Abstract

For any set $\mathcal{B} \subseteq \mathbb{N} = \{1, 2, \ldots\}$ one can define its set of multiples $M_{\mathcal{B}} := \bigcup_{b \in \mathcal{B}} b\mathbb{Z}$ and the set of $\mathcal{B}$-free numbers $\mathcal{F}_{\mathcal{B}} := \mathbb{Z} \setminus M_{\mathcal{B}}$. Tautness of the set $\mathcal{B}$ is a basic property related to questions around the asymptotic density of $M_{\mathcal{B}} \subseteq \mathbb{Z}$. From a dynamical systems point of view (originated in [11]) one studies $\eta$, the indicator function of $\mathcal{F}_{\mathcal{B}} \subseteq \mathbb{Z}$, its shift-orbit closure $X_\eta \subseteq \{0, 1\}^\mathbb{Z}$ and the stationary probability measure $\nu_\eta$ defined on $X_\eta$ by the frequencies of finite blocks in $\eta$. In this paper we prove that tautness implies the following two properties of $\eta$:

- The measure $\nu_\eta$ has full topological support in $X_\eta$.
- If $X_\eta$ is proximal, i.e. if the one-point set $\{\ldots 000 \ldots\}$ is contained in $X_\eta$ and is the unique minimal subset of $X_\eta$, then $X_\eta$ is hereditary, i.e. if $x \in X_\eta$ and if $w$ is an arbitrary element of $\{0, 1\}^\mathbb{Z}$, then also the coordinate-wise product $w \cdot x$ belongs to $X_\eta$.

This strengthens two results from [2] which need the stronger assumption that $\mathcal{B}$ has light tails for the same conclusions.

1 Introduction and results

For any given set $\mathcal{B} \subseteq \mathbb{N} = \{1, 2, \ldots\}$ one can define its set of multiples

$$M_{\mathcal{B}} := \bigcup_{b \in \mathcal{B}} b\mathbb{Z}$$

and the set of $\mathcal{B}$-free numbers

$$\mathcal{F}_{\mathcal{B}} := \mathbb{Z} \setminus M_{\mathcal{B}}.$$ 

The investigation of structural properties of $M_{\mathcal{B}}$ or, equivalently, of $\mathcal{F}_{\mathcal{B}}$ has a long history (see the monograph [5] and the recent paper [2] for references). Properties of $\mathcal{B}$ are closely related to properties of the shift dynamical system generated by the two-sided sequence $\eta \in \{0, 1\}^\mathbb{Z}$, the characteristic function of $\mathcal{F}_{\mathcal{B}}$. Indeed, topological dynamics and ergodic theory provide a wealth of concepts to describe various aspects of the structure of $\eta$, see [11] which originated this point of view by studying the set of square-free numbers, and also [10], [11], [2], [7] for later contributions.

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1.1 A new characterization of tautness

In this note we always assume that $B$ is primitive, i.e. that there are no $b, b' \in B$ with $b \mid b'$. We recall some notions from the theory of sets of multiples [5] and also from [7].

- For a set of multiples $M_B$ denote by
  \[
  d(M_B) := \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{B} M_B(k)
  \]
  the lower and upper density, respectively, and by
  \[
  \delta(M_B) := \lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} M_B(k)
  \]
  the logarithmic density. Davenport and Erdös [3, 4] showed that the logarithmic density always exists, that $\delta(M_B) = d(M_B)$.

- The set $B \subseteq \mathbb{N}$ is a Behrend set, if $\delta(M_B) = 1$ (in which case also $d(M_B) = 1$).

- The set $B$ is taut, if
  \[
  \delta(M_B \setminus \{b\}) < \delta(M_B) \quad \text{for each } b \in B.
  \]
  So a set is taut, if removing any single point from it changes its set of multiples drastically and not only by “a few points”.

- It is known [5] that $B$ is not taut if and only if it contains a scaled copy of a Behrend set, i.e. if there are $r \in \mathbb{N}$ and a Behrend set $A$ such that $rA \subseteq B$.

The logarithmic density of sets of multiples has the following continuity property from below, which is a by-product of the proof of the Davenport-Erdös theorem:

\[
\delta(M_B) = \lim_{K \to \infty} d(M_B \cap \{1, \ldots, K\}). \tag{1}
\]

At a first glance this property may seem rather close to the following one

\[
\lim_{K \to \infty} d(M_{\{b \in B : b > K\}}) = 0, \tag{2}
\]

which was introduced in [2] under the name light tails in order to prove two subtle dynamical properties of the dynamical system associated in a natural way to the set $B$ - see the next section for details. However it turns out that light tails is definitively a stronger property than (1). Indeed, the authors of [2] show that each set $B$ with light tails is actually taut and satisfies $d(B) = \overline{d}(B)$, but that the converse does not hold [2, Th. 4.20]. They conjecture that tautness might be a sufficient assumption to prove the two dynamical properties alluded to above. In this note we will show that this is indeed the case. A key ingredient to our proof is an apparently new equivalent characterization of Behrend sets in terms of a dichotomy:

**Theorem 1.** Let $B \subseteq \mathbb{N}$ be primitive and denote $\widehat{B}^{(N)} := \{b \in B : \text{Spec}(b) \cap \{1, \ldots, N\} = \emptyset\}$.

