SMOOTHING EFFECTS FOR NAVIER-STOKES EQUATIONS

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ABSTRACT. We prove some smoothing effects for the 3-D Navier-Stokes equations for initial data belonging to the critical Sobolev space $H^{1/2}(\mathbb{R}^3)$. Asymptotic behavior of the global solution when the time goes to infinity is studied. We also obtain a new energy estimate. Other results in this direction and with different methods can be found in [9].

1. Introduction

The purpose of this text is to establish some regularity results for the 3-D incompressible Navier-Stokes equations on the whole space $\mathbb{R}^3$. Throughout this paper we consider the three-dimensional incompressible Navier-Stokes equations

$$\begin{cases}
\partial_t u - \nu \Delta u + (u, \nabla)u = -\nabla p, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
\operatorname{div}(u) = 0 & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
u u_{|t=0} = u^0 & \text{on } \mathbb{R}^3,
\end{cases}
$$

(NS)\)

where $\nu > 0$ is the viscosity of the fluid, and $u = u(t, x) = (u_1, u_2, u_3)$ and $p = p(t, x)$ denote respectively the unknown velocity and the unknown pressure of the fluid at the point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$. While $u^0 = (u^0_1(x), u^0_2(x), u^0_3(x))$ is a given initial velocity. If $u^0$ is quite regular, the divergence free condition determine the pressure $p$. Moreover, $p$ can be expressed as follows

$$p = -\Delta^{-1} \sum_{j,k} \partial_j \partial_k (u_j u_k).$$

The above problem has been studied by many authors like [4], [10], [16],... Using compactness methods, Leray proved in 1934 ([16]) for $u^0 \in L^2(\mathbb{R}^3)$ an existence result in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))$ for the problem (NS). Also, in two dimension space, Leray proved the existence and uniqueness in $C_0(\mathbb{R}^+, L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^2))$ for the same problem. In [10] Fujita-Kato proved that for $u^0 \in \dot{H}^{1/2}(\mathbb{R}^3)$ there exist $T^* \in (0, +\infty)$ and at least one solution

$$u \in C([0, T^*), \dot{H}^{1/2}(\mathbb{R}^3)) \text{ and } t^{1/4} \nabla u \in C([0, T^*), L^2(\mathbb{R}^3)).$$

Moreover, if $\|u^0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < \nu$ we have $u \in C_0(\mathbb{R}^+, \dot{H}^{1/2}) \cap L^2(\mathbb{R}^+, \dot{H}^{3/2}).$

In what follows, we summarize some classical and useful results.

**Theorem 1.** [10] Let $u^0 \in \dot{H}^{1/2}(\mathbb{R}^3)$ be a divergence-free vectors field. There exists $T > 0$ and a unique solution $u \in C([0, T], \dot{H}^{1/2}(\mathbb{R}^3)) \cap L^2([0, T], \dot{H}^{3/2}(\mathbb{R}^3))$. Moreover, if $\|u^0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \leq \nu$ we have $u \in C_0(\mathbb{R}^+, \dot{H}^{1/2}) \cap L^2(\mathbb{R}^+, \dot{H}^{3/2}).$

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In this work, we investigate some effects of the elliptic operator $-\nu \Delta$ on the solution, we prove that if $u^0 \in H^{1/2}$, then $u \in C([0,T]\cap H^s)$, for all $s \in \mathbb{R}$, and we present some asymptotic behavior near 0 and $+\infty$ for the global solutions. The first result in this direction is due to J.-Y. Chemin in [9]. Precisely

**Theorem 1.2.** [11] If $u^0 \in H^s(\mathbb{R}^3)$, with $s > 5/2$, a divergence-free vectors field, then there exists a time $T > 0$ and a strong solution $u$ of $(NS\nu)$ in $C([0,T];H^s) \cap C^1([0,T];H^{s-2})$.

With similar hypothesis, we have the blow-up result for the maximal solutions.

**Theorem 1.3.** [11][13] Let $s > 5/2$ and $u \in C([0,T];H^s)$ a solution of $(NS\nu)$ with $u \in C([0,T^*];H^s)$. Then

$$\int_0^{T^*} \parallel \omega(t) \parallel_{L^\infty} dt = +\infty,$$

where $\omega = \text{curl}u \simeq \frac{1}{2}(\nabla u - \frac{1}{t} \nabla u)$.

Our main results are the following

**Theorem 1.4.** [9] Let $u^0 \in H^{1/2}(\mathbb{R}^3)$ be a divergence-free vector field. There exists a time $T > 0$ and a unique solution $u$ to $(NS\nu)$ satisfying

$$\forall 0 \leq t \leq T, \int_0^t \int_{\mathbb{R}^3} |\xi|^2 e^{(\nu t)^{1/2}|\xi|} |\hat{u}(\tau,\xi)|^2 d\tau d\xi \leq 4 \parallel \nabla e^{\nu t} \Delta u^0 \parallel^2_{L^2([0,t]\times \mathbb{R}^3)}.$$

Moreover, a constant $c$ exist such that, if $u^0$ satisfy $\parallel u^0 \parallel_{H^{1/2}} < cu$, then

$$\int_{\mathbb{R}^+ \times \mathbb{R}^3} |\xi|^2 e^{(\nu t)^{1/2}|\xi|} |\hat{u}(t,\xi)|^2 d\nu d\xi \leq \frac{4}{\nu} \parallel u^0 \parallel^2_{L^2(\mathbb{R}^3)}.$$

Our main results are the following

**Theorem 1.5.** (Small initial data) Let $u^0 \in H^{1/2}(\mathbb{R}^3)$ a divergence-free vector field, such that $\parallel u^0 \parallel_{H^{1/2}} \leq cu$, then there exists a unique $u_\nu \in C_b(\mathbb{R}^+,H^{1/2}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+,H^{3/2}(\mathbb{R}^3))$.

