Light Nuclei as Quantized Skyrmions

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Abstract

We consider the rigid body quantization of Skyrmions with topological charges 1 to 8, as approximated by the rational map ansatz. Novel, general expressions for the elements of the inertia tensors, in terms of the approximating rational map, are presented and are used to determine the kinetic energy contribution to the total energy of the ground and excited states of the quantized Skyrmions. Our results are compared to the experimentally determined energy levels of the corresponding nuclei, and the energies and spins of a few as yet unobserved states are predicted.

1 Introduction

The Skyrme model [24] is a nonlinear effective theory of mesons, specifically pions. Its nonlinearity allows for the existence of topological soliton solutions, labelled by an integer-valued topological charge, $B$. A quantized Skyrmion of topological charge $B$ is interpreted as a nucleus with baryon number $B$.

The $B = 1$ Skyrmion was first quantized by Adkins, Nappi and Witten [2, 1]. They provided the first calibration of the Skyrme model, by fitting the model to the proton and delta masses. The toroidal $B = 2$ Skyrmion was quantized in [16, 7], and the energies corresponding to the ground state, representing the deuteron, and excited states were calculated. This analysis was extended in [20], to allow the two single Skyrmions to separate in the most attractive channel. This led to a more accurate determination of the mean charge radius, as the deuteron is rather loosely bound.

The interpretation of the nuclei helium-3 and hydrogen-3 (triton) as quantized states of the $B = 3$ Skyrmion was considered by Carson [8], and the spins and energies were calculated. This analysis was extended in [9] by a computation of the static electroweak properties of the quantized Skyrmion.

In [28], the $B = 4$ Skyrmion was semiclassically quantized, and the ground state (corresponding to the alpha particle) and first excited state were determined, and their energies calculated. The results,
though novel, involved consideration of selected vibrational modes as well as rigid body motion, and are difficult to generalize to higher baryon numbers. Here we will consider the $B = 4$ case from a different perspective, which may easily be generalized to higher baryon numbers, and enables us to compute the excitation energies of further excited states.

Further results on the allowed spin and isospin states of quantized Skyrmions for $B$ up to 8 and beyond have been obtained by Irwin [15], and taken further by Krusch [18]. However, in this work, there were no estimates of the energies of the states.

It is not easy to assess the qualitative success of the Skyrme model just from the results for $B \leq 4$. The nuclei have the correct spin and isospin quantum numbers, but on the whole the ground states represent nuclei which are too small and too tightly bound. Nuclei with $B = 2$ or $B = 3$ have no excited states, experimentally, so Skyrmion excited states based on rigid body quantization are not meaningful, and one expects them to break up into individual nucleons if further degrees of freedom are included.

There have been a number of developments which make it worthwhile to reassess these results on quantized Skyrmions, and it is also possible to extend them to the range of baryon numbers $1 \leq B \leq 8$. First, it has been noted that a reparametrization of the Skyrme model is desirable to achieve a better fit to nuclear sizes and related quantities like moments of inertia [23]. The Skyrme length scale should be roughly doubled, and consequently the dimensionless pion mass parameter also doubled (to keep the physical pion mass fixed). Doubling the pion mass parameter has little effect on the qualitative character of classical Skyrmion solutions up to $B = 7$, but for $B \geq 8$ there is a clear difference [6]. The stable solutions are no longer the hollow polyhedra found earlier for $B$ up to 22 and beyond, but instead more dense structures closer to what one expects for nuclei. In particular, for $B$ a multiple of four, there are stable solutions which look like bound states of two or more of the cubically symmetric $B = 4$ Skyrmions [6]. We shall analyse below the quantum states of the $B = 8$ Skyrmion, which is made up of two $B = 4$ cubes, and compare with the states of nuclei with $B = 8$, including beryllium-8.

Another development is a better understanding of the quantization rules for Skyrmions, the so-called Finkelstein-Rubinstein (FR) constraints [11], which encode the requirement that a quantized $B = 1$ Skyrmion is a spin $\frac{1}{2}$ fermion. The FR constraints combine the symmetry of a Skyrmion, for any value of $B$, with the topology of the Skyrme model, to constrain the spins and isospins of quantum states. Here the rational map ansatz comes in [14]. This gives a separation of variables between the angular and radial dependence of the Skyrme field. True solutions do not exactly exhibit this separation, but they do so approximately. The ansatz gives a simple closed formula for the angular dependence of a Skyrme field, and rotational symmetries are easier to find than if one just has a numerical Skyrmion solution. The optimised rational map ansatz gives good approximations to true solutions up to $B = 7$ (and far beyond for zero or small pion mass). Even if it is a poor approximation, it can still be helpful in the numerical search for true solutions, and more importantly here, it is helpful in determining the effect of the FR constraints. Krusch has recently found a simple formula for determining the crucial signs that occur in the FR constraints [18]. This formula requires knowledge of the rational map approximating the Skyrmion.

The rational map ansatz allows a further simplification, valid to the extent that a Skyrmion is well approximated by the ansatz. Kopeliovich noted that the moments of inertia (both rotational and isorotational) of a Skyrme field described by the ansatz are rather simpler than for a general Skyrme field [17], since the effect of a rotation is just to rotate the map, leaving the radial profile function invariant. We simplify Kopeliovich’s formulae further, taking advantage of the complex analytic character of a rational map, and obtain formulae for the 36 components of the spin/isospin inertia tensor. These can be accurately evaluated, and it is easy to recognize if certain components vanish because of symmetry. Using these moments of inertia, we estimate anew the energies of ground and excited states of quantized Skyrmions over the range of baryon numbers $1 \leq B \leq 4$, and for the first time those in the range $5 \leq B \leq 8$. The quantization is based on the established method of rigid-body quantization of
rotations and isospin rotations. Particularly interesting for us are the states of the \( B = 6 \) Skyrmion, since our reparametrization of the Skyrme model \[23\] was based on the mass and charge radius of the lithium-6 nucleus. Also interesting are the states for \( B = 8 \), since the double cube \( B = 8 \) Skyrmion has not previously been quantized.

A problem for the Skyrme model that emerged in the work of Irwin \[15\], is that the spin states of the \( B = 5 \) and \( B = 7 \) Skyrmions disagree with those of the corresponding nuclei in their ground states. It has been suggested more than once (see e.g. \[21\]) that it might be appropriate, for these baryon numbers, to quantize a deformed Skyrmion with different symmetry. This would make sense, especially if the allowed spins were thereby reduced, making the spin energy smaller. The smaller spin energy might more than compensate the increased classical energy of the deformed Skyrmion. In this paper we are able to quantitatively assess this idea. For \( B = 7 \) it looks reasonable. A ground state with the correct spin \( \frac{3}{2} \) for the lithium-7/beryllium-7 isodoublet can be obtained, and the previously found spin \( \frac{7}{2} \) state interpreted as the observed, relatively low-lying second excited state. The spin \( \frac{1}{2} \) first excited state is still problematic, however. For \( B = 5 \) the situation is less satisfactory.

In the next section we review the Skyrme model, and briefly describe the recent reparametrization of the Skyrme model using the lithium-6 nucleus \[23\]. Although this is very important, we show that a reparametrization alone cannot solve all the problems of the Skyrme model. In section 3 we describe the rational map ansatz for Skyrmions. Section 4 deals with the quantization of Skyrmions, which proceeds by parametrizing time-dependent solutions through collective coordinates. Here, we recall how the model is fermionically quantized by the imposition of FR constraints. In section 5 we present expressions for the inertia tensors which appear in the formula for the kinetic energy operator, in terms of the approximating rational map. Sections 6 to 13 deal with the energy levels of quantized Skyrmions of baryon numbers 1 to 8 respectively. In section 14 we provide a conclusion.

## 2 The Skyrme Model

The Skyrme model is defined in terms of an \( SU(2) \)-valued scalar, the Skyrme field \[24, 22\]. It is a low energy effective theory of QCD, becoming exact as the number of quark colours becomes large \[29, 30\]. We call the topological soliton solutions which it admits Skyrmions.

The Lagrangian density is given by

\[
\mathcal{L} = \frac{F_\pi^2}{16} \text{Tr} \partial_\mu U \partial^\mu U^{-1} + \frac{1}{32e^2} \text{Tr} [\partial_\mu U U^{-1}, \partial_\nu U U^{-1}][\partial^\mu U U^{-1}, \partial^\nu U U^{-1}] + \frac{1}{8} m_\pi^2 F_\pi^2 \text{Tr} (U - 1_2),
\]

where \( U(t, x) \) is the Skyrme field, \( F_\pi \) is the pion decay constant, \( e \) is a dimensionless parameter and \( m_\pi \) is the pion mass.

Using energy and length units of \( F_\pi/4e \) and \( 2/eF_\pi \) respectively, we may express the Lagrangian as follows:

\[
L = \int \left\{ -\frac{1}{2} \text{Tr} (R_\mu R^\mu) + \frac{1}{16} \text{Tr} ([R_\mu, R_\nu][R^\mu, R^\nu]) + m^2 \text{Tr} (U - 1_2) \right\} d^3 x,
\]

where we have introduced the \( \mathfrak{su}(2) \)-valued current \( R_\mu = (\partial_\mu U)U^{-1} \), and defined the dimensionless pion mass parameter \( m = 2m_\pi/eF_\pi \).

Field configurations of finite energy must satisfy the boundary condition \( U \to 1_2 \) as \( |x| \to \infty \). This compactifies \( \mathbb{R}^3 \) to a 3-sphere of infinite size, and so topologically \( U : S^3 \to S^3 \) at a fixed time. Field configurations \( U \) therefore lie in topological sectors labelled by their topological degree

\[
B = \int B_0(x) \, d^3 x,
\]
where

$$B_\mu(x) = \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} \text{Tr} \partial^\nu U U^{-1} \partial^\alpha U U^{-1} \partial^\beta U U^{-1}.$$  

(4)

The degree $B$, which takes integer values, is identified with the baryon number. We refer to $B_0$ as the baryon density.

The kinetic part of the Lagrangian $L$ is

$$T = \int \left\{ -\frac{1}{2} \text{Tr} (R_0 R_0) - \frac{1}{8} \text{Tr} ([R_i, R_0][R_i, R_0]) \right\} d^3 x ,$$

(5)

and this is quadratic in the time derivative of the Skyrme field. The rest of the Lagrangian (2) is (minus) the potential energy:

$$E = \int \left\{ -\frac{1}{2} \text{Tr} (R_i R_i) - \frac{1}{16} \text{Tr} ([R_i, R_j][R_i, R_j]) - m^2 \text{Tr}(U - 1_2) \right\} d^3 x .$$

(6)

Static Skyrmion solutions can be obtained by solving the variational equations derived from $E$, or in practice by numerically minimising $E$ in the sector with given $B$.

The parameters $e$ and $F_\pi$ can be fixed in a number of ways. It has been common practice to use the set of parameters given in [1] with the physical pion mass taken into account, specifically

$$e = 4.84, \quad F_\pi = 108 \text{ MeV} \quad \text{and} \quad m_\pi = 138 \text{ MeV} \quad (\text{which implies} \quad m = 0.528) .$$

(7)

In [1], the values of $e$ and $F_\pi$ were tuned to reproduce the masses of the proton and the delta resonance. This parameter set was adjusted to optimise the predictions of the model in the $B = 1$ sector at the expense of the $B = 0$ sector, which requires $F_\pi = 186 \text{ MeV}$. It is not, therefore, the optimal parameter set globally.