(i) $B$ is Behrend if and only if $\widehat{B}^{(N)}$ is Behrend (i.e. $\delta(M_{\widehat{B}^{(N)}}) = 1$) for all $N \in \mathbb{N}$.

(ii) $B$ is not Behrend if and only if

\[
\lim_{N \to \infty} \delta(M_{\widehat{B}^{(N)}}) = 0. \tag{3}
\]
The proof, which we present in section\textsuperscript{2} relies on a version of Kolmogorov’s 0-1-law, that is behind Lemma\textsuperscript{2} below. Stanisław Kasjan found a purely number theoretic proof of this lemma and was so kind to allow a reproduction of his proof in this paper \textsuperscript{6}.

A rather immediate corollary to this theorem characterizes taut sets. We use the following notation: For a primitive set \( B \subseteq \mathbb{N} \) and any positive integer \( q \) let

\[
B/q := \{ b/q : b \in B \text{ and } q \mid b \}.
\]

**Corollary 1.** A primitive set \( B \subseteq \mathbb{N} \) is taut if and only if \( \lim_{N \to \infty} \delta(M_{B/q^{(N)}}) = 0 \) for all \( q \in \mathbb{N} \).

**Proof.** Suppose first that \( B \) is taut. As \( q \cdot B/q \subseteq B \), the set \( B/q \) is not Behrend, and Theorem\textsuperscript{1} implies \( \lim_{N \to \infty} \delta(M_{B/q^{(N)}}) = 0 \). Conversely, if \( B \) is not taut, then there are \( r \in \mathbb{N} \) and a Behrend set \( \mathcal{A} \) such that \( r \mathcal{A} \subseteq B \). In particular \( \mathcal{A} \subseteq B/r \), so that also \( B/r \) is Behrend. But then also all sets \( \tilde{B}/r^{(N)} \) are Behrend in view of Theorem\textsuperscript{1} so that \( \lim_{N \to \infty} \delta(M_{\tilde{B}/r^{(N)}}) = 1 \neq 0 \).

\( \square \)

### 1.2 Consequences for the dynamics of \( B \)-free systems

For a given set \( B \subseteq \mathbb{N} \) denote by \( \eta \in \{0,1\}^\mathbb{Z} \) the characteristic function of \( \mathcal{T}_B \), i.e. \( \eta(n) = 1 \) if and only if \( n \in \mathcal{T}_B \), and consider the orbit closure \( X_\eta \) of \( \eta \) in the shift dynamical system \((\{0,1\}^\mathbb{Z}, \sigma)\), where \( \sigma \) stands for the left shift. Topological dynamics and ergodic theory provide a wealth of concepts to describe various aspects of the structure of \( \eta \), see \textsuperscript{11} which originated this point of view by studying the set of square-free numbers, and also \textsuperscript{11}, \textsuperscript{2}, \textsuperscript{7}, \textsuperscript{8} which continued this line of research. We collect some facts from these references:

- (A) \( \eta \) is quasi-generic for a natural ergodic shift invariant probability measure \( \nu_\eta \) on \( \{0,1\}^\mathbb{Z} \), called the **Mirsky measure** of \( B \) \textsuperscript{2} Prop. E], in particular \( \text{supp}(\nu_\eta) \subseteq X_\eta \). The Mirsky measure can be characterized as the unique shift invariant probability measure \( P \) on \( X_\eta \subseteq \{0,1\}^\mathbb{Z} \) with the property that \( \lim_{n \to \infty} n^{-1} \sum_{k=1}^n x_k = \overline{d}(\mathcal{T}_B) \) for \( P \)-a.a. \( x \in X_\eta \) (while \( \limsup_{n \to \infty} n^{-1} \sum_{k=1}^n x_k \leq \overline{d}(\mathcal{T}_B) \) for all \( x \in X_\eta \), see \textsuperscript{8} Cor. 3 and 4].

- (B) If \( B \) has light tails, then \( B \) is taut, but the converse does not hold \textsuperscript{2} Sect. 4.3 and Cor. 4.19].

- (C) If \( B \) has light tails, then \( \eta \) is generic for \( \nu_\eta \) \textsuperscript{2} Prop. E and Rem. 2.24].

- (D) If \( B \) has light tails, then \( \text{supp}(\nu_\eta) = X_\eta \) \textsuperscript{2} Thm. G].

- (E) If \( B \) has light tails and if \( B \) contains an infinite pairwise coprime subset, then \( X_\eta \) is **hereditary**, i.e. \( y \in \{0,1\}^\mathbb{Z} \) belongs to \( X_\eta \) whenever there is \( x \in X_\eta \) with \( y \leq x \) coordinate-wise \textsuperscript{2} Thm. D].

One may ask, whether implications (C) - (E) continue to hold if only tautness of the set \( B \) is assumed. For Implication (C) this is not true \textsuperscript{2} Prop. 4.17], but for the other two implications this remained open in \textsuperscript{2}. Here we prove that it suffices indeed to assume tautness for the conclusions of (D) and (E) to hold true:

**Theorem 2.** Suppose that the primitive set \( B \subseteq \mathbb{N} \) is taut. Then \( \text{supp}(\nu_\eta) = X_\eta \).