Moreover, for all $T > 0$, there exists $\varepsilon = \varepsilon(T,\nu,u^0) > 0$, such that

$$\forall 0 \leq t \leq T, \int_{\mathbb{R}^3} |\xi|^2 e^{\nu t |\xi|} |\hat{u}(t,\xi)|^2 d\xi + \nu \int_0^t \int_{\mathbb{R}^3} |\xi|^2 e^{\nu t |\xi|} |\hat{u}(t,\xi)|^2 d\tau d\xi \leq 2 \parallel u^0 \parallel^2_{H^{1/2}}.$$

In addition, we have

\begin{align*}
(\alpha_1) \forall s \in \mathbb{R}, \ u_\nu \in C(\mathbb{R}^+,H^s), \\
(\alpha_2) \forall s > 1/2, \ t^{s-\frac{1}{2}} \parallel u_\nu(t) \parallel_{H^s} \leq C_s, \ t \to +\infty, \\
(\alpha_3) \forall 0 < q < 1/8, \ || u_\nu(t) ||_{H^{1/2}} \leq C_q t^{-q}, \ t \to +\infty, \\
(\alpha_4) \forall s > 1/2, \ || u_\nu(t) ||_{H^s} \leq C_s t^{-(s-\frac{1}{2})/2}, \ t \to +\infty.
\end{align*}

**Remark 1.1.** The property $(\alpha_1)$ imply that $\forall t > 0, \ u_\nu(t,.) \in C^\infty(\mathbb{R}^3)$.

**Theorem 1.6.** (Large initial data) For all vector field $u^0 \in H^{1/2}(\mathbb{R}^3)$ divergence-free, there exists $T > 0$ and a unique $u_\nu \in C([0,T],H^{1/2}(\mathbb{R}^3)) \cap L^2([0,T],H^{3/2}(\mathbb{R}^3))$. Moreover, there exists $\varepsilon = \varepsilon(T,\nu,u^0) > 0$, such that

$$\forall 0 \leq t \leq T, \int_{\mathbb{R}^3} |\xi|^2 e^{\nu t |\xi|} |\hat{u}(t,\xi)|^2 d\xi + \nu \int_0^t \int_{\mathbb{R}^3} |\xi|^2 e^{\nu t |\xi|} |\hat{u}(t,\xi)|^2 d\tau d\xi \leq 2 \parallel u^0 \parallel^2_{H^{1/2}}.$$
In addition, we have

\[(\beta_1) \forall s \in \mathbb{R}, \ u_\nu \in \mathcal{C}((0, T], H^s),\]

\[(\beta_2) \forall s > 1/2, \ \|u_\nu(t)\|_{H^s} \leq \frac{C}{t^{s/2}}, \ t \to 0^+,\]

\[(\beta_3) \forall 0 < t \leq T, \ u_\nu(t, \cdot) \in \mathcal{C}^\infty(\mathbb{R}^3).\]

The rest of this paper is organized as follows. Section 2 contains some notations and definitions. In Section 3, we prove Theorem 1.5. The proof is based on Friedrich methods, classical product law in the homogeneous Sobolev spaces, classical compactness methods and elementary technical results. Section 4 is devoted to prove Theorem 1.6. The proof is inspired from the previews section. In last Section we derive some general proprieties of strong solutions of Navier-Stokes equations (\(NS_\nu\)).

2. Notations

In this short section we collect some notations and definitions that will be used later on.

- The Fourier transformation is normalized as
  \[\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^3} \exp(-ix.\xi)f(x)dx, \ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.\]

- The inverse Fourier formula is
  \[\mathcal{F}^{-1}(g)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \exp(i\xi.x)f(\xi)d\xi, \ x = (x_1, x_2, x_3) \in \mathbb{R}^3.\]

- For \(s \in \mathbb{R}\), \(H^s(\mathbb{R}^3)\) denotes the usual non homogeneous Sobolev space on \(\mathbb{R}^3\) and \(<\cdot, \cdot>_{H^s(\mathbb{R}^3)}\) denotes the usual scalar product on \(H^s(\mathbb{R}^3)\).

- For \(s \in \mathbb{R}\), \(\dot{H}^s(\mathbb{R}^3)\) denotes the usual homogeneous Sobolev space on \(\mathbb{R}^3\) and \(<\cdot, \cdot>_{\dot{H}^s(\mathbb{R}^3)}\) denotes the usual scalar product on \(\dot{H}^s(\mathbb{R}^3)\).

- The convolution product of a suitable pair of functions \(f\) and \(g\) on \(\mathbb{R}^3\) is given by
  \[(f \ast g)(x) := \int_{\mathbb{R}^3} f(y)g(x - y)dy.\]

- For any Banach space \((B, \|\cdot\|)\), any real number \(1 \leq p \leq \infty\) and any time \(T > 0\), we will denote by \(L^p_T(B)\) the space of all measurable functions
  \[t \in [0, T] \rightarrow f(t) \in B\]
such that
  \[(t \rightarrow \|f(t)\|) \in L^p([0, T]).\]

- If \(f = (f_1, f_2, f_3)\) and \(g = (g_1, g_2, g_3)\) are two vector fields, we set
  \[f \otimes g := (g_1f, g_2f, g_3f),\]
  and
  \[\text{div}(f \otimes g) := (\text{div}(g_1f), \text{div}(g_2f), \text{div}(g_3f)).\]
• For any subset $X$ of a set $E$, the symbol $1_X$ denote the characteristic function of $X$
defined by

$$1_X(x) = 1 \text{ if } x \in X, \quad 1_X(x) = 0 \text{ elsewhere.}$$

3. Proof of Theorem [1.5]

We begin by recalling a fundamental lemma concerning some product laws in Sobolev
spaces.

