REILLY-TYPE INEQUALITIES FOR SUBMANIFOLDS IN CARTAN-HADAMARD MANIFOLDS

HANG CHEN AND XUDONG GUI

Abstract. Let $M$ be an $m$-($\geq 2$)-dimensional closed orientable submanifold in an $n$-dimensional complete simply-connected Riemannian manifold $N$, where the sectional curvature of $N$ is bounded above by $\delta$. When $\delta < 0$, inspired by Niu-Xu (arXiv:2106.01912), we give new upper bounds for the first nonzero eigenvalues of the $p$-Laplacian and the $L_T$ operator, respectively. These generalize Niu-Xu’s work for the Laplacian (arXiv:2106.01912) and improve the estimates due to Chen (Nonlinear Anal., 196, 111833, 2020) for the $p$-Laplacian and Grosjean (Hokkaido Math. J., 33(2), 319–339, 2004) for the $L_T$ operator, respectively. We also obtain several Reilly-type inequalities for the weighted manifolds and some boundary value problems.

1. Introduction

In geometry of submanifolds, Reilly-type inequality is an important estimate for the first non-zero eigenvalue of Laplacian, which gives the upper bound in terms of the mean curvature. It originated from Reilly’s work [21] in 1977, where closed (orientable) submanifolds in the Euclidean space were considered and a special case says that

$$\lambda_1^\Delta \leq \frac{m}{\text{vol}(M)} \int_M |H|^2. \tag{1.1}$$

Here $m$ and $H$ are the dimension and the mean curvature vector of the submanifold respectively, and $\lambda_1^\Delta$ represents the first non-zero eigenvalue of the Laplacian on the submanifold. Later, Heintze [15] considered a more general case that the sectional curvature of the ambient manifold $N$ is bounded above by a constant $\delta$, and showed that

$$\lambda_1^\Delta \leq \begin{cases} 
\frac{m}{\text{vol}(M)} \int_M (\delta + |H|^2) & \text{for } \delta \geq 0, \\
\frac{m}{\text{vol}(M)} \int_M (\delta + \max |H|^2) & \text{for } \delta < 0,
\end{cases} \tag{1.2}
$$

here it is required that $N$ is simply-connected when $\delta \leq 0$ and $M$ lies in a convex ball of $N$ of radius $\leq \pi/4\sqrt{\delta}$ when $\delta > 0$. Notice that the approach of Heintze cannot give the upper bound involving $L^2$-norm of the mean curvature when $\delta < 0$ while there is a restriction on the radius of the convex ball when $\delta > 0$. However, when $N$ is the space form of constant curvature $\delta$, El Soufi and Ilias [11] removed the restrictions and gave a sharp upper bound

$$\lambda_1^\Delta \leq \frac{m}{\text{vol}(M)} \int_M \delta + |H|^2 \tag{1.4}$$

by using a conformal map method whenever $\delta > 0$ and $\delta < 0$. 

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Very recently, Niu-Xu did nice work which says the estimate (1.3) can be improved to
\[
\lambda_1^A \leq \frac{m}{\text{vol}(M)} \int_M (\delta + |H|^2) \tag{1.5}
\]
when \(\delta < 0\) and \(m \geq 2\) ([20, Theorem 1.1]).

Note that the original Reilly’s inequalities in [21] actually involve not only the mean curvature, but also the higher order mean curvatures. More generally, given a symmetric \((1,1)\)-tensor \(T = (T_{ij})\) on \(M\), we can define a normal vector field \(H_T\) associated with \(T\) by
\[
H_T = \sum_{i=1}^m A(Te_i, e_i) = \sum_{m+1 \leq \alpha \leq n} \sum_{1 \leq i,j \leq m} h_{ij}^\alpha T_{ij}e_\alpha,
\]
where \(\{e_1, \ldots, e_n\}\) is a local orthonormal frame of \(N\) such that \(\{e_1, \ldots, e_m\}\) are tangent to \(M\) and \(\{e_{m+1}, \ldots, e_n\}\) are normal to \(M\), and \(A = (h_{ij}^\alpha)\) the second fundamental form of \(M\) in \(N\). Actually, \(H_T\) can be viewed as a natural generalization of the mean curvature and the higher order mean curvature, and has been involved in a lot of results ([2, 5, 14]). We call \(M\) is \(T\)-minimal if \(H_T\) vanishes.

To generalize the Reilly-type inequalities, a common approach is to consider the other operators instead of the Laplacian. Usually, there are two classes of the elliptic operators as alternatives: one is the \(p\)-Laplacian and the other one is the \(L_T\) operator. We states them as follows respectively.

1.1. \(p\)-Laplacian. The \(p\)-Laplacian \((p > 1)\) on a compact \(M\) is defined by
\[
\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u).
\]
It is a second order quasilinear elliptic operator for \(p \neq 2\) and can be viewed as a generalization of the Laplacian (corresponding to \(p = 2\)). A real number \(\lambda\) is called an eigenvalue if there exists a non-zero function \(u\) satisfying the following eigenvalue equation
\[
\Delta_p u = -\lambda |u|^{p-2}u \quad \text{on} \ M
\]
with appropriate boundary conditions. Although the regularity theory of the \(p\)-Laplacian is very different from the usual Laplacian, there is the smallest positive eigenvalue (i.e., the first non-zero eigenvalue) (cf. [12, 17]), denoted by \(\lambda_{1,p}\). When \(p = 2\), \(\lambda_{1,2}\) is just the first non-zero eigenvalue \(\lambda_1^A\) of the Laplacian. Moreover, \(\lambda_{1,p}\) has a Rayleigh-type variational characterization. For instance, if \(M\) is closed, then (cf. [24])
\[
\lambda_{1,p} = \inf \left\{ \frac{\int_M |\nabla u|^p}{\int_M |u|^p} \left| u \in W^{1,p}(M) \setminus \{0\}, \int_M |u|^{p-2}u = 0 \right. \right\}. \tag{1.6}
\]

Many estimates for the first non-zero eigenvalue of Laplacian in Riemannian geometry have been generalized to \(p\)-Laplacian (e.g., [4] and the references therein). In particular, the inequalities mentioned above have been extended from Laplacian to \(p\)-Laplacian. For example, Du-Mao [10] generalized (1.1) in the Euclidean space case; Wei and the first author [6] generalized (1.4) by the conformal map method in the space form case.

Recently, the first author generalized (1.2) and (1.3) and proved

**Theorem 1.1** (cf. [4, Theorem 1.3]). *Let \(M\) be an \(m(\geq 2)\)-dimensional closed orientable submanifold in an \(n\)-dimensional Riemannian manifold \(N\) of sectional curvature \(K_N \leq \delta\). If \(\delta \leq 0\), we assume that \(N\) is simply-connected; if \(\delta > 0\), we assume that \(M\) is contained in a convex ball of radius \(\leq \pi/4\sqrt{\delta}\); if \(\delta < 0\), we assume that \(M\) is contained in a convex ball...*
of radius $\leq \frac{1}{\sqrt{n-\delta}} \arcsinh(\sqrt{\delta})$. Let $T$ be a symmetric, positive definite and divergence-free $(1,1)$-tensor on $M$.