**Theorem 3.** Suppose that the primitive set \( B \subseteq \mathbb{N} \) is taut and contains an infinite co-prime subset. Then \( X_\eta \) is hereditary.
Remark 1. It was proved in [2, Th. B] that \( B \) contains an infinite co-prime subset if and only if the subshift \( X_\eta \) is proximal, i.e. if and only if it has a fixed point as its unique minimal subset (the point \((\ldots 000 \ldots)\) in this case).

The proofs of both theorems rely on substantial parts of the proofs of the corresponding results from [2]. We strengthen some of the lemmas from that paper in such a way that light tails are no longer needed to conclude, but the new characterization of tautness from Corollary [1] suffices.

Theorem 1 is a 0-1-law that we prove in a measure theoretic and probabilistic framework, which is borrowed from previous publications [2, 7, 8, 9]:

- \( \Delta : \mathbb{Z} \to \prod_{b \in B} \mathbb{Z}/b\mathbb{Z}, \Delta(n) = (n, n, \ldots) \), denotes the canonical diagonal embedding.
- \( H := \overline{\Delta(\mathbb{Z})} \) is a compact abelian group, and we denote by \( m_H \) its normalised Haar measure.
- The window associated to \( B \) is defined as \( W := \{ h \in H : h_b \neq 0 \ (\forall b \in B) \} \).

\( \bullet \) For an arbitrary subset \( A \subseteq H \) we define the coding function \( \varphi_A : H \to \{0, 1\}^\mathbb{Z} \) by \( \varphi_A(h)(n) = 1 \) if and only if \( h + \Delta(n) \in A \). Of particular interest is the coding functions \( \varphi := \varphi_W \).

\( \bullet \) Observe that \( \varphi(h)(n) = 1 \) if and only if \( h_b + n \neq 0 \mod b \) for all \( b \in B \).

\( \bullet \) With this notation \( \eta = \varphi(\Delta(0)) \) and \( X_\eta = \varphi(\Delta(\mathbb{Z})) \), so that \( X_\eta \subseteq X_\varphi := \varphi(H) \).

Our proof yields indeed the following sharpening of Theorem 2:

Theorem 4. Suppose that the primitive set \( B \subseteq \mathbb{N} \) is taut. Then \( \text{supp}(\nu_\eta) = X_\eta = X_\varphi \).

In [7, Prop. 2.2] (the second part of) this conclusion was proved under the assumption that \( B \) has light tails.

Remark 2. a) We recall from [7, Theorem A] a purely measure theoretic characterization of tautness: The primitive set \( B \) is taut if and only if the window \( W \) associated to \( B \) is Haar-regular, i.e. if \( \text{supp}(m_H|_W) = W \).

b) Also proximality of \( X_\eta \) (which is equivalent to \( B \) having no infinite co-prime subset) can be characterized in terms of the window: \( X_\eta \) is proximal if and only if \( W \) has no interior point [7, Th. C].

Acknowledgement The approach taken in this note occurred while I was supervising the MSc thesis of Jakob Seifert [12], who proved the identity \( \text{supp}(\nu_\eta) = X_\eta \) under an assumption on the set \( B \) which implies tautness and is strictly weaker than light tails, but does not seem to be equivalent to tautness, namely: for any finite set \( A \subseteq \mathcal{P} \) there is a thin set \( P \subseteq \mathcal{P} \setminus A \) such that the set \( B \setminus M_P \) has light tails. \( (P \) is thin if \( \sum_{p \in P} \frac{1}{p} \) converges.\)

2 Proof of Theorem 1

For any subset \( B' \subseteq B \) we denote the corresponding objects defined as above by \( \Delta', H', m_{H'}, W' \) and \( \varphi' \). On the other hand one can consider the window corresponding to \( B' \) as a subset of \( H \), namely \( W_{B'} := \{ h \in H : h_b \neq 0 \ (b \in B') \} \).

Lemma 1. With the previous notation, \( m_H(W_{B'}) = m_H(W') \).
Proof. Denote by $\pi$ the natural projection from $[\prod_{b \in B} \mathbb{Z} / b \mathbb{Z}]$ to $[\prod_{b \in B'} \mathbb{Z} / b \mathbb{Z}]$. Then $\Delta'(\mathbb{Z}) = \pi(\Delta(\mathbb{Z}))$, and as $\pi$ is continuous between compact metric spaces, it follows that $\pi(H) = H'$ so that $m_H = m_{H'} = m_H \circ \pi^{-1}$. Quite obviously, $\pi^{-1}(W') \subseteq W_{\mathbb{B}'}$. For the converse inclusion let $h \in W_{\mathbb{B}'} \subseteq H$. Then $(\pi(h))_b = h_b \neq 0$ for all $b \in B'$ so that $\pi(h) \in W'$. Hence $m_H(W_{\mathbb{B}'}) = m_H(\pi^{-1}(W')) = m_H(W')$. \hfill $\Box$

Lemma 2. Let $B \subseteq \mathbb{N}$ be primitive. Then either $\delta(M_{\mathbb{B}^N}) = 1$ for all $N \in \mathbb{N}$ or $\lim_{N \to \infty} \delta(M_{\mathbb{B}^N}) = 0$.