**Lemma 3.1.** (see [8]) Let $s, s'$ tow reals numbers such that

$$s < 3/2 \quad \text{and} \quad s + s' > 0.$$ 

There exists a positive constant $C := C(s, s')$, such that for all $f, g \in \dot{H}^s(\mathbb{R}^3) \cap \dot{H}^{s'}(\mathbb{R}^3)$,

$$\|fg\|_{\dot{H}^{s+s'-\frac{3}{2}}(\mathbb{R}^3)} \leq C \left(\|f\|_{\dot{H}^s(\mathbb{R}^3)}\|g\|_{\dot{H}^{s'}(\mathbb{R}^3)} + \|f\|_{\dot{H}^{s'}(\mathbb{R}^3)}\|g\|_{\dot{H}^s(\mathbb{R}^3)}\right).$$

If $s, s' < 3/2$ and $s + s' > 0$, there exist a constant $c = c(s, s')$,

$$\|fg\|_{\dot{H}^{s+s'-\frac{3}{2}}(\mathbb{R}^3)} \leq c\|f\|_{\dot{H}^s(\mathbb{R}^3)}\|g\|_{\dot{H}^{s'}(\mathbb{R}^3)}.$$

For a strictly positive integer $n$, the Friedrich’s operator $J_n$ is defined by

$$J_n(f) := \mathcal{F}^{-1}\left(1_{\{|\xi|<n\}}\mathcal{F}(f)\right).$$

Consider the following approximate Navier-Stokes system $\left(NS_{n,\nu}\right)$ on $\mathbb{R}_+ \times \mathbb{R}^3$,

$$\begin{cases}
\partial_t u - \nu \Delta J_n u + J_n \text{div}(J_n u \otimes J_n u) = \nabla \Delta^{-1} J_n \text{div}(J_n u \otimes J_n u), \\
u|t=0 = J_n u_0.
\end{cases}$$

Then by the ordinary differential equations theory the system $\left(NS_{n,\nu}\right)$ has a unique maximal solution $u_{n,\nu}$ in the space $C^1([0, T^*_n), L^2(\mathbb{R}^3))$. Using the uniqueness and the fact $J_{n}^2 = J_{n}$ we obtain

$$\begin{cases}
J_n u_{n,\nu} = u_{n,\nu} \\
\text{div}(u_{n,\nu}(t)) = 0, \quad \forall t \in [0, T^*_n),
\end{cases}$$

hence $u_{n,\nu}$ satisfies

$$\begin{cases}
\partial_t u_{n,\nu} - \nu \Delta u_{n,\nu} + J_n \text{div}(u_{n,\nu} \otimes u_{n,\nu}) = \nabla \Delta^{-1} J_n \text{div}(u_{n,\nu} \otimes u), \\
\nu|t=0 = J_n u_0.
\end{cases}$$

Taking the scalar product in $L^2(\mathbb{R}^3)$, we obtain, for $t \in [0, T^*_n)$,

$$\partial_t \|u_{n,\nu}\|_{L^2}^2 + 2\nu \|\nabla u_{n,\nu}\|_{L^2}^2 \leq 0,$$

then it follows for all $t \in [0, T^*_n)$,

$$\|u_{n,\nu}(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2,$$

which implies $T^*_n = +\infty$, and the following estimate holds for all $t \geq 0$

$$\|u_{n,\nu}(t)\|_{L^2}^2 + 2\nu \|\nabla u_{n,\nu}\|_{L^2(L^2)}^2 \leq \|u_0\|_{L^2}^2. \quad (3.1)$$
Let $T > 0$ a fixed time, and let
\[(C_1) \quad \varepsilon := \min \left( \frac{1}{2}, c\nu, (c\nu)^2, CT^{-1/2}, CT^{-1/3} \right),\]
with $C$ depending only of $\nu$, $u^0$.

Using Fourier transformation we obtain from the above system
\[(E_1) : \partial_t \hat{v}_{n,\nu,\varepsilon} + \nu|\xi||(|\xi| - \varepsilon)\hat{v}_{n,\nu,\varepsilon} + \varepsilon^{|\xi|\varepsilon} \mathcal{F} \left( J_n(u_{n,\nu}, \nabla u_{n,\nu}) \right) = \varepsilon^{|\xi|\varepsilon} \mathcal{F} \left( \nabla \Delta^{-1}J_n(u_{n,\nu}, \nabla u_{n,\nu}) \right),\]
where $v_{n,\nu,\varepsilon} := \mathcal{F}^{-1}\left( \varepsilon^{|\xi|\varepsilon} \hat{v}_{n,\nu,\varepsilon} \right)$. Take $E_1, \hat{v}_{n,\nu,\varepsilon} + E_1, \overline{\hat{v}_{n,\nu,\varepsilon}}$, we obtain
\[\partial_t|\hat{v}_{n,\nu,\varepsilon}|^2 + 2\nu|\xi||(|\xi| - \varepsilon)|\hat{v}_{n,\nu,\varepsilon}|^2 \leq -2\Re \left( \varepsilon^{|\xi|\varepsilon} \mathcal{F} \left( J_n(u_{n,\nu}, \nabla u_{n,\nu}) \right) \overline{\hat{v}_{n,\nu,\varepsilon}} \right).\]

Using the following elementary inequality
\[e^{a|\xi|} \leq e^{a[|\xi| - \eta]|\eta|}, \quad \forall a \in \mathbb{R}^+, \quad \forall \xi, \eta \in \mathbb{R}^3,\]
we obtain
\[(E_2) : \partial_t|\hat{v}_{n,\nu,\varepsilon}|^2 + 2\nu|\xi||(|\xi| - \varepsilon)|\hat{v}_{n,\nu,\varepsilon}|^2 \leq |\xi| \hat{v}_{n,\nu,\varepsilon} |\nabla \hat{v}_{n,\nu,\varepsilon}|.\]