Then the first non-zero eigenvalue $\lambda_{1,p}$ of the $p$-Laplacian ($p > 1$) on $M$ satisfies:

(1) For $\delta = 0$,
$$
\lambda_{1,p} \left( \int_M \text{tr} \, T \right)^p \leq n \frac{[p-2]}{2} m^{p/2} \text{vol}(M) \left( \int_M |H_T|^{p} \right)^{p-1}, \quad \text{for } p > 1.
$$

In particular, when $T = I$ the identity operator, we have
$$
\lambda_{1,p} \leq n \frac{[p-2]}{2} \left( \frac{m}{\text{vol}(M)} \right)^{p-1} \left( \int_M |H|^{p} \right)^{p-1}, \quad \text{for } p > 1.
$$

(2) For $\delta > 0$,
$$
\lambda_{1,p} \leq (n+1) \frac{[p-2]}{2} m^{p/2} \delta^{p/2} \left( \frac{1}{\text{vol}(M)} \int_M (1 + |H/\sqrt{\delta}|^2) \right)^{\max\{1,p/2\}}.
$$

(3) For $\delta < 0$,
$$
\lambda_{1,p} \leq n \frac{[p-2]}{2} m (\delta + \max |H|^2) \quad \text{for } 1 < p \leq 2.
$$

The estimate is not good enough in the case of $\delta < 0$ for at least two reasons: (i) there is a restriction on the radius of the convex ball for $1 < p < 2$, but it is not necessary for $p = 2$ (cf. Heintze’s upper bound (1.3)); (ii) the estimate is just valid for $1 < p < 2$ and no information is given for $p > 2$.

Inspired by Niu-Xu [20], we extend (1.5) to $p$-Laplacian and give a new upper bound of $\lambda_{1,p}$ without previous restrictions for $\delta < 0$. Precisely, we prove the following theorem.

**Theorem 1.2.** Let $M$ be an $m(\geq 2)$-dimensional closed orientable submanifold in an $n$-dimensional complete simply-connected Riemannian manifold $N$ of sectional curvature $K_N \leq \delta < 0$. Then the first non-zero eigenvalue $\lambda_{1,p}$ of $p$-Laplacian on $M$ satisfies

$$
\lambda_{1,p} \leq \begin{cases} 
  n \frac{[p-2]}{2} \left( \frac{m}{\text{vol}(M)} \right)^{p/2} \left( \int_M |H|^2 \right)^{p/2-1} \int_M (\delta + |H|^2), & \text{for } p \geq 2; \\
  n \frac{[p-2]}{2} \frac{m^{p/2} \delta^{p/2-1}}{\text{vol}(M)} \int_M \left( \frac{p}{2} \delta + |H|^2 \right), & \text{for } 1 < p \leq 2.
\end{cases}
$$

Moreover, equality implies $p = 2$ and $M$ is minimally immersed into a geodesic sphere of radius $\arcsinh_{\delta} \sqrt{\frac{m}{n-1}}$, where $\arcsinh_{\delta}$ is the inverse function of the $\delta$-hyperbolic sine $\sinh_{\delta}(y) = \frac{1}{\sqrt{\delta}} \sinh(\sqrt{\delta}y)$.

More generally, we can obtain the upper bound involving $H_T$.

**Theorem 1.3.** Assumptions as in Theorem 1.2. Let $T$ be a symmetric, positive definite and divergence-free $(1,1)$-tensor on $M$. Suppose the one of the followings holds:

(a) $(\text{tr} \, T) I - 2T \geq 0$, i.e., positive semidefinite;

(b) $M$ is contained in a convex ball of radius $\leq \frac{1}{\sqrt{\delta}} \text{arcosh} \sqrt{2}$.

Then the first non-zero eigenvalue $\lambda_{1,p}$ of $p$-Laplacian on $M$ satisfies

$$
\lambda_{1,p} \leq \begin{cases} 
  n \frac{[p-2]}{2} m^{p/2} Q(T)^{p/2-1} (\delta + Q(T)), & \text{for } p \geq 2; \\
  n \frac{[p-2]}{2} m^{p/2} (-\delta)^{p/2-1} \left( \frac{p}{2} \delta + Q(T) \right), & \text{for } 1 < p \leq 2,
\end{cases}
$$

where $Q(T) = \text{tr} \, T - \text{tr} \, \text{tr} \, T$.
where
\[ Q(T) = \frac{\text{vol}(M) \sup_M \text{tr} T}{(\int_M \text{tr} T)^2} \int_M |H_T|^2. \]
Moreover, equality implies \( p = 2 \), \( \text{tr} T \) is constant, and \( M \) is \( T \)-minimally immersed into a geodesic sphere of radius \( \text{arsinh} \frac{\sqrt{m}}{\lambda} \).

By taking \( p = 2 \), we immediately obtain the following

**Corollary 1.4.** Assumptions as in Theorem 1.3. We have
\[ \lambda_1^A \leq m \left( \delta + \frac{\text{vol}(M) \sup_M \text{tr} T}{(\int_M \text{tr} T)^2} \int_M |H_T|^2 \right). \] (1.11)

**Remark 1.5.**
(1) The condition \((\text{tr} T)I - 2T \geq 0\) is also required in [5, Theorem 1.1].

(2) When \( m \geq 2 \), \((\text{tr} T)I - 2T \geq 0\) always holds for \( T = I \) the identity operator, and then Theorem 1.3 recovers Theorem 1.2; furthermore, this recovers [20, Theorem 1.1] when \( p = 2 \). However, when \( m = 1 \), even for the Laplacian operator, \((\text{tr} T)I - 2T \geq 0\) fails, while an analogous estimate involving the curvature of the curve also fails unless “smallness” assumed (see [20] for details).

1.2. \( L_T \) operator. Consider a symmetric \((1,1)\)-tensor \( T = (T_{ij}) \) on \( M \) as previously mentioned. If we additionally assume that \( T \) is positive definite and divergence-free, then we can define a self-adjoint elliptic operator \( L_T \) associated to \( T \) by
\[ L_T u = \text{div}(T \nabla u). \]
The eigenvalue equation is given by \( L_T u = -\lambda u \). It is also a generalization of the Laplacian since \( L_T = \Delta \) when \( T = I \).

The \( L_T \) operator has been intensively studied. Regarding the Reilly-type inequality, Grosjean proved that

**Theorem 1.6** ([14, Theorem 1]). Let \( M \) be a \( m(\geq 2) \)-dimensional closed orientable submanifold in an \( n \)-dimensional Riemannian manifold \( N \) of sectional curvature \( K_N \leq \delta \). If \( \delta \leq 0 \), we assume that \( N \) is simply-connected; if \( \delta > 0 \), we assume that \( M \) is contained in a convex ball of radius \( \leq \pi/\sqrt{\delta} \). Let \( T \) be a symmetric, positive definite and divergence-free \((1,1)\)-tensor on \( M \). Then the first non-zero eigenvalue \( \lambda_{1,T} \) of \( L_T \) on \( M \) satisfies
\[ \lambda_{1,T} \leq \frac{\sup_M |H_T|^2 + \sup_M \delta(\text{tr} T)^2}{\inf_M \text{tr} T}. \] (1.12)

If the equality holds, then \( M \) is contained in a geodesic sphere.

When \( N \) is the space form with constant curvature \( \delta \), the first author and Wang [5] showed that the upper bound in (1.12) can be improved by the \( L_2 \)-norm of \( H_T \), precisely,
\[ \lambda_{1,T} \leq \frac{1}{\text{vol}(M)} \int_M \left( \delta \text{tr} T + \frac{|H_T|^2}{\text{tr} T} \right). \] (1.13)
We remark that in this case, there is no restriction on the radius of \( M \), but it is required that \((\text{tr} T)I - 2T \geq 0 \), i.e., positive semidefinite. The sufficient and necessary conditions for the equality holding are also given.

Motivated by Niu-Xu [20] again, we can also improve the estimate (1.12) for \( \delta < 0 \). Precisely, we prove the following theorem.
Theorem 1.7. Let $M$ be an $m(\geq 2)$-dimensional closed orientable submanifold in an $n$-dimensional complete simply-connected Riemannian manifold $N$ of sectional curvature $K_N \leq \delta < 0$. Let $T$ be a symmetric, positive definite and divergence-free $(1,1)$-tensor on $M$. Suppose the one of the followings holds:

(a) $(\text{tr} T) I - 2T \geq 0$, i.e., positive semidefinite;

(b) $M$ is contained in a convex ball of radius $\frac{1}{2\sqrt{-\delta}} \arcsinh \sqrt{2}$.