In [23], we proposed that in order for the Skyrme model to more closely model nuclear properties, a reparametrization would be desirable. We performed such a reparametrization by matching the model in the $B = 6$ sector with properties of the lithium-6 nucleus, obtaining

$$e = 3.26, \quad F_\pi = 75.2 \text{ MeV} \quad \text{and} \quad m_\pi = 138 \text{ MeV} \quad (\text{which implies} \quad m = 1.125) .$$

(8)

Figure 1 shows graphs of nuclear masses and static Skyrmion masses per unit baryon number, using this new parameter set. The Skyrme quantum energies are not included. We observe that the graphs intersect at $B = 6$, as expected. It is clear that it is not possible by a single parameter choice to correctly match nuclear and Skyrmion masses for all baryon numbers and this remains the case when the quantum spin and isospin energies are included. We believe that calibrating the model in the $B = 6$ sector is a promising way to describe the properties of nuclei with $B \geq 4$. For $B = 1, 2, 3$ the Skyrme energies are now too high, and neither the nucleon mass nor delta resonance will be accurately fitted.

For consistency, in the following sections we use the new parameter set (8) throughout, making a few remarks about the old parameters in Appendix B.
Figure 1: Nuclear masses per unit baryon number \( (M/B) \) (solid), compared with static Skyrmion masses per unit baryon number (dotted).

3 The Rational Map Ansatz

We describe here the ansatz for Skyrme fields which uses rational maps between Riemann spheres to describe their angular behaviour \[14\]. This has been shown to give good approximations to several known Skyrmions, including all the minimal-energy solutions up to \( B = 7 \) (and much higher \( B \) when \( m = 0 \)). The rational maps have exactly the same symmetries as the numerically known Skyrmions in almost all cases (\( B = 14 \) is an exception \[5\]).

Via stereographic projection, the complex coordinate \( z \) encodes the conventional polar coordinates as \( z = \tan(\theta/2)e^{i\phi} \). Equivalently, the point \( z \) on a sphere corresponds to the radial unit vector

\[
\mathbf{n}_z = \frac{1}{1 + |z|^2} (z + \bar{z}, i(\bar{z} - z), 1 - |z|^2),
\]

and inversely

\[
z = \frac{(\mathbf{n}_z)_1 + i(\mathbf{n}_z)_2}{1 + (\mathbf{n}_z)_3}.
\]

The ansatz for the Skyrme field depends on a rational map \( R(z) = p(z)/q(z) \), where \( p \) and \( q \) are polynomials in \( z \), and a radial profile function \( f(r) \). The target value \( R \) is associated with a point in the unit 2-sphere of the Lie algebra of \( SU(2) \), given by the unit vector

\[
\mathbf{n}_R = \frac{1}{1 + |R|^2} (R + \bar{R}, i(\bar{R} - R), 1 - |R|^2).
\]

The ansatz is then

\[
U(r, z) = \exp \left( if(r)\mathbf{n}_{R(z)} \cdot \tau \right),
\]

where \( \tau_1, \tau_2 \) and \( \tau_3 \) are Pauli matrices and \( f(r) \) satisfies \( f(0) = \pi \) and \( f(\infty) = 0 \).

Using this ansatz, the baryon number is given by

\[
B = \int \frac{-f'}{2\pi^2} \left( \frac{\sin f}{r} \right)^2 \left( 1 + |z|^2 \right)^2 \frac{dR}{dz} \left( 1 + |R|^2 \right)^2 \frac{2i d\bar{z} dz}{(1 + |z|^2)^2} r^2 dr,
\]

and it can be shown that this is an integer equal to the degree of the rational map \( R \).
The energy $E$ for a field of form (12) is

$$E = 4\pi \int_0^\infty \left( v^2 f'{}^2 + 2B \sin^2 f (f'{}^2 + 1) + \mathcal{I} \frac{\sin^4 f}{r^2} + 2m^2 r^2 (1 - \cos f) \right) dr ,$$

in which $\mathcal{I}$ denotes the angular integral

$$\mathcal{I} = \frac{1}{4\pi} \int \left( 1 + |z|^2 \left| \frac{dR}{dz} \right| \right)^4 \frac{2i d\bar{z} dz}{(1 + |z|^2)^2} .$$

To minimise $E$ one first minimise $\mathcal{I}$ over all maps of degree $B$. The profile function $f(r)$ is then found by solving the second order ODE that is the Euler-Lagrange equation for the expression (14) with $B$ and $\mathcal{I}$ as fixed parameters. Given the profile function, the energy is determined by numerical integration. This gives the optimised rational map ansatz, and we denote the minimised energy by $\mathcal{M}_B$. This is our estimate for the true Skyrmion mass, for baryon number $B$. To obtain a physical value to compare to a nuclear mass, one multiplies by the energy unit $F_\pi/\Delta = 5.76 \text{MeV}$, obtaining a classical Skyrmion mass in MeV.

4 Quantizing the Skyrme Model

Given a static Skyrmion $U_0(x)$, there is generically a nine-parameter set of solutions, each with the same energy, obtained by acting with the Euclidean group and isorotations:

$$U(x) = A_1 U_0(D(A_2)(x - X)) A_1^{-1} \quad (16)$$

where $A_1, A_2$ are $SU(2)$ matrices and $A_2$ is recast in the $SO(3)$ form $D(A_2)_{ij} = \frac{1}{2} \text{Tr}(\tau_i A_2 \tau_j A_2^{-1})$. Semi-classical quantization is performed by promoting the collective coordinates $A_1, A_2, X$ to dynamical degrees of freedom [7]. As we shall only be concerned with the computation of spin and isospin, we shall ignore the translational degrees of freedom $X$ and quantize the solitons in their zero-momentum frame.

Making the replacement $U(x) \to \bar{U}(x, t) = A_1(t) U_0(D(A_2(t)) x) A_1(t)^{-1}$, and inserting this into the Skyrme Lagrangian, one obtains the kinetic contribution to the total energy

$$T = \frac{1}{2} \bar{\epsilon}_{ijl} a_i b_j - a_i W_{ij} b_j + \frac{1}{2} b_i V_{ij} b_j , \quad (17)$$

where

$$a_j = -i \text{Tr} \tau_j A_1^{-1} \dot{A}_1 , \quad b_j = i \text{Tr} \tau_j A_2 A_2^{-1} . \quad (18)$$

$b$ is the angular velocity in physical space, and $a$ is the angular velocity in isospace. The inertia tensors $U_{ij}, V_{ij}$ and $W_{ij}$ are given by:

$$U_{ij} = -\int \text{Tr} \left( T_i T_j + \frac{1}{4}[R_k, T_i][R_k, T_j] \right) d^3 x , \quad (19)$$

$$V_{ij} = -\int \epsilon_{ilm} \epsilon_{jnp} x_l x_n \text{Tr} \left( R_m R_p + \frac{1}{4}[R_k, R_m][R_k, R_p] \right) d^3 x , \quad (20)$$

$$W_{ij} = \int \epsilon_{ilm} x_l \text{Tr} \left( T_i R_m + \frac{1}{4}[R_k, T_i][R_k, R_m] \right) d^3 x , \quad (21)$$

where $R_k = (\partial_k U_0) U_0^{-1}$ is the right invariant $su(2)$ current defined previously and $T_i = \frac{1}{2} [\tau_i, U_0] U_0^{-1}$ is also an $su(2)$ current. The total energy, in terms of collective coordinates, is just $T$ plus the constant $\mathcal{M}_B$, the static mass of the Skyrmion.
We may write
\[ T = \frac{1}{2} c^T \mathcal{W} c, \]  
where \( c^T = (a_1, a_2, a_3, b_1, b_2, b_3) \), and the 6 \times 6 symmetric matrix \( \mathcal{W} \) is given by
\[ \mathcal{W} = \left( \begin{array}{cc} U & -W \\ -W^T & V \end{array} \right). \]

The momenta corresponding to \( b_i \) and \( a_i \) are the body-fixed spin and isospin angular momenta \( \mathbf{L}_i \) and \( \mathbf{K}_i \) [2]:
\[ L_i = -W_{ij} a_j + V_{ij} b_j, \]
\[ K_i = U_{ij} a_j - W_{ij} b_j. \]

The usual space-fixed spin and isospin angular momenta \( \mathbf{J}_i \) and \( \mathbf{I}_i \) are related to the body-fixed momenta by
\[ J_i = -D(A_2)_{ij} L_j, \quad I_i = -D(A_1)_{ij} K_j. \]

Defining \( H^T = (K_1, K_2, K_3, L_1, L_2, L_3) \), and using the relation \( H^T = c^T \mathcal{W} \), we find, provided \( \det \mathcal{W} \neq 0 \),
\[ T = \frac{1}{2} H^T \mathcal{W}^{-1} H. \]

We now promote the four sets of classical momenta introduced above to quantum operators, each individually satisfying the \( \mathfrak{su}(2) \) commutation relations. The Casimir invariants satisfy \( \mathbf{J}^2 = \mathbf{L}^2 \) and \( \mathbf{I}^2 = \mathbf{K}^2 \).

The basic FR constraints, which apply to any Skyrmion, are that physical quantum states \( |\Psi\rangle \) should satisfy
\[ e^{2\pi i n \mathbf{L}} |\Psi\rangle = e^{2\pi i n \mathbf{K}} |\Psi\rangle = (-1)^B |\Psi\rangle, \]
for any unit vector \( \mathbf{n} \), which implies that for even \( B \) the spin and isospin are integral, and for odd \( B \) they are half-integral. There are further FR constraints on states if the Skyrmion has symmetries, and these are simple to determine if the Skyrmion is described by the rational map ansatz. A rational map, and hence the corresponding Skyrmion, has a rotational symmetry if it satisfies an equation of the form
\[ R(M_2(z)) = M_1(R(z)), \]
for some combination of \( SU(2) \) Möbius transformations \( M_2 \) and \( M_1 \). \( M_2 \) corresponds to a rotation in physical space, and \( M_1 \) to an isorotation. In general there will be a group \( \mathcal{S} \) of such symmetries. We say that the map \( R \) is \( \mathcal{S} \)-symmetric if for each \( M_2 \in \mathcal{S} \), there exists an \( M_1 \) such that (20) holds. For consistency, pairs \( \langle M_2, M_1 \rangle \) must have the same composition rule as in \( \mathcal{S} \), so \( R(M_2 M_2'(z)) = M_1 M_1'(R(z)) \). The map \( M_2 \rightarrow M_1 \) is therefore a homomorphism. Note that it is not possible to construct such a map from \( M_1 \) to \( M_2 \). This is related to the fact that a Skyrmion may be invariant under a rotation alone, but cannot be invariant under an isorotation alone.

Consider a rotation in physical space by an angle \( \theta_2 \) about an axis \( \mathbf{n}_2 \), and an isorotation by an angle \( \theta_1 \) about an axis \( \mathbf{n}_1 \). We recall that under such a rotation, \( z \) transforms to \( M_2(z) \), given by [18]:
\[ M_2(z) = \frac{\cos \theta_2 + i \mathbf{n}_2 \sin \theta_2}{\cos \theta_2 - i \mathbf{n}_2 \sin \theta_2} z + \frac{(\mathbf{n}_2)_2 - i (\mathbf{n}_2)_1 \sin \theta_2}{\cos \theta_2 + i \mathbf{n}_2 \sin \theta_2} + \frac{\cos \frac{\theta_2}{2} + i \mathbf{n}_2 \sin \frac{\theta_2}{2}}{\cos \frac{\theta_2}{2} - i \mathbf{n}_2 \sin \frac{\theta_2}{2}}. \]

Similarly, under such an isorotation, \( R \) transforms to \( M_1(R) \), given by:
\[ M_1(R) = \frac{\cos \theta_1 + i \mathbf{n}_1 \sin \theta_1}{\cos \theta_1 - i \mathbf{n}_1 \sin \theta_1} R + \frac{(\mathbf{n}_1)_2 - i (\mathbf{n}_1)_1 \sin \theta_1}{\cos \theta_1 + i \mathbf{n}_1 \sin \theta_1} + \frac{\cos \frac{\theta_1}{2} + i \mathbf{n}_1 \sin \frac{\theta_1}{2}}{\cos \frac{\theta_1}{2} - i \mathbf{n}_1 \sin \frac{\theta_1}{2}}. \]
So given a specific symmetry \((29)\) of a rational map, we use the above formulae to determine the corresponding angles and axes of rotation and isorotation. This is the data that is used in the conventional formulae describing the effect of rotations on quantized rigid bodies \([19]\).