To make good estimates, we decompose $v_{n,\nu,\varepsilon}$ as $v_{n,\nu,\varepsilon} = X_{n,\nu,\varepsilon} + Y_{n,\nu,\varepsilon}$, with
\[X_{n,\nu,\varepsilon} = \mathcal{F}^{-1}\left( 1_{\{|\xi| > 2\varepsilon\}} \hat{v}_{n,\nu,\varepsilon} \right),\]
\[Y_{n,\nu,\varepsilon} = \mathcal{F}^{-1}\left( 1_{\{|\xi| \leq 2\varepsilon\}} \hat{v}_{n,\nu,\varepsilon} \right).\]

**Estimate of** $\|X_{n,\nu,\varepsilon}\|^2_{\dot{H}^{1/2}}$: Multiply $(E_2)$ by $|\xi|$ and integrate over $\{|\xi| > 2\varepsilon\}$, we obtain
\[\partial_t\|X_{n,\nu,\varepsilon}\|^2_{\dot{H}^{1/2}} + \nu\|\nabla X_{n,\nu,\varepsilon}\|^2_{\dot{H}^{1/2}} \leq \|\xi\|_{L^2}\|\hat{v}_{n,\nu,\varepsilon}\| L^2\|\nabla X_{n,\nu,\varepsilon}\|_{\dot{H}^{1/2}} \leq \sum_{j=1}^{3} I_j,\]
where
\[I_1 = \|\xi\|_{L^2}\|\hat{X}_{n,\nu,\varepsilon}\| L^2\|\nabla X_{n,\nu,\varepsilon}\|_{\dot{H}^{1/2}},\]
\[I_2 = 2\|\xi\|_{L^2}\|\hat{X}_{n,\nu,\varepsilon}\| L^2\|\nabla X_{n,\nu,\varepsilon}\|_{\dot{H}^{1/2}},\]
\[I_3 = \|\xi\|_{L^2}\|\hat{Y}_{n,\nu,\varepsilon}\| L^2\|\nabla X_{n,\nu,\varepsilon}\|_{\dot{H}^{1/2}}.\]

Using now Lemma 3.1 we get
\[I_1 \leq C\|X_{n,\nu,\varepsilon}\|_{\dot{H}^{1/2}}\|\nabla X_{n,\nu,\varepsilon}\|^2_{\dot{H}^{1/2}}.\]
Once again, Lemma \[3.1\] combined together with the energy estimate \(3.1\), give
\[
I_2 \leq C\|X_{n,\nu,\varepsilon}\|_{H^{1/2}}\|\nabla Y_{n,\nu,\varepsilon}\|_{H^{1/2}}\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}} + C\|Y_{n,\nu,\varepsilon}\|_{H^{1/2}}\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2
\]
\[
\leq C\varepsilon^{3/2}\|X_{n,\nu,\varepsilon}\|_{L^2}\|X_{n,\nu,\varepsilon}\|_{H^{1/2}}\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}} + C\varepsilon^{1/2}\|Y_{n,\nu,\varepsilon}\|_{L^2}\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2
\]
\[
\leq C\varepsilon^{3/4}\|u^0\|_{L^2}\|X_{n,\nu,\varepsilon}\|_{H^{1/2}}\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}} + C\varepsilon^{1/2}\|u^0\|_{L^2}\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2
\]
\[
\leq C\varepsilon^{3}\|u^0\|_{L^2}\|X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 + \left(\frac{\nu}{20} + C\varepsilon^{1/2}\|u^0\|_{L^2}\right)\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2
\]
\[
\leq \frac{1}{10}\|X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 + \frac{\nu}{10}\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2.
\]

Similarly, we obtain
\[
I_3 \leq C\varepsilon^2\|Y_{n,\nu,\varepsilon}\|_{H^{1/2}}\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}
\]
\[
\leq C\varepsilon^2\|u^0\|_{L^2}^2\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}
\]
\[
\leq C\varepsilon^4\|u^0\|_{L^2}^4 + \frac{\nu}{10}\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2
\]
\[
\leq \frac{\|u^0\|_{H^{1/2}}^2}{10T} + \frac{\nu}{10}\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2,
\]

which leads to
\[
\partial_t\|X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 + \nu\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 \leq C\|X_{n,\nu,\varepsilon}\|_{H^{1/2}}\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2
\]
\[
+ C_2\varepsilon^3\nu^{-1}\|X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 + (C_2\varepsilon^{1/2} + \frac{\nu}{10})\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2
\]
\[
+ C_2\varepsilon^2,
\]

and hence
\[
\partial_t\|X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 + \frac{\nu}{2}\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 \leq C\|X_{n,\nu,\varepsilon}\|_{H^{1/2}}\|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 + C_2\varepsilon^3\nu^{-1}\|X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 + C_2\varepsilon^2.
\]

Let a time \(T_{n,\nu,\varepsilon}\) define by
\[
T_{n,\nu,\varepsilon} := \sup\{t \geq 0, \|X_{n,\nu,\varepsilon}\|_{L^\infty_t(H^{1/2})} < 2c\nu\}.
\]

