297-300.

[27] D. McDuff, *Quantum homology of fibrations over $S^2$*, International Journal of Mathematics 11(2000), no.5, 665-721.

[28] D. McDuff and D. Salamon, *J-holomorphic curves and quantum cohomology*, University Lecture Series, vol.6, American Mathematical Society, Rhode Island, 1994.
[29] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, 2nd edition, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1998.

[30] D. McDuff and J. Slimowitz, *Hofer-Zehnder capacity and length minimizing Hamiltonian paths*, Geometry and Topology 5(2001), no.25, 799–830.

[31] L. Polterovich, *Gromov’s K-area and symplectic rigidity*, Geometric and Functional Analysis 6(1996), no.4, 726–739.

[32] L. Polterovich, *Symplectic aspects of the first eigenvalue*, Journal für die reine und angewandte Mathematik (Crelle’s journal) 502(1998), 1-17.

[33] L. Polterovich, *The geometry of the group of symplectic diffeomorphisms*, Lectures in Mathematics ETH Zürich. Birkhäuser, Basel, 2001.

[34] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, Journal of Differential Geometry 42 (1995), no.2, 259–367.

[35] F. Schlenk, *Applications of Hofer’s geometry to Hamiltonian dynamics*, Commentarii Mathematici Helvetici 81(2006), 105-121.

[36] M. Schwarz, *On the action spectrum for closed symplectically aspherical manifolds*, Pacific Journal of Mathematics 193(2000), no.2, 419–461.

[37] R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Annals of Mathematics. Second Series 134 (1991), no. 2, 375–422.

[38] M. Struwe, *Existence of periodic solutions of Hamiltonian systems on almost every energy surface*, Boletim da Sociedade Brasileira de Matemática 20(1990), no.2, 49-58.

[39] C. Viterbo, *A proof of the Weinstein conjecture in \( \mathbb{R}^{2n} \)*, Annales de l’Institut Henri Poincaré. Analyses Nonlinéaires 4(1987), no.4, 337–356.

[40] A. Weinstein, *On the hypotheses of Rabinowitz’s periodic orbit theorems*, Journal of Differential Equations 33(1979), no.3, 353–358.

**DEPARTMENT OF MATHEMATICS, BEIJING NORMAL UNIVERSITY, BEIJING 100875, P.R.CHINA**

*E-mail address: gclu@bnu.edu.cn*  
**URL:** http://math.bnu.edu.cn/~gclu