For \(\theta_2\) not an integer multiple of \(2\pi\), \(M_2\) only leaves the points
\[
z_{n_2} = \frac{(n_2)_1 + i(n_2)_2}{1 + (n_2)_3} \quad \text{and} \quad z_{-n_2} = \frac{-(n_2)_1 - i(n_2)_2}{1 - (n_2)_3}
\]
(32)
fixed. Similarly, \(M_1\) only leaves \(R_{\pm n_1}\) fixed, where \(R_{\pm n_1}\) are defined similarly. Therefore, for the symmetry \((29)\) to hold, we have
\[
R(z_{-n_2}) = R_{n_1} \quad \text{or} \quad R_{-n_1}.
\]
(33)
Krusch showed that to correctly determine the FR constraint it is important to choose the direction of the axis \(n_1\) so as to satisfy the base point condition\(^1\)
\[
R(z_{-n_2}) = R_{-n_1}.
\]
(34)
The symmetry \((29)\) then leads to the following FR constraint on the wavefunction:
\[
e^{i\theta_2 n_2 \cdot L} e^{i\theta_1 n_1 \cdot K} \Psi = \chi_{FR} \Psi,
\]
(35)
a representation-independent statement, in which \(L\) and \(K\) are the body-fixed spin and isospin operators respectively, and the FR sign \(\chi_{FR} = \pm 1\). The combined rotation and isorotation corresponding to \(M_2\) and \(M_1\) respectively at most change the state by a sign factor, \(\chi_{FR}\). It was proved in \([18]\) that the value of \(\chi_{FR}\) for a given symmetry of a rational map only depends on \(\theta_2\) and \(\theta_1\), where the angles have unambiguous signs because of the base point condition \((34)\), and is given by
\[
\chi_{FR} = (-1)^N, \quad \text{where} \quad N = \frac{B}{2\pi}(\theta_2 - \theta_1).
\]
(36)
The FR signs \(\chi_{FR}\) form a 1-dimensional representation of the symmetry group \(S\) of the Skyrmion.

A basis for the wavefunctions is given by \(|J, L_3\rangle \otimes |I, K_3\rangle\), the tensor product of states of a rigid body in space and a rigid body in isospace. Here we suppress the additional labels \(J_3\) and \(I_3\) which can take any values in the usual ranges allowed by \(J\) and \(I\). \(J_3\) is the physically meaningful projection of spin on the third space axis, and \(I_3\) is the conventional third component of isospin. When we come to consider specific rational maps and the associated FR constraints, we seek low-energy states which are allowed by the FR constraints, and may represent the spin and isospin operators appearing on the left-hand side of \([35]\) by Wigner \(D\)-matrices, acting on the \((2J + 1) \times (2I + 1)\)-dimensional space of wavefunctions.

Another advantage of the rational map ansatz is that it clearly illustrates any reflection symmetries of the Skyrmion, which enables one to determine the effect of the inversion operator \(\mathcal{P}\), which acts as
\[
\mathcal{P} : U(x) \to U^\dagger(-x),
\]
(37)
and is baryon number preserving. For rational maps, the inversion \(x \to -x\) corresponds to \(z \to -1/\bar{z}\), and the inversion \(U \to U^\dagger\) corresponds to \(R \to -1/R\). A rational map, and hence the corresponding Skyrmion, has a reflection symmetry if it satisfies an equation of the form
\[
-1/R(M_2(z)) = M_1(R(-1/\bar{z})).
\]
(38)
In this case, \(\mathcal{P}\) is equivalently given by the combination of rotation and isorotation corresponding to \(M_2\) and \(M_1\) occurring here. The parity of a quantum state is then the eigenvalue of the state when acted

\(^1\)We note that this differs slightly from the condition given in \([18]\). This is because we are working with the inverse of the isospacial Möbius transformation which was considered there.
upon by the operator $e^{i\theta_2n_2\mathbf{L}}e^{i\theta_1n_1\cdot\mathbf{K}}$ derived from $M_2$ and $M_1$. There is, however, an ambiguity in the definition of $\mathcal{P}$ which was first explained in \cite{14}. Given a candidate parity operator $\mathcal{P}_0$ for a given Skyrmion, we can also represent the operator by $\mathcal{P}_0$ times any element of the symmetry group of the classical solution. If the FR sign of a particular symmetry element is $-1$, then these two choices for $\mathcal{P}$ give different results. In particular, there is this problem for odd $B$: given a parity operator $\mathcal{P}_0$, we can also represent the operator by $\mathcal{P}_0e^{2\pi i\mathbf{n}\cdot\mathbf{L}}$, where $\mathbf{n}$ is any unit vector. As $2\pi$ rotations have associated FR signs of $-1$ for odd $B$, we see that these two choices differ when acting on states. For each of the cases $B=1$ to $8$, we make particular choices for the parity operators, which we believe to be the most natural. We note that despite the ambiguity in the definition of $\mathcal{P}$ for a given Skyrmion, the relative parities of the Skyrmion’s quantum states are fixed.

5 Tensors of Inertia for Rational Map Skyrmions

Kopeliovich \cite{17} first presented general formulae for the inertia tensors of rational map Skyrmions. Writing $U_0 = \exp(if(r)\mathbf{n}\cdot\mathbf{\tau})$, these can be expressed as follows \cite{17}:

$$U_{ij} = 2\int \sin^2 f \left( (\delta_{ij} - n_in_j)(1 + f^2) + \sin^2 f \partial_{kn_i}\partial_{kn_j} \right) d^3x, \quad (39)$$

$$V_{ij} = 2\int \sin^2 f \left( (1 + f^2 + \sin^2 f \partial_{kn_i}\partial_{kn_j}) \left( \partial_m n_r \partial_m n_r (r^2\delta_{ij} - x_ix_j) - \partial_{kn_r}\partial_{kn_r} r^2 \right) - \sin^2 f \partial_{kn_i}\partial_{kn_j}\partial_{kn_r}(r^2\delta_{ij} - x_ix_j) - r^2 \partial_{kn_r}\partial_{kn_r}\partial_{kn_i}\partial_{kn_j} \right) d^3x, \quad (40)$$

$$W_{ij} = 2\int \epsilon_{jlm}\epsilon_{isp}x_in_s \sin^2 f \left( (1 + f^2) \partial_{mp} + \sin^2 f \partial_{kn_r}\partial_{kn_r}\partial_{kn_p} - \partial_{kn_r}\partial_{kn_r}\partial_{kn_p} \right) d^3x. \quad (41)$$

These formulae for the inertia tensors assume that $f$ depends only on $r$, and $\mathbf{n}$ depends only on the angular coordinates $\theta$, $\phi$; further simplifications can be made if we assume that $\mathbf{n}$ depends just on a rational function $R(z)$ as in \cite{11}. In order to obtain these simplified formulae, we find it helpful to write the $\mathbb{R}^3$ metric and volume element in terms of $r$, $z$ and $\bar{z}$:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 = dr^2 + \frac{4r^2 \frac{dz}{1 + |z|^2} d\bar{z}}{1 + |z|^2} = g_{\alpha\beta}dx^\alpha dx^\beta, \quad (42)$$

$$d^3x = \frac{4r^2 dr \frac{dz}{1 + |z|^2} d\bar{z}}{(1 + |z|^2)^2}, \quad (43)$$

and to replace Cartesian derivatives with derivatives with respect to $r$, $z$ and $\bar{z}$. The products of commutators in \cite{19,20,21} may then be rewritten in these coordinates:

$$[R_k, \cdots ||R_k, \cdots] = g^{\alpha\gamma}[R_r, \cdots ||R_r, \cdots] + g^{z\bar{z}}[R_z, \cdots ||R_z, \cdots] + g^{\bar{z}z}[R_{\bar{z}}, \cdots ||R_{\bar{z}}, \cdots], \quad (44)$$

where $R_z = (\partial_z U_0)U_0^{-1}$ etc. We also have

$$-i\epsilon_{jlm}x_lR_m = (l_jU_0)U_0^{-1} = \mu_j R_z - \bar{\mu}_j R_{\bar{z}}, \quad (45)$$

where

$$l_1 = -\frac{1}{2} \left( (1 - z^2) \frac{\partial}{\partial z} - (1 - \bar{z}^2) \frac{\partial}{\partial \bar{z}} \right), \quad (46)$$

$$l_2 = -\frac{i}{2} \left( (1 + z^2) \frac{\partial}{\partial z} + (1 + \bar{z}^2) \frac{\partial}{\partial \bar{z}} \right), \quad (47)$$

$$l_3 = z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}}. \quad (48)$$
\[
\mu_j = \left( -\frac{1}{2}(1-z^2), \frac{i}{2}(1+z^2), z \right). \quad (49)
\]

We ultimately find that for the rational map ansatz, the tensors of inertia \( U_{ij}, V_{ij} \) and \( W_{ij} \) can be expressed in the following form:

\[
\Sigma_{ij} = 2 \int \sin^2 f \frac{C_{\Sigma_{ij}}}{(1+|R|^2)^2} \left( 1 + f'^2 + \frac{\sin^2 f}{r^2} \left( \frac{1+|z|^2}{1+|R|^2} \right) \right)^2 d^3x, \quad (50)
\]

where \( \Sigma = (U, V, W) \) and the quantities \( C_{\Sigma_{ij}} \) (which are given explicitly in Appendix A) are functions of the variables \( z \) and \( \bar{z} \) only. In what follows, we use the above formula to numerically determine the elements of the inertia tensors, for a given rational map and profile function. The numerical values we obtain are, of course, in Skyrme units. To convert to physical values we must multiply these by the mass scale and by the square of the length scale: \( \frac{F_\pi}{4e} \times \frac{(2/eF_\pi)^2}{e^3F_\pi} = 1/e^3F_\pi \), obtaining quantities in inverse MeV. With the new parameter set \( \tilde{\lambda} \), \( e^3F_\pi = 2613 \text{ MeV} \). Although our optimal rational maps are familiar \cite{14}, the profile functions have all been calculated anew using a shooting method\cite{8}. In Fig. 2 we plot the profile functions for \( B = 1 \) to 8, using the new dimensionless pion mass parameter \( m = 1.125 \).

\begin{center}
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graph.png}
\caption{The profile functions \( f(r) \) for \( B = 1 \) to 8. \( B \) increases from left to right.}
\end{figure}
\end{center}

6  \( B = 1 \)

The rational map describing the single Skyrmion is given by \( R(z) = z \), which is \( O(3) \) symmetric. We find that the inertia tensors are each proportional to the unit matrix, satisfying \( U_{ij} = V_{ij} = W_{ij} = \lambda \delta_{ij} \), where \( \lambda \) is given by

\[
\lambda = \frac{16\pi}{3} \int r^2 \sin^2 f \left( 1 + f'^2 + \frac{\sin^2 f}{r^2} \right) dr. \quad (51)
\]

Numerically we compute \( \lambda = 45.1 \).

\footnotetext{2}{due to Bernard M. A. G. Piette, University of Durham, UK}
The FR constraints associated with spherical symmetry are
\[ e^{iθn}L e^{iθn}K |Ψ \rangle = |Ψ \rangle , \]
where \( θ \) and \( n \) are arbitrary, leading to the following constraint on the space of physical states:
\[ (L + K)|Ψ \rangle = 0. \] (53)

The “grand spin” \( M = L + K \), or its components, appears quite frequently in what follows. The states satisfying (53) are linear combinations of the states \( |J, L_3 \rangle \otimes |I, K_3 \rangle \), whose grand spin is zero. These are of the form \( |J, I; M, M_3 \rangle = |J, J; 0, 0 \rangle \), in the standard notation for adding angular momenta, where \( J = L \) and \( I = K \). So the spin \( J \) and isospin \( I \) have to have the same magnitude. In addition, \( J \) must be half-integral.