Then we have

$$\lambda_{1,T} \leq \sup_M \text{tr} T \int_M \left( \delta \text{tr} T + \frac{|H_T|^2}{\text{tr} T} \right).$$

Moreover, equality implies $\text{tr} T$ is constant and $M$ is $T$-minimally immersed into a geodesic sphere of radius $\arcsinh \delta \sqrt{\frac{\text{tr} T}{\lambda_{1,T}}}$.

There are other Reilly-type inequalities in various situations, for example, when the ambient Riemannian manifolds $(N, g_N)$ is endowed with a density $e^{-f}$ for certain $f \in C^\infty(N)$, one can consider the weighted $L_T$ operator, denoted by $L_{T,f}$; when the submanifold has a non-empty boundary, one can consider the first (non-zero) eigenvalues of the Steklov-type problems (see [7, 9, 16, 18, 22, 23, 25]). We also obtain some new estimates and list them in the last section.

This paper is organized as follows. In Sect. 2, we list some notations and show some lemmas and propositions, some of which are extensions from the mean curvature to $T$. In Sect. 3 and Sect. 4, we give the proofs for the $p$-Laplacian and $L_T$, respectively. To estimate the upper bound of the first (non-zero) eigenvalues, a starting point is the Rayleigh-type variational characterization (cf. Eqs. (1.6) and (4.3)). One of key ingredients is to find suitable test functions (Sect. 2.2), and the other one is an estimate of $L^2$ lower bound of $H_T$, which extends Niu-Xu’s estimate for the mean curvature (Sect. 2.3). In Sect. 5, we give some Reilly-type estimates when the ambient manifold is a weighted manifold.

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2. Preliminaries

In this section, we introduce some notations and review some lemmas.

2.1. Some estimates for radial vector fields. Let $M$ be an $m$-dimensional submanifold of an $n$-dimensional Riemannian manifold $N$ whose sectional curvature $K_N \leq \delta < 0$. We denote the Levi-Civita connection on $M$ and $N$ by $\nabla$ and $\nabla^N$ respectively. For any $q \in N$, we consider its normal coordinates $(x_1, \ldots, x_n)$ on $N$ centered at some fixed point $q_0 \in N$.

Let $\text{sh}_\delta(y) = \frac{1}{\sqrt{-\delta}} \sinh(\sqrt{-\delta} y)$, $\text{ch}_\delta(y) = \text{sh}_\delta'(y)$ and $\text{th}_\delta(y) = \text{sh}_\delta(y)/\text{ch}_\delta(y)$. It is easy to check that $\text{ch}_\delta^2 + \delta \text{sh}_\delta^2 = 1$ and $1/\text{ch}_\delta^2 = 1 + \delta \text{th}_\delta^2$.

Consider the vector field $Z = \phi(r) \nabla^N r$, where $r = r(q)$ is the distance function from $q_0$ to $q$ and $\phi(r)$ be a positive function with $\phi(r) \sim r$ as $r \to 0$. Then the coordinates of $Z$ in the normal local frame are $(\frac{\phi(r)}{r} x_i)_{1 \leq i \leq n}$. By using basic properties of the exponential map and the comparison for Jacobi fields, we can prove the following lemma.
Lemma 2.1 (cf. [14, 20]). Let $T$ be a symmetric, positive definite and divergence-free $(1,1)$-tensor on $M$, and $Z^\top$ be the tangential projection of $Z = \phi(r)\nabla^N r$ to $M$. Then we have

$$(1) \sum_{i=1}^n \langle T\nabla \left(\frac{\phi(r)}{r} x_i\right), \nabla \left(\frac{\phi(r)}{r} x_i\right) \rangle \leq (\text{tr } T) \frac{\phi^2}{\text{sh}_\delta^2} + \left((\phi')^2 - \frac{\phi^2}{r^2}\right) \langle T\nabla r, \nabla r \rangle,$$

and equality holds if $N$ has constant sectional curvature $\delta$.

$$(2) \text{div}_M(TZ^\top) \geq (\text{tr } T) \frac{\phi'}{\text{th}_\delta} + \left((\phi')^2 - \frac{\phi^2}{r^2}\right) \langle T\nabla r, \nabla r \rangle + \langle Z, H_T \rangle.$$

Proof. When $\phi = \text{sh}_\delta$, Items (1) and (2) go back to Lemma 1 and Lemma 2 in [14] respectively, and the proof of this lemma follows from the same steps with similar calculations in [14]. Here we give a quick proof by directly using the results in [14].

Since

$$\nabla \left(\frac{\phi(r)}{r} x_i\right) = \left(\frac{\phi'}{r}\right) x_i \nabla r + \frac{\phi}{r} \nabla x_i, \quad \sum_{i=1}^n (x_i)^2 = r^2, \quad \sum_{i=1}^n x_i \nabla x_i = r \nabla r,$$

we have

$$\sum_{i=1}^n \langle T\nabla \left(\frac{\phi(r)}{r} x_i\right), \nabla \left(\frac{\phi(r)}{r} x_i\right) \rangle = \sum_{i=1}^n \frac{\phi^2}{r^2} \langle T\nabla x_i, \nabla x_i \rangle + \left((\phi')^2 - \frac{\phi^2}{r^2}\right) |\sqrt{T}\nabla r|^2$$

$$= \sum_{i=1}^n \frac{\phi^2}{r^2} \langle T\nabla x_i, \nabla x_i \rangle + \left((\phi')^2 - \frac{\phi^2}{r^2}\right) |\sqrt{T}\nabla r|^2$$

$$+ \frac{\phi}{r^2} \left[\text{sh}_\delta^2 \sum_{i=1}^n \langle T\nabla x_i, \nabla x_i \rangle + (1 - \text{sh}_\delta^2) |\sqrt{T}\nabla r|^2\right]$$

$$\leq (\phi')^2 |\sqrt{T}\nabla r|^2 - \frac{\phi^2}{\text{sh}_\delta^2} |\sqrt{T}\nabla r|^2 + \frac{\phi}{r^2} (\text{tr } T),$$

where we used [14, Eqs. (5) and (6)] in the last inequality.

$$\text{div}_M(TZ^\top) = \text{div}_M \left(\frac{\phi}{\text{sh}_\delta} T(\text{sh}_\delta \nabla r)\right)$$

$$= \frac{\phi}{\text{sh}_\delta} \text{div}_M(T(\text{sh}_\delta \nabla r)) + \left(\frac{\phi}{\text{sh}_\delta}\right)' \langle \nabla r, T(\text{sh}_\delta \nabla r) \rangle$$

$$\geq \frac{\phi}{\text{sh}_\delta} \left((\text{tr } T) \text{ch}_\delta + (\text{sh}_\delta \nabla^N r, H_T)\right) + \frac{\phi' \text{sh}_\delta - \phi \text{ch}_\delta}{\text{sh}_\delta} \langle \nabla r, T\nabla r \rangle$$

$$= (\text{tr } T) \frac{\phi}{\text{th}_\delta} + \left((\phi')^2 - \frac{\phi^2}{r^2}\right) \langle T\nabla r, \nabla r \rangle + \langle Z, H_T \rangle,$$

where we used [14, Lemma 2] in the inequality. \hfill \Box

Remark 2.2. When $T = I$, Lemma 2.1 recovers [20, Lemma 2.3].