Proof. In [7, Lemma 4.1] it was proved that $m_H(W) = 1 - d(M_B)$ and, analogously, $m_H(W') = 1 - d(M_{\mathbb{B}'})$ for each $B' = \mathbb{B}^{(N)}$. Hence $\delta(M_{\mathbb{B}'}) = d(M_B) = 1 - m_H(W)$, and Lemma1 implies

$$\delta(M_{\mathbb{B}'}) = 1 - m_H(W') = 1 - m_H(W_{\mathbb{B}'}) \quad \text{for all } B' = \mathbb{B}^{(N)}.$$

Observing that $(W_{\mathbb{B}^N})_N$ is an increasing sequence of sets and denoting $W_\infty := \bigcup_{N \in \mathbb{N}} W_{\mathbb{B}^N}$, we thus conclude that

$$\lim_{N \to \infty} \delta(M_{\mathbb{B}^N}) = 1 - \lim_{N \to \infty} m_H(W_{\mathbb{B}^N}) = 1 - m_H(W_\infty),$$

and, in order to prove the lemma, we must show that either $m_H(W_{\mathbb{B}^N}) = 0$ for all $N \in \mathbb{N}$, or $m_H(W_\infty) = 1$. This will result from a variant of Kolmogorov’s 0-1-law.

For $b \in B$ define the random variable $Z_b : H \to \mathbb{Z}$ by $Z_b(h) = h_b$. If $\mathcal{A} \subseteq B$ and $C \subseteq B$ are co-prime to each other, i.e. if $\gcd(a, c) = 1$ for all $a \in \mathcal{A}$ and all $c \in C$, then the families $(Z_b)_{b \in \mathcal{A}}$ and $(Z_b)_{b \in C}$ are independent from each other. For $B' \subseteq B$ denote by $\Pi_{B'}$ the $\sigma$-algebra generated by the random variables $Z_b$ ($b \in B'$). Then $W_\infty \in \Pi_{\mathbb{B}^N}$ for all $N \in \mathbb{N}$, because $W_\infty = \bigcup_{N' \geq N} W_{\mathbb{B}^{(N')}}$, and $W_{\mathbb{B}^{(N')}} \in \Pi_{\mathbb{B}^{(N')}} \subseteq \Pi_{\mathbb{B}^N}$ whenever $N' \geq N$.

Let $\varepsilon > 0$.

- As $\Pi_B$ is generated by the algebras $\Pi_{B^{(N)}}$ ($N \in \mathbb{N}$) where $B^{(N)} := \{b \in B : \text{Spec}(b) \subseteq \{1, \ldots, N\}\}$, there are $N_1 \in \mathbb{N}$ and a set $V_1 \in \Pi_{B^{(N_1)}}$ such that $m_H(W_\infty \Delta V_1) < \varepsilon$. Note that all $B^{(N)}$ are finite, because $B$ is primitive [2, Lemma 5.14].

- As $\Pi_{B^{(N)}}$ is generated by the algebras $\Pi_{B'}$ ($B' \subseteq B^{(N)}$ finite), there are $N_2 \in \mathbb{N}$, a finite set $B'_2 \subseteq B^{(N_1)}$, and a set $V_2 \in \Pi_{B'_2}$ such that $m_H(W_\infty \Delta V_2) < \varepsilon$.

- As $B^{(N_1)}$ and $B'_2 \subseteq B^{(N)}$ are co-prime to each other, the corresponding $\sigma$-algebras $\Pi_{B^{(N_1)}}$ and $\Pi_{B'_2}$ are independent from each other, in particular $m_H(V_1 \cap V_2) = m_H(V_1) \cdot m_H(V_2)$.

Hence

$$|m_H(W_\infty) - m_H(W_\infty) \cdot m_H(W_\infty)| \leq |m_H(V_1 \cap V_2) - m_H(V_1) \cdot m_H(V_2)| + 4 \varepsilon = 4 \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this shows that $m_H(W_\infty) \in \{0, 1\}$. Finally note that if $m_H(W_\infty) = 0$, then also $m_H(W_{\mathbb{B}^N}) = 0$ for all $N$. \hfill $\Box$

Proof of Theorem[7] (i) Suppose first that all $\mathbb{B}^{(N)}$ are Behrend. Then $B$ is Behrend, because $\mathbb{B}^{(N)} \subseteq B$. If, conversely, there is a non-Behrend set $\mathbb{B}^{(N)}$, then $B$ is contained in the finite union $\mathbb{B}^{(N)} \cup \bigcup_{p \in \mathbb{P} \cap \{1, \ldots, N\}} p \cdot \mathbb{Z}$ of non-Behrend sets, and hence $B$ is not Behrend [5, Cor. 0.14].