The kinetic energy operator is given by
\[ T = \frac{1}{2λ} J^2 = \frac{1}{2λ} I^2 . \] (54)

The eigenvalue of \( J^2 \) in states of spin \( J \) is \( J(J+1) \), a standard result we will use frequently. Similarly \( I^2 \) has eigenvalues \( I(I+1) \). For the lowest energy states, the nucleons with spin/isospin \( \frac{1}{2} \), the spin energy in physical units is
\[ \frac{1}{2(45.1)^4} e^{3F_π} = 21.7 \text{ MeV} , \] (55)
and the total energy is
\[ E_{J=1/2, I=1/2} = M_1 + 21.7 \text{ MeV} = 986.2 \text{ MeV} + 21.7 \text{ MeV} = 1008 \text{ MeV} . \] (56)

This is not a bad fit to the nucleon mass 939 MeV. The spin energy is approximately one quarter of its value with the old parameters, but the higher classical Skyrmeon mass makes the total energy too high. The spin/isospin \( \frac{3}{2} \) delta resonances come out with the too low energy 1095 MeV. Of course, using the old parameter set \( \tilde{\gamma} \), the nucleon and delta masses are exactly right.

As the rational map \( R(z) = z \) satisfies \(-1/R(z) = R(-1/z)\), the parity operator \( P \) is naturally represented by the identity operator. We then find that each of the states described above has positive parity, in agreement with experiment.

7 \( B = 2 \)

The symmetry of the \( B = 2 \) Skyrmeon is \( D_{oh} \), and the rational map which approximates this Skyrmeon is \( R(z) = z^2 \). The tensors of inertia \( U_{ij} \), \( V_{ij} \) and \( W_{ij} \) are all diagonal, with \( U_{11} = U_{22} \), \( V_{11} = V_{22} \) and \( W_{11} = W_{22} = 0 \). We also have that \( U_{33} = \frac{1}{4} W_{33} = \frac{1}{4} V_{33} \), relations which make the inertia tensor degenerate, a consequence of the axial symmetry. The degeneracy is resolved by imposing the following FR constraint on physical states:
\[ (L_3 + 2K_3)|Ψ \rangle = 0. \] (57)

The discrete symmetry \( R(1/z) = 1/R(z) \) leads to the FR constraint
\[ e^{iπL_3} e^{iπK_3} |Ψ \rangle = -|Ψ \rangle . \] (58)

The ground state is then the \( J = 1, I = 0 \) state \( |1,0 \rangle \otimes |0,0 \rangle \), which has the quantum numbers of the deuteron. The first excited state \( |0,0 \rangle \otimes |1,0 \rangle \) may be identified with the isovector \(^1S_0 \) state of the two-nucleon system.
Using the expressions for the inertia tensors given in Appendix A, we find that numerically,
\[ U_{11} = 96.58, \quad U_{33} = 62.94 \quad \text{and} \quad V_{11} = 160.61. \] 
(59)
The kinetic energy operator is given by \[ T = \frac{1}{2V_{11}} J^2 + \frac{1}{2U_{11}} I^2 - \left( \frac{1}{2U_{11}} + \frac{2}{V_{11}} - \frac{1}{W_{33}} \right) K_3^2. \] 
(60)
For the ground state, we find (with the conversion factor \( e^3 F_\pi \) implied from now on)
\[ E_{J=1, I=0} = M_2 + 16.3 \, \text{MeV} = 1949.3 \, \text{MeV} + 16.3 \, \text{MeV} = 1966 \, \text{MeV}. \] 
(61)
For the first excited state, we find
\[ E_{J=0, I=1} = M_2 + 27.1 \, \text{MeV} = 1976 \, \text{MeV}. \] 
(62)
The experimentally determined mass of the deuteron is 1876 MeV, with the proton and neutron constituents only very weakly bound by 2 MeV. The \( ^1S_0 \) state is marginally unbound, with a mass of 1880 MeV \([7,3]\). As our energies (61) and (62) exceed the sum of the masses of a proton and a neutron, it would appear that we have predicted states that are unbound. However, when we compare \( E_{J=1, I=0} \) and \( E_{J=0, I=1} \) to the sum of the masses of two quantized single Skyrmions with spin \( \frac{1}{2} \) (calculated in the previous section), these states appear bound (with binding energies 50 MeV and 39 MeV respectively).

While the new parameters are clearly not ideal in the \( B = 2 \) sector, they predict results that are quantitatively quite accurate. The old parameters more strongly overestimate the binding energies of these two states \([7]\). Also, we calculate the excitation energy of the \( ^1S_0 \) state to be 11 MeV relative to the deuteron, which is of the correct order of magnitude, and better than that obtained in \([7]\) (35 MeV).

To determine the parities of these two states we observe that the rational map \( R(z) = z^2 \) has the reflection symmetry \( -1/R(z) = -R(-1/z) \), and so \( P = e^{i\pi K_3} \). Applying \( P \) to the allowed states, we find that both have positive parity, in agreement with experiment.

8 \( B = 3 \)

The tetrahedrally symmetric \( B = 3 \) Skyrmion was first quantized in \([8]\). Here we use the rational map ansatz to simplify the analysis. The Skyrmion is approximated using the map
\[ R(z) = \frac{\sqrt{3}iz^2 - 1}{z^3 - \sqrt{3}iz}. \] 
(63)
The symmetry group is generated by two elements. These correspond to the following symmetries of the rational map:
\[ R(-z) = -R(z), \] 
(64)
\[ R\left( \frac{iz + 1}{-iz + 1} \right) = \frac{iR(z) + 1}{-iR(z) + 1}. \] 
(65)
A \( \pi \) rotation about the \( x_3 \)-axis in space is equivalent to a \( \pi \) isorotation about the 3-axis in isospace; and a \( 2\pi/3 \) rotation about the \((x_1 + x_2 + x_3)\)-axis in space is equivalent to a \( 2\pi/3 \) isorotation about the \((1 + 2 + 3)\)-axis in isospace. This leads to the FR constraints
\[ e^{i\pi L_3} e^{i\pi K_3} |\Psi\rangle = |\Psi\rangle, \] 
(66)
\[ e^{i\frac{2\pi}{3} (L_1 + L_2 + L_3)} e^{i\frac{2\pi}{3} (K_1 + K_2 + K_3)} |\Psi\rangle = |\Psi\rangle. \] 
(67)
There is a spin $\frac{1}{2}$, isospin $\frac{1}{2}$ (unnormalised) solution of these constraints,

$$|\Psi\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle - \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle .$$  \hspace{1cm} (68)

This is the unique state with the same quantum numbers as the hydrogen-3/helium-3 isodoublet of nuclei in their ground states. The FR constraints also allow for two distinct states with spin $\frac{1}{2}$ and isospin $\frac{1}{2}$, given by

$$|\Psi\rangle = \left| \frac{3}{2}, \frac{3}{2} \right\rangle \otimes \left| \frac{3}{2}, -\frac{3}{2} \right\rangle - \left| \frac{3}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{3}{2}, -\frac{1}{2} \right\rangle + \left| \frac{3}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{3}{2}, -\frac{3}{2} \right\rangle - \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \otimes \left| \frac{3}{2}, \frac{3}{2} \right\rangle ,$$  \hspace{1cm} (69)

and

$$|\Psi\rangle = \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \otimes \left| \frac{3}{2}, -\frac{1}{2} \right\rangle - \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \otimes \left| \frac{3}{2}, \frac{1}{2} \right\rangle - \left| \frac{3}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{3}{2}, -\frac{3}{2} \right\rangle + \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \otimes \left| \frac{3}{2}, \frac{3}{2} \right\rangle .$$  \hspace{1cm} (70)

The first of these has the correct quantum numbers to allow for its interpretation as a nucleus in which one of the nucleons is excited to a delta isobar $^{[10]}$.

The inertia tensors have been numerically determined for the rational map given above. They are all diagonal and proportional to the unit matrix: $U_{ij} = u\delta_{ij}$, $V_{ij} = v\delta_{ij}$ and $W_{ij} = w\delta_{ij}$. This was to be expected due to the irreducibility of the action of the tetrahedral group on $\mathbb{R}^3$. Numerically,

$$u = 121.80, \ v = 418.83 \ \text{and} \ w = -80.34.$$  \hspace{1cm} (71)

The kinetic energy operator then takes the following form:

$$T = \frac{1}{2} \left[ \frac{1}{uv-w} \left[ (u-w)J^2 + (v-w)I^2 + wM^2 \right] \right] ,$$  \hspace{1cm} (72)

where $M = L + K$. Each of the three states $^{[68][69][70]}$ given above can be rewritten in terms of the basis states $|J, L; M, M_3\rangle$: the first is proportional to $|\frac{3}{2}, \frac{3}{2}; 0, 0\rangle$, the second to $|\frac{3}{2}, \frac{3}{2}; 0, 0\rangle$ and the third to $|\frac{3}{2}, \frac{3}{2}; 3, 2\rangle - |\frac{3}{2}, \frac{3}{2}; 3, -2\rangle$. They are thus eigenstates of $\mathbf{M}^2$ with eigenvalues 0, 0 and 12 respectively. The energies of the three states are then:

$$E_{J=\frac{1}{2}, I=\frac{1}{2}, M=0} = M_3 + \frac{3}{8} \frac{u+v-2w}{uv-w^2} = M_3 + 15.4 \text{ MeV} = 2895 \text{ MeV} ,$$  \hspace{1cm} (73)

$$E_{J=\frac{3}{2}, I=\frac{3}{2}, M=0} = M_3 + \frac{15}{8} \frac{u+v-2w}{uv-w^2} = M_3 + 77.1 \text{ MeV} = 2957 \text{ MeV} ,$$  \hspace{1cm} (74)

$$E_{J=\frac{3}{2}, I=\frac{3}{2}, M=3} = M_3 + \frac{3}{8} \frac{5u+5v+6w}{uv-w^2} = M_3 + 48.8 \text{ MeV} = 2929 \text{ MeV} .$$  \hspace{1cm} (75)

These formulae are identical to those obtained in $^{[3]}$, although the numerical values of $u$, $v$ and $w$ are different because of the rational map approximation. The average mass of a helium-3 nucleus and a hydrogen-3 nucleus is 2809 MeV. Our ground state comes to within 4% of this value. However, our second state, with an excitation energy of 62 MeV, is rather too low in energy to have an NN$\Delta$ interpretation, which would require a mass splitting of approximately 300 MeV with the spin $\frac{1}{2}$ ground state. Using the old parameter set, one obtains closer agreement with experiment.