For \(0 \leq t < \min(T, T_{n,\nu,\varepsilon})\), we have
\[
\|X_{n,\nu,\varepsilon}(t)\|_{H^{1/2}}^2 + \frac{\nu}{4} \int_0^t \|\nabla X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 \leq \|X_{n,\nu,\varepsilon}(0)\|_{H^{1/2}}^2 + C\varepsilon^2t + C_2\varepsilon^3\nu^{-1} \int_0^t \|X_{n,\nu,\varepsilon}\|_{H^{1/2}}^2,
\]

and by Gronwall’s lemma we get
\[
\|X_{n,\nu,\varepsilon}\|_{L^\infty_t(H^{1/2})}^2 \leq (\|u^0\|_{H^{1/2}}^2 + C\varepsilon^2T)e^{C\varepsilon^3\nu^{-1}T}
\]
\[
\leq \frac{3}{2}\|u^0\|_{H^{1/2}}^2 < (2c\nu)^2,
\]
that is \( T_{n,\nu,\varepsilon} > T \), and for all \( 0 \leq t \leq T \),

\[
(3.2) \quad \|X_{n,\nu,\varepsilon}\|_{L^2(L^2(H^{1/2}))}^2 + \frac{\nu}{4}\|\nabla X_{n,\nu,\varepsilon}\|_{L^2(H^{1/2})}^2 \leq \frac{3}{2}\|u^0\|_{H^{1/2}}^2.
\]

**Estimate of \( \|Y_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 \):** To estimate \( \|Y_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 \) we integrate \((E_2)\) over \( \{|\xi| < 2\varepsilon\} \) to obtain

\[
\partial_t \|Y_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 + \nu\|\nabla Y_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 \leq 2\|\xi\|^{1/2}|\hat{u}_{n,\nu}| * |\hat{u}_{n,\nu,\varepsilon}| \|L^2(B(0,2\varepsilon))\|\nabla Y_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 \\
\leq C\left(\int_{B(0,2\varepsilon)}|\xi|\right)^{1/2}d\xi \|\hat{u}_{n,\nu}| * |\hat{u}_{n,\nu,\varepsilon}| \|L^\infty\|\nabla Y_{n,\nu,\varepsilon}\|_{H^{1/2}},
\]

and by Young inequality it follows

\[
\partial_t \|Y_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 + 2\nu\|\nabla Y_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 \leq C\varepsilon^2 \|\hat{u}_{n,\nu}\|_{L^2}^2 \|\nabla Y_{n,\nu,\varepsilon}\|_{H^{1/2}} \\
\leq C\|u^0\|_{L^2}^2 \varepsilon^2 \|\nabla Y_{n,\nu,\varepsilon}\|_{H^{1/2}} \\
\leq C\nu^{-1}\|u^0\|_{L^2}^4 + \nu\|\nabla Y_{n,\nu,\varepsilon}\|_{H^{1/2}}^2.
\]

An easy computation shows that for all \( t \in [0, T] \)

\[
\|Y_{n,\nu,\varepsilon}(t)\|_{H^{1/2}}^2 + \nu\|\nabla Y_{n,\nu,\varepsilon}\|_{L^\infty(H^{1/2})}^2 \leq \|Y_{n,\nu,\varepsilon}(0)\|_{H^{1/2}}^2 + C\varepsilon^4 t \\
\leq \varepsilon\|u_0\|_{L^2}^2 + C\varepsilon^4 T \\
\leq \frac{1}{4}\|u_0\|_{L^2}^2.
\]

Thanks to equations \((3.2)-(3.3)\) we obtain for all \( t \in [0, T] \),

\[
\|v_{n,\nu,\varepsilon}(t)\|_{H^{1/2}}^2 + \nu \int_0^t \|\nabla v_{n,\nu,\varepsilon}\|_{H^{1/2}}^2 \leq 2\|u^0\|_{H^{1/2}}^2.
\]

Finally, a standard compactness argument gives the global existence result, precisely:

There exists \( u_\nu \in C_0(\mathbb{R}^+, H^{1/2}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, H^{3/2}(\mathbb{R}^3)) \) of \((NS_\nu)\), such that for all \( T > 0 \), and \( \varepsilon = \varepsilon_T := \min(1/2, c\nu, (c\nu)^2, CT^{-1/2}, CT^{-1/3}) \) we have

\[
(3.4) \quad \forall t \in [0, T], \int_\varepsilon e^{2\pi r|\xi|} |\xi| |\hat{u}_\nu(t, \xi)|^2 d\xi + \nu \int_0^t \int_\varepsilon e^{2\pi r|\xi|} |\xi|^3 |\hat{u}_\nu(\tau, \xi)|^2 d\xi d\tau \leq 2\|u^0\|_{H^{1/2}}^2.
\]

The equation \((3.1)\) yields

\[
(3.5) \quad \forall t \geq 0, \quad \|u_\nu(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u_\nu\|_{L^2}^2 \leq \|u^0\|_{L^2}^2,
\]

which implies \((a_1) - (a_2)\).
We introduce the real number $\varepsilon$ defined by
\[ \varepsilon := \min \left( \frac{1}{2}, \frac{\nu}{3} \right). \]

For large enough time $T$, we have $\varepsilon_T = CT^{-1/2}$ and choosing $a = T^{-r}$ with $0 < r < \frac{1}{4}$, it follows
\[ \|u_\nu(T)\|_{H^{1/2}} \leq \frac{\|u_0\|_{L^2}^2}{T^{r/2}} + e^{-CT^{1/2}} \sqrt{3}\|u_0\|_{H^{1/2}}. \]
as desired.

The case of $(\alpha_4)$, follows as well by choosing $a := \frac{C}{T^{r/2}}$. The proof of theorem 1.5 is completed. \qed

4. PROOF OF THEOREM 1.6

We apply the Friedrich’s method’s, we obtain the existence and uniqueness of solution $u_{n,\nu} \in C^1(\mathbb{R}^+, L^2(\mathbb{R}^3))$ of the following system
\[
\begin{aligned}
&\partial_t u - \nu \Delta u + J_\nu \text{div} (u \otimes u) = \nabla \Delta^{-1} J_\nu \text{div} (u \otimes u), \\
&u|_{t=0} = J_\nu u_0.
\end{aligned}
\]

For simplification we don’t well noting the index $n$.