2.2. Test functions. The next lemma will provide suitable test functions for the upper bound estimates of the first eigenvalue, and the basic idea can be found in [3, 15].

Lemma 2.3. Under the same assumptions as in Theorem 1.2 or Theorem 1.3, we have for $p > 1$, there exists a point $q_0 \in N$ such that

$$\int_M \left|\frac{\text{th}_\delta(r)}{r} x_i\right|^{p-2} \frac{\text{th}_\delta(r)}{r} x_i \text{d}v_M = 0 (i = 1, \ldots, n).$$

(2.1)
Here \((x_1, \cdots, x_n)\) is the normal coordinate around \(q_0\), and \(r(q) = d(q_0, q)\) is the geodesic distance from \(q_0\) to \(q\) on \(N\).

**Proof.** We can assume \(M\) lies in a convex ball \(B \subset N\). For each \(q \in N\), we define a vector \(Y_q \in T_qN\) under the normal coordinate system around \(q\) as follows:

\[
Y_q := \left( \int_M \frac{\theta_\delta(r)}{r} x_i \, dv_M, \cdots, \int_M \frac{\theta_\delta(r)}{r} x_n \, dv_M \right).
\]

Hence, we obtain a vector field \(Y\) in a neighborhood of \(B\) which points at the boundary into the interior of \(B\).

Now suppose \(Y(q) \neq 0\) for all \(q \in B\), which means \(\text{Ind}(Y) = 0\). Since \(Y\) is a continuous vector field on \(B\) and \(Y(q)\) points inside for all \(q \in \partial B\), we have (cf. [13, 19])

\[
\text{Ind}(Y) + \chi(\partial B) = \chi(B),
\]

so \(\text{Ind}(Y) = 1 - (1 + (-1)^{n-1}) = (-1)^n \neq 0\), a contradiction. The proof is complete. \(\Box\)

**Remark 2.4.** From the proof we see that \(q_0 \in B\). If the radius of \(B \leq \frac{1}{2\sqrt{-\delta}} \arccosh \sqrt{2}\), then \(M\) lies in a ball of radius \(\leq \frac{1}{\sqrt{-\delta}} \arccosh \sqrt{2}\) centered at \(q_0\), which implies \(\text{ch}_\delta^2(r) \leq 2\) on \(M\).

### 2.3. \(L^2\) lower bound of \(H_T\)

Recall the following key estimate used to prove (1.5).

**Proposition 2.5 ([20, Proposition 2.1]).** Under the same assumptions as in Theorem 1.2, we have

\[
\int_M |H|^2 \int_M \theta_\delta^2(r) \geq (\text{vol}(M))^2. \tag{2.2}
\]

Moreover, equality in (2.2) implies that \(M\) is minimally immersed in a geodesic sphere.

We can extend Proposition 2.5 to \(H_T\) as follows.

**Proposition 2.6.** Under the same assumptions as in Theorem 1.3, we have

\[
\int_M \frac{|H_T|^2}{\text{tr} T} \int_M \text{tr} T \theta_\delta^2(r) \geq \left( \int_M \text{tr} T \right)^2. \tag{2.3}
\]

Moreover, equality in (2.3) implies \(M\) is \(T\)-minimally immersed into a geodesic sphere.

**Proof.** By taking \(\phi(r) = \theta_\delta(r)\) in Lemma 2.1, we have \(Z = \theta_\delta(r) \nabla^N r\) and

\[
\text{div}_M(TZ^\top) \geq \text{tr} T + \delta \sqrt{|TZ^\top|^2 + |Z^\perp, H_T|} \tag{2.4}
\]

Integrating (2.4) over \(M\) and using the divergence theorem, the Cauchy-Schwarz inequality and the Hölder inequality, we have

\[
\|\sqrt{\text{tr} T} Z^\perp\|_2 \left\| \frac{H_T}{\sqrt{\text{tr} T}} \right\|_2 \geq \int_M -\langle Z^\perp, H_T \rangle \geq \int_M \text{tr} T + \delta \|\sqrt{\text{tr} T} Z^\top\|_2^2, \tag{2.5}
\]

where \(\|Y\|_2 := (\int_M |Y|^2)^{1/2}\) for some vector field \(Y\), and \(^\perp\) represents the normal part of the vector field.

We discuss each case.

**Case (a):** \((\text{tr} T)I - 2T \succeq 0\). Let \(\Lambda_T\) be the largest eigenvalue of \(T\), then \(\Lambda_T \leq \frac{1}{2} \text{tr} T\), which implies \(|\sqrt{\text{tr} T} Z^\top|^2 \leq \Lambda_T |Z^\top|^2 \leq \frac{1}{2} \text{tr} T |Z^\top|^2\), and it follows from (2.5) that

\[
\|\sqrt{\text{tr} T} Z^\top\|_2 \left\| \frac{H_T}{\sqrt{\text{tr} T}} \right\|_2 \geq \int_M \text{tr} T + \frac{\delta}{2} \|\sqrt{\text{tr} T} Z^\top\|_2^2. \tag{2.6}
\]
Denote $X = \sqrt{\text{tr}TZ}$ and note that $\|X^T\|_2^2 = \|X\|_2^2 - \|X^\perp\|_2^2$, then (2.6) becomes
\[
\left(\left\|\frac{H_T}{\sqrt{\text{tr}T}}\right\|_2 + \frac{\delta}{2}\right\|X^\perp\|_2\right)\|X^\perp\|_2 \geq \int_M \text{tr}T + \frac{\delta}{2}\|X\|_2^2 .
\] (2.7)

Now we consider the quadratic function $\phi_h(x) = (h + \frac{\delta}{2}x)x$ with $h = \left\|\frac{H_T}{\sqrt{\text{tr}T}}\right\|_2$. Note that (2.3) is equivalent to
\[
\phi_h(\|X\|_2) = \left(h + \frac{\delta}{2}\|X\|_2\right)\|X\|_2 \geq \int_M \text{tr}T + \frac{\delta}{2}\|X\|_2^2 ,
\] hence, to prove the lemma, it is sufficient to show that
\[
\phi_h(\|X\|_2) \geq \phi_h(\|X^\perp\|_2) .
\] (2.8)

Next we verify (2.8). Firstly, we have
\[
2\|X\|_2^2 = 2\int_M \text{tr}T \text{th}_\delta^2 = \frac{1}{-\delta}\int_M 2\text{tr}T \left(1 - \frac{1}{\text{ch}_\delta^2}\right) \leq \frac{2}{-\delta}\int_M \text{tr}T .
\] (2.9)

Therefore, we obtain that
\[
\left(\|X\|_2 + \|X^\perp\|_2\right)\|X^\perp\|_2
= \|X\|_2^2 + \|X\|_2^2 - \|X^\perp\|_2^2 - \|X^\perp\|_2^2 \leq 2\|X\|_2^2 - \|X^\perp\|_2^2
\leq \frac{2}{-\delta}\int_M \text{tr}T - \|X^\perp\|_2^2 = \frac{2}{-\delta}\left(\int_M \text{tr}T + \frac{\delta}{2}\|X^\perp\|_2^2\right)
\leq \frac{2h}{-\delta}\|X^\perp\|_2
\] (2.6)

Since $\|X^\perp\|_2 \neq 0$, we conclude that $2\|X^\perp\|_2 \leq \|X\|_2 + \|X^\perp\|_2 \leq \frac{2h}{-\delta}$. Hence, (2.8) holds by using the monotonicity of $\phi_h = (h + \frac{\delta}{2}x)x$.