To determine the parities of these three states we observe that there is the reflection symmetry $-1/R(iz) = iR(-1/z)$, and so $P = e^{i\frac{\pi}{2}(L_3+K_3)}$. Applying $P$ to the allowed states $^{[68][69][70]}$, we find that they have parities $+$, $+$ and $-$, respectively. We note that the helium-3 and hydrogen-3 ground states have positive parity.
The minimal-energy $B = 4$ Skyrmion has $O_h$ symmetry and a cubic shape, and is described by the rational map

$$R(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}. \quad (76)$$

This map has the generating symmetries

$$R(iz) = 1/R(z), \quad (77)$$

$$R\left(\frac{iz + 1}{-iz + 1}\right) = e^{i\frac{2\pi}{3}} R(z), \quad (78)$$

which lead to the FR constraints

$$e^{i\frac{\pi}{2}L_3} e^{i\pi K_1} |\Psi\rangle = |\Psi\rangle, \quad (79)$$

$$e^{i\frac{2\pi}{3}(L_1 + L_2 + L_3)} e^{i\frac{2\pi}{3} K_3} |\Psi\rangle = |\Psi\rangle. \quad (80)$$

Seeking simultaneous solutions of these, we obtain the ground state $|0, 0\rangle \otimes |0, 0\rangle$. There exists a spin 2, isospin 1 state given by

$$\left(|2, 2\rangle + \sqrt{2} |2, 0\rangle + |2, -2\rangle\right) \otimes |1, 1\rangle - \left(|2, 2\rangle - \sqrt{2} |2, 0\rangle + |2, -2\rangle\right) \otimes |1, -1\rangle, \quad (81)$$

and a spin 4, isospin 0 state given by

$$\left(|4, 4\rangle + \sqrt{\frac{14}{5}} |4, 0\rangle + |4, -4\rangle\right) \otimes |0, 0\rangle. \quad (82)$$

The cubic symmetry excludes a spin 2, isospin 0 state.

The tensors of inertia are found to be diagonal, satisfying $U_{11} = U_{22}$, $V_{ij} = v\delta_{ij}$ and $W_{ij} = 0$. Although the cubic group acts irreducibly on spatial $\mathbb{R}^3$, the associated isospin rotations are reducible, with the $\mathbb{R}^3$ of isospace decomposing into a 2-dimensional and a 1-dimensional subspace. This is why the inertia tensor $U$ has two independent diagonal entries, whereas $V$ only has one, and why the cross term $W$ vanishes. Numerically,

$$U_{11} = 142.84, \quad U_{33} = 169.41 \quad \text{and} \quad v = 663.16. \quad (83)$$

The kinetic energy operator is given by

$$T = \frac{1}{2v} J^2 + \frac{1}{2U_{11}} I^2 + \frac{1}{2} \left(\frac{1}{U_{33}} - \frac{1}{U_{11}}\right) K^2. \quad (84)$$

For the spin 0, isospin 0 ground state, the energy is simply the static mass of the Skyrmion, $M_4 = 3679$ MeV. Comparing this to the mass of the helium-4 nucleus, 3727 MeV, we see that our prediction comes to within 2% of the experimental value. The classical binding energy of the $B = 4$ Skyrmion is significantly larger than that of the $B = 3$ or $B = 5$ Skyrmion (see next section). The mean charge radius of the quantized $B = 4$ Skyrmion was calculated using the new parameter set in [23] to be 2.13 fm, which agrees reasonably well with the experimental value of 1.71 fm. Wallhout [28] calculated this quantity using the old parameter set and taking into account a number of the vibrational modes, obtaining 1.58 fm.

For the state (81) with spin 2 and isospin 1, the energy is

$$E_{J=2, I=1} = M_4 + 28.7 \text{ MeV} = 3679.0\text{ MeV} + 28.7\text{ MeV} = 3707.7\text{ MeV}. \quad (85)$$
We note here that hydrogen-4, helium-4 and lithium-4 form an isospin triplet, whose lowest energy state has spin 2, and average excitation energy 23.7 MeV relative to the ground state of helium-4 \cite{25}, so here the Skyrmion picture works well.

Finally, for the predicted spin 4, isospin 0 state, we find

$$E_{J=4, I=0} = M_4 + 39.4 \text{ MeV} = 3679.0 \text{ MeV} + 39.4 \text{ MeV} = 3718 \text{ MeV}. \quad (86)$$

Such a state of helium-4 has not yet been experimentally observed. However, predictions for such a state with an excitation energy of 24.6 MeV have been made \cite{12, 13}. Our calculation suggests a slightly larger energy, in the range 30-40 MeV (allowing for the discrepancy between our calculation and the data for the isospin 1 state). The energy levels are summarized in Fig. 3.

Figure 3: Energy level diagram for the quantized $B = 4$ Skyrmion. Solid lines indicate experimentally observed states, while dashed lines indicate our predictions.

To determine the parities of these three states we observe that the rational map (76) has the reflection symmetry $-1/R(z) = -R(-1/z)$, and so $P = e^{i\pi K_3}$. By acting with this operator on the physical states, we find that the spin 0, isospin 0 state, and the spin 4, isospin 0 state both have positive parity. On the other hand the spin 2, isospin 1 state has negative parity, and so we find no contradiction with experiment.

10 $B = 5$

Finding a quantized Skyrmion description of the ground and first excited states of the helium-5/lithium-5 isodoublet, with spins $\frac{3}{2}$ and $\frac{1}{2}$, has proved difficult. Physically, these states are not bound, and they may best be described as a cubic $B = 4$ Skyrmion loosely attracted to a single Skyrmion.

It still remains to determine the symmetries and FR constraints that might give a lowest energy state of spin $\frac{3}{2}$. Here we explore in detail the idea floated in \cite{21}, that one should consider variants of the rational map, and not just the one that optimises the classical Skyrmion energy. The minimal-energy $B = 5$ Skyrmion has $D_{2d}$ symmetry, and it can be approximated by the rational map

$$R(z) = \frac{z(z^4 + ibz^2 + a)}{az^4 + ibz^2 + 1}, \quad a = -3.07, \quad b = 3.94. \quad (87)$$

The ground state obtained from this map \cite{13} has spin $\frac{1}{2}$ and isospin $\frac{1}{2}$, which is inconsistent with the observed spin $\frac{3}{2}$ ground states of helium-5 and lithium-5. The Skyrmion has this shape up to a pion mass
When \( b = 0 \) the rational map \((87)\) has \(D_{4h}\) symmetry, and it acquires octahedral symmetry when in addition \( a = -5 \). Octahedral symmetry has been previously considered \([18]\) and it leads to a ground state with spin \( \frac{5}{2} \) and isospin \( \frac{1}{2} \). Let us therefore consider the \( D_{4h}\)-symmetric map (which could in fact be restricted to \( C_4 \) symmetry)

\[
R(z) = \frac{z(z^4 + a)}{a z^4 + 1}, \quad a \neq -5.
\]  

This map has the generating symmetries

\[
R(iz) = iR(z), \quad R(1/z) = 1/R(z),
\]

which lead to the FR constraints

\[
e^{i\frac{\pi}{2}L_3}e^{i\frac{\pi}{2}K_3}|\Psi\rangle = -|\Psi\rangle, \\
e^{i\pi L_3}e^{i\pi K_3}|\Psi\rangle = |\Psi\rangle.
\]

Seeking simultaneous solutions, we obtain a ground state with \( J = \frac{3}{2} \) and isospin \( I = \frac{1}{2} \), given by

\[
|\Psi\rangle = \begin{pmatrix} 3 & 3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \otimes \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \otimes \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}.
\]

This is the spin we are looking for. There are two excited states with \( J = \frac{5}{2} \) and \( I = \frac{1}{2} \), most easily written in terms of \( |J, I; M, M_3\rangle \), where \( M = \mathbf{L} + \mathbf{K} \):

\[
|\Psi\rangle = \left( \begin{pmatrix} 5 & 1 \\ 2 & 2 \end{pmatrix}; 3, 2 \right) + c_\pm \left( \begin{pmatrix} 5 & 1 \\ 2 & 2 \end{pmatrix}; 2, -2 \right),
\]

with \( c_\pm \) evaluated in Appendix C. The FR constraints allow for a further excited state with \( J = \frac{7}{2} \) and \( I = \frac{3}{2} \).

\( D_{4h}\) symmetry implies that the tensors of inertia are diagonal, with \( U_{11} = U_{22}, \ V_{11} = V_{22} \) and \( W_{11} = W_{22} \), which leads to the expression for the kinetic energy operator

\[
T = \frac{1}{2} \left\{ \frac{1}{(U_{11}V_{11} - W_{11}^2)} \left[ U_{11}(J^2 - L_3^2) + V_{11}(I^2 - K_3^2) + W_{11}(M^2 - J^2 - I^2 - 2L_3K_3) \right] \right. \\
+ \frac{1}{(U_{33}V_{33} - W_{33}^2)} \left[ U_{33}L_3^2 + V_{33}K_3^2 + 2W_{33}L_3K_3 \right] \right\}.
\]

The energy of the ground state is therefore

\[
E_{J=3/2, I=1/2} = M_5 + \frac{3U_{11} + V_{11}}{4(U_{11}V_{11} - W_{11}^2)} + \frac{9U_{33} + V_{33} + 6W_{33}}{8(U_{33}V_{33} - W_{33}^2)}.
\]

The numerical value of the energy depends on the parameter \( a \) in the rational map, which has yet to be determined.

We now argue that the \( D_{4h}\) symmetry which we are considering is justified even if octahedral symmetry (\( a = -5 \)) provides us with a slightly lower classical energy. The dependence of the classical energy on \( a \) is shown in Fig. 4, whereas the quantum energy is a strictly increasing function of \( a \) near \( a = -5 \) (see
Therefore the total energy achieves its minimum just below \( a = -5 \) (see Fig. 6), the quantum energy being much smaller than the classical one. Taking \( a = -5.0025 \) we find that

\[
U_{11} = 203.41, \quad U_{33} = 203.36, \quad V_{11} = 1333.49, \quad V_{33} = 1332.96, \quad W_{11} = -186.54, \quad W_{33} = -186.61. 
\] (97)

The static Skyrmion mass \( \mathcal{M}_5 \), for this value of \( a \), is calculated to be 5101 MeV. The energies of the four states given above are then:

\[
E_{J=3/2, I=1/2} = \mathcal{M}_5 + 8.2 \text{ MeV} = 5109 \text{ MeV},
\] (98)

\[
E_{J=5/2, I=1/2, c_-} = \mathcal{M}_5 + 12.5 \text{ MeV} = 5114 \text{ MeV},
\] (99)

\[
E_{J=5/2, I=1/2, c_+} = \mathcal{M}_5 + 12.9 \text{ MeV} = 5114 \text{ MeV},
\] (100)

\[
E_{J=1/2, I=3/2} = \mathcal{M}_5 + 26.9 \text{ MeV} = 5128 \text{ MeV}.
\] (101)

So, the achievement of the correct spin \( \frac{3}{2} \) for the ground state comes at a price. Firstly, the slightly excited \( J = \frac{1}{2} \) state is not allowed by the FR constraints. Secondly, by comparing \( E_{J=3/2, I=1/2} \) to the average mass of the helium-5 and lithium-5 nuclei (4668 MeV), we see that our prediction is some 10\% from the experimental value, whereas for the \( D_{2d} \)-symmetric Skyrmion there was an almost exact match to the helium-5/lithium-5 ground state energy.

The \( D_{4h} \)-symmetric map \((S)\) satisfies \(-1/R(z) = R(-1/z)\). The parity operator could therefore be represented by the identity operator. However, we may also choose \( P = e^{2\pi i n \cdot L} \), where \( n \) is any unit vector. If we make this choice, then each of the states described above has negative parity. We note that the ground states of helium-5 and lithium-5 have negative parities.

Figure 4: Classical energy of the \( B = 5 \) Skyrmion as a function of \( a \).

Figure 5: Quantum energy of the \( B = 5 \) Skyrmion as a function of \( a \).
The minimal-energy $B = 6$ Skyrmion has $D_{4d}$ symmetry, and is well-approximated using the rational map
\[ R(z) = \frac{z^4 + ia}{z^2(iaz^4 + 1)}. \] (102)

This map has the generating symmetries
\[ R(iz) = -R(z), \quad R(1/z) = 1/R(z), \] (103)

which lead to the FR constraints
\[ e^{i\frac{\pi}{2}L_3}e^{i\pi K_3}|\Psi\rangle = |\Psi\rangle, \] (105)
\[ e^{i\pi L_3}e^{i\pi K_3}|\Psi\rangle = -|\Psi\rangle. \] (106)

These constraints allow for the existence of states $|1, 0 \rangle \otimes |0, 0 \rangle$, $|3, 0 \rangle \otimes |0, 0 \rangle$, $|0, 0 \rangle \otimes |1, 0 \rangle$, $|2, 0 \rangle \otimes |1, 0 \rangle$ and $|5, 0 \rangle \otimes |0, 0 \rangle$.