(ii) This follows from assertion (i) in view of Lemma[2]. \hfill $\Box$

Stanislaw Kasjan provided another, purely arithmetic proof of Lemma[2]. I am indebted to him for the permission to reproduce it here [6]:

\footnote{1This is a consequence of the generalized Chinese Remainder Theorem, which guarantees that each cylinder set determined by a finite index set $\mathbb{B}'$ has Haar measure $1 / \text{lcm}(\mathbb{B}')$.}
Alternative proof of Lemma 2. First observe that if \( \mathcal{A}, \mathcal{C} \subseteq \mathbb{N} \) are such that \( \gcd(a, c) = 1 \) for every \( a \in \mathcal{A}, c \in \mathcal{C} \), then
\[
\delta(\mathcal{M}_a \cap \mathcal{M}_c) = \delta(\mathcal{M}_a) \cdot \delta(\mathcal{M}_c).
\]  (5)

For finite \( \mathcal{A}, \mathcal{C} \) this is proved in [2, Lemma 4.22], the general case is then derived using the Davenport-Erdős formula (1).

Assume now that \( \lim_{N \to \infty} \delta(\mathcal{M}_{\mathbb{N}^N}) \neq 0 \). Then
\[
\delta(\mathcal{M}_{\mathbb{N}^N}) \geq \varepsilon
\]  (6)
for every \( N \) and some \( \varepsilon > 0 \). Note that by (1),
\[
\lim_{N \to \infty} \delta(\mathcal{M}_{\mathbb{N}^N} \setminus \mathcal{M}_{\mathbb{N}^N}) \leq \lim_{N \to \infty} \delta(\mathcal{M}_{\mathcal{B}^N} \setminus \mathcal{M}_{\mathbb{N}^N}) = 0,
\]
and by (5),
\[
\delta(\mathcal{M}_{\mathbb{N}^N}) = \delta(\mathcal{M}_{\mathbb{N}^N} \setminus \mathcal{M}_{\mathbb{N}^N}) + \delta(\mathcal{M}_{\mathbb{N}^N}) \cdot \delta(\mathcal{M}_{\mathbb{N}^N}).
\]

Hence
\[
\lim_{N \to \infty} \delta(\mathcal{M}_{\mathbb{N}^N}) \cdot (1 - \delta(\mathcal{M}_{\mathbb{N}^N})) = \lim_{N \to \infty} \delta(\mathcal{M}_{\mathbb{N}^N} \setminus \mathcal{M}_{\mathbb{N}^N}) = 0.
\]
Together with (5) this yields
\[
\lim_{N \to \infty} (1 - \delta(\mathcal{M}_{\mathbb{N}^N})) = 0.
\]

Invoking Eq. (1) once more, this implies \( \delta(\mathcal{M}_{\mathbb{N}^N}) = 1 \). Finally \( \delta(\mathcal{M}_{\mathbb{N}^N}) = 1 \) for every \( N \) follows from Theorem 1, the simple proof of which is purely arithmetic and does not rely on Lemma 2. \( \square \)

Remark 3. In the present context of \( \mathcal{B} \)-free dynamics the purely arithmetic proof is certainly the more direct (and hence preferable) one. Having in mind that the sets \( \mathcal{F}_{\mathcal{B}} \) are very special example of model sets (see e.g. [9, Sec. 3.3] for a detailed discussion), the probabilistic proof might indicate how to use 0-1-laws for the investigation of more geometrically defined model sets.

3 Proof of Theorems 2, 3 and 4

Denote by \( \mathcal{P} \) the set of prime numbers. Recall that \( \mathcal{B}^{(n)} := \{ b \in \mathcal{B} : \text{Spec}(b) \subseteq \{1, \ldots, n\} \} \). For a finite set \( A \subseteq \mathcal{P} \) denote
\[
\mathcal{B}_A := \{ b \in \mathcal{B} : \text{Spec}(b) \subseteq A \}.
\]

Lemma 3. Suppose that the primitive set \( \mathcal{B} \) is taut. Then for each finite set \( A \subseteq \mathcal{P} \) and each \( \varepsilon > 0 \) there is a finite set \( P \subseteq \mathcal{P} \) such that
\[
P \cap A = \emptyset \quad \text{and} \quad \delta(\mathcal{M}_{\mathcal{B}^P}((\mathcal{B}_A \cup \mathcal{M}_P))) < \varepsilon.
\]  (7)

Proof. Denote \( a := \text{card} A \) and \( K := \sum_{p \in A} \frac{1}{p} \). Choose \( L \in \mathbb{N} \) large enough that \( \sum_{p \in A} \frac{1}{p} < \varepsilon \) and let
\[
Q := \left\{ \prod_{p \in A} p^k : k_p \in \mathbb{N}_0 \right\} \quad \text{and} \quad Q_0 := \left\{ \prod_{p \in A} p^{k_p} \in Q : k_p < L (p \in A) \right\}.
\]

