The antisymmetric tensor propagator in $AdS$

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Abstract

In this brief note we construct the propagator for the antisymmetric tensor in $AdS_{d+1}$. We check our result using the Poincaré duality between the antisymmetric tensor and the gauge boson in $AdS_5$. This propagator was needed for a computation which turned out to be too hard. It can be used for computing various other things in $AdS$. 
I. INTRODUCTION

In [1,2], a lot of effort was put into finding the AdS propagators for the graviton and the gauge boson. Their methods can be used straightforwardly for the $B_{\mu\nu}$ propagators. An ansatz can be made for bitensor propagators [3]. This ansatz contains both gauge artifacts and gauge invariant parts. Upon using the equation of motion for $B_{\mu\nu}$ we obtain an equation for the gauge invariant part of the propagator, whose solution is hypergeometric. For $d=5$ it simplifies to an algebraic function of the chordal distance. As explained in [1], working on the subspace of conserved sources makes gauge fixing unnecessary. We check our result by verifying the 5-dimensional Poincaré duality between $A_\mu$ and $B_{\mu\nu}$.

II. THE $B_{\mu\nu}$ PROPAGATOR

In Euclidean $AdS_{d+1}$, with the metric

$$ds^2 = \frac{1}{z^2_0}(dz_0^2 + \sum_{i=1}^{d} dz_i^2),$$

the easiest way to express invariant functions and tensors is in terms of the chordal distance:

$$u \equiv \frac{(z_0 - w_0)^2 + (z_i - w_i)^2}{2z_0 w_0}. \quad (2)$$

The action for an antisymmetric 2-tensor coupled to a conserved source $S_{\mu\nu}$ is:

$$S_B = \int d^{d+1}z \sqrt{g} \left[ \frac{1}{2} \cdot \frac{1}{3!} H^{\mu\nu\rho} H_{\mu\nu\rho} - \frac{1}{2} B_{\mu\nu} S^{\mu\nu} \right], \quad (3)$$

where

$$H_{\mu\nu\rho} = D_\mu B_{\nu\rho} + D_\nu B_{\rho\mu} + D_\rho B_{\mu\nu}. \quad (4)$$

The Euler Lagrange equation has a solution of the form:

$$B_{\mu\nu}(z) = \frac{1}{2} \int d^{d+1}w \sqrt{g} G_{\mu\nu;\mu'\nu'}(z, w) S^{\mu'\nu'}(w), \quad (5)$$

where $G_{\mu\nu;\mu'\nu'}$ is the bitensor propagator. To simplify notation, the $D$'s with unprimed indices mean covariant derivatives with respect to $z$, and those with primed indices with respect to $w$. The equation $G_{\mu\nu;\mu'\nu'}$ satisfies:

$$D^\rho(D_\mu G_{\nu\rho;\mu'\nu'} + D_\nu G_{\rho\mu;\mu'\nu'} + D_\rho G_{\mu\nu;\mu'\nu'}) = -\delta(z, w)(g_{\mu\nu} g_{\mu'\nu'} - g_{\mu\nu'} g_{\nu\mu'}) +$$

$$+D_{\mu'} \Lambda_{\nu;\mu'} - D_{\nu'} \Lambda_{\mu;\nu'}, \quad (6)$$

where $\Lambda_{\mu;\nu'}$ is a diffeomorphism whose contribution vanishes when integrated against the covariantly conserved source $S^{\mu\nu}$. We can see that all of our bitensors are antisymmetric at both points.

Similarly to the methods in [1] we observe that a suitable basis for antisymmetric bitensors is given by:

$$T^1_{\mu\nu;\mu'\nu'} = \partial_\mu \partial_{\mu'} u \partial_\nu \partial_{\nu'} u - \partial_\mu \partial_{\nu'} u \partial_\nu \partial_{\mu'} u \quad (7)$$
Thus, an ansatz for $G$ is $G = T^1F^1(u) + T^2F^2(u)$. Nonetheless, we use a different decomposition, which illustrates better the gauge artifacts

$$G_{\mu\nu;\mu'\nu'} = T^1_{\mu\nu;\mu'\nu'}H(u) + D_\mu V_{\nu;\mu'\nu'} - D_\nu V_{\mu;\mu'\nu'},$$

where $V_{\mu;\mu'\nu'} = Y(u)[\partial_\mu\partial_\nu u\partial_{\mu'}u - \partial_\mu\partial_{\nu'} u\partial_{\mu'}u - \partial_\nu\partial_{\mu'} u\partial_\nu u + \partial_\nu\partial_{\nu'} u\partial_\nu u].$ Also, an antisymmetric $\Lambda_{\mu\nu;\mu'}$ can be expressed as

$$\Lambda_{\mu\nu;\mu'} = A(u)[\partial_\nu\partial_{\mu'} u\partial_\mu u - \partial_\nu\partial_\mu u\partial_{\nu'} u].$$

We can now substitute (8) and (10) in (6), and after a long computation we obtain

$$D^\rho(D_\mu G_{\nu\rho;\mu'\nu'} + D_\nu G_{\rho\mu;\mu'\nu'}) - D_\mu\Lambda_{\mu\nu;\mu'} + D_\nu\Lambda_{\mu\nu;\mu'} =$$

$$= T^1[H''u(u + 2) + H'(1 + u)(d - 1) - 2A] - T^2[H''(1 + u) + H'(d - 1) + A'].$$