A numerical search over the parameter $a$ in the rational map shows that the integral $I$ is minimized at $a = 0.16$. However, it was suggested in [23] that allowing a slight deformation of the rational map would lead to more accurate predictions. In particular, it was found that by setting $a = 0.1933$, and using the new parameter set, one obtains a quantum quadrupole moment in agreement with experiment. In what follows, we set $a = 0.1933$.

The inertia tensors have been computed for this rational map, and are each found to be diagonal, satisfying $U_{11} = U_{22}, V_{11} = V_{22}$ and $W_{11} = W_{22} = 0$. Numerically,
\[ U_{11} = 215.84, U_{33} = 230.77, V_{11} = 1525.99, V_{33} = 1493.66 \text{ and } W_{33} = -105.45. \] (107)

The kinetic energy operator is given by:
\[ T = \frac{1}{2V_{11}}[J^2 - L_3^2] + \frac{1}{2U_{11}}[I^2 - K_3^2] + \frac{1}{2(U_{33}V_{33} - W_{33}^2)}[U_{33}L_3^2 + V_{33}K_3^2 + 2W_{33}L_3K_3]. \] (108)
The static Skyrmion mass, $M_6$, is calculated to be 5601 MeV, which is precisely equal to the mass of the lithium-6 nucleus (the new parameter set was determined such that this would be the case – in [23] we estimated the spin energy for spin 1, isospin 0 to be approximately 1 MeV, and then neglected this small quantity). The energy eigenvalues corresponding to the five states given above are then:

\[
E_{J=1, I=0} = M_6 + \frac{1}{V_{11}} = M_6 + 1.7 \text{ MeV} = 5602 \text{ MeV},
\]

(109)

\[
E_{J=3, I=0} = M_6 + \frac{6}{V_{11}} = M_6 + 10.3 \text{ MeV} = 5611 \text{ MeV},
\]

(110)

\[
E_{J=0, I=1} = M_6 + \frac{1}{U_{11}} = M_6 + 12.1 \text{ MeV} = 5613 \text{ MeV},
\]

(111)

\[
E_{J=2, I=1} = M_6 + \frac{3}{V_{11}} = M_6 + 17.2 \text{ MeV} = 5618 \text{ MeV},
\]

(112)

\[
E_{J=5, I=0} = M_6 + \frac{15}{V_{11}} = M_6 + 25.7 \text{ MeV} = 5626 \text{ MeV}.
\]

(113)

We may identify these with isospin 0 states of lithium-6, and with states of the helium-6, lithium-6 and beryllium-6 nuclei, which together form an isospin triplet (see Fig. 7). The assumption in [23] that the spin kinetic energy of the state $|1,0\rangle \otimes |0,0\rangle$ is of order 1 MeV is clearly justified.

We may identify these with isospin 0 states of lithium-6, and with states of the helium-6, lithium-6 and beryllium-6 nuclei, which together form an isospin triplet (see Fig. 7). The assumption in [23] that the spin kinetic energy of the state $|1,0\rangle \otimes |0,0\rangle$ is of order 1 MeV is clearly justified.

The splitting between the various spin and isospin states of the Skyrmion is clearly too large; the predicted quantum energies are roughly four times the experimental values. This may be connected to the fact that lithium-6 is an odd-odd nucleus. However, we have performed the same calculation using the old parameter set, and have found that this gives even wider gaps between the energy levels. The new parameter set is therefore not perfect, but is certainly an improvement. Furthermore, the ratios of...
the relative excitation energies given by

$$\frac{E_{J=0, I=1} - E_{J=1, I=0}}{E_{J=3, I=0} - E_{J=1, I=0}} = 1.2$$

and

$$\frac{E_{J=2, I=1} - E_{J=1, I=0}}{E_{J=0, I=1} - E_{J=1, I=0}} = 1.5$$

correspond well to experimental data for these nuclei, for which the first ratio is $3.6/2.2 = 1.6$ and the second is $5.4/3.6 = 1.5$.

To determine the parities of the states given above we firstly observe the reflection symmetry

$$-1/R(e^{i\pi z}) = -iR(-1/\bar{z}).$$

The parity operator can therefore be represented as $\mathcal{P} = e^{iL^3}e^{-iK^3}$. If we make this choice, then each of the states given above has positive parity, in agreement with experiment.

12 $B = 7$

Here, as for $B = 5$, quantizing the Skyrmion of lowest energy gives states with the wrong spins to match the nuclear data. The minimal-energy $B = 7$ Skyrmion has icosahedral symmetry, and is described by the rational map

$$R(z) = \frac{7z^5 + 1}{z^2(z^5 - 7)}.$$  

(117)

This map leads to a ground state with $J = \frac{7}{2}, I = \frac{1}{2}$, a spin which appears experimentally as the second excited state of the lithium-7/beryllium-7 isospin doublet. Experimentally, the ground state has spin $\frac{3}{2}$.

There are many ways in which the icosahedral symmetry might be broken, allowing for the appearance of a $J = \frac{5}{2}, I = \frac{3}{2}$ state in the spectrum. The most interesting possibility, in our opinion, is the breaking of the $C_3$ symmetry, while preserving $D_5$ symmetry. This leads to a ground state with $J = \frac{5}{2}$ and $I = \frac{1}{2}$. So let us consider the $D_5$-symmetric map

$$R(z) = \frac{az^5 + 1}{z^2(z^5 - a)}, \quad a \neq 7.$$  

(118)
where $a = 7$ restores the icosahedral symmetry. The generating symmetries of this map are

$$
R(e^{i\frac{2\pi}{5} z}) = e^{-i\frac{2\pi}{5}} R(z),
$$

$$
R(-1/z) = -1/R(z),
$$

which lead to the FR constraints

$$
e^{i\frac{2\pi}{5} L_3} e^{-i\frac{2\pi}{5} K_3} |\Psi\rangle = -|\Psi\rangle,
$$

$$
e^{i\pi L_2} e^{i\pi K_2} |\Psi\rangle = -|\Psi\rangle.
$$

The ground state with $J = \frac{3}{2}$ and $I = \frac{1}{2}$ is

$$
|\Psi\rangle = \left| \frac{3}{2}, \frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle,
$$

the first excited state with $J = \frac{5}{2}$ and $I = \frac{1}{2}$ is

$$
|\Psi\rangle = \left| \frac{5}{2}, \frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle - \left| \frac{5}{2}, -\frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle,
$$

and there exist two further excited states with $J = \frac{7}{2}$, $I = \frac{1}{2}$, given by

$$
|\Psi^1\rangle = \left| \frac{7}{2}, \frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \left| \frac{7}{2}, -\frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle,
$$

$$
|\Psi^2\rangle = \left| \frac{7}{2}, \frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle - \left| \frac{7}{2}, -\frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle.
$$

States with $I = \frac{3}{2}$ are also allowed. In particular, there is one spin $\frac{1}{2}$ state:

$$
|\Psi\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{3}{2}, \frac{3}{2} \right\rangle - \left| \frac{1}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{3}{2}, -\frac{3}{2} \right\rangle,
$$

and two spin $\frac{3}{2}$ states:

$$
|\Psi^1\rangle = \left| \frac{3}{2}, \frac{3}{2} \right\rangle \otimes \left| \frac{3}{2}, \frac{1}{2} \right\rangle - \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \otimes \left| \frac{3}{2}, \frac{1}{2} \right\rangle,
$$

$$
|\Psi^2\rangle = \left| \frac{3}{2}, \frac{3}{2} \right\rangle \otimes \left| \frac{3}{2}, \frac{3}{2} \right\rangle + \left| \frac{3}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{3}{2}, -\frac{3}{2} \right\rangle.
$$

The inertia tensors are found to be diagonal, with $U_{11} = U_{22}$, $V_{11} = V_{22}$ and $W_{11} = W_{22} = 0$, leading to the kinetic energy operator:

$$
T = \frac{1}{2V_{11}} \left[ J^2 - L_3^2 \right] + \frac{1}{2U_{11}} \left[ Y^2 - K_3^2 \right] + \frac{1}{2(U_{33} V_{33} - W_{33}^2)} \left[ U_{33} L_3^2 + V_{33} K_3^2 + 2W_{33} L_3 K_3 \right].
$$

The energy of the ground state is given by

$$
E_{J=3/2, I=1/2} = \mathcal{M}_7 + \frac{3}{4V_{11}} + \frac{1}{4U_{11}} + \frac{9U_{33} + V_{33} - 6W_{33}}{8(U_{33} V_{33} - W_{33}^2)}.
$$

The static Skyrmion mass, $\mathcal{M}_7$, is found to be close to 6328 MeV. The dependence of the classical and quantum energies on $a$ are shown in Fig. 9 and Fig. 10 respectively. Looking for the value of $a$ giving the minimum of the total energy, we obtain $a = 7.002$ (see Fig. 11). For this value of $a$, we find numerically

$$
U_{11} = 246.27, \quad U_{33} = 246.26, \quad V_{11} = 1873.03, \quad V_{33} = 1872.76, \quad W_{33} = 0.04.
$$
Figure 9: Classical energy of the $B = 7$ Skyrmion as a function of $a$.

Figure 10: Quantum energy of the $B = 7$ Skyrmion as a function of $a$.

Figure 11: Total energy of the $B = 7$ Skyrmion as a function of $a$.

and

$$E_{J=3/2, I=1/2} = M_7 + 6.6 \text{ MeV} = 6335 \text{ MeV},$$

(133)
to be compared to the average mass of the lithium-7 and beryllium-7 nuclei which is 6534 MeV.

For the excited states we find that:

\[
E_{j=5/2, I=1/2} = \mathcal{M}_7 + 10.1 \text{ MeV} = 6338 \text{ MeV},
\]

(134)

\[
E_{j=7/2, I=1/2} = \mathcal{M}_7 + 15.0 \text{ MeV} = 6343 \text{ MeV},
\]

(135)

\[
E_{j=7/2, I=1/2}^2 = \mathcal{M}_7 + 15.0 \text{ MeV} = 6343 \text{ MeV},
\]

(136)

\[
E_{j=1/2, I=3/2} = \mathcal{M}_7 + 20.4 \text{ MeV} = 6348 \text{ MeV},
\]

(137)

\[
E_{j=3/2, I=3/2}^1 = \mathcal{M}_7 + 22.5 \text{ MeV} = 6351 \text{ MeV},
\]

(138)

\[
E_{j=3/2, I=3/2}^2 = \mathcal{M}_7 + 22.5 \text{ MeV} = 6351 \text{ MeV}.
\]

(139)

There are two main problems with the above spectrum. One is the absence of the \( J = \frac{1}{2}, I = \frac{1}{2} \) state, and the other is the appearance of the \( J = \frac{5}{2}, I = \frac{1}{2} \) state as the first excitation. We could try to overcome this problem by noticing that the first two excited states in the experimental lithium-7 and beryllium-7 spectra are in a sense anomalous: they have very low excitation energy, and the spin-energy correspondence is reversed. As was discussed in the introduction it is possible that such excitations cannot be described by our usual approach, and we need to allow for some vibrational modes or consider a Skyrmion of a different shape. Possibly, the states we find above could correspond to the ones lying above the lowest energy \( J = \frac{5}{2}, I = \frac{1}{2} \) excited state. This interpretation fits rather well to the experimental data. The second problem is more difficult to tackle within this framework. The value of \( a \) being very close to 7 leads to a configuration which is nearly \( C_3 \)-symmetric. This fact is reflected in the spectrum: we have two spin \( \frac{5}{2} \) and two isospin \( \frac{3}{2} \) states whose energies are almost indistinguishably close. This is not reflected in the experimental data. Let us therefore consider a smaller \( a \) and see if there is a better fit to the spectrum. Another advantage of this approach is that it helps to partially overcome the first problem as well. Indeed, by looking through a large range of \( a \) we find that at \( a = 2 \) the energies of the states given above are, in increasing order,

\[
E_{j=3/2, I=1/2} = \mathcal{M}_7 + 6.3 \text{ MeV},
\]