In view of Corollary 1 we can fix \( N \in \mathbb{N} \) large enough that \( \delta(\mathcal{M}_{\mathcal{B}/q}(N)) < \varepsilon/L^n \) for all \( q \in Q_0 \).
Let \( P := (\mathcal{P} \cap \{1, \ldots, N\}) \setminus A \). Then

\[
\mathcal{B}(\mathcal{B}_A \cup M_P) \subseteq \bigcup_{q \in Q} q \cdot \mathcal{B}/q^{(N)} \subseteq \bigcup_{q \in Q_0} q \cdot \mathcal{B}/q^{(N)} \cup \bigcup_{q \in Q \setminus Q_0} q \cdot \mathbb{Z}
\]

so that

\[
\mathcal{M}_{\mathcal{B}(\mathcal{B}_A \cup M_P)} \subseteq \bigcup_{q \in Q_0} q \cdot \mathcal{M}_{\mathcal{B}_q^{(N)}} \cup \bigcup_{p \in A} p^L \cdot \mathbb{Z}.
\]

Hence

\[
\delta(\mathcal{M}_{\mathcal{B}(\mathcal{B}_A \cup M_P)}) \leq \sum_{q \in Q_0} \frac{1}{q} \delta(\mathcal{M}_{\mathcal{B}_q^{(N)}}) + \sum_{p \in A} \frac{1}{p^L} \leq \text{card } Q_0 \cdot \frac{\varepsilon}{L^a} + \varepsilon = 2\varepsilon.
\]

\[\square\]

Next we prove a strengthening of Lemma 5.20 from [7].

**Lemma 4.** Let \( \beta, r, n \in \mathbb{N} \) and \( C \subseteq \mathbb{N} \). Assume that \( P \subseteq \{n + 1, n + 2, \ldots\} \) is a finite set of prime numbers co-prime to \( \beta \). Then

\[
\delta \left( \beta \mathbb{Z} + r \cap \bigcap_{i=1}^n (\mathcal{F}_C - i) \right) \geq \prod_{p \in \mathcal{P}} \left( 1 - \frac{n}{p} \right) \cdot \delta \left( \beta \mathbb{Z} + r \cap \bigcap_{i=1}^n (\mathcal{F}_{C \setminus \mathcal{M}_P} - i) \right)
\]

**Proof.** Fix \( M \in \mathbb{N} \). As in the proof of [7, Lemma 5.18] one shows that

\[
d \left( \beta \mathbb{Z} + r \cap \bigcap_{i=1}^n (\mathcal{F}_C \cap [1, \ldots, M] - i) \right) \geq \left( 1 - \frac{n}{p} \right) \cdot d \left( \beta \mathbb{Z} + r \cap \bigcap_{i=1}^n (\mathcal{F}_{(C \setminus \mathcal{M}_P) \cap [1, \ldots, M]} - i) \right)
\]

for each \( p \in P \). Applying this inductively to all \( p \in P \) (replacing \( C \) by \( C \setminus p\mathbb{Z} \) etc.), this yields

\[
d \left( \beta \mathbb{Z} + r \cap \bigcap_{i=1}^n (\mathcal{F}_C \cap [1, \ldots, M] - i) \right) \geq \prod_{p \in P} \left( 1 - \frac{n}{p} \right) \cdot d \left( \beta \mathbb{Z} + r \cap \bigcap_{i=1}^n (\mathcal{F}_{(C \setminus \mathcal{M}_P) \cap [1, \ldots, M]} - i) \right),
\]

and the same holds, of course, for the logarithmic density \( \delta \). As the (logarithmic) density is monotone, we obtain

\[
\delta \left( \beta \mathbb{Z} + r \cap \bigcap_{i=1}^n (\mathcal{F}_C \cap [1, \ldots, M] - i) \right) \geq \prod_{p \in P} \left( 1 - \frac{n}{p} \right) \cdot \delta \left( \beta \mathbb{Z} + r \cap \bigcap_{i=1}^n (\mathcal{F}_{(C \setminus \mathcal{M}_P)} - i) \right)
\]

for all \( M \in \mathbb{N} \). In order to pass to the limit \( M \to \infty \) on the l.h.s. of this inequality, note first that the
logarithmic density is finitely (sub-)additive and invariant under shifts by some integer. Hence

\[
\delta \left( (\beta \mathbb{Z} + r) \cap \bigcap_{i=1}^{n} (F_{C \cap [1, \ldots, M]} - i) \right) - \delta \left( (\beta \mathbb{Z} + r) \cap \bigcap_{i=1}^{n} (F_{C} - i) \right) \\
= \delta \left( \beta \mathbb{Z} \setminus \bigcup_{i=1}^{n} (M_{C \cap [1, \ldots, M]} - i) \right) - \delta \left( \beta \mathbb{Z} \setminus \bigcup_{i=1}^{n} (M_{C} - i) \right) \\
\leq \delta \left( \left( \bigcup_{i=1}^{n} (M_{C} - i) \right) \setminus \left( \bigcup_{i=1}^{n} (M_{C \cap [1, \ldots, M]} - i) \right) \right) \\
\leq \sum_{i=1}^{n} \delta \left( (M_{C} - i) \setminus (M_{C \cap [1, \ldots, M]} - i) \right) \\
= \sum_{i=1}^{n} \left( \delta (M_{C} - i) - \delta (M_{C \cap [1, \ldots, M]} - i) \right) \\
= \sum_{i=1}^{n} \left( \delta (M_{C}) - \delta (M_{C \cap [1, \ldots, M]}) \right).
\]

and for fixed \( n \) this tends to 0 as \( M \to \infty \) by equation (1). This finishes the proof of the lemma. \( \square \)