Let $N \in \mathbb{N}$, such that
\[ \|\mathcal{F}^{-1}(1_{\{|x|<2^{-N}\}}) \mathcal{F}(u_0)\|_{H^{1/2}} < \min(\nu, \nu(2^{-3/2})^2). \]

And define $T, u_0^N, v_{N,L}, w_{N,\nu}, u_{n,\nu}$, by
\[
(C_2) \quad T = T(\nu, u_0) := -\nu^{-1}2^{-N} \log \left( 1 - \min(1/2, \nu \|u_0\|_{H^{1/2}}^{-4} \min(\nu, (\nu(2^{-3/2}))^2)) \right) > 0.
\]
\[
\begin{aligned}
&u_0^N : = \mathcal{F}^{-1}(1_{\{|x|<2^{-N}\}}) \mathcal{F}(u_0), \\
v_{N,L} : = e^{\nu L} u_0^N, \\
w_{N,\nu} : = u_{n,\nu} - v_{N,L}.
\end{aligned}
\]

We have
\[ \partial_t w_{N,\nu} - \nu \Delta w_{N,\nu} + w_{N,\nu} \cdot \nabla w_{N,\nu} + v_{N,L} \cdot \nabla w_{n,\nu} + w_{n,\nu} \cdot \nabla v_{N,L} = -v_{N,L} \cdot \nabla v_{N,L}. \]

We introduce the real number $\varepsilon$ defined by
\[ (C_3) \quad \varepsilon := \min \left( \frac{1}{2}, \frac{\nu}{3} (10CT)^{-1/3}, \min(\nu, (\nu(2^{-3/2}))^2 \right)^{1/4} (10CT)^{-1/4} > 0, \]
and
\[ V_{N,L} = e^{\nu t}|D| \varphi_{N,L} \]
\[ U_\nu = e^{\nu t}|D| u_\nu \]
\[ W_{N,\nu} = e^{\nu t}|D| w_{N,\nu} \]
\[ \alpha_{N,\nu,\varepsilon} = 1_{\{|D| > 2\varepsilon\}} W_{N,\nu} \]
\[ \beta_{N,\nu,\varepsilon} = 1_{\{|D| \leq 2\varepsilon\}} W_{N,\nu}. \]

Arguing as in the last section, we obtain
\[ \partial_t \|\alpha_{N,\nu,\varepsilon}\|^2_{H^{1/2}} + \nu \|\nabla \alpha_{N,\nu,\varepsilon}\|^2_{H^{1/2}} \leq \sum_{j=1}^3 K_j, \]
with
\[ K_1 = \int_\xi |\nabla_{N,L}| * |\nabla \alpha_{N,\nu,\varepsilon}|, \]
\[ K_2 = \int_\xi \frac{1}{2} \left( |\nabla_{N,L}| * |\nabla \nabla_{N,L}| + |\nabla \nabla_{N,L}| * |\nabla_{N,L}| \right) \|\alpha_{N,\nu,\varepsilon}\|^2, \]
\[ K_3 = \int_\xi \frac{1}{2} \left( |\nabla_{N,L}| * |\nabla \nabla_{N,L}| \right) \|\alpha_{N,\nu,\varepsilon}\|^2. \]

Using the Lemma 3.1, we obtain
\[ K_1 \leq C \|W_{N,\nu}\|_{H^{1/2}} \|\nabla W_{N,\nu}\|_{H^{1/2}} \|\nabla \alpha_{N,\nu,\varepsilon}\|_{H^{1/2}}, \]
\[ K_2 \leq C \|\nabla V_{N,L}\|_{L^2} \|\nabla W_{N,\nu}\|_{L^2} \|\nabla \alpha_{N,\nu,\varepsilon}\|_{H^{1/2}}, \]
\[ K_3 \leq C \|\nabla V_{N,L}\|_{L^2}^2 \|\nabla \alpha_{N,\nu,\varepsilon}\|_{H^{1/2}}. \]

**Estimate of \( K_1 \):** We have
\[ \|W_{N,\nu}\|_{H^{1/2}} \leq \|\alpha_{N,\nu}\|_{H^{1/2}} + \|\beta_{N,\nu}\|_{H^{1/2}} \leq \|\alpha_{N,\nu}\|_{H^{1/2}} + C\varepsilon^{1/2} \]
\[ \|\nabla W_{N,\nu}\|_{H^{1/2}} \leq \|\nabla \alpha_{N,\nu}\|_{H^{1/2}} + \|\nabla \beta_{N,\nu}\|_{H^{1/2}} \leq \|\nabla \alpha_{N,\nu}\|_{H^{1/2}} + C\varepsilon^{3/2}, \]
then
\[ K_1 \leq C\varepsilon^4 + C\varepsilon^3 \|\alpha_{N,\nu}\|^2_{H^{1/2}} + \left( C\varepsilon^{1/2} + \frac{\nu}{10} + C \|\alpha_{N,\nu}\|_{H^{1/2}} \right) \|\nabla \alpha_{N,\nu}\|^2_{H^{1/2}}. \]

**Estimate of \( K_2 \):** Similarly, we obtain
\[ K_2 \leq \|\nabla V_{N,L}\|^4_{H^{1/2}} + C\varepsilon^4 + C\varepsilon^3 \|\alpha_{N,\nu}\|^2_{H^{1/2}} + \left( C\varepsilon^{1/3} + \frac{\nu}{10} + C \|\alpha_{N,\nu}\|^2_{H^{1/2}} \right) \|\nabla \alpha_{N,\nu}\|^2_{H^{1/2}}. \]

**Estimate of \( K_3 \):** We have
\[ K_3 \leq C \|\nabla V_{N,L}\|^4_{L^2} + \frac{\nu}{10} \|\nabla \alpha_{N,\nu}\|^2_{H^{1/2}}. \]