(140)

\[
E_{j=7/2, I=1/2}^2 = \mathcal{M}_7 + 9.3 \text{ MeV},
\]

(141)

\[
E_{j=5/2, I=1/2} = \mathcal{M}_7 + 9.4 \text{ MeV},
\]

(142)

\[
E_{j=7/2, I=1/2}^1 = \mathcal{M}_7 + 13.7 \text{ MeV},
\]

(143)

\[
E_{j=1/2, I=3/2} = \mathcal{M}_7 + 19.7 \text{ MeV},
\]

(144)

\[
E_{j=3/2, I=3/2}^1 = \mathcal{M}_7 + 19.9 \text{ MeV},
\]

(145)

\[
E_{j=3/2, I=3/2}^2 = \mathcal{M}_7 + 21.6 \text{ MeV}.
\]

(146)

The energy of the \( J = \frac{5}{2}, I = \frac{1}{2} \) state is now higher than the energy of one of the \( J = \frac{5}{2}, I = \frac{1}{2} \) states, and lower than that of the other, in agreement with experiment. This achievement, however, comes at a price as the classical energy is now some 10% higher than the experimental value. Figs. 12 and 13 are energy level diagrams for the quantized \( D_5 \)-symmetric \( B = 7 \) Skyrmion, with \( a = 2 \), and for the \( B = 7 \) nuclei, respectively.

The symmetry breaking we have just considered is only one of the ways in which icosahedral symmetry might be broken. It is possible that a different breaking has to be considered in order to better understand the spectrum of the excited states, in particular the low-lying \( J = \frac{1}{2}, I = \frac{1}{2} \) state. It is also possible that with the increase of the pion mass the configuration will eventually break up into a \( B = 4 \) and a \( B = 3 \) part, which is suggested by the very low energy for break up of lithium-7 into helium-4 plus a triton.

The \( D_5 \)-symmetric map \( \Gamma(138) \) satisfies \( -1/R(z) = R(-1/\bar{z}) \). As for \( B = 5 \), we find it favourable to choose \( \mathcal{P} = e^{\pm \pi m \hat{n}} \), where \( \hat{n} \) is any unit vector. If we make this choice, then each of the states described above has negative parity, in agreement with experiment.

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Figure 12: Energy level diagram for the quantized $B = 7$ Skyrmion. A putative \( J = \frac{1}{2}^- \) isoquartet is represented by dashed lines.

Figure 13: Energy level diagram for nuclei with $B = 7$.

13 \( B = 8 \)

In this section we introduce some new ideas for estimating the moments of inertia of the $B = 8$ Skyrmion, and hence the excitation energies of the quantum states. It is believed that for our new parameter set, the minimal-energy classical solution resembles two touching $B = 4$ cubes (see Fig. 14) \cite{4}. Here the rational map ansatz is not a good approximation, so our previous methods of calculation are no longer valid. Despite this, it is convenient to note that a field which is qualitatively of the right form, with the correct symmetries, can be obtained from a rational map, and this enables one to determine the allowed spin/isospin/parity states. There are also classical Skyrmion solutions which are well approximated by the rational map ansatz, and have only very slightly greater energy than the double cube. We consider these first.
For pion mass parameter between 0 and approximately 1, the minimal-energy $B = 8$ Skyrmion has $D_{6d}$ symmetry, and is well-approximated by the rational map

$$R(z) = \frac{z^6 - ia}{z^2(iaz^6 - 1)}, \quad a = 0.14,$$

which has the symmetries

$$R\left(e^{i\pi z}\right) = e^{-i\pi R(z)}, \quad (148)$$
$$R(1/z) = 1/R(z), \quad (149)$$

leading to the FR constraints

$$e^{i\pi L_3}e^{-i\pi K_3}|\Psi\rangle = |\Psi\rangle, \quad (150)$$
$$e^{i\pi L_1}e^{i\pi K_1}|\Psi\rangle = |\Psi\rangle. \quad (151)$$

The ground state is then determined to be $|0, 0\rangle \otimes |0, 0\rangle$, and the first excited state is $|2, 0\rangle \otimes |0, 0\rangle$, in agreement with states of the beryllium-8 nucleus. However, this Skyrmion becomes unstable once the pion mass parameter exceeds 1. The true minimum is then described by two $B = 4$ cubes placed together, and as a first approximation to this in terms of a rational map we consider the $O_h$-symmetric map (whose Wronskian vanishes on the 14 faces of a truncated octahedron):

$$R(z) = \frac{z^8 + 4\sqrt{3}z^6 - 10z^4 + 4\sqrt{3}z^2 + 1}{z^8 - 4\sqrt{3}z^6 - 10z^4 - 4\sqrt{3}z^2 + 1}, \quad (152)$$
whose symmetries

$$R(iz) = 1/R(z), \quad (153)$$
$$R\left(\frac{iz + 1}{-iz + 1}\right) = -\sqrt{3} + R(z) \frac{1 + \sqrt{3} R(z)}{1}, \quad (154)$$

lead to the FR constraints

$$e^{i\pi L_3}e^{-i\pi K_3}|\Psi\rangle = |\Psi\rangle, \quad (155)$$
$$e^{i\pi (L_1 + L_2 + L_3)}e^{-i\pi K_2}|\Psi\rangle = |\Psi\rangle. \quad (156)$$

Here the ground state is again $|0, 0\rangle \otimes |0, 0\rangle$, but the $|2, 0\rangle \otimes |0, 0\rangle$ state is not allowed. The inertia tensors for this rational map are found to be diagonal, satisfying $U_{11} = U_{33}$, $V_{11} = V_{22} = V_{33}$ and $W_{ij} = 0$.

However, the $O_h$ symmetry is too strong for the description of two cubes, and has to be relaxed to $D_{4h}$ symmetry. Therefore we consider next

$$R(z) = \frac{z^8 + b^2 - a z^4 + b z^2 + 1}{z^8 - b^2 - a z^4 - b z^2 + 1}, \quad (157)$$
where $a = 10$ and $b = 4\sqrt{3}$ restores the $O_h$ symmetry. The rational map ansatz then gives a better approximation to the double cube Skyrmion, but only slightly because, for example, $U = -1$ at the origin with the rational map ansatz, whereas for the true solution, $U = -1$ at points near the cube centres. However, it has the right symmetry, and is good enough to determine the allowed spin/isospin states. The FR constraints are now

$$e^{i\pi L_3}e^{-i\pi K_1}|\Psi\rangle = |\Psi\rangle, \quad (158)$$
$$e^{i\pi L_1}|\Psi\rangle = |\Psi\rangle. \quad (159)$$
simplifies (160) to
\[ U = \frac{1}{2V_{11}} [J^2 - L_3^2] + \frac{L_4^2}{2V_{33}} + \frac{K_1^2}{2U_{11}} + \frac{K_2^2}{2U_{22}} + \frac{K_3^2}{2U_{33}}. \]  
(160)

To determine the parities of states, we observe the rational map (157) has the reflection symmetry
\[ -1/R(z) = -1/R(-1/\pi). \]  
The parity operator can therefore be represented by \( P = e^{i\pi K_2}. \)

To progress, we now work directly with two cubic \( B = 4 \) Skyrmions separated along the \( x_3 \)-axis, and find the moments of inertia of the resulting structure using the parallel axis theorem (ignoring the interaction of the cubes). The top cube is rotated by \( \frac{\pi}{4} \) about the \( x_3 \)-axis relative to the standard orientation. The bottom cube is rotated by \( -\frac{\pi}{4} \) about the \( x_3 \)-axis relative to the standard orientation. One difficulty here is in determining the separation of the cubes. The picture in Fig. 14 suggests that the separation is the value of \( r \) where the profile function becomes close to zero. From Fig. 2 we see that it is reasonable to take \( r = 1.8 \) leading to the separation in question being
\[ d = r/\sqrt{3} = 1.04 \] in dimensionless units. Then
\[ V_{11}^{(B=8)} = V_{22}^{(B=8)} = 2V_{11}^{(B=4)} + \mathcal{M}d^2 = 2706, \]  
(161)
\[ V_{33}^{(B=8)} = 2V_{33}^{(B=4)} = 1326. \]  
(162)
where \( \mathcal{M} = 1277 \) (in dimensionless units) is the classical mass of two \( B = 4 \) Skyrmions. The isospin moments of inertia are simply given by
\[ U_{11}^{(B=8)} = U_{22}^{(B=8)} = 2U_{11}^{(B=4)} = 286, \]  
(163)
\[ U_{33}^{(B=8)} = 2U_{33}^{(B=4)} = 339. \]  
(164)

The equality of \( U_{11} \) and \( U_{22} \), which we do not expect to be exactly satisfied by the true \( B = 8 \) solution, simplifies (160) to
\[ T = \frac{1}{2V_{11}} [J^2 - L_3^2] + \frac{1}{2U_{11}} [I^2 - K_3^2] + \frac{L_4^2}{2V_{33}} + \frac{K_2^2}{2U_{33}}. \]  
(165)

The ground state has quantum energy zero, so its total energy is simply the classical Skyrmion mass.

It remains worthwhile to find the optimal values of \( a \) and \( b \) in the rational map (157). Ideally, the classical mass should not be very far away from the experimental mass of the beryllium-8 ground state which is 7455 MeV, and the moments of inertia should be comparable with the ones we get from the double cube approach. The second condition is more difficult to achieve since the rational map is defined on a sphere, and cannot exactly reproduce a double cube configuration. Looking through a range of possible \( a \) and \( b \) values we find that the optimal map is given approximately by
\[ R(z) = \frac{z^8 + \frac{13\sqrt{2}}{2}z^6 - 20z^4 + \frac{13\sqrt{2}}{2}z^2 + 1}{z^8 - \frac{13\sqrt{2}}{2}z^6 - 20z^4 - \frac{13\sqrt{2}}{2}z^2 + 1}. \]  
(166)
leading to the following moments of inertia

\[ V_{11} = V_{22} = 2901, \quad (167) \]
\[ V_{33} = 2214, \quad (168) \]
\[ U_{11} = 308, \quad (169) \]
\[ U_{22} = 268, \quad (170) \]
\[ U_{33} = 283. \quad (171) \]

We have recalculated the energies of the states in Table 1, using the kinetic energy operator (160) and formulae for the energy levels of an asymmetrical top [19]. The energy of the first excited state is

\[ E_{J=2, I=0} = M_8 + \frac{3}{V_{11}} = 7529\text{MeV} + 2.7\text{MeV} = 7531\text{MeV}, \quad (172) \]

which is only slightly worse than the double cube approach. However, for further excited states the discrepancy in results increases, making the advantages of the double cube approach more evident.

Figure 14: Baryon density isosurface for the numerically relaxed \( B = 8 \) Skyrmion with \( m \approx 1 \), resembling two touching \( B = 4 \) Skyrmions.