**Case (b):** $M \subset B(R), R \leq \frac{1}{2\sqrt{-\delta}} \text{arcosh} \sqrt{2}$. We still denote $X = \sqrt{\text{tr}TZ}$, then (2.5) becomes
\[
\left(\left\|\frac{H_T}{\sqrt{\text{tr}T}}\right\|_2 + \delta\|X^\perp\|_2\right)\|X^\perp\|_2 \geq \int_M \text{tr}T + \delta\|X\|_2^2 .
\] (2.10)

Next we can use the analogous argument as in Case (a) for the quadratic function $\psi_h(x) = (h + \delta x)x$ with $h = \left\|\frac{H_T}{\sqrt{\text{tr}T}}\right\|_2$.

By Remark 2.4 we have $\text{ch}_\delta^2 \leq 2$ on $M$, then
\[
2\|X\|_2^2 = 2\int_M \text{tr}T \text{th}_\delta^2 = \frac{1}{-\delta}\int_M 2\text{tr}T \left(1 - \frac{1}{\text{ch}_\delta^2}\right) \leq \frac{1}{-\delta}\int_M \text{tr}T .
\] (2.11)

Consequently, $2\|X^\perp\|_2 \leq \|X\|_2 + \|X^\perp\|_2 \leq \frac{h}{-\delta}$ follows from
\[
(\|X\|_2 + \|X^\perp\|_2)\|X^\perp\|_2 \leq 2\|X\|_2^2 - \|X^\perp\|_2^2 \leq \frac{1}{-\delta}\left(\int_M \text{tr}T + \delta\|X^\perp\|_2^2\right) \leq \frac{h}{-\delta}\|X^\perp\|_2
\] (2.5)

Now we derive that
\[
\left(\left\|\frac{H_T}{\sqrt{\text{tr}T}}\right\|_2 + \delta\|X\|_2\right)\|X\|_2 = \psi_h(\|X\|_2) \geq \psi_h(\|X^\perp\|_2) \geq \int_M \text{tr}T + \delta\|X\|_2^2 ,
\] which is equivalent to (2.3).

Finally, we check the necessity for equality in (2.3). In both cases (a) and (b), we must have $\|X^\perp\|_2 = \|X\|_2$, which implies $X \equiv X^\perp$ and $X^\top \equiv 0$ along $M$. Further, the
first equality in (2.5) gives \( X = \sqrt{\text{tr} T Z} \) and \( \frac{H_T}{\sqrt{\text{tr} T}} \) are parallel, and \( |H_T| = k \text{tr} T |Z| \) for certain positive constant \( k \). On the other hand, equality in (2.4) implies \( \text{tr} T = |H_T||Z| = k \text{tr} T |Z|^2 \). Hence, \( \theta_\delta(r) = |Z| \) is constant on \( M \), from which we conclude that \( M \) lies in a geodesic sphere. Since \( H_T \) is parallel to \( Z \) along \( M \), we obtain that \( M \) is \( T \)-minimal in this geodesic sphere. We complete the proof. \( \square \)

**Remark 2.7.** Assume that equality in (2.3) holds. If we additionally assume that \( \text{tr} T \) is constant, then \( |H_T| = k \text{tr} T \theta_\delta(r) \) is also constant. Hence, from (2.3) we obtain \( k = 1/\theta_\delta^2(r) \) and \( |H_T| = \text{tr} T/\theta_\delta(r) \).

3. Proofs for \( p \)-Laplacian with \( \delta < 0 \)

In this section, we prove Theorems 1.2 and 1.3. Firstly, we give the following estimate for \( \lambda_{1,p} \).

**Lemma 3.1.** Let \( M \) be an \( m(\geq 2) \)-dimensional closed orientable submanifold in an \( n \)-dimensional Riemannian manifold \( N \). Then we have, for all \( p > 1 \),

\[
\lambda_{1,p} \int_M \theta_\delta^p(r) \leq n \frac{(n-1)}{2} m^p \int_M \left( 1 + \delta \theta_\delta^2(r) \right)^{p/2}.
\]  

(3.1)

**Proof.** We choose \( \frac{\theta_\delta(r)}{r} x_i \) in Lemma 2.3 as the test functions. Recall the Rayleigh-type quotient (1.6), then we obtain

\[
\lambda_{1,p} \int_M \left| \frac{\theta_\delta(r)}{r} x_i \right|^p \leq \int_M \left| \nabla \left( \frac{\theta_\delta(r)}{r} x_i \right) \right|^p \quad \text{for each } i.
\]  

(3.2)

When \( 1 < p \leq 2 \),

\[
\theta_\delta^p(r) = (\theta_\delta^2(r))^{p/2} = \left( \sum_{i=1}^n \left| \frac{\theta_\delta(r)}{r} x_i \right|^2 \right)^{p/2} \leq \sum_{i=1}^n \left| \frac{\theta_\delta(r)}{r} x_i \right|^p.
\]  

(3.3)

By taking \( T = I, f(r) = \theta_\delta(r) \) in Item (1) of Lemma 2.1 and using the H"older inequality, we derive that

\[
n^{1-2/p} \left( \sum_{i=1}^n \left| \nabla \left( \frac{\theta_\delta(r)}{r} x_i \right) \right|^p \right)^{2/p} \leq \sum_{i=1}^n \left| \nabla \left( \frac{\theta_\delta(r)}{r} x_i \right) \right|^2
\]

\[
\leq m \left( \frac{1}{\chi_\delta^2} + \delta \frac{\theta_\delta^2}{\chi_\delta^2} \right) \left| \nabla r \right|^2
\]

\[
\leq m \left( 1 + \delta \theta_\delta^2(r) \right),
\]  

(3.4)

where we used \( \delta < 0 \) and \( 1 = \chi_\delta^2 + \delta \chi_\delta^2 \). Then (3.1) is from (3.2), (3.3) and (3.4).

When \( p > 2 \), the H"older inequality gives

\[
\theta_\delta^2(r) = \sum_{i=1}^n \left| \frac{\theta_\delta(r)}{r} x_i \right|^2 \leq n^{1-2/p} \left( \sum_{i=1}^n \left| \frac{\theta_\delta(r)}{r} x_i \right|^p \right)^{2/p},
\]

so we have

\[
\lambda_{1,p} \int_M \theta_\delta^p(r) \leq n^{p/2-1} \left( \sum_{i=1}^n \lambda_{1,p} \int_M \left| \frac{\theta_\delta(r)}{r} x_i \right|^p \right).
\]  

(3.5)
On the other hand, we have
\[
\sum_{i=1}^{n} \left| \nabla \left( \frac{\theta(r)}{r} x_{i} \right) \right|^p \leq \left( \sum_{i=1}^{n} \left| \nabla \left( \frac{\theta(r)}{r} x_{i} \right) \right|^2 \right)^{p/2} \leq m^2 \left( 1 + \delta \theta^2(r) \right)^{p/2}.
\] (3.6)

Hence, (3.1) follows from (3.2), (3.5) and (3.6).

Now we are ready to prove our theorems. Since Theorem 1.2 is a special case of Theorem 1.3, we only prove the latter.