Let
\[ T^* = \sup \{ t > 0, \|\alpha_{N,\nu}\|_{L^\infty(H^{1/2})} < 2 \min(c\nu, c\nu^{3/2}) \}. \]
For $0 \leq t < \min(T, T^*)$, we have
\[
\|\alpha_{N,\nu,\epsilon}\|^2_{L_t^\infty(\dot{H}_{1/2}^2)} + \frac{\nu}{2}\|\nabla\alpha_{N,\nu,\epsilon}\|^2_{L_t^2(\dot{H}_{1/2}^1)} \leq \|\alpha_{N,\nu}(0)\|^2_{\dot{H}_{1/2}^1} + C\epsilon^3 T + C\|\nabla V_N L\|_{L_t^1(L^2)}^4 + C\epsilon^3 T\|\alpha_{N,\nu,\epsilon}\|^2_{L_t^\infty(\dot{H}_{1/2}^1)}.
\]

By inequalities (4.7), (4.8), (4.9) and the choices $(C_2), (C_3)$, we obtain
\[
\forall t \in (0, \min(T, T^*)) , \quad \|\alpha_{N,\nu,\epsilon}\|^2_{L_t^\infty(\dot{H}_{1/2}^2)} + \nu\|\nabla\alpha_{N,\nu,\epsilon}\|^2_{L_t^2(\dot{H}_{1/2}^1)} \leq 2\min(c\nu, c\nu^{3/2})^2.
\]

Then $T^* > T$, in particular
\[
\|\alpha_{N,\nu,\epsilon}\|^2_{L_t^\infty(\dot{H}_{1/2}^2)} + \nu\|\nabla\alpha_{N,\nu,\epsilon}\|^2_{L_t^2(\dot{H}_{1/2}^1)} \leq 2\min(c\nu, c\nu^{3/2})^2,
\]
\[
\|\alpha_{N,\nu}\|^2_{L_t^\infty(\dot{H}_{1/2}^2)} + \nu\|\nabla\alpha_{N,\nu}\|^2_{L_t^2(\dot{H}_{1/2}^1)} \leq 3\min(c\nu, c\nu^{3/2})^2,
\]
and we can deduce
\[
\|U_\nu\|^2_{L_t^\infty(\dot{H}_{1/2}^2)} + \nu\|\nabla U_\nu\|^2_{L_t^2(\dot{H}_{1/2}^1)} \leq 2\|u_0\|^2_{\dot{H}_{1/2}^1}.
\]
Finally, a standard compactness argument gives the local existence result. Moreover the solution satisfies $(\beta_1) - (\beta_2)$. This achieved the proof of Theorem 1.6. ■

5. General Properties Of Strong Solutions

This section combines the previous results and Theorems 1.2, 1.3 to derive some properties of any strong solutions of Navier-Stokes equations. The precise statements are the following.

**Theorem 5.1.** If $u \in C([0, T_0], H^{1/2}) \cap L^2([0, T_0], H^{3/2})$ is a solution of $(NS_\nu)$, then

$(\beta_1') \forall s \in \mathbb{R}, \quad u \in C([0, T_0], H^s),$

$(\beta_2') \forall s > 1/2, \quad t^{s-\frac{1}{2}}\|u(t)\|_{H^s} \leq C_s, \quad t \to 0^+$,

$(\beta_3') \forall t \in ]0, T_0[, \quad u(t, \cdot) \in C^\infty(\mathbb{R}^3).$

**Theorem 5.2.** If $u \in C(\mathbb{R}^+, H^{1/2}) \cap L^2(\mathbb{R}^+, H^{3/2})$ is a solution of $(NS_\nu)$, then $u \in L^\infty(\mathbb{R}^+, H^{1/2})$, and

$(\alpha_1') \forall s \in \mathbb{R}, \quad u \in C(\mathbb{R}^+, H^s),$

$(\alpha_2') \forall s > 1/2, \quad t^{s-\frac{1}{2}}\|u(t)\|_{H^s} \leq C_s, \quad t \to 0^+$,

$(\alpha_3') \forall 0 < q < 1/8, \quad \|u(t)\|_{H^{1/2}} \leq C_q t^{-q}, \quad t \to +\infty,$

$(\alpha_4') \forall s > 1/2, \quad \|u(t)\|_{H^s} \leq C_s t^{-(s-\frac{1}{2})/2}, \quad t \to +\infty.$

**Remark 5.1.** $(\beta_3')$ is an easy consequence of $(\beta_1')$. 

5.1. Proof of Theorem 5.1 Using Theorem 1.5, there exists $T > 0$ (suppose that $T < T_0$), and $v_1 \in C([0, T], H^{1/2}) \cap \mathcal{L}^2_f(H^{3/2})$, satisfying $(\beta_1)$. By the uniqueness, we have $v_1 = u$ on $[0, T]$. Let $s > 5/2$, and we consider the following system

\[
\begin{aligned}
&\begin{cases}
\begin{aligned}
\partial_t v - \nu \Delta v + v.\nabla v &= -\nabla p, &\text{on } \mathbb{R}_+ \times \mathbb{R}^3, \\
\text{div } v &= 0 &\text{on } \mathbb{R}_+ \times \mathbb{R}^3,
\end{aligned}
\end{cases}
\end{aligned}
\]

\[ (NS_{\nu,T}) \]

By Theorem 1.3 there exists a unique $v \in C([0, T^*), H^s)$ solution of $(NS_{\nu,T})$, satisfying

\[ T^* < \infty \implies \int_0^{T^*} \|v\|_{\dot{H}^s}^2 = +\infty. \]