Figure 15: Energy level diagram for the quantized \( B = 8 \) Skyrmion, using the double cube approach. A putative \( J = 0^- \) isotriplet is represented by dashed lines.
Table 1: Energies and parities of the obtained using the double cube approach and rational map ansatz respectively.

| J, I | Wavefunction | Parity | $E_{dc}$ (MeV) | $E_{rm}$ (MeV) |
|------|--------------|--------|----------------|----------------|
| 0, 0 | $(0, 0) \otimes (0, 0)$ | + | 0 | 0 |
| 2, 0 | $(2, 0) \otimes (0, 0)$ | + | 2.9 | 2.7 |
| 4, 0 | $(4, 0) \otimes (0, 0)$ | + | 9.7 | 9.0 |
| | $(4, 4) + |4, -4|) \otimes (0, 0)$ | + | 17.7 | 11.2 |
| 0, 1 | $(0, 0) \otimes ((1, 1) - |1, -1|)$ | − | 8.4 | 9.5 |
| 2, 1 | $(2, 0) \otimes ((1, 1) - |1, -1|)$ | − | 11.3 | 12.2 |
| | $(2, 2) + |2, -2| \otimes ((1, 1) + |1, -1|)$ | + | 13.3 | 12.1 |
| | $(2, 2) + |2, -2| \otimes |1, 0)$ | − | 14.1 | 12.4 |
| 3, 1 | $(|3, 2| - |3, -2|) \otimes ((1, 1) + |1, -1|)$ | + | 16.2 | 14.8 |
| | $(|3, 2| - |3, -2|) \otimes |1, 0)$ | − | 16.9 | 15.1 |
| 4, 1 | $(|4, 0| \otimes ((1, 1) - |1, -1|)$ | − | 18.1 | 18.5 |
| | $(|4, 2| + |4, -2|) \otimes ((1, 1) + |1, -1|)$ | + | 20.1 | 18.4 |
| | $(|4, 2| + |4, -2|) \otimes |1, 0)$ | − | 20.8 | 18.7 |
| | $(|4, 4| + |4, -4|) \otimes ((1, 1) - |1, -1|)$ | − | 26.1 | 20.7 |
| 0, 2 | $(0, 0) \otimes (|2, 2| + |2, -2|)$ | + | 24.6 | 26.3 |
| | $(|0, 0| \otimes (|2, 2| + |2, -2|)$ | − | 26.7 | 26.4 |
| | $(|0, 0| \otimes |2, 0)$ | + | 27.4 | 28.6 |
| 2, 2 | $(|2, 0| \otimes (|2, 2| + |2, -2|)$ | + | 27.5 | 29.0 |
| | $(|2, 2| + |2, -2|) \otimes |2, 2| - |2, -2|)$ | − | 29.5 | 30.8 |
| | $(|2, 0| \otimes (|2, 1| + |2, -1|)$ | − | 29.6 | 29.1 |
| | $(|2, 0| \otimes |2, 0)$ | + | 30.3 | 31.3 |
| | $(|2, 2| + |2, -2|) \otimes (|2, 1| - |2, -1|)$ | + | 31.6 | 31.6 |

Figure 16: Energy level diagram for nuclei with $B = 8$. 
The calculations presented here are subject to a number of limitations. Firstly, we consider the semiclassical quantization, in which only the collective coordinates for rotations and isospin rotations are considered. A more accurate procedure would have to take into account further degrees of freedom, which we refer to as vibrational modes. Our current understanding is that the Skyrme model provides a description of nuclear physics in which nucleons are partially merged, and their orientations in space and isospace are highly correlated. In a sense, this is the opposite of a naive shell model, in which nucleons move in a potential, and are to first approximation uncorrelated. A more realistic model would possibly lie somewhere between these two extremes. Allowing the individual Skyrmions, or subclusters of Skyrmions, to move relative to each other, and performing a quantization of these degrees of freedom, would be a significant refinement to our approach. In so doing, some missing low-lying experimentally observed states of nuclei may appear. These include the low-lying excited states with $J = \frac{1}{2}$ and $I = \frac{1}{2}$ that are present for $B = 5$ and $B = 7$.

The work here should be taken further by working with the exact Skyrmion solutions, and not just the rational map approximation to these solutions. Classical energies and moments of inertia will change, though we hope not drastically. Further investigation of the effect of varying the dimensionless pion mass parameter is also warranted. The length scale of the Skyrmions is quite sensitive to this. Possibly, an increased parameter will create an instability in the Skyrmions with $B = 5$ or $B = 7$, thereby justifying our arguments for changing the symmetries.
Appendix: Inertia Tensors

The tensors of inertia for rational map Skyrmions may be expressed in the form:

\[ \Sigma_{ij} = 2 \int \sin^2 f \frac{C_{\Sigma_{ij}}}{(1 + |R|^2)^2} \left( 1 + f'^2 + \frac{\sin^2 f}{r^2} \left( \frac{1 + |z|^2}{1 + |R|^2} \frac{dR}{dz} \right)^2 \right) d^3x, \tag{173} \]

where \( \Sigma = (U, V, W) \) and the quantities \( C_{U_{ij}} \) are given by

\[
C_{U_{11}} = |1 - R^2|^2, \tag{174}
C_{U_{22}} = |1 + R^2|^2, \tag{175}
C_{U_{33}} = 4|R|^2, \tag{176}
C_{U_{12}} = C_{U_{21}} = -2 \Im R^2, \tag{177}
C_{U_{13}} = C_{U_{31}} = 2 (|R|^2 - 1) \Re R, \tag{178}
C_{U_{23}} = C_{U_{32}} = 2 (|R|^2 - 1) \Im R, \tag{179}
\]

the quantities \( C_{V_{ij}} \) are given by

\[
C_{V_{11}} = |1 - z^2|^2 \left| \frac{dR}{dz} \right|^2, \tag{180}
C_{V_{22}} = |1 + z^2|^2 \left| \frac{dR}{dz} \right|^2, \tag{181}
C_{V_{33}} = 4|z|^2 \left| \frac{dR}{dz} \right|^2, \tag{182}
C_{V_{12}} = C_{V_{21}} = -2 \Im z^2 \left| \frac{dR}{dz} \right|^2, \tag{183}
C_{V_{13}} = C_{V_{31}} = 2 \Re \left( |z|^2 z - \bar{z} \right) \left| \frac{dR}{dz} \right|^2, \tag{184}
C_{V_{23}} = C_{V_{32}} = 2 \Im \left( |z|^2 z + \bar{z} \right) \left| \frac{dR}{dz} \right|^2, \tag{185}
\]
and finally, the quantities \( C_{W_{ij}} \) are given by

\[
C_{W_{11}} = \Re \left( (1 - z^2)(1 - R^2) \frac{dR}{dz} \right),
\]

(186)

\[
C_{W_{22}} = \Re \left( (1 + z^2)(1 + R^2) \frac{dR}{dz} \right),
\]

(187)

\[
C_{W_{33}} = 4 \Re \left( \bar{R} \frac{dR}{dz} \right),
\]

(188)

\[
C_{W_{12}} = -\Im \left( (1 + z^2)(1 - R^2) \frac{dR}{dz} \right),
\]

(189)

\[
C_{W_{13}} = -2 \Re \left( z(1 - \bar{R}^2) \frac{dR}{dz} \right),
\]

(190)

\[
C_{W_{23}} = -2 \Im \left( z(1 + \bar{R}^2) \frac{dR}{dz} \right),
\]

(191)

\[
C_{W_{21}} = \Im \left( (1 - z^2)(1 + \bar{R}^2) \frac{dR}{dz} \right),
\]

(192)

\[
C_{W_{31}} = -2 \Re \left( \bar{R}(1 - z^2) \frac{dR}{dz} \right),
\]

(193)

\[
C_{W_{32}} = 2 \Im \left( \bar{R}(1 + z^2) \frac{dR}{dz} \right).
\]

(194)

\[B\quad \text{Appendix: Old Parameters}\]

Here we collect some data on moments of inertia, in Skyrme units, calculated with the dimensionless pion mass parameter \( m = 0.528 \) that emerges from the calibration of [1]. The following results are novel, as they were obtained using the rational map ansatz and the formulae in Appendix A, and extend from \( B = 1 \) up to \( B = 4 \). For \( B = 1 \) the rational map ansatz is exact, so our result should agree with that of [1], and indeed it does. For \( B = 2, 3 \) our results can be compared with the moments of inertia calculated from the exact Skyrmion solutions (with the same \( m \)) as given by [7, 8]. This allows us to investigate the accuracy of the rational map ansatz for these Skyrmions.

The notation is as in the Sections 6 to 9 above. For \( B = 1 \)

\[
\lambda = 62.85.
\]

(195)

For \( B = 2 \)

\[
U_{11} = 135.43, \; U_{33} = 86.59 \; \text{and} \; V_{11} = 221.88.
\]

(196)

Comparing these numbers to those obtained in [7] using the exact numerical solution \((U_{11} = 127.8, \; U_{33} = 86.9 \; \text{and} \; V_{11} = 200.2)\), we see that the rational map ansatz has enabled us to obtain quite accurate moments of inertia. We recall that the old parameter set led to a model of the deuteron which was much too tightly bound.

For \( B = 3 \)

\[
u = 170.01, \; v = 576.09 \; \text{and} \; w = -109.47.
\]

(197)

These were evaluated in [8], using the exact numerical solution \((u = 136, \; v = 435 \; \text{and} \; w = -91)\).

For \( B = 4 \)

\[
U_{11} = 197.60, \; U_{33} = 236.49 \; \text{and} \; v = 911.45.
\]

(198)

31
These numbers were calculated using the same procedure that we have used throughout, but with the old parameters. Without \[28\] performed a different style of analysis for the $B = 4$ Skyrmion, and unfortunately we are unable to directly compare our results for the individual components of the inertia tensors.

C Appendix: Coefficients of Wavefunctions

The FR constraints do not determine all coefficients in the wavefunctions. Usually finding these constants is trivial, but in some cases (as in the $B = 5$ first and second excited states) one has to be more careful. As an illustration let us consider the constants $c_{\pm}$ in \[94\]. The solutions of \(91\,92\) form a subspace of Hilbert space, which is transformed into itself when acted upon by the operator of the kinetic energy \(10\). Therefore, the eigenvectors of the operator will define the wavefunctions we are looking for. In terms of the moments of inertia, $c_{\pm}$ is given by

$$c_{\pm} = \frac{b_2 + 5b_1 - a_1 - 5a_2 \pm \sqrt{(b_2 + 5b_1 - a_1 - 5a_2)^2 + 20(a_1 - a_2)(b_1 - b_2)}}{2\sqrt{5}(b_1 - b_2)}$$ \(199\)

where

$$a_1 = \frac{1}{8} \left( \frac{10U_{11} + 2V_{11} + 20W_{11}}{U_{11}V_{11} - W_{11}^2} + \frac{25U_{33} + V_{33} - 10W_{33}}{U_{33}V_{33} - W_{33}^2} \right),$$ \(200\)

$$a_2 = \frac{1}{8} \left( \frac{26U_{11} + 2V_{11} + 4W_{11}}{U_{11}V_{11} - W_{11}^2} + \frac{9U_{33} + V_{33} + 6W_{33}}{U_{33}V_{33} - W_{33}^2} \right),$$ \(201\)

$$b_1 = \frac{1}{8} \left( \frac{10U_{11} + 2V_{11} - 4W_{11}}{U_{11}V_{11} - W_{11}^2} + \frac{25U_{33} + V_{33} - 10W_{33}}{U_{33}V_{33} - W_{33}^2} \right),$$ \(202\)

$$b_2 = \frac{-1}{8} \left( \frac{26U_{11} + 2V_{11} - 20W_{11}}{U_{11}V_{11} - W_{11}^2} + \frac{9U_{33} + V_{33} + 6W_{33}}{U_{33}V_{33} - W_{33}^2} \right).$$ \(203\)

The quantum energy of these states is given by

$$E = a_1 + 5a_2 + c_\pm \sqrt{5(b_1 - b_2)}.$$ \(204\)

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