\textbf{Proof of Theorem 1.3.} We have \(\delta \theta^2(r) = -\theta^2(\sqrt{-\delta r}) \in (-1, 0]\) since \(\delta < 0\). From (2.3) we have
\[
\sup M \tr T \int M |H_r|^2 \int M |H_T|^2 \geq \int M \tr T \theta^2 \int M |H_T|^2 \geq \left( \int M \tr T \right)^2.
\] (3.7)

\textbf{Case 1.} When \(p \geq 2\), we have
\[
\lambda_{1,p} \int M \theta^p(r) \leq n \left( \frac{|p-2|}{2} m^2 \right) \int M \left( 1 + \delta \theta^2(r) \right)^{p/2}
\leq n \left( \frac{|p-2|}{2} m^2 \right) \int M \left( 1 + \delta \theta^2(r) \right)
\]
(3.8)
\[= n \left( \frac{|p-2|}{2} m^2 \right) \left( \operatorname{vol}(M) + \delta \int M \theta^2(r) \right)
\leq n \left( \frac{|p-2|}{2} m^2 \right) \left( \operatorname{vol}(M) \sup M \tr T \left( \int M |H_T|^2 \right)^2 \int M |H_T|^2 + \delta \int M \theta^2(r) \right),
\] (3.9)
where we used (3.7) in the last inequality. By the H"older inequality, we have
\[
\int M \theta^2(r) \leq \left( \int M \theta^p(r) \right)^{2/p} \left( \operatorname{vol}(M) \right)^{1-2/p},
\]
which is equivalent to
\[
(\operatorname{vol}(M))^{1-p/2} \left( \int M \theta^2(r) \right)^{p/2} \leq \int M \theta^p(r). \] (3.10)

Combining (3.10) with (3.9), we have
\[
\lambda_{1,p} \leq n \left( \frac{|p-2|}{2} m^2 \right) \left( \operatorname{vol}(M) \sup M \tr T \left( \int M |H_T|^2 \right)^2 \int M |H_T|^2 + \delta \int M \theta^2(r) \right)^{p/2-1}
\leq n \left( \frac{|p-2|}{2} m^p/2 \right) \left( \operatorname{vol}(M) \sup M \tr T \left( \int M |H_T|^2 \right)^2 \int M |H_T|^2 + \delta \int M \theta^2(r) \right)^{p/2-1},
\]
where we used (3.7) again. This is just (1.9).
Case 2. When $1 < p \leq 2$, we have
\[
\lambda_{1,p} \int_M \theta^p_\delta(r) \leq n \frac{|p-2|}{2} m^p \int_M \left( 1 + \delta \theta^2_\delta(r) \right)^{p/2}
\leq n \frac{|p-2|}{2} m^p \int_M \left( 1 + \frac{p}{2} \delta \theta^2_\delta(r) \right)
= n \frac{|p-2|}{2} m^p \left( \text{vol}(M) + \frac{p}{2} \delta \int_M \theta^2_\delta(r) \right)
\leq n \frac{|p-2|}{2} m^p \left( \frac{\text{vol}(M) \sup_M \tr T}{(\int_M \tr T)^2} \int_M \frac{|H_T|^2}{\tr T} + \frac{p}{2} \delta \right) \int_M \theta^2_\delta(r),
\] (3.11)
where we used (3.7) in the last inequality. Notice that
\[
\theta^p_\delta(r) = \theta^p(\sqrt{-\delta} r)/(-\delta)^{p/2}
\]
and
\[
\theta^2_\delta(r) = \theta^2(\sqrt{-\delta} r)/(-\delta) \leq \theta^p(\sqrt{-\delta} r)/(-\delta),
\]
so
\[
\int_M \theta^2_\delta(r) \leq (-\delta)^{p-1} \int_M \theta^p_\delta(r).
\]
Inserting this into (3.12) we derive (1.10).

Finally, we check what happens when equality holds in (1.9) or (1.10). Equality in either (3.8) or (3.11) implies $p = 2$; the first equality in (3.7) implies that $\tr T$ is constant, while the second equality in (3.7) implies that $M$ is $T$-minimally immersed into a geodesic sphere (cf. Proposition 2.6), and the radius can be determined by Remark 2.7. Hence, we complete the whole proof. \hfill \Box

4. Proof for $L_T$ Operators with $\delta < 0$

In this section, we prove Theorem 1.7. Actually, we will prove the following stronger result.

Theorem 4.1. Let $M$ be an $m(\geq 2)$-dimensional closed orientable submanifold in an $n$-dimensional complete simply-connected Riemannian manifold $N$ of sectional curvature $K_N \leq \delta < 0$. Let $T, S$ be a symmetric, positive definite and divergence-free $(1,1)$-tensor on $M$. Suppose the one of the followings holds:

(a) $(\tr S)I - 2S \geq 0$, i.e., positive semidefinite;

(b) $M$ is contained in a convex ball of radius $\leq \frac{1}{2\sqrt{-\delta}} \arccosh \sqrt{2}$.

Then we have the following two estimates for the upper bound of the first non-zero eigenvalue $\lambda_{1,T}$ of $L_T$:

\[
\lambda_{1,T} \leq \sup_M \left( \frac{\tr T}{\tr S} \right) \sup_M (\tr S) \left( \delta + \frac{1}{\int_M \tr S} \int_M \frac{|H_S|^2}{\tr S} \right),
\] (4.1)

and

\[
\lambda_{1,T} \leq \sup_M (\tr T) \left( \delta + \frac{\text{vol}(M) \sup_M \tr S}{(\int_M \tr S)^2} \int_M \frac{|H_S|^2}{\tr S} \right).
\] (4.2)

Moreover, equality in either (4.1) or (4.2) implies that both $\tr S$ and $\tr T$ are constant, and that $M$ is $S$-minimally immersed into a geodesic sphere of radius $\arcsinh \sqrt{\frac{\tr T}{\lambda_{1,T}}}$. 
Proof. We also have the Rayleigh-type quotient (cf. [14])

\[
\lambda_{1,T} = \inf \left\{ -\frac{\int_M uLTu}{\int_M u^2} \mid u \in W^{1,2}(M) \setminus \{0\}, \int_M u = 0 \right\}.
\] (4.3)

By slightly modifying Lemma 2.3, we can prove that there exists a point \(q_0 \in N\) such that

\[
\int_M \frac{th_\delta(r)}{r} x_i dv_M = 0 \quad (i = 1, \cdots, n),
\] (4.4)

where \((x_1, \cdots, x_n)\) is the normal coordinate around \(q_0\), and \(r(q) = d(q_0, q)\) is the geodesic distance from \(q_0\) to \(q\) on \(N\). Here we omit the detail.

Now by using Lemma 2.1, we have

\[
\lambda_{1,T} \int_M \frac{th_\delta^2(r)}{r} \leq -\int_M \left( \frac{th_\delta(r)}{r} x_i \right) L_T \left( \frac{th_\delta(r)}{r} x_i \right) = \sum_{i=1}^n \int_M \left\langle T \nabla \left( \frac{th_\delta(r)}{r} x_i \right), \nabla \left( \frac{th_\delta(r)}{r} x_i \right) \right\rangle \leq \int_M (tr T) \left( 1 + \delta th_\delta^2(r) \right).
\] (4.5)

In order to introduce \(tr S\) and \(H_S\) and cancel the term of \(\int_M th_\delta^2(r)\), we have two methods.

Method 1.

\[
\int_M (tr T) \left( 1 + \delta th_\delta^2(r) \right) \leq \sup_M \left( \frac{tr T}{tr S} \right) \int_M (tr S) \left( 1 + \delta th_\delta^2(r) \right) = \sup_M \left( \frac{tr T}{tr S} \right) \left( \frac{\int_M tr S}{\int_M tr S \cdot th_\delta^2(r)} + \delta \right) \int_M tr S \cdot th_\delta^2(r) \leq \sup_M \left( \frac{tr T}{tr S} \right) \sup_M (tr S) \left( \frac{1}{\int_M tr S} \int_M \frac{|H_S|^2}{tr S} + \delta \right) \int_M \cdot th_\delta^2(r),
\] (4.6)

where we used Proposition 2.6 for the tensor \(S\) in the penultimate inequality. Putting (4.6) into (4.5), we obtain (4.1).