**Proposition 1.** Let \( \beta, r, n \in \mathbb{N} \) and assume that the primitive set \( \mathcal{B} \subseteq \mathcal{P} \) is taut and denote \( A := \text{Spec}(\beta) \). Then

\[
\delta \left( (\beta \mathbb{Z} + r) \cap \bigcap_{i=1}^{n} (F_{\mathcal{B} \backslash \mathcal{B}_{A}} - i) \right) > 0. \tag{8}
\]

**Proof.** Apply Lemma 3 with \( \varepsilon := \frac{1}{2 \beta} \). This produces a finite set \( P \subseteq \mathcal{P} \setminus A \), hence co-prime to \( \beta \), with \( \delta \left( \mathcal{M}_{\mathcal{B}(\mathcal{B}_{A} \cup \mathcal{M}_{P})} \right) < \varepsilon \). Hence

\[
\delta \left( (\beta \mathbb{Z} + r) \cap \bigcap_{i=1}^{n} (F_{\mathcal{B} \backslash (\mathcal{B}_{A} \cup \mathcal{M}_{P})} - i) \right) \leq \delta \left( \bigcup_{i=1}^{n} (M_{\mathcal{B}(\mathcal{B}_{A} \cup \mathcal{M}_{P})} - i) \right) \leq \sum_{i=1}^{n} \delta \left( M_{\mathcal{B}(\mathcal{B}_{A} \cup \mathcal{M}_{P})} - i \right) < n \varepsilon = \frac{1}{2 \beta}.
\]

Combining this with Lemma 4 (applied with \( C = \mathcal{B}_{A} \cup \mathcal{M}_{P} \)) yields

\[
\delta \left( (\beta \mathbb{Z} + r) \cap \bigcap_{i=1}^{n} (F_{\mathcal{B} \backslash \mathcal{B}_{A}} - i) \right) \geq \prod_{p \in P} \left( 1 - \frac{n}{p} \right) \cdot \delta \left( (\beta \mathbb{Z} + r) \cap \bigcap_{i=1}^{n} (F_{\mathcal{B} \backslash (\mathcal{B}_{A} \cup \mathcal{M}_{P})} - i) \right) \\
= \prod_{p \in P} \left( 1 - \frac{n}{p} \right) \cdot \left( \delta (\beta \mathbb{Z} + r) - \delta \left( (\beta \mathbb{Z} + r) \cap \bigcap_{i=1}^{n} (F_{\mathcal{B} \backslash (\mathcal{B}_{A} \cup \mathcal{M}_{P})} - i) \right) \right) \\
\geq \frac{1}{2 \beta} \cdot \prod_{p \in P} \left( 1 - \frac{n}{p} \right) > 0.
\]

\( \square \)

Next we turn to Proposition 5.11 of [7] and provide a proof of the same assertion under the sole assumption that the set \( \mathcal{B} \) is taut.

**Proposition 2.** Assume that the primitive set \( \mathcal{B} \) is taut and that \( \mathcal{B}(n) \subseteq \mathcal{A} \subseteq \mathcal{B} \) for some \( n > 0 \). Suppose that

\[
\{r + 1, \ldots, r + n\} \cap M_{\mathcal{A}} = r + I \text{ for some } r \in \mathbb{N} \text{ and some set } I \subseteq \{1, \ldots, n\}. \tag{9}
\]
Then the density of the set of all \( k \in \mathbb{N} \) for which

\[
\{k + 1, \ldots, k + n\} \cap \mathcal{M}_B = k + I
\]

is strictly positive.

**Proof.** The proof is strongly inspired by the proof of Proposition 5.11 in [7]: For \( u \in I \) let \( j_u \) be such that \( b_{j_u} \mid r + u \). Without loss of generality we may assume that \( \mathcal{A} = \{b_{j_u} : u \in I\} \cup \mathcal{B}^{(n)} \). Then, by [7, Lemma 5.14], \( \mathcal{A} \) is finite, and we set \( \beta := \text{lcm}(\mathcal{A}) \).

By definition of the set \( \mathcal{A} \), we have for all \( i \in \{1, \ldots, n\} \)

\[
i \in I \iff r + i \in \mathcal{M}_\mathcal{A} \iff b_{j_i} \mid r + i.
\]

Let \( i \in \{1, \ldots, n\} \).

- If \( i \in I \), then \( b_{j_i} \mid r + i \), i.e. \( r + i \in b_{j_i} \mathbb{Z} \). As \( b_{j_i} \mid \text{lcm}(\mathcal{A}) = \beta \), it follows that \( r + \beta \mathbb{Z} + i \subseteq b_{j_i} \mathbb{Z} \subseteq \mathcal{M}_B \).