Suppose that $T^* \leq T_0 - \frac{T}{2}$. By uniqueness we have

\[ \forall t \in [T/2, T^*), \quad u(t) = v(t - \frac{T}{2}). \]

Taking the scalar product in $\dot{H}^s$, and using lemma 5.1, we obtain \( \forall t \in [0, T^*) \)

\[
\begin{aligned}
\partial_t \|v(t)\|_{\dot{H}^s}^2 + 2\nu \|\nabla v(t)\|_{L^2}^2 &\leq C \|\nabla v(t)\|_{L^2} \|v(t)\|_{\dot{H}^s} \|\nabla v(t)\|_{\dot{H}^s}^{3/2} \\
&\leq C \|\nabla v(t)\|_{L^2}^2 \|v(t)\|_{\dot{H}^s}^2 + \nu \|\nabla v(t)\|_{\dot{H}^s}^2.
\end{aligned}
\]

By Gronwall’s lemma

\[
\begin{aligned}
\|v(t)\|_{\dot{H}^s}^2 &\leq \|v(0)\|_{\dot{H}^s}^2 \exp \left( C \int_0^t \|\nabla v(t)\|_{L^2}^2 \right) \\
&\leq \|u(T/2)\|_{\dot{H}^s}^2 \exp \left( C \int_{T/2}^{t+\frac{T}{2}} \|\nabla u\|_{L^2}^2 \right) \\
&\leq \|u(T/2)\|_{\dot{H}^s}^2 \exp \left( C \|u\|_{L^\infty(\mathcal{L}^{1/2})} \|u\|_{L^2(\mathcal{L}^{3/2})}^2 \right).
\end{aligned}
\]

Then $T^* > T_0 - \frac{T}{2}$, we obtain $(\beta_1')$ consequently $(\beta_2')$.

Combines the first step and Theorems 1.5 we obtain $(\beta_2')$. This completes the proof. 

5.2. Proof of Theorem 5.2 The proprieties $(\alpha_1')$ and $(\alpha_2')$ are an easy consequences of Theorem 5.1.

Proof of $(\alpha_3') - (\alpha_4')$: If we prove the existence of a time $T \geq 0$ such that $\|u(T)\|_{\dot{H}^{1/2}} = c \nu$, we can apply Theorem 1.5 on $[T, +\infty)$, by the uniqueness we obtain the desired results. Then, for simplification, we begin by proving the following assertion.

\[ \forall t \geq 0, \quad \|u\|_{L^\infty(\mathcal{L}^2)} \leq \|u^0\|_{\mathcal{L}^2}. \]

Let $t^* := \sup\{t \geq 0, \quad \|u\|_{L^\infty(\mathcal{L}^2)} \leq \|u^0\|_{\mathcal{L}^2} \} \in [0, +\infty]$. Suppose that $t^* < \infty$, by continuity of $u$ show that $\|u(t^*)\|_{\mathcal{L}^2} = \|u^0\|_{\mathcal{L}^2}$. Applying Theorem 1.5 to the following
we integrate on \( R \) completes the proof.

Using Hölder inequality, we infer \( \alpha \) satisfying (5.11)

\[
\exists t \in [0, T_1], \quad \|v(t)\|_{L^2}^2 + 2\nu \|
abla v\|_{L^2(L^2)}^2 \leq \|v(0)\|_{L^2}^2 = \|u^0\|_{L^2}^2.
\]

Using the uniqueness, we obtain

\[
u(t) = v(t - t^*), \quad \forall t \in [t^*, t^* + T_1],
\]

then

\[
\|u(t)\|_{L^2} \leq \|u^0\|_{L^2}, \quad \forall t \in [0, t^* + T_1].
\]

Hence \( t^* = +\infty \) and the assertion (5.10) is proved.

Now, we have to prove

(5.11)

\[
\exists t \geq 0 \quad \text{s.t.} \quad \|u(t)\|_{H^{1/2}} \leq cv.
\]

Let

\[
A := \{ t \geq 0, \quad \|u(t)\|_{H^{1/2}} > cv \}.
\]

Using Hölder inequality, we infer

\[
\forall t \geq 0, \quad (cv)^6 1_A(t) \leq \|u(t)\|_{H^{1/2}}^6 \leq \|u(t)\|_{L^2}^4 \|u(t)\|_{H^{1/2}}^2 \leq \|u^0\|_{L^2}^4 \|u(t)\|_{H^{1/2}}^2.
\]

We integrate on \( R^+ \), we obtain

\[
\lambda_1(A) \leq \frac{\|u^0\|_{L^2}^4 \|u\|_{L^2(H^{1/2})}^2}{(cv)^6} := t_0 < +\infty,
\]

where \( \lambda_1 \) is the Lebesgue measure on \( R \).

Then, for \( \mu > 0 \), there exist \( t_\mu \in (0, t_0 + \mu) \), such that \( \|u(t_\mu)\|_{H^{1/2}} \leq cv \).

Now, we consider the following system

\[
\begin{aligned}
\partial_t v - \nu \Delta v + v \nabla v &= -\nabla p, \quad \text{on} \quad R^+ \times \mathbb{R}^3, \\
div (v) &= 0 \quad \text{on} \quad R^+ \times \mathbb{R}^3, \\
v_{|t=0} &= u(t_1) \quad \text{on} \quad \mathbb{R}^3.
\end{aligned}
\]

(\( NS^e_{t_1} \))

By Theorem [1,5] there exists a unique \( v_1 \in C_0([0, T_1], H^{1/2}) \cap L^2([0, T_1], \dot{H}^{3/2}) \), solution of (\( NS^e_{t_1} \)) satisfying \((\alpha_3) - (\alpha_4)\). The uniqueness imply \( u(t) = v_1(t - t_1) \) for all \( t \in [t_1, +\infty) \). This completes the proof. 

\[ \blacksquare \]
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