Method 2.

\[
\int_M (tr T) \left( 1 + \delta th_\delta^2(r) \right) \leq \sup_M (tr T) \int_M \left( 1 + \delta th_\delta^2(r) \right) = \sup_M (tr T) \left( \text{vol}(M) \times 1 + \delta \int_M th_\delta^2(r) \right) \leq \sup_M (tr T) \left( \text{vol}(M) \frac{\int_M tr S \cdot th_\delta^2(r)}{(\int_M tr S)^2} \int_M \frac{|H_S|^2}{tr S} + \delta \right) \int_M \cdot th_\delta^2(r) \leq \sup_M (tr T) \left( \frac{\text{vol}(M) \sup_M tr S}{(\int_M tr S)^2} \int_M \frac{|H_S|^2}{tr S} + \delta \right) \int_M \cdot th_\delta^2(r),
\] (4.7)

where we used Proposition 2.6 for the tensor \(S\) in the penultimate inequality. Putting (4.7) into (4.5), we obtain (4.2).
Finally, the necessity for equality in either (4.1) or (4.2) follows from (4.6) or (4.7) combining with Proposition 2.6 and Remark 2.7.

\[\square\]

**Remark 4.2.** We briefly discuss the upper bounds in Eqs. (4.1) and (4.2).

1. When \( S = T \), (4.1) is better than (4.2), and Theorem 4.1 reduces to Theorem 1.7.
2. When \( \text{tr} S \) is constant, (4.1) is the same as (4.2).
3. Generally, \( \sup_M \left( \frac{\text{tr} T}{\text{tr} S} \right) \sup_M (\text{tr} S) \geq \sup_M \left( \frac{\text{vol}(M) \sup_M (\text{tr} S)}{f_M \text{tr} S} \right) \). For example, when \( \text{tr} T \) is constant, we have \( \sup_M \left( \frac{\text{tr} T}{\text{tr} S} \right) \geq \frac{\text{vol}(M) \sup_M (\text{tr} S)}{f_M \text{tr} S} \), so we cannot compare these two upper bounds since \( \delta < 0 \).
4. When \( T = I \), (4.1) recovers (1.11).

5. **Weighted manifolds and boundary value problems**

In this section, we consider the weighted manifolds and some boundary value problems and give the corresponding Reilly-type inequalities. Since most of the discussion is similar to the previous section, we will highlight the differences and omit other details.

5.1. **Basic settings and notations.** Firstly we recall some concepts of weighted manifolds. Let \((N, \bar{g})\) be a Riemannian manifold equipped with the metric \( \bar{g} \). We denote by \( dv_{\bar{g}} \) by the usual (non-weighed) volume form on \( N \). For a smooth function \( f \) on \( N \), we can consider the weighted volume form (or weighted measure) \( \bar{\mu}_f := e^{-f} dv_{\bar{g}} \). The triplet \((N, \bar{g}, \bar{\mu}_f)\) is called a weighted manifold. It is also called a metric measure space, or a manifold with density in some literature.

Now let \( M \) be a submanifold of \((N, \bar{g}, \bar{\mu}_f)\). Then \( M \) has an induced metric \( g \) and a weighted measure \( \mu_f = e^{-f} dv_g \). If \( M \) has non-empty boundary \( \partial M \), then we denote by \( \tilde{g} \) and \( \tilde{\mu}_f = e^{-f} dv_{\tilde{g}} \) on \( \partial M \) the induced metric and the weighted measure on \( \partial M \), respectively.

Denote \( \dim M = m \), \( \dim N = n \). The corresponding \( n \)-volume \( \text{vol}_f(N) \), \( m \)-volume \( \text{vol}_f(M) \) and \((m-1)\)-volume \( \text{vol}_f(\partial M) \) (when \( \partial M \neq \emptyset \)) are respectively given by

\[
\text{vol}_f(N) = \int_{N} \bar{\mu}_f, \quad \text{vol}_f(M) = \int_{M} \mu_f, \quad \text{vol}_f(\partial M) = \int_{\partial M} \mu_f.
\]

For a symmetric \((1,1)\)-tensor \( T = (T_{ij}) \) on \( M \), we can define the weighted divergence (or \( f \)-divergence) on \( M \) by

\[
\text{div}_{f, M} = \text{div}_M - \langle \nabla f, \cdot \rangle.
\]

Then the weighted \( L_T \) operator, denoted by \( L_{T,f} \), can be defined by

\[
L_{T,f} u := \text{div}_{f, M}(T \nabla u) = \text{div}_M(T \nabla u) - \langle \nabla f, T \nabla u \rangle.
\]

One can easily obtain the Stokes’ formula for the weighted operators with respect to the weighted measure \( \mu_f \). When \( T \) is divergence-free, \( L_{T,f} \) is self-adjoint with respect to the weighted measure \( \mu_f \). In particular, the case of \( T = I \) has been widely studied, and in this situation, \( L_{T,f} \) becomes \( \Delta u - \langle \nabla f, \nabla u \rangle \), which is usually called the weighted Laplacian or \( f \)-Laplacian.
5.2. The case $\delta = 0$. Let $\sn(t) = \frac{1}{\sqrt{\nu_0}} \sin(\sqrt{\nu_0} t)$ and $\frac{1}{\sqrt{-\nu}} \sinh(\sqrt{-\nu} t)$ when $\delta > 0, \delta = 0$ and $\delta < 0$, respectively. Very recently, Manfio, Roth and Upadhyay [18] studied various extrinsic upper bounds in weighted manifolds and proved the following three results, which are restated by using the notations and settings in Sect. 5.1 as follows.$^1$

**Theorem 5.1** (cf. [18, Theorem 1.1]). Let $M$ be a closed submanifold immersed in a weighted manifold $(N, \tilde{g}, \tilde{\mu}_f)$ with sectional curvature $K_N \leq \delta \leq 0$. Let $T, S$ be symmetric, divergence-free and positive definite $(1, 1)$-tensors on $M$. Then the first positive eigenvalue of $L_{T,f}$, denoted by $\lambda_1$, satisfies

$$\lambda_1 \leq \sup_M \left[ \delta \operatorname{tr} T + \sup_M \left( \frac{|H_T - T\nabla f|}{\operatorname{tr} T} \right) |H_S - S\nabla f| \right]. \quad (5.1)$$

**Theorem 5.2** (cf. [18, Theorem 1.2]). Let $M$ be a compact submanifold with non-empty boundary $\partial M$ immersed in a weighted manifold $(N, \tilde{g}, \tilde{\mu}_f)$ with sectional curvature $K_N \leq \delta \leq 0$. Let $T, S$ be symmetric, divergence-free and positive definite $(1, 1)$-tensors on $M$ and $\partial M$, respectively, and denote by $\sigma_1$ the first eigenvalue of the generalized weighted Steklov problem

$$(f\text{-Stek}) \begin{cases} \tilde{L}_{T,f} u = 0, & \text{in } M; \\ \tilde{\partial}_u \frac{\partial u}{\partial T} = \sigma u, & \text{on } \partial M, \end{cases}$$

where $\tilde{\partial}_u = (T(\tilde{\nabla} u), \nu)$, $\tilde{\nabla}$ and $\nu$ are the Levi-Civita connection and unit normal on $\partial M$, respectively.