Hence

\[
\{k \in r + \beta \mathbb{Z} : \{k + 1, \ldots, k + n\} \cap \mathcal{M}_B = k + I\}
= \{k \in r + \beta \mathbb{Z} : k \in \bigcap_{i \in I} (\mathcal{M}_B - i) \cap \bigcap_{i \in \{1, \ldots, n\}\setminus I} (\mathcal{F}_B - i)\}
= \{k \in r + \beta \mathbb{Z} : k \in \bigcap_{i \in \{1, \ldots, n\}\setminus I} (\mathcal{F}_B - i)\}
= (r + \beta \mathbb{Z}) \cap \bigcap_{i \in \{1, \ldots, n\}\setminus I} (\mathcal{F}_B - i)
\]

- Denote \( A := \text{Spec}(\beta) \) and recall that

\[
\mathcal{B}_A = \{b \in \mathcal{B} : \text{Spec}(b) \subseteq \text{Spec}(\beta)\}.
\]

Notice that \( \mathcal{B}_A \) is finite [7, Lemma 5.14]. As \( \text{Spec}(\mathcal{B}^{(n)}) \subseteq \text{Spec}(\mathcal{A}) = \text{Spec}(\beta) \), we have \( \mathcal{B}^{(n)} \subseteq \mathcal{B}_A \). Let \( b \in \mathcal{B}_A \setminus \mathcal{B}^{(n)} \) and take a prime \( p \in \text{Spec}(b) \). By the definition of \( \mathcal{B}_A \), we have \( p \mid \beta \), whence \( p \leqslant n \) or \( p \mid b_{j_u} \) for some \( u \in I \). As \( b \notin \mathcal{B}^{(n)} \), only the second possibility can occur. It follows that if \( b \mid r + \beta \ell + i \) for some \( 1 \leqslant i \leqslant n \) and \( \ell \in \mathbb{Z} \), then \( p \mid ((r + \beta \ell + i) - \beta \ell - (r + u)) \), because \( p \mid b_{j_u} \mid r + u \). This implies \( p \mid i - u \), so that \( i = u \in I \), because \( p > n \mid i - u \). Thus we have shown that if \( b \mid r + \beta \ell + i \) for some \( b \in \mathcal{B}_A \setminus \mathcal{B}^{(n)} \) and \( \ell \in \mathbb{Z} \), then \( i \in I \). Equivalently, if \( i \in \{1, \ldots, n\} \setminus I \), then \( r + \beta \mathbb{Z} + i \subseteq \mathcal{F}_B \triangle \mathcal{B}^{(n)} \). Hence, \( r + \beta \mathbb{Z} \subseteq \bigcap_{i \in \{1, \ldots, n\}\setminus I} (\mathcal{F}_B \setminus \mathcal{B}^{(n)} - i) \), and we can continue the chain of identities from (10) by

\[
= (r + \beta \mathbb{Z}) \cap \bigcap_{i \in \{1, \ldots, n\}\setminus I} (\mathcal{F}_B \setminus \mathcal{B}^{(n)} - i) \cap \bigcap_{i \in \{1, \ldots, n\}\setminus I} (\mathcal{F}_B \setminus \mathcal{B}_A - i)
\]

Finally, if \( r + \beta \ell + i \in \mathcal{M}_{\mathcal{B}^{(n)}} \) for some \( \ell \in \mathbb{Z} \), then there is \( b \in \mathcal{B}^{(n)} \subseteq \mathcal{A} \) such that \( b \mid r + \beta \ell + i \) and \( b \mid \beta \). Hence \( b \mid r + i \), so that \( i \in I \). Equivalently, if \( i \in \{1, \ldots, n\} \setminus I \), then \( r + \beta \mathbb{Z} + i \subseteq \mathcal{F}_{\mathcal{B}^{(n)}} \), and we can finish the above identities by

\[
= (r + \beta \mathbb{Z}) \cap \bigcap_{i \in \{1, \ldots, n\}\setminus I} (\mathcal{F}_B \setminus \mathcal{B}_A - i) \supseteq (r + \beta \mathbb{Z}) \cap \bigcap_{i \in \{1, \ldots, n\}\setminus I} (\mathcal{F}_B \setminus \mathcal{B}_A - i).
\]
In view of Proposition 1, the logarithmic density of the latter set is strictly positive. This finishes the proof of the proposition. \qed

**Proof of Theorem 2.** We must show that \( X_\eta \subseteq \text{supp}(\nu_\eta) \) or, equivalently, that each block \((\eta_{r+1}, \ldots, \eta_{r+n})\) occurs in \( \eta \) with strictly positive frequency (observe that \( \eta \) is quasi-generic for \( \nu_\eta \)). But this is just a rewording of Proposition 2. \qed

**Proof of Theorem 3.** The heridity of \( X_\eta \) was proved in [2, sec. 5] under the additional assumption that \( \mathcal{B} \) has light tails. This assumption enters the proof only via Proposition 5.11 of that reference, so replacing it by our Proposition 2 leads to the heredity of \( X_\eta \) under the present assumptions. \qed

**Proof of Theorem 4.** The identity \( X_\varphi = X_\eta \) was proved in [7, Prop. 2.2] under the assumption that \( \mathcal{B} \) has light tails. Again, this assumption entered only via a reference to Proposition 5.11 from [2], which, once more, can be replaced by the present Proposition 2. The identity \( \text{supp}(\nu_\eta) = X_\eta \) was proved in Theorem 2. \qed

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