For $z \neq w$, we obtain 2 equations by setting the scalar coefficients of the two tensors to 0. We can observe that the $V_{\mu;\mu'\nu'}$ part which was a gauge artifact dropped out as expected. Thus, for $u \neq 0$ we have the equations:

$$H''u(u + 2) + H'(1 + u)(d - 1) - 2A = 0 \quad (12a)$$

$$H'(1 + u) + H'(d - 1) + A' = 0. \quad (12b)$$

The second equation can be integrated once, with the integration constant chosen so that $A$ and $H$ vanish as $u \to \infty$. Combining this with (12a) we find the differential equation obeyed by $H$:

$$u(2 + u)H''(u) + (d + 1)(u + 1)H'(u) + 2(d - 2)H = 0. \quad (13)$$

This equation is hypergeometric, but the solution which vanishes as $u \to \infty$ is rational:

$$H(u) = \frac{\Gamma((d - 1)/2)}{4\pi^{(d+1)/2}} \frac{u + 1}{[u(u + 2)]^{(d-1)/2}}, \quad (14)$$

properly normalized to take care of the $\delta$ function in (6).

### III. POINCARÉ DUALITY

In 5 dimensions a 2-form is Poincaré dual with a gauge boson, by the relation:

$$H_{\mu\nu\rho\epsilon^{\mu\nu\rho\sigma\lambda}} = 3! F^{\sigma\lambda} \quad (15)$$

Therefore, we expect:

$$\langle F^{\sigma\lambda}(z)F^{\sigma'\lambda'}(w)\rangle = \frac{1}{(3!)^2} \epsilon^{\mu\nu\rho\sigma\lambda} \epsilon_{\mu'\nu'\rho'\sigma'\lambda'} \langle H_{\mu\nu\rho}(z)H_{\mu'\nu'\rho'}(w)\rangle. \quad (16)$$

Checking (16) is a verification that our result is true. We use the fact that

$$\langle B_{\mu\nu}B_{\mu'\nu'}\rangle = G_{\mu\nu;\mu'\nu'} \quad (17)$$
and
\[ \langle A_\mu A_{\mu'} \rangle = G_{\mu \mu'}, \] (18)
where the second propagator was found in [2]. We could check the tensor equality (16) term by term, but it is messy. We rather observe that the right hand side of (16) is a bitensor antisymmetric at both ends, and thus it will have the structure
\[ \epsilon^{\mu \nu \rho \lambda} \epsilon^{\mu' \nu' \rho' \lambda'} \langle H_{\mu \nu \rho \lambda}(z) H_{\mu' \nu' \rho' \lambda'}(w) \rangle = F_1(u) T_1^{\mu \nu \rho \lambda} + F_2(u) T_2^{\mu \nu \rho \lambda}. \] (19)
Concentrating on the components of \( \langle F_{z_0 z_i}(z) F_{z_0 z_i}(w) \rangle \) we obtain:
\[ 2F_1 + F_2(1 + u) = H'', \] (20a)
\[ F_2(1 + u)^2 + 2F_1(1 + u) = 2H''(1 + u) + 3H', \] (20b)
which give the same \( F_1 \) and \( F_2 \) as the ones obtained from the gauge propagator derived in [2].

IV. CONCLUSION

We computed the propagator for \( B_{\mu \nu} \) in \( AdS_{d+1} \) and checked our result by using Poincaré duality for \( d = 4 \). This propagator can be used for computing various quantities having to do with \( B_{\mu \nu} \) charged objects (like strings or D-branes with electric flux) in \( AdS \). The propagators for higher form fields can also be found by using Poincaré duality [4] or by explicit calculation [5].

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APPENDIX A: SEVERAL USEFUL IDENTITIES INVOLVING THE CHORDAL DISTANCE

In the computations the following identities were useful:
\[ \partial_\mu \partial_{\nu'} u = -\frac{1}{z_0 w_0} \left[ \delta_{\mu \nu'} + \frac{(z - w)_\mu \delta_{\nu' 0}}{w_0} + \frac{(w - z)_{\nu'} \delta_{\mu 0}}{z_0} - u \delta_{\mu 0} \delta_{\nu' 0} \right] \] (A1)
\[ \partial_\mu u = \frac{1}{z_0} [(z - w)_\mu / w_0 - u \delta_{\mu 0}] \] (A2)
\[ \partial_{\nu'} u = \frac{1}{w_0} [(w - z)_{\nu'} / z_0 - u \delta_{\nu' 0}] \] (A3)
\[ D^\mu \partial_\mu u = (d + 1)(u + 1) \] (A4)
\[ \partial^\mu u \partial_\mu u = u(u + 2) \] (A5)
\[ D_\mu \partial_\nu u = g_{\mu \nu} (u + 1) \] (A6)
\[ (\partial^\mu u)(D_\mu \partial_\nu \partial_{\nu'} u) = \partial_\nu u \partial_{\nu'} u \] (A7)
\[ (\partial^\mu u)(\partial_\mu \partial_{\nu'} u) = (u + 1) \partial_{\nu'} u \] (A8)
\[ D_\mu \partial_\nu \partial_{\nu'} u = g_{\mu \nu} \partial_{\nu'} u \] (A9)
REFERENCES

[1] Eric D’Hoker, Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, Leonardo Rastelli, Nucl.Phys. B562 (1999) 330-352; hep-th/9902042
[2] Eric D’Hoker, Daniel Z. Freedman, Nucl.Phys. B544 (1999) 612-632; hep-th/9809179
[3] B. Allen and T. Jacobson, Commun. Math. Phys. 103 (1986) 669.
[4] Iosif Bena; hep-th/9911073
[5] Asad Naqvi, JHEP 9912 (1999) 025; hep-th/9911182