Assume that $M$ is contained in the geodesic ball $B(p, R)$ of radius $R$, where $p$ is the center of mass of $\partial M$ for the measure $\tilde{\mu}_f$, then

$$\sigma_1 \leq \sup_M \left[ \delta \operatorname{tr} T + \sup_M \left( \frac{|H_T - T\nabla f|}{\operatorname{tr} T} \right) |H_T - T\nabla f| \right] \times \left[ \delta + \frac{\sup_{\partial M} |H_S - S(\tilde{\nabla} f)|^2}{\inf_{\partial M} (\operatorname{tr} S)^2} \right] \frac{\operatorname{vol}(M)}{\operatorname{vol}(\partial M)} \sn^2(R). \quad (5.2)$$

**Theorem 5.3** (cf. [18, Theorem 1.3]). Let $M$ be a compact submanifold with non-empty boundary $\partial M$ immersed in a Riemannian manifold $(N, \tilde{g})$ with sectional curvature $K_N \leq \delta \leq 0$. Let $S$ be a symmetric, divergence-free and positive definite $(1, 1)$-tensors on $\partial M$, and denote by $\alpha_1$ the first non-zero eigenvalue of the Steklov-Wentzell problem

$$(SW) \begin{cases} \Delta u = 0, & \text{in } M; \\ -b \tilde{\Delta} u - \tilde{\partial}_u \frac{\partial u}{\partial T} = \alpha u, & \text{on } \partial M, \end{cases}$$

where $b$ is a given positive constant and $\tilde{\Delta}$ is the Laplacian on $\partial M$.

Assume that $M$ is contained in the geodesic ball $B(p, R)$ of radius $R$, where $p$ is the center of mass of $\partial M$, then

$$\alpha_1 \leq \left[ m \frac{\operatorname{vol}(M)}{\operatorname{vol}(\partial M)} + b(m - 1) - \delta \sn^2(R) \left( \frac{\operatorname{vol}(M)}{\operatorname{vol}(\partial M)} + b \right) \right] \times \left( \delta + \frac{\sup_{\partial M} |S|^2}{\inf_{\partial M} (\operatorname{tr} S)^2} \right). \quad (5.3)$$

When $\delta = 0$, we can obtain upper bounds in terms of the integrals. Precisely, we have

$^1$In fact, the estimates in the case $K_N \leq \delta$ with $\delta > 0$ are also obtained, but here we focus on the case $K_N \leq 0$ and assume that $N$ is simply-connected.
Theorem 5.4. Settings and notations as in Theorem 5.1 with $\delta = 0$. Then
\[
\lambda_1 \left( \int_M \text{tr} S \mu_f \right)^2 \leq \int_M \text{tr} T \mu_f \int_M |H_S - S \nabla f|^2 \mu_f.
\] (5.4)

Theorem 5.5. Settings and notations as in Theorem 5.2 with $\delta = 0$. Then
\[
\sigma_1 \left( \int_{\partial M} \text{tr} \tilde{S} \tilde{\mu}_f \right)^2 \leq \int_M \text{tr} T \mu_f \int_{\partial M} |H_S - S \nabla f|^2 \tilde{\mu}_f.
\] (5.5)

Theorem 5.6. Settings and notations as in Theorem 5.3 with $\delta = 0$. Then
\[
\alpha_1 \left( \int_{\partial M} \text{tr} S \right)^2 \leq \left( m \text{vol}(M) + b(m - 1) \text{vol}(\partial M) \right) \int_{\partial M} |H_S|^2.
\] (5.6)

Proofs of Theorems 5.4, 5.5 and 5.6. We have the variational characterizations (cf. [1,8,16,18,23]):
\[
\lambda_1 = \inf \left\{ \int_M (T \nabla u, \nabla u) \mu_f \left| \int_M u \mu_f = 0 \right. \right\},
\] (5.7)
\[
\sigma_1 = \inf \left\{ \int_M (T \nabla u, \nabla u) \tilde{\mu}_f \left| \int_{\partial M} u \tilde{\mu}_f = 0 \right. \right\},
\] (5.8)
\[
\alpha_1 = \inf \left\{ \int_M |\nabla u|^2 + b \int_{\partial M} |\tilde{\nabla} u|^2 \left| \int_{\partial M} u = 0 \right. \right\}.
\] (5.9)

When $\delta = 0$, consider $Z = r \nabla N r$, we have (cf. [18, Lemma 2.1])
\[
\text{div}_{f,M} (T Z^T) \geq \text{tr} T + \langle Z, H_T - T \nabla f \rangle,
\] (5.10)
which implies
\[
\int_M \text{tr} T \mu_f \leq -\int_M \langle Z, H_T - T \nabla f \rangle \mu_f \leq \int_M |H_T - T \nabla f| \mu_f
\] (5.11)
if $M$ has no boundary.

Similar to the proof of Lemma 2.3, there exists a point $q_0 \in N$ such that
\[
\int_M x_i \mu_f = 0 (i = 1, \cdots, n),
\] (5.12)
or
\[
\int_{\partial M} x_i \tilde{\mu}_f = 0 (i = 1, \cdots, n),
\] (5.13)

By taking $\phi(r) = r$ in Item (1) of Lemma 2.1, we have $\sum_{i=1}^n \langle T \nabla x_i, \nabla x_i \rangle \leq \text{tr} T$. In particular, $\sum_{i=1}^n \langle \nabla x_i, \nabla x_i \rangle \leq m$ for $T = I$. We also have $\sum_{i=1}^n \langle \tilde{\nabla} x_i, \tilde{\nabla} x_i \rangle \leq m - 1$ when we consider the boundary $\partial M$.

We use (5.7), (5.8) and (5.9) respectively. Considering (5.12), we have
\[
\lambda_1 \int_M r^2 \mu_f = \lambda_1 \int_M \sum_{i=1}^n (x_i)^2 \mu_f \leq \int_M \sum_{i=1}^n \langle T \nabla x_i, \nabla x_i \rangle \mu_f \leq \int_M \text{tr} T \mu_f.
\]
Multiplying the both sides by \( \int_M |H_S - S\nabla f|^2 \mu_f \), we derive that
\[
\left( \int_M |H_S - S\nabla f|^2 \mu_f \right) \left( \int_M \text{tr} T \mu_f \right) \geq \lambda_1 \left( \int_M r^2 \mu_f \right) \left( \int_M |H_S - S\nabla f|^2 \mu_f \right)
\geq \lambda_1 \left( \int_M r|H_S - S\nabla f| \mu_f \right)^2
\geq \lambda_1 \left( \int_M \text{tr} S \mu_f \right)^2,
\]
where we used (5.11) for the tensor \( S \) and the Hölder inequality. This is (5.4).

Considering (5.13), we have
\[
\sigma_1 \int_{\partial M} r^2 \tilde{\mu}_f \leq \int_M \sum_{i=1}^n \langle T \nabla x_i, \nabla x_i \rangle \mu_f \leq \int_M \text{tr} T \mu_f.
\]
Multiplying the both sides by \( \int_{\partial M} |H_S - S\nabla f|^2 \tilde{\mu}_f \), we can obtain (5.5) by using the similar arguments.

Considering (5.13) for \( f \equiv 0 \), we have
\[
\alpha_1 \int_{\partial M} r^2 \leq \int_M \sum_{i=1}^n \langle \nabla x_i, \nabla x_i \rangle + b \int_{\partial M} \sum_{i=1}^n \langle \nabla x_i, \tilde{\nabla} x_i \rangle \leq m \text{vol}(M) + b(m - 1) \text{vol}(\partial M).
\]
Multiplying the both sides by \( \int_{\partial M} |H_S|^2 \), we can obtain (5.6) by using the similar arguments. \( \square \)

**Remark 5.7.** Our upper bounds are formally the same as the previous results such as in [7, 22, 23]. The main differences are:

1. In [7, 22, 23], the ambient manifold is the Euclidean space. Here we extend it to a Cartan-Hadamard manifold.

2. When the ambient manifold is the Euclidean space, one can choose test functions by moving the mass center to the origin and use the Hsiung–Minkowski formulas; when the ambient manifold is a Cartan-Hadamard manifold, we choose test functions by (a modified version of) Lemma 2.3, and use the inequality (5.10).

3. When the ambient space is a Cartan-Hadamard manifold, giving the sufficient and necessary conditions for the equalities become more difficult.

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